NONUNIFORM $\mu$-DICHOTOMY SPECTRUM AND KINEMATIC SIMILARITY

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Abstract. For linear nonautonomous differential equations we introduce a new family of spectrums defined with general nonuniform dichotomies: for a given growth rate $\mu$ in a large family of growth rates, we consider a notion of spectrum, named nonuniform $\mu$-dichotomy spectrum. This family of spectrums contain the nonuniform dichotomy spectrum as the very particular case of exponential growth rates. For each growth rate $\mu$, we describe all possible forms of the nonuniform $\mu$-dichotomy spectrum, relate its connected components with adapted notions of Lyapunov exponents, and use it to obtain a reducibility result for nonautonomous linear differential equations. We also give an illustrative examples where the spectrum is obtained, including a situation where a normal form is obtained for polynomial behavior.

1. Introduction

The dichotomy spectrum (also called dynamical spectrum and Sacker-Sell spectrum), defined with exponential dichotomies, was introduced by Sacker and Sell and used to study linear skew product flows with compact base [48]. The usefulness of the dichotomy spectrum has been widely proved in several contexts [23, 35, 30]. In particular, the dichotomy spectral theory has proved to be an important tool in the obtention of normal forms for nonautonomous differential equations [44, 45]. A version of dichotomy spectrum for nonautonomous linear difference equations was considered and a spectral theorem in that context was proved in [11]. An infinite-dimensional version of the dichotomy spectrum was considered by Sacker and Sell [49] and by Chow and Leiva [24, 26, 25]. We also refer the interesting papers [42, 43] by Pötzche, where the relation of the dichotomy spectrum with a weighted shift operator on some sequence space is explored. The book [23] by Chicone and Latushkin is a central reference in the theory of differential equations in Banach spaces via spectral properties of the associated evolution semigroup.

Despite its importance in the theory, the notion of (uniform) exponential dichotomy is sometimes too restrictive and it is important to consider more general hyperbolic behavior. The concept of nonuniform exponential dichotomy generalizes the concept of (uniform) exponential dichotomy by allowing some exponential loss of hyperbolicity along the trajectories. A
very complete theory of nonuniform exponential dichotomies is being developed by Barreira and Valls and a vast amount of results have already been published by those authors concerning this subject. Still in the context of nonuniform exponential dichotomies, we refer the books [14], by Barreira and Valls, concerning stability theory, and [3], by Barreira, Dragićević and Valls, concerning the relation of admissibility with hyperbolic behavior.

To the best of our knowledge, a version of dichotomy spectrum defined with nonuniform exponential dichotomies, the so called nonuniform dichotomy spectrum, was considered for the first time by Zhang in [51] and by Chu, Liao, Siegmund, Xia and Zhang in [27], with a slightly different definition. In both papers the authors describe the topological structure of the nonuniform dichotomy spectrum and use this spectrum to prove a result on the kinematic similarity of nonautonomous linear equations and block diagonal systems. Additionally, in [51], the author obtained normal forms for nonautonomous nonlinear systems using the spectrum of the linear part. Corresponding results in the context of discrete dynamics were obtained in a recent paper by Chu, Liao, Siegmund, Xia and Zhu [28].

Another way to generalize the notion of exponential dichotomy is to assume that the asymptotic behavior is not exponential. This reasoning leads to uniform dichotomies with asymptotic behavior given by general growth rates, a notion considered in the work of Naulin and Pinto [38, 40]. Proceeding in the direction of more general behavior, we can consider dichotomies that are both nonuniform and do not necessarily have exponential growth.

In [13, 19] a very general notion of nonuniform dichotomy, the notion of nonuniform \((\mu, \nu)\)-dichotomy, was considered and stable manifold theorems were established for perturbations of nonautonomous linear equations admitting this type of hyperbolic behavior, both in the continuous and discrete settings. In the present work we consider nonuniform \(\mu\)-dichotomies, a general notion of dichotomy obtained by assuming in the definition of \((\mu, \nu)\)-dichotomy that \(\mu = \nu\). Note that the notion of \(\mu\)-dichotomy includes the usual nonuniform exponential dichotomies in [5, 4] as well as the nonuniform polynomial dichotomies, introduced independently in [17] and [13], as very particular cases.

There is already a considerable amount of papers devoted to the study of difference and differential equations under the hypothesis of existence of a generalized notion of nonuniform dichotomy: in [22], Chang, Zhang and Qin discussed the robustness of nonuniform \((\mu, \nu)\)-dichotomies and, in [29], Chu addressed the same problem for the discrete-time case (see also Crai [32] for related results); in [39], Pan discussed the existence of stable manifolds for delay equations with nonuniform \((\mu, \nu)\)-dichotomies; in [16], Bento, Lupa, Megan and Silva obtained integral conditions for the existence of a nonuniform \(\mu\)-dichotomy (see also [21] where Boruga, Megan and Toth obtained integral characterizations of a generalized notion of uniform stability, in the spirit of Barbashin and Datko); in [34] Dragićević, Peček and Lupa introduced the notion of a \(\mu\)-dichotomy with respect to a family of norms (a notion that generalizes the notion of nonuniform \(\mu\)-dichotomy) and characterized it in terms of two admissibility conditions (see also [50] for related results in discrete time context); in [20] Bento and Vilarinho proved the
existence of measurable invariant manifolds for small perturbations of linear random dynamical systems admitting a general type of dichotomy; in \[15\] Bento and Costa established conditions for the existence of global Lipschitz invariant center manifolds for perturbations of linear nonautonomous equations admitting a very general form of nonuniform trichotomy; in \[2\] Backes and Dragi će vić obtained a shadowing result for perturbations of linear equations admitting a nonuniform \((\mu, \nu)\)-dichotomy.

There are two main objectives in this paper. The first objective is to define and characterize a new notion a nonuniform spectrum, defined with nonuniform \(\mu\)-dichotomies. To the extent of our knowledge, a notion of dichotomy spectrum defined with dichotomies with nonexponential behavior (even in the uniform case) is considered for the first time in the present work (see the definitions of nonuniform \(\mu\)-dichotomy spectrum and of \(\mu\)-dichotomy spectrum in section \(2\)). We note that the equations involved in the definition of the new spectrum depend on the growth rates of the dichotomy and therefore the new definition is not evident from the definition of nonuniform dichotomy spectrum. The second objective is related to normal form theory: we use our notion of spectrum to obtain a reducibility theorem for linear nonautonomous differential equations and after apply our results to obtain the normal form of a triangular system with polynomial behavior. We also relate our notions of spectrum to the notion of Lyapunov exponent adapted to a growth rate \(\mu\), a notion already considered in the literature \([13, 11]\). We emphasise that our notion of spectrum allows us to obtain normal forms in situations where the usual Lyapunov exponents are zero. This is illustrated in the triangular example referred.

The theory on normal form can be traced back to Poincaré \([41]\). The aim of this theory is to find a suitable change of variables that transforms a system of ordinary differential equations in another system that is simpler to analyse and has the same qualitative behavior. In the context of nonautonomous systems, reducibility results were obtained in \([44]\) using the dichotomy spectrum and in \([51, 27, 28]\) using the nonuniform dichotomy spectrum. We also refer the series of papers \([6, 7, 8, 9]\), by Barreira and Valls, where several results on the topological conjugacy between systems with nonuniformly exponential behaviour were obtained. Finally, we mention the interesting work of Li, Llibre and Wu \([36]\) concerning normal forms of almost periodic differential systems and the related work \([37]\), by the same authors, devoted to the study of normal forms of almost periodic difference systems.

We now briefly describe the structure of the paper. In section \(2\) we present our setting and define the new family of spectrums. In section \(3\) we describe the topological structure of the new spectrums, relate them with adapted notions of Lyapunov exponents and present explicit examples of linear nonautonomous differential equations for which the new spectrums can be computed. In section \(4\) we obtain a reducibility theorem on the existence of normal forms for linear nonautonomous differential equations, using the new nonuniform \(\mu\)-dichotomy spectrum. Finally, in section \(5\) we rewrite our results in the half-line setting and present an example of a family
of nonautonomous triangular systems for which we can compute the nonuniform polynomial dichotomy spectrum and use it to obtain a normal form, highlighting the existence of linear integral manifolds where polynomial contraction and expansion occur.

2. Nonuniform \( \mu \)-dichotomy spectrum

Let \( M_n(\mathbb{R}) \) be the set of square matrix functions of \( n \)th order defined in \( \mathbb{R} \) and let \( A(t) \in M_n(\mathbb{R}) \) for each \( t \in \mathbb{R} \). In this paper we consider nonautonomous linear systems

\[
x' = A(t)x,
\]

\( t \in \mathbb{R} \). We assume that \( t \mapsto A(t) \) is continuous. As a consequence, all solutions of system (1) are defined on the whole \( \mathbb{R} \). We denote by \( \Phi(t, s) \), \( t, s \in \mathbb{R} \), the corresponding evolution operator.

We say that a function \( \mu : \mathbb{R} \rightarrow \mathbb{R}^+ \) is a growth rate if it is strictly increasing, \( \mu(0) = 1 \), \( \lim_{t \to +\infty} \mu(t) = +\infty \) and \( \lim_{t \to -\infty} \mu(t) = 0 \). If, additionally, \( \mu \) is differentiable, we say that it is a differentiable growth rate.

Denote the sign of \( a \) by \( \text{sgn}(a) \). We say that system (1) admits a nonuniform \( \mu \)-dichotomy (N\( \mu \)D) if there is a family of projections \( P(t) \in M_n(\mathbb{R}) \), \( t \in \mathbb{R} \), such that, for all \( t, s \in \mathbb{R} \),

\[
\Phi(t, s) = \Phi(t, s)P(s),
\]

and there are constants \( K \geq 1 \), \( \alpha < 0 \), \( \beta > 0 \) and \( \theta, \nu \geq 0 \), with \( \alpha + \theta < 0 \) and \( \beta - \nu > 0 \), such that

\[
\|\Phi(t, s)P(s)\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \mu(s)^{\text{sgn}(s)\theta} \text{ for } t \geq s,
\]

\[
\|\Phi(t, s)Q(s)\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(s)^{\text{sgn}(s)\nu} \text{ for } t \leq s,
\]

where \( Q(s) = I - P(s) \) is the complementary projection. When \( \theta = \nu = 0 \) we say that system (1) admits a uniform \( \mu \)-dichotomy (or more simply a \( \mu \)-dichotomy (\( \mu \)D)).

If \( \mu(t) = e^t \) we obtain the usual notion of nonuniform exponential dichotomy (NED).

Given a strictly increasing function \( \nu : \mathbb{R}_0^+ \rightarrow [1, +\infty[ \) such that \( \nu(0) = 1 \) and \( \lim_{t \to +\infty} \nu(t) = +\infty \), we can define a growth rate \( \mu : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\mu(t) = \nu(|t|)^{\text{sgn}(t)} = \begin{cases} \nu(t) & t \geq 0 \\ \frac{1}{\nu(|t|)} & t < 0 \end{cases}
\]

and obtain the corresponding nonuniform \( \mu \)-dichotomy notion. Notice that when \( \nu(t) = e^t \) we obtain the already mentioned notion of nonuniform exponential dichotomy and when \( \nu(t) = t + 1 \) we obtain the growth rate \( p : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
p(t) = \begin{cases} t + 1 & t \geq 0 \\ \frac{1}{t - 1} & t < 0 \end{cases}
\]
and the corresponding nonuniform p-dichotomy that we call a nonuniform polynomial dichotomy (NPD). Note also that if ν is differentiable then μ is differentiable. In particular, the polynomial growth rate p is differentiable.

**Remark 1.** Consider the family of projections $P(t) \in M_n(\mathbb{R})$ in (2)–(3) and set

$$\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

for all $t \in \mathbb{R}$, where $I$ denotes the identity of order $\dim \text{Im } P(t)$. A similar argument to that in Lemma 2.2 in [27], shows that a fundamental matrix of (1), $X(t)$, can be chosen appropriately so that

$$\|\Phi(t, s)P(s)\| = \|X(t)\tilde{P}X(s)^{-1}\| \quad \text{and} \quad \|\Phi(t, s)Q(s)\| = \|X(t)\tilde{Q}X(s)^{-1}\|.$$

Thus, inequalities (2)–(3) can be written as

(5) \hspace{1cm} \|X(t)\tilde{P}X(s)^{-1}\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^{\alpha} \mu(s)^{\text{sgn}(s)\theta} \quad \text{for } t \geq s,

(6) \hspace{1cm} \|X(t)\tilde{Q}X(s)^{-1}\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^{\beta} \mu(s)^{\text{sgn}(s)\nu} \quad \text{for } t \leq s,

where $\tilde{Q} = I - \tilde{P}$ is the complementary projection of $\tilde{P}$.

For the sake of completeness, we reproduce the argument in our context. Fix $\tau \in \mathbb{R}$. Then there is a non-singular matrix $T$ such that $TP(\tau)T^{-1} = \tilde{P}$ with

$$\tilde{P} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where $I$ denotes the identity of order $\dim \text{Im } P(t)$. For each $t \in \mathbb{R}$, let $X(t) = \Phi(t, \tau)T^{-1}$. We have

$$\Phi(t, s)P(s) = \Phi(t, \tau)\Phi(\tau, s)P(s) = \Phi(t, \tau)P(\tau)\Phi(\tau, s) = \Phi(t, \tau)T^{-1}\tilde{P}\Phi(\tau, s) = \Phi(t, \tau)T^{-1}\tilde{P}(\Phi(\tau, s)T^{-1})^{-1} = X(t)\tilde{P}X(s)^{-1}$$

and thus

(7) \hspace{1cm} \Phi(t, s)Q(s) = \Phi(t, s)I - \Phi(t, s)P(s) = \Phi(t, \tau)\Phi(\tau, s)I - X(t)\tilde{P}X(s)^{-1} = X(t)TT^{-1}X(s)^{-1} - X(t)\tilde{P}X(s)^{-1} = X(t)(I - \tilde{P})X(s)^{-1} = X(t)\tilde{Q}X(s)^{-1}.

