Calculation and Estimation of the Poisson Kernel

by

Steven G. Krantz

Abstract: We provide a simple method for obtain boundary asymptotics of the Poisson kernel on a domain in $\mathbb{R}^N$.

0 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set—called a domain. It is a matter of considerable interest to estimate the size of the Poisson kernel $P_\Omega(x,t) = P(x,t)$ of $\Omega$. Here, and throughout this paper, $x \in \Omega$ and $t \in \partial \Omega$.

In case $\Omega$ has a large group of symmetries, then it is often possible to calculate $P_\Omega$ explicitly. For example,

- The Poisson kernel of the disc $D \subseteq \mathbb{R}^2$ is
  
  $$P_D(x,t) = \frac{1}{2\pi} \cdot \frac{1 - |x|^2}{|x-t|^2}.$$  

- The Poisson kernel for the upper half-plane $U^2 = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\}$ is given by
  
  $$P_{U^2}(x,t) = \frac{1}{\pi} \cdot \frac{x_2}{(x_1-t)^2 + x_2^2}.$$  

- The Poisson kernel for the unit ball $B \subseteq \mathbb{R}^N$ is given by
  
  $$P_B(x,t) = \frac{\Gamma(N/2)}{2\pi^{N/2}} \cdot \frac{1 - |x|^2}{|x-t|^N}.$$  

Here $\Gamma$ is the classical gamma function.

1 Author supported in part by NSF Grant DMS-9988854.
The Poisson kernel for the upper halfspace $U^{N+1} \equiv \{ x = (x_1, \ldots, x_{N+1}) \in \mathbb{R}^{N+1} : x_{N+1} > 0 \}$ (with $x = (x_1, \ldots, x_{N+1}) = (x', x_{N+1})$) is given by

$$P_{U^{N+1}}(x, t) = c_N \frac{x_{N+1}}{(|x' - t|^2 + x_{N+1}^2)^{(N+1)/2}}$$

where

$$c_N = \frac{\Gamma([N + 1]/2)}{\pi^{[N+1]/2}}.$$ 

For purposes of studying the Schauder estimates for the Dirichlet problem, for studying the (nontangential) boundary behavior of harmonic functions, and for studying potential theory, one needs to have size estimates for the Poisson kernel on a fairly general domain (say a bounded domain with $C^2$ boundary).

The standard asymptotic is

$$P_{\Omega}(x, y) \approx \frac{\delta(x)}{|x - y|^N}.$$  \hfill (*)

Here $\delta(x) \equiv \delta_\Omega(x)$ is the distance from $x \in \Omega$ to $\partial \Omega$. This estimate, together with analogous estimates for the derivatives of $P_{\Omega}$, suffices for most applications. It is our purpose in this paper to give an efficient and elementary method for proving (*). At the end of the paper we shall also sketch an argument for obtaining the cognate estimate for derivatives of the Poisson kernel.

There are a number of methods for deriving estimates as we have described, though none of them is well known. After all, the harmonic analysis of domains in space is a fairly new field, and many of the techniques are only recently born. Classical studies, in dimension two only, appear in [KEL]. In the reference [KRA1], we present an argument based on Kelvin reflection of harmonic functions and comparisons by way of the maximum principle. These arguments were developed by Norberto Kerzman (personal communication). They were presented with Kerzman’s permission. They are intricate, and we shall not repeat them here.

Another natural method for developing an asymptotic expansion for the Poisson kernel is to use Fourier integral operators. To wit, let us suppose for simplicity that $\Omega$ is topologically trivial and has smooth boundary. Let $\Phi : \overline{\Omega} \to \overline{B}$ be a diffeomorphism of the closure of $\Omega$ with the closure of $B$. 

---

2
Then one can compare the true Poisson kernel on $\Omega$ with the pullback of the Poisson kernel from $B$ under the mapping $\Phi$. The result is the required asymptotic expansion (see [BMS], where a similar technique is used to obtain an asymptotic expansion for the Bergman kernel and the Szeg"o kernel).

