THETA-FUNCTIONS ON $T^2$-BUNDLES OVER $T^2$ WITH THE EULER CLASS ZERO

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Abstract. We construct an analogue of the classical theta-function on an Abelian variety for closed 4-dimensional symplectic manifolds which are $T^2$-bundles over $T^2$ with the zero Euler class. We use our theta-functions for a canonical symplectic embedding of these manifolds into complex projective spaces (an analogue of the Lefschetz theorem).

1. Introduction

We proposed in [1] a construction of theta functions on the Kodaira-Thurston manifold. Here we present a different approach to their construction on closed 4-dimensional symplectic manifolds, namely $T^2$-bundles over $T^2$ with the zero Euler class. From the geometric viewpoint, the classical theta function on an Abelian variety is a section of a holomorphic line bundle over the complex torus. The Lefschetz theorem states that a section of a sufficiently high tensor power of the bundle determines a complex-analytic embedding of the Abelian variety into some complex projective space. We aim at generalizing this construction to the bundles whose fibers and bases are 1-dimensional complex tori, while the Euler class is zero. We introduce some analogues of the classical theta-functions as sections of complex line bundles over these bundles. We have to relinquish the holomorphic embedding since these bundles, generally speaking, lack not only a Kähler structure, but even a complex structure. Nevertheless, they are symplectic manifolds. We will construct theta functions so that a symplectic analogue of the Lefschetz theorem holds for them: theta functions with characteristics, which are sections of tensor powers of the defining line bundle, determine a symplectic embedding of the manifold into $\mathbb{C}P^k$ (for sufficiently high tensor powers).

The Kodaira-Thurston manifold can be viewed as a $T^2$-bundle over $T^2$ in two distinct ways (the bundles are not isomorphic), one of which we consider in the article. Therefore, the theta functions on the Kodaira-Thurston manifold introduced in this article differ by construction from the theta functions of [1], and it turns out that they coincide with the theta functions of Kirwin and Uribe [2], whose approach is representation-theoretic. We discuss this coincidence in more detail in Subsection 4.4. In Section 2 we recall the necessary facts of the classical theory of theta-functions and in Section 3 we describe the bundles with which we are to work. In Section 4 we define theta-functions on bundles and study some of their properties, in Section 5 we construct an embedding of those bundles into complex projective spaces (Theorem 1), and in Section 6 we prove that the embedding is symplectic (Theorem 2). It would be interesting to find out how far the analogy

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with the classical theta-functions goes; for instance, whether the constructed theta-
functions are related to number theory (see [3] for instance) or nonlinear equations
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2. The Classical Theta-Function

Recall some useful facts concerning the theta-function on the 1-dimensional complex
torus.
Consider the formal series
\[
\theta(z, \tau) = e^{\pi i z + \pi i \tau/4 + \pi i/2} \sum_{k \in \mathbb{Z}} e^{2\pi i k z + \pi i (k+1) \tau + \pi i k}.
\]
For \(\text{Im} \tau > 0\) this series converges in every compact domain in \(\mathbb{C}\) and defines an
entire function. With respect to translations the theta function behaves as
\[
\theta(z + 1, \tau) = -\theta(z, \tau),
\]
\[
\theta(z + \tau, \tau) = -e^{-2\pi i z - \pi i \tau} \theta(z, \tau).
\]
Our choice of this particular theta function, which is \(\theta_{11}(z, \tau)\) in the notation of
Mumford, is related to the fact that it becomes multiplied by an exponential under
all modular transformations:
\[
\theta\left(\frac{z}{c\tau + d}, \frac{a \tau + b}{c\tau + d}\right) = \zeta(c\tau + d)^{1/2} \exp\left(\frac{cz^2}{c\tau + d}\right) \cdot \theta(z, \tau), \quad ad - bc = 1.
\]
Here the nonzero constant \(\zeta\) depends on the matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\).

A generalization of the concept of theta-function is the concept of theta-function
of degree \(k\), where \(k\) is some positive integer. A theta-function of degree \(k\) is an
entire function on \(\mathbb{C}\) with the periodicity properties
\[
\theta_k(z + 1, \tau) = (-1)^k \theta_k(z, \tau),
\]
\[
\theta_k(z + \tau, \tau) = (-1)^k e^{-2\pi i z - \pi i \tau} \theta_k(z, \tau).
\]
It is not difficult to verify that theta-functions of degree \(k\) span a linear space of
dimension \(k\) denoted by \(L_k\).

Multiplying theta functions, we obtain a theta-function of a higher degree. Take
a tuple \(\{\alpha_i\}_{i=1}^k\) of constants whose sum is equal to zero. Then
\[
\prod_{i=1}^k \theta(z + \alpha_i, \tau) \in L_k
\]
The theta function is equal to zero at the point \(z = 0\) modulo the lattice \(\mathbb{Z} + \tau\mathbb{Z}\). The fundamental domain of the lattice contains a unique zero counting the
multiplicity.

The theta function satisfies the following partial differential equation:
\[
\frac{\partial \theta(z, \tau)}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \theta(z, \tau)}{\partial z^2}.
\]
3. Bundles

The manifolds that are the total spaces of the bundles whose fiber and base are 2-dimensional tori are classified in [5]. Let us describe the bundles with the zero Euler class.

