Classical dynamics on Snyder spacetime

S. Mignemi

Dipartimento di Matematica e Informatica,
Università di Cagliari, viale Merello 92,
09123 Cagliari, Italy

May 11, 2014

Abstract

We study the classical dynamics of a particle in Snyder spacetime, adopting the formalism of constrained Hamiltonian systems introduced by Dirac. We show that the motion of a particle in a scalar potential is deformed with respect to special relativity by terms of order $\beta E^2$. An important result is that in the relativistic Snyder model a consistent choice of the time variable must necessarily depend on the dynamics.
1 Introduction

The interest on noncommutative geometries has greatly increased during last years, because they may describe the structure of spacetime at Planck scales, where the effects of quantum gravity are sensible and the location of particles in space and time may become fuzzy [1].

Historically, the first example of noncommutative geometry was proposed by Snyder [2], and was based on a deformation of the Heisenberg algebra of quantum mechanics. In spite of the presence of a fundamental length scale, his model is invariant under Lorentz transformations, and only the action of the translations is not trivial [3, 4]. The possibility of preserving the Lorentz invariance is due to the fact that the deformed Heisenberg algebra is not a Lie algebra, as in simpler noncommutative models [1], but rather a function algebra, the structure constants being dependent on position and momentum.

Although the investigation of Snyder spacetime has been neglected for many years, recently its implications have been studied from several points of view [5, 6]. In particular, in order to clarify the physical properties of the Snyder model, it can be useful to start from the investigation of its classical limit. This is described by a phase space with noncanonical symplectic structure, and its dynamics must therefore necessarily be investigated using Hamiltonian methods.

In this framework, the classical motion of a nonrelativistic particle in Snyder space has been studied in detail [5], and the exact solutions of the equations of motion have been found in the case of a free particle and of a harmonic potential. It results that, while the free motion is trivial, the classical dynamics is modified in the presence of external forces. For example, the motion of a harmonic oscillator is still periodic, but no longer given by a simple trigonometric function as in classical mechanics, and the frequency of oscillation acquires a dependence on the energy, like in special relativity.

It is interesting to investigate if these features extend to the relativistic dynamics. The problem is not trivial, because it is known that in the relativistic domain the Hamiltonian dynamics of a particle is constrained, and to treat the problem one must employ for example the Dirac formalism [7, 8]. Moreover, due to the nontrivial Poisson brackets between time and spatial coordinates of the relativistic Snyder model, its nonrelativistic limit does not necessarily coincide with the nonrelativistic theory.

While the study of the motion of a free relativistic particle presents no problems and reproduces the results of special relativity, the dynamics of a particle coupled to an external potential of scalar type in Hamiltonian form is not well known even in standard special relativity [9]. In particular, it is not obvious how to construct a consistent Hamiltonian formulation for a particle in a scalar potential in a covariant way. However, as mentioned above, the classical Snyder dynamics can be formulated only in Hamiltonian form. To overcome this difficulty, in this paper we adopt a recently proposed formalism [9] for the coupling of a particle to a scalar potential that permits the definition of a covariant Hamiltonian dynamics and the use of the Dirac procedure to eliminate the constraints. As it is well known, Dirac’s method requires the choice of a time variable, analogous to a gauge fixing, in order to eliminate the freedom in reparametrization. An important result of our paper is that in the interacting Snyder case a consistent choice must necessarily depend on the dynamics of the model.
2 Free particle

For simplicity we consider the motion in a (1+1)-dimensional spacetime. Since the structure of classical Snyder spacetime is expressed in terms of its noncanonical symplectic structure, its dynamics must be written in Hamiltonian form. For relativistic models, Hamiltonian dynamics can be described in a covariant way by using the Dirac formalism for constrained systems [7, 8].

