Abstract

Let $g$ be a semi-simple Lie algebra. For a $g$-valued 1-form $A$, consider the Yang-Mills action

$$S_{YM}(A) = \int_{\mathbb{R}^4} |dA + A \wedge A|^2$$

using the standard metric on $T^*\mathbb{R}^4$. Using axial gauge fixing, we want to make sense of the following path integral,

$$\text{Tr} \int_{A \in A_{\mathbb{R}^4}/G} \mathcal{T} \exp \left[ \int_C A \right] e^{-\frac{1}{2}S_{YM}(A)} DA,$$

whereby $DA$ is some Lebesgue type of measure on the space of $g$-valued 1-forms, modulo gauge transformations $A_{\mathbb{R}^4}/G$. Here $\mathcal{T}$ is the time ordering operator.

We will construct an Abstract Wiener space for which we can define the Yang-Mills path integral rigorously, both in the Abelian and non-Abelian cases. Subsequently, we will then derive the Wilson Area Law formula, in both the Abelian and non-Abelian cases from these definitions. One of the most important applications of the Area Law formula will be to explain why the potential measured between a quark and antiquark is a linear potential.

MSC 2010: 81T13, 81T08

Keywords: Yang-Mills measure, axial gauge fixing, area law, quark confinement, vortex

1 Preliminaries

Consider a 4-manifold $M$ and a principal bundle $P$ over $M$, with structure group $G$. We assume that $G$ is compact and semi-simple. Without loss of generality we will assume that $G$ is a Lie subgroup of $U(\bar{N})$, $\bar{N} \in \mathbb{N}$ and $P \to M$ is a trivial bundle. We will identify the Lie algebra $g$ of $G$ with a Lie subalgebra of the Lie algebra $u(\bar{N})$ of $U(\bar{N})$ throughout this article. Suppose we write $\text{Tr} \equiv \text{Tr}_{\text{Mat}(\bar{N},\mathbb{C})}$. Then we can define a positive, non-degenerate bilinear form by

$$\langle A, B \rangle = -\text{Tr}_{\text{Mat}(\bar{N},\mathbb{C})}[AB]$$

for $A, B \in g$. Let $\{E^\alpha\}^{\bar{N}}_{\alpha=1}$ be an orthonormal basis in $g$, which will be fixed throughout this article.

The vector space of all smooth $g$-valued 1-forms on the manifold $M$ will be denoted by $A_{M,g}$. Denote the group of all smooth $G$-valued mappings on $M$ by $\mathcal{G}$, called the gauge group. The gauge group induces a gauge transformation on $A_{M,g}$, $A_{M,g} \times \mathcal{G} \to A_{M,g}$ given by

$$A \cdot \Omega := A^\Omega = \Omega^{-1}d\Omega + \Omega^{-1}A\Omega$$
for \(A \in \mathcal{A}_{M,g}, \Omega \in \mathcal{G}\). The orbit of an element \(A \in \mathcal{A}_{M,g}\) under this operation will be denoted by \([A]\) and the set of all orbits by \(\mathcal{A}/\mathcal{G}\).

Let \(\Lambda^q(T^*M)\) be the \(q\)-th exterior power of the cotangent bundle over \(M\). Fix a Riemannian metric \(g\) on \(M\) and this in turn defines an inner product \((\cdot, \cdot)_q\) on \(\Lambda^q(T^*M)\), for which we can define a volume form \(\omega\) on \(M\). This allows us to define a Hodge star operator \(*\) acting on \(k\)-forms, \(\Lambda : \Lambda^k(T^*M) \to \Lambda^{4-k}(T^*M)\) such that for \(u, v \in \Lambda^k(T^*M)\), we have

\[
    u \wedge *v = (u, v)_q \, d\omega. \tag{1.2}
\]

An inner product on the set of smooth sections \(\Gamma(\Lambda^k(T^*M))\) is then defined as

\[
    \langle u, v \rangle = \int_M (u, v)_q \, d\omega. \tag{1.3}
\]

Given \(u \otimes E \in \Lambda^q(T^*M) \otimes g\), we write

\[
    |u \otimes E|^2 = -\text{Tr} [E \cdot E] \quad u \wedge *u = -\text{Tr} [E \cdot E] \langle u, u \rangle_q \, d\omega.
\]

Hence for \(A \in \mathcal{A}_{M,g}\), the Yang-Mills action is given by

\[
    S_{YM}(A) = \int_M |dA + A \wedge A|^2. \tag{1.4}
\]

Note that this action is invariant under gauge transformations.

Let \(C\) be a simple closed curve in the manifold \(M\). The holonomy operator of \(A\), computed along the curve \(C\), is given by

\[
    \mathcal{H} \exp \left[ \int_C A \right],
\]

whereby \(\mathcal{H}\) is the time ordering operator. See Definition 6.6 for the definition of \(\mathcal{H}\).

It is of interest to make sense of the following path integral,

\[
    \frac{1}{Z} \text{Tr} \int_{A \in \mathcal{A}_{M,g} / \mathcal{G}} \mathcal{H} \exp \left[ \int_C A \right] e^{-\frac{1}{2} S_{YM}(A)} \, DA, \tag{1.5}
\]

whereby \(DA\) is some Lebesgue type of measure on the space of \(g\)-valued 1-forms, modulo gauge transformations and

\[
    Z = \int_{A \in \mathcal{A}_{M,g} / \mathcal{G}} e^{-\frac{1}{2} S_{YM}(A)} \, DA.
\]

**Notation 1.1** Let \(\Lambda^p(\mathbb{R}^n)\) be the \(p\)-th exterior power of the vector space \(\mathbb{R}^n\), \(n = 3, 4\). We also denote the smooth sections of a bundle \(P\) by \(\Gamma(P)\). In this article, when \(n = 3\), \(p = 1\); and when \(n = 4\), \(p = 2\).

From now on, we only consider \(M = \mathbb{R}^4\) and take the principal bundle \(P\) over \(\mathbb{R}^4\) to be the trivial bundle. On \(\mathbb{R}^4\), fix the coordinate axis and choose global coordinates \(\{x^0, x^1, x^2, x^3\}\) and let \(\{e_a\}_{a=0}^3\) be the standard orthonormal basis in \(\mathbb{R}^4\). We will also choose the standard Riemannian metric on \(\mathbb{R}^4\). Now let \(T^*\mathbb{R}^4 \to \mathbb{R}^4\) denote the trivial cotangent bundle over \(\mathbb{R}^4\), i.e. \(T^*\mathbb{R}^4 \cong \mathbb{R}^4 \times \Lambda^1(\mathbb{R}^4)\) and \(\Lambda^1(\mathbb{R}^3)\) denote the subspace in \(\Lambda^1(\mathbb{R}^4)\) spanned by \(\{dx^1, dx^2, dx^3\}\). There is an obvious inner product defined on \(\Lambda^1(\mathbb{R}^3)\), i.e. \(\langle dx^i, dx^j \rangle = 0\) if \(i \neq j\), \(1\) otherwise, which it inherits from the standard metric on \(T^*\mathbb{R}^4\).
Using axial gauge fixing, every \( A \in \mathcal{A}_{\mathbb{R}^4\langle g/S} \) can be gauge transformed into a \( g \)-valued 1-form, of the form \( A = \sum_\alpha \sum_{j=1}^3 a_{j,\alpha} \otimes dx^j \otimes E^\alpha \), subject to the conditions
\[
a_{1,\alpha}(0, x^1, 0, 0) = 0, \quad a_{2,\alpha}(0, x^1, x^2, 0) = 0, \quad a_{3,\alpha}(0, x^1, x^2, x^3) = 0.
\]
Hence it suffices to consider the Yang-Mills integral in Expression (1.5) to be over the space of \( g \)-valued 1-forms of the form \( A = \sum_\alpha \sum_{j=1}^3 a_{j,\alpha} \otimes dx^j \otimes E^\alpha \), whereby \( a_{j,\alpha} : \mathbb{R}^4 \to \mathbb{R} \) is smooth.

Its curvature is then given by
\[
dA + A \wedge A = \sum_\alpha \sum_{1 \leq i < j \leq 3} a_{ij,\alpha} \otimes dx^i \wedge dx^j \otimes E^\alpha + \sum_\alpha \sum_{1 \leq i < j \leq 3} a_{i,\alpha} a_{j,\beta} \otimes dx^i \wedge dx^j \otimes [E^\alpha, E^\beta] \\
+ \sum_\alpha \sum_{j=1}^3 a_{0,j,\alpha} \otimes dx^0 \wedge dx^j \otimes E^\alpha,
\]
for \( a_{ij,\alpha} := (-1)^i [\partial_i a_{j,\alpha} - \partial_j a_{i,\alpha}], \quad a_{0,j,\alpha} := \partial_0 a_{j,\alpha}. \)

**Notation 1.2** Note that \( \Lambda^2(T^*\mathbb{R}^4) \cong \mathbb{R}^4 \times \Lambda^2(\mathbb{R}^4) \). Using the global coordinates \( \{x^0, x^1, x^2, x^3\} \), we fix an orthonormal basis \( \{dx^i \otimes dx^j, dx^1 \otimes dx^2, dx^1 \otimes dx^3, dx^0 \otimes dx^1, dx^0 \otimes dx^2, dx^0 \otimes dx^3\} \) in \( \Lambda^2(\mathbb{R}^4) \). Using the standard metric on \( \mathbb{R}^4 \), the corresponding volume form is given by \( d\omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \).

From the Hodge star operator and the above volume form, we can define an inner product on the set of smooth sections \( \Gamma(\Lambda^2(T^*\mathbb{R}^4)) \) as in Equation (1.3).

For \( i, j, k = 1, 2, 3 \), we will define \( \epsilon_{ijk} = (-1)^{\sigma(ijk)} \) if all are distinct, \( \sigma(ijk) \) is a permutation in \( S_3 \), and \( |\sigma(ijk)| \) is the number of transpositions; 0 otherwise.

**Definition 1.3** Define
\[
c_{\gamma}^{\alpha\beta} = -\text{Tr} \left[ E^\gamma [E^\alpha, E^\beta] \right].
\]

Then,
\[
dA + A \wedge A = \sum_\gamma \left[ \sum_{1 \leq i < j \leq 3} a_{ij,\gamma} \otimes dx^i \wedge dx^j + \sum_\alpha \sum_{1 \leq i < j \leq 3} a_{i,\alpha} a_{j,\beta} c_{\gamma}^{\alpha\beta} \otimes dx^i \wedge dx^j \\
+ \sum_{j=1}^3 a_{0,j,\gamma} \otimes dx^0 \wedge dx^j \right] \otimes E^\gamma.
\]

Thus,
\[
\int_{\mathbb{R}^4} |dA + A \wedge A|^2 = \sum_{i<j} \int_{\mathbb{R}^4} \left[ \sum_\alpha a_{i,j,\alpha}^2 + \sum_\gamma \sum_\alpha a_{i,\alpha} a_{j,\beta} a_{i,\alpha} a_{j,\beta} c_{\gamma}^{\alpha\beta} c_{\gamma}^{\alpha\beta} \\
+ 2 \sum_{\alpha<\beta,\gamma} a_{i,j,\gamma} a_{i,\alpha} a_{j,\beta} c_{\gamma}^{\alpha\beta} \right] d\omega + \sum_j \int_{\mathbb{R}^4} \sum_\alpha a_{0,j,\alpha}^2 d\omega. \tag{1.6}
\]

It is conventional wisdom to interpret \( \exp \left[ -\frac{1}{2} \sum_{i<j} \int_{\mathbb{R}^4} d\omega \sum_\alpha \left( a_{i,j,\alpha}^2 + a_{0,j,\alpha}^2 \right) \right] DA \) as a Gaussian measure.
1.1 Abelian Case

Consider first a 2-dimensional Euclidean space $\mathbb{R}^2$, with Abelian group $G = U(1)$, and $u(1) \cong \mathbb{R} \otimes \sqrt{-1}$. Let $S$ be a surface with boundary $C = \partial S$. Write $A = \sum_{i=1}^{2} A_i \otimes dx^i \in \Gamma(A^1(T^*\mathbb{R}^2))$, $A_i : \mathbb{R}^2 \to \mathbb{R}$. Using Stokes’ Theorem,

$$\int_{C} \sum_{i=1}^{2} A_i \otimes dx^i = \int_{\partial S} \sum_{i=1}^{2} A_i dx^i = \int_{S} dA = \int_{\mathbb{R}^2} dA \cdot 1_S = \langle dA, 1_S \rangle .$$

Here, $\langle \cdot, \cdot \rangle$ is the $L^2$ inner product on the space of Lebesgue integrable functions on $\mathbb{R}^2$. Thus, the Yang-Mills path integral becomes

$$\frac{1}{\int_{A} e^{-|dA|^2/2} D[A]} \int_{A} e^{\sqrt{-1}(dA, 1_S)} e^{-\frac{1}{2}|dA|^2} D[A].$$

Now, make a heuristic change of variables, $A \mapsto dA$, hence we have

$$\frac{1}{\det d^{-1} \int_{A} e^{-|dA|^2/2} D[dA]} \det d^{-1} \int_{A} e^{\sqrt{-1}(dA, 1_S)} e^{-\frac{1}{2}|dA|^2} D[dA]$$

$$= \int_{A} e^{-|dA|^2/2} D[dA] \int_{A} e^{\sqrt{-1}(dA, 1_S)} e^{-\frac{1}{2}|dA|^2} D[dA].$$

This is a Gaussian integral of the form given in Lemma A.\ref{lemma} hence we can define the path integral as $\exp\left[-|1_S|^2/2\right] = \exp\left[-|S|^2/2\right]$, whereby $|S|$ is the area of the surface $S$.

Now, we move up to $\mathbb{R}^4$, still using $G = U(1)$. The above argument still applies, except that now $1_S$ has norm 0. This will yield 1 for any surface $S$, which means we have to redefine our path integral to obtain non-trivial results.

Let $\chi_x$ be the evaluation map, i.e. given a (smooth) function $f : \mathbb{R}^4 \to \mathbb{R}$, we will write $f(x) = \langle f, \chi_x \rangle$. To the physicists, $\chi_x$ is just the Dirac Delta function. Furthermore, given a 2-form written in the form $F \equiv \sum_{0 \leq i < j \leq 3} f_{ij} dx^i \wedge dx^j$, with $f_{ij}$ being smooth functions on $\mathbb{R}^4$; when we write $\langle F, \chi_x \otimes dx^a \wedge dx^b \rangle$, we mean

$$\langle F, \chi_x \otimes dx^a \wedge dx^b \rangle = \left\langle \sum_{0 \leq i < j \leq 3} f_{ij} dx^i \wedge dx^j, \chi_x \otimes dx^a \wedge dx^b \right\rangle$$

$$= \langle f_{ab} dx^a \wedge dx^b, \chi_x \otimes dx^a \wedge dx^b \rangle = f_{ab}(x) \in \mathbb{R}. \quad (1.7)$$

Choose a surface $S$ such that $\partial S = C$. Suppose $\sigma : [0,1]^2 \to \mathbb{R}^4$ is any parametrization for $S$. Using axial gauge fixing, we only consider 1-forms of the form $A = \sum_{i=1}^{3} A_i \otimes dx^i$. Write $A_{i;j} := (-1)^{i+1}[\partial_i A_j - \partial_j A_i]$, $A_{0; j} = \partial_j A_0$. We also write $A_{0; j} = A_{0; j}$. By Stokes’ Theorem,

$$\int_{C} \sum_{i=1}^{3} A_i \otimes dx^i = \int_{\partial S} \sum_{i=1}^{3} A_i \otimes dx^i = \int_{S} dA$$

$$= \int_{[0,1]^2} ds dt \left[ \sum_{1 \leq i < j \leq 3} [A_{i;j}(\sigma)|J_{ij}^\sigma]|(s, t) + \sum_{j=1}^{3} [A_{0; j}(\sigma)|J_{0j}^\sigma]|(s, t) \right]$$

$$= \int_{[0,1]^2} ds dt \left\langle dA, \sum_{0 \leq i < j \leq 3} \chi_{s(t)} |J_{ij}^\sigma|(s, t) \otimes dx^i \wedge dx^j \right\rangle := \langle dA, \tilde{v}_S \rangle . \quad (1.8)$$
where
\[ \tilde{\nu}_S = \int_{[0,1]^2} \sum_{0 \leq i < j \leq 3} ds \, dt \, \chi_{\sigma(s,t)} |J_{ij}^\sigma|(s,t) \otimes dx^i \wedge dx^j, \]
is a linear functional on the space of smooth sections of 2-forms in \( \Lambda^2(T^*\mathbb{R}^4) \). And \( J_{ij}^\sigma \) is defined in Definition A.3.

Here, we wish to point out that the inner product in question is given by Equation (1.7) and therefore,
\[ \left\langle dA, \sum_{0 \leq i < j \leq 3} \chi_{\sigma(s,t)} |J_{ij}^\sigma|(s,t) \otimes dx^i \wedge dx^j \right\rangle = \langle dA, \sum_{0 \leq i < j \leq 3} \chi_{\sigma(s,t)} \otimes dx^i \wedge dx^j \rangle |J_{ij}^\sigma|(s,t) \] (1.9)
\[ = \sum_{0 \leq i < j \leq 3} \langle A_{ij}, \chi_{\sigma(s,t)} \rangle |J_{ij}^\sigma|(s,t) \]
\[ = \sum_{0 \leq i < j \leq 3} A_{ij}(\sigma(s,t)) |J_{ij}^\sigma|(s,t). \]

Thus, we want to make sense of
\[ \frac{1}{Z} \int_A e^{\sqrt{-1} \langle dA, \tilde{\nu}_S \rangle} e^{-\frac{1}{2} |dA|^2} DA \]
for some normalization constant. We have to do a change of variables, i.e. \( A \mapsto dA \). Then we have
\[ \frac{1}{Z} \int_A e^{\sqrt{-1} \langle dA, \tilde{\nu}_S \rangle} e^{-\frac{1}{2} |dA|^2} \det(d^{-1}) D[dA], \]
which is some undefined constant \( \det(d^{-1}) \). After dividing away this constant, we are left with
\[ \frac{1}{\bar{Z}} \int_A e^{\sqrt{-1} \langle dA, \tilde{\nu}_S \rangle} e^{-\frac{1}{2} |dA|^2} D[dA], \]
which we can define as a Gaussian integral, for some normalization constant \( \bar{Z} \). Now, apply a similar argument as in the \( \mathbb{R}^2 \) case, the Yang-Mills path integral will yield \( \exp \left[-\frac{1}{2} |\tilde{\nu}_S|^2 / 2 \right] \).

There are a few problems with this argument. First of all, \( \chi_x \) is not an honest function, in fact it is a generalized function. Secondly, when we construct a Gaussian measure on \( \Gamma(\Lambda^2(T^*\mathbb{R}^4)) \), we need to enlarge the space and consider generalized sections in \( \Lambda^2(T^*\mathbb{R}^4) \). As such, the term \( |\nu_S|^2 \) will face serious problems as it is not clear how to define a product of generalized functions.

Because of the above problems, if we wish to carry out the above heuristic argument, we cannot define the path integral over real-valued 2-forms. We will instead construct an Abstract Wiener space, consisting of \( \Lambda^1(\mathbb{R}^3) \)-valued holomorphic functions in \( \mathbb{C}^4 \). The path integral given in Equation (1.5) depends on the curve \( C \subset M \equiv \mathbb{R}^3 \), or more correctly a choice of a surface \( S \) whose boundary is \( \partial S = C \). Hence, we need to embed the surface \( S \subset \mathbb{R}^4 \) inside \( \mathbb{C}^4 \), by embedding \( \mathbb{R}^4 \) inside \( \mathbb{C}^4 \) in the standard way, so that we can define the path integral using the Abstract Wiener space setting. Such an approach was used in [Lim11] and [Lim12] to construct the Chern-Simons Path integral and obtain link invariants in \( \mathbb{R}^3 \) respectively.

This work is motivated by trying to apply the Abstract Wiener Space construction used in [Lim11]. However, there are significant differences in the Chern-Simons Path integral and the Yang-Mills Path integral. Hence, we need to modify the construction used in [Lim11] accordingly, so that a rigorous definition can be given to the Yang-Mills Path integral. See Definition 3.2.
The non-Abelian case will be dealt with in Section 4. See Definition 7.6 for a rigorous definition of the Yang-Mills Path integral for the non-Abelian case. From both the Abelian and non-Abelian cases, we will then derive the Wilson Area Law Formulas from these definitions. The results are given by Theorems 3.3 and 8.3. Finally, Section 9 will focus on applications of the Area Law formula.

We wish to make the following final remark. Dimension 3 is key for the Chern-Simons Path integral formalism to work effectively. In the Yang-Mills Path integral formalism, dimension 4 is the key. So where did we use the fact that the dimension is 4? It is actually used in Appendix A.2.

**Notation 1.4** Suppose we have two Hilbert spaces, $H_1$ and $H_2$. We consider the tensor product $H_1 \otimes H_2$. The inner product on the tensor product $H_1 \otimes H_2$ is given by

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{H_1 \otimes H_2} = \langle u_1, v_1 \rangle_{H_1} \langle u_2, v_2 \rangle_{H_2}.$$ 

This definition of the inner product on the tensor product of Hilbert spaces will be assumed throughout this article. See also subsection 2.4.

Finally, we always use $\langle \cdot, \cdot \rangle$ to denote an inner product.

### 2 Construction of Wiener Measure

Throughout the rest of this article, we adopt the following notation.

**Notation 2.1** For $x \in \mathbb{R}^4$, we let $\phi_\kappa(x) = \kappa^4 e^{-\kappa^2 |x|^2 / (2\pi)^2}$, which is a Gaussian function with variance $\kappa^{-2}$. Define a function $\psi_z \equiv \psi(z)$, where $\psi(z) = \frac{1}{\sqrt{2\pi}} e^{-\sum_{i=1}^{4} |z_i|^2 / 2}$ and $z \equiv (z_0, z_1, z_2, z_3) \in \mathbb{C}^4$.

**Remark 2.2** Throughout this article, this Gaussian function will play a key role in our calculations. As mentioned earlier on, the Dirac Delta function is a generalized function, but we can approximate it using a Gaussian function $\phi_\kappa$. The larger the value of $\kappa$, the better is this approximation. The variance $1/\kappa^2$, denotes how well we can resolve a point in $\mathbb{R}^4$. In our construction of the Yang-Mills path integral, we will make use of this Gaussian function in this construction, thus all our Yang-Mills path integral define later, will depend on the parameter $\kappa$.

#### 2.1 Schwartz Space

**Notation 2.3** We let $p_r$ denote the $n$-tuple $(m_1, m_2, \ldots, m_n)$, $m_1, \ldots, m_n \geq 0$ are integers with $\sum_{j=1}^{n} m_j = r$. And we write $p_r! := m_1! m_2! \cdots m_n!$. For $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, $z^{p_r} := z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$. Let $\mathcal{P}_r$ denote the set of all such $n$-tuples, i.e.

$$\mathcal{P}_r = \left\{ (m_1, m_2, \ldots, m_n) \mid \sum_{j=1}^{n} m_j = r \right\}.$$ 

Let $\mathcal{P} = \bigcup_{r=0}^{\infty} \mathcal{P}_r$. There is an ordering which we will adopt in the rest of the article. We will write $p_r \leq p_{\bar{r}}$, if in the order of priority, $r \leq \bar{r}$, followed by $m_1 \leq \bar{m}_1$, $m_2 \leq \bar{m}_2$, ..., $m_n \leq \bar{m}_n$. 


