CLUSTER ALGEBRAS OF INFINITE RANK AS COLIMITS

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ABSTRACT. We formalize the way in which one can think about cluster algebras of infinite rank by showing that every rooted cluster algebra of infinite rank can be written as a colimit of rooted cluster algebras of finite rank. Relying on the proof of the positivity conjecture for skew-symmetric cluster algebras (of finite rank) by Lee and Schiffler, it follows as a direct consequence that the positivity conjecture holds for cluster algebras of infinite rank. Furthermore, we give a sufficient and necessary condition for a ring homomorphism between cluster algebras to give rise to a rooted cluster morphism without specializations. Assem, Dupont and Schiffler proposed the problem of a classification of ideal rooted cluster morphisms. We provide a partial solution by showing that every rooted cluster morphism without specializations is ideal, but in general rooted cluster morphisms are not ideal.

1. INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky [FZ1] to provide an algebraic framework for the study of total positivity and canonical bases in algebraic groups. They are certain commutative rings with a combinatorial structure on a distinguished set of generators, called cluster variables, which can be grouped into overlapping sets of a given cardinality, called clusters. Classically, clusters are finite. However the theory can be extended to allow countable clusters, giving rise to cluster algebras of infinite rank. Recently, cluster algebras and cluster categories of infinite rank have received more and more attention, for example in work by Igusa and Todorov ([IT1] and [IT2]) and Hernandez and Leclerc [HL], as well as in joint work with Grabowski [GG].

All the data needed to construct a cluster algebra is contained in an initial seed, consisting of an initial cluster and a rule, encoded in a skew-symmetric matrix, which allows us to obtain all other cluster variables and all the relations between them from the initial cluster. Every seed can be mutated at certain cluster variables (called exchangeable cluster variables), giving rise to new seeds with new clusters. Two seeds are called mutation equivalent if they are connected by a finite sequence of mutation of seeds and any two mutation equivalent seeds give rise to the same cluster algebra.

Assem, Dupont and Schiffler introduced the category $\text{Clus}$ of rooted cluster algebras in [ADS]. The objects of $\text{Clus}$ are what can be thought of as pointed versions of cluster algebras; they are pairs consisting of a cluster algebra and a fixed initial seed. Fixing the distinguished initial seed allows for the definition of natural maps between cluster algebras, so-called rooted cluster morphisms, which are ring homomorphisms commuting with mutation and which provide the morphisms for the category $\text{Clus}$.

If the image of a rooted cluster morphism coincides with the rooted cluster algebra generated by the image of the initial seed, it is called an ideal rooted cluster morphism. Assem, Dupont and Schiffler ask in Problem 2.12 of [ADS] for a classification of ideal rooted cluster morphisms. We answer part of this question by showing that not every rooted cluster morphism is necessarily ideal in Theorem 3.13. The counterexample we provide is a rooted cluster morphism with specialization, that is, some cluster variables get sent to integers. Rooted cluster
morphisms without specializations are more nicely behaved and we characterize them by a necessary and sufficient combinatorial condition and we show that every rooted cluster morphism without specializations is ideal.

Since we are interested in cluster algebras of infinite rank, we study colimits and limits in the category $\text{Clus}$. We show that the category $\text{Clus}$ is neither complete nor cocomplete, that is, limits and colimits do not in general exist. Still, it allows for a formal way to think about cluster algebras as infinite versions of cluster algebras of finite rank, as we show in our main result (cf. Theorem 4.6).

**Theorem.** Every rooted cluster algebra of infinite rank can be written as a colimit of rooted cluster algebras of finite rank in the category $\text{Clus}$.

We expect this statement to be a useful tool to extend results that are known for (certain) cluster algebras of finite rank to cluster algebras of infinite rank. For example, it is a direct consequence of our result that the positivity conjecture for skew-symmetric cluster algebras as proved by Lee and Schiffler in [LS] holds for cluster algebras of infinite rank.

An important source of (rooted) cluster algebras are triangulations of marked surfaces, as studied for triangulations of surfaces with finitely many marked points by Fomin, Shapiro and Thurston [FST]. Motivated by the study of cluster structures on a category of infinite Dynkin type by Holm and Jørgensen [HJ], in a joint paper with Grabowski [GG] cluster algebras arising from triangulations of the closed disc with infinitely many marked points on the boundary with one limit point have been classified.

In [IT2], Igusa and Todorov studied more general cluster categories of infinite Dynkin type $A$. The underlying idea is to take the limit of the cluster category of Dynkin type $A_n$ as $n$ goes to infinity. This yields the continuous cluster category $\mathcal{C}_n$ of Dynkin type $A$. The category $\mathcal{C}_n$ has a cluster structure encoded by certain countable triangulations of the disc with marked points lying densely on the boundary. Furthermore, in a preprint version of [IT1], that to our knowledge is no longer available, Igusa and Todorov constructed discrete versions of infinite cluster categories of Dynkin type $A$, where cluster structures are encoded by certain triangulations of the closed disc with a fixed discrete set of marked points on the boundary. The method for their construction is a direct application of the detailed theory they introduce in the same paper [IT1]. The simplest case of these discrete infinite cluster categories, where the marked points have one limit point is just the category studied by Holm and Jørgensen in [HJ]. We show that every rooted cluster algebra arising from a countable triangulation of the closed disc can be written as a colimit of finite rooted cluster algebras of Dynkin type $A$, thus providing an algebraic interpretation for the cluster categories of infinite Dynkin type $A$.

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2. Rooted cluster algebras

Cluster algebras have been introduced by Fomin and Zelevinsky [FZ1]. Throughout this paper we work with cluster algebras of geometric type and we consider their rooted versions, which we obtain by fixing an initial seed. Rooted cluster algebras are the objects in the category $\text{Clus}$ we want to work in, and which was introduced by Assem, Dupont and Schiffler [ADS].

2.1. Seeds. All the information we need to construct a (rooted) cluster algebra is contained in a so-called seed. Along with a distinguished subset of generators for our cluster algebra, it contains a rule that describes how all other generators and the relations between them can
be obtained. This rule can be encoded in a skew-symmetrizeable integer matrix. A skew-
symmetrizeable (integer) matrix is a square matrix $B$ such that there exists a diagonal matrix 
$D$ with positive integer entries and a skew-symmetrizeable integer matrix $S$ with $S = DB$.

**Definition 2.1** (Seed, [FZ2]). A seed is a triple $\Sigma = (X, ex, B)$, where

- $X$ is a countable set of indeterminates over $\mathbb{Z}$, such that the field $\mathcal{F}_X = \mathbb{Q}(x \mid x \in X)$ of rational functions in $X$ is a purely transcendental field extension of $\mathbb{Q}$. The set $X$ is called the cluster of $\Sigma$.
- $ex \subseteq X$ is a subset of the cluster. The elements of $ex$ are called the exchangeable variables of $\Sigma$. The elements $X \setminus ex$ are called the coefficients of $\Sigma$.
- $B = (b_{vw})_{v,w \in X}$ is a skew-symmetrizeable integer matrix with rows and columns labelled by $X$, which is locally finite at exchangeable variables, i.e. for every $v \in ex$ there are only finitely many non-zero entries $b_{uv}$ and $b_{uw}$. The matrix $B$ is called the exchange matrix of $\Sigma$.

The field $\mathcal{F}_\Sigma = \mathbb{Q}(x \mid x \in X)$ is called the ambient field associated to the seed $\Sigma$. Two seeds $\Sigma = (X, ex, B = (b_{vw})_{v,w \in X})$ and $\Sigma' = (X', ex', B' = (b'_{vw})_{v,w \in X'})$ are called isomorphic, and we write $\Sigma \cong \Sigma'$, if there exists a bijection $f: X \to X'$ inducing a bijection $f: ex \to ex'$ such that for all $v, w \in X$ we have $b_{vw} = b'_{f(v)f(w)}$.

Note that usually, even in the case where the cluster $X$ is allowed to be infinite (e.g. in [ADS] or [GG]) the exchange matrix $B$ in a given seed $\Sigma = (X, ex, B)$ is assumed to be locally finite at every element of the cluster $X$, not merely at the exchangeable variables. While it is necessary to impose this condition at exchangeable variables, as will become apparent in the definition of mutation (cf. Definition 2.9) we don’t really need local finiteness elsewhere.

Often when giving examples it is more intuitive to think about the combinatorics of a seed to be encoded in a quiver instead of in a matrix. This is possible if the exchange matrix is skew-symmetric.

**Remark 2.2.** If the matrix $B$ in the seed $\Sigma = (X, ex, B)$ is skew-symmetric, we can express it via a quiver $Q_B$. The vertices in $Q_B$ are labelled by elements in the cluster $X$ and there are $b_{vw}$ arrows from $v$ to $w$ whenever $b_{vw} \geq 0$. The quiver $Q_B$ is locally finite at exchangeable vertices, i.e. there are only finitely many arrows incident with every vertex labelled by an exchangeable variable of $\Sigma$. For a seed whose exchange matrix is skew-symmetric by abuse of notation we will often write $\Sigma = (X, ex, Q_B)$ for the seed $\Sigma = (X, ex, B)$.

To any seed $\Sigma = (X, ex, B)$ we can naturally associate its opposite seed $\Sigma^{op} = (X, ex, -B)$, by reversing all signs in the exchange matrix $B$. If $B$ is skew-symmetric, this corresponds to reversing all arrows in the associated quiver $Q_B$ which gives rise to the opposite quiver $Q_B^{op}$.

### 2.1.1. Seeds from triangulations of the closed disc.

An important source of seeds is provided by triangulations of surfaces with (possibly infinitely many) marked points. Throughout this paper we will follow the example of countable triangulations of the closed disc with marked points on the boundary. This provides a connection to the work of Igusa and Todorov ([IT1] and [IT2]), where they introduced infinite cluster categories of type $A$ associated to the same combinatorial model. Let us start by defining what we mean by a triangulation of the closed disc. We cover the boundary $\partial D_2 = S^1$ of the closed disc by $\mathbb{R}$ in the usual way: $e: \mathbb{R} \to S^1, x \mapsto e^{ix}$.

**Definition 2.3** (Order on $S^1$). For any two elements $a, b \in S^1$ choose a lifting $\tilde{a} \in \mathbb{R}$ of $a$ and $\tilde{b} \in \mathbb{R}$ of $b$ under the map $e$ such that $\tilde{a} \leq \tilde{b} < \tilde{a} + 2\pi$. Then we denote by $[a, b]$ the image $[a, b] = e([\tilde{a}, \tilde{b}])$.

Analogously we define the open interval $(a, b)$ and the half-open intervals $[a, b)$ and $(a, b]$. 

We view $\overline{D}_2 \subseteq \mathbb{R}^2$ as a topological space with the standard topology. Let $\mathcal{Z} \subseteq S^1$ be a subset of the boundary of $\overline{D}_2$ and throughout we assume any such subset to contain at least two elements, $|\mathcal{Z}| \geq 2$.

**Definition 2.4** (Triangulation of the closed disc). An *arc* of $\mathcal{Z}$ is a two-element subset of $\mathcal{Z}$, i.e. $\{x_0, x_1\} \subseteq \mathcal{Z}$ with $x_0 \neq x_1$. An arc $\{x_0, x_1\}$ of $\mathcal{Z}$ is called an *edge* of $\mathcal{Z}$ if $(x_0, x_1) \cap \mathcal{Z} = \emptyset$ or $(x_1, x_0) \cap \mathcal{Z} = \emptyset$. An arc of $\mathcal{Z}$ that is not an edge of $\mathcal{Z}$ is called *internal arc* of $\mathcal{Z}$.

Two arcs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are said to *cross* if either $y_0 \in (x_0, x_1)$ and $y_1 \in (x_1, x_0)$ or $y_1 \in (x_0, x_1)$ and $y_0 \in (x_1, x_0)$, i.e. if the straight line connecting $x_0$ and $x_1$ crosses the straight line connecting $y_0$ and $y_1$ in the unit disc.

A *triangulation of the closed disc with marked points* $\mathcal{Z}$ is a maximal collection $\mathcal{T}$ of pairwise non-crossing arcs of $\mathcal{Z}$, i.e. a collection $\mathcal{T}$ of non-crossing arcs of $\mathcal{Z}$ such that every arc of $\mathcal{Z}$ that is not contained in $\mathcal{T}$ crosses at least one arc in $\mathcal{T}$. We call a triangulation $\mathcal{T}$ a *countable triangulation of the closed disc*, if the set $\mathcal{T}$ is countable.

**Remark 2.5.** Note that for a triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z}$ to be countable, the set $\mathcal{Z} \subseteq S^1$ does not need to be countable as well. Consider for example $\mathcal{Z} = S^1$ and the triangulation

$$\mathcal{T}_{st} = \left\{ \left( m \pi \frac{n}{2}, \frac{(m + 1) \pi}{2n} \right) \right\}_{n \geq 0, 0 \leq m < 2^n + 1}$$

with marked points $\mathcal{Z} = S^1$, where the endpoints of the arcs in $\mathcal{T}$ are a countable dense subset of $\mathcal{Z} = S^1$. Similarly, any subset $\mathcal{Z} \subseteq S^1$ allows a countable triangulation of the closed disc with marked points $\mathcal{Z}$.

**Remark 2.6.** An edge of a subset $\mathcal{Z} \subseteq S^1$ crosses no other arcs of $\mathcal{Z}$. Thus by definition every triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$ must contain all edges of $\mathcal{Z}$.

To any countable triangulation of the closed disc we can associate a seed, with the same method that has been introduced by Fomin, Shapiro and Thurston [FST] for finite triangulations of surfaces.

**Definition 2.7** (Seed associated to a countable triangulation of the closed disc). Let $\mathcal{T}$ be a countable triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$. The *seed* $\Sigma_\mathcal{T}$ associated to $\mathcal{T}$ is the triple $\Sigma_\mathcal{T} = (\mathcal{T}, \text{ex}_\mathcal{T}, Q_\mathcal{T})$ defined as follows.

- The cluster consists of the arcs in $\mathcal{T}$.
- An arc $\{x_0, x_1\} \in \mathcal{T}$ is called *exchangeable in* $\mathcal{T}$, if it is the diagonal of a quadrilateral in $\mathcal{T}$, i.e. if there exist vertices $y_0, y_1 \in \mathcal{Z}$ with $y_0 \in (x_0, x_1)$ and $y_1 \in (x_1, x_0)$ such that $\{x_0, y_0\}, \{y_0, x_1\}, \{x_1, y_1\}, \{y_1, x_0\} \subseteq \mathcal{T}$. The set of exchangeable variables $\text{ex}_\mathcal{T}$ is the set of arcs that are exchangeable in $\mathcal{T}$.
- The vertices of the quiver $Q_\mathcal{T}$ are labelled by the arcs in $\mathcal{T}$, and for $\alpha, \alpha' \in \mathcal{T}$ there is an arrow $\alpha \rightarrow \alpha'$ in $Q_\mathcal{T}$ if and only if the arcs $\alpha$ and $\alpha'$ are sides of a common triangle in $\mathcal{T}$ and $\alpha'$ lies in a clockwise direction from $\alpha$.

Because every arc in $\mathcal{T}$ is the side of at most two triangles in $\mathcal{T}$, the quiver $Q_\mathcal{T}$ is locally finite and the seed $\Sigma_\mathcal{T}$ associated to a triangulation $\mathcal{T}$ of the disc is indeed a seed in the sense of Definition 2.1 and Remark 2.2.
Remark 2.8. An exchangeable arc in a triangulation $T$ is always internal, as every edge is adjacent to at most one triangle in $T$ and hence cannot be the diagonal of a quadrilateral in $T$. However, not every internal arc is necessarily exchangeable. Consider for example the subset $Z = \{e(\pi k) | k \in \mathbb{Z}\} \subseteq S^1$ which has exactly one limit point at 1, and the triangulation $T = \{e(\pi/2), e(\pi/2)|k \in \mathbb{Z}_{>2}\} \cup \{e(-\pi/2), e(-\pi/2)|k \in \mathbb{Z}_{>2}\} \cup \{e(\pi/2), e(-\pi/2)\}$ with marked points $Z$ (cf. Figure 1). The arc $\{e(\pi/2), e(-\pi/2)\} \in T$ is internal. However, it is not exchangeable: If it was, then it would have to be contained in a quadrilateral in $T$, so there would exist a $z \in (e(-\pi/2), e(\pi/2)) \cap Z$ with $\{e(\pi/2), z\}, \{z, e(-\pi/2)\} \in T$. However, if $z \in (0, e(\pi/2))$ then the arc $\{z, e(-\pi/2)\}$ intersects infinitely many of the arcs in $\{e(\pi/2), e(\pi/2)|k \in \mathbb{Z}_{>2}\} \subseteq T$ and otherwise, if $z \in (e(-\pi/2), 0)$, the arc $\{e(\pi/2), z\}$ intersects infinitely many of the arcs in $\{e(-\pi/2), e(\pi/2)|k \in \mathbb{Z}_{<2}\} \subseteq T$. This leads to a contradiction, since arcs in $T$ have to be pairwise non-crossing.

