THE GEOMETRIC KANNAN-LOVÁSZ-SIMONOVITS LEMMA,
DIMENSION-FREE ESTIMATES FOR VOLUMES OF
SUBLEVEL SETS OF POLYNOMIALS, AND DISTRIBUTION
OF ZEROES OF RANDOM ANALYTIC FUNCTIONS.

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ABSTRACT. The goal of this paper is to attract the attention of the reader to a sim-
ple dimension-free geometric inequality that can be proved using the classical needle
decomposition technique. This inequality allows us to derive sharp dimension-free
estimates for the distribution of values of polynomials in convex subsets in $\mathbb{R}^n$ in a
simple and elegant way. Such estimates, in their turn, lead to a surprising result
about the distribution of zeroes of random analytic functions; informally speaking,
we show that for simple families of analytic functions, there exists a “typical” distri-
bution of zeroes such that the portion of the family occupied by the functions whose
distribution of zeroes deviates from that typical one by some fixed amount is about
$\text{Const} \exp\{-\text{size of the deviation}\}$.

The paper is essentially self-contained. When choosing the style, we tried to make
it an enjoyable reading for both a senior undergraduate student and an expert.

As to the standard question “What is new in the paper?”, one is supposed to
address in the abstract, we believe that the answer to it is a function of two variables,
the first being “what is written” and the second being “who is reading”. Since we
have no knowledge of the value of the second variable, we can only give the range
of answers with the first variable fixed. We believe that for the targeted audience it
will be the standard range [Nothing, Everything] (with both endpoints included).

§1. THE GEOMETRIC KANNAN-LOVÁSZ-SIMONOVITS LEMMA

By this name we will call the following

Proposition:

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Let $\mathcal{F}$ be a compact convex subset of $\mathbb{R}^n$ with non-empty interior, and let $\mathcal{E} \subset \mathcal{F}$ be a closed set. Let $\lambda > 1$, and let

$$E_{\lambda,\mathcal{F}} := \left\{ x \in \mathcal{E} : \text{for every interval } J \text{ such that } x \in J \subset \mathcal{F}, \frac{|E \cap J|}{|J|} \geq \frac{\lambda - 1}{\lambda} \right\}.$$

Then

$$\frac{\text{Vol}(E_{\lambda,\mathcal{F}})}{\text{Vol}(\mathcal{F})} \leq \left[ \frac{\text{Vol}(\mathcal{E})}{\text{Vol}(\mathcal{F})} \right]^\lambda.$$

**Remark:**
In the definition of the “core” $E_{\lambda,\mathcal{F}}$ it is enough to consider only the intervals $J$ that have $x$ as one of their endpoints. Indeed, if $x$ is an interior point of $J$, and the condition $\frac{|E \cap J|}{|J|} \geq \frac{\lambda - 1}{\lambda}$ is satisfied for each of the two subintervals into which $x$ splits $J$, then it is satisfied for the entire interval $J$.

**Proof of the Geometric KLS Lemma:**
Consider first the following special case: let $\mathcal{L} \subset \mathbb{R}^n$ be a line and let $\mathbb{P}$ be the orthogonal projection to $\mathcal{L}$. Let $I = \mathbb{P} \mathcal{F} \subset \mathcal{L}$. Assume that $\mathcal{E} = \{x \in \mathcal{F} : \mathbb{P}x \in E\}$ where $E$ is some closed subset of $I$.

**Claim:**

$$E_{\lambda,\mathcal{F}} = \mathcal{F} \cap \mathbb{P}^{-1}E_{\lambda,I}.$$

Informally, this means that the set $E_{\lambda,\mathcal{F}}$ is determined by its projection onto the line $\mathcal{L}$ (as the maximal subset of $\mathcal{F}$ with given projection) and that this projection is

$$E_{\lambda,I} := \left\{ x \in E : \text{for every interval } J \text{ such that } x \in J \subset I, \frac{|E \cap J|}{|J|} \geq \frac{\lambda - 1}{\lambda} \right\}.$$

Speaking even more vaguely, one may say that the statement of the geometric KLS lemma for such “simple” sets is “essentially one-dimensional”.

**Proof of the claim:**
Since this claim is a simple exercise in geometry, we shall present only the proof of that part of it that we really need, namely, of that the set on the left is contained in the set on the right. Suppose that $x \in \mathcal{E}$ and $x = \mathbb{P}x \notin E_{\lambda,I}$. Then there exists an interval $J \subset I$ such that $x$ is one of its endpoints and $\frac{|J \cap E|}{|J|} < \frac{\lambda - 1}{\lambda}$. Let
$y \in I$ be the other end of $J$. There exists a point $y \in F$ such that $y = P_y$. Since $F$ is convex, the entire interval $J = xy$ is contained in $F$. It is easy to check that
\[
\frac{|J \cap E|}{|J|} = \frac{|J \cap E|}{|J|} < \frac{\lambda - 1}{\lambda}
\]
and thereby $x \notin E_{\lambda,F}$. □

Once the claim has been proved, we are ready to reformulate the statement of the geometric KLS lemma for this special case as a one-dimensional problem. Let $f(x)$ ($x \in I$) be the $(n-1)$-dimensional volume of the cross-section of the convex set $F$ by the hyperplane orthogonal to the line $L$ and containing the point $x$. We have
\[
\text{Vol}(E_{\lambda,F}) = \int_{E_{\lambda,I}} f(x) \, dx;
\]
\[
\text{Vol}(F) = \int_I f(x) \, dx;
\]
and
\[
\text{Vol}(E) = \int_E f(x) \, dx.
\]
Using these three formulae, we see that the statement of the geometric KLS lemma for our special case can be rewritten as
\[
\frac{\int_{E_{\lambda,I}} f}{\int_I f} \leq \left[ \frac{\int_E f}{\int_I f} \right]^\lambda.
\]

The best thing one can hope for is that this inequality is valid for an arbitrary non-negative continuous function $f$ and an arbitrary set $E \subset I$. It doesn’t take a long time to see that it is not the case, so the next natural question to ask is “What is so special about the functions that express the volumes of cross-sections of convex bodies?”. The answer is given by the classical Brunn-Minkowski theorem, one of several equivalent formulations of which is that the function $f(x)^{\frac{1}{n-1}}$ is concave, i.e.,
\[
f(tx + (1-t)y)^{\frac{1}{n-1}} \geq tf(x)^{\frac{1}{n-1}} + (1-t)f(y)^{\frac{1}{n-1}} \quad \text{for all } x, y \in I, \ t \in [0,1].
\]

This property is for each $n$ stronger than and for large $n$ almost equivalent to logarithmic concavity of the function $f$, i.e., to the inequality $f(tx + (1-t)y) \geq f(x)^t f(y)^{1-t}$. Thus, our special case is covered by the following

**Lemma:**
Let $I \subset \mathbb{R}$ be an interval and let $f : I \to [0, +\infty)$ be a logarithmically concave function that does not vanish at interior points of $I$. Let $E \subset I$ be a measurable set. Fix $\lambda > 1$ and define

$$E_{\lambda,I} := \left\{ x \in E : \text{for every interval } J \text{ such that } x \in J \subset I, \quad \frac{|E \cap J|}{|J|} \geq \frac{\lambda - 1}{\lambda} \right\}.$$

Then

$$\frac{\int_{E_{\lambda,I}} f}{\int_I f} \leq \left[ \frac{\int_E f}{\int_I f} \right]^\lambda.$$

If at this stage the reader has the feeling that, once formulated, this statement requires only some routine techniques he already knows to prove it, he is probably right. We offer such a reader to try to prove the lemma by himself before reading our proof in the Appendix in the hope that he might be able to come up with a nicer proof than that of ours, which, though completely natural, lacks in elegance.