By (2) and (11), we get, for $t \geq s$,

$$\|X(t)\tilde{P}X(s)^{-1}\| = \|\Phi(t, s)P(s)\| = K \left( \frac{\mu(t)}{\mu(s)} \right)^{\alpha} \mu(s)^{\text{sgn}(s)\theta}$$

and by (3) and (8), we obtain, for $t \leq s$,

$$\|X(t)\tilde{Q}X(s)^{-1}\| = \|\Phi(t, s)Q(s)\| = K \left( \frac{\mu(t)}{\mu(s)} \right)^{\beta} \mu(s)^{\text{sgn}(s)\nu},$$

obtaining (5) and (6).
Let $\mu : \mathbb{R} \to \mathbb{R}^+$ be a differentiable growth rate. We define the nonuniform $\mu$-dichotomy spectrum of system (1) by

$$\Sigma_{\mu}^{ND}(A) = \{ \gamma \in \mathbb{R}; \ x' = \left( A(t) - \frac{\mu'(t)}{\mu(t)} I \right) x \text{ admits no NPD} \}$$

and refer to the complement of this set, $\rho_{\mu}^{ND}(A) = \mathbb{R} \setminus \Sigma_{\mu}^{ND}(A)$, as nonuniform $\mu$-resolvent set of system (1). We also define the $\mu$-dichotomy spectrum of system (1) by

$$\Sigma_{\mu}^{D}(A) = \{ \gamma \in \mathbb{R}; \ x' = \left( A(t) - \frac{\gamma}{1 + |t|} I \right) x \text{ admits no no D} \}$$

and call the complement of this set, denoted $\rho_{\mu}^{D}(A) = \mathbb{R} \setminus \Sigma_{\mu}^{D}(A)$, by $\mu$-resolvent set of system (1). Since a $\mu$-dichotomy is also a nonuniform $\mu$-dichotomy (with $\nu = \theta = 0$), we have $\Sigma_{\mu}^{ND}(A) \subseteq \Sigma_{\mu}^{D}(A)$. Following [27], when $\mu = (e^n)_{n \in \mathbb{N}}$ we denote the nonuniform $\mu$-dichotomy spectrum obtained by nonuniform dichotomy spectrum. When $p$ is the growth rate in (1) we obtain the new notion of spectrum

$$\Sigma_{\mu}^{ND}(A) = \{ \gamma \in \mathbb{R}; \ x' = \left( A(t) - \frac{\gamma}{1 + |t|} I \right) x \text{ admits no NPD} \}$$

that we call nonuniform polynomial dichotomy spectrum.

**Remark 2.** In the very particular case $\mu(t) = e^t$, $\alpha = -\beta$ and $\theta = \nu$, we recover the notion of nonuniform dichotomy spectrum considered in [27].

On the other hand, to recover the notion of nonuniform dichotomy spectrum in [51] when $\mu(t) = e^t$, one needs to add the extra condition $\max\{\theta, \nu\} \leq \min\{-\alpha, \beta\}$ to notion of nonuniform exponential dichotomy. Thus, in the particular case of nonuniform exponential dichotomies, our spectrum is contained in the one considered in [51].

A nonempty set $\mathcal{W}$ of $\mathbb{R} \times \mathbb{R}^n$ such that $\{(t, \Phi(t, s)\xi) : t \in \mathbb{R}\} \subset \mathcal{W}$ for each $(s, \xi) \in \mathcal{W}$ is called a linear integral manifold of system (1). For each $s \in \mathbb{R}$, the set

$$\mathcal{W}(s) = \{ \xi \in \mathbb{R}^n; \ (s, \xi) \in \mathcal{W} \}$$

is a linear subspace of $\mathbb{R}^n$ called a fiber of the linear integral manifold $\mathcal{W}$. Note that all the fibers of a linear integral manifold have the same dimension and form a vector bundle over $\mathbb{R}$. We define the rank of $\mathcal{W}$, denoted by $\text{rank } \mathcal{W}$, as the dimension of each of the fibers of $\mathcal{W}$. We also define the intersection and the sum of linear integral manifolds of (1), $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively by

$$\mathcal{W}_1 \cap \mathcal{W}_2 = \{(s, \xi) \in \mathbb{R} \times \mathbb{R}^n; \xi \in \mathcal{W}_1(s) \cap \mathcal{W}_2(s)\}$$

and

$$\mathcal{W}_1 + \mathcal{W}_2 = \{(s, \xi) \in \mathbb{R} \times \mathbb{R}^n; \xi \in \mathcal{W}_1(s) + \mathcal{W}_2(s)\}.$$

If $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{R} \times \{0\}$ for $1 \leq i \neq j \leq k$, the sum of the linear integral manifolds $\mathcal{W}_1, \ldots, \mathcal{W}_k$ is called the Whitney sum of those linear integral manifolds and is denoted by $\mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_k$. 
3. Spectral theorems

Given \( \gamma \in \mathbb{R} \), define the sets

\[
U_\gamma = \left\{ (s, \xi) \in \mathbb{R} \times \mathbb{R}^n : \sup_{t \geq 0} \| \Phi(s, t) \xi \| \mu(t)^{-\gamma} < \infty \right\}
\]

and

\[
V_\gamma = \left\{ (s, \xi) \in \mathbb{R} \times \mathbb{R}^n : \sup_{t \leq 0} \| \Phi(s, t) \xi \| \mu(t)^{-\gamma} < \infty \right\}.
\]

These sets will be used in this section to discuss the topological structure of the nonuniform \( \mu \)-dichotomy spectrum of system (1). Our first result includes Theorem 1.1 in [51] as the particular case of exponential growth rates.

**Theorem 3.** Let \( \mu : \mathbb{R} \to \mathbb{R}^+ \) be a differentiable growth rate. The following statements hold for system (1):

1) There is an \( m \in \{0, \ldots, n\} \) such that the nonuniform \( \mu \)-dichotomy spectrum \( \Sigma^{\text{ND}}_\mu(A) \) of system (1) is the union of \( m \) disjoint closed intervals in \( \mathbb{R} \):
- a) if \( m = 0 \) then \( \Sigma^{\text{ND}}_\mu(A) = \emptyset \);
- b) if \( m = 1 \) then \( \Sigma^{\text{ND}}_\mu(A) = (-\infty, b_1] \) or \( \Sigma^{\text{ND}}_\mu(A) = [a_1, b_1] \) or \( \Sigma^{\text{ND}}_\mu(A) = [a_1, \infty) \);
- c) if \( 1 < m \leq n \) then
  \[
  \Sigma^{\text{ND}}_\mu(A) = I_1 \cup [a_2, b_2] \cup \ldots \cup [a_{m-1}, b_{m-1}] \cup I_m
  \]
  with \( I_1 = [a_1, b_1] \) or \( (-\infty, b_1] \), \( I_m = [a_m, b_m] \) or \( [a_m, \infty) \) and \( a_i \leq b_i < a_{i+1} \) for \( i = 1, \ldots, m-1 \).

2) Assume \( m \geq 1 \), write

\[
\Sigma^{\text{ND}}_\mu(A) = I_1 \cup [a_2, b_2] \cup \ldots \cup [a_{m-1}, b_{m-1}] \cup I_m.
\]

and, for \( i = 0, \ldots, m + 1 \), define

\[
\begin{align*}
\mathcal{U}_{\gamma_0} & = \{ \mathbb{R} \times \{0\} \} \quad \text{if } i = 0 \text{ and } I_1 = (-\infty, b_1] \\
\mathcal{U}_{\gamma_i} & = \mathcal{U}_{\gamma_{i+1}} \cap \mathcal{V}_{\gamma_{i+1}} \quad \text{for some } \gamma_i \in (b_i, a_{i+1}) \quad \text{if } i = 1, \ldots, m \\
\mathcal{V}_{\gamma_m} & = \{ \mathbb{R} \times \{0\} \} \quad \text{if } i = m + 1 \text{ and } I_m = [a_m, b_m] \\
\end{align*}
\]

Then, the sets \( \mathcal{U}_{\gamma_i}, \mathcal{V}_{\gamma_i} \) and \( \mathcal{W}_i, i = 0, \ldots, m + 1 \), are integral manifolds, rank \( \mathcal{W}_i \geq 1 \) for \( i = 1, \ldots, m \) and

\[
\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_{m+1} = \mathbb{R} \times \mathbb{R}^n.
\]

We call the intervals in \( \Sigma^{\text{ND}} \) spectral intervals and each linear integral manifold \( \mathcal{W}_i, i = 0, \ldots, m + 1 \), a spectral manifold.

**Proof.** To prove our result we begin by establishing some lemmas.

**Lemma 4.** The sets \( \mathcal{U}_\gamma \) in (9) and \( \mathcal{V}_\gamma \) in (10) are linear integral manifolds of system (1). Additionally, if \( \gamma \leq \tilde{\gamma} \) then \( \mathcal{U}_\gamma \subseteq \mathcal{U}_{\tilde{\gamma}} \) and \( \mathcal{V}_\gamma \subseteq \mathcal{V}_{\tilde{\gamma}} \).
Thus we conclude that for any \((s, \xi) \in \mathcal{U}_\gamma\) we have \((\tau, \Phi(\tau, s)\xi) \in \mathcal{U}_\gamma\) for all \(\tau \in \mathbb{R}\). Thus \(\mathcal{U}_\gamma\) is a linear integral manifold of system \((1)\). Using a similar argument we conclude that \(\mathcal{V}_\gamma\) is also a linear integral manifold of system \((1)\).

Since \(\mu\) is increasing, \(\gamma \leq \tilde{\gamma}\) implies \(\mu(t)^{-\gamma} \geq \mu(t)^{-\tilde{\gamma}}\) and we have immediately \(\mathcal{U}_\gamma \subseteq \mathcal{U}_{\tilde{\gamma}}\) and \(\mathcal{V}_\gamma \supseteq \mathcal{V}_{\tilde{\gamma}}\). \(\square\)

**Lemma 5.** Let \(\gamma \in \mathbb{R}\). If \(x' = \left(A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I\right)x\) admits a nonuniform \(\mu\)-dichotomy with the invariant projection \(P\), then we have \(\mathcal{U}_\gamma = \text{Im}P\), \(\mathcal{V}_\gamma = \text{Ker}P\) and \(\mathcal{U}_\gamma \oplus \mathcal{V}_\gamma = \mathbb{R} \times \mathbb{R}^n\).

**Proof.** Let \(\Phi(t, s)\) be the evolution operator of the linear equation \(x' = A(t)x\) and

\[
\Phi_\gamma(t, s) = \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} \Phi(t, s).
\]

Noting that

\[
\Phi_\gamma(t, s)' = -\gamma \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma-1} \frac{\mu'(t)}{\mu(s)} \Phi(t, s) + \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} \Phi(t, s)'
\]

\[
= \left(A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I\right) \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} \Phi(t, s)
\]

\[
= \left(A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I\right) \Phi_\gamma(t, s)
\]

and \(\Phi_\gamma(s, s) = \left(\frac{\mu(s)}{\mu(s)}\right)^{-\gamma} \Phi(s, s) = I\), we conclude that \(\Phi_\gamma(t, s)\) is the evolution operator of the linear equation \(x' = \left(A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I\right)x\). Moreover, the projection \(P\) still commutes with \(\Phi_\gamma(t, s)\):

\[
\Phi_\gamma(t, s)P(s) = \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} \Phi(t, s)P(s) = \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} P(t)\Phi(t, s) = P(t)\Phi_\gamma(t, s).
\]

By the assumption, there exist \(K_{\gamma} \geq 1\), \(\alpha_{\gamma} < 0\), \(\beta_{\gamma} > 0\) and \(\theta_{\gamma}, \nu_{\gamma} \geq 0\) with \(\alpha + \theta_{\gamma} < 0\) and \(\beta - \nu_{\gamma} > 0\) such that

\[
\|\Phi_\gamma(t, s)P(s)\| \leq K_{\gamma} \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha_{\gamma}} \mu(s)^{\text{sgn}(s)\theta_{\gamma}} \text{ for all } t \geq s,
\]

\[
\|\Phi_\gamma(t, s)Q(s)\| \leq K_{\gamma} \left(\frac{\mu(t)}{\mu(s)}\right)^{\beta_{\gamma}} \mu(s)^{\text{sgn}(s)\nu_{\gamma}} \text{ for all } t \leq s.
\]

First we prove \(\mathcal{U}_\gamma \subseteq \text{Im}P\). Let \((\tau, \xi) \in \mathcal{U}_\gamma\). By definition there exists a constant \(c_{\gamma}\) such that

\[
\|\Phi(t, \tau)\xi\| \leq c_{\gamma} \mu(t)^{\gamma} \text{ for all } t \geq 0.
\]

It follows that

\[
\|\Phi_\gamma(t, \tau)\xi\| = \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\gamma} \|\Phi(t, \tau)\xi\| \leq c_{\gamma} \mu(\tau)^{\gamma} \text{ for all } t \geq 0.
\]
Let $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \text{Im} P(\tau)$ and $\xi_2 \in \text{Ker} P(\tau)$. We have

$$\xi_2 = (I - P(\tau))\xi = (I - \Phi_\tau(\tau, t)\Phi_\gamma(t, \tau)P(\tau))\xi$$

$$= (I - \Phi_\tau(\tau, t)P(t)\Phi_\gamma(t, \tau))\xi = \Phi_\gamma(t, \tau)Q(t)\Phi_\tau(t, \tau)\xi.$$ 

Hence, using (13), we have, for $t \geq \theta$, $\nu$

$$\|\xi_2\| \leq K_\gamma \left(\frac{\mu(\tau)}{\mu(t)}\right)^{\beta_\gamma} \mu(t)^{\nu_\gamma} \|\Phi_\gamma(t, \tau)\xi\| \leq K_\gamma c_\gamma \mu(t)^{-\nu_\gamma} \mu(\tau)^{\beta_\gamma + \gamma}.$$

Therefore, $\xi_2 = 0$ because $\beta_\gamma - \nu_\gamma > 0$. Thus $\xi = \xi_1 \in \text{Im} P(\tau)$. This proves that $\mathcal{U}_\tau \subseteq \text{Im} P$.

To establish that $\text{Im} P \subseteq \mathcal{U}_\tau$, assume that $\tau \in \mathbb{R}$, $\xi \in \text{Im} P(\tau)$. Then $P(\tau)\xi = \xi$. By (12), we have, for $t \geq \max\{\tau, 0\}$,

$$\|\Phi(t, \tau)\xi\|\mu(t)^{-\gamma} = \|\Phi_\tau(\tau, t)P(\tau)\xi\|\mu(\tau)^{-\gamma} \leq K_\gamma \mu(t)^{\alpha_\gamma} \mu(\tau)^{-(\gamma + \alpha_\gamma)} \mu(\tau)^{\theta_\gamma} \|\xi\|.$$ 

This implies that $(t, \tau, \xi) \in \mathcal{U}_\tau$, because $\alpha_\gamma < 0$. Hence $\text{Im} P \subseteq \mathcal{U}_\tau$. We obtain $\text{Im} P = \mathcal{U}_\tau$.

Using $\alpha_\gamma + \theta_\gamma < 0$, we can apply similar arguments to prove that $\mathcal{V}_\tau = \text{Ker} P$. Since $\mathcal{U}_\tau = \text{Im} P$ and $\mathcal{V}_\tau = \text{Ker} P$, the identity $\mathcal{U}_\tau \cap \mathcal{V}_\tau = \mathbb{R} \times \mathbb{R}^n$ follows.

