Dimensional Regularization and Dispersive Two-Loop Calculations

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The two-loop contributions are now often required by the precision experiments, yet are hard to express analytically while keeping precision. One way to approach this challenging task is via the dispersive approach, allowing to replace sub-loop diagram by effective propagator. This paper builds on our previous work, where we developed a general approach based on representation of many-point Passarino-Veltman functions in two-point function basis. In this work, we have extracted the UV-divergent poles of the Passarino-Veltman functions analytically and presented them as the dimensionally-regularized and multiply-subtracted dispersive sub-loop insertions, including self-energy, triangle, box and pentagon type.

I. INTRODUCTION

The electroweak precision searches for the physics beyond the Standard Model (BSM) frequently demand a sub-percent level of accuracy from both experiment and theory. For the new-generation precision experiments such as MOLLER [1] and P2 [2], for example, that means evaluating electroweak radiative corrections up to two-loop level with massive propagators and control of kinematics, which is a highly challenging task. In some cases, it may not possible to express the final results analytically, so one would have to use approximations and/or numerical methods. See, for example, an overview of numerical loop integration techniques in [3], a general case of the two-loop two-point function for arbitrary masses in [4], and a method of calculating scalar propagator and vertex functions based on a double integral representation in [5] and [6]. The more recent developments on analytical evaluation of two-loop self-energies can be found in [7–12], and on numerical evaluation of general n-point two-loop integrals using sector decomposition in [13, 14]. The idea of the sub-loop insertions with the help of the dispersive approach was implemented for the self-energies [15], [18] and partially for the vertex graphs with the help of Feynman parametrization [19]. A somewhat relevant case of the self-energy dispersive insertions for Bhabha scattering in QED was considered in [20] and [21].
In [22, 23], we have developed a general approach in calculations of the two-loops diagrams, which is based on the representation of many-point Passarino-Veltman (PV) functions in two-point function basis. As a result, we were able to replace a sub-loop integral by the dispersive representation of the two-point function. In that case, the second loop received an additional propagator and we were able to use the PV basis for the second loop integration in the final stage of the calculations. The final results were presented in a compact analytic form suitable for numerical evaluation. Since in the majority of applications such two-loops integrals are either ultraviolet or infrared (IR) divergent, a regularization scheme is required. In case of the IR-divergence, the regularization can be done by introducing a small mass of the photon which is later removed by a contribution of a combination of one-photon bremsstrahlung from one-loop and two-photon bremsstrahlung from tree level diagrams. Since the IR-divergence does not impact convergence of the dispersion sub-loop integral, the mass of the photon in the insertion could be carried into second loop without an additional complication. If necessary, the dependence on the photon mass can be extracted analytically. For the UV-divergent two-loops diagrams, the regularization of the sub-loop insertion is done by an introduction of a cut-off parameter for the divergent dispersive integral. The second-loop regularization is done by dimensional regularization, but in this case, when counter terms are added, one set of renormalization constants is evaluated in dispersive approach with a cut-off parameter, and another set of the constants is calculated using dimensional regularization. In this case, the independence of the final results from the regularization parameters could be confirmed numerically only. That can result in additional complications, since the two-loops integrals could suffer from a number of the numerical instabilities. In some simple cases, when sub-loop renormalization is possible (for ex. box diagram with self-energy insertion), one can represent the sub-loop by doubly-subtracted dispersive integrals and carry on the second-loop integration using the PV-function basis without dealing with additional UV divergences. In this paper, we follow a general approach developed in [22] and extract the UV-divergent parts of the two-loop integrals analytically. For that, we need to represent the UV-divergent dispersive sub-loop insertion using dimensional regularization and extract UV poles analytically. Since in [22, 23] the two-loop integrals where all reduced to the two-point PV-function basis, we start with the outline of the ideas on how to express the two-point sub-loop insertion with UV divergent part written out in the dimensional regularization and the UV-finite part represented
by a multiply-subtracted dispersive integral. Later, we extend this approach to triangle-, box- and pentagon-type of insertions.

