TIME DEPENDENCE OF BRANS-DICKE PARAMETER $\omega$
FOR AN EXPANDING UNIVERSE

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Abstract

We have studied the time dependence of $\omega$ for an expanding universe in the generalised B-D theory and have obtained its explicit dependence on the nature of matter contained in the universe in different era.Lastly, we discuss how the observed accelerated expansion of the present universe can be accommodated in the formalism.

Key words: Generalised Brans-Dicke theory, time-dependent $\omega$, accelerated expansion of the Universe.

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1 INTRODUCTION

The Brans-Dicke theory is defined by a constant coupling function $\omega$ and a scalar field $\phi$. The relative importance is determined by the arbitrary coupling function $\omega$ [1]. The generalised B-D theory [2] involves an extension of original theory to the case of a time dependent coupling function $\omega = \omega(t)$. Generalised B-D model is special for more than one reason. It appears naturally in super gravity theory, Kaluza-Klein theories and in all the known effective string actions. It is perhaps the most natural extension of General Relativity [3], which may explain its ubiquitous appearance in fundamental theories. In the generalised B-D theory, sometimes referred to as graviton-dilaton theory, $\omega$ is an arbitrary function of the scalar field $\phi$ (dilaton). Hence it includes a number of models, one for every function $\omega$. GR is obtained when the field $\phi = \text{constant}$ and $\omega(t) = \infty$ [4, 5].

Brans-Dicke theory with a constant $\omega$ is a successful theory because it explains almost all the important features of the evolution of the Universe. Some of the problems like inflation [6], early and late time behaviour of Universe [7], cosmic acceleration and structure formation [8], cosmic acceleration, quintessence and coincidence problem [9], self-interacting potential and cosmic acceleration [10] can be explained in the B-D formalism. For large $\omega$ B-D theory gives the correct amount of inflation and early and late time behaviour, and for small negative $\omega$ it correctly explains cosmic acceleration, structure formation and coincidence problem.

Brans-Dicke theory with a time dependent $\omega$ is also an interesting theory in its own right. Not only it gets a strong support from string and Kaluza-Klein theories, a few attempts have also been taken using this formalism to study the dynamics of the Universe. However, all these attempts address the problems of evolution of universe [11], cosmic acceleration and quintessence [9] etc. in a qualitative way without exhibiting the explicit time dependence of $\omega$. Also, Alimi and Serna [5] have shown that this time varying $\omega$ theory includes a number of models; one each for every time dependence of $\omega$. So it is of natural interest to seek exact derivation of time dependence of $\omega$ from the dynamical equations of the Universe.

Our aim, in this paper therefore, is to derive explicit time dependence of $\omega$ for simple expanding solutions of field and wave equations in the Brans-Dicke theory for all era which can give some information regarding the early and late time behaviour of the universe. This will also provide the opportu-
nity to examine if the derived time-dependence of $\omega$ can be used to explain at least some of the presently observed properties of the universe like cosmic acceleration, structure formation etc.

2 THE GENERAL $\omega$

For a Universe filled with perfect fluid and described by Friedmann-Robertson-Walker space-time with scale factor $a(t)$ and spatial curvature index $k$, the gravitational field equations in B-D theory with time dependent $\omega$, are

\[
\begin{align*}
\frac{\ddot{a}^2 + k}{a^2} + \frac{\dot{a}\dot{\phi}}{a\phi} - \frac{\omega\dot{\phi}^2}{6\phi^2} &= \frac{\rho}{3}\phi \\
2\ddot{a} + \frac{\ddot{a}^2 + k}{a^2} + \frac{\omega\dot{\phi}^2}{2\phi^2} + 2\frac{\dot{a}\dot{\phi}}{a\phi} + \frac{\ddot{\phi}}{\phi} &= -\frac{P}{\phi}
\end{align*}
\]

(1)

(2)

where $\rho$ and $P$ are respectively the energy density and pressure of the fluid distribution. The equation of state of the fluid is given by $P = \gamma \rho$. Some of the values of $\gamma$ for typical cases are -1 (vacuum), 0 (dust), 1/3 (radiation), 1 (massless scalar field).

