The General Decomposition Theory of $SU(2)$ Gauge Potential, Topological Structure and Bifurcation of $SU(2)$ Chern Density

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Abstract

By means of the geometric algebra the general decomposition of $SU(2)$ gauge potential on the sphere bundle of a compact and oriented 4-dimensional manifold is given. Using this decomposition theory the $SU(2)$ Chern density has been studied in detail. It shows that the $SU(2)$ Chern density can be expressed in terms of the $\delta$-function $\delta(\phi)$. And one can find that the zero points of the vector fields $\phi$ are essential to the topological properties of a manifold. It is shown that there exists the crucial case of branch process at the zero points. Based on the implicit function theorem and the taylor expansion, the bifurcation of the Chern density is detailed in the neighborhoods of the bifurcation points of $\phi$. It is pointed out that, since the Chern density is a topological invariant, the sum topological chargers of the branches will remain constant during the bifurcation process.

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I. INTRODUCTION

The topological properties of physics systems play important roles in studying some physical problem. It is well known that the gauge potential (connection) and gauge field (curvature) is essential to establish direct relationship between differential geometry and topological invariants. The decomposition theory of gauge potential provides a powerful method in researching some topological properties. It has been effectively used to study the topological gauge theory of dislocation and disclinations in condensed matter physics\textsuperscript{1}, the geometrization of Planck constant in terms of the space time defect in General Relativity\textsuperscript{2,3}, the space-time dislocations in the early universe\textsuperscript{4}, and the Gauss-Bonnet-Chern theorem\textsuperscript{3,5}. The essential feature of the decomposition shows that the gauge potential have inner structure\textsuperscript{6,1}. Generally, the topological characteristics of a manifold are represented by the properties of a smooth vector field on it, or in other words, the smooth vector fields carry the topological information of a manifold, which inspire us to study the decomposition theory of the gauge potential in terms of the unit vector field on the manifold.

It is well known that the $SU(2)$ gauge theory and the second Chern class has been widely used in discussing many physical problem. Such as the magnetic monopole\textsuperscript{6–9}, the anomaly in nonlinear $\sigma$-models\textsuperscript{10}. Especially the problems on four-manifold\textsuperscript{11,12}, the instantons\textsuperscript{13–15}, the merons\textsuperscript{16,17}, the Donsdson theory\textsuperscript{18}, and so on. It urges us to study the decomposition theory of the $SU(2)$ gauge potential and the topological properties on four-manifold.

In this paper, we will establish a general decomposition theory of the $SU(2)$ gauge potential in terms of the unit vector $\vec{n}$ on the compact and oriented 4-dimensional manifold. And by means of geometric algebra, we can describe the unit vector $\vec{n}$ as an element of $Spin(3)$\textsuperscript{19}. One can show that the general decomposition formula of $SU(2)$ gauge potential has a global property. The Chern density of the principal $P(\pi, M, SU(2))$ will be studied by using the decomposition formula. One shows that the $SU(2)$ Chern density takes the form of the $\delta$–function $\delta(\vec{\phi})$. The topological structure of the $SU(2)$ Chern density can be labeled by the Brouwer degrees and the Hopf index. And the further research shows that there exist
the crucial cases of branch process in the topological density, when the Jacobian \( D(\dot{x}) = 0 \). We calculate out the different branches of the Chern density by using the implicit function theorem. It is pointed out that the topological charges will be splitted at the critical points.

This paper is arranged in five sections. In section 2 we will study a general decomposition theory of \( SU(2) \) gauge potential on a sphere bundle. In section 3 we will study the topological structure of the \( SU(2) \) Chern density by using the decomposition expression given in section 2. The bifurcation of the Chern density will be studied in section 4. Then there will be a conclusion at last.

II. THE DECOMPOSITION THEORY OF \( SU(2) \) GAUGE POTENTIAL

In this section we will give the decomposition theory of \( SU(2) \) gauge potential in terms of the sphere bundle on a compact and oriented 4-dimensional manifold. Firstly we have to give the basic notions which are necessary for our discussions.

Let \( V \) be a unit \( SU(2) \) Clifford vector

\[ V = V^a \sigma_a, \quad a = 1, 2, 3 \; \]  

and

\[ V^a V^a = 1. \]  

In which \( \sigma_a \) are Pauli matrixes, and the base of \( SU(2) \) Clifford algebra\(^20\).

The covariant derivative 1-form of \( V \) is given by

\[ DV = dV - [A, V] \]  

where \( A \) is the \( SU(2) \) gauge potential 1-form:

\[ A = i/2A^a \sigma_a , \]  

and
\[ A^a = A_\mu^a dx^\mu \quad \mu = 0, 1, 2, 3. \quad (5) \]

In gauge theory, the potential 1-form undergoes the gauge transformation:
\[ A' = SAS^{-1} + dSS^{-1}. \quad (6) \]

In our viewpoint\(^6\), the gauge potential 1-form \( A \) can be decomposed and has inner structure. The main feature of the decomposition theory of the gauge potential is that the gauge potential \( A \) can be generally decomposed as
\[ A = a + b, \quad (7) \]

where \( a \) and \( b \) are required to satisfy the gauge transformation and vector covariant transformation rules, i.e.,\(^19\)
\[ a' = SaS^{-1} + dSS^{-1}, \quad (8) \]

and
\[ b = SbS^{-1}. \quad (9) \]

From (8) and (9), one can show that the gauge potential \( A \) rigorously satisfies the gauge transformation
\[ A' = a' + b' = S(a + b)S^{-1} + dSS^{-1} \]

