A reduction theorem for AH algebras with the ideal property

Guihua Gong, Chunlan Jiang, Liangqing Li and Cornel Pasnicu

Abstract

Let $A$ be an AH algebra, that is, $A$ is the inductive limit $C^*$-algebra of

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \rightarrow \cdots \rightarrow A_n \rightarrow \cdots$$

with $A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i}$, where $X_{n,i}$ are compact metric spaces, $t_n$ and $[n,i]$ are positive integers, and $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections. Suppose that $A$ has the ideal property: each closed two-sided ideal of $A$ is generated by the projections inside the ideal, as a closed two-sided ideal. Suppose that $\sup_{n,i} \dim(X_{n,i}) < +\infty$. In this article, we prove that $A$ can be written as the inductive limit of

$$B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow B_n \rightarrow \cdots,$$

where $B_n = \bigoplus_{i=1}^{s_n} Q_{n,i} M_{\{n,i\}}(C(Y_{n,i})) Q_{n,i}$, where $Y_{n,i}$ are $\{pt\}, [0,1], S^1, T_{II,k}, T_{III,k}$ and $S^2$ (all of them are connected simplicial complexes of dimension at most three), $s_n$ and $\{n,i\}$ are positive integers and $Q_{n,i} \in M_{\{n,i\}}(C(Y_{n,i}))$ are projections. This theorem unifies and generalizes the reduction theorem for real rank zero AH algebras due to Dadarlat and Gong ([D], [G3] and [DG]) and the reduction theorem for simple AH algebras due to Gong (see [G4]).

Keywords: $C^*$-algebra, AH algebra, ideal property, Elliott intertwining, Reduction theorem

AMS subject classification: Primary: 46L05, 46L35.
§1. Introduction

Successful classification results have been obtained for real rank zero AH algebras (see [Ell1], [Lin1-3], [EG1-2], [EGLP], [D], [G1-3], [DG]) and simple AH algebras (see [Ell2-3], [Li1-3], [G4], [EGL1-2]) in the case of no dimension growth (this condition can be relaxed to a certain slow dimension growth condition). To unify these two classification theorems, we will consider AH algebras with the ideal property (see [Ji-Jiang] and [GJLP]). This article is a continuation of the paper [GJLP]—we obtain the reduction theorem for arbitrary AH algebras A (of no dimension growth) with the ideal property. That is, we remove the restriction that $K_*(A)$ is torsion free in the paper [GJLP]. Since we do not assume that $K_*(A)$ is torsion free, we must involve higher dimensional spaces such as $T_{II,k}$, $T_{III,k}$, and $S^2$ in our reduction theorem (see below). This makes the main result of this paper much more difficult to prove than the one in [GJLP], as $C(X)$ is not stably generated when dim$(X) \geq 2$.

The ideal property is a property of structural interest for a $C^*$-algebra. Many interesting and important $C^*$-algebras have the ideal property. It was proved by Cuntz-Echterhoff-Li that semigroup $C^*$-algebras of $ax + b$-semigroups over Dedekind domains have the ideal property ([CEL]). A generalization of this result appeared in [L]. Interesting examples of crossed product $C^*$-algebras with the ideal property could be found, e.g., in [Pa-Ph1] and [Pa-Ph2]. Other important results involving the ideal property have been proved, e.g., in [Pa-R1-2], [Pa-Ph1] and [Pa-Ph2]. Many $C^*$-algebras coming from $\mathbb{Z}$ dynamical systems on compact metric spaces are AH algebras (see [EE], [Phi1-2], [Lin4] and [LinP]).

An AH algebra is a nuclear $C^*$-algebra of the form $A = \lim\lim(A_n, \phi_{n,m})$ with $A_n = \bigoplus_{i=1}^{n_i} P_{n,i}M[n,i](C(X_{n,i}))P_{n,i}$, where $X_{n,i}$ are compact metric spaces, $t_n$, $[n, i]$ are positive integers, $M[n,i](C(X_{n,i}))$ are algebras of $[n, i] \times [n, i]$ matrices with entries in $C(X_{n,i})$—the algebra of complex-valued continuos functions on $X_{n,i}$—, and finally $P_{n,i} \in M[n,i](C(X_{n,i}))$ are projections (see [Bla]). Let $T_{II,k}$ (and $T_{III,k}$) be a connected finite simplicial complex with $H^1(T_{II,k}) = 0$ and $H^2(T_{II,k}) = \mathbb{Z}/k\mathbb{Z}$ (and $H^1(T_{III,k}) = 0 = H^2(T_{III,k})$ and $H^3(T_{III,k}) = \mathbb{Z}/k\mathbb{Z}$, respectively). Recall that the unit circle is denoted by $S^1$ and the 2-dimensional unit sphere is denoted by $S^2$. In this article, we will prove that an AH algebra with ideal property and no dimension growth can be rewritten as an AH inductive limit with the spaces $X_{n,i}$ being $\{pt\}, [0, 1], S^1, T_{II,k}, T_{III,k}$ and $S^2$. (For the background information, we refer the readers to [GJLP].) The result in this paper plays an essential role in the classification of the AH algebras with the ideal property and with no dimension growth (see [GJL]).
§2. Notation and preliminary

The following notations are quoted from [G4] or [GJLP].

2.1. If $A$ and $B$ are two $C^*$-algebras, we use $\text{Map}(A,B)$ to denote the space of all linear, completely positive $*$-contractions from $A$ to $B$. If both $A$ and $B$ are unital, then $\text{Map}(A,B)_1$ will denote the subset of $\text{Map}(A,B)$ consisting of unital maps. By word "map", we shall mean a linear, completely positive $*$-contraction between $C^*$-algebras, or else we shall mean a continuous map between topological spaces, which one will be clear from the context.

By a homomorphism between $C^*$-algebras, will be meant a $*$-homomorphism. Let $\text{Hom}(A,B)$ denote the space of all the homomorphisms from $A$ to $B$. Similarly, if both $A$ and $B$ are unital, let $\text{Hom}(A,B)_1$ denote the subset of all the unital homomorphisms.

Definition 2.2. Let $G \subset A$ be a finite set and $\delta > 0$. We shall say that $\phi \in \text{Map}(A,B)$ is $G - \delta$ multiplicative if

$$\| \phi(ab) - \phi(a)\phi(b) \| < \delta$$

for all $a, b \in G$.

2.3. In the notation for an inductive system $(A_n, \phi_{n,m})$, we understand that $\phi_{n,m} = \phi_{m-1,m} \circ \phi_{m-2,m-1} \circ \cdots \circ \phi_{n,n+1}$, where all $\phi_{n,m} : A_n \to A_m$ are homomorphisms.

We shall assume that, for any summand $A^i_n$ in the direct sum $A_n = \bigoplus_{i=1}^{t_n} A^i_n$, necessarily $\phi_{n,n+1}(1_{A^i_n}) \neq 0$, since, otherwise, we could simply delete $A^i_n$ from $A_n$ without changing the limit algebra.

2.4. If $A_n = \bigoplus_i A^i_n$ and $A_m = \bigoplus_j A^j_m$, we use $\phi^i_{n,m}$ to denote the partial map of $\phi_{n,m}$ from the $i$-th block $A^i_n$ of $A_n$ to the $j$-th block $A^j_m$ of $A_m$.

2.5. By 2.3 of [Bla] and Theorem 2.1 of [EGL2], we know that any $AH$ algebra can be written as an inductive limit $A = \lim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}C(X_{n,i})P_{n,i}, \phi_{n,m})$, where $X_{n,i}$ are finite simplicial complexes and $\phi_{n,m}$ are injective. In this article we will assume that $X_{n,i}$ are (path) connected finite simplicial complexes and $\phi_{n,m}$ are injective.

It is well known that, for any connected finite simplicial complex $X$, there is a metric $d$ on $X$ with the following property: for any $x \in X$ and $\eta > 0$, the $\eta$-ball centered in $x$, $B_\eta(x) = \{ x' \in X \mid d(x',x) < \eta \}$ is path connected. So in this article, we will always assume that the metric on a connected simplicial complex has this property.
2.6. In this article, we assume that the inductive limit system satisfies the no dimension growth condition: there is an $M \in \mathbb{N}$ such that for any $n, i$,

$$\text{dim}(X_{n,i}) \leq M.$$ 

The condition could be relaxed to a so called very slow dimension growth for our main theorem. Since the proof of this slightly more general case is quite tedious, we will leave it to a subsequent paper.

2.7. By 1.3.3 of [G4], any $AH$ algebra

$$A = \lim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))) P_{n,i}, \phi_{n,m})$$

is isomorphic to a limit corner subalgebra (see Definition 1.3.2 of [G4] for this concept) of $\tilde{A} = \lim(\tilde{A}_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \tilde{\phi}_{n,m})$—an inductive limit of full matrix algebras over $X_{n,i}$. Once we prove that $\tilde{A}$ is an inductive limit of homogeneous algebras over the spaces in the following list: $\{pt\}, [0, 1], S^1, T_{II,k}, T_{III,k}, S^2$, then $A$ itself is also an inductive limit of such kind. Therefore in this article, we will assume $A$ itself is an inductive limit of finite direct sum of full matrix algebras over $X_{n,i}$. That is $P_{n,i} = 1_{M_{[n,i]}(C(X_{n,i}))}$.

2.8. Let $Y$ be a compact metrizable space. Let $P \in M_{k_1}(C(Y))$ be a projection with $\text{rank}(P)=k \leq k_1$. For each $y$, there is a unitary $u_y \in M_{k_1}(\mathbb{C})$ (depending on $y$) such that

$$P(y) = u_y \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} u_y^*,$$

where there are $k$ 1’s on the diagonal. If the unitary $u_y$ can be chosen to be continuous in $y$, then $P$ is called a trivial projection.

It is well known that any projection $P \in M_{k_1}(C(Y))$ is locally trivial. That is, for any $y_0 \in Y$, there is an open set $U_{y_0} \supseteq y_0$, and there is a continuous unitary-valued function

$$u : U_{y_0} \longrightarrow M_{k_1}(\mathbb{C})$$

such that the above equation holds for $u(y)$ (in place of $u_y$) for any $y \in U_{y_0}$.
If $P$ is trivial, then $PM_{k_1}(C(X))P \cong M_k(C(X))$.

**2.9.** Let $X$ be a compact metrizable space and $\psi : C(X) \to PM_{k_1}(C(Y))P$ be a unital homomorphism. For any given point $y \in Y$, there are points $x_1(y), x_2(y), \cdots, x_k(y) \in X$, and a unitary $U_y \in M_{k_1}(\mathbb{C})$ such that

$$\psi(f)(y) = P(y)U_y \begin{pmatrix} f(x_1(y)) & \cdots & f(x_k(y)) \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} U_y^*P(y) \in P(y)M_{k_1}(\mathbb{C})P(y),$$

for all $f \in C(X)$. Equivalently, there are $k$ rank one orthogonal projections $p_1, p_2, \cdots, p_k$ with $\sum_{i=1}^k p_i(y) = P(y)$ and $x_1(y), x_2(y), \cdots, x_k(y) \in X$, such that

$$\psi(f)(y) = \sum_{i=1}^k f(x_i(y))p_i(y), \quad \forall f \in C(X).$$

Let us denote the set $\{x_1(y), x_2(y), \cdots, x_k(y)\}$, counting multiplicities, by $SP\psi_y$ (see [Pa1]). In other words, if a point is repeated in the diagonal of the above matrix, it is included with the same multiplicity in $SP\psi_y$. **We shall call $SP\psi_y$ the spectrum of $\psi$ at the point $y$.** Let us define the **spectrum** of $\psi$, denoted by $SP\psi$, to be the closed subset

$$SP\psi := \bigcup_{y \in Y} SP\psi_y (= \bigcup_{y \in Y} SP\psi_y) \subset X.$$

Alternatively, $SP\psi$ is the complement of the spectrum of the kernel of $\psi$, considered as a closed ideal of $C(X)$. The map $\psi$ can be factored as

$$C(X) \xrightarrow{i^*} C(SP\psi) \xrightarrow{\psi_1} PM_{k_1}(C(Y))P$$

with $\psi_1$ an injective homomorphism, where $i$ denotes the inclusion $SP\psi \hookrightarrow X$.

Also, if $A = PM_{k_1}(C(Y))P$, then we shall call the space $Y$ the spectrum of the algebra $A$, and write $SPA = Y (= SP(id))$.

**2.10.** In 2.9, if we group together all the repeated points in $\{x_1(y), x_2(y), \cdots, x_k(y)\}$, and
sum their corresponding projections, we can write

\[ \psi(f)(y) = \sum_{i=1}^{t} f(\lambda_i(y))P_i, \quad (l \leq k), \]

where \( \{\lambda_1(y), \lambda_2(y), \ldots, \lambda_l(y)\} \) is equal to \( \{x_1(y), x_2(y), \ldots, x_k(y)\} \) as a set, but \( \lambda_i(y) \neq \lambda_j(y) \) if \( i \neq j \); and each \( P_i \) is the sum of the projections corresponding to \( \lambda_i(y) \). If \( \lambda_i(y) \) has multiplicity \( m \) (i.e., it appears \( m \) times in \( \{x_1(y), x_2(y), \ldots, x_k(y)\} \)), then \( \text{rank}(P_i) = m \).

2.11. Set \( P^k(X) = X \times X \times \cdots \times X / \sim \), where the equivalence relation \( \sim \) is defined by \((x_1, x_2, \ldots, x_k) \sim (x'_1, x'_2, \ldots, x'_k)\) if there is a permutation \( \sigma \) of \( \{1, 2, \ldots, k\} \) such that \( x_i = x'_{\sigma(i)} \), for each \( 1 \leq i \leq k \). A metric \( d \) on \( X \) can be extended to a metric on \( P^k(X) \) by

\[ d([x_1, x_2, \ldots, x_k], [x'_1, x'_2, \ldots, x'_k]) = \max_{1 \leq i \leq k} d(x_i, x'_{\sigma(i)}), \]

where \( \sigma \) is taken from the set of all permutations, and \([x_1, x_2, \ldots, x_k]\) denotes the equivalence class in \( P^k(X) \) of \((x_1, x_2, \ldots, x_k)\). Two \( k \)-tuples of (possible repeating) points \( \{x_1, x_2, \ldots, x_k\} \subset X \) and \( \{x'_1, x'_2, \ldots, x'_k\} \subset X \) are said to be paired within \( \eta \) if

\[ d([x_1, x_2, \ldots, x_k], [x'_1, x'_2, \ldots, x'_k]) < \eta \]

when one regards \((x_1, x_2, \ldots, x_k)\) and \((x'_1, x'_2, \ldots, x'_k)\) as two points in \( P^k(X) \).

