Some new $k$-Riemann–Liouville fractional integral inequalities associated with the strongly $\eta$-quasiconvex functions with modulus $\mu \geq 0$

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Abstract
A new class of quasiconvexity called strongly $\eta$-quasiconvex function was introduced in (Awan et al. in Filomat 31(18):5783–5790, 2017). In this paper, we obtain some new $k$-Riemann–Liouville fractional integral inequalities associated with this class of functions. For specific values of the associated parameters, we recover results due to Dragomir and Pearce (Bull. Aust. Math. Soc. 57:377–385, 1998), Ion (Ann. Univ. Craiova, Math. Sci. Ser. 34:82–87, 2007), and Alomari et al. (RGMIA Res. Rep. Collect. 12(Supplement):Article ID 14, 2009).

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1 Introduction
Let $I \subset \mathbb{R}$ be an interval, and let $I^\circ$ denote the interior of $I$. We say that a function $g : I \to \mathbb{R}$ is quasiconvex if

$$g(tx + (1 - t)y) \leq \max\{g(x), g(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For functions that are quasiconvex on $[a, b]$, Dragomir and Pearce [5] established the following inequality of the Hermite–Hadamard type.

Theorem 1 Let $g : [a, b] \to \mathbb{R}$ be a quasiconvex positive function. If $g \in L^1([a, b])$, then we have the following succeeding inequality:

$$\frac{1}{b - a} \int_a^b g(t) \, dt \leq \max\{g(a), g(b)\}. \quad (1)$$

Ion [8] obtained the following two results in the same direction.
Theorem 2 Let \( g : [a, b] \to \mathbb{R} \) be a differentiable function on \((a, b)\). If, in addition, the absolute value function \(|g'|\) is quasiconvex on \([a, b]\), then we have the following succeeding inequality:

\[
\frac{g(a) + g(b)}{2} - \frac{1}{b - a} \int_a^b g(t) \, dt \leq \frac{b - a}{4} \max \{|g'(a)|, |g'(b)|\}.
\] (2)

Theorem 3 Let \( g : [a, b] \to \mathbb{R} \) be a differentiable function on \((a, b)\). If, in addition, the absolute value function \(|g'|^{\frac{p}{2}}\) is quasiconvex on \([a, b]\) with \( p > 1 \), then we have the following succeeding inequality:

\[
\frac{g(a) + g(b)}{2} - \frac{1}{b - a} \int_a^b g(t) \, dt \leq \frac{b - a}{4(p + 1)} \left[ \max\{ |g'(a)|^\frac{p}{2}, |g'(b)|^\frac{p}{2} \} \right]^{\frac{p}{p+1}}.
\] (3)

Subsequently, Alomari et al. [2] obtained the following generalization of Theorem 2.

Theorem 4 Let \( g : I \to \mathbb{R} \) be a differentiable function on \( I \) with \( a, b \in I \) and \( a < b \). If, in addition, the absolute value function \(|g'|^q\) is quasiconvex on \([a, b]\), \( q \geq 1 \), then we have the following succeeding inequality:

\[
\frac{g(a) + g(b)}{2} - \frac{1}{b - a} \int_a^b g(t) \, dt \leq \frac{b - a}{4} \max\{ |g'(a)|^q, |g'(b)|^q \}^{\frac{1}{q}}.
\] (4)

Recently, Gordji et al. [6] introduced a new class of functions, called the \( \eta \)-quasiconvex functions. We present the definition for completeness.

Definition 5 A function \( g : I \subset \mathbb{R} \to \mathbb{R} \) is said to be an \( \eta \)-quasiconvex function with respect to \( \eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) if

\[
g(tx + (1-t)y) \leq \max\{g(y), g(y) + \eta(g(x), g(y))\}
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

For some results concerning the \( \eta \)-convex functions and related results, we refer the interested reader to the papers [4, 7, 9, 10, 12, 13, 15–17] and the references therein. Recently, Awan et al. [3] proposed the following definition, which gives a further generalization of Definition 5.

