Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ under gradient perturbation

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Abstract

For $d \geq 2$, $\alpha \in (0,2)$ and $M > 0$, we consider the gradient perturbation of a family of nonlocal operators $\{\Delta + a^\alpha \Delta^{\alpha/2}, a \in (0,M]\}$. We establish the existence and uniqueness of the fundamental solution $p(t,x,y)$ for

$$L^{a,b} = \Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla,$$

where $b$ is in Kato class $K^d$ on $\mathbb{R}^d$. We show that $p(t,x,y)$ is jointly continuous and derive its sharp two-sided estimates. The kernel $p(t,x,y)$ determines a conservative Feller process $X$. We further show that the law of $X$ is the unique solution of the martingale problem for $(L^{a,b}, C^\infty_0(\mathbb{R}^d))$ and $X$ can be represented as

$$X_t = X_0 + Z^n_t + \int_0^t b(X_s)ds, \quad t \geq 0,$$

where $Z^n_t = B_t + a Y_t$ for a Brownian motion $B$ and an independent isotropic $\alpha$-stable process $Y$. Moreover, we prove that the above SDE has a unique weak solution.

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1 Introduction

Let $B$ be a Brownian motion on $\mathbb{R}^d$ with $\mathbb{E}[(B_t - B_0)^2] = 2t$, and $Y$ be a rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$ that is independent of $B$. Here $d \geq 1$ and $\alpha \in (0,2)$. Then $B + Y$ is a symmetric Lévy process that has both diffusive and jumping components. Let $b$ be a bounded $\mathbb{R}^d$-valued function on $\mathbb{R}^d$. Using Girsanov transform, it is easy to show that for every $a > 0$, there is a strong Markov process $X^{a,b}$ on $\mathbb{R}^d$ so that

$$dX^{a,b}_t = dZ^n_t + b(X^{a,b}_t)dt,$$

where $Z^n_t$ is a Lévy process that has the same distribution as $B + a Y$. The goal of this paper is to study the transition density function $p^{a,b}(t,x,y)$ of the strong Markov process $X^{a,b}$ and its two-sided sharp estimates.

Recall that a rotationally symmetric $\alpha$-stable process on $\mathbb{R}^d$ is a Lévy process $Y$ so that

$$\mathbb{E}_x[e^{i\xi(Y_t - Y_0)}] = e^{-t||\xi||^\alpha} \text{ for every } x, \xi \in \mathbb{R}^d \text{ and } t > 0.$$
The infinitesimal generator of $Y$ is $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$, which is a prototype of nonlocal operator and can be written in the form

$$\Delta^{\alpha/2} f(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} A(d, -\alpha) \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy, \quad f \in C^2_c(\mathbb{R}^d). \quad (1.2)$$

Here $A(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma((d+\alpha)/2) \Gamma(1-\alpha/2)^{-1}$ is a normalizing constant, with $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$. Using Itô's formula, one can see that the infinitesimal generator of $X^{a,b}$ is

$$\mathcal{L}^{a,b} = \Delta + a^\alpha \Delta^{\alpha/2} + b \cdot \nabla.$$

In this paper we will in fact study heat kernel estimates of $X^{a,b}$ not only for bounded drift function $b$ but also for $b$ in certain Kato class $\mathbb{K}_{d,1}$ which can be unbounded; see Definition 11. When $b$ is in Kato class $\mathbb{K}_{d,1}$, one can not obtain the strong Markov process $X^{a,b}$ from $B + aY$ through Girsanov transform. So we will do it in the other way around. We first construct and establish in Theorem 1.2 the uniqueness of the fundamental solution through Girsanov transform. So we will do it in the other way around. We first construct and establish in Theorem 1.2 the uniqueness of the fundamental solution $p^{a,b}(t,x,y)$ for operator $\mathcal{L}^{a,b}$, and obtain its two-sided sharp estimates in Theorem 1.3. The heat kernel $p^{a,b}(t,x,y)$ determines a conservative Feller process $X^{a,b}$. We then show in Theorem 1.5 that $X^{a,b}$ satisfies (1.1) through establishing the well-posedness of the martingale problem for $(\Delta^{\alpha/2} + b \cdot \nabla, C^\infty_c(\mathbb{R}^d))$ in Theorem 1.3. Moreover, we derive sharp two-sided estimates for $p^{a,b}(t,x,y)$ in such a way that gives the explicit dependence on $a$ so that when $a \to 0$, we can recover the sharp two-sided heat kernel estimates for Brownian motion with drift obtained in Zhang [15, 16].

Brownian motions with drifts, which have $\Delta + b \cdot \nabla$ as their infinitesimal generators, have been studied by many authors under various conditions; see [12, 15, 16] and the references therein, where $b$ belongs to some suitable Kato class. In [3], a fundamental solution to $\Delta^{\alpha/2} + b \cdot \nabla$ on $\mathbb{R}^d$ with $d \geq 2$ is constructed and its two-sided estimates derived. The uniqueness of the fundamental solution, the well-posedness of the martingale problem for $(\Delta^{\alpha/2} + b \cdot \nabla, C^\infty_c(\mathbb{R}^d))$ and its connection to stochastic differential equations are recently settled in [10]. We also mention that relativistic stable processes with drifts have recently been studied in [11].

We now describe the main results of this paper in more details. The Lévy process $Z^a$ has infinitesimal generator $\mathcal{L}^a := \Delta + a^\alpha \Delta^{\alpha/2}$, and Lévy intensity kernel

$$J^a(x, y) = a^\alpha A(d, -\alpha)|x-y|^{-(d+\alpha)}, \quad (1.3)$$

The kernel $J^a(x, y)$ determines a Lévy system for $X^a$, which describes the jumps of the process $X^a$. Let $p^a(t, x, y) = p^a(t|x-y|$ be the transition density function of $Z^a$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Clearly, $p^a(t, z)$ is the smooth function determined by

$$\int_{\mathbb{R}^d} p^a(t, z) e^{iz \cdot \xi} d\xi = e^{-t(\xi^2 + \alpha^2|\xi|^\alpha)}, \quad \xi \in \mathbb{R}^d. \quad (1.4)$$

The following sharp two-sided estimates on $p^a(t, z)$, as stated in [3] Theorem 1.1, follows directly from [3] Theorem 1.4 (see also [14] Theorem 2.13) by scaling. There exist constants $C_i \geq 1$, $i = 1, 2$, so that for all $\alpha \in (0, \infty)$ and $(t, z) \in (0, \infty) \times \mathbb{R}^d$,

$$C_1(t^{-d/2} \wedge (a^\alpha t)^{-d/\alpha}) \wedge (t^{-d/2} e^{-C_2|z|^2/t} + (a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|z|^{d+\alpha}}) \leq p^a(t, z) \leq C_1(t^{-d/2} \wedge (a^\alpha t)^{-d/\alpha}) \wedge (t^{-d/2} e^{-|z|^2/(C_2 t)} + (a^\alpha t)^{-d/\alpha} \wedge \frac{a^\alpha t}{|z|^{d+\alpha}}). \quad (1.5)$$

We can view $\mathcal{L}^{a,b}$ as the perturbation of $\mathcal{L}^a$ by $b \cdot \nabla$. So intuitively, the fundamental solution $p^{a,b}(t, x, y)$ of $\mathcal{L}^{a,b}$ should be related to the fundamental solution $p^a(t, x - y)$ by the following formula

$$p^{a,b}(t, x, y) = p^a(t, x, y) + \int_0^t \int_{\mathbb{R}^d} p^{a,b}(t - s, x, z) b(z) \nabla_z p^a(s, z, y) dz ds \quad (1.6)$$
for \( t > 0 \) and \( x, y \in \mathbb{R}^d \). The above relation is a folklore and is called Duhamel’s formula in literature. Just as in [3, 10], applying (1.6) recursively, it is reasonable to conjecture that 
\[
\sum_{k=0}^{\infty} p_{k}^{a,b}(t,x,y), \text{ if convergent, is a solution of (1.6)},
\]
where \( p^{a,b}_0(t,x,y) = p^a(t,x,y) \) and
\[
p_k^{a,b}(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p_{k-1}^{a,b}(t-s,x,z)\nabla_z p^a(s,z,y)dzds \text{ for } k \geq 1. \tag{1.7}
\]

We now give the definition of Kato class \( K_{d,1} \). For a function \( f = (f_1, \ldots, f_k) : \mathbb{R}^d \to \mathbb{R}^k \)
and \( d \geq 2 \), define
\[
M_f(r) = \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{d-1}}dy \quad \text{for } r > 0.
\]

**Definition 1.1.** A function \( f = (f_1, \ldots, f_k) : \mathbb{R}^d \to \mathbb{R}^k \) is said to be in Kato class \( K_{d,1} \) if
\[
\lim_{r \to 0} M_f(r) = 0 \quad \text{when } d \geq 2, \text{ and bounded if } d = 1.
\]

It is easy to see that any bounded function is in Kato class \( K_{d,1} \) and, for \( d \geq 2 \), \( L^p(\mathbb{R}^d) \subset K_{d,1} \)
for any \( p > d \) by Hölder inequality. On the other hand, any function in \( K_{d,1} \) is locally integrable
on \( \mathbb{R}^d \).

For an integer \( k \geq 1 \), let \( C^k_c(\mathbb{R}^d) \) denote the space of all continuous functions on \( \mathbb{R}^d \) with
compact supports that have continuous derivatives up to and including \( k \)-th order, and set
\[
C^\infty_c(\mathbb{R}^d) = \bigcap_{k=1}^{\infty} C^k_c(\mathbb{R}^d).
\]
Denote by \( C^\infty(\mathbb{R}^d) \) the space of continuous functions on \( \mathbb{R}^d \) vanishing at the infinity, equipped with supremum norm. The followings are the first two main results of this paper.

**Theorem 1.2.** Suppose that \( M > 0 \) and \( b = (b_1, \ldots, b_d) \in K_{d,1} \). For every \( a \in (0, M] \), there is a unique positive jointly continuous function \( p^{a,b}(t,x,y) \) on \( (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) that satisfies (1.6) with \( p^{a,b}(t,x,y) \leq c_1 p^a(t,x,y) \) both on \( (0,t_0] \times \mathbb{R}^d \times \mathbb{R}^d \) for some constants \( c_1, t_0 > 0 \), and that
\[
p^{a,b}(t+s,x,y) = \int_{\mathbb{R}^d} p^{a,b}(t,x,z)p^{a,b}(s,z,y)dz \quad \text{for } t,s > 0, x,y \in \mathbb{R}^d. \tag{1.8}
\]

Moreover, the followings hold.

(i) There is a constant \( t_* = t_*(d,a,M,b) > 0 \), depending on \( b \) only via the rate at which
\( M_b(r) \) goes to zero, such that
\[
p^{a,b}(t,x,y) = \sum_{k=0}^{\infty} p_k^{a,b}(t,x,y) \quad \text{on } (0,t_*) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{1.9}
\]

where \( p_k^{a,b}(t,x,y) \) is defined by (1.7).

(ii) \( p^{a,b}(t,x,y) \) satisfies (1.6) on \( (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \).

(iii) (Conservativeness) \( \int_{\mathbb{R}^d} p^{a,b}(t,x,y)dy = 1 \) for every \( t > 0 \) and \( x \in \mathbb{R}^d \).

(iv) for every \( f \in C^\infty_c(\mathbb{R}^d) \) and \( g \in C^\infty(\mathbb{R}^d) \),
\[
\lim_{t \to 0} \int_{\mathbb{R}^d} \frac{P^a_b f(x) - f(x)}{t} g(x)dx = \int_{\mathbb{R}^d} C^{ab} f(x)g(x)dx, \tag{1.10}
\]
where \( P^a_b f(x) = \int_{\mathbb{R}^d} p^{a,b}(t,x,y)f(y)dy \).
Here and after, the meaning of the phrase "depending on \( b \) only via the rate at which \( M_b(r) \) goes to zero” is that the statement is true for any \( \mathbb{R}^d \)-valued function \( b \) on \( \mathbb{R}^d \) with \( M_b(r) \leq M_b(r) \) for all \( r > 0 \). In this paper, we use := as a way of definition. For \( a, b \in \mathbb{R} \), \( a \wedge b := \min\{a, b\} \) and \( a \vee b := \max\{a, b\} \). For constants \( a, \beta > 0 \), we define

\[
q_{a,\beta}^d(t, z) = t^{-d/2} \exp \left( -\frac{\beta |z|^2}{t} \right) + t^{-d/2} \wedge \frac{a^t}{|z|^{d+\alpha}} \quad \text{for } t > 0, \ z \in \mathbb{R}^d. \quad (1.11)
\]

**Theorem 1.3.** For every \( M > 0 \) and \( T > 0 \), there are constants \( C_i = C_i(d, \alpha, M), i = 4, 6 \) and \( C_j = C_j(d, \alpha, M, T, b), j = 3, 5 \) depending on \( b \) only via the rate at which \( M_b(r) \) goes to zero, such that for all \( a \in (0, M] \) and \( (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
C_3 q_{a,\beta}^d(t, x - y) \leq p_{a,b}^d(t, x, y) \leq C_5 q_{a,\beta}^d(t, x - y).
\]

The heat kernel upper bound estimate of \( p_{a,b}^d(t, x, y) \) is obtained by estimating each \( p_{a,b}^d(t, x, y) \) in \( (1.9) \). It relies on a key estimate obtained in Theorem 1.2, which can be regarded as an analogy of the so called 3P estimate in [16, Lemma 3.1] and [3, Lemma 13]. However, unlike the case in [16] where there is only Gaussian term coming from Brownian motion and the case in [3] where there is only polynomial term coming from symmetric stable process, there are many new difficulties to overcome as we have to deal with a mixture of them. It seems to be difficult to establish the positivity of \( p_{a,b}^d(t, x, y) \) directly from the estimates of \( p_{a,b}^d(t, x, y) \) as did in [3] for the symmetric stable process case. Following [3], we derive the positivity of \( p_{a,b}^d(t, x, y) \) by using the Hille-Yosida-Ray theorem when \( b \) is bounded and continuous. For general \( b \) in Kato class \( \mathbb{K}_{d,1} \), we approximate \( b \) by a sequence of smooth \( b_n \). For the lower bound of \( p_{a,b}^d(t, x, y) \) in Theorem 1.3 we identify and use the \( \text{Lévy} \) system of the Feller process \( \{X_t^{a,b}, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d \} \) associated with \( \{F_t^{a,b}, t \geq 0\} \) to get the polynomial part (see Lemma 5.6), and use a chaining argument to get the Gaussian part (see Lemma 5.7).

Let \( \mathbb{D}([0, \infty) : \mathbb{R}^d) \) be the space of right continuous \( \mathbb{R}^d \)-valued functions on \( [0, \infty) \) having left limits equipped with Skorokhod topology, and let \( X_t \) be the coordinate map on \( \mathbb{D}([0, \infty) : \mathbb{R}^d) \). A probability measure \( \mathbb{Q} \) on \( \mathbb{D}([0, \infty) : \mathbb{R}^d) \) is said to be a solution to the martingale problem for \( (\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d)) \) with initial value \( x \in \mathbb{R}^d \) if \( \mathbb{Q}(X_0 = x) = 1 \) and for every \( f \in C_c^\infty(\mathbb{R}^d) \) and \( t > 0 \), \( \int_0^t |\mathcal{L}^{a,b} f(X_s)| ds < \infty \) \( \mathbb{Q} \)-a.s. and

\[
M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}^{a,b} f(X_s) ds
\]

is a \( \mathbb{Q} \)-martingale. The martingale problem for \( (\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d)) \) with initial value \( x \in \mathbb{R}^d \) is said to be well-posed if it has a unique solution.

**Theorem 1.4.** The martingale problem for \( (\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d)) \) is well-posed for every initial value \( x \in \mathbb{R}^d \). These martingale problem solutions \( \{\mathbb{P}^x, x \in \mathbb{R}^d \} \) form a strong Markov process \( X \), which has \( p_{a,b}^d(t, x, y) \) of Theorem 1.3 as its transition density function with respect to the Lebesgue measure on \( \mathbb{R}^d \).

We now connect the strong Markov process in Theorem 1.4 to solution of SDE (1.1).

**Theorem 1.5.** For each \( x \in \mathbb{R}^d \), SDE (1.1) has a unique weak solution with initial value \( x \). Moreover, weak solutions with different starting points can be constructed on \( \mathbb{D}([0, \infty) : \mathbb{R}^d) \), and the process \( Z^a \) in (1.1) can be chosen in such a way that it is the same for all starting point \( x \in \mathbb{R}^d \). The law of the weak solution to (1.1) is the unique solution to the martingale problem for \( (\mathcal{L}^{a,b}, C_c^\infty(\mathbb{R}^d)) \).

