Inversion of the linearized Korteweg-deVries equation at the multi-soliton solutions

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Abstract

Uniform estimates for the decay structure of the n-soliton solution of the Korteweg-deVries equation are obtained. The KdV equation, linearized at the n-soliton solution is investigated in a class \( W \) consisting of sums of travelling waves plus an exponentially decaying residual term. An analog of the kernel of the time-independent equation is proposed, leading to solvability conditions on the inhomogeneous term. Estimates on the inversion of the linearized KdV equation at the n-soliton are obtained.

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1 Introduction

A little over 100 years ago Korteweg and deVries [10] derived their now-famous equation

\[ u_t + u_{xxx} - 6uu_x = 0, \]

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in order to explain the observations of solitary waves in barge canals by J. S. Russell [16]. Their equation exhibited solitary waves, and so showed that such phenomena can plausibly be explained by the Euler equations governing the motion of gravity waves in an inviscid fluid. This not only resolved a long-outstanding controversy, but eventually led to fundamental new developments in mathematics.

In 1967 Gardner, Greene, Kruskal, and Miura [6] showed that the KdV equation is in fact a Hamiltonian equation possessing remarkable properties. It could be solved by the method of inverse scattering, using the scattering Schrödinger equation

$$\psi_{xx} + k^2 \psi - u(x, t) \psi = 0.$$  

As $u(x, t)$ evolves according to the nonlinear KdV equation, the scattering data of the associated Schrödinger equation evolves linearly.

It is still an open problem as to the extent to which solutions of the KdV equation approximate solutions of the full Euler equations. There is an extensive literature on the validity of the KdV approximation to solitary waves of the full Euler equations; cf. for example the works of Amick and Kirchgässner [1], Kirchgässner [9]; see also [2], [3]. However, in a moving reference frame, the solitary wave appears as a time independent solution, and is considerably easier to treat mathematically than perturbations of fully time dependent solutions of the KdV equation.

Craig [5] has shown that the KdV equation is a good approximation to the full Euler equations over a time period of order $\epsilon^{-3}$, where $\epsilon^2 = \lambda - 1$, $\lambda$ being the inverse square of the Froude number (see also Kano and Nishida [7]). It is highly unlikely, however, that the KdV equation is an accurate approximation to the Euler equations over an infinite time scale.

One natural question which one may pose is the following. The KdV equation possesses the so-called $n$-soliton (multi-soliton) solution

$$u(x, t) = -2\frac{d^2}{dx^2} \log \det \left( \delta_{jk} + \frac{e^{-(\theta_j + \theta_k)}}{\omega_j + \omega_k} \right), \quad \theta_j = \omega_j(x - \alpha_j - 4\omega_j^2 t),$$  \hspace{1cm} (1.1)

\footnote{Recently Pego and Weinstein [14] have discovered that a simple equation equivalent to the KdV equation appears in Boussinesq’s treatise [4], and, moreover, that Boussinesq had noted the existence of solitary waves.}
where \(0 < \omega_1 < \cdots < \omega_n\) and \(\alpha_j \in \mathbb{R}, \ j = 1, \ldots, n\).

One may ask, do the multi-soliton solutions of the KdV equation extend to full, stable solutions of the Euler equations, say on a semi-infinite time axis \(0 \leq t \leq \infty\)? A positive answer to this question would, as a by product, prove the existence of non-trivial time dependent solutions to the Euler equations on a semi-infinite time interval \(0 < t < \infty\), something which has not yet been done. It would also show that the full Euler equations possess solutions which behave like elastically scattering solitary waves.

The neutral, orbital stability of the multi-soliton solutions of the KdV equation has been shown by Maddocks and Sachs [11] based on the observation that the \(n\)-soliton can be obtained by minimizing the \(n^{th}\) conservation law of the KdV equation subject to the constraints that the first \(n-1\) integrals of the motion are held fixed. On the other hand, the asymptotic stability of the solitary wave has been demonstrated by Pego and Weinstein [14], based on a spectral analysis of the linearized Korteweg-deVries equation. General arguments, based on the integration of the Korteweg-deVries equation by the inverse scattering method, imply that the \(n\)-soliton solution is in some sense asymptotically stable. If so, such a result would be based on the analysis of the linearized KdV equation at the \(n\)-soliton solution.

In this paper we analyze the linearized KdV equation at the multi-soliton solution. Such an investigation was begun by Sachs [17], who constructed a representation of the linearized KdV equations using the completeness of the squared eigenfunctions of the associated Schrödinger operator. We extend his analysis here in the case of the \(n\)-soliton solution to obtain estimates in norms suitable for studying perturbations of the KdV equation.

In the case of the solitary wave, one may work in a reference frame moving with the wave; the result is that the linearized operator has time independent coefficients, and the methods of classical spectral theory of linear operators can be applied. In the multi-soliton case this can no longer be carried out, and we are forced to consider time dependent operators. Nevertheless, it is possible to formulate an analog of the single-soliton case, as follows.

The \(n\)-soliton solution is a \(2n\) parameter family of solutions of the KdV equation. Differentiation with respect to those parameters yields a \(2n\)-dimensional subspace of solutions of the homogeneous linearized KdV equation which decay exponentially in \(x\) for fixed time. This subspace plays the role of the kernel of the infinitesimal generator in the case of an evolution equation with time-independent coefficients, as occurs when the KdV equation is linearized about the solitary wave. The presence of this "kernel" leads
to $2n$ solvability conditions.

We introduce the space $\mathcal{W}$ consisting of functions of the form

$$u = \sum_{j=1}^{n} f_j(x - 4\omega_j^2 t) + R(x,t);$$

the $f_j$ are “solitary”-like wave forms which decay exponentially as $x \to \pm \infty$; $f_j$ and $R(x,t)$ are analytic in a strip in the complex $x$ plane; and $R$ decays exponentially in time as $t \to \infty$, uniformly in $x$. The class $\mathcal{W}$ is closed under differentiation and multiplication, an important property when working with nonlinear equations. The main purpose of this paper is to invert the KdV equation linearized at the $n$-soliton solution (1.1) in $\mathcal{W}$.

In §2 we prove that the $n$-soliton solution belongs to $\mathcal{W}$. In fact, for the $n$-solitons, $R$ decays exponentially as $|x| \to \infty$ as well. This exponential decay is not preserved under inversion of the linearized KdV equation. Nevertheless, by considering weighted spaces, we can show that the exponential decay as $x \to \infty$ is preserved. In §3 we derive the properties of the wave functions of the associated Schrödinger operator. In §4 we give a proof of Sachs’ completeness theorem, including an extension to a completeness theorem for the squared eigenfunctions themselves.

In §5 we construct the propagator of the linearized KdV equation. The basic estimates on the propagator are obtained in §§6,8. In §6 we obtain estimates in a Hilbert space of functions analytic in a strip containing the real $x$-axis. In §8 we obtain estimates in a weighted norm, analogous to the estimates in §3. In §7 we discuss the inversion of the linearized KdV equation in the space $\mathcal{W}$ with suitable linear solvability constraints.

The perturbation scheme of the Euler equations which leads formally to the KdV approximation loses derivatives, whereas the inversion of the linearized KdV equation is neutral: it is bounded in $L_2$, but gains no derivatives. In a situation such as [13] the loss of one derivative was compensated by the use of weighted norms. Pego and Weinstein used a global existence theorem for the generalized KdV equation proved by Kato [8], together with estimates in weighted norms to gain minimal regularity and decay in time. The weighted norms exact a toll, however; they do not give rise to Banach algebras, and so are difficult to work with in nonlinear problems.

The loss of derivatives is more severe in the case of the Euler equations, and one cannot expect the method in [13] to work. It seems probable that some form of hard-implicit function theorem will be needed, such as that
described by Moser [12] or Scheurle [18]. A natural space to work in is the space of functions analytic in a strip in the complex $x$ plane.

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2 Asymptotics of the $n$–soliton solution

We study in this section the asymptotic properties of the $n$-soliton solution \((1.1)\) of the KdV equation, as $t \to \infty$. The $n$-soliton solution is a function of $2n$ parameters $\omega_1, \ldots, \omega_n$, $\alpha_1, \ldots, \alpha_n$. Throughout this paper we fix $0 < \omega_1 < \cdots < \omega_n$ and $\alpha_1, \ldots, \alpha_n$. The speeds of the individual solitary waves are $4\omega_1^2 < \cdots < 4\omega_n^2$; the $\alpha_1, \ldots, \alpha_n$ are called the phases. The determinant in \((1.1)\) is called the tau function of order $n$ and is denoted by $\tau$.

**Theorem 2.1** The $n$-soliton solution of the KdV equation can be written in the form

$$u(x,t) = -2 \sum_{j=1}^{n} \omega_j^2 \text{sech}^2(\theta_j + \gamma_j) - 2 \frac{d^2}{dx^2} \log(1 + R),$$

where

$$\gamma_n = \frac{1}{2} \log(2\omega_n), \quad \gamma_j = \frac{1}{2} \log(2\omega_j) + \sum_{k=j+1}^{n} \log\left(\frac{\omega_k + \omega_j}{\omega_k - \omega_j}\right), \quad 1 \leq j \leq n - 1,$$

and

$$\sup_{x,t>0} |\cosh(ax)R(x,t)| \leq Ce^{-bt},$$

for some $a, b > 0$ and some positive constant $C$.

A similar result is true as $t \to -\infty$, but with different phase shifts $\gamma_j$.  

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Proof: We will prove that the tau function of order $n$ can be factored
\[
\tau(\theta_1, \ldots, \theta_n) = 2e^{-(\theta_n + \gamma_n)} \cosh(\theta_n + \gamma_n) \tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n)(1 + R_n),
\]
where $\tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n)$ is the tau function of order $n - 1$,
\[
\beta_j^n = \log \left( \frac{\omega_n + \omega_j}{\omega_n - \omega_j} \right) > 0,
\]
and $R_n$ satisfies (2.2). Then by induction,
\[
\tau = \prod_{j=1}^n 2e^{-(\theta_j + \gamma)} \cosh(\theta_j + \gamma_j)(1 + R_j),
\]
where each of the $R_j$ satisfies (2.2). The result then follows for the KdV solution $u$ upon taking the second logarithmic derivative and letting
\[
1 + R = \prod_{j=1}^n (1 + R_j). \tag{2.3}
\]

We begin by writing
\[
\tau = \det(I + C_n), \quad C_n = \begin{bmatrix} e^{-(\theta_j + \theta_k)} \\ \omega_j + \omega_k \end{bmatrix}_{1 \leq j, k \leq n}.
\]
As observed in [4] $\tau$ can be expanded as a sum of all the principal minors of $C_n$. Moreover, each of these principal minors is of the same form. We may write
\[
C_n = \Lambda_n K_n \Lambda_n, \quad \Lambda_n = \text{diag}(e^{-\theta_1}, \ldots, e^{-\theta_n}), \quad K_n = \begin{bmatrix} 1 \\ \omega_j + \omega_k \end{bmatrix}_{1 \leq j, k \leq n}.
\]
Thus,
\[
\tau(\theta_1, \ldots, \theta_n) = 1 + \sum_{j=1}^n e^{-2\theta_j} + \sum_{1 \leq j < k \leq n} e^{-2(\theta_j + \theta_k)} K_{jk}^{(2)} + \cdots + e^{-2(\theta_1 + \cdots + \theta_n)} K^{(n)}, \tag{2.4}
\]
where $K_{j_1, \ldots, j_\ell}^{(\ell)}$ is the $\ell \times \ell$ principal minor on the rows $j_1, \ldots, j_\ell$. 

