Quantum singularities in the BTZ spacetime

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The spinless Bañados-Teitelboim-Zanelli (BTZ) spacetime is considered in the quantum theory context. Specially, we study the case of negative mass parameter using quantum test particles obeying the Klein-Gordon and Dirac equations. We study if this classical singular spacetime, with a naked singularity at the origin, remains singular when tested with quantum particles. The need of additional information near the origin is confirmed for massive scalar particles and all the possible boundary conditions necessary to turn the spatial portion of the wave operator self-adjoint are found. When tested by massless scalar particles or fermions, the singularity is “healed” and no extra boundary condition are needed. Near infinity, no boundary conditions are necessary.

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I. INTRODUCTION

In the 2 + 1 dimensional Einstein theory of gravitation without cosmological constant, the spacetime is necessarily flat and only its global topological properties [1] makes it different from the trivial 2 + 1 dimensional Minkowski spacetime. In the simplest case of a point particle at the origin, the resulting spacetime is conic [2] (the usual plane with a slice removed and identified edges). It is the 2 + 1 dimensional analog of the 3 + 1 dimensional cosmic string (for a classical treatment see [3] and for a quantum treatment [4]).

When a negative cosmological constant, \( \Lambda \), is considered, the Einstein equations admit a black hole solution \( \mathbb{R}^2 \). The lower dimension of the BTZ solution makes it a particularly simple example of a spacetime with the main properties of the usual 3 + 1 black hole.

The negative cosmological constant gives us a asymptotically anti-de Sitter spacetime, instead of a flat one. In fact, the BTZ spacetime is locally anti-de Sitter, differing only by its global topological properties \( \mathbb{R}^2 \).

There are three different kinds of spacetimes (solutions of Einstein equations) depending on a mass parameter \( m \), which has been adjusted so that the mass vanishes when the horizon size goes to zero, that is, \( m = 0 \) for the vacuum state. For \( m > 0 \), there is a continuum black hole spectrum with a singularity of the Taub-NUT type at the origin, hidden by an event horizon given by \( r_+ = \sqrt{m} \), where \( l^{-2} = -\Lambda \). This spacetime does not violate the cosmic censorship hypothesis since the singularity is hidden. It is a reasonable classical spacetime and quantum mechanical considerations are not needed in this case.

As \( m \) takes values smaller than or equal to zero, there appears a continuous sequence of naked singularities (point particle sources) at the origin. The singularities do not come from any curvature scalar divergence, but rather from a topological obstruction of the spacetime continuation, since the Ricci tensor has a term proportional to the Dirac distribution \( \delta \) in addition to the constant curvature. This last term is due to the presence of the cosmological constant. Near the origin, where the curvature can be neglected, the spacetime is conic, so it must be excluded by the cosmic censorship hypothesis. It is in this classical background that the quantum test particles will be studied in order to see if the spacetime remains singular when considered in the quantum theory context. In this paper we adopt the definition of quantum singularity due to Horowitz and Marolf [10], which says that a spacetime is quantum mechanically nonsingular if the time evolution of any wave packet is uniquely determined by the initial wave function.

When \( m \) takes the value \(-1\), the spacetime does not present an event horizon, but there is no singularity to hide either, so this solution (a true anti-de Sitter spacetime) is again permissible and it is the ground state of the theory (for a discussion of the importance of naked singularities to establish the ground state in any gravitation theory, see [5]).

For the mass parameter \( m < -1 \) the spacetime represents point sources with negative mass without physical meaning.

The purpose of this work is to study the naked singularities for the continuous sequences of spacetimes separating the black hole like spectrum from the ground state anti-de Sitter spacetime. We shall use quantum test particles to determine if these spacetimes are quantum mechanically singular.

