SIMPLE INFINITE PRESENTATIONS FOR THE MAPPING
CLASS GROUP OF A COMPACT NON-ORIENTABLE SURFACE

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Abstract. Omori and the author [5] have given an infinite presentation for
the mapping class group of a compact non-orientable surface. In this paper,
we give more simple infinite presentations for this group.

1. Introduction

For \( g \geq 1 \) and \( n \geq 0 \), we denote by \( N_{g,n} \) a surface obtained by removing \( n \) disjoint
open disks from a connected sum of \( g \) real projective planes, and call this surface
a compact non-orientable surface of genus \( g \) with \( n \) boundary components. We
can regard \( N_{g,n} \) as a surface obtained by attaching \( g \) Möbius bands to \( g \) boundary
components of a sphere with \( g + n \) boundary components, as shown in Figure 1.
We call these attached Möbius bands crosscaps.

Figure 1. A model of a non-orientable surface \( N_{g,n} \).

The mapping class group \( \mathcal{M}(N_{g,n}) \) of \( N_{g,n} \) is defined as the group consisting of
isotopy classes of all diffeomorphisms of \( N_{g,n} \) that fix the boundary pointwise.
\( \mathcal{M}(N_{1,0}) \) and \( \mathcal{M}(N_{1,1}) \) are trivial (see [2]). Finite presentations for \( \mathcal{M}(N_{2,0}) \),
\( \mathcal{M}(N_{2,1}) \), \( \mathcal{M}(N_{3,0}) \) and \( \mathcal{M}(N_{4,0}) \) ware given by [8], [13], [1] and [15] respectively.
Paris-Szepietowski [12] gave a finite presentation of \( \mathcal{M}(N_{g,n}) \) with Dehn twists and
crosscap transpositions for \( g + n > 3 \) with \( n \leq 1 \). Stukow [14] gave another finite
presentation of \( \mathcal{M}(N_{g,n}) \) with Dehn twists and one crosscap slide for \( g + n > 3 \)
with \( n \leq 1 \), applying Tietze transformations for the presentation of \( \mathcal{M}(N_{g,n}) \) given
in [12]. Omori [11] gave an infinite presentation of \( \mathcal{M}(N_{g,n}) \) with all Dehn twists
and all crosscap slides for \( g \geq 1 \) and \( n \leq 1 \), using the presentation of \( \mathcal{M}(N_{g,n}) \)
given in [12], and then, following this work, Omori and the author [5] gave an
infinite presentation of \( \mathcal{M}(N_{g,n}) \) with all Dehn twists and all crosscap slides for
\( g \geq 1 \) and \( n \geq 2 \). In this paper, we give four more simple infinite presentations of
\( \mathcal{M}(N_{g,n}) \) with all Dehn twists and all crosscap slides, and with all Dehn twists and
all crosscap transpositions.

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Through this paper, the product gf of mapping classes f and g means that we apply f first and then g. Moreover we do not distinguish a simple closed curve from its isotopy class.

