Optimal design and performance evaluation of sequentially planned hypothesis tests

Andrey Novikov
Metropolitan Autonomous University, Mexico City, Mexico

Abstract: In this paper, we propose a method of construction of optimal sequentially planned tests. In particular, for independent and identically distributed observations we obtain the form of optimal sequentially planned tests which turn to be a particular case of sequentially planned probability ratio tests (SPPRTs). Formulas are given for computing the numerical characteristics of general SPPRTs, like error probabilities, average sampling cost, etc. A numerical method of designing the optimal tests and evaluation of the performance characteristics is proposed, and computer algorithms of its implementation are developed. For a particular case of sampling from a Bernoulli population, the proposed method is implemented in R programming language, the code is available at a public GitHub repository. The proposed method is compared numerically with other known sampling plans.

Keywords: sequential analysis; sequentially planned procedure; hypothesis test; optimal sampling; optimal stopping

Subject Classifications: 62L10, 62L15, 62F03, 60G40

1. INTRODUCTION

Sequentially planned hypothesis tests have been proposed as a theoretical framework for some practical situations when sequential statistical methods are applied to sampling data in groups (group-sequential schemes) [Ehrenfeld, 1972]. In many occasions, the overall cost of sampling in groups includes, additionally to the unitary cost (say, \( c \)) of any collected data item, also some cost (\( c_0 \)) related to the group (the overall cost thus being \( c_0 + cn \), where \( n \) is the number of observations in the group). Substantial discussions on the related models and the problems arising with respect to them can be found in Schmitz [1993]. For the problem of testing two simple hypotheses, Schmegner and Baron [2007] studied a class of tests called sequentially planned probability ratio tests (SPPRT). Under an additional assumption that the log-likelihood ratio takes its values on a lattice, they obtained analytical expressions for characteristics of the SPPRTs and, in the particular case of sampling from a Bernoulli population, evaluated them for various sampling plans, aiming at minimisation of the average cost of observations. Schmegner [2009] studied the case of truncated sequentially planned tests, when no more than a given fixed number of groups is allowed.

In this paper, we propose a method of construction of optimal sequentially planned tests and develop the whole set of computer algorithms for their design and performance evaluation. For the case of Bernoulli observations, we implement the algorithms in R programming language and numerically compare the obtained plans with those of Schmegner and Baron [2007]. The relative efficiency of the optimal test with respect to the one-stage plan with the same levels of error probabilities is evaluated.

Address correspondence to A. Novikov, Universidad Autónoma Metropolitana, Unidad Iztapalapa, Avenida Ferrocarril San Rafael Atlixco, 186, col. Leyes de Reforma 1A Sección, C.P. 09310, Cd. de México, Mexico, E-mail: an@xanum.uam.mx
In Section 2, general results on optimal sequentially planned tests are placed.
In Section 3, optimal sequentially planned tests for independent and identically distributed observations are characterized and formulas for performance characteristics are obtained.
In Section 4, a numerical method of optimal sequential planning and evaluation of performance characteristics is proposed. A numerical example is presented.
Section 5 contains a brief summary of results and conclusions.

2. GENERAL RESULTS

In this section we adapt the results of [Novikov, 2008] to the context of sequentially planned tests.

2.1. Definitions and preliminaries

We assume that a sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ will be available on the group-by-group basis for testing two simple hypotheses $H_0$ and $H_1$ about their distribution.

A sequentially planned hypothesis test consists of two elements: sampling/stopping plan and (terminal) decision function.

A sequential sampling/stopping plan is a sequence of random variables $N_1, N_2, \ldots$, taking non-negative integer values, where each $N_i$ is the size of group number $i$, $i = 1, 2, \ldots$. Let $M_i = N_1 + N_2 + \cdots + N_i$ be the number of observations in the first $i$ groups. It is assumed that each $N_i$ is a function of observations in previously observed groups, that is, $N_i = N_i(X_1, X_2, \ldots, X_{M_{i-1}})$. Respectively, $N_1$ does not depend on any observation and is chosen before the experiment starts, it is required to be positive. The experiment stops after stage $T$ when, for the first time, $N_{T+1}$ equals 0.

The second element of a sequentially planned test is its (terminal) decision function, i.e., a function $D_n = D_n(X_1, \ldots, X_n)$ to be applied when the experiment stops (with a total number $n = M_T$ of data items observed). When $D_n = i$, hypothesis $H_i$ is accepted, $i = 0$ or 1. Here $n = 1, 2, \ldots$

Any element of the test can be randomised, that is, take its values with some probabilities, after the data is observed, conditionally on the data.

We assume throughout the paper that the set of eligible group sizes at every stage is the same and is finite. It will be denoted $G$.