The rest of this paper is organized as follows. In Section 2, we recall some properties of \( p^a(t, x, y) \) and derive its gradient estimates, as well as properties of functions in Kato class \( \mathbb{K}_{d,1} \).
In Section 3, we construct \( p^{a,b}(t, x, y) \) using the series of \( p^{a,b}_k(t, x, y) \) and prove Theorem 1.2 through a series of lemmas except the positivity of \( p^{a,b}(t, x, y) \). In addition, we derive the upper bound of \( |p^{a,b}(t, x, y)| \). The positivity of \( \gamma^{a,b}(t, x, y) \) is shown in Section 4, where we use the fact that \( \{ P^{a,b}_t, t \geq 0 \} \) is Feller semigroup, that is, a strongly continuous semigroup in \( C_{\infty}(\mathbb{R}^d) \). In Section 5, we determine the Lévy system of the Feller process \( X^{a,b} \) associated with the Feller semigroup \( \{ P^{a,b}_t, t \geq 0 \} \). We then use it to derive the lower bound estimate of \( p^{a,b}(t, x, y) \). In Section 6, we prove Theorem 1.3 and Theorem 1.5.

For convenience, in the rest of this paper, we assume \( d \geq 2 \). When \( d = 1 \), it can be treated in a similar but simpler way as the drift \( b \) would be bounded. Throughout this paper, unless stated otherwise, we use \( C_1, C_2, \ldots \), to denote positive constants whose value are fixed throughout the paper, while using \( c_1, c_2, \ldots \), to denote positive constants whose exact value are unimportant and whose value can change from one appearance to another. We use notation \( c = c(d, \alpha, \cdots) \) to indicate that this constant depends only on \( d, \alpha, \cdots \). For two non-negative functions \( f, g \), the notation \( f \lesssim g \) means that \( f \leq cg \) on their common domains of definition while \( f \asymp g \) means that \( c^{-1} g \leq f \leq c g \). We also write mere \( \lesssim \) and \( \asymp \) if \( c \) is unimportant or understood. For reader's convenience, we summarize the notation of functions that will appear many times throughout this paper. For \( t > 0 \) and \( x, y \in \mathbb{R}^d \),

\[
p^a(t, x, y) = p^a(t, x - y) : \text{ the transition density function of } B + a Y \]

\[
g_{d, \beta}(t, x, y) = g_{d, \beta}(t, x - y) := t^{-d/2} \exp \left( -\frac{\beta|x - y|^2}{t} \right), \quad (1.12)
\]

\[
g_d(t, x, y) = g_d(t, x - y) := (4\pi)^{-d/2} g_{d,1/4}(t, x - y),
\]

\[
q^{a,\beta}_{d,\theta}(t, x, y) = q^{a,\beta}_{d,\theta}(t, x - y) := g_{d,\beta}(t, x - y) + t^{-d/2} \wedge \frac{a^\alpha t}{|x - y|^{d+\alpha}}. \quad (1.13)
\]

## 2 Preliminaries

The following is a direct consequence of (1.5); see [4 Corollary 1.2].

**Theorem 2.1.** For any \( M > 0 \) and \( T > 0 \), there exist constants \( C_i, i = 8, 10 \) and \( C_j = C_j(d, \alpha, M, T), j = 7, 9 \) such that for all \( a \in (0, M] \) and \( (t, x) \in (0, T] \times \mathbb{R}^d \),

\[
\mathcal{C}_M q^{a}_{d,\alpha,C_8}(t, x) \leq p^a(t, x) \leq \mathcal{C}_9 q^{a}_{d,\alpha,C_9}(t, x).
\]

It is easy to see that for any \( \theta > 0 \), there is a positive constant \( c_1 = c_1(d, \beta, \theta) \) such that

\[
g_{d,\beta}(t, x) \leq t^{-d/2} + \frac{c_1 t^\theta}{|x|^{d+2\theta}}, \quad t > 0 \text{ and } x \in \mathbb{R}^d,
\]

(2.1) which will be frequently used in the rest of this paper.

Recall the definition of \( q^{a}_{d,\beta}(t, x) \) in (1.11). There is a constant \( C_11 = C_{11}(\alpha, M, T, \beta) \) such that for all \( a \in (0, M] \) and all \( (t, z) \in (0, T] \times \mathbb{R}^d \),

\[
q^{a}_{d,\beta}(t, z) \leq g_{d,\beta}(t, z) + \frac{a^\alpha t}{|z|^{d+\alpha}} 1_{\{|z|^2 \geq t\}}.
\]

(2.2)

Indeed, \( t^{-d/2} + \frac{a^\alpha t}{|z|^{d+\alpha}} \leq t^{-d/2} \leq c^\beta g_{d,\beta}(t, z) \) when \( |z|^2 < t \). Thus

\[
q^{a}_{d,\beta}(t, z) \lesssim g_{d,\beta}(t, z) + \frac{a^\alpha t}{|z|^{d+\alpha}} 1_{\{|z|^2 \geq t\}} \quad \text{for } a, t > 0 \text{ and } z \in \mathbb{R}^d.
\]

(2.3)

On the other hand, for \( a \in (0, M] \) and \( t \in (0, T] \),

\[
\frac{a^\alpha t}{|z|^{d+\alpha}} \leq M^\alpha t^{-d/2+1-\alpha/2} \leq M^\alpha T^{1-\alpha/2} t^{-d/2} \quad \text{if } |z|^2 \geq t,
\]

(2.4)
and so
\[ gd_\beta(t, z) + \frac{a^\alpha t}{|z|^{d+\alpha}} 1_{\{|z|^2 \geq t\}}^{M^\alpha T^{1-\alpha/2} v^1} \lesssim q^\alpha_{d, \beta}(t, z). \]  
(2.4)

The claim (2.2) now follows from (2.3) and (2.4) with \( C \) and so 

When there is no danger of confusion, for \( x \in \mathbb{R}^d \) and integer \( k \geq 1 \), for simplicity, we write \( q_{d+k, \beta}(t, x) \) for \( q_{d+k, \beta}(t, \tilde{x}) \), where \( \tilde{x} := (x, 0, \ldots, 0) \in \mathbb{R}^{d+k} \). Same convention will apply to function \( g_{d, \beta}(t, x) \).

The following theorem gives the two-sided estimate of \(|\nabla_x p^\alpha(t, x)|\). In this paper, only its upper bound will be used.

**Theorem 2.2.** For any \( M > 0 \) and \( T > 0 \), there is a positive constant \( C_{12} = C_{12}(d, \alpha, M, T) \) such that for all \( a \in (0, M] \) and \( (t, x) \in (0, T] \times \mathbb{R}^d \),

\[ 2\pi C_{7} q^\alpha_{d+2, C_{8}}(t, x)|x| \leq |\nabla_x p^\alpha(t, x)| \leq C_{12} q^\alpha_{d+1, 3C_{10}/4}(t, x). \]

**Proof.** It is well-known that, for each \( t > 0 \), \( x \mapsto p^\alpha(t, x) \) attains maximum at \( x = 0 \) so we have \( \nabla p^\alpha(t, 0) = 0 \). So it suffices to consider \( x \in \mathbb{R}^d \setminus \{0\} \). Set \( g_d(t, z) := g_{d, 1/4}(t, z) = (4\pi t)^{-d/2} e^{-|z|^2/(4t)} \), which is the transition density function of Brownian motion \( B \). Let \( S_t \) be the \( \alpha/2 \)-stable subordinator at time \( t \), independent of \( B \), and \( \eta^\alpha(u) \) be the density function of \( \alpha^2 S_t \). The Lévy process \( Z^\alpha \) can be realized as a subordination of Brownian motion \( B \); that is, \( \{Z^\alpha_t; t \geq 0\} \) has the same distribution as \( \{B_t/S_t; t \geq 0\} \). Thus

\[ p^\alpha(t, x) = \int_t^{\infty} g_d(u, x) \mathbb{P}(t + \alpha^2 S_t \in du) = \int_t^{\infty} g_d(u, x) \eta^\alpha(u - t) du, \]

and so

\[ \nabla_x p^\alpha(t, x) = \nabla_x \int_t^{\infty} g_d(u, x) \eta^\alpha(u - t) du. \]

Let \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \), where 1 is on \( j \)th place. Let \( x \in \mathbb{R}^d \setminus \{0\} \) and set \( s \in (-|x|/2, |x|/2) \). By the mean-value theorem, there exists \( \xi \in (-|s|, |s|) \) such that

\[ \left| \frac{g_d(u, x + se_j) - g_d(u, x)}{s} \right| = \left| \frac{\partial}{\partial x_j} g_d(u, x + \xi e_j) \right| = \left| \frac{x_j + \xi}{2u} g_d(u, x + \xi e_j) \right| \leq \frac{|x_j| g_d(u, x/2)}{u} \leq c(d)|x|^{-d-1}, \]

where \( c(d) \) is a positive constant depending only on \( d \). Since \( \int_t^{\infty} c(d)|x|^{-d-1} \eta^\alpha(u - t) du < \infty \), we have by the dominated convergence theorem

\[ \nabla_x p^\alpha(t, x) = \int_t^{\infty} \nabla_x g_d(u, x) \eta^\alpha(u - t) du = \int_t^{\infty} \frac{-x g_d(u, x)}{2u} \eta^\alpha(u - t) du = -2\pi x P^\alpha(d+2)(t, \tilde{x}), \]

where \( \tilde{x} := (x, 0) \in \mathbb{R}^{d+2} \) and \( P^\alpha(d+2)(t, \tilde{x}) \) is the transition density function of \( Z^\alpha \) in dimension \( d + 2 \). Thus, by Theorem 2.1 we have

\[ 2\pi C_{7} q^\alpha_{d+2, C_{8}}(t, x)|x| \leq |\nabla_x p^\alpha(t, x)| \leq 2\pi C_{9} q^\alpha_{d+2, 3C_{10}/4}(t, x)|x|. \]

(2.6)

Note that for all \( t > 0 \) and \( x \in \mathbb{R}^d \),

\[ t^{-(d+2)/2} \exp \left( -\frac{C_{10} |x|^2}{t} \right) \leq t^{-(d+1)/2} \exp \left( -\frac{3C_{10} |x|^2}{4t} \right) \cdot \frac{|x|}{t^{1/2}} \exp \left( -\frac{C_{10} |x|^2}{4t} \right) \]

\[ t^{-(d+2)/2} \exp \left( -\frac{C_{10} |x|^2}{t} \right) \leq t^{-(d+1)/2} \exp \left( -\frac{3C_{10} |x|^2}{4t} \right) \cdot \frac{|x|}{t^{1/2}} \exp \left( -\frac{C_{10} |x|^2}{4t} \right) \]
\[ \leq \sqrt{\frac{2}{C_{10}^2}} t^{-(d+1)/2} \exp \left( -\frac{3C_{10} |x|^2}{4} \right). \]

This together with (2.2) and (2.6) proves the theorem with \( C_{12} := 2\pi C_9 C_{11} \left( \sqrt{2/(C_{10} e)} + 1 \right). \]

\[ \square \]

For \( \beta > \frac{1}{2} \) and a function \( f \) on \( \mathbb{R}^d \), define for \( r > 0 \) and \( x \in \mathbb{R}^d \),

\[ H^\beta(r, x) = \frac{1}{|x|^{d-1}} \wedge \frac{r^\beta}{|x|^{d-1+2\beta}} \quad \text{and} \quad H^\beta_f(r, x) = \int_{\mathbb{R}^d} |f(y)| H^\beta(x - y) dy. \]

**Lemma 2.3.** Assume \( \beta > \frac{1}{2} \). There is a constant \( C_{13} = C_{13}(d, \beta) \) so that

\[ M_f(\sqrt{r}) \leq H^\beta_f(r, x) \leq C_{13} M_f(\sqrt{r}), \]  

for every \( r > 0, x \in \mathbb{R}^d \) and for every \( f \) on \( \mathbb{R}^d \). Consequently, \( f \in K_{d, 1} \) if and only if

\[ \lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} H^\beta_f(r, x) = 0. \]

The lower bound in (2.7) is trivial. The proof of the upper bound in (2.7) is almost the same as that for [13, Lemma 11 and Corollary 12] except with 2 in place of \( \alpha \) there. So we omit its details.

Let

\[ N^\beta(r, x) = \int_0^r g_{d+1, \beta}(s, x, y) ds = \int_0^r s^{-(d+1)/2} \exp \left( -\frac{\beta |x|^2}{s} \right) ds, \quad r > 0, x \in \mathbb{R}^d. \]

**Lemma 2.4.** \( f \in K_{d, 1} \) if and only if

\[ \lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |f(y)| N^\beta(r, x - y) dy = 0 \text{ for all } \beta > 0. \]  

**Proof.** Condition (2.8) is introduced in [13]. Its equivalence to the \( K_{d, 1} \) condition is proved in [13, Proposition 2.3]. For reader’s convenience, we give a short proof here.

By a change of variable \( t = \beta |x - y|^2/s \), we have

\[ N^\beta(r, x) = \frac{1}{\beta^{(d-1)/2} |x|^{d-1}} \int_{\beta |x|^2/r}^\infty t^{(d-3)/2} e^{-t} dt. \]  

Thus

\[ c_1(d, \beta) \frac{1}{|x|^{d-1}} 1_{\{|x| \leq \sqrt{r} \}} \leq N^\beta(r, x) \leq c_2(d, \beta) H^1(r, x) \]  

The equivalence now follows from Lemma 2.3 \( \square \)

### 3 Construction and upper bound estimates

By [16, Lemma 3.1] and its proof, we have the following lemma. Recall that \( g_{d, \beta}(t, x - y) \) is defined by (1.12), and define \( H^\beta(r, x, y) = H^\beta(r, x - y) \).

**Lemma 3.1.** For any \( 0 < \beta_1 < \beta_2 < \infty \), there exist constants \( C_g = C_g(d, \beta_1/\beta_2) \) and \( C_\beta = \min \{ \beta_2 - \beta_1, \beta_1/2 \} \) such that for all \( t > 0 \) and \( x, y, z \in \mathbb{R}^d \),

\[ \int_0^t g_{d, \beta_1}(t - s, x, z) s^{-1/2} g_{d, \beta_2}(s, z, y) ds \leq C_g(N_{C_\beta}(t, x, z) + N_{C_\beta}(t, z, y)) g_{d, \beta_1}(t, x, y). \]
In the rest of this paper, we assume \( b \in \mathbb{K}_{d,1} \) and let \( \gamma = (1 + \alpha \wedge 1)/2 \). The following lemma plays an important role in this paper and it is an analogy of \([3 \text{ Lemma 13}] \) or \([16 \text{ Lemma 3.1}] \).

**Lemma 3.2.** Suppose \( M > 0 \) and \( T > 0 \). For any \( 0 < \beta_1 < \beta_2 < \infty \), there is a positive constant \( C_{14} = C_{14}(d, \alpha, M, T, \beta_1, \beta_2) \) such that for all \( a \in (0, M] \) and \((t, x, y, z) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
\int_0^t q_{d,\beta_1}(t-s,x,z) q_{d+1,\beta_2}(s,z,y) ds \leq C_{14}(H^\gamma(t,x,z) + H^\gamma(t,y,z)) q_{d,\beta_1}(t, x, y). \tag{3.1}
\]

Consequently, there is a positive constant \( C_{15} = C_{15}(d, \alpha, M, T) \) such that for all \( a \in (0, M] \) and \((t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
\int_0^t \int_{\mathbb{R}^d} q_{d,\beta_1}(t-s,x,z) |b(z)| q_{d+1,\beta_2}(s,z,y) dz ds \leq C_{15} M_b(\sqrt{t}) q_{d,\beta_1}(t, x, y), \tag{3.2}
\]

**Proof.** We first verify \((3.1)\). By \((2.2)\), for all \( (c_1 := 1) \),

\[
\int_{\mathbb{R}^d} q_{d,\beta_1}(t-s,x,z) q_{d+1,\beta_2}(s,z,y) ds \\
\leq \int_0^t \int_{\mathbb{R}^d} q_{d,\beta_1}(t-s,x,z) + \frac{a^\alpha(t-s)}{|x-z|^{d+\alpha}} 1_{\{|x-z|^2 \geq t-s\}} \left( \frac{g_{d,\beta_2}(s,z,y)}{s^{1/2}} + \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} 1_{\{|z-y|^2 \geq s\}} \right) ds \\
= \int_0^t g_{d,\beta_1}(t-s,x,z) g_{d+1,\beta_2}(s,z,y) \frac{s^{1/2}}{s^{1/2}} ds + \int_0^t g_{d,\beta_1}(t-s,x,z) \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} 1_{\{|z-y|^2 \geq s\}} ds \\
\quad + \int_0^t \frac{a^\alpha(t-s)}{|x-z|^{d+\alpha}} 1_{\{|x-z|^2 \geq t-s\}} g_{d+1,\beta_2}(s,z,y) \frac{s^{1/2}}{s^{1/2}} ds \\
\quad + \int_0^t \frac{a^\alpha(t-s)}{|x-z|^{d+\alpha}} 1_{\{|x-z|^2 \geq t-s\}} \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} 1_{\{|z-y|^2 \geq s\}} ds \\
=: I_1 + I_2 + I_3 + I_4.
\]

