SOME COHOMOTOPY GROUPS OF SUSPENDED QUATERNIONIC PROJECTIVE PLANES

JIN HO LEE AND KEE YOUNG LEE

Abstract. In this paper we present the computation of two kinds of cohomotopy groups $[\Sigma^{n+4}H^P2, S^n]$ and $[\Sigma^{n+5}H^P2, S^n]$ for a non-negative integer $n$, where $\Sigma^kH^P2$ is the $k$-fold suspension of quaternionic projective plane $H^P2$.

1. Introduction

Let $X$ and $Y$ be based topological spaces, and let $[X, Y]$ denote the set of homotopy classes of base point preserving continuous maps from $X$ to $Y$. Given a space $X$ and an $n$-dimensional sphere $S^n$, the set $[X, S^n]$ has been studied by many authors [1, 2, 3, 5, 6, 7]. This set is known as the $n$-th cohomotopy set of $X$, and in particular, the $n$-th cohomotopy group of $X$ if it has a group structure, which is the case when $X$ is a suspension of a space. The cohomotopy groups $[\Sigma^mX, S^n]$ for the $m$-fold suspension $\Sigma^mX$ of a projective space $X$ have been studied and computed by many authors [2, 3, 7] using the exact sequence associated with the canonical cofiber sequence and a formula for a multiple of the identity class of the suspended projective plane. The cohomotopy groups $[\Sigma^{n+k}H^P2, S^n]$ for the quaternionic projective space $H^P2$, in particular, were computed by Kachi, Mukai, and colleagues on the condition that $|k| \leq 3$ [2].

The purpose of the present paper is to compute the cohomotopy groups $[\Sigma^{n+4}H^P2, S^n]$ and $[\Sigma^{n+5}H^P2, S^n]$ for each $n \geq 2$. The computation will be done as follows. As is well-known, the quaternionic projective plane $H^P2$ is defined by the mapping cone $S^4 \cup \nu_4 e^8$, where $\nu_4 : S^7 \to S^4$ is the Hopf fibering. Consider a Puppe sequence

\[ S^7 \xrightarrow{\nu_4} S^4 \xrightarrow{i} \mathbb{H}P^2 \xrightarrow{p} S^8 \xrightarrow{\nu_5} S^5 \xrightarrow{\Sigma} \ldots, \]

where $i : S^4 \to \mathbb{H}P^2$ is the inclusion map, $p : \mathbb{H}P^2 \to S^8$ is the collapsing map of $S^4$ to a point $*$, and $\nu_k = \Sigma^{k-4}\nu_4$ for $k \geq 4$. This gives a long exact sequence

Received October 14, 2015; Revised January 26, 2016.

2010 Mathematics Subject Classification. 55P15, 55Q05, 55Q40, 55Q55.

Key words and phrases. cohomotopy group, quaternionic projective plane, suspension, Toda bracket.

Supported by a Korea University Grant.
of homotopy sets

\[ \pi_{m+5}(S^n) \xrightarrow{\nu_{m+5}^*} \pi_{m+8}(S^n) \xrightarrow{\Sigma^m P^*} [\Sigma^m \mathbb{H}P^2, S^n] \]

and gives rise to the short exact sequence

\[ 0 \to \ker \nu_{m+5}^* \xrightarrow{\Sigma^m i^*} [\Sigma^m \mathbb{H}P^2, S^n] \xrightarrow{\Sigma^m i^*} \coker \nu_{m+5}^* \to 0. \]

This sequence is referred as the \((m, n)\)-type short exact sequence throughout this paper.

For \(n \geq 2\) and \(s = 9\) or \(10\), we determine \(\coker \nu_{n+s}^*\) and \(\ker \nu_{n+s}^*\) using the formulas of Toda brackets \([8]\), some results in \([2]\), and the Frudenthal suspension theorem. We also investigate the splitting properties of the \((n+k, n)\)-type short exact sequences for \(k = 4\) or \(5\). As a result, we obtain the following theorem.

**Theorem 1.** For each \(n \geq 2\), \([\Sigma^{n+4} \mathbb{H}P^2, S^n]\) and \([\Sigma^{n+5} \mathbb{H}P^2, S^n]\) are determined as follows:

| case \(n\) | 2   | 3   | 4   | 5   | 6   |
|----------|-----|-----|-----|-----|-----|
| \([\Sigma^{n+4} \mathbb{H}P^2, S^n]\) | \(4 + 2 + 3 + 35\) | \(2\) | \((2)^3\) | \((2)^3 + 63\) | \(16 + 2 + (3)^2 + 5\) |
| \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) |
| case \(n\) | 7   | 8   | 9   | 10  | 11  |
| \([\Sigma^{n+4} \mathbb{H}P^2, S^n]\) | \((2)^2\) | \((2)^2\) | \((2)^2\) | \((2)^2 + 3\) | \((2)^3\) |
| \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) | \(n \geq 14\) |

where the integer "s" denotes the cyclic group \(\mathbb{Z}_s\), "+" denotes the direct sum of abelian groups, and "\((s)^k\)" is the \(k\)-times direct sum of \(\mathbb{Z}_s\).

Throughout this paper, we follow Toda’s notation [8] for elements of homotopy groups of spheres. If \(G\) is a finitely generated abelian group generated by \(a_1, \ldots, a_n\), then we denote the group \(G\) by \(G\{a_1, \ldots, a_n\}\). We also denote the \(t\)-times direct sum of \(\mathbb{Z}_s\) by \(\mathbb{Z}_s^t\).

**Acknowledgment.** We are very grateful to the referee(s) whose constructive remarks considerably improved the original manuscript.
2. Preliminaries

In this section, we present selected basic principles of composition methods [8].

When $G$ is an abelian group and $p \geq 2$ is a prime number, we denote the $p$-primary parts of $G$ by $G_{(p)}$.

For $p \geq 5$, we have an isomorphism

$$[\Sigma^n \mathbb{H}P^2, S^k]_{(p)} \cong \pi_{n+4}(S^k)_{(p)} \oplus \pi_{n+8}(S^k)_{(p)},$$

because $\pi_{n+3}(S^n)$ has order 24 for $n \geq 5$ [8, Proposition 5.6].

Moreover, there is an isomorphism [8, (13.1)]

$$\pi_{i-1}(S^{2m-1})_{(p)} \oplus \pi_i(S^{4m-1})_{(p)} \cong \pi_i(S^{2m})_{(p)}$$

given by the correspondence $(\alpha, \beta) \mapsto \Sigma \alpha + [\nu, \chi] \circ \beta$, where $[\cdot, \cdot]$ is the Whitehead product. This is known as Serre’s isomorphism.

It is well known that the Hopf fibrations $\eta_2 : S^3 \to S^2$, $\nu_4 : S^7 \to S^4$, and $\sigma_8 : S^{15} \to S^8$ induce the isomorphisms

$$[X, S^3] \to [X, S^2], \quad \alpha \mapsto \eta_2 \circ \alpha,$$

$$[X, S^3] \oplus [\Sigma X, S^7] \to [\Sigma X, S^4], \quad (\alpha, \beta) \mapsto \Sigma \alpha + \nu_4 \circ \beta,$$

$$[X, S^7] \oplus [\Sigma X, S^{15}] \to [\Sigma X, S^8], \quad (\alpha, \beta) \mapsto \Sigma \alpha + \sigma_8 \circ \beta$$

respectively.

Consider elements $\alpha \in [Y, Z]$, $\beta \in [X, Y]$, and $\gamma \in [W, X]$ satisfying $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. Let $C_\beta$ be the mapping cone of $\beta$, and $i : Y \to C_\beta$, $p : C_\gamma \to \Sigma X$ be the inclusion and the shrinking map, respectively. We denote an extension of $\alpha$ satisfying $i^*(\overline{\alpha}) = \alpha$ by $\overline{\alpha} \in [C_\beta, Z]$, and a coextension of $\gamma$ satisfying $p_*(\overline{\gamma}) = \Sigma \gamma$ by $\overline{\gamma} \in [\Sigma W, C_\beta]$ [8].

We recall some relations between (co)extensions and Toda brackets [8].

**Theorem 2.** Let $\alpha \in [Y, Z]$, $\beta \in [X, Y]$, and $\gamma \in [W, X]$ be elements such that $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. Let $\{\alpha, \beta, \gamma\}$ be the Toda bracket, and $i : Y \to C_\beta$, $p : C_\gamma \to \Sigma W$ be the inclusion and the shrinking map, respectively. Then, we have $\overline{\alpha} \circ \overline{\gamma} \in \{\alpha, \beta, \gamma\}$ and $\alpha \circ \overline{\beta} \in \{\alpha, \beta, \gamma\} \circ p$.

The following is useful for determining 3-primary parts of the class $[\Sigma^n \mathbb{H}P^2, S^m]$ [2].

**Theorem 3.**

$$24\Sigma t_{24} = \Sigma i \circ 24t_{24} + 24t_{s} \circ \Sigma p$$

on $[\mathbb{H}P^2, \mathbb{H}P^2]$, where $t_{24} : \mathbb{H}P^2 \to \mathbb{H}P^2$ and $t_{s} : S^8 \to S^8$ are the identity maps on $\mathbb{H}P^2$ and $S^8$, respectively.
3. Basic computations

In this section, we describe the basic computation of two 3-dimensional cohomotopy groups, and apply them to the computation of other cohomotopy groups.

By [8, (5.9)], we have $\eta_n \circ \nu_{n+1} = 0$ for $n \geq 5$; thus, there is an extension $\overline{\nu}_n \in [\Sigma^{n-3}\mathbb{H}P^2, S^n]$ of $\eta_n$ for $n \geq 5$. Moreover, by [4, Proposition (2.2)], we have $\epsilon' \circ \nu_3 = 0$; thus, there is an extension $\overline{\epsilon'} \in [\Sigma^3\mathbb{H}P^2, S^3]$ of $\epsilon'$.