II. METHODOLOGY

Generally, a two-point function of an arbitrary rank could be written in the dimensional regularization as:

\[ B_{0\ldots01\ldots1} (p^2, m_1^2, m_2^2) \equiv B_{\{2l,n\}} (p^2, m_1^2, m_2^2) = \mu^{2\epsilon} e^{\gamma_E \epsilon} \frac{(-1)^{2+n+l}}{2^l \Gamma (\epsilon - l)} \]

(1)

\[ \times \lim_{\epsilon \to 0^+} \int_0^1 dx x^n (p^2 x^2 + m_1^2 + x (m_2^2 - m_1^2 - p^2) - i\epsilon)^{-\epsilon + l} \]

Here, \( \epsilon = \frac{4-D}{2} \) is the dimensional regularization and \( \mu \) is the mass-scale parameter. The UV-divergent part Eq.1 can be expressed as a polynomial in \( p^2 \) multiplied by \( \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_2^2} \right) \) term. A linear term in \( \epsilon \) will give rise to the local terms after taking the second-loop integration, and can be considered as a finite part of the two-point functions which has dependence on \( \ln \frac{\mu^2}{m_2^2} \). Hence, the regularized one-loop UV-divergent part has the following form:

\[ B_{\{2l,n\}}^{UV} (p^2, m_1^2, m_2^2) = \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_2^2} \right) \sum_{i=0}^l a_{\{2l,n\}} (2i) \epsilon^{2i} \]

(2)

Here, coefficients \( a_{\{2l,n\}} \) are the functions of masses \( m_1^2, m_2^2 \) with structure provided in Tbl.1.

In order to satisfy the definition given in Eq.1, the UV-divergent pole \( 1/\epsilon \) in Eq.2 should be treated as \( \frac{1}{\epsilon} \to \frac{1}{\epsilon} - \gamma_E + \ln (4\pi) \). In the case of sub-loop insertion, the UV part represented by Eq.2 can be easily carried into the second-loop integral. Here, the momentum \( p^2 \) could depend on the momentum of the second loop and Feynman parameters used in \([22]\). In order to keep the UV-divergent term presented in Eq.2 as simple as possible, we will treat masses as constants. In the case where masses depend on the Feynman and mass shift parameters (see \([22]\)), a simple transformation \( \ln \frac{m_1^2}{m_2^2} \to \ln \frac{m_1^2}{m_0^2} + \ln \frac{m_2^2}{m_0^2} \) can be used, where \( m_0 \) is the arbitrary constant mass. A term proportional to \( \ln \frac{m_2^2}{m_0^2} \) is UV-finite and scale-parameter independent, and hence can be moved to the UV-finite part of Eq.1 for which we will construct a dispersive representation. The UV-finite part could be presented through
Here, $B^{\text{fin}}_{\{2l,n\}} (p^2, m_1^2, m_2^2)$ is the UV-finite part of Eq. 1. $B_{\{2l,n\}} (p^2, m_1^2, m_2^2) = B^{\text{UV}}_{\{2l,n\}} (p^2, m_1^2, m_2^2) + B^{\text{fin}}_{\{2l,n\}} (p^2, m_1^2, m_2^2)$. The function $B^{\text{fin}}_{\{2l,n\}} (p^2, m_1^2, m_2^2)$ consists of the finite part of the two-point function, $b^{\text{fin}}_{\{2l,n\}} (p^2, m_1^2, m_2^2)$, which is free from any of the regularization parameters plus an additional terms linear in $\epsilon$, which are also finite. More
Table II: Coefficients $d_{il}$ used in the representation of the linear in $\epsilon$ term in Eq.4.

