In this work, we carefully analyze the macroscopic observables in the 1+1D gapped phases with abelian onsite symmetries, and show that the spacetime observables for each gapped phase form a clear structure that can be mathematically described by enriched fusion categories, which uncovers the behavior of non-local excitations that were blurry in traditional Landau paradigm. These categorical descriptions not only generate the known classification results for symmetry preserving/breaking phases, but also unifies lattice dualities in a broader picture. After analyzing the general lattice model together with their boundaries, we give explicit examples including non-trivial SPT phase, where non-trivial boundaries can be given directly through our classification. Using enriched categorical descriptions, the lattice dualities and their gapped phases are unified under a holographic duality between an 2d topological order with gapped 1d boundaries and 1+1D gapped quantum liquids with a categorical symmetry, which shed light on a unified definition of all quantum phases.
I. INTRODUCTION

The description of phases is a central question in condensed matter physics. Since a “phase” by itself, is a macroscopic notion, it should have a mathematical description which can be obtained by collecting all physical observables in the long wave length limit (LWLL). However, for a long period of time, the pursuing of this structure has been put aside, especially for phases within Landau’s paradigm like the symmetry breaking phases.

Recently, in virtue of category theory, a unified description is proposed for all gapped/gapless quantum liquid phases [1, 2]. It states that a quantum liquid phase \( \mathcal{X} \) can be described by a pair \( (\mathcal{X}_{iqs}, \mathcal{X}_{sk}) \), where \( \mathcal{X}_{iqs} \) is the local quantum symmetry and \( \mathcal{X}_{sk} \) is the topological skeleton consisting of all topological defects. For gapped quantum liquid phases, the information of local quantum symmetry is encoded in the topological skeletons. Therefore, a gapped quantum liquid phase can be described by a topological skeleton, which is mathematically an enriched (higher) category.

In [3], the validity of this proposal has been checked in the 1+1D Ising chain and Kitaev chain, which are the first concrete examples in the lattice models to show the spacetime observables indeed form an enriched categories. So in this work, we generalize this result to 1+1D bosonic gapped quantum phases with a finite abelian on-site symmetry \( G \). Our categorical descriptions also give the classification of these bosonic gapped phases manifestly.

So what are the macroscopic observables in a quantum phase and why do they form an enriched category? It follows from [3] that, for a 1d lattice model, there are two kinds of observables in LWLL: topological sectors of operators and topological sectors of states.

1. The topological sectors of operators are the spaces of non-local operators that are invariant under the action of local operators. An operator between two topological sectors of operators that intertwines the action of local operators is called a morphism. We can say the topological sectors of operators form a category, denoted by \( \mathcal{B} \). Moreover, these operators can be fused and braided in 1+1D spacetime. In particular, the sector consisting of local operators is the tensor unit of \( \mathcal{B} \), denoted by \( 1_\mathcal{B} \). In consequence, \( \mathcal{B} \) is expected to be a braided fusion category \( \mathcal{B} \), which is often called the categorical symmetry [4, 5].

2. The topological sectors of states are the topological defect lines (also known as topological excitations) in 1+1D, which are the subspaces of the total Hilbert space that are invariant under the action of local operators. The sector generated by the vacuum state (i.e. the ground state) is called the vacuum sector or the trivial sector, we denote by \( \mathcal{1} \). The topological defect lines can be mapped to each other or fused together (see figure 4), which indicates that the topological sectors of states can be characterized by a fusion category, denoted by \( \mathcal{S} \).

Under categorical notations, the space of (possibly non-local) operators which maps a topological sector of states \( a \) to another one \( b \) is denoted by \( \text{Hom}(a, b) \). Since the topological sectors of operators act on those of states, \( \text{Hom}(a, b) \) should be an object in the category \( \mathcal{B} \) of topological sectors of operators. Then the topological sectors of states and the hom spaces between them should form a \( \mathcal{B} \)-enriched category \( \mathcal{B}_\mathcal{S} \) (\( \mathcal{B} \) is also called background category and \( \mathcal{S} \) is also called underlying category. For a detailed explanation, see appendix C).

Furthermore, the fusion behavior of non-local operators should be compatible with the fusion of topological sectors of states. This fusion compatibility can be summarized mathematically by the condition that \( \mathcal{S} \) is equipped with a braided functor \( \phi: \mathcal{B} \rightarrow \mathcal{S}_1(\mathcal{S}) \), where \( \mathcal{S}_1(\mathcal{S}) \) is the Drinfeld center (also called the monoidal center) of \( \mathcal{S} \) [6, 7]. In addition, since all excitations on a 1+1D anomaly-free lattice model can be created from the ground state by non-local operators, and these excitations should be detectable by the double braiding of the non-local operators that create them from the vacuum [3], \( \phi \) need to be an equivalence \( \phi: \mathcal{B} \simeq \mathcal{S}_1(\mathcal{S}) \).

In other words, the topological sectors of operators form the monoidal center of the category of the topological sectors of states. And the topological skeleton that captures the macroscopic observables of a gapped quantum phase is \( \mathcal{B}_\mathcal{S} \simeq \mathcal{S}_1(\mathcal{S}) \).

Remark I.1. Note that non-local operators will be confined due to arbitrary perturbations. For example, if we

1 Throughout this work, we use nd to represent the spatial dimension and nD to represent the spacetime dimension.

2 Indeed, the canonical construction requires a braided functor \( \mathcal{B} \rightarrow \mathcal{S}_1(\mathcal{S}) \) where \( \mathcal{B} \) is the time-reversal of \( \mathcal{B} \) obtained by reversing the braiding. In this work we only consider the case that \( \mathcal{B} = \mathcal{S}_1(\text{Rep}(G)) \) for some finite group \( G \) and thus \( \mathcal{B} \simeq \mathcal{B} \). So for simplicity we just take a braided functor \( \phi: \mathcal{B} \rightarrow \mathcal{S}_1(\mathcal{S}) \) and consider its “canonical construction”.

C. Enriched categories

References
do not impose any symmetry, then all non-local operators are confined because arbitrary perturbations lead to infinitely large energy. In this case $\mathcal{B}$ can only be Vec, i.e. there is only one sector consisting of local operators. Since the system is anomaly-free, then $\mathcal{S}$ is also Vec and $\mathfrak{a} \mathcal{S} = \text{Vec}$ is the topological skeleton of the trivial 1 + 1D topological order. So we only consider non-local operators that will not be confined by any symmetry-allowed perturbations.

Remark I.2. Here we offer a beginner’s guide tailored for readers with a physics background to understand categorical language in the context of topological phases of matter. In his seminal papers [8, 9], Kitaev introduces the modular tensor category (MTC) descriptions of anyonic excitations, establishing a foundational link between category theory and quantum models. Levin and Wen [10] investigate “string-net condensation” as a physical mechanism underlying topological phases, demonstrating how tensor category theory provides a unifying framework for various quantum states. Reference [11], along with [12], serves as an accessible introduction to MTCs and their applications in quantum computing. Further developments discussed in [13–19] cover topics such as gapped boundaries and domain walls, ground-state degeneracy, symmetry-protected topological phases, fermionic topological orders, tensor network applications, and dualities in topological phases related to fusion categories. Recently, [2] offers a modern and comprehensive guide on applying categorical language to topological orders, which is valuable for building a foundational understanding of how enriched fusion categories can be used to describe various phases.

A. Holographic duality and topological Wick rotation

Though finding the observables in the 1d quantum phases does not need to appeal to higher dimension, it would be more clear to understand the relation between the category $\mathcal{B}$ of topological sectors of operators and the category $\mathcal{S}$ of topological sectors of states via a holographic duality\(^3\). [1, 5].

In this work, by a holographic duality (or topological holographic duality) we mean a duality between an $(n+1)d$ topological order with an $nd$ gapped boundary and $n + 1D$ quantum liquids [1, 5]. This holographic duality can be realized by the so-called topological Wick rotation [1, 20]; given an $(n+1)d$ topological order with an $nd$ gapped boundary, we can intuitively ‘rotate’ the $(n+1)d$ bulk phase via its $nd$ boundary to the time direction to get an anomaly-free $n + 1D$ quantum liquid phase (see figure 1). This process allow us to modify an nd boundary phase $\mathcal{S}$ of an $n + 1d$ topological order $\mathcal{B}$ into an $n + 1D$ quantum liquid $\mathfrak{a} \mathcal{S}$.

\[ \mathcal{B} \rightarrow \mathcal{S} \rightarrow \mathfrak{a} \mathcal{S} \]

FIG. 1. The left hand side depicts an $(n+1)d$ topological order with an nd gapped boundary. After the topological Wick rotation, we get its holographic dual as depicted in the right hand side, which is an anomaly-free $n + 1D$ quantum liquid phase that can be described by $\mathfrak{a} \mathcal{S}$.

We mainly focus on the case that $n = 1$. In this case, the topological excitations in the 2d bulk form a unitary modular tensor category (UMTC) $\mathcal{B}$ and the topological excitations on the 1d boundary form a unitary fusion category $\mathcal{S}$. By the boundary-bulk relation \([21, 22]\) we have $\mathcal{B} \simeq \mathcal{Z}_1(\mathcal{S})$. After the topological Wick rotation, we get an anomaly-free 1 + 1D quantum liquid phase. The bulk excitations in $\mathcal{B} \simeq \mathcal{Z}_1(\mathcal{S})$ before the rotation are replaced by the topological sectors of operators in the 1+1D spacetime after the rotation, and the topological excitations in $\mathcal{S}$ become topological sectors of states. Then we have an enriched fusion category $\mathfrak{a} \mathcal{B} \simeq \mathcal{Z}_1(\mathfrak{a} \mathcal{S})$, which is just the topological skeleton of the obtained anomaly-free 1 + 1D phase.

Remark I.3. The anomaly-free property for the 1 + 1D phase after the rotation means $\mathfrak{a} \mathcal{S}$ is not a boundary of a one dimensional higher bulk. Mathematically, it is to say enriched fusion category $\mathfrak{a} \mathcal{S}$ has trivial Drinfeld center $\mathcal{Z}_1(\mathfrak{a} \mathcal{S}) \simeq \text{Vec}$.\(^\diamond\)

Remark I.4. These kind of topological holographic phenomena have been studied by various groups of people and in different contexts. The holographic duality based on topological Wick rotation was proposed by [1, 5]. The variations of this duality were also appeared in [1, 3–5, 20, 23–35].\(^\diamond\)

B. Classification of gapped phases and anyon condensation

Now if we fix a categorical symmetry $\mathcal{B}$ in the 1+1D anomaly-free gapped quantum liquid, how can we find different topological sectors of states $\mathcal{S}$, so as to classify these 1+1D gapped phases $\{\mathfrak{a} \mathcal{S}_i\}$ with the same categorical symmetry? Since anyon condensation theory can be used to classify gapped boundaries of a 2d UMTC $\mathcal{B}$, and $\mathcal{S}$ correspond to the 1d gapped boundaries of $\mathcal{B}$, we can use anyon condensation to give a classification of 1+1D gapped phases with categorical symmetry $\mathcal{B}$ through holographic duality.

\(^3\) Note the difference between holographic duality and lattice duality based on lattice symmetries.
So we briefly review the mathematical theory of 2d anyon condensation [36]. Suppose a 2d topological order (with excitations form a UMTC) \( \mathcal{C} \) is obtained from another 2d topological order \( \mathcal{B} \) via a 2d condensation, and a 1d gapped domain wall with the wall excitations form a unitary fusion category \( \mathcal{S} \) is produced as a result of this condensation process. Then the vacuum \( 1_\mathcal{C} \) of \( \mathcal{C} \)-phase is naturally a condensable algebra \( A \) in \( \mathcal{B} \) (see appendix A 2 for the definition of a condensable algebra). The excitations \( \mathcal{S} \) on the 1d domain wall can be identified with the category \( \mathcal{B}_A \) of all right \( A \)-modules in \( \mathcal{B} \). The bulk-to-wall map from the original phase \( \mathcal{B} \) to \( \mathcal{S} \) is given by

\[
L = - \otimes A : \mathcal{B} \rightarrow \mathcal{B}_A \simeq \mathcal{S}, \\
\quad a \mapsto a \otimes A, \quad \forall a \in \mathcal{B}.
\]

We mainly focus on the special case that the condensed phase \( \mathcal{C} \) is trivial, namely \( \mathcal{C} \simeq \text{Vec} \) (figure 2(a)). In this case the corresponding condensable algebra \( A \) is called Lagrangian. Also the domain wall \( \mathcal{S} \) becomes a boundary of the \( \mathcal{B} \)-phase and we have the boundary-bulk relation \( \mathcal{B} \simeq 3_1(\mathcal{S}) \) [37]. Hence the gapped boundaries of the \( \mathcal{B} \)-phase are classified by the Lagrangian algebras in \( \mathcal{B} \).

Now we apply the topological Wick rotation to the 2d topological order \( \mathcal{B} \) with boundary \( \mathcal{S} \) and get an anomaly-free 1+1D quantum liquid phase. Then we can view the condensation process as happening in space-time. See figure 2(b). The Lagrangian algebra \( A \in \mathcal{B} \) is condensed on the vacuum sector of states \( 1_\mathcal{S} \), in the sense that it consists of the operators that act on the topological sector of the ground state invariably. Therefore, we call this Lagrangian algebra \( A \) the *ground state algebra*\(^4\) (ground state algebra). Mathematically, this means that the ground state algebra \( A \) is the internal hom \( \text{Hom}_\mathcal{S}(1_\mathcal{S}, 1_\mathcal{S}) \) [38]. On the other hand, as we have known in the anyon condensation theory, the fusion category \( \mathcal{S} \) can be recovered as the module category \( \mathcal{B}_A \). Thus the topological skeleton \( \mathcal{B} \mathcal{S} \) of the phase can be completely determined by the categorical symmetry \( \mathcal{B} \) and the ground state algebra \( A \).

\[
^4 \text{We use Lagrangian algebra in the context of 2d anyon condensation theory and use ground state algebra in 1+1D topological sectors of states.}
\]

We mainly focus on the case that the 2d topological order is realized by the Kitaev quantum double model [8], which is the topological order that exhibits gauge symmetry associated with a finite group \( G \). Let \( \text{Rep}(G) \) denote the category of \( G \)-linear representation of \( G \) (or \( G \)-charges, physically). The topological excitations of this 2d topological order form the UMTC \( 3_1(\text{Rep}(G)) \) [39]. The Lagrangian algebras in \( 3_1(\text{Rep}(G)) \) are classified by pairs \((H, \omega)\), where \( H \) is a subgroup of \( G \) and \( \omega \in H^2(H, \text{U}(1)) \) is a 2-cohomology class [40]. We use \( A(H, \omega) \) to denote such a Lagrangian algebra. Readers may check appendix A 2 for the case that \( G \) is abelian. Equivalently, the gapped boundaries of the Kitaev quantum double model is also classified by the pair \((H, \omega)\) [41], and the topological excitations on the boundary is \( 3_1(\text{Rep}(G))_{A(H, \omega)} \). These classification results exactly meet with a known classification of 1d bosonic gapped phases with an onsite \( G \)-symmetry [42, 43].

After topological Wick rotation, \( 3_1(\text{Rep}(G))_{A(H, \omega)} \) becomes the topological sector of states, and the topological features of the quantum double model captured by \( 3_1(\text{Rep}(G)) \) becomes the categorical symmetry that acts on those states (see figure 3 (b)), and the ground state algebra is exactly \( A(H, \omega) \).

**Example I.5.** In particular, when \( H = G \), the 1+1D phases after the rotation are 1d symmetry protected topological (SPT) orders with symmetry \( G \). The topological sectors of states are all given by the symmetry charges which form the fusion category \( \text{Rep}(G) \), and the 2d bulk \( 3_1(\text{Rep}(G)) \) before the rotation is the gauged phase of the 2d trivial G-SPT phase [26, 44, 45]. The 2-cohomology classes \( \omega \in H^2(G; \text{U}(1)) \) determine different actions of the categorical symmetry \( 3_1(\text{Rep}(G)) \) on the category \( \text{Rep}(G) \) of topological sectors of states. Therefore, the SPT orders can be distinguished by the actions of the topological sectors of operators on the topological sectors of states. In particular, it suffices to use the ground state algebra \( A(G, \omega) \) consisting of the operators that act on the ground state invariantly to distinguish the SPT orders. We discuss the example with \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \) in Section V D. This gives a method to distinguish different

![FIG. 2. (a) depicts a normal 2d anyon condensation process from a 2d topological order \( \mathcal{B} \) to vacuum Vec, which generates a boundary phase described by \( \mathcal{S} \simeq \mathcal{B}_A \). After topological Wick rotation, we get (b), an anomaly-free 1+1D phase described by \( \mathcal{S} \simeq 3_1(\mathcal{S}) \). The original boundary phase \( \mathcal{S} \) now becomes the topological sector of states, and the original 2d bulk phase \( \mathcal{B} \) becomes the categorical symmetry that acts on those states.](image-url)
1d SPT orders without seeking a boundary or gauging the bulk. We believe that this concept can be generalized to higher dimensions.

Lattice models constructed in this work are mainly for trivial 2-cohomology. We expect that, with different choices of $H \subseteq G$, we obtain different symmetry breaking phases with the global symmetry $G$ on the 1+1D gapped phase breaking to its subgroup $H$. And the topological features on these different symmetry breaking phases should be captured by the enriched fusion category $\mathcal{Z}_{\mathcal{A}(H)}$. Also, in this case $\mathcal{A}(H)$ is the full center [46] of the function algebra $F_{\mathcal{H}} \coloneqq \text{Fun}(G/H) \in \text{Rep}(G)$, and $\mathcal{Z}_{\mathcal{A}(H)}$ is equivalent to the bimodule category $F_{\mathcal{H}} \text{Rep}(G)_{F_{\mathcal{H}}} \coloneqq \text{BMod}_{F_{\mathcal{H}}/F_{\mathcal{H}}} \text{Rep}(G)$). Or to say a boundary phase of $\mathcal{Z}_{\mathcal{A}(H)}$ can be described by $\mathcal{Z}_{\mathcal{A}(H)}$ from the 2d condensation perspective, or as $F_{\mathcal{H}} \text{Rep}(G)_{F_{\mathcal{H}}}$ form the 1d condensation perspective [47]. Thus, in this work we check the following theorem, which was originally proposed in [1, 3], in concrete lattice models:

**Theorem I.6.** A 1+1D bosonic gapped phase with an onsite symmetry $G$ can be described by the enriched fusion category $\mathcal{Z}_{\mathcal{A}(H)} \simeq \mathcal{Z}_{\mathcal{A}(H)}$ if the symmetry spontaneously breaks to a subgroup $H \subseteq G$.

The simplest example to illustrate theorem I.6 is for $G = \mathbb{Z}_2$, namely, we have the holographic duality between two 1d gapped boundaries of 2d toric code model [8] and the two 1+1D gapped phases of transverse Ising chain, see figure 3 (a). The process of demonstrating that the $\mathbb{Z}_2$ SPT phase can be described by $\mathcal{Z}_{\mathcal{A}(H)} \simeq \mathcal{Z}_{\mathcal{A}(H)}$ and the spontaneous symmetry-breaking phase can be described by $\mathcal{Z}_{\mathcal{A}(H)} \simeq \mathcal{Z}_{\mathcal{A}(H)}$ given in [3]. We would retell this story in section V.A. Conceptually, we generalize the 2d topological order of $\mathcal{B}$ from the toric code model $\mathcal{Z}_{\mathcal{A}(H)}$ to the quantum double model $\mathcal{Z}_{\mathcal{A}(H)}$. In order to show that the gapped boundaries of 2d quantum double model is one to one correspond to the 1+1D gapped phases after topological Wick rotation, we explicitly construct a 1+1D lattice model with onsite symmetry $G$ and read off its observables. It turns out that, for each symmetry breaking phase where $G$ breaks to its subgroup $H$, the topological sectors of operators and states together form enriched fusion category $\mathcal{Z}_{\mathcal{A}(H)} \simeq \mathcal{Z}_{\mathcal{A}(H)}$, as the holographic duality indicates. We depict this correspondence in figure 3 (b), in which $H_1, H_2, H_3 \ldots$ label different subgroups of $G$.

Note that though model III.2 is constructed artificially as a concrete example to check Theorem I.6, what matters here is indeed the categorical symmetry that defines the macroscopic observables (even lattices with different global symmetries as long as they are Morita equivalent as fusion categories). Namely, we can also have other models to get the same categorical description as long as they realize the same phases. We invent a general method in finding the descriptions of topological sectors of states with onsite $G$-symmetry in the next section.

Here is the layout of this paper: In section II, we analyze the categorical structure of the topological defects with onsite $G$-symmetry and show the topological sectors of states respect symmetry can be obtained from the so-called equivariantization. In section III, we construct lattice models with a finite abelian onsite symmetry $G$ and demonstrate that the categorical symmetry is $\mathcal{Z}_{\mathcal{A}(H)}$. We then analyze topological skeletons of each phase and prove our main result Theorem I.6. In section IV, we further check the enriched category descriptions of the 0d boundaries and domain walls of these 1d gapped phases and show they also agree with the predictions of holographic duality. In section V, we perform some physical examples including transverse field Ising model, its Kramers-Wannier dual, clock model and cluster model with non-trivial SPT order. In section VI, we discuss the physical perspectives in regard of ground state algebra. We also discuss the unification power of enriched categories in lattice dualities in section VII. Some background knowledge and calculations are performed in the appendices.

II. TOPOLOGICAL SECTORS OF STATES WITH ONSITE $G$-SYMMETRY

In this section, we translate the behavior of excitations in a topological order with an onsite $G$-symmetry into categorical language, in which we invent a general method to find the topological sectors of states with onsite symmetry $G$ through considering those states without symmetry. Analyses throughout this section do not depend on $G$ abelian and 2-cocycle trivial or not. This technique is also valid in higher dimensions. For readability, we hide some of the categorical details, which can be found in appendix B.

A. The category of topological sectors of states and group actions

First we consider observables in topological orders without symmetry. In LWLL, the coarse-graining process integrates and averages out the microscopic degrees of freedom by the action of local operators. Thus given a local operator $A$, the states $\{A|\psi\}$ and $\{\psi\}$ should be viewed as the same after coarse-graining. These states form a subspace of the total Hilbert space:

$$\{A|\psi\} \text{ or } \{\psi\}$$

This subspace is called the topological sector of states (or topological excitation) generated by $|\psi\rangle$, which can be viewed as a macroscopic observable [48].

Given an nd topological order $\mathcal{C}$, all (particle-like, i.e. 0+1D) topological sectors of states form a category $\mathcal{C}$. The hom space between two topological sectors of states
consists of non-local operators that map one topological sector of states to another [49]. Since topological sectors of states are invariant under the action of local operators, the hom spaces between them are also invariant under the action of local operators. A space of non-local operators that is closed under the action of local operators is called a topological sector of operators. It is clear that each hom space fall into a topological sector of operators.

**Remark II.1.** When \( n \geq 1 \), the topological sectors of states in \( \mathbb{C} \) can be fused together in space: if two topological sectors of states \( x \) and \( y \) are closed enough to each other, they can be viewed as a single topological sector of states. The non-local operators can also be fused together because they are localized in the spacetime and behave like a 0D defect. Figure 4 depicts the fusion of topological sectors of states in a 1+1D topological order. Then \( \mathcal{C} \) is a (multi-)fusion category with the tensor product given by the fusion of topological sectors of states.\( \diamond \)

Moreover, if \( B \) is an operator that maps a topological sector of states \( x \) to another one \( y \), then \( U(g)BU(g)^{-1} \) is an operator that maps \( T_g(x) \) to \( T_g(y) \). Hence we find the symmetry induces a functor \( T_g: \mathcal{C} \to \mathcal{C} \) for every \( g \in G \). Note that the functors \( T_g \circ T_h \) and \( T_{gh} \) are not necessarily equal on the nose because the states \( U(g)U(h)|\psi_x\rangle \) and \( U(gh)|\psi_x\rangle \) may be differed by a phase factor \( \omega_x(g,h) \). Such phase factors induce a natural isomorphism \( \gamma_{g,h}: T_g \circ T_h \Rightarrow T_{gh} \), in which the component \( \gamma_{g,h}|_x \) is given by the phase factor \( \omega_x(g,h) \) for all \( x \in \mathcal{C} \). By computing \( U(g)U(h)(U(k)|\psi_x\rangle \) in two different ways we obtain the following equation:

\[
\omega_{T_k(x)}(g,h)\omega_x(g,k) = \omega_x(h,k)\omega_x(g,hk), \quad g,h,k \in G.
\]

This equation translates to commutative diagram B.1. In other words, the \( G \)-symmetry on the topological order \( \mathbb{C} \) equips the category \( \mathcal{C} \) with a \( G \)-action.
Remark II.2. Recall that $C$ is a (multi-)fusion category when $n \geq 1$. Since $G$ is an onsite symmetry, the symmetry action operators $\{U(g)\}_{g \in G}$ preserve the fusion of topological sectors of states. Thus the $G$-action on $C$ is indeed a monoidal action, i.e., $T$ is a monoidal functor $T: G \rightarrow \text{Aut}_G(C)$ (see appendix B1 for details).

