EXPONENTIAL SUMS OVER PRIMES IN SHORT INTERVALS
AND APPLICATION IN WARING-GOLDBACH PROBLEM

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Abstract. Let $\Lambda(n)$ be the von Mangoldt function, $x$ real and $2 \leq y \leq x$. This paper improves the estimate on the exponential sum over primes in short intervals

$$S_k(x, y; \alpha) = \sum_{x < n \leq x + y} \Lambda(n)e(n^k \alpha)$$

when $k \geq 3$ for $\alpha$ in the minor arcs. And then combined with the Hardy-Littlewood circle method, this enables us to investigate the Waring-Goldbach problem of representing a positive integer $n$ as the sum of $s$ $k$th powers of almost equal prime numbers, which improves the results in Wei and Wooley [12].

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1. INTRODUCTION

Let $\Lambda(n)$ be the von Mangoldt function, $k \geq 1$ an integer, $x$ real and $2 \leq y \leq x$. The estimate of the exponential sum over primes in short intervals

$$S_k(x, y; \alpha) = \sum_{x < n \leq x + y} \Lambda(n)e(n^k \alpha) \quad (1.1)$$

was first studied by I. M. Vinogradov [11] in 1939 with his elementary method. Since then this topic has attracted the interest of quite a number of authors. These sums arise naturally and play important roles when solving the Waring-Goldbach problems in short intervals by the circle method. In particular, the case $k = 1$, i.e., the linear exponential sum over primes in short intervals, was studied quite extensively, because of

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its applications to the study of the Goldbach-Vinogradov theorem with three almost equal prime variables (see [13] and the references therein).

For the case $k = 2$, Liu and Zhan [8] first established a non-trivial estimate of $S_2(x, y; \alpha)$ for all $\alpha$ and all published results before their result are valid only for $\alpha$ in a very thin subset of $[0, 1]$. In [9], Lü and Lao improved the results in [8] to be as good as what was previously derived from the Generalized Riemann Hypothesis.

In this paper we deal with $S_k(x, y; \alpha)$ in the general case $k \geq 3$. Let $y = x^\theta$ with $3/4 < \theta \leq 1$. We set

$$P = x^{2K\delta}, \quad \text{and} \quad Q = x^{k-2}y^2P^{-1},$$

(1.2)

where

$$K = 2t_k(t_k + 2), \quad t_k = k(k - 1)$$

(1.3)

are defined as in [12, equations (1.2) and (2.1)] and $\delta$ is a positive parameter which may depend on $k$. By Dirichlet’s theorem on Diophantine approximations, every real number $\alpha$ has a rational approximation $a/q$, where $a$ and $q$ are integers subject to

$$1 \leq q \leq Q, \quad (a, q) = 1, \quad |q\alpha - a| \leq 1/Q.$$  

(1.4)

We denote by $M$ the union of the major arcs

$$M(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \leq Q^{-1}\},$$

with $0 \leq a \leq q \leq P$ and $(a, q) = 1$, and let $m = [0, 1) \setminus M$ for the set of minor arcs complementary to the set $M$. In Liu and Zhan [7] and Huang and Wang [2], they handled with $S_k(x, y; \alpha)$ for all $\alpha \in [0, 1]$ by dividing this into several parts and then estimating them by different methods. In Liu, Lü and Zhan [6], Kumchev [5] and Wei and Wooley [12], the authors mainly bounded $S_k(x, y; \alpha)$ for $\alpha \in m$. Liu, Lü and Zhan [6] used methods from multiplicative number theory, and then Kumchev [5] used the sieve method to improve their results. Recently, Wei and Wooley [12] gave a substitute for a Weyl-type estimate for $S_k(x, y; \alpha)$, which made use of Daemen’s estimates via a bilinear form treatment motivated by analogous arguments making use of Vinogradov’s mean value theorem. We first state our main result for exponential sums.

**Theorem 1.** Let $k \geq 3$. Let $\theta$ be a real number with $3/4 < \theta \leq 1$ and suppose that $0 < \rho < \rho_k(\theta)$, where

$$\rho_k(\theta) = \min \left\{ \frac{\sigma_k(\theta - 3/4)}{8}, \delta \right\},$$

(1.5)

with $\sigma_k = 1/(2t_k)$. Then, for any fixed $\varepsilon > 0$, we have

$$S_k(x, y; \alpha) \ll y^{1-\rho+\varepsilon} + \left(\frac{yx^\varepsilon}{(q + y^2x^{k-2}|q\alpha - a|)^{1/2k}}\right).$$

(1.6)

To prove Theorem 1 we first use Vaughan’s identity to divide this to two types of sums, and then estimate the exponential sums of type I and type II respectively (see [5]). The exponential sums of type I will be estimated in §3 and the other one will be treated in §4. To do this we mainly follow the method of Kumchev [5], and combine with the results in Daemen [1] (see §2-4).
In the last section, we apply the circle method to give an application of this exponential sum to Waring-Goldbach problem in short intervals.