Our next task will be to reduce the full statement of the Geometric KLS lemma to this special case. We will do it using the classical needle decomposition.

First of all, let us remind/tell the reader what the classical needle decomposition is. Given a compact convex body $F \in \mathbb{R}^n$ and $\delta > 0$, we can perform the following construction. Take any 2-dimensional plane $K$ that intersects $F$ and choose a $\delta$-net in the set $F \cap K$. For each point in this $\delta$-net, take an $(n - 2)$-dimensional plane that is orthogonal to $K$ and intersects $K$ at the corresponding point. Clearly, for any 2-dimensional plane $K'$ sufficiently close to $K$ in some natural metric\(^1\) these planes are transversal to $K'$ and their intersections with $K'$ form a $2\delta$-net in $K' \cap F$. Therefore, since the set of all 2-dimensional planes intersecting $F$ is compact in any natural metric, we can find finitely many $(n - 2)$-dimensional spaces $M_1, \ldots, M_N$ such that for every 2-dimensional plane $K$ intersecting $F$, the set of points at which $K$ is intersected by those of the planes $M_1, \ldots, M_N$ that are transversal to it, forms a $2\delta$-net in $K \cap F$.

Carry out the following algorithm:

**Step 1:**

Choose a hyperplane $H \supseteq M_1$. It splits $F$ into two compact convex subsets $F^+$ and $F^-$.

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\(^1\)One possible way to introduce a “natural distance” between two planes $K_1$ and $K_2$ of the same dimension is the following. Consider all isometric motions of $\mathbb{R}^n$ that map $K_1$ to $K_2$. Every such isometric motion is of the kind $x \mapsto Ux + a$ where $a \in \mathbb{R}^n$ and $U$ is a unitary operator. Define $\text{dist}(K_1, K_2) := \inf(||U - I|| + |a|)$. To check the axioms of distance is left to the reader as an exercise.
**Step 2:**
Choose a hyperplane $H^+ \supset M_2$ and split $F^+$ into 2 compact convex subsets (one of which may be empty) by this hyperplane. Then choose a hyperplane $H^- \supset M_2$ and split $F^-$ into 2 compact convex subsets by that hyperplane.

**Step $k$:**
After completion of Step $k-1$, we have a decomposition of $F$ into $2^{k-1}$ subsets. Split each of those subsets into two smaller ones by a hyperplane containing $M_k$ (so, Step $k$ consists of $2^{k-1}$ substeps).

After completing all $N$ steps in this algorithm, we obtain a decomposition of $F$ into $2^N$ compact convex subsets $F_j$ some of which may be empty.

**Definition:**
Let $\gamma > 0$. A convex set $F$ is called a $\gamma$-needle if there is a line in $\mathbb{R}^n$ such that the distance from every point of $F$ to this line is not greater than $\gamma$.

**Claim:**
Every set $F_j$ is an $8\delta$-needle.

**Proof of the claim:**
Let us first show that for every two-dimensional plane $K$, the set $F_j \cap K$ contains no disk $D$ of radius $2\delta$. Indeed, otherwise there would exist an $(n-2)$-dimensional plane $M_k$ transversal to $K$ such that $M_k$ intersects $K$ at some point inside the disk $D$. But then the set $F_j$ cannot be contained entirely in any half-space bounded by any hyperplane containing $M_k$. On the other hand, Step $k$ provides such a half-space, and we get a contradiction.

Now, let $a$ and $b$ be the endpoints of the longest interval contained in $F_j$. Note that for every point $c \in F_j$, the angles $\hat{a}$ and $\hat{b}$ of the triangle $abc$ are less than $\pi/2$.

If $\text{dist}(a,b) \leq 8\delta$, then $F_j$ lies in a $8\delta$-neighborhood of any line containing the point $a$. Otherwise, consider any point $c \in F_j$. If the distance from the point $c$ to the line $ab$ is greater than $8\delta$, then the triangle $abc$ contains a rectangle both sides of which are greater than $4\delta$ and, thereby, a disk of radius $2\delta$, which is impossible. Thus, $F_j$ lies entirely in a $8\delta$-neighborhood of the line $ab$. □

This construction can be used (and/or generalized) in many different ways. Since we are not after that extremely elusive thing known by the name “full generality” in this note, we shall only show how this construction can be used to fit our purposes. For other usages see the papers [ND1] by Gromov-Milman, [ND2] by Lovász-Simonovits, and [ND3] by Kannan-Lovász-Simonovits where this elementary idea
was developed into a powerful tool in “high-dimensional” geometry, especially in the study of isoperimetric inequalities.

The only freedom we have in the algorithm described above is the choice of the hyperplanes containing given \((n - 2)\)-dimensional planes. That is one degree of freedom at each substep and we can use it to “solve one equation”.

Now take some small \(\delta > 0\). Let \(\tilde{E} := \{ x \in F : \text{dist}(x, \mathcal{E}) \leq 16\delta \}\) and let \(\alpha = \frac{\text{Vol}(\tilde{E})}{\text{Vol}(F)}\). Let us look at the first step in the needle decomposition construction.

To choose a hyperplane \(H \supset M_1\) is the same as to choose a unit vector \(v \perp M_1\) (the unit vector orthogonal to \(H\)). Since \(\dim M_1 = n - 2\), the set of such vectors \(v\) is a unit circumference. Let’s adopt the natural agreement that \(F_j^+ = F_j^+(v)\) is the part of \(F\) contained in the half-space that lies in the direction of the vector \(v\) from \(H\), i.e., \(F_j^+ = \{ x \in F : \langle x - y, v \rangle \geq 0 \text{ for all } y \in H \}\), and that \(F_j^-\) is the other part.

Suppose that \(\text{Vol}(\tilde{E}^+(v)) > \alpha \text{Vol}(F_j^+(v))\). Then, obviously,
\[
\text{Vol}(\tilde{E}^+(-v)) = \text{Vol}(\tilde{E}^-(v)) < \alpha \text{Vol}(F_j^-(v)) = \alpha \text{Vol}(F_j^+(-v)).
\]

Thus, since the continuous function \(v \mapsto \text{Vol}(\tilde{E}^+(v)) - \alpha \text{Vol}(F_j^+(v))\) attains both positive and negative values on the unit circumference, it must vanish somewhere, i.e., there exists a hyperplane \(H \supset M_1\) such that \(\text{Vol}(\tilde{E}^+) = \alpha \text{Vol}(F_j^+)\) (this is exactly that “one equation” we solve using one degree of freedom we have in Step 1). Obviously, for such a hyperplane, we also have \(\text{Vol}(\tilde{E}^-) = \alpha \text{Vol}(F_j^-)\). It is easy to check that two other possible assumptions \(\text{Vol}(\tilde{E}^+(v)) < \alpha \text{Vol}(F_j^+(v))\) and \(\text{Vol}(\tilde{E}^+(v)) = \alpha \text{Vol}(F_j^+(v))\) result in the same conclusion.