**Lemma 6.** The resolvent set $\rho^{\text{ND}}(A)$ is open. Moreover, if $\gamma \in \rho^{\text{ND}}(A)$ and $J \subseteq \rho^{\text{ND}}(A)$ is an interval containing $\gamma$, then

$$\mathcal{U}_\eta = \mathcal{U}_\gamma, \quad \mathcal{V}_\eta = \mathcal{V}_\gamma \quad \text{for all } \eta \in J.$$

**Proof.** Let $\gamma \in \rho^{\text{ND}}(A)$. Then $x' = \left(A(t) - \gamma \frac{\mu'(t)}{\mu(t)}I\right)x$ admits a N\textmu D: there is some family of projections $P(t)$ and constants $K \geq 1$, $\alpha < 0$, $\beta > 0$ and $\theta, \nu \geq 0$, with $\alpha + \theta < 0$ and $\beta - \nu > 0$ such that

$$\|\Phi_\gamma(t, s)P(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha} \mu(s)^{\gamma(s)\theta} \quad \text{for } t \geq s,$$

$$\|\Phi_\gamma(t, s)Q(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\beta} \mu(s)^{\gamma(s)\nu} \quad \text{for } t \leq s.$$ 

Let $0 < \sigma < \min\{(-\beta - \nu)/2, -(\alpha + \theta)/2\}$. It follow from the proof of Lemma 5 that $P(t)$ is an invariant projection for the evolution operator $\Phi_\gamma(t, s) = \left(\frac{\mu(s)}{\mu(t)}\right)^{\eta} \Phi(t, s)$ of system $x' = \left(A(t) - \eta \frac{\mu'(t)}{\mu(t)}I\right)x$. Additionally, for $\eta \in (-\sigma, \gamma + \sigma)$ we have

$$\|\Phi_\eta(t, s)P(s)\| = \left(\frac{\mu(s)}{\mu(t)}\right)^{-\eta} \|\Phi_\gamma(t, s)P(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha - \eta} \mu(s)^{\gamma(s)\theta} \quad \text{for } t \geq s,$$

$$\|\Phi_\eta(t, s)Q(s)\| = \left(\frac{\mu(s)}{\mu(t)}\right)^{-\eta} \|\Phi_\gamma(t, s)Q(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{-\beta - \eta} \mu(s)^{\gamma(s)\nu} \quad \text{for } t \leq s.$$ 

This proves that \( x' = \left( A(t) - \eta u(\eta, t) I \right) x \) admits a \( N \mu \Delta \) for all \( \eta \in (\gamma - \sigma, \gamma + \sigma) \). Consequently \( (\gamma - \sigma, \gamma + \sigma) \subset \rho_\mu^{\text{ND}}(A) \), for sufficiently small \( \sigma > 0 \). Hence, \( \rho_\mu^{\text{ND}}(A) \) is an open set. We also conclude that systems \( x' = \left( A(t) - \eta u(\eta, t) I \right) x \) and \( x' = \left( A(t) - \gamma u(\gamma, t) I \right) x \) admit a \( N \mu \Delta \) with the same projection, \( P(t) \), when \( \eta \in (\gamma - \sigma, \gamma + \sigma) \). According to Lemma 6, we have \( U_{\gamma} = U_{\eta} = \text{Im} P \) and \( V_{\gamma} = V_{\eta} = \text{Ker} P \).

Let \( J \subset \rho_\mu^{\text{ND}}(A) \) be an interval containing \( \gamma \) and \( \gamma_0 \). Assuming \( \gamma_0 \leq \gamma \), we have \( [\gamma_0, \gamma] \subset J \). For each \( a \in [\gamma_0, \gamma] \) there exists \( \sigma_a > 0 \) such that \( U_{\gamma} = U_a \) and \( V_{\gamma} = V_a \) for all \( \zeta \in (a - \sigma_a, a + \sigma_a) \). Since these open intervals cover \([\gamma_0, \gamma] \), we get that \( U_{\gamma} = U_{\gamma_0} \) and \( V_{\gamma} = V_{\gamma_0} \). The same argument leads to the conclusion that the same property holds when \( \gamma_0 > \gamma \). Since \( \gamma_0 \in J \) is arbitrary, we obtain the result.

The next lemma characterizes the intersection of the linear integral manifolds \( U_{\gamma_1} \) and \( V_{\gamma_1} \) with \( \gamma_1, \gamma_2 \in \rho_\mu^{\text{ND}}(A) \).

**Lemma 7.** Let \( \gamma_1, \gamma_2 \in \rho_\mu^{\text{ND}}(A) \) and \( \gamma_1 < \gamma_2 \). The following conditions are equivalent.

a) \( U_{\gamma_1} \cap V_{\gamma_1} \neq \mathbb{R} \times \{0\} \);

b) \([\gamma_1, \gamma_2] \cap \Sigma_\mu^{\text{ND}}(A) \neq \emptyset \);

c) \( \text{rank} \; U_{\gamma_1} < \text{rank} \; U_{\gamma_2} \);

d) \( \text{rank} \; V_{\gamma_1} > \text{rank} \; V_{\gamma_2} \).

**Proof.** By Lemma 6, we have

\[
\text{rank} \; U_{\gamma_1} + \text{rank} \; V_{\gamma_1} = \text{rank} \; U_{\gamma_2} + \text{rank} \; V_{\gamma_2} = n
\]

and we immediately conclude the equivalence between c) and d).

Assume that b) holds. By Lemma 4, taking into account that \( U_{\gamma_1} \subseteq U_{\gamma_2} \), we have \( U_{\gamma_1} \subseteq U_{\gamma_2} \cup V_{\gamma_1} \subseteq \mathbb{R} \times \mathbb{R}^{n} \) and therefore

\[
\text{rank} \; (U_{\gamma_2} \cap V_{\gamma_1}) \geq \text{rank} \; U_{\gamma_2} + \text{rank} \; V_{\gamma_1} - n > \text{rank} \; U_{\gamma_1} + \text{rank} \; V_{\gamma_1} - n = 0.
\]

So \( U_{\gamma_2} \cap V_{\gamma_1} \neq \mathbb{R} \times \{0\} \), which proves a).

Assume that a) holds and assume, by contradiction, that \([\gamma_1, \gamma_2] \cap \Sigma_\mu^{\text{ND}}(A) = \emptyset \) and therefore that \([\gamma_1, \gamma_2] \subset \rho_\mu^{\text{ND}}(A) \). It follows from Lemma 6 that

\[
U_{\gamma_2} \cap V_{\gamma_1} = U_{\gamma_1} \cap V_{\gamma_1} = \mathbb{R} \times \{0\},
\]

a contradiction with a). We conclude that b) holds.

Next we assume, by contradiction, that c) holds but d) doesn’t hold, that is, \( c \) holds but rank \( U_{\gamma_1} \geq \text{rank} \; U_{\gamma_2} \). Since \( \gamma_1 < \gamma_2 \), by Lemma 4, we must have rank \( U_{\gamma_1} \leq \text{rank} \; U_{\gamma_2} \). Thus \( \text{rank} \; U_{\gamma_1} = \text{rank} \; U_{\gamma_2} \). By the equivalence of c) and d) we also have rank \( V_{\gamma_1} = \text{rank} \; V_{\gamma_2} \). Since \( U_{\gamma_1}(t) \) and \( V_{\gamma_1}(t) \), \( i = 1, 2 \), are linear subspaces of \( \mathbb{R}^{n} \), we must have \( U_{\gamma_1}(t) = U_{\gamma_2}(t) \) and \( V_{\gamma_1}(t) = V_{\gamma_2}(t) \). Thus \( U_{\gamma_1} = U_{\gamma_2} \) and \( V_{\gamma_1} = V_{\gamma_2} \). As a consequence, by Lemma 6, the nonuniform \( \mu \)-dichotomies of \( x' = \left( A(t) - \gamma u(\gamma, t) I \right) x \) and \( x' = \left( A(t) - \gamma u(\gamma, t) I \right) x \) have the same invariant projection \( P(t) \). Let
\[ K_i \geq 1, \alpha_i < 0, \beta_i > 0, \theta_i \geq 0 \text{ and } \nu_i \geq 0, \text{ satisfying } \alpha_i + \theta_i < 0 \text{ and } \beta_i - \nu_i > 0, \text{ for } i = 1, 2, \text{ be such that} \]
\[
\| \Phi_{\gamma_1}(t, s)P(s) \| \leq K_1 \left( \frac{\mu(t)}{\mu(s)} \right)^{\alpha_1} \mu(s)^{\sgn(s)\theta_1} \text{ for } t \geq s
\]
and
\[
\| \Phi_{\gamma_1}(t, s)Q(s) \| \leq K_1 \left( \frac{\mu(t)}{\mu(s)} \right)^{\beta_1} \mu(s)^{\sgn(s)\nu_1} \text{ for } t \leq s.
\]

For \( \gamma \in [\gamma_1, \gamma_2] \), we have for \( t \geq s \)
\[
\| \Phi_{\gamma}(t, s)P(s) \| = \left( \frac{\mu(t)}{\mu(s)} \right)^{\gamma_1 - \gamma} \| \Phi_{\gamma_1}(t, s)P(s) \| \leq \| \Phi_{\gamma_1}(t, s)P(s) \| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^{\alpha_1} \mu(s)^{\sgn(s)\theta_1},
\]
and, for \( t \leq s \),
\[
\| \Phi_{\gamma}(t, s)Q(s) \| = \left( \frac{\mu(t)}{\mu(s)} \right)^{\gamma_2 - \gamma} \| \Phi_{\gamma_2}(t, s)Q(s) \| \leq \| \Phi_{\gamma_2}(t, s)Q(s) \| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^{\beta_2} \mu(s)^{\sgn(s)\nu_2},
\]
where \( K = \max\{K_1, K_2\} \). Since \( \alpha_1 + \theta_1 < 0 \) and \( \beta_2 - \nu_2 > 0 \), we conclude that \( \Phi_\gamma \) admits a N\(^\mu\)D with constants \( \alpha_1, \beta_2, \theta_1, \nu_2 \) and \( K \). Thus \( \gamma \in \rho_\mu^{\text{ND}}(A) \) and consequently \( [\gamma_1, \gamma_2] \subset \rho_\mu^{\text{ND}}(A) \). This contradicts \([\text{a}]\) We conclude that \([\text{c}]\) hold and the lemma is established. \( \square \)

Now we will use the auxiliary results obtained above to prove the theorem. According to Lemma\([\text{b}]\) \( \rho_\mu^{\text{ND}}(A) \) is a nonempty open subset of \( \mathbb{R} \) and therefore it can be written as a finite or countable union of open mutually disjoint intervals. Thus, \( \Sigma_\mu^{\text{ND}}(A) = \mathbb{R} \setminus \rho_\mu^{\text{ND}}(A) \) is either empty or consists of a finite or countable union of closed intervals with vanishing intersection. Let \( m \in \mathbb{N} \cup \{+\infty\} \) be the number of disjoint closed intervals whose union is \( \Sigma_\mu^{\text{ND}} \):
\[
\Sigma_\mu^{\text{ND}}(A) = \cdots \cup [a_{i-1}, b_{i-1}] \cup [a_i, b_i] \cup \cdots,
\]
with \( a_{i-1} \leq b_{i-1} < a_i \leq b_i \). Assume, by contradiction, that \( m > n \). Choose \( n + 1 \) consecutive disjoint intervals, \([a_{k_0}, b_{k_0}], \ldots, [a_{k_0+n}, b_{k_0+n}]\), and, for \( j = k_0, \ldots, k_0 + n - 1 \), let \( \gamma_j \in (b_j, a_{j+1}) \). By the equivalence of \([\text{a}]\) and \([\text{c}]\) in Lemma\([\text{b}]\) we have
\[
0 \leq \text{rank } U_{\gamma_{k_0}} < \text{rank } U_{\gamma_{k_0+1}} < \cdots < \text{rank } U_{\gamma_{k_0+n}} \leq n,
\]
which allows us to conclude that \( \text{rank } U_{\gamma_{k_0+n}} = n \) or \( \text{rank } U_{\gamma_{k_0}} = 0 \).

If \( \text{rank } U_{\gamma_{k_0+n}} = n \), we must have \( P = I, U_{\gamma_{k_0+n}} = \mathbb{R} \times \mathbb{R}^n \) and \( V_{\gamma_{k_0+n}} = \mathbb{R} \times \{0\} \). By definition, \( x' = \left( A(t) - \gamma \frac{\mu(t)}{\mu(s)} \right) x \) admits a N\(^\mu\)D with invariant projection \( P = I \) for \( \gamma = \gamma_{k_0+n} \). Thus, for \( \gamma > \gamma_{k_0+n} \) and \( t \geq s \) we have
\[
\| \Phi_\gamma(t, s) \| = \left( \frac{\mu(t)}{\mu(s)} \right)^{\gamma_{k_0+n} - \gamma} \| \Phi_{\gamma_{k_0+n}}(t, s) \| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^{\gamma_{k_0+n} - \gamma + \alpha} \mu(s)^{\sgn(s)\theta},
\]
for some $K \geq 1$, $\alpha < 0$ and $\theta \geq 0$, with $\alpha + \theta < 0$ (which implies that $\gamma_{\theta} = \gamma + \alpha + \theta < 0$). This shows that $x' = \left( A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I \right) x$ admits a NMD with the invariant projection $P = I$ for all $\gamma > \gamma_{\theta} + \alpha + \theta$.

If rank $U_{\theta} = 0$, we must have $P = 0, U_{\theta} = \mathbb{R} \times \{0\}$ and $\mathcal{N}_{\theta} = \mathbb{R} \times \mathbb{R}^n$. By definition, $x' = \left( A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I \right) x$ admits a NMD with the invariant projection $P = 0$ for $\gamma = \gamma_{\theta}$. Thus, for $\gamma < \gamma_{\theta}$, and $t \leq s$ we have

$$\| \Phi_\gamma(t, s) \| = \left( \frac{\mu(t)}{\mu(s)} \right)^{\gamma_{\theta} - \gamma} \| \Phi_{\gamma_0}(t, s) \| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^{\gamma_{\theta} - \gamma + \beta} \mu(s)^{\text{sgn}(s) \nu},$$

for some $K \geq 1$, $\beta > 0$ and $\nu \geq 0$, with $\beta - \nu > 0$ (which implies that $\gamma_{\theta} = \gamma + \beta - \mu > 0$). This shows that $x' = \left( A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I \right) x$ admits a NMD with the invariant projection $P = 0$ for all $\gamma < \gamma_{\theta}$.

This is in contradiction with the assumption that $m > n$. We obtain [1].

We now prove [2]. We begin by noting that, by Lemma [3] the spectral manifolds $W_\gamma$ are independent of the choice of $\gamma$.

Assuming that $I_1 = [a_1, b_1]$, take $\gamma_1 \in (-\infty, a_1)$ and $x_1 \in (b_1, a_2)$. Then $\gamma_0, \gamma_1 \in \rho^N_\mu(A)$ and $[\gamma_0, \gamma_1] \cap \Sigma^N_\mu(A) \neq \emptyset$. By Lemmas [4] and [6] we have $U_{\gamma_0} \subset U_{\gamma_1}$. Since $U_{\gamma_0} \oplus U_{\gamma_1} = \mathbb{R} \times \mathbb{R}^n$, we must have $U_{\gamma_0} \cap V_{\gamma_0} \subset U_{\gamma_1} \cap V_{\gamma_0} = W_{\gamma_1}$. Since $W_{\gamma_1}$ is a linear integral manifold, we conclude that rank $W_{\gamma_1} \geq 1$.

If $I_1 = (-\infty, b_1]$ then $U_{\gamma_1} = \mathbb{R} \times \{0\}$ which implies $V_{\gamma_1} = \mathbb{R} \times \mathbb{R}^n$. Thus $W_{\gamma_1} = U_{\gamma_1} \cap V = U_{\gamma_1}$. By contradiction, if rank $W_{\gamma_1} = 0$ we would have $W_{\gamma_1} = \mathbb{R} \times \{0\}$ and thus we would have a NMD for $x' = \left( A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I \right) x$ with the invariant projection $P \equiv 0$. Proceeding as in the proof of [1] we conclude that $(-\infty, \gamma_1] \subset \rho^N_\mu(A)$, a contradiction with the choice of $\gamma_1$. We conclude that rank $W_{\gamma_1} \geq 1$ also in this case.