The paper is organized as follows, in section II we present a brief review of quantum singularities in a general static spacetime. In section III we apply the formalism presented in the previous section to the case of scalar particles. We also consider the boundary conditions studied by Kay and Studer [11] and adopt them in the context of the BTZ spacetime. In section IV we extend the formalism to particles with spin. Finally, in section V we discuss the results presented in this work.

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II. QUANTUM SINGULARITIES

In general relativity, a spacetime singularity is indicated by incomplete geodesics or, more precisely, by incompleteness, i.e., incomplete curves of boundary acceleration \[14\]. At the singular points, an extra information must be added, since we lose the capacity to predict the future of a particle following an incomplete worldline.

In order to generalize this concept to quantum mechanics, Horowitz and Marolf proposed a simple definition of singularity. They stated that a spacetime is nonsingular if the time evolution of any wave packet is uniquely determined by the initial wave data.

To be more precise, let \((M, g_{\mu\nu})\) be a static spacetime with a timelike Killing vector field \(\xi^\mu\), \(t\) be the Killing parameter and \(\Sigma\) a static spatial slice orthogonal to \(\xi^\mu\).

The Klein-Gordon equation on this spacetime, \[\Box \Psi = M^2 \Psi, \tag{1}\]
can be split in a temporal and a spatial part, \[\frac{\partial^2 \Psi}{\partial t^2} = -A \Psi = V D^i (V D_i \Psi) + M^2 V^2 \Psi, \tag{2}\]
where \(V^2 = -\xi^\mu \xi_\mu\) and \(D_i\) is the spatial covariant derivative on \(\Sigma\).

To avoid the singular points, we take \(C^\infty_0 (\Sigma)\), the set of all smooth functions of compact support on \(\Sigma\), as the domain \(D(A)\) of the operator \(A\) defined in equation \(\ref{eq:Klein-Gordon}\).

With this domain, \(A\) is a well-defined positive symmetric operator on the Hilbert space \(H = L^2 (\Sigma, V^{-1} \, d\mu)\), where \(d\mu\) is the usual measure on \(\Sigma\).

The chosen domain is so small, i.e., the restrictions on functions are so strong, that the domain of the Hilbert adjoint operator \(A^\star\) is extremely large and it is composed of all functions \(\psi\) in \(L^2 (\Sigma, V^{-1} \, d\mu)\) such that \(A^\star \psi \in L^2\). Then, \(A\) is not self-adjoint.

Hence, we are face with the problem of searching for self-adjoint extensions of \(A\) and to discover if it has only one or many of such extensions.

If \(A\) has only one self-adjoint extension (its closure \(\overline{A}\)), then \(A\) is said essentially self-adjoint \([13, 16, 17]\). Since we are worried with the one particle description, not a field theory, the positive frequency solution satisfies \[i \frac{\partial \Psi}{\partial t} = (\overline{A})^{1/2} \Psi, \tag{3}\]
and the evolution of the wave packet is uniquely determined by the initial data, \[\Psi(t, x) = e^{-i A \frac{1}{2}} \Psi(0, x). \tag{4}\]

In this case, we say that the spacetime is quantum mechanically non-singular.

Now, if \(A\) has many self-adjoint extensions \(A_\alpha\), where \(\alpha\) is a real parameter, we must choose one in order to evolve the wave packet. Any solution of the form \[\Psi(t, x) = e^{-it (A_\alpha)^{1/2}} \Psi(0, x), \tag{5}\]
is a good solution and an extra information must be given to tell us which one has to be chosen. In this case we say that the spacetime is quantum mechanically singular.

The criterion used to determine the number of self-adjoint extensions of \(A\) (Theorem X.2 on reference \([16]\)) is to solve the equations \[A^\star \psi \pm i \psi = 0, \tag{6}\]
and to count the number of linear independent solutions in \(L^2\), i.e., the dimension of \(\ker (A^\star \pm i)\).