Proposition 1. (1) $[\Sigma^8\mathbb{H}P^2, S^3] = Z_2^2\{\nu' \circ \eta_6 \circ \mu_4 \circ \Sigma^8 p, \epsilon_3 \circ \overline{\eta}_{11}\} \oplus Z_3\{\alpha_1(3) \circ \beta_1(6) \circ \Sigma^8 p\}$.

(2) $[\Sigma^3\mathbb{H}P^2, S^3] \cong Z_4\{\overline{\epsilon'}\} \oplus Z_2\{\mu_3 \circ \overline{\eta}_{11}\}$.

Proof. (1) Consider the (8,3)-type short exact sequence

$$0 \to \text{Coker}\nu_1^3 \xrightarrow{\Sigma^8 p'} [\Sigma^8\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^8 \epsilon'} \text{Ker}\nu_2 \to 0$$

where $\nu_1^3 : \pi_{13}(S^3) \to \pi_{16}(S^3)$ and $\nu_2^3 : \pi_{12}(S^3) \to \pi_{15}(S^3)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$\nu_1^3 : Z_4\{\epsilon'\} \oplus Z_2\{\eta_3 \circ \mu_4\} \oplus Z_3 \to Z_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus Z_3\{\alpha_1(3) \circ \beta_1(6)\}$

and

$\nu_2^3 : Z_2^2\{\mu_3, \eta_3 \circ \epsilon_4\} \to Z_2^2\{\nu' \circ \mu_6, \nu' \circ \eta_6 \circ \epsilon_7\}$

respectively.

Then, we have $\nu_1^3(\epsilon') = 0$ by [4, (2.2)], and $\nu_1^3(\eta_3 \circ \mu_4) = 0$ by [8, (5.9)] and [4, (2.2)]. Thus, we have $\nu_1^3(\mu_3) = \nu' \circ \eta_6 \circ \epsilon_7$ by [4, (2.2)], and $\nu_1^3(\eta_3 \circ \epsilon_4) = 0$ by [8, (5.9)] and [4, (2.1)]. Thus, we have $\text{Coker}\nu_1^3 = Z_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus Z_3\{\alpha_1(3) \circ \beta_1(6)\}$ and $\text{Ker}\nu_2^3 = Z_2\{\eta_3 \circ \epsilon_4\}$. This gives the following short exact sequence

$$0 \to Z_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus Z_3\{\alpha_1(3) \circ \beta_1(6)\} \xrightarrow{\Sigma^8 p'} [\Sigma^8\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^8 \epsilon'} Z_2\{\eta_3 \circ \epsilon_4\} \to 0.$$

By [4, (2.1)], we know that $\eta_3 \circ \epsilon_4 = \epsilon_3 \circ \eta_{11}$. Consider the extension $\epsilon_3 \circ \eta_{11}$ of $\epsilon_3$ of $\eta_{11}$. By [2, Proposition 4.1], the order of $\eta_{11}$ is two; thus, the order of $\epsilon_3 \circ \eta_{11}$ is two, and therefore the short exact sequence is split.

(2) Consider the (9,3)-type short exact sequence

$$0 \to \text{Coker}\nu_1^4 \xrightarrow{\Sigma^9 p'} [\Sigma^9\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^9 \epsilon'} \text{Ker}\nu_3 \to 0,$$

where $\nu_1^4 : \pi_{14}(S^3) \to \pi_{17}(S^3)$ is the homomorphism induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. This homomorphism can be restated as follows:

$\nu_1^4 : Z_4\{\mu'\} \oplus Z_2^2\{\epsilon_3 \circ \nu_{11}, \nu' \circ \epsilon_6\} \oplus Z_3 \oplus Z_7 \to Z_2\{\epsilon_3 \circ \nu_{11}\} \oplus Z_3 \oplus Z_5$.

Then, we have $\nu_1^4(\mu') = 0$ by [4, (2.4)], and $\nu_1^4(\nu' \circ \epsilon_6) = \nu' \circ \epsilon_6 \circ \nu_4 = 0$ since $\epsilon_6 \circ \nu_4 = (E^3 \nu') \circ \overline{\nu}_9 = (2 \nu_9) \circ \overline{\nu}_9 = 0$ by [4, (2.1)]. Thus, we have $\text{Coker}\nu_1^4 = Z_2\{\epsilon_3 \circ \nu_{11}\} \oplus Z_3 \oplus Z_5$ and $\text{Ker}\nu_3 = Z_4\{\epsilon'\} \oplus Z_2\{\eta_3 \circ \mu_4\} \oplus Z_3$ by (1). This gives the following two-primary short exact sequence

$$0 \xrightarrow{\Sigma^9 p'} [\Sigma^9\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^9 \epsilon'} Z_4\{\epsilon'\} \oplus Z_2\{\eta_3 \circ \mu_4\} \to 0.$$
4. Computation of $[\Sigma^{n+4} \mathbb{H}P^2, S^n]$ for $n \geq 2$

In this section, we compute the $n$-th cohomotopy groups of $(n+4)$-fold suspended quaternionic projective planes.

**Proposition 2.** $[\Sigma^6 \mathbb{H}P^2, S^2] = \mathbb{Z}_4 \{\eta_2 \circ \mu' \circ \Sigma^6 p\} \oplus \mathbb{Z}_2 \{\eta_2 \circ \nu' \circ \epsilon_6 \circ \Sigma^6 p\} \oplus \mathbb{Z}_3 \{\eta_2 \circ \alpha_3(3) \circ \Sigma^6 p\} \oplus \mathbb{Z}_{35}$.

**Proof.** Since $\eta_2 : S^3 \to S^2$ is a fibration with fiber $S^1$, $\eta_2 : [\Sigma^6 \mathbb{H}P^2, S^3] \to [\Sigma^6 \mathbb{H}P^2, S^2]$ is an isomorphism. Thus, by [2, Theorem 4.7 (2)], this completes the proof.

**Proposition 3.** $[\Sigma^7 \mathbb{H}P^2, S^3] \cong \mathbb{Z}_2 \{\nu' \circ \mu_6 \circ \Sigma^7 p\}$.

**Proof.** Consider the $(7, 3)$-type short exact sequence:

$$0 \to \text{Coker} \nu_{12}^* \xrightarrow{\Sigma^7 \nu'} [\Sigma^7 \mathbb{H}P^2, S^3] \xrightarrow{\Sigma^7 \epsilon'} \text{Ker} \nu_{11}^* \to 0,$$

where $\nu_{12}^* : \pi_{12}(S^3) \to \pi_{15}(S^3)$ and $\nu_{11}^* : \pi_{11}(S^3) \to \pi_{14}(S^3)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{12}^* : \mathbb{Z}_2 \{\mu_3\} \oplus \mathbb{Z}_2 \{\eta_3 \circ \epsilon_4\} \to \mathbb{Z}_2 \{\nu' \circ \mu_6\} \oplus \mathbb{Z}_2 \{\nu' \circ \epsilon_6 \circ \nu_{11}\} \oplus \mathbb{Z}_{21},$$

and

$$\nu_{11}^* : \mathbb{Z}_2 \{\epsilon_3\} \to \mathbb{Z}_4 \{\mu'\} \oplus \mathbb{Z}_2 \{\nu' \circ \epsilon_6\} \oplus \mathbb{Z}_2 \{\nu' \circ \epsilon_7\},$$

respectively. Then, we have $\nu_{12}^* \circ \eta_3 \circ \epsilon_4 = \eta_3 \circ \epsilon_4 \circ \nu_{12} = \epsilon_3 \circ \eta_1 \circ \nu_{12} = 0$ [4, (2.1)], [8, (5.9)], and $\nu_{11}^* \circ \epsilon_3 = \epsilon_3 \circ \nu_{11}$. Thus, we have $\text{Coker} \nu_{12}^* = \mathbb{Z}_2 \{\nu' \circ \mu_6\}$ and $\text{Ker} \nu_{11}^* = 0$. From the short exact sequence, we have

$$[\Sigma^7 \mathbb{H}P^2, S^3] = \mathbb{Z}_2 \{\nu' \circ \mu_6 \circ \Sigma^7 p\}.$$  

**Proposition 4.** $[\Sigma^8 \mathbb{H}P^2, S^4] = \mathbb{Z}_4 \{\nu_4 \circ \sigma' \circ \eta_{14}^2 \circ \Sigma^8 p, \nu_4 \circ \mu_7 \circ \Sigma^8 p, \nu_4 \circ \eta_7 \circ \epsilon_8 \circ \Sigma^8 p, (E\nu') \circ \mu_7 \circ \Sigma^8 p\}$.

**Proof.** Consider the $(8, 4)$-type short exact sequence

$$0 \to \text{Coker} \nu_{13}^* \xrightarrow{\Sigma^8 \nu'} [\Sigma^8 \mathbb{H}P^2, S^4] \xrightarrow{\Sigma^8 \epsilon'} \text{Ker} \nu_{12}^* \to 0,$$

where $\nu_{13}^* : \pi_{13}(S^4) \to \pi_{16}(S^4)$ and $\nu_{12}^* : \pi_{12}(S^4) \to \pi_{15}(S^4)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{13}^* : \mathbb{Z}_2 \{\nu_4^3, \mu_4, \eta_4 \circ \epsilon_5\} \to \mathbb{Z}_2 \{\nu_4 \circ \sigma' \circ \eta_{14}^2, \nu_4^3, \nu_4 \circ \mu_7, \nu_4 \circ \eta_7 \circ \epsilon_8, (E\nu') \circ \mu_7, (E\nu') \circ \eta_7 \circ \epsilon_8\}$$

and

$$\nu_{12}^* : \mathbb{Z}_2 \{\epsilon_4\} \to \mathbb{Z}_4 \oplus \mathbb{Z}_3^5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7,$$

respectively. Then, we have $\nu_{12}^* (\nu_{12}^*) = \nu_{12}^* \circ \nu_{12}^* = (E\nu') \circ \eta_7 \circ \epsilon_8$. 


