Periodic orbits of Linear and Invariant flows on Semisimple Lie groups

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Abstract

Our main is to study periodic orbits of linear or invariant flows on a real, connected, semisimple Lie group. Since there exist a derivation of Lie algebra to linear or invariant flow, we show that a periodic orbit that is not fixed point of a linear or invariant flow is periodic if and only the eingevalues of derivation is 0 or $\pm \alpha i$ for an unique $\alpha \neq 0$ and they are semisimple. We apply this result in noncompact case through Iwasawa’s decomposition. Furthermore, we present a version of Poincaré-Bendixon’s Theorem for periodic orbits.

Keywords: periodic orbits, linear flow, invariant flow.

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1 Introduction

In [6], the author study periodic orbits of a linear or invariant flow on a real, connected, compact, semisimple Lie group. Our wish is extend the study founded there to a real, connected, semisimple Lie group.

We begin by recalling some facts. Let $G$ be a real, connected, semisimple Lie group and denote by $\mathfrak{g}$ its Lie algebra. We recall that a linear vector field $\mathcal{X}$ on $G$ is called linear if its flow $\varphi_t$ is a family of automorphism of the Lie group $G$. Namely, the linear flow $\varphi_t$ is solution of dynamical system

$$\dot{g} = \mathcal{X}(g), \; g \in G. \tag{1}$$

An important remark is that there exists an one-to-one correspondence between linear vector field and right invariant vector fields, that is, for each linear vector field $\mathcal{X}$ there exists a unique right invariant vector field $X$ in correspondence. From
this it is possible to show that the linear flow $\varphi_t$ and invariant flow $\exp(tX)$ are in an one-to-one correspondence. Furthermore, for each linear vector field $\mathcal{X}$ or right invariant vector field $X$ it is possible to associate the derivation of Lie algebra $\mathcal{D} = -\text{ad}(\mathcal{X}) = -\text{ad}(X)$. The association between linear and right invariant vector field is the key to characterize periodic orbits of linear or invariant flow.

Our work is to show that an orbit that is not a fixed point of linear or invariant flow is periodic if and only if eigenvalues of derivation $\mathcal{D}$ associated to they are semisimple and they are $0$ or $\pm\alpha i$ for an unique $\alpha \neq 0$. It is the result of our main Theorem. In the case of compact, semisimple Lie group the hypothesis that eigenvalues are semisimple is fulfilled naturally (see Theorem 3 in [6]). If we assume that $G$ is noncompact, semisimple Lie groups, then using the Iwasawa’s decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ we show that orbits that are nor fixed points of linear or invariant flows that are associated to derivations $\mathcal{D} = -\text{ad}(aH + bX)$ are not periodic, where $H \in \mathfrak{a}, X \in \mathfrak{n}$ and $a, b \in \mathbb{R}$. Thus, a natural question arises: take $Y \in \mathfrak{k}$ and $c \in \mathbb{R}$, is the family of derivations of the form $\mathcal{D} = -\text{ad}(cY)$ the unique possibility to find periodic orbits? The answer is no. To view this we study periodic orbits on the special linear group $\text{Sl}(2, \mathbb{R})$. We show in Theorem 4.1 that may exist periodic orbits with influence of three components of Iwasawa’s decomposition.

Finally, assuming that eigenvalues of derivation $\mathcal{D}$ are semisimple and they are $0$ or $\pm\alpha i$ for an unique $\alpha \neq 0$ we obtain a version of Poincaré-Bendixon’s Theorem.

This paper is organized as follows. Section 2 briefly reviews the notions of linear vector fields. Section 3 works with periodic orbits. Section 4 studies periodic orbits on $\text{Sl}(2, \mathbb{R})$.

2 Linear vector fields

In this section we recall some basic facts about linear vector field. For a fuller treatment we refer to [1] and [3]. Let $G$ be a connected Lie group and let $\mathfrak{g}$ denote its Lie algebra. A vector field $\mathcal{X}$ on $G$ is called linear if its flow $(\varphi_t)_{t \in \mathbb{R}}$ are automorphisms of Lie group. For any linear vector field it is possible to associated the following derivation of Lie algebra $\mathfrak{g}$:

$$\mathcal{D}(Y) = -[\mathcal{X}, Y], \ Y \in \mathfrak{g}.$$

For the convenience of the reader we resume some facts about a linear vector field $\mathcal{X}$ and its flow $\varphi_t$. The proof of these facts can be found in [3].