For $i = \{1, \ldots, m\}$, we have $\gamma_{i-1}, \gamma_i \in \rho^N_\mu(A)$ and $[\gamma_{i-1}, \gamma_i] \cap \Sigma^N_\mu(A) \neq \emptyset$. Thus, by Lemma [7] we have $U_{\gamma_i} \cap V_{\gamma_{i-1}} = W_i \neq \mathbb{R} \times \{0\}$ and consequently rank $W_i \geq 1$. Additionally, since $V_{\gamma_{i+1}} \subset V_{\gamma_i}$, we have, for $i = 0, \ldots, m - 1$,

$$W_{i+1} + V_{\gamma_{i+1}} = \{ (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^n : \eta \in V_{\gamma_{i+1}}(\tau) + V_{\gamma_i}(\tau) \} = \{ (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^n : \eta \in V_{\gamma_i}(\tau) \cap U_{\gamma_{i+1}}(\tau) + V_{\gamma_i}(\tau) \} = \{ (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^n : \eta \in V_{\gamma_i}(\tau) \} = V_{\gamma_i}.$$

As a consequence, we have

$$W_0 + W_1 + \ldots + W_m + W_{m+1} = W_0 + W_1 + \ldots + W_m + V_{\gamma_m} = W_0 + W_1 + \ldots + W_{m-1} + V_{\gamma_{m-1}} = \ldots = W_0 + W_1 + V_{\gamma_1} = W_0 + V_{\gamma_0} = U_{\gamma_0} + V_{\gamma_0} = \mathbb{R} \times \mathbb{R}^n.$$

By Lemma [8] we have

$$W_i \cap W_j \subseteq U_{\gamma_j} \cap V_{\gamma_{j-1}} \subseteq U_{\gamma_i} \cap V_{\gamma_i} = \mathbb{R} \times \{0\},$$

for $0 \leq i < j \leq m + 1$. This proves that $\mathbb{R} \times \mathbb{R}^n = W_0 \oplus W_1 \oplus \ldots \oplus W_{m+1}$.

We conclude that [2] holds and the theorem is proved. □
Given $\varepsilon \geq 0$ and a growth rate $\mu$, we say that the evolution operator $\Phi(t, s)$ of $x' = A(t)x$ has nonuniformly bounded growth with respect to $(\mu, \varepsilon)$ if there are $K \geq 1$ and $a \geq 0$ such that

$$\|\Phi(t, s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\text{sgn}(t-s)\varepsilon} \mu(s)^{\text{sgn}(s)\varepsilon}, \quad t, s \in \mathbb{R}. \quad (16)$$

When $\varepsilon = 0$ the evolution operator is said to have bounded growth with respect to $\mu$.

The next result shows that nonuniformly bounded growth with respect to the growth rate $\mu$ is a sufficient condition for the $\mathbb{N}\mu\mathbb{D}$ spectrum to be nonempty and bounded.

**Theorem 8.** If the evolution operator of system (1) has nonuniformly bounded growth with respect to the growth rate $\mu$ then the nonuniform $\mu$-dichotomy spectrum $\Sigma_{\mu}(A)$ of system (1) is $\Sigma_{\mu}(A) = [a_1, b_1] \cup \ldots \cup [a_m, b_m]$ with $a_i, b_i \in \mathbb{R}$, for some $i \in \{1, \ldots, m\}$, and $b_i < a_{i+1}$, for $i = 1, \ldots, m - 1$.

**Proof.** By the assumption the evolution operator $\Phi(t, s)$ of system (1) has a nonuniformly bounded growth with respect to the growth rate $\mu$: there are $K \geq 1$, $a \geq 0$ and $\varepsilon \geq 0$ such that (16) holds. By (16), for $t \geq s$ we have

$$\|\Phi(t, s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\varepsilon} \|\Phi(t, s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\varepsilon}. \quad (16)$$

For $\gamma > a + \varepsilon \implies -\gamma + a + \varepsilon < 0$, system $x' = \left(A(t) - \frac{\mu(t)}{\mu(s)}I\right)x$ admits a $\mathbb{N}\mu\mathbb{D}$ with the invariant projection $P = I$. We conclude that $(a + \varepsilon, +\infty) \subseteq \lambda_{\mu}^{\mathbb{D}}(A)$. Again by (16), for $t \leq s$ we have

$$\|\Phi(t, s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{-\varepsilon} \|\Phi(t, s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{-\varepsilon}. \quad (16)$$

For $\gamma < -a - \varepsilon \implies -\gamma - a - \varepsilon > 0$, system $x' = \left(A(t) - \frac{\mu(t)}{\mu(s)}I\right)x$ admits a $\mathbb{N}\mu\mathbb{D}$ with the invariant projection $P \equiv 0$. We conclude that $(-\infty, -a - \varepsilon) \subseteq \lambda_{\mu}^{\mathbb{D}}(A)$. Finally, since

$$\Sigma_{\mu}(A) = \mathbb{R} \setminus \lambda_{\mu}^{\mathbb{D}}(A) \subseteq \mathbb{R} \setminus \{(-\infty, -a - \varepsilon) \cup (a + \varepsilon, +\infty)\} = [-a - \varepsilon, a + \varepsilon],$$

we conclude that $\Sigma_{\mu}(A)$ is bounded.

By the above proof, for $\gamma > a + \varepsilon$, we have $U_\gamma = \text{Im} I = \mathbb{R} \times \mathbb{R}^n$ and $V_\gamma = \text{Ker} I = \mathbb{R} \times \{0\}$ and for $\gamma < -a - \varepsilon$, we have $U_\gamma = \text{Im} 0 = \mathbb{R} \times \{0\}$ and $V_\gamma = \text{Ker} 0 = \mathbb{R} \times \mathbb{R}^n$. Let

$$\gamma^* = \sup\{\gamma \in \lambda_{\mu}^{\mathbb{D}}(A) : V_\gamma = \mathbb{R} \times \mathbb{R}^n\}.$$

We have $\gamma^* \in [-a - \varepsilon, a + \varepsilon]$. If $\gamma^* \in \lambda_{\mu}^{\mathbb{D}}(A)$, by Lemma 9 there would be $\delta > 0$ such that $(\gamma^* - \delta, \gamma^* + \delta) \subseteq \lambda_{\mu}^{\mathbb{D}}(A)$ and thus we would have $V_\gamma = V_{\gamma^*}$ for any $\gamma \in (\gamma^* - \delta, \gamma^* + \delta)$, contradicting the definition of $\gamma^*$. Therefore, $\gamma^* \in \Sigma_{\mu}(A)$ and we conclude that $\Sigma_{\mu}(A) \neq \emptyset$. Taking into account the conclusions of Theorem 8 the result is established. \hfill \Box

Next, present a family of simple examples of nonautonomous systems of linear differential equations for which we are able to compute the nonuniform $\mu$-dichotomy spectrum.
Example 9. Let \( \alpha < 0, \beta > 0, \theta, \nu \geq 0 \) with \( \alpha + \theta < 0 \) and \( \beta - \nu > 0 \) and let \( \mu : \mathbb{R} \to \mathbb{R}^+ \) be a differentiable growth rate. Define

\[ A(t) = \begin{bmatrix} a_1(t) & 0 \\ 0 & a_2(t) \end{bmatrix}, \]

where

\[ a_1(t) = \alpha \frac{\mu'(t)}{\mu(t)} + t \text{sgn}(t) \left( \frac{\mu'(t) \cos t - 1}{\mu(t)} - \log \mu(t) \frac{\sin t}{2} \right) \]

and

\[ a_2(t) = \beta \frac{\mu'(t)}{\mu(t)} + \nu \text{sgn}(t) \left( \frac{\mu'(t) \cos t - 1}{\mu(t)} - \log \mu(t) \frac{\sin t}{2} \right). \]

Consider the differential equation in \( \mathbb{R}^2 \) given by

\[ (17) \quad \begin{bmatrix} u' \\ v' \end{bmatrix} = A(t) \begin{bmatrix} u \\ v \end{bmatrix}. \]

The evolution operator of this equation is given by

\[ \Phi(t, s)(u, v) = (U(t, s)u, V(t, s)v), \]

where

\[ U(t, s) = \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \frac{\mu(t)}{\mu(s)}^{\text{sgn}(t)(\cos t - 1)/2} \]

and

\[ V(t, s) = \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \frac{\mu(t)}{\mu(s)}^{\text{sgn}(s)(\cos s - 1)/2}. \]

For the projections \( P_1(t) : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( P_1(t)(u, v) = (u, 0) \) we have

\[ \|\Phi(t, s)P_1(s)\| = |U(t, s)| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \mu(s)^{\text{sgn}(s)^\theta}, \quad \text{for} \quad t \geq s \]

\[ \|\Phi(t, s)Q_1(t)\| = |V(t, s)| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(s)^{\text{sgn}(s)^\nu}, \quad \text{for} \quad t \leq s \]

and thus the equation admits a N\( \mu \)D. Moreover, if \( t = 2k\pi \) and \( s = (2k - 1)\pi, k \in \mathbb{N}, \) then

\[ \|\Phi(t, s)P_1(s)\| = \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \mu(s)^{\text{sgn}(s)^\theta} \]

and if \( t = (2k - 1)\pi \) and \( s = 2k\pi, k \in \mathbb{N}, \) then

\[ \|\Phi(t, s)Q_1(t)\| = \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(t)^{\text{sgn}(t)^\nu}. \]

Note now that

\[ \Phi_\gamma(t, s)(u, v) = (U_\gamma(t, s)u, V_\gamma(t, s)v), \]

where

\[ U_\gamma(t, s) = \left( \frac{\mu(t)}{\mu(s)} \right)^{-\gamma} U(t, s) \quad \text{and} \quad V_\gamma(t, s) = \left( \frac{\mu(t)}{\mu(s)} \right)^{-\gamma} V(t, s). \]

Thus

\[ \|\Phi_\gamma(t, s)P_1(s)\| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^{-\gamma} \mu(s)^{\text{sgn}(s)^\theta}, \quad \text{for} \quad t \geq s \]

\[ \|\Phi_\gamma(t, s)Q_1(t)\| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^{-\gamma} \mu(s)^{\text{sgn}(s)^\nu}, \quad \text{for} \quad t \leq s. \]

We conclude that if \( \alpha - \gamma + \theta < 0 \Leftrightarrow \gamma > \alpha + \theta \) and \( \beta - \gamma - \nu > 0 \Leftrightarrow \gamma < \beta - \nu, \) that is if \( \gamma \in (\alpha + \theta, \beta - \nu), \) then equation (17) admits a N\( \mu \)D.
Letting $P_2(t) = I$, where $I$ the identity, for all $t \in \mathbb{R}$, we have

$$
\| \Phi_\gamma(t, s)P_2(s) \| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^{\beta - \gamma} \mu(s)^{\text{sgn}(s) \nu}, \quad \text{for } t \geq s
$$

and we conclude that if $\beta - \gamma + \nu < 0 \iff \gamma > \beta + \nu$, that is, if $\gamma \in (\beta + \nu, +\infty)$, then equation (17) admits a $N\mu D$.

Letting $P_3(t) = 0$, where $0$ the null projection, for all $t \in \mathbb{R}$, we have

$$
\| \Phi_\gamma(t, s)Q_3(t) \| \leq \left( \frac{\mu(t)}{\mu(s)} \right)^{\alpha - \gamma} \mu(s)^{\text{sgn}(s) \theta}, \quad \text{for } t \leq s
$$

and we conclude that if $\alpha - \gamma - \theta > 0 \iff \gamma < \alpha - \theta$, that is if $\gamma \in (-\infty, \alpha - \theta)$, then equation (17) admits a $N\mu D$. By the above, we can conclude that

$$
\Sigma^\mu_\nu(A) = [\alpha - \theta, \alpha + \theta] \cup [\beta - \nu, \beta + \nu].
$$

Letting $\gamma_0 \in (-\infty, \alpha - \theta)$, $\gamma_1 \in (\alpha + \theta, \beta - \nu)$ and $\gamma_2 \in (\beta + \nu, +\infty)$, we have $U_{\gamma_0} = V_{\gamma_2} = \mathbb{R} \times \{(0, 0)\}$, $U_{\gamma_2} = V_{\gamma_0} = \mathbb{R} \times \mathbb{R}^2$, $U_{\gamma_1} = \mathbb{R} \times \{(r, 0) \in \mathbb{R}^2 : r \in \mathbb{R}\}$ and $V_{\gamma_1} = \mathbb{R} \times \{0, r \in \mathbb{R} : r \in \mathbb{R}\}$. Thus

$$
W_0 = U_{\gamma_0} = \mathbb{R} \times \{(0, 0)\},
$$

$$
W_1 = U_{\gamma_1} \cap V_{\gamma_0} = U_{\gamma_1} = \mathbb{R} \times \text{span}(1, 0),
$$

$$
W_2 = U_{\gamma_2} \cap V_{\gamma_0} = V_{\gamma_1} = \mathbb{R} \times \text{span}(0, 1),
$$

and

$$
W_3 = V_{\gamma_2} = \mathbb{R} \times \{(0, 0)\}.
$$

Finally,

$$
\mathbb{R} \times \mathbb{R}^n = W_0 \oplus W_1 \oplus W_2 \oplus W_3 = W_1 \oplus W_2.
$$

Note that $W_1$ is the integral manifold corresponding to forward nonuniform contraction with respect to the growth rate $\mu$ and $W_2$ is the integral manifold corresponding to backward nonuniform contraction with respect to the growth rate $\mu$. Furthermore, by (18)–(19), there is no (uniform) $\mu$-dichotomy for any $\gamma$ and we have $\Sigma^\mu_\nu(A) = \mathbb{R}$.

Notice also that, if

$$
\lim_{t \to +\infty} \frac{\mu(t)}{e^{at}} = 0,
$$

for each $a > 0$, then equations (18)–(19) imply that $\Sigma^\nu_\mu(A) = \mathbb{R}$. In particular, the nonuniform polynomial dichotomy spectrum for equation (17), with $\mu$ given by (1), is different from the nonuniform dichotomy spectrum for equation (17) with $\mu$ given by (1) (that in this case is the whole $\mathbb{R}$).

To better understand the information one can obtain from the nonuniform $\mu$-dichotomy spectrum, we consider a notion of Lyapunov exponent for nonuniform $\mu$-dichotomies. This generalized type of Lyapunov exponents were introduced by Barreira and Valls in [13] (in the particular case of polynomial growth). In that paper, the authors show that the notion of polynomial Lyapunov exponent is in fact a Lyapunov exponent in the sense of the abstract theory [12]. See also [11].

With the usual convention that $\log 0 = -\infty$, define the $\mu$-Lyapunov exponent associated with system (1) as the function $\lambda^+: \mathbb{R}^n \to [-\infty, +\infty]$ given by

$$
\lambda^+(v) = \limsup_{t \to +\infty} \frac{\log \| \Phi(t, s)v \|}{\log \mu(t)}.
$$
Also consider the function $\lambda^- : \mathbb{R}^n \to [-\infty, +\infty]$ given by
\[
\lambda^-(v) = \liminf_{t \to +\infty} \frac{\log \| \Phi(t, s)v \|}{\log \mu(t)}.
\]

The next result relates the numbers above with the nonuniform $\mu$-spectrum of system $\mathcal{P}$.