We define a Dehn twist, a crosscap slide and a crosscap transposition which are elements of $\mathcal{M}(N_{g,n})$. For a simple closed curve $c$ of $N_{g,n}$, a regular neighborhood of $c$ is either an annulus or a Möbius band. We call $c$ a two sided or a one sided simple closed curve respectively. For a two sided simple closed curve $c$, we can take two orientations $+c$ and $-c$ of a regular neighborhood of $c$. The right handed Dehn twist $t_{c;\theta}$ about a two sided simple closed curve $c$ with respect to $\theta \in \{+c, -c\}$ is the isotopy class of the map as shown in Figure 2 (a). Note that for a n oriented two sided simple closed curve $\alpha$ and $\theta \in \{+\alpha, -\alpha\}$, we regard $t_{\alpha;\theta} = t_{\alpha^{-1};\theta}$, where $\alpha^{-1}$ is the inverse loop of $\alpha$. We write $t_{c;\theta} = t_c$ if the orientation $\theta$ is given explicitly, that is, the direction of the twist is indicated by an arrow written beside $c$ as shown in Figure 2 (a). For a one sided simple closed curve $\mu$ of $N_{g,n}$ and an oriented two sided simple closed curve $\alpha$ of $N_{g,n}$ such that $N_{g,n} \setminus \{\alpha\}$ is non-orientable when $g \geq 3$ and that $\mu$ and $\alpha$ intersect transversely at only one point, the crosscap slide $Y_{\mu,\alpha}$ about $\mu$ and $\alpha$ is the isotopy class of the map described by pushing the crosscap which is a regular neighborhood of $\mu$ once along $\alpha$, as shown in Figure 2 (b). The crosscap transposition $U_{\mu,\alpha;\theta}$ about $\mu$ and $\alpha$ with respect to $\theta \in \{+\alpha, -\alpha\}$ is defined as $U_{\mu,\alpha;\theta} = t_{\alpha;\theta}Y_{\mu,\alpha}$, as shown in Figure 2 (b). Note that $Y_{\mu,\alpha}$ and $U_{\mu,\alpha;\theta}$ can not be defined when $g = 1$. We have $Y_{\mu,\alpha}(\alpha) = \alpha$ and $(Y_{\mu,\alpha})*(\pm\alpha) = \mp\alpha$, where $f*(\theta)$ is the orientation of a regular neighborhood of $f(c)$ induced from $\theta \in \{+c, -c\}$, for a two sided simple closed curve $c$ of $N_{g,n}$ and $f \in \mathcal{M}(N_{g,n})$. Since $t_{\alpha;\theta}(\alpha) = \alpha$ and $(t_{\alpha;\theta})*(\pm\alpha) = \mp\alpha$, we also have $U_{\mu,\alpha;\theta}(\alpha) = \alpha$ and $(U_{\mu,\alpha;\theta})*(\pm\alpha) = \mp\alpha$.

Figure 2.
Let $c_1, \ldots, c_k$, $c_0$, $c'_0$ and $d_1, \ldots, d_7$ be simple closed curves with arrows of a surface as shown in Figure 3. $M(N_{g,n})$ admits following relations.

- $(t_{c_1} t_{c_2} \cdots t_{c_k})^{k+1} = t_{c_0} t_{c'_0}$ if $k$ is odd,
- $(t_{c_1} t_{c_2} \cdots t_{c_k})^{2k+2} = t_{c_0}$ if $k$ is even.

$M(N_{g,n})$ admits following relations.

- $t_{d_1} t_{d_2} t_{d_3} = t_{d_4} t_{d_5} t_{d_6} t_{d_7}$.

These relations are called a $k$-chain relation and a lantern relation respectively.

**Figure 3.** Simple closed curves $c_1, \ldots, c_k$, $c_0$, $c'_0$ and $d_1, \ldots, d_7$ with arrows of a surface.

Let $T$, $Y$ and $U \subset M(N_{g,n})$ denote the sets consisting of all Dehn twists, all crosscap slides and all crosscap transpositions respectively. Our main results are as follows.

**Theorem 1.1.** For $g \geq 1$ and $n \geq 0$, $M(N_{g,n})$ admits a presentation with a generating set $T \cup Y$. The defining relations are

1. $t_{c,\theta} = 1$ if $c$ bounds a disk or a Möbius band,
2. (a) $t_{c}^{-1} = t_{c^{-1}}$ for any $t_{c} \in T$,
   (b) $Y_{\mu,\alpha}^{-1} = Y_{\mu,\alpha}$ for any $Y_{\mu,\alpha} \in Y$ and $f \in T$,
3. (a) $f t_{c,\theta} f^{-1} = t_{f(c),f(\theta)}$ for any $t_{c,\theta} \in T$ and $f \in T \cup Y$,
   (b) $f Y_{\mu,\alpha} f^{-1} = Y_{f(\mu),f(\alpha)}$ for any $Y_{\mu,\alpha} \in Y$ and $f \in T$,
4. all the 2-chain relations,
5. all the lantern relations and
6. $Y_{\mu,\alpha}^2 = t_{\delta(\mu,\alpha)}$ for any $Y_{\mu,\alpha} \in Y$, where $\delta(\mu,\alpha)$ is a simple closed curve with an arrow determined by $\mu$ and $\alpha$ as shown in Figure 3 (b).

