A NOTE ON HELFFER-SJÖSTRAND REPRESENTATION FOR A GINZBURG-LANDAU PROCESS

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Abstract. In this work, we explore a link between an unbounded spin system and a random walk. This allows us to study the decay of the (co)variance of functions with respect to time. We extend here the previous work of T. Bodineau and G. Graham [1] to a more general class of graph and potential.

1. Introduction

A model of Ginzburg-Landau is a conservative model defined from a system of stochastic differential equations whose drift is given by a gradient (in the discrete sense) of a function, called the potential function. On each vertex $x$ of a graph $G$, we assign a real value $\eta_x$ called a mass which evolves according to the value its neighbors and random part given by Brownian motion. These kind of dynamics can be seen as hydrodynamic limit of particle system, see [24], and in this context has been studied in the 80’s with a series of article [8, 9, 10, 11, 12]. Central limit theorem has been broadly discussed, and we refer the reader to the Ph.D. thesis of J. Sheriff [22] which is fully dedicated to this subject. The spectral gap and the log-Sobolev inequality is also studied in particular in [1, 5, 18, 19].

In this work, following the approach of [1, 2], we use the Helffer-Sjöstrand representation to link the evolution of the spin system with a random walk in random dynamic environment. This representation, see [14, 15, 16, 17], relies on the commutation of an operator with a gradient giving rise to a second operator, called the Witten-Laplacian, which has, under some assumptions, a probabilistic interpretation. A closely related model, where this method has been used, is the $\nabla \varphi$ interface model, in [7], which is the same model as ours in dimension 1.

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Furthermore, the space-time correlation in \( \mathbb{Z}^d \) between two masses, in [24], is conjectured to behave in the following way:

\[
\text{Cov} (\eta_x (0) ; \eta_y (t)) \simeq \frac{C_1}{t^{d/2}} \exp \left( -\frac{|x - y|^2}{C_2 t} \right)
\]

where \( C_1 \) and \( C_2 \) are explicit. Using the connection between time-evolution of the covariance of masses and a random walk should lead to these estimates. As a short intuition, one can understand that the behavior of a random walk is encoded by the behavior of the diffusion of the mass; therefore, the behavior of the masses should be encoded by the behavior of a particle (of the masses).

In this work, we solve the difficulty encountered in [1] while trying to use the Helffer-Sjöstrand representation on general graph by introducing a site approach. The paper is organized in the following way. In the second section, we introduce the notations, the model and the results. In the third section, we give the representation and prove the main theorem. In the last section, we prove the auxiliary results.

2. Notations and Results

In this section, we introduce the notations, the model and the result.

Let \( G = (V, E) \) be a connected graph, where \( V \) is the set of vertices and \( E \) the set of unoriented edges. We assume that the degree of each vertex is uniformly bounded by a constant \( d \geq 0 \). Furthermore, we fix a vertex and call it origin, denoted \( 0 \). We note \( d_G \) the graph distance and \( |\cdot| \) the distance of a point to the origin. For practical reasons, we give an arbitrary orientation to the edges and note \( \overrightarrow{B} \) the set of these edges; we note \( \overleftarrow{B} \) the set of reversed oriented edges, i.e. \( \overleftarrow{B} = \{ (y, x) : (x, y) \in \overrightarrow{B} \} \). Furthermore, we introduce \( B = \overrightarrow{B} \cup \overleftarrow{B} \).

To standardize the notations, we write \( e = \{x, y\} \) for an unoriented edges and \( b = (x, y) \) for an oriented one (belonging to \( \overrightarrow{B} \) or \( \overleftarrow{B} \)). For two vertices \( x \) and \( y \), we note \( x \sim y \) iff the edge \( \{x, y\} \in E \). Moreover, for all \( x \in V \), we define \( sgn_x : B \to \{-1; 0; 1\} \) the function which assign to an oriented edge \( b = (y, z) \) associates :

\[
sgn_x (b) = \begin{cases} 
-1 & \text{if } y = x \\
1 & \text{if } z = x \\
0 & \text{else}
\end{cases}
\]

