Invariant states on the wreath product

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Abstract

Let $\mathcal{S}_\infty$ be the infinity permutation group and $\Gamma$ be a separable topological group. The wreath product $\Gamma \wr \mathcal{S}_\infty$ is the semidirect product $\Gamma_e^\infty \rtimes \mathcal{S}_\infty$ for the usual permutation action of $\mathcal{S}_\infty$ on $\Gamma_e^\infty = \{[\gamma_i]_n^\infty : \gamma_i \in \Gamma, \text{only finitely many } \gamma_i \neq e\}$. In this paper we obtain the full description of indecomposable states $\varphi$ on the group $\Gamma \wr \mathcal{S}_\infty$, satisfying the condition:

$$\varphi (sgs^{-1}) = \varphi (g) \text{ for each } g \in \Gamma \wr \mathcal{S}_\infty, s \in \mathcal{S}_\infty.$$ 

1 Introduction

1.1 The wreath product and $\mathcal{S}_\infty$-central states. Let $\mathbb{N}$ be the set of the natural numbers. By definition, a bijection $s : \mathbb{N} \to \mathbb{N}$ is called finite if the set $\{i \in \mathbb{N} | s(i) \neq i\}$ is finite. Define a group $\mathcal{S}_\infty$ as the group of all finite bijections $\mathbb{N} \to \mathbb{N}$ and set $\mathcal{S}_n = \{s \in \mathcal{S}_\infty | s(i) = i \text{ for each } i > n\}$. Given a group $\Gamma$ identify element $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Gamma^e_n$ with $(\gamma_1, \gamma_2, \ldots, \gamma_n, e) \in \Gamma^{n+1}$, where $e$ is the identity element of $\Gamma$. The group $\Gamma_e^\infty$ is defined as an inductive limit of sets

$$\Gamma \hookrightarrow \Gamma^2 \hookrightarrow \Gamma^3 \hookrightarrow \cdots \hookrightarrow \Gamma^n \hookrightarrow \cdots.$$ 

The wreath product $\Gamma \wr \mathcal{S}_\infty$ is the semidirect product $\Gamma_e^\infty \rtimes \mathcal{S}_\infty$ for the usual permutation action of $\mathcal{S}_\infty$ on $\Gamma_e^\infty$. Using the imbeddings $\gamma \in \Gamma_e^\infty \to (\gamma, \text{id}) \in \Gamma \times \mathcal{S}_\infty$, $s \in \mathcal{S}_\infty \to (e(\infty), s) \in \Gamma \times \mathcal{S}_\infty$, where $e(\infty) = (e, e, \ldots, e, \ldots)$ and $\text{id}$ is the identical bijection, we identify $\Gamma_e^\infty$ and $\mathcal{S}_\infty$ with the corresponding subgroups of $\Gamma \times \mathcal{S}_\infty$. Therefore, each element $g$ of $\Gamma \wr \mathcal{S}_\infty$ is of the form $g = s\gamma$, with $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ and $s \in \mathcal{S}_\infty$. Furthermore, it is assumed that $s(\gamma_1, \gamma_2, \ldots) s^{-1} = (\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots)$.

If $\Gamma$ is a topological group, then we will equip $\Gamma^n$ with the natural product-topology. Furthermore, we will always consider $\Gamma_e^\infty$ as a topological group with the inductive limit topology. The group $\Gamma \wr \mathcal{S}_\infty$ is isomorphic to $\Gamma_e^\infty \times \mathcal{S}_\infty$, as a set. Therefore, we will equip the group $\Gamma \wr \mathcal{S}_\infty$ with the product-topology, considering $\mathcal{S}_\infty$ as a discrete topological space. From now on we assume that $\Gamma$ is a separable topological group.

1.2 The basic definitions. Let $\mathcal{H}$ be a Hilbert space, let $B(\mathcal{H})$ be the set of all bounded operators in $\mathcal{H}$ and let $I_\mathcal{H}$ be the identity operator in $\mathcal{H}$. We
denote by $\mathcal{U}(\mathcal{H})$ the unitary subgroup in $\mathcal{B}(\mathcal{H})$. By a unitary representation of the topological group $G$ we will always mean a continuous homomorphism of $G$ into $\mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ is equipped with the strong operator topology. For unitary representation $\pi$ of the group $G$ we denote $\mathcal{M}_{\pi}$ the $W^*$-algebra $\pi(G)^\prime\prime$, which is generated by the operators $\pi(g)$ $(g \in G)$.

**Definition 1.** An unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of the group $G$ is called a factor-representation if $\mathcal{M}_{\pi}$ is a factor. A positive definite function $\varphi$ on group $G$ is called an indecomposable, if the corresponding GNS-representation is a factor-representation.

Further, an element $\Gamma \in \mathcal{S}_\infty$ can always be written as the product of an element from $\mathcal{S}_\infty$ and an element from $\Gamma^\infty$. The commutation rule between these two kinds of elements is

$$s\gamma = s(\gamma_1, \gamma_2, \ldots) = (\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots) s,$$

where $s \in \mathcal{S}_\infty, \gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^\infty$. Let $\mathbb{N}/s$ be the set of orbits of $s$ on the set $\mathbb{N}$. Note that for $p \in \mathbb{N}/s$ permutation $s_p$, which is defined by the formula

$$s_p(k) = \begin{cases} s(k) & \text{if } k \in p \\ k & \text{otherwise} \end{cases},$$

is a cycle of the order $|p|$, where $|p|$ denotes the cardinality of $p$. For $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^\infty$ we define the element $\gamma(p) = (\gamma_1(p), \gamma_2(p), \ldots) \in \Gamma^\infty$ as follows

$$\gamma_k(p) = \begin{cases} \gamma_k & \text{if } k \in p \\ e & \text{otherwise}. \end{cases}$$

Thus, using (2), we have

$$s\gamma = \prod_{p \in \mathbb{N}/s} s_p \gamma(p).$$

Element $s_p \gamma(p)$ is called the generalized cycle of $s\gamma$.

Denote by $(n, k) \in \mathcal{S}_\infty$ the transposition of numbers $k$ and $n$. Following Olshanski (see [3]) we introduce permutations $\omega_n = \omega_n(0) \in \mathcal{S}_\infty$ by the next formula:

$$\omega_n(i) = \begin{cases} i, & \text{if } 2n < i, \\ i+n, & \text{if } i \leq n, \\ i-n, & \text{if } n < i \leq 2n. \end{cases}$$

For the element $g = s\gamma$ we call support of $g$ the set $\text{supp}(g) = \{i : s(i) \neq i \text{ or } \gamma_i \neq e\}$. Note that $\text{supp}(g)$ is always finite subset of $\mathbb{N}$. If $\text{supp}(g_1) \cap \text{supp}(g_2) = \emptyset$ then elements $g_1$ and $g_2$ commute.

**Definition 2.** Let $G$ be a group and let $H$ be a subgroup of $G$. A positive definite function $\varphi$ on $G$ is called $H$-central if $\varphi(gh) = \varphi(hg)$ for all $h \in H$ and $g \in G$. We say that $\varphi$ is a state on $G$, if $\varphi(e) = 1$, where $e$ is the identical element of $G$. A state $\varphi$ is called indecomposable, if the corresponding GNS-representation $\pi_\varphi$ is a factor representation.
Let $\mathcal{M}$ denote the space of all $\sigma$-weakly continuous functional on $w^*$-algebra $\mathcal{M}$.

Now we fix a $\mathcal{S}_\infty$-central state $\varphi$ on $\Gamma \wr \mathcal{S}_\infty$, and denote by $\pi_\varphi$ the corresponding GNS-representations.

**Theorem 3.** Let $\pi_\varphi (\Gamma \wr \mathcal{S}_\infty)^{''}$ be a $w^*$-algebra generated by operators $\pi_\varphi (\Gamma \wr \mathcal{S}_\infty)$ and let $\mathcal{C} (\pi_\varphi (\Gamma \wr \mathcal{S}_\infty))$ be the center of $\pi_\varphi (\Gamma \wr \mathcal{S}_\infty)^{''}$. Suppose that the positive functionals $\varphi_1$ and $\varphi_2$ from $\pi_\varphi (\Gamma \wr \mathcal{S}_\infty)^{''}$ satisfy the next conditions:

1. $\varphi_k (\pi_\varphi (s) a) = \varphi_k (a \pi_\varphi (s))$ for all $s \in \mathcal{S}_\infty$ and $a \in \pi_\varphi (\Gamma \wr \mathcal{S}_\infty)^{''}$, $(k = 1, 2)$;
2. $\varphi_1 (c) = \varphi_2 (c)$ for all $c \in \mathcal{C} (\pi_\varphi (\Gamma \wr \mathcal{S}_\infty))$.

Then $\varphi_1 (a) = \varphi_2 (a)$ for all $a \in \pi_\varphi (\Gamma \wr \mathcal{S}_\infty)$.

Recall that representations $\pi_1$ and $\pi_2$ of the group $G$ are called quasiequivalent if there exists isomorphism $\theta : \pi_1 (G)^{''} \rightarrow \pi_2 (G)^{''}$ with the property

$$\theta (\pi_1 (g)) = \pi_2 (g) \text{ for all } g \in G. \quad (7)$$

The following corollary is immediate consequence of the above theorem.

**Corollary 4.** If $\varphi_1$ and $\varphi_2$ are indecomposable $\mathcal{S}_\infty$-central states on $\Gamma \wr \mathcal{S}_\infty$ such that the corresponding GNS-representations $\pi_{\varphi_1}$ and $\pi_{\varphi_2}$ are quasiequivalent, then $\varphi_1 = \varphi_2$.

### 1.3 The natural examples.

For any state $\varphi$ on $\Gamma$ define two $\mathcal{S}_\infty$-central states $\varphi_{sp}$ and $\varphi_{reg}$ on $\Gamma \wr \mathcal{S}_\infty$ as follows

$$\varphi_{sp} (s \gamma) = \prod \varphi (\gamma_k) \text{ for all } \gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^{\infty} \text{ and } s \in \mathcal{S}_\infty; \quad (8)$$

$$\varphi_{reg} (s \gamma) = \begin{cases} \prod \varphi (\gamma_k) & \text{if } s = e \\ 0 & \text{if } s \neq e. \end{cases} \quad (9)$$

We have the following result:

**Proposition 5.** For GNS-representations $\pi_{\varphi_{sp}}$ and $\pi_{\varphi_{reg}}$ the next properties hold:

- (i) If $\pi_{\varphi_{sp}}$ acts in Hilbert space $\mathcal{H}_{\varphi_{sp}}$, and $\mathcal{H}_{\varphi_{sp}}^{\mathcal{S}_\infty} = \{ \eta \in \mathcal{H}_{\varphi_{sp}} : \pi_{sp} (s) \eta = \eta \text{ for all } s \in \mathcal{S}_\infty \}$, then $\dim \mathcal{H}_{\varphi_{sp}}^{\mathcal{S}_\infty} = 1$. In particular, $\pi_{\varphi_{sp}}$ is irreducible.

- (ii) $\pi_{\varphi_{reg}}$ is a factor representation.

- (iii) $w^*$-algebra $\pi_{\varphi_{reg}} (\Gamma \wr \mathcal{S}_\infty)^{''}$ is a factor of the type II or III.

**Proof.** Let $\xi_{\varphi_{sp}}$ ($\xi_{\varphi_{reg}}$) be the cyclic vector for representation $\pi_{sp}$ ($\pi_{reg}$) with the property

$$\varphi_{sp} (g) = (\pi_{sp} (g) \xi_{\varphi_{sp}}, \xi_{\varphi_{sp}}) \quad (\varphi_{reg} (g) = (\pi_{reg} (g) \xi_{\varphi_{reg}}, \xi_{\varphi_{reg}}))$$

for all $g \in \Gamma \wr \mathcal{S}_\infty$. 

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Set $\Gamma_e^\infty = \{\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty \mid \gamma_k = e \text{ for all } k \leq n\}$, $\mathcal{S}_n^\infty = \{s \in \mathcal{S}_\infty \mid s(k) = k \text{ for all } k \leq n\}$. Denote by $\Gamma \wr \mathcal{S}_n^\infty$ the subgroup of $\Gamma \wr \mathcal{S}_\infty$ generated by $\Gamma_e^\infty$ and $\mathcal{S}_n^\infty$.

To the proof point (i), first we note that, by definition GNS-construction, $\xi_{\varphi_{reg}}$ lies in $\mathcal{H}_{\varphi_{reg}}^\infty$. Further we will use the important mixing-property. Namely, denote by $\omega_n$ a bijection which acts as follows

$$
\omega_n(i) = \begin{cases} 
  i, & \text{if } 2n < i, \\
  i + n, & \text{if } i \leq n, \\
  i - n, & \text{if } n < i \leq 2n.
\end{cases}
$$

Then for any $\eta \in \mathcal{H}_{\varphi_{reg}}^\infty$, using (\ref{eq:omega_n}), we obtain

$$
\lim_{n \to \infty} \left(\pi_{sp}(\omega_n) \eta, \eta\right) = \left(\xi_{\varphi_{reg}}, \eta\right) \left(\eta, \xi_{\varphi_{reg}}\right).
$$

This implies (i).

A property (ii) follows from Proposition \ref{prop:property_ii} (below). Nevertheless, using the explicit realizations of $\pi_{\varphi_{reg}}$, we give another proof. We begin with the GNS-representation $T$ of $\Gamma$ which acts in Hilbert space $\mathcal{H}_T$ with cyclic vector $\xi_\varphi$: $\varphi(\gamma) = (T(\gamma)\xi_\varphi, \xi_\varphi)$ for all $\gamma \in \varphi$. Further, using embedding $\mathcal{H}_T^\infty \ni \eta \mapsto \eta \otimes \xi_\varphi \in \mathcal{H}_T^{\infty+1}$, define Hilbert space $\mathcal{H}_T^\infty$ and corresponding representation $T^\otimes \infty$ of $\Gamma_e$:

$$
T^\otimes \infty(\gamma) (\xi_1 \otimes \xi_2 \otimes \ldots) = T(\gamma_1) \xi_1 \otimes T(\gamma_2) \xi_2 \otimes \ldots, \text{ where } \gamma = (\gamma_1, \gamma_2, \ldots).
$$

The action $U$ of $\mathcal{S}_\infty$ on $\mathcal{H}_T^\infty$ is given by the formula

$$
U(s)(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_k \otimes \ldots) = \xi_{s^{-1}(1)} \otimes \xi_{s^{-1}(2)} \otimes \ldots \otimes \xi_{s^{-1}(k)} \otimes \ldots
$$

Now we define operator $\Pi(g)$ ($g \in \Gamma \wr \mathcal{S}_\infty$) in $l^2 (\mathcal{S}_\infty, \mathcal{H}_T^\infty)$ as follows

$$
(\Pi(\gamma) \eta)(s) = U(s) T^\otimes \infty(\gamma) U^*(s) \eta(s) \quad (\gamma \in \Gamma_e^\infty, \eta \in l^2 (\mathcal{S}_\infty, \mathcal{H}_T^\infty)),
$$

$$
(\Pi(t) \eta)(s) = \eta(st) \quad (t \in \mathcal{S}_\infty).
$$

Since for any $s \in \mathcal{S}_\infty$ and $g = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e^\infty$ and $s^{-1} = (\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots)$, $\Pi$ extends by multiplicativity to the representation of $\Gamma \wr \mathcal{S}_\infty$.

If $\xi_\varphi^\otimes = \xi_\varphi \otimes \xi_\varphi \otimes \ldots \in \mathcal{H}_T^\infty$ and $\hat{\xi}_\varphi(g) = \left\{ \begin{array}{ll} 
\xi_\varphi^\otimes, & \text{if } g = e, \\
0, & \text{if } g \neq e
\end{array} \right.$ then we have

$$
\varphi_{reg}(s) = \left(\Pi(s) \xi_\varphi, \xi_\varphi\right) (s \in \mathcal{S}_\infty, \gamma \in \Gamma_e^\infty).
$$

Therefore, without loss generality we can assume that $\pi_{\varphi_{reg}} = \Pi$.

Let $\Pi'$ denote the representation of $\mathcal{S}_\infty$ which acts on $l^2 (\mathcal{S}_\infty, \mathcal{H}_T^\infty)$ by

$$
(\Pi'(t) \eta)(s) = U(t) \eta(t^{-1}s).
$$
Obvious, $\Pi'(\mathcal{S})$ is contained in commutant $\Pi(\Gamma \wr \mathcal{S})'$ of $\Pi(\Gamma \wr \mathcal{S})$.

Let us prove that center $\mathcal{C} = \Pi(\Gamma \wr \mathcal{S})'' \cap \Pi(\Gamma \wr \mathcal{S})'$ of $\Pi(\Gamma \wr \mathcal{S})''$ is trivial.

Our proof starts with the observation that

$$\Pi(g)\Pi'(g)\hat{\xi}_\varphi = \hat{\xi}_\varphi \text{ for all } g \in \mathcal{S}. \tag{14}$$

Hence for $c \in \mathcal{C}$ we have

$$\Pi(g)\Pi'(g)c\hat{\xi}_\varphi = c\hat{\xi}_\varphi \text{ for all } g \in \mathcal{S}. \tag{15}$$

In particular, this gives

$$\|c\hat{\xi}_\varphi(s)\| = \|c\hat{\xi}_\varphi(gs^{-1})\| \text{ for all } g, s \in \mathcal{S}. \tag{16}$$

Since every conjugacy class $C(s) = \{gs^{-1} : g \in \mathcal{S}\}$ is infinite except $s = e$, we have

$$c\hat{\xi}_\varphi(s) = 0 \text{ for all } s \neq e. \tag{17}$$

It follows from (15) that

$$U(s)\left(c\hat{\xi}_\varphi(e)\right) = c\hat{\xi}_\varphi(e) \text{ for all } s \in \mathcal{S}. \tag{18}$$

As in the proof of the point (i), this gives that $c\hat{\xi}_\varphi(e) = \alpha\xi_\varphi^\otimes(e)$ ($\alpha \in \mathbb{C}$). Since $\hat{\xi}_\varphi$ is cyclic, we have $c = \alpha I$. Therefore, $w^*$-algebra $\Pi(\Gamma \wr \mathcal{S})''$ is a factor.

