A More Accurate Half-Discrete Hardy-Hilbert-Type Inequality with the Best Possible Constant Factor Related to the Extended Riemann-Zeta Function

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Abstract

By the method of weight coefficients, techniques of real analysis and Hermite-Hadamard’s inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of the hyperbolic cosecant function with the best possible constant factor expressed in terms of the extended Riemann-zeta function is proved. The more accurate equivalent forms, the operator expressions with the norm, the reverses and some particular cases are also considered.

Key words: Hardy-Hilbert-type inequality; extended Riemann-zeta function; Hurwitz zeta function; Gamma function; weight function; equivalent form; operator

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0, f \in L^p(\mathbb{R}_+), g \in L^q(\mathbb{R}_+)$,

$$||f||_p = \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} > 0,$$

and $||g||_q > 0$, then we have the following Hardy-Hilbert’s integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}||f||_p||g||_q,$$ (1.1)
we have the following discrete analogue of (1.1) with the same best constant

\[ a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, ||a||_p = \left( \sum_{m=1}^\infty a_m^p \right)^{\frac{1}{p}} > 0, ||b||_q > 0, \]

we have the following discrete analogue of (1.1) with the same best constant \( \frac{\pi}{\sin(\pi/p)} \) (cf. 1):

\[
\sum_{m=1}^\infty \sum_{n=1}^\infty a_mb_n / m + n < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{1.2}
\]

Inequalities (1.1) and (1.2) are important in Mathematical Analysis and its applications (cf. 1, 2, 3, 4, 5).

Suppose that \( \mu_i, \nu_j > 0 \) (\( i, j \in \mathbb{N} = \{1, 2, \ldots\} \)),

\[
U_m := \sum_{i=1}^m \mu_i, V_n := \sum_{j=1}^n \nu_j \ (m, n \in \mathbb{N}). \tag{1.3}
\]

Then we have the following inequality (cf. 1, Theorem 321, replacing \( \mu_m^{1/p} a_m \) and \( \nu_n^{1/q} b_n \) by \( a_m \) and \( b_n \) :)

\[
\sum_{m=1}^\infty \sum_{n=1}^\infty a_mb_n / U_m + V_n < \frac{\pi}{\sin(\pi/p)} \left( \sum_{m=1}^\infty a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^\infty b_n^q \right)^{\frac{1}{q}}. \tag{1.4}
\]

For \( \mu_i = \nu_j = 1 \) (\( i, j \in \mathbb{N} \)), inequality (1.4) reduces to (1.2). We call (1.4) Hardy–Hilbert-type inequality.

**Note.** The authors of [1] did not prove that (1.4) is valid with the best possible constant factor.

In 1998, by introducing an independent parameter \( \lambda \in (0, 1] \), Yang [6] obtained an extension of (1.1) with the kernel \( \frac{1}{(x+y)^\lambda} \) for \( p = q = 2 \). Refining the method applied in [6], Yang [5] provided extensions of (1.1) and (1.2) as follows:

Assuming that \( \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x,y) \) is a non-negative homogeneous function of degree \( -\lambda \), with

\[
k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in \mathbb{R}_+,
\]

\[
\phi(x) = x^{\theta(1-\lambda_1)-1}, \quad \psi(x) = x^{\theta(1-\lambda_2)-1}, f(x), g(y) \geq 0,
\]

\[
f \in L_{p,\phi}(\mathbb{R}_+) = \left\{ f; ||f||_{p,\phi} := \left\{ \int_0^\infty \phi(x)||f(x)||^pdx \right\}^{\frac{1}{p}} < \infty \right\},
\]

where \( g \in L_{q,\psi}(\mathbb{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0 \), we have

\[
\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.5}
\]

where, the constant factor \( k(\lambda_1) \) is the best possible. Moreover, if \( k_\lambda(x,y) \) keeps finite and \( k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1}) \) is decreasing with respect to \( x > 0 \) \( (y > 0) \), then for \( a_m, b_n \geq 0, \)

\[
a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := \left( \sum_{n=1}^\infty \phi(n)||a_n||^p \right)^{\frac{1}{p}} < \infty \right\},
\]
\[ b = \{ b_n \}_{n=1}^{\infty} \in L_{q,\psi}, \| a \|_{p,\phi}, \| b \|_{q,\psi} > 0, \text{ we have} \]
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_\lambda(m,n)a_mb_n < k(\lambda_1)\| a \|_{p,\phi}\| b \|_{q,\psi}, \quad (1.6) \]

where, the constant factor \( k(\lambda_1) \) is still the best possible.

For \( 0 < \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda \), we set
\[ k_\lambda(x,y) = \frac{1}{(x+y)^\lambda} \quad ((x,y) \in \mathbb{R}^2_+). \]

Then by (1.6), we have
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_mb_n}{(m+n)^\lambda} < B(\lambda_1, \lambda_2)\| a \|_{p,\phi}\| b \|_{q,\psi}, \quad (1.7) \]

where, the constant \( B(\lambda_1, \lambda_2) \) is the best possible, and
\[ B(u, v) = \int_0^\infty \frac{1}{(1+t)^{u+v-1}} dt \quad (u, v > 0) \]

is the beta function. Clearly, for \( \lambda = 1, \lambda_1 = \frac{1}{\eta}, \lambda_2 = \frac{1}{\eta} \), inequality (1.7) reduces to (1.2).

In 2015, by adding some conditions, Yang [7] extended (1.7) and (1.4) as follows:
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_mb_n}{(U_m+V_n)^\lambda} \leq B(\lambda_1, \lambda_2) \left[ \sum_{m=1}^{\infty} \frac{U_m^{\rho(1-\lambda_1)-1}a_m^{p-1}}{\mu_m^{p-1}} \right]^\frac{1}{p} \left[ \sum_{n=1}^{\infty} \frac{V_n^{\rho(1-\lambda_2)-1}b_n^{q-1}}{\nu_n^{q-1}} \right]^\frac{1}{q}, \quad (1.8) \]

where, the constant \( B(\lambda_1, \lambda_2) \) is still the best possible.

Some other results including multidimensional Hilbert-type inequalities are provided in [8]-[30].

Related to the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [11]. But they did not prove that the constant factors are the best possible. However, Yang [31] established a result with the kernel \( \frac{1}{(1+x+y)^\lambda} \) by introducing a variable and proved that the constant factor is the best possible. In 2011 Yang [32] proved the following half-discrete Hardy-Hilbert’s inequality with the best possible constant factor \( B(\lambda_1, \lambda_2) \):
\[ \int_0^\infty f(x) \left[ \sum_{n=1}^{\infty} \frac{a_n}{(x+n)^\lambda} \right] dx < B(\lambda_1, \lambda_2)\| f \|_{p,\phi}\| a \|_{q,\psi}, \quad (1.9) \]

where, \( \lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda \). Zhong et al. [33]-[39] investigated several half-discrete Hilbert-type inequalities with particular kernels. Applying the method of weight functions, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree \(-\lambda \in \mathbb{R}\) and a best constant factor \( k(\lambda_1) \) is obtained as follows:
\[ \int_0^\infty f(x) \sum_{n=1}^{\infty} k_\lambda(x,n)a_n dx < k(\lambda_1)\| f \|_{p,\phi}\| a \|_{q,\psi}, \quad (1.10) \]
which is an extension of (1.9) (cf. [40]). At the same time, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [41]. In 2012-2014, Yang et al. published three books [42], [43] and [44] extensively presenting the framework of half-discrete Hilbert-type inequalities.

In this paper, by the method of weight coefficients, techniques of real analysis and Hermite-Hadamard’s inequality, a half-discrete Hardy-Hilbert-type inequality related to the kernel of the hyperbolic cosecant function with a best possible constant factor expressed by the extended Riemann-zeta function is proved, which is an extension of (1.10) for $\lambda = 0$ in the following particular kernel:

$$k_0(x,n) = \frac{\csc h(p(\frac{x}{n}))}{e^{\alpha(\frac{x}{n})}}(p > \max\{0, -\alpha\}, 0 < \gamma < 1).$$

Furthermore, the more accurate equivalent forms, the operator expressions with the norm, the reverses and some particular cases are also considered.

## 2 Some Lemmas

In the sequel, we shall assume that $\nu_n > 0 \ (n \in \mathbb{N}), \{\nu_n\}_{n=1}^\infty$ is decreasing, $V_n = \sum_{j=1}^n \nu_j$, $\mu(t)$ is a positive continuous function in $\mathbb{R}_+ = (0, \infty)$,

$$U(0) := 0; \ U(x) := \int_0^x \mu(t) dt < \infty (x \in (0, \infty)),$$

$$v(t) := v_n, \ t \in (n-1, n] \ (n \in \mathbb{N}),$$

and

$$V(0) := 0; \ V(y) := \int_0^y v(t) dt (y \in (0, \infty)),$$

$p \neq 0, 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\delta \in \{-1, 1\}$, $\beta \leq \frac{\nu_i}{2}$, $f(x), a_n \geq 0 \ (x \in \mathbb{R}_+, N \in \mathbb{N})$,

$$||f||_{p, \Phi_\delta} = \left( \int_0^\infty \Phi_\delta(x) f^p(x) dx \right)^{\frac{1}{p}},$$

$$||a||_{q, \Psi_\beta} = \left( \sum_{n=1}^\infty \Psi_\beta(n) a_n^q \right)^{\frac{1}{q}},$$

where,

$$\Phi_\delta(x) := \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} (x \in \mathbb{R}_+),$$

$$\Psi_\beta(n) := \frac{(V_n - \beta)^{q(1-\sigma)-1}}{V_n^{q-1}} (n \in \mathbb{N}).$$

**Lemma 2.1.** If $a \in \mathbb{R}$, $f(x)$ is continuous in $[a - \frac{1}{2}, a + \frac{1}{2}]$, $f'(x)$ is strictly increasing in $(a - \frac{1}{2}, a)$ and $(a, a + \frac{1}{2})$ respectively, as well as

$$\lim_{x \to a^-} f'(x) = f'(a - 0) \leq f'(a + 0) = \lim_{x \to a^+} f'(x),$$

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then \( f(x) \) is strictly convex in \([a - \frac{1}{2}, a + \frac{1}{2}]\), and we have the following Hermite-Hadamard’s inequality (cf. [48]):

\[
f(a) < \int_{a - \frac{1}{2}}^{a + \frac{1}{2}} f(x) \, dx. \tag{2.1}
\]

**Proof.** Since \( f'(a - 0) \) (\( \leq f'(a + 0) \)) is finite, we define a function \( g(x) \) as follows:

\[ g(x) := f'(a - 0)(x - a) + f(a), x \in [a - \frac{1}{2}, a + \frac{1}{2}]. \]

In view of \( f'(x) \) being strictly increasing in \((a - \frac{1}{2}, a)\), then for \( x \in (a - \frac{1}{2}, a) \),

\[
(f(x) - g(x))' = f'(x) - f'(a - 0) < 0.
\]

Since \( f(a) - g(a) = 0 \), it follows that \( f(x) - g(x) > 0, x \in (a - \frac{1}{2}, a) \). Similarly, we can obtain \( f(x) - g(x) > 0, x \in (a, a + \frac{1}{2}) \). Hence, \( f(x) \) is strictly convex in \([a - \frac{1}{2}, a + \frac{1}{2}]\), and therefore

\[
\int_{a - \frac{1}{2}}^{a + \frac{1}{2}} f(x) \, dx > \int_{a - \frac{1}{2}}^{a + \frac{1}{2}} g(x) \, dx = f(a),
\]

namely, (2.1) follows.

