The area minimizing problem in conformal cones, II

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Abstract In this paper we continue to study the connection among the area minimizing problem, certain area functional and the Dirichlet problem of minimal surface equations in a class of conformal cones with a similar motivation from [15]. These cones are certain generalizations of hyperbolic spaces. We describe the structure of area minimizing n-integer multiplicity currents in bounded $C^2$ conformal cones with prescribed $C^1$ graphical boundary via a minimizing problem of these area functionals. As an application we solve the corresponding Dirichlet problem of minimal surface equations under a mean convex type assumption. We also extend the existence and uniqueness of a local area minimizing integer multiplicity current with star-shaped infinity boundary in hyperbolic spaces into a large class of complete conformal manifolds.

Keywords area minimizing problem, conformal cones, mean curvature equation

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1 Introduction

In this paper we continue to study the area minimizing problem with prescribed boundary in a class of conformal cones similar to [15]. A conformal cone in this paper is defined as follows.

Definition 1.1. Let $N$ be an $n$-dimensional open Riemannian manifold with a metric $\sigma$, $\mathbb{R}$ be the real line with the metric $dr^2$ and $\phi(x)$ be a $C^2$ positive function on $N$. In this paper we call

$$M_\phi := \{N \times \mathbb{R}, \phi^2(x)(\sigma + dr^2)\}$$

(1.1)

as a conformal product manifold. Let $\Omega$ be a $C^2$ bounded domain with compact closure in $N$. We refer $\Omega \times \mathbb{R}$ in $M_\phi$ as a conformal cone, denoted by $Q_\phi$.

Let $\psi(x)$ be a $C^1$ function on $\partial \Omega$ and $\Gamma$ be its graph in $\partial \Omega \times \mathbb{R}$. The area minimizing problem in a conformal cone $Q_\phi$ is to find an $n$-integer multiplicity current in $\bar{Q}_\phi$, the closure of $Q_\phi$, to realize

$$\min\{M(T) \mid T \in \mathcal{G}, \partial T = \Gamma\},$$

(1.2)
where $M$ is the mass of integer multiplicity currents in $M_\phi$, and $\mathcal{G}$ denotes the set of $n$-integer multiplicity currents with compact support in $Q_\phi$, i.e., for any $T \in \mathcal{G}$, its support $\text{spt}(T)$ is contained in $\Omega \times [a, b]$ for some finite numbers $a < b$ (see Subsection 4.1 for more details).

The main reason to study the conformal product manifold $M_\phi$ in Definition 1.1 is that the hyperbolic space is a special case of $M_\phi$ (see Remark 6.5). With this model in [23, Theorem 4.1], Hardt and Lin showed that there is a unique local area minimizing $n$-integer multiplicity current for any prescribed infinity $C^1$ star-shaped boundary. Moreover, it is a radial minimal graph over $S^n$ in hyperbolic spaces. On the other hand in [27, Subsection 4.1], Lin described the solution to the area minimizing problem (1.2) with $C^1$ graphical boundary in a bounded $C^2$ cylinder via a minimizing problem of an area functional of bounded variation (BV) functions. Motivated by Lin’s idea we studied the area minimizing problem (1.2) in a conformal product manifold $M_h = \{N \times \mathbb{R}, h^2(r)(\sigma + dr^2)\}$ (see [15]). Note that there $h(r)$ only depends on $r \in \mathbb{R}$. Based on these preceding results, it is natural to consider the corresponding area minimizing problem (1.2) in conformal cones of $M_\phi$. In particular it is desirable to know how many phenomena in the hyperbolic space do not really rely on the hyperbolic structures. We refer the readers to [2] for some historical remarks on such kinds of area minimizing problems.

For any open set $W$ we denote the set of all bounded variation functions on $W$ by $BV(W)$. A key concept for our study of the problem (1.2) is an area functional in $BV(W)$ defined as follows:

$$
\tilde{\mathcal{F}}(u, W) = \sup \left\{ \int_{\Omega} \{\phi^n(x)h + u\text{div}(\phi^n(x)X)\}d\text{vol} \middle| h \in C_0(W), X \in T_0(W) \text{ and } h^2 + \langle X, X \rangle \leq 1 \right\},
$$

(1.3)

Here, $d\text{vol}$ and $\text{div}$ are the volume form and the divergence of $N$ respectively, and $C_0(W)$ and $T_0(W)$ denote the set of smooth functions and vector fields with compact support in $W$ respectively. Note that when $u \in C^2(\Omega)$, $\tilde{\mathcal{F}}(u, W)$ is the area of the graph of $u(x)$ in $M_\phi$.

Let $\Omega$ be the $C^2$ domain in Definition 1.1. Suppose that $\Omega'$ is a $C^2$ domain in $N$ satisfying $\Omega \subset \subset \Omega'$, i.e., the closure of $\Omega$ is a compact set in $\Omega'$. Suppose $\psi(x) \in C^1(\Omega' \setminus \Omega)$. The following minimizing problem:

$$
\min \{\tilde{\mathcal{F}}(v, \Omega') \mid v(x) \in BV(\Omega'), v(x) = \psi(x) \text{ on } \Omega' \setminus \Omega\}
$$

(1.4)

also plays an important role to solve (1.2).

In fact, the key idea to solve (1.2) is to establish the connection among the problem (1.2), the area functional minimizing problem (1.4) and the Dirichlet problem of minimal surface equations in $M_\phi$. This can be easily seen when $\Sigma$ is a minimal graph of $u(x)$ in $M_\phi$ over $\Omega$ with $C^1$ boundary $\psi(x)$ on $\partial \Omega$. From Theorem 2.3 one has (1) $u(x)$ should satisfy

$$
\text{div} \left( \frac{Du}{\omega} \right) + n \left( D \log \phi, \frac{Du}{\omega} \right) = 0 \text{ on } \Omega
$$

(1.5)

with $u(x) = \psi(x)$ on $\partial \Omega$; (2) the area of $\Sigma$ is less than that of any compact $C^2$ surface $S$ with $\partial S$ as the graph of $\psi(x)$ contained in $\overline{\Omega} \times \mathbb{R}$; (3) $u(x)$ realizes the minimum of (1.4) if one requires all $v(x) \in C^2(\Omega) \cap BV(\Omega') \cap C(\Omega)$.

In $M_\phi$ a similar connection between the problem (1.2) and the problem (1.4) is obtained in Theorem 4.1. It says that if $u(x)$ is the solution to the problem (1.4), then $T = \partial \{[U]\} |_{Q_\phi}$ solves the problem (1.2) in $M_\phi$ where $U$ is the subgraph of $u(x)$ and $|[U]|$ is the corresponding integer multiplicity current. This generalizes Lin’s result [27] into conformal cones in $M_\phi$. Its proof is based on the following three observations: (1) by Theorem 3.9, $\tilde{\mathcal{F}}(u, \cdot)$ is just the perimeter of its subgraph $U$; (2) if $T$ solves (1.2) and $\Gamma = (x, \psi(x))$ for $\psi(x) \in C^1(\partial \Omega)$, then $T$ is a boundary of a Caccioppoli set (see Lemma 4.11); (3) by Theorem 3.10, for any Caccioppoli set $F$ in $Q \subset M_\phi$ with $\partial F \subset \Omega \times [a, b]$, there is a $u(x) \in BV(\Omega)$ such that $\tilde{\mathcal{F}}(u, \Omega)$ is less than the perimeter of $F$. Note that the observation (3) is a very general phenomenon in area-type functionals [7, 16, 19, 28, 31, 32] (see Remark 3.11). We refer it as the Miranda’s observation.

A direct application of Theorem 4.1 is the Dirichlet problem of minimal surface equations in $M_\phi$. 

In Theorem 5.3 we show that if $\Omega$ is $\phi$-mean convex, then the Dirichlet problem (1.5) with continuous boundary data has a unique solution in $C^2(\Omega) \cap C(\Omega)$. Here, $\Omega$ is $\phi$-mean convex if the mean curvature of $\partial \Omega$ satisfies that $H_{\partial \Omega} + n(\vec{\gamma}, D \log \phi) \geq 0$ on $\partial \Omega$ where $\vec{\gamma}$ is the outward normal vector of $\partial \Omega$ and $H_{\partial \Omega} = \text{div}(\vec{\gamma})$. A more general form of the Dirichlet problem (1.5) was already considered by Casteras et al. [8]. To obtain the $C^0$ estimate they relied on a lower bound of the Ricci curvature $\Omega$ and the positive mean curvature of $\partial \Omega$ (see Remark 5.4). But our result in Theorem 5.3 is independent of the curvature of $\Omega$ and also cannot be obtained by the classical continuous method in [17, Chapters 11 and 17] (see Remark 5.5). A consequence of Theorem 5.3 is that if $\Omega$ is $\phi$-mean convex, the area minimizing integer multiplicity current in (1.2) is unique as the graph of a $C^2$ function to solve the Dirichlet problem (see Theorem 5.6).

At last we consider the existence and uniqueness of local area minimizing integer multiplicity current in $M_\phi$ with infinity boundary $\Gamma$ when $\phi(x)$ can be written as $\phi(d(x, \partial N))$ which goes to $+\infty$ as $d(x, \partial N)$ goes to zero in $N$. Here, $N$ is a compact Riemannian manifold with $C^2$ boundary and $d$ is the distance function in $N$. Let $N_r$ be the set
\[ \{ x \in N : d(x, \partial N) > r \} . \]
In Theorem 6.4 we obtain that if there is $r_1$ such that for any $r \in (0, r_1)$ $N_r$ is $\phi$-mean convex, then for any $\psi(x) \in C(\partial N)$ and $\Gamma = (x, \psi(x))$ there is a unique local area minimizing integer multiplicity current $T$ with infinity boundary $\Gamma$. Moreover, $T$ is a minimal graph in $M_\phi$ over $N$. This illustrates that [23, Theorem 4.1] does not depend on the hyperbolic structure (see Remark 6.5). Unlike the case in hyperbolic spaces our existence in Theorem 6.4 is from the Dirichlet problem of minimal surface equations in Theorem 5.3, not from the results in [5, 6] by Anderson via geometric measure theory.

For the idea of radial graphs in the hyperbolic space we refer to [9, 20–22, 28, 29, 32, 33, 38]. For the Dirichlet problem of minimal surface equations in Riemannian manifolds, we refer to [1, 3, 4, 8, 37] and the references therein. As for the variational method to study the Dirichlet problem of minimal surface equations in $R$iemannian manifolds for later use.

The rest of this paper is organized as follows. In Section 2, we show three properties of $C^2$ minimal graphs in $M_\phi$ and collect preliminary facts on BV functions. In Section 3, we show the $C^\infty$ approximation theorem of $\mathcal{F}_\phi(u, \Omega)$ (see Theorem 3.8) and the Miranda’s observation (see Theorems 3.9 and 3.10). In Section 4, we consider the connection between the problems (1.2) and (1.4) (see Theorem 4.1). In Section 5, we discuss the Dirichlet problem of minimal surface equations in $M_\phi$ on a $\phi$-mean convex domain (see Theorem 5.3). As an application we obtain the uniqueness of the problem (1.2) under the $\phi$-mean convex assumption (see Theorem 5.6). In Section 6 we discuss the existence and uniqueness of the local area minimizing $n$-integer multiplicity current with infinity graphical boundary (see Theorem 6.4). In Appendix A, we give a proof of the interior estimate of minimal surface equations in (1.5) (see Theorem A.3).

## 2 Preliminaries

Throughout this section we assume $N$ is a manifold with a metric $\sigma$, $\phi(x) > 0$ is a positive $C^2$ function defined on $N$ and $M_\phi$ is the conformal product manifold $N \times \mathbb{R}$ with the metric $\phi^2(\sigma + dr^2)$. We show three properties of $C^2$ minimal graphs in $M_\phi$ and collect some results on BV functions in Riemannian manifolds for later use.

### 2.1 Three properties

**Definition 2.1.** Let $S$ be a $C^2$ orientable hypersurface in a Riemannian manifold $M$ with a normal vector $\vec{v}$. We call $\text{div}(\vec{v})$ as the mean curvature of $S$ with respect to $\vec{v}$.

Let $\Omega$ be a bounded domain in $N$ and $u \in C^2(\Omega)$. Let $\Sigma$ be the graph of $u(x)$. Then its upward normal vector in the product manifold $N \times \mathbb{R}$ is $\vec{v}_u := \frac{\partial N - Du}{\omega}$, where $D$ denotes the gradient of $N$ and $\omega = \sqrt{1 + |Du|^2}$. Then we have the following lemma.
Lemma 2.2. The mean curvature of $\Sigma$ with respect to its upward normal vector in $M_\phi$ is

$$H_\Sigma = \frac{1}{\phi(x)} \left( - \text{div} \left( \frac{Du}{\omega} \right) - n \left( D \log \phi, \frac{Du}{\omega} \right) \right), \tag{2.1}$$

where $\omega = \sqrt{1 + |Du|^2}$, $n$ is the dimension of $N$ and $\text{div}$ is the divergence of $N$.

Proof. We write the metric of $M_\phi$ as $e^{2f}(g + dr^2)$ where $f = \log \phi$. The upward normal vector of $\Sigma$ in $M_\phi$ is $e^{-f}\vec{v}_\Sigma$. By [43, Lemma 3.1], the mean curvature of $\Sigma$ in $M_\phi$ with respect to $e^{-f}\vec{v}_\Sigma$ is

$$H_\Sigma = e^{-f}(H + n df(\vec{v}_\Sigma)),$$ \tag{2.2}

where $H$ is the mean curvature of $S$ with respect to $\vec{v}_\Sigma$ in the product manifold $N \times \mathbb{R}$ and $n$ is the dimension of $N$. With a straightforward computation $H = -\text{div} \left( \frac{Du}{\omega} \right)$ where $\text{div}$ is the divergence of $N$. Putting $\vec{v}_\Sigma$ and $f = \log \phi$ into (2.2), we obtain the conclusion. \hfill $\Box$

Now we summarize three properties of minimal graphs in $M_\phi$ as follows.

Theorem 2.3. Suppose that $\Sigma$ is a minimal graph of $u(x)$ over $\Omega$ with $C^1$ boundary

$$\Gamma := \{(x, \psi(x)) : x \in \partial \Omega, \psi(x) \in C^1(\partial \Omega)\}.$$ 

Then we have the following conclusions.

1. $u(x)$ satisfies that

$$- \text{div} \left( \frac{Du}{\omega} \right) - n \left( D \log \phi(x), \frac{Du}{\omega} \right) = 0. \tag{2.3}$$

2. Let $\Omega'$ be a domain such that $\Omega \subset \subset \Omega'$ and $\psi(x) \in C^1(\Omega' \setminus \Omega)$. Furthermore, assume $u(x) = \psi(x)$ on $\Omega' \setminus \Omega$. Then $u(x)$ realizes

$$\min \{ \overline{\mathcal{S}}\phi(v, \Omega') \mid v(x) \in C^2(\Omega), \psi(x) = \psi(x) \text{ on } \Omega' \setminus \Omega \}, \tag{2.4}$$

where $\overline{\mathcal{S}}\phi(v, \Omega') = \int_{\Omega'} \phi^n(x) \sqrt{1 + |Du|^2} d\text{vol}$ and $d\text{vol}$ is the volume form of $N$.

3. The area of $\Sigma$ achieves the minimum of the area of all $C^2$ compact orientable hypersurfaces $S$ in $\bar{Q}_\phi := \bar{\Omega} \times \mathbb{R}$ with $\partial S = \Gamma$.

Proof. Note that the property (1) is from Lemma 2.2.

For any $C^2$ graph $S = (x, v(x))$ over $\Omega$, its area in $M_\phi$ is

$$\int_{\Omega} \phi^n(x) \sqrt{1 + |Du|^2} d\text{vol}, \tag{2.5}$$

where $d\text{vol}$ is the volume form on $\Omega$. Thus the property (2) follows from the property (3). It is sufficient to show the property (3).

Recall that $\vec{v}_\Sigma := \frac{\partial_x - Du}{\omega}$. For any $t \in \mathbb{R}$, define a map $T_t : N \times \mathbb{R} \to N \times \mathbb{R}$ as $T_t(x, r) = (x, r - t)$. We define a new vector field in $\bar{Q}_\phi$ as

$$X(x, u(x) - t) = \frac{1}{\phi(x)} T_t(\vec{v}_\Sigma), \tag{2.6}$$

where $T_t$ is the pushforward of $T_t$. Thus $X$ is a smooth unit vector field in the tangent bundle of $\bar{Q}_\phi$. By the definition of divergence (see [25, p. 423]), we define an $n$-form as

$$\xi = X \wedge (\phi^{n+1} d\text{vol} \wedge dr) = X d\text{vol}_{M_\phi}. \tag{2.7}$$

A key fact is

$$d\xi = \text{div}_{M_\phi}(X) d\text{vol}_{M_\phi} = H_{T_t(\Sigma)} d\text{vol}_{M_\phi}, \tag{2.8}$$

where $\text{div}_{M_\phi}$ and $d\text{vol}_{M_\phi}$ are the divergence and volume form of $M_\phi$. Note that $T_t$ is an isometry of $M_\phi$ for every $t \in \mathbb{R}$. Because $\Sigma$ is minimal in $M_\phi$, so is $T_t(\Sigma)$. Then $d\xi = 0$ on the whole $Q_\phi$. Thus
$X^\perp := \{ Y \in TQ_\phi, (Y, X) = 0 \}$ is an involutive distribution.