(iii) We begin by recalling the notion of a central sequence in a factor $\mathcal{M}$. A bounded sequence $\{a_n\} \subset \mathcal{M}$ is called central if

$$s - \lim_{n \to \infty} (a_n m - ma_n) = 0 \text{ and } s - \lim_{n \to \infty} (a_n^* m - ma_n^*) = 0 \text{ for all } m \in \mathcal{M}.$$ 

A central sequence is called trivial if there exists sequence $\{c_n\} \subset \mathbb{C}$ such that

$$s - \lim_{n \to \infty} (a_n - c_n I) = 0 \text{ and } s - \lim_{n \to \infty} (a_n^* - c_n I) = 0.$$ 

Let $s_k$ be the transposition interchanging $k$ and $k+1$. We claim that $\{\pi_{\text{reg}}(s_n)\}$ is non trivial central sequence. Indeed, since $\varphi_{\text{reg}}$ is a $\mathcal{S}$-central state, we have

$$\lim_{n \to \infty} (m\pi_{\text{reg}}(s_n) - \pi_{\text{reg}}(s_n) m)\xi_{\varphi_{\text{reg}}} = 0 \text{ for all } m \in \Pi(\Gamma \wr \mathcal{S})''.$$ 

It follows that

$$\lim_{n \to \infty} (m\pi_{\text{reg}}(s_n) - \pi_{\text{reg}}(s_n) m) x\xi_{\varphi_{\text{reg}}} = 0 \text{ for all } m, x \in \Pi(\Gamma \wr \mathcal{S})''.$$ 

Since $\xi_{\varphi_{\text{reg}}}$ is cyclic and $\varphi_{\text{reg}}(s_n) = 0$, then $\{\pi_{\text{reg}}(s_n)\}$ is non trivial central sequence.
It remains to prove that each central sequence in factor $\mathcal{M}$ of type I is trivial. Suppose that $\mathcal{M}$ is a factor of type I. Let $\{\epsilon_{kl} : k, l \in \mathbb{N}\}$ be a matrix unit in $\mathcal{M}$. This means that the next relations hold

$$\epsilon_{kl}^* = \epsilon_{lk}, \quad \epsilon_{kl} \epsilon_{pq} = \delta_{lp} \epsilon_{kq}, \quad \sum_{k \in \mathbb{N}} \epsilon_{kk} = I. \quad (19)$$

Let $\left\{a_n = \sum_{k,l} c_{kl}(n) \epsilon_{kl} : c_{kl}(n) \in \mathbb{C}\right\}$ be a central sequence in $\mathcal{M}$. Set $c_{pq}(n) = a_n \epsilon_{pq} - \epsilon_{pq} a_n$. An easy computation shows that

$$e_{qq} (c_{pq}(n))^* c_{pq}(n) e_{qq} = \left[|c_{pp}(n) - c_{qq}(n)|^2 - |c_{pp}(n)|^2 + \sum_k |c_{kp}(n)|^2\right] e_{qq},$$

$$e_{pp} c_{pq}(n) (c_{pq}(n))^* e_{pp} = \left[|c_{pp}(n) - c_{qq}(n)|^2 - |c_{qq}(n)|^2 + \sum_k |c_{qk}(n)|^2\right] e_{pp}.$$

Using the fact that $\{a_n\}$ is a central sequence, we deduce from this that

$$\lim_{n \to \infty} \sum_{k : k \neq q} |c_{qk}(n)|^2 = 0, \quad \lim_{n \to \infty} \sum_{k : k \neq q} |c_{kq}(n)|^2 = 0,$$

$$\lim_{n \to \infty} |c_{11}(n) - c_{qq}(n)|^2 = 0 \text{ for all } q.$$

This means that $s - \lim_{n \to \infty} (a_n - c_{11}(n)I) = 0$ and $s - \lim_{n \to \infty} \left(a_n^* - c_{11}(n)I\right) = 0$. Thus $\{a_n\}$ is trivial.

The goal of this paper is to give the full description of indecomposable $S_\infty$-central states on $\Gamma \wr S_\infty$ (see definition 2). The character theory of infinite wreath product in the case of finite $\Gamma$ is developed by R. Boyer [6]. In this case $\Gamma \wr S_\infty$ is inductive limit of finite groups, their finite characters can be obtained as limits of normalized characters of prelimit finite groups, and Boyer’s method is a direct generalization of Vershik’s-Kerov’s asymptotic approach [4]. The characters of $\Gamma \wr S_\infty$ for general separable group $\Gamma$ were found by authors in [9], [10]. Our method has been based on the ideas of Okounkov, which he has developed for the proof of Thoma’s theorem [13], [7], [8].

A finite character is a $\Gamma \wr S_\infty$-central positive definite function on $\Gamma \wr S_\infty$. In this paper we study the more general class of the $S_\infty$-central states on $\Gamma \wr S_\infty$. Our results provide a complete classification such indecomposable states. The set of all indecomposable $S_\infty$-central states have very important property. Namely, if for for two indecomposable $S_\infty$-central states $\varphi_1$ and $\varphi_2$ the corresponding GNS-representations $\pi_{\varphi_1}$ and $\pi_{\varphi_2}$ are quasiequivalent, then $\varphi_1 = \varphi_2$ (theorem [3] corollary [4]).

The papers is organized as follows. Below we give a brief description of the general properties of the $S_\infty$-central states. The key results are lemma [6] and proposition [7]. Here we also recall the classification of the traces (central
states) on $\Gamma \wr \mathcal{S}_\infty$ (theorem 9). In section 2 we present the full collection of factor-representations, which define the $\mathcal{S}_\infty$-central states (proposition 10). Each such state is parametrized by pair $(A, \rho)$, where $A$ is self-adjoint operator, $\rho$ is the unitary representation of $\Gamma$ (paragraph 2.1). In proposition 11 we prove that the unitary equivalence of pairs $(A_1, \rho_1)$ and $(A_2, \rho_2)$ is equivalent to the equality of the corresponding $\mathcal{S}_\infty$-central states. In section 3 we discuss about physical KMS-condition (see [15]) for these states (theorem 15). In section 4 we prove the classification theorem 18.

1.4 The multiplicativity. Let $\varphi$ be an indecomposable $\mathcal{S}_\infty$-central state on the group $\Gamma \wr \mathcal{S}_\infty$. Then it defines according to GNS-construction a factor-representation $\pi_\varphi$ of the group $\Gamma \wr \mathcal{S}_\infty$ with cyclic vector $\xi_\varphi$ such that $\pi_\varphi(g) = (\pi_\varphi(g) \xi_\varphi, \xi_\varphi)$ for each $g \in \Gamma \wr \mathcal{S}_\infty$. The next lemma shows, that different indecomposable $\mathcal{S}_\infty$-central states define representations which are not quasi-equivalent. Let $w - \lim$ stand for the limit in the weak operator topology.

**Lemma 6.** Let $\varphi$ be an indecomposable $\mathcal{S}_\infty$-central state on the group $\Gamma \wr \mathcal{S}_\infty$. Then for each $g \in \Gamma \wr \mathcal{S}_\infty$ there exists $w - \lim_{n \to \infty} \pi_\varphi(\omega_n g \omega_n)$ and the next equality holds:

$$w - \lim_{n \to \infty} \pi_\varphi(\omega_n g \omega_n) = \varphi(g)I. \quad (20)$$

**Proof.** Let $h_1, h_2 \in \Gamma \wr \mathcal{S}_\infty$. Fix $k$ such that

$$\text{supp}(h_1), \text{supp}(h_2), \text{supp}(g) \subset \{1, 2, \ldots, k\}. \quad (21)$$

For each $n \in \mathbb{N}$ there exists elements $g(n,k), h(n,k) \in \mathcal{S}_\infty$ such that

$$\text{supp}(g(n,k)), \text{supp}(h(n,k)) \subset \{k + 1, k + 2, \ldots\} \quad (22)$$

and $\omega_{n+k} = g(n,k) \omega_k h(n,k)$ (see (9)). Permutations $g(n,k), h(n,k)$ can be defined as follows:

$$g(n,k)(i) = \begin{cases} 
  i, & \text{if } i \leq k \text{ or } 2k + 2n < i, \\
  i + n, & \text{if } k < i \leq 2k + n, \\
  i - k - n, & \text{if } 2k + n < i \leq 2k + 2n.
\end{cases}$$

$$h(n,k)(i) = \begin{cases} 
  i, & \text{if } i \leq k \text{ or } 2k + n < i, \\
  i + k, & \text{if } k < i \leq k + n, \\
  i - n, & \text{if } k + n < i \leq 2k + n.
\end{cases}$$

By (21) and (22), the elements $g(n,k)$ and $h(n,k)$ commutes with the elements $h_1, h_2$ and $g$. Therefore

$$h_2^{-1} \omega_{n+k} g \omega_{n+k} h_1 = h_2^{-1} (g(n,k) \omega_k h(n,k))^{-1} gg(n,k) \omega_k h(n,k) h_1$$

$$= h_1^{-1} h_2^{-1} \omega_k g \omega_k h_1 h(n,k). \quad (23)$$
As $\varphi$ is $\mathcal{G}_\infty$-central, one has:

\[
(\pi_{\varphi}(\omega_n g \omega_n) \pi_{\varphi}(h_1) \xi_{\varphi}, \pi_{\varphi}(h_2) \xi_{\varphi}) = \varphi\left(h_2^{-1} \omega_n g \omega_n h_1\right) = \\
\varphi\left(h_2^{-1} \omega_k g \omega_k h_1\right) = (\pi_{\varphi}(\omega_k g \omega_k) \pi_{\varphi}(h_1) \xi_{\varphi}, \pi_{\varphi}(h_2) \xi_{\varphi}).
\]

(24)

As $\xi_{\varphi}$ is cyclic, by (24), there exists the limit

\[
w - \lim_{n \to \infty} \pi_{\varphi}(\omega_n g \omega_n).
\]

For each $h \in \Gamma \wr \mathcal{S}_\infty$ for large enough $n$ one has $\text{supp}(\omega_n g \omega_n) \cap \text{supp}(h) = \emptyset$. Therefore $\pi_{\varphi}(\omega_n g \omega_n) \pi_{\varphi}(h) = \pi_{\varphi}(h) \pi_{\varphi}(\omega_n g \omega_n)$. This involves that the weak limit $w - \lim_{n \to \infty} \pi_{\varphi}(\omega_n g \omega_n)$ lies in the center of the algebra $M_{\pi_{\varphi}}$, generated by operators of the representation $\pi_{\varphi}$. Thus $\lim_{n \to \infty} \pi_{\varphi}(\omega_n g \omega_n)$ is scalar. By $\mathcal{G}_\infty$-centrality of $\varphi$,

\[
w - \lim_{n \to \infty} \pi_{\varphi}(\omega_n g \omega_n) \xi_{\varphi}, \xi_{\varphi} = \lim_{n \to \infty} \varphi(\omega_n g \omega_n) = \varphi(g),
\]

which finishes the proof. \hfill \square

The following claim gives a useful characterization of the class of the indecomposable $\mathcal{G}_\infty$-central states:

**Proposition 7.** The following conditions for $\mathcal{G}_\infty$-central state $\varphi$ on the group $\Gamma \wr \mathcal{S}_\infty$ are equivalent:

(a) $\varphi$ is indecomposable;

(b) $\varphi(gg') = \varphi(g)\varphi(g')$ for each $g, g' \in \Gamma \wr \mathcal{S}_\infty$ with $\text{supp}(g) \cap \text{supp}(g') = \emptyset$;

(c) $\varphi(g) = \prod_{p \in \mathbb{N}/s} \varphi(s_p \gamma(p))$ for each $g = s\gamma \prod_{p \in \mathbb{N}/s} s_p \gamma(p)$ (see [3]).

**Proof.** The equivalence of (b) and (c) is obvious. We prove the equivalence of (a) and (b). Using GNS-construction, we build the representation $\pi_{\varphi}$ of the group $\Gamma \wr \mathcal{S}_\infty$ which acts in the Hilbert space $\mathcal{H}_{\varphi}$ with cyclic vector $\xi_{\varphi}$ such that

$\varphi(g) = (\pi_{\varphi}(g) \xi_{\varphi}, \xi_{\varphi})$ for each $g \in \Gamma \wr \mathcal{S}_\infty$.

Suppose that the property (a) holds. Consider two elements $g = s\gamma$ and $g' = s'\gamma'$ from $\Gamma \wr \mathcal{S}_\infty$ satisfying $\text{supp}(g) \cap \text{supp}(g') = \emptyset$. Then there exists a sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_\infty$ such that for each $n$

\[
\text{supp}(s_n) \cap \text{supp}(g) = \emptyset \quad \text{and} \quad \text{supp}(s_n g' s_n^{-1}) \subset \{n+1, n+2, \ldots\}. \quad (25)
\]

For example we can put $s_n = \prod_{i \in \text{supp}(g')} (i, i + k + n)$, where $k$ is fixed number such that $\text{supp}(g) \cup \text{supp}(g') \subset \{1, 2, \ldots, k\}$. Using the ideas of the proof of the
lemma \[6\] we obtain, that the limit \( \lim_{n \to \infty} \pi_{\varphi}(s_{n}g's_{n}) \) exists in the weak operator topology and the next equality holds:

\[
\varphi(g') = \lim_{n \to \infty} \varphi(g_{n}g's_{n}^{-1}) = \lim_{n \to \infty} (\pi_{\varphi}(g)\pi_{\varphi}(s_{n}g's_{n}^{-1}) \xi_{\varphi}, \xi_{\varphi}) = \varphi(g)\varphi(g').
\]

Thus (b) follows from (a).

Further suppose that the condition (b) holds. If \( \pi_{\varphi}(\Gamma \cap \mathcal{S}_{\infty})' \cap \pi_{\varphi}(\Gamma \cap \mathcal{S}_{\infty})'' = Z \) is larger than the scalars, then it contains a pair of orthogonal projections \( E \) and \( F \) satisfying the condition:

\[
EF = 0. \tag{27}
\]

Fix arbitrary \( \varepsilon > 0 \). By the von Neumann Double Commutant Theorem there exist \( g_{k},h_{k} \in \Gamma \cap \mathcal{S}_{\infty} \) and complex numbers \( c_{k},d_{k} \) \((k = 1,2,\ldots,N < \infty)\) such that

\[
\left| \sum_{k=1}^{N} c_{k}\pi_{\varphi}(g_{k})\xi_{\varphi} - E\xi_{\varphi} \right| < \varepsilon, \tag{28}
\]

\[
\left| \sum_{k=1}^{N} d_{k}\pi_{\varphi}(h_{k})\xi_{\varphi} - F\xi_{\varphi} \right| < \varepsilon.
\]

Fix \( n \) such that \( \text{supp}(g_{k}) \subset \{1,2,\ldots,n\} \) and \( \text{supp}(h_{k}) \subset \{1,2,\ldots,n\} \) for each \( k \). As \( \varphi \) is \( \mathcal{S}_{\infty} \)-central, using \[25\], we obtain

\[
\left| \sum_{k=1}^{N} c_{k}\pi_{\varphi}(\omega_{n}g_{k}\omega_{n})\xi_{\varphi} - E\xi_{\varphi} \right| < \varepsilon, \tag{29}
\]

Now, using \[27\], \[28\] and \[29\], we have

\[
\left| \left( \sum_{k=1}^{N} c_{k}\pi_{\varphi}(\omega_{n}g_{k}\omega_{n}) \right) \left( \sum_{k=1}^{N} d_{k}\pi_{\varphi}(h_{k})\xi_{\varphi}, \xi_{\varphi} \right) \right| < 2\varepsilon + \varepsilon^{2}. \tag{30}
\]

Note, that \( \text{supp}(\omega_{n}g_{k}\omega_{n}) \subset \{n+1,n+2,\ldots\} \) for each \( k \). Therefore, by the property (b), \[28\] and \[29\], one has:

\[
\left| \left( \sum_{k=1}^{N} c_{k}\pi_{\varphi}(\omega_{n}g_{k}\omega_{n}) \right) \left( \sum_{k=1}^{N} d_{k}\pi_{\varphi}(h_{k})\xi_{\varphi}, \xi_{\varphi} \right) \right| = \left( E\xi_{\varphi}, \xi_{\varphi} \right) (F\xi_{\varphi}, \xi_{\varphi}) - \varepsilon ((E\xi_{\varphi}, \xi_{\varphi}) + (F\xi_{\varphi}, \xi_{\varphi})) - \varepsilon^{2}. \tag{31}
\]
Note that, as $\xi_\varphi$ is cyclic, $E\xi_\varphi \neq 0$ and $F\xi_\varphi \neq 0$. Therefore, taking in view (30) and (31), we arrive at a contradiction.

Denote the element $\sigma_n \in \mathcal{S}_\infty$ by the formula:

$$
\sigma_n(i) = \begin{cases} 
  i+1 & \text{if } i < n, \\
  1 & \text{if } i = n, \\
  i & \text{if } i > n.
\end{cases}
$$

(32)

**Corollary 8.** Each indecomposable $\mathcal{S}_\infty$-central state $\varphi$ on the group $\Gamma \wr \mathcal{S}_\infty$ is defined by its values on the elements of the form $\sigma_n \gamma$, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, e, e, \ldots)$ and $n \in \mathbb{N}$.

**Proof.** By the proposition 7, $\varphi$ is defined by its values on the elements of the view $s_p \gamma(p)$ (see (5)). Fix an element $s_p \gamma(p)$. Let $n = |p|$. Then there exists a permutation $h \in \mathcal{S}_\infty$ such that $hs_p h^{-1} = \sigma_n$. Therefore $\varphi(s_p \gamma(p)) = \varphi(hs_p \gamma(p)h^{-1}) = \varphi(\sigma_n h \gamma(p)h^{-1})$, which proves the corollary.