B. Equivariantization and $G$-symmetric topological sectors of states

Not all topological sector of states are observables in LWLL after imposing a symmetry $\{U(g)\}_{g \in G}$. An operator is $G$-symmetric if it commutes with each $U(g)$. An observable in LWLL for a topological order with $G$-symmetry should be a subspace $x$ of the total Hilbert space that is invariant under both the action of $G$-symmetric local operators and the $G$-action.

Since we only know how to express topological sectors of states without symmetry in $C$, so in order to formulate $x$, we first apply all local operators on $x$ to obtain a larger subspace $\tilde{x}$ that is an object in $C$. Note that $x$ is a $G$-invariant subspace (i.e., subrepresentation) of $\tilde{x}$, and this invariant subspace is characterized by its behavior under the $G$-action. To restore the $G$-action of $x$ on $\tilde{x}$, we act the operator $U(g)$, the subspace $x$ is invariant, but the states in $x$ may be changed by a (possibly non-abelian) phase. Categorically, such a phase under the $U(g)$-action is described by an isomorphism

$$u_g: T_g(\tilde{x}) \xrightarrow{\sim} \tilde{x}, \quad g \in G.$$

By comparing the action of $U(g)U(h)$ and $U(gh)$ for $g, h \in G$, we get commutative diagram B.2 for $\tilde{x}$. Then the $G$-symmetric topological sector of states $x$ is characterized by the pair $\langle \tilde{x}, \{u_g\}_{g \in G} \rangle$.

In addition, an operator between two such topological sectors of states $\langle \tilde{x}, \{u_g\}_{g \in G} \rangle$ and $\langle \tilde{y}, \{v_g\}_{g \in G} \rangle$ should also be symmetric, i.e., commutes with operators $U(g)$ for all $g \in G$. Categorically, it means diagram B.3 commutes. Hence, the category of $G$-symmetric topological sector of states is an equivariantization (see Definition B.2). Thus we have the following physical theorem.

Theorem II.3. Let $C_n$ be an nd topological order with a unitary onsite $G$-symmetry. Suppose all particle-like topological sectors of states (after ignoring the symmetry) form a category $\mathcal{C}$.

(1) The $G$-symmetry induces a $G$-action on the category $\mathcal{C}$.

(2) The category of (particle-like) $G$-symmetric topological sectors of states is equivalent to the equivariantization $\mathcal{C}^G$.

As a result, for a 1+1D gapped phase with onsite symmetry $G$, $\mathcal{C}^G$ is the underlying category of its topological skeleton. We will demonstrate these structures in a 1+1D lattice model in the next section.

III. GENERAL ANALYSIS FOR LATTICE MODEL WITH ABELIAN ONSITE $G$-SYMMETRY

In this section, we construct a lattice model exhibits an onsite abelian $G$-symmetry to analyze its macroscopic observables in both symmetric and symmetry breaking cases. This model construction can be generalized to any 1+1D gapped phases with fusion categorical symmetries [30, 50].

A. The lattice model and ground states

Consider the 1+1D lattice (defined on an infinitely long open interval) with the total Hilbert space $\mathcal{H}_{\text{tot}} = \bigotimes_{i \in Z} \mathcal{H}_i$, where each local Hilbert space $\mathcal{H}_i$ is the group algebra $\mathbb{C}[G]$ spanned by an orthonormal basis $\{g\}_{g \in G}$. For each site $i$ and $g \in G$, we define an operator $L_i^g$ acting on $\mathcal{H}_i$ as follows:

$$L_i^g|\psi\rangle := |gh\rangle_i, \quad \forall h \in G.$$

The global $G$-symmetry action is defined by

$$U(g) := \bigotimes_i L_i^g, \quad \forall g \in G.$$

In order to construct the Hamiltonian, we need to define some operators. Recall that all group homomorphisms $G \rightarrow U(1)$ form a finite abelian group $\hat{G}$, called the dual group of $G$. Equivalently, $\hat{G}$ is also the group of equivalence classes of irreducible $G$-representations. For each site $i$ and $\rho, \sigma \in \hat{G}$, we define an operator $Z^\rho_i$ acting on $\mathcal{H}_i$ by

$$Z^\rho_i|\psi\rangle := \rho(g)|\psi\rangle.$$

It is clear that the following equations hold for all $g, k \in G$ and $\rho, \sigma \in \hat{G}$:

$$L_i^g L_i^k = L_i^{gk}, \quad (L_i^g)^\dagger = (L_i^g)^{-1} = L_i^{-g^{-1}},$$

$$Z^\rho_i Z^\rho_i = Z^{\rho \sigma}_i, \quad (Z^\rho_i)^\dagger = (Z^\rho_i)^{-1} = Z^{\rho^{-1}}_i,$$

$$L_i^g Z^\rho_i (L_i^{-g})^{-1} = \rho(g)^{-1} Z^\rho_i. \tag{III.1}$$

Let $H \subseteq G$ be a subgroup of $G$. Note that $\hat{G}/H$ can be naturally viewed as a subgroup of $\hat{G}$: every group homomorphism $\lambda: G/H \rightarrow U(1)$ induces a group homomorphism $G \rightarrow G/H \xrightarrow{\lambda} U(1)$, which is nontrivial if $\lambda$ is nontrivial. Then for each site $i$ we define two Hermitian operators:

$$X_i^H := \frac{1}{|H|} \sum_{h \in H} L_i^h, \quad Z_i^{H^i+1} := \frac{|H|}{|G|} \sum_{\rho \in G/H} Z^\rho_i (Z_i^{\rho+1})^\dagger.$$

It is easy to verify that these operators are mutually commuting projectors and $G$-symmetric (i.e. commute with $U(g)$ for all $g \in G$).
The lattice model we considered depends on the symmetry group $G$ and a chosen subgroup $H \subseteq G$. The Hamiltonian is defined as follows:

$$\mathcal{H} := \sum_i (1 - X_{h,i}^j) + \sum_i (1 - Z_{h,i}^{i+1}).$$  \hfill (III.2)

**Remark III.1.** When $G = \mathbb{Z}_n$, the Hamiltonian equivalently realizes the bosonic quantum clock model [51], where $L^i$ and $Z^i$ are $n \times n$ generalizations of the Pauli matrices $\sigma_x$ and $\sigma_z$. In particular, when $G = \mathbb{Z}_2$, it recovers the well-known 1d quantum Ising model. We illustrate both cases explicitly in section V.

Since the Hamiltonian (III.2) is the sum of commuting projectors, it is not hard to verify that the ground state subspace has dimension $|G|/|H|$ and there is an orthonormal basis labeled by coset $x \in G/H$:

$$|\psi_x\rangle := \bigotimes_i \frac{1}{\sqrt{|H|}} \sum_{y \in x} |y\rangle_i.$$ \hfill (III.3)

Moreover, we have $U(g)|\psi_x\rangle = |\psi_{gx}\rangle$ for $g \in G$ and $x \in G/H$. Thus the ground state $|\psi_H\rangle$ (indeed, every ground state $|\psi_x\rangle$) is stable under the action of $U(h)$ for $h \in H$. Hence the lattice model (III.2) spontaneously breaks the $G$-symmetry to the subgroup $H$.

**B. Topological sectors of operators**

Now let us analyse the topological sector of operators within this model. There are two kinds of $G$-symmetric non-local operators:

$$M_g^i := \prod_{j \leq i} L_g^j, \forall g \in G, \quad E_\rho^i = \prod_{j \geq i} Z^{i}_\rho^{j+1}, \forall \rho \in \hat{G}.$$  

Since $E_\rho^i$ is the product of infinitely many $G$-symmetric operators, it should be viewed as a $G$-symmetric non-local operator. On the other hand, the single operator $Z^i_\rho$ is a non-symmetric local operator. Note that $E_\rho^i = Z^i_\rho$ as an operator that catches all the corrected properties near site $i$.

**Remark III.2.** These two operators are also known as fluxions and chargeons in a quantum double model [41]. In some literatures [4, 31], they are called “patch operators”.

Apparently, the product $M_g^i E_\rho^j$ is also a symmetric non-local operator. We denote the topological sector of operators generated by $M_g^i E_\rho^j$ by $\mathcal{O}_{(g,\rho)}$. They form the simple objects of the background category.

By (III.1) we have

$$M_g^i E_\rho^j = \begin{cases} E_\rho^i M_g^j, & i < j, \\ \rho(g)^{-1} E_\rho^i M_g^j, & i \geq j. \end{cases}$$

Thus, $M_g^i E_\rho^j M_h^k E_\sigma^l$ and $M_g^i M_h^k E_\rho^j E_\sigma^l$ differ by a coefficient at most. It follows that the fusion products of these topological sectors of operators are given by

$$\mathcal{O}_{(g,\rho)} \otimes \mathcal{O}_{(h,\sigma)} = \mathcal{O}_{(gh,\rho\sigma)}.$$ \hfill (III.4)

This recovers the fusion structure in $\mathcal{Z}_1(\text{Rep}(G))$.

Moreover, for $i < k < j$ we have

$$M_g^i M_g^j E_\rho^k M_g^j M_g^i = \rho(g) \cdot E_\rho^k.$$ 

This means that the double braiding of two topological sectors $\mathcal{O}_{(g,1)}$ and $\mathcal{O}_{(e,\rho)}$ is

$$\mathcal{O}_{(g,1)} \otimes \mathcal{O}_{(e,\rho)} \xrightarrow{\rho(g)^{-1}(e)} \mathcal{O}_{(g,1)} \otimes \mathcal{O}_{(e,\rho)}.$$ 

Conceptually, we can regard this process as first creating a pair of fluxions at sites $i$ and $j$, then winding a chargeon around one of the fluxions (where the chargeon would cross in between the fluxions on site $k$), after this process, we annihilate the pair of fluxions and obtain a phase $\rho(g)$.

For a general braiding process, we have

$$M_g^i E_\rho^j E_\rho^i M_g^j M_g^i E_\rho^k M_g^j E_\rho^i M_g^j M_g^i,$$

and this equation gives the double braiding

$$\mathcal{O}_{(h,\sigma)} \otimes \mathcal{O}_{(g,\rho)} \xrightarrow{\sigma(h)^{-1}(\rho)} \mathcal{O}_{(h,\sigma)} \otimes \mathcal{O}_{(g,\rho)}. \hfill (III.5)$$

The topological sectors of operators $\{\mathcal{O}_{(g,\rho)}\}_{g \in G, \rho \in \hat{G}}$, together with the fusion and braiding properties (III.4) (III.5), form a braided fusion category equivalent to $\mathcal{Z}_1(\text{Rep}(G))$ (see appendix A1 and example A3 for details).

**Theorem III.3.** In a 1d lattice model with an on-site abelian $G$-symmetry, the topological sectors of operators $\{\mathcal{O}_{(g,\rho)}\}_{g \in G, \rho \in \hat{G}}$, together with the fusion and braiding properties (III.4) (III.5), form a braided fusion category equivalent to $\mathcal{Z}_1(\text{Rep}(G))$.

**C. The topological skeletons**

In order to convince the readers of the categorical structure stated in Theorem III.6, we would like to give the exact categorical description for each phase step by step. It would be beneficial to first check the symmetry preserving case and the symmetry completely broken case.

1. **Symmetry preserving case (the trivial SPT)**

Consider the phase realized by (III.2) with $H = G$. In other words, the Hamiltonian is

$$\mathcal{H} = \sum_i (1 - X_{i}^i). \hfill (III.6)$$
In this case there is no symmetry breaking and the unique ground state is

$$|\Omega_G\rangle := \bigotimes_i \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle_i.$$  

Since the ground state $|\Omega_G\rangle$ is a product state, the Hamiltonian (III.6) realizes the trivial $G$-SPT order.

We use $S_1$ to denote the topological sector of states generated by the ground state $|\Omega_G\rangle$ (i.e. the trivial topological sector of states). Note that $M_g^j|\Omega_G\rangle = |\Omega_G\rangle$ for all sites $i$ and group $g \in G$, so the non-local operator $M_g^j$ does not generate a new topological sector of states.

Then we consider the topological sector of states generated by $E^j_{\rho}$. For each $\rho \in \hat{G}$ and site $i$, we define

$$|\rho, i\rangle := E^j_{\rho} |\Omega_G\rangle.$$  

In particular, $|1, i\rangle = |\Omega_G\rangle$ where 1 $\in \hat{G}$ is the trivial $G$-representation. By the orthogonality of characters, one can verify that $X_G^j|\rho, i\rangle = 0$ if $\rho \neq 1$ and $X_G^j|1, i\rangle = X_G^j|\Omega_G\rangle = |\Omega_G\rangle$. Moreover, for two sites $i$ and $j$, the states $|\rho, i\rangle$ and $|\rho, j\rangle$ generate the same $G$-symmetric topological sector of states because they are differed by a symmetric local operator $Z^j_{\rho}(Z^i_{\rho})^\dagger$:

$$|\rho, j\rangle = Z^j_{\rho}(Z^i_{\rho})^\dagger |\rho, i\rangle.$$  

Therefore, we use $S_\rho$ to denote the $G$-symmetric topological sector of states generated by $|\rho, i\rangle$, they form simple objects of the underlying category.

**Remark III.4.** The state $|\rho, i\rangle$ can also be regarded as created from the ground state by a non-symmetric local operator $Z^i_{\rho}$. So if we ignore the $G$-symmetry, the state $|\rho, i\rangle$ belongs to the trivial topological sector of states $S_1$, in this situation the underlying category is just Vec. $\diamondsuit$

Intuitively, the $G$-symmetric topological sector of states $S_\rho \otimes S_\sigma$ (i.e. the fusion of $S_\rho$ and $S_\sigma$) should be generated by two non-local operators $E^i_{\rho^j}$, $E^j_{\sigma^i}$, i.e. consider the following state:

$$|\rho, i; \sigma, j\rangle := E^i_{\rho^j} E^j_{\sigma^i} |\Omega_G\rangle \in S_\rho \otimes S_\sigma,$$

where $\rho, \sigma \in \hat{G}$ and $i < j$. On the other hand, by acting the symmetric local operator $Z^i_{\rho}(Z^j_{\rho})^\dagger$ on this state we find that $|\rho, i; \sigma, j\rangle$ generates the same $G$-symmetric topological sector of states as $|\rho\sigma, j\rangle$:

$$Z^i_{\rho}(Z^j_{\rho})^\dagger |\rho, i; \sigma, j\rangle = E^i_{\rho^j} E^j_{\sigma^i} |\Omega_G\rangle = E^i_{\rho\sigma} |\Omega_G\rangle = |\rho\sigma, j\rangle.$$  

It follows that the fusion rules of topological sectors of states of the phase realized by (III.6) are given by

$$S_\rho \otimes S_\sigma = S_{\rho\sigma}.$$  

Therefore, the fusion category of topological sectors of states is equivalent to the fusion category $\text{Rep}(G)$ of finite-dimensional $G$-representations.

From another perspective, if we view (III.6) as a system without symmetry, it realizes the trivial 1d topological order (without symmetry) because the ground state $|\Omega_G\rangle$ is a product state. Thus the topological sectors of states form a fusion category equivalent to $\text{Vec}$ (see also remark III.4). By Theorem II.3, if we view (III.6) as a system with $G$-symmetry, the $G$-symmetric topological sectors of states form a fusion category equivalent to the equivariantization $\text{Vec}^G \simeq \text{Rep}(G)$ (see example B.4), which coincide with the upper analysis.

Now we have the topological sectors of operators $\mathcal{Z}_1(\text{Rep}(G))$ and the topological sectors of states $\text{Rep}(G)$. We want to figure out the action of $\mathcal{Z}_1(\text{Rep}(G))$ on $\text{Rep}(G)$ on the macroscopic level.

Since $M_g^j$ only has trivial actions on the ground state $|\Omega_G\rangle$, the operator $M_g^j E^j_{\rho}$ also maps the sector $S_\sigma$ to $S_{\rho\sigma}$ for all $g \in G$ and $\rho, \sigma \in \hat{G}$. Equivalently, $M_g^j E^j_{\rho\sigma^{-1}}$ maps the topological sector $S_\rho$ to $S_\sigma$. Hence, the action of $\mathcal{Z}_1(\text{Rep}(G))$ on $\text{Rep}(G)$ can be expressed by

$$\mathcal{O}_{(g, \rho\sigma^{-1})} \otimes S_\rho = S_\sigma \quad (\text{III.7})$$

This action can also be characterized by the hom spaces between topological sectors of states:

$$\text{Hom}(S_\rho, S_\sigma) = \bigoplus_{g \in G} \mathcal{O}_{(g, \rho\sigma^{-1})}. \quad (\text{III.8})$$

This is indeed the internal hom of $\mathcal{Z}_1(\text{Rep}(G))\text{Rep}(G)$.

Equivalently, this action can be determined by the ground state algebra consisting of the operators that act on the topological sector of the ground state invariably. In this case, the ground state algebra is

$$A(G) := \text{Hom}(S_1, S_1) = \bigoplus_{g \in G} \mathcal{O}_{(g, 1)} \in \mathcal{Z}_1(\text{Rep}(G)).$$

In the view of holographic duality, the trivial action of $A(G)$ or the operators $M_g^j$ on the ground state can be interpreted as the condensation of fluxions.

Hence we have proved the following physical theorem within a concrete model step by step:

**Theorem** III.5. The macroscopic observables of the $1 + 1$D trivial SPT order with the bosonic onsite abelian symmetry $G$ form enriched fusion category $\mathcal{Z}_1(\text{Rep}(G))\text{Rep}(G) \simeq \mathcal{Z}_1(\text{Rep}(G)) \mathcal{Z}_1(\text{Rep}(G))_{A(G)}$.

2. **Symmetry completely broken case**

The procedures of finding the underlying category when $G$ is completely broken (i.e., $H = \{e\}$ is the trivial group) follows a similar path. The Hamiltonian now is

$$\mathcal{H} = \sum_i (1 - Z^i_{\{e\}}). \quad (\text{III.9})$$
In this case, the ground state subspace has dimension \(|G|\) and there is an orthonormal basis labeled by group elements \(g \in G\):

\[
|\psi_g\rangle := \bigotimes_i |g\rangle_i.
\]

Clearly we have \(U(g)|\psi_h\rangle = |\psi_{gh}\rangle\) for \(g, h \in G\).

First we view (III.9) as a system without symmetry. Since each ground state \(|\psi_g\rangle\) is a product state, (III.9) realizes the direct sum of \(|G|\) copies of the trivial 1d topological order (physically we need a fine tuning to realize this phase). There are \(|G|^2\) simple topological sectors of states (domain walls) \(\{T_{g,h}\}_{g,h \in G}\) generated by the following states:

\[
|\psi_{g,h,i}\rangle := \left(\bigotimes_{j \leq i} |g\rangle_j\right) \otimes \left(\bigotimes_{j > i} |h\rangle_j\right), \quad g, h \in G.
\]

FIG. 5. A topological sector of states without symmetry \(T_{g,h}\) can be generated by the state \(|\psi_{g,h,i}\rangle\), which can be interpreted as a 0+1D domain wall on 1+1D lattice model.

Clearly the fusion rules of these topological sectors of states are given by

\[
T_{g,h} \otimes T_{k,l} = \delta_{h,k} T_{g,l}, \quad g, h, k, l \in G.
\]

Therefore, these topological sectors of states form a multi-fusion category equivalent to \(\text{Mat}_{|G|}(\text{Vec})\) of \(|G|\)-by-\(|G|\) matrices valued in \(\text{Vec}\), or equivalently, \(\text{Fun}(\text{Vec}_G, \text{Vec}_G)\) of linear functors.

If we view (III.9) as a system with \(G\)-symmetry, the symmetry group \(G\) acts on the multi-fusion category \(\text{Fun}(\text{Vec}_G, \text{Vec}_G)\) of topological sectors of states, and the fusion category of \(G\)-symmetric topological sector of states is equivalent to the equivariantization \(\text{Fun}(\text{Vec}_G, \text{Vec}_G)^G\) by Theorem\(\text{\textsuperscript{ph II.3}}\). The \(G\)-action on states is as follows:

\[
U(g)|\psi_{h,k,i}\rangle = |\psi_{gh,k,i}\rangle, \quad g, h, k \in G.
\]

Thus the \(G\)-action on \(\text{Fun}(\text{Vec}_G, \text{Vec}_G) \cong \text{Mat}_{|G|}(\text{Vec})\) is given by

\[
g(T_{h,k}) = T_{gh,\cdot k}.
\]

Mathematically, the equivariantization \(\text{Fun}(\text{Vec}_G, \text{Vec}_G)^G\) is equivalent to \(\text{Vec}_G\) as fusion categories. This is a special case of Proposition III.7 below (see also remark B.11).

\[
\begin{array}{cccccccc}
\cdots & |g\rangle & |g\rangle & |g\rangle & |g\rangle & |g\rangle & |g\rangle & |g\rangle \\
\cdots & i-2 & i-1 & i & i+1 & i+2 & \cdots & \cdots
\end{array}
\]

FIG. 6. When symmetry \(G\) is imposed on the system, the global symmetry \(U(g)\) does not generate a new symmetric sector, e.g. \(T_{gh,ak}\) and \(T_{hk,\cdot k}\) both belong to the \(G\)-symmetric topological sector of states \(S_{k+i-1}\).

The mathematical proof can also be reformulated in physical language by explicitly finding the \(G\)-symmetric topological sectors of states in this lattice model. We consider the ground states first. Since the \(G\)-action permutes the ground states \(|\psi_g\rangle\), each ground state (and hence the ground state subspace) generates the same \(G\)-symmetric topological sector of states:

\[
S_c := \bigoplus_{g \in G} T_{g,b}.
\]

Similarly, given \(g \in G\), the states \(M^g_{ij} |\psi_h\rangle = |\psi_{gh,\cdot j\cdot i}\rangle\) for all \(h \in G\) and sites \(i\) generate the same \(G\)-symmetric topological sector of state (note that \(E^g_{ij}|\psi_h\rangle = \hat{\rho}(h)|\psi_h\rangle\) does not create a new topological sector of states):

\[
S_g := \bigoplus_{h \in G} T_{gh,\cdot j}\.
\]

These \(S_g\)’s form the simple objects of the underlying category. It is not hard to see that the fusion products of \(G\)-symmetric topological sectors of states are given by

\[
S_g \otimes S_h = S_{gh}.
\]

Hence the fusion category of \(G\)-symmetric topological sectors of states is equivalent to \(\text{Vec}_G\).

Now we have the topological sectors of operators \(3_1(\text{Rep}(G))\) and the topological sectors of states \(\text{Vec}_G\). Since \(M^g_{ij}\) maps the sector \(S_h\) to \(S_{gh}\), and \(E^g_{ij}\) has only trivial actions on these sectors, the operator \(M^g_{ij} E^g_{ij}\) maps the sector \(S_h\) to \(S_{gh}\) for all \(g, h \in G\) and \(\rho \in G\). The action of \(3_1(\text{Rep}(G))\) on \(\text{Vec}_G\) on the macroscopic level is given by

\[
O_{(h\cdot g^{-1}, \rho)} \circ S_h = S_h \quad \text{(III.10)}
\]

This action can also be characterized by the hom spaces between \(G\)-symmetric topological sectors of states:

\[
\text{Hom}(S_g, S_h) = \bigoplus_{\rho \in G} O_{(h\cdot g^{-1}, \rho)}. \quad \text{(III.11)}
\]

This is indeed the internal hom of \(3_1(\text{Rep}(G))\). In particular, the ground state algebra is

\[
\mathcal{A}(\{e\}) := \text{Hom}(S_e, S_e) = \bigoplus_{\rho \in G} O_{(e, \rho)} \in 3_1(\text{Rep}(G)).
\]

In the view of holographic duality, the trivial action of \(\mathcal{A}(\{e\})\) or the operators \(E^g_{ij}\) on the sector of ground state can be interpreted as the condensation of chargeons.

Hence we have proved the following physical theorem:
Let $G$ be a finite group and $H \subseteq G$ be a subgroup. Then the fusion category $\text{Rep}(G)_H$ is monoidally equivalent to the equivariantization $\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H})^G$. Also we have the following mathematical result:

**Proposition III.7.** Let $G$ be a finite group and $H \subseteq G$ be a subgroup. Then the fusion category $\text{Rep}(G)_H$ is monoidally equivalent to the equivariantization $\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H})^G$. Thus the fusion category of $G$-symmetric topological sectors of states is also equivalent to the fusion category $F_{\text{ph}} \text{Rep}(G)_F \text{Rep}(G)_F$ of $(F_H, F_H)$-bimodules in $\text{Rep}(G)$. As in the symmetry completely broken case, we translate the mathematical proof of Proposition III.7 in appendix B.2 to physical language by explicitly finding the $G$-symmetric topological sectors of states.

