ON ALGEBRAS OF STRONGLY DERIVED UNBOUNDED TYPE

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Abstract. Let $A$ be a finite-dimensional algebra over an algebraically closed field. We prove $A$ is a strongly derived unbounded algebra if and only if there exists an integer $m$, such that $C_m(\text{proj } A)$, the category of all minimal projective complexes with degree concentrated in $[0, m]$, is of strongly unbounded type, which is also equivalent to the statement the repetitive algebra $\hat{A}$ is of strongly unbounded representation type. As a corollary, we can establish the dichotomy on the representation type of $C_m(\text{proj } A)$, the homotopy category $K^b(\text{proj } A)$ and the repetitive algebra $\hat{A}$.

Introduction

Throughout this article, $k$ is an algebraically closed field and all the algebras are associative finite dimensional connected basic $k$-algebras with identity. During the research of representation theory of algebras, one of the main topics is to study the representation type. As early as 1940s, Brauer and Thrall began the investigation of representation type of finite dimensional algebras [10, 25]. Jans formulated the first and second Brauer-Thrall conjectures for finite dimensional algebras in his paper [20], roughly speaking, the first Brauer-Thrall conjecture says that an algebra is of bounded representation type if and only if it is of finite representation type, whereas the second Brauer-Thrall conjecture states that the algebras of unbounded representation type are of strongly unbounded representation type. Here, we say an algebra is of bounded representation type if the dimensions of all indecomposable modules have a common upper bound, and of strongly unbounded representation type if there are infinitely many $d \in \mathbb{N}$ such that for each $d$, there exist infinitely many isomorphism classes of indecomposable modules of dimension $d$. The study of the Brauer-Thrall conjectures, to a large extent, stimulated the development of representation theory [3, 4, 21, 23, 24].

During the last years, the bounded derived categories of algebras have been studying extensively and play an important role in representation theory of finite-dimensional algebras. By a theorem from [18], there is a full embedding from the bounded derived category of a finite-dimensional algebra to the stable module category over its repetitive algebra, which is an equivalence if and only if the global dimension is finite. The theorem bridged together the bounded derived category and the module category, and hence provided a method to explore the property of bounded derived category of algebras in terms of their repetitive algebra, like the derived representation type [12, 13]. Moreover, the classification and distribution of indecomposable objects in the bounded derived category of an algebra are still important themes in representation theory of algebras. In this context, the definitive work was due to Vossieck [26]. He introduced and classified derived discrete algebras, i.e., the algebras whose bounded derived categories admit only finitely many isomorphism classes of indecomposable objects of arbitrarily given
cohomology dimension vector, and proved an algebra is derived discrete if and only if its repetitive algebra is discrete. Bautista [5] generalized the definition of derived discrete for the artin algebras. Motivated by Vossieck’s work, Han and Zhang introduced the cohomological range of a bounded complex, which leads to the concept of strongly derived unbounded algebras naturally. We say an algebra is strongly derived unbounded if there are infinitely many $r \in \mathbb{N}$ such that for each $r$, there exist infinitely many isomorphism classes of indecomposable object of cohomological range $r$ in its bounded derived category. Moreover, the authors proved an algebra is either derived discrete or strongly derived unbounded [17].

During the research of bounded derived category of algebras, a high emphasis has been placed another category, i.e., the category of all minimal complexes of finitely generated projective modules with degree concentrated in $[0, m]$, for any fixed integer $m \geq 0$, and we denote it by $C_m(\text{proj} \ A)$. Bautista, Souto Salorio and Zuazua described the AR-triangles in $C_m(\text{proj} \ A)$, and also observed their relation with the AR-triangles in $K^{-,b}(\text{proj} \ A)$, the homotopy category of all right bounded projective complexes with bounded cohomology [5]. Moreover in [5], Bautista established that, if $k$ is infinite, then a finite-dimensional $k$-algebra is derived discrete if and only if for any integer $m$, the category $C_m(\text{proj} \ A)$ does not contain generic objects. For the representation type, Bautista defined the finite, tame and wild representation type for $C_m(\text{proj} \ A)$, and then proved that $C_m(\text{proj} \ A)$ is either of tame representation type or of wild representation type [9]. Furthermore, $A$ is derived discrete if and only if $C_m(\text{proj} \ A)$ is of finite representation type for all $m$. In present paper, we first define the strongly unboundedness of the category $C_m(\text{proj} \ A)$ for any fixed integer $m$, and study the strongly unbounded algebras in terms of the associated category $C_m(\text{proj} \ A)$ and the representation type of repetitive algebras.

We prove the following

**Theorem.** Let $A$ be a finite-dimensional algebra. Then the following statements are equivalent

1. $A$ is strongly derived unbounded;
2. There exists an integer $m \geq 1$, such that the category $C_m(\text{proj} \ A)$ is of strongly unbounded type.
3. $K^b(\text{proj} \ A)$ is of strongly unbounded type;
4. The repetitive algebra $\hat{A}$ is of strongly unbounded representation type.

Consider the dichotomy theorem from [17], we know any algebra $A$ is derive discrete or strongly derived unbounded. Combined with the equivalent characterizations of derived discrete algebras from [6, 20], we can establish the dichotomy on the representation type of $C_m(\text{proj} \ A)$, the homotopy category $K^b(\text{proj} \ A)$ and the repetitive algebra $\hat{A}$ as a corollary.