The wave equation for Brans-Dicke scalar field when $\omega$ is a function of time is [2, 3, 4, 5,]

\[
\ddot{\phi} + 3\frac{\dot{\phi}}{a} = \rho - 3P \frac{\dot{\omega}}{2\omega + 3} - \frac{\omega\dot{\phi}}{2\omega + 3}.
\]

(3)

Energy conservation equation which can be obtained from eqs. (1), (2), and (3) is,

\[
\dot{\rho} + 3\frac{\dot{a}}{a} (\rho + P) = 0.
\]

(4)

One important property of Brans-Dicke theory is that it gives simple expanding solutions for field $\phi(t)$ and scale factor $a(t)$ which are compatible with solar system experiments [12]. To derive the time dependence of $\omega$ which satisfies both field and wave equations, we assume the time dependence of the scale factor and scalar field in the following form which provides simple expanding solutions,
\[ a = a_0 \left( \frac{t}{t_0} \right)^\alpha. \]  
\[ \phi = \phi_0 \left( \frac{t}{t_0} \right)^\beta. \]  
\[ a_0 \text{ and } \phi_0 \text{ are the present values of } a(t) \text{ and } \phi(t). \]  
From eq.(1) and taking \( k = 0 \) to be consistent with the paradigm of inflation, we get,
\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} - \left( \frac{\rho}{3\phi} + \frac{\omega(t)\dot{\phi}^2}{6\phi^2} \right) = 0. \]  
This immediately leads to
\[ \frac{\dot{a}}{a} = -\frac{\dot{\phi}}{2\phi} \pm \left[ \frac{(2\omega+3)\dot{\phi}^2}{12\phi^2} + \frac{\rho}{3\phi} \right]^{1/2}. \]  
This equation can alternatively be written in the form
\[ \left[ \left( \frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi} \right)^2 - \frac{\Omega(t)\dot{\phi}^2}{12\phi^2} \right] 3\phi = \rho, \]  
where \( \Omega(t) = 2\omega(t) + 3 \).  
Solving eq.(4) one gets
\[ \rho = \rho_0 a^{-3(1+\gamma)}. \]  
Using eq.(5), (6) and (8) in (7) we get
\[ \left[ \left( \frac{2\alpha+\beta}{2t} \right)^2 - \frac{\Omega(t)\dot{\phi}^2}{12\phi^2} \right] 3\phi_0 \left( \frac{t}{t_0} \right)^\beta = \rho_0 a_0^{-3(1+\gamma)} \left( \frac{t}{t_0} \right)^{-3\alpha(1+\gamma)}. \]  
From the above equation, one immediately obtains,
\[ \Omega(t) = -\frac{4\rho_0 t_0^2 a_0^{-3(1+\gamma)}}{\beta^2 \phi_0} \left( \frac{t}{t_0} \right)^{-3\alpha(1+\gamma)-\beta+2} + 3 \left( \frac{2\alpha + \beta}{\beta} \right)^2. \]  
Since \( \Omega(t) = 2\omega(t) + 3 \), we are led to the time dependence of \( \omega \) of the form,
\[ \omega(t) = -\frac{2}{\beta^2} \frac{\rho_0 t_0^2 a_0^{-3(1+\gamma)}}{\phi_0} \left( \frac{t}{t_0} \right)^{-3\alpha(1+\gamma)-\beta+2}. \]
We now proceed to check the consistency of above time-dependence of \( \omega \) with eq. (3) which is the wave equation for the scalar field \( \phi(t) \). Using eqs. (5), (6), (8) and (9), eq. (3) reduces to

\[
[\beta (\beta - 1) + 3\alpha \beta] t^{\beta - 2} = \frac{[(1-3\gamma) - \frac{3(3\alpha(1+\gamma)+\beta-2)}{\beta^2} t^{-3\alpha(1+\gamma)-\beta +2}] t^{-3\alpha(1+\gamma)}}{t^{-3\alpha(1+\gamma)-\beta +2}}.
\]