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Lemma 2.2 We have
\[
\tau(\theta_1, \ldots, \theta_n) = \tau(\theta_1, \ldots, \theta_{n-1}) + e^{-2(\theta_n + \gamma_n)} \tau(\theta_1 + \beta_{1n}^{n}, \ldots, \theta_{n-1} + \beta_{(n-1)n}^{n}).
\]

Proof: In (2.4) the only terms which do not contain the factor \(e^{-2\theta_n}\) are the principal minors of the matrix \(C_{n-1}\), so we have the factorization
\[
\tau(\theta_1, \ldots, \theta_n) = \tau(\theta_1, \ldots, \theta_{n-1}) + e^{-2\theta_n} \left( \frac{1}{2\omega_n} + \sum_{j=1}^{n-1} e^{-2\theta_j} K_{jn}^{(2)} \right) + \sum_{1 \leq j < k \leq n-1} e^{-2(\theta_j + \theta_k)} K_{jkn}^{(3)} + \cdots + e^{-2(\theta_1 + \cdots + \theta_{n-1})} K_{(n)}^{(n)}).
\]

The following formula has been proved in [6], p.121:
\[
\det K_n = \frac{1}{2\omega_n} \prod_{j=1}^{n-1} \left( \frac{\omega_n - \omega_j}{\omega_n + \omega_j} \right)^2 \det K_{n-1}, \quad (2.5)
\]
so
\[
K_{j_1 \ldots j_l}^{(l+1)} = \frac{1}{2\omega_n} \prod_{a=1}^{l} \left( \frac{\omega_n - \omega_{ja}}{\omega_n + \omega_{ja}} \right)^2 K_{j_1 \ldots j_l}^{(l)}.
\]

Then
\[
\tau(\theta_1, \ldots, \theta_n) = \tau(\theta_1, \ldots, \theta_{n-1}) + \frac{e^{-2\theta_n}}{2\omega_n} \left( 1 + \sum_{j=1}^{n-1} \left( \frac{\omega_n - \omega_j}{\omega_n + \omega_j} \right)^2 e^{-2\theta_j} K_j^{(1)} \right)
\]
\[
+ \sum_{1 \leq j < k \leq n-1} \left( \frac{\omega_n - \omega_j}{\omega_n + \omega_j} \right)^2 \left( \frac{\omega_n - \omega_k}{\omega_n + \omega_k} \right)^2 e^{-2(\theta_j + \theta_k)} K_{jk}^{(2)}
\]
\[
+ \cdots + \prod_{k=1}^{n-1} \left( \frac{\omega_n - \omega_k}{\omega_n + \omega_k} \right)^2 e^{-2\theta_k} K_{(n-1)}^{(n-1)}\right)
\]
\[
= \tau(\theta_1, \ldots, \theta_{n-1}) + e^{-2(\theta_n + \gamma_n)} \tau(\theta_1 + \beta_{1n}^{n}, \ldots, \theta_{n-1} + \beta_{(n-1)n}^{n}),
\]
since
\[
e^{-2\gamma_n} = \frac{1}{2\omega_n}, \quad e^{-2\beta_j^{n}} = \left( \frac{\omega_n - \omega_j}{\omega_n + \omega_j} \right)^2, \quad 1 \leq j \leq n-1.
\]
From this lemma, we can write the $\tau$-function in the form, $\tilde{\theta}_n = \theta_n + \gamma_n$,

$$\tau(\theta_1, \ldots, \theta_n) = 2e^{-\tilde{\theta}_n} \cosh \tilde{\theta}_n \left( \frac{e^\tilde{\theta}_n}{2 \cosh \tilde{\theta}_n} \tau(\theta_1, \ldots, \theta_{n-1}) + \frac{e^{-\tilde{\theta}_n}}{2 \cosh \tilde{\theta}_n} \tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n) \right)$$

$$= 2e^{-\tilde{\theta}_n} \cosh \tilde{\theta}_n \left( \frac{1 + \tanh \tilde{\theta}_n}{2} \tau(\theta_1, \ldots, \theta_{n-1}) + \frac{1 - \tanh \tilde{\theta}_n}{2} \tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n) \right)$$

$$= 2e^{-\tilde{\theta}_n} \cosh \tilde{\theta}_n \tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n)(1 + R_n),$$

where

$$1 + R_n = \frac{1 - \tanh \tilde{\theta}_n}{2} + \frac{1 + \tanh \tilde{\theta}_n}{2} \frac{\tau(\theta_1, \ldots, \theta_{n-1})}{\tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n)}$$

$$= 1 + \frac{1 + \tanh \tilde{\theta}_n}{2} \left[ \frac{\tau(\theta_1, \ldots, \theta_{n-1})}{\tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n)} - 1 \right].$$

To complete the proof of the theorem we have to show that $R_n$ satisfies (2.2). Remark first that

$$1 \leq \tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n) \leq \tau(\theta_1, \ldots, \theta_{n-1}),$$

and that the ratio

$$\frac{\tau(\theta_1, \ldots, \theta_{n-1})}{\tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n)}$$

is bounded for all values of $\theta_1, \ldots, \theta_{n-1}$ (it is the ratio of two positive polynomials of $e^{-\theta_1}, \ldots, e^{-\theta_{n-1}}$ having the same degree).
Then
\[ 0 \leq R_n = \frac{1 + \tanh \tilde{\theta}_n}{2} \left[ \frac{\tau(\theta_1, \ldots, \theta_{n-1})}{\tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n)} - 1 \right] \]

\[ \leq C_1 \frac{1 + \tanh \tilde{\theta}_n}{2} \leq C_1 e^{2\tilde{\theta}_n}, \]
and the required inequality is immediate if \( x \leq 0 \).

For \( 0 \leq x \leq 2(\omega_{n-1}^2 + \omega_n^2)t \), we find, for all \( a \in \mathbb{R} \),
\[ 0 \leq R_n \leq C_1 e^{2\tilde{\theta}_n} = \frac{C_1}{2\omega_n} e^{-2\omega_n \alpha_n} e^{-ax+(2\omega_n+a)x} - 8\omega_n^3 t \]
\[ \leq \frac{C_1}{2\omega_n} e^{-2\omega_n \alpha_n} e^{-ax+[2(\omega_{n-1}^2+\omega_n^2)(2\omega_n+a)-8\omega_n^3]t} \]
\[ \leq \frac{C_1}{2\omega_n} e^{-2\omega_n \alpha_n} e^{-ax-[4\omega_n(\omega_n^2-\omega_{n-1}^2)-2a(\omega_n^2+\omega_{n-1}^2)]t} \]

Choose \( a > 0 \), \( a < 2\omega_n(\omega_n^2 - \omega_{n-1}^2)/(\omega_n^2 + \omega_{n-1}^2) \), and (2.2) follows with
\[ b = 4\omega_n(\omega_n^2 - \omega_{n-1}^2) - 2a(\omega_n^2 + \omega_{n-1}^2). \]

Now, for \( x \geq 2(\omega_{n-1}^2 + \omega_n^2)t \) we have \( \theta_j = \omega_j(x - \alpha_j - 4\omega_j^3 t) > -\omega_j \alpha_j \), if \( j = 1, \ldots, n-1 \); and, for \( 0 < a < 2\omega_1 \),
\[ 0 \leq R_n \leq \tau(\theta_1, \ldots, \theta_{n-1}) - \tau(\theta_1 + \beta_1^n, \ldots, \theta_{n-1} + \beta_{n-1}^n) \]
\[ = \sum_{j=1}^{n-1} \left( 1 - e^{-2\beta_j^n} \right) \frac{e^{-2\beta_j^n}}{2\omega_j} + \sum_{1 \leq j < k \leq n-1} \left( 1 - e^{-2(\beta_j^n + \beta_k^n)} \right) e^{-2(\theta_j + \theta_k)} K_{jk}^{(2)} \]
\[ + \cdots + \left( 1 - e^{-2(\beta_1^n + \cdots + \beta_{n-1}^n)} \right) e^{-2(\theta_1 + \cdots + \theta_{n-1})} K^{(n-1)} \]
\[ \leq C_2 \sum_{j=1}^{n-1} e^{-2\beta_j^n} \leq C_3 \sum_{j=1}^{n-1} e^{-2\omega_j x + 8\omega_j^3 t} = C_3 \sum_{j=1}^{n-1} e^{-ax-(2\omega_j-a)x + 8\omega_j^3 t} \]
\begin{align*}
&\leq C_3 \sum_{j=1}^{n-1} e^{-ax - 2(\omega_{n-1}^2 + \omega_n^2)(2\omega_j - a)t + 8\omega_j^2 t} \\
&\leq C_3 \sum_{j=1}^{n-1} e^{-ax - |4\omega_j(\omega_n^2 + \omega_{n-1}^2 - 2\omega_j^2) - 2a(\omega_1^2 + \omega_n^2)|t}.
\end{align*}

As before (2.2) follows if

\[0 < a < 2\omega_j(\omega_n^2 + \omega_{n-1}^2 - 2\omega_j^2) / (\omega_n^2 + \omega_{n-1}^2).\]

This theorem shows that the \(n\)-soliton solution is asymptotic to a sum of \(n\) travelling solitary waves plus a remainder term that decays exponentially fast to zero as \(t \to \infty\), uniformly in \(x\).