**Example 2.2.** For \( \rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1 \),

\[
\csc h(u) = \frac{2}{e^u - e^{-u}} \quad (u > 0)
\]

is called hyperbolic cosecant function (cf. [43]), we set

\[
h(t) = \frac{\csc h(\rho t^\gamma)}{e^{\alpha t}} = \frac{2}{e^{(\alpha + \rho)\gamma}(1 - e^{-2\rho t})} \quad (t \in \mathbb{R}_+).
\]

(i) Setting \( u = \rho t^\gamma \), we find

\[
k(\sigma) := \int_{0}^{\infty} \frac{\csc h(\rho t^\gamma)}{e^{\alpha t}} t^{\sigma-1} \, dt
\]

\[
= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_{0}^{\infty} \frac{\csc h(u)}{e^{\frac{\alpha}{\rho} u}} u^{\sigma-1} \, du
\]

\[
= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_{0}^{\infty} e^{-\frac{\alpha}{\rho} u} u^{\frac{\sigma-1}{\gamma}-1} \, du
\]

\[
= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_{0}^{\infty} e^{-(\frac{\rho}{\alpha} + 1) u} u^{\frac{\sigma-1}{\gamma}-1} \, du
\]

\[
= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_{0}^{\infty} \sum_{k=0}^{\infty} e^{-(2k+\frac{\rho}{\alpha} + 1) u} u^{\frac{\sigma-1}{\gamma}-1} \, du.
\]
By Lebesgue’s term by term theorem (cf. [45]), setting \( v = (2k + \frac{a}{p} + 1)u \), we have

\[
k(\sigma) = \int_0^\infty \frac{\csc h(\rho t^\sigma)}{e^{\rho t}} t^{\sigma-1} dt
\]

\[
= \frac{2}{\gamma^p \sigma \gamma} \sum_{k=0}^\infty \int_0^\infty e^{-(2k + \frac{a}{p} + 1)u} \frac{1}{u^{\sigma-1} du}
\]

\[
= \frac{2}{\gamma^p \sigma \gamma} \sum_{k=0}^\infty \frac{1}{(2k + \frac{a}{p} + 1)^{\sigma \gamma}} \int_0^\infty e^{-v} v^{\sigma \gamma - 1} dv
\]

\[
= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2p)^{\sigma \gamma}} \sum_{k=0}^\infty \frac{1}{(k + \frac{a}{2p})^{\sigma \gamma}}
\]

\[
= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2p)^{\sigma \gamma}} \zeta(\frac{\sigma}{\gamma}, \frac{\alpha + p}{2p}) \in \mathbb{R}_+, \tag{2.2}
\]

where

\[
\zeta(s, a) := \sum_{k=0}^\infty \frac{1}{(k + a)^s} \quad (\text{Re}(s) > 1, a > 0)
\]

is called the extended Riemann-zeta function (also known as the Hurwitz zeta function)\(^1\) and

\[
\Gamma(y) := \int_0^\infty e^{-v} v^{y-1} dv \quad (y > 0)
\]

is called Gamma function (cf. [46]). In particular, for \( \alpha = \rho \), we have

\[
h(t) = \frac{\csc h(\rho t^\gamma)}{e^{\rho t}} \quad \text{and} \quad k(\sigma) = k_1(\sigma) := \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2p)^{\sigma \gamma}} \zeta(\frac{\sigma}{\gamma}).
\]

In this case, for \( \gamma = \frac{\sigma}{2} \), we have

\[
h(t) = \frac{\csc h(\rho t^{\sigma/2})}{e^{\rho t^{\sigma/2}}} \quad \text{and} \quad k(\sigma) = \frac{\pi^2}{6\sigma^2}.
\]

(ii) We obtain for \( u > 0 \) that

\[
1 - e^{-2u} > 0, \quad \left( \frac{1}{1 - e^{-2u}} \right)' = -\frac{2e^{-2u}}{(1 - e^{-2u})^2} < 0,
\]

and

\[
\left( \frac{1}{1 - e^{-2u}} \right)'' = \frac{4e^{-2u}}{(1 - e^{-2u})^3} + \frac{8e^{-4u}}{(1 - e^{-2u})^5} > 0.
\]

(iii) If \( g(u) > 0, g'(u) < 0, g''(u) > 0 \), then for \( 0 < \gamma \leq 1 \), we find that \( g(\rho t^\gamma) > 0 \),

\[
\frac{d}{dt} g(\rho t^\gamma) = \rho \gamma \rho^\gamma t^{-2} g'(\rho t^\gamma) < 0 \quad \text{and}
\]

\[
\frac{d^2}{dt^2} g(\rho t^\gamma) = \rho \gamma^2 (\gamma - 1) \rho t^{-2} g''(\rho t^\gamma) + (\rho \gamma^2 t^{-1})^2 g''(\rho t^\gamma) > 0.
\]

\(^1\)Clearly \( \zeta(s, 1) = \zeta(s) \), where \( \zeta(s) \) is the Riemann-zeta function.
Then we find that for $y \in (n - \frac{1}{2}, n)$,

$$\frac{d}{dy} g(V(y) - \beta) > 0, \quad \frac{d^2}{dy^2} g(V(y) - \beta) = g''(V(y) - \beta) v_n < 0,$$

and

$$\frac{d^2}{dy^2} g(V(y) - \beta) = g''(V(y) - \beta) v_n^2 > 0 \ (n \in \mathbb{N});$$

for $y \in (n, n + \frac{1}{2})$,

$$\frac{d}{dy} g(V(y) - \beta) > 0, \quad \frac{d^2}{dy^2} g(V(y) - \beta) = g''(V(y) - \beta) v_{n+1} < 0,$$

and

$$\frac{d^2}{dy^2} g(V(y) - \beta) = g''(V(y) - \beta) v_{n+1}^2 > 0 \ (n \in \mathbb{N}).$$

If $g_1(u) > 0, g'_1(u) < 0, g''_1(u) > 0, g'_2(u) > 0, g''_2(u) \leq 0, g''_2(u) \geq 0$, then we find for $u > 0$ that

$$g_1(u) g_2(u) > 0, (g_1(u) g_2(u))' = g'_1(u) g_2(u) + g_1(u) g'_2(u) < 0,$$

and

$$(g_1(u) g_2(u))'' = g''_1(u) g_2(u) + 2 g'_1(u) g'_2(u) + g_1(u) g''_2(u) > 0.$$

(iv) For $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1$, we have

$$h(t) > 0, h'(t) < 0, \ h''(t) > 0, \ \text{with } k(\sigma) \in \mathbb{R}_+,$$

and then for $c > 0, \beta \leq \frac{\sigma}{2}, y \geq \frac{1}{2}, n \in \mathbb{N}$, we have

$$h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1} > 0, \quad \frac{d}{dy} [h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}] < 0,$$

and

$$\frac{d^2}{dy^2} [h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}] > 0 \ (y \in (n - \frac{1}{2}, n) \cup (n, n + \frac{1}{2})).$$

Setting $f(y) = h(c(V(y) - \beta))(V(y) - \beta)^{\sigma-1}$, it follows that $f'(y)(< 0)$ is strictly increasing in $(n - \frac{1}{2}, n)$ and

$$\lim_{x \to n^-} f'(y) = f'(n - 0) = [ch'(c(V_n - \beta))(V_n - \beta)^{\sigma-1} + (\sigma - 1)h(c(V_n - \beta))(V_n - \beta)^{\sigma-2}]v_n.$$

In the same way, for $x \in (n, n + \frac{1}{2})$, we find that $f'(y)(< 0)$ is strictly increasing and

$$\lim_{x \to n^+} f'(y) = f'(n + 0) = [ch'(c(V_n - \beta))(V_n - \beta)^{\sigma-1} + (\sigma - 1)h(c(V_n - \beta))(V_n - \beta)^{\sigma-2}]v_{n+1}.$$
In view of \( v_{n+1} \leq v_n \), it follows that
\[
\lim_{x \to n^+} f'(x) = f'(n + 0) \geq f'(n - 0) = \lim_{x \to n^-} f'(x).
\]
Then by (2.1), for \( n \in \mathbb{N} \), we have
\[
f(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(y)dy = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(c(V(y) - \beta))(V(y) - \beta)^{\sigma - 1}dy.
\]

**Lemma 2.3.** If \( g(t) > 0 \) is a strictly decreasing continuous function in \((\frac{1}{2}, \infty)\), which is strictly convex satisfying
\[
\int_{\frac{1}{2}}^{\infty} g(t)dt \in \mathbb{R}_+,
\]
then we have
\[
\int_1^{\infty} g(t)dt < \sum_{n=1}^{\infty} g(n) < \int_{\frac{1}{2}}^{\infty} g(t)dt. 
\]

**Proof.** By (2.1) and the decreasing property, we have
\[
\int_n^{n+1} g(t)dt < \int_n^{n+1} g(n)dt = g(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)dt \quad (n \in \mathbb{N}),
\]
and for \( n_0 \in \mathbb{N} \), it follows that
\[
\int_1^{n_0+1} g(t)dt < \sum_{n=1}^{n_0} g(n) < \sum_{n=1}^{n_0} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} g(t)dt = \int_{\frac{1}{2}}^{n_0+\frac{1}{2}} g(t)dt,
\]
\[
\int_{n_0+1}^{\infty} g(t)dt \leq \sum_{n=n_0+1}^{\infty} g(n) \leq \int_{n_0+\frac{1}{2}}^{\infty} g(t)dt < \infty.
\]
Hence, we obtain (2.4). \( \square \)