A formal application of the circle method suggests that whenever \( s \) and \( k \) are natural numbers with \( s \geq k + 1 \), then all large integers \( n \) satisfying appropriate local conditions should be represented as the sum of \( s \) \( k \)-th powers of prime numbers. With this expectation in mind, consider a natural number \( k \) and prime \( p \), take \( \tau = \tau(k, p) \) to be the integer with \( p^\tau | k \) but \( p^{\tau+1} \nmid k \), and the define \( \eta = \eta(k, p) \) by putting \( \eta(k, p) = \tau + 2 \), when \( p = 2 \) and \( \tau > 0 \), and otherwise \( \eta(k, p) = \tau + 1 \). We then define \( R = R(k) \) by putting \( R(k) = \prod p^\eta \), where the product is taken over primes \( p \) with \( (p - 1) | k \). Write \( X = (n/s)^{1/k} \). We say that the exponent \( \Delta_{k,s} \) is admissible when, provided that \( \Delta \) is a positive number with \( \Delta < \Delta_{k,s} \), then for all sufficiently large positive integers \( n \) with \( n \equiv s \ (\text{mod} \ R) \), the equation

\[
p_1^k + p_2^k + \cdots + p_s^k = n
\]

has a solution in prime numbers \( p_j \) satisfying \( |p_j - X| \leq X^{1-\Delta} \) \( (1 \leq j \leq s) \). We refer the reader to [12] for more details.

Together the circle method used in [12] with our estimate of exponential sums in Theorem 2, we show that there are larger admissible exponents \( \Delta_{k,s} \) as soon as \( s > 2t_k \).

**Theorem 2.** Let \( s \) and \( k \) be integers with \( k \geq 3 \) and \( s > 2t_k \). Suppose that \( \varepsilon > 0 \), that \( n \) is a sufficiently large number satisfying \( n \equiv s \ (\text{mod} \ R) \), and write \( X = (n/s)^{1/k} \). Then the equation \( n = p_1^k + p_2^k + \cdots + p_s^k \) has a solution in prime numbers \( p_j \) with \( |p_j - X| \leq X^{19/24+\varepsilon} \) \( (1 \leq j \leq s) \).

In [12], Wei and Wooley gave the same exponent \( 19/24 \) for the case \( k = 2 \). They remark that since \( \frac{19}{24} = \frac{1}{2} \left( 1 + \frac{7}{12} \right) \), this exponent is in some sense half way between the trivial exponent \( 1 \) and the exponent \( \frac{7}{12} \) that, following the work of Huxley [3], represents the effective limit of our knowledge concerning the asymptotic distribution of prime numbers in short intervals.

By the same argument in [12, §9], we obtain the following almost-all result. (The history of this kind of problem can be seen in [12, §1].)

**Theorem 3.** Let \( s \) and \( k \) be integers with \( k \geq 3 \) and \( s > t_k \). Suppose that \( \varepsilon > 0 \). Then for almost all positive integers \( n \) with \( n \equiv s \ (\text{mod} \ R) \), \( (\text{and, in case } k = 3 \text{ and } s = 7, \text{satisfying also } 9 \nmid n) \), the equation \( n = p_1^k + p_2^k + \cdots + p_s^k \) has a solution in prime numbers \( p_j \) with \( |p_j - X| \leq X^{19/24+\varepsilon} \) \( (1 \leq j \leq s) \), where \( X = (n/s)^{1/k} \).

**Notation.** Throughout the paper, the letter \( \varepsilon \) denotes a sufficiently small positive real number which may be different at each occurrence. For example, we may write \( x^\varepsilon \ll y^\varepsilon \). Any statement in which \( \varepsilon \) occurs holds for each positive \( \varepsilon \), and any implied constant in such a statement is allowed to depend on \( \varepsilon \). The letter \( p_j \), with or without subscripts, is reserved for prime numbers. In addition, as usual, \( e(z) \) denotes \( e^{2\pi iz} \). We write \( (a, b) = \gcd(a, b) \), and we use \( m \sim M \) as an abbreviation for the condition \( M < m \leq 2M \).
2. Auxiliary results

The following lemma is useful to give an estimate for exponential sums of type I and type II which is an improvement of [5 Lemma 2.2].

**Lemma 1.** Let \( k \geq 3 \) be an integer and \( \gamma \geq 3 \) be a real number. Let \( 0 < \rho \leq \frac{\sigma_k}{\gamma} \), where \( \sigma_k = \frac{1}{2t_k} \). Suppose that \( y \leq x \), and \( y \geq x^{\frac{\gamma - \sigma_k - 1}{\gamma}} \). Then either

\[
\sum_{x < n \leq x+y} e\left( n^k \alpha \right) \ll y^{1-\rho+\varepsilon},
\]

(2.1)

or there exist integers \( a \) and \( q \) such that

\[
1 \leq q \leq y^{k\rho}, \quad (a, q) = 1, \quad |q\alpha - a| \leq x^{1-k}y^{k\rho-1},
\]

(2.2)

and

\[
\sum_{x < n \leq x+y} e\left( n^k \alpha \right) \ll y^{1-\rho+\varepsilon} + \frac{y}{(q + yx^{k-1}|q\alpha - a|)^{1/k}}.
\]

(2.3)

**Proof.** Take

\[
P_0 = y^{1/\gamma} \quad \text{and} \quad Q_0 = x^{k-2}y^2/P_0.
\]

