Exact critical exponents for vector operators in the 3d Ising model and conformal invariance

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It is widely expected that the realization of scale invariance in the critical regime implies conformal invariance for a large class of systems. This is known to be true if there exist no integrated operator which transforms like a vector under rotations and which has scaling dimension \(-1\). In this article we give exact expressions for the critical exponents of some of these vector operators. In particular, we show that one operator has scaling dimension exactly 3 in any space dimension. This operator turns out to be the leading operator (i.e. the operator with the smallest scaling dimension) at least in \(d = 2\) and \(d = 4\). Moreover, we prove that the operator previously considered in Monte-Carlo simulations has also scaling dimension exactly 3 in any dimension.

I. INTRODUCTION

The vicinity of a second order transition is remarkable because the long-distance properties are invariant under dilatations, even though the underlying microscopic model involves some typical scales, such as a lattice spacing or a typical inter-particle distance. This emergent symmetry under dilatation is best described in the framework of the renormalization group. To each microscopic model one can associate an effective action \(S_k\) which describes the dynamics of the coarse-grained model, where the short-distance degrees of freedom (as compared to the length scale \(k^{-1}\)) have been integrated out. Scale invariance shows up in this framework as a fixed point of the renormalization-group flow.

Soon in the ’70s it was conjectured that other emergent symmetries may occur in the critical regime. In fact, it may be that the whole conformal group is realized \(1, 2\). This was proven to be true in bidimensional systems under quite general conditions \(3\) but the situation is more intricate in the case of \(d > 2\). The issue of the validity of conformal invariance above two dimensions became of the utmost importance in the last few years given the success of the conformal bootstrap program in \(d = 3\) \(4, 5\).

Polchinski \(8\) showed that a model with translation, rotation and scale invariance also presents conformal invariance if there exists no virial current. That is, there exists no local operator \(V_\mu\) which

- transforms as a vector under space rotations,
- is a scalar under the internal symmetries of the problem (e.g. symmetric under \(\phi \rightarrow -\phi\) in the Ising universality class),
- is not a total derivative,

- has scaling dimension \(d - 1\).

An equivalent sufficient condition is that there exists no integrated vector operator \(\int d^d x V_\mu\) of scaling dimension \(-1\). A similar sufficient condition was derived in a different setting \(9\), with slightly different prerequisites. In particular, Polchinski’s sufficient condition requires the existence of a local energy-momentum tensor which is not always granted, as for example, in the long-range Ising model. The sufficient condition proposed in \(8\) does not require the existence of a local energy-momentum tensor and therefore generalizes, for example, to the case of mild long-range interactions.

These sufficient conditions indicate a path to prove that conformal invariance is indeed realized in the critical domain of a particular system. We need to find a bound on the scaling dimensions of the vector operators of the kind described above. One strategy followed in \(9\) for the Ising universality class, on which we concentrate from now on, consists in using Griffiths and Lebowitz inequalities on correlation functions \(10, 12\) in order to prove that any integrated vector operator invariant under \(\mathbb{Z}_2\) symmetry has scaling dimension greater than \(-1\).

Another strategy consists in computing explicitly the lowest scaling dimension of such vector operators. Several attempts have been performed in this direction during the last few years. In \(13\), a Monte-Carlo simulation was performed with the aim of determining the scaling dimension of the vector operator \(\int d^d x \phi \partial_\mu \phi (\partial_\nu \phi)^2\) appropriately discretized on a 3d lattice. The result quoted for the scaling dimension of the integrated vector operator is \(3 \pm 1\) (i.e., \(5 \pm 1\) for the local vector operator). Since the discretized operator considered in \(12\) is quite generic, it is natural to believe that it couples to the operator of lowest scaling dimension. Under this assumption, and invoking the aforementioned sufficient condition, this result strongly indicates that conformal invariance is present...
in the critical regime of the \(d = 3\) Ising model. From the analytic side, a 1-loop calculation\(^[9]\) performed in \(d = 4 - \epsilon\) shows that the integrated vector operator of lowest scaling dimension is \(O^4_3 = \int d^d x \phi^3 \partial_\mu \Delta \phi\) where \(\Delta\) is the Laplacian. This operator turns out to be the same (up to integration by parts) as the one employed in the 3d case in Ref.\(^{[13]}\). It has dimension \(3 + O(\epsilon^2)\) (i.e., the correction linear in \(\epsilon\) vanishes). Finally, the scaling dimension of the leading vector operator is found to be 3 again in the bidimensional case\(^{[13]}\). All these results indicate that the smallest scaling dimension of an integrated vector operator is close to 3 in all dimensions between 2 and 4. The aim of this article is to show that the scaling dimension of the integrated vector operator studied previously is actually exactly 3 in any dimension. This eliminates the uncertainties coming from the numerical simulation in\(^{[13]}\) and, accordingly, under the same assumptions made in that reference that scale invariance implies conformal invariance. We stress that the determination of a critical exponent is remarkable because very few exact results are known for critical exponents in \(d = 3\).