2.12. Let \( \psi : C(X) \rightarrow \text{PM}_{k_1}(C(Y))P \) be a unital homomorphism as in 2.9. Then

\[ \psi^* : y \mapsto SP\psi_y \]

defines a map \( Y \rightarrow P^k(X) \), if one regards \( SP\psi_y \) as an element of \( P^k(X) \). This map is continuous. In terms of this map and the metric \( d \), let us define the spectral variation of \( \psi \):

\[ SPV(\psi) := \text{the diamenter of image of } \psi^*. \]

**Definition 2.13** We shall call \( P_i \) in 2.10 the spectral projection of \( \phi \) at \( y \) with respect to the spectral element \( \lambda_i(y) \). For a subset \( X_1 \subset X \), we shall call

\[ \sum_{\lambda_i(y) \in X_1} P_i \]

the spectral projection of \( \phi \) at \( y \) corresponding to the subset \( X_1 \) (or with respect to the subset \( X_1 \)).
2.14. Let $\phi : M_k(C(X)) \to PM_l(C(Y))P$ be a unital homomorphism. Set $\phi(e_{11}) = p$, where $e_{11}$ is the canonical matrix unit corresponding to the upper left corner. Set

$$
\phi_1 = \phi|_{e_{11}M_k(C(X))e_{11}} : C(X) \to PM_l(C(Y))p.
$$

Then $PM_l(C(Y))P$ can be identified with $PM_l(C(Y))p \otimes M_k$ in such a way that

$$
\phi = \phi_1 \otimes id_k.
$$

Let us define

$$
SP\phi_y := SP(\phi_1)_y, \\
SP\phi := SP\phi_1, \\
SPV(\phi) := SPV(\phi_1).
$$

Suppose that $X$ and $Y$ are connected. Let $Q$ be a projection in $M_k(C(X))$ and $\phi : QM_k(C(X))Q \to PM_l(C(Y))P$ be a unital map. By the Dilation lemma (Lemma 2.13 of [EG2]), there are an $n$, a projection $P_1 \in M_n(C(Y))$, and a unital homomorphism

$$
\widetilde{\phi} : M_k(C(X)) \to P_1 M_n(C(Y))P_1
$$

such that

$$
\phi = \widetilde{\phi}|_{QM_k(C(X))Q}.
$$

(Note that this implies that $P$ is a subprojection of $P_1$.) We define:

$$
SP\phi_y := SP\widetilde{\phi}_y, \\
SP\phi := SP\widetilde{\phi}, \\
SPV(\phi) := SPV(\widetilde{\phi}).
$$

(Note that these definitions do not depend on the choice of the dilation $\widetilde{\phi}$.)

2.15. Let $\phi : M_k(C(X)) \to PM_l(C(Y))P$ be a (not necessarily unital) homomorphism, where $X$ and $Y$ are connected finite simplicial complexes. Then

$$
#(SP\phi_y) = \frac{rank\phi(1_k)}{rank(1_k)}, \text{ for any } y \in Y,
$$

where again $#(\cdot)$ denotes the number of elements in the set counting multiplicity. It is also true that for any nonzero projection $p \in M_k(C(X))$, $#(SP\phi_y) = \frac{rank\phi(p)}{rank(p)}$. 

7
2.16. Let $X$ be a compact connected space and let $Q$ be a projection of rank $n$ in $M_N(C(X))$. The weak variation of a finite set $F \subset QM_N(C(X))Q$ is defined by

$$\omega(F) = \sup_{\Pi_1, \Pi_2} \inf_{u \in U(n)} \max_{a \in F} \| u \Pi_1(a) a^* - \Pi_2(a) \|,$$

where $\Pi_1, \Pi_2$ run through the set of irreducible representations of $QM_N(C(X))Q$ into $M_n(\mathbb{C})$.

Let $X_i$ be compact connected spaces and $Q_i \in M_{n_i}(C(X_i))$ be projections. For a finite set $F \subset \bigoplus_i Q_i M_{n_i}(C(X_i))Q_i$, define the weak variation $\omega(F)$ to be $\max_i \omega(\pi_i(F))$, where $\pi_i : \bigoplus_j Q_j M_{n_j}(C(X_j))Q_j \to Q_i M_{n_i}(C(X_i))Q_i$ is the natural projection map onto the $i$-th block.

The set $F$ is said to be weakly approximately constant to within $\varepsilon$ if $\omega(F) < \varepsilon$.

2.17. The following notations will be frequently used in this article.

(a) As in 2.15, we use notation $\#(\cdot)$ to denote the cardinal number of the set counting multiplicity.

(b) For any metric space $X$, any $x_0 \in X$ and any $c > 0$, let

$$B_c(x_0) := \{ x \in X \mid d(x, x_0) < c \}$$

denote the open ball with radius $c$ and center $x_0$.

(c) Suppose that $A$ is a $C^*$-algebra, $B \subset A$ is a sub-$C^*$-algebra, $F \subset A$ is a (finite) subset and let $\varepsilon > 0$. If for each element $f \in F$, there is an element $g \in B$ such that $\| f - g \| < \varepsilon$, then we shall say that $F$ is approximately contained in $B$ to within $\varepsilon$, and denote this by $F \subset_\varepsilon B$.

(d) Let $X$ be a compact metric space. For any $\delta > 0$, a finite set $\{x_1, x_2, \ldots, x_n\}$ is said to be $\delta$-dense in $X$, if for any $x \in X$, there is $x_i$ such that $\text{dist}(x, x_i) < \delta$.

(e) We shall use $\bullet$ to denote any possible positive integer. To save notation, $a_1, a_2, \ldots$ may be used for a finite sequence if we do not care how many terms are in the sequence. Similarly, $A_1 \cup A_2 \cup \cdots$ or $A_1 \cap A_2 \cap \cdots$ may be used for a finite union or a finite intersection. If there is a danger of confusion with an infinite sequence, union or intersection, we will write them as $a_1, a_2, \ldots, a_\bullet$, $A_1 \cup A_2 \cup \cdots \cup A_\bullet$, $A_1 \cap A_2 \cap \cdots \cap A_\bullet$.

(f) In this paper, we often use $1$ to denote the units of different unital $C^*$-algebras. In particular, if $1$ appears in $\phi(1)$, where $\phi$ is a homomorphism, then $1$ is the unit of the domain algebra. For example for a homomorphism $\phi : \bigoplus_{i=1}^r A_i \to B$, then $1$ in $\phi(1)$ means $1_{\bigoplus_{i=1}^r A_i}$ and $1$ in $\phi(1)$ means $1_{A_i}$.

(g) For any two projections $p, q \in A$, we use the notation $[p] \leq [q]$ to denote that $p$ is unitarily equivalent to a subprojection of $q$. And we use $p \sim q$ to denote that $p$ is unitarily equivalent to $q$. 

8
2.18. For any $\eta > 0$, $\delta > 0$, a unital homomorphism $\phi : PM_k(C(X))P \to QM_{k'}(C(Y))Q$ is said to have the property $sdp(\eta, \delta)$ (spectral distribution property with respect to $\eta$ and $\delta$) if for any $\eta$-ball $B_\eta(x)$ and any point $y \in Y$,

$$\#(SP\phi_y \cap B_\eta(x)) \geq \delta \#(SP\phi_y)$$

counting multiplicity. Any homomorphism

$$\phi : \bigoplus_i P_iM_k(C(X_i))P_i \to \bigoplus_j Q_jM_{l_j}(C(Y_j))Q_j$$

is said to have the property $sdp(\eta, \delta)$ if each partial map

$$\phi^{i,j} : P_iM_k(C(X_i))P_i \to \phi^{i,j}(P_i)M_{l_j}(C(Y_j))\phi^{i,j}(P_i)$$

has the property $sdp(\eta, \delta)$ as a unital homomorphism. Note that by definition, a nonunital homomorphism $\phi : M_k(C(X)) \to M_l(C(Y))$ has the property $sdp(\eta, \delta)$ if the corresponding unital map

$$\phi : M_k(C(X)) \to \phi(1_k)M_l(C(Y))\phi(1_k)$$

has the property $sdp(\eta, \delta)$.

The following Lemma is Lemma 2.8 of [GJLP].

**Lemma 2.19.** Let $A = \lim \to A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i}))$, $\phi_{n,m}$ be an AH algebra with the ideal property. For $A_n$ and any $\eta > 0$, there exist a $\delta > 0$, a positive integer $m > n$, connected finite simplicial complexes $Z_1^i, Z_2^i, \cdots, Z_t^i \subset X_{n,i}, i = 1, 2, \cdots, t_n$, and a homomorphism

$$\phi : B = \bigoplus_{s=1}^{t_n} \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(Z_s^i)) \to A_m$$

such that

(1) $\phi_{n,m}$ factors as $A_n \xrightarrow{\pi} B \xrightarrow{\phi} A_m$, where $\pi$ is defined by

$$\pi(f) = (f|_{Z_1^i}, f|_{Z_2^i}, \cdots, f|_{Z_t^i}) \in \bigoplus_{s} M_{[n,i]}(C(Z_s^i)) \subset B,$$

for any $f \in M_{[n,i]}(C(X_{n,i}))$.

(2) The homomorphism $\phi$ satisfies the dichotomy condition $(*):$ for each $Z_s^i$, the partial map

$$\phi^{(i,s),j} : M_{[n,i]}(C(Z_s^i)) \to A_m^j$$

either has the property $sdp(\frac{\eta}{32}, \delta)$ or is the zero map. Furthermore for any $m' > m$, each partial map of $\phi_{m,m'} \circ \phi$ satisfies the dichotomy condition $(*):$ either it has the property $sdp(\frac{\eta}{32}, \delta)$ or is the zero map.
The following result is quoted from [Pa2].

**Lemma 2.20.** Let \( A = \lim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}) \) be an AH inductive limit with the ideal property and with no dimension growth (as in 2.5, we assume \( X_{n,i} \) path connected). For any \( A_n \), finite set \( F_n = \bigoplus F_n^i \subset A_n, \varepsilon > 0 \), and positive integer \( L \), there is an \( A_m \), such that for each pair \((i,j)\), one of the following conditions holds

(i) \( \frac{\text{rank}(\psi_{n,m}^i(A_n^i))}{\text{rank}(A_m^j)} \geq L \), or

(ii) there is a homomorphism

\[
\psi : A_n^i \to \phi_{n,m}^j(1_{A_n^i})A_n^j\psi_{n,m}^i(1_{A_n^i})
\]

with finite dimensional image such that \( \psi \) is homotopic to \( \phi_{n,m}^j \) and

\[
\|\psi(f) - \phi_{n,m}^j(f)\| \leq \varepsilon \quad \forall f \in F_n^i.
\]

The following proposition is Theorem 2.12 of [GJLP].

**Proposition 2.21.** Let \( A = \lim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}) \) be an AH algebra with the ideal property and with no dimension growth. For any \( A_n \), finite set \( F = \bigoplus F_n^i \subset A_n \), positive integer \( J \) and \( \varepsilon > 0 \), there exist \( m \) and there exist projections \( Q_0, Q_1, Q_2 \in A_m \) with \( Q_0 + Q_1 + Q_2 = \phi_{n,m}(1_{A_n}) \), a unital map \( \psi_0 \in \text{Map}(A_n, Q_0A_mA_0) \) and two unital homomorphisms \( \psi_1 \in \text{Hom}(A_n, Q_1A_mA_1), \psi_2 \in \text{Hom}(A_n, Q_2A_mA_2) \) such that the following statements are true

1. \( \|\phi_{n,m}(f) - (\psi_0(f) \oplus \psi_1(f) \oplus \psi_2(f))\| < \varepsilon \), for all \( f \in F \)
2. \( \omega((\psi_0 \oplus \psi_1)(F)) < \varepsilon \)
3. The homomorphism \( \psi_2 \) factors through \( C \)—a finite direct sum of matrix algebras over \( C[0,1] \), or \( C \) as

\[
\psi_2 : A_n \xrightarrow{\xi_1} C \xrightarrow{\xi_2} Q_2A_mQ_2
\]

where \( \xi_1, \xi_2 \) are unital homomorphisms

4. Let \( \psi_0^{i,j} : A_n^i \to \psi_0^j(1_{A_n^i})A_n^j\psi_0^{i,j}(1_{A_n^i}) \) and \( \psi_1^{i,j} : A_n^i \to \psi_1^j(1_{A_n^i})A_n^j\psi_1^{i,j}(1_{A_n^i}) \) be the corresponding partial maps of \( \psi_0 \) and \( \psi_1 \). For each pair \((i,j)\), one of the following is true.