1.5 **The characters of the group $\mathcal{S}_\infty$ and $\Gamma \wr \mathcal{S}_\infty$.** In the paper [13], E.Thoma obtained the following remarkable description of all indecomposable character ($\mathcal{S}_\infty$-central states) of the group $\mathcal{S}_\infty$. Characters of the group $\mathcal{S}_\infty$ are labeled by a pair of non-increasing positive sequences of numbers $\{\alpha_k\}, \{\beta_k\}$ ($k \in \mathbb{N}$), such that

$$
\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k \leq 1.
$$

(33)

The value of the corresponding character on a cycle of length $l$ is

$$
\sum_{k=1}^{\infty} \alpha_k l + (-1)^{l-1} \sum_{k=1}^{\infty} \beta_k l.
$$

Its value on a product of several disjoint cycles equals to the product of values on each of cycles.

In [9] authors described all indecomposable characters on the group $\Gamma \wr \mathcal{S}_\infty$. Before to formulate the main result of [9] we introduce some more notations.

We call an element $g = s \gamma$ a generated cycle if either $s$ is a cycle and $\text{supp}(\gamma) \subset \text{supp}(s)$ or $s = e$ and $\text{supp}(\gamma) = \{n\}$ for some $n$. For an element $g = s \gamma$ and an orbit $p \in \mathbb{N}/s$ choose the minimal number $k \in p$ and denote

$$
\tilde{\gamma}(p) = \gamma_k \gamma_s(-1)(k) \cdots \gamma_s(-1)(k) \cdots \gamma_s(-|p|+1)(k).
$$

(34)

For a factor-representation $\tau$ of the finite type let $\chi_{\tau}$ be its normalized character. That is $\chi_{\tau}(g) = tr_M(\tau(g))$, where $tr_M$ stands for the unique normal, normalized ($tr_M(I) = 1$) trace on the factor $M$ of the finite type. Note that $\chi_{\tau}(e) = 1$. Let $tr$ be the ordinary matrix normalized trace.
**Theorem 9** ([9], [10]). Let \( \varphi \) be a function on the group \( \Gamma \) \( \in \mathfrak{S}_\infty \). Then the following conditions are equivalent.

a) \( \varphi \) is an indecomposable character.

b) There exist a representation \( \tau \) of the finite type of the group \( \Gamma \), two non-increasing positive sequences of numbers \( \{\alpha_k\}, \{\beta_k\} \quad (k \in \mathbb{N}) \) and two sequences \( \{\rho_k\}, \{\hat{\rho}_k\} \) of finite-dimensional irreducible representations of \( \Gamma \) with properties

- (i) \( \delta = 1 - \sum_k \alpha_k \dim \rho_k - \sum_k \beta_k \dim \hat{\rho}_k \geq 0 \);

- (ii) if \( s \) is cycle, \( g = s\gamma \quad (\gamma \in \Gamma^\infty) \), \( p = \mathrm{supp} = \mathrm{supp}(s\gamma) \), then

\[
\varphi(g) = \begin{cases} 
\sum_k \alpha_k \tr(\rho_k(\gamma_n)) + \sum_k \beta_k \tr(\hat{\rho}_k(\gamma_n)) + \delta \chi_\tau(\gamma_n), & \text{if } p = \{n\}, \\
\sum_k \alpha_k |p| \tr(\rho_k(\gamma(p))) + (-1)^{|p| - 1} \sum_k \beta_k |p| \tr(\hat{\rho}_k(\gamma(p))), & \text{if } |p| > 1;
\end{cases}
\]

- (iii) if \( g = s\gamma = \prod_{p \in \mathbb{N} / s} s_p \gamma(p) \) (see [9], then \( \varphi(g) = \prod_{p \in \mathbb{N} / s} \varphi(s_p \gamma(p)) \).

2 Examples of representations.

2.1 Parameters of states. Let \( A \) be a self-adjoint operator of the trace class (see [12]) from \( \mathcal{B(H)} \) with the property:

\( \mathrm{Tr}(|A|) \leq 1 \), where \( \mathrm{Tr} \) is ordinary trace\(^1\) on \( \mathcal{B(H)} \).

Further we fix vector \( \hat{\xi} \in \mathrm{Ker} \) and the unitary representation \( \rho \) of \( \Gamma \) in \( \mathcal{H} \), which satisfies the conditions:

- (1) if \( \mathrm{Tr}(|A|) = 1 \), then subspace \( \mathrm{Ker} \) is cyclic for \( w^* \)-algebra \( \mathfrak{A} \) generated by \( A \) and \( \rho(\Gamma) \);

- (2) if \( \mathrm{Tr}(|A|) < 1 \), subspace \( \mathcal{H} \) is generated by \( \{\mathfrak{A}v, v \in (\mathrm{Ker} A)^\perp\} \) and \( \mathcal{H}_{\mathrm{reg}} = \mathcal{H} \oplus \mathcal{H} \), then \( \dim \mathcal{H}_{\mathrm{reg}} = \infty \);

- (3) if \( P_{[0,1]} \) and \( P_{[1,0]} \) are the spectral projections of \( A \), then subspaces \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) generated by vectors \( \{\mathfrak{A}v, v \in P_{[0,1]} \mathcal{H}\} \) and \( \{\mathfrak{A}v, v \in P_{[1,0]} \mathcal{H}\} \), respectively, are orthogonal;

- (4) there exist \( \mathfrak{I} \)-factor \( N_{\mathrm{reg}} \subset \left( \rho(\Gamma) \bigg|_{\mathcal{H}_{\mathrm{reg}}} \right)' \) with matrix unit \( \{\hat{e}_{kl}, \quad k, l \in \mathbb{N}\} \) such that \( \hat{\xi} \in \mathfrak{e}_{11} \mathcal{H}_{\mathrm{reg}}, \|\hat{\xi}\| = 1 \) and \( \mathfrak{e}_{11} \mathcal{H}_{\mathrm{reg}} \) is generated by \( \left\{ \rho(\Gamma) \hat{\xi} \right\} \). In particular, if \( \mathrm{Tr}(|A|) = 1 \) then \( \hat{\xi} = 0 \). When \( \mathrm{Tr}(|A|) < 1 \) we assume for convenience that \( \|\hat{\xi}\| = 1 \).

---

\(^1\)If \( p \) is the minimal projection from \( \mathcal{B(H)} \), then \( \mathrm{Tr}(p) = 1 \).
2.2 Hilbert space $\mathcal{H}_A^\rho$. Define a state $\psi_k$ on $\mathcal{B}(\mathcal{H})$ as follows

$$\psi_k(v) = \text{Tr}(v|A\rangle\langle A|) + (1 - \text{Tr}(|A\rangle\langle A|))(v\xi_k^i, \xi_k^i), \quad v \in \mathcal{B}(\mathcal{H}).$$  \hfill (35)

Let $1\psi_k$ denote the product-state on $\mathcal{B}(H)^\otimes k$:

$$1\psi_k(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = \prod_{j=1}^{k} \psi_j(v_j).$$  \hfill (36)

Now define inner product on $\mathcal{B}(H)^\otimes k$ by

$$(v, u)_k = 1\psi_k(u^* v).$$  \hfill (37)

Let $\mathcal{H}_k$ denote the Hilbert space obtained by completing $\mathcal{B}(H)^\otimes k$ in above inner product norm. Now we consider the natural isometrical embedding

$$v \ni \mathcal{H}_k \mapsto v \otimes I \in \mathcal{H}_k+1,$$  \hfill (38)

and define Hilbert space $\mathcal{H}_A^\rho$ as completing $\bigcup_{k=1}^{\infty} \mathcal{H}_k$.

2.3 The action $\Gamma \wr \mathfrak{S}_\infty$ on $\mathcal{H}_A^\rho$. First, using the embedding $a \in \mathcal{B}(\mathcal{H})^\otimes k \mapsto a \otimes I \in \mathcal{B}(\mathcal{H})^\otimes k+1$, we identify $\mathcal{B}(\mathcal{H})^\otimes k$ with subalgebra $\mathcal{B}(\mathcal{H})^\otimes k \otimes \mathbb{C} \subset \mathcal{B}(\mathcal{H})^\otimes k+1$. Therefore, algebra $\mathcal{B}(\mathcal{H})^\otimes \infty = \bigcup_{n=1}^{\infty} \mathcal{B}(\mathcal{H})^\otimes n$ is well defined.

Further we give the explicit embedding $\mathfrak{S}_\infty$ into unitary group of $\mathcal{B}(\mathcal{H})^\otimes \infty$. First fix the matrix unit $\{e_{pq}: p, q = 1, 2, \ldots, n = \dim \mathcal{H}\} \subset \mathcal{B}(\mathcal{H})$ with the properties:

- (i) projection $e_{kk}$ is minimal and $e_{kk}A = e_{kk}e_{kk}$ ($e_{kk} \in \mathbb{C}$) for all $k = 1, 2, \ldots, n$;
- (ii) $e_{kk}\mathcal{H}_+ \subset \mathcal{H}_+$ and $e_{kk}\mathcal{H}_- \subset \mathcal{H}_-$ for all $k = 1, 2, \ldots, n$.

Put $X = \{1, 2, \ldots, n\}^\times \infty$. For $x = (x_1, x_2, \ldots, x_l, \ldots) \in X$ we set $\iota_A(x) = |\{i : e_{x_i, x_i} \in \mathcal{H}_-\}|$. Define subsequence $x_A = (x_{i_1}, x_{i_2}, \ldots) \in \{1, 2, \ldots, n\}^{\iota_A(x)}$ by induction

$$i_1 = \min \{i : e_{x_i, x_i} \in \mathcal{H}_-\} \quad \text{and} \quad i_k = \min \{i > i_{k-1} : e_{x_i, x_i} \in \mathcal{H}_-\}. \hfill (39)$$

For $s \in \mathfrak{S}_\infty$ denote by $c(x, s)$ the unique permutation from $\mathfrak{S}_{\iota_A(x)} \subset \mathfrak{S}_\infty$ such that

$$s^{-1}(i_{c(x, s)(1)}) < s^{-1}(i_{c(x, s)(2)}) < \ldots \quad \text{and} \quad s^{-1}(i_{c(x, s)(l)}) < \ldots \hfill (40)$$

Let $\mathfrak{S}_\infty$ acts on $X$ as follows

$$X \times \mathfrak{S}_\infty \ni (x, s) \mapsto sx = (x_{s(1)}, x_{s(2)}, \ldots) \in X. \hfill (41)$$
By definition, \((sx)_A = (x_{\iota(sx)(1)}, x_{\iota(sx)(2)}, \ldots, x_{\iota(sx)(1)}, \ldots)\). Therefore,
\[
c(x, ts) = c(sx, t)c(x, s) \quad \text{for all} \quad t, s \in \Sigma; x \in X. \quad (42)
\]
Given any \(s \in \Sigma\), put
\[
U_N(s) = \sum_{x_1, x_2, \ldots, x_N = 1}^n \text{sign} \ (c(x, s)) \ e_{x_1} \otimes e_{x_2} \otimes \cdots \otimes e_{x_N} \in H^N_x,
\]
where \(N < \infty\) satisfies the condition: \(s(i) = i\) for all \(i \geq N\), \(x = (x_1, x_2, \ldots, x_N, \ldots)\). We see at once that for \(L > N\)
\[
U_N(s) \otimes I \otimes \cdots \otimes I = U_L(s).
\]
Thus operator \(U(s) = U_N(s) \otimes I \otimes \cdots \in B(H)^{\otimes \infty} = \bigcup_{n=1}^{\infty} B(H)^{\otimes n}\) is well defined. It follows from (42) that
\[
U(t)U(s) = U(ts) \quad \text{for all} \quad t, s \in \Sigma. \quad (43)
\]
It is clear that
\[
\text{sign} \ (c(x, s)c(y, s)) (U(s) (e_{x_1} \otimes e_{x_2} \otimes \cdots \otimes e_{x_N} \otimes I \otimes I \otimes \cdots) U(s)^*)
\]
\[
= e_{x_{s-1}(1)} \otimes e_{x_{s-2}(2)} \otimes \cdots \otimes e_{x_{s-N}(N)} \otimes I \otimes I \otimes \cdots
\]
If \(x, y\) satisfies the condition:
\(e_{x, y} \in H \subset H_+\) if and only if, when \(e_{y_1, y_2} \in H \subset H_+\),
then, by definition cocycle \(c\), we have \(c(x, s) = c(y, s)\).
Therefore,
\[
U(s) (e_{x_1} \otimes e_{x_2} \otimes \cdots \otimes e_{x_N} \otimes I \otimes I \otimes \cdots) U(s)^*
\]
\[
= e_{x_{s-1}(1)} \otimes e_{x_{s-2}(2)} \otimes \cdots \otimes e_{x_{s-N}(N)} \otimes I \otimes I \otimes \cdots
\]
Hence, using properties (2)-(3) on the page 11, we obtain
\[
U(s) (\rho (\gamma_1) \otimes \rho (\gamma_2) \otimes \ldots \otimes \rho (\gamma_N) \otimes \ldots) U(s)^*
\]
\[
= \rho (\gamma_{s-1}(1)) \otimes \rho (\gamma_{s-2}(2)) \otimes \ldots \otimes \rho (\gamma_{s-N}(N)) \otimes \ldots
\]
for all \(s \in \Sigma, \gamma \in \Gamma\).
Now we define the operators \(\Pi_A^\rho(s), (s \in \Sigma)\) and \(\Pi_A^\rho(\gamma), (\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma^\infty)\) on \(H_A^\rho\) as follows
\[
\Pi_A^\rho(s)v = U(s)v, \quad v \in H_A^\rho;
\]
\[
\Pi_A^\rho(\gamma)v = (\rho (\gamma_1) \otimes \rho (\gamma_2) \otimes \ldots) v.
\]
By (45), \(\Pi_A^\rho\) can be extended to the unitary representation of \(\Gamma \otimes \Sigma\).

The next proposition follows from the definition of Hilbert space \(H_A^\rho\) (see paragraph 2.2) and proposition 7.
Proposition 10. Let $I$ be the unit in $\mathcal{B}(\mathcal{H})^{\otimes \infty}$. Identify the elements of $\mathcal{B}(\mathcal{H})^{\otimes \infty}$ with the corresponding vectors in $\mathcal{H}_A^\rho$. Put $\psi_A^\rho(s \gamma) = (\Pi_A^\rho(s) \Pi_A^\rho(\gamma) I) I$. Then $\phi_A^\rho$ is indecomposable $\mathcal{S}_\infty$-central state on $\Gamma \wr \mathcal{S}_\infty$ (see definitions 7 and 8). Let $A_1, A_2$ be the self-adjoint operators of the trace class (see [12]) from $\mathcal{B}(\mathcal{H})$ with the property $\text{Tr}([A_j]) \leq 1$, $(j = 1, 2)$, and let $\rho_1, \rho_2$ be the unitary representations of $\Gamma$: $\rho_i : \gamma \in \Gamma \mapsto \rho_i(\gamma) \in \mathcal{B}(\mathcal{H})$. 

Proposition 11. Let $(\mathcal{H}_i, A_i, \rho_i, \xi_i)$, $i = 1, 2$ satisfy assumptions (1)-(4) (paragraph 2.7). Equality $\psi_{A_1}^{\rho_1} = \psi_{A_2}^{\rho_2}$ holds if and only if there exists isometry $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$\hat{\xi}_2 = U\hat{\xi}_1, A_2 = U A_1 U^{-1} \quad \text{and} \quad \rho_2(\gamma) = U \rho_1(\gamma) U^{-1} \text{ for all } \gamma \in \Gamma. \quad (47)$$

Proof. Assume $(47)$ hold. It follows from $(35)$ and proposition 10 that $\psi_{A_1}^{\rho_1} = \psi_{A_2}^{\rho_2}$.

Conversely, suppose that $\psi_{A_1}^{\rho_1} = \psi_{A_2}^{\rho_2}$.

Denote by $\Pi_{A_0}^\rho$ the restriction $\Pi_A^\rho$ to subspace $[\Pi_A^\rho(\Gamma \wr \mathcal{S}_\infty) I]$ generated by the vectors $[\Pi_A^\rho(\Gamma \wr \mathcal{S}_\infty) I]$. Let $(l k)$ be the transposition interchanging $l$ and $k$. According to the construction of representation $\Pi_{A}^\rho$ and properties (i)-(ii) from paragraph 2.3, there exists operator

$$\mathcal{O}_l = w - \lim_{k \rightarrow \infty} \Pi_A^\rho ((l k)) \quad (48)$$

and

$$\mathcal{O}_l (a_1 \otimes a_2 \otimes \ldots) = b_1 \otimes b_2 \otimes \ldots, \text{ where } b_k = \begin{cases} a_k, & \text{if } k \neq l, \\ A a_k, & \text{if } k = l. \end{cases} \quad (49)$$

Let $\mathfrak{A}_l^A \rho$ be $w^*$-algebra in $\Pi_{A}^\rho(\Gamma \wr \mathcal{S}_\infty)$ generated by $\mathcal{O}_l$ and $I \otimes \ldots \otimes I \otimes \rho(\gamma) \otimes I \otimes I \otimes \ldots, \gamma \in \Gamma$. Denote by $\mathcal{P}_0$ the orthogonal projection $\mathcal{H}_A^\rho$ onto $[\mathfrak{A}_l^A \rho I]$. First we prove that $w^*$-algebra $\{A, \rho(\Gamma)\}'' \subset \mathcal{B}(\mathcal{H})$ generated by $A$ and $\rho(\Gamma)$ is isomorphic to $w^*$-algebra $\mathfrak{A}_l^A \rho \mathcal{P}_0$. Namely, the map

$$m_l : A \mapsto \mathcal{O}_l \mathcal{P}_0,$$

$$m_l : \rho(\gamma) \mapsto \left(I \otimes \ldots \otimes I \otimes \rho(\gamma) \otimes I \otimes I \otimes \ldots \right) \mathcal{P}_0 \quad (50)$$

extends to an isomorphism of $\{A, \rho(\Gamma)\}''$ onto $\mathfrak{A}_l^A \rho \mathcal{P}_0$.