**II. NONRENORMALIZATION THEOREM**

We consider a description of the Ising universality class in terms of continuous fields. We can choose the Hamiltonian (or action) to be of the Ginzburg-Landau type:

\[
S[\phi] = \int_x \left( \frac{1}{2} \nabla \phi \right)^2 + \frac{1}{2} \mu \phi^2 + \frac{u}{4!} \phi^4,
\]

where \(x = \int d^d x\). We consider the model with an appropriate ultraviolet regulator at some scale \(\Lambda\). In Eq. (1), the subscript \(\Lambda\) indicates that the coupling constants are defined at the microscopic scale \(\Lambda\). Following Polchinski and Wetterich\(^{[14, 15]}\), we add a quadratic regulator to the theory:

\[
\Delta S_k[\phi] = \frac{1}{2} \int_{x,y} \phi(x) R_k(|x - y|) \phi(y)
\]

which regularizes the theory in the infrared. The properties of the regulating function are more conveniently discussed in Fourier space. The so-called regulating function \(R_k(q)\) is chosen to approach zero exponentially fast for \(q \gg k\) and to saturate at a value which scales as \(k^{2-\eta}\) when \(q \ll k\). This ensures that the fluctuations of the long-distance modes (i.e. whose typical length scale are greater than \(k^{-1}\)) are effectively suppressed while the short-distance ones are kept unchanged. In what concerns the ultraviolet, we can regularize the theory either by modifying the regulating function \(R_k(q)\)\(^{[14]}\) or by considering the model on a hypercubic lattice with lattice spacing \(\pi/\Lambda\) at the price of introducing a discretization of the field derivatives.

Following Wilson, a convenient strategy for determining the scaling dimension of an operator consists in studying the evolution of the corresponding coupling under the renormalization-group flow in the vicinity of the fixed point. To this end, we add to the action a part which couples to a vector operator:

\[
S_V[\phi] = \int_x \frac{a^{\mu}_k}{3} \phi^3 \partial_\mu \Delta \phi.
\]

Up to integrations by parts, this operator is the same as the one considered in\(^{[13]}\). Moreover, it has been proved to be the most relevant integrated vector operator invariant under \(Z_2\) symmetry near \(d = 4\)\(^{[9]}\). Baring coincidences (or superselection rules) we expect this operator to couple to all \(Z_2\) symmetric vector operators, in particular to the most relevant one.

The critical Ising model is invariant under (space) rotations from which we conclude that, at the Wilson-Fisher fixed point, the dimensionless, renormalized, counterpart of \(a^\mu\) vanishes. Moreover, since we are only interested in the scaling dimension of the vector operator around the Wilson-Fisher fixed point, we concentrate on infinitesimally small \(a^\mu_\Lambda\).

The regularized partition function in presence of a source \(J(x)\) then reads:

\[
e^{W_k[J, a^\mu_\Lambda]} = \int \mathcal{D}\phi e^{-S - S_V - \Delta S_k + \int_x J \phi}
\]

We now perform an infinitesimal transformation of the integration variable: \(\phi \to \phi - a^\mu_\Lambda/\mu \partial_\mu \Delta \phi\) in the path integral. It is readily found that the quadratic pieces in the action, including the regulating term \(\Delta S_k\), are invariant under this transformation. The variation of the quartic part of the action is found to compensate exactly \(S_V\). We thus find that

\[
W_k[J, a^\mu_\Lambda] = W_k[J + \frac{a^\mu_\Lambda}{\mu} \partial_\mu \Delta J, 0] + O(a^{\mu}_\Lambda a^{\nu}_\Lambda)
\]

We now introduce the scale dependent effective action as the (modified) Legendre transform\(^{[13]}\):

\[
\Gamma_k[\phi] = -W_k[J] + \int_x J \phi - \Delta S_k[\phi]
\]

and check easily that

\[
\Gamma_k[\phi, a^\mu_\Lambda] = \Gamma_k[\phi + a^\mu_\Lambda \partial_\mu \Delta \phi, 0] + O(a^{\mu}_\Lambda a^{\nu}_\Lambda).
\]