**Lemma 2.4.** If \( \rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1 \), define the following weight coefficients:
\[
\omega_{\delta}(\sigma, x) := \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta \gamma}(x)(V_n - \beta)^{\gamma}) U^{\delta \sigma}(x) V_{n+1} \mu(x)}{e^{\alpha U^{\delta \gamma}(x)(V_n - \beta)^{\gamma}} (V_n - \beta)^{1-\sigma}}, \quad x \in \mathbb{R}_+, 
\]
\[
\sigma_{\delta}(\sigma, n) := \int_0^{\infty} \frac{\csc h(\rho U^{\delta \gamma}(x)(V_n - \beta)^{\gamma}) (V_n - \beta)^{\sigma \mu(x)}}{e^{\alpha U^{\delta \gamma}(x)(V_n - \beta)^{\gamma}} U^{1-\delta \sigma}(x)} dx, \quad n \in \mathbb{N}.
\]
Then, we have the following inequalities:
\[
\omega_{\delta}(\sigma, x) < k(\sigma) \quad (x \in \mathbb{R}_+), \quad (2.7)
\]
\[
\sigma_{\delta}(\sigma, n) \leq k(\sigma) \quad (n \in \mathbb{N}), \quad (2.8)
\]
where, \( k(\sigma) \) is indicated by (2.2).
Proof. Since \( V_n = V(n) \), and for \( t \in (n - \frac{1}{2}, n) \),

\[
v_{n+1} \leq v_n = V'(t);
\]

for \( t \in (n, n + \frac{1}{2}) \),

\[
v_{n+1} = V'(t),
\]

by (2.3) (for \( c = U^\delta(x) \)), we have

\[
\frac{\csc h(\rho U^\delta(x)(V_n - \beta)^\gamma)}{e^{U^\delta(x)(V_n - \beta)^\gamma}} \frac{U^\delta(x)}{(V_n - \beta)^{1-\sigma}} = \frac{\csc h(\rho U^\delta(x)(V(t) - \beta)^\gamma)}{e^{U^\delta(x)(V(t) - \beta)^\gamma}} \frac{U^\delta(x)}{(V(t) - \beta)^{1-\sigma}} < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^\delta(x)(V(t) - \beta)^\gamma)}{e^{U^\delta(x)(V(t) - \beta)^\gamma}} \frac{U^\delta(x)}{(V(t) - \beta)^{1-\sigma}} dt \ (n \in \mathbb{N}),
\]

\[
\omega_6(\sigma, x) < \sum_{n=1}^{\infty} v_{n+1} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^\delta(x)(V(t) - \beta)^\gamma)}{e^{U^\delta(x)(V(t) - \beta)^\gamma}} \frac{U^\delta(x)}{(V(t) - \beta)^{1-\sigma}} dt \leq \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{\csc h(\rho U^\delta(x)(V(t) - \beta)^\gamma)}{e^{U^\delta(x)(V(t) - \beta)^\gamma}} \frac{U^\delta(x)}{(V(t) - \beta)^{1-\sigma}} dt
\]

Setting \( u = U^\delta(x)(V(t) - \beta) \), by (2.2), we obtain

\[
\omega_6(\sigma, x) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{U^\delta(x)(V(t) - \beta)^\gamma}{e^{U^\delta(x)(V(t) - \beta)^\gamma}} \frac{U^\delta(x)}{(V(t) - \beta)^{1-\sigma}} dt.
\]

Hence, (2.7) follows.

Setting \( u = (V_n - \beta)U^\delta(x) \) in (2.6), we find \( du = (V_n - \beta)U^\delta(x)(V_n - \beta)\sigma^{-1}(V_n - \beta)^{-1}u^{\delta-1} du \) and

\[
\omega_0(\sigma, n) = \frac{1}{\delta} \int_{(V_n - \beta)U^\delta(0)}^{(V_n - \beta)U^\delta(n)} \frac{\csc h(\rho u^\gamma)}{e^{\rho u^\gamma}} \frac{(V_n - \beta)\sigma^{-1}(V_n - \beta)^{-1}u^{\delta-1}}{[(V_n - \beta)^{-1}u]^\frac{1}{\delta}-\sigma} du
\]

If \( \delta = 1 \), then

\[
\omega_0(\sigma, n) = \int_{0}^{(V_n - \beta)U^\delta(n)} \frac{\csc h(\rho u^\gamma)}{e^{\rho u^\gamma}} u^{\sigma-1} du \leq \int_{0}^{\infty} \frac{\csc h(\rho u^\gamma)}{e^{\rho u^\gamma}} u^{\sigma-1} du;
\]

if \( \delta = -1 \), then

\[
\omega_0(-\sigma, n) = -\int_{(V_n - \beta)U^{-\delta}(0)}^{(V_n - \beta)U^{-\delta}(n)} \frac{\csc h(\rho u^\gamma)}{e^{\rho u^\gamma}} u^{\sigma-1} du \leq \int_{0}^{\infty} \frac{\csc h(\rho u^\gamma)}{e^{\rho u^\gamma}} u^{\sigma-1} du.
\]

Hence, by (2.2), we have (2.8).
Remark 2.5. We do not need the condition of $\sigma \leq 1$ in obtaining \((2.8)\). If $U(\infty) = \infty$, then we have

$$\omega_{\delta}(\sigma, n) = k(\sigma) \quad (n \in \mathbb{N}). \quad (2.9)$$

For example, we set $\mu(t) = \frac{1}{(1+t)^{\rho}} \quad (t > 0; 0 \leq \rho \leq 1)$, then for $x \geq 0$, we find

$$U(x) = \int_{0}^{x} \frac{dt}{(1+t)^{\rho}} = \begin{cases} \frac{(1+x)^{1-\rho}-1}{1-\rho} \ln(1+x), & 0 \leq \rho < 1, \\ \infty, & \rho = 1. \end{cases}$$

$U(0) = 0$ and $U(\infty) = \int_{0}^{\infty} \frac{dt}{(1+t)^{\rho}} = \infty$.

Lemma 2.6. If $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1, V(\infty) = \infty$, then, (i) for $x \in \mathbb{R}_{+}$, we have

$$k(\sigma)(1 - \theta_{\delta}(\sigma, x)) < \omega_{\delta}(\sigma, x), \quad (2.10)$$

where,

$$\theta_{\delta}(\sigma, x) := \frac{1}{k(\sigma)} \int_{0}^{U^{\delta}(x)(V(t) - \beta)^{\gamma}} \frac{\csc h(\rho u^{\gamma})}{e^{au^{\gamma}}} u^{\sigma-1} du$$

$$= O((U(x))^{\gamma}(\sigma - \gamma)) \in (0, 1);$$

(ii) for any $b > 0$, we have

$$\sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_{n} - \beta)^{1+b}} = \frac{1}{b} \left[ \frac{1}{(V_{1} - \beta)^{b}} + bO(1) \right]. \quad (2.11)$$

Proof. By \((2.4)\), we find

$$\omega_{\delta}(\sigma, x) \geq \sum_{n=1}^{\infty} V_{n+1} \int_{0}^{V_{n+1}} \frac{\csc h(\rho U^{\delta}(x)(V(t) - \beta)^{\gamma})}{e^{aU^{\delta}(x)(V(t) - \beta)^{\gamma}}} U^{\delta}(x)dt$$

$$= \sum_{n=1}^{\infty} \int_{0}^{V_{n+1}} \frac{\csc h(\rho U^{\delta}(x)(V(t) - \beta)^{\gamma})}{e^{aU^{\delta}(x)(V(t) - \beta)^{\gamma}}} U^{\delta}(x)V(t)U^{\delta}(x)(V(t) - \beta)^{\gamma}$$

$$= \int_{1}^{\infty} \frac{\csc h(\rho U^{\delta}(x)(V(t) - \beta)^{\gamma})}{e^{aU^{\delta}(x)(V(t) - \beta)^{\gamma}}} U^{\delta}(x)V(t)U^{\delta}(x)(V(t) - \beta)^{\gamma}.$$
is continuous in \((0, \infty)\) satisfying \(u^{\frac{1}{2}(\sigma + \gamma)}F(u) \to 0 (u \to 0^+)\), and \(u^{\frac{1}{2}(\sigma + \gamma)}F(u) \to 0 (u \to \infty)\), there exists a constant \(L > 0\), such that \(u^{\frac{1}{2}(\sigma + \gamma)}F(u) \leq L\), namely,

\[
\frac{\csc h\left(\rho u^\gamma\right)}{\epsilon^{\alpha u^\gamma}} \leq Lu^{\frac{1}{2}(\sigma + \gamma)} (u \in (0, \infty)).
\]

Hence we find

\[
0 < \theta_0(\sigma, x) \leq \frac{L}{k(\sigma)} \int_0^{U^\delta(x)} u^{\frac{1}{2}(\sigma - \gamma) - 1} du = \frac{2L[U^\delta(x)(1 - \beta)]^{\frac{1}{2}(\sigma - \gamma)}}{k(\sigma)(\sigma - \gamma)} (x \in \mathbb{R}_+),
\]

and then (2.10) follows.

For \(b > 0\), we find

\[
\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1+b}} < \frac{v_2}{(V_1 - \beta)^{1+b}} + \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{V'(x)}{(V(x) - \beta)^{1+b}} dx
\]

\[
= \frac{v_2}{(V_1 - \beta)^{1+b}} + \int_{1}^{\infty} \frac{V'(x)}{(V(x) - \beta)^{1+b}} dx
\]

\[
= \frac{v_2}{(V_1 - \beta)^{1+b}} + \int_{1}^{\infty} \frac{du}{u^b}
\]

\[
\leq \frac{1}{b} \left[ \frac{1}{(V_1 - \beta)^{p}} + b \frac{v_2}{(V_1 - \beta)^{1+b}} \right],
\]

\[
\sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1+b}} = \sum_{n=1}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{v_{n+1}}{(V(n) - \beta)^{1+b}} dx > \sum_{n=1}^{\infty} \int_{n}^{n+1} \frac{V'(x)dx}{(V(x) - \beta)^{1+b}}
\]

\[
= \int_{1}^{\infty} \frac{V'(x)dx}{(V(x) - \beta)^{1+b}} = \frac{1}{b(V_1 - \beta)^b}.
\]

Hence we have (2.11). \(\square\)

**Note.** For example, \(v_n = \frac{1}{n^a} (n \in \mathbb{N}; 0 \leq a \leq 1)\) satisfies the condition that \(v_n > 0 (n \in \mathbb{N})\), \(\{v_n\}_{n=1}^{\infty}\) is decreasing, and \(V(\infty) = \infty\).