Definition 6 A function \( g : I \subset \mathbb{R} \to \mathbb{R} \) is said to be a strongly \( \eta \)-quasiconvex function with respect to \( \eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and modulus \( \mu \geq 0 \) if

\[
g(tx + (1-t)y) \leq \max\{g(y), g(y) + \eta(g(x), g(y))\} - \mu t(1-t)(y-x)^2
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

Example 7 The function \( g(x) = x^2 \) is strongly \( \eta \)-quasiconvex with respect to the bifunction \( \eta(x, y) = 2x + y \) and modulus \( \mu = 1 \). To see this, let \( t \in [0, 1] \). Then

\[
\max\{g(y), g(y) + \eta(g(x), g(y))\} - \mu t(1-t)(y-x)^2
\]
\[
\begin{align*}
&\geq g(y) + \eta(g(x),g(y)) - t(1-t)(y-x)^2 \\
&\geq y^2 + t(2x^2 + y^2) - t(1-t)(y-x)^2 \\
&= t^2x^2 + 2xty(1-t) + (1-t)^2y^2 + t(x^2 + 2y^2) \\
&\geq t^2x^2 + 2xty(1-t) + (1-t)^2y^2 \\
&= g(tx + (1-t)y).
\end{align*}
\]

**Remark 8** If \(g\) is strongly \(\eta\)-quasiconvex with respect to \(\eta(x,y) = x-y\) and modulus \(\mu = 0\), then Definition 6 reduces to the classical definition of the quasiconvexity.

Our purpose in this paper is to prove analogues of inequalities (1)–(4) for the strongly \(\eta\)-quasiconvex functions via the \(k\)-Riemann–Liouville fractional integral operators. We recapture these inequalities as particular cases of our results (see Remark 20).

We close this section by presenting the definition of the \(k\)-Riemann–Liouville fractional integral operators.

**Definition 9** (See [11]) The left- and right-sided \(k\)-Riemann–Liouville fractional integral operators \(k^\alpha \mathcal{I}_a^+\) and \(k^\alpha \mathcal{I}_b^-\) of order \(\alpha > 0\), for a real-valued continuous function \(g(x)\), are defined as

\[
k^\alpha \mathcal{I}_a^+ g(x) = \frac{1}{k \Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k} - 1} g(t) \, dt, \quad x > a, \tag{5}
\]

and

\[
k^\alpha \mathcal{I}_b^- g(x) = \frac{1}{k \Gamma_k(\alpha)} \int_b^x (t-x)^{\frac{\alpha}{k} - 1} g(t) \, dt, \quad x < b, \tag{6}
\]

where \(k > 0\), and \(\Gamma_k\) is the \(k\)-gamma function given by

\[
\Gamma_k(x) := \int_0^\infty t^{x-1} e^{-\frac{t^k}{x}} \, dt, \quad \text{Re}(x) > 0,
\]

with the properties \(\Gamma_k(x + k) = x\Gamma_k(x)\) and \(\Gamma_k(k) = 1\).

This paper is made up of two sections. In Sect. 2, our main results are framed and justified. Some new inequalities are also obtained as corollaries of the main results.

**2 Main results**

In what follows, we will use the following notation (where convenient): for \(g : [a,b] \to \mathbb{R}\) and \(\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), we define

\[
\mathcal{M}(g;\eta) := \max\{g(b), -g(b) + \eta(g(a),g(b))\}
\]

and

\[
\mathcal{N}(g;\eta) := \max\{g(a), -g(a) + \eta(g(b),g(a))\}.
\]