   (i) Both \( \psi_0^{i,j} \) and \( \psi_1^{i,j} \) are zero, or
   (ii) \( \psi_1^{i,j} \) is a homomorphism with finite dimensional image and for each non zero projections \( e \in A_n^i \) (including any rank 1 projection)

\[
[\psi_1^{i,j}(e)] > J[\psi_0^{i,j}(1_{A_n^i})](\in K_0(A_n^j)).
\]

Furthermore, we can assume that \( Q_0^i, Q_1^i \) are trivial projections in \( A_n^i \).
The following Corollary is also in [GJLP],

**Corollary 2.22.** We use the notation from 2.21. For any $A_n$, any projection $P = \bigoplus P^i \in \bigoplus A^n_i$, any finite set $F = \bigoplus F^i \in \bigoplus P^n A^n_i P^i = PA_nP$, any positive integer $J$, and any number $\varepsilon > 0$, there are an $A_m$, mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_m$ with $Q_0 + Q_1 + Q_2 = \phi_{n,m}(1_{A_n})$, a unital map $\psi_0 \in \text{Map}(A_n, Q_0 A_m Q_0)_1$ and two unital homomorphisms $\psi_1 \in \text{Hom}(A_n, Q_1 A_m Q_1)_1$, $\psi_2 \in \text{Hom}(A_n, Q_2 A_m Q_2)_1$ such that for each pair $(i, j)$, $\psi_0^{i,j}(P^i)$ and $\psi_0^{i,j}(1_{A^n} - P^i)$ are mutually orthogonal projections and there is an approximate decomposition of $\phi_{n,m} := \phi_{n,m} \mid_{PA_nP}$ as a direct sum of $\psi_0' := \psi_0 \mid_{PA_nP}$, $\psi_1' := \psi_1 \mid_{PA_nP}$ and $\psi_2' := \psi_2 \mid_{PA_nP}$ satisfying the following conditions:

1. $\|\phi_{n,m}(f) - (\psi_0'(f) \oplus \psi_1'(f) \oplus \psi_2'(f))\| < \varepsilon$, for all $f \in F$.
2. $\psi_1'$ has finite dimensional image and $\psi_2'$ factors through a finite direct sum of matrix algebras over $C[0, 1]$ or $\mathbb{C}$.
3. If $\psi_0^{i,j} \neq 0$, then for any nonzero projection $e \in P^i A^n_i P^i$, $[\psi_1^{i,j}(e)] > J[\psi_0^{i,j}(P^i)](\in K_0(A^n_m))$.
4. $\psi_0'$ is $F - \varepsilon$ multiplicative.

**2.23.** Let $X$ be a connected finite simplicial complex, $A = M_k(C(X))$. A unital *-monomorphism $\phi : A \rightarrow M_l(A)$ is called a (unital) simple embedding if it is homotopic to the homomorphism $id \oplus \lambda$, where $\lambda : A \rightarrow M_{l-1}(A)$ is defined by

$$\lambda(f) = \text{diag}(f(x_0), f(x_0), \cdots, f(x_0)),$$

for a fixed base point $x_0 \in X$.

Let $A = \bigoplus_{i=1}^n M_{k_i}(C(X_i))$, where $X_i$ are connected finite simplicial complexes. A unital *-monomorphism $\phi : A \rightarrow M_l(A)$ is called a (unital) simple embedding, if $\phi$ is of the form $\phi = \oplus \phi^i$ defined by

$$\phi(f_1, f_2, \cdots, f_n) = (\phi^1(f_1), \phi^2(f_2), \cdots, \phi^n(f_n)),$$

where the homomorphisms $\phi^i : A^i(= M_{k_i}(C(X_i))) \rightarrow M_l(A^i)$ are unital simple embeddings.

§3. Additional decomposition theorems and factorization theorems

In this section, we will prove certain decomposition theorems which say that the map $\phi_{n,m}$ (in an $AH$ inductive limit with the ideal property) can be decomposed into two parts roughly described as below:
(a) the major part factors through as
\[ A_n \xrightarrow{\xi_1} C \xrightarrow{\xi_2} A_m, \]
where \( C \) is a direct sum of matrix algebras over interval \([0,1]\) or \( \{pt\} \) and \( \xi_1, \xi_2 \) are * homomorphisms, and
(b) the other part factors through as
\[ A_n \xrightarrow{\beta} B \xrightarrow{\alpha} A_m, \]
where \( B \) is a direct sum of matrix algebras over \( C(S^1), C(T_{II,k}), C(T_{III,k}) \) or \( C(S^2) \), \( \alpha \) is a * homomorphisms, but \( \beta \) is a sufficiently multiplicative completely positive contraction.

These theorems will play the roles of Theorem 5.32a and Theorem 5.32b in [G4] in our proof.

Since our limit algebra \( A \) is no longer simple, we cannot make each partial map to satisfy the dichotomy condition: each partial map \( \phi_{n,m}^{i,j} \) is either injective or has finite dimensional image.

3.1. Recall that the total K-theory of \( A \) is defined by
\[ K(A) = K_\ast(A) \oplus \bigoplus_{n=2}^{\infty} K_\ast(A, \mathbb{Z}/n). \]
Any \( KK \) element \( \alpha \in KK(A,B) \) determine a homomorphism
\[ \alpha : K(A) \rightarrow K(B). \]
Now we let \( W_k = T_{II,k} \) and let \( \mathcal{P} \subset \bigcup_k M_\ast(A \otimes C(W_k \times S^1)) \) be any finite set of projections. Then each element \( p \in \mathcal{P} \) defines an element \([p] \in K(A)\). We will use \( \mathcal{P}K(A) \) to denote the subgroup of \( K(A) \) generated by \( \{[p] \in K(A), p \in \mathcal{P}\} \). For each finite set \( \mathcal{P} \), there is a finite set \( G(\mathcal{P}) \subset A \) (large enough) and \( \delta(\mathcal{P}) > 0 \) (small enough) such that if \( \phi : A \rightarrow B \) is \( G(\mathcal{P}) - \delta(\mathcal{P}) \) multiplicative completely positive contraction, then \( \phi \) induces
\[ \phi_* : \mathcal{P}K(A) \rightarrow K(B). \]
(See [GL].)

Let \( A = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})) \), where \( X_{n,i} \) are connected finite simplicial complexes. Then \( K_\ast(A) \) is finitely generated, and there is a finite set \( \mathcal{P} \subset \bigcup_k M_\ast(A \otimes C(W_k \times S^1)) \) such that if two element \( \alpha, \beta \in KK(A,B) \) satisfy
\[ \alpha \mid_{\mathcal{P}K(A)} = \beta \mid_{\mathcal{P}K(A)} \]
then \( \alpha = \beta \in KK(A,B) \).
The following is 5.24 of [G4].

**Definition 3.2.** For any finite set of projections $\mathcal{P} \subset \bigcup_{k=2}^{\infty} M_\bullet(A \otimes C(W_k \times S^1))$, let $G(\mathcal{P}), \delta(\mathcal{P})$ be as in 3.1. A $G(\mathcal{P}) - \delta(\mathcal{P})$ multiplicative map $\phi : A \rightarrow B$ is called quasi-$\mathcal{P} \mathcal{K}$-homomorphism if there is a homomorphism $\psi : A \rightarrow B$ with $\phi(1_A) = \psi(1_A)$ such that $[\phi]_* = [\psi]_* : \mathcal{P} \mathcal{K}(A) \rightarrow \mathcal{K}(B)$.

### 3.3
Let $B = M_\bullet(C(Y))$, $Y$ is one of the space $T_{II,k}, T_{III,k}, S^2$, let $\mathcal{P} \subset \bigcup_{k=2}^{\infty} M_\bullet(B \otimes C(W_k \times S^1))$ be a finite set as in the end of 3.1 (one can choose $\mathcal{P}$ as in 5.16 of [G4]). Let $\phi : B \rightarrow A$ be a homomorphism and let $\mathcal{P}_1 \subset \bigcup_{k=2}^{\infty} M_\bullet(A \otimes C(W_k \times S^1))$, where $A$ and $\mathcal{P}_1$ satisfy the condition in 3.1. Furthermore we assume $\mathcal{P}_1 \supseteq \phi(\mathcal{P})$. Then there is a finite set $G \subset A$ and $\delta > 0$, such that if a $G - \delta$ multiplicative map $\psi : A \rightarrow A_1$ is a quasi-$\mathcal{P}_1 \mathcal{K}$-homomorphism, then $\psi \circ \phi : B \rightarrow A_1$ is a quasi-$\mathcal{P}_1 \mathcal{K}$-homomorphism.

**Lemma 3.4.** Let $A = \bigoplus_{i=1}^{n} M_{[n,i]}(C(X_{n,i}))$. For any $\mathcal{P}_1 \subset \bigcup_{k=2}^{\infty} M_\bullet(A \otimes C(W_k \times S^1))$, there is a finite set $F \subset A$ and $\varepsilon > 0$, such that the following is true.

If $\phi : A \rightarrow A' = PM_\bullet(C(X))P$ is a unital homomorphism, $Q_1, Q_2 \in PM_\bullet(C(X))P$ are two projections with $Q_1 + Q_2 = P$, and $\phi_1 \in Map(A, Q_1 A' Q_1)$, $\phi_2 \in Hom(A, Q_2 A' Q_2)$ satisfy

(i) $\| \phi(f) - (\phi_1 \oplus \phi_2)(f) \| < \varepsilon \quad \forall f \in F$ and

(ii) for each $i$, $\phi_i \in Map(A^i, Q_1 A' Q_1)$ is either zero, or

$$\text{rank}(\phi_i(1_{A^i})) \geq 3(\text{dim}(X) + 1)\text{rank}(1_{A^i})$$

then $\phi_1$ is a quasi-$\mathcal{P}_1 \mathcal{K}$-homomorphism, where $A^i = M_{[n,i]}(C(X_{n,i}))$.

**Proof.** First, by Lemma 4.40 of [G4], if $F$ is large enough and $\varepsilon > 0$ is small enough, then the condition (i) implies that $\phi_1$ is $G(\mathcal{P}_1) - \delta(\mathcal{P}_1)$ multiplicative and induces maps on $\mathcal{P}_1 \mathcal{K}(A)$.

We can assume that $A$ has only one block $M_{[n,1]}(C(X_{n,1}))$. Using Lemma 1.6.8 of [G4], one can assume that $\phi_1 |_{M_{[n,1]}(C)}$ is a homomorphism, where $M_{[n,1]}(C) \subset M_{[n,1]}(C(X_{n,1}))$. Then one can reduce the proof of the lemma to the case that $[n,1] = 1$, that is, $A = C(X_{n,1})$. Then both $\phi \in Hom(C(X_{n,1}), PM_\bullet(C(X))P)$ and $\phi_2 \in Hom(C(X_{n,1}), Q_2 M_\bullet(C(X))Q_2)$ induce $[\phi] \in kk(X, X_{n,1})$ and $[\phi_2] \in kk(X, X_{n,1})$ (see 2.8 of [EG2], also see [DN] for notation $kk$). Since $kk(X, X_{n,1})$ is an Abelian group, and $\text{rank}(Q_1) \geq 3(\text{dim}(X) + 1)\text{rank}(1_{A^i})$, by 6.4.4 of [DN] (see also Proposition 3.16 of [EG2]), there is a unital homomorphism $\psi_1 \in Hom(A, Q_1 A' Q_1)$.
such that
\[ [\psi_1] = [\phi] - [\phi_2] \in \text{kk}(X, X_{n,1}). \]
Hence \([\psi_1] |_{\mathcal{P}_K(A)} = [\phi_1] |_{\mathcal{P}_K(A)}\), which implies \(\phi_1\) is a quasi-\(\mathcal{P}_K\)-homomorphism.

\[ \square \]

**Lemma 3.5.** Let \(A = PM_\bullet(C(X))P\), where \(X\) is a connected finite simplicial complex. Let \(F \subset A\) be approximately constant to within \(\varepsilon\) (i.e. \(\omega(F) < \varepsilon\)). Then for any two homomorphisms \(\phi, \psi : A \to B = QM_lC(Y)Q\) defined by point evaluations with \(K_0\phi = K_0\psi\) and assuming that for any \(p \in A\), \(\text{rank}(\phi(p)) \geq \text{rank}(p) \cdot \text{dim}(Y)\), there exists a unitary \(u \in B\) such that
\[ ||\phi(f) - u\psi(f)u^*|| < 2\varepsilon, \quad \forall f \in F. \]

**Proof.** Let \(x_0 \in X\) be a base point of \(X\). There are finitely many points \(\{x_1, x_2, \ldots, x_n\} \subset X\) such that \(\phi\) factors through as
\[ A \xrightarrow{\pi} A |_{\{x_1, x_2, \ldots, x_n\}} = \bigoplus_{i=1}^n M_{\text{rank}(P)}(\mathbb{C}) \xrightarrow{\phi''} B. \]
But for each \(i\), there is a unitary \(u_i \in M_{\text{rank}(P)}(\mathbb{C})\) such that
\[ ||f(x_0) - u_if(x_i)u_i^*|| < \varepsilon, \quad \forall f \in F, \]
since \(\omega(F) < \varepsilon\). Let \(\pi' : A \to \bigoplus_{i=1}^n M_{\text{rank}(P)}(\mathbb{C})\) defined by
\[ \pi'(f) = (u_1^*f(x_0)u_1, u_2^*f(x_0)u_2, \ldots, u_n^*f(x_0)u_n). \]
Then \(||\pi(f) - \pi'(f)|| < \varepsilon\) for all \(f \in F\). Evidently there is a homomorphism \(\phi' : M_{\text{rank}(P)}(\mathbb{C}) \to B\) such that \(||\phi(f) - \phi'(f(x_0))|| < \varepsilon, \forall f \in F\). Similarly there is a \(\psi' : M_{\text{rank}(P)}(\mathbb{C}) \to B\) with \(||\psi(f) - \psi'(f(x_0))|| < \varepsilon, \forall f \in F\). On the other hand \(K_0\phi = K_0\psi\), since \(e_\pi : K_0(PM_\bullet(C(X))P) \to K_0(M_{\text{rank}}(\mathbb{C}))\) is surjective for \(e : PM_\bullet(C(X))P \to P(x_0)M_\bullet(\mathbb{C})P(x_0) \cong M_{\text{rank}(P)}(\mathbb{C})\). Furthermore by the condition that \(\text{rank}(\phi(P)) \geq \text{rank}(P) \cdot \text{dim}(Y)\) and the fact that for any two projections \(Q_1, Q_2\) (of rank at least \(\text{dim}(Y)\)) over \(Y\), \([Q_1] = [Q_2] \in K_0(M_\bullet(C(Y)))\) implies \(Q_1\) unitarily equivalent to \(Q_2\), we know that there is a unitary \(u \in B\) such that \(\phi' = u\psi'u^*\).

\[ \square \]

**Lemma 3.6.** Fix a positive integer \(M \geq 3\), suppose that \(B = \bigoplus_{i=1}^s M_{l_i}(C(Y_i))\), where \(Y_i\) are the spaces: \(\{pt\}, [0,1], S^1, T_{11,k}, T_{111,k}, S^2\). Let \(\varepsilon > 0\) and
\[ \tilde{G}(= \bigoplus \tilde{G}^i) \subset G(= \bigoplus G^i) \subset \bigoplus B^i. \]
be two finite sets satisfying that if \( Y_i \) is one of \( \{ T_{II,k} \}_{k=1}^{\infty}, \{ T_{III,k} \}_{k=1}^{\infty} \) and \( S^2 \), then \( \omega(\tilde{G}^i) < \varepsilon \), and if \( Y_i \) is one of \( \{ pt \}, [0,1], S^1 \), then \( \tilde{G}^i = G^i \). Then there is a subset \( G_1 \subseteq B \) with \( G_1 \supseteq G(\mathcal{P}) \), \( \mathcal{P} \subseteq \bigcup_{k=2}^{\infty} M_k(B \otimes C(W_k \times S^1)) \) as in the end of 3.1) and \( \delta_1 > 0 \) with \( \delta_1 < \delta(\mathcal{P}) \), and a positive integer \( L > 0 \) such that the following is true.

