Yukawa textures in modular symmetric vacuum of magnetized orbifold models

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Abstract

We study quark mass matrices derived from magnetized $T^2/Z_2$ orbifold models. Yukawa matrices at three modular fixed points, $\tau = i, e^{2\pi i/3}$ and $i\infty$ are invariant under $S$, $ST$ and $T$-transformations. We study these invariances on $T^2/Z_2$ twisted orbifold. We find that Yukawa matrices have a kind of texture structures although ones at $\tau = i\infty$ are not realistic. We classify Yukawa textures at $\tau = i$ and $e^{2\pi i/3}$. Moreover we investigate the conditions such that quark mass matrix constructed by Yukawa textures becomes approximately rank one matrix, which is favorable to lead to hierarchical masses between the third generation and the others. It is found that realistic quark mass matrices can be obtained around the $S$-invariant vacuum and $ST$-invariant vacuum. As an illustrating example, we show the realization of the quark mass ratios and mixing based on Fritzch and Fritzch-Xing mass matrices.
1 Introduction

The origin of the flavor structure such as the mass hierarchy and the flavor mixing is one of the unsolved mysteries in present day particle physics. In the Standard Model (SM), quark flavor observables have been described by 10 real parameters: 6 quark masses, 3 mixing angles and 1 CP violating phase. Similarly, lepton flavor observables need 12 real parameters: 6 lepton masses, 3 mixing angles and 3 Dirac and Majorana CP violating phases. To understand the origin of this large number of parameters, two types of approaches, bottom-up approach and top-down approach, have been carried out. In bottom-up approach, non-Abelian discrete flavor models have been proposed where $S_N$, $A_N$, $\Delta(3N^2)$, $\Delta(6N^2)$ and so on are assumed as flavor symmetries of quarks and leptons \cite{1–6}. Then such symmetries are broken by the vacuum expectation values (VEVs) of gauge singlet scalars so-called flavons but they become complicated.

As another bottom-up approach, it is essential idea to limit the number of parameters in the fermion mass matrices. For example, in \cite{7}, Fritzch proposed the idea of texture-zero for quark mass matrices where some of entries are zero, and it was extended in \cite{8} as the Fritzch-Xing mass matrix. (See for a review Ref. \cite{9}.) Moreover, several types of texture structures were studied \cite{10}. Actually, phenomenologically viable four zero textures of Hermitian quark mass matrices have been investigated and it has been found that there are several possibilities. (See e.g. Ref. \cite{11} and references therein.)

On the other hand, superstring theory is a promising candidate for the unified theory. Superstring theory predicts ten dimensions. Low-energy effective field theory of superstring theory can be described by ten-dimensional (10D) super Yang-Mills theory. Compactification of 10D superstring theory as well as super Yang-Mills theory can lead to a variety of phenomena in particle physics, e.g. the flavor structure. Among various compactifications, torus and orbifold compactifications with magnetic flux background are one of simplest ones, but have interesting structure. They lead to four-dimensional chiral theory and the generation number is determined by the size of magnetic fluxes \cite{12–15}. Furthermore, their Yukawa couplings depend on moduli and can be suppressed. Indeed, realistic mass matrices can be realized \cite{16–19}.

One of important aspects is that the torus compactification and its orbifolding have the modular symmetry $\Gamma \equiv SL(2, \mathbb{Z})$ as well as $\tilde{\Gamma} \equiv SL(2, \mathbb{Z})/\mathbb{Z}_2$, which is a geometrical symmetry. Moreover, zero-mode wavefunctions in magnetized torus and orbifold models transform non-trivially under the modular symmetry \cite{20–26}. In this context the modular symmetry is regarded as the flavor symmetry. Indeed, three-generation magnetized orbifold models lead to covering groups of $A_4$, $S_4$, $A_5$, $\Delta(98)$, $\Delta(384)$ with center extensions as flavor symmetries \cite{25}. In addition, Yukawa couplings also transform non-trivially under the modular symmetry. In this sense, the modular symmetry is not a simple symmetry, under which coupling constants and masses are invariant, but Yukawa couplings are spurion fields, which transform non-trivially under the modular symmetry.

Recently, the modular symmetry has been attracting attention from the bottom-up ap-
proach. Interestingly the finite modular subgroups $\Gamma_N$ for $N = 2, 3, 4$ and 5 are isomorphic to $S_3, A_4, S_4$ and $A_5$, respectively [27]. Motivated by this point and string compactification, in the bottom-up approach, flavor models with $\Gamma_N$ were studied intensively to lead to realistic quark and lepton mass matrices. (See e.g. Refs. [28–76].) In these modular flavor symmetric models, Yukawa couplings as well as masses are modular forms, which are functions of the modulus $\tau$. When we choose proper values of $\tau$, we can realize quark and lepton masses and their mixing angles as well as CP phases. Stabilization of the modulus $\tau$ was also studied. The modulus can be stabilized at fixed points, $\tau = i, e^{2\pi i/3}$ with a certain probability [77–79]. The $\mathbb{Z}_2$ and $\mathbb{Z}_3$ residual symmetries remain at these fixed points $\tau = i$ and $e^{2\pi i/3}$, respectively, and they are generated by $S$ and $ST$, while at the fixed point $\tau = i\infty$, $T$-symmetry remains. Because of residual symmetries, mass matrices have specific patterns. Indeed, realistic results were obtained at nearby fixed points [49, 52, 59, 71, 72].

In this paper, we revisit the structure of Yukawa matrices in magnetized orbifold models. Generic string compactifications including magnetized models lead to more than one candidates for the Higgs modes, which have the same quantum numbers under the $SU(3) \times SU(2) \times U(1)$ SM gauge group and can couple with quarks and leptons. They are massless at perturbative level. They may gain mass terms by non-perturbative effects, i.e. $\mu$-term in supersymmetric models, and the lightest direction of multi-Higgs modes may be determined. However, such analyses are not straightforward in explicit models, and the lightest direction is not clear. Thus, in analysis of Refs. [16–19], the lightest direction is parametrized in the multi-Higgs field space. By use of those parameters, the possibility to derive realistic quark masses and mixing angles was examined. We follow the same procedure. In addition, we emphasize the modular symmetry of Higgs modes. Multi-Higgs modes are a (reducible) multiplet of the modular symmetry in magnetized orbifold models. As mentioned above, the $\mathbb{Z}_2$ ($\mathbb{Z}_3$) residual symmetries generated by $S$ ($ST$) remain at these fixed points $\tau = i$ ($\tau = e^{2\pi i/3}$). Each of Higgs modes has a definite $\mathbb{Z}_2$ ($\mathbb{Z}_3$) charge at $\tau = i$ ($e^{2\pi i/3}$). We can realize a specific pattern of Yukawa matrix at these fixed points of $\tau$, depending on $\mathbb{Z}_N$ charges of Higgs modes. That is, texture structures are realized. We classify them. We show that $S$-invariant vacua at $\tau = i$ and $ST$-invariant vacua at $\tau = e^{2\pi i/3}$ are useful to realize a large hierarchy in quark masses. However, we need small deviations from $S$-invariant and $ST$-invariant vacua to derive realistic results fixing $\tau = i$ and $\tau = e^{2\pi i/3}$. For example, the Fritzch mass matrix and the Fritzch-Xing mass matrix can be realized from these textures by taking appropriate Higgs VEV directions.

This paper is organized as follows. In section 2 we review the zero-modes wavefunctions and Yukawa couplings on torus and orbifold with magnetic fluxes. In section 3 we review the three-generation fermion models on the orbifold. In section 4 we study and classify the structure of Yukawa matrices at three modular fixed points. In section 5 we show the condition such that quark mass matrices become rank one matrix, hence large hierarchy of quarks is realized. In section 6 we give examples of numerical studies for the quark mass matrices in our models. In section 7 we conclude this study. In Appendix A and B we give the proofs of the rank one conditions shown in section 5.
2 Orbifold compactification with magnetic fluxes

The 10D super Yang-Mills theory is the low-energy effective theory of superstring theory. We compactify the six dimensions, which includes the orbifold $T^2/\mathbb{Z}_2$ and four-dimensional compact space. We assume the flavor structure originated from $T^2/\mathbb{Z}_2$, although four-dimensional compact space may contribute to an overall factor of Yukawa matrices. Thus, we concentrate on two-dimensional orbifold $T^2/\mathbb{Z}_2$ with magnetic flux, and give a review of zero-mode wavefunctions and Yukawa couplings on these backgrounds [13–15].

2.1 Torus compactification

First, we briefly review zero-mode wavefunctions on magnetized $T^2$ [12]. For simplicity, we concentrate on $U(1)$ background magnetic flux given by

\[ F = dA = \frac{\pi i M}{\text{Im}\tau} dz \wedge d\bar{z}, \]

where $z$ is the complex coordinate on $T^2$ and $\tau$ is the complex structure modulus. The flux $M$ is induced by the following vector potential one-form,

\[ A = \frac{\pi M}{\text{Im}\tau} \text{Im}((\bar{z} + \bar{\zeta}) dz). \]

In what follows we consider vanishing Wilson line $\zeta = 0$. Then the torus identification $z \sim z + m + n \tau$, $m,n \in \mathbb{Z}$, gives the Dirac quantization condition, $M \in \mathbb{Z}$. Furthermore, the two-dimensional spinor with $U(1)$ unit charge $q = 1$, $\psi = (\psi_+, \psi_-)^T$, must fulfill the boundary conditions,

\[ \psi(z + 1) = e^{i\pi M \text{Im}z} \psi(z), \quad \psi(z + \tau) = e^{i\pi M \text{Im}(\bar{\tau} z)} \psi(z). \]

By solving the massless Dirac equation, $i \not{D}\psi = 0$, under above conditions, it is found that only positive (negative) chiral zero-mode wavefunctions have the $|M|$ number of degenerate solutions for $M > 0$ ($M < 0$); the $j$-th zero-mode is expressed as

\[ \psi^{j,|M|}_+(z, \tau) = \left( \frac{|M|}{A} \right)^{1/4} e^{i\pi |M| \text{Im}z} \sum_{\ell \in \mathbb{Z}} e^{i\pi |M| \tau (|M| + \ell)^2} e^{2\pi i |M| z (|M| + \ell)} \]

\[ \psi^{j,|M|}_-(z, \tau) = \left( \frac{|M|}{A} \right)^{1/4} e^{i\pi |M| \text{Im}z} \theta \begin{bmatrix} j \\ 0 \end{bmatrix} (|M| z, |M| \tau), \]

\[ \psi^{j,|M|}_+(z, \tau) = (\psi^{j,|M|}_-(z, \tau))^*, \quad j = 0, 1, ..., |M| - 1, \]

where $A$ denotes the area of $T^2$ and $\theta$ denotes the Jacobi theta function defined by

\[ \theta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{\ell \in \mathbb{Z}} e^{\pi i (a + \ell)^2 \tau} e^{2\pi i (a + \ell)(\nu + b)}. \]
This function has the property
\[ \vartheta \left[ \frac{j}{M_1} \right] (\nu_1, M_1\tau) \times \vartheta \left[ \frac{k}{M_2} \right] (\nu_2, M_2\tau) = \sum_{m \in \mathbb{Z}_{M_1+M_2}} \vartheta \left[ \frac{j+k+M_1m}{M_1+M_2} \right] (\nu_1 + \nu_2, (M_1 + M_2)\tau) \]
\[ \times \vartheta \left[ \frac{M_1j-M_1k+M_1M_2m}{M_1M_2(M_1+M_2)} \right] (\nu_1 M_2 - \nu_2 M_1, M_1M_2(M_1+M_2)\tau). \]
\[ (8) \]

Consequently we find the normalization and product expansions of the zero-modes:
\[ \int d^2 z \psi_{\pm}^{j,|M|}(z, \tau) \left( \psi_{\pm}^{j,|M|}(z, \tau) \right)^* = (2\text{Im}\tau)^{-1/2} \delta_{i,j}, \]
\[ (9) \]
\[ \psi_{\pm}^{i,|M_1|}(z, \tau) \cdot \psi_{\pm}^{j,|M_2|}(z, \tau) = \sum_{k \in \mathbb{Z}_{M_1+M_2}} Y^{ijk} \psi_{\pm}^{k,|M_1+|M_2|}(z, \tau), \]
\[ (10) \]
where
\[ Y^{ijk} = \int d^2 z \psi_{\pm}^{i,|M_1|}(z, \tau) \psi_{\pm}^{j,|M_2|}(z, \tau) \left( \psi_{\pm}^{k,|M_1+|M_2|}(z, \tau) \right)^* \]
\[ = A^{-1/2} \left| \frac{M_1 M_2}{M_1 + M_2} \right|^{1/4} \vartheta \left[ \frac{|M_2||M_1|+|M_2|}{|M_1M_2(M_1+M_2)|} \right] (0, |M_1M_2(M_1+M_2)|). \]
\[ (11) \]
\[ (12) \]

Hereafter, we omit the chirality sign ± from the zero-modes.

