GODSIL–MCKAY SWITCHING AND TWISTED GRASSMANN GRAPHS

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Dedicated to Andries E. Brouwer on the occasion of his 65th birthday

Abstract. We show that the twisted Grassmann graphs introduced by Van Dam and Koolen are obtained by Godsil–McKay switching applied to the Grassmann graphs. The partition for the switching is constructed by a polarity of a hyperplane.

1. Introduction

The twisted Grassmann graphs introduced by Van Dam and Koolen [7], are the first family of non-vertex-transitive distance-regular graphs with unbounded diameter. We refer the reader to [4, 5, 8] for an extensive discussion of distance-regular graphs, to [12] for a characterization of Grassmann graphs, and to [3, 9] for more information on the twisted Grassmann graphs.

Let $V$ be a $(2e+1)$-dimensional vector space over a finite field $\text{GF}(q)$. If $W$ is a subset of $V$ closed under multiplication by the elements of $\text{GF}(q)$, then we denote by $[W]$ the set of 1-dimensional subspaces contained in $W$. We also denote by $[W]^k$ the set of $k$-dimensional subspaces of $W$, when $W$ is a vector space. The Grassmann graph $J_q(2e+1, e+1)$ is the graph with vertex set $[V]^{e+1}$, where two vertices $W_1, W_2$ are adjacent whenever $\dim W_1 \cap W_2 = e$.

Let $H$ be a fixed hyperplane of $V$. The twisted Grassmann graph $\tilde{J}_q(2e + 1, e)$ has $A \cup B$ as the set of vertices, where

$$ A = \{ W \in \begin{bmatrix} V \end{bmatrix}^{e+1}_{e+1} | W \not\subset H \}, $$

$$ B = \begin{bmatrix} H \end{bmatrix}^{e-1}_{e-1}. $$
The adjacency is defined as follows:

\[ W_1 \sim W_2 \iff \begin{cases} \dim(W_1 \cap W_2) = e & \text{if } W_1 \in \mathcal{A}, W_2 \in \mathcal{A}, \\ W_1 \supseteq W_2 & \text{if } W_1 \in \mathcal{A}, W_2 \in \mathcal{B}, \\ \dim(W_1 \cap W_2) = e - 2 & \text{if } W_1 \in \mathcal{B}, W_2 \in \mathcal{B}. \end{cases} \]

Let \( \sigma \) be a polarity of \( H \). That is, \( \sigma \) is an inclusion-reversing permutation of the set of subspaces of \( H \), such that \( \sigma^2 \) is the identity. The pseudo-geometric design constructed by Jungnickel and Tonchev [11] has \([V]\) as the set of points, and \( \mathcal{A}' \cup \mathcal{B}' \) as the set of blocks, where

\[ \mathcal{A}' = \{ [\sigma(W \cap H)] \cup [W \setminus H] \mid W \in \mathcal{A} \}, \]
\[ \mathcal{B}' = \{ [W] \mid W \in \left[\begin{array}{c} H \\ e+1 \end{array}\right] \}. \]

It is shown in [11] that the incidence structure \(([V], \mathcal{A}' \cup \mathcal{B}')\) is a 2-(\( v, k, \lambda \)) design, where

\[ v = \frac{q^{2e+1} - 1}{q - 1}, \quad k = \frac{q^{e+1} - 1}{q - 1}, \quad \lambda = \frac{(q^{2e-1} - 1) \cdots (q^{e+1} - 1)}{(q^{e-1} - 1) \cdots (q - 1)}. \]

Recall that the geometric design \( \text{PG}_e(2e, q) \) has \([V]\) as the set of points, and \([V_{e+1}]\) as the set of blocks. Jungnickel and Tonchev [11] describe this design as the one obtained from the geometric design \( \text{PG}_e(2e, q) \) by distorting with the help of a polarity acting on a fixed hyperplane in \( \text{PG}(2e, q) \). The sizes of the intersections of pairs of blocks are

\[ \frac{q^i - 1}{q - 1} \quad (i = 1, \ldots, e), \]

in both geometric design and pseudo-geometric design. The block graph of \( \text{PG}_e(2e, q) \), where two distinct blocks are adjacent whenever their intersection has size \((q^e - 1)/(q - 1)\), is nothing but the Grassmann graph \( J_q(2e + 1, e + 1) \). The block graph \( \Delta(e, q) \), defined in a similar manner for the pseudo-geometric design, is shown to be isomorphic to the twisted Grassmann graph \( \tilde{J}_q(2e + 1, e) \) by the author and Tonchev [13].