[4, Proposition (2.2)(4)], and
\[ \nu_{13}(\eta_4 \circ \epsilon_5) = \eta_4 \circ \epsilon_5 \circ \nu_{13} = \epsilon_4 \circ \eta_2 \circ \nu_{13} = 0 \]
by [4, (2.1)]. Thus, it follows that
\[ \text{Coker} \nu_{13} = Z_2^4 \{ \nu_4 \circ \sigma' \circ \eta_{14}, \nu_4 \circ \mu_7, \nu_4 \circ \epsilon_8, (E\nu') \circ \mu_7 \}. \]
Since the element \( \nu_{12}^* (\epsilon_4) = \epsilon_4 \circ \nu_{12} \) has order 2 in \( \pi_{15}(S^4) \), \( \nu_{12}^* \) is injective. Hence, we have the short exact sequence
\[ 0 \to Z_2^4 \{ \nu_4 \circ \sigma' \circ \eta_{14}, \nu_4 \circ \mu_7, \nu_4 \circ \epsilon_8, (E\nu') \circ \mu_7 \} \xrightarrow{\Sigma^8 p^*} [\Sigma^8 \mathbb{H}P^2, S^4] \xrightarrow{\Sigma^8 i^*} 0. \]

By [4, (1.1) (7)], \( H(\epsilon') = \epsilon_5 \), and from [4, (2.2) (7)], we have \( \epsilon' \circ \nu_{13} = 0 \). Thus, there is an extension \( \sigma \in [\Sigma^9 \mathbb{H}P^2, S^3] \) of \( \epsilon' \). Denote \( \sigma = H(\sigma) \) to be an extension of \( \epsilon_5 \). We denote \( \sigma_n = \Sigma^{-n} \sigma \) for \( n \geq 5 \).

**Proposition 5.** \([\Sigma^9 \mathbb{H}P^2, S^5] = Z_2^5 \{ \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p, \nu_5 \circ \mu_8 \circ \Sigma^9 p, \sigma \} \oplus Z_9 \oplus Z_7 \).

**Proof.** Consider the \((9, 5)\)-type short exact sequence:
\[ 0 \to \text{Coker} \nu_{14}^* \xrightarrow{\Sigma^9 p^*} [\Sigma^9 \mathbb{H}P^2, S^5] \xrightarrow{\Sigma^9 i^*} \ker \nu_{13}^* \to 0, \]
where \( \nu_{14}^* : \pi_{14}(S^5) \to \pi_{17}(S^9) \) and \( \nu_{13}^* : \pi_{13}(S^5) \to \pi_{16}(S^9) \) are the homomorphisms induced originally by the Hopf fibration \( \nu_4 : S^7 \to S^9 \). These homomorphisms can be restated as follows:
\[ \nu_{14}^* : Z_2^3 \{ \nu_5^3, \nu_5 \circ \mu_8, \nu_5 \circ \eta_8 \circ \epsilon_9 \} \xrightarrow{\Sigma^3 \nu^*} Z_2^3 \{ \nu_5^3, \nu_5 \circ \mu_8, \nu_5 \circ \eta_8 \circ \epsilon_9 \}, \]
and
\[ \nu_{13}^* : Z_2 \{ \epsilon_5 \} \to Z_8 \{ \chi_8 \} \oplus Z_2^2 \{ \nu_5 \circ \sigma_8, \nu \circ \epsilon_9 \} \oplus Z_9 \oplus Z_7, \]
respectively. Then, we have \( \nu_{14}^* (\nu_5^3) = \nu_5^3, \nu_{14}^* (\eta_8 \circ \epsilon_9) = 0, \) and
\[ \nu_{14}^* (\mu_8) = \mu_8 \circ \nu_{14} = \Sigma^2 (\nu' \circ \eta_8 \circ \epsilon_7) = \Sigma^2 (\nu' \circ \eta_8 \circ \epsilon_7) = \nu_8 \circ \eta_8 \circ \epsilon_9 = (2\nu_5) \circ \eta_8 \circ \epsilon_9 = 0 \]
by [4, Proposition 2.2(4)]. Thus, we have
\[ \text{Coker} \nu_{14}^* = Z_2^2 \{ \nu_5 \circ \eta_8 \circ \epsilon_9, \nu_5 \circ \mu_8 \} \oplus Z_9 \oplus Z_7. \]
Additionally, we have
\[ \nu_{13}^* (\epsilon_5) = \Sigma^2 (\nu' \circ \sigma_8) = (\Sigma^2 \nu') \circ \sigma_8 = 2 (\nu_5 \circ \sigma_8) = 0 \]
since \( \nu_5 \circ \sigma_8 \in \pi_{16}(S^9) \) has order 2 [8]. Thus, we have \( \ker \nu_{13}^* = Z_2 \{ \epsilon_5 \} \) and therefore, we have the short exact sequence
\[ (\ast) \quad 0 \to Z_2^2 \{ \nu_5 \circ \eta_8 \circ \epsilon_9, \nu_5 \circ \mu_8 \} \xrightarrow{\Sigma^3 \nu^*} [\Sigma^9 \mathbb{H}P^2, S^5] \xrightarrow{\Sigma^9 i^*} Z_2 \{ \epsilon_5 \} \to 0. \]

If \([\Sigma^9 \mathbb{H}P^2, S^5]\) is isomorphic to \( Z_4 \oplus Z_2 \), we only have three cases:
\[ 2\sigma_5 = \left\{ \begin{array}{ll} \nu_5 \circ \mu_8 \circ \Sigma^9 p, & \\
 \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p, & \\
 \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p. & 
\end{array} \right. \]
First, we assume that $2\overline{\nu_5} = \nu_5 \circ \mu_9 \circ \Sigma^9 p$. Then, we have $0 = \Delta \circ H(2(\overline{\nu})) = \Delta(2H(\overline{\nu})) = \Delta(2\overline{\nu_5}) = \Delta(\nu_5) = \nu_5 \circ \mu_9 \circ \Sigma^9 p = \nu_2 \circ \nu' \circ \mu_9 \circ \Sigma^7 p$, since $\Delta(\nu_5) = \nu_2 \circ \nu'$ [8]. However, we know that

$$[\Sigma^7 H P^2, S^2] = Z_2\{\nu_2 \circ \nu' \circ \mu_9 \circ \Sigma^7 p\}.$$ 

This is a contradiction. Thus, $2\overline{\nu_5} \neq \nu_5 \circ \mu_9 \circ \Sigma^9 p$. Next, we assume that $2\overline{\nu_5} = \nu_5 \circ \mu_8 \circ \Sigma^9 p + \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$. Then, we have $0 = \Delta \circ H(2\overline{\nu}) = \Delta(2H(\overline{\nu})) = \Delta(\nu_5) + \nu_5 \circ \mu_9 \circ \Sigma^7 p + \Delta(\nu_5) \circ \eta_9 \circ \epsilon_7 \circ \Sigma^7 p = \nu_2 \circ \nu' \circ \mu_9 \circ \Sigma^7 p + \nu_2 \circ \nu' \circ \eta_9 \circ \epsilon_7 \circ \Sigma^7 p = \nu_2 \circ \nu' \circ \mu_9 \circ \Sigma^7 p,$ since $\Delta(\nu_5) = \nu_2 \circ \nu'$ [8] and $\nu' \circ \eta_9 \circ \epsilon_7 \circ \Sigma^7 p = 0 \in [\Sigma^7 H P^2, S^3]$. As this is also a contradiction, we have $2\overline{\nu_5} = \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$.

Consider a suspension homomorphism $E : [\Sigma^9 H P^2, S^2] \to [\Sigma^9 H P^2, S^3]$ where

$$[\Sigma^9 H P^2, S^2] = Z_2(\nu_2 \circ \nu' \circ \eta_9 \circ \mu_7 \circ \Sigma^8 p, \eta_2 \circ \epsilon_3 \circ \eta_1 \mu_{11})$$

and

$$[\Sigma^9 H P^2, S^3] = Z_4(\eta_2 \circ \epsilon_3 \circ \eta_1 \mu_{10}).$$

Now, we consider an element $E(\eta_2 \circ \epsilon_3 \circ \eta_1 \mu_{10}) \in [\Sigma^9 H P^2, S^3]$. We have $\Sigma^9 i^*: [\Sigma^9 H P^2, S^3] \to Kerv_{13}i_{13}^*$ is an isomorphism, we have $E(\eta_2 \circ \eta_2 \circ \eta_1 \circ \eta_1 \mu_{10}) = \eta_2 \circ \epsilon_3 \circ \epsilon_2 \circ \eta_1 \mu_{10}$. By exactness and $E(\eta_2 \circ \epsilon_3 \circ \eta_1 \mu_{10}) = 2\mu_7^p$, we have $0 = H \circ E(\eta_2 \circ \epsilon_3 \circ \eta_1 \mu_{10}) = H(2\mu_7^p) = \mu_7 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$. This contradicts to the statement that $\nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$ has order 2. Thus, the short exact sequence (*) splits. □

**Proposition 6.** $[\Sigma^{10} H P^2, S^6] = Z_{16}(\Delta(\sigma_{11}) \circ \Sigma^{10} p) \oplus Z_2(\tau_6) \oplus Z_2(\overline{\Delta(\sigma_{11})}) \oplus Z_2(\overline{\Delta(\sigma_{11})}) \oplus \Sigma^5 p \oplus Z_5$.

