A Discipline of Programming with Quantities

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Abstract

In scientific and engineering applications, physical quantities embodied as units of measurement (UoM) are frequently used. The loss of the Mars climate orbiter, attributed to a confusion between the metric and imperial unit systems, popularised the disastrous consequences of incorrectly handling measurement values. Dimensional analysis can be used to ensure expressions containing annotated values are evaluated correctly. This has led to the development of a large number of libraries, languages and validators to ensure developers can specify and verify UoM information in their designs and codes. Many tools can also automatically convert values between commensurable UoM, such as yards and metres. However these systems do not differentiate between quantities and dimensions. For instance torque and work, which share the same UoM, can not be interchanged because they do not represent the same entity. We present a named quantity layer that complements dimensional analysis by ensuring that values of different quantities are safely managed. Our technique is a mixture of analysis and discipline, where expressions involving multiplications are relegated to functions, in order to ensure that named quantities are handled soundly.

Keywords: quantities, units of measurement, quantity checking, dimensional analysis

1 Introduction

Humans have used local units of measurement since the days of early trade, enhanced over time to fulfil the accuracy and interoperable needs of science and technology. The technical definition of a physical quantity is a “property
Dimensions are physical quantities that can be measured, while units are arbitrary labels that correspond to a given dimension to make it relative. For example a dimension is length, whereas a metre is a relative unit that describes length. Units of measure can be defined in the most generic form as either base quantities or derived quantities. The base quantities are the basic building blocks, and the derived quantities are built from these. The base quantities and derived quantities together form a way of describing any part of the physical world [4]. For example length (metre) is a base quantity, and so is time (second). If these two base quantities are combined they express velocity (metre/second or metre × second\(^{-1}\)) which is a derived quantity. The International System of Units (SI) defines seven base quantities (length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity) as well as a corresponding unit for each quantity [5]. Some popular examples of both base and derived units are shown in Table 1. It is common for quantities to be declared as a number (the magnitude of the quantity) with an associated unit [6].

There are many ways in which software processes and development can accommodate units of measurement in this manner [7]. Adding units to conventional programming languages goes back to the 1970s [8] and early 80s with...
proposals to extend Fortran [9] and then Pascal [10]. Hilfinger [11] showed how to exploit Ada’s abstraction facilities, namely operator overloading and type parameterisation, to assign attributes for UoM to variables and values. The emergence of object oriented programming languages enabled developers to implement UoM either through a class hierarchy of units and their derived forms, or through the Quantity pattern [12]. There are a large number of libraries for all popular object oriented programming languages [13] that support this approach [14].

Applying UoM annotations requires an advanced checker to ensure variables and method calls are handled soundly. Maxwell introduced the notion of a system of quantities with a corresponding system of units. This approach allows scientists working with different measurement systems to communicate more easily [15]. Two units are compatible if they both can be represented as the same derived quantity. For instance degrees Celsius is compatible with Fahrenheit. Values in Celsius can be converted to values in Fahrenheit, and vice versa. This notion of interoperability allows equations to be stated in terms of their dimensions and not their base values. These can be computed when the equation is evaluated. Two values can be added or subtracted only if their units are the same. Multiplication and division either add or subtract the two units product of power representations, assuming both values are compatible. Although, converting values to ensure compatibility can create round-off errors. Once a variable has been defined to be of a given unit, then it should remain as such. Checking that all annotated entities behave according to these rules ensures both completeness and correctness of the program, and can be undertaken before the code is run.

However two values that share the same UoM might not represent the same kinds of quantities (KOQ) [16]. For example, torque is a rotational force which causes an object to rotate about an axis while work is the result of a force acting over some distance. Surface tension can be described as newtons per meter or kilogram per second squared, and even though they equate, they represent different quantities. Our focus is to present a simple set of rules for arithmetic and function calls that allow quantities to be named and handled safely. This is not as straightforward as preserving the names of quantities throughout the programme text. Multiplication will generate a new quantity so it is very likely that information is lost in intermediate stages of a calculation. We propose that functions, whose return KOQ are known, are used to regain information when calculations use multiplication. In this manner a discipline of programming with quantities is suggested, one in which equations involving multiplication are defined as functions to ensure quantities in the main block of code are always known.

This paper extends [17] to include a notion of safe KOQ arithmetic and a demonstration of how this discipline deals with information loss, along with a streamlined functional presentation of the checking algorithm. In Section 2 we describe how UoM are typically implemented and argue for a more comprehensive representation, while acknowledging the drawbacks of adding complexity
to the development process. In Section 3 we introduce unit expressions and dimensional analysis. In Section 4 we describe a simple algebra of named quantities, and show how they can be maintained while evaluating unit assignments and function calls. We also discuss some of the obstacles to implementing named quantities and UoM in general. Finally, in Section 5 we summarise quantity validation and describe avenues of current research.

2 Background

One can assert the physical dimension of length with the unit metre and the magnitude 10 (10m). However, the same length can also be expressed using other units such as centimetres or kilometres, at the same time changing the magnitude (1000cm or 0.01km). Although these examples are all based on the International System of Units (SI) there exists several other systems, such as the Imperial system where yards and miles would be used. On this basis a very simple object oriented design would entail a superclass for each dimension, such as Length, and then specific subclasses for the various units, each of which would contain overloaded operators to ensure unit based arithmetic could be performed correctly.

