DIFFERENTIABLE STRATIFIED GROUPOIDS AND A DE RHAM THEOREM FOR INERTIA SPACES

CARLA FARSI, MARKUS J. PFLAUM, AND CHRISTOPHER SEATON

ABSTRACT. We introduce the notions of differentiable groupoids and differentiable stratified groupoids, generalizations of Lie groupoids in which the spaces of objects and arrows have the structures of differentiable spaces, respectively differentiable stratified spaces, compatible with the groupoid structure. After studying basic properties of these groupoids including Morita equivalence, we prove a de Rham theorem for proper locally contractible differentiable stratified groupoids. We then focus on the study of the inertia groupoid associated to a proper Lie groupoid. We show that the loop and the inertia space of a proper Lie groupoid can be endowed with a natural Whitney (b)-regular stratification, which we call the orbit Cartan type stratification. Endowed with this stratification, the inertia groupoid of a proper Lie groupoid becomes a locally contractible differentiable stratified groupoid.

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1. Introduction

The theory of Lie group actions on smooth manifolds is fundamental for several areas in mathematics and has a long tradition. But examples of natural Lie group actions do not only comprise actions on smooth manifolds but also on singular spaces such as orbifolds [LT97, Sch01, Ver96] or manifolds with boundary or corners [MROD93, Mel91]. Lie group actions on singular spaces arise also in singular symplectic reduction [SL91] and the transverse cotangent bundle of a $G$-manifold [DCPV13, PV09]. In these cases, the corresponding translation groupoid, while not a Lie groupoid, has significantly more structure than merely that of a topological groupoid. In particular, the spaces of objects and arrows inherit differentiable stratified space structures compatible with the groupoid structure maps. One of the main goals of this paper is to define and study the category of such groupoids and their singular structures.

In this paper, we introduce differentiable stratified groupoids, a class of groupoids within the category of differentiable stratified spaces that share many of the properties of Lie groupoids [CM18]. The definition is designed so that, under mild hypotheses, the restriction of a differentiable stratified groupoid to a stratum of the orbit space is a Lie groupoid. The definitions of a differentiable groupoid and of a differentiable stratified groupoid we propose are deliberately broad so as to be easily verified and to include many interesting cases. Examples of such groupoids arise as restrictions or quotients of Lie groupoid actions or in other contexts from a related Lie groupoid. Often more specific properties are inherited from the context. To take advantage of these we define the subclasses of sliceable differentiable (stratified) groupoids in Definitions 2.7 and 2.31 and introduce a local contractibility condition in Definition 5.7.

One major motivating example is that of a compact Lie group acting differentiably on a differentiable stratified space in such a way that the strata are permuted by the action. We will show that the corresponding translation groupoid is a differentiable stratified groupoid. We study the properties of these groupoids and their object spaces including the appropriate notion of Morita equivalence. Moreover, we prove a de Rham theorem which relates the singular cohomology of the orbit space of a sliceable differentiable stratified groupoid fulfilling a local contractibility condition to the cohomology of basic differential forms on the object space; see Theorem 5.10. To prove this theorem, we first localize to groupoid charts and then prove a Poincaré lemma for basic forms in the context of sliceable differentiable stratified groupoids.

A particularly important example which we consider is that of the inertia groupoid of a proper Lie groupoid $G$. The inertia groupoid is presented as the translation groupoid of $G$ acting on the so-called loop space $\Lambda_0 G \subset G$ which consist of all arrows having the same source and target. When $G$ is an orbifold groupoid, the loop space $\Lambda_0 G$ is a smooth manifold so that the inertia groupoid presents an orbifold, the inertia orbifold. The inertia orbifold plays an important role in the geometry and index theory of orbifolds; see [ALR07, PPT07]. However, when $G$ is not an orbifold groupoid, the loop space $\Lambda_0 G$ becomes singular, and very little is known about its singularity structure in general. One of the goals of this paper is to deepen the understanding of inertia groupoids by exhibiting explicit stratifications for them. Motivation for such a study comes from the abundance and diversity where differentiable stratified groupoids appear, often in a somewhat hidden way. In addition to the mentioned orbifolds, loop spaces and their associated inertia groupoids appear naturally in geometry e.g. as commuting varieties.
when the adjoint action of a Lie group $G$ on itself is considered. The loop space corresponds then to the set of pairs of commuting elements in $G$. Similarly, spaces of (conjugacy classes of) commuting $m$-tuples of elements of $G$ as studied e.g. in provide related examples of differentiable stratified groupoids. Important for the understanding of moduli spaces of flat connections and for representation theory are sets of conjugacy classes of homomorphisms $\pi \to G$, where $\pi$ is a finitely generated discrete group and $G$ a compact connected Lie group. Such spaces can also be naturally interpreted as differentiable stratified groupoids.

When $G$ is the translation groupoid $G \ltimes M$ associated to the smooth action of a compact Lie group $G$ on a manifold $M$, the orbit space of the $G$-action on the corresponding loop space - here this orbit space is called the inertia space - was first considered in. By virtue of its structure as a semialgebraic $G$-set, the loop space is a Whitney (b)-regular stratified space, see Example 3.4, but the explicit structure of this stratification is not well understood. In the previous paper, we introduced an explicit Whitney (b)-regular stratification of the loop space of $G \ltimes M$ and its associated orbit space. Applying this stratification to local charts of the form $G \ltimes M$ for a proper Lie groupoid $G$ does in general not yield a well-defined global stratification of the loop space of $G$. Specifically, the local stratifications from that paper may not coincide on intersections of charts. Here, we will give a significant modification of the stratification in that can be patched together on charts to yield a well-defined stratification of the loop and inertia spaces of an arbitrary proper Lie groupoid. We call this stratification the orbit Cartan type stratification. For the translation groupoid of a compact group action the orbit Cartan type stratification presented here is in general coarser than the one considered in our previous paper. Because the construction is local, we carry it out on a single groupoid chart, which, up to Morita equivalence, is given by the translation groupoid associated to a finite-dimensional $G$-representation $V$ where $G$ is a proper Lie groupoid, see Theorem 6.4. For this case, the main ideas of our construction rely on providing a stratification of slices, see Section 6.2 for details. To achieve this, we first decompose the Cartan subgroups of the stabilizers into equivalence classes determined by their fixed point sets in $V$, which motivates the name of the stratification. We then use these decompositions together with the fixed point sets of the stabilizers to stratify the slice and take saturations to stratify the loop and orbit spaces. In Section 6.3 we demonstrate that our orbit Cartan type decomposition indeed satisfies all the properties of a stratification and turns the loop and the inertia space into differentiable stratified spaces; see Theorem 6.6. Moreover, in Section 6.5 we show that the orbit Cartan type stratification is Whitney (b)-regular. That this stratification is given explicitly allows us finally to verify in Theorem 6.19 that the local contractibility condition of Definition 5.7 is fulfilled for the inertia space, hence the de Rham theorem holds in this case.

It appears that the stratification and the de Rham theorem for the inertia space could provide a decisive tool to verify Brylinski’s claim, which is not fully proven in his paper, that for a given transformation groupoid $G \ltimes M$ the complex of basic differential forms on the loop space is quasi-isomorphic to the Hochschild chain complex of the convolution algebra on $G \ltimes M$. Our results from have been used in a proof of Brylinski’s conjecture for the case of an $S^1$-action, see. Work on the general case is in progress.

The outline of this paper is as follows. In Section 2 we introduce the fundamental concepts of differentiable (stratified) groupoids we will rely on throughout this paper,
demonstrate several basic properties, and consider morphisms of differentiable (stratified) groupoids and Morita equivalence. In Section 3 we present several examples of differentiable stratified groupoids and continue in Section 4 with the definition of the algebroid of a differentiable stratified groupoid assumed to fulfill some additional technical hypotheses. The algebroid is then used in Section 5 to present the de Rham theorem for sliceable differentiable stratified groupoids. In Section 6 we establish that the inertia space of a proper Lie groupoid admits a locally contractible Whitney (b)-regular stratification, the orbit Cartan type stratification. The result is stated in Theorem 6.6 and relies on Theorem 6.4 which provides the result for the local case, that is, for linear actions of compact Lie groups. In Proposition 6.18 we demonstrate Whitney (b)-regularity for the orbit Cartan type stratification. The appendix contains a recollection of the the basics of differentiable spaces and describes the category of differentiable stratified spaces used throughout this paper.

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2. Fundamentals

In this section, we give the definitions and basic properties of the groupoids under consideration. See Appendices A.1 and A.2 for a review of the notions of differentiable and differentiable stratified spaces used in this paper.

2.1. Topological groupoids. Recall that by a groupoid \( G \) one understands a small category with object set \( G_0 \) and arrow set \( G_1 \) such that all arrows are invertible. We use \( s, t: G_1 \to G_0 \) to denote the source and target maps, respectively, write \( u: G_0 \to G_1 \) for the unit map, \( i: G_1 \to G_1 \) for the inverse map, and finally denote by \( m: G_1 \times G_1 \to G_1 \) the multiplication or composition map. The maps \( s, t, u, i, \) and \( m \) are collectively referred to as the structure maps of the groupoid. If \( G \) and \( H \) are groupoids, a morphism \( f: G \to H \) is a functor, i.e. a pair of functions \( f_0: G_0 \to H_0 \) and \( f_1: G_1 \to H_1 \) that commute with each of the structure maps.

The orbit through a point \( x \in G_0 \) is defined as the set of all \( y \in G_0 \) for which there exists a \( g \in G_1 \) such that \( s(g) = x \) and \( t(g) = y \). It is denoted by \( G_x \). Obviously, the object space is partitioned into orbits. We denote the set of orbits of a groupoid \( G \) by \( |G| \), and the canonical projection from the object to the orbit space by \( \pi: G_0 \to |G| \).

A groupoid \( G \) is called a topological groupoid if \( G_0 \) and \( G_1 \) are topological spaces and each of the structure maps are continuous. This implies in particular that the unit map is a homeomorphism onto its image. If the topological groupoid \( G \) is Hausdorff, which means that \( G_1 \) is Hausdorff, the image \( u(G_0) \) is closed in \( G_1 \), cf. [Ren80], Chap. 1.
Sec. 2. The orbit space of a topological groupoid $G$ is always assumed to carry the quotient topology with respect to the canonical projection $\pi : G_0 \to |G|$. If $G$ and $H$ are topological groupoids and $f : G \to H$ a morphism of groupoids, then $f$ is a morphism of topological groupoids if in addition $f_0$ and $f_1$ are continuous functions.

If the source map of a topological groupoid $G$ is an open map one says that $G$ is an open topological groupoid. Note that as $i$ is a homeomorphism and $t = s \circ i$, the target map of an open topological groupoid is open as well. A topological groupoid $G$ for which the source map $s$ is a local homeomorphism is called an étale groupoid. $G$ is called a quasi-proper groupoid if $s \times t : G_1 \times G_1 \to G_0$ is a proper map, and a proper groupoid if it is Hausdorff and quasi-proper. Finally, we say that a topological groupoid $G$ is compact or locally compact if the topological space $G_1$ has the respective property. Note that then $G_0$ is compact or locally compact, respectively, as well by the proof of [Tu04, Prop. 2.5].

**Remark 2.1.** In this paper we follow the definitions of quasi-compact, compact, and locally compact spaces by Bourbaki [Bou98]. That means, a topological space $X$ is called quasi-compact if every open cover of $X$ admits a finite subcover, compact if $X$ is both Hausdorff and quasi-compact, and finally locally compact if it is Hausdorff and every point $x \in X$ has a compact neighborhood. If each point of a topological space $X$ possesses a Hausdorff (respectively quasi-compact) neighborhood, we say that $X$ is locally Hausdorff (respectively locally quasi-compact). A continuous map $f : X \to Y$ between not necessarily Hausdorff spaces $X$ and $Y$ is called proper if $f \times \text{id}_Z : X \times Z \to Y \times Z$ is a closed map for every topological space $Z$, cf. [Bou98, I.10.1, Def. 1] or [Tu04, Sec. 1.3]. By [Bou98, I.10.2, Thm. 1], properness of $f$ is equivalent to the property that $f$ is closed and $f^{-1}(y)$ is quasi-compact for each $y \in Y$.

**Proposition 2.2.** Let $G$ be a topological groupoid. Then the following holds true:

1. If $G$ is proper and the object space $G_0$ is locally compact, then $G$ is a locally compact groupoid.
2. The quotient map $\pi : G_0 \to |G|$ is an open map of topological spaces if $G$ is an open groupoid.
3. The orbit space of $G$ is Hausdorff if $G$ is open, $G_0$ is locally compact, and $(s,t)(G_1)$ is closed in $G_0 \times G_0$. In particular, this is the case if $G$ is a proper open groupoid.
4. The orbit space of $G$ is locally compact if $G$ is a locally compact open groupoid and $(s,t)(G_1)$ is locally closed in $G_0 \times G_0$.

**Proof.** Let $g \in G_1$, $K$ be a compact neighborhood of $s(g)$, and $L$ be a compact neighborhood of $t(g)$. Then $(s,t)^{-1}(K \times L)$ is a compact neighborhood of $g$. Since $G_1$ is Hausdorff, the first claim is proved. Point (2) is an immediate consequence of [Tu04, Prop. 2.11]. The claims (3) and (4) follow from [Tu04, Prop. 2.12].

2.2. Differentiable groupoids. We now consider topological groupoids endowed with a differentiable structure as defined in Appendix A.

**Definition 2.3.** Let $G$ be an open topological groupoid. We say that $G$ is a differentiable groupoid if $G_0$ and $G_1$ are differentiable spaces and the structure maps $s$, $t$, $i$, $u$, and $m$ are morphisms of differentiable spaces. We say that $G$ is a reduced differentiable groupoid if $G_0$ and $G_1$ are reduced differentiable spaces. A morphism of topological groupoids $f : G \to H$ between differentiable groupoids $G$ and $H$ is called a morphism of differentiable groupoids if the functions $f_0$ and $f_1$ are both morphisms of differentiable spaces.
The requirement that the structure maps \( s : G_1 \to G_0 \) and \( t : G_1 \to G_0 \) are morphisms of the differentiable spaces \( G_0 \) and \( G_1 \) implies by [NGSdS03, Thm. 7.6] that the fibered product \( G_1 \times_t G_1 \) inherits the structure of a differentiable space. It is with respect to this structure that we require that \( m : G_1 \times_t G_1 \to G_1 \) is a morphism of differentiable spaces. As the inverse map \( i : G_1 \to G_1 \) is clearly invertible with \( i^{-1} = i \), it follows that \( i \) is an isomorphism of differentiable spaces. Similarly, \( u : G_0 \to G_1 \) is an embedding of differentiable spaces.

**Remark 2.4.** If \( G \) is a differentiable groupoid, the underlying topological spaces \( G_0 \) and \( G_1 \) need not be Hausdorff. However, because both carry the structure of a differentiable space, \( G_0 \) and \( G_1 \) are locally Hausdorff and locally quasi-compact, see [NGSdS03, Chap. 4].

**Example 2.5.** (a) Let \( G \) be a compact Lie group and let \( X \) be a Hausdorff differentiable space with a differentiable \( G \)-action, cf. [NGSdS03, Sec. 11.2]. Then the transformation groupoid \( G \times X \) is a differentiable groupoid with space of objects \( X \) and space of arrows \( G \times X \), the latter being a differentiable space by [NGSdS03, Thm. 7.6]. The source map \( s : G \times X \to X \) is the projection, hence smooth and open, and the target map \( t : G \times X \to X \) is given by \( (g, x) \mapsto gx \), which is differentiable by the definition of a differentiable action. The unit map \( u : x \mapsto (e, x) \) is easily seen to be a closed embedding since \( \{e\} \) is closed in \( G \). The domain of the product map \( m \) is a differentiable subspace of \( G \times X \times G \times X \) by [NGSdS03, Thm. 7.6] and the proof of [NGSdS03, Rem. 7.5]. Moreover, the map \( G \times G \times X \to (G \times X) \times_t (G \times X), (g, h, x) \mapsto ((g, hx), (h, x)) \) is an isomorphism of differentiable spaces onto the domain of \( m \). Since the pullback of \( m \) by this isomorphism is the smooth map \( G \times G \times X \to G \times X, (g, h, x) \mapsto (gh, x) \), multiplication in \( G \times X \) is smooth. Similarly one shows that the inverse map \( i : G \times X \to G \times X, (g, x) \to (g^{-1}, gx) \) is smooth.