By Dirichlet’s theorem on Diophantine approximation, there exists integers \( a \) and \( q \) with

\[
1 \leq q \leq Q_0, \quad (a, q) = 1, \quad |q\alpha - a| \leq 1/Q_0.
\]

(2.4)

When \( q > P_0 \), we rewrite the sum on the left of (2.1) as

\[
\sum_{1 \leq n \leq z} e(\alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_0),
\]

where \( z \leq y \) and \( \alpha_j = \binom{k}{j} \alpha u^{k-j} \), with \( u \) a fixed integer. Hence, it follows from the argument underlying the proof of [11 eq. (3.5)] and [12 eq. (4.23)] that

\[
\sum_{x < n \leq x+y} e\left( n^k \alpha \right) \ll yP_0^{-1/(2t_k)+\varepsilon} \ll y^{1-\rho+\varepsilon}.
\]

(2.5)

When \( q \leq P_0 \), from [11 eq. (5.1)-(5.5) and §6], we deduce

\[
\sum_{1 \leq n \leq y} e(\alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_0) \ll \frac{y}{(q + yx^{k-1}|q\alpha - a|)^{1/k}} + \Delta,
\]

where

\[
\Delta \ll P_0^{1/2+\varepsilon} \left(1 + \frac{P_0 x^k}{x^{k-2}y^2}\right)^{1/2} \ll P_0^{1+\varepsilon}x/y \ll y^{1-\rho+\varepsilon},
\]

provided that \( y \geq x^{\frac{\gamma - \sigma_k - 1}{\gamma}} \). Thus, at least one of (2.1) and (2.3) holds. The lemma follows on noting that when conditions (2.2) fail, inequality (2.1) follows from (2.3). □

The next lemma gives some inequalities which will be used in the following sections.
Lemma 2. We have
\[ \sum_{n \sim N} (r, n^k)^{1/k} \leq N \tau(r), \]  
where \( \tau(r) \) is the divisor function; and for any \( \varepsilon > 0 \), we have
\[ \sum_{n \sim N} (r, R(n, h))^{1/k} \ll Nr^\varepsilon + r^{1/k + \varepsilon}, \]  
where \( R(n, h) = ((n + h)^k - n^k)/h \).

Proof. We have
\[ \sum_{n \sim N} (r, n^k)^{1/k} \leq \sum_{n \sim N} (n, r) \leq \sum_{d \mid r} \sum_{n \sim N/d} 1 \leq N \tau(r). \]

To prove (2.7), see the inequality (3.11) in Kawada and Wooley [4]. □

3. Type I estimate

The following proposition treats the exponential sums of type I which is an improvement of [2, Lemma 8]

Proposition 1. Let \( k \geq 3 \), \( 0 < \rho < \min\{\sigma_k/(2\gamma), \delta\} \). Suppose that \( \alpha \) is real that there exist integers \( a \) and \( q \) such that (1.4) holds with \( Q \) given by (1.2). Let \( a(m) \ll m^\varepsilon \), and define
\[ T_1 = \sum_{m \sim M} a(m) \sum_{x < mn \leq x + y} e((mn)^k \alpha). \]

Then
\[ T_1 \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + yx^{k-1}|q\alpha - a|)^{1/k}}, \]
provided that
\[ M \ll y \left( \frac{y}{x} \right)^{\frac{1}{\gamma - \sigma_k}}, \quad M \ll yx^{-\rho/\sigma_k}, \quad M^{2k} \ll yx^{k-1-2k\rho}. \]  

Proof. Set
\[ S_m = \sum_{X < n \leq X + Y} e\left(m^k n^k \alpha\right), \]
where \( X = x/m, Y = y/m, \) with \( m \sim M \). Define \( \nu \) by \( Y^\nu = x^\rho L^{-1} \). Note that, by (3.1), we have
\[ \nu < \sigma_k/\gamma. \]
We denote by \( M \) the set of integers \( m \sim M \), for which there exist integers \( b_1 \) and \( r_1 \) with
\[ 1 \leq r_1 \leq Y^{k\nu}, \quad (b_1, r_1) = 1, \quad |r_1 m^k \alpha - b_1| \leq X^{1-k} Y^{k\nu-1}. \]  
We apply Lemma 1 to the summation over \( n \) and get
\[ S_m \ll Y^{1-\nu+\varepsilon} + \frac{Y}{(r_1 + YX^{k-1}|r_1 m^k \alpha - b_1|)^{1/k}}, \]
for \( m \in \mathcal{M} \). So

\[
T_1 \ll \sum_{m \sim M} a(m)Y^{1-\nu+\varepsilon} + \sum_{m \in \mathcal{M}} \frac{a(m)Y}{(r_1 + YX^{k-1}|r_1m^k\alpha - b_1|)^{1/k}}.
\]

Then, we have

\[
T_1 \ll y^{1-\rho+\varepsilon} + T_1(\alpha),
\]

where

\[
T_1(\alpha) = \sum_{m \in \mathcal{M}} \frac{a(m)Y}{(r_1 + YX^{k-1}|r_1m^k\alpha - b_1|)^{1/k}}.
\]