Length 11 = new LengthMetre (5.0);
Length 12 = new LengthYard (4.0);
Length 13 = 11.addlength (12);

The addlength command would convert 12 into metres and perform the addition. We could extend our object oriented design to create a class hierarchy for each base type and use a tree structure to construct derived types. However this would result in hundreds of units and thousands of conversions.

Fortunately, a normal form exists which makes storage and comparison a lot easier. Any system of units can be derived from the base units as a product of powers of those base units: \( \text{base}^{e_1} \times \text{base}^{e_2} \times \ldots \times \text{base}^{e_n} \), where the exponents \( e_1, \ldots, e_n \) are rational numbers. Thus an SI unit can be represented as a 7-tuple \( \langle e_1, \ldots, e_7 \rangle \) where \( e_i \) denotes the \( i \)-th base unit; or in our case \( e_1 \) denotes length, \( e_2 \) mass, \( e_3 \) time and so on.

Performing calculations in relation to quantities, dimensions and units is subtle and can easily lead to mistakes. A dimensional analysis needs to ensure that (1) two physical quantities can only be equated if they have the same dimensions; (2) two physical quantities can only be added if they have the same dimensions (known as the Principle of Dimensional Homogeneity); (3) the dimensions of the multiplication of two quantities is given by the addition of the dimensions of the two quantities. If we only consider the three common dimensions of length, mass and time then we can capture the rules for addition and multiplication.

\[
(l_1, m_1, t_1) \hat{+} (l_2, m_2, t_2) = (l_1, m_1, t_1), \text{if } l_1 = l_2 \land m_1 = m_2 \land t_1 = t_2 \\
(l_1, m_1, t_1) \hat{\times} (l_2, m_2, t_2) = (l_1 + l_2, m_1 + m_2, t_1 + t_2)
\]
The full 7-tuple can be reflected in a typical programming language as an array of integers. Within an object oriented class structure the array can be coupled with conversion, equality and numeric operators to form a Unit abstract data type which ensures only UoM correct arithmetic is undertaken. In Java this would be represented as:

```java
class Unit {
    private int [7] dimension;
    private float [7] conversionFactor;
    private int [7] offset;
    ...
    boolean isCompatibleWith (Unit u);
    boolean equals (Unit u);
    Unit multiplyUnits (Unit u);
    Unit divideUnits (Unit u);
}
```

This is the basis of the Quantity pattern [12] in which quantity values are represented as a pair: the numerical value, \( \{Q\} \), and the unit of measure, \([Q]\), such that \( Q = \{Q\} \cdot [Q] \).

```java
class Quantity {
    private float value;
    private Unit unit;
    ....
}
```

The Quantity pattern provides a means of annotating variable declarations and method signatures with behavioural UoM specifications. Most libraries for modern programming languages implement this approach but, as was found in the survey of [18], do not satisfy the core requirements of the scientific programming community. Interview subjects felt that UoM libraries were inconvenient: they did not interact well with the eco-system, incurred additional and often cumbersome syntax, had unwanted performance costs as the checking was undertaken at run-time, required effort to learn and costly rewrites to support.

A value is almost always represented as a float in the Quantity pattern so UoM checking will only apply to that number representation. Other number representations will need to be converted and rounding errors absorbed. Similarly the exponents are typically represented as integers, but there are rare instances when fractional exponents might be required for intermediate results even when there are no SI units that requires them. This inflexibility coupled with an unwieldy syntax and the need for manual conversions are reasons for the poor adoption of such libraries.

Increasing uptake for quantity aware code requires a language neutral interface that allows programmers to manage UoM in an indistinguishable language agnostic fashion, either as part of the core language (e.g. Swift [19] and F# [20]) or through the use of a separate validator [21–24]). Moreover, language based solutions enable UoM checking to be undertaken at compile-time, detecting
errors early while ensuring no run-time overheads are required. What these systems fail to provide is a system that checks for kinds of quantities and ensures entities that have the same UoM but different KOQ are managed safely.

3 Unit Expressions

Performing calculations in relation to quantities, dimensions and units is often complex and can easily lead to mistakes. We shall begin by defining dimensional analysis for a hypothetical programming language extended with unit variable declarations, udecs, and only consider the three common dimensions, dims. Hence velocity, namely length × time⁻¹, is represented as (1, 0, −1). We use the standard float implementation to approximate for real numbers but as we are not performing arithmetic any representation would suffice. A program consists of a sequence of declarations followed by a sequence of statements.

\[
\begin{align*}
\text{prog} &::= \text{begin} \ udecs \ \text{in} \ \text{ustmts} \ \text{end} \\
udecs &::= \ udec_1 ; \ldots ; \ udec_m \\
dims &::= (\text{int} , \text{int} , \text{int}) \\
udec &::= w : \text{float of} \ dims
\end{align*}
\]

It has the standard statement constructs, ustmt, and boolean expressions, bexp, but we will only focus on quantity variable assignments and conditionals as these affect unit variables, uv. Further constructs such as while loops would create a more complete programming language but not add to the presentation. Unit arithmetic expressions, uexp, impose syntactic restrictions so that their soundness can be inferred using the algebra of quantities. By creating a separate syntax for unit expressions we can distinguish between scalar values and unitless quantities, namely values that have the dimensions (0, 0, 0) such as moisture content.