### 3  Main Results and Operator Expressions

**Theorem 3.1.** If \(\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1, k(\sigma)\) is indicated by (2.2), then for \(p > 1, 0 < ||f||_{p, \Phi_b}, ||a||_q, \Psi_b < \infty\), we have the following equivalent inequalities:

\[
I := \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h\left(\rho U^\delta(x)(V_n - \beta)^\gamma\right)}{\epsilon^{\alpha U^\delta(x)(V_n - \beta)^\gamma}} a_n f(x) dx < k(\sigma)||f||_{p, \Phi_b} ||a||^p_q \Psi_b, \quad (3.1)
\]

\[
J_1 := \sum_{n=1}^{\infty} \frac{v_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\csc h\left(\rho U^\delta(x)(V_n - \beta)^\gamma\right)}{\epsilon^{\alpha U^\delta(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p
\]

\[
< k(\sigma)||f||_{p, \Phi_b}, \quad (3.2)
\]
Proof. By the weighted Hölder inequality (cf. [48]), we have

\[
J_2 := \left\{ \int_0^\infty \frac{\mu(x)}{U^{1-q} \sigma(x)} \left[ \sum_{n=1}^\infty \frac{\csc h(\rho U^{\delta}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta}(x)(V_n - \beta)^\gamma}} a_n \right]^q \, dx \right\}^{\frac{1}{q}}
\]  

\[
< k(\sigma) ||a||_{q, \psi, n^\beta}. \tag{3.3}
\]

In view of (2.38) and the Lebesgue term by term integration theorem (cf. [47]), we find

\[
J_1 \leq (k(\sigma))^{\frac{1}{p}} \left[ \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(\rho U^{\delta}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta}(x)(V_n - \beta)^\gamma}} U^{(1-\delta)(p-1)}(x) V_{n+1} f^p(x) \, dx \right]^{\frac{1}{p}}
\]

\[
= (k(\sigma))^{\frac{1}{p}} \left[ \int_0^\infty \frac{\sum_{n=1}^\infty \csc h(\rho U^{\delta}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta}(x)(V_n - \beta)^\gamma}} U^{(1-\delta)(p-1)}(x) V_{n+1} f^p(x) \, dx \right]^{\frac{1}{p}}
\]

\[
= (k(\sigma))^{\frac{1}{p}} \left[ \int_0^\infty \omega_0(\sigma, x) U^{(1-\delta)(p-1)}(x) \frac{f^p(x)}{\mu^p(x)} \, dx \right]^{\frac{1}{p}}. \tag{3.5}
\]

Then by (2.7), we derive (3.2).

By Hölder’s inequality (cf. [48]), we have

\[
I = \sum_{n=1}^\infty \left[ \frac{V_{n+1}}{(V_n - \beta)^{\frac{1}{p}}} \right] \int_0^\infty \frac{\csc h(\rho U^{\delta}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta}(x)(V_n - \beta)^\gamma}} f(x) \, dx
\]

\[
= \left[ \frac{(V_n - \beta)^{\frac{1}{p}-\sigma} a_n}{V_{n+1}^\sigma} \right] \leq J_1 ||a||_{q, \psi, n^\beta}. \tag{3.6}
\]
Then by (3.2), we obtain (3.1). On the other hand, assuming that (3.1) is valid, we set
\[ a_n := \frac{V_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_0^\infty \csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma} f(x) dx \right]^{p-1}, \quad n \in \mathbb{N}. \]

Then we find \( J_1^p = ||a||^{q}_{q, q_p}. \) If \( J_1 = 0, \) then (3.2) is trivially valid; if \( J_1 = \infty, \) then (3.2) is still not valid. Suppose that \( 0 < J_1 < \infty. \) By (3.1), we have
\[ ||a||^{q}_{q, q_p} = J_1^p = I < k(\sigma)||f||_{p, \phi_\beta}||a||_{q, q_p}, \]
\[ ||a||^{q-1}_{q, q_p} = J_1 < k(\sigma)||f||_{p, \phi_\beta}, \]
and then (3.2) follows, which is equivalent to (3.1).

Still by the weighted Hölder inequality (cf. (48)), we have
\[
\left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma}}{e^{aU^{\delta x}(x)(V_n - \beta)^{\gamma}}} a_n \right]^q \leq \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma}}{e^{aU^{\delta x}(x)(V_n - \beta)^{\gamma}}} \left( \frac{U^{1-\delta x}(x)}{V_n^{\delta x}} \right)^{\frac{q}{p}} \left( \frac{1-\delta x}{V_n^{\delta x}} \right)^{\frac{q}{p}} a_n \times \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma}}{e^{aU^{\delta x}(x)(V_n - \beta)^{\gamma}}} \left( \frac{U^{1-\delta x}(x)}{V_n^{\delta x}} \right)^{\frac{q}{p}} \left( \frac{1-\delta x}{V_n^{\delta x}} \right)^{\frac{q}{p}} a_n \left( \frac{\omega_0(\sigma, x)^{q(1-\sigma)}}{U^{q(1-\sigma)}(x)^{q-1}} \right)^{q-1} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma}}{e^{aU^{\delta x}(x)(V_n - \beta)^{\gamma}}} \left( \frac{U^{1-\delta x}(x)}{V_n^{\delta x}} \right)^{\frac{q}{p}} \left( \frac{1-\delta x}{V_n^{\delta x}} \right)^{\frac{q}{p}} a_n. \]

Then by (2.7) and the Lebesgue term by term integration theorem (cf. (47)), it follows that
\[
J_2 < (k(\sigma))^\frac{1}{q} \left[ \int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma}}{e^{aU^{\delta x}(x)(V_n - \beta)^{\gamma}}} \left( \frac{V_n - \beta}{U^{1-\delta x}(x)} \right)^{q(1-\sigma)} a_n dx \right]^{\frac{1}{q}}.
\]
\[
= (k(\sigma))^\frac{1}{q} \left[ \sum_{n=1}^{\infty} \int_0^\infty \frac{\csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma}}{e^{aU^{\delta x}(x)(V_n - \beta)^{\gamma}}} \left( \frac{V_n - \beta}{U^{1-\delta x}(x)} \right)^{q(1-\sigma)} a_n dx \right]^{\frac{1}{q}}.
\]
\[ = (k(\sigma))^\frac{1}{q} \left[ \sum_{n=1}^{\infty} \omega_0(\sigma, n) \frac{(V_n - \beta)^{q(1-\sigma)-1}}{V_n^{q-1}} a_n \right]^{\frac{1}{q}}. \tag{3.8}
\]

Then by (2.8), we derive (3.3).

By Hölder’s inequality (cf. (48)), we have
\[
I = \int_0^\infty \left( \frac{U^{\delta x}(x)}{\mu_1(x)} \right)^{\frac{1}{q}} f(x) \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta x}(x))(V_n - \beta)^{\gamma}}{e^{aU^{\delta x}(x)(V_n - \beta)^{\gamma}}} a_n \right] dx \leq ||f||_{p, \phi_\beta} J^2. \tag{3.9}
\]
Then by (3.3), we obtain (3.1). On the other hand, assuming that (3.3) is valid, we set
\[ f(x) := \frac{\mu(x)}{U^{-q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \csc \left( \rho U^{-q\gamma}(x)(V_n - \beta)^\gamma \right) \alpha_n \right]^{q-1}, \quad x \in \mathbb{R}_+. \]

Then we find \( J_2^q = \|f\|_{p, \Phi_\delta}^p \). If \( J_2 = 0 \), then (3.3) is trivially valid; if \( J_2 = \infty \), then (3.3) remains impossible. Suppose that \( 0 < J_2 < \infty \). By (3.1), we have
\[
\begin{align*}
\|f\|_{p, \Phi_\delta}^p &= J_2 = I < k(\sigma) \|f\|_{p, \Phi_\delta}^p \|a\|_{q, \Psi_\beta}, \\
\|f\|_{p, \Phi_\delta}^{p-1} &= J_2 < k(\sigma) \|a\|_{q, \Psi_\beta},
\end{align*}
\]
and then (3.3) follows, which is equivalent to (3.1).

Therefore, (3.1), (3.2) and (3.3) are equivalent. \( \square \)

**Theorem 3.2.** With the assumptions of Theorem 3.1, if \( U(\infty) = V(\infty) = \infty \), then the constant factor \( k(\sigma) \) in (3.1), (3.2) and (3.3) is the best possible.

**Proof.** For \( \varepsilon \in (0, \frac{q(\sigma - \gamma)}{2}) \), we set \( \tilde{\sigma} = \sigma - \frac{\varepsilon}{q} \), and \( \tilde{f} = \tilde{f}(x), \ x \in \mathbb{R}_+ \), \( \tilde{a} = \{\tilde{a}_n\}_{n=1}^{\infty}, \)
\[
\tilde{f}(x) = \begin{cases} 
U^{\delta(\tilde{\sigma} + \varepsilon)}(x)\mu(x), & 0 < x^\delta \leq 1 \\
0, & x^\delta > 0 \end{cases}, \quad \tilde{a}_n = (V_n - \beta)^{\tilde{\sigma} - 1}V_{n+1} = (V_n - \beta)^{\sigma - \varepsilon}V_{n+1}, \quad n \in \mathbb{N}. \quad (3.10)
\]

Then for \( \delta = \pm 1 \), since \( U(\infty) = \infty \), we find
\[
\int_{\{x > 0: 0 < x^\delta \leq 1\}} \frac{\mu(x)}{U^{1-\delta}(x)} dx = \frac{1}{\varepsilon} U^\delta(1). \quad (3.12)
\]

By (2.11), (3.12) and (2.10), we obtain
\[
\| \tilde{f} \|_{p, \Phi_\delta} \| \tilde{a} \|_{q, \Psi_\beta} = \left( \int_{\{x > 0: 0 < x^\delta \leq 1\}} \frac{\mu(x) dx}{U^{1-\delta}(x)} \right)^{\frac{1}{q}} \left[ \sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1+\varepsilon}} \right]^{\frac{1}{q}} = \frac{1}{\varepsilon} U^\delta(1) \left[ \frac{1}{(V_1 - \beta)^{\varepsilon}} + \varepsilon O(1) \right]^{\frac{1}{q}}, \quad (3.13)
\]
\[ \tilde{I} := \int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(pU^\delta(x)(V_n - \beta)^\gamma)}{e^{aU^\delta(x)(V_n - \beta)^\gamma}} \mu(x) dx \]

\[ = \int_{\{x:0<0<\delta^2 \leq 1\}} \sum_{n=1}^{\infty} \frac{\csc h(pU^\delta(x)(V_n - \beta)^\gamma)(V_n - \beta)^{\gamma - 1}v_{n+1}^x \mu(x)}{U^{1-\delta(\sigma + \epsilon)}(x)} dx \]

\[ = \int_{\{x:0<0<\delta^2 \leq 1\}} \omega_{\delta}(\sigma, x) \frac{\mu(x)}{U^{1-8\epsilon(x)}(x)} dx \]

\[ \geq k(\sigma) \int_{\{x:0<0<\delta^2 \leq 1\}} (1 - \theta_\delta(\sigma, x)) \frac{\mu(x)}{U^{1-8\epsilon(x)}(x)} dx \]

\[ = k(\sigma) \left[ \int_{\{x:0<0<\delta^2 \leq 1\}} (1 - O(U(x))^{\delta(\sigma + \epsilon)})) \frac{\mu(x)}{U^{1-8\epsilon(x)}(x)} dx \right] \]

\[ = k(\sigma) \left[ \int_{\{x:0<0<\delta^2 \leq 1\}} \frac{\mu(x)}{U^{1-8\epsilon(x)}(x)} dx - \int_{\{x:0<0<\delta^2 \leq 1\}} O \left( \frac{\mu(x)}{U^{1-8(\epsilon + \sigma + \epsilon)}(x)} dx \right) \right] \]

\[ = \frac{1}{\epsilon} k(\sigma - \epsilon)(U^{\delta}(1) - \epsilon O(1)). \]

If there exists a positive constant \( K \leq k(\sigma) \), such that (3.1) is valid when replacing \( k(\sigma) \) to \( K \), then in particular, by the Lebesgue term by term integration theorem, we have

\[ \epsilon \tilde{I} < \epsilon K \| f \|_{p, \Phi_\delta} \| \tilde{g} \|_{q, \Psi_\beta}, \]

namely,

\[ k(\sigma - \epsilon)(U^{\delta}(1) - \epsilon O(1)) \leq K \cdot U^{\delta}(1) \left( \frac{1}{(1 - \beta)^{\epsilon}} + \epsilon O(1) \right)^{\frac{1}{\epsilon}}. \]

It follows that \( k(\sigma) \leq K(\epsilon \to 0^+) \). Hence, \( K = k(\sigma) \) is the best possible constant factor of (3.1).