**Proposition 2.1** Let $\mathcal{X}$ be a linear vector field and let $\varphi_t$ denote its flow. The following assertions hold:

(i) $\varphi_t$ is an automorphism of Lie groups for each $t$;

(ii) $\mathcal{X}$ is linear iff $\mathcal{X}(gh) = R_{b\cdot} \mathcal{X}(g) + L_{g\cdot} \mathcal{X}(h)$;

(iii) $(d\varphi_t)_e = e^{t\mathcal{D}}$ for all $t \in \mathbb{R}$. 

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It is known that $G = \mathbb{R}^n$ is a Euclidean Lie Group. For any $n \times n$ matrix $A$ it is true that $\mathcal{X} = A$ is a linear vector field. Furthermore, $D_x(b) = -[Ax, b] = Ab$. In this sense, we can view the dynamical system

$$\dot{g} = \mathcal{X}(g), \quad g \in G,$$

on $G$ as a generalization of dynamical system on $\mathbb{R}^n$ given by

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n.$$

### 3 Periodic orbits on semisimple Lie groups

In this section, we assume that $G$ is a connected, semisimple Lie group. Let $\mathcal{X}$ be a linear vector field on $G$, and let us denote by $\varphi_t$ its flow. Since $G$ is semisimple, there exists a right invariant vector field $X$ on $G$ such that $\mathcal{X} = X + I_\ast X$, where $I_\ast X$ is the left invariant vector field associated to $X$. Here, $I_\ast$ is the differential of inverse map $i(g) = g^{-1}$ (more details is founded in [5]). A simple account shows that $D = -\text{ad}(\mathcal{X}) = -\text{ad}(X)$. Furthermore, the differential equation (1) is written as

$$\dot{g} = X(g) + (I_\ast X)(g).$$

For this it is possible to show that linear flow $\varphi_t$ is solution of (1) if and only if $\varphi_t(g) \cdot \exp(tX)$ is solution of $\dot{g} = X(g)$. In consequence, as it shows on [6], periodic orbits of invariant and linear dynamical system are related. For the convenience of the reader, we repeat the Proposition 8 in [6] without proof.

**Proposition 3.1** Let $\mathcal{X}$ be a linear vector field on a semisimple Lie group $G$. The following sentences are equivalent:

i) for every $g \in G$, the invariant flow $\exp(tX)g$ is periodic;

ii) the identity $e$ is a periodic point of invariant flow $\exp(tX)$;

iii) any $g \in G$ is a periodic point of linear flow $\varphi_t$;

iv) for every $g \in G$, $\text{Ad}(g)$ is a periodic point of the flow $e^{tD}$.

Before we prove our main theorem we need to introduce some concepts. Following [5], if for an eigenvalue $\mu$ all complex Jordan blocks are one-dimensional, i.e., a complete set of eigenvectors exists, it is called semisimple. Equivalently, the corresponding real Jordan blocks are one-dimensional if $\mu$ is real, and two-dimensional if $\mu$ and $\bar{\mu} \in \mathbb{C} \setminus \mathbb{R}$.

**Theorem 3.2** Let $G$ be a semisimple Lie group. The following sentences are equivalent:

i) there exists a periodic orbit for the linear flow $\varphi_t$;

ii) there exists a periodic orbit for the right invariant flow $\exp(tX)$;
iii) the derivation $D$ of $X$ has only semisimple eigenvalues of the form $0$ or $\mu = \pm \alpha i$ with unique $\alpha \in \mathbb{R}^*$.

Furthermore, if there exists a periodic orbit, then its period is $T = 2\pi/\alpha$.

**Proof:** We first observe that i) is equivalent to ii) by Proposition 3.1. We are going to show that ii) is equivalent to iii). For this, it is sufficient to consider the identity $e$ as a periodic point to the flow $\exp(tX)$ with period $T > 0$. Then, for all $t \in \mathbb{R}$,

$$\exp((t + T)X) = \exp(tX) \Leftrightarrow \exp(TX) = e \Leftrightarrow e^{-TD} = I_n.$$ 

Take the Jordan form $J$ of $D$. A simple account shows that $e^{-TJ} = I_d$.