Using $(19)$ and definition of $\Pi_A^\rho$, we can consider $m_l$ as the GNS-representation of $\{A, \rho(\Gamma)\}'' \subset \mathcal{B}(\mathcal{H})$ corresponding to $\psi_l$ (see $(35)$). Thus
\[ \text{Ker } m_l = \{ a \in \{ A, \rho(\Gamma) \}'': m_l(a) = 0 \} \text{ is weakly closed two-sided ideal. Therefore, there exists unique orthogonal projection } e \text{ from the center of } \{ A, \rho(\Gamma) \}'' \text{ such that} \]

\[ \text{Ker } m_l = e \{ A, \rho(\Gamma) \}'' \text{ (see (51)).} \]

Let us prove that \( e = 0 \).

Denote by \( c \left( \bar{P} \right) \) central support of orthogonal projection \( \bar{P} \in \{ A, \rho(\Gamma) \}' \):

\[ \bar{P} \mathcal{H} = \bar{\mathcal{H}} \text{ (see property (2) from paragraph 2.1).} \]

Let us first show that \( e c \left( \bar{P} \right) = 0 \).

Conversely, suppose that \( e c \left( \bar{P} \right) \neq 0 \). Hence, since the map \( \{ A, \rho(\Gamma) \}'' c \left( \bar{P} \right) \ni a \mapsto a\bar{P} \in \{ A, \rho(\Gamma) \}'' \bar{P} \) is isomorphism, we obtain \( e\bar{P} \neq 0 \). It follows from properties (1)-(3) (paragraph 2.1) that \( e \left( F_{[0,1]} + F_{[-1,0]} \right) \neq 0 \). Thus, by (35), \( \psi_l(e) \neq 0 \). Therefore, \( e \notin \text{Ker } m_l \). This contradicts property (51).

Now, using (52) and property (2) (paragraph 2.1), we have

\[ e \left( I - c \left( \bar{P} \right) \right) \mathcal{H} \subseteq \mathcal{H}_{reg}. \]

Therefore, if \( e \left( I - c \left( \bar{P} \right) \right) \neq 0 \), then, using property (4) (paragraph 2.1), we obtain

\[ e \left( I - c \left( \bar{P} \right) \right) \psi_l \xi \neq 0. \]

Again, by (35), \( \psi_l(e) \neq 0 \) and \( e \notin \text{Ker } m_l \). It follows from (51) that

\[ e \left( I - c \left( \bar{P} \right) \right) = 0. \]

Hence, using (52), we obtain

\[ \text{Ker } m_l = 0. \]

Now we suppose that \( \phi_{A_1}^{(1)} = \phi_{A_2}^{(2)} \). Let \( \mathcal{O}_l^{(1)} \) and \( \mathcal{O}_l^{(2)} \) be the operators, which are defined by formula (8) for representations \( \Pi_{A_1}^{(1)} \) and \( \Pi_{A_2}^{(2)} \) respectively. If \( \mathcal{I}_l \) is the extension the map

\[ \mathcal{I}_l: \prod_{i-1}^l I \otimes \cdots \otimes I \otimes \rho_1(\gamma) \otimes I \otimes I \otimes \cdots \mapsto \prod_{i-1}^l I \otimes \cdots \otimes I \otimes \rho_2(\gamma) \otimes I \otimes I \otimes \cdots \]

by multiplication, then

\[ (\mathcal{I}_l(a)I, \mathcal{I}_l(b)I) = (aI, bI) \text{ for all } a, b \in \mathbb{A}_l^{A_1, \rho}. \]
Therefore, there exists isomorphism $U \in \{A_1, \rho_1(\Gamma)\}^{\prime\prime}$ such that
\[ \{A_1, \rho_1(\Gamma)\}^{\prime\prime} \ni a \mapsto m_1^{-1} \circ J_i \circ m_i(a) \in \{A_2, \rho_2(\Gamma)\}^{\prime\prime} \]  
(58)
is an isomorphism. Since $\phi^{\rho_1}_{A_1} = \phi^{\rho_2}_{A_2}$; then, using definition of $\phi^{\rho}_{A}$, in particular (55), obtain for all $v \in \{A_1, \rho_1(\Gamma)\}^{\prime\prime}$:
\begin{align*}
\Tr (v|A_1|) + (1 - \Tr (|A_1|)) \left(\hat{v}\xi_1, \xi_1\right) \\
= \Tr (\theta(v)|A_2|) + (1 - \Tr (|A_2|)) \left(\theta(v)\hat{\xi}_2, \hat{\xi}_2\right).
\end{align*}
(59)

Without loss of generality we can assume that $\{A_1, \rho_1(\Gamma)\}^{\prime\prime}, \{A_2, \rho_2(\Gamma)\}^{\prime\prime} \subset B(\mathcal{H})$. Let $P^{(i)}_{[-1,0]}, P^{(i)}_{[0,1]}$ be the spectral projections of $A_i$ $(i = 1, 2)$. Put $P^{(i)}_{\pm} = P^{(i)}_{[-1,0]} + P^{(i)}_{[0,1]}$. It is clear $(\Ker A_i)^\perp = P^{(i)}_{\pm} \mathcal{H}$. Denote by $\mathcal{H}_i$ subspace
\[ \{A_1, \rho_1(\Gamma)\}^{\prime\prime} P^{(i)}_{\pm} \mathcal{H}_1 \].
Let $P_i$ be the orthogonal projection of $\mathcal{H}_i$ onto $\mathcal{H}_i$. Put $P_{reg} = I - P_i$. For $\alpha \in \text{ Spectrum } A_i$ denote by $P^{(i)}_{\alpha}$ the corresponding spectral projection.

Now, using properties of $(A_i, \rho_i)$ (see paragraph 2.1), we have
\[ \dim P^{(i)}_{\alpha} \mathcal{H} < \infty \text{ and } P^{(i)}_{\pm} = \sum_{\alpha \in \text{ Spectrum } A_i; \alpha \neq 0} P^{(i)}_{\alpha}. \]  
(60)

Therefore, there exists collection $\left\{c^{(j)}_j\right\}_{j=1}^N$ of pairwise orthogonal projections from the center of $\omega^*$-algebra $P^{(i)}_{\pm} \{A_1, \rho_1(\Gamma)\}^{\prime\prime} P^{(i)}_{\pm}$ with properties
\[ \theta \left(c^{(1)}_j\right) = c^{(2)}_j \text{ (see (58))} ; \quad \sum_{j=1}^N c^{(j)}_j = P^{(i)}_{\pm}; \]  
(61)
\[ c^{(j)}_j P^{(i)}_{\pm} \{A_1, \rho_1(\Gamma)\}^{\prime\prime} c^{(i)}_j \text{ is a factor of type } I_{n_j}. \]

Fix matrix unit $\left\{f^{(j)}_{k,l}\right\}_{k,l=1}^{n_j} \subset c^{(i)}_j P^{(i)}_{\pm} \{A_1, \rho_1(\Gamma)\}^{\prime\prime} c^{(i)}_j$, which is a linear basis in $c^{(i)}_j P^{(i)}_{\pm} \{A_1, \rho_1(\Gamma)\}^{\prime\prime} c^{(i)}_j$, minimal projections $\left\{f^{(j)}_{k,l}\right\}_{k,l=1}^{n_j}$ satisfy condition
\[ P^{(1)}_{\alpha} f^{(j)}_{k,k} = f^{(j)}_{k,k} P^{(1)}_{\alpha} \text{ for all } \alpha \in \text{ Spectrum } A_1; k, j \in \mathbb{N}. \]  
(62)

Now, using (57), (58), (59) and definition of $\Pi^{\rho}_{A}$ (see paragraphs 2.1, 2.2, 2.3), we have
\[ \Tr \left(f^{(j)}_{k,k}\right) = \Tr \left(\theta \left(f^{(j)}_{k,k}\right)\right) \text{ for all } k, j \in \mathbb{N}. \]  
(63)

Therefore, there exists isometry $U : P^{(1)}_{\pm} \mathcal{H}_1 \rightarrow P^{(1)}_{\pm} \mathcal{H}_2$ such that $U P^{(1)}_{\pm} \mathcal{H}_1 = P^{(1)}_{\pm} \mathcal{H}_2$ and
\[ U f^{(j)}_{k,k} U^{-1} = \theta \left(f^{(j)}_{k,k}\right) \text{ for } k = 1, 2, \ldots n_j; j = 1, 2, \ldots, N. \]  
(64)
Let $C_i$ be the center of $w^*$-algebra $\{A_i, \rho_i(\Gamma)\}''$ and let $c\left(P_{\pm}^{(i)}\right) \in C_i$ be the central support of $P_{\pm}^{(i)}$. It follows from this and (61) that there exist pairwise orthogonal projections $\left\{C_j^{(i)}\right\}_{j=1}^{N} \subset c\left(P_{\pm}^{(i)}\right) \cdot C_i$ with the next properties

$$c_j^{(i)} = C_j^{(i)} \cdot P_{\pm}^{(i)}, \quad \sum_{j=1}^{N} C_j^{(i)} = c\left(P_{\pm}^{(i)}\right),$$

(65)

$C_j^{(i)} \{A_i, \rho_i(\Gamma)\}''$ $C_j^{(i)}$ is a factor of type $I_{N_j}$.

In $C_j^{(1)} \{A_i, \rho_i(\Gamma)\}''$ $C_j^{(1)}$ there exists matrix unit $\left\{f_{kl}^{(j)}\right\}_{k,l=1}^{N_j}$ ($n_j \geq N_j$). Now, applying (64), we obtain that

$$\widetilde{U} = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \theta\left(f_{kl}^{(j)}\right) U f_{1k}$$

(66)

is an isometry of $c\left(P_{\pm}^{(1)}\right) H_1$ onto $c\left(P_{\pm}^{(2)}\right) H_2$. An easy computation shows that $\widetilde{U} f_{kl}^{(j)} \widetilde{U}^{-1} = \theta\left(f_{kl}^{(j)}\right)$ for $k, l = 1, 2, \ldots, N_j; j = 1, 2, \ldots, N$. Thus

$$\theta(a) = \widetilde{U} a \widetilde{U}^{-1} \quad \text{for all} \quad a \in c\left(P_{\pm}^{(1)}\right) \{A_1, \rho_1(\Gamma)\}''.$$

(67)

Hence, using (69) and relations $\theta(\big|A_1\big|) = \big|A_2\big|$, $\theta\left(c\left(P_{\pm}^{(1)}\right)\right) = c\left(P_{\pm}^{(2)}\right)$, which follows from the definition of $\theta$ (see (68)), we have

$$\left((I - c\left(P_{\pm}^{(2)}\right)) \theta(v) \hat{\xi}_2, \hat{\xi}_2\right) = \left((I - c\left(P_{\pm}^{(1)}\right)) v \hat{\xi}_1, \hat{\xi}_1\right).$$

(68)

Since $\tilde{P}_i \leq c\left(P_{\pm}^{(i)}\right)$, then

$$I - c\left(P_{\pm}^{(i)}\right) \leq P_{\text{reg}}^{(i)}, \quad i = 1, 2.$$

(69)

Denote by $\{\xi_{kl}^{(i)}, k, l \in \mathbb{N}\}$ ($i = 1, 2$) the matrix unit from property (4) of paragraph 2.1. Now we define map $V$ as follows

$$a \left(I - c\left(P_{\pm}^{(1)}\right)\right) \hat{\xi}_1 \xrightarrow{V} \theta(a) \left(I - c\left(P_{\pm}^{(2)}\right)\right) \hat{\xi}_2, \quad \text{where} \quad a \in \{A_1, \rho_1(\Gamma)\}''.$$

By (68) and (69), $V$ extends to isometry $V$ of $\left(I - c\left(P_{\pm}^{(1)}\right)\right) \xi_{11}^{(1)'} H_1 \subset P_{\text{reg}}^{(1)} H_1$ onto $\left(I - c\left(P_{\pm}^{(2)}\right)\right) \xi_{11}^{(1)'} H_2 \subset P_{\text{reg}}^{(2)} H_2$ and for all $a \in \{A_1, \rho_1(\Gamma)\}''$

$$V(I - c\left(P_{\pm}^{(1)}\right)) a \xi_{11}^{(1)'} V^{-1} = (I - c\left(P_{\pm}^{(2)}\right)) \theta(a) \xi_{11}^{(2)'}.$$
It follows from this that $\widetilde{V} = \sum_{k=1}^{\infty} \epsilon_{k1}^{(2)'} V \left( I - c \left( P_{\pm}^{(1)} \right) \right) \epsilon_{k1}^{(1)'}$ is an isometry of $(I - c \left( P_{\pm}^{(1)} \right)) \mathcal{H}_1$ onto $(I - c \left( P_{\pm}^{(2)} \right)) \mathcal{H}_2$, satisfying the next relation

$$\widetilde{V} \left( I - c \left( P_{\pm}^{(1)} \right) \right) a \widetilde{V}^{-1} = \left( I - c \left( P_{\pm}^{(2)} \right) \right) \theta(a) \quad \left( a \in \{A_1, \rho_1(\Gamma)\}' \right).$$

Hence, using (71), we obtain that $W = \widetilde{U} c \left( P_{\pm}^{(1)} \right) + \widetilde{V} \left( I - c \left( P_{\pm}^{(1)} \right) \right)$ is an isometry of $\mathcal{H}_1$ onto $\mathcal{H}_2$ and

$$W a W^{-1} = \theta(a) \text{ for all } a \in \{A_1, \rho_1(\Gamma)\}'. \quad (70)$$

Now, on account of definition of $\theta$ and (69) one can easy to check that

$$W \hat{\xi}_1 \bot \{(A_2, \rho_2(\Gamma))'' P_{\pm}^{(2)} \mathcal{H}_2 = \mathcal{H}_2 \quad \text{and} \quad \left(aW \hat{\xi}_1, W \hat{\xi}_1\right) = \left(a \hat{\xi}_2, \hat{\xi}_2\right) \text{ for all } a \in \{A_2, \rho_2(\Gamma)\}''. \quad (71)$$

Define linear map $K$ by $K(v) = \left\{ \begin{array}{ll} a \hat{\xi}_2, & \text{if } v = a W \hat{\xi}_1, \quad a \in \{A_2, \rho_2(\Gamma)\}'', \\ 0, & \text{if } v \in \mathcal{H}_2 \ominus \{(A_2, \rho_2(\Gamma))'' \hat{\xi}_2\}. \end{array} \right.$

It follows from (71) that $K$ extends to the partial isometry from $\{A_2, \rho_2(\Gamma)\}'$.

Therefore, there exists unitary $\tilde{K} \in \{A_2, \rho_2(\Gamma)\}'$ with the property: $\tilde{K} v = K v$ for all $v \in \{(A_2, \rho_2(\Gamma))'' W \hat{\xi}_1\}$. Thus $\mathcal{U} = \tilde{K} W$ satisfies the conditions of proposition 11. \hfill $\Box$

2.4 The parameters of the states from paragraph 1.3 Here we follow the notation of paragraphs 1.3 and 2.1

2.4.1 State $\varphi_{sp}$. Below we find parameters $(\mathcal{H}, A, \tilde{\mathcal{H}}, \rho)$ from paragraph 2.1 such that $\varphi_{sp} = \psi_{A}^{\rho}$, where $\psi_{A}^{\rho}$ defined in proposition 10.

Let $(\rho, \mathcal{H}_\varphi, \xi_\varphi)$ be GNS-representation of group $\Gamma$ corresponding to $\varphi$, where $\varphi(\gamma) = \langle \rho(\gamma) \xi_\varphi, \xi_\varphi \rangle$ for all $\gamma \in \Gamma$ and $\mathcal{H}_\varphi = [\rho(\Gamma) \xi_\varphi]$. An easy computation shows that $\mathcal{H} = \mathcal{H}_\varphi$, $A$ acts by

$$A \xi = \langle \xi, \xi_\varphi \rangle \xi_\varphi \quad (\xi \in \mathcal{H}), \quad (72)$$

and $\tilde{\mathcal{H}} = \mathcal{H}$. It is clear $\mathcal{H}_{reg} = 0$.

2.4.2 State $\varphi_{reg}$. As above $(\rho_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ is GNS-representation of $\Gamma$. If $(\rho_\varphi^{(k)}, \mathcal{H}_\varphi^{(k)}, \xi_\varphi^{(k)})$ is $k$-th copy of $(\rho_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ then

$$\mathcal{H} = \mathcal{H}_{reg} = \bigoplus_{k=1}^{\infty} \left( \rho_\varphi^{(k)}, \mathcal{H}_\varphi^{(k)}, \xi_\varphi^{(k)} \right).$$
It is obvious, $A \equiv 0$. Now define $\epsilon'_{kl}$ by

$$
\epsilon'_{kl} (\xi_1, \xi_2, \ldots) = \begin{pmatrix}
0, \ldots, 0, \xi_l, 0, 0, \ldots \\
k-1
\end{pmatrix}.
$$

Put $\rho = \bigoplus_{k=1}^{\infty} \rho^{(k)}_{\xi}, \xi = (\xi_1, 0, 0, \ldots)$. It is easy to check that $\varphi_{reg} = \psi^0_{\rho}$.

### 2.5 $\mathcal{S}_\infty$-invariance of $\psi^0_{\rho}$

**Proposition 12.** Let $s \in \mathcal{S}_\infty$, $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_0^\infty$. If $s \gamma = \prod_{p \in \mathbb{N} \setminus s} s_p \gamma(p)$, where $s_p \gamma(p)$ is generalized cycle of $s \gamma$ (see (2)), then $\psi^0_{\rho_A} (s \gamma) = \prod_{p \in \mathbb{N} \setminus s} \psi^0_{\rho_A} (s_p \gamma(p))$. In particular, it follows from Proposition 7 that $\psi^0_{\rho_A}$ is indecomposable state on $\Gamma \mathcal{S}_\infty$.