| $l$ | $i = 1$ | $i = 2$ | $i = 3$ |
|-----|---------|---------|---------|
| 0   | $\frac{\pi^2}{12}$ | $\frac{1}{3}$ | 0       |
| 1   | $\frac{12 + \pi^2}{24}$ | $-\frac{1}{2}$ | $\frac{1}{4}$ |
| 2   | $\frac{21 + \pi^2}{96}$ | $-\frac{3}{16}$ | $\frac{1}{16}$ |
| 3   | $\frac{85 + 3\pi^2}{1728}$ | $-\frac{11}{288}$ | $\frac{1}{15}$ |

Specifically, we can write:

$$B_{(2l,n)}^{fin} \left( p^2, m_1^2, m_2^2 \right) = b_{(2l,n)}^{fin} \left( 1 + \epsilon \ln \frac{\mu^2}{m_2^2} \right) + (-1)^n \epsilon \left( d_{11}I_1 + d_{21}I_2 + d_{31}I_3 + d_{31}I_1 \ln^2 \frac{\mu^2}{m_2^2} \right)$$

where

$$I_1 = \int_0^1 dx x^n A^l \left( p^2, m_1^2, m_2^2 \right)$$

$$I_2 = \int_0^1 dx x^n A^l \left( p^2, m_1^2, m_2^2 \right) \ln \frac{m_2^2}{A \left( p^2, m_1^2, m_2^2 \right)}$$

$$I_3 = \int_0^1 dx x^n A^l \left( p^2, m_1^2, m_2^2 \right) \ln^2 \frac{m_2^2}{A \left( p^2, m_1^2, m_2^2 \right)}$$

$$A \left( p^2, m_1^2, m_2^2 \right) = p^2 x^2 + m_1^2 + x \left( m_2^2 - m_1^2 - p^2 \right) - i\epsilon.$$
on any scale, and hence where will be an additional terms to remove any dependence. To remove the UV-part of Eq.\(1\) we can easily generalize this procedure by using the following subtractions:

\[
B_{\{2l,n\}}^{\text{sub}} \left( p^2, m_1^2, m_2^2, \Lambda^2 \right) = B_{\{2l,n\}} - \sum_{i=0}^{l} \frac{1}{i!} \left( \frac{\partial^i B_{\{2l,n\}}}{\partial (p^2)^i} \right) \bigg|_{p^2=\Lambda^2} (p^2 - \Lambda^2)^i. \tag{5}
\]

Here, \(B_{\{2l,n\}} \equiv B_{\{2l,n\}} \left( p^2, m_1^2, m_2^2 \right)\) and \(B_{\{2l,n\}}^{\text{sub}} \left( p^2, m_1^2, m_2^2, \Lambda^2 \right)\) is multiply-subtracted Eq.\(1\).

Now, we will subtract and add the finite part of the second term of Eq.\(5\) to Eq.\(3\), and use the subtracted terms to construct the multiply-subtracted dispersive integral of Eq.\(3\). As a result, we can write the following:

\[
B_{\{2l,n\}}^{\text{fin}} \left( p^2, m_1^2, m_2^2 \right) = \frac{(p^2 - \Lambda^2)^{l+1}}{\pi} \int_{(m_1 + m_2)^2}^{\infty} ds \frac{\Im B_{\{2l,n\}}^{\text{fin}} \left( s, m_1^2, m_2^2 \right)}{(s - p^2 - i\varepsilon) (s - \Lambda^2 - i\varepsilon)^{l+1}} \tag{6}
\]

\[
+ \sum_{i=0}^{l} \frac{1}{i!} \left( \frac{\partial^i B_{\{2l,n\}}^{\text{fin}} \left( p^2, m_1^2, m_2^2 \right)}{\partial (p^2)^i} \right) \bigg|_{p^2=\Lambda^2} (p^2 - \Lambda^2)^i.
\]