We break the proof in two steps:

i) real eigenvalues: Let $\mu$ be a real eigenvalue of derivation $D$. Then for its $m$-dimensional Jordan block $J_\mu$ we have $e^{-TJ_\mu} = I_m$. If $m > 1$ then for Jordan block $J_\mu$ we have $e^{-T \mu}(-T) = 0$. It implies that $T = 0$, a contradiction. Therefore $m = 1$. It means that $\mu$ is a semisimple eigenvalue. In this case, for $e^{-TJ_\mu} = I_1$ we obtain $e^{(-T)\mu} = 1$. We thus get $(-T)\mu = 0$. Since $T > 0$, it follows that $\mu = 0$. It means that unique real eigenvalue of derivation $D$ is 0.

ii) complex eigenvalues: suppose that $\mu = \alpha \pm i\beta$ are conjugate, complex eigenvalues of derivation $D$. Let us denote $R = R(t) = \begin{pmatrix} \cos(t\beta) & -\sin(t\beta) \\ \sin(t\beta) & \cos(t\beta) \end{pmatrix}$

and denote $J_\mu$ its $2m$-dimensional Jordan block. Suppose that $m > 1$. From $e^{-TJ_\mu} = I_{2m}$ we have that

$$e^{\alpha(-T)}(-T)R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

A simple accounts shows that $T = 0$, a contradiction. Therefore $m = 1$. It means that $\mu$ is semisimple. Hence the Jordan block $J_\mu$ is two dimensional. It gives $e^{(-T)J_\mu} = I_2$. From this equality we obtain

$$e^{\alpha(-T)} \cos(\beta(-T)) = 1 \quad \text{and} \quad e^{\alpha(-T)} \sin(\beta(-T)) = 0$$

The last equality implies that $\sin(\beta(-T)) = 0$. It follows that $\beta(-T) = n\pi$ for any $n \in \mathbb{Z}$. Substituting this in the first equation we obtain

$$e^{\alpha(-T)} \cos(n\pi) = 1$$

Because $e^{\alpha(-T)} > 0$, it follows that $n = 2m$ for some $m \in \mathbb{Z}$. Hence $e^{\alpha(-T)} = 1$. So $\alpha(-T) = 0$. Since $T > 0$, it follows that $\alpha = 0$. It means that complex eigenvalues of derivation $D$ are of the form $\mu = \pm \beta i$.

Now we proved that complex eigenvalues of $D$ are $\pm \lambda i$ for a unique $\lambda \in \mathbb{R}$. As proved above for any eigenvalues of $D$ its real Jordan Block has dimension 1 or 2 if it is real or complex, respectively. Furthermore, we showed that unique
real eigenvalue of $D$ is 0. Therefore its real Jordan block is written as $J_0 = [0]$. Thus $e^{tJ_0}$ is constant. It means that in direction of 0 the $e^{tJ}$ is constant. Consequently, solutions associated to 0 are trivially periodic. On the other hand, take two complex, conjugate eigenvalues of $D$ of the form $\pm \lambda i$ and $\pm \nu i$. By proved above, its real Jordan blocks are, respectively,

$$
\begin{pmatrix}
\cos(t\lambda) & -\sin(t\lambda) \\
\sin(t\lambda) & \cos(t\lambda)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\cos(t\nu) & -\sin(t\nu) \\
\sin(t\nu) & \cos(t\nu)
\end{pmatrix}.
$$

As $e^{(-T)J} = Id$ we have $-T = \frac{2\pi}{\lambda}$ and $-T = \frac{2\pi}{\nu}$. This clearly forces $\lambda = \nu$.

