On the geometry of the characteristic class of a star product on a symplectic manifold

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Abstract

The characteristic class of a star product on a symplectic manifold appears as the class of a deformation of a given symplectic connection, as described by Fedosov. In contrast, one usually thinks of the characteristic class of a star product as the class of a deformation of the Poisson structure (as in Kontsevich’s work). In this paper, we present, in the symplectic framework, a natural procedure for constructing a star product by directly quantizing a deformation of the symplectic structure. Basically, in Fedosov’s recursive formula for the star product with zero characteristic class, we replace the symplectic structure by one of its formal deformations in the parameter $\hbar$. We then show that every equivalence class of star products contains such an element. Moreover, within a given class, equivalences between such star products are realized by formal one-parameter families of diffeomorphisms, as produced by Moser’s argument.

1 Introduction

The characteristic class of a star product on a symplectic manifold appears as the class of a deformation of a given symplectic connection, as described by Fedosov [Fed96, Fed94]. In contrast, one usually thinks of the characteristic class of a star product as the class of a deformation of the Poisson structure [Kon97]. In this paper, we present, in the symplectic framework, a natural procedure for constructing a star product by directly quantizing a deformation of the symplectic structure. Basically, in Fedosov’s recursive formula for the star product with zero characteristic class, we replace the symplectic structure by one of its formal deformations in the parameter $\hbar$. We then show that every equivalence class of star products contains such an element. Moreover, within a given class, equivalences between such star products are realized by formal one-parameter families of diffeomorphisms, as produced by Moser’s argument.

2 Fedosov construction on regular Poisson manifolds

We present Fedosov star products on regular Poisson manifolds [Fed96, Fed94] by mean of a partial connection defined (only) on the characteristic distribution of the Poisson structure. By this we avoid considering Poisson affine

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connections (cf. Lemma 2.4). This little point excepted, there is essentially nothing new in the present section. But it sets the notations and presents Fedosov’s construction in a completely intrinsic way.

2.1 Linear Weyl algebra

Let \((V, \omega)\) be a real symplectic vector space and consider the associated Heisenberg Lie algebra \(H\) over the dual space \(V^*\). That is \(H = V^* \oplus \mathbb{R}h\) where \(h\) is central and where the Lie bracket of two elements \(y, y' \in V^*\) is defined by \([y, y'] = y'(\omega)h\). The map \(V^* \xrightarrow{\omega} V\) being the isomorphism induced by \(\omega\). Denote by \(S(H)\) (resp. \(U(H)\)) the symmetric (resp. the universal enveloping) algebra of \(H\) and consider the complete symmetrization map \(S(H) \xrightarrow{\omega} U(H)\) given by the Poincaré-Birkhoff-Witt theorem. The symmetric product on \(S(H)\) will be denoted by \(\bullet\), while \(\star\) will denote the product on \(S(H)\) transported via \(\varphi\) of the universal product on \(U(H)\).

**Lemma 2.1** There exists one and only one grading \(S(H) =: \oplus_{r \geq 0} S^{(r)}(H)\) on \(S(H)\) such that:

\[
\begin{align*}
(i) & \quad S^r(V^*) \subset S^{(r)}(H) \\
(ii) & \quad S^{(r)}(H) \ast S^{(s)}(H) \subset S^{(r+s)}(H),
\end{align*}
\]

where \(S^r(V^*)\) denotes the \(r\)-th symmetric power of \(V^*\). This grading is compatible with the symmetric product \(\bullet\) as well.

One then defines the linear Weyl algebra \(W(H)\) as the direct product \(W(H) := \prod_{r = 0}^{\infty} S^{(r)}(H)\) endowed with the extended product \(\ast\). Note that the symmetric product \(\bullet\) extends to \(W(H)\) as well. The center \(Z W(H)\) of \((W(H), \ast)\) is canonically isomorphic to the space of power series \(\mathbb{R}[[H]]\).

By using the symplectic structure, one gets an identification between the Lie algebra \(sp(V, \omega)\) and the second symmetric power \(S^2(V^*)\):

\[
sp(V, \omega) \xrightarrow{A} \quad S^2(V^*) \subset W(H)
\]

**Lemma 2.2** For all \(a \in W(H)\) and \(A \in sp(V, \omega)\), one has

\[
[A, a] = 2h A(a),
\]

where \([,\] denotes the Lie bracket on \(W(H)\) induced by the associative product \(\ast\).

**Proof.** Both \(\text{ad}(A)\) and \(h A\) are derivations of \((W(H), \ast)\). It is therefore sufficient to verify formula (i) on generators.

The isomorphism \(V \xrightarrow{h} V^*\) defines an injection \(V \xrightarrow{\mu} W(H)\) which we call the linear moment. Observe that, viewed as an element of \(W(H) \otimes V^*\), \(\mu\) is fixed under the action of the symplectic group \(Sp(V, \omega)\).

Both products \(\ast\) and \(\bullet\) extend naturally to the space \(W(H) \otimes \Lambda^\bullet(V^*)\) of multilinear forms on \(V\) valued in \(W(H)\). We define the total degree \(t\) of an element \(a \otimes \omega, a \in S^{(r)}(H), \omega \in \Lambda^p(V^*)\) by \(t = p + r\). With respect to this degree on \(\Lambda^\bullet(V^*)\), the extended multiplications, again denoted by \(\ast\) and \(\bullet\), are graded. The bracket \([,\] mentioned in Lemma 2.2 therefore extends to \(W(H) \otimes \Lambda^\bullet(V^*)\) as well, and, \((W(H) \otimes \Lambda^\bullet(V^*), [,\])\) is a graded Lie algebra.