**Corollary** Let $A$ be an algebra. Then we have

1. $C_m(\text{proj} \ A)$ is of finite representation type for any $m$, or there exists an integer $m' \geq 1$, such that $C_{m'}(\text{proj} \ A)$ is of strongly unbounded type.
2. $K^b(\text{proj} \ A)$ is either discrete or of strongly unbounded type;
3. The repetitive algebra $\hat{A}$ is either of discrete representation type or strongly unbounded representation type.

The present paper is organized as follows. In the first section, we define the strongly unboundedness of $C_m(\text{proj} \ A)$ and prove some basic lemmas. In section 2, we observe the strongly unboundedness of $C_m(\text{proj} \ A)$ under the derived equivalences and cleaving functors. Moreover, we study $C_m(\text{proj} \ A)$ for representation-infinite algebras, simply connected algebras and finally prove the main theorem.
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1. The strongly unboundedness of $C_m(\text{proj} \, A)$

1.1. Notations and definitions. Let $A$ be an algebra, and $\text{mod} \, A$ be the category of all finite-dimensional right $A$-modules and $\text{proj} \, A$ be its full subcategory consisting of all finitely generated projective right $A$-modules. Denote by $C(A)$ the category of all complexes of finite-dimensional right $A$-modules, and by $C^b(A)$ and $C^{-b}(A)$ its full subcategories consisting of all bounded complexes and right bounded complexes with bounded cohomology respectively. Denote by $\mathbb{C}^b(\text{proj} \, A)$ and $\mathbb{C}^{-b}(\text{proj} \, A)$ the full subcategories of $C^b(A)$ and $C^{-b}(A)$ respectively consisting of all complexes of finitely generated projective modules. Denote by $K(A)$, $K^b(\text{proj} \, A)$ and $K^{-b}(\text{proj} \, A)$ the homotopy categories of $C(A)$, $C^b(\text{proj} \, A)$ and $C^{-b}(\text{proj} \, A)$ respectively. Moreover, $D^b(A)$ is the bounded derived category of $A$.

From [17], for any complex $X^\bullet \in D^b(A)$, the cohomological length is
\[\text{hl}(X^\bullet) := \max\{\dim H^i(X^\bullet) \mid i \in \mathbb{Z}\},\]
the cohomological width of $X^\bullet$ is
\[\text{hw}(X^\bullet) := \max\{|j - i + 1| \mid H^j(X^\bullet) \neq 0 \neq H^i(X^\bullet)\},\]
and the cohomological range of $X^\bullet$ is
\[\text{hr}(X^\bullet) := \text{hl}(X^\bullet) \cdot \text{hw}(X^\bullet).\]
Note that these numerical invariants preserve under shifts and isomorphisms. Moreover, the dimension of an $A$-module $M$ is equal to the cohomological range of the stalk complex with $M$ in degree 0.

Definition 1.1. [17] Def.5 An algebra $A$ is said to be strongly derived unbounded or of strongly derived unbounded type if there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each $r_i$, up to shifts and isomorphisms, there are infinitely many indecomposable objects in $D^b(A)$ of cohomological range $r_i$.

Recall that a complex $X^\bullet = (X^i, d^i) \in C^b(A)$ is said to be minimal if $\text{Im} \, d^i \subseteq \text{rad} \, X^{i+1}$ for all $i \in \mathbb{Z}$, and the width of $X^\bullet$ is\[w(X^\bullet) := \max\{|j - i + 1| \mid X^j \neq 0 \neq X^i\}.\]
For any integer $m \geq 0$, $C_m(\text{proj} \, A)$ is the subcategory of $C^b(\text{proj} \, A)$ consisting of all minimal complexes $P^\bullet = (P^i, d^i)$ such that $P^i = 0$ for any $i \notin \{0, 1, \cdots, m\}$. Following [5] [6], for $P^\bullet \in C_m(\text{proj} \, A)$, we put the dimension of $P^\bullet$ is
\[\dim(P^\bullet) = \sum_{i=0}^m \dim P^i.\]
Now we shall define the strongly unboundedness of $C_m(\text{proj} \, A)$.

Definition 1.2. Let $A$ be an algebra and $m \geq 1$ be an integer. The category $C_m(\text{proj} \, A)$ is said to be strongly unbounded or of strongly unbounded type if there is an increasing sequence $\{d_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each $d_i$, up to isomorphisms, there are infinitely many indecomposable objects in $C_m(\text{proj} \, A)$ of dimension $d_i$.

Remark 1.3. Since for any algebra $A$ and fixed integer $m$, there is a full embedding from the category $C_m(\text{proj} \, A)$ to $C_{m+1}(\text{proj} \, A)$, the strongly unboundedness of $C_m(\text{proj} \, A)$ implies that the category $C_{m+1}(\text{proj} \, A)$ is of strongly unbounded type.
In particular, the statement $C_m(\text{proj } A)$ is strongly unbounded for some integer $m$ is equivalent to that $C_m(\text{proj } A)$ is of strongly unbounded type for all but finitely many $m$.

We need two lemmas in the following.

**Lemma 1.4.** (See [5 Lemma 2.2]) Let $A$ be an algebra with $\dim A = d$ and $P^\bullet \in C_m(\text{proj } A)$ such that $\text{hl}(P^\bullet) = c$. Then for any $i \in [0, m]$, we have
\[
\dim P^i \leq c(d + d^2 + \cdots + d^{m-i+1}).
\]

*Proof.* Since $K$ have an isomorphism $\text{Hom}_K$ is indecomposable in $D$, which implies we have a triangle in $\text{Hom}_K$.\hfill $\square$

We need two lemmas in the following.