If there is no square-integrable solutions, the operator possess a unique self-adjoint extension and it is essentially self-adjoint. If there is one solution in \(L^2\) to each equation in \(\ref{eq:Klein-Gordon}\), a one-parameter family of self-adjoint extensions exists and its extension is not unique. The theory of deficiency indices of von Neumann says that these self-adjoint extensions are represented by the one-parameter family of the extended domains of the operator \(A\) given by \([16]\)

\[D^\omega = \{ \psi = \phi + \phi^+ + e^{i \omega} \phi^- : \omega \in \mathbb{R}, \phi \in D(A)\}, \tag{7}\]
where \[A^\star \phi^\pm = \pm i \phi^\mp \tag{8}\]
and \(\phi^\pm \in L^2\). The term \(e^{i \omega}\) in \(\ref{eq:Klein-Gordon}\) appears because the theory says that the self-adjoint extensions of the operator \(A\) are in one-to-one correspondence with the isometries from \(\ker (A^\star - i)\) to \(\ker (A^\star + i)\), i.e., the isometries given by \(\phi^+ \mapsto e^{i \omega} \phi^-\).

III. SCALAR FIELDS

The metric for the spinless BTZ spacetime \([5]\) is \[ds^2 = -V(r)^2 dt^2 + V(r)^{-2} dr^2 + r^2 d\theta^2, \tag{9}\]
with the usual ranges of the cylindrical coordinates and \[V(r)^2 = -m + \frac{r^2}{l^2}, \tag{10}\]
where \(m\) is the mass parameter.

After separating variables, \(\psi = R(r) e^{i n \theta}\), the radial portion of equation \(\ref{eq:Klein-Gordon}\) can be cast as \[R'' + \frac{(V^2 - n^2)}{V^2} R' - n^2 R - i \frac{R}{V^4} R_n = 0. \tag{11}\]
To consider the case \(r \to \infty\), we note that for large values of \(r\) the metric takes the form, \[ds^2 \approx -(\frac{r^2}{l^2}) dt^2 + (\frac{r^2}{l^2})^{-1} dr^2 + r^2 d\theta^2. \tag{12}\]
This spacetime is asymptotically anti-de Sitter. The measure on the slice $\Sigma$ is $d\mu = r dr$. From the definition of Horowitz and Marolf we find that the measure of our Hilbert space $H$ is $V^{-1} d\mu = ld r$.

Then, the equation (11) takes the form

$$R'' + \frac{3}{r} R' + \frac{n^2 \ell^2}{r^2} R_n - \frac{M^2 \ell^2}{r^2} R_n \pm i \frac{\ell}{r^4} R_n = 0. \quad (13)$$

For very large values of $r$, we can consider only the first two terms of the equation (13). Then we have

$$R'' + \frac{3}{r} R' = 0, \quad (14)$$

whose solution is

$$R_n(r) = C_{1n} + C_{2n} r^{-2}, \quad (15)$$

where $C_{1n}$ and $C_{2n}$ are arbitrary constants. $R(r) \in L^2$ if and only if $C_{1n} = 0$. Then, for each mode we have only one solution in $L^2$. Let us now, analyze the case $r \rightarrow 0$.

The metric in this case is approximately given by

$$ds^2 \approx -\alpha^2 dt^2 + \alpha^{-2} dr^2 + r^2 d\theta^2, \quad (16)$$

where $\alpha^2 = -m$ (remember we are interested in the case $-1 < m < 0$).

Redefining the coordinates $(t \rightarrow \alpha, r \rightarrow \alpha^{-1} r)$, we have

$$ds^2 \approx -dt^2 + dr^2 + \alpha^2 r^2 d\theta^2. \quad (17)$$

The metric (17) tells us that near the singularity $r = 0$, where curvature effects are negligible, the BTZ spacetime is conic.

The parameter $m = -\alpha^2$ is related to the mass of the point particle source by $\alpha = 1 - 4Gm_{\text{source}}$ and to the deficit angle by $\Delta = 2\pi (1 - \alpha)$.

