ON SOME SPECTRAL PROPERTIES OF THE WEIGHTED $\bar{\partial}$-NEUMANN OPERATOR

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Abstract. We study necessary conditions for compactness of the weighted $\bar{\partial}$-Neumann operator on the space $L^2(\mathbb{C}^n, e^{-\varphi})$ for a plurisubharmonic function $\varphi$. Under the assumption that the corresponding weighted Bergman space of entire functions has infinite dimension, a weaker result is obtained by simpler methods. Moreover, we investigate (non-) compactness of the $\bar{\partial}$-Neumann operator for decoupled weights, which are of the form $\varphi(z) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)$. More can be said if every $\Delta \varphi_j$ defines a nontrivial doubling measure.

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1. Introduction

We consider the Cauchy-Riemann complex (or Dolbeault complex)

$$0 \to C^\infty(\mathbb{C}^n) \xrightarrow{\partial} \Omega^{0,1}(\mathbb{C}^n) \xrightarrow{\partial} \Omega^{0,2}(\mathbb{C}^n) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^{0,n}(\mathbb{C}^n) \to 0,$$

where $\Omega^{0,q}(\mathbb{C}^n) := C^\infty(\mathbb{C}^n, \Lambda^{0,q}T^*(\mathbb{C}^n))$ denotes the space of smooth $(0,q)$-forms on $\mathbb{C}^n$, and the operator $\bar{\partial}$ is defined by

$$\bar{\partial}u := \sum_{|J|=q} \sum_{j=1}^n \frac{\partial u_j}{\partial z_j} d\bar{z}_j \wedge d\bar{z}_J$$

for $u = \sum_{|J|=q} u_J d\bar{z}_J \in \Omega^{0,q}(\mathbb{C}^n)$, and the primed sum indicates that summation is done only over increasing multiindices $J = (j_1, \ldots, j_q)$ of length $q$, and $d\bar{z}_J := d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$.

If $\varphi: \mathbb{C}^n \to \mathbb{R}$ is a $C^2$ function, we denote by $L^2_{\bar{\partial},q}(\mathbb{C}^n, e^{-\varphi})$ the Hilbert space completion of $\Omega^{0,q}_c(\mathbb{C}^n)$ (compactly supported elements of $\Omega^{0,q}(\mathbb{C}^n)$) with respect to the inner product

$$(u, v)_\varphi := \sum_{|J|=q} \int_{\mathbb{C}^n} u_J \bar{v}_J e^{-\varphi} d\lambda,$$  \hspace{1cm} (1.1)
with $\lambda$ the Lebesgue measure on $\mathbb{C}^n$. Let $\| \cdot \|_\varphi$ be the associated norm. The $\overline{\partial}$-operator extends maximally to a closed operator on $L^2_{0,q}(\mathbb{C}^n, e^{-\varphi})$ by letting it act in the sense of distributions on
\[
\text{dom}(\overline{\partial}) := \{ f \in L^2_{0,q}(\mathbb{C}^n, e^{-\varphi}) : \overline{\partial}f \in L^2_{0,q+1}(\mathbb{C}^n, e^{-\varphi}) \}.
\]
In other words, $\overline{\partial}f$ is defined to be the distributional derivative of $f$, viewed as an element of $L^2_{0,q+1}(\mathbb{C}^n, e^{-\varphi})$. Note that the symbol of its formal adjoint with respect to (1.1), which we denote by $\overline{\partial}^*_\varphi$, depends on the weight $\varphi$, and we denote the Hilbert space adjoint of $\overline{\partial}$ by $\overline{\partial}^*_{\varphi}$. For more details on the weighted $\overline{\partial}$-problem, see [12, 11].

The complex Laplacian is the Laplacian of the $\overline{\partial}$-complex,
\[
\Box^\varphi_{0,q} := \overline{\partial}\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial} : \Omega^{0,q}(\mathbb{C}^n) \to \Omega^{0,q}(\mathbb{C}^n).
\]
This is an elliptic, formally self-adjoint differential operator, and its Gaffney extension, which we also denote by $\Box^\varphi_{0,q}$, is then
\[
\Box^\varphi_{0,q} := \overline{\partial}\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial},
\]
understood as an unbounded operator on $L^2_{0,q}(\mathbb{C}^n, e^{-\varphi})$ with domain
\[
\text{dom}(\Box^\varphi_{0,q}) := \{ u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*_\varphi) : \overline{\partial}u \in \text{dom}(\overline{\partial}^*_\varphi) \text{ and } \overline{\partial}^*_\varphi u \in \text{dom}(\overline{\partial}) \},
\]
where $\overline{\partial}^*_\varphi$ is the Hilbert space adjoint of $\overline{\partial}$ for the inner product (1.1), and $\text{dom}(\overline{\partial}^*_\varphi)$ is its domain. It is a positive self-adjoint operator on $L^2_{0,q}(\mathbb{C}^n, e^{-\varphi})$ by techniques going back to [9], see also [11]. By completeness of $\mathbb{C}^n$, (1.2) is essentially self-adjoint, so that the Gaffney extension is its only closed symmetric extension, see for instance [11, 16].