Now let $S$ be any compact $C^2$ orientable hypersurface in $Q_\phi$, i.e., $\Omega \times \mathbb{R}$ satisfying $\partial S = \Gamma$. Without loss of generality we can assume $S$ and $\Sigma$ enclose a domain $W$ with compact closure in $Q_\phi$ (see Remark 4.10 for a proof). Suppose that the outward normal vector of $W$ on $\Sigma$ and $S$ are denoted by $\vec{n}_\Sigma$ and $\vec{n}_S$, the volume form of $\partial W$ on $\Sigma$ and $S$ are denoted by $dvol_\Sigma$ and $dvol_S$, respectively. If not the case, we reverse the orientation of $W$. It is clear that the upward normal vector of $\Sigma$ in $M_\phi$, $\vec{n}_\Sigma$, is $\frac{1}{\rho}\vec{n}_S$. Applying the divergence theorem in the domain $W$, we obtain

$$0 = \int_W \partial_t \xi = \int_\Sigma \langle (X, \vec{n}_\Sigma)_M \rho \rangle dvol_\Sigma - \int_S \langle X, \vec{n}_S \rangle _M dvol_S \geq \text{Area}(\Sigma) - \text{Area}(S).$$

If the equality holds, we have $X\mid_S = \vec{n}_S$ on the whole $S$. Then $S$ is an integral hypersurface of the distribution $X^\perp$. By the definition of $X$, $T_t(S)$ is also an integral hypersurface of $X^\perp$ for any $t \in \mathbb{R}$. Suppose that $S$ is not equal to $\Sigma$. Because $\partial S = \partial \Sigma$ is a graph over $\partial \Omega$, by the continuity there is a $t \neq 0$ such that there is a point $p \in T_t(S) \cap \Sigma$ in $\Omega \times \mathbb{R}$. But for any fixed point in $\Omega \times \mathbb{R}$, there is a unique integral hypersurface of $X^\perp$ at $p$. This implies that $T_t(S) = \Sigma$. This leads to a contradiction because $\partial T_t(S) \cap \partial \Sigma = \emptyset$ when $t \neq 0$.

As a result we obtain the property (3). The proof is completed. $\square$

2.2 BV functions

Now we collect some preliminary facts for BV functions in Riemannian manifolds. We refer the readers to the books of Giusti [19], Evans and Gariepy [13] and the papers of McGonagle and Xiao [30], and Zhou [42]. Recall that $\Omega$ is a bounded domain in $\mathbb{R}^n$ and $dvol$ is the volume form of $\Omega$. Let $T_0\Omega$ be the set of smooth vector fields with compact support in $\Omega$.

Now we define BV functions and Caccioppoli sets as follows:

**Definition 2.4.** Let $u \in L^1(\Omega)$. Define

$$\|Du\|(\Omega) = \sup \left\{ \int_{\Omega} u\text{div}(X) dvol \ \bigg| \ X \in T_0\Omega \text{ and } \langle X, X \rangle \leq 1 \right\},$$

(2.9)

where $dvol$ is the volume form of $\Omega$. If $\|Du\|(\Omega) < \infty$, we say that $u$ has bounded variation in $\Omega$. The set of all functions with bounded variation is denoted by $BV(\Omega)$. If $u$ belongs to $BV(W)$ for any bounded domain $W \subset \subset \Omega$, we say $u \in BV_{\text{loc}}(\Omega)$.

Let $E$ be a Borel set in $\Omega$ and $\lambda_E$ be the characteristic function of $E$. If $\lambda_E \in BV_{\text{loc}}(\Omega)$, then $E$ is called a Caccioppoli set and $\|D\lambda_E\|(\Omega)$ is called as the perimeter of $E$. In the rest of this paper, we also write it as $P(E, \Omega)$.

**Remark 2.5.** In some settings, in order to emphasize the ambient manifold $M$, we use the notation $\|Du\|_M(\Omega)$ instead of $\|Du\|(\Omega)$.

For a Caccioppoli set $E$ all properties are unchanged if we make alterations of any Hausdorff measure zero set. Arguing exactly as [19, Proposition 3.1], we can always choose a set $E'$ differing by a Hausdorff measure zero set with $E$ and satisfying for any $x \in \partial E'$,

$$0 < \text{vol}(E' \cap B(x, \rho)) < \text{vol}(B(x, \rho)),$$

(2.10)

where $\rho$ is sufficiently small. From now on, we always assume that the condition (2.10) holds for any Caccioppoli set $E$.

**Definition 2.6.** We say a sequence of measurable functions $\{u_k\}_{k=1}^\infty$ locally converges to $u$ in $L^1(\Omega)$ if for any open set $W \subset \subset \Omega$ we have

$$\lim_{k \to +\infty} \int_W |u_k - u| dvol = 0.$$
Remark 2.7. If \( \Omega \) is compact and a uniformly bounded sequence \( \{u_k\}_{k=1}^{\infty} \) converges to \( u(x) \) a.e., then this sequence locally converges to \( u(x) \) in \( L^1(\Omega) \). By the definition in (2.9) it is easy to see the following theorem.

**Theorem 2.8** (Lower-semicontinuity). Let \( \{u_k\}_{k=1}^{\infty} \) be a sequence of functions in \( BV(\Omega) \) locally converging to a function \( u \) in the \( L^1(\Omega) \). Then

\[
\|Du\|(\Omega) \leq \liminf_{k \to +\infty} \|Du_k\|(\Omega). \tag{2.12}
\]

The following \( C^\infty \) approximation result is not trivial when \( \Omega \) is not contained in a simply-connected domain of a Riemannian manifold. Because in this case no global symmetric mollifiers exist for this domain as those in [19, p. 15] for Euclidean spaces. For a complete proof, we refer to [42, Section 3].

**Theorem 2.9** (See [42, Theorem 3.6]). Suppose \( u \in BV(\Omega) \). Then there exists a sequence of functions \( \{u_k\}_{k=1}^{\infty} \) in \( C^\infty(\Omega) \) such that

\[
\lim_{k \to +\infty} \int_{\Omega} |u - u_k|dvol = 0, \tag{2.13}
\]

\[
\lim_{k \to +\infty} \|Du_k\|(\Omega) = \|Du\|(\Omega). \tag{2.14}
\]

As a conclusion, we can obtain the following well-known compactness result.

**Theorem 2.10** (The compactness theorem). Let \( \Omega \subset M \) be a bounded domain with Lipschitz boundary. Suppose that there is a constant \( C > 0 \) such that for \( \{u_k\}_{k=1}^{\infty} \subset BV(\Omega) \),

\[
\|u_k\|_{L^1} + \|Du_k\|(\Omega) \leq C \quad \text{for any } k. \tag{2.15}
\]

Then there is a function \( u \in BV(\Omega) \) such that there is a subsequence of \( \{u_k\}_{k=1}^{\infty} \) converging to \( u(x) \) in \( L^1(\Omega) \).

We can also view a \( BV_{loc} \) function as a Radon measure.

**Theorem 2.11** (See [42, Theorem 2.6]). Let \( \Omega \subset M \) be a bounded domain. Suppose that \( u \in BV_{loc}(\Omega) \).

1. There exists a Radon measure \(|Du|\) on \( \Omega \) such that

\[
\int_{\Omega} f d|Du| = \sup \left\{ \int_{\Omega} u \text{div}(X)dvol \left| X \in T_0\Omega' \right. \right. \left. \left. \text{and } \langle X, X \rangle \leq f^2 \right\} \tag{2.16}
\]

for any open set \( \Omega' \subset \subset \Omega \) and any nonnegative function \( f \in L^1(|Du|) \).

2. There exists a vector field \( \vec{v} \) on \( \Omega \) satisfying

\[
\int_{\Omega} \text{div}(X)dvol = - \int_{\Omega} \langle X, \vec{v} \rangle d|Du|, \tag{2.17}
\]

where \( \langle \vec{v}, \vec{v} \rangle = 1 |Du| \cdot a.e. \) for any \( X \in T_0\Omega \).

3 The area functional

In this section we define a new area functional \( \mathfrak{F}_\phi(u, \Omega) \) for BV functions to generalize the area of the graph of \( C^1 \) functions in \( M_\phi \) (see (2.5)). Suppose that \( N \) is a fixed Riemannian manifold with metric \( \sigma \) and \( \phi(x) \) is a \( C^2 \) positive function on \( N \). Here, \( M_\phi \) is the set \( N \times \mathbb{R} \) equipped with \( \phi^2(x)(\sigma + dr^2) \). We shall establish the \( C^\infty \) approximation of \( \mathfrak{F}_\phi(u, \Omega) \) and the Miranda’s observation mentioned in the introduction.

Recall that \( \Omega \) is a bounded domain in \( N \), \( T_0(\Omega) \) and \( C_0(\Omega) \) denote the sets of all smooth vector fields and smooth functions with compact support in \( \Omega \), respectively. Let \( dvol \) denote the volume form of \( N \).

**Definition 3.1.** Let \( u \) be a measurable function in \( \Omega \). Define

\[
\mathfrak{F}_\phi(u, \Omega) = \sup \left\{ \int_{\Omega} (\phi^n h + u \text{div}(\phi^n X))dvol \left| h \in C_0(\Omega), X \in T_0(\Omega) \right. \right. \left. \left. \text{and } h^2 + \langle X, X \rangle \leq 1 \right\}. \tag{3.1}
\]
Lemma 3.4. There is a positive constant $h$. We choose $\nu$.

Proof. Since $\bar{\Omega}$ is compact, there is a positive constant $h$. Now let $\nu$.

Lemma 3.3 (Lower-semicontinuity). Let $\{u_k\}_{k=1}^\infty$ be a sequence converging to $u(x)$ in $L^1(\Omega)$. Then

$$\mathcal{F}_\phi(u, \Omega) \leq \liminf_{k \to +\infty} \mathcal{F}_\phi(u_k, \Omega).$$

The following lemma establishes a relation between $\mathcal{F}_\phi(u, \Omega)$ and $\|Du\|(\Omega)$.

Lemma 3.4. There is a positive constant $\mu_0 := \mu_0(\Omega, \phi)$ such that for any $u \in L^1(\Omega)$,

$$\frac{1}{\mu_0} \max\{\|Du\|/(\Omega), \text{vol}(\Omega)\} \leq \mathcal{F}_\phi(u, \Omega) \leq \mu_0(\|Du\|/(\Omega) + \text{vol}(\Omega)).$$

Proof. Since $\bar{\Omega}$ is compact, there is a positive constant $\mu_0$ such that

$$\frac{1}{\mu_0} \phi^n(x) \leq \mu_0 \quad \text{on} \quad \bar{\Omega}.$$

By the conclusion (1) in Theorem 2.11 we have

$$\sup_{(x,x) \leq 1, x \in T_0 \Omega} \int_\Omega u \text{div}(\phi^n X) d\text{vol} = \int_\Omega \phi^n(x) d|Du|.$$ (3.4)

We choose $h \in C_0(\Omega)$ and $X \in T_0(\Omega)$ satisfying $h^2 + \langle X, X \rangle \leq 1$. By the definition in (3.1),

$$\mathcal{F}_\phi(u, \Omega) \leq \mu_0(\text{vol}(\Omega)) \sup_{|X|^2 \leq 1} \int_\Omega u \text{div}(\phi^n X) d\text{vol} \leq \mu_0(\text{vol}(\Omega) + \|Du\|(\Omega)) \text{ by (3.4).}$$ (3.5)

Let $X = 0$ and $h = 1$. (3.1) implies that

$$\mathcal{F}_\phi(u, \Omega) \geq \frac{1}{\mu_0} \text{vol}(\Omega).$$

Now let $h = 0$. (3.4) implies that

$$\mathcal{F}_\phi(u, \Omega) \geq \frac{1}{\mu_0} \|Du\|(\Omega).$$

The proof is completed.

The area functional $\mathcal{F}_\phi(u, \Omega)$ induces a Radon measure on $\Omega$ as follows.

Theorem 3.5. Suppose $u \in BV(\Omega)$. Then there is a Radon measure $\nu$ on $\Omega$ such that for any open set $W$ in $\Omega$,

$$\nu(W) = \mathcal{F}_\phi(u, W).$$

Proof. For any nonnegative function $f \in C_0(\Omega)$, define

$$\lambda(f) = \sup \left\{ \int_\Omega \phi^n h + u \text{div}(\phi^n X) d\text{vol} \left| h \in C_0(\Omega), X \in T_0(\Omega) \text{ and } h^2 + \langle X, X \rangle \leq f^2 \right. \right\}.$$ (3.7)

It is easy to see that $\lambda(cf) = c\lambda(f)$ for any positive constant $c > 0$ and $\lambda(f + g) = \lambda(f) + \lambda(g)$. Thus $\lambda$ is a positive linear functional on $C_0(\Omega)$. By [36, Remark 4.3] (see also [42, Theorem 2.5]), there is a Radon measure $\nu$ on $\Omega$ such that

$$\nu(W) = \sup\{\lambda(f) \mid \text{spt}(f) \subset W, 0 \leq f \leq 1\}$$ (3.8)

for any open set $W \subset \Omega$. Here, $\text{spt}(f)$ denotes the closure of $\{f(x) \neq 0 : x \in \Omega\}$. From the definition it is clear that $\mathcal{F}_\phi(u, W) = \sup\{\lambda(f) \mid \text{spt}(f) \subset W, 0 \leq f \leq 1\}$. The proof is completed.
Fix any \( p \in N \). Let \( \exp_p \) be the exponential map at \( p \). Let \( \text{inj}_Q \) be the finite number given by
\[
\min\{\text{the injective radius of } x \text{ in } N : x \in \bar{\Omega}, 1\}.
\] (3.9)
For any \( r < \text{inj}_Q \), let \( B_r(0) \) be the Euclidean ball centering at 0 with radius \( r \). Thus \( \exp_p : B_r(0) \to B_r(p) \) is a diffeomorphism. Via this exponential map we identify \( \Omega \) with \( B_r(p) \). Hence \( g = g_{ij}dx^idx^j \). Then \( B_r(p) \) is called a normal ball. A vector field \( X \) along \( B_r(p) \) can be represented by
\[
X = X^i \frac{\partial}{\partial x^i}.
\] (3.10)
Let \( \varphi(x) \) be a symmetric mollifier in \( \mathbb{R}^n \), i.e., a function satisfying \( \varphi(x) = \varphi(-x) \), \( \text{spt}(\varphi) \subset B_1(0) \) and \( \int_{\mathbb{R}^n} \varphi(x) d\text{vol}_{\mathbb{R}^n} = 1 \). Here, \( \text{vol}_{\mathbb{R}^n} \) denotes \( dx^1 \wedge \cdots \wedge dx^n \). If \( u \in L^1(\mathbb{R}^n) \), let \( U \ast \varphi \) denote the convolution of \( u \), where \( \varphi = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right) \) namely,
\[
u \ast \varphi_\varepsilon(y) = \int_{\mathbb{R}^n} \varphi_\varepsilon(x - y)u(x) d\text{vol}_{\mathbb{R}^n}.
\] (3.11)
The convolution of a vector field \( X = X^i \frac{\partial}{\partial x^i} \) and \( \varphi_\varepsilon \) on \( B_r(p) \) is defined by
\[
X \ast \varphi_\varepsilon := X^i \ast \varphi_\varepsilon \frac{\partial}{\partial x^i}.
\] (3.12)
Note that \( d\text{vol} = \sqrt{\det g} d\text{vol}_{\mathbb{R}^n} \). Suppose \( f, h \in C_0(B_r(p)) \). By a direct computation we have
\[
\int_{B_r(p)} (\varphi_\varepsilon \ast f) h d\text{vol} = \int_{B_r(p)} \frac{f}{\sqrt{\det g}} \varphi_\varepsilon \ast (h \sqrt{\det g}) d\text{vol},
\] (3.13)
\[
\int_{B_r(p)} (\varphi_\varepsilon \ast h) \text{div}(X) d\text{vol} = \int_{B_r(p)} h \text{div} \left( \frac{1}{\sqrt{\det g}} \varphi_\varepsilon \ast (\sqrt{\det g} X) \right) d\text{vol}.
\] (3.14)
To prove the \( C^\infty \) approximation property of the area functional \( \mathcal{F}_\phi(u, \Omega) \), we need the following two technical lemmas from [42]. Both of their proofs follow from those in [42] with minor modifications. So we skip them here.

