Statistics of narrow-band partially polarized light

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Abstract
A complete single-point statistical description of a narrow-band partially polarized optical field is developed in terms of the 2D period-averaged probability density function (PA-PDF) of the electrical field vector. This statistic can be measured using the coherent (heterodyne) detection. PA-PDF carries more information about the partially polarized light than the traditional Stokes vector. For a simple Gaussian partially polarized field the PA-PDF depends on 13 real parameters in contrast to the four parameters of the Stokes vector or coherence tensor. We show on several examples that the polarization state of the wave, as described by PA-PDF can vary significantly even while Stokes vector remains fixed.

Keywords: polarization, statistical optics, Stokes vector

(Some figures may appear in colour only in the online journal)

1. Introduction

For more than 150 years, the Stokes vector [1] has been exclusively used to describe the local polarization state of electromagnetic radiation. The recent review paper on the subject [2] still states that ‘polarization ... is fully described by the ... Stokes vector.’ The popularity of the Stokes parameters is based on the fact that a linear polarizer and retardation plate are all that is needed to measure them [3]. The more recent development of the so-called unified theory of coherence and polarization of stochastic electromagnetic beams, see for example [4], added spatial coherence to the polarization theory and made it possible to investigate the propagation of partially polarized beam waves. However at each point the coherence tensor of the field, which is called a cross-spectral density matrix in [4], contains the same information as the Stokes vector.

The objective of this paper is to develop a complete statistical description of a partially polarized quasi-monochromatic field at a single point. For this purpose, we treat the electrical field as a two-dimensional random process. The quasi-monochromatic nature of the field implies that this process is non-stationary. The non-stationarity manifests itself as high-frequency harmonic oscillations, and this necessitates a modification of the traditional probability theory treatment, which mostly deals with the stationary processes, or non-stationary processes with slowly varying parameters. It is clear from the beginning that the Stokes vector, which is comprised only from the second statistical moments of the field, cannot provide exhaustive information about the statistics of the polarized field. Even when only the first and the second statistical moments of the field are considered, there are, as we will show, as many as 13 parameters that characterize the statistics.

In section 1 we present statistics of the narrow band partially polarized field in terms of the non-stationary oscillating probability density function (PDF), introduce the more perceivable concept of the period-averaged PDF (PA-PDF), and discuss the relationship between the classic Stokes vector and the statistical moments of our description. In section 2, we apply our approach to the simplest case of the oscillating bivariate Gaussian PDF and show some examples of the ensuing PA-PDFs. In section 3, based on several numerical examples, we show that the Stokes vector does not fully characterize the state of the partially polarized wave. The discussion in section 4 primarily addresses the measurements of the PA-PDF and associated statistical moments.

2. General statistics

Electric field $E(t)$ of the transverse electromagnetic wave with carrier frequency $\omega$ with local wave front normal along the $z$-axis at some point in the $(x, y)$ plane can be presented as

$$E(t) = \begin{pmatrix} E_x(t) \\ E_y(t) \end{pmatrix} = \text{Re} \begin{pmatrix} u(t)e^{i\omega t} \\ v(t)e^{i\omega t} \end{pmatrix}.$$  \hspace{1cm} (1)
Here $E_x$ and $E_y$ are orthogonal components of an electrical field in the $x$ and $y$ directions, and $u(t)$ and $v(t)$ are random complex amplitudes of these components. For the quasi-monochromatic waves considered here, correlation time $t_C$ of $u(t)$ and $v(t)$ is much larger than the carrier oscillation period

$$\omega t_C \gg 1.$$  \hspace{1cm} (2)

Field $\mathbf{E}(t)$ is a real-valued two-dimensional non-stationary random vector that can be represented as

$$\mathbf{E}(t) = \begin{pmatrix} u_R(t) \cos(\omega t) - u_I(t) \sin(\omega t) \\ v_R(t) \cos(\omega t) - v_I(t) \sin(\omega t) \end{pmatrix}. \hspace{1cm} (3)$$

where subscripts $R$ and $I$ stand for the real and imaginary parts (in-phase and quadrature components) of corresponding

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example1.png}
\caption{Two examples of the electric vector hodographs for partially polarized waves. (a) Mean field is larger than the fluctuating component. (b) Mean field is smaller than the fluctuating component.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example2.png}
\caption{Examples of the non-stationary PDF changes over the oscillation period. (a) Rotating scattering ellipse with constant principal axes magnitudes. (b) Principal axes maintain direction but change the magnitude.}
\end{figure}
complex amplitudes. For time intervals less than $t_C$ vector $E(t)$ inscribes an ellipse at the $(E_x, E_y)$ plane. For a narrowband polarized wave this ellipse randomly changes with time. These changes are slow in the carrier oscillations time scale $t_0 = 2\pi/\omega$. The concept of instantaneous ellipse for electrical field was discussed in chapter 1 of [5] and in section 3.1 of [6]. Freund [7] discussed Lissajous figures traced by polychromatic harmonically modulated electrical field. Figure 1 shows two examples of hodographs of vector $E(t)$. Dark ellipses represent the mean or coherent component $(E(t))$. Here and further on the angular brackets indicate the statistical ensemble averaging over the field fluctuations. For the ease of graphical presentation a relatively short correlation time $t_C/t_0 = 3$ was chosen here, and 10 periods of carrier oscillations were traced. In the case of relatively small field fluctuations with respect to the mean shown at the left panel, $E(t)$ remains close to the polarization ellipse of the mean field. The right panel’s field fluctuations are larger than the mean field and $E(t)$ outlines ellipses, which shape, size and orientation slowly evolve with time. This polarization dynamics can be described in terms of the instantaneous Stokes vector movement on the Poincare sphere [8]. Here we assume that the fluctuations of the local normal to the wave front are negligibly small and Cartesian basis used in equation (3) remain fixed. Note also that, due to the random modulation, hodographs in figure 1 have no resemblance to Lissajous figures of [7].

The complete statistic of the random vector $E(t)$ at any fixed moment is given by non-stationary PDF

$$P(E_x, E_y, t)de_xde_y = \text{Prob}(E_x < E_x(t) < E_x + de_x, E_y < E_y(t) < E_y + de_y).$$

(4)

Obviously $P(E_x, E_y, t)$ is a $t_0$—periodic function of time. Typically, the carrier frequency is known, and carries no useful information. The classical parametrization of a polarized field, such as complex Jones vector,

$$J \equiv \begin{pmatrix} u_x \\ v_x \\ u_y \\ v_y \end{pmatrix} = \begin{pmatrix} u_x + i u_y \\ v_x + i v_y \end{pmatrix},$$