The Snyder fundamental Poisson brackets are defined as

\[
\{x_\mu, p_\nu\} = \eta_{\mu\nu} + \beta p_\mu p_\nu, \quad \{x_\mu, x_\nu\} = \beta J_{\mu\nu}, \quad \{p_\mu, p_\nu\} = 0,
\]

where \(\eta_{\mu\nu}\) is the flat metric and \(J_{\mu\nu} = x_\mu p_\nu - p_\mu x_\nu\) is the generator of the Lorentz transformations. The parameter \(\beta\) has dimensions of inverse mass square and is usually assumed to be of Planck scale. It can be either positive or negative. In the latter case, the allowed value of the mass are bounded, \(m^2 < |\beta|^{-1}\), as in doubly special relativity [10]. The Poisson brackets behave covariantly under Lorentz transformations, while the action of the translations, generated by \(p_\mu\), on spacetime coordinates is nonlinear [3, 4].

The dynamics of a free particle in Snyder spacetime is trivial. In fact, since the Lorentz invariance is preserved, the Hamiltonian can be chosen as in special relativity,

\[
H = \frac{\lambda}{2} (p^2 - m^2),
\]

with \(p^2 = p_0^2 - p_1^2\) and \(\lambda\) a Lagrange multiplier enforcing the mass shell constraint \(\chi_1 = p^2 - m^2 = 0\). The Hamilton equations that follow from the nontrivial symplectic structure are

\[
\dot{x}_\mu = \{x_\mu, H\} = \lambda (1 + \beta p^2) p_\mu = \lambda (1 + \beta m^2) p_\mu, \quad \dot{p}_\mu = \{p_\mu, H\} = 0,
\]

where a dot denotes the derivative with respect to the evolution parameter. The constraint \(\chi_1 = 0\) is first class, and according to Dirac, one must impose a further constraint to eliminate the redundant degrees of freedom \(x_0\) and \(p_0\) and reduce the system to the motion in one spatial dimension with external time.

For the standard choice \(\chi_2 = x_0 - t = 0\), which corresponds to the identification of the evolution parameter with the coordinate time, one has

\[
C \equiv \{\chi_2, \chi_1\} = (1 + \beta m^2)p_0.
\]

It follows from the requirement \(\dot{\chi}_2 = 0\) that \(\lambda = 1/C\) [8]. On the constraint surface, the dynamics is dictated by the Dirac brackets, defined as

\[
\{A, B\}^* = \{A, B\} + \{A, \chi_2\} C^{-1}\{\chi_1, B\} - \{A, \chi_1\} C^{-1}\{\chi_2, B\}
\]

For the independent variables \(x_1, p_1\), they read

\[
\Delta \equiv \{x_1, p_1\}^* = -1,
\]

as in special relativity. Moreover, the reduced Hamiltonian \(K\) results in

\[
K = p_0 = \sqrt{p_1^2 + m^2},\]

\(^1\)We use units in which \(c = 1\) and metric signature \((+,-)\).
and the Hamilton equations following from (5) and (6) are

\[
\frac{dx_1}{dt} = \frac{p_1}{\sqrt{p_1^2 + m^2}}, \quad \frac{dp_1}{dt} = 0, \tag{7}
\]

which coincide with the equations of motion of a free particle in special relativity. In the case of a free particle the motion in Snyder spacetime is therefore trivial.

In view of the the results on the interacting particle of the following section, it appears however that a more physical choice of gauge is given by a constant rescaling of time,

\[
t = \sqrt{1 + \beta m^2} x_0 = \sqrt{1 + \beta p^2} x_0.
\]

A justification for this choice is that the natural metric of spacetime, invariant under Snyder transformations is \(ds^2 = (1 + \beta p^2)dx^2\), with \(dx^2\) the Minkowski metric \([4]\).

In this gauge, \(\{\chi_2, \chi_1\} = (1 + \beta m^2)^{3/2}p_0\), but the Dirac brackets are still given by (5). The reduced Hamiltonian is now

\[
K = \frac{p_0}{\sqrt{1 + \beta m^2}} = \sqrt{\frac{p_1^2 + m^2}{1 + \beta m^2}}, \tag{8}
\]

and the Hamilton equations read

\[
\frac{dx_1}{dt} = \frac{p_1}{\sqrt{(1 + \beta m^2)(p_1^2 + m^2)}}, \quad \frac{dp_1}{dt} = 0. \tag{9}
\]

Of course, the only difference from (7) is a rescaling of the momenta. In this gauge the rest energy of a particle is \(m_0 = \frac{m}{\sqrt{1 + \beta m^2}}\).