As noted by Horowitz and Marolf, the case of the massive test particles need not to be considered. The additional term $-\frac{\ell^2}{r^2} \frac{d^2}{dr^2}$ in equation (11) acts as a repulsive potential, increasing the rate at which the non-square integrable solution diverges at the origin, and driving the square integrable solution more quickly to zero. Then, if the operator $A$ defined in equation (20) is essentially self-adjoint for $M = 0$, it is also essentially self-adjoint for $M > 0$. Therefore, we need only to analyze the massless case $M = 0$.

From (17) we find that equation (11) reduces to

$$R'' + \frac{1}{r} R' + \left[ \pm i - \frac{n^2 \ell^2}{\alpha^2 r^2} \right] R_n = 0, \quad (18)$$

whose general solution is

$$R_n(r) = A_n J_{|n/\alpha|}(kr) + B_n N_{|n/\alpha|}(kr), \quad (19)$$

where $J_{\nu}(kr)$ and $N_{\nu}(kr)$ are the $\nu$th order Bessel and Neumann functions, respectively, and $k = \sqrt{r}$.

Near $r = 0$, $J_{|n/\alpha|}(x) \sim x^{|n/\alpha|} \sqrt{x}$, while $N_{|n/\alpha|}(x) \sim x^{|n/\alpha|}$, except for $n = 0$, when $N_{|n/\alpha|}(x) \sim \ln x$. From the behavior of the Bessel and Neumann functions near the origin, it is easy to show that $J_{|n/\alpha|}(kr)$ is square-integrable near $r = 0$ for all $n = 0, 1, 2, \ldots$, while $N_{|n/\alpha|}(kr)$ is square-integrable near $r = 0$ only if $|n/\alpha| \leq 1$, or $|n| \leq \alpha < 1$. So $N_n(r)$ belongs to $L^2$ near $r = 0$ only if $n = 0$. In this case, we can adjust the constants $A_0$ and $B_0$ in equation (19) to meet the asymptotic behavior at infinity, $R(r) \sim 1/r^2$.

Then, for $n = 0$, there is a solution of equation (6) in $L^2(\mathbb{R}^+, V^{-1} d\mu)$. Therefore, there is a one-parameter family of self-adjoint extensions of $A$ and the spacetime is quantum mechanically singular.

Near $r = 0$, the negative mass BTZ spacetime is similar to a conic spacetime [see equation (17)]. Positive-frequency solutions of the Klein-Gordon equation in this spacetime satisfies

$$i \frac{\partial \Psi}{\partial t} = (\mu^2 - \Delta)^{1/2} \Psi, \quad (20)$$

where $\mu$ is the particle mass and $\Delta$ is the Laplace-Beltrami operator on the cone.

The boundary conditions necessary to turn the operator $(\mu^2 - \Delta)^{1/2}$ self-adjoint, in this case, are already known for a scalar test particle [11]. They are obtained using Neumann’s theory of deficiency indices (see [16]) and are given by

$$\lim_{r \rightarrow 0} \left\{ \ln qr/2 + \gamma \right\} R_0'(r) - R_0 = 0, \quad q \in (0, \mu],$$

$$\lim_{r \rightarrow 0} r R'_0(r) = 0, \quad q = 0, \quad (21)$$

where $\gamma$ is Euler-Mascheroni constant. Since $-q^2$ is an eigenvalue of $-\Delta_0$, the quantity $q$ is restricted by $0 \leq q \leq \mu$ in order that the operator $(\mu^2 - \Delta)^{1/2}$ makes sense. Note that for a massless particle ($\mu = 0$) we must take the boundary condition for $q = 0$. Thus the spacetime is non-singular when tested by massless particles.

Because we are interested only in local conditions at $r = 0$, we take these boundary conditions as the boundary conditions of our problem and, given one of the conditions in equation (21), the evolution of the wave packet is uniquely determined by the initial data. Different choices give us different theories.