To an element \(a \otimes x \in W(H) \otimes \Lambda^p(V)\), one can associate the operator

\[
i_a \otimes x : W(H) \otimes \Lambda^\bullet(V^*) \rightarrow W(H) \otimes \Lambda^{\bullet-p}(V^*),
\]

defined by

\[
i_a \otimes x (b \otimes \omega) := a \bullet b \otimes i_x \omega,
\]

where \(i_x \omega\) denotes the usual interior product. Using the universal property, one gets a map

\[
(W(H) \otimes V) \times W(H) \otimes \Lambda^\bullet(V^*) \quad \rightarrow \quad W(H) \otimes \Lambda^{\bullet-p}(V^*)
\]

\[
(X, s) \quad \mapsto \quad i_X s.
\]

In the case where \(p\) is odd, since \(i_X\) acts “symmetrically” on the “Weyl part” and “anti-symmetrically” on the “form part”, one has \(i_X^2 = 0\).
In the same way, if \( Y \subset \mathcal{W}(\mathcal{H}) \) is a subspace such that \([Y, Y] \subset \mathcal{Z} \mathcal{W}(\mathcal{H})\) (e.g. \( Y = S^{(1)}(\mathcal{H}) \)), to any element \( U \in Y \otimes \Lambda^p(V^*) \), one can associate the operator

\[
\text{ad}(U) : \mathcal{W}(\mathcal{H}) \otimes \Lambda^p(V^*) \rightarrow \mathcal{W}(\mathcal{H}) \otimes \Lambda^{p+1}(V^*).
\]

Using Jacobi identity on the “Weyl part”, one observes that, if \( p \) is odd, one has \( \text{ad}(U)^2 = 0 \).

**Definition 2.1** Using the duality

\[
\frac{S^{(1)}(\mathcal{H})}{U} \otimes V^* \rightarrow \frac{S^{(1)}(\mathcal{H})}{U} \otimes V
\]

one defines the cohomology (resp. homology) operator \( \delta \) (resp. \( \delta^* \)) by

\[
\delta := \text{ad}(\mu),
\]

\[
\delta^* := \iota_\mu^*,
\]

where the linear moment \( \mu \) is viewed as an element of \( S^{(1)}(\mathcal{H}) \otimes V^* \). For a form \( a \in \mathcal{W}(\mathcal{H}) \otimes \Lambda^p(V^*) \) with total degree \( t \), we set

\[
\delta^{-1}a := \begin{cases} \frac{1}{t} \delta^*a & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}
\]

One extends this definition to the whole \( \mathcal{W}(\mathcal{H}) \otimes \Lambda^p(V^*) \).

**Lemma 2.3** (“Hodge decomposition”)

\[
\delta \delta^{-1} + \delta^{-1} \delta = \text{Id} - pr_0
\]

where \( pr_0 \) is the canonical projection \( \mathcal{W}(\mathcal{H}) \otimes \Lambda^p(V^*) \rightarrow \mathcal{Z} \mathcal{W}(\mathcal{H}) \).

**Proof.** We observe that \( \delta \) and \( \delta^* \) are anti-derivations of degree \(+1\) of \( (\mathcal{W}(\mathcal{H}) \otimes \Lambda^*(V^*) \), \( \bullet \)). Their anti-commutator being a derivation of degree \( 0 \), it is therefore sufficient to check the formula on generators. \( \blacksquare \)

Observe that \( \delta \) is an anti-derivation of degree \(+1\) of \( (\mathcal{W}(\mathcal{H}) \otimes \Lambda^*(V^*) \), \( \star \)).

### 2.2 The Weyl bundle

Let \( (N, \Lambda) \) be a regular Poisson manifold. The Poisson bivector \( \Lambda \) induces a short sequence of vector bundles over \( N \):

\[
0 \rightarrow \text{rad}(\Lambda) \rightarrow T^*(N) \rightarrow D^* \rightarrow 0
\]

where \( D \rightarrow T(N) \) denotes the characteristic distribution associated to \( \Lambda \) [Vai94], and where \( \text{rad}(\Lambda) \) is the radical of \( \Lambda \) in \( T^*(N) \). One therefore gets a non-degenerate foliated 2-form \( \omega^D \in \Omega^2(D) \), dual to the canonical one on the quotient \( T^*(N)/\text{rad}(\Lambda) = D^* \). Fix a \( \text{rank}(D) \)-dimensional symplectic vector space \( (V, \omega) \), and, for all \( x \in N \), define

\[
P_x = \{ b \in \text{Hom}_{\mathbb{R}}(V, D_x) | b^* \omega^D_x = \omega \}.
\]

Then \( P = \bigcup_{x \in N} P_x \) is naturally endowed with a structure of \( Sp(V, \omega) \)-principal bundle over \( N \) (analogous to the symplectic frames in the symplectic case, except that here, one does not have a \( G \)-structure in general).

**Definition 2.2** The Weyl bundle is the associated bundle

\[
\mathcal{W} = P \times_{Sp(V, \omega)} \mathcal{W}(\mathcal{H}),
\]

where \( \mathcal{W}(\mathcal{H}) \) is the vector space underlying the linear Weyl algebra defined from the data of \( (V, \omega) \).

The space of \( p \)-forms with values in the sections of \( \mathcal{W} \) is denoted by \( \Omega^p(\mathcal{W}) \); it is canonically isomorphic to the space of sections of the associated bundle \( P \times_{Sp(V, \omega)} (\mathcal{W}(\mathcal{H}) \otimes \Lambda^p(V^*)) \). The \( Sp(V, \omega) \)-invariance, at the linear level, of
both product $\star$ and $\bullet$ on $\mathcal{W}(\mathcal{H}) \otimes \Lambda^\bullet(V^*)$ provides graded products, again denoted by $\star$ and $\bullet$, on $\Omega^\bullet(\mathcal{W})$. In the same way, the operators $\delta$ and $\delta^{-1}$ on $\mathcal{W}(\mathcal{H}) \otimes \Lambda^\bullet(V^*)$ define operators on sections:

$$
\begin{align*}
\Omega^\bullet(\mathcal{W}) &\to \Omega^{\bullet+1}(\mathcal{W}), \\
\delta^{-1} &\to \delta
\end{align*}
$$

leading to a Hodge decomposition of sections as in Lemma 2.3. Notes that the bundle $\mathcal{ZW} = \mathcal{P} \times_{Sp(V,\omega)} \mathcal{ZW}(\mathcal{H})$ being trivial, its space of sections is isomorphic to $C^\infty(N)[[\hbar]]$.

**Remark 2.1** Observe that, as a vector bundle, $\mathcal{W}$ is defined as soon as the distribution $\mathcal{D}$ is given (cf. Lemma 2.1). The full data of the Poisson tensor $\Lambda$ is only needed to define the algebra structure on its space of sections.

### 2.3 Fedosov-Moyal star products

**Definition 2.3** A foliated connection is a linear map

$$
\nabla : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}
$$

verifying ($f \in C^\infty(N)$)

(i) $\nabla f u v = f \nabla u v$,

(ii) $\nabla u f v = f \nabla u v + L_{\iota(u)} f v$.