We will treat each term separately. First, by Lemma \((3.1)\), there are constants \( c_2 = c_2(d, \beta_1/\beta_2) \) and \( c_3 = c_3(\beta_2 - \beta_1, \beta_1/2) \) such that \( I_1 \leq c_2 (N^{c_3}(t, x, z) + N^{c_3}(t, y, z)) g_{d,\beta_1}(t, x, y) \), while

\[
I_2 = \left( \int_0^{t/2} + \int_0^{t/2} \right) g_{d,\beta_1}(t-s,x,z) \frac{a^\alpha s}{|z-y|^{d+1+\alpha}} 1_{\{|z-y|^2 \geq s\}} ds \\
\leq 2^{d+\alpha} M^\alpha \int_0^{t/2} \int_0^{t/2} \frac{s}{|z-y|^{d+1+\alpha}} ds \\
\quad + t^{-d/2} \frac{1}{|z-y|^{d-3+\alpha}} \int_0^{t/2} s^{-d/2} \exp \left( -\frac{\beta_1 |x-z|^2}{s} \right) \frac{t^{1-\alpha/2}}{|z-y|} ds \\
\leq 2^{d/2} \left( \frac{1}{|z-y|^{d-3+\alpha}} \wedge \frac{t^2}{|z-y|^{d+1+\alpha}} \right) \\
+ t^{-d/2+1-\alpha/2} \int_0^{t/2} s^{-d+1/2} \exp \left( -\frac{\beta_1 |x-z|^2}{s} \right) ds \\
\leq t^{-d/2} \left( N^{\beta_1}(t, x, z) + H^{(1+\alpha)/2}(t, z, y) \right). \tag{3.3}
\]

On the other hand, if \( |x-z| \geq |z-y| \), then \( 2|x-z| \geq |x-z| + |z-y| \geq |x-y| \), and so

\[
I_2 \lesssim \int_0^t |z-y|^2 \frac{(t-s)^{\alpha/2}}{|x-z|^{d+\alpha} |z-y|^{d+1+\alpha}} ds,
\]

On the other hand, if \( |x-z| \geq |z-y| \), then \( 2|x-z| \geq |x-z| + |z-y| \geq |x-y| \).
If \(|x - z| < |z - y|\), then \(2|z - y| \geq |x - y|\) and
\[
I_2 \lesssim \frac{a^t}{|x - y|^{d+\alpha}} \int_0^t g_{d,\beta_1}(t - s, x, z)\sqrt{s}ds
= \frac{a^t}{|x - y|^{d+\alpha}} \int_0^t \sqrt{\frac{s}{t}} \sqrt{t - s} (t - s)^{-(d+1)/2} \exp\left(-\frac{\beta_1}{t} |x - z|^2\right)ds
\leq \frac{a^t}{|x - y|^{d+\alpha}} N_{\beta_1}(t, x, z).
\]

Thus we have by (3.3)–(3.5)
\[
I_2 \overset{c_6(d,\alpha,\beta_1,M,T)}{\lesssim} \left(t^{-d/2} \wedge \frac{a^t}{|x - y|^{d+\alpha}}\right) \left(N_{\beta_1}(t, x, z) + H^{(1+\alpha)/2}(t, z, y)\right).
\]

Similarly, we have
\[
I_3 \overset{c_7(d,\alpha,\beta_2,M,T)}{\lesssim} \left(t^{-d/2} \wedge \frac{a^t}{|x - y|^{d+\alpha}}\right) \left(H^{1+\alpha/2}(t, x, z) + N_{\beta_2}(t, z, y)\right).
\]

It remains to estimate \(I_4\). If \(|x - z| \geq t - s\) and \(|z - y| \geq s\), then \(|x - z| \vee |z - y| \geq \sqrt{t/2}\).

Since \(|x - z| \vee |z - y| \geq |x - y|/2\), we have \(|x - z| \vee |z - y| \geq \frac{1}{2}(\sqrt{t} \vee |x - y|)\). Therefore
\[
\frac{t - s}{|x - z|^{d+\alpha}} \frac{s}{|z - y|^{d+\alpha}} = \frac{1}{(|x - z| \vee |z - y|)^{d+\alpha}} \frac{1}{(|x - z| \wedge |z - y|)^{d+\alpha}} \frac{(t - s)s}{|z - y|^2}
\leq \frac{1}{(t - s)s}
\leq \frac{t}{|x - y|^{d+\alpha}} \left(t - s\right) \left(\frac{1}{|x - z|^{d+\alpha} + \frac{1}{|z - y|^{d+\alpha}}}\right)
\]

Thus,
\[
I_4 \overset{c_8(d,\alpha,T)}{\lesssim} \frac{a^{2t}}{1_{\{|x - z|^2 \geq t - s\}} 1_{\{|z - y|^2 \geq s\}}} \frac{t - s}{|x - z|^{d+\alpha}} \frac{s}{|z - y|^{d+\alpha}} ds
\]
\[
\overset{M^\alpha(M\vee 1)^\alpha}{\lesssim} \frac{a^{2t}}{1_{\{|x - z|^2 \geq t - s\}} 1_{\{|z - y|^2 \geq s\}}} \frac{t - s}{|x - z|^{d+\alpha}} \frac{s}{|z - y|^{d+\alpha}} ds
\]
Notice that
\[
\int_0^t \frac{t-s}{|x-z|^{d+1+\alpha} + s} 1_{|x-z|^2 \geq t-s} ds = \frac{1}{|x-z|^{d+1+\alpha}} \int_0^{t \wedge |x-z|^2} r \ dr \\
\leq 2^{-1} t^{1-\alpha/2} \left( \frac{1}{|x-z|^{d-1}} \wedge \frac{t^{1+\alpha/2}}{|x-z|^{d+1+\alpha}} \right) \\
\leq 2^{-1} T^{1-\alpha/2} H^{1+\alpha/2}(t, z, x, y).
\] (3.7)

Similarly,
\[
\int_0^t \frac{s}{|x-y|^{d+1+\alpha} + s} 1_{|x-y|^2 \geq s} ds \leq 2^{-1} T^{1-\alpha/2} H^{1+\alpha/2}(t, z, y).
\] (3.8)

We have by (3.6), (3.7) and (3.8),
\[
I_4 \leq C_9(d, \alpha, M, T) \left( t^{-d/2} \wedge \frac{\alpha \tau}{|x-y|^{d+\alpha}} \right) \left( H^{1+\alpha/2}(t, x, z) + H^{1+\alpha/2}(t, z, y) \right).
\]

Hence by (2.10) and the fact that $\beta \mapsto H^{\beta}(t, x, y)$ is decreasing, we have
\[
I_{C_{14}(d, \alpha, \beta_1, \beta_2, M, T)} \lesssim I_4 \lesssim 2^{-1} \left( H^{\gamma}(t, x, z) + H^{\gamma}(t, z, y) \right) \left( g_{d, \beta_1}(t, x, y) + t^{-d/2} \wedge \frac{\alpha \tau}{|x-y|^{d+\alpha}} \right).
\]

This completes the proof of (3.11) by $|b(z)|$, we get
\[
\int_0^t \int_{\mathbb{R}^d} q_{d, \beta_1}^a(t-s, x, z) |b(z)| q_{d+1, \beta_2}^a(s, z, y) \ dz \ ds \\
\leq 2 \int_{\mathbb{R}^d} q_{d, \beta_1}^a(t, x, y) \sup_{x \in \mathbb{R}^d} H^{\gamma}_b(t, x) \lesssim M_b(\sqrt{t}) q_{d, \beta_1}^a(t, x, y).
\]

This proves the lemma with $C_{15} = 2C_{14}C_{13}$.

For $t > 0$ and $x, y \in \mathbb{R}^d$, we define
\[
|p_k^{a,b}(t, x, y)| = p^a(t, x, y), \\
|p_k^{a,b}(t, x, y)| = \int_0^t \int_{\mathbb{R}^d} |p_{k-1}^{a,b}(t-s, x, z)| |b(z)| \ |\nabla_z p^a(s, z, y)| \ dz \ ds, \ \text{for} \ k \geq 1.
\]

For every $M > 0$ and $T > 0$, we can verify by induction that
\[
|p_k^{a,b}(t, x, y)| \leq C_9(C_{12} M_b(\sqrt{T}))^{k} q_{d,C_{10}/2}^a(t, x, y), \ \ a \in (0, M], (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (3.9)
\]

Indeed, (3.9) holds for $k = 0$. Assume (3.9) holds for $k$. Then by assumption and (3.2),
\[
|p_{k+1}^{a,b}(t, x, y)| \leq C_9(C_{12} C_{15} M_b(\sqrt{T}))^{k} C_{12} \int_0^t \int_{\mathbb{R}^d} q_{d,C_{10}/2}^a(t-s, x, z) |b(z)| q_{d+1,3C_{10}/4}^a(s, z, y) \ dz \ ds \\
\leq C_9(C_{12} C_{15} M_b(\sqrt{T}))^{k} q_{d,C_{10}/2}^a(t, x, y) \\
\leq C_9(C_{12} C_{15} M_b(\sqrt{T}))^{k+1} q_{d,C_{10}/2}^a(t, x, y).
\]

Thus for every $k \geq 1$, $t \in (0, T]$, $p_k^{a,b}(t, x, y)$ of (1.7) is well defined and has bound
\[
|p_k^{a,b}(t, x, y)| \leq |p_k^{a,b}(t, x, y)| \leq C_9(C_{12} C_{15} M_b(\sqrt{T}))^{k} q_{d,C_{10}/2}^a(t, x, y) < \infty. \quad (3.10)
\]
Lemma 3.3. Suppose $M > 0$. For every $a \in (0, M]$ and $k \geq 0$, $p_k^{a,b}(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. We will use induction in $k$ to prove this lemma. Obviously, $p_0^{a,b}(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Assume $p_k^{a,b}(t, x, y)$ is jointly continuous. By (2.11) with $\theta = 1$,

$$q_d^{a, C_{10}/2}(t, x, y) \leq t^{-d/2} \sqrt{\frac{c_1 t}{|x-y|^{d+\alpha}}} + t^{-d/2} \sqrt{\frac{a^d t}{|x-y|^{d+\alpha}}}, \quad t > 0, x, y \in \mathbb{R}^d,$$

(3.11)

for some positive constant $c_1$ depending only on $d$. Suppose $T > 1$ and $0 < \varepsilon < 1/(2T)$. For $t \in [T^{-1}, T]$ and $s \in [\varepsilon, t - \varepsilon]$, we have by (3.10) and (3.11) that there is a constant $c_2 = c_2(d, \alpha, M, T, b)$ such that

$$|p_k^{a,b}(t - s, x, z)| \leq C_9(C_{12}C_{15}M_b(\sqrt{t}))^k q_d^{a, C_{10}/2}(t - s, x, z) \leq 2c_2(t - s)^{-d/2} \leq 2c_2\varepsilon^{-d/2}, \quad (3.12)$$

$$|\nabla z p^a(s, z, y)| \leq C_{12}q_d^{a, 12C_{10}/4}(s, z, y) \leq 2C_{12}s^{-(d+1)/2} \leq 2C_{12}\varepsilon^{-(d+1)/2}, \quad (3.13)$$

$$|p_k^{a,b}(t - s, x, z)| \leq 2c_2 \frac{1}{|x-z|^{d+\alpha}}, \quad \text{if } |x-z| \geq 1.$$

Then for $R \geq 1$,

$$\sup_{x \in \mathbb{R}^d} \int_{t-\varepsilon}^{t} \int_{|x-z| \geq R} |p_k^{a,b}(t - s, x, z)||b(z)||\nabla z p^a(s, z, y)|dzds \leq \sup_{x \in \mathbb{R}^d} 4c_2C_{12}\varepsilon^{-(d+1)/2}T \int_{|x-z| \geq R} \frac{|b(z)|}{|x-z|^{d+\alpha}}dz,$$

which goes to zero as $R \to \infty$. Moreover, for any $r > 0$, by (3.12) and (3.13), the local integrability of $b$ and the dominated convergence theorem, we have

$$x \mapsto \int_{t-\varepsilon}^{t} \int_{|x-z| < R} p_k^{a,b}(t - s, x, z)b(z)\nabla z p^a(s, z, y)dzds$$

is continuous on $B(0, r)$. Thus, we can conclude that

$$(t, x, y) \mapsto \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^d} p_k^{a,b}(t - s, x, z)b(z)\nabla z p^a(s, z, y)dzds$$

(3.14)

is jointly continuous on $[T^{-1}, T] \times B(0, r) \times B(0, r)$. Since $r$ is arbitrary, (3.14) is jointly continuous on $[T^{-1}, T] \times \mathbb{R}^d \times \mathbb{R}^d$. On the other hand, by (3.13) and (3.10),

$$\sup_{t \in [1/T, T]} \sup_{x, y \in \mathbb{R}^d} t \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^d} |p_k^{a,b}(t - s, x, z)||b(z)||\nabla z p^a(s, z, y)|dzds \leq \sup_{t \in [1/T, T]} \sup_{x \in \mathbb{R}^d} \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^d} |p_k^{a,b}(t - s, x, z)||b(z)||\nabla z p^a(s, z, y)|dzds \leq 2C_{12}(2T)^{(d+1)/2} \sup_{x \in \mathbb{R}^d} \left( \int_{\mathbb{R}^d} |b(z)| q_d^{a, C_{10}/2}(s, z, x, x) s^{d/2}dz + \int_{|x-z| \geq s} \frac{|b(z)|}{|x-z|^{d+\alpha}}dz \right) \leq 2C_{12}(2T)^{(d+1)/2} \left( \sqrt{\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |b(z)| N^{C_{10}/2}(\varepsilon, x, z)dz} + \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{a^\alpha x^2}{|x-z|^{d+\alpha}}dz \right) \leq 2C_{12}(2T)^{(d+1)/2} \left( c_3C_{13}\varepsilon M_b(\sqrt{\varepsilon}) + a^\alpha \varepsilon^{(3-\alpha)/2}C_{13}M_b(\sqrt{\varepsilon}) \right),$$

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which goes to zero as \( \varepsilon \to 0 \). Similarly, by (3.12),

\[
\sup_{t \in [1/T, T]} \sup_{x,y \in \mathbb{R}^d} \int_0^\varepsilon \int_{\mathbb{R}^d} |p_k^{a,b}(t - s, x, z)||b(z)||\nabla_z p^a(s, z, y)|dzds \\
\leq 2c_2(2T)^{d/2} \left( c_3 C_{13} M_b(\sqrt{\varepsilon}) + a^\alpha \varepsilon^{1 - \alpha/2} C_{13} M_b(\sqrt{\varepsilon}) \right),
\]

which goes to zero as \( \varepsilon \to 0 \). Therefore,

\[
p_k^{a,b}(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_k^{a,b}(t - s, x, z)b(z)\nabla_z p^a(s, z, y)dzds
\]

is jointly continuous on \([T^{-1}, T] \times \mathbb{R}^d \times \mathbb{R}^d\) for every \( T > 0 \). This completes the proof. \( \square \)

**Lemma 3.4.** Suppose \( M > 0 \). There are two positive constants \( t_*(d, \alpha, M, b) > 0 \) depending on \( b \) only via the rate at which \( M_b(\alpha) \) goes to zero and \( C_{16} = C_{16}(d, \alpha, M) > 0 \) such that for all \( t \in (0, t_*) \) and \( x, y \in \mathbb{R}^d \),

\[
\sum_{k=0}^\infty |p_k^{a,b}(t, x, y)| \leq \sum_{k=0}^\infty |p_k^{a,b}(t, x, y)| \leq C_{16} q_{d,C_{10}/2}^a(t, x, y).
\]

\[\text{(3.15)}\]

Moreover, for all \( |x - y|^2 < t \leq t_* \),

\[
\sum_{k=0}^\infty p_k^{a,b}(t, x, y) \geq C_{16}^{-1} t^{-d/2}.
\]

\[\text{(3.16)}\]

**Proof.** By (3.10) with \( T = 1 \), there is a constant \( 0 < t_* < 1 \) such that for all \( t \in (0, t_*) \)

\[
C_{12} C_{15} M_b(\sqrt{T}) \leq \frac{1}{2} \wedge \frac{C_7 e^{-C_8}}{4},
\]

and so

\[
\sum_{k=1}^\infty |p_k^{a,b}(t, x, y)| \leq C_9 C_{12} C_{15} M_b(\sqrt{T}) q_{d,C_{10}/2}^a(t, x, y) \\
\leq 2C_9 C_{12} C_{15} M_b(\sqrt{T}) q_{d,C_{10}/2}^a(t, x, y), \quad x, y \in \mathbb{R}^d.
\]

\[\text{(3.17)}\]

Thus, by Lemma 2.1 with \( T = 1 \) and (3.17), we have for all \( (t, x, y) \in (0, t_*) \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
\sum_{k=0}^\infty |p_k^{a,b}(t, x, y)| \leq \sum_{k=0}^\infty |p_k^{a,b}(t, x, y)| \leq 2C_9 q_{d,C_{10}/2}^a(t, x, y),
\]

which gives (3.15). On the other hand, if \( |x - y|^2 < t \leq t_* \), then

\[
p^a(t, x, y) \geq C_7 e^{-C_8} t^{-d/2} \text{ and } q_{d,C_{10}/2}^a(t, x, y) \leq 2t^{-d/2}.
\]

Thus, we have for \( (t, x, y) \in (0, t_*) \times \mathbb{R}^d \times \mathbb{R}^d \) with \( |x - y|^2 \leq t \),

\[
\sum_{k=0}^\infty p_k^{a,b}(t, x, y) \geq p^a(t, x, y) - \sum_{k=1}^\infty |p_k^{a,b}(t, x, y)| \geq C_7 e^{-C_8} t^{-d/2} - \frac{C_7 e^{-C_8}}{2} t^{-d/2} \\
= \frac{C_7 e^{-C_8}}{2} t^{-d/2}.
\]

\( \square \)
In the remainder of this paper, we fix \( t_\star \). By Lemma 3.3, the series \( \sum_{k=0}^\infty p_k^{a,b}(t,x,y) \) absolutely converges on \((0,t_\star] \times \mathbb{R}^d \times \mathbb{R}^d\). For every \( a \in (0,M] \), define

\[
p^{a,b}(t,x,y) = \sum_{k=0}^\infty p_k^{a,b}(t,x,y), \quad 0 \leq t \leq t_\star \text{ and } x,y \in \mathbb{R}^d.
\] (3.18)

**Lemma 3.5.** Suppose \( M > 0 \). For every \( a \in (0,M] \), \( p^{a,b}(t,x,y) \) is jointly continuous on \((0,t_\star] \times \mathbb{R}^d \times \mathbb{R}^d\).