The inverse of the restriction of $\Box^\varphi_{0,q}$ to $\text{dom}(\Box^\varphi_{0,q}) \cap \ker(\Box^\varphi_{0,q})$ is called the $\overline{\partial}$-Neumann operator and is denoted by
\[
N^\varphi_{0,q} := (\Box^\varphi_{0,q}|_{\text{dom}(\Box^\varphi_{0,q}) \cap \ker(\Box^\varphi_{0,q})})^{-1} : \text{img}(\Box^\varphi_{0,q}) \to L^2_{0,q}(\mathbb{C}^n, e^{-\varphi}).
\]
While $N^\varphi_{0,q}$ is initially only defined on $\text{img}(\Box^\varphi_{0,q})$, we can extend it to a bounded operator on $L^2_{0,q}(\mathbb{C}^n, e^{-\varphi})$ if and only if $\Box^\varphi_{0,q}$ has closed range. This is equivalent to $\text{img}(\overline{\partial}) \cap L^2_{0,q}(\mathbb{C}^n, e^{-\varphi})$ and $\text{img}(\overline{\partial}) \cap L^2_{0,q+1}(\mathbb{C}^n, e^{-\varphi})$ both being closed, and also to the existence of $C > 0$ such that
\[
\|u\|_\varphi^2 \leq C(\|\overline{\partial}u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2)
\]
holds for all $u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*_\varphi) \cap \ker(\Box^\varphi_{0,q})$. We are interested in spectral properties of $N^\varphi_{0,q}$, in particular the problem of deciding its compactness in terms of easily accessible properties of $\varphi$. We denote by $\sigma(\Box^\varphi_{0,q})$ and $\sigma_e(\Box^\varphi_{0,q})$ the spectrum and the essential spectrum of $\Box^\varphi_{0,q}$, respectively. The latter set consists of all points in $\sigma(\Box^\varphi_{0,q})$ which are accumulation points of the spectrum or eigenvalues of infinite multiplicity. If $N^\varphi_{0,q}$ is a bounded operator, then it follows from general operator theory that its compactness is equivalent to $\sigma_e(\Box^\varphi_{0,q}) \subseteq \{0\}$. Equivalently, the spectrum of $\Box^\varphi_{0,q}$ may only consist of discrete eigenvalues of finite multiplicity, except possibly allowing for an infinite dimensional kernel.

One important property of the $\overline{\partial}$-Neumann operator lies in the fact that it can be used to compute the norm-minimal solution to the inhomogeneous equation $\overline{\partial}u = f$, for a given $\overline{\partial}$-closed $(0,q+1)$-form $f$. Indeed, this solution is then given by $u = \overline{\partial}^*_\varphi N_{0,q+1} f$, and many operator theoretic properties such as compactness transfer from $N$ to the minimal solution operator $\overline{\partial}^*_\varphi N$. We refer the reader to the literature, for example [23, 11].
An important tool is the (weighted) Kohn-Morrey-Hörmander formula,
\[
\|\partial u\|_{q}^{2} + \|\nabla_{\varphi} u\|_{q}^{2} = \sum_{|j|=q} \sum_{k=1}^{n} \left( \frac{\partial^{2} \varphi}{\partial z_{j} \partial \bar{z}_{k}} \right)_{j,k=1}^{n} \int_{\mathbb{C}^{n}} \frac{\partial^{2} \varphi}{\partial z_{j} \partial \bar{z}_{k}} u_{j,k} \, d\nu_{K} e^{-\varphi} d\lambda, \tag{1.4}
\]
see [13], [11, p. 109], or [23, Proposition 2.4], which is valid a priori for \( u \in \Omega_{0,q}^{0}(\mathbb{C}^{n}) \) with \( 1 \leq q \leq n \), and by extension also for \( u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_{\varphi}) \), since \( \Omega_{0,q}^{0}(\mathbb{C}^{n}) \) is dense in the latter space for the norm \( u \mapsto (\|u\|_{q}^{2} + \|\partial u\|_{q}^{2} + \|\nabla_{\varphi} u\|_{q}^{2})^{1/2} \). Again, the reason for this is that \( \mathbb{C}^{n} \) is complete for the Euclidean metric.

**Remark 1.1.** (i) From (1.4), one derives a sufficient condition for boundedness of \( N_{0,q}^{\varphi} \), which is that \( \varphi \) be plurisubharmonic, and
\[
\lim \inf_{z \to \infty} s_{q}(z) > 0,
\]
where \( s_{q}(z) \) is the sum of the \( q \) smallest eigenvalues of the complex Hessian
\[
M_{\varphi}(z) := \left( \frac{\partial^{2} \varphi}{\partial z_{j} \partial \bar{z}_{k}} (z) \right)_{j,k=1}^{n}
\]
of \( \varphi \). We refer to [11] for the details. From this condition it also follows that \( \Box_{0,q} \) is injective, so that \( N_{0,q}^{\varphi} \) is the bounded inverse of \( \Box_{0,q}^{-1} \). (ii) A sufficient condition for \( N_{0,q}^{\varphi} \), being compact, is the following: Suppose that \( \varphi \in C^{2}(\mathbb{C}^{n}, \mathbb{R}) \) and that the sum \( s_{q}(z) \) of the smallest \( q \) eigenvalues of \( M_{\varphi}(z) \) has the property
\[
\lim_{z \to \infty} s_{q}(z) = +\infty. \tag{1.5}
\]
Then \( N_{0,q}^{\varphi} \) is compact. Indeed, (1.4) together with some linear algebra implies that \( \|\partial u\|_{q}^{2} + \|\nabla_{\varphi} u\|_{q}^{2} \geq (s_{q} u, v)_{\varphi} \) for all \( u, v \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_{\varphi}) \cap L^{2}_{0,q}(\mathbb{C}^{n}, e^{-\varphi}) \), and hence the diverging of \( s_{q} \) implies compactness, see [10, 11] or also [20] for the details. Further recent results concerning compactness of the \( \overline{\partial} \)-Neumann operator for the weighted problem can be found in [6, 7].