By definition, the quotient space \( X/G \) of the \( G \)-space \( X \) coincides with the orbit space of the groupoid \( |G \times X| \). Hence, if \( X \) is Hausdorff, the orbit space becomes a Hausdorff space by Prop. 2.2 and even is a differentiable space by [NGSdS03, Thm. 11.17] when equipped with the sheaf of \( G \)-invariant smooth functions on \( X \) as the structure sheaf.

(b) Let \( X \) be a Hausdorff differentiable space. Then the pair groupoid \( X \times X \rightrightarrows X \) with source and target maps being projection onto the first and second factor, respectively, is a differentiable groupoid. It has only one orbit, hence the orbit space is naturally a differentiable space as well.

(c) In general, the orbit space \( |G| \) of a differentiable groupoid \( G \) need not admit a differentiable structure with respect to which the quotient map \( G_0 \to |G| \) is a smooth map. As a well known example, consider the action of \( \mathbb{Z} \) on the circle \( S^1 \) by an irrational rotation. Then the translation groupoid \( \mathbb{Z} \times S^1 \) is a (non-quasi-proper) differentiable groupoid whose orbit space \( |\mathbb{Z} \times S^1| \) is not locally Hausdorff. Hence \( |\mathbb{Z} \times S^1| \) does not carry the structure of a differentiable space.

Let \( G \) be a reduced differentiable groupoid and \( Y \) a differentiable subspace of \( G_0 \). A bisction of \( G \) over \( Y \) then is a continuous map \( \sigma : Y \to G_1 \) such that \( s \circ \sigma = \text{id}_Y \), and such that \( t \circ \sigma \) is an isomorphism of \( Y \) onto the differentiable subspace \( t \circ \sigma(Y) \) of \( G_0 \), cf. [Mac05, Def. 1.4.8]. Note that we will consider bisections over sets \( Y \) which may not be open in \( G_0 \).
In the case that $G$ is proper, the question of whether $|G|$ admits a differentiable structure remains open in general. However, from Example 2.5 (a), it is clear that the quotient map of the action groupoid of a differentiable group action on a Hausdorff differentiable space is a morphism of differentiable spaces. Many of the examples we consider will be of this form locally, which will allow us to similarly conclude that $|G|$ admits a compatible differentiable space structure; see Proposition 2.12. Because of their importance for our further considerations and their interesting topological structure we characterize such groupoids in Definition 2.7 below. To be able to formulate it, we need the following observation.

**Proposition 2.6.** Let $G$ be a proper reduced differentiable groupoid. Then for every $x \in G_0$ the following holds true.

(LT0) The isotropy group $G_x$ becomes a compact Lie group with the induced topological and differentiable structures.

**Proof.** The isotropy group $G_x$ is compact by properness of the groupoid. Hence it has finitely many connected components. By [Spa91, Cor. 4.2], $G_x$ becomes a Lie group with the induced reduced differentiable structure. □

**Definition 2.7.** We say that a proper reduced differentiable groupoid $G$ is **sliceable** if the following conditions are satisfied for every $x \in G_0$.

(LT1) Locally, the groupoid is isomorphic to the product of a translation groupoid with a pair groupoid. More precisely, there is an open Hausdorff neighborhood $U_x \subset G_0$ of $x$ and a relatively closed connected reduced differentiable subspace $Y_x \subset U_x$ containing $x$ together with a smooth $G_x$-action on $Y_x$ such that $x$ is a fixed point of the $G_x$-action and such that the restriction $G|_{U_x}$ is isomorphic as a differentiable groupoid to the product of the translation groupoid $G_x \times Y_x$ and the pair groupoid $O_x \times O_x$, where $O_x$ is an open neighborhood of $x$ in its orbit. Such a neighborhood $U_x$ is called a **trivializing neighborhood** of $x$, and $Y_x$ a $G$-slice or a **groupoid-slice** at $x$.

(LT2) For each $z \in Y_x$, one may choose a trivializing neighborhood $U_z$ of $z$ with corresponding $G$-slice $Y_z$ such that $U_z \subset U_x$ and $Y_z \subset Y_x$.

(LT3) The trivializing neighborhoods and $G$-slices of two objects $x$ and $y$ in the same orbit can be chosen to be compatible with translation by a smooth bisection in the following sense. For each arrow $g \in G$ with $s(g) = x$ and $y = t(g)$ there exist trivializing neighborhoods $U_x$ of $x$ and $U_y$ of $y$ and a smooth bisection $\sigma$ over $U_x$ such that $\sigma(x) = g$ and such that the morphism of differentiable groupoids $f: G|_{U_x} \to G|_{U_y}$ with components $f_0 := t \circ \sigma: U_x \to U_y$ and $f_1: G|_{U_x,1} \to G|_{U_y,1}$, $h \mapsto \sigma(t(h))h(\sigma(s(h)))^{-1}$ is an isomorphism.

**Example 2.8.** Every proper Lie groupoid is sliceable by e.g. [Zun06, Thm. 2.3] or [PPT14, Cor. 3.11]. In Section 6 we show that the inertia groupoid of a proper Lie groupoid is always sliceable even when it is singular; see Theorem 6.19.

**Remark 2.9.** Clearly, a non-quasi-proper reduced differentiable groupoid may have non-compact isotropy groups and hence fail to be sliceable; the groupoid $\mathbb{Z} \times \{pt\}$ is such an example. An interesting question which we could not answer in this paper is whether a proper reduced differentiable groupoid is necessarily sliceable. In other words, this question essentially asks whether there is a slice theorem for proper reduced
differentiable groupoids. We are not aware of a proper reduced differentiable groupoid which is not sliceable.

As noted above, a related open question is whether the orbit space of a proper (reduced) differentiable groupoid inherits a (reduced) differentiable structure in such a way that the orbit map is a morphism of differentiable spaces. This would follow in the reduced case from a slice theorem as above, as it holds for sliceable differentiable groupoids by Proposition 2.12 below.

**Remark 2.10.** By the following observation one can assume, possibly after shrinking, that a $G_x$-slice $Y_x$ of $x$ possesses a $G_x$-equivariant singular chart $\iota : Y_x \hookrightarrow T_x Y_x \cong \mathbb{R}^{rk x}$ with $\iota(x) = 0$; see Appendix A.1 for the definition of a singular chart and Appendix A.4 for the definition of the Zariski tangent space $T_x Y_x$. Note that $T_x Y_x$ carries a natural $G_x$-action inherited from the one on $Y$.

**Proposition 2.11.** Let $G$ be a compact Lie group, $Y$ a reduced differentiable space carrying a differentiable $G$-action, and $x \in Y$ a fixed point. Then the Zariski tangent space $T_x Y$ inherits a natural $G$-action from the $G$-action on $Y$. Moreover, there exists an open $G$-invariant neighborhood $W$ of $x$ in $Y$ and a $G$-equivariant singular chart $\iota : W \hookrightarrow T_x Y$ mapping $x$ to the origin.

**Proof.** By [Pfl01, Prop. 1.3.10] there exists a singular chart $\kappa : W \hookrightarrow \mathbb{R}^{rk x} \cong T_x Y$, where $rk x = \dim T_x Y$. Since $x$ is assumed to be a fixed point, one can achieve after possibly shrinking $W$ that $W$ is $G$-invariant. Now the proof is literally identical to the proof of [NW14, Lem. 5.2.6] when replacing “holomorphic” with “smooth”. □

**Proposition 2.12.** Let $G$ be a sliceable differentiable groupoid. Then the orbit space $|G|$ inherits the structure of a differentiable space with respect to which the quotient map $\pi : G_0 \to |G|$ is a smooth map. Specifically, the structure sheaf of $|G|$ is given by the differentiable functions on $G_0$ that are constant on orbits.

**Proof.** We define the structure sheaf of $|G|$ to be the sheaf of continuous functions on $|G|$ which pull back under the projection $\pi$ to $G$-invariant smooth functions on $G_0$. For each orbit $G_x \in |G|$, we may use condition (L1) in Definition 2.7 to identify a neighborhood of $G_x$ with $|G_x \times_Y X|$, a Hausdorff differentiable space as explained in Example 2.5 (a). That these local identifications are well defined and isomorphisms of Hausdorff differentiable spaces are consequences of (L12) and (L13). □

### 2.3. Differentiable stratified groupoids.

Recall that by a stratified submersion (respectively stratified immersion), one understands a morphism $f : X \to Y$ of reduced differentiable stratified spaces such that the restriction of $f$ to a connected component of a stratum of the maximal decomposition of $X$ is a submersion (respectively immersion), cf. [Pfl01, 1.2.10]. A stratified surjective submersion is a stratified submersion that maps connected components of strata onto connected components of strata, and a stratified embedding is a stratified immersion that is injective on connected components of strata.

We will now state the definition of a differentiable stratified groupoid, the primary object of investigation in this paper. Let us first give a brief explanation of this definition. A differentiable stratified groupoid is a reduced differentiable groupoid such that the spaces of objects and arrows are differentiable stratified spaces, required by condition (DSC1), and the structure maps are stratified mappings, see condition (DSC2). In order for the requirement that multiplication $m : G_1 \times_l G_1 \to G_1$ is a stratified mapping to
make sense, the fiber product $G_1 \times_t G_1$ has to be a stratified space. By condition \((\text{DSG3})\) we impose requirements on the structure maps to ensure this; see Remark \(2.14(\text{b})\). Condition \((\text{DSG4})\) is a kind of equivariance condition for the stratification. It entails that every set germ of a source fiber is contained in a stratum of the arrow space, so that the stratification of the arrow space is not unnecessarily fine. This will be used to show that $G$-orbits are locally contained in strata; see Lemma \(2.17\) below. Likewise, condition \((\text{DSG5})\) requires that smooth bisections defined on open subsets of strata act on $G_1$ in a way compatible with its stratification.

**Definition 2.13.** A differentiable stratified groupoid is a reduced differentiable groupoid $G$ such that the following properties hold true.

\((\text{DSG1})\) $G_0$ and $G_1$ are differentiable stratified spaces with respective stratifications $S^0$ and $S^1$.

\((\text{DSG2})\) The structure maps $s$, $t$, $i$, $u$, and $m$ are stratified mappings.

\((\text{DSG3})\) The maps $s$ and $t$ are stratified surjective submersions, and $u$ is a stratified embedding.

\((\text{DSG4})\) For every $x \in G_0$ and arrow $g \in s^{-1}(x)$ the set germ $[s^{-1}(S^0_g)]_g$ is a subgerm of $S^1_g$.

\((\text{DSG5})\) Let $x \in G_0$, $g \in G_1$ with $t(g) = x$, and $U$ be an open connected neighborhood of $x$ within the stratum of $G_0$ containing $x$. Assume that $\sigma : U \to G_1$ is a smooth bisection of $G$. Then the map $L_\sigma : t^{-1}(U) \to G_1$ defined by $h \mapsto \sigma(t(h))h$ satisfies $L_\sigma(S^1_g) = S^1_{L_\sigma(g)}$.

If $G$ and $H$ are differentiable stratified groupoids, a morphism of differentiable stratified groupoids is a morphism of differentiable groupoids $f : G \to H$ such that $f_0$ and $f_1$ are in addition stratified mappings.

A differentiable stratified groupoid $G$ is called a Lie groupoid if in addition the following axiom holds true.

\((\text{DSG6})\) The stratifications $S^0$ and $S^1$ are induced by $G_0$ and $G_1$, respectively, which in other words means that both $G_0$ and $G_1$ are smooth manifolds and their stratifications have only one stratum.

A structurally (b)-regular differentiable stratified groupoid is a differentiable stratified groupoid $G$ such that the stratifications of $G_0$ and $G_1$, as well as the induced stratification of $G_1 \times_s G_1$, are Whitney (b)-regular. Similarly, $G$ is structurally locally trivial if the stratifications of $G_0$, $G_1$, and the induced stratification of $G_1 \times_s G_1$ are topologically locally trivial (cf. [Pfl01, Sec. 1.4], Def. A.3).

**Remark 2.14.** (a) In the definition, $[s^{-1}(S^0_g)]_g$ of course means the set germ of $s^{-1}(S)$ at $g$, where $S$ is a set defining the germ $S^0_x$ at $x$. Note that a smooth bisection $\sigma : S \supset U \to G_1$ as in \((\text{DSG4})\) has image in the stratum of $G_1$ through $g = \sigma(x)$ by \((\text{DSG2})\) and \((\text{DSG4})\). The existence of such bisections $\sigma$ is demonstrated by Lemma \(2.18\) and Corollary \(2.19\) below.

(b) Though $G_0$ and $G_1$ are locally Hausdorff spaces, we do not require that they are Hausdorff. See Appendix A for stratifications of locally Hausdorff spaces.

(c) One readily checks that Definition \(2.13\) reduces to the standard definition of a Lie groupoid if \((\text{DSG5})\) is fulfilled. Observe that in the case of a Lie groupoid, conditions \((\text{DSG4})\) and \((\text{DSG5})\) become trivial.

(d) We will always let $G_1 \times_s G_1$ carry the stratification induced by the stratification of $G_1$, the existence of which is guaranteed by Lemma A.4.
Our definition of a structurally (b)-regular differentiable stratified groupoid is stronger than the one of a stratified Lie groupoid given in [FOR09, Def. 4.16] in that we require conditions (DSG4) and (DSG5). The following examples illustrate the kinds of behavior we preclude by requiring these conditions.

**Example 2.15.** (a) Consider the translation groupoid $G = S^1 \times S^1$ where the action is by left-translation. We decompose $G_0 = S^1$ into pieces $\{1\}$ and $S^1 \smallsetminus \{1\}$, and $G_1 = S^1 \times S^1$ into pieces $\{(1, 1)\}$, $\{(a, 1) \mid a \neq 1\}$, $\{(a, a^{-1}) \mid a \neq 1\}$, and $\{(a, b) \mid b \neq 1, a \neq b^{-1}\}$. Then it is immediate to check that (DSG4), (DSG5), and (DSG6) are satisfied. However, condition (DSG6) fails, as the germ of $s^{-1}(1) = \{(a, 1) \mid a \in S^1\}$ is not contained in the stratum $\{(1, 1)\}$. Note that $G_0$ consists of a single connected orbit that is given an “artificial” stratification that is too fine. Of course, $G$ is a Lie groupoid when given the trivial stratifications of $G_0$ and $G_1$.

(b) Let $G$ be a compact Lie group of positive dimension and let $G = G \times \{p\}$ be the translation groupoid associated to the trivial action of $G$ on a one-point space. Define a stratification of $G_1 = G \times \{p\}$ by the decomposition into $\{(1, p)\}$ and $\{(g, p) \mid g \neq 1\}$. Obviously, the source and target maps are stratified submersions, the unit map is a stratified embedding, and the inverse map is a stratified mapping. Though the multiplication map $G_1 \times G_1 = G_1 \times G_1 \to G_1$ given by $((g, p), (h, p)) \mapsto (gh, p)$ is not a stratified mapping with respect to the induced stratification, the space $G_1 \times G_1$ does admit a stratification with respect to which the multiplication map $m$ is stratified. Specifically, we may decompose $G_1 \times G_1$ into the pieces

$\{(g^{-1}, p), (g, p) \mid g \in G\}$ and $\{((g, p), (h, p)) \mid g \neq h^{-1}\}$,

and then $m$ becomes stratified. In this case, while $G_0$ consists of a single stratum, the stratification of $G_1$ is “too fine” so that (DSG6) fails; the preimage $s^{-1}(p) = G_1$ is not contained in the stratum $\{(1, p)\}$ through $(1, p)$.

(c) Let $G$ be the pair groupoid on the topological disjoint union $\mathbb{R} \sqcup \{p\}$. Give $G_0$ and $G_1$ the stratifications by connected components and natural differentiable structures. One checks that (DSG4), (DSG5), (DSG6), and (DSG7) are satisfied. However, let $\sigma: \{p\} \to G_1$ be the smooth bisection with the single value $\sigma(p) = (0, p)$. For any arrow of the form $(p, y)$ one has $\sigma(t(p, y))(p, y) = (0, y)$. Hence the mapping $h \mapsto \sigma(t(h))h$ maps the stratum $\{(p, y) \mid y \in \mathbb{R}\}$ of $G_1$ into the stratum $\mathbb{R}^2$ as the $y$-axis. Thus (DSG6) fails.

