UNIQUENESS OF SCHRODINGER FLOW VIA ENERGY INEQUALITY

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Abstract. In this short note, we show a uniqueness result of the energy solutions for the Cauchy problem of Schrodinger flow in the whole space $\mathbb{R}^n$ provided there is a smooth solution in the energy class.

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1. Introduction

Uniqueness problem for weak solutions of nonlinear evolution system is an important and hard topic. In this short note, we study the uniqueness result for Schrodinger (map) flow from $\mathbb{R}^2 \times [0, +\infty) \to S^2$ (since we have no intention to make survey on this topic, we refer to [5], [2], and [1] for more physical background and results from K.Uhlenbeck, C.Terng, J.Shatah, C.E.Kenig, T.Tao, D.Tataru, A.D.Ionescu, and others). Here $S^2$ is equipped with the standard metric. By definition, the Cauchy problem of the Schrodinger map flow is a smooth mapping

$$u : \mathbb{R}^2 \times [0, +\infty) \to S^2$$

satisfying

$$u_t = u \times \Delta u, \quad \text{in } R^2 \times [0, +\infty),$$

with the Cauchy data

$$u|_{t=0} = u_0,$$

where $\Delta u = \sum \frac{\partial^2 u}{\partial x_i^2}$ is the usual Laplacian operator in $R^2$. Here and thereafter, we use the sum convention. From the equation (1) we can easily see that for the regular solution $u$, the energy

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(t)|^2 dx = E(u(0))$$

is conserved. In fact, from the fact

$$(u_t, \Delta u) = (u \times \Delta u, \Delta u) = 0,$$
we derive that
\[
\int \frac{d}{dt} E(u(t)) = \int_{\mathbb{R}^2} (\nabla u, \nabla u_t) = -\int_{\mathbb{R}^2} (\Delta u, u_t) = 0.
\]

Another useful form of the equation (1) is
\[
-\Delta u - u|\nabla u|^2 = u \times u_t.
\]
Some people like to use another form. If we compose \( u \) with a stereographic projection \( \pi \) such that \( z(x, t) = \pi \circ u(x, t) \in \mathbb{R}^2 \), then we have
\[
i z_t - \Delta z = \frac{2\bar{z}}{1 + |z|^2} \partial_j z \partial_j z.
\]
We say that the mapping \( v \in L^1_{loc}(\mathbb{R}^2 \times [0, +\infty); S^2) \)
is an energy class solution (and also called the weak Schrodinger flow) if \( v \) satisfies (1) in the distributional sense and the energy inequality
\[
E(v(t)) \leq E(v(s)), \quad \text{for all } t > s \geq 0.
\]

We denote by \( \mathbb{R}_+ = [0, +\infty) \), \( Du = (\nabla u, u_t) \), and \((A, B, C)\) the determinant of the matrix formed by the ordered column vectors \( A, B, C \). We have the following uniqueness result.

**Theorem 1.** Let
\[
u \in C^\infty(\mathbb{R}^2 \times [0, +\infty); S^2)
\]
with
\[
|Du| \in L^\infty_{loc}([0, \infty), L^\infty(\mathbb{R}^2))
\]
and
\[
v \in L^1_{loc}(\mathbb{R}^2 \times [0, +\infty); S^2), \text{ with } |Dv| \in L^\infty_{loc}([0, +\infty); L^\infty(\mathbb{R}^2))
\]
be (weak) solutions to (1) with the smooth Cauchy data \( u_0 \).
Assume that \( |\nabla u_0| \) has compact support. Then \( u = v \). Furthermore, the same result is true for any Euclidean space \( \mathbb{R}^n \) in place of the plane \( \mathbb{R}^2 \).

We remark that the assumption about the \( L^\infty \) bound of the t-derivative \( v_t \) is equivalent to \( |\Delta v|(t) \in L^\infty(\mathbb{R}^2) \) by using the Schrodinger map flow equation (2). We remark that Theorem 1 in the plane case can also be derived from a result in [3]. It seems to us that \( L^\infty \) bound of the t-derivative \( v_t \) is very stronger. However, our assumption in dimension two can be weaken by using the Ladyzhenskaya’s inequality. Recall here that the Ladyzhenskaya’s inequality (4) is the following one. For any \( w \in C^1_0(\mathbb{R}^2) \),
\[
||w||_{L^2(\mathbb{R}^2)} \leq C||w||_{L^2(\mathbb{R}^2)}|\nabla w||_{L^2(\mathbb{R}^2)}.
\]

We point out that similar results for wave equation and wave maps have been obtained by M.Struwe [7] and here we use the similar Gronwall inequality method. One may also see the paper [7] for more related results. Similar result for Schrodinger system has been obtained in [6]. The new ingredient in our proof is the simple \( L^2 \) growth of \( v - u \) and its derivatives.
In the following, we denote by $C$ the various constants, which do not depend on time variable $t$.

2. Proof of Theorem

For simple presentation, we just give the proof for $n = 2$. The general case can be treated in the same way.

Write $v = u + w$. Note that $|u|^2 = 1 = |v|^2$. So, $u_t \cdot u = 0 = v \cdot v_t$.

