An entanglement measure for n qubits

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Abstract

In Phys. Rev. A 61, 052306 (2000), Coffman, Kundu and Wootters introduced the residual entanglement for three qubits. In this paper, we present the entanglement measure \( \tau(\psi) \) for even \( n \) qubits; for odd \( n \) qubits, we propose the residual entanglement \( \tau^{(i)}(\psi) \) with respect to qubit \( i \) and the odd \( n \)-tangle \( R(\psi) \) by averaging the residual entanglement with respect to each qubit. In this paper, we show that these measures are \( LU \)-invariant, entanglement monotones, invariant under permutations of the qubits, and multiplicative in some cases.

Keywords: entanglement measure, entanglement monotone, residual entanglement

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1 Introduction

Entanglement plays an important role in quantum computation and quantum information \cite{1} \cite{2}. Many researchers in quantum information theory show interests in entanglement measures. Wootters introduced the idea of concurrence for two qubits to quantify entanglement \cite{3}. Subsequently, the concurrence was further developed in \cite{4} \cite{5} \cite{6}. Recently, Coffman, Kundu, and Wootters presented the residual entanglement which measures the amount of entanglement between subsystem \( A \) and subsystems \( BC \) for a tripartite state and

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gave an elegant expression for computing the residual entanglement for three qubits via the concurrence [7].

Vidal proposed entanglement monotone in [8]. It was later proved that the residual entanglement for three qubits is an entanglement monotone [9]. Recently, many authors have studied the residual entanglement.

Wong and Christensen defined even $n$-tangle for even $n$ qubits which is invariant under permutations of the qubits and demonstrated that the even $n$-tangle for even $n$ qubits is an entanglement monotone [10].

Their even $n$-tangle for even $n$ qubits is listed as follows. See (2) in [10].

$$\tau_{1...n} = 2 \sum a_{\alpha_1...\alpha_n} a_{\beta_1...\beta_n}^* a_{\gamma_1...\gamma_n}^* a_{\delta_1...\delta_n}^* \times \epsilon_{\alpha_1\beta_1} \epsilon_{\alpha_2\beta_2} \epsilon_{\alpha_3\beta_3} \epsilon_{\alpha_{n-1}\beta_{n-1}} \epsilon_{\gamma_1\delta_1} \epsilon_{\gamma_2\delta_2} \epsilon_{\gamma_{n-1}\delta_{n-1}} \epsilon_{\alpha_n\gamma_n} \epsilon_{\beta_n\delta_n}.$$ 

The even $n$-tangle is quartic and requires $3 \times 2^{4n}$ multiplications. Our entanglement measure $\tau(\psi)$ for even $n$ qubits is quadratic and requires $2^{n-1}$ multiplications [11]. Furthermore, Wong and Christensen indicated that the even $n$-tangle for even $n$ qubits is not a measure of $n$-way entanglement [10]. The $n$-way entanglement is the entanglement that critically involves all $n$ particles [10]. For odd $n$ qubits, they said that the $n$-tangle is undefined for odd $n > 3$, see their abstract in [10].

In a separate work [12], Yu and Song defined the residual entanglement for $n$ qubits as follows.

$$\tau_{ABC...N} = \min \{\tau_{\alpha} | \alpha = 1, 2, ..., \frac{N}{2} \},$$ \hspace{1cm} (1.1)

where $\alpha$ corresponds to all the possible foci and $C_N^i = n!/[(n-i)!i!]$. However, they did not show whether the residual entanglement is $LU$-invariant, or invariant under permutations of the qubits. Nor did they show that the residual entanglement is an entanglement monotone.

In another paper, Osterloh and Siewert constructed an $n$-qubit entanglement monotone from antilinear operators [13].

In an interesting work [14], Ou and Fan found that the monogamy of concurrence implies the monogamy of negativity, and that the resulting residual entanglement obtained through symmetrization all possible subsystem permutation gives rise to an entanglement monotone. In [14], they defined the negativity $N = (||\rho^{T_A}|| - 1)/2$, where $\rho^{T_A}$ is the partial transpose with respect to the subsystem $A$. Then, they defined the residual entanglements $\pi_A = N_{A(BC)}^2 - N_{AB}^2 - N_{AC}^2$, $\pi_B = N_{B(AC)}^2 - N_{BA}^2 - N_{BC}^2$, and
\[ \pi_C = N_C^{(AB)} - N_C^{(A)} - N_C^{(B)}. \]
However, they indicated that the residual entanglement corresponding to the
different focus varies under permutations of the qubits, i.e., generally \( \pi_A \neq \pi_B \neq \pi_C \).

Entanglement monotone is an important quality for entanglement measures. Any increase in correlations
achieved by LOCC should be naturally classical. In other words, entanglement should be non-increasing
under LOCC. Therefore monotonicity for entanglement measure under LOCC is considered as the natural
requirement [8]. The symmetry of entanglement measure under permutations implies that the measure
represents a collective property of the qubits which is unchanged by permutations [7]. In this paper, we
present entanglement measures for \( n \) qubits, and demonstrate that the entanglement measures in question
are (i) entanglement monotones, i.e., non-increasing on average under LOCC in all the \( n \) qubits, (ii) invariant
under permutations of the qubits, and (iii) multiplicative in some cases.

In this paper, in Sec. 2 we study the entanglement measure \( \tau(\psi) \) for even \( n \) qubits. In Sec. 3, we
investigate the residual entanglement \( \tau^{(i)}(\psi) \) with respect to qubit \( i \) and the odd \( n \)-tangle \( R(\psi) \) for odd \( n \)
qubits. \( \tau(\psi), \tau^{(i)}(\psi), \) and \( R(\psi) \) only require \(+, -, \) and \( \ast \) operations.

Notations: (1). Let \( |\psi\rangle = \sum_{i=0}^{2^{n-1}-1} a_i |i\rangle \) and \( |\psi'\rangle = \sum_{i=0}^{2^{n-1}-1} a'_i |i\rangle \) be states of \( n \) qubits in this paper.

(2). Let \( i_{n-1}...i_1i_0 \) be an \( n \)-bit binary representation of \( i \). That is, \( i = i_{n-1}2^{n-1} + ... + i_12^1 + i_02^0 \).
Then, let \( N(i) \) be the number of the occurrences of “1” in \( i_{n-1}...i_1i_0 \) and \( N^*(i) \) be the number of the
occurrences of “1” in \( i_{n-2}...i_1i_0 \), respectively.

## 2 Entanglement measure for even \( n \) qubits

In our previous work [11], we defined the entanglement measure of the state \( |\psi\rangle \) of even \( n \) qubits as

\[ \tau(\psi) = 2 |I^*(a, n)|, \tag{2.1} \]

where

\[ I^*(a, n) = \sum_{i=0}^{2^{n-2}-1} sgn^*(n, i)(a_{2i}a_{(2^{n-1})-2i} - a_{2i+1}a_{(2^{n-2})-2i}). \tag{2.2} \]

The functions \( sgn \) and \( sgn^* \) have been defined previously in [11]. To facilitate reading, we have listed
the definitions of \( sgn \) and \( sgn^* \) in Appendix A. When \( n = 2 \), \( \tau(\psi) = 2 |a_0a_3 - a_1a_2| \), which is just the
concurrence for two qubits.

Theorem 1 in [11] implies that \( I^*(a, n) \) and \( \tau(\psi) \) for even \( n \) qubits are invariant under \( SL \) (determinant-one) operators, especially under \( LU \) (local unitary) operators. In order to argue below that \( \tau(\psi) \) for even \( n \) qubits is an entanglement monotone, we need the following result. If the states \( |\psi\rangle \) and \( |\psi\rangle \) are related by a local operator as

\[
|\psi\rangle = \alpha \otimes \beta \otimes \gamma \cdots |\psi\rangle,
\]

then

\[
I^*(a', n) = I^*(a, n) \det(\alpha) \det(\beta) \det(\gamma)\ldots
\]

and

\[
\tau(\psi') = \tau(\psi) |\det(\alpha) \det(\beta) \det(\gamma)\ldots|.
\]

It is easy to see that Eq. (2.5) follows Eqs. (2.1) and (2.4). The proof of Eq. (2.4) is found in part A of Appendix D in [15] in which the condition that \( \alpha, \beta, \ldots \) are invertible was not used. Following this result, we have the following two results. (1). That the states \( |\psi\rangle \) and \( |\psi\rangle \) are connected by SLOCC, i.e., \( \alpha, \beta, \ldots \) are invertible, becomes a special case of Eq. (2.3). Hence Eq. (2.5) holds. (2). Eq. (2.5) is true even if the states \( |\psi\rangle \) and \( |\psi\rangle \) are connected by general LOCC, i.e., by non-invertible operators (see [9]).

2.1 Invariance under permutations of the \( n \) qubits

For a state of even \( n \) qubits, \( |\psi\rangle \), we show in this section the invariance of \( \tau(\psi) \) under permutations of the qubits. To this end, we first prove following propositions.

Remark 2.1 Each term of \( I^*(a, n) \) in Eq. (2.4) takes the form \((-1)^{N(k)}a_ka_{2^n-1-k}\).