In the case of 2-solitons, a simple computation shows that

\[\tau = 1 + e^{-2\theta_1} \left( 1 + \tanh(\theta_2 + \gamma_2) \right) + e^{-2\theta_2} \left( 1 + \tanh(\theta_1 + \gamma_1) \right) + \frac{1}{2\omega_1} \left( \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} \right)^2 e^{-2\theta_1} \left( 1 - \tanh(\theta_2 + \gamma_2) \right).\]

We may factor \(\tau\) as

\[\tau = 2e^{-\theta_2 + \gamma_2} \cos(\theta_2 + \gamma_2) \tau_1,\]

where \(\gamma_2 = \frac{1}{2} \log(2\omega_2)\) and

\[\tau_1 = 1 + \frac{1}{4\omega_1} e^{-2\theta_1} \left( 1 + \tanh(\theta_2 + \gamma_2) \right) + \frac{1}{4\omega_1} \left( \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} \right)^2 e^{-2\theta_1} \left( 1 - \tanh(\theta_2 + \gamma_2) \right).\]

As \(t \to \infty\),

\[(1 - \tanh(\theta_2 + \gamma_2)) \to 2, \quad (1 + \tanh(\theta_2 + \gamma_2)) \to 0;\]

(2.6)

hence

\[\tau_1 \sim 1 + \frac{1}{2\omega_1} \left( \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} \right)^2 e^{-2\theta_1} = 2e^{-(\theta_1 + \gamma_1)} \cos(\theta_1 + \gamma_1),\]

where

\[\gamma_1 = \frac{1}{2} \log(2\omega_1) + \log \left( \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1} \right).\]

This leads, ultimately, to the factorization

\[\tau = 4e^{-(\theta_1 + \theta_2 + \gamma_1 + \gamma_2)} \cos(\theta_2 + \gamma_2) \cos(\theta_1 + \gamma_1)(1 + R),\]
with
\[
R = \frac{\omega_1\omega_2}{(\omega_2 - \omega_1)^2}(1 + \tanh(\theta_2 + \gamma_2))(1 - \tanh(\theta_1 + \gamma_1)).
\] (2.7)

Hence, the 2-soliton solution of the KdV equation can be written
\[
u(x, t) = -2\omega_1^2 \sech^2(\theta_1 + \gamma_1) - 2\omega_2^2 \sech^2(\theta_2 + \gamma_2) - 2\frac{d^2}{dx^2} \log(1 + R).
\]

The diagram below shows the negative of the two-soliton solution (1.1), with \(\omega_1, \omega_2 = .5, .75\), and \(\alpha_1 = \alpha_2 = 0\) (solid lines). The negative is the leading term in the approximation of the free surface in the full Euler equations. The first diagram shows the two-soliton solution in the middle of the interaction. The diagram on the right shows the two-soliton solution after the interaction. The boxed line shows the superposition of two solitary (sech$^2$) waves. These two solitary waves fit the two-soliton solution exactly before the interaction. The displacement is the scattering of the solitary waves due to the non-linear interaction.

**Theorem 2.3** The \(n\)-soliton solution (1.1) is analytic in the strip \(|\Im x| < \pi/2\omega_n\). Moreover, (2.1) is valid for \(|\Im x| < \pi/2\omega_n\) and (2.2) holds for \(|\Im x| \leq \eta_0\), for any \(0 < \eta_0 < \pi/2\Omega\), where \(\Omega = \max\{2\sum_{j=1}^{n-1} \omega_j, \omega_n\}\). The constant \(C\) in (2.2) depends only on \(\eta_0\).

**Proof:** The Gel’fand-Levitan-Marcenko (GLM) equation for the \(n\)-soliton solutions is (cf. [3])
\[
K(x, y, t) + f(x + y, t) + \int_x^\infty K(x, s, t)f(s + y, t) \, ds = 0, \quad (2.8)
\]
where
\[ f(\xi, t) = \sum_{j=1}^{n} e^{-\omega_j \xi + 8\omega_j t + 2\omega_j \alpha_j}. \]

We need to prove its invertibility for complex \( x \). The integral in (2.8) should be understood as a complex integral over \( \Gamma = \{ z \in \mathbb{C} : \Re z > \Re x, \Im z = 3x \} \). The \( n \)-soliton solution is then given by
\[ u(x, t) = -2 \frac{d}{dx} K(x, x, t). \]

Since (2.8) is a Fredholm integral equation, its solvability follows from uniqueness. We first make a transformation and rewrite the homogeneous equation as
\[ K_{x,t}(y) + \int_{0}^{\infty} K_{x,t}(s) f(s + y + 2x, t) \, ds = 0 \quad K_{x,t}(y) = K(x, x + y, t). \] (2.9)

Now it is clear that \( x \) appears analytically in the equation, and so the solutions of (2.8) will be analytic in \( x \) wherever we can prove uniqueness.

To prove uniqueness, multiply (2.9) by \( K_{x,t}(y) \) and integrate over \( (0, \infty) \).

This leads to
\[ 0 = \int_{0}^{\infty} |K_{x,t}(y)|^2 \, dy + \int_{0}^{\infty} \int_{0}^{\infty} f(s + y + 2x, t) K_{x,t}(s) \overline{K_{x,t}(y)} \, dy \, ds \]
\[ = \int_{0}^{\infty} |K_{x,t}(y)|^2 \, dy + \sum_{j=1}^{n} e^{-2\omega_j x + 8\omega_j t + 2\omega_j \alpha_j} \left| \int_{0}^{\infty} e^{-\omega_j s} K_{x,t}(s) \, ds \right|^2. \]

The above expression can vanish only when the convex hull of the complex numbers \( e^{-2\omega_j \eta} \), where \( x = \xi + i\eta \), contains the negative real axis, hence only when \( |\eta| \geq \pi/2\omega_n \). The first part of the theorem is now proved.

The waveform \( \text{sech}^2(\omega_j x) \) is analytic in the strip \( |\Im x| < \pi/2\omega_j \): it has poles on the imaginary axis at the points \( x = i(\frac{\pi}{2} + k\pi)/\omega_j, k \in \mathbb{R} \). Hence the left hand side and the sum in the right hand side of (2.1) are analytic in the strip \( |\Im x| < \pi/2\omega_n \), so (2.1) holds in this strip.

To obtain the estimate (2.2) remark that the arguments in the proof of Theorem 2.1 remain valid as long as all the exponential terms \( e^{-2(\theta_1 + \cdots + \theta_j)}, j = 1, \ldots, n-1 \) have positive real parts, and \( 1 + \tanh \theta_n \) is uniformly bounded. These properties hold in any strip \( |\Im x| \leq \eta_0 \), for \( 0 < \eta_0 < \pi/2\Omega \).
Remark. The argument in the last part of the proof of this theorem implies also the analyticity of the $n$-soliton solution in the strip $|\Im x| < \pi/2\Omega$. However, this result is weaker than the one above. We conjecture that (2.2) holds in fact in any strip $|\Im x| \leq \eta_0$, for $\eta_0 < \pi/2\omega_n$.

3 Wave functions

We study in this section some of the properties of the wave functions of the Schrödinger operator

$$ (D^2 + k^2 - u(x,t))\psi(x,k) = 0, \quad (3.1) $$

when $u$ is an $n$-soliton solution of the KdV equation

$$ u_t + u_{xxx} - 6uu_x = 0. \quad (3.2) $$

The Lax pair for this equation is

$$ L = -D^2 + u, \quad B = -4D^3 + 3(uD + Du). \quad (3.3) $$

Substituting $u = \varphi_x$ into the above form of the KdV equation, we obtain

$$ \varphi_{x,t} + \varphi_{xxxx} - 6\varphi_x\varphi_{xx} = 0, $$

or after integration, the potential KdV equation

$$ \varphi_t + \varphi_{xxx} - 3\varphi_x^2 = 0. \quad (3.4) $$

The linearized KdV equation is

$$ \nu_t + \nu_{xxx} - 6(\nu\nu)_x = 0, \quad (3.5) $$

while the linearization of (3.4) is

$$ \psi_t + \psi_{xxx} - 6u\psi_x = 0. \quad (3.6) $$

Thus the linearized potential KdV and the KdV equations are formal adjoints of one another.

Lemma 3.1 Let $q$ be a solution of (3.2), and let $f_1$, $f_2$ be any pair of solutions of $Lf = k^2f$, $f_t = Bf$, where $L$ and $B$ are given in (3.3). Then $\psi(x,t,k) = (f_1f_2)(x,t,k)$ satisfies the linearized potential KdV equation (3.4), and $\psi_x$ satisfies the linearized KdV equation (3.3).
Proof: Straightforward calculation. \[ \square \]

The asymptotic behaviour of the wave functions is determined by the large \( x \) behaviour of the Lax pair \( L \) and \( B \). It is thus given by the simultaneous equations

\[
-D^2 \psi = k^2 \psi, \quad \psi_t = -4D^3 \psi
\]

This leads to

\[
\varphi_+(x, t, k) = m(x, t, k)e^{-i(kx+4k^3t)}, \quad \psi_+(x, t, k) = r(x, t, k)e^{i(kx+4k^3t)},
\]

(3.7)

The reduced wave functions \( m(x, t, k) \), \( r(x, t, k) \) are normalized by

\[
\lim_{x \to -\infty} m(x, t, k) = 1, \quad \lim_{x \to \infty} r(x, t, k) = 1.
\]

(3.8)

Lemma 3.2 The reduced wave functions \( m(x, t, k) \) and \( r(x, t, k) \) are analytic in \( |\Im x| < 2\pi/\Omega \), and \( \Im k > 0 \). Moreover, \( m \), \( r \) and their derivatives with respect to \( x \) are uniformly bounded for \( |\Im x| \leq \eta_0 \), \( t \in \mathbb{R} \), \( \Im k \geq \varepsilon \), for any \( 0 < \eta_0 < 2\pi/\Omega \) and \( \varepsilon > 0 \).

Proof: The reduced wave functions \( m(x, t, k) \), \( r(x, t, k) \) are obtained as solutions of Volterra integral equations

\[
m(x, t, k) = 1 - \int_{-\infty}^{x} \frac{1 - e^{2ik(x-y)}}{2ik} u(y, t)m(y, t, k) \, dy,
\]

\[
r(x, t, k) = 1 - \int_{x}^{\infty} \frac{1 - e^{-2ik(x-y)}}{2ik} u(y, t)r(y, t, k) \, dy.
\]

(3.9)

From Theorem 2.3 it follows that the integrals

\[
\int_{-\infty}^{x} \frac{|1 - e^{2ik(x-y)}|}{2|k|} |u(y, t)| \, dy, \quad \int_{x}^{\infty} \frac{|1 - e^{-2ik(x-y)}|}{2|k|} |u(y, t)| \, dy,
\]

are uniformly bounded for \( |\Im x| \leq \eta_0 \), \( t \in \mathbb{R} \), and \( \Im k \geq \varepsilon \), for any \( 0 < \eta_0 < 2\pi/\Omega \) and \( \varepsilon > 0 \). Then, the existence, boundedness, and analyticity of \( m \), \( r \) follow easily from standard arguments for Volterra integral equations.
The uniform boundedness of the derivatives of $m$ and $r$ with respect to $x$ follows from the uniform boundedness of $m$ and $r$ by using the Volterra integral equations (3.9) in differentiated form, e.g.

$$m_x(x, t, k) = \int_{-\infty}^{x} e^{2ik(x-y)}u(y, t)m(y, t, k) dy.$$

The conjugate wave functions are

$$\varphi_{-}(x, t, k) = \varphi_{+}(x, t, \bar{k}), \quad \text{and} \quad \psi_{-}(x, t, k) = \psi_{+}(x, t, \bar{k}).$$

Since the $n$-soliton solutions are reflectionless potentials they satisfy

$$\varphi_{+} = a(k)\varphi_{-}, \quad \psi_{+} = a(k)\psi_{-}, \quad (3.10)$$

where

$$a(k) = \frac{W(\varphi_{+}(x, t, k), \psi_{+}(x, t, k))}{2ik}.$$

For the $n$-soliton solution of the KdV equation both wave functions $\varphi_{+}$ and $\psi_{+}$ are in fact meromorphic functions of $k$ with simple poles at $-i\omega_1, \ldots, -i\omega_n$. From (3.7) and (3.8) we deduce the following principal parts expansion of the wave functions:

$$\varphi_{+}(x, t, k) = e^{-i(kx+4k^3t)} \left[ 1 + \sum_{j=1}^{n} \frac{n_j(x, t)}{\omega_j - ik} \right], \quad (3.11)$$

$$\psi_{+}(x, t, k) = e^{i(kx+4k^3t)} \left[ 1 + \sum_{j=1}^{n} \frac{p_j(x, t)}{\omega_j - ik} \right], \quad (3.12)$$

where $n_j, p_j$ are the residues of the reduced wave functions at the poles $k = -i\omega_j$. Then, from (3.12) and from Lemma 3.2 follows.