**Lemma 3.6** (See [42, Theorem 2.12]). Let \( B \) be a normal ball in \( \Omega \) with a metric \( g = g_{ij}dx^idx^j \). Let \( f \) be a nonnegative continuous function and \( K \subset \subset B \) be a compact subset. Then for any \( \delta > 0 \), there exists an \( \varepsilon_0 = \varepsilon_0(\Omega, K, g, f) \) such that for all \( \varepsilon < \varepsilon_0 \), any continuous function \( h \) and any vector field \( X \) satisfying
\[
h^2 + \langle X, X \rangle \leq f^2 \quad \text{in } \Omega,
\] (3.15)
we have
\[
h^2 + \langle X', X' \rangle \leq (f + \delta)^2 \quad \text{in } K,
\] (3.16)
where \( X' \) is defined by
\[
X' = \frac{1}{\sqrt{\det g}} \frac{\varphi_\varepsilon}{\phi^n}(x) \ast (\sqrt{\det g} \phi^n(X)).
\] (3.17)

**Lemma 3.7** (See [42, Lemma 2.13]). Let \( B \) be a normal ball in \( \Omega \) with a metric \( g = g_{ij}dx^idx^j \). Suppose \( u \in BV(B) \) and \( q(x) \) is a smooth function with compact support in \( B \). Then for any \( \varepsilon > 0 \) there is a \( \sigma_0 = \sigma_0(u, g, q, K) \) such that for all \( \sigma \in (0, \sigma_0) \) and any smooth vector field \( X \) on \( B \) satisfying \( \langle X, X \rangle \leq 1 \), we have
\[
\int_B \varphi_\sigma \ast (qu) \text{div}(\phi^n X) d\text{vol} \leq \int_B u \text{div}(q\phi^n Y_\sigma) d\text{vol} - \int_B \langle \phi^n X, \nabla q \rangle d\text{vol} + \varepsilon,
\] (3.18)
where \( \nabla \) is the covariant derivative of \( N \),
\[
Y_\sigma = \frac{1}{\sqrt{\det g}} \phi^n \varphi_\sigma \ast (\sqrt{\det g} \phi^n X)
\]
and assume \( X = 0 \) outside \( B \).
The $C^\infty$ approximation property of $\mathcal{F}_\phi(u, \Omega)$ is stated as follows.

**Theorem 3.8.** Let $\Omega$ be a bounded domain in $N$ and $u \in BV(\Omega)$. Then there is a sequence

$$\{u_k\}_{k=1}^\infty \in C^\infty(\Omega)$$

such that

$$\lim_{k \to +\infty} \int_\Omega |u - u_k| \text{dvol} = 0, \tag{3.19}$$

$$\lim_{k \to +\infty} \mathcal{F}_\phi(u_k, \Omega) = \mathcal{F}_\phi(u, \Omega). \tag{3.20}$$

**Proof.** By Lemma 3.4, $u \in BV(\Omega)$ and that $\Omega$ is bounded imply that $\mathcal{F}_\phi(u, \Omega)$ is finite.

By Theorem 3.5, there is a Radon measure $\nu$ on $\Omega$ such that $\nu(W) = \mathcal{F}_\phi(u, W)$ for any open set $W \subset \Omega$. Fix any $\varepsilon > 0$ and $r_0 > 0$. By [42, Theorem A.4], there exist a countable open cover $\{B_k\}_{k=1}^\infty$ of $\Omega$ and a positive integer $\kappa_0$ such that

1. each $B_k$ is a normal ball such that $\nu(\partial B_k) = 0$ and $\text{diam}(B_k) \leq 2r_0$;
2. $\{B_1, \ldots, B_{\kappa_0}\}$ is a pairwise disjoint subcollection with

$$\mathcal{F}_\phi(u, \Omega) - \varepsilon \leq \kappa_0 \sum_{k=1}^{\kappa_0} \mathcal{F}_\phi(u, B_k) \leq \mathcal{F}_\phi(u, \Omega); \tag{3.21}$$
3. the subcollection $\{B_k\}_{k=\kappa_0+1}^\infty$ satisfies

$$\sum_{k=\kappa_0+1}^{\infty} \mathcal{F}_\phi(u, B_k) \leq \kappa_1 \varepsilon, \tag{3.22}$$

where $\kappa_1$ is a positive integer only depending on $\Omega$ and $n$.

Take a partition of unity $\{\zeta_k\}$ subordinate to the open cover $\{B_k\}$, i.e.,

$$\zeta_k \in C_0^\infty(B_k), \quad 0 \leq \zeta_k \leq 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \zeta_k = 1 \quad \text{on} \quad \Omega. \tag{3.23}$$

For any $x$ in $\Omega$ there is a compact neighborhood $V$ of $x$ such that on $V$ the summation above is finite.

Now fix any $h \in C_0(\Omega)$ and any $X \in T_0\Omega$ satisfying $h^2 + \langle X, X \rangle^2 \leq 1$. For each $k$ by Lemmas 3.6 and 3.7 we can choose $\varepsilon_k$ so small and independent of $X$ and $h$ such that $\text{spt}(\varphi_{\varepsilon_k} * (u\zeta_k)) \subset B_k$,

$$\int_{B_k} |\varphi_{\varepsilon_k} * (u\zeta_k) - u\zeta_k| \text{dvol} < \frac{\varepsilon}{2^k}, \tag{3.24}$$

$$(\zeta_k h)^2 + \zeta_k^2 \langle Y_{\varepsilon_k}, Y_{\varepsilon_k} \rangle \leq 1 + \varepsilon, \tag{3.25}$$

and

$$\int_{B_k} \varphi_{\varepsilon_k} * (u\zeta_k) \text{div}(\phi^n(x)X) \text{dvol} \leq \int_{B_k} u \text{div}(\phi^n \zeta_k Y_{\varepsilon_k}) \text{dvol} - \int_{B_k} u \langle \phi^n X, \nabla \zeta_k \rangle \text{dvol} + \frac{\varepsilon}{2^k}. \tag{3.26}$$

Here,

$$Y_{\varepsilon_k} = \frac{1}{\sqrt{\det(g)} \phi^n} \varphi_{\varepsilon_k} * (\sqrt{\det(g)} \phi^n X).$$

Now we define $u_\varepsilon$ by

$$u_\varepsilon = \sum_{k=1}^{\infty} \varphi_{\varepsilon_k} * (u\zeta_k). \tag{3.27}$$
Thus $u_\epsilon \in C^\infty(\Omega)$. By (3.26) we obtain
\[
\int_{\Omega} \{\phi^n h + u_\epsilon \text{div}(\phi^n X)\} \text{dvol}
= \sum_{k=1}^{\infty} \int_{B_k} \{\phi^n \zeta_k h + \varphi_{\epsilon_k} \ast (u_\epsilon \zeta_k) \text{div}(\phi^n X)\} \text{dvol}
\leq \sum_{k=1}^{\infty} \int_{B_k} \{\phi^n \zeta_k h + u \text{div}(\phi^n \zeta_k Y_{\epsilon_k}) - u(\phi^n X, \nabla \zeta_k)\} \text{dvol} + \varepsilon \quad \text{by (3.26)}
= \sum_{k=1}^{\infty} \int_{B_k} \{\phi^n \zeta_k h + u \text{div}(\phi^n \zeta_k Y_{\epsilon_k})\} \text{dvol} + \varepsilon.
\]

In the last line we use the fact that
\[
\sum_{k=1}^{\infty} \int_{B_k} -u(\phi^n X, \nabla \zeta_k) \text{dvol} = 0.
\]

By definition we have
\[
\int_{B_k} \{\phi^n \zeta_k h + u \text{div}(\phi^n \zeta_k Y_{\epsilon_k})\} \text{dvol} \leq (1 + \varepsilon)\mathfrak{H}_\phi(u, B_k).
\]

Combining this with (3.21) and (3.22) we obtain
\[
\int_{\Omega} \{\phi^n h + u_\epsilon \text{div}(\phi^n X)\} \text{dvol} \leq (1 + \varepsilon) \sum_{k=1}^{\infty} \mathfrak{H}_\phi(u, B_k) \leq (1 + \varepsilon) \{\mathfrak{H}_\phi(u, \Omega) + \kappa \varepsilon\}.
\]

Since $h$ and $X$ are chosen arbitrarily satisfying $h \in C_0(\Omega), X \in T_0(\Omega)$ and $h^2 + \langle X, X \rangle \leq 1$, we conclude that
\[
\mathfrak{H}_\phi(u_\epsilon, \Omega) \leq (1 + \varepsilon)(\mathfrak{H}_\phi(u, \Omega) + \kappa \varepsilon).
\]

As a result we can choose $\varepsilon_i \to 0$ such that
\[
\lim_{\varepsilon_i \to 0} \sup \mathfrak{H}_\phi(u_{\varepsilon_i}, \Omega) \leq \mathfrak{H}_\phi(u, \Omega).
\]

On the other hand by (3.24) $u_{\varepsilon_i}$ converges to $u$ in $L^1(\Omega)$. Thus by Lemma 3.3 we have
\[
\liminf \mathfrak{H}_\phi(u_{\varepsilon_i}, \Omega) \geq \mathfrak{H}_\phi(u, \Omega).
\]

Thus we obtain the conclusion. The proof is completed. \hfill \Box

### 3.2 The Miranda’s observation

In this subsection we show the Miranda’s observation for $\mathfrak{H}_\phi(u, \Omega)$ mentioned in Section 1.

Let $u(x)$ be a measurable function on $\Omega$. The set $\{(x, t) \mid x \in \Omega, t < u(x)\}$ is called as the subgraph of $u$, written as $U$. For any Borel set $E$ we denote by $\lambda_E$ its characterization function.

Let $P_\phi$ denote the perimeter of Caccioppoli sets in $M_\phi$. Its connection with the area functional $\mathfrak{H}_\phi(u, \Omega)$ can be summarized as follows.

**Theorem 3.9.** Suppose $u \in BV(\Omega)$, $U$ is its subgraph and $\Omega$ is a bounded domain. Then
\[
P_\phi(U, \Omega \times \mathbb{R}) = \mathfrak{H}_\phi(u, \Omega).
\]

**Proof.** By Lemma 3.4, $\mathfrak{H}_\phi(u, \Omega)$ is finite. If $u$ is a smooth function, the equality is obvious since both sides equal the area of the graph of $u$ in $M_\phi$.

Suppose $u \in BV(\Omega)$. By Theorem 3.8 there is a sequence of smooth functions $\{u_k\}_{k=1}^{\infty}$ such that $\{u_k\}$. 


converges to \( u \) in \( L^1(\Omega) \) and \( \mathcal{F}_\phi(u_k, \Omega) \) converges to \( \mathcal{F}_\phi(u, \Omega) \) as \( k \to \infty \). Let \( U_k \) be the subgraph of \( u_k \). Note that \( \{\lambda_{U_k}\} \) converges locally to \( \lambda_U \) in \( L^1 \). By Theorem 2.8 we have

\[
P_\phi(U, \Omega \times \mathbb{R}) \leq \liminf_{k \to \infty} P_\phi(U_k, \Omega \times \mathbb{R}) = \liminf_{k \to \infty} \mathcal{F}_\phi(u_k, \Omega) = \mathcal{F}_\phi(u, \Omega).
\]

(3.32)

To prove the converse first suppose that \( |u| \leq \mu \) for some positive constant \( \mu > 1 \). Observe that we can add a constant to \( u \) without changing \( P_\phi(U, \Omega \times \mathbb{R}) \) and \( \mathcal{F}_\phi(u, \Omega) \). Thus without loss of generality we assume that \( u \geq 1 \). Let \( X \in T_0 \Omega \) and \( h(x) \in C_0(\Omega) \) satisfy

\[
\langle X, X \rangle + h^2(x) \leq 1.
\]

Let \( \eta(t) \) be a smooth nonnegative function with compact support in \((0, \mu + 1)\) such that \( \eta \equiv 1 \) in \([1, \mu]\) and \( \eta \leq 1 \) on \((0, \mu + 1)\). Define

\[
X' = \frac{1}{\phi(x)} \eta(r)(X + h(x)\partial_r).
\]

(3.33)

Then \( X' \in T_0(\Omega \times \mathbb{R}) \) and \( \langle X', X' \rangle \leq 1 \) in \( M_\phi \). Observe that for each \( x \in \Omega \), we have

\[
\int_{\mathbb{R}} \lambda_U(x, r) \eta dr = u(x) + C, \quad \int_{\mathbb{R}} \lambda_U(x, r) \eta' dr = 1,
\]

where \( C \) is a constant depending only on \( \eta \). We denote the product manifold \( N \times \mathbb{R} \) with the metric \( \sigma + dr^2 \) by \( M \). Let \( \text{div}_{M_\phi} \) and \( \text{div}_M \) be the divergence on \( M_\phi \) and \( M \), respectively. Note that

\[
\text{div}_{M_\phi}(X') = \frac{1}{\phi^{n+1}} \text{div}_M(\phi^{n+1}X')
\]

(see, for example, [42, (2.11)]). Then by the definition of perimeter, we have

\[
P_\phi(U, \Omega \times \mathbb{R}) \geq \int_{\Omega \times \mathbb{R}} \lambda_U \text{div}_{M_\phi}(X') d\text{vol}_{M_\phi}
\]

\[
= \int_\Omega \left\{ \text{div}(\phi^n X) \int_{\mathbb{R}} \lambda_U \eta dr + h \phi^n \left( \int_{\mathbb{R}} \lambda_U \eta' dr \right) \right\} d\text{vol}
\]

\[
= \int_\Omega \left\{ u \text{div}(\phi^n X) + h \phi^n \right\} d\text{vol}.
\]

Taking supremum over both sides for all \( h \in C_0(\Omega) \), \( X \in T_0 \Omega \) with \( h^2 + \langle X, X \rangle \leq 1 \) yields that

\[
P_\phi(U, \Omega \times \mathbb{R}) \geq \mathcal{F}_\phi(u, \Omega).
\]

With (3.32) this yields (3.31) in the case of \( |u| \leq \mu \).

As for the general case we consider an approximation procedure based on the finite case. Let \( u_T \) be the truncation of \( u \) by \( T \), i.e.,

\[
u_T = \max\{\min\{u, T\}, -T\}.
\]

Let \( U_T \) be the subgraph of \( u_T \). Thus \( \lambda_{U_T} \) converges to \( \lambda_U \) a.e. in \( \Omega \times \mathbb{R} \). By (3.32) and the lower semicontinuity in Lemma 3.3, \( P_\phi(U_\Omega, \Omega \times \mathbb{R}) \) is finite. Note that

\[
\lim_{T \to +\infty} P_\phi(U, \Omega \times (-T, T)) = P_\phi(U, \Omega \times \mathbb{R}),
\]

\[
P_\phi(U_T, \Omega \times (-T, T)) = P_\phi(U_T, \Omega \times (-T, T)),
\]

(3.35)

and by [19, Proposition 2.8] and the definition of \( u_T \) we have

\[
\|D\lambda_{U_T}\|_{M_\phi}(\Omega \times \{\pm T\}) \leq \max_{\Omega} |\phi(x)| \int_{|u| \geq T} d\text{vol}.
\]
Since \( u \in L^1(\Omega) \), we have \( \int_{|u| > T} d\text{vol} \) converges to 0 as \( T \to +\infty \). We obtain that \( \|D\lambda_{U_T}\|_{M_\phi}(\Omega \times \{-T\}) \) converges to 0 as \( T \to +\infty \). Therefore
\[
\lim_{T \to +\infty} P_\phi(U_T, \Omega \times \mathbb{R}) = P_\phi(U, \Omega \times \mathbb{R}). \tag{3.36}
\]
As a result we conclude
\[
P_\phi(U, \Omega \times \mathbb{R}) = \lim_{T \to +\infty} P_\phi(U_T, \Omega \times \mathbb{R}) = \lim_{T \to +\infty} \mathcal{F}_\phi(u_T, \Omega) \geq \mathcal{F}_\phi(u, \Omega). \tag{3.37}
\]
Combining this with (3.32) yields the conclusion. The proof is completed. \( \square \)

As an application we obtain a decreasing property of certain Caccioppoli sets in \( M_\phi \).