Let’s denote $\langle N, D \rangle$ the sequentially planned test with sampling plan $N = (N_1, N_2, \ldots)$ and decision function $D = (D_1, D_2, \ldots)$.

A sequentially planned test is truncated if no more than a fixed number $K$ of groups can be taken ($N_{K+1} \equiv 0$).

A classical sequential test corresponds to $N_i = 1$ or 0 (one observation at a time is taken, if any) for all $i = 1, 2, \ldots$

Suppose there is some cost, say, $c(n)$ we should pay for any group of $n$ items to be observed (for obvious reasons, we can assume that $c(n) > 0$ for all $n \in G$; another natural assumption is that $c(n)$ is a strictly increasing function of $n$). Then the average sampling cost (ASC), under $H_j$, of carrying out a test based on sampling plan $N$ is

$$\text{ASC}_j(N) = \sum_{i=1}^{\infty} E_j(c(N_1) + \cdots + c(N_i)) I_{\{N_1 > 0, \ldots, N_i > 0, N_{i+1} = 0\}}$$

$$= E_j(c(N_1)I_{\{N_1 > 0\}} + c(N_2)I_{\{N_1 > 0, N_2 > 0\}} + \cdots + c(N_i)I_{\{N_1 > 0, \ldots, N_i > 0\}} + \cdots),$$

where $E_j$ is the symbol of the expectation calculated under hypothesis $H_j$, $j = 0, 1$.

In order that (2.1) is well defined, the sampling plan $N$ should satisfy

$$\lim_{k \to \infty} P_j(N_1 > 0, \ldots, N_k > 0) = 0.$$

(2.2)
Let us denote $F_j$ the set of sampling plans $N$ satisfying (2.2), $j = 0, 1$, and let $F = F_0 \cap F_1$.

Let us define the following events:

- $S_N(i) = \{N_1 > 0, \ldots, N_i > 0, N_{i+1} = 0\}$ (stop with $i$ groups observed),
- $R_{N,D}(i) = \{D_{N_1 + \cdots + N_i} = 1\}$ (reject $H_0$ with $i$ groups observed),
- $A_{N,D}(i) = \{D_{N_1 + \cdots + N_i} = 0\}$ (accept $H_0$ with $i$ groups observed).

Error probabilities of the first and the second kind are defined, respectively, as

$$\alpha(N, D) = \sum_{i=1}^{\infty} P_0(S_N(i) \cap R_{N,D}(i))$$
$$\beta(N, D) = \sum_{i=1}^{\infty} P_1(S_N(i) \cap A_{N,D}(i))$$

The usual context for hypothesis testing is to minimise the average experimental cost under the restriction that

$$\alpha(N, D) \leq \alpha \quad \text{and} \quad \beta(N, D) \leq \beta,$$

where $\alpha$ and $\beta$ some numbers between 0 and 1.

In this paper, we want to minimise a weighted average

$$\text{ASC}_\gamma(N) = (1 - \gamma)\text{ASC}_0(N) + \gamma \text{ASC}_1(N)$$

under condition (2.3), where $\gamma \in [0, 1]$ is a given fixed number.

It is easy to see that the problem of minimisation of (2.4) under restrictions (2.3) reduces to the problem of minimisation of

$$\text{ASC}_\gamma(N) + \lambda_0 \alpha(N, D) + \lambda_1 \beta(N, D)$$

with some non-negative $\lambda_0, \lambda_1$ [see Novikov, 2008, Section 2]. Essentially, this is a straightforward application of the Lagrange method to a problem of constrained minimisation. From this point of view, (2.5) is interpreted as a Lagrangian function with constant multipliers $\lambda_0, \lambda_1$.

On the other hand, if $0 < \gamma < 1$ then (2.5) can be seen as Bayesian risk [cf. (2.1) Ehrenfeld, 1972] in the Bayes formulation. In this case $\gamma$ can be interpreted as an a priori probability of hypothesis $H_1$, and $\lambda_0, \lambda_1$ as some characteristics related to the loss due to incorrect decisions.

In a very usual way, it can be shown that there is a unique form for the decision function to be used in (2.5) for its minimisation.

We need some additional notation for this. Let $f^n_j = f^n_j(x_1, \ldots, x_n)$ be the Radon-Nikodym density of the distribution of $(X_1, X_2, \ldots, X_n)$ under $H_j$, $j = 0, 1$, with respect to a product-measure $\mu^n$ ($n$ times $\mu$ by itself), and let $f^n_\gamma = (1 - \gamma)f^n_0 + \gamma f^n_1$.