**Theorem 3.3** The reduced wave functions $m$, $r$ and their derivatives with respect to $x$ are uniformly bounded in $|\Re x| \leq \eta_0$, $t \in \mathbb{R}$, $\Re k \geq 0$, for any $0 < \eta_0 < 2\pi/\Omega$.  

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We define the eigenfunctions corresponding to the bound states $k = i\omega_j$ by

$$\varphi_j(x, t) = \varphi_+(x, t, i\omega_j), \quad \psi_j(x, t) = \psi_+(x, t, i\omega_j).$$  \hfill (3.13)

They can be calculated explicitly by solving the GLM equation, cf. \cite{GGKM}. In \cite{GGKM} the bound state eigenfunctions are normalized differently, namely so that $\int \psi_j^2 = 1$; this leads to their constraint (2.13). Comparing (2.13) in \cite{GGKM} and (3.8), we see that

$$\psi_j^{[GGKM]} = d_j \psi_j, \quad d_j = e^{\omega_j \alpha_j}.$$  \hfill (3.14)

Then, according to (3.3) in \cite{GGKM}, the kernel $K$ in the GLM equation is of the form

$$K(x, y, t) = -\sum_{j=1}^{n} e^{\omega_j \alpha_j + 4\omega_j^2 t} d_j \psi_j(x, t) e^{-\omega_j y},$$  \hfill (3.15)

with $d_j$ the constants determined above. The substitution of (3.14) into (2.8) leads to a linear system for $\psi_j$:

$$d_j \psi_j + \sum_{m=1}^{n} \frac{e^{-(\theta_j + \theta_m)} \omega_m}{\omega_j + \omega_m} d_m \psi_m = e^{-\theta_j}.$$  \hfill (3.16)

Furthermore, from Theorem 3.4 in \cite{GGKM} we deduce the following relation between the $n$-soliton solution and the bound state eigenfunctions

$$u(x, t) = -4 \sum_{j=1}^{n} \omega_j e^{2\omega_j \alpha_j} \psi_j^2(x, t).$$  \hfill (3.17)

We now look for the wave function $\psi_\pm$. For this we determine $p_j$ in (3.12) in terms of $\psi_j$. From (3.12) and (3.13) we find

$$\psi_j(x, t) = e^{-\omega_j x + 4\omega_j^3 t} \left[ 1 + \sum_{m=1}^{n} \frac{p_m(x, t)}{\omega_m + \omega_j} \right].$$

Multiply this equation by $d_j = e^{\omega_j \alpha_j}$, and recall that $\theta_j = \omega_j(x - \alpha_j - 4\omega_j^2 t)$. Comparing the resulting system with (3.13) we see that

$$p_j(x, t) = -e^{-\theta_j} d_j \psi_j(x, t) = -e^{-\omega_j x + 4\omega_j^3 t + 2\omega_j \alpha_j} \psi_j(x, t).$$  \hfill (3.18)
Hence

\[
\psi_+(x, t, k) = e^{i(kx + 4k^3t)} \left[ 1 - \sum_{j=1}^{n} \frac{e^{-\omega_j x + 4\omega_j^3 t + 2\omega_j \alpha_j}}{\omega_j - ik} \psi_j(x, t) \right].
\] (3.18)

We can also obtain an explicit formula for \(\varphi_+\) from (3.10). For this we need \(a(k)\). First note that

\[
\varphi_-(x, t, k) = \varphi_+(x, t, k) \sim e^{i(kx + 4k^3t)}, \quad \text{as} \quad x \to -\infty.
\]

Equations (3.10) and (3.18) then imply

\[
a(k) = \lim_{x \to -\infty} e^{-i(kx + 4k^3t)} \varphi_+(x, t, k) = 1 - \lim_{x \to -\infty} \sum_{j=1}^{n} \frac{e^{\omega_j \alpha_j - \theta_j}}{\omega_j - ik} \psi_j(x, t).
\] (3.19)

The limiting values of \(e^{-\theta_j} \psi_j\) as \(x \to -\infty\) can be determined from (3.15). Multiply (3.15) by \(e^{\theta_j}\) and note that \(e^{\theta_j}\) tends to zero exponentially as \(x \to -\infty\). Moreover, \(\psi_m\) is the eigenfunction corresponding to the eigenvalue \(-\omega_m^2\) and is in \(L^2(\mathbb{R})\). (In fact, it also tends to zero exponentially as \(x \to \pm \infty\).) Hence, these limits satisfy a linear system with constant coefficients, so

\[
a(k) = 1 - \sum_{j=1}^{n} \frac{a_j}{\omega_j - ik} = \frac{P_n(k)}{\prod_{j=1}^{n} (\omega_j - ik)},
\]

with \(P_n(k)\) a polynomial of \(k\) of degree \(n\). But \(a(i\omega_j) = 0\), for \(j = 1, \ldots, n\), so

\[
P_n(k) = \alpha \prod_{j=1}^{n} (\omega_j + ik).
\]

The constant \(\alpha\) is determined from the asymptotic properties of \(a(k)\) for large \(k\). From (3.19) we find

\[
\lim_{k \to \infty} a(k) = 1,
\]

so \(\alpha = (-1)^n\), and we conclude

\[
a(k) = \prod_{j=1}^{n} \frac{k - i\omega_j}{k + i\omega_j}.
\] (3.20)
Now, (3.10) yields
\[
\varphi_+(x, t, k) = e^{-(kx + 4k^3t)} \prod_{j=1}^{n} \frac{k - i\omega_j}{k + i\omega_j} \left[ 1 - \sum_{j=1}^{n} e^{\omega_j \alpha_j - \theta_j} \psi_j(x, t) \right].
\]
\[(3.21)\]

Finally, we compute the coupling coefficients \(c_j\) between \(\varphi_j\) and \(\psi_j\), defined by \(\varphi_j = c_j \psi_j\). From (3.18) a straightforward calculation yields
\[
c_j = \frac{e^{2\alpha_j \omega_j}}{2\omega_j} \prod_{m \neq j} \frac{\omega_j - \omega_m}{\omega_j + \omega_m}.
\]
\[(3.22)\]

Note that, with our choice of normalization of the wave functions, the coupling coefficients for the bound state wave functions are time independent!

### 4 Completeness theorem

Sachs [17] proved a completeness theorem for the derivatives of the squared eigenfunctions of the Schrödinger operator (3.1), and used this to construct the inverse operator for the linearized KdV equation. We describe in this section a modification of Sachs’ result that gives a direct extension to a result for the squared eigenfunctions themselves. For simplicity we assume that the potential \(u\) is a multi-soliton solution, though the results are still valid when \(u\) is not a reflectionless potential.

The squared eigenfunctions satisfy a third order linear homogeneous differential equation, which we write in the form
\[
M\varphi = (D^3 - 2(uD + Du) + 4k^2D)\varphi(x, k) = 0,
\]
\[(4.1)\]
where \(\varphi\) is the product of any pair of solutions of (3.1). The function \(u\) here is the \(n\)-soliton solution in §2 and §3 (below \(t\) will appear only as a parameter, and for simplicity we do not write the dependence of the functions on \(t\)).

Using the wave functions \(\varphi_+(x, k)\) and \(\psi_+(x, k)\) as a basis of solutions of (4.1) we obtain
\[
\varphi_+^2(x, k), \quad \psi_+^2(x, k), \quad \varphi_+(x, k)\psi_+(x, k),
\]
as a basis of solutions of (4.1).
The completeness theorem is based on the following formal calculation of the contour integral of the resolvent of the operator $M$. Let $\Gamma_R$ be the semicircle of radius $R$ in the upper half plane traversed from -1 to 1. Formally

$$\lim_{R \to \infty} \frac{1}{2\pi i} \int_{\Gamma_R} (D^3 - 2(uD + Du) + 4k^2)^{-1} 8k \, dk = -\lim_{R \to \infty} \frac{1}{2\pi i} \int_{C_R} D^{-1}(D^2 - 2(u + DuD^{-1}) - \lambda)^{-1} d\lambda \quad (\lambda = -4k^2) = D^{-1},$$

where $C_R$ is the circle of radius $4R^2$ in the $\lambda$-plane oriented in the counterclockwise direction.

We construct the inverse of $M$ using as a basis of homogeneous solutions the above set of squared eigenfunctions. Since $M$ is skew symmetric, we seek a kernel $K(x, y, k)$ satisfying

$$MK(x, y, k) = \delta(x - y), \quad K(x, y, k) = -K(y, x, k).$$

Thus

$$[K] = 0, \quad [D_x K] = 0, \quad [D_x^2 K] = 1,$$

where $[K]$ denotes the jump of $K$ across the singularity $x = y$ (from $x < y$ to $x > y$), etc. We also require $K(x, y, k)$ to be meromorphic in $\Im k > 0$ and bounded as $x \to -\infty$ and $y \to \infty$.