**Theorem 10.** Let $\mu : \mathbb{R} \to \mathbb{R}^+$ be a differentiable growth rate, $m > 1$, and let $\Sigma_{\mu}^{ND}(A)$ be the nonuniform $\mu$-dichotomy spectrum of system $\mathcal{P}$:
\[
\Sigma_{\mu}^{ND}(A) = I_1 \cup [a_2, b_2] \cup \cdots \cup [a_{m-1}, b_{m-1}] \cup I_m,
\]
where $I_1 = [a_1, b_1]$ or $I_1 = (-\infty, b_1]$ and $I_m = [a_m, b_m]$ or $I_m = [a_m, +\infty)$.

Given a bounded connected component $[a_i, b_i]$ of $\Sigma_{\mu}^{ND}(A)$, the following holds: if $(s, v) \in \mathcal{W}_i = \mathcal{U}_i \cap \mathcal{V}_{i+1} \setminus \{0\}$, we have
\[
a_i \leq \lambda^-(v) \leq \lambda^+(v) \leq b_i,
\]
for $i = 1, \ldots, m$.

**Proof.** Let $\gamma_i \in (b_i, a_{i+1})$, $i = 1, \ldots, m$. Since $\gamma_i \notin \Sigma_{\mu}^{ND}(A)$, $x' = \left(A(t) - \gamma_i \frac{\mu'(t)}{\mu(t)} t\right)x$ admits a nonuniform $\mu$-dichotomy. Thus, there is a family of projections $P(t) \in M_n(\mathbb{R})$, $t \in \mathbb{R}$, such that, for all $t, s \in \mathbb{R}$,
\[
P(t)\Phi(t, s) = \Phi(t, s)P(s),
\]
and there are constants $K \geq 1$, $\alpha < 0$, $\beta > 0$ and $\theta, \nu \geq 0$, with $\alpha + \theta < 0$ and $\beta - \nu > 0$, such that
\[
\|\Phi_{\gamma_i}(t, s)P(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha} \mu(s)^{\theta} \text{ for } t \geq s,
\]
\[
\|\Phi_{\gamma_i}(t, s)Q(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\beta} \mu(s)^{\nu} \text{ for } t \leq s,
\]
and thus
\[
\|\Phi(t, s)P(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha + \gamma_i} \mu(s)^{\theta} \text{ for } t \geq s,
\]
\[
\|\Phi(t, s)Q(s)\| \leq K \left(\frac{\mu(t)}{\mu(s)}\right)^{\beta + \gamma_i} \mu(s)^{\nu} \text{ for } t \leq s,
\]
By Lemma 5 we have $\text{Im}P = \mathcal{U}_i$ and $\text{Ker}P = \mathcal{V}_{i-1}$. Therefore, for each $v \in \mathcal{U}_i$, we have
\[
\lambda^+(v) = \limsup_{t \to +\infty} \frac{\log \| \Phi(t, s)v \|}{\log \mu(t)} = \limsup_{t \to +\infty} \frac{\log \| \Phi(t, s)P(s)v \|}{\log \mu(t)}
\]
\[
\leq \limsup_{t \to +\infty} \frac{\log K}{\log \mu(t)} \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha + \gamma_i} \mu(s)^{\theta} = \alpha + \gamma_i.
\]
Letting $\gamma_i \to b_i$, we conclude that $\lambda^+(v) \leq \alpha + b_i < b_i$. Additionally, for each $w \in \mathcal{V}_{i-1}$ and $s \geq t$, we have
\[
\|\Phi(s, t)w\| \geq \frac{1}{K} \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma_i - \beta} \mu(s)^{-\theta} \|w\|.
\]
and thus
\[
\lambda^-(v) = \liminf_{s \to +\infty} \frac{\log \|\Phi(s, t)w\|}{\log \mu(s)} \geq \liminf_{s \to +\infty} \frac{\log \left( \frac{1}{\mu(s)} \left( \frac{\mu(t)}{\mu(s)} \right)^{-\gamma_i-1-\beta} \mu(s)^{-\text{sgn}(s)v} \right)}{\log \mu(s)} = \beta + \gamma_i - \nu.
\]

Letting \( \gamma_i - 1 \to a_i \) and since \( \beta - \nu > 0 \), we have \( \lambda^-(v) \geq \beta + a_i - \nu > a_i \). The result follows.

**Example 11.** In example 9, consider the particular case of the polynomial grow rates: take \( \mu(t) \) to be the growth rate \( p(t) \) in (4), \( \alpha = -2, \beta = 2 \) and \( \theta = \nu = 1 \). We get the system:
\[
\begin{bmatrix}
  u' \\
  v'
\end{bmatrix} = A(t) \begin{bmatrix}
  u \\
  v
\end{bmatrix} \quad \text{and} \quad A(t) = \begin{bmatrix}
  a_1(t) & 0 \\
  0 & a_2(t)
\end{bmatrix}.
\]

and, for \( i = 1, 2, \)
\[
a_i(t) = \frac{(-1)^i 4 + \text{sgn}(t)(\cos t + 1)}{2(1 + |t|)} - \text{sgn}(t) \log(1 + |t|) \sin t.
\]

It follows from Example 9 that, for the system above, the nonuniform polynomial spectrum and the nonuniform exponential spectrum are, respectively,
\[
\Sigma^\text{ND}_p(A) = [-3, -1] \cup [1, 3] \quad \text{and} \quad \Sigma^\text{ND}_e(A) = \mathbb{R}.
\]

With respect to the spectrum \( \Sigma^\text{ND}_p \), note that, by Theorem 10, the interval \([-3, -1]\) indicates the existence of a linear integral manifold where we have nonuniform polynomial contraction (since the polynomial Lyapunov exponents satisfy \(-3 \leq \lambda^-(v) \leq \lambda^+(v) \leq -1 \) for \((s, v) \in W_1\)) and the interval \([1, 3]\) indicates the existence of a linear integral manifold where we have nonuniform polynomial expansion (since the polynomial Lyapunov exponents satisfy \(1 \leq \lambda^-(v) \leq \lambda^+(v) \leq 3 \) for \((s, v) \in W_2\)). Recalling that the present system is a particular case of the family of systems in Example 4 we confirm once again that the linear integral manifold where we have nonuniform polynomial contraction is the linear manifold \( W_1 = \mathbb{R} \times \text{span}\{(1, 0)\} \) and the linear integral manifold where we have nonuniform polynomial expansion is the linear manifold \( W_2 = \mathbb{R} \times \text{span}\{(0, 1)\} \).

Once again we stress that the above information about polynomial contraction and polynomial expansion along the linear manifolds can not be obtained from the nonuniform (exponential) dichotomy spectrum for equation (17), that in this case is the whole \( \mathbb{R} \), as already explained in Example 4.

Note that the computations we undertake in this example can immediately be done considering other growth rates in Example 4 illustrating the potential use of the nonuniform \( \mu \)-dichotomy spectrum to search for nonuniform contraction and expansion with respect to growth rates \( \mu \) that are different from exponential ones.

### 4. Kinematic similarity

In [11] the dichotomy spectrum is used to establish the existence of a normal forms for nonautonomous linear systems. A version of this result is obtained in [51, 28] using the nonuniform dichotomy spectrum. The objective
of this section is to obtain a version of these results for the nonuniform $\mu$-spectrum, obtaining the results mentioned above as a very particular case.

Given $\varepsilon \geq 0$ and a growth rate $\mu$, we say that systems $x' = A(t)x$ and $y' = B(t)y$ are nonuniformly $(\mu, \varepsilon)$-kinematically similar if there exists a differentiable matrix function $S : \mathbb{R} \to \text{GL}_n(\mathbb{R})$ and a constant $M_\varepsilon > 0$ such that, for all $t \in \mathbb{R}$, we have

$$\|S(t)\| \leq M_\varepsilon \mu(t) \text{sgn}(t)^\varepsilon \quad \text{and} \quad \|S(t)^{-1}\| \leq M_\varepsilon \mu(t)^{-1} \text{sgn}(t)^\varepsilon,$$

and the change of variables $x(t) = S(t)y(t)$ transforms $x' = A(t)x$ into $y' = B(t)y$. If $\varepsilon = 0$ the systems $x' = A(t)x$ and $y' = B(t)y$ are said nonuniformly kinematically similar, a notion considered in [43].

Each $S : \mathbb{R} \to \text{GL}_n(\mathbb{R})$ satisfying (20) for some $\varepsilon \geq 0$ is called a nonuniform Lyapunov matrix function with respect to $\mu$ and the change of variables $x(t) = S(t)y(t)$ is said a nonuniform Lyapunov transformation with respect to $\mu$.

The previous definition is a generalization of the definitions of nonuniformly kinematically similar, considered in [43, 51] and corresponding to the special case $\mu(t) = e^t$.

In the next result we obtain a characterization of the normal forms of nonautonomous linear systems using the nonuniform $\mu$-dichotomy spectrum. This result contains Theorem 1.3 in [51] as the very particular case $\mu(t) = e^t$, corresponding to the nonuniform dichotomy spectrum.

**Theorem 12.** Assume that $A(t)$ is differentiable and that the evolution operator of system (1) has nonuniformly bounded growth with respect to $(\mu, \varepsilon)$. Assume also that system (1) has a nonuniform $\mu$-dichotomy with parameters $\alpha, \beta, \theta$ and $\nu$ such that

$$3 \max\{\theta, \nu\} - \min\{-\alpha - \theta, \beta - \nu\} \leq 0.$$

Let the nonuniform $\mu$-dichotomy spectrum be

$$\Sigma^\mu_{\text{ND}}(A) = [a_1, b_1] \cup \cdots \cup [a_m, b_m],$$

with $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ for any $i, j \in \{1, \ldots, m\}$ such that $i \neq j$. Then system (1) is $(\mu, \varepsilon)$-nonuniformly kinematically similar to system $y' = B(t)y$, where

$$B(t) = \text{diag}(B_0(t), B_1(t), \cdots, B_m(t), B_{m+1}(t)),$$

the matrices $B_i(t) : \mathbb{R} \to \mathbb{R}^{n_i \times n_i}$, with $n_i = \text{rank} W_i$, are differentiable and

$$\Sigma^\mu_{\text{ND}}(B_i) = \begin{cases} \emptyset & \text{for } i \in \{0, m+1\} \\ [a_i, b_i] & \text{for } i \in \{1, \ldots, m\}. \end{cases}$$

**Proof.** To prove the result we begin by establishing some lemmas.

The proof of the next lemma is obtained using arguments borrowed from the proof of Lemma 3.1 in [51], that in turn is based on the arguments in the proof of Lemma 2.1 in page 158 of Daleckii and Krein [33]. We present a full proof for the sake of completeness.

**Lemma 13.** Let $S(t)$ be a nonuniform Lyapunov matrix function with respect to $\mu$. Then, the following statements are equivalent.

a) For some $\varepsilon \geq 0$, systems $x' = A(t)x$ and $y' = B(t)y$ are nonuniformly $(\mu, \varepsilon)$-kinematically similar via the transformation $x = S(t)y$. 
b) We have $\Phi_A(t, s)S(s) = S(t)\Phi_B(t, s)$ for all $t, s \in \mathbb{R}$, where $\Phi_A$ and $\Phi_B$ are the evolution operators of systems $x' = A(t)x$ and $y' = B(t)y$, respectively.

c) $S(t)$ is a solution of $S' = A(t)S - SB(t)$.

Proof. Assume that systems $y' = A(t)y$ and $y' = B(t)y$ are nonuniformly $(\mu, \varepsilon)$-kinematically similar via $x = S(t)y$, where $S : \mathbb{R} \to \text{GL}_n(\mathbb{R})$ is a differentiable matrix function such that (20) holds. Let $s_0 \geq 0$, $y_0 \in \mathbb{R}^n$ and $y(t)$ be a solution of $y' = B(t)y$ with $y(s) = y_0$. Let $x(t)$ denote the solution of $x' = A(t)x$ with $x(s) = S(s)y_0$. Denoting by $\Phi_A$ and $\Phi_B$ the evolution operators of systems $x' = A(t)x$ and $y' = B(t)y$, respectively, we have

$$
\Phi_A(t, s)S(s)y_0 = \Phi_A(t, s)x(s) = x(t) = S(t)y(t) = S(t)\Phi_B(t, s)y_0.
$$

Since $y_0$ is arbitrary, we conclude that $\Phi_A(t, s)S(s) = S(t)\Phi_B(t, s)$, for all $s, t \in \mathbb{R}$. We conclude that (2) implies (b).

Assume now that $\Phi_A(t, s)S(s) = S(t)\Phi_B(t, s)$ for all $t, s \in \mathbb{R}$. We have in particular $\Phi_A(t, 0)S(0)\Phi_B(t, 0)^{-1} = S(t)$. Since $\Phi_A(t, 0)S(0)$ and $\Phi_B(t, 0)$ are fundamental matrices, respectively, of systems $x' = A(t)x$ and $y' = B(t)y$, we conclude that $S(t)$ is differentiable and thus

$$
S'(t) = \frac{d}{dt} \Phi_A(t, 0)S(0)\Phi_B(t, 0)^{-1}
= A(t)\Phi_A(t, 0)S(0)\Phi_B(t, 0)^{-1} - \Phi_A(t, 0)S(0)\Phi_B(t, 0)^{-1}B(t)\Phi_B(t, 0)\Phi_B(t, 0)^{-1}
= A(t)\Phi_A(t, 0)\Phi_A(t, 0)^{-1}S(t) - S(t)\Phi_B(t, 0)\Phi_B(t, 0)^{-1}B(t)
= A(t)S(t) - S(t)B(t)
$$

We conclude that (b) implies (c).

Assume now that $S'(t) = A(t)S(t) - S(t)B(t)$ and let $S(t_0) = C$, where $C$ is an invertible matrix. Defining $Z(t) = \Phi_A(t, t_0)C\Phi_B(t, t_0)^{-1}$ we get

$$
\frac{d}{dt} Z(t) = \frac{d}{dt} (\Phi_A(t, t_0)C\Phi_B(t, t_0)^{-1})
= A(t)\Phi_A(t, t_0)C\Phi_B(t, t_0)^{-1} - \Phi_A(t, t_0)C\Phi_B(t, t_0)^{-1}B(t)
= A(t)Z(t) - Z(t)B(t)
$$

and $Z(t_0) = C$. We conclude that $S(t) = Z(t) = \Phi_A(t, t_0)C\Phi_B(t, t_0)^{-1}$. Let now $x(t) = S(t)y(t)$, where $y$ is the solution of $y' = B(t)y$ with $y(t_0) = y_0$. We have

$$
x' = (S(t)y)' = S'(t)y + S(t)y' = A(t)S(t)y - S(t)B(t)y + S(t)B(t)y = A(t)x
$$

and $x(t_0) = S(t_0)y(t_0) = S(t_0)y_0$. Since $S(t)$ is a nonuniform Lyapunov matrix function with respect to $\mu$, we conclude that $y' = A(t)y$ and $y' = B(t)y$ are nonuniformly $(\mu, \varepsilon)$-kinematically similar via a transformation $x = S(t)y$ and thus (c) implies (a). □

Lemma 14. Let $y' = A(t)y$ and $y' = B(t)y$ be nonuniformly $(\mu, \varepsilon)$-kinematically similar and

$$
\varepsilon \leq \frac{1}{3} \min\{-\alpha - \theta, \beta - \nu\}.
$$

Then $\Sigma_{\mu, \varepsilon}^n(A) = \Sigma_{\mu, \varepsilon}^n(B)$. 