Then for any given sampling plan $N$, (2.5) is minimised by the decision function $D$ defined as

$$D_n = I_{\{\lambda_0 f^n_0 \leq \lambda_1 f^n_1\}}$$

$n = 1, 2, \ldots$

The corresponding minimum value of (2.5) is

$$L(N) = \sum_{n=1}^{\infty} \int I_{S_N(n)} \left( (c(N_1) + \cdots + c(N_n))f^M_n + \min\{\lambda_0 f^M_0, \lambda_1 f^M_1\} \right) d\mu^M_n,$$

where $M_n = N_1 + \cdots + N_n$ as before. The proof is along the lines of the proof of Theorem 3.1 in Novikov [2008].
2.2. Optimal truncated plans

Let $\mathcal{F}^K$ be the set of sampling plans $N$ taking at most $K$ groups (such that $N_{K+1} \equiv 0$). For $N \in \mathcal{F}^K$, let us denote

$$L_K(N) = \sum_{n=1}^{K} \int I_{S(n)} \left( (c(N_1) + \cdots + c(N_n)) f_n M_n + \min\{\lambda_0 f_0 M_n, \lambda_1 f_1 M_n\} \right) d\mu M_n.$$ 

For any set of group sizes $n_1, \ldots, n_i$ let us denote $f_{n_1,\ldots,n_i} = f_n = f_n(x_1, \ldots, x_n)$ where $n = n_1 + n_2 + \cdots + n_i$, $i = 1, 2, \ldots$

Starting from

$$V^K_{n_1,\ldots,n_k} = \min\{\lambda_0 f_0^{n_1,\ldots,n_k}, \lambda_1 f_1^{n_1,\ldots,n_k}\}$$

define recursively over $i = K - 1, K - 2, \ldots, 1$

$$V^K_{n_1,\ldots,n_i} = \min\{\lambda_0 f_0^{n_1,\ldots,n_i}, \lambda_1 f_1^{n_1,\ldots,n_i}, \min_m\{c(m) f^K_{n_1,\ldots,n_i} + \int V^K_{n_1,\ldots,n_i,m} d\mu m\}\}$$

Then for any sampling plan $N \in \mathcal{F}^K$

$$L_K(N) \geq \min_m\{c(m) + \int V^K_m d\mu m\} \quad \text{(2.7)}$$

There is an equality in (2.7) if the sampling plan $N$ is such that for $i = 1, 2, \ldots, K - 1$

$$N_{i+1} = \begin{cases} 0 & \text{if } \min_m\{\lambda_0 f_0^{n_1,\ldots,n_i}, \lambda_1 f_1^{n_1,\ldots,n_i}\} \leq \min_m\{c(m) f^K_{n_1,\ldots,n_i} + \int V^K_{n_1,\ldots,n_i,m} d\mu m\}, \\ \arg\min_m\{c(m) f^K_{n_1,\ldots,n_i} + \int V^K_{n_1,\ldots,n_i,m} d\mu m\}, & \text{otherwise}, \end{cases}$$

$$N_1 = \arg\min_m\{c(m) + \int V^K_m d\mu m\}, \text{ and } N_{K+1} \equiv 0.$$ 

It follows from (2.7) that this is an optimal sampling plan minimizing $L_K(N)$.

This result can be obtained in essentially the same way as Corollary 4.4 in Novikov [2008].

2.3. Optimal non-truncated plans

The treatment of the general case is essentially the same as in Section 5 in Novikov [2008].

First, it can easily be shown that $V^{K+1}_{n_1,\ldots,n_i} \leq V^K_{n_1,\ldots,n_i}$ for any fixed $n_1, \ldots, n_i$, so there exists

$$V_{n_1,\ldots,n_i} = \lim_{K \to \infty} V^K_{n_1,\ldots,n_i}.$$ 

Furthermore, $L_K(N) \to L(N)$ as $K \to \infty$ for all $N \in \mathcal{F}$.

Therefore, it follows from (2.7) that for all $N \in \mathcal{F}$

$$L(N) \geq \min_m\{c(m) + \int V^K_m d\mu m\}. \quad \text{(2.8)}$$

And similarly to the proof of Theorem 5.5 in Novikov [2008] it is shown that there is an equality in (2.8) if a sampling plan $N$ is such that for $i = 1, 2, \ldots$

$$N_{i+1} = \begin{cases} 0 & \text{if } \min_m\{\lambda_0 f_0^{n_1,\ldots,n_i}, \lambda_1 f_1^{n_1,\ldots,n_i}\} \leq \min_m\{c(m) f^K_{n_1,\ldots,n_i} + \int V^K_{n_1,\ldots,n_i,m} d\mu m\}, \\ \arg\min_m\{c(m) f^K_{n_1,\ldots,n_i} + \int V^K_{n_1,\ldots,n_i,m} d\mu m\}, & \text{otherwise}, \end{cases} \quad \text{(2.9)}$$
and $N_1 = \arg\min_m \{c(m) + \int V_m d\mu^m \}$.