Eq.\(6\) has no dependence on the scale \(\Lambda\) and its second term is finite with a polynomial structure in \(p^2\), which can be easily evaluated in the second-loop integration. Finally, we can write dimensionally regularized sub-loop insertion as:

\[
B_{\{2l,n\}} \left( p^2, m_1^2, m_2^2 \right) =
\]

\[
\sum_{i=0}^{l} \left[ \left( \frac{1}{\varepsilon} + \ln \frac{\mu^2}{m_2^2} \right) a_{\{2l,n\}}^{(2l+1)} p^{2i} + \frac{1}{i!} \left( \frac{\partial^i B_{\{2l,n\}}^{\text{fin}} \left( p^2, m_1^2, m_2^2 \right)}{\partial (p^2)^i} \right) \bigg|_{p^2=\Lambda^2} (p^2 - \Lambda^2)^i \right] \tag{7}
\]

\[
+ \frac{(p^2 - \Lambda^2)^{l+1}}{\pi} \int_{(m_1 + m_2)^2}^{\infty} ds \frac{\Im B_{\{2l,n\}}^{\text{fin}} \left( s, m_1^2, m_2^2 \right)}{(s - p^2 - i\varepsilon) (s - \Lambda^2 - i\varepsilon)^{l+1}}.
\]

The first term of Eq.\(7\) will contribute to the numerator algebra and the second term will add an additional propagator \(\frac{(p^2 - \Lambda^2)^{l+1}}{s - p^2 - i\varepsilon}\) to the second-loop integral.

In the case of the triangle insertion, the three-point PV functions which can be written in the form of the derivatives of the two-point functions. To begin with, the scalar three-point
function is given by:

\[
C_0 \equiv C_0 \left( p_1^2, p_2^2, (p_1 + p_2)^2, m_1^2, m_2^2, m_3^2 \right) = \\
\frac{\mu^{4-D}}{i\pi^{D/2}} \int d^D q \frac{1}{[q^2 - m_1^2] \left[ (q + p_1)^2 - m_2^2 \right] \left[ (q + p_1 + p_2)^2 - m_3^2 \right]}. 
\]  

(8)

With Feynman’s trick, we can join the first two propagators in Eq.8 and after shifting momentum \( q = \tau - p_1 - p_2 \), we can write:

\[
C_0 = \frac{\mu^{4-D}}{i\pi^{D/2}} \int_0^1 dx \int d^D \tau \frac{1}{\left[ (\tau - (p_1 \bar{x} + p_2))^2 - m_{12}^2 \right]^2 \left[ \tau^2 - m_3^2 \right]}. 
\]  

(9)

\[m_{12}^2 = m_1^2 \bar{x} + m_2^2 x - p_1^2 x \bar{x}.\]

Here, \( \bar{x} = 1 - x \), and momentum \( p_1 \) does not enter the second loop integral and is treated as a combination of the external momenta of the two-loop graph. Term \( \left( (\tau - (p_1 \bar{x} + p_2))^2 - m_{12}^2 \right)^{-2} \) can be replaced after shifting mass \( m_{12}^2 \) by a small parameter \( \phi \):

\[
\frac{1}{\left( (\tau - (p_1 \bar{x} + p_2))^2 - m_{12}^2 \right)^2} = \lim_{\phi \to 0} \frac{\partial}{\partial \phi} \left[ \frac{1}{(\tau - (p_1 \bar{x} + p_2))^2 - (m_{12}^2 + \phi)} \right]. 
\]  

(10)

As a result, Eq.9 can be represented in the form of

\[
C_0 = \frac{\mu^{4-D}}{i\pi^{D/2}} \lim_{\phi \to 0} \frac{\partial}{\partial \phi} \int_0^1 dx \int d^D \tau \frac{1}{\left[ (\tau - (p_1 \bar{x} + p_2))^2 - (m_{12}^2 + \phi) \right] \left[ \tau^2 - m_3^2 \right]} = \\
\lim_{\phi \to 0} \frac{\partial}{\partial \phi} \int_0^1 dx B_0 \left( (p_1 \bar{x} + p_2)^2, m_3^2, m_{12}^2 + \phi \right). 
\]  

(11)