We now state and prove our first result of this paper.
Theorem 10  Let $\alpha, k > 0$, and let $g : [a, b] \to \mathbb{R}$ be a positive strongly $\eta$-quasiconvex function with modulus $\mu \geq 0$. If $g \in L_1([a, b])$, then we have the following inequality:

$$\frac{\Gamma_k^\alpha(a + k)}{2(b - a)^\frac{\alpha}{k}} \left[ L_{\alpha^+}^b g(b) + L_{\alpha^-}^a g(a) \right] \leq \frac{\mathcal{M}(g; \eta) + \mathcal{N}(g; \eta)}{2} - \alpha \mu (b - a)^2 \left( \frac{1}{\alpha + k} - \frac{1}{\alpha + 2k} \right).$$

Proof  The function $g$ is strongly $\eta$-quasiconvex on $[a, b]$ with $\mu \geq 0$. This implies that

$$g(ta + (1 - t)b) \leq \max \left\{ g(b), g(b) + \eta(g(a), g(b)) \right\} - \mu t (1 - t)(b - a)^2$$

$$= \mathcal{M}(g; \eta) - \mu t (1 - t)(b - a)^2 \tag{7}$$

and

$$g((1 - t)a + tb) \leq \max \left\{ g(a), g(a) + \eta(g(b), g(a)) \right\} - \mu t (1 - t)(b - a)^2$$

$$= \mathcal{N}(g; \eta) - \mu t (1 - t)(b - a)^2 \tag{8}$$

for all $t \in [0, 1]$.

By adding (7) and (8) we obtain

$$g(ta + (1 - t)b) + g((1 - t)a + tb)$$

$$\leq \mathcal{M}(g; \eta) + \mathcal{N}(g; \eta) - 2\mu t (1 - t)(b - a)^2. \tag{9}$$

Now, multiplying both sides of (9) by $t^{\frac{\alpha}{k} - 1}$ and thereafter integrating the outcome with respect to $t$ over the interval $[0, 1]$ give

$$\int_0^1 t^{\frac{\alpha}{k} - 1} g(ta + (1 - t)b) \, dt + \int_0^1 t^{\frac{\alpha}{k} - 1} g((1 - t)a + tb) \, dt$$

$$\leq \mathcal{M}(g; \eta) \int_0^1 t^{\frac{\alpha}{k} - 1} \, dt + \mathcal{N}(g; \eta) \int_0^1 t^{\frac{\alpha}{k} - 1} \, dt - 2\mu (b - a)^2 \int_0^1 t^{\frac{\alpha}{k} - 1} t(1 - t) \, dt$$

$$= \frac{2k}{\alpha} \left[ \mathcal{M}(g; \eta) + \mathcal{N}(g; \eta) \right] - \alpha \mu (b - a)^2 \left( \frac{1}{\alpha + k} - \frac{1}{\alpha + 2k} \right). \tag{10}$$

Using the substitutions $x = ta + (1 - t)b$ and $y = (1 - t)a + tb$ in the definition of the $k$-Riemann–Liouville fractional integrals, we obtain

$$\int_0^1 t^{\frac{\alpha}{k} - 1} g(ta + (1 - t)b) \, dt = \frac{1}{(b - a)^\frac{\alpha}{k}} \int_a^b (b - x)^{\frac{\alpha}{k} - 1} g(x) \, dx$$

$$= \frac{k \Gamma_k^\alpha(a)}{(b - a)^\frac{\alpha}{k}} \times L_{\alpha^+}^b g(b) \tag{11}$$

and

$$\int_0^1 t^{\frac{\alpha}{k} - 1} g((1 - t)a + tb) \, dt = \frac{1}{(b - a)^\frac{\alpha}{k}} \int_a^b (y - a)^{\frac{\alpha}{k} - 1} g(y) \, dy$$

$$= \frac{k \Gamma_k^\alpha(a)}{(b - a)^\frac{\alpha}{k}} \times L_{\alpha^-}^a g(a). \tag{12}$$
Employing (11) and (12) in (10), we get
\[
\frac{k \Gamma_k(\alpha)}{(b - a)^2} \left[ \mathcal{J}^\alpha_a g(b) + \mathcal{J}^\alpha_b g(a) \right]
\leq \frac{2k}{\alpha} \left[ \mathcal{M}(g; \eta) + \mathcal{N}(g; \eta) \right] - \alpha \mu (b - a)^2 \left( \frac{1}{\alpha + k} - \frac{1}{\alpha + 2k} \right).
\]

Hence the intended inequality is reached. \(\square\)

Setting \(\mu = 0\) in Theorem 10, we get the following corollary.