(5)

is a PA, possibly random, but not oscillating parameter. In contrast, Stokes vector

$$S \equiv \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} \langle |u_x|^2 \rangle + \langle |v_x|^2 \rangle \\ \langle |u_y|^2 \rangle - \langle |v_y|^2 \rangle \\ 2 \Re \langle u_x v_y \rangle - 2 \Im \langle u_y v_x \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle u_x^2 \rangle + \langle u_y^2 \rangle + \langle v_x^2 \rangle + \langle v_y^2 \rangle \\ \langle u_x^2 \rangle + \langle u_y^2 \rangle - \langle v_x^2 \rangle - \langle v_y^2 \rangle \\ 2 \langle u_x v_y \rangle + 2 \langle u_y v_x \rangle - 2 \langle u_y v_y \rangle \end{pmatrix},$$

(6)

and correlation tensor

$$W \equiv \begin{pmatrix} \langle |u_x|^2 \rangle + \langle |v_x|^2 \rangle & \langle u_x u_y \rangle & \langle u_x v_y \rangle & \langle u_x v_x \rangle \\ \langle u_x u_y \rangle & \langle |v_x|^2 \rangle + \langle |v_y|^2 \rangle & \langle u_y v_y \rangle & \langle u_y v_x \rangle \\ \langle u_x v_y \rangle & \langle u_y v_y \rangle & \langle |v_y|^2 \rangle + \langle |u_x|^2 \rangle & \langle u_x u_y \rangle \\ \langle u_x v_x \rangle & \langle u_y v_x \rangle & \langle u_x u_y \rangle & \langle |u_x|^2 \rangle + \langle |v_x|^2 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle u_x^2 \rangle + \langle u_y^2 \rangle & \langle u_x u_y \rangle & \langle u_x v_y \rangle & \langle u_x v_x \rangle \\ \langle u_x u_y \rangle & \langle |v_x|^2 \rangle + \langle |v_y|^2 \rangle & \langle u_y v_y \rangle & \langle u_y v_x \rangle \\ \langle u_x v_y \rangle & \langle u_y v_y \rangle & \langle |v_y|^2 \rangle + \langle |u_x|^2 \rangle & \langle u_x u_y \rangle \\ \langle u_x v_x \rangle & \langle u_y v_x \rangle & \langle u_x u_y \rangle & \langle |u_x|^2 \rangle + \langle |v_x|^2 \rangle \end{pmatrix}$$

(7)

are both ensemble-averaged statistics, as indicated by the angular brackets that neither oscillate, nor fluctuate. The Stokes vector and correlation tensor hold the same information about the statistics of partially polarized field, and the Stokes vector can be represented in terms of the components of the correlation tensor and vice versa. It is possible to form the random, instantaneous Stokes vector and correlation tensor, but they carry the same information as Jones vector.

Correlation tensor $W$ does not define all possible second moments of the Jones vector components, however after supplementing it with the second correlation tensor

$$B \equiv \begin{pmatrix} \langle u_x^2 \rangle & \langle u_x v_y \rangle \\ \langle u_x v_y \rangle & \langle v_y^2 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle u_x^2 \rangle - \langle u_y^2 \rangle + 2i \langle u_x u_y \rangle & \langle u_x v_y \rangle - \langle u_y v_y \rangle \\ \langle u_x v_y \rangle + \langle u_y v_y \rangle & \langle v_y^2 \rangle + 2 \langle v_y v_y \rangle \end{pmatrix}$$

(8)

it is possible to retrieve a full set of the second moments of the Jones vector as follows

$$\langle u_x^2 \rangle = \frac{1}{2} \Re (W_{uu} + B_{uu}), \langle u_y^2 \rangle = \frac{1}{2} \Re (W_{uu} - B_{uu}),$$

$$\langle v_y^2 \rangle = \frac{1}{2} \Re (W_{vv} - B_{vv}), \langle v_y v_y \rangle = \frac{1}{2} \Re (W_{vv} + B_{vv}),$$

$$\langle u_x v_y \rangle = \frac{1}{2} \Im (W_{uv} + W_{vu}), \langle u_x v_y \rangle = \frac{1}{2} \Im (B_{uv} + W_{vu}),$$

$$\langle u_x v_x \rangle = \frac{1}{2} \Im (B_{uu} + W_{uu}), \langle u_x v_x \rangle = \frac{1}{2} \Im (B_{uu} - W_{uu}).$$

(9)

Full statistical description of the partially polarized field in space and time requires a complete set of multiple point and multiple time moments PDFs of the electrical field, as was pointed out in section 3.1 of [6]. Here we consider only the simplest case of a single point and single time moment statistics. In addition, we assume that it is possible to measure the instantaneous components of the field, but the exact time of the measurement is undetermined.
at the carrier oscillation scale $t_0$. In other words, the absolute phase of the field is unknown or irrelevant. Alternatively, the multiple measurements of $(E_x, E_y)$ are performed at the time moments $t_n$ that are uniformly distributed over the carrier oscillation period $t_0$. In this case the probability distribution of the measurement outcomes is the PA-PDF

$$F(E_x, E_y) \equiv \frac{1}{t_0} \int_0^{t_0} dP(E_x, E_y, t).$$

(10)

PA-PDF contains complete single-point statistics of the real electrical field vector $\mathbf{E}(t)$, except the part that is related to the absolute phase, or clock zero. For example, PA-PDF cannot provide probability distribution of the absolute phase, or clock zero. For example, PA-PDF gives the probable distribution of the measurement outcomes is the PA-PDF.

Given the joint probability distribution of the quadrature components $w(\xi R, \xi I, \eta R, \eta I)$ it is possible to calculate the non-stationary PDF of the field as

$$P(E_x, E_y, t) = \langle \delta(E_x - \xi R \cos \omega t + \xi I \sin \omega t) \delta(E_y - \eta R \cos \omega t + \eta I \sin \omega t) \rangle$$

$$= \int d\xi R d\xi I d\eta R d\eta I P(E_x, \xi R, \xi I, \eta R, \eta I)$$

$$= \int d\xi R d\xi I d\eta R d\eta I P(E_x, \xi R, \xi I, \eta R, \eta I)$$

$$= \int d\xi R d\xi I d\eta R d\eta I P(E_x, \xi R, \xi I, \eta R, \eta I)$$

$$= \int d\xi R d\xi I d\eta R d\eta I P(E_x, \xi R, \xi I, \eta R, \eta I)$$

and then use equation (10) to calculate the PA-PDF.

Obviously, the Stokes vector does not exhaust the statistics of the field at a point. For instance, the Stokes vector does not include the statistical moments of the orders higher than two that can possibly be important for the light from non-thermal sources [6]. But, even when only the moments up to the second order are concerned, the Stokes vector still does not capture all the information. The deficiency of the Stokes vector description was noticed before [5], but no attempt was made to develop the comprehensive statistics.