Consider the Schwartz space $S_\kappa(\mathbb{R}^4)$, with the Gaussian function $\phi_\kappa$, where $\sqrt{\phi_\kappa}(x) = \kappa^2 e^{-\kappa^2|x|^2/4}/(2\pi)$. Any $f \in S_\kappa(\mathbb{R}^4)$ can be written in the form

$$f(x) = p(x) \sqrt{\phi_\kappa(x)},$$

whereby $p$ is a polynomial in $(x^0, x^1, x^2, x^3)$. The inner product $\langle \cdot, \cdot \rangle$ defined on the Schwartz space is given by

$$\langle f, g \rangle = \int_{\mathbb{R}^4} f \cdot g \, d\lambda,$$  \hspace{1cm} (2.1)

d$\lambda$ is Lebesgue measure on $\mathbb{R}^4$. Let $\mathfrak{S}_\kappa(\mathbb{R}^4)$ be the smallest Hilbert space containing $S_\kappa(\mathbb{R}^4)$, using this inner product.

The Hermite polynomials $\{h_i\}_{i \geq 0}$ form an orthogonal set in $L^2(\mathbb{R}, d\mu)$ with the Gaussian measure $d\mu(x_0) \equiv e^{-x_0^2/2}dx_0/\sqrt{2\pi}$. Let

$$H_{p_r}(x) := h_i(x^0)h_j(x^1)h_k(x^2)h_l(x^3), \quad p_r = (i, j, k, l) \in P_r,$$

be a product of Hermite polynomials and $H_{p_r}^\kappa := H_{p_r}(\kappa \cdot)$.

We have the normalized Hermite polynomials $H_{p_r}/\sqrt{p_r!}$ with respect to the Gaussian measure $e^{-\sum_{i=0}^3 |x_i|^2/2}d\lambda/(2\pi)^2$. Then

$$\bigcup_{r=0}^\infty \left\{ H_{p_r}(\kappa x^0, \kappa x^1, \kappa x^2, \kappa x^3) \sqrt{\phi_\kappa}/\sqrt{p_r!} : p_r \in P_r \right\}$$

is an orthonormal basis for $\mathfrak{S}_\kappa(\mathbb{R}^4)$.

**Notation 2.4** We will write

$$S_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3) = \left\{ \sum_{a=1}^3 f_a \otimes dx^a : f_a \in S_\kappa(\mathbb{R}^4) \right\}$$

and

$$\mathfrak{S}_\kappa(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4) = \left\{ \sum_{0 \leq a < b \leq 3} f_{ab} \otimes dx^a \wedge dx^b : f_{ab} \in \mathfrak{S}_\kappa(\mathbb{R}^4) \right\}.$$

Here, $S_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$ is the space of smooth 1-forms over $\mathbb{R}^4$ and this space consists of $\Lambda^1(\mathbb{R}^3)$-valued functions in $\mathbb{R}^4$, integrable with respect to Lebesgue measure.

In a similar fashion, $\mathfrak{S}_\kappa(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4)$ is the space of smooth 2-forms over $\mathbb{R}^4$ and this space consists of $\Lambda^2(\mathbb{R}^4)$-valued functions in $\mathbb{R}^4$, integrable with respect to Lebesgue measure.

Note that these Schwartz spaces are dependent on $\kappa$. Recall we use the standard metric on $T\mathbb{R}^4$ and the volume form on $\mathbb{R}^4$ is given by $d\omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. Using the Hodge star operator and the above volume form, we will define an inner product on $\mathfrak{S}_\kappa(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4)$ by

$$\left\langle \sum_{0 \leq a < b \leq 3} f_{ab} \otimes dx^a \wedge dx^b, \sum_{0 \leq a < b \leq 3} \bar{f}_{ab} \otimes dx^a \wedge dx^b \right\rangle = \sum_{0 \leq a < b \leq 3} \left\langle f_{ab}, \bar{f}_{ab} \right\rangle.$$
Definition 2.5 Define a bilinear form on $S_n(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$, by $\langle df, dg \rangle$, $f, g \in S_n(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$. Note that $df, dg \in S_n(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4)$, and we compute the bilinear form of $df$ and $dg$ using the inner product in $S_n(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4)$.

Proposition 2.6 The bilinear form $\langle d, d \cdot \rangle$ on $S_n(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$ is an inner product.

Proof. The only thing we have to show is that $\langle d\omega, d\omega \rangle = 0$ imply $\omega = 0$. Now, recall using axial gauge fixing, $\omega = \sum_{i=1}^3 \omega_i \otimes dx^i$. Therefore, $0 = |d\omega|^2 \geq |\partial_0 \omega|^2$ and thus $|\partial_0 \omega_i|^2 = 0$ for each $i = 1, 2, 3$. This means that $\omega_i$ is independent of $x_0$. But since $\omega_i$ is $L^2$ integrable over $x_0$, hence $\omega_i \equiv 0$ for each $i$. Therefore $\omega \equiv 0$.

Complete $S_n(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$ into a Hilbert space, denoted by $\overline{S_n(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)}$. In future, we will denote the inner product on this Hilbert space by $\langle d, d \cdot \rangle$.

2.2 Segal Bargmann Transform

In the construction of the Abstract Wiener space, we need to enlarge the space $\overline{S_n(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)}$, by choosing a suitable measurable norm.

But by doing so, we will end up with distribution valued forms. We do not want to consider this, as this will lead to difficulties in defining $\tilde{\nu}_g$ given in Equation (1.8). To avoid this problem, we will use the Segal Bargmann Transform and map the Schwartz space to the space of $\Lambda^1(\mathbb{R}^3)$-valued holomorphic functions, over in $\mathbb{C}^4$.

Thus, path integral is not defined on the space of $\Lambda^1(\mathbb{R}^3)$-valued Schwartz functions in $\mathbb{R}^4$, but rather, over the space of $\Lambda^1(\mathbb{R}^3)$-valued holomorphic functions in $\mathbb{C}^4$.

Let $z = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4$, each $z_i \in \mathbb{C}$. Consider the real vector space spanned by $\{z^n : z \in \mathbb{C}\}_{n=0}^{\infty}$, integrable with respect to the Gaussian measure, equipped with a sesquilinear complex inner product, given by

$$\langle z^n, z' \rangle = \frac{1}{\pi} \int_\mathbb{C} z^n \overline{z'} e^{-|z|^2} \, dx \, dp, \quad z = x + \sqrt{-1} p.$$  \hspace{1cm} (2.2)

Note that $\overline{z'}$ means complex conjugate. Denote this (real) inner product space by $H^2(\mathbb{C})$, which consists of polynomials in $z$. An orthonormal basis is given by

$$\left\{ \frac{z^n}{\sqrt{n!}} : n \geq 0 \right\}.$$

Notation 2.7 Let $H^2(\mathbb{C}^4) \equiv H^2(\mathbb{C})^4$, which consists of polynomials in $z = (z_0, z_1, z_2, z_3)$. Write

$$H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3) = \left\{ \sum_{a=1}^3 f_a \otimes dx^a : f_a \in H^2(\mathbb{C}^4) \right\}.$$

Note that this is a space of complex-valued holomorphic sections in the trivial bundle $\Lambda^1(\mathbb{R}^3) \rightarrow \mathbb{C}^4$, integrable using a Gaussian measure.

Let $\mathcal{H}^2(\mathbb{C}^4)$ be the smallest Hilbert space containing $H^2(\mathbb{C}^4)$. Also denote

$$\mathcal{H}^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^4) = \left\{ \sum_{0 \leq a < b \leq 3} f_{ab} \otimes dx^a \wedge dx^b : f_{ab} \in \mathcal{H}^2(\mathbb{C}^4) \right\}.$$
Note that this is a space of complex valued holomorphic sections in the trivial bundle $\Lambda^2(\mathbb{R}^4) \to \mathbb{C}^4$, integrable using a Gaussian measure.

We will further define an inner product on $\mathcal{H}^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^4)$ by

$$\left\langle \sum_{0 \leq a < b \leq 3} f_{ab} \otimes dx^a \wedge dx^b, \sum_{0 \leq a < b \leq 3} \hat{f}_{ab} \otimes dx^a \wedge dx^b \right\rangle = \sum_{0 \leq a < b \leq 3} \left\langle f_{ab}, \hat{f}_{ab} \right\rangle.$$ 

Here, $f_{ab}$ and $\hat{f}_{ab}$ are in $\mathcal{H}^2(\mathbb{C}^4)$.

We will continue to use $\langle \cdot, \cdot \rangle$ to denote the inner product on $\mathcal{H}^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^4)$ and write $d\lambda_4 = \frac{1}{\pi^2} e^{-\sum_{i=0}^{3} |x_i|^2} \prod_{i=0}^{3} dx^i dp^i$. Note that

$$\left\langle f_{ab}, \hat{f}_{ab} \right\rangle = \int_{\mathbb{C}^4} f_{ab} \hat{f}_{ab} d\lambda_4.$$ 

Recall the inner product we are using in $\mathcal{H}(\mathbb{R}^4) \otimes \Lambda^4(\mathbb{R}^3)$ is $\langle d\cdot, d\cdot \rangle$. We are now going to construct an isometry between $\mathcal{H}(\mathbb{R}^4) \otimes \Lambda^4(\mathbb{R}^3)$ and $H^2(\mathbb{C}^4) \otimes \Lambda^4(\mathbb{R}^3)$, using $\langle d\cdot, d\cdot \rangle$.

The Hermite polynomials satisfy the following property

$$h_{n+1}(x) = xh_n(x) - h'_n(x).$$

Therefore,

$$\frac{d}{dx} \left( h_n(x) e^{-x^2/4} \right) = \left[ xh_n(x) - h_{n+1}(x) - \frac{x}{2} h_n(x) \right] e^{-x^2/4}.$$ 

But we also have $h'_n = nh_{n-1}$, which means that $xh_n(x) = h_{n+1}(x) + nh_{n-1}(x)$. Thus,

$$\frac{d}{dx} \left( h_n(x) e^{-x^2/4} \right) = \left[ \frac{1}{2} h_{n+1}(x) + \frac{n}{2} h_{n-1}(x) - h_{n+1}(x) \right] e^{-x^2/4} = \left[ \frac{n}{2} h_{n-1}(x) - \frac{1}{2} h_{n+1}(x) \right] e^{-x^2/4}.$$ 

Definition 2.8 Define an operator $\partial_a$ by

$$\partial_a x^n = \frac{n}{2} x^{n-1} - \frac{1}{2} x^{n+1}, \quad (2.3)$$

for $a = 0, 1, 2, 3$.

Definition 2.9 For any $f_a \in H^2(\mathbb{C}^4)$, define

$$\partial \sum_{a=1}^{3} f_a \otimes dx^a := \sum_{a=1}^{3} \partial_0 f_a dx^0 \wedge dx^a + \sum_{1 \leq i < j \leq 3} (-1)^{ij} [\partial_i f_j - \partial_j f_i] dx^i \wedge dx^j.$$ 

Furthermore, define a bilinear form $\langle \cdot, \cdot \rangle_{\partial, \kappa}$ in $H^2(\mathbb{C}^4) \otimes \Lambda^4(\mathbb{R}^3)$ by

$$\left\langle \sum_{a} f_a \otimes dx^a, \sum_{b} g_b \otimes dx^b \right\rangle_{\partial, \kappa} = \kappa^2 \left\langle \partial \sum_{a} f_a \otimes dx^a, \partial \sum_{b} g_b \otimes dx^b \right\rangle. \quad (2.4)$$
Remark 2.10 Just as we have the differential operator $d$ acting on 1-forms in $S_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$, the analogous operator will be $\partial$ acting on $\Lambda^1(\mathbb{R}^3)$-valued holomorphic functions, which are in $H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3)$.

The Segal Bargmann transform $\Psi_\kappa$ actually maps Hermite polynomials in $\mathbb{R}^4$ to complex holomorphic functions in $\mathbb{C}^4$, equipped with the Gaussian measure as defined in Equation (2.2).

Definition 2.11 (Segal Bargmann Transform)
Refer to Notation 2.3. Using the Segal Bargmann Transform $\Psi_\kappa : S_\kappa(\mathbb{R}^4) \rightarrow \mathcal{H}^2(\mathbb{C}^4)$, we map

$$\Psi_\kappa : H_p(\kappa x^0, \kappa x^1, \kappa x^2, \kappa x^3) \sqrt{\phi_\kappa}/\sqrt{p_r}! \rightarrow \frac{z_p}{\sqrt{p_r}!} = \frac{z_0^i z_1^1 z_2^2 z_3^3}{\sqrt{i_0! i_1! i_2! i_3!}}$$

if $p_r = (i_0, i_2, i_2, i_3)$. Clearly, it is an isometry.

We can extend this isometry to tensor and direct products and by abuse of notation, use the same symbol. That is, $\Psi_\kappa : S_\kappa(\mathbb{R}^4) \otimes \Lambda^2(\mathbb{R}^4) \rightarrow \mathcal{H}^2(\mathbb{C}^4) \otimes \Lambda^2(\mathbb{R}^4)$ by

$$\Psi_\kappa \left[ \sum_{0 \leq a < b \leq 3} f_{a,b} \otimes dx^a \wedge dx^b \right] = \sum_{0 \leq a < b \leq 3} \Psi_\kappa[f_{a,b}] \otimes dx^a \wedge dx^b.$$  

By abuse of notation, we use the same symbol to define an isometry that sends this Schwartz space $S_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$ into $H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3)$, over in $\mathbb{C}^4$. Note that $H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3)$ is complex-valued holomorphic sections of the trivial bundle $\Lambda^1(\mathbb{R}^3) \rightarrow \mathbb{C}^4$, integrable using a Gaussian measure. Alternatively, one can view this space as $\Lambda^1(\mathbb{R}^3)$-valued holomorphic functions in $\mathbb{C}^4$, which are integrable with respect to the Gaussian measure.

By definition of $\partial$, we note that

$$\Psi_\kappa(df) = \kappa \partial(\Psi_\kappa f), \ f \in S_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3).$$

Thus we have the following result.

Proposition 2.12 We have $\langle \cdot, \cdot \rangle_{\partial, \kappa}$ is an inner product. Thus, we let $| \cdot |_{\partial, \kappa}$ denote the norm using this inner product and $(H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3), \langle \cdot, \cdot \rangle_{\partial, \kappa})$ denote this inner product space.

Proof. It follows from $\langle \Psi_\kappa \omega, \Psi_\kappa \omega \rangle_{\partial, \kappa} = \langle d\omega, d\omega \rangle$.

Henceforth, complete $(H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3), \langle \cdot, \cdot \rangle_{\partial, \kappa})$ into a Hilbert space, denoted by $\overline{H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3)}$. Note that the Segal Bargmann Transform $\Psi_\kappa$ extends into an isometry between $S_\kappa(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3)$ and $\overline{H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3)}$.

2.3 Abstract Wiener space

Let $(H, \langle \cdot, \cdot \rangle)$ be a (real) infinite dimensional Hilbert space.

Definition 2.13
1. Let $\mathcal{F}$ be a partial ordered set of finite dimensional orthogonal projections onto $H$, i.e. $P > Q$ if $QH \subseteq PH$.

2. Let $P \in \mathcal{F}$. Given any Borel subset $F \subseteq PH$, define for $\kappa > 0$,
\[
\mu_\kappa \left( x \in P^{-1}(F) \right) = \left( \frac{\kappa}{2\pi} \right)^{1/2} \int_{y \in F} e^{-\kappa |y|^2/2} dy,
\]
where $l$ is the dimension of $PH$.

3. A semi-norm $\| \cdot \|$ in $H$ is called measurable if for every $\epsilon > 0$, there exists a $P_0 \in \mathcal{F}$ such that
\[
\mu_\kappa \left( \| Px \| > \epsilon \right) < \epsilon
\]
for all $P \perp P_0$ and $P \in \mathcal{F}$.

The Hilbert space we are going to complete will be $\overline{H^2(\mathbb{C}^+)} \otimes \Lambda^1(\mathbb{R}^3)$. Recall the inner product in this Hilbert space is given by Equation (2.4). As explained in [Kuo79], an infinite dimensional Gaussian measure does not exist in $H^2(\mathbb{C}^+ \otimes \Lambda^1(\mathbb{R}^3))$. We need to complete the space $\overline{H^2(\mathbb{C}^+ \otimes \Lambda^1(\mathbb{R}^3))}$ into a Banach space, denoted $B(\mathbb{R}^4; \delta)$, using a measurable norm. This space $B(\mathbb{R}^4; \delta)$ is an Abstract Wiener space in the sense of Gross and it supports a Gaussian measure required for our definition of the path integral.

Refer to Notation 2.14 Let
\[
V_{a,r} = \text{span} \left\{ z^{p_r} \otimes dx^a : s \leq r - 1, p_s \in \mathcal{P}_s \right\}.
\]
The following observations are useful.

1. For $a = 1, 2, 3$, $\delta[f \otimes dx^a] = \sum_{i \neq a} \delta_i f \otimes dx^i \wedge dx^a$.

2. For any $p_r = (i_0, i_1, i_2, i_3)$, it is clear that the only vectors in $V_{a,r}$ which are not orthogonal to $z^{p_r}$, are of the form $z^{q_{r-2}} \otimes dx^a$, where $q_{r-2}$ are of the form

   $$(i_0 - 2, i_1, i_2, i_3), (i_0, i_1 - 2, i_2, i_3), (i_0, i_1, i_2 - 2, i_3) \text{ and } (i_0, i_1, i_2, i_3 - 2).$$

   If $i_k - 2 < 0$, we take it to be 0.

3. Last but not least, $z^{p_r} \otimes dx^a$ is orthogonal to $z^{q_r} \otimes dx^a$ for any $p_r \neq q_r$.

Notation 2.14 Let $\hat{z}^{p_r} \otimes dx^a$ denote the projection of the vector $z^{p_r} \otimes dx^a$ which is orthogonal to $V_{a,r}$ using the inner product $\langle \cdot, \cdot \rangle_{\delta, \kappa}$. And for $p_r = (i, j, k, l) \in \mathcal{P}_r$, we write $p_{r, \pm}^{i, j, k} \in \mathcal{P}_{r \pm 2}$ by adding $\pm 2$ to the $\alpha$-entry in $p_r$. If the $\alpha$-entry is 0 or 1, then $p_{r, \pm}^{i, j, k}$ is the empty set.

Let $a = 1, 2, 3$. For any orthogonal projection $P$, we let $P_a$ be the orthogonal projection onto the range of $P$ which intersects $H^2(\mathbb{C}^+) \otimes dx^a$, closed using $| \cdot |_{\delta, \kappa}$.

Note that
\[
\left\{ \hat{z}^{p_r} \otimes dx^a \mid \hat{z}^{p_r} \otimes dx^a |_{\delta, \kappa} : p_r \in \mathcal{P}_r, r \geq 0 \right\}
\]
is an orthonormal set and we can write
\[
\hat{z}^{p_r} \otimes dx^a = z^{p_r} \otimes dx^a - \sum_{\alpha = 0}^{3} \left\langle \hat{z}^{p_r} \otimes dx^a, \hat{z}^{p_r}_{\alpha}^{-} \otimes dx^a \right\rangle |_{\delta, \kappa} \hat{z}^{p_r}_{\alpha}^{-} \otimes dx^a - \sum_{\alpha = 0}^{3} \left\langle \hat{z}^{p_r} \otimes dx^a, \hat{z}^{p_r}_{\alpha}^{+} \otimes dx^a \right\rangle |_{\delta, \kappa} \hat{z}^{p_r}_{\alpha}^{+} \otimes dx^a.
\]

(2.6)
Lemma 2.15  
1. We have 
\[ |z^{pr} \otimes dx^a|_{\mathcal{D}, \kappa} \geq \frac{K}{2} \sqrt{pr!}. \]
Similarly, 
\[ |\hat{z}^{pr} \otimes dx^a|_{\mathcal{D}, \kappa} \geq \frac{K}{2} \sqrt{pr!}. \] (2.7)

2. We have 
\[ |z^{pr} \otimes dx^a|_{\mathcal{D}, \kappa} \leq 3(r+1)\kappa \sqrt{pr!}. \] (2.8)
Thus, 
\[ |\hat{z}^{pr}| \leq |z^{pr}| + 6(r+1)^2 \sum_{\alpha=0}^{3} |z^{pr_{\alpha}}|. \] (2.9)

Here, \( |\hat{z}^{pr}| \) refers to the absolute value of \( \hat{z}^{pr} \).

Proof. By definition of \( \mathcal{D}_0 \), we see that 
\[ |\mathcal{D}_0 z^0|^2 = \frac{n^2}{4} (n-1)! + \frac{1}{4} (n+1)!. \] (2.10)
Hence, \( |z^{pr} \otimes dx^a|_{\mathcal{D}, \kappa} \geq \frac{K^2}{4} \sqrt{pr!} \). From Equation (2.3), we see that \( \mathcal{D}(z^{pr} \otimes dx^a) \) will contribute a term \( f \equiv z^{i_0+1}_0 z^{i_1}_1 z^{i_2}_2 z^{i_3}_3 \otimes dx^0 \wedge dx^a \) when we apply the operator \( \mathcal{D} \), if \( pr = (i_0, i_1, i_2, i_3) \). Note that \( \mathcal{D}(z^{pr_{\alpha}} \otimes dx^a) \) will not contribute any term that will cancel the term \( f \). Using Equation (2.10),
\[ |z^{pr_{\alpha}} \otimes dx^a|_{\mathcal{D}, \kappa}^2 \geq \frac{K^2}{4} \sqrt{pr!}. \]
This gives us the lower bound. When we apply \( \mathcal{D} \) to \( z^{pr} \otimes dx^a \), we will obtain 3 terms, and from Equation (2.3), we obtain
\[ |z^{pr} \otimes dx^a|_{\mathcal{D}, \kappa} \leq 3 \kappa \times \left[ \frac{r}{2} \sqrt{pr!} + \frac{\sqrt{pr!} \cdot (r+1)}{2} \right] \leq 3\kappa (r+1) \sqrt{pr!}. \]
Together with Equation (2.7), we see that
\[ \left| \frac{z^{pr} \otimes dx^a}{|\hat{z}^{pr_{\alpha}} \otimes dx^a|_{\mathcal{D}, \kappa}} \right| \leq \frac{3\kappa (r+1) \sqrt{pr!}}{\kappa \sqrt{pr!} / r^2 / 2} \leq 6(r+1)^2. \]
Therefore, 
\[ |\hat{z}^{pr}| \leq |z^{pr}| + 6(r+1)^2 \sum_{\alpha=0}^{3} |z^{pr_{\alpha}}|. \]

Now, we are ready to choose a measurable norm, so as to define an infinite dimensional Gaussian measure on \( H^2({\mathbb{C}}^4) \otimes \Lambda^1({\mathbb{R}}^3) \).