2.2. Mutation. A seed $\Sigma = (X, ex, B)$ contains all the data that is needed to construct the associated (rooted) cluster algebra. In order to actually obtain all generators of the cluster algebra, a combinatorial process is applied, which is called mutation. The information needed to perform mutation is encoded in the exchange matrix $B$.

Definition 2.9 (Mutation, [FZ1]). Let $\Sigma = (X, ex, B)$ be a seed and let $x \in ex$ be an exchangeable variable in $\Sigma$. We denote the mutation of $\Sigma$ in the direction of $x$ by $\mu_x(\Sigma) = (\mu_x(X), \mu_x(ex), \mu_x(B))$. It is defined by the following data.

- For any $y \in X$ the mutation of $y$ in the direction of $x$ is defined by
  $$\mu_x(y) = y, \text{ if } y \neq x$$

  and

  $$\mu_x(x) = \frac{\prod_{v \in X: b_{xv} > 0} v^{b_{xv}} + \prod_{v \in X: b_{xv} < 0} v^{-b_{xv}}}{x} \in F_\Sigma.$$  

  The equations of the form (1) are called exchange relations. The cluster, respectively the exchangeable variables, in the seed $\mu_x(\Sigma)$ thus are

  $$\mu_x(X) = \{\mu_x(y) | y \in X\} = (X \setminus x) \cup \mu_x(x)$$

  and

  $$\mu_x(ex) = \{\mu_x(y) | y \in ex\} = (ex \setminus x) \cup \mu_x(x).$$
The matrix $\mu_x(B) = (\tilde{b}_{v\tilde{w}})_{v,\tilde{w} \in \mu_x(X)}$ is given by matrix mutation of $B$ at $x$: For $\tilde{v} = \mu_x(v)$ and $\tilde{w} = \mu_x(w)$ set

$$
\tilde{b}_{v\tilde{w}} = \mu_x(b_{vw}) = \begin{cases} 
-b_{vw} & \text{if } v = x \text{ or } w = x, \\
 b_{vw} + \frac{1}{2}(b_{vw} + b_{xw}b_{xw}) & \text{otherwise.}
\end{cases}
$$

**Remark 2.10.**

- Mutation is involutive, i.e. for a seed $\Sigma = (X, \text{ex}, B)$ and any $x \in \text{ex}$ we have $\mu_{\mu_x(x)} \circ \mu_x(\Sigma) = \Sigma$. In particular, mutation defines an equivalence relation on the class of all seeds.
- It is well-known that every cluster in a seed which is mutation equivalent to $\Sigma$ is a transcence basis of the ambient field $F_\Sigma = \mathbb{Q}(X)$ associated to $\Sigma$.

Consider our standard example of a seed $\Sigma_T = (X_T, \text{ex}_T, Q_T)$ associated to a countable triangulation $T$ of the closed disc. Geometrically, mutation $\Sigma_T$ at an exchangeable variable in $\text{ex}_T$ can be represented by a so-called diagonal flip of $T$. Every exchangeable arc $\{x_0, x_1\} \in \text{ex}_T$ is the diagonal of a unique quadrilateral with vertices $x_0, x_1, x_0', x_1'$, with sides $\{x_0, x_0'\}$, $\{x_1, x_1'\}$ and $\{x_0', x_1\}$ all contained in $T$. The diagonal flip of $T$ at $\alpha = \{x_0, x_1\}$ is the map $f_\alpha: T \to T \setminus \alpha \cup \overline{\alpha}$ which replaces the arc $\alpha$ by the arc $\overline{\alpha} = \{x_0', x_1\}$ and leaves all other arcs invariant, cf. Figure 2.

It is well-known for finite triangulations of the closed disc that for any exchangeable arc $\alpha$ in $T$ we have $\mu_\alpha(Q_T) = Q_{f_\alpha(T)}$. Since mutations of quivers and diagonal flips are defined locally, i.e. there are only finitely many vertices, respectively arcs, involved, only a finite subquiver of $Q_T$ is affected by the mutation at $\alpha$ and the equality remains true for infinite triangulations.

### 2.3. Rooted cluster algebras

Mutation of a seed at any exchangeable variable in its cluster yields another seed, which again can be mutated at any exchangeable variable in its respective cluster. Thus we can successively mutate a seed $\Sigma$ along what are called $\Sigma$-admissible sequences. Mutation along all possible $\Sigma$-admissible sequences will provide all generators of the cluster algebra associated to the seed $\Sigma$, the definition of which we will recall in this section.

**Definition 2.11** ($\Sigma$-admissible sequence, [ADS]). Let $\Sigma = (X, \text{ex}, B)$ be a seed and let $l \geq 0$. A sequence $(x_1, \ldots, x_l)$ is called $\Sigma$-admissible if $x_1 \in \text{ex}$ and for every $2 \leq k \leq l$, we have $x_k \in \mu_{x_{k-1}} \circ \ldots \circ \mu_{x_1}(\text{ex})$. The empty sequence of length $l = 0$ is $\Sigma$-admissible for any seed $\Sigma$ and mutation of $\Sigma$ along the empty sequence leaves $\Sigma$ invariant. We denote by

$$
\text{Mut}(\Sigma) = \{\mu_{x_l} \circ \ldots \circ \mu_{x_1}(\Sigma) \mid l \geq 0, (x_1, \ldots, x_l) \Sigma\text{-admissible}\}
$$

the set of all seeds which can be reached from $\Sigma$ by iterated mutation along $\Sigma$-admissible sequences and call it the mutation class of $\Sigma$.

By mutating a seed $\Sigma$ along all possible $\Sigma$-admissible sequences we obtain the collection $\text{Mut}(\Sigma)$ of seeds and with it a collection of overlapping clusters. Let $P(F_\Sigma)$ denote the power-set (i.e. the set of all subsets) of the ambient field $F_\Sigma$, and let

$$
\text{cl}_\Sigma: \text{Mut}(\Sigma) \to P(F_\Sigma), (\tilde{X}, \text{ex}, \tilde{B}) \mapsto \tilde{X}
$$
be the map assigning to each seed in the mutation class of $\Sigma$ its cluster.

**Definition 2.12** (Rooted cluster algebra, [FZ1], [ADS]). Let $\Sigma$ be a seed. The *rooted cluster algebra with initial seed* $\Sigma$ is the pair $(A(\Sigma), \Sigma)$, where $A(\Sigma)$ is the $\mathbb{Z}$-subalgebra of the ambient field $F_\Sigma$ given by

$$A(\Sigma) = \mathbb{Z}[x \mid x \in cl_\Sigma(\text{Mut}(\Sigma))] \subseteq F_\Sigma.$$  

The elements of $cl_\Sigma(\text{Mut}(\Sigma))$ are called the *cluster variables* of the rooted cluster algebra $(A(\Sigma), \Sigma)$.

**Remark 2.13.** The coefficients $X \setminus \text{ex}$ of the seed $\Sigma$ appear in the cluster $cl_\Sigma(\Sigma)$ of every seed $\Sigma$ in the mutation equivalence class $\text{Mut}(\Sigma)$. For a seed $\Sigma = (X, \emptyset, B)$ with no exchangeable cluster variables, we have $\text{Mut}(\Sigma) = \{\Sigma\}$ and the algebra $A(\Sigma)$ is isomorphic to the polynomial algebra $\mathbb{Z}[x \mid x \in X]$. The empty seed $\Sigma_0 = (\emptyset, \emptyset, \emptyset)$ gives rise to the rooted cluster algebra $(A(\Sigma_0), \Sigma_0)$ with $A(\Sigma_0) \cong \mathbb{Z}$ as a ring.

By abuse of notation we will generally write $A(\Sigma)$, meaning the rooted cluster algebra $(A(\Sigma), \Sigma)$.

**2.3.1. Rooted cluster algebras associated to triangulations of the closed disc.** Let $T$ be a countable triangulation of the closed disc with marked points $Z \subseteq S^1$. Recall that the cluster variables in the associated seed $\Sigma_T$ are labelled by the arcs of $T$ and mutation is represented by diagonal flips. We denote by $R_T$ the set of arcs

$$R_T = \{\mu_{\alpha_l} \circ \cdots \circ \mu_{\alpha_1}(\alpha) \mid l \geq 0, \alpha \in T, (\alpha_{1}, \ldots, \alpha_{l}) \Sigma_T - \text{admissible}\}$$

and call its elements the *arcs that can be reached from $T$*. These are all the arcs of $Z$ we obtain from $T$ by successive diagonal flips.

**Remark 2.14.** If $T$ is finite, then all arcs of $Z$ can be reached from $T$. However, this is not necessarily the case if $T$ is infinite. For example, as in Remark 2.8 consider the subset

$$Z = \{e\left(\frac{\pi}{k}\right) \mid k \in \mathbb{Z}\} \subseteq S^1$$

and the triangulation

$$T = \left\{e\left(\frac{\pi}{2}\right), e\left(\frac{\pi}{k}\right)\right\} \mid k \in \mathbb{Z}_{>2}\right\} \cup \left\{e\left(-\frac{\pi}{2}\right), e\left(\frac{\pi}{k}\right)\right\} \mid k \in \mathbb{Z}_{<2}\right\} \cup \left\{e\left(\frac{\pi}{2}\right), e\left(-\frac{\pi}{2}\right)\right\}$$

with marked points $Z$ (cf. Figure 1). The arc $\{e\left(-\frac{\pi}{2}\right), e\left(\frac{\pi}{4}\right)\}$ cannot be reached from $T$. If it could, then there would be an $l \geq 0$ and a $\Sigma_T$-admissible sequence $(\alpha_l, \ldots, \alpha_1)$ of arcs of $Z$ and an arc $\alpha \in T$, such that $\{e\left(-\frac{\pi}{2}\right), e\left(\frac{\pi}{4}\right)\} = \mu_{\alpha_l} \circ \cdots \circ \mu_{\alpha_1}(\alpha)$. However, the infinite triangulations $\mu_{\alpha_l} \circ \cdots \circ \mu_{\alpha_1}(T)$ and $T$ differ only by finitely many elements. Since $\{e\left(-\frac{\pi}{2}\right), e\left(\frac{\pi}{4}\right)\}$ crosses infinitely many arcs in $T$ it also crosses infinitely many arcs in $\mu_{\alpha_l} \circ \cdots \circ \mu_{\alpha_1}(T)$. This contradicts the fact that $\mu_{\alpha_l} \circ \cdots \circ \mu_{\alpha_1}(T)$ is a triangulation.

The exchange relations (cf. Definition 2.9) for mutation of seeds in $\text{Mut}(\Sigma_T)$ are the *Plücker relations in $T$*:

$$\{x_0, x_1\} \{y_0, y_1\} = \{x_0, y_0\} \{x_1, y_1\} + \{x_0, y_1\} \{x_1, y_0\}$$

for any two crossing arcs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ in $R_T$. We denote the ideal generated by the Plücker relations in $T$ by $J_T$. Then the cluster algebra $A(\Sigma_T)$ is the ring

$$A(\Sigma_T) = \mathbb{Z}[\alpha \mid \alpha \in R_T]/J_T.$$  

**Remark 2.15.** If $T$ is a finite triangulation of the closed disc with marked points $Z \subseteq S^1$ of cardinality $|Z| = n + 3$, then the ring $A(\Sigma_T)$ is a cluster algebra of Dynkin type $A_n$. We then say that the rooted cluster algebra $A(\Sigma_T)$ is of *finite Dynkin type $A$*. In the case where $Z \subseteq S^1$ is discrete with exactly one limit point the cluster algebras of the rooted cluster
algebras associated to triangulations of $\mathcal{Z}$ have been classified in [GG]. They are subrings of the homogeneous coordinate ring of the doubly infinite Grassmannian of planes.

3. Rooted cluster morphisms and the category of rooted cluster algebras

When working with cluster algebras, it is interesting to ask how natural maps between cluster algebras should look. Intuitively, we want such maps to be ring homomorphisms commuting with mutation. In [FZ2], Section 1.2, Fomin and Zelevinsky considered what they called strong isomorphisms. These are isomorphisms of rings between cluster algebras that map each seed to an isomorphic seed. This idea was generalized by Assem, Schiffler, and Shramchenko in [ASS] via the new notion of cluster automorphisms. A cluster automorphism is a ring automorphism of a cluster algebra which sends a distinguished seed $\Sigma$ to another seed $f(\Sigma)$ in the mutation class of $\Sigma$, such that $f$ commutes with mutation at every variable in the two clusters. Again, only ring homomorphisms between isomorphic rings are considered. Furthermore, cluster algebras with coefficients are not considered and also, cluster automorphisms always bijectively map clusters to clusters, such that there is no way to "delete" cluster variables.

3.1. Rooted cluster morphisms. In [ADS] Assem, Dupont, and Schiffler introduced the notion of rooted cluster morphisms. Passing from a cluster algebra to a rooted cluster algebra by fixing an initial seed allows for a rigorous definition of what one means for a ring homomorphism between not necessarily ring isomorphic cluster algebras to commute with mutation.

**Definition 3.1** (Rooted cluster morphism, [ADS]). Let $\Sigma = (X, ex, B)$ and $\Sigma' = (X', ex', B')$ be seeds and let $A(\Sigma)$ and $A(\Sigma')$ be the corresponding rooted cluster algebras, cf. Definition 2.12. A rooted cluster morphism from $A(\Sigma)$ to $A(\Sigma')$ is a ring homomorphism $f: A(\Sigma) \to A(\Sigma')$ of unital rings, i.e. a ring homomorphism with $f(1) = 1$, satisfying the following conditions:

- **CM1** $f(X) \subseteq X' \cup \mathbb{Z}$.
- **CM2** $f(ex) \subseteq ex' \cup \mathbb{Z}$.
- **CM3** Let $(x_1, \ldots, x_l)$ be a $\Sigma$-admissible sequence such that the sequence $(f(x_1), \ldots, f(x_l))$ is $\Sigma'$-admissible. Then the homomorphism $f$ commutes with mutation along $(x_1, \ldots, x_l)$:

$$f(\mu_{x_1} \circ \ldots \circ \mu_{x_l}(y)) = \mu_{f(x_1)} \circ \ldots \circ \mu_{f(x_l)}(f(y))$$

for all $y \in X$ with $f(y) \in X'$.

**Remark 3.2.** Every cluster automorphism in the sense of Assem, Schiffler, and Shramchenko [ASS] can be viewed as a rooted cluster isomorphism from a rooted cluster algebra $A(\Sigma)$ to itself, where $\Sigma$ is a seed with a finite cluster, no coefficients, and a skew-symmetric exchange matrix (as are the assumptions in [ASS] for the definition of a cluster automorphism). Thus rooted cluster morphisms really provide a generalization of the concept of cluster automorphisms.

**Definition 3.3** (Biadmissible sequences, [ADS]). Let $f: A(\Sigma) \to A(\Sigma')$ be a rooted cluster morphism. A $\Sigma$-admissible sequence $(x_1, \ldots, x_l)$ whose image $(f(x_1), \ldots, f(x_l))$ is $\Sigma'$-admissible is called $(f, \Sigma, \Sigma')$-biadmissible.

The following example includes some of the more interesting things that can happen with rooted cluster morphisms: Firstly, they may exist between non-isomorphic rings, further we may "delete" cluster variables by sending them to integers and we may "defreeze" coefficients by sending them to exchangeable cluster variables.