The constant factor \( k(\sigma) \) in (3.2) (3.3) is still the best possible. Otherwise, we would reach a contradiction by (3.6) (3.9) that the constant factor in (3.1) is not the best possible.

For \( p > 1 \), we obtain

\[ \Psi_\beta^{1-p}(n) = \frac{\nu^{n+1}}{(V_n - \beta)^{1-p \sigma}} (n \in \mathbb{N}), \quad \Phi_\delta^{1-q}(x) = \frac{\mu(x)}{U^{1-q \delta \sigma}(x)} (x \in \mathbb{R}^+), \]

and define the following real normed spaces:

\[ L_{p, \Phi_\delta}(\mathbb{R}^+) = \{ f; f = f(x), x \in \mathbb{R}^+, \| f \|_{p, \Phi_\delta} < \infty \}, \]

\[ l_{q, \Psi_\beta} = \{ a; a = \{ a_n \}_{n=1}^{\infty}, \| a \|_{q, \Psi_\beta} < \infty \}, \]

\[ L_{q, \Phi_\delta^{1-q}}(\mathbb{R}^+) = \{ h; h = h(x), x \in \mathbb{R}^+, \| h \|_{q, \Phi_\delta^{1-q}} < \infty \}, \]

\[ l_{p, \Psi_\beta^{1-p}} = \{ c; c = \{ c_n \}_{n=1}^{\infty}, \| c \|_{p, \Psi_\beta^{1-p}} < \infty \}. \]

Assuming that \( f \in L_{p, \Phi_\delta}(\mathbb{R}^+) \) and setting

\[ c = \{ c_n \}_{n=1}^{\infty}, \quad c_n := \int_0^\infty \frac{\csc h(pU^\delta(x)(V_n - \beta)^\gamma)}{e^{aU^\delta(x)(V_n - \beta)^\gamma}} f(x) dx, n \in \mathbb{N}, \]

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we can rewrite (3.2) as
\[ ||c||_{p, \psi_{b}^{1-p}} < k(\sigma)||f||_{p, \Phi_{b}} < \infty, \]
namely, \( c \in \ell_{p, \psi_{b}^{1-p}}. \)

**Definition 3.3.** Define a half-discrete Hardy-Hilbert-type operator \( T_{1} : L_{p, \Phi_{b}}(\mathbb{R}+) \rightarrow \ell_{p, \psi_{b}^{1-p}} \) as follows: For any \( f \in L_{p, \Phi_{b}}(\mathbb{R}+) \), there exists a unique representation \( T_{1}f = c \in \ell_{p, \psi_{b}^{1-p}}. \)

Define the formal inner product of \( T_{1}f \) and \( a = \{a_{n}\}_{n=1}^{\infty} \in \ell_{q, \psi_{b}} \) as follows:
\[
(T_{1}f, a) := \sum_{n=1}^{\infty} \left[ \int_{0}^{\infty} \frac{\csc h(\rho U^{q}(x)(V_{n} - \beta)^{\gamma})}{e^{\alpha U^{q}(x)(V_{n} - \beta)^{\gamma}}} f(x)dx \right] a_{n}. \tag{3.14}
\]

Then we can rewrite (3.1) and (3.2) as follows:
\[
\begin{align*}
(T_{1}f, a) &< k(\sigma)||f||_{p, \Phi_{b}}||a||_{q, \psi_{b}}, \tag{3.15} \\
||T_{1}f||_{p, \psi_{b}^{1-p}} &< k(\sigma)||f||_{p, \Phi_{b}}. \tag{3.16}
\end{align*}
\]

Define the norm of operator \( T_{1} \) as follows:
\[
||T_{1}|| := \sup_{f(\neq 0) \in L_{p, \Phi_{b}}(\mathbb{R}+)} \frac{||T_{1}f||_{p, \psi_{b}^{1-p}}}{||f||_{p, \Phi_{b}}}. \tag{3.18}
\]

Then by (3.16), it follows that \( ||T_{1}|| \leq k(\sigma) \). Since by Theorem 3.2, the constant factor in (3.16) is the best possible, we have
\[
||T_{1}|| = k(\sigma) = \frac{2\Gamma\left(\frac{\sigma}{\gamma}\right)\zeta\left(\frac{\sigma}{\gamma}, \frac{\alpha + \rho}{2\rho}\right)}{2\Gamma\left(\frac{\sigma}{\gamma}\right)\zeta\left(\frac{\sigma}{\gamma}, \frac{\alpha + \rho}{2\rho}\right)}.
\]  

Assuming that \( a = \{a_{n}\}_{n=1}^{\infty} \in \ell_{q, \psi_{b}} \) and setting
\[
h(x) := \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{q}(x)(V_{n} - \beta)^{\gamma})}{e^{\alpha U^{q}(x)(V_{n} - \beta)^{\gamma}}} a_{n}, \quad x \in \mathbb{R}+,
\]
we can rewrite (3.3) as \( ||h||_{q, \Phi_{b}^{1-q}} < k(\sigma)||a||_{q, \psi_{b}} < \infty \), namely, \( h \in L_{q, \Phi_{b}^{1-q}}(\mathbb{R}+) \).

**Definition 3.4.** Define a half-discrete Hardy-Hilbert-type operator \( T_{2} : L_{q, \Phi_{b}} \rightarrow L_{q, \Phi_{b}^{1-q}}(\mathbb{R}+) \) as follows: For any \( a = \{a_{n}\}_{n=1}^{\infty} \in \ell_{q, \psi_{b}} \), there exists a unique representation
\[
T_{2}a = h \in L_{q, \Phi_{b}^{1-q}}(\mathbb{R}+). \tag{3.18}
\]

Define the formal inner product of \( T_{2}a \) and \( f \in L_{p, \Phi_{b}}(\mathbb{R}+) \) as follows:
\[
(T_{2}a, f) := \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{q}(x)(V_{n} - \beta)^{\gamma})}{e^{\alpha U^{q}(x)(V_{n} - \beta)^{\gamma}}} a_{n} f(x)dx. \tag{3.18}
\]
Then we can rewrite (3.1) and (3.3) as follows:

$$\langle T_2 \alpha, f \rangle < k(\sigma) \| f \|_{\rho, \Phi_{\sigma}} \| \alpha \|_{q, \Psi_{\beta}},$$

$$\| T_2 \alpha \|_{q, \Phi_{q_{\beta}^{-1}}} < k(\sigma) \| \alpha \|_{q, \Psi_{\beta}}. \quad (3.19)$$

Define the norm of operator $T_2$ as follows:

$$\| T_2 \| := \sup_{\alpha \neq 0, \| \alpha \|_{q, \Psi_{\beta}}} \frac{\| T_2 \alpha \|_{q, \Phi_{q_{\beta}^{-1}}}}{\| \alpha \|_{q, \Psi_{\beta}}}. \quad (3.20)$$

Then by (3.20), we find $\| T_2 \| \leq k(\sigma)$. Since by Theorem 3.2, the constant factor in (3.20) is the best possible, we obtain

$$\| T_2 \| = k(\sigma) = \frac{2 \Gamma(\frac{q}{p})}{\gamma(2\rho)^{\sigma/\gamma}} \left( \frac{\sigma + p}{2\rho} \right) = \| T_1 \|. \quad (3.21)$$

### 4 Some Equivalent Reverse Inequalities

In the following, we also set

$$\check{\Phi}_{\beta}(x) := (1 - \theta_{\delta}(\sigma, x)) \frac{U^p(1-\delta\sigma)-1(x)}{\mu^{p-1}(x)} \quad (x \in \mathbb{R}_+).$$

For $0 < p < 1$ or $p < 0$, we still use the formal symbols $\| f \|_{\rho, \Phi_{\sigma}}$, $\| f \|_{\rho, \check{\Phi}_{\beta}}$ and $\| \alpha \|_{q, \Psi_{\beta}}$ et al.

**Theorem 4.1.** If $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1, k(\sigma)$ is indicated by (2.1), and $U(\infty) = V(\infty) = \infty$, then for $p < 0$, $0 < \| f \|_{\rho, \check{\Phi}_{\beta}}, \| \alpha \|_{q, \Psi_{\beta}} < \infty$, we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$I = \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho U^{0}(x)(V_n - \beta)^{\gamma})}{\mu(x)} \frac{d_n f(x) dx}{\mu^{p-1}(x)} > k(\sigma) \| f \|_{\rho, \check{\Phi}_{\beta}} \| \alpha \|_{q, \Psi_{\beta}}, \quad (4.1)$$

$$J_1 = \sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1-\rho\sigma}} \left[ \int_{0}^{\nu} \frac{\csc h(\rho U^{0}(x)(V_n - \beta)^{\gamma})}{\mu(x)} \frac{f(x) dx}{\mu^{p-1}(x)} \right]^p > k(\sigma) \| f \|_{\rho, \check{\Phi}_{\beta}} \quad (4.2)$$

$$J_2 = \left\{ \int_{0}^{\nu} \frac{\mu(x)}{U^{1-q\rho\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{0}(x)(V_n - \beta)^{\gamma})}{\mu^{p-1}(x)} \frac{d_n}{\mu^{p-1}(x)} \right]^q \right\}^{\frac{1}{q}}$$

$$> k(\sigma) \| \alpha \|_{q, \Psi_{\beta}}. \quad (4.3)$$

**Proof.** By the reverse weighted Hölder inequality (cf. [48]), since $p < 0$, similarly to the way we obtained (3.4) and (3.5), we have

$$\left[ \int_{0}^{\nu} \frac{\csc h(\rho U^{0}(x)(V_n - \beta)^{\gamma})}{\mu^{p-1}(x)} f(x) dx \right]^p \leq \frac{(\check{\Phi}_{\beta}(\sigma, n))^{p-1}}{(V_n - \beta)^{\rho\sigma-1} \nu_{n+1}^p} \int_{0}^{\nu} \frac{\csc h(\rho U^{0}(x)(V_n - \beta)^{\gamma})}{\mu^{p-1}(x)} U^{(1-\delta\sigma)(p-1)}(x) V_{n+1}^{\gamma} \nu_{n+1} \nu_{n+1} f^p(x) dx.$$
Then by (2.9) and the Lebesgue term by term integration theorem, it follows that

\[
J_1 \geq (k(\sigma))^\frac{1}{p} \left[ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\csc h(p U^{\delta}(x)(V_n - \beta))}{e^{a U^{\delta}(x)(V_n - \beta)^\gamma}} \frac{U^{(1-\delta)(p-1)}(x)V_{n+1}}{(V_n - \beta)^{1-\sigma} U^{p-1}(x)} f^p(x) \, dx \right]^{\frac{1}{p}} \\
= (k(\sigma))^\frac{1}{p} \left[ \int_0^{\infty} \omega_b(\sigma, x) \frac{U^{(1-\delta)(p-1)}(x)}{U^{p-1}(x)} f^p(x) \, dx \right]^{\frac{1}{p}}.
\]

Then by (2.7), we have (4.2).