Definition 2.16 (Measurable norm)
Let 
\[ x_a \otimes dx^a = \sum_r \sum_{pr \in pr} c_{a,pr} \frac{z^{pr}}{|z^{pr} \otimes dx^a|_{\mathcal{D}, \kappa}} \otimes dx^a \]
and \(c_{a,p} \in \mathbb{R}\). Introduce a norm \(\| \cdot \|\) by setting
\[
\left\| \sum_{a=1}^{3} x_a \otimes dx^a \right\| = \sup_{z \in B(0,1/2)} \sum_{a=1}^{3} \sum_{r \in \mathcal{P}_r} \sum_{p \in \mathcal{P}_r} |c_{p,r,a}| \left( |z^{p,r}| + 6(r+1)^2 \sum_{a=0}^{3} |z^{p,r,a}| \right).
\] (2.11)

Here, \(B(0,1/2)\) is the ball with radius \(1/2\), center 0 in \(\mathbb{C}^4\).

**Proposition 2.17** The norm \(\| \cdot \|\) is measurable in \((H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3), \langle \cdot, \cdot \rangle_{\beta,\kappa})\).

**Proof.** Now \(H^2(\mathbb{C}^4) \otimes dx^b \equiv \{ f \otimes dx^b : f \in H^2(\mathbb{C}^4) \}\) is a subspace in \(H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3)\) and we denote its closure, using \(\langle \cdot, \cdot \rangle_{\beta,\kappa}\), by \(H^2(\mathbb{C}^4) \otimes dx^b\).

It suffices to show that for any projection \(P = \sum_{a=1}^{3} P_a\) orthogonal to \(P_0\),
\[
\mu_\kappa(\| P_a x \| \geq \epsilon) \leq \epsilon / 3.
\]

We will determine \(P_0\) later. Let \(\epsilon > 0\) and choose a \(N\) large such that
\[
\sum_{k \geq N} 102k^3(k+1)^{2-k/2} < \frac{k^{1/4} \epsilon}{\sqrt{3}}
\]
and let \(P_{a,b}\) be a finite dimensional orthogonal projection onto the span of
\[
\{ z^{p,r} \otimes dx^b : p_r \in \mathcal{P}_r, r \leq N \}.
\]

Let \(V \subseteq H^2(\mathbb{C}^4) \otimes dx^b\) be a finite dimensional subspace in the orthogonal complement of the range of \(P_{a,b}\) and let \(\{ \beta_1, \ldots, \beta_l \}\) be an orthonormal basis for \(V\). Let \(P_b\) denote the projection onto \(V \subseteq H^2(\mathbb{C}^4) \otimes dx^b\). Now, there are at most \(k^3\) 4-tuples \((m_1, m_2, m_3, m_4)\) with \(\sum_{j=1}^{4} m_j = k\). It is possible to write each basis vector as a linear combination
\[
\beta_{s,b} = \sum_{p_r \geq q^1} a_{p_r,b} z^{p_r} \otimes dx^b |_{\beta_{s,b}}
\]
with \(q^1 < q^2 < \ldots < q^4\) and each \(q^i\) is a 4-tuple \((m_1^i, m_2^i, m_3^i, m_4^i)\) with \(\sum_{j=1}^{4} m_j^i \geq N\). Observe that \(|a_{p_r,b}| \leq 1\).

Any projected vector \(P_b x\) can be written as \(P_b x = \sum_s c_s(x) \beta_{s,b}\). By definition (See Equation (2.5)), \(c_s(\cdot) : H^2(\mathbb{C}^4) \otimes dx^b \to \mathbb{R}\) is a Gaussian random variable with variance \(1/\kappa\). Let \(\mathbb{E}\) denote
the expectation of a Gaussian random variable. Then

\[ \mu_\kappa (\| P_\theta x \| > \epsilon) \leq \mu_\kappa \left( \sup_{z \in B(0,1/2)} \sum_s \sum_{p^r \geq q^r} |c_s a_{p^r,b}^s| \left[ |z^{p^r}| + 6(r + 1)^2 \sum_{\alpha = 0}^3 |z^{p^r_{\alpha -}}| > \epsilon \right] \right) \]

\[ \leq \frac{1}{\epsilon} \sup_{z \in B(0,1/2)} \sum_s \sum_{p^r \geq q^r} E|c_s a_{p^r,b}| \left[ |z^{p^r}| + 6(r + 1)^2 \sum_{\alpha = 0}^3 |z^{p^r_{\alpha -}}| \right] \]

\[ \leq \frac{1}{\sqrt{N}} \sum_{s \geq N} \frac{1}{\sqrt{2^k}} \sum_{k \geq N} \frac{17k^36(k + 1)^2}{\sqrt{2^k}} \]

\[ \leq \frac{1}{\sqrt{N}} \left( \sum_{k \geq N} 102(k + 1)^2k^32^{-k/2} \right)^2 < \epsilon/3. \]

The proof is thus complete by choosing \( P_o = \sum_{n=1}^3 P_{o,n}. \)

Complete \((H^2(C^4) \otimes \Lambda^1(\mathbb{R}^3), \langle \cdot, \cdot \rangle_{B^o})\) into a Banach space \( B(\mathbb{R}^4; \mathfrak{d}) \), and form a triple, \((i, H^2(C^4) \otimes \Lambda^1(\mathbb{R}^3), B(\mathbb{R}^4; \mathfrak{d}))\), an Abstract Wiener Space. Identify

\[ y \in B(\mathbb{R}^4; \mathfrak{d})^* \subseteq H^2(C^4) \otimes \Lambda^1(\mathbb{R}^3) \subseteq B(\mathbb{R}^4; \mathfrak{d}) \]

with an element in \( H^2(C^4) \otimes \Lambda^1(\mathbb{R}^3) \) and denote the pairing \( \langle x, y \rangle = y(x) \). Here, \( y \in B(\mathbb{R}^4; \mathfrak{d})^* \).

**Remark 2.18** We will always write \( \langle \cdot, \cdot \rangle_{B} \) to denote a pairing in this article.

**Definition 2.19** (Gaussian measure on Abstract Wiener Space)

1. Define \( \tilde{\mu}_\theta \), a measure on \( B(\mathbb{R}^4; \mathfrak{d}) \) with variance \( 1/\theta \), by

\[ \tilde{\mu}_\theta \left\{ x \in B(\mathbb{R}^4; \mathfrak{d}) : \left( \langle x, y_1 \rangle, \ldots, \langle x, y_k \rangle \right) \in F \right\} = \mu_\theta \left\{ x \in H^2(C^4) \otimes \Lambda^1(\mathbb{R}^3) : \left( \langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle \right) \in F \right\}. \]

The \( y_i \)'s are in \( B(\mathbb{R}^4; \mathfrak{d})^* \).

2. \( \left\{ x \in B(\mathbb{R}^4; \mathfrak{d}) : \left( \langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle \right) \in F \right\} \) is called a cylinder set in \( B(\mathbb{R}^4; \mathfrak{d}) \). Let \( \mathcal{R}_{B(\mathbb{R}^4; \mathfrak{d})} \) be the collection of cylinder sets in \( B(\mathbb{R}^4; \mathfrak{d}) \).

**Remark 2.20**

1. It was shown by Gross that \( \tilde{\mu}_\theta \) is \( \sigma \)-additive in the \( \sigma \)-field generated by \( \mathcal{R}_{B(\mathbb{R}^4; \mathfrak{d})} \).

2. Extend \( \tilde{\mu}_\theta \) over the Borel field of \( B(\mathbb{R}^4; \mathfrak{d}) \).

3. It can be shown that the \( \sigma \)-field generated by \( \mathcal{R}_{B(\mathbb{R}^4; \mathfrak{d})} \) is equal to the Borel field of \( B(\mathbb{R}^4; \mathfrak{d}) \).
Now, any \( x \in B(\mathbb{R}^4; \mathfrak{d}) \) can be written as
\[
x = \sum_a \sum_r \sum_{p_r \in \Omega_r} e_{p_r,a} \hat{w}^{p_r} \otimes dx^a |_{\mathfrak{d},\kappa}, \tag{2.12}
\]
convergence in the sense of \( \| \cdot \| \). This space \( B(\mathbb{R}^4; \mathfrak{d}) \) can be described explicitly. Let
\[
\mathcal{H}^2(\mathbb{C}^4)_C = \mathcal{H}^2(\mathbb{C}^4) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad B(\mathbb{R}^4; \mathfrak{d})_C^* = B(\mathbb{R}^4; \mathfrak{d})^* \otimes_{\mathbb{R}} \mathbb{C}.
\]

**Proposition 2.21** Let \( x \) be given by Equation (2.12).

1. For \( w \in \mathbb{C}^n \), define
\[
\zeta_a(w) : x = \sum_a x_a \otimes dx^a \in B(\mathbb{R}^4; \mathfrak{d}) \mapsto \langle x(w), dx^a \rangle = x_a(w).
\]
Then \( \zeta_a(w) \) is in \( B(\mathbb{R}^4; \mathfrak{d})_C^* \).

2. For \( w \in \mathbb{C}^n \), define
\[
\hat{\zeta}_{ab}^\kappa(w) : x = \sum_a x_a \otimes dx^a \in B(\mathbb{R}^4; \mathfrak{d}) \mapsto \langle \kappa \delta x(w), dx^a \wedge dx^b \rangle = \kappa |\delta_a x - \delta_b x_a|.
\]
Then \( \hat{\zeta}_{ab}^\kappa(w) \) is in \( B(\mathbb{R}^4; \mathfrak{d})_C^* \).

**Proof.**

1. Let \( R \geq 2|w| + 1 \). Choose \( M > 0 \) such that for all \( r > M, R^2 \leq \sqrt{[(r/4)]]} \). Note that \( p_r! \geq [(r/4)!] \). Then,
\[
|\langle x(w), dx^a \rangle| \leq \left| \sum_r \sum_{p_r \in \Omega_r} e_{p_r,a} \hat{w}^{p_r} \otimes dx^a |_{\mathfrak{d},\kappa} \right| \\
\leq R^M \sum_{r \leq M} \sum_{p_r \in \Omega_r} |e_{p_r,a}| \left( \left| \left( \frac{w}{R} \right)^{p_r} \right| + 6(r + 1)^2 \sum_{\alpha = 0}^3 \left| \left( \frac{w}{R} \right)^{p^{\alpha \cdot -}} \right| \right) \frac{2}{\kappa \sqrt{p_r!}} \\
+ \sum_{r > M} \sum_{p_r \in \Omega_r} |e_{p_r,a}| \left( \left| \left( \frac{w}{R} \right)^{p_r} \right| + 6(r + 1)^2 \sum_{\alpha = 0}^3 \left| \left( \frac{w}{R} \right)^{p^{\alpha \cdot -}} \right| \right) \frac{2R^2}{\kappa \sqrt{p_r!}} \\
< \frac{2(R^M + 1)}{\kappa} \| x \| . \tag{2.13}
\]
This shows that \( \zeta(w) : x \to \langle x(w), dx_a \rangle \) is a bounded complex functional on \( B(\mathbb{R}^4; \mathfrak{d})_C \).

2. Let \( R \geq 2|w| + 1 \). Choose \( M > 0 \) such that for all \( r > M, 4(r + 1)R^2 \leq \sqrt{[(r/4)]]} \). Note that \( p_r! \geq [(r/4)!] \). From Equation (2.13), note that
\[
|\delta_a z_a^n| \leq (n + 1)|z_a^{n-1}| \leq 2(n + 1)|z_a|^n, \quad |z_a| < 1/2.
\]
Then,
\[
|\langle \kappa \partial x(w), dx^a \wedge dx^b \rangle | \\
\leq \kappa \left| \sum_{r \leq M} \sum_{p_r \in Q_r} c_{p_r, b} |\partial_a \hat{w}^{p_r} | \right| \left[ \langle \frac{w}{R} \rangle^{p_r} \right] + \kappa \left| \sum_{r \leq M} \sum_{p_r \in Q_r} c_{p_r, a} |\partial_b \hat{w}^{p_r} | \right| \left[ \langle \frac{w}{R} \rangle^{p_r} \right] \\
\leq R^M \sum_{r \leq M} \sum_{p_r \in Q_r} \left[ |c_{p_r, b}| + |c_{p_r, a}| \right] \left[ \langle \frac{w}{R} \rangle^{p_r} \right] + 6(r+1) \sum_{a=0}^{3} \left[ \langle \frac{w}{R} \rangle^{p_r} \right] \left( 4(r+1) \right) \frac{4(r+1)}{\sqrt{pr^r}} \\
+ \sum_{r>M} \sum_{p_r \in Q_r} \left[ |c_{p_r, a}| + |c_{p_r, b}| \right] \left[ \langle \frac{w}{R} \rangle^{p_r} \right] + 6(r+1) \sum_{a=0}^{3} \left[ \langle \frac{w}{R} \rangle^{p_r} \right] \left( 4(r+1) \right) \frac{4(r+1)}{\sqrt{pr^r}} \\
< (R^M + 1) \| x \| .
\] (2.14)

This shows that \( \hat{\xi}_{ab}(w) : x \to \langle \partial x(w), dx^a \wedge dx^b \rangle \) is a bounded complex functional on \( B(\mathbb{R}^4; \mathfrak{d}) \).

\[ \blacksquare \]

**Remark 2.22** Both \( \zeta_a(w) \) and \( \hat{\xi}_{ab}(x) \) are like the Dirac Delta functions, except that over in \( (H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3), \langle \cdot, \cdot \rangle_{\lambda, \kappa}) \), they are honest linear functionals in \( B(\mathbb{R}^4; \mathfrak{d}) \). We refer \( \zeta_a(w) \) as an evaluation form, for the rest of this article. This functional will not be used for the Abelian case; it will only be used in the non-Abelian case.

The other linear functional \( \hat{\xi}_{ab}(x) \) plays an important role in the definition of the Yang-Mills path integral, which we will in future refer to it as an evaluation differential form. We just showed that it lies in the Hilbert space \( H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3) \).

This evaluation differential form does the following. For each \( x \in \mathbb{C}^4 \), it sends
\[
\hat{\xi}_{ab}(x) : \Psi_{\kappa} \left[ \sum_{1 \leq i \leq 3} g_i \otimes dx^i \right] \in H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^3) \to \Psi_{\kappa}[\partial_a g_b - \partial_b g_a](x) \in \mathbb{C}.
\]

Here, \( \sum_{1 \leq i \leq 3} g_i \otimes dx^i \in S_{\kappa}(\mathbb{R}^4) \otimes \Lambda^1(\mathbb{R}^3) \). We are going to use it to construct \( \hat{\nu}_S \), which we talked about earlier in Equation (1.3).

**Proposition 2.23** The support of \( \hat{\nu}_\theta \) is on continuous \( \Lambda^1(\mathbb{R}^3) \)-valued holomorphic functions on \( \mathbb{C}^4 \).

**Proof.** It suffices to prove for the case when \( x = f \otimes dx^a \). Fix a \( w_0 = (t_1, t_2, t_3, t_4) \) Now for any \( w \in \mathbb{C} \),
\[
\langle x(w_0 + w) - x(w_0), dx^a \rangle = (x, \zeta_a(w_0 + w) - \zeta_a(w_0))_z
\]
for \( x \in B(\mathbb{R}^4; \mathfrak{d}) \). From Equation (2.13), it is clear that \( \zeta_a(\cdot) \) is continuous at 0 using the operator norm.

Fix a \( z_0 \neq 0 \in \mathbb{C} \) and let \( |z_0| \leq R \). Let \( D = \{ z \in \mathbb{C} : |z - z_0| < R/4 \} \subseteq \mathbb{C} \) and \( \gamma \) be the boundary of \( D \). By Cauchy integral formula,
\[
z_0^k = \int_\gamma \frac{z^k}{z - z_0} dz.
\]

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Thus, for \( z_1 \) close to \( z_0 \), i.e. \(|z_0 - z_1| < R/8\),
\[
|z_1^k - z_0^k| = \left| \int_\gamma \frac{z_1(z_1 - z_0)}{(z - z_1)(z - z_0)} dz \right| \leq 2\pi(2R + 1)^k|z_1 - z_0|/(R/8)^2.
\]
Thus, if write \( p_r = (k_1, k_2, k_3, k_4) \) and \( w + w_0 = (y_1, y_2, y_3, y_4) \), then
\[
|(w + w_0)^{p_r} - w_0^{p_r}| = \left| (t_1^{k_1} - y_1^{k_1})t_2^{k_2}t_3^{k_3}t_4^{k_4} + y_1^{k_1}(t_2^{k_2} - y_2^{k_2})t_3^{k_3}t_4^{k_4} + \cdots + y_1^{k_1}y_2^{k_2}y_3^{k_3}(t_4^{k_4} - y_4^{k_4}) \right|
\leq 2\pi \cdot 4(2|w_0| + 1)^2|w|
\]
\[
\frac{(|w_0|/8)^2}{2}. \tag{2.15}
\]
Let \( x = \sum_r \sum_{p_r \in Q_r} c_{p_r} \tilde{w}^{p_r} \otimes dx^a / \tilde{w}^{p_r} \otimes dx^a \). From Equation (2.15), choose \( R > 0 \) such that
\[
|(w + w_0)^{p_r} - w_0^{p_r}| + 6(r + 1)^2 \sum_{a=0}^3 |(w + w_0)^{p_{r,-}} - w_0^{p_{r,-}}| \leq R|w|
\]
for all \( w \in B(w_0, 2|w_0|) \). Then, if \( |w| < \epsilon \),
\[
\sup_{w \in B(0, \epsilon)} |\langle x(w + w_0) - x(w_0), dx^a \rangle| \leq 2\epsilon \sum_{p_r \in \mathbb{P}_r} \frac{|c_{p_r}|}{\kappa^{p_r!}} \frac{2\pi |w_0|}{\kappa^{p_r!}} \frac{6(r + 1)^2}{K}
\]
\[
\leq \frac{2\epsilon}{R(\log R)^2} \kappa \frac{|w_0|}{\kappa^{p_r!}} \frac{2\pi |w_0|}{\kappa^{p_r!}} \frac{6(r + 1)^2}{K}
\]
for some constant \( c(w_0) \).

Thus for any \( \epsilon > 0 \)
\[
\mu_\kappa \left( \sup_{w \in B(w_0, 1/k)} |\langle x(w + w) - x(w_0), dx^a \rangle| > \epsilon \right) = \mu_\kappa \left( \sup_{w \in B(w_0, 1/k)} |\langle x, \zeta_a(w + w) - \zeta_a(w_0) \rangle| > \epsilon \right)
\leq \mu_\kappa \left( \frac{2\epsilon |w_0|}{\kappa k} \sup_{w \in B(w_0, 1/k)} \|x\| > \epsilon \right) \rightarrow 0
\]
as \( k \) goes to infinity. Let
\[
E_k = \left\{ x : \sup_{w \in B(w_0, 1/k)} |\langle x, \zeta_a(w + w) - \zeta_a(w_0) \rangle| > \epsilon \right\}.
\]

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Choose an increasing subsequence \( \{r_k\}_{k=1}^{\infty} \) in \( \mathbb{N} \) such that \( \sum_k \hat{\mu}(E_{r_k}) < \infty \). Then, by the choice of \( r_k, \sum_k \hat{\mu}(E_{r_k}) < \infty \) and hence by the Borel Cantelli Lemma,

\[
\bigcap_{q=1}^{\infty} \bigcup_{p=q}^{\infty} E_{r_p}
\]

is a null set with respect to the measure \( \hat{\mu} \). Hence \( \hat{\mu} \)-almost surely, for each \( x \), there exists a \( r_k(x) \) such that

\[
\sup_{w \in B(w_0,1/r_k)} |\langle x(w_0 + w) - x(w_0), dx^\kappa \rangle| < \varepsilon.
\]

\( \blacksquare \)

**Corollary 2.24** Let \( \chi_w(z) = e^{iwz} \), where \( w, z \in \mathbb{C}^4 \). We have

\[
|\zeta_\alpha(w)\|_{\beta,\kappa} \leq \frac{2}{\kappa} e^{w/2}.
\]

**Proof.** Now by definition of \( \delta_n \), we note that for any \( n \geq 0 \),

\[
\left\langle \frac{\delta_n}{\sqrt{n!}}, \delta_n \right\rangle \geq n + 1 \left( \frac{1}{4} \right) \left( \frac{z_n^{n+1}}{(n+1)!} \right).
\]

Thus, it follows that \( |\langle A, A \rangle|^{1/2} \leq \frac{2}{\kappa} |A|_{\beta,\kappa} \), if \( A = f \otimes dx^\alpha \) for \( f \in H^2(\mathbb{C}^4) \).

A simple computation will show that \( \langle A, \zeta_\alpha(w) \rangle_{\beta,\kappa} = \langle A(w), dx^\alpha \rangle = \langle f, \chi_w \rangle = f(w) \). Then we have

\[
|A(w)| = |\langle f, \chi_w \rangle| \leq |f| \chi_w \leq \frac{2}{\kappa} |A|_{\beta,\kappa} |\chi_w|.
\]

A direct computation will show that \( |\chi_w| = (\chi_w, \chi_w)^{1/2} = e^{w/2} \) and this completes the proof. \( \blacksquare \)

### 2.4 A note on Tensor Products

Let \( H_i \equiv (H_i, \langle \cdot, \cdot \rangle_i) \) be Hilbert spaces, \( i = 1, 2 \). When we write \( H_1 \otimes H_2 \), we define the tensor product of Hilbert spaces, with the inner product

\[
\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle = \langle f_1, f_2 \rangle_1 \langle g_1, g_2 \rangle_2.
\]

Now suppose \( H_2 \) is finite dimensional with dimension \( n \). Then, \( H_1 \otimes H_2 \cong H_1^{\otimes n} \). Suppose \( B_1^* \subset H_1 \subset B_1 \) is an Abstract Wiener space containing \( H_1 \) as a dense subspace, with Wiener measure \( \hat{\mu} \). We will construct an Abstract Wiener space \( B_1 \otimes H_2 \), containing \( H_1 \otimes H_2 \), with the product measure \( \hat{\mu}^{\otimes n} \).

Suppose we have an Abstract Wiener space, \( B \), with a Hilbert space \( H \subset B \) which is dense and let \( B^* \subset H \) be the dual space to \( B \). Let \( f \in B^* \), \( (x, f)_H \in \mathbb{R} \). Complexify \( B \) into \( B \otimes \mathbb{C} \), and extend \( f \) to be a complex valued linear functional on \( B \otimes \mathbb{C} \).

When we write \( B^{\otimes k} \), we mean the \( k \) tensor product of \( B \). An element \( x \) in \( B^{\otimes k} \) can be written in the form \( \sum_\alpha x^{(1)}_\alpha \otimes x^{(2)}_\alpha \otimes \cdots \otimes x^{(k)}_\alpha \), \( x^{(i)}_\alpha \in B \). Likewise, an element \( z \) in \( B^{\otimes k,\ast} \) can be written in the form \( \sum_\beta z^\beta_1 \otimes z^\beta_2 \otimes \cdots \otimes z^\beta_k \), where \( z^\beta_i \in B^* \).
When we write \((x, z)\)\(_2\), we mean
\[
\sum_\alpha \sum_\beta (x_1^\alpha \otimes x_2^\alpha \otimes \cdots \otimes x_k^\alpha, z_1^\beta \otimes z_2^\beta \otimes \cdots \otimes z_k^\beta)\)\(_2 = \sum_\alpha \sum_\beta (x_1^\alpha, z_1^\beta)\)\(_3 (x_2^\alpha, z_2^\beta)\)\(_4 \cdots (x_k^\alpha, z_k^\beta)\)\(_4.
\]

By abuse of notation, we will use the same symbol \((\cdot, \cdot)\)_2 to denote the pairing used in the \(k\) tensor product of \(B\).