**Proof.** For any \( 0 < t_1 < t_\star \), we have

\[
\sup_{|t_1,t_\star| \times \mathbb{R}^d \times \mathbb{R}^d} q_{d,C_{10}/a}(t,x,y) \leq 2t_1^{-d/2} < \infty.
\]

By Lemma 3.3 and inequality (3.10), the series \( \sum_{k=0}^\infty p_k^{a,b}(t,x,y) \) converges uniformly on \([t_1,t_\star] \times \mathbb{R}^d \times \mathbb{R}^d\). Since \( t_1 \) is arbitrary, the result follows from Lemma 3.3.

**Theorem 3.6.** Suppose \( M > 0 \). For every \( a \in (0,M] \), \( 0 < s,t \leq t_\star \) with \( s + t \leq t_\star \) and \( x,y \in \mathbb{R}^d \), we have

\[
p^{a,b}(t + s,x,y) = \int_{\mathbb{R}^d} p^{a,b}(t,x,z)p^{a,b}(s,z,y)dz.
\] (3.19)

**Proof.** Note that for \( s,t > 0 \) with \( s + t \leq t_\star \),

\[
p^{a,b}(t,x,z)p^{a,b}(s,z,y) = \left( \sum_{m=0}^\infty p_m^{a,b}(t,x,z) \right) \left( \sum_{k=0}^\infty p_k^{a,b}(s,z,y) \right)
\]

\[
= \sum_{k=0}^\infty \sum_{m=0}^k p_m^{a,b}(t,x,z)p_k^{a,b}(s,z,y).
\]

So it suffices to prove that for any \( k \geq 0 \),

\[
p^{a,b}_k(t + s,x,y) = \sum_{m=0}^k \int_{\mathbb{R}^d} p_m^{a,b}(t,x,z)p_k^{a,b}(s,z,y)dz,
\] (3.20)

which will be done inductively. When \( k = 0 \), (3.20) is clearly true since \( p_0^{a,b}(t,x,y) = p^a(t,x,y) \).

Suppose (3.20) holds when \( k = l \) and we have

\[
p^{a,b}_{l+1}(t + s,x,y) = \int_{0}^{t+s} \int_{\mathbb{R}^d} p_{l+1}^{a,b}(t + s - \tau,x,w)b(w)\nabla_w p_0^{a,b}(\tau,x,y)dw d\tau
\]

\[
= \left( \int_{0}^{s} + \int_{s}^{t+s} \right) \int_{\mathbb{R}^d} p_{l+1}^{a,b}(t + s - \tau,x,w)b(w)\nabla_w p_0^{a,b}(\tau,x,y)dw d\tau
\]

\[
= \int_{0}^{s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{m}^{a,b}(t,x,z)p_{l-m}^{a,b}(s-\tau,z,w)dzb(w)\nabla_w p_0^{a,b}(\tau,x,y)dw d\tau
\]

\[
+ \int_{s}^{t+s} \int_{\mathbb{R}^d} p_{m}^{a,b}(t + t-s,x,w)b(w)\nabla_w p_0^{a,b}(\tau,x,y)dw d\tau
\]

\[
= \int_{m=0}^{l} \int_{\mathbb{R}^d} p_{m}^{a,b}(t,x,z)\int_{0}^{s} p_{l-m}^{a,b}(s-\tau,z,w)b(w)\nabla_w p_0^{a,b}(\tau,x,y)dw d\tau dz
\]

\[
+ \int_{\mathbb{R}^d} \int_{0}^{t} p_{l+1}^{a,b}(t,x,w)b(w)\nabla_w p_0^{a,b}(\tau,x,y)dw d\tau p_0^{a,b}(s,z,y)dz
\]

\[
= \int_{m=0}^{l} \int_{\mathbb{R}^d} p_{m}^{a,b}(t,x,z)p_{l+1-m}^{a,b}(s,z,y)dz + \int_{\mathbb{R}^d} p_{l+1}^{a,b}(t,x,z)p_0^{a,b}(s,z,y)dz,
\]

which is what we wanted.
where in the third to the last equality, we used Fubini's theorem as for \((t, x, y) \in (0, t_\ast) \times \mathbb{R}^d \times \mathbb{R}^d\) and any \(m, l \in \mathbb{Z}_+\), by (3.9) and lemma 3.2.

\[
\int_0^s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |p_{m}^{a,b}(t, x, w)||p_{l}^{a,b}(s - \tau, w, z)||b(z)||\nabla_z p^a(\tau, z, y)dwdzd\tau
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |p_{m}^{a,b}(t, x, w)||p_{l}^{a,b}(s - \tau, w, z)||b(z)||\nabla_z p^a(\tau, z, y)dzd\tau dw
\]
\[
\leq \int_{\mathbb{R}^d} |p_{m}^{a,b}(t, x, w)||p_{l+1}^{a,b}(s, w, y)dwy < \infty.
\]

We have also used the fact that due to Lemma 2.1 and the dominated convergence theorem, \(\nabla_z p^a(\tau, z, y) = \int_{\mathbb{R}^d} \nabla_z p^a(\tau - s, z, w)p^a(s, w, y)dw\).

In view of Theorem 3.6, the definition of \(p_{0}^{a,b}(t, x, y)\) can be uniquely extended to all \(t > 0\) so that (1.8) holds for all \(s, t > 0\). Suppose \(p_{0}^{a,b}(t, x, y)\) has been well defined on \((0, kt_\ast) \times \mathbb{R}^d \times \mathbb{R}^d\) for integer \(k \geq 0\) and (1.8) holds for all \(s, t > 0\) with \(s + t \leq kt_\ast\). For \(t \in (kt_\ast, (k+1)t_\ast]\), we define

\[
p_{0}^{a,b}(t, x, y) = \int_{\mathbb{R}^d} p_{0}^{a,b}(kt_\ast, x, z)p_{0}^{a,b}(t - kt_\ast, z, y)dz, \quad x, y \in \mathbb{R}^d.
\]

One can verify easily that the Chapman-Kolmogorov equation (1.8) holds for every \(t > 0\) with \(t + s \leq (k+1)t_\ast\). This proves that (1.8) holds for all \(t, s > 0\).

**Theorem 3.7.** Suppose \(M > 0\). For every \(a \in (0, M]\), \(p_{0}^{a,b}(t, x, y)\) is continuous on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\) and \(\int_{\mathbb{R}^d} p_{0}^{a,b}(t, x, y)dy = 1\) for every \(t > 0\) and \(x \in \mathbb{R}^d\).

**Proof.** The continuity of \(p_{0}^{a,b}(t, x, y)\) for all \(t > 0\) follows from Lemma 3.5 (3.21) and the dominated convergence theorem.

It follows from (2.5) that \(\int_{\mathbb{R}^d} \nabla_z p^a(t, x, y)dy = 0\) for all \(t > 0\) and \(x \in \mathbb{R}^d\). Thus for every \(k \geq 1\), by Lemma 3.2 (1.7), (3.10) and Fubini’s theorem,

\[
\int_{\mathbb{R}^d} p_{k}^{a,b}(t, x, y)dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{t} p_{k-1}^{a,b}(t - s, x, z)b(z) \cdot \nabla_z p^a(s, z, y)dsdzdy
\]
\[
= \int_{\mathbb{R}^d} \int_{0}^{t} p_{k-1}^{a,b}(t - s, x, z)b(z) \cdot \int_{\mathbb{R}^d} \nabla_z p^a(s, z, y)dydsdz = 0.
\]

In view of (3.10) and the dominated convergence theorem, we have for all \(t \in (0, t_\ast]\) and \(x \in \mathbb{R}^d\),

\[
\int_{\mathbb{R}^d} p^{a,b}(t, x, y)dy = \sum_{k=0}^{\infty} \int_{\mathbb{R}^d} p_{k}^{a,b}(t, x, y)dy = \int_{\mathbb{R}^d} p^{a}(t, x, y)dy = 1,
\]

which extends to all \(t > 0\) by (3.21).

For bounded measurable function \(f\) on \(\mathbb{R}^d\), \(t > 0\) and \(x \in \mathbb{R}^d\), we define operator \(P_{t}^{a,b}\)

\[
P_{t}^{a,b}f(x) = \int_{\mathbb{R}^d} p_{t}^{a,b}(t, x, y)f(y)dy.
\]

It follows from (3.19) that \(P_{t}^{a,b}P_{s}^{a,b} = P_{t+s}^{a,b}\).

The following theorem tells us that the generator of \(\{P_{t}^{a,b}, t \geq 0\}\) is \(\mathcal{L}^{a,b}\) in the weak sense. The proof is almost the same to part of the proof of [3] Theorem 1. We give the details of the proof for completeness. For any compact set \(K \subset \mathbb{R}^d\) and \(r > 0\), let \(K^r = \{y \in \mathbb{R}^d : \exists x \in K\) such that \(|x - y| < r\}\) be the \(r\)-neighborhood of \(K\).
Theorem 3.8. Suppose $M > 0$. For every $a \in (0, M]$ and for all $f \in C_c^\infty(\mathbb{R}^d)$, $g \in C_\infty(\mathbb{R}^d)$,
\[
\lim_{t \to 0} \int_{\mathbb{R}^d} \frac{P^{a,b}_t f(x) - f(x)}{t} g(x) dx = \int_{\mathbb{R}^d} L^{a,b} f(x) g(x) dx.
\]

Proof. Note that for all $t \in (0, t_*)$, 
\[
\int_{\mathbb{R}^d} \frac{P^{a,b}_t f(x) - f(x)}{t} g(x) dx = \frac{1}{t} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} p^{a,b}_0(t, x, y) f(y) dy - f(x) \right) g(x) dx
\]
\[+ \frac{1}{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( p^{a,b}_1(t, x, y) + \sum_{k=2}^{\infty} p^{a,b}_k(t, x, y) \right) f(y) g(x) dy dx.
\]
Since $p^{a,b}_0(t, x, y) = p^a(t, x, y)$ we have 
\[
\lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} p^a_0(t, x, y) f(y) dy - f(x) \right) g(x) dx = \int_{\mathbb{R}^d} \left( \Delta + a^\alpha \Delta^{\alpha/2} \right) f(x) g(x) dx.
\]
For $t \in (0, t_*)$, let $I(t) = t^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P^{a,b}_1(t, x, y) f(y) g(x) dx$. We claim that $I(t)$ converges to $\int_{\mathbb{R}^d} b(x) \cdot \nabla f(x) dx$ as $t \to 0$. By (1.7), Fubini’s theorem and integration by parts,
\[
I(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{t} \frac{1}{t} p^a(t-s, x, z) p^a(s, z, y) ds db(z) \cdot \nabla g(y) g(x) dz dy dx.
\]
Since $\nabla f(x) g(x)$ is uniformly continuous and bounded, for every $\varepsilon > 0$, there is $\delta > 0$ so that $|\nabla f(y) g(x) - \nabla f(w) g(z)| < \varepsilon$ for $|x - z| < \delta$ and $|y - w| < \delta$. Let $M_0 = \sup_{x, y \in \mathbb{R}^d} |\nabla f(y) g(x)|$, and $K$ be the support of $\nabla f$. Recall that $K^1$ denotes the 1-neighborhood of $K$. Clearly
\[
|I(t) - \int_{\mathbb{R}^d} b(z) \cdot \nabla f(z) g(z) dz| \\
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{t} \frac{1}{t} p^a(t-s, x, z) p^a(s, z, y) ds db(z) ||\nabla f(y) g(x) - \nabla f(z) g(z)|| dx dy dz
\]
\[= \left( \int_{(K^1)^c} \int_{K} \int_{0}^{t} \int_{(B(z, \delta) \times B(z, \delta))^c} + \int_{K} \int_{B(z, \delta)} \int_{0}^{t} \int_{B(z, \delta)} \cdots ds dxdydz \right) dz
\]
\[= J_1 + J_2 + J_3.
\]
We estimate $J_1, J_2$ and $J_3$ separately. Note that if $x \in K$ and $z \in (K^1)^c$, then $|x - z| \geq 1$ and so by Theorem 2.1 for $x, y \in \mathbb{R}^d$ and $0 < s < t$,
\[
p^a(t-s, x, z) \leq C_g q^\alpha_d C_{10} (t-s, x, z) \leq C_g c_1 \frac{t-s}{|x-z|^{d+\alpha}}.
\]
where $c_1$ is a positive constant depending only on $d, \alpha, M$. Thus,
\[
J_1 \leq 2M_0 \int_{(K^1)^c} \int_{K} \int_{0}^{t} \left( \int_{\mathbb{R}^d} p^a(s, z, y) dy \right) \frac{1}{t} p^a(t-s, x, z) Db(z) |dz ds dx
\]
\[\leq 2M_0 C_g c_1 \int_{(K^1)^c} \int_{K} \int_{0}^{t} \frac{1}{t} \frac{t-s}{|x-z|^{d+\alpha}} Db(z) |dz ds dx
\]
\[\leq tM_0 C_g c_1 |K| \sup_{x \in \mathbb{R}^d} \int_{|x-z| \geq 1} \frac{|Db(z)|}{|x-z|^{d+\alpha}} dz
\]
\[\leq tM_0 C_g c_1 |K| \sup_{x \in \mathbb{R}^d} H^\alpha_b (3-\alpha/2, 1, x) \to 0,
\]
as \( t \) goes to zero. Similarly, if \((x, y) \in (B(z, \delta) \times B(z, \delta))^c\), then \(|x - z| \geq \delta\) or \(|y - z| \geq \delta\). Since \( b \) is locally integrable, we have

\[
J_2 \leq 2tM_0 C_9 C_1 \int_{K^1} \int_{|x-z| \geq \delta} |b(z)| \frac{1}{|x-z|^{d+\alpha}} dx dz \leq 2tM_0 C_9 C_1 \delta^{-d-2} \int_{K^1} |b(z)| dz \to 0,
\]
as \( t \to 0 \).

\[
J_3 \leq \varepsilon \int_{K^1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{t} \frac{1}{t} p^{a}(t-s, x, z) p^{a}(s, z, y) ds |b(z)| dx dy dz \leq \varepsilon \int_{K^1} |b(z)| dz.
\]

Since \( \varepsilon \) is arbitrary, we have \( \lim_{t \to 0} I(t) = \int_{\mathbb{R}^d} b(z) \cdot \nabla f(z) g(z) dz \).