Therefore, this is a form of a sampling plan which minimises the Lagrangian function $L(N)$, over all $N \in \mathcal{F}$. All other optimal sampling plans can be obtained from this one by randomisation, similarly to Novikov [2008]. The randomisation can be applied in any case of equality between the respective elements in (2.9), including those participating in the argmin definition. Obviously, the randomisation is irrelevant for Bayesian set-up but may be useful in the conditional setting.

3. The i.i.d. case

In the case of independent and identically distributed (i.i.d.) observations the constructions of Section 2 acquire a much simpler form.

Let $f_j(x)$ be the Radon-Nikodym derivative (with respect to $\mu$) of the distribution of $X_i$ under hypothesis $H_j$, and assume $X_i$ are all independent. For simplicity, let us assume that

$$\mu(x : f_0(x) \neq f_1(x)) = 0.$$ 

3.1. Optimal sequentially planned tests

For any bounded measurable non-negative function $U(z)$, $z \geq 0$, let us define the operator

$$J_m U(z) = E_0 \left( U \left( z \frac{f^m_1(X_1, \ldots, X_m)}{f^m_0(X_1, \ldots, X_m)} \right) \right), \quad z \geq 0.$$

Let us denote $g(z) = g(z; \lambda_0, \lambda_1) = \min \{\lambda_0, \lambda_1 z \}$ for $z \geq 0$.

Starting from

$$\rho_0(z) = g(z)$$

(3.1)

define recursively over $i = 1, 2, \ldots$

$$\rho_i(z) = \min \{g(z), \min_m \{c(m)(1 + \gamma(z - 1)) + J_m \rho_{i-1}(z)\} \}.$$  (3.2)

It is easy to see, by induction, that

$$V^K_{n_1, \ldots, n_i} = \rho_{K-i}(z_{n_1, \ldots, n_i}) f^{n_1, \ldots, n_i}_0$$

where

$$z_{n_1, \ldots, n_i} = \frac{f^{n_1, \ldots, n_i}_1}{f^{n_1, \ldots, n_i}_0}$$

is the likelihood ratio, and that the optimal sampling rule $N \in \mathcal{F}^K$ depends on $z = z_{n_1, \ldots, n_{i-1}}$ in such a way that for $i = 2, 3, \ldots, K$

$$N_i = \begin{cases} 0 & \text{if } g(z) \leq \min_m \{c(m)(1 + \gamma(z - 1)) + J_m \rho_{K-i}(z)\}, \\ \arg\min_m \{c(m)(1 + \gamma(z - 1)) + J_m \rho_{K-i}(z)\}, & \text{otherwise}, \end{cases}$$

(3.3)

and $N_1 = \arg\min_m \{c(m) + J_m \rho_{K-1}(1)\}$.

In particular, we obtain here a characterisation of truncated Bayesian sequentially planned test (properly saying, of its sampling-plan part, but the decision function to apply with any sampling plan is universal and is defined in (2.6), so we obtain a complete sequentially planned test $(N, D)$ which is Bayes-optimal).
In the same way, to obtain optimal truncated sequentially planned tests in the conditional set-up we only need to satisfy \( \Delta_{2.3} \) choosing appropriate Lagrangian multipliers \( \lambda_0 \) and \( \lambda_1 \).

At last, a concluding remark on the form of continuation regions of optimal truncated tests. Using the concavity of functions \( \rho_i, J_m \rho_i \) and other involved, it is not difficult to see that the “continuation region” \( \{ z : N_{i+1}(z) > 0 \} \) (see the first line of \( (3.3) \)) always has a form of an interval \( (a_i, b_i) \) (if not empty). We used this technique in Novikov and Popoca Jiménez [2022] for a related problem, when the group sizes are independent of the observations. In addition, it can be shown that \( a_1 \leq a_2 \leq \cdots \leq a_i \) and \( b_1 \geq b_2 \geq \cdots \geq b_i \), where \( (a_i, b_i) \) is the last non-empty continuation interval.

The optimal non-truncated tests are obtained by passing to the limit as \( K \to \infty \). It is easy to see that \( \rho_K(z) \geq \rho_{K+1}(z) \) for all \( z \geq 0 \) (see \( (3.2) \)). So there exists \( \rho(z) = \lim_{K \to \infty} \rho(z), z \geq 0 \). It is a concave non-decreasing function with \( \rho(0) = 0 \).