By the reverse Hölder inequality (cf. [48]), we have

\[
I = \sum_{n=1}^{\infty} \left[ \frac{V_{n+1}^{\frac{1}{p}}}{(V_n - \beta)^{\frac{1}{p} - \sigma}} \int_0^{\infty} \frac{\csc h(p U^{\delta}(x)(V_n - \beta))^\gamma}{e^{a U^{\delta}(x)(V_n - \beta)^\gamma}} f(x) \, dx \right] \left[ \frac{(V_n - \beta)^{\frac{1}{p} - \sigma} a_n}{V_{n+1}^{\frac{1}{p}}} \right]^{\frac{1}{p}} \\
\geq J_1 |a|_{q, \Psi_b}.
\]

(4.4)

Then by (4.2), we derive (4.1). On the other hand, assuming that (4.1) is valid, we set \(a_n\) as in Theorem 3.1. Then we obtain

\[
J_1^p = |a|_{q, \Psi_b}.
\]

If \(J_1 = \infty\), then (4.2) is trivially valid. If \(J_1 = 0\), then (4.2) is still not valid. Suppose that \(0 < J_1 < \infty\). By (4.1), it follows that

\[
|a|_{q, \Psi_b} = J_1^p = I > k(\sigma) \|f\|_{p, \Phi_b} |a|_{q, \Psi_b}, \\
|a|_{q-1, \Psi_b} = J_1 > k(\sigma) \|f\|_{p, \Phi_b},
\]

and then (4.2) follows, which is equivalent to (4.1).

Applying again the weighted reverse Hölder inequality (cf. [48]), since \(0 < q < 1\), similarly to how we obtained (3.7) and (3.8), we have

\[
\left[ \sum_{n=1}^{\infty} \frac{\csc h(p U^{\delta}(x)(V_n - \beta))^\gamma}{e^{a U^{\delta}(x)(V_n - \beta)^\gamma}} a_n \right]^{q} \\
\geq \frac{(\omega_b(\sigma, x))^{q-1}}{U^{q \delta - 1}(x) \mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(p U^{\delta}(x)(V_n - \beta))^\gamma (V_n - \beta)^{(1-\sigma)(q-1)} \mu(x)}{e^{a U^{\delta}(x)(V_n - \beta)^\gamma} U^{1-\delta}(x) V_{n+1}^{q-1}} a_n^{q}.
\]

Then, by (2.7) and the Lebesgue term by term integration theorem, it follows that

\[
J_2 > (k(\sigma))^\frac{1}{p} \left[ \int_0^{\infty} \frac{\csc h(p U^{\delta}(x)(V_n - \beta))^\gamma (V_n - \beta)^{(1-\sigma)(q-1)} \mu(x)}{e^{a U^{\delta}(x)(V_n - \beta)^\gamma} U^{1-\delta}(x) V_{n+1}^{q-1}} a_n^{q} \, dx \right]^{\frac{1}{q}} \\
= (k(\sigma))^\frac{1}{p} \left[ \sum_{n=1}^{\infty} \omega_b(\sigma, n) \frac{(V_n - \beta)^{(1-\sigma)(q-1)}}{V_{n+1}^{q-1}} a_n^{q} \right]^{\frac{1}{q}}.
\]

Hence, by (2.9), we have (4.3).
By the reverse Hölder inequality (cf. [48]), we get

\[
I = \int_0^\infty \left( \frac{U^{\frac{1}{2}-\delta}(x)}{\mu^{\frac{1}{2}}(x)} \right) \left[ \frac{\mu^{\frac{1}{2}}(x)}{U^{\frac{1}{2}-\delta}(x)} \sum_{n=1}^\infty \frac{\csc h(pU^{\delta}(x)(V_n - \beta)^\gamma)}{e^{\alpha U^{\delta}(x)(V_n - \beta)^\gamma}} a_n \right] dx \\
\geq ||f||_{p,\Phi_n} J_2.
\]

(4.5)

Thus by (4.3), we obtain (4.1). On the other hand, assuming that (4.3) is valid, we set \( f(x) \) as in Theorem 4.1. Then we derive that

\[
J_2^q = ||f||_{p,\Phi_n}^q.
\]

If \( J_2 = \infty \), then (4.3) is trivially valid. If \( J_2 = 0 \), then (4.3) remains impossible. Suppose that \( 0 < J_2 < \infty \). By (4.1), it follows that

\[
||f||_{p,\Phi_n} = J_2^q = I > k(\sigma)||f||_{p,\Phi_n} ||a||_{q,\psi_\beta},
\]

\[
||f||_{p,\Phi_n}^{q-1} = J_2 > k(\sigma)||a||_{q,\psi_\beta},
\]

and then (4.3) follows, which is equivalent to (4.1).

Therefore, inequalities (4.1), (4.2) and (4.3) are equivalent.

For \( \epsilon \in (0, \frac{q(\sigma-\gamma)}{2}) \), we set \( \tilde{\sigma} = \sigma - \frac{\epsilon}{q} \), and \( \tilde{f} = \tilde{f}(x) \), \( x \in \mathbb{R}_+ \), \( \tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty \),

\[
\tilde{f}(x) = \left\{ \begin{array}{ll}
U^{(\tilde{\sigma}+\epsilon)-1}(x)\mu(x), & 0 < x^{\tilde{\sigma}} \leq 1 \\
0, & x^{\tilde{\sigma}} > 0
\end{array} \right.
\]

\[
\tilde{a}_n = (V_n - \beta)^{\tilde{\sigma}-1} V_{n+1} = (V_n - \beta)^{\sigma - \frac{\epsilon}{q} - 1} V_{n+1}, \quad n \in \mathbb{N}.
\]

By (2.11), (3.12) and (2.7), we obtain

\[
||\tilde{f}||_{p,\Phi_n} ||\tilde{a}||_{q,\psi_\beta} = \frac{1}{\epsilon} U^{\frac{\epsilon}{p}}(1) \left[ \frac{1}{(V_1 - \beta)^{\epsilon}} + \epsilon O(1) \right]^{\frac{1}{p}},
\]

\[
\tilde{I} = \sum_{n=1}^\infty \int_0^\infty \frac{\csc h(pU^{\delta}(x)(V_n - \beta)^\gamma) \mu(x)}{e^{\alpha U^{\delta}(x)(V_n - \beta)^\gamma}} \tilde{a}_n \tilde{f}(x) dx
\]

\[
= \int \{x > 0, 0 < x^{\tilde{\sigma}} \leq 1\} \omega_0(\tilde{\sigma},x) \frac{\mu(x)}{U^{1-\delta}(x)} dx
\]

\[
\leq k(\tilde{\sigma}) \int \{x > 0, 0 < x^{\tilde{\sigma}} \leq 1\} \frac{\mu(x)}{U^{1-\delta}(x)} dx
\]

\[
= \frac{1}{\epsilon} k(\sigma - \frac{\epsilon}{q}) U^{\delta}(1).
\]

If there exists a positive constant \( K \geq k(\sigma) \), such that (4.1) is valid when replacing \( k(\sigma) \) to \( K \), then in particular, we have

\[
\epsilon \tilde{I} > \epsilon K ||\tilde{f}||_{p,\Phi_n} ||\tilde{a}||_{q,\psi_\beta},
\]
Therefore, namely,

\[ k(\sigma - \frac{\varepsilon}{q}) U^\delta (1) > K \cdot U^\delta (1) \left[ \frac{1}{(V_1 - \beta)^e} + \varepsilon O(1) \right]^{\frac{1}{\beta}}. \]

It follows that \( k(\sigma) \geq K(\varepsilon \to 0^+) \). Hence, \( K = k(\sigma) \) is the best possible constant factor of (4.1).

The constant factor \( k(\sigma) \) in (4.2) (4.3) is still the best possible. Otherwise, we would reach a contradiction by (4.4) (4.5) that the constant factor in (4.1) is not the best possible.

\[ \square \]

**Theorem 4.2.** With the assumptions of Theorem 4.1, if

\[ 0 < p < 1, \ 0 < ||f||_{p, \Phi_3}, ||a||_{q, \Psi_{\beta}} < \infty, \]

then we have the following equivalent inequalities with the best possible constant factor \( k(\sigma) \):

\[ I = \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho U^{\delta y}(x)(V_n - \beta)^\gamma)}{e^{U^{\delta y}(x)(V_n - \beta)^\gamma}} \cdot a_n f(x) dx > k(\sigma)||f||_{p, \Phi_3}||a||_{q, \Psi_{\beta}} \quad (4.6) \]

\[ J_1 = \sum_{n=1}^{\infty} \frac{\nu_{n+1}}{\nu_n} \left[ \int_{0}^{\infty} \frac{\csc h(\rho U^{\delta y}(x)(V_n - \beta)^\gamma)}{e^{U^{\delta y}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k(\sigma)||f||_{p, \Phi_3} \quad (4.7) \]

\[ J := \left\{ \int_{0}^{\infty} (1 - \theta_\delta(\sigma, x))^{1-q} \mu(x) \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U^{\delta y}(x)(V_n - \beta)^\gamma)}{e^{U^{\delta y}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma)||a||_{q, \Psi_{\beta}}. \quad (4.8) \]

**Proof.** By the reverse weighted Hölder inequality (cf. [48]), since \( 0 < p < 1 \), similarly to as we obtained (3.4) and (3.5), we have

\[ \left[ \int_{0}^{\infty} \frac{\csc h(\rho U^{\delta y}(x)(V_n - \beta)^\gamma)}{e^{U^{\delta y}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p \geq \frac{(\omega_\delta(\sigma, n))^{p-1}}{(V_n - \beta)^{p\sigma-1} \nu_{n+1} \nu_n} \int_{0}^{\infty} \frac{\csc h(\rho U^{\delta y}(x)(V_n - \beta)^\gamma)}{e^{U^{\delta y}(x)(V_n - \beta)^\gamma}} \cdot U_{(1-\delta\sigma)(p-1)}(x) \nu_n \cdot f^p(x) dx. \]