Proof It is easy to see that binary representations of \( k \) and \( 2^n-1-k \) are complementary. So, \( N(k) + N(2^n-1-k) = n \). Hence, \((-1)^{N(k)} = (-1)^{N(2^n-1-k)}\). By the definition for \( sgn^* \) in Appendix A, \( sgn^*(n, i) = (-1)^{N(i)} \) when \( 0 \leq i \leq 2^{n-3}-1 \) and \( sgn^*(n, i) = (-1)^{n+N(i)} \) when \( 2^{n-3} \leq i \leq 2^{n-2}-1 \). Therefore, \( sgn^*(n, i) = (-1)^{N(i)} \) when \( n \) is even and \( 0 \leq i \leq 2^{n-2}-1 \). Next there are two cases.
1. Consider term $\text{sgn}^*(n, i)a_{2i}a_{(2^n-1)-2i}$. Since $N(2i) = N(i)$, this remark is true for case 1.

2. Consider term $-\text{sgn}^*(n, i)a_{2i+1}a_{(2^n-2)-2i}$. Since $N(2i+1) = N(i) + 1$, this remark is true for case 2.

**Lemma 2.2** The term $I^*(a, n)$ in Eq. (2.2) does not vary under any permutation of the $n$ qubits.

**Proof** By remark 2.1, each term of $I^*(a, n)$ is of the form $(-1)^{N(k)}a_k a_{2^n-1-k}$. Let the binary number for $k$ correspond to the binary number for $k'$ under permutation $\pi$ of the qubits. Then, the binary number for $2^n - 1 - k$ corresponds to the binary number for $2^n - 1 - k'$ under $\pi$. That is, $\pi(2^n - 1 - k) = 2^n - 1 - k'$. Obviously, $a_k = a_{k'}$, $a_{2^n-1-k} = a_{2^n-1-k'}$, and $N(k) = N(k')$. Thus, $(-1)^{N(k')}a_{k'} a_{2^n-1-k'}$. Therefore, $I^*(a, n)$ does not vary under any permutation of the qubits.

From lemma 2.2 and Eq. (2.1), we have the following corollary 1.

**Corollary 2.3** The residual entanglement $\tau(\psi)$ does not vary under any permutation of the $n$ qubits.

### 2.2 Product states

For product states, the residual entanglement $\tau(\psi)$ either vanishes or is multiplicative. In this section, we state an important theorem and refer the reader to the Appendix B for a detailed proof.

**Theorem 2.4** Let $|\psi\rangle$ be a state of even $n$ qubits which can be expressed as a tensor product state of state $|\phi\rangle$ of the first $l$ qubits and state $|\omega\rangle$ of the rest $(n-l)$ qubits. Let $|\phi\rangle = \sum_{i=0}^{2^l-1} b_i |i\rangle$, where $1 \leq l < n$, and $|\omega\rangle = \sum_{i=0}^{2^{n-l}-1} c_i |i\rangle$. Then, $\tau(\psi) = \tau(\phi)\tau(\omega)$ for even $l$ while $\tau(\psi) = 0$ for odd $l$.

**Proof** See Appendix B for a detailed proof.

It is instructive to look at several examples to see the usefulness of this theorem. In example 1, we show a four-qubit state in which $\tau(\psi) = 1$ and in example 2, we look at a case of a six-qubit state in which $\tau(\psi) = 0$.

**Example** For four qubits, $\tau((1/2)((|00\rangle + |11\rangle)_{12} \otimes (|00\rangle + |11\rangle)_{34})) = 1$. 
Example For six qubits, $\tau((1/2)((|00\rangle + |11\rangle)_{123} \otimes (|00\rangle + |11\rangle)_{456})) = 0$.

It is possible to extend theorem 1 further. From theorem 1 and corollary 1, we have the following corollary 2.5:

**Corollary 2.5** *(An extension of theorem 1):* (1). If $|\psi\rangle$ is a tensor product state of state $|\phi\rangle$ of even qubits and state $|\omega\rangle$ of even qubits, then $\tau(|\psi\rangle) = \tau(|\phi\rangle) \tau(|\omega\rangle)$. That is, $\tau(|\psi\rangle)$ is multiplicative. (2). If $|\psi\rangle$ is a tensor product state of state $|\phi\rangle$ of odd qubits and state $|\omega\rangle$ of odd qubits, then $\tau(|\psi\rangle) = 0$.

The corollary 2.5 argues that $\tau(|\psi\rangle)$ for even $n$ qubits is not a measure of $n$-way entanglement. Note that the conjecture for even $n$ qubits in [11] is the same as Corollary 2. At this juncture, it is probably interesting to note some examples for six-qubit states.

Example For six qubits, $\tau((1/2)((|00\rangle + |11\rangle)_{1456} \otimes (|00\rangle + |11\rangle)_{23})) = 1$.

Example For six qubits, $\tau((1/2)((|00\rangle + |11\rangle)_{135} \otimes (|00\rangle + |11\rangle)_{246})) = 0$.

In [9] SLOCC classes of three qubits are related by means of non-invertible operators, i.e., of general LOCC, see Fig.1 in [9]. Unfortunately, we can not derive a nice result for four qubits. For example, for four qubits, no non-invertible operators can transform the state $|GHZ\rangle$ to a state within $|GHZ\rangle_{12} \otimes |GHZ\rangle_{34}$ SLOCC class. Assume that the states $|\phi\rangle$ and $|GHZ\rangle$ are connected by a non-invertible operator as $|\phi\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta |GHZ\rangle$. Then by Eq. (2.5), $\tau(|\phi\rangle) = \tau(|GHZ\rangle) |\det(\alpha) \det(\beta) \det(\gamma) \det(\delta)| = 0$. However, for any state $|\phi\rangle$ in $|GHZ\rangle_{12} \otimes |GHZ\rangle_{34}$ SLOCC class, $\tau(|\phi\rangle) \neq 0$ by Eq. (2.5) and Example 1.

### 2.3 Entanglement monotone

As indicated in [8], a natural measure of entanglement should also be an entanglement monotone. Let us follow the idea in [8] to prove that $\tau(|\psi\rangle)$ for $n$ qubits is an entanglement monotone. Based on the work in [9], it is enough to consider two-outcome POVM’s and apply POVM’s to one party. For example, we can simply apply a local POVM to qubit $k$. Let $A_1$ and $A_2$ be the two POVM elements such that $A_1^+ A_1 + A_2^+ A_2 = I$. By the singular value decomposition, there are unitary matrices $U_i$ and $V_i$ and diagonal matrices $D_i$ with...
non-negative entries such that \( A_i = U_i D_i V_i \) \[^2\], where \( D_1 = \text{diag}(a, b) \) and \( D_2 = \text{diag}((1-a^2)^{1/2}, (1-b^2)^{1/2}) \) \[^9\]. Let \(|\psi\rangle\) be an initial state and

\[
|\tilde{\phi}_i\rangle = I \otimes \ldots \otimes I \otimes A_i \otimes I \otimes \ldots \otimes I |\psi\rangle
\]

be the states after the application of the POVM for any \( n \) qubits, where \( I \) is an identity. To normalize \(|\tilde{\phi}_i\rangle\), let

\[
|\phi_i\rangle = |\tilde{\phi}_i\rangle / \sqrt{p_i}, \quad \text{where} \quad p_i = \langle \tilde{\phi}_i | \tilde{\phi}_i \rangle. \]

Clearly \( p_1 + p_2 = 1 \) \[^9\]. As discussed in \[^9\], next we can consider

\[
\langle \tau^n \rangle = p_1 \tau^n(\phi_1) + p_2 \tau^n(\phi_2), \quad \text{where} \quad 0 < \eta \leq 1,
\]

and prove

\[
\langle \tau^n \rangle \leq \tau^n(\psi)
\]

to show that \( \tau \) is an entanglement monotone.

It is intuitive that \( \tau(\phi_i) = \tau(\tilde{\phi}_i)/p_i \) because \( \tau \) is a quadratic function with respect to its coefficients in the standard basis, see Eq. \((2.2)\). Note that \( \tau \) is a quartic function in \[^9\] \[^10\]. By Eqs. \((2.5)\) and \((2.6)\),

\[
\tau(\tilde{\phi}_i) = \tau(\psi) |\det(A_i)| = \tau(\psi) |\det(D_i)|.
\]

So, it is trivial to get \( \tau(\tilde{\phi}_1) = ab\tau(\psi) \) and \( \tau(\tilde{\phi}_2) = [(1-a^2)(1-b^2)]^{1/2}\tau(\psi) \). By substituting \( \tau(\tilde{\phi}_1) \) and \( \tau(\tilde{\phi}_2) \) into Eq. \((2.7)\), we get

\[
\langle \tau^n \rangle = \{p_1 \left( \frac{ab}{p_1} \right)^{\eta} + p_2 \left( \frac{(1-a^2)(1-b^2)^{1/2}}{p_2} \right)^{\eta/2} \} \tau^n(\psi).
\]

When \( \eta = 1 \),

\[
\langle \tau \rangle = \{ab + [(1-a^2)(1-b^2)]^{1/2} \} \tau(\psi).
\]

As discussed in \[^9\], it is easy to derive \( \langle \tau \rangle \leq \tau(\psi) \). Thus, this means when \( \eta = 1 \), \( \tau \) is an entanglement monotone. Finally, as pointed out in \[^9\], when \( 0 < \eta \leq 1 \), it is easy to show that \( \tau \) is an entanglement monotone.