**Corollary 11** Let \(\alpha, k > 0\), and let \(g : [a, b] \to \mathbb{R}\) be a positive strongly \(\eta\)-quasiconvex function with modulus 0. If \(g \in L_1([a, b])\), then we have the following inequality:
\[
\frac{\Gamma_k(\alpha + k)}{2(b - a)^2} \left[ \mathcal{J}^\alpha_a g(b) + \mathcal{J}^\alpha_b g(a) \right] \leq \frac{\mathcal{M}(g; \eta) + \mathcal{N}(g; \eta)}{2}.
\]

The following lemmas will be useful in the proof of the remaining results of this paper.

**Lemma 12** Let \(\alpha, k > 0\), and let \(g : [a, b] \to \mathbb{R}\) be a differentiable function on the interval \((a, b)\). If \(g' \in L_1([a, b])\), then we have the following equality for the \(k\)-fractional integral:
\[
\frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^2} \left[ \mathcal{J}^\alpha_a g(b) + \mathcal{J}^\alpha_b g(a) \right] = \frac{b - a}{2} \int_0^1 \left[ (1 - t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right] g'(ta + (1 - t)b) \, dt.
\]

**Proof** The identity is achieved by setting \(s = 0\) in [1, Lemma 2.1]. \(\square\)

**Lemma 13** (See [14, 18]) If \(\sigma \in (0, 1]\) and \(0 \leq x < y\), then
\[
|x^\sigma - y^\sigma| \leq (y - x)^\sigma.
\]

**Theorem 14** Let \(\alpha, k > 0\), and let \(g : [a, b] \to \mathbb{R}\) be a differentiable function on \((a, b)\). If \(|g'|\) is strongly \(\eta\)-quasiconvex on \([a, b]\) with modulus \(\mu \geq 0\) and \(g' \in L_1([a, b])\), then we have the following inequality:
\[
\left| \frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^2} \left[ \mathcal{J}^\alpha_a g(b) + \mathcal{J}^\alpha_b g(a) \right] \right|
\leq \frac{b - a}{\frac{\alpha}{k} + 1} \left( 1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \mathcal{M}(|g'|; \eta)
\leq \mu (b - a)^3 \left[ \frac{1}{(\frac{\alpha}{k} + 2)} \left( 1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) - \frac{1}{(\frac{\alpha}{k} + 3)} \left( 1 - \frac{1}{2^{\frac{\alpha}{k}}} \right) \right] - \mu (b - a)^2 \left( \frac{1}{\alpha + k} - \frac{1}{\alpha + 2k} \right).
\]

**Proof** We start by making the following observations: for \(t \in [0, 1]\), we obtain
\[
(1 - t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \begin{cases} \geq 0, & 0 \leq t \leq \frac{1}{2}, \\ < 0, & \frac{1}{2} < t \leq 1 \end{cases}
\]

(14)
\[ \int_0^1 |(1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} | \, dt = \int_0^1 \left[ (1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} \right] \, dt + \int_0^1 \left[ t \frac{2}{(1 + t)} - (1 - t) \frac{2}{(1 + t)} \right] \, dt = \frac{2}{(\frac{2}{(1 + t)} + 1)} \left( 1 - \frac{1}{2}\right). \]  