In this paper, we show that \( \Delta(e, q) \) is obtained from the Grassmann graph \( J_q(2e + 1, e + 1) \) via Godsil–McKay switching. The following commutative diagram illustrates the situation.

\[
\begin{array}{ccc}
\text{PG}_e(2e, q) & \xrightarrow{\text{block graph}} & J_q(2e + 1, e + 1) \\
\text{pseudo-geometric design} & \xrightarrow{\text{block graph}} & \Delta(e, q) \cong \tilde{J}_q(2e + 1, e)
\end{array}
\]
It is worth mentioning that Van Dam, Haemers, Koolen and Spence [6] constructed a large number of graphs cospectral with distance-regular graphs including the Grassmann graphs, by Godsil–McKay switching and distorting the set of lines of a partial linear space. The contribution of the present paper is simply that distorting which leads to the construction of the twisted Grassmann graph can also be described by Godsil–McKay switching.

2. Godsil–McKay switching

Let $\Gamma$ be a graph with vertex set $X$, and let $\{C_1, \ldots, C_t, D\}$ be a partition of $X$ such that $\{C_1, \ldots, C_t\}$ is an equitable partition of the subgraph induced on $X \setminus D$. This means that the number of neighbors in $C_i$ of a vertex $x$ depends only on $j$ for which $x \in C_j$ holds, and independent of the choice of $x$ as long as $x \in C_j$. Assume also that for any $x \in D$ and $i \in \{1, \ldots, t\}$, $x$ has either $0, \frac{1}{2}|C_i|$ or $|C_i|$ neighbors in $C_i$. The graph $\tilde{\Gamma}$ obtained by interchanging adjacency and nonadjacency between $x \in D$ and the vertices in $C_i$ whenever $x$ has $\frac{1}{2}|C_i|$ neighbors in $C_i$, is cospectral with $\Gamma$ (see [10]). The operation of constructing $\tilde{\Gamma}$ from $\Gamma$ is called Godsil–McKay switching.

Godsil–McKay switching has been known for years, as a method to construct cospectral graphs. However, finding an instance for which the hypotheses of Godsil–McKay switching are satisfied, is nontrivial. We mention recent work [1, 2], but mainly the case $t = 1$ has been treated. In our work, $t$ will be unbounded.

We now take $\Gamma$ to be the Grassmann graph $J_q(2e+1, e+1)$, and keep the same notation as in Section 1. Recall that $H$ is a fixed hyperplane of $V$, and the set $A$ is defined in (1).

**Lemma 1.** Let $\{C_1, C_2, \ldots, C_t\}$ be an equitable partition of the graph $J_q(2e, e)$ with vertex set $\left[\begin{array}{c} V \\ e \end{array}\right]$. Let

$$\tilde{C}_i = \{W \in \left[\begin{array}{c} V \\ e + 1 \end{array}\right] | W \cap H \subseteq C_i\} \quad (1 \leq i \leq t).$$

Then $\{\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_t\}$ is an equitable partition of the subgraph $J_q(2e + 1, e + 1)$ induced on $A$.

**Proof.** By the assumption, for $1 \leq i, j \leq t$, there exists an integer $m_{ij}$ such that

$$|\{U \in C_i \mid \dim U \cap U' = e - 1\}| = m_{ij}$$

whenever $U' \in C_j$. Suppose $U \in C_i$ and $U' \in C_j$ satisfy $\dim U \cap U' = e - 1$. For $W' \in \tilde{C}_j$ with $W' \cap H = U'$, counting the number of elements
\{(Z, W) \in [W' \setminus H] \times \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = U, Z \subset W\}

in two ways, we find
\[ q^e = q^{e-1} \{W \in \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = U, W \cap W' \not\subset H\} \].