It is worthwhile pointing out that in \[^9\] the authors simplified the calculation for \( \tau(\tilde{\phi}_i) \) \[^9\] \[^10\] by using the restriction \( V_1 = V_2 \) since they apparently thought the fact that \( A_1 \) and \( A_2 \) constitute a POVM implies \( V_1 = V_2 \). The authors in \[^9\] and \[^10\] used the invariance of the 3-tangle in \[^9\] and the even \( n \)-tangle in \[^10\]
under permutations of the qubits, respectively, to consider a local POVM in party A only. Moreover, they also used the invariance of the 3-tangle and the even n-tangle under \( LU \) respectively to obtain \( \tau(U,D,V\psi) = \tau(D,V\psi) \) in [9][10].

### 3 Residual entanglement for odd \( n \) qubits and the odd \( n \)-tangle

In this section, we propose the residual entanglement with respect to each qubit. We consider \( \tau(\psi) \) for odd \( n \) qubits in [11] as the residual entanglement with respect to qubit 1. Let \((1,i)\) be the transposition of qubits 1 and \( i \), and \((1,i)|\psi\rangle\) be the state obtained from \(|\psi\rangle\) under the transposition \((1,i)\). Let \( \tau^{(i)}(\psi) = \tau((1,i)\psi) \), \( i = 2, 3, ..., n \) and \( \tau^{(1)}(\psi) = \tau(\psi) \). Then, we propose \( \tau^{(i)}(\psi) \) as the residual entanglement with respect to qubit \( i \), where \( i = 1, ..., n \). It seems that the residual entanglement \( \tau^{(1)}(\psi) \) with respect to qubit 1 is transferred to qubit \( i \) under the transposition \((1,i)\). By averaging the residual entanglement with respect to each qubit, we propose the following \( R(\psi) \) as the odd \( n \)-tangle.

\[
R(\psi) = \frac{1}{n} \sum_{i=1}^{n} \tau^{(i)}(\psi).
\] (3.1)

First, we study the properties of \( \tau(\psi) \). Then, by means of the properties of \( \tau(\psi) \), we investigate the residual entanglement \( \tau^{(i)}(\psi) \) with respect to qubit \( i \) and the odd \( n \)-tangle \( R(\psi) \).

In [11], we defined the entanglement measure for the state \(|\psi\rangle\) of odd \( n \) qubits as

\[
\tau(\psi) = 4|\overline{I}(a,n)|^2 - 4I^{*}(a,n-1)I^{*}_{+2n-1}(a,n-1)|,
\] (3.2)

where

\[
\overline{I}(a,n) = \sum_{i=0}^{2^{n-3}-1} \text{sgn}(n,i) [(a_{2i}a_{(2^n-1)}-2i) - a_{2i+1}a_{(2^n-2)-2i}]
\]

\[
- (a_{(2^n-1)}-2i) - a_{(2^n-1)-2i}a_{2^n-1+2i}],
\] (3.3)
\[ \mathcal{I}_{+2^{n}-1}^{*}(a, n-1) = \sum_{i=0}^{2^{n}-3-1} \text{sgn}^{*}(n-1, i) \times (a_{2^{n}-1+2i}(2^{n}-1)-2i - a_{2^{n}-1+1+2i}(2^{n}-2)-2i), \]  

(3.4)

\[ \mathcal{I}^{*}(a, n-1) = \sum_{i=0}^{2^{n}-3-1} \text{sgn}^{*}(n-1, i) \times (a_{2i}(2^{n}-1)-2i - a_{2i+1}(2^{n}-2)-2i). \]  

(3.5)

For \( n = 3 \), \( \tau(\psi) \) in Eq. (3.2) is just simply the residual entanglement for three qubits [9], i.e., 3 tangle, which is \( \tau_{ABC} = 4|d_{1} - 2d_{2} + 4d_{3}| \), where the expressions for \( d_{i} \) are omitted here.

Theorem 2 in [11] implies that \((\mathcal{I}^{*}(a, n-1))^{2} - 4\mathcal{I}^{*}(a, n-1)\mathcal{I}_{+2^{n}-1}^{*}(a, n-1)\) and \( \tau(\psi) \) are invariant under \( SL \)-operators, especially under \( LU \)-operators. We argue below that the entanglement measure \( \tau \) for odd \( n \) qubits is an entanglement monotone, using the following result.

If the states \( |\psi'\rangle \) and \( |\psi\rangle \) are connected by a local operator as

\[ |\psi'\rangle = \alpha \otimes \beta \otimes \gamma \cdots |\psi\rangle, \]  

(3.6)

then

\[ (\mathcal{IV}(a', n))^{2} - 4\mathcal{IV}^{*}(a', n-1)\mathcal{IV}_{+2^{n}-1}^{*}(a', n-1) = \]  

\[ [(\mathcal{IV}(a, n))^{2} - 4\mathcal{IV}^{*}(a, n-1)\mathcal{IV}_{+2^{n}-1}^{*}(a, n-1)] \times \]  

\( (\det(\alpha) \det(\beta) \det(\gamma)\cdots)^{2} \),  

(3.7)

and

\[ \tau(\psi') = \tau(\psi) |\det(\alpha) \det(\beta) \det(\gamma)\cdots|^{2}. \]  

(3.8)

It is easy to know that Eq. (3.8) follows Eqs. (3.2) and (3.7). For the proof of Eq. (3.7), see the proof in part B of Appendix D in [15] in which the condition that \( \alpha, \beta, \ldots \) are invertible was not used. Following
this result, for odd \( n \) qubits we also have the following two results. (1). That the states \(|\psi'\rangle\) and \(|\psi\rangle\) are connected by SLOCC, i.e., \( \alpha, \beta, \ldots \) are invertible, becomes a special case of Eq. (3.6). Hence Eq. (3.8) holds.

(2). Eq. (3.8) is true even if the states \(|\psi'\rangle\) and \(|\psi\rangle\) are connected by general LOCC, i.e., by non-invertible operators (see [9]).

3.1 Invariance of \( \tau(\psi) \) under any permutation of the qubits 2, 3, ..., \( n \).

The residual entanglement \( \tau(\psi) \) is invariant under permutation of these qubits. To prove the invariance, we prove the following remark 3.1 and lemma 3.2, and corollary 3.3 stated below.

**Remark 3.1** Let \(|\psi\rangle\) be a state of odd \( n \) qubits. Then each term of \( \overline{I}(a,n) \) in Eq. (3.3) is of the form

\[
(-1)^{N^*(k)} a_k a_{2^n-1-k}.
\]

**Proof** Since the binary representations of \( k \) and \( 2^n - 1 - k \) are complementary, \( N(k) + N(2^n - 1 - k) = n \) and \( N^*(k) + N^*(2^n - 1 - k) = n - 1 \). Hence, \( (-1)^{N^*(k)} = (-1)^{N^*(2^n-1-k)} \). Note that \( sgn(n,i) = (-1)^{N(i)} \) when \( 0 \leq i \leq 2^{n-3} - 1 \) by the definition for \( sgn^* \) in Appendix A. Next there are four cases.

1. Term \( sgn(n,i)a_{2i}a_{(2^n-1)-2i} \). Since \( 0 \leq 2i \leq 2^{n-2} - 2 \), \( N^*(2i) = N(2i) = N(i) \).

2. Term \(-sgn(n,i)a_{2i+1}a_{(2^n-2)-2i} \). Since \( 1 \leq 2i + 1 \leq 2^{n-2} - 1 \), \( N^*(2i + 1) = N(2i + 1) = N(i) + 1 \).

3. Term \(-sgn(n,i)a_{(2^n-1-2)}a_{(2^n-1)+2i} \). Clearly, \( N^*(2^n-1+1+2i) = N^*(1+2i) = N(i) + 1 \).

4. Term \( sgn(n,i)a_{(2^n-1-1)-2}a_{2^n-1+2i} \). It is trivial that \( N^*(2^n-1+2i) = N(2i) = N(i) \).

Since the above four cases exhaust all possibilities, the remark holds.

**Lemma 3.2** Let \(|\psi\rangle\) be a state of odd \( n \) qubits. Then, \( \overline{I}(a,n) \) in Eq. (3.3) does not vary under any permutation of the qubits 2, 3, ..., \( n \).
Proof By remark 3.1, each term of $I(a,n)$ in Eq. (3.3) is of the form $(-1)^{N^*(k)}a_k a_{2^n - 1 - k}$. Let the binary number for $k$ correspond to the binary number for $k'$ under permutation $\pi$ of the qubits 2, 3, ..., and $n$. Then, the binary number for $2^n - 1 - k$ corresponds to the binary number for $2^n - 1 - k'$ under $\pi$. That is, $\pi(2^n - 1 - k) = 2^n - 1 - k'$. Obviously, $a_k = a_{k'}$, $a_{2^n - 1 - k} = a_{2^n - 1 - k'}$, and $N^*(k) = N^*(k')$. Thus, $(-1)^{N^*(k)}a_k a_{2^n - 1 - k} = (-1)^{N^*(k')}a_{k'} a_{2^n - 1 - k'}$. Therefore, $I(a,n)$ does not vary under any permutation of the qubits 2, 3, ..., and $n$.