Since \( C_0 \) function is UV finite, its dispersive representation will be given by a singly sub-
tracted integral:

\[
C_0 = \lim_{\phi \to 0} \frac{\partial}{\partial \phi} \int_0^1 dx \ln \frac{m_3^2}{m_{12}^2 + \phi} + B_0^{fin}(\Lambda^2, m_3^2, m_{12}^2 + \phi)
\]

\[
+ \left( \frac{(p_1 \bar{x} + p_2)^2 - \Lambda^2}{\pi} \right) \int_0^\infty ds \frac{\Im B_0^{fin}(s, m_3^2, m_{12}^2 + \phi)}{(s - (p_1 \bar{x} + p_2)^2 - i\epsilon)(s - \Lambda^2 - i\epsilon)}.
\]

(12)

In this representation of \( C_0 \) function, we have momentum \( p_2 \) as a combination of the second-loop and external momenta. When taking a derivative with respect to the mass shift parameter \( \phi \), we use transformation \( \ln \frac{m^2}{m_{12}^2 + \phi} \to \ln m_3^2 + \ln \frac{m^2}{m_{12}^2 + \phi} \) in order to remove \( \mu \)-scale dependence from the Feynman integral. The finite part of the \( B_0 \) function has a rather simple analytical structure:

\[
B_0^{fin}(p_2, m_1^2, m_2^2) = 2 + \frac{\kappa^{1/2}(p_2, m_1^2, m_2^2)}{p^2} \ln \left( \frac{\kappa^{1/2}(p_2, m_1^2, m_2^2) + m_1^2 + m_2^2 - p^2}{2m_1 m_2} \right)
\]

\[
- \frac{(m_1^2 - m_2^2 + p^2)}{2p^2} \ln \left( \frac{m_1^2}{m_2^2} \right).
\]

(13)

Here, \( \kappa(p_2, m_1^2, m_2^2) \) is a Källen function, \( \kappa(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac) \). In the case of the higher rank three-point tensor coefficient functions, we can represent them through a combinations of \( B_{(2l,n)} \) functions following the prescription of [22]:

\[
C_{0...01...12...2} \equiv C_{(2l,n,m)} = \lim_{\phi \to 0} \frac{\partial}{\partial \phi} \int_0^1 dx x^n \sum_{i=0}^m b_i^{(m)} B_{(2l,i+n)}.
\]

(14)

Here, \( B_{(2l,i+n)} \equiv B_{(2l,i+n)}((p_1 \bar{x} + p_2)^2, m_3^2, m_{12}^2 + \phi) \), and the UV-divergent three-point functions have \( l \geq 1 \). Coefficients \( b_i^{(m)} \) are given in the Tbl.III. Using Eq.7 in Eq.14, we can write the generalized three-point function dispersive with dimensionally regularized
Table III: Expansion coefficients \( b_i^{\{m\}} \) for many-points Passarino-Veltman functions.

UV-divergence:

\[
C_{(2l,n,m)} = \lim_{\phi \to 0} \frac{\partial}{\partial \phi} \int_0^1 dx \sum_{i=0}^m b_i^{\{m\}} \left( \sum_{j=0}^l \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_{12}^2 + \phi} \right) a_j^{(2l,i+n)} \right) p_{12x}^{2j} \\
+ \frac{1}{j!} \left( \frac{\partial^j B_{(2l,i+n)}^{\text{fin}}(p^2, m_3^2, m_{12}^2 + \phi)}{\partial (p^2)^j} \right) \bigg|_{p^2=\Lambda^2} (p_{12x}^2 - \Lambda^2)^j \\
+ \frac{(p_{12x}^2 - \Lambda^2)^{l+1}}{\pi} \int_{\left( m_3 + \sqrt{m_{12}^2 + \phi} \right)^2}^\infty ds \frac{\Im B_{(2l,i+n)}^{\text{fin}}(s, m_3^2, m_{12}^2 + \phi)}{(s - p_{12x}^2 - i\varepsilon)(s - \Lambda^2 - i\varepsilon)^{l+1}},
\]