**Theorem 3.10.** Let \( E \subset \Omega \times \mathbb{R} \) be a Caccioppoli set in \( M_\phi \) with the following assumption: for almost every \( x \in \Omega \), there exists a \( T_x > 0 \) such that \( \lambda_E(x, t) = 0 \) for all \( t > T_x \) and \( \lambda_E(x, t) = 1 \) for all \( t < -T_x \). Then the function
\[
w(x) = \lim_{k \to +\infty} \left( \int_{-k}^k \lambda_E(x, t) dt - k \right) \tag{3.38}
\]
is well defined and
\[
\mathcal{F}_\phi(w, \Omega) \leq P_\phi(E, \Omega \times \mathbb{R}). \tag{3.39}
\]

**Remark 3.11.** This result should be firstly observed by Miranda [31] in the case of \( \phi(x) \equiv 1 \) and \( N = \mathbb{R}^n \) (see also [2]). It is a generalization of [19, Theorem 14.8] and the symmetrization of hyperbolic spaces in [27, Remark 2.3]. It is also similar to the arrangement of constant mean curvature functionals in [9, Theorem 3.1], the decreasing perimeter property of singular area functionals in [7, Lemma 9], [16, Lemma 3.3] and [42, Lemma 5.8].

**Proof of Theorem 3.10.** By the assumption on \( E \) it is not hard to see that \( \omega(x) \) is well defined. Moreover we assume that \( E \) has finite perimeter in \( M_\phi \). Otherwise nothing needs to be proved.

First we assume that there is a \( T > 0 \) such that
\[
\partial E \subset \Omega \times (-T, T).
\]
Let \( \eta(r) \) be a compactly supported smooth function on \( \mathbb{R} \) such that \( \eta(t) \equiv 1 \) on \((-1, 1)\) and \( \eta(r) \leq 1 \). For each positive integer \( k \), define
\[
\eta_k(r) = \begin{cases} 
\eta(r + k), & r \in (-\infty, -k), \\
1, & r \in [-k, k], \\
\eta(r - k), & r \in (k, +\infty).
\end{cases} \tag{3.40}
\]
Choose \( h \in C_0(\Omega) \) and \( X \in T_0(\Omega) \) satisfying \( h^2 + \langle X, X \rangle \leq 1 \). Define
\[
X' = \frac{1}{\phi(x)} \eta_k(r)(X + h(x)\partial_r). \tag{3.41}
\]
Thus \( \langle X', X' \rangle \leq 1 \). Then we have
\[
P_\phi(E, \Omega \times \mathbb{R}) \geq \int_{\Omega \times \mathbb{R}} \lambda_E \text{div}_{M_\phi}(X') d\text{vol}_{M_\phi}
\]
\[
= \int_{\Omega \times \mathbb{R}} \lambda_E \frac{1}{\phi^{n+1}} \text{div}_{M}(\phi^n \eta_k(r)(X + h\partial_r)) \phi^{n+1} dr d\text{vol}
\]
\[
= \int_{\Omega} \left\{ \text{div}(\phi^n X) \int_\mathbb{R} \lambda_E \eta_k(r) dr + \phi^n h \int_\mathbb{R} \lambda_E \eta'_k(r) dr \right\} d\text{vol}, \tag{3.42}
\]
where $d\text{vol}_{M_\phi}$ and $\text{div}_{M_\phi}$ denote the volume form and the divergence of $M_\phi$ respectively, and $\text{div}_M$ is the divergence of the product manifold

$$M := (N \times \mathbb{R}, \sigma + dr^2).$$

By the assumption on $E$ and $\partial E \subset \Omega \times [-T,T]$ for almost every $x \in \Omega$ we have

$$\lim_{k \to +\infty} \int_{\mathbb{R}} \lambda_E(x,r)\eta_k(r)dr = 1, \quad (3.43)$$

$$\lim_{k \to +\infty} \int_{\mathbb{R}} \eta_k(r)\lambda_E(x,r)dr = w(x) + C, \quad (3.44)$$

where $C$ is a constant depending only on $\eta$ and $T$.

Let $k \to \infty$. We get

$$P_\phi(E,\Omega \times \mathbb{R}) \geq \int_{\Omega} \text{div}(\phi^n X)w + \phi^n hd\text{vol}. \quad (3.45)$$

Because $h(x) \in C_0(\Omega)$ and $X \in T_0\Omega$ are arbitrarily chosen satisfying $h^2 + \langle X,X \rangle \leq 1$, we obtain (3.39) in the case of $\partial E \subset \Omega \times (-T,T)$.

As for a general Caccioppoli set $E$ with finite perimeter, we set

$$E_T = E \cup \Omega \times (-\infty,-T)\setminus \Omega \times [T,\infty).$$

By the assumption of $E$, the sequence of sets

$$\Omega_T = \{ x \in \Omega : \lambda_{E_T}(x,r) \neq \lambda_{E}(x,r), \text{ for some } |r| > T \}$$

satisfies

$$\lim_{T \to +\infty} \text{vol}(\Omega_T) = 0.$$  

Here, we use the fact $\Omega$ is bounded. By [19, Proposition 2.8] and the definition of $E_T$ we have

$$\|D\lambda_{E_T}\|_{M_\phi}(\Omega \times \{-T, T\}) \leq \max_{\Omega} |\phi(x)|^n \int_{\Omega_T} d\text{vol}. $$

This implies that $\|D\lambda_{E_T}\|_{M_\phi}(\Omega \times \{-T, T\})$ converges to 0 as $T \to +\infty$. Note that

$$\lim_{T \to +\infty} P_\phi(E,\Omega \times (-T, T)) = P_\phi(E,\Omega \times \mathbb{R}),$$

$$P_\phi(E_T,\Omega \times (-T, T)) = P_\phi(E,\Omega \times (-T, T)).$$

By the lower semicontinuity in Lemma 3.3, we obtain

$$\lim_{T \to +\infty} P_\phi(E_T,\Omega \times \mathbb{R}) = P_\phi(E,\Omega \times \mathbb{R}). \quad (3.46)$$

Define

$$w_T(x) = \lim_{k \to +\infty} \left( \int_{-k}^{k} \lambda_{E_T}(x,t)dt - k \right).$$

By the assumption of $E$, $w_T(x)$ converges to $w(x)$ a.e. on $\Omega$ as $T \to +\infty$. Again by Lemma 3.3,

$$\mathcal{T}_\phi(w(x),\Omega) \leq \liminf \mathcal{T}_\phi(w_T(x),\Omega) \leq \liminf P_\phi(E_T,\Omega \times \mathbb{R}) = P_\phi(E,\Omega \times \mathbb{R}). \quad (3.47)$$

The proof is completed.

Theorem 3.10 and the following result together are referred as the Miranda’s observation.
Theorem 3.12. Let \( u \in BV(\Omega) \) with \( \mathcal{F}_\phi(u, \Omega) < \infty \). Suppose \( u(x) \) locally minimizes \( \mathcal{F}_\phi(\cdot, \Omega) \), i.e., \( \mathcal{F}_\phi(u, \Omega) \leq \mathcal{F}_\phi(v, \Omega) \) if \( U \Delta V \subset \subset \Omega \times \mathbb{R} \) where \( U \) and \( V \) are the subgraphs of \( u(x) \) and \( v(x) \), respectively. Then \( U \) locally minimizes the perimeter in \( M_\phi \), i.e.,

\[
P_\phi(U, \Omega \times \mathbb{R}) \leq P_\phi(F, \Omega \times \mathbb{R})
\]

for any Caccioppoli set \( F \) satisfying \( F \Delta U \subset \subset \Omega \times \mathbb{R} \).

Proof. Let \( F \subset \Omega \times \mathbb{R} \) be a Caccioppoli set satisfying \( F \Delta U \subset \subset \Omega \times \mathbb{R} \). Since \( U \) is a subgraph of \( u(x) \), it is easy to see that \( F \) satisfies the condition in Theorem 3.10. Let \( w(x) \) be defined as in Theorem 3.10. Then

\[
P_\phi(U, \Omega \times \mathbb{R}) = \mathcal{F}_\phi(u, \Omega) \leq \mathcal{F}_\phi(w(x), \Omega) \leq P_\phi(F, \Omega \times \mathbb{R}) \quad (3.48)
\]

The proof is completed. \( \Box \)

4 The area minimizing problem

In this section we consider the area minimizing problem in the conformal product manifold \( M_\phi \) (see Definition 1.1) similar to that in [15]. Throughout this section suppose \( \Omega \) and \( \Omega' \) are two bounded \( C^2 \) domains in \( N \) satisfying \( \Omega \subset \subset \Omega' \). Let \( Q_\phi \) be the set \( \Omega \times \mathbb{R} \) in \( M_\phi \) and \( \bar{Q}_\phi \) be its closure \( \bar{\Omega} \times \mathbb{R} \). Suppose \( \psi(x) \in C^1(\Omega' \setminus \Omega) \). We denote by \( n \) the dimension of \( N \). Let \( G \) denote the set of \( n \)-integer multiplicity currents with compact support in \( \bar{Q}_\phi \), i.e., for any \( T \in G \), its support \( \text{spt}(T) \) is contained in \( \Omega \times [a, b] \) for some finite numbers \( a < b \). Let \( \Gamma \) be the graph of \( \psi(x) \) on \( \partial \Omega \times \mathbb{R} \). The area minimizing problem in this section is to find a solution to attain the value

\[
\min \{ M(T) : T \in G, \partial T = \Gamma \},
\]

where \( M \) is the mass of \( T \) in \( M_\phi \) (see Subsection 4.1).

The main result of this section is stated as follows.

Theorem 4.1. Suppose that \( \phi(x) \) is a \( C^2 \) positive function on \( N \). Then there is a \( u(x) \in BV(\Omega') \) satisfying \( u(x) = \psi(x) \) outside \( \Omega \) with the following three properties:

1. \( u(x) \) realizes the minimum

\[
\min \{ \mathcal{F}_\phi(v(x), \Omega') : v(x) \in BV(\Omega'), v(x) = \psi(x) \text{ outside } \Omega \}. \quad (4.2)
\]

2. \( T = \partial [U] |_{\bar{Q}_\phi} \) solves the area minimizing problem (4.1) where \( U \) is the subgraph of \( u(x) \) in \( \Omega' \times \mathbb{R} \).

3. \( u(x) \in C^{2,\beta}(\Omega) \) satisfies (2.3) and \( \min_{\partial \Omega} \psi(x) \leq u(x) \leq \max_{\partial \Omega} \psi(x) \) for any \( x \in \Omega \) and any \( \beta \in (0, 1) \).

Remark 4.2. In the case of \( \phi(x) \equiv 1 \), the above theorem is obtained by Lin [27, Subsection 4.1]. But we do not understand the proof of its uniqueness just assuming \( \Omega \) is bounded and Lipschitz.

Remark 4.3. We show that for \( T \) in the property (2) \( \partial T = \Gamma \). For \( r > 0 \) we denote the set

\[
\{ x \in \Omega' : \text{dist}(x, \Omega) < r \}
\]

by \( \Omega_r \). When \( r \) is sufficiently small, \( \Omega_r \) is a \( C^2 \) domain. We define \( T_r := \partial [U] |_{\bar{\Omega}_r \times \mathbb{R}} \). It is obvious that \( T_r \) is a \( C^2 \) graph near \( \partial \Omega_r \). Thus \( \partial T_r = \text{graph}(\psi(x) |_{\partial \Omega_r}) \). In addition, \( T_r \) converges to \( T \) and \( \partial T_r \) converges to \( \partial T \) in the sense of current as \( r \to 0 \). Moreover \( \text{graph}(\psi(x) |_{\partial \Omega_r}) \) converges to \( \Gamma \) in the \( C^1 \) sense as \( r \to 0 \). By the definition of the current \( \partial T = \Gamma \). This is the reason that we assume \( \psi(x) \in C^1(\Omega' \setminus \Omega) \).

4.1 Integer multiplicity currents

Here, we collect some necessary facts on integer multiplicity currents. Our main references are [36, Section 27] and [26].
Let $U$ be an open domain in a Riemannian manifold $M$ with dimension $m$ and $H^j$ denote the $j$-dimensional Hausdorff measure in $M$ for any $j > 0$. Suppose that $k$ is an integer in $[0, m]$. Let $D^k(U)$ be the set of all $k$-smooth forms with compact support in $U$. A $k$-current in $M$ is a linear functional in $D^k(M)$.

**Definition 4.4.** A set $E \subset M$ is said to be countable $k$-rectifiable if

$$E \subset E_0 \bigcup_{j=1}^{\infty} F_j(E_j),$$

where $H^k(E_0) = 0$ and $F_j : E_j \subset \mathbb{R}^k \rightarrow M$ is a Lipschitz map for each $j$.

Now we can define a $k$-integer multiplicity rectifiable current.

**Definition 4.5.** Let $T$ be a $k$-current in $M$. We say that $T$ is a $k$-integer multiplicity current if

$$T(\omega) = \int_S \langle \omega, \eta \rangle \theta(x) dH^k(x), \quad (4.3)$$

where $S$ is a countable $k$-rectifiable subset of $M$, $\theta$ is a positive locally $H^k$-integrable function which is integer-valued, and $\eta$ is a $k$-form $\tau_1 \wedge \cdots \wedge \tau_k$ oriented by the tangent space of $S$ a.e. $H^k$. $T$ is also written as $\tau(S, \theta, \eta)$.

The mass of $T$ in $U$ is

$$M_U(T) := \sup \{ T(\omega) : \langle \omega, \omega \rangle \leq 1, \omega \in D^k(U) \}, \quad (4.4)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing of $k$-form. The boundary of $T$ is defined by $\partial T(\omega) = T(d\omega)$ for any $\omega \in D^{k-1}(U)$.

**Remark 4.6.** For any $k$-submanifold $M'$, $\lfloor [M'] \rfloor$ is a $k$-integer multiplicity current just by choosing $\eta$ as its orientation which is equal to $\tau(M', 1, \eta)$.

If the dimension of an integer multiplicity current $T$, written as $\tau(S, \theta, \eta)$, is equal to the dimension of $M$ we always choose $\eta$ as the volume form of $M$. In this case $T$ is written as $\tau(S, \theta)$.

A good property of integer multiplicity currents is their compactness obtained by Federer and Fleming [14] (see also [36, Theorem 27.3]).

**Theorem 4.7.** Suppose that $\{T_j\}_{j=1}^{\infty}$ is a sequence of $k$-integer multiplicity currents with

$$\sup \{ M_W(T_j) + M_W(\partial T_j) \} < \infty$$

for any open set $W \subset \subset M$. Then there is a $k$-integer multiplicity current $T$ such that $T_j$ converges weakly to $T$ and $M_W(T) \leq \lim_{j \rightarrow +\infty} \inf_{i \geq j} M_W(T_i)$.

A useful way to construct integer multiplicity currents is the pushforward of local Lipschitz maps.

**Definition 4.8.** Let $U$ and $V$ be two open sets in (different) Riemannian manifolds. Suppose that $f : U \rightarrow V$ is local Lipschitz, $T = \tau(S, \theta, \eta)$ is a $k$-integer multiplicity current and $f \vert_{spt T}$ is proper. We define $f_# T$ by

$$f_# T(\omega) = \int_S \langle \omega \vert_{f(x)}, d f_# \eta \rangle \theta(x) dH^k(x). \quad (4.5)$$

**Theorem 4.9.** Let $Q_\phi$ be the conformal cone in $M_\phi$ (see Definition 1.1) and $n = \dim N$. Let $k < n + 1$ be a positive integer. Let $T$ be a $k$-integer multiplicity current in $\tilde{G}$ satisfying $\partial T = 0$. Then there is a $(k + 1)$-integer multiplicity current $R$ in $Q_\phi$ such that $\partial R = T$. Here, $spt(R)$ may be noncompact in $Q_\phi$.

**Proof.** The proof is exactly the same as that of [15, Theorem 3.9]. Note that even with the metric $\phi^2(x)(\sigma + dr^2)$ the map $h : M_\phi \times \mathbb{R} \rightarrow M_\phi$ as $h((x, r), t) = (x, r + t)$ is still proper and local Lipschitz. Thus $h_#([[(-\infty, 0)] \times T)$ is well defined. Moreover,

$$\partial h_#([[(-\infty, 0)] \times T) = h_#(\partial([[(-\infty, 0)] \times T)).$$
Now set \( R \) as \( h_\#([(-\infty,0)]\times T) \). The conclusion follows from that \( \partial T = 0 \). \( \square \)

**Remark 4.10.** A similar proof also appears in [36, Subsection 26.26].

There is one case such that \( \text{spt}(R) \) is compact in \( \bar{Q}_\phi \) stated as follows.

Let \( \Sigma \) be a \( C^2 \) graph over \( \Omega \) with the \( C^1 \) boundary \( \Gamma = (x, \psi(x)) \) and \( S \) be a compact hypersurface in \( \bar{Q}_\phi \) with \( \partial S = \Gamma \). Let \( \pi \) be the projection from \( \bar{Q}_\phi \) into \( \bar{\Omega} \) as \( \pi(x,r) = x \). Thus \( \pi(S) \) is a compact set in \( \bar{\Omega} \). Set \( \Omega' := \bar{\Omega} \setminus \pi(S) \) to be an open set in \( \bar{\Omega} \). Let \( T = S - \Sigma \) and \( R \) be the current \( h_\#([(-\infty,0)]\times T) \).