Denote by $(n_1, n_2, \ldots, n_k)$ cycle $\{n_1 \mapsto n_2 \mapsto \ldots \mapsto n_k \mapsto n_1\} \in \mathcal{S}_\infty$. Suppose that $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_0^\infty$ satisfies the condition: $\gamma_i = e$ for all $i \notin \{n_1, n_2, \ldots, n_k\}$. If $\text{Tr} (|A|) = 1$, $c_k = (n_1, n_2, \ldots, n_k)$ then, using (35), we have

$$
\psi^0_{\rho_A} (c_k \gamma) = \text{Tr}^{\otimes N} (U (c_k) (\rho (\gamma_1) \otimes \rho (\gamma_2) \otimes \ldots \otimes \rho (\gamma_N)) A^{\otimes N})
$$

(73)

for all $N \geq \max \{n_1, n_2, \ldots, n_k\}$, where $\text{Tr}^{\otimes N}$ is the ordinary trace on $\mathcal{B} (\mathcal{H})^{\otimes N}$, $A^{\otimes N} = A \otimes \ldots \otimes A$. The next lemma extends formula (73) on the general case.

**Lemma 13.** If $k > 1$ then

$$
\psi^0_{\rho_A} (c_k \gamma) = \text{Tr}^{\otimes N} (U ((n_1, n_2, \ldots, n_k)) (\rho (\gamma_{n_1}) \otimes \rho (\gamma_{n_2}) \otimes \ldots \otimes \rho (\gamma_{n_k})) A^{\otimes k}).
$$

**Proof.** Let $\widetilde{P}$ be an orthogonal projection on subspace $\overline{\mathcal{H}} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (see paragraph 2.1). Let $E = E_1 \otimes E_2 \otimes \ldots \otimes E_N \otimes \ldots$, where $E_i = \begin{cases}
\widetilde{P} + \epsilon''_{\xi_i}, & \text{if } i = n_j; \\
I_{\mathcal{H}_i}, & \text{if } i \neq n_j \text{ for all } j \in \{1, 2, \ldots, k\}. \end{cases}$ Considering identical operator $I \in \mathcal{B} (\mathcal{H})$ as element of $\mathcal{H}^0_{\rho_A}$, we obtain from (35), (36), (37)

$$
EI = I.
$$

(74)

It follows from (44) that

$$
\widetilde{E} = U (c_k) E U (c_k)^* E = \widetilde{E}_1 \otimes \widetilde{E}_2 \otimes \ldots \otimes \widetilde{E}_N \otimes \ldots,
$$

(75)

where $\widetilde{E}_i = \begin{cases}
\widetilde{P}, & \text{if } i = n_j; \\
I_{\mathcal{H}_i}, & \text{if } i \neq n_j \text{ for all } j \in \{1, 2, \ldots, k\}. \end{cases}$ By properties (1)-(4) from paragraph 2.1 using (46) and (44), we obtain

$$
\Pi^0_{\rho_A} (\gamma) E = E \Pi^0_{\rho_A} (\gamma), \Pi^0_{\rho_A} (\gamma) \widetilde{E} = \widetilde{E} \Pi^0_{\rho_A} (\gamma).
$$

(76)
Thus
\[
\psi^\rho_A(c_k \gamma) = (\Pi^\rho_A(c_k) \Pi^\rho_A(\gamma) I, I) = (\Pi^\rho_A(c_k) \Pi^\rho_A(\gamma) EI, EI) \\
= (\Pi^\rho_A(c_k) \Pi^\rho_A(\gamma) [\Pi^\rho_A(c_k) E \Pi^\rho_A(c_k)]* \Pi^\rho_A(c_k) I, EI) \\
= (\Pi^\rho_A(c_k) \Pi^\rho_A(\gamma) \Pi^\rho_A(c_k)* \Pi^\rho_A(c_k) I, [\Pi^\rho_A(c_k) E \Pi^\rho_A(c_k)]* EI) \\
= (\Pi^\rho_A(c_k) \Pi^\rho_A(\gamma) E I, EI).
\]

Hence, applying (35), (36), (37), obtain for \( N \geq \max \{n_1,n_2,\ldots,n_k\} \)
\[
\psi^\rho_A(c_k \gamma) = 1\psi_N\left( EU(c_k) (\rho(\gamma_1) \otimes \rho(\gamma_2) \otimes \cdots \otimes \rho(\gamma_N)) E \right). \tag{77}
\]
for all \( k \), then \( 1\psi_N\left( EU(c_k) (\rho(\gamma_1) \otimes \rho(\gamma_2) \otimes \cdots \otimes \rho(\gamma_N)) E \right) = \text{Tr}^\otimes_N (U (n_1 \ n_2 \ \ldots \ n_k) (\rho(\gamma_{n_1}) \otimes \rho(\gamma_{n_2}) \otimes \cdots \otimes \rho(\gamma_{n_k})) A^{\otimes k}). \]

\[\square\]

**Remark 1.** One should notice that in the case in which \( c_k = 1 \),
\[
\psi^\rho_A(\gamma) = \prod_{n=1}^{\infty} \left[ \text{Tr} (\rho(\gamma_n) |A|) + (1 - \text{Tr} (|A|)) \left( \rho(\gamma_n) \hat{\xi} \right) \right]. \tag{78}
\]

Hence, taking into account Proposition \[12\] Lemma \[13\] and \[73\], we obtain the next important property
\[
\psi^\rho_A(sgs^{-1}) = \psi^\rho_A(g) \text{ for all } s \in G_\infty, g \in \Gamma \l G_\infty. \tag{79}
\]

3 **KMS-condition for the \( G_\infty \)-central states.**

3.1 **KMS-condition for \( \psi^\rho_A \).** To the general definition of the KMS-condition we refer the reader to the book \[15\]. Here we introduce the definition of the KMS-condition for the indecomposable states only.

**Definition 14.** Let \( \varphi \) be an indecomposable state on the group \( G \). Let \((\pi_\varphi, H_\varphi, \xi_\varphi)\) be the corresponding GNS-construction, where \( \xi_\varphi \) is such that\( \varphi(g) = (\pi_\varphi(g) \xi_\varphi, \xi_\varphi) \) for each \( g \in G \). We say that \( \varphi \) satisfies the KMS-condition or \( \varphi \) is KMS-state, if \( \xi_\varphi \) is separating\(^2\) for the \( w^* \)-algebra \( \pi_\varphi(G)^w \), generated by operators \( \pi_\varphi(G) \).

The main result of this paragraph is the following:

**Theorem 15.** Let \( (A, \hat{\xi}, H_{reg}, \epsilon_{11}) \) satisfy the conditions (1)-(4) from paragraph \[27\]. State \( \psi^\rho_A \) satisfies the KMS-condition if and only if \( \text{Ker} A = H_{reg} \) and \( \hat{\xi} \) is cyclic and separating for the restriction \( \rho_{11} = \rho|_{\epsilon_{11}}^\epsilon_{11} \) of representation \( \rho \) to subspace \( \epsilon_{11}H \).

As a preliminary to the proof of the theorem, we will discuss two auxiliary lemmas.

---

\(^2\)This means that for every \( a \in \pi_\varphi(G)^w \) the conditions \( a\xi_\varphi = 0 \) and \( a = 0 \) are equivalent.
Lemma 16. Let \((\pi_{\psi_k}, \mathcal{H}_{\psi_k}, \xi_{\psi_k})\) be GNS-representation of \(\mathcal{B}(\mathcal{H})\) corresponding to state \(\psi_k\) (see (39)). Fix any \(\epsilon > 0\) and denote by \(P_{[\epsilon, 1]}\) the spectral projection of \(|A|\). Then for each \(a \in \mathcal{B}(\mathcal{H})\) the map

\[ \mathfrak{R}_{P_{[\epsilon, 1]}aP_{[\epsilon, 1]}} : x \mapsto x \cdot P_{[\epsilon, 1]} a P_{[\epsilon, 1]} \]

may be extended by continuous to the bounded operator on \(\mathcal{H}_{\psi_k}\) and

\[ \|\mathfrak{R}_{P_{[\epsilon, 1]}aP_{[\epsilon, 1]}}\|_{\mathcal{H}_{\psi_k}} \leq \frac{|a|}{\sqrt{\epsilon}}. \]

Proof. Put \(b = P_{[\epsilon, 1]} a P_{[\epsilon, 1]}\). Then

\[
(\mathfrak{R}_b x, \mathfrak{R}_b x)_{\mathcal{H}_{\psi_k}} = \text{Tr} (b |A| b^* x^* x) \leq \|b |A| b^*\| \cdot \|\text{Tr} (P_{[\epsilon, 1]}x^* x)\|
\]

\leq \epsilon^{-1} \cdot \|b |A| b^*\| \cdot \|\text{Tr} (|A| P_{[\epsilon, 1]} x^* x)\| \leq \epsilon^{-1} \cdot \|b |A| b^*\| \cdot \|\text{Tr} (|A| x^* x)\|
\]

\[ \leq \epsilon^{-1} \cdot \|b |A| b^*\| \psi_k (x^* x) \leq \epsilon^{-1} \cdot \|b\|^2 (x^* x)_{\mathcal{H}_{\psi_k}}. \]

\[ \square \]

Lemma 17. Suppose that for \(\left(A, \xi, \mathcal{H}_{\text{reg}}, \xi_{\text{kl}}\right)\) the conditions (1)-(4) from paragraph [2.1] hold. Denote by \(P_0\) and \(P_{\text{reg}}\) the orthogonal projections onto \(\ker A\) and \(\mathcal{H}_{\text{reg}}\) respectively. Let \([\Pi_{A}^0 (\Gamma \setminus \mathcal{G}_\infty)] I\) be the subspace in \(\mathcal{H}_A^0\) (see paragraphs [2.2] [2.3]), generated by \(\Pi_{A}^0 (\Gamma \setminus \mathcal{G}_\infty) I\). For \(m \in \{\rho(\Gamma)\}' \subset \mathcal{B}(\mathcal{H})\) define the linear map \(\mathfrak{M}_m : \mathcal{B}(\mathcal{H})^{\otimes \infty} \mapsto \mathcal{B}(\mathcal{H})^{\otimes \infty}\) as follows

\[ \mathfrak{M}_m (a_1 \otimes \ldots \otimes a_k \otimes a_{k+1} \otimes \ldots) = a_1 \otimes \ldots \otimes a_k \cdot \xi_{\text{kl}} m \cdot \xi_{\text{kl}} \otimes a_{k+1} \otimes \ldots. \]

(80)

If \(P_0 = P_{\text{reg}}\) then

\[ a_{\epsilon} (I \otimes \ldots \otimes I \otimes P_{\text{reg}} a_{\epsilon} P_{\text{reg}} \otimes I \otimes \ldots) \]

as the elements from \(\mathcal{H}_{A}^0\), we have

\[ \|\mathfrak{M}_m (I) - a_{\epsilon} (I)\|_{\mathcal{H}_{A}^0} < \epsilon. \]

(81)
It follows from (48) and (49), that operator of the left multiplication on $I \otimes \ldots \otimes I \otimes P_0 \otimes I \otimes \ldots$ lies in $\Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty)''$. Hence, since $P_0 = P_{reg}$, we get $a(k) \in \Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty)''$. Therefore, using (51), we obtain $\mathcal{B}_m^{(k)}(I) \in [\Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty) I]$.

Let us prove statement (ii). Put $\mathfrak{S}_\infty^{(k)} = \{ s \in \mathfrak{S}_\infty : s(k) = k \}$. First, using (79), we observe that

$$
(a_1 b_1 I, a_2 b_2 I)_{\mathcal{B}_A^p} = (a_1 b_1 b_2 0 I, a_2 I)_{\mathcal{B}_A^p}
$$

for all $a_1, a_2 \in \Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty)''$ and $b_1, b_2 \in \Pi_A^p (\mathfrak{S}_\infty)''$. (82)

Denote be $\mathcal{L}_{P_0}^{(k)}$ operator of the left multiplication on $I \otimes \ldots \otimes I \otimes P_0 \otimes I \otimes \ldots$. By (48) and (49), $\mathcal{L}_{P_0}^{(k)} \in \Pi_A^p (\mathfrak{S}_\infty)''$. Therefore, $[\Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty) \left( I - \mathcal{L}_{P_0}^{(k)} I \right), \mathcal{H}_l = \left[ \Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty) \left( I - \mathcal{L}_{P_0}^{(k)} I \right) \right]$ are the subspaces in $[\Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty) I]$. and, according to (82), we have

$$
\left[ \Pi_A^p (\Gamma \cdot \mathfrak{S}_\infty) \left( I - \mathcal{L}_{P_0}^{(k)} I \right) \right] \perp \mathcal{H}_l \text{ for all } l \in \mathbb{N}. \quad (83)
$$

Now we prove that subspaces $\{ \mathcal{H}_l \}_{l \in \mathbb{N}}$ are pairwise orthogonal. For convenience we assume that $k = 1$. Denote by $E_m$ the orthogonal projection on subspace $\mathcal{C} \subset \mathcal{H}$ ($m \in \mathbb{N}$). Put $A_m = A + (I - \mathcal{T}) E_m$, $\mathcal{C}^{(i)} I \otimes \ldots \otimes I \otimes \mathcal{C}^{(i)} I \otimes \ldots$. By definition,

$$
E_m^{(i)} \mathcal{C}^{(i)} I = \delta_{ml} E_m^{(i)} I, \text{ where } \delta_{ml} \text{ is Kronecker's delta.} \quad (84)
$$

It follows from the definition of $A_m$ that for $s^{-1}(1) \neq 1$ and $n > s^{-1}(1)

$$
\mathcal{C}^{(s^{-1}(1))} I \cdot \prod_{m=1}^n A_m = 0. \quad (85)
$$

Fix any $\bar{\gamma}, \bar{\gamma} \in \Gamma, s_1 \in (1 l_1) \mathfrak{S}_\infty^{(1)}$ and $s_2 \in (1 l_2) \mathfrak{S}_\infty^{(1)}$. Let us show that for $l_1 \neq l_2$

$$
\kappa = \left( \Pi_A^p (s_1 \bar{\gamma}) \mathcal{L}_{P_0}^{(i)} I, \Pi_A^p (s_2 \bar{\gamma}) \mathcal{L}_{P_0}^{(i)} I \right)_{\mathcal{B}_A^p} = 0. \quad (86)
$$

Let $\mathcal{T} \otimes_n$ be the ordinary trace on $w^*-\text{factor } \mathcal{B} (\mathcal{H}) \otimes_n$. If $s = s_2^{-1}s_1$, $\gamma_m = \bar{\gamma}_{s(m)}$, $\bar{\gamma}_m \in \Gamma$, $\gamma = (\gamma_1, \gamma_2, \ldots)$ and $n > \max \{ \max \{ i : \gamma_i \neq e \}, \max \{ i : s(i) \neq i \} \}$ then, using definition of $\Pi_A^p$ (see (46)), we have

$$
\kappa = \mathcal{T} \otimes_n \left( E_1^{(i)} \cdot U_n(s) \cdot \prod_{m=1}^n \rho (\gamma_m) \cdot E_1^{(i)} \cdot \prod_{m=1}^n A_m \right), \quad (87)
$$
where $U_n(s)$ is defined in paragraph 2.3. Hence, applying property (4) from paragraph 2.1 (84) and (44), we obtain

$$
\kappa = \text{Tr} \otimes^n \left( E_1^{(1)} \cdot U_n(s) \cdot (U_n(s))^* \cdot \mathcal{E}_1^{(1)} \cdot U_n(s) \cdot \bigotimes_{m=1}^n \rho(\gamma_m) \cdot E_1^{(1)} \cdot \bigotimes_{m=1}^n A_m \right)
$$

This is equivalent to

$$
\text{Tr} \otimes^n \left( E_1^{(1)} \cdot U_n(s) \cdot (s^{-1}(1))^{(1)} \cdot \bigotimes_{m=1}^n \rho(\gamma_m) \cdot E_1^{(1)} \cdot \bigotimes_{m=1}^n A_m \right)
$$

property (4)

$$
\text{Tr} \otimes^n \left( E_1^{(1)} \cdot U_n(s) \cdot \bigotimes_{m=1}^n \rho(\gamma_m) \cdot E_1^{(1)} \cdot \bigotimes_{m=1}^n A_m \right) = \text{0}.
$$

Therefore,

$$
H_l \perp H_m \text{ for all } l \neq m.
$$

As in the proof of (i), $\mathcal{K}_m^{(1)}(I) = \mathcal{E}_1^{(1)} \otimes I \otimes I \otimes \ldots$ lies in subspace $[\Pi_A^\rho (\Gamma_{e^\infty}) L_{P_0}^{(1)}] \subset H_I$. Therefore,

$$
\Pi_A^\rho \left( (1 \ l) \cdot \mathcal{E}_1^{(1)} \right) \Pi_A^\rho (\Gamma_{e^\infty}) L_{P_0}^{(1)} \mathcal{K}_m^{(1)}(I) \subset H_I.
$$

Further, using (44) and relation

$$
\mathcal{K}_m^{(1)} \Pi_A^\rho ((1 \ l) \cdot s) \Pi_A^\rho (\gamma) L_{P_0}^{(1)} I = \Pi_A^\rho ((1 \ l) \cdot s) \Pi_A^\rho (\gamma) L_{P_0}^{(1)} I,
$$

where $s \in \mathcal{S}_e^{(1)}$, $\gamma \in \Gamma_{e^\infty}$, we obtain that $\mathcal{K}_m^{(1)}$ is the bounded operator on $H_I$ and $\|\mathcal{K}_m^{(1)}\|_{H_I} \leq \|\mathcal{E}_1^{(1)} \otimes I\|_{H_I}$. Since, by (83) and (88),

$$
[\Pi_A^\rho (\Gamma \otimes \mathcal{S}_e) I] = \left[ \Pi_A^\rho (\Gamma \otimes \mathcal{S}_e) \left( I - L_{P_0}^{(1)} \right) I \right] \bigoplus_{m=1}^\infty H_m,
$$

and $[\Pi_A^\rho (\Gamma \otimes \mathcal{S}_e) \left( I - L_{P_0}^{(1)} \right) I] \subset \text{Ker} \mathcal{K}_m^{(1)}$, operator $\mathcal{K}_m^{(1)}$ is bounded on subspace $[\Pi_A^\rho (\Gamma \otimes \mathcal{S}_e) I]$.