Presenting the in-phase and quadrature parts of the complex components of Jones vector as sums of mean and the fluctuating components

$$u_R(t) = \langle u_R \rangle + \tilde{u}_R(t), \quad u_I(t) = \langle u_I \rangle + \tilde{u}_I(t),$$

$$\eta_R(t) = \langle \eta_R \rangle + \tilde{\eta}_R(t), \quad \eta_I(t) = \langle \eta_I \rangle + \tilde{\eta}_I(t),$$

(12)

we limit our attention to the mean values $\langle u_R \rangle, \langle u_I \rangle, \langle \eta_R \rangle$ and $\langle \eta_I \rangle$, and the covariance matrix of the central second moments

$$C \equiv \begin{pmatrix} \langle \tilde{u}_R^2 \rangle & \langle \tilde{u}_R \tilde{u}_I \rangle & \langle \tilde{u}_R \tilde{\eta}_R \rangle & \langle \tilde{u}_R \tilde{\eta}_I \rangle \\ \langle \tilde{u}_R \tilde{u}_I \rangle & \langle \tilde{u}_I^2 \rangle & \langle \tilde{u}_I \tilde{\eta}_R \rangle & \langle \tilde{u}_I \tilde{\eta}_I \rangle \\ \langle \tilde{u}_R \tilde{\eta}_R \rangle & \langle \tilde{u}_I \tilde{\eta}_R \rangle & \langle \tilde{\eta}_R^2 \rangle & \langle \tilde{\eta}_R \tilde{\eta}_I \rangle \\ \langle \tilde{u}_R \tilde{\eta}_I \rangle & \langle \tilde{u}_I \tilde{\eta}_I \rangle & \langle \tilde{\eta}_R \tilde{\eta}_I \rangle & \langle \tilde{\eta}_I^2 \rangle \end{pmatrix}.$$  

(13)

The Stokes vector can be represented in terms of the mean values and covariances as

$$\mathbf{S} = \begin{pmatrix} \langle u_R^2 \rangle + \langle u_I^2 \rangle + \langle \eta_R^2 \rangle + \langle \eta_I^2 \rangle \\ \langle u_R \rangle (\langle u_R \rangle + \langle u_I \rangle) + \langle \eta_R \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) \\ 2\langle u_R \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) + 2\langle u_I \rangle (\langle u_R \rangle + \langle u_I \rangle) + 2\langle \eta_R \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) + 2\langle \eta_I \rangle (\langle u_R \rangle + \langle u_I \rangle) \\ 2\langle u_I \rangle (\langle u_R \rangle + \langle u_I \rangle) + 2\langle \eta_I \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) + 2\langle \eta_R \rangle (\langle u_R \rangle + \langle u_I \rangle) + 2\langle \eta_I \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) \end{pmatrix}.$$  

(14)

It is immediately noticeable that the covariances $\langle \tilde{u}_R \tilde{u}_I \rangle$ and $\langle \tilde{\eta}_I \tilde{\eta}_R \rangle$ do not affect the Stokes vector, and the parameters’ count suggests that only four combinations of the possible 14 values of the four means and ten components of the symmetric semi-positive definite (SPD) covariance matrix $C$ enter the Stokes vector. Note that the 14 components are essential for the oscillating non-stationary PDF, equation (4). Since the field state at $t = 0$ is not relevant for the PA-PDF, equation (10), we can always eliminate one of the 14 parameters, e.g. by setting $\langle E_x(0) \rangle = 0$ or, equivalently, $\langle \eta_R \rangle = 0$. This leaves 13 parameters that fully define the first and second-order statistical moments of the polarization state of the partially polarized wave in contrast to the four parameters offered by the Stokes vector’s description.

Correlation tensor $\mathbf{W}$, equation (7) can be presented in terms of the mean values and covariances as

$$\mathbf{W} = \begin{pmatrix} \langle u_R^2 \rangle + \langle u_I^2 \rangle + \langle \tilde{u}_R^2 \rangle + \langle \tilde{u}_I^2 \rangle \\ \langle u_R \rangle (\langle u_R \rangle + \langle u_I \rangle) + \langle \eta_R \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) \\ 2\langle u_R \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) + 2\langle u_I \rangle (\langle u_R \rangle + \langle u_I \rangle) + 2\langle \eta_R \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) + 2\langle \eta_I \rangle (\langle u_R \rangle + \langle u_I \rangle) \\ 2\langle u_I \rangle (\langle u_R \rangle + \langle u_I \rangle) + 2\langle \eta_I \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) + 2\langle \eta_R \rangle (\langle u_R \rangle + \langle u_I \rangle) + 2\langle \eta_I \rangle (\langle \eta_R \rangle + \langle \eta_I \rangle) \end{pmatrix}.$$  

(15)
Figure 3. Examples of PA-PDFs. (a) Almost perfectly elliptically polarized wave. (b) Depolarized wave with non-Gaussian PA-PDF.

Figure 4. Examples of PA-PDFs with different mean fields but identical Stokes vectors.

Figure 5. Examples of PA-PDFs with identical mean fields and Stokes vectors but varying quadrature components.
It is customary to present the Stokes vector as a sum of polarized and non-polarized parts

\[
S = S_{\text{POL}} + S_{\text{DEP}} = \begin{pmatrix}
\sqrt{Q^2 + U^2 + V^2} \\
Q \\
U \\
V
\end{pmatrix}
+ \begin{pmatrix}
I - \sqrt{Q^2 + U^2 + V^2} \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Equation (14) suggests an alternative presentation of the Stokes vector as a sum of the two terms determined by the mean and fluctuating components of the field

\[
S = S_M + S_F = \begin{pmatrix}
\langle u_R \rangle^2 + \langle u_I \rangle^2 + \langle v_R \rangle^2 + \langle v_I \rangle^2 \\
\langle u_R \rangle^2 + \langle u_I \rangle^2 - \langle v_R \rangle^2 - \langle v_I \rangle^2 \\
\langle \tilde{u}_R \rangle + \langle \tilde{u}_I \rangle + \langle \tilde{v}_R \rangle + \langle \tilde{v}_I \rangle \\
\langle \tilde{u}_R \rangle + \langle \tilde{u}_I \rangle - \langle \tilde{v}_R \rangle - \langle \tilde{v}_I \rangle \\
\langle \tilde{u}_R \rangle + \langle \tilde{u}_I \rangle + 2\langle \tilde{v}_R \rangle + 2\langle \tilde{v}_I \rangle \\
2\langle \tilde{u}_R \rangle - 2\langle \tilde{v}_R \rangle
\end{pmatrix},
\]

(17)

The mean component is always fully polarized. The fluctuating component may or may not have a polarized part.