**Proof.** Consider the (10,6)-type short exact sequence:

$$0 \to Coker\nu_{15}^* \xrightarrow{\Sigma^{10} p} [\Sigma^{10} H P^2, S^6] \xrightarrow{\Sigma^{10} i^*} Kerv\nu_{14}^* \to 0,$$

where $\nu_{15}^*: \pi_{15}(S^6) \to \pi_{18}(S^6)$ and $\nu_{14}^*: \pi_{14}(S^6) \to \pi_{17}(S^6)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^9$. These homomorphisms can be restated as follows:

$$\nu_{15}^*: Z_2(\nu_3^6 \circ \mu_6, \eta_9 \circ \epsilon_7) \to Z_{16}(\Delta(\sigma_{11})) \oplus Z_3(\nu_5, \nu_6, \alpha_{11}) \oplus Z_5,$$

and

$$\nu_{14}^*: Z_2(\tau_6) \oplus Z_2(\nu_6) \oplus Z_3(\nu_5, \nu_6, \alpha_{11}) \to Z_{16}(\Delta(\sigma_{11})) \oplus Z_3(\nu_5, \nu_6, \alpha_{11}) \oplus Z_7 \oplus Z_5$$

by [8, p. 61, p. 66, p. 74, p. 186], respectively. Then, we have

$$\nu_{15}^*(\nu_5^6) = 0^6 \circ E(\nu_5^6) = E(\Delta(\sigma_{11})) = 0$$

by [8, p. 77],

$$\nu_{15}^*(\mu_5) = E(\mu_5 \circ \nu_{14}) = 0,$$

by Proposition 5, and $\nu_{15}^*(\eta_8 \circ \epsilon_7) = 0$. Thus, we have Coker$\nu_{15}^* = \pi_{18}(S^6)$. In addition, we have $\nu_{14}^*(\tau_6) = \tau_6 \circ \nu_{14}, \nu_{14}^*(\nu_6) = \nu_6 \circ \nu_{14} = E(\nu_6 \circ \tau_6) = (E^3 \nu_6 \circ \nu_6) = (2(\nu_6) \circ \nu_6) = 2(\nu_6 \circ \nu_6) = 4(\nu_6 \circ \nu_14) = 0$ by [4, (2,1)]
and [8, Theorem 7.4]. Thus, we have \( \ker \nu_{14} = \mathbb{Z}_2^2 \{ 4\eta_6, \epsilon_6 \} \oplus \mathbb{Z}_3 \{ [\iota_6, \iota_6] \circ \alpha_1(11) \} \). This gives the short exact sequence

\[
0 \to \mathbb{Z}_{16} \{ \Delta(\sigma_{13}) \} \oplus \mathbb{Z}_3 \{ [\iota_6, \iota_6] \circ \alpha_2(11) \} \oplus \mathbb{Z}_5 \overset{\Sigma^{10}\rho^*}{\longrightarrow} \mathbb{Z}^{10}_{\mathbb{H} P^2, S^6} \cong \mathbb{Z}_2^2 \{ 4\eta_6, \epsilon_6 \} \oplus \mathbb{Z}_3 \{ [\iota_6, \iota_6] \circ \alpha_1(11) \} \to 0.
\]

In Proposition 5, we showed that \( \overline{\iota_6} \) has order 2. Thus \( \overline{\iota_6} \) has order 2. By [4, (2.1)] and [2, Proposition 4.1], we have \( 4\eta_6 = \sigma'' \circ \eta_{13} \) and \( \overline{\iota_{13}} \) has order 2. Therefore, we conclude that \( 4\eta_6 = \sigma'' \circ \eta_{13} \) has order 2. It implies that the 2-primary exact sequence

\[
0 \to \mathbb{Z}_{16} \{ \Delta(\sigma_{13}) \} \overset{\Sigma^{10}\rho^*}{\longrightarrow} \mathbb{Z}^{10}_{\mathbb{H} P^2, S^6} \overset{\Sigma^{10}\rho^*}{\longrightarrow} \mathbb{Z}_2^2 \{ 4\eta_6, \epsilon_6 \} \to 0
\]
is split. Now consider the 3-primary parts

\[
0 \to \mathbb{Z}_3 \{ [\iota_6, \iota_6] \circ \alpha_2(11) \} \overset{\Sigma^{10}\rho^*}{\longrightarrow} \mathbb{Z}^{10}_{\mathbb{H} P^2, S^6} \overset{\Sigma^{10}\rho^*}{\longrightarrow} \mathbb{Z}_3 \{ [\iota_6, \iota_6] \circ \alpha_1(11) \} \to 0.
\]

By Theorem 3, we have \( 3\alpha_1(11) = \alpha_1(11) \circ 24 \Sigma^{10}_{\mathbb{H} P^2} = \alpha_1(11) \circ \Sigma^{10} \circ 24 \iota_{14} + \alpha_1(11) \circ 24 \iota_{17} \circ \Sigma^{10} \circ p = \alpha_1(11) \circ 24 \iota_{14} + \alpha_1(11) \circ 24 \iota_{17} \circ \Sigma^{10} \circ p \).

By [2, Theorem 2.7], we have \( \alpha_1(11) \circ 24 \iota_{14} \in \{ \alpha_1(11), 24 \iota_{14}, \alpha_1(14) \} \circ \Sigma^{10} \circ p \).

Since \( \alpha_2(11) \) has order 3, we have

\[
-\alpha_2(11) = 8\alpha_2(11) \in 8 \{ \alpha_1(11), 3 \iota_{14}, \alpha_1(14) \} = \{ \alpha_1(11), 24 \iota_{14}, \alpha_1(14) \}
\]

by [8, Lemma 13.5]. Then, we have

\[
\alpha_1(11) \circ 24 \iota_{14} \in \{ \alpha_1(11), 24 \iota_{14}, \alpha_1(14) \} \circ \Sigma^{10} \circ p \ni -\alpha_2(11) \circ \Sigma^{10} \circ p \mod 0,
\]

that is, \( \alpha_1(11) \circ 24 \iota_{14} = -\alpha_2(11) \circ \Sigma^{10} \circ p \).

By Theorem 2, we have \( \alpha_1(11) \circ 24 \iota_{14} \in \{ \alpha_1(11), \alpha_1(14), 24 \iota_{17} \} \). From [8, (13.8)], we have \( (1/2) \alpha_2(11) \in \{ \alpha_1(11), \alpha_1(14), 3 \iota_{17} \} \) and so, we have

\[
\alpha_2(11) = 4\alpha_2(11) \in 8 \{ \alpha_1(11), \alpha_1(14), 3 \iota_{17} \} = \{ \alpha_1(11), \alpha_1(14), 24 \iota_{17} \}.
\]

Thus, we have

\[
3\alpha_1(11) \circ 24 \iota_{17} \in \{ \alpha_1(11), \alpha_1(14), 24 \iota_{17} \} \ni \alpha_1(11) \mod 0,
\]

that is, \( \overline{\alpha_1(11)} \circ 24 \iota_{17} = \overline{\alpha_2(11)} \). Thus, we have

\[
3\alpha_1(11) = \alpha_1(11) \circ 24 \iota_{14} + \alpha_1(11) \circ 24 \iota_{17} \circ \Sigma^{10} \circ p
\]

\[
= -\alpha_2(11) \circ \Sigma^{10} \circ p + \alpha_2(11) \circ \Sigma^{10} \circ p
\]

\[
= 0.
\]

Consequently, \( \overline{\alpha_1(11)} \) has order 3, and so has \( [\iota_6, \iota_6] \circ \overline{\alpha_1(11)} \). Therefore, we have

\[
\{ \Sigma^{10}_{\mathbb{H} P^2, S^6} \} = \mathbb{Z}_2^2 \{ [\iota_6, \iota_6] \circ \overline{\alpha_1(11)}, [\iota_6, \iota_6] \circ \alpha_2(11) \circ \Sigma^{10} \circ p \}.
\]

\[\square\]

**Proposition 7.** (1) \( \{ \Sigma^{11}_{\mathbb{H} P^2, S^7} \} = \mathbb{Z}_2^2 \{ \sigma' \circ \overline{\eta}_{14}, \overline{\epsilon}_7 \} \).

(2) \( \{ \Sigma^{12}_{\mathbb{H} P^2, S^8} \} = \mathbb{Z}_2^2 \{ \sigma_8 \circ \overline{\eta}_{15}, (E\sigma') \circ \overline{\eta}_{15}, \overline{\epsilon}_8 \} \).

(3) \( \{ \Sigma^{13}_{\mathbb{H} P^2, S^9} \} = \mathbb{Z}_2^2 \{ \sigma_9 \circ \overline{\eta}_{16}, \overline{\epsilon}_9 \} \).
Proof. (1) Consider the homomorphisms $\nu_{16} : \pi_{16}(S^7) \to \pi_{19}(S^7)$ and $\nu_{15} : \pi_{15}(S^7) \to \pi_{18}(S^7)$ related to the $(11,7)$-type short exact sequence. Since $\pi_{19}(S^7) = 0$, $\text{Coker}\nu_{16} = 0$. Moreover, for the homomorphism

$$\nu_{15} : Z_2^2\{\sigma' \circ \eta_{14}, \tau_7, \epsilon_7\} \to Z_8\{\zeta_7\} \oplus \mathbb{Z}_2\{\tau_7 \circ \nu_{15}\} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9,$$

we have $\nu_{15}(\sigma' \circ \eta_{14}) = 0$, $\nu_{15}(\tau_7) = \tau_7 \circ \nu_{15}$, and $\nu_{15}(\epsilon_7) = E(\epsilon_6 \circ \nu_{14}) = E = 0$ from Proposition 6. Therefore, we have $\text{Ker}\nu_{15} = Z_2^2\{\sigma' \circ \eta_{14}, \epsilon_7\}$. Thus, we obtain the following short exact sequence form the $(11,7)$-type short exact sequence

$$0 \to \left[\Sigma^{11}H^P^2, S^7\right] \xrightarrow{\Sigma^{11}\epsilon_7} Z_2^2\{\sigma' \circ \eta_{14}, \epsilon_7\} \to 0.$$

(2) Consider the homomorphisms $\nu_{17} : \pi_{17}(S^8) \to \pi_{20}(S^8)$ and $\nu_{16} : \pi_{16} \to \pi_{19}(S^9)$ related to the $(12,8)$-type short exact sequence. Since $\pi_{20}(S^8) = 0$, $\text{Coker}\nu_{17} = 0$. For the homomorphism

$$\nu_{16} : Z_2^2\{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{13}, \tau_8, \epsilon_8\} \to Z_8\{\zeta_8\} \oplus \mathbb{Z}_2\{\tau_8 \circ \nu_{16}\} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9,$$

we have $\nu_{16}(\sigma_8 \circ \eta_{15}) = 0$, $\nu_{16}((E\sigma') \circ \eta_{13}) = 0$, $\nu_{16}(\tau_8) = \tau_8 \circ \nu_{16}$, and $\nu_{16}(\epsilon_8) = \epsilon_8 \circ \nu_{16} = E^2(\epsilon_6 \circ \nu_{14}) = 0$ from Proposition 6. Thus, we have $\text{Ker}\nu_{16} = Z_2^2\{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{13}, \epsilon_8\}$, which leads to the short exact sequence

$$0 \to \left[\Sigma^{12}H^P^2, S^8\right] \xrightarrow{\Sigma^{12}\epsilon_8} Z_2^2\{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{13}, \epsilon_8\} \to 0.$$