By (4.17), Theorem 3.4 and the dominated convergence theorem, we have

\[
\sum_{k=2}^{\infty} p_{k}^{a,b}(t, x, y) = \int_{\mathbb{R}^d} \int_{0}^{t} \left( \sum_{k=1}^{\infty} p_{k}^{a,b}(t-s, x, z) \right) b(z) \cdot \nabla_{z} p^{a}(s, z, y) ds dz.
\]

Similar to the estimate of \( I(t) \), by Fubini’s theorem, integration by parts and (5.17), we have for all \( t \in (0, t_*] \)

\[
\frac{1}{t} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \sum_{k=2}^{\infty} p_{k}^{a,b}(t, x, y) \right) f(y) g(x) dy dx \right|
\]

\[
= \frac{1}{t} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{t} \left( \sum_{k=1}^{\infty} p_{k}^{a,b}(t-s, x, z) \right) p^{a}(s, z, y) b(z) \cdot \nabla f(y) g(x) ds dz dy dx \right|
\]

\[
\leq \frac{2C_9 C_{12} C_{15} M_b(\sqrt{t})}{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{t} \sum_{k=1}^{\infty} q_{k}^{a,C_{10}}(t-s, x, z) p^{a}(s, z, y) b(z) \left| \nabla f(y) g(x) \right| ds dz dy dx
\]

\[
\leq 2C_9 C_{12} C_{15} M_b(\sqrt{t}) \left( \frac{2C_8}{C_{10}} \right)^{d/2} \frac{1}{C_7} p^{a}(t-s, x, y) \left| \nabla f(y) g(x) \right| ds dz dy dx,
\]

which goes to zero as \( t \to 0 \). This completes the proof. \( \square \)

## 4 Uniqueness and Positivity

**Theorem 4.1.** Suppose \( M > 0 \). There are constants \( C_{17} = C_{17}(d, \alpha, M) \), \( C_{18} = C_{18}(d, \alpha, M, b) \) such that for all \( a \in (0, M) \),

\[
|p^{a,b}(t, x, y)| \leq C_{17} e^{C_{17} t} p^{a}(2C_8 t / C_{10}, x, y), \quad t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{4.1}
\]

Consequently, for any \( T > 0 \), there is a constant \( C_{19} = C_{19}(d, \alpha, M, T) \) such that

\[
|p^{a,b}(t, x, y)| \leq C_{19} e^{C_{17} t} q_{d,C_{10}/2}^{a}(t, x, y), \quad t \in (0, T] \text{ and } x, y \in \mathbb{R}^d.
\]

**Proof.** Note that by the expression of \( q_{d,C_{10}/2}^{a}(t, x, y) \) and the lower bound of \( p^{a}(t, x, y) \) in Theorem 2.1 with \( T = 1 \),

\[
q_{d,C_{10}/2}^{a}(t, x, y) \leq \left( \frac{2C_8}{C_{10}} \right)^{d/2} q_{d,C_{10}}^{a}(2C_8 t / C_{10}, x, y) \leq \left( \frac{2C_8}{C_{10}} \right)^{d/2} C_{17}^{-1} p^{a}(2C_8 t / C_{10}, x, y). \tag{4.2}
\]

Recall that \( t_* \) is the constant in Lemma 3.4. If \( t < t_* \), by (3.15) and Lemma 2.1,

\[
|p^{a,b}(t, x, y)| \leq C_{16} q_{d,C_{10}/2}^{a}(t, x, y) \leq c_1 p^{a}(2C_8 t / C_{10}, x, y),
\]

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where \( c_1 = \frac{C_8t(2C_9)^{d/2}}{C_9C_{10}^{d/2}} \) depends only on \( d, \alpha, M \). It remains to consider the case \( t > t_* \). Let \( k = \lfloor t/t_* \rfloor + 1 \), then \( t/k \in (0, t_*) \). Combining (4.2), (1.8) and (3.15), we have
\[
\begin{align*}
\left| p^{a,b}(t, x, y) \right| &\leq \int_{R^d(k-1)} c_1^k \mathcal{P}(2C_8 t/C_9)^k, x, x_1) \cdots \mathcal{P}(2C_8 t/C_9, x_{k-1}, y) dx_1 \cdots dx_{k-1} \\
&= c_1^k \mathcal{P}(2C_8 t/C_9, x, y) \\
&\leq c_1 c_1^k \mathcal{P}(2C_8 t/C_9, x, y),
\end{align*}
\]
which gives the first conclusion with \( C_{17} = c_1 \) and \( C_{18} = \frac{1}{t_*} \ln c_1 \). Furthermore, by the upper bound of \( p^a(t, x, y) \) in Theorem 2.1 for \( t \in (0, T) \) and \( x, y \in R^d \)
\[
p^a(2C_8 t/C_9, x, y) \leq C_9 q^d_{d,C_9/2}(2C_8 t/C_9, x, y) \leq \frac{2C_8 C_9}{C_9} q^d_{d,C_{10}^d/(2C_9)}(t, x, y).
\]
Combining the last two displays, we finish the proof by setting \( C_{19} = c_1 ((2C_8 C_9/C_9) \vee 1) \).

\[ \square \]

**Theorem 4.2.** Suppose that \( M > 0 \) and \( b \in K_{d,1} \). For every \( a \in (0, M) \), \( p^{a,b}(t, x, y) \) satisfies (1.4) for all \( t > 0 \) and \( x, y \in R^d \).

**Proof.** Recall that \( t^* \) is the constant in Lemma 3.4. We first prove that \( p^{a,b}(t, x, y) \) satisfies (1.6) for all \( t \in (0, t^*) \) and \( x, y \in R^d \). Indeed, by (3.18), (3.17), Theorem 2.2 (3.2) and the dominated convergence theorem, we have for all \( t \in (0, t^*) \)
\[
p^{a,b}(t, x, y) = \sum_{n=0}^{\infty} p^{a,b}_n(t, x, y)
\]
\[
= p^{a,b}_0(t, x, y) + \sum_{n=1}^{\infty} \int_0^t \int_{R^d} p^{a,b}_{n-1}(t-s, x, z) b(z) \nabla p^a(s, z, y) dz ds
\]
\[
= p^{a,b}_0(t, x, y) + \int_0^t \int_{R^d} \sum_{n=1}^{\infty} p^{a,b}_{n-1}(t-s, x, z) b(z) \nabla p^a(s, z, y) dz ds
\]
\[
= p^{a,b}_0(t, x, y) + \int_0^t \int_{R^d} p^{a,b}(t-s, x, z) b(z) \nabla p^a(s, z, y) dz ds.
\]
Now, we use induction in \( k \) to prove (1.6) for all \( t > 0 \). Suppose that (1.6) is true for \( t \in (0, 2^k t^*) \), then for all \( x, y \in R^d \) and \( s = t/2 \in (2^{k-1} t^*, 2^k t^*) \). By (1.8), Theorem 4.1 (3.2), and Fubini’s theorem, we have
\[
p^{a,b}(t, x, y) = \int_{R^d} \int_{R^d} p^{a,b}(s, x, z) p^{a,b}(s, z, y) dz
\]
\[
= \int_{R^d} p^{a,b}(s, x, z) \left( p_0(s, z, y) + \int_0^s \int_{R^d} p^{a,b}(s-r, z, w) \nabla w p^a(r, w, y) dw dr \right) dz
\]
\[
= \int_{R^d} p_0(s, x, z) p_0(s, z, y) dz
\]
\[
+ \int_{R^d} \left( \int_0^s \int_{R^d} p^{a,b}(s-r, x, u) b(u) \nabla u p^a(r, u, z) dw dr \right) p_0(s, z, y) dz
\]
\[
+ \int_{R^d} p^{a,b}(s, x, z) \left( \int_0^s \int_{R^d} p^{a,b}(s-r, z, w) b(w) \nabla w p^a(r, w, y) dw dr \right) dz
\]
\[
= p_0(t, x, y) + \int_0^s \int_{R^d} p^{a,b}(s-r, x, u) b(u) \left( \int_{R^d} \nabla u p_0(r, u, z) p_0(s, z, y) dz \right) dw dr
\]
\[
+ \int_0^s \int_{R^d} \left( \int_{R^d} p^{a,b}(s, x, z) p^{a,b}(s-r, z, w) dz \right) b(w) \nabla w p^a(r, w, y) dw dr
\]
\[ 17 \]
Similar to the arguments that lead to (3.10), by (4.3), we can recursively verify that
\[ \frac{n}{n} \text{ which goes to zero as } \lim_{n \to \infty} \]

Theorem 4.3. Suppose that \( M > 0 \) and \( b \in \mathbb{R}^{d,1} \). For every \( a \in (0,M] \), \( p^{a,b}(t,x, y) \) is the unique continuous heat kernel that satisfies the Chapman-Kolmogorov equation (1.8) on \((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d\), Duhamel’s formula (1.6) on \((0,t_0] \times \mathbb{R}^d \times \mathbb{R}^d\) for some constant \( t_0 \geq 0 \) and that for some \( c_1 > 0 \),

\[ \left| p^{a,b}(t,x, y) \right| \leq c_1 p^a(t,x,y) \quad \text{for } t \in (0,t_0] \text{ and } x,y \in \mathbb{R}^d. \]  

(4.3)

Proof. Suppose that \( p(t, x, y) \) is any continuous heat kernel that satisfies Duhamel’s formula (1.6) and (1.3) for \((t, x, y) \in (0,t_0] \times \mathbb{R}^d \times \mathbb{R}^d\). Without loss of generality, we may and do assume that \( t_0 \leq t^* \). Firstly, let \( R_n(t, x, y), n \geq 2 \),

\[ R_n(t, x, y) = \int_{t_0}^t \int_{\mathbb{R}^d} R_n(t, s, x, z) \nabla z p_0(s, z, y) dz ds, \]

where \( R_n(t, x, y) \) is well defined. Furthermore, we have the upper bound of \( |R_n(t, x, y)| \):

\[ |R_n(t, x, y)| \leq c_1(C_{12}C_{15}M_b(\sqrt{t}))^k q_{d,C_{10}/2}^a(t, x, y). \]

On the other hand, using Duhamel’s formula (1.6) inductively, we have for every \( n \geq 1 \),

\[ p(t, x, y) = \sum_{j=0}^{n-1} p_j^{a,b}(t, x, y) + R_n(t, x, y), \]

where \( p_j^{a,b}(t, x, y) \) is defined by (1.7). Note that for all \((t, x, y) \in (0,t_0] \times \mathbb{R}^d \times \mathbb{R}^d\), by Lemma 3.4, \( C_{12}C_{15}M_b(\sqrt{t}) \leq 1/2 \) and so

\[ |R_n(t, x, y)| \leq c_1 2^{-k} q_{d,C_{10}/2}^a(t, x, y) < \infty, \]

which goes to zero as \( n \to \infty \). Thus, we have

\[ q(t, x, y) = \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) = p^{a,b}(t, x, y), \quad \text{for all } (t, x, y) \in (0,t_0] \times \mathbb{R}^d \times \mathbb{R}^d. \]

Since both \( q \) and \( p^{a,b} \) satisfy the Chapman-Kolmogorov equation (1.8) on \((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d\), we have \( q = p^{a,b} \) on \((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d\). 

\[ \square \]
Unlike that in [3], it is not easy to show the positivity of \( p^{a,b}(t, x, y) \) directly from its construction. We show \( p^{a,b}(t, x, y) \geq 0 \) by adopting the approach from [3], using Hille-Yosida-Ray theorem when \( b \) is bounded continuous and then using approximation for general \( b \).

**Lemma 4.4.** Suppose \( M > 0 \). For every \( a \in (0, M] \) and every \( t > 0 \), \( P_t^{a,b} \) maps bounded functions to continuous functions. Furthermore, \( \{P_t^{a,b}, t \geq 0\} \) is a strongly continuous semigroup in \( C_{\infty}(\mathbb{R}^d) \).

**Proof.** By Lemma 4.1 and Lemma 3.7 one easily verify that \( P_t^{a,b} \) maps bounded functions to continuous functions for every \( t > 0 \). For every \( f \in C_{\infty}(\mathbb{R}^d) \) and \( t > 0 \), by Lemma 4.1

\[
\lim_{|x| \to \infty} \left| P_t^{a,b} f(x) \right| \leq \lim_{|x| \to \infty} \int_{\mathbb{R}^d} C_{17} e^{C_{18} t} s^{q_{d,C_{10}} / (2C_{8})} (t, x, y) f(y) dy \leq \lim_{|x| \to \infty} \int_{\mathbb{R}^d} C_{17} e^{C_{18} t} s^{q_{d,C_{10}} / (2C_{8})} (0, 0) f(x + y) dy = 0,
\]

which shows \( P_t^{a,b} f \in C_{\infty}(\mathbb{R}^d) \). Moreover, since \( f \) is uniformly continuous on \( \mathbb{R}^d \), for every \( \varepsilon > 0 \), there is a constant \( \delta > 0 \) such that \( |f(x) - f(y)| \leq \varepsilon \) for all \( x, y \in \mathbb{R}^d \) with \( |x - y| \leq \delta \). And so by (3.11),

\[
\limsup_{t \to 0} \sup_{s \leq t} \int_{|x - y| \geq \delta} |p^{a,b}(s, x, y)| dy \leq \limsup_{t \to 0} \sup_{s \leq t} \int_{|x - y| \geq \delta} C_{17} e^{C_{18} s} q_{d,C_{10}} / (2C_{8}) (s, x, y) dy \leq \limsup_{t \to 0} \sup_{s \leq t} \int_{|x - y| \geq \delta} C_{17} e^{C_{18} t} c_1 \left( \frac{t}{|x - y|^d + 2} + \frac{t}{|x - y|^d + \alpha} \right) dy = 0,
\]

where \( c_1 \) is some positive constant depending only on \( d, \alpha, M \). Thus, we have

\[
\limsup_{t \to 0} \sup_{s \leq t} \left| P_s^{a,b} f(x) - f(x) \right| = \limsup_{t \to 0} \sup_{s \leq t} \left| \int_{\mathbb{R}^d} p^{a,b}(s, x, y) f(y) dy - f(x) \right| \leq \limsup_{t \to 0} \sup_{s \leq t} \int_{|x - y| < \delta} \left| p^{a,b}(s, x, y) \right| |f(x) - f(y)| dy \leq \limsup_{t \to 0} \sup_{s \leq t} \int_{|x - y| < \delta} C_{17} e^{C_{18} s} p^a(2C_{8} s / C_{10}, x, y) |f(x) - f(y)| dy \leq \varepsilon C_{17},
\]

which shows that \( \lim_{t \to 0} \| P_t^{a,b} f - f \|_{\infty} = 0 \). \hfill \Box

**Lemma 4.5.** Suppose \( M > 0 \) and the function \( b \) is bounded and continuous on \( \mathbb{R}^d \). Then, for every \( a \in (0, M] \), \( p^{a,b}(t, x, y) \geq 0 \), \( t > 0 \) and \( x, y \in \mathbb{R}^d \).

**Proof.** Denote the Feller generator of \( \{P_t^{a,b}, t \geq 0\} \) in \( C_{\infty}(\mathbb{R}^d) \) by \( \hat{L}^{a,b} \), which is a closed operator. For every \( f \in C_{2}^{2}(\mathbb{R}^d) \), since \( b \) is continuous, it is easy to see that \( L^{a,b} f \in C_{\infty}(\mathbb{R}^d) \). Similar to Theorem 3.8 we claim that \( (P_t^{a,b} f - f) / t \) uniformly converges to \( L^{a,b} f \) as \( t \to 0 \). Indeed, for any \( t \in (0, t_*] \),

\[
\left\| (P_t^{a,b} f - f) / t - L^{a,b} f \right\|_{\infty} = \sup_{x \in \mathbb{R}^d} \left| \frac{1}{t} \left( \int_{\mathbb{R}^d} \sum_{k=0}^{\infty} p_k^{a,b}(t, x, y) f(y) dy - f(x) \right) - \left( \Delta + a^\alpha \Delta^{\alpha / 2} + b \cdot \nabla \right) f(x) \right|
\]
It follows that $I_1$ goes to zero as $t \to 0$ since $\Delta + a^a\Delta^{a/2}$ is the generator of $Z^a$. We next treat $I_2$ as we did with $I$ in the proof of Theorem 3.5.3. Let $M_0 = \sup_{x \in R^d}|b(x)\nabla f(x)|$. Since $f \in C^2_c(R^d)$, for any $\varepsilon > 0$, there is constant $\delta > 0$ such that $|b(z)\nabla f(y) - b(x)\nabla f(x)| < \varepsilon$ for all $x \in R^d$ and $(z, y) \in B(x, \delta) \times B(x, \delta)$. while, if $(z, y) \in (B(x, \delta) \times B(x, \delta))^c$ then $|z - x| \geq \delta$ or $|z - y| \geq \delta$. Then, by (3.11), we have

$$I_2 \leq \sup_{x \in R^d} \int_0^t \int_{R^d} \int_{R^d} \frac{1}{t} p^a(t - s, x, z) p^a(s, z, y) |b(z)\nabla f(y) - b(x)\nabla f(x)| \ dz\ dy\ ds,$$

where $c_1$ is some positive constant depending only on $d, a, M$. Since $\varepsilon$ is arbitrary, $I_2$ goes to zero as $t \to 0$. Similar to $I_2$, we can prove that $I_3$ goes to zero as $t \to 0$. Thus we have

$$C^2_c(R^d) \subset D(\hat{L}^{a,b}) \text{ and } \hat{L}^{a,b}f = \mathcal{L}^{a,b}f \text{ for all } f \in C^2_c(R^d).$$

(4.4)

On the other hand, for $\lambda > C_{18}$, by Theorem 3.5.3

$$\sup_{x \in R^d} \int_0^\infty e^{-\lambda t} |p^{a,b}_t (x,y)| \ dt \leq \sup_{x \in R^d} \int_0^\infty e^{-\lambda t} |p^{a,b}_t (x,y)| \ dy \ dt \leq \|f\|_\infty \int_0^\infty C_{17} e^{-\lambda t} e^{-C_{18}t} \ dt = c_\lambda \|f\|_\infty,$$

(4.5)

where $c_\lambda = C_{17}/(\lambda - C_{18})$. Consider the strongly continuous semigroup $\{e^{-C_{18}t}p^{a,b}_t, t \geq 0\}$ with its generator $\hat{L}^{a,b} - C_{18}$. By (3.5), the residual set $\rho(\hat{L}^{a,b} - C_{18})$ of $\hat{L}^{a,b} - C_{18}$ contains $(0, \infty)$. Moreover, $\hat{L}^{a,b} - C_{18}$ satisfies the positive maximum principle in view of (3.11) and [1] Theorem 3.5.3. Therefore, $\{e^{-C_{18}t}p^{a,b}_t, t \geq 0\}$ is a positive preserving semigroup on $C^\infty_c(R^d)$ by Hille-Yosida-Ray (10 Theorem 3.5.1). Since $\{e^{-C_{18}t}p^{a,b}_t, t \geq 0\}$ has a continuous kernel $e^{-C_{18}t}p^{a,b}_t(t, x, y)$, we have $p^{a,b}_t(t, x, y) \geq 0$ for all $(t, x, y) \in (0, \infty) \times R^d \times R^d$.