Finally, we have the following corollary concerning the invariance of the entanglement measure $\tau(\psi)$ under any permutations of the qubits 2, 3, ..., and $n$.

**Corollary 3.3** Let $|\psi\rangle$ be a state of odd $n$ qubits. Then, $\tau(\psi)$ does not vary under any permutation of the qubits 2, 3, ..., and $n$.

Proof Note that a binary representation of each subscript in each term of $I^*(a,n-1)$ in Eq. (3.3) is of the form $0k_{n-2}...k_1 k_0$ and a binary representation of each subscript in each term of $I^*_{+2n-1}(a,n-1)$ in Eq. (3.4) is of the form $1k_{n-2}...k_1 k_0$. Under any permutation of the qubits 2, 3, ..., and $n$, by lemma 2.2 either $I^*(a,n-1)$ or $I^*_{+2n-1}(a,n-1)$ does not vary and by lemma 3.2 $I(a,n)$ does not vary. Hence, by the definition in Eq. (3.2), $\tau(\psi)$ does not vary under any permutation of the qubits 2, 3, ..., and $n$.

To see the usefulness of the results that we have shown, it is instructive to study an example:

**Example** Let $|\psi\rangle = (1/2)(|00\rangle + |11\rangle)_{12} \otimes (|00\rangle + |11\rangle)_{1345}$. Then, by Eq. (3.2), a simple calculation shows that $\tau(\psi) = 0$. Under the transposition (1, 5) of the qubits 1 and 5, $|\psi\rangle$ becomes $|\psi'\rangle = (1/2)(|00\rangle + |11\rangle)_{25} \otimes (|00\rangle + |11\rangle)_{134}$. By Eq. (3.2), $\tau(\psi') = 1$.

3.2 Product states

For product states, $\tau(\psi)$ vanishes or is multiplicative. To prove this statement, we have the following theorem:
**Theorem 3.4** Let $|\psi\rangle$ be a state of odd $n$ qubits and a tensor product state of the state $|\phi\rangle$ of the first $l$ qubits and the state $|\omega\rangle$ of the rest $(n-l)$ qubits. Let $|\phi\rangle = \sum_{i=0}^{2^l-1} b_i |i\rangle$, where $1 \leq l < n$, and $|\omega\rangle = \sum_{i=0}^{2^{n-l}-1} c_i |i\rangle$. Then, $\tau(\psi) = \tau(\phi)\tau^2(\omega)$ for odd $l$, while $\tau(\psi) = 0$ for even $l$.

**Proof** See Appendix C for the detailed proof.

It is interesting to study some examples to see some application of the theorem.

**Example** For five qubits, $\tau((1/2)((|000\rangle + |111\rangle)_{123} \otimes (|00\rangle + |11\rangle)_{45})) = 1$.

**Example** For five qubits, $\tau((1/2)((|00\rangle + |11\rangle)_{12} \otimes (|000\rangle + |111\rangle)_{345})) = 0$.

Moreover, from theorem 3.4 and corollary 3.3, we have the following corollary as an extension of theorem 3.4.

**Corollary 3.5** Theorem 3.4 holds under any permutation $\pi$ of the qubits 2, 3, ..., and $n$. That is, let $|\phi\rangle$ be a state of $l$ qubits including qubit 1 and state $|\omega\rangle$ be a state of the rest $(n-l)$ qubits, then, $\tau(\psi) = \tau(\phi)\tau^2(\omega)$ for odd $l$ while $\tau(\psi) = 0$ for even $l$. Hence, $\tau(\psi)$ can be considered to be multiplicative for odd $l$.

The corollary 3.5 implies that for odd $n$ qubits, $\tau(\psi)$ is not a measure of $n$-way entanglement. In [11], we conjectured that $\tau(\psi) = 0$ whenever $\psi$ is a product states of the odd $n$ qubits. This corollary indicates that the conjecture is not always true.

**Example** For five qubits, $\tau((1/2)((|000\rangle + |111\rangle)_{125} \otimes (|00\rangle + |11\rangle)_{34})) = 1$ and $\tau((1/2)((|00\rangle + |11\rangle)_{15} \otimes (|000\rangle + |111\rangle)_{234})) = 0$.

For five qubits, by resorting to the iterative formula about the number of the degenerate SLOCC classes in [16], there are $5 \times t(4) + 66$ degenerate SLOCC classes, where $t(4)$ is the number of true SLOCC entanglement classes for four qubits. In [16], 28 true SLOCC classes for four qubits were found. Hence, in total, there are at least 206 degenerate SLOCC classes for five qubits. Note that degenerate SLOCC classes are SLOCC classes of product states.
By corollary 3.5, for five qubits, $\tau$ always vanishes for all the product states except for the states within the following SLOCC classes:

- $|(000) + (111)\rangle_{123} \otimes (|000) + (111)\rangle_{45}$,
- $|(000) + (111)\rangle_{125} \otimes (|000) + (111)\rangle_{34}$,
- $|(000) + (111)\rangle_{134} \otimes (|000) + (111)\rangle_{25}$,
- $|(000) + (111)\rangle_{135} \otimes (|000) + (111)\rangle_{24}$,
- $|(000) + (111)\rangle_{145} \otimes (|000) + (111)\rangle_{23}$.

As discussed in [9], SLOCC classes of three qubits are related by means of non-invertible operators, i.e., of general LOCC, see Fig.1 in [9]. Here, we want to show that it is not true for five qubits. For example, no non-invertible operators can transform the state $|\text{GHZ}\rangle$ to a state within $|\text{GHZ}\rangle_{123} \otimes |\text{GHZ}\rangle_{45}$ SLOCC class.

Assume that the states $|\phi\rangle$ and $|\text{GHZ}\rangle$ are connected by a non-invertible operator as $|\phi\rangle = \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \eta |\text{GHZ}\rangle$. Then by Eq. (3.8),

$$\tau(\phi) = \tau(\psi) |\det(A_i)|^2 = \tau(\psi) |\det(D_i)|^2.$$  

(3.9)

So, $\tau(\phi_1) = (ab)^2 \tau(\psi)$ and $\tau(\phi_2) = (1 - a^2)(1 - b^2) \tau(\psi)$. By substituting $\tau(\phi_1)$ and $\tau(\phi_2)$ into Eq. (2.7), we get

$$\langle \tau^\eta \rangle = \left\{p_1 \frac{(ab)^{2\eta}}{p_{1\eta}} + p_2 \frac{(1 - a^2)(1 - b^2)^\eta}{p_{2\eta}}\right\} \tau(\psi).$$  

(3.10)

Eq. (3.10) was also obtained in [9]. Therefore the rest of the proof is the same as the one in [9].

Note that in the above proof, we do not use the restriction $V_1 = V_2$, the invariance of $\tau$ under permutations of the qubits, or the invariance of $\tau$ under $LU$. Therefore, it is not necessary to establish a relation between the invariance of a measure under permutations of the qubits and an entanglement monotone.
3.4 The residual entanglement with respect to each qubit and the odd $n$-tangle

3.4.1 The residual entanglement $\tau^{(i)}(\psi)$ with respect to qubit $i$

It is plain to derive that $\tau^{(i)}(\psi)$ satisfy Eq. (3.8). From the properties of $\tau(\psi)$, one can obtain that (1). $0 \leq \tau^{(i)}(\psi) \leq 1$; (2). $\tau^{(i)}(\psi)$ are $SL$-invariant, especially $LU$-invariant; (3). $\tau^{(i)}(\psi)$ are entanglement monotones. (4). $\tau^{(i)}(\psi)$, $i = 1, 2, \ldots, n$, are invariant under permutations of the qubits: $1, \ldots, (i-1)$, $(i+1)$, $\ldots, n$; (5). When $|\psi\rangle$ is a product state of odd $n$ qubits, that is, $|\psi\rangle = |\phi\rangle \otimes |\omega\rangle$, where $|\phi\rangle$ is a state of $l$ qubits including qubit $i$, and $|\omega\rangle$ is a state of $(n-l)$ qubits, then $\tau^{(i)}(\psi) = \tau^{(i)}(\phi)(\tau^{(i)}(\omega))^2$ for odd $l$ while $\tau^{(i)}(\psi) = 0$ for even $l$.

We argue that the above (5) holds as follows. Let $(1, i)$ be a transposition of qubits 1 and $i$, and the state $(1, i)|\psi\rangle$ be obtained from $|\psi\rangle$ under the transposition $(1, i)$. It is not hard to see that $\tau^{(i)}(\psi) = \tau((1, i)|\psi\rangle) = \tau(((1, i)|\phi\rangle \otimes (1, i)|\omega\rangle))$. There are two cases. Case 1. In this case, qubit 1 occurs in $|\phi\rangle$. Under the transposition $(1, i)$, qubits 1 and $i$ occur in $(1, i)|\phi\rangle$, and $(1, i)|\omega\rangle = |\omega\rangle$. Case 2. In this case, qubit 1 occurs in $|\omega\rangle$. Under the transposition $(1, i)$, qubit 1 occurs in $(1, i)|\phi\rangle$ while qubit $i$ occurs in $(1, i)|\omega\rangle$. In either case, by corollary 4, $\tau(((1, i)|\phi\rangle \otimes (1, i)|\omega\rangle)) = \tau((1, i)|\phi\rangle)\tau^2((1, i)|\omega\rangle) = \tau^{(i)}(\phi)(\tau^{(i)}(\omega))^2$ for odd $l$ while $\tau(((1, i)|\phi\rangle \otimes (1, i)|\omega\rangle)) = 0$ for even $l$. For the proofs of (1), (2), (3), and (4), see [17].