This last equation is the main result of our article. It states that the evolution of the effective action with an infinitesimal \(a^\mu_\Lambda\) is related to the effective action at vanishing \(a^\mu_\Lambda\), up to a modification of the field. This can be used in the following way. Defining the running coupling constants \(u_k\) and \(a^\mu_k\) as the prefactors of, respectively, \(\int_x \phi^4\) and \(\int_x \phi^3 \partial_\mu \Delta \phi\) in \(\Gamma_k\), we obtain that \(a^{\mu}_k/\mu_k\) is constant along the flow. To obtain the scaling dimension of the vector operator, we introduce dimensionless, renormalized quantities (denoted with tilde) as

\[
\tilde{x} = k x
\]

\[
\tilde{\phi}(\tilde{x}) = k^{-(d-2)/2} Z_k^{1/2} \phi(x),
\]

where the scaling dimension of the vector operator is found in the 3d case in Ref.\(^{[13]}\). It has dimension \(3 + \frac{1}{2} \phi^4\).
where $Z_k$ scales as $k^{-\eta}$ at the Wilson-Fisher fixed point with $\eta$ the anomalous dimension. The renormalized coupling constants are thus:

\[
\tilde{u}_k = k^{d-4} Z_k^{-2} u_k \\
\tilde{u}^\mu_k = k^{d-1} Z_k^{-2} u^\mu_k.
\]

At the critical point, $\tilde{u}$ flows to a fixed point value $u_*$. Consequently, when $k \to 0$,

\[
\tilde{a}^\mu_k \sim a^\mu_k u_* k^3
\]

which shows that the scaling dimension of $a^\mu$ is exactly 3.

The proof given above relies strongly on the particular microscopic action given in Eq. (11). This gives interesting non-universal information on the flow of the coupling $a^\mu$, but confers a preeminent role to the peculiar form of the Hamiltonian. To overcome this issue, we now present an alternative proof of the same result. To this end, we first recall the exact Wetterich flow equation (15) for the effective average action, expressed in terms of dimensionless, renormalized fields:

\[
\partial_t \tilde{\Gamma}_k[\tilde{\phi}] = \frac{\delta}{\delta \tilde{\phi}(\tilde{x})} \left[ \int \frac{d^d \phi}{(2\pi)^d} \frac{\delta \tilde{\phi}(\tilde{x})}{\delta \phi(\tilde{x})} \right] \left( R(\tilde{x}, \tilde{y}) - \frac{1}{2} \frac{\delta^2 \tilde{\phi}(\tilde{x})}{\delta \phi(\tilde{y}) \delta \phi(\tilde{z})} \right) = \delta(\tilde{x} - \tilde{z})
\]

We now identify an exact eigenvector of the linearized flow. To this end, we add to the Wilson-Fisher fixed-point effective action $\tilde{\Gamma}_*$ a small perturbation

\[
\Gamma_k = \tilde{\Gamma}_* + \tilde{\Gamma}_k[\tilde{\phi} - \tilde{\phi}^*]
\]

and compute the flow of this functional at linear order in $\tilde{\phi}^*$.

Combining the two equations, we obtain:

\[
\partial_t \tilde{\Gamma}_k[\tilde{\phi}] = 3 \partial_\mu \tilde{\phi} = \frac{\delta}{\delta \tilde{\phi}(\tilde{x})} \left[ \partial_\mu \tilde{\phi}(\tilde{x}) \right] (\tilde{x}, \tilde{y}) = \frac{1}{2} \frac{\delta^2 \tilde{\phi}(\tilde{x})}{\delta \phi(\tilde{y}) \delta \phi(\tilde{z})}
\]

The commutator is easily evaluated to be equal to $3 \partial_\mu \tilde{\phi}$. From this we deduce that the small perturbation introduced in Eq. (15) is an exact eigenvector of the flow around the fixed point, with eigenvalue 3. This is consistent with the result found in the one-loop calculation of (4) and with the exact result of (12) in $d = 2$. This is also consistent with a Monte-Carlo simulation performed in $d = 3$ (13).