Recall that, for $\Im k > 0$,

$$\varphi_+^2(x, k) \to 0, \quad as \quad x \to -\infty,$$

$$\psi_+^2(x, k) \to 0, \quad as \quad x \to \infty,$$

$$\varphi_+ \psi_+ = a(k) \psi_+ \psi_+ \to a(k), \quad as \quad x \to \infty.$$

We try

$$K(x, y, k) = \varphi_+^2(x, k)C_1(y, k) + \varphi_+(x, k)\psi_+(x, k)C_2(y, k), \quad x < y.$$

The jump conditions across $x = y$ give three equations for the two unknowns $C_1$ and $C_2$, but one finds that the equations are consistent. After some computations, we obtain

$$K(x, y, k) = \varphi_+(x, k)\psi_+(y, k)\frac{R(x, y, k)}{-8k^2a^2(k)}, \quad x < y, \quad (4.2)$$

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where

\[ R(x, y, k) = \varphi_+(x, k)\psi_+(y, k) - \varphi_+(y, k)\psi_+(x, k), \quad x < y. \]  (4.3)

Based on these considerations, we now prove

**Lemma 4.1** For \( f \in L^1(\mathbb{R}) \) we have

\[
\int_{x_0}^{x} f(y) \, dy = \lim_{R \to \infty} \int_{\Gamma_R} \int_{-\infty}^{\infty} [\mathcal{K}(x, y, k) - \mathcal{K}(x_0, y, k)] f(y) \, dy \frac{8k \, dk}{2\pi i}. \]  (4.4)

**Proof:** From the asymptotic properties of the wave functions (3.7) it is easy to see that

\[
8k\mathcal{K}(x, y, k) = \frac{1 - e^{2ik(y-x)}}{k} \left(1 + O\left(\frac{1}{|k|}\right)\right), \quad \text{as} \quad |k| \to \infty,
\]

if \( x < y \). So for \( x_1 < x_2 \),

\[
8k[\mathcal{K}(x_2, y, k) - \mathcal{K}(x_1, y, k)] \sim \begin{cases} 
\frac{e^{2ik(y-x_1)} - e^{2ik(y-x_2)}}{k} & x_1 < x_2 < y, \\
\frac{e^{2ik(x_2-y)} + e^{2ik(y-x_1)} - 2}{k} & x_1 < y < x_2, \\
\frac{e^{2ik(x_2-y)} - e^{2ik(x_1-y)}}{k} & y < x_1 < x_2,
\end{cases}
\]

as \( k \to \infty \). For large \( R \), we may replace the integrand in the integral over \( \Gamma_R \) by these asymptotic values. Since there is no singularity at \( k = 0 \) we may deform the contour to an integral over \((-R, R)\) on the real \( k \)-axis. The real parts, involving cosines, are odd in \( k \), and hence their contribution vanishes. We are therefore left with the identity

\[
\lim_{R \to \infty} \frac{1}{2\pi i} \int_{\Gamma_R} [\mathcal{K}(x_2, y, k) - \mathcal{K}(x_1, y, k)] \, 8k \, dk \]

\[
= \lim_{R \to \infty} \int_{-R}^{R} \frac{\sin 2k(y - x_1) - \sin 2k(y - x_2)}{2\pi k} \, dk
\]

\[
= \frac{1}{2}[\text{sgn}(y - x_1) - \text{sgn}(y - x_2)].
\]
The proof of the lemma follows immediately when this result is substituted into the right side of (4.4). \square

The completeness theorem is proved by deforming the integral over $\Gamma_R$ in (4.4) to the real line. We begin by analyzing the poles of

$$8kK = -\varphi_+(x,k)\psi_+(y,k)\frac{R(x,y,k)}{k\alpha^2(k)}, \quad x < y.$$  \hfill (4.5)

Denote by $'$ differentiation with respect to $k$.

**Lemma 4.2** The expression (4.5) has a removable singularity at $k = 0$ and simple poles at $k = i\omega_j, j = 1, \ldots, n$, with residues

$$K_j(x,y) = -\frac{1}{2\omega_j(a'(i\omega_j))^2}(F_j(x)G_j(y) - G_j(x)F_j(y)),$$ \hfill (4.6)

for all $x, y$, where

$$F_j(x) = \psi_+^2(x,i\omega_j),$$

$$G_j(y) = i\varphi_+(y,i\omega_j)\frac{d}{dk}(\varphi_+(y,k) - c_j\psi_+(y,k)) \bigg|_{k=i\omega_j},$$ \hfill (4.7)

with $c_j$ such that $\varphi_+(x,i\omega_j) = c_j\psi_+(x,i\omega_j)$.

*Proof:* For $k = 0$ the wave functions $\varphi_+, \psi_+$ are real so

$$\varphi_-(x,0) = \varphi_+(x,0), \quad \psi_-(x,0) = \psi_+(x,0).$$

Then by (3.10) we obtain $R(x,y,0) = 0$, hence (4.5) is regular at $k = 0$.

The poles of $8kK$ coincide with the zeros of $a$. Hence, by (3.20), it has $n$ poles, $i\omega_j, j = 1, \ldots, n$. From the relation $\varphi_+(x,i\omega_j) = c_j\psi_+(x,i\omega_j)$, where $c_j$ is the coupling coefficient defined by (3.22), we see that $R(x,y,i\omega_j) = 0$ and $R'(x,y,i\omega_j) \neq 0$. Since $a^2(k)$ has double zeroes at $k = i\omega_j$, $8kK$ has simple poles at $i\omega_j$. A simple calculation shows that the residues there are given by

$$K_j(x,y) = \begin{cases} -M_j(x,y), & x < y \\ M_j(y,x), & y < x \end{cases}$$ \hfill (4.8)
where

\[ M_j(x, y) = \frac{\varphi_+(x, i\omega_j)\psi_+(y, i\omega_j)}{2i\omega_j[a'(i\omega_j)]^2}B_j(x, y), \quad x < y, \quad (4.9) \]

and

\[ B_j(x, y) = \left. \frac{d}{dk} (\varphi_+(x, k)\psi_+(y, k) - \psi_+(x, k)\varphi_+(y, k)) \right|_{k=i\omega_j} \quad x < y. \quad (4.10) \]

Using the relationship \( \varphi_+(x, i\omega_j) = c_j\psi_+(x, i\omega_j) \), this form of \( K_j \) can be rewritten in the form (4.6), as originally obtained by Sachs. \( \square \)

We now deform the contour \( \Gamma_R \) to the real axis, picking up the residues at \( i\omega_j \). By (4.2), the contribution from the real line is

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x, y, k)f(y) \, dy \, \frac{dk}{8k} =
\]

\[
\int_{-\infty}^{\infty} \left\{ \frac{\varphi_+(x, k)\psi_+(y, k)}{ka^2(k)} \left[ \int_{x}^{\infty} \varphi_+(y, k)\psi_+(y, k)f(y) \, dy \right. \right.
\]

\[
- \left. \int_{-\infty}^{x} \varphi_+(y, k)\psi_+(y, k)f(y) \, dy \right\] \frac{\psi_+^2(x, k)}{ka^2(k)} \int_{-\infty}^{x} \varphi_+^2(y, k)f(y) \, dy \frac{dk}{2\pi i}. \]

From the equalities \( \psi_-(x, k) = \overline{\psi_+(x, k)} = \psi_+(x, -k) \) which hold for \( k \in \mathbb{R} \), by Sachs’ argument \( \{17\} \), p. 678, this simplifies to

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\psi_+^2(x, k)\psi_+^2(y, k) - \psi_-^2(x, k)\psi_-^2(y, k))f(y) \, dy \frac{dk}{4\pi ik}. \quad (4.11) \]

This gives the contribution from the real line; the contribution from the residues is straightforward. Hence
**Theorem 4.3** Let \( u \) be an \( n \)-soliton potential. Then for \( f \in L_1(\mathbb{R}) \),

\[
\int_{x_0}^{x} f(y) \, dy = \sum_{j=1}^{n} \int_{-\infty}^{\infty} (\mathcal{K}_j(x_0, y) - \mathcal{K}_j(x, y)) f(y) \, dy
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (\psi_+^2(x, k) - \psi_+^2(x_0, k)) \psi_-^2(y, k)
- (\psi_-^2(x, k) - \psi_-^2(x_0, k)) \psi_+^2(y, k) \right] f(y) \, dy \, \frac{dk}{4\pi ik}, \tag{4.12}
\]

where \( \mathcal{K}_j \) are given in (4.3).

**Theorem 4.4** Let \( f \) and its Fourier transform both be in \( L_1 \). Then

\[
f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ D\psi_+^2(x, k)\psi_-^2(y, k) - D\psi_-^2(x, k)\psi_+^2(y, k) \right] f(y) \, dy \, \frac{dk}{4\pi ik}
+ \sum_{j=1}^{n} C_j \int_{-\infty}^{\infty} (DF_j(x)G_j(y) - DG_j(x)F_j(y)) f(y) \, dy, \tag{4.13}
\]

where \( F_j, G_j \) are given in (4.7) and

\[
C_j = \frac{1}{2\omega_j(a'(i\omega_j))^2}. \tag{4.14}
\]

**Proof:** Since \( \tilde{f}(k) \in L_1 \), \( f \) is continuous everywhere, and the derivative of the left side of (4.12) is \( f(x) \) everywhere. When \( f \) and \( \tilde{f} \) are in \( L_1 \), the integrals on the right side of (4.13) converge absolutely. We may divide both sides of (4.12) by \( (x - x_0) \) and let \( x \to x_0 \), thereby obtaining (4.13). \( \square \)

### 5 Propagator of the linearized KdV equation

Consider the homogeneous initial value problem

\[
v_t + v_{xxx} - 6(uv)_x = 0, \quad v(x, s) = \phi(x). \tag{5.1}
\]
The propagator for this time dependent equation is the operator \( T_{t,s} \) such that \( v(t) = T_{t,s} \phi \). By Theorem 4.4 the solution of the initial value problem (5.1) is

\[
v(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{4\pi ik} [D\psi^2_+(x,t,k)\psi^2(y,s,k) - \\
D\psi^2_+(x,t,k)\psi^2(y,s,k)] \phi(y) dy dk
\]

\[+ \sum_{j=1}^{n} C_j \int_{-\infty}^{\infty} [DF_j(x,t)G_j(y,s) - DG_j(x,t)F_j(y,s)] \phi(y) dy =: T_{t,s} \phi, \quad (5.2)
\]

where

\[F_j(x,t) = \psi^2_+(x,t,i\omega_j),
\]

\[G_j(y,s) = i\varphi_+(y,s,i\omega_j) \frac{d}{dk}(\varphi_+(y,s,k) - c_j\psi_+(y,s,k)) \bigg|_{k=i\omega_j},
\]

and \( C_j \) is given by (4.14).

A direct computation shows that the functions

\[g_j(x,t) = \frac{d}{dk}(\varphi_+(x,t,k) - c_j\psi_+(x,t,k)) \bigg|_{k=i\omega_j},
\]

satisfy both the Schrödinger equation and the equation \( \psi_t = B\psi \) (cf. [14]). Then by Theorem 3.1 \( G_j \) satisfies the linearized potential KdV equation (3.4) and \( DG_j \) satisfies the linearized KdV equation (3.5). The same is true for \( F_j \). Hence the last term in (5.2) takes the form of a projection onto the kernel of the linearized KdV equation.

