Upper and Lower Solutions for Second-Order with M-Point Impulsive Boundary Value Problems

Ilkay Yaslan Karaca, Dondu Oz

Department of Mathematics, Ege University, 35100, Bornova, Izmir, Turkey

Abstract. In this paper, we consider a second-order m-point impulsive boundary value problem. By applying the upper and lower solutions method and the Schauder’s fixed point theorem, we obtain the existence of at least one positive solution. We also give an example to illustrate our main result.

1. Introduction

The theory of impulsive differential equations is a new and significant subsection of differential equation theory, which has an extensive physical ecology, biological systems, population dynamics, and engineering background. Moreover, impulsive differential equations advance a more realistic approach to modeling many tangible problems encountered in various fields such as control theory, electronics, mechanics, economics, electrical circuits and medicine. For the introduction of the fundamental theory of impulsive equations, see \cite{1, 4, 6, 22, 24} and the papers \cite{2, 21}.

Some authors in the literature have obtained results about the solutions of second-order impulsive boundary value problems, for some see \cite{5, 10–14, 16, 19, 20, 23, 27} in the references. Recently, some works, such as Liu and Yu \cite{3}, Meiqiang and Dongxiu \cite{8}, Zhao et al. \cite{18} and Tian and Liu \cite{26}, deal with second-order m-point impulsive boundary value problems. Yet, many of the results are usually obtained using fixed point theorems on the cone. There are also other methods to prove the existence of solutions. The upper and lower solutions method, in particular, is a powerful method for proving the existence of results for boundary value problems, for some see \cite{7, 9, 17, 18, 25}. There is no study on second-order with m-point impulsive boundary value problems using the method upper and lower solutions, except that in \cite{18}.

In \cite{18}, Zhao and Ge considered

\[
\begin{align*}
\left(\phi_p(u'(t))\right)' + q(t)f(t, u(t), u'(t)) &= 0, \quad t \in J' = [0, 1]\setminus\{t_1, t_2, ..., t_n\} \\
\Delta u|_{t=t_k} &= I_1(k u(t_k)), \quad k = 1, 2, ..., n \\
\Delta \phi_p(u')|_{t=t_k} &= I_2(k u(t_k), u'(t_k)), \quad k = 1, 2, ..., n \\
u(0) &= \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u'(1) = 0
\end{align*}
\]

**Mathematics Subject Classification.** Primary 34B37; Secondary 34B15, 34B18, 34B40.

**Keywords.** Impulsive boundary value problems, positive solutions, m-point, upper and lower solutions, fixed point theorems, Nagumo’s Conditions.

Received: 17 May 2021; Accepted: 05 June 2021

Communicated by Maria Alessandra Ragusa

The second author was granted a fellowship by the Scientific and Technological Research Council of Turkey (TUBITAK-2211-A).

Email addresses: ilkay.karaca@ege.edu.tr (Ilkay Yaslan Karaca), dondu.oz@ege.edu.tr (Dondu Oz)
by using the upper and lower solutions method, the authors have existence of solutions to the above boundary value problem.

Motivated by the mentioned above result, in this study, we consider the following second-order m-point impulsive boundary value problem (IBVP)

\[
\begin{align*}
\Delta z(t) + f(t, z(t), z'(t)) &= 0, \quad t \in J = [0, 1], t \neq t_k, k = 1, 2, \ldots, n, \\
\Delta z|_{t=t_k} &= I_k(z(t_k)), \\
\Delta z'|_{t=t_k} &= -\tilde{I}_k(z(t_k), z'(t_k)), \\
\end{align*}
\]

(1)

where

\[
\begin{align*}
\|z\| &= \sup_{t \in [0, 1]} |z(t)|, \\
\|z\|_{PC} &= \sup_{t \in J} |z(t)|, \\
\|z\|_{C^1} &= \sup_{t \in J} |z'(t)|, \\
\|z\|_{C^2} &= \sup_{t \in J} |z''(t)|. \\
\end{align*}
\]

We assume that following conditions are provided throughout this paper.

(K1) \(a, b \in [0, \infty), \; \gamma_j, \delta_j \in [0, \infty), \; \varsigma_j \in (0, 1), \; \) for \(j \in [1, m-2]\),

(K2) \(f \in C(J \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)\),

(K3) \(I_k \in C(\mathbb{R}^+, \mathbb{R}^+)\) is a bounded function, \(\tilde{I}_k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)\) such that

\[
b + a(1 - t_k)\tilde{I}_k(z(t_k), z'(t_k)) > aI_k(z(t_k)), \quad t < t_k, \; k = 1, 2, \ldots, n.
\]

Our aim in this article is to investigate the existence of positive solutions for the IBVP (1). For this, Schauder’s Fixed Point Theorem and the upper and lower solutions method will be the main tool.

The main structure of this paper is as follows. In Section 2, we provide some definitions and preliminary lemmas which will be used later. In Section 3, we prove our main result. In Section 4, we give an example to demonstrate our main result.
The function \( \overline{z} \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \) is called an upper solution for the IBVP (1) if
\[
\begin{align*}
\left[ \overline{z}'(t) + f(t, \overline{z}(t), \overline{z}'(t)) \right] & \leq 0, \quad t \in J = [0, 1], t \neq t_k, k = 1, 2, ..., n, \\
\Delta \overline{z}|_{t=t_k} & = I_k(\overline{z}(t_k)), \\
\Delta \overline{z}'|_{t=t_k} & = -I_k(\overline{z}(t_k), \overline{z}'(t_k)), \\
\overline{z}(0) & \geq \sum_{j=1}^{m-2} \gamma_j \overline{z}(\zeta_j), \quad a\overline{z}(1) + b\overline{z}'(1) \geq \sum_{j=1}^{m-2} \delta_j \overline{z}(\zeta_j).
\end{align*}
\]

For \( \underline{z}, \overline{z} \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \), we can write \( \underline{z} \leq \overline{z} \) if \( \underline{z}(t) \leq \overline{z}(t) \), \( \forall t \in J \).