We next collect some useful consequences of Definition 2.13 indicating how the set germs defining the stratifications of $G_0$ and $G_1$ are related via the source and target maps of $G$ and hence determine one another. We hereby assume for the remainder of this section that $G$ denotes a differentiable stratified groupoid.

**Lemma 2.16.** Let $g \in G_1$ with $s(g) = x$ and $t(g) = y$. Then one has

\begin{align*}
(2.1) \quad & S^1_g = [s^{-1}(S^0_x)]_g, \text{ i.e. } S^1 \text{ is the pullback of } S^0 \text{ via } s, \\
(2.2) \quad & S^0_x = [s(S^1_x)]_x, \\
(2.3) \quad & S^1_g = [t^{-1}(S^0_y)]_g, \\
(2.4) \quad & S^0_y = [t(S^1_y)]_y, \text{ and,} \\
(2.5) \quad & S^0_y = [t(s^{-1}(S^0_x))]_y.
\end{align*}

In particular, the stratifications $S^0$ of $G_0$ and $S^1$ of $G_1$ determine one another.
Proof. Let $R_g$ be the connected component of the stratum of $G_1$ containing $g$, and let $S_x$ and $S_y$ be the connected components of the strata of $G_0$ containing $x$ and $y$, respectively. Then $s_{|R_g}$ and $t_{|R_g}$ are, respectively, surjective submersions onto $S_x$ and $S_y$. By (DSG4), there exists a relatively open neighborhood $U_x$ of $x$ in $S_x$ and an open neighborhood $V_g$ of $g$ in $G_1$ such that $V_g \cap s^{-1}(U_x) \subset R_g$. As $s_{|R_g}$ is a smooth map onto $S_x$, $s^{-1}(U_x)$ is a relatively open neighborhood of $g$ in $R_g$, proving (2.1). Since $s_{|R_g}$ is a surjection, hence an open map, $(V_g \cap s^{-1}(U_x))$ is an open neighborhood of $x$ in $S_x$, which gives (2.2). Then (2.3) and (2.4) follow from the fact that $t = s \circ i$ and that $i$ is a stratified mapping with $i^2 = \operatorname{id}_{G_1}$. Finally, (2.5) is a consequence of (2.1) and (2.4). □

Lemma 2.17. Let $G$ be a proper differentiable stratified groupoid and let $x \in G_0$ be a point. Then the connected component of the orbit $Gx$ containing $x$ is contained in the stratum of $G_0$ containing $x$.

Proof. Let $S_x$ be the connected component of the stratum of $G_0$ containing $x$. Suppose for contradiction that the germ of $Gx$ at $x$ is not a subgerm of $S_x^1$. Then one may construct a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $Gx$ such that $\lim_{n \to \infty} x_n = x$, yet each $x_n$ is not contained in $S_x$. Since the set $K = \{(x, x_n)\}_{n \in \mathbb{N}} \cup \{(x, x)\}$ is compact and $G$ is proper, the preimage $C := (s, t)^{-1}(K)$ is a quasi-compact subset of $G_1$.

For each $n$, choose an arrow $g_n \in G_1$ with source $x$ and target $x_n$. As each $g_n$ is in the quasi-compact set $C$, there is a subsequence $(g_{n_k})_{k \in \mathbb{N}}$ with limit $g \in G_1$. However, (DSG4) entails the existence of an open neighborhood $U_x$ of $x$ in $S_x$ and a neighborhood $V_g$ of $g$ in $G_1$ such that $V_g \cap s^{-1}(U_x)$ is contained in the connected component $R_g$ of the stratum of $G_1$ containing $g$. Infinitely many of the $g_{n_k}$ must be contained in $V_g$ and each $g_{n_k}$ is an element of $s^{-1}(U_x)$ by construction, so infinitely many of the $g_{n_k}$ are contained in $R_g$. But since $t$ is a stratified mapping, infinitely many of the $x_{n_k}$ are contained in $S_x$, which is a contradiction. It follows that the germ of $Gx$ at $x$ is a subgerm of $S_x$, hence that the connected component of $Gx$ containing $x$ is a subset of $S_x$. □

The following property is important in realizing the consequences of (DSG5) and is proven as in the case of a Lie groupoid; cf. [Mac05, Prop. 1.4.9].

Lemma 2.18. Let $x, y$ be two points of the differentiable stratified groupoid $G$ and $g$ an arrow with $s(g) = x$ and $t(g) = y$. Denote by $S_x$ and $S_y$ the connected components of the strata of $G_0$ containing $x$ and $y$, respectively. If $\dim S_x \leq \dim S_y$, then there exists a relatively open neighborhood $U_x$ of $x$ in $S_x$ and a smooth bisection $\sigma$ of $G$ on $U_x$ such that $\sigma$ is a stratified mapping and $\sigma(x) = g$.

Of course, since $\sigma$ is only defined on a subset of the stratum through $x$, $\sigma$ being a stratified mapping means that its image is contained in the stratum containing $g$. The hypothesis that $\dim S_x \leq \dim S_y$ will be seen to be unnecessary below.

Proof. Let $R_g$ be the connected component of the stratum of $G$ containing $g$. Then $s_{|R_g}$ and $t_{|R_g}$ are surjective submersions onto $S_x$ and $S_y$, respectively. Moreover, there are relatively open neighborhoods of $g$ in $s^{-1}(x)$ and $t^{-1}(y)$ contained in $R_g$ by (DSG4) and Lemma 2.16 (2.3). Then there are subspaces $E \subset F$ of the tangent space $T_g R_g$ such that $T_g R_g = T_g(s^{-1}(x)) \times E = T_g(t^{-1}(y)) \times F$. Choose a smooth local section $\sigma : U_x \to R_g$ such that $\sigma(x) = g$ and such that the image of $T_x \sigma$ is $E$. Then $T_x(t \circ \sigma)$ is injective so that we can shrink $U_x$ in a way that $t \circ \sigma$ is a diffeomorphism onto its image. □
As an important consequence, we may now conclude that the strata of $G_0$ that meet the orbit $Gx$ of a point $x \in G_0$ must all have the same dimension. The proof follows [Ma05, Corollaries 1.4.11 & 1.4.12].

**Corollary 2.19.** Let $G$ be a differentiable stratified groupoid, and let $S$ be a connected component of a stratum of $G_0$. Then the following holds true.

1. Each stratum of $G_1$ contained in $s^{-1}(S)$ has the same dimension.
2. The rank of $t$ on $s^{-1}(S)$ is constant.
3. Each connected component $S'$ of a stratum of $G_0$ such that $s^{-1}(S) \cap t^{-1}(S') \neq \emptyset$ has the same dimension as $S$.

**Proof.** Let $S_1$ and $S_2$ be (not necessarily distinct) connected components of strata of $G_0$ such that $s^{-1}(S_1) \cap t^{-1}(S_2) \neq \emptyset$. We assume $\dim S_1 \leq \dim S_2$. Otherwise, we may switch roles and apply the inverse map, so this hypothesis introduces no loss of generality. As $s$ and $t$ are stratified surjective submersions, $s^{-1}(S_1)$ and $t^{-1}(S_2)$ are both unions of connected components of strata of $G_1$. Hence their intersection is a union of connected components of strata. Let $g \in G_1$ with $s(g) = x \in S_1$ and $t(g) = y \in S_2$. Then there exists by Lemma 2.18 a smooth bisection $\sigma$ of $G$ on a relatively open neighborhood $U_x$ of $x$ in $S_1$ such that $\sigma(x) = g$. Let $R_u(x)$ and $R_g$ denote the connected components of the strata of $G_1$ containing $u(x)$ and $g$, respectively. By (DSC), there is a relatively open neighborhood $V_u(x)$ of $u(x)$ in $R_u(x)$ such that $L_\sigma(V_u(x))$ is a relatively open neighborhood of $g$ in $R_g$. The requirement that $t \circ \sigma$ is injective implies that $L_\sigma$ is injective, hence $R_u(x)$ and $R_g$ have the same dimension. Since $S_2$ and $g$ were arbitrary, (1) follows. Moreover, from the definition of $L_\sigma$, we have $t|_{L_\sigma(V_u(x))} \circ L_\sigma = (t \circ \sigma) \circ t|_{V_u(x)}$. Then, as $L_\sigma(V_u(x))$ is an open neighborhood of $g$ in $R_g$ and $t \circ \sigma$ is a diffeomorphism onto its image, the ranks of $t|_{R_u(x)}$ at $u(x)$ and $t|_{R_g}$ at $g$ coincide, yielding (2). Since $t$ is a stratified surjective submersion, (3) is immediate. □

**Remark 2.20.** In particular, the hypothesis $\dim S_x \leq \dim S_y$ in Lemma 2.18 is superfluous by Corollary 2.19. Indeed, given the other hypotheses, we always have $\dim S_x = \dim S_y$.

**Example 2.21.** Note that if $S_1$ and $S_2$ are connected components of strata of $G_0$, even connected components of the same stratum, it need not be the case that the strata of $s^{-1}(S_1)$ and $s^{-1}(S_2)$ have the same dimension. As an example, let $G$ and $H$ be compact Lie groups, and let $G$ be the disjoint union of $G \times \{p\}$ and $H \times \{q\}$. Then $G_0$ is the discrete set $\{p, q\}$. The maximal stratification of $G_0$ contains a single stratum with two one-point connected components, yet the space of arrows of $G_{\{p\}}$ and $G_{\{q\}}$ have dimensions $\dim G$ and $\dim H$, respectively, which clearly need not coincide.

**Proposition 2.22.** Assume that $G_0$ and $G_1$ are topologically locally trivial, and let $S \subset G_0$ be a connected component of a stratum of $G_0$. Let $\mathcal{P}$ be a collection of connected components of strata of $G_0$ such that $S \in \mathcal{P}$, and such that for each $S' \in \mathcal{P}$ the relation $s^{-1}(S) \cap t^{-1}(S') \neq \emptyset$ is satisfied. Letting $P = \bigcup_{S' \in \mathcal{P}} S'$, the restriction $G_1|P$ then is a Lie groupoid.

**Proof.** By Corollary 2.19 each element of $\mathcal{P}$ has the same dimension as $S$, hence each connected component of a stratum of $G_1$ contained in $s^{-1}(P)$ has the same dimension as well. The hypothesis that $G_0$ is a topologically locally trivial implies that if $S' < S''$ cannot occur for $S', S'' \in \mathcal{P}$ so that $P$ is a manifold. The same holds for $G_1$ so that
$s^{-1}(P) \cap t^{-1}(P)$ is a union of strata of the same dimension and therefore a manifold. Hence $P$ and $s^{-1}(P) \cap t^{-1}(P)$ are manifolds, which can both be given their trivial stratifications. The claim follows.

Of course, the hypothesis that $G_0$ and $G_1$ are topologically locally trivial is only required so that the strata of $G_0$ and $G_1$ of the same dimension are not contained in one another’s closures. Any other hypothesis that ensures this is as well sufficient.

We also note the following, which will be useful in the sequel.

**Corollary 2.23.** Assume that $G$ is proper and let $x \in G_0$. Then each connected component of the orbit $Gx$ is a smooth submanifold of $G_0$. If $G_0$ and $G_1$ are in addition topologically locally trivial, then $Gx$ is a smooth submanifold of $G_0$.

**Proof.** Let $S_x$ be the connected component of the stratum containing $x$. Then the connected component of $Gx$ containing $x$ is contained in $S_x$ by Lemma 2.17. The restricted groupoid $G|_{S_x}$ is a Lie groupoid by Proposition 2.22. Since $Gx \cap S_x = (G|_{S_x})x$ this implies that the connected components of $Gx$ contained in $S_x$ are smooth submanifolds of $S_x$, hence of $G_0$. If $G_0$ and $G_1$ are topologically locally trivial, then the restriction of $G$ to the saturation of $S_x$ is as well a Lie groupoid, and the same argument applies.

Finally, we include the following example to demonstrate that the hypothesis that $G_0$ and $G_1$ are topologically locally trivial (or a similar requirement) is necessary in Proposition 2.22 and Corollary 2.23, cf. [Pfl01, 1.1.12].

**Example 2.24.** Let $G_0 = S_1 \cup S_2 \subset \mathbb{R}^2$ where $S_1 = \{0\} \times (-1, 1)$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin 1/x\}$, and let $G$ be the pair groupoid on $G_0$. Then $S_1$ and $S_2$, both of dimension 1, are the pieces of a decomposition of $G_0$ with $S_1 \subset S_2$. The decomposition of $G_1$ is into pieces of the form $S_i \times S_j$, each of dimension 2. Then $G$ is a differentiable stratified groupoid. However, the single orbit $G_0$ is not locally connected and hence not a manifold, and $G$ is not a Lie groupoid. Clearly, however, the restriction of $G$ to either stratum of $G_0$ is a Lie groupoid.

2.4. **Morita equivalence.** Let $G$ be a topological groupoid, $Y$ a topological space, and $f : Y \to G_0$ a continuous function. Following [Tu04], we denote by $G[Y]$ the groupoid with object space $G[Y]_0 := Y$, arrow space $G[Y]_1 := (Y \times Y)_{(f,f)\times(t,s)}G_1 = \{(y, z, g) \in Y \times Y \times G_1 \mid t(g) = f(y), s(g) = f(z)\}$, and structure maps given as follows. The source of $(y, z, g) \in G[Y]_1$ is $y$, its target is $z$. Multiplication maps $(y, w, g), (w, z, h) \in G[Y]_1$ with $s(g) = t(h)$ to

$$(y, w, g) \cdot (w, z, h) := (y, z, gh).$$

The unit map is $G[Y]_0 \to G[Y]_1, y \mapsto (y, y, u(f(y)))$, and the inverse map is $G[Y]_1 \to G[Y]_1, (y, z, g) \mapsto (z, y, g^{-1})$. Then $G[Y]$ is a subgroupoid of $Y \times Y \times G$ where $Y \times Y$ denotes the pair groupoid and we identify $G[Y]_0 = Y$ with $Y_{(f \times u)G_0}$. By [Tu04 Prop. 2.7], $G[Y]$ is a locally closed subgroupoid of $Y \times Y \times G$ if $G_0$ is locally Hausdorff. Moreover, $G[Y]$ is locally compact if $G$ and $T$ are locally compact. Finally, by [Tu04 Prop. 2.22], $G[Y]$ is proper if $G$ is proper.

Now suppose that $G$ is a differentiable groupoid, $Y$ a differentiable space, and $f$ a differentiable map. Then it is straightforward to see that $G[Y]$ is a differentiable subgroupoid of $Y \times Y \times G$. In particular, both $G[Y]_0$ and $G[Y]_1$ are defined as fibered products. Similarly, if $G$ and $H$ are differentiable stratified groupoids, $Y$ is a differentiable
stratified space, and \( f \) a differentiable stratified surjective submersion, then \( G[Y] \) is a differentiable stratified groupoid as well where \( G[Y]_0 \) and \( G[Y]_1 \) are given the induced stratifications.

**Definition 2.25.** Two open topological groupoids \( G \) and \( H \) are called *Morita equivalent as topological groupoids* if there exists a topological space \( Y \) together with open surjective continuous functions \( f \colon Y \to G_0 \) and \( g \colon Y \to H_0 \) such that \( G[Y] \) and \( H[Y] \) are isomorphic as topological groupoids. If \( G \) and \( H \) are differentiable groupoids, \( Y \) is a differentiable space, and \( f \) and \( g \) are differentiable maps, then \( G \) and \( H \) are called *Morita equivalent as differentiable groupoids* if \( G[Y] \) and \( H[Y] \) are isomorphic as differentiable groupoids. Similarly, if \( G \) and \( H \) are differentiable stratified groupoids, \( Y \) is a differentiable stratified space, and \( f \) and \( g \) are differentiable stratified surjective submersions, then \( G \) and \( H \) are called *Morita equivalent as differentiable stratified groupoids* if \( G[Y] \) and \( H[Y] \) are isomorphic as differentiable stratified groupoids.