**IV. DIRAC FIELDS**

In a $2 + 1$ dimensional spacetime, fermions have only one spin polarization [18], hence spinors have only two components and the Dirac equation consists of a set of two coupled partial differential equations. The constant Dirac matrices, $\gamma^{(j)}$, in flat spacetimes are replaced by the Pauli matrices [19], i.e.,

$$\gamma^{(j)} = (\sigma^{(3)}, i\sigma^{(1)}, i\sigma^{(2)}), \quad (22)$$
where Latin indices represent internal (local) indices. In this way
\begin{equation}
\{ \gamma^{(i)}, \gamma^{(j)} \} = 2 \eta^{(ij)} \mathbb{I}_{2 \times 2},
\end{equation}
where $\eta^{(ij)}$ is the Minkowski metric in $2 + 1$ dimensions, i.e., $\eta^{(ij)} = \text{diag}(-1,1,1)$ and $\mathbb{I}_{2 \times 2}$ is the $2 \times 2$ identity matrix.

The coordinate dependent metric $g_{\mu \nu}(x)$ and matrices $\sigma^{\mu}(x)$ (Greek indices representing external, or global, indices) are related to the dreibein $e^{(i)}_\mu$ by
\begin{equation}
\begin{aligned}
g_{\mu \nu}(x) &= e^{(i)}_\mu(x)e^{(j)}_\nu(x) \eta^{(ij)}, \\
\sigma^{\mu}(x) &= e^{(i)}_\mu \gamma^{(i)}.
\end{aligned}
\end{equation}

In the general $2 + 1$ dimensional spacetime with metric $g_{\mu \nu}(x)$, the Dirac equation for a free particle (with mass $M$) can be cast as
\begin{equation}
i \sigma^{\mu}(x) [\partial_{\mu} - \Gamma_{\mu}(x)] \Psi(x) = M \Psi,
\end{equation}
where $\Gamma_{\mu}(x)$ is the spinorial affine connection and it is given by
\begin{equation}
\Gamma_{\mu}(x) = \frac{1}{4} \theta_{\alpha \lambda} [e^{(i)}_{\nu \mu}(x) e^{(j)}_{\nu}(x) - e^{(j)}_{\nu \mu}(x) e^{(i)}_{\nu}(x)] s^{\nu \lambda}(x),
\end{equation}
with
\begin{equation}
s^{\nu \lambda}(x) = \frac{1}{2} [\sigma^{\nu}(x), \sigma^{\lambda}(x)].
\end{equation}

As in the case of scalar particles, we are interested in the two singular cases ($r \to \infty$ and $r = 0$). When $r \to \infty$, we take the metric \cite{12}. For this metric, we choose
\begin{equation}
\begin{aligned}
e^{(i)}_{\mu}(t,r,\theta) &= \text{diag}(r/l,1/r,l,r), \\
e^{(i)}_{(i)}(t,r,\theta) &= \text{diag}(l/r,l/r,1/r).
\end{aligned}
\end{equation}

The coordinate dependent gamma matrices and the spinorial affine connection are given by
\begin{equation}
\begin{aligned}
\sigma^{\mu}(x) &= \left( \frac{1}{r} \alpha^{(3)}, \frac{i r}{l} \alpha^{(1)}, \frac{i}{r} \alpha^{(2)} \right), \\
\Gamma_{\mu}(x) &= \left( \frac{1}{2 l^2} \alpha^{(2)}, 0, \frac{i r}{2 l} \alpha^{(3)} \right).
\end{aligned}
\end{equation}