(iii) Boundedness and compactness of the \( \overline{\partial} \)-Neumann operator “percolate” up the \( \overline{\partial} \)-complex: if \( N_{0,q}^{\varphi} \) is bounded or compact, then \( N_{0,q+1}^{\varphi} \) has the same property, see [11].

2. **Overview of the results**

In this article we will first derive a necessary condition for compactness to hold, which is weaker than (1.5): Under the assumption that the weighted Bergman space
\[
A^{2}(\mathbb{C}^{n}, e^{-\varphi}) := \left\{ f : \mathbb{C}^{n} \to \mathbb{C} : f \text{ entire and } \int_{\mathbb{C}^{n}} |f|^{2} e^{-\varphi} d\lambda < \infty \right\}
\]
has infinite dimension (see Remark 3.2 for a known sufficient condition for this) and \( N_{0,q}^{\varphi} \) is compact for some \( 1 \leq q \leq n \), then
\[
\lim \sup_{z \to \infty} \text{tr}(M_{\varphi}(z)) = +\infty, \tag{2.1}
\]
\( ^{1} \)Actually, for the Laplacian to be injective it suffices to have \( \varphi \) plurisubharmonic and \( M_{\varphi}(z) \neq 0 \) for some \( z \in \mathbb{C}^{n} \), compare with the arguments in the proof of Theorem 5.1.
or, in other words, $\text{tr}(M_\varphi)$ is unbounded. A stronger result, obtained with different methods, is presented in section 4: it says that if $N_{0,q}^\varphi$ is compact (regardless of dimensionality of the Bergman space), then

$$\lim_{z \to \infty} \int_{B_1(z)} \text{tr}(M_\varphi)^2 d\lambda = +\infty.$$  

Of course, this also implies (2.1).

In section 5 we will consider weights which are decoupled, i.e., of the form $\varphi(z_1, \ldots, z_n) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)$. Decoupled weights are known from [12] to be an obstruction to compactness of the Bergman space (of entire functions on $\mathbb{C}$). By using results of [2, 17, 18], we will characterize compactness of $N_{0,q}^\varphi$ for this class of weights, under the additional assumption that each $\varphi_j: \mathbb{C} \to \mathbb{R}$ is a subharmonic function such that $\Delta \varphi_j$ defines a nontrivial doubling measure. This means that $\varphi_j$ is not harmonic and there is $C > 0$ such that

$$\int_{B_{2r}(z)} \Delta \varphi_j d\lambda \leq C \int_{B_{r}(z)} \Delta \varphi_j d\lambda$$

for all $z \in \mathbb{C}$ and $r > 0$. As an example, $\varphi(z) := |z|^\alpha$ for $\alpha > 0$ satisfies these conditions. It turns out that $N_{0,q}^\varphi$ is never compact for $0 \leq q \leq n - 1$, and compactness of $N_{0,n}^\varphi$ depends on the asymptotics of the integral $\int_{B_1(z)} \text{tr}(M_\varphi) d\lambda$ as $z \to \infty$. This will also give examples of weights $\varphi$ such that (2.1) is not sufficient for compactness of $N^\varphi$.

3. A NECESSARY CONDITION FOR COMPACTNESS

To derive a necessary condition for compactness of $N_{0,q}^\varphi$, we will apply $\square_{0,q}^\varphi$ to $(0, q)$-forms with coefficients (for the standard basis) belonging to $A^2(\mathbb{C}^n, e^{-\varphi})$. The point is that $\square_{0,q}^\varphi$ restricts to a multiplication operator on this space.

**Theorem 3.1.** Let $\varphi: \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic $C^2$ function and suppose that the corresponding weighted space $A^2(\mathbb{C}^n, e^{-\varphi})$ of entire function is infinite dimensional. If there is $1 \leq q \leq n$ such that the $\bar{\partial}$-Neumann operator $N_{0,q}^\varphi$ is compact, then

$$\limsup_{z \to \infty} \text{tr}(M_\varphi(z)) = +\infty.$$  

**Proof.** Let $g$ be a smooth $(0, q)$-form, $g = \sum'_{|J| = q} g_J d\zeta_J \in \Omega^{0,q}(\mathbb{C}^n)$. If the coefficients $g_J$ of $g$ are holomorphic, then $g \in \ker(\bar{\partial}; \Omega^{0,q}(\mathbb{C}^n) \to \Omega^{0,q+1}(\mathbb{C}^n))$, and $\square_{0,q}^\varphi g$ as in (1.2) is given by

$$\boldsymbol{\bar{\partial}\bar{\partial}_c}^\varphi g = \bar{\partial} \left( - \sum'_{|K| = q-1} \sum_{k=1}^n \left( \frac{\partial}{\partial z_k} - \frac{\partial \varphi}{\partial z_k} \right) g_{kK} d\zeta_K \right)$$

$$= - \sum_{j=1}^n \sum'_{|K| = q-1} \sum_{k=1}^n \left( \frac{\partial^2}{\partial \zeta_j \partial z_k} - \frac{\partial \varphi}{\partial z_k} \frac{\partial}{\partial \zeta_j} - \frac{\partial^2 \varphi}{\partial \zeta_j \partial z_k} \right) g_{kK} d\zeta_j \wedge d\zeta_K,$$

which reduces to

$$\sum'_{|K| = q-1} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial \zeta_j \partial z_k} g_{kK} d\zeta_j \wedge d\zeta_K,$$