As the end of this subsection, we also give a review of the modular symmetry for wavefunctions [23]. The modular group \( \Gamma = SL(2,\mathbb{Z}) \) is generated by two generators, \( S \) and \( T \)-transformations, and defined as
\[ \Gamma \equiv \langle S, T | S^2 = Z, S^4 = (ST)^3 = Z^2 = \mathbb{I} \rangle. \]
\[ (13) \]

Then, the modular transformation for \( (z, \tau) \) is given by
\[ S : (z, \tau) \rightarrow \left( -\frac{z}{\tau}, -\frac{1}{\tau} \right), \quad T : (z, \tau) \rightarrow (z, \tau + 1), \]
\[ (14) \]
and under these two transformations the wavefunctions in Eq. (5) behave as the modular forms of weight 1/2 transformed by \( \tilde{\Gamma}_{2|M|} \):
\[ \psi_{\pm}^{j,|M|}(\tilde{\gamma}(z, \tau)) = \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \sum_{k=0}^{[M]-1} \tilde{\rho}(\tilde{\gamma})_{jk} \psi_{\pm}^{k,|M|}(z, \tau), \quad \tilde{\gamma} \in \tilde{\Gamma}, \]
\[ (15) \]
where \( \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \) is the automorphy factor, \( \tilde{\Gamma} \) is the double covering group of \( \Gamma \) generated by two generators, \( \tilde{S} \) and \( \tilde{T} \)-transformations (which are the double covering of \( S \) and \( T \)), and defined as
\[ \tilde{\Gamma} \equiv \langle \tilde{S}, \tilde{T} | \tilde{S}^2 = \tilde{Z}, \tilde{S}^4 = (\tilde{ST})^3 = \tilde{Z}^2, \tilde{S}^8 = (\tilde{ST})^6 = \tilde{Z}^4 = \mathbb{I}, \tilde{Z}\tilde{T} = \tilde{T}\tilde{Z} \rangle, \]
\[ (16) \]
and $\tilde{\rho}$ is the unitary representation of $\tilde{\Gamma}_{2|\mathcal{M}|}$ generated by following $\tilde{S}$ and $\tilde{T}$-transformations:

$$\tilde{\rho}(\tilde{S})_{jk} = e^{i\pi/4} \frac{1}{\sqrt{|\mathcal{M}|}} e^{2\pi i jk/M}, \quad \tilde{\rho}(\tilde{T})_{jk} = e^{i\pi (2j/M) \delta_{j,k}}. \quad (17)$$

$\tilde{\Gamma}_{2|\mathcal{M}|}$ is defined as

$$\tilde{\Gamma}_{2|\mathcal{M}|} \equiv \langle \tilde{S}, \tilde{T} | \tilde{S}^2 = \tilde{Z}, \tilde{S}^4 = (\tilde{S} \tilde{T})^3 = \tilde{Z}^2 = -\mathbb{I}, \tilde{Z} \tilde{T} = \tilde{T} \tilde{Z}, \tilde{T}^{2M} = \mathbb{I} \rangle. \quad (18)$$

That is, $\tilde{\rho}$ satisfies the following algebraic relations:

$$\tilde{\rho}(\tilde{S})^2 = \tilde{\rho}(\tilde{Z}), \quad \tilde{\rho}(\tilde{S})^4 = [\tilde{\rho}(\tilde{S})\tilde{\rho}(\tilde{T})]^3 = \tilde{\rho}(\tilde{Z})^2 = -\mathbb{I}, \quad \tilde{\rho}(\tilde{Z})\tilde{\rho}(\tilde{T}) = \tilde{\rho}(\tilde{T})\tilde{\rho}(\tilde{Z}), \quad \tilde{\rho}(\tilde{T})^{2M} = \mathbb{I}. \quad (19)$$

We note that $T$-transformation for the wavefunctions can be defined with vanishing Wilson line only if $M \in 2\mathbb{Z}$ because of the consistency with the boundary conditions. The $T$-transformation can be consistent for non-vanishing Wilson lines when $M \in 2\mathbb{Z} + 1 \quad [25]$.  

### 2.2 Orbifold compactification

Second, we briefly review zero-mode wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold with magnetic flux $M \quad [13]$. The $T^2/\mathbb{Z}_2$ twisted orbifold is obtained by further identifying $\mathbb{Z}_2$ twisted point $-z$ with $z$, i.e. $z \sim -z$. In addition to the torus boundary conditions in Eq. (3), the wavefunctions on magnetized $T^2/\mathbb{Z}_2$ twisted orbifold are required to fulfill,

$$\psi_{T^2/\mathbb{Z}_2^m}(-z) = (-1)^m \psi_{T^2/\mathbb{Z}_2^m}(z), \quad m \in \mathbb{Z}_2. \quad (20)$$

Hence, they can be expressed by the wavefunctions on magnetized $T^2$; actually zero-modes are expressed as

$$\psi_{T^2/\mathbb{Z}_2^m}^{j,|\mathcal{M}|}(z) = \mathcal{N}^j \left( \psi_{T^2}^{j,|\mathcal{M}|}(z) + (-1)^m \psi_{T^2}^{j,|\mathcal{M}|}(-z) \right) = \mathcal{N}^j \left( \psi_{T^2}^{j,|\mathcal{M}|}(z) + (-1)^m \psi_{T^2}^{|\mathcal{M}|-j,|\mathcal{M}|}(z) \right), \quad (21)$$

where

$$\mathcal{N}^j = \begin{cases} 1/2 & (j = 0, |\mathcal{M}|/2) \\ 1/\sqrt{2} & \text{(otherwise)} \end{cases} \quad (22)$$

In Table 1 we show the number of zero-modes on magnetized $T^2/\mathbb{Z}_2$ twisted orbifold for vanishing discrete Wilson lines and Sherk-Shcwarz phases $[^1]$.  

Next, we review the modular symmetry of zero-modes on the orbifold. The zero-modes in Eq. (21) behave as the modular forms of weight $1/2$ transformed by $\tilde{\Gamma}_{2|\mathcal{M}|}$ under the modular transformation:

$$\psi_{T^2/\mathbb{Z}_2^m}(\tilde{\gamma}(z, \tau)) = \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \sum_k \tilde{\rho}_{T^2/\mathbb{Z}_2^m}(\tilde{\gamma})_{jk} \psi_{T^2/\mathbb{Z}_2^m}^{k,|\mathcal{M}|}(z, \tau), \quad (23)$$

$^1$See for zero-modes with non-vanishing discrete Wilson lines and Sherk-Shcwarz phases Refs. 14, 15.
Table 1: The number of zero-modes on magnetized $T^2/Z_2$ twisted orbifold.

| $|M|$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z_2$-even | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 6 | 7 | 7 |
| $Z_2$-odd | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |

where $\tilde{\rho}_{T^2/Z_2}^m$ is the unitary representation of $\tilde{\Gamma}_{2|M|}$ generated by following $\tilde{S}$ and $\tilde{T}$-transformations:

$$
\tilde{\rho}_{T^2/Z_2}^0(\tilde{S})_{jk} = N^j N^k \frac{4e^{\pi i/4}}{\sqrt{|M|}} \cos \left( \frac{2\pi jk}{|M|} \right), \quad \tilde{\rho}_{T^2/Z_2}^0(\tilde{T})_{jk} = e^{i\pi \frac{2}{|M|}} \delta_{j,k},
$$

(24)

$$
\tilde{\rho}_{T^2/Z_2}^1(\tilde{S})_{jk} = N^j N^k \frac{4ie^{\pi i/4}}{\sqrt{|M|}} \sin \left( \frac{2\pi jk}{|M|} \right), \quad \tilde{\rho}_{T^2/Z_2}^1(\tilde{T})_{jk} = e^{i\pi \frac{2}{|M|}} \delta_{j,k}.
$$

(25)

We again note that the $T$-transformation is consistent for vanishing discrete Wilson lines only if $M \in 2\mathbb{Z}$. The $T$-transformation can be consistent for non-vanishing discrete Wilson lines when $M \in 2\mathbb{Z} + 1$ [25].

### 3 Three-generation models

#### 3.1 Classification for three-generation models

In this subsection, we review the classification of the three-generation models which lead to non-vanishing Yukawa coupling in the $T^2/Z_2$ twisted orbifolds. (See for details Refs. [80, 81].) Yukawa coupling for 4D effective theory is given by the overlap integral of zero-modes on the orbifold:

$$
Y_{ijk} = \int_{6D} d^6z \psi_L^i(z) \psi_R^j(z) \psi_H^k(z)^*,
$$

(26)

where $\psi_L^i$, $\psi_R^j$ and $\psi_H^k$ are zero-modes for left-handed fermion, right-handed fermion and Higgs fields. We focus on the case that the flavor structure comes from only $T^2/Z_2$, although other 4-dimensional compact space contributes an overall factor of Yukawa matrices. Then Yukawa couplings relevant to the flavor structure are written as

$$
Y_{T^2/Z_2}^{ijk} = \int_{T^2/Z_2} d^2z \psi_L^i |M_L|_{T^2/Z_2} (z) \psi_R^j |M_R|_{T^2/Z_2} (z) \psi_H^k |M_H|_{T^2/Z_2} (z)^*,
$$

(27)

where $M_L$, $M_R$ and $M_H$ are the magnetic fluxes for left-handed fermion, right-handed fermion and Higgs fields, respectively. To preserve the gauge invariance, these fluxes must satisfy the following flux condition:

$$
|M_H| = ||M_L| \pm |M_R||.
$$

(28)
Moreover, Yukawa coupling in Eq. (27) should be invariant under $Z_2$ twist. Thus, non-vanishing Yukawa coupling must satisfy the following $Z_2$ parity condition:

$$\ell + m + n = 0 \pmod{2}.$$  

(29)

By these flux and parity conditions, the flux and parity for Higgs fields are fixed once we choose ones for left- and right-handed fermions such that three generations of fermions are realized. In Table 2, we show all the possible three-generation models with non-vanishing Yukawa couplings when $|M_H| = |M_L| + |N_R|$. Here, we ignore the three-generation models with the flux $|M_H| = |M_L| - |N_R|$ because such models do not lead to realistic results.

| $M_L$ (parity) | $M_R$ (parity) | $M_H$ (parity) | number of Higgs modes | Model name |
|----------------|----------------|----------------|-----------------------|------------|
| 4 (even)       | 4 (even)       | 8 (even)       | 5                     | 4-4-8, (e,e,e), 5H |
| 4 (even)       | 5 (even)       | 9 (even)       | 5                     | 4-5-9, (e,e,e), 5H |
| 5 (even)       | 5 (even)       | 10 (even)      | 6                     | 5-5-10, (e,e,e), 6H |
| 4 (even)       | 7 (odd)        | 11 (odd)       | 5                     | 4-7-11, (e,o,o), 5H |
| 4 (even)       | 8 (odd)        | 12 (odd)       | 5                     | 4-8-12, (e,o,o), 5H |
| 5 (even)       | 7 (odd)        | 12 (odd)       | 5                     | 5-7-12, (e,o,o), 5H |
| 5 (even)       | 8 (odd)        | 13 (odd)       | 6                     | 5-8-13, (e,o,o), 6H |
| 7 (odd)        | 7 (odd)        | 14 (even)      | 8                     | 7-7-14, (o,o,e), 8H |
| 7 (odd)        | 8 (odd)        | 15 (even)      | 8                     | 7-8-15, (o,o,e), 8H |
| 8 (odd)        | 8 (odd)        | 16 (even)      | 9                     | 8-8-16, (o,o,e), 9H |

Table 2: Possible three-generation models with non-vanishing Yukawa couplings on the $T^2/Z_2$ twisted orbifold when $|M_H| = |M_L| + |N_R|$. There are additional possible models obtained by left ($L$) and right ($R$) flipping although we omitted them in this table.