Let \( V_i \) \((i = 1, 2, 3)\) be an orthonormal basis of \( SU(2) \) Clifford vector with the orthogonal relations:
\[ V_i \cdot V_j = \delta_{ij} \quad (10) \]

i.e.
\[ V_i^a V_j^a = \delta_{ij} \cdot \]

where \( V_i \cdot V_j \) is the Clifford scalar product defined by\(^19\)
\[ V_i \cdot V_j = \frac{1}{2}(V_i V_j + V_j V_i). \quad (11) \]
Since the potential $A$ is also a $SU(2)$ Clifford vector, we have the projection formula

$$A = (A \cdot V_{(i)}) V_{(i)}. \quad (12)$$

Substituting this formula into (3), we obtain

$$DV_{(i)} = dV_{(i)} - (A \cdot V_{(j)}) [V_{(j)}, V_{(i)}], \quad (13)$$

Using (10), (11) and (13), and considering that

$$[V_{(j)}, V_{(i)}] = 2V_{(j)} V_{(i)} - 2V_{(i)} \cdot V_{(j)}, \quad (14)$$

we can rewrite $A$ as

$$A = 1/4dV_{(j)} V_{(i)} - 1/4DV_{(j)} V_{(i)}. \quad (15)$$

According to (6), we define that

$$a = \frac{1}{4}dV_{(j)} V_{(i)}, \quad (16)$$

and

$$b = \frac{1}{4}DV_{(j)} V_{(i)}. \quad (17)$$

It is easy to prove that the decomposition formula above satisfies the requirement in (6), even has global property and is independent from the local coordinates.

Let a family \{W, V, U, \ldots\} be an open cover of $M$ and $S_{uw}$ be the transition matrix function which satisfy the following condition\textsuperscript{21}

$$S_{uu} = 1, \quad S_{vu}^{-1} = S_{uv}, \quad S_{wu}S_{wu}S_{uw} = I; \quad W \cap V \cap U \neq \emptyset. \quad (18)$$

For any two open neighborhoods $V$ and $U$, if $V \cap U \neq \emptyset$, then

$$V_{(j)u} = S_{vu} V_{(i)} S_{wu}^{-1}, \quad (19)$$

where the subscripts "$u$" and "$v$" are represent the open covers $U$ and $V$ correspondingly, and we know that the gauge potential $A$ undergoes the gauge transformation
\[ A_v = S_{vu} A_u S_{vu}^{-1} + dS_{vu} S_{vu}^{-1}, \]  

which is the fundamental condition for the existence of the gauge potential on the principal \( P(\pi, M, SU(2)) \). In the physics terminology \( S_{vu} \) is just the gauge transformation in the gauge theory. In the following, for abbreviation, we shall use the notation \( S_{vu} = S \).

According to (13), we deduce that

\[ dV^{(i)}_v V^{(i)}_v = S dV^{(i)}_u V^{(i)}_u S^{-1} + dSS^{-1} - V^{(i)}_v dSS^{-1} V^{(i)}_v. \]  

(21)

Considering that \( A \) is a vector of Clifford algebra, we can see that \( dSS^{-1} \) is a vector of Clifford algebra also, then there exist the following formula

\[ V^{(i)}_v dSS^{-1} V^{(i)}_v = -3dSS^{-1}. \]  

(22)

Noticing that

\[ DV^{(i)}_v = SDV^{(i)}_u S^{-1}, \]

from (14), and making use of (19), (21) and (22), one can easily obtain

\[ A_v - \frac{1}{4} (dV^{(i)}_v V^{(i)}_v - DV^{(i)}_v V^{(i)}_v) = S[A_u - \frac{1}{4} (DV^{(i)}_u V^{(i)}_u - DV^{(i)}_u V^{(i)}_u)] S^{-1}. \]  

(23)

The expression above shows that if the decomposition formula on the open neighborhood \( U \)

\[ A_v = \frac{1}{4} (dV^{(i)}_v V^{(i)}_v - DV^{(i)}_v V^{(i)}_v) \]

holds true, the decomposition formula on the open neighborhood \( U \)

\[ A_u = \frac{1}{4} (dV^{(i)}_u V^{(i)}_u - DV^{(i)}_u V^{(i)}_u) \]

must holds true, too. This means that the general decomposition formula (13) has a global property and is independent from the choice of the local coordinates.

We know that the characteristic class is the fundamental topological property, and it is independent of the gauge potential. So, to discuss the Chern class, we can take \( A \) as

\[ A = \frac{1}{4} dU^{(i)} U^{(i)}. \]  

(24)
One can regard it as a special gauge. By use of this formula, the magnetic monopole can be studied as in \(^6\to^8\). In this paper we want to discuss the topological properties of a 4-dimensional manifold, a decomposition formula in terms of the sphere bundle over the manifold will be convenient.

Now, let us return to discuss the decomposition theory on the sphere bundle of a compact and oriented 4-dimensional manifold. Let \(M\) be a compact and oriented 4-dimensional manifold and \(P(\pi, M, G)\) be a principal bundle with the structure group \(G = SU(2)\). A smooth vector field \(\phi^A(A = 0, 1, 2, 3)\) can be found on the base manifold \(M\). We define a unit vector field on \(M\) as

\[
n^A = \frac{\phi^A}{||\phi||} \quad A = 0, 1, 2, 3;
\]

\[
||\phi|| = \sqrt{\phi^A \phi^A},
\]

in which the superscripts “\(A\)” are the local orthonormal frame index.