In view of (2.9) and the Lebesgue term by term integration theorem, we find

\[ J_1 \geq (k(\sigma))^\frac{1}{p} \left[ \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho U^{\delta y}(x)(V_n - \beta)^\gamma)}{e^{U^{\delta y}(x)(V_n - \beta)^\gamma}} \cdot U_{(1-\delta\sigma)(p-1)}(x) \nu_n \cdot f^p(x) dx \right]^\frac{1}{p} \]

\[ = (k(\sigma))^\frac{1}{p} \left[ \int_{0}^{\infty} \omega_\delta(\sigma, x) \cdot U_{(p-1-\delta\sigma-1)}(x) \cdot f^p(x) dx \right]^\frac{1}{p}. \]

Then by (2.10), we have (4.7).
By the reverse Hölder inequality (cf. \[48\]), we have

\[
I = \sum_{n=1}^{\infty} \left[ \frac{v_{n+1}^{\frac{1}{p}}}{(V_n - \beta)^{\frac{1}{p} - \sigma}} \int_0^\infty \frac{csc h(pU^{-\beta}(x)(V_n - \beta)^\gamma)}{e^{U^{-\beta}(x)(V_n - \beta)^\gamma}} f(x) \, dx \right] \left[ (V_n - \beta)^{\frac{1}{p} - \sigma} a_n \right]
\geq J_1 \|a\|_{q, \Psi_\beta}.
\tag{4.9}
\]

Then by (4.7), we have (4.6). On the other hand, assuming that (4.6) is valid, we set \(a_n\) as in Theorem 3.1. Then we find \(J_1^p = \|a\|^q_{q, \Psi_\beta}\). If \(J_1 = \infty\), then (4.7) is trivially valid; if \(J_1 = 0\), then (4.7) keeps impossible. Suppose that \(0 < J_1 < \infty\). By (4.6), it follows that

\[
\|a\|^q_{q, \Psi} = J_1^p = I > k(\sigma) \|f\|_{p, \Phi_\beta} \|a\|_{q, \Psi_\beta},
\]

and then (4.7) follows, which is equivalent to (4.6).

Similarly, by the reverse weighted Hölder inequality (cf. \[48\]), since \(q < 0\), we have

\[
\sum_{n=1}^{\infty} \frac{csc h(pU^{-\beta}(x)(V_n - \beta)^\gamma)}{e^{U^{-\beta}(x)(V_n - \beta)^\gamma}} a_n\]

\[
\leq \frac{\omega_n(\sigma, x)^{q-1}}{U^{q\delta-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\csc h(pU^{-\beta}(x)(V_n - \beta)^\gamma)}{e^{U^{-\beta}(x)(V_n - \beta)^\gamma}} (V_n - \beta)^{1-\sigma}(q-1)\mu(x) \frac{\mu(x)^{\frac{1}{q}}}{U^{1-\delta}(x)\nu_{n+1}^{q-1}} a_n^q.
\]

Therefore, by (2.10) and the Lebesgue term by term integration theorem, it follows that

\[
J > (k(\sigma))^{\frac{1}{p}} \left[ \int_0^\infty \sum_{n=1}^{\infty} \frac{\csc h(pU^{-\beta}(x)(V_n - \beta)^\gamma)}{e^{U^{-\beta}(x)(V_n - \beta)^\gamma}} (V_n - \beta)^{1-\sigma}(q-1)\mu(x) \frac{\mu(x)^{\frac{1}{q}}}{U^{1-\delta}(x)\nu_{n+1}^{q-1}} a_n^q \, dx \right]^{\frac{1}{q}}
\]

\[
= (k(\sigma))^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \frac{\omega_n(\sigma, x)^{q-1}}{U^{q\delta-1}(x)\mu(x)} (V_n - \beta)^{q(1-\sigma)-1} \frac{\mu(x)}{\nu_{n+1}^{q-1}} a_n^q \right]^{\frac{1}{q}}.
\]

Hence, by (2.9), we have (4.8).

By the reverse Hölder inequality (cf. \[48\]), we have

\[
I = \int_0^\infty \left[ (1 - \theta(\sigma, x))^{\frac{1}{p}} U^{-\frac{1}{p} - \delta}(x) f(x) \right] \frac{1}{\mu(x)} dx \geq \|f\|_{p, \Phi_\beta} J.
\tag{4.10}
\]

Then by (4.8), we have (4.6). On the other hand, assuming that (4.6) is valid, we set \(f(x)\) as in Theorem 3.1. Then we derive that \(J^q = \|f\|^p_{p, \Phi_\beta}\). If \(J = \infty\), then (4.8) is trivially valid; if \(J = 0\), then (4.8) is still not valid. Suppose that \(0 < J < \infty\). By (4.6), it follows that

\[
\|f\|^p_{p, \Phi_\beta} = J^q = I > k(\sigma) \|f\|_{p, \Phi_\beta} \|a\|_{q, \Psi_\beta},
\]

\[
\|f\|^{p-1}_{p, \Phi_\beta} = J > k(\sigma) \|a\|_{q, \Psi_\beta},
\]
and then (4.8) follows, which is equivalent to (4.6).

Therefore, inequalities (4.6), (4.7) and (4.8) are equivalent.

For \( \varepsilon \in (0, \frac{\rho_{(\sigma-\eta)}}{p}) \), we set \( \tilde{\sigma} = \sigma + \frac{\varepsilon}{p} \), and \( \tilde{f} = \tilde{f}(x), x \in \mathbb{R}_+, \tilde{a} = \{ \tilde{a}_n \}_{n=1}^{\infty} \),

\[
\tilde{f}(x) = \begin{cases} 
U^{\delta \tilde{a} - 1}(x) \mu(x), 0 < x^\delta \leq 1 \\
0, x^\delta > 0 \end{cases},
\]

\( \tilde{a}_n = (V_n - \beta)^{\delta - \varepsilon - 1} V_{n+1} = (V_n - \beta)^{\sigma - \frac{1}{\delta}} - 1 V_{n+1}, n \in \mathbb{N} \).

By (2.10), (2.11) and (3.12), we obtain

\[
\sum_{n=1}^{\infty} \left( \int_{s > 0, 0 < \sigma \leq 1} (1 - O(U(x))^{\frac{1}{\delta}}(\sigma - \eta)) \frac{\mu(x) dx}{U^{1 - \delta \overline{\sigma}(x)}} \right)^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1+\varepsilon}} \right] = \frac{1}{\varepsilon} \left( U^{\delta \varepsilon}(1) - \varepsilon O_1(1) \right)^{\frac{1}{p}} \left[ \frac{1}{(V_1 - \beta)^{\varepsilon}} + \varepsilon O(1) \right].
\]

\[
\mathcal{I} = \sum_{n=1}^{\infty} \int_0^\infty \csc h \left( \rho U^{\delta \varepsilon}(x) \right) (V_n - \beta)^{\gamma} \tilde{a}_n \tilde{f}(x) dx
\]

\[
= \sum_{n=1}^{\infty} \left[ \int_{s > 0, 0 < \sigma \leq 1} \csc h \left( \rho U^{\delta \varepsilon}(x) \right) (V_n - \beta)^{\gamma} \left( V_n - \beta \right)^{\delta \varepsilon}(x) \frac{dx}{U^{1 - \delta \overline{\sigma}(x)}} \right] \frac{V_{n+1}}{(V_n - \beta)^{1+\varepsilon}}
\]

\[
\leq \sum_{n=1}^{\infty} \left[ \sum_{n=1}^{\infty} \csc h \left( \rho U^{\delta \varepsilon}(x) \right) (V_n - \beta)^{\gamma} \left( V_n - \beta \right)^{\delta \varepsilon}(x) \frac{dx}{U^{1 - \delta \overline{\sigma}(x)}} \right] \frac{V_{n+1}}{(V_n - \beta)^{1+\varepsilon}}
\]

\[
= \frac{1}{\varepsilon} k(\sigma + \frac{\varepsilon}{p}) \left[ \frac{1}{(V_1 - \beta)^{\varepsilon}} + \varepsilon O(1) \right].
\]

If there exists a positive constant \( K \geq k(\sigma) \), such that (4.1) is valid when replacing \( k(\sigma) \) by \( K \), then in particular, we have

\[
\varepsilon \mathcal{I} > \varepsilon K \left( \int_{s > 0, 0 < \sigma \leq 1} (1 - O(U(x))^{\frac{1}{\delta}}(\sigma - \eta)) \frac{\mu(x) dx}{U^{1 - \delta \overline{\sigma}(x)}} \right)^{\frac{1}{p}} \left[ \frac{1}{(V_1 - \beta)^{\varepsilon}} + \varepsilon O(1) \right].
\]

It follows that \( k(\sigma) \geq K(\varepsilon \to 0^+) \). Hence, \( K = k(\sigma) \) is the best possible constant factor of (4.6).

The constant factor \( k(\sigma) \) in (4.7) (4.8) is still the best possible. Otherwise, we would reach a contradiction by (4.9) (4.10) that the constant factor in (4.6) is not the best possible. \(\Box\)
5 Some Corollaries

For $\delta = 1$ in Theorem 3.2, Theorem 4.1 and Theorem 4.2, the following inequalities with the non-homogeneous kernel hold true:

**Corollary 5.1.** If $\rho > \max\{0, -\alpha\}$, $0 < \gamma < \sigma \leq 1$, $k(\sigma)$ is indicated by (5.2), and $U(\infty) = V(\infty) = \infty$, then

(i) for $p > 1$, $0 < ||f||_{p, \Phi_1}, ||a||_{q, \Psi_\beta} < \infty$, we have the following equivalent inequalities:

\[
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} a_n f(x) dx < k(\sigma) ||f||_{p, \Phi_1} ||a||_{q, \Psi_\beta}, \quad (5.1)
\]

\[
\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1 - \rho \sigma}} \left[ \int_{0}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p < k(\sigma) ||f||_{p, \Phi_1}, \quad (5.2)
\]

\[
\left\{ \int_{0}^{\infty} \frac{\mu(x)}{U^{1 - \rho ^\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\sigma) ||a||_{q, \Psi_\beta}; \quad (5.3)
\]

(ii) for $p < 0$, $0 < ||f||_{p, \Phi_1}, ||a||_{q, \Psi} < \infty$, we have the following equivalent inequalities:

\[
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k(\sigma) ||f||_{p, \Phi_1} ||a||_{q, \Psi_\beta}, \quad (5.4)
\]

\[
\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1 - \rho \sigma}} \left[ \int_{0}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k(\sigma) ||f||_{p, \Phi_1}, \quad (5.5)
\]

\[
\left\{ \int_{0}^{\infty} \frac{\mu(x)}{U^{1 - \rho ^\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) ||a||_{q, \Psi_\beta}; \quad (5.6)
\]

(iii) for $0 < p < 1$, $0 < ||f||_{p, \Phi_1}, ||a||_{q, \Psi} < \infty$, we have the following equivalent inequalities:

\[
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k(\sigma) ||f||_{p, \tilde{\Phi}_1} ||a||_{q, \Psi_\beta}, \quad (5.7)
\]

\[
\sum_{n=1}^{\infty} \frac{\nu_{n+1}}{(V_n - \beta)^{1 - \rho \sigma}} \left[ \int_{0}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k(\sigma) ||f||_{p, \tilde{\Phi}_1}, \quad (5.8)
\]

\[
\left\{ \int_{0}^{\infty} \frac{(1 - \theta_1(x, \sigma))^{1 - q_1}}{U^{1 - q_1}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc h(\rho U'(x)(V_n - \beta)^\gamma)}{e^{au(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) ||a||_{q, \Psi_\beta}. \quad (5.9)
\]

The above inequalities involve the best possible constant factor $k(\sigma)$.