Passing to the limit in \( (3.2) \) as \( i \to \infty \), we obtain

\[
\rho(z) = \min \{ g(z), \min_m \{ c(m)(1 + \gamma(z - 1)) + J_m \rho(z) \} \}.
\]

Now, passing to the limit in \( (3.3) \), as \( K \to \infty \) we see that the optimal sampling plan depends on \( z = z_{n_1, \ldots, n_i-1} \) in such a way that for \( i = 2, 3, \ldots \),

\[
N_i = \begin{cases} 
0 & \text{if} \quad g(z) \leq \min_m \{ c(m)(1 + \gamma(z - 1)) + J_m \rho(z) \}, \\
\arg\min_m \{ c(m)(1 + \gamma(z - 1)) + J_m \rho(z) \}, & \text{otherwise},
\end{cases}
\]

(3.4)

and \( N_1 = \arg\min_m \{ c(m) + J_m \rho(1) \} \).

Once again, for all \( i = 2, 3, \ldots \) the continuation region \( \{ N_i(z) > 0 \} = (a, b) \) is an interval with some \( 0 < a < b < \infty \) (if not empty).

The resulting sequentially planned test \( \langle N, D \rangle \) is a particular case of Sequentially Planned Probability Ratio Tests (SPPRTs) [see (3.5) in Schmitz [1993]], more specifically, the one with \( \hat{t}_i(z) = N_i(z) \) defined in \( (3.4) \) for \( i \geq 2, \hat{t}_1 = N_1, \) and \( k_1 = a, k_2 = b \).

In fact, the formal definition of SPPRT in Schmitz [1993] requires that \( k_1 < 1 < k_2 \). Our construction \( (3.4) \) may result in \( 1 \notin (a, b) \), but it is still optimal in the class of sequentially planned tests we consider.

### 3.2. Performance characteristics

In this part, we obtain formulas for calculating error probabilities, average sampling cost and some related probabilities, for truncated sequentially planned probability ratio tests (TSPPRT, see Schmeegn [2009]).

Let \( N \in \mathcal{F}^K \) be such that \( N_i = N_i(z) \) where \( z = z_{n_1, \ldots, n_i-1} \), and \( D_i = D_i(z) \) where \( z = z_{n_1, \ldots, n_i}, i = 1, \ldots, K \). The test \( \langle N, D \rangle \) will be held fixed within this subsection, so we will suppress it in the notation.

Let us denote \( A^K_i = A^K_i(N, D) \) the event meaning “\( H_0 \) is accepted at stage \( i \) or thereafter” (in accordance with the rules of the test \( \langle N, D \rangle \)).

**Proposition 3.1.** Let

\[
d^K_i(z) = I_{\{D_K=0\}}(z), \quad z \geq 0
\]

(3.5)

and, recursively over \( i = K - 1, K - 2, \ldots, 1, \)

\[
d^K_i(z) = I_{\{N_{i+1}=0\}}(z)I_{\{D_i=0\}}(z) + \sum_{m>0} I_{\{N_{i+1}=m\}}(z)Ed^K_{i+1}(zZ_m), \quad z \geq 0.
\]

(3.6)

Then for any \( 1 \leq i \leq K \)

\[
d^K_i(Z_{N_1, \ldots, N_i}) = P(A^K_i | Z_{N_1, \ldots, N_i}),
\]

(3.7)
Proposition 3.2. Let
\[ l^K_i(z) = c(N_K(z)) I_{\{N_K>0\}}(z), \quad z \geq 0 \]
and, recursively over \( i = K - 1, K - 2, \ldots, 2, \)
\[ l^K_i(z) = \sum_{m>0} I_{\{N_i=m\}}(z) c(m) + E l^K_{i+1}(z Z_m), \quad z \geq 0. \]
(3.9)

Then for all \( 2 \leq i \leq K \)
\[ l^K_i(Z_{N_1,\ldots,N_{i-1}}) = E\{ (c(N_i) I_{\{N_i>0\}} + \cdots + c(N_K) I_{\{N_K>0\}}) | Z_{N_1,\ldots,N_{i-1}} \} \]
(3.10)

In particular,
\[ E(c(N_2) I_{\{N_2>0\}} + \cdots + c(N_K) I_{\{N_K>0\}}) = E l^K_2(Z_{N_1}) \]
(3.11)
Remark 3.3. For non-truncated SPPRTs $i$ which proves (3.10) also for $I = ASC$ the average sampling cost $\mathbb{E}(N_{i+1} > 0, N_{i+2} > 0, \ldots, N_{i+n} > 0)$ for any $N$ is $\mathbb{E}(N_{i+1} > 0, N_{i+2} > 0, \ldots, N_{i+n} > 0)$ for all $N \in \mathcal{F}_K$.

Again, arbitrary distribution can be used within Proposition 3.2 (and consequently in (3.12)). In particular, using the distribution under $H_0$ and under $H_1$ when evaluating (3.9), we obtain from (3.12), respectively, the average sampling cost $ASC_0(N)$ and $ASC_1(N)$ and thus the value of

$$ASN_{\gamma}(N) = (1 - \gamma)ASC_0(N) + \gamma ASC_1(N)$$

for any $N \in \mathcal{F}_K$.