\[
\begin{align*}
\text{ustmts} &::= \ ustmt_1 ; \ldots ; \ ustmt_m \\
\text{ustmt} &::= wv := \text{uexp} \\
&\quad | \ \text{if} \ bexp \ \text{then} \ \text{ustmts}_1 \ \text{else} \ \text{ustmts}_2 \\
\text{uexp} &::= wv \ | \ \text{uexp}_1 + \text{uexp}_2 \ | \ r \times \text{uexp} \ | \ \text{uexp}_1 \times \text{uexp}_2
\end{align*}
\]

In Figure 1 we present the dimensional analysis rules for programs. The rules for declarations, DS, build an environment, \( \varrho \), mapping variables to their dimensions. The environment will not change throughout the lifetime of the block. Thus, once a variable has been defined to be of a given quantity, then it will remain as such. Many library based systems allow programmers to change the dimensions of unit variables as they are objects of type Quantity, namely a mutable array. Once \( \varrho \) has been built, it will be used to perform dimensional analysis on the statements. The rules for statements, SM, return either DimValid or DimFail depending on whether a given statement uses quantities correctly or not. An assignment statement is valid only if the quantity of the unit expression is dimensionally homogeneous with the unit variable that
\[ P \begin{align*} \text{begin } & udecs \text{ in } \text{ustmts} \text{ end} = \mathcal{SM}[\text{ustmts}]_{(\mathcal{DS}[\text{udecs}]}) \end{align*} \]

\[ \mathcal{DS}[udec_1; \ldots; udec_m] = \text{let } \varrho_1 = \mathcal{D}[udec_1] \{ \} \\
& \vdots \\
\mathcal{D}[uv: \text{float of } d]_\varrho = \varrho \oplus \{ uv \mapsto d \} \]

\[ \mathcal{SM}[\text{ustmt}_1; \ldots; \text{ustmt}_n]_\varrho = \begin{cases} \text{DimValid}, & \text{if } \mathcal{S}[\text{ustmt}_1]_\varrho = \text{DimValid} \land \ldots \land \mathcal{S}[\text{ustmt}_n]_\varrho = \text{DimValid} \\ \text{DimFail}, & \text{otherwise} \end{cases} \]

\[ S[uv := uexp]_\varrho = \begin{cases} \text{DimValid}, & \text{if } \mathcal{UE}[uexp]_\varrho = (\varrho \ uv) \\ \text{DimFail}, & \text{otherwise} \end{cases} \]

\[ S[\text{if } bexp \text{ then } \text{ustmts}_1 \text{ else } \text{ustmts}_2]_\varrho = \begin{cases} \text{DimValid}, & \text{if } \mathcal{SM}[\text{ustmts}_1]_\varrho = \text{DimValid} \land \mathcal{SM}[\text{ustmts}_2]_\varrho = \text{DimValid} \\ \text{DimFail}, & \text{otherwise} \end{cases} \]

\[ \mathcal{UE} : uexp \rightarrow (w \rightarrow \text{dims}) \rightarrow \text{dims} \]

\[ \mathcal{UE}[uv]_\varrho = \varrho \ uv \]

\[ \mathcal{UE}[uexp_1 + uexp_2]_\varrho = \mathcal{UE}[uexp_1]_\varrho \hat{+} \mathcal{UE}[uexp_2]_\varrho \]

\[ \mathcal{UE}[r \ast uexp]_\varrho = \mathcal{UE}[uexp]_\varrho \hat{\times} \]

\[ \mathcal{UE}[uexp_1 \ast uexp_2]_\varrho = \mathcal{UE}[uexp_1]_\varrho \hat{\times} \mathcal{UE}[uexp_2]_\varrho \]

Fig. 1 Dimensional Analysis rules for declarations, statements and expressions.

it is being assigned to. The rule for conditionals checks the dimensional validity of both true and false statements. The rules for unit expressions, \( \mathcal{UE} \), are partial and might not have a solution in the case of trying to add quantities that have different dimensions. The rule for unit variables is just a lookup on the quantity environment \( \varrho \). The rule for addition ensures that both the left hand and right hand side subexpressions have the same quantities as enforced by the operator \( \hat{+} \). The rules for multiplication allow constants to be applied, and multiplying two unit expressions will create a combined quantity, where each dimension is summed as defined by the operator \( \hat{\times} \).

Dimensional analysis would be sufficient if only one unit system, such as the SI system, was required. In such cases the base units of metre, kilogram and second could be implicit in implementations. Dimensionally correct unit expressions can be evaluated in much the same way as normal arithmetic expressions. As this is rarely the case in scientific applications where a myriad of unit systems and magnitudes are used, we need to perform unit conversions before evaluating the arithmetic expression. This can be undertaken at compile-time [25] or at run-time. Moreover we need to declare the units alongside their dimensions. A variable denoting torque would be stored as \( \{t \mapsto ((\text{Metre}, 2), (\text{Kilogram}, 1), (\text{Second}, -2))\} \) in the environment.
4 Quantity Rules

This section introduces named quantities, their rules and how they are supported in our typical programming language. We include functions that provide a clear interface and the potential for more comprehensive checking.

4.1 Expressions

We adopt a similar approach to [26, 27] in that quantities should be represented as a 3-tuple, and not as a 2-tuple mentioned previously. Consequently we add a quantity name, \( \langle Q \rangle \), to the numerical value, \( \{Q\} \), and the unit of measure, \( [Q] \), such that \( Q = \langle Q \rangle \cdot \{Q\} \cdot [Q] \).