In fact \(\vec{n}\) is identified as a section of the sphere bundle over \(M\) (or a partial section of the vector bundle over \(M\))\(^5\). We see that the zeros of \(\vec{\phi}\) are just the singular points of \(\vec{n}\). Since the global property of a manifold has close relation with zeros of a smooth vector fields on it, this expression of the unit vector \(\vec{n}\) is a very powerful tool in the discussion of the global topology. It naturally guarantee the constraint

\[
n^A n^A = 1. \quad (26)
\]

We can express the unit vector \(\vec{n}\) in terms of Clifford algebra\(^{23}\) as

\[
n = n^A s_A, \quad A = 0, 1, 2, 3. \quad (27)
\]

where

\[
s = (I, i\vec{\sigma}), \quad s^\dagger = (I, -i\vec{\sigma}). \quad (28)
\]

We know that \(n\) is just an element of \(Spin(3)\) in terms of geometric algebra\(^{19}\). And we can rewrite (26) as
\[ nn^\dagger = 1. \quad (29) \]

It is easy to see that \( n \) has three independent components. Since the orthonormal basis of \( SU(2) \) Clifford one-vector just have three independent components, we can express \( U_{(i)} \) in terms of an element of \( Spin(3) \). From gauge transformations of the orthonormal basis given in \(^{24}\), and the spinorial transformation given in \(^{25}\), one can expand \( U_{(i)} \) in terms of an element of \( Spin(3) \) as:

\[
U^a_{(i)} = 2(n^0)^2 \delta^a_i + 2n^0 n^b \epsilon_{iba} + 2n^i n^a - \delta^a_i. \quad (30)
\]

It is easy to prove

\[
U^a_{(i)} U^b_{(j)} = \delta_{ij}. \quad (31)
\]

Using \(^{30}\) and \(^{24}\), as a result, the gauge potential \( A \) will be expressed as

\[
A = dn n^\dagger. \quad (32)
\]

Because \( n \) is an unit element of the sphere bundle over \( M \), we will discuss the topological property of the principle \( P(\pi, M, SU(2)) \) directly by using this formula.

**III. THE SU(2) CHERN DENSITY AND ITS INNER STRUCTURE**

It is well known that gauge potential(connection) and gauge field(curvature) play essential roles in discussing the topological properties of a manifold. For the principal \( P(\pi, M, SU(2)) \) the \( SU(2) \) Chern density is a important topological characteristic. In this section we will discuss the \( SU(2) \) Chern density by using the decomposition formula \(^{32}\) which we have just given in the last section.

We know that the second Chern class is the fundamental characteristic of the principal \( P(\pi, M, SU(2)) \), it is denoted as:

\[
C_2(P) = \frac{1}{8\pi^2} Tr(F \wedge F). \quad (33)
\]
The curvature $F$ is defined as

$$ F = dA - A \wedge A. \quad (34) $$

One can show that the gauge field $F$ is generalized function when there exit $m$ singular points $z_i \ (i = 1, 2, \cdots, m)$ in the unit vector field $n$. Let us substitute the formula (32) into (34), then

$$ F \begin{cases} = 0 & \text{when } x \neq z_i \\ \neq 0 & \text{when } x = z_i \end{cases}. \quad (35) $$

where $z_i$ are the singular points of $n$. This means that the gauge field $F$ vanishes at the region where $n$ has no singular points, but at the singular points of $n$ the gauge field $F$ does not vanish. This feature we will show soon is due to the non-triviality property of the principal $P(\pi, M, SU(2))$, and is essential to study some physical problem.

The second $SU(2)$ Chern class can be rewrite in terms of Chern-Simon$$C_2(P) = \frac{1}{8\pi^2} d\Omega, \quad (36)$$

and

$$ \Omega = \frac{1}{8\pi^2} \text{Tr} \left( A \wedge dA - \frac{2}{3} A \wedge A \wedge A \right) \quad (37)$$

which is known as Chern-Simon form.

Substituting (32) into (36) and considering (29), we obtain

$$ C_2(P) = \frac{1}{24\pi^2} \text{Tr} (dn \wedge dn^\dagger \wedge dn \wedge dn^\dagger). \quad (38) $$

In detail, substituting (27) into the formula above

$$ C_2(P) = \frac{1}{24\pi^2} \varepsilon^{\mu\nu\lambda\rho} \partial_{\mu} n^A \partial_{\nu} n^B \partial_{\lambda} n^C \partial_{\rho} n^D \text{Tr} \left( s_A s_B^\dagger s_C s_D^\dagger \right) dx^4$$

$$ = \frac{1}{12\pi^2} \varepsilon^{\mu\nu\lambda\rho} \varepsilon_{ABCD} \partial_{\mu} n^A \partial_{\nu} n^B \partial_{\lambda} n^C \partial_{\rho} n^D dx^4. \quad (39)$$

By substituting (25) into (39), considering
\[ \frac{dn^A}{||\phi||} + \phi^A d \left( \frac{1}{||\phi||} \right), \]  

we have

\[ C_2(P) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \phi^A \partial \phi^A} \left( \frac{1}{||\phi||^2} \right) D(\phi/x) \, d^4x, \]  

where \( D(\phi/x) \) is the Jacobian defined as

\[ \varepsilon^{ABCD}(\phi/x) =\varepsilon^{\mu\nu\lambda\rho} \partial_\mu \phi^A \partial_\nu \phi^B \partial_\lambda \phi^C \partial_\rho \phi^D. \]  