In the rest of this section, we show by an approximation argument that Lemma 3.1 continues to hold for $b \in K_{d,1}$. Let $\varphi$ be a non-negative function in $C^\infty_c(R^d)$ with $\text{supp}(\varphi) \subset B(0, 1)$ and $\int_{R^d} \varphi(x) \ dx = 1$. For $n \geq 1$, define $\varphi_n(x) := n^d \varphi(nx)$ and

$$b_n(x) = \int_{R^d} \varphi_n(x - y) b(y) \ dy, \ x \in R^d.$$
For any compact set $K \subset \mathbb{R}^d$ and $r > 0$, recall that $K^r$ is the $r$-neighborhood of $K$. For any $0 \leq r_1 \leq r_2 \leq +\infty$ and $\beta \geq 0$, we have

$$\sup_{x \in K} \int_{|x-y| \in [r_1, r_2]} \frac{|b_n(y)|}{|x-y|^{d-1+2\beta}} dy \leq \sup_{x \in K} \int_{|x-y| \in [r_1, r_2]} \int_{\mathbb{R}^d} \frac{\varphi_n(y-z)|b(z)|}{|x-y|^{d-1+2\beta}} dxdy$$

$$= \sup_{x \in K} \int_{|x-y| \in [r_1, r_2]} \varphi_n(z) \int_{|z| < 1/n} \frac{|b(y)|}{|x-y|^{d-1+2\beta}} dzdy$$

$$= \sup_{x \in K} \int_{|x-y| \in [r_1, r_2]} \frac{\varphi_n(z)}{|x-y|^{d-1+2\beta}} dzdy \leq \int_{|x| < 1/n} \varphi_n(z) \sup_{x \in K^1} \int_{|x-y| \in [r_1, r_2]} \frac{|b(y)|}{|x-y|^{d-1+2\beta}} dzdy$$

$$= \sup_{x \in K^1} \int_{|x-y| \in [r_1, r_2]} \frac{|b(y)|}{|x-y|^{d-1+2\beta}} dy. \tag{4.6}$$

In particular, for every $r > 0$ and $n \geq 1$, by setting $r_1 = 0, r_2 = r$ and $\beta = 0$, we have

$$M_{b_n}(r) \leq M_b(r). \tag{4.7}$$

Recall that $\gamma = (1 + \alpha \wedge 1)/2$.

**Lemma 4.6.** $H^\gamma_{b-b_n}(t, x)$ converges to 0 uniformly on compact subsets of $(0, +\infty) \times \mathbb{R}^d$ as $n \to \infty$.

**Proof.** Let $[t_0, T_0] \times K \subset (0, +\infty) \times \mathbb{R}^d$ be an arbitrary compact set. Then, we have

$$\sup_{(t,x) \in [t_0, T_0] \times K} H^\gamma_{b-b_n}(t, x) \leq \sup_{x \in K} \int_{\mathbb{R}^d} \left( \frac{1}{|x-y|^{d-1}} \wedge \frac{T_0^\gamma}{|x-y|^{d-1+2\gamma}} \right) |b(y) - b_n(y)| dy$$

$$\leq \sup_{x \in K} \left( \int_{|x-y|^2 < r} + \int_{r \leq |x-y|^2 < R} + \int_{|x-y|^2 \geq R} \right) \cdots dy$$

$$= : I_1 + I_2 + I_3,$$

where $0 < r < R < \infty$ are undetermined. By setting $r_1 = 0, r_2 = \sqrt{r}$ and $\beta = 0$ in (4.6), we have

$$I_1 \leq 2 \sup_{x \in K^1} \int_{|x-y|^2 < r} \frac{|b(y)|}{|x-y|^{d-1}} dy.$$

Since $b \in K_{d,1}$, for any $\varepsilon > 0$, we can choose $r$ small enough such that

$$I_1 \leq 2 \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

By setting $r_1 = \sqrt{R}, r_2 = \infty$ and $\beta = \gamma$ in (4.6), we have

$$I_3 \leq 2T_0^\gamma \sup_{x \in K^1} \int_{|x-y|^2 \geq R} \frac{|b(y)|}{|x-y|^{d-1+2\gamma}} dy.$$

Fix a point $x_0 \in K^1$. Note that for $x \in K^1$, $|x-y|^2 \geq R$ and $\sqrt{R} > diam(K^1)$, we have

$$\frac{|x_0 - y|}{|x-y|} \leq \frac{|x_0 - x| + |x-y|}{|x-y|} \leq \frac{diam(K^1)}{\sqrt{R}} + 1 \leq 2,$$

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and so
\[ I_3 \leq 2T_0^\gamma \sup_{x \in K^1} \int_{|x-y|^2 \geq (\sqrt{R} \cdot \text{diam}(K^1))^2} |b(y)| \frac{|x_0 - y|^{d-1+2\gamma}}{|x-y|^{d-1+2\gamma}} dy \]
\[ \leq 2^{d+2}T_0^\gamma \int_{|x-y|^2 \geq (\sqrt{R} \cdot \text{diam}(K^1))^2} |b(y)| \frac{|x_0 - y|^{d-1+2\gamma}}{|x-y|^{d-1+2\gamma}} dy. \]

By lemma 2.3 and the dominated convergence theorem, we can choose \( R \) large enough such that
\[ I_3 < \frac{\varepsilon}{2}. \]

Now, we fix the above \( r, R \). Note that \( b_n \to b, a.s. \) By the dominated convergence theorem
\[ \lim_{n \to \infty} I_2 = \lim_{n \to \infty} \int_{|x-y|^2 < R} |b(y) - b_n(y)| dy = 0. \]

Then, we have
\[ \lim_{n \to \infty} \sup_{(t,x) \in [0,T] \times K} H_{b-b_n}^\gamma (t, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0. \]
This proves the lemma since \( \varepsilon \) is arbitrary.

**Lemma 4.7.** Suppose \( M > 0 \) and \( T > 0 \). There exist positive constants \( C_{20} = C_{20}(d, \alpha, M, T) \) and \( C_{21} = C_{21}(d, \alpha, M, T) \) so that for every \( n \geq 1 \), \( j \geq 1 \) and all \( a \in (0, M] \), \((t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \),
\[ |p_j^{a,b_n}(t, x, y) - p_j^{a,b}(t, x, y)| \leq C_{20} \left( C_{21} M_b(\sqrt{t}) \right)^{j-1} \left( (H_{b-b_n}^\gamma (t, x) + H_{b-b_n}^\gamma (t, y)) q_{d,C_{10}/2}^a (t, x, y) \right). \] (4.8)

**Proof.** We prove (4.8) inductively in \( j \). It holds when \( j = 1 \), since by (1.7) and (3.1),
\[ |p_1^{a,b_n}(t, x, y) - p_1^{a,b}(t, x, y)| \]
\[ \leq \int_0^t \int_{\mathbb{R}^d} p^a(t-s, x, z) |b(z) - b_n(z)| \|\nabla_z p^a(s, z, y)\| dz ds \]
\[ \leq C_{20} C_{12} \int_0^t \int_{\mathbb{R}^d} q_{d,C_{10}/2}^a(t-s, x, z) |b(z) - b_n(z)| q_{d+1,3C_{10}/4}^a (s, z, y) dz ds \]
\[ \leq C_{14} \int_0^t \int_{\mathbb{R}^d} (H_{b-b_n}^\gamma (t, x, z) + H_{b-b_n}^\gamma (t, y)) |b(z) - b_n(z)| dz \]
\[ \leq (H_{b-b_n}^\gamma (t, x) + H_{b-b_n}^\gamma (t, y)) q_{d,C_{10}/2}^a (t, x, y). \]

Assume (4.8) is true for \( j = k \geq 1 \). By (1.7),
\[ |p_k^{a,b_n}(t, x, y) - p_k^{a,b}(t, x, y)| \]
\[ \leq \int_0^t \int_{\mathbb{R}^d} |p_k^{a,b_n}(t-s, x, z) - p_k^{a,b}(t-s, x, z)||b_n(z)| \|\nabla_z p^a(s, z, y)\| dz ds \]
\[ + \int_0^t \int_{\mathbb{R}^d} |p_k^{a,b}(t-s, x, z)||b(z) - b_n(z)| \|\nabla_z p^a(s, z, y)\| dz ds \]
\[ = I_1 + I_2. \]

Let \( C_{20} = C_{20} C_{12} C_{14} \) and \( C_{21} = 2^{d+3} C_{13} C_{12} C_{14} \). Then
\[ I_1 \leq \int_0^t \int_{\mathbb{R}^d} \left( C_{21} M_b(\sqrt{t-s}) \right)^{k-1} \left( (H_{b-b_n}^\gamma (t-s, x) + H_{b-b_n}^\gamma (t-s, y)) q_{d,C_{10}/2}^a (t, x, y) \right). \]
Thus

Similarly,

Therefore

Note that

\[ H^\gamma(t, z, w) \land H^\gamma(t, x, z) = \left( \frac{1}{|w - z|^{d-1}} \land \frac{t^\gamma}{|w - z|^{d-1+2\gamma}} \right) \land \left( \frac{1}{|z - x|^{d-1}} \land \frac{t^\gamma}{|z - x|^{d-1+2\gamma}} \right) \]

\[ 2^{d+1} \left( \frac{1}{|w - x|^{d-1}} \land \frac{t^\gamma}{|w - x|^{d-1+2\gamma}} \right) = H^\gamma(t, x, w). \]

Similarly,

\[ H^\gamma(t, z, w) \land H^\gamma(t, x, z) \leq H^\gamma(t, y, w). \]

Thus

\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(w) - b_n(w)||b_n(z)||H^\gamma(t, z, w)(H^\gamma(t, x, z) + H^\gamma(t, z, y))|\ dz\ dw \]

\[ \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(w) - b_n(w)||b_n(z)|H^\gamma(t, x, w)(H^\gamma(t, z, w) + H^\gamma(t, x, z))\ dz\dw \]

\[ + H^\gamma(t, y, w)(H^\gamma(t, z, w) + H^\gamma(t, y, z)) \]

\[ = \int_{\mathbb{R}^d} |b(w) - b_n(w)|H^\gamma(t, x, w)(H^\gamma(t, z, w) + H^\gamma(t, x, z))\ dz\dw \]

\[ + H^\gamma(t, y, w)(H^\gamma(t, z, w) + H^\gamma(t, y, z)) \]

\[ \leq \int_{\mathbb{R}^d} |b(w) - b_n(w)|(H^\gamma(t, x, w) + H^\gamma(t, y, w))\ dw \]

\[ = M_6(\sqrt{t}) \int_{\mathbb{R}^d} |b(w) - b_n(w)|(H^\gamma(t, x, w) + H^\gamma(t, y, w))\ dw \]

Therefore

\[ I_1 \leq 2^{d+1} \left( \frac{1}{|w - x|^{d-1}} \land \frac{t^\gamma}{|w - x|^{d-1+2\gamma}} \right) \land \left( \frac{1}{|z - x|^{d-1}} \land \frac{t^\gamma}{|z - x|^{d-1+2\gamma}} \right) \]

\[ 2^{d+1} \left( \frac{1}{|w - x|^{d-1}} \land \frac{t^\gamma}{|w - x|^{d-1+2\gamma}} \right) = H^\gamma(t, x, w). \]
On the other hand, by (3.10),
\[
I_2 \lesssim C_{a,b} \int_0^t \int_{\mathbb{R}^d} q_{b_0, a}^2(t-s, x, z) |b(z) - b_n(z)|^2 \, dz \, ds
\]
Thus
\[
|p_{b+1}(t, x, y) - p_{b+1}(t, x, y)| \leq C_{a, b} \int_0^t \int_{\mathbb{R}^d} q_{b_0, a}^2(t-s, x, z) |b(z) - b_n(z)|^2 \, dz \, ds
\]
\[
\leq C_{a, b} \int_0^t \int_{\mathbb{R}^d} q_{b_0, a}^2(t-s, x, z) |b(z) - b_n(z)|^2 \, dz \, ds
\]
\[
= \left( \int_0^t \int_{\mathbb{R}^d} q_{b_0, a}^2(t-s, x, z) |b(z) - b_n(z)|^2 \, dz \, ds \right)^{1/2}
\]
This completes the proof of the lemma.

Lemma 4.8. Suppose \( M > 0 \) and \( T > 0 \). For every \( a \in (0, M] \), \( 0 < T_0 < T \) and compact set \( K \subset \mathbb{R}^d \), \( p^{a, b}(t, x, y) \) converges to \( p^{a, b}(t, x, y) \) uniformly in \( [0, T] \times K \times K \) as \( n \to \infty \).

Proof. By (3.10) and Lemma 4.7 with \( T = 1 \), for all \( t \in (0, t_*) \) and \( x, y \in K \),
\[
|p^{a, b}(t, x, y) - p^{a, b}(t, x, y)| \leq \sum_{k=1}^\infty |p_{k+1}^{a, b}(t, x, y) - p_k^{a, b}(t, x, y)|
\]
\[
\leq C_{a, b} \sum_{k=1}^{\infty} \left( C_{a, b} \right)^k \left( H_{b-1}^\gamma(t, x) + H_{b-1}^\gamma(t, y) \right) q_{b, a}^2(t, x, y).
\]
Since \( b \in \mathbb{R}_{d, 1} \), there is a constant \( 0 < T_1 < t_* \) so that \( C_{a, b} \) \( \leq 1/2 \). Then for all \( t \leq T_1 \) and \( x, y \in \mathbb{R}^d \),
\[
|p^{a, b}(t, x, y) - p^{a, b}(t, x, y)| \leq C_{a, b} \sum_{k=1}^{\infty} 2^{-(k-1)} \left( H_{b-1}^\gamma(t, x) + H_{b-1}^\gamma(t, y) \right) q_{b, a}^2(t, x, y)
\]
\[
\leq 2C_{a, b} \left( H_{b-1}^\gamma(t, x) + H_{b-1}^\gamma(t, y) \right) q_{b, a}^2(t, x, y).
\]
Without loss of generality, we may and do assume \( T_0 < T_1 / 2 \). Note that \( q_{b, a}^2(t, x, y) \leq 2T_0^{-d/2} \) for \( T_0 \leq t \leq T_1 \) and \( x, y \in \mathbb{R}^d \). By (4.9) and Lemma 4.6,
\[
\lim_{n \to \infty} \sup_{t \in [T_0, T_1]} \sup_{x, y \in K} |p^{a, b}(t, x, y) - p^{a, b}(t, x, y)|
\]
\[
\leq 4T_0^{-d/2} \lim_{n \to \infty} \sup_{x, y \in K} \left( H_{b-1}^\gamma(T_1, x) + H_{b-1}^\gamma(T_1, y) \right) = 0.
\]
For \( t \in [T_1, 3T_1/2] \), let \( t_1 = T_1 / 2 \). By Chapman-Kolmogorov equation (1.8),
\[
|p^{a, b}(t, x, y) - p^{a, b}(t, x, y)| \leq \int_{\mathbb{R}^d} |p^{a, b}(t - t_1, x, z) - p^{a, b}(t - t_1, x, z)| |p^{a, b}(t, z, y)| \, dz
\]
\[
+ \int_{\mathbb{R}^d} |p^{a, b}(t - t_1, x, z)| |p^{a, b}(t, z, y) - p^{a, b}(t, z, y)| \, dz
\]
\[
= : I_1 + I_2.
\]
By Theorem 1.1 and 4.7,
\[
|p^{a, b}(t - t_1, x, z)| \leq C_{19} e^{C_{18}(t - t_1)} q_{b, a}^{2}(t - t_1, x, z) \leq 2C_{19} e^{C_{18}T_1 / 2} \leq 2C_{19} e^{C_{18}T_1 (T_1 / 2)^{-d/2}},
\]
Similarly, we can get
\[
|p^{a,b}(t - t_1, x, z)| \leq 2C_{19}e^{C_{18}T_1}(T_1/2)^{-d/2},
\]
and, for any \( \varepsilon > 0 \), there is a constant \( R_0 > 0 \) such that for all \( n \geq 1 \) and \( y \in \mathbb{R}^d \),
\[
\int_{|z-y| > R_0} |p^{a,b_n}(t_1, z, y)|dz < \frac{\varepsilon}{4C_{19}e^{C_{18}T_1}(T_1/2)^{-d/2}}.
\]
On the other hand, when \( n \) large enough, it follows from Lemmas 4.6 and 4.7 that
\[
\sup_{x, z \in K_{R_0}} |p^{a,b_n}(t - t_1, x, z) - p^{a,b}(t - t_1, x, z)| < \frac{\varepsilon}{2C_{17}e^{C_{18}T}},
\]
while by Theorem 4.1,
\[
\int_{\mathbb{R}^d} |p^{a,b_n}(t_1, z, y)|dz \leq C_{17}e^{C_{18}T} \int_{\mathbb{R}^d} P^a(2C_{8}t/C_{101}, z, y)dz = C_{17}e^{C_{18}T}
\]
for all \( y \in \mathbb{R}^d \). Thus we have
\[
I_1 \leq \left( \int_{|z-y| > K_{R_0}} + \int_{|z-y| < K_{R_0}} \right) |p^{a,b_n}(t - t_1, x, z) - p^{a,b}(t - t_1, x, z)||p^{a,b_n}(t_1, z, y)|dz
\]
\[
\leq 2C_{19}e^{C_{18}T_1}(T_1/2)^{-d/2} \leq \frac{\varepsilon}{4C_{19}e^{C_{18}T_1}(T_1/2)^{-d/2}} + C_{17}e^{C_{18}T} \frac{\varepsilon}{2C_{17}e^{C_{18}T}}
\]
\[
= \varepsilon.
\]
Similarly, we can get \( I_2 < \varepsilon \) for large enough \( n \). Thus we have proved that \( p^{a,b_n}(t, x, y) \) converges to \( p^{a,b}(t, x, y) \) uniformly in \([T_1, 3T_1/2] \times K \times K\) as \( n \to \infty \). We can finish the proof by repeating the above arguments for \([2T/T_1 - 2]\) times. \(\square\)

Lemma 4.9 and Lemma 4.8 immediately yield the following.