Take two commuting matrices $A$ and $B$ in $SL(2, \mathbb{Z})$. Denote by $\begin{bmatrix} s \\ t \end{bmatrix}$ the point of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ corresponding to $\begin{bmatrix} s \\ t \end{bmatrix} \in \mathbb{R}^2$. Then the bundle corresponding to $\{A, B\}$ arises upon taking the quotient of $T^2 \times \mathbb{R}^2/\sim$ by the action of $A$ and $B$:

$$\left( \begin{bmatrix} s \\ t \end{bmatrix}, (x + 1, y) \right) \sim \left( A \begin{bmatrix} s \\ t \end{bmatrix}, (x, y) \right)$$

and

$$\left( \begin{bmatrix} s \\ t \end{bmatrix}, (x, y + 1) \right) \sim \left( B \begin{bmatrix} s \\ t \end{bmatrix}, (x, y) \right).$$

A complete list of these bundles, up to a diffeomorphism of the total space, is shown in the table, where $I$ is the identity matrix.

|   | Notes                           |
|---|---------------------------------|
| (A) $\{I, I\} = T^4$            | 4-torus                         |
| (B) $\{0 -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, I\}$ | Hyperelliptic surface          |
| (2) $\{0 -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, I\}$ |                                   |
| (3) $\{0 -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, I\}$ |                                   |
| (4) $\{-I, I\}$                     |                                   |
| (C) $\{1 -1 \begin{bmatrix} k \\ 1 \end{bmatrix}, I\}$, $k \neq 0$ | Kodaira–Thurston manifold         |
| (D) $\{-1 -1 \begin{bmatrix} k \\ 0 \end{bmatrix}, I\}$, $k \neq 0$ |                                   |
| (E) $\{1 -1 \begin{bmatrix} k \\ 0 \end{bmatrix}, -I\}$, $k \neq 0$ |                                   |
| (F) $\{A, I\}$, $-\text{tr} \ A > 2$ |                                   |
| (G) $\{A, -I\}$, $\text{tr} \ A > 2$ |                                   |

A survey devoted to the manifolds of type (B) can be found in [6] for instance.

Verify that all these bundles are homogeneous spaces of real Lie groups. Take a real Lie group $G$ with coordinates $(x, y, s, t)$ and the linear representation

$$\rho : (x, y, s, t) \rightarrow \begin{pmatrix} A^x B^y & s \\ 0 & t \end{pmatrix}.$$ 

Denote by $\Gamma$ the discrete subgroup of elements with integer coordinates. Then the homogeneous space $\Gamma \backslash G$ is the total space of the bundle corresponding to $\{A, B\}$. Observe that $\rho$ is a faithful representation not of $G$, but rather of $\Gamma \backslash G$. 
It is obvious that we can proceed to define these bundles as the quotient manifolds of $\mathbb{R}^4$ by the action of the group $\Gamma$ with the following generators (when $B = I$): 

\begin{align*}
(5) & \quad a : (x + 1, y, \alpha s + \beta t, \gamma s + \delta t), \\
(6) & \quad b : (x, y + 1, s, t), \\
(7) & \quad c : (x, y, s + 1, t), \\
(8) & \quad d : (x, y, s + 1, t + 1),
\end{align*}

where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. If $B = -I$, then 

\begin{equation}
(9) \quad b : (x, y + 1, -s, -t).
\end{equation}

Knowing a faithful representation $\Gamma \backslash G$, we can calculate the generators of the algebra of left-invariant forms. As usual 

\[
\rho^{-1}d\rho = \begin{pmatrix} Cdx + Ddy & A^{-x}B^{-y}(ds) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} ds \\ dt \end{pmatrix},
\]

where $C$ and $D$ are some constant matrices. Therefore, 1-forms $dx$, $dy$, $\omega_1$, $\omega_2$, where 

\[
\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A^{-x}B^{-y} \begin{pmatrix} ds \\ dt \end{pmatrix},
\]

are the generators.

4. Theta-functions on bundles

4.1. The definition of theta-function. Henceforth we denote by $M$ the total space of bundles whose the fiber and base are 2-dimensional tori and whose Euler class is zero.

Introduce a function on the universal covering $\mathbb{R}^4$ of $M$ formally:

\begin{equation}
(10) \quad \vartheta_M(x, y, s, t) = \theta(s + \omega t, \omega) \cdot \theta(x + iy, i).
\end{equation}

Here $\theta(z, \tau)$ is the classical theta function of the variable $z$ with period $\tau$ (see Section 2). Specify the function $\omega$ depending on the bundle (writing $i = \sqrt{-1}$):

| $\omega$ | (B1), (B3) | $(-1 + \sqrt{-3})/2$ |
|----------|------------|-------------------|
| (B2), (B4) | $i$ |
| (C), (E) | $-kx + i$ |
| (D) | $kx + i$ |
| (F), (G) | $(\lambda^{-x}v^+ + i\lambda^xv^-)/(\lambda^{-x}u^+ + i\lambda^xu^-)$ |

Here $\lambda$ and $\lambda^{-1}$ are the eigenvalues of $A^T$, while $(u^+, v^+)^T$ and $(u^-, v^-)^T$ are eigenvectors of the transposed matrix $A$:

\[
A^T \begin{pmatrix} u^+ \\ v^+ \end{pmatrix} = \lambda \begin{pmatrix} u^+ \\ v^+ \end{pmatrix}, \quad A^T \begin{pmatrix} u^- \\ v^- \end{pmatrix} = \lambda^{-1} \begin{pmatrix} u^- \\ v^- \end{pmatrix},
\]

the vectors are normalized so that $u^+v^- - u^-v^+ = 1$.

