Towards the biconjugate of bivariate piecewise quadratic functions* **

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Abstract. Computing the closed convex envelope or biconjugate is the core operation that bridges the domain of nonconvex with convex analysis. We focus here on computing the conjugate of a bivariate piecewise quadratic function defined over a polytope. First, we compute the convex envelope of each piece, which is characterized by a polyhedral subdivision such that over each member of the subdivision, it has a rational form (square of a linear function over a linear function). Then we compute the conjugate of all such rational functions. It is observed that the conjugate has a parabolic subdivision such that over each member of its subdivision, it has a fractional form (linear function over square root of a linear function). This computation of the conjugate is performed with a worst-case linear time complexity algorithm.

Our results are an important step toward computing the conjugate of a piecewise quadratic function, and further in obtaining explicit formulas for the convex envelope of piecewise rational functions.

Keywords: Conjugate · Convex envelope · Piecewise quadratic function.

1 Introduction

Computational convex analysis (CCA) focuses on creating efficient algorithms to compute fundamental transforms arising in the field of convex analysis. Computing the convex envelope or biconjugate is the core operation that bridges the...
domain of nonconvex analysis with convex analysis. Development of most of the
algorithms in CCA began with the Fast Legendre Transform (FLT) in [5], which
was further developed in [6,18], and improved to the optimal linear worst-case
time complexity in [19] and then [10,20]. More complex operators were then
considered [3,4,22] (see [21] for a survey including a list of applications).

Piecewise Linear Quadratic (PLQ) functions (piecewise quadratic functions
over a polyhedral partition) are well-known in the field of convex analysis [24]
with the existence of linear time algorithms for various convex transforms [22,4].
Computing the full graph of the convex hull of univariate PLQ functions is
possible in optimal linear worst-case time complexity [9].

For a function $f$ defined over a region $P$, the pointwise supremum of all its
convex underestimators is called the convex envelope and is denoted $\text{conv} f_P(x, y)$. Computing the convex envelope of a multilinear function over a unit hypercube
is NP-Hard [7]. However, the convex envelope of functions defined over a polytope $P$ and restricted by the vertices of $P$ can be computed in finite time using
a linear program [20,27]. A method to reduce the computation of convex enve-
lope of functions that are one lower dimension($\mathbb{R}^{n-1}$) convex and have indefinite
Hessian to optimization problems in lower dimensions is discussed in [14].

Any general bivariate nonconvex quadratic function can be linearly trans-
formed to the sum of bilinear and a linear function. Convex envelopes for bilinear
functions over rectangles have been discussed in [23] and validated in [1]. The
convex envelope over special polytopes (not containing edges with finite positive
slope) was derived in [25] while [15] deals with bilinear functions over a triangle
containing exactly one edge with finite positive slope. The convex envelope over
general triangles and triangulation of the polytopes through doubly nonnegative
matrices (both semidefinite and nonnegative) is presented in [2].

In [16], it is shown that the analytical form of the convex envelope of some
bivariate functions defined over polytopes can be computed by solving a con-
tinuously differentiable convex problem. In that case, the convex envelope is
characterized by a polyhedral subdivision.

The Fenchel conjugate $f^*(s) = \sup_{x \in \mathbb{R}^n} [(s, x) - f(x)]$ (we note $(s, x) = s^T x$)
of a function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is also known as the Legendre-Fenchel
Transform or convex conjugate or simply conjugate. It plays a significant role in duality
and computing it is a key step in solving the dual optimization problem [24].
Most notably, the biconjugate is also the closed convex envelope.

A method to compute the conjugate known as the fast Legendre transform
was introduced in [5] and studied in [6,18]. A linear time algorithm was later
introduced by Lucet to compute the discrete Legendre transform [19]. Those
algorithms are numeric and do not provide symbolic expressions.

Computation of the conjugate of convex univariate PLQ functions have been
well studied in the literature and linear time algorithms have been developed in [11,18]. Recently, a linear time algorithm to compute the conjugate of convex
bivariate PLQ functions was proposed in [12].

Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a piecewise function, i.e. $f(x) = f_i(x)$ if $x \in
P_i$ for $i = 1, \ldots, N$. From [13] Theorem 2.4.1, we have $(\inf_i f_i)^* = \sup_i f_i^*$,
and from Proposition 2.6.1, \(\text{conv}(\inf_i (f_i + I_{P_i})) = \text{conv}(\inf_i [\text{conv}(f_i + I_{P_i})])\) where \(I_{P_i}\) is the indicator function for \(P_i\). Hence, \(\text{conv}(\inf_i (f_i + I_{P_i})) = (\sup_i [\text{conv}(f_i + I_{P_i})]^*)^*\). This provides an algorithm to compute the closed convex envelope: (1) compute the convex envelope of each piece, (2) compute the conjugate of the convex envelope of each piece, (3) compute the maximum of all the conjugates, and (4) compute the conjugate of the function obtained in (3) to obtain the biconjugate. The present work focuses on Step (2).

Recall that given a quadratic function over a polytope, the eigenvalues of its symmetric matrix determine how difficult its convex envelope is to compute (for computational purposes, we can ignore the affine part of the function). If the matrix is semi-definite (positive or negative), the convex envelope is easily computed. When it is indefinite, a change of coordinate reduces the problem to finding the convex envelope of the function \((x, y) \mapsto xy\) over a polytope, for which step (1) is known \cite{17}.

The paper is organized as follow. Section 3 focuses on the domain of the conjugate while Section 4 determines the symbolic expressions. Section 5 concludes the paper with future work.

2 Preliminaries and notations

The subdifferential \(\partial f(x)\) of a function \(f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) at any \(x \in \text{dom}(f) = \{x : f(x) < \infty\}\) is \(\partial f(x) = \{s : f(y) \geq f(x) + \langle s, y - x \rangle, \forall y \in \text{dom}(f)\}\). \(\partial f(x) = \{\nabla f(x)\}\) when \(f\) is differentiable at \(x\). We note \(I_P\) the indicator function of the set \(P\), i.e. \(I_P(x) = 0\) when \(x \in P\) and \(I_P(x) = +\infty\) when \(x \notin P\).

A parabola is a two dimensional planar curve whose points \((x, y)\) satisfy the equation \(ax^2 + bxy + cy^2 + dx + ey + f = 0\) with \(b^2 - 4ac = 0\). A parabolic region is formed by the intersection of a finite number of parabolic inequalities, i.e. \(P_r = \{x \in \mathbb{R}^2 : C_r^i(x) \leq 0, i \in \{1, \cdots, k\}\}\) where \(C_r^i(x) = a_i x_1^2 + b_i x_1 x_2 + c_i x_2^2 + d_i x_1 + e_i x_2 + f_i\) and \(b_i^2 - 4a_i c_i = 0\). The set \(P_r^* = \{x \in \mathbb{R}^2 : C_r^i(x) \leq 0\}\) is convex, but \(P_r^* = \{x \in \mathbb{R}^2 : C_r^i(x) \geq 0\}\) is not.

A convex set \(R = \bigcup_{i \in \{1, \cdots, m\}} R_i, R \subseteq \mathbb{R}^2\), defined as the union of a finite number of parabolic regions is said to have a parabolic subdivision if for any \(j, k \in \{1, \cdots, m\}, j \neq k, R_j \cap R_k\) is either empty or is contained in a parabola.

3 The domain of the conjugate

Given a nonconvex PLQ function, we first compute the closed convex envelope of each piece and obtain a piecewise rational function \cite{17}. We now compute the conjugate of such a rational function over a polytope by first computing its domain, which will turn out to be a parabolic subdivision. Recall that for PLQ functions, \(\text{dom } f^* = \partial f(\text{dom } f)\). We decompose the polytope \(\text{dom } f = P\) into its interior, its vertexes, and its edges.

Following \cite{17}, we write a rational function as

\[
r(x, y) = \frac{(\xi_1(x, y))^2}{\xi_2(x, y)} + \xi_0(x, y),
\]  

(1)
where $\xi_i(x, y)$ are linear functions in $x$ and $y$.

**Proposition 1 (Interior).** Consider $r$ defined by 1, there exists $c_{ij}$ such that 
\[ \bigcup_{x \in \text{dom}(r)} \partial r(x) = \{ s : C_r(s) = 0 \} \]
where 
\[ C_r(s) = \alpha_{11}s_1^2 + \alpha_{12}s_1s_2 + \alpha_{22}s_2^2 + \alpha_{10}s_1 + \alpha_{02}s_2 + \alpha_{00} \]
and \( \{ s : C_r(s) = 0 \} \) is a parabolic curve.