First note that $M_g^1|\psi_{x,x,j}\rangle$ generates the topological sector of states $T_{gx,x}$ without symmetry. For each $g \in G$, $x \in G/H$. Then for each $\rho \in \hat{G}$, the following states generate a simple $G$-symmetric topological sector of states, denoted by $S_{g,\rho}$:

$$\{E^i_g M^j_g |\psi_{x,x,j}\rangle\}_{x \in G/H}$$

FIG. 8. When symmetry $G$ is imposed on the system, the global symmetry $U(g)$ does not generate a new symmetric sector, e.g. $T_{gx,gy}$ and $T_{x,\varphi}$ both belong to the $G$-symmetric topological sector of states. Furthermore, the $G$-action would also be distinguished by the action of $E^i_g$, which leads to another structure $\rho$ on the symmetric topological sector.

Or we can write a $G$-symmetric topological sector of states as

$$S_{g,\rho} = (\bigoplus_{x \in G/H} T_{gx,x,\rho})$$

But there are some redundancy in counting of $S_{g,\rho}$ for each $g \in G$ and $\rho \in \hat{G}$. Note that

1. for $g, g' \in G$ satisfying $g^{-1} g' \in H$, we have $M^1_g |\psi_{x,x,j}\rangle = M^1_{g'} |\psi_{x,x,j}\rangle$;

2. for $\rho, \rho' \in \hat{G}$ satisfying $\rho^{-1} \rho' \in \hat{G}/H$, the states $E^{1}_g |\psi_{x,x,j}\rangle$ and $E^{1}_{\rho'} |\psi_{x,x,j}\rangle$ are differed by a nonzero factor, thus correspond to the same quantum state.

Therefore, $S_{g,\rho}$ only depends on the equivalence classes (cosets) $[g] \in G/H$ and $[\rho] \in \hat{G}/\hat{H} \cong \hat{H}$. So we denote these $G$-symmetric topological sectors of state by $\{S_{[g],[\rho]}\}_{[g] \in G/H, [\rho] \in \hat{H}}$. They form distinguished simple objects of the underlying category.

The fusion products of topological sectors of states are given by

$$S_{[g],[\rho]} \otimes S_{[h],[\sigma]} = S_{[gh],[\rho\sigma]}, \quad g, h \in G, \rho, \sigma \in \hat{G}.$$ Hence the fusion category of $G$-symmetric topological sectors of states is equivalent to $F_{\text{ph}} \text{Rep}(G)_F \text{Rep}(G)_F$. Now we have the topological sectors of operators $\mathcal{M}(\text{Rep}(G))$ and the topological sectors of states $F_{\text{ph}} \text{Rep}(G)_F \text{Rep}(G)_F$. It is not hard to see that the operator
$M^i_j E^j_i$ maps the sector $S_{[g],[\rho]}$ to $S_{[g],[\rho]}$ for all $g, h \in G$ and $\rho, \sigma \in \hat{G}$. The action of $\mathcal{Z}_1(\text{Rep}(G))$ on $F_H \text{Rep}(G)_{F_H}$ on the macroscopic level is given by

$$\mathcal{O}(k, \lambda) \otimes S_{[g],[\rho]} = S_{[g],[\lambda\rho]}$$

Thus the hom spaces between $G$-symmetric topological sectors of states are given by

$$\text{Hom}(S_{[g],[\rho]}, S_{[k],[\sigma]}) = \bigoplus_{k \in [h^{-1}g], \lambda \in [\sigma\rho^{-1}]} \mathcal{O}(k, \lambda).$$

By comparing with example C.5 we see that this enriched category $\mathcal{Z}_1(\text{Rep}(G))_{F_H} \text{Rep}(G)_{F_H}$ is precisely the one obtained from the canonical construction of the obvious action of $\mathcal{Z}_1(\text{Rep}(G))$ on $F_H \text{Rep}(G)_{F_H}$.

Hence we have proved the following through concrete models:

**Theorem III.8.** The macroscopic observables of the $1 + 1$D gapped phase with an onsite abelian symmetry $G$ form enriched fusion category $\mathcal{Z}_1(\text{Rep}(G))_{F_H} \text{Rep}(G)_{F_H}$ if the symmetry spontaneously breaks to a subgroup $H \subseteq G$.

**Remark III.9.** The categorical description $\mathcal{Z}_1(\text{Rep}(G))_{F_H} \text{Rep}(G)_{F_H}$ is anomaly-free in the sense that the Drinfeld center of this category is trivial:

$$\mathcal{Z}_1(\mathcal{Z}_1(\text{Rep}(G))_{F_H} \text{Rep}(G)_{F_H}) \simeq \text{Vec}.$$ 

See [7, section 5.3] for the definition of the Drinfeld center ($E_i$-center) of an enriched fusion category and the proof of the above equivalence.

**Remark III.10.** In the view of holographic duality, we may say the operators are ‘partially condensed’, with $E^i_\rho$ acting trivially on the trivial $G$-symmetric topological sector of states $S_{[\rho],[1]}$ for each $\rho \in \hat{G}/H$, and $M^i_j$ acting trivially on $S_{[\rho],[1]}$ for each $g \in H$. More precisely, these operators form the ground state Lagrangian algebra

$$A(H) := \text{Hom}(S_{[\rho],[1]}, S_{[\rho],[1]}) = \bigoplus_{g \in H, \rho \in \hat{G}/H} \mathcal{O}(g, \rho),$$

in a 2d quantum double model $\mathcal{Z}_1(\text{Rep}(G))$.

In this way the underlying bimodule category $F_H \text{Rep}(G)_{F_H}$ corresponds to the 1d gapped boundary $\mathcal{Z}_1(\text{Rep}(G))_{A(H)}$ of $A(H)$-modules in $\mathcal{Z}_1(\text{Rep}(G))$. Or to say, a $1+1$D gapped phase with an onsite symmetry $G$ can be equivalently written as $\mathcal{Z}_1(\text{Rep}(G)) \mathcal{Z}_1(\text{Rep}(G))_{A(H)}$ if the symmetry spontaneously breaks to a subgroup $H \subseteq G$.

By now we have finished the proof of our main theorem I.6 in concrete models.

**IV. 0+1D boundaries and domain walls with abelian onsite symmetry**

In this section we discuss the $0+1$D open boundaries with an onsite abelian symmetry $G$ explicitly broken and domain walls of those $1+1$D bosonic gapped phases.

**A. Boundaries**

In principle, for a $0d$ topological phase, it is not precise to talk about particle-like excitations, since each excitation can be viewed as a $0d$ phase itself, but there is still a $0+1$D categorical structure given by the ‘category of boundary conditions’.

More precisely, the category of boundary conditions is defined as follows:

- The objects are $0d$ boundary phases, which can be identified with the topological sector of states generated by the ground state.
- The morphisms are topological sectors of operators between objects, living on the worldline of boundary phases.

In this section we consider $0+1$D open boundary that explicitly breaks the bulk symmetry $G$ to its subgroup $H_b \subseteq G$. That is, all $H_b$-symmetric $0d$ boundary phases form a category $\mathcal{X}$. The topological sectors of operators can be fused along the world line. So they form a fusion category $A$. We call this fusion category $A$ the categorical symmetry of the boundaries. Therefore, the category of boundary conditions is an $A$-enriched category $\mathcal{X}^A$.

In practice, we can just write down a boundary Hamiltonian that preserves the symmetry we want. This would give us a distinguished object $x \in \mathcal{X}$. Then we can act the non-local operators on it to get other boundary conditions.

1. $0+1$D open boundaries of $1+1$D trivial SPT bulk

In order to see the categorical structures of the boundary phases in lattice models, we again start from the most obvious case, where we choose the $1+1$D bulk to be the trivial SPT order, whose topological skeleton is $\mathcal{Z}_1(\text{Rep}(G))_{\text{Rep}(G)}$ as theorem III.5 shows. We pick the boundary phase on the left side, i.e. $\mathcal{H}_{\text{tot}} = \bigotimes_{i \geq 0} \mathcal{H}_i$ (the right side is similar) and analyze its boundary conditions:

**G-symmetric boundary conditions:** First we consider the boundary conditions that preserve $G$-symmetry. For example, the following boundary Hamiltonian preserves the $G$-symmetry on the boundary:

$$\mathcal{H} = \sum_{i \geq 0} (1 - X^i_H).$$ (IV.1)
Its ground state is simply $\bigotimes_{i \geq 0} \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g \rangle_i$. This Hamiltonian is the most trivial case that preserves $G$-symmetry on the boundary (where we can obtain it by taking $\rho$ to be the trivial representation 1 in equation (IV.2)), we may call this Hamiltonian the canonical choice. Adding a $G$-symmetric operator to this Hamiltonian also gives a $G$-symmetric boundary condition. For example, for every $\rho \in G$ the following Hamiltonian realizes a $G$-symmetric boundary condition:

$$\mathcal{H} = \sum_{i > 0} (1 - X^{i}_{g}) + (1 - \frac{1}{|G|} \sum_{g \in G} \rho(g)L^{0}_{g}).$$  \hfill (IV.2)

Its ground state is $\frac{1}{\sqrt{|G|}} \sum_{g \in G} \rho(g)|g \rangle_{0} \otimes (\bigotimes_{i > 0} \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g \rangle_{i})$, and the corresponding boundary phase is denoted by $S_{\rho}$. Then the underlying category of the enriched category of $G$-symmetric boundary conditions matches with $\text{Rep}(G)$.

**Remark IV.1.** We can also apply the equivariantization technique introduced in section II B in 0+1D case. Since the 1d SPT bulk without symmetry is $\text{Vec}$ III.4, its open boundary when viewed as a 0d topological order can only be $\text{Vec}$. After imposing symmetry $G$ on the boundary, the topological sectors of states form $\text{Vec}^G \simeq \text{Rep}(G)$ rightfully so.

When acting $E^{0}_{\sigma}$ on the ground state, site 0 becomes $\frac{1}{\sqrt{|G|}} \sum_{g \in G} \sigma^{g}(g)\rho(g)|g \rangle_{0} \in S_{\sigma_0}$. Or to say, $E^{0}_{\sigma}$ maps $S_{\rho}$ to $S_{\sigma \rho}$. Note that for every $g \in G$, the operator $M^{0}_{g} = \prod_{k \leq 1} L^{k}_{g}$ is the product of symmetric local operators. So it should be viewed as ‘local’ on $G$-symmetric boundaries. Therefore, the topological sectors of operators mapping one $G$-symmetric boundary conditions to another are invariant under the action of $M^{0}_{g}$. Mathematically, this means that the topological sectors of operators are the modules over the algebra $A(G) = \bigoplus_{g \in G} O_{(g, 1)} \in Z_{1}(\text{Rep}(G))$. Therefore, the categorical symmetry of symmetry preserving boundary conditions is equivalent to the category $Z_{1}(\text{Rep}(G))_{A(G)}$. More explicitly,

$$\text{Hom}(S_{\rho}, S_{\sigma}) = \bigoplus_{g \in G} O_{(g, \sigma \rho^{-1})} = O_{(\epsilon, \sigma \rho^{-1})} \otimes \bigoplus_{g \in G} O_{(g, 1)} = O_{(\epsilon, \sigma \rho^{-1})} \otimes A(G) \ni \sigma \rho^{-1} \in \text{Rep}(G),$$

In other words, the 1+1D bulk topological sector of operators $Z_{1}(\text{Rep}(G))$ is condensed to be a 0+1D topological sector of operators $Z_{1}(\text{Rep}(G))_{A(G)} \simeq \text{Rep}(G)$.

And the action of categorical symmetry $\text{Rep}(G)$ on the underlying category $\text{Rep}(G)$ is given by $\sigma \rho^{-1} \otimes S_{\rho} = S_{\sigma}$.

So the topological skeleton of this $G$-symmetric 0+1D boundary is enriched category $\text{Rep}(G)^{\text{Rep}(G)}$.

**Remark IV.2.** $\text{Rep}(G)^{\text{Rep}(G)}$ can also be understood as $\text{Rep}(G)^{\text{Rep}(G)}_{F_{G}}$ through 1d condensation, in which $F_{G}$ is just the tensor unit 1 in $\text{Rep}(G)$. The 0+1D boundary description $\text{Rep}(G)^{\text{Rep}(G)}_{F_{G}}$ is parallel to the 1+1D bulk description $Z_{1}(\text{Rep}(G))_{A(G)}$ in the sense that the full center of $F_{G}$ is $A(G)$.

**Symmetry completely broken boundary conditions:** Then we consider the boundary conditions that do not preserve any symmetry, i.e. $H_{b} = \{ \epsilon \}$. For example, we can choose the boundary Hamiltonian to be

$$\mathcal{H} = \sum_{i > 0} (1 - X^{i}_{g}) + (1 - Z^{0}_{\{\epsilon\}}),$$

where $Z^{0}_{\{\epsilon\}} := \frac{1}{|G|} \sum_{g \in G} Z^{0}_{g}$. Its ground state is $|\epsilon \rangle_{0} \otimes (\bigotimes_{i > 0} \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g \rangle_{i})$, which completely breaks the $G$-symmetry on site 0.

For each $g \in G$, $M^{0}_{g}$ is a local operator, but $U(g^{-1})M^{0}_{g}$ is a non-local operator and defines a non-trivial topological sector of operators because the $G$-symmetry is broken on the boundary. Since $E^{0}_{\rho}$ acts on $|\epsilon \rangle_{0}$ trivially, there is only one simple boundary condition that completely breaks the symmetry, denoted by $S$. Therefore, the underlying category of the symmetry completely broken boundary conditions matches with $\text{Vec}$.

**Remark IV.3.** We can also check that the equivariantization technique holds. Unlike in section III C 2 where the 1+1D bulk is spontaneously broken, here the 0+1D open boundary is explicitly broken. Or to say, there is no symmetry on this boundary. So we have $\text{Vec}^{\{\epsilon\}} \simeq \text{Vec}$. Note that $Z^{0}_{\{\epsilon\}}$ is a local operator. On symmetry completely broken boundaries, the operator $E^{0}_{\rho}$ for every $\rho \in \hat{G}$ is also ‘local’ since $Z^{0}_{\{\epsilon\}}E^{0}_{\rho}$ is an identity operator. These $E^{0}_{\rho}$ operators form the Lagrangian algebra $A(\{\epsilon\}) = \bigoplus_{\rho \in \hat{G}} O_{(\epsilon, \rho)}$, and the hom spaces between symmetry completely broken boundary conditions are (right) $A(\{\epsilon\})$-modules. Therefore, the categorical symmetry of symmetry completely broken boundary conditions is equivalent to the category $Z_{1}(\text{Rep}(G))_{A(\{\epsilon\})}$. More explicitly,

$$\text{Hom}(S, S) = \bigoplus_{\rho \in \hat{G}} O_{(\rho, \rho)} = \bigoplus_{g \in G} O_{(g, 1)} \otimes A(\{\epsilon\}) \ni \epsilon \in \text{Vec}_{G}.$$
where \( C_g \) is the complex number field with \( g \)-grading. In other words, the 1+1D bulk topological sector of operators \( 3_1(\text{Rep}(G)) \) is condensed to be a 0+1D topological sector of operators \( 3_1(\text{Rep}(G))_{\text{Alt}(e_1)} \cong \text{Vec}_G \).

And the action of categorical symmetry \( \text{Vec}_G \) on the underlying category \( \text{Vec} \) is given by

\[
C_g \otimes S = S
\]

So the topological skeleton of this \( G \)-symmetry completely broken 0+1D boundary is enriched category

\[
\text{Vec}_G \text{Vec}
\]

Symmetry partially broken boundary conditions: Fix a subgroup \( H_b \subseteq G \). Now we consider the boundary conditions that only preserve the subgroup \( H_b \). The canonical choice of Hamiltonian that break the ground state symmetry on site 0 from \( G \) to \( H_b \) can be written as

\[
\mathcal{H} = \sum_{i>0} (1 - X_G^i) + (1 - Z_H^0) + (1 - X^0_{H_b}),
\]

where \( Z^0_{H_b} = \frac{|H_b|}{|G|} \sum_{\rho \in G/H_b} Z^0_{\rho} \). The ground state is then given by

\[
\frac{1}{\sqrt{|H_b|}} \sum_{g \in H_b} |g\rangle_0 \otimes \left( \bigotimes_{i>0} \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g_i\rangle \right).
\]

For every \( \sigma \in \hat{G} \), we can generate the upper Hamiltonian to be the Hamiltonian which realizes each \( H_b \)-symmetric boundary condition, for example:

\[
\mathcal{H} = \sum_{i>0} (1 - X_G^i) + (1 - Z^0_{H_b}) + (1 - |H_b|) \sum_{h \in H_b} \sigma(h) L^0_h,
\]

Its ground state is given by

\[
\frac{1}{\sqrt{|H_b|}} \sum_{g \in H_b} \sigma(g)|g\rangle_0 \otimes \left( \bigotimes_{i>0} \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g_i\rangle \right).
\]

Note that it only depends on the equivalence class (coset) \( [\sigma] \in \hat{G}/H_b \), so we denote the \( H_b \)-symmetric boundary condition by \( S[\sigma] \). Therefore, the underlying category of the enriched category of \( H_b \)-symmetric boundary conditions matches with \( \text{Rep}(H_b) \).

Remark IV.4. The underlying category can also be obtained from equivariantization, in which the category of \( H_b \)-symmetric boundary conditions should be the equivariantization \( \text{Vec}^{H_b} \cong \text{Rep}(H_b) \).

Similarly to above two cases, for \( H_b \)-symmetric boundary conditions, clearly the operators \( M^0_g \) for all \( g \in H_b \) do not change the boundary condition. Dually, since these boundary conditions can be viewed as \( H_b \)-representations, the operators \( E^0_{\rho} \) for \( \rho \in G/H_b \) also do not change the boundary conditions. Thus the operators \( M^0_g \) and \( E^0_{\rho} \) for \( g \in H_b \) and \( \rho \in G/H_b \) should be viewed as partially localized (condensed) because each hom space between two boundary conditions contains them. These ‘local operators’ form a Lagrangian algebra (see Remark III.10)

\[
A(H_b) = \bigoplus_{g \in H_b, \rho \in G/H_b} \mathcal{O}_{(g,\rho)}
\]

and the hom spaces between completely broken boundary conditions are \( A(H_b) \)-modules. Therefore, the categorical symmetry of symmetry partially broken boundary conditions is equivalent to the category \( 3_1(\text{Rep}(G))_{A(H_b)} \). More explicitly,

\[
\text{Hom}(S[\rho], S[\sigma]) = \bigoplus_{g \in G} \mathcal{O}_{(g,\rho,\sigma^i_{\rho-1})}
\]

\[
= (\bigoplus_{[k] \in G/H_b} \mathcal{O}_{(k,\rho^i_{\rho-1})}) \otimes (\bigoplus_{[k] \in G/H_b} \mathcal{O}_{(g,\rho)}),
\]

\[
\rightarrow \bigoplus_{[k] \in G/H_b} \mathcal{M}_{([k],[\rho^i_{\rho-1}]}) \in F_{H_b} \text{Rep}(G)F_{H_b}
\]

where \( [\rho], [\sigma] \in \hat{H}_b \) and \( [k] \in G/H_b \) (see Remark B.8). In other words, the 1+1D bulk topological sector of operators \( 3_1(\text{Rep}(G)) \) is condensed to be a 0+1D topological sector of operators \( 3_1(\text{Rep}(G))_{A(H_b)} \cong F_{H_b} \text{Rep}(G)F_{H_b} \).

And the action of categorical symmetry \( F_{H_b} \text{Rep}(G)F_{H_b} \) on the underlying category \( \text{Rep}(H_b) \) is given by

\[
\mathcal{M}_{([k],[\rho^i_{\rho-1}]}) \otimes S[\rho] = S[\sigma]
\]

So the topological skeleton of this \( G \)-symmetry completely broken 0+1D boundary is enriched category

\[
\text{Rep}(G)F_{H_b} \text{Rep}(H_b)
\]

Remark IV.5. The invariant operators acting on the 0+1D boundary sectors of states are given by \( \text{Hom}(S[1], S[1]) \), which form a 1d condensable algebra in \( F_{H_b} \text{Rep}(G)F_{H_b} \). Its full center is just the ground state algebra \( A(G) := \text{Hom}(S[1], S[1]) \in 3_1(\text{Rep}(G)) \) of the symmetry preserving 1+1D bulk phase. This relation corresponds to open-closed duality discussed in VI.

We can also give a picturesque explanation of the above boundary phases through holographic duality (see figure 9). In figure 9 (a), the 0d ‘corner’ described by \( \text{Rep}(H_b) \) is
the invertible domain wall between two 1d boundaries of the quantum double $\mathcal{Z}_1(\text{Rep}(G))$, which can be obtained through 1d condensation $\text{Rep}(G)_{F_{H_b}}$. $L$ is the functor of 2d condensation which maps the topological sector of operators in $\mathcal{Z}_1(\text{Rep}(G))$ to its boundary, as introduced in (I.1).

After topological Wick rotation, this 2d condensation process corresponds to the "shrink" of operator space from $\mathcal{Z}_1(\text{Rep}(G))$ to $\mathcal{Z}_1(\text{Rep}(G))_{A(H_b)}$. And the invertible 0d domain wall $\text{Rep}(H_b)$ becomes the underlying category of the enriched category $\mathcal{R}_{\omega_0} \text{Rep}(G)_{o_0} \text{Rep}(H_b)$ of boundary conditions, as illustrated in figure 9 (b).

Remark IV.6. The bulk-to-wall map $L$ acts as the background changing functor [7], which changes the categorical symmetry (the background category) from $\mathcal{Z}_1(\text{Rep}(G))$ to $\text{Fun}(\text{Vec}_{H_b})$. 

Remark IV.7. The intuition of topological Wick rotation suggests that all possible enriched category of boundary conditions of the trivial 1d $G$-SPT order are determined by indecomposable $\text{Rep}(G)$-modules, which are classified by pairs $(H_b, [\omega])$ where $H_b \subseteq G$ is a subgroup and $[\omega] \in H^2(H_b, U(1))$ is a 2-cohomology classes [52]. More precisely, the indecomposable module corresponding to $(H_b, [\omega])$ is the category $\text{Rep}(H_b, \omega)$ of finite-dimensional projective $H_b$-representations twisted by $\omega$. Hence, for a subgroup $H_b \subseteq G$ and a 2-cohomology class $[\omega]$, there are $H_b$-symmetric boundary conditions that are ‘twisted’ by $\omega$. Such twisted boundary conditions should form an enriched category $\mathcal{R}_{\omega_0} \text{Rep}(G)_{o_0} \text{Rep}(H_b, \omega)$. In a concrete lattice model, such twisted boundary conditions can be realized by attaching a projective representation to the boundary. We discuss the example $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ in Section V D.

2. 0+1D open boundaries of 1+1D symmetry breaking bulks

Now we consider the general case, where the 1+1D bulk $\mathcal{Z}_1(\text{Rep}(G))_{F_{H_b}} \text{Rep}(G)_{F_{H_b}}$ is chosen to spontaneously break the symmetry $G$ to a subgroup $H_b$ (see section III.C.3), and we choose its boundary conditions explicitly breaking to a subgroup $H_b \subseteq G$. We can pick the canonical Hamiltonian that realizes $H_b$ symmetry on its boundary:

$$\mathcal{H} := \sum_{i>0} (1 - X_H^{i+1}) + \sum_{i>0} (1 - Z_H^{i+1}) + (1 - Z_H^0) + (1 - X_H^0),$$

and the ground state is

$$\frac{1}{\sqrt{|H_b|}} \sum_{h \in H_b} |h\rangle_0 \otimes |\psi_x\rangle_{i>0}$$

$\forall x \in G/H$. Just like the above case, on $H_b$-symmetric boundaries, the bulk operators $M_h$ and $E_x$ are partially localized (condensed) on the boundary up to $g \in H_b$ and $\rho \in G/H_b$, therefore the background category should be the category $\mathcal{Z}_1(\text{Rep}(G))_{A(H_b)}$ of $A(H_b)$ modules, which is equivalent to $\text{Fun}(\text{Vec}_{H_b})$. 

As the equivariantization technique introduced in section II has nothing to do with dimensions, we can view the underlying categories of these boundary phases as a 0d topological orders imposed with symmetries. Since $G$ explicitly breaks to the subgroup $H_b$ on the boundary, the category of symmetric boundary conditions should be an $H_b$-equivariantization of the category of boundary conditions without symmetry.

Following a similar procedure in section III.C, we first view equation (IV.3) as systems without symmetry. By ignoring the symmetry, the topological skeleton of the fine tuned bulk is $\text{Fun}(\text{Vec}_{G/H_b}, \text{Vec}_{G/H_b})$, as illustrated in section III.C.3. There are $n := |G|/|H_b|$ topological sectors of states $\{\mathcal{T}_x\}_{x \in G/H_b}$ in the boundary. Then the category of boundary conditions (without symmetry) of the fine tuned bulk is given by the category $\text{Vec}_{G/H_b}$. If we choose the boundary on the left side (the right side is similar) of $\text{Fun}(\text{Vec}_{G/H_b}, \text{Vec}_{G/H_b})$, we have the following fusion rules,

$$\mathcal{T}_x \otimes \mathcal{T}_{y,z} = \delta_{x,y} \mathcal{T}_z', \quad \forall x, y, z \in G/H.$$

which endows the category $\text{Vec}_{G/H}$ with a structure of right $\text{Fun}(\text{Vec}_{G/H_b}, \text{Vec}_{G/H_b})$-module. See the bottom left figure.