Proof of Proposition 3.2

By definition,

$$l^K_i(Z_{N_1, \ldots, N_{K-1}}) = c(N_K)I_{\{N_K > 0\}} = \mathbb{E}\{c(N_K)I_{\{N_K > 0\}}|Z_{N_1, \ldots, N_{K-1}}\}.$$ 

Let us suppose now that (3.10) is satisfied for some $2 \leq i \leq K$. Then, by definition,

$$l^K_i(z) = \sum_{m>0} I_{\{N_{i-1} = m\}}(z)(c(m) + \mathbb{E}\{l^K_{i-1}(z|Z_m)\})$$

and

$$l^K_{i-1}(Z_{N_1, \ldots, N_{i-2}}) = I_{\{N_{i-1} > 0\}}(c(N_{i-1}) + \mathbb{E}\{l^K_{i-1}(Z_{N_1, \ldots, N_{i-2}}|Z_{N_1, \ldots, N_{i-2}})\})$$

$$= I_{\{N_{i-1} > 0\}}(c(N_{i-1}) + \mathbb{E}\{c(N_i)I_{\{N_i > 0\}} + \cdots + c(N_k)I_{\{N_k > 0\}}|Z_{N_1, \ldots, N_{i-2}}\}|Z_{N_1, \ldots, N_{i-2}})$$

$$= \mathbb{E}\{I_{\{N_{i-1} > 0\}}c(N_{i-1}) + c(N_i)I_{\{N_i > 0\}}|Z_{N_1, \ldots, N_{i-2}}\} + \cdots + c(N_k)I_{\{N_k > 0\}}|Z_{N_1, \ldots, N_{i-2}}\},$$

which proves (3.10) also for $i - 1$. $\Box$

Remark 3.3. For non-truncated SPPRTs $(N, D)$ we can also make use of Propositions 3.1 and 3.2 by truncating the sampling plan $N$.

Let us define, for any $N \in \mathcal{F}$, $N^K$ as the sampling plan $N$ with redefined $N_{K+1} \equiv 0$ (keeping intact all other components of $N$).

Using the fact that $ASC_j(N^K) \rightarrow ASC_j(N)$, as $K \rightarrow \infty$, for $j = 0, 1$, we obtain from (3.13) a numerical approximation for $ASC_j(N)$.

Also $\alpha(N, D) = \lim_{K \rightarrow \infty}(1 - P_0(A^K_1))$ and $\beta(N, D) = \lim_{K \rightarrow \infty}P_1(A^K_1)$, so we obtain from Proposition 3.4 a numerical approximation for the error probabilities $\alpha(N, D)$ and $\beta(N, D)$.

Remark 3.4. Proposition 3.2 can also be used for calculating other sampling characteristics of the test.

For example, taking in Proposition 3.2 $c(m) = 1$, for all $m$, one obtains, in place of the ASC, the average number of groups taken, $ET$.

Employing $c(m) = m$, for all $m$, provides the average number of observations taken, $E(N_1 + N_2 + \cdots + N_T)$.

4. Numerical algorithms

In this section we propose numerical algorithms for optimal design of sequentially planned tests and their performance evaluation.
4.1. Optimal design and performance evaluation

We propose a numerical method based on the optimal sampling plan described in (3.3).

The sampling plan is entirely based on the sequence of functions $\rho_0, \rho_1, \ldots, \rho_{K-1}$ defined in (3.1) and (3.2) for $z \geq 0$. The idea of the method is a numerical approximation of every $\rho_i$ on the continuation interval by a piecewise-linear function based on a grid of $z$-values. So instead of functions $\tilde{\rho}_i$ we will work with functions $\cdot \tilde{\rho}_i$ defined as follows: let $\tilde{\rho}_0 = g$, and define recursively for $i = 1, 2, \ldots$

$$\tilde{\rho}_i(z) = \min \{g(z), \min_m \{c(m)(1 + \gamma(z - 1)) + J_m \tilde{\rho}_{i-1}(z)\}\},$$

where $\tilde{\rho}_i$ is calculated by interpolation between the grid points on the continuation interval.

Formally, the proposed algorithm is as follows (applicable for $K \geq 2$)

Step 1. Start from $n = 1$

Step 2. Find a minimum ($a_n$) and a maximum ($b_n$) value of $z$ for which

$$g(z) > \min_m \{c(m)(1 + \gamma(z - 1)) + J_m \tilde{\rho}_{n-1}(z)\}$$

(4.1)

If no such $z$ exist, declare Emergency Exit and Stop.