Reciprocally, suppose that the eigenvalues of $D$ are 0 or $\pm \alpha i$ and they are semisimple. As noted above, in direction of 0 the solution is constant. Consider thus semisimple, complex eigenvalues $\pm \alpha i$ with $\alpha \neq 0$. It entails that every real Jordan block associated to it has the dimension two and the solution applied at this block gives the following matrix

$$
\begin{pmatrix}
\cos(t\alpha) & -\sin(t\alpha) \\
\sin(t\alpha) & \cos(t\alpha)
\end{pmatrix}.
$$

It follows easily that the matrix above is periodic with period $T = \frac{2\pi}{\alpha}$. In consequence, $e$ is periodic point of $e^{TJ}$ with period $T = \frac{2\pi}{\alpha}$, which is equivalent $Ad(e) = e$ to be periodic point of $e^{tD}$. Consequently, by Proposition 3.1, the right invariant flow $\exp(tX)$ is periodic with period of $T = \frac{2\pi}{\alpha}$.

A natural question arises from Theorem above: is it possible $D$ has only real eigenvalues? We answer in next result.

**Corollary 3.3** Let $G$ be a semisimple Lie group. If a derivation $D$ on $\mathfrak{g}$ has only real eigenvalue then orbits of the linear flow $\varphi_t$ or invariant flow $\exp(tX)$ that are not fixed point are not periodic, where $X \in \mathfrak{g}$ is associated to $D$.

**Proof:** It is sufficient to show Corollary for the linear flow $\varphi_t$. Let $g \in G$ and suppose that $\varphi_t(g)$ is a periodic orbits that is not a fixed point with period $T > 0$. From Theorem above eigenvalues $D$ are semisimple. Furthermore, the unique real eigenvalue of $D$ is 0. These two facts show that $D$ is null. Since $G$ is semisimple, it follows that $Ad$ is a bijection. It implies that, for any $t \in \mathbb{R}$,

$$
Ad(\varphi_t(g)) = Ad(\exp(tX)g\exp(-tX)) = e^{-tD}Ad(g)e^{tD} = Ad(g)
$$

It shows that $\varphi_t(g) = g$ for every $t \in \mathbb{R}$. It means that every $g$ is a fixed point, which is a contradiction.

Assume that $G$ is a noncompact, semisimple Lie group. From Iwasawa’s decomposition there exists three Lie subalgebra $\mathfrak{t}, \mathfrak{a}$ and $\mathfrak{n}$ such that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Our next result study the behavior of linear flows or invariant flows associated to family of derivation $D = -ad(aH + bX)$ with $H \in \mathfrak{a}$, $X \in \mathfrak{n}$ and $a, b \in \mathbb{R}$.

**Corollary 3.4** Let $G$ be a noncompact, semisimple Lie group such that its Lie algebra is decomposed as $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$. If $H \in \mathfrak{a}$ and $X \in \mathfrak{n}$, and if $D =$
− ad(aH + bX) is a derivation for some \( a, b \in \mathbb{R} \), then orbits that are not fixed points of the linear flow \( \varphi_t \) or invariant flow associated to \( D = -\text{ad}(aH + bX) \) for some \( a, b \in \mathbb{R} \) are not periodic.

**Proof:** Let \( D = -\text{ad}(aH + bX) \) be a derivation for some \( a, b \in \mathbb{R}, H \in \mathfrak{a} \) and \( X \in \mathfrak{n} \). From Lemma 6.4.5 in [2], there exists a basis of \( \mathfrak{g} \) such that the matrix of \( \text{ad}(H) \) is a diagonal with real entries and \( \text{ad}(X) \) is an upper triangular with 0’s on the diagonal. Hence the matrix of \( D \) is an upper triangular matrix with real entries on diagonal. In consequence, eigenvalues of \( D \) are real. From Corollary above it follows that orbits that are not fixed points of linear or invariant flow associated to \( D \) are not periodic. □

Other consequence of Theorem 3.2 is a version of Poincaré-Bendixson’s Theorem for semisimple Lie groups.

**Theorem 3.5 (Poincaré - Bendixson)** Let \( G \) be a semisimple Lie group and \( \mathcal{X} \) a linear vector field on \( G \). Suppose that eigenvalues of a derivation \( D = -\text{ad}(\mathcal{X}) \) are 0 or \( \pm \alpha i \) and they are semisimple. If \( \Omega \) is a nonempty compact \( \omega \)-limit set for the linear flow \( \varphi_t \), and if it does not contain a fixed point of \( \varphi_t \), then \( \Omega \) is a periodic orbit. Furthermore, if \( \mathcal{X} \) is the right vector field associated to \( \mathcal{X} \), if \( \Omega \) is a nonempty compact \( \omega \)-limit set for the invariant flow \( \exp(t\mathcal{X}) \), and if it does not contain a fixed point of \( \exp(t\mathcal{X}) \), then \( \Omega \) is a periodic orbit.