Since there is a \( \kappa > 0 \) such that \( \text{spt}(S) \cup \text{spt}(\Sigma) \subset \bar{\Omega} \times [-\kappa, +\infty] \), the definition of \( h_\# \) gives that

\[
\bar{\Omega} \times (-\infty, -\kappa) \subset h_\#([(-\infty,0)]\times \Sigma),
\]

\[
\pi(S) \times (-\infty, -\kappa) \subset h_\#([(-\infty,0)]\times S).
\]

By the definition of integer multiplicity currents, the only possible case is that

\[
\text{spt}(R) \cap \bar{\Omega} \times (-\infty, -\kappa) = \Omega' \times (-\infty, -\kappa).
\]

This implies that

\[
\partial \Omega' \times (-\infty, -\kappa) \subset \partial R \cap \bar{\Omega} \times (-\infty, -\kappa) = \emptyset.
\]

Thus \( \Omega' \) is an empty set and \( \pi(S) = \bar{\Omega} \). This implies \( \text{spt}(R) \cap \bar{\Omega} \times (-\infty, -\kappa) = \emptyset \). As a result \( \text{spt}(R) \) is compact.

Fix two numbers \( a < b \) such that \( \Gamma \subset \partial \Omega \times (a,b) \). Now consider an auxiliary problem of (4.1) as follows:

\[
A_{a,b} := \min \{M(T) : T \in \mathcal{G}, \partial T = \Gamma, \text{spt}(T) \subset \bar{\Omega} \times [a,b] \}.
\]

By Theorem 4.7 there is an \( n \)-integer multiplicity current \( T_{a,b} \) contained in \( \bar{\Omega} \times [a,b] \) with \( \partial T_{a,b} = \Gamma \) satisfying \( M(T_{a,b}) = A_{a,b} \). We denote \( \Omega' \) to be an open domain in \( N \) satisfying \( \Omega \subset \subset \Omega' \). Similar to [15, Lemma 4.6] we have the following lemma.

**Lemma 4.11.** There is a Caccioppoli set \( F \) in \( \Omega' \times \mathbb{R} \) such that \( T_{a,b} = \partial [[F]] |_{\bar{Q}_\phi} \) and \( F \) is the subgraph of \( \psi(x) \) outside \( \bar{Q}_\phi \).

**Remark 4.12.** Because \( T_{a,b} = \partial [[F]] |_{\bar{Q}_\phi} \) is contained in \( \bar{\Omega} \times [a,b] \) one sees that for each \( x \in \Omega \) it holds that \( \lambda_F(x,t) = 1 \) for \( t < a \) and \( \lambda_F(x,t) = 0 \) for \( t > b \). If not, we can replace \( F \cap \Omega \times \mathbb{R} \), the complement of \( F \), with \( F' \cap \Omega \times \mathbb{R} \) without any change of the perimeter.

**Proof.** Since \( \psi(x) \in C^1(\Omega' \setminus \bar{\Omega}) \) we can extend \( \psi(x) \) as a \( C^1 \) function on \( \Omega' \) satisfying \( a < \psi(x) < b \) on \( \Omega' \) and its subgraph \( E \) is a Caccioppoli set in \( \Omega' \times \mathbb{R} \). Define \( S := \partial [[E]] |_{\bar{\Omega} \times [a,b]} \). Then by Remark 4.3, \( \partial S = \Gamma \).

From Theorem 4.9 there is an \( (n+1) \)-integer multiplicity current \( R \) in \( \Omega' \times (-\infty, b) \) such that \( T_{a,b} - S = \partial R \). Then we have

\[
T_{a,b} = \partial [[E]] |_{\bar{Q}_\phi} + \partial R = \partial [[E]] |_{\bar{\Omega} \times [a,b]} + \partial R |_{\bar{\Omega} \times [a,b]}.
\]

Observe that \( [[E]] + R \) can be represented as \( \tau(\Omega' \times (-\infty, b), \theta) \), where \( \theta \) is some integer value measurable function on \( N \times \mathbb{R} \). Since \( spt(R) \subset \bar{\Omega} \times (-\infty, b] \), \( \theta = \lambda_E \) outside \( \bar{Q}_\phi \), i.e.,

\[
\theta \text{ equals } 1 \text{ or } 0 \text{ outside } \bar{Q}_\phi.
\]

Now define \( U_j = \{ p \in \Omega' \times (-\infty, b) : \theta(p) \geq j \} \) for any integer \( j \). By the definition of \( E \), we have \( spt(\partial [[U_j]]) \subset Q_\phi \) for any \( j \neq 1 \). Note that \( spt(T_{a,b}) \subset \bar{\Omega} \times [a,b] \subset \bar{Q}_\phi \). Applying the decomposition theorem of integer multiplicity currents [36, Corollary 27.8] we obtain

\[
\mu_{T_{a,b}} = \sum_{j=-\infty}^{-1} \mu_{\partial [[U_j]]} + \mu_{\partial [[U_1]]} |_{\bar{Q}_\phi}
\]
This implies that
\[ \text{spt}(\mu_{\partial[U_j]}) \subset \Omega \times [a, b], \quad j \neq 1, \quad \text{spt}(\mu_{\partial[U_j]})|_{\partial F} \subset \Omega \times [a, b]. \] (4.9)

Since \( T_{a,b} \) solves (4.1), \( \{\partial[U_j], j \neq 1\} \) are closed area minimizing currents in \( \Omega \times [a, b] \). Fix any \( j \neq 1 \) such that \( \partial[U_j] \) is not an empty set. For each \( t \in \mathbb{R} \), \( \Omega' \times \{t\} \) is a minimal hypersurface in \( M_p \). Thus
\[ t_0 := \inf \{s, (s, \infty) \cap \partial[U_j] = \emptyset\} \]
is well defined. Let \( p = \langle x, t_0 \rangle \) be the tangent point in \( \Omega' \times \{t\} \cap \partial[U_j] \). Note that \( \Omega \) is a \( C^2 \) domain. By Theorem 4.17, \( \partial[U_j] \) is \( C^{1,\alpha} \) near \( p \). Thus \( H^{\partial[U_j]} \leq 0 \) with respect to the outward normal vector. On the other hand, \( H^{\Omega' \times \{t\}} = 0 \). By the maximum principle, \( \partial[U_j] \) should coincide with \( \Omega' \times \{t\} \) near \( p \) (see, for example, [15, Appendix A]). Due to the connectedness, this implies that \( \Omega' \times \{r\} \subset \partial[U_j] \). We obtain the contradiction since \( \Omega \subset \subset \Omega' \). Thus for all \( j \neq 1 \) \( \partial[U_j] \) is an empty set.

As a result \( T_{a,b} = \partial[U_1]|_{\partial F} \). Let \( F \) be the set \( U_1 \). The proof is completed. \[ \square \]

### 4.2 Regularity of almost minimal boundary

In this subsection we recall some results on the regularity of the almost minimal boundary from [15, Subsection 3.2]. All of their proofs are skipped here. We refer the readers to [10,19,39] for more details. Throughout this subsection let \( M \) be an \((n+1)\)-dimensional Riemannian manifold.

Suppose that \( T \) is an \((n+1)\)-dimensional integer multiplicity current in \( M \), represented as \( \tau(V, \theta) \) where \( V \) is an \( L^{n+1} \) measurable set of \( M \). The relationship between the mass and BV functions can be summarized as follows:
\[ M(W)(\tau) = \|D\theta\|_{M}(W) \] (4.10)
for any open set \( W \subset M \). In particular if \( E \) is a Caccioppoli set in \( W \) and \( T = [\|E\|] \) we have
\[ M(W)(\|E\|) = \|D\lambda_{E}\|_{M}(W) = P(E, W), \] (4.11)
where \( P \) is the perimeter in \( M \). For a derivation, see [36, Subsection 27.7].

For a point \( p \in M \) and \( r > 0 \) we denote by \( B_{r}(p) \) the open ball centered at \( p \) with radius \( r \). Now we define almost minimal sets in an open set and in a closed set, respectively.

**Definition 4.13.** Let \( \Omega \) be a domain. Fix \( \mu_0 \geq 0 \) and \( \alpha \in (0, \frac{1}{2}] \). Suppose that \( E \subset \Omega \) is a Caccioppoli set in \( M \).

1. We say that \( E \) is a \((\mu_0, \alpha)\)-almost minimal set in \( \Omega \) if there are an \( r_0 > 0 \) and a constant \( \mu_0 \) with the property that for any \( r < r_0, x \in \Omega \) with \( B_{r}(x) \subset \subset \Omega \),
\[ P(E, B_{r}(x)) \leq P(F, B_{r}(x)) + \mu_0 r^{n+2\alpha}, \] (4.12)
where \( F \) is any Caccioppoli set satisfying \( E \Delta F \subset B_{r}(x) \). In particular if \( \mu_0 = 0 \), we say that \( E \) is a minimal set in \( \Omega \).

2. We say that \( E \) is a \((\mu_0, \alpha)\)-almost minimal set in \( \bar{\Omega} \) if there are an \( r_0 > 0 \) and a constant \( \mu_0 \) with the property that for any \( r < r_0, x \in \bar{\Omega} \),
\[ P(E, B_{r}(x)) \leq \|D\lambda_{E\cap \Omega}\|_{M}(B_{r}(x)) + \mu_0 r^{n+2\alpha}, \] (4.13)
where \( F \) is any Caccioppoli set satisfying \( E \Delta F \subset B_{r}(x) \). In particular if \( \mu_0 = 0 \), we say that \( E \) is a minimal set in \( \bar{\Omega} \).

3. The regular set of \( \partial E \) is the set \( \{p \in \partial E : \partial E \text{ is a } C^{1,\alpha} \text{ graph in a ball containing } p \} \). The singular set of \( \partial E \) is the complement of the regular set in \( \partial E \).
Remark 4.14. By [42, Lemma 7.6] all $C^2$ bounded domains are almost minimal sets in an open neighborhood of their boundaries.

A good property of almost minimal sets in a domain is their boundary regularity.

Theorem 4.15 (See [39, Theorem 1] and [10, Theorem 5.6]). Suppose that a Caccioppoli set $E$ is a $(\mu_0, \alpha)$-almost minimal set in a domain $\Omega$. Let $S$ be the singular set of $\partial E$ in $\Omega$. Then

1. if $m \leq 7$, $S = \emptyset$;
2. if $m = 7$, $S$ consists of isolated points;
3. if $m > 7$, $H^t(S) = 0$ for any $t > m - 7$. Here, $H^t$ denotes the Hausdorff measure in $M$, where $m = \dim M$.

Remark 4.16. Note that in general the boundary of almost minimal sets in closed sets does not have such good regularity.

The following result is very important.

Theorem 4.17 (See [15, Theorem 3.19]). Let $\Omega_1$ and $\Omega_2$ be two $C^2$ domains in $\mathbb{M}$. Define $\Omega' = \Omega_1 \cap \Omega_2$. Fix a point $p$ in $\partial \Omega_1 \cap \partial \Omega_2$. Suppose that $E \subset \Omega'$ is a $(\mu_0, \alpha_0)$-almost minimal set in $\Omega'$ (the closure of $\Omega'$) and $\partial E$ passes through $p$. Then $\partial E$ is a $C^1, \alpha'$ graph in an open ball containing $p$ for some $\alpha' \in (0, 1)$.

Remark 4.18. In general the boundary of $\Omega'$ is not $C^2$.

4.3 The proof of Theorem 4.1

Proof of Theorem 4.1. Set $\alpha_0 := \min_{\partial \Omega} \psi(x)$ and $\alpha_1 := \max_{\partial \Omega} \psi(x)$. Fix $n > 0$ such that $-n < \alpha_0 \leq \alpha_1 < n$. By Lemma 4.11 there is a Caccioppoli set in $\Omega' \times \mathbb{R}$ such that $T_n = \partial ([F_n])_{\bar{T} \times [-n,n]}$ realizes the minimum in (4.6) and $F_n$ is the subgraph of $\psi(x)$ outside $\bar{Q}_\phi$.

We claim that $T_n$ has to be contained in the closed set $\bar{T} \times [\alpha_0, \alpha_1]$. If not, assume

$$r_0 := \max \{ r \in \bar{T} : p = (x, r) \in T_n = \partial ([F_n])_{\bar{T} \times [-n,n]} \} \in (\alpha_1, n).$$

Let $p_0$ be the point on $T_n$ achieving this maximum. Since $r_0 > \alpha_1$, there is an embedded ball $B_{r_0}(p_0)$ such that $B_{r_0}(p_0) \cap \Gamma = \emptyset$.

The first case is that $p_0 = (x_0, r_0)$ for some $x_0 \in \partial \Omega$. Note that $F_n$ is a minimal set in $\bar{T} \times \mathbb{R} \cap B_{r_0}(p_0)$. By Theorem 4.17 $\partial F_n$ is still a $C^{1, \alpha'}$ graph near $p_0$ for some $\alpha' \in (0, 1)$. Moreover $\partial F_n$ is tangent to $\partial \Omega \times \mathbb{R}$ and $\Omega \times \{r_0\}$ at $p_0$. But $\partial \Omega \times \mathbb{R}$ and $\Omega \times \{r_0\}$ are orthogonally transverse at $p_0$. This leads to a contradiction and the first case is impossible.

The second case is $p_0 = (x_0, r_0)$ for some $x_0 \in \Omega$. By Remark 4.12, $\partial ([F_n])_{\bar{T} \times [-n,n]} \subset \bar{T} \times [-n,n]$. Thus $F_n$ is a minimal set in $\bar{T} \times [-n,n]$ away from $\Gamma$. Note that $\Omega \times [-n,n]$ is the intersection of two $C^2$ domains. Again by Theorem 4.17 $\partial F_n$ is a $C^{1, \alpha'}$ graph in a neighborhood of $p_0$ contained in $B_{r_0}(p_0)$. But we should observe that $\Omega' \times \{r_0\}$ is also minimal in $M_\phi$ by Theorem 2.3. With respect to the upward normal vector, $H_{\partial F_n} \leq 0$ near $p_0$ in the Lipschitz sense. By the maximum principle in [15, Theorem A.1], $\partial F_n$ coincides with $\Omega' \times \{r_0\}$ near $p_0$. From the connectedness of $\Omega \times \{r_0\}$ we obtain $\Omega' \times \{r_0\} \subset \partial F_n$. This is impossible since $\partial F_n$ is contained in $\bar{T} \times [-n,n]$. Thus the second case is also impossible. This means $r_0 \leq \alpha_1$. Arguing a similar derivation we obtain

$$\min \{ r \in \bar{T} : p = (x, r) \in T_n = \partial ([F_n])_{\bar{T} \times [-n,n]} \} \geq \alpha_0.$$  

Thus the above claim is true.

For each $n$ with $-n < \alpha_0 < \alpha_1 < n$, $T_n$ is contained in $\bar{T} \times [\alpha_0, \alpha_1]$ and $\mathcal{M}(T_n)$ is uniformly bounded and $\partial T_n = \Gamma$. By Theorem 4.7 as $n$ goes to $\infty$ the sequence $\{T_n\}_{n \geq \max(\{\alpha_0, \alpha_1\})}$ will converge to a $T_\infty$ such that $T_\infty |_{\bar{Q}_\phi} \subset \bar{T} \times [\alpha_0, \alpha_1]$ and

$$\mathcal{M}(T_\infty) := \min \{ \mathcal{M}(T) : \partial T = \Gamma, T \in \mathcal{G} \}.$$

Moreover from the compactness of BV functions in Theorem 2.10 there is a Caccioppoli set $F_\infty$ such that $T_\infty = \partial ([F_\infty])_{\bar{T} \times [-n,n]}$ which is contained in $\bar{T} \times [\alpha_0, \alpha_1]$ and $F_\infty$ is the subgraph of $\psi(x)$ outside $\bar{Q}_\phi$. By
Theorem 3.10 there is a BV function \( u(x) \in BV(\Omega') \) such that \( u(x) = \psi(x) \) outside \( \Omega \) and
\[
P_\phi(U, \Omega' \times \mathbb{R}) \leq P_\phi(F_\infty, \Omega' \times \mathbb{R}),
\] (4.16)
where \( U \) is the subgraph of \( u(x) \) and
\[
\alpha_0 \leq u(x) \leq \alpha_1 \text{ on } \Omega, \quad u(x) = \psi(x) \text{ on } \Omega' \setminus \Omega.
\] (4.17)
Moreover, \( \partial(\partial[U]|_{\bar{Q}_\phi}) = \Gamma \) by Remark 4.3. Since \( F_\infty \) coincides with \( U \) outside \( \bar{Q}_\phi \), we have
\[
\mathcal{M}(\partial[U]|_{\bar{Q}_\phi}) \leq \mathcal{M}(T_\infty).
\] (4.18)
Thus \( \partial[U]|_{\bar{Q}_\phi} \) also solves the area minimizing problem (4.1). This shows the existence of \( u(x) \) with the property (2).