**Theorem 1.2.** For $g \neq 2$ and $n \geq 0$, $M(N_{g,n})$ admits a presentation with a generating set $T \cup Y$. The defining relations are

1. $t_{c,\theta} = 1$ if $c$ bounds a disk or a Möbius band,
2. (a) $t_{c}^{-1} = t_{c^{-1}}$ for any $t_{c} \in T$,
   (b) $Y_{\mu,\alpha}^{-1} = Y_{\mu,\alpha}$ for any $Y_{\mu,\alpha} \in Y$,
3. (a) $f t_{c,\theta} f^{-1} = t_{f(c),f(\theta)}$ for any $t_{c,\theta} \in T$ and $f \in T \cup Y$,
   (b) $f Y_{\mu,\alpha} f^{-1} = Y_{f(\mu),f(\alpha)}$ for any $Y_{\mu,\alpha} \in Y$ and $f \in T$,
4. all the 2-chain relations and
5. all the lantern relations.

**Theorem 1.3.** For $g \geq 1$ and $n \geq 0$, $M(N_{g,n})$ admits a presentation with a generating set $T \cup U$. The defining relations are

1. $t_{c,\theta} = 1$ if $c$ bounds a disk or a Möbius band,
For Theorem 1.4. generating set presentations for [11] and [5]. In this sense one could say that we have given more simple infinite some problems on presentations for \( M \), special forms of a relation on crosscap slides and Dehn twists of \( M \).

Proof of Theorem 1.1 and the relation (2) (b) of Theorem 1.2 are all the lantern relations and -chain relations, \( \gamma \) with an arrow determined by \( \mu \) and \( \alpha \) as shown in Figure 2 (b).

The relation (6) of Theorem 1.1 and the relation (2) (b) of Theorem 1.2 are special forms of a relation on crosscap slides and Dehn twists of \( M \) given in [11] and [5]. In this sense one could say that we have given more simple infinite presentations for \( M(N_0, n) \).

In Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.2. Finally, in Appendix A we introduce some problems on presentations for \( M(N_0, n) \).

2. PROOF OF THEOREM 1.1

We denote by \( T(N_0, n) \) the subgroup of \( M(N_0, n) \) generated by \( T \), and call the twist subgroup of \( M(N_0, n) \). Omori and the author [6] gave an infinite presentation for \( T(N_0, n) \) as follows.

Theorem 2.1 (6). For \( g \geq 1 \) and \( n \geq 0 \), \( T(N_0, n) \) admits a presentation with a generating set \( T \). The defining relations are

1. \( t_{c, \theta} = 1 \) if \( c \) bounds a disk or a Möbius band,
2. \( t_{c, e}^{-1} = t_{c, e} \) for any \( t_{c, e} \in T \),
3. \( f t_{c, \theta} f^{-1} = t_{f(c), f(\theta)} \) for any \( t_{c, \theta} \in T \) and \( f \in T \cup U \),
4. all the 2-chain relations,
5. all the lantern relations and
6. \( U_{\mu, \alpha, \theta} = t_{\delta(\mu, \alpha)} \) for any \( U_{\mu, \alpha, \theta} \in U \), where \( \delta(\mu, \alpha) \) is a simple closed curve.

As a corollary of Theorem 2.1 we obtain the following.