Step 3. For a grid $\{z_i\}$ of values on $[a_n, b_n]$ calculate and store the respective values $\{v_i\}$ of the function on the right in (4.1). Take note, for future use, that $\tilde{\rho}_n(z)$ will be calculated as $\tilde{\rho}(z)$ for $z \notin [a_n, b_n]$ and using an interpolation between the respective grid points for $z \in [a_n, b_n]$

Step 4. Set $n = n + 1$. If $n = K$ then Stop, else go to Step 2.

Emergency Exit means that only one-stage sampling plans come into question in the hypothesis testing problem with given input parameters (for example, when the cost of data is too high).

After a normal exit, we obtain a way to calculate the functions $\tilde{\rho}_1, \ldots, \tilde{\rho}_{K-1}$ and to apply them in definitions of optimal sampling plans in (3.3) (where $i = 2, 3, \ldots, K$ and $z = z_n, \ldots, z_{n-1}$):

$$\tilde{N}_i = \begin{cases} 0 & \text{if } z \notin [a_{K-i+1}, b_{K-i+1}], \\ \arg\min_m \{c(m)(1 + \gamma(z - 1)) + J_m \tilde{\rho}_{K-i}(z)\}, & \text{otherwise,} \end{cases}$$

and

$$\tilde{N}_1 = \arg\min_m \{c(m) + J_m \tilde{\rho}_{K-1}(1)\}, \tilde{N}_{K+1} \equiv 0$$

Respectively, we can use Proposition 3.1 for approximate evaluation of error probabilities by substituting $\tilde{N}_i$ for $N_i$ in (3.6) for $i = K - 1, \ldots, 1$:

$$\tilde{d}\tilde{a}_i^K(z) = I_{\{\tilde{N}_{i+1}=0\}}(z) I_{\{D_i=0\}}(z) + \sum_{m>0} I_{\{\tilde{N}_{i+1}=m\}}(z) E_1 d_m^K(z Z_m), \ z \geq 0,$$

(4.3)

with $\tilde{d}_K^K \equiv d_K^K$, and finally from (3.8)

$$\beta(\tilde{N}, D) = E_1 d_1^K(Z_{\tilde{N}_1})$$

(4.4)

as an approximate value of $\beta(N, D)$.

Analogously, using $E_0$ instead of $E_1$ in (4.3) for $i = K - 1, \ldots, 1$ we get an approximation for $\alpha(N, D)$ in the form of

$$\alpha(\tilde{N}, D) = 1 - E_0 \tilde{d}_K^K(Z_{\tilde{N}_1}).$$

In the same way, substituting $\tilde{N}_i$ for $N_i$ (and $\tilde{t}_i^K$ for $t_i^K$) in Proposition 3.2 we obtain from (3.12) an approximation for the average sampling number:

$$E(c(\tilde{N}_1) I_{\{\tilde{N}_1>0\}} + \cdots + c(\tilde{N}_K) I_{\{\tilde{N}_1>0, \tilde{N}_2>0, \ldots, \tilde{N}_K>0\}}) = c(\tilde{N}_1) + E_2^K(Z_{\tilde{N}_1}),$$

(4.5)

whatever the distribution of the i.i.d. observations is used for calculations in Proposition 3.2.
4.2. Example

We implemented the algorithms above for the problem of testing a simple hypotheses $H_0 : \theta = \theta_0$ vs. a simple alternative $H_0 : \theta = \theta_1$ about the success probability $\theta$ of a Bernoulli distribution. The program code in R programming language [R Core Team, 2013] can be downloaded from a public GitHub repository at https://github.com/HOBuKOB-MEX/SPPRT.

We present here the results of numerical evaluations for the particular case of testing $\theta_0 = 0.52$ vs. $\theta_1 = 0.48$ seen in Schmegner and Baron [2007]. We use the same cost function $c(m) = c_0 + cm$ they used for this example, where $c = 10$ (cost per observation) and $c_0 = 1000$ (cost per group). Various sampling plans were evaluated by Schmegner and Baron [2007], all of them based on a continuation interval which guarantees that the error probabilities of the first and second kind are at most $\alpha = \beta = 0.05$. The best value of ASC found was 18254, with an average number of groups of $ET = 9.3$ and an average total number of observations of $EM = 892$ [see Schmegner and Baron, 2007, Example 4.6].

For our implementation of the algorithms of Subsection 4.1, we used, in each continuation interval, a uniform grid formed by equidistant points on the logarithmic scale of $z$ with step $h = 0.1$. We used our truncated SPPRT (4.2) with $K = 15$ (maximum number of groups to be observed). Also we employed as the set $G$ of possible group sizes the set $G = \{10i + 10\}_{i=0,\ldots,59}$. We used $ASC_\gamma$ with $\gamma = 0.5$ as a criterion of minimisation. The Lagrange multipliers $\lambda_0 = \lambda_1 = 44$ were chosen in such a way that $\alpha(\tilde{N}, D) = 0.05$ and $\beta(\tilde{N}, D) = 0.05$.