3. PA-PDF for a Gaussian polarized field

3.1. Oscillating and PA-PDFs

In this section, we illustrate the conceptual development of the previous section by, possibly, the simplest example of normal distribution of the complex amplitudes of the field. Namely, equation (3), the field components \(E_x\) and \(E_y\) are linear combinations of the normal complex amplitudes, they also have normal distribution. Parameters of this normal distribution can be calculated as follows. The mean field is

\[
\langle \mathbf{E}(t) \rangle = \begin{pmatrix}
\langle E_x(t) \rangle \\
\langle E_y(t) \rangle
\end{pmatrix} = \begin{pmatrix}
\langle u_R \rangle C - \langle u_I \rangle S \\
\langle v_R \rangle C - \langle v_I \rangle S
\end{pmatrix}.
\]

(19)

and the covariance tensor is

\[
\begin{pmatrix}
\langle \tilde{E}_x(t) \rangle & \langle \tilde{E}_y(t) \rangle \\
\langle \tilde{E}_y(t) \rangle & \langle \tilde{E}_x(t) \rangle
\end{pmatrix} = \begin{pmatrix}
\langle \tilde{u}_R \rangle^2 C^2 + \langle \tilde{u}_I \rangle^2 S^2 & -2\langle \tilde{u}_R \rangle \langle \tilde{u}_I \rangle CS - [\langle \tilde{u}_R \rangle \langle \tilde{v}_R \rangle + \langle \tilde{u}_I \rangle \langle \tilde{v}_R \rangle ]CS \\
-2\langle \tilde{u}_R \rangle \langle \tilde{u}_I \rangle CS + 2\langle \tilde{v}_R \rangle \langle \tilde{v}_I \rangle CS & \langle \tilde{v}_R \rangle^2 C^2 + \langle \tilde{v}_I \rangle^2 S^2 -2\langle \tilde{v}_R \rangle \langle \tilde{v}_I \rangle CS - 2\langle \tilde{v}_R \rangle \langle \tilde{v}_I \rangle CS
\end{pmatrix}.
\]

(20)

Here we used the shorthand notations

\[
C \equiv \cos(\omega t), \quad S \equiv \sin(\omega t).
\]

(21)

Mean field and covariance tensor are all that is necessary to present the non-stationary bivariate Gaussian PDF \(P(E_x, E_y, t)\) as

\[
P(E_x, E_y, t) = \frac{1}{2\pi\sqrt{\langle \tilde{E}_x(t) \rangle \langle \tilde{E}_y(t) \rangle [1 - C^2(t)]}} \exp \left[ -\frac{\langle (E_x - \langle E_x \rangle) \rangle^2}{2[1 - C^2(t)]\langle \tilde{E}_x(t) \rangle} - \frac{\langle (E_y - \langle E_y \rangle) \rangle^2}{2[1 - C^2(t)]\langle \tilde{E}_y(t) \rangle} - \frac{\rho(t)(E_x - \langle E_x \rangle)(E_y - \langle E_y \rangle)}{\sqrt{\langle \tilde{E}_x(t) \rangle \langle \tilde{E}_y(t) \rangle [1 - C^2(t)]}} \right].
\]

(22)

where the correlation coefficient of the fluctuating field components \(\tilde{E}_x\) and \(\tilde{E}_y\) is

\[
\rho(t) = \frac{\langle \tilde{E}_x(t) \tilde{E}_y(t) \rangle}{\sqrt{\langle \tilde{E}_x(t) \rangle \langle \tilde{E}_y(t) \rangle}} = \frac{\langle \tilde{u}_R \tilde{v}_R \rangle C^2 - [\langle \tilde{u}_R \rangle \langle \tilde{v}_R \rangle + \langle \tilde{u}_I \rangle \langle \tilde{v}_I \rangle ]CS + \langle \tilde{u}_I \rangle^2 S^2 - 2\langle \tilde{v}_R \rangle \langle \tilde{v}_I \rangle CS + \langle \tilde{v}_I \rangle^2 S^2}{\sqrt{\langle \tilde{u}_R \rangle^2 C^2 - 2\langle \tilde{u}_R \rangle \langle \tilde{u}_I \rangle CS + \langle \tilde{u}_I \rangle^2 S^2 - 2\langle \tilde{v}_R \rangle \langle \tilde{v}_I \rangle CS + \langle \tilde{v}_I \rangle^2 S^2}}.
\]

(23)

we assume that

\[
w(u_R, u_I, v_R, v_I) = \frac{1}{4\pi}\frac{1}{\sqrt{|C|}} \exp \left[ \frac{1}{2} \begin{pmatrix}
\frac{u_R - \langle u_R \rangle}{u_I - \langle u_I \rangle} \\
\frac{v_R - \langle v_R \rangle}{v_I - \langle v_I \rangle}
\end{pmatrix} C^{-1} \begin{pmatrix}
\frac{u_R - \langle u_R \rangle}{u_I - \langle u_I \rangle} \\
\frac{v_R - \langle v_R \rangle}{v_I - \langle v_I \rangle}
\end{pmatrix} \right].
\]

(18)

Non-stationary PDF \(P(E_x, E_y, t)\), equation (4), can be calculated using equation (11). However, since, according to it is remarkable that only the sum of covariances \(\langle \tilde{u}_R \tilde{v}_R \rangle\) and \(\langle \tilde{u}_I \tilde{v}_I \rangle\) enters the covariance tensor of the oscillating field components, equation (20), and the non-stationary Gaussian PDF, equation (22). The difference \(\langle \tilde{u}_R \tilde{v}_I \rangle - \langle \tilde{u}_I \tilde{v}_R \rangle\), or equivalently, the fourth component \(F_T\) of the fluctuation-related Stokes vector \(S_F\), in equation (17), does not affect the oscillating Gaussian PDF, and only 13 independent combinations out of the total of 14 first and second moments of the Jones vector enter equation (22). The \(\langle \tilde{u}_R \tilde{v}_I \rangle - \langle \tilde{u}_I \tilde{v}_R \rangle\) is related to the difference of the right and left-handed circular polarizations of the electrical field fluctuations, and appears to have no effect when field is
normally distributed. Of course, the \(\langle u_R v_R \rangle - \langle u_I v_I \rangle\) can affect the higher non-central moments for non-Gaussian PDFs.

Non-stationary two-dimensional PDF \(P(E_x, E_y, t)\), equation (22) is a periodic function of time with period \(t_o\). The mean electric vector \(\langle E(t) \rangle\) traces out a mean polarization ellipse with period \(t_o\) while, according to equation (20), the scattering ellipse of the Gaussian PDF changes its shape, size and orientation with period \(t_o/2\). Figure 2 shows two examples of \(P(E_x, E_y, t)\) at five time moments equally spaced over the period \(t_o\) starting with \(t = 0\). For both examples the mean field is

\[
\begin{bmatrix}
\langle u_R \rangle \\
\langle u_I \rangle \\
\langle v_R \rangle \\
\langle v_I \rangle
\end{bmatrix} = \begin{bmatrix}
10 \\
2 \\
20
\end{bmatrix},
\]

and the mean field ellipse is the same for both panels. At the left panel the scattering ellipse keeps the constant size and shape while rotating 360°. Statistical parameters for the left panel of figure 2 are

\[
C = \begin{bmatrix}
10 & 1 & -1 & 4 \\
4 & 1 & -1 & 10 \\
1 & 27.3 & 3.3 & 1 \\
-1 & 3.3 & 27.3 & 1
\end{bmatrix},
\]

\[
S = \begin{bmatrix}
530.6 & 0 & 0 & 0 \\
0 & -296 & 0 & 0 \\
0 & 0 & -402.6 & 0 \\
0 & 0 & 0 & -80
\end{bmatrix},
\]