**The proof of Theorem 15** Let $\Pi_A^\rho$ be the restriction $\Pi_A^\rho$ to subspace $[\Pi_A^\rho (\Gamma \otimes \mathcal{S}_e) I]$. Obvious, $\Pi_A^\rho$ and GNS-representation of $\Gamma \otimes \mathcal{S}_\infty$, corresponding to $\psi_A^\rho$, are naturally unitary equivalent. Let us prove that $I$ is the cyclic vector for $\Pi_A^\rho (\Gamma \otimes \mathcal{S}_\infty)$.

For any $n \in \mathbb{N}$ fix $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n, e, e, \ldots) \in \Gamma_{e^\infty}$ and $s \in \mathcal{S}_n$. Put

$$
\eta = \Pi_A^\rho(\gamma) I = \left( \bigotimes_{m=1}^n \rho(\gamma_m) \right) \otimes I \otimes I \otimes \ldots \in \left[ \Pi_A^\rho (\Gamma_{e^\infty}) I \right] \subset \left[ \Pi_A^\rho (\Gamma \otimes \mathcal{S}_\infty) I \right].
$$

If $P_{[\gamma, 1]}$ is the spectral projection of $[A]$ then, by (18), (19) and lemma 17 (i), for every $m_j' \in \rho(\Gamma')$, $j = 1, \ldots, n$

$$
a_\varepsilon = \left( \bigotimes_{j=1}^n (P_{[\gamma, 1]} \rho(\gamma_j) P_{[\gamma, 1]} + \mathcal{E}_{jj} m_j' \mathcal{E}_{jj}) \right) \otimes I \otimes I \otimes \ldots \in \left[ \Pi_A^\rho (\Gamma \otimes \mathcal{S}_\infty) I \right].
$$

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Since \( \hat{\xi} \) is cyclic and separating for the representation \( \rho_{11} = \rho \big|_{\mathcal{H}_{\text{reg}}} \) and \( \text{Ker} \, A = \mathcal{H}_{\text{reg}} \), then for any \( \delta > 0 \) there exist \( \epsilon > 0 \) and \( \{ m_j \} _{j=1} ^{n} \subset \rho(\Gamma) \) such that
\[
\| \Pi_{\rho} (\gamma) I - a_\epsilon \| _{\mathcal{H}_{\rho}^\prime} < \delta.
\]
But, by lemmas 16-17, operator \( \mathfrak{R}_{\rho} \) of right multiplication on \( a_\epsilon \) lies in \( \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})' \). Therefore,
\[
\Pi_{\rho}^0 (\gamma) I \in \left[ \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})' I \right].
\]
Now we note that, by (79), the right multiplication on \( \mathcal{H} \) defines the unitary operator \( \mathfrak{R}_{U(s)} \in \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})' \). It follows from (91) that \( \Pi_{\rho}^0 (\gamma) I = \mathfrak{R}_{U(s)}(\Pi_{\rho}^0(\gamma)I) \in \left[ \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})' I \right] \). Therefore \( I \) is the cyclic vector for \( \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})' \).

Conversely, suppose that \( \psi_{\rho}^o \) is KMS-state on \( \Gamma \wr \mathcal{S}_{\infty} \). Define state \( \hat{\psi}_{\rho}^o \in \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})'' \) as follows
\[
\hat{\psi}_{\rho}^o (a) = (aI, I)_{\mathcal{H}_{\rho}^\prime}.
\]
Then, by propositions 7 and 12 \( \hat{\psi}_{\rho}^o \) is faithful state. This means that for every \( a \in \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})'' \) the conditions \( \hat{\psi}_{\rho}^o (a^* a) = 0 \) and \( a = 0 \) are equivalent.

Let us prove that \( \text{Ker} \, A = \mathcal{H}_{\text{reg}} \). If \( \mathcal{H}_{\text{reg}} \not\subset \text{Ker} \, A \) then, by properties (1)-(4) from paragraph 2.1, there exists \( \gamma \in \Gamma \) such that
\[
\rho(\gamma) (P_{[0,1]} + P_{[-1,0]}) \neq (P_{[0,1]} + P_{[-1,0]}) \rho(\gamma).
\]
It follows from this
\[
Q = (P_{[0,1]} + P_{[-1,0]}) \lor \rho(\gamma) (P_{[0,1]} + P_{[-1,0]}) \rho(\gamma)^* - (P_{[0,1]} + P_{[-1,0]}) \neq 0.
\]
Since \( Q \in \mathfrak{A} \), where \( \mathfrak{A} \) is defined in property (1) from paragraph 2.1, then, by (48)-(49), operator \( \mathfrak{L}_Q^{(k)} \) of the left multiplication on \( (\otimes_{m=1}^{k-1} I) \otimes Q \otimes I \otimes \ldots \) lies in \( \Pi_{\rho}^0 (\Gamma \wr \mathcal{S}_{\infty})'' \). Thus \( \hat{\psi}_{\rho}^o (\mathfrak{L}_Q^{(k)}) = \text{Tr} (Q \cdot |A|) = 0 \). But this contradicts the faithfulness of \( \hat{\psi}_{\rho}^o \).

Now we prove that \( \hat{\xi} \) is cyclic and separating for the representation \( \rho_{11} = \rho \big|_{\mathcal{H}_{\text{reg}}} \). Denote by \( E_{11} \) the projection onto \( \left[ \rho_{11} (\Gamma) \hat{\xi} \right] \) and suppose \( \left[ \rho_{11} (\Gamma) \hat{\xi} \right] \not\subset \left[ \rho_{11} (\Gamma) \hat{\xi} \right] \). It follows from this that
\[
E_{11} \in \rho_{11} (\Gamma)'' \quad \text{and} \quad E_{11} \hat{\xi} = 0.
\]
Denote by \( P_{\text{reg}} \) the orthogonal projection onto \( \mathcal{H}_{\text{reg}} \). Since \( \text{Ker} \, A = \mathcal{H}_{\text{reg}} \), then
\[
P_{\text{reg}} \in \mathfrak{A} \quad \text{and} \quad P_{\text{reg}} \cdot (\Gamma)'' \cdot P_{\text{reg}} \subset \mathfrak{A}.
\]
Hence, by properties (2) and (4) from paragraph 2.1 we obtain
\[ F = \sum_{n=1}^{\infty} \epsilon_{m} \cdot F_{11} \cdot \epsilon_{m} \in P_{\text{reg}} \cdot \rho(\Gamma)'' \]

Hence, using (48)-(49), we obtain that operator \( \hat{\rho} \) of the left multiplication on \( (\otimes_{n=1}^{\infty} I) \otimes F \otimes I \otimes \ldots \) lies in \( \Pi_{A}^{0} (\Gamma \uparrow \mathcal{G}_{\infty})'' \). It follows from this and (47) that \( \hat{\rho} \) lies in \( (\Pi_{A}^{0} (\Gamma \uparrow \mathcal{G}_{\infty})'' \bigcap \Gamma \uparrow \mathcal{G}_{\infty})'' \).

\[ \square \]

44 The main result.

In this section we prove the main result of this paper:

**Theorem 18.** Let \( \varphi \) be any indecomposable \( \mathcal{G}_{\infty} \)-central state on the group \( \Gamma \uparrow \mathcal{G}_{\infty} \). Then there exist self-adjoint operator \( A \) of the trace class (see [12]) from \( \mathcal{B}(\mathcal{H}) \) and unitary representation \( \rho \) with the properties (1)-(4) (paragraph 2.1) such that \( \varphi = \hat{\rho} \) (see Proposition 17).

We have divided the proof into a sequence of lemmas and propositions. First we introduce some new objects and notations.

4.1 Asymptotical transposition. Let \((\pi_{\varphi}, H_{\varphi}, \xi_{\varphi})\) be GNS-representation of \( \Gamma \uparrow \mathcal{G}_{\infty} \) associated with \( \varphi \), where \( \varphi(g) = (\pi_{\varphi}(g)\xi_{\varphi}, \xi_{\varphi}) \) for all \( g \in \Gamma \uparrow \mathcal{G}_{\infty} \). In the sequel for convenience we denote group \( \Gamma \uparrow \mathcal{G}_{\infty} \) by \( G \). Put
\[ G_{n}(\infty) = \left\{ s \in G | s \in \mathcal{G}_{\infty}, \gamma = (\gamma_{1}, \gamma_{2}, \ldots) \in \Gamma_{\infty}, s(l) = l \text{ and } \gamma_{l} = e \text{ for } l = 1, 2, \ldots, n \right\}, \]
\[ G_{n} = \left\{ s \gamma \in G | s(l) = l \text{ and } \gamma_{l} = e \text{ for all } l > n \right\}, \]
\[ G^{(k)} = \left\{ s \gamma \in G | s(k) = k \text{ and } \gamma_{k} = e \right\}. \]

It is clear that \( G_{0}(\infty) = G \).

**Proposition 19.** Let \((i, j)\) denotes the transposition exchanging \( i \) and \( j \). In the weak operator topology there exists \( \lim_{j \to \infty} \pi_{\varphi} ((i, j)) \).

**Proof.** It is suffices to show that for any \( g, h \in G \) there exists \( \lim_{j \to \infty} \pi_{\varphi} ((i, j)) \pi_{\varphi}(g)\xi_{\varphi}, \pi_{\varphi}(g)\xi_{\varphi}) \). Find \( N > i \) such that \( g, h \in G_{n} \) for all \( n \geq N \).

Since \( \varphi \) is \( \mathcal{G}_{\infty} \)-central, then
\[ (\pi_{\varphi}((i, N)) \pi_{\varphi}(g)\xi_{\varphi}, \pi_{\varphi}(g)\xi_{\varphi}) = (\pi_{\varphi}((i, j)) \pi_{\varphi}(g)\pi_{\varphi}((n, N))\xi_{\varphi}, \pi_{\varphi}(g)\pi_{\varphi}((n, N))\xi_{\varphi}) = (\pi_{\varphi}((i, n)) \pi_{\varphi}(g)\xi_{\varphi}, \pi_{\varphi}(g)\xi_{\varphi}). \]

Thus \( \lim_{j \to \infty} (\pi_{\varphi}((i, j)) \pi_{\varphi}(g)\xi_{\varphi}, \pi_{\varphi}(g)\xi_{\varphi}) = (\pi_{\varphi}((i, N)) \pi_{\varphi}(g)\xi_{\varphi}, \pi_{\varphi}(g)\xi_{\varphi}). \)

We will call \( O_{i} = \lim_{j \to \infty} \pi_{\varphi} ((i, j)) \) the asymptotical transposition.
4.2 The properties of the asymptotical transposition.

Lemma 20. Let \( g, h \in G^{(n)} \). Then for each \( k \neq n \) the next relation holds:

\[
(\pi_{\varphi}(g \cdot (n k) \cdot h) \xi_{\varphi}, \xi_{\varphi}) = (\pi_{\varphi}(g) \sigma_{k} \pi_{\varphi}(h) \xi_{\varphi}, \xi_{\varphi})
\]  
(95)

Proof. Fix \( N \in \mathbb{N} \) such that \( g, h \in G_N \cap G^{(n)} \). Then for each \( m > N \) we have:

\[
(n m) \cdot g = g \cdot (n m), \quad (n m) \cdot h = h \cdot (n m).
\]

Hence, by the \( \mathcal{S}_\infty \)-centrality of \( \varphi \), we obtain

\[
(\pi_{\varphi}(g \cdot (n k) \cdot h) \xi_{\varphi}, \xi_{\varphi}) = \varphi(g \cdot (n k) \cdot h) = \varphi((n m) \cdot g \cdot (n k) \cdot h \cdot (n m)) =
(\pi_{\varphi}((n m) \cdot g(n k) \cdot h \cdot (n m)) \xi_{\varphi}, \xi_{\varphi}) = (\pi_{\varphi}(g \cdot (m k) \cdot h) \xi_{\varphi}, \xi_{\varphi}).
\]

Approaching the limit as \( m \to \infty \) we obtain the required assertion. \qed

Lemma 21. The next relations hold true:

1. \( \sigma_{k} \sigma_{n} = \sigma_{n} \sigma_{k} \) for all \( k, n \in \mathbb{N} \);
2. \( \sigma_{k} \varphi(\gamma) = \varphi(\gamma) \sigma_{k} \) for all \( \gamma = (\gamma_{1}, \gamma_{2}, \ldots) \in \Gamma_{\mathcal{S}}^\infty \) such that \( \gamma_{k} = e \);
3. \( \varphi(s) \sigma_{k} = \sigma_{s(k)} \varphi(s) \) for all \( s \in \mathcal{S}_\infty \).

The proof follows immediately from definition \( \sigma_{k} \) (Proposition 19). The details are left the reader. \qed

We will use the notation \( \mathfrak{A}_{j} \) for the \( W^* \)-algebra generated by the operators \(\pi_{\varphi}(\gamma)\), where \(\gamma = (e, \cdots, e, \gamma_{j}, e, \cdots)\) and \(\sigma_{j}\). There is the natural isomorphism \(\phi_{j,k}\) between \(\mathfrak{A}_{j}\) and \(\mathfrak{A}_{k}\) for any \(k\) and \(j\):

\[
\phi_{j,k} : \mathfrak{A}_{k} \to \mathfrak{A}_{j}, \quad \phi_{j,k}(a) = \pi_{\varphi}(\gamma)(k j) a \pi_{\varphi}(\gamma)(k j).
\]  
(96)

Observe that \(\phi_{j,k}(a) \xi_{\varphi}, \xi_{\varphi}) = (a \xi_{\varphi}, \xi_{\varphi})\) for all \(k, j\) and \(a \in \mathfrak{A}_{k}\).

The next statement is the simple technical generalization of proposition 7.

Lemma 22. Let \( s = \prod_{p \in \mathbb{N}/s} s_{p} \) be the decomposition of \( s \in \mathcal{S}_\infty \) into the product of cycles \( s_{p} \), where \( p \subset \mathbb{N} \) is the corresponding orbit. Fix any finite collection \(\{U_{j}\}_{j=1}^{N} \) of the elements from \( \pi_{\varphi}(G)^{\infty} \). If \( U_{j} \in \mathfrak{A}_{j} \), then

\[
(\pi_{\varphi}(s) \prod_{j=1}^{N} U_{j} \xi_{\varphi}, \xi_{\varphi}) = \prod_{p \in \mathbb{N}/s} (\pi_{\varphi}(s_{p}) \prod_{j \in p} U_{j} \xi_{\varphi}, \xi_{\varphi}).
\]  
(97)

Proposition 23. Let \( s_{p} \in \mathcal{S}_\infty \) be the cyclic permutation on the set \( p = \{k_{1}, k_{2}, \ldots, k_{|p|}\} \subset \mathbb{N} \), where \( k_{i} = s^{i-1}(k_{1}) \). If \( U_{k_{i}} \in \mathfrak{A}_{k_{i}} \) for all \( k_{i} \in p \) then

\[
(\pi_{\varphi}(s_{p}) U_{k_{1}} U_{k_{2}} \cdots U_{k_{|p|}} \xi_{\varphi}, \xi_{\varphi}) = (\phi_{k_{|p|}k_{1}}(U_{k_{1}}) \mathcal{O}_{k_{|p|}} \phi_{k_{|p|}k_{2}}(U_{k_{2}}) \mathcal{O}_{k_{|p|}} \cdots \mathcal{O}_{k_{|p|}} U_{k_{|p|}} \xi_{\varphi}, \xi_{\varphi}).
\]  
(98)
Proof. For convenience we suppose that \( p = \{1, 2, \ldots, n\} \) and
\[
s_p(k) = \begin{cases} 
  k - 1, & \text{if } k > 1 \\
  n, & \text{if } k = 1
\end{cases}.
\]
Since \( s_p = (1)(2) \cdots (n - 1) \), we obtain
\[
(\pi_\varphi (s_p) U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi)
= (\pi_\varphi ((1)(2n) \cdots (n - 2n)) U_1 U_2 \cdots \pi_\varphi ((n - 1)(1) U_{n-1} U_n \xi_\varphi, \xi_\varphi)
= (\pi_\varphi ((1)(2n) \cdots (n - 2n)) U_1 U_2 \cdots \phi_{n,n-1} (U_{n-1}) \pi_\varphi ((n - 1)(1) U_{n} \xi_\varphi, \xi_\varphi).
\]
Hence, using \( \mathcal{S}_\infty \)-invariance of \( \varphi \) and lemma [21] for any \( N > n \) we have
\[
(\pi_\varphi (s_p) U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi) = (\pi_\varphi ((n - 1)N) s_p(n - 1)N) U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi)
= (\pi_\varphi ((1)(2n) \cdots (n - 2n)) U_1 U_2 \cdots \phi_{n,n-1} (U_{n-1}) \pi_\varphi ((N)(1) U_{n} \xi_\varphi, \xi_\varphi).
\]
Approaching the limit as \( N \to \infty \), we obtain
\[
(\pi_\varphi (s_p) U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi)
= (\pi_\varphi ((1)(2n) \cdots (n - 2n)) U_1 U_2 \cdots U_{n-2} \phi_{n,n-1} (U_{n-1}) \circ_n U_n \xi_\varphi, \xi_\varphi).
\]
Since \( \phi_{n,n-1} (U_{n-1}) \circ_n \), then, by the obvious induction, we have
\[
(\pi_\varphi (s_p) U_1 U_2 \cdots U_n \xi_\varphi, \xi_\varphi)
= (\phi_{n,1} (U_1) \circ_n \phi_{n,2} (U_2) \circ_n \cdots \phi_{n,n-2} (U_{n-2}) \circ_n \phi_{n,n-1} (U_{n-1}) \circ_n U_n \xi_\varphi, \xi_\varphi).
\]

The next statement is an analogue of Theorem 1 from [8].