FIG. 10. The illustration of equivariantization process of the topological sector of states of a boundary phase. In which we have $\text{Vec}_{G/H_b}^{H_b}$ equivalent to $\text{Fun}(\text{Vec}_{G/H_b}, \text{Vec}_{G/H_b})$.

Now we consider $H_b$-symmetric boundaries. The $H_b$-action on $\text{Vec}_{G/H}$ is induced from the $G$-action on $\text{Vec}_{G/H}$:

$$U(h)|\psi_x\rangle = |\psi_{hx}\rangle, \quad h \in H_b, x \in G/H.$$ 

In other words, we have

$$h \mathcal{T}_x' = \mathcal{T}_{hx}'.$$

Then the category of $H_b$-symmetric boundaries is the equivariantization $(\text{Vec}_{G/H}^{H_b})$, which is equivalent to $\text{BMod}_{\text{Fun}(\text{Vec}_{G/H_b})}$, and is equivalent to $\text{Fun}(\text{Vec}_{G/H_b})$. As a category it is equivalent to $\text{Vec}_{G/H}$, so we use $\text{Vec}_{G/H}$ in order to simplify the notation.
Proposition IV.8. Let $G$ be a finite group and $H, H_b \subseteq G$ be two subgroups of $G$. Then the category $\text{Rep}(G)_{F_{H_b}}$ is equivalent to the equivariantization $(\text{Vec}(G/H))^H_b$.

In order to see the action of background category $\text{Rep}(G)_{F_{H_b}}$ on the underlying category $F_{H_b} \text{Rep}(G)_{F_H}$ (which is the black arrow drawn in figure 11 (a)), we use three known actions (drawn in gray).

We label simple objects in $F_{H_b} \text{Rep}(G)_{F_H}$ by $P_{[g],[\rho]}$. The action of $\text{Fun}(\text{Vec}(G/H), \text{Vec}(G/H))$ on $\text{Vec}(G/H)$ (equation (IV.4)) induces action on their equivariantization, i.e. an action of $F_{H_b} \text{Rep}(G)_{F_H}$ on $F_{H_b} \text{Rep}(G)_{F_H}$. By composing with the action of topological sectors of operators $3_1(\text{Rep}(G))$ on $F_{H_b} \text{Rep}(G)_{F_H}$ in the 1+1D bulk, we obtain the action from $3_1(\text{Rep}(G))$ onto $F_{H_b} \text{Rep}(G)_{F_H}$.

Moreover, since the topological sector of operators $F_{H_b} \text{Rep}(G)_{F_H} \simeq 3_1(\text{Rep}(G))_{A(H_b)}$ on the boundary is obtained from condensing the sectors of operators in the bulk, we can eventually obtain an categorical symmetry $F_{H_b} \text{Rep}(G)_{F_H}$ action on the category of boundary conditions $F_{H_b} \text{Rep}(G)_{F_H}$:

$$O_{[h,\sigma]} \circ P_{[g],[\rho]} = P_{[hg],[\sigma \rho]}$$

for all $g \in G$ and $\rho \in \hat{G}$, and $M_{[k],[\lambda]} \in F_{H_b} \text{Rep}(G)_{F_{H_b}}$. Thus the hom spaces between $H_b$-symmetric topological sectors of states are given by

$$\text{Hom}(P_{[g],[\rho]}, P_{[h],[\sigma]}) = \bigoplus_{k \in [h g^{-1}]} \bigoplus_{\lambda \in [\sigma^{-1}]} M_{[k],[\lambda]}$$

where $[h g^{-1}]$ is the equivalence class of $G/H_b$ and $[\sigma^{-1}]$ is the equivalence class of $\hat{H}_b$.

It turns out that the observables of a 1d bulk phase with symmetry $H \subseteq G$ with its boundary conditions that preserve a subgroup $H_b$ form the enriched category $c_{F_{H_b}} \text{Rep}(G)_{F_H}$.

Thus we have proved the following physical theorem:

Theorem IV.9. The $H_b$-symmetric boundary conditions of the 1d bulk phase $3_1(\text{Rep}(G))_{F_H} \text{Rep}(G)_{F_H}$ forms an enriched category $c_{F_{H_b}} \text{Rep}(G)_{F_H} \text{Rep}(G)_{F_{H_b}}$.

We depict a $H_b$-symmetric boundary condition of the $H$-partially broken 1d bulk phase in figure 11 (b), and figure 11 (a) is the corresponding 2d topological order $3_1(\text{Rep}(G))$ with 1d boundaries before topological Wick rotation.

Remark IV.10. The 0d invertible domain wall between $F_{H_b} \text{Rep}(G)_{F_H}$ and $F_{H_b} \text{Rep}(G)_{F_H}$ is given by $\text{Rep}(G)_{F_H} \boxtimes \text{Rep}(G)_{F_H} \simeq F_{H_b} \text{Rep}(G)_{F_{H_b}}$.

Remark IV.11. We can check that these 0d boundary categorical descriptions are compatible with the 1d bulk descriptions from the boundary-bulk relation:

$$3_0(c_{F_{H_b}} \text{Rep}(G)_{F_{H_b}} \text{Rep}(G)_{F_{H_b}}) \simeq 3_1(\text{Rep}(G))_{F_H} \text{Rep}(G)_{F_H}$$

See [7, section 4.3] for the definition of the $E_0$-center of an enriched category and the proof of the above equivalence.

B. Domain walls

One can also consider a domain wall connecting two 1+1D gapped phases that break to subgroups $H_1$ (described by $3_1(\text{Rep}(G))_{F_{H_1}} \text{Rep}(G)_{F_{H_1}}$) and $H_2$ (described by $3_1(\text{Rep}(G))_{F_{H_2}} \text{Rep}(G)_{F_{H_2}}$) respectively.

We first consider an invertible domain wall between these two bulk, for which we mean there is no explicit symmetry breaking happens on site 0. We choose a canonical Hamiltonian constructed as

$$H = \sum_{i \leq 0} (1 - X_i H_1) + \sum_{i < 0} (1 - Z_{H_1}^{i,-1}) + \sum_{i \geq 0} (1 - X_i H_2) + \sum_{i \geq 0} (1 - Z_{H_2}^{i,1})$$

(IV.6)

The Deligne’s tensor product $\boxtimes$ physically corresponds to stacking two phases together.
For equation (still the right ground state). So the categorical symmetry is placed by the trivial action $A$ left and right sides. Or to say, the trivial action of this domain wall anymore. There are just some in-boundaries, and the bulk excitations have a natural action on the 0d domain wall with a boundary that breaks to $H \in G$, as depicted in (b). After topological Wick rotation, we get the categorical description of the 1d bulk phase $\mathcal{Z}(\text{Rep}(G))_{F_H}$ with a boundary that breaks to $H \in G$, as depicted in (b).

$$Z_{H_1}^{i,i-1} = \frac{1}{|H_1|} \sum_{g \in G/H_1} E_i^g (E_i^{-1})^g. \text{ The ground state is}$$

$$|\psi_{x_1,x_2} \rangle := \left( \bigotimes_{i \in \mathbb{O}} \frac{1}{\sqrt{|H_1|}} \sum_{g \in x_1} |g \rangle_i \right) \otimes \left( \bigotimes_{i \geq 0} \frac{1}{\sqrt{|H_2|}} \sum_{g \in x_2} |g \rangle_i \right),$$

$\forall x_1 \in G/H_1, x_2 \in G/H_2$.

Since the 1d Hilbert space extends to infinity on both sides, $E_i^g$ and $M_i^g$ are not localized (or condensed) on this domain wall anymore. There are just some interchange of topological sectors of operators between left and right sides. Or to say, the trivial action of $A(H_1) = \bigoplus g \in H \delta_{O(g,\rho)}$ on the left ground state is replaced by the trivial action $A(H_2) = \bigoplus g \in H \delta_{O(g,\rho)}$ on the right ground state. So the categorical symmetry is still $\mathcal{Z}(\text{Rep}(G))$ as in the bulk cases.

In order to find the underlying category, it is also straightforward to use the equivariantization technique. For equation (IV.6), the bulk excitations without symmetry on each side is $\text{Fun}(\text{Vec}(G/H_1), \text{Vec}(G/H_1))$ and $\text{Fun}(\text{Vec}(G/H_2), \text{Vec}(G/H_2))$. And $\text{Fun}(\text{Vec}(G/H_1), \text{Vec}(G/H_2))$ is the 0d invertible domain wall between them. There are $\sum_{i} n_i = |G|/|H_1| \times |G|/|H_2|$ simple topological sectors of state $\{\mathcal{R}_{x,y} \otimes \mathcal{T}_{x,y} \in G/H_1 \times G/H_2 \}$. Moreover, we have the following fusion rules:

$$\mathcal{R}_{x,y} \otimes \mathcal{T}_{x,w} = \delta_{y,z} \mathcal{R}_{x,w}, \\forall x, y, z \in G/H_1, w \in G/H_2,$$

$$\mathcal{T}_{x,y} \otimes \mathcal{T}_{x,w} = \delta_{y,z} \mathcal{R}_{x,w}, \\forall x, y, z \in G/H_1, w \in G/H_2.$$

These fusion rules endow the category $\text{Fun}(\text{Vec}(G/H_1), \text{Vec}(G/H_2))$ with a structure of $\text{Fun}(\text{Vec}(G/H_1), \text{Vec}(G/H_1))$-$\text{Fun}(\text{Vec}(G/H_2), \text{Vec}(G/H_2))$-bimodule.

Note that Hamiltonian (IV.6) is spontaneously breaking. So we can view the domain wall as a 0+1D topological order with $G$-symmetry. The $G$ action on $\text{Fun}(\text{Vec}(G/H_1), \text{Vec}(G/H_2))$ is induced by $\text{Fun}(\text{Vec}(G/H_1), \text{Vec}(G/H_2))$ on states:

$$U(g)|\psi_{x_1,x_2} \rangle = |\psi_{gx_1,gx_2} \rangle, \\forall x_1 \in G/H_1, x_2 \in G/H_2.$$
and the ground state is
\[|\psi_x\rangle_{i<0} \otimes \frac{1}{\sqrt{|H_0|}} \sum_{h \in H_0} |h\rangle_0 \otimes \frac{1}{\sqrt{|H'_b|}} \sum_{h' \in H'_b} |h'_1 \otimes |\psi_y\rangle_{i>1},\]

\[\forall x \in G/H_1 \text{ and } \forall y \in G/H_2, \text{ which can be viewed as piecing together two of the symmetric explicit breaking 0+1D boundaries, see bottom figure:}\]

FIG. 12. It is obvious to see that this domain wall is just the stacking of left and right gapped open boundaries analyzed in section IV A 2.

Except for this invertible domain wall, there are other kinds of domain walls controlled by \( K \subset G \). Or to say, there are some categorical entanglement \([47]\) induced by the topological sectors of operators. The 2d topological order before topological Wick rotation is depicted in the follow figure (see the dashed frame part). In which the 'interconnected' part can be obtained through a 2d condensation from \( Z_1(\text{Rep}(G)) \) to \( Z_1(\text{Rep}(K)) \).

For example, if we choose \( Z_1(\text{Rep}(K)) = \text{Vec} \). This domain wall after topological Wick rotation is just the above 'purely explicit breaking one. If we choose \( Z_1(\text{Rep}(K)) = Z_1(\text{Rep}(G)) \), this would become an invertible domain wall.

V. PHYSICAL EXAMPLES

A. Transverse field Ising chain

We first revisit the 1+1D Ising model, in which Kong, Wen and Zheng directly find the observables to show that the \( \mathbb{Z}_2 \) SPT phase can be described by \( Z_1(\text{Rep}(\mathbb{Z}_2)) \) and the spontaneous symmetry-breaking phase can be described by \( Z_1(\text{Rep}(\mathbb{Z}_2)) \) in \([3]\). We will first review their methods, then show the another way in finding the topological sectors of states through the equivariantization method invented in section II B.

So consider a 1d Ising chain: \( \mathcal{H}_{\text{tot}} = \otimes_{i \in \mathbb{Z}} \mathbb{C}^2_i \), with the Hamiltonian

\[\mathcal{H}^{\text{sing}} = - \sum_i g X^i - \sum_i J Z^i Z^{i+1}, \quad (V.1)\]

where \( X^i \) and \( Z^i \) are Pauli matrices, i.e. \( Z^i |\uparrow\rangle_i = |\uparrow\rangle_i \), \( Z^i |\downarrow\rangle_i = -|\downarrow\rangle_i \), and \( X^i |\pm\rangle_i = \pm |\pm\rangle_i \) for \( |\pm\rangle_i = \frac{1}{\sqrt{2}}(|\uparrow\rangle_i \pm |\downarrow\rangle_i) \). This model is equivalent to taking \( G = \mathbb{Z}_2 := \{e, a\} \) for the model (III.2) we constructed in section III. It has a global \( \mathbb{Z}_2 \) onsite symmetry:

\[U = \bigotimes_{i \in \mathbb{Z}} X^i\]

A symmetric operator should commute with \( U = \otimes_{i \in \mathbb{Z}} X^i \), (i.e. \( [P, U] = 0 \)). We can explicitly find four \( U \)-symmetric topological sectors of operators listed as follows:

1. \( 1 := \mathcal{O}_{e, 1} \) consists of \( U \)-symmetric local operators;
2. \( m := \mathcal{O}_{e, 1} \) is generated by \( M^i = \prod_{k \leq i} X^k \);
3. \( e := \mathcal{O}_{e, -1} \) is generated by \( E^i = \prod_{k \geq i} (Z^k Z_{k+1}) \);
4. \( f := \mathcal{O}_{e, -1} \) is generated by \( M^i E^j \).

\( M^i \) and \( E^i \) here are just \( M^i_g \) and \( E^i_p \) in section III B by taking \( G = \mathbb{Z}_2 \).

Notice that these topological sectors of \( U \)-symmetric operators automatically satisfy the same fusion rule as the excitations of the 2d toric code model, i.e.

\[e \otimes e = 1, \quad m \otimes m = 1, \quad e \otimes f = m, \quad f \otimes f = 1\]

Moreover, if we first create a pair of "m particles" at site \( i \) and \( j \) for \( i < j \) then apply \( E_k \) for \( i < k < j \), then annihilate two \( m \)-particles to obtain \( M_j M_i E_k M_j = -E_k \), we can recover the double braiding between \( e \) and \( m \) in the 2d toric code model, i.e.

\[m \otimes e \xrightarrow{\mathcal{M}} m \otimes e.\]

As a consequence, these topological sectors of operators \( 1, e, m, f \) provide a physical realization of the category of 2d toric code model \( Z_1(\text{Rep}(\mathbb{Z}_2)) = \mathcal{C} \), as we expected.

Now we consider the two cases that can be realized by tuning parameters \( J \) and \( B \).

1. Ising chain realizing \( \mathbb{Z}_2 \) SPT order

First consider \( J = 0 \) and \( g \approx 1 \). In this case, the Hamiltonian is

\[\mathcal{H} = - \sum_i g X^i.\]

The ground state is the product state

\[|\Omega\rangle = |\cdots + + + + \cdots\rangle,\]

which corresponding to the trivial SPT phase discussed in section III.6.

Now we act the four topological sectors of operators on the ground state \( |\Omega\rangle \), we see the total Hilbert space splits into two irreducible topological sectors of states (note that \( M^i = \otimes_{k \leq i} X^k \) only has a trivial action on \( |\Omega\rangle \)): 
1. The trivial sector $1$ is generated by the vacuum state $|\Omega\rangle$.

2. A non-trivial sector $E$: the lowest energy states are given by

$$E^i|\Omega\rangle = |\cdots + + -i + + \cdots\rangle, \quad \forall i \in \mathbb{Z},$$

For two $e$ topological sectors of operators acting on site $i$ and $j$, we have $E^iE^j|\Omega\rangle = Z^iZ^j|\Omega\rangle \in 1$ since $Z^iZ^j$ is a local operator. This implies the following fusion rules: $1 \otimes E = E \otimes 1 = E, E \otimes E = 1$, which coincide with those in the fusion category $\text{Rep}(Z_2)$.

It is also straightforward to see that the space of $U$-symmetric operators which map between these two sectors of states and $E$ are given by:

$$\text{Hom}_{\text{Ising}}^{SPT}(1, 1) = 1 \oplus m$$

This result is just equation (III.8) for $Z_2$ case, which exactly matches the enriched fusion category $S_1(\text{Rep}(Z_2))\text{Rep}(Z_2)$ obtained from the canonical construction. This convinces theorem III.5 in case of $G = Z_2$.

**Theorem V.1** ([3]). The 1d $Z_2$ SPT order of Ising chain can be mathematically described by the enriched fusion category $S_1(\text{Rep}(Z_2))\text{Rep}(Z_2)$.

**Remark V.2.** We can also check the topological sectors of states form fusion category $\text{Rep}(Z_2)$ using the equivariantization technique.

Note that without symmetry, $|\Omega\rangle$ realizes a trivial 1d topological order, whose topological excitations form a fusion category equivalent to Vec. If we act a local operator $Z^j, Z^j|\Omega\rangle = |\cdots + + -i + + \cdots\rangle$ generates the same topological sector of state as $|\Omega\rangle$ (i.e. the tensor unit $1$ in Vec).

Now we view this system as a topological order with the $Z_2$-symmetry realized by $U := \otimes_j X^i$. We can obtain $Z^j|\Omega\rangle$ by adding a symmetric local trap $2X_j$ at a site $j$, i.e. $Z^j|\Omega\rangle$ is the ground state of $H' = X_j - \sum_{i \neq j} X^i$. But since $X_j$ anti-commutes with $Z^j$, we have

$$UZ^j|\Omega\rangle = -Z^jU|\Omega\rangle = -Z^j|\Omega\rangle.$$

In other words, the topological excitation generated by $Z^j|\Omega\rangle$ should be described not only by $1 \in \text{Vec}$ but also by a morphism $u = -1: 1 \rightarrow 1$, which measures the nontrivial $Z_2$-charge of $Z^j|\Omega\rangle$. As a consequence, there are two simple topological excitations $\{1, u = 1\}$ and $\{1, u = -1\}$ in this topological order with the $Z_2$-symmetry, and they form a fusion category equivalent to $\text{Vec}^{Z_2} \simeq \text{Rep}(Z_2)$.

**Remark V.3.** From the holographic point of view, $\text{Rep}(Z_2)$ is just the smooth boundary of toric code model $\mathcal{T}C$. The trivial action of $M^i$ on $|\Omega\rangle$ can be interpreted as a condensation of the ”m-particles” (or equivalently, the ground state algebra $A_m = 1 \oplus m = \text{Hom}_{\text{Ising}}^{SPT}(1, 1)$) in the categorical symmetry $\mathcal{T}C$. In other words, the ground state algebra is $A_m$. In this way we get the underlying category $\text{Rep}(Z_2) \simeq S_1(\text{Rep}(Z_2))A_m$. See the following figure.

![FIG. 13. By condensing 1 \oplus m in 2d toric code model \mathcal{T}C, we can have a 1d smooth gapped boundary Rep(Z_2) \simeq S_1(\text{Rep}(Z_2))1_{1\oplus m}. After topological Wick rotation, the excitations 1, e, m, f \in \mathcal{T}C become the topological sectors of operators, and the excitations 1, E \in \text{Rep}(Z_2) becomes the topological sectors of states of 1+1D Ising SPT. Their mapping behaviors exactly meet with enriched fusion category S_1(\text{Rep}(Z_2))\text{Rep}(Z_2), leading to a macroscopic description of the Ising SPT phase.](image)

2. Ising chain realizing $Z_2$ symmetry breaking order

On the other hand, we can consider the case when $g = 0$ and $J \approx 1$, so the Hamiltonian is

$$H = -\sum_i JZ^iZ^{i+1}$$

which realizes the symmetry breaking case. The global symmetry is still $U = \otimes_i X^i$, and

$$|\Omega_\uparrow\rangle = |\cdots \uparrow \uparrow \cdots\rangle$$

and $|\Omega_\downarrow\rangle = |\cdots \downarrow \downarrow \cdots\rangle$

are two-fold degenerate ground states representing $U$-symmetry broken phases.

If we do not impose symmetry and ignore perturbations, the total Hilbert space splits into four sectors $\mathcal{H}_{ab}$ for $a, b = \uparrow, \downarrow$, where $\mathcal{H}_{ab}$ is spanned by states $(\otimes_{k<i} |a\rangle_k)(\otimes_{k>1} |b\rangle_k)$ for $i \in \mathbb{Z}$. We denote the topological sector associated to $\mathcal{H}_{ab}$ by $s_{ab}$, $s_{ab}$ has fusion rule

$$s_{ab} \otimes s_{cd} = \delta_{bc}s_{ad}.$$

The topological sectors of states form a multi-fusion 1-category

$$\text{Mat}_2(\text{Vec}) = \text{Fun}(\text{Vec}_{Z_2}, \text{Vec}_{Z_2}).$$

Now we view this system as a topological order with the $Z_2$-symmetry realized by $U := \otimes_j X^i$. In this case, each $s_{ab}$ can not be a topological excitation because symmetry $U$ would flip each ground state and the domain wall in between, i.e. $U(s_{\uparrow\uparrow}) = s_{\downarrow\downarrow}, U(s_{\uparrow\downarrow}) = s_{\downarrow\uparrow}$ and vice versa. However,

$$1 := s_{\uparrow\uparrow} \oplus s_{\downarrow\downarrow}.$$
is a topological excitation.

After acting topological sectors of operators 1, e, m, f on vacuum 1, there emerges a non-trivial sector

\[ M := s_{1\downarrow} \oplus s_{1\uparrow}. \]

Note that here e only has a trivial action on the vacuum, so the morphism \( u_e : U(1) := 1 \rightarrow 1 \) is identity. Similarly, M has identity morphism \( u_M : U(M) = M \rightarrow M \). Thus \( \{ 1, u_1 = 1 \} \) and \( \{ M, u_M = 1 \} \) exhaust all symmetric topological excitations. The fusion rules are given by

\[ 1 \otimes M = M \otimes 1 = M, M \otimes M = 1, \]

which coincides with those in Vec_{Z_2}.

The above analysis is just equivariantization technique introduced in section II B. As a consequence, the topological excitations \( \{ 1, u_1 = 1 \} \) and \( \{ M, u_M = 1 \} \) form a fusion category equivalent to the equivariantization \( \text{Fun}(\text{Vec}_{Z_2}, \text{Vec}_{Z_2}) \).\(^{\text{II B}}\)

The space of \( U \)-symmetric operators map between these two sectors of sates 1 and M are clearly given by:

\[
\text{Hom}_{I_{\text{Ising}}}^B(1, 1) = 1 \oplus e \\
\text{Hom}_{I_{\text{Ising}}}^B(1, M) = \text{Hom}_{I_{\text{Ising}}}^B(M, 1) = m \oplus f \\
\text{Hom}_{I_{\text{Ising}}}^B(M, M) = 1 \oplus e
\]

Again, this result is just equation (III.11) for \( Z_2 \) case, which exactly matches the enriched fusion category \( 3_1(\text{Rep}(Z_2)) \text{Vec}_{Z_2} \) obtained from the canonical construction. This converges theorem III.6 in case of \( G = Z_2 \).

**Theorem V.4 ([3])**. The \( Z_2 \) symmetry breaking phase of Ising chain can be described mathematically by the enriched fusion category \( 3_1(\text{Rep}(Z_2)) \text{Vec}_{Z_2} \).

**Remark V.5.** From the holographic point of view, Vec_{Z_2} is just the smooth boundary of toric code model \( \mathcal{T} \mathcal{C} \). The trivial action of \( E^i \) on \( Q \) can be interpreted as a condensation of the "e-particles" (or equivalently, the ground state algebra \( A_e = 1 \oplus e = \text{Hom}_{I_{\text{Ising}}}^B(1, 1) \)) in the categorical symmetry. In other words, the Lagrangian algebra is \( A_e \). In this way we get the underlying category \( \text{Rep}(Z_2) \cong 3_1(\text{Rep}(Z_2))A_e \). See the following figure.

**Remark V.6.** Using method discussed in section IV, we can interpret the 0+1D gapped boundaries of this phases. Detailed analysis can be found in [3], here we show an alternative way of finding the topological sectors of states using equivariantization:

- If we view the Ising SPT system as 1d topological order without symmetry, which is Vec, its 0d open boundaries should also be Vec. But if we impose \( Z_2 \) symmetry on one of its boundaries, we got the sectors of states from Vec_{Z_2} \( \cong \text{Rep}(Z_2) \), similar to remark V.2. For symmetry breaks on the boundary the topological sectors of states still form Vec_{Z_2} \( \cong \text{Vec} \).