\[
S_{POL} = \begin{bmatrix}
506.1 & 0 & 0 & 0 \\
0 & -296 & 0 & 0 \\
0 & 0 & -402.6 & 0 \\
0 & 0 & 0 & -80
\end{bmatrix},
\]

\[
S_{DEP} = \begin{bmatrix}
26.6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2.6 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

and the polarization degree is 0.954. Statistical parameters for the right panel of figure 2 are

\[
C = \begin{bmatrix}
10 & 0 & 0 & 0 \\
0 & 0.31 & 0 & 0 \\
0 & 0 & 0.31 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
S = \begin{bmatrix}
516.6 & 0 & 0 & 0 \\
0 & -288 & 0 & 0 \\
0 & 0 & -400 & 0 \\
0 & 0 & 0 & -80
\end{bmatrix},
\]

\[
S_{POL} = \begin{bmatrix}
499.3 & 0 & 0 & 0 \\
0 & -288 & 0 & 0 \\
0 & 0 & -400 & 0 \\
0 & 0 & 0 & -80
\end{bmatrix},
\]

\[
S_{DEP} = \begin{bmatrix}
12.6 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

and the polarization degree is 0.970. For this example the principal axes of the scattering ellipse maintain their orientation while the major and minor axes change according to equation (20). Note that, following the convention described earlier, \(\langle v_R \rangle = 0\), applies to this and all examples that follow. For figure 2 this simply implies that at the moment \(t = 0\) the center of the scattering ellipse is located at the \(x\)-axis.

Figure 3 shows two contrasting examples of the PA-PDF calculated by numerical time averaging of the non-stationary Gaussian PDF, equation (22), over the oscillation period \(t_o\), as prescribed by equation (10). The left panel depicts the PA-PDF for the almost fully elliptical polarized field with very small isotropic fluctuating component. Statistical parameters for the left panel of figure 3 are

\[
\begin{bmatrix}
\langle u_R \rangle \\
\langle u_I \rangle \\
\langle v_R \rangle \\
\langle v_I \rangle
\end{bmatrix} = \begin{bmatrix}
-1 \\
0 \\
0 \\
2
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0.001 & 0 & 0 & 0 \\
0 & 0.001 & 0 & 0 \\
0 & 0 & 0.001 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
S = \begin{bmatrix}
5.004 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

and the degree of polarization in this case is 0.9992. Here the field vector resides mostly on the ellipse corresponding to the elliptically polarized mean field as should be expected. Somewhat unexpected is a higher probability to register the field vector in the \(y\)-polarized state than in the \(x\)-polarized state. This rather counterintuitive result can be readily explained by calculating the angular velocity of the elliptically polarized field vector and discovering that it is not constant, and is lowest at the larger semi-axle position. Note also that the polarized part of the Stokes vector in this case is the same as the Stokes vector of the mean field and that the depolarized part of the Stokes vector is the same as the Stokes vector for the fluctuating field component.

The right panel of the figure 3 shows the opposite case of the fully random (diffuse) field. Statistical parameters for the right panel of figure 3 are

\[
\begin{bmatrix}
\langle u_R \rangle \\
\langle u_I \rangle \\
\langle v_R \rangle \\
\langle v_I \rangle
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
S = \begin{bmatrix}
2.002 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

and the Stokes vectors of the depolarized and fluctuating components are equal to the field Stokes vector while the polarized and mean parts of the Stokes vector are zero as is the degree of polarization. For this unpolarized wave one would expect to have a bivariate Gaussian distribution of the field, similar to one described in [6] and discussed in details in the next subsection. However, the right panel of figure 3 shows that this is not the case. Notwithstanding the very simple structure of the Stokes vector, the PA-PDF shows a sharp peak and four ridges that are related to the disparity of the components of the covariance matrix \(C\) of the fluctuating component of the field.
3.2. Large phase fluctuations. Brosseau PDF

In section 3.3 of [6] Brosseau presents PDF for the components of Jones vector, introduced here in equation (5). In our notations equation (3.3.1) of [6] has the form:

\[
w(u_R, u_t, v_R, v_t) = \frac{1}{\pi^2|\textbf{W}|} \exp[-\textbf{J}^* \cdot \textbf{W}^{-1} \cdot \textbf{J}] = \frac{1}{\pi^2(W_{uu}W_{vv} - |W_{uv}|^2)} \times \exp\left[\frac{(u_R^2 + u_t^2)W_{uv} - 2(u_Rv_R + u_tv_t)Re(W_{uv}) + 2(u_Rv_t - u_tv_R)Im(W_{uv}) + (v_R^2 + v_t^2)W_{vv}}{(W_{uu}W_{vv} - |W_{uv}|^2)}\right]
\]

Brosseau uses the analytic signal formulation for the complex narrow-band field throughout his fundamental text, and assumes zero-mean Gaussian distribution for this specific case. No derivation of equation (3.3.1) is given besides the references to three texts on random processes and signal processing. Covariance matrix of the zero-mean components of Jones vector can be calculated from equation (29) as follows:

\[
\mathbf{C} = \begin{pmatrix}
\frac{1}{2}W_{uu} & 0 & \frac{1}{2}Re(W_{uv}) & \frac{1}{2}Im(W_{uv}) \\
0 & \frac{1}{2}W_{uu} & \frac{1}{2}Im(W_{uv}) & \frac{1}{2}Re(W_{uv}) \\
\frac{1}{2}Re(W_{uv}) & \frac{1}{2}Im(W_{uv}) & \frac{1}{2}W_{vv} & 0 \\
\frac{1}{2}Im(W_{uv}) & \frac{1}{2}Re(W_{uv}) & 0 & \frac{1}{2}W_{vv}
\end{pmatrix}
\]

Corresponding covariance tensor of the electrical field vector, as follows from equation (20) is

\[
\begin{pmatrix}
\langle \dot{E}_x^2(t) \rangle & \langle \dot{E}_x(t)\dot{E}_y(t) \rangle \\
\langle \dot{E}_x(t)\dot{E}_y(t) \rangle & \langle \dot{E}_y^2(t) \rangle
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}W_{uu} & \frac{1}{2}Re(W_{uv}) \\
\frac{1}{2}Re(W_{uv}) & \frac{1}{2}W_{vv}
\end{pmatrix},
\]

and PDF for the real electrical field components, as introduced by equation (20) is

\[
P(E_x, E_y, t) = \frac{1}{\pi \sqrt{W_{uu}W_{vv} - [Re(W_{uv})]^2}} \times \exp\left(-\frac{E_x^2W_{vv} - 2E_xE_yRe(W_{uv}) + E_y^2W_{uu}}{W_{uu}W_{vv} - [Re(W_{uv})]^2}\right)
\]

Several comments can be made in order to relate the classical Brosseau results to the model presented here.

Equation (29) assumes zero mean values for the Jones vector and, consequently, for the electrical field. This is not the most general case of the partially polarized field. In particular, Brosseau distribution cannot describe fully polarized field or mixture of polarized and depolarized field, similar to our examples in figure 2.