**Lemma 1.** $\text{Im} \omega > 0$ for all bundles.
Proof. Only the case of the last function is not obvious:

\[
\text{Im} \omega = \frac{u^+v^- - u^-v^+}{(\lambda^2u^+)^2 + (\lambda^2u^-)^2} > 0.
\]

The proof of the lemma is complete.

By Lemma 1 the series defining the theta function converges, and the function \( \theta_M \) is well-defined.

**Lemma 2.** Under the action of the generator (5) of \( \Gamma \):

\[
(x, y, s, t) \rightarrow (x + 1, y, \alpha s + \beta t, \gamma s + \delta t)
\]

functions \( \omega \) and \( s + \omega t \) transform as:

\[
\omega \rightarrow \frac{\alpha \omega - \beta}{-\gamma \omega + \delta}, \quad s + \omega t \rightarrow \frac{s + \omega t}{-\gamma \omega + \delta}.
\]

**Proof.** Direct calculations.

Take a basis \( \{\theta^k_p(z, \tau)\}_{p=1}^k \) for the space of classical theta-functions of degree \( k \) (see Section 2). Define the space of theta-functions of degree \( k \) on the manifold \( M \) as the linear span of pairwise products of ordinary basis theta-functions of degree \( k \) on the fiber and base:

\[
\theta^k_p(s + \omega t, \omega) \cdot \theta^k_q(x + iy, i), \quad p, q = 1 \ldots k
\]

Denote this space by \( L_k \). Observe that the dimension of \( L_k \) is equal to \( k^2 \). The theta function of degree 1 is precisely \( \vartheta_M \).

4.2. \( \vartheta_M \) is a section of a complex line bundle. In order to show that the function \( \vartheta_M \) is a section of a complex line bundle over \( M \), recall that the sections are in a bijective correspondence with the functions \( f \) on the universal covering such that \( f(\lambda \cdot u) = \exp(\lambda e_u) \cdot f(u) \), where \( \lambda \) is an element of the lattice \( \Gamma \) and \( e_u : \mathbb{R}^4 \rightarrow \mathbb{C}^* \) are the multipliers; i.e., nonzero functions \( e_\lambda : \mathbb{R}^4 \rightarrow \mathbb{C}^* \) such that

\[
e_\lambda(\mu \cdot u) \cdot e_\mu(u) = e_{\lambda \mu}(u), \quad \lambda, \mu \in \Gamma
\]

and \( e_0(u) = 1 \).

The multipliers determine a complex line bundle over \( M \) such that the direct product \( \mathbb{R}^4 \times \mathbb{C} \) can be modded out by the action of \( \Gamma \):

\[
(u, w) \sim (\lambda \cdot u, e_\lambda(u)w), \quad u \in \mathbb{R}^4, w \in \mathbb{C}, \lambda \in \Gamma.
\]

Consider the behavior of \( \vartheta_M \) under the action of the generators (5)-(8) of \( \Gamma \) (with \( B = I \)):

\[
\vartheta_M(x + 1, y, \alpha s + \beta t, \gamma s + \delta t) = \zeta(-\gamma \omega + \delta)^\rho \exp(-\gamma(s + \omega t)^2 - \gamma \omega + \delta) \cdot \vartheta_M(x, y, s, t),
\]

\[
\vartheta_M(x, y + 1, s, t) = -\exp(-2\pi i(x + iy) + \pi) \cdot \vartheta_M(x, y, s, t),
\]

\[
\vartheta_M(x, y, s + 1, t) = -\vartheta_M(x, y, s, t),
\]

\[
\vartheta_M(x, y, s, t + 1) = -\exp(-2\pi i(s + \omega t) - \pi i \omega) \cdot \vartheta_M(x, y, s, t).
\]

In the case \( B = -I \) we should replace (12) by

\[
\vartheta_M(x, y + 1, -s, -t) = \exp(-2\pi i(x + iy) + \pi) \cdot \vartheta_M(x, y, s, t).
\]
We used the periodicity properties (1)-(3) of the classical theta function and Lemma 2. These formulas imply that \( \vartheta_M \) is a section of the bundle determined by the multipliers (11) - (15).

In order to verify that the construction of the bundle is well-defined, given these multipliers, we must verify that the nontrivial relations (5)-(9):

\[
[a, c] = c^{1-\delta} d^\gamma, \\
[a, d] = c^{2d-\alpha}, \\
[g, h] = g^{h^{-1}gh},
\]

between the generators (5)-(9) of \( \Gamma \) imply identities on the multipliers. This is obvious considering that all multipliers are determined by the behavior of the same function.

4.3. **A multiplicative property of** \( \vartheta_M \). Introduce the action of \( \zeta = (\lambda, \mu) \in \mathbb{C}^2 \) on \( \vartheta_M \) as

\[
(\zeta \cdot \vartheta_M)(x, y, s, t) = \theta(s + \omega t + \lambda, \omega) \cdot \theta(x + iy + \mu, i).
\]

Take a tuple \( \zeta_i = (\lambda_i, \mu_i) \), \( i = 1, \ldots, k \) of constant vectors in \( \mathbb{C}^2 \) whose sum is equal to zero. As for the classical theta function, it is desirable that the product

\[
\prod_{i=1}^{k} (\zeta_i \cdot \vartheta_M)(x, y, s, t)
\]

be a theta function of degree \( k \). This property of the theta function is key for the proof of our theorem on the embedding into a complex projective space. It is easy to verify that the multiplier (11) of the form \( \exp(-\gamma(s + \omega t)^2) \) is an obstruction. If this multiplier is nontrivial, i.e., \( \gamma \neq 0 \), then we require also that

\[
\sum_{i=1}^{k} \lambda_i^2 = 0.
\]

It is not difficult to verify then that the product (17) is a theta function of degree \( k \).