If a map \( \alpha = \alpha_0 \oplus \alpha_1 : B \longrightarrow A = \bigoplus_{j=1}^{t} M_{k_j}(C(X_j)) \), where \( X_j \) are connected finite simplicial complexes with \( \dim(X_j) \leq M \), satisfying the following conditions:

1. \( \alpha_0 \) is \( G_1 - \delta_1 \) multiplicative, \( \{ \alpha_0(1_{B^i}) \}_{i=1}^{s} \) are mutually orthogonal projections, and for any block \( B^i \) with \( Y_i = T_{II,k}, T_{III,k}, S^2 \) and any block \( A^j \), the partial map \( \alpha_0^{ij} \) is quasi-\( \mathcal{P}K \)-homomorphism; and

2. \( \alpha_1 \) is a homomorphism defined by point evaluations and for each block \( B^i \), with \( Y_i = T_{II,k}, T_{III,k}, S^2 \) and any block \( A^j \),

\[
\alpha_1^{ij}(1_{B^i}) \geq L \alpha_0^{ij}(1_{B^i}),
\]

then there is a unital homomorphism \( \alpha' : B \longrightarrow \alpha(1_B)A\alpha(1_B) \) such that

\[
\| \alpha'(g) - \alpha(g) \| < 3\varepsilon \quad \forall g \in \tilde{G}.
\]

**Proof.** Without loss of generality, we can assume \( B \) has only one block: \( B = M_l(C(Y)) \). If \( Y = \{ pt \}, [0,1], S^1 \), the Lemma is true since \( B \) is stably generated (see Lemma 1.6.1 of [G4]).

Suppose that \( Y = T_{II,k}, T_{III,k} \) or \( S^2 \). Let \( G_1 \) and \( \delta_1 \), and \( \eta \) be as Lemma 5.30 of [G4] (note that, for a positive integer \( L \) and positive number \( \eta \), the notion \( PE(L, \eta) \) (used in Lemma 5.30 of [G4]) is defined in Definition 4.38 of [G4]). Let \( \{ y_j \}_{j=1}^{K} \) be an \( \eta \)-dense subset of \( Y \). Let \( L = 4KM \). Let us verify the conclusion holds for such a choice. If \( \alpha_0 = 0 \), then one can choose \( \alpha' = \alpha_1 \) as desired. Suppose \( \alpha_0 \neq 0 \) and therefore \( rank \alpha_0(1_B) > 0 \). Using \( rank \alpha_1(1_B) \geq L \eta \cdot rank(\alpha_0(1_B)) \), there is a unital homomorphism \( \alpha'_1 : B \longrightarrow \alpha_1(1_B)A\alpha_1(1_B) \) which satisfies the following conditions

(i) \( \alpha'_1 \) is homotopy to \( \alpha_1 \)

(ii) \( \alpha'_1 \) is defined by direct sum of point evaluations at different points

(iii) all the points in the \( \eta \)-dense set \( \{ y_j \}_{j=1}^{K} \) are among those point evaluations that define \( \alpha'_1 \) and each point evaluation at \( y_i \) satisfies that the rank of the image of \( 1_B \) is at least rank \( \alpha_0(1_B) \)—that is \( \alpha'_1 \) has property \( PE(rank \alpha_0(1_B), \eta) \).

By 5.30 of [G4], there is a homomorphism \( \alpha'' : B \longrightarrow \alpha(1_B)A\alpha(1_B) \) such that

\[
\| \alpha''(f) - (\alpha_0 \oplus \alpha'_1(f)) \| < \varepsilon, \quad \forall f \in \tilde{G}.
\]

15
On the other hand by Lemma 3.5, there is a unitary $u \in \alpha'_1(1_B)A_1(1_B)$ (note that $\alpha'_1(1_B) = \alpha_1(1_B)$) such that
\[ \| u^* \alpha'_1(f)u - \alpha_1(f) \| < 2\varepsilon, \quad \forall f \in \tilde{G}. \]
Let $\alpha'(t) = \text{diag}(\alpha_0(1_B), u^* \alpha''(f) \text{diag}(\alpha_0(1_B), u)$. Evidently, we have
\[ \| \alpha'(f) - (\alpha_0 \oplus \alpha_1)(f) \| < 3\varepsilon, \quad \forall f \in \tilde{G}. \]

**Lemma 3.7.** Let $M$ be a fixed positive integer. Let $B = M_1(C(Y)), Y = T_{II,k}, T_{III,k}$ or $S^2$. Let the set of projection $P \subset M_1(B \otimes C(W_k \times S^1))$ be as in 3.1 (also see 5.16 of [G4]).

Let $A = RM_1(C(X))R$ with $X$ a connected finite simplicial complex and let $\alpha : B \to A$ be a homomorphism. Let $\mathcal{P}' \subset \bigcup_{k=2}^{\infty} M_1(A \otimes C(W_k \times S^1))$ be a set of projections (chosen for $A$) as in the end of 3.1 and $\mathcal{P}' \supseteq (\alpha \otimes \text{id})(\mathcal{P})$.

For any finite sets $\tilde{G}_1 \subseteq G_1 \subseteq B$ and numbers $\varepsilon_1 > 0, \delta_1 > 0$, with $\omega(\tilde{G}_1) < \varepsilon_1$, there are a finite set $G_2 \subseteq A$ a number $\delta_2 > 0$, and positive integer $L$, such that the following are true. Let $C = M_1(C(Z))$ with $Z$ connected simplicial complex and $\dim(Z) \leq M$, and let $Q_0, Q_1 \in C$ be two orthogonal projections.

(1) If $\psi_0 : A \to Q_0 C Q_0$ is $G_2 - \delta_2$ multiplicative quasi- $\mathcal{P}'K$-homomorphism and $\psi_0(\alpha(1_B))$ is a projection, then $\psi_0 \circ \alpha$ is a $G_1 - \delta_1$ multiplicative quasi- $\mathcal{P}K$-homomorphism.

(2) If $\psi_0$ as in (1) and $\psi_1 : A \to Q_1 C Q_1$ is defined by point evaluations with rank $(\psi_1(1_A)) \geq L\text{rank}(\psi_0(1_A))$, then there is a homomorphism $\psi : B \to (Q_0 \oplus Q_1)C(Q_0 \oplus Q_1)$ such that
\[ \| \psi(g) - (\psi_0 \oplus \psi_1)(\alpha(g)) \| < 3\varepsilon. \]

**Proof.** The proof is the same as the proof of Lemma 5.31 of [G4] using Lemma 3.6 above to replace 5.30 of [G4].

**Theorem 3.8.** Let $M > 3$ be a positive integer. Let $\lim_{\beta \to \infty}(A_n = \bigoplus_{i=1}^{k_n} M_{n,i}(C(X_{n,i})), \phi_{n,m})$ be an $AH$ inductive limit such that the limit algebra has the ideal property, where $X_{n,i}$ are connected simplicial complexes with $\dim(X_{n,i}) \leq M$, for all $n, i$. Let $B = \bigoplus_{i=1}^{t_n} M_{i}(C(Y_i))$, where $Y_i$ are spaces $\{pt\}, [0,1], S^1, T_{II,k}, T_{III,k}$, and $S^2$. Suppose that $\tilde{G} = \bigoplus G^i \subset G(= \bigoplus G^i) \subset B(= \bigoplus B^i)$ are two finite sets and $\varepsilon_1 > 0$ is a positive number with $\omega(\tilde{G}^i) < \varepsilon_1$, if $Y_i = T_{II,k}, T_{III,k}$, or $S^2$. Suppose that $L$ is any positive integer. Let $\alpha : B \to A_n$ be any
homomorphism. Denote $\alpha(1_B) := R(= \bigoplus R^i) \in A_n(= \bigoplus A_n^i)$. Let $F \subset RA_n R$ be any finite set and $\varepsilon < \varepsilon_1$ be any positive number.

It follows that there are $A_m$, and mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_m$ with $\phi_{n,m}(R) = Q_0 + Q_1 + Q_2$, a unital map $\theta_0 \in Map(RA_n R, Q_0 A_m Q_0)$, two unital homomorphisms $\theta_1 \in Hom(RA_n R, Q_1 A_m Q_1)$, $\xi \in Hom(RA_n R, Q_2 A_m Q_2)$ such that

(1) $\| \phi_{n,m}(f) - (\theta_0(f) \oplus \theta_1(f) \oplus \xi(f)) \| < \varepsilon \quad \forall f \in F$

(2) there is a homomorphism $\alpha : B \rightarrow (Q_0 + Q_1) A_m (Q_0 + Q_1)$ such that

$$\| \alpha(g) - (\theta_0 \oplus \theta_1) \circ \alpha(g) \| < 3\varepsilon_1 \quad \forall g \in \tilde{G^i}, \text{ if } Y_i = T_{I1,k}, T_{III,k} \text{ or } S^2$$

and

$$\| \alpha(g) - (\theta_0 \oplus \theta_1) \circ \alpha(g) \| < \varepsilon \quad \forall g \in G^i, \text{ if } Y_i = \{pt\}, [0,1] \text{ or } S^1.$$

(3) $\theta_0$ is $F - \varepsilon$ multiplicative and $\theta_1$ satisfies that for any nonzero projections $e \in R^i A_n^i R^i$

$$\theta_1^{ij}([e]) \geq L \cdot [\theta_0^{ij}(R^i)].$$

(4) $\xi$ factors through a $C^*$-algebra $C$—a direct sum of matrix algebras over $C[0,1]$ or $C$ as

$$\xi : RA_n R \xrightarrow{\xi_1} C \xrightarrow{\xi_2} Q_2 A_m Q_2.$$  

**Proof.** Let $D \subset A_n = \bigoplus A_n^i$ be defined by

$$D = \bigoplus_j \bigoplus_i \alpha^{ij}(C \cdot 1_{B^i}) \subset \bigoplus_j A_n^i$$

which is a finite dimensional subalgebra of $A_n$ containing the set of mutually orthogonal projections $\{E^{ij} = \alpha^{ij}(1_{B^i})\}_{i,j}$.

Apply Corollary 2.22 for sufficiently large set $F' \subset RA_n R$, sufficiently small number $\varepsilon' > 0$ and sufficiently large integer $J > 0$, to obtain $A_m$ and the decomposition $\theta_0 \oplus \theta_1 \oplus \xi$ of $\phi_{n,m} |_{RA_n R}$ as $\psi_0' \oplus \psi_1' \oplus \psi_2'$ in the corollary. By Lemma 1.6.8 of [G4], we can assume $\theta_0 |_D$ is a homomorphism. The condition (1) follows if we choose $F' \supset F$, and $\varepsilon' < \varepsilon$. The $F - \varepsilon$ multiplicative of $\theta_0$ in (3) follows from Lemma 4.40 of [G4], if $F'$ is large enough and $\varepsilon'$ is small enough; and property of $\theta_1$ in (3) follows if we choose $J > L$.

To construct $\alpha_1$ as desired in the condition (2), we need to construct

$$\alpha_1^{ij,k} : B^i \rightarrow \theta^{ij,k}(E^{ij}) A_m^k \theta^{ij,k}(E^{ij})$$
where \( \theta = \theta_0 \oplus \theta_1 \), to satisfy
\[
\| \alpha_1^{i,j,k}(g) - \theta^{i,k} \circ \alpha_1^{i,j}(g) \| < 3\varepsilon_1 \quad \forall g \in \widetilde{G}^i, \text{ if } Y_i = T_{II,k}, T_{III,k}, S^2
\]
and
\[
\| \alpha_1^{i,j,k}(g) - \theta^{i,k} \circ \alpha_1^{i,j}(g) \| < \varepsilon \quad \forall g \in G^i, \text{ if } Y_i = \{pt\}, [0, 1], S^1.
\]

For the case of \( Y_i = \{pt\}, [0, 1], S^1 \), the existence of \( \alpha_1^{i,j,k} \) follows from Lemma 1.6.1 and Lemma 4.40 of [G4]—that is \( \theta_0^{i,k} \circ \alpha_1^{i,j} \) itself can be perturbed to a homomorphism as \( B^i \) is stably generated.

For the case \( Y_i = T_{II,k}, T_{III,k}, S^2 \), the existence of \( \alpha_1^{i,j,k} \) follows from Lemma 3.7 and \( \omega(\widetilde{G}^i) < \varepsilon_1 \) provided \( J \) is large enough, \( F' \) large enough, \( \varepsilon' \) small enough.

\[ \square \]

**Theorem 3.9.** Let \( M \) be a positive integer. Let \( \lim_{n \to \infty} (A_n = \bigoplus_{i=1}^{b_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m}) \) be an AH inductive limit such that the limit algebra has the ideal property, where \( X_{n,i} \) are connected simplicial complexes with \( \text{dim}(X_{n,i}) \leq M \), for all \( n, i \). Let \( B = \bigoplus_{i=1}^{b_m} M_{[i,m]}(C(Y_i)) \), where \( Y_i \) are spaces \( \{pt\}, [0, 1], S^1, T_{II,k}, T_{III,k} \) and \( S^2 \). Suppose that \( \widetilde{G} (= \bigoplus \widetilde{G}^i) \subset G (= \bigoplus G^i \subset B (= \bigoplus B^i) \) are two finite subsets and \( \varepsilon_1 \) is a positive number such that \( \omega(\widetilde{G}^i) < \varepsilon_1 \), if \( Y_i \) is one of \( T_{II,k}, T_{III,k} \), or \( S^2 \). Let \( \alpha : B \to A_n \) be a homomorphism and \( F(\supset \alpha(G)) \) be a finite subset of \( A_n \) and \( \varepsilon < \varepsilon_1 \) be any positive number.

