Proofs of ”LQG Control For MIMO Systems Over Multiple TCP-like Erasure Channels”

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Here we will provide the proofs of the results stated in the Infinite Horizon LQG Control section of [1] by focusing on the control law and the related MARE. The analysis of the observation case can be achieved in a dual way and it is partially covered by [2].

Let us recall the control MARE:

\[
S_{k+1} = \Pi_c \left( S_k, A, B, U, W, \bar{N} \right) = A^T S_k A + W - A^T S_k B \bar{N} \left[ \sum_{I \in 2^\mathcal{I}} \left[ \eta_1^2 \left( N_I \left( U + B^T S_k B \right) N_I \right) \right] \right]^{-1} \bar{N} B^T S_k A,
\]

where \( \eta_I \in \mathbb{R} \) is

\[
\eta_I = \sqrt{\left( \prod_{i \in I} \bar{\nu}_i \right) \left( \prod_{i \notin I} (1 - \bar{\nu}_i) \right)},
\]

for every set \( I \in \mathcal{I} = \{1, ..., m\} \). Moreover, let us recall the auxiliary function

\[
\phi \left( K, X \right) = \sum_{I \in 2^\mathcal{I}} \left[ \eta_I^2 \left( F_I X F_I^T + V_I \right) \right],
\]

where

\[
F_I \triangleq A^T + K \left( N_I B^T \right),
\]

\[
V_I \triangleq W + KN_I U N_I^T K^T.
\]
The following results can be proved.

**Proposition (P1)** - If \( K \) is chosen such that
\[
K = \tilde{K}_X = -A^T XBN \left[ \sum_{I \in 2^3} \eta_I^2 \left( N_I \left( U + B^T XB \right) N_I \right) \right]^{-1},
\]
then
\[
\phi \left( \tilde{K}_X, X \right) = \Pi_c \left( X, A, B, U, W, \bar{N} \right).
\]

**Proof** - Consider the operator (2) together with (3)-(4)
\[
\phi \left( K, X \right) = \sum_{I \in 2^3} \eta_I^2 \left( \left( A^T + KN_I B^T \right) X \left( A + BN_I K^T \right) + W + KN_I UN_I K^T \right)
\]
Let us expand the product and group the terms of the latter equation
\[
\phi \left( K, X \right) = \sum_{I \in 2^3} \left[ \eta_I^2 \left( A^T XBN_I K^T + KN_I B^T XA + KN_I B^T XB N_I + A^T XA + W + KN_I UN_I K^T \right) \right] = \sum_{I \in 2^3} \left[ \eta_I^2 \left( A^T XBN_I K^T + KN_I B^T XA + KN_I B^T XB + U \right) N_I K^T \right] + \eta_I^2 \left( A^T XA + W \right)
\]
By splitting the summation and exploiting the fact that \( \sum_{I \in 2^3} \eta_I^2 = 1 \), we obtain
\[
\phi \left( K, X \right) = \sum_{I \in 2^3} \left[ \eta_I^2 \left( A^T XBN_I K^T + KN_I B^T XA + KN_I \left( B^T XB + U \right) N_I K^T \right) \right] + \left( \sum_{I \in 2^3} \eta_I^2 \right) \left( A^T XA + W \right) = \sum_{I \in 2^3} \left[ \eta_I^2 \left( A^T XBN_I K^T + KN_I B^T XA + KN_I \left( B^T XB + U \right) N_I K^T \right) \right] + A^T XA + W.
\]
Moreover, because \( \sum_{I \in 2^3} \eta_I^2 N_I = \bar{N} \) we have
\[
\phi \left( K, X \right) = \sum_{I \in 2^3} \left[ \eta_I^2 \left( K \left( N_I \left( B^T XB + U \right) N_I K^T \right) \right) + A^T XB \bar{N} K^T + K \bar{N} B^T XA + A^T XA + W \right] = KL^{-1}(X)K^T + A^T XB \bar{N} K^T + K \bar{N} B^T XA + A^T XA + W.
\]
where \( L(X) \) is a shorthand defined as follows
\[
L(X) = \left[ \sum_{I \in 2^3} \eta_I^2 \left( N_I \left( U + B^T XB \right) N_I \right) \right]^{-1}.
\]
Finally, by substituting $K = K_x = -ATXB\bar{N}(L(X))$ we obtain
\[
\phi(K, X) = 
= A^TXB\bar{N}(L(X))NBTSA - A^TXB\bar{N}(L(X))\bar{N}BTXA - A^TXB\bar{N}(L(X))\bar{N}BXA + ATXA + W = 
= A^TXA + W - A^TXB\bar{N}(L(X))\bar{N}BTXA,
\]
which concludes the proof. \(\square\)