*Proof.* Note $\xi_1(x) = \xi_{11}x_1 + \xi_{12}x_2 + \xi_{10}$, $\xi_2(x) = \xi_{21}x_1 + \xi_{22}x_2 + \xi_{20}$, and $\xi_0(x) = \xi_{01}x_1 + \xi_{02}x_2 + \xi_{00}$. Since $r$ is differentiable everywhere in $\text{dom}(r) = \mathbb{R}^2/\{z : \xi_2(z) = 0\}$, for any $x \in \text{dom}(r)$ we compute $s = \nabla r(x)$ as 
\[ s_i = 2\xi_{1i}t - \xi_{2it}^2 + \xi_{0i} \]
for $i = 1, 2$, where $t = (\xi_{11}x_1 + \xi_{12}x_2 + \xi_{10})/(\xi_{21}x_1 + \xi_{22}x_2 + \xi_{20})$. Hence, $s = \nabla r(x)$ represents the parametric equation of a conic section, and by eliminating $t$, we get $C_r(s) = 0$ where
\[ C_r(s) = \alpha_{11}s_1^2 + \alpha_{12}s_1s_2 + \alpha_{22}s_2^2 + \alpha_{10}s_1 + \alpha_{02}s_2 + \alpha_{00}, \]
with $\alpha_{11} = \xi_{21}^2\xi_{22}$, $\alpha_{12} = -2\xi_{21}\xi_{22}$, $\alpha_{22} = \xi_{22}^2$ and other $c_{ij}$ are functions of the coefficients of $r$. We check that $\alpha_{11}^2 - 4\alpha_{11}\alpha_{22} = (-2\xi_{21}\xi_{22})^2 - 4\xi_{21}^2\xi_{22}^2 = 0$, so the conic section is a parabola. Consequently, for all $x \in \text{dom}(r)$, $\partial r(x)$ is contained in the parabolic curve $C_r(s) = 0$, i.e.
\[ \bigcup_{x \in \text{dom}(r)} \partial r(x) \subset \{ s : C_r(s) = 0 \}. \]

Conversely, any point $s_r$ that satisfies $C_r(s_r) = 0$, satisfies the parametric equation as well, so the converse inclusion is true.

**Corollary 1 (Interior).** For a bivariate rational function $r$, and a polytope $P$, define $f(x) = r(x) + I_P(x)$, then for all $x \in \text{int}(P)$, the set $\bigcup_{x \in \text{int}(P)} \partial f(x)$ is contained inside a parabolic arc.

*Proof.* We have, $\bigcup_{x \in \text{int}(P)} \partial f(x) \subseteq \bigcup_{x \in \text{int}(P)} \partial r(x)$ and $\bigcup_{x \in \text{int}(P)} \partial r(x) \subset P$ where $P \subset \mathbb{R}^2$ is a parabolic curve (from Proposition 1). Since $P$ is connected, we obtain that $\bigcup_{x \in \text{int}(P)} \partial r(x)$ is contained in a parabolic arc.

Next we compute the subdifferential at any vertex in the smooth case (the proof involves a straightforward computation of the normal cone).

**Lemma 1 (Vertices).** For $g \in C^1$, $P$ a polytope, and $v$ vertex. Let $f(x) = g(x) + I_P(x)$. Then $\partial f(v)$ is an unbounded polyhedral set.

There is one vertex at which both numerator and denominator equal zero although the rational function can be extended by continuity over the polytope; we conjecture the result based on numerous observations.

**Conjecture 1 (Vertex).** Let $r$ as in 1, $f(x) = r(x) + I_P(x)$ and $v$ be a vertex of $P$ with $\xi_1(v) = \xi_2(v) = 0$. Then $\partial f(v)$ is a parabolic region.