**Lemma 4.9.** Let \( M > 0 \). For every \( a \in (0, M] \),
\[
p^{a,b}(t, x, y) \geq 0, \quad t > 0 \text{ and } x, y \in \mathbb{R}^d.
\]

**Proof of Theorem 1.2.** Theorem 1.2 follows from (3.18), Lemma 3.7, Lemma 4.9, Theorem 3.8 and Theorem 4.8. \(\square\)

### 5 Lower bound estimates

In this section, we derive the sharp lower bound of the heat kernel \( p^{a,b}(t, x, y) \). We know from Lemma 4.4 and Lemma 4.9 that \( P^{a,b} \) is a Feller semigroup in \( C_\infty(\mathbb{R}^d) \). Therefore in view of Theorem 1.2(iv), there exists a conservative Feller process \( X^{a,b} = \{X^{a,b}_t, t \geq 0, \mathbb{P}, x \in \mathbb{R}^d\} \) such that \( \mathbb{P}^{a,b}_t f(x) = \mathbb{E}_x[f(X^{a,b}_t)] \) for every \( x \in \mathbb{R}^d \) and \( f \in C_\infty(\mathbb{R}^d) \). Moreover, the process \( X^{a,b} \) has strong Feller property and has \( p^{a,b}(t, x, y) \) as its transition density. By Lemma 4.1 and Lemma 4.9, \( p^{a,b}(t, x, y) \) has the following upper bound
\[
p^{a,b}(t, x, y) \leq C_{17}e^{C_{18}t}p^a(2C_{8}t/C_{101}, x, y), \quad t > 0 \text{ and } x, y \in \mathbb{R}^d
\]
and for every \( T > 0 \),
\[
p^{a,b}(t, x, y) \leq C_{19}e^{C_{18}t a q_{d}(2C_{8}t/C_{101})(t, x, y), \quad t \in (0, T] \text{ and } x, y \in \mathbb{R}^d
\]
where \( C_{17}, C_{18} \) and \( C_{19} \) are constants in Lemma 4.1.

The following lemmas will be used to derive the Lévy system of \( X^{a,b} \).
Lemma 5.1. For every $f \in \mathbb{K}_{d,1}$, $\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t P_s^{a,b} |f|(x)ds = 0$.

Proof. By (5.2) with $T = 1$ and (2.2), for $0 < t < 1$ and $x \in \mathbb{R}^d$,

$$
\int_0^t P_s^{a,b} |f|(x)ds \leq C_{19}e^{C_{19}t} \int_0^t \int_{\mathbb{R}^d} \frac{a}(x,y)f(y)dyds
$$

$$
\leq C_{11}C_{19}e^{C_{19}t} \left( \sqrt{t} \sup_{x \in \mathbb{R}^d} |f(y)|\sum_{\tau=1}^{(2)(2)C_{19}}(t,x,y)dy \right) + \int_0^t \int_{|x-y|^2 \geq \epsilon} |f(y)|dyds
$$

Thus, by Lemma 2.3

$$
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t P_s^{a,b} |f|(x)ds \leq C_{11}C_{19} \left( \sqrt{t}N_{f}^{C_{19}/(2C_{19})}(t) + t^{(3-\alpha)/2} \right)
$$

Similar to [3] Theorem 2.5], by Lemma 4.4, Lemma 5.1 and Theorem 3.8, we have the

Lemma 5.2. Suppose $M > 0$. For every $a \in (0,M]$, $x \in \mathbb{R}^d$ and every $f \in C^\infty_c(\mathbb{R}^d)$,

$$
M_t^{f} := f(X_t^{a,b}) - f(X_0^{a,b}) - \int_0^t L^{a,b}f(X_s^{a,b})ds
$$

is a martingale under $\mathbb{P}_x$.

The following Lévy system of $X^{a,b}$ follows from the similar arguments in [6] Lemma 4.7] and
[7] Appendix A. See also [5] Theorem 2.6].

Theorem 5.3. For $M > 0$ and every $a \in (0,M]$, $X^{a,b}$ has the same Lévy system as $Z^a$, that is for any $x \in \mathbb{R}^d$, any non-negative measure function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on

$$
\{ (s,x,y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y \} \quad \text{and stopping time } S \quad \text{(with respect to the filtration of } X^{a,b})
$$

$$
\mathbb{E}_x \left[ \sum_{s \leq S} f(s,X^{a,b}_s, X^{a,b}_s) \right] = \mathbb{E}_x \left[ \int_0^S \int_{\mathbb{R}^d} f(s,X^{a,b}_s, y)\mathbb{E}(X^{a,b}_s, y)dyds \right]
$$

For an open set $U \subset \mathbb{R}^d$, define $\tau^{a,b}_U := \inf \{ t > 0 : X^{a,b}_t \notin U \}$.

Lemma 5.4. For each $M > 0$ and $R_0 > 0$, there is a constant $\kappa = \kappa(d,\alpha,M,R_0,b) < 1$
depending on $b$ only via the rate at which $M_b(r)$ goes to zero such that for all $a \in (0,M],

$$
P_x \left( \tau_{B(x,r)} \leq \kappa r^2 \right) \leq \frac{1}{2}.
$$

Proof. By the strong Markov property of $X^{a,b}$ (See [2] Exercise (8.17), pp. 43-44)], for $x \in \mathbb{R}^d$
and $t > 0$, we have

$$
P_x \left( \tau_{B(x,r)} \leq t \right) \mathbb{P}_x \left( \tau_{B(x,r)} \leq t, X^{a,b}_t \in B(x,r/2) \right) \mathbb{P}_x \left( \tau_{B(x,r)} \leq t, X^{a,b}_t \in B(x,r/2) \right)
$$

$$
\mathbb{E}_x \left[ P_{X^{a,b}_{B(x,r)}} \left( X^{a,b}_{t-B(x,r)} \geq r/2, \tau_{B(x,r)} \leq t \right) \right]
$$

$$
+ \mathbb{P}_x \left( \left| X^{a,b}_t - X^{a,b}_0 \right| \geq r/2 \right)
$$
\[
\leq 2 \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x \left( \left| X^{a,b}_s - X^{a,b}_0 \right| \geq r/2 \right).
\]

By (5.2) with \( T = R_0^2 \), for \( t \in (0, R_0^2) \), there are positive constants \( c_i, i = 1, 2, 3 \) depending only on \( d, \alpha, M, R_0 \) such that
\[
\begin{align*}
\sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x \left( \left| X^{a,b}_s - X^{a,b}_0 \right| \geq r/2 \right) \lesssim & c_1 c_2 R_0^2 \\
\leq & \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \geq r/2} \left( s^{-d/2} \exp \left( -c_3 |x-y|^2 \right) + s^{-d/2} \wedge \frac{\alpha s}{|x-y|^{d+\alpha}} \right) dy \\
\lesssim & \sup_{s \leq t} \int_{\mathbb{R}^d} e^{-c_3 \rho^2} + 1 \wedge \frac{M^\alpha s^{1-\alpha/2}}{\rho^{d+\alpha}} \rho^{-1} d\rho \\
\leq & \int_{\mathbb{R}^d} e^{-c_3 \rho^2} + 1 \wedge \frac{M^\alpha R_0^{2-\alpha}}{\rho^{d+\alpha}} \rho^{-1} d\rho.
\end{align*}
\]

Setting \( t = \kappa r^2 \) in the last display, where \( \kappa \in (0,1) \) is undetermined, we have
\[
\mathbb{P}_x \left( \tau^{a,b}_{B(x,r)} \leq \kappa r^2 \right) \leq 2 c_1 c_2 R_0^2 \omega_d \int_{\mathbb{R}^d} e^{-c_3 \rho^2} + 1 \wedge \frac{M^\alpha R_0^{2-\alpha}}{\rho^{d+\alpha}} \rho^{-1} d\rho,
\]
which goes to 0 as \( \kappa \to 0 \). Thus we can choose \( \kappa < 1 \) so that (5.3) holds.

For any open set \( U \subset \mathbb{R}^d \), define \( \sigma_{U}^{a,b} = \inf \{ t \geq 0 : X^{a,b}_t \in U \} \).

**Lemma 5.5.** For each \( M > 0 \) and \( R_0 > 0 \), there is a constant \( c_1 = c_1(d, \alpha, M, R_0, b) \) depending on \( b \) only via the rate at which \( M_b(r) \) goes to zero, such that for all \( r \in (0, R_0] \) and \( x, y \in \mathbb{R}^d \) with \( |x-y| \geq 2r \),
\[
\mathbb{P}_x \left( \sigma_{B(x,r)}^{a,b} < \kappa r^2 \right) \geq c_1 r^{d+2} \frac{\alpha^\alpha}{|x-y|^{d+\alpha}}.
\]

**Proof.** By Lemma 5.4,
\[
\mathbb{E}_x \left[ \kappa r^2 \wedge \tau_{B(x,r)}^{a,b} \right] \geq \frac{\kappa r^2}{2} \mathbb{P}_x \left( \tau_{B(x,r)}^{a,b} \geq \kappa r^2 / 2 \right) \geq \frac{\kappa r^2}{4}.
\]

By Lemma 5.3, we have
\[
\begin{align*}
\mathbb{P}_x \left( \sigma_{B(y,r)}^{a,b} < \kappa r^2 \right) & \geq \mathbb{P}_x \left( X^{a,b}_{\kappa r^2 \wedge \tau_{B(y,r)}^{a,b}} \in B(y, r) \right) \\
& = \mathbb{E}_x \left( \int_0^{\kappa r^2 \wedge \tau_{B(y,r)}^{a,b}} \int_{B(y,r)} J^a(X^{a,b}_s, u) duds \right) \\
& \geq 2^{-(d+\alpha)} \mathbb{E}_x \left[ \frac{\kappa r^2}{2} \wedge \tau_{B(y,r)}^{a,b} \right] \int_{B(y,r)} \frac{\alpha^\alpha}{|x-y|^{d+\alpha}} du \\
& \geq \frac{V_d}{4 \cdot 2^{d+\alpha} \kappa^{d+2}} \frac{\alpha^\alpha}{|x-y|^{d+\alpha}},
\end{align*}
\]

where in the second to the last inequality, we have used the fact that for \( u \in B(y, r) \), \( |u - X^{a,b}_s| \leq 2r + |x-y| \leq 2|x-y| \).

**Lemma 5.6.** For every \( M > 0 \), there is a constant \( C_{22} = C_{22}(d, \alpha, M, b) \) depending on \( b \) only via the rate at which \( M_b(r) \) goes to zero, such that for all \( t \in (0, t_*], a \in (0, M] \) and \( x, y \in \mathbb{R}^d \),
\[
p^{a,b}(t, x, y) \geq C_{22} \left( t^{-d/2} \wedge \frac{\alpha^a t}{|x-y|^{d+\alpha}} \right).
\]

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Proof. By (3.16), for \( t \in (0, t_*) \) and \( x, y \in \mathbb{R}^d \), with \( |x - y|^2 \leq t \)

\[
p^{a,b}(t, x, y) \geq C_{16}^{-1} t^{-d/2} \geq C_{16}^{-1} \left( t^{-d/2} \land \frac{a^\alpha t}{|x - y|^{d+\alpha}} \right).
\]

It remains to consider the case \( |x - y|^2 > t \). For any \( t \in (0, t_*) \), by the strong Markov property, Lemma 5.4 and Lemma 5.5 with \( R_0 = \sqrt{t} \) and \( r = \sqrt{t}/4 \), we have \( |x - y| > \sqrt{t} > 2r \) and

\[
\mathbb{P}_x \left( X^{a,b}_{\kappa t/16} \in B(y, \sqrt{t}/2) \right) \geq \mathbb{P}_x \left( X^{a,b} \text{ hits } B(y, \sqrt{t}/2) \text{ before time } \kappa t/16 \text{ and stays there for at least } \kappa t/16 \text{ units of time} \right)
\]

\[
\geq \mathbb{P}_x \left( \sigma_{B(y, \sqrt{t}/4)} < \kappa t/16 \right) \text{ and } \mathbb{P}_x \left( \tau_{B(y, \sqrt{t}/2)} > \kappa t/16 \right)
\]

\[
\geq \mathbb{P}_x \left( \sigma_{B(y, \sqrt{t}/4)} < \kappa t/16 \right) \text{ and } \mathbb{P}_x \left( \tau_{B(z, \sqrt{t}/4)} > \kappa t/16 \right)
\]

\[
\geq c_1 t^{(d+2)/2} \frac{a^\alpha}{|x - y|^{d+\alpha}}
\]

for some constant \( c_1 = c_1(d, \alpha, M, b) > 0 \). Combining this with Lemma 5.4 and Chapman-Kolmogorov equation (1.8), we have for \( t \in (0, t_*) \)

\[
p^{a,b}(t, x, y) = \int_{\mathbb{R}^d} p^{a,b}(kt/16, x, z) p^{a,b}((1 - \kappa/16)t, z, y) dz
\]

\[
\geq \int_{B(y, \sqrt{t}/2)} p^{a,b}(kt/16, x, z) p^{a,b}((1 - \kappa/16)t, z, y) dz
\]

\[
\geq \inf_{z \in B(y, \sqrt{t}/2)} p^{a,b}((1 - \kappa/16)t, z, y) \mathbb{P}_x \left( X^{a,b}_{\kappa t/16} \in B(y, \sqrt{t}/2) \right)
\]

\[
\geq c_2 t^{-d/2} t^{(d+2)/2} \frac{a^\alpha}{|x - y|^{d+\alpha}} = c_2 \frac{a^\alpha t}{|x - y|^{d+\alpha}} \geq c_2 \left( t^{-d/2} \land \frac{a^\alpha t}{|x - y|^{d+\alpha}} \right),
\]

where \( c_2 = c_2(d, \alpha, M, b) \) is a positive constant and in the third to the last inequality, we have used the fact that \( \kappa < 1 \) and for \( z \in B(y, \sqrt{t}/2) \), \( |z - y|^2 < \sqrt{t} < (1 - \kappa/16)t \). \( \square \)

Lemma 5.7. Suppose \( M > 0 \). For all \( a \in (0, M], t \in (0, t_*) \) and \( x, y \in \mathbb{R}^d \), there are constants \( C_i = C_i(d, \alpha, M) > 0, i = 23, 24 \) such that

\[
p^{a,b}(t, x, y) \geq C_{23} t^{-d/2} \exp \left( - \frac{C_{24} |x - y|^2}{t} \right).
\]

Proof. By (3.16), for all \( t \in (0, t_*) \) and \( x, y \in \mathbb{R}^d \) with \( |x - y|^2 < t \), we have

\[
p^{a,b}(t, x, y) \geq C_{16}^{-1} t^{-d/2}.
\]

Next, we consider the case \( |x - y|^2 > t \). We fix \( x, y \in \mathbb{R}^d \) with \( |x - y|^2 \geq t \). Let \( k \) be the smallest integer such that \( 9|x - y|^2/t < k \). Set \( \xi_j = x + \frac{j - 1}{k}(y - x), 1 \leq j \leq k-1 \) and \( A = \prod_{j=1}^{k-1} B(\xi_j, \sqrt{\tau}/3\sqrt{k}) \).