Using a similar line of arguments (as previously), we obtain

\[ \int_0^1 t(1 - t) \left| (1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} \right| \, dt = \frac{2}{(\frac{2}{(1 + t)} + 2)} \left( 1 - \frac{1}{2}\right) - \frac{2}{(\frac{2}{(1 + t)} + 3)} \left( 1 - \frac{1}{2}\right). \]

Now, using the fact that \(|g'|\) is strongly \(\eta\)-quasiconvex with \(\mu \geq 0\) and then applying Lemma 12, the properties of the modulus, and identities (15) and (16), we obtain:

\[ \frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\alpha - 2)} \left[ \left| \int_a^b g(b) + b \int_a^b g(a) \right| \right] \leq \frac{b - a}{2} \int_0^1 \left| (1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} \right| |g'(ta + (1 - t)b)| \, dt \leq \frac{b - a}{2} \int_0^1 \left| (1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} \right| \left[ \max \{|g'(b)|, |g'(b)| + \eta(|g'(b)|, |g'(b)|)\} \right] \, dt \leq \frac{\mu t(1-t)(b-a)^2}{2} \, dt \leq \frac{b - a}{2} \max \{|g'(b)|, |g'(b)| + \eta(|g'(b)|, |g'(b)|)\} \int_0^1 \left| (1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} \right| \, dt \leq \frac{\mu (b-a)^3}{2} \int_0^1 \left| (1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} \right| \, dt \leq \frac{b - a}{2} \max \{|g'(b)|, |g'(b)| + \eta(|g'(b)|, |g'(b)|)\} \int_0^1 \left| (1 - t) \frac{2}{(1 + t)} - t \frac{2}{(1 + t)} \right| \, dt \leq \frac{\mu (b-a)^3}{2} \left[ \frac{2}{(\frac{2}{(1 + t)} + 2)} \left( 1 - \frac{1}{2}\right) - \frac{2}{(\frac{2}{(1 + t)} + 3)} \left( 1 - \frac{1}{2}\right) \right]. \]

Hence the result follows. \(\square\)

Putting \(\mu = 0\) in Theorem 14, we obtain the following result.

**Corollary 15** Let \(\alpha, k > 0\), and let \(g : [a, b] \to \mathbb{R}\) be a differentiable function on \((a, b)\). If \(|g'\) is strongly \(\eta\)-quasiconvex on \([a, b]\) with modulus \(0\) and \(g' \in L_1([a, b])\), then we have the following inequality:

\[ \frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(\alpha - 2)} \left[ \left| \int_a^b g(b) + b \int_a^b g(a) \right| \right] \leq \frac{b - a}{(\frac{2}{(1 + t)} + 1)} \max \{|g'(b)|, |g'(b)| + \eta(|g'(b)|, |g'(b)|)\}. \]

**Theorem 16** Let \(\alpha, k > 0, q > 1\), and let \(g : [a, b] \to \mathbb{R}\) be a differentiable function on \((a, b)\). If \(|g'|^q\) is strongly \(\eta\)-quasiconvex on \([a, b]\) with modulus \(\mu \geq 0\) and \(g' \in L_1([a, b])\), then we
have the following inequality:
\[
\left| \frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^\alpha} \left[ g^\alpha a + \frac{\Gamma_k}{\Gamma_k} g^\alpha b - g^\alpha a \right] \right| \\
\leq \frac{b-a}{2} \left( \frac{1}{\xi} \frac{\rho p + 1}{\xi} \right)^\frac{1}{\xi} \left( M(\eta) \frac{(b-a)^2}{6} \right)^\frac{1}{\xi},
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \frac{q}{\xi} \in (0, 1) \).