**Example 3.4.** From now on, when we consider seeds with a skew-symmetric exchange matrix pictured as a quiver, we will mark vertices associated to coefficients by squares. Consider the
seeds 
\[ \Sigma = (\{x_1, x_2, x_3\}, \{x_2, x_3\}, x_1 \xrightarrow{} x_2 \xleftarrow{} x_3) \]

and 
\[ \Sigma' = (\{y_1, y_2\}, \{y_1, y_2\}, y_1 \xrightarrow{} y_2). \]

The associated cluster algebras are as rings isomorphic to 
\[ A(\Sigma) \cong \mathbb{Z}[x_1, x_2, x_3, \frac{x_1 x_3 + 1}{x_2}, \frac{x_2 + 1}{x_3}, \frac{x_1 x_3 + x_2 + 1}{x_2 x_3}] \]
\[ A(\Sigma') \cong \mathbb{Z}[y_1, y_2, \frac{y_1 + 1}{y_2}, \frac{y_2 + 1}{y_1}, \frac{y_1 + y_2 + 1}{y_1 y_2}]. \]

Consider the ring homomorphism \( f : A(\Sigma) \to A(\Sigma') \) we obtain from the projection of \( x_3 \) to 1, thus on the cluster variables it acts as \( x_i \mapsto y_i \) for \( i = 1, 2 \) and \( x_3 \mapsto 1 \). This ring homomorphism satisfies axioms CM1 and CM2 by definition. The only exchangeable cluster variable in \( \Sigma \) whose image is exchangeable in \( \Sigma' \) is \( x_2 \) with \( f(x_2) = y_2 \), so the first entry of any \((f, \Sigma, \Sigma')\)-biadmissible sequence has to be \( x_2 \). Indeed we have 
\[ f(\mu_{x_2}(x_2)) = f\left(\frac{x_1 x_3 + 1}{x_2}\right) = \frac{y_1 + 1}{y_2} = \mu_{y_2}(y_2) = \mu_f(x_2)(f(x_2)). \]

Furthermore, the only exchangeable cluster variable in \( \mu_{x_2}(\Sigma) \) whose image is exchangeable in \( \mu_{y_2}(\Sigma') \) is \( \mu_{x_2}(x_2) \) with \( f(\mu_{x_2}(x_2)) = \mu_{y_2}(y_2) \), so all \((f, \Sigma, \Sigma')\)-biadmissible sequences have alternating entries \( x_2 \) and \( \mu_{x_2}(x_2) \). Since mutation is involutive (cf. Remark 2.10), the ring homomorphism \( f \) commutes with mutation along any of these sequences. Thus axiom CM3 is satisfied and \( f \) is a rooted cluster morphism.

We show in Lemma 3.7 that any rooted cluster morphism is quite restrictive in its action on exchangeable variables: it has to be injective on the exchangeable variables that are not being sent to integers. Furthermore, we also cannot map any coefficients to the same cluster variable to which we map an exchangeable variable. We may however send two coefficients to the same cluster variable, as long as we are careful about their exchangeable neighbors.

**Definition 3.5 (Neighbor).** Let \( \Sigma = (X, \text{ex}, B = (b_{vw})_{v,w\in X}) \) be a seed and let \( x \in X \) be a cluster variable in \( \Sigma \). We call a cluster variable \( y \in X \) a neighbor of \( x \) in \( \Sigma \), if \( b_{xy} \neq 0 \).

**Remark 3.6.** Note that being neighbors is a symmetric relation: for a given seed \( \Sigma = (X, \text{ex}, B = (b_{vw})_{v,w\in X}) \) a cluster variable \( x \in X \) is a neighbor of \( y \in X \) in \( \Sigma \) if and only if \( y \) is a neighbor of \( x \) in \( \Sigma \). We then say that \( x \) and \( y \) are neighbors in \( \Sigma \).

**Lemma 3.7.** Let \( \Sigma = (X, \text{ex}, B) \) and \( \Sigma' = (X', \text{ex}', B') \) be seeds and let \( f : A(\Sigma) \to A(\Sigma') \) be a rooted cluster morphism. If \( x \neq y \) are cluster variables in \( X \) with \( f(x) = f(y) \), then both \( x \) and \( y \) are coefficients. Furthermore, for any \((f, \Sigma, \Sigma')\)-biadmissible sequence \((x_1, \ldots, x_l)\), and any \( z \in \mu_{x_l} \circ \ldots \circ \mu_{x_1}(\text{ex}) \) with \( f(z) \in \mu_f(x_l) \circ \ldots \circ \mu_f(x_1)(\text{ex}') \) the entries \( \mu_{x_l} \circ \ldots \circ \mu_{x_1}(b_{xz}) \) and \( \mu_{x_l} \circ \ldots \circ \mu_{x_1}(b_{yz}) \) of the matrix \( \mu_{x_l} \circ \ldots \circ \mu_{x_1}(B) \) cannot have opposite signs.

**Proof.** Let \( x \in \text{ex} \) with \( f(x) \in \text{ex} \). We want to show that \( f(x) \neq f(y) \) for any cluster variable \( y \in X \). By axiom CM2 for rooted cluster morphisms we have \( f(x) \in \text{ex}' \) and the sequence \((x)\) is \((f, \Sigma, \Sigma')\)-biadmissible. Let \( y \in X \) with \( y \neq x \). If \( f(y) \in \mathbb{Z} \) then we have \( f(x) \neq f(y) \), thus assume \( f(y) \in X' \). By axiom CM3 we obtain 
\[ f(y) = f(\mu_x(y)) = \mu_f(x)(f(y)) \]

Assume by contradiction that \( f(y) = f(x) =: z \in \text{ex}' \). This would imply \( z = \mu_x(z) \). Writing \( B' = (b'_{xy})_{x,y\in X} \) we thus would have 
\[ z^2 = z \mu_x(z) = \prod_{w\in X' : b'_{zw} > 0} v^{b'_{zw}} + \prod_{w\in X' : b'_{zw} < 0} w^{-b'_{zw}}, \]
which contradicts algebraic independence of the cluster variables in $X'$.

Let now $x \neq y \in X \setminus \text{ex}$ be coefficients of $\Sigma$ and and let $(x_1, \ldots, x_l)$ be a $(f, \Sigma, \Sigma')$-biadmissible sequence. Set
\[
\mu_{x_1} \cdots \mu_{x_l} (\Sigma) =: \hat{\Sigma} = (X, \tilde{e}x, \hat{B} = (b_{vw})_{v,w \in \hat{X}})
\]
and
\[
\mu_{f(x_1)} \cdots \mu_{f(x_l)} (\Sigma') =: \hat{\Sigma}' = (X', \tilde{e}x', \hat{B}' = (b'_{vw})_{v,w \in \hat{X}'}).
\]

We have $x, y \in \hat{X}$, cf. Remark 2.13. Assume by contradiction that $f(x) = f(y) = a \in \hat{X}'$ and there exists a $z \in \tilde{e}x$ with $f(z) \in \tilde{e}x'$ such that $b_{xz} \neq \hat{b}'_{x'y}$ and $\hat{b}'_{x'y}$ have opposite sign – without loss of generality assume $b_{xz} > 0$ and $\hat{b}'_{x'y} < 0$. Then we have
\[
f(z \mu_z (z)) = \prod_{v \in X : b_{zv} > 0} f(v)^{b_{zv}} + \prod_{w \in X : b_{zw} < 0} f(w)^{-b_{zw}} = a \left( \prod_{v \neq x, b_{zv} > 0} f(v)^{b_{zv}} + \prod_{w \neq y \in X : b_{zw} < 0} f(w)^{-b_{zw}} \right).
\]

By axiom CM3 this has to be equal to
\[
f(z \mu_{f(z)} (f(z)) = \prod_{v \in X' : b'_{f(z)v} > 0} \hat{b}'_{f(z)v} + \prod_{w \in X' : b'_{f(z)w} < 0} w^{-\hat{b}'_{f(z)w}},
\]
and since we either have $\hat{b}'_{f(z)a} \geq 0$ or $\hat{b}'_{f(z)a} < 0$ the cluster variable $a$ cannot devide both
\[
\prod_{v \in X' : b'_{f(z)v} > 0} \hat{b}'_{f(z)v}
\]
and
\[
\prod_{w \in X' : b'_{f(z)w} < 0} w^{-\hat{b}'_{f(z)w}}.
\]

This contradicts algebraic independence of the variables in $\hat{X}'$. \hfill $\Box$

3.2. Ideal rooted cluster morphisms. An ideal rooted cluster morphism is a rooted cluster morphism $f : A(\Sigma) \to A(\Sigma')$ whose image $f(A(\Sigma))$ is a rooted cluster algebra associated to the image of the initial seed $f(\Sigma)$. In the discussion before Problem 2.12 of [ADS] (which asks for a characterization of all ideal rooted cluster morphisms), the authors asked the question whether every rooted cluster morphism was ideal. In this section we answer the question by showing that not every rooted cluster morphism is necessarily ideal.

**Definition 3.8** (Image of a seed). Let $\Sigma = (X, \text{ex}, B)$ and $\Sigma' = (X', \text{ex}', B')$ be seeds and let $f : A(\Sigma) \to A(\Sigma')$ be a rooted cluster morphism. Then the image $f(\Sigma)$ of the seed $\Sigma$ under the morphism $f$ is the seed $f(\Sigma) = (f(X) \cap X', f(\text{ex}) \cap \text{ex}', f(B) = (b'_{vw})_{v,w \in f(X) \cap X'})$.

**Example 3.9.** Consider the seeds
\[
\Sigma = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, \ \{x_1, x_2, x_3\}, \ x_1 \to x_2 \equiv x_4 \equiv x_7 \ni x_5 \ni x_3 \ni x_6
\]
and
\[
\Sigma = \{y_1, y_2, y_3, z_1, z_2, a\}, \ \{y_1, y_2, y_3, z_1, z_2a\}, \ y_1 \to y_2 \equiv z_1 \equiv y_3 \equiv z_2 \equiv a
\]
and the map \( f : \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \to \{y_1, y_2, y_3, z_1, z_2, a\} \) which maps
\[
\begin{align*}
x_i &\mapsto y_i \text{ for } i = 1, 2, 3 \\
x_i &\mapsto z_1 \text{ for } i = 4, 5, 7 \\
x_6 &\mapsto z_2.
\end{align*}
\]
As we will see in Example 3.33 this map induces a rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \). The image \( f(\Sigma) \) is the seed
\[
\{(y_1, y_2, y_3, z_1, z_2), (y_1, y_2, y_3), y_1 \leftrightarrow z_1 \leftarrow y_2 \leftarrow z_2 \).
\]
If \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) is a rooted cluster morphism, then the seed \( f(\Sigma) \) is an example of what is called a full subseed of the seed \( \Sigma' \).

**Definition 3.10** (Full subseed, [ADS]). Let \( \Sigma' = (X', \text{ex}', B' = (b'_{vw})_{v,w \in X'} \) be a seed. A full subseed of \( \Sigma' \) is a seed \( \Sigma = (X, \text{ex}, B = (b_{vw})_{v,w \in X}) \) such that
\[
\begin{itemize}
  \item \( X \subseteq X' \),
  \item \( \text{ex} \subseteq \text{ex}' \),
  \item \( B \) is the submatrix of \( B' \) formed by the entries labelled by \( X \times X \), i.e. for all \( v, w \in X \) we have \( b_{vw} = b'_{vw} \).
\end{itemize}
\]

**Remark 3.11.** Note that while all exchangeable variables in a full subseed of \( \Sigma' \) have to be exchangeable in \( \Sigma' \), cluster variables which are coefficients in the full subseed are not necessarily coefficients in \( \Sigma' \).

**Definition 3.12** (Ideal rooted cluster morphism, [ADS]). A rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) is called ideal if its image is the rooted cluster algebra associated to the seed \( f(\Sigma) \), i.e. if \( \mathcal{A}(\Sigma) = \mathcal{A}(f(\Sigma)) \).

In Lemma 2.10 of [ADS] the authors showed that the inclusion \( \mathcal{A}(f(\Sigma)) \subseteq f(\mathcal{A}(\Sigma)) \) holds for any rooted cluster morphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \). We can show that the converse is not true in general.

**Theorem 3.13.** Not every rooted cluster morphism is ideal.

**Proof.** We give an example of a rooted cluster morphism that is not ideal. Consider the seeds
\[
\Sigma = (\{a_1, a_2, x\}, \{x\}, \xymatrix{a_1 & x & a_2})
\]
and
\[
\Sigma' = (\{y_1, y_2\}, \{y_1, y_2\}, \xymatrix{y_1 & y_2}).
\]
As rings, the cluster algebras are isomorphic to
\[
\mathcal{A}(\Sigma) \cong \mathbb{Z}[a_1, a_2, x, x']/\langle xx' = a_1 + a_2 \rangle
\]
and
\[
\mathcal{A}(\Sigma') \cong \mathbb{Z} \left[ y_1, y_2, \frac{1 + y_1}{y_2}, \frac{1 + y_2}{y_1}, \frac{1 + y_1 + y_2}{y_1 y_2} \right].
\]
Consider the ring homomorphism \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) defined by the algebraic extension of the map which sends
\[
\begin{align*}
a_1 &\mapsto 1 \\
a_2 &\mapsto -1 \\
x &\mapsto 0 \\
x' &\mapsto y_2
\end{align*}
\]
Because \( f(xx') = 0 = f(a_1 + a_2) \) this is well-defined. Furthermore, it satisfies the axioms CM1 and CM2 for a rooted cluster morphism and because there are no \( (f, \Sigma, \Sigma') \)-biadmissible sequences it trivially satisfies CM3 and thus is a rooted cluster morphism. The image of the seed \( \Sigma \) is \( f(\Sigma) = (\emptyset, \emptyset, \emptyset) \) and thus as a ring we have \( \mathcal{A}(f(\Sigma)) \cong \mathbb{Z} \). However, the image of the cluster algebra \( \mathcal{A}(\Sigma) \) is as a ring isomorphic to \( f(\mathcal{A}(\Sigma)) \cong \mathbb{Z}[y_1] \). \( \square \)
3.3. **The category of rooted cluster algebras.** Considering rooted cluster algebras and rooted cluster morphisms gives rise to a category.

**Definition 3.14** (Category of rooted cluster algebras, [ADS]). The *category of rooted cluster algebras* $\text{Clus}$ is the category which has as objects rooted cluster algebras and as morphisms rooted cluster morphisms.

In Section 2 of [ADS] it was shown that $\text{Clus}$ satisfies the axioms of a category. In particular, axiom CM2 for rooted cluster morphisms is necessary to ensure that compositions of rooted cluster morphisms are again rooted cluster morphisms, as the following example illustrates.

**Example 3.15.** Consider the seeds

$$
\Sigma_1 = (\{x_1, x_2, x_3\}, \{x_2\}, x_1 \rightarrow x_2 \rightarrow x_3), \\
\Sigma_2 = (\{z\}, \emptyset, z) \\
\Sigma_3 = (\{y_1, y_2\}, \{y_1, y_2\}, y_1 \rightarrow y_2)
$$

with associated cluster algebras $\mathcal{A}(\Sigma_1) = \mathbb{Z}[x_1, x_2, x_3, \frac{x_1 + x_2}{x_2}], \mathcal{A}(\Sigma_2) = \mathbb{Z}[z]$ and $\mathcal{A}(\Sigma_3) = \mathbb{Z}[y_1, y_2, \frac{1 + y_1}{y_1}, \frac{1 + y_2}{y_2}, \frac{1 + y_1 + y_2}{y_1 y_2}]$ and the ring homomorphisms $f: \mathcal{A}(\Sigma_1) \rightarrow \mathcal{A}(\Sigma_2)$, which is defined by sending $x_i \mapsto z$ for all $i = 1, 2, 3$, and $g: \mathcal{A}(\Sigma_2) \rightarrow \mathcal{A}(\Sigma_3)$ defined by $z \mapsto y_1$. Both $f: \mathcal{A}(\Sigma_1) \rightarrow \mathcal{A}(\Sigma_2)$ and $g: \mathcal{A}(\Sigma_2) \rightarrow \mathcal{A}(\Sigma_3)$ satisfy axiom CM1. Axiom CM3 is satisfied trivially by both $f$ and $g$, since there are neither $(f, \Sigma_1, \Sigma_2)$ nor $(g, \Sigma_2, \Sigma_3)$-biadmissible sequences. However, the composition $g \circ f$ does not satisfy axiom CM3: Consider the $(g \circ f, \Sigma_1, \Sigma_3)$-biadmissible sequence $(x_2)$. We have

$$
g \circ f(\mu_{x_2}(x_2)) = g \circ f\left(\frac{x_1 + x_3}{x_2}\right) = g(2) = 2
$$

but

$$
\mu_{g \circ f(x_2)}(g \circ f(x_2)) = \mu_{y_1}(y_1) = \frac{1 + y_2}{y_1}.
$$

3.4. **Connectedness of seeds and coproducts.** Assem, Dupont and Schiffler showed in [ADS] that countable coproducts exist in the category of rooted cluster algebras. Taking coproducts of a countable family $\{\mathcal{A}(\Sigma_i)\}_{i \in I}$ of rooted cluster algebras amounts to taking what can be intuitively described as the discrete union $\Sigma$ of their seeds. The seeds $\Sigma_i$ will be full subseeds of the seed $\Sigma$ which are mutually disconnected. This means that mutation within each of the seeds $\Sigma_i$ is independent of what happens in any of the other seeds $\Sigma_j$ for $j \neq i$.