In the rest of the article, we equip \(B(\mathbb{R}^4; \mathfrak{d})\) with a Wiener measure \(\tilde{\mu}\) with variance 1. Note that \(B(\mathbb{R}^4; \mathfrak{d})\) is actually a probability space with Wiener measure as the probability measure. We will often write for a continuous function \(f : B(\mathbb{R}^4; \mathfrak{d}) \to \mathbb{R}\), the expectation of \(f\) as
\[
\mathbb{E}[f] \equiv \int_{B(\mathbb{R}^4; \mathfrak{d})} f \, d\tilde{\mu}.
\]

### 3 Definition of the Yang-Mills Path Integral in Abelian Case

Consider \(G\) Abelian, i.e. \(G = U(1)\) The Lie algebra \(\mathfrak{u}(1)\) will be isomorphic to \(\mathbb{R} \otimes \sqrt{-1}\). Let \(A = \sum_{j=1}^3 A_j \otimes dx^j, A_j \in C^\infty(\mathbb{R}^4)\). In this section, we want to define a functional integral of the form
\[
\frac{1}{Z} \int_A e^{iC \sum_{j=1}^3 A_j \otimes dx^j} e^{-|\Lambda|^2/2} d\Lambda, \quad i = \sqrt{-1},
\]
in the Abelian case, since \(A \wedge A = 0\). Let \(\sigma : [0, 1]^2 \equiv I^2 \to \mathbb{R}^4\) be any parametrization of the surface \(S\) such that \(\partial S = C, C\) is a simple closed curve.

Using Stokes Theorem, \(\int_S \sum_{j=1}^3 A_j \otimes dx^j = \int_A d\Lambda\), where \(A = \sum_{j=1}^3 A_j \otimes dx^j\) and \(C = \partial S\). Motivated by the case in \(\mathbb{R}^2\), we will make sense of
\[
\frac{1}{Z} \int_A e^{iS \sum_{j=1}^3 A_j \otimes dx^j} e^{-|\Lambda|^2/2} d\Lambda
\]
for a surface \(S\). However, there are many surfaces \(S\) with \(\partial S = C\). And depending on the surface \(S\), we will obtain a different result. Thus, unless we specify a surface \(S\), the Yang-Mills path integral is not well-defined for a closed loop \(C\).

Now recall our construction of \(H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^4)\), which is contained inside \(B(\mathbb{R}^4; \mathfrak{d})\) from Subsection 2.3. Our path integral will be made sense of over \(B(\mathbb{R}^4; \mathfrak{d})\). To do that, we embed \(\mathbb{R}^4 \hookrightarrow \mathbb{C}^4\) in the standard way. Given the curve \(C \in \mathbb{R}^4\), we scale it by a factor of \(\kappa/2\) inside \(\mathbb{C}^4\), hence the surface \(S\) is scaled by a factor of \((\kappa/2)^2\). And we replace \(d\) by \(\kappa \mathfrak{d}\). Therefore, the Yang-Mills path integral we wish to make sense of is of the form
\[
\frac{1}{Z} \int_{A \in H^2(\mathbb{C}^4) \otimes \Lambda^1(\mathbb{R}^4)} e^{iS \kappa \mathfrak{d} A} e^{-|\kappa \mathfrak{d} A|^2/2} dA
\]
for a surface \(S\).

Recall the linear functional \(\tilde{\nu}_S\) defined in Equation (18), which is essential in defining the path integral. Using Proposition 2.21 there exists a \(\tilde{\nu}_S\) such that
\[
\int_S \kappa \mathfrak{d} A = (A, \tilde{\nu}_S)_4.
\]
Explicitly, if we let $\sigma$ be a parametrisation of $S$, then
\[
\tilde{\nu}_S^\kappa = \int_{I^2} dsdt \sum_{0 \leq a < b \leq 3} \frac{\kappa^2}{4} |J^{\sigma}_{ab}|(s,t)\tilde{\xi}^{\kappa}_{ab}(\kappa(\sigma s,t)/2),
\]
whereby $\left(\sum_{1 \leq i \leq 3} f_i \otimes dx^i, \tilde{\xi}^{\kappa}_{ab}(w)\right)_f = \kappa(d_a f_b - d_b f_a)(w)$ and $|J^{\sigma}_{ab}|$ is defined in Definition A.3. The factor $\kappa^2/4$ comes from the scaling of the curve $C$ in $\mathbb{C}^4$ by $\kappa/2$.

However, instead of using $\tilde{\xi}^{\kappa}_{ab}(w)$, we replace it with $\xi^{\kappa}_{ab}(w)$, where $\xi^{\kappa}_{ab}(w) = \psi(w)\tilde{\xi}^{\kappa}_{ab}(w)$. So define
\[
\nu_S^\kappa = \int_{I^2} dsdt \sum_{0 \leq a < b \leq 3} \frac{\kappa^2}{4} |J^{\sigma}_{ab}|(s,t)\xi^{\kappa}_{ab}(\kappa(\sigma s,t)/2).
\tag{3.2}
\]

The factor $\psi(w)$ can be regarded as a form of normalization constant, necessary to obtain the Area Law Formula when we take the limit as $\kappa$ goes to infinity.

\textbf{Remark 3.1} In the definition of the Chern Simons path integral in [Lim11], this factor $\psi(w)$ was also used.

\textbf{Definition 3.2} (Definition of Yang-Mills path integral in the Abelian case)
Let $\nu_S^\kappa$ be given by Equation (3.2). Hence, we now define the Yang-Mills path integral in the Abelian case in the Quantum space $B(\mathbb{R}^4; \Theta) \otimes i_1$, as
\[
Y(\mathbb{R}^4, \kappa; S, i) := E_{YM} \left[ \exp \left( \frac{i}{\kappa} \nu_S^\kappa \right) \right]
:= \int_{A \in B(\mathbb{R}^4; \Theta)} \exp \left( \frac{1}{\kappa} \left( A, \nu_S^\kappa \otimes i_1 \right) \right) d\mu(A). \tag{3.3}
\]
The factor $\frac{1}{\kappa}$ is a constant required to obtain non-trivial limits as $\kappa$ goes to infinity.

We will now take the limit as $\kappa$ goes to infinity, which gives us the Wilson Area Law formula.

\textbf{Theorem 3.3} Consider the Abelian Lie group $U(1)$. Define the Yang-Mills path integral using Equation (3.3). Therefore,
\[
E_{YM} \left[ \exp \left( \frac{1}{\kappa} \sum_{0 \leq a < b \leq 3} \int_{I^2} \frac{\kappa^2}{4} |J^{\sigma}_{ab}|(s,t) \left( \cdot, \xi^{\kappa}_{ab}(\kappa(\sigma s,t)/2) \otimes i_1 \right)_f dsdt \right) \right]
= \exp \left[ -\frac{1}{2} \sum_{0 \leq a < b \leq 3} \int_{I^2} dsdt \frac{\kappa}{4} |J^{\sigma}_{ab}|(s,t)\xi^{\kappa}_{ab}(\kappa(\sigma s,t)/2)^2 \right]_{b_0, \kappa}
\rightarrow \exp \left[ \frac{-1}{8} \int_{S} \rho S \right], \tag{3.4}
\]
as $\kappa$ goes to infinity.
Proof. Recall \( \psi_w = \psi(w) = e^{-|w|^2/2\sqrt{2\pi}} \) and \( \chi_w(z) = e^{\overline{w}z} \). For \( v, w \in \mathbb{R}^4 \subset \mathbb{C}^4 \),

\[
\langle \psi_w \chi_w, \psi_v \chi_v \rangle = \psi_w \psi_v e^{wv} = \frac{1}{2\pi} e^{-|w-v|^2/2}.
\]

And we can write

\[
\exp \left[ \sum_{i=1}^{3} \sum_{r} c_{p_r} \int dx^i, \xi^i_{ab}(w) \right] = \kappa^2 \exp \left[ \sum_{i=1}^{3} \sum_{r} c_{p_r} \int dx^i, \partial \xi^i_{ab}(w) \right] = \kappa \sum_{r} \psi(w) [c_{p_r} \partial_a w^{pr} - c_{p_r,a} \partial_b w^{pr}].
\]

We leave to the reader to check that for any \( f, \chi_w \in \mathcal{H}^2(\mathbb{C}^4) \), \( \chi_w(z) = e^{\overline{w}z} \) is the unique vector such that \( \langle f, \chi_w \rangle = f(w) \) for any \( w \in \mathbb{C}^4 \). And the subspace spanned by

\[ \{ \partial_a z^{pr} : p_r \in \mathcal{P}_r, r \geq 0 \}, \]

is dense inside \( \mathcal{H}^2(\mathbb{C}^4) \). Therefore,

\[
\partial \xi^i_{ab}(w) = \frac{1}{\kappa} \psi_w \chi_w \otimes dx^a \wedge dx^b.
\]

Thus,

\[
\langle \partial \xi^i_{ab}(w), \partial \xi^i_{ab}(\hat{w}) \rangle = \frac{1}{2\pi \kappa^2} e^{-|w-\hat{w}|^2/2}, \quad (3.5)
\]

From Lemma \textbf{A1}, RHS of Equation (3.3) is equal

\[
\exp \left[ -\frac{1}{\kappa} \sum_{0 \leq a < b \leq 3} \int I^2 ds dt |J^\sigma_{ab}(s,t)|^2 \frac{\kappa^2}{4} \xi^i_{ab}(\kappa \sigma(s,t)/2)^2 \right] = \exp \left[ -\frac{1}{2} \sum_{0 \leq a < b \leq 3} \int I^2 \times I^2 ds d\sigma d\bar{\sigma} \frac{\kappa^2}{16} |J^\sigma_{ab}(s,t)|^2 J^\sigma_{ab}(\bar{s}, \bar{t}) \kappa^2 \langle \partial \xi^i_{ab}(\kappa \sigma(s,t)/2), \partial \xi^i_{ab}(\kappa \sigma(\bar{s}, \bar{t})/2) \rangle \right] = \exp \left[ -\frac{1}{8(2\pi)} \sum_{0 \leq a < b \leq 3} \int I^2 \times I^2 ds d\sigma d\bar{\sigma} \frac{\kappa^2}{4} |J^\sigma_{ab}(s,t)|^2 J^\sigma_{ab}(\bar{s}, \bar{t}) e^{-\kappa^2 |\sigma(s,t)-\sigma(\bar{s}, \bar{t})|^2/8} \right] \rightarrow \exp \left[ -\frac{1}{8} \sum_{0 \leq a < b \leq 3} \int I^2 ds d|J^\sigma_{ab}(s,t)|^2 \rho_S^a \rho_S^b (\sigma(s,t)) \right] = e^{-\frac{1}{8} \int_S \rho_S^a \rho_S^b},
\]

the limit now follows from Corollary \textbf{A5} \QED

From the discussion in Appendix \textbf{A2}, we note that \( \int_S \rho_S \) is the area of the surface \( S \). Thus, in the Abelian case, our definition of the Yang-Mills path integral is actually dependent on the surface \( S \) chosen, since different surfaces with the same boundary will have different areas.

Now that we have settled the Abelian case, we will now move on to the non-Abelian case.
4 Non-Abelian Case

Let $G$ be a Lie group and $\mathfrak{g}$ be its Lie algebra. We will define any inner product on $\mathfrak{g}$ using Equation (1.1). Write $F = dA + A \wedge A$, whereby $A \in \Gamma(\Lambda^1(T^*\mathbb{R}^3) \otimes \mathfrak{g})$. In the non-Abelian case, we want to make sense of

$$\mathcal{T} \int_A e^{\int_C \sum_{i=1}^3 A_i \otimes dx^i} e^{-\frac{i}{\hbar} |F|^2} DA,$$

where $\mathcal{T}$ is the time ordering operator. From the Abelian case, we know that we need to consider a surface $S$ such that the boundary of $S$ is $C$. Therefore, our approach would be to rewrite $\int_C \sum_{i=1}^3 A_i \otimes dx^i$ in terms of a surface $S$. Unfortunately, we cannot apply Stokes Theorem here, unless $G$ is Abelian group. This is one major obstacle that we need to overcome.

Now recall that $\mathcal{T} \exp[\int_C \sum_{i=1}^3 A_i \otimes dx^i]$ gives us the holonomy operator by parallel translating a frame along the closed loop $C$. Given a surface $S$, we break up $S$ into small little surfaces $\{S_i\}$, such that each surface $S_i$ has area approximately $\Delta_i s^2$. Furthermore, let $C_i$ be the boundary of $S_i$. Let $s_i$ be some point in $C_i$. Since $\mathcal{T}$ commutes all the holonomy operators, using such an approximation gives

$$\mathcal{T} \exp \left[ \int_{C_i} \sum_{i=1}^3 A_i \otimes dx^i \right] = \mathcal{T} \exp \left[ \sum_{i} \int_{C_i} \sum_{j=1}^3 A_j \otimes dx^j \right] = \mathcal{T} \bigotimes_i \exp \left[ \int_{C_i} \sum_{j=1}^3 A_j \otimes dx^j \right].$$

Now recall that $\mathcal{T} \exp[\int_C \sum_{i=1}^3 A_i \otimes dx^i]$ gives us the holonomy operator by parallel translating a frame along the closed loop $C$. Given a surface $S$, we break up $S$ into small little surfaces $\{S_i\}$, each with a boundary curve $C_i$. Thus, the time ordering operator does not know how to order the operators indexed by internal curves $C_i$. As such, the above heuristic argument is seriously flawed.

Notice that we are now able to write the holonomy operator as a surface integral. Unfortunately, there is a problem with the above argument. The time ordering operator $\mathcal{T}$ only orders the operators indexed by points along the curve $C$. However, on the surface $S$, we approximate the surface by a finite number of surfaces $\{S_i\}$, each with a boundary curve $C_i$. Thus, the time ordering operator does not know how to order the operators indexed by internal curves $C_i$. As such, the above heuristic argument is seriously flawed.

Fortunately, the above argument can be made rigorous if we can extend the definition of $\mathcal{T}$ on $C$ to the surface $S$. If we can order the operators indexed by points in $S$, then we can apply the above argument and rewrite the holonomy operator in terms of the surface $S$. We will devote one full section, Section 5, to explain how we can extend the definition of the time ordering operator $\mathcal{T}$. The correct expression for the holonomy operator, written as a surface integral, is given by Corollary 6.8 in Section 6.

5 Time ordering operator

In this and the next section, we allow $M$ to be any 4-manifold. We kick off this section with the definition of the time ordering operator.
**Definition 5.1** (Time ordering operator)
For any permutation $\sigma \in S_r$,
\[
\mathcal{T}(A(s_{\sigma(1)}) \cdots A(s_{\sigma(r)})) = A(s_1) \cdots A(s_r), \ s_1 > s_2 > \ldots > s_r.
\]

**Definition 5.2** (Grid $Z_n$)
Consider the grid $Z_n := \{(i,j) : i,j = 0,1/2,1 \}$, which we identify it to be a graph. The vertex set $V(Z_n)$ consists of points labeled in the grid, $(i/n,j/n), 0 \leq i,j \leq n$. The horizontal edges are the lines joining $(x,y)$ to $(x+1/n,y)$ and the vertical edges are lines joining $(x,y)$ to $(x,y+1/n)$. Let $\partial Z_n$ denote the boundary of $Z_n$, a subgraph of $Z_n$ that contains the vertices $(x,y)$ such that either $x = 0,1$ or $y = 0,1$. Note that $\partial Z_n$ is a cycle.

Define a directed cycle $\partial Z_n^+$ by assigning arrows to the edges in $\partial Z_n$ in a counterclockwise direction. Further define a directed graph on $Z_n$, denoted $\overrightarrow{Z_n}$ as follows. Let $p,q$ denote vertices in $Z_n$. The set of vertices in $\overrightarrow{Z_n}$ is the same as the set of vertices in $Z_n$. The set of directed edges consist of all the edges going from $p$ to $q$, and back, if either $p$ or $q$ is not a vertex inside $\partial Z_n$, union all the directed edges in $\overrightarrow{Z_n}$.

**Notation 5.3** To simplify notations, we will write $e_{i,j}^+, (e_{i,j}^-)$ to denote a directed edge joining $(i/n,j/n)$ to $(i+1/n,j/n)$, $(i/n,j/n)$ to $(i-1/n,j/n)$. Similarly, we will write $e_{i,j}^+$ to denote a directed edge joining $(i/n,j/n)$ to $(i/n,(j+1)/n)$, $(i/n,(j-1)/n)$.

Let $C$ be a simple closed curve and $S$ be a surface such that $\partial S = C$. Let $\sigma : [0,1] \times [0,1] \rightarrow M$ be any parametrization for the surface $S$. An edge $x$ to $y$ in $Z_n$ is mapped under $\sigma$ to the corresponding edge joining $\sigma(x)$ to $\sigma(y)$ in $\sigma(Z_n) \subset S$. Thus using this parametrization, we can form a grid $\sigma(Z_n)$ on the surface $S$, with $\sigma(\partial Z_n)$ as the boundary. We will also write $\sigma(\overrightarrow{Z_n})$ to denote the directed graph in the surface $S$ using $\sigma$. Finally when we say we move from a point $x$ to $y$ in $Z_n$, we mean moving from $\sigma(x)$ to $\sigma(y)$ in $\sigma(Z_n)$.

We want to move on this grid $\sigma(Z_n)$, starting from the vertex $\sigma(0,0)$, subject to the condition that we traverse the edges in $\sigma(\partial Z_n)$ once and only once, in a counterclockwise direction, and all other edges twice, in opposite directions. To describe this movement, we first define 4 basic moves.

**Definition 5.4**
1. Let $R_n(x,y)$ denote moving from $\sigma(x,y)$ to $\sigma(x+1/n,y)$.
2. Let $U_n(x,y)$ denote moving from $\sigma(x,y)$ to $\sigma(x,y+1/n)$.
3. Let $\Lambda_n(x,y)$ denote moving from $\sigma(x,y)$ upwards to $\sigma(x,y+1/n)$, then left to $\sigma(x-1/n,y+1/n)$, then downwards to $\sigma(x-1/n,y)$.
4. Let $C_n(x,y)$ denote moving from $\sigma(x,y)$ to the left to $\sigma(x-1/n,y)$, then downwards to $\sigma(x-1/n,y-1/n)$, then to the right to $\sigma(x,y-1/n)$.

Finally, we write $U_n\Lambda_n^{-1}R_n^{-1}(1/n,y)$ to mean starting from $\sigma(1/n,y)$, we first apply $R_n$ a total of $n-1$ times, followed by $\Lambda_n$ a total of $n-1$ times, then $U_n$ once. The final point is at $\sigma(1/n,y+1/n)$.

On $\overrightarrow{Z_n}$, there is an obvious ordering on the edges, by letting the directed edge joining $\sigma(0,0)$ to $\sigma(1/n,0)$ to be the first edge. Now, we will show how to extend this ordering to all the directed edges in $\overrightarrow{Z_n}$. We will order the edges in the order we move along the edges in the grid, as described below:

\[
1. \text{Let } \mathcal{T}(A(s_{\sigma(1)}) \cdots A(s_{\sigma(r)})) = A(s_1) \cdots A(s_r), \ s_1 > s_2 > \ldots > s_r.
\]
1. Starting from $\sigma(0,0)$, we apply Move $R_n$ once, so we end up at $\sigma(1/n,0)$.
2. At $\sigma(1/n,0)$, we apply $U_n \Lambda_n^{n-1} R_n^{n-1}(1/n,0)$, so we end up at $\sigma(1/n,1/n)$.
3. Repeat $U_n \Lambda_n^{n-1} R_n^{n-1}$ a total of $n-2$ times, so we end up at $\sigma(1/n, (n-1)/n)$.
4. Now we apply $U_n$, so we move one unit up to $\sigma(1/n,1)$.
5. Apply Move $C_n$ a total of $n-1$ times, until we end up at $\sigma(1/n,1/n)$.
6. Finally, move left to $\sigma(0,1/n)$ and then down to $\sigma(0,0)$.

**Definition 5.5** (Time ordering operator $\hat{T}$) We now extend our time ordering operator $T$ to $\hat{T}$ on the directed graph $\sigma(Z_n)$, and order the edges according to how we move along the grid described above.

**Notation 5.6** Let $C$ be a simple closed curve and $S$ be a surface such that $\partial S = C$. On the surface $S$, let $\sigma : I^2 \to M$ be any parametrization of $S$. Let $A_{i,j}^\pm$ denote parallel translation along the path $\sigma(i,j)$ to $\sigma(i \pm 1,j)$. Likewise, let $A_i^\pm$ denote parallel translation along the path $\sigma(i,j)$ to $\sigma(i, j \pm 1)$. The parallel translation in question here is prescribed by a choice of $A \in A_{M,g}$.

Suppose we have a set of matrices $B_i$ indexed by $i$. We will write

$$\prod_{i=1}^m B_i = B_1 B_{t+1} \cdots B_m,$$
$$\prod_{i=1}^m B_i = B_m B_{m-1} \cdots B_1.$$  

Recall the holonomy operator, defined by $T_{e^L_c} A$, whereby $A \in A_{M,g}$. Using the directed graph $\sigma(\partial Z_n)$, we will write it as

$$T_{e^L_c} A = \prod_{i=0}^{n-1} A_{i+}^{0} \cdot \prod_{i=1}^{n} A_{i-}^{n} \cdot \prod_{i=0}^{n-1} A_{i+}^{1+n} \cdot \prod_{i=0}^{n-1} A_{i-}^{1-n}.$$  

We remark that $\prod_{i=0}^{n-1} A_{i+}^{0}$ means parallel translation from $\sigma(0,0)$ to $\sigma(1,0)$, $\prod_{i=1}^{n} A_{i-}^{n}$ means parallel translation from $\sigma(1,0)$ to $\sigma(1,1)$, $\prod_{i=0}^{n-1} A_{i+}^{1+n}$ means parallel translation from $\sigma(1,1)$ to $\sigma(0,1)$ and finally $\prod_{i=1}^{n} A_{i-}^{1-n}$ means parallel translation from $\sigma(0,1)$ down to $\sigma(0,0)$.

**Lemma 5.7** We have that

$$T_{e^L_c} A = \prod_{i=0}^{n-1} A_{i+}^{0} \cdot \prod_{i=1}^{n} A_{i-}^{n} \cdot \prod_{i=0}^{n-1} A_{i+}^{1+n} \cdot \prod_{i=0}^{n-1} A_{i-}^{1-n} = \hat{T} \prod_{i=0}^{n-1} A_{i+}^{0} \prod_{j=0}^{n-1} A_{j-}^{n} \prod_{i=0}^{n-1} A_{i+}^{1+n} \prod_{j=1}^{n-1} A_{j-}^{1-n}.$$  

**Proof.** We need to show that parallel translation along the directed edges in $\sigma(Z_n)$ will give the same holonomy operator as parallel translation along $\sigma(\partial Z_n)$.

First, observe that we begin by parallel translation from $\sigma(0,0)$ using move $R_n$, applied $n$ times until we arrive at $\sigma(1,0)$. The parallel translation operator can be written as

$$\prod_{i=0}^{n-1} A_{i+}^{0}, \quad (5.1)$$  

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From $\sigma(1, 0)$, we apply moves $\Lambda_n$ a total of $n - 1$ times. Now, from any point $\sigma(i/n, 0)$, when we parallel translate using Move $\Lambda_n$, the parallel translation operator is given by $A_i^{-1}A_i^1A_i^{0+}$. Therefore, from $\sigma(1, 0)$, when we parallel translate along the edges using Moves $\Lambda_n$ applied $n - 1$ times, the translation operator is given by

$$\prod_{i=2}^{n=1}A_i^{-1}A_i^1A_i^{0+} = A_i^{-1} \cdot \prod_{i=2}^{n=1}A_i^1 \cdot A_i^{0+},$$

(5.2)

since $A_i^{0+}A_i^{-1}$ is the identity.