We apply Dirichlet’s theorem on Diophantine approximation to find integers \( b \) and \( r \) with

\[
1 \leq r \leq x^{-k\rho}X^{k-1}, \quad (b, r) = 1, \quad |r\alpha - b| \leq x^{k\rho}Y^{-1}X^{1-k}.
\]

(3.3)

By (3.1), (3.2) and (3.3), we have

\[
|b_1r - bm^kr_1| = |r(b_1 - r_1m^k\alpha) + r_1m^k(r\alpha - b)|
\]

\[
\leq x^{-k\rho}YX^{k-1}X^{1-k}Y^{-1}X^{k\rho} + Y^{k\rho}(2M)^kx^{k\rho}Y^{-1}X^{1-k}
\]

\[
\ll L^{-k} + M^{2k}L^{-k}x^{2k\rho-k+1}y^{-1} \ll L^{-k} < 1,
\]

whence

\[
\frac{b_1}{r_1} = \frac{m^kb}{r}, \quad r_1 = \frac{r}{(r, m^k)}.
\]

Thus, by Lemma 2, we have

\[
T_1(\alpha) \ll \sum_{m \in \mathcal{M}} \frac{a(m)YM^{-1}r_1^{-1/k}}{(1 + YX^{k-1}|m^k\alpha - m^kb/r|)^{1/k}}
\]

\[
\ll \frac{YM^{-1+\varepsilon}}{(1 + yx^{k-1}(|\alpha - b/r|)^{1/k})} \sum_{m \sim M} \left( \frac{r}{(r, m^k)} \right)^{-1/k}
\]

\[
\ll \frac{y^x^{-\varepsilon}}{(r + yx^{k-1}|r\alpha - b|)^{1/k}}.
\]

Recall that \( b \) and \( r \) satisfy the conditions (3.3). We now consider three cases depending on the size of \( r \) and \(|r\alpha - b|\).

Case 1: If \( r > x^{k\rho} \), then \( T_1(\alpha) \ll y^{1-\rho+\varepsilon} \).

Case 2: If \( r \leq x^{k\rho} \) and \(|r\alpha - b| > y^{-1}x^{1-k}x^{k\rho} \), then \( T_1(\alpha) \ll y^{1-\rho+\varepsilon} \).

Case 3: If \( r \leq x^{k\rho} \) and \(|r\alpha - b| \leq y^{-1}x^{1-k}x^{k\rho} \), we have

\[
|ra - bq| = |r(a - q\alpha) + q(r\alpha - b)|
\]

\[
\leq x^{k\rho} \frac{1}{Q} + Qy^{-1}x^{1-k}x^{k\rho}
\]

\[
\leq \frac{x^{k\rho}P}{x^{k-2}y^2} + \frac{yx^{k\rho}}{xP}.
\]
Since $\rho < \delta$, we have $|ra - bq| < 1$, hence
\[ a = b, \quad q = r. \]

Then
\[ T_1(\alpha) \ll \frac{yx^\varepsilon}{(q + yx^{k-1}|q\alpha - a|)^{1/k}}. \]

So we prove
\[ T_1 \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + yx^{k-1}|q\alpha - a|)^{1/k}}. \]

\[ \square \]

**Remark 1.** Let
\[ T_1^* = \sum_{m \sim M} a(m) \sum_{x < mn \leq x+y} e\left((mn)^k\alpha\right) \log n. \] (3.4)

Under the condition of Proposition 1 we have
\[ T_1^* \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + yx^{k-1}|q\alpha - a|)^{1/k}}. \]

**Remark 2.** One can estimate the exponential sums
\[ \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} a(m_1, m_2) \sum_{x < m_1m_2n \leq x+y} e\left((m_1m_2n)^k\alpha\right) \]

with some suitable conditions on $M_1$ and $M_2$ as [5, Lemma 3.2] and [12, Lemma 4.2] did, and then may give a better result than Proposition 1. Since it has no influence on our main results, we will not do it.

### 4. Type II Estimate

To prove Theorem 1, we also need to handle the exponential sums of type II. Let $a(m)$ and $b(n)$ be arithmetic functions satisfying the property that for all natural numbers $m$ and $n$, one has
\[ a(m) \ll m^\varepsilon \quad \text{and} \quad b(n) \ll n^\varepsilon. \] (4.1)

Let $M$ and $N$ be positive parameters, and define the exponential sum $T_2 = T_2(\alpha; M)$ by
\[ T_2(\alpha; M) := \sum_{M < m \leq 2M} a(m) \sum_{x < mn \leq x+y} b(n)e\left((mn)^k\alpha\right). \] (4.2)

The following proposition gives an estimate for $T_2$ which is an improvement of [5, Lemma 3.1]

**Proposition 2.** Let $k \geq 3$, $0 < \rho < \min\{\sigma_k/(8\gamma), \delta\}$. Suppose that $\alpha$ is real that there exist integers $a$ and $q$ such that (1.4) holds with $Q$ given by (1.2). And let $x$ and $y$ be positive numbers with
\[ y = x^\theta, \quad \frac{1}{1-2\rho} \frac{3\gamma - \sigma_k - 1}{2(2\gamma - \sigma_k - 1)} \leq \theta \leq 1. \] (4.3)
Then
\[ T_2 \ll y^{1-\rho+\epsilon} + \frac{yx^\epsilon}{(q + y^2x^{k-2}|q\alpha - a|)^{1/(2k)}}, \]
provided that
\[ x^{1/2} \leq M \ll y^{1-2\rho}. \]  