A foliated connection is said to be symplectic if

(iii) $\nabla u v - \nabla v u - [u, v] = 0$,

(iv) $\nabla \omega = 0$.

**Lemma 2.4** On a regular Poisson manifold, a symplectic foliated connection always exists.

**Proof.** Choose any linear connection $\nabla^0$ in the vector bundle $\mathcal{D} \to N$. Since $\mathcal{D}$ is an involutive tangent distribution, the torsion $T^0$ of the connection is well defined as a section of $\mathcal{D}^* \otimes \text{End} \mathcal{D}$. One then obtains a “torsion-free” connection $\nabla^1 = \nabla^0 - \frac{1}{2} T^0$ in $\mathcal{D}$. Now, the formula

$$
\omega^D(S(u, v), w) = \frac{1}{3} \left( \nabla^1 \omega^D(v, w) + \nabla^1 \omega^D(u, w) \right)
$$

defines a tensor $S$, section of $\mathcal{D}^* \otimes \mathcal{D}^* \otimes \mathcal{D}$ such that $\nabla = \nabla^1 + S$ is as desired. 

Now, fix such a foliated symplectic connection $\nabla$ in $\mathcal{D}$ and consider its associated covariant exterior derivative

$$
\Omega^p(\mathcal{W}) \to \Omega^{p+1}(\mathcal{W}),
$$

declared by

$$
\partial s(u_1, ..., u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} (\nabla u_i, s)(u_1, ..., \hat{u}_i, ..., u_{p+1}).
$$

Lemma 2.3 then provides a 2-form $R \in \Omega^2(\mathcal{D} \otimes \mathcal{D}) \subset \Omega^2(\mathcal{W})$ defined by the formula

$$
2\hbar \partial^2 = \text{ad}(R).
$$

Inductively on the degree, one sees ([Fed96] (Theorem 5.2.2)) that the equation

$$
R + 2\hbar (\partial \gamma - \delta \gamma + \gamma^2) = 0.
$$

4
has a unique solution $\gamma \in \Omega^1(W)$ such that $\delta^{-1}\gamma = 0$.

This implies that the graded derivation

$$D = \partial - \delta + \text{ad}(\gamma)$$

of $(\Omega^*(W), \ast)$ is flat i.e. $D^2 = 0$. One then proves, again inductively, that the projection

$$W_D \xrightarrow{pr} C^\infty(N)[[\hbar]],$$

where $W_D$ is the kernel of $D$ restricted to the sections of $W$, is a linear isomorphism. The space of flat sections $W_D$ being a subalgebra of the sections of $W$ with respect to the product $\ast$ ($D$ is a derivation), the above linear isomorphism yields a star product on $C^\infty(N)$ called Fedosov-Moyal star product on $(N, \Lambda)$.

### 3 The main construction

#### 3.1 A particular Poisson manifold—Notations

Let $(M, \omega)$ be a compact symplectic manifold. Let $\Omega \in C^\infty([-\epsilon, \epsilon], \Omega^2(M))$ be a smooth path of symplectic structures on $M$ such that $\Omega(0) = \omega$. The smooth family $\{\Omega(t)\}_{t \in [-\epsilon, \epsilon]}$ then canonically defines on $\tilde{M} := M \times [-\epsilon, \epsilon]$ a Poisson structure $\tilde{\Omega}$ whose symplectic leaves are $\{(M \times \{t\}, \Omega(t))\}$. We will denote by $\mathcal{D} \subset T\tilde{M}$ the characteristic distribution of the Poisson structure $\tilde{\Omega}$ (i.e. $\mathcal{D}(x) = T(x)(M \times \{t\})$).

The spaces $C^\infty(\tilde{M})[[\hbar]]$ (resp. $C^\infty(M)[[\hbar]]$) of power series in $\hbar$ with values in the algebra of smooth functions on $\tilde{M}$ (resp. $M$) are $\mathbb{R}[[\hbar]]$-algebras. The quotient $\mathbb{R}[[\hbar]]$-algebra $C^\infty(\tilde{M})[[\hbar]]/\hbar^{n+1}C^\infty(\tilde{M})[[\hbar]]$ will be denoted by $\bar{C}^\infty(\tilde{M})[[\hbar]]_n$. It will often be identified with the space of polynomials in $\hbar$ of degree at most $n$ with values in $C^\infty(M)$. We will consider the natural inclusion $\iota : \bar{C}^\infty(\tilde{M})[[\hbar]]_n \hookrightarrow \bar{C}^\infty(\tilde{M})[[\hbar]]$ defined by $i(f)(x, t) = f(x) \forall t \in [-\epsilon, \epsilon]$. We will often denote $i(f)$ by $\tilde{f}$.

By $DO_{\mathcal{D}}(\tilde{M})$ we will denote the algebra of tangential (with respect to the distribution $\mathcal{D}$) differential operators on $\tilde{M}$, i.e. the set of all differential operators on $\tilde{M}$ vanishing on leafwise constant functions. By $DO(M)$ we will denote the algebra of differential operators on $M$. As above we can consider the $\mathbb{R}[[\hbar]]$-algebras, $DO_{\mathcal{D}}(\tilde{M})[[\hbar]]$ and $DO(M)[[\hbar]]/\hbar^{n+1}DO(M)[[\hbar]]$ (abbreviated by $DO(M)[[\hbar]]_n$). When dealing with bidifferential operators, we will use the prefix "biDO".

#### 3.2 Taylor expansions

We have $C^\infty(\tilde{M}) \simeq C^\infty([-\epsilon, \epsilon], C^\infty(M))$ seeing every element $a \in C^\infty(\tilde{M})$ as a function of one variable with values in a Fréchet space. We can therefore consider [Bou67] its Taylor expansion of order $n$ at $0$:

$$a(t) = \sum_{k=0}^{n} \frac{t^k}{k!} a^{(k)}(0) + t^n R_n(u)(t) \text{ with } R_n(u)(t) \rightarrow 0 \text{ as } t \rightarrow 0.$$ 

We define the $\mathbb{R}$-linear map, $j^n : C^\infty(\tilde{M}) \rightarrow C^\infty(M)[[\hbar]]_n$ by $j^n a = \sum_{k=0}^{n} h^k \frac{1}{k!} a^{(k)}(0)$. It is extended to $C^\infty(\tilde{M})[[\hbar]]$ in the following way:

$$C^\infty(\tilde{M})[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]_n$$

$$a = \sum_{i \geq 0} h^i a_i \mapsto j^n a = \sum_{k=0}^{n} h^k j^n a_i = \sum_{0 \leq j \leq n} h^{j+i} \frac{1}{j!} a_i^{(k)}(0).$$

One then has

**Lemma 3.1**

1. $j^n a = j^n (a \mod h^{n+1})$.
2. $j^n$ is an $\mathbb{R}[[\hbar]]$-algebra homomorphism.
We now extend the map \( j_n^h \) to \( DO_\mathcal{D}(\hat{M})[[h]] \) in the natural way.