By means of the general Green function formula

\[ \frac{\partial^2}{\partial \phi^A \partial \phi^A} \left( \frac{1}{||\phi||^2} \right) = -4\pi \delta^4(\vec{\phi}), \]  

we have

\[ C_2(P) = \delta^4(\vec{\phi}) \, D(\phi/x) \, d^4x. \]  

Suppose \( \phi^A(x) \) \((A = 0, 1, 2, 3)\) have \( m \) isolated zeros at \( x_\mu = z_\mu^i \) \((i = 1, 2, \ldots, m)\), according to the \( \delta - Function \) theory\(^{28}\), \( \delta(\vec{\phi}) \) can be expressed by

\[ \delta(\vec{\phi}) = \sum_{i=1}^{m} \beta_i \delta(\vec{x} - \vec{z}_i), \]  

and one then obtains

\[ C_2(P) = \sum_{i=1}^{m} \eta_i \beta_i \delta^4(x - z_i) \, d^4x, \]  

where \( \beta_i \) is a positive integer (the Hopf index of the \( i \)th zeros) and \( \eta_i \) is the Brouwer degree\(^{29}\):

\[ \eta_i = \frac{D(\phi/x)|_{x=z_i}}{|D(\phi/x)|} = sgn[D(\phi/x)|_{x=z_i}] = \pm 1. \]  

The meaning of the Hopf index \( \beta_i \) is that the vector field function \( \vec{\phi} \) covers the corresponding region \( \beta_i \) times while \( \vec{x} \) covers the region neighborhood of zero \( z_i \) once. From above discussion, the Chern density \( \rho(M) \) is defined as:

\[ \rho(M) = \sum_{i=1}^{m} \eta_i \beta_i \delta^4(x - z_i), \]
which shows that the topological structure of Chern density $\rho$ is labeled by the Brouwer degrees and the Hopf index. The integration of $\rho(M)$ on $M$

$$C_2 = \int_M \rho(M) d^4x = \sum_{i=1}^{m} \eta_i \beta_i$$

is integer called Chern number which is a topological invariant of $M$.

The result (49) suggest that the zeros points of the smooth vector $\vec{\phi}$ are essential to the topological properties of the base manifold $M$. On the other hand, the density (48) can be regarded as the density of a system of $k$ classical point-like particles with topological invariant charges $g_i = \eta_i \beta_i$ on the 4-dimensional manifold.

**IV. THE BIFURCATION OF CHERN DENSITY**

As being discussed before, the zeros of the smooth vector $\vec{\phi}$ play important roles in studying the Chern Class of the manifold $M$. In this section, we will study the properties of the zero points, in other words, the properties of the following equations solutions

$$\begin{cases}
\phi^0(x^0, x^1, x^2, x^3) = 0 \\
\phi^1(x^0, x^1, x^2, x^3) = 0 \\
\phi^2(x^0, x^1, x^2, x^3) = 0 \\
\phi^3(x^0, x^1, x^2, x^3) = 0
\end{cases}$$

As we knew before, if the Jacobian determinant

$$D(\frac{\phi}{x}) = \frac{\partial(\phi^0, \phi^1, \phi^2, \phi^3)}{\partial(x^0, x^1, x^2, x^3)} \neq 0,$$

we will have the isolated solutions of (50). The isolated solution are called regular points. It is easy to see that the result in section 3 is based on this condition. However, when this condition fails, the above results will change in some way, and will lead to the branch process of topological density and give rise to the bifurcation.

In order to show this case easily, we want to have some suppose. It is convinient to let $x^0 = t$, and suppose $\phi^0 = \phi^0(t)$. We denote one of zero points as $(t^*, \vec{z}_i)$. We know that if

$$D(\frac{\phi}{x})|_{(t^*, \vec{z}_i)} = \frac{\partial(\phi^1, \phi^2, \phi^3)}{\partial(x^1, x^2, x^3)}|_{(t^*, \vec{z}_i)} = 0,$$
the Jacobian $D(\vec{\phi})(t^*, \vec{z}_i) = 0$ will be obtained automatically. We will show that this case will lead to the branch process of topological density. In this case, the equations (50) will be rewritten as

$$\phi^0(t) = 0,$$

(51)

and

$$\begin{cases}
\phi^1(t, x^1, x^2, x^3) = 0 \\
\phi^2(t, x^1, x^2, x^3) = 0 \\
\phi^3(t, x^1, x^2, x^3) = 0
\end{cases}$$

(52)

It is well-known that when the Jacobian $D(\vec{\phi})(t^*, \vec{z}_i) = 0$, the usual implicit function theorem is no use. But if the Jacobian

$$D^1(\vec{\phi})(t^*, \vec{z}_i) = 0,$$

we can use the Jacobian $D^1(\vec{\phi})(t^*, \vec{z}_i)$ instead of $D(\vec{\phi})(t^*, \vec{z}_i) = 0$, for the purpose of using the implicit function theorem. Then we have an unique solution of the equations (52) in the neighborhood of the points $(t^*, \vec{z}_i)$

$$t = t(x^1)$$

$$x^i = x^i(x^1) \quad i = 2, 3.$$

(53)

with $t^* = t(x^1)$. And we call the critical points $(t^*, \vec{z}_i)$ the limit points. In the present case, it is easy to know that

$$\frac{dx^1}{dt}(t^*, \vec{z}_i) = \frac{D^1(\vec{\phi})(t^*, \vec{z}_i)}{D(\vec{\phi})(t^*, \vec{z}_i)} = \infty$$

i.e.