### 3.2 Yukawa couplings

Here, we review how to calculate Yukawa couplings in the three-generation models. First of all, we calculate ones on torus which is given by

$$Y_{T^2}^{ijk} = \int_{T^2} d^2z \psi^{i,|M_L|}_{T^2}(z)\bar{\psi}^{j,|M_R|}_{T^2}(z)\left(\psi^{k,|M_H|}_{T^2}(z)\right)^*. \quad (30)$$

Using the normalization in Eq. (9) and the product expansion in Eq. (10), we find

$$Y_{T^2}^{ijk} = (2A\text{Im}\tau)^{-1/2} \left|\frac{M_L M_R}{M_H}\right|^{1/4} |M_H|^{-1} \sum_{m=0}^{\left\lfloor |M_R|/|M_L|\right\rfloor} \theta \left[\frac{|M_R| - |M_L| |j + |M_L M_R| m|}{|M_L M_R M_H|} \right] (0, |M_L M_R M_H| \tau) \cdot \delta_{i+j+k, |M_H|\ell-|M_L| m} \quad (31)$$

$$= c \sum_{m=0}^{\left\lfloor |M_H|/|M_R|\right\rfloor} \eta_{|M_R| - |M_L| |j + |M_L M_R| m|} \cdot \delta_{i+j+k, |M_H|\ell-|M_L| m} \quad (32)$$
where $\ell \in \mathbb{Z}$, $c = (2A \text{Im} \tau)^{-1/2} \left| \frac{M_L M_R}{M_H} \right|^{1/4}$ and we have used the notation,

$$\eta N = \vartheta \begin{pmatrix} N \\ M \\ 0 \end{pmatrix} (0, M \tau), \quad M = |M_L M_R M_H|.$$  \hspace{1cm} (33)

Then, Yukawa couplings on $T^2/\mathbb{Z}_2$ twisted orbifold can be expressed by ones on torus, because zero-modes on the orbifold can be expressed by ones on torus. Inserting zero-modes on the orbifold in Eq. (21) to Yukawa couplings on the orbifold in Eq. (27), we find

$$Y_{ijk}^{T^2/\mathbb{Z}_2} = \sum_{i',j',k'} O_{ij|^M_L |O^M_R |O^M_H}^{i'j'|M_L |M_R |M_H} Y_{i'j'k'}^{T^2},$$ \hspace{1cm} (34)

where

$$O_{mj}^{j,k,M} = N^j (\delta_{j,k} + (-1)^m \delta_{j,M-j}).$$ \hspace{1cm} (35)

We also study the modular symmetry of Yukawa couplings on the orbifold. Since Yukawa couplings are written by the overlap integral of zero-modes, from the transformation law for zero-modes, we find that Yukawa couplings are transformed as

$$Y_{ijk}^{T^2/\mathbb{Z}_2}(\tilde{\gamma}\tau) = \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \tilde{J}_{1/2}(\tilde{\gamma}, \tau) \tilde{\rho}^{T^2/\mathbb{Z}_2}(\tilde{\gamma}) i^{i'i'j'} \rho^{T^2/\mathbb{Z}_2}(\tilde{\gamma}) j^{j'j'k'} \rho^{* T^2/\mathbb{Z}_2}(\tilde{\gamma}) k^{k'k'} Y_{i'j'k'}^{T^2/\mathbb{Z}_2}(\tau).$$ \hspace{1cm} (36)

## 4 Yukawa textures by modular symmetry

In this section, we study the restrictions on Yukawa matrices by modular symmetry. We will see that modular symmetry at its fixed points restrict the structure of Yukawa matrices and then Yukawa matrices have a kind of texture structures. The fixed points for the modular transformation are as follows:

I. $\tau = i$ is invariant under $S$-transformation.

II. $\tau = e^{2\pi i/3} = \omega$ is invariant under $ST$-transformation.

III. $\tau = i\infty \ (\text{Im} \tau = \infty)$ is invariant under $T$-transformation.

Hereafter, we investigate the structure of Yukawa matrices at above three fixed points. We note that we write Yukawa matrices on $T^2/\mathbb{Z}_2$ twisted orbifold as $Y_{ijk}$ instead of $Y_{ijk}^{T^2/\mathbb{Z}_2}$.

### 4.1 $S$-invariance

Only if $\tau = i$, the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold can be expanded by $\mathbb{Z}_4$ twist eigenstates. (See for $\mathbb{Z}_4$ twist eigenstates Refs. [14][15][23][32].) The $\mathbb{Z}_4$ twist is defined by the following transformation of the complex coordinate on $T^2$:

$$z \rightarrow iz.$$ \hspace{1cm} (37)
The number of each $Z_4$ eigenstate in the wavefunctions on the $T^2/Z_2$ twisted orbifold is shown in Table 3. Note that the $S$-transformation eigenstates and eigenvalues are the same as ones for $Z_4$; under $S$-transformation the wavefunctions on $Z_4$ eigenbasis are transformed by diagonalized matrix composed of $Z_4$ eigenvalues.

| $Z_2$ parity, number of generation | Number of $Z_4$ ($S$) eigenstates |
|-----------------------------------|-----------------------------------|
| even, $2n$                        | $n$  $n$  $0$  $0$               |
| even, $2n+1$                      | $n+1$ $n$  $0$  $0$              |
| odd, $2n$                         | $0$   $0$  $n$  $n$              |
| odd, $2n+1$                       | $0$   $0$  $n+1$ $n$             |

Table 3: Number of each $Z_4$ eigenstate in wavefunctions on the $T^2/Z_2$ twisted orbifold at $\tau = i$. $\eta$ denotes the eigenvalues of $Z_4$ twist. The $S$-transformation eigenstates and eigenvalues are same as ones for $Z_4$.

At $\tau = i$, Yukawa matrices are invariant under $S$-transformation because $S : \tau = -1/\tau$. This $S$-invariance is written as

$$Y^{ijk} = \tilde{J}_{1/2}(\tilde{S}, i)\tilde{\rho}_L(\tilde{S})_{ii'} \cdot \tilde{J}_{1/2}(\tilde{S}, i)\tilde{\rho}_R(\tilde{S})_{jj'} \cdot (\tilde{J}_{1/2}(\tilde{S}, i)\tilde{\rho}_H(\tilde{S})_{kk'})^* \cdot Y^{i'j'k'}, \quad (38)$$

with

$$\tilde{J}_{1/2}(\tilde{S}, \tau) = (-\tau)^{1/2}. \quad (39)$$

On the $Z_4$ eigenstates, that is, on $S$-transformation eigenstates, the transformation matrix, $\tilde{\rho}(\tilde{S})$, is given by a diagonalized matrix composed of $Z_4$ eigenvalues. The number of each $Z_4$ eigenvalue in the diagonalized matrix can be read from Table 3. Then, $S$-invariance in Eq. (38) restricts the structure of Yukawa matrices to two types as shown in Table 4.

As a simple example, we show a restriction on Yukawa matrices in the model “4-4-8, (e,e,e), 5H” in Table 2. Five Higgs modes in this model, whose flux is eight and parity is even, are transformed by

$$\tilde{J}_{1/2}(\tilde{S}, i)\tilde{\rho}_H(\tilde{S}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad (40)$$

under $S$-transformation. On the other hand, three generations of fermions, whose flux is four and parity is even, are transformed by

$$\tilde{J}_{1/2}(\tilde{S}, i)\tilde{\rho}_L(\tilde{S}) = \tilde{J}_{1/2}(\tilde{S}, i)\tilde{\rho}_R(\tilde{S}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}. \quad (41)$$
The structures of Yukawa matrices for each S-eigenstate Higgs mode

\[
\begin{array}{c|cc|cc|cc}
\text{Z}_2 \text{parities of} & \text{The structures of Yukawa matrices for each S-eigenstate Higgs mode} \\
(L, R, H) & 1 & -1 & i & -i \\
\hline
\text{even, even, even} & \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} & \begin{pmatrix} 0 & 0 & * \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} & \text{None} & \text{None} \\
\text{even, odd, odd} & \text{None} & \text{None} & \begin{pmatrix} * & * & 0 \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & * \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \\
\text{odd, even, odd} & \text{None} & \text{None} & \begin{pmatrix} 0 & 0 & * \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} & \begin{pmatrix} * & * & 0 \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \\
\text{odd, odd, even} & \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} & \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} & \text{None} & \text{None} \\
\end{array}
\]

Table 4: The structures of Yukawa matrices for each S-eigenstate Higgs mode. The Yukawa matrices are S-transformation eigenstates and then they are restricted to two types of structures by S-invariance. The symbol “∗” denotes nonzero elements of matrices.

Then the S-invariance on Yukawa matrices is written as

\[
Y^{ijk} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{ii'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{jj'} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}_{kk'} \ Y^{i'j'k'}. \tag{42}
\]

Thus Yukawa matrices for S-invariant Higgs modes, \(Y^{ij0}, Y^{ij1}, Y^{ij2}\), and ones for S-variant Higgs modes, \(Y^{ij3}, Y^{ij4}\), are restricted to the following two structures, respectively,

\[
Y^{ij0,1,2} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad Y^{ij3,4} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}, \tag{43}
\]

where the symbol “∗” denotes nonzero elements of matrices.