Making an analogous choice during each (sub)step, we shall arrive at the decomposition of \(F\) into \(8\delta\)-needles \(F_j\) such that the volumes of the corresponding parts \(\tilde{E}_j = \tilde{E} \cap F_j\) of the set \(\tilde{E}\) satisfy \(\text{Vol}(\tilde{E}_j) = \alpha \text{Vol}(F_j)\).

Let \(L_j \subset \mathbb{R}^n\) be some line in whose \(8\delta\)-neighborhood the set \(F_j\) is contained. Let \(P_j\) be the orthogonal projection onto \(L_j\). Let \(I_j = P_j F_j\). At last, let \(E_j = \tilde{E} \cap F_j\). Denote by \(G_j\) the maximal subset of \(F_j\) whose orthogonal projection to the line \(L_j\) coincides with that of \(E_j\). Formally, it means that \(G_j = F_j \cap \bigcap_{I_j}(P_j E_j)\). Clearly, \(E_j \subset G_j \subset \tilde{E}_j\). We have
\[
F_j \cap E_{\lambda,F} \subset (G_j)_{\lambda,F} \subset (G_j)_{\lambda,F}
\]

Applying the special case of the geometric KLS lemma to the sets \(G_j\) and \(F_j\) and recalling that \(\text{Vol}(G_j) \leq \text{Vol}(\tilde{E}_j) = \alpha \text{Vol}(F_j)\), we obtain
\[
\text{Vol}(F_j \cap E_{\lambda,F}) \leq \alpha^3 \text{Vol}(F_j).
\]
Adding these estimates for all $j$, we arrive at the inequality $\text{Vol}(\mathcal{E}_{\lambda, F}) \leq \alpha^\lambda \text{Vol}(F)$ or, equivalently,

$$\frac{\text{Vol}(\mathcal{E}_{\lambda, F})}{\text{Vol}(F)} \leq \left\{ \frac{\text{Vol}(\mathcal{\tilde{E}})}{\text{Vol}(F)} \right\}^\lambda.$$ 

Now, to finish the proof, it remains only to note that $\text{Vol}(\mathcal{\tilde{E}}) \to \text{Vol}(\mathcal{E})$ as $\delta \to 0$. □

If the reader wants to understand this proof better and to see how neatly the needle decomposition works, we recommend him to consider the convex set $F = \{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |\mathbf{x}| \leq 1 \} \subset \mathbb{R}^2$ with subsets $\mathcal{E}_1 = \{ \mathbf{x} \in F : x_2 \geq 0 \}$ and $\mathcal{E}_2 = \{ \mathbf{x} \in F : |\mathbf{x}| \geq r \}$ ($0 < r < 1$), and draw all the corresponding pictures and write the corresponding inequalities for these two cases.

**Remark:**

An expert may observe here that, instead of volume, we might consider an arbitrary logarithmically concave measure $\mu$ in $\mathbb{R}^n$, i.e., a measure of the kind $d\mu(\mathbf{x}) = p(\mathbf{x}) \, d\mathbf{x}$ where the density $p : \mathbb{R}^n \to [0, +\infty)$ is a logarithmically concave function. (as above, we call $p$ logarithmically concave if $p(t\mathbf{x} + (1 - t)\mathbf{y}) \geq p(\mathbf{x})^t p(\mathbf{y})^{1-t}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $0 \leq t \leq 1$). When $p \equiv 1$, we get the volume. Another interesting example coming from the probability theory is $p(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|\mathbf{x}|^2}{2}}$, the density of the standard Gaussian distribution in $\mathbb{R}^n$.

A version of the Brunn-Minkowski theorem asserts that the class of logarithmically concave measures is closed under projections of $\mathbb{R}^n$ to affine subspaces (see, e.g., [ND1]). This allows us to extend the inequality of the geometric KLS lemma verbatim to arbitrary finite logarithmically concave measures:

$$\frac{\mu(\mathcal{E}_{\lambda, F})}{\mu(F)} \leq \left[ \frac{\mu(\mathcal{E})}{\mu(F)} \right]^\lambda$$

for every convex set $F$ with $0 < \mu(F) < +\infty$, closed subset $\mathcal{E} \subset F$ and $\lambda > 1$.

On the other hand, our whole point was to re-emphasize the geometric nature of the Lovasz-Simonovits localization technique and to somewhat counterbalance the tendency to present the needle decomposition as a statement about two (or four) integrals rather than a geometric partition algorithm. So, we preferred to use “purely geometric” terminology and to restrict ourselves to “volumes” and “convex sets” in the main text. At last, it may be worth mentioning that the class of logarithmically concave measures is only marginally wider than the class of convex sets: every logarithmically concave measure in $\mathbb{R}^n$ can be obtain as the limit when $m \to \infty$ of projections to $\mathbb{R}^n$ of volumes of convex sets in $\mathbb{R}^m \supset \mathbb{R}^n$. This allows one to extend many statements about convex sets to the case of logarithmically
concave measures more or less automatically. The Gaussian measure, for example, can be
viewed as the limit of projections of the volume measures of balls in large
dimensions.

\section{Dimension-free estimates for volumes of sublevel sets polynomials}

Let us start with recalling the classical 1-dimensional Remez inequality:

\textbf{Remez inequality:}

Let \( P \) be a polynomial of degree \( d \) in \( \mathbb{R}^1 \). Then for every interval \( J \subset \mathbb{R}^1 \) and for every measurable subset \( E \subset J \),

\[
\max_J |P| \leq \left[ \frac{A|J|}{|E|} \right]^d \sup_E |P|
\]

where \( A > 0 \) is an absolute constant (whose best possible value is \( A = 4 \)).

The proof (with a worse constant \( A = 2e \)) follows by a straightforward application of the Lagrange interpolation formula with \( d + 1 \) nodes on \( E \) spaced by at least \( \frac{|E|}{d} \). The sharp constant can obtained by a Markov-type “moving zeroes” argument, which shows that the worst case is attained when \( E \) is a sub-interval of \( J \) with a common end-point with \( J \) and \( P \) is the (properly renormalized) Chebyshev polynomial.

There is no hope for a dimension-free Remez type inequality with the \( L^\infty \) norm on the left hand side. This can be already seen when \( \mathcal{F} \) is a unit ball in \( \mathbb{R}^n \) and \( P(x) = 1 - |x|^2 \) (the reason is that, for large \( n \), most of the volume of \( \mathcal{F} \) is concentrated in a small neighborhood of the unit sphere where \( P(x) \) is very small). So, we have to confine ourselves to weaker distribution estimates.