(3) Consider the homomorphisms $\nu_{18} : \pi_{18}(S^9) \to \pi_{21}(S^9)$ and $\nu_{17} : \pi_{17}(S^9) \to \pi_{20}(S^9)$ related to the $(13,9)$-type short exact sequence. Since $\pi_{21}(S^9) = 0$, $\text{Coker}\nu_{18} = 0$. Moreover, for the homomorphism

$$\nu_{17} : Z_2^2\{\sigma_9 \circ \eta_{16}, \tau_9, \epsilon_9\} \to Z_8\{\zeta_9\} \oplus \mathbb{Z}_2\{\tau_9 \circ \nu_{17}\} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9,$$

we have $\nu_{17}(\sigma_9 \circ \eta_{16}) = 0$, $\nu_{17}(\tau_9) = \tau_9 \circ \nu_{17}$, and $\nu_{17}(\epsilon_9) = \epsilon_9 \circ \nu_{17} = E(\epsilon_8 \circ \nu_{16}) = E = 0$ from Proposition 6. Thus, we have $\text{Ker}\nu_{17} = Z_2^2\{\sigma_9 \circ \eta_{16}, \epsilon_9\}$, and consequently, we obtain the following short exact sequence from the $(13,9)$-type short exact sequence:

$$0 \to \left[\Sigma^{13}H^P^2, S^9\right] \xrightarrow{\Sigma^{13}\epsilon_9} Z_2^2\{\sigma_9 \circ \eta_{16}, \epsilon_9\} \to 0.$$
by [8, Proposition 2.5, Theorem 7.2, Proposition 13.6] and

$$\nu_{18} : Z_2^2(\mathcal{V}_{10}, \epsilon_{10}) \rightarrow Z_8(\zeta_{10}) \oplus Z_9 \oplus Z_7,$$

respectively. Then, we have $$\nu_{18}(\Delta(\iota_{21})) = \Delta(\iota_{21}) \circ \nu_{19} = \Delta(\iota_{21} \circ \nu_{21}) = \Delta(\nu_{21})$$

by [8, Proposition 2.5]. Moreover, since $$\nu_{10}^2 \circ \nu_{19} = 0$$, $$\mu_{10} \circ \nu_{19} = 0$$, and

$$\eta_{10} \circ \iota_{11} \circ \nu_{19} = 0,$$

we have Coker$$\nu_{19} = Z_4(\Delta(\alpha_1(21))).$$ Since $$\Delta(\nu_{19}) = \mathcal{T}_9 \circ \nu_{17}

by [8, (7.22)], $$\mathcal{T}_9 \circ \nu_{18} = E(\nu(\nu_{19}) = 0$$, and $$\nu_{18}(\epsilon_{10}) = 0$$, we have Ker$$\nu_{18} = Z_2^2(\mathcal{V}_{10}, \epsilon_{10}).$$ Consequently, we have the split short exact sequence

$$0 \rightarrow Z_3(\Delta(\alpha_1(21))) \xrightarrow{\Sigma_{14}} [\Sigma^{14}_H P^2, S^{10}] \xrightarrow{\Sigma_{14}} Z_2^2(\mathcal{V}_{10}, \epsilon_{10}) \rightarrow 0.$$ (2)

(2) We can show that $$\nu_{20}^2 : \pi_{20}(S^{11}) \rightarrow \pi_{23}(S^{11})$$ and $$\nu_{19}^2 : \pi_{19}(S^{11}) \rightarrow \pi_{22}(S^{11})$$ are trivial by using approaches similar to those used for (1). Hence, we obtain the following short exact sequence from the (15,11)-type short exact sequence:

$$0 \rightarrow Z_2(\theta') \xrightarrow{\Sigma_{15}} [\Sigma^{15}_H P^2, S^{11}] \xrightarrow{\Sigma_{15}} Z_2(\mathcal{V}_{11}, \epsilon_{11}) \rightarrow 0.$$ Consider the following commutative diagram:

$$\begin{array}{ccc}
0 & \rightarrow & Z_3(\Delta(\alpha_1(21)))[2] \xrightarrow{\Sigma_{14}} [\Sigma^{14}_H P^2, S^{10}] \xrightarrow{\Sigma_{14}} Z_2^2(\mathcal{V}_{10}, \epsilon_{10}) \rightarrow 0 \\
\downarrow{\Sigma_1} & & \downarrow{\Sigma_2} \\
0 & \rightarrow & Z_2(\theta') \xrightarrow{\Sigma_{15}} [\Sigma^{15}_H P^2, S^{11}] \xrightarrow{\Sigma_{15}} Z_2(\mathcal{V}_{11}, \epsilon_{11}) \rightarrow 0,
\end{array}$$

where $$\Sigma_1$$, $$\Sigma_2$$, and $$\Sigma_3$$ are homomorphisms induced by Freudenthal’s homomorphisms.

Since the first row splits and $$\Sigma_3$$ is an isomorphism, the second row also splits.

(3) By using the same approach as in (1) and (2), we obtain the following split short exact sequence from the (16,12)-type short exact sequence:

$$0 \rightarrow Z_2^2(\theta, E\theta') \xrightarrow{\Sigma_{16}^r} [\Sigma^{16}_H P^2, S^{12}] \xrightarrow{\Sigma_{16}^r} Z_2^2(\mathcal{V}_{12}, \epsilon_{12}) \rightarrow 0.$$ (4) By using the same approach as in (1)~(3), we obtain the following split short exact sequence from the (17,13)-type short exact sequence:

$$0 \rightarrow Z_2^2(E\theta) \xrightarrow{\Sigma_{17}^r} [\Sigma^{17}_H P^2, S^{13}] \xrightarrow{\Sigma_{17}^r} Z_2^2(\mathcal{V}_{13}, \epsilon_{13}) \rightarrow 0.$$ (5) Consider the homomorphisms $$\nu_{23} : \pi_{23}(S^{11}) \rightarrow \pi_{26}(S^{14})$$ and $$\nu_{22} : \pi_{22}(S^{14}) \rightarrow \pi_{25}(S^{14})$$ related to the (18,14)-type short exact sequence. Since Coker$$\nu_{23} = 0$$ and $$\nu_{22}$$ is a trivial homomorphism, we obtain the following short exact sequence from the (18,14)-type short exact sequence:

$$0 \rightarrow [\Sigma^{18}_H P^2, S^{14}] \xrightarrow{\Sigma_{18}^r} Z_2^2(\mathcal{V}_{14}, \epsilon_{14}) \rightarrow 0.$$ The suspension homomorphism

$$\Sigma : [\Sigma^{n+4}_H P^2, S^n] \rightarrow [\Sigma^{n+5}_H P^2, S^{n+1}]$$
is isomorphic for $n + 12 < 2n - 1$ (that is, $13 < n$), since $\Sigma^{n+4}\mathbb{H}P^2$ is an $(n+12)$-dimensional CW-complex and $S^n$ is $(n-1)$-connected. Thus, the proof is complete according to above fact. □

5. Computation of $[\Sigma^{n+5}\mathbb{H}P^2, S^n]$ for $n \geq 2$

In this section, we compute the $n$-th cohomotopy groups of $(n + 5)$-fold suspended quaternionic projective planes.

Proposition 9. $[\Sigma^7\mathbb{H}P^2, S^2] = \mathbb{Z}_2 \{\eta_2 \circ \nu' \circ \mu_6 \circ \Sigma^7 p\}.$

Proof. By Proposition 1(1) and (2.2), the proof is complete. □

Proposition 10. (1) $[\Sigma^8\mathbb{H}P^2, S^3] = \mathbb{Z}_2 \{\nu' \circ \eta_6 \circ \mu_7 \circ \Sigma^8 p\} \oplus \mathbb{Z}_2 \{\epsilon_3 \circ \eta_{11}\} \oplus \mathbb{Z}_3 \{\alpha_2(3) \circ \Sigma^8 p\}.$

(2) $[\Sigma^8\mathbb{H}P^2, S^4] = \mathbb{Z}_2 \{\nu_4 \circ \eta_7 \circ \mu_8 \circ \Sigma^9 p, (E\nu') \circ \eta_7 \circ \mu_8 \circ \Sigma^9 p, \epsilon_4 \circ \eta_{11}\} \oplus \mathbb{Z}_3 \{\alpha_4(4) \circ \beta_1(7) \circ \Sigma^9 p, [\epsilon_4, \epsilon_4] \circ \beta_1(7) \circ \Sigma^9 p\}.$