4.2 \textit{ST}-invariance

Only if \(\tau = e^{2\pi i/3} \equiv \omega\) and flux \(M = \text{even}\), the wavefunctions on the \(T^2/Z_2\) twisted orbifold can be expanded by \(Z_6\) twist eigenstates. (See for \(Z_6\) twist eigenstates Refs. [14, 15, 23, 82].) The \(Z_6\) twist is defined by the following transformation of the complex coordinate on \(T^2\):

\[
z \rightarrow e^{\pi i/3} z. \tag{44}
\]
The number of each $\mathbb{Z}_6$ eigenstate in the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold is shown in Table 5. Note that the $ST$-transformation eigenstates are the same as ones for $\mathbb{Z}_6$. The $ST$-transformation eigenvalues are given by the square of $\mathbb{Z}_6$ eigenvalues since $ST$-transformation at its fixed point is equivalent to $\mathbb{Z}_3$ twist. Under the $ST$-transformation, hence, the wavefunctions on $\mathbb{Z}_6$ eigenbasis are transformed by diagonalized matrix composed of the square of $\mathbb{Z}_6$ eigenvalues.

| $\mathbb{Z}_2$ parity, number of generation | Number of $\mathbb{Z}_6$ eigenstates |
|--------------------------------------------|-----------------------------------|
| even, $3n$                                  | $n$                               |
| even, $3n + 1$                              | $n + 1$                           |
| even, $3n + 2$                              | $n + 1$                           |
| odd, $3n$                                   | $0$                               |
| odd, $3n + 1$                               | $0$                               |
| odd, $3n + 2$                               | $0$                               |

Table 5: Number of each $\mathbb{Z}_6$ eigenstate in wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold at $\tau = e^{2\pi i/3} = \omega$. $\eta$ denotes the eigenvalues of $\mathbb{Z}_6$ twist. The $ST$-transformation eigenstates are same as ones for $\mathbb{Z}_6$. The $ST$-transformation eigenvalues are given by the square of $\mathbb{Z}_6$ eigenvalues.

At $\tau = \omega$, Yukawa matrices are invariant under the $ST$-transformation because $ST : \tau = -1/(\tau + 1)$. Only if fluxes $M_L$, $M_R$ and $M_H$ are all even integers, this $ST$-invariance is written as

$$Y^{ijk} = \tilde{J}_{1/2}(\tilde{ST}, \omega)\tilde{\rho}_L(\tilde{ST})_{ii'} \cdot \tilde{J}_{1/2}(\tilde{ST}, \omega)\tilde{\rho}_R(\tilde{ST})_{jj'} \cdot (\tilde{J}_{1/2}(\tilde{ST}, \omega)\tilde{\rho}_H(\tilde{ST})_{kk'})^* \cdot Y^{i'j'k'},$$  \hspace{1cm} (45)

with

$$\tilde{J}_{1/2}(\tilde{ST}, \tau) = (-\tau + 1)^{1/2}.$$  \hspace{1cm} (46)

On the $\mathbb{Z}_6$ eigenstates, that is, on $ST$-transformation eigenstates, the transformation matrix, $\tilde{\rho}(\tilde{ST})$, is given by a diagonalized matrix composed of the square of $\mathbb{Z}_6$ eigenvalues. The number of each $\mathbb{Z}_6$ eigenvalue in the diagonalized matrix can be read from Table 5. Then, $ST$-invariance in Eq. (45) restricts Yukawa matrices to three types of structures as shown in Table 6.

As a simple example, we show a restriction on Yukawa matrices in the model “4-4-8, (e,e,e), 5 H”. Five Higgs modes in this model, whose flux is eight and parity is even, are transformed by

$$\tilde{J}_{1/2}(\tilde{ST}, \omega)\tilde{\rho}_H(\tilde{ST}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix},$$  \hspace{1cm} (47)
under $ST$-transformation. On the other hand, three-generation fermions, whose flux is four and parity is even, are transformed by

$$
\tilde{J}_{1/2}(\tilde{S}_T, \omega)\tilde{\rho}_L(\tilde{S}_T) = \tilde{J}_{1/2}(\tilde{S}_T, \omega)\tilde{\rho}_R(\tilde{S}_T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.
$$

(48)

Then $ST$-invariance on Yukawa matrices are written as

$$
Y^{ijk} = \tilde{J}_{1/2}(\tilde{S}_T, \omega)\tilde{\rho}_L(\tilde{S}_T) \cdot \tilde{J}_{1/2}(\tilde{S}_T, \omega)\tilde{\rho}_R(\tilde{S}_T) \cdot \tilde{J}_{1/2}(\tilde{S}_T, \omega)\tilde{\rho}_H(\tilde{S}_T) \cdot Y^{i'j'k'}.
$$

(49)

Thus Yukawa matrices for $ST$-invariant Higgs, $Y^{ij0}$, $Y^{ij1}$, ones for $\omega$ eigenstates Higgs, $Y^{ij2}$, $Y^{ij3}$, and ones for $\omega^2$ eigenstates Higgs, $Y^{ij4}$, are restricted to the following three structures, respectively

$$
Y^{ij0.1} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}, \quad Y^{ij2.3} = \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}, \quad Y^{ij4} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}.
$$

(50)

### 4.3 $T$-invariance

Only if flux $M =$even, the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold can be expanded by $T$-transformation eigenstates.

At $\text{Im}\tau = \infty$, Yukawa matrices are invariant under the $T$-transformation because $T : \tau = \tau + 1$. Only if fluxes $M_L$, $M_R$ and $M_H$ are all even integers, this $T$-invariance is written as

$$
Y^{ijk} = \tilde{J}_{1/2}(\tilde{T}, i\infty)\tilde{\rho}_L(\tilde{T})_{ii'} \cdot \tilde{J}_{1/2}(\tilde{T}, i\infty)\tilde{\rho}_R(\tilde{T})_{jj'} \cdot (\tilde{J}_{1/2}(\tilde{T}, i\infty)\tilde{\rho}_H(\tilde{T})_{kk'})^* \cdot Y^{i'j'k'},
$$

(51)
with
\[ J_{1/2}(\tilde{T}, \tau) = 1, \quad \tilde{\rho}(\tilde{T})_{jk} = e^{i\pi j^2/M} \delta_{j,k}. \] (52)

This leads to
\[ Y_{ijk} = Y_{ijk} \exp \left[ \pi i \left( \frac{i^2}{M_L} + \frac{j^2}{M_R} - \frac{k^2}{M_H} \right) \right], \] (53)

and we find the nonzero elements condition,
\[ \left( \frac{i^2}{M_L} + \frac{j^2}{M_R} - \frac{k^2}{M_H} \right) \mod 2 = 0, \quad \text{otherwise} \quad Y_{ijk} = 0, \] (54)

which makes almost elements of Yukawa matrices vanish. For example, in the model “4-4-8, (e,e,e), 5 H.”, only three combinations of indices,
\[ (i, j, k) = (0, 0, 0), (1, 1, 2), (2, 2, 4), \] (55)

can satisfy the nonzero elements condition in Eq. (54), and Yukawa matrices are restricted to the following four structures,
\[ Y_{ij0} = \begin{pmatrix} \ast & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_{ij2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \ast & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_{ij4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \ast \end{pmatrix}, \quad Y_{ij1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (56)

We cannot realize flavor mixing from these Yukawa matrices. Similarly, in other three-generation models, we cannot realize mass matrices for up and down sectors consistent with observations. Therefore, hereafter we avoid discussion of $T$-invariance in Yukawa matrices.

4.4 Classification for textures in three-generation models

As the end of this section, we classify the number of each texture structure in three-generation models on the $T^2/Z_2$ twisted orbifold. We show the result in Table 7 Note that we ignore the textures by $T$-invariance at $\text{Im} \tau = \infty$.

5 Rank one structures in mass matrix

Once the lightest Higgs field develops its VEV, Yukawa couplings give a fermion mass term:
\[ M^{ij} = Y^{ijk} \langle H^k \rangle, \] (57)

where we have assumed that $\langle H^k \rangle$ are given by the direction of the lightest Higgs mode. By using texture structures, here we investigate the Higgs VEV direction such that quark mass
matrix has rank one. Since quark mass ratios have a large hierarchy, we can approximately regard it as rank one matrix:

\[
\begin{pmatrix}
  m_u \\ m_c \\ m_t
\end{pmatrix} = m_t \begin{pmatrix} O(10^{-6}) \\ O(10^{-3}) \\ 1 \end{pmatrix} \sim m_t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (58)
\]

\[
\begin{pmatrix}
  m_d \\ m_s \\ m_b
\end{pmatrix} = m_b \begin{pmatrix} O(10^{-4}) \\ O(10^{-2}) \\ 1 \end{pmatrix} \sim m_b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (59)
\]

Thus the mass ratios consistent with observations would be realized near the Higgs VEV directions leading to rank one quark mass matrix. In other words, if there is no direction leading to rank one mass matrix, it is difficult to reproduce the observation values of quark mass ratios.

In this section, we show the conditions that such rank one mass matrix can be realized by textures in the three-generation magnetized orbifold models.

Table 7: The number of each texture structure matrix in three-generation models. The first column shows three-generation models classified and named in Table 2. Other columns shows the number of each texture at \( \tau = i \) and \( \tau = \omega \). The values in parentheses denote the eigenvalues of corresponding Higgs modes under \( S \) (at \( \tau = i \)) and \( ST \) (at \( \tau = \omega \)) -transformations.
5.1 Higgs VEV directions at $\tau = i$

In this subsection, we investigate the Higgs VEV directions leading to rank one fermion mass matrix at $\tau = i$. In this case, fermion mass matrix can be expanded by textures as

$$M^{ij} = \sum_m \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}^{ijm} \langle H^m \rangle + \sum_n \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}^{ijn} \langle H^n \rangle. \quad (60)$$

Suppose that non-vanishing elements have generic values, but not specific relations among elements. Then rank one matrix can be realized in the following cases:

I. If mass matrix includes three or more of $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$, then the Higgs VEV directions leading to rank one exist in $S$-eigenstate directions.

II. Besides the case of I, if mass matrix is symmetric (non-symmetric) and includes one (two) or more of $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$, then the Higgs VEV directions leading to rank one exist in not $S$-eigenstate directions, too.

III. If mass matrix is symmetric and includes two or more of both types of textures respectively, then the Higgs VEV directions leading to rank one exist in not $S$-eigenstate directions.

IV. If mass matrix is non-symmetric and includes two or more of $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$ and three or more of $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$, then the Higgs VEV directions leading to rank one exist in not $S$-eigenstate directions.

V. If mass matrix is non-symmetric and includes three or more of $\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$, then the Higgs VEV directions leading to rank one exist in $S$-eigenstate directions.

The proofs of the above are shown in Appendix A. We show which Higgs VEV directions leading to rank one exist in three-generation models in Table 8. There are four models where rank one directions exist on $S$-invariant directions. In these four models, we have a possibility to realize realistic quark mass matrix if we assume almost $S$-invariant vacuum.
Three-generation models | The Higgs VEV directions leading to rank one
--- | ---
4-4-8, (e,e,e), 5H | $S$-invariant, not $S$-eigenstate
4-5-9, (e,e,e), 5H | $S$-invariant, not $S$-eigenstate
5-5-10, (e,e,e), 6H | $S$-invariant, not $S$-eigenstate
4-7-11, (e,o,o), 5H | $i$ eigenstate, not $S$-eigenstate
4-8-12, (e,o,o), 5H | $i$ eigenstate, not $S$-eigenstate
5-7-12, (e,o,o), 5H | $i$ eigenstate, not $S$-eigenstate
5-8-13, (e,o,o), 6H | $i$ eigenstate, $-i$ eigenstate, not $S$-eigenstate
7-7-14, (o,o,e), 8H | $-1$ eigenstate, not $S$-eigenstate
7-8-15, (o,o,e), 8H | $S$-invariant, $-1$ eigenstate, not $S$-eigenstate
8-8-16, (o,o,e), 9H | $-1$ eigenstate, not $S$-eigenstate

Table 8: The Higgs VEV directions leading to rank one mass matrix at $\tau = i$.