**Proof** As a consequence of Lemma 13, we have that
\[
|x^\xi - y^\xi| \leq |x - y|^\frac{q}{\xi}
\]
for all \( x, y \in [0, 1] \) with \( \frac{q}{\xi} \in (0, 1) \). Using the above information, we make the following computations:
\[
\int_0^1 |(1-t)^\frac{q}{\xi} - t^\frac{q}{\xi}|^p dt \leq \int_0^1 |1 - 2t|^\frac{q}{\xi}^p dt \\
= \int_0^{\frac{1}{2}} |1 - 2t|^\frac{q}{\xi}^p dt + \int_{\frac{1}{2}}^1 |1 - 2t|^\frac{q}{\xi}^p dt \\
= \int_0^{\frac{1}{2}} (1 - 2t)^\frac{q}{\xi}^p dt + \int_{\frac{1}{2}}^1 (2t - 1)^\frac{q}{\xi}^p dt \\
= \frac{1}{\xi} \frac{\rho p + 1}{\xi}.
\]
(18)

Since the function \( |g'|^\eta \) is strongly \( \eta \)-quasiconvex on \([a, b]\) with modulus \( \mu \geq 0 \), we have
\[
|g'(ta + (1 - t)b)|^\eta \leq \max\left\{ |g'(b)|^\eta, |g'(b)|^\eta + \eta (|g'(a)|^\eta, |g'(b)|^\eta) \right\} - \mu t(1 - t)(b - a)^2.
\]
(19)

Now, applying Lemma 12, the Hölder inequality, the properties of absolute values, and inequalities (18) and (19), we obtain
\[
\left| \frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b-a)^\alpha} \left[ g^\alpha a + \frac{\Gamma_k}{\Gamma_k} g^\alpha b - g^\alpha a \right] \right| \\
\leq \frac{b-a}{2} \int_0^1 |(1-t)^\frac{q}{\xi} - t^\frac{q}{\xi}| |g'(ta + (1-t)b)|^\eta dt \\
\leq \frac{b-a}{2} \left( \frac{1}{\xi} \frac{\rho p + 1}{\xi} \right)^\frac{1}{\xi} \left( \int_0^1 |g'(ta + (1-t)b)|^\eta dt \right)^\frac{1}{\xi} \\
\leq \frac{b-a}{2} \left( \frac{1}{\xi} \frac{\rho p + 1}{\xi} \right)^\frac{1}{\xi} \left( \int_0^1 \max\left\{ |g'(b)|^\eta, |g'(b)|^\eta + \eta (|g'(a)|^\eta, |g'(b)|^\eta) \right\} \\
- \mu t(1 - t)(b - a)^2 dt \right)^\frac{1}{\xi} \\
= \frac{b-a}{2} \left( \frac{1}{\xi} \frac{\rho p + 1}{\xi} \right)^\frac{1}{\xi} \left( \max\left\{ |g'(b)|^\eta, |g'(b)|^\eta + \eta (|g'(a)|^\eta, |g'(b)|^\eta) \right\} - \mu \frac{(b-a)^2}{6} \right)^\frac{1}{\xi}.
\]

This completes the proof. \( \square \)
Taking $\mu = 0$ in Theorem 16, we get the following:

**Corollary 17** Let $\alpha, k > 0, q > 1$, and let $g : [a, b] \to \mathbb{R}$ be a differentiable function on $(a, b)$. If $|g'|^q$ is strongly $\eta$-quasiconvex on $[a, b]$ with modulus $0$ and $g' \in L^1([a, b])$, then we have the following inequality:

$$
\frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^\frac{\alpha}{k}} \left[ I^\alpha_{a^+} g(b) + I^\alpha_{b^-} g(a) \right]
\leq \frac{b - a}{2} \left( \frac{1}{2^\frac{1}{q} + 1} \right) \left( \max\{|g'(b)|^q, |g'(b)|^q + \eta|g(a)|^q, |g'(b)|^q\} \right)^{\frac{1}{q}},
$$

(20)

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{q}{k} \in (0, 1]$.

Finally, we present the following result.