Now fix any \( v(x) \in BV(\Omega') \) satisfying \( v(x) = \psi(x) \) outside \( \Omega \). Let \( V \) be the subgraph of \( v(x) \). By Remark 4.3, \( \partial(\partial[V]|_{\bar{Q}_\phi}) = \Gamma \). By the property (2),
\[
\mathcal{M}(\partial[U]|_{\bar{Q}_\phi}) \leq \mathcal{M}(\partial[V]|_{\bar{Q}_\phi}).
\] (4.19)
Because \( u(x) = \psi(x) = v(x) \) outside \( \bar{\Omega} \), by Theorem 3.9 and (4.11) we obtain
\[
\mathcal{F}_\phi(u, \Omega') \leq \mathcal{F}_\phi(v, \Omega').
\] (4.20)
Thus \( u(x) \) realizes the minimum
\[
\min \{ \mathcal{F}_\phi(v(x), \Omega') : v(x) \in BV(\Omega'), \ v(x) = \psi(x) \text{ outside } \bar{\Omega} \}.
\] (4.21)
We conclude that \( u(x) \) satisfies the property (1).

At last we show that \( u(x) \) has the property (3). By the property (2) \( U \) is a minimal set in \( \Omega \times \mathbb{R} \). By Theorem 4.15, except a closed set \( S \) with \( H^{n-6}(S) = 0, \) \( \partial U \setminus S \) is connected and \( C^{1,\alpha} \) for some \( \alpha \in (0,1) \). Since \( \partial U \setminus S \) is minimal in \( \Omega \times \mathbb{R} \subset M_\phi \) and \( \phi(x) \) is \( C^2 \), the regularity of the minimal surface equation (2.3) implies that it is \( C^{2,\beta} \) for any \( \beta \in (0,1) \). Let \( \bar{v} \) be the normal vector of \( \partial U \setminus S \) in \( \Omega \times \mathbb{R} \) pointing to the positive infinity. Define \( \Theta = \langle \bar{v}, \partial_r \rangle \). Thus \( \Theta \equiv 0 \) on \( \partial U \setminus S \) in \( \Omega \times \mathbb{R} \) which is connected and \( C^{2,\beta} \).

By the Harnack principle in Corollary A.2, we have \( \Theta \equiv 0 \) or \( \Theta > 0 \) on \( \partial U \setminus S \). Since \( \partial U|_{\bar{Q}_\phi} \) is contained in \( \Omega \times [\alpha_0, \alpha_1] \) and \( H^{n-1}(S) = 0 \), only \( \Theta > 0 \) can happen in \( \partial U \setminus S \).

Let \( S' \) be the orthonormal projection of \( S \) in \( \Omega \). Therefore \( H^{n-1}(S') = 0 \) and \( u \in C^{2,\beta}(\Omega \setminus S') \).

Moreover
\[
\int_{\Omega \setminus S'} \phi^n(x)\sqrt{1 + |Du|^2}dvol \leq \mathcal{F}_\phi(u, \Omega) < \infty.
\] (4.22)
This implies that \( u \in W^{1,1}(\Omega) \). By Theorem 4.1(1), \( u(x) \) is also the critical point of the functional
\[
\int_\Omega \phi^n(x)\sqrt{1 + |Dv|^2}dvol \quad \text{for } v \in W^{1,1}(\Omega).
\] (4.23)
From the removable singularity result of Simon [35, Theorem 1], \( u(x) \) is regular at every \( x \) in \( \Omega \). Thus \( u(x) \) satisfies (2.3) in the Lipschitz sense. Since \( \phi(x) \in C^2 \), by the classical regularity of uniformly elliptic equations, \( u(x) \) is \( C^{2,\beta}(\Omega) \) for any \( \beta \in (0,1) \). The proof is completed.

5 The Dirichlet problem

In this section we apply Theorem 4.1 to solve the Dirichlet problem of minimal surface equations in \( M_\phi \).

**Definition 5.1.** We say that \( \Omega \) is \( \phi \)-mean convex if the mean curvature of \( \partial \Omega \) satisfies
\[
H_{\partial \Omega} + n(D \log \phi(x), \gamma) \geq 0.
\]
Here, \( D \) denotes the covariant derivative of \( N \) and \( H_{\partial \Omega} = \text{div}(\gamma) \) for the outward normal vector \( \gamma \) on \( \partial \Omega \).

**Remark 5.2.** By (2.2) (see also [43, Lemma 3.1]), \( \Omega \) is \( \phi \)-mean convex if and only if \( \Omega \times \mathbb{R} \) is mean convex in \( M_\phi \) with respect to the metric \( \phi^2(x)(\sigma + dr^2) \) for its outward normal vector.
5.1 The Dirichlet problem on the $\phi$-mean convex domain

The first result of this subsection is given as follows.

**Theorem 5.3.** Let $\Omega$ be a $C^2$ bounded $\phi$-mean convex domain in $N$ and let $\phi(x)$ be a $C^2$ positive function in $N$. For any $\psi(x) \in C(\partial \Omega)$ the Dirichlet problem

$$
\begin{align*}
L(u) & = -\text{div} \left( \frac{Du}{\omega} \right) + n \left( D\log \phi(x), -\frac{Du}{\omega} \right) = 0 \quad \text{on} \quad \Omega, \\
u(x) & = \psi(x) \quad \text{on} \quad \partial \Omega
\end{align*}
$$

(5.1)

has a unique solution in $C(\bar{\Omega}) \cap C^2(\Omega)$. Here $\omega = \sqrt{1 + |Du|^2}$.

**Remark 5.4.** Suppose that $\Omega$ is $C^2,\alpha$ for some $\alpha \in (0, 1)$. Casteras et al. [8, Theorem 2] showed that if there is a positive constant $F > 0$ such that

1. $m(x) \in C^2(N)$, $r(t) \in C^1(\mathbb{R})$ satisfying
   $$\max_{\bar{\Omega}} |Dm(x)| + \max_{t \in \mathbb{R}} |r'(t)| \leq F;$$

2. the Ricci curvature of $\Omega$ satisfies $\text{Ric}_\Omega \geq -\frac{F^2}{n}$ and $H_{\partial \Omega} \geq F$,

then the Dirichlet problem

$$
-\text{div} \left( \frac{Du}{\omega} \right) + \left( D(m(x)), -\frac{Du}{\omega} \right) + r'(u(x)) \frac{1}{\omega} = 0 \quad \text{on} \quad \Omega
$$

(5.2)

with $u(x) = \psi(x)$ on $\partial \Omega$ for any $\psi(x) \in C(\partial \Omega)$ has a solution in $C^2,\alpha(\Omega) \cap C(\bar{\Omega})$.

In the case of $r(t) \equiv 0$ Theorem 5.3 removes the curvature assumption on $\Omega$ in [8, Theorem 2]. Our mean curvature assumption on $\partial \Omega$ should be optimal. For example when $\phi(x) \equiv 1$ this is confirmed by Serrin [34] in Euclidean spaces.

**Remark 5.5.** The above result cannot be obtained by the continuous method in [17, Subsections 11.2 and 11.3 and Chapter 18]. The reason is that the boundary assumption may not be preserved by these methods.

**Proof of Theorem 5.3.** The uniqueness of the Dirichlet problem (5.1) is obvious by the maximum principle. We only need to show the existence.

First we assume that $\psi(x) \in C^2(\partial \Omega)$. Without loss of generality we can assume $\psi(x) \in C^1(\Omega \setminus \Omega')$ for some $\Omega'$ strictly containing $\Omega$. Let $Q_\phi$ be the set $\Omega \times \mathbb{R}$ and $Q_{\phi}$ be its closure. By Theorem 4.1 there is a $u(x) \in BV(\Omega')$ with $u(x) = \psi(x)$ outside $\Omega$ such that $T = \partial |U|$ realizes the area minimizing problem (4.1). Moreover $u(x) \in C^2(\Omega)$. As a result $\partial U \cap Q_{\phi}$ is a minimal graph in $Q_{\phi}$. By Theorem 2.3(1), $u(x)$ satisfies $Lu = 0$ on $\Omega$. It only suffices to show that $u(x) \in C(\Omega)$.

Fix any $z \in \partial \Omega$. Now define

$$
A := \sup \left\{ \limsup_{n \to \infty} u(x_k) : \{x_n\}_{n=1}^\infty \in \Omega, \lim_{n \to \infty} \text{dist}(x_n, z) = 0 \right\},
$$

(5.3)

where dist is the distance function in $N$. By Theorem 4.1 $\min_{\partial \Omega} \psi \leq u(x) \leq \max_{\partial \Omega} \psi$ for any $x \in \Omega$. Thus $A$ is a finite number. Suppose $A > \psi(z)$. There is a sequence $\{x_n\}_{n=1}^\infty$ in $\Omega$ such that $\lim_{n \to +\infty} x_n = z$ and $\lim_{n \to +\infty} \psi(x_n) = A$. Since $\psi(x)$ is continuous at $z$, there is a neighborhood $V$ of the point $(z, A)$ such that $V$ is disjoint with $\Gamma$, the graph of $\psi(x)$ in $\partial \Omega \times \mathbb{R}$.

By Theorem 4.1, $U$ is a minimal set in $V \cap Q_{\phi}$ and passes through the point $(z, A)$. Since $\partial \Omega \times \mathbb{R}$ is $C^2$, Theorem 4.17 implies that $\partial U$ is a $C^{1,\alpha}$ graph near $(z, A)$. Since $U$ is a minimal set in $V \cap Q_{\phi}$, $H_{\partial U} \leq 0$ near $(z, A)$ with respect to the outward normal vector of $\Omega \times \mathbb{R}$ in the Lipschitz sense. Since $\Omega$ is $\phi$-mean convex, by Remark 5.2, $\Omega \times \mathbb{R}$ is a mean convex domain in $M_{\phi}$. Because $U \cap V \subset \Omega \times \mathbb{R}$ and $\partial U$ is tangent to $\partial \Omega \times \mathbb{R}$, from the maximum principle in [15, Theorem A.1], $\partial U$ coincides with $\partial \Omega \times \mathbb{R}$ in $V$. This contradicts the definition of $A$. Thus

$$
\limsup_{n \to \infty} u(x_n) \leq \psi(z), \quad \forall \{x_n\}_{n=1}^\infty \in \Omega, \quad \lim_{n \to \infty} x_n = z.
$$

(5.4)
With a similar derivation we also obtain
\[
\liminf_{n \to \infty} u(x_n) \geq \psi(z), \quad \forall \{x_n\}_{n=1}^{\infty} \in \Omega, \quad \lim_{n \to \infty} x_n = z.
\] (5.5)

Combining the above two facts yields that \(u(x)\) is continuous for any fixed \(z \in \partial \Omega\), thus \(u(x) \in C(\Omega)\). As a result we obtain the existence of the Dirichlet problem (5.1) when \(\psi(x) \in C^2(\partial \Omega)\).

Now we consider the general case of \(\psi(x) \in C(\partial \Omega)\). Then there are two monotone sequences \(\{\psi_k^\pm(x)\}_{k=1}^{\infty}\) in \(C^2(\partial \Omega)\) such that
\[
\cdots \leq \psi_k^-(x) \leq \psi_{k+1}^-(x) \leq \cdots \leq \psi_k^+(x) \leq \psi_{k+1}^+(x) \leq \cdots
\] (5.6)
and both of them converge to \(\psi(x)\) in the \(C^0(\partial \Omega)\). By the previous argument for any positive integer \(k\) let \(u_k^\pm (x)\) be the solutions of the Dirichlet problem (5.1) on \(\Omega\) with \(u_k(x) = \psi_k^\pm (x)\) on \(\partial \Omega\). Applying the maximum principle on (5.1), we obtain

1. \(\max_{x \in \Omega} |u_k(x)| \leq C\) where \(C = \max_{k=1, \ldots, x \in \partial \Omega} \{\psi(x), \psi_k^\pm (x)\}\);
2. \(u_k^-(x) \leq u_{k+1}^-(x) \leq \cdots \leq u_l^+(x) \leq \cdots \) on \(\Omega\).

According to Theorem A.3, the sequence \(\{u_k^\pm (x)\}_{k=1}^{\infty}\) locally has a uniformly bound of their \(C^2\) norms in \(\Omega\). Now we can choose a subsequence, still written as \(\{\psi_k^\pm (x)\}_{k=1}^{\infty}\) such that \(\lim_{k \to \infty} u_k^-(x) = u(x)\) on \(\Omega\) and \(u(x) \in C^2(\Omega)\) and satisfies \(Lu = 0\) on \(\Omega\).

The only thing left is to show that \(u(x) \in C(\overline{\Omega})\) and \(u(x) = \psi(x)\) on \(\partial \Omega\). By the conclusion (2) above applying the maximum principle on \(Lu = 0\) yields that
\[
u_k^-(x) \leq u(x) \leq u_l^+(x) \quad \text{on} \quad \Omega
\] (5.7)
for any positive integers \(k\) and \(l\). This implies that for any \(z \in \partial \Omega\),
\[
\psi_k^-(z) \leq \lim_{x \to z, x \in \Omega} \inf u(x) \leq \lim_{x \to z, x \in \Omega} \sup u(x) \leq \psi_l^+(z).
\] (5.8)

Let \(k\) and \(l\) go to the positive infinity. We obtain \(\lim_{x \to z, x \in \Omega} u(x) = \psi(z)\) for any \(z \in \partial \Omega\). Thus we define \(u(z) = \psi(z)\) on \(\partial \Omega\) and \(u(x) \in C(\overline{\Omega})\). The proof is completed. \(\square\)

### 5.2 The uniqueness of area minimizing currents

Now by Theorem 5.3 we shall obtain a uniqueness result for the area minimizing problem (4.1).

**Theorem 5.6.** Suppose that \(\Omega\) is a \(\phi\)-mean convex \(C^2\) domain and \(\psi(x) \in C^1(\partial \Omega)\). Set
\[
\Gamma = \{(x, \psi(x)) : x \in \partial \Omega\}.
\]

Let \(u(x)\) be the solution of the Dirichlet problem (5.1) with boundary data \(\psi(x)\). Let \(U(x)\) be the subgraph of \(u(x)\).

1. Then \(T = \partial ||U|| \mid_{Q_+}^\phi\) is the unique \(n\)-integer multiplicity current in \(G\) to realize
\[
\min\{M(T) : T \in \mathcal{G}, \partial T = \Gamma\}.
\] (4.1)

2. Let \(\Omega'\) be a domain with \(\Omega \subset \subset \Omega'\). Extend \(\psi(x)\) as a function \(C^1(\Omega' \setminus \Omega)\). Let \(u(x) = \psi(x)\) outside \(\Omega\). Then \(u(x)\) is a unique function to realize
\[
\min\{\mathcal{F}_\phi(v(x), \Omega) : v(x) = \psi(x) \text{ outside } \Omega, \ v(x) \in BV(\Omega')\}.
\] (4.2)

**Remark 5.7.** This theorem is a finite version of [23, Theorem 4.1] in finite conformal cones in \(M_\phi\).

**Proof of Theorem 5.6.** By Theorem 5.3, let \(T\) be an \(n\)-integer multiplicity current with compact support in \(G\) to realize (4.1). For any \(t \in \mathbb{R}\), let \(U_t\) be the subgraph of \(u(x) + t\) over \(\Omega\). By (5.1), \(\partial ||U_t||\) is a minimal graph over \(\Omega\) in \(M_\phi\) and the boundary of \(\partial ||U_t||\) is disjoint with \(\Gamma\) if \(t \neq 0\).

Now the following number:
\[
t_0 = \inf\{t > 0 : \partial ||U_t|| \cap T = \emptyset \forall s \in [t, +\infty)\}
\] (5.9)
is well defined since \(\text{spt}(T)\) is compact. Suppose \(t_0 > 0\). Then \(\partial([U_{t_0}])\) is tangent to \(T\) at some point \(p \in \Omega \times \mathbb{R}\).

By Lemma 4.11, \(T = (\partial([U]))|_{\tilde{Q}_p}\) where \(F\) is a minimal set in the set \(\bar{\Omega} \times \mathbb{R}\). From the definition of \(t_0\), \(F \cap \Omega \times \mathbb{R} \subset U_{t_0}\). By Theorem 4.17, \(\partial F\) is \(C^{1,\alpha}\) near \(p\). In the following, let \(H\) denote the mean curvature. Thus \(H_{\partial F} \leq 0\) near \(p\) with respect to the upward normal vector in the Lipschitz sense. Note that \(H_{\partial U_{t_0}} = 0\) near \(p\). Then the maximum principle in [15, Theorem A.1] implies that \(\partial F\) coincides with \(\partial U_{t_0}\) near \(p\). By the connectedness of \(\partial U_{t_0}\), we have \(\partial U_{t_0} \subset T\). Thus for some \(x_0 \in \partial \Omega\), the point

\[
p_0 = (x_0, u(x_0) + t_0)
\]

is contained in \(T\).