**Lemma 2 (Edges).** For $g \in C^1$, a polytope $P$, and an edge $E = \{ x : x_2 = mx_1 + c, x_1 \leq x_1 \leq x_2 \} \subset P$, let $f(x) = g(x) + I_P(x)$, then 
\[ \bigcup_{x \in \text{ri}(E)} \partial f(x) = \bigcup_{x \in \text{ri}(E)} \{ s + \nabla g(x) : s_1 + ms_2 = 0, s_2 \geq 0 \}. \]
Proof. For all \( x \in \text{ri}(E) \), \( \partial f(x) = \partial g(x) + N_P(x) \). Let \( L(x) = x_2 - mx_1 - c \) be the expression of the line joining \( x_1 \) and \( x_u \) such that \( P \subset \{ x : L(x) \leq 0 \} \). (The case \( P \subset \{ x : L(x) \geq 0 \} \) is analogous.)

Since \( P \subset \mathbb{R}^2 \) is a polytope, for all \( x \in \text{ri}(E) \), \( N_P(x) = \{ s : s = \lambda \nabla L(x), \lambda \geq 0 \} \) is the normal cone of \( P \) at \( x \) and can be written \( N_P(x) = \{ s : s_1 + ms_2 = 0, s_2 \geq 0 \} \). In the special case when \( E = \{ x : x_1 = d, x_1 \leq x_1 \leq x_u \} \), \( L(x) = x_1 - d \) and \( N_P(x) = \{ s : s_2 = 0, s_1 \geq 0 \} \). Now for any \( x \in \text{ri}(E) \),

\[
\partial f(x) = \partial g(x) + N_P(x) = \{ s + \nabla g(x) : s_1 + ms_2 = 0, s_2 \geq 0 \}.
\]

**Proposition 2 (Edges).** For \( r \) as in [1], a polytope \( P \) and an edge \( E = \{ x : x_2 = mx_1 + c, v^- \leq x_1 \leq v^+ \} \) between vertices \( v^- \) and \( v^+ \), let \( f(x) = r(x) + I_P(x) \), then \( \bigcup_{x \in \text{ri}(E)} \partial f(x) \) is either a parabolic region or a ray.

**Proof.** From Corollary 1 there exists \( l, u \in \mathbb{R}^2 \) such that \( \bigcup_{x \in \text{ri}(E)} \partial r(x) = \bigcup_{x \in \text{ri}(E)} \{ s : C_r(s) = 0, t_1 \leq s_1 \leq t_u \} \). So computing \( \bigcup_{x \in \text{ri}(E)} \partial f(x) \) leads to the following two cases:

**Case 1** \((l = u)\). Same case as when \( r \) is quadratic (known result).

**Case 2** \((l \neq u)\). By setting \( g = r \) in Lemma 2 for any \( x \in \text{ri}(E) \), \( \partial f(x) = \{ s : s_1 + ms_2 = 0, s_2 \geq 0 \} \). Similar to the quadratic case, when \( \nabla r(x) = l \), \( \partial f(x) = \{ s : s_1 + ms_2 - (l_1 + ml_2) = 0, s_2 \geq l_2 \} \) and when \( \nabla r(x) = u \), \( \partial f(x) = \{ s : s_1 + ms_2 - (u_1 + mu_2) = 0, s_2 \geq u_2 \} \). Assume \( \partial f(x) \subset \{ s : C_r(s) \leq 0 \} \) (the case \( \partial f(x) \subset \{ s : C_r(s) \geq 0 \} \) is analogous). Then

\[
\bigcup_{x \in \text{ri}(E)} \partial f(x) = \bigcup_{x \in \text{ri}(E)} \{ s + \nabla r(x) : s_1 + ms_2 = 0, s_2 \geq 0 \}
\]

\[
= \{ s : l_1 + ml_2 \leq s_1 + ms_2 \leq u_1 + mu_2, C_r(s) \leq 0 \}
\]

is a parabolic region.

By gathering Lemma 1, Proposition 2 and Corollary 1 we obtain.

**Theorem 1 (Parabolic domain).** Assuming Conjecture 1 holds, \( r \) as in [1], \( P \) is a polytope, and \( f(x) = r(x) + I_P(x) \). Then \( \bigcup_{x \in E} \partial f(x) \) has a parabolic subdivision.