The \( n \)-soliton solutions satisfy the KdV equation identically in the \( 2n \) parameters \( \alpha_1, \ldots, \alpha_n \) and \( \omega_1, \ldots, \omega_n \). Differentiating the KdV equation with respect to each of the \( \alpha_1, \ldots, \alpha_n \) and \( \omega_1, \ldots, \omega_n \) we obtain \( 2n \) solutions of the linearized KdV equation (5.3):

\[u_{\alpha_j} = \frac{\partial u}{\partial \alpha_j}, \quad u_{\omega_j} = \frac{\partial u}{\partial \omega_j}, \quad j = 1, \ldots, n. \quad (5.3)
\]

It is easily seen that the \( u_{\omega_j} \) grow linearly with time.
We conjecture that the $2n$ functions $DF_j$ and $DG_j$ are linear combinations of (5.3). For the $DF_j$, from (3.16), we have

$$Du = -4 \sum_{j=1}^{n} \omega_j e^{2\omega_j \alpha_j} D(\psi_j^2) = -4 \sum_{j=1}^{n} \omega_j e^{2\omega_j \alpha_j} DF_j,$$

and by (1.1)

$$Du = -\sum_{j=1}^{n} \frac{\partial u}{\partial \alpha_j}.$$  

Hence,

$$\sum_{j=1}^{n} \omega_j e^{2\omega_j \alpha_j} DF_j = \frac{1}{4} \sum_{j=1}^{n} \frac{\partial u}{\partial \alpha_j}.$$  

We conjecture that in fact

$$DF_j = \frac{1}{4\omega_j} e^{-2\omega_j \alpha_j} \frac{\partial u}{\partial \alpha_j},$$

holds for each $j = 1, \ldots, n$. For $n = 1, 2$ we have confirmed this relationship by Maple calculations.

We expect each of the $DG_j$, $j = 1, \ldots, n$, to be a linear combination of all the functions (5.3). For the one-soliton solution we found, again by Maple calculations,

$$8\omega_1^2 e^{-2\omega_1 \alpha_1} DG_1 = -\frac{\partial u}{\partial \omega_1} + \left( \frac{\alpha_1}{\omega_1} - \frac{1}{2\omega_1^2} \right) \frac{\partial u}{\partial \alpha_1}. \quad (5.4)$$

For the two-soliton solution the relationship between $DG_j$ and (5.3) seems already to be far more complicated, and we were unable to determine it with Maple.

We see from (5.4) that $DG_1$ grows linearly in $t$, since the derivative of the $n$-soliton solution with respect to $\omega_1$ has this property. In fact, the same is true for all the $DG_j$ in the $n$-soliton case, as can be seen by differentiating (3.12) with respect to $k$. This implies that we obtain secular growth terms in the sum appearing in (5.2). In order to eliminate this secular growth in the propagator, we must impose the orthogonality conditions

$$\int_{-\infty}^{\infty} G_j(y,t) \phi(y) dy = \int_{-\infty}^{\infty} F_j(y,t) \phi(y) dy = 0, \quad j = 1, \ldots, n. \quad (5.5)$$
Recall that \( F_j \) and \( G_j \) are solutions of the linearized potential KdV operator, which is the adjoint of the linearized KdV operator. So, the orthogonality conditions (5.3) are analogous to solvability conditions in the Fredholm alternative for the time dependent operator occurring in (5.1). (cf. Lemma 7.2 below.)

In the case of a single soliton, one may evaluate the linearized KdV equation in a frame moving with the solitary wave. In that case the linearized equation is time independent, as are the pair of solutions \( u_{\alpha 1} \) and \( u_{\omega 1} \). This pair of functions spans the generalized kernel of the linearized operator \( L_v := v_{xxx} - 4\omega_1^2 v_x - 6(uv)_x \).

They are both exponentially decaying as \( x \to \pm \infty \), hence are in \( L_2(\mathbb{R}) \).

In the present case, the linearized equation is no longer time independent, and we cannot talk about the kernel of a linear time independent operator. Nevertheless, the \( 2n \) functions (5.3) play a similar role in the analysis. They are exponentially decaying in \( x \) for fixed time.

### 6 Estimates on the propagator

In this section we estimate the propagator \( T_{t,s} \) in spaces of functions analytic on the strip \( |\Im x| < \pi / \Omega \). Given a function \( \phi \) analytic in this strip, define \( \phi_{\alpha}(x) = \phi(x + i\alpha) \), for \( x \in \mathbb{R} \). Consider the space

\[
\mathcal{A}_\alpha^m := \{ \phi \text{ analytic in } |\Im x| < \alpha, \; \phi_{\pm \alpha} \in H^m(\mathbb{R}), \phi_y \to 0, \text{ uniformly in } |y| < \alpha, \text{ as } |x| \to \infty \}
\]

where \( 0 \leq \alpha < 2\pi / \Omega \) and \( m \) is some positive integer. If \( \alpha = 0 \) this space is the Sobolev space \( H^m(\mathbb{R}) \). The norm in \( \mathcal{A}_\alpha^m \) is defined to be

\[
\| \phi \|_{\alpha,m} := \max\{ |\phi_{\alpha}|_m, |\phi_{-\alpha}|_m \},
\]

where \( |\cdot|_m \) denotes the usual norm in \( H^m(\mathbb{R}) \).

The following result follows from the properties of the Fourier transform.

**Lemma 6.1** If \( \phi \in \mathcal{A}_\alpha^m \) then the Fourier transform \( \widehat{\phi} \) belongs to

\[
\mathcal{A}_\alpha^m := \{ \widehat{\phi}(k) : (1 + |k|^m e^{\alpha |k|}) \widehat{\phi}(k) \in L_2(\mathbb{R}) \}.
\]

Conversely, if \( \widehat{\phi} \in \mathcal{A}_\alpha^m \) then its inverse Fourier transform \( \phi \) belongs to \( \mathcal{A}_\alpha^m \). Moreover, the norms of \( \phi \) in \( \mathcal{A}_\alpha^m \) and of \( \widehat{\phi} \) in \( \mathcal{A}_\alpha^m \) are equivalent.
The norm of \( \hat{\phi} \in \mathcal{A}_\alpha^m \) is the \( L_2 \)-norm of the function \((1 + |k|)^m e^{\alpha |k|} \hat{\phi}(k)\), and we shall denote it also by \( \| \cdot \|_{\alpha,m} \).

To analyze the propagator we write

\[
\psi_2^+ = e^{2i(kx + 4k^3 t)}[1 + w_+(x, t, k)], \quad \psi_2^- = e^{-2i(kx + 4k^3 t)}[1 + w_-(x, t, k)].
\]

By the results in §3, \( w_+ \) is a meromorphic function of \( k \), with poles in the lower half plane at \( -i\omega_j, \ j = 1, \ldots, n \), and it decays like \( 1/k \) as \( k \) tends to infinity. Moreover, \( w_+ \) is analytic in \( x \) in the strip \( \Im x < 2\pi/\Omega; \) it is uniformly bounded in \( \Im x \leq \alpha, t \in \mathbb{R}, \Im k \geq -\omega_1 + \varepsilon \), for any \( 0 < \alpha < 2\pi/\Omega, \varepsilon > 0 \), and \( w_-(x, t, k) = w_+(x, t, k) \).

Substituting the expressions for \( \psi_2^\pm \) above into the integrand in the first term of (5.2), we find

\[
\frac{1}{4\pi i k} [D\psi_2^+(x, t, k)\psi_2^-(y, s, k) - D\psi_2^-(x, t, k)\psi_2^+(y, s, k)]
\]

\[
= \frac{1}{\pi} \left[ \cos 2(k(x - y) + 4k^3(t - s)) + K_1(x, y, t, s, k) + K_2(x, y, t, s, k) \right],
\]

where

\[
K_1(x, y, t, s, k) = \frac{H_1(x, y, t, s, k) - H_1(y, x, s, t, k)}{2},
\]

\[
H_1(x, y, t, s, k) = e^{2i(k(x - y) + 4k^3(t - s))} (w_+(x, t, k) + w_-(y, s, k) + w_+(x, t, k)w_-(y, s, k)),
\]

\[
K_2(x, y, t, s, k) = \frac{1}{4\pi i k} [e^{2i(k(x - y) + 4k^3(t - s))} Dw_+(x, t, k)(1 + w_-(y, s, k))
\]

\[
- e^{-2i(k(x - y) + 4k^3(t - s))} Dw_-(x, t, k)(1 + w_+(y, s, k))].
\]

Since \( \psi_+(x, t, 0) = \psi_-(x, t, 0) \), \( w_+(x, t, 0) = w_-(x, t, 0) \), and \( K_2 \) has a removable singularity at \( k = 0 \).

The propagator \( T_{t,s} \) is thus a sum of 4 terms:

\[
T = T_0 + T_1 + T_2 + T_3,
\]

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where

\[ T_0(t, s) \phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos 2(k(x-y) + 4k^3(t-s)) \phi(y) \, dy \, dk, \quad (6.1) \]

\[ T_1(t, s) \phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_1(x, y, t, s, k) \phi(y) \, dy \, dk, \quad (6.2) \]

\[ T_2(t, s) \phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_2(x, y, t, s, k) \phi(y) \, dy \, dk, \quad (6.3) \]

\[ T_3(t, s) \phi = \sum_{j=1}^{n} C_j \int_{-\infty}^{\infty} (DF_j(x, t)G_j(y, s) - DG_j(x, t)F_j(y, s)) \phi(y) \, dy. \quad (6.4) \]

We begin by estimating \( T_0 \). We first note that \( T_0 \) can be written

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2i(k(x-y)+4k^3(t-s))} \phi(y) \, dy \, dk \]

\[ + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2i(k(x-y)+4k^3(t-s))} \phi(y) \, dy \, dk. \]

Changing \( k \to -k \) in the second integral, and replacing \( 2k \) by \( k \) we get

\[ T_0(t, s) \phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(kx+k^3(t-s))} \int_{-\infty}^{\infty} e^{-iky} \phi(y) \, dy \, dk. \quad (6.5) \]

The contribution in the solution of (7.1) coming from \( T_0 \) is

\[ v_0(x, t) = - \int_{t}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{i(k^3(t-s))} \int_{-\infty}^{\infty} e^{-iky} f(y, s) \, dy \, dk \, ds, \]

or in terms of Fourier transforms

\[ \hat{v}_0(k, t) = - \int_{t}^{\infty} e^{ik^3(t-s)} \hat{f}(k, s) \, ds. \quad (6.6) \]
The action of $T_0(t, s)$ on $\phi$ is multiplication of the Fourier transform $\hat{\phi}$ by $e^{ik^3(t-s)}$. This is a unitary operator on $\mathcal{A}_\alpha^m$, so by Lemma 6.1 $T_0$ is a bounded operator on $\mathcal{A}_\alpha^m$, with bound independent of $t$ and $s$.

We next show that the operators $T_j(t, s)$, $j = 1, 2$ are smoothing operators. The first term in $T_1$ is in fact a pseudo-differential operator

$$T_{1,1}(t, s)\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2i(kx+4k^3(t-s))} w_+(x, t, k) \hat{\phi}(2k) \, dk.$$ 

From the structure and decay properties in $k$ of $w_+$, $T_{1,1}$ is a bounded map from $\mathcal{A}_\alpha^0$ to $\mathcal{A}_\alpha^1$. In fact, by (3.12), $T_{1,1}$ is a sum of operators such as

$$\frac{1}{\pi} p_j(x, t) \int_{-\infty}^{\infty} e^{2i(kx+4k^3(t-s))} \frac{1}{k-i\omega_j} \hat{\phi}(2k) \, dk$$

as well as others which are order $k^{-2}$. But multiplication of $\hat{\phi}$ by terms such as $(k-i\omega)^{-1}$ is a smoothing operator; it acts as an integration. Furthermore, multiplication by the analytic functions $p_j(x, t)$ is also a bounded operation on $\mathcal{A}_\alpha^m$.