Corollary 2.2. Fix one crosscap slide \( Y_{\mu_0, \alpha_0} \in \mathcal{Y} \). For \( g \geq 2 \) and \( n \geq 0 \), \( M(N_0, n) \) admits a presentation with a generating set \( \mathcal{T} \cup \{ Y_{\mu_0, \alpha_0} \} \). The defining relations are

1. \( t_{c, \theta} = 1 \) if \( c \) bounds a disk or a Möbius band,
2. \( t_{c, e}^{-1} = t_{c, e} \) for any \( t_{c, e} \in T \),
3. \( f t_{c, \theta} f^{-1} = t_{f(c), f(\theta)} \) for any \( t_{c, \theta} \in T \) and \( f \in T \cup \{ Y_{\mu_0, \alpha_0} \} \),
4. all the 2-chain relations,
5. all the lantern relations and
6. \( Y_{\mu_0, \alpha_0}^2 = t_{\delta(\mu_0, \alpha_0)} \).
Proof. For \( g \geq 2 \), we have the short exact sequence
\[
1 \to \mathcal{T}(N_g,n) \to \mathcal{M}(N_g,n) \to \langle Y_{\mu_0,\alpha_0} | Y_{\mu_0,\alpha_0}^2 \rangle \to 1
\]
(see [3][4]). As basics on combinatorial group theory, a presentation for \( \mathcal{M}(N_g,n) \) is obtained by adding the generator \( Y_{\mu_0,\alpha_0} \) and the relations \( Y_{\mu_0,\alpha_0}^t = t Y_{\mu_0,\alpha_0}(\cdot) \) and \( Y_{\mu_0,\alpha_0}^2 = t \delta(\mu_0,\alpha_0) \) to the presentation for \( \mathcal{T}(N_g,n) \) given in Theorem 2.1 (for details, for instance see [3]). Thus we obtained the claim. \( \Box \)

We now prove Theorem 1.1

Let \( G \) be the group presented in Theorem 1.1. When \( g = 1 \), since a crosscap slide can not be defined we see \( \mathcal{M}(N_g,n) = \mathcal{T}(N_g,n) \). In addition, since \( \mathcal{Y} \) is the empty set, \( G \) is isomorphic to \( \mathcal{T}(N_g,n) \) by Theorem 2.1 and so \( \mathcal{M}(N_g,n) \). Therefore we suppose \( g \geq 2 \).

Let \( \varphi : G \to \mathcal{M}(N_g,n) \) be the natural homomorphism and \( \psi : \mathcal{M}(N_g,n) \to G \) the homomorphism defined as \( \psi(f) = f \) for any \( f \in \mathcal{T} \cup \{ Y_{\mu_0,\alpha_0} \} \). Since any relation of \( \mathcal{M}(N_g,n) \) in Corollary 2.2 is satisfied in \( G \), we see that \( \psi \) is well-defined. Since \( \varphi \circ \psi \) is the identity map clearly, it suffices to show that \( \psi \circ \varphi \) is the identity map.

For any \( t_{c,\theta} \in \mathcal{T} \), it is clear that \( \psi(\varphi(t_{c,\theta})) = t_{c,\theta} \). For any \( Y_{\mu,\alpha} \in \mathcal{Y} \), there is \( f \in \mathcal{M}(N_g,n) \) such that \( f Y_{\mu_0,\alpha_0} f^{-1} = Y_{\mu,\alpha} \), that is, \( f(\mu_0) = \mu \) and \( f(\alpha_0) = \alpha \), in \( \mathcal{M}(N_g,n) \). If \( f \) is in \( \mathcal{T}(N_g,n) \), since \( f \) can be represented as a word \( f_1 \cdots f_k \) on \( \mathcal{T} \), repeating the relations (2) (a) and (3) (b) of \( G \), we calculate
\[
\psi(\varphi(Y_{\mu,\alpha})) = \psi(Y_{\mu,\alpha}) = \psi(f Y_{\mu_0,\alpha_0} f^{-1}) = \psi(f_1 \cdots f_k Y_{\mu_0,\alpha_0} f_1^{-1} \cdots f_k^{-1}) = f_1 \cdots f_k Y_{\mu_0,\alpha_0} f_1^{-1} \cdots f_k^{-1} = Y_{f_1 \cdots f_k(\mu_0), f_1 \cdots f_k(\alpha_0)} = Y_{\mu,\alpha}.
\]