One verifies immediately that Morita equivalence is transitive. Specifically, if the maps \( G_0 \overset{f_1}{\leftarrow} Y \overset{g_1}{\to} H_0 \) realize a Morita equivalence between (topological, differentiable, or differentiable stratified) groupoids \( G \) and \( H \), and \( H_0 \overset{f_2}{\leftarrow} Y' \overset{g_2}{\to} K_0 \) is a Morita equivalence between \( H \) and \( K \), then \( G_0 \overset{f_{1\text{opr}}}{\leftarrow} Y \times_{f_1} \times_{g_1} Y' \overset{g_{2\text{opr}}}{\to} K_0 \) induces a Morita equivalence between \( G \) and \( K \).

**Proposition 2.26.** If \( G \) and \( H \) are Morita equivalent open topological groupoids, then the orbit spaces \( |G| \) and \( |H| \) are homeomorphic. Moreover, if \( G \) and \( H \) are Morita equivalent differentiable groupoids such that \( |G| \) and \( |H| \) admit the structure of differentiable spaces and the quotient maps are differentiable, then \( |G| \) and \( |H| \) are isomorphic as differentiable spaces.

**Proof.** Because the orbit spaces \( |G[Y]| \) and \( |H[Y]| \) are clearly homeomorphic, it is sufficient to show that \(|G| \) is homeomorphic to \(|G[Y]| \). To see this, define the map \( G[Y]_0 \to G_0 \) by \((y,x)\mapsto x\). Given an arrow \( g \in G_1 \) from \( x \) to \( x' \), there exists, by the surjectivity of \( f \), a \( y' \) such that \( f(y') = x' \) and hence an arrow \((y,y',g)\) from \((y,x)\) to \((y',x')\). Conversely, if \((y,y',g)\) is an arrow from \((y,x)\) to \((y',x')\), then \( g \) is by definition an arrow from \( x \) to \( x' \). Hence \((y,x)\mapsto x\) maps orbits to orbits. In the differentiable case, this map is obviously differentiable, so that if the quotient map \( G_0 \to |G| \) is differentiable, then its composition with \((y,x)\mapsto x\) is also differentiable. \( \square \)

For Lie groupoids, the notion of Morita equivalence is often defined in terms of morphisms called *weak equivalences*. We introduce a similar notion as follows.

**Definition 2.27.** A morphism \( f \colon G \to H \) of differentiable stratified groupoids is called a *weak equivalence* if it is essentially surjective and fully faithful, i.e. if the following two conditions are satisfied.

- **(ES) (Essential Surjectivity)** The map \( t \circ \text{pr}_1 \colon H_{s\times f_0} G_0 \to H_0 \) is a stratified surjective submersion.
- **(FF) (Full Faithfulness)** The arrow space \( G_1 \) is a fibered product via the diagram

\[
\begin{array}{ccc}
G_1 & \xrightarrow{f_1} & H_1 \\
(s,t) \downarrow & & \downarrow (s,t) \\
G_0 \times G_0 & \xrightarrow{(f_0,f_0)} & H_0 \times H_0.
\end{array}
\]
Remark 2.28. There is an analogous notion of a weak equivalence between differentiable groupoids, where axiom (ES) is replaced with the requirement that $t \circ \text{pr}_1$ is a surjective map that admits smooth local sections.

Note that if the object and arrow spaces of $G$ and $H$ are topologically locally trivial, then the restrictions of $G$ and $H$ to connected components of strata are Lie groupoids by Proposition 2.27. One immediately checks that the restriction of a weak equivalence to strata yields a weak equivalence of Lie groupoids. We have the following.

**Proposition 2.29.** Let $G$ and $H$ be differentiable stratified groupoids. The following are equivalent.

1. $G$ and $H$ are Morita equivalent as differentiable stratified groupoids.
2. There is a differentiable stratified groupoid $K$ together with weak equivalences $h: K \to G$ and $k: K \to H$.

**Proof.** Assume that (1) holds true. Then there exists a differentiable stratified space $Y$ together with open surjective differentiable stratified submersions $f: Y \to G_0$ and $g: Y \to H_0$ such that $G[Y]$ and $H[Y]$ are isomorphic. Define the groupoid $K$ by setting $K_0 = Y$ and $K_1 = G/(f_0, f_G) \times (g_0, g_G)$, and put $t_K := f \circ t_G = g \circ t_H$ and $s_K := f \circ s_G = s_H$ and $t_K := f \circ t_G = g \circ t_H$. The unit, inverse, and multiplication maps are defined component-wise. Then it is straightforward to see that $K$ is a differentiable stratified groupoid, where $K_1$ is given the induced stratification. We define a morphism $h: K \to G$ by setting $h_0 = f$ and $h_1 = \text{pr}_1$. Then $h_0 = f$ is in fact surjective by hypothesis so that $t_G \circ \text{pr}_1: G_1 \times_{h_0} K_0 \to G_0$ is a composition of stratified surjective submersions, and $K_1$ is a fibered product by construction, so $h$ is a weak equivalence. The weak equivalence $k: K \to H$ is defined identically.

Now assume that we have a weak equivalence $h: K \to G$. Let $Y = G_1 \times_{h_0} K_0$ with the induced stratification, and let $f = \text{pr}_2: Y \to K_0$ and $g = s_G \circ \text{pr}_1: Y \to G_0$. Then $f$ is obviously an open surjective differentiable stratified submersion, while the same for $g$ follows from condition (ES) in Definition 2.27 (along with applying the inverse map). By definition, $K[Y]$ and $G[Y]$ both have $Y$ as object space. The arrow spaces are given by

$$K[Y]_1 = (G_1 \times_{h_0} K_0 \times G_1 \times_{h_0} K_0) \times_{(\text{pr}_2, \text{pr}_1 \times (t_K, s_K)} K_1,$$

and

$$G[Y]_1 = (G_1 \times_{h_0} K_0 \times G_1 \times_{h_0} K_0) \times_{(s_G \circ \text{pr}_1, s_G \circ \text{pr}_1)} G_1.$$

We define an isomorphism $K[Y] \to G[Y]$ as the identity on objects and by applying $h_1$ to the last factor on arrows. This is obviously a differentiable stratified mapping. The fact that it is an isomorphism follows from condition (FF) in Definition 2.27.

We will need the following, whose proof is that of [PPT14, Prop. 3.7] with minor modifications.

**Lemma 2.30.** Let $G$ be a proper differentiable stratified groupoid. Suppose $Y$ is a locally closed differentiable stratified subspace of $G_0$ such that $G[Y]$ is a differentiable stratified subgroupoid. Then the inclusion map $\iota: G[Y] \to G|_{\text{Sat}(Y)}$ is a weak equivalence, where $\text{Sat}(Y) := \{ y \in G_0 \mid y = t(g) \text{ for some } g \in G_1 \text{ with } s(g) \in Y \}$

denotes the saturation of $Y$. 

Proposition 2.32. Let $G$ be a sliceable differentiable stratified groupoid, and suppose that for each $x \in G_0$, the set $Y_x$ as in (L1) of Definition 2.7 can be chosen so that on each stratum of $Y_x$ the $G_x$-orbit type is constant. Then the assignment

$$|G| \ni Gx \mapsto S_{Gx} = \pi(S^0_x)$$

defines a stratification of $|G|$ with respect to which the orbit map $\pi: G_0 \to |G|$ is a stratified surjective submersion.

Proof. Note that $|G|$ is a differentiable space by Proposition 2.12. Choose $U_x$, $Y_x$, etc. as in Definitions 2.7 and 2.31, and note that we may shrink $U_x$ if necessary to assume that the stratification of $Y_x$ consists of a finite number of strata. By Lemma 2.23 and Proposition 2.24, the inclusion of $Y_x$ into its saturation in $G_0$ induces an isomorphism of the differentiable spaces $Y_x/G_x$ and $|G_{\text{Sat}(Y_x)}|$. As $\text{Sat}(Y_x)$ contains $U_x$ by definition, and as the orbit map is open by Proposition 2.24, it follows that $Y_x/G_x$ is isomorphic as a differentiable space to an open neighborhood of $G_x$ in $|G|$. Moreover, by (L1), the embedding of $Y_x$ into its saturation in $G_0$ preserves strata, so it is sufficient to show that the stratification of $Y_x$ induces a stratification of the orbit space $Y_x/G_x$.

Now, each stratum $S$ of $Y_x$ is a smooth manifold with $G_x$-action so that $S/G_x$ is stratified by $G_x$-orbit types. Since each $S$ has a single orbit type by hypothesis, the stratification of $S/G_x$ is trivial, i.e. $S/G_x$ is a smooth submanifold of $Y_x/G_x$. As $Y_x$ is assumed to have finitely many strata, the resulting stratification of $Y_x/G_x$ is clearly finite. As $G_x$ is compact, the orbit map $Y_x \to Y_x/G_x$ is closed so that $(S/G_x) = \overline{S}/G_x$. Therefore, if $S/G_x \cap \overline{S'}/G_x \neq \emptyset$ for strata $S$ and $S'$ of $Y_x$, then $S \cap S' \neq \emptyset$, implying $S \subset \overline{S'}$ and hence $S/G_x \subset \overline{S'}/G_x$. The pieces $S/G_x$ therefore satisfy the condition of frontier and define a stratification of $Y_x/G_x$. □

Remark 2.33. Note that the hypotheses of Proposition 2.32 need not be satisfied even in the case of a proper Lie groupoid. For instance, when $G$ is a compact Lie group, a smooth $G$-manifold equipped with the trivial stratification satisfies these hypotheses if and only if $G$ acts with a single orbit type.
3. Examples of differentiable stratified groupoids

Many examples of differentiable stratified groupoids arise naturally from differentiable actions of Lie groupoids on stratified differentiable spaces. Recall that if $G$ is a topological groupoid and $X$ is a topological space, an action of $G$ on $X$ is given by a continuous anchor map $\alpha : X \to G_0$ together with a continuous map $\cdot : \mathcal{G}_1 \times_{\alpha} X \to X$ such that for all $x \in X$ and $g, h \in G_1$ with $s(g) = \alpha(x)$ and $t(g) = s(h)$ the relations

$$\alpha(g \cdot x) = t(g), \quad h \cdot (g \cdot x) = (hg) \cdot x, \quad \text{and} \quad u(\alpha(x)) \cdot x = x$$

hold true. As in the case of group actions, the $G$-orbit of $x \in X$ is defined to be $\{ g \cdot x \mid s(g) = x \}$. The translation groupoid $G \ltimes X$ associated to the action has object space $(G \ltimes X)_0 = X$ and arrow space $(G \ltimes X)_1 = \mathcal{G}_1 \times_{\alpha} X$. The structure maps are given by

$$s_{G \ltimes X}(g, x) = x, \quad t_{G \ltimes X}(g, x) = g \cdot x, \quad u_{G \ltimes X}(x) = (u \circ \alpha(x), x),$$

$$i_{G \ltimes X}(g, x) = (g^{-1}, g \cdot x), \quad m_{G \ltimes X}((h, g \cdot x), (g, x)) = (hg, x).$$

(3.1)

If $G$ is a differentiable groupoid and $X$ a differentiable space, we say that the action is differentiable if $\alpha$ and $\cdot$ are morphisms of differentiable space. In this case, the arrow space $(G \ltimes X)_1 = \mathcal{G}_1 \times_{\alpha} X$ inherits a differentiable structure by [NCSDS03] Thm. 7.6. Then, as each of the structure maps of $G \ltimes X$ is defined in terms of the structure maps of $G$ and the differentiable maps $\alpha$ and $\cdot$, $G \ltimes X$ is a differentiable groupoid. For actions of differentiable stratified groupoids, one even has the following.

**Proposition 3.1.** Let $X$ be a differentiable stratified space that is topologically locally trivial. Further let $G$ be a differentiable stratified groupoid with $G_0$ and $G_1$ topologically locally trivial that acts differentiably on $X$ in such a way that the $G$-orbit of each $x \in X$ is a subset of the stratum containing $x$. Then the translation groupoid $G \ltimes X$ is a differentiable stratified groupoid.

**Proof.** Let $Z$ denote the maximal decomposition of $X$, and endow $(G \ltimes X)_1 = \mathcal{G}_1 \times_{\alpha} X$ with the stratification induced by those of $G_1$ and $X$. Then by Lemma A.4, $(G \ltimes X)_1$ is a differentiable stratified space. By definition of the induced stratification, the germ of the stratum of $(G \ltimes X)_1$ at the point $(g, x)$ is given by the fibered product over $s$ and $\alpha$ of neighborhoods of strata in $G_1$ and $X$. Since $i$, $u$, $m$, and $\alpha$ are stratified mappings, the structure maps $s_{G \ltimes X}, t_{G \ltimes X}, u_{G \ltimes X}, i_{G \ltimes X}$, and $m_{G \ltimes X}$ defined by Equation (3.1) are as well stratified mappings, so that (DSC1) and (DSC2) are satisfied. To see that $s_{G \ltimes X}$ is a stratified surjective submersion, let $(g, x) \in (G \ltimes X)_1$. Note that as $s$ is a stratified surjective submersion, the restriction of $s$ to the connected component $R_g$ of the stratum of $G_1$ containing $g$ is a surjective submersion onto the connected component $S_{\alpha(x)}$ of the stratum of $G_0$ containing $\alpha(x)$. As $\alpha$ is a stratified mapping, it maps the connected component $P_x$ of the stratum of $X$ containing $x$ into $S_{\alpha(x)}$. Hence the restriction of $s_{G \ltimes X}$ to $R_g \times_{\alpha} P_x$ is a stratified surjective submersion onto $P_x$. It follows that $s_{G \ltimes X}$ is a stratified surjective submersion. The proof for $t_{G \ltimes X}$ is identical. In a similar fashion one shows that $u_{G \ltimes X}$ is a stratified embedding, because $\alpha$ is a stratified map and $u$ a stratified embedding, so (DSC3) is satisfied. Property (DSC4) follows from the definition of the induced stratification, as the connected component of the stratum of $(G \ltimes X)_1$ containing the arrow $(g, x)$ equals the fibered product $R_g \times_{\alpha} P_x$ with $R_g$ and $P_x$ as above. Therefore, the germ at $(g, x)$ of the set of points $(h, y) \in (G \ltimes X)_1$ such that $y \in P_x$ is contained in the germ $(R_g \times_{\alpha} P_x)_{(g, x)}$ of the stratum through $(g, x)$. 


To verify (DSG5), consider again $P_x \subset X$, the connected component of the stratum through $x$. Since by assumption on $X$ the $G$-orbit of each point in $P_x$ is a subset of the stratum containing $P_x$, the saturation $P := \text{Sat} P_x$ has to be a union of connected components of strata which do not intersect one another’s closures by topological local triviality, see Proposition 2.22. Assume that $\sigma : P \to (G \ltimes X)_1$ is a bisection as in (DSG5). Since $\alpha$ is a stratified mapping, $\alpha(P_x)$ is contained in a connected component $S$ of a stratum of $G_0$. The saturation $\text{Sat} S$ of $S$ by $G$ now is given by $t(s^{-1}(S))$ and hence is a union of connected components of strata of $G_0$ such that for each such connected component $S'$, we have $s^{-1}(S) \cap t^{-1}(S') \neq \emptyset$ by construction. Then by Proposition 2.22, the restriction $G|_{\text{Sat} S}$ is a Lie groupoid. Moreover, it follows from (DSG4) that $\sigma(P) \subset (G|_{\text{Sat} S} \ltimes P)_1$. Therefore, the map $L_\sigma$ described in (DSG5) is a left translation of the Lie groupoid $G|_{\text{Sat} S} \ltimes P$, see [Mac05, page 22], implying in particular that it is a diffeomorphism of the stratum $s^{-1}_{G \ltimes X}(P)$ onto itself. Hence (DSG5) holds, completing the proof.

\hfill $\square$

A particularly important special case appears when a Lie group $G$ acts differentiably on a differentiable stratified space $X$ in such a way that the action restricts to a smooth action on each stratum. The resulting translation groupoid $G \ltimes X$ then is a differentiable stratified groupoid by Proposition 3.1. Many significant and naturally occurring examples of differentiable stratified groupoids are constructed that way. In the following, we will provide a few.