We therefore see without difficulty that

$$\|T_{1,1}(t, s)\phi\|_{\alpha,1} \leq C\|\phi\|_{\alpha,0}, \quad (6.7)$$

for some constant $C$. Thus, by the smoothness of $w_+\pm$ in $x$ and $y$ we see that $T_{1,1}$ maps continuously $\mathcal{A}_\alpha^m$ to $\mathcal{A}_\alpha^{m+1}$. The other two terms can be treated in exactly the same way. Actually, the third term in $T_1$ is a bounded map from $\mathcal{A}_\alpha^m$ to $\mathcal{A}_\alpha^{m+2}$; though this fact is of no real use. Similarly the operator $T_2$ is a bounded map from $\mathcal{A}_\alpha^m$ to $\mathcal{A}_\alpha^{m+2}$, since its kernel is regular at $k = 0$ and decays like $k^{-2}$ as $k \to \infty$.

The terms arising from $T_3$ grow linearly in time, as we observed in the previous section; and so we have

**Theorem 6.2** Let $T_{t,s}$ be the propagator defined by the initial value problem (5.1), and let $\phi$ satisfy the orthogonality conditions (5.5). Then

$$\|T_{t,s}\phi\|_{\alpha,m} \leq C_{\alpha,m}\|\phi\|_{\alpha,m}.$$
7 Inversion of the linearized KdV equation

We now turn to the inhomogeneous equation

\[ v_t + v_{xxx} - 6(uv)_x = f(x, t), \]  

(7.1)

where \( u \) is a multi-soliton solution of the KdV equation. By Duhamel’s principle, solutions of this inhomogeneous equation are given by

\[ v(\cdot, t) = -\int_t^\infty T_{t,s}f(\cdot, s) \, ds. \]

The solution of the inhomogeneous equation on the semi infinite interval \( t > 0 \) is of course not unique. We have chosen this form so that we preserve the class of functions decaying exponentially as \( t \to \infty \).

Let

\[ \mathcal{R}_{\alpha,b}^m = \{ R(x, t) : \sup_{t \geq 0} e^{bt} \| R(\cdot, t) \|_{\alpha,m} \leq \infty \}, \]

with norm

\[ \| R \|_{\alpha,b,m} = \sup_{t \geq 0} e^{bt} \| R(\cdot, t) \|_{\alpha,m}, \]

where \( \alpha, m \) are as in the previous section and \( b \) is a positive constant. Denote by \( \mathcal{W} \) the class of all functions \( f(x, t) \) of the form

\[ f(x, t) = \sum_{j=1}^n f_j(x - 4\omega_j^2 t) + R_f(x, t), \]  

(7.2)

where \( R_f \in \mathcal{R}_{\alpha,b}^m \) and \( f_j \in \mathcal{A}_{\alpha,\mu}^m \). In fact we assume that \( f_j \) decays exponentially to zero as \( |x| \to \infty \), in the strip \( |\Im x| \leq \alpha \), i.e. \( f_j \in \mathcal{A}_{\alpha,\mu}^m \), where

\[ \mathcal{A}_{\alpha,\mu}^m = \{ f : \cosh(\mu x) f \in \mathcal{A}_\alpha^m \}. \]

For simplicity we choose the same analyticity domain and the same exponential decay rate for all \( f_j \), though this is not necessary. In this section we solve the inhomogeneous linearized KdV equation in the space \( \mathcal{W} \).

The class \( \mathcal{W} \) corresponds, roughly, to a decomposition of the space of functions defined on the half plane \( t > 0 \) into exponentially decaying functions and persisting functions.
Recall (Theorem 2.1) that the \( n \)-soliton solution of the KdV equation is of the form

\[
u(x, t) = \sum_{j=1}^{n} u_j(x - 4\omega_j^2 t) + R_u(x, t),\tag{7.3}\]

where \( u_j(z) = -2\omega_j^2 \text{sech}^2(\omega_j z - \omega_j \alpha_j + \gamma_j) \) and \( R_u \) satisfies (2.2). Hence, it belongs to \( W \) for any \( 0 < \alpha < \pi/2\Omega, 0 \leq \mu < 2\omega_1 \), for some \( b > 0 \) as in (2.2), and any positive integer \( m \).

We solve (7.1) in the space \( W \) above. For \( f \in W \) we look for solutions \( v \in W \),

\[
v(x, t) = \sum_{j=1}^{n} v_j(x - 4\omega_j^2 t) + V(x, t),\tag{7.4}\]

where \( V \in R^m_{a,b} \) and \( v_j \in A^m_{a,\mu} \). Since \( u_j \) and \( v_k \) decay exponentially in \( x \), and are waveforms moving to infinity at different speeds, \( u_j v_k \in R^m_{a,b} \), for some \( b > 0 \) related to the difference \( \omega_j^2 - \omega_k^2 \) of the speeds of \( u_j \) and \( v_k \); thus waveforms moving at different velocities decouple, and their product lies in \( R^m_{a,b} \) for some \( b \).

The \( v_j \) are determined independently by solving the ordinary differential equation

\[
-4\omega_j^2 v'_j + v''_j - 6(u_j v_j)' = f_j.\tag{7.5}\]

Then for \( V \) we solve

\[
V_t + V_{xxx} - 6(uV)_x = G(x, t),\tag{7.6}\]

where

\[
G(x, t) = 6 \sum_{j \neq k} (u_j v_k)_x + 6 \sum_{k=1}^{n} (R_u v_k)_x + R_f.\tag{7.7}\]

**Lemma 7.1** Assume \( f_j \in A^m_{a,\mu} \), with \( 0 < \alpha < \pi/2\Omega, 0 \leq \mu < 2\omega_1 \), and

\[
\int_{-\infty}^{\infty} f_j(z) \, dz = 0.\tag{7.8}\]
Then, (7.3) has a unique solution $v_j \in A_{\alpha,\mu}^{m+3}$ if and only if $f_j$ satisfies the orthogonality condition

$$\int_{-\infty}^{\infty} u_j(z) f_j(z) \, dz = 0. \quad (7.9)$$

In particular, if $f_j(z) = \tilde{f}_j(\omega_j z - \omega_j \alpha_j + \gamma_j)$, and $\tilde{f}_j$ is an even function in $\mathbb{R}z$, then (7.3) has a unique solution $v_j(z) = \tilde{v}_j(\omega_j z - \omega_j \alpha_j + \gamma_j)$ with $\tilde{v}_j$ an even function in $\mathbb{R}z$.

Proof: From (7.8) it follows that there exists $g_j \in A_{\alpha,\mu}^{m+1}$ such that $f_j = g_j'$. Then by integrating (7.5) once, we get

$$v_j'' - 4\omega_j^2 v_j - 6u_j v_j = g_j. \quad (7.10)$$

Recall that $u_j(z) = -2\omega_j^2 \text{sech}^2(\omega_j z - \omega_j \alpha_j + \gamma_j)$. Then, set $y = \omega_j z - \omega_j \alpha_j + \gamma_j$ and (7.10) reads

$$v_j'' - 4v_j + 12 \text{sech}^2(y) v_j = h_j, \quad (7.11)$$

with $h_j(y) = g_j(z)/\omega_j^2$, $h_j \in A_{\alpha',\mu'}^{m+1}$, $\alpha' = \omega_j \alpha < \pi/2$, $\mu' = \mu/\omega_j < 2$.

In the space above the operator $D^2 - 4 + 12 \text{sech}^2 y$ is well-studied in connection with the KdV solitons (cf. [15]). In $L^2(\mathbb{R})$ its discrete spectrum consists of simple eigenvalues 5, 0, -3, and its continuous spectrum occupies the interval $(-\infty, -4]$. Due to translation invariance the derivative of the solitary wave is a homogeneous solution of (7.11), i.e.,

$$\phi_0 = \text{sech}^2 x \tanh x,$$

is a null function for (7.11). It spans the kernel of $D^2 - 4 + 12 \text{sech}^2 y$ in $L^2(\mathbb{R})$. But $\phi_0$ belongs to $A_{\alpha',\mu'}^{m+1} \subset L^2(\mathbb{R})$, so the kernel of $D^2 - 4 + 12 \text{sech}^2 y$ in this space is also one dimensional.

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2 This is precisely the situation which arises in our perturbation scheme (unpublished here) of the Euler equations for surface gravity waves which leads to the KdV approximation.
From these considerations it follows that (7.11) has a unique solution provided $h_j$ is orthogonal to $\phi_0$. The same is true for the unscaled equation (7.10). A simple calculation shows that the orthogonality condition for (7.10) is in fact (7.9) and the first part of the theorem is proved.

The final part is a consequence of the fact that (7.11) can be solved in spaces of even functions, and there it always has a unique solution. The eigenfunction $\phi_0$ is odd, so the operator $D^2 - 4 + 12 \text{sech}^2 y$ restricted to the subspace of even functions has a trivial kernel.

We substitute now the $v_j$ obtained in this lemma into (7.7). Then $G \in \mathcal{R}^m_{\alpha,b}$, where $b$ is less than the one given in (2.2) and

$$\min\{4\omega_j (\omega^2_{j+1} - \omega^2_j), 4\mu (\omega^2_{j+1} - \omega^2_j), \ j = 1, \ldots, n\}.$$  

(7.12)

**Lemma 7.2** Assume $0 < \alpha < \pi/2\Omega$, and $G \in \mathcal{R}^m_{\alpha,b}$ satisfies the orthogonality conditions

$$\int_{-\infty}^{\infty} G_j(y,t)G(y,t) \, dy = 0, \quad \int_{-\infty}^{\infty} F_j(y,t)G(y,t) \, dy = 0, \quad j = 1, \ldots, n,$$

(7.13)

where $F_j$ and $G_j$ are the functions in (4.7). Then (7.6) has a unique solution $V \in \mathcal{R}^m_{\alpha,b}$,

$$V(\cdot, t) = -\int_t^\infty (T_0 + T_1 + T_2)_{t,s}G(\cdot, s) \, ds,$$

(7.14)

where $T_j, \ j = 0, 1, 2$ are defined in (6.1)–(6.3).

**Proof:** Since $G$ satisfies (7.13), we deduce from the results in §8 that the solution $V$ of (7.6) is given by (7.14). By the estimates in §8 we have

$$\|T_j(t, s)G(\cdot, s)\|_{\alpha,m} \leq C_j \|G(\cdot, s)\|_{\alpha,m}, \quad j = 0, 1, 2,$$

and the constants $C_j$ are independent of $t$ and $s$. Then

$$\|V\|_{\alpha,b,m} = \sup_{t \geq 0} e^{bt} \int_t^\infty \|T_0 + T_1 + T_2\|_{t,s}G(\cdot, s) \|_{\alpha,m} \, ds$$

$$\leq C \sup_{t \geq 0} e^{bt} \int_t^\infty e^{-bs} \|G\|_{\alpha,b,m} \, ds = \frac{C}{b} \|G\|_{\alpha,b,m},$$

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hence $V \in \mathbb{R}_{\alpha,b}^m$. \[\square\]

The orthogonality conditions \((7.13)\) are needed to eliminate terms which grow linearly in time. Since $F_j$ and $G_j$ are solutions of the adjoint linearized KdV equation, \((7.13)\) may be viewed as Fredholm solvability conditions for the inversion of the linearized KdV equation on a half space $t > 0$.