However, not all quantity variables in a programme will have a name such as Torque or Work. Some might denote an entity such as length that could be in metres or yards, while another might be a variable used to store some temporary value. Neither of these need to be named. Using an algebraic data type, we define named quantities as:

\[
\text{type quantname} = \text{Named of string} | \text{Nameless}
\]

We are now in a position to define the rules for adding and multiplying named quantities. In both cases we assume that the unit expression is dimensionally correct, our concern is to define how named quantities conduct themselves. The operator \( \diamond \) takes two named quantities and states the conditions under which they can be summed: two named quantities can be added together only if they represent the same entity, if one quantity is named but the other is not then it is necessary for the result to be named, and if both are unnamed then the result will be too:

\[
\begin{align*}
\text{Named } n_1 \diamond \text{Named } n_2 &= \text{Named } n_1, \text{ if } n_1 = n_2 \\
\text{Named } n \diamond \text{Nameless} &= \text{Named } n \\
\text{Nameless} \diamond \text{Named } n &= \text{Named } n \\
\text{Nameless} \diamond \text{Nameless} &= \text{Nameless}
\end{align*}
\]

Our comparison rules cast upwards from Nameless to Named, so as to assume a named quantity whenever possible. This is required to ensure named quantities behave correctly. If we cast downwards then we would have the alternative rule \( \text{Named } n \diamond \text{Nameless} = \text{Nameless} \), that would allow Work to be added to Torque through associativity: (\( \text{Named } "\text{Work}" \diamond (\text{Named } "\text{Torque}" \diamond \text{Nameless}) \)) \( \Rightarrow \) (\( \text{Named } "\text{Work}" \diamond \text{Nameless} \)) \( \Rightarrow \) Nameless.

For multiplication the rules are simpler. The operator \( \triangle \) takes in two named quantities and defines how they behave over the multiplication operator. As multiplication sums the dimensions of the two operands, the value will
be different to either and so the result will always be Noname.

\[
\begin{align*}
\text{Named } n_1 \triangle \text{ Named } n_2 &= \text{Noname} \\
\text{Named } n \triangle \text{ Noname} &= \text{Noname} \\
\text{Noname} \triangle \text{ Named } n &= \text{Noname} \\
\text{Noname} \triangle \text{ Noname} &= \text{Noname}
\end{align*}
\]

Fig. 2 Named quantity rules for unit expression.

The named quantity algebra can be incorporated into our language as shown in Figure 2. The language rules for scaler multiplication do not change the named quantity, the scaler value only affects the quantity value when evaluating expressions. Consider the example where \( \tau = \{ t \mapsto \text{Named } "\text{Torque}" , w \mapsto \text{Named } "\text{Work}" \} \). If we were to try to validate \( t + w \) with \( \mathcal{NE} \) then the rule for addition will not succeed as there is no case for \( (\text{Named } "\text{Torque}" \diamond \text{Named } "\text{Work}" ) \).

**Theorem 1.** For a given unit expression, \( uexp \), which does not include general multiplication, and an environment, \( \tau \), binding unit variables to named quantities; \( \mathcal{NE}[uexp]_\tau \) will only succeed if all named subexpressions represent the same entity.

*Proof* by induction on unit expressions. The cases for unit variables, \( uv \), and scaler multiplication, \( r \ast uexp \), are straightforward. The important cases revolve around addition as we need to consider the potential associative effect of nested subexpressions. Consider \( uexp_1, uexp_2 \) and \( uexp_3 \) where:

\[
\mathcal{NE}[uexp_1 + uexp_2 + uexp_3]_\tau = (\mathcal{NE}[uexp_1]_\tau \diamond \mathcal{NE}[uexp_2]_\tau) \diamond \mathcal{NE}[uexp_3]_\tau
\]

This will result in the following indicative 5 cases:

**Case 1:** \( (\text{Named } n_1 \diamond \text{Named } n_2) \diamond \text{Named } n_3 \)

\[ \Rightarrow \text{Named } n_1 \diamond \text{Named } n_3, \text{ where } n_1 = n_2 \text{ by definition of } \diamond \]

\[ \Rightarrow \text{Named } n_1, \text{ where } n_1 = n_2 = n_3 \text{ by definition of } \diamond \]

**Case 2:** \( (\text{Named } n_1 \diamond \text{Named } n_2) \diamond \text{Noname} \)

\[ \Rightarrow \text{Named } n_1 \diamond \text{Noname}, \text{ where } n_1 = n_2 \text{ by definition of } \diamond \]

\[ \Rightarrow \text{Named } n_1, \text{ where } n_1 = n_2 \text{ by definition of } \diamond \]
Case 3: $(\text{Named } n_1 \bowtie \text{Name}) \bowtie \text{Named } n_3$
\[ \Rightarrow \text{Named } n_1 \bowtie \text{Named } n_3, \text{ by definition of } \bowtie \]
\[ \Rightarrow \text{Named } n_1, \text{ where } n_1 = n_3 \text{ by definition of } \bowtie \]

Case 4: $(\text{Name}) \bowtie (\text{Name}) \bowtie \text{Named } n_3$
\[ \Rightarrow \text{Name} \bowtie \text{Named } n_3, \text{ by definition of } \bowtie \]
\[ \Rightarrow \text{Named } n_3, \text{ by definition of } \bowtie \]

Case 5: $(\text{Name}) \bowtie (\text{Name}) \bowtie \text{Name}$
\[ \Rightarrow \text{Name} \bowtie \text{Name}, \text{ by definition of } \bowtie \]
\[ \Rightarrow \text{Name}, \text{ by definition of } \bowtie \]

which ensure that addition maintains the property that named subexpressions must represent the same entity for evaluation to succeed. We will see how to ensure multiplication can be made safe in Section 4.3.