**Definition 3.1**

1. For \( \Phi \in DO_\mathcal{D}(\hat{M})[[h]] \), we define the operator \( j_n^h \Phi : C^\infty(M)[[h]] \to C^\infty(M)[[h]] \) by
   \[
   j_n^h \Phi \cdot f = j_n^h (\Phi, \hat{f}), \quad \forall f \in C^\infty(M)[[h]].
   \]
   Similarly, for \( B \in biDO_\mathcal{D}(\hat{M})[[h]] \), we set \( j_n^h B : (f, g) = j_n^h (B, (\hat{f}, \hat{g})) \), \( \forall f, g \in C^\infty(M)[[h]] \).

**Lemma 3.2**

1. One has \( j_n^h \Phi \in DO(M)[[h]] \) and \( j_n^h B \in biDO(M)[[h]] \).
2. For all \( a, b \in C^\infty(M)[[h]] \) one has \( j_n^h (\Phi, a) = j_n^h \Phi \cdot j_n^h a \) and \( j_n^h (B, (a, b)) = j_n^h B \cdot (j_n^h a, j_n^h b) \).

**Proof.** We will show that \( j_n^h \Phi \) and \( j_n^h B \) are local hence differential by Peetre’s theorem [Pe55, Pe60, CGDW80]. Let \( f \in C^\infty(M) \) and \( U \) be an open set in \( M \) such that \( f_U \equiv 0 \). Since \( \hat{f}(x, t) = 0 \) \( \forall (x, t) \in U \times \epsilon, \epsilon \) and \( \Phi \) is differential, one has \( (\Phi, \hat{f})_U \equiv 0 \). Hence
   \[
   (j_n^h \Phi, f)_U = (j_n^h (\Phi, \hat{f}))_U = \sum_{0 \leq k+l \leq n} \frac{\hbar^{k+l}}{l!} (\Phi, \hat{f})^{(l)}_U - \epsilon, \epsilon(0) = 0.
   \]

The bidifferential case follows in the same way. This proves the first part of the lemma. The second one follows from simple computations.

**Remark 3.1** Lemma 3.2 implies that \( j_n^h \), defined as a map from \( DO_\mathcal{D}(\hat{M})[[h]] \) to \( DO(M)[[h]] \), is an \( \mathbb{R}[[h]] \)-algebra homomorphism for the composition product on both algebras.

### 3.3 Induced star-products

Let now \( \hat{*} \) be any tangential star product on \((\hat{M}, \hat{\Omega})\); for instance consider the Moyal-Fedosov star product defined in Section 2.

**Definition 3.2**

1. We define \( \ast_n \) to be the operation from \( C^\infty(M)[[h]] \times C^\infty(M)[[h]] \) to \( C^\infty(M)[[h]] \) given by
   \[
   f \ast_n g = j_n^h (\hat{f} \hat{*} \hat{g}).
   \]
   Equivalently (by Lemma 3.2), seeing \( \hat{*} \) as an element of \( biDO_\mathcal{D}(\hat{M})[[h]] \), one has \( \ast_n = j_n^h \hat{*} \).
2. We define \( \ast \) to be the operation from \( C^\infty(M)[[h]] \times C^\infty(M)[[h]] \) to \( C^\infty(M)[[h]] \) given by
   \[
   f \ast g \mod \hbar^{n+1} = f \mod \hbar^{n+1} \ast_n g \mod \hbar^{n+1}
   \]
   for all \( n \) in \( \mathbb{N} \).

**Lemma 3.3**

1. \( \ast_n \) is an associative product on the \( \mathbb{R}[[h]] \)-algebra \( C^\infty(M)[[h]] \).
2. \( \ast \) is a star-product on \( M \), called the **induced star-product** on \( M \) by \( \hat{*} \).

**Proof.** For \( f, g, h \in C^\infty(M)[[h]] \), one has \( (\hat{f} \hat{*} \hat{g}) \hat{*} \hat{h} = \hat{f} \hat{*} (\hat{g} \hat{*} \hat{h}) \). Therefore, \( j_n^h (\hat{f} \hat{*} \hat{g}) \hat{*} \hat{h} = j_n^h \hat{f} \hat{*} (\hat{g} \hat{*} \hat{h}) \) if and only if
   \[
   \left( j_n^h \hat{f} \hat{*} \hat{g}, \hat{h} \right) = j_n^h \left( \hat{f} \hat{*} \hat{g}, \hat{h} \right) \quad (\text{reformulation})
   \]
   \[
   \Leftrightarrow \left( j_n^h \hat{f} \hat{*} \hat{g}, \hat{h} \right) = j_n^h \hat{f} \hat{*} \hat{g}, \hat{h} \quad \text{(by Lemma 3.2)}
   \]
   \[
   \Leftrightarrow \left( j_n^h \hat{f} \hat{*} \hat{g}, \hat{h} \right) = j_n^h \hat{f} \hat{*} \hat{g}, \hat{h} \quad \text{(by Lemma 3.2)}
   \]
   \[
   \Leftrightarrow \left( j_n^h \ast_n g, h \right) = j_n^h \ast (g, h) \quad \text{by Definition 3.2}
   \]
   This proves item 1 which is a classical way to show that a star-product is associative.
**Corollary 3.1** If $\hat{\ast}_1$ and $\hat{\ast}_2$ are tangentially equivalent tangential star products on $(\hat{M}, \hat{\Omega})$, then the induced star products $\ast_1$ and $\ast_2$ on $(M, \omega)$ are equivalent.