(3.2)
see [11] for the computation of \( \partial_{d}^{\varphi} \). If \( \tilde{\square}^{\varphi}_{0,q} \) denotes the maximal closed extension of (1.2) to an operator on \( L^{2}_{0,q}(\mathbb{C}^{n}, e^{-\varphi}) \), defined in the sense of distributions on

\[
\text{dom}(\tilde{\square}^{\varphi}_{0,q}) := \{ u \in L^{2}_{0,q}(\mathbb{C}^{n}, e^{-\varphi}) : \partial_{d}^{\varphi} u \in L^{2}_{0,q}(\mathbb{C}^{n}, e^{-\varphi}) \},
\]

then \( \tilde{\square}^{\varphi}_{0,q} \) is symmetric and hence must equal the Gaffney extension.

Now assume that (3.1) is wrong, so that there exists \( C \geq 0 \) such that

\[
\text{tr}(M_{\varphi}(z)) \leq C
\]

for all \( z \in \mathbb{C}^{n} \). Since \( \varphi \) is assumed to be plurisubharmonic, \( M_{\varphi} \) is nonnegative and hence (3.3) is equivalent to all entries of \( M_{\varphi} \) being bounded. Indeed, \( \text{tr}(M_{\varphi}^{*}M_{\varphi}) = \text{tr}(M_{\varphi}^{2}) \leq \text{tr}(M_{\varphi})^{2} \), and the Hilbert-Schmidt norm \( A \mapsto \text{tr}(A^{*}A)^{1/2} \) on the space of complex \( n \times n \) matrices is equivalent to the max norm (i.e., the maximum of the entries). It now follows from (3.2) that all \((0, q)\)-forms \( g \) with coefficients in \( A^{2}(\mathbb{C}^{n}, e^{-\varphi}) \) belong to \( \text{dom}(\tilde{\square}^{\varphi}_{0,q}) = \text{dom}(\tilde{\square}^{\varphi}_{0,q}) \), and

\[
\tilde{\square}^{\varphi}_{0,q}g = \sum_{|K|=q-1}^{n} \sum_{j,k=1}^{n} \partial_{z_{j}}^{2} \varphi \frac{\partial^{2} \varphi}{\partial z_{j} \partial z_{k}} g_{kK} dz_{j} \wedge dz_{K}.
\]

Hence, if all such \( g \) belong to a bounded set of \( L^{2}_{0,q}(\mathbb{C}^{n}, e^{-\varphi}) \), the image under \( \tilde{\square}^{\varphi}_{0,q} \) of this bounded set will be again bounded in \( L^{2}_{0,q}(\mathbb{C}^{n}, e^{-\varphi}) \). In addition we have

\[
N_{0,q}^{\varphi} \left( \sum_{|K|=q-1}^{n} \sum_{j,k=1}^{n} \partial_{z_{j}}^{2} \varphi \frac{\partial^{2} \varphi}{\partial z_{j} \partial z_{k}} g_{kK} dz_{j} \wedge dz_{K} \right) = g,
\]

so that the identity on the (infinite dimensional) space of \((0, q)\)-forms with coefficients in \( A^{2}(\mathbb{C}^{n}, e^{-\varphi}) \) is the composition of the compact operator \( N_{0,q}^{\varphi} \) and a bounded operator, hence it is compact. This contradicts \( \text{dim}(A^{2}(\mathbb{C}^{n}, e^{-\varphi})) = \infty \). \( \square \)

**Remark 3.2.** (i) To obtain Theorem 3.1, one could also use the fact that compactness of \( N_{0,q}^{\varphi} \) “percolates up” the weighted \( \partial_{d} \)-complex, see [11]: if \( 1 \leq q \leq n-1 \) and \( N_{\varphi,q}^{0,1} \) is compact, then \( N_{\varphi,q+1}^{0} \) is also compact. At the upper end of the \( \partial_{d} \)-complex, one has

\[
\tilde{\square}^{\varphi}_{0,n}g = \partial_{d}^{2} \varphi g
\]

\[
= \partial_{d} \left( - \sum_{|K|=n-1}^{n} \sum_{k=1}^{n} \left( \frac{\partial \varphi}{\partial z_{k}} g_{kK} d_{z_{k}} \right) dz_{K} \right)
\]

\[
= - \frac{1}{4} \Delta g + \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_{j}} \frac{\partial g}{\partial z_{j}} + \frac{1}{4} (\Delta \varphi)g
\]

(3.4)

for \( g \in \Omega^{0,n}(\mathbb{C}^{n}) \), where we identify the \((0, n)\)-form

\[
g = \tilde{g} d z_{1} \wedge \cdots \wedge d z_{n} \in \Omega^{0,n}(\mathbb{C}^{n})
\]

with its coefficient \( \tilde{g} \in C^{\infty}(\mathbb{C}^{n}) \). If we consider \((0, n)\)-forms \( g \) with coefficient belonging to \( A^{2}(\mathbb{C}^{n}, e^{-\varphi}) \), then \( g \in \Omega^{0,n}(\mathbb{C}^{n}) \cap L^{2}_{0,n}(\mathbb{C}^{n}, e^{-\varphi}) \) and, by (3.4), we have

\[
\tilde{\square}^{\varphi}_{0,n}g = \frac{1}{4} (\Delta \varphi)g = \text{tr}(M_{\varphi})g,
\]
and one can now use the same argument as before to reach a contradiction.