From $\sigma(1/n, 0)$, we move up to $\sigma(1/n, 1/n)$, then move right along the edges until we arrive at $\sigma(1, 1/n)$. The parallel translation operator is given by

$$\prod_{i=1}^{n-1}A_i^+ \cdot A_i^{0+}.$$  

(5.3)

Combine Equations (5.2) and (5.3) together, from $\sigma(1, 0)$, we move along the internal edges in $Z_n$ and arrive at $\sigma(1, 1/n)$, the parallel translation operator is given by

$$\prod_{i=1}^{n-1}A_i^+ \cdot A_i^{0+} \cdot A_1^{-1} \cdot \prod_{i=2}^{n}A_i^1 \cdot A_i^{0+} = A_0^{0+}.$$  

(5.4)

Now, we repeat the same argument, from $\sigma(1, j/n)$, as we move along the internal edges in $Z_n$ using moves $R_n^{-1}U_nA_n^{-1}$ and arrive at $\sigma(1, (j+1)/n)$, the parallel translation operator is given by

$$\prod_{i=1}^{n-1}A_i^{j+1} \cdot A_i^{j+1} \cdot A_i^{(j+1)-} \cdot \prod_{i=2}^{n}A_i^{j+1} \cdot A_i^{j+1} = A_0^{j+1}.$$  

(5.5)

Hence, from the point $\sigma(1, 0)$, we move along the internal edges in $Z_n$ and end up at $\sigma(1, (n-1)/n)$, apply Equation (5.4) a total of $n - 1$ times, we obtain $\prod_{i=0}^{n-2}A_i^{j+1}$. Now, from $\sigma(1, (n-1)/n)$, we apply Move $\Lambda_n$ a total of $n - 1$ times to arrive at $\sigma(1/n, (n-1)/n)$, we obtain the translation operator

$$\prod_{i=2}^{n}A_i^{-1}A_i^{n}A_i^{(n-1)+} = A_1^{-} \cdot \prod_{i=2}^{n}A_i^1 \cdot A_i^{(n-1)+},$$

(5.6)

Next move up one unit to $\sigma(1/n, n)$. Thus, the translation operator from $\sigma(1, 0)$ to $\sigma(1/n, 1)$, moving along the internal edges, is given by

$$A_i^{(n-1)+} \cdot \prod_{i=2}^{n}A_i^{-1}A_i^{n}A_i^{(n-1)+} \cdot \prod_{i=0}^{n-2}A_i^{0+} = A_1^{(n-1)+} \cdot \prod_{i=2}^{n}A_i^{-} \cdot \prod_{i=0}^{n-1}A_i^{0+},$$

using Equation (5.5).

From $\sigma(1/n, 1)$, we apply Move $C_n$ a total of $n - 1$ times. From $\sigma(1/n, j/n)$, the translation operator along Move $C_n$ is given by $A_0^{-1}A_i^0A_i^{-1}$, ending at the point $\sigma(1/n, (j-1)/n)$. Thus, from $\sigma(1/n, 1)$, parallel transport to $\sigma(1/n, 1/n)$ by applying Move $C_n$ successively, we obtain the parallel transport operator

$$\prod_{i=2}^{n}A_i^{-1}A_i^0A_i^{-1} = A_0^{-1} \cdot \prod_{i=2}^{n}A_i^{0+} \cdot A_i^{-1}.$$ 

(5.7)

Finally, we move left from $\sigma(1/n, 1/n)$ to $\sigma(0, 1/n)$ and down to $\sigma(0, 0)$. The parallel translation operator along these 2 paths is given by $A_0^{-1}A_1^{-1}$. Now put Equations (5.1), (5.6), (5.7) together, we obtain the translation operator

$$A_0^{-1}A_1^{-1} \cdot A_0^1 \cdot \prod_{i=2}^{n}A_i^{-1}A_i^0A_i^{-1} \cdot \prod_{i=2}^{n}A_i^0A_i^{-1} \cdot \prod_{i=0}^{n-1}A_i^0A_i^{0+} \cdot \prod_{i=0}^{n-1}A_i^0A_i^{-1} \cdot \prod_{i=0}^{n-1}A_i^0 \cdot \prod_{i=0}^{n-1}A_i^0A_i^{-1}.$$ 

This completes the proof.
6 Holonomy operator on a Grid

In this section, we need not restrict ourselves to $\mathbb{R}^4$ and the following discussion holds in general on any Riemannian 4-manifold $M$. Let $\pi : P \to M$ be a trivial principal bundle over $M$ with a structure group $G$. For any $g$-valued 1-form $A$ on $M$, let $R = dA + A \wedge A$ denote its curvature, which is a $g$-valued 2-form. Recall we define $\sigma : [0, 1]^2 \to M$ to be a parametrization of a surface $S$ in $M$. Suppose we choose a group element $u \in G$ at the point $m \in \sigma([0, 1]^2) \subset M$. We will write $\Omega_u := u^{-1}R(\sigma', \sigma')u$, where $\sigma' = d\sigma/ds$ and $\sigma = d\sigma/dt$.

The directed graph $\sigma(Z^n)$ was defined in Section 5. For some $A \in A_{M,g}$, $\mathcal{J}\exp [j_G A]$ represents the parallel translation along a closed curve $C$. We will now try to rewrite this expression in terms of its curvature, by first doing a Riemannian sum approximation over $\sigma(Z^n)$, and then taking the limit as $n$ goes to infinity. To simplify a bit of writing, we will let $\epsilon = 1/n$ in this section.

**Definition 6.1 (Frame and Curvature)** Fix a group element $u_0 \equiv u_0^0 \in G$ at $\sigma(0,0)$. Now we will write

$$B^j_i \equiv \prod_{i=2}^{n} A^{j+1}_{i-1} - A^{j+1}_{i} + \prod_{i=1}^{n} A^{j}_i .$$

Define for $i \geq 1, 0 \leq j \leq n$, a group element $u^j_i \in G$ at $\sigma(i/n,j/n)$ by

$$u^j_i := \prod_{k=1}^{i-1} A^j_k + \prod_{i=1}^{j-1} B^j_1 \cdot A^0_{i+1} u_0 .$$

Let $\Omega^j_{i,ab}$ denote the curvature at $\sigma(i/n,j/n)$, $\Omega^j_{i,ab} := u_{i-1}^j R(\sigma'(i/n,j/n), \sigma(i/n,j/n)) u_i^j$, $i \geq 1$. We will also let $\nabla^j_i$ denote the directed square spanned by the edges $\{e^j_{i+1}, e^j_{i+1}, e^j_{i+1} \}$.

The curvature $R$ can be written as $R = \sum_{a < b \leq 3} R_{ab} dx^a \wedge dx^b$ in local coordinates $\{x^a\}_{a=0}^3$. Given a group element $u^j_i$ at the point $\sigma(i/n,j/n)$, we will write

$$\Omega^j_{i,ab} = u_{i-1}^j R_{ab}(\sigma'(i/n,j/n), \sigma(i/n,j/n)) u_i^j .$$

To ease the notation in the next few paragraphs, we will write

$$J\Omega^j_i \equiv \sum_{0 \leq a < b \leq 3} [J^a_{ab}] \Omega^j_{i,ab}(\sigma'(i/n,j/n), \sigma(i/n,j/n)) .$$

By definition of curvature,

$$A^{j+1}_{i} - A^{j+1}_{i+1} A^{j+1}_{i+1} u_i^j = u_i^j + \epsilon^2 \sum_{0 \leq a < b \leq 3} [J^a_{ab}] R_{ab}(\sigma'(i/n,j/n), \sigma(i/n,j/n)) u_i^j + O(\epsilon^3) .$$

That is, when we parallel translate a group element $u_i^j$ along $\nabla^j_i$, we obtain $u_i^j + \epsilon^2 u_i^j J\Omega^j_i + O(\epsilon^3)$.

Fix a $0 \leq j \leq n - 1$. From the fixed group element $u_0$ at $\sigma(0,0)$, we parallel translate $u_0$ all the way to the right to $\sigma(1,0)$, then move up $j$ times, to $\sigma(1,j/n)$, the group element obtained is equivalent to $u_j^j$. Recall from $\sigma(1,j/n)$, we apply $\Lambda_n$ a total of $n - 2$ times, to arrive at $\sigma(1/n,j/n)$. Our goal is to obtain an expression for the group element at $\sigma(1/n,j/n)$, in terms of $u^j_i$. 
Lemma 6.2 Let \( \hat{u}_i^j \) be the group element at \( \sigma(i/n, j/n) \), obtained by parallel translating the group element \( u_n^j \) by Move \( \Lambda_n \) a total of \( n - i \) times, defined by

\[
\hat{u}_i^j = \prod_{k=i+1}^{n} A_{k-1}^{(j+1)} A_k^{j+1} u_n^j. \tag{6.1}
\]

Then, we can write

\[
\hat{u}_i^j = u_i^j \prod_{k=i+1}^{n-1} \left[ I + J \Omega_k^j e^2 + O(e^3) \right], \tag{6.2}
\]

for \( 1 \leq i \leq n - 1 \) and \( 0 \leq j \leq n - 1 \).

**Proof.** The proof is by induction. We begin at \( \sigma(n/n, j/n) \). Next, we apply \( A_n^{(j+1)} A_n^{j+1} \) to it. Because \( u_n^j = A_n^{(j+1)} A_n^{j+1} u_{n-1}^j \) by definition,

\[
\begin{align*}
& \quad \quad A_{n-1}^{(j+1)} A_{n-1}^{j+1} u_{n-1}^j = A_{n-1}^{(j+1)} A_{n-1}^{j+1} A_{n-1}^j u_{n-1} \\
&= u_{n-1}^j + e^2 u_{n-1}^j J \Omega_{n-1}^j + O(e^3) \\
&= u_{n-1}^j \left[ 1 + e^2 J \Omega_{n-1}^j + O(e^3) \right].
\end{align*}
\]

Suppose we have a group element \( \hat{u}_{i+1}^j \) at \( \sigma((i+1)/n, j/n) \). Apply \( A_i^{(j+1)} A_i^{j+1} A_i^j \) to \( \hat{u}_{i+1}^j \) defined by Equation (6.2), we get

\[
\begin{align*}
& \quad \quad A_{i+1}^{(j+1)} A_{i+1}^{j+1} A_i^{j+1} u_i^j \prod_{k=i+1}^{n-1} \left[ 1 + e^2 J \Omega_k^j + O(e^3) \right] \\
&= A_{i+1}^{(j+1)} A_{i+1}^{j+1} A_{i+1}^j u_i^j \prod_{k=i+1}^{n-1} \left[ 1 + e^2 J \Omega_k^j + O(e^3) \right] \\
&= u_i^j \left[ 1 + e^2 J \Omega_k^j + O(e^3) \right] \prod_{k=i+1}^{n-1} \left[ 1 + e^2 J \Omega_k^j + O(e^3) \right].
\end{align*}
\]

\[\blacksquare\]

Lemma 6.3 By Definition (6.1), \( u_i^j = \prod_{l=0}^{i-1} A_i^l B_l A_0 + u_0 \). From the definition of \( \hat{u}_i^j \) defined in Equation (6.1), we have

\[
\hat{u}_i^j = \prod_{l=0}^{i-1} A_i^l A_0 + u_0 \prod_{l=0}^{n-1} \prod_{k=1}^{n-1} \left[ 1 + e^2 J \Omega_k^j + O(e^3) \right],
\]

for \( 0 \leq j \leq n - 1 \).

**Proof.** The proof is by induction. When \( j = 0 \), the result is trivial. Now using Equation (6.2) and the fact that \( u_i^{j+1} = \prod_{l=1}^{i-1} A_i^{j+1} A_i^j \hat{u}_i^j \),

\[
\begin{align*}
& \hat{u}_i^{j+1} = A_i^j A_i^{j+1} \prod_{k=1}^{n-1} \left[ 1 + J \Omega_k^{j+1} e^2 + O(e^3) \right] \\
&= A_i^j \cdot \prod_{l=0}^{i-1} A_i^l \cdot A_0 + u_0 \prod_{l=0}^{j+1} \prod_{k=1}^{n-1} \left[ 1 + e^2 J \Omega_k^j + O(e^3) \right] \cdot \prod_{k=1}^{n-1} \left[ 1 + J \Omega_k^{j+1} e^2 + O(e^3) \right] \\
&= \prod_{l=0}^{i+1} A_i^l \cdot A_0 + u_0 \prod_{l=0}^{j+1} \prod_{k=1}^{n-1} \left[ 1 + e^2 J \Omega_k^j + O(e^3) \right].
\end{align*}
\]

\[\blacksquare\]
In particular, at \(\sigma(1/n, (n - 1)/n)\), the group element \(\tilde{u}_{1}^{n-1}\) is given by

\[
\tilde{u}_{1}^{n-1} = \prod_{l=0}^{n-2} A_{l+1}^{0+} \cdot A_{l}^{0} + u_{0} \prod_{j=1}^{n-2} \prod_{k=1}^{n-1} [1 + \epsilon^2 J\tilde{\Omega}_{k} + O(\epsilon^3)] .
\]  

(6.3)

From \(\sigma(1/n, (n - 1)/n)\), we want to parallel translate \(\tilde{u}_{1}^{n-1}\) back to \(\sigma(0, 0)\), by first translate it up to \(\sigma(1/n, 1)\), then apply Move \(C_{n}\) a total of \(n - 1\) times, by at \(\sigma(1/n, 1/n)\), then translate to the left and then down to \(\sigma(0, 0)\). Let \(J\tilde{\Omega}_{j}^{1}\) denote the curvature at \(\sigma(1/n, j/n)\),

\[
J\tilde{\Omega}_{j}^{1} := \sum_{0 \leq a < b \leq 3} [\|\mathcal{J}_{ab}\| R_{ab}] (\sigma(1/n, j/n), \sigma'(1/n, j/n)) \tilde{u}_{j}^{1}, \quad 0 \leq j \leq n - 1 .
\]

Lemma 6.4 Let \(\tilde{u}_{j}^{1}\) be the group element at \(\sigma(1/n, j/n)\), obtained by parallel translating the group element \(A_{1}^{(n-1)+} \tilde{u}_{1}^{n-1}\) by Move \(C_{n}\) a total of \(n - j\) times, defined by

\[
\tilde{u}_{j}^{1} = \prod_{l=j+1}^{j+1} A_{l+1}^{0+} A_{l}^{0} + u_{0} \prod_{j=1}^{n-2} \prod_{k=1}^{n-1} [1 + \epsilon^2 J\tilde{\Omega}_{k} + O(\epsilon^3)] .
\]

Then, we can write

\[
\tilde{u}_{j}^{1} = \prod_{l=j+1}^{j+1} A_{l+1}^{0+} A_{l}^{0} + u_{0} \prod_{j=1}^{n-2} \prod_{k=1}^{n-1} [1 + \epsilon^2 J\tilde{\Omega}_{k} + O(\epsilon^3)] ,
\]

(6.4)

for \(1 \leq j \leq n - 1\).

Proof. Let \(\tilde{u}_{1}^{1} = \prod_{l=0}^{j-1} A_{l+1}^{0+} A_{l}^{0} + u_{0}\). The proof is by induction. We begin at \(\sigma(1/n, 1)\) with group element \(\tilde{u}_{1}^{n}\). Next, we apply \(A_{1}^{n-1} A_{0}^{-1} A_{0}^{n-2}\) to \(\tilde{u}_{1}^{n}\). Then we have from Equation (6.3),

\[
A_{0+}^{n-1} A_{0}^{n-1} A_{1}^{(n-1)+} \tilde{u}_{1}^{n-1} = A_{0+}^{0} A_{0}^{n-1} A_{1}^{(n-1)+} \tilde{u}_{1}^{0} \prod_{l=0}^{n-1} \prod_{k=1}^{n-1} [1 + \epsilon^2 J\tilde{\Omega}_{k} + O(\epsilon^3)] .
\]

Suppose we have a group element \(\tilde{u}_{j+1}^{1}\) at \(\sigma(1/n, (j + 1)/n)\). Apply \(A_{0+}^{j+1} A_{0}^{(j+1)-} A_{1}^{(j+1)+}\) to \(\tilde{u}_{j+1}^{1}\) defined by Equation (6.3), we get

\[
A_{0+}^{j+1} A_{0}^{(j+1)-} A_{1}^{(j+1)+} \tilde{u}_{j+1}^{1} \prod_{l=j+1}^{j+1} \prod_{j=1}^{n-1} [1 + \epsilon^2 J\tilde{\Omega}_{k} + O(\epsilon^3)] .
\]

In particular, we have at \(\sigma(1/n, 1/n)\),

\[
\tilde{u}_{1}^{1} = A_{1}^{0+} A_{0}^{0} \prod_{l=0}^{n-1} [1 + \epsilon^2 J\tilde{\Omega}_{1}^{1} + O(\epsilon^3)] .
\]

Let the final group element at \(\sigma(0, 0)\) be \(\tilde{u}_{0}\).

Corollary 6.5 We have

\[
\tilde{u}_{0} = u_{0} \prod_{l=0}^{n-1} [1 + \epsilon^2 J\tilde{\Omega}_{1}^{1} + O(\epsilon^3)] .
\]

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Thus we need to apply $A_0^1 - A_1^1$ to $\tilde{u}_1^1$ to the left to $\sigma(0,1/n)$, then down to $\sigma(0,0)$. Thus we need to apply $A_0^1 - A_1^1$ to $\tilde{u}_1^1$. But note that

$$A_0^1 - A_1^1 = A_0^0 + u_0 \prod_{l=0}^{n-1} \left[ 1 + e^2 J \hat{\Omega}_l^1 + O(\epsilon^3) \right] \cdot \prod_{l=0}^{n-1} \prod_{k=1}^{n-1} \left[ 1 + e^2 J \Omega_k^1 + O(\epsilon^3) \right]$$

and this completes the proof.

**Definition 6.6 (Time ordering operator on $I^2$)** Let $\tilde{T}$ be a time ordering operator on $I^2$,

$$\tilde{T}A(s_1,t_1)A(s_2,t_2) \cdots A(s_m,t_m) = A(s_{\tau(1)},t_{\tau(1)})A(s_{\tau(2)},t_{\tau(2)}) \cdots A(s_{\tau(m)},t_{\tau(m)})$$

where we order first according to $t_{\tau(1)} > t_{\tau(2)} > \cdots > t_{\tau(m)}$, followed by $s_{\tau(1)} \leq s_{\tau(2)} \leq \cdots \leq s_{\tau(m)}$.

**Definition 6.7 (Parallel translation of $u_{s,t}$)** Let $\sigma: I^2 \to M$ be a parametrization of a surface $S$. Given $(s,t) \in I^2$, $0 < s, t < 1$, we define a path $P_{s,t}$ from $\sigma(0,0)$ to $\sigma(s,t)$ as follows: Starting from $\sigma(0,0)$, we move along the path $\sigma(\cdot,0)$ until $\sigma(1,0)$. Then from $\sigma(1,0)$, we move along the path $\sigma(1,\cdot)$ to $\sigma(1,t)$. Then from $\sigma(1,t)$, we move along $\sigma(1-\cdot,t)$ until we arrive at $\sigma(s,t)$. Note that $P_{s,t}$ is an open path.

Define $u_{s,t}: \mathbb{R}^d \to \pi^{-1}(\sigma(s,t))$ as a parallel translation of a group element $u_0 \in G$ from $\sigma(0,0)$ to $\sigma(s,t)$ along the path $P_{s,t}$. Note that we can also write it as $u_{s,t} = \tilde{T} e^{\int_{P_{s,t}} A} u_0$, whereby $A \in \mathcal{A}_{M,g}$.

Define $\Omega_{ab}$ as

$$\Omega_{ab}(s,t) = u^{-1}_{s,t} R_{ab}(\sigma'(s,t), \dot{\sigma}(s,t)) u_{s,t}.$$  

**Corollary 6.8** Let $C$ be a closed curve and $S$ be a surface such that $\partial S = C$. Let $\sigma: I^2 \to M$ be a parametrization of $S$. Refer to Definitions 5.2, 6.4, and Notation 5.6. Then we have

$$\mathcal{T}_C e^{\int_C A} u_0 = u_0 \prod_{l=0}^{n-1} \left[ 1 + e^2 J \hat{\Omega}_l^1 + O(\epsilon^3) \right] \cdot \prod_{l=0}^{n-1} \prod_{k=1}^{n-1} \left[ 1 + e^2 J \Omega_k^1 + O(\epsilon^3) \right]$$

$$= u_0 \prod_{l=0}^{n-1} e^{2 J \hat{\Omega}_l^1 + O(\epsilon^3)} \cdot \prod_{l=0}^{n-1} \prod_{k=1}^{n-1} e^{2 J \Omega_k^1 + O(\epsilon^3)}$$

$$\to u_0 \cdot \tilde{T} \exp \left[ \int_{I^2} \sum_{0 \leq a < b \leq 3} |J_{ab}^c| (s,t) \Omega_{ab}^c(s,t) ds dt \right].$$  

Hence the holonomy operator is given as

$$\text{Tr} \mathcal{T}_C e^{\int_C A} = \text{Tr} \tilde{T} \exp \left[ \int_{I^2} \sum_{0 \leq a < b \leq 3} |J_{ab}^c| (s,t) \Omega_{ab}^c(s,t) ds dt \right].$$  

(6.5)

**7 Yang-Mills Path Integral**

In the Abelian case, we defined a measure $\tilde{\mu}$, given in Equation 6.3, over a Banach space $B(\mathbb{R}^4; \mathfrak{d})$. Recall that $B(\mathbb{R}^4; \mathfrak{d})_C = B(\mathbb{R}^4; \mathfrak{d}) \otimes \mathbb{C}$. In the non-Abelian case, we will construct a measure over...
where the integrand should be of the form $e^{\int_{\gamma} w^{-1}[dA + \Lambda \wedge A]u}$, whereby $u$ is parallel translation, even though the correct form is given by Equation (6.5).

Let $H^2(\mathbb{C}^4)_{\mathbb{R}} \otimes \mathbb{C} = H^2(\mathbb{C}^4)_{\mathbb{C}}$. On $H^2(\mathbb{C}^4)_{\mathbb{C}} \otimes \Lambda^1(\mathbb{R}^4) \otimes \mathfrak{g}$, the Gaussian term in the Yang-Mills integral in Equation (1.5) is replaced by

\[
\exp \left( -\frac{1}{2} \int_{\mathbb{C}^4} d\lambda_4 (\kappa \partial A + A \wedge A)^2 \right) \, dA
\]

Using the construction of the Abstract Wiener space in Section 2, we will interpret $e^{\frac{1}{2} \int_{\gamma} d\lambda_4 (\kappa \partial A)^2 \, dA}$ as the Wiener measure over $B(\mathbb{R}^4; \partial C) \otimes \mathfrak{g}$. We will now explain how to interpret the remaining term

\[
\exp \left( -\frac{1}{2} \int_{\mathbb{C}^4} d\lambda_4 (\kappa \partial A, A \wedge A) + (A \wedge A, \kappa \partial A) + |A \wedge A|^2 \right), \tag{7.1}
\]

which will be shown later to be integrable with respect to Wiener measure.

