On an application of multidimensional arrays

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Abstract

This article discusses some difficulties in the implementation of combinatorial algorithms associated with the choice of all elements with certain properties among the elements of a set with great cardinality. The problem has been resolved by using multidimensional arrays. Illustration of the method is a solution of the problem of obtaining one representative from each equivalence class with respect to the described in the article equivalence relation in the set of all \( m \sim n \) binary matrices. This equivalence relation has an application in the mathematical modeling in the textile industry.

Keywords: binary matrix; equivalence relation; factor-set; cardinality; multidimensional array

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1 Introduction and task formulation

The following problem often occurs in computer science:

Problem 1.1. Let \( M \) be a finite set and let \( \sim \) be an equivalence relation in \( M \). Describe and implement an algorithm that receives exactly one representative from each equivalence class with respect to \( \sim \).

As a consequence of this problem follows the combinatorial problem of finding the cardinality of the factor set \( \tilde{M} = M/\sim \) consisting of all equivalence classes of \( M \) with respect of \( \sim \).

We assume that for every \( x \in M \), there is a procedure \( K(x) \) which receives all elements of \( M \), which are equivalent to \( x \).

Since \( M \) is a finite set, then there exists bijective mapping

\[
 b : \leftrightarrow \{1, 2, \ldots, |M|\} ,
\]

which will call numbering function. Thus, each element of \( M \) uniquely corresponds to an element of Boolean array \( H[\cdot] \) with size equal to the cardinality \(|M|\) of the set \( M \). Moreover, the element \( x \in M \) is selected if \( H[b(x)] = 1 \) and \( x \) is not selected if \( H[b(x)] = 0 \).

The next algorithm is a modification of the well-known method, known as "Sieve of Eratosthenes" [Reingold, Nievergeld and Deo (1977); Yordzhev and Markovska (2007)] solves Problem 1.1.

Algorithm 1.2. Receives exactly one representative of each equivalence class of the factor-set \( \tilde{M} = M/\sim \).

Input: Finite set \( M \)
Output: Set \( N \subseteq M \)

1. \( N := \emptyset \);

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2. Declare a Boolean array \( H[\ ] \) with size equal to the cardinality \(|M|\) of the set \( M \) and put \( H[b(x)] := 0 \) for all \( x \in M \);

3. For every \( x \in M \) such that \( H[b(x)] = 0 \) do
   
   \[
   \begin{aligned}
   &\text{Begin of loop 1} \\
   &N := N \cup \{x\}; \\
   &H[b(x)] := 1; \\
   &\text{Using the procedure } K(x) \text{ obtain the set } P_x = \{y \in M \mid y \sim x\}; \\
   &\text{For every } y \in P_x \text{ obtained in step 6 do} \\
   &\quad \text{Begin of loop 2} \\
   &\quad H[b(y)] := 1; \\
   &\quad \text{End of loop 2} \\
   &\text{End of loop 1}
   \end{aligned}
   \]

4. \( \text{End of the algorithm.} \)

Algorithm 1.2 has a number of disadvantages, the main of which is that it is practically inapplicable for programs when a sufficiently great number of elements is present in the base set \( M \). This limitation comes from the maximum integer, which can be used in the corresponding programming environment. For example, by standard in the C++ language the biggest number of the type \texttt{unsigned long int} is equal to \(2^{32} - 1\), which in a number of cases is insufficient for the previously defined array \( H[\ ]\) to be completely addressed. The purpose of this article is to avoid this problem by using a multidimensional Boolean array, the elements of which have a one-to-one correspondence to the elements of the base set, with a much smaller range of the indices. There are many publications related to multidimensional arrays, for example [Mishra (2014)], but they are not used for our specific goals and objectives. Another solution to the problem is the use of dynamic data structures or other special programming techniques [Collins (2002); Sutter (2002); Tan, Steeb and Hardy (2001)] but it is not the subject of consideration in this article.