**Proof.** The hypothesis implies that there exists an equivalence $\Phi \in DO_D(\hat{M})[[h]]$ such that $\Phi.(a \hat{\ast}_1 b) = \Phi.a \hat{\ast}_2 \Phi.b$ for all $a, b \in C^\infty(M) [[h]]$. We then check, as in the proof of Lemma 3.3, that the operator $\Psi \mod h^n+1 := j_n^\Phi$, $n \in \mathbb{N}$ defines an equivalence between $\ast_1$ and $\ast_2$. \[
\]

## 4 Characteristic classes

Let $\Omega^h = \sum_{k \geq 0} \hbar^k \omega^k \in Z^2(M) [[\hbar]]$ be a formal power series of closed 2-forms on $M$. A refinement of the classical Borel lemma (see the appendix) yields

**Lemma 4.1** Let $\Omega^h_i \in Z^2(M) [[\hbar]]$ (i = 1, 2). Assume that $[\Omega^h_1] = [\Omega^h_2]$ in $H^2(M) [[\hbar]]$ or, equivalently, that there exists $\nu^h \in \Omega^1(M) [[\hbar]]$ such that $\Omega^h_2 - \Omega^h_1 = d\nu^h$. Then there exists smooth functions $\Omega_i \in C^\infty([-\epsilon, \epsilon], \Omega^2(M))$ and $\nu \in C^\infty([-\epsilon, \epsilon], \Omega^1(M))$ such that

(i) $\frac{1}{\hbar^{\beta}} \frac{d}{dt} \Omega_i |_{t=0} = \omega^h_i$;

(ii) $\forall t, \Omega_i(t)$ is symplectic;

(iii) $\forall t, \Omega_2(t) - \Omega_1(t) = d(\nu(t))$ or, equivalently, $[\Omega_1(t)] = [\Omega_2(t)]$.

**Definition 4.1** Let us fix a connection $\nabla^0$ in the vector bundle

$D \to \hat{M} = M \times [-\epsilon, \epsilon]$.

Let $\Omega^h \in \Omega^2(M) [[\hbar]]$ be a series of closed 2-forms on $M$ such that $\Omega^h \mod \hbar = \omega$. Let $\Omega \in C^\infty([-\epsilon, \epsilon], \Omega^2(M))$ be a smooth family of symplectic structures on $M$ admitting $\Omega^h$ as $\infty$-jet (cf. Lemma 3.7). Let $\nabla$ be the symplectic foliated connection on $\hat{M}$ obtained from the data of $\nabla^0$ and $\Omega$ (cf. Section 3). Let $\ast$ be the Moyal-Fedosov star product on $(\hat{M}, \hat{\Omega})$ associated to $\nabla$. The star product $\ast_{\Omega^h}$ on $(M, \omega)$ induced by $\ast$ will be called the **star product associated to the series** $\Omega^h$.

**Proposition 4.1** Let $\Omega^h_i$ (i = 1, 2) be two series of closed 2-forms on $M$ such that $\Omega^h_i \mod \hbar = \omega$. Denote by $\ast_i$ (i = 1, 2) the associated star products on $(M, \omega)$. Then $\ast_1$ and $\ast_2$ are equivalent star products if and only if $[\Omega^h_1] = [\Omega^h_2]$ in $H^2_{\text{de Rham}} [[\hbar]]$.

The proof of Proposition 4.1 is postponed to the end of this section.

**Definition 4.2** A diffeomorphism $\hat{\varphi} : \hat{M} \to \hat{M}$ preserves the foliation if

(i) $\hat{\varphi}(M_t) \subset M_t$ \quad \forall t and

(ii) $\hat{\varphi}|_{M_0} = id_{M_0}$.

We first adapt Moser’s lemma to our parametric situation.

**Lemma 4.2** Let $\{\Omega_i(t)\}_{t \in [-\epsilon, \epsilon]}$ (i = 1, 2) be two smooth families of symplectic structures on $M$ such that $\Omega_1(0) = \Omega_2(0) = \omega$. Assume that, for all $t \in [-\epsilon, \epsilon]$ they have the same de Rham class : $[\Omega_1(t)] = [\Omega_2(t)]$ in $H^2(M)$. Then there exists a Poisson diffeomorphism $\hat{\varphi} : (\hat{M}, \hat{\Omega}) \to (M, \Omega_1)$ which preserves the foliation.

**Proof.** By Hodge’s theory one has that $\Omega_1(t) - \Omega_2(t) = dv^t$ where $v^t \in \Omega^1(M)$ is smooth in $t$. Set $\omega^h_1 = \Omega_1(t) + s dv^t$, $s \in [0, 1]$. The form $\omega^h_1$ is symplectic on $M$ for all $s \in [0, 1]$; hence by compactness, one can choose $\epsilon > 0$ such that $\omega^h_1$ is symplectic for all $t \in [-\epsilon, \epsilon]$, and $s \in [0, 1]$.
Consider \( N = M \times [0, 1] \) endowed with the natural foliation \( \mathcal{F} = \{ M \times \{ s \} \} \). Define the following smooth families of 2-forms on \( N \):

\[
(\tilde{\omega}_t)_{(x,s)} := (\omega^t)_x \quad \text{and} \quad (\omega_t)_{(x,s)} := (\tilde{\omega}_t)_{(x,s)} - (\nu^t)_x \wedge ds.
\]

Then \( d_N(\omega_t) = d_N(\tilde{\omega}) - d_M(\nu^t) \wedge ds = 0 \) for all \( t \). Moreover, \( \operatorname{rad}_{\mathcal{F}}(N)(\omega_t) \) is not entirely contained in \( T(\mathcal{F}) \); hence one can find a smooth family of vector fields of the form : \( X_t = \frac{\partial}{\partial s} + Y_t \) (\( Y_t \in \Gamma(T(\mathcal{F})) \)) generating the smooth family of smooth distributions : \( \operatorname{rad}(\omega_t) \).