Observing that a restriction of a polynomial of degree \( d \) to any line in \( \mathbb{R}^n \) is again a polynomial of degree (not exceeding) \( d \) and combining the one-dimensional Remez inequality with the geometric KLS lemma, we obtain the following

\textbf{Comparison lemma:}

Let \( P \) be a polynomial of degree \( d \) in \( \mathbb{R}^n \), and let \( \mathcal{F} \) be a convex compact set of volume one. Then for any \( c > 0, \lambda \geq 1 \)

\[
\text{Vol}\{x \in \mathcal{F} : |P(x)| \geq (A\lambda)^d c\} \leq [\text{Vol}\{x \in \mathcal{F} : |P(x)| \geq c\}]^\lambda.
\]

\textbf{Proof:}

If \( P \) is constant, the estimate is trivial. Otherwise, let \( \mathcal{E} = \{x \in \mathcal{F} : |P(x)| \geq c\} \). For each \( x \notin \mathcal{E}_{\lambda, \mathcal{F}} \), we can find an interval \( J \subset \mathcal{F} \) containing the point \( x \) and such
that the length of the set $\mathcal{J} \setminus \mathcal{E}$ is at least $\lambda^{-1} |\mathcal{J}|$. Then, according to the Remez inequality,

$$|P(x)| \leq \max_{\mathcal{J}} |P| \leq \left[ \frac{A |\mathcal{J}|}{|\mathcal{J} \setminus \mathcal{E}|} \right]^d \sup_{\mathcal{J} \setminus \mathcal{E}} |P| \leq (A\lambda)^d c$$

and, thereby, $\{x \in \mathcal{F} : |P(x)| > (A\lambda)^d c\} \subset \mathcal{E}_{\lambda, \mathcal{F}}$. It remains to observe that the strict inequality $|P(x)| > \ldots$ can be replaced by a non-strict one $|P(x)| \geq \ldots$ because the volume of any level set of a non-constant polynomial is 0. □

Let now $\mathcal{F}$ be a convex set in $\mathbb{R}^n$ of volume $\text{Vol}(\mathcal{F}) = 1$, and let $P$ be any (non-constant) polynomial in $\mathbb{R}^n$ of degree $d$. Let $M(P)$ be the unique positive number such that

$$\text{Vol}\{x \in \mathcal{F} : |P(x)| \geq M(P)\} = 1/e.$$

**Distribution inequalities:**

*For every* $\lambda > 1$,

$$\text{Vol}\{x \in \mathcal{F} : |P(x)| > (A\lambda)^d M(P)\} \leq e^{-\lambda}$$

*and*

$$\text{Vol}\{x \in \mathcal{F} : |P(x)| < (A\lambda)^{-d} M(P)\} \leq \frac{1}{\lambda}.$$

**Proof:**

The first inequality is just the comparison lemma applied to $c = M(P)$. To obtain the second one, let us denote the volume on the left by $V$. According to the comparison lemma applied to $c = (A\lambda)^{-d} M(P)$, we have

$$1/e \leq (1 - V)^\lambda$$

and thereby

$$V \leq 1 - e^{-1/\lambda} \leq \frac{1}{\lambda}. \quad \Box$$

**Remarks:**

The first distribution inequality (basically due to Bourgain [DI1]) can be viewed as (a kind of) *concentration phenomenon*. The second distribution inequality resembles a lot the classical Remez estimate (R): the only difference is that instead of the maximum over the *entire set* $\mathcal{F}$, we have the “median” $M(P)$ on the left hand side. We want to emphasize here that the comparison lemma and both distribution inequalities are derived directly from the one-dimensional Remez inequality and,
thereby, remain valid (together with all their corollaries below) for an arbitrary function (or class of functions) for which the one-dimensional result holds. For instance, instead of polynomials of degree $d$, we may consider exponential polynomials of order $d$, i.e., functions of the kind

$$P(x) = \sum_{k=1}^{d} c_k e^{i(x_k, x)}$$

with $c_k \in \mathbb{C}$, $x_k \in \mathbb{R}^n$, for which the Remez inequality (known in this case as Turan’s lemma) holds with $A = 316$, say.

It may also be worth mentioning that replacing the somewhat loose inequality (R) by the sharp one-dimensional Remez estimate coming from the consideration of Chebyshev polynomials, one can obtain the sharp dimension-free comparison lemma

$$\text{Vol}\{x \in F : |P(x)| \geq T_d(2\lambda - 1) c\} \leq \left[ \text{Vol}\{x \in F : |P(x)| \geq c\} \right]^{\lambda}$$

and the corresponding distribution inequalities

$$\text{Vol}\{x \in F : |P(x)| > T_d(2\lambda - 1) M(P)\} \leq e^{-\lambda}$$

and

$$\text{Vol}\{x \in F : |P(x)| < \frac{1}{T_d(2\lambda - 1)} M(P)\} \leq 1 - e^{-\frac{1}{\lambda}},$$

where

$$T_d(x) = \frac{1}{2} \left[ (x + \sqrt{x^2 - 1})^d + (x - \sqrt{x^2 - 1})^d \right]$$

is the Chebyshev polynomial of degree $d$.

**Digression: estimates for average values via distribution functions**

Since in what follows we shall have to calculate a few integrals and averages of real valued functions using the estimates for their distribution functions, let us remind the reader the corresponding general formulae.

Let $\mathcal{X}$ be a measure space with measure $\mu$. Let $g : \mathcal{X} \rightarrow \mathbb{R}$. Let $\mathcal{Y}$ be a measurable subset of $\mathcal{X}$. We want to construct a formula that would allow us to evaluate the integral $\int_{\mathcal{Y}} g \, d\mu$ or the average value $\langle g \rangle_{\mathcal{Y}} := \frac{1}{\mu(\mathcal{Y})} \int_{\mathcal{Y}} g \, d\mu$ of the function $g$ over the set $\mathcal{Y}$ using only the information about measures of sets of the kind $\{x \in \mathcal{X} : g(x) > t\}$.

Fix some “floor level” $L \in \mathbb{R}$ and consider the function $g^+ := \max(g, L)$. We have

$$g^+(x) = L + \int_L^{g^+(x)} dt.$$
Hence,
\[ \int_Y g \, d\mu \leq \int_Y g^+ \, d\mu \leq L\mu(Y) + \int_L^{+\infty} \mu\{x \in X : g(x) > t\} \, dt, \]
and, finally,
\[ \langle g \rangle_Y \leq L + \frac{1}{\mu(Y)} \int_L^{+\infty} \mu\{x \in X : g(x) > t\} \, dt. \]

For practical computations, we shall need the following modifications of these estimates. Let \( \varphi \) be any smooth, increasing to \(+\infty\) function on \((0, +\infty)\). Let \( \Lambda \) belong to the domain of \( \varphi \) and let \( L = \varphi(\Lambda) \). Making the change of variable \( t = \varphi(\lambda) \), we can rewrite the above estimates as
\[ \int_Y g \, d\mu \leq \varphi(\Lambda)\mu(Y) + \int_L^{+\infty} \mu\{x \in X : g(x) > \varphi(\lambda)\} \varphi'(\lambda) \, d\lambda, \]
\[ \langle g \rangle_Y \leq \varphi(\Lambda) + \frac{1}{\mu(Y)} \int_L^{+\infty} \mu\{x \in X : g(x) > \varphi(\lambda)\} \varphi'(\lambda) \, d\lambda. \]