**Binary (or Boolean, or \((0,1)\)-matrix)** is a matrix whose elements are equal to 0 or 1. Let \( B_{m \times n} \) be the set of all \( m \times n \) binary matrices. It is well known that

\[
|B_{m \times n}| = 2^{mn}
\]  \hspace{1cm} (1.1)

In this work, we will consider and solve the following special case of Problem 1.1:

**Problem 1.3.** Let \( B_{m \times n} \) be the set of all \( m \times n \) binary matrices and let \( X, Y \in B_{m \times n} \). We define an equivalence relation \( \rho \) as follows: \( X \rho Y \) if and only if we can obtain \( X \) from \( Y \) by a sequential moving of the last row or column to the first place. Find the cardinality \(|B_{m \times n}/\rho|\) of the factor-set \( \bar{M} = B_{m \times n}/\rho \) and receive a single representative of each equivalence class.

The proof that \( \rho \) is an equivalence relation is trivial and we will omit it here.

The equivalence classes of \( B_{m \times n} \) by the equivalence relation \( \rho \) are a particular kind of double coset [Bogopolski (2008); Curtis and Rainer (1962); De Vos (2010)]. They make use of substitutions group theory and linear representation of finite group theory [Curtis and Rainer (1962); De Vos (2010)].

When \( m = n \), the elements of the factor-set \( \bar{M} = B_{n \times n}/\rho \) put carry into practice in the textile technology [Borzunov (1983); Yordzhev and Kostadinova (2012)].

In [Yordzhev (2005)] an algorithm is shown, which utilizes theoretical graphical methods for finding the factor set \( \bar{S} = S_n/\rho \), where \( S_n \subset B_{n \times n} \) is a set of all permutation matrices, i.e. binary matrices having exactly one 1 on each row and each column. In [Yordzhev (2014)] we extended this problem in the case when \( \rho \) is an arbitrary permutation.
2 Description of an algorithm with the use of a multidimensional array

Theorem 2.1. Let us denote by $\mathcal{P}_n$ the set

$$\mathcal{P}_n = \{0, 1, \ldots, 2^n - 1\}$$

(2.1)

Then a one-to-one correspondence (bijection) between the elements of the Cartesian product $\mathcal{P}_n^m = \mathcal{P}_n \times \mathcal{P}_n \times \cdots \times \mathcal{P}_n$ and the elements of the set $\mathcal{B}_{m \times n}$ of all $m \times n$ binary matrices exists.

Proof. We consider the mapping $\alpha : \mathcal{P}_n^m \to \mathcal{B}_{m \times n}$, defined in the following way: If $\pi \in \mathcal{P}_n^m$ and $\pi = \langle p_1, p_2, \ldots, p_m \rangle$ then let us denote by $z_i$, $i = 1, 2, \ldots, m$, the representation of the integer $p_i$ in a binary notation, and if less than $n$ digits 0 or 1 are necessary, we fill $z_i$ from the left with insignificant zeros, so that $z_i$ will be written with exactly $n$ digits. Since by definition, $p_i \in \mathcal{P}_n$, i.e. $0 \leq p_i \leq 2^n - 1$, this will always be possible. Then we form an $m \times n$ binary matrix, so that the $i$-th row is $z_i$, $i = 1, 2, \ldots, m$. Apparently this is a correctly defined mapping of $\mathcal{P}_n^m$ to $\mathcal{B}_{m \times n}$. It is clear that for different $m$-tuples from $\mathcal{P}_n^m$ with the help of $\alpha$ we will obtain different matrices from $\mathcal{B}_{m \times n}$, i.e. $\alpha$ is an injection. Conversely, rows of each binary matrix can be considered as natural numbers, written in binary system by using exactly $n$ digits 0 or 1, eventually with insignificant zeros in the beginning, that is, these numbers belong to the set $\mathcal{P}_n = \{0, 1, \ldots, 2^n - 1\}$. Therefore each $m \times n$ binary matrix corresponds to an $m$-tuple of numbers $\langle p_1, p_2, \ldots, p_m \rangle \in \mathcal{P}_n^m$, that is, $\alpha$ is a surjection. Hence $\alpha$ is a bijection. \hfill \Box

It is easy to see the validity of the following statement, which in fact shows the meaning of our considerations.