**Proposition (P2)** - Let us define
\[
g_{\bar{N}}(X) = \min_K \phi(K, X), \tag{5}
\]
then
\[
g_{\bar{N}}(X) = \Pi_c(X, A, B, U, W, \bar{N}).
\]

**Proof** - By differentiate $\phi(K, X)$ with respect to $K$ we obtain:
\[
\frac{\partial \phi(K, X)}{\partial K} = 2K(L(X))^{-1} + 2A^TXB\bar{N} = 0.
\]
The minimizer is then
\[
K = -A^TXB\bar{N}(L(X)) = \bar{K}_X,
\]
and because of Proposition 1, the statement follows. \(\square\)

**Proposition (P3)** - If $X \leq Y$ then $g_{\bar{N}}(X) \leq g_{\bar{N}}(Y)$.

**Proof** - Being $\phi(K, X)$ affine in $X$, then
\[
g_{\bar{N}}(X) = \min_K \phi(K, X) \leq \phi(K_Y, X) \leq \phi(K_Y, Y) = g_{\bar{N}}(Y).
\]
\(\square\)

**Lemma (L1)** - Let us define the following operator:
\[
L(Y) = \sum_{I \in 2^\mathbb{I}} ||_I^2 \left( F_I Y F_I^T \right),
\]
If there exists a positive matrix $\bar{Y} > 0$ such that $\bar{Y} > L(\bar{Y})$ then:

a) $\forall W \geq 0, \lim_{k \to \infty} L^k(W) = 0,$

b) Let $U \geq 0$ and let $Y_{k+1} = L(Y_k) + U$ initialized at $Y_0$: then the sequence $Y_k$ is bounded.

**Proof** -

a) Let us choose two scalars $r \in [0, 1)$ and $m \geq 0$ such that
• $\mathcal{L}(\bar{Y}) < r \bar{Y}$,
• $W \leq m \bar{Y}$.

Being the operators $\mathcal{L}$ linear, crescent monotone, i.e. if $X > Y$ then $\mathcal{L}(X) > \mathcal{L}(Y)$, and because $W \geq 0$ implies $\mathcal{L}(W) \geq 0$, the following inequality results

$$0 \leq \mathcal{L}^k(W) \leq m \mathcal{L}^k(\bar{Y}) < mr^k \bar{Y}.$$ 

Then, for $k \to \infty$, we have $\mathcal{L}^k(W) \to 0$.

b) Let introduce two further scalars $m_U \geq 0$, $m_{Y_0} \geq 0$. By following the same lines of a) we obtain the following inequality

$$Y_k = \mathcal{L}^k(Y_0) + \sum_{t=0}^{k-1} \mathcal{L}^t(U) < \left( m_{Y_0} r^k + \sum_{t=0}^{k-1} m_U r^t \right) \bar{Y} = \left( m_{Y_0} r^k + m_U \frac{1 - r^k}{1 - r} \right) \bar{Y},$$

where the properties of the geometric series have been exploited. \( \square \)

**Lemma (L2)** - Let us suppose there exits a pair $(\bar{K}, \bar{S})$ of matrices such that:

$$\bar{S} \geq 0,$$
$$\bar{S} > \phi (\bar{K}, \bar{S}),$$

then, $\forall S_0$ the sequence $S_k = g^k_N(S_0)$ is bounded.

**Proof** - By using the operator

$$\mathcal{L}(Y) = \sum_{I \in \mathcal{I}_n} \eta^2_I \left( F_I Y F^T_I \right),$$

where

$$F_I \triangleq A^T + \bar{K} \left( N_I B^T \right),$$

we can write

$$\bar{S} > \phi (\bar{K}, \bar{S}) = \mathcal{L}(\bar{S}) + W + \sum_{I \in \mathcal{I}_n} \eta^2_I \left( \bar{K} N_I U N^T_I \bar{K}^T \right) \geq \mathcal{L}(\bar{S}).$$

By exploiting the definition of $g(S_k)$

$$S_{k+1} = g(S_k) \leq \phi (K, S_k) = \mathcal{L}(S_k) + W + \sum_{I \in \mathcal{I}_n} \eta^2_I \left( \bar{K} N_I U N^T_I \bar{K}^T \right) = \mathcal{L}(S_k) + U,$$

where

$$U = W + \sum_{I \in \mathcal{I}_n} \eta^2_I \left( \bar{K} N_I U N^T_I \bar{K}^T \right) \geq 0.$$
Then by resorting to Lemma L1 we can conclude that \( \{S_k\} \) is bounded.