3.4.2 The odd $n$-tangle

It is not difficult to show that $R(\psi)$ in Eq. (3.1) satisfies Eq. (3.8). Thus, from the properties of $\tau^{(i)}(\psi)$, one can derive that (1). $0 \leq R \leq 1$; (2). $R$ is invariant under $SL$-operators, especially $LU$-operators; (3). $R$ is an entanglement monotone; (4). $R(\psi)$ is invariant under any permutation of all the odd $n$ qubits. For the proofs of (1), (2), (3), and (4), see [17]. However, $R(\psi)$ is not multiplicative.

Next let us see the performance of $R(\psi)$ for three qubits. Let $n = 3$. As discussed before, $\tau(\psi)$ happens to be Coffman et al.’s residual entanglement for three qubits. From (5) of p. 429 in [18], $\tau(\psi) = \tau^{(1)}(\psi) = \tau^{(2)}(\psi) = \tau^{(3)}(\psi)$. Thus, $R(\psi) = \tau(\psi)$. That is, $R(\psi)$ is just Coffman et al.’s residual entanglement for three qubits.
4 Summary

We summarize this paper as follows. We demonstrate that the entanglement measure $\tau(\psi)$ for even $n$ qubits, the residual entanglement $\tau(i)(\psi)$ with respect to qubit $i$ and the odd $n$-tangle $R(\psi)$ for odd $n$ qubits satisfy the following properties. (1). $\tau(\psi)$, $\tau(i)(\psi)$, and $R(\psi)$ are between 0 and 1; (2). $\tau(\psi)$, $\tau(i)(\psi)$, and $R(\psi)$ are $SL$-invariant, especially $LU$-invariant; (3). $\tau(\psi)$, $\tau(i)(\psi)$, and $R(\psi)$ are entanglement monotones; (4). $\tau(\psi)$ for even $n$ qubits and the odd $n$-tangle $R(\psi)$ are invariant under permutations of all the qubits; however $\tau(i)(\psi)$ are invariant only under permutations of the qubits: $1,..., (i - 1), (i + 1), ..., n$. (5). For product states, i.e., $|\psi\rangle = |\phi\rangle \otimes |\omega\rangle$, for even $n$ qubits, if $|\phi\rangle$ is a state of even qubits then $\tau(\psi) = \tau(\phi)\tau(\omega)$ else $\tau(\psi) = 0$; for odd $n$ qubits, if $|\phi\rangle$ is a state of $l$ qubits including qubit $i$, then $\tau(i)(\psi) = \tau(i)(\phi)(\tau(i)(\omega))^2$ for odd $l$ while $\tau(\psi) = 0$ for even $l$.

Monotonicity is a natural requirement for entanglement measure. The symmetry of entanglement measure under permutations represents a collective property of the qubits. Therefore the entanglement measures presented in this paper are natural.

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Appendix A. Properties of $sgn$ and $sgn^*$

In order to show the invariance of the entanglement measure for even (odd) $n$ qubits, we need the following properties of the function $sgn$ ($sgn^*$). The functions $sgn$ and $sgn^*$ were recursively defined in [11]. For readability, we redefine $sgn$ and $sgn^*$ as follows.

Definition of $sgn$:


g_{n,i} = (-1)^{N(i)} \text{ when } 0 \leq i \leq 2^{n-3} - 1. \tag{A1}

The definition of $N(i)$ is given in the last paragraph of this introduction. Whereas, $N(i)$ is the number
proof of (ii): 

In fact, this definition of $sgn(n, i)$ can be derived from the recursive definition of $sgn(n, i)$ in [11] by using the following property 1 about $N(i)$.

**Definition of $sgn^*$:**

$$sgn^*(n, i) = \begin{cases} 
(-1)^{N(i)} & \text{for } 0 \leq i \leq 2^{n-3} - 1, \\
(-1)^{n+N(i)} & \text{for } 2^{n-3} \leq i \leq 2^{n-2} - 1.
\end{cases}$$

This definition of $sgn^*(n, i)$ can be derived from the recursive definition of $sgn^*$ in [11] by using the following property 1 about $N(i)$.

It is straightforward to derive the following property 1 about $N(i)$ by means of the definition of $N(i)$. The property 1 will be used in the proofs of the following properties 2-5.

**Property 1:**

(i). Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq j \leq 2^{l-2} - 1$. Then $N(k + j \times 2^{n-l-1}) = N(j) + N(k)$.

(ii). Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq t \leq 2^{l-3} - 1$. Then $N(k + 2t^{n-l}) = N(k) + N(t)$ and $N(k + (2t + 1)2^{n-l-1}) = N(k) + N(t) + 1$.

(iii). Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq j \leq 2^{l-2} - 1$. Then, $N(2^{n-l-1} - 1 - k) = n - l - 1 - N(k)$ and $N((j + 1) \times 2^{n-l-1} - 1 - k) = N(j) + n - l - 1 - N(k)$.

**Proof** Proof of (i):

Let the binary number of $j$ be $j_{l-3}j_{l-4}\ldots j_1j_0$, where $j_i \in \{0, 1\}$. That is, $j = j_{l-3} \times 2^{l-3} + \ldots + j_1 \times 2^1 + j_0 \times 2^0$. $j \times 2^{n-l-1} = j_{l-3} \times 2^{n-4} + \ldots + j_1 \times 2^{n-l} + j_0 \times 2^{n-l-1}$. Clearly, $N(j) = N(j \times 2^{n-l-1})$. Since $0 \leq k \leq 2^{n-l-2} - 1$, $N(k + j \times 2^{n-l-1}) = N(j \times 2^{n-l-1}) + N(k) = N(j) + N(k)$.

Proof of (ii):

Let the binary representation of $t$ be $t_{l-4}\ldots t_1t_0$ and the binary number of $k$ be $k_{n-l-3}\ldots k_1k_0$, where $t_i$, $k_i \in \{0, 1\}$, $k + (2t + 1)2^{n-l-1} = k + t2^{n-l} + 2^{n-l-1}$. The latter can be rewritten as $t_{l-4}2^{n-l} + \ldots + t_12^{n-l+1} + t_02^{n-l} + 2^{n-l-1} + k_{n-l-3}2^{n-l-3} + \ldots + k_02^0$. It is obvious that $N(k + t2^{n-l} + 2^{n-l-1}) = N(k) + N(t) + 1$. As well, $N(k + t2^{n-l}) = N(k) + N(t)$.
Proof of (iii):

Let us calculate \( N(2^{n-l-1} - 1 - k) \). The binary number of \( 2^{n-l-1} - 1 \) is \( 1 \cdots 1 \). That is, \( 2^{n-l-1} - 1 = 2^{n-l-2} + \cdots + 2^1 + 2^0 \). Let \( k_{n-l-3} \cdots k_1 k_0 \) be the binary number of \( k \), where \( k_i \in \{0, 1\} \). That is, \( k = k_{n-l-3} \times 2^{n-l-3} + \cdots + k_1 \times 2^1 + k_0 \times 2^0 \). Note that the binary numbers of \( 2^{n-l-1} - 1 - k \) and \( k \) are complementary. Hence, it is straightforward that \( N(2^{n-l-1} - 1 - k) = n - l - 1 - N(k) \).

\[
(j + 1) \times 2^{n-l-1} - 1 - k = j \times 2^{n-l-1} + (2^{n-l-1} - 1 - k).
\]
Notice that \( 2^{n-l-2} \leq 2^{n-l-1} - 1 - k \leq 2^{n-l-1} - 1 \). It is intuitive that \( N(j \times 2^{n-l-1} + (2^{n-l-1} - 1 - k)) = N(j) + N(2^{n-l-1} - 1 - k) \). Therefore,

\[
N((j + 1) \times 2^{n-l-1} - 1 - k) = N(j) + n - l - 1 - N(k).
\]

The following properties 2-5 are used in proofs of Theorems 1 and 2. The property 2 about \( \text{sgn} \) follows the property 1 and the definition for \( \text{sgn} \).

**Property 2:**

Assume that \( 0 \leq k \leq 2^{n-l-2} - 1 \) and \( 0 \leq j \leq 2^{l-2} - 1 \). Then \( \text{sgn}(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{n+l+1} \text{sgn}(n, k + j \times 2^{n-l-1}) \).

**Proof** (1). Compute \( \text{sgn}(n, k + j \times 2^{n-l-1}) \). Since \( k + j \times 2^{n-l-1} < 2^{n-3} - 1 \), by the definition for \( \text{sgn} \),

\[
\text{sgn}(n, k + j \times 2^{n-l-1}) = (-1)^{N(k + j \times 2^{n-l-1})}.
\]
By (i) of property 1, \( N(k + j \times 2^{n-l-1}) = N(k) + N(j) \).
Therefore \( \text{sgn}(n, k + j \times 2^{n-l-1}) = (-1)^{N(j) + N(k)} \).