Example 3.2 (Singular symplectic reduction). Suppose $(M, \omega)$ is a connected symplectic manifold equipped with a Hamiltonian $G$-action with moment map $J : M \to \mathfrak{g}^*$ such that $0$ is a singular value for $J$. By [SL91, Thm. 2.1], the zero level set $J^{-1}(0)$ and the symplectic quotient $J^{-1}(0)/G$ are both stratified by orbit types. Moreover, the corresponding stratifications are Whitney $(b)$-regular. In addition, $J^{-1}(0)$ inherits from the ambient manifold the structure of a differentiable stratified space, and the $G$-action preserves the stratification; see [OR04, Thm. 8.3.1]. Therefore, the singular symplectic quotient can be realized as the orbit space of the differentiable stratified groupoid $G \ltimes J^{-1}(0)$.

Example 3.3 (Lie Groupoid actions on manifolds with corners). Manifolds with corners and manifolds with boundary are in a natural way locally trivial differentiable stratified spaces, and even more $C^\infty$-cone spaces, see [Ph01, 1.1.19 & 3.10.3] and [SLV15]. Compact Lie group actions on manifolds with corners have been considered, e.g. in [MRD93, Mel91]. By Proposition 3.1, the corresponding translation groupoids are differential stratified groupoids under mild hypotheses.

Example 3.4 (Semialgebraic actions). Recall that a semialgebraic set is a locally closed subset of $\mathbb{R}^n$ locally given by the solution of a finite collection of polynomial equations and inequalities, see [Shi97]. Semialgebraic sets are differentiable stratified spaces in a natural way, since they admit a minimal Whitney $(b)$-regular, and hence topologically locally trivial, stratification into semialgebraic manifolds. If a compact Lie group acts on a semialgebraic set and preserves this stratification, the resulting translation groupoid is again a stratified differentiable groupoid by Proposition 3.1. Lie group actions on semialgebraic sets have been considered in [CPS04, Par13, PS02].
Example 3.5 (Transverse cotangent bundle). Let $G$ be a compact Lie group and $M$ a $G$-manifold. The transverse cotangent bundle $T^*_G M$ is the subspace of the cotangent bundle $T^* M$ consisting of elements that are conormal to the $G$-orbits in $M$. The transverse cotangent bundle appears in the study of transversally elliptic $G$-invariant pseudodifferential operators on $M$, see [DCPV11, Par12, PV09]. The action of $G$ on $M$ induces an action of $G$ on $T^*_G M$. It is not difficult to show that the stratification of $M$ by orbit types induces a stratification of $T^*_G M$ that is compatible with the smooth structure $T^*_G M$ inherits as a subset of $T^* M$. Hence the corresponding translation groupoid is a differentiable stratified groupoid.

If $G$ is a proper Lie groupoid, the transverse cotangent bundle $T^*_G G_0 \subset T^* G_0$ can be defined in a similar fashion as the subspace of all $\xi \in T^*_x G_0$ such that $\langle \xi, v \rangle = 0$ for all $v \in T_x O_x$; here, $x \in G_0$ with $\xi \in T^*_x G_0$ and $O_x$ is the orbit through $x$. By the slice theorem for groupoids as stated in [PPT14, Cor. 3.11] (also recalled in Section 6.2 below), it follows that in a neighborhood of a point $x \in G_0$ the transverse cotangent bundle $T^*_G G_0$ is isomorphic to the exterior tensor product of the transverse cotangent bundle $T^*_x Y_x$ of a slice $Y_x$ through $x$ with the cotangent bundle $T^*_O$ of an open connected neighborhood $O$ of $x$ in the orbit through $x$.

Example 3.6 (Singular riemannian foliations). Recall from [Mol88, Section 6.1] that a singular riemannian foliation is a pair $(M, F)$ where $M$ is a smooth, connected manifold and $F$ is a partition of $M$ into connected, immersed submanifolds called leaves such that the module of smooth vector fields on $M$ that are tangent to the leaves is transitive on each leaf and there is a riemannian metric on $M$ with respect to which every geodesic that is normal to a leaf is normal to every leaf it intersects. A singular riemannian foliation is an example of a singular Stefan–Sussmann foliation [Ste74, Sus73]. By [Mol88, Section 6.2], $M$ is stratified by unions of leaves of the same dimension; see also [RPSAW05, Sections 1.2–3] and [BM93].

Suppose $(M, F)$ is almost regular, meaning that the union of leaves of maximal dimension $k$ is an open, dense subset of $M$. Suppose further that the foliation $F$ can be defined by a Lie algebroid of dimension $k$ in the sense of [Deb01, page 472], i.e. the leaves of $F$ are the maximal connected integral manifolds of the distribution on $M$ induced by a Lie algebroid over $M$. In [Deb01], the holonomy groupoid $G$ of $(M, F)$ is constructed as a Lie groupoid with object space $G_0 = M$ and such that the orbits of $G$ correspond to the leaves $F$; see also [Pra85]. Giving $M$ the stratification by leaves of the same dimension described above and $G_1$ the stratification given by the pullback of this stratification via $s$, it is easy to see that $G$ has, along with its structure as a Lie groupoid, an alternate structure as a differential stratified groupoid which is finer than the original one. These two structures coincide if and only if the foliation $F$ is regular [Mol88, page 190 and Section 1.1], i.e. all leaves have the same dimension.

Note that the holonomy groupoid of a general singular Stefan–Sussmann foliation was constructed as a topological groupoid in [AS09] and coincides with the holonomy groupoid of [Deb01] when the latter is defined. Other hypotheses under which this holonomy groupoid naturally has the structure of a differentiable stratified groupoid are not yet clear.
4. THE ALGEBROID OF A DIFFERENTIABLE STRATIFIED GROUPOID

Given a reduced structurally (b)-regular differentiable stratified groupoid \( G \) with \( S^i \), \( i = 0, 1 \) the decomposition of \( G_1 \) into its strata we obtain the so-called stratified tangent bundles

\[
T^{st}G_0 := \bigcup_{S \in S^0} T S \quad \text{and} \quad T^{st}G_1 := \bigcup_{S \in S^1} T R = \bigcup_{S \in S^1} T S^1,
\]

where \( S^1 := s^{-1}(S) \) for \( S \in S^0 \).

Since, by assumption, the stratifications of \( G_0 \) and \( G_1 \) are Whitney (b)-regular, hence (a)-regular, the stratified tangent bundles \( T^{st}G_0 \) and \( T^{st}G_1 \) inherit the structures of differentiable stratified spaces by [Ph01, Thm. 2.1.2]. Moreover, we have tangent maps \( T_s: T^{st}G_1 \rightarrow T^{st}G_0 \) and \( T_t: T^{st}G_1 \rightarrow T^{st}G_0 \). Now we can define the algebroid of \( G \).

**Definition 4.1.** Given a structurally (b)-regular differentiable stratified groupoid \( G \), the differentiable stratified algebroid \( A \) of \( G \) is defined as the space

\[
A := \bigcup_{S \in S^0} A_S,
\]

where \( A_S := u^*_S \ker T_{|S^1 S} \) denotes the Lie algebroid of the Lie groupoid \( G_{S_1} \). In other words, \( A \) can be identified with \( u^*_S \ker T_s \), the restriction of \( \ker T_s \) to \( G_0 \). We define the anchor map of the algebroid \( A \) to be the map

\[
\varrho: A \rightarrow T^{st}G_0, \quad v \mapsto T_t(v).
\]

**Proposition 4.2.** Let \( G \) be a proper reduced structurally (b)-regular differentiable stratified groupoid. Then the differentiable stratified algebroid \( A \) of \( G \) is a reduced differentiable stratified space, where the differentiable structure is that inherited from \( T^{st}G_1 \) and the stratification is that induced by the decomposition in Equation (4.1). The anchor map \( \varrho: A \rightarrow T^{st}G_0 \) is a differentiable stratified map. If \( G \) is in addition a sliceable differentiable stratified groupoid, then there is a bracket

\[
[-, -]: \Gamma^\infty(G_0, A) \times \Gamma^\infty(G_0, A) \rightarrow \Gamma^\infty(G_0, A)
\]

which satisfies \( \varrho \circ [-, -] = [\varrho(-), \varrho(-)] \) and which coincides over each stratum \( S \in S^0 \) with the bracket of the Lie algebroid \( A_S \).

By \( \Gamma^\infty(G_0, A) \), we mean the sections of the stratified vector bundle \( A \) that are as well morphisms of reduced differentiable spaces.

**Proof.** Since \( G_1 \) is locally compact, \( u(G_0) \) is locally closed in \( G_1 \) by [Tu04, Prop. 2.5]. For each stratum \( S \) of \( G_0 \) the source map \( s \) restricts to \( S_1 \) as a submersion which implies that \( \ker T_{|S_1 S} \) is a subbundle, hence closed subset, of \( T S_1 \). Hence, if \( U \) is an open subset of \( G_1 \) intersecting finitely many strata, \( A \cap T^{st}U \) is a finite union of locally closed sets, therefore locally closed itself. Then \( A \) inherits the structure of a reduced differentiable space from that of \( T^{st}G_1 \), see [NGSdS03, Ex. 3.21].

Since each \( A_S \) is a subbundle of the restriction of \( T S_1 \) to the smooth submanifold \( S_1 \cap u(G_0) \) of \( G_1 \), one concludes that each \( A_S \) is a smooth submanifold of \( A \). Note that the projection \( \pi: T^{st}G_1 \rightarrow G_1 \) is clearly open as its restriction to an element of \( S^1 \) is a bundle map. The fact that \( S^1 \) is locally finite implies then that \( (A_S)_{S \in S^0} \) is a locally finite decomposition of \( A \).
To verify the condition of frontier, suppose $A_S \cap \overline{A_{S'}} \neq \emptyset$ for $S, S' \in S^0$, and let $S_1 = s^{-1}(S)$ and $S'_1 = s^{-1}(S')$. By [Pll01] Thm. 2.1.2, $π: T^*G_1 \to G_1$ is a topological projection, hence $S_1 \subset S'_1$ and $TS_1 \subset TS'_1$ follow.

Choose a tangent vector $v \in \ker T_{|S_1} s$ and assume for simplicity that $v$ is a unit vector. Let $x \in S$ be the footpoint of $v$, i.e. $x$ is the unique object such that $π(v) = u(x)$. Choosing a singular chart for $G_1$ at $u(x)$, we may reduce to the case where $S_1$ and $S'_1$ are closed subsets of $\mathbb{R}^n$. Let $\bar{s}$ denote a smooth function from $\mathbb{R}^n$ into a singular chart of $G_0$ at $x$ that extends $s$. Note that $\bar{s}$ may not be a submersion. Let $p(t) : [-1, 1] \to S_1$ be a smooth path in $S_1$ with $p(0) = u(x)$ and tangent $p'(0) = v$, and put $α_i = p(1/i)$ for $i \in \mathbb{N}$. As $α_1 \in S_1 \subset \overline{S'_1}$, we may choose a sequence $(β_{1,j})_{j \in \mathbb{N}} \subset S'_1$ such that $\lim_{j \to \infty} β_{1,j} = α_1$. We then set $x_j = s(β_{1,j}) \in S'$ for each $j$. By continuity of $u$ and $s$, we have $\lim_{j \to \infty} x_j = x$ and $\lim_{j \to \infty} u(x_j) = u(x)$. As $s|_{S_1}$ is a submersion, each $s^{-1}(x_j)$ is a closed submanifold of $\mathbb{R}^n$. Then the intersection of each $s^{-1}(x_j)$ with a closed ball in $\mathbb{R}^n$ is compact. Hence we may define, for each $i > 1$ and each $j \geq 1$, the element $β_{i,j}$ of $S'_1$ to be the point on the connected component of $s^{-1}(x_j)$ containing $u(x_j)$ intersected with a closed annulus of external radius $1/j$ about $u(x_j)$ that minimizes the distance to $α_i$. In particular, for each $j$, $\lim_{i \to \infty} β_{i,j} = u(x_j)$, and for $i$ sufficiently large, $\lim_{j \to \infty} β_{i,j} = α_i$; if necessary, we restrict to a subsequence in the $j$ direction so that this intersection is not empty. With this, we set

$$v_j := \lim_{i \to \infty} \frac{β_{i,j} - u(x_j)}{∥β_{i,j} - u(x_j)∥},$$

and then $v_j \in T_{x_j} S'_1$ for each $j$. Moreover, as $β_{i,j} \in s^{-1}(x_j)$, it follows that $v_j \in \ker T_{|S'_1} s$. However,

$$\lim_{j \to \infty} v_j = \lim_{i \to \infty} \lim_{j \to \infty} \frac{β_{i,j} - u(x_j)}{∥β_{i,j} - u(x_j)∥} = \lim_{i \to \infty} \frac{α_i - u(x)}{∥α_i - u(x)∥} = v,$$

so that $v \in \ker T_{|S'_1} s$. It follows that the decomposition of $A$ into the $A_S$ satisfies the condition of frontier.

The anchor map is a differentiable stratified morphism by Eq. (2.5) in Lemma 2.16 and because $t$ is a stratified map.

Now, let $X, Y \in Γ^∞(G_0, A)$. We define $[X, Y]$ locally as follows. For $x \in G_0$, choose a neighborhood $U_x$ with slice $Y_x$ as in (Ll4) of Definition 2.31. By shrinking $U_x$ if necessary so that we may extend the $G_x$-action to the closure of $Y_x$ in $G_0$, we have by the Equivariant Embedding Theorem of [NGSdS03] Section 11.2 a $G_x$-equivariant closed embedding $Y_x \to E$ where $E$ is a finite-dimensional representation of $G_x$. Let $f_0 : U_x = Y_x \times O_x \to E \times O_x$ denote the corresponding embedding which is the identity on the $O_x$-factor. By Corollary 2.23, $O_x$ is a smooth manifold so that that $G_x \ltimes E$ and the pair groupoid $O_x \times O_x$ are both Lie groupoids. Hence, $f_0$ is the map on objects of an embedding of $G|_{U_x}$ into the Lie groupoid $H := (G_x \ltimes E) \times (O_x \times O_x)$. Roughly speaking, we will define $[X, Y]$ following the definition of the bracket for the Lie algebroid of a Lie groupoid taking advantage of the structure of $H$; see [Mar08] Section 3.2, and note that we use right-invariant vector fields as in [Mar08] Remark 3.2.4. By shrinking $O_x$ if necessary, we may assume that $f_0$ maps $U_x$ into an open subset $V$ of $H_0$ small enough so that the Lie algebroid $B$ of $H$, a vector bundle in the usual sense and hence locally trivial, restricts to a trivial vector bundle over $V$. Fixing a trivialization of $B|_V$, the section $u^*_{\phi}(f_1 \circ u)_s X$, which is defined only on $f_0(U_x)$, can be described by coordinate
functions with respect to this trivialization, which are smooth by construction. Hence each coordinate function extends to a smooth function on a neighborhood $W$ of $Y_x \times O_x$ in $H_0$ yielding a smooth section of $B$ on $W$. We let $X$ denote the unique right-invariant vector field on $\text{Sat}(uH(W))$ in $H_1$ generated by this extension and define $\hat{X}$ similarly. Then we define $[X,Y]$ to be $(f_1 \circ u)^* [\hat{X}, \hat{Y}]_H \in \Gamma^\infty(U_x,A)$, where $[-,-]_H$ denotes the ordinary bracket of vector fields on $H_1$. Note that $[X,Y]$ is a morphism of differentiable spaces as it is defined as the pullback of the bracket $[\hat{X}, \hat{Y}]_H$ on the manifold $H_1$, which is smooth. That $[X,Y]$ is a section of $A$ over $U_x$ follows from the fact that $[\hat{X}, \hat{Y}]_H$ is a section of $B$ and $s_B \circ f_1 = f_0 \circ s_G$.