One has therefore

\[
L_{X_t} \omega_t = d(i_{X_t} \omega_t) + i_{X_t} d\omega_t = 0.
\]

Denoting by \( \{ \varphi^N_x \} \) the flow of \( X_t \), one has:

\[
(\varphi^N_{X_t})^* \omega_t = \omega_t \quad \text{and} \quad \varphi^N_{X_t}(M \times \{ s \}) = M \times \{ s + u \}.
\]

One then gets a smooth family \( \{ \varphi_t \} \) of diffeomorphisms of \( M \) defined by

\[
\varphi^1_{X_t} \circ i_0 = i_1 \circ \varphi_t
\]

such that \( \varphi^1_{X_t}(\Omega_1(t)) = \Omega_2(t) \) \((i_s : M \rightarrow N \) denotes the natural inclusion \( i_s(x) = (x, s)\)). Shrink \( \epsilon \) once more if necessary, one gets the desired Poisson map by setting \( \tilde{\varphi}(x, t) = (\varphi_t(x), t) \). Observe that \( X_0 = \partial_s \), hence \( \varphi_0 = \text{id}_M \).

**Lemma 4.3** Let \( \ast_i \quad (i = 1, 2) \) be tangential star products on \( \hat{M} \). Suppose there exists a diffeomorphism \( \tilde{\varphi} : \hat{M} \rightarrow \hat{M} \) preserving the foliation such that \( \ast^1_i = \tilde{\varphi} \) mod \( (h^n) \). Then, \( \ast_1 \) and \( \ast_2 \) are equivalent star products up to order \( n \).

**Proof.** The right action of the diffeomorphism group, \( C^\infty(\hat{M}) \times \operatorname{Diff}(\hat{M}) \xrightarrow{\rho^\hat{M}} C^\infty(\hat{M}) \), \( \rho(\tilde{\varphi})u = \tilde{\varphi}^*u \) yields a map:

\[
\rho^h_n : \operatorname{Diff}(\hat{M}) \rightarrow \operatorname{Hom}(C^\infty(M), C^\infty(M)[[h]]_n) : \rho^h_n(\tilde{\varphi})f = \hat{f}_1^n(\tilde{\varphi}^*\hat{f}).
\]

Definition \( 4.2 \) implies that if \( \tilde{\varphi} \) preserves the foliation, then \( \rho^h_n(\tilde{\varphi}) \in \operatorname{DO}(M)[[h]]_n \) and \( \rho^h_n(\tilde{\varphi}) = \text{id} \). Therefore an argument similar to the one used for Lemma \( 3.3 \) yields the conclusion.

**Corollary 4.1** Within the notations of Proposition \( 4.7 \), if \( \Omega^1_h \) and \( \Omega^2_h \) are cohomologous in \( H^2(M)[[h]] \), then the star products \( \ast_1 \) and \( \ast_2 \) are equivalent.

**Proof.** The first \( N \) cochains of a Fedosov star product are entirely determined by the \( N \) first terms of its Weyl curvature. Therefore, the above lemmas imply that \( \ast_1 \) and \( \ast_2 \) are equivalent up to any order. It is then classical that they are equivalent [BFF+77].

**Proof of Proposition 4.4**

We first consider a particular case. Let \( \alpha^h = \alpha^0 + h \alpha^1 + \ldots \in Z^2(M)[[h]] \) be a sequence of closed 2-forms on \( M \). Set \( \Omega^h = \Omega^h + h \alpha^h \). Denote by \( \Omega, \alpha \) and \( \Omega' = \Omega + t^k \alpha \) respectively the smooth functions associated to the series \( \Omega^h, \alpha^h \) and \( \Omega^h \) as in Lemma \( 4.1 \). The function \( \Omega' \) defines a Poisson structure on \( M \). We denote by \( \Lambda' \) (resp. \( \nu' \)) the corresponding bivector field (resp. \( D^2 \)-form). One has

\[
\omega'^t = \omega^t + t^k \alpha^t \quad \text{and} \quad \Lambda'^t = \Lambda' - t^k \nu \omega^0 + t^{k+1} \lambda,
\]

where we denote again by \( \alpha^t \) the \( D^2 \)-form corresponding to \( \alpha^t \) and where \( \lambda \) is an element of \( C^\infty([-\epsilon, \epsilon], \Gamma \Lambda^2 \mathcal{D}) \). Let \( \ast' \) be the star-product on \( M \) induced by the Moyal-Fedosov star-product \( \ast' \) on \( (\hat{M}, \Lambda') \). We now define a specific foliated symplectic connection \( \nabla' \) adapted to \( \omega'^t \). Let us look for \( \nabla' \) of the form \( \nabla + S \) where \( S \) is a symmetric 2-\( \mathcal{D} \)-tensor field. We set

\[
\omega'^t(\nabla'_{u'}v, w) = \omega'^t(\nabla u v, w) + \frac{1}{3} (\nabla_{u} \omega'^t)(v, w) + \frac{1}{3} (\nabla_{v} \omega'^t)(u, w).
\]
This leads to \((\omega^t + t^k \alpha) (S(u, v), w) = t^k \frac{1}{t} \left[ \nabla_u \alpha \right] (v, w) + (\nabla_v \alpha) (u, w) \) as \(\nabla \omega^t = 0\). By construction \(\omega^t + t^k \alpha^t\) is invertible, so \(S(u, v)\) is completely determined and of the form \(S(u, v) = t^k s(u, v)\). We thus have

\[
\nabla' = \nabla + t^k s. \tag{2}
\]

Let now \(o^t\) (resp. \(o'^t\)) be the associative product on the sections of the Weyl bundle \(\mathcal{W}\) over \(\tilde{M}\) determined by the data of \(\Lambda\) (resp. \(\Lambda'\)) (cf. Section \ref{sec:pre} and Remark \ref{rem:assoc}). By construction, we then get \(\forall u, v \in \mathcal{W},\)

\[
\frac{d^l}{dt^l} (u \circ^t v - u \circ'^t v)(0) = 0 \quad \forall l \leq k - 1. \tag{3}
\]

Similarly for Moyal-Fedosov star products, \(\ast\) and \(\ast',\) associated to \((\Omega, \nabla)\) and \((\Omega', \nabla'),\) \ref{eq:dfdg} and \ref{eq:dfdg'} yield

\[
\frac{d^l}{dt^l} (\ast b - \ast' b)(0) = 0 \quad \forall l \leq k - 1. \tag{4}
\]