The main obstruction to define the Yang-Mills measure for the non-Abelian group is to define the cubic and quartic terms. Before we explain how we are going to make sense of the cubic and quartic terms

\[
\int_{\mathbb{C}^4} d\lambda_4 (\kappa \partial A, A \wedge A), \quad \int_{\mathbb{C}^4} d\lambda_4 (A \wedge A, \kappa \partial A), \quad \int_{\mathbb{C}^4} d\lambda_4 |A \wedge A|^2
\]

respectively, we need to recall Proposition 2.21.

Recall there is an evaluation differential form $\xi_{ab}(w)$ that we used to define $\partial A$, $A = \sum_{i=1}^3 A_i \otimes dx^i$, i.e. $\kappa(\partial_A A_b, \partial_A A_a)(w) = (A, \xi_{ab}(w))_2$, or

\[\langle \kappa \partial A(w), dx^a \wedge dx^b \rangle = (A, \tilde{\xi}_{ab}(w))_2.\]

So how are we going to make sense of $A_i(w)$? From Proposition 2.21 there is an evaluation form $\zeta_i(w)$, that sends $\sum_{i=1}^3 A_i \otimes dx^i \in B(\mathbb{R}^4; \partial C)$ to $A_i(w)$. That is, for $w \in \mathbb{C}^4$, we write $A_i(w) = (A, \zeta_i(w))_2$. Observe that we can now write the product

\[\left[ A_i, A_j \right](w) = A_i(w)A_j(w) = (A \otimes A, \zeta_i(w) \otimes \zeta_j(w))_2.\]

We can now explain how to define the quartic term, given by $A_{i,\alpha} A_{j,\beta} \overline{A_{i,\alpha} A_{j,\beta}}$, whereby $A_{i,\alpha}$ is a holomorphic function over $\mathbb{C}^4$ and $\overline{A_{i,\alpha}}$ denotes its complex conjugate. The trick is to write

\[\left[ A_{i,\alpha} A_{j,\beta} \right](w) = \left( A \otimes A \otimes A \otimes A, \chi_{i,\alpha, w} \otimes \chi_{j,\beta, w} \otimes \chi_{i,\alpha, w} \otimes \chi_{i,\beta, w} \right),\]

where $\chi_{i,\alpha, w} = \zeta_i(w) \otimes E^\alpha$, $\{ E^\alpha \}_{\alpha=1}^N$ is an orthonormal basis in $\mathfrak{g}$ and

\[A = \sum_{i=1}^3 \sum_{\alpha=1}^N A_{i,\alpha} \otimes dx^i \otimes E^\alpha, \quad \overline{A} = \sum_{i=1}^3 \sum_{\alpha=1}^N \overline{A_{i,\alpha}} \otimes dx^i \otimes E^\alpha.\]
Thus,
\[ \int_{w \in \mathbb{R}^4} [A_{i,\alpha} A_{j,\beta} A_{i,\bar{\alpha}} A_{j,\bar{\beta}}](w) = \left( A^{\otimes 2} \otimes \bar{A}^{\otimes 2} , \int_{w \in \mathbb{R}^4} d\omega \chi_{i,\alpha,w} \otimes \chi_{i,\beta,w} \otimes \chi_{i,\bar{\alpha},w} \otimes \chi_{i,\bar{\beta},w} \right). \]

This means that we have to consider the tensor products of \( A \) in order to define the Yang-Mills integral. A similar approach will be used to define the cubic term.

**Notation 7.1** Write \( A_{i,\alpha} \equiv A_{i,\alpha} \otimes dx^i \otimes E^\alpha \) and \( A = \sum_{i=1}^{\delta} \sum_{\alpha=1}^{N} A_{i,\alpha} \).

To ease our notations, we will write for \( j = 1, 2, 3 \), \( \zeta_{j,w} \equiv \zeta_j(w) \) and

\[
(A_{i_1,\alpha_1} \otimes \cdots A_{i_4,\alpha_4} \otimes \tilde{w}^{\otimes 4}) = \prod_{j=1}^{4} (A_{i_j,\alpha_j}, \tilde{\pi}_{i_j,\alpha_j,w}),
\]

whereby \( \tilde{\pi}_{i_j,\alpha_j,w} = \zeta_{j,w} \otimes E^{\alpha_j} \). We will drop the subscripts \( i_j \) and \( \alpha_j \) and write as \( \tilde{\pi}_w \).

Similarly, we will write
\[
(A_{i_1,\alpha_1} \otimes \cdots A_{i_3,\alpha_3} \otimes (\tilde{\xi}_{ab}^\kappa(w) \otimes \tilde{\pi}_w^{\otimes 2})) = (A_{i_1,\alpha_1}, \tilde{\xi}_{ab}^\kappa(w) \otimes \tilde{\pi}_w^{\otimes 2}) \prod_{j=2}^{3} (A_{i_j,\alpha_j}, \tilde{\pi}_{i_j,\alpha_j,w}),
\]
and
\[
(A_{i_1,\alpha_1} \otimes \cdots A_{i_3,\alpha_3} \otimes \tilde{\pi}_w^{\otimes 2} \otimes (\tilde{\xi}_{ab}^\kappa(w) \otimes E^{\alpha_3})) = (A_{i_1,\alpha_1}, \tilde{\xi}_{ab}^\kappa(w) \otimes E^{\alpha_3}) \prod_{j=1}^{2} (A_{i_j,\alpha_j}, \tilde{\pi}_{i_j,\alpha_j,w}).
\]

But just like in the Abelian case, we do not consider \( \tilde{\xi}_{ab}^\kappa(w) \). Instead, we introduce a factor \( \psi_w \), defined in Notation (2.7). We will also write \( \pi_w = \psi_w \tilde{\pi}_w \), i.e. that is we replace \( \tilde{\xi}_{ab}^\kappa(w) \) and \( \tilde{\pi}_w \) with

\[ \xi_{ab}^\kappa(w) = \psi_w \tilde{\xi}_{ab}^\kappa(w) \quad \text{and} \quad \pi_w = \psi_w \tilde{\pi}_w \]

respectively. This factor \( \psi_w \) is required to obtain the necessary convergence as we will see later on.

Let
\[ d\lambda_4(w) = \frac{1}{\pi^4} e^{-|w|^2} \prod_{i=1}^{4} dp_i dq_i, \]
\[ w_i = p_i + \sqrt{-1} q_i. \]

Refer to Equation (4.6). For
\[ A = \sum_{i,\alpha} A_{i,\alpha} \in B(\mathbb{R}^4; \mathfrak{g}) \otimes \mathfrak{g}, \]
we will write \( \int_{\mathbb{C}^4} d\lambda_4 |A \wedge A|^2 \) as
\[
\int_{\mathbb{C}^4} d\lambda_4 |A \wedge A|^2 := \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \sum_{\alpha, \beta, \gamma} \frac{1}{2} \left( C_{\gamma,\alpha,\beta} A_{i,\alpha} \otimes A_{j,\beta} \otimes A_{i,\bar{\alpha}} \otimes A_{j,\bar{\beta}} \left( \int_{w \in \mathbb{C}^4} d\lambda_4(w) \tilde{\pi}_w^{\otimes 4} \right)_{\gamma,34} \right.
\]
\[
+ \left. (A_{i,\alpha} \otimes A_{j,\beta} \otimes A_{i,\bar{\alpha}} \otimes A_{j,\bar{\beta}} \left( \int_{w \in \mathbb{C}^4} d\lambda_4(w) \tilde{\pi}_w^{\otimes 4} \right)_{\gamma,12}, \right) \right), \quad (7.2)
\]
where the complex-valued linear functionals \((\cdot, \cdot)_{z,12}, (\cdot, \cdot)_{z,34}\) on \((B(\mathbb{R}^4; \mathfrak{d})_\mathbb{C} \otimes \mathfrak{g})^{\otimes 4}\) are defined as

\[
(a_1 \otimes a_2 \otimes a_3 \otimes a_4, x_1 \otimes x_2 \otimes x_3 \otimes x_4)_{z,34} := \prod_{i=1}^{2} (a_i, x_i)_{z,2} \times \prod_{i=3}^{4} (a_i, x_i)_{z,2},
\]

\[
(a_1 \otimes a_2 \otimes a_3 \otimes a_4, x_1 \otimes x_2 \otimes x_3 \otimes x_4)_{z,12} := \prod_{i=1}^{2} (a_i, x_i)_{z,2} \times \prod_{i=3}^{4} (a_i, x_i)_{z,2},
\]

and \((a_i, x_i)_{z,2}\) means complex conjugate of \((a_i, x_i)_{z,2}\).

Note that in writing Equation (72), we use the fact that for a set of complex numbers \(z_i,\)

\[
\sum_{i \neq j} z_i \bar{z_j} = \sum_{i \neq j} \frac{1}{2} (z_i \bar{z_j} + \bar{z}_i z_j).
\]

Similarly, we interpret

\[
\int_{\mathbb{C}^4} d\lambda_4 \langle \kappa \delta A, A \wedge A \rangle
\]

\[
:= \sum_{\gamma} \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{\alpha, \beta} \sum_{\gamma} \epsilon_{\alpha \beta} (A_{k, \gamma} \otimes A_{i, \alpha} \otimes A_{j, \beta}, \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \tilde{\zeta}^\gamma_{ij} (w) \otimes \tilde{\pi}_w^{\otimes 3} \right)),
\]

\[
\int_{\mathbb{C}^4} d\lambda_4 \langle A \wedge A, \kappa \delta A \rangle
\]

\[
:= \sum_{\gamma} \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{\alpha, \beta} \sum_{\gamma} \epsilon_{\alpha \beta} (A_{i, \alpha} \otimes A_{j, \beta} \otimes A_{k, \gamma}, \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \tilde{\pi}_w^{\otimes 3} \otimes \left( \tilde{\zeta}^\gamma_{ij} (w) \otimes E^\gamma \right) \right)),
\]

where the complex-valued linear functionals \((\cdot, \cdot)_{z,23}, (\cdot, \cdot)_{z,34}\) on \((B(\mathbb{R}^4; \mathfrak{d})_\mathbb{C} \otimes \mathfrak{g}))^{\otimes 3}\) are defined as

\[
(a_1 \otimes a_2 \otimes a_3, x_1 \otimes x_2 \otimes x_3)_{z,23} := (a_4, x_4)_{z,2} \times \prod_{i=2}^{3} (a_i, x_i)_{z,2},
\]

\[
(a_1 \otimes a_2 \otimes a_3, x_1 \otimes x_2 \otimes x_3)_{z,34} := \prod_{i=1}^{2} (a_i, x_i)_{z,2} \times (a_3, x_3)_{z,2}.
\]

**Definition 7.2** Refer to Definition 6.7. Recall

\[
\Omega^\alpha_{ab}(s, t) = u^{-1}_{s,t} R_{ab}(\sigma'(s, t), \dot{\sigma}(s, t)) u_{s,t} = u^{-1}_{s,t} [dA + A \wedge A]_{ab}(\sigma'(s, t), \dot{\sigma}(s, t)) u_{s,t}.
\]

Suppose \(P_{s,t} = (P_{s,t}^0, P_{s,t}^1, P_{s,t}^2, P_{s,t}^3)\) and \(P_{s,t}^i(\tau) = dP_{s,t}^i(\tau)/d\tau, i = 1, 2, 3.\) Define

\[
u_{s,t} \in \mathcal{T}_{s,t} \sum_{\alpha} \bar{A}_{i, \alpha} u_0 := \mathcal{T}_{s,t} \sum_{\alpha} \sum_{i} (A_{i, \alpha} \circ P_{s,t}^i(\tau) \circ E^\alpha) u_{s,t}.
\]

See Definition 6.7. Note that \(E^\alpha_{s,t} \equiv E^\alpha\) and is indexed by \((s, t)\) and we order these matrices according to the time ordering defined in Definition 6.6. And it is understood that

\[
(A_{i, \alpha}, \pi_P_{s,t}(\tau))_{z} \equiv (A_{i, \alpha} \otimes E^\alpha, \zeta_{i, \alpha} P_{s,t}(\tau) \otimes E^\alpha)_{z} = A_{i, \alpha}(P_{s,t}(\tau)).
\]
Let $S$ be a surface such that $\partial S = C$ and embed $C \subset \mathbb{R}^4$ inside $\mathbb{C}^4$ and scale it with a factor $\kappa/2$. Let $\sigma : I^2 \to \mathbb{R}^4$ be any parametrization of $S$. Let $\rho : g \to \text{End}(\mathbb{C}^N)$ be a representation of $g$.

Write $A_\alpha = \sum_{k=1}^{3} A_{k,\alpha}$. We interpret $\text{Tr} \exp \left[ \int_C \sum_{i,\alpha} A_{i,\alpha} \right]$ using Equation (6.5) as

$$\mathfrak{J}_S \left( \{ A_{i,\alpha} \}_{i,\alpha} \right) = \text{Tr} \exp \left[ \frac{1}{\kappa} \int_{I^2} ds dt \left( \sum_{0 \leq i < j \leq 3} |J^\sigma_{ij}(s, t)\sum_{\alpha} (A_{i,\alpha} \otimes \xi_{\alpha \kappa}(s, t)/2) \otimes \rho(E^\alpha) \right) u_{s,t} \right]$$

and Expression (7.1) is interpreted as

$$y^\kappa \left( \{ A_{i,\alpha} \}_{i,\alpha} \right) := \exp \left\{ -\frac{1}{2} \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \sum_{\alpha < \beta} \sum c_{\alpha \beta}^{\gamma} \left( A_{i,\alpha} \otimes A_{j,\beta} \right) \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \xi_{ij}(w) \otimes \pi_{w}^{\otimes 2} \right) \right\}$$

Note that we replace $\tilde{\pi}$ with $\pi$ and $\tilde{\xi}_{ab}$ with $\xi_{ab}$.

**Lemma 7.3** Consider the probability space $B(\mathbb{R}^4; \mathcal{B}) \otimes g$ equipped with Wiener measure $\tilde{\mu}^{\times 2N}$. For any $\kappa > 0$, $\mathbb{E}[y^\kappa] := \int_{B(\mathbb{R}^4; \mathcal{B}) \otimes g} y^\kappa d\tilde{\mu}^{\times 2N}$ is finite.
Proof. Write $A_\alpha = \sum_{k=1}^3 A_{k,\alpha}$. Observe that

$$
\sum_{\gamma} \sum_{1 \leq i < j \leq 3} \left\{ \sum_{k=1}^3 \sum_{\alpha < \beta} c_{\gamma k}^{\alpha \beta} \left( A_{k,\gamma} \otimes A_{i,\alpha} \otimes A_{j,\beta}, \left( \xi_{ij}^k(w) \otimes E^\gamma \right) \otimes \pi_w^{\otimes 2} \right)_{\sharp,23} \right. \\
+ \sum_{k=1}^3 \sum_{\alpha < \beta} c_{\gamma k}^{\alpha \beta} \left( A_{i,\alpha} \otimes A_{j,\beta} \otimes A_{k,\gamma}, \pi_w^{\otimes 2} \otimes \left( \xi_{ij}^k(w) \otimes E^\gamma \right) \right)_{\sharp,3} \right\} \\
+ \frac{1}{2} \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \sum_{\alpha < \beta} c_{\gamma}^{\alpha \beta} c_{\gamma}^{\alpha \beta} \left[ \left( A_{i,\alpha} \otimes A_{j,\beta} \otimes A_{i,\alpha} \otimes A_{j,\beta}, \int_{w \in C^4} d\lambda_4(w) \pi_w^{\otimes 4} \right)_{\sharp,34} \right] \\
+ \left( A_{i,\alpha} \otimes A_{j,\beta} \otimes A_{i,\alpha} \otimes A_{j,\beta}, \int_{w \in C^4} d\lambda_4(w) \pi_w^{\otimes 4} \right)_{\sharp,12} \\
= \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \left| \left( A_{\gamma}, \xi_{ij}^k(w) \otimes E^\gamma \right) \right|_{\sharp}^2 + \sum_{\alpha < \beta} c_{\gamma}^{\alpha \beta} \left( A_{i,\alpha} \otimes A_{j,\beta}, \pi_w \otimes \pi_w \right)_{\sharp}^2 \\
- \sum_{1 \leq i < j \leq 3} \sum_{\alpha} \left| \left( A_{\alpha}, \xi_{ij}^k(w) \otimes E^\alpha \right) \right|_{\sharp}^2.
$$

Thus,

$$
y^k \left( \{A_{i,\alpha}\}_{i,\alpha} \right) = \exp \left[ -\frac{1}{2} \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \int_{w \in C^4} d\lambda_4(w) \left( A_{\gamma}, \xi_{ij}^k(w) \otimes E^\gamma \right)_{\sharp}^2 + \sum_{\alpha < \beta} c_{\gamma}^{\alpha \beta} \left( A_{i,\alpha} \otimes A_{j,\beta}, \pi_w \otimes \pi_w \right)_{\sharp}^2 \right] \\
+ \frac{1}{2} \int_{w \in C^4} d\lambda_4(w) \sum_{1 \leq i < j \leq 3} \sum_{\alpha} \left| \left( A_{\alpha}, \xi_{ij}^k(w) \otimes E^\alpha \right) \right|_{\sharp}^2 \right] \right] \leq \exp \left[ \frac{1}{2} \int_{w \in C^4} d\lambda_4(w) \sum_{1 \leq i < j \leq 3} \sum_{\alpha} \left| \left( A_{\alpha}, \xi_{ij}^k(w) \otimes E^\alpha \right) \right|_{\sharp}^2 \right].
$$

Therefore it suffices to show that $\exp \left[ \frac{1}{2} \int_{w \in C^4} d\lambda_4(w) \left| \left( A_{\alpha}, \xi_{ij}^k(w) \otimes E^\alpha \right) \right|_{\sharp}^2 \right]$ is integrable.

First, note that $\left( \chi_w, \chi_w \right) = e^{-|w|^2}$. From Equation (3.3),

$$
\left| \sum_{1 \leq i < j \leq 3} \xi_{ij}^k(w) \right|_{\sharp,\gamma}^2 = \frac{1}{2\pi} e^{-|w|^2} \sum_{1 \leq i \leq 3} |\chi_w|^2 = 3/2\pi.
$$

Hence,

$$
\left| \int_{w \in C^4} d\lambda_4(w) \left( \sum_{1 \leq i < j \leq 3} \xi_{ij}^k(w) \right) \right|_{\sharp}^2 = \frac{3}{(2\pi)^{4}} \int_{w \in C^4} e^{-|w|^2} \left| \sum_{1 \leq i < j \leq 3} \xi_{ij}^k(w) \right|_{\sharp,\gamma}^2 \left| \sum_{1 \leq i < j \leq 3} \xi_{ij}^k(w) \right|_{\sharp,\gamma}^2 4 dp_i dq_i.
$$

Therefore we want to compute

$$
\mathbb{E} \left[ \exp \left[ \frac{1}{2} \frac{3}{(2\pi)^{4}} \int_{w \in C^4} e^{-|w|^2} |N_w|^2 \prod_{i=1}^4 dp_i dq_i \right] \right],
$$

34
whereby \( \{ N_w = (\cdot, \varphi_w) \}_{j} \) is a family of complex-valued random variables, \( N_w \) equal in distribution to \( X_w + iY_w \), \( X_w \) and \( Y_w \) are independent Normal distributions, with \( \mathbb{E}[X^2 + Y^2] = 1 \). And

\[
\varphi_w = \sum_{1 \leq i < j < k \leq 3} \epsilon_{ij}^{(w)} |_{\mathcal{R}_{ij}}.
\]

On

\[
\frac{3}{\pi^4} \int_{w \in \mathbb{C}^4} d\lambda_4(w) e^{-|w|^2} = \frac{3}{\pi^4} \int_{w \in \mathbb{C}^4} e^{-|w|^2} \prod_{i=1}^{4} dp_i dq_i = 3,
\]
we approximate it by a lower Riemann sum using some partition function \( \beta : \{1, 2, \cdots M(n)\} \rightarrow \mathbb{C}^4 \),

\[
3 \geq \frac{3}{\pi^4} \sum_{j=1}^{M(n)} e^{-|\beta(j)|^2} \frac{1}{n^8} \prod_{i=1}^{4} dp_i dq_i
\]
as \( n \to \infty \). Note that \( M(n) = O(n^{16}) \) and \( \beta(j) \in \mathbb{C}^4 \).

Now, let \( \{ \hat{N}_k \}_{j=1}^{M(n)} \) be equal in distribution to \( \{ \hat{N}_k \}_{j=1}^{M(n)} \), whereby \( \{ \hat{N}_k \}_{j=1}^{M(n)} \) is a finite set of independent random variables, with \( \hat{N}_k \) each equal to some \( X + iY \), \( X \) and \( Y \) are independent Normal distributions with the condition that \( \mathbb{E}[X^2 + Y^2] = 1 \).

Suppose \( N_{\beta(j)} = \sum_{k=1}^{M(n)} \alpha_{j,k} \hat{N}_k \), equality in terms of distribution and \( \sum_{k=1}^{M(n)} |\alpha_{j,k}|^2 = 1 \). Define a matrix \( A \) by

\[
A_{jk} = \sqrt{\frac{3}{(2\pi)^4} e^{-|\beta(j)|^2/2} \frac{1}{n^8} |\alpha_{j,k}|^2}.
\]

Then,

\[
\frac{3}{(2\pi)^4} \sum_{j=1}^{M(n)} e^{-|\beta(j)|^2} \frac{1}{n^8} \hat{N}_{\beta(j)}^2 = \frac{3}{(2\pi)^4} \sum_{j=1}^{M(n)} e^{-|\beta(j)|^2} \frac{1}{n^8} \left| \sum_{k=1}^{M(n)} \alpha_{j,k} \hat{N}_k \right|^2
\]

\[
= \left\langle A(\hat{N}_1, \ldots, \hat{N}_{M(n)})^T, A(\hat{N}_1, \ldots, \hat{N}_{M(n)})^T \right\rangle
\]

\[
= \left\langle A^* A(\hat{N}_1, \ldots, \hat{N}_{M(n)})^T, (\hat{N}_1, \ldots, \hat{N}_{M(n)})^T \right\rangle.
\]

All the equality is in terms of distributions.