- Similarly, if we view the Ising symmetry breaking order as 1d topological order without symmetry, which is \( \text{Fun}(\text{Vec}_{Z_2}, \text{Vec}_{Z_2}) \), its 0d open boundaries can be described by \( \text{Vec}_{Z_2} \) with objects \( s'a \) satisfying \( s'_e \otimes s_{bc} = \delta_{s'b',va}, b, c = 0, 1 \). If we impose \( Z_2 \) symmetry on one of its boundaries, after equivariantization, we are left with only one object \( (1 \oplus M, u = 1) \), which is \( \text{Vec}_{Z_2} \). For symmetry breaks on the boundary the topological sectors of states still form Vec_{Z_2} \( \cong \text{Vec} \).

This is an example to show that equivariantization technique introduced in section II B is also valid in 0d.

**B. Kramers-Wannier dual of Ising chain**

One can also start from a dual model and find the exact same topological skeleton for "dual phases". For abelian cases, these dualities are related to an automorphism (or a non-trivial invertible domain wall) within the categorical symmetry. Here we show the simplest example of 1+1D transverse field Ising model.

A nonlocal mapping of Pauli matrices known as the Kramers-Wannier duality transformation can be done as follows:

\[
X = Z^i Z^{i+1} \\
Z'^i = X^{i+1}
\]

Or we can also write \( Z' = \Pi_{j \leq i} X' \). This transformation is also known as the \( Z_2 \) gauging. The newly defined
Pauli matrices obey the same algebraic relations as the original Pauli matrices, in which we denote $|\uparrow_{\text{kw}}\rangle$ and $|\downarrow_{\text{kw}}\rangle$ as the eigenstates of $Z$, and $|+_{\text{kw}}\rangle$ and $|-_{\text{kw}}\rangle$ as the eigenstates of $X$. The Hamiltonian is simply:

$$\mathcal{H}^{\text{KW}} = -\sum_i g Z^i Z^{i+1} - \sum_i J X^{i+1},$$

in which the coupling parameter $g$ is dual to parameter $J$, and the critical Ising point, in which $g = J$ is a self-dual point. See figure 15. The degeneracy and $\mathbb{Z}_2$ symmetry properties of the spontaneously breaking and SPT phases are changed under the Kramers-Wannier duality.

The only difference is that the $\mathbb{Z}_2$ onsite symmetry of Ising chain becomes

$$U = \bigotimes_i X^i,$$

in KW model. In some literature, people understood it as $\text{Rep}(\mathbb{Z}_2)$ symmetry [55–57].

\[ \begin{array}{c|c|c}
\mathcal{H}^{\text{Ising}} & \mathbb{Z}_2 \text{SSB} & \mathbb{Z}_2 \text{symmetric} \\
\hline
\mathcal{H}^{\text{KW}} & \text{Rep}(\mathbb{Z}_2) \text{symmetric} & \text{Rep}(\mathbb{Z}_2) \text{SSB} \\
\end{array} \]

FIG. 15. The $\mathbb{Z}_2$ symmetry is explicit in the $\mathcal{H}^{\text{Ising}}$ description, while dual symmetry $\text{Rep}(\mathbb{Z}_2)$ is explicit in the $\mathcal{H}^{\text{KW}}$ description. These two models actually has both the $\mathbb{Z}_2$ symmetry and $\text{Rep}(\mathbb{Z}_2)$ symmetry, namely, we can start from either one to find the topological sectors of operators form $3_1(\text{Rep}(\mathbb{Z}_2))$.

It is clear that the topological sectors of operators are the same as Ising. In which we have:

1. $1$ consists of $U$-symmetric local operators;
2. $e=m_{\text{kw}}$ is generated by $M^i = \prod_{k \leq i} X^k = E^i$;
3. $m=e_{\text{kw}}$ is generated by $E^i = \prod_{k \geq i}(Z^k Z_{k+1}) = M^i$;
4. $f := \mathcal{O}_{n, -1}$ is generated by $M^i E^j$.

Note that $m_{\text{kw}}$ ($e_{\text{kw}}$) in KW model corresponds to the $e$ ($m$) in Ising model. This is related to the braided-equivalence ($e - m$ exchange) of toric code model before topological Wick rotation.

- In case of $J = 1, g = 0$, $\mathcal{H} = -\sum_i X^{i+1}$, we have the trivial sector of states generated by the symmetry preserving ground state:

$$|\Omega\rangle = |\cdots +_{\text{kw}} +_{\text{kw}} +_{\text{kw}} \cdots \rangle \in \mathbb{1}$$

Since $X^i = Z^i Z^{i+1}$, we have $|\uparrow^{i+1}\rangle = |+_{\text{kw}}\rangle$ and $|\downarrow^{i+1}\rangle = |-_{\text{kw}}\rangle$. And the non-trivial topological sector of states is generated by applying an $m$ particle on site $i$:

$$E^i |\Omega\rangle = |\cdots +_{\text{kw}} +_{\text{kw}} -_{\text{kw}} +_{\text{kw}} +_{\text{kw}} \cdots \rangle \in M$$

In this KW SPT case the topological sectors of states form fusion category $\text{Vec}_{\mathbb{Z}_2}$, and it is obvious that the enriched category should be $3_1(\text{Rep}(\mathbb{Z}_2)) \text{Vec}_{\mathbb{Z}_2}$.

- For $J = 0, g = 1$, we have $\mathcal{H} = -\sum_i Z^i Z^{i+1}$, and the ground states are two fold degenerate

$$|\Omega_{\uparrow_{\text{kw}}}\rangle = |\cdots \uparrow_{\text{kw}} \uparrow_{\text{kw}} \uparrow_{\text{kw}} \cdots \rangle$$

Consider $Z^i Z^{i+1}$, the state $|\uparrow_{\text{kw}}^{i+1}\rangle = |+_{\text{kw}}\rangle$, since $X^i = Z^i Z^{i+1}$, thus the state $|\uparrow_{\text{kw}}^{i+1}\rangle$ should be the eigenstate of $X^i$, i.e. $|\uparrow_{\text{kw}}^{i+1}\rangle = |+_{\text{kw}}\rangle$. Similarly, we should have $|\downarrow_{\text{kw}}^{i+1}\rangle = |-_{\text{kw}}\rangle$, which coincides with the ground state of the Ising SPT phase.

Again according to the analysis in previous subsection, we can find two topological sector of states 1 and $E$, which form the fusion category $\text{Rep}(\mathbb{Z}_2)$. It is also obvious that the topological skeleton of this SSB phase should be $3_1(\text{Rep}(\mathbb{Z}_2)) \text{Rep}(\mathbb{Z}_2)$.

**Remark V.7.** These topological sectors of states’ categorical descriptions should also be checkable using equivariantization. For example, the symmetry preserving case that has trivial product state before applying symmetry is just Vec, and we have $\text{Vec}_{\text{Rep}(\mathbb{Z}_2)} \simeq \text{Vec}_{\mathbb{Z}_2}$ after imposing $\text{Rep}(\mathbb{Z}_2)$ symmetry ($\text{Rep}(G)$ equivariantization is equivalent to $G$ de-equivariantization). Also for the symmetry breaking case, we have the topological sectors of states as $\text{Fun}(\text{Rep}(\mathbb{Z}_2), \text{Rep}(\mathbb{Z}_2))\text{Rep}(\mathbb{Z}_2) \simeq \text{Rep}(\mathbb{Z}_2)$. 

| Categorical symmetry | $\mathcal{TC} := 3_1(\text{Rep}(\mathbb{Z}_2))$ |
|---------------------|-----------------|
| $\text{Vec}_{\mathbb{Z}_2}$ | $\text{Rep}(\mathbb{Z}_2)$ |
| $\text{Rep}(\mathbb{Z}_2)$ | $\text{Ising SPT}$ | $\text{KW SPT}$ | $3_1(\text{Rep}(\mathbb{Z}_2))\text{Vec}_{\mathbb{Z}_2}$ |
| $\text{Rep}(\mathbb{Z}_2)$ | $\text{Ising SSB}$ | $\text{KW SSB}$ | $3_1(\text{Rep}(\mathbb{Z}_2))\text{Rep}(\mathbb{Z}_2)$ |

**TABLE I.** The $\mathbb{Z}_2$-action on transverse field Ising model can also be viewed as $\text{Vec}_{\mathbb{Z}_2}$-action. Under KW duality, the $\mathbb{Z}_2$ onsite symmetry becomes a $\text{Rep}(\mathbb{Z}_2)$ symmetry. The categorical symmetry is always $3_1(\text{Rep}(\mathbb{Z}_2))$, and spectrum in these two cases would be the same. An enriched category description not only describes the SPT (SSB) phase of Ising, but also the KW dual SSB (SPT) phase under duality.

See section VII for general discussion on duality and enriched categories.

**C. $\mathbb{Z}_n$ clock model**

The next example is the quantum clock model, which is a generalization of the transverse-field Ising chain. It is defined on a lattice with $n$ states on each site. The
Hamiltonian of this model is typically known as \([58, 59]\)

\[
\mathcal{H}_c = -g \sum_i \left( \hat{X}^i + (\hat{X}^i)^\dagger \right) - J \sum_i \left( \hat{Z}^i (\hat{Z}^{i+1})^\dagger + (\hat{Z}^i)^\dagger \hat{Z}^{i+1} \right).
\]

(V.3)

The clock operators \(\hat{X}^i\) and \(\hat{Z}^i\) are \(n \times n\) generalizations of the Pauli matrices \(X\) and \(Z\) satisfying

\[
\hat{Z}^i \hat{X}^k = e^{\frac{2\pi i}{n} \delta_{j,k}} \hat{X}^k \hat{Z}^i \quad (\hat{X}^i)^n = (\hat{Z}^i)^n = 1,
\]

where \(\delta_{j,k}\) is 1 if \(j\) and \(k\) are the same site and zero otherwise. The model obeys a global \(Z_n\) symmetry, which is generated by the unitary operator \(U_X = \prod_i \hat{X}^i\), where the product is over every site of the lattice.

Now we focus on one site \(i\) and omit the site index. We choose a basis \(|0\rangle, |1\rangle, \ldots, |n-1\rangle\) for \(\mathcal{H}_c^i\), such that

\[
\hat{X}|k\rangle = |k+1\rangle \quad \hat{Z}|k\rangle = (e^{\frac{2\pi i}{n}})^k |k\rangle.
\]

It is clear that these operators satisfy the relations of generalized Pauli matrices, i.e. \(\hat{X}Z|k\rangle = \hat{Z}|k+1\rangle = (e^{\frac{2\pi i}{n}})^k \hat{X}|k\rangle = (e^{\frac{2\pi i}{n}})^k \hat{Z}|k\rangle\).

The clock model has two gapped phases parameterized by \(g\) and \(J\). When \(g \approx 1\), \(J \approx 0\), it realizes the disordered phase. When \(g \approx 0\), \(J \approx 1\), it realizes the ordered phase.

To figure out the relation between the clock model and our constructed model (III.2), let \(a\) be the generator of \(Z_n\), then basis in our model are \(|e\rangle, |a\rangle, |a^2\rangle, \ldots, |a^{n-1}\rangle\).

Recall that \(L_{a^k} |a^k\rangle = |a^{k+1}\rangle\). Under the isomorphism \(|k\rangle \mapsto |a^k\rangle\), we see \(\hat{X}\) is indeed \(L_a\) in our model. Also \(\hat{X}^i = L_{a^{n-1}}\), since \(\hat{X}^i \dagger = \text{id}\). Note that the representation \(\rho_a \in \text{Rep}(Z_n)\) such that \(\rho_a(a) = e^{\frac{2\pi i}{n}}\) is the generator of \(\text{Rep}(Z_n)\). It is not hard to see that \(Z_{\rho_a} |a^k\rangle = \rho_a(a^k) |a^k\rangle\), and hence \(\hat{Z} = Z_{\rho_a}\). Therefore, \(\hat{Z}^i = (Z_{\rho_a})^{n-1}\).

Note that these two models have the same global symmetry, since \(\hat{X} = L_a\). As a consequence, the topological sectors of operators of \(Z_n\) clock model should also be \(\mathcal{S}_1(\text{Rep}(Z_n))\), because the \(Z_n\) clock model has the same generator of operators as our construction.

When \(n = 2\), the quantum clock model is identical to the transverse-field Ising model analyzed in the last subsection.

**Example V.8 (clock model with \(Z_3\) symmetry).**

When \(n = 3\), i.e. \(G = Z_3 := \{e, a, a^2\}\), the clock model is also known as the three-states Potts model \([60, 61]\). Hamiltonian (V.3) also coincides with our construction precisely:

- If we take \(H = Z_3\) in Hamiltonian (III.2), we have

\[
\mathcal{H} = \sum_i \left( 1 - \sum_{g \in Z_3} L_g^i \right).
\]

Note that \(L_c = 1\), so we have

\[
\mathcal{H} = -L_{a^1} - L_{a^2} = -\hat{X}^i - (\hat{X}^i)^\dagger = \mathcal{H}_c^i
\]

on each site. This reveals \(g \approx J\) in Hamiltonian (V.3)

This shows that the trivial \(Z_3\) SPT phase realized in (III.2) is just the disordered phase of quantum clock model.

- If we take \(H = \{e\}\) in Hamiltonian (III.2), we have

\[
\mathcal{H} = \sum_i \left( 1 - \sum_{g \in Z_3} L_g^i \right) = \mathcal{H}_c
\]

for \(g \ll J\). This show that the symmetry breaking phase realized in (III.2) is just the ordered phase of quantum clock model.

The process of finding the macroscopic observables for \(n = 3\) has no difference from section III for \(G = Z_3\), in which we have the enriched fusion category \(\mathcal{S}_1(\text{Rep}(Z_3))\) to describe the disordered phase and \(\mathcal{S}_1(\text{Rep}(Z_3))\text{Vec}_{Z_3}\) to describe the ordered phase.

**Example V.9 (clock model with \(Z_p\) symmetry).**

When \(n = p\) for \(p\) a prime number, even though Hamiltonian (V.3) lacks some operators comparing to our construction, they still realize the same phases (which are described by the same topological skeletons \(\mathcal{S}_1(\text{Rep}(Z_p))\text{Rep}(Z_p)\) and \(\mathcal{S}_1(\text{Rep}(Z_p))\text{Vec}_{Z_p}\)).

- For \(g \gg J\) in Hamiltonian (V.3), \(\mathcal{H}_c^i = -\hat{X}^i - (\hat{X}^i)^\dagger = -\sum_{k=1}^p L_{a^k}\) on each site. However, the ground state is only controlled by generator of operators \(\hat{X} = L_a\) for prime cases. Thus, even though \(\mathcal{H}_c\) has less operators compare to \(H\), these two Hamiltonians again has the same ground state \(|\Omega_{Z_p}\rangle\). And we can obtain the same topological sectoral states by applying the topological sectors of operators in \(\mathcal{S}_1(\text{Rep}(Z_p))\): trivial sector \(1\) generated by \(|\Omega_{Z_p}\rangle\) and sector \(E^k\) generated by states \(|\theta^k\rangle\), \(\forall k \in Z_p\), whose non-trivial sites are given by

\[
\frac{1}{\sqrt{p}} \left( \sum_{k=0}^{p-1} e^{2\pi i k} |a^k\rangle \right)
\]

respectively.

The sectors of states form fusion category \(\text{Rep}(Z_p)\). The topological skeleton of this phase is \(\mathcal{S}_1(\text{Rep}(Z_p))\text{Rep}(Z_p)\), in which the trivial \(Z_p\) SPT phase realized in (III.2) is just the disordered phase of p-states quantum clock model.

- when we take \(g \ll J\) in Hamiltonian (V.3), The p-fold degenerate ground states \(|\psi_{a^k}\rangle\) \(a^k \in Z_p\) of \(\mathcal{H}_c\) (recall that \(|\psi_g\rangle := \otimes |g\rangle_i\) are the same as those of \(\mathcal{H}_c\). According to a similar argument as the above case, the symmetry breaking phase realized in (III.2) is just the ordered phase of quantum clock model. And the topological skeleton of this phase is \(\mathcal{S}_1(\text{Rep}(Z_p))\text{Vec}_{Z_p}\), where the sectors of states form fusion category \(\text{Vec}_{Z_p}\). See the following picture.

![Diagram](https://via.placeholder.com/150)
Example V.10 (clock model with \(Z_4\) symmetry).
But the lack of operators in Hamiltonian (V.3) would result in the lack of partially symmetry breaking phases comparing with our Hamiltonian (III.2) for \(G = Z_4\) and other non-prime cases. This is because for non-prime cases, not only the generator of operators \(X = L_a\) determines the ground state, but also some generators of subgroup operators (e.g. \(L_{2a}\) can generate \(Z_2\) in \(Z_4\) case). Indeed, in \(Z_n\)-clock model, the clock model minimally add operators (i.e. Hermitian conjugation of \(X\) and \(\overline{Z}\)) in Hamiltonian such that it is well-defined. Our model admit all possible operators that are related to the symmetry.

We illustrate the phases in case of \(Z_4\):

- When we take \(g \gg J\) (\(g \ll J\)) in Hamiltonian (V.3), i.e. the disordered (ordered) phase of \(Z_4\) clock model, the ground state(s) is again determined by the generator of operators \(X = L_a\) (\(\overline{Z} = Z_{\rho_a}\)). So the analysis of \(Z_p\) case still work, and we again obtain the enriched category description as \(\mathfrak{C}(\text{Rep}(Z_4))\)\(^1\) \(\text{Rep}(Z_4)\) \(\text{Vec}_{Z_4}\).

- Note that the model (III.2) admits a \(Z_2\) partially symmetry breaking phase if we set \(H = Z_2 \subseteq Z_4\). The corresponding Hamiltonian is

\[
\mathcal{H} = \sum_i \left(1 - \frac{1}{2} \sum_{\rho \in Z_2} L^{\rho}_i \right) - \sum_i \left(1 - \frac{1}{2} \sum_{\rho \in Z_2} Z^{\rho}_i (Z^{\rho+1}_i)^\dagger \right),
\]

and the ground states are \(\{\bigotimes_i (|e\rangle_i + |a^2\rangle_i), \bigotimes_i (|a\rangle_i + |a^3\rangle_i)\}\), the topological sectors of states form \(\mathfrak{C}_2\) \(\text{Rep}(Z_4)\) \(\text{Vec}_{Z_2}\). See the dashed line part in figure 16. However, \(\mathcal{H}_c\) cannot be tuned to create such ground state, since \(\mathcal{H}_c\) miss operators \(L_{2a}\) and \(Z_{\rho_2}\). Hence the clock model (V.3) cannot realize this \(Z_2\) partially symmetry breaking phase. However this gapped phase is “hidden” inside the gapless phase transition point for \(g = J\), see outlook VII for some discussions.

![FIG. 16. The phase transition point between symmetry preserving case and symmetry completely broken phase can be understood in two ways: one is to completely break \(G\) (in this case \(Z_4\)), other one is to first partially break to some subgroup \(H \subseteq G\) (here \(Z_2\), then break \(H\) again.](image)

D. \(Z_2 \times Z_2\) SPT with non-trivial two-cocycle

1. Cluster state

When \(G = Z_2 \times Z_2 := \{e, a, b, ab\}\), it has five subgroups \(\{e\}, Z_2 \times \{e\}, \{e\} \times Z_2, Z_2^f := \{e, ab\}\) and \(Z_2 \times Z_2\). They correspond to the symmetry completely broken phase, three \(Z_2\) partially symmetry breaking phases and two SPT phases respectively. The description of four symmetry breaking phases and trivial SPT phase can be analyzed directly by the method introduced in section III C 3, so we omit here.

Now we focus on the non-trivial 1+1D SPT order, which can be understood as the non-trivial \(Z_2 \times Z_2\)-SPT phase of the Haldane chain, or the cluster model [62, 63].

First we consider the cluster model. The Hilbert space of the cluster model is \(\mathfrak{H} = \bigotimes_i \mathcal{V}_i\) where \(\mathcal{V}_i := \mathbb{C}^2\) and the Hamiltonian is given by

\[
\mathcal{H} = - \sum_i Z^i_1 X^i_1 Z^i_{1+1},
\]

here \(X^i\) and \(Z^i\) are Pauli matrices acting on each site. And the onsite \(Z_2 \times Z_2\) symmetry is given by \(U_o = \bigotimes_i X^{2i_1} \bigotimes_o = \bigotimes_i X^{2i_1+1}\), where the subscripts \(e\) and \(o\) denote the even and odd sites, respectively.

We define \(\mathcal{H}_i := \mathcal{V}_{2i-1} \bigotimes \mathcal{V}_{2i}\) then there is an isomorphism of \(Z_2 \times Z_2\)-representations

\[
\mathbb{C}[Z_2 \times Z_2] = \mathcal{H}_i \simeq \mathcal{V}_{2i-1} \bigotimes \mathcal{V}_{2i},
\]

\[
|\epsilon\rangle \rightarrow \ket{\uparrow \otimes \uparrow} |\alpha\rangle \rightarrow \ket{\downarrow \otimes \downarrow}
\]

where the \(Z_2 \times Z_2\)-action on \(\mathcal{H}_i\) is given by

\[
a \mapsto X_{2i-1}, \quad b \mapsto X_{2i}.
\]

Therefore, the Hilbert space and the \(G\)-action of the cluster model are the same as those in Section III. So the categorical symmetry, i.e., the category of topological sectors of operators, is still \(\mathfrak{C}_1\) \(\text{Rep}(Z_2 \times Z_2)\). Under the isomorphism \(\mathfrak{H}_i \simeq \mathcal{V}_{2i-1} \bigotimes \mathcal{V}_{2i}\), the operators \(L_a\) and \(L_b\) are identified with \(X_{2i-1}\) and \(X_{2i}\), respectively. The dual group of \(Z_2 \times Z_2\) is also isomorphic to \(Z_2 \times Z_2\), and we denote the two generators by \(\phi_a\) and \(\phi_b\), where \(\phi_b\) maps \(a\) to \(-1\) and \(b\) to 1, and \(\phi_a\) maps \(a\) to 1 and \(b\) to \(-1\). Then the operators \(Z^{2k-1}_a\) and \(Z^{2k}_a\) are identified with \(Z^{2k-1} \bigotimes Z^{2k}\). Hence, the topological sectors of operators are generated by the following symmetric non-local operators:

- **\(e_1\)** generated by \(E_{\phi_b}^i := \prod_{k \geq i} Z^{2k-1} Z^{2k+1} = Z^{2i-1}\);
- **\(e_2\)** generated by \(E_{\phi_a}^i := \prod_{k \geq i} Z^{2k} Z^{2k+2} = Z^{2i}\);
- **\(m_1\)** generated by \(M_a^i := \prod_{k < i} L_k^a = \prod_{k < i} X^{2k-1}\);
- **\(m_2\)** generated by \(M_b^i := \prod_{k < i} L_k^b = \prod_{k < i} X^{2k}\);
The simple topological sectors of operators are generated by the composition of above four kinds of symmetric non-local operators, e.g. $e_1 m_1 = f_1, e_2 m_2 = f_2, e_1 e_2 m_2$.

As we have seen in Section III, the topological sectors of operators that act on the (topological sector of the) ground state invariably form a Lagrangian algebra in the categorical symmetry. In the cluster model, the ground state $|\psi\rangle$ is the common eigenstate of operators $Z_i^{2i} X_i Z_i^{2i+1}$ for all i with eigenvalue 1. Therefore, an operator acts on $|\psi\rangle$ invariably if and only if it commutes with $Z_i^{2i} X_i Z_i^{2i+1}$ for all i. For example, the operator $Z_i^{2i}$ does not commute with $Z_i^{2i} X_i Z_i^{2i+1}$, thus $e_2$ does not act on the ground state invariably. It is not hard to see that the non-trivial topological sectors of operators that act on the ground state invariably are $1, e_1 m_2, e_2 m_1, f_1 f_2$. Therefore, the ground state algebra is the Lagrangian algebra spanned by $e_1 e_2 m_2$. The Lagrangian algebras in $3_1(\text{Rep}(Z_2 \times Z_2))$ are listed in Example A.8, and we see that the ground state algebra corresponds to the nontrivial cohomology class $\omega \in H^2(Z_2 \times Z_2; U(1)) \cong Z_2$. In other words, the ground state algebra is $A(Z_2 \times Z_2, \omega)$. This suggests that the cluster model realizes the nontrivial $Z_2 \times Z_2$ SPT phase. Thus we obtain the following result.

**Theorem** V.11. The non-trivial $Z_2 \times Z_2$ symmetry protected topological order can be described by the enriched fusion category $3_1(\text{Rep}(Z_2 \times Z_2))/A(Z_2 \times Z_2, \omega)$.

2. **Boundaries of Haldane chain**

To study the 0+1D boundary theories of this phase, it is simpler to consider the Haldane chain. First we note that there is a unique projective representation $(W, \rho)$ associated to $[\omega] \in H^2(Z_2 \times Z_2; U(1))$, where $W = \mathbb{C}^2$ with the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and $\rho: Z_2 \times Z_2 \to GL(W)$ is given by

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(ab) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The local Hilbert space of the Haldane chain is $\mathcal{H}_i := \mathbb{C}[Z_2 \times Z_2]$. We also have an isomorphism of $Z_2 \times Z_2$-representations

$$\mathcal{C}[Z_2 \times Z_2] = \mathcal{H}_i \cong W_{2i-1} \otimes W_{2i},$$

where $|e\rangle \mapsto |\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle$, $|a\rangle \mapsto |\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle$, $|b\rangle \mapsto |\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle$, $|ab\rangle \mapsto |\downarrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle$.