4.4. **Relation to the theta functions of Kirwin and Uribe.** Let us recall how theta functions on the Kodaira-Thurston manifold were introduced in [2]. Given a square-integrable function \( f : \mathbb{R}^2 \to \mathbb{C} \) for all \( m, n = 0, 1, \ldots, 2k - 1 \) the functions \( \vartheta_k^{m,n} : \mathbb{R}^4 \to \mathbb{C} \) are defined as

\[
(\vartheta_k^{m,n} f)(x, y, z, t) = e^{-2\pi i [my-n(z+xy)]-4\pi ikzx} \sum_{a, b \in \mathbb{Z}} e^{2\pi i ny - 4\pi ikz(yz-za-y(x+a)^2/2)} f(x + a, t + b).
\]

These functions satisfy the pseudo-periodicity conditions

\[
(\vartheta_k^{m,n} f)(x + 1, y, z, t) = (\vartheta_k^{m,n} f)(x, y, z, t),
\]

\[
(\vartheta_k^{m,n} f)(x, y + 1, z - x, t) = e^{-2\pi ikzx^2} (\vartheta_k^{m,n} f)(x, y, z, t),
\]

\[
(\vartheta_k^{m,n} f)(x, y, z + 1, t) = e^{4\pi ikx} (\vartheta_k^{m,n} f)(x, y, z, t),
\]

\[
(\vartheta_k^{m,n} f)(x, y, z, t + 1) = e^{4\pi iky} (\vartheta_k^{m,n} f)(x, y, z, t).
\]

Verify that for a certain choice \( f : \mathbb{R}^2 \to \mathbb{C} \) the function \( \vartheta_k^{m,n} f \) turns out with some degree of precision to be the theta function on the Kodaira-Thurston manifold defined in this article.
Take \( k = 1, m = 0, n = 0, f = g(t) \cdot h(x) \):
\[
g(t) = e^{-2\pi t^2}, \quad h(x) = e^{-2\pi x^2}.
\]

Then
\[
(19) \quad \left( \vartheta_{m,n}^k \right)(x, y, z, t) = e^{-4\pi i zx - 2\pi t^2 - 2\pi x^2} \cdot \theta(2(z + (y + i)x), 2(y + i)) \cdot \theta(2(-y + it), 2i),
\]
where \( \theta(z, \tau) \) is the classical theta function with characteristics \([0, 0]\).

Recall that the theta function on the Kodaira-Thurston manifold defined here in (10) is of the form
\[
\theta_{1/2, 1/2}(z + \omega t, \omega) \cdot \theta_{1/2, 1/2}(x + iy, i), \quad \omega = -kx + i, \quad k \in \mathbb{Z} \setminus \{0\},
\]
where \( \theta[a, b](z, \tau) \) is the classical theta function with characteristics \([a, b]\). We chose a somewhat different theta function because of its invariance under modular transformations. Kirwin and Uribe multiply the argument and the period by 2 for seemingly the same purpose. The change of variables
\[
x' = t, \quad y' = -x, \quad z' = s, \quad t' = y, \quad k = 1,
\]
carries the function
\[
\theta_{1/2, 1/2}(s + \omega t, \omega) \cdot \theta_{1/2, 1/2}(x + iy, i), \quad \omega = -kx + i
\]
into
\[
\theta_{1/2, 1/2}(z' + (y' + i)x', y' + i) \cdot \theta_{1/2, 1/2}(-y' + it', i),
\]
which coincides with (19) up to the translations of characteristics and multiplication by an exponential.

5. Embedding into a Complex Projective Space

Enumerate the basis theta functions \( \{\sigma_i\}_{i=1}^{k^2} \) in \( L_k \). Then
\[
\varphi_k = (\sigma_1, \sigma_2, \ldots, \sigma_{k^2})
\]
is a well-defined mapping of the manifold \( M \) into \( \mathbb{C}P^{k^2-1} \).

Recall that we denote the entries of the monodromy matrix \( A \) by
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix},
\]
and for \( \gamma \neq 0 \) we require the fulfilment of the additional condition (18).

**Theorem 1.** The mapping \( \varphi_k \) is an embedding provided that
(a) \( k \geq 4 \) when \( \gamma \neq 0 \);
(b) \( k \geq 3 \) when \( \gamma = 0 \).

**Proof.** Let us prove claim (a) of the theorem for \( k = 4 \). It should be clear from the proof how to deal with the remaining cases.

First establish the injectivity of \( \varphi_k \). We will follow the proof of the classical Lefschetz embedding theorem for Abelian varieties (see the exposition of it in [3 Chapter 2, Theorem 1.3]).