5.2 Higgs VEV directions at $\tau = \omega$

In this subsection, we investigate the Higgs VEV directions leading to rank one fermion mass matrix at $\tau = \omega$. In this case, fermion mass matrix can be expanded by textures as

$$M_{ij} = \sum_{\ell} \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}^{ij\ell} \langle H^\ell \rangle + \sum_{m} \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}^{ijm} \langle H^m \rangle + \sum_{n} \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{ijn} \langle H^n \rangle.$$ (61)

Suppose that non-vanishing elements have generic values, but not specific relations among elements. Then rank one matrix can be realized in the following cases:

I. If mass matrix is symmetric (non-symmetric) and includes two (three) or more of

$$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in $ST$-invariant directions.

II. If mass matrix is symmetric (non-symmetric) and includes two (three) or more of

$$\begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in $ST$-eigenstate directions corresponding to eigenvalue $\omega$.

III. If mass matrix is symmetric (non-symmetric) and includes two (three) or more of

$$\begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in $ST$-eigenstate directions corresponding to eigenvalue $\omega^2$. 

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IV. If mass matrix is symmetric (non-symmetric) and includes one (two) or more of two
types of textures and two (one) or more of other one type of texture, then the Higgs VEV
directions leading to rank one exist in not $ST$-eigenstate directions.

V. If non-symmetric mass matrix includes three or more of two types of textures, then the
Higgs VEV directions leading to rank one exist in not $ST$-eigenstate directions.

The proofs of the above are shown in Appendix B. We show which Higgs VEV directions leading
to rank one exist in three-generation models in Table 9. Note that we omit three-generation
models including odd integral flux since $ST$-transformation for Yukawa couplings cannot be
defined with vanishing Wilson lines. There are two models where rank one directions exist on
$ST$-invariant directions. In these two models, we have a possibility to realize realistic quark
mass matrix if we assume almost $ST$-invariant vacuum.

| Three-generation models | The Higgs VEV directions leading to rank one |
|-------------------------|-------------------------------------------|
| 4-4-8, (e,e,e), 5H      | $ST$-invariant, $\omega$ eigenstate, not $ST$-eigenstate |
| 4-8-12, (e,o,o), 5H     | not $ST$-eigenstate |
| 8-8-16, (o,o,e), 9H     | $ST$-invariant, $\omega$ eigenstate, $\omega^2$ eigenstate, not $ST$-eigenstate |

Table 9: Higgs VEV directions leading to rank one mass matrix at $\tau = \omega$.

6 Numerical example: the model “4-4-8, (e,e,e), 5H”

In this section, we study the model “4-4-8, (e,e,e), 5H”. We assume that both the up sector
and down sector correspond to this model. Then we show examples to realize the quark masses
and mixing angles.

6.1 Yukawa matrices

Here we show the Yukawa matrices in the model “4-4-8, (e,e,e), 5H”. Table 10 shows the zero-
mode assignments for left-handed fermions $L$, right-handed fermions $R$ and the Higgs fields $H$.

This model has five zero-modes for Higgs fields. Yukawa couplings $Y^{ijk} L^i R^j H^k$ are given by

$$Y^{ijk} H^k = Y^{ij0} H^0 + Y^{ij1} H^1 + Y^{ij2} H^2 + Y^{ij3} H^3 + Y^{ij4} H^4,$$
Table 10: Zero-mode wavefunctions in “4-4-8, (e,e,e), 5H.” model.

|   | $L^i(\lambda^{ab})$ | $R^j(\lambda^{ca})$ | $H^k(\lambda^{bc})$ |
|---|-----------------|-----------------|-----------------|
| 0 | $\psi_{T^2}^{0,4}$ | $\psi_{T^2}^{0,4}$ | $\psi_{T^2}^{0,8}$ |
| 1 | $\frac{1}{\sqrt{2}}(\psi_{T^2}^{1,4} + \psi_{T^2}^{3,4})$ | $\frac{1}{\sqrt{2}}(\psi_{T^2}^{1,4} + \psi_{T^2}^{3,4})$ | $\frac{1}{\sqrt{2}}(\psi_{T^2}^{1,8} + \psi_{T^2}^{7,8})$ |
| 2 | $\psi_{T^2}^{2,4}$ | $\psi_{T^2}^{2,4}$ | $\frac{1}{\sqrt{2}}(\psi_{T^2}^{2,8} + \psi_{T^2}^{6,8})$ |
| 3 | $\sqrt{2}(\psi_{T^2}^{1,4} \psi_{T^2}^{3,4} + \psi_{T^2}^{1,8} \psi_{T^2}^{7,8})$ | $\sqrt{2}(\psi_{T^2}^{1,4} \psi_{T^2}^{3,4} + \psi_{T^2}^{1,8} \psi_{T^2}^{7,8})$ | $\sqrt{2}(\psi_{T^2}^{2,8} \psi_{T^2}^{6,8})$ |
| 4 | $\sqrt{2}(\psi_{T^2}^{2,4} \psi_{T^2}^{4,8} + \psi_{T^2}^{4,8} \psi_{T^2}^{4,8})$ | $\sqrt{2}(\psi_{T^2}^{2,4} \psi_{T^2}^{4,8} + \psi_{T^2}^{4,8} \psi_{T^2}^{4,8})$ | $\sqrt{2}(\psi_{T^2}^{2,8} \psi_{T^2}^{6,8})$ |

where

\[
Y^{ij0} = c_{4-4-8} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad Y^{ij1} = c_{4-4-8} \begin{pmatrix} X_3 \\ X_4 \end{pmatrix}, \quad Y^{ij2} = c_{4-4-8} \begin{pmatrix} \sqrt{2}X_1 \\ \frac{1}{\sqrt{2}}(X_0 + X_2) \end{pmatrix}, \quad Y^{ij3} = c_{4-4-8} \begin{pmatrix} X_4 \\ X_3 \end{pmatrix}, \quad Y^{ij4} = c_{4-4-8} \begin{pmatrix} X_2 \\ X_1 \\ X_0 \end{pmatrix},
\]

with

\[
X_0 = \eta_0 + 2\eta_{32} + \eta_{64}, \\
X_1 = \eta_8 + \eta_{24} + \eta_{40} + \eta_{50}, \\
X_2 = 2(\eta_{16} + \eta_{48}), \\
X_3 = \eta_4 + \eta_{28} + \eta_{36} + \eta_{60}, \\
X_4 = \eta_{12} + \eta_{20} + \eta_{44} + \eta_{52}.
\]

Here, we have used the notation,

\[
\eta_N = \vartheta \left[ \frac{N}{128} \right] (0, 128\tau).
\]

Under modular transformation, these Yukawa couplings $Y^{ijk}$ are transformed as follows:

\[
Y^{ijk} \rightarrow \left( J_{1/2}(\gamma, \tau) \tilde{\rho}_{1}^{i}(\gamma) \right) \left( J_{1/2}(\gamma, \tau) \tilde{\rho}_{2}^{j}(\gamma) \right) \left( J_{1/2}(\gamma, \tau) \tilde{\rho}_{3}^{k}(\gamma) \right)^* Y^{i'j'k'}, \tag{63}
\]
where \( \tilde{\gamma} \in \tilde{\Gamma} \) and the unitary representations \( \tilde{\rho}_4 \) and \( \tilde{\rho}_8 \) are generated by

\[
\tilde{\rho}_4(\tilde{S}) = \frac{e^{\pi i/4}}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}, \quad \tilde{\rho}_8(\tilde{S}) = \frac{e^{\pi i/4}}{2\sqrt{2}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 0 & -2 & 0 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} \\ 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 1 \end{pmatrix},
\]

(64)

\[
\tilde{\rho}_4(\tilde{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\pi i/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \tilde{\rho}_8(\tilde{T}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\pi i/8} & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & e^{-\pi i/8} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\]

(65)

In what follows, we assume both up and down Yukawa matrices for quarks are given by Eq. (62). We also assume Higgs VEV directions for up and down sectors are independent. Otherwise, we cannot derive realistic results. In particular, the quark mixing can be realized by taking different Higgs VEV directions for the up and down sectors.

### 6.2 Quark flavors at \( \tau = i \)

In this subsection, we show numerical studies on the model “4-4-8, (e,e,e), 5H” at \( \tau = i \) where Yukawa matrices are restricted by \( S \)-invariance. First we assume that the vacuum is \( S \)-invariant. Then we search the Higgs VEV directions leading to rank one quark mass matrix on \( S \)-invariant vacuum. The rank one matrix is favorable in the limit that we neglect masses of the first and second generations. However, we need a small deviation from the \( S \)-invariant vacuum to realize non-vanishing masses of two light generations. That is, we could realize quark masses and mixing angles at a point close to the \( S \)-invariant vacuum. As an illustrating example, we show that the Fritzch-Xing mass matrix can be realized on such a vacuum. We also show numerical results.

#### 6.2.1 \( S \)-invariance and rank one directions

At \( \tau = i \), \( S \)-transformations for Yukawa couplings in Eq. (64) are diagonalized into

\[
O_4^T \tilde{\rho}_4(\tilde{S})O_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad O_8^T \tilde{\rho}_8(\tilde{S})O_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},
\]

(66)

On rank one directions, we can also realize small but nonzero up (down) and charm (strange) quarks masses by slightly shifting the value of the modulus \( \tau \) from fixed points instead of the shifting of the directions of Higgs VEVs.
where $O_4$ and $O_8$ are orthogonal matrices to diagonalize $\tilde{\rho}_4$ and $\tilde{\rho}_8$. These diagonalizations are consistent with the transformation in Eq. (42). Note that there are degrees of freedom on the choice of $S$-transformation eigenbasis because of its degeneracy. Without loss of generality, it is possible to choose $S$-transformation eigenbasis such that Yukawa matrices,

$$\hat{Y}^{ijk} = [O^T_4]^{i i'}[O^T_4]^{j j'}[O^T_8]^{k k'}Y^{i' j' k'},$$

are expressed as

$$\hat{Y}^{ij0} = \begin{pmatrix} 1.00 & -0.0839 & 0 \\ -0.0839 & 0.00704 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{Y}^{ij1} = \begin{pmatrix} -0.0572 & -0.248 & 0 \\ -0.248 & -0.943 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{Y}^{ij2} = \begin{pmatrix} 0.0683 & -0.301 & 0 \\ -0.301 & 0.281 & 0 \\ 0 & 0 & 0.844 \end{pmatrix}, \quad \hat{Y}^{ij3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.636 \\ 0 & -0.636 & 0 \end{pmatrix},$$

$$\hat{Y}^{ij4} = \begin{pmatrix} 0 & 0 & 0.602 \\ 0 & 0 & -0.158 \\ 0.602 & -0.158 & 0 \end{pmatrix}.$$  

As shown in Table 8, this model has the Higgs VEV directions leading to rank one mass matrix in both of $S$-invariant and not $S$-eigenstates directions. In our numerical studies, we assume an almost $S$-invariant vacuum. We calculate the absolute values of the CKM matrix elements as well as the mass ratios of the quarks near the $S$-invariant Higgs VEV direction which lead to rank one mass matrix. On the $S$-transformation eigenbasis in Eq. (68), we can find that one of such $S$-invariant Higgs VEV direction is given by

$$\langle \hat{H}^k \rangle \equiv [O^T_8]^{kk'}\langle H^{k'} \rangle = (1, 0, 0, 0).$$

### 6.2.2 Illustrating example: Fritzch-Xing mass matrix

In the model “4-4-8, (e,e,e), 5H”, the mass matrix is symmetric. Here, we assume the mass matrix such as

$$M_u = \begin{pmatrix} A & B & 0 \\ B & D & C \\ 0 & C & 0 \end{pmatrix}, \quad M_d = \begin{pmatrix} A' & B' & 0 \\ B' & D' & C' \\ 0 & C' & 0 \end{pmatrix},$$

where $A-D$ and $A'-D'$ are real values. Such mass matrices can be realized by the appropriate linear combination of Yukawa matrices in Eq. (68). Note that we have used the flavor basis such that the (1,1) entry is the largest. For convenience, we redefine the mass matrix for up sector, $M_u$, as

$$M_u \to M_u^{(h)} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} M_u \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C & 0 \\ C & D & B \\ 0 & B & A \end{pmatrix}.$$
As the same way, we can obtain

$$M_d^{(h)} = \begin{pmatrix} 0 & C' & 0 \\ C' & D' & B' \\ 0 & B' & A' \end{pmatrix},$$

(72)

for down sector. These redefined mass matrices are the so-called Fritzch-Xing mass matrices.