**Definition 3.16** (Connected seed). We call two cluster variables $x, y \in \Sigma$ connected in $\Sigma$, if there exists a $k \geq 0$ and a sequence $x = x_0, x_1, \ldots, x_k = y$ of cluster variables in $\Sigma$ such that for any $0 \leq i < k$, the cluster variables $x_i$ and $x_{i+1}$ are neighbors in $\Sigma$. We call the seed $\Sigma$ *connected* if any two cluster variables $x, y \in \Sigma$ are connected by a finite sequence of cluster variables in $\Sigma$. We call a rooted cluster algebra $\mathcal{A}(\Sigma)$ *connected* if its initial seed $\Sigma$ is connected.

**Remark 3.17.** If $\Sigma = (X, \text{ex}, B)$ is a seed with a skew-symmetric matrix $B$ it is connected if and only if the underlying graph of the associated quiver $Q_B$ is connected.

We can decompose any seed into its connected components. Conversely we can build a new seed from a countable collection of seeds by taking the discrete union of the clusters and the exchangeable variables and constructing a big matrix which contains all of their exchange matrices as block-diagonal entries: Let $\{\Sigma_j = (X_j, \text{ex}_j, B_j = (b_{vw})_{v, w \in X_j})\}_{j \in I}$ be a countable collection of seeds. Denote by $\bigsqcup$ the disjoint union and set

$$
\bigsqcup_{j \in I} \Sigma_j := (\bigsqcup_{j \in I} X_j, \bigsqcup_{j \in I} \text{ex}_j, B),
$$

where $B = (b_{vw})_{v, w \in \bigsqcup_{j \in I} X_j}$ is given by

$$
b_{vw} = b_{u_1 v}, w = (b_{u_2 v}, w)_{u_1, u_2 \in \bigsqcup_{j \in I} X_j} = (b_{u_1 v}, w)_{u_1 \in X_j, u_2 \in X_{j'}},
$$

for $u_1, u_2 \in \bigsqcup_{j \in I} X_j$. Then $\bigsqcup_{j \in I} \Sigma_j$ is a connected seed.
where \( B = (b_{vw})_{v,w \in \bigsqcup_{i \in I} X_i} \) is the block-diagonal matrix with blocks \( B_j \) for \( j \in I \), i.e. \( b_{vw} = b^j_{vw} \) if \( v, w \in X_j \) for some \( j \in I \) and \( b_{vw} = 0 \) otherwise. The analogous construction for rooted cluster algebras is taking coproducts. By Lemma 5.1 in [ADS], the category \( \text{Clus} \) of rooted cluster algebras admits countable coproducts \( \bigsqcup \) and for a countable index set \( I \) we have
\[
\prod_{j \in I} A(\Sigma_j) \cong A(\bigsqcup_{j \in I} \Sigma_j).
\]

The seeds \( \Sigma_j \) for \( j \in I \) are mutually disconnected full subseeds of \( \Sigma \). On the other hand, since we can decompose any given seed into its mutually disconnected connected components and there are only countably many cluster variables, given a rooted cluster algebra \( A(\Sigma) \) we can write it as a countable coproduct of connected rooted cluster algebras.

### 3.4.1. Connected components of a triangulation of the closed disc

Let us consider again the example of a countable triangulation \( T \) of the closed disc with marked points \( \mathbb{Z} \subseteq S^1 \). Note that by definition of the seed \( \Sigma_T \) associated to \( T \) two cluster variables \( \alpha, \beta \in T \) are neighbors in \( \Sigma_T \) if and only if the arcs \( \alpha \) and \( \beta \) are sides of a common triangle in \( T \) and they are connected in \( \Sigma_T \) if and only if there exists a \( k \geq 0 \) and sequences of arcs \( \gamma_0, \ldots, \gamma_k \), such that \( \alpha = \gamma_0 \) and \( \beta = \gamma_k \) and for all \( 0 \leq i \leq k \) the arcs \( \gamma_i \) and \( \gamma_{i+1} \) are sides of a common triangle in \( T \).

It turns out that the connected components of \( \Sigma_T \) depend on the behaviour of arcs in \( T \) in the neighbourhood of limit points of \( \mathbb{Z} \).

**Definition 3.18** (Fountains and nests of a countable triangulation). Let \( T \) be a countable triangulation of the closed disc with marked points \( \mathbb{Z} \subseteq S^1 \). We say that a sequence \( \{z_i\}_{i \in \mathbb{Z}_{\geq 0}} \) of points in \( \mathbb{Z} \) converges to \( z \in S^1 \) *from the right*, if for any \( x \in S^1 \) the set \([x, z) \cap \mathbb{Z} \) is infinite and the set \([z, x] \cap \mathbb{Z} \) is finite, we say that it converges to \( z \in S^1 \) *from the left*, if for any \( x \in S^1 \) the set \([x, z) \cap \mathbb{Z} \) is finite and the set \([z, x] \cap \mathbb{Z} \) is infinite and we say that it converges to \( z \in S^1 \) *from both sides*, if for any \( x \in S^1 \) both the set \([x, z) \cap \mathbb{Z} \) and the set \([z, x] \cap \mathbb{Z} \) are infinite.

Let \( \{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) be a sequence of arcs of \( \mathbb{Z} \) and let the sequence of endpoints converge to \( a = \lim_{i \to \infty} a_i \in S^1 \) and \( b = \lim_{i \to \infty} b_i \in S^1 \). If both sequences of endpoints \( \{a_i\}_{i \in \mathbb{Z}_{\geq 0}} \) and \( \{b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) are non-constant, we say that the sequence \( \{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) of arcs is a nest if \( a = b \) and we say that it is a half-nest if \( a \neq b \).

If the sequence \( \{a_i\}_{i \in \mathbb{Z}_{\geq 0}} \) is constant and the sequence \( \{b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) is non-constant, we say that the sequence \( \{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) of arcs is a right-fountain at a converging to \( b \), if \( b_i \) converges to \( b \) from the right, we say that it is a left-fountain at a converging to \( b \), if \( b_i \) converges to \( b \) from the left and that it is a fountain at a converging to \( b \), if \( b_i \) converges to \( b \) from both sides. We say that there is a split fountain converging to a limit point \( b \) if there are \( a_1 \neq a_r \in \mathbb{Z} \) such that there is a fountain at \( a_1 \) and a right-fountain at \( a_r \) both converging to \( b \).

To figure out the connected components of the seed \( \Sigma_T \) associated to a given countable triangulation \( T \) of the closed disc with marked points \( \mathbb{Z} \) it is helpful to view any half-nest, fountain and right-or left-fountain \( \{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) as converging to an arc \( \{a, b\} \) of the topological closure \( \overline{\mathbb{Z}} \) of \( \mathbb{Z} \subseteq S^1 \), where \( a = \lim_{i \to \infty} a_i \) and \( b = \lim_{i \to \infty} b_i \). Figure 3 provides an illustration of a left-fountain, a right-fountain and a fountain, while Figure 4 illustrates a nest and a half-nest.

**Lemma 3.19.** Let \( T \) be a countable triangulation of the closed disc with marked points \( \mathbb{Z} \subseteq S^1 \) and topological closure \( \overline{\mathbb{Z}} \subseteq S^1 \). Then two arcs \( \{x_0, x_1\} \neq \{y_0, y_1\} \) are connected in \( \Sigma_T \) if and only if there is no half-nest, fountain or right-or left-fountain \( \{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) in \( T \) converging to an arc \( \{a, b\} \) of \( \overline{\mathbb{Z}} \) such that \( x_0, x_1 \in [a, b] \) and \( y_0, y_1 \in [b, a] \) or vice-versa.

**Proof.** First assume that there is a half-nest, fountain or right-or left-fountain \( \{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}} \) converging to an arc \( \{a, b\} \) of \( \overline{\mathbb{Z}} \). It is straightforward to convince oneself that there cannot be finite sequence of arcs connecting any arc with endpoints in \( \{a, b\} \) with an arc with endpoints in
Figure 3. A right fountain at $a \in \mathcal{Z}$, a left fountain at $a$ and a fountain at $a$, all converging to the arc $\{a, b\}$ of $\mathcal{Z}$

Figure 4. A half nest converging to $\{a, b\}$ and a nest where the sequence of endpoints converges to $a$

Figure 5. Partition of a triangulation $\mathcal{T}$ of the closed disc with marked points $\mathcal{Z} \subseteq S^1$ into mutually disconnected components. The arcs in $\mathcal{T}$ are drawn in grey, the arcs $\{a, b\}$ of $\overline{\mathcal{Z}}$ such that there is a half-nest, fountain or right-or left-fountain $\{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in $\mathcal{T}$ converging to $\{a, b\}$ are drawn in black. Limit points of $\mathcal{Z}$ are marked by a bullet $(b, a)$, since there are infinitely many arcs from the right-or left-fountain, fountain or half-nest in between. On the other hand, assume that $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are not connected. Without loss of generality let $(x_0, x_1) \subseteq (x_0, y_0) \subseteq (x_0, y_1)$. We can construct a sequence of arcs $\{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ by setting $a_0 = x_0$ and $b_0 = x_1$ and for $i \geq 1$ setting $\{a_i, b_i\}$ such that $\{a_{i-1}, b_{i-1}\}$ and $\{a_i, b_i\}$ are sides of a common triangle in $\mathcal{T}$ and such that $a_i \in [y_1, a_{i-1}]$ and $b_i \in [b_{i-1}, y_0]$. Both sequences of endpoints $\{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\{b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ are monotonous and bounded above and below and because $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are not connected they are both infinite and non-constant, thus $\{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is a half-nest, fountain or right-or left-fountain converging to an arc $\{a, b\}$ of $\overline{\mathcal{Z}}$ such that $x_0, x_1 \in [a, b]$ and $y_0, y_1 \in [b, a]$. \[\square\]

Remark 3.20. For a given countable triangulation $\mathcal{T}$ with marked points $\mathcal{Z} \subset S^1$ the arcs $\{a, b\}$ of $\overline{\mathcal{Z}}$ such that there is a half-nest, fountain or right-or left-fountain $\{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in $\mathcal{T}$ converging to $\{a, b\}$ partition the seed $\Sigma_{\mathcal{T}}$ associated to $\mathcal{T}$ into mutually disconnected connected components. If $\{a, b\}$ is not an arc of $\mathcal{Z}$ (in particular if $\mathcal{Z}$ is discrete) then it divides $\Sigma_{\mathcal{T}}$ into two mutually disconnected components. If $\{a, b\}$ is an arc of $\mathcal{Z}$, then it provides an additional connected component, consisting only of the arc $\{a, b\}$ itself. Figure 5 provides an illustration of the partition of a triangulation into mutually disconnected connected components.
Lemma 3.21. Let $\mathcal{T}$ be a countable triangulation of the closed disc with marked points $\mathcal{Z} \subseteq S^1$ and let $\Sigma_\mathcal{T}$ be the associated seed. Then the rooted cluster algebra $\mathcal{A}(\Sigma_\mathcal{T})$ is isomorphic to a countable coproduct of connected rooted cluster algebras associated to countable triangulations of the closed disc.

Proof. The arcs $\{a, b\}$ of $\mathcal{Z}$ such that there is a half-nest, fountain or right-or left-fountain $\{a_i, b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in $\mathcal{T}$ converging to an arc $\{a, b\}$ partition the seed $\Sigma_\mathcal{T}$ associated to $\mathcal{T}$ into mutually disconnected connected components. There are countably many such arcs, and hence countably many connected components $\{\Sigma_{\mathcal{T}_i}\}_{i \in I}$ of $\Sigma_\mathcal{T}$. Thus by the discussion in Section 3.4 the rooted cluster algebra $\mathcal{A}(\Sigma_\mathcal{T})$ is isomorphic to the coproduct $\coprod_{i \in I} \mathcal{A}(\Sigma_{\mathcal{T}_i})$. \hfill $\Box$

3.5. Isomorphisms of rooted cluster algebras. For two rooted cluster algebras to be isomorphic in $\text{Clus}$ means that their initial seeds are combinatorially closely connected. First, we introduce some useful terminology.

Let $\Sigma = (X, \text{ex}, B)$ be a seed. We say that two variables $x, y \in X$ are connected by exchangeable variables if there exists a $n \geq 0$ and a sequence $(x_0, x_1, \ldots, x_n)$ of variables in $X$ with $x = x_0$ and $y = x_n$, such that $x_i$ and $x_{i+1}$ are neighbors for $0 \leq i \leq n - 1$ and such that $x_1, \ldots, x_{n-1}$ lie in $\text{ex}$. Further, if $n \in \{0, 1\}$ then at least one of $x_0$ and $x_n$ has to lie in $\text{ex}$. Thus, two coefficients that are neighbors are not necessarily connected by exchangeable variables and a coefficient is not necessarily connected to itself by exchangeable variables. For every exchangeable variable $x \in \text{ex}$ define the exchangeably connected component of $x$ in $\Sigma$ to be the full subseed $\Sigma_x = (X_x, \text{ex} \cap X_x, B_x = (b_{vw})_{v, w \in X_x})$ of $\Sigma$ where

$$X_x = \{y \in X \mid x, y \text{ are connected by exchangeable variables}\}.$$ 

To partition a seed into its exchangeably connected components can be useful when studying mutations of a seed, as mutation in an exchangeably connected component leaves all other exchangeably connected components invariant.

Remark 3.22. We can decompose any seed $\Sigma = (X, \text{ex}, B)$ into its exchangeably connected components $\{\Sigma^i = (X^i, \text{ex}^i, B^i = (b^i_{vw})_{v, w \in X^i})\}_{i \in I}$, where $I$ is a countable index set and $\text{ex} = \bigcup_{i \in I} \text{ex}^i$. Mutation along a $\Sigma^i$-admissible sequence leaves all other exchangeably connected components invariant (up to entries of the exchange matrix labelled by coefficients), i.e. if $(x_1, \ldots, x_l)$ is a $\Sigma^i$-admissible sequence and $y \in \text{ex}^i$ with $i \neq j \in I$ we have

$$(\mu_{x_1} \circ \ldots \circ \mu_{x_l})_y (\Sigma) = (X^j, \text{ex}^j, B^j = (b^j_{vw})_{v, w \in X^j}),$$

where $\mu_{x^i_{xy}} = b^i_{xy}$ for all $x, y \in \text{ex}^i$. This follows directly from the fact that mutation at an exchangeable variable $x$ only affects entries in the exchange matrix that are labelled by neighbors of $x$, hence in particular no entries that are labelled exchangeable variables in any other exchangeably connected component than $x$.