Now, if $S \in S^0$ is a stratum that intersects $U_x$, then by Proposition 2.22, $G_{|S \cap U_x}$ is a Lie groupoid. The morphism $f$ restricts to an isomorphism of $G_{|S \cap U_x}$ with a Lie subgroupoid of $H$ so that the unique right-invariant vector field on the object space of $G_{|S \cap U_x}$ generated by $X_{|S \cap U_x}$ pushes forward via $f_1$ to the restriction of $X$, and similarly for $\hat{X}$. Hence, on $S \cap U_x$, $[X,Y]$ restricts to the usual bracket of $X_{|S \cap U_x}$ and $Y_{|S \cap U_x}$ in the Lie algebroid of $G_{|S \cap U_x}$. Therefore, $[-,-]$ coincides on each stratum with the bracket of the corresponding Lie algebroid. In particular, this demonstrates that the value of $[X,Y]$ on each stratum, and hence on $G_0$, does not depend on the various choices made. Moreover, as the anchor map $\rho$ restricts to each stratum as the anchor map of the corresponding Lie algebroid by definition, the relation $\rho \circ [-,-] = [\rho(-), \rho(-)]$ follows. This completes the proof. 

\section{A de Rham theorem}

In this section, we prove a de Rham theorem for sliceable proper reduced differentiable stratified groupoids which satisfy a certain local contractibility condition. As we will see below, this condition is satisfied in a number of the examples we have considered.

Throughout this section let $G$ be a proper reduced structurally (b)-regular differentiable stratified groupoid. Let $\Omega^\bullet$ be the sheaf complex of abstract forms on $G_0$ as constructed in Section A.5. Recall from [PPT14, Def. 8.1] that a differential form on the object space of a proper Lie groupoid is called basic if contraction with any smooth section of the Lie algebroid vanishes and if the form is invariant under the conormal action of the Lie groupoid.

\begin{definition}
Let $U \subset |G|$ be an open subset of the orbit space of $G$, and $U_0$ its preimage under the canonical projection $\pi : G_0 \to |G|$. One calls an abstract $k$-form $\omega \in \Omega^k(U_0)$ $G$-horizontal or simply horizontal, if for each smooth section $\xi : U_0 \to A$ of the differentiable stratified algebroid $A$ of $G$ given by Definition 1.1, the stratewise contracted form $(\rho \circ \xi)_\omega \omega$ vanishes.

Next let $S \subset G_0$ be a (relatively closed and open) component of a stratum of $G_0$ such that the projection $\pi(S)$ to the orbit space is connected. For $x \in G_0$ consider the Zariski tangent space $T_x G_0$; see Appendix A.6. Note that $T_x G_0$ does in general not coincide with $T^g_x G_0$, but that the latter is always contained in the former.

\begin{lemma}
Let $G$ be sliceable. Then the space $T_S G_0 := \bigcup_{x \in S} T_x G_0$ naturally carries the structure of a smooth vector bundle over $S$.
\end{lemma}

\begin{proof}
Given $x \in S$ choose a trivializing neighborhood $U_x$ and a groupoid slice $Y_x$ with a $G_x$-action such that there exists an isomorphism of differentiable groupoids $G_{|U_x} \to (O_x \times$
descends to \( \Omega \), where \( O_x \) is an open contractible neighborhood of \( x \) in its orbit. Such \( U_x \), \( Y_x \), and \( O_x \) exist according to (LIT). After possibly shrinking these data, we can assume by (LIT) that the slice \( Y_x \) is stratified and is isomorphic as a differentiable stratified space to the product space \( Z_x \times R_x \), where \( Z_x \subset Y_x \) is a \( G_x \)-invariant subspace, and \( R_x \subset Y_x \) is the stratum through \( x \). Hence \( \bigcup_{y \in O_x \times R_x} T_y G_0 \) is a vector bundle isomorphic to \( T(O_x \times R_x) \times T_x Z_x \). But the set \( O_x \times R_x \) contains \( x \) and is relatively open in the stratum \( S \). This means that, locally, \( T|_S G_0 \) is a vector bundle. By construction, the transition maps arise from the local isomorphisms \( G_{U_x} \to (O_x \times O_x) \times (G_x \times Y_x) \), and are also vector bundle isomorphisms, hence the claim follows. \( \square \)

We can now define basic forms on \( G \). Note that we will apply the pull-back morphisms constructed in Appendix A.5.

**Definition 5.3.** Let \( U \subset |G| \) be an open subset of the orbit space, and \( U_0 := \pi^{-1}(U) \).

One calls an abstract \( k \)-form \( \omega \in \Omega^k(U_0) \) \( G \)-basic or simply basic if it is horizontal and if for every \( x \in U_0 \) and every smooth bisection \( \sigma : U_x \to G \) defined on an open neighborhood \( U_x \subset U_0 \) of \( x \) the equality \( (t \circ \sigma)^* \omega = \omega|_{U_x} \) holds true. Let \( \Omega^k_{\text{basic}}(U) \) denote the set of \( G \)-basic \( k \)-forms on \( U \).

Observe that by definition, \( \Omega^0_{\text{basic}}(U) \) coincides with the space of smooth functions on \( U_0 \) which are invariant under the \( G \)-action, hence \( \Omega^0_{\text{basic}}(U) \) can be naturally identified with \( \mathcal{C}^\infty(U) \). Moreover, we will now show that the spaces \( \Omega^k_{\text{basic}}(U) \), where \( U \) runs through the open sets of \( |G| \), form the space of sections of a sheaf on \( |G| \) which will be denoted by \( \Omega^k_{\text{basic}} \).

**Proposition 5.4.** The sheaves of basic \( k \)-forms \( \Omega^k_{\text{basic}} \) are fine. Moreover, the exterior differential on \( \Omega^*_{\text{basic}} \) descends to a differential \( d \) on \( \Omega^*_{\text{basic}} \) turning \( (\Omega^*_{\text{basic}}, d) \) into a sheaf of commutative differential graded algebras over the orbit space \( |G| \).

**Proof.** By construction the sheaf \( \Omega^k_{\text{basic}} \) is a \( \mathcal{C}^\infty_{|G|} \)-module. It remains to show that \( d \) descends to \( \Omega^*_{\text{basic}} \). But that follows from the fact that \( d \) commutes with the pull-back morphism \( (t \circ \sigma)^* \) for every bisection \( \sigma : U_x \to G \) defined on an open neighborhood \( U_x \) of \( x \in G_0 \). \( \square \)

The next result will be needed for the proof of a Poincaré Lemma for basic forms.

**Proposition 5.5.** Assume \( G \) is sliceable. Let \( V \) be an open subset of \( G_0 \) and let \( \omega \in \Omega^k(V) \) be an abstract \( k \)-form which is \( G|_V \)-basic. Then there exists a unique form \( \tilde{\omega} \in \Omega^k(\text{Sat}(V)) \) which is \( G \)-basic and whose restriction to \( V \) coincides with \( \omega \). We call \( \tilde{\omega} \) the basic extension of \( \omega \).

**Proof.** Let \( x \in \text{Sat}(V) \), and choose a bisection \( \sigma : U_x \to G_1 \) defined on an open neighborhood of \( x \) such that \( (t \circ \sigma)(U_x) \subset V \). Then put \( \omega|_{U_x} := (t \circ \sigma)^* \omega \). If \( \eta : U_x \to G_1 \) is another bisection with \( U_z := (t \circ \eta)(U_x) \subset V \), where \( z := (t \circ \eta)(x) \), then the bisection \( \mu : U_z \to G_1 \) defined by

\[
\mu(y) = \sigma((t \circ \eta)^{-1}(y)) \cdot \eta^{-1}((t \circ \eta)(y)) \quad \text{for } y \in U_z
\]

starts and ends in \( V \). Hence, by assumption, \( (t \circ \mu)^* \omega = \omega|_{U_z} \). But \( t \circ \mu = (t \circ \sigma) \circ (t \circ \eta)^{-1} \), which implies

\[
(t \circ \eta)^* \omega = (t \circ \eta)^*(t \circ \mu)^* \omega = (t \circ \sigma)^* \omega.
\]
This shows that $\omega_{U_x}$ does not depend on the particular choice of the bisection $\sigma : U_x \to t^{-1}(V)$. An analogous argument proves that for points $x, y \in \text{Sat}(V)$ the forms $\omega_{U_x}$ and $\omega_{U_y}$ coincide over $U_x \cap U_y$. Let $\tilde{\omega} \in \Omega^k(\text{Sat}(V))$ be the abstract $k$-form such that $\tilde{\omega}_{U_x} = \omega_{U_x}$ for all $x \in \text{Sat}(V)$. Obviously, $\tilde{\omega}$ is horizontal and basic by construction. Uniqueness of $\tilde{\omega}$ is clear since the form has to be basic. □

Since the kernel of the sheaf morphism $d : \Omega^0_{\text{basic}} \to \Omega^1_{\text{basic}}$ can be naturally identified with $\mathbb{R}|G|$, the sheaf of locally constant real-valued functions on $|G|$, we obtain a sheaf complex

$$0 \longrightarrow \mathbb{R}|G| \longrightarrow \Omega^\bullet_{\text{basic}}$$

which is exact at $\mathbb{R}|G|$ and at $\Omega^0_{\text{basic}}$. Now observe that the orbit space $|G|$ is paracompact, locally contractible and locally path connected by [PPT14]. Since the sheaves $\Omega^k_{\text{basic}}$ are fine, the following is a consequence of [God58, Sec. 3.9].

**Theorem 5.6.** Let $G$ be a proper reduced structurally ($b$)-regular differentiable stratified groupoid. If the sheaf complex $(\Omega^\bullet_{\text{basic}}, d)$ of basic forms on $G$ is exact, then the basic cohomology $H^\bullet_{\text{basic}}(G) := H^\bullet(\Omega^\bullet_{\text{basic}}(|G|))$ of $G$ naturally coincides with the real singular cohomology of $|G|$.

In the remainder of this section we will show that the sheaf complex of basic forms on a groupoid satisfies a Poincaré Lemma, i.e. is exact, if a certain local contractibility condition is satisfied. Before stating the local contractibility condition, recall that by Proposition 2.11 every $G$-slice can be, possibly after shrinking, equivariantly embedded into the Zariski tangent space around the fixed point.

**Definition 5.7.** Let $G$ be a sliceable proper differentiable stratified groupoid. We then say that $G$ fulfills the local contractibility condition if the following holds true.

(LC) For each $x \in G_0$ there exists a groupoid slice $Y_x$ as in (LT4), a linear $G_x$-action on some $\mathbb{R}^n$ together with a singular $G_x$-equivariant chart $\iota : Z_x \to \bar{V}_x \subset \mathbb{R}^n$, $y \mapsto \bar{y}$, and a smooth homotopy $h : \bar{V}_x \times [0, 1] \to \bar{V}_x$ having the following properties:

1. The chart $\iota$ maps $x$ to 0 and the stratum $R_x$ through $x$ to the subspace of $\mathbb{R}^n$ fixed by $G_x$. Moreover, $\bar{V}_x$ is an open neighborhood of 0 in $\mathbb{R}^n$, and $\bar{Y}_x := \iota(Y_x)$ is relatively closed in $\bar{V}_x$.
2. One has $h_0 = \{0\}$ and $h_1 = \text{id}_{\bar{V}_x}$.
3. The homotopy $h$ is a homotopy along $\bar{Y}_x$ which means that $h(\bar{y}, t) \in \bar{Y}_x$ for all $y \in Z_x$ and $t \in [0, 1]$.
4. The homotopy $h$ is $G_x$-equivariant.
5. The homotopy $h$ preserves the stratification which means for each $y \in Y_x$ and $t \in (0, 1]$ the points $\iota^{-1}h(\bar{y}, t)$ and $y$ are in the same stratum.

**Example 5.8.** Let $G$ be a compact Lie group and $M$ is a $G$-manifold. The transverse cotangent bundle described in Example 3.5 satisfies the local contractibility condition of Definition 5.7. To verify this, observe first that $T^*_G M$ is equivariantly embedded in the $G$-manifold $T^* M$, and that $\mathbb{R}_{\geq 0}$ acts on $T^* M$ by fiberwise homotheties. Since the transverse cotangent bundle is invariant under these homotheties, it contracts smoothly to the 0-section which is diffeomorphic to $M$. But $M$ is a smooth manifold, so is clearly locally smoothly contractible. Hence $T^*_G M$ is locally smoothly contractible as well.
Example 5.9. Let \((M, \omega)\) be a connected symplectic Hamiltonian \(G\)-manifold with \(0\) a singular value of the moment map \(J: M \to \mathfrak{g}^*\) so that the singular symplectic quotient is the orbit space of the differentiable stratified groupoid \(G \times J^{-1}(0)\); see Example 3.2. By the Marle–Guillemin–Sternberg normal form of the moment map \([OR04\text{ Thm. 7.5.5}]\), the contractibility condition (LC) in Definition 5.7 is satisfied for the zero level set \(J^{-1}(0)\). Theorem 5.10 below reduces to the de Rham theorem of \([Sja05]\) in this case.

As our primary example of a differentiable stratified groupoid satisfying the locally contractibility condition, we will show in Section 6 that the inertia groupoid of a proper Lie groupoid fulfills the local contractibility condition as well.

We now prove that the sheaf complex of basic forms on a sliceable proper reduced differentiable stratified groupoid fulfilling the contractibility condition is exact, or in other words satisfies Poincaré’s Lemma.

**Theorem 5.10.** Let \(G\) be a sliceable proper reduced structurally \((b)\)-regular differentiable stratified groupoid satisfying the local contractibility condition. The complex of sheaves \((\Omega^*_\text{basic}, d)\) on \(|G|\) then is a fine resolution of the sheaf of locally constant \(\mathbb{R}\)-valued functions on \(|G|\). In particular this implies that the cohomology of the complex \((\Omega^*_\text{basic}, d)\) of basic differential forms on \(G\) is naturally isomorphic to the singular cohomology of \(|G|\) with coefficients in \(\mathbb{R}\).

**Proof.** We first consider the case where the groupoid \(G\) is of the form \((O \times O) \times (G \times Y) \rightrightarrows O \times Y\), where \(O\) is an open contractible set in some \(\mathbb{R}^m\), \(O \times O\) denotes the corresponding pair groupoid, \(G\) is a Lie group acting linearly on some \(\mathbb{R}^n\), and \(Y \subset \mathbb{R}^n\) is an affine \(G\)-invariant differentiable stratified space on which \(G\) acts by strata preserving maps. In addition we assume that \(0 \in Y\) and that there exists a smooth homotopy \(h: \tilde{V} \times [0, 1] \to \tilde{V}\) defined on an open \(G\)-invariant subset \(\tilde{V} \subset \mathbb{R}^n\) such that the five conditions of the local contractibility condition are satisfied with \(t: Y \to \mathbb{R}^n\) being the identical embedding. Denote by \(I \subset C^\infty(O \times \tilde{V})\) the vanishing ideal of \(O \times Y\). Observe that by \([DJZ04\text{ Sec. 2}]\) and the smooth contractibility of \(Y\), we have that the subcomplex \(I^* \subset \Omega^*(O \times \tilde{V})\) defined by

\[
I^k := \begin{cases} I, & \text{for } k = 0, \\ I\Omega^k(O \times \tilde{V}) + dI \wedge \Omega^{k-1}(O \times \tilde{V}), & \text{for } k = 1, \ldots, n + m \end{cases}
\]

is contractible. More precisely, a contraction is given by

\[
K\omega = \begin{cases} 0, & \text{for } \omega \in \Omega^0(O \times \tilde{V}) = C^\infty(O \times \tilde{V}), \\ \int_0^1 H_t^*(\xi_t \omega) \, dt, & \text{for } \omega \in \Omega^k(O \times \tilde{V}), t \in \mathbb{N}^*, \end{cases}
\]

where the homotopy \(h\) has been extended to a homotopy on \(O \times \tilde{V}\) by putting \(H_t(v, x) = (v, h_t(x))\) for \(v \in O, x \in \tilde{V}, t \in [0, 1]\), and \(\xi_t : O \times \tilde{V} \to T\tilde{V}\) is the vector field defined by \(\xi_t := \partial_t H_t\). Cartan’s magic formula now implies that

\[
\omega - H_0^*\omega = dK\omega + Kd\omega, \quad \text{for } \omega \in \Omega^k(O \times \tilde{V}), k \in \mathbb{N}.
\]

But this implies that the restriction of \(K\) to \(I^*\) is an algebraic contraction indeed, since every form \(\omega \in I^k\) satisfies the relation \(H_0^*\omega = 0\).