**Definition 2.2.** Let \( \underline{z}, \overline{z} \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \) be such that \( \underline{z} \leq \overline{z} \) on \( J \). We express that \( f \) provides the Nagumo condition with respect to \( \underline{z}, \overline{z} \) if for
\[
\varrho = \max_{1 \leq k \leq n} \left\{ \frac{\overline{z}(t_{k+1}) - \overline{z}(t_k)}{t_{k+1} - t_k}, \frac{\underline{z}(t_{k+1}) - \underline{z}(t_k)}{t_{k+1} - t_k} \right\},
\]
there exists a constant \( \sigma \) such that
\[
\sigma > \max_{1 \leq k \leq n} \| \varrho \|_{PC}, \| \overline{z} \|_{PC}, \| \underline{z} \|_{PC},
\]
a continuous function \( \varphi : [0, \infty) \to [0, \infty) \) and constants \( D \geq 0, E \geq 0 \) such that
\[
|f(t, z, z')| \leq D|z'|\varphi(|z'|) + E, \quad t \in J, \underline{z} \leq z, \overline{z} \in \mathbb{R}^+
\]
and
\[
\int_{t}^{t'} \frac{1}{\varphi(s)} ds > D[\max_{t \in J} \overline{z}(t) - \min_{t \in J} \underline{z}(t)] + E \max_{z \leq \overline{z}} 1 \frac{1}{\varphi(s)}.
\]

Also, suppose that the following conditions are provided:

(K4) \( f \) provides the Nagumo condition with respect to \( \underline{z}, \overline{z} \);

(K5) \( I_k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+), \tilde{I}_k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+) \), \( \tilde{I}_k(z, z') \) is nondecreasing in \( z' \in [-\sigma, \sigma] \) for all \( 1 \leq k \leq n; \)

(K6) \( \sum_{j=1}^{m-2} \delta_j < a, \sum_{j=1}^{m-2} \gamma_j < 1. \)

We consider the modified problem
\[
\begin{align*}
\left[ z''(t) + f'(t, z(t), \frac{d}{dt}m_{z}(t, z)) \right] & \leq 0, \quad t \in J = [0, 1], t \neq t_k, k = 1, 2, ..., n, \\
\Delta z|_{t=t_k} & = I_k(m_{z}(t_k, z(t_k))), \\
\Delta z'|_{t=t_k} & = -I_k(m_{z}(t_k, z(t_k)), n(t_k, z'(t_k))), \\
z(0) & = \sum_{j=1}^{m-2} \gamma_j z(\zeta_j), \quad az(1) + bz'(1) = \sum_{j=1}^{m-2} \delta_j z(\zeta_j)
\end{align*}
\]
where
\[
f'(t, z, z') = f(t, m_{z}(t, z), n(t, z')) + \frac{m_{z}(t, z) - z}{1 + (z - m_{z}(t, z))^2}
\]
and \( m_{z}(t, z) = \max(\overline{z}(t), \min(z, \underline{z}(t))), n(t, z) = \max(-\sigma, \min(z, \sigma)). \)
Lemma 2.3. [23] For each $z \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$, the following two properties hold:

(i) $\frac{d}{dt}m_z(t, z)$ exists for a.e. $t \in J'$;

(ii) if $z_n \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ and $z_n \to z \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$, then $\frac{d}{dt}m_z(t, z_n(t)) \to \frac{d}{dt}m_z(t, z(t))$ for a.e. $t \in J'$.

Set

$$\rho = a + b,$$

and

$$\Delta = \left| \begin{array}{cc}
- \sum_{j=1}^{m-2} \gamma_j \zeta_j & \rho - \sum_{j=1}^{m-2} \gamma_j [b + a(1 - \zeta_j)] \\
\rho - \sum_{j=1}^{m-2} \delta_j \zeta_j & - \sum_{j=1}^{m-2} \delta_j [b + a(1 - \zeta_j)]
\end{array} \right|. \quad (4)$$

Lemma 2.4. Let (K1)-(K3) hold. Assume that (K7) $\Delta \neq 0$.

If $z \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is a solution of the equation

$$z(t) = \int_0^1 G(t, s) f(s, z(s), z'(s)) ds + \sum_{k=1}^n W_k(t, z) + tA(f) + (b + a(1-t))B(f), \quad (5)$$

where

$$W_k(t, z) = \frac{1}{\rho} \left\{ \begin{array}{ll}
[t - a_k(z(t_k))] + (b + a(1 - t_k))\tilde{z}_k(z(t_k), z'(t_k)), & t < t_k, \\
(b + a(1 - t)[t_k(z(t_k)) + t_l \tilde{z}_k(z(t_k), z'(t_k))], & t_k \leq t,
\end{array} \right. \quad (6)$$

$$G(t, s) = \frac{1}{\rho} \left\{ \begin{array}{ll}
s[a(1-t) + b], & s \leq t, \\
t[a(1-s) + b], & t \leq s,
\end{array} \right. \quad (7)$$

$$A(f) = \frac{1}{\Delta} \left| \begin{array}{cc}
\sum_{j=1}^{m-2} \gamma_j K(z_j, z) & \rho - \sum_{j=1}^{m-2} \gamma_j [b + a(1 - \zeta_j)] \\
\sum_{j=1}^{m-2} \delta_j K(z_j, z) & - \sum_{j=1}^{m-2} \delta_j [b + a(1 - \zeta_j)]
\end{array} \right|, \quad (8)$$

$$B(f) = \frac{1}{\Delta} \left| \begin{array}{cc}
- \sum_{j=1}^{m-2} \gamma_j \zeta_j & \sum_{j=1}^{m-2} \gamma_j K(z_j, z) \\
\rho - \sum_{j=1}^{m-2} \delta_j \zeta_j & \sum_{j=1}^{m-2} \delta_j K(z_j, z)
\end{array} \right|, \quad (9)$$