It follows that there are \( A_m \) and mutually orthogonal projections \( P, Q \in A_m \) with \( \phi_{n,m}(1_{A_n}) = P+Q \), a unital map \( \theta \in \text{Map}(A_n, PA_mP)_1 \), and a unital homomorphism \( \xi \in \text{Hom}(A_n, QA_mQ)_1 \) such that

1. \( \| \phi_{n,m}(f) - (\theta(f) \oplus \xi(f)) \| < \varepsilon \quad \forall f \in F \)
2. there is a homomorphism \( \alpha_1 : B \to PA_mP \) such that
   \[
   \| \alpha_1^{i,j}(g) - (\theta \circ \alpha)^{i,j}(g) \| < 3\varepsilon_1 \quad \forall g \in \widetilde{G}^i, \text{ if } Y_i = T_{II,k}, T_{III,k}, S^2,
   \]
   \[
   \| \alpha_1^{i,j}(g) - (\theta \circ \alpha)^{i,j}(g) \| < \varepsilon \quad \forall g \in G^i, \text{ if } Y_i = \{pt\}, [0, 1], S^1.
   \]
3. \( \omega(\theta(F)) < \varepsilon \) and \( \theta \) is \( F - \varepsilon \) multiplicative.
4. \( \xi \) factors through a \( C^* \)-algebra \( C \)—a direct sum of matrix algebras over \( C[0, 1] \) or \( C \) as
   \[
   \xi : A_n \xrightarrow{\xi_1} C \xrightarrow{\xi_2} QA_mQ.
   \]

The proof is similar to but slightly easier than that of Theorem 3.8, and we omit it.

18
The following is Corollary 1.6.15 of [G4].

**Proposition 3.10.** Let \( A = \bigoplus_{k=1}^{t} M_{s(k)}(C(X_k)) \), where \( X_k \) are connected finite simplicial complexes and \( s(k) \) are positive integers. Let \( F \subset A \) be a finite set and \( \varepsilon > 0 \). There are a finite set \( G \subset A \) and a number \( \delta > 0 \) with the following property. If \( B \) is a unital \( C^* \)-algebra, \( p \in B \) is a projection, \( \phi_t : A \rightarrow pBp, 0 \leq t \leq 1 \) is a continuous path of \( G - \delta \) multiplicative maps, then there are a positive integer \( L \), and \( \eta > 0 \) such that for a homomorphism \( \lambda : A \rightarrow B \otimes \mathcal{K} \) (with finite dimensional image), there is a unitary \( u \in (p \oplus \lambda(1))(B \otimes \mathcal{K})(p \oplus \lambda(1)) \) satisfying
\[
\| \phi_0(f) \oplus \lambda(f) - u(\phi_1(f) \oplus \lambda(f))u^* \| < \varepsilon, \quad \forall f \in F,
\]
provided that \( \lambda \) is of the following form: there are an \( \eta \)-dense subset \( \{x_1, x_2, \ldots, x_\bullet\} \subset \Pi_{k=1}^{t} X_k(= Sp(A)) \), and a set of mutually orthogonal projections
\[
\{p_1, p_2, \ldots, p_\bullet\} \subset \lambda(\bigoplus_{k}^{1} e_{11}^{k})(B \otimes \mathcal{K})\lambda(\bigoplus_{k}^{1} e_{11}^{k})
\]
with \( [p_i] \geq L \cdot [p] \), such that
\[
\lambda(f) = \sum_{i=1}^{n} p_i \otimes f(x_i) \oplus \lambda_1(f), \quad \forall f \in A,
\]
where \( \lambda_1 \) is also a homomorphism, under the identification
\[
\lambda(1_{A^k})B\lambda(1_{A^k}) \cong \lambda(e_{11}^{k})B\lambda(e_{11}^{k}) \otimes M_{s(k)}(\mathbb{C}).
\]

**Corollary 3.11.** Let \( A = \bigoplus_{k=1}^{t} M_{s(k)}(C(X_k)) \), where \( X_k \) are connected finite simplicial complexes and \( s(k) \) are positive integers. Let \( \varepsilon > 0 \) and \( F \subset A \) be a finite set with \( \omega(F) < \varepsilon \). There are a finite set \( G \subset A \) and a number \( \delta > 0 \) with the following property. If \( B = \bigoplus_{j=1}^{r} M_{r(j)}(C(Z_j)) \), where \( Z_j \) are connected finite simplicial complexes and \( p \in B \) is a projection, \( \phi_t : A \rightarrow pBp, 0 \leq t \leq 1 \), is a continuous path of \( G - \delta \) multiplicative maps, then there is a positive integer \( L \) such that for a homomorphism \( \lambda : A \rightarrow B \otimes \mathcal{K} \) with finite dimensional image, there is a unitary \( u \in (p \oplus \lambda(1))(B \otimes \mathcal{K})(p \oplus \lambda(1)) \) satisfying
\[
\| \phi_0(f) \oplus \lambda(f) - u(\phi_1(f) \oplus \lambda(f))u^* \| < 5\varepsilon, \quad \forall f \in F,
\]
provided \( \lambda^k = \lambda |_{A^k} : M_{s(k)}(C(X_k)) \rightarrow B \otimes \mathcal{K} \) has finite dimensioned image (or equivalently, is defined by point evaluation) with
\[
[\lambda^k(1_{A^k})] \geq L \cdot [p] \in K_0(B).
\]
Lemma 3.5. Let $L'$ and $\eta$ be $L$ and $\eta$ as in Proposition 3.10. Let $\{x_1, x_2, \cdots, x_K\} \subset \bigoplus_{k=1}^j X_k$ be a $\eta$-dense subset. Choose $L = 4L' \cdot K \max(dim(Z_j) + 1)$. If $\lambda$ satisfies the condition in our corollary then it is easy to find $\lambda' : A \longrightarrow \lambda(1)(B \otimes K)\lambda(1)$ satisfies the condition in Proposition 3.10 and $K_0\lambda = K_0\lambda'$. Then the corollary follows from Proposition 3.10 and Lemma 3.5.

\[\square\]

Theorem 3.12. Let $B_1 = \bigoplus_{j=1}^s M_{k(j)}(C(Y_j))$, where $Y_j$ are spaces $\{pt\}$, $[0,1]$, $S^1$, $\{T_{II,k}\}_{k=2}^\infty$, $\{T_{III,k}\}_{k=2}^\infty$ and $S^2$. Let $X$ be a connected finite simplicial complex and let $A = M_N(C(X))$. Let $\tilde{G}_1(= \bigoplus \tilde{G}_1^i) \subset G_1(= \bigoplus G_1^i) \subset B_1(= \bigoplus B_1^i)$ be two finite sets with $\omega(\tilde{G}_1^i) < \varepsilon_1$ for certain $\varepsilon_1 > 0$ and any $i$ with $Y_i$ being $T_{II,k}$, $T_{III,k}$ or $S^2$. Let $\alpha_1 : B_1 \rightarrow A$ be a homomorphism, and let $F_1 \subset A$ be a finite set and any positive number $\varepsilon < \varepsilon_1$ and $\delta > 0$. Then there exists a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & A' \\
\downarrow{\alpha_1} & & \downarrow{\alpha_2} \\
B_1 & \xrightarrow{\psi} & B_2
\end{array}
\]

where $A' = M_L(A)$, and $B_2$ is a direct sum of matrix algebras over space: $\{pt\}$, $[0,1]$, $S^1$, $T_{II,k}$, $T_{III,k}$ and $S^2$, with the following conditions

1. $\psi$ is a homomorphism, $\alpha_2$ is a unital injective homomorphism and $\phi$ is a unital simple embedding (see 2.23 for the definition of unital simple embedding);
2. $\beta \in \text{Map}(A, B_2)_1$ is $F_1 - \delta$ multiplicative;
3. $\|\psi(g) - \beta \circ \alpha_1(g)\| < 5\varepsilon_1$ $\forall g \in \tilde{G}_1^i$, if $Y_i = T_{II,k}, T_{III,k}, S^2$,
4. $\|\psi(g) - \beta \circ \alpha_1(g)\| < \varepsilon$ $\forall g \in G_1^i$, $\text{if} \ Y_i = \{pt\}, [0,1], S^1$,

and
5. $\|\phi(f) - \alpha_2 \circ \beta(f)\| < \varepsilon$ $\forall f \in F_1$; and

$\beta(F_1) \cup \psi(G_1) \subset B_2$ satisfies that $\omega(\beta(F_1) \cup \psi(G_1)) < \varepsilon$.

\[\textbf{Proof.}\] The proof is similar to the proof of Theorem 1.6.26 (see 1.6.25) of [G4]. The differences are the following. First, we do not have the condition that $\alpha_1$ is injective, so we need to use Corollary 3.11 to deal with blocks $B_1^i$ with $Y_i = T_{II,k}, T_{III,k}$ or $S^2$. For those block $B_1^i$ with $Y_i = \{pt\}, [0,1], S^1$, we use the condition that $B_1^i$ is stably generated (see [G4] 1.6.1). Secondly, we need to make the condition (4) hold.

20
Without loss of generality, we can assume \( \alpha_1(B_i^1) \neq 0 \) for each block \( B_i^1 \), otherwise we simply take \( \psi \) to be zero on this block.

Apply Corollary 3.11 to \( \tilde{G}_i^1 \subset B_i^1 \) for blocks \( Y_i = T_{II,k}, T_{III,k} \) or \( S^2 \), to obtain \( G_i \subset B_i^1 \) and \( \delta_1 \) as in the corollary. Apply Lemma 1.6.1 of [G4] to \( G_i \subset B_i^1 \) for \( Y_i = \{pt\}, [0,1], S^1 \), to obtain \( G_i \) and \( \delta_i' \). We assume \( G_i \supset G_i^1 \) and \( \delta_1 \) as in the corollary. Apply Lemma 1.6.1 of [G4] to \( G_i^1 \subset B_i^1 \) for \( Y_i = \{pt\}, [0,1], S^1 \), to obtain \( G_i \) and \( \delta_i' \). We assume \( G_i \supset G_i^1 \) and \( \delta_1 \) as in the corollary. Apply Proposition 3.10 to \( F_1' \cup F_1 \) (as \( F \subset A \)) and \( \min(\delta_2, \delta) \) (as \( \varepsilon > 0 \)) to obtain \( F \subset A \) and \( \delta_3 \) (in place of the set \( G \) and the number \( \delta \) there). We can assume \( \delta_3 < \min(\delta_2, \delta, \delta_1, \delta_i') \) and \( F \supset F_1' \cup F_1 \).

Apply 1.6.24 of [G4] to obtain the following diagram

\[
\begin{array}{ccc}
A & \overset{\phi}{\longrightarrow} & A' \\
\alpha_1 & \downarrow{\beta} & \alpha_2 \\
B_1 & \overset{\psi}{\rightarrow} & B_2
\end{array}
\]

where \( A' = M_L(A) \), and \( B_2 \) is a direct sum of matrix algebras over the spaces \( \{pt\}, [0,1], S^1 \), \( \{T_{II,k}\}, \{T_{III,k}\} \) and \( S^2 \) with the following conditions:

(i) \( \psi \) is homomorphism, \( \alpha_2 \) is a unital injective homomorphism and \( \phi \) is a unital simple embedding

(ii) \( \beta \in \text{Map}(A, B_2) \) is \( F - \delta_3 \) multiplicative and therefor \( \beta \circ \alpha_1 \) is \( G - \min(\delta_1, \delta_i') \) multiplicative

(iii) there exist two homotopies \( \Psi \in \text{Map}(B_1, B_2[0,1]) \) and \( \Phi \in \text{Map}(A, A'[0,1]) \) such that \( \Psi \) is \( G - \delta_3 \) multiplicative and \( \Phi \) is \( F - \delta_3 \) multiplicative.

Note that we can choose a simple embedding \( \xi : B_2 \to M_k(B_2) \) such that \( \omega(\xi(\beta(F) \cup \psi(G))) < \varepsilon \) by adding to the identity the homomorphism defined by point evaluations at all points in a sufficiently dense finite set. Let \( \xi' : A' \to M_k(A') \) be any simple embedding. Since \( \alpha_2 \) take trivial projections to trivial projections (see Remark 1.6.20 of [G4]), we know that

\[
\begin{array}{ccc}
A' & \overset{\xi'}{\rightarrow} & M_k(A') \\
\alpha_2 & \downarrow{\alpha_2 \circ \text{id}_k} & \\
B_2 & \overset{\xi}{\rightarrow} & M_k(B_2)
\end{array}
\]
commutes up to homotopy. Therefor replacing $\beta$, $\phi$, $\psi$, and $\alpha_2$, by $\xi \circ \beta$, $\xi' \circ \phi$, $\xi \circ \psi$, and $\alpha_2 \otimes id_k$, respectively, we can assume our original diagram (I) also satisfies

(iv) $\omega(\beta(F) \cup \psi(G)) < \varepsilon$.

Without loss of generality, we can assume $B_1$ has only one block $Y = T_{II,k}, T_{III,k}$ or $S^2$. Regarding the homotopy $\Psi$ as the homotopy path $\phi_1$ in Corollary 3.11, we can obtain $L_1$ as the number $L$ in Corollary 3.11. Similarly regarding the homotopy $\Phi$ as $\phi_1$ in Proposition 3.10 we obtain $L_2$ (as $L$) and $\eta$.