with \( p_{12x} \) is defined as \( p_{12x} = p_1 \bar{x} + p_2 \). As an example, let’s consider expression for \( C_{001} \)
where UV-divergent pole is extracted explicitly:

\[
C_{001} = -\frac{1}{12} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_3^2} \right)
\]

\[
+ \lim_{\phi \to 0} \frac{\partial}{\partial \phi} \int_0^1 dx x \left( \frac{1}{12} \left( \frac{1}{2} p_{12x}^2 - m_3^2 - 2 (m_{12}^2 + \phi) \right) \ln \frac{m_3^2}{m_{12}^2 + \phi} \right)
\]

\[
+ B_{001}^{fin} (\Lambda^2, m_3^2, m_{12}^2 + \phi) + \left( \frac{\partial B_{001}^{fin} (p^2, m_3^2, m_{12}^2 + \phi)}{\partial p^2} \right) \bigg|_{p^2 = \Lambda^2} \left( p_{12x}^2 - \Lambda^2 \right)
\]

\[
+ \frac{(p_{12x}^2 - \Lambda^2)^2}{\pi} \int_0^\infty ds \frac{\Im B_{001}^{fin} (s, m_3^2, m_{12}^2 + \phi)}{(s - p_{12x}^2 + \imath \epsilon) (s - \Lambda^2 - \imath \epsilon)^2}.
\]

To derive expressions for the four-point PV functions in the two-point function basis, we can use the ideas outlined in Eqns. 8-11.

\[
D_{0...01...12...23...3} \equiv D_{\{2l, n, k, m\}} = \lim_{\phi \to 0} \frac{\partial^2}{\partial \phi^2} \int_0^1 dx x^n \int_0^{1-x} dy y^k \sum_{i=0}^m b_i^{(m)} B_{\{2l, i+n+k\}}
\]

where \(D_{\{2l, n, k, m\}} \equiv D_{\{2l, n, k, m\}} (p_1^2, p_2^2, p_3^2, p_4^2, (p_1 + p_2)^2, (p_2 + p_3)^2, m_1, m_2, m_3, m_4)\) and \(B_{\{2l, i+n+k\}} \equiv B_{\{2l, i+n+k\}} [(p_1 (\bar{x} - y) + p_2 \bar{y} + p_3)^2, m_4, m_{123} + \phi]\) with \(m_{123} = m_1 (\bar{x} - y) + m_2 x + m_3 y - p_1^2 x \bar{x} - p_2^2 \bar{y} + 2 xy (p_1 p_2)\) and \(p_{12} = p_1 + p_2\). As a result, the dispersive generalization can be written as:

\[
D_{\{2l, n, k, m\}} = \lim_{\phi \to 0} \frac{\partial^2}{\partial \phi^2} \int_0^1 dx x^n \int_0^{1-x} dy y^k \sum_{i=0}^m b_i^{(m)} \left( \sum_{j=0}^{l \geq 0} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_{123}^2 + \phi} \right) a_j^{\{2l, i+n+k\}} P_{123xy}^{2j} \right)
\]

\[
+ \frac{1}{j!} \left( \frac{\partial^j B_{\{2l, i+n+k\}}^{fin} (p^2, m_3^2, m_{123}^2 + \phi)}{\partial (p^2)^j} \right) \bigg|_{p^2 = \Lambda^2} \left( p_{123xy}^2 - \Lambda^2 \right)^j
\]

\[
+ \frac{(p_{123xy}^2 - \Lambda^2)^{l+1}}{\pi} \int_0^\infty ds \frac{\Im B_{\{2l, i+n+k\}}^{fin} (s, m_4^2, m_{123}^2 + \phi)}{(s - p_{123xy}^2 + \imath \epsilon) (s - \Lambda^2 + \imath \epsilon)^{l+1}}.
\]
Here, we have $p_{123xy} = p_1(\vec{x} - y) + p_2\vec{y} + p_3$. Eq[18] shows that the UV-divergent four-point functions show up at $l \geq 2$. The five-point function also can be easily expressed in two-point function basis:

$$E_{0\ldots01\ldots12\ldots23\ldots34\ldots4} \equiv E_{(2l,n,k,r,m)} = \ldots$$

$$\lim_{\phi \to 0} \frac{\partial^3}{\partial \phi^3} \int_0^1 dx x^n \int_0^1 dy y^k \int_0^1 dz z^r \sum_{i=0}^{m} b_i^{(m)} B_{(2l,i+n+k+r)}.$$ 

Here, $E_{(2l,n,k,r,m)} = E_{(2l,n,k,r,m)} (p_1^2, p_2^2, p_3^2, p_4^2, p_{12}, p_{23}, p_{34}, p_5^2, m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$ with $p_{ij} = (p_i + p_j)^2$, $p_{ijk} = (p_i + p_j + p_k)^2$, and $B_{(2l,i+n+k+r)} = B_{(2l,i+n+k+r)} ((p_1 (\vec{x} - y - z) + p_2 (\vec{y} - z) + p_3 \vec{z} + p_4)^2, m_5^2, m_{1234}^2 + \phi)$ with $m_{1234}^2 = m_1^2 (\vec{x} - y - z) + m_2^2 x + m_3^2 y + m_4^2 z - p_1^2 \vec{x} - p_2^2 \vec{y} - p_3^2 \vec{z} + 2xy (p_1 p_{12}) + 2xz (p_1 p_{13}) + 2yz (p_1 p_{14})$. The dispersive generalization of the five-point function is given in a similar way:

$$E_{(2l,n,k,r,m)} = \lim_{\phi \to 0} \frac{\partial^3}{\partial \phi^3} \int_0^1 dx x^n \int_0^1 dy y^k \int_0^1 dz z^r \sum_{i=0}^{m} b_i^{(m)} \left( \sum_{j=0}^{l} \left[ \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m_{1234}^2 + \phi} \right) a_j^{(2l,i+n+k+r)} \right] \right) \frac{2j}{p_{1234}^2 m_{1234}^2 \phi^2}$$

$$+ \frac{1}{j!} \left( \left. \frac{\partial^j B_{(2l,i+n+k+r)}^{fin}}{\partial (p^2)^j} \right|_{p^2 = \Lambda^2} (p_{1234}^2 - \Lambda^2)^j \right)$$

$$+ \frac{(p_{1234}^2 - \Lambda^2)^{l+1}}{\pi} \int_0^\infty ds \frac{3B_{(2l,i+n+k+r)}^{fin} (s, m_2^2, m_{1234}^2 + \phi)}{(s - p_{1234}^2 m_{1234}^2 \phi^2) (s - \Lambda^2 - i\epsilon)^{l+1}},$$

where momentum $p_{1234}^2$ is defined as $p_{1234}^2 = p_1 (\vec{x} - y - z) + p_2 (\vec{y} - z) + p_3 \vec{z} + p_4$. 


III. CONCLUSION

In this work, we have extracted the UV-divergent poles of the Passarino-Veltman functions analytically and presented them as the dimensionally-regularized and multiply-subtracted dispersive sub-loop insertions. We have also retained the terms linear in $\epsilon$, which are required to produce local terms for the second-loop integration. Finally, all sub-loop insertions are conveniently expressed in the two-point function basis, which allows to carry out the calculations analytically, with numerical integration done only over the Feynman and dispersion parameters. As a result, this approach will allow to speed up calculations for the two-loop radiative corrections and to better account for the experiment-specific kinematics.

Acknowledgments

The authors are grateful to A. Davydychev, H. Spiesberger and M. Vanderhaeghen for the fruitful and exciting discussions. We would also like to express special thanks to the Institut für Kernphysik of Johannes Gutenberg-Universität Mainz for hospitality and support. This work was funded by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

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