Observe that the space of theta functions of degree \( k \) consists of the global sections of the \( k \)th tensor power of the bundle determined by the multipliers (11-16).
If it is true that for all points \( u \neq v \in M \) there exists a section \( \sigma \in \mathcal{L}_k \) such that \( \sigma(u) = 0 \) and \( \sigma(v) \neq 0 \), then \( \varphi_k \) is injective. Indeed, suppose that the mapping \( \varphi_k \) glues together \( u \) and \( v \). Since \( \varphi_k \) is made up from the basis sections of \( \mathcal{L}_k \), it follows that \( \sigma(v) = \zeta \cdot \sigma(u) \) for every section \( \sigma \in \mathcal{L}_k \), where \( \zeta \) is some non-zero constant. If \( \sigma \) is a section satisfying the condition indicated above then we arrive at a contradiction. Observe also that, when this condition is fulfilled at every point \( u \in M \) not all sections vanish at \( u \).

We will seek a theta function of degree 4 as a product \( \sigma = f \cdot g \) of two functions:

\[
(20) \quad f(s + \omega(x)t, x, \alpha, \beta) = \theta(s + \omega t + \alpha, \omega)\theta(s + \omega t + \beta, \omega) \times
\]
\[
\quad \times \theta(s + \omega t + \gamma, \omega)\theta(s + \omega t + \delta, \omega),
\]
\[
(21) \quad g(x + iy, \alpha', \beta', \gamma') = \theta(x + iy + \alpha', i)\theta(x + iy + \beta', i) \times
\]
\[
\quad \times \theta(x + iy + \gamma', i)\theta(x + iy + \delta', i).
\]

Here

\[
(22) \quad \gamma = \frac{1}{4} \left(-2(\alpha + \beta) + \sqrt{-4(\alpha + \beta)^2 - \alpha^2 - \beta^2}\right),
\]
\[
(23) \quad \delta = \frac{1}{4} \left(-2(\alpha + \beta) - \sqrt{-4(\alpha + \beta)^2 - \alpha^2 - \beta^2}\right),
\]
\[
(24) \quad \delta' = -\alpha' - \beta' - \gamma'.
\]

In Subsection 4.3 we showed that \( f \cdot g \) is indeed a theta-function of degree 4 on \( M \).

Denote the coordinates of \( u \) and \( v \) by \((x, y, s, t)\) and \((x', y', s', t')\) respectively. Choose \( \alpha' \) so that \( \theta(x + iy + \alpha') = 0 \). Now choose \( \beta', \gamma', \delta' \) so that the remaining factor in the definition of \( g \) does not vanish at \( v \):

\[
\theta(x' + iy' + \beta')\theta(x' + iy' + \gamma')\theta(x' + iy' + \delta') \neq 0.
\]

We can achieve this since the zeros of the theta function are isolated. By a small perturbation of \( \alpha, \beta, \gamma, \delta \) we can also achieve the non-vanishing of \( f \) at \( v \).

Observe that now it is easy to explain why in claim (a) \( k \) must be at least 4. For \( k = 3 \), when there is no \( \delta' \), the constants \( \beta', \gamma' \) are functions of \( \alpha' \) because of [18], and it is impossible to achieve the non-vanishing indicated above.

The constructed section would solve the problem provided that \( \theta(x' + iy' + \alpha') \neq 0 \). Suppose that it is so. Since the classical theta function has a unique zero in the fundamental domain of the lattice formed by its periods, it follows that \( x = x', y = y' \). The equality is understood modulo the lattice, but without restricting the generality we may assume that \( u = v \) lie in the fundamental domain; i.e., the unit cube \( 0 \leq x, y, s, t < 1 \).

Choose \( \alpha \) so that \( \theta(s + \omega(x)t + \alpha, \omega(x)) = 0 \). Observe that then \( \theta(s' + \omega(x)t' + \alpha, \omega(x')) \neq 0 \), for otherwise \( u = v \). Pick \( \beta, \gamma, \delta \) so that \( f(v) \neq 0 \), while \( \alpha', \beta', \gamma', \delta' \) so that \( g(v) \neq 0 \).

Therefore, we have constructed a required section and proved the injectivity of \( \varphi_k \).

Let us now prove that the rank of \( \varphi_k \) is maximal. We will follow the proof of the Lefschetz theorem in [3]. To start off show that the rank of the mapping is
maximal provided that the rank (over \( \mathbb{C} \)) of the following matrix is maximal

\[
J = \begin{pmatrix}
\sigma_1 & \ldots & \sigma_{k^2} \\
\partial_x \sigma_1 & \ldots & \partial_x \sigma_{k^2} \\
\partial_y \sigma_1 & \ldots & \partial_y \sigma_{k^2} \\
\partial_s \sigma_1 & \ldots & \partial_s \sigma_{k^2} \\
\partial_t \sigma_1 & \ldots & \partial_t \sigma_{k^2}
\end{pmatrix}.
\]

Observe that \( \varphi_k \), expressed in the homogeneous coordinates, is the composition of a mapping \( \tilde{\varphi}_k \) into \( \mathbb{C}^{k^2} \) and the subsequent projection \( \pi : \mathbb{C}^{k^2} \setminus \{0\} \rightarrow \mathbb{P}^{k^2-1} \). It is obvious that the differential of \( \tilde{\varphi}_k \) coincides with the submatrix of \( J \) resulting upon the removal of the first row.

Suppose now that at a point \( u^* \in M \) the first row of \( J \) is a linear combination of the remaining rows. This means that the radius-vector \( \tilde{\varphi}_k(u^*) \) is collinear to the image of some tangent vector at \( u^* \). Since \( \pi \) projects along the complex lines passing through the origin, it follows that the kernel of the differential of \( \pi \) consists of precisely those vectors. Consequently, the maximality of the rank of \( J \) is a necessary and sufficient condition for the maximality of the rank of \( \varphi_k \).