(2). Compute \( \text{sgn}(n, (j + 1) \times 2^{n-l-1} - 1 - k) \). Since \( (j + 1) \times 2^{n-l-1} - 1 - k \leq 2^{n-3} - 1 \), by the definition for \( \text{sgn} \),

\[
\text{sgn}(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{N((j+1) \times 2^{n-l-1} - 1 - k)}.
\]
By (iii) of property 1,

\[
\text{sgn}(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{N(j) + n - l - 1 - N(k)}.
\]

Conclusively, \( \text{sgn}(n, (j + 1) \times 2^{n-l-1} - 1 - k) = (-1)^{n+l+1} \text{sgn}(n, k + j \times 2^{n-l-1}) \).

The property 3 about \( \text{sgn} \) and \( \text{sgn}^* \) can be shown by means of the property 1 and the definitions for \( \text{sgn} \) and \( \text{sgn}^* \).

**Property 3:**

Assume that \( 0 \leq k \leq 2^{n-l-2} - 1 \) and \( 0 \leq t \leq 2^{l-3} - 1 \). Then

(i) \( \text{sgn}(n, k + (2t + 1) \times 2^{n-l-1}) = -\text{sgn}(n, k + t \times 2^{n-l}) \).
(ii) $sgn^*(n-1, k + (2t + 1) \times 2^{n-l-1}) = -sgn^*(n-1, k + t \times 2^{n-l})$.

**Proof** Proof of (i): Since $k + (2t+1) \times 2^{n-l-1} \leq 2^{n-3} - 2^{n-l-2} - 1$, by the definition for $sgn$, $sgn(n, k + (2t+1) \times 2^{n-l-1}) = (-1)^{N(k+(2t+1)\times 2^{n-l-1})}$. By (ii) of property 1, $sgn(n, k + (2t+1) \times 2^{n-l-1}) = (-1)^{N(k)+N(t)+1}$. Similarly, $sgn(n, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}$.

Proof of (ii):

1. $0 \leq t \leq 2^{l-4} - 1$. Thus, $k + (2t+1) \times 2^{n-l-1} \leq 2^{n-4} - 1$ and $k + t \times 2^{n-l} \leq 2^{n-4} - 1$. By the definition for $sgn^*$ and (ii) of property 1, $sgn^*(n-1, k + (2t+1) \times 2^{n-l-1}) = (-1)^{N(k)+N(t)+1}$ and

$$sgn^*(n-1, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}. \quad (A2)$$

2. $2^{l-4} \leq t \leq 2^{l-3} - 1$. Thus, $2^{n-4} \leq k + t \times 2^{n-l} < 2^{n-3} - 1$ and $2^{n-4} < k + (2t+1) \times 2^{n-l-1} < 2^{n-3} - 1$. By the definition for $sgn^*$ and (ii) of property 1, $sgn^*(n-1, k + (2t+1) \times 2^{n-l-1}) = (-1)^{n-1}(-1)^{N(k)+N(t)+1}$ and

$$sgn^*(n-1, k + t \times 2^{n-l}) = (-1)^{n-1}(-1)^{N(k)+N(t)}. \quad (A3)$$

The property 4 about $sgn^*$ can be obtained from the property 1 and the definition for $sgn^*$.

**Property 4:**

Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq j \leq 2^{l-2} - 1$. Then $sgn^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{n+l+1}sgn^*(n-1, k + j \times 2^{n-l-1})$.

**Proof** 1. $0 \leq j \leq 2^{l-3} - 1$. Since $(j+1) \times 2^{n-l-1} - 1 - k \leq 2^{n-4} - 1$, by the definition for $sgn^*$, $sgn^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{N((j+1)\times 2^{n-l-1}-1-k)}$. By (iii) of property 1, $sgn^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{N(j)+n-l-1-N(k)}$. As well, since $k + j \times 2^{n-l-1} < 2^{n-4} - 2^{n-l-2} - 1$, by the definition for $sgn^*$, $sgn^*(n-1, k + j \times 2^{n-l-1}) = (-1)^{N(k+j\times 2^{n-l-1})}$. By (i) of property 1, $sgn^*(n-1, k + j \times 2^{n-l-1}) = (-1)^{N(k)+N(j)}$. Therefore, the property holds for this case.
2. $2^{l-3} \leq j \leq 2^{l-2} - 1$. Thus, $2^{n-4} + 2^{n-l-2} \leq (j+1) \times 2^{n-l-1} - 1 - k \leq 2^{n-3} - 1$ and $2^{n-4} + 2^{n-l-2} - 1 \leq k + j \times 2^{n-l-1} \leq 2^{n-3} - 1$. By the definition for $sgn^*$ and (iii) of property 1, $sgn^*(n-1, (j+1) \times 2^{n-l-1} - 1 - k) = (-1)^{n-1}(-1)^{N(j)+n-l-1-N(k)}$. By the definition for $sgn^*$ and (i) of property 1, $sgn^*(n-1, k + j \times 2^{n-l-1}) = (-1)^{n-1}(-1)^{N(k)+N(j)}$. Therefore, this property holds for this case.

It is not hard to derive the property 5 by means of the property 1 and the definitions for $sgn$ and $sgn^*$.

Property 5:

Assume that $0 \leq k \leq 2^{n-l-2} - 1$ and $0 \leq t \leq 2^{l-3} - 1$. When $n$ is odd and $l$ is odd or $n$ is even and $l$ is even, then the following statements are true:

(i). $sgn^*(n-l, k) = (-1)^{N(k)}$,

(ii). $sgn(n, k + t \times 2^{n-l}) = sgn^*(n-l, k)sgn(l, t),$

(iii). $sgn^*(n-1, k + t \times 2^{n-l}) = sgn^*(n-l, k)sgn^*(l-1, t)$.

Proof Proof of (i):

1. $0 \leq k \leq 2^{n-l-3} - 1$. By the definition for $sgn^*$, $sgn^*(n-l, k) = (-1)^{N(k)}$.

2. $2^{n-l-3} \leq k \leq 2^{n-l-2} - 1$. By the definition for $sgn^*$, $sgn^*(n-l, k) = (-1)^{n-l}(-1)^{N(k)}$. When $n$ is odd and $l$ is odd or $n$ is even and $l$ is even, clearly $sgn^*(n-l, k) = (-1)^{N(k)}$.

From cases 1 and 2, this statement follows.

Proof of (ii):

Step 1. Compute $sgn(n, k + t \times 2^{n-l})$. Since $0 \leq k + t \times 2^{n-l} \leq 2^{n-3} - 2^{n-l} + 2^{n-l-2} - 1$, by the definition for $sgn$ and (ii) of property 1, $sgn(n, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}$.

Step 2. Compute $sgn(l, t)$. By the definition for $sgn$, $sgn(l, t) = (-1)^{N(t)}$. From (i) of this property and steps 1 and 2, we can conclude that (ii) holds.

Proof of (iii):

1. $0 \leq t \leq 2^{l-4} - 1$. By the definition for $sgn^*$, $sgn^*(l-1, t) = (-1)^{N(t)}$. By Eq. (A2), $sgn^*(n-1, k + t \times 2^{n-l}) = (-1)^{N(k)+N(t)}$. Therefore, by (i) of this property, (iii) is true for this case.
2. \(2^{l-4} \leq t \leq 2^{l-3} - 1\). By the definition for \(sgn^*\), \(sgn^*(l-1, t) = (-1)^{l-1}(-1)^{N(t)}\). By Eq. (A3),
\[
sgn^*(n-1, k + t \times 2^{n-l}) = (-1)^{n-1}(-1)^{N(k)+N(t)}.
\]
By (i) of this property, it is not hard to see that (iii) holds for this case.

**Appendix B. The proof of Theorem 1**

We show this theorem in three cases: case 1, \(l = 1\); case 2, \(l = 2\); case 3, \(l \geq 3\).

**Proof of \(l = 1\):**

**Proof** When \(l = 1\), \(|\phi\rangle = b_0|0\rangle + b_1|1\rangle\). By solving \(|\psi\rangle = |\phi\rangle_1 \otimes |\omega\rangle_{2,...,n}\), we obtain the following amplitudes:

\[
a_i = b_0c_i, \quad a_{2^n-1+i} = b_1c_i, \quad 0 \leq i \leq 2^{n-1} - 1.
\]

By substituting the amplitudes in Eq. (B1) into \(I^*(a, n)\) in Eq. (2.2),
\[
I^*(a, n) = b_0b_1 \sum_{i=0}^{2^{n-2}-1} sgn^*(n, i)(c_2i(c_{2^{n-1}-1}-2i) - c_{2i+1}c(c_{2^n-2}-2i))
\]
\[
= b_0b_1 \sum_{i=0}^{2^{n-2}-1} sgn^*(n, i)(c_2i(c_{2^{n-1}-1}-2i) - c_{2i+1}c(c_{2^n-2}-2i)) +
\]
\[
b_0b_1 \sum_{i=2^n-3}^{2^{n-2}-1} sgn^*(n, i)(c_2i(c_{2^{n-1}-1}-2i) - c_{2i+1}c(c_{2^n-2}-2i)).
\]

Let \(k = 2^{n-2} - 1 - i\). Then the last sum can be rewritten as
\[
- b_0b_1 \sum_{k=2^n-3}^{2^{n-2}-1} sgn^*(n, 2^{n-2} - 1 - k)(c_{2k}(c_{2^n-1}-2k) - c_{2k+1}c(2^n-2)-2k).
\]

It is easy to demonstrate \(sgn^*(n, 2^{n-2} - 1 - k) = sgn^*(n, k)\) by the definition of \(sgn^*\). Thus, \(I^*(a, n) = 0\) and \(\tau(\psi) = 0\).