Choose $\{x_1, x_2, \cdots, x_m\} \subset X$ to be a $\eta$-dense subset. Define

$$\lambda_1 : A(= M_N(C(X))) \to M_nM(B_2)$$

by

$$\lambda_1(f) = \text{diag}(1_{B_2} \otimes f(x_1), 1_{B_2} \otimes f(x_2), \cdots, 1_{B_2} \otimes f(x_m))$$

Let $L' = \max\{L_1, L_2\}$ and $n = mNL'$. Define

$$\lambda : A \to M_n(B_2) = M_{L'}(M_nM(B_2)) \text{ by } \lambda = \text{diag}(\lambda_{11}, \lambda_{12}, \cdots, \lambda_{1L'})$$

Then $\lambda \circ \alpha_1 : B_1 \to M_n(B_2)$ satisfies the condition for $\lambda$ in Corollary 3.11 for the homotopy $\Psi$ and the positive integer $L_1$. Also $(\alpha_2 \otimes id_n) \circ \lambda : A \to M_n(A')$ satisfies the condition for $\lambda$ in Proposition 3.10 for the homotopy $\Phi, L_2$ and $\eta$. Therefor there are $u_1 \in M_{n+1}(B_2)$ and $u_2 \in M_{n+1}(A')$ such that

$$\|((\beta \oplus \lambda) \circ \alpha_1(g) - u_1((\psi \oplus \lambda \circ \alpha_1)(g)))u_1^*\| \leq 5\varepsilon_1, \quad \forall g \in \tilde{G}_1 \quad (*)$$

and

$$\|((\phi \oplus ((\alpha_2 \otimes id_n) \circ \lambda))(f) - u_2((\alpha_2 \otimes id_{n+1}) \circ (\beta \oplus \lambda))(f)))u_2^*\| < \varepsilon, \quad \forall f \in F_1$$

Note that if $Y_i = \{pt\}, [0, 1]$ or $S^1$, then $\beta \circ \alpha_1$ itself is close to a homomorphism $\psi' : B_1 \to B_2$. That is, we can replace the above $(*)$ by

$$\|\beta \circ \alpha_1(g) - \psi'(g)\| < \varepsilon, \quad \forall g \in G_1. \quad (**)$$

In the diagram (I) if we replace $B_2$ by $M_{n+1}(B_2)$, $A'$ by $M_{n+1}(A')$, $\psi$ by $Adu_1 \circ (\psi \oplus \lambda \circ \alpha_1)$ (or $\psi$ by $\psi' \oplus (\lambda \circ \alpha_1)$ for the case $Y_i = \{pt\}, [0, 1], S^1$ using $(**)$), $\beta$ by $\beta \oplus \lambda, \alpha_2$ by $Adu_2 \circ (\alpha_2 \otimes id_{n+1})$ and finally $\phi$ by $\phi \oplus ((\alpha_2 \otimes id_n) \circ \lambda)$, we get the desired diagram. \qed
3.13 Recall that in 1.1.7(h) of [G4], for \( A = \bigoplus_{i=1}^{t} M_{k_{i}}(C(X_{i})) \), where \( X_{i} \) are path connected simplicial complexes, we used the notation \( r(A) \) to denote \( \bigoplus_{i=1}^{t} M_{k_{i}}(\mathbb{C}) \), which could be considered to be the subalgebra consisting of \( t \)-tuples of constant functions from \( X_{i} \) to \( M_{k_{i}}(\mathbb{C})(i = 1, 2, \ldots, t) \). Fixed a base point \( x_{i}^{0} \in X_{i} \) for each \( X_{i} \), one defines a map \( r : A \to r(A) \) by

\[
    r(f_{1}, f_{2}, \ldots, f_{t}) = (f_{1}(x_{i}^{0}), f_{2}(x_{i}^{0}), \ldots, f_{t}(x_{i}^{0})) \in r(A)
\]

We have the following corollary.

**Corollary 3.14** Let \( B_{1} = \bigoplus_{j=1}^{s} M_{k(j)}(C(Y_{j})) \), where \( Y_{j} \) are the spaces \( \{pt\}, [0, 1], S^{1}, \{T_{II,k}\}_{k}, \{T_{III,k}\}_{k} \) and \( S^{2} \). Let \( A = \bigoplus_{j=1}^{t} M_{l(j)}(C(X_{j})) \), where \( X_{j} \) are connected simplicial complexes. Let \( \alpha_{1} : B_{1} \to A \) be any homomorphism. Let \( \varepsilon_{1} > \varepsilon_{2} > 0 \) be any two positive numbers. Let \( \tilde{E} (= \bigoplus \tilde{E}^{i}) \subset E (= \bigoplus E^{i}) \subset B_{1}(= \bigoplus B_{1}^{i}) \) be two finite subsets with the condition

\[
    \omega(\tilde{E}^{i}) < \varepsilon_{1} \text{ for all } Y_{i} = T_{II,k}, T_{III,k} \text{ or } S^{2}.
\]

Let \( F \subset A \) be any finite subset, \( \delta > 0 \). Then there exists a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi \oplus r} & A' \oplus r(A) \\
\downarrow{\alpha_{1}} & & \downarrow{\alpha_{2} \oplus \text{id}} \\
B_{1} & \xleftarrow{\psi \oplus (r \circ \alpha_{1})} & B_{2} \oplus r(A)
\end{array}
\]

where \( A' = M_{L}(A), B_{2} \) is a direct sum of matrix algebras over the spaces \( \{pt\}, [0, 1], S^{1}, \{T_{II,k}\}, \{T_{III,k}\} \) and \( S^{2} \), with the following properties.

1. \( \psi \) is a homomorphism, \( \alpha_{2} \) is injective homomorphism and \( \phi \) is a unital simple embedding (see 2.23).
2. \( \beta \in \text{Map}(A, B_{2}) \) is \( F - \delta \) multiplicative.
3. For \( g \in \tilde{E}^{i} \) with \( Y_{i} = T_{II,k}, T_{III,k} \) or \( S^{2} \),

\[
    \|(\beta \oplus r)(\alpha_{1}(g)) - (\psi \oplus (r \circ \alpha_{1}))(g)\| < 5\varepsilon_{1};
\]

for \( g \in E^{i}(\supset \tilde{E}^{i}) \) with \( Y_{i} = \{pt\}, [0, 1], S^{1}, \)

\[
    \|(\beta \oplus r)(\alpha_{1}(g)) - (\psi \oplus r \circ \alpha_{1})(g)\| < \varepsilon_{1};
\]

for all \( f \in F, \)

\[
    \|(\alpha_{2} \oplus \text{id}) \circ (\beta \oplus r)(f) - (\phi \oplus r)(f)\| < \varepsilon_{1}.
\]

4. \( \omega(\beta(F) \cup \psi(E)) < \varepsilon_{2}. \)
The following lemma will be used in the proof of our main theorem.

**Lemma 3.15** Let \( \lim(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]} C(X_{n,i}), \phi_{n,m}) \) be an AH inductive system for which the limit \( C^* \)-algebra has the ideal property, with \( X_{n,i} \) connected simplicial complexes and with \( \sup \dim(X_{n,i}) < +\infty \). Let \( P_0 \in A_n \) be a trivial projection. Let \( A = P_0 A_n P_0 = \bigoplus_{i} P_0^n_i A_i^n P_0^n \). Let \( A' = M_L(A), r(A) \) be as in 3.13 and 3.14 with \( r : A \to r(A) \). Let \( \phi : A \to A' \) be a unital simple embedding (see 2.23). Then there exist \( m > n \), and a unital homomorphism \( \lambda : M_L(A) \oplus r(A) \to \phi_{n,m}(P_0)A_m \phi_{n,m}(P_0) \) such that \( \lambda \circ (\phi \oplus r) \) is homotopic to \( \phi_{n,m}|_{P_0 A_n P_0} : P_0 A_n P_0 \to \phi_{n,m}(P_0)A_m \phi_{n,m}(P_0) \). That is the following diagram commutes up to homotopy

\[
\begin{array}{ccc}
A(= P_0 A_n P_0) & \xrightarrow{\phi_{n,m}} & \phi_{n,m}(P_0)A_m \phi_{n,m}(P_0) \\
& \phi \oplus r & \\
& \lambda & \\
& M_L(A) \oplus r(A) \\
\end{array}
\]

**Proof.** One can construct the maps inside each block for each \( \phi_{n,m}^i,j : P_0^i \phi_{n,m}^i,j P_0^i \to \phi_{n,m}^i,j(P_0^i)A_m \phi_{n,m}^i,j(P_0^i) \). Then the lemma follows from Lemma 1.6.30 of [G4] and Lemma 2.20. Note that we use the fact that if \( \xi : M_\bullet(C(X)) \to B \) has finite dimensional image, then there is a map \( \xi' : r(M_\bullet C(X)) \to B \) such that \( \xi' \circ r \) is homotopic to \( \xi \).

\[\square\]

§4. The proof of the main theorem

The following lemma is one of the versions of the Elliott intertwining argument (see [Ell1]) or Proposition 3.1 of [D].

**Proposition 4.1** Consider the diagram

\[
\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\phi_{1,2}} & A_2 & \xrightarrow{\phi_{2,3}} & A_3 & \cdots & A_n & \xrightarrow{\phi_{n,n+1}} & \cdots \\
\downarrow{\alpha_1} & & \downarrow{\beta_1} & & \downarrow{\alpha_2} & & \downarrow{\beta_2} & & \downarrow{\alpha_n} & & \downarrow{\beta_n} \\
B_1 & \xrightarrow{\psi_{1,2}} & B_2 & \xrightarrow{\psi_{2,3}} & B_3 & \cdots & B_n & \xrightarrow{\psi_{n,n+1}} & \cdots \\
\end{array}
\]
where $A_n, B_n$ are $C^*$-algebras $\phi_{n,n+1}, \psi_{n,n+1}$ are homomorphisms and $\alpha_n, \beta_n$ are linear $*$-contraction. Suppose that $F_n \subset A_n, \tilde{E}_n \subset E_n \subset B_n$ are finite sets satisfying the following condition
\[
\phi_{n,n+1}(F_n) \cup \alpha_{n+1}(E_{n+1}) \subset F_{n+1}, \psi_{n,n+1}(E_n) \cup \beta_n(F_n) \subset \tilde{E}_{n+1}
\]
and $\bigcup_{n=1}^{\infty} \phi_{n,\infty}(F_n)$ and $\bigcup_{n=1}^{\infty} \psi_{n,\infty}(E_n)$ are the unit balls of $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$, respectively. Suppose that there is a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of positive numbers with $\sum \varepsilon_n < +\infty$ such that $\alpha_n$ and $\beta_n$ are $E_n - \varepsilon_n$ and $F_n - \varepsilon_n$ multiplicative, respectively, and
\[
\|\phi_{n,n+1}(f) - \alpha_{n+1} \circ \beta_n(f)\| < \varepsilon_n,
\]
and
\[
\|\psi_{n,n+1}(g) - \beta_n \circ \alpha_n(g)\| < \varepsilon_n,
\]
for all $f \in F_n$ and $g \in \tilde{E}_n$. Then $A$ is isomorphic to $B$.

The following is the main theorem of this article.

**Theorem 4.2** Suppose that $\lim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}C(X_{n,i}), \phi_{n,m})$ is an $AH$ inductive limit with $\text{dim}(X_{n,i}) \leq M$ for a fixed positive integer $M$ such that the limit algebra has the ideal property. Then there is another inductive limit system $(B_n = \bigoplus_{i=1}^{s_n} M_{(n,i)}C(Y_{n,i}), \psi_{n,m})$ with the same limit algebra as the above system, where all the $Y_{n,i}$ are spaces of the form $\{pt\}, [0, 1], S^1, S^2, T_{II,k}, T_{III,k}$.

**Proof.** Without loss of generality, assume that the spaces $X_{n,i}$ are connected finite simplicial complexes and $\phi_{n,m}$ are injective (see [EGL2]).

Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > 0$ be a sequence of positive numbers satisfying $\sum \varepsilon_n < +\infty$. We
need to construct the intertwining diagram

\[
\begin{array}{ccccccc}
F_1 & F_2 & \cdots & F_{n-1} & F_n & F_{n+1} \\
\cap & \cap & \cdots & \cap & \cap & \cap \\
A_{s(1)} \alpha_1 \downarrow & A_{s(2)} \beta_1 \psi_{1,2} \alpha_2 \downarrow & \cdots & A_{s(n)} \beta_n \psi_{n,n+1} \alpha_{n+1} \downarrow & B_n & B_{n+1} \\
B_1 & B_2 & \cdots & B_{n-1} & B_n & \cdots \\
E_1 & E_2 & \cdots & E_{n-1} & E_n & \cdots \\
\tilde{E}_1 & \tilde{E}_2 & \cdots & \tilde{E}_{n-1} & \tilde{E}_n & \cdots
\end{array}
\]

satisfying the following conditions

(0.1) \((A_{s(n)}, \phi(s(n),s(m)))\) is a subinductive system of \((A_n, \phi_n, m)\), and \((B_n, \psi_n, m)\) is an inductive system of matrix algebras over the spaces \(\{pt\}, \{0, 1\}, S^1, T_{II,k}, T_{III,k}, S^2\).