Definition 3.23 (Similar seeds). We call two seeds $\Sigma = (X, \text{ex}, B = (b_{vw})_{v, w \in X})$ and $\Sigma' = (X', \text{ex}', B' = (b'_{vw})_{v, w \in X'})$ similar, if there exists a bijection $\varphi: X \rightarrow X'$ restricting to a bijection $\varphi: \text{ex} \rightarrow \text{ex}'$ such that for every exchangeable variable $x \in X$ the exchangeably connected component $\Sigma_x$ of $x$ in $\Sigma$ is isomorphic (cf. Definition 2.1) to the exchangeably connected component $\Sigma'_{\varphi(x)}$ of $\varphi(x)$ in $\Sigma'$ or to its opposite seed $\Sigma'^{op}_{\varphi(x)}$.

Example 3.24. Consider the seeds

$$\Sigma_1 = (\{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_2, x_4, x_5\}, x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5),$$
Proof. However that we consider the case where a seed might consist of several exchangeably connected components, which is isomorphic to $\Sigma_1$. It is straightforward to check that $\Sigma_2$ and $\Sigma_3$ and we see that $\Sigma_1$ and $\Sigma_2$ are similar ($\Sigma_1 \sim (\Sigma_2)_{y_1}$ and $\Sigma_1 \sim (\Sigma_2)^\text{op}_{y_1}$), but they are both not similar to $\Sigma_3$: there is neither an exchangeably connected component of $\Sigma_2$ nor of $\Sigma_3$ that is isomorphic to $\Sigma_1$, or $\Sigma_3$ is $\text{op}$.

Theorem 3.25. The rooted cluster algebras $A(\Sigma)$ and $A(\Sigma')$ are isomorphic if and only if the associated seeds $\Sigma$ and $\Sigma'$ are similar.

The statement can be derived from Section 3, in particular Theorem 3.9, of [ADS]. Note however that we consider the case where a seed might consist of several exchangeably connected components, for the convenience of the reader we give a short proof.

Proof. Let $\Sigma = (X, \text{ex}, B = b_{vw})_{v,w \in X}$ and $\Sigma' = (X', \text{ex}', B' = b'_{vw})_{v,w \in X'}$ be similar via a bijection $\varphi \colon X \to X'$. It involves some calculations to check that $\varphi$ induces a rooted cluster morphism, but in the interest of not giving a rather technical argument twice, we refer to a result that will be proved in Section 3.6: In Theorem 3.30 we give three necessary and sufficient conditions for a map between clusters of initial seeds to give rise to a rooted cluster morphism. It is straightforward to check that $\varphi$ satisfies all of these, since it is a bijection and for every two exchangeable cluster variables $x$ and $y$ in the same exchangeably connected component of $\Sigma$ we have $b'_{f(x)f(w)} = b_{xw}$ and $b'_{f(y)f(w)} = b_{yw}$ for all $w \in X$ or $b'_{f(x)f(w)} = -b_{xw}$ and $b'_{f(y)f(w)} = -b_{yw}$ for all $w \in X$. Thus by Theorem 3.30 it induces a rooted cluster morphism $f : A(\Sigma) \to A(\Sigma')$. For the same reasons, the inverse $\varphi^{-1} : X' \to X$ induces a rooted cluster morphism $g : A(\Sigma') \to A(\Sigma)$. It remains to check that $f$ and $g$ are mutual inverses as rooted cluster morphisms.

Let $x$ be a cluster variable in $\Sigma$. It is of the form $x = \mu_{x_1} \circ \ldots \circ \mu_{x_1}(y)$ for some $y \in X$ and a $\Sigma$-admissible sequence $(x_1, \ldots, x_l)$. By induction over the length $l$ of the admissible sequence, we show that $g \circ f(x) = x$ and thus $g \circ f$ is the identity on $A(\Sigma)$: If $l = 0$ then have $g \circ f(x) = \varphi^{-1} \circ \varphi(x) = x$. If $g \circ f$ is the identity on all cluster variables which can be written as a mutation along a $\Sigma$-admissible sequence of length $l - 1$, then in particular $g \circ f(\mu_{x_{l-1}} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{x_{l-1}} \circ \ldots \circ \mu_{x_1}(y)$ and $g \circ f(x_i) = x_i$. Thus by axiom CM3 for $g \circ f$
Lemma 3.27. Let \( f : \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma') \) be a rooted cluster morphism without specializations. Then every \( \Sigma \)-admissible sequence is \((f, \Sigma, \Sigma')\)-biadmissible.

Proof. We prove the claim by induction over the length \( l \) of a \( \Sigma \)-admissible sequence. It is satisfied by axiom CM2 for sequences of length \( l = 1 \). Assume now that it is satisfied for all \( \Sigma \)-admissible sequences of length at most \( l \). Let \( (x_1, \ldots, x_{l+1}) \) be a \( \Sigma \)-admissible sequence of length \( l + 1 \). By Definition 2.11 we have \( x_{l+1} = \mu_{x_1} \circ \cdots \circ \mu_{x_2}(y) \) for some \( y \in \mathrm{ex} \) and thus \( f(x_{l+1}) = f(\mu_{x_2} \circ \cdots \circ \mu_{x_1}(y)) = \mu_f(x_2) \circ \cdots \circ \mu_f(x_1)(y) \) by axiom CM3. Because \( f(y) \in \mathrm{ex}' \) we have \( f(x_{l+1}) \in \mu_f(x_{l+1}) \circ \cdots \circ \mu_f(x_1)(\mathrm{ex}') \) and thus \( (x_1, \ldots, x_{l+1}) \) is \((f, \Sigma, \Sigma')\)-biadmissible. \( \square \)
In particular, this helps us to further understand ideal rooted cluster morphisms, partially answering Problem 2.12 from [ADS], through the following consequence.

**Corollary 3.28.** Every rooted cluster morphism without specializations is ideal.

**Proof.** This follows directly from Lemma 3.27 and Definition 3.12 \(\square\)

Generally, if we have a rooted cluster morphism \(f: \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')\) the combinatorial structures of the two seeds \(\Sigma\) and \(\Sigma'\) are linked via those exchangeable cluster variables in the cluster of \(\Sigma\) that do not get sent to integers. This provides a particularly strong combinatorial link of two rooted cluster algebras between which there exists a rooted cluster morphism without specializations.

**Lemma 3.29.** Let \(\Sigma = (X, \text{ex}, B = (b_{vw})_{v,w \in X})\) and \(\Sigma' = (X', \text{ex}', B' = (b'_{vw})_{v',w' \in X'})\) be seeds and let \(f: \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')\) be a rooted cluster morphism. Let \(x \in \text{ex}\) be an exchangeable cluster variable with \(f(x) \in \text{ex}'\). Any neighbor of \(x\) in \(\Sigma\) either gets mapped to a cluster variable or to \(\pm 1\). Furthermore, consider the exchangeably connected component \((f(\Sigma))(f(x)), \Sigma = (X, \text{ex}, B = (b_{vw})_{v,w \in X})\) of \(f(x)\) in the full subseed \(f(\Sigma) \subseteq \Sigma'\). Then we have

\[
\begin{align*}
b'_{f(v)w} &= \sum_{f(w) = w'} b_{vw} \quad \text{for all } f(v) \in f'(\text{ex})f(x) \text{ and all } w' \in X' \text{ or} \\
b'_{f(v)w} &= -\sum_{f(w) = w'} b_{vw} \text{ for all } f(v) \in f(\text{ex})f(x) \text{ and all } w' \in X',
\end{align*}
\]

where the empty sum is assumed to be 0. In particular, if \(v, w \in \text{ex} \) with \(f(v), f(w) \in \text{ex}'\), then we have \(b'_{f(v)w} = \pm b_{vw}\).

It is helpful to visualize this statement for rooted cluster algebras associated to seeds with skew-symmetric exchange matrices which can be represented as quivers. Let \(\Sigma = (X, \text{ex}, Q)\) and \(\Sigma' = (X', \text{ex}', Q')\) be seeds with their combinatorial data encoded in the quiver \(Q\), respectively \(Q'\), and let \(f: \mathcal{A}(\Sigma) \to \mathcal{A}(\Sigma')\) be a rooted cluster morphism. Then Lemma 3.29 implies that for every \(x \in \text{ex}\) with \(f(x) \in \text{ex}'\) the number of arrows starting, respectively ending, in \(x\) as a vertex of \(Q\) is equal to the number of arrows starting, respectively ending, in \(f(x)\) as a vertex of \(Q'\) or vice versa the arrows incident with \(x\) "change direction under \(f\)", i.e. the number of arrows starting, respectively ending, in \(x\) is equal to the number of arrows ending, respectively starting, in \(f(x)\). Furthermore, if the images \(f(x), f(y) \in \text{ex}'\) of two exchangeable cluster variables \(x, y \in \text{ex}\) lie in the same exchangeably connected component of the image seed \(f(\Sigma)\), then the number of arrows starting, respectively ending, in \(x\) as a vertex of \(Q\) is equal to the number of arrows starting, respectively ending, in \(f(x)\) as a vertex of \(Q'\) if and only if the equivalent statement holds for \(y\) and \(f(y)\) (or, equivalently, the arrows incident with \(x\) "change direction under \(f\)" if and only if the arrows incident with \(y\) do so.)

**Proof.** Let \(\tilde{X} = \{x \in X \mid f(x) \in X'\}\) be the set of cluster variables in \(\Sigma\) that get mapped to cluster variables in \(\Sigma'\) and let \(x \in \tilde{X} \cap \text{ex} \) with \(f(x) = z\). Because \(f\) is a ring homomorphism we have

\[
f(\mu_x(x)) = \left(\prod_{w \in \tilde{X}: b_{xw} > 0} w^{b_{xw}} + \prod_{w \in \tilde{X}: b_{xw} < 0} w^{-b_{xw}}\right) = k_1 \prod_{w \in \tilde{X}: b_{xw} > 0} f(w)^{b_{xw}} + k_2 \prod_{w \in \tilde{X}: b_{xw} < 0} f(w)^{-b_{xw}}
\]

for some \(k_1, k_2 \in \mathbb{Z}\). By axiom CM3 this has to be equal to

\[
\mu_{f(x)}(f(x))f(x) = \mu_z(z) = \prod_{w' \in X': b'_{xw'} > 0} (w')^{b'_{xw'}} + \prod_{w' \in X': b'_{xw'} < 0} (w')^{-b'_{xw'}}.
\]
By algebraic independence of the cluster variables in $X'$ this implies $k_1 = k_2 = 1$, thus any neighbor of $x$ gets mapped into $X' \cup \{\pm 1\}$. Furthermore, we obtain

$$\prod_{w \in \tilde{X} : b_{zw} > 0} f(w)^{b_{zw}} = \prod_{w' \in \tilde{X}' : b'_{zw'} > 0} (w')^{b'_{zw'}}$$

$$\prod_{w \in \tilde{X} : b_{zw} < 0} f(w)^{-b_{zw}} = \prod_{w' \in \tilde{X}' : b'_{zw'} < 0} (w')^{-b'_{zw'}}$$

or vice versa

$$\prod_{w \in \tilde{X} : b_{zw} > 0} f(w)^{b_{zw}} = \prod_{w' \in \tilde{X}' : b'_{zw'} > 0} (w')^{-b'_{zw'}}$$

$$\prod_{w \in \tilde{X} : b_{zw} < 0} f(w)^{-b_{zw}} = \prod_{w' \in \tilde{X}' : b'_{zw'} < 0} (w')^{b'_{zw'}}.$$  

Equations (2) and (3) imply

$$b'_{f(x)w'} = b'_{zw'} = \sum_{f(w) = w'} b_{zw},$$

and equations (4) and (5) imply

$$b'_{f(x)w'} = b'_{zw'} = -\sum_{f(w) = w'} b_{zw}.$$  

In particular this implies that if $w' = f(w)$ for some $w \in \text{ex}$ then we have $b'_{f(x)f(w)} = \pm b_{zw}$.

Let now $x, v \in \tilde{X} \cap \text{ex}$ be cluster variables such that their images $f(x)$ and $f(v)$ are cluster variables in the same exchangeably connected component of $f(\Sigma)$ and let them be connected by the finite sequence of cluster variables $f(x) = f(x_0), f(x_1), \ldots, f(x_k) = f(v)$ with $x_1, \ldots, x_{k-1} \in \text{ex}$. Applying the above argument iteratively to the sequence $f(x_0), \ldots, f(x_k)$ shows that $b'_{f(x)z'} = \sum_{f(z) = z'} b_{xz}$ for all $z' \in \tilde{X}'$ if and only if $b'_{f(x_i)f(x_{i+1})} = b_{x_ix_{i+1}}$ for all $0 \leq i < k - 1$, if and only if $b'_{f(v)z'} = \sum_{f(z) = z'} b_{vz}$ for all $z' \in \tilde{X}'$ and thus proves the claim. \(\square\)

In fact, this combinatorial property on the entries of the exchange matrices is almost sufficient for a map to give rise to a rooted cluster morphism without specializations. More precisely, the following theorem classifies rooted cluster morphisms without specializations. Note that if we do not have any specializations, a ring homomorphism $f : A(\Sigma) \to A(\Sigma')$ is uniquely determined by its values on the initial cluster.

**Theorem 3.30.** Let $\Sigma = (X, \text{ex}, B = (b_{vw})_{v,w \in X})$ and $\Sigma' = (X', \text{ex}', B' = (b'_{vw})_{v,w \in X'})$ be seeds and let $f : X \to X'$ be a map. Then the algebraic extension of $f$ on $A(\Sigma)$ gives rise to a rooted cluster morphism $f : A(\Sigma) \to A(\Sigma')$ if and only if the following holds.

1. The map $f$ restricts to an injection $f |_{\text{ex}} : \text{ex} \to \text{ex}'$.
2. If $f(x) = f(y)$ for some $x \neq y \in X$ then both $x$ and $y$ are coefficients of $\Sigma$. In that case for any $\Sigma$-admissible sequence $(x_1, \ldots, x_i)$ setting $\mu_{x_1} \circ \ldots \circ \mu_{x_1}(\Sigma) =: (\tilde{X}, \tilde{e}X, \tilde{B} = (\tilde{b}_{vw})_{v,w \in X})$ and for any $z \in \tilde{e}X$ the entries $\tilde{b}_{xz}$ and $\tilde{b}_{zy}$ cannot have opposite signs.
3. Let $x \in \text{ex}$ and consider the exchangeably connected component

$$f(\Sigma)f(x) = (f(X)f(x), f(\text{ex})f(x), (b'_{vw})_{v,w \in f(X)}f(x))$$

of $f(x)$ in the full subseed $f(\Sigma) \subseteq \Sigma'$. Then we have
\[
\begin{align*}
b'_f(v)w' &= \sum_{f(w)=w'} b_{vw} \text{ for all } f(v) \in f(ex) f(x) \text{ and all } w' \in X' \text{ or} \\
b'_f(v)w' &= -\sum_{f(w)=w'} b_{vw} \text{ for all } f(v) \in f(ex) f(x) \text{ and all } w' \in X',
\end{align*}
\]

where the empty sum is assumed to be 0.

Note that condition (3) of Theorem 3.30 is, as a necessary condition, the statement of Lemma 3.29.

**Remark 3.31.** Condition (2) of Theorem 3.30 is not always easy to check for two given seeds \(\Sigma = (X, ex, B)\) and \(\Sigma' = (X', ex', B')\) and a map \(f: X \to X'\). However, it is useful for checking when such a map does not induce a rooted cluster morphism. Furthermore, if for all \(x, y \in X\setminus ex\) with \(f(x) = f(y)\) we have \(b_{xy} = b_{yx}\) for all \(v \in ex\) then it is straightforward to check using Definition 2.9 that condition (2) is satisfied.

**Proof.** Assume first that the map \(f\) extends to a rooted cluster morphism. By axiom CM2 and Lemma 3.7 point (1) holds. By Lemma 3.27 every \(\Sigma\)-admissible sequence is \((f, \Sigma, \Sigma')\)-biadmissible and thus again by Lemma 3.7 point (2) holds. By Lemma 3.29 point (3) is satisfied.

Assume on the other hand that \(f: X \to X'\) is a map satisfying points (1) to (3). The map \(f: A(\Sigma) \to F_{\Sigma'}\) is a ring homomorphism and axioms CM1 and CM2 for rooted cluster morphisms are satisfied by definition and assumption (1). It remains to check axiom CM3 and that the image \(f(A(\Sigma))\) is contained in \(A(\Sigma')\).