(ii) Let \( n = 1 \) and suppose that \( \varphi : \mathbb{C} \to \mathbb{R} \) is a subharmonic \( C^2 \) function such that the Bergman space \( A^2(\mathbb{C}, e^{-\varphi}) \) is infinite dimensional. The complex Laplacian on \( (0, 1) \)-forms equals \( \square_0^\varphi = \partial \bar{\partial}^\varphi \), and the necessary condition (3.1) for compactness of the \( \bar{\partial} \)-Neumann operator becomes

\[
\limsup_{z \to \infty} \Delta \varphi(z) = +\infty. \tag{3.5}
\]

If one supposes that \( \Delta \varphi \) defines a doubling measure, the condition

\[
\lim_{z \to \infty} \int_{B_1(z)} \Delta \varphi \, d\lambda = +\infty \tag{3.6}
\]

is known to be both necessary and sufficient for compactness of the \( \bar{\partial} \)-Neumann operator, see [12] and [18]. Of course, (3.6) implies (3.5), as it should.

(iii) A sufficient condition for the Bergman space \( A^2(\mathbb{C}^n, e^{-\varphi}) \) to have infinite dimension is given in [21, Lemma 3.4]: If \( \varphi : \mathbb{C}^n \to \mathbb{R} \) is a smooth function such that

\[
\lim_{z \to \infty} |z|^2 s_1(z) = +\infty, \tag{3.7}
\]

where \( s_1 \) is the smallest eigenvalue of \( M_\varphi \), then \( A^2(\mathbb{C}^n, e^{-\varphi}) \) has infinite dimension. Condition (3.7) is not sharp: the function

\[
\varphi(z_1, z_2) = |z_1|^4 + |z_1 z_2|^2
\]

on \( \mathbb{C}^2 \) does not satisfy (3.7). Nevertheless, the corresponding space \( A^2(\mathbb{C}^2, e^{-\varphi}) \) is of infinite dimension since it contains all polynomials in \( z_1 \), see the computation in [11, p. 125].

4. A STRONGER RESULT OBTAINED FROM SCHRODINGER OPERATOR THEORY

Let \( u = u \, dz_1 \wedge \cdots \wedge dz_n \) be a smooth \((0, n)\)-form belonging to the domain of \( \square_0^\varphi \). For \( 1 \leq j \leq n \) denote by \( K_j \) the increasing multiindex \( K_j := (1, \ldots, j-1, j+1, \ldots, n) \) of length \( n - 1 \). Then

\[
\bar{\partial}^j u = \sum_{j=1}^n (-1)^{j+1} \left( \frac{\partial \varphi}{\partial z_j} u - \frac{\partial u}{\partial z_j} \right) dz_{K_j},
\]

Hence

\[
\bar{\partial} \partial^{j} u = \left[ \sum_{j=1}^n \frac{\partial \varphi}{\partial z_j} \left( \frac{\partial \varphi}{\partial z_j} u - \frac{\partial u}{\partial z_j} \right) dz_{K_j} \right] d z_1 \wedge \cdots \wedge d z_n.
\]

Conjugation with the unitary operator \( L^2(\mathbb{C}^n, e^{-\varphi}) \to L^2(\mathbb{C}^n) \) of multiplication by \( e^{-\varphi/2} \) gives

\[
e^{-\varphi/2} \square_0^\varphi e^{\varphi/2} f = \sum_{j=1}^n \left( - \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} - \frac{\partial f}{\partial z_j \partial \bar{z}_j} + \frac{1}{2} \frac{\partial \varphi}{\partial z_j} \frac{\partial f}{\partial \bar{z}_j} + \frac{1}{4} \frac{\partial \varphi}{\partial z_j} \frac{\partial \varphi}{\partial \bar{z}_j} f + \frac{1}{2} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j} f \right), \tag{4.1}
\]

where \( f \in L^2(\mathbb{C}^n) \) and we just wrote down the coefficient of the corresponding \((0, n)\)-form. This operator can be expressed by real variables in the form

\[
e^{-\varphi/2} \square_0^\varphi e^{\varphi/2} f = \frac{1}{4} (-\Delta_A + V) f, \tag{4.2}
\]
Theorem 4.1. Let \( \varphi: \mathbb{C}^n \to \mathbb{R} \) be smooth and plurisubharmonic. Then \( \Box_{0,n}^\varphi \) has compact resolvent on \( L^2_{0,n}(\mathbb{C}^n, e^{-\varphi}) \) if and only if the magnetic Schrödinger operator \(-\Delta_A + V\) has compact resolvent on \( L^2(\mathbb{R}^{2n})\). In particular, if there is \( 1 \leq q \leq n \) such that \( \Box_{0,q}^\varphi \) has compact resolvent, then

\[
\lim_{z \to \infty} \int_{B(z,1)} \operatorname{tr}(M_\varphi)^2 \, d\lambda = +\infty \tag{4.3}
\]