**Theorem 18** Let $\alpha, k > 0, q \geq 1$, and let $g : [a, b] \to \mathbb{R}$ be a differentiable function on $(a, b)$. If $|g'|^q$ is strongly $\eta$-quasiconvex on $[a, b]$ with modulus $\mu \geq 0$ and $g' \in L^1((a, b))$, then we have the following inequality:

$$
\frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^\frac{\alpha}{k}} \left[ I^\alpha_{a^+} g(b) + I^\alpha_{b^-} g(a) \right]
\leq \frac{b - a}{2} \left( P(\alpha; k) \right) \left( \mathcal{M}(|g'|^q; \eta) \mathcal{P}(\alpha; k) - \mu(b - a)^2 \mathcal{Q}(\alpha; k) \right)^{\frac{1}{q}},
$$

where

$$
P(\alpha; k) = \frac{2}{\left( \frac{q}{k} + 1 \right)} \left( 1 - \frac{1}{2^\frac{1}{q}} \right),
$$

and

$$
\mathcal{Q}(\alpha; k) = \frac{2}{\left( \frac{q}{k} + 2 \right)} \left( 1 - \frac{1}{2^\frac{1}{q} + 1} \right) - \frac{2}{\left( \frac{q}{k} + 3 \right)} \left( 1 - \frac{1}{2^\frac{1}{q} + 2} \right).
$$

**Proof** We follow similar arguments as in the proof of the previous theorem. For this, we use again Lemma 12, the Hölder inequality, and the properties of the absolute values to obtain

$$
\frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^\frac{\alpha}{k}} \left[ I^\alpha_{a^+} g(b) + I^\alpha_{b^-} g(a) \right]
\leq \frac{b - a}{2} \int_0^1 |(1-t)^\frac{\alpha}{k} - t^\frac{\alpha}{k}| |g'(ta + (1-t)b)| \, dt
\leq \frac{b - a}{2} \left( \int_0^1 |(1-t)^\frac{\alpha}{k} - t^\frac{\alpha}{k}| \, dt \right)^{\frac{1}{2}} \left( \int_0^1 |(1-t)^\frac{\alpha}{k} - t^\frac{\alpha}{k}| |g'(ta + (1-t)b)|^q \, dt \right)^{\frac{1}{q}}
\leq \frac{b - a}{2} \left( \int_0^1 |(1-t)^\frac{\alpha}{k} - t^\frac{\alpha}{k}| \, dt \right)^{\frac{1}{2}}
\times \left( \int_0^1 |(1-t)^\frac{\alpha}{k} - t^\frac{\alpha}{k}| \left[ \max\{|g'(b)|^q, |g'(b)|^q + \eta|g(a)|^q, |g'(b)|^q\} \right] - \mu t(1-t)(b-a)^2 \right) \, dt^{\frac{1}{q}}.
$$
The desired inequality follows by appealing to identities (15) and (16).

Taking \( \mu = 0 \) in Theorem 18, we get the succeeding corollary.

**Corollary 19** Let \( \alpha, k > 0, q \geq 1, \) and let \( g : [a, b] \to \mathbb{R} \) be a differentiable function on \( (a, b) \). If \( |g'|^q \) is strongly \( \eta \)-quasiconvex on \( [a, b] \) with modulus 0 and \( g' \in L_1([a, b]) \), then we have the following inequality:

\[
\left| \frac{g(a) + g(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^{\frac{1}{\alpha}}} \left[ \int_a^b |g'(b)|^q + k \int_a^b |g'(a)|^q \right] \right| \\
\leq \frac{b - a}{2} P(\alpha; k) \left( \max \left\{ \frac{|g'(b)|^q}{\eta(\alpha)}, \frac{|g'(b)|^q}{\eta(\alpha)} + \eta \left( \frac{|g'(a)|^q}{\eta(\alpha)}, \frac{|g'(b)|^q}{\eta(\alpha)} \right) \right\} \right)^{\frac{1}{q}},
\]

where

\[
P(\alpha; k) = \frac{2}{(\alpha k + 1)} \left( 1 - \frac{1}{2^\alpha} \right)
\]
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