and

$$K(z_j, z) = \int_0^1 G(z_j, s) f(s, z(s), z'(s)) ds + \sum_{k=1}^n W_k(z_j, z), \quad (10)$$

then $z$ is a solution of the IBVP (1).
Proof. Let \( z \) satisfies the integral equation (5), then we get
\[
z(t) = \int_0^t G(t, s)f(s, z(s), z'(s))ds + \sum_{k=1}^n W_k(t, z) + tA(f) + (b + a(1 - t))B(f),
\]
i.e.,
\[
z(t) = \frac{1}{\rho} \int_0^t s(b + a(1 - t))f(s, z(s), z'(s))ds + \frac{1}{\rho} \int_t^1 \rho(s(1 - s))f(s, z(s), z'(s))ds
\]
\[
+ \frac{1}{\rho} \sum_{0 < i < j < t} (b + a(1 - t))[I_k(z(t_k)) + t_k I_k(z(t_k), z'(t_k))]
\]
\[
+ \frac{1}{\rho} \sum_{t_k < t} \rho(b + a(1 - t_k))I_k(z(t_k), z'(t_k))
\]
\[
+ tA(f) + (b + a(1 - t))B(f),
\]
\[
z'(t) = \frac{1}{\rho} \int_0^t (-as)f(s, z(s), z'(s))ds + \frac{1}{\rho} \int_t^1 (b + a(1 - s))f(s, z(s), z'(s))ds
\]
\[
+ \frac{1}{\rho} \sum_{0 < i < j < t} (-a)[I_k(z(t_k)) + t_k I_k(z(t_k), z'(t_k))]
\]
\[
+ \frac{1}{\rho} \sum_{t_k < t} [(-a)I_k(z(t_k)) + (b + a(1 - t_k))I_k(z(t_k), z'(t_k))]
\]
\[
+ A(f) + (-a)B(f).
\]
So that
\[
z''(t) = \frac{1}{\rho} (-at - (b + a(1 - t)))f(t, z(t), z'(t))
\]
\[
= -f(t, z(t), z'(t)),
\]
\[
z''(t) + f(t, z(t), z'(t)) = 0.
\]
Since
\[
z(0) = \rho B(f),
\]
we have that
\[
z(0) = \rho B(f)
\]
\[
= \sum_{j=1}^{n-2} \gamma_j \left[ \int_0^1 G(z_j, s)f(s, z(s), z'(s))ds + \sum_{k=1}^n W_k(z_j, z) + z_j A(f) + (b + a(1 - z_j))B(f) \right].
\]
Since
\[
z(1) = \frac{b}{\rho} \int_0^1 z f(s, z(s), z'(s))ds + \frac{b}{\rho} \sum_{k=1}^n t_k I_k(z(t_k), z'(t_k)) + \frac{b}{\rho} \sum_{k=1}^n I_k(z(t_k)) + A(f) + bB(f)
\]
and
\[ z'(1) = -\frac{n}{\rho} \int_0^1 t_k f(s, z(s), z'(s)) ds + \frac{n}{\rho} \sum_{k=1}^n t_k \tilde{k}(z(t_k), z'(t_k)) + \frac{n}{\rho} \sum_{k=1}^n k(z(t_k)) + A(f) + (-a)B(f), \]

we get that
\[ az(1) + bz'(1) = \rho A(f) \]
(12)

From (11) and (12), we have that
\[ \left\{ \begin{array}{c}
- \sum_{j=1}^{m-2} \gamma_j \zeta_j \right\} A(f) + \left[ \rho - \sum_{j=1}^{m-2} \gamma_j (b + a(1 - \zeta_j)) \right] B(f) = \sum_{j=1}^{m-2} \gamma_j K(\zeta_j, z), \\
\left\{ \rho - \sum_{j=1}^{m-2} \delta_j \zeta_j \right\} A(f) + \left[ - \sum_{j=1}^{m-2} \delta_j (b + a(1 - \zeta_j)) \right] B(f) = \sum_{j=1}^{m-2} \delta_j K(\zeta_j, z)
\end{array} \right. \]
which yields that \( A(f) \) and \( B(f) \) satisfy (8) and (9), respectively. □

**Lemma 2.5.** Let (K1)-(K3) hold. Then for all \( t, s \in J \), we have
\[ G(t, s) \leq G(s, s). \]

**Lemma 2.6.** Let (K1)-(K3) hold. Assume that
\[ (K8) \triangleq < 0, \quad \rho - \sum_{j=1}^{m-2} \delta_j \zeta_j > 0, \quad \rho - \sum_{j=1}^{m-2} \gamma_j (b + a(1 - \zeta_j)) > 0 \]
holds. Then for \( z \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \), the solution \( z \) of the IBVP (1) satisfies
\[ z(t) \geq 0 \quad \text{for} \quad t \in J. \]

**Proof.** It can be easily seen from the facts of \( G(t, s) \geq 0 \) on \( J \times J \) and \( A(f) \geq 0, B(f) \geq 0 \). □

**Lemma 2.7.** If \( z \) is a solution of the IBVP (2), \( \underline{z}(t) \) and \( \overline{z}(t) \) are lower and upper solutions of the IBVP (1), respectively, \( \tilde{z} \leq \underline{z} \), and
\[ l_k(z(t_k)) \leq l_k(z) \leq l_k(\overline{z}(t_k)), \quad k = 1, ..., n, \]
(13)

then
\[ \underline{z}(t_k) \leq z \leq \overline{z}(t_k), \quad k = 1, ..., n, \]
\[ \underline{z}(t) \leq z \leq \overline{z}(t), \quad t \in J. \]

**Proof.** Indicate \( \delta(t) = z(t) - \overline{z}(t) \), we will just prove that \( z(t) \leq \overline{z}(t) \) for all \( t \in J \). With the similar technique it can be shown that \( z(t) \geq \underline{z}(t) \) for all \( t \in J \) is provided. Let \( z(t) > \overline{z}(t), t \in J \) holds. Then, \( \sup_{t \in J}(z(t) - \overline{z}(t)) > 0 \), there are three cases.
Case 1. Assume that \( \max_{t \in J} \delta(t) = \sup_{t \in J} (z(t) - \overline{z}(t)) = \delta(0) \), or \( \max_{t \in J} \delta(t) = \delta(1) \), we only see that \( \max_{t \in J} \delta(t) = \delta(1) \).