**Lemma 1.5.** Let $A$ be an algebra and $m \geq 0$ be an integer. Suppose $P^\bullet, Q^\bullet$ are two objects in $C_m(\text{proj } A)$. Then

1. $P^\bullet$ is indecomposable in $C_m(\text{proj } A)$ if and only if it is indecomposable as an object in $D^b(A)$.
2. $P^\bullet \cong Q^\bullet$ in $C_m(\text{proj } A)$ if and only if $P^\bullet \cong Q^\bullet$ as objects in $D^b(A)$.

*Proof.* (1) Since $D^b(A) \simeq K^{-b}(\text{proj } A)$, which is Krull-Schmidt, the complex $P^\bullet$ is indecomposable in $D^b(A)$ if and only if it is an indecomposable complex in $K^b(\text{proj } A)$, and if and only if its endomorphism algebra $\text{End}_{K(A)}(P^\bullet)$ is a local algebra. Moreover, since the complex $P^\bullet$ is minimal, all null homotopic cochain maps in $\text{End}_{C(A)}(P^\bullet)$ are in $\text{rad } \text{End}_{C(A)}(P^\bullet)$. Thus $\text{End}_{K(A)}(P^\bullet)/\text{rad } \text{End}_{C(A)}(P^\bullet)$ is strongly unbounded for some integer $m$. Conversely, suppose $P^\bullet \cong Q^\bullet$ in $D^b(A)$ and there is a quasi-isomorphism $f^*: P^\bullet \to Q^\bullet$. Then we have a triangle in $K(A)$
\[
P^\bullet \xrightarrow{f^*} Q^\bullet \to L^\bullet \to P^\bullet[1]
\]
such that $L^\bullet$ is an acyclic complex. Applying $\text{Hom}_{K(A)}(Q^\bullet, -)$ to the triangle, we have an isomorphism $\text{Hom}_{K(A)}(Q^\bullet, P^\bullet) \cong \text{Hom}_{K(A)}(Q^\bullet, Q^\bullet)$ induced by $f^*$ since $\text{Hom}_{K(A)}(Q^\bullet, L^\bullet) = 0$, which implies $f^*$ is a split epimorphism in $K(A)$. Note that $P^\bullet$ and $Q^\bullet$ are quasi-isomorphic. Then $f^*$ is a chain homotopy equivalence, i.e., there is a morphism $g^*$ such that $1 - g^* f^*$ and $1 - f^* g^*$ are null homotopic. Since $P^\bullet, Q^\bullet$ are minimal, all null-homotopic chain maps are nilpotent. Thus $f^*$ and $g^*$ are split monomorphisms in $C_m(\text{proj } A)$. Therefore, $P^\bullet \cong Q^\bullet$ in $C_m(\text{proj } A)$.\hfill $\square$

The following lemma implies the strongly unboundedness of $C_m(\text{proj } A)$ can be defined in terms of the cohomological range as well.
Lemma 1.6. Let $A$ be an algebra and $m \geq 1$ be an integer. The category $C_m(\text{proj} \ A)$ is strongly unbounded if and only if there is an increasing sequence \( \{r_i \mid i \in \mathbb{N} \} \subseteq \mathbb{N} \) such that for each $r_i$, up to isomorphisms, there are infinitely many indecomposable objects in $C_m(\text{proj} \ A)$ of cohomological range $r_i$.

Proof. Suppose there is an increasing sequence \( \{r_i \mid i \in \mathbb{N} \} \subseteq \mathbb{N} \) and pairwise non-isomorphic objects \( \{P_{ij}^* \mid i, j \in \mathbb{N} \} \) in $C_m(\text{proj} \ A)$ such that \( \text{hr}(P_{ij}^*) = r_i \). Note that for any object $P^* \in C_m(\text{proj} \ A)$, $\text{hr}(P^*) \leq (m + 1) \cdot \dim(P^*)$. Moreover by Lemma 1.4 \( \dim(P^*) \leq \text{hr}(P^*) \cdot (m + 1) \cdot (d + d^2 + \cdots + d^{m+1}) \). Set \( N = (m + 1) \cdot (d + d^2 + \cdots + d^{m+1}) \), then for any $i, j \in \mathbb{N}$, we have

\[
\frac{1}{m + 1} \cdot \text{hr}(P_{ij}^*) \leq \dim(P^*) \leq N \cdot \text{hr}(P_{ij}^*).
\]

In order to show $C_m(\text{proj} \ A)$ is of strongly unbounded type, we shall find inductively an increasing sequence \( \{d_i \mid i \in \mathbb{N} \} \subseteq \mathbb{N} \) and infinitely many indecomposable objects \( \{Q_{ij}^* \in C_m(\text{proj} \ A) \mid i, j \in \mathbb{N} \} \) which are pairwise different up to isomorphism such that $\dim(Q_{ij}^*) = d_i$ for all $j \in \mathbb{N}$. For $i = 1$, $0 < \dim(P_{1j}^*) \leq N r_1$. Then there is $0 < d_1 \leq N r_1$ and infinitely many indecomposable objects \( \{Q_{ij}^* \mid j \in \mathbb{N} \} \subseteq \{P_{ij}^* \mid j \in \mathbb{N} \} \) of dimension $d_1$. Assume that we have found $d_i$. We choose some $r_i$ with $r_i > (m + 1) \cdot d_i$. Since

\[
d_i < \frac{1}{m + 1} \cdot r_i = \frac{1}{m + 1} \cdot \text{hr}(X_{ij}^*) \leq \dim(P_{ij}^*) \leq N \cdot \text{hr}(X_{ij}^*) = N \cdot r_i,
\]

we can choose $d_i < d_{i+1} \leq N \cdot r_i$ and infinitely many indecomposable objects \( \{Q_{ij}^* \mid j \in \mathbb{N} \} \subseteq \{P_{ij}^* \mid j \in \mathbb{N} \} \) which are pairwise non-isomorphism such that $\dim(Q_{ij}^*) = d_{i+1}$ for all $j \in \mathbb{N}$.