(0.2) Choose \(\{a_{ij}\}_{j=1}^\infty \subset A_{s(i)}\) and \(\{b_{ij}\}_{j=1}^\infty \subset B_i\) to be countable dense subsets of the unit balls of \(A_{s(i)}\) and \(B_i\), respectively. \(F_n\) are subsets of the unit balls of \(A_{s(n)}\), and \(\tilde{E}_n \subset E_n\) are both subsets of the unit balls of \(B_n\) satisfying

\[
\phi_{s(n),s(n+1)}(F_n) \cup \alpha_{n+1}(E_{n+1}) \cup \bigcup_{i=1}^{k+1} \phi_{s(i),s(n+1)}(\{a_{i1}, a_{i2} \cdots a_{in+1}\}) \subset F_{n+1}
\]

\[
\psi_{n,n+1}(E_n) \cup \beta_n(F_n) \subset \tilde{E}_{n+1} \subset E_{n+1}
\]

and

\[
\bigcup_{i=1}^{n+1} \psi_{i,n+1}(\{b_{i1}, b_{i2} \cdots b_{in+1}\}) \subset E_{n+1}
\]

(0.3) \(\beta_n\) are \(F_n - 2\varepsilon_n\) multiplicative and \(\alpha_n\) are homomorphisms

(0.4) \(\|\psi_{n,n+1}(g) - \beta_n \circ \alpha_n(g)\| < 8\varepsilon_n\) for all \(g \in \tilde{E}_n\) and \(\|\phi_{s(n),s(n+1)}(f) - \alpha_{n+1} \circ \beta_n(f)\| < 14\varepsilon_n\) for all \(f \in F_n\)

(0.5) For any block \(B^i_n\) with spectrum \(T_{II,k}, T_{III,k}, S^2; \omega(\tilde{E}^i_n) < \varepsilon_n\), where \(\tilde{E}^i_n = \pi_i(\tilde{E}_n)\) for \(\pi_i : B_n \rightarrow B^i_n\) the canonical projections.

The diagram will be constructed inductively. First, let \(B_1 = \{0\}, A_{s(1)} = A_1, \alpha_1 = 0\). Let \(b_{1j} = 0 \in B_1\) for \(j = 1, 2, \ldots\) and let \(\{a_{1j}\}_{j=1}^\infty\) be a countable dense subset of the unit ball of \(A_{s(1)}\). And let \(\tilde{E}_1 = E_1 = \{b_{11}\} = B_1\) and \(F_1 = \bigoplus_{i=1}^{q_1} F^i_1\), where \(F^i_1 = \pi_i(\{a_{11}\}) \subset A^i_1\). As inductive assumption, assume that we already have the diagram

\[
\begin{array}{ccccccc}
F_1 & F_2 & \cdots & F_{n-1} & F_n & F_{n+1} \\
\cap & \cap & \cdots & \cap & \cap & \cap \\
A_{s(1)} \alpha_1 \downarrow & A_{s(2)} \beta_1 \psi_{1,2} \alpha_2 \downarrow & \cdots & A_{s(n)} \beta_n \psi_{n,n+1} \alpha_{n+1} \downarrow & B_n & B_{n+1} \\
B_1 & B_2 & \cdots & B_{n-1} & B_n & \cdots \\
E_1 & E_2 & \cdots & E_{n-1} & E_n & \cdots \\
\tilde{E}_1 & \tilde{E}_2 & \cdots & \tilde{E}_{n-1} & \tilde{E}_n & \cdots
\end{array}
\]

As inductive assumption, assume that we already have the diagram
and for each \( i = 1, 2, \ldots, n \), we have dense subsets \( \{a_{ij}\}_{j=1}^{\infty} \subset \text{the unit ball of } A_{s(i)} \) and \( \{b_{ij}\}_{j=1}^{\infty} \subset \text{the unit ball of } B_i \) satisfying the conditions (0.1)-(0.5) above. We have to construct the next piece of the diagram

\[
\begin{array}{cccccc}
F_1 & F_2 & \cdots & F_n \\
\cap & \cap & \cdots & \cap \\
A_{s(1)} & A_{s(2)} & \cdots & A_{s(n)} \\
\downarrow & \downarrow & \cdots & \downarrow \\
B_1 & B_2 & \cdots & B_n \\
\cup & \cup & \cdots & \cup \\
E_1 & E_2 & \cdots & E_n \\
\cap & \cap & \cdots & \cap \\
\tilde{E}_1 & \tilde{E}_2 & \cdots & \tilde{E}_n \\
\end{array}
\]

\( F_n \subset A_{s(n)} \xrightarrow{\phi_{s(n),s(n+1)}} A_{s(n+1)} \supset F_{n+1} \)

\( \tilde{E}_n \subset E_n \subset B_n \xrightarrow{\psi_{s,n+1}} B_{n+1} \supset E_{n+1} \supset \tilde{E}_{n+1} \)

to satisfy the conditions (0.1)-(0.5).

Among the conditions for the induction assumption, we will only use the conditions that \( \alpha_n \) is a homomorphism and (0.5) above.

**Step 1.** We enlarge \( \tilde{E}_n \) to \( \bigoplus_i \pi_i(\tilde{E}_n) \) and \( E_n \) to \( \bigoplus_i \pi_i(E_n) \). Then

\( \tilde{E}_n(= \bigoplus_i \tilde{E}_i) \subset E_n(= \bigoplus_i E_i) \)

and for each \( B_i \) with spectrum \( T_{II,k}, T_{III,k}, S^2 \), we have \( \omega(\tilde{E}_i) < \varepsilon_n \) from the induction assumption (0.5). By Theorem 3.9 applied to \( \alpha_n : B_n \rightarrow A_{s(n)} \), \( \tilde{E}_n \subset E_n \subset B_n \) and \( F_n \subset A_{s(n)} \) and \( \varepsilon_n > 0 \), there are \( A_{m_1} \) with \( \phi_{s(n),m_1}(1_{A_{s(n)}}) = P_0 + P_1 \) and \( P_0 \) trivial, a \( C^* \)-algebra \( C \) — a direct sum of matrix algebras over \( C[0,1] \) or \( C \), a unital map \( \theta \in \text{Map}(A_{s(n)}, P_0 A_{m_1} P_0) \), a unital homomorphism \( \xi_1 \in \text{Hom}(A_{s(n)}, C)_1 \), a unital homomorphism \( \xi_2 \in \text{Hom}(C, P_1 A_{m_1} P_1)_1 \) and a homomorphism \( \alpha \in \text{Hom}(B_n, P_0 A_{m_1} P_0) \) such that

(1.1) \( \| \phi_{s(n),m_1}(f) - \theta(f) \oplus (\xi_2 \circ \xi_1)(f) \| < \varepsilon_n \) for all \( f \in F_n \).

(1.2) \( \theta \) is \( F_n - \varepsilon_n \) multiplicative and \( F := \theta(F_n) \) satisfies \( \omega(F) < \varepsilon_n \).

(1.3) \( \| \alpha(g) - \theta \circ \alpha_n(g) \| < 3\varepsilon_n \) for all \( g \in \tilde{E}_n \).
Let all the blocks of $C$ be parts of the $C^*$-algebra $B_{n+1}$. That is

$$B_{n+1} = C \oplus \text{(some other blocks)}.$$  

The map $\beta_n : A_{s(n)} \to B_{n+1}$, and the homomorphism $\psi_{n,n+1} : B_n \to B_{n+1}$ are defined by

$\beta_n = \xi_1 : A_{s(n)} \to C(\subset B_{n+1})$ and $\psi_{n,n+1} = \xi_1 \circ \alpha_n : B_n \to C(\subset B_{n+1})$ for the blocks of $C(\subset B_{n+1})$. For this part, $\beta_n$ is also a homomorphism.

**Step 2.** Let $A = P_0A_{m_1}P_0, \ F = \theta(F_n)$. Since $P_0$ is a trivial projection, $A \cong \mathbb{M}_i(C(X_{m_1,i}))$.

Let $r(A) := \bigoplus M_i(C)$ and $r : A \to r(A)$ be as in 3.13. Applying Corollary 3.14 to $\alpha : B_n \to A, \ E_n \subset E_n \subset B_n$ and $F \subset A$, we obtain the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi \oplus r} & M(A) \oplus r(A) \\
\downarrow{\alpha} & & \downarrow{\alpha'} \\
B_n & \xrightarrow{\psi} & B
\end{array}
\]

such that

(2.1) $B$ is a direct sum of matrix algebras over $\{pt\}, [0, 1], S^1, T_{II,k}, T_{III,k}$ or $S^2$.

(2.2) $\alpha'$ is an injective homomorphism and $\beta$ is $F - \varepsilon_n$ multiplicative.

(2.3) $\phi : A \to M(A)$ is a unital simple embedding and $r : A \to r(A)$ is as in 3.13.

(2.4) $\|\beta \circ \alpha(g) - \psi(g)\| < 5\varepsilon_n$ for all $g \in \widetilde{E}_n$ and $\|((\phi \oplus r)(f) - \alpha' \circ \beta(f))\| < \varepsilon_n$ for all $f \in F (= \theta(F_n))$.

(2.5) $\omega(\psi(E_n) \cup \beta(F)) < \varepsilon_{n+1}$ (note that $\beta(F) = \beta \circ \theta(F_n)$).

Let all the blocks $B$ be also part of $B_{n+1}$, that is

$$B_{n+1} = C \oplus B \oplus \text{(some other blocks)}$$

The maps $\beta_n : A_{s(n)} \to B_{n+1}, \psi_{n,n+1} : B_n \to B_{n+1}$ are defined by

$\beta_n := \beta \circ \theta : A_{s(n)} \xrightarrow{\theta} A \xrightarrow{\beta} B(\subset B_{n+1})$

and

$\psi_{n,n+1} := \psi : B_n \to B(\subset B_{n+1})$

for the blocks of $B(\subset B_{n+1})$. This part of $\beta_n$ is $F_n - 2\varepsilon_n$ multiplicative, since $\theta$ is $F_n - \varepsilon_n$ multiplicative, $\beta$ is $F - \varepsilon_n$ multiplicative and $F = \theta(F_n)$.
**Step 3.** By Lemma 3.15 applied to $\phi \oplus r : A \to M_L(A) \oplus r(A)$, there is an $A_{m_2}$ and there is a unital homomorphism

$$\lambda : M_L(A) \oplus r(A) \to RA_{m_2}R,$$

where $R = \phi_{m_1,m_2}(P_0)$ (write $R$ as $\bigoplus_j R^j \in \bigoplus_j A^j_m$) such that the diagram

$$A (= P_0 A_m P_0) \xrightarrow{\phi_{m_1,m_2}} RA_{m_2}R \xrightarrow{\lambda} M_L(A) \oplus r(A)$$

satisfies the following condition:

(3.1) $\lambda \circ (\phi \oplus r)$ is homotopy equivalent to

$$\phi' := \phi_{m_1,m_2}|A.$$

**Step 4.** Applying Theorem 1.6.9 of [G4] to the finite set $F \subset A$ (with $\omega(F) < \varepsilon_n$) and to the two homotopic homomorphisms $\phi'$ and $\lambda \circ (\phi \oplus r) : A \to RA_{m_2}R$ (with $RA_{m_2}R$ in place of $C$ in Theorem 1.6.9 of [G4]), we obtain a finite set $F' \subset RA_{m_2}R, \delta > 0$ and $L > 0$ as in the theorem.

Let $G := \psi(E_n) \cup \beta(F)$. From (2.5), we have $\omega(G) < \varepsilon_{n+1} < \varepsilon_n$. By Theorem 3.8 applied to $RA_{m_2}R$ and

$$\lambda \circ \alpha' : B \to RA_{m_2}R$$

finite set $G \subset B$, $F' \cup \phi'(F) \subset RA_{m_2}R$, $\min\{\varepsilon_n, \delta\} > 0$ (in place of $\varepsilon$) and $L > 0$, there are $A_{s(n+1)}$, mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_{s(n+1)}$ with $\phi_{m_2,s(n+1)}(R) = Q_0 \oplus Q_1 \oplus Q_2$, a $C^*$-algebra $D$—a direct sum of matrix algebras over $C[0,1]$ or $C_1$, a unital map $\theta_0 \in \text{Map}(RA_{m_2}R, Q_0 A_{s(n+1)} Q_0)$, and four unital homomorphisms

$$\theta_1 \in \text{Hom}(RA_{m_2}R, Q_1 A_{s(n+1)} Q_1), \xi_3 \in \text{Hom}(RA_{m_2}R, D)_{1}, \xi_4 \in \text{Hom}(D, Q_2 A_{s(n+1)} Q_2)_{1}$$

and $\alpha'' \in \text{Hom}(B, (Q_0 + Q_1) A_{s(n+1)} (Q_0 + Q_1)_{1})$ such that the following statements are true.

1. $\|\phi_{m_2,s(n+1)}(f) - ((\theta_0 + \theta_1) \circ \xi_4 \circ \xi_3)(f)\| < \varepsilon_n$ for all $f \in \phi_{m_1,m_2}|A(F) \subset RA_{m_2}R$.
2. $\|\alpha''(g) - (\theta_0 + \theta_1) \circ \lambda \circ \alpha'(g)\| < 3\varepsilon_{n+1} < 3\varepsilon_n$ for all $g \in G$.
3. $\theta_0$ is $F' - \min\{\varepsilon_n, \delta\}$ multiplicative and $\theta_1$ satisfies that

$$\theta_1^{i,j}([q]) > L \cdot [\theta_0^{i,j}(R^i)]$$

for any non zero projection $q \in R^i A_{m_1} R^i$.

By Theorem 1.6.9 of [G4], there is a unitary $u \in (Q_0 \oplus Q_1) A_{s(n+1)} (Q_0 \oplus Q_1)$ such that

$$\|((\theta_0 + \theta_1) \circ \phi'(f) - Adu \circ (\theta_0 + \theta_1) \circ \lambda \circ (\phi \oplus r)(f))\| < 8\varepsilon_n$$

29
for all $f \in F$.

Combining with second inequality of (2.4), we have

$$(4.4) \|((\theta_0 + \theta_1) \circ \phi'(f) - Adu \circ (\theta_0 + \theta_1) \circ \lambda \circ \alpha' \circ \beta(f))\| < 9\varepsilon_n$$ for all $f \in F$.

**Step 5.** Finally let all blocks of $D$ be the rest of $B_{n+1}$. Namely, let

$$B_{n+1} = C \oplus B \oplus D,$$

where $C$ is from Step 1, $B$ is from Step 2 and $D$ is from Step 4.

We already have the definition of $\beta_n : A_{s(n)} \to B_{n+1}$ and $\psi_{n,n+1} : B_n \to B_{n+1}$ for those blocks of $C \oplus B \subset B_{n+1}$ (from Step 1 and Step 2). The definition of $\beta_n$ and $\psi_{n,n+1}$ for blocks of $D$ and the homomorphism $\alpha_{n+1} : C \oplus B \oplus D \to A_{s(n+1)}$ will be given below.