We neglect \( 3\left(\frac{2\alpha+\beta}{\beta}\right)^2 \) in the denominator as we are interested in time dependence only. Thus above equation reduces to

\[
\beta \left[ \frac{3(1-\gamma)}{4} \beta + \frac{3\alpha(1-\gamma)}{2} \right] = 0.
\]

Eq. (11) has obvious solution \( \beta = 0 \) or \( \beta = -2\alpha \). Thus, eq. (9) will satisfy both field and wave equations when \( \beta = 0 \) or, \( \beta = -2\alpha \). It may be noted in passing that for these two cases in particular the neglect of the term \( 3\left(\frac{2\alpha+\beta}{\beta}\right)^2 \) as noted earlier is evidently justified. We now consider these two cases one by one.

**CASE - 1:** \( \beta = 0 \).

For \( \beta = 0 \) eq. (10) gives \( \omega = \infty \), eq. (7) gives \( \phi = \phi_0 = \text{constant} \) and \( a = a_0 \left( \frac{t}{t_0} \right)^\alpha \). Thus, for \( \beta = 0 \) Brans-Dicke model goes over to General Relativity [5]. Here, \( \alpha \) is not related to \( \beta \) and its value can be obtained by solving equations of General relativity [13].

**CASE - 2:** \( \beta = -2\alpha \).

Eq. (10) in this case reduces to,

\[
\omega(t) = -\frac{1}{2\alpha^2} \frac{\rho_0 t_0^2 a_0^{-3(1+\gamma)}}{\phi_0} \left( \frac{t}{t_0} \right)^{-3\alpha(1+\gamma)+2\alpha+2}.
\]

For completeness we also write,

\[
a = a_0 \left( \frac{t}{t_0} \right)^{\alpha}.
\]

\[
\phi = \phi_0 \left( \frac{t}{t_0} \right)^{-2\alpha}.
\]
Thus, to work within the B-D model with a time-dependent $\omega$ we have to take $\beta = -2\alpha$ leading to the above solutions. Clearly, for a given $\alpha > 0$ (non-negative and non-zero) which can be obtained from the observational data [12] the time dependence of $a(t)$, $\phi(t)$ and $\omega(t)$ are fixed through the above equations.

3 $\omega$ FOR DIFFERENT ERA

It is clear from eq.(12) that, the parameter $\gamma$ which takes different values in different era, controls the time dependence of $\omega$ in the respective era as detailed below.

3.1 Vacuum dominated era: $\gamma = -1$

Here,

$$\omega = -\frac{1}{2\alpha^2} \frac{\rho_0 t_0^2}{\phi_0} \left( \frac{t}{t_0} \right)^{2\alpha + 2}. \quad (15)$$

Since for an expanding universe $\alpha > 0$, the time dependence of $\omega$ will always be governed by power of time greater than 2. Hence, $\omega$ decreases faster than $-t^2$ with time.

3.2 Radiation dominated era: $\gamma = 1/3$

Here,

$$\omega = -\frac{1}{2\alpha^2} \frac{\rho_0 t_0^2}{a_0^3\phi_0} \left( \frac{t}{t_0} \right)^{-2\alpha + 2}. \quad (16)$$

Here, again $\omega$ is a decreasing function of time as the Universe undergoes a deccelerated expansion in this era with $0 < \alpha < 1$.

3.3 Matter dominated era : $\gamma = 0$

Here,
The implications of this time dependence is dicussed in the next section.

3.4 Massless scalar field dominated era : $\gamma = 1$

Here,

$$\omega = -\frac{1}{2\alpha^2 a_0^3 \phi_0} \left( \frac{t}{t_0} \right)^{-\alpha + 2}. \quad (17)$$

4 $\omega$ FOR PRESENT UNIVERSE.