In the present paper we use a method that has come to be known as “scaling” to produce the estimates (*) for the Poisson kernel. This is a methodology that has been developed extensively in the study of automorphism groups of domains in $\mathbb{C}^n$—see [ISK]. It has also been used in harmonic analysis to obtain information about reproducing kernels (see [NRSW], where it was used to study the Szeg"o kernel). The advantages of this approach are that (i) it is quite elementary and straightforward and (ii) it can be applied to a variety of reproducing kernels in many different circumstances. Thus the techniques presented here should find utility in a number of different contexts.

It is a pleasure to thank Kang-Tae Kim, Richard Rochberg, and Norm Levenberg for helpful conversations. Kim has taught me much of what I know about scaling. Coifman and Rochberg [COR] prove estimates much like the ones presented here, but on the ball for a Bergman space with weights. Levenberg and Yamaguchi [LEY] use a scaling method similar to the one here to estimate a reproducing kernel from another context.

1 The Main Result

For the remainder of the paper, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with $C^2$ boundary. This means that there is a $C^2$, real-valued function $\rho$ such that

$$\Omega = \{ x \in \mathbb{R}^N : \rho(x) < 0 \}$$

and $\nabla \rho \neq 0$ on $\partial \Omega$. Thus $\partial \Omega$ is a regularly imbedded $C^2$ hypersurface in $\mathbb{R}^N$.

**Theorem 1** Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with $C^2$ boundary. Let $P : \Omega \times \partial \Omega \to \mathbb{R}^+$ be the Poisson kernel for $\Omega$. Then there are constants $c_1, c_2 > 0$ such that

$$c_1 \cdot \frac{\delta(x)}{|x - y|^N} \leq P(x, y) \leq c_2 \cdot \frac{\delta(x)}{|x - y|^N}.$$  

(*)

3
The remainder of this section is devoted to the proof of this theorem. In the last section of the paper we shall remark on how to obtain a similar asymptotic for the derivatives of $P$. For convenience, we write

$$P(x, y) \approx \frac{\delta(x)}{|x - y|^N}$$

instead of (⋆).

Notice before we begin that, if $K$ is a compact set in $\Omega$, then the estimate we seek is trivial for $x \in K$ and $y \in \partial \Omega$. For then $|x - y| \geq c > 0$, $\delta(x)$ is bounded above, and we get a universal bound above and below on $\delta(x)/|x - y|^N$. A similar comment applies if $x$ is near the boundary and $y$ is far from $x$. So we may concentrate our attention on $x$ near the boundary and $y$ near $x$.

Now fix a point $P \in \partial \Omega$ and a point $P^0 \in \Omega$ such that the segment $\overline{P^0P}$ is normal to the boundary at $P$. We shall dilate coordinates with center $P^0$. We assume that $P^0$ is close to $\partial \Omega$—within a tubular neighborhood of the boundary—and we set $\epsilon = \text{dist}(P^0, P)$. We assume that coordinates have been rotated and centered so that

(a) The point $P$ is the origin $(0, 0, \ldots, 0)$;

(b) The normal direction $\overrightarrow{P^0P}$ is the positive $x_N$-direction.

For a point $x \in \mathbb{R}^N$, we write $x = (x_1, \ldots, x_N)$. We set $P^0 = (P^0_1, \ldots, P^0_N)$. With the normalization of coordinates, in fact $P^0 = (P^0_1, P^0_2, \ldots, P^0_N) = (0, \ldots, 0, +\epsilon)$. Now define

$$\Phi_\epsilon(x) = \left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}, \ldots, \frac{x_N}{\epsilon}\right).$$

Observe particularly that the mapping $\Phi_\epsilon$ sends the point $P^0$ to $(1, 0, \ldots, 0)$.