If \( f \) is not in \( \mathcal{T}(N_g,n) \), there exists \( h \in \mathcal{T}(N_g,n) \) such that \( f = h Y_{\mu_0,\alpha_0} \), by the sequence in the proof of Corollary 2.2. Since \( h \) can be represented as a word \( h_1 \cdots h_k \) on \( \mathcal{T} \), repeating the relations (2) (a) and (3) (b) of \( G \), we calculate
\[
\psi(\varphi(Y_{\mu,\alpha})) = \psi(Y_{\mu,\alpha}) = \psi(f Y_{\mu_0,\alpha_0} f^{-1}) = \psi(h Y_{\mu_0,\alpha_0} Y_{\mu_0,\alpha_0}^{-1} h^{-1}) = \psi(h_1 \cdots h_k Y_{\mu_0,\alpha_0} h_1^{-1} \cdots h_k^{-1}) = h_1 \cdots h_k Y_{\mu_0,\alpha_0} h_1^{-1} \cdots h_k^{-1} = Y_{h_1 \cdots h_k(\mu_0), h_1 \cdots h_k(\alpha_0)} = Y_{h(\mu_0), h(\alpha_0)} = Y_{h(Y_{\mu_0,\alpha_0}(\mu_0)), h(Y_{\mu_0,\alpha_0}(\alpha_0))} = Y_{f(\mu_0), f(\alpha_0)} = Y_{\mu,\alpha}.
\]

Therefore we conclude that \( \psi \circ \varphi \) is the identity map.
Thus we have that $G$ is isomorphic to $\mathcal{M}(N_{g,n})$, and hence the proof of Theorem 1.1 is completed.

3. Proof of Theorem 1.2

Any relation which appeared in Theorem 1.2 is clearly satisfied in $\mathcal{M}(N_{g,n})$. In addition, the relations (1)-(5) in Theorem 1.1 are included in the relations in Theorem 1.2. Hence it suffices to show that the relation (6) in Theorem 1.1 is obtained from relations in Theorem 1.2 when $g \geq 3$.

Remark 3.1. • In the relation (3) (a) in the main theorems, if $f = t_{c',\varrho'}$, $|c \cap c'| = 0$ or 1, and the orientations $\varrho$ and $\varrho'$ are compatible, then the relation can be rewritten as a commutativity relation $t_{c,\varrho} t_{c',\varrho'} = t_{c',\varrho'} t_{c,\varrho}$ or a braid relation $t_{c,\varrho} t_{c',\varrho} t_{c,\varrho} = t_{c',\varrho'} t_{c,\varrho} t_{c',\varrho}$ respectively. We assign the label (3)' to all the commutativity relations and all the braid relations.

• It is known that any chain relation is obtained from the relations (1), (3) (a), (4) and (5) in the main theorems (see [10]). We assign the label (4)' to all the chain relations.

For any $Y_{\mu,\alpha} \in \mathcal{Y}$, let $\beta, \gamma, \delta, A, B, C, D$ and $E$ be simple closed curves with arrows as shown in Figure 4. By Figure 4(a), we have $t_{\beta} t_{\alpha} t_{\gamma} t_{\alpha} t_{\beta}(\mu) = \mu$ and $t_{\beta} t_{\alpha} t_{\gamma} t_{\alpha} (\alpha^{-1}) = \alpha$. By Figure 4(b) and the relation (2) (a), we have $t_{Y_{\mu,\alpha}^{-1}(\beta)} = t_{\delta}$ and $t_{Y_{\mu,\alpha}^{-1}(\gamma)} = t_{\beta}$. Note that $t_{Y_{\mu,\alpha}^{-1}(\alpha)} = t_{A}$. By Figure 4(c) and the relation (2) (a), we have $t_{t_{\gamma}(\alpha)} = t_{A}$ and $t_{t_{\gamma}(\delta)} = t_{B}$. By Figure 4(d) and the relation (4)', we have $(t_{\delta} t_{\alpha} t_{\beta})^{4} = t_{C} t_{D}$. By Figure 4(e) and the relations (2) (a) and (5), we have $t_{B}^{-1} t_{A}^{-1} t_{t_{\delta}(\mu,\alpha)} = t_{C} t_{E} t_{\alpha} t_{\alpha}^{-1}$.