We call an environment, an element \( \eta = (\eta_x)_{x \in V} \in \Omega = \mathbb{R}^V \) and introduce the partial order \( \geq : \) for any two environments \( \sigma \) and \( \eta \),
\[ \sigma \geq \eta \] if and only if \( \forall x \in V, \sigma_x \geq \eta_x \). Thereby, we define the notion of increasing function, meaning that \( f \) is an increasing function if \( \forall \eta, \sigma : \)
\[ \sigma \geq \eta \Rightarrow f(\sigma) \geq f(\eta) \]
Let \((V_x)_{x \in V}\) be a family of functions such that \( \int \exp(-V_x(t)) dt = 1 \) which satisfied the following assumptions:

**Assumption 1** (Mean). There exists a constant \( M \) such that for all \( x \in V, \int_{\mathbb{R}} t \times \exp(-V_x(t)) dt = M. \)

**Assumption 2** (Potential \( C^2 \)). for all \( x \in V, V_x \in C^2. \)

**Assumption 3** (Strict convexity). There exists two constants \( C_+ \geq C_- > 0 \) such that for all \( \eta \in \Omega, x \in V, C_+ \geq V_x''(\eta) \geq C_- \).

We define the function \( H : \Omega \rightarrow \mathbb{R} \), called the potential function, given by \( \forall \eta, H(\eta) = \sum_{x \in V_x(\eta)}. \)

Finally, we introduce operators of partial derivatives for all \( x \in V, \partial_x = \frac{d}{d\eta_x} \), and for all \( b = (x, y) \in B, \partial_b = \partial_x - \partial_y. \) The dynamic of Ginzburg-Landau is defined by the following system of differential equations:

\[ d\eta_x(t) = \sum_{b \in \overrightarrow{B}} \text{sgn}_x(b) \left( \partial_b H(\eta) dt + \sqrt{2} dB_b(t) \right), x \in S \]

where \((B_b(t))_{b \in \overrightarrow{B}}\) is a family of independent Brownian motion indexed by the set of oriented edges. From these SDE, we extract the infinitesimal generator \( \mathcal{L}_e \) which describes the evolution of an environment. The subscript \( e \) has been chosen to signify that the operator describes the evolution of the environment. The generator is given by:

\[ \mathcal{L}_e f(\eta) = \frac{1}{2} \sum_{x,y:x \sim y} (\partial_x - \partial_y)^2 f(\eta) + \frac{1}{2} \sum_{x,y:x \sim y} (\partial_x - \partial_y) H(\eta) \times (\partial_x - \partial_y) f(\eta) \]
\[ = - \sum_{x,y:x \sim y} \partial_x (\partial_x - \partial_y) f(\eta) + \sum_{x,y:x \sim y} \partial_x H(\eta) \times (\partial_x - \partial_y) f(\eta) \]
\[ = - \sum_{b \in \overrightarrow{B}} \partial_b \partial_b f(\eta) + \partial_b H(\eta) \partial_b f(\eta) \]

where \( f \) is a local function, twice differentiable and with a finite norm \( \|f\| = \sum_{x \in S} \|\partial_x f\|_\infty; \) these three assumptions on the functions will
always be assumed in the rest of the article. We note for all generator $L$, the associated semigroup $\left( P^L_t \right) = \left( e^{-Lt} \right)_{t \geq 0}$, in particular, when there is no ambiguity, we will omit the superscript for the semigroup associated to the generator $L_e$, meaning that $\left( P^L_t \right)_{t \geq 0} = \left( P^L_{e}t \right)_{t \geq 0}$. A reversible measure of the Ginzburg-Landau process on a finite graph is the Gibbs measure $\mu$ given by:

$$d\mu(\eta) = \exp\left(-H(\eta)\right)d\eta = \prod_{x \in S} \exp\left(-V_x(\eta_x)\right)d\eta$$

This definition can be easily extended in the case of infinite graph under assumption 1, 2 and 3. For simplicity, we will use the probabilistic notations, meaning that $\mathbb{P}$ is the reversible measure, $\mathbb{E}$ the associated mean, and $\text{Cov}(f;g) = \mathbb{E}\left[(f - \mathbb{E}[f])(g - \mathbb{E}[g])\right]$. Since the model is conservative, note that $\mathbb{P}_\rho$, the law conditioned on a mass density $\rho := \rho(\eta) = \frac{1}{|V|} \sum_x \eta_x$ (defined for finite graph), is also a reversible measure.