$$\frac{dt}{dx^1}(t^*, \vec{z}_i) = 0.$$

Then we have the Taylor expansion of (53) at the point $(t^*, \vec{z}_i)$

$$t = t^* + \frac{dt}{dx^1}(t^*, \vec{z}_i)(x^1 - z^1_i) + \frac{1}{2} \frac{d^2t}{(dx^1)^2}(t^*, \vec{z}_i)(x^1 - z^1_i)^2$$

$$= t^* + \frac{1}{2} \frac{d^2t}{(dx^1)^2}(t^*, \vec{z}_i)(x^1 - z^1_i)^2.$$
Therefore
\[ t - t^* = \frac{1}{2} \frac{d^2 t}{(dx^1)^2} |(t^*, \vec{z}_i)(x^1 - z_i^1)^2 \] (54)

which is a parabola in the \( x^1 - t \) plane. From (54), we can obtain the two solutions \( x_1^1(t) \) and \( x_2^1(t) \), which give the branch solutions of the system (50). If \( \frac{d^2 t}{(dx^1)^2} |(t^*, \vec{z}_i) > 0 \), we have the branch solutions for \( t > t^* \), otherwise, we have the branch solutions for \( t > t^* \). These two condition are related to the origin and annihilation of topological charges\(^{31}\). Since the Chern number (49) is a topological invariant, the topological number of these two must be opposite at the zero point, i.e.
\[ \beta_{i1} \eta_{i1} = - \beta_{i2} \eta_{i2}. \]

For a limit point, it also requires the \( D^1(\frac{\partial}{\partial x}) |(t^*, \vec{z}_i) \neq 0 \). As to a bifurcation point\(^{32}\), it must satisfy a more complement condition. This case will be discussed in the following subsections in detail.

**IV.1 The branch process at the bifurcation point.**

In this subsection, we have the restrictions of the system (50) at the bifurcation point \((t^*, \vec{z}_i)\)
\[
\begin{align*}
D(\frac{\partial}{\partial x}) |(t^*, \vec{z}_i) &= 0 \\
D^1(\frac{\partial}{\partial x}) |(t^*, \vec{z}_i) &= 0
\end{align*}
\] (55)

These will lead to an important fact that the function relationship between \( t \) and \( x^1 \) is not unique in the neighborhood of the bifurcation point \((\vec{z}_i, t^*)\). It is easy to see from the equation
\[
\frac{dx^1}{dt} |(t^*, \vec{z}_i) = \frac{D^1(\frac{\partial}{\partial x}) |(t^*, \vec{z}_i)}{D(\frac{\partial}{\partial x}) |(t^*, \vec{z}_i)}
\] (56)
which under the restraint (54) directly shows that the direction of the integral curve of the equation (56) is indefinite at the point \((\vec{z}_i, t^*)\). This is why the very point \((\vec{z}_i, t^*)\) is called a bifurcation point of the system (50).
Next, we will find a simple way to search for the different directions of all branch curves at the bifurcation point. Assume that the bifurcation point \((z_i, t^*)\) has been found from (52). We know that, at the bifurcation point \((z_i, t^*)\), the rank of the Jacobian matrix \(\frac{\partial \phi}{\partial x}\) is smaller than 3. First, we suppose the rank of the Jacobian matrix \(\frac{\partial \phi}{\partial x}\) is 2 (the case of a more smaller rank will be discussed later). Suppose that the \(2 \times 2\) submatrix \(J_1(\frac{\phi}{x})\) is

\[
J_1(\frac{\phi}{x}) = \begin{pmatrix}
\frac{\partial \phi^1}{\partial x^1} & \frac{\partial \phi^1}{\partial x^3} \\
\frac{\partial \phi^2}{\partial x^1} & \frac{\partial \phi^2}{\partial x^3}
\end{pmatrix},
\]

and its determinant \(D_1(\frac{\phi}{x})\) does not vanish. The implicit function theorem says that there exist one and only one function relation

\[
x^i = f^i(x^1, t), \quad i = 2, 3
\]

We denoted the partial derivatives as

\[
f_1^i = \frac{\partial f^i}{\partial x^1}, \quad f_1^i = \frac{\partial f^i}{\partial t}, \quad f_{11}^i = \frac{\partial^2 f^i}{\partial x^1 \partial x^1}, \quad f_{1t}^i = \frac{\partial^2 f^i}{\partial x^1 \partial t}, \quad f_{tt}^i = \frac{\partial f^i}{\partial t \partial t}
\]

From (50) and (58) we have for \(a = 1, 2, 3\)

\[
\phi^a = \phi^a(x^1, f^2(x^1, t), f^3(x^1, t), t) = 0
\]

which give

\[
\frac{\partial \phi^a}{\partial x^1} = \phi_1^a + \sum_{j=2}^{3} \frac{\partial \phi^a}{\partial f^j} \frac{\partial f^j}{\partial x^1} = 0
\]

\[
\frac{\partial \phi^a}{\partial t} = \phi_t^a + \sum_{j=2}^{3} \frac{\partial \phi^a}{\partial f^j} \frac{\partial f^j}{\partial t} = 0.
\]

from which we can get the first order derivatives of \(f^i\): \(f_1^i\) and \(f_1^i\). Differentiating (50) with respect to \(x^1\) and \(t\) respectively we get