Here we realize quark masses and mixing angles based on the Fritzch-Xing mass matrix. To realize the Fritzch-Xing mass matrix, first, we parametrize the Higgs VEV direction by polar coordinates ($\theta, \phi$) as

$$\langle \hat{H}^k_{u,d} \rangle = v_{u,d}(\cos \theta_{u,d}, \sin \theta_{u,d} \cos \phi_{u,d}, \sin \theta_{u,d} \sin \phi_{u,d}, 0).$$

(73)

Note that we take the third and fifth VEVs into zero to construct Fritzch-Xing mass matrix. Then, quark mass matrices take the forms as in Eq. (70).

Next, to realize the quark flavors at $\tau = i$, we choose the following parameters:

$$\begin{cases} 
\theta_u, \phi_u = (0.00838, -0.0251) \\
\theta_d, \phi_d = (-0.0427, 0.346)
\end{cases}$$

(74)

The Higgs VEV direction is given by

$$\begin{cases} 
\langle \hat{H}_u^k \rangle = v_u(1.00, 0.00838, 0, -0.000211, 0) \\
\langle \hat{H}_d^k \rangle = v_d(0.999, -0.0402, 0, -0.0145, 0)
\end{cases}$$

(75)

which are the directions very close to the rank one in Eq. (69). Then mass matrices for up and down quarks are given by

$$M_u^{ij} = y^{ijk}\langle \hat{H}_u^k \rangle = \begin{pmatrix}
1.00 & -8.60 \times 10^{-2} & 0 \\
-8.60 \times 10^{-2} & -8.53 \times 10^{-4} & 1.34 \times 10^{-4} \\
0 & 1.34 \times 10^{-4} & 0
\end{pmatrix},$$

(76)

$$M_d^{ij} = y^{ijk}\langle \hat{H}_d^k \rangle = \begin{pmatrix}
1.00 & -7.39 \times 10^{-2} & 0 \\
-7.39 \times 10^{-2} & 4.49 \times 10^{-2} & 9.20 \times 10^{-3} \\
0 & 9.20 \times 10^{-3} & 0
\end{pmatrix}.$$

(77)

We can obtain the mass ratios of the quarks and the absolute values of the CKM matrix elements as shown in Table [1].

---

3The Fritzch-Xing mass matrix can be obtained by another type of string compactification [83, 84].
Table 11: The mass ratios of the quarks and the absolute values of the CKM matrix elements at $\tau = i$ under the Higgs vacuum in Eq. (75). Comparison values of mass ratios are shown in Ref [86]. Ones of the CKM matrix elements are shown in Ref [87].

### 6.3 Quark flavors at $\tau = \omega$

In this subsection, we show another numerical example on the model “4-4-8, (e,e,e), 5H” at $\tau = \omega$ where Yukawa matrices are restricted by $ST$-invariance. First we assume that the vacuum is $ST$-invariant. Then we search the Higgs VEV directions leading to rank one quark mass matrix on $ST$-transformation invariant vacuum. The rank one matrix is favorable in the limit that we neglect masses of the first and second generations. However, as same as the studies at $\tau = i$, we need a small deviation from the $ST$-invariant vacuum to realize non-vanishing masses of two light generations. That is, we could realize quark masses and mixing angles at a point close to the $ST$-invariant vacuum. As an illustrating example, we show that the Fritzch mass matrix can be realized on such a vacuum. We also show numerical results.

#### 6.3.1 $ST$-invariance and rank one directions

At $\tau = \omega$, $ST$-transformations for Yukawa couplings which are given by a product of Eqs. (64) and (65) are diagonalized into

$$
U_4^\dagger \tilde{\rho}_4 (\tilde{S}\tilde{T}) U_4 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}, 
U_8^\dagger \tilde{\rho}_8 (\tilde{S}\tilde{T}) U_8 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \omega & 0 \\
0 & 0 & 0 & \omega^2
\end{pmatrix},
$$

(78)

where $U_4$ and $U_8$ are unitary matrices to diagonalize $\tilde{\rho}_4$ and $\tilde{\rho}_8$. These diagonalizations are consistent with the transformation in Eq. (49). Note that there are degrees of freedom on the choice of $ST$-transformation eigenbasis because of its degeneracy. Without loss of generality, it is possible to choose $ST$-transformation eigenbasis such that Yukawa matrices,

$$
\hat{Y}^{ijk} = [U_4^\dagger]^{ii'}[U_4^\dagger]^{jj'}[U_8^T]^{kk'} Y^{i'i'j'k'},
$$

(79)
are expressed as

\[
\hat{Y}_{ij0} = \begin{pmatrix}
0.9535 + 0.04357i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{Y}_{ij1} = \begin{pmatrix}
0.2852 - 0.1027i & 0 & 0 \\
0 & 0.8093 - 0.0005968i & 0 \\
0 & 0.8093 - 0.0005968i & 0
\end{pmatrix},
\]

\[
\hat{Y}_{ij2} = \begin{pmatrix}
-0.6454 - 0.06436i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{Y}_{ij3} = \begin{pmatrix}
0.1615 + 0.1576i & 0 & 0 \\
0 & 0 & -0.6802 - 0.5248i \\
0 & 0 & 0.4039 + 0.08034i
\end{pmatrix},
\]

\[
\hat{Y}_{ij4} = \begin{pmatrix}
0.4039 + 0.08034i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

(80)

As shown in Table 9, this model has the Higgs VEV directions leading to rank one mass matrix in both of ST-invariant and \(\omega\)-eigenstates directions. In our numerical studies, we assume an almost ST-invariant vacuum. We calculate the absolute values of the CKM matrix elements as well as the mass ratios of the quarks close to the ST-invariant Higgs VEV direction which lead to rank one mass matrix. On the ST-transformation eigenbasis in Eq. (80), we can find that one of such ST-invariant Higgs VEVs is given by

\[
\langle \hat{H}^k \rangle \equiv [U^\dagger_8]^{kk'} \langle H^{k'} \rangle = (1, 0, 0, 0, 0).
\]

(81)

### 6.3.2 Illustrating example: the Fritzch mass matrix

Here, we assume the mass matrix such as

\[
M_u = \begin{pmatrix} A & B & 0 \\ B & 0 & C \\ 0 & C & 0 \end{pmatrix}, \quad M_d = \begin{pmatrix} A' & B' & 0 \\ B' & 0 & C' \\ 0 & C' & 0 \end{pmatrix},
\]

(82)

where \(A-C\) and \(A'-C'\) are complex values. Such mass matrices can be realized by the appropriate linear combination of Yukawa matrices in Eq. (80). Note again that we have used the flavor basis such that the (1,1) entry is the largest. For convenience, we redefine the mass matrix for up sector, \(M_u\), as

\[
M_u \rightarrow M_u^{(h)} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} M_u \begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{iy} & 0 \\ 0 & 0 & e^{iz} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C e^{iy} & 0 \\ C e^{iz} & 0 & B e^{ix} \\ 0 & B e^{iy} & A e^{iz} \end{pmatrix},
\]

(83)
where \( x, y \) and \( z \) are fixed by
\[
    x = -\text{Arg}(A), \quad y = \text{Arg}(A) - 2\text{Arg}(B), \quad z = -\text{Arg}(A) + 2\text{Arg}(B) - 2\text{Arg}(C).
\] (84)

Then, redefined mass matrix is given by
\[
    M_u^{(h)} = \begin{pmatrix}
        0 & Ce^{i\text{Arg}(A) - 2i\text{Arg}(B)} & 0 \\
        (Ce^{i\text{Arg}(A) - 2i\text{Arg}(B)})^* & 0 & Be^{-i\text{Arg}(A)} \\
        0 & (Be^{-i\text{Arg}(A)})^* & |A|
    \end{pmatrix},
\] (85)
and this is a hermitian matrix. As the same way, we can obtain the hermitian mass matrix for down sector:
\[
    M_d^{(h)} = \begin{pmatrix}
        0 & C'e^{i\text{Arg}(A') - 2i\text{Arg}(B')} & 0 \\
        (C'e^{i\text{Arg}(A') - 2i\text{Arg}(B'))^* & 0 & B'e^{-i\text{Arg}(A')} \\
        0 & (B'e^{-i\text{Arg}(A')})^* & |A'|
    \end{pmatrix}.
\] (86)

These redefined mass matrices are the so-called Fritzch mass matrices.

Here we realize quark masses and mixing angles based on the Fritzch mass matrix. To obtain Fritzch mass matrices, first, we parametrize the Higgs VEV direction by polar coordinates \((\theta, \phi)\) as
\[
    \langle \hat{H}^k_{u,d} \rangle = v_{u,d}(\cos \theta_{u,d}, \sin \theta_{u,d} \cos \phi_{u,d}, \sin \theta_{u,d} \sin \phi_{u,d}, 0, 0).
\] (87)

Note that we take the fourth and fifth VEVs into zero to construct Fritzch mass matrix. Then, quark mass matrices take the forms as in Eq. (82) and they can always be rewritten as Fritzch mass matrices by the appropriate transformations.

Next, to realize the quark masses and mixing angles at \( \tau = \omega \), we choose the following parameters:
\[
    \left\{ \begin{array}{l}
        (\theta_u, \phi_u) = (0.07854, 1.574) \\
        (\theta_d, \phi_d) = (0.1414, 1.558)
    \end{array} \right. .
\] (88)

The Higgs VEV direction is given by
\[
    \left\{ \begin{array}{l}
        \langle \hat{H}^k_u \rangle = v_u(0.9969, -0.0002465, 0.07846, 0, 0) \\
        \langle \hat{H}^k_d \rangle = v_d(0.9900, 0.001771, 0.1409, 0, 0)
    \end{array} \right.,
\] (89)
which are the directions close to the rank one in Eq. (81). Then mass matrices for up and down
quarks are given by

\[ M^{ij}_u = \hat{Y}^{ijk}(\hat{H}^k_u) \]

\[
= \begin{pmatrix}
0.9505 + 0.04346i & -0.05064 - 0.005050i & 0 \\
-0.05064 - 0.005050i & 0 & -(1.995 - 0.001471i) \times 10^{-4} \\
0 & -(1.995 - 0.001471i) \times 10^{-4} & 0 \\
\end{pmatrix},
\]

(90)

\[ M^{ij}_d = \hat{Y}^{ijk}(\hat{H}^k_d) \]

\[
= \begin{pmatrix}
0.9445 + 0.04296i & -0.09093 - 0.009068i & 0 \\
-0.09093 - 0.009068i & 0 & (1.433 - 0.001057) \times 10^{-3} \\
0 & (1.433 - 0.001057) \times 10^{-3} & 0 \\
\end{pmatrix}.
\]

(91)

We can obtain the mass ratios of the quarks and the absolute values of the CKM matrix elements as shown in Table 12.

| (m_u, m_c, m_t) / m_t | Obtained values \( (1.52 \times 10^{-5}, 2.86 \times 10^{-3}, 1) \) | Comparison values \( (5.58 \times 10^{-6}, 2.69 \times 10^{-3}, 1) \) |
|----------------------|---------------------------------|---------------------------------|
| (m_d, m_s, m_b) / m_b | (2.37 \times 10^{-4}, 9.41 \times 10^{-3}, 1) | (6.86 \times 10^{-4}, 1.37 \times 10^{-2}, 1) |
| | \( |V_{CKM}| \equiv |(U^u_L)^\dagger U^d_L| \) | \( |V_{CKM}| \equiv |(U^u_L)^\dagger U^d_L| \) |
| | \( 0.974 \ 0.228 \ 0.00292 \) | \( 0.974 \ 0.227 \ 0.00361 \) |
| | \( 0.228 \ 0.973 \ 0.0421 \) | \( 0.226 \ 0.973 \ 0.0405 \) |
| | \( 0.00677 \ 0.0416 \ 0.999 \) | \( 0.00854 \ 0.0398 \ 0.999 \) |

Table 12: The mass ratios of the quarks and the absolute values of the CKM matrix elements at \( \tau = \omega \) under the vacuum alignments of Higgs fields in Eq. (89). Comparison values of mass ratios are shown in Ref \[86\]. Ones of the CKM matrix elements are shown in Ref \[87\].