Transform the matrix \( J \) into a form convenient for us. The rank of the matrix

\[
\tilde{J} = \begin{pmatrix}
\sigma_1 & \ldots & \sigma_{k^2} \\
(\partial_x - i\partial_y)\sigma_1 & \ldots & (\partial_x - i\partial_y)\sigma_{k^2} \\
(\partial_s + \frac{\partial}{\partial \sigma})\sigma_1 & \ldots & (\partial_s + \frac{\partial}{\partial \sigma})\sigma_{k^2} \\
(\partial_x + i\partial_y)\sigma_1 & \ldots & (\partial_x + i\partial_y)\sigma_{k^2} \\
(\partial_s - \frac{\partial}{\partial \sigma})\sigma_1 & \ldots & (\partial_s - \frac{\partial}{\partial \sigma})\sigma_{k^2}
\end{pmatrix},
\]

coincides with the rank of \( J \). Recall that for a holomorphic function \( f \) of a complex variable \( w = u + iv \) we have

\[
\frac{\partial f}{\partial w} = \frac{1}{2}(\partial_u - i\partial_v)f, \quad \frac{\partial f}{\partial \bar{w}} = \frac{1}{2}(\partial_u + i\partial_v)f = 0.
\]

The last two rows of \( \tilde{J} \) appear in the Cauchy-Riemann conditions. Since for every \( x \) the sections \( \sigma_j \) expand into series \( s + \omega(x)t \), the last row of \( \tilde{J} \) is always zero.

Observe that if \( \omega = \text{const} \) (which we have for the theta functions on the bundles of type (B)) then the sections are holomorphic, and we can use the proof of the classical Lefschetz theorem. We will assume that \( (\partial_x + i\partial_y)\theta_M \neq 0 \).

Assume that the rank of \( \tilde{J} \) (over \( \mathbb{C} \)) at some fixed point \( u^* = (x^*, y^*, s^*, t^*) \in M \) is less than 4. This means that there exists a nontrivial tuple \( a, b, c, d \) such that

\[
a\sigma_j(u^*) + \frac{b}{2}(\partial_x - i\partial_y)\sigma_j(u^*) + \frac{c}{2}(\partial_s + \frac{\partial}{\partial \sigma})\sigma_j(u^*) + \frac{d}{2}(\partial_x + i\partial_y)\sigma_j(u^*) = 0,
\]

\[
j = 1, \ldots, k^2.
\]

The function

\[
\sigma = f(s + \omega(x)t, x, \alpha, \beta) \cdot g(x + iy, \alpha', \beta', \gamma'),
\]

described by (20)-(24) lies in \( \mathcal{L}_k \) for all \( \alpha, \beta, \alpha', \beta', \gamma' \). Hence, it expands in terms of the basis \( \sigma_j \), and

\[
a\sigma + \frac{b}{2}(\partial_x - i\partial_y)\sigma + \frac{c}{2}(\partial_s + \frac{\partial}{\partial \sigma})\sigma + \frac{d}{2}(\partial_x + i\partial_y)\sigma = 0.
\]
at \( u^* \). Put \( L = \frac{\partial}{2}(\partial - i\partial_y) + \frac{\partial}{2}(\partial + i\partial_y) \) and rewrite (25) as

\[
(26) \quad L \log((\alpha, \alpha') \cdot \theta_M)(u^*) = -a - L \log((\beta, \beta') \cdot \theta_M)(u^*) - L \log((\gamma, \gamma') \cdot \theta_M)(u^*) - L \log((\delta, \delta') \cdot \theta_M)(u^*).
\]

Here \( ((\lambda, \mu) \cdot \theta_M) \) is the action described by (16). For all \( u, \alpha, \alpha' \) there exist \( \beta, \beta', \gamma', \) such that

\[
(27) \quad ((\beta, \beta') \cdot \theta_M)(u) \times ((\gamma, \gamma') \cdot \theta_M)(u) \times ((\delta, \delta') \cdot \theta_M)(u) \neq 0.
\]

It follows from (26) and (27) that

\[
(28) \quad \xi(\alpha, \alpha') = L \log((\alpha, \alpha') \cdot \theta_M)(u^*)
\]

is an entire function of \( (\alpha, \alpha') \in \mathbb{C}^2 \). By (11)-(15) the function \( \xi(\alpha, \alpha') \) satisfies the following periodicity conditions:

\[
(29) \quad \xi(\alpha + 1, \alpha') = \xi(\alpha, \alpha'),
\]

\[
(30) \quad \xi(\alpha + \omega(x^*), \alpha') = \xi(\alpha, \alpha') - 2\pi i c - \pi i \frac{(b + d)}{2} \omega(x^*),
\]

\[
(31) \quad \xi(\alpha, \alpha' + 1) = \xi(\alpha, \alpha'),
\]

\[
(32) \quad \xi(\alpha, \alpha' + i) = \xi(\alpha, \alpha') - 2\pi i b.
\]

Therefore, derivatives \( \partial_\alpha \xi \) and \( \partial_{\alpha'} \xi \) are doubly periodic entire functions. This means that they are constant and \( \xi = A\alpha + B\alpha' + C \). From (29) and (31) it follows that \( A = B = 0 \), and \( \xi \equiv C \). From (30), (32) it follows that

\[
(33) \quad b = 2\pi i c + \pi i \frac{(b + d)}{2} \omega(x^*) = 0.
\]