Conversely we suppose $C_m(\text{proj} \ A)$ is of strongly unbounded. Then we can construct an increasing sequence \( \{r_i \mid i \in \mathbb{N} \} \subseteq \mathbb{N} \) and pairwise non-isomorphic objects \( \{Q_{ij}^* \mid i, j \in \mathbb{N} \} \) such that $\text{hr}(Q_{ij}^*) = r_i$ in the similar way by the inequality

\[
\frac{1}{N} \cdot \dim(P^*) \leq \text{hr}(P^*) \leq (m + 1) \cdot \dim(P^*),
\]

for any $P^* \in C_m(\text{proj} \ A)$. $\square$

2. The proof of Theorem

2.1. Simply connected algebras. Simply connected algebras play an important role in the representation theory of algebras since any representation-finite algebra can be transformed to a simply connected algebra using covering technique. We first recall the definition of simply connected algebras from [2]. Fix a connected quiver $(Q, I)$ with $I$ admissible. For any $\alpha \in Q_1$, we write its formal inverse $\alpha^{-1}$ with source $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. A walk in $Q$ is a path \( w = w_1 w_2 \cdots w_n \) with $w_i \in Q_1$ or $w_i^{-1} \in Q_1$ such that $s(w_{i+1}) = t(w_i)$. An relation \( r = \sum_{i=1}^m t_i u_i \in I (m \geq 1) \) with $u_i$ pairwise distinct and $t_i \in k \setminus \{0\}$ is called minimal if $r = \sum_{i \in S} t_i u_i \notin I$ for any non-empty proper subset $S \subseteq \{1, 2, \cdots, m\}$. The homotopy relation is the smallest equivalence relation $\sim_I$ on the set of walks such that:

1. $\alpha \sim_I e_x$ and $\alpha^{-1} \sim_I e_y$ for any $x \sim_I y$;
2. $u_1 \sim_I u_2$ for any minimal relation $t_1 u_1 + t_2 u_2 + \cdots + t_m u_m$;
3. $u \sim_I v$ implies $uv \sim_I uv$ and $wu \sim_I wv$ for any $w$.

The fundamental group $\Pi_1(Q, I, x_0)$ of $(Q, I)$ is defined to the group consisting of homotopy classes of walks from $x_0$ to $x_0$ for any vertex $x_0 \in Q_0$ [13]. Note that the definition is independent of the choice of $x_0$, and we write $\Pi_1(Q, I)$ for
short. A triangular algebra $A$ is said to be simply connected if for any presentation $A \cong kQ/I$, the fundamental group $\Pi_1(Q, I)$ is trivial.

The following lemma implies that for a representation-infinite algebra $A$, the category $C_1(\text{proj} A)$ is of strongly unbounded type.

**Lemma 2.1.** Let $A$ be a representation-infinite algebra. Then $C_1(\text{proj} A)$ is of strongly unbounded type.

**Proof.** Since $A$ is representation-finite, $A$ is of strongly unbounded type, i.e., there is an infinite sequence $\{d_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and infinitely many indecomposable $A$-modules $\{M_{ij} \mid i,j \in \mathbb{N}\}$ which are pairwise different up to isomorphism such that $\dim(M_{ij}) = d_i$ for all $j \in \mathbb{N}$. For any $M_{ij}$, we can take a minimal presentation $P^{-1} \to P^0 \to M_{ij} \to 0$. Let

$$P^•_{ij} = \cdots \to 0 \to P^{-1} \to P^0 \to 0 \to \cdots$$

with $P^{-1}$ concentrated in degree 0. Then $P^•_{ij} \in C_1(\text{proj} A)$ is indecomposable by [17, Prop.2] with $\dim H^1(P^•_{ij}) = d_i$. Moreover, $P^•_{ij}$ are non-isomorphic for different $i,j \in \mathbb{N}$. Since $P^•_{ij}$ is a minimal presentation of $M_{ij}$, $\dim P^{-1} \leq (\dim A)^2 \cdot d_i$ and we have

$$d_i \leq \text{hr}(P^•_{ij}) \leq 2 \cdot (\dim A)^2 \cdot d_i.$$

With the similar argument in the proof of Lemma 1.6, we can construct a sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{Q^•_{ij} \mid i,j \in \mathbb{N}\}$ such that $\text{hr}(Q^•_{ij}) = r_i$. Thus $C_1(\text{proj} A)$ is of strongly unbounded type by Lemma 1.6. \qed

The following lemma observe the strongly unboundedness of $C_m(\text{proj} A)$ under the derived equivalences.

**Proposition 2.2.** Let $A$ be an algebra with $C_m(\text{proj} A)$ strongly unbounded for some integer $m$ and $\text{gl.dim} A < \infty$. If there is an algebra $B$ derived equivalent to $A$, then $C_{m'}(\text{proj} B)$ is of strongly unbounded type for some integer $m'$.