As the results, we can obtain realistic quark mass ratios and mixing on the model “4-4-8, (e,e,e), 5 H” at both of \( \tau = i \) and \( \tau = \omega \) by choosing appropriate Higgs VEV directions. As illustrating examples, we have used the Fritzch and Fritzch-Xing mass matrices, but we can obtain realistic values of quark masses and mixing angles with other forms of mass matrices around the \( S \)-invariant vacuum and \( ST \)-invariant vacuum. It is also possible to study other three-generation magnetized orbifold models.

7 Conclusion

In this paper, we have studied the forms of Yukawa matrices in magnetized orbifold models. In particular, we focus on the forms at three modular fixed points, \( \tau = i, \omega \) and \( i\infty \). Consequently
we have found that Yukawa matrices have a kind of texture structures although ones at \( \tau = i \omega \) are not realistic. Therefore we have classified Yukawa textures at \( \tau = i \) and \( \omega \).

By choosing appropriate Higgs VEV directions, Yukawa textures classified in this paper can lead to mass matrix whose rank is one. The rank one mass matrix is favorable in the limit that we neglect masses of the first and second generations. We have also investigated the conditions such that the quark mass matrix constructed by Yukawa textures becomes rank one matrix. Then we have found that rank one directions exist on \( S \)-invariant and \( ST \)-invariant vacua in several three-generation models. Thus it is possible to realize the large hierarchy of quark masses if we assume that vacuum has \( S \)-invariance or \( ST \)-invariance approximately. These invariances need to break slightly to shift the Higgs VEV directions from rank one directions since the first and second generation quarks have small but nonzero masses.

Here, we have given numerical studies on the model “4-4-8, (e,e,e), 5H” at both of \( \tau = i \) and \( \omega \), and assumed almost \( S \)-invariant and \( ST \)-invariant vacua to reproduce the quark masses and mixing angles. As illustrating examples, we have shown Fritzch-Xing and Fritzch mass matrices can be realized from Yukawa textures at \( \tau = i \) and \( \omega \), respectively. Not only these forms, but also other forms of quark mass matrices can lead to the realistic mass ratios of quarks and values of the CKM matrix elements around the \( S \) and \( ST \)-invariant vacua. Also, other three-generation magnetized orbifold models are interesting.

Also we can extend our studies to the realization of lepton flavors. The charged lepton masses are given by Dirac mass matrix as the quarks, but we need to study Majorana masses for the neutrino sector. For example, in [88], Majorana masses for right-handed neutrino induced by non-perturbative effects of D-brane instanton effects were studied systematically in magnetized orbifold models. We would also study it and examine the realization of both quark and lepton flavors elsewhere.

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Appendix

A Proof: rank one conditions at \( \tau = i \)

Here we prove the conditions that mass matrix becomes rank one at \( \tau = i \). As shown in section 5.1 there are five conditions denoted as I, II, III, IV and V to realize rank one mass matrix. Under each condition, we show the existences of Higgs VEVs \( \langle H^k \rangle = v^k \) such that mass matrix \( M^{ij} = Y^{ijk} v^k \) becomes rank one. Here and hereafter, we use \( c_k, k \in \mathbb{Z} \) as any constant value.

In Table B3 we show the forms of rank one mass matrices realized on each condition. This
Table 13: Rank one mass matrices realized on each condition. The second column shows one of realized rank one matrices whose elements satisfy Eqs. (92)-(95) to realize rank one, of course other rank one matrices can be constructed. The third column shows textures included in each condition.

| Condition | Mass Matrix | 1st Column | 2nd Column |
|-----------|-------------|------------|------------|
| I         | \[
\begin{pmatrix}
M^{00} & M^{01} \\
M^{10} & M^{11}
\end{pmatrix} = \begin{pmatrix} 0 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
* & * \\
0 & 0
\end{pmatrix}
\times 3,
\begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}\] | 1 (symmetric) |
| II (symmetric) | \[
\begin{pmatrix}
M^{00} & M^{01} & M^{02} \\
M^{10} & M^{11} & M^{12}
\end{pmatrix} = \begin{pmatrix} * & * & 0 \\
0 & 0 & *
\end{pmatrix}\times 3,
\begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}\times 2,
\begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}\times 1
\] | 2 (non-symmetric) |
| III | \[
\begin{pmatrix}
M^{00} & M^{01} & M^{02} \\
M^{10} & M^{11} & M^{12}
\end{pmatrix} = \begin{pmatrix} * & * & 0 \\
0 & 0 & *
\end{pmatrix}\times 2,
\begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}\times 2
\] | |
| IV | \[
\begin{pmatrix}
M^{00} & M^{01} & M^{02} \\
M^{10} & M^{11} & M^{12}
\end{pmatrix} = \begin{pmatrix} * & * & 0 \\
0 & 0 & *
\end{pmatrix}\times 2,
\begin{pmatrix}
0 & 0 \\
* & 0
\end{pmatrix}\times 3
\] | |
| V | \[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix} 0 & 0 \\
0 & 0
\end{pmatrix}\times 3
\] | |

In what follows, we will check the above equations are satisfied by the textures on each condition shown in Table 13. Note that then the normalization condition of Higgs VEVs, \( \sum_k |v_k|^2 = \langle H \rangle^2 \), is also satisfied.
A.1 Condition I

In this condition, mass matrix can be expanded as

\[ M^{ij} = Y^{ijk}v^k = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^0 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^1 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^2, \]

where Yukawa matrices \( Y^{ijk} \) correspond to S-even textures. The rank one equations in Eq. (92) require the following conditions:

\[ M^{22} = Y^{22k}v^k = 0, \quad M^{00}M^{11} - M^{01}M^{10} = (Y^{00k}v^k)(Y^{11k}v^k) - (Y^{01k}v^k)(Y^{10k}v^k) = 0. \]

The first equation means that \( v^2 \) is given by the linear combination of \( v^0 \) and \( v^1 \). Then second equation becomes the quadratic equation for \( v^1/v^0 \in \mathbb{C} \) and we can always find the solution to this equation. Thus we can obtain \( (v^0, v^1, v^2) \) satisfying the normalization condition and rank one condition.

A.2 Condition II (symmetric), III

First we consider the condition II (symmetric). In this condition, mass matrix can be expanded as

\[ M^{ij} = Y^{ijk}v^k = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^0 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^1 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} v^3, \]

where Yukawa matrices \( Y^{ij0}, Y^{ij1}, Y^{ij2} \) correspond to S-even textures and \( Y^{ij3} \) corresponds to S-odd texture. The rank one equations in Eq. (93) require the following conditions:

\[ Y^{123}(Y^{000} + Y^{001}(v^1/v^0) + Y^{002}(v^2/v^0)) + Y^{023}(Y^{100} + Y^{101}(v^1/v^0) + Y^{102}(v^2/v^0)) = 0, \]

\[ Y^{123}(Y^{010} + Y^{011}(v^1/v^0) + Y^{012}(v^2/v^0)) + Y^{023}(Y^{110} + Y^{111}(v^1/v^0) + Y^{112}(v^2/v^0)) = 0. \]

The first and second equations are linear equations for \( (v^1/v^0) \) and \( (v^2/v^0) \) and we can always find the solutions. The third equation leads to \( v^0 = c_1 v^3 \) and \( v^3 \) is determined by the normalization condition. Thus we can obtain \( (v^0, v^1, v^2, v^3) \) satisfying the normalization condition and rank one condition.

Next we consider the condition III. In this condition, mass matrix can be expanded as

\[ M^{ij} = Y^{ijk}v^k = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^0 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^1 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} v^3. \]
where Yukawa matrices $Y^{ij0}$, $Y^{ij1}$ correspond to $S$-even textures and $Y^{ij2}$, $Y^{ij3}$ correspond to $S$-odd textures. The rank one equations in Eq. require the following conditions:

\[\begin{align*}
(Y^{000} + Y^{001}(v^1/v^0))(Y^{122} + Y^{123}(v^3/v^2)) &= (Y^{022} + Y^{023}(v^3/v^2))(Y^{100} + Y^{101}(v^1/v^0)), \\
(Y^{000} + Y^{001}(v^1/v^0))(Y^{110} + Y^{111}(v^1/v^0)) &= (Y^{100} + Y^{101}(v^1/v^0))(Y^{100} + Y^{101}(v^1/v^0)), \\
(Y^{000}v^0 + Y^{001}v^1)(Y^{222} + Y^{223}(v^3/v^2)) &= v^2(Y^{202} + Y^{203}(v^3/v^2))(Y^{022} + Y^{023}(v^3/v^2)).
\end{align*}\]

The first equation is a quadratic equation for $v^1/v^0 \in \mathbb{C}$ and it is possible to find the solution $v^1 = c_1 v^0$. The second equation is a linear equation for $v^3/v^2 \in \mathbb{C}$ and the solution $v^3 = c_2 v^2$ exists. The third equation leads to the solution $v^0 = c_3 v^2$ and $v^2$ is determined by the normalization condition. Thus we can obtain $(v^0, v^1, v^2, v^3)$ satisfying the normalization condition and rank one condition.

### A.3 Condition II (non-symmetric), IV

First we consider the condition II (non-symmetric). In this condition, the mass matrix can be expanded as

\[M^{ij} = Y^{ijk}v^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v^0 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v^1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v^3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v^4,\]

where Yukawa matrices $Y^{ij0}$, $Y^{ij1}$, $Y^{ij2}$ correspond to $S$-even textures and $Y^{ij3}$, $Y^{ij4}$ correspond to $S$-odd textures. Note that we have chosen two of three Higgs basis corresponding to $S$-invariant textures and two fermion basis corresponding to $S$-invariant states to make $(1,1)$, (1,2) and (2,1) elements of the first Yukawa matrix be zero. The rank one equations in Eq. require the following conditions:

\[\begin{align*}
Y^{001} + Y^{002}(v^2/v^1) &= Y^{011} + Y^{012}(v^2/v^1) \\
Y^{101} + Y^{102}(v^2/v^1) &= Y^{110}(v^0/v^1) + Y^{111} + Y^{112}(v^2/v^1) \\
Y^{001} + Y^{002}(v^2/v^1) &= Y^{023} + Y^{024}(v^4/v^3) \\
Y^{101} + Y^{102}(v^2/v^1) &= Y^{123} + Y^{124}(v^4/v^3) \\
Y^{001} + Y^{002}(v^2/v^1) &= Y^{011} + Y^{012}(v^2/v^1) \\
Y^{203} + Y^{204}(v^4/v^3) &= Y^{213} + Y^{214}(v^4/v^3) \\
(v^1/v^3)Y^{001} + Y^{002}(v^2/v^1) &= (v^3/v^1)Y^{023} + Y^{024}(v^4/v^3) \\
Y^{203} + Y^{204}(v^4/v^3) &= Y^{220}(v^0/v^1) + Y^{221} + Y^{222}(v^2/v^1).
\end{align*}\]

The first equation means that $(v^0/v^1)$ is determined by $(v^2/v^3)$. The second and third equations lead to

\[\frac{(v^2/v^1)}{c_3 + c_4(v^4/v^3)} = \frac{c_5 + c_6(v^4/v^3)}{c_7 + c_8(v^4/v^3)}.\]
This is a quadratic equation for \((v^4/v^3) \in \mathbb{C}\) and it is possible to find the solution. That is, we can obtain \((v^4/v^3), (v^2/v^1)\) and \((v^0/v^1)\). Then the fourth equation leads to \(v^3 = c_0 v^1\) and \(v^1\) is determined by the normalization condition. Thus we can obtain \((v^0, v^1, v^2, v^3, v^4)\) satisfying the normalization condition and rank one condition.