**Estimates for \( L^q \)-norms:**

We shall need the following trivial observation: for every \( \sigma \geq 1 \),
\[ 1 + \sigma \int_0^{+\infty} \lambda^{\sigma-1} e^{-\lambda} \, d\lambda = 1 + 2^{\sigma-1} \sigma^{\sigma} \int_0^{+\infty} \left[\frac{\lambda}{2\sigma}\right]^{\sigma-1} e^{-\lambda} \, d\lambda \]
\[ \leq 1 + 2^{\sigma-1} \sigma^{\sigma} \int_0^{+\infty} e^{-\lambda/2} \, d\lambda = 1 + (2\sigma)^\sigma \leq (3\sigma)^\sigma. \]

Let now \( q \geq \frac{1}{d} \). Applying the estimates (*) with \( X = Y = \mathcal{F}, \mu = \text{Vol}, g = \left[\frac{|P|}{A^d M(P)}\right]^q, \varphi(\lambda) = \lambda^{qd}, \Lambda = 1 \) and using the estimate \( \mu\{g > \varphi(\lambda)\} \leq e^{-\lambda} \) (which is equivalent to the first distribution inequality), we get
\[ \int_{\mathcal{F}} \left[\frac{|P|}{A^d M(P)}\right]^q \leq 1 + qd \int_1^{+\infty} \lambda^{qd-1} e^{-\lambda} \, d\lambda \leq (3qd)^qd. \]

Therefore,
\[ \|P\|_{L^q(\mathcal{F})} \leq (3Aqd)^d M(P) \quad \text{for every } q \geq \frac{1}{d}. \]

Using the monotonicity of the function \( q \to \|P\|_{L^q(\mathcal{F})} \), we immediately derive from here that
\[ \|P\|_{L^q(\mathcal{F})} \leq (3A)^d M(P) \quad \text{for every } q \leq \frac{1}{d}. \]
Estimates for $L^{-q}$-norms:

Let $0 < q < \frac{1}{d}$. Applying the estimates $(\ast)$ with $\mathcal{X} = \mathcal{Y} = \mathcal{F}$, $\mu = \text{Vol}$, $g = \left[ \frac{|P|}{A^{-d}M(P)} \right]^{-q}$, $\varphi(\lambda) = \lambda^{qd}$, $\Lambda = 1$ and using the estimate $\mu \{ g > \varphi(\lambda) \} \leq \frac{1}{\lambda}$ (which is equivalent to the second distribution inequality), we get

$$\int_{\mathcal{F}} \left[ \frac{A^d|P|}{M(P)} \right]^{-q} \leq 1 + qd \int_{1}^{\infty} \lambda^{qd-1} \frac{1}{\lambda} d\lambda = \frac{1}{1 - qd}.$$

Therefore,

$$\|P\|_{L^{-q}(\mathcal{F})} \geq A^{-d}(1 - qd)^{1/q}M(P) \quad \text{for every } 0 < q < \frac{1}{d}.$$

The geometric mean:

The above inequalities immediately imply that

$$(eA)^{-d}M(P) \leq \|P\|_{L^0(\mathcal{F})} \leq (3A)^dM(P).$$

Inverse Hölder inequalities:

We shall start with the following simple

Observation:

Let $A_+$ be the best constant such that

$$\text{Vol}\{|P| \geq (A_+\lambda)^dM(P)\} \leq e^{-\lambda} \quad \text{for all } \lambda \geq 1.$$

Let $A_-$ be the best constant such that

$$\text{Vol}\{|P| < (A_-\lambda)^{-d}M(P)\} \leq \frac{1}{\lambda} \quad \text{for all } \lambda \geq 1.$$

Then $A_+A_- \leq A$.

Proof of the observation:

Let $0 < a < A_-$. According to the definition of $A_-$, there exists $\lambda_- \geq 1$ such that

$$\text{Vol}\{|P| < (a\lambda_-)^{-d}M(P)\} \geq \frac{1}{\lambda_-}.$$
and, thereby,
\[ \text{Vol}\{|P| \geq (a \lambda_\downarrow)^{-d} M(P)\} \leq 1 - \frac{1}{\lambda_\downarrow} \leq e^{-\frac{1}{\lambda_\downarrow}}. \]

Then, for every \( \lambda \geq 1 \), we have
\[ \text{Vol}\{|P| \geq (\frac{A}{a} \lambda)^d M(P)\} = \text{Vol}\{|P| \geq (A[\lambda \lambda_\downarrow])^d (a \lambda_\downarrow)^{-d} M(P)\} \]
\[ \leq \text{Vol}\{|P| \geq (a \lambda_\downarrow)^{-d} M(P)\}^{\lambda \lambda_\downarrow} \leq \left[e^{-\frac{1}{\lambda_\downarrow}}\right]^{\lambda \lambda_\downarrow} = e^{-\lambda} \]

according to the comparison lemma applied with \( c = (a \lambda_\downarrow)^{-d} \). Thus, \( A_+ \leq \frac{A}{a} \) and, since \( a < A_\downarrow \) was arbitrary, we are done. \( \square \)

Applying the above estimates for the \( L^q \) and \( L^{-r} \)-norms with \( A_\pm \) in place of \( A \), we conclude that
\[ \|P\|_{L^q(\mathcal{F})} \cdot \|1/P\|_{L^r(\mathcal{F})} \leq \frac{(3A \max\{1, qd\})^d}{(1 - rd)^{1/r}} \text{ for all } q \geq 0, 0 \leq r < 1/d. \]

The BMO - norm of \( \log |P| \):

We shall use the following definition of the BMO-norm of a function \( u : \mathbb{R}^n \to \mathbb{R} \):
\[ \|u\|_{BMO} = \sup_{\mathcal{F} \subseteq \mathbb{R}^n \text{ is convex}} \inf_{C \in \mathbb{R}} \frac{1}{\text{Vol}(\mathcal{F})} \int_{\mathcal{F}} |u - C|. \]

Since the class of polynomials is dilation-invariant, it is enough to obtain an estimate for convex sets \( \mathcal{F} \) of volume 1. Choosing \( C = \log M(P) + d \frac{\log A_+ - \log A_\downarrow}{2} \), applying the estimates (\( * \)) with \( \mathcal{X} = \mathcal{Y} = \mathcal{F} \), \( \mu = \text{Vol} \), \( g = \log |P| - C \), \( \varphi(\lambda) = d[\log \lambda + \log(\sqrt{A})] \), \( \Lambda = 1 \), and using the estimate \( \mu\{g > \varphi(\lambda)\} \leq e^{-\lambda + \frac{1}{\lambda}} \) for the distribution function (which is the combination of both estimates in the observation), we get
\[ \int_{\mathcal{F}} |\log |P| - C| \leq d \left[ \frac{\log A}{2} + \int_1^\infty \frac{1}{\lambda} \left(e^{-\lambda + \frac{1}{\lambda}}\right) d\lambda \right] \leq \frac{4 + \log A}{2} d. \]

§3. Estimates for distribution of zeroes of “random” analytic functions

An estimate for the averages of \( \log |P| \) over subsets of a compact convex set:

The purpose of this subsection is to prove the following

Claim:
Let \( F \subset \mathbb{R}^n \) be a compact convex set and let \( P : \mathbb{R}^n \to \mathbb{R} \) be a polynomial of degree \( d \). Then for any measurable \( E \subset F \),

\[
|\langle \log |P| \rangle_E - \langle \log |P| \rangle_F| \leq d \log \frac{e^2 A \text{Vol}(F)}{\text{Vol}(E)},
\]

where the averages are taken with respect to the \( n \)-dimensional Lebesgue measure (volume) in \( \mathbb{R}^n \) and \( A \) is the constant in the (one-dimensional) Remez inequality.