**Lemma (L3)** - Let \( X_{t+1} = h(X_t), Y_{t+1} = h(X_t) \), if \( h \) is a monotone function, then:

\[
\begin{align*}
X_1 &\leq X_0 \Rightarrow X_{t+1} \leq X_t \quad \forall t, \\
X_1 &\geq X_0 \Rightarrow X_{t+1} \geq X_t \quad \forall t, \\
Y_0 &\leq X_0 \Rightarrow Y_t \leq X_t \quad \forall t.
\end{align*}
\]

**Proof** - Consider the first condition. It is true \( t=0 \). Then, by induction:

\[
X_{t+2} = h(X_{t+1}) \leq h(X_t) = X_t
\]

The other two cases are similar.

**Lemma (L4)** - Consider the operator \((2)\) together with \((3)-(4)\) If it exists a couple of matrices \((\tilde{K}, \tilde{S})\) such that

\[
\begin{align*}
\tilde{S} &> 0, \\
\tilde{S} &> \phi (\tilde{K}, \tilde{S}),
\end{align*}
\]

then:

1) for each initial condition \( \bar{S}_0 \geq 0 \), the MARE \((1)\) converges and moreover

\[
\lim_{t \to \infty} S_k = \lim_{k \to \infty} g_N^k (S_0) = \bar{S}
\]

does not depend on the initial condition,

2) \( \bar{S} \) is the unique positive definite fixed point of the MARE \((1)\)

**Proof** -

1) First let us prove that the MARE initialized at \( S_0 = Q_0 = 0 \) converges. Let \( Q_k = g_N^k (0) \).

Because \( Q_0 \leq Q_1 \), from Proposition 3 it follows

\[
Q_1 = g_N (Q_0) \leq g_N (Q_1) = Q_2.
\]

By induction we have a monotonic nondecreasing sequence of matrix. Because of Lemma L2 this sequence is bounded as follows

\[
0 \leq Q_1 \leq Q_2 \leq \ldots \leq M_{Q_0},
\]

Then, the sequence converges to one of the positive semidefinite fixed point of the MARE.

\[
\lim_{k \to \infty} Q_k = \bar{S},
\]
such that $S = g_N(\bar{S})$.

The next step is to show that the sequence converges to the same point $\forall S_0 = R_0 \geq \bar{S}$.

By resorting to Lemma L2 notation, we can observe that

$$\bar{S} = g_N(\bar{S}) = \mathcal{L}(\bar{S}) + W + \sum_{I \in 2^3} \eta_I^2 (K_N U N_I^T K^T) > \mathcal{L}(\bar{S})$$

The latter implies, as a consequence of Lemma L1, that $\lim_{k \to \infty} \mathcal{L}^k(X) = 0, \forall X \geq 0$.

Let us assume $S_0 = R_0 \geq \bar{S}$. Then

$$\mathcal{L}(R_0) = R_1 \geq \mathcal{L}(\bar{S}) = \bar{S}.$$ 

By induction it yields $R_k \geq \bar{S}, \forall k$.

Notice that:

$$0 \leq R_{k+1} - \bar{S} = g_N(R_k) - g_N(\bar{S}) = \phi(\bar{K}R_k, R_k) - \phi(\bar{K}, \bar{S}) \leq \phi(\bar{K}, R_k) - \phi(\bar{K}, \bar{S}) = \sum_{I \in 2^3} \eta_I^2 (F_I(R_k - S) F_I^T + V_I - V_I) = \mathcal{L}(R_k - \bar{S}).$$

Then, finally

$$0 \leq R_k - \bar{S} \leq \lim_{k \to \infty} \mathcal{L}(R_k - \bar{S}) = 0.$$ 

The last thing to show is that the Riccati equation converges to $\bar{S}, \forall \bar{S}_0 \geq 0$. Let us define

$$Q_0 = 0,$$

$$R_0 = S_0 + \bar{S}.$$

Then, $Q_0 \leq S_0 \leq R_0$. By resorting to Lemma L3 we obtain:

$$Q_k \leq S_k \leq R_k \ \forall k.$$ 

Then, finally

$$\bar{S} = \lim_{k \to \infty} Q_k \leq \lim_{k \to \infty} S_k \leq \lim_{k \to \infty} R_k = \bar{S}.$$ 