The first thing to notice is that, in a natural sense,

$$\lim_{\epsilon \to 0^+} \Phi_\epsilon(\Omega) = U^N.$$

To see this, we first check that if the defining function $\rho$, expanded about the point $P$, is given by

$$\rho(x) = \sum_{j=1}^{N} a_j^1 x_j + \sum_{j,k=1}^{N} a_{jk}^2 x_j x_k + \cdots = -x_N + \sum_{j,k=1}^{N} a_{jk}^2 x_j x_k.$$
(of course note that $\rho(P) = 0$) then

$$\rho_\epsilon(s) \equiv \frac{1}{\epsilon} \left[ \rho \circ \Phi_\epsilon^{-1}(s) \right] = \frac{1}{\epsilon} \left[ -\epsilon s_N + \sum_{j,k=1}^N a_{jk}^2 \epsilon^2 s_j s_k + \cdots \right] = -s_N + \epsilon \sum_{j,k=1}^N a_{jk}^2 s_j s_k + \cdots .$$

Plainly, as $\epsilon \to 0$, the transferred defining function $\rho_\epsilon$ tends to the linear defining function $\rho_0(s) \equiv -s_N$. In other words, the domains $\Phi_\epsilon(\Omega) \equiv \Omega_\epsilon$ converge (in an appropriate sense) to the standard halfspace. This last information is useful because we know the Poisson kernel for a halfspace.

Now we may take advantage of the facts accrued by setting $\Omega_\epsilon = \Phi_\epsilon(\Omega)$, letting $d\sigma$ be $(N - 1)$-dimensional area measure on $\partial\Omega$, $d\sigma_\epsilon$ to be $(N - 1)$-dimensional area measure on $\partial\Omega_\epsilon$, and taking $f$ to be a function that is continuous on $\Omega_\epsilon$ and harmonic on $\Omega_\epsilon$. Further, we let $x \in \Omega$ and set $s = \Phi_\epsilon(x)$. Then we calculate that

$$f(s) = f(\Phi_\epsilon(x))$$

$$= \int_{\partial\Omega_\epsilon} P_{\Omega_\epsilon}(\Phi_\epsilon(x), t) f(t) d\sigma_\epsilon(t)$$

$$= \int_{\partial\Omega} P_{\Omega}(\Phi_\epsilon(x), \Phi_\epsilon(\tau)) f(\Phi_\epsilon(\tau)) \det \text{Jac}\Phi_\epsilon(\tau) d\sigma(\tau) .$$

It is crucial to note here that the integral is over an $(N - 1)$-dimensional hypersurface, and hence the Jacobian determinant is that of an $(N - 1) \times (N - 1)$ matrix.

Now let us write

$$K_\epsilon(x, \tau) = P_{\Omega_\epsilon}(\Phi_\epsilon(x), \Phi_\epsilon(\tau)) \cdot \det \text{Jac}\Phi_\epsilon(\tau) = \epsilon^{-(N-1)} \cdot P_{\Omega_\epsilon}(\Phi_\epsilon(x), \Phi_\epsilon(\tau)) .$$

We thus have the equation

$$f \circ \Phi_\epsilon(x) = \int_{\partial\Omega} P_{\Omega}(x, \tau)[f \circ \Phi_\epsilon(\tau)] d\sigma(\tau) = \int_{\partial\Omega} K_\epsilon(x, \tau)[f \circ \Phi_\epsilon(\tau)] d\sigma(\tau) . \quad (\dagger)$$

Since this identity holds true for any choice of continuous $f$ on the boundary of $\Omega_\epsilon$ (with unique harmonic extension to $\Omega_\epsilon$), we may conclude that

$$P_{\Omega}(x, \tau) = K_\epsilon(x, \tau) . \quad (\ddagger)$$

The identity $(\ast)$ is the key to our result, for we know asymptotically what $K_\epsilon$ looks like. In particular, we know (see [KRA1, Section 1.3]) on any
smoothly bounded domain $U$ that the Poisson kernel is a normal derivative of the Green’s function:

$$P_U(x, y) = \frac{\partial}{\partial \nu_y} G_U(x, y).$$

And the Green’s function, in turn, is the solution on $U$ of the Dirichlet problem with boundary data the Newton potential $\Gamma_N(\cdot - x)$.

Now with $P, P^0$ fixed as before, let $W$ be a small, smoothly bounded, topologically trivial domain with these properties:

(a) $W \subseteq \Omega$;

(b) $P^0 \in W, P \in \partial W$;

(c) $\partial W \cap \partial \Omega$ is a relative neighborhood of $P$ in $\partial \Omega$.