## 3 Harmonic oscillator

A more interesting problem occurs when the particle is subject to an external force generated by a potential. We shall consider in particular the case of a harmonic potential, which depends only on the spatial position of the particle, \(V = V(x_1)\). To our knowledge, the coupling of a particle with a scalar potential in classical special relativity has not been discussed in depth. Here, we adopt the proposal of \([9]\), that preserves the reparametrization invariance of the theory and hence permits the use of the Dirac formalism. According to it, a consistent Hamiltonian for a particle coupled to a scalar potential is given by

\[
H = \frac{\lambda}{2}[p^2 - (m + V)^2] = 0, \tag{10}
\]

enforcing the constraint \(\chi_1 = p^2 - (m + V)^2 = 0\). The equations of motion derived from (10) with the help of the Poisson brackets (1) read

\[
\dot{x}_0 = \lambda [(1 + \beta p^2)p_0 + \beta J(m + V)V'], \quad \dot{p}_0 = \lambda p_0 (m + V)V', \\
\dot{x}_1 = \lambda (1 + \beta p^2)p_1, \quad \dot{p}_1 = -\lambda (1 - \beta p_1^2)(m + V)V'. \tag{11}
\]

where a prime denotes a derivative with respect to \(x_1\) and \(J \equiv J_{10}\) is the generator of the Lorentz transformations. Note in particular that, because of the nontrivial Poisson brackets between \(x_0\) and \(x_1\), an additional term proportional to \(\beta\) appears in the \(\dot{x}_0\) equation in
comparison with special relativity. Moreover, due to the nontrivial symplectic structure, in
the limit $c \to \infty$, with $\beta$ constant, the coordinate $x_0$ does not coincide with the nonrelativistic
time, and hence in that limit the relativistic Snyder dynamics does not go into the
nonrelativistic Snyder dynamics.

In the interacting case, it is difficult to find a gauge fixing compatible with the nontrivial
symplectic structure. For example, the gauge choice $t = x_0$ leads to inconsistencies. One is
forced to make a choice of time that depends on the dynamics of the model. We choose
\[ \chi = Sx_0 - t = 0, \] (12)
with
\[ S = \sqrt{1 + \beta(m + V)^2} = \sqrt{1 + \beta p^2}. \] (13)
As explained before, this choice can be understood considering the natural metric of the
Snyder spacetime [4]. This choice will make the Dirac brackets independent of $x_0$.

The Poisson bracket of the constraints $\chi_1$ and $\chi_2$ reads
\[ C = \{\chi_2, \chi_1\} = S(1 + \beta p^2)p_0 + \beta SJ(m + V)V' + \beta S^{-1}(1 + \beta p^2)J(m + V)V'p_1x_0 \]
\[ = S[S^2 + (m + V)V'x_1]p_0. \] (14)
The Lagrange multiplier resulting from this gauge choice is therefore $\lambda = 1/C$, and the Dirac
brackets of the independent variables $x_1$ and $p_1$ read
\[ \{x_1, p_1\}^* = -\frac{S^2}{S^2 + (m + V)V'x_1}. \] (15)

The reduced Hamiltonian $K$ must be chosen so that it generates the motion on the
reduced phase space induced by the Dirac brackets (15). The correct choice is
\[ K = \frac{p_0}{S} = \sqrt{\frac{p_1^2 + (m + V)^2}{1 + \beta(m + V)^2}}. \] (16)
This quantity is conserved under the time evolution dictated by $H$ and represents the energy
of the system in the laboratory frame.

The Hamilton equations derived from (15) and (16) or equivalently from (11) are
\[ \frac{dx_1}{dt} = \frac{1}{B} \frac{p_1}{K}, \quad \frac{dp_1}{dt} = -\frac{1 - \beta p_1^2}{(1 - \beta m^2)B} \frac{(m + V)V'}{K}, \] (17)
where $B = S^2 + (m + V)V'x_1$.