The present observable universe contains cold matter of negligible pressure (dust). We, therefore, take the time dependence of $\omega$ as given by eq. (17) i.e,

$$\omega = -\frac{1}{2\alpha^2 a_0^3 \phi_0} \left( \frac{t}{t_0} \right)^{-\alpha + 2}. \quad (18)$$

For present time, taking $t = t_o$, one gets,

$$\omega = -\frac{1}{2\alpha^2 a_0^3 \phi_0} \left( \frac{t}{t_0} \right)^{-\alpha + 2} = \omega_0, \quad (19)$$

where $\omega_o$ represents present value of $\omega$. $\rho_0, \phi_0, a_0$ are positive non-zero constants. However, they can all be set equal to 1 with no loss of generality if time $t$ is measured in units of $\sqrt{\phi_0 a_0^3 \rho_0}$. Therefore, setting $\frac{\rho_0 t_0^2}{a_0^3 \phi_0} = 1$, we get,

$$\omega_0 = -\frac{1}{2\alpha^2}. \quad (20)$$

For the presently observed acceleration to be accomodated $\alpha$ needs to be greater than 1. In that case $\omega_0$ as given by eq.(21) has the minimum value of $-\frac{1}{2}$. This result is in agreement with the observation made by Banerjee and Pavon [9] that $\omega$ value must be greater than $-\frac{3}{2}$ for Newtonian constant of gravitation G and the scalar field energy density to remain positive.
We now recall that the present observational data for deceleration parameter[12] is
\[ -1 < q_0 < 0. \] (22)
Since by definition,
\[ q_0 = -\frac{\ddot{a}a}{\dot{a}^2}. \] (23)
Using eq.(5) we get,
\[ q_0 = -\frac{\alpha - 1}{\alpha}. \] (24)
We see from eq. (22) and eq. (24) that in order to obtain \( q_0 \) between -1 and 0, \( \alpha \) need only be greater than 1. Putting \( \alpha = 1 + \epsilon \), where \( \epsilon > 0 \) but small, the present day solutions of a \( \omega \)-varying B-D theory can be taken as
\[ a = a_0 \left( \frac{t}{t_0} \right)^{1+\epsilon}. \] (25)
\[ \phi = \phi_0 \left( \frac{t}{t_0} \right)^{-2(1+\epsilon)}. \] (26)
\[ \omega = -\frac{1}{2(1+\epsilon)^2} \left( \frac{t}{t_0} \right)^{1-\epsilon}. \] (27)
Thus, we find that time dependence of \( \omega \) obtained through consistent solutions of B-D field equations and the wave equation leads to a negative constant \( \omega \) at the present epoch. This result is consistent with conclusions arrived at by Bertolami and Martins [8], N.Banerjee and D.Pavon [9], Sen and Seshadri [10] that \( \omega \) should possess a low negative value for a satisfactory explanation of structure formation, cosmic acceleration, coincidence problem, and to avoid the problems of quintessence etc within the formalism of B-D theory.
5 DISCUSSIONS AND CONCLUSIONS

In this work, we wish to emphasize that we have, for the first time derived the explicit time dependence of the Brans-Dicke parameter $\omega$ by solving gravitational field and wave equations of generalised B-D theory consistently, assuming power law behaviour for the scale factor $a(t)$ and scalar field $\phi(t)$. Interestingly, we find two consistent solutions of the field and wave equations. One solution leads to General Relativity with the implication that B-D theory is a more general formulation than GR. The other solution, which is of greater interest to us, leads to a time-dependent $\omega(t)$ whose time dependence is governed by the EOS parameter $\gamma$. Consequently, $\omega$ exhibits different temporal behaviour in different epochs of the evolving Universe characterised by its dominant matter/radiation component. This, we believe, is an important result which can be used to study various characteristics of an evolving Universe within the generalised B-D formalism. In particular, for an accelerated expanding universe, the present value of $\omega$ comes out to be negative with a minimum value of $-\frac{1}{2}$. This result, once again, nicely agrees with the conclusions of the earlier works [8, 9, 10] carried within the formalism of constant-$\omega$ B-D theory that $\omega$ needs to be negative for a successful explanations of the various observed characteristic of the evolving universe.

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