Even in the case of zero-mean, and completely random field Brosseau distribution is not the most general case of possible PDFs. Equation (32) describes standard, arguably featureless, bivariate Gaussian distribution, characterized by three parameters. PA-PDF of zero-mean field is characterized by nine parameters, and as our example, equation (28) shows, can have rather unusual shape.

Brosseau PDF for harmonically oscillating electrical field, equation (32), as well as covariance matrix \(\mathbf{C}\) of the Jones vector components, equation (30), and covariance tensor of the electrical field, equation (31) are all stationary. This is in stark contrast to the general case when all statistics oscillate with period \(t_0 = 2\pi/\omega\). It can also be seen from equation (20) that covariance matrix of the form of equation (30) is the sole case when covariance matrix of the oscillating field components remains stationary.

PDF given by equation (32) describes the bivariate circular Gaussian variable [9], when the phases of both components of the Jones vector are \((-\pi, \pi)\) uniformly distributed. Direct calculations of the phase PDF in [6], equation (3.3.26) support this observation. Also, in this case the second correlation tensor \(\mathbf{B}\), equation (8) is identically zero. Here we consider a general non-circular case.

Following the general case discussed in the previous subsection, the Jones matrix PDF, equation (29) depends on four parameters, \(W_{uu}, W_{uv}, W_{vv}, Re(W_{uv})\) and \(Im(W_{uv})\), while electrical field distribution is not sensitive to \(Im(W_{uv})\).

4. Diversity of polarized waves with fixed Stokes vector

As was mentioned earlier, only four combinations of the 14 first and second moments of the Jones vector enter equation (14) for the Stokes vector. This indicates that the Stokes vector provides incomplete characterization of the partially polarized waves statistics and opens up the possibility of altering the PA-PDF without changing the Stokes
vector. Three fairly straightforward ways to do this are discussed here. However, a full examination of the nine-dimensional $S = \text{const}$ manifold in the 13-dimensional space of the PA-PDF parameters is outside the scope of this paper.

4.1. Mean values split

Examination of equations (14) or (15) suggests that the products of the mean values of the quadratic components of the field always enter the components of the Stokes vector as a sum with the corresponding covariances, e.g.

$$\langle u_R \rangle^2 + \langle u_R \rangle \langle v_I \rangle + \langle u_R \rangle \langle v_I \rangle, \ldots \quad (33)$$

This allows a simultaneous change to the mean values and the covariances while maintaining the fixed value of the Stokes vector. Of course, these manipulations should preserve the SPD property of the covariance matrix. Without getting into the details of the pertinent limitations, we show an example of such a transformation in figure 4. For both panels of figure 4, the Stokes vector and its classical polarized and depolarized components are

$$S = \begin{pmatrix} 30.7 \\ 20.5 \\ 0 \end{pmatrix}$$

and for the right panel of figure 4 statistical parameters are

$$\begin{pmatrix} \langle u_R \rangle \\ \langle u_I \rangle \\ \langle v_R \rangle \\ \langle v_I \rangle \end{pmatrix} = \begin{pmatrix} 4.55 \\ 0 \\ 0 \\ 2.22 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 5.25 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.0617 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_M = \begin{pmatrix} 25.2 \\ 15.3 \\ 0 \end{pmatrix}^T, S_F = \begin{pmatrix} 5.5 \\ 5.2 \\ 0 \end{pmatrix}^T \quad (36)$$

Changes to the mean field and covariance matrix preserve the Stokes vector and its polarized and depolarized components, but the mean and fluctuating components, of the Stokes vector, equation (17), vary.

The dramatic change of the PA-PDF is obvious. In particular, the most probable state of the field changes from the $y$-polarized to $x$-polarized while the Stokes vector remains unchanged.

4.2. Quadrature components split

It follows from equation (14) that the coherence tensor, and, hence the Stokes vector remain unchanged when $\langle u_R^2 \rangle + \langle u_I^2 \rangle$, $\langle u_R \langle v_I \rangle \rangle + \langle u_I \langle v_I \rangle \rangle$, and $\langle u_R \langle v_I \rangle \rangle - \langle u_I \langle v_I \rangle \rangle$ remain constant, even when the individual components of the mean field and the covariance matrix $C$, equation (13) vary. There are many different ways to do this, and without getting into the details of the all possible transformations, we present two examples here. For all charts in the figure 5 the mean field, Stokes vector and the Stokes vector components are

$$S_M = \begin{pmatrix} 29.7 \\ -21.3 \end{pmatrix}, S_F = \begin{pmatrix} 29.4 \\ -21.3 \end{pmatrix}$$

and

$$S_{\text{DEP}} = \begin{pmatrix} 0.3 \\ 0 \end{pmatrix}, S_M = \begin{pmatrix} 29 \\ -21 \end{pmatrix}, S_F = \begin{pmatrix} 0.7 \\ -0.3 \end{pmatrix} \quad (37)$$

and the polarization degree is 0.990.
For the left panel of figure 5 the covariance matrix of the fluctuating components is

$$C = \begin{pmatrix} 0.21 & 0 & 0 & 0.16 \\ 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0.16 & 0 \\ 0.16 & 0 & 0 & 0.51 \end{pmatrix}.$$  (38)

and PA-PDF has four modes of equal height at approximately $\pm 80^\circ$ and $\pm 100^\circ$.

For the right panel of figure 5 the covariance matrix of the fluctuating component is

$$C = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 \end{pmatrix}.$$  (39)

and PA-PDF has four modes of unequal height at $0^\circ$, $180^\circ$ and $\pm 90^\circ$. Note that the three combination of the covariances $(\bar{u}_R \bar{u}_I)$, $(\bar{v}_R \bar{v}_I)$ and $(\bar{u}_I \bar{v}_R)$ are the same for both equations (38) and (39).

4.3. Covariance of the in-phase and quadrature components

It is clear from the equation (14) that covariances $(\bar{u}_R \bar{u}_I)$ and $(\bar{v}_R \bar{v}_I)$ do not affect the Stokes vector. Therefore it is possible to change the shape of the PA-PDF while maintaining the fixed Stokes vector and degree of polarization by altering these covariances. It is still necessary to maintain the SPD property of the covariance matrix $C$, which puts certain limitations on the choice of $(\bar{u}_R \bar{u}_I)$ and $(\bar{v}_R \bar{v}_I)$. For all charts in the figure 6 the mean field, the Stokes vector and its components are

$$\begin{pmatrix} \langle u_R \rangle \\ \langle u_I \rangle \\ \langle v_R \rangle \\ \langle v_I \rangle \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 5 \\ 5 \end{pmatrix}, \quad S = \begin{pmatrix} 62 \\ 0 \\ -50 \\ 0 \end{pmatrix}, \quad S_{POL} = S_M = \begin{pmatrix} 50 \\ 0 \\ 0 \\ -50 \end{pmatrix}, \quad S_{DEP} = S_F = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$  (40)

and the degree of polarization is 0.81. Based on the equation (40), the mean field is circular polarized, and there is a depolarized fluctuating component.