By induction over the length \(l\) we show the following points for every \(\Sigma\)-admissible sequence \((x_1, \ldots, x_l)\).

(a) The sequence \((x_1, \ldots, x_l)\) is \((f, \Sigma, \Sigma')\)-biadmissible.

(b) For any \(y \in X\) we have

\[
\mu_{f(x)} \circ \ldots \circ \mu_{x_1}(y) = \mu_{f(x)} \circ \ldots \circ \mu_{f(x)}(f(y)).
\]

(c) Set

\[
\mu_{f(x)} \circ \ldots \circ \mu_{x_1}(\Sigma) =: \tilde{\Sigma} = (\tilde{X}, \tilde{e}x, \tilde{B} = (\tilde{b}_{vw})_{v,w \in \tilde{X}})
\]

to be the mutations of the seed \(\Sigma\) along \((x_1, \ldots, x_l)\) and

\[
\mu_{f(x)} \circ \ldots \circ \mu_{f(x)}(\Sigma') =: \tilde{\Sigma}' = (\tilde{X}', \tilde{e}x', \tilde{B}' = (\tilde{b}_{vw}')_{v,w \in \tilde{X}'})
\]

to be the mutation of \(\Sigma'\) along \((f(x_1), \ldots, f(x_l))\). Then for any \(x \in \tilde{e}x\) and any \(y \in \tilde{X}\) with \(x \neq y\) we have \(f(x) \neq f(y)\).

(d) For every \(v \in \tilde{e}x\) we have

\[
\tilde{b}'_{f(v)w'} = \sum_{f(w)=w'} \tilde{b}_{vw} \text{ for all } w' \in X' \text{ or}
\]

\[
\tilde{b}'_{f(v)w'} = -\sum_{f(w)=w'} \tilde{b}_{vw} \text{ for all } w' \in X',
\]

and for all \(y \in \tilde{e}x\) such that \(f(x)\) and \(f(y)\) lie in the same exchangeably connected component of \(f(\Sigma)\) we have \(\tilde{b}'_{f(x)w'} = \sum_{f(w)=w'} \tilde{b}_{xw} \) for all \(w' \in \tilde{X}'\) if and only if \(\tilde{b}'_{f(y)w'} = \sum_{f(w)=w'} \tilde{b}_{yw} \) for all \(w' \in \tilde{X}'\).
If these conditions are satisfied for every $\Sigma$-admissible sequence $(x_1, \ldots, x_l)$, then by condition (b) axiom CM3 is satisfied and by condition (a) and (b) the image of $T$ lies in $A(\Sigma')$. Conditions (c) and (d) merely help with the proof of conditions (a) and (b). We check conditions (a) to (d) for arbitrary $\Sigma$-admissible sequences by induction over their length $l$. For a $\Sigma$-admissible sequence of length $l = 0$ conditions (a) and (b) are satisfied trivially and conditions (c) and (d) are satisfied by assumption (2) and (3) respectively. Assume now that they are satisfied for all $\Sigma$-admissible sequences of length $\leq l$ and let $(x_1, \ldots, x_{l+1})$ be a $\Sigma$-admissible sequence of length $l + 1$ and set

$$\mu_{x_1} \circ \ldots \mu_{x_1}(\Sigma) =: \bar{\Sigma} = (\bar{X}, \bar{e}x, \bar{B} = (\bar{b}_{vw})_{v, w \in \bar{X}})$$

and

$$\mu_{f(x_1)} \circ \ldots \mu_{f(x_1)}(\Sigma') =: \bar{\Sigma}' = (\bar{X}', \bar{e}x', \bar{B}' = (\bar{b}'_{vw})_{v, w \in \bar{X}'})$$
as above.

We have $x_{l+1} = \mu_{x_1} \circ \ldots \circ \mu_{x_1}(y)$ for some $y \in \bar{e}x$ and thus $f(x_{l+1}) = f(\mu_{x_1} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{f(x_1)} \circ \ldots \circ \mu_{f(x_1)}(f(y))$ by induction assumption (b). By assumption (1) we have $f(y) \in \bar{e}x'$ and thus $f(x_{l+1}) \in \mu_{f(x_1)} \circ \ldots \circ \mu_{f(x_1)}(\bar{e}x')$ and $(x_1, \ldots, x_{l+1})$ is $(f, \Sigma, \Sigma')$-biadmissible. Thus condition (a) is satisfied.

Let $y \in X$. By definition of the exchange relations (see Definition 2.9) we have

$$f(\mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(y)) = \begin{cases} f(\mu_{x_{l+1}}(x_{l+1})), & \text{if } \mu_{x_1} \circ \ldots \circ \mu_{x_1}(y) = x_{l+1} \\ f(\mu_{x_1} \circ \ldots \circ \mu_{x_1}(y)), & \text{otherwise.} \end{cases}$$

If we have $\mu_{x_1} \circ \ldots \circ \mu_{x_1}(y) \neq x_{l+1} \in \bar{e}x$ this implies $\mu_{f(x_1)} \circ \ldots \circ \mu_{f(x_1)}(f(y)) = f(\mu_{x_1} \circ \ldots \circ \mu_{x_1}(y)) \neq f(x_{l+1})$ by induction assumption (c) and thus

$$f(\mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(y)) = f(\mu_{x_1} \circ \ldots \circ \mu_{x_1}(y)) = \mu_{f(x_1)} \circ \ldots \circ \mu_{f(x_1)}(f(y)) = \mu_{f(x_{l+1})} \circ \ldots \circ \mu_{f(x_{l+1})}(f(y)).$$

If on the other hand $\mu_{x_1} \circ \ldots \circ \mu_{x_1}(y) = x_{l+1}$ then we have

$$f(\mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(y)) = f(\mu_{x_{l+1}}(x_{l+1})) = \prod_{v > 0} f(v)^{b_v x_{l+1}} + \prod_{w < 0} f(w)^{-b_w x_{l+1}}$$

By assumption (2), for any $v, w \in \bar{X}$ with $f(v) = f(w)$ the entries $b_{wx_{l+1}}$ and $b_{wx_{l+1}}$ cannot have opposite sign. This and induction assumption (d) for the sequence $(x_1, \ldots, x_l)$ implies that the above is equal to

$$\prod_{v > 0} (v')^{b'_{f(x_{l+1})}} + \prod_{w < 0} (v')^{-b'_{f(x_{l+1})}}$$

By definition of mutation and induction assumption (b) this is equal to

$$\mu_{f(x_{l+1})}(f(x_{l+1})) = \mu_{f(x_1)}(f(y)).$$

This proves condition (b) for $(x_1, \ldots, x_{l+1})$. Let now $x \in \mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(\bar{e}x)$ and $y \in \mu_{x_{l+1}} \circ \ldots \circ \mu_{x_1}(X) = \mu_{x_{l+1}}(\bar{X})$ with $x \neq y$. Thus we have $x = \mu_{x_{l+1}}(\bar{x})$ and $y = \mu_{x_{l+1}}(\bar{y})$ for some $\bar{x} \in \bar{e}x$ and $\bar{y} \in \bar{X}$ with $\bar{x} \neq \bar{y}$. If both $\bar{x} \neq x_{l+1}$ and $\bar{y} \neq x_{l+1}$, then $x = \bar{x} \in \bar{e}x$ and $y = \bar{y} \in \bar{X}$ and by induction assumption
we have \( f(x) \neq f(y) \). Thus assume without loss of generality that \( \tilde{x} = x_{l+1} \) and \( \tilde{y} \neq x_{l+1} \). Then we have

\[
f(x)f(x_{l+1}) = f(\mu_{x_{l+1}}(x_{l+1}))f(x_{l+1}) = \prod_{v \in \epsilon: b_{uv}x_{l+1} > 0} f(v)^{b_{uv}x_{l+1}} + \prod_{w \in \epsilon: b_{w}x_{l+1} < 0} f(w)^{-b_{w}x_{l+1}}
\]

and thus \( f(x) \) divides the right hand side of the equation. On the other hand, we have \( f(y) = f(\mu_{x_{l+1}}(\tilde{y})) = f(\tilde{y}) \in \tilde{X}' \). Assume by contradiction that \( f(x) = f(\tilde{y}) \). In particular, this implies \( f(x) \in \tilde{X}' \). By algebraic independence of the elements of \( \tilde{X}' \), \( f(x) \) must divide both

\[
\prod_{v \in \epsilon: b_{uv}x_{l+1} > 0} f(v)^{b_{uv}x_{l+1}} \quad \text{and} \quad \prod_{w \in \epsilon: b_{w}x_{l+1} < 0} f(w)^{-b_{w}x_{l+1}}.
\]

However this is a contradiction to assumption (2) of the lemma and algebraic independence of \( \tilde{X}' \). Thus condition (c) holds for the sequence \( (x_1, \ldots, x_{l+1}) \).

Set now

\[
\mu_{x_{l+1}}(\tilde{B}) =: \mathcal{B} = (\beta_{uv})_{v, w \in \mu_{x_{l+1}}(\tilde{X})}
\]

and

\[
\mu_{f(x_{l+1})}(\tilde{B}') =: \mathcal{B}' = (\beta'_{uv})_{v, w \in \mu_{f(x_{l+1})}(\tilde{X}')},
\]

By definition of matrix mutation we have for every \( x = \mu_{x_{l+1}}(\tilde{x}) \in \mu_{x_{l+1}}(\tilde{e}x) \) and every \( w = \mu_{x_{l+1}}(\tilde{w}) \in \mu_{x_{l+1}}(\tilde{X}) \)

\[
\beta_{xw} = \mu_{x_{l+1}}(\tilde{b}_{xw}) = \begin{cases} 
\tilde{b}_{xw}, & \text{if } \tilde{x} = x_{l+1} \text{ or } \tilde{w} = x_{l+1} \\
\tilde{b}_{xw} + \frac{1}{2}(\tilde{b}_{xw} + \tilde{b}_{xw} + \tilde{b}_{xw} + b_{xx_{l+1}}|\tilde{b}_{xw}|), & \text{else}.
\end{cases}
\]

By condition (b) for the sequence \( (x_1, \ldots, x_{l+1}) \) we have \( f(x) = f(\mu_{x_{l+1}}(\tilde{x})) = \mu_{f(x_{l+1})}(f(\tilde{x})) \).

Thus for every \( w' = \mu_{f(x_{l+1})}((\tilde{w'}) \in \mu_{f(x_{l+1})}(\tilde{X}') \) we have

\[
\beta'_{f(x)w'} = \mu_{f(x_{l+1})}(\tilde{b}'_{(\tilde{x})}(\tilde{w'})) = \begin{cases} 
\tilde{b}'_{f(x)}(\tilde{w'}), & \text{if } f(\tilde{x}) = f(x_{l+1}) \text{ or } \tilde{w'} = x_{l+1} \\
\tilde{b}'_{f(x)}(\tilde{w'}) + \frac{1}{2}(\tilde{b}'_{f(x)}(\tilde{w'})|\tilde{b}'_{f(x)}(\tilde{w'}) + \tilde{b}'_{f(x)}(\tilde{w'})|\tilde{b}'_{f(x)}(\tilde{w'})|), & \text{else}.
\end{cases}
\]

By induction assumption (d) we have

\[
(6) \quad \tilde{b}'_{f(\tilde{x})}\tilde{u}' = \sum_{f(\tilde{u})=\tilde{u}', \tilde{u} \in \tilde{X}} \tilde{b}_{\tilde{x}\tilde{u}} \quad \text{and} \quad \tilde{b}'_{f(x_{l+1})}\tilde{u}' = \sum_{f(\tilde{u})=\tilde{u}', \tilde{u} \in \tilde{X}} \tilde{b}_{x_{l+1}\tilde{u}}
\]

for all \( \tilde{u}' \in \tilde{X}' \) and the signs of the two sums are the same if \( f(x) \) and \( f(x_{l+1}) \) are connected by a sequence of variables in \( f(\tilde{e}x) \), hence in particular if \( \tilde{b}'_{f(x)}(\tilde{w}_{a_{l+1}}) \neq 0 \). Note further that \( f(\tilde{x}) = f(x_{l+1}) \) if and only if \( \tilde{x} = x_{l+1} \) by assumption (c). Thus by substituting equation (6)

into the formula for $\beta'_f(x)w'$ we obtain

$$
\beta'_f(x)w' = \mu f(x_{i+1}) \left( \tilde{b}_f'(\tilde{z}) \right) = \begin{cases} 
-(\pm \sum_{\tilde{w} \in X} \tilde{b}_{x\tilde{w}}) & \text{if } \tilde{x} = x_{i+1} \text{ or } \tilde{w}' = f(x_{i+1}) \\
\pm \sum_{\tilde{w} \in X} \tilde{b}_{x\tilde{w}} + \frac{1}{2} \left( (\pm \sum_{\tilde{w} \in X} \tilde{b}_{x_{i+1}\tilde{w}}) \right) & \text{else.}
\end{cases}
$$

where the last equality holds because by condition (b) we have $f(w) = f(\mu f(x_{i+1})(\tilde{w})) = \mu f(x_{i+1})(f'(\tilde{w}))$ for every $w = \mu x_{i+1}(\tilde{w}) \in \mu x_{i+1}(X)$ and thus $f(w) = w' = f(x_{i+1})(\tilde{w}')$ if and only if $f(\tilde{w}) = \tilde{w}'$.

Observe that by definition of matrix mutation, if two exchangeable variables $f(\mu x_{i+1}(x)) = \mu f(x_{i+1}')(x') \in \mu f(x_{i+1}')(\tilde{w})$ and $f(y) = \mu f(x_{i+1}')(f'(\tilde{y})) \in \mu f(x_{i+1}')(\tilde{w}')$ are neighbors in $\mu f(x_{i+1})(\tilde{Sigma})'$ then $f(x')$ and $f(y)$ are neighbors in $\tilde{Sigma}'$ or both $f(x')$ and $f(y)$ are neighbors of $f(x_{i+1}' \tilde{w})$ in $\tilde{Sigma}'$. Thus if two exchangeable variables $f(x_{i+1}')(x') = f(x_{i+1}')(f'(\tilde{w})) \in f(x_{i+1}')(\tilde{Sigma})'$ and $f(y) = f(x_{i+1}')(f'(\tilde{y})) \in f(x_{i+1}')(\tilde{Sigma})'$ are connected by a finite sequence of exchangeable variables in $f(\mu x_{i+1})(\tilde{Sigma})'$ then $f(x') \in \tilde{Sigma}'$ and $f(y') \in \tilde{Sigma}'$ are connected by a finite sequence of exchangeable variables in $f(\tilde{Sigma})$. Therefore the signs of the sums in a given exchangeably connected component of $f(\tilde{Sigma})$ carry over from $B'$ to $\mu f(x_{i+1})(B) = B$ so we obtain by induction assumption (d) that

$$
\beta'_f(x)w' = \sum_{f(w)=w', \tilde{w} \in x_{i+1}(X)} \beta_{x\tilde{w}}
$$

for all $w' \in \mu f(x_{i+1})(\tilde{X}')$ if and only if

$$
\beta'_f(y)w' = \sum_{f(w)=w', \tilde{w} \in x_{i+1}(X)} \beta_{y\tilde{w}}
$$

for all $w' \in \mu f(x_{i+1})(\tilde{X}')$. This shows that condition (d) is satisfied for any $\tilde{Sigma}$-admissible sequence of length $l + 1$.

Remark 3.32. Theorem 3.30 implies that for a rooted cluster morphism $f : A(\Sigma) \to A(\Sigma')$ without specializations the full subseed the full subseed $(ex, ex, (b_{vw})_{v,w\in ex})$ of exchangeable variables in $\Sigma = (X, ex, B = (b_{vw})_{v,w\in X})$ is similar to a full subseed of $\Sigma'$.

Theorem 3.30 shows that we do not have much choice as to what rooted cluster morphisms without specializations can look like. It is helpful to visualize this via rooted cluster algebras associated to seeds with an skew symmetric exchange matrix that can be encoded in a quiver. The equivalent statements (that would be more cumbersome to state and not provide much
added insight) hold for rooted cluster algebras associated to seeds with skew-symmetric exchange matrices.