Figure 4. Simple closed curves $\mu, \alpha, \beta, \gamma, \delta, A, B, C, D$ and $E$ of $N_{g,n}$.
Figure 5.
We calculate

\[ Y_{\mu, \alpha}^2 = Y_{t_\beta t_\gamma t_\delta t_\beta t_\delta (\mu), t_\beta t_\gamma t_\delta t_\beta (\alpha)} \]

\[ \equiv (3)(b)(2)(b) \]

\[ t_\beta t_\gamma t_\delta t_\beta Y_{\mu, \alpha}^{-1} t_\beta^{-1} t_\gamma^{-1} t_\delta^{-1} Y_{\mu, \alpha}^{-1} \]

\[ \equiv (3)(a) \]

\[ t_\beta t_\gamma t_\delta t_\beta Y_{\mu, \alpha}^{-1} (\beta) Y_{\mu, \alpha}^{-1} (\alpha) Y_{\mu, \alpha}^{-1} (\gamma) Y_{\mu, \alpha}^{-1} (\alpha) Y_{\mu, \alpha}^{-1} (\beta) \]

Therefore the group presented in Theorem 1.2 is isomorphic to the group presented in Theorem 1.1, and so \( M(N, g, n) \). Thus we finish the proof of Theorem 1.2.

4. Proof of Theorems 1.3 and 1.4

We first note that Theorems 1.3 and 1.4 can be shown by the arguments similar to Sections 2 and 3 respectively. However we prove these by giving isomorphisms from the groups presented in Theorems 1.3 and 1.4 to the groups presented in Theorems 1.1 and 1.2 respectively.

Let \( H \) be the group presented in either Theorems 1.3 or 1.4. When \( g = 1 \), since \( U \) is the empty set, the presentation of \( H \) is same to the presentation of the group presented in Theorems 1.1 and 1.2. Hence \( H \) is isomorphic to \( M(N_1, n) \). Therefore we suppose \( g \geq 2 \).
Let $\eta : H \rightarrow \mathcal{M}(N_{g,n})$ be the natural homomorphism and $\nu : \mathcal{M}(N_{g,n}) \rightarrow H$ the homomorphism defined as $\nu(t_{c,\theta}) = t_{c,\theta}$ and $\nu(Y_{\mu,\alpha}) = t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta}$ for any $t_{c,\theta} \in T$ and $Y_{\mu,\alpha} \in Y$. If $\nu$ is well-defined, $\nu$ is the inverse map of $\eta$ clearly. So it suffices to show well-definedness of $\nu$, that is, we show that the correspondence $\nu(Y_{\mu,\alpha}) = t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta}$ does not depend on the choice of $\theta \in \{+\alpha,-\alpha\}$, and that any relation of $\mathcal{M}(N_{g,n})$ is satisfied in $H$.

First we show that $t_{\alpha,\beta}^{-1}U_{\mu,\alpha;+\alpha} = t_{\alpha,\beta}^{-1}U_{\mu,\alpha;-\alpha}$ in $H$. By the relations (2) (a) and (3) (a) of $H$, we calculate

$$
U_{\mu,\alpha,\theta}t_{\alpha,\beta}U_{\mu,\alpha,\theta}^{-1} = t_{\nu_{\mu,\alpha,\theta}(\alpha);(U_{\mu,\alpha,\theta})}(\theta)
$$

$$
= t_{\alpha,\theta'}
$$

$$
= t_{\alpha,\theta'}^{-1},
$$

where $\theta'$ is the inverse orientation of $\theta$, and so $t_{\alpha,\theta}^{-1}U_{\mu,\alpha,\theta} = U_{\mu,\alpha,\theta}t_{\alpha,\theta}$. Since $t_{\alpha,\beta}^{-1}(\mu) = t_{\alpha,\beta}^{-1}(Y_{\mu,\alpha}(\mu)) = U_{\mu,\alpha,\theta}(\mu)$, $t_{\alpha,\beta}^{-1}(\alpha) = t_{\alpha,\beta}^{-1}(Y_{\mu,\alpha}(\alpha)) = U_{\mu,\alpha,\theta}(\alpha)$ and $(t_{\alpha,\beta}^{-1})(\theta) = (t_{\alpha,\beta}^{-1})(U_{\mu,\alpha,\theta}(\theta)) = (U_{\mu,\alpha,\theta})^{-1}$, by the relations (2) (a) and (3) (b) of $H$, we calculate