2) Let consider a certain $S' = g_N(S')$. If the Riccati equation is initialized at $S_0 = S'$, a constant sequence will results. Because each Riccati equation converges to $\bar{S}$, then $S' = \bar{S}$. \hfill \Box

**Theorem (LMI)** - The following statements are equivalent:

1) $\exists (\bar{K}, \bar{S}) : \bar{S} > 0, \ \bar{S} > \phi(\bar{K}, \bar{S})$
2) \( \exists Z, Y > 0 \) such that:

\[
\begin{bmatrix}
Y & Y & \eta_0 (YA^T + ZN_0B) & \eta_0 ZN_0U^{1/2} & \cdots & \eta_3 (YA^T + ZN_3B) & \eta_3 ZN_3U^{1/2} \\
Y & W^{-1} & 0 & 0 & \cdots & 0 & 0 \\
... & 0 & Y & 0 & \cdots & ... & ... \\
* & ... & 0 & I & & & \\
* & ... & ... & ... & & & \\
* & ... & Y & ... & & & \\
0 & ... & ... & 0 & I & & \\
\end{bmatrix} > 0
\]

where the sequence \( \emptyset, \ldots, 3 \) represents all possible sets belonging to \( 2^3 \).

**Proof** - Let us consider conditions 1) in the statement:

\[
S > 0,
\]

\[
S > \phi(K, S) = \sum_{i \in 2^3} \eta_i^2 (F_i X F_i^T + V_i),
\]

where \( F_i \) and \( V_i \) are introduced in (3)-(4). By expanding the second condition, we obtain

\[
S > \sum_{i \in 2^3} \eta_i^2 \left( (A^T + KN_iB) S (A^T + KN_iB^T)^T + W + KN_iUN_i^T K^T \right)
\]

which is equivalent to

\[
S - W - \sum_{i \in 2^3} \eta_i^2 \left( (A^T + KN_iB) S (A^T + KN_iB^T)^T + KN_iUN_i^T K^T \right) > 0
\]

By iteratively applying the Schur’s complements we obtain the following Matrix Inequality

\[
\begin{bmatrix}
S & I & \eta_0 (A^T + KN_0B) & \eta_0 ZN_0U^{1/2} & \cdots & \eta_3 (A^T + KN_3B) & \eta_3 ZN_3U^{1/2} \\
* & W^{-1} & 0 & 0 & \cdots & 0 & 0 \\
... & 0 & S^{-1} & 0 & \cdots & ... & ... \\
* & ... & 0 & I & & & \\
* & ... & ... & ... & & & \\
* & ... & S^{-1} & ... & & & \\
0 & ... & ... & 0 & I & & \\
\end{bmatrix} > 0
\]
that, by means of the congruence transformation $\text{diag}\{S^{-1}, I, I, \ldots, I\}$, simplifies into

$$
\begin{bmatrix}
S^{-1} & S^{-1} & \eta_0(S^{-1}A^T + S^{-1}KN_0B) & \eta_0S^{-1}KN_0U^{1/2} & \eta_3(S^{-1}A^T + S^{-1}KN_3B) & \eta_3S^{-1}KN_3U^{1/2} \\
S^{-1} & W^{-1} & 0 & 0 & \ldots & 0 & 0 \\
\ldots & 0 & S^{-1} & 0 & \ldots & \ldots & \ldots \\
* & \ldots & 0 & I & \ldots & \ldots & \ldots \\
* & \ldots & \ldots & \ldots & S^{-1} & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & 0 & I & \ldots \\
\end{bmatrix} > 0.
$$

By substituting $Y = S^{-1}, Z = S^{-1}K$ and by noticing that

$$S > 0 \iff S^{-1} = Y > 0,$$

the proof is completed.

REFERENCES

[1] E. Garone, B. Sinopoli, A. Goldsmith and A. Casavola, "LQG control for MIMO systems over multiple tcp-like erasure channels", Under Review, 2009.

[2] X. Liu and A. Goldsmith, "Kalman filtering with artial observation losses", in Proceedings of IEEE Conference on Decision and Control, Vol.4, Bahamas, Deceber 2005, pp.4180-4186.