Assignment statements have to satisfy the named quantity of the variable being assigned to, specifically the left hand side, and can therefore either succeed or fail. The $\triangleleft$ rules specify that one can assign a named quantity to a variable that has the same named quantity but not otherwise. One can assign a Noname value to a named quantity variable as it will have the same dimensions. However, one cannot allow Noname $\triangleleft$ Named $n$ to Succeed as this would allow one to assign a Torque value to a Work variable through the intermediary of a local unnamed variable. The solution is to create new variable bindings and update $\tau$ accordingly.

\[
\text{type assignstate} = \text{Succeed of } (uv \to \text{quantname}) \mid \text{Fail}
\]

Where $uv$ denotes the variable being assigned to, the rules for $\triangleleft^T_{uv}$ will override the existing binding for $uv$ in the case where we try to assign a named entity to an unnamed variable:

\[
\begin{align*}
\text{Named } n_1 & \triangleleft^T_{uv} \text{Named } n_2 = \text{Succeed } \tau, & \text{if } n_1 = n_2 \\
\text{Named } n_1 & \triangleleft^T_{uv} \text{Named } n_2 = \text{Fail}, & \text{if } n_1 \neq n_2 \\
\text{Named } n & \triangleleft^T_{uv} \text{Noname} = \text{Succeed } \tau \\
\text{Noname} & \triangleleft^T_{uv} \text{Noname} = \text{Succeed } \tau \\
\text{Noname} & \triangleleft^T_{uv} \text{Named } n = \text{Succeed } \tau \oplus \{ uv \mapsto \text{Named } n \}
\end{align*}
\]

This has a distinct effect on how we define our programming language rules to support the named quantity algebra. One must update the environment $\tau$ to reflect that the named quantity assignment has taken place, as shown in both $\mathcal{NSM}$ and $\mathcal{NS}$ rules of Figure 3. For instance, the program:

\[
\begin{align*}
\text{begin} \\
& \text{t1 : float of Noname;} \\
& \text{t2 : float of Named T} \\
& \ldots \\
& \text{t1 := t2}
\end{align*}
\]
will update $\tau$ so that $t_1$ also has the kind Named "T". This ensures that the bindings of unnamed values will reflect their usage and protect the code from erroneous assignments. This is unlike the rules for dimensional analysis where the environment mapping variables to their dimensions does not change over their lifetime. In order to guarantee coherence of potential changes to bindings in $\tau$, the environment is threaded through the rule for statements and conditionals. If an initial Noname unit variable is assigned a Name then any subsequent attempt to redefine it will Fail.

$$\begin{align*}
\text{NSM}[\text{ustmt}_1; \ldots; \text{ustmt}_m]_{\tau} &= \text{Succeed } \tau_m, \text{ if } \text{NSM}[\text{ustmt}_1]_{\tau} = \text{Succeed } \tau_1 \land \cdots \land \text{NSM}[\text{ustmt}_{m-1}]_{\tau_{m-1}} = \text{Succeed } \tau_m \\
&= \text{Fail, otherwise}
\end{align*}$$

$$\begin{align*}
\text{NS}[uv := uexp]_{\tau} &= (\tau uv) \triangleright_{uv} (\text{NE}[uexp]_{\tau}) \\
\text{NS}[\text{if } bexp \text{ then } \text{ustmts}_1 \text{ else } \text{ustmts}_2]_{\tau} &= \text{Succeed } \tau_2, \text{ if } \text{NSM}[\text{ustmts}_1]_{\tau} = \text{Succeed } \tau_1 \land \text{NSM}[\text{ustmts}_2]_{\tau_1} = \text{Succeed } \tau_2 \\
&= \text{Fail, otherwise}
\end{align*}$$

Fig. 3 Named quantity rules for statements, assignments and conditionals.

### 4.2 Function Calls

Low coupling is generally desirable, especially in large complicated programs which are common nowadays. Functions enable one to specify a simple interface, to be self-contained, and to be reused. Functions are a convenient construct for making pieces of code written by different people or different groups interoperable.

Quantity functions, $ufun$, differ from normal functions in that they can take a number of quantity arguments, and return a quantity, if the function body satisfies the quantity algebra. Two new syntactic constructs are required:

$$
ufun ::= \text{fun } ufn(\text{uv}_1:qn_1, \ldots, \text{uv}_m:qn_m):qn_{\text{out}} \\
\text{is } uexp
$$

$$
uexp ::= \ldots | ufn(uexp_1, \ldots, uexp_m)
$$

Both a definition mechanism and an invocation mechanism are provided for named quantity functions, as shown in Figure 4. Function definitions are stored in a new environment, $\sigma$, binding function names to their input parameters, expression body and returning quantity. Hence our rules for expressions, $\text{NE}$, and assignments will need to be extended to pass this second environment.
around but otherwise stay unchanged. On invocation we retrieve the definition from \( \sigma \), and then build a local unit variable environment, \( \tau' \), which binds the parameters to their quantities. We must then make sure that named quantities, \( qn_i \), are safely assigned. To be specific, that the named quantity of each argument matches that of its parameter using the \( \vartriangleright \{ \} \) operator. This will create a single binding if need be, say a parameter is defined in the interface as \text{Noname} but called with a \text{Name} "\text{T}" value. The initial binding will be overridden to become a value of kind \text{Name} "\text{T}". On completing the expression body, the derived named quantity will be compared with that in the definition, using \( \diamond \). Consequently, if the derived quantity has a different name to that of the function definition, then the analysis cannot proceed. More importantly though, is if the derived quantity has a \text{Noname} kind then it can be given a KOQ on return.