For any \( (x_1, \cdots, x_{k-1}) \in A \), we have \( |x - x_1| < \sqrt{\tau}/3\sqrt{k} < \sqrt{\tau}/k \).

\[
\max_{1 \leq j < k-1} |x_j - x_{j-1}| = \max_{1 \leq j < k-1} |x_j - \xi_j + \xi_{j-1} - x_{j-1} + y - x/k| < \frac{\sqrt{t}}{3\sqrt{k}} + \frac{\sqrt{t}}{3\sqrt{k}} + \frac{\sqrt{t}}{3\sqrt{k}} = \frac{\sqrt{t}}{\sqrt{k}}.
\]
and \(|x_{k-1} - y| = |x_{k-1} - \xi_{k-1} + \xi_{k-1} - y| < \frac{\sqrt{2}}{N^2}\). Hence by Lemma 4.9, Chapman-Kolmogorov equation (18) and (3.4),

\[
p^{a,b}(t, x, y) = \int_{\mathbb{R}^d} p^{a,b}(\frac{t}{k}, x, x_1) \cdots p^{a,b}(\frac{t}{k}, x_{k-1}, y) dx_1 dx_2 \cdots dx_{k-1}
\]

\[
\geq \int_{A} p^{a,b}(\frac{t}{k}, x, x_1) \cdots p^{a,b}(\frac{t}{k}, x_{k-1}, y) dx_1 dx_2 \cdots dx_{k-1}
\]

\[
\geq C_{16}^{-k} \left(\frac{t}{k}\right)^{-dk/2} \Omega_d^{k-1} \left(\frac{\sqrt{t}}{3\sqrt{k}}\right)^{d(k-1)}
\]

\[
= t^{-d/2} \frac{k^{d/2}}{C_{16}^{d/2}} \left(\frac{\Omega_d}{C_{16}^{d/2}}\right)^{k-1}
\]

\[
= \frac{k^{d/2}}{C_{16}} t^{-d/2} \exp \left(-\ln \frac{C_{16}^{d/2}}{\Omega_d} - \frac{3|a-b|^2}{t}\right),
\]

where \(\Omega_d\) is the volume of unit ball in \(\mathbb{R}^d\). This together with (5.4) proves the lemma with \(C_{23} := \frac{3^d}{C_{16}}\) and \(C_{24} := 9 \ln \frac{4^{d/2}}{\Omega_d}\). \(\square\)

**Proof of Theorem 1.3** The upper bound of \(p^{a,b}(t, x, y)\) is shown by (5.2). We need only to show the lower bound. Without loss of generality, we assume \(T > t_\ast\). If \(t \in (0, t_\ast]\), by Lemma 5.3 and Lemma 5.4 there is a constant \(c_1 = c_1(d, \alpha, M, b) > 0\) such that for any \(t, x, y \in \mathbb{R}^d\),

\[
p^{a,b}(t, x, y) \geq \frac{1}{2} \left(C_{23} t^{-d/2} \exp \left(-\frac{C_{24}|x-y|^2}{t}\right) + C_{22} \left(t^{-d/2} \wedge \frac{|x-y|^{d+\alpha}}{t}\right)\right)
\]

\[
\geq c_1 \left(t^{-d/2} \exp \left(-\frac{C_{24}|x-y|^2}{t}\right) + t^{-d/2} \wedge \frac{a^\alpha}{|x-y|^{d+\alpha}}\right) \tag{5.5}
\]

If \(t \geq t_\ast\), we let \(k\) be the smallest integer such that \(t_\ast k \geq t > (k-1)t_\ast\). Note that by Theorem 2.1 for \(t \in (0, T]\) and \(x, y \in \mathbb{R}^d\),

\[
g_{a,C_{24}}^d(t, x, y) \geq \left(\frac{C_{10}}{C_{24}}\right)^{d/2} g_{a,C_{10}}^d(C_{10} t / C_{24}, x, y) \geq (C_{10} / C_{24})^{d/2} C_{24}^{-1} p^a(C_{10} t / C_{24}, x, y).
\]

Using this, (18) and (5.5), we have

\[
p^{a,b}(t, x, y) \geq c_1^{-k} \int_{\mathbb{R}^d(k-1)} g_{a,C_{24}}^d(t / k, x, x_1) \cdots g_{a,C_{24}}^d(t / k, x_{k-1}, y) dx_1 \cdots dx_{k-1}
\]

\[
\geq c_1^{-k} \left(\frac{C_{10}}{C_{24}}\right)^{d/2} C_{24}^{-k} \int_{\mathbb{R}^d(k-1)} p^a(C_{10} t / C_{24}, k, x, x_1) \cdots p^a(C_{10} t / C_{24}, k, x_{k-1}, y) dx_1 \cdots dx_{k-1}
\]

\[
= c_1^{-k} \left(\frac{C_{10}}{C_{24}}\right)^{d/2} C_{24}^{-k} p^a(C_{10} t / C_{24}, x, y)
\]

\[
= c_1^{-k} \left(\frac{C_{10}}{c_k \bar{C}}\right)^{d/2} q_{a,C_{24}}^d(C_{10} t / C_{24}, x, y)
\]

\[
\geq c_1^{-k} \left(\frac{C_{10}}{C_{24}}\right)^{d/(2t_\ast)} q_{a,C_{24}}^d / C_{10} (t, x, y)
\]

where \(c_2 = c_2(d, \alpha, M, b)\) is a positive constant. Combining this and (5.5) completes the proof. \(\square\)
6 Martingale problem and Lévy process with drift

Following the approach in [10], we can show that the martingale problem for \((\mathcal{L}^a, C_c^\infty(\mathbb{R}^d))\) is well-posed, and there is a unique weak solution to SDE (1.1).

For \(a > 0\) and \(\lambda > 0\), define

\[
u^a_\lambda(x) = \int_0^\infty e^{-\lambda t} p^a(t, x) dt, \quad x \in \mathbb{R}^d.
\]

Lemma 6.1. There is a constant \(C_{25} = C_{25}(d)\) such that for all \(a > 0\), \(\lambda \geq 1\) and \(x \in \mathbb{R}^d\), we have

\[
u^a_\lambda(x) \leq C_{25}(1 \vee a^\alpha) \begin{cases} \frac{1}{|x|^{d-1}} \wedge \frac{\lambda^{-1}}{|x|^{d+\alpha}}, & d = 2, \\
\frac{1}{|x|^{d-2}} \wedge \frac{\lambda^{-1}}{|x|^{d+\alpha}}, & d > 2,
\end{cases}
\]

and

\[
|\nabla \nu^a_\lambda(x)| \leq C_{25}(1 \vee a^\alpha) \left( \frac{1}{|x|^{d-1}} \wedge \frac{\lambda^{-1}}{|x|^{d+\alpha}} \right).
\]

Proof. Note that for each \(\theta > 0\), the function \(\psi(t) = t^\theta e^{-t}\) on \([0, \infty)\) is bounded by \(\theta^\theta e^{-\theta}\). By (1.3), we have for \(|x| \geq 1/\lambda,\)

\[
u^a_\lambda(x) \leq C_1 \int_0^\infty e^{-\lambda t} \left( t^{-d/2} e^{-C_2|x|^2/t} + (a^\alpha t)^{-d/2} \wedge \frac{a^\alpha}{|x|^{d+\alpha}} \right) dt
\]

\[
\leq C_1 \int_0^\infty e^{-\lambda t} \left( \frac{t}{|x|^{d+2}} + \frac{a^\alpha t}{|x|^{d+\alpha}} \right) dt
\]

\[
= C_1 \lambda^{-2} \left( \frac{1}{|x|^{d+2}} + \frac{a^\alpha}{|x|^{d+\alpha}} \right)
\]

\[
\leq C_1 (1 \vee a^\alpha) \lambda^{-2} \left( \frac{1}{|x|^{d+2}} + \frac{1}{|x|^{d+\alpha}} \right).
\]

Since \(\lambda \geq 1\), if \(|x|^2 \geq 1/\lambda,\)

\[
u^a_\lambda(x) \leq 2C_1 (1 \vee a^\alpha) \lambda^{-2} \frac{2^\alpha}{|x|^{d+\alpha}}.
\]

When \(|x|^2 < 1/\lambda,\) similar to (6.3), we have

\[
\int_0^{|x|^2} e^{-\lambda t} p^a(t, x) dt \leq C_1 \int_0^{|x|^2} \left( \frac{t}{|x|^{d+2}} + \frac{a^\alpha t}{|x|^{d+\alpha}} \right) dt \leq C_1 \frac{1}{2} \left( \frac{1}{|x|^{d-2}} + \frac{a^\alpha}{|x|^{d-4+\alpha}} \right) \leq \frac{C_1 (1 \vee a^\alpha)}{|x|^{d-2}}
\]

and

\[
\int_{|x|^2}^\infty e^{-\lambda t} p^a(t, x) dt \leq C_1 \int_{|x|^2}^\infty e^{-\lambda t^{-d/2}} dt
\]

\[
\leq C_1 \begin{cases} \frac{1}{|x|} \int_{|x|^2}^\infty e^{-t^{-1/2}} dt \leq \frac{\sqrt{\pi}}{|x|} & \text{if } d = 2, \\
\int_{|x|^2}^\infty t^{-d/2} dt = \frac{2}{d-2} \frac{1}{|x|^{d-2}} & \text{if } d > 2.
\end{cases}
\]

Therefore, (6.1) follows from (6.4)-(6.6). Finally, (6.2) follows from (2.5) and (6.1). \[\square\]
For $a > 0$ and $\lambda > 0$, define the resolvent operator $R_\lambda^a$ by
\[
U_\lambda^a(x) = \int_{\mathbb{R}^d} u_\lambda^a(x - y)g(y)dy = \int_{\mathbb{R}^d} u_\lambda^a(y)g(x - y)dy, \quad g \in C_0(\mathbb{R}^d), \, x \in \mathbb{R}^d.
\]

Let $C_0^\infty(\mathbb{R}^d)$ be the collection of the smooth functions on $\mathbb{R}^d$ that vanish at infinity.

**Lemma 6.2.** For every $a > 0$ and $\lambda \geq 1$, $R_\lambda^a$ and $\nabla R_\lambda^a$ are bounded operators on $C_0^\infty(\mathbb{R}^d)$. Moreover, $R_\lambda^a f \in C_0^\infty(\mathbb{R}^d)$ for every $f \in C_0^\infty(\mathbb{R}^d)$.

**Proof.** By (6.2), we have for every $a > 0$, $\lambda \geq 1$, $f \in C_0^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} |\nabla u_\lambda^a(x)||f(y)|dy \leq C_{25}(1 \vee a^\alpha)\|f\|_\infty \int_{\mathbb{R}^d} \frac{1}{|y|^{d-1}} \wedge \frac{\lambda^{-\frac{\alpha+2}{d}+\alpha}}{|y|^{d+\alpha}}dy < \infty.
\]
Combining this with the fact that $u_\lambda^a$ in continuously differentiable off the origin and the dominated convergence theorem, we have
\[
\nabla U_\lambda^a f(x) = \int_{\mathbb{R}^d} \nabla u_\lambda^a(x - y)f(y)dy = \int_{\mathbb{R}^d} \nabla u_\lambda^a(y)(f(x - y)dy).
\]
Since both $u_\lambda^a$ and $\nabla u_\lambda^a$ are integrable over $\mathbb{R}^d$ and $f(x - y)$ converges to 0 as $|x| \to \infty$, we have that both $U_\lambda^a f$ and $\nabla U_\lambda^a f$ are in $C_0^\infty(\mathbb{R}^d)$ and
\[
\|U_\lambda^a f\|_\infty \leq C_{25}(1 \vee a^\alpha)\|f\|_\infty, \quad \text{and} \quad \|\nabla U_\lambda^a f\|_\infty \leq C_{25}(1 \vee a^\alpha)\|f\|_\infty,
\]
where $C_{25}$ is the constant from Lemma (6.1). Similarly, by the dominated convergence theorem, for $f \in C_0^\infty(\mathbb{R}^d)$, we have
\[
\partial_{x_1} \cdots \partial_{x_d} U_\lambda^a f(x) = \int_{\mathbb{R}^d} u_\lambda^a(y)\partial_{x_1} \cdots \partial_{x_d} f(x - y)dy,
\]
which shows that $U_\lambda^a f \in C_0^\infty(\mathbb{R}^d)$. \[\square\]

**Lemma 6.3.** Suppose that $M > 0$ and $b \in \mathbb{K}_{d,1}$. There is a constant $\lambda_0 = \lambda_0(d, \alpha, M, b) \geq 1$ with the dependence on $b$ only via the rate at which $M_b(r)$ goes to zero such that for every $a \in (0, M]$, $\lambda \geq \lambda_0$ and $f \in C_0^\infty(\mathbb{R}^d)$,
\[
\|\nabla U_\lambda^a(bf)\|_\infty \leq \left(\frac{1}{2}\right)^\gamma \|f\|_\infty.
\]

**Proof.** By (2.31) (with $\beta = \frac{\alpha+2}{d}$) and (6.2), we have for $a \in (0, M]$, $\lambda \geq \lambda_0$ and $f \in C_0^\infty(\mathbb{R}^d)$,
\[
\|\nabla U_\lambda^a(bf)\|_\infty \leq C_{25}(1 \vee a^\alpha) \int_{\mathbb{R}^d} \left(\frac{1}{|x|^{d-1}} \wedge \frac{\lambda^{-\frac{\alpha+2}{d}+\alpha}}{|x|^{d+\alpha}}\right)|b(y)|dy \leq C_{25}C_1(1 \vee M^\alpha)M_b(\lambda^{-1/2}).
\]
Since $b \in \mathbb{K}_{d,1}$, we can choose $\lambda_0 \geq 1$ such that $C_{25}C_1(1 \vee M^\alpha)M_b(\lambda^{-1/2}) \leq 1/2$ for every $\lambda > \lambda_0$. This completes the proof. \[\square\]

By (6.1), for $\lambda > C_{18}C_{10}/(2C_8)$,
\[
E_x \left[\int_0^\infty e^{-\lambda t}|b(X_t)|dt\right] \leq C_{17}C_{10}/(2C_8) \int_{\mathbb{R}^d} \int_0^\infty e^{-(\lambda - C_{18}C_{10}/(2C_8))t}p^a(t, x, y)dt|b(y)|dy
\]
\[
= C_{17}C_{10}/(2C_8) \int_{\mathbb{R}^d} u_\lambda^a C_{18}C_{10}/(2C_8)(x - y)b(y)|dy.
\]
Similar to Lemma 6.3 by (6.1), there is a constant $C_{26} > C_{18}C_{10}/(2C_8) \vee 1$ so that for every $a \in (0, M]$ and $\lambda > C_{26}$,
\[
\sup_{x \in \mathbb{R}^d} U_\lambda^a |b|(x) = \sup_{x \in \mathbb{R}^d} E_x \left[\int_0^\infty e^{-\lambda t}|b(X_t)|dt\right] < \infty.
\]
By increasing the value of $\lambda_0$ in Lemma 6.3 if needed, we may and do assume that $\lambda_0 \geq C_{26}$. 31
Theorem 6.4 (Uniqueness). For each \( x \in \mathbb{R}^d \) and \( a \in (0, M] \), \( \mathbb{P}_x \) is the unique solution to the martingale problem for \((\mathcal{L}^{a,b}, C^\infty_c(\mathbb{R}^d))\) with initial value \( x \).

Proof. Using Lemma 6.1, Lemma 6.2, Lemma 6.3 and (6.7), we can finish the proof by repeating the arguments in the proof of [10, Theorem 2.3] except using the following Itô’s formula in place of that in Step (ii) of [10, Theorem 2.3]:

\[
e^{-\lambda t} f(X_t) = f(X_0) + \int_0^t e^{-\lambda s} dM^f_s + \int_0^t e^{-\lambda s} \left( \Delta f(X_s) + a^\alpha \Delta^{\alpha/2} f(X_s) + b(X_s) \cdot \nabla f(X_s) \right) ds
- \lambda \int_0^t e^{-\lambda s} f(X_s) ds.
\]

Proof of Theorem 1.4. Theorem 6.4 implies that the martingale problem for \((\mathcal{L}^{a,b}, C^\infty_c(\mathbb{R}^d))\) is well-posed. The rest follows from Theorem 1.2.

The following theorem establishes the existence of the weak solution of SDE (1.1).

Theorem 6.5 (Existence). For every \( a > 0 \), there is a process \( Z^a \) defined on \( \Omega \) so that all its paths are right continuous and admit left limits, and

\[ X^{a,b}_t = x + Z^a_t + \int_0^t b(X^{a,b}_s) ds, \quad t \geq 0. \]

Proof. The proof is almost the same to that of [10, Theorem 3.1], except that we use the following arguments instead of those at the beginning of Page 13 in [10]: for any \( f \in C^\infty_c(\mathbb{R}^d) \),

\[
\int_0^t b(X^{a,b}_s) \nabla f(X^{a,b}_s) ds + \int_0^t \Delta f(X^{a,b}_s) ds
= \int_0^t \nabla f(X^{a,b}_s) dA_s + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial^2 f \partial x_i \partial x_j (X^{a,b}_s) d\langle M^i, M^j \rangle_s,
\]

which implies that

\[ A_t = \int_0^t b(X^{a,b}_t) ds \quad \text{and} \quad \langle M^i, M^j \rangle_t = \delta_{ij} t. \]

Here \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

Proof of Theorem 1.5. The existence of weak solution to SDE (1.1) follows from Lemma 6.5. Every weak solution to (1.1) solves the martingale problem for \((\mathcal{L}^{a,b}, C^\infty_c(\mathbb{R}^d))\) by Itô’s formula. Then, the rest follows from Theorem 1.4.

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