From these two lemmas we have

**Theorem 7.3** Assume $u$ in \((1.1)\) is an $n$-soliton solution of the KdV equation. Then $u$ is of the form \((7.3)\), and $u \in W$, for any $0 < \alpha < \pi/2\Omega$, $0 \leq \mu < 2\omega_1$, $b > 0$ as in \((2.2)\), and any positive integer $m$.

Assume $b$ is less than the value in \((7.12)\), and $f \in W$ is of the form \((7.2)\) where each $f_j$ satisfies the orthogonality conditions \((7.8)\) and \((7.9)\). Then the inhomogeneous linearized KdV equation \((7.1)\) has a solution $v \in W$, given by \((7.4)\), with $v_j$ solutions of \((7.5)\), provided that

$$G(x, t) = 6 \sum_{j \neq k} (u_j v_k)_x + 6 \sum_{k=1}^n (Rv_k)_x + Rf,$$

satisfies \((7.13)\).

In their analysis of the stability of the solitary wave for the generalized KdV equation, Pego and Weinstein \([13]\) decomposed the equations for the perturbation into the domain and range of the linearized operator, and obtained the orthogonality conditions by imposing modulation equations on the phase $\alpha$ and speed $\omega$ of the solitary wave. Such a perturbation scheme is in principle feasible in the case of the multi-solitons; but the fact that the perturbation scheme loses derivatives make the problem considerably more complicated, as we indicated in the introduction.

### 8 Estimates in weighted spaces

In \([13]\) weighted spaces have been considered to prove the asymptotic stability of the solitary wave solutions of the KdV equation. In these spaces the inverse of the linearized KdV operator for solitary waves was a smoothing operator. In this section we consider similar weighted spaces but for functions which are analytic in a strip. We estimate the propagator $T$ and then the solution of \((7.1)\) in these spaces.
Define the weighted space
\[ \mathcal{B}_{m}^{\alpha,\eta} := \{ \phi : e^{\eta x} \phi \in A_{\alpha}^{m}(\mathbb{R}) \} \],
(8.1)
with norm \( \| \phi \|_{\alpha,\eta,m} := \| e^{\eta x} \phi \|_{\alpha,m} \). Consider also
\[ \mathcal{B}_{m}^{\alpha,\eta,b} := \{ f(x,t) : \sup_{t \geq 0} e^{bt} \| f(\cdot,t) \|_{\alpha,\eta,m} < \infty \} \].
(8.2)
Denote by \( \| \cdot \|_{\alpha,\eta,m,b} \) the norm in this space.

From the properties of the Fourier transform one obtains the following result.

**Lemma 8.1** If \( \phi \in \mathcal{B}_{m}^{\alpha,\eta} \) then its Fourier transform belongs to \( \widehat{\mathcal{B}}_{m}^{\alpha,\eta} := \{ \hat{\phi}(k) : (1 + |k|)^m e^{\alpha |k|} \hat{\phi}(k + i\eta) \in L_2(\mathbb{R}) \} \). The spaces \( \widehat{\mathcal{B}}_{m}^{\alpha,\eta} \) and \( \mathcal{B}_{m}^{\alpha,\eta} \) are isomorphic by the Plancherel theorem.

The norm in \( \widehat{\mathcal{B}}_{m}^{\alpha,\eta} \), being equivalent with the one in \( \mathcal{B}_{m}^{\alpha,\eta} \), is also denoted by \( \| \cdot \|_{\alpha,\eta,m} \).

To obtain estimates on the propagator \( T \) in these spaces we have to consider the KdV equation in a moving frame. We shall take the following form of the KdV equation
\[ u_t - \frac{1}{6} u_{xxx} + \frac{1}{2} u_x + \frac{3}{2} uu_x = 0. \]
(8.3)
This is the actual equation we obtained in the long wave approximation of the Euler equations for water waves.

The inhomogeneous linearized equation we solve is
\[ v_t - \frac{1}{6} v_{xxx} + \frac{1}{2} v_x + \frac{3}{2} (uv)_x = f(x,t), \]
(8.4)
with \( u \) an \( n \)-soliton solution of (8.3). The results in the previous sections are all valid for (8.4).

The solution of (8.4) is given by
\[ v(\cdot,t) = - \int_{t}^{\infty} T_{t,s} f(\cdot,s) \, ds, \]
where \( T \) is the propagator of the homogeneous equation. It is defined as for (7.1) but with different wave functions \( \psi_+ \) and \( \varphi_+ \),

\[
\varphi_+(x, t, k) = e^{-i(kx-\sigma t)} (1 + m(x, t, k)), \quad \psi_+(x, t, k) = e^{i(kx-\sigma t)} (1 + r(x, t, k)),
\]

where

\[
\sigma = \frac{2}{3} k^3 + \frac{1}{2} k.
\]

The reduced wave functions \( m \) and \( r \) are normalized by (3.8). They have the same properties as the reduced wave functions in \( \S 3 \).

The propagator \( T_0 \) is

\[
T_0(t, s)\phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos 2(k(x - y) - \sigma(t - s))\phi(y) \, dy \, dk \tag{8.5}
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(kx+\theta(s-t))} \int_{-\infty}^{\infty} e^{-iky} \phi(y) \, dy \, dk, \tag{8.6}
\]

where \( \sigma = \frac{k^3}{6} + \frac{k}{2} \). The other terms in \( T, T_1, T_2 \) and \( T_3 \), have the same structure as before.

Choose \( \alpha \) such that the decomposition (2.1) and the estimate (2.2) hold in the strip \( |\Im x| \leq \alpha \).

**Theorem 8.2** Assume \( 0 < \eta < \sqrt{3} \), \( b > -\frac{1}{6} \eta(3 - \eta^2) \), and \( m \geq 1 \) is an integer. Then the propagator \( T_0 \) satisfies:

\[
\left\| \int_t^\infty T_0(t, s)f(\cdot, s) \, ds \right\|_{\alpha,\eta, m, b} \leq C_1(\eta, b) \| f \|_{\alpha, \eta, m-1, b}, \tag{8.7}
\]

where \( C_1(\eta, b) \) is a positive constant, and \( C_1(\eta, b) \to \infty \), as \( \eta \to 0 \) and \( \eta \to \sqrt{3} \).

**Proof:** The propagator \( T_0 \) from (8.5) is the multiplication of the Fourier transform \( \hat{\phi} \) by \( e^{i\sigma(s-t)} \). Then, to obtain (8.7) we first need bounds for \( e^{i\sigma(s-t)} \) and \( ke^{i\sigma(s-t)} \), for \( \Im k = \eta \).

Let \( k = \xi + i\eta \), \( 0 < \eta < \sqrt{3} \), and \( s > 0 \). We have

\[
|e^{i\sigma s}| = e^{-3\sigma s} = e^{-\frac{1}{6} \eta(3(\xi^2+1)-\eta^2)s} \leq e^{-\frac{1}{6} \eta(3-\eta^2)s},
\]

and

\[
|ke^{i\sigma s}| = \sqrt{\xi^2 + \eta^2} e^{-\frac{1}{6} \eta(3(\xi^2+1)-\eta^2)s} \leq c_0(\eta, s),
\]

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where
\[ c_0(\eta, s) = \begin{cases} \frac{1}{\sqrt{e^{\eta} s}} e^{-\frac{1}{4} \eta (3 - 4\eta^2) s}, & 0 < s < \frac{1}{\eta^3} \\ \eta e^{-\frac{1}{6} \eta (3 - \eta^2) s}, & s \geq \frac{1}{\eta^3}. \end{cases} \]

Hence, we find, for \( s \geq t \),
\[ \| \hat{T}_0(t, s) f(s) \|_{\alpha, \eta, m} \leq c_0(\eta, s - t) c_1 \| \hat{f}(s) \|_{\alpha, \eta, m - 1}, \]
for some \( c_1 > 0 \), and by Lemma 8.1
\[ \| T_0(t, s) f(s) \|_{\alpha, \eta, m} \leq c_0(\eta, s - t) c_2 \| f(s) \|_{\alpha, \eta, m - 1}. \]

Denote
\[ v_0(x, t) = \int_t^\infty T_0(t, s) f(s) \, ds. \]

Then
\[ e^{bt} \| v_0(t) \|_{\alpha, \eta, m} \leq c_2 \int_t^\infty e^{bs} c_0(\eta, s - t) \| f(s) \|_{\alpha, \eta, m - 1} \, ds \]
\[ \leq c_2 \int_t^\infty e^{b(t-s)} c_0(\eta, s - t) \, ds \| f \|_{\alpha, \eta, m - 1, b} \]
\[ = c_2 \int_0^\infty e^{-bs} c_0(\eta, s) \, ds \| f \|_{\alpha, \eta, m - 1, b} = C_1(\eta, b) \| f \|_{\alpha, \eta, m - 1, b}. \]

It is easily seen that \( C_1(\eta, b) < \infty \), for \( 0 < \eta < \sqrt{3} \), and \( b \) as in the theorem. Furthermore, \( C_1(\eta, b) \to \infty \) as \( \eta \to 0, \sqrt{3} \).

The operators \( T_j, j = 1, 2 \) are again smoothing operators. In fact by arguing as in 8.4 we can prove
\[ \| \int_t^\infty T_j(t, s) f(s) \, ds \|_{\alpha, \eta, m, b} \leq C_1(\alpha, \eta, b) \| f \|_{\alpha, \eta, m - j - 1, b}. \] (8.8)

From these results follows.

**Theorem 8.3** Assume \( \eta \) and \( b \) are as in Theorem 8.2. If \( f \in B^{m}_{\alpha, \eta, b} \) satisfies the orthogonality conditions (5.3), then (8.4) has a solution \( v \in B^{m+1}_{\alpha, \eta, b} \),
\[ v(t) = -\sum_{j=0}^{2} \int_t^\infty T_j(t, s) f(s) \, ds, \]

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and

$$\|v\|_{\alpha,\eta,b,m+1} \leq C\|f\|_{\alpha,\eta,b,m}.$$ 

This theorem is the analog of Lemma 7.2 for the weighted spaces. A result similar to the one in Theorem 7.3 holds then for a space $\mathcal{W}_\eta$, defined as $\mathcal{W}$ but with $R_f \in \mathcal{B}_{\alpha,\eta,b}^m$ instead of $R_f \in \mathcal{A}_{\alpha,b}^m$. 

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