Proof. It remains to prove the last assertion. If \( \Box_{0,q}^\varphi \) has compact resolvent, then so does \( \Box_{0,n}^\varphi \), see Remark 1.1, and if the magnetic Schrödinger operator has compact resolvent, then

\[
\int_{B(z,1)} (|M_\varphi|^2 + 2 \operatorname{tr}(M_\varphi)) \, d\lambda \to \infty \tag{4.4}
\]

as \( z \to \infty \) by [15, Theorem 5.2], where \( |B|^2 := \sum_{j,k=1}^n |B_{j,k}|^2 \), since the magnetic field is \( i\partial\bar{\varphi} \).

Because \( M_\varphi \geq 0 \), the above is equivalent to \( \int_{B(z,1)} (\operatorname{tr}(M_\varphi)^2 + \operatorname{tr}(M_\varphi)) \, d\lambda \to \infty \) as \( z \to \infty \). By Hölder’s inequality,

\[
\int_{B(z,1)} \operatorname{tr}(M_\varphi) \, d\lambda \leq |B(z,1)|^{1/2} \left( \int_{B(z,1)} \operatorname{tr}(M_\varphi)^2 \, d\lambda \right)^{1/2},
\]

and we see that (4.4) implies \( f(\int_{B(z,1)} \operatorname{tr}(M_\varphi)^2 \, d\lambda) \to \infty \) as \( z \to \infty \), where \( f: [0, \infty) \to [0, \infty) \) is \( f(t) := t + t^{1/2} \). Because \( f \) is a monotone bijection, this is equivalent to \( \int_{B(z,1)} \operatorname{tr}(M_\varphi)^2 \, d\lambda \to \infty \) as \( z \to \infty \).

\[\square\]

Remark 4.2. (i) The above condition (4.3) is not a sufficient condition for compactness of \( N_{0,q}^\varphi \) in general, see Remark 5.2 below.

(ii) If \( \varphi: \mathbb{C}^n \to \mathbb{R} \) is such that \( \operatorname{tr}(M_\varphi) \) satisfies the reverse Hölder condition,

\[
\left( \frac{1}{|B|} \int_B \operatorname{tr}(M_\varphi)^r \, d\lambda \right)^{1/r} \leq C \frac{1}{|B|} \int_B \operatorname{tr}(M_\varphi) \, d\lambda
\]

for some \( r \geq 2 \), some \( C > 0 \), and all balls \( B \subseteq \mathbb{C}^n \), where \( \frac{1}{|B|} \int_B \) denotes the average over \( B \) for the Lebesgue measure, then Hölder’s inequality implies that (4.3) can be replaced by the formally weaker condition of

\[
\lim_{z \to \infty} \int_{B(z,1)} \operatorname{tr}(M_\varphi) \, d\lambda = +\infty. \tag{4.5}
\]
Theorem 5.1. Let \( A_\infty := \bigcup_{p \geq 1} A_p \), where \( A_p \) are the Muckenhoupt classes, see [22, Theorem 3, p. 212]. Every positive polynomial belongs to \( A_\infty \). In fact, \( |P|^p \in A_p \) for \( p > 1 \) if \(-1 < ad < p - 1\), where \( d \) is the degree of \( P \), see [22, 6.5, p 219].

5. Decoupled weights

We will now consider weight functions \( \varphi \) which are decoupled, meaning that
\[
\varphi(z) = \varphi_1(z_1) + \cdots + \varphi_n(z_n)
\]
for functions \( \varphi_j : \mathbb{C} \to \mathbb{R} \). Then
\[
\sigma(\Box^\varphi) = \bigcup_{q_1 + \cdots + q_n = q} \left( \sigma(\Box^{\varphi_1}) + \cdots + \sigma(\Box^{\varphi_n}) \right)
\]
and
\[
\sigma_\epsilon(\Box^\varphi) = \bigcup_{q_1 + \cdots + q_n = q} \left( \sigma_\epsilon(\Box_j^\varphi) + \sum_{k \neq j} \sigma(\Box^\varphi_k) \right).
\]

Here, \( \Box_j^\varphi \) denotes the complex Laplacian for the weight \( \varphi_j \) (of one variable) on \( \mathbb{C} \). This can be seen as a special case of [2, Theorem 5.5]. Indeed, the \( \bar{\partial} \)-operator acting on \( L^2(\mathbb{C}^n, e^{-\varphi}) \) can be understood geometrically as the \( \bar{\partial}E \)-operator for the trivial line bundle \( E := \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \), and with Hermitian metric \( \langle (z, v_1), (z, v_2) \rangle := v_1 \overline{v_2} e^{-\varphi(z)} \) on \( E \). If now \( E_j \) is \( \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) with metric given by \( \varphi_j : \mathbb{C} \to \mathbb{R} \), then it is easy to see that under the isomorphism
\[
(\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}) \cong \pi_1^* E_1 \otimes \cdots \otimes \pi_n^* E_n
\]
the trivial line bundle on \( \mathbb{C}^n \) carries the metric given by (5.1), where \( \pi_j : \mathbb{C}^n \to \mathbb{C} \) is the projection onto the \( j \)th factor. For the details on the general definition of \( \bar{\partial}E \) and its properties, we refer to [24, 16, 14, 8]. We note that equation (5.2) has also appeared in [4].