**Proof of the claim:**

Without loss of generality, we may assume that \( \text{Vol}(F) = 1 \) and \( M(P) = 1 \). Let, as before, \( A_+ \) and \( A_- \) be the best constants in the inequalities

\[
\text{Vol}\{ |P| \geq (A_+ \lambda)^d \} \leq e^{-\lambda} \leq \frac{1}{\lambda} \quad (\lambda \geq 1)
\]

and

\[
\text{Vol}\{ |P| < (A_- \lambda)^{-d} \} \leq \frac{1}{\lambda} \quad (\lambda \geq 1).
\]

Taking \( X = F, \ Y = E, \ g = \log |P|, \ \mu = \text{Vol}, \ \varphi(\lambda) = d(\log A_+ + \log \lambda) \), and using the inequality \( \mu\{ g > \varphi(\lambda) \} \leq \frac{1}{\lambda} \), we conclude that for every \( \Lambda \geq 1 \),

\[
\langle \log |P| \rangle_E \leq d \left[ \log A_+ + \log \Lambda + \frac{1}{\text{Vol}(E) \Lambda} \right].
\]

Substituting \( \Lambda = \frac{1}{\text{Vol}(E)} \), we get

\[
\langle \log |P| \rangle_E \leq d \log \frac{e A_+}{\text{Vol}(E)}.
\]

Analogously, taking \( X = Y = F, \ g = -\log |P|, \ \mu = \text{Vol}, \ \varphi(\lambda) = d(\log A_- + \log \lambda) \), we conclude that for every \( \Lambda \geq 1 \),

\[
\langle \log |P| \rangle_F \geq -d \left[ \log A_- + \log \Lambda + \frac{1}{\Lambda} \right].
\]

Substituting \( \Lambda = 1 \), we get

\[
\langle \log |P| \rangle_F \geq -d \log (e A_-).
\]

Combining these two estimates, we obtain

\[
\langle \log |P| \rangle_E - \langle \log |P| \rangle_F \leq d \log \frac{e^2 A_+ A_-}{\text{Vol}(E)} \leq d \log \frac{e^2 A}{\text{Vol}(E)}.
\]
The inequality $\langle \log |P| \rangle_{E} - \langle \log |P| \rangle_{F} \geq -d \log \frac{e^{2A}}{\text{Vol}(E)}$ can be proved in a similar way. □

**The Offord estimate:**

Fix some open domain $G \subset \mathbb{C}$ and consider a family of analytic functions $f(x; \cdot) : G \to \mathbb{C}$, where $x$ runs over some parameter set $X$ endowed with a finite measure $\mu$. Let

$$\nu_{x} := \sum_{w : f(x; w) = 0} \delta_{w}$$

be the *counting measure* of zeroes of the function $f(x; \cdot)$ where $\delta_{w}$ stands for the Dirac measure at $w \in G$ and each zero is counted with its multiplicity. For each $x \in X$, the measure $\nu_{x}$ is a locally finite measure in $G$.

Consider the average measure

$$\nu(U) := \frac{1}{\mu(X)} \int_{X} \nu_{x}(U) d\mu(x), \quad U \subset G.$$ 

The measure $\nu$ gives a “typical” (average) distribution of zeroes of the “random” function $f(x; \cdot)$ in $G$. Let $\psi \in C_{0}^{\infty}(G)$ and let $\lambda > 0$. Define the exceptional set $E_{+} = E_{+}(\psi, \lambda)$ by

$$E_{+}(\psi, \lambda) := \left\{ x \in X : \int_{G} \psi d\nu_{x} - \int_{G} \psi d\nu \geq \lambda \right\}.$$ 

Note that, since for each $x \in X$, the measure $\nu_{x}$ is $\frac{1}{2\pi}$ times the distributional Laplacian of the function $\log |f(x; \cdot)|$, we have

$$\int_{G} \psi d\nu_{x} = \frac{1}{2\pi} \int_{G} \Delta \psi(z) \log |f(x; z)|\, dm_{2}(z),$$

where $m_{2}$ is the area measure on the complex plane $\mathbb{C}$. Averaging over $X$, we get

$$\int_{G} \psi d\nu = \frac{1}{2\pi} \int_{G} \Delta \psi(z) \langle \log |f(\cdot; z)| \rangle_{X}\, dm_{2}(z).$$

Averaging the difference of these identities with respect to the parameter $x$ over the set $E_{+} = E_{+}(\psi, \lambda)$, we obtain the inequality

$$\lambda \leq \frac{1}{2\pi} \int_{G} \Delta \psi(z) \cdot \left[ \langle \log |f(\cdot; z)| \rangle_{E_{+}} - \langle \log |f(\cdot; z)| \rangle_{X} \right]\, dm_{2}(z)$$

$$\leq \frac{1}{2\pi} \| \Delta \psi \|_{L^{1}(G)} \cdot \sup_{z \in G} \left| \langle \log |f(\cdot; z)| \rangle_{E_{+}} - \langle \log |f(\cdot; z)| \rangle_{X} \right|. $$
Almost exactly the same argument shows that the same inequality holds for the set
\[ \mathcal{E}_- = \mathcal{E}_-(\psi, \lambda) := \{ x \in \mathcal{X} : \int_G \psi \, d\nu_x - \int_G \psi \, d\nu \leq -\lambda \}. \]

Combining these estimates with the claim, we obtain the following

**Theorem (Offord’s estimate):**

If \( \mathcal{X} = \mathcal{F} \) is a convex set in \( \mathbb{R}^n \), \( \mu \) is the Lebesgue measure in \( \mathbb{R}^n \), and \( f(x; z) \) depends on \( x \) as a polynomial of degree \( d \) for each \( z \), then
\[
\frac{\text{Vol}(\mathcal{E}(\psi, \lambda))}{\text{Vol}(\mathcal{F})} \leq 2Ae^2 \exp \left\{ -\frac{2\pi \lambda}{d \| \Delta \psi \|_{L^1(G)}} \right\},
\]

where
\[
\mathcal{E}(\psi, \lambda) := \mathcal{E}_+(\psi, \lambda) \cup \mathcal{E}_-(\psi, \lambda) = \{ x \in \mathcal{X} : \left| \int_G \psi \, d\nu_x - \int_G \psi \, d\nu \right| \geq \lambda \}.
\]