$$
t_{\alpha,\beta}^{-1}U_{\mu,\alpha;+\alpha} = t_{\alpha,\beta}^{-1}U_{\mu,\alpha;+\alpha}
$$

$$
= t_{\alpha,\beta}^{-1}U_{\mu,\alpha;+\alpha}
$$

$$
= t_{\alpha,\beta}^{-1}U_{\mu,\alpha;+\alpha}
$$

$$
= U_{\mu,\alpha,\theta}t_{\alpha,\theta}^{-1}U_{\mu,\alpha,\theta}
$$

$$
= U_{\mu,\alpha,\theta}
$$

and so $t_{\alpha,\beta}^{-1}U_{\mu,\alpha;+\alpha} = t_{\alpha,\beta}^{-1}U_{\mu,\alpha;-\alpha}$. Therefore we conclude that the correspondence $\nu(Y_{\mu,\alpha}) = t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta}$ does not depend on the choice of $\theta \in \{+\alpha,-\alpha\}$.

Next we show that any relation appearing in Theorems 1.1 and 1.2 is satisfied in $H$. The relations (1), (2) (a), (3) (a) with $f \in T$, (4) and (5) in Theorems 1.1 and 1.2 are satisfied in $H$ clearly. By the relation (2) (b) of $H$ and the equality $t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta} = U_{\mu,\alpha,\theta}t_{\alpha,\theta}$ shown above, we calculate

$$
\nu(Y_{\mu,\alpha}^{-1}) = (t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta})^{-1}
$$

$$
= (U_{\mu,\alpha,\theta})^{-1}
$$

$$
= t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta}
$$

$$
= t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta}
$$

$$
= \nu(Y_{\mu,\alpha}^{-1}).
$$

Hence the relation (2) (b) in Theorem 1.2 is satisfied in $H$. By the relations (2) (a) and (3) (a) of $H$, we calculate

$$
\nu(Y_{\mu,\alpha}^{-1}Y_{\mu,\alpha}^{-1}) = \nu(Y_{\mu,\alpha}^{-1}Y_{\mu,\alpha}^{-1})
$$

$$
= (t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta})(t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta})^{-1}
$$

$$
= (t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta})(t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta})^{-1}
$$

$$
= t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta}
$$

$$
= t_{\alpha,\beta}^{-1}U_{\mu,\alpha,\theta}
$$

$$
= \nu(Y_{\mu,\alpha}^{-1}).
$$
where $\theta'$ is the inverse orientation of $\theta$. Hence the relations (3) (a) with $f \in \mathcal{Y}$ in Theorems 1.1 and 1.2 are satisfied in $H$. By the relations (3) (a) and (3) (b) of $H$, we calculate

\[
\nu(f_{Y,\alpha}^{-1}) = f(t_{\alpha,\theta}^{-1}U_{\alpha,\theta})^{-1}
= (f_{\alpha,\theta}^{-1}U_{\alpha,\theta})^{-1}(f_{\alpha,\theta}^{-1}U_{\alpha,\theta})
= t_{\alpha,\theta}^{-1}f(\alpha); f(\alpha); f_{\alpha,\theta})
= \nu(Y_{f(\alpha,\theta)}).
\]

Hence the relations (3) (b) in Theorems 1.1 and 1.2 are satisfied in $H$. By the relation (6) of $H$ and the equality $t_{\alpha,\theta}^{-1}U_{\alpha,\theta} = U_{\alpha,\theta}t_{\alpha,\theta}$, we calculate

\[
\nu(Y_{\mu,\alpha}^2) = (t_{\alpha,\theta}^{-1}U_{\alpha,\theta})^2
= (U_{\alpha,\theta}t_{\alpha,\theta})(t_{\alpha,\theta}^{-1}U_{\alpha,\theta})
= U_{\alpha,\theta}
= t_{\delta(\mu,\alpha)}
= \nu(t_{\delta(\mu,\alpha)}).
\]