Now, for the spinor
\begin{equation}
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\end{equation}
we can write the Dirac equation in the spacetime \cite{12} as,
\begin{equation}
\begin{aligned}
\frac{i l}{r} \frac{\partial \psi_1}{\partial t} - \frac{r}{l} \frac{\partial \psi_2}{\partial t} + \frac{i}{r} \frac{\partial \psi_2}{\partial \theta} - \frac{1}{l} \psi_2 - M \psi_1 &= 0, \\
- \frac{i l}{r} \frac{\partial \psi_2}{\partial t} - \frac{r}{l} \frac{\partial \psi_1}{\partial t} + \frac{i}{r} \frac{\partial \psi_1}{\partial \theta} - \frac{1}{l} \psi_1 - M \psi_2 &= 0.
\end{aligned}
\end{equation}

For the positive frequency solutions we shall use the ansatz,
\begin{equation}
\Psi_{n,E}(t,x) = \left( \begin{array}{c} R_{1n}(r) \\ R_{2n}(r) e^{-i\theta} \end{array} \right) e^{in\theta} e^{-iEt}.
\end{equation}
Note that $\Psi(t,x)$ is an eigenfunction of the total angular momentum $J_z = L_z + S_z$, with $J_z = -i \frac{\partial}{\partial \theta}$ and $S_z = \sigma^{(3)}/2$, with eigenvalue $n + \frac{1}{2}$ \cite{20}. We have for the radial part of the Dirac equation,
\begin{equation}
\begin{aligned}
R_{1n}'(r) + \left( \frac{1 - 4 n l}{r^2} \right) R_{1n}(r) + \left( \frac{2 M l}{r^2} + \frac{E^2}{r^2} \right) R_{2n}(r) &= 0, \\
R_{2n}'(r) + \left( \frac{1 + 4 n l}{r^2} \right) R_{2n}(r) + \left( \frac{2 M l}{r^2} - \frac{E^2}{r^2} \right) R_{1n}(r) &= 0.
\end{aligned}
\end{equation}
By neglecting the lower order terms, since we are interested in the $r \to \infty$ case, we have:
\begin{equation}
\begin{aligned}
R_{1n}'(r) + \left( \frac{1}{r} \right) R_{1n}(r) + \left( \frac{M l}{r} \right) R_{2n}(r) &= 0, \\
R_{2n}'(r) + \left( \frac{1}{r} \right) R_{2n}(r) + \left( \frac{M l}{r} \right) R_{1n}(r) &= 0.
\end{aligned}
\end{equation}
Therefore, for both components we have the same equation,
\begin{equation}
R_j''(r) + \frac{3}{r} R_j(r) + \frac{1}{r^2}(1 - M^2 l^2) R_j(r) = 0 \quad (j = 1, 2).
\end{equation}
Again, neglecting lower order terms we obtain
\begin{equation}
R_j''(r) + \frac{3}{r} R_j(r) = 0 \quad (j = 1, 2).
\end{equation}
Hence, asymptotically, the radial portion of the spinor $\Psi$ behaves as
\begin{equation}
R(r) = A r^{-2} + B,
\end{equation}
where $A$ and $B$ are constant spinors.

The solution \cite{37} is square-integrable only if $B = 0$. Only one constant must be specified. Then our solution is well-behaved near infinity and no extra boundary conditions are necessary.

The metric near $r = 0$ is very close to the conic background \cite{17}. This problem has already been dealt with in references \cite{12} and \cite{13}. The metric (after changing the variable $r \to \alpha r$) is,
\begin{equation}
ds^2 = -dt^2 + \alpha^{-2} dr^2 + r^2 d\theta^2,
\end{equation}
and the measure on the slice $\Sigma$ is $\alpha^{-1} r dr$.