Now consider the subcomplex \(\Omega^*_{\text{r-basic}}(O \times \tilde{V}) \subset \Omega^*(O \times \tilde{V})\) of relative basic forms, or more precisely of relative \(O \times Y\) basic forms on \(O \times \tilde{V}\), defined as follows. The subcomplex \(\Omega^*_{\text{r-basic}}(O \times \tilde{V})\) is the set of all \(\omega \in \Omega^k(O \times \tilde{V})\) which are invariant under
the $G$-action and which have the property that for each stratum $S$ of $O \times Y$ the form $\iota^*_S \omega$ is a basic form for the restricted Lie groupoid $G|_S$. In particular, this implies that $H_0^* \omega = \iota^*_O \omega$ vanishes for each restricted basic form $\omega$. Since $H_t$ commutes with the $G$-action and maps fibers $O \times \{y\}$ to $O \times \{h_t(y)\}$, the algebraic contraction $K$ maps basic forms to basic forms. One concludes that the complex $\Omega^{\bullet}_{r\text{-basic}}(O \times \tilde{V})$ is exact, and that the subcomplex $I^{\bullet}_{r\text{-basic}} := I^\bullet \cap \Omega^{\bullet}_{r\text{-basic}}(O \times \tilde{V})$ is contractible. Hence the quotient complex $\Omega^{\bullet}_{r\text{-basic}}(O \times \tilde{V})/I^{\bullet}_{r\text{-basic}}$ is exact. Moreover, one has

$$\Omega^{\bullet}_{\text{basic}}(Y/G) = \Omega^{\bullet}_{r\text{-basic}}(O \times \tilde{V})/I^{\bullet}_{r\text{-basic}},$$

which can be seen by averaging a representative of an element $[\omega] \in \Omega^k_{\text{basic}}(Y/G)$ over the orbits of the $G$-action using a bi-invariant Haar measure on $G$. The resulting new representative $\omega$ then is $G$-invariant. Since $[\omega]$ is basic, the pull-back of such a representative $\omega$ to each stratum $S$ of $O \times Y$ has to be basic as well, hence $\omega \in \Omega^{\bullet}_{r\text{-basic}}(O \times \tilde{V})$. So we have shown Poincare’s Lemma in the special case where the groupoid $G$ is of the form $(O \times O) \times (G \times Y) \supset O \times Y$ with $Y$ and $O$ as stated above.

Next let us consider the general case of a proper sliceable structurally (b)-regular differentiable stratified groupoid $G$ which fulfills the local contractibility condition. Let $x \in G_0$ be a point in the object space. Since $G$ is sliceable, we can choose a trivializing neighborhood $U_x \subset G_0$ of $x$, an open contractible neighborhood $O_x$ in the orbit through $x$ and a groupoid slice $Y_x \subset U_x$ with a $G_x$-action such that $G_{U_x}$ as a differentiable stratified groupoid is isomorphic to the groupoid $(O_x \times O_x) \times (G_x \times Y_x)$. We will show that $\Omega^{\bullet}_{r\text{-basic}}(\pi(U_x))$ is exact. To this end let $\omega \in \Omega^k(\Sat(U_x))$ be a closed basic $k$-form. The restriction of $\omega$ to $U_x$ is a closed and $G_{U_x}$-basic form. By the proceeding considerations there exists a $G_{U_x}$-basic $(k-1)$-form $\eta \in \Omega^{k-1}_{r\text{-basic}}(U_x)$ such that $d\eta = \omega|_{U_x}$. By Proposition 5.5 $\eta$ has a unique extension to a basic form $\tilde{\eta} \in \Omega^{k-1}_{\text{basic}}(\pi(U_x))$. By construction of $\tilde{\eta}$ we have $d\tilde{\eta} = \omega$, since $d$ commutes which each of the isomorphisms $t \circ \sigma$, where $\sigma : U \rightarrow G$ is a isection. This proves exactness of $\Omega^{\bullet}_{\text{basic}}(\pi(U_x))$. \hfill \Box

Remark 5.11. If $G$ is a proper Lie groupoid, then the complex of sheaves $(\Omega^{\bullet}_{\text{basic}}, d)$ coincides with the sheaf of basic differential forms defined in [PPT14, Def. 8.1] so that Theorem 5.10 reduces to [PPT14, Prop. 8.6 & Cor. 8.7].

If $G$ is a sliceable groupoid, then $(s, t)(G_1)$ is necessarily locally closed in $G_0 \times G_0$. Specifically, using the local model of $G$ given by condition (L1), the condition that $(s, t)(G_1)$ is locally closed is equivalent to the requirement that the $G_x$-action on $Y_x$ is a proper group action, which is automatically satisfied as $G_x$ is compact by [Tu04, Prop. 2.10(ii)]. Therefore, $|G|$ is locally compact by Proposition 2.32. If we assume further that $|G|$ carries a stratification according to Proposition 2.32 and that this stratification is Whitney (b)-regular, then $|G|$ is a differentiable stratified space with control data in the sense of Mather by [PH01, Thm. 3.6.9]. Hence $|G|$ admits in this case a triangulation subordinate to its stratification, cf. [PPT14, Thm. 7.1]. Therefore, given an open covering of $|G|$, there exists a subordinate good covering. See [PPT14 Sec. 7] for more details. As in the case of Lie groupoids [PPT14, Cor. 8.8] one concludes that the singular cohomology of $|G|$ with real coefficients coincides with the Čech cohomology under these hypotheses, and that it is finite-dimensional if $|G|$ is compact.
6. The inertia groupoid of a proper Lie groupoid as a differentiable stratified groupoid

The goal of this section is to construct an explicit Whitney (b)-regular stratification of the loop space $\Lambda_0 G$ of a proper Lie groupoid $G$ so that the inertia groupoid $\Lambda G$ becomes a reduced differentiable stratified groupoid which is sliceable and satisfies the local contractibility condition. Our strategy employs the slice theorem for proper Lie groupoids which enables us to describe the groupoid $G$ locally in terms of translation groupoids by compact Lie groups. This will allow us to describe $\Lambda G$ locally in terms of the inertia groupoid associated to such a translation groupoid. Using isotropy types and an equivalence relation on the group defined in terms of this action, we will construct stratifications for inertia groupoids of such translation groupoids that patch together to a well-defined stratification of $\Lambda_0 G$.

6.1. The inertia groupoid of a proper Lie groupoid. Let $G$ be a proper Lie groupoid. Define the loop space of $G$ to be

$$\Lambda_0 G := \{ h \in G_1 \mid s(h) = t(h) \}.$$  

Since the loop space is closed in $G_1$ it inherits the structure of a reduced differentiable space from the ambient manifold $G_1$. The map $s = t : \Lambda_0 G \to G_0$ serves as an anchor map for the action of $G$ on $\Lambda_0 G$ by conjugation. More precisely, the action of $g \in G_1$ on $h \in \Lambda_0 G$ with $s(g) = s(h)$ is given by

$$g \cdot h := ghg^{-1}. \tag{6.1}$$

**Definition 6.1.** The inertia groupoid $\Lambda G$ of a proper Lie groupoid $G$ is the action groupoid $\Lambda G := G \rtimes \Lambda_0 G$. The space of its objects is the loop space $\Lambda_0 G$, its space of arrows is $G_1 \times \Lambda_0 G$. The inertia space of $G$ then is the orbit space $|\Lambda G|$.

**Remark 6.2.** Note that, while $\Lambda_0 G$ is a differentiable subspace of the smooth manifold $G_1$, the action of $G$ on $\Lambda_0 G$ does not necessarily extend to an action on $G_1$ or an open neighborhood of $\Lambda_0 G$ in $G_1$.

6.2. The stratification of the loop space: Statement of the results. In this section, we state our main results about the stratifications of the loop and inertia spaces of a proper Lie groupoid. The proofs will be given in Sections 6.3, 6.4, and 6.5. We begin with the case of a translation groupoid.

The compact Lie group action case. Assume that the compact Lie group $G$ acts by diffeomorphisms on the smooth manifold $M$ without boundary. The loop space $\Lambda_0 (G \ltimes M)$ coincides in this case with the union $\bigcup_{g \in G} \{ g \} \times M^0 \subseteq G \times M$, where $M^0$ denotes the fixed point space of $g \in G$. The $G$-action on $\Lambda_0 (G \ltimes M)$ is the diagonal action, acting by conjugation on the first factor. To describe our stratification of $\Lambda_0 (G \ltimes M)$ recall first [BtD95] IV Def. 4.1] that a closed subgroup $T$ of the Lie group $G$ is called a Cartan subgroup if it is closed, topologically cyclic, and of finite index in its normalizer; topologically cyclic means that it contains a dense cyclic subgroup, and implies by [BtD95] I Cor. 4.14] that $T$ is isomorphic to the product of a torus and a finite cyclic group. As in [FPS15], we say $T$ is associated to an element $h \in G$ if $h \in T$, and $T / T^0$ is generated by $h T^0$ where $T^0$ denotes the connected component of the identity. Now let $(h, x) \in \Lambda_0 (G \ltimes M) \subset G \times M$, and choose a slice $Y_x$ at $x$ for the $G$-action on $M$, see [DK00] Def. 2.3.1]. By shrinking $Y_x$ if necessary and choosing an appropriate $G$-invariant riemannian metric on $M$, we can
assume that $Y_x$ is the image under the exponential map of an open ball $B_x \subset N_x$ around the origin of the normal space $N_x := T_x M / T_x (Gx)$ to the tangent space of the orbit through $x$. Let $H = G_{(h, x)}$, and note that $H = G_x \cap Z_G(h) = Z_{G_x}(h)$ is the centralizer of $h$ in $G_x$. Let $T_{(h, x)}$ be a Cartan subgroup of $H$ associated to $h$. Note that if $G_x$ is connected, the relation $h \in (Z_{G_x}(h))^0 = H^0$ holds true by [DK00] Thm. 3.3.1 (i)], so that $T_{(h, x)}$ is a maximal torus of $H^0$ containing $h$. Define an equivalence relation $\simeq$ on $T_{(h, x)}$ by $s \simeq t$ if $N^s_x = N^t_x$, and let $T^*_{(h, x)}$ denote the connected component of the $\simeq$ class containing $h$. Note that by construction, $s \simeq t$ if and only if the germs of the sets $Y^s_x$ and $Y^t_x$ at $x$ coincide.

By [DK00] Prop. (3.1.1)(i), $H$ is a slice at $h$ for the action of $G_x$ on itself by conjugation. Because $G_x$ acts on $Y_x$ and fixes $x$, it follows that $H \times Y_x$ is a slice at $(h, x)$ for the action of $G_x$ on $G_x \times Y_x$ which is given by $g(k, y) = (gkg^{-1}, g_y)$, i.e. by the diagonal action with conjugation on the $G_x$-factor. Hence, because $T_{(h, x)} \leq H$, we have that $T_{(h, x)} \times Y_x$ is contained in the slice $H \times Y_x$ at the point $(h, x)$, which has $G_x$-isotropy group $H$. It follows that $(T^*_{(h, x)} \times Y^G_x)^H = (T^*_{(h, x)} \times Y^G_x)^H$, where $S_H$ denotes the set of points in a set $S$ with isotropy group equal to $H$, i.e. that each point in $T^*_{(h, x)} \times Y^G_x$ fixed by $H$ has $G_x$-isotropy $H$.

**Remark 6.3.** Note that if $(k, y) \in (T^*_{(h, x)} \times Y^G_x)^H$, then $y \in Y^G_x$ so that, as $Y_x$ is a slice for the $G$-action on $M$, $G_y = G_x$. Hence, $G_{(k, y)} \leq G_x$ so that $G_{(k, y)} = H$. That is, a point in $T^*_{(h, x)} \times Y^G_x$ has $G_x$-isotropy $H$ if and only if it has $G$-isotropy $H$, and so the notation $(T^*_{(h, x)} \times Y^G_x)^H$ is unambiguous.

With these observations, assign to $(h, x)$ the germ

$$S_{(h, x)} = \left[ G \left( T_{(h, x)} \times Y^G_x \right) \right]_{(h, x)} = \left[ G \left( T^*_{(h, x)} \times Y^G_x \right)^H \right]_{(h, x)}.$$

That is, near $(h, x)$, the stratum of $(h, x)$ consists of points $(k, y)$ such that $G_{(k, y)}$ is conjugate to $H$ in $G$, and the conjugation isomorphism $G_{(k, y)} \to H$ maps $G_y$ onto $G_x$ and as well maps $k$ to a point in $T_{(h, x)}$ that fixes the same subspace of $Y_x$ as $h$. It will be demonstrated below that Equation (6.2) yields a stratification of the loop space $\Lambda_0(G \ltimes M)$ that induces a stratification of the inertia space $|A_0(G \ltimes M)|$.

We refer to the stratification of Equation (6.2) as the orbit Cartan type stratification of the loop space of the Lie groupoid $G \ltimes M$. Note that this stratification differs from that given in [FPS15] in that we require the germ $S_{(h, x)}$ to be a subset of $Y^G_x$, and we use a different (weaker) definition of the equivalence relation $\simeq$ defining $T^*_{(h, x)}$.

The germ $S_{(h, x)}$ is obviously $G_x$-invariant, and hence, if intersected with $A_0(G_x \ltimes Y_x)$ (i.e. take $G_x$-orbits rather than $G$-orbits) defines a germ in the quotient $|A(G_x \ltimes Y_x)|$. This stratification depends only on the $G$-orbit of $(h, x)$, and hence defines a germ in the quotient $|A(G \ltimes M)|$ as well. To see this, note that if $g \in G$, then $gy_x$ is a slice at $gx$ for the $G$-action on $M$, and conjugation by $g$ maps $G_x$ onto $G_{gy_x}$. Choosing $T_{g(h, x)}$ to be the image of $T_{(h, x)}$ under the induced isomorphism $G_x \ltimes Y_x \to G_{gy_x} \ltimes gY_x$, the germ $S_{g(h, x)}$ coincides with $S_{(h, x)}$. Hence, the stratification given in Equation (6.2) induces a stratification of the inertia space $|A_0(G \ltimes M)|$, the orbit Cartan type stratification, given
by
\begin{equation}
R_{(h, x)} = G \setminus \left[ G \left( T^*_x \times Y^{G_x}_x \right)^H \right]_{(h, x)}.
\end{equation}

We have the following result which we will prove in Section 6.3.

**Theorem 6.4.** Let $\mathcal{G}$ be a compact Lie group and $M$ a smooth $\mathcal{G}$-manifold. Then Equation \((6.2)\) defines a Whitney \((b)\)-regular $\mathcal{G}$-invariant stratification of the loop space $\Lambda_0(G \ltimes M)$ with respect to which $\Lambda_0(G \ltimes M)$ is a differentiable stratified space such that the $\mathcal{G}$-orbits are subsets of strata. Using the induced stratifications on the space of arrows and corresponding fibered product, the inertia groupoid $\Lambda(G \ltimes M)$ is a proper reduced structurally \((b)\)-regular differentiable stratified groupoid satisfying the local contractibility condition. Moreover, Equation \((6.3)\) defines a stratification of the inertia space $|\Lambda_0(G \ltimes M)|$ with respect to which the orbit map is a differentiable stratified surjective submersion.

**The proper Lie groupoid case.** We now turn to the case of a proper Lie groupoid $\mathcal{G}$, and endow $\mathcal{G}$ with a transversally invariant riemannian metric. Recall \[PPT14\] Cor. 3.11 that for each point $x \in \mathcal{G}_0$, there is an open neighborhood $U$ of $x$ in $\mathcal{G}_0$ diffeomorphic to $O \times B_x$ such that $\mathcal{G}|_U$ is isomorphic to the product of the pair groupoid over $O$ and $\mathcal{G}_x \ltimes B_x$. Here, $O$ is an open ball around $x$ in the orbit of $x$ and $B_x$ is a $\mathcal{G}_x$-invariant open ball around the origin in the normal space $N_x = T_x\mathcal{G}_0/T_xO_x$ to the tangent space of the orbit through $x$. According to \[PPT14\] Thm. 4.1, one can assume that the corresponding diffeomorphism $O \times B_x \to \mathcal{G}_0$ is given, over the factor $B_x$, by the exponential map with respect to the chosen transversally invariant riemannian metric. We let $Y_x \subset \mathcal{G}_0$ denote the image of $\{x\} \times B_x$ under this diffeomorphism and call it a slice for $\mathcal{G}$ at $x$; note that $Y_x$ is a slice in the sense of (LT1) in Definition 2.7. In particular, by \[PPT14\] Thm. 4.1, $\mathcal{G}|_{Y_x}$ is isomorphic to $\mathcal{G}_x \ltimes Y_x$. Since the latter is isomorphic to $\mathcal{G}_x \ltimes Y_x$, we obtain an isomorphism between $\mathcal{G}|_{Y_x}$ and $\mathcal{G}_x \ltimes Y_x$ which is induced by the exponential map and the canonical action of $\mathcal{G}_x$ on $N_x$. This isomorphism gives rise to an inclusion $\mathcal{G}_x \ltimes Y_x \hookrightarrow \mathcal{G}$ of differentiable stratified groupoids. Note that the metric is used only to define the exponential map. Moreover, we will see in Proposition 6.17 below that the stratification does not depend on the choice of metric.