For figure 6(a) the covariance matrix of the fluctuating component is

$$C = \begin{pmatrix} 3.1 & 0 & 0 & 0 \\ 3.1 & 0 & 0 & 0 \\ 0 & 0 & 3.1 & 0 \\ 0 & 0 & 0 & 3.1 \end{pmatrix}.$$  (41)

and PA-PDF is circular-symmetric and has a toroidal shape. This is probably the simplest example of the field satisfying equation (40) where the fluctuating component has an isotropic and stationary covariance tensor, equation (20).

For figure 6(b), the covariance matrix is:

$$C = \begin{pmatrix} 3 & -2.97 & 0 & 0 \\ -2.97 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2.7 \\ 0 & 0 & 2.7 & 3 \end{pmatrix}.$$  (42)

and the covariance tensor is non-stationary which causes the scattering ellipse to change the major and minor axes, while maintaining the axes direction. The resulting PA-PDF has an unusual, four-mode diamond shape.

For figure 6(c) the covariance matrix is:

$$C = \begin{pmatrix} 3 & 2.85 & 0 & 0 \\ 2.85 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2.85 \\ 0 & 0 & 2.85 & 3 \end{pmatrix}.$$  (43)

and the covariance tensor, equation (20), again is non-stationary causing the scattering ellipse to change its size, while maintaining the circular symmetry (see figure 2 for reference). The resulting PA-PDF has two well-defined modes, and is very dissimilar to the both of the previous examples.

The examples shown in figures 4–6 clearly demonstrate that the Stokes vector does not provide the full information regarding the field statistics. Three approaches presented here by no means exhaust all the possible variations of statistics of the partially polarized wave with fixed Stokes vector.

4.4. Practical example

Here we present a practical example of situation where standard Stokes vector apparatus fails to recover the details of the polarization state of the field. Consider narrowband field in the form of a sum of monochromatic elliptically polarized field and random partially polarized ‘noise’ field. Corresponding random Jones vector can be presented as

$$J = \begin{pmatrix} u_R + iu_I \\ v_R + iv_I \end{pmatrix} = \begin{pmatrix} a_x \sigma_x \exp(\xi + i\chi) \\ a_y \sigma_y \exp(\psi + i\eta) \end{pmatrix}.$$  (44)

Here $\xi$, $\chi$, $\psi$, $\eta$ are zero-average, independent random variables with unit variance. First term in the right-hand part of equation (44) represents monochromatic component elliptically polarized component, and second term is a random polarized ‘noise’ component.

Mean value and covariance matrix of the Jones vector components are

$$\begin{pmatrix} \langle u_R \rangle \\ \langle u_I \rangle \\ \langle v_R \rangle \\ \langle v_I \rangle \end{pmatrix} = \begin{pmatrix} a_x \sigma_x^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & 0 \\ 0 & 0 & 0 & \sigma_y^2 \end{pmatrix}.$$  (45)
Random component, as evident from comparison covariance matrices in equations (30) and (45) is a simple case of a field described by Brosseau distribution with Jones vector components being independent circular complex random variables. State of the field is characterized by four real parameters: $a_2$, $a_s$, $\sigma_x$, and $\sigma_y$. Knowledge of these four parameters, in particular degree of polarization of the random component, can be crucial to some remote sensing applications when random component is associated with the scattered field and carries information about the scattering medium. Measurements of the complete second-order statistics, performed as described in the following section, provide complete information about the average values and covariance matrix, equation (45) of Jones vector components. This gives not only the degree of polarization of the ‘noise’, but also allows to check if principal polarization axes of monochromatic and noise parts are aligned with each other, as is suggested by equation (45).

Correlation tensor and Stokes vector for this field are

$$W = \begin{pmatrix} a_2^2 + 2\sigma_x^2 & -ia_xa_y \\ ia_xa_y & a_y^2 + 2\sigma_y^2 \end{pmatrix},$$

$$S = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} a_x^2 + a_y^2 + 2\sigma_x^2 + 2\sigma_y^2 \\ a_x^2 - a_y^2 + 2\sigma_x^2 - 2\sigma_y^2 \\ 0 \\ 2a_xa_y \end{pmatrix},$$

(46)

Classic Stokes vector description provides three real parameters: $I$, $Q$, and $V$, while the third Stokes parameter $U = 0$. Since $I^2 > U^2 + V^2$, it is clear that field is partially polarized and contains monochromatic and random components. However, Stokes vector does not have enough information to identify these components. Under the most natural, but in general unjustified assumption, that the noise component is unpolarized, $\sigma_x = \sigma_y = \sigma$, it is possible to find the parameters of the monochromatic component and ‘noise’ strength from the measured Stokes vector as

$$a_x^2 = \frac{1}{2}(\sqrt{Q^2 + V^2 + Q}), a_y^2 = \frac{1}{2}(\sqrt{Q^2 + V^2 - Q}),$$

$$\sigma^2 = \frac{1}{4}(I - \sqrt{Q^2 + V^2}).$$

(47)

Alternatively, one can assume that the random component is horizontally polarized: $\sigma_x = 0$, and arrive at the following estimates of the monochromatic and random components:

$$a_x^2 = \frac{V^2}{2(I - Q)}, a_y^2 = \frac{(I - Q)}{2}, \sigma_x^2 = \frac{I^2 - Q^2 - V^2}{4(I - Q)}.$$

(48)

Clearly neither monochromatic nor random component are estimated correctly in both cases of equations (47) and (48) that are based on additional unwarranted assumptions of either unpolarized or fully polarized noise. It is simply not possible to find the polarization components of the coherent part of the wave and the state of polarization of the noise part, which is actually partially polarized, as shown in equation (44) based only on the Stokes vector measurements. Full statistics given by equation (45) does not require any additional assumptions, and provides the complex Cartesian components of the monochromatic part of the field and the variances of the noise components. It also shows that both monochromatic and noise parts are aligned with the reference coordinate system of the coherent detector.

5. Discussion

Analytical development illustrated by several examples in the previous section demonstrated that the Stokes vector provides only partial information about the state of the partially polarized narrow-band electromagnetic field. This is in stark contrast with Van de Hulst’s [10] Principle of Optical Equivalence: ‘It is impossible by means of any instruments to distinguish between various incoherent sums of simple waves that may together form a beam with the same Stokes parameters.’

The explanation for this apparent disagreement is very simple: Stokes and his countless followers considered a simple incoherent measurement system, typically including phase retarder, polarizer and a photodetector [3, 11]. The measurements of the 13 statistical parameters introduced here requires coherent (heterodyne) detection.