In other words, under this isomorphism, the operator $L_i^e$ is identified with $\rho(g_{2i-1}) \rho(g)_{2i}$ for all $g \in Z_2 \times Z_2$. Also, it is not hard to see that the operator $Z_i^{\phi_h}$ and $Z_i^{\phi_\omega}$ are identified with $\rho(b)_{2i-1}$ and $\rho(a)_{2i}$, respectively. Therefore, the topological sectors of operators in the Haldane chain are generated by the following symmetric non-local operators:

- $e_1$: generated by $E_i^{e_1} := \Pi_{k \geq i} Z_k^b Z_{k+1}^b = Z_i^{\phi_h}$.
- $e_2$: generated by $E_i^{e_2} := \Pi_{k \geq i} Z_k^a Z_{k+1}^a = Z_i^{\phi_\omega}$.
- $m_1$: generated by $M_i^a := \Pi_{k \leq i} L_k^a = \Pi_{k \leq i} \rho(a)_{2k-1} \rho(a)_{2k}$.
- $m_2$: generated by $M_i^b := \Pi_{k \leq i} L_k^b = \Pi_{k \leq i} \rho(b)_{2k-1} \rho(b)_{2k}$.

Note that $\rho(b)_{2i}$ is also contained in $e_1$ because $\rho(b)_{2i-1} \rho(b)_{2i} = L_i^b$ is a symmetric local operator. Similarly, $\rho(a)_{2i-1}$ is contained in $e_2$.

Define an operator $P_{i+1}$ acting on $\mathcal{H}_i \otimes \mathcal{H}_{i+1}$ by

$$P_{i+1} = \begin{pmatrix} \mathcal{H}_i \otimes \mathcal{H}_{i+1} = W \otimes W \otimes W \otimes W \otimes W \otimes W \end{pmatrix}$$

where $X_G = (1 + L_a + L_b + L_{ab})/4$ is the projector to the state $|\uparrow\rangle + |\downarrow\rangle + |\downarrow\rangle + |\downarrow\rangle + |\downarrow\rangle + |\downarrow\rangle$. The Hamiltonian is defined by

$$H = -\sum_i P_{i, i+1}.$$
state subspace can be identified with $W_1$ because the projectors $P_{i,i+1}$ only act on the other local Hilbert spaces. This suggests that the category of symmetry preserving boundary conditions is equivalent to $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2, \omega)$ and $W_1$ corresponds the unique simple objects. This category can also be obtained from equivariantization $\text{Vec}(\tilde{\mathbb{Z}_2} \times \mathbb{Z}_2) \simeq \text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2, \omega)$, where the $\tilde{\mathbb{Z}_2} \times \mathbb{Z}_2$-action on $\text{Vec}$ is twisted by $\omega$ because the boundary Hilbert space $W_1$ is a projective representation.

Also, on this boundary, the operators $M_1^i = \prod_{k \leq i} L_k^i$ for all $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ is the product of finitely many symmetric local operators. So they are local on this boundary. These operators form the topological sectors of operators $1 \oplus m_1 \oplus m_2 \oplus m_1 m_2$, which is the Lagrangian algebra $A(\mathbb{Z}_2 \times \mathbb{Z}_2) \in \mathcal{Z}_1(\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2))$. Therefore, the categorical symmetry on the symmetry preserving boundary is given by

$$\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2))_{A(\mathbb{Z}_2 \times \mathbb{Z}_2)} \simeq \text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2).$$

Thus, the topological skeleton of this boundary should be described by $\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2))_{A(\mathbb{Z}_2 \times \mathbb{Z}_2, \omega)} \mathcal{Z}_1(\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2))_{A(\mathbb{Z}_2 \times \mathbb{Z}_2, \omega)}$.

VI. GROUND STATE ALGEBRA AS THE GROUND STATE SUBSPACE ON A CIRCLE

In this section we discuss the notion of the ground state algebra and show that it is indeed the ground state subspace on a circle $S^1$ (with all possible twisted periodic boundary conditions). This is also the reason why we call it the ground state algebra.

Recall the ground state algebra $A$ of a 1+1D quantum liquid phase $\mathcal{B}$ is defined in Section 1B as the operators that acts on the topological sector of the ground state invariably. The holographical duality (or the intuition of topological Wick rotation) and the anyon condensation theory implies that $A$ is a Lagrangian algebra in the categorical symmetry $\mathcal{B}$ and the fusion category $\mathcal{S}$ is equivalent to the module category $\mathcal{B}_A$.

In the anyon condensation theory, it is known that the Lagrangian algebra $A \in \mathcal{B}$ can be obtained by shrinking a hole in the topological order $\mathcal{B}$ with boundary $\mathcal{S}$ [36, 64, 65] (see the following figure).

![FIG. 17. The Lagrangian algebra $A \in \mathcal{B}$ can be obtained by shrinking a hole with boundary $\mathcal{S}$.](image)

After the topological Wick rotation, this hole becomes a circle $S^1$. So we obtain a 1+1D phase $\mathcal{B}_S$ on a circle (see the following figure). Then the shrinking process becomes shrinking the 1+1D phase on the circle. The result is a 0+1D phase, that is, a quantum mechanic system, which is just the ground state subspace of the phase on the circle.

![FIG. 18. The topological Wick rotation suggests that the ground state algebra $A$ can be viewed as the ground state subspace on the circle $S^1$.](image)

We consider the 1+1D gapped phases with onsite $G$ symmetries and thus $\mathcal{B} = \mathcal{Z}_1(\text{Rep}(G))$. To put such a phase on a global manifold $M$, we also need to specify a $G$-bundle or a $G$-connection on $M$. Physically the $G$-connection is the same as a $G$-gauge field, and we need to couple the system with $G$-symmetry to the fixed gauge...
field to put it on \( M \). For \( M = S^1 \), the \( G \)-bundles on \( S^1 \) are classified by the conjugacy classes of \( G \), i.e., \( G \)-fluxes. Indeed, the \( G \)-gauge fields on \( S^1 \) can be distinguished by their holonomies, which are elements in \( G \). However, the holonomy depends on the reference point, and if we change the reference point the holonomy may changed by a conjugation. Equivalently, if we do a gauge transformation of \( G \)-gauge fields, the holonomy may be changed by a conjugation. So the different \( G \)-bundles can be viewed as the insertions of \( G \)-fluxes and are usually called the twisted periodic boundary conditions.

Here the ground state subspace on \( S^1 \), which is still denoted by \( A \), is the ground state subspace with all possible \( G \)-gauge fields or twisted periodic boundary conditions. In other words, there is a \( G \)-grading

\[
A = \bigoplus_{g \in G} A_g,
\]

where \( A_g \) is the ground state subspace on \( S^1 \) with a \( G \)-gauge field whose holonomy is \( g \). As we discussed above, \( A_h \) and \( A_{gh^{-1}} \) are isomorphic.

Moreover, the global \( G \)-symmetry operators \( U(g) \) can also act on the ground state subspace. However, if there is a \( G \)-gauge field with holonomy \( h \), after a global action of \( U(g) \) we obtain a gauge field with holonomy \( gh^{-1} \). In other words, \( U(g) \) changes the twisted periodic boundary condition from \( h \) to \( gh^{-1} \). Therefore, \( U(g) \) maps the space \( A_h \) to \( A_{gh^{-1}} \).

Mathematically, an object in \( \mathfrak{Z}_1(\text{Vec}_G) \cong \mathfrak{Z}_1(\text{Rep}(G)) \) is a vector space \( V \) equipped with a \( G \)-grading \( V = \bigoplus_{g \in G} V_g \) and a \( G \)-representation \( \rho \in \text{GL}(V) \) such that \( \rho(g)(V_h) \subseteq V_{gh^{-1}} \) for all \( g, h \in G \). Hence, we see that the ground state subspace \( A \) of a 1+1D gapped phase with \( G \)-symmetry is naturally an object in \( \mathfrak{Z}_1(\text{Rep}(G)) = \mathfrak{B} \), where the \( G \)-grading is given by the fluxes or the twisted periodic boundary conditions, and the \( G \)-action is given by the global \( G \)-action.

**Example VI.1.** Let us consider the Ising chain. For the \( \mathbb{Z}_2 \)-symmetric phase, the ground state is \( | \cdots + + \cdots \rangle \). This ground state can be put on \( S^1 \) with arbitrary fluxes. Indeed, the anti-periodic boundary condition does not affect the Hamiltonian \( \mathcal{H} = - \sum_i X^i \). Therefore, the ground state subspace \( A \) is given by \( A_e = \mathbb{C} \) and \( A_a = \mathbb{C} \). The global \( \mathbb{Z}_2 \)-symmetry acts on the ground state invarially, so \( A \) is equipped with the trivial \( \mathbb{Z}_2 \)-action. This gives \( A = 1 \oplus m \in \mathfrak{Z}_1(\text{Rep}(\mathbb{Z}_2)) \).

For the symmetry breaking phase, the ground state is two-fold degenerate. In this case we can not put the ground state on \( S^1 \) with the anti-periodic boundary condition, because the anti-periodic boundary condition is equivalent to add a domain wall \( M \), which can not live alone on a closed manifold. So in this case \( A_e = 0 \) and \( A_a = \mathbb{C}_2 \) with the \( \mathbb{Z}_2 \)-symmetry permuting two basis. This gives \( A = 1 \oplus e \in \mathfrak{Z}_1(\text{Rep}(\mathbb{Z}_2)) \).

So we see that in this example the two ground state subspaces on \( S^1 \) (with all twisted periodic boundary conditions) are exactly the ground state algebra discussed in the previous sections.

**Example VI.2.** Let us consider the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) SPT orders. For the trivial SPT, similar to the Ising chain, the ground state is invariant under the symmetry action and can be put on the circle with arbitrary twisted periodic boundary conditions. Thus we obtain the ground state space \( 1 \oplus m_1 \oplus m_2 \oplus m_1 m_2 \), which is the same as \( A(\mathbb{Z}_2 \times \mathbb{Z}_2) \).

For the nontrivial SPT order, we first consider the Haldane chain on a circle with an \( a \)-twisted periodic boundary condition. Such a twisted periodic boundary condition is equivalent to inserting an \( a \)-flux or adding an \( m_i \) defect, which can be realized by the string operator \( M^j_2 = \prod_{k \leq i} L^j_k \). This string operator commutes with the usual Hamiltonian of the Haldane chain, except on the end of the string. Indeed, recall that the Hamiltonian is

\[
\mathcal{H} = - \sum_i P_{i,i+1}
\]

acting on \( \mathcal{H}_i \otimes \mathcal{H}_{i+1} \cong W \otimes \mathbb{C}[G] \otimes W \). For the projective representation \( W = (W, \rho) \), we have

\[
\rho(a) \rho(g) = \begin{cases} \rho(g) \rho(a), & g = e, a; \\ -\rho(g) \rho(a), & g = b, ab. \end{cases}
\]

Therefore, we have \( L^j_g P_{i,i+1} = P^\rho_{i,i+1} L^j_g \), where \( P^\rho_{i,i+1} = 1 \otimes X^\rho_G \otimes 1 \) with

\[
X^\rho_G = (1 + L_a - L_b - L_{ab})/4 = \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} L_g/4.
\]

It is easy to see that \( X^\rho_G \) is a projector to the state \( | e \rangle + | a \rangle - | b \rangle - | ab \rangle \). Under the isomorphism \( \mathbb{C}[\mathbb{Z} \times \mathbb{Z}_2] \cong W \otimes W \), it is the projector to \( | \uparrow \rangle \otimes | \downarrow \rangle + | \downarrow \rangle \otimes | \uparrow \rangle \). The original ground state \( | \Omega \rangle \) of the Haldane chain is the common eigenstate of \( P^\rho_{j,j+1} \) for all \( j \) with eigenvalue 1. Then we see that \( M^j_2 | \Omega \rangle \) is not an eigenstate of \( P^\rho_{i,i+1} \), but the eigenstate of \( P^\rho_{i,i+1} \). So the \( a \)-twisted periodic boundary condition in the Haldane chain can be realized by replacing \( P_{i,i+1} \) in the Hamiltonian by \( P^\rho_{i,i+1} \). If we take the total Hilbert space to be \( \bigotimes_{j=1}^2 W_{2j} \otimes W_{2j+1} \), the \( a \)-twisted ground state \( | \Omega_a \rangle \) is also a ‘tensor product state’ with \( | \uparrow \rangle \otimes | \downarrow \rangle + | \downarrow \rangle \otimes | \uparrow \rangle \) on \( W_{2i} \otimes W_{2i+1} \) and \( | \uparrow \rangle \otimes | \uparrow \rangle + | \downarrow \rangle \otimes | \downarrow \rangle \) on \( W_{2j} \otimes W_{2j+1} \) for \( j \neq i \). It is easy to see that

\[
U(g) | \Omega_a \rangle = \begin{cases} | \Omega_a \rangle, & g = e, a; \\ -| \Omega_a \rangle, & g = b, ab. \end{cases}
\]

Hence, the \( a \)-twisted ground state spans a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-representation \( \phi_a \). In other words, the \( a \)-twisted ground state carries the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-charge \( e_2 \).

Similarly, we can check that the \( b \)-twisted ground state is unique and carries the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-charge \( e_1 \), and the \( ab \)-twisted ground state is also unique and carries the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-charge \( e_1 e_2 \). So we obtain the ground state space of the nontrivial \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) SPT on the circle is \( 1 \oplus e_1 m_2 \oplus e_2 m_1 \oplus f_1 f_2 \). This is exactly the Lagrangian algebra \( A(\mathbb{Z}_2 \times \mathbb{Z}_2, \omega) \).
Remark VI.3. The dimension of a vector space $V \in \mathcal{Z}_1(\text{Rep}(G))$ is equal to its Frobenius-Perron dimension [66]. Moreover, the Frobenius-Perron dimension of a Lagrangian algebra $A \in \mathcal{Z}_1(\text{Rep}(G))$ is always equal to $|G|$. Thus we see that the ground state degeneracy of a 1+1D gapped quantum phase with symmetry $G$ on a circle (with all twisted periodic boundary conditions) is equal to $G$.

Furthermore, when $G$ is abelian, by the construction of the Lagrangian algebra $A(H, \omega)$ (see Example A.7), we have
\[
\dim A(H, \omega)_g = \begin{cases} 
|G|/|H|, & g \in H, \\
0, & \text{otherwise}.
\end{cases}
\]

This gives the ground state degeneracy on the circle with the $g$-twisted periodic boundary condition. \hfill ◊

Remark VI.4. We have not discussed the algebra structure of the ground state space on the circle. In the language of open-closed TQFT [67], the ground state space on $S^1$ is the closed TQFT (or closed string algebra), and its algebra structure is obviously given by the cobordism of pants.

Moreover, given a $0+1D$ boundary, we can also consider the ground state space on a finite interval $[0, 1]$. In the language of open-closed TQFT, the ground state space on $[0, 1]$ is the open TQFT (or open string algebra). In a 2D open-closed TQFT or CFT, the open-closed duality [67–69] states that the closed string algebra is the full center [46] of the open string algebra.

Furthermore, if we fix the categorical symmetry on the boundary (in our case this means that we need to specify that the symmetry on the boundary is explicitly broken to a subgroup $H$, possibly with a 2-cocycle twist), we can also consider the ground state spaces on $[0, 1]$ with different boundary conditions. These ground state spaces are not only open string algebras, but also realize the whole enriched category of boundary conditions with the given categorical symmetry discussed in Section IV A. \hfill ◊

Remark VI.5. The finite group symmetry $G$ in the above discussion can also be generalized. For example, a 1+1D modular invariant CFT can be viewed as a quantum liquid phase with a gapless non-chiral symmetry, usually realized by a non-chiral algebra $V$, and the categorical symmetry is the UMTC $\mathcal{B} := \text{Mod}_V$ of $V$-modules [20, 25]. In this case, the ground state subspace on $S^1$ is usually known as the Hilbert space on $S^1$ of the CFT (or simply the closed CFT). It has been proved [69] that the Hilbert space on $S^1$ of a modular invariant CFT is a Lagrangian algebra in $\text{Mod}_V$. \hfill ◊

VII. DISCUSSIONS AND OUTLOOKS

Category theory is becoming an indispensable tool in studying the quantum many-body systems. However, it has been mostly used in the study of topological phases. In this work, we have shown the capability of enriched fusion categories in describing 1d gapped phases with symmetries, including symmetry breaking phases within traditional Landau’s paradigm.

For a 1+1D gapped phase with abelian onsite symmetry $G$, its macroscopic properties can be summarized by the categorical symmetry $\mathcal{Z}_1(\text{Rep}(G))$ and a Lagrangian algebra $A(H, \omega)$, in which $A(H, \omega)$ plays the role of invariant operators acting on the sector of ground state, such that the enriched fusion category $\mathcal{Z}_1(\text{Rep}(G)) \mathcal{Z}_1(\text{Rep}(G))_{A(H, \omega)}$ can be read off from lattice model directly. Under this observation, we can distinguish different symmetry breaking phases as well as the trivial and non-trivial SPT orders.

We construct general lattice models with abelian onsite symmetry to show that the topological sectors of operators form a braided fusion category $\mathcal{Z}_1(\text{Rep}(G)) \simeq \mathcal{Z}_1(\text{Vec}_G)$, and we invent equivariantization technique in section II to find different topological sectors of state, which turns out to be $\mathcal{Z}_1(\text{Rep}(G))_{A(H)}$ (depending on which subgroup $H \subset G$ the symmetry breaks to). We also demonstrate that the 0d boundaries of these 1d phases form enriched categories which are compatible with the 1d bulk descriptions. Well-known physical examples are also been performed. These results meet with the predictions of holographic duality. But it is just a tip of the iceberg. For example, the idea of holographic duality can be expanded to other kinds of symmetries such as non-invertible ones. Here we briefly explain how lattice duality should be unified under the same enriched categorical description.

A. lattice duality under holographic duality

As the enriched category description only cares about observables in the LWLL (or to say, it really captures what a gapped phase is in the macroscopic level), for a given lattice model with onsite symmetry $G$, we should also have the same categorical descriptions appear in the phases of its dual lattices, in which their energy spectrum are the same. Dual models have equivalent but distinct realizations of symmetries, characterized by $S^*_\mathcal{M}$ [30] with different choices of indecomposable module category $\mathcal{M}$. On the other hand, the indecomposable module categories of $\mathcal{B}$ one to one correspond to the gapped boundaries of $\mathcal{Z}_1(\mathcal{B})$ [37] with different choices of condensable algebras $A_i$. So just like the topological sectors of states, the global lattice symmetries form Morita equivalent fusion categories that can be obtained by condensing Lagrangian algebras.
FIG. 19. Actually, the topological skeletons we obtain here describes a set of phases that can be achieved by dual models. Suppose there are 4 Lagrangian algebras $A_1$ to $A_4$ of the categorical symmetry $3_1(S)$, then we would have 4 dual models with lattice symmetries $A_i$. Then the permutation of topological sectors of states $3_1(S)_A$ defines the symmetry preserving / symmetry breaking properties of each model, leading to seemingly 16 gapped phases, but each four of them can be unified under one enriched fusion categories.

Indeed, everytime we talk about the (onsite) symmetry of a concrete lattice model, it is that we fix a specific boundary (or Lagrangian algebra equivalently) from the categorical symmetry which forms a modular tensor category. In this paper, we implicitly fix the global onsite symmetry $G$ which can be regarded as fusion category $Vec_G \simeq 3_1(\text{Rep}(G))_{A(\{e\})}$, where $A(\{e\})$ is the Lagrangian algebra in $3_1(\text{Rep}(G))$ contains all chargeons.

If we choose other Lagrangian algebras in $3_1(\text{Rep}(G))$, we can have different lattice symmetries. Note that the lattice symmetries may be non-invertible and non-onsite (e.g. $\text{Rep}(G)$ for non-abelian cases). Concrete lattice constructions with these kind of symmetries can be found in literatures such as $[30, 55, 56]$. A lattice model $\mathcal{H}$’s dual models and $\mathcal{H}$’s gapped phases are actually ‘dual concepts’ which are both aspects of the total categorical symmetry, and every enriched category describes a bunch of representations in dual lattices realizing equivalent phases.

In the example of Kramers-Wannier duality of transverse field Ising model $V_B$, we give a taste of this. We can also consider the Kennedy-Tasaki duality $[70]$.KT transformation provides a map between a model with symmetry protected topological order in the ground state, such as the Haldane phase, to a model without symmetry protected topological order. And the four degenerate edge modes becomes four bulk degenerate ground states in the symmetry breaking phase afterwards. See the two frames in table II, this two phases are both controlled by the ground state algebra $A(G, \omega)$. We also list other kinds of probabilities in $Z_2 \times Z_2$’s dual models. $0+1D$ boundaries can also be understood through a similar procedure, which may lead to non-trivial conclusions.

Other than explaining interesting phenomena emerge in dualities, the immediate follow-up for Theorem$^{36}$ is to check that the 1d gapped phases with onsite non-abelian symmetries can also be unified through enriched fusion categories with ground state algebras. In fact, as equivariantization does not depend on symmetry $G$ abelian or not, the only difficulty remains is to check the topological sector of operators, namely, to see the non-local operators in a system of a general onsite symmetry $G$ also form $3_1(\text{Rep}(G))$.

Then one can try to replace the holographic bulk phase of figure 3 by the Levin-Wen model $[10]$, which should give interesting fusion category symmetries. Actually, there are already constructions of the general 1d lattice models with arbitrary fusion category symmetries $[50]$. The generalization of our construction using quantum current is given recently in $[71]$. One can also think of the fermionic cases, the starting point might be to think of the fermion condensation $[72]$ in 2d topological order $3_1(\text{Rep}(G, z)) = 3_1(\text{Rep}(G))$ ($z \in G$ is the fermion parity). Different from the bosonic cases, an interchange between two topological excitations should have a non-trivial phase factor $-1$. 

### TABLE II. The modular tensor category $3_1(\text{Rep}(Z_2 \times Z_2))$ has six Lagrangian algebra, which we label by $A_1$ through $A_6$ $[32]$. Through lattice models can have different symmetries up to Morita equivalence, phases in LWLL are classified by ground state algebras. For example, the non-trival SPT order and the symmetry completely broken phase connected by Kennedy-Tasaki transformation can be both described by $3_1(\text{Rep}(Z_2 \times Z_2))^{\text{Rep}(Z_2 \times Z_2)_{A(G, \omega)}}$

| top. states | lattice sym. | $\mathcal{H}^{Z_2 \times Z_2}$ | $A_1(\text{Rep}(G))$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|------------|-------------|------------------|------------------|-------|-------|-------|-------|-------|
| $A_1 = 1 \oplus e_1 \oplus e_2 \oplus e_1 e_2$ | SCB | $Z_2$ | $Z_2$ | $Z_2$ | SPT | SPT | $3_1(\text{Rep}(Z_2 \times Z_2))^{\text{Vec}(Z_2 \times Z_2)}$ |
| $A_2 = 1 \oplus e_1 \oplus m_1 \oplus e_1 m_2$ | $Z_2$ | SCB | SPT | SPT | $Z_2$ | $Z_2$ | $3_1(\text{Rep}(Z_2 \times Z_2))^{3_1(\text{Rep}(Z_2 \times Z_2))_{A_2}}$ |
| $A_3 = 1 \oplus m_1 \oplus e_2 \oplus m_1 e_2$ | $Z_2$ | SPT | SCB | SPT | $Z_2$ | $Z_2$ | $3_1(\text{Rep}(Z_2 \times Z_2))^{3_1(\text{Rep}(Z_2 \times Z_2))_{A_3}}$ |
| $A_4 = 1 \oplus e_2 e_1 \oplus m_1 m_2 \oplus f_1 f_2$ | $Z_2$ | SPT | SCB | $Z_2$ | $Z_2$ | $3_1(\text{Rep}(Z_2 \times Z_2))^{3_1(\text{Rep}(Z_2 \times Z_2))_{A_4}}$ |
| $A_5 = 1 \oplus m_1 \oplus m_2 \oplus e_1 m_2$ | SPT | $Z_2$ | $Z_2$ | SCB | SPT | $3_1(\text{Rep}(Z_2 \times Z_2))^{3_1(\text{Rep}(Z_2 \times Z_2))_{A_5}}$ |
| $A_6 = 1 \oplus e_1 m_2 \oplus e_2 m_1 \oplus f_1 f_2$ | SPT | $Z_2$ | $Z_2$ | SPT | SCB | $3_1(\text{Rep}(Z_2 \times Z_2))^{3_1(\text{Rep}(Z_2 \times Z_2))_{A_6}}$ |
Moreover, the enriched category description is valid in describing phase transition point [20, 73]. It would be interesting to describe the phase transition points between 1d gapped quantum phases. In the clock model example, we roughly discuss the 'step-like' phase transition behavior (see Figure 16). The topological skeleton of phase transition points in partially symmetry breaking cases may be 'fused' in an interesting way to become the phase transition point of larger symmetry breaking subgroup.