We define the random walk $(X(t))_{t \geq 0}$ on the vertices whose jump rates are dependent of the environment $(\eta(t))_{t \geq 0}$ which evolve according to the Ginzburg-Landau process. For all time $t$, the random walk $X(t) \in S$ and the jump rate from its position to one of its neighbor is given by $V''(\eta_x(t))_{t \geq 0}$. The law describing the joint evolution of the walker and the environment is noted $\mathbb{P}_x^\sigma(\cdot) = \mathbb{P}(\cdot|X(0) = x; \eta(0) = \sigma)$. We write $L^p$ the generator of this random walk is therefore given by:

$$L^p f(x) = \sum_{z \sim x} V''(\eta_x)(f(z) - f(x)) \quad \forall (x, \eta) \in S \times \Omega$$

The subscript $p$ has been chosen to signify that the generator acts on the position. Finally, we define the generator $L = \text{Id} \otimes L_e + L_p \otimes \text{Id}$ describing the joint evolution of the walker and the environment id acting on the functions $F : S \times \Omega \to \mathbb{R}$ in the following way:

$$LF(x, \eta) = (\text{Id} \otimes L_e) F(x, \eta) + \left( L_p \otimes \text{Id} \right) F(x, \eta), \forall (x, \eta) \in S \times \Omega$$

(2.3)

To simplify the notations, we will write $L_e$ instead of $\text{Id} \otimes L_e$ and $L_p$ instead of $L_p \otimes \text{Id}$ so that we can write $L = L_e + L_p$.

The main theorem of this article is the following:
Theorem 2.1. Let $G = (V, E)$ be an infinite graph with bounded degree. Under assumptions $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$, $\forall x, y \in V$:

$$\text{Cov}(\eta_x; P_t \eta_y) \leq \frac{1}{C_-} \mathbb{E}[\mathbb{P}_x^n(X(t) = y)]$$

$$\text{Cov}(\eta_x; P_t \eta_y) \geq \frac{1}{C_+} \mathbb{E}[\mathbb{P}_x^n(X(t) = y)]$$

Remark 2.2. If we have space and time bounds on the coefficients $\mathbb{P}_x^n(X(t) = y)$, then, these bounds apply immediately to the Ginzburg-Landau model. Note that bounds can be (and some have been obtained in [13]) on $\mathbb{Z}^d$ following [4, 20]. Furthermore, if $C_- = C_+$, so that the potential is Gaussian, we obtain the equality instead of inequality above, and the walk is a simple random walk.

Remark 2.3. The uniform bounded degree assumption on the graph may not be necessary as well as the assumptions $\mathbf{1}$ and $\mathbf{3}$. Indeed, they guarantee the well definition of the model but they can be extended. The main result has the following corollary:

Corollary 2.4. Under the assumptions $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$, then for all function $f : \Omega \rightarrow \mathbb{R}$ such that

$$(2.4) \quad \|f\| = \sum_{x \in S} \|\partial_x f\|_{\infty} < +\infty$$

the following inequality holds:

$$\text{Cov}(f; P_t f) \leq \frac{1}{C_-} \|f\|^2 \sup_{x \in S} \mathbb{E}[\mathbb{P}_x^n(X_t = x)]$$

Furthermore, if $f$ and $g$ are two increasing functions then,

$$\text{Cov}(f; P_t g) \leq \frac{\|f\| \times \|g\|}{C_-} \sup_{x \in \text{supp}(f)} \sup_{y \in \text{supp}(g)} \mathbb{E}[\mathbb{P}_x^n(X_t = y)]$$

The next proposition, taken from [1], compares the spectral gap of the Ginzburg-Landau process with the spectral gap of the random walk. The proof is given for the sake of completeness.

Proposition 2.5. The spectral gap $\lambda^*_e$ of the Ginzburg-Landau Process given by the generator $\mathcal{L}_e$ is lower bounded by the spectral gap $\lambda^*_L$ of the random walk given by the generator $L$:

$$\lambda^*_e \geq \lambda^*_L$$

The spectral gap has been derived especially in [3, 5, 18]. In particular, in $(\mathbb{Z}/N\mathbb{Z})^d$, one can easily obtain a spectral gap of order $C/N^2$ with
a constant $C$ that depends on $C_-$ and $C_+$ using the Helffer-Sjöstrand representation.

3. THE REPRESENTATION OF HELFFER-SJÖSTRAND BY SITE

In this section, we prove start by proving the main theorem using two key lemmas which will be proved later.