Since \(t_0 > 0\), there is a neighborhood \(V\) of \(p_0\) disjoint with \(\Gamma\). Then \(F\) is a minimal set in \(V \cap \bar{\Omega} \times \mathbb{R}\). By Remark 5.2, \(\Omega \times \mathbb{R}\) is mean convex in \(M_{\phi}\). Again by [15, Theorem A.1], \(\partial \Omega \times \mathbb{R}\) coincides with \(T\) near \(p_0\). With the connectedness, the set \(\{(x, r) : x \in \partial \Omega, r > u(x) + t_0\}\) is contained in \(T\). This is impossible since \(\text{spt}(T)\) is compact. Thus \(t_0 = 0\) and \(\text{spt}(T) \subset \{(x, r) : x \in \bar{\Omega}, r \leq u(x)\}\). With a similar derivation, we can show that

\[
\text{spt}(T) \subset \{(x, r) : x \in \bar{\Omega}, r \geq u(x)\}.
\]

Finally combining the above two facts yields that \(T = \partial([U])|_{\tilde{Q}_p}\), where \(U\) is the subgraph of \(u(x)\). We obtain the conclusion (1).

Suppose that there are two functions \(u_1\) and \(u_2\) in \(BV(\Omega')\) to solve (4.2). Let \(U_i\) be the subgraph of \(u_i(x)\) and \(T_i = \partial([U_i])|_{\tilde{Q}_p}\) for \(i = 1, 2\). By Theorem 4.1(2), \(T_i\) should solve (4.1). By the conclusion (1) we just obtained, \(T_1 = T_2\) and \(u_1(x) = u_2(x)\) on \(\Omega\). We arrive at the conclusion (2). The proof is completed.

\[\Box\]

### 6 Infinity boundary cases

In this section, we apply the results in the previous sections to generalize the existence and the uniqueness of area minimizing graphs with the infinity star-shaped boundary in hyperbolic spaces.

Throughout this section, we assume that \(N\) is a compact \(n\)-dimensional Riemannian manifold with its metric \(\sigma\) and compact embedded \(C^2\) boundary \(\partial N\). We define

\[
N_r := \{d(x) > r : x \in N\},
\]

where \(d(x)\) denotes \(d(x, \partial N)\), the distance between \(x\) and \(\partial N\). The following lemma is obvious.

**Lemma 6.1.** Since \(\partial N\) is \(C^2\) embedded and compact, there is an \(r_0 > 0\) such that for any \(x \in N \setminus N_{r_0}\), there is a unique \(y \in \partial N\) such that \(d(x) = \text{dist}(x, y)\).

From now on fix \(r_0\). Thus \(d(x)\) is \(C^2\) on \(N \setminus N_{r_0}\). Suppose \(\phi(x)\) is a positive \(C^2\) function on \(N\) with

\[
\phi(x) = h(d(x)) \quad \text{on} \quad N \setminus N_{r_0}, \quad \lim_{d(x) \to 0} \phi(x) = \lim_{d(x) \to 0} h(d(x)) = +\infty,
\]

where \(h(r) : (0, r_0) \to \mathbb{R}^+\) is a positive \(C^2\) function. Then

\[
M_{\phi} := (N \times, \phi^2(\sigma + dr^2))
\]

is a Riemannian manifold (possibly incomplete). A natural compactification is

\[
\tilde{M}_{\phi} = M_{\phi} \cup (\partial N \times \mathbb{R})
\]

equipped with the product metric topology of \((N \times \mathbb{R}, \sigma + dr^2)\).

**Definition 6.2.** Let \(S\) be a complete \(k\)-integer multiplicity current in \(M_{\phi}\). Its infinity asymptotic boundary in \(\partial N \times \mathbb{R}\) is the set \(\tilde{S} \setminus S\), where \(\tilde{S}\) is the closure of \(S\) in the product metric topology.
Definition 6.3. Let $M$ be the mass of currents in $M_\phi$. We say that $T$ is a local area minimizing $n$-integer multiplicity current in $M_\phi$ such that if spt$(T)$ is contained in $N \times [a, b]$ for some finite interval $[a, b]$ for any $n$-integer multiplicity current $T'$ satisfying $T = T'$ outside a compact set in $M_\phi$, then $M(T) \leq M(T')$.

The main result of this section is given as follows.

Theorem 6.4. Let $N_r, \phi, r_0$ be given in (6.1), (6.2) and Lemma 6.1, respectively. Suppose for any $r \in (0, r_0)$, $N_r$ is $\phi$-mean convex. Fix any $\psi(x) \in C^1(\partial N)$. Let $\Gamma$ denote $\{(x, \psi(x)) : x \in \partial N\}$ in $\partial N \times \mathbb{R}$. Then there is a unique local area minimizing $n$-integer multiplicity current $T$ in $M_\phi$ with the infinity boundary $\Gamma$. Moreover, $T$ is a graph over $N$.

Remark 6.5. Recall that $\mathbb{H}^{n+1}$ is the upper half space $\{(x, y) : x \in \mathbb{R}^n, y > 0\}$ equipped with the metric
\[
ds_H^2 = \frac{dx^2 + dy^2}{y^2}, \quad dx^2 \text{ is the Euclidean metric in } \mathbb{R}^n.
\]
Let $S^n_+$ be the upper half hemisphere in $\mathbb{R}^{n+1}$ with the induced metric $\sigma_n$. We introduce $(\theta, \varphi) \in S^{n-1} \times [0, \frac{\pi}{2})$ as the polar coordinate on $S^n_+$. Thus $\sigma_n = d\varphi^2 + \sin^2(\varphi)d\theta^2$. Note that $dx^2 + dy^2 = s^2\sigma_n + ds^2$ for $s \in (0, \infty)$ and $x_{n+1} = s\cos(\varphi)$. Letting $r = \ln s$, we can represent hyperbolic spaces $\mathbb{H}^{n+1}$ as
\[
\left(S^n_+ \times \mathbb{R}, \frac{1}{\cos^2(\varphi)}(\sigma_n + dr^2)\right),
\]
which is a special case of (1.1).

The above theorem is to generalize Hardt-Lin’s result in hyperbolic spaces $\mathbb{H}^{n+1}$ [23, Theorem 4.1]. We verify this claim as follows. Fixing any number $\varphi_0 \in (0, \frac{\pi}{2})$, we define a domain
\[
S_{\varphi_0} := \{(\theta, \varphi) \in S^n_+ : \varphi \in [0, \varphi_0)\}.
\]
By [41, Proposition 2.1], the mean curvature of $\partial S_{\varphi_0}$ with respect to $\frac{\partial}{\partial \varphi}$ is $(n - 1)\frac{\cos(\varphi_0)}{\sin(\varphi_0)}$. By (2.2), the mean curvature of $\partial S_{\varphi_0} \times \mathbb{R}$ in $\mathbb{H}^{n+1}$ is
\[
H_{\partial S_{\varphi_0} \times \mathbb{R}} = \cos(\varphi)\left\{(n - 1)\frac{\cos(\varphi)}{\sin(\varphi)} + n\left<D\log\frac{1}{\cos(\varphi)}, \frac{\partial}{\partial \varphi}\right>\right\}_{\varphi = \varphi_0} = \cos(\varphi_0)\left\{(n - 1)\frac{\cos(\varphi_0)}{\sin(\varphi_0)} + n\frac{\sin(\varphi_0)}{\cos(\varphi_0)}\right\}.
\]
Thus $S_{\varphi_0} \times \mathbb{R}$ is mean convex in $M_\phi$ and $S_{\varphi_0}$ is $\phi$-mean convex by Remark 5.2. Then $\mathbb{H}^{n+1}$ satisfies the condition in the above theorem. Thus [23, Theorem 4.1] is a special case of Theorem 6.4.

Now we are ready to show Theorem 6.4.

Proof of Theorem 6.4. First, we show the existence of a minimal graph in $M_\phi$ with the infinity boundary $\Gamma$. The local area minimizing property can be easily obtained by Theorem 4.1.

Step 1 (The existence). First assume $\psi(x) \in C^2(\partial N)$. Since $\partial N$ is compact, by Lemma 6.1 we extend $\psi(x)$ into a $C^2$ function on $N$ such that
\[
\psi(x) = \psi(y) \quad \text{for any } x \in N \setminus N_r \text{ with } d(x, y) = d(x, \partial N).
\]
Because $N_r$ for each $r < r_0$ is $\phi$-mean convex, by Theorem 5.3 there is a $u_r(x)$ in $C^2(\Omega) \cap C(\bar{\Omega})$ to solve the Dirichlet problem
\[
\begin{cases}
Lu := -\text{div}\left(\frac{Du}{\omega}\right) + n\left<D\log \phi(x), -\frac{Du}{\omega}\right> = 0 \quad \text{on } N_r, \\
u(x) = \psi(x) \quad \text{on } \partial N_r.
\end{cases}
\]
Let $\mu$ be a positive constant such that
\[
\max_{x \in N \cup \partial N} |\psi(x)| \leq \mu.
\]
By the maximum principle, for each \( r \in (0, r_0) \),
\[
\max_{N_r} |u_r| \leq \mu. \tag{6.11}
\]

For any fixed embedded open ball \( B_\varepsilon(x) \) with \( \text{dist}(x, \partial N) > 2\varepsilon \), Theorem A.3 implies that
\[
\max_{B_\varepsilon(x)} \{|u_r|(x), |Du_r|(x), |D^2u_r|(x)\} \leq \mu_1, \tag{6.12}
\]
where \( \mu_1 \) is a constant depending on \( \psi \) and \( \phi \) on \( B_2(x) \) and \( \mu \). Thus after one chooses a subsequence from \( \{u_r(x)\}_{r<r_0} \), denoted by \( u_k(x) \), \( u_k(x) \) converges uniformly to \( u_\infty(x) \) in the \( C^2 \) sense on any compact set of \( N \) disjoint with \( \partial N \). Thus
\[
Lu_\infty(x) = 0 \quad \text{on } N. \tag{6.13}
\]

Next, we show
\[
\lim_{x \to z, x \in N} u_\infty(x) = \psi(z)
\]
for any \( z \in \partial N \). To achieve this goal we construct the supersolution and subsolution of \( Lu = 0 \) in (6.10) on \( N_r \) for some \( r \in (0, r_0) \).

**Lemma 6.6.** Let \( \psi(x) \) satisfy (6.9), \( \mu \) be a constant given in (6.11) and \( r_0 \) be given in Definition 6.2. Suppose that for any \( r \in (0, r_0) \), \( N_r \) is \( \phi \)-mean convex. Then there are three positive constants \( r_1 \in (0, r_0) \), \( \nu > 0 \) and \( \kappa > 0 \) such that
\[
\begin{align*}
(u_+(x) = \psi(x) + \varphi(d(x)) & \quad \text{on } N \setminus N_{r_1}, \quad \varphi(r) := \frac{1}{\nu} \log(1 + \kappa r), \tag{6.14} \\
Lu_+(x) & \geq 0, \quad Lu_-(x) \leq 0 \quad \text{on } N \setminus N_{r_1}, \tag{6.15} \\
u u_+(x) & \geq 2\mu, \quad u_-(x) \leq -2\mu \quad \text{on } \{x \in N : d(x, \partial N) = r_1\}. \tag{6.16}
\end{align*}
\]

**Proof.** Let \( \{x_1, \ldots, x_n\} \) be a local coordinate in \( N \) and \( \{\partial_i\}_{i=1}^n \) be the corresponding frame. We write \( \langle \partial_i, \partial_j \rangle \) as \( \sigma_{ij} \) and \( (\sigma^{ij}) = (\sigma_{ij})^{-1} \). For any smooth function \( f, f_i \) and \( f_{ij} \) denote the first and second covariant derivatives of \( f \) and \( f^i = \sigma^i_k f_k \) with the sum over \( k \).

Note that
\[
\mathcal{E}_u := \omega^3 Lu = (1 + |Du|^2)\{-\Delta u + n\langle D \log \phi(x), -Du \rangle\} + u^i u^j u_{ij}. \tag{6.17}
\]
Here, \( \omega = \sqrt{1 + |Du|^2} \), \( \Delta u = \text{div}(Du) \) and \( D \) denote the Laplacian and the covariant derivative of \( N \), respectively. Now assume \( u_\pm(x) \) are given in (6.14), where \( \nu \) and \( \kappa \) are determined later.

By (6.2) and (6.9) on \( N \setminus N_{r_0} \) we have
\[
\langle D\psi, Dd \rangle = 0
\]
and
\[
\langle D \log \phi(x), -Du_+(x) \rangle = \varphi'(D \log \phi(x), -Dd), \tag{6.18}
\]
where \( d(x) = d(x, \partial N) \) and \( \varphi' = \frac{1}{\nu(1 + \kappa d(x))} \). With (6.18) we compute \( \mathcal{E}u_+ \) on \( N \setminus N_{r_0} \) as follows:
\[
\mathcal{E}(u_+) = \varphi'(1 + |Du_+|^2)(-\Delta d + n\langle D \log \phi, -Dd \rangle) - (1 + |D\psi|^2)\Delta \psi
\]
\[
+ \psi^i \psi^j \psi_{ij} + (\varphi')^2(-\Delta \psi + d^i d^j \psi_{ij}) - (1 + |D\psi|^2)\varphi''.
\]
Since \( N_r \) is \( \phi \)-mean convex for each \( r \in (0, r_0) \) and \( \varphi' > 0 \), the first term above is nonnegative. By (6.14), \( \varphi'' = -\nu \varphi'^2 \). Now assuming \( \varphi' \geq 1 \), \( \mathcal{E}(u_+) \) on \( N \setminus N_{r_0} \) satisfies
\[
\mathcal{E}u_+ \geq \nu \varphi'^2 - C(\varphi'^2) \tag{6.19}
\]
for some positive constant \( C \) only depending on \( \psi(x) \). Now take \( \nu = C \). We obtain \( \mathcal{E}u_+ \geq 0 \) on \( N \setminus N_{r_0} \). This shows (6.15) for \( u_+ \) and we have to determine \( \kappa \) and \( r_1 \in (0, r_0) \) such that \( \varphi'(r) \geq 1 \) and \( \varphi(r_1) \geq \mu_1 := 2\mu + \max_N \psi(x) \). We have
\[
\varphi'(r) = \frac{1}{\nu} \frac{\kappa}{1 + \kappa r} > \frac{1}{\nu} \frac{\kappa}{1 + \kappa r_1} \geq 1, \tag{6.20}
\]
\[ \varphi(r_1) = \frac{1}{\nu} \log(1 + \kappa r_1) \geq \mu_1. \]  
(6.21)

All the conditions are satisfied provided we take \( r_1 \) small enough and large \( \kappa \) satisfying

\[ \kappa \geq \max \left\{ \frac{\nu}{1 - r_1 \nu}, \frac{e^{\nu + \nu}}{r_1} \right\}. \]  
(6.22)

This gives (6.16) for \( u_+(x) \).

A similar derivation yields the conclusions for \( u_-(x) \). The proof is completed. \( \square \)

Now we continue to show Theorem 6.4. Now for any \( r \in (0, r_1) \), with the maximum principle, (6.10) and (6.11), Lemma 6.6 implies that

\[ u_-(x) \leq u_r(x) \leq u_+(x) \quad \text{on} \quad N_r \setminus N_{r_0}. \]

Let \( r \) go to 0. We obtain that

\[ u_-(x) \leq u_\infty(x) \leq u_+(x) \quad \text{on} \quad N \setminus N_{r_1}. \]  
(6.23)

Since both \( u_+(x) \) and \( u_-(x) \) are continuous on \( N \cup \partial N \) and equal to \( \psi(x) \) on \( \partial N \), we have

\[ \lim_{x \in N, x \rightarrow z \in \partial N} u_\infty(x) = \psi(z). \]  
(6.24)

Let \( U \) be the subgraph of \( u_\infty(x) \) in \( M_\phi \). Let \( T = \partial([U]) \) be the corresponding integer multiplicity current. Thus with respect to the product topology of \( N \times \mathbb{R}, \partial T = \Gamma \).

Now we obtain the existence of a local integer multiplicity current in \( M_\phi \) with the desirable infinity boundary \( \Gamma \) for any \( \psi \in C^2(\partial N) \). Moreover it is a minimal graph over \( N \) in \( M_\phi \).