Next we consider the condition IV. In this condition, similar to Eq. (107), the mass matrix can be expanded as

\[
M^{ij} = Y^{ijk}v^k = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} v^0 + \begin{pmatrix}
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
\end{pmatrix} v^1 + \begin{pmatrix}
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
\end{pmatrix} v^2 + \begin{pmatrix}
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
\end{pmatrix} v^3 + \begin{pmatrix}
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
\end{pmatrix} v^4,
\]

(113)

where Yukawa matrices \(Y^{ij0}, Y^{ij1}\) correspond to \(S\)-even textures and \(Y^{ij2}, Y^{ij3}, Y^{ij4}\) correspond to \(S\)-odd textures. The rank one equations in Eq. (94) require the following conditions:

\[
\begin{align*}
Y^{001} &= Y^{011} - Y^{110}(v^0/v^1) - Y^{111} \\
Y^{001} &= Y^{022} + Y^{023}(v^3/v^2) + Y^{024}(v^4/v^2) \\
Y^{101} &= Y^{122} + Y^{123}(v^3/v^2) + Y^{124}(v^4/v^2) \\
Y^{202} + Y^{203}(v^3/v^2) + Y^{204}(v^4/v^2) &= Y^{212} + Y^{213}(v^3/v^2) + Y^{214}(v^4/v^2) \\
(v^1/v^2)Y^{202} + Y^{203}(v^3/v^2) + Y^{204}(v^4/v^2) &= (v^2/v^1)Y^{022} + Y^{023}(v^3/v^2) + Y^{024}(v^4/v^2) \\
\end{align*}
\]

(114) (115) (116) (117)

The first equation determines \((v^0/v^1)\). The second and third equations determine \((v^3/v^2)\) and \((v^4/v^2)\). Then the third equation leads to \(v^1 = c_1 v^2\) and \(v^2\) is determined by the normalization condition. Thus we can obtain \((v^0, v^1, v^2, v^3, v^4)\) satisfying the normalization condition and rank one condition.

### A.4 Condition V

In this condition, mass matrix can be expanded as

\[
M^{ij} = Y^{ijk}v^k = \begin{pmatrix}
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
\end{pmatrix} v^0 + \begin{pmatrix}
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
\end{pmatrix} v^1 + \begin{pmatrix}
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
\end{pmatrix} v^2,
\]

(118)

where Yukawa matrices \(Y^{ijk}\) correspond to \(S\)-odd textures. The rank one equations in Eq. (95) require the following conditions:

\[
\begin{align*}
M^{02} &= Y^{020}v^0 + Y^{021}v^1 + Y^{022}v^2 = 0, \\
M^{12} &= Y^{120}v^0 + Y^{121}v^1 + Y^{122}v^2 = 0.
\end{align*}
\]

(119) (120)

The first equation means that \(v^2\) is given by the linear combination of \(v^0\) and \(v^1\). Then second equation leads to \(v^1 = c_1 v^0\) and \(v^0\) is determined by the normalization condition. Thus we can obtain \((v^0, v^1, v^2)\) satisfying the normalization condition and rank one condition.
B Proof: rank one conditions at $\tau = \omega$

As shown in section 5.2, there are five conditions denoted as I, II, III, IV and V to realize rank one mass matrix at $\tau = \omega$. We prove these rank one conditions in a way similar to Appendix A.

In Table 14, we show the form of rank one mass matrices realized on each condition. This table shows there are one (I, II, III (symmetric)), two (I, II, III (non-symmetric)) and four (IV, V) equations in each condition as follows,

$$I : M_{12} = M_{21} = 0,$$
$$II : M_{01} = M_{10} = 0,$$
$$III : M_{02} = M_{20} = 0,$$
$$IV (symmetric) : M_{00} M_{11} = M_{01} M_{02} = M_{20} = M_{22},$$
$$V : \begin{cases} M_{12} = M_{21} = M_{01} = M_{22} = 0, \\ M_{12} = M_{21} = M_{02} = M_{11} = 0, \\ M_{01} = M_{22} = M_{02} = M_{11} = 0. \end{cases}$$

B.1 Condition I, II, III

Here we prove only the condition I because the conditions II and III can be proved in a similar way. In the condition I, the mass matrix can be expanded as

$$M_{ij} = Y_{ijk} v^k = \begin{cases} \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 & \text{(symmetric)} \\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^2 & \text{(non-symmetric)} \end{cases},$$

where Yukawa matrices $Y_{ijk}$ correspond to $ST$-invariant textures. The rank one equations in Eq. (121) require the following conditions:

$$\begin{cases} M_{12} = M_{21} = Y_{120} v^0 + Y_{121} v^1 = 0 \quad \text{(symmetric)} \\ M_{12} = Y_{120} v^0 + Y_{121} v^1 + Y_{122} v^2 = 0, \quad M_{21} = Y_{210} v^0 + Y_{211} v^1 + Y_{212} v^2 = 0 \quad \text{(non-symmetric)} \end{cases}.$$

These are linear equations for $v^k$ and we can find their solutions and the normalization condition. Thus we can obtain $v^k$ satisfying the normalization condition and rank one condition.
B.2 Condition IV (symmetric)

Here we prove only one of three condition IV (symmetric) cases in Table 14 because other two cases can be proved in a similar way. We prove the first case. In this case, the mass matrix can be expanded as

\[ M^{ij} = Y^{ijk} v^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} v^3, \quad (129) \]

where Yukawa matrices \( Y^{ij0} \), \( Y^{ij1} \) correspond to \( ST \)-invariant textures, \( Y^{ij2} \) corresponds to \( \omega \)-eigenstate texture and \( Y^{ij3} \) corresponds to \( \omega^2 \)-eigenstate texture. Note that we have chosen two Higgs basis corresponding to \( ST \)-invariant textures to make \((1,1)\) elements of the first Yukawa matrix be zero. The rank one equations in Eq. (124) require the following conditions,

\[ \begin{align*}
Y^{001} v^1 + Y^{102} v^2 &= Y^{012} v^3, \\
Y^{001} v^1 + Y^{023} v^3 &= Y^{120} v^0 + Y^{121} v^1, \\
Y^{001} v^1 + Y^{203} v^3 &= Y^{222} v^2.
\end{align*} \quad (130) \]

The first and second equations lead to \( v^1 = c_1 v^3 \) and \( v^2 = c_2 v^3 \). Then the third equation leads to \( v^3 = c_3 v^0 \) and \( v^0 \) is determined by the normalization condition. Thus we can obtain \((v^0, v^1, v^2, v^3)\) satisfying the normalization condition and rank one condition.

B.3 Condition IV (non-symmetric)

Here we prove only one of three condition IV (non-symmetric) cases in Table 14 because other two cases can be proved in a similar way. We prove the first case. In this case, similar to Eq. (129), the mass matrix can be expanded as

\[ M^{ij} = Y^{ijk} v^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} v^3, \quad (133) \]

where Yukawa matrices \( Y^{ij0} \), \( Y^{ij1} \) correspond to \( ST \)-invariant textures, \( Y^{ij2} \), \( Y^{ij3} \) correspond to \( \omega \)-eigenstate textures and \( Y^{ij4} \) corresponds to \( \omega^2 \)-eigenstate texture. The rank one equations
in Eq. (125) require the following conditions:

\[
\begin{align*}
Y^{001}v^1 &= Y^{012}v^2 + Y^{013}v^3, \\
Y^{102}v^2 + Y^{103}v^3 &= Y^{114}v^4, \\
Y^{001}v^1 &= Y^{012}v^2 + Y^{013}v^3, \\
Y^{102}v^2 + Y^{103}v^3 &= Y^{120}v^0 + Y^{121}v^1, \\
Y^{001}v^1 &= Y^{012}v^2 + Y^{013}v^3, \\
Y^{204}v^4 &= Y^{210}v^0 + Y^{211}v^1, \\
Y^{001}v^1 &= Y^{024}v^4, \\
Y^{204}v^4 &= Y^{223}v^3.
\end{align*}
\]

(134) (135) (136) (137)

The first and second equations lead to

\[
\frac{(v^2/v^3)}{c_3 + c_4(v^0/v^1)} = \frac{(v^4)^2}{v^1} = v^3(c_5(v^2/v^3) + c_6)(c_7(v^0/v^1) + c_8). \tag{138}
\]

On the other hand, the fourth equation leads to \(v^3 = c_9(v^4)^2/v^1\). Combining both results, we obtain a quadratic equation for \((v^0/v^1) \in \mathbb{C}\) and it is possible to find the solution \(v^0 = c_{10}v^1\). Then the third equation leads to \(v^1 = c_{11}v^4\) and \(v^4\) is determined by the normalization condition. Thus we can obtain \((v^0, v^1, v^2, v^3, v^4)\) satisfying the normalization condition and rank one condition.

**B.4 Condition V**

Here we prove only one of three condition V cases in Table 14 because other two cases can be proved in a similar way. We prove the first case. In this case, we can choose \(ST\)-eigenbasis on wavefunctions such that the mass matrix is expanded as

\[
M^{ij} = Y^{ijk}v^k = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^2 \\
+ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^3 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^4 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^5,
\]

(139)

where Yukawa matrices \(Y^{ij0}, Y^{ij1}, Y^{ij2}\) correspond to \(ST\)-invariant textures, \(Y^{ij3}, Y^{ij4}, Y^{ij5}\) correspond to \(\omega\)-eigenstate textures. The rank one equations in Eq. (126) require the following conditions:

\[
\begin{align*}
Y^{120}v^0 + Y^{121}v^1 + Y^{122}v^2 &= 0, \\
Y^{210}v^0 + Y^{211}v^1 + Y^{212}v^2 &= 0, \\
Y^{013}v^3 + Y^{014}v^4 + Y^{015}v^5 &= 0, \\
Y^{223}v^3 + Y^{224}v^4 + Y^{225}v^5 &= 0.
\end{align*}
\]

(140) (141) (142) (143)

There are four linear equations for six VEVs \((v^0, v^1, v^2, v^3, v^4, v^5)\). Thus we can obtain \((v^0, v^1, v^2, v^3, v^4, v^5)\) satisfying the normalization condition and rank one condition.
Table 14: Rank one mass matrices realized on each condition. The second column shows one of realized rank one matrices whose elements satisfy Eqs. (121)-(126) to realize rank one, of course other rank one matrices can be constructed. The third column shows textures included in each condition.
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