Proof. Assume that \( x' = A(t)x \) and \( y' = B(t)y \) are nonuniformly \((\mu, \varepsilon)\)-kinematically similar with \( \varepsilon \) satisfying (22). Let \( S : \mathbb{R} \to \text{GL}_n(\mathbb{R}) \) be a differentiable matrix function such that \( y(t) = S(t)x(t) \) transforms \( x' = A(t)x \) into \( y' = B(t)y \) and satisfies \( \|S(t)\| \leq M_\mu \mu(t)^{\varepsilon} \) and \( \|S(t)^{-1}\| \leq M_\mu \mu(t)^{\varepsilon} \), for all \( t \in \mathbb{R} \) with \( \varepsilon \) satisfying (22). Define \( Y(t) = S(t)X(t) \), where \( X(t) \) is a fundamental matrix of \( x' = A(t)x \). Assume that \( x' = A(t)x \) satisfies (20–23). The proof of Lemma 13 showed immediately that \( Y(t) \) is a fundamental matrix of \( y' = B(t)y \). Moreover, for \( t \geq 0 \) and \( t \geq s \), we get

\[
\|Y(t)\bar{P}Y(s)^{-1}\| \leq \|S(t)\|\|X(t)\bar{P}X(s)^{-1}\|\|S(s)^{-1}\|
\]

\[
\leq M_\mu \mu(t)^{\varepsilon}K \left( \mu(t) \mu(s) \right)^\alpha \mu(s)^{\varepsilon} M_\mu \mu(s)^{\varepsilon}
\]

\[
\leq K(M_\varepsilon)^2 \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \mu(s)^{\varepsilon} M_\mu \mu(s)^{\varepsilon}
\]

\[
= K(M_\varepsilon)^2 \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \mu(s)^{\varepsilon} \mu(s)^{\varepsilon}
\]

where \( \alpha = \alpha + \varepsilon \) and \( \beta = \theta + 2\varepsilon \) (since \( \mu \) is increasing and \( \mu(0) = 1 \), we have \( \mu(s) \leq \mu(s)^{\varepsilon} \) for all \( s \in \mathbb{R} \). Assuming now that \( t < 0 \) and \( t \geq s \), we have

\[
\|Y(t)\bar{P}Y(s)^{-1}\| \leq K(M_\varepsilon)^2 \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \mu(s)^{\varepsilon} M_\mu \mu(s)^{\varepsilon}
\]

According to (22), \( \alpha + \beta < 0 \).

Similarly, for \( t < 0 \) and \( s \geq t \) we have

\[
\|Y(t)\bar{Q}Y(s)^{-1}\| \leq \|S(t)\|\|X(t)\bar{Q}X(s)^{-1}\|\|S(s)^{-1}\|
\]

\[
\leq M_\mu \mu(t)^{\varepsilon} K \left( \mu(t) \mu(s) \right)^\beta \mu(s)^{\varepsilon} M_\mu \mu(s)^{\varepsilon}
\]

\[
\leq K(M_\varepsilon)^2 \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(s)^{\varepsilon} M_\mu \mu(s)^{\varepsilon}
\]

\[
= K(M_\varepsilon)^2 \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(s)^{\varepsilon} \mu(s)^{\varepsilon}
\]

where \( \beta = \beta - \varepsilon \) and \( \nu = \nu + 2\varepsilon \). Also, for \( t \geq 0 \) and \( s \geq t \) we obtain

\[
\|Y(t)\bar{Q}Y(s)^{-1}\| \leq K(M_\varepsilon)^2 \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(s)^{\varepsilon} \mu(s)^{\varepsilon}
\]

\[
\leq K(M_\varepsilon)^2 \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(s)^{\varepsilon} \mu(s)^{\varepsilon}
\]

By (22) we have \( \beta - \nu > 0 \). We conclude that the existence of a N\( \mu \)D for \( x' = A(t)x \) implies the existence of a N\( \mu \)D for \( x' = B(t)x \). Reciprocally, since \( H : \mathbb{R} \to \text{GL}_n(\mathbb{R}) \) given by \( H(t) = S^{-1}(t) \) is a differentiable matrix function such that \( x(t) = H(t)y(t) \) transforms \( y' = B(t)y \) into \( x' = A(t)x \) and satisfies
\[ \|H(t)\| \leq M_\mu(t)^{\text{sgn}(t)} \text{ and } \|H(t)^{-1}\| \leq M_\mu(t)^{\text{sgn}(t)^2}, \]

we conclude that there is a N\muD for \( x' = A(t)x \) if and only if there is a N\muD for \( x' = B(t)x \).

Thus \( \Sigma^{\text{ND}}_\mu(A) = \Sigma^{\text{ND}}_\mu(B) \).

\[ \square \]

The proof of next lemma is contained in the proof of Lemma A.5 of \[51\] and was already used in the present form in \[51\].

**Lemma 15** (Lemma 3.3 of [51]). Let \( P_0 \in M_n(\mathbb{R}) \) be a symmetric projection and \( X : \mathbb{R} \to \text{GL}_n(\mathbb{R}) \). Then:

a) For every \( t \in \mathbb{R} \), the matrices \( Q(t) \in M_n(\mathbb{R}) \) given by
\[ (23) \quad Q(t) = P_0 X(t)^T X(t)P_0 + (I - P_0) X(t)^T X(t)(I - P_0) \]

are positively definite and symmetric. Moreover, there exists a unique positively definite and symmetric matrix \( R(t) \) such that \( R(t)^2 = Q(t) \) and \( P_0 R(t) = R(t) P_0 \) for every \( t \in \mathbb{R} \);

b) The matrix \( S(t) = X(t)R(t)^{-1} \) is invertible, \( S(t) P_0 S(t)^{-1} = X(t) P_0 X(t)^{-1} \) and we have, for every \( t \in \mathbb{R} \),
\[ \|S(t)\| \leq \sqrt{2}, \quad \|S(t)^{-1}\| \leq \sqrt{\|X(t) P_0 X(t)^{-1}\|^2 + \|X(t)(I - P_0) X(t)^{-1}\|^2}. \]

**Lemma 16.** Assume that system (11) has a N\muD with an invariant projection \( P : \mathbb{R} \to M_n(\mathbb{R}) \) satisfying \( P(t) \neq 0, I \). Then there exists a differentiable nonuniform Lyapunov matrix function with respect to \( \mu \), \( S : \mathbb{R} \to \text{GL}_n(\mathbb{R}) \), such that, for all \( t \in \mathbb{R} \), we have \( \|S(t)\|, \|S(t)^{-1}\| \leq K \mu(t)^{\text{sgn}(t)\max\{\theta, \nu\}} \), where \( K, \theta \) and \( \nu \) are the constants in (22)–(23), and
\[ S(t)^{-1} P(t) S(t) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \]

**Proof.** All the projections \( P(t) \) have the same rank since \( P(t) \Phi(t, s) = \Phi(t, s) P(s) \), for \( t, s \in \mathbb{R} \), and this property implies that, for all \( t, s \in \mathbb{R} \), the matrices \( P(t) \) and \( P(s) \) are similar. By Remark 1 for any given \( s \in \mathbb{R} \) there exists a \( T(s) \in \text{GL}_n(\mathbb{R}) \) such that, for all \( s \in \mathbb{R} \),
\[ (24) \quad T(s) P(s) T(s)^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} =: P_0, \]

where \( I \) is the identity of dimension \( \dim \text{Im} P \). Let \( X(t) = \Phi(t, s) T(s)^{-1} \). By Lemma 15 for all \( t \in \mathbb{R} \), there is a uniquely positive definite and symmetric matrix \( R(t) \) satisfying \( P_0 R(t) = R(t) P_0 \) and \( R(t)^2 = Q(t) \) with \( Q(t) \) given by (23). Letting \( S(t) = \Phi(t, s) T(s)^{-1} R(t)^{-1} \), using the invariance of \( P(t) \) and (24) we have
\[ S(t)^{-1} P(t) S(t) = R(t) T(s) \Phi(t, s)^{-1} P(t) \Phi(t, s) T(s)^{-1} R(t)^{-1} = R(t) T(s) P(s) T(s)^{-1} R(t)^{-1} = R(t) P_0 R(t)^{-1} = P_0. \]

Finally, by Lemma 15 we have \( \|S(t)\| \leq \sqrt{2} \) and, using the fact that system (11) has a N\muD with an invariant projection \( P : \mathbb{R} \to M_n(\mathbb{R}) \) satisfying \( P(t) \neq 0, I \) and estimates (5–6), we conclude that
\[ \|S(t)^{-1}\| \leq \sqrt{\|X(t) P_0 X(t)^{-1}\|^2 + \|X(t)(I - P_0) X(t)^{-1}\|^2} \leq \sqrt{K^2 \mu(t)^{2 \text{sgn}(t)\theta} + K^2 \mu(t)^{2 \text{sgn}(t)\nu}} \leq \sqrt{2 K \mu(t)^{\text{sgn}(t)\max\{\theta, \nu\}}}. \]
Finally, note that the differentiability of $S(t)$ follows from the following facts: $R(t)$ is the positive square root of a differentiable, positive definite and symmetric matrix valued function; positive square roots of differentiable, positive definite and symmetric matrix valued functions are differentiable and the inverse of an invertible differentiable matrix valued function is still differentiable (see Coppel [31], Lemma 1, page 39).

Thus $S(t)$ is a nonuniform Lyapunov matrix function with respect to $\mu$ and we establish the lemma. \hfill $\Box$

Next we will use the auxiliary lemmas to establish our result. According to Theorem 3 we have $\mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_m \oplus \mathcal{W}_{m+1} = \mathbb{R} \times \mathbb{R}^n$ with rank $\mathcal{W}_i \geq 1$ for $i = 1, \ldots, m$.

Choose $\gamma_0 \in (-\infty, a_1)$, $\gamma_m \in (b_m, +\infty)$ and $\gamma_i \in (b_i, a_{i+1})$ for $i = 1, \ldots, m - 1$. According to Theorems 3 and 5 we have $\mathcal{W}_0 = \mathcal{U}_{\gamma_0}$, $\mathcal{W}_{m+1} = \mathcal{V}_{\gamma_m}$ and $\mathcal{W}_i = \mathcal{U}_{\gamma_i} \cap \mathcal{V}_{\gamma_i-1}$ for $i = 1, \ldots, m$.

For $\gamma_0 \in (-\infty, a_1) \subset \rho_\mu^{\mathbb{D}}(A)$, the system

$$\tag{25} x' = \left(A(t) - \frac{\mu(t)}{\mu(s)} I\right) x,$$

with $\gamma = \gamma_0$, admits a $\mathbb{N}_\mu \mathbb{D}$ with some invariant projection $\widetilde{P}_0$: if $\Phi_{\gamma_0}(t, s)$ denotes the evolution operator of (25) with $\gamma = \gamma_0$, then there are $K_0 \geq 1$, $\alpha_0 < 0 < \beta_0$ and $\theta_0, \nu_0 \geq 0$ satisfying $\alpha_0 + \theta_0 < 0$ and $\beta_0 - \nu_0 > 0$ such that (2) - (3) hold with $\Phi = \Phi_{\gamma_0}$. The argument used to obtain (11) shows that the evolution operator of (25), $\Phi_\gamma(t, s)$, is given by $\Phi_\gamma(t, s) = \left(\frac{\mu(t)}{\mu(s)}\right)^{\gamma - \gamma_0} \Phi_{\gamma_0}(t, s)$. Thus, for $t \geq s$ we have

$$\tag{26} \|\Phi_\gamma(t, s) \tilde{P}_0(s)\| = \left(\frac{\mu(t)}{\mu(s)}\right)^{\gamma - \gamma_0} \|\Phi_{\gamma_0}(t, s) \tilde{P}_0(s)\| \leq K_0 \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha_0 + \gamma - \gamma_0} \mu(s)^{\text{sgn}(s)\nu_0}\theta_0$$

and, similarly, for $s \geq t \geq 0$ we have

$$\tag{27} \|\Phi_\gamma(t, s) \tilde{Q}_0(s)\| = \left(\frac{\mu(t)}{\mu(s)}\right)^{\gamma - \gamma_0} \|\Phi_{\gamma_0}(t, s) \tilde{Q}_0(s)\| \leq K_0 \left(\frac{\mu(t)}{\mu(s)}\right)^{\beta_0 + \gamma - \gamma_0} \mu(s)^{\text{sgn}(s)\nu_0},$$

where $\tilde{Q}_0(s) = I - \tilde{P}_0(s)$. Additionally,

$$\tilde{P}_0(t) \Phi_\gamma(t, s) = \left(\frac{\mu(t)}{\mu(s)}\right)^{\gamma - \gamma_0} \tilde{P}_0(t) \Phi_{\gamma_0}(t, s) = \left(\frac{\mu(t)}{\mu(s)}\right)^{\gamma - \gamma_0} \Phi_{\gamma_0}(t, s) \tilde{P}_0(s) = \Phi_\gamma(t, s) \tilde{P}_0(s)$$

and $\tilde{P}_0$ is an invariant projection for $\Phi_\gamma$. In particular, $\tilde{P}_0$ is an invariant projection for $\Phi$. By Lemma 16 there exists a differentiable nonuniform Lyapunov matrix function with respect to $\mu$, $S_0 : \mathbb{R} \to \text{GL}_n(\mathbb{R})$, such that
\[ \|S(t)\|, \|S(t)^{-1}\| \leq M\mu(t)^{\text{sgn}(t) \max\{|\theta|,|\nu|\}} \] and
\[
S_0(t)^{-1}\tilde{P}_0(t)S_0(t) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} := P_0,
\]
where \( I \) denotes the identity of order \( \dim \text{Im}\tilde{P}_0(t) \). For \( t \in \mathbb{R} \), define
\[
B(t) := S_0(t)^{-1}(A(t)S_0(t) - S_0'(t)) \quad \iff \quad S_0'(t) = A(t)S_0(t) - S_0(t)B(t).
\]
By \( \mathcal{P} \) in Lemma 13 and the identity above we conclude that system \( \Pi \) is nonuniformly \((\mu,\varepsilon_0)\)-kinematically similar to system \( y' = B(t)y \) via the transformation \( x(t) = S_0(t)y(t) \), where \( \varepsilon_0 = \max\{\theta,\nu\} \). Moreover, according to \( \mathcal{P} \) in Lemma 13, \( \Phi_B(t,s) = S_0(t)^{-1}\Phi(t,s)S_0(s) \) is the evolution operator of \( y' = B(t)y \).