\[
\sum_{j=2}^{3} \phi_j^a f_{11}^j = - \sum_{j=2}^{3} [2\phi_j^a f_1^j + \sum_{k=2}^{3} (\phi_j^a f^k_1) f_{11}^j] - \phi_1^a \quad a = 1, 2, 3
\]

\[
\sum_{j=2}^{3} \phi_j^a f_{1t}^j = - \sum_{j=2}^{3} [\phi_j^a f_{1}^j + \phi_j^a f_{1}^j + \sum_{k=2}^{3} (\phi_j^a f^k_1) f_{1t}^j] - \phi_{1t}^a \quad a = 1, 2, 3
\]
And the differentiation of (61) with respect to $t$ gives
\[ \sum_{j=2}^{3} \phi^a_j f^j_t = - \sum_{j=2}^{3} [2 \phi^a_j f^j_t + \sum_{k=2}^{3} (\phi^a_{jk} f^k_t)] - \phi^a_{tt} \quad a = 1, 2, 3 \] (64)

where
\[ \phi^a_{jk} = \frac{\partial^2 \phi^a}{\partial x^j \partial x^k}, \quad \phi^a_{jt} = \frac{\partial^2 \phi^a}{\partial x^j \partial t}. \] (65)

The differentiation of (61) with respect to $x^1$ gives the same expression as (63). By making use of the Gaussian elimination method to (63), (64) and (65) we can find the second order derivatives $f^1_{11}, f^1_{1t}$ and $f^1_{tt}$. The above discussion does no matter to the last component $\phi^3(\vec{x}, t)$. In order to find the different values of $dx^1/dt$ at the bifurcation point $(\vec{z}_i, t^*)$, let us investigate the Taylor expansion of $\phi^3(\vec{x}, t)$ in the neighborhood of $(\vec{z}_i, t^*)$. Substituting (58) into $\phi^3(\vec{x}, t)$ we have the function of two variables $x^1$ and $t$
\[ F(x^1, t) = \phi^3(x^1, f^2(x^1, t), f^3(x^1, t), t) \] (66)

which according to (52) must vanish at the bifurcation point
\[ F(z^1_i, t^*) = 0. \] (67)

From (60) we have the first order partial derivatives of $F(x^1, t)$
\[ \frac{\partial F}{\partial x^1} = \phi_1^3 + \sum_{j=2}^{3} \phi_j^3 f^j_1, \quad \frac{\partial F}{\partial t} = \phi_t^3 + \sum_{j=2}^{3} \phi_j^3 f^j_t. \] (68)

Using (60) and (61) the first equation of (55) is expressed as
\[ D(\vec{\phi}_x)|_{(\vec{z}, t^*)} = \begin{vmatrix} - \sum_{j=2}^{3} \phi_j^1 f^j_1 & \phi_1^1 & \phi_1^3 \\ - \sum_{j=2}^{3} \phi_j^2 f^j_1 & \phi_2^2 & \phi_2^3 \\ \phi_1^3 & \phi_2^3 & \phi_3^3 \end{vmatrix}_{(\vec{z}, t^*)} = 0 \] (69)

which by Cramer’s rule can be written as
\[ D(\vec{\phi}_x)|_{(\vec{z}, t^*)} = \frac{\partial F}{\partial x^1} \det J_1(\vec{\phi}_x)|_{(\vec{z}, t^*)} = 0 \]
Since \( det.J_1(\vec{x})|_{(\vec{z}, t^*)} \neq 0 \), the above equation gives
\[
\left. \frac{\partial F}{\partial x^1} \right|_{(\vec{z}, t^*)} = 0.
\] (70)

With the same reasons, we have
\[
\left. \frac{\partial F}{\partial t} \right|_{(\vec{z}, t^*)} = 0.
\] (71)

The second order partial derivatives of the function \( F \) are easily to find out to be
\[
\frac{\partial^2 F}{(\partial x^1)^2} = \phi_{11}^3 + \sum_{j=2}^{3} [2\phi_{1j}^3 f_{1j}^1 + \phi_{1j}^3 f_{1j}^1 + \sum_{k=2}^{3} (\phi_{kj}^3 f_{k}^1)]
\] (72)
\[
\frac{\partial^2 F}{\partial x^1 \partial t} = \phi_{1t}^3 + \sum_{j=2}^{3} [\phi_{1j}^3 f_{1j}^t + \phi_{1j}^3 f_{1j}^t + \sum_{k=2}^{3} (\phi_{kj}^3 f_{k}^t)]
\] (73)
\[
\frac{\partial^2 F}{\partial t^2} = \phi_{tt}^3 + \sum_{j=2}^{3} [2\phi_{jt}^3 f_{jt}^t + \phi_{jt}^3 f_{jt}^t + \sum_{k=2}^{3} (\phi_{jk}^3 f_{k}^t)]
\] (74)

which at \((\vec{z}, t^*)\) are denoted by
\[
A = \left. \frac{\partial^2 F}{(\partial x^1)^2} \right|_{(\vec{z}, t^*)}, \quad B = \left. \frac{\partial^2 F}{\partial x^1 \partial t} \right|_{(\vec{z}, t^*)}, \quad C = \left. \frac{\partial^2 F}{\partial t^2} \right|_{(\vec{z}, t^*)}.
\] (75)