**Example 1.** For \( r = \frac{36x_1^2 + 21x_1x_2 + 36x_2^2 - 81x_1 + 24x_2 - 252}{-12x_1 + 9x_2 + 75} \) and polytope \( P \) formed by vertices \( v_1 = (-1,1), v_2 = (-3,-3) \) and \( v_3 = (-4,-3) \), let \( f(x) = r(x) + I_P(x) \). We have \( \bigcup_{x \in \text{dom}(r)} \partial r(x) = \{ s : C(s) = 0 \} \) where \( C(s) = 9s_1^2 + 24s_1s_2 - 34s_1 + 16s_2^2 + 200s_2 - 527 \). The parabolic subdivision for this example is shown in Figure 1.
4 Conjugate Expressions

Now that we know \( \text{dom } f^* \) as a parabolic subdivision, we turn to the computation of its expression on each piece. We note

\[
\begin{align*}
g_f(s_1, s_2) &= \frac{\psi_1(s_1, s_2)}{\zeta_{00} \sqrt{\psi_{1/2}(s_1, s_2)}} + \psi_0(s_1, s_2) \quad (2) \\
g_q(s_1, s_2) &= \zeta_{11}s_1^2 + \zeta_{12}s_1s_2 + \zeta_{22}s_2^2 + \zeta_{10}s_1 + \zeta_{01}s_2 + \zeta_{00} \quad (3) \\
g_l(s_1, s_2) &= \zeta_{10}s_1 + \zeta_{01}s_2 + \zeta_{00} \quad (4)
\end{align*}
\]

where \( \psi_0, \psi_{1/2} \) and \( \psi_1 \) are linear functions in \( s \), and \( \zeta_{ij} \in \mathbb{R} \).

**Theorem 2.** Assume Conjecture 1 holds. For \( r \) as in (1), a polytope \( P \), and \( f(x) = r(x) + I_P(x) \), the conjugate \( f^*(s) \) has a parabolic subdivision such that over each member of its subdivision it has one of the forms in (2)-(4).

**Proof.** We compute the critical points for the optimization problem defining \( f^* \).

**Case 1 (Vertices)** For any vertex \( v \), \( f^*(s) = s_1v_1 + s_2v_2 - r(v) \) is a linear function of form (4) defined over an unbounded polyhedral set (from Lemma 1). In the special case, when \( \partial f(v) \) is a parabolic region (Conjecture 1), the conjugate would again be a linear function but defined over a parabolic region.
Case 2 (Edges) Let \( F \) be the set of all the edges, and \( E = \{ x : x_2 = \langle s, x \rangle \} \in F \) be an edge between vertices \( l \) and \( u \), then \( f^*(s) = \sup_{x \in \bar{r}(E)} \{ (s, x) - (r(x) + I_P(x)) \} \). By computing the critical points, we have \( s - (\nabla r(x) + N_P(x)) = 0 \) where \( N_P(x) = \{ s : s = \lambda(-m, 1), \lambda \geq 0 \} \) with \( m \) the slope of the edge. So

\[
\begin{align*}
    s_1 &= -\xi_{21}t^2 + 2\xi_{11}t + \xi_{01} - m\lambda \\
    s_2 &= -\xi_{22}t^2 + 2\xi_{12}t + \xi_{02} + \lambda
\end{align*}
\]

where \( t = \frac{\xi_{11}s_1 + \xi_{12}s_2 + \xi_{10}}{\xi_{21}s_1 + \xi_{22}s_2 + \xi_{20}} \). Since \( x \in \bar{r}(E) \), we have

\( x_2 = mx_1 + c \),

which with (5) gives

\[
x_1 = \begin{cases} 
    \gamma_{10}s_1 + \gamma_{01}s_2 + \gamma_{00} & \text{when } \xi_{21} + m\xi_{22} = 0 \\
    \frac{\gamma_{00} + \xi_{21}}{\xi_{22}} \sqrt{\gamma_{10}/2s_1 + \gamma_{01}/2s_2 + \gamma_{00}/2} & \text{otherwise},
\end{cases}
\]

where all \( \gamma_{ij} \) and \( \gamma_{ij/k} \) are defined in the coefficients of \( r \), and parameters \( m \) and \( c \). When \( \xi_{21} + m\xi_{22} \neq 0 \), solving (5) and (6), leads to a quadratic equation in \( t \) with coefficients as linear functions in \( s \).