Proof. Set \( N = x/M, X = x/N \) and \( Y = y/N = yM/x \). Define \( \nu \) by \( Y^\nu = x^{2\rho}L^{-1} \). By (4.4), we have
\[ \nu < \sigma_k/\gamma. \]

For \( n_1, n_2 \leq 2N \), let
\[ \mathcal{M}(n_1, n_2) = \{ m \in (M, 2M) : x < mn_1, mn_2 \leq x + y \}. \]

By Cauchy's inequality and an interchange of the order of summation, we have
\[ T_2^2 \ll y^{1+\epsilon}M + Mx^\epsilon T_1(\alpha), \]  
where
\[ T_1(\alpha) = \sum_{n_1 < n_2} \left| \sum_{m \in \mathcal{M}(n_1, n_2)} e \left( \alpha(n_2^k - n_1^k)m^k \right) \right|. \]

Let \( N \) denote the set of pairs \((n_1, n_2)\) with \( n_1 < n_2 \) and \( \mathcal{M}(n_1, n_2) \neq \emptyset \) for which there exist integers \( b \) and \( r \) such that
\[ 1 \leq r \leq Y^{\kappa\nu}, \quad (b, r) = 1, \quad |r(n_2^k - n_1^k)\alpha - b| \leq Y^{\kappa\nu-1}X^{1-k}. \]  

Since \( N/2 < n_1 < n_2 \leq 2N \) and \( \mathcal{M}(n_1, n_2) \neq \emptyset \), we have \( n_2 - n_1 \leq yx^{-1}n_1 \). Hence \( \#N \ll xyM^{-2} \). In order to handle the inner summation in \( T_1(\alpha) \), we set
\[ X_1 = \max \left\{ M, \frac{x}{n_1} \right\} = M = \frac{x}{N} = X, \]
\[ Y_1 = \min \left\{ 2M, \frac{x+y}{n_2} \right\} - \max \left\{ M, \frac{x}{n_1} \right\} \ll \frac{y}{N} = Y. \]

If \( Y_1 < X_1 \gamma/(2\gamma - \sigma_k - 1) \), by (4.3) and (4.4), the contribution to \( T_1(\alpha) \) is
\[ \ll xyM^{-2}M^\gamma/(2\gamma - \sigma_k - 1) \ll y^{2-2\rho+\epsilon}M^{-1}. \]

If \( Y_1 \geq X_1 \gamma/(2\gamma - \sigma_k - 1) \), since \( \nu < \sigma_k/d \), we can apply Lemma \( \Pi \) with \( \rho = \nu, x = X_1 \) and \( y = Y_1 \) to the inner summation in \( T_1(\alpha) \). We get
\[ T_1(\alpha) \ll y^{2-2\rho+\epsilon}M^{-1} + T_2(\alpha), \]  
where
\[ T_2(\alpha) = \sum_{(n_1, n_2) \in N} \frac{Y}{(r + YX^{k-1}|r(n_2^k - n_1^k)\alpha - b|)^{1/k}}. \]

We now change the summation variables in \( T_2(\alpha) \) to
\[ d = (n_1, n_2), \quad n = n_1/d, \quad h = (n_2 - n_1)/d. \]
We obtain
\[ T_2(\alpha) \ll \sum_{dh \leq y/M} \frac{Y}{(r + Y X^{k-1} |rhd^k R(n, h) \alpha - b|)^{1/k}}, \]  
(4.8)
where \(R(n, h) = ((n + h)^k - n^k)/h\) and the inner summation is over \(n\) with \((n, h) = 1\) and \((nd, (n + h)d) \in \mathcal{N}\). For each pair \((d, h)\) appearing in the summation on the right side of (4.8), Dirichlet’s theorem on Diophantine approximation yields integers \(b_1\) and \(r_1\) with
\[ 1 \leq r_1 \leq x^{2k\rho} Y X^{k-1}, \quad (b_1, r_1) = 1, \quad |r_1hd^k \alpha - b_1| \leq x^{2k\rho} Y^{1-1/k}. \]  
(4.9)
As \(R(n, h) \leq 4^k (N/d)^{k-1}\), combining (4.4), (4.6) and (4.9), we have
\[ |b_1 R(n, h) - br_1| = |r R(n, h) (b_1 - r_1 hd^k \alpha) + r_1 (rhd^k R(n, h) \alpha - b)| \]
\[ \leq r_1 Y^{k-1} X^{1-k} + r R(n, h) x^{2k\rho} Y^{-1} X^{1-k} \]
\[ \leq L^{-k} + 4^k N^{k-1} x^{2k\rho} L^{-k} x^{2k\rho} Y^{-1} X^{1-k} < 1. \]
Hence,
\[ \frac{b}{r} = \frac{b_1 R(n, h)}{r_1}, \quad r = \frac{r_1}{(r_1, R(n, h))}. \]
(4.10)
Combining (4.8) and (4.10), we obtain
\[ T_2(\alpha) \ll \sum_{dh \leq y/M} \frac{Y}{(r_1 + Y X^{k-1} N_d^{k-1} |r_1 hd^k \alpha - b_1|)^{1/k}} \sum_{n \sim N_d, (n, h) = 1} (r_1, R(n, h))^{1/k}, \]
where \(N_d = N/d\). By Lemma 2, we deduce that
\[ T_2(\alpha) \ll y^2 x^{-1+\varepsilon} + T_3(\alpha), \]  
(4.11)
where
\[ T_3(\alpha) = \sum_{dh \leq y/M} \frac{yx^{\varepsilon}/d}{(r_1 + Y X^{k-1} N_d^{k-1} |r_1 hd^k \alpha - b_1|)^{1/k}}. \]
We now write \(\mathcal{H}\) for the set of pairs \((d, h)\) with \(dh \leq y/M\) for which there exist integers \(b_1\) and \(r_1\) subject to
\[ 1 \leq r_1 \leq x^{2k\rho}, \quad (b_1, r_1) = 1, \quad |r_1 hd^k \alpha - b_1| \leq x^{-k+1+2k\rho} Y^{-1}. \]  
(4.12)
We have
\[ T_3(\alpha) \ll y^{2-2\rho+\varepsilon} M^{-1} + T_4(\alpha), \]  
(4.13)
where
\[ T_4(\alpha) = \sum_{(d, h) \in \mathcal{H}} \frac{yx^{\varepsilon}/d}{(r_1 + Y X^{k-1} N_d^{k-1} |r_1 hd^k \alpha - b_1|)^{1/k}}. \]
For each \(d \leq y/M\), Dirichlet’s theorem on Diophantine approximation yields integers \(b_2\) and \(r_2\) with
\[ 1 \leq r_2 \leq x^{k-1-2k\rho} Y/2, \quad (b_2, r_2) = 1, \quad |r_2 d^k \alpha - b_2| \leq 2x^{-k+1+2k\rho} Y^{-1}. \]  
(4.14)
Combining (4.12) and (4.14), we obtain