It is clear that \( \delta(1) > 0 \). By virtue of definition 2.1 and (K6), we get

\[
\delta(1) = z(1) - \overline{z}(1) \leq \frac{1}{a} \sum_{j=1}^{m-2} \delta_j [z(\zeta_j) - \frac{b}{a} z'(1)] - \frac{1}{a} \sum_{j=1}^{m-2} \delta_j [\overline{z}(\zeta_j) - \frac{b}{a} \overline{z}'(1)]
\]

\[
= \frac{1}{a} \sum_{j=1}^{m-2} \delta_j[z(\zeta_j) - \frac{b}{a} z'(1)] - \frac{b}{a} \delta'(1)
\]

\[
\leq \max_{t \in J} \delta(t) \frac{1}{a} \sum_{j=1}^{m-2} \delta_j
\]

\[
< \max_{t \in J} \delta(t).
\]

This is a contradiction. So, our assumption is wrong.

Case 2. Assume that there exist \( k \in \{0, 1, ..., n\} \) and \( \zeta \in (t_k, t_{k+1}) \) such that \( \sup_{t \in (t_k, t_{k+1})} \delta(t) = \delta(\zeta) = z(\zeta) - \overline{z}(\zeta) > 0 \).

Then \( \delta''(\zeta) = 0 \) and \( \delta''(t) \leq 0 \). On the other hand,

\[
\delta''(\zeta) = z''(\zeta) - \overline{z}''(\zeta) \geq -f'(\zeta, z(\zeta), \frac{d}{dc} m z(\zeta, z)) + f(\zeta, \overline{z}(\zeta), \overline{z}'(\zeta))
\]

\[
= -f(\zeta, m(\zeta, z(\zeta)), \frac{d}{dc} m z(\zeta, z)) - \frac{m z(\zeta, z(\zeta)) - z(\zeta)}{1 + (z(\zeta) - m z(\zeta, z(\zeta)))^2}
\]

\[
+ f(\zeta, \overline{z}(\zeta), \overline{z}'(\zeta))
\]

\[
= -f(\zeta, \overline{z}(\zeta), \overline{z}'(\zeta)) - \frac{m z(\zeta, z(\zeta)) - z(\zeta)}{1 + (z(\zeta) - m z(\zeta, z(\zeta)))^2}
\]

\[
+ f(\zeta, \overline{z}(\zeta), \overline{z}'(\zeta))
\]

\[
= \frac{z(\zeta) - m z(\zeta, z(\zeta))}{1 + (z(\zeta) - m z(\zeta, z(\zeta)))^2}
\]

\[
= \frac{\delta(\zeta)}{1 + \delta^2(\zeta)} > 0.
\]

This is a contradiction. So, our assumption is wrong. Consequently, the function \( \delta \) cannot have any positive maximum in the interval \( (t_k, t_{k+1}) \) for \( k = 1, 2, ..., n \).

Case 3. In accordance with Case 2, if \( \sup_{t \in J} \delta(t) > 0 \), then \( \sup_{t \in J} \delta(t) = \delta(t^*_k) \) or \( \sup_{t \in J} \delta(t) = \delta(t_k) = \delta(t_k) \), we just prove that \( \sup_{t \in J} \delta(t) = \delta(t_k^*), k = 1, 2, ..., n \). Assume that \( \sup_{t \in J} \delta(t) = \delta(t^*_1), \delta'(t_1) = \delta'(t_1) \geq 0 \).
From (2) and (13), we can get
\[
\begin{align*}
    z(t_1^+ - z(t_1) &= \bar{z}(t_1^+) - z(t_1)) = I_1(m_\varpi z(t_1, z(t_1))) \\
                        &= I_1(z(t_1)) = \bar{z}(t_1^+) - \bar{z}(t_1) = \bar{z}(t_1^+) - \bar{z}(t_1), \\
    z^\prime(t_1^+) - z^\prime(t_1) &= \bar{z}(t_1^+) - \bar{z}(t_1^_) = \bar{z}(t_1^+) - \bar{z}(t_1^_) = \bar{z}(t_1^+) - \bar{z}(t_1^_).
\end{align*}
\]

Hence
\[
\begin{align*}
    \vartheta(t_1^+) &= z(t_1^+) - \bar{z}(t_1^+) = z(t_1) - \bar{z}(t_1) > 0, \\
    \vartheta^\prime(t_1^+) &= z^\prime(t_1^+) - \bar{z}^\prime(t_1^+) = z^\prime(t_1) - \bar{z}^\prime(t_1) \geq 0.
\end{align*}
\]

Assume that \( \vartheta^\prime(t_1^+) = 0 \) and \( \vartheta \) is nonincreasing on some interval \((t_1, t_1 + \kappa) \subset (t_1, t_2)\), where \( \kappa > 0 \) is small enough such that \( \vartheta(t) > 0 \) on \( t \in (t_1, t_1 + \kappa) \). For \( t \in (t_1, t_1 + \kappa) \),
\[
\begin{align*}
    z^\prime(t) - \bar{z}^\prime(t) &\geq -f(t, z(t)) \frac{d}{dt} m_\varpi z(t, z) - \frac{m_\varpi z(t, z) - z}{1 + (z - m_\varpi z(t, z))^2} \\
                                &\quad + f(t, z(t), \bar{z}(t)) \\
    &= \frac{z - m_\varpi z(t, z)}{1 + (z - m_\varpi z(t, z))^2} > 0.
\end{align*}
\]