More generally, since our work is an example of the n+1D gauge theory with gapped boundaries that maps to nD gapped quantum liquid with global symmetries, we want to see more works that fulfill the territory of holographic dualities, especially in the realm of high energy physics. We should even look for a theory unifying both AdS/CFT dualities and the holographic dualities in condensed matter. For the adventurers, there is still a continent to be discovered.

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Appendix A: Pointed braided fusion categories

1. Pointed braided fusion categories and pre-metric groups

In this section we briefly review the relation between pointed braided fusion categories and pre-metric groups.

Definition A.1. Let $\mathcal{C}$ be a fusion category. An object $x \in \mathcal{C}$ is called invertible if it is invertible under the tensor product, i.e. there exists an object $y \in \mathcal{C}$ such that $x \otimes y \cong 1 \cong y \otimes x$. We say $\mathcal{C}$ is pointed if every simple object in $\mathcal{C}$ is invertible. ■

Definition A.2. Let $G$ be an abelian group. A quadratic form on $G$ is a map $q: G \to \mathbb{C}^\times$ such that $q(g) = q(g^{-1})$ and the symmetric function $b: G \times G \to \mathbb{C}^\times$ defined by

$$b(g, h) := \frac{q(gh)}{q(g)q(h)}$$

is a bicharacter, i.e., $b(g_1g_2, h) = b(g_1, h)b(g_2, h)$ for all $g_1, g_2, h \in G$. The symmetric bicharacter $b$ is called the associated bicharacter of $q$. We say $q$ is nondegenerate if $b$ is, i.e., $b(g, h) = 1$ for all $h \in G$ if and only if $g = e$ is the unit.

A pre-metric group is a finite abelian group $G$ equipped with a quadratic form $q$. If $q$ is nondegenerate, then $(G, q)$ is called a metric group.

Let $\mathcal{C}$ be a pointed braided fusion category. Then isomorphism classes of simple objects of $\mathcal{C}$ form a finite abelian group under the tensor product, denoted by $\mathcal{G}$. In other words, $\mathcal{C} \cong \text{Vec}_\mathcal{G}$ as categories. For $g \in G$, define

$$q(g) := c_{x, x} \in \text{Aut}_\mathcal{C}(x \otimes x) \cong \mathbb{C}^\times,$$

where $x$ is a simple object of $\mathcal{C}$ whose isomorphism class is $g$. It follows from the pentagon and hexagon equations that $q$ is a quadratic form. The associated bicharacter of $q$ is equal to the double braiding:

$$b(g, h) = c_{y, x} \circ c_{x, y} \in \text{Aut}_\mathcal{C}(x \otimes y) \cong \mathbb{C}^\times,$$

where $x$ and $y$ are simple objects of $\mathcal{C}$ whose isomorphism classes are $g$ and $h$, respectively. We call $(G, q)$ the associated pre-metric group of $\mathcal{C}$.

Conversely, there is a mathematical theorem [74, 75] states that two pointed braided fusion categories are equivalent if and only if their associated pre-metric groups are isomorphic. In other words, for any pre-metric group $(G, q)$ there exists a unique (up to equivalence) pointed braided fusion category, denoted by $\mathcal{C}(G, q)$, such that its associated pre-metric group is isomorphic to $(G, q)$. Clearly the pointed braided fusion category $\mathcal{C}(G, q)$ is nondegenerate if and only if the pre-metric group $(G, q)$ is nondegenerate, i.e., $(G, q)$ is a metric group.

Moreover, since $\mathcal{C}(G, q)$ is pseudo-unityary, there is a unique spherical structure on $\mathcal{C}(G, q)$ such that the quantum dimension of every simple object is 1 [66]. Then $\mathcal{C}(G, q)$ is a pre-modular category with this spherical structure, and it is modular if and only if $(G, q)$ is a metric group. Its $S$ matrix is given by

$$S_{g, h} = b(g, h^{-1}) = b(g^{-1}, h),$$

and the $T$ matrix is given by

$$T_g = q(g).$$

Example A.3. Let $G$ be a finite abelian group. Then the Drinfeld center $Z_1(\text{Rep}(G)) \cong Z_1(\text{Vec}_G)$ is a nondegenerate pointed braided fusion category and the associated metric group is $(G \times G, q)$, where $G$ is the dual group of $G$ and the quadratic form $q$ is defined by $q(g, \rho) := \rho(g)$. We use $\mathcal{O}_{(g, \rho)}$ to denote a representative of isomorphism classes of simple objects of $Z_1(\text{Rep}(G))$. Then the double braiding of two simple objects $\mathcal{O}_{(g, \rho)}$ and $\mathcal{O}_{(h, \sigma)}$, by (A.1), is equal to $\rho(h)\sigma(g)$.

2. Condensable algebras in pointed braided fusion categories

In this section we briefly review the classification of condensable algebras (in particular, Lagrangian algebras)
in pointed braided fusion categories.

**Definition A.4.** Let $\mathcal{C}$ be a braided fusion category and $A = (A, \mu, \eta)$ be an algebra in $\mathcal{C}$. Then

(a) $A$ is **separable** if there exists an $(A, A)$-bimodule map $e: A \to A \otimes A$ such that $\mu \circ e = \text{id}_A$;

(b) $A$ is **connected** if $\dim \text{Hom}_\mathcal{C}(\mathbb{I}, A) = 1$;

(c) $A$ is **commutative** if $\mu \circ c_{A,A} = \mu$.

A **condensable algebra** is a commutative connected separable algebra. Moreover, a **Lagrangian algebra** in a non-degenerate braided fusion category $\mathcal{C}$ is a condensable algebra $A$ such that $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$, or equivalently, the category $\mathcal{C}^0_A$ of local right $A$-modules in $\mathcal{C}$ is equivalent to $\text{Vec}$.

**Definition A.5.** Let $(G, q)$ be a pre-metric group. The associated bicharacter of $q$ is denoted by $b$. For any subgroup $H \subseteq G$, define the **orthogonal complement** of $H$ as

$$H^\perp := \{ g \in G \mid b(g, h) = 1, \forall h \in H \}.$$ We say a subgroup $H \subseteq G$ is **isotropic** if $q(h) = 1$ for all $h \in H$. It follows that an isotropic subgroup $H$ satisfies $H \subseteq H^\perp$. A **Lagrangian subgroup** of a metric group is an isotropic subgroup $H$ such that $H = H^\perp$.

**Theorem A.6** ([37, 75–77]). Let $(G, q)$ be a pre-metric group and $A$ be a condensable algebra in $\mathcal{C}(G, q)$.

1. Suppose $\{ C_g \}_{g \in G}$ is a representative of isomorphism classes of simple objects of $\mathcal{C}(G, q)$. Then $\dim \text{Hom}_{\mathcal{C}(G, q)}(C_g, A)$ is either 0 or 1.

2. Define the **support** of $A$ to be the set

$$\text{Supp } A := \{ g \in G \mid \dim \text{Hom}_{\mathcal{C}(G, q)}(C_g, A) = 1 \}.$$ Then the support $\text{Supp } A$ is an isotropic subgroup of $(G, q)$.

3. For any isotropic subgroup $H$ of $(G, q)$, there exists a unique (up to isomorphism) condensable algebra $A_H$ in $\mathcal{C}(G, q)$ such that $\text{Supp } A_H = H$. In other words, there is a one-to-one correspondence between isomorphism classes of condensable algebras in $\mathcal{C}(G, q)$ and isotropic subgroups of $(G, q)$. As an object $A_H$ is isomorphic to $\bigoplus_{h \in H} C_h$, and the multiplication is induced by the group multiplication:

$$A \otimes A \simeq \left( \bigoplus_{g \in H} C_g \right) \otimes \left( \bigoplus_{h \in H} C_h \right) = \bigoplus_{g,h \in H} C_g \otimes C_h \oplus \delta_{gh,k} C_k + \bigoplus_{k \in H} C_k = A.$$ (4) For any isotropic subgroup $H \subseteq G$, the braided fusion category $\mathcal{C}(G, q)^0_H$ of local right $A_H$-modules is pointed, and the associated pre-metric group is the quotient group $H^\perp / H$ equipped with the quadratic form $\bar{q}([g]) = q(g)$. In particular, there is a one-to-one correspondence between isomorphism classes of Lagrangian algebras in $\mathcal{C}(G, q)$ and Lagrangian subgroups of $(G, q)$.

**Example A.7.** Let $G$ be a finite abelian group. Recall example A.3 that the metric group associated to $3_1(\text{Rep}(G)) \simeq 3_1(\text{Vec}_G)$ is $(G \times G, q)$ where $q(g, \rho) = \rho(g)$. For any subgroup $H \subseteq G$, the dual group $G / H$ naturally embeds into $\hat{G}$ and $H \times \hat{G}$ is a Lagrangian subgroup of $(G \times \hat{G}, q)$. We denote the corresponding Lagrangian algebra by $A(H) \in 3_1(\text{Rep}(G))$.

More generally, suppose $H \subseteq G$ is a subgroup and $\tau: H \times H \to \mathbb{C}^\times$ is an anti-symmetric bicharacter, i.e. a bicharacter satisfying $\tau(h, h) = 1$ for all $h \in H$. Then $S(H, \tau) := \{ (h, \rho) \in G \times \hat{G} \mid h \in H, \rho(k) = \tau(h, k), \forall k \in H \}$ is a Lagrangian subgroup of $(G \times \hat{G}, q)$. Conversely, every Lagrangian subgroup of $(G \times \hat{G}, q)$ is equal to $S(H, \tau)$ for some subgroup $H \subseteq G$ and anti-symmetric bicharacter $\tau$.

For any finite abelian group $H$ and 2-cocycle $\omega \in \text{Z}^2(H; \text{U}(1))$, one can verify that

$$H \times H \to \mathbb{C}^\times$$

$$(g, h) \mapsto \omega(g, h) / \omega(h, g)$$
defines an anti-symmetric bicharacter on $H$. Also, this anti-symmetric bicharacter only depends on the cohomology class of $\omega$. Thus we have a group homomorphism from $\text{H}^2(H; \text{U}(1))$ to the group of anti-symmetric bicharacters on $H$. It is well-known that this group homomorphism is indeed an isomorphism (because every symmetric 2-cocycle of a finite group is a coboundary). Thus for every subgroup $H$ and $\omega \in \text{H}^2(H; \text{U}(1))$ there is a Lagrangian algebra $A(H, \omega)$ defined by the Lagrangian subgroup $S(H, \tau)$.

As a conclusion, there is a one-to-one correspondence between isomorphism classes of Lagrangian algebras in $3_1(\text{Rep}(G)) \simeq 3_1(\text{Vec}_G)$ and pairs $(H, [\omega])$, where $H \subseteq G$ is a subgroup and $[\omega] \in \text{H}^2(H; \text{U}(1))$ is a 2-cohomology class. This coincides with the general classification result in [40].

**Example A.8.** As an example, we list the Lagrangian algebras in $3_1(\text{Rep}(G))$ when $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. We denote the elements in $G$ by $1, m_1, m_2, m_1m_2$, and denote the elements in $\hat{G}$ by $1, e_1, e_2, e_1e_2$ with $(m_1, e_j) = (-1)^{i+j}$ for $i, j = 1, 2$. Then there are 6 Lagrangian algebras in $3_1(\text{Rep}(G))$ (i.e., 6 Lagrangian subgroups in $(G \times \hat{G}, q)$):

- The Lagrangian algebra corresponding to the trivial subgroup $H = \{1\}$ is $1 \oplus e_1 \oplus e_2 \oplus e_1e_2$. 

\[\text{\textcircled{}}\]
• There are 3 $\mathbb{Z}_2$ subgroups of $G$ generated by $m_1, m_2, m_1m_2$, respectively. Note that $H^2(\mathbb{Z}_2; U(1)) = 0$ is trivial. Then the Lagrangian algebra corresponding to these $\mathbb{Z}_2$ subgroups are $I \oplus m_1 \oplus e_2 \oplus m_1e_2, I \oplus m_2 \oplus e_1 \oplus m_2e_1$, and $I \oplus m_1m_2 \oplus e_1e_2 \oplus f_1f_2$, respectively, where $f_i := e_im_i$ for $i = 1, 2$.

• We have $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1)) = 0$. Then the Lagrangian algebra $A(\mathbb{Z}_2 \times \mathbb{Z}_2)$ corresponding to $H = G$ and the nontrivial 2-cohomology class is given by $1 \oplus m_1 \oplus m_2 \oplus m_1m_2$, and the Lagrangian algebra $A(\mathbb{Z}_2 \times \mathbb{Z}_2, \omega)$ corresponding to $H = G$ and the nontrivial 2-cohomology class is given by $1 \oplus m_1e_2 \oplus m_2e_1 \oplus f_1f_2$.

Appendix B: Equivariantization

In this appendix we briefly review the notion of equivariantization and prove Proposition III.7.

1. Group actions on categories and equivariantization

Let $\mathcal{C}$ be a category. The autoequivalences of $\mathcal{C}$ and natural isomorphisms between them form a monoidal category, denoted by $\text{Aut}(\mathcal{C})$. An action of a group $G$ on $\mathcal{C}$ is defined to be a monoidal functor $T: G \to \text{Aut}(\mathcal{C})$, where $G$ is viewed as a monoidal category with only identity morphisms.

More precisely, by the definition of a monoidal functor, a $G$-action on $\mathcal{C}$ consists of a set of equivalences $\{T_g: \mathcal{C} \to \mathcal{C}\}_{g \in G}$ and a set of natural isomorphisms $\{\gamma_{g,h}: T_g \circ T_h \Rightarrow T_{gh}\}_{g,h \in G}$, such that the following diagram commutes:

\[
\begin{array}{ccc}
(T_gT_h)T_k & \overset{\gamma_{g,h}}{\Rightarrow} & T_g(T_hT_k) \\
T_{gh}T_k & \overset{\gamma_{g,h,k}}{\Leftarrow} & T_{ghk} \overset{\gamma_{g,h,k}}{\Leftarrow} T_gT_{hk}
\end{array}
\]  

Remark B.1. If $\mathcal{C}$ is a monoidal category, a monoidal action of a group $G$ on $\mathcal{C}$ is a monoidal functor from $G$ to the monoidal category $\text{Aut}(\mathcal{C})$ of monoidal autoequivalences of $\mathcal{C}$. If $\mathcal{C}$ is a braided monoidal category, a braided action of a group $G$ on $\mathcal{C}$ is a monoidal functor from $G$ to the monoidal category $\text{Aut}^\text{br}(\mathcal{C})$ of braided monoidal autoequivalences of $\mathcal{C}$.\]

Definition B.2. Let $G$ be a group and $\mathcal{C}$ be a category equipped with a $G$-action. The equivariantization $\mathcal{C}^G$ of $\mathcal{C}$ is the category consisting of the following data:

- The objects of $\mathcal{C}^G$ are pairs $(x, \{u_g\}_{g \in G})$, where $x \in \mathcal{C}$ and $u_g: T_g(x) \to x$ is an isomorphism such that the following diagram commutes:

\[
\begin{array}{ccc}
T_gT_h(x)(\gamma_{g,h}) & \overset{u_{gh}}{\Rightarrow} & T_{gh}(x) \\
T_g(u_h) & \overset{u_{gh}}{\Leftarrow} & u_h
\end{array}
\]  

(B.2)

- A morphism $f: (x, \{u_g\}_{g \in G}) \to (y, \{v_g\}_{g \in G})$ is a morphism $f: x \to y$ in $\mathcal{C}$ such that the following diagram commutes:

\[
\begin{array}{ccc}
T_g(x) & \overset{T_g(f)}{\Rightarrow} & T_g(y) \\
\downarrow u_g & \overset{v_g}{\Leftarrow} & \downarrow \circ v_h \\
x & \overset{f}{\Rightarrow} & y
\end{array}
\]  

(B.3)

There is an obvious forgetful functor $\mathcal{C}^G \to \mathcal{C}$ defined by $(x, \{u_g\}_{g \in G}) \mapsto x$.

Remark B.3. Let $G$ be a group. If $\mathcal{C}$ is a (braided) monoidal category equipped with a (braided) monoidal $G$-action, then the equivariantization $\mathcal{C}^G$ is also a (braided) monoidal category and the forgetful functor $\mathcal{C}^G \to \mathcal{C}$ is also a (braided) monoidal functor.\]

Example B.4. Let $G$ be a group. Consider the trivial $G$-action on the category $\text{Vec}$ of finite-dimensional vector spaces. More precisely, all functors $T_g$ are identity functors and all natural transformations $\gamma_{g,h}$ are identity natural transformations. Then an object $(V, \{v_g\}_{g \in G})$ is a finite-dimensional vector space $V$ equipped with linear isomorphisms $v_g: V \to V$ such that $v_{gh} = v_g \circ v_h$. In other words, $(V, \{v_g\}_{g \in G})$ is a finite-dimensional $G$-representation. Similarly one can verify that morphisms between equivariant objects are homomorphisms between $G$-representations. Hence the equivariantization $\text{Vec}^G$ is equivalent to the symmetric monoidal category $\text{Rep}(G)$ of finite-dimensional $G$-representations, and the forgetful functor $\text{Rep}(G) \simeq \text{Vec}^G \to \text{Vec}$ is the usual forgetful functor.\]

2. The proof of Proposition III.7

Here we give a proof of Proposition III.7. To be precise, we reformulate the proposition as follows.

Let $G$ be a (not necessarily abelian) finite group and $H \subseteq G$ be a subgroup. We equip the quotient set $G/H$ with the left translation $G$-action: $(g, x) \mapsto gx$ for $g \in G$ and $x \in G/H$. It induces a $G$-action on the function algebra $F_H := \text{Fun}(G/H)$:

$$(g \cdot f)(x) := f(g^{-1}x), \quad g \in G, \quad f \in F_H, \quad x \in G/H.$$
If we choose a basis \( \{ \delta_x \}_{x \in G/H} \) of \( F_H \) consisting of delta functions
\[
\delta_x(y) := \delta_{x,y}, \quad x, y \in G/H,
\]
then the \( G \)-action on \( F_H \) is given by \( g \cdot \delta_x = \delta_{g \cdot x} \). Moreover, this \( G \)-action is compatible with the algebra structure so that \( \text{Fun}(G/H) \) is a condensable algebra in \( \text{Rep}(G) \).

This \( G \)-action also induces a \( G \)-action on the category \( \text{Vec}_{G/H} \) of finite-dimensional \( G/H \)-graded vector spaces:
\[
T_g(V)_x := V_{g^{-1} \cdot x},
\]
for \( g \in G, x \in G/H, \) and \( V \in \text{Vec}_{G/H} \). It induces a monoidal \( G \)-action on the multi-fusion category \( \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H}) \) defined by conjugation:
\[
g \cdot F := T_g \circ F \circ T_{g^{-1}},
\]
for \( g \in G, F \in \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H}) \). Also we know that \( \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H}) \) is monoidally equivalent to the multi-fusion category \( \text{Mat}_n(\text{Vec}) \) of \( n \)-by-\( n \) matrices valued in \( \text{Vec} \), where \( n := |G|/|H| \). An object in \( \text{Mat}_n(\text{Vec}) \) is a finite-dimensional bi-graded vector space
\[
V = \bigoplus_{x, y \in G/H} V_{x, y}.
\]
The tensor product in \( \text{Mat}_n(\text{Vec}) \) is defined by
\[
(V \otimes W)_{x, y} = \bigoplus_{z \in G/H} V_{x, z} \otimes W_{z, y}.
\]
Under the monoidal equivalence \( \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H}) \simeq \text{Mat}_n(\text{Vec}) \), the monoidal \( G \)-action on \( \text{Mat}_n(\text{Vec}) \) is given by
\[
g(V)_{x, y} = V_{g^{-1} \cdot x, g^{-1} \cdot y},
\]
for \( g \in G, x, y \in G/H, V \in \text{Mat}_n(\text{Vec}) \). More generally, let \( K \subseteq G \) be another subgroup. Then there is also a conjugation \( G \)-action on the finite semisimple category \( \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/K}) \). Under the equivalence \( \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/K}) \simeq \text{Mat}_{n \times m}(\text{Vec}) \) (where \( m := |G|/|K| \)), the \( G \)-action is given by
\[
g(V)_{x, y} = V_{g^{-1} \cdot x, g^{-1} \cdot y},
\]
for \( g \in G, x \in G/K, y \in G/H, V \in \text{Mat}_{n \times m}(\text{Vec}) \).

**Proposition B.5.** The category \( F_K \text{Rep}(G)_{F_H} \) is equivalent to \( \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/K})^G \). When \( K = H \), this is an equivalence of fusion categories.

The main idea is to show that both two categories are equivalent to the third one.

**Definition B.6.** Let \( G \) be a group and \( X \) be a set equipped with a left \( G \)-action. The action groupoid \( \mathcal{G}(X, G) \) is defined by the following data:

- \( \text{ob}(\mathcal{G}(X, G)) := X \).
- \( \text{Hom}_{\mathcal{G}(X, G)}(x, y) := \{ g \in G \mid g \cdot x = y \} \) for \( x, y \in X \).
- The composition and identity morphisms are induced by the multiplication and unit of \( G \).

Let \( \mathcal{G} \) be a finite groupoid, i.e., a groupoid with finitely many objects and morphisms. Recall that a finite-dimensional \( \mathcal{G} \)-representation is a functor \( F : \mathcal{G} \to \text{Vec} \). More precisely, a finite-dimensional \( \mathcal{G} \)-representation is a collection \( \{ F(x) \in \text{Vec} \}_{x \in \mathcal{G}} \) of finite-dimensional vector spaces, together with linear isomorphisms \( F(f) : F(x) \to F(y) \) labeled by morphisms \( f : x \to y \) in \( \mathcal{G} \), such that \( F(g) \circ F(f) = F(g \circ f) \) whenever the composition of morphisms \( f, g \) in \( \mathcal{G} \) can be defined. All finite-dimensional \( \mathcal{G} \)-representations and natural transformations between them form a finite semisimple category \( \text{Rep}(\mathcal{G}) \).

**Example B.7.** Consider the action groupoid \( \mathcal{G}(G/K \times G/H, G) \), where \( G \) acts on \( G/K \times G/H \) diagonally: \( g \cdot (x, y) := (gx, gy) \). Then a finite-dimensional \( \mathcal{G}(G/K \times G/H, G) \)-representation \( F \) is equivalent to a collection of the following data:

- finite-dimensional vector spaces \( \{ F(x, y) \in \text{Vec} \}_{x \in G/K, y \in G/H} \);
- linear isomorphisms \( F(x, y) : F(x, y) \to F(x, y) \), such that the following equation holds for every \( x \in G/K, y \in G/H \) and \( g, h \in G \):
\[
(F(x, y) \xrightarrow{F(h, x, y)} F(hx, hy) \xrightarrow{F(g, hx, hy)} F(ghx, ghy)) = F(gh, x, y).
\]

**Remark B.8.** The connected components of \( \mathcal{G}(G/K \times G/H, G) \) are one-to-one corresponds to double cosets in \( K \backslash G/H \). In particular, if \( G \) is a finite abelian group and \( K = H \), the category \( \text{Rep}(\mathcal{G}(G/K \times G/H, G)) \) is equivalent to the direct sum of \( n = |G|/|H| \) copies of \( \text{Rep}(H) \). A simple object \( F \in \text{Rep}(\mathcal{G}(G/K \times G/H, G)) \) only supports on a single connected component and every nonzero \( F(x, y) \) is an irreducible \( H \)-representation. In other words, the simple objects are labeled by pairs \( (x, \rho) \) where \( x \in G/H \) and \( \rho \in H \). More precisely, the simple object corresponding to \( (x, \rho) \), denoted by \( \mathcal{M}(x, \rho) \), is the induced representation \( \text{Ind}^G_H(\rho) \) equipped with the \( G/H \times G/H \)-grading
\[
\langle \mathcal{M}(x, \rho) \rangle_{y, z} = \begin{cases} \langle \text{Ind}^G_H(\rho) \rangle_z, & y = xz, \\ 0, & \text{otherwise} \end{cases}
\]
which is determined by \( x \).

**Lemma B.9.** The category \( F_K \text{Rep}(G)_{F_H} \) is equivalent to \( \text{Rep}(\mathcal{G}(G/K \times G/H, G)) \) as categories.

**Proof.** By definition, an \( (F_K, F_H) \)-bimodule \( M \in F_K \text{Rep}(G)_{F_H} \) in \( \text{Rep}(G) \) is nothing but
• an \((F_K, F_H)\)-bimodule \(M \in F_K \text{Vec} F_H\) in Vec,
• equipped with a \(G\)-action, such that the left \(F_K\)-
  and right \(F_H\)-actions on \(M\) are \(G\)-equivariant, i.e.,
\[
(g \cdot a) \triangleright (g \cdot m) \triangleleft (g \cdot b) = g \cdot (a \triangleright m \triangleleft b)
\]
\[(B.4)\]
for all \(g \in G\), \(m \in M\) and \(a \in F_K\), \(b \in F_H\).