Let \( A(\Sigma) \) and \( A(\Sigma') \) be rooted cluster algebras associated to seeds \( \Sigma = (X, \text{ex}, Q) \) and \( \Sigma' = (X', \text{ex}', Q') \) with skew-symmetric exchange matrices which we express via the quivers \( Q \), respectively \( Q' \) and let \( f : A(\Sigma) \to A(\Sigma') \) be a rooted cluster morphism without specializations. By Remark 3.32 every connected component of the full subquiver of exchangeable variables of \( Q \) is isomorphic to a full subquiver, or to the opposite of a full subquiver, of \( Q' \). Furthermore, two coefficients \( x \neq y \in X \setminus \text{ex} \) may get sent to the same cluster variable if and only if there is no path of length two between \( x \) and \( y \) that passes through an exchangeable variable. If two coefficients \( x \) and \( y \) do get sent to the same cluster variable, then in the image, the number of arrows between \( x \) and any exchangeable variable \( v \) gets added up with the number of arrows between \( y \) and \( v \). The following example includes all of the more interesting things that can happen in a rooted cluster morphism without specializations: we may glue on vertices to images of coefficients and we may glue together coefficients of \( \Sigma \) while keeping track of all the arrows going into (or out of) exchangeable variables. Furthermore, we can always add or remove arrows between coefficients of \( \Sigma \) and we can turn any coefficient into an exchangeable variable.

**Example 3.33.** Consider the seeds

\[
\Sigma = (\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}, \{x_1, x_2, x_3\}, \ x_1 \to x_2 \leftarrow x_4 \xrightarrow{x_5} \ x_6 \xrightarrow{x_7} )
\]

and

\[
\Sigma' = (\{y_1, y_2, y_3, z_1, z_2, a\}, \{y_1, y_2, y_3, a\}, \ y_1 \to y_2 \equiv z_1 \xrightarrow{y_3} z_2 \to a)
\]

and the map \( f : \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \to \{y_1, y_2, y_3, z_1, z_2, a\} \) which maps

\[
\begin{align*}
x_i &\mapsto y_i \text{ for } i = 1, 2, 3 \\
x_i &\mapsto z_1 \text{ for } i = 4, 5, 7 \\
x_6 &\mapsto z_2.
\end{align*}
\]

We check that \( f \) satisfies conditions (1), (2) and (3) from Theorem 3.30. By definition of \( f \), the restriction of \( f \) to the exchangeable variables of \( \Sigma \) is an injection that maps into the exchangeable variables of \( \Sigma' \), thus condition (1) is satisfied.

The variables \( x_4, x_5 \) and \( x_7 \) all get mapped to the same variable, so we have to check condition (2) for those. Indeed, they are all coefficients. Let now \( (x_1, \ldots, x_l) \) be a \( \Sigma \)-admissible sequence and set \( \Sigma = \mu_{x_1} \circ \cdots \circ \mu_{x_l}(\Sigma) \) with \( \tilde{\Sigma} = (\tilde{X}, \text{ex}, \tilde{Q}) \). We have to check that in \( \tilde{Q} \) there are no paths of length 2 passing through an exchangeable vertex \( v \in \text{ex} \) from any of \( x_4, x_5 \) and \( x_7 \) to any of \( x_4, x_5 \) or \( x_7 \) (i.e. no paths of the form \( x_i \to v \to x_j \) for \( i, j \in \{4, 5, 7\} \)). Since \( x_7 \) is its own connected component in \( \Sigma \), and therefore also in \( \tilde{\Sigma} \), we have no arrow between \( x_7 \) and any \( v \in \text{ex} \) in \( \tilde{Q} \). Furthermore, we can check the condition for the two exchangeably connected components \( x_{21} = (X_{x_1}, \text{ex}_{x_1}, Q_{x_1}) \) and \( x_{23} = (X_{x_3}, \text{ex}_{x_3}, Q_{x_3}) \) individually, by Remark 3.22. It is straightforward to check that both \( Q_{x_1} \) and \( Q_{x_3} \) are mutation finite along \( x_{21}, x_{23} \)-, respectively \( x_{23} \)-admissible sequences, with ten quivers in the mutation class of \( Q_{x_1} \) and two quivers in the mutation class of \( Q_{x_3} \), and that the condition holds for all of them.

Finally, we can see that in the exchangeably connected components of the image of the seed (cf. Example 3.9) we have

\[
\begin{align*}
b'_{f(v)w} &= \sum_{f(w) = w'} b_{vw} \text{ for all } w' \in X' \text{ and all } v \in \text{ex} \text{ with } f(v) \in f(X)_{f(x_1)} \\
b'_{f(v)w} &= -\sum_{f(w) = w'} b_{vw} \text{ for all } w' \in X' \text{ and all } v \in \text{ex} \text{ with } f(v) \in f(X)_{f(x_3)}.
\end{align*}
\]
4. Rooted Cluster Algebras of Infinite Rank as Colimits of Rooted Cluster Algebras of Finite Rank

In this section, we show that every rooted cluster algebra of infinite rank can be written as a linear colimit of rooted cluster algebras of finite rank. This yields a formalized way to think of cluster algebras of infinite rank by viewing them locally as cluster algebras of finite rank.

4.1. Colimits and limits in \textit{Clus}. We start by recalling the notion of limit and colimit. Let \( \mathcal{C} \) and \( \mathcal{J} \) be categories and let \( F : \mathcal{J} \to \mathcal{C} \) be a diagram of type \( \mathcal{J} \) in the category \( \mathcal{C} \), i.e. a functor from \( \mathcal{J} \) to \( \mathcal{C} \).

The \textit{limit} \( \lim(F) \) of \( F \) (if it exists) is an object in \( \mathcal{C} \) together with a family of morphisms \( f_i : \lim(F) \to F(i) \) in \( \mathcal{C} \) indexed by the objects \( i \in \mathcal{J} \) such that for any morphism \( f_{ij} : i \to j \) in \( \mathcal{J} \) we have \( F(f_{ij}) \circ f_i = f_j \) and \( \lim(F) \) is universal with this property. I.e. for any object \( C \in \mathcal{C} \) with a family of morphisms \( g_i : C \to F(i) \) in \( \mathcal{C} \) labelled by the objects \( i \in \mathcal{J} \) such that \( F(f_{ij}) \circ g_i = g_j \) for all morphisms \( f_{ij} : i \to j \) in \( \mathcal{J} \) there exists a unique morphism \( h : C \to \lim(F) \) such that the following diagram commutes.

\[
\begin{tikzcd}
& \lim(F) \\
F(i) \\
\end{tikzcd}
\]

The dual notion of the limit of \( F \) is the \textit{colimit} \( \text{colim}(F) \) of \( F \). If it exists, it is an object in \( \mathcal{C} \) together with a family of morphisms \( f_i : F(i) \to \text{colim}(F) \) in \( \mathcal{C} \) indexed by the objects \( i \in \mathcal{J} \) such that for any morphism \( f_{ij} : i \to j \) in \( \mathcal{J} \) we have \( f_j \circ F(f_{ij}) = f_i \) and for any object \( C \in \mathcal{C} \) with a family of morphisms \( g_i : F(i) \to C \) in \( \mathcal{C} \) labelled by the objects \( i \in \mathcal{J} \) such that \( g_j \circ F(f_{ij}) = g_i \) for all morphisms \( f_{ij} : i \to j \) in \( \mathcal{J} \) there exists a unique morphism \( h : \text{colim}(F) \to C \) such that the following diagram commutes.

\[
\begin{tikzcd}
& \text{colim}(F) \\
F(i) \\
\end{tikzcd}
\]

A limit \( \lim(F) \) or colimit \( \text{colim}(F) \) is called \textit{finite}, respectively \textit{small} if the index category \( \mathcal{J} \) in the diagram \( F : \mathcal{J} \to \mathcal{C} \) is finite, respectively small. A category is called \textit{complete}, respectively \textit{cocomplete}, if it has all small limits, respectively colimits.

\textbf{Remark 4.1.} Products are an example of limits. They are limits of diagrams \( F : \mathcal{J} \to \mathcal{C} \), where \( \mathcal{J} \) is a discrete category, i.e. a category with no morphisms but the identity morphisms. Dually, coproducts are an example of colimits.

Examples for finite colimits are coequalizers. They are colimits of diagrams \( G : \mathcal{J} \to \mathcal{C} \), where \( \mathcal{J} \) is the category with two objects \( i_1 \) and \( i_2 \) and two parallel morphisms \( i_1 \Rightarrow i_2 \) in addition to the identity morphisms. Dually, equalizers are an example of finite limits.

In fact, these are rather important examples as having equalizers and products is necessary and sufficient for a category to be complete, and dually a category is cocomplete if and only if it has coequalizers and coproducts, cf. for example Chapter V of [ML].
**Theorem 4.2.** The category $\text{Clus}$ is neither complete nor cocomplete.

**Proof.** If the category $\text{Clus}$ were complete, then in particular finite products would exist, cf. Remark 4.1. However, by Proposition 5.4 in [ADS], the category $\text{Clus}$ does not generally admit finite products, hence it cannot be complete.

Furthermore if $\text{Clus}$ were cocomplete then in particular coequalizers would exist. However, consider the seeds

$$\Sigma_0 = (\{x_0, x_1\}, \{x_0, x_1\}, x_0 \to x_1) \text{ and } \Sigma_1 = (\{y_0, y_1\}, \{y_0, y_1\}, y_0 \to y_1)$$

and the parallel rooted cluster isomorphisms defined by the algebraic extensions of

$$f: A(\Sigma_0) \to A(\Sigma_1) \quad \text{and} \quad g: A(\Sigma_0) \to A(\Sigma_1)$$

for $i = 0, 1$. Assume by contradiction that there exists a coequalizer for $f$ and $g$, i.e. a rooted cluster algebra $A(\Sigma)$ associated to a seed $\Sigma = (X, \text{ex}, B)$ with a rooted cluster morphism $\varphi: A(\Sigma_1) \to A(\Sigma)$ such that $\varphi \circ f = \varphi \circ g$ and it is universal with this property. Because $\varphi$ is a rooted cluster morphism and $\varphi \circ f = \varphi \circ g$ we have $\varphi(y_0) = \varphi(y_1) \in \text{ex} \cup \mathbb{Z}$. By Lemma 3.7 two distinct exchangeable variables cannot be sent to the same exchangeable variable via a rooted cluster morphism. Thus we must have $\varphi(y_0) = \varphi(y_1) \in \mathbb{Z}$. Consider the empty seed $\Sigma_0 = (\emptyset, \emptyset, \emptyset)$. As a ring, we have $A(\Sigma_0) \cong \mathbb{Z}$. Consider the rooted cluster morphisms $\psi_1: A(\Sigma_1) \to A(\Sigma_0)$, defined by sending all cluster variables to 0, and $\psi_2: A(\Sigma_1) \to A(\Sigma_0)$ defined by evaluating both $y_0$ and $y_1$ at 1. Because a rooted cluster morphism is a ring homomorphism between unital rings, any rooted cluster morphism from $A(\Sigma)$ to $A(\Sigma_0)$ acts as the identity on the subring $\mathbb{Z}$. Thus, if $\varphi(y_0) = \varphi(y_1) \neq 0$, then $\psi_1$ does not factor through $\varphi$ and if $\varphi(y_0) = \varphi(y_1) = 0$, then $\psi_2$ does not factor through $\varphi$. Therefore there exists no coequalizer for $f$ and $g$ and $\text{Clus}$ is not cocomplete. \qed

### 4.2. Rooted cluster algebras of infinite rank as colimits.

Even though colimits do not in general exist in $\text{Clus}$, we can show that every rooted cluster algebra of infinite rank is isomorphic to a colimit of rooted cluster algebras of finite rank.

**Definition 4.3** (Rooted cluster algebras of finite and infinite rank). A rooted cluster algebra $A(\Sigma)$ is called **rooted cluster algebra of infinite rank**, respectively **rooted cluster algebra of finite rank** if the cluster in $\Sigma$ is infinite, respectively finite.

We want to show that every rooted cluster algebra, and in particular rooted cluster algebras of infinite rank, can be written as a colimit of rooted cluster algebras of finite rank. More precisely, we can write them as linear colimits. Recall that a colimit $\text{colim}(F)$ in a category $C$ is called **linear**, if the index category $\mathcal{J}$ of the diagram $F: \mathcal{J} \to C$ is a set endowed with a linear order. A diagram $F: \mathcal{J} \to C$ where $\mathcal{J}$ is equipped with a linear order $\leq$ is just a linear system of objects in $C$, that is a family of objects $\{C_i\}_{i \in \mathcal{J}}$ and a family of morphisms $\{f_{ij}\}_{i \leq j \in \mathcal{J}}$ such that $f_{jk} \circ f_{ki} = f_{ij}$ and $f_{ii} = id_{C_i}$ for all $i \leq k \leq j$ in $\mathcal{J}$. In order to explicitly construct a suitable linear system of rooted cluster algebras of finite rank, we use the fact that in certain nice cases the inclusions of subseeds give rise to rooted cluster morphisms.

In general if $\Sigma$ is a full subseed of $\Sigma'$ (cf. Definition 3.10), the natural inclusion of rings $A(\Sigma) \to A(\Sigma')$ does not give rise to a rooted cluster morphism, cf. [ADS], Remark 4.10. However we can fix this with an additional condition which has to do with how the subseed is connected to the bigger seed.

**Definition 4.4** (Subseed connected only by coefficients). Let $\Sigma' = (X', \text{ex}', B')$ be a seed with a full subseed $\Sigma = (X, \text{ex}, B = (b_{uv})_{v,u \in X})$ such that for every $x \in X$ with a neighbor $y \in X' \setminus X$ in $\Sigma'$ we have $x \in X \setminus \text{ex}$, i.e. $x$ is a coefficient in $\Sigma$. Then we say that $\Sigma$ is **connected to $\Sigma'$ only by coefficients in $\Sigma$.**
If a full subseed $\Sigma$ of $\Sigma'$ is connected only by coefficients in $\Sigma$ to $\Sigma'$, this means that mutation of $\Sigma'$ at exchangeable variables in $\Sigma$ is completely unaffected by anything in the complement of $\Sigma$ in $\Sigma'$. In particular, if $\Sigma$ is a full subseed of $\Sigma'$, such that the seeds are connected only by coefficients in $\Sigma$, then the inclusion of $\Sigma$ in $\Sigma'$ induces a rooted cluster morphism.

**Lemma 4.5.** Let $\Sigma = (X, \text{ex}, B)$ be a full subseed of $\Sigma' = (X', \text{ex}', B')$ such that $\Sigma$ and $\Sigma'$ are connected only by coefficients in $\Sigma$. Then the canonical homomorphism

$$f: \begin{cases} \mathbb{Q}(X) \to \mathbb{Q}(X') \\ x \mapsto x \text{ for all } x \in X \end{cases}$$

restricts to a rooted cluster morphism $f: A(\Sigma) \to A(\Sigma')$.

**Proof.** This follows directly from Theorem 3.30. □

For any given rooted cluster algebra $A(\Sigma)$ we can build a linear system $\{A(\Sigma_i)\}_{i \in \mathbb{Z}}$ of rooted cluster algebras associated to finite full subseeds $\Sigma_i$ of $\Sigma$ that are only connected to $\Sigma$ via coefficients, such that for all $i \leq j$ the seed $\Sigma_i$ is a subseed of $\Sigma_j$ which is connected to $\Sigma_j$ only by coefficients. This will yield a linear system of rooted cluster algebras of finite rank which has the desired rooted cluster algebra $A(\Sigma)$ as a colimit.

**Theorem 4.6.** Every rooted cluster algebra is isomorphic to a linear colimit of rooted cluster algebras of finite rank in the category $\text{Clus}$ of rooted cluster algebras.