**Proof.** Since $C_m(\text{proj} A)$ is strongly unbounded, by Lemma 1.6, there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{P^•_{ij} \mid i,j \in \mathbb{N}\}$ in $C_m(\text{proj} A)$ such that $\text{hr}(P^•_{ij}) = r_i$. Moreover, since $A$ and $B$ are derived equivalent, there is a two-sided tilting complex in $D^b(A^{op} \otimes B)$

$$A T^•_B = 0 \to T^{−1} \to T^{−1+l+1} \to \cdots \to T^{−1} \to P^0 \to 0,$$

such that $F = \bigoplus A T^•_B : D^b(A) \to D^b(B)$ is a derived equivalence [22]. Note that $\text{gl.dim} A < \infty$ implies $\text{gl.dim} B < \infty$ [18]. We assume $\text{gl.dim} B = n$, and we can take a minimal projective $B$-$B$-bimodule resolution of $B$

$$R^• = 0 \to R^{-n} \to R^{-n+1} \cdots \to R^{-1} \to R^{0} \to 0.$$

Then for any $i,j \in \mathbb{N}$, $F(P^•_{ij}) = P^•_{ij} \otimes_A T^•_B \cong P^•_{ij} \otimes_A T^•_B \otimes_B R^•_B$, which is a projective $B$-module complex of width less than $m+l+n$. Thus, without loss of generality, we can assume $F(P^•_{ij}) \in C_{m+l+n}(\text{proj} B)$ with suitable shifts and isomorphisms for any $i,j \in \mathbb{N}$. By [17, Prop.1(3)], we have two integers $N, N'$, such that

$$\frac{1}{N'} \cdot \text{hr}(P^•_{ij}) \leq \text{hr}(F(P^•_{ij})) \leq N \cdot \text{hr}(P^•_{ij}).$$

With a similar discussion as in the proof of Lemma 1.6, we shall find inductively an increasing sequence $\{r'_s \mid s \in \mathbb{N}\}$ and infinitely many indecomposable pairwise non-isomorphic objects $\{Q^•_{ij} \in C_{m+l+n}(\text{proj} B) \mid s,t \in \mathbb{N}\} \subseteq \{F(P^•_{ij}) \mid i,j \in \mathbb{N}\}$ such that $\text{hr}(Q^•_{ij}) = r'_s$. Thus the lemma follows by Lemma 1.6. \qed
Corollary 2.3. Let $A$ be a simply connected algebra. If $A$ is strongly derived unbounded, then there exists an integer $m$ such that $C_m(\text{proj} \ A)$ is of strongly unbounded type.

Proof. By [17] Lemma 2, any simply connected algebra is tilting equivalent to a hereditary algebra of Dynkin type or a representation-infinite algebra. If $A$ is strongly derived unbounded, then $A$ is tilting equivalent to a representation-infinite algebra. Since simply connected algebras are triangular algebras and then of finite global dimension, by the previous proposition and Lemma 2.1 there exists an integer $m$ such that $C_m(\text{proj} \ A)$ is of strongly unbounded type. \hfill \Box

2.2. Cleaving functors and the strongly unboundedness of $C_m(\text{proj} \ A)$. In the context of cleaving functors, bound quiver algebras are viewed as bounded categories, see [15] for details. In the rest of this paper, we will replace bound quiver algebras by bounded categories.

A $k$-linear category $A$ is a category together with $k$-vector space structure on the set $A(x,y)$ of all morphisms from $x \in A$ to $y \in A$ such that the composition of morphisms is bilinear. We say a $k$-linear category $A$ is a locally bounded category if

1. different objects in $A$ are non-isomorphic;
2. for any $a \in A$, the endomorphism algebra $A(a,a)$ is local;
3. $\dim_k \sum_{a \in A} A(a,a) < \infty$ and $\dim_k \sum_{a \in A} A(a,a) < \infty$ for all $a \in A$.

A locally bounded category is a bounded category if it has only finitely many objects.

Note that a bound quiver algebra $A = kQ/I$ with $I$ admissible can be viewed as a bounded category by seeing the vertexes $i \in Q_0$ as objects and the combinations of paths in $kQ/I$ as morphisms. Conversely, a bounded category $A$ admits a presentation $A \cong kQ_A/I$ with $Q_A$ finite and $I$ admissible.

Let $A$ be a locally bounded category. A right $A$-module $M$ is just a covariant $k$-linear functor from $A$ to the category of $k$-vector spaces. Denote by $\text{Mod} \ A$ the category of all right $A$-modules $M$ with $\dim M(a) < \infty$ for any $a \in A$. For any $M \in \text{Mod} \ A$, the dimension vector of $M$ is $\text{dimv} M := (\dim M(a))_{a \in A}$, and the support of $M$ is $\text{Supp} \ M := \{a \in A \mid M(a) \neq 0\}$. Denote by $\text{mod} \ A$ the full subcategory of $\text{Mod} \ A$ consisting of all $A$-modules $M$ such that $\text{Supp} \ M$ is finite. The dimension of $M \in \text{mod} \ A$ is $\dim M := \sum_{a \in A} \dim_k M(a)$. The indecomposable projective $A$-modules are $P_a = A(a,-)$ and indecomposable injective $A$-modules are $I_a = DA(-,a)$ for all $a \in A$, where $D = \text{Hom}_A(\cdot, k)$. Moreover, all the concepts and notations defined for a bound quiver algebra make sense for a bounded category.