**Step 2** (The general case). As for \( \psi(x) \in C(\partial N) \), we can construct two monotone sequences \( \{u_1^k(x)\}_{k=1}^\infty \) and \( \{u_2^k(x)\}_{k=1}^\infty \) in \( C^2(\partial N) \) such that the former one converges increasingly to \( \psi(x) \) and the latter one converges decreasingly to \( \psi(x) \) in the \( C^0(\partial N) \) sense. Let \( \{u_1^k(x)\}_{k=1}^\infty \) be the solutions of \( Lu = 0 \) on \( N \) with the asymptotic value \( \{\psi_i^k(x)\}_{k=1}^\infty \) for \( i = 1, 2 \). From the maximum principle and the translating invariant of minimal graphs, we have that \( \{u_1^k(x)\} \) is an increasing sequence on \( N \) and \( u_2^k(x) \leq u_1^k(x) \) for any \( k \) and \( l \). Note that both sequences are uniformly bounded. By the interior estimate of \( Lu = 0 \) in \( N \) (see Theorem A.3), \( u_k(x) \) locally converges to a \( C^2 \) function \( u_\infty(x) \) in \( N \) satisfying \( Lu_\infty = 0 \). Thus for any \( k \) and \( l \), on \( N \) we obtain

\[ u_1^k(x) \leq u_\infty(x) \leq u_2^l(x). \]  
(6.25)

Thus (6.24) still holds for \( \psi(x) \in C(\partial N) \). As a result for any \( \psi(x) \in C(\partial N) \) we show that there is a minimal graph \( \Sigma = (x, u(x)) \) in \( M_\phi \) with the infinity boundary \( \Gamma \). It is also a local area minimizing integer multiplicity current according to Theorem 4.1.

**Step 3** (The uniqueness). Suppose that \( T \) is a local area minimizing \( n \)-integer multiplicity current with infinity boundary \( \Gamma = (x, \psi(x)) \). For any \( t \in \mathbb{R} \), we define

\[ f_t(x, t) = (x, r + t) \]

for any \( x \in N \) and \( t \in \mathbb{R} \). By (1.1), \( f_t \) is an isometry of \( M_\phi \). Thus, \( f_t \# T \) is also a local area minimizing \( n \)-integer multiplicity current in \( M_\phi \) with infinity boundary

\[ \Gamma_t = (x, \psi(x) + t). \]

Recall that \( \text{spt}(T) \subset N \times [a, b] \). We claim that \( f_t \#(T) \cap T = \emptyset \) for any \( t \neq 0 \). Otherwise there is a \( t \neq 0 \) such that the regular part of \( f_t \#(T) \) and \( T \) intersect transversely. Since \( \Gamma_t \cap \Gamma = \emptyset \) for any \( t \neq 0 \), by the area minimizing property, we can replace a piece of \( T \) by the piece of \( f_t \# \) that is cut off by \( T \) and still have a local area minimizing property. But then the singular set of the resulting integer multiplicity current would include the intersection of \( T \) and \( f_t \#(T) \) in \( n - 1 \) dimensions. This contradicts the \( n - 7 \)
dimensional singular set of local area minimizing integer multiplicity currents. Thus $T$ is a graph over $N$. By the arguments similar to those in the proof of Theorem 4.1, $T$ is a $C^2$ minimal graph in $M_\phi$ with the infinity boundary $\Gamma$. Since $N \cup \partial N$ is compact, applying the maximum principle into $Lu = 0$ yields the uniqueness of the minimal graph.

The proof is completed.

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Appendix A  The interior estimate of mean curvature equations

Throughout this section let $N$ be a complete Riemannian manifold with a metric $\sigma$ and $f(x)$ be a $C^2$ function in $N$. Let $M_0$ be the product manifold $N \times \mathbb{R}$ equipped with the metric $\sigma + dr^2$. Let $D$ be the covariant derivative of $N$. Let $S$ be an orientable $C^2$ hypersurface in $M_0$ with the normal vector $\vec{v}$ for any $\beta_i \in (0, 1)$. We have the following theorem about its angle function $\langle \vec{v}, \partial_r \rangle$. Compared with [8, Subsection 2.3] we use the maximum principle method of Korevaar and Simon [24] to obtain the interior estimate of the type of mean curvature equations in this paper.

Lemma A.1. Let $\Theta = \langle \vec{v}, \partial_r \rangle$. Suppose the mean curvature of $S$ with respect to $\vec{v}$ satisfies that
\[
H_S + \langle Df, \vec{v} \rangle = 0, \tag{A.1}
\]
where $D$ is the gradient of $f$ on $N$. Then in the Lipschitz sense it holds that
\[
\Delta \Theta + (|A|^2 + \bar{\text{Ric}}(\vec{v}, \vec{v}) - \text{Hess}(f)(\vec{v}, \vec{v}))\Theta + \langle \nabla \Theta, Df \rangle = 0, \tag{A.2}
\]
where $|A|^2$ is the second fundamental form of $S$, $\nabla$ is the covariant derivative of $S$, $\bar{\text{Ric}}$ is the Ricci curvature of the product manifold $M_0$ and $\text{Hess}$ is the Hessian of $f$ in $M_0$.

Proof. Notice that $S$ is a $C^3$ hypersurface. We denote $\{e_i\}_{i=1}^n$ to be a local orthonormal frame of $S$. Note that the derivation of [42, Lemma 2.2] is true for any dimension, i.e., by [42, Lemma 2.2] we have
\[
\Delta \Theta + (|A|^2 + \bar{\text{Ric}}(\vec{v}, \vec{v}))\Theta - \langle \nabla H, \partial_r \rangle = 0, \tag{A.3}
\]
where $H$ is the mean curvature of $S$. Let $\bar{D}$ be the covariant derivative of $M_0$. Since $-H = \langle Df, \vec{v} \rangle$,
\[
- \langle \nabla H, \partial_r \rangle = \langle e_i, \partial_r \rangle \langle \bar{D}_e, Df, \vec{v} \rangle + \langle e_i, \partial_r \rangle \langle Df, e_k \rangle h_{ik}, \tag{A.4}
\]
where $\bar{D}_e\bar{v} = h_{ik}e_k$. Note that
\[
\langle \partial_r, e_i \rangle e_i = \partial_r - \langle \bar{v}, \partial_r \rangle \bar{v}.
\]
Thus
\[
\langle e_i, \partial_r \rangle \langle \bar{D}_e, Df, \bar{v} \rangle = -\Theta \text{Hess}(f)(\bar{v}, \bar{v}).
\]
(A.5)

Here, we use the fact $\bar{D}_{\partial_r} Df = 0$. Since the metric of $M_0$ is the product metric, $\bar{D}_X \partial_r = 0$ for any tangent vector field $X$. Thus
\[
\langle \nabla \Theta, Df \rangle = \langle Df, e_k \rangle h_{ik} \langle e_i, \partial_r \rangle.
\]
Combining this with (A.4) and (A.5), we obtain
\[
-\langle \nabla H, \partial_r \rangle = -\Theta \text{Hess}(f)(\bar{v}, \bar{v}) + \langle \nabla \Theta, Df \rangle.
\]
(A.6)

Putting this into (A.3), we obtain the conclusion in the case of $f \in C^3(N)$. In the case of $f \in C^2(N)$, $S$ is a $C^{2,\beta}$ hypersurface for any $\beta$ in $(0,1)$. The result comes from the classical approximation method. The proof is completed.

As a corollary we obtain the following Harnack type result.

**Corollary A.2.** Let $\Omega$ be a bounded domain in $N$ and $f$ be a $C^2$ function in $N$. Suppose that $S$ is a $C^2$ connected orientable hypersurface satisfying (A.1) with its normal vector $\bar{v}$. If $\Theta = \langle \bar{v}, \partial_r \rangle \geq 0$ on $S$, $\Theta \equiv 0$ or $\Theta > 0$ on the whole $S$.

**Proof.** Suppose that $f$ is $C^3$. Then $S$ is $C^3$. Because $\Omega$ is bounded, there is a positive constant $C > 0$ such that
\[
\Delta \Theta - C \Theta + \langle \nabla \Theta, Df \rangle \leq 0.
\]
(A.7)

Since $\Theta \geq 0$ on $S$, the weak maximum principle implies that $\Theta \equiv 0$ or $\Theta > 0$ on the whole $S$.

When $f$ is $C^2$, $S$ is $C^{2,\beta}$. Then $\Theta$ satisfies (A.7) in the Lipschitz sense. By the weak Harnack inequality in [40], $\Theta > 0$ or $\Theta \equiv 0$ on the whole $S$. The proof is completed.

We have the following interior estimate of mean curvature equations.

**Theorem A.3.** Let $f \in C^2(N)$. Fix $x_0 \in N$ and let $B_\rho(x_0)$ be an embedded ball centered at $x_0$ with radius $\rho$. Suppose that $u(x) \in C^2(B_\rho(x_0))$ satisfying
\[
-\text{div} \left( \frac{Du}{\omega} \right) + \left\langle Df, \frac{Du}{\omega} \right\rangle = 0,
\]
(A.8)

where $Du$ is the gradient of $u$, $\omega = \sqrt{1 + |Du|^2}$ and $\text{div}$ is the divergence of $N$. If
\[
\max_{x \in B_\rho(x_0)} |u(x)| \leq c_0
\]
for some positive constant $c_0$, then
\[
\max_{B_{\frac{\rho}{2}}(x_0)} |Du| \leq C,
\]
(A.9)

where $C$ is a constant only depending on the Ricci curvature, the $C^2$ norm of $f$, $c_0$ and $\rho$.

**Proof.** The proof uses the maximum principle method in [24] (see also [38, Section 2], [11, Lemma 2.1] and [12, Appendix A]).

First, we assume $f(x)$ is a $C^3$ function in $N$. By the classical Schauder estimate $u(x)$ is $C^{3,\beta}$ for any $\beta \in (0,1)$. We denote by $\Lambda$ the $C^2$ norm of $f$ on $\bar{\Omega}$, i.e.,
\[
\Lambda := \max_{x \in \bar{\Omega}} \{|f(x)|, |Df(x)|, |D^2f(x)|\}.
\]
(A.10)

Let $\Sigma$ be the graph of $u(x)$ with its upward normal vector
\[
\bar{v} = \frac{\partial_r - Du}{\omega} \quad \text{in} \quad M_0.
By (A.8), the mean curvature of $\Sigma$ with respect to $\vec{v}$, $H_\Sigma$, in $M_0$ satisfies that

$$H_\Sigma + \langle Df, \vec{v} \rangle = 0.$$ 

Let $\Theta = \langle \vec{v}, \partial_r \rangle = \frac{1}{\omega}$, which is a $C^{2,\beta}$ function. By Lemma A.1, we have

$$\Delta \Theta + (|A|^2 + \tilde{\text{Ric}}(\vec{v}, \vec{v}) - \text{Hess}(f)(\vec{v}, \vec{v}))\Theta - \langle \nabla \Theta, Df \rangle = 0. \tag{A.11}$$

Thus $\omega$ satisfies that

$$\Delta \omega - 2\frac{|
abla \omega|^2}{\omega} - c_1 \geq \langle \nabla \omega, Df \rangle - \omega \langle \nabla \eta, Df \rangle, \tag{A.12}$$

where $c_1 \geq 0$ is a constant only depending on the Ricci curvature and $\Lambda$.

Let $d(x, x_0)$ be the distance function between $x_0$ and $x$. Now we define

$$q(x) := 1 + \frac{u(x)}{2c_0} - \frac{3}{2\rho^2}d^2(x, x_0). \tag{A.13}$$

Define $B := \{x \in N : q(x) > 0\}$. Thus $B_{\frac{q}{2}}(x_0) \subset B \subset B_{\rho}(x_0)$. Set

$$\eta(x) := e^{Kq(x)} - 1,$$

where $K$ is a positive constant determined later. Thus the maximum of $\eta \omega$ is obtained at a point in $B$, for example $x_1 \in B_{\rho}(x_0)$. At this point,

$$\eta \Delta(\eta \omega) + \omega \langle \nabla \eta, Df \rangle = 0, \tag{A.14}$$

Observe that

$$\langle \nabla \eta, Df \rangle \geq e^{Kq(x)} \left( - \frac{K^2|\nabla q(x)|^2}{2} - 4\Lambda^2 \right). \tag{A.15}$$

Since $\Delta \eta = K\Delta q(x) + K^2|\nabla q(x)|^2$, from (A.14) and (A.15) we obtain

$$0 \geq e^{Kq(x)} \left( K\Delta q(x) + \frac{1}{2}K^2|\nabla q(x)|^2 - 4\Lambda^2 - c_1 \right). \tag{A.16}$$

Note that for any $C^2$ function $h(x)$,

$$\Delta h = \text{Hess}(h)(e_i, e_i) - H_\Sigma(h, \vec{v}),$$

where $\{e_i\}_{i=1}^n$ is the orthonormal frame on $S$. Thus

$$\Delta u = \frac{1}{\omega} \left( Df, -\frac{Du}{\omega} \right),$$

$$\Delta d^2(x, x_0) = \text{Hess}(d^2)(e_i, e_i) + \left( Df, -\frac{Du}{\omega} \right) \left( Dd^2(x, x_0), -\frac{Du}{\omega} \right).$$

As a result,

$$\Delta q(x) \geq c_2, \tag{A.17}$$

where $c_2$ is a constant depending on $\Lambda$, the Ricci curvature on $B_{\rho}(x_0)$. Let $D$ be the covariant derivative on $M$. Note that

$$|\nabla q(x)|^2 \geq \frac{1}{4c_0}|Du|^2 - c_3. \tag{A.18}$$
Here, \( c_3 \) is a fixed constant depending on \( \rho \). Now combining (A.16), (A.17), and (A.18) together we obtain

\[
0 \geq \frac{c_3}{K} + \frac{1}{4c_0} |Du|^2 - c_2 - \frac{1}{K^2}(4\Lambda^2 - c_1).
\]  

(A.19)

When taking \( K \) sufficiently large, we obtain that at \( x_1 \), \(|Du|^2 \leq \frac{1}{2}c_3 \). Note that \( K \) only depends on \( \Lambda \), \( c_1 \) and \( c_2 \), the Ricci curvature of \( M \) on \( B_\rho(x_0) \). Thus

\[
(e^{Kq(x)} - 1)\omega \leq e^{\frac{3K}{2}} \left( 1 + \frac{c_3}{2} \right)
\]  

(A.20)

on \( B_\rho(x_0) \). Note that for any \( x \in B_{\frac{\rho}{2}}(x_0) \), by (A.13), \( q(x) \geq \frac{1}{8} \). Thus

\[
\max_{B_{\frac{\rho}{2}}(x_0)} |Du| \leq C.
\]  

Here, \( C \) is a constant only depending on \( c_0 \), \( \Lambda \) and \( \rho \).

Now we show the case that \( f(x) \in C^2(N) \). Suppose that the conclusion is not true. There is a sequence \( \{x_k\}_{k=1}^\infty \) in \( B_\frac{\rho}{2}(x_0) \) such that \(|Du(x_k)|\) goes to +\( \infty \). Without loss of generality we assume \( \{x_k\}_{k=1}^\infty \) converges to \( x_\infty \). Taking \( r < \rho \) sufficiently small such that the mean curvature of the boundary of \( B_r(x_\infty) \) satisfies

\[
H_{\partial B_r(x_\infty)} + \langle Df, \gamma \rangle > 0,
\]  

(A.21)

where \( \gamma \) is the outward normal vector of \( \partial B_r(x_\infty) \).

Now \( \{f_k(x)\}_{k=1}^\infty \) is a sequence of \( C^3 \) functions such that \( \{f_k(x)\}_{k=1}^\infty \) uniformly converges to \( f(x) \) in \( B_\rho(x_0) \) in the \( C^2 \) norm. As a result the \( C^2 \) norm of \( f_k \), \( \Lambda_k \) (defined similar to that in (A.10)), is uniformly bounded. By the continuity, we can assume (A.21) holds if we replace \( f \) with any \( f_k \). By Theorem 5.3 there is a unique

\[
u_k(x) \in C^2(B_r(x_\infty)) \cap C(\bar{B}_r(x_\infty))
\]  

solving

\[
-\text{div} \left( \frac{Du}{\omega} \right) + \left( Df_k - \frac{Du}{\omega} \right) = 0 \quad \text{on} \quad B_r(x_\infty)
\]  

(A.22)

and \( \nu_k(x) = u(x) \) on \( \partial B_r(x_\infty) \). By applying the first case of the proof and the uniqueness of the Dirichlet problem above, it is easy to see that \( \{\nu_k(x)\} \) locally converges to \( u(x) \) in the \( C^2 \) uniform. Since \( \Lambda_k \) is uniformly bounded and \( f_k \in C^3(B_r(x_\infty)) \), the first case implies that there is a positive constant \( C \) independent of \( k \) such that

\[
\max_{B_{\frac{\rho}{2}}(x_\infty)} |Du_k| \leq C.
\]  

(A.23)

As a result,

\[
\max_{B_{\frac{\rho}{2}}(x_\infty)} |Du| \leq C
\]

by the local \( C^2 \) convergence. This is a contradiction to the definition of \( x_\infty \).

The proof is completed. \( \square \)