**Proof of \(l = 2\):**

**Proof** In this case, \(|\phi\rangle\) is a state of the first two qubits and \(|\phi\rangle = \sum_{i=0}^{3} b_i|i\rangle\), \(|\omega\rangle\) is a state of the last \((n-2)\)-qubits and \(|\omega\rangle = \sum_{i=0}^{2^{n-2}-1} c_i|i\rangle\). By the definition (11), \(\tau(\phi) = 2|b_0b_3 - b_1b_2|\), \(\tau(\omega) = 2|I^*(c, n-2)|\), and \(\tau(\psi) = 2|I^*(a, n)|\). We can write
\[
|\psi\rangle = |\phi\rangle_{1,2} \otimes |\omega\rangle_{3,...,n}\]

By solving Eq. (B2), we obtain the following amplitudes:
\[ a_j = b_0c_j, \quad a_{2^n-j} = b_1c_j, \quad a_{2^{n-1}-j} = b_2c_j, \quad a_{3 \times 2^{n-2}+j} = b_3c_j \] (B3)

, where \( 0 \leq j \leq 2^{n-2} - 1 \).

We rewrite \( I^*(a, n) = E_1 + E_2 \), where

\[ E_1 = \sum_{i=0}^{2^{n-3}-1} sgn^*(n, i)(a_{2i}a_{(2^n-1)-2i} - a_{2i+1}a_{(2^n-2)-2i}) \] (B4)

and

\[ E_2 = \sum_{i=2^{n-3}}^{2^{n-2}-1} sgn^*(n, i)(a_{2i}a_{(2^n-1)-2i} - a_{2i+1}a_{(2^n-2)-2i}) \] (B5)

Let us compute \( E_1 \) as follows. Since \( 0 \leq i \leq 2^{n-3} - 1 \), by Eq. (B3)

\[ a_{2i} = b_0c_{2i}, \quad a_{(2^n-1)-2i} = b_3c_{(2^n-2)-2i}, \]

\[ a_{2i+1} = b_0c_{2i+1}, \quad a_{(2^n-2)-2i} = b_3c_{(2^n-2)-2i}. \] (B6)

By substituting the amplitudes in Eq. (B6) into \( E_1 \), \( E_1 \) becomes

\[ E_1 = b_0b_3 \sum_{i=0}^{2^{n-3}-1} sgn^*(n, i)(c_{2i}c_{(2^n-2)-2i} - c_{2i+1}c_{(2^n-2)-2i}) \] (B7)

In Eq. (B7) let \( E_1 = E_1^{(1)} + E_1^{(2)} \), where

\[ E_1^{(1)} = b_0b_3 \sum_{i=0}^{2^{n-4}-1} sgn^*(n, i)(c_{2i}c_{(2^n-2)-2i} - c_{2i+1}c_{(2^n-2)-2i}) \] (B8)

and

\[ E_1^{(2)} = b_0b_3 \sum_{i=2^{n-4}}^{2^{n-3}-1} sgn^*(n, i)(c_{2i}c_{(2^n-2)-2i} - c_{2i+1}c_{(2^n-2)-2i}). \] (B9)

Let us demonstrate \( E_1^{(2)} = E_1^{(1)} \). Let \( k = (2^{n-3} - 1) - i \). Then

\[ E_1^{(2)} = -b_0b_3 \sum_{k=2^{n-4}-1}^{0} sgn^*(n, 2^{n-3} - 1 - k)(c_{2k}c_{(2^n-2)-2k} - c_{2k+1}c_{(2^n-2)-2k}). \] (B10)

When \( 0 \leq k \leq 2^{n-4} - 1 \), by the definition for \( sgn^* \) and (iii) of property 1 in Appendix A, then \( sgn^*(n, 2^{n-3} - 1 - k) = -sgn^*(n, k) \). Thus, \( E_1^{(2)} = E_1^{(1)} \) and \( E_1 = 2E_1^{(1)} \).
Next we show $E_1 = 2b_0b_3 I^*(c, n - 2)$. For this purpose, we only need to show $\text{sgn}^*(n, i) = \text{sgn}^*(n - 2, i)$ provided that $0 \leq i \leq 2^{n-4} - 1$. The definition for $\text{sgn}^*$ in Appendix A asserts this.

Similarly, we can derive $E_2 = -2b_1b_2 I^*(c, n - 2)$. Thus, $I^*(a, n) = 2(b_0b_3 - b_1b_2) I^*(c, n - 2)$. Conclusively, $\tau(\psi) = \tau(\phi)\tau(\omega)$.

Proof for $l \geq 3$:

**Proof** We write

$$|\psi\rangle = |\phi\rangle_{1,...,l} \otimes |\omega\rangle_{(l+1),...,n}. \tag{B11}$$

By solving equation Eq. (B11), we obtain the following amplitudes:

$$a_k \times 2^{n-l-1} = b_k c_i, k = 0, 1, ..., (2^l - 1), i = 0, 1, ..., (2^{n-l} - 1). \tag{B12}$$

We rewrite $I^*(a, n)$ as $I^*(a, n) = \sum_{j=0}^{2^{l+1} - 1} \Delta_j$, where

$$\Delta_j = \sum_{i=j \times 2^{n-l-2}}^{(j+1)\times 2^{n-l-2} - 1} \text{sgn}(n, i) [(a_2a_{(2^n-1)-2i} - a_{2i+1}a_{(2^n-2)-2i})

\quad + (a_{(2^n-1)-2i}^2a_{(2^n-1)-1} + 2a_{(2^n-2)-2i}a_{(2^n-1)+2i} - a_{(2^n-2)-2i}a_{(2^n-1)+2i})]. \tag{B13}$$

By substituting the amplitudes in Eq. (B12) into $\Delta_{2j}$ and $\Delta_{2j+1}$, we get

$$\Delta_{2j} = \sum_{k=0}^{2^{n-l-2} - 1} \text{sgn}(n, k + j \times 2^{n-l-1}) \times

\quad (b_jb_{2l-1-j} - b_{2l-1+j}b_{2^{l-1}-1-j}) \times

\quad [(c_{2k}c_{(2^n-1)-2k} - c_{2k+1}c_{(2^n-2)-2k})], \tag{B14}$$

and
\[
\Delta_{2j+1} = - \sum_{k=2^{n-l-2}-1}^{0} \text{sgn}(n, (j+1) \times 2^{n-l-1} - 1 - k) \times 
(b_j b_{2j+1-j} - b_{2j+2j+1-j}) \times 
[(c_{2k} c_{(2^n-j-1)-2k} - c_{2k+1} c_{(2^n-j-2)-2k})]. \tag{B15}
\]

When \( l \) is odd, by property 2 in Appendix A, then \( \Delta_{2j+1} = -\Delta_{2j}, \ j = 0, 1, ..., 2^{l-2} - 1 \). Hence, \( \mathcal{I}^*(a, n) = 0 \). Thus, \( \tau(\psi) = 0 \). When \( l \) is even, by property 2 in Appendix A, then \( \Delta_{2j+1} = \Delta_{2j}, \ j = 0, 1, ..., 2^{l-2} - 1 \). Therefore

\[
\mathcal{I}^*(a, n) = 2 \sum_{j=0}^{2^{l-2} - 1} \Delta_{2j} = 2 \sum_{t=0}^{2^{l-3} - 1} (\Delta_{4t} + \Delta_{4t+2}). \tag{B16}
\]

By (i) of property 3 in Appendix A, from Eq. (B16),

\[
\mathcal{I}^*(a, n) = 2^{2^{l-3} - 1} \sum_{t=0}^{2^{l-3} - 1} \{ [(b_{2t} b_{2t+1} - b_{2t+2} b_{2t+2}) + 
(b_{2t+2} b_{2t+2t+1} b_{2t+2t+2} b_{2t+2t+2})] \times 
(2^{n-l-2}) \times 
\text{sgn}(n, k + t \times 2^{n-l}) \times 
[c_{2k} c_{(2^n-j-1)-2k} - c_{2k+1} c_{(2^n-j-2)-2k}]. \tag{B17}
\]

By (ii) of property 5 in Appendix A, from Eq. (B17), we obtain

\[
\mathcal{I}^*(a, n) = 2 \mathcal{I}^*(b, l) \mathcal{I}^*(c, n-l). \tag{B18}
\]

Therefore, \( \tau(\psi) = \tau(\phi) \tau(\omega) \).

**Appendix C. The proof of theorem 2**

When \( l = 1 \), see [11]. When \( l = 2 \), the proof is omitted. Next let us consider that \( l \geq 3 \).