Proof. Now, we consider the $(8, 3)$-type short exact sequence:

$$0 \rightarrow \text{Coker} \nu_{13}^* \rightarrow [\Sigma^8\mathbb{H}P^2, S^3] \rightarrow \text{Ker} \nu_{12}^* \rightarrow 0,$$

where $\nu_{13}^* : \pi_{13}(S^3) \rightarrow \pi_{16}(S^3)$ and $\nu_{12}^* : \pi_{12}(S^3) \rightarrow \pi_{15}(S^3)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \rightarrow S^3$. These homomorphisms can be restated as follows:

$$\nu_{13}^* : Z_4 \{\nu' \circ \eta_6 \circ \mu_7\} \oplus Z_2 \{\eta_3 \circ \mu_4\} \oplus Z_3 \{\alpha_2(3)\} \rightarrow Z_2 \{\nu' \circ \eta_6 \circ \mu_7\} \oplus Z_3 \{\alpha_3(3) \circ \beta_1(6)\}$$

and

$$\nu_{12}^* : Z_2 \{\mu_3\} \oplus Z_2 \{\eta_3 \circ \epsilon_4\} \rightarrow Z_2 \{\nu' \circ \mu_6\} \oplus Z_2 \{\nu' \circ \eta_6 \circ \epsilon_7\},$$

respectively. Then, we have $\nu_{13}^*(\nu') = 0$ by [4, Proposition 2.2] and $\nu_{13}^*(\eta_3 \circ \mu_4) = \eta_3 \circ \mu_4 \circ \eta_1 = \mu_3 \circ \eta_1 \circ \eta_3 \circ \eta_1 = 0$ by [4, (2.1)] and [8, (5.9)], and $\nu_{13}^*(\alpha_2(3)) = 0$. Moreover, $\nu_{12}^*(\mu_3) = \nu' \circ \eta_6 \circ \epsilon_7$ by [4, Proposition 2.2], and $\nu_{12}^*(\eta_3 \circ \epsilon_4) = \eta_3 \circ \epsilon_4 \circ \eta_1 \circ \epsilon_3 \circ \eta_1 = 0$ by [4, (2.1)] and [8, (5.9)]. Thus, we have

$$\text{Coker} \nu_{13}^* = Z_2 \{\nu' \circ \eta_6 \circ \mu_7\} \oplus Z_2 \{\alpha_1(3) \circ \beta_1(6)\} \text{ and } \text{Ker} \nu_{12}^* = Z_2 \{\eta_3 \circ \epsilon_4\}.$$ 

Then, we have the short exact sequence

$$0 \rightarrow Z_2 \{\nu' \circ \eta_6 \circ \mu_7\} \oplus Z_3 \{\alpha_1(3) \circ \beta_1(6)\} \rightarrow [\Sigma^8\mathbb{H}P^2, S^3] \rightarrow Z_2 \{\eta_3 \circ \epsilon_4\} \rightarrow 0.$$ 

Consider an extension $\epsilon_3 \circ \eta_{11} \circ \eta_1 = \eta_3 \circ \epsilon_4$ [4, (2.1)]. Since $\eta_{11}$ has order 2, $\epsilon_3 \circ \eta_{11}$ has order 2 [2, Proposition 41.1]. Therefore, the short exact sequence splits.

(2) Consider the $(9, 4)$-type short exact sequence:

$$0 \rightarrow \text{Coker} \nu_{14}^* \rightarrow [\Sigma^9\mathbb{H}P^2, S^4] \rightarrow \text{Ker} \nu_{13}^* \rightarrow 0,$$

where $\nu_{14}^* : \pi_{14}(S^4) \rightarrow \pi_{17}(S^4)$ and $\nu_{13}^* : \pi_{13}(S^4) \rightarrow \pi_{16}(S^4)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \rightarrow S^3$. These homomorphisms can be restated as follows:

$$\nu_{14}^* : Z_8 \{\mu_4 \circ \sigma'\} \oplus Z_4 \{E\nu'\} \oplus Z_2 \{\eta_4 \circ \mu_5\} \oplus Z_3^2 \oplus Z_5 \rightarrow \cdots.$$
Thus, Coker \(\nu_1\) is odd by [4, (2.1)]; thus, it has order 8. Moreover,

\[ \nu_1^*(E') = E(\nu' \circ \nu_1) = 0 \]

by [4, (2.2) (7)] and \(\nu_1^*(\eta_4 \circ \epsilon_5) = 0\). Thus, we have

\[ \text{Coker} \nu_1^* = \mathbb{Z}_2^2\{\nu_4 \circ \eta \circ \mu_8, (E \nu') \circ \eta \circ \mu_8\} \oplus \mathbb{Z}_2^2\{\alpha_1(4) \circ \beta_1(7), [\iota_4, \iota_4] \circ \beta_1(7)\}. \]

Proposition 11.

\[ [\Sigma^{10}H^2, S^0] = \mathbb{Z}_2^3\{\nu_4 \circ \eta \circ \mu_8 \circ \Sigma^{10}p, \overline{\mu_5}, \epsilon_5 \circ \overline{\mu_13}\} \oplus \mathbb{Z}_4\{\alpha_1(5) \circ \beta_1(8) \circ \Sigma^{10}p\}. \]

**Proof.** Consider the \((10,5)\)-type short exact sequence:

\[ 0 \rightarrow \text{Coker} \nu_1^* \xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}H^2, S^0] \xrightarrow{\Sigma^{10}p^*} \text{Ker} \nu_1^* \rightarrow 0, \]

where \(\nu_1^* : \pi_{15}(S^5) \rightarrow \pi_{18}(S^5)\) and \(\nu_1^* : \pi_{14}(S^5) \rightarrow \pi_{17}(S^5)\) are the homomorphisms induced originally by the Hopf fibration \(\nu_4 : S^7 \rightarrow S^0\). These homomorphisms can be restated as follows:

\[ \nu_1^* : \mathbb{Z}_2\{\nu_5 \circ \sigma_8 \circ \mu_8 \circ \Sigma^{10}p, \overline{\mu_5}, \epsilon_5 \circ \overline{\mu_13}\} \oplus \mathbb{Z}_4\{\alpha_1(5) \circ \beta_1(8)\}. \]

and

\[ \nu_1^* : \mathbb{Z}_2^2\{\nu_5^2, \mu_5, \eta_8 \circ \epsilon_6\} \rightarrow \mathbb{Z}_2^2\{\nu_5^4, \nu_5 \circ \mu_3, \nu_5 \circ \eta_8 \circ \epsilon_9\}, \]

respectively. Then, we have \(\nu_1^*(\nu_5 \circ \sigma_8) = \nu_5 \circ \sigma_8 \circ \nu_15\) and \(\nu_1^*(\eta_5 \circ \mu_6) = 0\). Thus, \(\text{Coker} \nu_1^* = \mathbb{Z}_2\{\nu_5 \circ \eta_8 \circ \mu_5\} \oplus \mathbb{Z}_4\{\alpha_1(5) \circ \beta_1(8)\}\). In addition, we have \(\nu_1^*(\epsilon_8^2) = E^2(\mu_3 \circ \nu_12) = E^2(\nu' \circ \eta_6 \circ \epsilon_7) = (E^2\nu') \circ \eta_8 \circ \epsilon_9 = \).
2ν₅ ∘ η₈ ∘ ε₉ = 0 and ν₄ᵢ(η₅ ∘ ε₆) = 0. Thus, Kerv₁₄ = Z₂[ν₅, η₅ ∘ ε₆], which
gives the short exact sequence

0 → Z₂[ν₅ ∘ η₈ ∘ µ₉] ⊕ Z₃[α₂(5)] \overset{\Sigma¹¹π}{\longrightarrow} [Σ¹¹H²P², S⁶] \overset{\Sigma¹¹i}{\longrightarrow} Z₂[ν₅, η₅ ∘ ε₆] → 0.

Since the extension \( \overline{\mu} \) has order 2 [2, Proposition 4.1(4)], the extension \( \epsilon_5 \circ \overline{\eta}_{13} \) of \( \epsilon_5 \circ \eta_{13} = \eta_5 \circ \epsilon_6 \) also has order 2. We know that \( H(\mu') = H[8] \) and \( \mu' \circ \nu_{14} = 0 \) by [4, Proposition (2.4)(1)]. Thus, we have an extension \( \overline{\mu}' \in [Σ¹¹H²P², S⁶] \).

Proposition 12. (1) \([Σ¹¹H²P², S⁶] = Z³[ν₀, µ₆, ε₆ ∘ \overline{η}_{13}] \oplus Z₃[α₁(6) \circ β₁(9) ∘ Σ¹¹p] \).

(2) \([Σ¹²H²P², S⁷] = Z³[σ', ∂_{14}ν₅, µ₇, ε₇ ∘ \overline{η}_{13}] \oplus Z₄[α₁(7) ∘ β₁(10) ∘ Σ¹²p] \).

(3) \([Σ¹³H²P², S⁸] = Z³[σ₈ ∘ \overline{η}_{13}, (Eσ') ∘ η₃, ν₃, µ₈, ε₈ ∘ \overline{η}_{13}] \oplus Z₃[α₁(8) ∘ β₁(11) ∘ Σ¹³p] \).

(4) \([Σ¹⁴H²P², S⁹] = Z³[σ₉ ∘ \overline{η}_{13}, ν₄, \mu₉, ε₉ ∘ \overline{η}_{13}] \oplus Z₃[α₁(9) ∘ β₁(12) ∘ Σ¹⁴p] \).

(5) \([Σ¹⁵H²P², S¹⁰] = Z⁴(\Delta(σ_{21})) \oplus Z³[ν₅, µ₆, ε₆ ∘ \overline{η}_{13}] \oplus Z₃[α₁(10) ∘ β₁(13) ∘ Σ¹⁵p] \).

Proof. (1) Consider the (11, 6)-type short exact sequence:

\[
0 \to \text{Coker}ν₁₆ \overset{\Sigma¹¹π}{\longrightarrow} [Σ¹¹H²P², S⁶] \overset{\Sigma¹¹i}{\longrightarrow} \text{Kerv}_₁₅ \to 0,
\]

where \( ν₁₆ : π₁₆(S⁶) → π₁₉(S⁶) \) and \( ν₁₅ : π₁₅(S⁶) → π₁₈(S⁶) \) are the homomorphisms induced originally by the Hopf fibration \( ν₄ : S⁵ → S⁴ \). These homomorphisms can be restated as follows:

\( ν₁₆ : Z₃[ν₀ ∘ σ₉] ⊕ Z₂[η₀ ∘ µ₇] \oplus Z₃[β₁(6)] \to Z₂[ν₀ ∘ σ₉ ∘ ν₁₆] \oplus Z₃[α₁(6) ∘ β₁(9)] \)

and

\( ν₁₅ : Z³[ν₆, µ₆ ∘ ε₇] → Z₁₆[\Delta(σ_{13})] \oplus Z₃[α₁(6) ∘ β₁(9)], \)

respectively. Then, we have \( ν₁₆(ν₀ ∘ σ₉) = ν₀ ∘ σ₉ ∘ ν₁₆ \) and \( ν₁₆(η₀ ∘ µ₇) = 0 \).