This system of equations can be solved in two steps. First define an auxiliary time
variable $\tau$, such that $dt = Bd\tau$, so that the equations (17) take the form
\[ \frac{dx_1}{d\tau} = \frac{p_1}{K}, \quad \frac{dp_1}{d\tau} = -\frac{1 - \beta p_1^2}{1 - \beta m^2} \frac{(m + V)V'}{K}. \] (18)
The equations (18) can be solved in a standard way by exploiting the conservation of the
reduced Hamiltonian $K$, from which follows
\[ p_1^2 = K^2 - (1 - \beta K^2)(m + V)^2 \] (19)
and then, using the first of eqs. (18)

\[
\frac{dx_1}{d\tau} = \sqrt{1 - \left(\frac{1 - \beta K^2}{K^2}\right)(m + V)^2}.
\]  

(20)

A redefinition of the energy,

\[
E = \frac{K}{\sqrt{1 - \beta K^2}},
\]  

(21)

reduces this equation to that of classical special relativity. In particular, in the case of the harmonic oscillator with potential \(V = \frac{\kappa}{2}x_1^2\), the solution is [9, 11]

\[
x_1 = \sqrt{\frac{E^2 - m^2}{\kappa E}} \, \text{sd}(\omega \tau, q),
\]  

(22)

where

\[
\omega^2 = \frac{\kappa}{E} = \frac{\kappa\sqrt{1 - \beta K^2}}{K}, \quad q = \frac{E - m}{2E} = \frac{K - m\sqrt{1 - \beta K^2}}{2K},
\]

and \(\text{sd}(\omega \tau, q)\) is a Jacobian elliptic function. The period of oscillation \(T_0\) can be written in terms of the complete elliptic integral \(K(q)\) as

\[
T_0 = \frac{4}{\omega} \, K(q) \sim \frac{2\pi}{\omega_0} \left(1 - \frac{3}{8} \frac{E - m}{m}\right),
\]  

(23)

where \(\omega_0 = \sqrt{\kappa/m}\) is the frequency of the nonrelativistic oscillator and we have written the first terms of a low-energy expansion.

After some calculations, one gets for the momentum

\[
p_1 = \sqrt{\frac{E^2 - m^2}{1 + \beta E^2}} \, \text{cd}(\omega \tau, q) \, \text{nd}(\omega \tau, q),
\]  

(24)

with \(\text{cd}(\omega \tau, q)\) and \(\text{nd}(\omega \tau, q)\) Jacobian elliptic functions.

One can now write down the solution of the Snyder oscillator in terms of the physical time variable \(t\), substituting (22) and (24) in its definition and integrating,

\[
t = \int [1 + \beta(m + V)(m + V) + V'x_1] d\tau = \int \left[m^2 + 2mkx_1^2 + \frac{3}{4} \kappa^2 x_1^4\right] d\tau
\]

\[
= (1 + \beta E^2) \tau - \frac{\beta(E^2 - m^2)}{\omega} \, \text{sd}(\omega \tau, q) \, \text{cd}(\omega \tau, q) \, \text{nd}(\omega \tau, q).
\]  

(25)

Eqns. (22), (24) and (25) give the exact solution of the relativistic Snyder oscillator in parametric form. The period of oscillation is given by \(T = t(T_0)\), i.e.

\[
T = \frac{4(1 + \beta E^2)}{\omega} \, K(q) = \frac{4}{(1 - \beta K^2)\omega} \, K(q),
\]  

(26)

and contains energy-dependent contributions coming from both special relativity and Snyder dynamics.
It may also be interesting to consider the limit of the solution for small energy, namely \( K - m_0 \ll m \ll \beta^{-1/2} \). In this limit one can make an expansion in powers of \( \frac{K - m_0}{m_0} \). However, to keep the formulae simple we shall write the expansion in terms of \( E \) and \( m \). It is easy to check that \( \frac{E - m}{m} \sim \frac{K - m_0}{m_0} (1 + \beta m_0^2) \).