**Corollary:**

Denote by \( D_r \) the disk of radius \( r \) centered at the origin. Let \( G = D_1 \). We shall call a value \( x \in \mathcal{F} \) *exceptional* if the function \( f(x; \cdot) \) does not vanish in \( G \). Let \( \mathcal{E}^* \subset \mathcal{F} \) be the set of all exceptional values. If \( \text{Vol}(\mathcal{E}^*) > 0 \), we can estimate the growth of the (average) counting function \( r \mapsto \nu(D_r) \):
\[
\nu(D_r) \leq \lambda \leq \frac{4d}{1 - r} \log \frac{Ae^2 \text{Vol}(\mathcal{F})}{\text{Vol}(\mathcal{E}^*)}, \quad 0 < r < 1.
\]

**Proof of the Corollary:**

Fix \( r \) and choose a test function \( \psi(z) = \Psi(|z|) \) where \( \Psi \in C^\infty_0[0, 1) \), \( \Psi \geq 0 \), and \( \Psi(t) = 1 \) for \( 0 \leq t \leq r \). Let \( \lambda := \int_G \psi \, d\nu \geq \nu(D_r) \). Note that for such choice of \( \lambda \), we obviously have \( \mathcal{E}^* \subset \mathcal{E}_-(\psi; \lambda) \) and, therefore,
\[
\frac{\text{Vol}(\mathcal{E}^*)}{\text{Vol}(\mathcal{F})} \leq Ae^2 \exp \left\{ -\frac{2\pi \lambda}{d \| \Delta \psi \|_{L^1(G)}} \right\}.
\]

We can rewrite it as
\[
\nu(D_r) \leq \lambda \leq \frac{d}{2\pi} \| \Delta \psi \|_{L^1(G)} \log \frac{Ae^2 \text{Vol}(\mathcal{F})}{\text{Vol}(\mathcal{E}^*)}.
\]

Note that
\[
\frac{1}{2\pi} \| \Delta \psi \|_{L^1(G)} = \int_r^1 \left| t \Psi''(t) + \Psi'(t) \right| dt.
\]

Choosing \( \Psi \) sufficiently close to the quadratic spline whose second derivative is \(-\frac{4}{(1-r)^2}\) between \( r \) and \( \frac{1+r}{2} \) and \(+\frac{4}{(1-r)^2}\) between \( \frac{1+r}{2} \) and 1, we observe that the right hand side can always be made less than \( \frac{4}{1-r} \). \( \Box \)
Appendix: Proof of the lemma

Before starting the proof, we will make several simple observations about numerical inequalities that we shall use in the course of the proof.

Observation 1:

For all $X > 0, Y \geq 0$,

$$(X + Y)^\lambda \geq X\left(X + \frac{\lambda}{X-Y}Y\right)^{\lambda-1}.$$  

Indeed, we have an identity for $Y = 0$, and, obviously, for each $Y \geq 0$,

$$\frac{\partial}{\partial Y} \log(\text{LHS}) = \frac{\lambda}{X + Y} \geq \frac{\lambda}{X + \frac{\lambda}{X-Y}Y} = \frac{\partial}{\partial Y} \log(\text{RHS})$$

where, as usual, L(R)HS stands for the Left (Right) Hand Side of the inequality. □

Observation 2:

If the inequality $(X + Y)^\lambda \geq X(X + Z)^{\lambda-1}$ holds for some $X > 0, Y, Z \geq 0$, then for each $T \geq 0$,

$$(X + Y + T)^\lambda \geq X(X + Z + \frac{\lambda}{X-Y}T)^{\lambda-1}.$$  

Indeed, if $Z \geq Y$, we may repeat the proof of Observation 1 with $\frac{\partial}{\partial T}$ instead of $\frac{\partial}{\partial Y}$. If $Z < Y < \frac{\lambda}{X-Y}Y$, then the desired inequality immediately follows from Observation 1. □

Observation 3:

If $(X + Y)^\lambda \geq X(X + Z)^{\lambda-1}$ for some $X, Y, Z > 0$, then

$$(x + Y)^\lambda \geq x(x + Z)^{\lambda-1} \quad \text{for all } x \in [0, X].$$

This is the least trivial of our observations. Rewrite the inequality in the form

$$\frac{x}{x + Y} \leq \left[\frac{x + Y}{x + Z}\right]^{\lambda-1}$$

which is equivalent to

$$\left[\frac{x}{x + Y}\right]^{\lambda-1} \leq \frac{x + Z}{x + Y}.$$  

Denote $\beta := \frac{1}{\lambda-1}, \theta := \frac{x}{x+Y}, \Theta := \frac{X}{X+Y}$. Then $\frac{\lambda+Z}{\lambda+Y} = \frac{\lambda}{\lambda+Y} - 1 \theta = L(\theta)$ is a linear function. We want to show that if the inequality $\theta^{-\beta} \geq L(\theta)$ holds at $\theta = \Theta$, 17
then it holds on the entire interval \([0, \Theta]\). The desired inequality obviously holds for \(\theta\) sufficiently close to 0. Therefore, if it were false for at least one \(\theta \in [0, \Theta]\), the graphs of functions \(\theta^{-\beta}\) and \(L(\theta)\) would intersect at at least two points on the interval \((0, \Theta]\). Since they also intersect at \(\theta = 1\), we would then have at least three points common for a convex curve (the graph of \(\theta^{-\beta}\)) and a line (the graph of \(L(\theta)\)), which is impossible. \(\square\)

**Observation 4:**

If \((X + Y)^\lambda \geq X(X + Y + Z)^{\lambda - 1}\) for some \(X, Y, Z > 0\), then

\[(x + y)^\lambda \geq x(x + y + z)^{\lambda - 1}\]

for all \(x \leq X, y \geq Y, z \leq Z\).

Indeed, we obviously can replace \(Z\) by \(z\). After that, Observation 3 allows us to change \(X\) to \(x\). It remains to observe that for fixed \(x\) and \(z\),

\[
\frac{\partial}{\partial y} \log(\text{LHS}) = \lambda \frac{x + y}{x + y + z} \geq \lambda - 1 \frac{x + y + z}{x + y} = \frac{\partial}{\partial y} \log(\text{RHS}).
\]

Now we are ready to start proving the lemma. Since the problem is invariant with respect to linear change of variable, we may assume that \(I = [0, 1]\) We may also assume without loss of generality that the function \(f\) is continuous, strictly logarithmically concave and satisfies \(f(0) = f(1) = 0\) (if it isn’t so, just consider the family of functions \(f_\varepsilon(x) = [x(1 - x)]^\varepsilon f(x)\), apply the statement to each of them, and pass to the limit as \(\varepsilon \to 0\)).

Clearly, \(E_{\lambda, I}\) is a closed set. If \(E_{\lambda, I}\) is empty, there is nothing to prove. Otherwise, \((0, 1) \setminus E_{\lambda, I} = \bigcup I_j\) where \(I_j\) are disjoint open intervals each of which is shorter than the entire interval \((0, 1)\). Consider one of these intervals \(I_j = (a, b)\). We shall call it *regular* if either \(a > 0\) and \(f\) is decreasing on \((a, b)\), or \(b < 1\) and \(f\) is increasing on \((a, b)\). Otherwise we shall call the interval \(I_j\) *exceptional*. Clearly, there may be not more than one exceptional interval. If such an interval exists, we shall assign the index 0 to it. Let \(E_j = E \cap I_j\). We claim that for each regular interval, one has

\[
\int_{E_j} f \geq \frac{\lambda - 1}{\lambda} \int_{I_j} f.
\]

Indeed, if, say, \(I_j = (a, b)\) and \(a > 0\), then \(a \in E_{\lambda, I}\) and, thereby, \(|E_j \cap (a, t)| \geq \frac{\lambda - 1}{\lambda} |(a, t)|\) for each \(a < t < b\), which, together with the fact that \(f\) is decreasing on \((a, b)\), is enough to ensure the desired estimate.