3.1. Proof of the Theorem [2,1]. The two following key lemmas are inspired by [1]:

Lemma 3.1. For all $t \geq 0$ and all $x, y \in S$, we have the following equality:
\[
\text{Cov} (V'_x (\eta_x) ; P_t \eta_y) = \mathbb{E} [P^n_t (X (t) = y)]
\]

Lemma 3.2. For all increasing functions $f$ and $g$, we have the:
\[
\text{Cov} (f ; P_t g) \geq 0
\]

From these two key lemmas, the proof of the main theorem is immediate:

Proof of the theorem [2,1]. Consider two vertices $x$ and $y$ of $G$ and the functions $f : \Omega \rightarrow \mathbb{R}$ and $g : \Omega \rightarrow \mathbb{R}$ defined by:
\[
\begin{align*}
    f (\eta) &= \frac{1}{C_-} V' (\eta_x) - \eta_x \\
    g (\eta) &= \eta_y
\end{align*}
\]

Since $f$ and $g$ are increasing, using lemma 3.2 we obtain:
\[
\text{Cov} (f ; P_t g) \geq 0 \Rightarrow \text{Cov} (\eta_x ; P_t \eta_y) \leq \frac{1}{C_-} \text{Cov} (V' (\eta_x) ; P_t \eta_y)
\]

Then, using lemma 3.1 which concludes the proof of the first part of the theorem. Using the same reasoning with $f : \eta \mapsto \eta_x - \frac{1}{C} V'' (\eta_x)$, we obtain the second inequality which concludes the proof of [2,1].

3.2. Proof of Lemma [3,1]. In this section, we introduce the Helffer-Sjöstrand representation, a probabilistic interpretation of the intertwining technique. We recall briefly that the intertwining technique is the commutation of a generator and an operator. Here, we will commute $\mathcal{L}_e$ and $\partial_x$ so we get a second generator satisfying (informally) the relation $\partial_x \mathcal{L}_e = L \partial_x$. Recall that $L$ is the infinitesimal generator describing the joint evolution of the random walk and the environment. This relation is the key to interpret the evolution of the mass with the evolution of the random walk.

The Helffer-Sjöstrand representation is the following:
Lemma 3.3. Define $G(x, \eta) = \partial_x g(\eta)$ of some function $g$. We have the following inequality:

$$\partial_x \mathcal{L}_e g(\eta) = LG(x, \eta)$$

where $L$ is defined by (2.3). Consequently:

$$\partial_x P^L_t g(\eta) = P_t^L G(x, \eta)$$

Proof. By definition:

$$\partial_x \mathcal{L}_e g(\eta) = -\sum_b \partial_b \partial_b \partial_x g(\eta) + \sum_b \partial_b H(\eta) \times \partial_b \partial_x g(\eta)$$

$$+ \sum_b \partial_b \partial_x H(\eta) \times \partial_b g(\eta)$$

$$= L_c \partial_x g(\eta) + \sum_{y \sim x} V''(\eta_x) \times (\partial_x g(\eta) - \partial_y g(\eta))$$

$$= (L_c + L_p) \partial_x g(\eta) = LG(x, \eta)$$

The consequence is due to the fact $P_t^L = \exp(-Lt)$. □

We begin the proof of the lemma by recalling the formula of integration by part. :

$$E[f \times \mathcal{L}_e g] = E\left[ \sum_{x, y: x \sim y} \partial_x f \times (\partial_x - \partial_y) g \right]$$

Applying this formula with the functions $f : \eta \rightarrow V'_x(\eta_x)$ and $g : \eta \rightarrow \eta_y - E[\eta_y]$, defining the function $G : (x, \eta) \rightarrow \partial_x g(\eta)$ and noting that for all $h : \Omega \rightarrow \mathbb{R}$ we have $E[\mathcal{L}_e h] = 0$, we obtain:

$$\partial_t E[f \times P_t g] = E\left[ \sum_{y: x \sim y} V''(\eta_x) \times (P_t^L G(x, \eta) - P_t^L G(y, \eta)) \right]$$