Now let us see what happens for \(\ast\) and \(\ast'.\) Let \(f, g \in C^\infty(M)\) and write \(\ast = \sum_{i \geq 0} \hbar^i \hat{C}_i\) and \(\ast' = \sum_{i \geq 0} \hbar^i \hat{C}'_i.\) Setting \(u^{(i)} := \frac{d}{dt^i} u,\) we have

\[
\begin{align*}
    f \ast g - f \ast' g &= \\
    = \sum_{j \geq 0} \frac{\hbar^j}{j!} \left( \sum_{i \geq 0} \frac{1}{j!} \left( \sum_{k \geq 0} \hat{C}_i(f, g) - \hat{C}'_i(f, g) \right) \right)^{(j)} (0) \quad \text{(cf. equation \ref{eq:dfdg})} \\
    = \sum_{j \geq k, i \geq 0} \frac{\hbar^{i+j}}{j!} \left( \sum_{k \geq 0} \hat{C}_i(f, g) - \hat{C}'_i(f, g) \right)^{(j)} (0) \\
    = \sum_{m \geq k} \frac{\hbar^m}{m!} \sum_{m-i, j \geq k, i, j \geq 0} \frac{1}{j!} \left( \sum_{k \geq 0} \hat{C}_i(f, g) - \hat{C}'_i(f, g) \right)^{(j)} (0) \\
    = \frac{\hbar^k}{k!} (fg - gf) + \frac{\hbar^{k+1}}{k!} (fg - gf) + \frac{\hbar^{k+1}}{k!} \left( \sum_{m \geq k+2} \frac{1}{m!} \left( \sum_{m-i, j \geq k, i, j \geq 0} \frac{1}{j!} \left( \sum_{k \geq 0} \hat{C}_i(f, g) - \hat{C}'_i(f, g) \right) \right)^{(j)} (0) \\
    = \frac{\hbar^{k+1}}{k!} (\Lambda^t(df, dg) - \Lambda'^t(df, dg))^{(k)} (0) + o(\hbar^{k+1}) \\
    = \frac{\hbar^{k+1}}{k!} \left( \sum_{m \geq k+2} \frac{1}{m!} \left( \sum_{m-i, j \geq k, i, j \geq 0} \frac{1}{j!} \left( \sum_{k \geq 0} \hat{C}_i(f, g) - \hat{C}'_i(f, g) \right) \right)^{(j)} (0) \\
    = \frac{\hbar^{k+1}}{k!} (\Lambda^t(df, dg) - \Lambda'^t(df, dg) + \hbar^k \alpha \omega^t(df, dg) - \hbar^{k+1} \lambda(df, dg))^{(k)} (0) \\
    & \quad + o(\hbar^{k+1})
\end{align*}
\]

Then, setting \(\ast = \sum_{i \geq 0} \hbar^i C_i\) and \(\ast' = \sum_{i \geq 0} \hbar^i C'_i,\) we have

\[
C'_i = C_i, \quad i = 0, \ldots, k \quad \text{and} \quad C'_{k+1} = C_{k+1} + \hbar \omega \alpha. \tag{5}
\]

Let us pass to the general case. Suppose that \([\Omega^1] \neq [\Omega^2].\) We denote by \(k\) the smallest integer such that \([\omega^1] \neq [\omega^2].\)

Let us consider \(\Omega^k = h \omega^1 + h^2 \omega^2 + \cdots + h^{k-1} \omega^{k-1} + h^k \omega^k + h^{k+1} \omega^k + \cdots.\) We have \([\Omega^1] = [\Omega^2]\) and \([\Omega^1] = \Omega^2 + h^k (\omega^1 - \omega^2) + h^{k+1} \omega^k + \cdots.\) Denoting by \(\sum_{k \geq 0} \hbar^k \omega\alpha\) the product associated with \(\omega\alpha,\) we know that \(\ast_2\) and \(\ast_3\) are equivalent. What has been done previously implies \(\ast_1 = \ast_3 \mod h^{k+1}\) and \(C_{k+1}^{(1)} = C_{k+1}^{(3)} \pm \hbar \omega \alpha\) with \(\omega \alpha = \omega^1 - \omega^2.\) But in this case, we know that \(\ast_1 = \ast_3 \mod h^{k+2}\) if and only if \(\omega \alpha\) is exact \([\text{BCG97}].\) Since \(\omega^1 - \omega^2\) is not exact by hypothesis, \(\ast_1 \not\equiv \ast_3\) and thus \(\ast_1 \not\equiv \ast_2.\)
5 Appendix: Borel's Lemma

Proposition 5.1 (Borel's Lemma)

Let $M$ be a compact smooth manifold of dimension $d$ and $\{\alpha_n \in \Omega^q(M) \mid n \in \mathbb{N}\}$ be a sequence of $q$-forms on $M$. Then there exists $f \in C^\infty(\mathbb{R}, \Omega^q(M))$ such that $\frac{d^n f}{dt^n}(0) = \alpha_n$.

Proof. Let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $\varphi(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\varphi(t) = 0$ for $|t| \geq 1$ and set $f_n : \mathbb{R} \to \Omega^q(M)$, $f_n(t) = \frac{\alpha_n}{q!} \varphi(\lambda_n t)$ where the numbers $\{\lambda_n\}$ will be defined later. Let $\{V_i, \psi_i, i = 1, \ldots, N\}$ be a finite (M is compact) atlas of $M$ trivializing the bundle $\wedge^q T^* M \to M$. Restricted to such a chart, we can view a section of $\wedge^q T^* M \to M$ as a function from $\psi_i(V_i) \to \mathbb{R}^q$ with $s = (q \choose d)$. We denote by $|| \cdot ||$ the Euclidean norm on $\mathbb{R}^q$. One can make a choice of $\{\lambda_n\}$ is such a way that, for all $k \in \mathbb{N}$ such that $0 \leq k \leq n - 1$, one has $\sup \{||D^n \frac{\partial}{\partial t} f_n(x,t)|| \mid (x,t) \in V_i \times \mathbb{R}, \ |\nu| + l \leq k, \ i = 1, \ldots, N\} \leq \frac{1}{\lambda_n}$ where $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d$, $|\nu| = \nu_1 + \ldots + \nu_d$ and $D^\nu = \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} \ldots \frac{\partial^{\nu_d}}{\partial x_d^{\nu_d}}$. Indeed, let us fix a $V_i$. In this chart we have

$$D^\nu, \frac{\partial^l}{\partial t^l} f_n(x,t) = \sum_{p=0}^{l} \left( \begin{array}{c} l \\ p \end{array} \right) \frac{n!}{(n-p)!} \frac{D^{\nu} \alpha_n}{n!} t^{n-p} \lambda_n^{-p} \varphi((l-p))(\lambda_n t).$$

Define $K_n = \sum_{|\nu|=1}^{n} \sup_{x \in V_i} ||D^\nu \alpha_n(x)||$ and $M_n = \sum_{j=1}^{n} \sup_{t \in \mathbb{R}} |\varphi^{(j)}(t)|$. On the support of $\varphi^{(l-p)}(\lambda_n t)$ we have $\lambda_n t \leq 1$.