\[ |b_2 r_1 h - b_1 r_2| = |r_1 h(b_2 - r_2 d^k \alpha) + r_2 (r_1 h d^k \alpha - b_1)| \leq r_1 h |r_2 d^k \alpha - b_2| + r_2 |r_1 h d^k \alpha - b_1| \leq 1/2 + 2 x^{-k+2+4k \rho} M^{-2} < 1, \]

whence

\[ \frac{b_1}{r_1} = \frac{h b_2}{r_2}, \quad r_1 = \frac{r_2}{(r_2, h)}. \]

We write \( Z_d = Y X^{k-1} N_d^{k-1} |r_2 d^k \alpha - b_2| \) and by Lemma 2, we get

\[ T_4(\alpha) = \sum_{(d, h) \in H} \frac{y x^\varepsilon}{d} (r_1 + Z_d h)^{1/k} \ll \sum_{d \leq y/M} \frac{y^2 x^\varepsilon M^{-1}}{d^2 (r_2 + y (Md)^{-1} Z_d)^{1/k}}, \]

Hence

\[ T_4(\alpha) \ll y^{2-2\rho+\varepsilon} M^{-1} + T_5(\alpha), \quad (4.15) \]

where

\[ T_5(\alpha) = \sum_{d \in D} \frac{y^2 x^\varepsilon M^{-1}}{d^2 (r_2 + y (Md)^{-1} Z_d)^{1/k}}, \]

and \( D \) is the set of integers \( d \leq x^{2 \rho} \) for which there exist integers \( b_2 \) and \( r_2 \) with

\[ 1 \leq r_2 \leq x^{2 \rho}, \quad (b_2, r_2) = 1, \quad |r_2 d^k \alpha - b_2| \leq y^{-2} x^{2-k+2k \rho}. \quad (4.16) \]

Combining (1.2), (1.4) and (4.16), we deduce that

\[ |r_2 d^k a - b_2 q| = |r_2 d^k (a - q \alpha) + q (r_2 d^k \alpha - b_2)| \leq r_2 d^k Q^{-1} + q |r_2 d^k \alpha - b_2| \leq x^{4k \rho} Q^{-1} + y^{-2} x^{2-k+2k \rho} Q < 1, \]

whence

\[ \frac{b_2}{r_2} = \frac{d^k a}{q}, \quad r_2 = \frac{q}{(q, d^k)}. \]

Thus, recalling Lemma 2, we get

\[ T_5(\alpha) \ll \frac{y^2 x^\varepsilon M^{-1}}{(q + y^2 x^{k-2} |q \alpha - a|)^{1/k}} \sum_{d \leq x^{2 \rho}} \frac{(q, d^k)^{1/k}}{d^2} \ll \frac{y^2 x^\varepsilon M^{-1}}{(q + y^2 x^{k-2} |q \alpha - a|)^{1/k}}. \quad (4.17) \]

The desired estimate follows from (4.3), (4.4), (4.5), (4.7), (4.11), (4.13), (4.15) and (4.17).
5. Proof of Theorem 1

In this section we deduce Theorem 1 from Propositions 1 and 2 and Vaughan’s identity for $\Lambda(n)$. We put

$$U = x^{\theta/2 - \rho}, \quad V = x^{1 - \theta + 2\rho}. \quad (5.1)$$

By (1.5), we have

$$UV \asymp (x + y)/U \asymp x^{1 - \theta/2 + \rho} \ll y^{1 - 2\rho}. \quad (5.2)$$