To a $k$-linear functor $F : B \to A$ between bounded categories, we associates a restriction functor $F_* : \text{mod} \ A \to \text{mod} \ B$, which is given by $F_*(M) = M \circ F$ and exact. The restriction functor $F_*$ admits a left adjoint functor $F^*$, called the extension functor, which sends a projective $B$-module $B(b,-)$ to a projective $A$-module $A(Fb,-)$. Moreover, $F_*$ extends naturally to a derived functor $F_* : D^b(A) \to D^b(B)$, which has a left adjoint $\text{LF}^* : D^b(B) \to D^b(A)$. Note that $\text{LF}^*$ is the left derived functor associated with $F^*$ and maps $K^b(\text{proj} \ B)$ into $K^b(\text{proj} \ A)$. We refer to [27] for the definition of derived functors.

A $k$-linear functor $F : B \to A$ between bounded categories is called a cleaving functor [17] if it satisfies the following equivalent conditions:

1. The linear map $B(b,b') \to A(Fb, Fb')$ associated with $F$ admits a natural retraction for all $b, b' \in B$;
2. The adjunction morphism $\phi_M : M \to (F_* \circ F^*)(M)$ admits a natural retraction for all $M \in \text{mod} \ B$;
3. The adjunction morphism $\Phi_{X^*} : X^* \to (F_* \circ \text{LF}^*)(X^*)$ admits a natural retraction for all $X^* \in D^b(B)$.
**Proposition 2.4.** Let $B$ be a bounded category of finite global dimension and $C_m(\text{proj } B)$ be of strongly unbounded type for some $m$. If there is a cleaving functor $F : B \to A$, then $C_m(\text{proj } A)$ is of strongly unbounded type.

**Proof.** Suppose there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{P_{ij}^* \mid i, j \in \mathbb{N}\}$ in $C_m(\text{proj } B)$ such that $\text{hr}(P_{ij}^*) = r_i$. Since $F$ is a cleaving functor, for any $i, j \in \mathbb{N}$, $LF^*(P_{ij}^*) = F^*(P_{ij}^*)$, which is projective $A$-module complex of width less than $m$ by the definition of $F^*$. Then, with suitable isomorphisms, we can assume $LF^*(P_{ij}^*)$ lies in $C_m(\text{proj } A)$. Moreover, for any $i, j \in \mathbb{N}$, $P_{ij}^*$ is a direct summand of $(F \circ LF^*)(P_{ij}^*)$. Thus for any $P_{ij}^*$, we can choose an indecomposable direct summand $Q_{ij}^*$ of $LF^*(P_{ij}^*)$, such that $P_{ij}^*$ is a direct summand of $F_*(Q_{ij}^*)$. Note that for any $i \in \mathbb{N}$, the set $\{Q_{ij}^* \mid j \in \mathbb{N}\}$ contains infinitely many elements which are pairwise non-isomorphic since the set $\{P_{ij}^* \mid j \in \mathbb{N}\}$ contains infinitely many pairwise non-isomorphic elements. Moreover, by the proof of [17], Prop.5(1)], there exist two integers $N, N'$, such that for any $i, j \in \mathbb{N}$, we have the inequality $\frac{1}{N} \cdot \text{hr}(P_{ij}^*) \leq \text{hr}(Q_{ij}^*) \leq N \cdot \text{hr}(P_{ij}^*)$. Thus $C_m(\text{proj } A)$ is of strongly unbounded with a similar discussion as in the Lemma 1.6 \hspace{1cm} \Box

2.3. The proof of the main theorem. Let $A$ be a bounded category. Recall that the repetitive category $A$ has the pairs $(a, i)$ as objects, where $a \in A$ and $i \in \mathbb{Z}$, while the morphisms from $(a, i)$ to $(b, i)$ and $(b, i+1)$ are determined by $A(a, b)$ and $A(b, a)$ respectively, and zero else [19]. Note that $A$ is self-injective locally bounded category. Moreover, there is a full embedding triangulated functor $F : D^b(A) \to \text{mod } \hat{A}$ [18].

Recall from [26], $A$ is said to be derived discrete if for any $d \in \mathbb{N}$, there are only finitely many indecomposables in $D^b(A)$ with cohomological range $d$. Moreover, $K^b(\text{proj } A)$ is discrete if for any $d \in \mathbb{N}$, there are only finitely many indecomposables in $K^b(\text{proj } A)$ of cohomological range $d$.

**Definition 2.5.** A locally bounded category $B$ is said to be of discrete representation type if for any $d \in \mathbb{N}^{\# B}$, there are only finitely many indecomposable objects $M \in \text{mod } A$ with $\text{dim} M = d$. Moreover, we say $B$ is of strongly unbounded representation type if there are infinitely many $d \in \mathbb{N}^{\# B}$ such that for each $d$, there are infinitely many indecomposables in $\text{mod } A$ with dimension vector $d$.

The following lemma is the classification of derived discrete algebras due to Vossieck [26, Theorem].

**Lemma 2.6.** Let $A$ be a bounded category. Then the following statements are equivalent

1. $A$ is of discrete representation type;
2. $A$ is derived discrete;
3. $K^b(\text{proj } A)$ is discrete;
4. $A$ is piecewise hereditary of Dynkin type or admits a presentation $\mathbb{Q}$/I with $\mathbb{Q}$ one-cycle gentle quiver such that the numbers of clockwise and of counterclockwise paths of length two which belongs to $I$ are different.

**Definition 2.7.** Let $A$ be a bounded category. $K^b(\text{proj } A)$ is said to be of strongly unbounded type if there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that for each $r_i$, up to shifts and isomorphisms, there are infinitely many indecomposable objects in $K^b(\text{proj } A)$ of cohomological range $r_i$.