For $\delta = -1$ in Theorem 3.2, Theorem 4.1 and Theorem 4.2, we have the following inequalities with the homogeneous kernel of degree 0:
Corollary 5.2. If $\rho > \max\{0, -\alpha\}, 0 < \gamma < \sigma \leq 1, k(\sigma)$ is indicated by (2.2), and $U(\infty) = V(\infty) = \infty$, then

(i) for $p > 1$, $0 < ||f||_{p, \Phi^{-1}}, ||a||_{q, \Psi_\beta} < \infty$, we have the following equivalent inequalities:

$$
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho(V_n - \beta)^{\gamma})}{e^{\alpha(V_n - \beta)^{\gamma}}} a_n f(x) dx < k(\sigma) ||f||_{p, \Phi^{-1}}, ||a||_{q, \Psi_\beta},
$$

(ii) for $p < 0$, $0 < ||f||_{p, \Phi^{-1}}, ||a||_{q, \Psi_\beta} < \infty$, we have the following equivalent inequalities:

$$
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho(V_n - \beta)^{\gamma})}{e^{\alpha(V_n - \beta)^{\gamma}}} a_n f(x) dx > k(\sigma) ||f||_{p, \Phi^{-1}}, ||a||_{q, \Psi_\beta},
$$

(iii) for $0 < p < 1$, $0 < ||f||_{p, \Phi^{-1}}, ||a||_{q, \Psi_\beta} < \infty$, we have the following equivalent inequalities:

$$
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc h(\rho(V_n - \beta)^{\gamma})}{e^{\alpha(V_n - \beta)^{\gamma}}} a_n f(x) dx > k(\sigma) ||f||_{p, \Phi^{-1}}, ||a||_{q, \Psi_\beta},
$$

The above inequalities involve the best possible constant factor $k(\sigma)$. For $\alpha = \rho$ in Theorem 3.2, Theorem 4.1 and Theorem 4.2, we have
Corollary 5.3. If $p > 0$, $0 < \gamma < \sigma \leq 1$, and $U(\infty) = V(\infty) = \infty$, then

(i) for $p > 1$, $0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $k_1(\sigma)$:

$$k_1(\sigma) = \frac{2\Gamma\left(\frac{\sigma}{\gamma}\right)\xi\left(\frac{\sigma}{\gamma}\right)}{\gamma(2p)^{\sigma/\gamma}} ;$$

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx < k_1(\sigma)||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta}, \quad (5.19)$$

$$\sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_{0}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p < k_1(\sigma)||f||_{p,\Phi_\delta}, \quad (5.20)$$

$$\left\{ \int_{0}^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k_1(\sigma)||a||_{q,\Psi_\beta}; \quad (5.21)$$

(ii) for $p > 0, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $k_1(\sigma)$:

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k_1(\sigma)||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta}, \quad (5.22)$$

$$\sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_{0}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k_1(\sigma)||f||_{p,\Phi_\delta}, \quad (5.23)$$

$$\left\{ \int_{0}^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k_1(\sigma)||a||_{q,\Psi_\beta}; \quad (5.24)$$

(iii) for $0 < p < 1, 0 < ||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta} < \infty$, we have the following equivalent inequalities with the best possible constant factor $k_1(\sigma)$:

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n f(x) dx > k_1(\sigma)||f||_{p,\Phi_\delta}, ||a||_{q,\Psi_\beta}, \quad (5.25)$$

$$\sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1-p\sigma}} \left[ \int_{0}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} f(x) dx \right]^p > k_1(\sigma)||f||_{p,\Phi_\delta}, \quad (5.26)$$

$$\left\{ \int_{0}^{\infty} \frac{(1 - \theta_\delta(\sigma, x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\csc(pU^{\delta\gamma}(x)(V_n - \beta)^\gamma)}{e^{pU^{\delta\gamma}(x)(V_n - \beta)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k_1(\sigma)||a||_{q,\Psi_\beta}; \quad (5.27)$$

For $\gamma = \frac{\sigma}{\gamma}$ in Corollary 5.3, we obtain the following:
Corollary 5.4. If $p > 0, 0 < \sigma \leq 1$, and $U(\infty) = V(\infty) = \infty$, then
(i) for $p > 1, 0 < ||f||_{p, \Phi\delta}, ||a||_{q, \Psi\rho} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma \rho^2}$:

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) a_n f(x) dx < \frac{\pi^2}{6\sigma \rho^2} ||f||_{p, \Phi\delta} ||a||_{q, \Psi\rho}, \tag{5.28}$$

$$\sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1-\rho \sigma}} \left[ \int_{0}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) f(x) dx \right]^p < \frac{\pi^2}{6\sigma \rho^2} ||f||_{p, \Phi\delta}, \tag{5.29}$$

$$\left\{ \int_{0}^{\infty} \mu(x) \left[ \sum_{n=1}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) a_n \right]^q dx \right\}^{\frac{1}{q}} < \frac{\pi^2}{6\sigma \rho^2} ||a||_{q, \Psi\rho}, \tag{5.30}$$

(ii) for $p < 0, 0 < ||f||_{p, \Phi\delta}, ||a||_{q, \Psi\rho} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma \rho^2}$:

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) a_n f(x) dx > \frac{\pi^2}{6\sigma \rho^2} ||f||_{p, \Phi\delta} ||a||_{q, \Psi\rho}, \tag{5.31}$$

$$\sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1-\rho \sigma}} \left[ \int_{0}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) f(x) dx \right]^p > \frac{\pi^2}{6\sigma \rho^2} ||f||_{p, \Phi\delta}, \tag{5.32}$$

$$\left\{ \int_{0}^{\infty} \mu(x) \left[ \sum_{n=1}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\pi^2}{6\sigma \rho^2} ||a||_{q, \Psi\rho}, \tag{5.33}$$

(iii) for $0 < p < 1, 0 < ||f||_{p, \Phi\delta}, ||a||_{q, \Psi\rho} < \infty$, we have the following equivalent inequalities with the best possible constant factor $\frac{\pi^2}{6\sigma \rho^2}$:

$$\sum_{n=1}^{\infty} \int_{0}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) a_n f(x) dx > \frac{\pi^2}{6\sigma \rho^2} ||f||_{p, \Phi\delta} ||a||_{q, \Psi\rho}, \tag{5.34}$$

$$\sum_{n=1}^{\infty} \frac{V_{n+1}}{(V_n - \beta)^{1-\rho \sigma}} \left[ \int_{0}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) f(x) dx \right]^p > \frac{\pi^2}{6\sigma \rho^2} ||f||_{p, \Phi\delta}, \tag{5.35}$$

$$\left\{ \int_{0}^{\infty} (1 - \theta_\delta(\sigma, x))^{1-q} \mu(x) \left[ \sum_{n=1}^{\infty} \csc h(pU^{\delta \sigma}/2(x)(V_n - \beta^{\sigma}/2)) a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\pi^2}{6\sigma \rho^2} ||a||_{q, \Psi\rho}. \tag{5.36}$$
Remark 5.5.
(i) For $\beta = 0$ in (3.1), the following inequality holds true:

$$
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc \left( \rho \left( U \delta (x) V_n \right) \right)}{e^{\alpha \left( U \delta (x) V_n \right)}} a_n f(x) \, dx < k(\sigma) \| f \|_{\rho, \Phi_0} \| a \|_{q, \Psi_0}.
$$

(5.37)

Hence, (3.1) is a more accurate inequality of (5.37) for $0 < \beta \leq \frac{\nu}{2}$.

(ii) For $\mu(x) = \nu_n = 1$ in (5.37), we have the following inequality with the best possible constant factor $k(\sigma)$:

$$
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc \left( \rho \left( x^\delta n \right) \right)}{e^{\alpha \left( x^\delta n \right)}} a_n f(x) \, dx
\leq k(\sigma) \left[ \int_{0}^{\infty} x^{p(1-\delta \sigma)-1} f^p(x) \, dx \right] \frac{1}{p} \left[ \sum_{n=1}^{\infty} n^q(1-\sigma)^{-1} a_n^q \right]^{\frac{1}{q}}.
$$

(5.38)

In particular, for $\delta = 1$, we have the following inequality with the non-homogeneous kernel:

$$
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc \left( \rho \left( x \right) \right)}{e^{\alpha \left( x \right)}} a_n f(x) \, dx
\leq k(\sigma) \left[ \int_{0}^{\infty} x^{p(1-\sigma)-1} f^p(x) \, dx \right] \frac{1}{p} \left[ \sum_{n=1}^{\infty} n^q(1-\sigma)^{-1} a_n^q \right]^{\frac{1}{q}};
$$

(5.39)

for $\delta = -1$, we have the following inequality with the homogeneous kernel of degree 0:

$$
\sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{\csc \left( \rho \left( \frac{x}{n} \right) \right)}{e^{\alpha \left( \frac{x}{n} \right)}} a_n f(x) \, dx
\leq k(\sigma) \left[ \int_{0}^{\infty} x^{p(1+\sigma)-1} f^p(x) \, dx \right] \frac{1}{p} \left[ \sum_{n=1}^{\infty} n^q(1-\sigma)^{-1} a_n^q \right]^{\frac{1}{q}}.
$$

(5.40)

Acknowledgements

The authors wish to express their thanks to the referees for their careful reading of the manuscript and for their valuable suggestions.

We would like to thank Professors J. C. Kuang and M. Krnić for their very useful comments.

B. Yang: This work is supported by the National Natural Science Foundation of China (No. 61370186), and 2013 Knowledge Construction Special Foundation Item of Guangdong Institution of Higher Learning College and University (No. 2013KJCX0140). We are grateful for their help.

M. Th. Rassias: This work is supported by the SNF grant: SNF PP00P2_138906. I would like to express my gratitude to Professor P. -O. Dehaye and the Swiss National Science Foundation for providing me with financial support to conduct postdoctoral research at the University of Zurich during the academic year 2015-2016.
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