Thus,
\[ \left| \{W \in \tilde{C}_i \mid \dim W \cap W' = e\} \right| = \sum_{U \in \mathcal{C}_i} \left| \{W \in \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = U, \dim W \cap W' = e\} \right| = \sum_{U \in \mathcal{C}_i} \left| \{W \in \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = U, W \cap W' \not\subset H\} \right| \]
\[ + \delta_{ij} \left| \{W \in \left[ \begin{array}{c} V \\ e + 1 \end{array} \right] \mid W \cap H = W \cap W'\} \right| \]
\[ = qm_{ij} + \delta_{ij} \left| \{\tilde{W} \in [V/U'] \mid \tilde{W} \not\subset [H/U'] \cup [W'/U']\} \right| \]
\[ = qm_{ij} + \delta_{ij} (|[V/U']| - |[H/U']| - 1) \]
\[ = qm_{ij} + \delta_{ij} (q^e - 1). \]

Therefore, every vertex in \( \tilde{C}_i \) has exactly \( qm_{ij} + \delta_{ij} (q^e - 1) \) neighbors in \( \tilde{C}_j \). \( \square \)

Let
\[ C_U = \{W \in \mathcal{A} \mid W \cap H = U\} \quad (U \in \left[ \begin{array}{c} H \\ e \end{array} \right]), \]
\[ D = \left[ \begin{array}{c} H \\ e + 1 \end{array} \right], \]
\[ \mathcal{C} = \{C_U \cup C_{\sigma(U)} \mid U \in \left[ \begin{array}{c} H \\ e \end{array} \right]\}, \]
where \( \sigma \) is a polarity of \( H \). Then
\[ \mathcal{A} = \bigcup_{U \in \left[ \begin{array}{c} H \\ e \end{array} \right]} C_U \quad \text{(disjoint).} \]

**Lemma 2.** With the above notation, \( \mathcal{C} \) is an equitable partition of the subgraph of \( J_q(2e + 1, e + 1) \) induced on \( \mathcal{A} \).
Proof. Observe that, the partition \( \{ \{ U, \sigma(U) \} \mid U \in \left[ \frac{H}{e} \right] \} \) of the graph \( J_q(2e, e) \) with vertex set \( \left[ \frac{H}{e} \right] \), is equitable. This is because \( \dim U \cap U' = \dim \sigma(U) \cap \sigma(U') \) for any \( U, U' \in \left[ \frac{H}{e} \right] \). The result follows from Lemma 1. \( \square \)

Let \( U \in \left[ \frac{H}{e} \right] \) and \( W \in D \). Since \( \{ W_1 \in C_U \mid \dim W_1 \cap W = e \} = \begin{cases} C_U & \text{if } W \supset U, \\ \emptyset & \text{otherwise,} \end{cases} \)

we have

\[
\left| \left\{ W_1 \in C_U \cup C_{\sigma(U)} \mid \dim W_1 \cap W = e \right\} \right| = \begin{cases} |C_U \cup C_{\sigma(U)}| & \text{if } W \supset U + \sigma(U), \\ |C_U| & \text{if } W \supset U \text{ and } W \not\supset \sigma(U), \\ |C_{\sigma(U)}| & \text{if } W \not\supset U \text{ and } W \supset \sigma(U), \\ 0 & \text{otherwise} \end{cases},
\]

\[
el \in \{|C_U \cup C_{\sigma(U)}|, \frac{1}{2}|C_U \cup C_{\sigma(U)}|, 0\}.
\]

This implies that the partition \( \{ C_U \cup C_{\sigma(U)} \mid U \in \left[ \frac{H}{e} \right] \} \cup \{ D \} \)\]

of \( \left[ \frac{V}{e+1} \right] \) satisfies the hypothesis of Godsil–McKay switching. Let \( \tilde{\Gamma} \) denote the resulting graph. Then for \( W_1 \in C_U \) and \( W_2 \in D \),

\[ W_1 \sim W_2 \text{ in } \tilde{\Gamma} \iff W_2 \supset \sigma(U). \] (3)

### 3. The Isomorphism

In this section, we prove our main result.