Then taking notice of (77), (70), (71) and (75) the Taylor expansion of \( F(x^1, t) \) in the neighborhood of the bifurcation point \((\vec{z}, t^*)\) can be expressed as
\[
F(x^1, t) = \frac{1}{2} A(x^1 - z^1_1)^2 + B(x^1 - z^1_1)(t - t^*) + \frac{1}{2} C(t - t^*)^2
\] (76)

which by (56) is the expression of \( \phi^3(\vec{x}, t) \) in the neighborhood of \((\vec{z}, t^*)\). The expression (76) is reasonable, which shows that at the bifurcation point \((\vec{z}, t^*)\) one of the equations (50), \( \phi^3(\vec{x}, t) = 0 \), is satisfied, i.e.
\[
A(x^1 - z^1_1)^2 + 2B(x^1 - z^1_1)(t - t^*) + C(t - t^*)^2 = 0.
\] (77)

Dividing (77) by \((t - t^*)^2\) and taking the limit \( t \to t^* \) as well as \( x^1 \to z^1_1 \) respectively we get
\[
A \left( \frac{dx^1}{dt} \right)^2 + 2B \frac{dx^1}{dt} + C = 0.
\] (78)
In the same way we have

\[ C \left( \frac{dt}{dx^1} \right)^2 + 2B \frac{dt}{dx^1} + A = 0. \]  \hspace{1cm} (79)

The different directions of the branch curves at the bifurcation point are determined by (78) or (79). It is easy to see that there are at most two different branches.

IV.2 The branch process at a higher degenerated point.

In the subsection 3.2, we have studied the case that the rank of the Jacobian matrix \( \left[ \frac{\partial \phi}{\partial x} \right] \) of the equations (50) is \( 2 = 3 - 1 \). In this subsection, we consider the case that the rank of the Jacobian matrix is \( 1 = 3 - 2 \). Let the \( J_2(\vec{\phi}) = \frac{\partial \phi^1}{\partial x^2} \) and suppose that \( \text{det}J_2 \neq 0 \). With the same reasons of obtaining (58), we can have the function relations

\[ x^3 = f^3(x^1, x^2, t) \]  \hspace{1cm} (80)

Substituting the relations (80) into the last two equations of (50), we have the following two equations with three arguments \( x^1, x^2, t \)

\[
\begin{align*}
F_1(x^1, x^2, t) &= \phi^2(x^1, x^2, f^3(x^1, x^2, t), t) = 0 \\
F_2(x^1, x^2, t) &= \phi^3(x^1, x^2, f^3(x^1, x^2, t), t) = 0.
\end{align*} \]  \hspace{1cm} (81)

Calculating the partial derivatives of the function \( F_1 \) and \( F_2 \) with respect to \( x^1, x^2 \) and \( t \), taking notice of (80) and using six similar expressions to (70) and (71), i.e.

\[
\left. \frac{\partial F_j}{\partial x^1} \right|_{(\vec{z}, t^\ast)} = 0, \quad \left. \frac{\partial F_j}{\partial x^2} \right|_{(\vec{z}, t^\ast)} = 0, \quad \left. \frac{\partial F_j}{\partial t} \right|_{(\vec{z}, t^\ast)} = 0, \quad j = 1, 2, \]  \hspace{1cm} (82)

we have the following forms of Taylor expressions of \( F_1 \) and \( F_2 \) in the neighborhood of \( (\vec{z}, t^\ast) \)

\[
F_j(x^1, x^2, t) \approx A_{j1}(x^1 - z^1_i)^2 + A_{j2}(x^1 - z^1_i)(x^2 - z^2_i) + A_{j3}(x^1 - z^1_i) \]

\[
(t - t^\ast) + A_{j4}(x^2 - z^2_i)^2 + A_{j5}(x^2 - z^2_i)(t - t^\ast) + A_{j6}(t - t^\ast)^2 = 0 \]

\[
\begin{align*}
&j = 1, 2. \hspace{1cm} (83)
\end{align*}
\]
In case of $A_{j1} \neq 0, A_{j4} \neq 0$, dividing (83) by $(t - t^*)^2$ and taking the limit $t \to t^*$, we obtain two quadratic equations of $\frac{dx^1}{dt}$ and $\frac{dx^2}{dt}$

$$A_{j1}(\frac{dx^1}{dt})^2 + A_{j2}\frac{dx^1}{dt} \frac{dx^2}{dt} + A_{j3} \frac{dx^1}{dt} + A_{j4}(\frac{dx^2}{dt})^2 + A_{j5} \frac{dx^2}{dt} + A_{j6} = 0 \quad (84)$$

$$j = 1, 2.$$  

Eliminating the variable $dx^1/dt$, we obtain an equation of $dx^2/dt$ in the form of a determinant

$$\begin{vmatrix}
A_{11} & A_{12} & A_{14}v^2 + A_{15}v + A_{16} & 0 \\
0 & A_{11} & A_{12}v + A_{13} & A_{14}v^2 + A_{15}v + A_{16} \\
A_{21} & A_{22}v + A_{23} & A_{24}v^2 + A_{25}v + A_{26} & 0 \\
0 & A_{21} & A_{22}v + A_{23} & A_{24}v^2 + A_{25}v + A_{26}
\end{vmatrix} = 0 \quad (85)$$

where $v = \frac{dx^2}{dt}$, that is a 4th order equation of $dx^2/dt$

$$a_0(\frac{dx^2}{dt})^4 + a_1(\frac{dx^2}{dt})^3 + a_2(\frac{dx^2}{dt})^2 + a_3\frac{dx^2}{dt} + a_4 = 0. \quad (86)$$

Therefore we get different directions at the bifurcation point corresponding to different branch curves. The number of different branch curves is at most four. At the end of this section, we conclude that in our theory of topological density there exist the crucial case of branch process. This means that a normal point-like topological charge, when moves through the bifurcation point, may split into several point-like topological charges moving along different branch curves. Since the topological density is a invariant, the total charge of the splitting topological particles must precisely equal to the topological charge of the original particle.