Lemma 24. Let \([a, b]\) belongs to \([-1, 0]\) or \([0, 1]\), with the property. Denote by \( E_{[a,b]}^{(i)} \) the spectral projection of self-adjoint operator \( \mathcal{O}_i \). If \( \min \{|a|, |b|\} > \varepsilon > 0 \) then \( \left( E_{[a,b]}^{(i)} \xi_\varphi, \xi_\varphi \right)^2 \geq \varepsilon \left| E_{[a,b]}^{(i)} \xi_\varphi, \xi_\varphi \right| \).

This result may be proved in much the same way as theorem 1 from [8]. For convenience we give below the full proof of lemma 24.

Proof. Using Lemma [20], we have
\[
\left| \left( \pi_\varphi ((i, i + 1)) E_{[a,b]}^{(i)} \xi_\varphi, E_{[a,b]}^{(i)} \xi_\varphi \right) \right| =
\left| \left( \mathcal{O}_i E_{[a,b]}^{(i)} \xi_\varphi, E_{[a,b]}^{(i)} \xi_\varphi \right) \right| \geq \varepsilon \left| E_{[a,b]}^{(i)} \xi_\varphi, \xi_\varphi \right|.
\]
Hence, applying (99) and lemma 24, we obtain
\[
E_{[a,b]}^{(i)} \pi_\varphi ((i, i + 1)) E_{[a,b]}^{(i)} = E_{[a,b]}^{(i)} E_{[a,b]}^{(i+1)} \pi_\varphi ((i, i + 1)) = E_{[a,b]}^{(i)} E_{[a,b]}^{(i+1)} \pi_\varphi ((i, i + 1)) E_{[a,b]}^{(i+1)}.
\]
Therefore,
\[
\left| \left( \pi_\varphi \left( (i, i+1) \right) E_{[a,b]}(i) \xi_\varphi, E_{[a,b]}(i+1) \xi_\varphi \right) \right| \\
= \left| \left( \pi_\varphi \left( (i, i+1) \right) E_{[a,b]}(i) E_{[a,b]}(i+1) \xi_\varphi, E_{[a,b]}(i+1) \xi_\varphi \right) \right| \\
\leq \left| \left( E_{[a,b]}(i) E_{[a,b]}(i+1) \xi_\varphi, \xi_\varphi \right) \right|^2 \tag{Lemma 22}.
\]

Hence, using (99), we obtain the statement of lemma 24. \(\square\)

Let \(P_0^{(i)}\) be the orthogonal projection on \(\text{Ker} \, O_i\). Put \(P^{(i)}_\pm = I - P_0^{(i)}\).

**Lemma 25.** Vector \(\xi_\varphi\) is separating for \(w^*\)-algebra \(P^{(j)}\mathfrak{A}_jP^{(j)}\).

**Proof.** Let \(V \in P^{(j)}\mathfrak{A}_jP^{(j)}\) and let \(V\xi_\varphi = 0\). It suffices to show that
\[
(\pi_\varphi (g) \xi_\varphi, O_j V^* \pi_\varphi (h) \xi_\varphi) = 0 \quad \text{for all} \quad g, h \in G. \tag{100}
\]

First we note that, by \(S_\infty\)-invariance \(\varphi\),
\[
\pi_\varphi (s) V \pi_\varphi (s^{-1}) \xi_\varphi = 0 \quad \text{for all} \quad s \in S_\infty. \tag{101}
\]

Further, if \(g \in G_N\) then for all \(n > N\)
\[
\pi_\varphi ((j n)) V^* \pi_\varphi ((j n)) \pi_\varphi (g) = \pi_\varphi (g) \pi_\varphi ((j n)) V^* \pi_\varphi ((j n)).
\]

Hence, using definition of \(O_j\) (see proposition 19),
\[
(\pi_\varphi (g) \xi_\varphi, O_j V^* \pi_\varphi (h) \xi_\varphi) = \lim_{n \to \infty} (\pi_\varphi (g) \xi_\varphi, \pi_\varphi ((j n)) V^* \pi_\varphi (h) \xi_\varphi)
\]
\[
= \lim_{n \to \infty} (\pi_\varphi ((j n)) V \pi_\varphi ((j n)) \xi_\varphi, \pi_\varphi (g^{-1}) \pi_\varphi ((j n)) \pi_\varphi (h) \xi_\varphi) \tag{101}
\]

Thus (100) is proved. \(\square\)

The following statement is well known for the case of separating vector \(\xi_\varphi\) (see [28]). In our case it follows from lemmas 24 and 25.

**Corollary 26.** There exist at most countable set of numbers \(\alpha_i\) from \([-1, 0) \cup (0, 1]\) and a set of the pairwise orthogonal projections \(\{P^{(j)}_{\alpha_i}\}\) \(\subset \mathfrak{A}_j\) such that
\[
O_j = P^{(j)}_0 + \sum_i \alpha_i P^{(j)}_{\alpha_i}. \tag{102}
\]

**Lemma 27.** Let \(\alpha, \beta \in \text{Spectrum} \, O_j\). If \(\alpha \beta < 0\) then \(P^{(j)}_{\alpha} \mathfrak{A}_j P^{(j)}_{\beta} = 0\).
Proof. By lemma 23 it suffices to show that
\[ P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi} = 0 \quad \text{for all } U \in \mathfrak{A}_j. \] (103)

First we note that
\[ \| P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi} \|^2 = \left( P^{(j)}_{\beta} U^{*} P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi}, \xi_{\varphi} \right) = \frac{1}{\alpha} \left( P^{(j)}_{\beta} U^{*} P^{(j)}_{\alpha} O_j U P^{(j)}_{\beta} \xi_{\varphi}, \xi_{\varphi} \right). \]

Hence, using proposition 23 we receive
\[ \| P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi} \|^2 = \frac{1}{\alpha} \left( P^{(j)}_{\beta} U^{*} P^{(j)}_{\alpha} (\xi_{j + 1}) P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi}, \xi_{\varphi} \right). \] (104)

It follows from lemma 21 that
\[ \| P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi} \|^2 = \frac{1}{\alpha} \left( P^{(j)}_{\beta} U^{*} P^{(j)}_{\alpha} \phi_{j + 1,j} \left( P^{(j)}_{\alpha} U P^{(j)}_{\beta} \right) \pi_{\varphi} ((j + 1)) \xi_{\varphi}, \xi_{\varphi} \right) = \frac{1}{\alpha} \left( \phi_{j + 1,j} \left( P^{(j)}_{\alpha} U P^{(j)}_{\beta} \right) \pi_{\varphi} ((j + 1)) \phi_{j + 1,j} \left( P^{(j)}_{\alpha} U P^{(j)}_{\beta} \right) \xi_{\varphi}, \xi_{\varphi} \right) = \frac{1}{\alpha} \left( P^{(j)}_{\alpha} U P^{(j)}_{\beta} \pi_{\varphi} ((j + 1)) P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi}, \xi_{\varphi} \right) \]

proposition 23 that
\[ \| P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi} \|^2 = \frac{1}{\alpha} \left( P^{(j)}_{\alpha} U P^{(j)}_{\beta} O_j P^{(j)}_{\alpha} U P^{(j)}_{\beta} \xi_{\varphi}, \xi_{\varphi} \right) \leq 0. \]

Therefore, (103) holds true. \qed

The next assertion is an analogue of the theorem 2 from [8].

Lemma 28. Let \( \alpha \neq 0 \) be the eigenvalue of operator \( O_j \) and let \( P^{(j)}_{\alpha} \) be the corresponding spectral projection. Take any orthogonal projection \( P \in P^{(j)}_{\alpha} \mathfrak{A}_j P^{(j)}_{\alpha} \) and put \( \nu(P) = (P \xi_{\varphi}, \xi_{\varphi}) / |\alpha|. \) Then \( \nu(P) \in \mathbb{N} \cup \{0\}. \)

Proof. We use the arguments of Kerov, Olshanski, Vershik [1] and Okounkov [8]. Let \( j = 1. \)

First consider the case \( \alpha > 0. \) For \( n \in \mathbb{N} \) put \( \eta_n = \prod_{m=0}^{n-1} \phi_{1+m,1}(P) \xi_{\varphi}. \) Let \( s \in \mathcal{G}_n. \) In each orbit \( p \in \mathbb{N} / s \) fix number \( s(p). \) Since \( \prod_{m=0}^{n-1} \phi_{1+m,1}(P) \) is an orthogonal projection and
\[ \pi_{\varphi}(s) \cdot \prod_{m=0}^{n-1} \phi_{1+m,1}(P) = \prod_{m=0}^{n-1} \phi_{1+m,1}(P) \cdot \pi_{\varphi}(s), \]

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then we have

$$(\pi \varphi(s) \eta_n, \eta_n) = \left(\pi \varphi(s) \prod_{m=0}^{n-1} \phi_{1+m,1}(P)\xi_{\varphi}, \xi_{\varphi}\right)$$

Lemma 22

$$\prod_{p \in \{N/s \subset [1,n]\}} \left(\pi \varphi(p) \prod_{k \in p} \phi_{k,j}(P)\xi_{\varphi}, \xi_{\varphi}\right)$$

Prop 23

$$\prod_{p \in \{N/s \subset [1,n]\}} \left(\phi_{s(p),1}(P) \cdot \mathcal{O}_{s(p)} \cdot \phi_{s(p),1}(P) \cdot \mathcal{O}_{s(p)} \cdots \mathcal{O}_{s(p)} \cdot \phi_{s(p),1}(P)\xi_{\varphi}, \xi_{\varphi}\right) = \prod_{p \in \{N/s \subset [1,n]\}} \right)^{|p|-1} \left(\phi_{s(p),1}(P)\xi_{\varphi}, \xi_{\varphi}\right) = \alpha^n \nu^{l(s)},$$

where $l(s)$ is the number of cycles in the decomposition of permutation $s$.

Now define orthogonal projection $\text{Alt}(n) \in \pi_\varphi(\mathfrak{S}_\infty)^\prime \subset \pi_\varphi(G)^\prime$ by

$$\text{Alt}(n) = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \text{sign}(s) \pi_\varphi(s).$$

Using (106), we obtain:

$$(\text{Alt}(n)\eta_n, \eta_n) = \alpha^n \sum_{s \in \mathfrak{S}_n} \text{sign}(s) \nu^{l(s)}.$$ (107)

In the same way as in [8], applying equality:

$$\sum_{s \in \mathfrak{S}_n} \text{sign}(s) \nu^{l(s)} = \nu(\nu - 1) \cdots (\nu - n + 1),$$

we have

$$0 \leq (\pi_\varphi(s)\eta_n, \eta_n) = \nu(\nu - 1) \cdots (\nu - n + 1).$$ (108)

Therefore, $\nu \in \mathbb{N} \cup \{0\}$.

The same proof remains for $\alpha < 0$. In above reasoning operator $\text{Alt}(n)$ it is necessary to replace by $\text{Sym}(n) = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \pi_\varphi(s)$.

For $\alpha \in \text{Spectrum } \mathcal{O}_j$ denote by $P^{(j)}_\alpha$ the corresponding spectral projection (see corollary 26). It follows from lemmas 25 and 28 that for $\alpha \neq 0$ $w^*$-algebra $P^{(j)}_\alpha \mathfrak{A}^{(j)}_j P^{(j)}_\alpha$ is finite dimensional. Therefore, there exists finite collection $\{P^{(j)}_{\alpha,i}\}_{i=1}^{n_\alpha} \subset P^{(j)}_\alpha \mathfrak{A}^{(j)}_j P^{(j)}_\alpha$ of the pairwise orthogonal projections with the properties:

$$P^{(j)}_{\alpha,i}\xi_{\varphi} \neq 0$$ and $P^{(j)}_{\alpha,i}$ is minimal for all $i = 1, 2, \ldots, n_\alpha$;

$$\sum_{i=1}^{n_\alpha} P^{(j)}_{\alpha,i} = P^{(j)}_\alpha.$$ (109)
Proof. It is suffice to prove that $P_0^{(j)} \xi_\varphi = 0$ for all nonzero $\alpha \in \text{Spectrum } \mathcal{O}_j$. But this fact follows from the next relations:

$$
\left( P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi \right) = \left( P_0^{(j)} U^* P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, \xi_\varphi \right) = \frac{1}{\alpha} \left( P_0^{(j)} U^* \mathcal{O}_j P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, \xi_\varphi \right)
$$

**Lemma 20**

$$
= \frac{1}{\alpha} \left( P_0^{(j)} U^* \pi_\varphi ((j - j + 1)) P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, \xi_\varphi \right)
$$

$$
= \frac{1}{\alpha} \left( P_0^{(j)} U^* \pi_\varphi ((j - j + 1)) P_\alpha^{(j)} U P_0^{(j)} \xi_\varphi, \xi_\varphi \right)
$$

$$
= \frac{1}{\alpha} \left( P_\alpha^{(j+1)} \cdot \phi_{j+1,j}(U) \cdot P_0^{(j+1)} \cdot P_0^{(j)} U^* \pi_\varphi ((j - j + 1)) P_\alpha^{(j)} \cdot \phi_{j+1,j} (U^*) \xi_\varphi, \xi_\varphi \right)
$$

**Lemma 20**

$$
= \frac{1}{\alpha} \left( P_\alpha^{(j+1)} \cdot \phi_{j+1,j}(U) \cdot P_0^{(j+1)} \cdot P_0^{(j)} U^* \pi_\varphi ((j - j + 1)) P_\alpha^{(j)} \cdot \phi_{j+1,j} (U^*) \xi_\varphi, \xi_\varphi \right) = 0.
$$

Put $\mathbb{H}_\text{reg}^{(j)} = \left[ \mathfrak{A}_j, P_0^{(j)} \xi_\varphi \right]$ and $\mathbb{H}_\pm^{(j)} = \left[ \mathfrak{A}_j, P_\pm^{(j)} \xi_\varphi \right]$. The next assertion follows from the previous proposition.

**Corollary 30.** (a) Subspaces $\mathbb{H}_\text{reg}^{(j)}$ and $\mathbb{H}_\pm^{(j)}$ are orthogonal for each $j \in \mathbb{N}$;

(b) if $\sum_{\alpha \in \text{Spectrum } \mathcal{O}_j : \alpha \neq 0} |\alpha| \cdot \nu \left( P_\alpha^{(j)} \right) = 1$ (see lemma 20) then $P_0^{(j)} \xi_\varphi = 0$.

**Proof.** Property (a) at once follows from proposition 20. To prove (b) we note that $1 = \left\| P_0^{(j)} \xi_\varphi \right\|^2 + \sum_{\alpha \in \text{Spectrum } \mathcal{O}_j : \alpha \neq 0} \left\| P_\alpha^{(j)} \xi_\varphi \right\|^2$ **Lemma 20** $\left\| P_0^{(j)} \xi_\varphi \right\|^2 = 0 + \sum_{\alpha \in \text{Spectrum } \mathcal{O}_j : \alpha \neq 0} \alpha \cdot \nu \left( P_\alpha^{(j)} \right)$. Therefore, $\left\| P_0^{(j)} \xi_\varphi \right\|^2 = 0$. □

**Lemma 31.** $\left( U \mathcal{O}_j V P_0^{(j)} \xi_\varphi, P_0^{(j)} \xi_\varphi \right) = 0$ for all $U, V \in \mathfrak{A}_j$. 

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Corollary 33. Let

\[ U_1 V P_0^{j+1} \xi_\varphi, P_0^{(j)} \xi_\varphi \]

The statement follows from the next relations:

Proof. The proof follows from the next relations:

\[
\begin{aligned}
&\left( U_1 V P_0^{(j)} \xi_\varphi, P_0^{(j)} \xi_\varphi \right) \\
&= \left( P_0^{(j)} \cdot U \cdot \phi_{j+1,j}(V) \cdot P_0^{(j+1)} \cdot \pi_\varphi (j+1) \xi_\varphi, \xi_\varphi \right) \\
&= \left( \phi_{j+1,j}(V) \cdot P_0^{(j+1)} \cdot U \cdot \pi_\varphi (j+1) \xi_\varphi, \xi_\varphi \right) \\
&= \left( \phi_{j+1,j}(V) \cdot P_0^{(j+1)} \cdot P_0^{(j)} \cdot \phi_{j+1,j}(U) \xi_\varphi, \xi_\varphi \right)
\end{aligned}
\]

Proposition 32. Let \( \{ P_{\alpha,i}^{(j)} \}_{i=1}^{n_\alpha} \) \((\alpha \in \{ \text{Spectrum } O_j \} \setminus 0)\) are the same as in \((109)\). If \( P_{\alpha,i}^{(j)} \cdot P_{\beta,k}^{(j)} = 0 \) then \( P_{\alpha,i}^{(j)} \cdot U \cdot P_{\beta,k}^{(j)} \xi_\varphi, \xi_\varphi \) = 0 for all \( U \in A_j \).

Proof. The statement follows from the next relations:

\[
\begin{aligned}
&\left( P_{\alpha,i}^{(j)} \cdot U \cdot P_{\beta,k}^{(j)} \xi_\varphi, \xi_\varphi \right) \\
&= \frac{1}{\alpha} \left( P_{\alpha,i}^{(j)} \cdot \phi_{j+1,j}(U) \cdot P_{\beta,k}^{(j+1)} \cdot \pi_\varphi (j+1) \xi_\varphi, \xi_\varphi \right)
\end{aligned}
\]

Now we give important

Corollary 33. Let \( P_+^{(j)} \) and \( P_-^{(j)} \) are the same as in proposition \(39\). Then subspaces \([A_j, P_+^{(j)} \xi_\varphi] \) and \([A_j, P_-^{(j)} \xi_\varphi] \) are orthogonal.