And then we apply Vaughan’s identity (see [10]) in the following form

$$\Lambda(n) = \sum_{md = n} \mu(d) \log m - \sum_{1 \leq d \leq V} \sum_{1 \leq m \leq U} \mu(d) \Lambda(m) - \sum_{\lambda d > V} \mu(d) \Lambda(m). \quad (5.3)$$

Thus we deduce that

$$S_k(x, y; \alpha) = S_1 - S_2 - S_3, \quad (5.4)$$

where

$$S_1 = \sum_{1 \leq d \leq V} \mu(d) \sum_{x < md \leq x + y} (\log m) e\left((md)^k \alpha\right),$$

$$S_2 = \sum_{1 \leq v \leq UV} \lambda_0(v) \sum_{x < lv \leq x + y} e\left((lv)^k \alpha\right),$$

$$S_3 = \sum_{V < u \leq (x + y)/U} \lambda_1(u) \sum_{x < mu \leq x + y} \Lambda(m) e\left((mu)^k \alpha\right),$$

in which

$$\lambda_0(v) = \sum_{md = v} \mu(d) \Lambda(m) \quad \text{and} \quad \lambda_1(u) = \sum_{d | u} \mu(d).$$

We begin with estimating the sum $S_3$. Take

$$\gamma = (\theta - 3/4)^{-1}. \quad (5.5)$$

Since $3/4 < \theta \leq 1$, by (1.5), we have

$$\frac{1}{(1 - 2\rho)} 3\gamma - \sigma_k - 1 \leq \theta \leq 1.$$ 

To apply Proposition 2, we further divide $S_3$ into two parts

$$S_{31} = \sum_{x^{1/2} \leq u \leq (x + y)/U} \lambda_1(u) \sum_{x < mu \leq x + y} \Lambda(m) e\left((mu)^k \alpha\right),$$

and

$$S_{32} = \sum_{V < u < x^{1/2}} \lambda_1(u) \sum_{x < mu \leq x + y} \Lambda(m) e\left((mu)^k \alpha\right).$$
On noting that (5.1), (5.2) and $\lambda_1(u) \leq \tau(u)$, we can divide the summation over $u$ into dyadic intervals to deduce from Proposition 2 that

$$S_{31} \ll (\log x) \max_{x^{1/2} \leq M \leq (x+y)/U} \sum_{u \sim M} a(u) \sum_{x \leq M} b(m) e((mu)^k \alpha)$$

$$\ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + y^2 x^{k-2} |q\alpha - a|)^{1/(2k)}},$$

where $a(u) = \lambda_1(u)$, and $b(m) = \Lambda(m)$ if $m > U$ and is 0 if else. For $S_{32}$, we first interchange the order of summation, and then by the same argument as above, we obtain

$$S_{32} \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + y^2 x^{k-2} |q\alpha - a|)^{1/(2k)}}.$$

Hence we get

$$S_3 \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + y^2 x^{k-2} |q\alpha - a|)^{1/(2k)}}, \quad (5.6)$$

Next we estimate $S_2$. Write

$$S_4(Z, W) = \sum_{Z < v \leq W} \lambda_0(v) \sum_{x < l v \leq x + y} e((lv)^k \alpha).$$

Then we find that

$$S_2 = S_4(0, V) + S_4(V, UV). \quad (5.7)$$

Note that (5.1), (5.2) and the bound $|\lambda_0(v)| \leq \log v$, we deduce from Proposition 2 that

$$S_4(V, UV) \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + y^2 x^{k-2} |q\alpha - a|)^{1/(2k)}}. \quad (5.8)$$

We then estimate $S_4(0, V)$. Since $3/4 < \theta \leq 1$, by (1.5), (5.1) and (5.5), we have

$$V \ll y \left(\frac{y}{x}\right)^{-\gamma_0/\sigma k}, \quad V \ll y x^{-\gamma/\sigma k}, \quad V^{2k} \ll y x^{k-1-2\rho}.$$

So we can divide the summation over $v$ into dyadic intervals to deduce from Proposition 11 that

$$S_4(0, V) \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + y^{2k-1} |q\alpha - a|)^{1/k}}. \quad (5.9)$$

Thus, by combining (5.8) and (5.9), we deduce form (5.7) that

$$S_2 \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + y^2 x^{k-2} |q\alpha - a|)^{1/(2k)}}. \quad (5.10)$$

Finally, in order to estimate $S_1$, we apply (3.4) directly, proceeding as in the treatment of $S_4(0, V)$. Thus we again obtain the bound

$$S_1 \ll y^{1-\rho+\varepsilon} + \frac{yx^\varepsilon}{(q + y^2 x^{k-2} |q\alpha - a|)^{1/(2k)}}. \quad (5.11)$$