Then

\[
(33) \quad \xi = C = c \left[ \frac{(\partial + \frac{\partial}{\omega}) (s + \omega(x)t + \alpha, \omega(x))}{\theta(s + \omega(x)t + \alpha, \omega(x))} \right]_{u = u^*} + \frac{d}{2} \left[ \frac{((\partial_\theta)(s + \omega(x)t + \alpha, \omega(x))}{\theta(s + \omega(x)t + \alpha, \omega(x))} \right]_{u = u^*}.
\]

Here we used implicitly the Cauchy–Riemann conditions

\[
(\partial_x + i\partial_y)\theta(x + iy, i) = 0.
\]

Denote by \( D \) the differentiation with respect to \( s + \omega t \):

\[
D = \frac{1}{2} \left( \partial_s + \frac{\partial_t}{\omega} \right).
\]

It follows from (11) that

\[
(34) \quad \partial_s \theta(s + \omega t, \omega) = \frac{1}{4\pi i} (D^2 \theta)(s + \omega t, \omega) + t(D\theta)(s + \omega t, \omega).
\]

Inserting (34) into (33) and taking

\[
(D\theta)(s + \omega t + \alpha, \omega) = \partial_s \theta(s + \omega t + \alpha, \omega),
\]

into account, we find that \( \theta(s^* + \omega t^* + \alpha, \omega) \) as a function of \( \alpha \) satisfies a linear ordinary differential equation with constant coefficients:

\[
\frac{d}{2} \frac{\partial \omega(x^*)}{\partial x} \left( \frac{1}{4\pi i} \theta''' + t^* \theta' \right) + c \theta' - C \theta = 0.
\]
Writing down its general solution, we can easily verify that this leads to a contradiction with the periodicity conditions (1)-(2) for the theta function; thereby, \( c = d = C = 0 \). It follows from (25) that \( a = 0 \).

We find that the tuple \( a, b, c, d \) of constants is trivial, and the matrix \( \tilde{J} \) is of maximal rank. Since the point \( u^\ast \) is chosen arbitrarily, the rank of \( \varphi_k \) is equal to 4 everywhere. The proof of the theorem is complete.

6. Embedding is symplectic

For all \( \{A, B\} \) the total space \( M \) of the bundle is a symplectic manifold; a symplectic form can be, for instance, \( \omega_M = dx \wedge dy + ds \wedge dt \). In this section we prove the following statement.

**Theorem 2.**

1. If the mapping \( \varphi_k \) is an embedding then it induces a symplectic structure on \( M \).
2. The induced symplectic form is cohomologous to \( k \cdot \omega_M \).

In the definition of \( \varphi_k \) choose the functions

\[
\theta^p_k(s + \omega t, \omega) \cdot \theta^q_k(x + iy, i); \quad p, q = 1, \ldots, k.
\]

as a basis for the space \( L_k \) of theta functions.

Observe that \( \varphi_k \) is the composition of the Segre mapping \( \sigma_k : \mathbb{CP}^{k-1} \times \mathbb{CP}^{k-1} \rightarrow \mathbb{CP}^{k^2-1} \), which is defined in the homogeneous coordinates as

\[
\sigma_k ([z^1 : \ldots : z^k], [w^1 : \ldots : w^k]) = [z^1 w^1 : z^1 w^2 : \ldots : z^k w^{k-1} : z^k w^k]
\]

and the map \( \psi_k : M \rightarrow \mathbb{CP}^{k-1} \times \mathbb{CP}^{k-1} \), \( \psi_k = (\psi'_k, \psi''_k) \), with

\[
\psi'_k(x, s, t) = [\theta^1_k(s + \omega t, \omega) : \ldots : \theta^k_k(s + \omega t, \omega)]
\]

\[
\psi''_k(x, y) = [\theta^1_k(x + iy, i) : \ldots : \theta^k_k(x + iy, i)].
\]

Thus, \( \varphi_k = \sigma_k \circ \psi_k \). Denote by \( \Omega' \) the symplectic form (associated to the Fubini-Study metric) on the first factor of \( \mathbb{CP}^k \times \mathbb{CP}^k \), by \( \Omega'' \) on the second factor. Then \( \Omega' + \Omega'' \) is a symplectic form on the product. Since the Segre mapping is a holomorphic embedding, it suffices to prove that the induced form \( \psi''_k(\Omega' + \Omega'') \) is symplectic.

Recall that in Section 3 we calculated the generators \( dx, dy, \omega_1, \omega_2 \) for the algebra of left-invariant forms. The mapping \( \psi''_k \) is a holomorphic embedding of the complex torus into \( \mathbb{CP}^k \) by the classical Lefschetz theorem. Hence,

\[
(\psi''_k)^*(\omega_M) = \mu \cdot dx \wedge dy,
\]

where \( \mu \neq 0 \) everywhere on \( M \). Put

\[
(\psi'_k)^*(\Omega') = f \cdot \omega_1 \wedge dx + g \cdot \omega_2 \wedge dx + h \cdot ds \wedge dt
\]

for some functions \( f, g, h \) on \( M \); this is the general form of a 2-form induced by the map \( \psi'_k \), which depends on \( x, s, t \).