To illustrate how these collection of rules enable named quantity checking we consider a dimensionally correct assignment of \( nt := 2 \ast \text{addtq}(t1,t2) \) with differing named quantity definitions. In the first case we consider \( t1 \) and \( t2 \) to both represent torque values, so that environment \( \tau \) is \( \{ nt \mapsto \text{Named "T"}, t1 \mapsto \text{Named "T"}, t2 \mapsto \text{Named "T"} \} \). We also define \text{addtq} to expect two torque quantities. It will be stored in the function environment \( \sigma \) as \( \{ \text{addtq} \mapsto ((x,\text{Named "T"}),(y,\text{Named "T"}),(x+y,\text{Named "T"})) \} \). The code fragment would look like this:

```plaintext
begin
nt : float of Named T;
t1 : float of Named T;
t2 : float of Named T;
fun addtq (x:Named T,y:Named T):Named T = x+y
...
nt := 2 * addtq(t1,t2)
end
```
Quantity checking the assignment would succeed as follows:

\[ NS[nt := 2*addtq(t1,t2)]_{\{nt\mapsto Named \ "T"},t1\mapsto Named \ "T"},t2\mapsto Named \ "T"} \sigma \]
\[ (\tau nt) \triangleleft^\tau_{nt} (NE[2*addtq(t1,t2)]_{\tau}) \]
\[ (Named \ "T") \triangleleft^\tau_{nt} (NE[addtq(t1,t2)]_{\tau}) \]
\[ (Named \ "T") \triangleleft^\tau_{nt} (Named \ "T") \]
\[ \Rightarrow \]
\[ Named \ "T" \triangleleft^\tau_{nt} (Named \ "T") \]
\[ \Rightarrow \]
\[ \Rightarrow \]
\[ \Rightarrow \]
\[ \Rightarrow \]

Alternatively, if we try a similar assignment, \( nt := 2 * addtq(t,w) \), but with quantities denoting torque and work:

\[ begin \]
\[ nt : float of Named T; \]
\[ t : float of Named T; \]
\[ w : float of Named W; \]
\[ fun addtq (x:Named T,y:Named T):Named T = x+y \]
\[ ...
\[ nt := 2 * addtq(t,w) \]
\[ end \]

Such that \( \tau = \{nt \mapsto Named \ "T"},t \mapsto Named \ "T"},w \mapsto Named \ "W"} \} then the derivation cannot be completed as the parameter \( w \) has the named quantity \( Named \ "W" \) where a \( Named \ "T" \) was expected:

\[ NS[nt := 2*addtq(t,w)]_{\{nt\mapsto Named \ "T"},t\mapsto Named \ "T"},w\mapsto Named \ "W"} \sigma \]
\[ (\tau nt) \triangleleft^\tau_{nt} (NE[2*addtq(t,w)]_{\tau}) \]
\[ (Named \ "T") \triangleleft^\tau_{nt} (NE[addtq(t,w)]_{\tau}) \]
\[ (Named \ "T") \triangleleft^\tau_{nt} (Named \ "T") \]
\[ \Rightarrow \]
\[ Named \ "T" \triangleleft^\tau_{nt} (Named \ "T") \]
\[ \Rightarrow \]
\[ \Rightarrow \]
\[ \Rightarrow \]

This form of named quantity error detection is labeled Type 1 KOQ error [16].
We can use Noname quantities in function interfaces to avoid having to commit to a given name, such as torque or work:

```algebra
begin
  nt : float of Named T;
  t : float of Named T;
  w : float of Named W;
  fun addtq (x:Noname,y:Noname):Noname = x+y
  ...
  nt := 2 * addtq(t,w)
end
```

In this case our function `addtq` accepts Noname quantities so \( \sigma = \{ addtq \mapsto ((x,\text{Noname}),(y,\text{Noname}),((x+y),\text{Noname})) \} \). However, the function body will not proceed as both arguments to \( x+y \) need to follow the named quantity rules for addition:

\[
\mathcal{NS}[nt := 2*addtq(t,w)]_{nt\mapsto\text{Named } "T" , t\mapsto\text{Named } "T" , w\mapsto\text{Named } "W"} \sigma
\[
\Rightarrow (\tau nt) \prec_{nt} (\mathcal{NE}[2*addtq(t,w)]_{\tau})
\[
\Rightarrow (\text{Named } "T") \prec_{nt} (\mathcal{NE}[\text{addtq}(t,w)]_{\tau})
\[
\Rightarrow (\text{Named } "T") \prec_{nt}
\quad (\text{Noname} \diamond (\mathcal{NE}[x + y]_{\{x\mapsto\text{Noname},y\mapsto\text{Noname}\}} + \tau_1 + \tau_2) \sigma), \text{ if}
\quad \text{Noname} \prec_{\lambda_1} (\mathcal{NE}[t]_{\tau} \sigma) = \text{Succeed } \tau_1 \land
\quad \text{Noname} \prec_{\lambda_2} (\mathcal{NE}[w]_{\tau} \sigma) = \text{Succeed } \tau_2)
\[
\Rightarrow (\text{Named } "T") \prec_{nt}
\quad (\text{Noname} \diamond (\mathcal{NE}[x + y]_{\{x\mapsto\text{Named } "T" , y\mapsto\text{Named } "W"\}} \sigma), \text{ if}
\quad \text{Noname} \prec_{\lambda_1} \text{Named } "T" = \text{Succeed } \{x \mapsto \text{Named } "T"\} \land
\quad \text{Noname} \prec_{\lambda_2} \text{Named } "W" = \text{Succeed } \{y \mapsto \text{Named } "W"\}
\[
\Rightarrow (\text{Named } "T") \prec_{nt}
\quad (\text{Noname} \diamond (\mathcal{NE}[x + y]_{\{x\mapsto\text{Named } "T" , y\mapsto\text{Named } "W"\}} \sigma)
\[
\Rightarrow (\text{Named } "T") \prec_{nt} (\text{Noname} \diamond (\text{Named } "T" \diamond \text{Named } "W"))
```

This example does show how the local function environment is updated to reflect the kinds of entities passed into it.

### 4.3 Safe Multiplication Through Function Calls

The real potential of quantity functions is that they can re-establish a named quantity. This is of particular interest when evaluating expressions containing multiplication as the \( \triangle \) rules lose information on the kind of quantity generated. An example of this Type 2 KOQ error [16] is the incorrect analysis of a turbine, of moment-of-inertia \( I \) (SI unit of \( kg \cdot m^2 \)) rotating with an angular velocity of \( \omega_1 \) (\( s^{-1} \)) with a torque \( T \) (\( kg \cdot m^2 \cdot s^{-2} \)) applied for duration \( t \) in seconds. The initial kinetic energy \( E_1 \) is defined as \( E_1 = 0.5 * I * \omega_1^2 \). It is easy to code this quantity equation incorrectly as \( E_1 = 0.5 * I/t^2 \), where the units of both sides of the assignment (\( kg \cdot m^2 \cdot s^{-2} \)) are compatible but the kind of quantity of the unit expression is Noname:
Our dimensional analysis rules would notice the UoM compatibility of the assignment, while our quantity checking rules would evaluate the assignment expression, \(0.5 * i / (t*t)\), as having Noname, so the assignment to \(e\), through \(\bowtie_{\tau}\), would Succeed.

However if we demand a discipline of programming with quantities where expressions involving multiplication are promoted to functions then we can ensure that results have a known named quantity:

\[
\begin{align*}
\text{begin} & \\
  e & : \text{float of Named T} \\
  i & : \text{float of Named MI} \\
  t & : \text{float of Named S} \\
  e & := 0.5 * i / (t*t) \\
\text{end}
\end{align*}
\]

In this second case, the arguments to \texttt{kin\_energy} have to represent moment of inertia, \texttt{Named MI}, and angular velocity, \texttt{Named AV}, kinds of quantities. On completion the function will return a torque quantity, \texttt{Named T}, so subsequent calculations can be undertaken safely. The function behaves like a contract even through we cannot ascertain the KOQ of \(0.5*I*(w*w)\) directly using our algebra.

**Theorem 2.** For a given unit expression, \(uexp\), in which general multiplication is undertaken within a function that returns a named quantity, along with a variable environment, \(\tau\), and a function environment \(\sigma\); \(\mathcal{N}[\{uexp\}]_{\tau \sigma}\) will only succeed if all named subexpressions represent the same entity.

**Proof** is an extension of Theorem 1. The additional base case of the function call \(\mathcal{N}[\{uexp1, \ldots, uexp_m\}]_{\tau \sigma}\) will, by definition, yield a named quantity. We therefore need to show the two successful cases that arise when such a function is used within a nested subexpression involving another \(uexp:\)

\[
\begin{align*}
\mathcal{N}[\{uexp + ufn(uexp1, \ldots, uexp_m)\}]_{\tau \sigma} &= (\mathcal{N}[\{uexp\}]_{\tau \sigma}) \diamond (\mathcal{N}[\{ufn(uexp1, \ldots, uexp_m)\}]_{\tau \sigma})
\end{align*}
\]

**Case 1:** \(\text{Named } n_1 \diamond \text{Named } n_2\)

\(\Rightarrow\) \(\text{Named } n_1\), where \(n_1 = n_2\) by definition of \(\diamond\)

**Case 2:** \(\text{Noname} \diamond \text{Named } n\)

\(\Rightarrow\) \(\text{Named } n\), by definition of \(\diamond\)
Functions that return a named quantity are safe in their usage of KOQ as subsequent arithmetic can only proceed if both arguments represent the same quantity.

“We must not forget that it is not our business to make programs, it is our business to design classes of computations that will display a desired behaviour.” (Dijkstra 1972)

Functions lend themselves towards the development of programming libraries, collections of behaviour, useful for specific applications which can be used by multiple programs that have no connection to each other. By extending the interface to include units and kind of quantity information we can ensure that dimensional analysis is performed and quantities are handled safely. Thereby suggesting that scientific programmers should develop libraries of functions that are correctly annotated such that checkers can detect misuse of quantities in their code. Moreover by encouraging library development, the subsequent annotation burden is reduced.

Explicitly named quantity parameters protect the function body but do not allow commonalities to be exploited. Having to explicitly name each parameter quantity is cumbersome and minimises reusability as many functions would have to be duplicated. We can use Noname quantities to avoid having to commit to a given name, such as torque or work, but this defeats the purpose of our approach. Extending named quantities to include named quantity variables, Quantvar, so that we could write generic quantity functions is preferable as equivalences could be easily defined:

```latex
fun add (x:Quantvar q,y:Quantvar q):Quantvar q = x+y
```

In this case the variable q could be assigned to any named quantity but both x and y would have the same named quantity. Our function invocation rule can be extended to ensure named quantity variables are uniquely assigned, and the named quantity of each argument that shared the same quantity variable were equal using a union-find data structure.

4.4 Discussion

Our algebra of named quantities is intended to be implemented as part of the static analysis phase of a compiler for an existing programming language or scientific domain specific language. Incorporating named quantities into software models would expand their use, and increase the robustness of designs. Although named quantity and dimensional analysis are effective at discovering errors early, there are three concerns that impede their adoption [28].

- Lack of Awareness: many developers are totally unaware of software solutions that deal with quantities and UoM. Inertia arises from factors like tradition, fear of change and effort of learning something new. Our approach is intended for language extensions or a pluggable type system, but could equally be included in popular UoM libraries, both of which elicit change and adaptation.
• *Technical Internal Factors:* many solutions are awkward and imprecise, introducing a loss of precision and struggling at times with dimensional consistencies. The strength of a language based solution, versus a library one, is that these issues are reduced.

• *External Factors:* modern systems are not built in a vacuum but form part of an eco-system [29]. It is harder to argue for quantity annotations when values pass through numerous generic components that do not support them, such as legacy systems, databases, spreadsheets, graphics tools and many other components that are unlikely to support quantities without costly updates. Efforts are underway to address this essential issue [30] but ensuring that units are routinely documented for easy, unambiguous exchange of information requires a multi-stakeholder approach [31].

A lightweight and comprehensive language solution is fundamental to adoption. However, a relevant observation from both a survey of UoM libraries [13] and interviews with practitioners [18] was the need for quantities to be available at run-time. There are many mature and active UoM libraries for popular dynamic object oriented programming languages, such as Python and Ruby, in which no static checking will occur. If we allow dimensions to be only available at run-time then quantity checking, and also unit conversion, will have to be undertaken while the program is executing. Faults can be avoided but testing is required to ensure annotations are congruent, a feature that a static programme analysis will uncover prior to evaluation.

### 5 Conclusion

We have developed a simple algebra of named quantities and shown how it can be incorporated into an existing programming language. Thereby ensuring quantities that share the same units of measure are handled separately, if required. Our algebra is safe and allows a degree of flexibility through casting upwards from unnamed quantities to named ones, thus enabling greater code reuse and a more practical implementation of the concept. Unnamed quantities can be assigned to named one’s but not the other way around as this breaks substitutability. Our theorems address both KOQ errors discussed in [16] that are not detectable by conventional UoM checkers. The algebra is distinct to that of dimensional analysis but they can be combined to form a single pass validator, allowing aliases (such as J for work or N for force) and standard named mathematical functions to be part of a prelude.

We lack an unqualified estimate of how frequently unit inconsistencies occur or their cost. Anecdotally we can glean that it is not negligible from experiments described in the literature [25, 32–34]. These focus on checking existing scientific repositories and are not representative of any quantity adhering software discipline. They have been applied post-development whereas we are seeking to support developers by ensuring their code bases model their scientific domain.
Quantity checking, much like type checking, is extremely useful when developing as errors are found before any code is run. Data on existing systems that have been extensively tested are less indicative of coding practices. However annotating all unit variables in a programme is costly. Ore [35] found subjects choose a correct UoM annotation only 51% of the time and take an average of 136 seconds to make a single correct annotation. This explains the appeal of systems that try to lower the burden while still ensuring coverage through unit type variables and a solver [21, 23, 24, 36], or compromise coverage through a component based approach [33, 37]. A component based discipline means that the consequences of local unit mistakes are underestimated. On the other hand, it allows diverse teams to collaborate even if their domain specific environments or choice of quantity systems are dissimilar.

If all quantity type variables are resolved by the static checker then dimensional correctness can be shown. A corpus of quantity errors, their programming language and associated software systems is required to assess the significance of annotations and the cost of developing without. We have no idea of how much time is spent chasing incompatible quantity assignments, dimension errors or incorrect unit conversions. We do know that there are many reasons for not adopting a quantity discipline based approach [18]. Different stakeholders will have different robustness concerns and willingness to compromise on the proportion of quantity and unit annotations required. Addressing usability concerns is an important aspect of our research.

Our tool currently performs static named quantity and dimension analysis for a simple imperative language. It also searches for the least number of unit conversions to reduce round-off errors in generated code. The main design principle was based on extending the Quantity pattern with added functionality. However, we feel that this strategy does not address the usability concerns of many scientific coders who require a lightweight bespoke solution to managing UoM as opposed to a multi-purpose and inherently cumbersome one. Consequently we are working on a customisable approach leveraging generics and staged computation.

Declarations
Not applicable.

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