This contradicts the assumption of monotonicity of \( \vartheta \). Therefore, we have
\[
0 < \vartheta(t_1^+) < \vartheta(t_2) = \vartheta(t_2^+),
\]
\[
\vartheta^\prime(t_2) = \vartheta^\prime(t_2^+) \geq 0.
\]

We practice the previous method and conclude by induction that
\[
\vartheta(t_k) > 0, \quad \vartheta^\prime(t_k) \geq 0, \quad k = 1, 2, ..., n + 1.
\]

This is a contradiction. Because \( \vartheta(t) \) is not the maximum point. With the same analysis, we can obtain that \( \vartheta(t_k) > 0, \quad k = 2, 3, ..., n, \) cannot hold. \( \square \)

**Lemma 2.8.** \(-\sigma \leq z^\prime(t) \leq \sigma \) on \( J \), where \( z(t) \) is the solution of the IBVP (2).

**Proof.** We just show \( z^\prime(t) \leq \sigma \). Assume that there exists \( \rho \in (t_k, t_{k+1}) \) with
\[
    z^\prime(\rho) = \frac{z(t_{k+1}) - z(t_k)}{t_{k+1} - t_k}, \quad k = 1, 2, ..., n,
\]
and as a result,
\[
-\sigma < -\rho \leq \frac{z(t_{k+1}) - z(t_k)}{t_{k+1} - t_k} \leq \rho \leq \frac{z(t_{k+1}) - z(t_k)}{t_{k+1} - t_k} \leq \sigma.
\]

Therefore, there exist \( \eta_1, \eta_2 \in (t_k, t_{k+1}) \) such that
\[
z^\prime(\eta_1) = \rho, \quad z^\prime(\eta_2) = \sigma \quad \text{and either}
\]
\[
\rho \leq z^\prime(t) \leq \sigma, \quad t \in (\eta_1, \eta_2),
\]
or
\[
\rho \leq z^\prime(t) \leq \sigma, \quad t \in (\eta_2, \eta_1).
\]
we will only handle the first situation, as the other situation can be handled similarly. The following statement is due to the assumption
\[ |z''(t)| = |f(t, z(t), z'(t))| \leq D |z'(t)|q(|z'(t)|) + E, \quad \text{for } t \in (\eta_1, \eta_2). \]

This implies that
\[
\int_{\eta_1}^{\eta_2} \frac{|z''(t)|}{q(|z'(t)|)} \, dt \leq D \int_{\eta_1}^{\eta_2} |z'(t)| \, dt + E \int_{\eta_1}^{\eta_2} \frac{dt}{q(|z'(t)|)} = D(z(\eta_2) - z(\eta_1)) + E \int_{\eta_1}^{\eta_2} \frac{dt}{q(|z'(t)|)},
\]

which yields
\[
\int_{\eta}^{t} \frac{ds}{q(s)} = \int_{z(\eta_1)}^{z(\eta_2)} \frac{ds}{q(s)} \leq D(z(\eta_2) - z(\eta_1)) + E(\eta_2 - \eta_1) \max_{s \geq \eta} \frac{1}{q(s)} \leq D(\max_{t \in J} \Xi(t) - \min_{t \in J} \Xi(t)) + E \max_{s \geq \eta} \frac{1}{q(s)}.
\]

This contradicts the choice of \( \sigma \). Thus, it is obtained that \( z'(t) \leq \sigma \) is.

\[ \Box \]

3. Main Result

In this section, we will obtain the existence of at least one positive solution for the IBVP (1). The theorem, which is fundamental and important for the proof of our main result, is the fixed point theorem below.

**Lemma 3.1.** [6] (Schauder’s fixed point theorem) Let \( K \) be a convex subset of a normed linear space \( E \). Each continuous, compact map \( L : K \to K \) has a fixed point.

**Theorem 3.2.** Assume that conditions (K1)-(K8) hold. Then the IBVP (1) has at least one positive solution \( z \in PC^1([J, \mathbb{R}^+]) \cap C^2(J', \mathbb{R}) \) such that
\[ z(t) \leq \Xi(t), \quad -\sigma \leq z'(t) \leq \sigma, \quad t \in J. \]

**Proof.** Solving the IBVP (2) is equivalent to finding \( z \in PC^1([J, \mathbb{R}^+]) \cap C^2(J', \mathbb{R}) \) which satisfies
\[
\begin{align*}
\frac{dz}{dt} &= G(t, z) + \sum_{k=1}^{n} W_k(t, z) + tA(f^*(s, z(s), \frac{dz}{ds}, m_z(s, z), \frac{d^2z}{ds^2} m_{zz}(s, z))) + (b + a(1 - t))B(f^*(s, z(s), \frac{dz}{ds}, m_z(s, z), \frac{d^2z}{ds^2} m_{zz}(s, z))),
\end{align*}
\]

where \( I_k'(z(t_k)) = I_k(m_z(z(t_k), z(t_k))), \quad \tilde{I}_k'(z(t_k), z'(t_k)) = \tilde{I}_k(m_z(z(t_k), z(t_k)), n(t_k, z(t_k)) \) and
\[
W_k'(t, z) = \begin{cases} 
|1 - al_k'(z(t_k)) + (b + a(1 - t))b_k'(z(t_k), z'(t_k))|, & t < t_k, \\
(b + a(1 - t))[I_k'(z(t_k)) + t_m \tilde{I}_k'(z(t_k), z'(t_k))], & t_k < t.
\end{cases}
\]
Now, define the following operator \( T : PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \rightarrow PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \) by

\[
(Tz)(t) = \int_0^1 G(t, s) f^*(s, z(s), \frac{d}{ds} m_z(z(s))) ds + \sum_{k=1}^n W_k(t, z) + tA(f^*(s, z(s), \frac{d}{ds} m_z(z(s)))) + (b + a(1 - t))B(f^*(s, z(s), \frac{d}{ds} m_z(z(s))).
\]