By substituting (7) and (6) in \( f^*(s) \), when \( \xi_{21} + m\xi_{22} \neq 0 \), we have

\[
f^*(s) = \frac{-\psi_1(s_1, s_2)}{\xi_{00} \psi_{1/2}(s_1, s_2)} + \psi_0(s_1, s_2),
\]

and when \( \xi_{21} + m\xi_{22} = 0 \),

\[
f^*(s) = \xi_{11}s_1^2 + \xi_{12}s_1s_2 + \xi_{22}s_2^2 + \xi_{10}s_1 + \xi_{01}s_2 + \xi_{00},
\]

where all \( \xi_{ij}, \psi_1 \) and \( \psi_{ij} \) are defined in the coefficients of \( r \), and parameters \( m \) and \( c \), with \( \psi_i(s) \) and \( \psi_{ij}(s) \) linear functions in \( s \).

From Proposition 2 \( \bigcup_{x \in \bar{r}(E)} \partial f(x) \) is either a parabolic region or a ray. So for any \( E \), the conjugate is a fractional function of form (2) defined over a parabolic region. When \( \bigcup_{x \in \bar{r}(E)} \partial f(x) \) is a ray, the computation of the conjugate is deduced from its neighbours by continuity.

Case 3 (Interior) Since \( \bigcup_{x \in \text{int}(P)} \partial f(x) \) is contained in a parabolic arc (from Corollary 1), the computation of the conjugate is deduced by continuity.

Example 2. For a bivariate rational function \( r(x) = \frac{x_2^2}{x_2 - x_1 + 1} \) defined over a polytope \( P \) with vertices \( v_1 = (1, 1), v_2 = (1, 0) \) and \( v_3 = (0, 0) \), let \( f(x) = r(x) + I_P(x) \).
The conjugate (shown in Figure 2) can be written

\[ f^*_{P}(s) = \begin{cases} 
0 & s \in R_1 \\
 s_1 & s \in R_2 \\
 s_1 & s \in R_3 \\
 s_1 + s_2 - 1 & s \in R_4 \\
 \frac{1}{4}(s_1 + s_2)^2 & s \in R_5 
\end{cases} \]

where

\[ R_1 = \{ s : s_2 \geq -s_1 + 2, s_2 \geq 1 \} \]
\[ R_2 = \{ s : s_2 \geq s_1, s_1^2 + 2s_1s_2 - 4s_1 + s_2^2 \leq 0 \} \]
\[ R_3 = \{ s : s_2 \leq s_1, s_2 \leq 1, s_1 \geq 0 \} \]
\[ R_4 = \{ s : 0 \leq s_1, s_2 \leq -s_1 \} \]
\[ R_5 = \{ s : s_2 \geq -s_1, s_2 \leq -s_1 + 2, s_2 \geq s_1, s_1^2 + 2s_1s_2 - 4s_1 + s_2^2 \geq 0 \}. \]

5 Conclusion and future work

Figure 3 summarizes the strategy. Given a PLQ function, for each piece, its convex envelope is computed as the convex envelope of a quadratic function over a polytope using [16]. This is the most time consuming operation since the known algorithms are at least exponential. For each piece, we obtain a piecewise rational function. Then we take each of those pieces, and compute its conjugate to obtain
a fractional function over a parabolic subdivision. That computation is complete except for Conjecture 1. Note that there is only a single problematic vertex \( v \) and since the conjugate is full domain, we can deduce \( \partial f(v) \) by elimination.

Future work will focus on Step 3, which will give the conjugate of the original PLQ function. This will involve solving repeatedly the map overlay problem and is likely to take exponential time. From hundred of examples we ran, we expect the result to be a fractional function of unknown kind over a parabolic subdivision; see Figure 3, bottom row, middle figure. The final step will be to compute the biconjugate (bottom-left in Figure 3). We know it is a piecewise function over a polyhedral subdivision but do not know the formulas.