Easy Schauder estimates, and the discussion in the preceding paragraph, show that we may obtain our estimate ($\star$) by studying the cognate question on $W$ (details of this type of argument may be found in [APF]).

Now the key observation at this point is that, when $\epsilon > 0$ is small, then the Poisson kernel for $\Phi_\epsilon(W)$ at interior points of the line segment $\Phi_\epsilon(P \overline{P^0})$ is very near to the Poisson kernel of the upper half space $U^N$ at those same points. The reason, of course, is that if $\rho_1$ is the defining function for $U^N$ and $\rho_2$ is the defining function for $\Phi_\epsilon(W)$ then there is a diffeomorphism $\lambda$ so that $\rho_1 = \rho_2 \circ \lambda$ near 0 and $\|\lambda - \text{id}\|_{C^1}$ is small. Referring again to the construction of the Poisson kernel above, the claim follows.

As a result, we may calculate the Poisson kernel on $\Omega$ by instead calculating the kernel on $W$. In turn, it then suffices to calculate the kernel on $U^N$. Thus we see that, for $x$ on the interior of the line segment $P \overline{P^0}$,

$$K_\epsilon(x, \tau) = \epsilon^{-(N-1)} \cdot P_{U^N}(\Phi_\epsilon(x), \Phi_\epsilon(\tau))$$

$$\approx \epsilon^{-(N-1)} \Phi_\epsilon(x)^N \left( |\Phi_\epsilon(x) - \Phi_\epsilon(\tau)|^2 + |\Phi_\epsilon(x)|^2 \right)^{N/2}$$

$$= \epsilon^{-(N-1)} \cdot \frac{x_N/\epsilon}{\left( |x'/\epsilon - \tau/\epsilon|^2 + |x_N/\epsilon|^2 \right)^{N/2}}$$

Unraveling the notation, we find that we have proved the approximation ($\star$).
2 Estimates for the Derivatives of the Poisson Kernel

The argument to obtain an asymptotic for a derivative of the Poisson kernel is nearly the same as the ones used above. The result we seek is

$$\nabla^k P(x, y) \approx \frac{\delta(x)}{|x - y|^{N+k}}. \quad (***)$$

The crux of the argument is the analog of equation (†). For the present application, that equation now becomes

$$\nabla^k f \circ \Phi_\epsilon(x) = \nabla^k \int_{\partial \Omega} P_\Omega(x, \tau)[f \circ \Phi_\epsilon(\tau)] d\sigma(\tau) = \int_{\partial \Omega} \nabla^k K_\epsilon(x, \tau)[f \circ \Phi_\epsilon(\tau)] d\sigma(\tau).$$

It follows as before that we have the identity

$$\nabla^k P_\Omega(x, \tau) = \nabla^k K_\epsilon(x, \tau) + \mathcal{E},$$

where $\mathcal{E}$ is an error term that is a polynomial (and hence is of no interest for our estimates). The remainder of the derivation of (**) is as before.

References

[APF] L. Apfel, Localization properties and boundary behavior of the Bergman kernel, thesis, Washington University in St. Louis, 2003.

[BMS] L. Boutet de Monvel and J. Sjöstrand, Sur la singularité des noyaux de Bergman et Szegö, Soc. Mat. de France Asterisque 34-35(1976), 123-164.

[COR] R. Coifman and R. Rochberg, Representation Theorems for Holomorphic and Harmonic Functions in $L^p$. Representation Theorems for Hardy Spaces, pp. 11–66, Asterisque 77, Société Mathématique de France, Paris, 1980.

[ISK] A. Isaev and S. G. Krantz, Domains with non-compact automorphism group: a survey, Advances in Math. 146(1999), 1–38.

[KEL] Keldych Lavrentieff, Sur une evaluation de la fonction de Green, Doklady Acad. USSR 24(1939), 102.
[KRA1] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[LEY] N. Levenberg and H. Yamaguchi, The metric induced by the Robin function, *Mem. Amer. Math. Soc.* 92(1991), viii+156.

[NRSW] A. Nagel, J. P. Rosay, E. M. Stein, and S. Wainger, Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$, *Ann. Math.* 129(1989), 113-149.