Let's show the completely continuous of operator \( T : PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \rightarrow PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \) in 2 steps. Define the following for convenience:

\[
H(z_j) = \int_0^1 G(z_j, s) ds + 2n,
\]

\[
\bar{A} = \frac{1}{\Delta} \begin{vmatrix} \sum_{j=1}^{m-1} \gamma_j H(z_j) & \sum_{j=1}^{m-1} \gamma_j H(z_j) \\ \sum_{j=1}^{m-2} \delta_j H(z_j) & \sum_{j=1}^{m-2} \delta_j H(z_j) \end{vmatrix}, \quad \bar{B} = \frac{1}{\Delta} \begin{vmatrix} \sum_{j=1}^{m-1} \gamma_j \bar{c}_j & \sum_{j=1}^{m-1} \gamma_j \bar{c}_j \\ \sum_{j=1}^{m-2} \delta_j \bar{c}_j & \sum_{j=1}^{m-2} \delta_j \bar{c}_j \end{vmatrix}.
\]

Step 1: In this step show that the operator \( T : PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \rightarrow PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \) is continuous. Assume that \( z_n \) be a sequence in \( PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \) such that \( z_n \rightarrow z_0 \in PC^1(J, \mathbb{R}^+ \cap C^2(J', \mathbb{R}) \) as \( n \rightarrow \infty \). Thus, we can write \( \|z_n - z_0\|_{PC} \rightarrow 0 \) \( (n \rightarrow \infty) \). From Lemma 2.3, we obtain that \( \|\frac{d}{dt} m_z(t, z_n) - \frac{d}{dt} m_z(t, z_0)\|_{PC} \rightarrow 0 \) \( (n \rightarrow \infty) \). Since \( \|z_n - z_0\|_{PC} \rightarrow 0 \) \( (n \rightarrow \infty) \) and \( \|\frac{d}{dt} m_z(t, z_n) - \frac{d}{dt} m_z(t, z_0)\|_{PC} \rightarrow 0 \) \( (n \rightarrow \infty) \), from the definition of the norm in \( PC^1(J, \mathbb{R}^+) \), we can write \( \|z_n - z_0\|_{PC} \rightarrow 0 \) \( (n \rightarrow \infty) \) and \( \|\frac{d}{dt} m_z(t, z_n) - \frac{d}{dt} m_z(t, z_0)\|_{PC} \rightarrow 0 \) \( (n \rightarrow \infty) \).

Also, from the continuity of the \( f, I_k, \tilde{I}_k \) functions, we conclude that

\[
|f^*(s, z_n(s), \frac{d}{ds} m_z(s, z_n)) - f^*(s, z_0(s), \frac{d}{ds} m_z(s, z_0))| \rightarrow 0 \quad (n \rightarrow \infty),
\]

\[
|I_k'(z_n(t_k)) - I_k'(z_0(t_k))| \rightarrow 0 \quad (n \rightarrow \infty),
\]

\[
|\tilde{I}_k(z_n(t_k), \frac{d}{dt_k} m_z(t_k, z_n(t_k))) - \tilde{I}_k(z_0(t_k), \frac{d}{dt_k} m_z(t_k, z_0(t_k)))| \rightarrow 0 \quad (n \rightarrow \infty).
\]

Therefore, considering Lemma 2.5 and with the help of Lebesgue Dominated Convergence Theorem, we can
obtain

\[
||Tz_n - Tz_0||_{PC} \leq \int_0^1 G(s, s) |f'(s, z_n(s), \frac{d}{ds} m_z(s, z_n)) - f'(s, z_0(s), \frac{d}{ds} m_z(s, z_0))| ds
\]

\[
+ 2n \max \left\{ |I_k'(z_n(t_k)) - I_k'(z_0(t_k))| \right\}
\]

\[
|I_k'(z_n(t_k), \frac{d}{dt_k} m_z(t_k, z_n(t_k))) - \bar{T}_k(z_0(t_k), \frac{d}{dt_k} m_z(t_k, z_0(t_k)))|
\]

\[
+ (\tilde{A} + \rho B) \max \left\{ |I_k'(z_n(t_k)) - I_k'(z_0(t_k))| \right\}
\]

\[
|f'(s, z_n(s), \frac{d}{ds} m_z(s, z_n)) - f'(s, z_0(s), \frac{d}{ds} m_z(s, z_0))| \rightarrow 0 \quad (n \rightarrow \infty),
\]

This yields that, \( ||Tz_n - Tz_0||_{PC} \rightarrow 0 \) as \( n \rightarrow \infty \). Hence, \( T \) is continuous.

Step 2: In order to prove the relatively compactness of operator \( T : PC^1(I, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \rightarrow PC^1(I, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \). Let \( B \) be any bounded subset of \( PC^1(I, \mathbb{R}^+) \cap C^2(J', \mathbb{R}) \). Then there exists \( M_0 > 0 \) such that \( ||z||_{PC^1} \leq M_0 \) for all \( z \in B \). Also, from \( f \in C(I \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}) \), \( I_k \in C(\mathbb{R}^+, \mathbb{R}^+) \), \( T_k \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+) \), we can write

\[
M_1 = \max_{s \in [0,1]} f'(s, z(s), \frac{d}{ds} m_z(s, z)) < \infty
\]

and

\[
M_2 = \max_{1 \leq k \leq n} \left\{ \max_{z \in [0, M_0]} I_k'(z), \max_{z \in [0, M_0]} T_k(z, \frac{d}{ds} m_z(s, z)) \right\} < \infty.
\]
From the above equation, we can obtain that
\[ ||Tz||_{PC} \leq \int_0^1 G(s,s)ds + 2n + \overline{A} + \rho \overline{B} \max\{M_1, M_2\} < \infty \]
and
\[ ||T'z||_{PC} \leq \int_0^1 G(t,s)|_{t=s} ds + 2n + \overline{A} + \rho \overline{B} \max\{M_1, M_2\} < \infty. \]
This yields that \( ||Tz||_{PC} < \infty. \) Hence, we obtain \( T(B) \) is uniformly bounded.