Now we can prove the Theorem.
Theorem 2.8. Let $A$ be a bounded category. Then the following statements are equivalent

(1) $A$ is strongly derived unbounded;

(2) There exists an integer $m \geq 1$, such that the category $C_m(\text{proj} A)$ is of strongly unbounded type.

(3) $K^b(\text{proj} A)$ is of strongly unbounded type;

(4) $A$ is of strongly unbounded representation type.

Proof. (1)$\Rightarrow$(2): We assume for any integer $m > 0$, $A_m(\text{proj} A)$ is not of strongly unbounded type. Then $A$ is representation-finite by Lemma 2.1. Thus for any $a \in A$, $A(a,a)$ is a uniserial local algebra, and then $A(a,a) \cong k$ or $A(a,a) \cong k[x]/(x^l)$ with $l \geq 2$. Moreover, we can exclude the cases $A(a,a) \cong k[x]/(x^l)$ for $l \geq 3$. Indeed, we consider the the functor $F : A^l_n \to A$ given by $F(i) = a$ and $F(aj) = x$, where $A^l_n$ is the bounded category defined by the quiver

$$\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& n & 0_{a_{n-1}} & n - 1 & 0_{a_2} & \cdots & 0_{a_2} & 2 & 0_{a_1} & 1 \\
\end{array}$$

and the admissible ideal generated by all paths of length $l$. Note that $F$ is a cleaving functor. By the construction in [17, Lemma 4.4], if $l \geq 3$, then $C_5(\text{proj} A^l_n)$ is strongly unbounded, and thus $C_5(\text{proj} A)$ is of strongly unbounded type by Proposition 2.4 which is a contradiction. Therefore, for any $a \in A$, $A(a,a) \cong k$ or $A(a,a) \cong k[x]/(x^l)$. By [7, Section 9], $A$ is standard since $A$ contains no Riedtmann contours.

If $A$ is simply connected, then $A$ is not strongly unbounded by Corollary 2.3. Suppose $A$ is not simply connected, then there is a Galois covering $\pi : \hat{A} \to A$ and $\pi$ is cleaving functor. Then $B$ is as constructed from $\hat{A}$ by Bieknert and Merklen’s classification of the indecomposable objects in the derived category of a gentle algebra [9], if $A$ contains a generalized band $w$, then we can construct a family of pairwise non-isomorphic indecomposables $P^*_w,f$ for $f = (x - \lambda)^d \in k[x]$ in $C_m(\text{proj} A)$ for some integer $m$, such that $P^*_w,f$ and $P^*_w,f'$ have the same dimension if and only if $\deg(f) = \deg(f')$, where $\lambda \in k \setminus \{0\}$ and $d > 0$. Then $C_m(\text{proj} A)$ is of strongly unbounded type, which is a contradiction. Thus $A$ contains no generalized bands and then $A$ is derived discrete by [9, Theorem 4]. Therefore, $A$ is not strongly unbounded.

(2)$\Rightarrow$(3): Suppose there exists an integer $m \geq 1$, such that $C_m(\text{proj} A)$ is of strongly unbounded type. Then by Lemma 1.6 there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic objects $\{P^*_{ij} \mid i,j \in \mathbb{N}\}$ in $C_m(\text{proj} A)$ such that $\text{hr}(P^*_{ij}) = r_i$. Since the elements in $\{P^*_{ij} \mid i,j \in \mathbb{N}\}$, seen as objects in $K^b(\text{proj} A)$, are also pairwise non-isomorphic indecomposables, $K^b(\text{proj} A)$ is strongly unbounded.

(3)$\Rightarrow$(1): Trivial.

(2)$\Rightarrow$(4): Suppose $C_m(\text{proj} A)$ is of strongly unbounded type, by Lemma 1.6 there is an increasing sequence $\{r_i \mid i \in \mathbb{N}\} \subseteq \mathbb{N}$ and pairwise non-isomorphic complexes $\{P^*_{ij} \mid i,j \in \mathbb{N}\}$ in $C_m(\text{proj} A)$ such that $\text{hr}(P^*_{ij}) = r_i$. Note that $\{P^*_{ij} \mid i,j \in \mathbb{N}\}$ are pairwise non-isomorphic indecomposables viewed as objects
in $D^b(A)$ by Lemma 1.3. Assume $\{S_a \mid a \in A\}$ and $\{S_h \mid h \in \hat{A}\}$ are the sets of all simple $A$-modules and $\hat{A}$-modules respectively.

Now we consider the full embedding $F : D^b(A) \to \text{mod} \hat{A}$. On one hand, for a fixed object $X^\bullet \in D^b(A)$, without loss of generality, we assume it concentrated in degree $[0, n]$. Note that $X^\bullet$ is generated by the cohomologies via triangles and the comhomologies can be also obtained by triangles with the simples. Since $F$ sends a triangle in $D^b(A)$ to a triangle in $\text{mod} \hat{A}$, by the additivity of dimension functor $\dim(-)$ in $\text{mod} \hat{A}$, we have the following estimate (see also [20])

$$\dim F(X^\bullet) \leq \sum_{a \in A} \sum_{l=0}^n \dim H^l(X^\bullet)(a) \cdot \dim F(S_a[l]) \leq \text{hr}(X^\bullet) \cdot \sum_{a \in A} \sum_{l=0}^n \dim F(S_a[l]).$$

Set $d = \sum_{a \in A} \sum_{l=0}^n \dim F(S_a[l])$. Then for any $i, j \in \mathbb{N}$, $\dim F(P^\bullet_{ij}) \leq r_1 \cdot d$.