![Fig. 3. Summary](image)

**References**

1. Al-Khayyal, F.A., Falk, J.E.: Jointly constrained biconvex programming. Mathematics of Operations Research 8(2), 273–286 (1983)
2. Anstreicher, K.M.: On convex relaxations for quadratically constrained quadratic programming. Mathematical Programming 136(2), 233–251 (2012)
3. Bauschke, H.H., Goebel, R., Lucet, Y., Wang, X.: The proximal average: basic theory. SIAM Journal on Optimization 19(2), 766–785 (2008)
4. Bauschke, H.H., Lucet, Y., Trienis, M.: How to transform one convex function continuously into another. SIAM Review 50(1), 115–132 (2008)
5. Brenier, Y.: Un algorithme rapide pour le calcul de transformées de Legendre-Fenchel discrètes. Comptes rendus de l’Académie des sciences. Série 1, Mathématique 308(20), 587–589 (1989)
6. Corrias, L.: Fast Legendre-Fenchel transform and applications to Hamilton-Jacobi equations and conservation laws. SIAM Journal on Numerical Analysis 33(4), 1534–1558 (1996)
7. Crama, Y.: Recognition problems for special classes of polynomials in 0–1 variables. Mathematical Programming 44(1-3), 139–155 (1989)
8. Gardiner, B., Jakee, K., Lucet, Y.: Computing the partial conjugate of convex piecewise linear-quadratic bivariate functions. Computational Optimization and Applications 58(1), 249–272 (2014)
9. Gardiner, B., Lucet, Y.: Convex hull algorithms for piecewise linear-quadratic functions in computational convex analysis. Set-Valued and Variational Analysis 18(3-4), 467–482 (2010)
10. Gardiner, B., Lucet, Y.: Graph-matrix calculus for computational convex analysis. In: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 243–259. Springer (2011)
11. Gardiner, B., Lucet, Y.: Computing the conjugate of convex piecewise linear-quadratic bivariate functions. Mathematical Programming 139(1-2), 161–184 (2013)
12. Haque, T., Lucet, Y.: A linear-time algorithm to compute the conjugate of convex piecewise linear-quadratic bivariate functions. Computational Optimization and Applications 70(2), 593–613 (2018)
13. Hiriart-Urruty, J.B., Lemaréchal, C.: Convex analysis and minimization algorithms II: Advanced Theory and Bundle Methods. Springer Science & Business Media (1993)
14. Jach, M., Michaels, D., Weismantel, R.: The convex envelope of (n–1)-convex functions. SIAM Journal on Optimization 19(3), 1451–1466 (2008)
15. Linderoth, J.: A simplicial branch-and-bound algorithm for solving quadratically constrained quadratic programs. Mathematical Programming 103(2), 251–282 (2005)
16. Locatelli, M.: A technique to derive the analytical form of convex envelopes for some bivariate functions. Journal of Global Optimization 59(2-3), 477–501 (2014)
17. Locatelli, M.: Polyhedral subdivisions and functional forms for the convex envelopes of bilinear, fractional and other bivariate functions over general polytopes. Journal of Global Optimization 66(4), 629–668 (2016)
18. Lucet, Y.: A fast computational algorithm for the Legendre-Fenchel transform. Computational Optimization and Applications 6(1), 27–57 (1996)
19. Lucet, Y.: Faster than the fast Legendre transform, the linear-time Legendre transform. Numerical Algorithms 16(2), 171–185 (1997)
20. Lucet, Y.: Fast Moreau envelope computation i: Numerical algorithms. Numerical Algorithms 43(3), 235–249 (2006)
21. Lucet, Y.: What shape is your conjugate? a survey of computational convex analysis and its applications. SIAM Review 52(3), 505–542 (2010)
22. Lucet, Y., Bauschke, H.H., Trienis, M.: The piecewise linear-quadratic model for computational convex analysis. Computational Optimization and Applications 43(1), 95–118 (2009)
23. McCormick, G.P.: Computability of global solutions to factorable nonconvex programs: Part i—convex underestimating problems. Mathematical Programming 10(1), 147–175 (1976)
24. Rockafellar, R.T., Wets, R.J.B.: Variational Analysis, vol. 317. Springer Science & Business Media (1998)
25. Sherali, H.D., Alameddine, A.: An explicit characterization of the convex envelope of a bivariate bilinear function over special polytopes. Annals of Operations Research 25(1), 197–209 (1990)
26. Tardella, F.: On the existence of polyhedral convex envelopes. In: Frontiers in Global Optimization, pp. 563–573. Springer (2004)
27. Tardella, F.: Existence and sum decomposition of vertex polyhedral convex envelopes. Optimization Letters 2(3), 363–375 (2008)