$$= E\left[ \mathcal{L}_p P_t^L G(x, \eta) \right]$$

$$= E\left[ LP_t^L G(x, \eta) \right]$$

Which immediately implies that $E[f \times P_t g] = E[P_t^L G(x, \eta)]$. By the definition of $G$, we have for all $z$ and $\eta$ that $G(z, \eta) = 1_{y=z}$, therefore $P_t^L G(x, \eta) = P_{\eta}^0(X(t) = y)$ which concludes the proof of the lemma. □
3.3. Proof of Lemma 3.2. In this section, we prove Lemma 3.2. Given the independence assumption on the potential \( H(\eta) = \sum V_x(\eta_x) \), the Harris FKG-inequality would be enough if one can prove that the partial order is preserved with time. Indeed, in this case, if \( g \) is an increasing function, then \( P_t g \) is an increasing function and therefore the lemma is proved. This lemma is already proven in [1, Lemma 4.8], so we give here an improved version on a more general class of potential for the time preservation of the partial order following the same ideas.

In the next lemma, we consider that the potential there exists two families of functions of \( C^2(\mathbb{R}, \mathbb{R}) \) \( (V_x)_x \in S \) and \( (V_{x,y})_{x,y \in V, y \sim x} \) such that \( H(\eta) = \sum_{x,y \sim x} V_x(\eta_x) + V_{x,y}(\eta_x + \eta_y) \).

**Lemma 3.4.** Assume that there exists four constants \( C_{1,-}, C_{2,-}, C_{1,+} \) and \( C_{2,+} \) such that:

\[
C_{1,-} \leq V''_x \leq C_{1,+} \quad \text{and} \quad C_{2,-} \leq V''_{x,y} \leq C_{2,+}
\]

Then the partial order is time preserved if:

\[
\inf_x \inf_{y \sim x} \left( \sum_{z \sim y} C_{2,-} \right) - C_{2,+} + C_{1,-} \geq 0
\]

\[
C_{2,-} \geq 0
\]

**Proof.** We define two solutions \((\eta(t))_{t \geq 0}\) and \((\sigma(t))_{t \geq 0}\) of the system of SDE (2.1) driven by the same family of Brownian motions and such that \( \eta(0) \geq \sigma(0) \). We define for all \( x \in V \) the function \( \phi_x : \mathbb{R}_+ \to \mathbb{R} \), defined by \( \phi_x(t) = \eta_x(t) - \sigma_x(t) \). Of course, this is a continuous function \( \phi_x \). Then, we define the function \( \Phi : \mathbb{R}_+ \to \mathbb{R} \) by \( \Phi(t) = \sum_{x \in S} (2d)^{-|x|} \phi_x^2(t) \mathbb{1}_{\{\phi_x(t) < 0\}} \) where \( d \) is the uniform upper bound on the degree of the vertices of \( V \). Note that \( \Phi \) is continuous, positive, \( \Phi(0) = 0 \) and well defined, see [23, Théorème 2.1]. Intuitively, this function \( \Phi \) measure the number of negative \( \phi_x \). Using Gronwall Lemma, we will show that the function \( \Phi \) is equal to the zero constant function.

To simplify the notations, we will omit to write the dependence on \( t \) of the processes \( \eta(t) \) and \( \sigma(t) \) when there are no ambiguity and rather write \( \eta \) and \( \sigma \). In the same way, we will write \( \eta_x \) and \( \sigma_x \) instead \( \eta_x(t) \) and \( \sigma_x(t) \).
To establish the comparison lemma, we differentiate each summand of $\Phi$:

$$d\phi_x(t) = d(\eta_x(t) - \sigma_x(t))$$

$$= \sum_{y \sim x} \left[ \sum_{z \sim y} V'_{y,z}(\eta_y + \eta_z) - V'_{y,z}(\sigma_y + \sigma_z) \right]$$

$$- \left( V'_{x,y}(\eta_x + \eta_y) - V'_{x,y}(\sigma_x + \sigma_y) \right)$$

$$+ \sum_{y \sim x} V'(\eta_y) - V'(\sigma_y) - \left( V'(\eta_x) - V'(\sigma_x) \right)$$