The appropriate dreibein is
\begin{equation}
\begin{pmatrix} e^{(i)}_{\mu} \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -r \sin \theta \cos \theta \end{pmatrix}.
\end{equation}
And by choosing positive energy solutions of the form,
\begin{equation}
\Psi_{n,E}(t,x) = \left( \begin{array}{c} R_{1n}(r) \\ i R_{2n}(r) \end{array} \right) e^{i(n + \frac{1}{2} - \frac{\theta}{2})} e^{-iEt},
\end{equation}
we obtain the system of equations,

\[
\begin{align*}
R'_{1n}(r) + \left[ \frac{1}{2r} - \frac{n + \frac{1}{2}}{\alpha r} \right] R_{1n}(r) + \frac{E + M}{\alpha} R_{2n}(r) &= 0, \\
R'_{2n}(r) + \left[ \frac{1}{2r} + \frac{n + \frac{1}{2}}{\alpha r} \right] R_{2n}(r) - \frac{E - M}{\alpha} R_{1n}(r) &= 0.
\end{align*}
\]

\tag{41}

where $\nu = [2n + (1 - \alpha)]/2\alpha$, $\kappa^2 = (E^2 - M^2)/\alpha^2$, $A_n$ and $B_n$ are arbitrary constant. The Bessel function $N_{\nu}$, is square-integrable for all $\lambda \notin \mathbb{R}$, which is not, except when $|\lambda| < 1$. Note that the second spinor in (42) is square-integrable when $|\nu| < 1$ and $|\nu + 1| < 1$, i.e., $-1 < \nu < 0$. It is easy to see that this condition does not hold for any value of $n \in \mathbb{Z}$. Therefore, an arbitrary wave packet can be described by

\[
\Psi(t, \mathbf{x}) = \left[ A_n \left( \frac{J_{\nu}|(\kappa r)}{iJ_{\nu+1}|(\kappa r)} \right) + B_n \left( \frac{N_{\nu}|(\kappa r)}{-iN_{\nu+1}|(\kappa r)} \right) \right] e^{i(n+\frac{1}{2}-\frac{1}{2}\sigma^3)\theta} e^{-iEt},
\]

\tag{42}

and the initial condition $\Psi(0, \mathbf{x})$ is sufficient to determine the time evolution of the particle. The Cauchy problem is well-posed and the spacetime is nonsingular when tested by fermions. It is interesting to note that this is the case only for the 2+1 dimensional spacetime, since for the 3+1 dimensional case (see [21]), the extra dimension adds a continuous parameter $k$ representing the wave vector in the Fourier transform of $\Psi$. The existence of this continuous parameter allows that an infinite number of normal modes to be singular. So the spacetime around a cosmic string remains singular when tested by fermions. The fact that in the 2+1 dimensional spacetime the spatial modes have only discrete indices excludes this possibility.

Thus, an arbitrary wave packet can be described by

\[
\Psi(t, \mathbf{x}) = \sum_{n=-\infty}^{+\infty} A_n \left( \frac{J_{\nu}(\kappa r)}{iJ_{\nu+1}(\kappa r)} \right) e^{i(n+\frac{1}{2}-\frac{1}{2}\sigma^3)\theta} e^{-iEt},
\]

\tag{43}

A complete set of solutions of the Dirac equation in the spacetime (38) is given by the normal modes, hypothesis is satisfied (the singularity is hidden). For the mass parameter interval $-1 < m < 0$, the classical singularity persists when tested by massive scalar fields. In principle, we do not have any reason to choose one of the boundary conditions in (21). As well as, it is uncertain the future of a classical particle moving along a geodesic which reaches the singularity after a finite time, it is uncertain the time evolution of the corresponding quantum particle, provided it obeys the Klein-Gordon equation. But, when tested by massless scalar bosons and by fermions, the singularity is “healed” and no extraboundary conditions are necessary. The spacetime is wave regular for these fields.

Because the BTZ spacetime with negative mass parameter is not regular for every quantum particle, it must be excluded by the weak cosmic censorship hypothesis. For this reason, the $m = -1$ case, which is the stable ground state of the BTZ spacetime when studied in the general relativity context, remains stable when studied in the quantum mechanics framework.

\section{V. CONCLUSION}

The BTZ spacetime with mass parameter $m \geq 0$ does not cause any problem since the weak cosmic censorship

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