Theorem 1 follows from (5.4), (5.6), (5.10) and (5.11).
6. Proof of Theorem 2

We outline our proof of Theorem 2 which proceeds via the circle method. Suppose that \( k \) and \( s \) are integers with \( k \geq 2 \) and \( s \geq t_k \), where \( t_k \) is defined as in (1.3). Let \( \theta \) be a real number with \( 3/4 < \theta < 1 \), and let \( \delta \) be a sufficiently small, but fixed, positive number with \( 4K\delta < \min\{\theta - 3/4, 1 - \theta\} \). Consider a sufficiently large natural number \( N \), put \( X = (N/s)^{1/k} \), and write \( Y = X^\theta \). When \( n \) is a natural number with \( N \leq n \leq N + X^{k-1}Y \), we denote

\[
\rho_s(n) = \sum_{|p_1 - X| \leq Y} \cdots \sum_{|p_s - X| \leq Y} (\log p_1) \cdots (\log p_s),
\]

which is the weighted number of solutions of the equation (1.7) with \( |p_i - X| \leq Y \) (\( 1 \leq i \leq s \)). Define

\[
f(\alpha) = \sum_{|p - X| \leq Y, p \text{ prime}} (\log p) e(p^k \alpha). \tag{6.1}
\]

Then it follows from orthogonality that

\[
\rho_s(n) = \int_0^1 f(\alpha)^s e(-n\alpha) d\alpha. \tag{6.2}
\]

Next we define the Hardy-Littlewood dissection. We rewrite \( P \) and \( Q \) to be \( P = X^{2K\delta} \), \( Q = X^{k-2}Y^2 P^{-1} \).

Then let \( \mathfrak{M} \) and \( \mathfrak{m} \) be the major arc and minor arc as in §1 respectively, with \( P \) and \( Q \) defined above. When \( \mathfrak{B} \) is a measurable subset of \([0, 1)\), we define

\[
\rho_s(n; \mathfrak{B}) = \int_{\mathfrak{B}} f(\alpha)^s e(-n\alpha) d\alpha. \tag{6.3}
\]

Thus, since \([0, 1)\) is the disjoint union of \( \mathfrak{M} \) and \( \mathfrak{m} \), we find from (6.2) that

\[
\rho_s(n) = \rho_s(n; \mathfrak{M}) + \rho_s(n; \mathfrak{m}). \tag{6.4}
\]

The major arc contribution can be summarised in the following proposition.

**Proposition 3.** Suppose that \( k \geq 2 \) and \( s \geq \min\{5, k + 2\} \). Then, whenever \( 19/24 < \theta < 1 \), \( Y = X^\theta \) and \( n \) is a natural number with \( N \leq n \leq N + X^{k-1}Y \), we have

\[
\rho_s(n; \mathfrak{M}) = \mathfrak{S}(n) \mathfrak{J}(n) + O(Y^{s-1}X^{1-k} (\log X)^{-1}), \tag{6.5}
\]

where the singular integral

\[
\mathfrak{J}(n) = \int_0^1 v(\beta)^s e(-\beta n) d\beta
\]

with

\[
v(\beta) = k^{-1} \sum_{(X-Y)^k \leq m \leq (X+Y)^k} m^{-1+1/k} e(\beta m),
\]

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and the singular series
\[ \mathcal{S}(n) = \sum_{q=1}^{\infty} \varphi(q)^{-s} \sum_{\substack{a=1 \\ (a,q)=1}}^q S(q,a)^s e(-na/q). \]

with
\[ S(q,a) = \sum_{\substack{r=1 \\ (q,r)=1}} e(ar^k/q). \]

Moreover, we have
\[ Y^{s-1}X^{1-k} \ll \mathcal{S}(n)J(n) \ll Y^{s-1}X^{1-k} \log X^{\eta}, \quad (6.6) \]
where \( \eta = \eta(s,k) \) is a positive number.

Proof. See [12, Proposition 2.1 and eq. (2.7)]. \( \square \)

In order to estimate the minor arc contribution, we have the following analogue of Hua’s lemma.

Proposition 4. Suppose that \( y \) is a real number with \( y \geq x^{1/2} \). Then whenever \( s \geq 2t_k \) and \( \varepsilon > 0 \), we have
\[ \int_0^1 |f(\alpha)|^s d\alpha \ll y^{s-1}x^{1-k+\varepsilon}. \quad (6.7) \]

Proof. See [12, Proposition 2.2]. \( \square \)

Next, by Theorem 1, we establish a non-trivial estimate for \( f(\alpha) \) throughout the set of minor arcs \( \mathfrak{m} \).

Proposition 5. Let \( k \geq 3 \). Let \( \theta \) be a real number with \( 3/4 < \theta \leq 1 \) and suppose that
\[ \varrho = \varrho_k(\theta) = \frac{1}{2} \min \left\{ \frac{\sigma_k(\theta - 3/4)}{8}, \delta \right\}, \quad (6.8) \]
where \( \sigma_k = 1/(2t_k) \). Then, for any fixed \( \varepsilon > 0 \), we have
\[ \max_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll Y^{1-\varepsilon+\varepsilon}. \quad (6.9) \]

Proof. Take \( x = X - Y \), \( y = 2Y \). Recall that for \( \alpha \in \mathfrak{m} \), we have \( q > P \). Proposition 5 is an easy consequence of Theorem 1. \( \square \)

Now following the same argument as in [12, §2], we give the proof of Theorem 2 by combining Propositions 3, 4 and 5.
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