Consider the case $\phi_x(t) < 0$, meaning that $\eta_x(t) \leq \sigma_x(t)$. Consequently, $d\phi_x(t)$ is an increasing function in $\sigma_x$ so we can bound it from below by setting $\sigma_x = \eta_x$. For each term, we measure its contribution in terms of $\phi_x(t)$. Observe that:

$$V'_{x,y}(\eta_x + \eta_y) - V'_{x,y}(\sigma_x + \sigma_y) = \int_{\sigma_y}^{\eta_y} V''_{x,y}(\eta_x + s) \, ds$$

$$V'_{y,z}(\eta_y + \eta_z) - V'_{y,z}(\sigma_y + \sigma_z) = V''_{y,z}(\eta_y + \eta_z) - V''_{y,z}(\sigma_y + \sigma_z)$$

$$+ \int_{\sigma_y}^{\eta_y} V''_{y,z}(s + \eta_z) \, ds + \int_{\sigma_z}^{\eta_z} V''_{y,z}(\sigma_y + s) \, ds$$

$$V'(\eta_y) - V'(\sigma_y) = \int_{\sigma_y}^{\eta_y} V''(s) \, ds$$

Thus:

$$d\phi_x(t) \geq \sum_{y \sim x} \int_{\sigma_y}^{\eta_y} \left[ \sum_{z \sim y} V''_{y,z}(s + \eta_z) \right] \, ds - V''_{x,y}(\eta_x + s) + V''_{y}(s) \, ds$$

$$+ \sum_{y \sim x} \sum_{z \sim y} \int_{\sigma_z}^{\eta_z} V''_{y,z}(\sigma_y + s) \, ds$$

Under the assumption of the lemma, the signs of the integrands are positive. Under the ellipticity assumptions on the potential, we get:

$$d\phi_x(t) \geq \sum_{y \sim x} \phi_y(t) \mathbf{1}_{\phi_y(t)<0} \left( \sum_{z \sim y} C_{2,+} - C_{2,-} + C_{1,+} \right)$$

$$+ \sum_{y \sim x} \sum_{z \sim y} \phi_z(t) \mathbf{1}_{\phi_z(t)<0} C_{2,+}$$
Recall that $\Phi$ is given by:

$$\Phi (t) = \sum_{x \in V} 2^{-|x|} \phi_x^2 (t) \mathbf{1}_{\{\phi_x (t) < 0\}}$$

By differentiating and writing $C_{\text{max}} = \max_{x,y : y \sim x} \left[ \sum_{z \sim y} C_{z,+} \right] - C_{2,-} + C_{1,+}$, we obtain:

$$\frac{d}{dt} \Phi (t) \leq \sum_{x \in V} 2^{1-|x|} \phi_x (t) \times \mathbf{1}_{\phi_x (t) < 0}$$

$$\times C_{\text{max}} \left( \sum_{y \sim x} \phi_y (t) \mathbf{1}_{\phi_y (t) < 0} + \sum_{z \sim y} \phi_z (t) \mathbf{1}_{\phi_z (t) < 0} \right)$$

Using the trivial inequality $\forall a, b \in \mathbb{R}, 2ab \leq a^2 + b^2$, then there exists a constant $C := C (C_{\text{max}}, d)$ such that:

$$\frac{d}{dt} \Phi (t) \leq C \times \sum_{x \in V} (2d)^{-|x|} \phi_x^2 (t) \mathbf{1}_{\phi_x (t) < 0}$$

$$\leq C \times \Phi (t)$$

Finally, using Gronwall Lemma, we obtain:

$$\Phi (t) \leq \Phi (0) \times \exp (C \times t)$$

Which concludes the proof of the lemma since $\Phi (0) = 0$. □

We can now prove the second key Lemma.

**Proof of Lemma 3.2.** As written above, it is enough to show that the Harris-FKG inequality holds. It relies on the sufficient condition of the theorem 3 of [21]. For any two configurations $\sigma$ and $\eta$, noting $(\eta \lor \sigma)_x = \max \{\eta_x, \sigma_x\}$ and $(\eta \land \sigma)_x = \min \{\eta_x, \sigma_x\}$, if

$$(3.2) \quad \mu (\eta \lor \sigma) \mu (\eta \land \sigma) \geq \mu (\sigma) \mu (\eta)$$

then the Harris-FKG inequality holds. This criterion if immediately verified since we are in the independent case, i.e.

$$V_x ((\eta \lor \sigma)_x) + V_x ((\eta \land \sigma)_x) = V_x (\eta_x) + V_x (\sigma_x), \quad \forall x \in V$$

Thus, the inequality $$(3.2)$$ is an equality in our case, which concludes the proof of the lemma. □

**Remark 3.5.** One can easily verified that the inequality $$(3.2)$$ is not true when $H$ can be written $H (\eta) = \sum_x V_x (\eta_x) + \sum_{y \sim x} V_{x,y} (\eta_x + \eta_y)$, where $V_{x,y}$ is a strictly convex function. Moreover, in this case, one can show that $\mathbb{E} [\eta_x \eta_y] < 0$ for $x \sim y$ and therefore the Harris-FKG can't be established.
4. Proof of Auxiliary Results

We start by proving corollary 2.4.