Next, we show that \( T \) is equicontinuous on \( J. \) For all \( t \in J, \) we can get
\[ (Tz)'(t) \leq \int_0^1 \frac{1}{\rho} (b + a(1 - s))ds + n + \overline{A} \max\{M_1, M_2\} \]
\[ := M_3. \]
For all \( t_1, t_2 \in J, t_1 < t_2, \) if we integrate the last inequality from \( t_1 \) to \( t_2, \) then we have \( (Tz)(t_2) - (Tz)(t_1) \leq M_3(t_2 - t_1). \) This means that
\[ (Tz)(t_2) - (Tz)(t_1) \rightarrow 0, \quad t_1 \rightarrow t_2. \] (14)

On the other hand, we know that
\[ (Tz)'(t) = \frac{1}{\rho} \int_0^1 (-as) f'(s, z(s), \frac{d}{ds} m_z(s, z))ds \]
\[ + \frac{1}{\rho} \int_1^t (b + a(1 - s))f'(s, z(s), \frac{d}{ds} m_z(s, z))ds \]
\[ + \frac{1}{\rho} \sum_{0 < k < cl} (-a)[I_k'(z(t_k)) + t_k I_k'(z(t_k), z'(t_k))] \]
\[ + \frac{1}{\rho} \sum_{1 < k < cl} [aI_k'(z(t_k)) + (b + a(1 - t_k))I_k'(z(t_k), z'(t_k))] \]
\[ + A(f'(s, z(s), \frac{d}{ds} m_z(s, z))) + (-a)B(f'(s, z(s), \frac{d}{ds} m_z(s, z))). \]
From the above equation, we can obtain that
\[ (Tz)'(t_2) - (Tz)'(t_1) \rightarrow 0, \quad t_1 \rightarrow t_2. \] (15)

Accordingly, from the equations (14) and (15), \( T(B) \) is equicontinuous on \( J. \) Consequently, from steps 1-2 by Arzela-Ascoli Theorem, \( T : PC^1(J, \mathbb{R}^n) \cap C^2(J', \mathbb{R}) \rightarrow PC^1(J, \mathbb{R}^n) \cap C^2(J', \mathbb{R}) \) is completely continuous operator. With the help of the Schauder’s Fixed Point Theorem, we get that \( T \) has a fixed point \( z \in PC^1(J, \mathbb{R}^n) \cap C^2(J', \mathbb{R}). \) Thus, it is obtained that \( z \) is a solution of the IBVP (2). From Lemma 2.7 and Lemma 2.8, we know that \( \bar{z}(t) \leq z(t) \leq \bar{z}(t), \quad -\sigma \leq z'(t) \leq \sigma, \) thereby IBVP (2) turns into the IBVP (1). Consequently, \( z(t) \) is a solution of the IBVP (1). \( \square \)

4. An Example

In this section, we give an example to demonstrate how our main result can be used in practice.
Example 4.1. Consider the following the IBVP:
\[
\begin{aligned}
\ddot{z}(t) + f(t, z(t), z'(t)) &= 0, \quad t \in J', \\
\Delta z|_{t=k} &= z(t_k), \\
\Delta z'|_{t=k} &= 4z(t_k) + z'(t_k), \\
\end{aligned}
\]
\[
\begin{aligned}
z(0) &= \frac{1}{5}z(\frac{1}{3}) + \frac{1}{6}z(\frac{2}{3}), \\
4z(1) + z'(1) &= z(\frac{1}{3}) + \frac{1}{2}z(\frac{2}{3}), \\
\end{aligned}
\]
where \( f(t, z(t), z'(t)) = z'(t). \) It is clear that
\[
\begin{aligned}
z(t) &= 0, \quad z(t) = \begin{cases} -t - \frac{1}{3}^2 + 2, & t \in \left[0, \frac{1}{3}\right], \\
-\left(t - \frac{2}{3}\right)^2 + 3, & t \in \left(\frac{1}{3}, \frac{2}{3}\right], \\
-\left(t - 1\right)^2 + 4, & t \in \left(\frac{2}{3}, 1\right]. 
\end{cases}
\end{aligned}
\]
are lower and upper solutions of the IBVP (16), respectively.

The figures of \( z(t) \) and \( \Xi(t) \) are given in Figure 1. It is clearly seen from the Figure 1 that \( z(t) \leq \Xi(t) \). Substituting \( z(t) \) and \( \Xi(t) \) in the equation \( \ddot{z}(t) + f(t, z(t), z'(t)) = 0, \quad t \in J' \), it can be seen from Figure 2 that \( t \in J', \quad \ddot{z}(t) + f(t, z(t), z'(t)) \leq 0 \) inequality and \( \ddot{z}(t) + f(t, z(t), z'(t)) \geq 0 \) inequality are obtained. It can easily be obtained that other inequalities are satisfied. In accordance with Definition 2.1, it is obtained that \( z(t) \) and \( \Xi(t) \) are upper and lower solutions of the IBVP (16), respectively.

Let \( \phi(s) = \sqrt{s + 1}; \) then \( |f(t, z, z')| = |z'| \leq D|z'|(|\sqrt{|z'|} + 1| + E, \quad t \in J, \quad z \leq z \leq \Xi, \quad z', \Xi' \in \mathbb{R}^+ \) \( D, E \geq 0. \) Noting that for any \( a > 0, \)
\[
\int_a^\infty \frac{ds}{\phi(s)} = \infty \quad \text{and} \quad \max_{s \geq a} \frac{1}{\phi(s)} \leq 1., \quad \text{there exists} \quad a \quad \text{such that} \quad f \quad \text{satisfies the Nagumo condition with respect to} \quad z \quad \text{and} \quad \Xi. \quad \text{Hence, the IBVP (16) has at least one positive solution} \quad z(t) \in [z, \Xi].
\]

References

[1] A. M. Samoilenko, N. A. Peretyuk, Impulsive Differential Equations of Word Scientific Series on Nonlinear Science Series A, Monographs and Treatises Word Scientific Vol. 14. River Edge, NJ, USA, 1995.
[2] R.P. Agarwal, S. Gala, M.A. Ragusa, A regularity criterion in weak spaces to Boussinesq equations, Mathematics 8 (6) art. n. 920 (2020) 11pp.