Consider

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |w|^2 dx = \int_{\mathbb{R}^2} w \cdot w_t.
$$

Note that

$$
w \cdot w_t = w \cdot v \times \Delta v - w \cdot u \times \Delta u = w \cdot u \times \Delta w.
$$

Then

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |w|^2 dx = \int_{\mathbb{R}^2} (w \times u) \cdot \Delta w.
$$

Using the integrating by part, the latter term can be bounded by

$$
\leq C \int_{\mathbb{R}^2} (|\nabla w|^2 + |w|^2).
$$

Hence, we have

$$
\int_{\mathbb{R}^2} |w|^2 dx(t) \leq C \int_0^t \int_{\mathbb{R}^2} (|\nabla w|^2 + |w|^2).
$$

We clearly have that

$$
E(v) = E(u) + I + E(w),
$$

where

$$
I = \langle dE(u), w \rangle = \int_{\mathbb{R}^2} \nabla u \cdot \nabla wdx.
$$

Hence,

$$
0 \geq E(v(t)) - E(v(0)) = I(t) - I(0) + E(w(t)) - E(w(0)).
$$

Consider

$$
I(t) - I(0) = \int_0^t \int_{\mathbb{R}^2} \partial_t (\nabla u \cdot \nabla w)dx.
$$

Then we have

$$
\mathbf{5} = - \int_0^t \int_{\mathbb{R}^2} [(\Delta u) \cdot w_t + (\Delta w) \cdot u_t],
$$

where $\mathbf{5}$ is the term contributed by the interaction between $u$ and $w$. 

which can be written as

\[
\begin{align*}
(5) & = \int_0^t d\tau \int_{\mathbb{R}^2} w_t \cdot [u|\nabla u|^2 - v|\nabla v|^2 + u \times u_t - v \times v_t] \\
& = \int_0^t d\tau \int_{\mathbb{R}^2} w_t \cdot [u|\nabla u|^2 - v|\nabla v|^2 - w \times u_t] \\
& = \int_0^t d\tau \int_{\mathbb{R}^2} [w_t \cdot (u|\nabla u|^2 - v|\nabla v|^2) + (w, w_t, u_t)]
\end{align*}
\]

after using the relations

\[
\begin{align*}
w_t \cdot \Delta u &= -w_t \cdot (u|\nabla u|^2 + u \times u_t), \\
u_t \cdot \Delta w &= u_t \cdot \Delta v - u_t \cdot \Delta u \\
&= -u_t \cdot (v|\nabla v|^2 + v \times v_t) \\
&= w_t \cdot (v|\nabla v|^2 + v \times v_t).
\end{align*}
\]

We now bound each term case by case. Using

\[
w_t = v \times \Delta w + w \times \Delta u,
\]

we have

\[
\begin{align*}
\int_0^t d\tau \int_{\mathbb{R}^2} (w, w_t, u_t) &= \int_0^t d\tau \int_{\mathbb{R}^2} (w \times \Delta w, u_t) \\
&= -\int_0^t d\tau \int_{\mathbb{R}^2} (w \times u_t, v, \Delta w)
\end{align*}
\]

Upon integrating by part, we have

\[
|\int_0^t d\tau \int_{\mathbb{R}^2} (w \times u_t, v, \Delta w)| \leq C \int_0^t d\tau \int_{\mathbb{R}^2} (|\nabla w|^2 + |w|^2) dx.
\]

To bound other terms, we notice the relation that

\[
v|\nabla v|^2 - u|\nabla u|^2 = w|\nabla v|^2 + u(|\nabla v|^2 - |\nabla u|^2).
\]

Using

\[
w_t = v \times \Delta w + w \times \Delta u,
\]

again, we have

\[
\begin{align*}
w_t \cdot (v|\nabla v|^2 - u|\nabla u|^2) &= (v \times \Delta w + w \times \Delta u) \cdot w|\nabla v|^2 + \\
&
\space (v \times \Delta w + w \times \Delta u) \cdot u(|\nabla v|^2 - |\nabla u|^2) \\
&= (v \times \Delta w + w \times \Delta u) \cdot w|\nabla v|^2 + \\
&
\space v \times \Delta w \cdot u(|\nabla v|^2 - |\nabla u|^2) \\
&
\space + w \times u \cdot u(\nabla w \cdot \nabla(u + v)) \\
&= (v \times \Delta w + w \times \Delta u) \cdot w|\nabla v|^2 + \\
&
\space w \times \Delta w \cdot u(|\nabla v|^2 - |\nabla u|^2) \\
&
\space + w \times u \cdot u(\nabla w \cdot \nabla(u + v)),
\end{align*}
\]
which implies that upon integrating by part, based on a regularization approach,

$$
| \int_{\mathbb{R}^2} w_t \cdot (v|\nabla v|^2 - u|\nabla u|^2) | \leq C(E(w) + |w|^2_2).
$$

Then using (4), we have

$$
|I(t) - I(0)| \leq C \int_0^t (E(w) + |w|^2_2),
$$

and we have

$$
E(w(t)) \leq E(w(0)) + C \int_0^t (E(w) + |w|^2).
$$

This estimate gives us the stability result for the Schrödinger map flow.

Note that $E(w(0)) = 0$. Combining this with (3), we have

$$
E(w(t)) + |w|^2_2(t) \leq C \int_0^t (E(w) + |w|^2_2).
$$

By the Gronwall inequality, we have

$$
w = 0, \quad \text{and} \quad u = v.
$$

This completes the proof of Theorem 1.

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