In the following Theorem 5.1, we will consider the case where all \( \Delta \varphi_j \) define nontrivial doubling measures. It is known from [18] (or [12, Theorem 2.3], with slightly stronger assumptions) that, under these conditions, \( N_{0,1}^{\varphi_j} \) (for the one-variable weights) is compact if and only if
\[
\lim_{z \to \infty} \int_{B_1(z)} \Delta \varphi_j d\lambda = +\infty.
\]
holds. Using this condition and the above results, we can characterize compactness of the \( \bar{\partial} \)-Neumann operator in terms of the decoupled weight:

**Theorem 5.1.** Let \( \varphi_j \in C^2(\mathbb{C}, \mathbb{R}) \) for \( 1 \leq j \leq n \) with \( n \geq 2 \), and set \( \varphi(z_1, \ldots, z_n) := \varphi_1(z_1) + \cdots + \varphi_n(z_n) \). Assume that all \( \varphi_j \) are subharmonic and such that \( \Delta \varphi_j \) defines a nontrivial doubling measure, then

(i) \( \dim(\ker(\Box^\varphi)) = \dim(A^2(\mathbb{C}^n, e^{-\varphi})) = \infty \),

(ii) \( \ker(\Box^\varphi) = 0 \) for \( q \geq 1 \),

(iii) \( N_{0,q}^\varphi \) is bounded for \( 0 \leq q \leq n \),

(iv) \( N_{\epsilon,q}^\varphi \) with \( 0 \leq q \leq n - 1 \) is not compact, and
(v) $N_{0,n}^{\varphi}$ is compact if and only if
\[
\lim_{z \to \infty} \int_{B_1(z)} \text{tr}(M_\varphi) \, d\lambda = \infty. \tag{5.5}
\]

Proof. From [17, Theorem C], it follows from our assumptions on $\varphi$ that $\bar{\partial}$ has closed range in $L^2_{0,j} (\mathbb{C}, e^{-\varphi})$ for all $j$. From this it also follows that $\bar{\partial}$ has closed range in $L^2_{0,q} (\mathbb{C}^n, e^{-\varphi})$ for all $0 \leq q \leq n$, see [5, Theorem 4.5] or [3, Corollary 2.15], which implies that the $\bar{\partial}$-Neumann operator is bounded.

Moreover, $\ker(\Box_{0,j}^{\varphi}) = 0$ for all $j$. In fact, in this simple case, we obtain from the Kohn-Morrey-Hörmander formula (1.4)
\[
\|\Box_{0,1}^{\varphi} u\|^2 = \|\partial_x^{\varphi} u\|^2 \geq \frac{1}{4} \int_{\mathbb{C}} \Delta \varphi_j |u_1|^2 e^{-\varphi_j} d\lambda \tag{5.6}
\]
for all forms $u = u_1 d\sigma \in \text{dom}(\partial_x^{\varphi})$. If $u \in \ker(\Box_{0,1}^{\varphi})$, then $u$ is smooth by elliptic regularity, and if $z_0$ is such that $\Delta \varphi_j(z_0) > 0$ (note that $\Delta \varphi_j \geq 0$ everywhere by subharmonicity), then $u = 0$ in a neighborhood of $z_0$. But then $u = 0$ everywhere since $\mathbb{C}$ is connected and by using a unique continuation principle of Aronszajn, see [1] or [8, p. 333]. We also have $\ker(\Box_{0,q}^{\varphi}) = 0$ for all $q \geq 1$, either by combining the above with the Künneth formula,
\[
\ker(\Box_{0,q}^{\varphi}) \cong \bigoplus_{q_1 + \cdots + q_n = q} \ker(\Box_{0,q_1}^{\varphi}) \otimes \cdots \otimes \ker(\Box_{0,q_n}^{\varphi}), \tag{5.7}
\]
see [3, Corollary 2.15] or [5, Theorem 4.5], where $\hat{\otimes}$ denotes the Hilbert space tensor product, or directly by using the same argument as above, i.e., applying (1.4) to the higher dimensional problem and using that $M_\varphi \neq 0$ at one point.