Hence $||D^n \frac{\partial}{\partial t} f_n(x,t)|| \leq \sum_{p=0}^{l} \left( \begin{array}{c} l \\ p \end{array} \right) K_n \frac{M_n}{(n-p)!} \lambda_n^{n-1} \leq K_n M_n \lambda_n \sum_{p=0}^{n-1} \left( \begin{array}{c} n-1 \\ p \end{array} \right) \frac{1}{(n-p)!}$ as $n-p \geq n-1 \geq n-k \geq 1$. Therefore $\lambda_n^{n-p} \geq \lambda_n$ if $\lambda_n \geq 1$ and $\left( \begin{array}{c} l \\ p \end{array} \right) \leq \left( \begin{array}{c} n-1 \\ p \end{array} \right)$. Thus a choice of the $\lambda_n$’s such that

$$\lambda_n \geq \max\{1, 2^n K_n M_n \sum_{p=0}^{n-1} \left( \begin{array}{c} n-1 \\ p \end{array} \right) \frac{1}{(n-p)!}\}$$

yields the assertion on $V_i$. The conclusion follows by finiteness of our atlas. In particular the function $f := \sum_{n=0}^{\infty} f_n$ is well defined. By the preceding lemma $f$ is well defined. Moreover, $f \in C^k(M \times \mathbb{R}, \wedge^q T^* M)$ for all $k \in \mathbb{N}$ as it appears when writing $f(x,t) = \sum_{n=0}^{k} f_n(x,t) + \sum_{n=k+1}^{\infty} f_n(x,t)$. Therefore $f \in C^\infty(M \times \mathbb{R}, \wedge^q T^* M)$. By definition of $f$, $f(x,t) \in \wedge^q T^*_x M$, hence $f(\cdot,t)$ is a smooth section of $\wedge^q T^* M \to M$ and $f : \mathbb{R} \to \Omega^q(M)$ with $f(t) = f(\cdot,t)$ belongs to $C^\infty(\mathbb{R}, \Omega^q(M))$. Moreover, we have

$$f^{(k)}(t) = \sum_{n=0}^{k} \frac{\alpha_n}{n!} \left( t^n \varphi(\lambda_n t) \right)^{(k)}(0) + \sum_{n=k+1}^{\infty} \sum_{p=0}^{\infty} \left( \begin{array}{c} k \\ p \end{array} \right) \frac{n!}{(n-p)!} \frac{\alpha_n}{n!} t^{n-p} \varphi^{(k-p)}(\lambda_n t).$$

In the second sum, we have $n-p \geq n-k \geq 1$. Thus it vanishes for $t = 0$. In the first sum, if $n \leq k-1$, $\varphi$ is differentiated at least once. As $\varphi^{(j)}(0) = 0$ for $j \geq 1$, it vanishes for $t = 0$. Therefore, we have $f^{(k)}(0) = \frac{\alpha_k}{k!} \left( t^k \varphi(\lambda_k t) \right)^{(0)}(0) = \frac{\alpha_k}{k!} \sum_{p=0}^{k} \left( \begin{array}{c} k \\ p \end{array} \right) \frac{k!}{(k-p)!} \lambda_k^{k-p} \varphi^{(k-p)}(\lambda_k t) |_{t=0}$.

For $p \leq k-1$, we have $k-p \geq 0$ and the corresponding term vanishes. Hence $f^{(k)}(0) = \frac{\alpha_k}{k!} k! \varphi^{(k)}(0) = \alpha_k$. □

Corollary 5.1 Let $(\alpha_n)_{n \in \mathbb{N}}, (\alpha_n^1)_{n \in \mathbb{N}}, (\alpha_n^2)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ be sequences of forms on $M$. Then there exist smooth functions $f^1$, $f^2$ and $f$ corresponding respectively to $(\alpha_n^1)_{n \in \mathbb{N}}, (\alpha_n^2)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ as in Proposition 5.1 such that
1. if \( d\alpha_n = 0 \), \( \forall n \in \mathbb{N} \) then, \( d(f(t)) = 0 \), \( \forall t \in \mathbb{R} \).

2. if \( \alpha^2_n - \alpha^1_n = d\nu_n \) \( \forall n \in \mathbb{N} \), then \( f^2(t) - f^1(t) = d(f(t)) \), \( \forall t \in \mathbb{R} \).

Proof. 1) We have \( f_n(t) = \frac{\alpha_n}{n!}\varphi(\lambda_n t) \) hence \( d(f_n(t)) = 0 \), \( \forall t \in \mathbb{R} \). For each \( t \), \( f(t) = \sum_{n=0}^{\infty} f_n(t) \) converges absolutely in \( \Gamma^1(M, \Lambda^2 T^* M) \).

2) Let \( \lambda^1_n, \lambda^2_n \) and \( \lambda^3_n \) be three real sequences defining smooth functions \( \tilde{f}^1, \tilde{f}^2 \) and \( \tilde{f} \) corresponding respectively to \( (\alpha^1_n)_{n \in \mathbb{N}}, (\alpha^2_n)_{n \in \mathbb{N}} \) and \( (\nu_n)_{n \in \mathbb{N}} \) as in the proof of Proposition 5.1. Consider the sequence \( \mu_n = \max\{\lambda^1_n, \lambda^2_n, \lambda^3_n\} \). Replacing \( \lambda^1_n, \lambda^2_n \) and \( \lambda^3_n \) by \( \mu_n \) we get new functions \( f^1, f^2 \) and \( f \) again corresponding respectively to \( (\alpha^1_n)_{n \in \mathbb{N}}, (\alpha^2_n)_{n \in \mathbb{N}} \) and \( (\nu_n)_{n \in \mathbb{N}} \) such that \( f^2_n - f^1_n = df_n \) \( \forall n \in \mathbb{N} \). Since for \( t \) fixed the series converge absolutely for the \( C^0 \) norm on the forms, we obtain the result.

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