### 4 Periodic orbits on \( \text{Sl}(2, \mathbb{R}) \)

In this section we study periodic orbits of linear or invariant flows on semisimple Lie group \( \text{Sl}(2, \mathbb{R}) \). Let us denote by \( \mathfrak{sl}(2, \mathbb{R}) \) the Lie algebra of \( \text{Sl}(2, \mathbb{R}) \). Since \( \text{Sl}(2, \mathbb{R}) \) is a semisimple non-compact Lie groups, its Lie algebra has the following Iwasawa’s decomposition:

\[
\mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\} \oplus \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, a \in \mathbb{R} \right\} \oplus \left\{ \begin{pmatrix} 0 & \nu \\ 0 & 0 \end{pmatrix}, \nu \in \mathbb{R} \right\}
\]

It is clear that

\[
\beta = \left\{ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}
\]

form a basis of \( \mathfrak{sl}(2, \mathbb{R}) \). Furthermore, they brackets are given by

\[
[A, H] = 2A + 4S, \ [A, S] = -H, \ [H, S] = 2S.
\]

Let \( \mathcal{X} \) be a linear vector field on \( \text{Sl}(2, \mathbb{R}) \) and \( D \) its derivation. Then there exist a right invariant vector field \( X \in \mathfrak{sl}(2, \mathbb{R}) \) such that \( D = -\text{ad}(X) \). Write \( X \) as

\[
X = aA + bH + cS, \ a, b, c \in \mathbb{R}.
\]

Considering the basis \( \beta \) above we obtain the following matrix of derivation

\[
D = -\text{ad}(X) = \begin{pmatrix}
2b & -2a & 0 \\
-c & 0 & a \\
4b & -4a + 2c & -2b
\end{pmatrix}.
\]
From this we see that eigenvalues of $D$ are
\[ \left\{ 0, -2\sqrt{-a^2 + ac + b^2}, 2\sqrt{-a^2 + ac + b^2} \right\}. \] (2)

We are in a position to show a condition to an orbit of linear or invariant flow be periodic.

**Theorem 4.1** Let $\mathcal{X}$ be a linear vector field on $\text{Sl}(2, \mathbb{R})$ and $X$ its associated right invariant vector field. Write $X = aA + bH + cS$ in the basis $\beta$ of $\text{sl}(2, \mathbb{R})$. Orbits of linear flow or invariant flow associated to linear vector field $\mathcal{X}$ that are not fixed points are periodic if and only if
\[ a^2 > ac + b^2. \]

**Proof:** It is a direct application of Theorem 3.2 with eigenvalues (2). □

Theorem above shows that there exist periodic orbits on $\text{Sl}(2, \mathbb{R})$ with influence of with contribution of compact, abelian and nilpotent parts of Iwasawa's decomposition. It says that to classify the periodic orbits on semisimple Lie groups it is necessary to consider the three components of Iwasawa’s decomposition. In other words, we need to consider a derivation $D = -\text{ad}(X)$ with $X \in \mathfrak{g}$, not only derivations $D = -\text{ad}(X)$ with $X \in \mathfrak{k}$.

**References**

[1] F. Cardetti and D. Mittenhuber, *Local controllability for linear control systems on Lie groups*, Journal of Dynamical and Control Systems; vol. 11, No. 3., 353-373

[2] A. W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics, 140, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1399083

[3] Ph. Jouan; *Equivalence of Control Systems with Linear Systems on Lie Groups and Homogeneous Spaces*, Dynamics and Control Systems, 17 (2011) 591-616.

[4] F. Colonius and W. Kliemman, *Dynamical System and Linear Algebra*, Graduate studies in mathematics; volume 158, AMS, 2014.

[5] L.B. A. San Martin, *Grupos de Lie*, Editora da Unicamp, Campinas, (2017).

[6] S.N. Stelmastchuk, *Linear flows on compact, semisimple Lie groups: stability, periodic orbits, and Poincaré-Bendixon’s Theorem*, https://arxiv.org/abs/1806.08026.