Proof. Because of the condition (2.4) on $f$, we have that $f$ is Lipschitz in all its coordinate. We write $L_f(x) = \|\partial_x f\|_\infty$ to get:

$$|f(\eta) - f(\eta^x)| \leq L_f(x) |\eta_x - \eta_x^x|$$

where $\eta^x$ is the configuration which all coordinates are equal to the ones of $\eta$ except (potentially) in $x$, i.e. $\forall y \neq x$ we have $\eta^x_y = \eta_y$. From this, we obtain for the functions $g_+$ and $g_-$ defined by $g_+(\eta) = \sum_x L_f(x) \eta_x + f(\eta)$ and $g_-(\eta) = \sum_x L_f(x) \eta_x - f(\eta)$ are increasing functions and using lemma 3.2:

$$\text{Cov}(P_t g_-; g_+) \geq 0 \iff \text{Cov} \left( P_t \sum_x L_f(x) \eta_x; \sum_y L_f(y) \eta_y \right) \geq \text{Cov}(P_t f; f)$$

which gives by the main theorem:

$$\text{Cov}(f; P_t f) \leq \sum_{x,y} L_f(x) L_f(y) \text{Cov}(\eta_x; P_t \eta_y)$$

$$\leq \sum_{x,y} L_f(x) L_f(y) \sup_x \text{Cov}(\eta_x; P_t \eta_x)$$

Using Cauchy-Schwartz inequality and concavity of the square root function, this proves the first part of the corollary. If $f$ and $g$ are two increasing functions, by lemma 3.2

$$\text{Cov} \left( \sum_x L_f(x) \eta_x; P_t \sum_y L_g(y) \eta_y \right) \geq \text{Cov} \left( f; P_t \sum_y L_g(y) V'_y(\eta_y) \right)$$

Expanding the sum, and taking the supremum over the $x$ and $y$ such that $L_f(x) \neq 0$ and $L_g(y) \neq 0$ and the main theorem, concludes the corollary. □

We then prove proposition 2.5.

Proof. Recall that $\lambda_L^*$ is the spectral gap of the random walk defined by:

$$\lambda_L^* = \inf_{f \neq 0} \frac{-\partial_{t=0} \|P_t f\|_2^2}{\|f\|_2^2}$$
where \( \|f\|_2^2 = \sum f^2(x) \). Therefore \( \|P^L_t f\|_2^2 \leq \|f\|_2^2 e^{-\lambda^*_L t} \), for all function \( f \). In this way,

\[
\mathbb{V}ar(P^L_t f) = \int_{\mathbb{R}^+} -\partial_s \mathbb{V}ar(P_s f) \, ds \\
= \int_{\mathbb{R}^+} \mathbb{E} \left[ \sum_{x,y} ((\partial_x - \partial_y) P_s f)^2 \right] \, ds \\
= \int_{\mathbb{R}^+} \mathbb{E} \left[ \sum_{x,y} (P^L_s (\partial_x - \partial_y) f)^2 \right] \, ds \\
\leq \int_{\mathbb{R}^+} e^{-\lambda^*_L s} \mathbb{E} \left[ \sum_{x,y} ((\partial_x - \partial_y) f)^2 \right] \, ds \\
\leq (\lambda^*_L)^{-1} \times (-\partial_t \mathbb{V}ar(P_t f))
\]

Taking the infimum over the function \( f \), one obtains:

\[
\lambda^*_L \leq \inf_f \frac{-\partial_{t=0} \mathbb{V}ar(P_t f)}{\mathbb{V}ar(f)} = \lambda^*_e
\]

Which ends the proof. \( \square \)