**Proof.** Let $\Sigma = (X, \text{ex}, B = (b_{uv})_{u,v \in X})$ be a seed and let $\Sigma = \bigsqcup_{j \in J} \Sigma^j$ be its decomposition into mutually disconnected connected seeds with $\Sigma^j = (X^j, \text{ex}^j, B^j)$ for $j \in J$, and $J$ is some countable index set (since the cluster $X$ is countable by Definition 2.1, there are only countably many connected components). We can thus write the rooted cluster algebra $A(\Sigma)$ as the countable coproduct of the connected rooted cluster algebras $A(\Sigma^j)$:

$$A(\Sigma) \cong \bigsqcup_{j \in J} A(\Sigma^j).$$

We construct a linear system of rooted cluster algebras as follows. For $j \in J$ let $x^j_0 \in X^j$ and for $i \in \mathbb{Z}_{\geq 0}$ let $\Sigma^j_i$ be the full subseed of $\Sigma$ given by

$$\Sigma^j_0 = (X^j_0, \text{ex}^j_0, B^j_0) = (\{x^j_0\}, \emptyset, [0])$$

$$\Sigma^j_{i+1} = (X^j_{i+1}, \text{ex}^j_{i+1}, B^j_{i+1})$$

$$= (X^j_i \cup \{ w \in X \mid b_{uv} \neq 0 \text{ for some } v \in X^j_i \}, X^j_{i+1} \cap \text{ex}^j, B^j_{i+1})$$

for $i \geq 1$

where $B^j_{i+1}$ is the full submatrix of $B$ formed by the entries labelled by $X^j_{i+1} \times X^j_{i+1}$. Note that because $B$ is skew-symmetric $b_{uv} \neq 0$ is equivalent to $b_{uv} \neq 0$. Because $B^j$ is locally finite, for all $i \geq 0$ the cluster $X^j_i$ is finite. We set

$$\tilde{\Sigma}_i := \Sigma^j_0 \bigsqcup \Sigma^j_{i-1} \bigsqcup \cdots \bigsqcup \Sigma^j_i$$

and write $\tilde{\Sigma}_i = (\tilde{X}_i, \tilde{\text{ex}}_i, \tilde{B}_i = ((\tilde{b}_{i})_{uv})_{u,v \in \tilde{X}_i}$). Because the cluster in each of the seeds $\Sigma^j_k$ for $0 \leq k \leq i$ is finite, so is the cluster $\tilde{X}_i$ of $\tilde{\Sigma}_i$. By definition, the seed $\tilde{\Sigma}_i$ is a full subseed of the seed $\tilde{\Sigma}_{i+1}$ for all $i \geq 0$ and all the seeds $\tilde{\Sigma}_i$ are full subseeds of $\Sigma$.

We now want to show that for all $i \geq 0$ the seeds $\tilde{\Sigma}_i$ and $\tilde{\Sigma}_{i+1}$ are connected only by coefficients in $\tilde{\Sigma}_i$. Because the subseeds $\Sigma^j_i$ and $\Sigma^j_{i'}$ are by definition mutually disconnected for $j \neq j'$ in $J$ and any $i, i' \in \mathbb{Z}_{\geq 0}$, it is enough to check that $\tilde{\Sigma}^j_i$ and $\tilde{\Sigma}^j_{i+1}$ are connected only by coefficients in $\Sigma^j_i$ for any $i \in \mathbb{Z}$ and $j \in J$. Let $x \in \text{ex}^j_i$ and $y \in X^j_{i+1}$ with $b_{xy} \neq 0$. We want to show that this implies $y \in X^j_i$. We have $i \geq 0$, since $\text{ex}^j_0 = \emptyset$ for all $i \geq 0$. It follows
that \( x \in \text{ex}_i = X^j_{i-1} \cap \text{ex} \subseteq X^j_{i-1} \) and thus \( y \in \{ w \in X \mid b_{vw} \neq 0 \text{ for some } v \in X^j_{i-1} \} \subseteq X^j_{i} \). Therefore \( \Sigma_i \) and \( \Sigma_{i+1} \) are connected only by coefficients in \( \Sigma_i \). The same argument shows that for any \( i \geq 0 \) the seeds \( \Sigma_i \) and \( \Sigma \) are connected only by coefficients in \( \Sigma_i \) for any \( i \geq 0 \).

By Lemma 4.5 for \( 0 \leq i \leq j \), the natural inclusion \( f_{ij} : \mathbb{Q}(\check{X}_i) \to \mathbb{Q}(\check{X}_j) \) gives rise to a rooted cluster morphism \( f_{ij} : A(\Sigma_i) \to A(\Sigma_j) \). For all \( 0 \leq i \leq j \leq k \) we have \( f_{kj} \circ f_{ij} = f_{ik} \) and \( f_{ii} = \text{id}_{A(\Sigma_i)} \), so the morphisms form a linear system of rooted cluster algebras of finite rank. Further, again by Lemma 4.5, for \( i \geq 0 \) the natural inclusions \( f_i : \mathbb{Q}(\check{X}_i) \to \mathbb{Q}(X) \) restrict to rooted cluster morphisms \( f_i : A(\Sigma_i) \to A(\Sigma) \). We show that \( A(\Sigma) \) together with the maps \( f_i : A(\Sigma_i) \to A(\Sigma) \) for \( i \geq 0 \) is in fact the colimit of this linear system in the category of rooted cluster algebras.

Because for any \( j \in J \), the seed \( \Sigma^j \) is connected, we have \( X^j = \bigcup_{i \geq 0} X^j_i \) and thus

\[
X = \bigcup_{j \in J} X^j = \bigcup_{i \geq 0, j \in J} X^j_i = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \check{X}_i.
\]

Because every exchange relation in \( A(\Sigma) \) lifts to an exchange relation in \( A(\Sigma_i) \) for all \( i \) big enough (by virtue of the exchange matrices \( \check{B}_i \) being arbitrarily large restrictions of the exchange matrix \( B \)), every element of \( A(\Sigma) \) is contained in all \( A(\Sigma_i) \) for \( i \) big enough.

Let \( \check{\Sigma} = (\check{X}, \check{e}_X, \check{Q}) \) be a seed such that for all \( i \geq 0 \) there are rooted cluster morphisms \( g_i : A(\Sigma_i) \to A(\Sigma) \) commuting with the linear system \( f_{ij} : A(\Sigma_i) \to A(\Sigma_j) \). We define the ring homomorphism \( f : A(\Sigma) \to A(\Sigma) \) by \( f(x) = g_i(x) \), whenever \( x \in A(\Sigma_i) \), i.e. it is the ring homomorphism making the following diagram commute.

\[
\begin{array}{ccc}
A(\Sigma) & \xrightarrow{f_i} & A(\Sigma) \\
\downarrow{g_i} & & \downarrow{g_j} \\
A(\Sigma_i) & \xrightarrow{f_{ij}} & A(\Sigma_j)
\end{array}
\]

For every \( x \in X \) (respectively \( x \in \text{ex} \)), there exists a \( k \geq 0 \) such that \( x \in X_i \) (respectively \( x \in \text{ex}_i \)) for all \( i \geq k \). Thus \( f(x) = g_i(x) \) for all \( i \geq k \) lies in \( \check{X} \) (respectively in \( \check{e}_X \)), because \( g_i \) is a rooted cluster morphism for all \( i \geq 0 \). Thus the ring homomorphism \( f \) satisfies axioms CM1 and CM2. Let now \((x_1, \ldots, x_l)\) be a \((f, \Sigma, \Sigma)\)-biadmissible sequence and let \( y \in X \) such that \( f(y) \notin X \). Then there exists a \( i \geq 0 \) such that \( y \in X_i \) and the sequence \((x_1, \ldots, x_l)\) is \((g_i, \Sigma_i, \Sigma)\)-biadmissible. Thus we get

\[
f(\mu_{x_l} \circ \cdots \circ \mu_{x_1}(y)) = f \circ f_i(\mu_{x_l} \circ \cdots \circ \mu_{x_1}(y)) = g_i(\mu_{x_l} \circ \cdots \circ \mu_{x_1}(y)) = \mu_{g_i(x_i)}(\mu_{g_i(x_i)}(g_i(y))) = \mu_{f(x_i)}(f(y)).
\]

Therefore the ring homomorphism \( f \) satisfies CM3 and is a rooted cluster morphism.

\( \square \)

**Remark 4.7.** Work in progress by Stovicek and van Roosmalen shows the analogue of Theorem 4.6 for cluster categories of infinite rank. However, their approach is different and it is not clear that either result can be easily obtained from the other.

4.3. **Positivity for cluster algebras of infinite rank.** Fomin and Zelevinsky showed in [FZ1] that every cluster variable of a (finite rank) cluster algebra is a Laurent polynomial in the variables of its initial cluster and they conjectured that the coefficients in this Laurent polynomial were positive. The so-called positivity conjecture has been a central problem in the theory of
Proof. Let $\Sigma = (X, e_x, Q)$ be a skew-symmetric cluster algebra of infinite rank. Using the construction in the proof of Theorem 4.6, the associated rooted cluster algebra $A(\Sigma)$ can be written as a linear colimit $A(\Sigma) = \operatorname{colim}(A(\Sigma_i))$ of a linear system $\{A(\Sigma_i)\}_{i \in \mathbb{Z}}$ of skew-symmetric rooted cluster algebras of finite rank with seeds $\Sigma_i = (X_i, e_x, Q_i)$ and with canonical inclusions $f_i: A(\Sigma_i) \to A(\Sigma)$ for $i \in \mathbb{Z}$. Let $\tilde{x} \in A(\Sigma)$ be a cluster variable, thus $\tilde{x} = \mu_{x_1} \circ \ldots \circ \mu_{x_l}(x)$ for some $x \in X$ and some $\Sigma$-admissible sequence $(x_1, \ldots, x_l)$. Then there exists a $k \in \mathbb{Z}$ such that $x \in f_i(X_i)$ and $(x_1, \ldots, x_l)$ is $\Sigma$-admissible for all $i \geq k$. By the construction of the colimit, the map $f_k$ is injective and $f_k^{-1}(X) = X_k$. Set $y = f_k^{-1}(x)$. By Theorem 4.2 in [LS], the cluster variable $y \in A(\Sigma_k)$ is a Laurent polynomial in $X_k$ over $\mathbb{Z}$ with positive coefficients. Since $f_k$ is ring homomorphism (with $f_k(1) = 1$), the image $x = f_k(y)$ is a Laurent polynomial in $f(X_k) \subseteq X$ over $\mathbb{Z}$ with positive coefficients. □

4.4. Rooted cluster algebras from infinite triangulations of the disc. It follows from Theorem 4.6 that every rooted cluster algebra arising from a countable triangulation of the closed disc can be written as a colimit of rooted cluster algebras of finite rank. Moreover, it can be written as a linear colimit of rooted cluster algebras that arise from finite triangulations of the disc. Thus we obtain a formal way of treating cluster algebras associated to infinite triangulations of the closed disc as infinite versions of cluster algebras of Dynkin type $A$. This provides the algebraic analogue of the work of Igusa and Todorov ([IT1], [IT2]), who introduced infinite versions of cluster categories of Dynkin type $A$.

Let $T$ be a connected triangulation of the closed disc with marked points $Z \subseteq S^1$. Then we can write $T$ as a countable union of finite triangulations ordered by inclusion:

$$T = \bigcup_{i \geq 0} T_i,$$

where $T_i$ is a finite triangulation of the closed disc. For example we could start with any arc $\{a_0, a_1\} \in T$ and set $T_0 = \{\{a_0, a_1\}\}$ and for all $i \geq 0$

$$T_{i+1} = T_i \cup \left\{ \alpha \in T \mid \text{there exists a } \beta \in T_i \text{ such that } \alpha \text{ and } \beta \text{ are sides of a common triangle in } T \right\}.$$

This corresponds to the construction in the proof of Theorem 4.6. Glueing on arcs to the edges of the triangulations corresponds to gluing on new cluster variables to coefficients.

Theorem 4.9. Let $T$ be a triangulation of the closed disc. Then the associated rooted cluster algebra $A(\Sigma_T)$ is isomorphic to a countable coproduct $A(\Sigma_T) \cong \bigsqcup_{j \geq 1} A(\Sigma_{T_j})$ of linear colimits $A(\Sigma_{T_j}) \cong \operatorname{colim}(A(\Sigma_{T_j}))$ of rooted cluster algebras $A(\Sigma_{T_j})$ of finite Dynkin type $A$.

Proof. We can directly translate the proof from Theorem 4.6 to this situation. Let first $T$ be a connected triangulation: As in the comment before Theorem 4.9, we can build a linear system of rooted cluster algebras associated to finite triangulations of the disc. Indeed, by Lemma 4.5, the natural inclusions $f_{ij}: A(\Sigma_{T_i}) \to A(\Sigma_{T_j})$ for $0 \leq i \leq j$ provide a linear system of rooted cluster algebras and following the lines of the proof of Theorem 4.6 there is an isomorphism of rooted cluster algebras

$$A(\Sigma_T) \cong \operatorname{colim}(A(\Sigma_{T_j})).$$
The rooted cluster algebras $A(\Sigma_T)$ are associated to finite triangulations of the closed disc and thus of finite Dynkin type $A$. By Lemma 3.21, every rooted cluster algebra associated to a triangulation of the closed disc is isomorphic to the coproduct of rooted cluster algebras associated to connected triangulations of the closed disc. This proves the claim.

By Theorem 4.9, classifying cluster algebras associated to infinite triangulations of the closed disc for a given set of marked points $Z \subseteq S^1$ boils down to partitioning every possible triangulation into mutually disconnected connected components. In the case where the set of marked points $Z \subseteq S^1$ has precisely one limit point, the cluster algebras associated to triangulations of $Z$ have been classified in [GG].

In the language of the category $\text{Clus}$ we can reformulate the main result from [GG] as follows.

**Theorem 4.10** ([GG], Theorems 3.11 and 3.16). Let $Z$ be a discrete subset of $S^1$ with exactly one limit point and let $T$ be a triangulation of $Z$. Then one of the following holds.

1. The triangulation $T$ has a nest and the rooted cluster algebra $A(\Sigma_T)$ is isomorphic to a colimit of rooted cluster algebras of finite Dynkin type $A$.
2. The triangulation $T$ has a fountain and the rooted cluster algebra $A(\Sigma_T)$ is isomorphic to the coproduct of two colimits of rooted cluster algebras of finite Dynkin type $A$.
3. The triangulation $T$ has a split fountain and the rooted cluster algebra $A(\Sigma_T)$ is isomorphic to the coproduct of a rooted cluster algebra of finite Dynkin type $A$ and two colimits of rooted cluster algebras of finite Dynkin type $A$.

**Remark 4.11.** In a similar fashion it is possible to classify (rooted) cluster algebras associated to arbitrary triangulations of the disc with marked points $Z \subseteq S^1$. It depends on the connected components of the triangulation $T$ which for any given triangulation $T$ can be worked out directly with the help of Lemma 3.19 and Remark 3.20.

Igusa and Todorov [IT2] considered the continuous cluster category of type $A$, which can be combinatorially described by triangulations of $S^1$. By [IT2], Theorem 4.2.1, any cluster of the continuous cluster category can be described through triangulations of the closed disc which are equivalent (that is they are in bijection under an orientation preserving homeomorphism of $S^1$) to what they call the standard cluster given by the triangulation

$$T_{st} = \left\{ e \left( \frac{m\pi}{2^n} \right), e \left( \frac{(m+1)\pi}{2^n} \right) \mid n \geq 0, 0 \leq m < 2^n + 1 \right\}.$$

This is a countable set of arcs, thus it gives rise to a (connected) seed $\Sigma_{T_{st}}$. The rooted cluster algebra $A(\Sigma_{T_{st}})$ is isomorphic to a linear colimit of rooted cluster algebras of finite Dynkin type $A$: We can construct the triangulation $T_{st}$ by starting with a triangulation of the closed disc with two marked points and successively glue on triangles to all the edges, cf. Figure 6. This corresponds to constructing a linear system of rooted cluster algebras $A(\Sigma_{T_n})$ for $n \geq 0$, where $T_n = \left\{ e \left( \frac{m\pi}{2^n} \right), e \left( \frac{(m+1)\pi}{2^n} \right) \mid 0 \leq m < 2^{n+1} \right\}$ and the rooted cluster morphism $f_{mn} : A(T_m) \to A(T_n)$ for $n \geq m$ is defined by the natural embedding.

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Figure 6. We can obtain the triangulation associated to the standard cluster of the continuous cluster category by starting with a finite triangulation and successively gluing on triangles to all edges.

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