Observe that for every fixed \( x \) the map \( \psi'_k \) is a holomorphic embedding as well; thereby,

\[
(\psi'_k)^*(\Omega') = \nu \cdot ds \wedge dt,
\]
where \( \nu \neq 0 \) everywhere on \( M \). This implies that \( h \equiv \nu \). Putting everything together, we obtain
\[
(\psi_k^*(\Omega' + \Omega''))^2 = \left( (\psi_k')^*(\Omega') + (\psi_k'')^*(\Omega'') \right)^2 =
(f \cdot \omega_1 \wedge dx + g \cdot \omega_2 \wedge dx + \nu \cdot ds \wedge dt + \mu \cdot dx \wedge dy)^2.
\]
and multiplying out
\[
(\psi_k^*(\Omega' + \Omega''))^2 = 2\mu \nu \cdot dx \wedge dy \wedge ds \wedge dt.
\]
The last equality is equivalent to the non-degeneracy condition for the induced forms. The closedness follows since the differential commutes with \( \psi^* \). Thus, \( \psi_k^*(\Omega' + \Omega'') \) is a symplectic form. We have proved claim (1) of the theorem.

Let us prove claim (2). Denote by \( L \) the bundle given by the multipliers (11)-(15). While establishing the embedding we observed that the theta functions of degree \( k \) are the sections of \( L \otimes k \). Recall that every complex line bundle over \( M \) is induced by the universal bundle over \( \mathbb{C}P^n \) via a mapping of \( M \) into the complex projective space. Consequently, the bundle \( L \otimes k \) and its curvature form are the images of the universal bundle and its curvature form, which is the Fubini-Study form. Recall also that the first Chern class of a line bundle is realized precisely by the curvature form. Hence, the cohomology class of the induced form coincides with the first Chern class \( c_1(L \otimes k) = k \cdot c_1(L) \) and we must prove that
\[
c_1(L) = [dx \wedge dy + ds \wedge dt].
\]
Use the Čech cohomology theory to calculate \( c_1(L) \). Cover \( \mathbb{R}^4 \) by the open sets
\[
U_\lambda = \lambda \cdot U_0, \quad \lambda \in \Gamma.
\]
To this end, dilate the set
\[
U_0 = \{ |u^k| < 3/4 \}.
\]
by the translations of \( \Gamma \). Observe that this is a good cover: all nonempty finite intersections are diffeomorphic to \( \mathbb{R}^4 \). Therefore, the cohomologies of the nerve of this cover are isomorphic to the cohomologies of \( M \).

We can express the transition functions \( g_{\lambda \mu} : U_\lambda \cap U_\mu \to \mathbb{C}^* \) in terms of the multipliers
\[
g_{\lambda \mu}(u) = e_\lambda(u) \cdot e_{\mu^{-1}}(\mu \cdot u); \quad \lambda, \mu \in \Gamma.
\]
The nerve \( N(\mathcal{U}) \) of a minimal subcover of \( U_\lambda \) is homeomorphic to \( M \), and its cohomology with coefficients in \( \mathbb{Z} \) coincides with \( H^*(M; \mathbb{Z}) \). The cocycle \( z_{\lambda \mu \nu} \in C^2(\mathcal{U}; \mathbb{Z}) \)
\[
z_{\lambda \mu \nu} = \frac{1}{2\pi i} (\log(g_{\lambda \mu}) + \log(g_{\mu \nu}) - \log(g_{\nu \lambda}))
\]
by definition realizes the first Chern class of the bundle \( L \). This formula defines the value of \( z \) on the 2-dimensional simplex \( (\lambda, \mu, \nu) \in N(\mathcal{U}) \).

It is easy to establish that
\[
(1) \quad H^2(M; \mathbb{R}) \cong \mathbb{R}^4, \text{ if } M \text{ is the Kodaira-Thurston manifold:}
\]
\[
A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad B = E,
\]
and the generators for \( H^2(M; \mathbb{R}) \) can be chosen to be \([dx \wedge dy], [ds \wedge dt], [dy \wedge dt], [(ds - \lambda dt) \wedge dx]\).
THETA-FUNCTIONS ON $T^2$-BUNDLES OVER $T^2$ WITH THE EULER CLASS ZERO

(2) in all remaining cases $H^2(M; \mathbb{R}) \cong \mathbb{R}^2$ and the generators are $[dx \wedge dy], [ds \wedge dt]$.

Therefore, for all manifolds $M$ tori $T_{ab}$ and $T_{cd}$ formed by the commuting translations (5)-(9) in $\Gamma$ are dual cycles to $[dx \wedge dy], [ds \wedge dt]$. In the case of the Kodaira-Thurston manifold we obtain two more cycles, $T_{bd}$ and $T_{ac}$.

Define the functions $f_{\lambda}(u)$ so as

$$e_{\lambda}(u) = e^{2\pi i f_{\lambda}(u)}.$$  (37)

By (35)- (37)

$$c_1([T_{\lambda\mu}]) = f_{\mu}(u) + f_{\lambda}(\mu \cdot u) - f_{\lambda}(u) - f_{\mu}(\lambda \cdot u).$$

Evaluating the first Chern class on $T_{ab}$ and $T_{cd}$, we obtain

$$c_1([T_{ab}]) = c_1([T_{cd}]) = 1;$$  (38)

In the case of the Kodaira-Thurston manifold,

$$c_1([T_{bd}]) = c_1([T_{ac}]) = 0.$$  (39)

Since the manifold $M$ is a homogeneous space of a real Lie group, all elements of $H^2(M; \mathbb{R})$ can be realized by left-invariant forms dual to the basis 2-cycles. From (38) and (39) it follows that $c_1(L) = [dx \wedge dy + ds \wedge dt]$. The proof of the theorem is complete.

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