Consider two-channel heterodyne detection system where in the $x$-channel the partially polarized narrow-band field is mixed with the $x$ and $y$-polarized local oscillator (LO) fields

$$E_{X}^{(i)}(t) = \begin{pmatrix} U \\ V \end{pmatrix} \cos[(\omega + \Omega)t + \phi],$$

$$E_{Y}^{(i)}(t) = \begin{pmatrix} 0 \\ V \end{pmatrix} \cos[(\omega + \Omega)t + \phi].$$

(49)

Here $\Omega \ll \omega$ is an intermediate (radio) frequency, $\Omega tc \gg 1$, and we assume that the LO phase $\phi$ is unknown but remains stable during the measurement time $T \gg tc$. The detector photocurrent for the $x$-channel is proportional to the square of the sum of the signal and LO fields

$$I^{(i)}(t) = R[U \cos[(\omega + \Omega)t + \phi] + u(t)\cos(\omega t) - u(t)\sin(\omega t)]^2$$

$$+ R[v(t)\cos(\omega t) - v(t)\sin(\omega t)]^2,$$

(50)

where $R$ is detector responsivity. The detector signal is comprised of a DC component proportional to $U^2$, slowly changing at $tc$ time scale component proportional to $u^2$, and slowly changing IF component $I^{(i)}(t)$ at the frequency $\Omega$ for the $x$-channel, and similar component $J^{(i)}(t)$ for the
The fast oscillating component at frequency $2\omega_0$ is not registered by detector, but we assume that the IF frequency signals $I^{(i)}(t)$ and $J^{(i)}(t)$ can be sampled properly. Analytical signals $I^{(AS)}(t)$ and $J^{(AS)}(t)$ with carrier frequency $\Omega$ corresponding to the real signals $I^{(i)}(t)$ and $J^{(i)}(t)$ can be recovered using the Hilbert transformation of the time series for $I^{(i)}(t)$ and $J^{(i)}(t)$ as

$$I^{(AS)}(t) = RU [u_R(t) \cos \varphi + u_I(t) \sin \varphi] \cos(\Omega t) - RU [u_I(t) \cos \varphi - u_R(t) \sin \varphi] \sin(\Omega t),$$
$$J^{(AS)}(t) = RV [v_R(t) \cos \varphi + v_I(t) \sin \varphi] \cos(\Omega t) - RV [v_I(t) \cos \varphi - v_R(t) \sin \varphi] \sin(\Omega t).$$

(51)

Note that the unknown LO phase $\varphi$ is assumed to be the same for both channels. Down converting these analytical signals to DC produces two random complex signals with $t_c$ time scale

$$I^{(C)}(t) = I_R(t) + iI_I(t) = RU [u_R(t) \cos \varphi + u_I(t) \sin \varphi] + iRU \times [u_I(t) \cos \varphi - u_R(t) \sin \varphi],$$
$$J^{(C)}(t) = J_R(t) + iJ_I(t) = RV [v_R(t) \cos \varphi + v_I(t) \sin \varphi] + iRV \times [v_I(t) \cos \varphi - v_R(t) \sin \varphi].$$

(53)

The time series for $I_R(t)$, $I_I(t)$, $J_R(t)$, and $J_I(t)$ can be used to estimate the statistics of these four real signals including the mean values and covariance matrix. Essentially, this requires time-averaging over time periods longer than correlation time $t_c$ of field fluctuations.

Formally, equation (53) can be solved for $u_R(t)$, $u_I(t)$, $v_R(t)$, and $v_I(t)$ as follows

$$u_R(t) = \frac{1}{RU} I_R(t) \cos \varphi - \frac{1}{RU} I_I(t) \sin \varphi,$$
$$u_I(t) = \frac{1}{RU} I_I(t) \cos \varphi + \frac{1}{RU} I_R(t) \sin \varphi,$$
$$v_R(t) = \frac{1}{RV} J_R(t) \cos \varphi - \frac{1}{RV} J_I(t) \sin \varphi,$$
$$v_I(t) = \frac{1}{RV} J_I(t) \cos \varphi + \frac{1}{RV} J_R(t) \sin \varphi.$$ 

(54)

However, the field components $u_R(t)$, $u_I(t)$, $v_R(t)$, and $v_I(t)$ are still not accessible since the LO phase $\varphi$ is unknown. The unknown phase can be eliminated if an additional assumption is imposed on the statistics of the field. This is equivalent to the uncertainty of the measurements timing discussed earlier that led to the PA-PDF concept and the reduction of the number of statistical parameters from 14 to 13.

Using the same, arbitrary, prescription $\langle v_R \rangle = 0$, we set

$$\varphi = \arctan \left( \frac{J_R}{J_I} \right).$$

(55)

This determines the sought quadrature components of the polarized field $u_R(t)$, $u_I(t)$, $v_R(t)$, and $v_I(t)$ in terms of the measured quadrature components of the IF photocurrent as

$$u_R(t) = \frac{1}{RU} \sqrt{\langle J_R^2 \rangle + \langle J_I^2 \rangle} J_R(t),$$
$$u_I(t) = \frac{1}{RU} \sqrt{\langle J_R^2 \rangle + \langle J_I^2 \rangle} J_I(t),$$
$$v_R(t) = \frac{1}{RV} \sqrt{\langle J_R^2 \rangle + \langle J_I^2 \rangle} J_R(t),$$
$$v_I(t) = \frac{1}{RV} \sqrt{\langle J_R^2 \rangle + \langle J_I^2 \rangle} J_I(t).$$

(56)

6. Conclusion

Considering the electrical field of a partially polarized wave as a two-dimensional real oscillating random vector, we introduced the PA-PDF that encompasses the complete statistics of the field at a single point.

For the simple case of a non-stationary oscillating Gaussian probability distribution of the field, we find that field statistics at a single point are completely described by 13 parameters. This is in contrast to the four parameters provided by the conventional Stokes vector or coherence tensor descriptions.

Using several examples, we illustrated that the partially polarized field state as described by PA-PDF can vary significantly when the Stokes vector remains fixed.

PA-PDF and nine statistical parameters supplementary to the Stokes vector can be measured by a two-channel heterodyne detectors and require a stable LO.

Our results suggest that it is possible to extract a lot more information from the partially polarized wave than is currently though of based on the Stokes vector description.

We believe that this new development can be important for a variety of remote sensing applications where medium or surface parameters are derived from the statistics of the scattered electromagnetic waves or where partially polarized waves are used as a probe.

For example, this new development can be used for remote sensing of particular matter, such as clouds [12] and aerosols, as well as for the radar and optical signatures of the randomly
rotating targets. Other applications are optical and radar surveillance of the sea surface, surface terrain and vegetation [13], as well as imaging and sensing of biological tissues [14].

This presented theory is not limited to the optical range, but can also be applied to the radar polarimetry and polarimetric SAR interferometry [12, 15].

Thirteen measurable degrees of freedom of the partially polarized wave, even when restricted by the SPD constraint, make this type of wave an attractive choice for the optical communication, but this topic is out of the scope of this paper.

This work addresses the statistics of the partially polarized field at a single point. The issues related to the spatial coherence of the field [16], which are essential for the development of the propagation model are not discussed. However, it becomes clear that the conventional, coherence tensor-based formulation, of the partially polarized waves' propagation, e.g. [17], cannot fully describe their statistics.

In this work Cartesian basis is used for the polarized waves, representation. It would be interesting to use the right/ left circular polarized waves as the basis.

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13