Thus, we have \( \text{Coker}ν₁₆ = Z₃[α₁(6) ∘ β₁(9)] \). Moreover, since \( ν₁₅(ν₀₉) = 0 \), \( ν₁₅(µ₉) = 0 \) and \( ν₁₅(η₀ ∘ ε₇) = 0 \), we have \( \text{Kerv}_₁₅ = Z³[ν₆, µ₆ ∘ ε₇] \).

We have a split short exact sequence

\[
0 \to Z₃[α₁(6) ∘ β₁(9)] \overset{\Sigma¹¹π}{\longrightarrow} [Σ¹¹H²P², S⁶] \overset{\Sigma¹¹i}{\longrightarrow} Z³[ν₆, µ₆ ∘ ε₇] → 0.
\]

(2) Consider the (12, 7)-type short exact sequence

\[
0 \to \text{Coker}ν₁₇ \overset{\Sigma¹²π}{\longrightarrow} [Σ¹²H²P², S⁷] \overset{\Sigma¹²i}{\longrightarrow} \text{Kerv}_₁₆ \to 0,
\]
where \( \nu_{17} : \pi_{17}(S^7) \to \pi_{20}(S^7) \) and \( \nu_{10} : \pi_{16}(S^7) \to \pi_{19}(S^7)(\cong 0) \) are the homomorphisms induced originally by the Hopf fibration \( \nu_{4} : S^7 \to S^3 \). \( \nu'_{17} \) can be restated as follows:

\[ \nu'_{17} : Z_8(\nu_{7} \circ \sigma_{10}) \oplus Z_2{\eta_7 \circ \mu_8} \oplus Z_3{\beta_1(7)} \to Z_2{\nu_{7} \circ \sigma_{10} \circ \nu_7} \oplus Z_3{\alpha_1(7) \circ \beta_1(10)}. \]

Then, we have \( \nu'_{17}(\nu_{7} \circ \sigma_{10}) = \nu_{7} \circ \sigma_{10} \circ \nu_7 \) and \( \nu'_{17}(\eta_7 \circ \mu_8) = 0 \). Hence, \( \text{Coker}\nu'_{17} = Z_3{\alpha_1(7) \circ \beta_1(10)} \) and \( \text{Ker}\nu'_{17} = \pi_{16}(S^7) = Z_4{\sigma' \circ \eta_{14}, \nu_3, \mu_7, \eta_7 \circ \epsilon_8} \). Thus, we have a split short exact sequence

\[ 0 \to Z_3{\alpha_1(7) \circ \beta_1(10)} \xrightarrow{\Sigma_{11}^{11}p^*} [\Sigma_{11}^{11}P^2, S^6] \xrightarrow{\Sigma_{11}^{11}r} Z_4{\sigma' \circ \eta_{14}, \nu_3, \mu_7, \eta_7 \circ \epsilon_8} \to 0. \]

(3) Consider the \((13, 8)\)-type short exact sequence:

\[ 0 \to \text{Coker}\nu'_{18} \xrightarrow{\Sigma_{13}^{13}p^*} [\Sigma_{13}^{13}P^2, S^8] \xrightarrow{\Sigma_{13}^{13}r} \text{Ker}\nu'_{17} \to 0, \]

where \( \nu_{18} : \pi_{18}(S^8) \to \pi_{21}(S^8) \) and \( \nu_{17} : \pi_{17}(S^8) \to \pi_{20}(S^8)(\cong 0) \) are the homomorphisms induced originally by the Hopf fibration \( \nu_{4} : S^7 \to S^3 \). \( \nu'_{18} \) can be restated as follows:

\[ \nu'_{18} : Z_2^2{\sigma_8 \circ \nu_{15}, \nu_8 \circ \sigma_{11}} \oplus Z_2{\eta_8 \circ \mu_9} \oplus Z_2^2 \to Z_2^2{\sigma_8 \circ \nu_{15}, \nu_8 \circ \sigma_{11}} \oplus Z_2{\alpha_1(8) \circ \beta_1(11)}. \]

Then, we have \( \nu'_{18}(\sigma_8 \circ \nu_{15}) = \sigma_8 \circ \nu_{15}, \nu'_{18}(\nu_8 \circ \sigma_{11}) = \nu_8 \circ \sigma_{11} \circ \nu_{15}, \) and \( \nu'_{18}(\eta_8 \circ \mu_9) = 0 \). Thus, we have \( \text{Coker}\nu'_{18} = Z_3{\alpha_1(8) \circ \beta_1(11)} \) and \( \text{Ker}\nu'_{17} = \pi_{17}(S^8) = Z_2^2{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{15}, \nu_3, \mu_8, \eta_8 \circ \epsilon_9} \). Therefore, we have a split short exact sequence

\[ 0 \to Z_3{\alpha_1(8) \circ \beta_1(11)} \xrightarrow{\Sigma_{11}^{11}p^*} [\Sigma_{11}^{11}P^2, S^6] \xrightarrow{\Sigma_{11}^{11}r} Z_2^2{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{15}, \nu_3, \mu_8, \eta_8 \circ \epsilon_9} \to 0. \]

(4) Consider the \((14, 9)\)-type short exact sequence:

\[ 0 \to \text{Coker}\nu'_{19} \xrightarrow{\Sigma_{14}^{14} p^*} [\Sigma_{14}^{14}P^2, S^9] \xrightarrow{\Sigma_{14}^{14} r} \text{Ker}\nu'_{18} \to 0, \]

where \( \nu'_{19} : \pi_{19}(S^9) \to \pi_{22}(S^9) \) and \( \nu'_{18} : \pi_{18}(S^9) \to \pi_{21}(S^9)(\cong 0) \) are the homomorphisms induced originally by the Hopf fibration \( \nu_{4} : S^7 \to S^3 \). \( \nu'_{19} \) can be restated as follows:

\[ \nu'_{19} : Z_8{\sigma_9 \circ \nu_{16}} \oplus Z_2{\eta_9 \circ \mu_{10}} \to Z_2{\sigma_9 \circ \nu_{16}} \oplus Z_4{\alpha_1(9) \circ \beta_1(12)}. \]

Then, we have \( \nu'_{19}(\sigma_9 \circ \nu_{16}) = \sigma_9 \circ \nu_{16} \) and \( \nu'_{19}(\eta_9 \circ \mu_{10}) = 0 \). Hence, \( \text{Coker}\nu'_{19} = Z_4{\alpha_1(9) \circ \beta_1(12)} \) and \( \text{Ker}\nu'_{18} = \pi_{18}(S^9) = Z_4{\sigma_9 \circ \eta_{16}, \nu_3, \mu_9, \eta_9 \circ \epsilon_{10}} \). Thus, we have a split short exact sequence

\[ 0 \to Z_4{\alpha_1(9) \circ \beta_1(12)} \xrightarrow{\Sigma_{11}^{11}p^*} [\Sigma_{11}^{11}P^2, S^6] \xrightarrow{\Sigma_{11}^{11}r} Z_2^2{\sigma_9 \circ \eta_{16}, \nu_3, \mu_9, \eta_9 \circ \epsilon_{10}} \to 0. \]

(5) Consider the \((15, 9)\)-type short exact sequence:

\[ 0 \to \text{Coker}\nu_{20} \xrightarrow{\Sigma_{15}^{15} p^*} [\Sigma_{15}^{15}P^2, S^{10}] \xrightarrow{\Sigma_{15}^{15} r} \text{Ker}\nu'_{19} \to 0, \]
where $\nu_{20} : \pi_{20}(S^{10}) \to \pi_{22}(S^{10})$ and $\nu_{19} : \pi_{19}(S^{10}) \to \pi_{22}(S^{10})$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows: 

$\nu_{20} : Z_4\{\sigma_{10} \circ \nu_{17}\} \oplus Z_2(\eta_{10} \circ \mu_{11}) \oplus Z_3(\beta_{1}(10)) \to Z_2(\sigma_{10} \circ \nu_{17}) \oplus Z_3(\alpha_1(10) \circ \beta_{1}(13))$ and

$\nu_{19} : Z_3(\Delta(\nu_{21})) \oplus Z_2^2(\nu_{10}, \mu_{10}, \eta_{10} \circ \epsilon_{11}) \to Z_4(\Delta(\nu_{21})) \oplus Z_3.

Then, we have $\nu_{20}(\sigma_{10} \circ \nu_{17}) = \sigma_{10} \circ \nu_{2}^2$ and $\nu_{20}(\eta_{10} \circ \mu_{11}) = 0$. Thus, we have $\text{Coker}\nu_{20} = Z_3(\alpha_1(10) \circ \beta_{1}(13))$.

In addition, we have $\nu_{19}(\Delta(\nu_{21})) = \Delta(\nu_{21}) \circ \nu_{19} = \Delta(\nu_{21})$, $\nu_{19}(\nu_{10}) = 0$, $\nu_{19}(\mu_{10}) = 0$, and $\nu_{19}(\eta_{10} \circ \epsilon_{11}) = 0$. Thus, we have $\text{Ker}\nu_{19} = Z_4(4\Delta(\nu_{21})) \oplus Z_3^2(\mu_{10}, \eta_{10} \circ \epsilon_{11})$. Therefore, we have a split short exact sequence

$0 \to Z_3(\alpha_1(10) \circ \beta_{1}(13)) \xrightarrow{\Sigma_{15}^3 r^*} [\Sigma_{15}^4\mathbb{H}P^2, S^{10}] \xrightarrow{\Sigma_{15}^3 r^*} \to 0$. \hfill $\square$

**Proposition 13.** (1) $[\Sigma_{16}^3\mathbb{H}P^2, S^{11}] = Z_2^2(\theta' \circ \eta_{23}, \theta_{16}, \mu_{11}, \eta_{11} \circ \eta_{12}) \oplus Z_3(\alpha_1(11) \circ \beta_{1}(14)) \oplus \Sigma_{16}^4 P_p$.