If the exceptional interval is absent, the inequality of the lemma is quite easy to prove. Indeed, it is equivalent to the estimate

\[
\left[\int_E f\right]^\lambda \geq \left[\int_{E_{\lambda, I}} f\right] \cdot \left[\int_{(0, 1)} f\right]^{\lambda - 1};
\]

\(18\)
i.e., to the inequality
\[ \left[ \int_{E_{\lambda,I}} f + \int_{\bigcup E_j} f \right]^\lambda \geq \left[ \int_{E_{\lambda,I}} f \right] \cdot \left[ \int_{E_{\lambda,I}} f + \int_{\bigcup I_j} f \right]^{\lambda-1}. \]

But \( \int_{\bigcup I_j} f \leq \frac{\lambda}{\lambda-1} \int_{\bigcup E_j} f \) and thereby the desired estimate follows from Observation 1.

Suppose now that \( I_0 = (a,b) \) is exceptional. Without loss of generality we may assume that \( f(b) \leq f(a) \) (otherwise just make the change of variable \( t \to 1 - t \), which leaves the problem invariant). Note that this automatically implies that \( a > 0 \) because otherwise we would have \( f(b) \leq f(a) = f(0) = 0 \), which, since the function \( f \) is strictly positive on \((0, 1)\), would imply that \( b = 1 \), \( I_0 = (0, 1) \), and, finally, that \( E_{\lambda,I} \) is empty.

If \( f(b) < f(a) \), let \( c \in (a,b) \) be the (unique) point such that \( f(a) = f(c) \). We are going to slightly modify the portion \( E_0 \) of the set \( E \). Observe again that, since \( a \in E_{\lambda,I}, |E \cap (a,c)| \geq \frac{\lambda-1}{\lambda} |(a,c)| \). Take an arbitrary portion of \( E \cap (a,c) \) of measure \( |E \cap (a,c)| - \frac{\lambda-1}{\lambda} |(a,c)| \) and replace it by a set of equal measure on \((c,b)\) using the points of \((c,b) \setminus E \) as close to the left end \( c \) as possible. If the measure of the entire set \((c,b) \setminus E \) is too small, just fill the entire interval \((c,b)\) and forget about lost measure. Let \( E' \) be the resulting set. We claim that

\[ |E' \cap (c,t)| \geq \frac{\lambda-1}{\lambda} |(c,t)| \]

for all \( c < t < b \). Indeed, the portion \( E' \cap (c,b) \) of the modified set \( E' \) starts with an interval. As long as \( t \) stays within this interval, there is nothing to prove. As soon as \( t \) leaves this interval, the length of the intersection \( E' \cap (a,t) \) coincides with the length of the intersection \( E \cap (a,t) \) and therefore is not less than \( \frac{\lambda-1}{\lambda} |(a,t)| \). But we also have \( |E' \cap (a,c)| = \frac{\lambda-1}{\lambda} |(a,c)| \), so we should have the desired inequality for the remaining portion. Also, \( f \) obviously decreases on \((c,b)\). So, we may treat the interval \((c,b)\) as a regular interval and to restrict our attention to \((a,c)\).

If we originally had the identity \( f(a) = f(b) \), this construction reduces to denoting the point \( b \) by the letter \( c \) and replacing the part \( E \cap (a,b) \) of the set \( E \) by its arbitrary subset of measure \( \frac{\lambda-1}{\lambda} |(a,b)| \).

On \((a,c)\), let us modify the set \( E' \) even further. Namely, let us replace the corresponding portion of \( E' \) by the level set of \( f \) of measure \( \frac{\lambda-1}{\lambda} |(a,c)| \) containing the small values of the function. Clearly, such modifications only decrease the integral of the function \( f \) over the set that undergoes them, so we have \( \int_{E'} f \leq \int_{E} f \). Thus, it will suffice to prove the inequality of the lemma with \( \int_{E} f \) replaced by \( \int_{E'} f \).

Now let us look at the picture we have obtained. We have one exceptional interval \( I_0' = (a,c) \) such that \( f(a) = f(c) \), \( |E_0'| = \frac{\lambda-1}{\lambda} |(a,c)| \), and \( E_0' \) is a level set of \( f \) on \( I_0' \).
containing the small values of the function. We have also some regular intervals $I_j'$ (original regular intervals plus, maybe, $(c, b)$) satisfying $\int_{E_j'} f \geq \frac{\lambda-1}{\lambda} \int_{I_j'} f$ for each $j$. We need to prove the estimate

$$\left[ \int_{E_{\lambda,i}} f + \int_{E_{0}'} f + \int_{\cup_{j>0} E_j'} f \right]^\lambda \geq \left[ \int_{E_{\lambda,i}} f + \int_{E_0'} f + \int_{\cup_{j>0} E_j'} f \right]^{\lambda-1}. $$

Using Observation 2, we see that it is enough to prove that

$$\left[ \int_{E_{\lambda,i}} f + \int_{E_0'} f \right]^\lambda \geq \left[ \int_{E_{\lambda,i}} f \right] \cdot \left[ \int_{E_{\lambda,i}} f + \int_{E_0'} f \right]^{\lambda-1}. $$

Observation 3 allows us to extend the set $E_{\lambda,i}$ in the last inequality to the entire set $(0, 1) \setminus I_0'$. Let now $|I_0'| = \lambda m$ and let $f^*$ be the decreasing rearrangement of $f$ on $(0, 1)$. It is obviously decreasing and logarithmically concave. We need to prove the inequality

$$\left[ \int_{\lambda m}^1 f^* + \int_{m}^\lambda f^* \right]^\lambda \geq \left[ \int_{\lambda m}^1 f^* \right] \cdot \left[ \int_{\lambda m}^1 f^* + \int_{m}^\lambda f^* + \int_{0}^m f^* \right]^{\lambda-1}. $$

According to Observation 4, if we modify $f^*$ in such a way that simultaneously the integrals $\int_0^m f^*$ and $\int_{\lambda m}^1 f^*$ become bigger while the integral $\int_{m}^\lambda f^*$ becomes smaller, we shall get a harder inequality to prove. Such modification can be done by replacing $\log f^*$ by a linear function interpolating it at the points $m$ and $\lambda m$. Using Observation 3 once more, we see that we may extend the integration to the entire right semi-axis. Finally, we need to prove that if $f^*$ is a decreasing exponential function, then

$$\left[ \int_{m}^\infty f^* \right]^\lambda \geq \left[ \int_{\lambda m}^\infty f^* \right] \cdot \left[ \int_{0}^\infty f^* \right]^{\lambda-1}. $$

But this is an identity! □

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