Hence the relation (6) in Theorem 1.1 is satisfied in $H$. Therefore we conclude that any relation of $\mathcal{M}(N_{g,n})$ is satisfied in $H$. So we obtain well-definedness of $\nu$, and hence it follows that $H$ is isomorphic to $\mathcal{M}(N_{g,n})$. Thus we complete the proof of Theorems 1.3 and 1.4.

APPENDIX A. PROBLEMS ON PRESENTATIONS FOR $\mathcal{M}(N_{g,n})$

In this appendix, we introduce three problems on presentations for $\mathcal{M}(N_{g,n})$ as follows.

In the main theorems, we can reduced relations. For example, in the relation (3) (a) in Theorems 1.1 and 1.2 (resp. Theorems 1.3 and 1.4), $T \cup \mathcal{Y}$ (resp. $T \cup \mathcal{U}$) can be reduced to a finite number of Dehn twists and one crosscap slide (resp. a finite number of Dehn twists and one crosscap transposition). Similarly, in the relation (3) (b) in Theorems 1.1 and 1.2 (resp. Theorems 1.3 and 1.4), $T$ (resp. $T \cup \{U_{\mu,\alpha,\theta}\}$) can be reduced to a finite number of Dehn twists (resp. a finite number of Dehn twists by adding the relation $U_{\mu,\alpha,\theta} = U_{\mu,\alpha,\theta}t_{\alpha,\theta}$). In addition, we can reduced the relation (6) in Theorem 1.1 (resp. Theorem 1.3) to one relation $Y_{\mu,\alpha,\theta} = t_{\delta(\mu,\alpha,\theta)}$ (resp. $U_{\mu,\alpha,\theta}^2 = t_{\delta(\mu,\alpha,\theta)}$) for some pair $(\mu, \alpha, \theta)$. On the other hand, the relation of the mapping class group of an orientable surface corresponding to the relation (3) (a) with $f \in T$ in the main theorems can be reduced to only all the commutativity relations and all the braid relations. However, in the non-orientable case, we do not know whether or not the same holds. So we have a natural problem as follows.

**Problem A.1.** Give an infinite presentation of $\mathcal{M}(N_{g,n})$ whose relations are more simple for $g \geq 1$ and $n \geq 0$.

It is known that $\mathcal{M}(N_{g,n})$ can not be generated by only either Dehn twists or crosscap slides for $g \geq 2$ and $n \geq 0$ (see [8, 9]). On the other hand, Leśniak-Szpejotowski [7] showed that $\mathcal{M}(N_{g,n})$ can be generated by only crosscap transpositions for $g \geq 7$ and $n \geq 0$. So we have a natural problem as follows.
Problem A.2. Give an infinite presentation of $\mathcal{M}(N_{g,n})$ with all crosscap transpositions for $g \geq 7$ and $n \geq 0$.

It is known that $\mathcal{M}(N_{1,0})$ and $\mathcal{M}(N_{1,1})$ are trivial (see [2]). In addition, for $g \geq 2$ and $n \leq 1$, a simple finite presentation of $\mathcal{M}(N_{g,n})$ was given (see [8, 1, 13, 15, 12]). Moreover, Omori and the author [5] gave a finite presentation of $\mathcal{M}(N_{g,n})$ for $g \geq 1$ and $n \geq 2$. However this presentation is very complicated. On the other hand, there is a simple finite presentation for the mapping class group of a compact orientable surface (see [3]). So we have a natural problem as follows.

Problem A.3. Give a simple finite presentation of $\mathcal{M}(N_{g,n})$ with Dehn twists and crosscap slides, or with Dehn twists and crosscap transpositions (resp. with crosscap transpositions) for $g \geq 1$ (resp. $g \geq 7$) and $n \geq 0$.

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