Evaluating the characteristics of the proposed SPPRT according to (4.4) - (4.5) we obtained $ASC_0 = ASC_1 = 11510$, with the average number of groups $ET = 2.07$ and the average number of observations $EM = 944$. Thus, the proposed method provides nearly 1.6 times lower sampling cost, in comparison with the best plan found in Schmegner and Baron [2007]. Taking into account that there are a number of ways to improve the numerical characteristics, namely, by 1) choosing a higher truncation level $K$, 2) making the grid size $h$ smaller, 3) making the set $G$ of possible group sizes “denser”, 4) adjusting the criterion of minimization by varying $\gamma$ as required by the practical context,- the real efficiency of the proposed method can turn even higher.

In Figure 1, the set of continuation intervals at each of the fourteen sequentially planned steps is presented. We know that theoretically the continuation interval gets closer to the one corresponding to the optimal non-truncated SPPRT, as $K \to \infty$. It appears that in this example the convergence is so fast that the interval reaches its limit after as few as some 4 steps (remember that the first interval found comes last).

In Figure 2, one can see the “nearly optimal” sampling plan $\tilde{N}_2(z)$ calculated according to (4.2). Again, because $N_K(z)$ converges to the optimal sampling plan $N(z)$ as $K \to \infty$, we may expect that this is approximately the sampling plan the optimal non-truncated SPPRT will use in each step.

Taking into account that the gain from using our method, with respect to other known methods, is comparable to the gain the classical SPRT provides with respect to one-sample (fixed sample size, FSS) test, we would like to examine the efficiency of our method with respect to the one-sample test, in various scenarios. As a reference, we want to use, for given $\theta_0$ and $\theta_1$, the average sampling cost of the one-step test with a minimum sample size $n(\alpha, \beta)$ that provides error probabilities not exceeding $\alpha$ and $\beta$, respectively. According to the definition of average sampling cost, the one-sample test has an ASC equal to $ASC_{FSS} = c_0 + cn(\alpha, \beta)$, which will be compared with ASC of the SPPRT we proposed.

It is interesting to note that for $\alpha = \beta = 0.05$ and $\theta_0 = 0.52$ and $\theta_1 = 0.48$ used in this example the FSS $n(\alpha, \beta)$ is equal to 1691, giving the average sampling cost $ASC_{FSS} = 17910$ for the one-sample plan, which outperforms all the SPPRTs examined in Schmegner and Baron [2007].

To compare the performance of our “nearly optimal” SPPRT with that of the one-sample test we ran our program for a series of $\lambda_0$ and $\lambda_1$ between 3 and 6.3 on the scale of natural logarithms, with a total of $9 \times 9$ points. For each one, we calculated the corresponding $\alpha$, $\beta$ and $ASC_0$ and $ASC_1$. The relative efficiency was calculated as $R_j = ASC_{FSS}/ASC_j$ under hypothesis $H_j$, $j = 0, 1$. 

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To get a compact visual representation of the results, we fit a local polynomial regression model (LOESS)\footnote{https://www.rdocumentation.org/packages/stats/versions/3.6.2/topics/loess} to the data obtained, to represent the relationship between the relative efficiency and $\alpha$ and $\beta$. We use decimal logarithms of $\alpha$ and $\beta$ as independent variables, and $R_j$ as response. The result of the model fitting for $R_0$ is shown in Figure 3. The graph of $R_1$ is perfectly symmetrical with respect to the diagonal $\alpha = \beta$ and is not shown.

We see from Figure 3 that the maximum of relative efficiency $R_0$ is attained in the asymmetric case when $\alpha$ is small and $\beta$ is relatively large, and is about 2.5. In the vicinity of the diagonal the maximum efficiency of approx. 2.1 is reached for small $\alpha \approx \beta$ with a clear tendency of increasing as $\alpha, \beta \to 0$. The minimum efficiency is about 1.3 and is attained whenever $\alpha$ is relatively large.

5. Conclusions

In this paper, we proposed a method of construction of optimal sequentially planned tests. In particular, for i.i.d. observations we obtained the form of optimal sequentially planned tests and formulas for computing their numerical characteristics like error probabilities, average sampling cost, average number of observations and the average number of groups.

A method of numerical evaluation of the performance characteristics is proposed and computer algorithms of their implementation are developed.

For a particular case of sampling from a Bernoulli population, the proposed method is implemented in
The proposed method is compared numerically with other known sampling plans. A numerical comparison of the proposed tests with one-sample tests having the same error probabilities has been carried out. The relative efficiency based on the average sampling cost compared to one-sample tests exhibits largely the same behaviour as that of the classical SPRT does, where the efficiency is based on comparison of the average sample number with the FSS.

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Figure 3. Relative efficiency $R_0$ under $H_0$

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