[3] B. Liu, J. Yu, Existence of Solution of M-Point Boundary Value Problems of Second-Order Differential Systems with Impulses, Appl. Math. Comput. 125 no. 2-3 (2002) 155–175.

[4] D. Bainov, P. Simeonov, Impulsive Differential Equations: Periodic Solutions and Applications of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific Technical, Vol. 66. Harlow, UK, 1993.

[5] D.J. Guo, Existence of Solutions of Boundary Value Problems For Nonlinear Second Order Impulsive Differential Equations in Banach Spaces, J. Math. Anal. Appl. 181 no.2 (1994) 407–421.

[6] D.R. Smart, Fixed Point Theorems, University Press, Cambridge, 1974.

[7] E.K. Lee, J.H. Lee, Multiple Positive Solutions of Singular Two Point Boundary Value Problems for Second Order Impulsive Differential Equations, Appl. Math. Comput. 158 no. 3 (2004) 745–759.

[8] F. Mei, Q. Dong, Multiple Positive Solutions of Multi-Point Boundary Value Problem For Second-Order Impulsive Differential Equations, J. Comput. Appl. Math. 223 (2009) 438–448.

[9] H. Pang, M. Lu, C. Cai, The Method of Upper and Lower Solutions to Impulsive Differential Equations with Integral Boundary Conditions, Adv. Difference Equ. (2014) 11pp.

[10] I. Yaslan, Positive Solutions for Multi Point Impulsive Boundary Value Problems on Time Scales, Journal of Nonlinear Functional Analysis Vol. 2019 Article ID 5 (2019) 1–11.

[11] I. Yaslan, Existence of Positive Solutions for Second-Order Impulsive Boundary Value Problems on Time Scales, Mediterr. J. Math. Vol. 13 No.4 (2016) 1613–1624.

[12] I.Y. Karaca, A. Sinanoglu, Positive Solutions of Impulsive Time-Scale Boundary Value Problems with p-Laplacian on the Half-Line, Filomat 33 (2019) 415–433.

[13] I.Y. Karaca, F.T. Fen, Multiple Positive Solutions for Nonlinear Second-Order m-Point Impulsive Boundary Value Problems on Time Scales, Filomat 29-4 (2015) 817–827.

[14] I. Yaslan, Z. Haznedar, Existence of Positive Solutions for Second-Order Impulsive Time Scale Boundary Value Problems on Infinite Intervals, Filomat Vol. 28 No. 10 (2014) 2163–2173.

[15] J.J. Nieto, J.M. Uzal, Nonlinear Second-Order Impulsive Differential Problems with Dependence on the Derivative via Variational Structure., J. Fixed Point Theory Appl. 22 (2020) 19pp.

[16] J.R. Graef, H. Kadari, A. Ouahab, A. Oumansour, Existence Results for Systems of Second-Order Impulsive Differential Equations, Acta Math. Univ. Comenian. (N.S.) 88 no.1 (2019) 51–66.

[17] J. Shen, W. Wang, Impulsive Boundary Value Problems with Nonlinear Boundary Conditions, Nonlinear Anal. 69 no.11 (2008) 4055–4062.

[18] J. Zhao, J. Zhao, W. Ge, Upper and Lower Solutions for M-Point Impulsive BVP with One-Dimensional p-Laplacian, Discrete Dyn. Nat. Soc. (2013) 9pp.

[19] K. Zhang, J. Xu, D. O’Regan, Weak Solutions for a Second Order Impulsive Boundary Value Problem, Filomat 31:20 (2017) 6431–6439.

[20] L. Hu, L. Liu, Y. Wu, Positive Solutions of Nonlinear Singular Two-Point Boundary Value Problems for Second-Order Impulsive Differential Equations, Appl. Math. Comput. 196 no.2 (2008) 550–562.

[21] M.A. Ragusa, A. Razani, F. Safari, Existence of radial solutions for a p(x)-Laplacian Dirichlet problem, Adv Diff Equ 215 (2021) 14pp.

[22] M. Benchohra, J. Henderson, S. Ntouyas, Impulsive Differential Equations and Inclusions vol. 2 of Contemporary Mathematics and Its Applications, Hindawi, Publishing Corporation, New York, NY, USA, 2006.

[23] M.X. Wang, A. Cabada, J.J. Nieto, Monotone Method for Nonlinear Second Order Periodic Boundary Value Problems with Caratheodory Functions, Ann. Polon. Math. 58 (1993) 221–235.

[24] V. Lakshmikantham, D.D. Bainov, P. Simeonov, Theory of Impulsive Differential Equations vol. 6 of Series in Modern Applied Mathematics, Word Scientific Publishing, Teaneck, NJ, USA, 1989.

[25] Y. Tian, C. Liu, Multiplicity of Positive Solutions to M-Point Boundary Value Problem of Second-Order Impulsive Differential Equations, Acta Math. Appl. Sin. Engl. Ser. 26 no.1 (2010) 145–158.

[26] Y. Zhao, H. Chen: Multiplicity of Solutions to Two-Point Boundary Value Problems for Second-Order Impulsive Differential Equations, Appl. Math. Comput. 206 no.2 (2008) 925–931.

[27] Z. Li, X. Shu, F. Xu, The Existence of Upper and Lower Solutions to second Order Random Impulsive Differential Equation With Boundary Value Problem, AIMS Math. 5(6) (2020) 6189–6210.