(2) $[\Sigma_{17}^3\mathbb{H}P^2, S^{12}] = Z_2^2(\theta_{15}, \mu_{12}) \oplus Z_2(\epsilon_{12} \circ \eta_{23}) \oplus Z_3(\alpha_1(12) \circ \beta_{1}(15) \circ \Sigma_{17}^8 p)$.

(3) $[\Sigma_{18}^3\mathbb{H}P^2, S^{13}] = Z_2^2(\mu_{15}, \epsilon_{13} \circ \eta_{23}) \oplus Z_3(\alpha_1(13) \circ \beta_{1}(16) \circ \Sigma_{18}^8 p)$.

(4) $[\Sigma_{19}^3\mathbb{H}P^2, S^{14}] = Z_2^2(\eta_{23}) \oplus Z_2^3(\mu_{14}, \epsilon_{14} \circ \eta_{23}) \oplus Z_3(\alpha_1(14) \circ \beta_{1}(17) \circ \Sigma_{19}^8 p)$.

(5) $[\Sigma_{11}^{n+3}\mathbb{H}P^2, S^n] = Z_2^3(\eta_{n+3}, \epsilon_{n} \circ \eta_{n+8}) \oplus Z_3(\alpha_1(n) \circ \beta_{1}(n+3) \circ \Sigma_{11}^{n+5} p)$ for $n \geq 15$.

**Proof.** (1) Consider the $(16, 11)$-type short exact sequence:

$0 \to \text{Coker}\nu_{21} \xrightarrow{\Sigma_{16}^3 p^*} [\Sigma_{16}^3\mathbb{H}P^2, S^{11}] \xrightarrow{\Sigma_{16}^3 r^*} \text{Ker}\nu_{20} \to 0$,

where $\nu_{21} : \pi_21(S^{11}) \to \pi_{22}(S^{11})$ and $\nu_{20} : \pi_{20}(S^{11}) \to \pi_{22}(S^{11})$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$\nu_{21} : Z_2^2(\sigma_{11} \circ \nu_{18}, \eta_{11} \circ \mu_{12}) \to Z_2^2(\theta' \circ \eta_{23}, \sigma_{11} \circ \nu_{18}) \oplus Z_3(\alpha_1(11) \circ \beta_{1}(14))$ and

$\nu_{20} : Z_2^3(\nu_{11}, \mu_{11}, \eta_{11} \circ \epsilon_{12}) \to Z_2(\theta')$.

Respectively. Then, we have $\nu_{21}(\alpha_1(11) \circ \nu_{18}) = \sigma_{11} \circ \nu_{18}$ and $\nu_{21}(\eta_{11} \circ \mu_{12}) = 0$. Thus, we have $\text{Coker}\nu_{21} = Z_2(\theta' \circ \eta_{23}) \oplus Z_3(\alpha_1(11) \circ \beta_{1}(14))$. Moreover, since $\nu_{20}(\nu_{11}^2) = 0$, $\nu_{20}(\mu_{12}) = 0$, and $\nu_{20}(\eta_{11} \circ \epsilon_{12}) = 0$, we have $\text{Ker}\nu_{20} = Z_2^3(\nu_{11}, \mu_{11}, \eta_{11} \circ \epsilon_{12})$. Therefore, we have the short exact sequence

$0 \to Z_2(\theta' \circ \eta_{23}) \oplus Z_3(\alpha_1(11) \circ \beta_{1}(14)) \xrightarrow{\Sigma_{16}^3 r^*} [\Sigma_{16}^3\mathbb{H}P^2, S^{11}] \xrightarrow{\Sigma_{16}^3 r^*} \text{Coker}\nu_{21} \to 0$.

We now consider a generalized EH-sequence

$[\Sigma_{17}^4\mathbb{H}P^2, S^{21}] \xrightarrow{\Delta} [\Sigma_{15}^4\mathbb{H}P^2, S^{10}] \xrightarrow{E} [\Sigma_{16}^3\mathbb{H}P^2, S^{11}] \xrightarrow{H} [\Sigma_{16}^3\mathbb{H}P^2, S^{21}]$. 

The suspension homomorphism \( \Sigma : [\Sigma_n] \rightarrow [\Sigma_{n+1}] \) is trivial.

Thus, we have

\[
[\Sigma_{15}^3 P^2, S^{11}]_{(2)} \cong [\Sigma_{15}^3 P^2, S^{10}]_{(2)} / \text{Ker} E
\]

Therefore, the above short exact sequence splits.

(2) Since two homomorphisms \( \nu_{25}^2 : \pi_{25}(S^{12}) \rightarrow \pi_{23}(S^{12}) \) and \( \nu_{25}^2 : \pi_{21}(S^{12}) \rightarrow \pi_{24}(S^{12}) \) are trivial, we obtain the following short exact sequence from the (17, 12)-type short exact sequence:

\[
0 \rightarrow \mathbb{Z}_2^2 \{ \theta \circ \eta_{24}, (E \theta') \circ \eta_{24} \} \oplus \mathbb{Z}_4 \{ \alpha_{1}(12) \circ \beta_1(15) \} \xrightarrow{\Sigma_{17}^+ \nu} [\Sigma_{17}^3 P^2, S^{12}] \xrightarrow{\Sigma_{17}^+ \nu} \mathbb{Z}_2^2 \{ \nu_{12}, \mu_{12} \circ \epsilon_{12} \} \rightarrow 0.
\]

Consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{Z}_2 \{ \theta \circ \eta_{25}, (E \theta') \circ \eta_{25} \} & \oplus & \mathbb{Z}_4 \{ \alpha_{1}(13) \circ \beta_1(16) \} & \xrightarrow{\Sigma_{18}^+ \nu} & [\Sigma_{18}^3 P^2, S^{13}] & \xrightarrow{\Sigma_{18}^+ \nu} & \mathbb{Z}_2^3 \{ \nu_{13}, \mu_{13} \circ \epsilon_{13} \} & \rightarrow 0.
\end{array}
\]

Since the first row splits, the second row also splits.

(3) By using the same approach as in (2), we obtain the following split short exact sequence from (18, 13)-type short exact sequence:

\[
0 \rightarrow \mathbb{Z}_2 \{ (E \theta) \circ \eta_{25} \} \oplus \mathbb{Z}_4 \{ \alpha_{1}(13) \circ \beta_1(16) \} \xrightarrow{\Sigma_{18}^+ \nu} [\Sigma_{18}^3 P^2, S^{13}] \xrightarrow{\Sigma_{18}^+ \nu} \mathbb{Z}_2^3 \{ \nu_{13}, \mu_{13} \circ \epsilon_{13} \} \rightarrow 0.
\]

(4) By using the same approach as in (2) and (3), we obtain the following split short exact sequence from the (19, 14)-type short exact sequence:

\[
0 \rightarrow \mathbb{Z} \{ \Delta(\epsilon_{29}) \} \oplus \mathbb{Z}_4 \{ \alpha_{1}(14) \circ \beta_1(17) \} \xrightarrow{\Sigma_{18}^+ \nu} [\Sigma_{18}^3 P^2, S^{13}] \xrightarrow{\Sigma_{18}^+ \nu} \mathbb{Z}_2^3 \{ \nu_{14}, \mu_{14} \circ \epsilon_{14} \} \rightarrow 0.
\]

(5) Since two homomorphisms \( \nu_{24}^2 : \pi_{24}(S^{15}) \rightarrow \pi_{25}(S^{15}) \) and \( \nu_{24}^2 : \pi_{24}(S^{15}) \rightarrow \pi_{27}(S^{15}) \cong 0 \) are trivial, we obtain the following split short exact sequence from the (20, 15)-type short exact sequence:

\[
0 \rightarrow \mathbb{Z}_4 \{ \alpha_{1}(15) \circ \beta_1(18) \} \xrightarrow{\Sigma_{20}^+ \nu} [\Sigma_{20}^3 P^2, S^{15}] \xrightarrow{\Sigma_{20}^+ \nu} \mathbb{Z}_2^3 \{ \nu_{15}, \mu_{15} \circ \epsilon_{15} \} \rightarrow 0.
\]

The suspension homomorphism \( \Sigma : [\Sigma_{n+5}^3 P^2, S^n] \rightarrow [\Sigma_{n+5}^3 P^2, S^{n+1}] \) is isomorphic for \( n + 13 < 2n - 1 \) (that is, \( 14 < n \)), since \( \Sigma_{n+5}^3 P^2 \) is an \( (n + 13) \)-dimensional CW complex and \( S^n \) is \( (n - 1) \)-connected. Thus, we have the desired result.
References

[1] K. Aoki, *On torus cohomotopy groups*, Proc. Japan Acad. **30** (1954), 694–697.

[2] H. Kachi, J. Mukai, T. Nozaki, Y. Sumita, and D. Tamaki, *Some cohomotopy groups of suspended projective planes*, Math. J. Okayama Univ. **43** (2001), 105–121.

[3] S. Kikkawa, J. Mukai, and D. Takaba, *Cohomotopy sets of projective planes*, J. Fac. Sci. Shinshu Univ. **33** (1996), no. 1, 1–7.

[4] K. Oguchi. *Generators of 2-primary components of homotopy groups of spheres, unitary groups and symplectic groups*, J. Fac. Sci. Univ. Tokyo Sect. I **11** (1964), 65–111.

[5] H. Oshima and Koji Takahara, *Cohomotopy of Lie groups*, Osaka J. Math. **28** (1991), no. 2, 213–221.

[6] R. Rubinstein, *Some remarks on the cohomotopy of $\mathbb{R}P^\infty$*, Quart. J. Math. Oxford Ser. (2) **30** (1979), no. 117, 119–128.

[7] Y. Sumita, *Master’s thesis*, Shinshu University, 1998.

[8] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J. 1962.

Jin Ho Lee  
Department of Mathematics  
Korea University  
Seoul 136-701, Korea  
E-mail address: sabforev@korea.ac.kr

Kee Young Lee  
Department of Mathematics  
Korea University  
Sejong City 339-700, Korea  
E-mail address: keyolee@korea.ac.kr