Since \(F_H\) and \(F_K\) are semisimple algebras, a bimodule \(M \in F_K \text{Vec} F_H\) can be decomposed as
\[
M = \bigoplus_{x \in G/K} \bigoplus_{y \in G/H} M_{x,y},
\]
where \(M_{x,y} := \delta_x \triangleright M \triangleleft \delta_y\). Taking an element \(m \in M_{x,y}\),
the condition \((B.4)\) implies that
\[
\delta_{g,z} \triangleright (g \cdot m) \triangleleft \delta_{g,w} = (g \cdot \delta_z) \triangleright (g \cdot m) \triangleleft (g \cdot \delta_w) = g \cdot (\delta_z \triangleright m \triangleleft \delta_w) = \delta_{x,z} \delta_{y,w} (g \cdot m).
\]

Thus \(g \cdot m \in M_{g \cdot x, g \cdot y}\). Hence an \((F_K, F_H)\)-bimodule \(M \in F_K \text{Rep}(G) F_H\) in \text{Rep}(G) is equivalent to a collection of the following data:
• finite-dimensional vector spaces \(
\{M_{x,y}\}_{x \in G/K, y \in G/H}\);
• linear isomorphism \(\rho_{g,x,y}: M_{x,y} \rightarrow M_{g \cdot x, g \cdot y}\), such that the following equation holds for every \(x \in G/K\), \(y \in G/H\) and \(g, h \in G\):
\[
(M_{x,y} \rho_{h,x,y}) = (M_{h \cdot x, h \cdot y} \rho_{g,h \cdot x, h \cdot y} \rho_{g,h \cdot x, h \cdot y}) = \rho_{g,h \cdot x, h \cdot y}.
\]

By comparing with example \(B.7\) the lemma is proved. □

**Lemma B.10.** The equivariantization \(\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/K})^G\) is equivalent to \(\text{Rep}(\mathcal{G}(G/K \times G/H, G))\) as categories.

**Proof.** An object in the equivariantization \(\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/K})^G\) is a pair \((V, \{v_g\}_{g \in G})\), where \(V\) is an object in \(\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/K})\) \(\simeq \text{Mat}_{\text{ax}}(\text{Vec})\) and \(v_g: g(V) \rightarrow V\) is an isomorphism such that the diagram \((B.2)\) commutes. By taking the \((x, y)\)-component for \(x \in G/K\) and \(y \in G/H\), we get the following commutative diagram:

\[
\begin{array}{ccc}
V_{h^{-1}g^{-1}x, h^{-1}g^{-1}y} & \xrightarrow{g(\cdot h)_x \cdot y = (v_g)_{-1}g^{-1}y} & V_{(gh)^{-1}x, (gh)^{-1}y} \\
\downarrow{g(\cdot y)_x} & & \downarrow{(v_{gh})_{-1}x, y} \\
V_{g^{-1}x, g^{-1}y} & \xrightarrow{(v_g)_x \cdot y} & V_{x,y}
\end{array}
\]

By comparing with example \(B.7\) the lemma is proved. □

Now we give the proof.

**Proof of Proposition B.5.** By Lemma B.9 and B.10 we see that \(F_K \text{Rep}(G) F_H\) is equivalent to the equivariantization \(\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/K})^G\) as categories. It suffices to show that this equivalence preserves the tensor product when \(K = H\).

First we consider the tensor product of \(F_K \text{Rep}(G) F_H\) (i.e., the relative tensor product \(\otimes_{F_H}\)). Since \(F_H\) is a special Frobenius algebra with \(\Delta(\delta_x) := \delta_x \otimes \delta_x\) and \(\varepsilon(\delta_x) = 1\) for \(x \in G/H\), the relative tensor product \(M \otimes_{F_H} N\) of two bimodules \(M, N \in F_H \text{Rep}(G) F_H\) is isomorphic to the image of the following idempotent:

\[
\begin{array}{ccc}
M \otimes_{F_H} N \xrightarrow{1 \otimes \eta_{\otimes 1}} M \otimes F_H \otimes_{F_H} N \xrightarrow{1 \otimes \Delta \otimes 1} \\
\end{array}
\]

\[
M \otimes F_H \otimes F_H \otimes N \xrightarrow{\rho_{M}^R \otimes \rho_{N}^L} M \otimes N,
\]

where \(\rho_{M}^R\) and \(\rho_{N}^L\) are right \(F_H\)-action on \(M\) and left \(F_H\)-action on \(N\), respectively. This idempotent maps \(m \otimes n \in M \otimes N\) to

\[
\sum_{x \in G/H} (m \otimes \delta_x) \otimes (\delta_x \triangleright n).
\]

Therefore, we have

\[
(M \otimes_{F_H} N)_{x,y} = \bigoplus_{z \in G/H} M_{x,z} \otimes N_{z,y},
\]

for \(x, y \in G/H\). The \(G\)-action on \(M \otimes_{F_H} N\) is induced from that of \(M \otimes N\), i.e., the diagonal action \(g \cdot (m \otimes n) := (g \cdot m) \otimes (g \cdot n)\).

On the other hand, the tensor product \((V, \{v_g\}_{g \in G}) \otimes (W, \{w_g\}_{g \in G})\) in \(\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H})^G\) is given by the object \(V \otimes W \in \text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H}) \simeq \text{Mat}_{\text{ax}}(\text{Vec})\) equipped with the \(G\)-action \(\{v_g \otimes w_g\}_{g \in G}\). This \(G\)-action on the \((x, y)\)-component is

\[
(g(V \otimes W))_{x,y} = \bigoplus_{z \in G/H} V_{g^{-1}x, g^{-1}z} \otimes W_{g^{-1}z, g^{-1}y} \otimes_{W_{z,y}} V_{x,y} = (V \otimes W)_{x,y}.
\]

Hence the tensor product of \(\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H})^G\) coincides with that of \(F_H \text{Rep}(G) F_H\). This completes the proof. □

**Remark B.11.** When \(K = H = G\), the algebra \(F_G = \text{Fun}(G/G) = \mathbb{C}\) is trivial and Proposition B.5 recovers example B.4. When \(K = H = \{e\}\), the algebra \(F_{\{e\}} = \text{Fun}(G)\) and \(F_{\{e\}} \text{Rep}(G)_{\{e\}} \simeq \text{Vec}_{G}\) as fusion categories. Thus in this case Proposition B.5 simply says that \(\text{Fun}(\text{Vec}_G, \text{Vec}_G)^G \simeq \text{Vec}_G\) as fusion categories. □

**Remark B.12.** The representation category \(\text{Rep}(\mathcal{G})\) of a finite groupoid \(\mathcal{G}\) is naturally a multi-fusion category. However, the “natural” monoidal structure on \(\text{Rep}(\mathcal{G}(G/H \times G/H, G))\) does not coincide with those of \(F_H \text{Rep}(G) F_H\) and \(\text{Fun}(\text{Vec}_{G/H}, \text{Vec}_{G/H})^G\). □
Remark B.13. Consider the action groupoid $\mathcal{G}(G/K, H)$, where $H$ acts on $G/K$ by left translation. There is an equivalence of action groupoids $\mathcal{G}(G/K, H) \simeq \mathcal{G}(G/K \times G/H, G)$ defined by $x \mapsto (x, H)$. It is straightforward to check that the equivariantization $\mathcal{V}e_{G/K}^H$ is equivalent to $\text{Rep}(\mathcal{G}(G/K, H))$, and hence equivalent to $F_K\text{Rep}(G)_{F_H}$.

\[ \square \]

Appendix C: Enriched categories

In this appendix we give some examples of enriched (fusion) categories.

Let $\mathcal{C}$ be a fusion category. By the so-called canonical construction [78], every finite semisimple left $\mathcal{C}$-module $M$ can be promoted to a $\mathcal{C}$-enriched category $\mathcal{C}M$:

- The objects in $\mathcal{C}M$ are objects in $\mathcal{M}$;
- The hom space $\text{Hom}_{\mathcal{C}M}(x, y)$ for $x, y \in \mathcal{M}$ is the internal hom $[x, y]$, which is defined by $\text{Hom}_{\mathcal{M}}(a, [x, y]) \simeq \text{Hom}_{\mathcal{M}}(a \otimes x, y)$, $\forall a \in \mathcal{C}$, $x, y \in \mathcal{M}$.
- The identity morphisms $\mathcal{C} \to [x, x]$ and composition of morphisms $[y, z] \otimes [x, y] \to [x, z]$ are induced by the universal property of the internal homs.

Moreover, if $\mathcal{C}$ is braided and $\mathcal{M}$ is a fusion left $\mathcal{C}$-module defined by a braided functor $\phi: \mathcal{C} \to \mathcal{Z}_1(\mathcal{M})$, then the canonical construction $\mathcal{C}M$ is a $\mathcal{C}$-enriched category [6, 79].

Example C.1. Let $\mathcal{C}$ be a fusion category and $A, B \in \mathcal{C}$ be separable algebras. Then $\mathcal{C}A = \mathcal{C}B$ is a finite semisimple left $\mathcal{A} \mathcal{C}B$-module with the module action defined by

$X \otimes M := X \otimes_A M$, $X \in \mathcal{A} \mathcal{C}B$, $M \in \mathcal{A} \mathcal{C}B$.

It induces an enriched category $\mathcal{A} \mathcal{C}B = \mathcal{A} \mathcal{C}B$ by the canonical construction. The internal hom (i.e., the hom space) $[M, N] \in \mathcal{A} \mathcal{C}B$ for $M, N \in \mathcal{A} \mathcal{C}B$ is given by

$[M, N] = (M \otimes_B N^R)^L$, \hspace{1cm} (C.2)

where both the left and right duals are taken in $\mathcal{C}$. In the case $A = 1$, this result first appeared in [54, example 3.19] (see also [53, Lemma 2.1.6] for a proof, which applies to our general case). Similarly, $\mathcal{A} \mathcal{C}B$ is a finite semisimple right $\mathcal{C}B$-module and can be viewed as a finite semisimple left $(\mathcal{B} \mathcal{C}B)^{rev}$-module. The internal hom $[M, N] \in (\mathcal{B} \mathcal{C}B)^{rev}$ for $M, N \in \mathcal{A} \mathcal{C}B$ is given by

$[M, N] = (N^L \otimes_A M)^R$, \hspace{1cm} (C.2)

where both the left and right duals are taken in $\mathcal{C}$.

Example C.2. Let $G$ be a finite group and $H, K \subseteq G$ be subgroups. By taking $\mathcal{C} = \text{Rep}(G)$, $A = F_H$ and $B = F_K$ in example C.1, we get an enriched category $\mathcal{C} = \text{Rep}(G)_{F_H}$, $\mathcal{C}B = \text{Rep}(G)_{F_K}$. Let us compute the internal hom $[M, N] \in \mathcal{C}$ for $M, N \in \mathcal{C}$.

By (C.2), we have $[M, N] = (M \otimes_{F_K} N)_{F_H}$. The right dual $N^R$ of $N$ in $\text{Rep}(G)$ is given by the dual representation $N^*$, which is automatically bi-graded:

$N^* = \bigoplus_{x \in G/K, y \in G/H} \bigoplus_{x \in G/K, y \in G/H} (N^*)_{x, y}$.

The left $F_K$-action on $N^R$ is induced by this $G/K$-grading, i.e., $\delta x$ acts as a projector onto $\bigoplus_{y \in G/H} (N^*)_{x, y}$ for every $x \in G/K$. The relative tensor product $\otimes_{F_K}$ has been computed in the proof of Proposition B.5. We have

$[M, N] = \bigoplus_{x, y \in G/K} [M, N]_{x, y} \simeq \bigoplus_{x, y \in G/K} \bigoplus_{z \in G/K} M_{x,z} \otimes (N^*)_{z,y}$.

Then by taking the left dual we have

$[M, N] = \bigoplus_{x, y \in G/K} [M, N]_{x, y} \simeq \bigoplus_{x, y \in G/K} \bigoplus_{z \in G/K} N_{x,z} \otimes (M_{y,z})^*$

equipped with the diagonal $G$-action on each direct summand.

Example C.3. Let us consider the special case that $K = G$ in example C.2. Then the bimodule category $F_H \text{Rep}(G)_{F_G} = F_H \text{Rep}(G)$ is equivalent to $\text{Rep}(H)$, and the equivalence $\text{Rep}(H) \to F_H \text{Rep}(G)$ is given by the induced representation functor $\text{Ind}_H^G$. Thus for two irreducible representations $\rho, \sigma \in \text{Rep}(H)$, their internal hom in $F_H \text{Rep}(G)_{F_G}$ is

$\langle \rho, \sigma \rangle = \text{Ind}_H^G(\sigma) \otimes \text{Ind}_H^G(\rho^*)$

equipped with the product $G/H \times G/H$-grading of the $G/H$-gradings on induced representations. By comparing with remark B.8, we see that

$\langle \rho, \sigma \rangle = \bigoplus_{x \in G/H} \text{M}(x, \sigma \rho^{-1})$.

Example C.4. Let $\mathcal{C}$ be a fusion category and $M, N$ be finite semisimple left $\mathcal{C}$-modules. Then the category $\text{Fun}_{\mathcal{C}}(M, N)$ of left $\mathcal{C}$-module functors is also finite semisimple [54, 66] and admits a left $\mathcal{Z}_1(\mathcal{C})$-module structure:

$\langle x, \gamma \rangle \otimes F := x \otimes F(-)$,

for $(x, \gamma) \in \mathcal{Z}_1(\mathcal{C})$, $F \in \text{Fun}_{\mathcal{C}}(M, N)$. The internal hom $[F, G]_{\mathcal{Z}_1(\mathcal{C})}$ of $F, G \in \text{Fun}_{\mathcal{C}}(M, N)$ in $\mathcal{Z}_1(\mathcal{C})$ is the end

$\int_{M \in \mathcal{C}} [F(m), G(m)]_{\mathcal{C}}$. 

$\square$
equipped with a canonical half-braiding induced by changing the variable in ends, where \( [F(m), G(m)]_\mathcal{C} \) is the internal hom in \( \mathcal{C} \) (see [80, Proposition 3.5] for more details). It induces an enriched category \( 3_{(\mathcal{C})} \mathcal{E}(\mathcal{M}, \mathcal{N}) \) by the canonical construction. If \( \mathcal{M} = \mathcal{C}_A \) and \( \mathcal{N} = \mathcal{C}_B \) where \( A, B \in \mathcal{C} \) are separable algebras, then the functor category \( \mathcal{E}(\mathcal{M}, \mathcal{N}) \) is equivalent to \( \mathcal{C}_B \).

In particular, when \( \mathcal{N} = \mathcal{M} \), the functor category \( \mathcal{E}(\mathcal{M}, \mathcal{M}) \) is a multi-fusion category [66], which is monoidally equivalent to \( (\mathcal{A}_\mathcal{G})_{rev} \) if \( \mathcal{M} = \mathcal{C}_A \). The above \( 3_{(\mathcal{C})} \)-module structure can be induced from a central functor \( 3_{(\mathcal{C})} \to \mathcal{E}(\mathcal{M}, \mathcal{M}) \) called the \( \alpha \)-induction. Moreover, the central structure \( 3_{(\mathcal{C})} \to 3_{(\mathcal{E}(\mathcal{M}, \mathcal{M}))} \) is a braided equivalence [54]. It follows that \( 3_{(\mathcal{C})} \mathcal{E}(\mathcal{M}, \mathcal{M}) \) is an enriched fusion category whose \( E_1 \)-center is trivial [7, Corollary 5.29]. Indeed, it is the \( E_0 \)-center of the enriched category \( \mathcal{A}_\mathcal{M} \) [7, Corollary 4.41].

Example C.5. Let \( G \) be a finite abelian group and \( H \subseteq \mathbb{C} \) be a subgroup. By taking \( \mathcal{C} = \text{Rep}(\mathcal{G}) \) and \( B = A = \mathcal{F}_H \) in example C.4, we get an enriched category \( 3_{(\text{Rep}(\mathcal{G}))} \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) (indeed, an enriched fusion category \( 3_{(\text{Rep}(\mathcal{G}))} \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \)) Let us compute the internal hom \( [M, N] \) in \( 3_{(\text{Rep}(\mathcal{G}))} \) for \( M, N \in \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \). We do not use the result in example C.4, but compute the internal hom by definition (C.1).

First we focus on the case that \( M = N = \mathcal{F}_H \). The general cases are discussed later. Since \( \mathcal{F}_H \) is the tensor unit of \( \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) and \( \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) is a closed left fusion \( 3_{(\text{Rep}(\mathcal{G}))} \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \)-module, the internal hom \( [\mathcal{F}_H, \mathcal{F}_H] \in 3_{(\text{Rep}(\mathcal{G}))} \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) is a Lagrangian algebra in \( 3_{(\text{Rep}(\mathcal{G}))} \) [37]. Thus it suffices to find its support (i.e., underlying object), which is a Lagrangian subgroup of \( (\mathcal{G} \times \mathcal{G}, \mathcal{q}) \) (see example A.3 and A.7). Recall that the forgetful functor \( F : 3_{(\text{Rep}(\mathcal{G}))} \to \text{Rep}(\mathcal{G}) \) is given by \( O_{(g, \phi)} \mapsto \phi \), and the half-braiding \( \beta_{\psi_{(g, \phi)}} \) of \( O_{(g, \phi)} \) is given by

\[
\psi \otimes F(O_{(g, \phi)}) = \psi \otimes \phi \xrightarrow{\psi(g)^{-1}} \phi \otimes \psi = F(O_{(g, \phi)}) \otimes \psi.
\]

Let us recall some basic facts about \( F_H \). As an object in \( \text{Rep}(\mathcal{G}) \),

\[
F_H = \text{Fun}(\mathcal{G}/\mathcal{H}) \simeq \bigoplus_{\psi \in \mathcal{G}/\mathcal{H}} \psi,
\]

and this isomorphism (i.e., the Fourier transform) maps \( \delta_x \in F_H \) to \( \sum_{\psi \in \mathcal{G}/\mathcal{H}} \psi(x) \cdot 1_\psi \). As an object in \( \mathcal{F}_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \), the tensor unit \( F_H \) is described by the following data (see the proof of Lemma B.9):

- Each \( (F_H)_{x,x} = \mathbb{C} \) (spanned by the delta function \( \delta_x \)) and \( (F_H)_{x,y} = 0 \) for \( x \not= y \).

- The \( \mathcal{G} \)-action \( \mathbb{C} = (F_H)_{x,x} \to (F_H)_{g \cdot x, g \cdot x} = \mathbb{C} \) is trivial for every \( g \in \mathcal{G} \) and \( x \in \mathcal{G}/\mathcal{H} \).

Now consider the bimodule \( O_{(g, \phi)} \otimes F_H \in F_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) for \( g \in \mathcal{G} \) and \( \phi \in \hat{\mathcal{G}} \).

(a) As an object in \( \text{Rep}(\mathcal{G}) \), \( O_{(g, \phi)} \otimes F_H \in F_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) is the tensor product \( \phi \otimes F_H \), which has the same underlying vector space as \( F_H \) (i.e., spanned by \( \{\delta_x\}_{x \in \mathcal{G}/\mathcal{H}} \)) but equipped with the \( \mathcal{G} \)-action

\[
h \cdot \delta_x := \phi(h)\delta_{hx}, \quad h \in \mathcal{G}, \quad x \in \mathcal{G}/\mathcal{H}.
\]

(b) The right \( F_H \)-action on \( O_{(g, \phi)} \otimes F_H \) is defined by

\[
(O_{(g, \phi)} \otimes F_H) \cdot F_H = \phi \otimes F_H \otimes F_H \xrightarrow{1 \otimes \mu} \phi \otimes F_H = O_{(g, \phi)} \otimes F_H,
\]

where \( \mu \) is the multiplication of \( F_H \). Thus we see the right \( F_H \)-action on \( O_{(g, \phi)} \otimes F_H \) is given by

\[
\delta_x \otimes \delta_y = \delta_{x,y} \delta_x.
\]

(c) The left \( F_H \)-action on \( O_{(g, \phi)} \otimes F_H \) is defined by

\[
F_H \otimes (O_{(g, \phi)} \otimes F_H) = F_H \otimes \phi \otimes F_H \xrightarrow{\beta_{F_H, \phi} \cdot 1} \phi \otimes F_H \otimes F_H \xrightarrow{1 \otimes \mu} \phi \otimes F_H = O_{(g, \phi)} \otimes F_H,
\]

where \( \beta_{F_H, \phi} \) is the half-braiding of \( O_{(g, \phi)} \) with \( F_H \):

\[
F_H \otimes O_{(g, \phi)} = \bigoplus_{\psi \in \mathcal{G}/\mathcal{H}} \psi \otimes \phi \xrightarrow{\Theta_{\psi(g)^{-1}}} \bigoplus_{\psi \in \mathcal{G}/\mathcal{H}} \phi \otimes \psi = O_{(g, \phi)} \otimes F_H.
\]

Under the isomorphism \( F_H \simeq \bigoplus_{\psi \in \mathcal{G}/\mathcal{H}} \psi \), the half-braiding \( \beta_{F_H, \phi} \) maps \( \delta_y \otimes 1_\phi \in F_H \otimes O_{(g, \phi)} \) to

\[
\sum_{\psi \in \mathcal{G}/\mathcal{H}} \psi(g)^{-1} \psi(y) 1_\phi \otimes 1_\psi = 1_\phi \otimes \sum_{\psi \in \mathcal{G}/\mathcal{H}} \psi(g^{-1}y) 1_\psi = 1_\phi \otimes \delta_{g^{-1}y}.
\]

Thus the left \( F_H \)-action on \( F_H \) is given by

\[
\delta_y \cdot \delta_x = \delta_{x,g^{-1}y} \delta_x = \delta_{gx,y} \delta_x.
\]

Therefore, the bimodule \( O_{(g, \phi)} \otimes F_H \in F_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) is described by the following data:

- Each \( O_{(g, \phi)} \otimes F_H \in F_H \text{Rep}(\mathcal{G}) \mathcal{F}_H \) is and other components are zero.

- The \( \mathcal{G} \)-action \( \mathbb{C} = (O_{(g, \phi)} \otimes F_H)_{g \cdot x, g \cdot x} \to (O_{(g, \phi)} \otimes F_H)_{g \cdot x, g \cdot x} = \mathbb{C} \) is given by \( \phi(h) \) for \( g, h \in \mathcal{G} \) and \( x \in \mathcal{G}/\mathcal{H} \).
It was proved that for an $1d$ condensable algebra $B$ in a fusion category $\mathcal{C}$, $A := \oplus_{\mathbb{F}_{G}} \mathbb{F}_{G}$ is a Lagrangian algebra in $\mathcal{Z}_{1}(\mathcal{C})$ such that $\mathcal{Z}_{1}(\mathcal{C}) A \simeq \mathbb{C} B$ [38]. Now for $\mathcal{C} = \text{Rep}(G)$ and $B = F_{H}$, $[F_{H}, F_{H}]$ is a Lagrangian algebra in $\mathcal{Z}_{1}(\text{Rep}(G))$.

By the definition of internal homs (C.1) and Remark B.8, we have

$$
\text{Hom}_{\mathcal{Z}_{1}(\text{Rep}(G))}(O_{g,\phi}, [F_{H}, F_{H}]) \simeq \\
\text{Hom}_{F_{H}\text{Rep}(G)F_{H}}(O_{g,\phi} \odot F_{H}, F_{H}) \simeq \\
\begin{cases} 
\mathcal{C}, \quad (g, \phi) \in H \oplus \hat{G}/H, \\
0, \quad \text{otherwise}.
\end{cases}
$$

Then by Schur’s lemma, we have

$$
[F_{H}, F_{H}] \simeq \bigoplus_{(g, \phi) \in H \oplus \hat{G}/H} O_{(g, \phi)}.
$$

Hence the support of the Lagrangian algebra $[F_{H}, F_{H}]$ is the Lagrangian subgroup $H \oplus \hat{G}/H \subset G \oplus \hat{G}$.

Now we discuss the general internal homs. By the basic properties of internal homs [52] we have

$$
P \otimes [F_{H}, F_{H}] \otimes Q^{L} \simeq [Q \otimes F_{H}, P \otimes F_{H}]
$$

for $P, Q \in \mathcal{Z}_{1}(\text{Rep}(G))$. By the above calculation it is not hard to see that every bimodule $M \in F_{H}\text{Rep}(G)F_{H}$ is isomorphic to $P \otimes F_{H}$ for some $P \in \mathcal{Z}_{1}(\text{Rep}(G))$. Indeed, every simple bimodule $M$ is isomorphic to $O_{(g, \phi)} \otimes F_{H}$ for some $(g, \phi) \in G \times \hat{G}$. Then the internal hom $[M, N]$ for simple bimodules $M, N \in F_{H}\text{Rep}(G)F_{H}$ can be obtained explicitly by using the above formula, and the support of this internal hom is a coset of the Lagrangian subgroup $H \oplus \hat{G}/H$. \hfill $\Box$

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