The part of $\beta_n : A_{s(n)} \to D(\subset B_{n+1})$ is defined by

$$\beta_n = \xi_3 \circ \phi' \circ \theta : A_{s(n)} \xrightarrow{\theta} A \xrightarrow{\phi'} RA_{m_2}R \xrightarrow{\xi_3} D$$

(Recall that $A = P_0A_{m_2}P_0$ and $\phi' = \phi_{m_1,m_2}|A$). Since $\theta$ is $F_n - \varepsilon_n$ multiplicative, and $\phi'$ and $\xi_3$ are homomorphisms, we know this part of $\beta_n$ is $F_n - \varepsilon_n$ multiplicative.

The part of $\psi_{n,n+1} : B_n \to D(\subset B_{n+1})$ is defined by

$$\psi_{n,n+1} = \xi_3 \circ \phi' \circ \alpha : B_n \xrightarrow{\alpha} A \xrightarrow{\phi'} RA_{m_2}R \xrightarrow{\xi_3} D$$

which is a homomorphism.

The homomorphism $\alpha_{n+1} : C \oplus B \oplus D \to A_{s(n+1)}$ is defined as following.

Let $\phi'' = \phi_{m_1,s(n+1)}|P_1A_{m_1}P_1 : P_1A_{m_1}P_1 \to \phi_{m_1,s(n+1)}(P_1)A_{s(n+1)}\phi_{m_1,s(n+1)}(P_1)$, where $P_1$ is from Step 1. Define

$$\alpha_{n+1}|_C = \phi'' \circ \xi_2 : C \xrightarrow{\xi_2} P_1A_{m_1}P_1 \xrightarrow{\phi''} \phi_{m_1,s(n+1)}(P_1)A_{s(n+1)}\phi_{m_1,s(n+1)}(P_1)$$

where $\xi_2$ is from Step 1.

$$\alpha_{n+1}|_B = Adu \circ \alpha'' : B \xrightarrow{\alpha''} (Q_0 \oplus Q_1)A_{s(n+1)}(Q_0 + Q_1) \xrightarrow{Adu} (Q_0 \oplus Q_1)A_{s(n+1)}(Q_0 + Q_1),$$

where $\alpha''$ is from Step 4, and define

$$\alpha_{n+1}|_D = \xi_4 : D \to Q_2A_{s(n+1)}Q_2.$$
Finally choose \( \{a_{n+1,j}\}_{j=1}^\infty \subset A_{s(n+1)} \) and \( \{b_{n+1,j}\}_{j=1}^\infty \subset B_{n+1} \) to be countable dense subsets of the unit balls of \( A_{s(n+1)} \) and \( B_{n+1} \), respectively. And choose

\[
F_{n+1}' = \phi_{s(n),s(n+1)}(F_n) \cup \alpha_{n+1}(E_{n+1}) \cup \bigcup_{i=1}^{n+1} \phi_{s(i),s(n+1)}(\{a_{i1}, a_{i2}, \ldots, a_{im}\})
\]

\[
E_{n+1}' = \psi_{n,n+1}(E_n) \cup \beta_n(F_n) \cup \bigcup_{i=1}^{n+1} \psi_{i,n+1}(\{b_{i1}, b_{i2}, \ldots, b_{im}\})
\]

\[
\tilde{E}_{n+1}' = \psi_{n,n+1}(E_n) \cup \beta_n(F_n) \subset E_{n+1}'.
\]

Define \( F_{n+1}' = \pi_1(F_{n+1}') \) and \( F_{n+1} = \bigoplus_1 F_{n+1}' \), \( E_{n+1}' = \pi_1(E_{n+1}') \) and \( E_{n+1} = \bigoplus_1 E_{n+1}' \). For those blocks \( B_{n+1}' \) inside the algebra \( B \) define \( \tilde{E}_{n+1}' = \pi_1(\tilde{E}_{n+1}') \). For those blocks inside \( C \) and \( D \), define \( \tilde{E}_{n+1} = \bigoplus_1 \tilde{E}_{n+1}' \). Note that all the blocks with spectrum \( T_1, T_2, T_3 \) and \( S^2 \) are in \( B \). And hence (2.5) tells us that for those blocks, \( \omega(\tilde{E}_{n+1}') < \varepsilon_{n+1} \).

Thus we obtain the following diagram

\[
\begin{array}{ccc}
F_n & \xrightarrow{\phi_{s(n),s(n+1)}} & A_{s(n+1)} \supset F_{n+1} \\
\alpha_n & \downarrow & \beta_n \\
\tilde{E}_n & \xrightarrow{\psi_{n,n+1}} & B_{n+1} \supset E_{n+1} \supset \tilde{E}_{n+1}.
\end{array}
\]

**Step 6.** Now we need to verify all the conditions (0.1)-(0.5) for the above diagram.

From the end of Step 5, we know (0.5) holds, (0.1)-(0.2) hold from the construction (see the construction of \( B, C, D \) in Step 1, 2 and 4, and \( \tilde{E}_{n+1} \subset E_{n+1}, F_{n+1} \) is the end of Step 5). (0.3) follows from the end of Step 1, the end of Step 2 and the part of definition of \( \beta_n \) for \( D \) from Step 5.

So we only need to verify (0.4).

Combining (1.1) with (4.1), we have

\[
\|\phi_{s(n),s(n+1)}(f) - [(\phi'' \circ \xi_2 \circ \xi_1) \oplus (\theta_0 + \theta_1) \circ \phi' \circ \theta \oplus (\xi_4 \circ \xi_3 \circ \phi' \circ \theta)](f)\| < \varepsilon_n + \varepsilon_n = 2 \varepsilon_n
\]

for all \( f \in F_n \) (recall that \( \phi'' = \phi_{m_1,s(n+1)}|_{P_1A_{m_1}P_1}, \phi' := \phi_{m_1,m_2}|_{P_0A_{m_1}P_0} \)).
Combining with (4.2) and (4.4), and the definitions of $\beta_n$ and $\alpha_{n+1}$, the above inequality yields

$$\|\phi_{s(n),s(n+1)}(f) - (\alpha_{n+1} \circ \beta_n)(f)\| < 9\varepsilon_n + 3\varepsilon_n + 2\varepsilon_n = 14\varepsilon_n, \quad \forall f \in F_n.$$ 

Combining (1.3), the first inequality of (2.4) and the definition of $\beta_n$ and $\psi_{n,n+1}$, we have

$$\|\psi_{n,n+1}(g) - (\beta_n \circ \alpha_n)(g)\| < 5\varepsilon_n + 3\varepsilon_n = 8\varepsilon_n, \quad \forall g \in \tilde{E}_n.$$ 

So we obtain (0.4).

The theorem follows from Proposition 4.1.

References

[Bla] B. Blackadar, Matricial and ultra-matricial topology, Operator Algebras, Mathematical Physics, and Low Dimensional Topology (R. H. Herman and B Tanbay eds) A K Peter, Massachusetts (1993), 11-38

[CEL] J. Cuntz, S. Echterhoff and X. Li, On the K-theory of the $C^*$-algebra generated by the left regular representation of an Ore semigroup, J. Eur. Math. Soc. (JEMS) 17 (2015), no. 3, 645-687

[D] M. Dadarlat, Reduction to dimension these of local spectra of Real rank zero $C^*$-algebras, J. Reine Angew. Math. 460(1995) 189-212

[DG] M. Dadarlat and G. Gong, A classification result for approximately homogeneous $C^*$-algebras of real rank zero, Geometric and Functional Analysis, 7(1997) 646-711

[DN] M. Dadarlat and A. Nemethi, Shape theory and (connective) K-theory, J. Operator Theory 23(1990) 207-291

[Ell1] G. A. Elliott, On the classification of $C^*$-algebras of real rank zero, J. Reine Angew. Math. 443(1993) 263-290

[Ell2] G. A. Elliott, A classification of certain simple $C^*$-algebras, in: H Arak: et al.(Eds.) Quantum and non commutative Analysis, Kluwer, Dordrecht, (1993), pp 373-385.

[Ell3] G. A. Elliott, A classification of certain simple $C^*$-algebras, II, J.Ramanujan Math. Soc. 12(1997) 97-134
[EE] G.A. Elliott and D.E. Evans, The structure of the irrational rotation $C^*$-algebra. Ann. of Math. 138, 477-501 (1993)

[EG1] G. A. Elliott and G. Gong, On the inductive limits of matrix algebras over two-tori, American. J. Math 118(1996) 263-290

[EG2] G. A. Elliott and G. Gong, On the classification of $C^*$-algebras of real rank zero, II. Ann. of Math 144(1996) 497-610

[EGL1] G. A. Elliott, G. Gong and L. Li, On the classification of simple inductive limit $C^*$-algebras, II; The isomorphism Theorem, Invent. Math. 168(2)(2007) 249-320

[EGL2] G. A. Elliott, G. Gong and L. Li, Injectivity of the connecting maps in AH inductive limit systems, Canand. Math. Bull. 26(2004) 4-10

[EGLP] G. A. Elliott, G. Gong, H. Lin, C. Pasnicu, Abelian $C^*$-subalgebras of $C^*$-algebras of real rank zero and inductive limit $C^*$-algebras, Duke Math. J. 83 (1996) 511-554

[G1] G. Gong, Approximation by dimension drop $C^*$-algebras and classification, C. R. Math. Rep. Acad. Sci Can. 16(1994) 40-44

[G2] G. Gong, Classification of $C^*$-algebras of real rank zero and unsuspended E-equivalent types, J. Funct. Anal. 152(1998) 281-329

[G3] G. Gong, On inductive limit of matrix algebras over higher dimension spaces, Part I, II, Math Scand. 80(1997) 45-60, 61-100

[G4] G. Gong, On the classification of simple inductive limit $C^*$-algebras, I: Reduction Theorems. Doc. Math. 7(2002) 255-461

[GJL] G. Gong, C. Jiang, L. Li, A classification of inductive limit $C^*$-algebras with ideal property, Preprint

[GJLP] G. Gong, C. Jiang, L. Li and C.Pasnicu, AT structure of AH algebras with the ideal property and torsion free $K$-theory, Journal of Functional Analysis, 258(2010) 2119-2143

[GL] G. Gong and H. Lin, Almost multiplicative morphisms and $K$-theory, International J. Math. 11 (2000) 983-1000

[Ji-Jiang] K. Ji and C. Jiang, A complete classification of AI algebra with ideal property, Canadian. J. Math, 63(2), (2011), 381-412

[Jiang] C. Jiang, A classification of non simple $C^*$-algebras of tracial rank one:Inductive limit of finite direct sums of simple TAI $C^*$-algebras, J. Topol. Anal. 3 No.3(2011), 385-404
[Li1] L. Li, On the classification of simple $C^*$-algebras: Inductive limit of matrix algebras over trees, Mem Amer. Math. Soc. 127(605) 1997

[Li2] L. Li, Simple inductive limit $C^*$-algebras: Spectra and approximation by interval algebras, J. Reine Angew Math 507(1999) 57-79

[Li3] L. Li, Classification of simple $C^*$-algebras: Inductive limit of matrix algebras over 1-dimensional spaces, J. Func. Anal. 192(2002) 1-51

[L] X. Li, Semigroup $C^*$-algebras of $ax+b$-semigroups, Trans. Amer. Math. Soc. 368 (2016), no. 6, 4417-4437

[Lin1] H. Lin, Approximation by normal elements with finite spectra in simple AF algebras, J. Operator Theory 31 (1994) 83-89

[Lin2] H. Lin, Approximation by normal elements with finite spectra in $C^*$-algebras of real rank zero, Pacific J. Math. 173 (1996) 443-489.

[Lin3] H. Lin, Homomorphisms from $C(X)$ into $C^*$-algebras, Canad. J. Math. 49 (1997) 963-1009.

[Lin4] H. Lin, Asymptotic unitary equivalence and classification of simple amenable $C^*$-algebras, Invent. Math. 183 (2011), 385–450.

[LinP] H. Lin and N. C. Phillips, Crossed products by minimal homeomorphisms, J. Reine Angew. Math. 641 (2010), 95–122.

[Pa1] C. Pasnicu, On inductive limit of certain $C^*$-algebras of the form $C(X) \otimes F$, Trans. Amer. Math. Soc. 310(2)(1988) 703-714

[Pa2] C. Pasnicu, Shape equivalence, nonstable K-theory and AH algebras, Pacific J. Math 192(2000) 159-182

[Pa-Ph1] C. Pasnicu and N. C. Phillips, Permanence properties for crossed products and fixed point algebras of finite groups, Trans. Amer. Math. Soc. 366 (2014), no. 9, 4625-4648

[Pa-Ph2] C. Pasnicu and N. C. Phillips, Crossed products by spectrally free actions, J. Funct. Anal. 269 (2015), 915-967

[Pa-R1] C. Pasnicu and M. Rordam, Tensor products of $C^*$-algebras with the ideal property, J. Funct. Anal. 177 (2000), no.1, 130-137

[Pa-R2] C. Pasnicu and M. Rordam, Purely infinite $C^*$-algebras of real rank zero, J. Reine Angew. Math. 613 (2007), 51-73
[Phi1] N. C. Phillips, How many exponentials?, American J. of Math. 116 (1994) 1513-1543

[Phi2] N. C. Phillips, Reduction of exponential rank in direct limits of $C^*$- algebras, Canad. J. Math. 46 (1994), 818-853

Guihua Gong, College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei, 050024, China, and Department of Mathematics, University of Puerto Rico at Rio Piedras, PR 00936, USA
email address: guihua.gong@upr.edu

Chunlan Jiang, College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei, 050024, China
email address: cljiang@hebtu.edu.cn

Liangqing Li, Department of Mathematics, University of Puerto Rico at Rio Piedras, PR 00936, USA
email address: liangqing.li@upr.edu

Cornel Pasnicu, Department of Mathematics, University of Texas at San Antonio, San Antonio, TX 78249, USA
email address: Cornel.Pasnicu@utsa.edu