Now,

\[
[A^* A]_i = \sum_k [A^*]_i^k A_k^i = \sum_k |A_k|^2 = \frac{3}{(2\pi)^4} \sum_k e^{-|\beta(k)|^2} \frac{1}{n^8} |\alpha_{k,i}|^2
\]

and thus

\[
\text{Tr}[A^* A] = \frac{3}{(2\pi)^4} \sum_i \sum_k e^{-|\beta(k)|^2} \frac{1}{n^8} |\alpha_{k,i}|^2 = \frac{3}{(2\pi)^4} \sum_k e^{-|\beta(k)|^2} \frac{1}{n^8}.
\]
From Equation \((7.6)\), we see that \(\| \mathcal{A}^* \mathcal{A} \| \leq 3/2\pi < 1\). Hence,

\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \frac{3}{(2\pi)^4} \sum_{j=1}^{M(n)} e^{-|\beta(j)|^2} \frac{1}{n^8} \mathcal{N}_j^2(\kappa) \right) \right] \\
= \exp \left[ \frac{1}{2} \left( \mathcal{A}^* \mathcal{A}(\hat{N}_1, \ldots, \hat{N}_{M(n)})^T, (\hat{N}_1, \ldots, \hat{N}_{M(n)})^T \right) \right] \\
= \left( \frac{1}{\det(1-\mathcal{A}^* \mathcal{A})} \right)^{1/2} \leq \exp \left[ \frac{c}{2} \text{Tr}[\mathcal{A}^* \mathcal{A}] \right] \\
= \exp \left[ \frac{1}{2} \frac{3c}{(2\pi)^4} \sum_{j=1}^{M(n)} e^{-|\beta(j)|^2} \frac{1}{n^8} \right] \longrightarrow \exp \left[ \frac{1}{2} \frac{3c}{(2\pi)^4} \int_{\mathbf{w} \in \mathbb{C}^4} e^{-|\mathbf{w}|^2} \prod_{i=1}^{4} dp_idq_i \right] < \infty.
\]

The first, second and third equality follows from Equations \((7.7)\), \((A.2)\) and \((7.8)\) respectively. The inequality follows from Lemma \ref{B.2}. Since this holds for any \(n\), we thus have

\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \frac{3}{(2\pi)^4} \int_{\mathbf{w} \in \mathbb{C}^4} e^{-|\mathbf{w}|^2} \prod_{i=1}^{4} dp_idq_i \right) \right] < \infty.
\]

By modifying the above proof, we can actually prove the following result.

**Corollary 7.4** For any \(p < 2\pi/3\), there exists a \(c \equiv c(p) = (1 - 3p/2\pi)^{-1}\) such that

\[
\mathbb{E} \left[ |Y|^p \right] \leq \exp \left[ \frac{1}{2} \frac{3c}{(2\pi)^4} \int_{\mathbf{w} \in \mathbb{C}^4} e^{-|\mathbf{w}|^2} \prod_{i=1}^{4} dp_idq_i \right]. \tag{7.9}
\]

**Theorem 7.5** Consider the probability space \(B(\mathbb{R}^4; \mathcal{O}) \otimes \mathfrak{g}\) equipped with Wiener measure \(\tilde{\mu}_{2N}\). Define \(\mathcal{G}_S^\kappa\) as in Equation \((7.3)\). For large \(\kappa\), we have

\[
\mathbb{E}_{Y_M} \left[ \mathcal{G}_S^\kappa \right] := \frac{1}{\mathbb{E}[|Y|^2]} \mathbb{E} \left[ \mathcal{G}_S^\kappa |Y|^2 \right] < \infty.
\]

**Proof.** By Holder’s Inequality and previous Lemma \ref{7.3} it suffices to show that \(\mathbb{E}[\mathcal{G}_S^\kappa] < \infty\) for \(\kappa\) sufficiently large. Now since \(S\) and \(G\) are compact, \(|u|\) is bounded above and below. Thus we will show that

\[
\exp \left[ \frac{p}{4} \int_{I} dsdt \sum_{0 \leq i < j \leq 3} \left| J^\kappa_{ij}(s,t) \right| \left| \sum_{\alpha} (A_{ij}, \xi^\kappa_{ij}(\kappa\sigma(s,t)/2) \otimes E_{ij})_t \right| \right] \\
+ \sum_{1 \leq i < j \leq 3} \left| J^\kappa_{ij}(s,t) \right| \sum_{\gamma} \sum_{\alpha < \beta} \left| c_{ij}^{\alpha\beta} \right| \left| (A_{i,\alpha} \otimes A_{j,\beta}, \pi_{\kappa\sigma(s,t)/2})_t \right| \right] \tag{7.10}
\]

is integrable for large values of \(p\). Because

\[
\exp \left[ \frac{p}{4} \int_{I} dsdt \sum_{0 \leq i < j \leq 3} \left| J^\kappa_{ij}(s,t) \right| \left| \sum_{\alpha} (A_{ij}, \xi^\kappa_{ij}(\kappa\sigma(s,t)/2) \otimes E_{ij})_t \right| \right]
\]
is definitely integrable for any $\kappa$ and for any large value of $p$, using Holder's Inequality again, it suffices to show that

$$
\exp \left[ \frac{pK}{4} \int_{I^2} ds dt \sum_{1 \leq i < j \leq 3} |J_{ij}^s|(s, t) \sum_{\gamma} \sum_{\alpha < \beta} |c_{\alpha \beta}^{\gamma}| \left| \left( A_{i,\alpha} \otimes A_{j,\beta}, \pi_{\kappa\sigma(s,t)/2} \right)_t \right| \right]
$$

is integrable if $\kappa$ is made large.

Thus

$$
|\pi_i(w)|_{\dot{b},\kappa} \leq \psi(w)|\zeta_i(w)|_{\dot{b},\kappa} \leq \frac{2}{\kappa \sqrt{2\pi}}.
$$

(7.12)

Let

$$
F_n := \frac{4(\kappa/4)}{(2\pi)^{K/2}} \sum_{1 \leq i < j \leq 3} \sum_{n=1}^{n} |J_{ij}^s|(p/n, q/n) \sum_{\gamma} \sum_{\alpha < \beta} |c_{\alpha \beta}^{\gamma}| \left| M_{i,\kappa\sigma(s,t)/2}^\alpha M_{j,\kappa\sigma(s,t)/2}^\beta \right|
$$

be a Riemannian sum approximation to

$$
\left( \frac{K}{4} \right) \frac{4}{(2\pi)^{K/2}} \int_{I^2} ds dt \sum_{1 \leq i < j \leq 3} |J_{ij}^s|(s, t) \sum_{\gamma} \sum_{\alpha < \beta} |c_{\alpha \beta}^{\gamma}| \left| M_{i,\kappa\sigma(s,t)/2}^\alpha M_{j,\kappa\sigma(s,t)/2}^\beta \right|.
$$

It suffices to show that $\sup_n E[F_n]$ is bounded above.

For each $\gamma$, define an upper triangular matrix $B(\gamma)$ by $B(\gamma) = |c_{\alpha \beta}^{\gamma}|$. Note that

$$
\sum_{\alpha < \beta} |c_{\alpha \beta}^{\gamma}| \left| M_{i,\kappa\sigma(s,t)/2}^\alpha M_{j,\kappa\sigma(s,t)/2}^\beta \right|
$$

$$
= \left( B(\gamma) (M_{i,\kappa\sigma(s,t)/2}^{N,1}, \ldots, M_{i,\kappa\sigma(s,t)/2}^{N,1})^T, M_{j,\kappa\sigma(s,t)/2}^{N,1}, \ldots, M_{j,\kappa\sigma(s,t)/2}^{N,1})^T \right).
$$

Hence,

$$
\sum_{\alpha < \beta} |c_{\alpha \beta}^{\gamma}| \left| M_{i,\kappa\sigma(s,t)/2}^\alpha M_{j,\kappa\sigma(s,t)/2}^\beta \right| \leq N \| B(\gamma) \| \sqrt{\sum_{\alpha} |M_{i,\kappa\sigma(s,t)/2}^{\alpha,2}|^2} \sqrt{\sum_{\alpha} |M_{j,\kappa\sigma(s,t)/2}^{\alpha,2}|^2}
$$

$$
\leq N \| B(\gamma) \| \left[ \sum_{\alpha} |M_{i,\kappa\sigma(s,t)/2}^{\alpha,2}| + \sum_{\alpha} |M_{j,\kappa\sigma(s,t)/2}^{\alpha,2}| \right].
$$
Then,

\[
\mathbb{E} \left[ \exp \left\{ \frac{4N(\kappa/4)}{(2\pi)^{N^2}} \sum_{1 \leq a < b \leq 3} \sum_{p,q=1}^{n} \frac{|J_{ab}^*(p/n,q/n)|}{n^2} \sum_{\gamma} \left| e_{\gamma}^{\alpha\beta} \right| M_{a,\kappa\sigma(p/n,q/n)/2}^{\alpha} \otimes M_{b,\kappa\sigma(p/n,q/n)/2}^{\beta} \right\} \right] 
\]

\[
\leq \mathbb{E} \left[ \exp \left\{ \frac{(\kappa)}{4} \frac{4Nc}{(2\pi)^{N^2}} \sum_{1 \leq a < b \leq 3} \sum_{p,q=1}^{n} \frac{1}{n^2} \frac{|J_{ab}^*(p/n,q/n)|}{2} \sum_{\gamma} \left\| B(\gamma) \right\| \left[ \sum_{\alpha} M_{a,\kappa\sigma(p/n,q/n)/2}^{\alpha,2} + \sum_{\alpha} M_{b,\kappa\sigma(p/n,q/n)/2}^{\alpha,2} \right] \right\} \right].
\]  

(7.13)

Choose \( \kappa \) large enough so that we can apply Lemma B.2. Then using a similar argument used in Lemma 7.3, the RHS of Equation (7.13) is less than or equal to

\[
\exp \left\{ \frac{(\kappa)}{4} \frac{8Nc}{(2\pi)^{N^2}} \sum_{1 \leq a < b \leq 3} \sum_{p,q=1}^{n} \frac{1}{n^2} \frac{|J_{ab}^*(p/n,q/n)|}{2} \sum_{\gamma} \left\| B(\gamma) \right\| \sum_{1 \leq a < b \leq 3} \int f^2 |\sigma_a' x_b - \sigma_b' x_a| (s,t) ds dt \right\} < \infty,
\]

as \( n \to \infty \). This completes the proof.

\[\blacksquare\]

**Definition 7.6** (Definition of the Yang-Mills Path integral.)
Let \( C \) be a closed curve in \( \mathbb{R}^4 \) and let \( S \) be a surface with \( \partial S = C \). Let \( \rho \) be a representation of \( \mathfrak{g} \). Consider the probability space \( B(\mathbb{R}^4; \mathcal{D}) \otimes \mathfrak{g} \) equipped with Wiener measure \( \tilde{\mu} \otimes \times ^{2N} \).

We define the Yang-Mills Path integral for the Wilson loop, as

\[
\frac{1}{Z} \text{Tr} \int_{A} \sum_{A} \mathfrak{g} \otimes dA^i e^{-\frac{1}{4} \int |dA + A \wedge A|^2} DA := \mathbb{E}_{YM} [\mathfrak{y}_S] = \frac{1}{\mathbb{E} [\mathfrak{y}_C]} \mathbb{E} [\mathfrak{y}_S \cdot \mathfrak{y}_C],
\]

where \( \mathfrak{y}_S \) and \( \mathfrak{y}_C \) were defined in Definition 7.2. Because the path integral will be dependent on the chosen surface \( S \), we will also write

\[
Y(\mathbb{R}^4, \kappa; S, \rho) = \mathbb{E}_{YM} [\mathfrak{y}_S].
\]

8 **Wilson Area Law Formula**

Recall we derive the Area Law Formula for the Abelian case, in Section 3. Now we will show how we can obtain the Area Law Formula for the non-Abelian case, using our definition of the Yang-Mills Path integral, given in Definition 7.0.

From the proof of Theorem 7.5, we see that Expression (7.11) converges pointwise to 1 as \( \kappa \to 0 \). We will now show that as \( \kappa \to \infty \), the Yang-Mills Path integral actually converges to a Gaussian type of integral, similar in form to Expression (3.1) in Section 3. To prove this, we will show that \( (\cdot, \pi_{w}) \to 0 \) as a linear functional on \( B(\mathbb{R}^4; \mathcal{D}) \otimes \mathfrak{g} \), hence implying \( \mathfrak{y}_C \to 1 \).
Refer to Equation (7.4). We will show that as $\kappa \to \infty$,

\[ Y_1^\kappa \left( \{ A_{i,\alpha} \}_{i,\alpha} \right) := \frac{1}{2} \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \sum_{\alpha < \beta} c^{\alpha,\beta}_{\gamma} \left( A_{i,\gamma} \otimes A_{i,\alpha} \otimes A_{j,\beta}, \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \xi^\kappa_{ij}(w) \otimes E^\gamma \right) \otimes \pi_w^{\otimes 3} \right) \]

\[ Y_2^\kappa \left( \{ A_{i,\alpha} \}_{i,\alpha} \right) := \frac{1}{2} \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \sum_{\alpha < \beta} c^{\alpha,\beta}_{\gamma} \left( A_{i,\alpha} \otimes A_{j,\beta} \otimes A_{k,\gamma}, \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \pi_w^{\otimes 2} \otimes (\xi^\kappa_{ij}(w) \otimes E^\gamma) \right) \right) \]

and

\[ Y_3^\kappa \left( \{ A_{i,\alpha} \}_{i,\alpha} \right) := \frac{1}{2} \sum_{\gamma} \sum_{1 \leq i < j \leq 3} \sum_{\alpha < \beta} \frac{1}{2} c^{\alpha,\beta}_{\gamma} c^{\alpha,\beta}_{\gamma} \left[ \left( A_{i,\alpha} \otimes A_{j,\beta} \otimes A_{i,\alpha} \otimes A_{j,\beta}, \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \pi_w^{\otimes 4} \right) \right) \right] \]

all tend to 0.

**Lemma 8.1** For $i = 1, 2, 3$, $Y_i^\kappa \to 0$ as $\kappa \to \infty$.

**Proof.** To show that $Y_1^\kappa$ and $Y_2^\kappa$ goes to 0, it suffices to show that

\[ \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \cdot, \sum_{1 \leq i < j \leq 3} \left( \xi^\kappa_{ij}(w) \otimes E^\gamma \right) \otimes \pi_w^{\otimes 2} \right) \]

all go to 0. From Equations (7.3) and (7.12), we have

\[ \left| \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \cdot, \sum_{1 \leq i < j \leq 3} \left( \xi^\kappa_{ij}(w) \otimes E^\gamma \right) \otimes \pi_w^{\otimes 2} \right) \right|_{\dot{2}, \kappa} \]

\[ \leq \int_{w \in \mathbb{C}^4} d\lambda_4(w) \left| \cdot, \sum_{1 \leq i < j \leq 3} \left( \xi^\kappa_{ij}(w) \otimes E^\gamma \right) \right|_{\dot{2}, \kappa} \leq \int_{w \in \mathbb{C}^4} d\lambda_4 \frac{3 \sqrt{2 \pi}}{2 \kappa \sqrt[4]{2 \pi}} \to 0 \]

as $\kappa \to 0$. The other term is similar.

To show $Y_3^\kappa$ converges to 0, it suffices to show that $\int_{w \in \mathbb{C}^4} d\lambda_4(w) \left( \pi_w^{\otimes 4} \right)$ goes to 0 in $| \cdot |_{\dot{2}, \kappa}$ as $\kappa \to \infty$. Using Equation (7.12), we have

\[ \left| \int_{w \in \mathbb{C}^4} d\lambda_4(w) \pi_w^{\otimes 4} \right|_{\dot{2}, \kappa} \leq \int_{w \in \mathbb{C}^4} d\lambda_4 \pi_w^{\otimes 4} \left| \cdot \right|_{\dot{2}, \kappa} = \int_{w \in \mathbb{C}^4} d\lambda_4 \left( \frac{2}{2 \kappa \sqrt[4]{2 \pi}} \right)^4 = \frac{16}{(2 \pi)^2 \kappa^4} \to 0 \]

as $\kappa$ goes to 0.

---

**Definition 8.2** Recall we fix an orthonormal basis $\{ E^\alpha \}_{\alpha=1}^N$ for $g$. Define a set $S = \{ E^\alpha_{s,t} \}_{(s,t) \in S, \alpha=1,...,N}$, $S$ is possibly uncountable set. Note that $S$ is only a set, and we have $E^\alpha_{s,t} = E^\alpha$. We will write $f \otimes E^\alpha_{s,t}$ to mean $f \otimes E^\alpha \otimes E^\alpha_{s,t}$, where $f \otimes E^\alpha \in (B(\mathbb{R}^4; \mathcal{D} \otimes g))^*$ and $E^\alpha_{s,t} \in S$. 

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In the following, we will extend our real-valued inner product to be a $S$-valued inner product. Thus,
\[
\langle f \otimes E^\alpha_{s,t}, g \otimes E^\beta_{u,v} \rangle_{\alpha,\beta} := \langle f \otimes E^\alpha, g \otimes E^\beta \rangle_{0,0} \otimes E^\alpha_{s,t} \otimes E^\beta_{u,v}.
\]
Likewise, we will also write
\[
\langle b \otimes E^\alpha_{s,t}, f \otimes E^\beta_{u,v} \rangle_{\alpha,\beta} := \langle b \otimes E^\alpha, f \otimes E^\beta \rangle_{0,0} \otimes E^\alpha_{s,t} \otimes E^\beta_{u,v}
\]
for $b \in B(\mathbb{R}^4; \mathcal{D})$ and $f \in B(\mathbb{R}^4; \mathcal{D})^*$. Similar definition for $\langle \cdot, \cdot \rangle$.

**Theorem 8.3** Let $C$ be a closed curve in $\mathbb{R}^4$ and fix a surface $S$ such that $\partial S = C$. Define $\mu(A \otimes B) = A \cdot B$, for $A, B \in M_n(\mathbb{R})$. As $\kappa \to \infty$, we have
\[
E_{YM} [\mathcal{J}_S] \rightarrow \text{Tr} \exp \left[ \frac{1}{8} \int_S \rho_S \otimes \mu \left( \sum_{\alpha} \rho(E^\alpha) \otimes \rho(E^\alpha) \right) \right].
\]
Here, $\rho$ is some representation of $g$.

**Proof.** Observe that $Y^\kappa = \exp \left[ -\frac{1}{2} (Y^1_1 + Y^2_2 + Y^3_3) \right]$ and by Equation (7.9), $\{Y^\kappa\}_{\kappa > 0}$ is actually $L^p$ integrable for some $p > 1$. Hence it is uniformly integrable and because of Lemma 3.1, $E \mathbb{E}(Y^\kappa)$ converges to 1. Hence it suffices to show that $E \mathbb{E}(\mathcal{J}^\kappa_S)$ converges to the desired limit.

Now for $p > 1$, $(1 + x)^p \leq 2^{p-1}(1 + |x|^p)$ by Jensen’s Inequality. Because $Y^\kappa \to 1$ pointwise and by Equation (7.9), we have that $E \mathbb{E}((1 - Y^\kappa)^p) \to 0$ by uniform integrability. For some $p > 1$, using Holder’s Inequality,
\[
E \mathbb{E}(\mathcal{J}^\kappa_S (1 - Y^\kappa)) \leq E \mathbb{E}((\mathcal{J}^\kappa_S)^{1/q}) E \mathbb{E}((1 - Y^\kappa)^{p/q}) \to 0
\]
as $\kappa \to \infty$ if $E \mathbb{E}(\mathcal{J}^\kappa_S)$ converges. The rest of the proof is now focus on proving this convergence, for $q = 1$ without any loss of generality.

Write $E^\alpha_{s,t} \equiv \rho(E^\alpha_{s,t})$ and
\[
V^\kappa_1 \left( \{A_{i,\alpha}\}_{i,\alpha} \right) = \frac{\kappa}{4} \int_{f^2} ds dt \sum_{0 \leq j < l \leq 3} |J^\kappa_{ij}(s,t)\sum_{\alpha} (A_{\alpha}, \xi^\kappa_{ij}(\kappa\sigma(s,t)/2) \otimes E^\alpha_{s,t})| \otimes E^\alpha_{s,t},
\]
\[
V^\kappa_2 \left( \{A_{i,\alpha}\}_{i,\alpha} \right) = \frac{\kappa}{4} \int_{f^2} ds dt \sum_{0 \leq j < l \leq 3} |J^\kappa_{ij}(s,t)\sum_{\alpha} \sum_{\beta} (u^{s,t}_{i,\alpha} E^\alpha_{s,t} (A_{\alpha}, \xi^\kappa_{ij}(\kappa\sigma(s,t)/2) \otimes E^\alpha_{s,t})| \otimes E^\alpha_{s,t},
\]
\[
V^\kappa_3 \left( \{A_{i,\alpha}\}_{i,\alpha} \right) = \frac{\kappa}{4} \int_{f^2} ds dt \sum_{1 \leq j < l \leq 3} |J^\kappa_{ij}(s,t)\sum_{\alpha} \sum_{\beta} (A_{i,\alpha} \otimes A_{j,\beta}, \pi^\alpha_{ij}(\kappa\sigma(s,t)/2) \otimes E^\gamma_{s,t})| \otimes E^\gamma_{s,t},
\]
where
\[
u_{s,t} = \mathcal{J} e^{P_{s,t}, \Sigma_{i,\alpha} A_{i,\alpha} u_0} := \mathcal{J} \exp \left[ \sum_{1 \leq j < l \leq 3} \sum_{\alpha} (A_{i,\alpha}, \int_f d\tau P^\kappa_{s,t}(\tau) \pi_{s,t}(\tau) \otimes dx^l \otimes E^\alpha_{s,t}) \right] u_0.
\]
Note that $\mathcal{J}^\kappa_S = \exp[V^\kappa_1 + V^\kappa_2 + V^\kappa_3]$. Apply Fubini’s Theorem,
\[
E \text{Tr} \mathcal{J} \left[ \exp[V^\kappa_1 + V^\kappa_2 + V^\kappa_3] \right] = \text{Tr} \mathcal{J} E \left[ \exp[V^\kappa_1 + V^\kappa_2 + V^\kappa_3] \right].
\]

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Since \( G \) is assumed to be compact, we have \( u_{s,t} = 1 + O(1/\kappa) \) from Equation (7.12). Thus, we have \( V_2^\kappa, V_3^\kappa \) all converge to 0 pointwise.

Apply Lemma B.3:

\[
\mathbb{E} \left[ e^{V_2^\kappa + V_3^\kappa} - e^{V_1^\kappa} \right] \leq \mathbb{E} \left[ |V_2^\kappa + V_3^\kappa| \exp \left[ 2|V_1^\kappa| + |V_2^\kappa| + |V_3^\kappa| \right] \right].
\]

Now, \( \exp \left[ 2|V_1^\kappa| + |V_2^\kappa| + |V_3^\kappa| \right] \) is less than or equal to Expression (7.10) for \( p = 4 \) and in the proof of Theorem 7.3 we showed that if \( \kappa \) is large enough, then Expression (7.10) is in fact integrable for any values of \( p \).

Thus, \( \{ |V_2^\kappa + V_3^\kappa| \exp \left[ 2|V_1^\kappa| + |V_2^\kappa| + |V_3^\kappa| \right] \}_{\kappa \geq 1} \)

is uniformly integrable and since \( V_2^\kappa, V_3^\kappa \) all converge to 0, we have \( \mathbb{E} \left[ e^{V_2^\kappa + V_3^\kappa} - e^{V_1^\kappa} \right] \) converge to 0 as \( \kappa \) goes to infinity.