Define \( R(t) = S_0(t)^{-1}\Phi(t,s)T^{-1}(s) \), where \( T(s) \in GL_n(\mathbb{R}) \) was defined in the proof of Lemma 16 (see (24)). Since \( R_0R(t) = R(t)P_0 \), we have
\[
R(t)^{-1}P_0 = P_0R(t)^{-1} \quad \text{and} \quad R'(t)P_0 = P_0R'(t).
\]
Using the fact that \( S_0(t)^{-1}\Phi(t,s)S_0(s) \) is a fundamental matrix solution of \( y' = B(t)y \), we have \( R(t)T(s)S_0(s) = \Phi_B(t,s) \) and thus
\[
R'(t)T(s)S_0(s) = B(t)\Phi_B(t,s) = B(t)S_0(t)^{-1}\Phi(t,s)S_0(s).
\]
Therefore
\[
R'(t)R(t)^{-1} = B(t)S_0(t)^{-1}\Phi(t,s)\Phi(t,s)^{-1}S_0(t) = B(t)
\]
and also, since \( P_0 \) commutes with \( R'(t) \) and \( R(t)^{-1} \),
\[
(28) \quad P_0R(t) = R_0R(t)^{-1} = R'(t)R(t)^{-1}P_0 = B(t)P_0.
\]
For \( t \in \mathbb{R} \) write
\[
B(t) = \begin{bmatrix} B_0(t) & C_0(t) \\ D_0(t) & E_0(t) \end{bmatrix},
\]
where \( B_0 : \mathbb{R} \to M_{n_0}(\mathbb{R}) \), \( D_0 : \mathbb{R} \to M_{n-n_0}(\mathbb{R}) \), \( C_0 : \mathbb{R} \to \mathbb{R}^{n_0 \times (n-n_0)} \), \( E_0 : \mathbb{R} \to \mathbb{R}^{(n-n_0) \times n_0} \) and \( n_0 = \dim \text{Im}\tilde{P}_0 \) (and thus \( \dim \text{Ker}\tilde{P}_0 = n - n_0 \)).

The identity \( 28 \) implies that \( C_0(t) = D_0(t) = 0 \) for all \( t \in \mathbb{R} \). We have established that system \( \Pi \) is nonuniformly \((\mu,\varepsilon_0)\)-kinematically similar to \( y' = D_0y \), where
\[
(29) \quad D_0 = \begin{bmatrix} B_0(t) & 0 \\ 0 & E_0(t) \end{bmatrix},
\]
\( B_0 : \mathbb{R} \to M_{n_0}(\mathbb{R}) \) and \( E_0 : \mathbb{R} \to M_{n-n_0}(\mathbb{R}) \) are differentiable, where \( n_0 = \dim \text{Im}\tilde{P}_0 \) and \( n_1 = \dim \text{Ker}\tilde{P}_0 \). Taking into account \( 21 \), we have \( \varepsilon_0 \leq \frac{1}{\gamma} \max\{-\alpha - \gamma, -\beta - \gamma\} \) and it follows from Lemma 14 that systems \( \Pi \) and \( \Sigma^\mu_\gamma \) have the same nonuniform \( \mu \)-dichotomy spectrum. Moreover, \( P_0 \) is an invariant projection for system \( 29 \). So we get from \( 20 \)(27) that \( \Sigma^\mu_\gamma(B_0) \subset (-\infty,a_1) \) and \( \Sigma^\mu_\gamma(E_0) = \Sigma^\mu_\gamma(A) \). This implies that \( \Sigma^\mu_\gamma(B_0) = \emptyset \). Note that, for each \( \lambda > a_1 \), equation \( 26 \) implies that
\[
z' = \left( B_0(t) - \lambda\frac{\mu(t)}{\mu(s)}I_{n_0} \right) z
\]
admits a \( N_\mu D \) with projection \( P = I_{n_0} \).

For \( \gamma_1 \in (b_1,a_2) \subset \rho^\mu_\gamma(E_0) = \rho^\mu_\gamma(A) \), system \( 25 \) with \( \gamma = \gamma_1 \) admits a \( N_\mu D \) with some invariant projection \( \tilde{P}_0 \); if \( \Phi_{\gamma_1}(t,s) \) denotes the evolution
operator of \([23]\) with \(\gamma = \gamma_1\), then there are \(K_1 \geq 1\), \(\alpha_1 < 0 < \beta_1\) and \(\theta_1, \nu_1 \geq 0\) satisfying \(\alpha_1 + \theta_1 < 0\) and \(\beta_1 - \nu_1 > 0\) such that \([2]-[3]\) hold with \(\Phi = \Phi_{\gamma_1}\).

Again, for \(t \geq s\) we have

\[
\|\Phi_{\gamma}(t, s)\tilde{P}_1(s)\| = \left(\frac{\mu(t)}{\mu(s)}\right)^{\gamma_1 - \gamma} \|\Phi_{\gamma_1}(t, s)\tilde{P}(s)\| \\
\leq K_0 \left(\frac{\mu(t)}{\mu(s)}\right)^{\alpha_1 + \gamma_1 - \gamma} \mu(s)^{\text{sgn}(s)\theta_1}
\]

and, similarly, for \(s \geq t\) we have

\[
\|\Phi_{\gamma}(t, s)\tilde{Q}_1(s)\| = \left(\frac{\mu(t)}{\mu(s)}\right)^{\gamma_1 - \gamma} \|\Phi_{\gamma_1}(t, s)\tilde{Q}(s)\| \\
\leq K_1 \left(\frac{\mu(t)}{\mu(s)}\right)^{\beta_1 + \gamma_1 - \gamma} \mu(s)^{\text{sgn}(s)\nu_1},
\]

where \(\tilde{Q}_1(t) = I - \tilde{P}_1(t)\) and \(\tilde{P}_1\) is an invariant projection for \(\Phi_{\gamma}\).

Reproducing the argument used to obtain \([24]\), we conclude that system \(z' = E_0 z\) is nonuniformly \((\mu, \varepsilon)\)-kinematically similar to system

\[
y' = \begin{bmatrix} B_1(t) & 0 \\ 0 & E_1(t) \end{bmatrix} y,
\]

with \(B_1 : \mathbb{R} \to M_{n_1}(\mathbb{R})\) and \(E_0 : \mathbb{R} \to M_{m_1-1}(\mathbb{R})\) differentiable and \(n_1 = \dim \tilde{P}_1\). Thus, by \([21]\), system \([1]\) is nonuniformly \((\mu, \varepsilon_0)\)-kinematically similar to \(y' = D_1 y\) where

\[
D_1 = \begin{bmatrix} B_0(t) & 0 & 0 \\ 0 & B_1(t) & 0 \\ 0 & 0 & E_1(t) \end{bmatrix}
\]

and \(B_0 : \mathbb{R} \to M_{n_0}(\mathbb{R}), B_1 : \mathbb{R} \to M_{n_1}(\mathbb{R})\) and \(E_1 : \mathbb{R} \to M_{m_1}(\mathbb{R})\), with \(p_1 = n - n_0 - n_1\), are differentiable. From \([30]\), we conclude that \(\Sigma_{\mu}^{\text{ND}}(B_1) \subset (-\infty, a_2)\) (note that when \(\gamma \geq a_2\) we have \(\gamma_1 - \gamma \leq 0\)) and from \([31]\), we have \(\Sigma_{\mu}^{\text{ND}}(E_1) \subset (b_1, +\infty)\) (note that when \(\gamma \leq b_1\) we have \(\gamma_1 - \gamma \geq 0\)). Thus \((-\infty, b_1) \subset \rho_{\mu}^{\text{ND}}(E_1)\) and we conclude that

\[
\Sigma_{\mu}^{\text{ND}}(B_1) = [a_1, b_1] \quad \text{and} \quad \Sigma_{\mu}^{\text{ND}}(E_1) = [a_2, b_2] \cup \ldots \cup [a_m, b_m].
\]

Iterating the process, we conclude that system \([1]\) is nonuniformly \((\mu, \varepsilon_0)\)-kinematically similar to \(y' = D_{m-1} y\) where

\[
D_{m-1} = \text{diag}(B_0(t), \ldots, B_{m-1}(t), E_{m-1}(t))
\]

and \(B_0 : \mathbb{R} \to M_{n_0}(\mathbb{R}), \ldots, B_{m-1} : \mathbb{R} \to M_{n_{m-1}}(\mathbb{R})\) and \(E_{m-1} : \mathbb{R} \to M_{p_{m-1}}(\mathbb{R})\), with \(p_{m-1} = n - \sum_{k=0}^{m-1} n_k\), are differentiable.

Proceeding like before and taking into account \([21]\), we conclude that \(\Sigma_{\mu}^{\text{ND}}(B_i) = [a_i, b_i], i = 1, \ldots, m - 1,\) and \(\Sigma_{\mu}^{\text{ND}}(E_{m-1}) = [a_m, b_m]\).

Finally, system \(z' = E_{m-1} z\) is nonuniformly \((\mu, \varepsilon_0)\)-kinematically similar to system

\[
y' = \begin{bmatrix} B_m(t) & 0 \\ 0 & B_{m+1}(t) \end{bmatrix} y,
\]
with $B_m : \mathbb{R} \to M_{n_m}(\mathbb{R})$ and $B_{m+1} : \mathbb{R} \to M_{n_{m+1}}(\mathbb{R})$, $n_{m+1} = n - \sum_{k=0}^{m} n_k$, differentiable. Thus, system (11) is nonuniformly $(\mu, \varepsilon)$-kinematically similar to $y' = D_m y$ where

$$D_m = \text{diag} \left( B_0(t), \ldots, B_{m+1}(t) \right)$$

and $B_i : \mathbb{R} \to M_{n_i}(\mathbb{R})$, $i = 0, \ldots, m + 1$, are differentiable.

For $\gamma_m \in (b_m, +\infty) \subset \rho^n_{\mu}(E_{m+1}) \subset \rho^n_{\mu}(A)$, system (25) with $\gamma = \gamma_m$, admits a $N\mu D$ with some invariant projection $\tilde{P}_m$: if $\Phi_{\gamma_m}(t, s)$ denotes the evolution operator of (25) with $\gamma = \gamma_m$, then there are $K_m \geq 1$, $\alpha_m < 0 < \beta_m$ and $\theta_m, \nu_m \geq 0$ satisfying $\alpha_m + \theta_m < 0$ and $\beta_m - \nu_m > 0$ such that (2)–(3) hold with $\Phi = \Phi_{\gamma_m}$. We conclude, using (21), that

$$\Sigma^\text{ND}_\mu(B_m) = [a_m, b_m \rangle \text{ and } \Sigma^\text{ND}_\mu(B_{m+1}) = \emptyset.$$

It remains to prove that $n_i = \text{rank } \mathcal{W}_i$. By Lemma 5 taking into account that $\mathcal{W}_0 = \mathcal{U}_{\gamma_0}$ for $\gamma_0 \in (-\infty, b_1)$, we have $\ker \text{Im} \tilde{P}_1 = \ker \mathcal{U}_{\gamma_0} = \ker \mathcal{W}_0$. Thus $n_0 = \text{dim } \mathcal{W}_0$. Since $\gamma_0 \in (-\infty, a_1)$, $\gamma_1 \in (b_1, a_2)$, $\mathcal{U}_{\gamma_0} \subset \mathcal{U}_{\gamma_1}$ and $\mathcal{W}_1 = \mathcal{U}_{\gamma_1} \cap \mathcal{V}_{\gamma_0}$, we get from Lemmas 5 and 7 that $n_0 + n_1 = \text{rank } \text{Im} \tilde{P}_1 = \text{rank } \mathcal{U}_{\gamma_1} = \text{rank } (\mathcal{U}_{\gamma_1} \cap (\mathcal{U}_{\gamma_0} \oplus \mathcal{V}_{\gamma_0})) = \text{rank } \mathcal{W}_0 + \text{dim } \mathcal{W}_1$ and we get $n_1 = \text{rank } \mathcal{W}_1$. Repeating the argument we immediately conclude that $n_i = \text{rank } \mathcal{W}_i$ for $i = 2, \ldots, m$. Using again Lemma 5 for $\gamma_m \in (b_m, \infty)$ we get $n_{m+1} = \text{rank } \text{Ker} \tilde{P}_m = \text{rank } \mathcal{V}_{\gamma_m} = \text{rank } \mathcal{W}_{m+1}$ and the theorem follows.

In the next remark we show that there is a conjugacy between the flow of an autonomous linear system associated to system $x' = A(t)x$ in Theorem 12 to the flow of an autonomous linear system associated to system $x' = B(t)x$ in that result.

**Remark 17.** Assume that the linear nonautonomous system $x' = A(t)x$ satisfy the conditions of Theorem 12 and that $x' = B(t)x$ is the nonuniformly $(\mu, \varepsilon)$-kinematically similar linear block diagonal system given by the referred theorem. Denote, respectively, the evolution operators of the systems by $\Phi_A$ and $\Phi_B$. Consider also the associated autonomous systems

$$\begin{cases} x' = A(t)x \\ t' = 1 \end{cases} \quad \text{and} \quad \begin{cases} x' = B(t)x \\ t' = 1 \end{cases}$$

and the corresponding flows, $\Psi^A$ and $\Psi^B$, given by

$$\Psi^A(\tau, (s, x)) = (s + \tau, \Phi_A(s + \tau, s)x) =: \Psi^A_\tau(s, x)$$

and

$$\Psi^B(\tau, (s, x)) = (s + \tau, \Phi_B(s + \tau, s)x) =: \Psi^B_\tau(s, x).$$

Define a linear map $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ by $H(s, x) = (s, S(s)x)$, where $S(t)$ is given in Lemma 12. It follows from (6) in Lemma 13 that we have

$$(H \circ \Psi^B_\tau)(s, x) = H(s + \tau, \Phi_B(s + \tau, s)x) = (s + \tau, S(s + \tau)\Phi_B(s + \tau, s)x)$$

$$(s + \tau, \Phi_A(s + \tau, s)S(s)x) = \Psi^A_\tau(s, S(s)x) = (\Psi^A_\tau \circ H)(s, x),$$
for all \( \tau, s, t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \). We conclude that
\[
H \circ \Psi^B = \Psi^A \circ H
\]
and the flows \( \Psi^A \) and \( \Psi^B \) are conjugated by \( H \).

5. SPECTRUM AND KINEMATIC SIMILARITY ON THE HALF-LINE

The purpose of this section is twofold: to remark that, with the natural adaptations, our results still hold on the half-line and to present a nontrivial example. We begin by enumerate briefly the adapted versions of the definitions that we will need.

Still assuming that \( t \mapsto A(t) \) is continuous, consider now that system (1) is only defined for \( t \in \mathbb{R}^+_0 \).

In the present context, a growth rate is a function \( \mu : \mathbb{R}^+_0 \to \mathbb{R}^+ \) that is strictly increasing and satisfies \( \mu(0) = 1 \) and \( \lim_{t \to +\infty} \mu(t) = +\infty \).

We say that system
\[
x' = A(t)x, \quad t \in \mathbb{R}^+_0
\]
(35) admits a nonuniform \( \mu \)-dichotomy on the half-line \((N\mu D+)\) if there is a family of projections \( P(t) \in M_n(\mathbb{R}) \), \( t \in \mathbb{R}^+_0 \), such that, for all \( t, s \in \mathbb{R}^+_0 \),
\[
P(t)\Phi(t, s) = \Phi(t, s)P(s),
\]
and there are constants \( K \geq 1, \alpha < 0, \beta > 0 \) and \( \theta, \nu \geq 0 \), with \( \alpha + \theta < 0 \) and \( \beta - \nu > 0 \), such that
\[
\|\Phi(t, s)P(s)\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^\alpha \mu(s)^\theta \text{ for } t \leq s \geq 0,
\]
(36)
\[
\|\Phi(t, s)Q(s)\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^\beta \mu(s)^\nu \text{ for } 0 \leq t \leq s,
\]
(37)
where \( Q(s) = I - P(s) \) is the complementary projection. When \( \theta = \nu = 0 \) we say that system (1) admits a (uniform) \( \mu \)-dichotomy on the half-line \((\mu D+)\).