On the other hand, for any object $X^\bullet \in D^b(A)$, we have

$$\text{hr}(X^\bullet) = \sum_{l \in \mathbb{Z}} \dim H^l(X^\bullet) = \sum_{l \in \mathbb{Z}} \dim \text{Hom}_{D^b(A)}(A[l], X^\bullet)$$

$$= \sum_{l \in \mathbb{Z}} \dim \text{Hom}_{\hat{A}}(F(A[l]), F(X^\bullet))$$

$$\leq \sum_{l \in \mathbb{Z}} \sum_{h \in \hat{A}} c_h(F(A[l])) \dim \text{Hom}_{\hat{A}}(S_h, F(X^\bullet))$$

$$\leq \sum_{l \in \mathbb{Z}} \sum_{h \in \hat{A}} c_h(F(A[l])) \dim \text{Hom}_{\hat{A}}(P_h, F(X^\bullet))$$

$$\leq \sum_{l \in \mathbb{Z}} \sum_{h \in \hat{A}} c_h(F(A[l])) \dim F(X^\bullet)(h),$$

where $c_h(F(A[l]))$ denotes the number of composition factors of $F(A[l])$ isomorphic to $S_h$, and $P_h$ is the indecomposable projective $\hat{A}$-module associated to $h \in \hat{A}$. Set $c = \sup\{c_h(F(A[l])) \mid 0 \leq l \leq m, h \in \hat{A}\}$. Then for any $i, j \in \mathbb{N}$, $\text{hr}(P^\bullet_{ij}) \leq c \cdot \text{hw}(P^\bullet_{ij}) \cdot \dim F(P^\bullet_{ij})$. Thus $\frac{1}{c} \cdot \text{hl}(P^\bullet_{ij}) \leq \dim F(P^\bullet_{ij})$.

To prove $\hat{A}$ is of strongly unbounded representation type, we shall find inductively infinitely many vectors $\{d_i \mid i \in \mathbb{N}\}$ and infinitely many indecomposable objects $\{M^\bullet_{ij} \in \text{mod} \hat{A} \mid i, j \in \mathbb{N}\}$ which are pairwise different up to isomorphism such that $\dim M^\bullet_{ij} = d_i$ for all $j \in \mathbb{N}$. For $i = 1$, we have $0 < \dim F(P^\bullet_{ij}) \leq r_1 \cdot d$. Then there is $0 < d_1 \leq r_1 \cdot d$ and infinitely many indecomposable objects $\{M_{ij} \mid j \in \mathbb{N}\} \subseteq \{F(P^\bullet_{ij}) \mid j \in \mathbb{N}\}$ of dimension vector $d_1$. Assume that we have done for $i$, and $d_i = \sum_{j \in \mathbb{N}} d_{ij}$. Then we choose some $r_i$ with $r_i > c(m+1) \cdot d_i$, and thus $\text{hl}(P^\bullet_{ij}) \leq c \cdot d_i$. Since $d_i < \frac{1}{c} \cdot \text{hl}(P^\bullet_{ij}) \leq \dim F(P^\bullet_{ij})$, and $\dim F(P^\bullet_{ij}) \leq r_1 \cdot d$, we can choose a vector $d_{i+1}$, which is different from $\{d_s \mid s = 1, 2, \cdots, i\}$, such that $d_{i+1} \leq r_i \cdot d$, and infinitely many pairwise non-isomorphism indecomposable objects $\{M_{i+1,j} \mid j \in \mathbb{N}\} \subseteq \{F(P^\bullet_{ij}) \mid j \in \mathbb{N}\}$ with $\dim M_{i+1,j} = d_{i+1}$ for all $j \in \mathbb{N}$.

(4)$\Rightarrow$(1): Suppose $\hat{A}$ is not strongly derived unbounded, then by [17] Theorem 2, $\hat{A}$ is derived discrete. Thus $\hat{A}$ is representation discrete by Lemma 2.6 which is a contradiction with the assumption. \hfill \Box

Recall from [6], for an algebra $A$ and a fixed integer $m$, the category $C_m(\text{proj} A)$ is said to be of finite representation type if $C_m(\text{proj} A)$ contains only finitely many indecomposables up to isomorphisms. As a corollary of the previous theorem, we obtain the dichotomy on the representation type of $C_m(\text{proj} A)$, $K^b(\text{proj} A)$ and also the repetitive algebra $\hat{A}$. 

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Corollary 2.9. Let $A$ be an algebra. Then we have

1. $C_m(\text{proj }A)$ is of finite representation type for any $m$, or there exists an integer $m' \geq 1$, such that $C_{m'}(\text{proj }A)$ is of strongly unbounded type.

2. $K^b(\text{proj }A)$ is either discrete or of strongly unbounded type;

3. The repetitive algebra $\hat{A}$ is either of discrete representation type or strongly unbounded representation type.

Proof. By [6, Theorem 2.4(1)], we know that $A$ is derived discrete if and only if any $C_m(\text{proj }A)$ is of finite representation type. Moreover, $A$ is strongly derived unbounded if and only if there exists an integer $m \geq 1$, such that $C_m(\text{proj }A)$ is of strongly unbounded type by the previous theorem. Since any algebra $A$ is either derived discrete or strongly derived unbounded by [17, Theorem 2], the statement (1) follows. Similarly, the statements (2) and (3) hold by the Lemma 2.6 and the previous theorem. □

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