Since the Chern density is identically conserved, the sum of the topological charges of these splitted topological density must be equal to that of the original current at the bifurcation point. We suppose that there exist $l$ different branches. Then the topological density of Chern class $\rho(M)$ changes in the following form

$$\rho(M) = \sum_{i=1}^k \rho_i = \sum_{i=1}^k \sum_{j=1}^l \rho_{ij}$$
where
\[ \rho_i = g_i \delta^4 (x - z_i), \quad \rho_{ij} = g_{ij} \delta^4 (x - z_{ij}), \quad (87) \]

\[ 1 \leq j \leq l. \]

With the same reason, the sum of the topological charges at the bifurcation points will be have the following form
\[ g_i = \sum_{i=1}^{l} g_{ij}. \]

for fixed \( i \).

\section*{V. CONCLUSION}

We have explicitly constructed the gauge potential decomposition theory of \( SU(2) \) gauge theory in terms of the sphere bundle on a 4-dimensional manifold. An important observation of the application of this theory is that the \( SU(2) \) Chern density takes the form of a general function. And we find that the Chern density have the bifurcation process. It is shown that the topological charges are splitted under this case, and the total charges is conserved. Those features are very important in discussing the topological problems on Four-Manifold. Since many problems, not only in theoretical physics, but also in differential geometry, are associated with the \( SU(2) \) gauge theory, the theory of \( SU(2) \) gauge potential decomposition, the inner structure of the \( SU(2) \) Chern density will provide an important and powerful methods in those fields.

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\(^1\)Y.S. Duan and S.L. Zhang, Int. J. Eng. Sci. \textbf{28} 689 (1990); \textbf{29} 153 (1991); Int. J. Eng. Sci. \textbf{30} 153 (1992).

\(^2\)Y.S. Duan, S.L. Zhang and S.H. Feng, J. Math.Phys. \textbf{35} 9 (1994).
3Y.S. Duan and X.H. Meng, J. Math. Phys. 34 1149 (1993); Y.S. Duan and Lee, X.G.: Helv. Phys. Acta. 68 513 (1995).

4Y.S. Duan, G.H. Yang and Y. Jing, Helv. Phys. Acta. (to be published).

5J.S. Dowker and J.P. Schofield, J. Math. Phys. 31 808 (1990).

6Y.S. Duan and M.L. Ge, Kexue Tongbao 26 282 (1976); Sci. Sinica 11 1072 (1979); Y.S. Duan and S.C. Zhao, Commun. Theory Phys. 2 1553 (1983).

7Y.S. Duan, SLAC-PUB-3301/84.

8Y.S. Duan and J.C. Liu, Proceedings of Johns Hopkins Workshop 11, edited by Y.S. Duan, G. Domokos and Kovesi-Domokos, S. (World Scientific, Singapore, 1988).

9J.M.F. Labastida and M. Marino, Nucl. Phys. B456 633 (1996).

10J. Wess and B. Zumino, Phys. Lett. B37 95 (1971).

11E. Witten, Monopoles and Four-Manifold, hep-th/9411102 IASSNS-HEP-94-96.

12E. Witten, J. Math. Phys. 35 5101 (1994).

13C. Michael and P. S. Spencer, Phys. Rev. D. 52 4691 (1996).

14A. Galperin and E. Sokatchev, Report BONN-TH-94-27 24 (1994).

15P. M. Sotcliffe, Nucl. Phys. B431 97 (1995).

16Alfred Actor, Rev. Mod. Phys. 51 461 (1979).

17V. Alfaro et al., Nucl. Phys. B73 463 (1978).

18S. Donaldson, Topology, 29 257 (1990).

19D. Hestenes and G. Sobczyk, Clifford Algebra to Geometric Calculus (Reidel, Dordrecht 1984).

20S. Okubo, J. Math. Phys. 32 1657 (1991).

21S. S. Chern, An. da Acad. Brasileira de Ciências, 35 17 (1963).

22S. Nash and S. Sen, Topological and Geometry of Physicists (Academic Pre. INC. London. 1983); M.W. Hirsch, Differential Topology (Springer Verlag. New York 1976).

23C. Doran, D. Hestenes, F. Sommen and N. V. Acker, J. Math. Phys. 34 3642 (1993); A. Lasenby, C. Doran and S. Gull, J. Math. Phys. 34 3683 (1993).

24H. Boermer, Representation of Groups (North-Holland Publishing Company 1963).
25 A. Dimakis and Müller-Hoisen, Class. Quantum Grav. 8 2093 (1991).

26 Eguchi, Gilkey and Hanson, *Gravitation, gauge theories and differential geometry*, Phys. Rep. 66 213 (1980).

27 S.S. Chern and J. Simons, Ann. Math. 99 48 (1974).

28 Albert S. Schwarz, *Topology For Physicist* (Springer Verlag Press 1994).

29 H. Hopf, Math. Ann. 96 209 (1929).

30 E. Goursat, *A Course in Mathematical Analysis Vol. 1* (transl. E. R. Hedrick).

31 Nobuyuki Sakai, Phys. Rev. D54 1548 (1996).

32 M. Kubicek, and M. Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures* (Springer-Verlag, New York 1983).