Proposition 2.1. Let us denote by $\mu$ the maximum integer, which we use when coding the elements of the set $\mathcal{B}_{m \times n}$, by means of the bijection, defined in Theorem 2.1. Then, for sufficiently great $m$ and $n$, the following is valid:

$$\mu = \max (2^n - 1, m) \ll |\mathcal{B}_{m \times n}| = 2^{mn}$$

(2.2)

Proof. Trivial. \hfill \Box

Let $a$ and $b$ be integers, $b \neq 0$. With $a/b$ we will denote the operation "integer division" of $a$ by $b$, i.e. if the division has a remainder, then the fractional part is cut, and with $a \% b$ we will denote the remainder when dividing $a$ by $b$. In other words, if $\frac{a}{b} = p + \frac{q}{b}$, where $p$ and $q$ are integers, $0 \leq q < b$ then by definition $a/b = p$, $a \% b = q$.

We consider the function

$$\xi(a) = (a \% 2) 2^{n-1} + a/2,$$

(2.3)

where $\%$ and $/$ are the defined in the above operations.

The author of this paper is not familiar with an existing a general formula expressed as a function of $m$ and $n$ for finding $|\mathcal{B}_{m \times n}|$. The goal of this paper is to describe an effective algorithm for finding the number of elements of the factor set $\bar{M} = \mathcal{B}_{m \times n}$, as well as finding a single representative of each equivalence class. Here we will describe an algorithm, which overcomes some difficulties, which would inevitably arise with sufficiently great $m$ and $n$ if we apply the classical algorithm (Algorithm 1.2). The main difficulty arises from the great number of elements of $\bar{M} = \mathcal{B}_{m \times n}$ with comparatively small integers $m$ and $n$, according to formula (1.1).

For undefined notions and definitions, we refer to [Aigner (1979); Sachkov and. Tarakanov (2002)].
Definition 2.1. Let $\alpha$ be the defined in the proof of Theorem 2.1 bijection and let the functions $f_r, f_c : \mathcal{P}_m \rightarrow \mathcal{P}_m$ be defined such that for every $\pi = \langle p_1, p_2, \ldots, p_m \rangle \in \mathcal{P}_m$

\begin{align}
  f_r(\pi) &= \langle p_m, p_1, p_2, \ldots, p_{m-1} \rangle \quad (2.4) \\
  f_c(\pi) &= \langle \xi(p_1), \xi(p_2), \ldots, \xi(p_m) \rangle, \quad (2.5)
\end{align}

where the function $\xi(\alpha)$ is the defined with (2.3).

Theorem 2.2. Let $A \in B_{m \times n}$ be an arbitrary $m \times n$ binary matrix and let $\alpha$ be the defined in the proof of Theorem 2.1 bijection. Let us to get the matrices

\begin{align}
  B &= \alpha \left( f_r \left( \alpha^{-1}(A) \right) \right) \quad (2.6) \\
  C &= \alpha \left( f_c \left( \alpha^{-1}(A) \right) \right) \quad (2.7)
\end{align}

Then $B$ is obtained from $A$ by moving the last row to the first place, and $C$ is obtained from $A$ by moving the last column to the first place (respectively the first row or column becomes the second, the second becomes the third respectively etc.).

Proof. Let $\pi = \langle p_1, p_2, \ldots, p_m \rangle = \alpha^{-1}(A) \in \mathcal{P}_m$. Then the integer $p_i$, $0 \leq p_i \leq 2^n - 1$, $i = 1, 2, \ldots, m$ will correspond to the $i$-th row of the matrix $A$. Then obviously, the matrix $B = \alpha(f_r(< p_1, p_2, \ldots, p_m >)) = \alpha(< p_m, p_1, p_2, \ldots, p_{m-1} >)$ is obtained from $A$ by moving the last row in the place of the first one, and moving the remaining rows one row below.