As a nontrivial doubling measure, $\Delta \varphi_j$ satisfies $\int_{\mathbb{C}} \Delta \varphi_j \, d\lambda = \infty$. Consequently, the weighted Bergman space $A^2_{\varphi} (\mathbb{C}, e^{-\varphi})$ has infinite dimension by [19, Theorem 3.2]. This also implies, by (5.7), that $\dim(\ker(\Box_{0,0}^{\varphi})) = \infty$. On the other hand, $\dim(A^2(\mathbb{C}, e^{-\varphi})) = \infty$ for at least one $1 \leq j \leq n$ implies that $N_{0,q}^{\varphi}$ cannot be compact for $0 \leq q \leq n - 1$, see [2, Theorem 5.6]: indeed, $0 \in \sigma_e(\Box_{0,0}^{\varphi})$ and hence (5.3) implies that the infinite sets $\sigma(\Box_{0,q}^{\varphi})$ for $j \neq k$ are contained in $\sigma_e(\Box_{0,q}^{\varphi})$. This finishes the proof of (i) to (iv). Again by [2, Theorem 5.6], $N_{0,n}^{\varphi}$ is compact if and only if all $N_{0,1}^{\varphi_j}$ are compact, which is the case if and only if (5.4) holds for all $1 \leq j \leq n$. It remains to show that this is equivalent to (5.5). By a simple scaling argument, the claim is equivalent to
\[
\int_{B_1(z_1) \times \cdots \times B_{n}(z_n)} \text{tr}(M_\varphi) \, d\lambda = \frac{\pi^{n-1}}{4} \sum_{j=1}^{n} \int_{B_1(z_j)} \Delta \varphi_j \, d\lambda \to \infty \quad \text{as} \quad z = (z_1, \ldots, z_n) \to \infty, \tag{5.8}
\]
and if (5.4) holds for all $1 \leq j \leq n$, then (5.8) is also satisfied. Conversely, if (5.8) is true, then choosing $z = \zeta e_k$ with $\zeta \in \mathbb{C}$ and $e_k$ the $k$th standard basis vector of $\mathbb{C}^n$ implies
\[
\int_{B_1(\zeta)} \Delta \varphi_k \, d\lambda + \sum_{j \neq k} \int_{B_{1}(0)} \Delta \varphi_j \, d\lambda \to \infty \quad \text{as} \quad \zeta \to \infty,
\]
so that $\lim_{\zeta \to \infty} \int_{B_1(\zeta)} \Delta \varphi_k \, d\lambda = \infty$ since the second term is bounded. This shows (v) and concludes the proof.

Remark 5.2. (i) The doubling condition is satisfied if the $\Delta \varphi_j$ belong to $A_{q_r}$, see [22, p. 196], where we recall from Remark 4.2 that $A_{q_r}$ is the union of the Muckenhoupt classes. As an example, $z \mapsto |z|^\alpha$ is in $A_p$ for $p > 1$ if and only if $-2 < \alpha < 2(p-1)$, and defines a doubling measure
for $-2 < \alpha$, cf. [22, 6.4, p. 218]. Since $\Delta |z|^\alpha = \alpha^2 |z|^{\alpha-2}$, we see that $\varphi_j(z) = |z|^\alpha$ satisfies the assumptions of Theorem 5.1 for all $\alpha \geq 4$ (so that $\varphi$ is at least $C^2$).

(ii) Let $\alpha \geq 0$ and consider the weight function

$$\varphi(z) = \sum_{j=1}^{n} |z_j|^{\alpha_j}, \quad (5.9)$$

where $\alpha_j \in \mathbb{R}$, $\alpha_j \geq 4$. Then $\varphi \in C^2(\mathbb{C}^n, \mathbb{R})$ and by the above

$$\lim_{z \to \infty} \text{tr}(M_\varphi(z)) = \lim_{z \to \infty} \frac{1}{4} \sum_{j=1}^{n} \alpha_j^2 |z_j|^{\alpha_j-2} = +\infty.$$

Therefore it follows from Theorem 5.1 that $N_{0,q}^{\varphi}$ with $0 \leq q \leq n-1$ is not compact while $N_{0,n}^{\varphi}$ is compact. Hence, the necessary conditions (5.1) and (4.3) fail to be sufficient for compactness of $N_{0,q}^{\varphi}$ for $0 \leq q \leq n-1$ in general. Of course, by Remark 4.2 and Theorem 5.1, the integral condition (4.3) is both necessary and sufficient for compactness of $N_{0,n}^{\varphi}$ for plurisubharmonic weights $\varphi$ with $\text{tr}(M_\varphi) \in A_\infty$, such as (5.9).

(iii) Using a variation of the above decoupled weights, one easily sees that, for $q > n/2$, there is a plurisubharmonic function $\varphi_q : \mathbb{C}^n \to \mathbb{R}$, such that $N_{0,k}^{\varphi_q}$ is compact precisely for $q \leq k \leq n$. Indeed, one may take

$$\varphi_q(z_1, \ldots, z_n) := |(z_1, \ldots, z_{q-1})|^4 + |(z_q, \ldots, z_n)|^4.$$

Then both of the spaces $A^2(\mathbb{C}^{q-1}, e^{-\varphi_1})$ and $A^2(\mathbb{C}^{n-q+1}, e^{-\varphi_2})$, where $\varphi_1 : \mathbb{C}^{q-1} \to \mathbb{R}$, $\varphi(z) := |z|^4$, and $\varphi_2 : \mathbb{C}^{n-q+1} \to \mathbb{R}$, $\varphi_2(z) := |z|^4$, have infinite dimension by the result of Shigekawa quoted in Remark 3.2. Moreover, the $\overline{\partial}$-Neumann operators $N_{0,q}^{\varphi_j}$ are compact for $q \geq 1$ and $j \in \{1, 2\}$, as is easily deduced by verifying (1.5). Since $n - q + 1 < n/2 + 1$ implies $n - q + 1 \leq n/2 \leq q - 1$, one obtains from [2, Theorem 5.2] that $N_{0,k}^{\varphi}$ is compact exactly for $k = q - 1 + j$ with $j \geq 1$, as claimed.

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