**Appendix A. Helffer-Sjöstrand Representation by edge**

In the appendix, we develop an extension of the original approach of [1], where the authors commuted the generator \( \mathcal{L}_e \) and \( \partial_b \) with \( b \in \mathbb{B} \). We give the approach, its difficulties and some associated results. A discussion of this approach can be found in the Ph.D. thesis of the author, see [6] (in french). Commuting \( \mathcal{L}_e \) and \( \partial_b \), one gets:

\[
\partial_b \mathcal{L}_e f = \partial_b \left( \sum_{b \in \mathbb{B}} -\partial_h \partial_b f + \partial_b H \times \partial_b f \right) \\
= \sum_{b \in \mathbb{B}} -\partial_h \partial_b \partial_b f + \partial_b \partial_b H \times \partial_b f + \partial_b H \times \partial_b \partial_b f \\
= \mathcal{L}_e \partial_b f + \sum_{b \in \mathbb{B}} \partial_b \partial_b H \times \partial_b f
\]

(4.1)

The authors argued that the second summand of (4.1) can only be interpreted as the (positive) generator of a random walk on the directed edges (of \( \mathbb{B} \)) on the torus (or \( \mathbb{Z} \)) since the Hessian of the potential function would have non negative off-diagonal term and positive diagonal term. However, noting for every \( b = (x, y) \in \mathbb{B} \) the edge with a
reversed orientation $\vec{b} = (y, x)$, one can note that $\partial_b = -\partial_{\vec{b}}$ so we get:

$$
\sum_{b \in B} \partial_b \partial_{b'} H \times \partial_b f = \sum_{\substack{b \in B \\ b \sim b'}} \partial_b \partial_{b'} H (\partial_b f - \partial_{b'} f) + (\partial_{b'} \partial_b H - \sum_{\substack{b \in B \\ b \sim b'}} \partial_b \partial_{b'} H) \partial_b f
$$

(4.2)

$$
= \mathcal{L}_p' \partial_{b'} f + (\partial_{b'} \partial_b H - \sum_{\substack{b \in B \\ b \sim b'}} \partial_b \partial_{b'} H) \partial_{b'} f
$$

Where $b \sim b'$ means $b' \neq \vec{b}$ and $\partial_b \partial_{b'} H < 0$. On $\mathbb{Z}$, note that $(\partial_{b'} \partial_b H - \sum_{\substack{b' \sim b \in B \\ b \sim b'}} \partial_b \partial_{b'} H) = 0$. Furthermore one can see that $\mathcal{L}_p'$ is the generator of a random walk on the set of oriented edges $B$. To continue the discussion, we can restrict ourselves to the case where the graph is $\mathbb{Z}^2$ and the Gaussian potential $"H(\eta) = \frac{1}{2} \sum \eta_x^2"$. In this case, the walker follows the law of a simple random walk on the oriented "kite" graph, drawn with unoriented edges in figure 4.1.

Figure 4.1. $\mathbb{Z}^2$ and the kite graph

We have the following relation:

$$
(4.2) = \mathcal{L}_p' \partial_{b'} f - 4\partial_b f
$$

So we obtain the following relation:

$$
\partial_b \mathcal{L}_e f = (\mathcal{L}_e + \mathcal{L}_p' - 4Id) \partial_b f = (L' - 4Id) F(b, \cdot)
$$

$$
\Rightarrow \partial_b P_t f = e^{4t} P_t^{L'} F(b, \cdot)
$$

Where $F(b, \cdot) = \partial_b f$. Before stating the next proposition, we define $(X'(t))$, the simple random walk on the oriented kite graph of generator $\mathcal{L}_p'$. A result obtained in [6], where the proof can be found, is the following:
Proposition 4.1. On $\mathbb{Z}^2$ with a Gaussian potential, for any oriented edge $b \in B$, we have the following relation:

$$
\mathbb{E} \left[ \frac{1}{2} \eta_x^2 \right] = e^{4t} \left( \mathbb{P}_b(X'(t) = b) - \mathbb{P}_b(X'(t) = \bar{b}) \right)
$$

The result can be generalized to a more general class of potential, i.e. the potential that can be written $H(\eta) = \sum V_0(\eta_x)$ where $V_0 \in C^2(\mathbb{R}, \mathbb{R})$, $V''_0 \geq C_- \in \mathbb{R}$ and $\lim_{|t| \to \infty} V_0(t) = +\infty$. However, one can see that this quantity is hard to exploit since we need to compensate the exponential term.

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