**Theorem 3.** The graph \( \tilde{\Gamma} \) obtained by Godsil–McKay switching to the Grassmann graph \( J_q(2e+1, e+1) \) with respect to the partition \( [2] \) is isomorphic to the twisted Grassmann graph \( \tilde{J}_q(2e+1, e+1) \).

**Proof.** Since \( \tilde{J}_q(2e+1, e+1) \) is isomorphic to the block graph \( \Delta(e, q) \) by [13], it suffices to give an isomorphism between \( \tilde{\Gamma} \) and \( \Delta(e, q) \). We claim that \( \phi : \left[ \frac{V}{e+1} \right] \to A' \cup B' \) defined by

\[
\phi(W) = \begin{cases} [\sigma(W \cap H)] \cup [W \setminus H] & \text{if } W \in A, \\ [W] & \text{otherwise} \end{cases}
\]

is an isomorphism from \( \tilde{\Gamma} \) to \( \Delta(e, q) \).
Let $W_1, W_2 \in \binom{V}{e+1}$. First suppose $W_1, W_2 \in \mathcal{A}$. Then
\[
|W_1 \cap W_2| = |W_1 \cap H| \cap |W_2 \cap H| + |W_1 \setminus H| \cap |W_2 \setminus H| \\
= |\sigma(W_1 \cap H)\cap \sigma(W_2 \cap H)| + |W_1 \setminus H| \cap |W_2 \setminus H| \\
= |(\sigma(W_1 \cap H)| \cup |W_1 \setminus H|) \cap ((\sigma(W_2 \cap H)) \cup |W_2 \setminus H|) \\
= |\phi(W_1) \cap \phi(W_2)|.
\]
We also have $|W_1 \cap W_2| = |\phi(W_1) \cap \phi(W_2)|$ if $W_1, W_2 \in D$. Therefore, for these two cases,
\[
W_1 \sim W_2 \text{ in } \bar{\Gamma} \iff W_1 \sim W_2 \text{ in } J_q(2e+1, e+1) \\
\iff \frac{q^e - 1}{q-1} \iff |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q-1} \iff \phi(W_1) \sim \phi(W_2) \text{ in } \Delta(e, q).
\]
Next suppose $W_1 \in \mathcal{A}$, $W_2 \in D$. Then there exists $U \in \binom{V}{e}$ such that $W_1 \in C_U$. By (3), we have
\[
W_1 \sim W_2 \text{ in } \bar{\Gamma} \iff W_2 \supset \sigma(U) \\
\iff [\sigma(U) \cap W_2] = \sigma(U) \\
\iff |(\sigma(W_1 \cap H) \cup |W_1 \setminus H|) \cap |W_2|| = |\sigma(U)| \\
\iff |\phi(W_1) \cap \phi(W_2)| = \frac{q^e - 1}{q-1} \iff \phi(W_1) \sim \phi(W_2) \text{ in } \Delta(e, q).
\]
Note that the Godsil–McKay switching we have described depends on a polarity of the hyperplane $H$. One might wonder whether different choice of a polarity gives rise to nonisomorphic graphs. This question has already been addressed in the context of pseudo-geometric designs in [11]. Since the composition of two polarities is a collineation of the projective space defined by $H$, and every collineation of $H$ extends to that of $V$, the resulting switched graphs are isomorphic. The fact that the resulting graph is not isomorphic to the original Grassmann graph is related to the existence of an extra automorphism, i.e., a polarity, of the Grassmann graph $J_q(2e, e)$ with vertex set $\binom{V}{e}$, which does not extend to an automorphism of $J_q(2e+1, e+1)$.

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