**Proof** Step 1. Compute \( \mathcal{I}(a, n) \).

We rewrite \( \mathcal{I}(a, n) \) in Eq. (3.3) as \( \mathcal{I}(a, n) = \sum_{j=0}^{2^{l-1}-1} \Omega_j \), where
\begin{equation}
\Omega_j = \sum_{i=j \times 2^{n-l-2}}^{(j+1) \times 2^{n-l-2}-1} \text{sgn}(n, i) \left[ (a_{2i} a_{(2^n-1)-2i} - a_{2i+1} a_{(2^n-2)-2i}) - (a_{(2^n-1)-2i} a_{(2^n-1)+1} + 2i - a_{(2^n-1)-2i} a_{2n-1}) \right].
\end{equation}

(C1)

By substituting the amplitudes in Eq. (B12) into \( \Omega_{2j} \) and \( \Omega_{2j+1} \), we obtain

\begin{align*}
\Omega_{2j} &= \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}(n, k + j \times 2^{n-l-1}) \times \\
& \quad (b_j b_{2^l-1-j} + b_{2^l-1-j} b_{2^l-1-j}) \times \\
& \quad \left[ (c_{2k} c_{(2^n-l-1)-2k} - c_{2k+1} c_{(2^n-l-2)-2k}) \right], \\
\Omega_{2j+1} &= -\sum_{k=2^{n-l-2}-1}^{0} \text{sgn}(n, (j+1) \times 2^{n-l-1} - 1 - k) \times \\
& \quad (b_j b_{2^l-1-j} + b_{2^l-1-j} b_{2^l-1-j}) \times \\
& \quad \left[ (c_{2k} c_{(2^n-l-1)-2k} - c_{2k+1} c_{(2^n-l-2)-2k}) \right].
\end{align*}

(C2)
(C3)

When \( l \) is even, by property 2 in Appendix A, then \( \Omega_{2j+1} = -\Omega_{2j}, j = 0, 1, \ldots, 2^l - 2 - 1 \). Hence, \( \overline{I}(a, n) = 0 \). When \( l \) is odd, by property 2 in Appendix A, then \( \Omega_{2j+1} = \Omega_{2j}, j = 0, 1, \ldots, 2^l - 2 - 1 \). Therefore

\begin{equation}
\overline{I}(a, n) = 2 \sum_{j=0}^{2^{l-2}-1} \Omega_{2j} = 2 \sum_{t=0}^{2^{l-3}-1} (\Omega_{4t} + \Omega_{4t+2}).
\end{equation}

(C4)

By (i) of property 3 in Appendix A, from Eq. (C4), we obtain

\begin{align*}
\overline{I}(a, n) &= 2 \sum_{t=0}^{2^{l-3}-1} \left\{ \left[ (b_{2t} b_{2^l-1-2t} - b_{2t+1} b_{2^l-2-2t}) - \\
& \quad \left( b_{2^l-1-2t} b_{2^l-1+1+2t} - b_{2^l-1-2t} b_{2^l-1+2t} \right) \right] \times \\
& \quad \sum_{k=0}^{2^{n-l-2}-1} \text{sgn}(n, k + t \times 2^{n-l}) \times \\
& \quad \left[ c_{2k} c_{(2^n-l-1)-2k} - c_{2k+1} c_{(2^n-l-2)-2k} \right] \right\}.
\end{align*}

(C5)
By (ii) of property 5 in Appendix A, from Eq. (C5), we obtain

\[ I(a,n) = 2 I(b,l) I^*(c,n-l). \]  

(C6)

Step 2. Compute \( I^*_{+2n-1}(a,n-1) \).

We can rewrite \( I^*_{+2n-1}(a,n-1) \) as \( I^*_{+2n-1}(a,n-1) = \sum_{j=0}^{2^{l-1}-1} Q_j \), where

\[ Q_j = \sum_{i=j \times 2^{n-l-2}}^{(j+1) \times 2^{n-l-2}-1} sgn^*(n-1,i)(a_{2n-1+2i}a_{2n-2i-2} - a_{2n-1+2i+1}a_{2n-2i-2}). \]  

(C7)

By substituting the amplitudes in Eq. (B12) into \( Q_{2j} \) and \( Q_{2j+1} \), we get

\[ Q_{2j} = \sum_{k=0}^{2^{n-l-2}-1} sgn^*(n-1,k+j \times 2^{n-l-1}) \times (b_{2^{l-1}+j}b_{2^{l-1}-1-j})(c_{2k}c_{2^{n-l-1}-2k} - c_{2k+1}c_{2^{n-l-2}-2k}). \]  

(C8)

and

\[ Q_{2j+1} = - \sum_{k=2^{n-l-2}}^{0} sgn^*(n-1,(j+1)2^{n-l-1} - 1 - k) \times (b_{2^{l-1}+j}b_{2^{l-1}+1-j})(c_{2k}c_{2^{n-l-1}-2k} - c_{2k+1}c_{2^{n-l-2}-2k}). \]  

(C9)

When \( l \) is even, by property 4 in Appendix A, then \( Q_{2j+1} = -Q_{2j} \), \( j = 0, 1, \ldots, 2^{l-2} - 1 \). Hence, \( I^*_{+2n-1}(a,n-1) = 0 \). When \( l \) is odd, by property 4 in Appendix A, then \( Q_{2j+1} = Q_{2j} \), \( j = 0, 1, \ldots, 2^{l-2} - 1 \). Therefore

\[ I^*_{+2n-1}(a,n-1) = 2 \sum_{j=0}^{2^{l-2}-1} Q_{2j} = 2 \sum_{t=0}^{2^{l-3}-1} (Q_{4t} + Q_{4t+2}). \]  

(C10)

By (ii) of property 3 in Appendix A, from Eq. (C10), we obtain
I^*_2n−1(a, n−1) =
2^{l−3} − 1 \sum_{t=0}^{2^n−1−2} [(b_{2^{l−1}−1−2t}b_{2^{l−1}−1−2t}−b_{2^{l−1}+2}b_{2^{l−1}−2−2t}) × \sum_{k=0}^{2n−l−2−1} sgn^*(n−1, k + t × 2^{n−l})(c_{2k}c_{2^{n−l}−2k} − c_{2k+1}c_{2^{n−l}−2−2k})]. \quad \text{(C11)}

By (iii) of property 5 in Appendix A, from Eq. (C11), we get

I^*_2n−1(a, n−1) = 2I^*_2n−1(b, l−1)I^*(c, n−l). \quad \text{(C12)}

Step 3. Compute I^*(a, n−1).

We rewrite I^*(a, n−1) as

I^*(a, n−1) = \sum_{j=0}^{2^{l−1}−1} R_j, \quad \text{where}

R_j = \sum_{i=j×2^{n−l−2}}^{j+1×2^{n−l−2}−1} sgn^*(n−1, i)(a_{2i}a_{2^{n−l}−2i} − a_{2i+1}a_{2^{n−l}−2i}). \quad \text{(C13)}

By substituting the amplitudes in Eq. (B12) into R_{2j} and R_{2j+1}, we get

R_{2j} = \sum_{k=0}^{2^{n−l−2}−1} sgn^*(n−1, k + j × 2^{n−l−1}) × 
(b_jb_{2^{−l}−1−j})[c_{2k}c_{2^{n−l}−2k} − c_{2k+1}c_{2^{n−l}−2−2k}], \quad \text{(C14)}

and

R_{2j+1} = \sum_{k=2^{n−l−2}−1}^{0} sgn^*(n−1, (j + 1) × 2^{n−l−1}−1 − k) × 
(b_jb_{2^{−l}−1−j})[c_{2k}c_{2^{n−l}−2k} − c_{2k+1}c_{2^{n−l}−2−2k}], \quad \text{(C15)}

When l is even, by property 4 in Appendix A, then R_{2j+1} = −R_{2j}, j = 0, 1, ..., 2^{l−2} − 1. Hence, I^*(a, n−1) = 0. When l is odd, by property 4 in Appendix A, then R_{2j+1} = R_{2j}, j = 0, 1, ..., 2^{l−2} − 1. Therefore

I^*(a, n−1) = 2 \sum_{j=0}^{2^{l−2}−1} R_{2j} = 2 \sum_{t=0}^{2^{l−3}−1} (R_{4t} + R_{4t+2}). \quad \text{(C16)}
By (ii) of property 3 in Appendix A, from Eq. (C16), we get

\[ I^*(a, n - 1) = 2^{l-3-1} \sum_{t=0}^{2^{l-2}-1} \left[ \left( b_{2t}b_{2t+1} - b_{2t+1}b_{2t-1} \right) \times \sum_{k=0}^{2^{n-l}-1} sgn^*(n - 1, k + t2^{n-l}) \times \right. \\
\left. \left( c_{2k}c_{2n-l-2k} - c_{2k+1}c_{2n-l-2-2k} \right) \right]. \]  

By (iii) of property 5 in Appendix A, from Eq. (C17), we get

\[ I^*(a, n - 1) = 2I^*(b, l - 1)I^*(c, n - l). \]  

From steps 1, 2 and 3, it is obvious that by the definition of \( \tau(\psi) \), \( \tau(\psi) = 0 \) whenever \( l \) is even. While \( l \) is odd, by substituting Eqs. (C6), (C12) and (C18) into \( \tau(\psi) \) in Eq. (3.2),

\[ \tau(\psi) = 16||I(b, l)||^2 - 4I^*(b, l - 1)I_{\perp 2l-1}(b, l - 1) || \times \]

\[ ||I^*(c, n - l)||^2 = \tau(\phi)\tau^2(\omega). \]  

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