At first order in \( \frac{E - m}{m} \), the corrections can be obtained from the nonrelativistic limit of eqns. (22) and (24),

\[
x_1 \sim \sqrt{\frac{2(E - m)}{\kappa}} \sin \omega_0 \tau, \quad p_1 \sim \sqrt{\frac{2m(E - m)}{1 + \beta m^2}} \cos \omega_0 \tau, \quad (27)
\]

with

\[
t \sim (1 + \beta m^2) \tau - \frac{\beta m(E - m)}{\omega_0} \sin \tau \cos \tau, \quad (28)
\]

and the period \( T \) of the oscillator is given by

\[
T \sim \frac{2\pi}{\omega_0} \left[ 1 + \beta m^2 - \left( \frac{3}{8} - 2\beta m^2 \right) \frac{E - m}{m} \right]. \quad (29)
\]

For positive \( \beta \) the period is increased with respect to special relativity. It may be compared with the exact nonrelativistic result [5], \( T = \frac{2\pi}{\omega_0} \left[ 1 - 2\beta m(E - m) \right]^{-1/2} \sim \frac{2\pi}{\omega_0} [1 + \beta m(E - m)] \).

The limit of (29) for \( c \to \infty \), with \( \beta \) fixed, does not coincide with this result. The reason is of course that the time variable of the relativistic model differs from that of the nonrelativistic model in this limit.

### 4 Conclusions

We have investigated the dynamics of the harmonic oscillator for the relativistic Snyder model and found an exact analytic solution of the equations of motion. The solution presents corrections of order \( \beta E^2 \) with respect to that of special relativity, as could have been predicted from dimensional considerations. In particular, for positive \( \beta \), the period of the harmonic oscillator is increased with respect to that of the relativistic oscillator.

An interesting result of our investigation is that in the interacting case the choice of the time variable must depend on the dynamics. This can be understood considering that in Snyder spacetime the natural metric is given by \( (1 + \beta p^2)(dx_0^2 - dx_1^2) \), and hence time in the laboratory frame is measured by \( t = \sqrt{1 + \beta p^2} x_0 \).

As for the nonrelativistic case [12], our results can easily be generalized to a curved background. A more challenging problem would be to extend our investigation to the quantum dynamics. This is of course not straightforward, since it would be necessary to define a quantum field theory framework.

### Acknowledgements

I wish to thank Stjepan Meljanac and Boris Ivetić for useful comments.
References

[1] For a review, see M.R. Douglas and A. Nekrasov, Rev. Mod. Phys. 73, 977 (2001); R.J. Szabo, Phys. Rep. 378, 207 (2003).

[2] H.S. Snyder, Phys. Rev. 71, 38 (1947).

[3] R. Banerjee, S. Kulkarni and S. Samanta, JHEP 05, 077 (2006).

[4] S. Mignemi, Phys. Lett. B672, 186 (2009).

[5] S. Mignemi, Phys. Rev. D84, 025021 (2011).

[6] J.M. Romero and A. Zamora, Phys. Rev. D70, 105006 (2004); Phys. Lett. B661, 11 (2008); M.V. Battisti and S. Meljanac, Phys. Rev. D79, 067505 (2009); Phys. Rev. D82, 024028 (2010); Lei Lu and A. Stern, Nucl. Phys. B852, 894 (2011); Nucl. Phys. B860, 186 (2012).

[7] P.A.M. Dirac, Lectures on quantum mechanics, Yeoshua University, New York 1964; A. Hanson, T. Regge and C. Teitelboim, Constrained Hamiltonian systems, Accademia Nazionale dei Lincei, Rome 1976.

[8] E.C.G. Sudarshan, N. Mukunda and J.N. Goldberg, Phys. Rev. D23, 2218 (1981).

[9] S. Mignemi, arXiv:1210.2707.

[10] G. Amelino-Camelia, Phys. Lett. B510, 255 (2001), Int. J. Mod. Phys. D11, 35 (2002); J. Magueijo and L. Smolin, Phys. Rev. Lett. 88, 190403 (2002).

[11] A.L. Harvey, Phys. Rev. D6, 1474 (1972).

[12] S. Mignemi, Class. Quantum Grav. 29, 215019 (2012).