Proposition 34. Let \( \{ P_{\alpha,i}^{(j)} \}_{i=1}^{n_\alpha} \) \((\alpha \in \{ \text{Spectrum } O_j \} \setminus 0)\) are the same as in \((109)\). If there exists unitary \( U \in A_j \) such that \( U \cdot P_{\alpha,i}^{(j)} \cdot U^* = P_{\beta,k}^{(j)} \) then

\[
\left( P_{\alpha,i}^{(j)} \xi_\varphi, \xi_\varphi \right)_{\alpha} = \left( P_{\beta,k}^{(j)} \xi_\varphi, \xi_\varphi \right)_{\beta}.
\]

Proof. Let \( \kappa_\alpha = \left( P_{\alpha,i}^{(j)} \xi_\varphi, \xi_\varphi \right)_{\alpha} \) and \( \kappa_\beta = \left( P_{\beta,k}^{(j)} \xi_\varphi, \xi_\varphi \right)_{\beta} \). By lemma \(28\) \( \kappa_\alpha, \kappa_\beta \in \mathbb{N} \). Suppose for the convenience that \( j = 1 \). For any \( n \in \mathbb{N} \), using \((105)\)
and \([107]\), we obtain

\[
\left(\text{Alt}(n) \prod_{m=1}^{n} \phi_{m,1} \left( P_{\alpha,i}^{(1)} \right) \xi_{\varphi} \right) \left( \prod_{m=1}^{n} \phi_{m,1} \left( P_{\alpha,i}^{(1)} \right) \xi_{\varphi} \right) = |\alpha|^n \prod_{m=0}^{n-1} \left( \kappa_{\alpha} - m \right);\\
\left(\text{Alt}(n) \prod_{m=1}^{n} \phi_{m,1} \left( P_{\beta,k}^{(1)} \right) \xi_{\varphi} \right) \left( \prod_{m=1}^{n} \phi_{m,1} \left( P_{\beta,k}^{(1)} \right) \xi_{\varphi} \right) = |\beta|^n \prod_{m=0}^{n-1} \left( \kappa_{\beta} - m \right).
\]

This implies for \(n = \kappa_{\alpha} + 1\) that

\[
\left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1} \left( P_{\alpha,i}^{(1)} \right) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1} \left( P_{\alpha,i}^{(1)} \right) \xi_{\varphi} \right) = 0.
\]

Further, applying relation

\[
\text{Alt}(n) \cdot \prod_{m=1}^{n} \phi_{m,1}(a) = \prod_{m=1}^{n} \phi_{m,1}(a) \cdot \text{Alt}(n) \text{ (for all } a \in \mathfrak{A}_1),
\]

we get

\[
0 \leq \left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m,1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m,1}(U^*) \xi_{\varphi} \right) = \left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m,1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1}(U^*) \xi_{\varphi} \right) = \frac{1}{|\alpha|^{\kappa_{\alpha} + 1}} \left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m,1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1}(U^*) \xi_{\varphi} \right)
\]

\[
= \frac{1}{|\alpha|^{\kappa_{\alpha} + 1}} \left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m+\kappa_{\alpha}+1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1}(U^*) \xi_{\varphi} \right)
\]

\[
= \frac{1}{|\alpha|^{\kappa_{\alpha} + 1}} \left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m+\kappa_{\alpha}+1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1}(U^*) \xi_{\varphi} \right)
\]

\[
\leq \frac{1}{|\alpha|^{\kappa_{\alpha} + 1}} \left\| \text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m+\kappa_{\alpha}+1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1}(U^*) \xi_{\varphi} \right\|^{1/2}
\]

\[
= \frac{1}{|\alpha|^{\kappa_{\alpha} + 1}} \left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m+\kappa_{\alpha}+1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1}(U^*) \xi_{\varphi} \right)^{1/2}
\]

\[
\Theta_{\infty}\text{-centrality of } \varphi = \frac{1}{|\alpha|^{\kappa_{\alpha} + 1}} \left(\text{Alt}(\kappa_{\alpha} + 1) \prod_{m=1}^{\kappa_{\alpha}+1} P_{\alpha,i}^{(m)} \phi_{m+\kappa_{\alpha}+1}(U^*) \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} \phi_{m,1}(U^*) \xi_{\varphi} \right)^{1/2} = 0.
\]
Hence, applying (110), we have

$$
\left( \text{Alt}(\kappa_{\alpha} + 1) \cdot \prod_{m=1}^{\kappa_{\alpha}+1} P_{\beta,k}^{(m)} \xi_{\varphi}, \prod_{m=1}^{\kappa_{\alpha}+1} P_{\beta,k}^{(m)} \xi_{\varphi} \right)
\tag{112}
$$

$$
= |\beta|^{\kappa_{\alpha}+1} \kappa_{\beta} (\kappa_{\beta} - 1) (\kappa_{\beta} - 2) (\kappa_{\beta} - \kappa_{\alpha}) = 0.
$$

Therefore, \( \kappa_{\alpha} \geq \kappa_{\beta} \). Similarly, \( \kappa_{\alpha} \leq \kappa_{\beta} \). \( \square \)

4.3 The proof of theorem\(^{18} \) Now we will give the description of parameters \((A, \rho)\) from paragraph \(^2 \) corresponding to \( \varphi \).

First we describe the structure of \( w^*\)-algebra \( P_{\pm}^{(j)} A_j \), \(\exists\) where \( P_{\pm}^{(j)} \) is the orthogonal projection of \( [A_j \xi_{\varphi}] \) onto \( \left[ A_j P_{\pm}^{(j)} \xi_{\varphi} \right] + \).

Let \( C_{\pm}^{(j)} \) be the center of \( P_{\pm}^{(j)} A_j \). Denote by \( c(P) \in C_{\pm}^{(j)} \) the central support of projection \( P \in P_{\pm}^{(j)} A_j \). Let us prove that

$$
c \left( P_{\pm}^{(j)} \right) = P_{\pm}^{(j)}. \tag{112}
$$

Indeed, if \( F = P_{\pm}^{(j)} - c \left( P_{\pm}^{(j)} \right) \), then for all \( B \in A_j \) we have \( F B P_{\pm}^{(j)} \xi_{\varphi} = F B P_{\pm}^{(j)} \xi_{\varphi} = 0 \). Therefore, \( F = 0 \).

Since for any nonzero \( \alpha \in \{ \text{Spectrum } O_j \} \setminus 0 \) in \( P_{\alpha}^{(j)} A_j \) there exists finite collection \( \left\{ P_{\alpha,i}^{(j)} \right\}_{i=1}^{n_{\alpha}} \) of the minimal projections with properties \(^{19} \), then \( w^*\)-algebra \( P_{\pm}^{(j)} A_j P_{\pm}^{(j)} \) is \( *\)-isomorphic to the direct sum of full matrix algebras.

Thus, using \(^{12} \), we find the collection \( \left\{ F_m \right\}_{m=1}^{N} \) of pairwise orthogonal projections from \( C_{\pm}^{(j)} \) such that \( F_m \cdot P_{\pm}^{(j)} A_j \cdot F_m \) is a factor of the type \( I_{k_m} \). Denote \( F_m \cdot P_{\pm}^{(j)} A_j \cdot F_m \) by \( M_{k_m} \). That is \( P_{\pm}^{(j)} A_j P_{\pm}^{(j)} \) is isomorphic to \( M_{k_1} \oplus M_{k_2} \oplus \ldots \).

Let \( \left\{ e_{pq}^{(m)} \right\}_{p,q=1}^{k_m} \) be the matrix unit of \( M_{k_m} \). Without loss of the generality we suppose that for certain \( l_m \leq k_m \)

$$
\bigcup_{m} \left\{ e_{pq}^{(m)} \right\}_{p=1}^{l_m} \subset \bigcup_{\alpha \in \text{Spectrum } O_j, \alpha \neq 0} \left\{ P_{\alpha,i}^{(j)} \right\}_{i=1}^{n_{\alpha}} \quad \text{and} \quad \left\{ \left\{ e_{pq}^{(m)} \right\}_{p=l_m+1}^{k_m} \right\} \bigcap \left\{ \left\{ P_{\alpha,i}^{(j)} \right\}_{i=1}^{n_{\alpha}} \right\} = \emptyset.
\tag{113}
$$

By lemmas \(^25 \) \(^{28} \) and propositions \(^32 \) \(^{34} \) minimal projections \( \bigcup_{m} \left\{ e_{pq}^{(m)} \right\}_{p=1}^{l_m} \) satisfy the next conditions

\(^3 \) see page \(^26 \) for the definition of \( \mathfrak{A}_j \)

\(^4 \) \( P_{\pm}^{(j)} \) is defined in proposition \(^{29} \)
• (a) if \( e_{pp}^{(m)} \cdot B_j = \alpha_p \cdot e_{pp}^{(m)} \), where \( \alpha_p \in \text{Spectrum} \setminus 0 \), then there exists natural \( q_m \) such that \( \frac{[e_{pp}^{(m)}]_{\alpha_p}}{[\alpha_p]} = q_m \) for all \( p = 1, 2, \ldots, l_m \);

• (b) if \( p \neq q \) then \( (e_{pq}^{(m)} \xi_\varphi, \xi_\varphi) = 0 \) for all \( p, q = 1, 2, \ldots, l_m; \; m = 1, 2, \ldots, N \).

Further, using (113), for \( p > l_m \) we have

\[
e_{pp}^{(m)} \cdot P_0^{(j)} = e_{pp}^{(m)}.
\]

It follows from this and proposition 29 that

\[
(e_{pq}^{(m)} \xi_\varphi, \xi_\varphi) = 0 \quad \text{for} \quad p = 1, 2, \ldots, l_m; \; q = l_m + 1, l_m + 2, \ldots, k_m. \tag{114}
\]

Let us prove that

\[
(e_{pq}^{(m)} \xi_\varphi, \xi_\varphi) = 0 \quad \text{for} \quad p, q = l_m + 1, l_m + 2, \ldots, k_m. \tag{115}
\]

For this it suffices to prove the next equality:

\[
(e_{pp}^{(m)} \xi_\varphi, \xi_\varphi) = 0 \quad \text{for} \quad p, q = l_m + 1, l_m + 2, \ldots, k_m. \tag{116}
\]

Fix \( p > l_m \). Applying proposition 20 we have

\[
\begin{align*}
\left( e_{pp}^{(m)} \xi_\varphi, \xi_\varphi \right) &= \frac{1}{\alpha_1} \left( e_{pp}^{(m)} \cdot O_j \cdot e_{1p}^{(m)} \xi_\varphi, \xi_\varphi \right) \\
&= \frac{1}{\alpha_1} \left( e_{pp}^{(m)} (j, j + 1) \cdot \phi_{j+1,j} \left( e_{pp}^{(m)} \cdot e_{1p}^{(m)} \right) \xi_\varphi, \xi_\varphi \right) \\
&= \frac{1}{\alpha_1} \left( e_{pp}^{(m)} (j, j + 1) \cdot \phi_{j+1,j} \left( e_{pp}^{(m)} \right) \xi_\varphi, \xi_\varphi \right) \\
&= \lim_{n \to \infty} \frac{1}{\alpha_1} \left( e_{pp}^{(m)} (j, j + 1) \cdot \phi_{j+1,j} \left( e_{pp}^{(m)} \right) \xi_\varphi, \xi_\varphi \right) \\
&= \frac{1}{\alpha_1} \left( e_{pp}^{(m)} \cdot O_{j+1} \cdot e_{1p}^{(m)} \xi_\varphi, \xi_\varphi \right). \\
\end{align*}
\]

Thus (116) and (115) are proved.

Define \( \tilde{\varphi} \in \pi_\varphi (G)'' \) by \( \tilde{\varphi}(a) = (a \xi_\varphi, \xi_\varphi) \). Denote by \( M_{q_m} \) the algebra of all complex matrices and put \( N_m = M_{k_m} \otimes M_{q_m}, \; A^{(m)} = \sum_{p=1}^{l_m} \alpha_p \cdot \).
Consider $\mathcal{A}_j = \left( \bigoplus_{m=1}^{N} F_m \mathfrak{A}_m \mathcal{M}_{q_m} \right) \oplus \left( I - \widetilde{P}_j \right) \mathfrak{A}_j$. Observe that there exists the natural embedding

$$e_j = \sum_{m=1}^{N} (F_m a F_m \otimes I) + \left( I - \widetilde{P}_j \right) a \in \mathfrak{A}_j.$$ (117)

Now, using properties (a)-(b), (114) and (115), we have for all $a \in \mathfrak{A}_j$

$$\hat{\varphi}(a) = \sum_{m=1}^{N} \text{Tr}_m \left( a \left| A^{(m)} \right\rangle \otimes I \right) + \left( a \left( I - \widetilde{P}_j \right) \xi, \xi \right),$$ (118)

where $\text{Tr}_m$ is ordinary trace on $\mathcal{N}_m$.

Now we define parameters $\{ \mathcal{H}, A, \rho, \hat{\xi} \}$ from paragraph 2.1 such that

$$\varphi = \psi_A^{\rho} \quad (\text{see proposition } 11).$$ (119)

For this purpose we fix in each $\mathcal{N}_m = \mathcal{M}_{k_m} \otimes \mathcal{M}_{q_m}$ minimal projection $e_m$.

Define state $f$ on $\mathfrak{A}_j$ by

$$f(\tilde{a}) = \sum_{m=1}^{N} \text{Tr}_m (e_m \tilde{a} e_m) \quad (\tilde{a} \in \mathfrak{A}_j).$$ (120)

Let $(R_f, \mathcal{H}_f, \xi_f)$ be the corresponding GNS-representation of $\mathfrak{A}_j$. Now we define $\mathcal{H}$ by

$$\mathcal{H} = \mathcal{H}_f \oplus \left[ \left( I - \widetilde{P}_j^{(1)} \right) \mathfrak{A}_1 \xi_f \right] \oplus \left[ \left( I - \widetilde{P}_j^{(2)} \right) \mathfrak{A}_2 \xi_f \right] \oplus \ldots.$$ (121)

Representation $\rho$ acts on $\eta_p \in \left[ \left( I - \widetilde{P}_j^{(p)} \right) \mathfrak{A}_p \xi_f \right]$ as follows

$$\rho(\gamma) \eta_p = \pi_\varphi \left( \left( e, \ldots, \gamma, e, \ldots \right) \right) \eta_p.$$ (122)

If $\eta \in \mathcal{H}_f$ then

$$\rho(\gamma) \eta = R_f \circ i \left( \left( e, \ldots, \gamma, e, \ldots \right) \right) \eta.$$ (123)

Operator $A$ is defined by

$$A \eta = \begin{cases} R_f \circ i \left( \sum_{m=1}^{N} A^{(m)} \right) \eta, & \text{if } \eta \in \mathcal{H}_f, \\ 0, & \text{if } \eta \in \left[ \left( I - \widetilde{P}_j^{(p)} \right) \mathfrak{A}_p \xi_f \right]. \end{cases}$$ (124)

If $e$ is minimal projection from $\mathcal{N}_m$ then $\text{Tr}_m(e) = 1$. 

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In the case \( \sum_{\alpha \in \text{Spectrum} \mathcal{O}_j, \alpha \neq 0} |\alpha| \nu(P^{(j)}_\alpha) = \sum_{m=1}^{N} \sum_{p=1}^{k_m} |\alpha_p| < 1 \) (see corollary 30 and property (a)) vector \( \hat{\xi} \) is defined by

\[
\hat{\xi} = \frac{(I - \overline{P}^{(1)}_\pm)}{\left\| (I - \overline{P}^{(1)}_\pm) \xi_\varphi \right\|} \xi_\varphi.
\] (125)

Now it follows from (118) that for \( a \in \mathcal{A}_j \)

\[
\tilde{\varphi}(a) = \text{Tr} \left( (R_f (i(a)) \cdot |A|) + \left\| (I - \overline{P}^{(1)}_\pm) \xi_\varphi \right\| (\pi_\varphi ((1j)) \cdot a \cdot \pi_\varphi ((1j)) \hat{\xi}, \hat{\xi}) \right). \] (126)

Hence, applying lemma 22, proposition 23 and definition of \( \psi^A_\rho \), we can to receive equality (119). In particular, lemma 27 implies property (3) from paragraph 2.1.

References

[1] S.Kerov, G.Olshanski, A.Vershik, Harmonic analysis on the infinite symmetric group, RT-0312270.

[2] G.Olshanski, An introduction to harmonic analysis on the infinite symmetric group, RT-0311369.

[3] G.Olshanski, Unitary representations of \((G,K)\)-pairs connected with the infinite symmetric group \(S(\infty)\), Algebra i Analiz 1 (1989), no. 4, 178-209 (Russian); English translation in Leningrad Math. J. 1 (1990), no. 4, 983-1014.

[4] A.M. Vershik and S.V. Kerov, Asymptotic theory of characters of the infinite symmetric group, Funct. Anal. Appl., 15 (1981), 246-255.

[5] A.M. Vershik and S.V. Kerov, Characters and factor representations of the infinite symmetric group, Soviet Math. Dokl., 23 (1981), no. 2, 389–392.

[6] R.Boyer, Character theory of infinite wreath products, Inter. Journal of Mathematics and Math. Sciences, 9 (2005),1365-1379.

[7] A.Okounkov, The Thoma theorem and representation of the infinite bisymmetric group, Funct. Anal. Appl. 28 (1994), no. 2, 100-107.

[8] A.Okounkov, On the representation of the infinite symmetric group, RT-9803037.

[9] Dudko, A.; Nessonov, N. A description of characters on the infinite wreath product, arXiv: math.RT/0510597, 33pp.
[10] Dudko A. V., Nessonov N. I. *A description of characters on the infinite wreath product*, Methods of functional analysis and topology, Volume 13 (2007), Number 4, 301-317.

[11] G.Olshanski and A.Vershik, *Ergodic unitary invariant measures on the space of infinite Hermitian matrices*, Contemporary Mathematical Physics (R.L. Dobrushin, R.A. Minlos, M.A. Shubin, A.M. Vershik, eds.), American Mathematical Society Translations, Ser. 2, Vol. 175, Amer. Math. Soc., Providence, 1996, pp. 137-175.

[12] M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. 1, 1980, ACADEMIC PRESS, INS.

[13] E.Thoma, *Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen symmetrischen Gruppe*, Math. Zeitschr. 85 (1964), no.1, 40-61.

[14] Takesaki M., *Theory of Operator Algebras, v. I*, Springer, 2005, 415 pp.

[15] Takesaki M., *Theory of Operator Algebras, v. II*, Springer, 2005, 518 pp.

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