Given a differentiable growth rate \( \mu : \mathbb{R}^+_0 \to \mathbb{R}^+ \). We define the nonuniform \( \mu \)-dichotomy spectrum on the half-line by
\[
\Sigma^{ND+}_\mu(A) = \left\{ \gamma \in \mathbb{R}; \ x' = \left( A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I \right)x \text{ admits no } N\mu D+ \right\}
\]
and the \( \mu \)-dichotomy spectrum on the half-line by
\[
\Sigma^{D+}_\mu(A) = \left\{ \gamma \in \mathbb{R}; \ x' = \left( A(t) - \gamma \frac{\mu'(t)}{\mu(t)} I \right)x \text{ admits no } \mu D+ \right\}.
\]

Naturally, we have \( \Sigma^{ND+}_\mu(A) \subseteq \Sigma^{D+}_\mu(A) \). We use the expressions nonuniform dichotomy spectrum on the half line and dichotomy spectrum on the half line to refer to the concepts obtained when the growth rate is given by \( \mu(t) = e^t \) and the expressions nonuniform polynomial spectrum on the half line and polynomial spectrum on the half line to refer to the concepts obtained when the growth rate is given by \( p(t) = 1 + t \). We use the notation \( \Sigma^{ND+}_p(A) \) and \( \Sigma^{ND+}_p(A) \), respectively, for the nonuniform dichotomy spectrum on the half line and the nonuniform polynomial dichotomy spectrum on the half line.

Given \( \varepsilon \geq 0 \) and a growth rate \( \mu \), we still say that systems \( x' = A(t)x \) and \( y' = B(t)y \) are nonuniformly \((\mu, \varepsilon)\)-kinematically similar if, for \( t \in \mathbb{R}^+_0 \),
they satisfy the definition of nonuniformly \((\mu, \varepsilon)\)-kinematic similarity given in section \(\S\). With the same immediate adaptation we can bring the concepts of nonuniform Lyapunov matrix function with respect to \(\mu\) and nonuniform Lyapunov transformation with respect to \(\mu\) to the half-line setting. We say that the evolution operator \(\Phi(t, s)\) of \(x' = A(t)x\) has nonuniformly bounded growth with respect to \((\mu, \varepsilon)\) on the half-line if there are \(K \geq 1\) and \(\alpha \geq 0\) such that

\[
\|\Phi(t, s)\| \leq K \left( \frac{\mu(t)}{\mu(s)} \right)^{\text{sgn}(t-s)\alpha} \mu(s)^\varepsilon, \quad t, s \in \mathbb{R}_0^+.
\]

When \(\varepsilon = 0\) the evolution operator is said to have bounded growth with respect to \(\mu\) on the half-line.

The next results are versions of Theorems 3 and 12 for linear equations defined on the half-line. Their proof can be obtained by simply rewriting the proofs of those theorems in the present setting and thus we omit them.

Given \(\gamma \in \mathbb{R}\), define the sets

\[
\mathcal{U}_{\gamma}^+ = \left\{ (s, \xi) \in \mathbb{R}_0^+ \times \mathbb{R}^n : \sup_{t \geq 0} \|\Phi(t, s)\xi\| \mu(t)^{-\gamma} < \infty \right\}
\]

and

\[
\mathcal{V}_{\gamma}^+ = \left\{ (s, \xi) \in \mathbb{R}_0^+ \times \mathbb{R}^n : \sup_{t \geq 0} \|\Phi(t, s)\xi\| \mu(t)^{-\gamma} < \infty \right\}.
\]

**Theorem 18.** Let \(\mu : \mathbb{R}_0^+ \to \mathbb{R}^+\) be a differentiable growth rate. The following statements hold for system \((\mathcal{H})\) on the half-line:

1) There is an \(m \in \{0, \ldots, n\}\) such that the nonuniform \(\mu\)-dichotomy spectrum on the half-line \(\Sigma_{\mu}^{ND+}(A)\) is the union of \(m\) disjoint closed intervals in \(\mathbb{R}\):

   a) if \(m = 0\) then \(\Sigma_{\mu}^{ND+}(A) = \emptyset\);

   b) if \(m = 1\) then

   \[\Sigma_{\mu}^{ND+}(A) = \mathbb{R} \text{ or } \Sigma_{\mu}^{ND+}(A) = (-\infty, b_1) \text{ or } \Sigma_{\mu}^{ND+}(A) = [a_1, b_1] \text{ or } \Sigma_{\mu}^{ND+}(A) = [a_1, \infty);\]

   c) if \(1 < m \leq n\) then

   \[\Sigma_{\mu}^{ND+}(A) = I_1 \cup [a_2, b_2] \cup \ldots \cup [a_{m-1}, b_{m-1}] \cup I_m\]

   with \(I_1 = [a_1, b_1]\) or \((-\infty, b_1]\), \(I_m = [a_m, b_m]\) or \([a_m, \infty)\) and \(a_i \leq b_i < a_{i+1}\) for \(i = 1, \ldots, m-1\).

2) Assume \(m \geq 1\), write

\[\Sigma_{\mu}^{ND+}(A) = I_1 \cup [a_2, b_2] \cup \ldots \cup [a_{m-1}, b_{m-1}] \cup I_m\]

and, for \(i = 0, \ldots, m+1\), define

\[
\mathcal{W}_i^+ = \begin{cases} 
\mathbb{R}_0^+ \times \{0\} & \text{if } i = 0 \text{ and } I_1 = (-\infty, b_1] \\
\mathcal{U}_{\gamma_0}^+ & \text{for some } \gamma_0 \in (-\infty, a_1) \\
\mathcal{U}_{\gamma_i}^+ \cap \mathcal{V}_{\gamma_{i+1}}^+ & \text{for some } \gamma_i \in (b_i, a_{i+1}) \\
\mathcal{V}_{\gamma_m}^+ & \text{for some } \gamma_m \in (b_m, +\infty) \\
\mathbb{R}_0^+ \times \{0\} & \text{if } i = m+1 \text{ and } I_m = [a_m, \infty) \\
\end{cases}.
\]
Then, the sets \( U_i^+ \), \( V_i^+ \) and \( W_i^+ \), \( i = 0, \ldots, m + 1 \), are integral manifolds, rank \( W_i^+ \geq 1 \) for \( i = 1, \ldots, m \) and
\[
W_0^+ \oplus W_1^+ \oplus \cdots \oplus W_{m+1}^+ = \mathbb{R}_0^+ \times \mathbb{R}^n.
\]

**Theorem 19.** Assume that \( A(t) \) is differentiable, that the evolution operator of system (1) on the half-line has nonuniform bounded growth with respect to \((\mu, \varepsilon)\), and that system (55) has a nonuniform \( \mu \)-dichotomy with parameters \( \alpha, \beta, \theta \) and \( \nu \) such that (21) holds. Let the nonuniform \( \mu \)-dichotomy spectrum on the half-line be
\[
\Sigma_{\mu}^{ND+}(A) = [a_1, b_1] \cup \ldots \cup [a_m, b_m],
\]
with \([a_i, b_i] \cap [a_j, b_j] = \emptyset\) for any \( i, j \in \{1, \ldots, m\} \) such that \( i \neq j \). Then system (1) on the half-line is \((\mu, \varepsilon)\)-nonuniformly kinematically similar to system \( y' = B(t)y \), where
\[
B(t) = \text{diag}(B_0(t), B_1(t), \cdots, B_m(t), B_{m+1}(t)),
\]
the matrices \( B_i(t) : \mathbb{R}_0^+ \to \mathbb{R}^{n_i \times n_i} \), with \( n_i = \text{rank } W_i \), are differentiable and
\[
\Sigma_{\mu}^{ND+}(B_i) = \begin{cases} 
\emptyset & \text{for } i \in \{0, m + 1\} \\
[a_i, b_i] & \text{for } i \in \{1, \ldots, m\}.
\end{cases}
\]

In the next example we consider a certain nonautonomous triangular system, compute its nonuniform \( \mu_0 \)-dichotomy spectrum, where \( \mu_0(t) = e^{\sqrt{1+t}-1} \) is a non-exponential growth rate, and obtain its normal form using Theorem 19. The normal form, that in this case is diagonal, highlights the contraction/expansion with growth rate \( \mu_0 \) along linear integral manifolds. We stress that, in the example we will describe, the behavior along integral linear manifolds (and elsewhere) is given by the growth rates \( \mu_0(t) \). Thus, this normal form cannot be obtained when we look for exponential behavior (even nonuniform), as confirmed by the fact that, in the present situation, the nonuniform (exponential) dichotomy spectrum doesn’t reveal any type of exponential behavior; as we shall see, the nonuniform dichotomy spectrum is \( \Sigma_{\mu_0}^{ND+}(A) = \mathbb{R} \) whereas the nonuniform \( \mu_0 \)-dichotomy spectrum \( \Sigma_{\mu_0}^{ND+} = \{-1/2, 1/2\} \).

**Example 20.** Consider the bidimensional system
\[
\begin{bmatrix}
u' \\
u'
\end{bmatrix} = A(t) \begin{bmatrix} u \\ v
\end{bmatrix}
\]
with \( A(t) = \begin{bmatrix} -2/4\sqrt{1+t} & e^{\sqrt{1+t}}-1/2 \\ 0 & 1/4\sqrt{1+t}
\end{bmatrix} \).

Let \( \mu_0 : \mathbb{R}_0^+ \to \mathbb{R}^+ \) be the growth rate given by \( \mu_0(t) = e^{\sqrt{1+t}-1} \) and consider in \( \mathbb{R}^2 \) the norm given by \( \|(u, v)\| = |u| + |v| \). The second equation of the system is independent of \( u \), which allows us to obtain explicitly the evolution operator (in matrix form):
\[
\Phi(t, s) = \begin{bmatrix}
\left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}\right)^{1/2} & \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}\right)^{1/2} - \left(\frac{e^{\sqrt{1+s}}}{e^{\sqrt{1+t}}}\right)^{1/2} \\
0 & \left(\frac{e^{\sqrt{1+s}}}{e^{\sqrt{1+t}}}\right)^{1/2}
\end{bmatrix}
\].
For the projections $P_1(t): \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$P_1(t) = \begin{bmatrix} 1 & e^{-\sqrt{1+t+1}} - 1 \\ 0 & 0 \end{bmatrix},$$

it is easy to check that $\Phi(t,s)P_1(s) = P_1(t)\Phi(t,s)$, for all $s, t \in \mathbb{R}_0^+$. We have, for $t \geq s \geq 0$,

$$\|\Phi(t,s)P_1(s)(u,v)\| = \left(\frac{e^{\sqrt{1+s}}}{e^{\sqrt{1+t}}}ight)^{1/2-\gamma} \| (u, (e^{-\sqrt{1+s+1}} - 1)v)\| \leq \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma - 1/2} \|(u,v)\|$$

and, since $Q_1(s)(u,v) = (I - P_1(s))(u,v) = (1 - e^{-\sqrt{1+s+1}}v,v)$, we have, for $0 \leq t \leq s$,

$$\|\Phi(t,s)Q_1(s)(u,v)\| \leq \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{1/2+\gamma} (2 + e^{1-\sqrt{1+t}})|v| \leq 3 \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{1/2+\gamma} \|(u,v)\|.$$

We conclude that, if $-1/2 - \gamma < 0$ and $1/2 - \gamma > 0$, the system above admits a nonuniform $\mu$-dichotomy (that in fact is a uniform $\mu_0$-dichotomy). Thus, $(-1/2, 1/2) \subseteq \rho_{\mu_0}^D(A)$.

For the projections $P_2(t): \mathbb{R}^2 \to \mathbb{R}^2$ defined by $P_2(t)(u,v) = (u,v)$ we have, for $t \geq s \geq 0$,

$$\|\Phi(t,s)P_2(s)(u,v)\| \leq \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma - 1/2} + \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma + 1/2} \|(u,v)\| \leq 2 \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma - 1/2} \|(u,v)\|.$$

and thus, for $0 \leq s \leq t$,

$$\|\Phi(t,s)P_3(s)\| \leq 2 \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma - 1/2}.$$

We conclude that, if $1/2 - \gamma < 0$ $\iff$ $\gamma \in (1/2, +\infty)$, we have a nonuniform $\mu_0$-dichotomy.

Finally, for the projections $P_3(t): \mathbb{R}^2 \to \mathbb{R}^2$ defined by $P_3(t)(u,v) = (0,0)$ we have, for $0 \leq t \leq s$,

$$\|\Phi(t,s)Q_3(s)(u,v)\| \leq \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma - 1/2} \|(u,v)\| + \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{1/2+\gamma} |v| = 2 \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma + 1/2} \|(u,v)\|.$$

and thus, for $0 \leq t \leq s$,

$$\|\Phi(t,s)Q_3(s)\| \leq 2 \left(\frac{e^{\sqrt{1+t}}}{e^{\sqrt{1+s}}}ight)^{\gamma + 1/2}.$$
We conclude that, if $-1/2 - \gamma > 0 \iff \gamma \in (-\infty, -1/2)$, we have a nonuniform $\mu_0$-dichotomy.

By the conclusions obtained, we know that $\Sigma_{\mu_0}^{ND+} \subseteq \{-1/2, 1/2\}$. On the other hand, the form of $\Phi_{\pm1/2}(t,s)$ for $t,s \in \mathbb{R}_0^+$, shows immediately that for $\gamma = \pm 1/2$ we don’t have a nonuniform $\mu_0$-dichotomy. Thus

$$\Sigma_{\mu_0}^{ND+} = \{-1/2, 1/2\}.$$

In our context, condition (21) holds with $\alpha = 1/2$ and $\varepsilon = 0$ and we have nonuniform bounded growth. Thus, Theorem 19 shows that system (1) in the half line is nonuniformly kinematically similar to a system $y' = B(t)y$, where $B(t) = \text{diag}(B_0(t), B_1(t))$ and $B_i(t) : \mathbb{R}_0^+ \rightarrow \mathbb{R}$, are differentiable,

$$\Sigma_{\mu_0}^{ND+}(B_0) = \{-1/2\} \quad \text{and} \quad \Sigma_{\mu_0}^{ND+}(B_1) = \{1/2\}.$$

We can confirm this conclusion by noting that the change of variables

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \sqrt{1+t} e^{-\sqrt{1+t}} \\ e^{\sqrt{1+t} - 1} 0 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}$$

transforms system $x' = A(t)x$ into the diagonal system

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = B(t) \begin{bmatrix} z \\ w \end{bmatrix} \quad \text{with} \quad B(t) = \begin{bmatrix} -1/4\sqrt{1+t} & 0 \\ 0 & 1/4\sqrt{1+t} \end{bmatrix},$$

where it becomes evident that we have polynomial contraction in $W_1 = \mathbb{R} \times \text{span}\{(1,0)\}$ and polynomial expansion in $W_2 = \mathbb{R} \times \text{span}\{(0,1)\}$.

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