We can generalize the previous result in different ways. First, we can change the power of the Laplacian in Eq. (15) from unity to a positive integer $n$. The main change appears at the level of Eq. (15), where the commutator is now $[\partial_\mu \Delta^n, \tilde{x}^\rho \partial_{\tilde{x}^\rho}] = (2n + 1) \partial_\mu \Delta^n$. This implies that the associated eigenvector has dimension $2n + 1$. As a check of this result, we have considered the vector eigenvectors compatible with the $Z_2$ symmetry whose scaling dimensions are 5 in $d = 4$ and we have computed their first correction in $\epsilon = 4 - d$. There are four (independent) such operators: one ($O^1_6$) with 6 powers of the field and 3 derivatives and three ($O^i_{4}$, with $i \in \{1, 2, 3\}$) with 4 powers of the field and 5 derivatives. A one-loop calculation shows that $O^1_6$ has scaling dimension $5 - 5\epsilon/3 + O(\epsilon^2)$. The eigenvectors $O^i_{4}$ have dimensions $5 + O(\epsilon^2)$, $5 - 4\epsilon/9 + O(\epsilon^2)$ and $5 - 2\epsilon/3 + O(\epsilon^2)$. The eigenvector with scaling dimension $5 + O(\epsilon^2)$ is found to be $\int \tilde{x}^\rho \tilde{\phi}(\tilde{x}) \partial_\mu \Delta^{2n} \tilde{\phi}$, in agreement with the general result mentioned above. Other relations can be obtained if we consider in Eq. (15) an odd number of derivatives,
with Lorentz indices not necessarily contracted together.

The present result also generalizes to the long-range
Ising model, where the interaction between spins is not
limited to nearest neighbors but decay as a power-law:

\[ H = - \sum_{i,j} J(i-j)S_iS_j \]  

(19)

where \( J(i-j) \sim |i-j|^{-\sigma} \) and \( \sigma \) is the exponent characterizing the decrease of the interactions. When \( 0 < \sigma < 2 - \eta \), the model still has an extensive free-energy but belongs to a different universality class than the local Ising model. The Ginzburg-Landau Hamiltonian is identical to the one given in Eq. (11) except that the quadratic part is now, in Fourier space,

\[ \int \frac{d^dq}{(2\pi)^d} \phi(-q)\eta^\sigma \phi(q). \]  

(20)

It is easy to verify that all the present analysis still applies to this case. We have checked that the one-loop calculation around the upper critical dimension \( d_c = 2\sigma \) gives that the most relevant integrated vector operator has scaling dimension \( 3+O(\epsilon^2) \). This result is important because it justifies the use of the conformal bootstrap program in this model [17].

We can also generalize the result to other internal groups. For \( O(N) \) theories, an exact eigenvector can be found by adding a common \( O(N) \) index on both the functional derivative and the field appearing in Eq. (15) and summing over this index. The associated eigenvalue is again 3 (or \( 2n+1 \), if we change the power of the Laplacian). In [3], we computed the scaling dimensions of the two vector operators of lowest dimension in an expansion in \( \epsilon \) and found 3 + \( O(\epsilon^2) \) and 3 – \( 6\epsilon/(N+8) + O(\epsilon^2) \). This result is consistent with the nonrenormalization theorem proven here. Let us stress, however, that in the \( O(N) \) model the non renormalization theorem does not constraint the leading vector operator but the next-to-leading, as can be seen already at one-loop level [9].

For completeness, we give other exact eigenvectors which can be obtained following the same idea. These are, however, not invariant under the internal symmetries of the theory, involve even derivatives and are not relevant to our discussion on conformal invariance. A well-known example is the eigenvector associated with the external magnetic field: \( \int x_a \delta \phi(x) \) with eigenvalue \( \alpha = -(d-2 + \eta)/2 \). Another exact eigenvector, valid for the \( O(N) \) model is \( \int x_a \epsilon^{ab} \partial_\phi \Delta^n \phi^a(x) \) where \( \epsilon^{ab} \) is antisymmetric, which is associated with the eigenvalue \( 2n \).

III. CONCLUSION

To conclude, we have shown that there exists a family of eigenvectors which transform as vectors under rotations, are scalars under the internal group, are not total derivatives and whose scaling dimension receive no loop correction. Among these operators lies the operator previously analyzed in a Monte-Carlo simulation in \( d = 3 \) [13]: \( O_3^1 = \int \Phi^a \partial_a \Delta \phi \) which has the lowest scaling dimension in \( d = 4 \) and which couples to the dominant operator in \( d = 2 \). This result is interesting at the light of the sufficient condition under which scale invariance implies conformal invariance, mentioned in the introduction. Indeed, as long as we admit (as is usually assumed) that this operator couples to the leading integrated operator, its scaling dimension being larger than \( -1 \), we would have an alternative proof to the one of [9] that scale invariance implies conformal invariance in all dimensions for the Ising universality class.

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