Thus it remains to compute the limit of \( \text{Tr} \tilde{T} \mathbb{E} \left[ e^{V_1^\kappa} \right] \). Write

\[
F^\kappa = \sum_{\alpha} \sum_{0 \leq a < b \leq 3} \int_{I^2} dsdt |J_{ab}^\sigma|(s,t) E_{ab}(\kappa\sigma(s,t)/2) \otimes \rho(E_{ab}^\alpha).
\]

Now apply Lemma A.1:

\[
\mathbb{E} \left[ e^{V_1^\kappa} \right] = \exp \left[ \frac{\kappa^2}{32} \langle F^\kappa, F^\kappa \rangle_{\mathfrak{h},\kappa} \right]
\]

\[
\longrightarrow \exp \left[ \frac{1}{8} \sum_{0 \leq a < b \leq 3} \int_{I^2} dsdt |J_{ab}^\sigma|(s,t) \rho_2^{ab}(\sigma(s,t)) \otimes \left[ \sum_{\alpha} \rho(E^\alpha) \otimes \rho(E^\alpha) \right] (s,t) \right],
\]

using calculations from Section 3. Hence,

\[
\mathbb{E} \left[ \exp \left[ V_1^\kappa + V_2^\kappa + V_3^\kappa \right] \right] \rightarrow \exp \left[ \frac{1}{8} \sum_{0 \leq a < b \leq 3} \int_{I^2} dsdt |J_{ab}^\sigma|(s,t) \rho_2^{ab}(\sigma(s,t)) \otimes \left[ \sum_{\alpha} \rho(E^\alpha) \otimes \rho(E^\alpha) \right] (s,t) \right].
\]

Write \( E = \sum_{\alpha} \rho(E^\alpha) \otimes \rho(E^\alpha), g(s,t) = \frac{1}{8} \sum_{0 \leq a < b \leq 3} \rho_2^{ab}(\sigma(s,t)) |J_{ab}^\sigma|(s,t) \).

By definition of \( \tilde{T} \) (See Definition 5.5),

\[
\tilde{T} \exp \left[ \frac{1}{8} \sum_{0 \leq a < b \leq 3} \int_{I^2} dsdt |J_{ab}^\sigma|(s,t) \rho_2^{ab}(\sigma(s,t)) \otimes \sum_{\alpha} \rho(E^\alpha) \otimes \rho(E^\alpha)(s,t) \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{n!}{n} \int_{I^2n} \prod_{i=1}^{n} g(s_i, t_i)^{ds dt} \otimes (E(s, t))^\otimes^n
\]

\[
= \sum_{n=0}^{\infty} \frac{n}{n!} \int_{I^2n} \prod_{i=1}^{n} ds_i \int_{t_{i+1} > \cdots > t_n} \prod_{i=1}^{n} g(s_i, t_i)^{ds dt} \otimes \mu(E)(s_1, t_1) \cdots \mu(E)(s_n, t_n)
\]

\[
= \sum_{n=0}^{\infty} \frac{n!}{n} \prod_{i=1}^{n} g(s_i, t_i)^{ds dt} \otimes [\mu(E)]^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{8} \int_S \rho \otimes \mu \left( \sum_{\alpha} \rho(E^\alpha) \otimes \rho(E^\alpha) \right)^n = \exp \left[ \frac{1}{8} \int_S \rho \otimes \mu \left( \sum_{\alpha} \rho(E^\alpha) \otimes \rho(E^\alpha) \right) \right].
\]
Hence we have
\[
\mathbb{E}^\kappa_{YM}[\mathcal{D}_S] \to \text{Tr} \exp \left[ \frac{1}{8} \int_S \rho_S \otimes \mu \left( \sum_\alpha \rho(E^\alpha) \otimes \rho(E^\alpha) \right) \right]
\]
as \kappa \text{ goes to infinity.}

8.1 Some Important Examples

The Yang-Mills path integral is dependent on the parameter \( \kappa \). We will now take the limit as \( \kappa \) go to infinity, in the cases of \( G = SU(N) \) and \( G = SO(N) \).

Definition 8.4 (Special elements in \( \bigotimes^2 M_N(\mathbb{C}) \).) Define \( I, J, K \) in \( \bigotimes^2 M_N(\mathbb{C}) \),
\[
I_{ab}^{cd} = \delta_a^c \delta_b^d, \quad J_{ab}^{cd} := \delta_a^d \delta_b^c, \quad K_{ab}^{cd} = \delta_a^b \delta_c^d.
\]

\( K \) commutes with \( J \). Note that if \( I \) is the identity matrix in \( M_N(\mathbb{C}) \), then \( I = I \otimes I \). Furthermore, \( J \cdot J = I, K \cdot K = NK \) and \( K \cdot J = K = J \cdot K \).

Example 1 (\( SU(N) \))

Suppose our Lie group is \( G = SU(N) \). Considering its standard representation, one shows that
\[
\sum_{\alpha \in \sigma} E^\alpha \otimes E^\alpha = \frac{1}{N} I - J.
\]
Hence, \( \mu(\sum_\alpha E^\alpha \otimes E^\alpha) = \frac{1}{N} \mathbb{I} - N \mathbb{I} \), and
\[
\exp \left[ \frac{1}{8} \int_S \rho_S \cdot \mu \left( \sum_\alpha E^\alpha \otimes E^\alpha \right) \right] = \text{Tr} \exp \left( \frac{1}{8} \left( \frac{1}{N} - N \right) \int_S \rho_S \right) I = N \exp \left( \frac{1}{8} \left( \frac{1}{N} - N \right) \int_S \rho_S \right).
\]
Therefore,
\[
\lim_{\kappa \to \infty} \mathbb{E}^\kappa_{YM}[\mathcal{D}_S] = N \exp \left( \frac{1}{8} \left( \frac{1}{N} - N \right) \int_S \rho_S \right). \quad (8.1)
\]

Example 2 (\( SO(N) \))

Now consider \( G = SO(N) \). Considering its standard representation, then
\[
\sum_{\alpha \in \sigma} E^\alpha \otimes E^\alpha = (K - J)/2.
\]
Hence, \( \mu(\sum_\alpha E^\alpha \otimes E^\alpha) = [1 - N] \mathbb{I}/2 \), and
\[
\exp \left[ \frac{1}{8} \int_S \rho_S \cdot \mu \left( \sum_\alpha E^\alpha \otimes E^\alpha \right) \right] = \text{Tr} \exp \left( \frac{1}{16} (1 - N) \int_S \rho_S \right) \mathbb{I} = N \exp \left( \frac{1}{16} (1 - N) \int_S \rho_S \right).
\]
Therefore,
\[
\lim_{\kappa \to \infty} \mathbb{E}^\kappa_{YM}[\mathcal{D}_S] = N \exp \left( \frac{1}{16} (1 - N) \int_S \rho_S \right). \quad (8.2)
\]
9 Applications

The main application of computing the Wilson Loop observable using Yang-Mills measure is to derive the Wilson Area Law formula. A quark is never observed in isolation due to quark confinement. A quark and an antiquark are bounded together in a meson via the strong force. Unlike the Coulomb potential, which decays inversely proportional to the distance, the strong force actually gets stronger as the separation between the quarks increases. It is confirmed experimentally that the potential varies proportionally to the distance between quarks.

The Wilson Loop observable using the Yang-Mills action can be used to explain why this potential is linear. However, to compute the Wilson Loop observable is highly nontrivial and difficult. Note that perturbative methods using Feynman diagrams cannot be used to compute it, lest derive the area formula.

Another important application for this is to explain superconductivity, by replacing quarks with magnetic monopoles. Due to our limited understanding on this subject, we prefer to refer the interested reader to [Nai05] for a more concise description on this subject matter.

9.1 Quark Confinement

Let $C$ be a rectangular contour with spatial length $R$ and timelike length $T$. Let $S$ be a minimal surface bounded by $C$, which in this case will be a rectangular region of area $RT$. Then, we will write $W(R,T;\rho) := \lim_{\kappa \to \infty} Y(\mathbb{R}^4, \kappa; S, \rho)$. From Theorem 8.3, we have

$$W(R,T;\rho) = \text{Tr} \exp \left[ \frac{RT}{8} \rho \left( \sum_{\alpha} \rho(E_{\alpha}) \otimes \rho(E_{\alpha}) \right) \right].$$

A meson is made up of a quark and antiquark. Now, these 2 quarks go in opposite direction along the spatial length $R$ and after time $T$, are attracted to each other along the spatial length $R$, hence tracing a rectangular contour $C$ described in the previous paragraph. We want to calculate the potential energy between the quark and antiquark, as a function of distance $R$. Using the Wilson Loop observable computed along the rectangular $C$, the potential is given by

$$V(R) = -\lim_{T \to \infty} \log \frac{W(R,T+1;\rho)}{W(R,T;\rho)}.$$  

This formula can be derived from Equation (19.11) in [Nai05].

When $G$ is Abelian, from Equation (8.2), $V(R) = R/8$. In the case of the standard representation of $SU(N)$, we can see from Equation (8.1) that

$$V(R) = \frac{R}{8} \left( N - \frac{1}{N} \right),$$

and in the case of the standard representation of $SO(N)$, we see that

$$V(R) = \frac{R}{16} (N - 1)$$

from Equation (8.2). Compare with Equation (19.6) in [Nai05]. Thus, we see that quark confinement implies that the potential energy between the quarks is linear in nature.
9.2 Superconductivity

In electromagnetism, we have the electric field \( E \equiv \sum_{i=1}^{3} E_i dx^i \land dx^j \) and the magnetic field \( B = \sum_{i=1}^{3} \epsilon_{ijk} B_k dx^i \land dx^j \). Now, we are going to interchange \( E \) with the \( B \) field, i.e. we define \( \bar{E} \) and \( \bar{B} \) as

\[
\bar{E} = \sum_{i=1}^{3} \epsilon_{ijk} E_k dx^i \land dx^j \quad \text{and} \quad \bar{B} = \sum_{i=1}^{3} B_i dx^0 \land dx^i.
\]

This electric-magnetic duality transformation motivates the next definition.

**Definition 9.1** Let \( S \) be a bounded and connected surface with boundary \( \partial S \), a simple connected closed curve. Recall \( S \) defines

\[
F^\kappa = \sum_{\alpha} \sum_{0 \leq a < b \leq 3} \int_{I^2} ds dt |J_{ab}^\kappa(s,t) \xi_{ab}^\kappa(\kappa \sigma(s,t)/2) \otimes E^\alpha_{s,t}.
\]

We define a vortex of \( S \) in \( \mathbb{R}^4 \), which is a surface \( \bar{S} \) defined by

\[
\bar{F}^\kappa = \sum_{\alpha} \sum_{0 \leq a,b,c,d \leq 3} \int_{I^2} ds dt \epsilon_{abcd} |J_{cd}^\kappa(s,t) \xi_{ab}^\kappa(\kappa \sigma(s,t)/2) \otimes E^\alpha_{s,t}.
\]

Here, \( \epsilon_{abcd} = (-1)^{|\sigma(abcd)|} \) if \( a, b, c, d \) all distinct for a permutation \( \sigma(abcd) \) in \( S_4 \), \( |\sigma(abcd)| \) counts the number of transpositions in the permutation; 0 otherwise.

Note that \( F^\kappa \) and \( \bar{F}^\kappa \) are \( g \)-valued evaluation differential forms and \( \bar{F}^\kappa \) should be interpreted as the dual of \( F^\kappa \). Furthermore, \( |F^\kappa|_{\partial,\kappa} = |\bar{F}^\kappa|_{\partial,\kappa} \), so both \( S \) and \( \bar{S} \) have the same area and we can write

\[
F^\kappa = (\cos \theta) \bar{F}^\kappa + (\sin \theta) \gamma,
\]

whereby \( |\gamma|_{\partial,\kappa} = 1 \) and

\[
\cos \theta = \frac{(F^\kappa, \bar{F}^\kappa)_{\partial,\kappa}}{|F^\kappa|_{\partial,\kappa}^2}, \quad \sin \theta = \frac{1}{|F^\kappa|_{\partial,\kappa} \sqrt{|F^\kappa|_{\partial,\kappa}^4 - (F^\kappa, \bar{F}^\kappa)_{\partial,\kappa}^2}}.
\]

**Definition 9.2** Let \( \text{lk}(S, \bar{S}) \) denote the linking number between a surface \( S \) and its vortex, which counts the total number of points the vortex intersect the surface \( S \).

It is straightforward to show that

\[
\left( \frac{\kappa}{2} \right)^4 (F^\kappa, \bar{F}^\kappa)_{\partial,\kappa} \to (2\pi)^2 \text{lk}(S, \bar{S}) \sum_{\alpha} E^\alpha \otimes E^\alpha.
\]

So, we have a dual statement to Theorem 8.3.

**Theorem 9.3** Let \( C \) be a closed curve in \( \mathbb{R}^4 \) and fix a surface \( S \) such that \( \partial S = C \). Define its vortex \( \bar{S} \), with \( \partial \bar{S} = \bar{C} \). As \( \kappa \to \infty \), we have

\[
E_{YM}^\kappa [\bar{\gamma}^\kappa_S] \to \text{Tr} \exp \left[ \frac{1}{8} \int_S \rho_S \otimes \mu \left( \sum_{\alpha} E^\alpha \otimes E^\alpha \right) \right].
\]

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Proof. We just note that from Equation (9.2), \( \kappa^2 \langle F^\kappa, \bar{F}^\kappa \rangle_{\delta, \kappa} \to 0 \). Then from Equation (9.1), it follows that the limit coincides with the limit of \( E^\gamma_M [J_S^\kappa] \). This completes the proof. 

In the case of quark confinement, the force between the pair of quarks is electric in nature. In the case of a type II superconductor, a similar phenomenon is present, except that we focus on the vortex, which is of magnetic in nature. Here, we have a magnetic monopole and an antimonopole pair which are connected via a vortex. From Theorem 9.3, we have an area law for the vortex, which implies that the interaction energy between the magnetic monopole and antimonopole pair increases linearly with their separation distance.

Thus, for a closed curve \( C \), the Wilson Loop observable along \( C \) is interpreted as electric confinement; for the boundary of a vortex \( \bar{C} \), the Wilson Loop observable along \( \bar{C} \) is interpreted as magnetic confinement. The former is due to magnetic superconductivity, the later is due to electric superconductivity.

A Useful Calculations

A.1 Gaussian Integrals

The following two lemmas are trivial, but are central to our calculations.

Lemma A.1 For \( \theta > 0 \),

\[
\frac{\theta}{\sqrt{2\pi}} \int e^{\alpha x} e^{-\theta^2 x^2/2} dx = e^{\alpha^2/2\theta^2}.
\]

Lemma A.2 Let \( A \) be a \( n \times n \) matrix and \( x = (x_1, \ldots, x_n)^T \). If \( \| A^* A \| < 1/2 \), then

\[
\left( \frac{1}{\sqrt{2\pi}} \right)^n \int \exp \left[ \langle A x, A x \rangle \right] e^{-|x|^2/2} \prod_{i=1}^n dx_i = \left( \frac{1}{\det[1 - 2A^* A]} \right)^{1/2}.
\]

A.2 Surface Integrals

Fix a closed curve \( C \) and let \( S \) be a surface embedded in \( \mathbb{R}^4 \) such that \( \partial S = C \). Let \( \sigma \equiv (\sigma_0, \sigma_1, \sigma_2, \sigma_3) : [0,1]^2 \to M \) be a parametrization of a surface \( S \). Here, \( \sigma' = \partial \sigma/\partial s \) and \( \dot{\sigma} = \partial \sigma/\partial t \). Let \( x = (x_0, x_1, x_2, x_3) \).

Definition A.3 Let \( \sigma : [0,1]^2 \equiv I^2 \to \mathbb{R}^4 \) be a parametrization of a surface \( S \subset \mathbb{R}^4 \). Define Jacobian matrices,

\[
J^\sigma_{ij}(s, t) = \begin{pmatrix} \sigma_i'(s, t) & \dot{\sigma}_i(s, t) \\ \sigma_j'(s, t) & \dot{\sigma}_j(s, t) \end{pmatrix},
\]

and write \( |J^\sigma_{ij}| = |\det J^\sigma_{ij}| \) and \( W^{cd}_{ab} := J_{cd} J^{-1}_{ab} \). \( a, b, c, d \) all distinct. Note that \( W^{cd}_{ab} = (W^{cd}_{ab})^{-1} \).

For \( a, b, c, d \) all distinct, define \( \rho_S^{ab} : S \to \mathbb{R} \) by

\[
\rho_S^{ab} = \frac{1}{\sqrt{\det \left[ 1 + W^{cd}_{ab} W^{cd}_{ab} \right]}}
\]
Lemma A.4 Assume that in a small neighborhood $U \subset [0, 1]^2$, $\sigma'_a, \dot{\sigma}_a, \sigma'_b, \dot{\sigma}_b \neq 0$ and let $\sigma(p) = x, p \in U$. We have for distinct $a, b, c, d$,

$$\lim_{\kappa \to \infty} \int_U \frac{\kappa^2}{4} e^{-\kappa^2|x|} (\sigma'_a \dot{\sigma}_b - \sigma'_b \dot{\sigma}_a)(s, t) ds dt = \frac{2\pi}{\sqrt{\det \begin{bmatrix} 1 + W_{ab}^{cd}T W_{ab}^{cd}(p) \end{bmatrix}}}.$$

Proof. Write $s = s_0, t = t_0$ and $J_{ab} \equiv J_{ab}^\sigma(p)$. Let $\sigma(s_0, t_0) = x_i$. Using Taylor’s theorem,

$$\sigma_i(s, t) = x_i + \mathbf{a}'_i(p) + \dot{\mathbf{a}}_i(p) + O((s_0 + |u|)^2).$$

Let

$$(s, t)^T = \left( \begin{array}{c} \xi \\ \eta \end{array} \right), (u, v)^T = \left( \begin{array}{c} u \\ v \end{array} \right).$$

Use a transformation, i.e. $u = \mathbf{a}'_a(p) + \dot{\mathbf{a}}_a(p), v = \mathbf{a}'_b(p) + \dot{\mathbf{a}}_b(p)$. And let $0 \in V$ be the range of $U$ under this transformation. Then, we have

$$\int_U \frac{\kappa^2}{4} e^{-\kappa^2|x|} (\sigma'_a \dot{\sigma}_b - \sigma'_b \dot{\sigma}_a)(s, t) ds dt$$

$$= \int_U \frac{\kappa^2}{4} e^{-\kappa^2|\mathbf{a}'(\mathbf{u})|^2} |J_{ab}(\mathbf{u})|^2 + O((|\mathbf{u}| + |v|)^3)) ds dt$$

$$= \int_U \frac{\kappa^2}{4} e^{-\kappa^2|\mathbf{a}'(\mathbf{u})|^2} |J_{ab}(\mathbf{u})|^2 + O((|\mathbf{u}| + |v|)^3)) ds dt$$

$$\to_{\kappa \to \infty} \frac{2\pi}{\sqrt{\det \begin{bmatrix} 1 + J_{ab}^{T_1}J_{cd}^{T_2}J_{ab}^{T_1} \end{bmatrix}}} = \frac{2\pi|J_{ab}|}{\sqrt{\det \begin{bmatrix} J_{ab}^{T_1}J_{ab} + J_{cd}^{T_2}J_{cd} \end{bmatrix}}}.$$

Corollary A.5 Let $\sigma$ be any parametrization for a surface $S \subset \mathbb{R}^4$. As $\kappa \to \infty$, we have

$$\sum_{0 \leq a < b \leq 3} \int_{f_2 \times f_2} ds dt d\tilde{v} e^{-\kappa^2|\sigma(s, t) - C(s, t)|^2/8} |J_{ab}^\sigma|(s, t)|J_{ab}^\sigma|(\tilde{s}, \tilde{t})$$

$$\to 2\pi \sum_{0 \leq a < b \leq 3} \int_{f_2} \rho_{ab}^S(\sigma(s, t)) |J_{ab}^\sigma|(s, t) ds dt := 2\pi \int_S \rho_S,$$

which is independent of the parametrization $\sigma$ used.

Proof. This follows from Lemma A.4 details omitted.
B Some Inequalities

Lemma B.1 For \( 0 < x < \epsilon \), \( \frac{1}{1 - x} \leq e^{cx} \) if we choose \( c = 1/(1 - \epsilon) \).

Proof. Let \( f(x) = e^{cx}(1 - x) \) such that for \( 0 < x < \epsilon \), \( f(x) \geq 1 \). Therefore,

\[
  f'(x) = ce^{cx}(1 - x) - e^{cx} = (c - 1)e^{cx} - cxe^{cx} > 0 \Rightarrow (c - 1) - \epsilon > 0 \Rightarrow c > 1/(1 - \epsilon).
\]

Lemma B.2 Let \( A^*A \) be a \( n \times n \) matrix such that \( 2 \| A^*A \| < \epsilon < 1 \). Then we have

\[
  \frac{1}{\det(1 - 2A^*A)} \leq e^{2\epsilon \text{Tr}[A^*A]}
\]

whereby \( c = (1 - \epsilon)^{-1} \).

Proof. Let \( \epsilon > \lambda_1 \geq \ldots \geq \lambda_n \geq 0 \) be the eigenvalues of \( A^*A \). Then, we have from Lemma B.1

\[
  \frac{1}{\det(1 - 2A^*A)} = \prod_{i=1}^{n} \frac{1}{1 - 2\lambda_i} \leq \prod_{i=1}^{n} e^{2\epsilon \lambda_i} = \exp \left( 2c \sum_{i=1}^{n} \lambda_i \right) = \exp [2c \text{Tr}[A^*A]]
\]

Lemma B.3 Let \( A, B \) be \( N \times N \) matrices. We have

\[
  \left| A^{\otimes n} - B^{\otimes n} \right| \leq n|A - B|[|A| + |B|]^{n-1}.
\]

Proof. Now

\[
  A^{\otimes n} - B^{\otimes n} = (A - B) \otimes A^{\otimes n-1} + B \otimes (A - B) \otimes A^{\otimes n-2} + \cdots + B \otimes (A - B) \otimes A + B^{\otimes n-1} \otimes (A - B).
\]

Hence,

\[
  \left| A^{\otimes n} - B^{\otimes n} \right| \leq |A - B| \left[ |A|^{n-1} + |A|^{n-2}|B| + \cdots + |B|^{n-1} \right] \leq n|A - B|[|A| + |B|]^{n-1}.
\]

Lemma B.4 Let \( F, G : I^2 \to M_{N \times N}(\mathbb{C}) \) and write

\[
  e^{\int_{I^2} F(s,t)dsdt} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_{I^2} F(s,t)dsdt \right)^{\otimes n}, \quad e^{\int_{I^2} G(s,t)dsdt} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_{I^2} G(s,t)dsdt \right)^{\otimes n}.
\]

Then,

\[
  \left| e^{\int_{I^2} F(s,t)dsdt} - e^{\int_{I^2} G(s,t)dsdt} \right| \leq \left| \int_{I^2} F(s,t)dsdt - \int_{I^2} G(s,t)dsdt \right| \exp \left[ \left| \int_{I^2} F(s,t)dsdt \right| + \left| \int_{I^2} G(s,t)dsdt \right| \right].
\]
Proof. Apply Lemma B.3

\[ |\int_{I^2} F(s,t)dsdt - \int_{I^2} G(s,t)dsdt| \leq \sum_{n=0}^{\infty} \frac{n!}{n!} \left| \int_{I^2} F(s,t)dsdt - \int_{I^2} G(s,t)dsdt \right| \left( \left| \int_{I^2} F(s,t)dsdt \right| + \left| \int_{I^2} G(s,t)dsdt \right| \right)^{n-1} \]

\[ \leq \sum_{n=0}^{\infty} \frac{1}{(n-1)!} \left| \int_{I^2} F(s,t)dsdt - \int_{I^2} G(s,t)dsdt \right| \exp \left[ \left| \int_{I^2} F(s,t)dsdt \right| + \left| \int_{I^2} G(s,t)dsdt \right| \right]. \]

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