Let $p_i \in \mathcal{P}_n = \{0, 1, \ldots, 2^n - 1\}$, $i = 1, 2, \ldots, m$. Then $d_i = p_i \% 2$ gives the last digit of the binary notation of the integer $p_i$. If $p_i$ is written in binary notation with precisely $n$ digits, optionally with insignificant zeros in the beginning, then by applying integer division of $p_i$ by 2, we practically remove the last digit $d_i$ and we move it to the first position, in case we multiply by $2^{n-1}$ and add it to $p_i/2$. This is, by definition, how the function $\xi(p_i)$ works. Hence, the $m \times n$ binary matrix $C = \alpha(f_c(< p_1, p_2, \ldots, p_m >)) = \alpha(< \xi(p_1), \xi(p_2), \ldots, \xi(p_m) >)$ is obtained from the matrix $A$ by moving the last column to the first position, and all the other columns are moved one column to the right. \hfill \Box

From the definitions of the functions $f_r$, according to (2.4) and $f_c$, according to (2.5) it is easy to verify the validity of the following

Proposition 2.2. If by definition

\begin{align}
  f^0_r(\pi) &= f^0_c(\pi) = \pi \quad (2.8) \\
  f^k_r(\pi) &= f_r \left( f^{k-1}_r(\pi) \right) \quad (2.9) \\
  f^k_c(\pi) &= f_c \left( f^{k-1}_c(\pi) \right) , \quad (2.10)
\end{align}

where $\pi \in \mathcal{P}_m$ and $k$ is a positive integer, then

\begin{align}
  f^m_r(\pi) &= \pi \quad (2.11) \\
  f^m_c(\pi) &= \pi. \quad (2.12)
\end{align}

Proof. Trivial. \hfill \Box
Algorithm 2.3. Receives exactly one representative of each equivalence class of the factor-set $\tilde{M} = M_\rho$, and calculates the cardinality of the factor set $M = M_\rho$ when $m$ and $n$ are given.

1. We declare the $m$-dimensional Boolean arrays $W_1$ and $W_2$ which we will be indexed by using the elements of the set $P_m$, i.e. $W_1[p_1, p_2, \ldots, p_m] \in P_m$. We proceed analogically with the array $W_2$.

2. Initially we take all elements of $W_1$ and $W_2$ to be 0. In $W_1$ we will remember all elements selected from $B_m \times n$ (one for each equivalence class) by changing $W_1[p_1, p_2, \ldots, p_m]$ to 1 if we have selected the element $\alpha(p_1, p_2, \ldots, p_m)$ for a representative of the respective equivalence class. We will change the elements of $W_2$ to 1 for each selection of an element from $B_m \times n$, i.e. for each $\pi'' \in P_m$, for which there exists $\pi' \in P_m$, such that $W_1[\pi'] = 1$ and $\alpha(\pi'') = \rho \alpha(\pi')$, or in other words, $\pi'$ and $\pi''$ encode two different matrices of the same equivalence class as we have chosen $\alpha(\pi')$ for a representative of this equivalence class.

3. We declare the counter $N$, which we initialize by 0. In case of normal ending of the algorithm, $N$ will be showing the cardinality of the factor set $B_m \times n_\rho$.

4. While a zero element exists in $W_2$ do

   { Begin of loop 1

5.     We choose the minimal $\pi = < p_1, p_2, \ldots, p_m > \in P_m$ according to the lexicographic order, for which $W_1[\pi] = 0$.

6.     $W_1[\pi] := 1$;

7.     $N := N + 1$;

8.     For $i = 1, 2, \ldots, m$ do

       { Begin of loop 2

9.         $\pi = f_i(\pi)$.

10.        For $j = 1, 2, \ldots, n$ do

             { Begin of loop 3

11.                $\pi := f_j(\pi)$;

12.                $W_2[\pi] := 1$;

             End of loop 3

       End of loop 2

   End of loop 1

13.   End of the algorithm.

3 CONCLUSIONS

Applying the above ideas, a computer program that receives a computer program that gets only one representative from each equivalence class of the factor-set $\tilde{B}_\rho = B_\rho / \rho$. The purpose of these calculations was to describe and classify some textile structures [Yordzhev and Kostadinova (2012)]. The results relate to obtaining quantitative estimation of all kinds of textile fabric.

In fact, the cardinality of the factor-set $M$ coincides with an integer sequence noted in On-Line Encyclopedia of Integer Sequences [Encyclopedia (2015)] as number A179043, namely

A179043={ 2, 7, 64, 4156, 1342208, 1908897152, 11488774559744, 288230376353050816, 2985002023798264483840, 12676506002282327791964489728, 21970710674130840874443091905462272, 154866286100907105149651981766316633972736, ... }
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