**Remark 6.5.** Note that if $\mathcal{G} = G \ltimes M$ is a translation groupoid, then a slice as defined here corresponds to a slice for the $G$-action on $M$, so using the same notation for both will cause no confusion.

To define a stratification of $\Lambda_0 \mathcal{G}$, we will employ the stratification constructed above of the loop space $\Lambda_0(\mathcal{G}_x \ltimes Y_x)$. To this end choose $g \in \Lambda_0 \mathcal{G}$ with $s(g) = t(g) = x \in \mathcal{G}_0$. We then define the germ $S^G_g$ of the stratification of $\Lambda_0 \mathcal{G}$ as follows. Let $h \in \mathcal{G}_x$ denote the element such that $(h, x)$ corresponds to the arrow $g$ under the isomorphism between $\mathcal{G}|_{Y_x}$ and $\mathcal{G}_x \ltimes Y_x$. Let $S_{(h, x)}$ be the germ of the orbit Cartan type stratification of $\Lambda_0(\mathcal{G}_x \ltimes Y_x)$, and let Sat($S_{(h, x)}$) denote its saturation within $\mathcal{G}$, i.e. the germ of the saturation of a defining set for $S_{(h, x)}$. Putting
\begin{equation}
S^G_g := \text{Sat}(S_{(h, x)})
\end{equation}
then defines the orbit Cartan type stratification of the loop space $\Lambda_0 \mathcal{G}$. Similarly, we define
\begin{equation}
R^G_{\Lambda \pi(g)} := \Lambda \pi(\text{Sat}(S_{(h, x)})),
\end{equation}
where $\Lambda \pi(g)$ is the orbit through $g$ of the canonical action of $\mathcal{G}$ on $M$.
where \( \Delta \tau : \Lambda_0 G \to |\Lambda G| \) is the orbit map of the inertia groupoid. That means that the germ \( R^G_{\Delta \tau(g)} \) in the orbit space \( |\Lambda G| \) is defined to be the projection of \( S^G_g \) to the orbit space.

With these definitions, we can state the next result. It will be proven in Sections 6.4 and 6.5.

**Theorem 6.6.** Let \( G \) be a proper Lie groupoid. Then Equation (6.4) defines a Whitney \((b)\)-regular stratification of the loop space \( \Lambda_0 G \) with respect to which the inertia groupoid \( \Lambda G \) is a sliceable structurally \((b)\)-regular differentiable stratified groupoid. Moreover, the inertia space \( |\Lambda G| \) inherits a differentiable structure, and Equation (6.5) defines a stratification with respect to which \( |\Lambda G| \) is a differentiable stratified space and the orbit map \( \Lambda_0 G \to |\Lambda G| \) a differentiable stratified surjective submersion.

6.3. **Proof of Theorem 6.4.** The stratification given by Equation (6.2) is a variation on the stratification given by [FPS15, Thm. 4.1], so the proof of Theorem 6.4 will follow a similar outline as a series of lemmata. The primary virtue of the revised definition given here is that it is invariant under Morita equivalences, and hence can be glued together on intersections of charts of a proper Lie groupoid. For the convenience of the reader, we present the complete argument.

We assume \( G \times M \) is equipped with a riemannian metric given by the product of a \( G \)-invariant metric on \( M \) and a bi-invariant metric on \( G \). For a point \( y \in M \), we will denote by \( Y_y \) a slice at \( y \) for the \( G \)-action on \( M \). We use \( H \) to denote the isotropy group \( G_{(h,x)} = \Lambda G_{(h)}(h,x) \in \Lambda_0 G \) of \( h \in \Lambda_0 G \) and the symbol \( N_{(h,x)} \) to denote the normal space \( T_{(h,x)}(G_x \times Y_x)/T_{(h,x)}(G_x(h,x)) \cong T_{(h,x)}(G \times M)/T_{(h,x)}(G(h,x)) \). Because \( H \times Y_x \) is a slice at \( (h,x) \) for the \( G_x \)-action on \( G_x \times Y_x \), the exponential map identifies a neighborhood of \( N_{(h,x)} \) with a neighborhood of \( (h,x) \) in \( G_x \times Y_x \).

In order to prove that the stratification does not depend on the choice of Cartan subgroup \( T_{(h,x)} \), it will be helpful to observe the following, which is a slight strengthening of a special case of [BtD95, IV Prop. 4.6].

**Lemma 6.7.** Let \( G \) be a compact Lie group, let \( h \) and \( k \) be elements of a single connected component of \( G \), and let \( T_h \) and \( T_k \) be Cartan subgroups of \( G \) associated to \( h \) and \( k \), respectively. Then there is an element \( g \in G^0 \) such that

\[
\begin{align*}
gT_kg^{-1} &= T_h & \text{and} & \quad g(T_k^0)g^{-1} = hT_h^0.
\end{align*}
\]

**Proof.** By [BtD95, IV Prop. 4.6], we know a priori that \( T_h \) and \( T_k \) are conjugate and hence isomorphic. Let \( T_k \cong \mathbb{T}^\ell \times \mathbb{Z}/r\mathbb{Z} \) for some \( \ell \) and \( r \). Then the topological generators of \( \mathbb{T}^\ell \times \mathbb{Z}/r\mathbb{Z} \), i.e. the elements that generate a dense subset of \( \mathbb{T}^\ell \times \mathbb{Z}/r\mathbb{Z} \), are given by \( (s,\alpha) \) where \( s \) is a topological generator of \( \mathbb{T}^\ell \) and \( \alpha \) is a generator of \( \mathbb{Z}/r\mathbb{Z} \); see [BtD95, proof of IV Prop. 4.6]. Since \( kT_k^0 \) generates \( T_k/T_k^0 \cong \mathbb{Z}/r\mathbb{Z} \), we may therefore choose a topological generator \( t \) of \( T_k^0 \) such that \( t \in kT_k^0 \). Note that \( t, h, \) and \( k \) are all in the same connected component of \( G \). By the same argument a topological generator of \( T_h \) can be chosen to be an element of \( hT_h^0 \). Hence there is an element \( g \in G^0 \) such that \( gT_kg^{-1} \in hT_h^0 \). But then \( gT_kg^{-1} \) generates a subgroup of \( T_h \) that is dense in \( gT_kg^{-1} \). Therefore \( gT_kg^{-1} \) is contained in \( T_h \). But as \( gT_kg^{-1} \) is isomorphic to \( T_k \), hence to \( T_h \), we have \( gT_kg^{-1} = T_h \). \( \Box \)

We can now prove the following using an argument similar to [FPS15, Lemma 4.14].
Lemma 6.8. Let \((h, x) \in G \times M\). The set germ \(S_{(h, x)}\) does not depend on the choice of the Cartan subgroup \(T_{(h, x)}\).

Proof. Suppose \(T_{(h, x)}\) and \(T'_{(h, x)}\) are two Cartan subgroups of \(H\) associated to \(h\). Applying Lemma 6.7 with \(k = h\), there is a \(g \in H\) such that \(gT_{(h, x)}g^{-1} = T'_{(h, x)}\) and \(g(hT^\circ_{(h, x)})g^{-1} = h(T'_{(h, x)})^\circ\); as \(g \in H = G_{(h, x)}\), \(ghg^{-1} = h\).

Choose a slice \(Y_x\) at \(x\) for the \(G\)-action on \(M\). Then as \(g \in H \subseteq G_x\) so that \(g^{-1}Y_x = Y_x\), we have that for \(t \in T_{(h, x)}\), \((g^{-1}Y_x)^t = (g^{-1}Y_x)^h\) if and only if \(Y_x = Y_x^h\); i.e. \(t \sim h\) if and only if \(gtg^{-1} \simeq ghg^{-1}\). Therefore, \(gT_{(h, x)}g^{-1} = (T'_{(h, x)})^*\). Finally, as \(g \in H \subseteq G_x\), we have both \(gY_x = Y_x\) and \(g(M_H) = M_H\) so that

\[
g \left( T^*_{(h, x)} \times Y_x^{G_x} \right)_H = \left( (T'_{(h, x)})^* \times Y_x^{G_x} \right)_H,
\]

implying

\[
G \left( T^*_{(h, x)} \times Y_x^{G_x} \right)_H = G \left( (T'_{(h, x)})^* \times Y_x^{G_x} \right)_H
\]

and completing the proof. \(\square\)

It will be helpful to observe the following simple fact.

Lemma 6.9. Let \((h, x) \in A_0(G \times M)\) and \(H = G_{(h, x)}\). A subgroup \(T_{(h, x)}\) of \(G_x\) is a Cartan subgroup of \(G_x\) associated to \(h\) if and only if \(T_{(h, x)}\) is a Cartan subgroup of \(H\) associated to \(h\).

Proof. Let \(T_{(h, x)}\) be a Cartan subgroup of \(G_x\) associated to \(h\). As \(T_{(h, x)}\) is abelian and contains \(h\), it is a subgroup of \(Z_{G_x}(h) = H\). As \(T_{(h, x)}\) has finite index in \(N_{G_x}(T_{(h, x)})\), it obviously has finite index in \(N_H(T_{(h, x)}) \leq N_{G_x}(T_{(h, x)})\). The fact that \(T_{(h, x)}\) is closed and topologically cyclic and that \(T_{(h, x)}/T^\circ_{(h, x)}\) is generated by \(hT^\circ_{(h, x)}\) does not depend on the ambient group so that \(T_{(h, x)}\) is a Cartan subgroup of \(H\) associated to \(h\).

Conversely, if \(T_{(h, x)}\) is a Cartan subgroup of \(H\) associated to \(h\), then as noted in the previous paragraph, we need only show that \(T_{(h, x)}\) has finite index in \(N_{G_x}(T_{(h, x)})\). Following the proof of [BtD95] IV Prop. 4.2, note that as \(T_{(h, x)}\) is isomorphic to the product of a torus and a finite cyclic group, its automorphism group is discrete. As \(N_{G_x}(T_{(h, x)})/Z_{G_x}(T_{(h, x)})\) acts on \(T_{(h, x)}\) as automorphisms and is compact, it must be finite. Moreover, \(Z_{G_x}(T_{(h, x)})\) is clearly contained in \(Z_{G_x}(h) = H\) and hence in \(N_H(T_{(h, x)})\) so that as \(T_{(h, x)}\) has finite index in \(N_H(T_{(h, x)})\), the index \([Z_{G_x}(T_{(h, x)}): T_{(h, x)}]\) is finite. Therefore,

\[
[N_{G_x}(T_{(h, x)}): T_{(h, x)}] = [N_{G_x}(T_{(h, x)}): Z_{G_x}(T_{(h, x)})][Z_{G_x}(T_{(h, x)}): T_{(h, x)}]
\]

is finite, completing the proof. \(\square\)

Recall that the equivalence relation \(\simeq\) on a Cartan subgroup \(T\) of the isotropy group \(G_x\) is defined by setting \(s \simeq t\) if and only if \(N^s = N^t\), where \(N_x\) denotes the normal space \(T_x M/T_x(G_x)\). We denote by \([s]\) the \(\simeq\) class of \(s \in T\). Note that by definition of a slice, \(s \simeq t\) holds true if and only \(Y^s_x = Y^t_x\) for one, hence all (sufficiently small) slices \(Y_x\) at \(x\). We recall the following properties of the relation \(\simeq\), whose proofs are similar to [FFS15] Lemmata 4.7, 4.8, & 4.10.

Proposition 6.10. Let \(Y\) be a slice for the \(G\)-action on \(M\) through a point \(x \in M\) and let \(T \subset G_x\) be a Cartan subgroup. Then the following holds true.
The group \( T \) is partitioned into a finite number of \( \simeq \) classes, each with a finite number of connected components. Each \( \simeq \) class \([t]\) is an open subset of the closed subgroup \( t^* \) of \( T \) defined by

\[
t^* := \bigcap_{t \in H_i} H_i = \bigcap_{y \in Y^t} T_y,
\]

where \( \{H_0, \ldots, H_r\} \) is the finite set of isotropy groups for the \( T \)-action on \( Y \) and \( T_y \) is the isotropy group of \( y \) in \( T \). The closure \([t]\) of \([t]\) consists of a union of connected components of \( t^* \).

If \( s, t \in T \) with \([s] \cap [t] \neq \emptyset \), then for each connected component \([s]^0\) of \([s]\) and \([t]^0\) of \([t]\) such that \([s]^0 \cap [t]^0 \neq \emptyset \), we have \([s]^0 \subset [t]^0\).

If \( s, t \in T \) such that \( s \not\in t \) and \([s]\) is diffeomorphic to \([t]\), then \([s] \cap [t] = \emptyset\).

Proof. The \( \simeq \) class \([t]\) of \( t \in T \) is given by

\[
[t] = \bigcap_{t \in H_i} H_i \cap \left( \bigcup_{t \in H_j} H_j \right)^c
\]

and hence determined by a subset of \( \{0, 1, \ldots, r\} \), yielding (1).

If \( u \in [s]^0 \cap [t]^0 \), then \( Y^s = Y^u \), and by continuity of the action, \( Y^t \subseteq Y^u \). Hence \( Y^t \subseteq Y^s \), and \( s^* = \bigcap_{y \in Y^s} T_y \subseteq \bigcap_{y \in Y^t} T_y = t^* \). As \([s]^0\) is contained in a connected component \((s^*)^0\) of \( s^*\), which is contained in a connected component \((t^*)^0\) of \( t^*\), and as \([t]\) consists of entire connected components of \( t^*\), we have \([s]^0 \subseteq (t^*)^0 = [t]^0\), proving (2).

For (3), if \( Y^s \subseteq Y^t \) and \( Y^t \not\subset Y^s \), then let \( l \in [s] \cap [t] \). By continuity of the action, \( l \) fixes \( Y^s \cup Y^t \neq Y_s \) so that \( l \not\in s \). So \( l \in [s] \setminus [s] \), and \([s] \cap [t] = \emptyset\).

We now demonstrate that \( S_{(h,x)} \) and \( R_{(h,x)} \) are the germs of smooth manifolds, see Appendix A. The proof follows that of [FPS15, Prop. 4.16].

**Proposition 6.11.** Each \( S_{(h,x)} \) is the germ of a smooth \( G \)-submanifold of \( G \times M \) that intersects \( G_x \times Y_x \) as a smooth submanifold. Each \( R_{(h,x)} \) is the germ of a smooth submanifold of the differentiable space \( G \setminus \langle G \times M \rangle \) that intersects \( G_x \setminus \langle G_x \times Y_x \rangle \) as a smooth submanifold.

**Proof.** Recall that \( G \times M \) is equipped with an invariant riemannian metric given by the product of metrics on \( M \) and \( G \) and that \( Y_x \) is the image under the exponential map for \( M \) of an open ball \( B_\epsilon \) about the origin in the normal space \( N_x = T_x M / T_x(Gx) \). Let \( N_{(h,x)} = T_{(h,x)}(G_x \times Y_x) / T_{(h,x)}(G_{(h,x)}) \), the normal space in \( T_{(h,x)}(G_x \times Y_x) \) to the tangent space of the \( G_x \)-orbit of \((h,x)\), let \( B_{(h,x)} \subset N_{(h,x)} \) be an open \( H \)-invariant neighborhood of the origin, and let \( V_{(h,x)} \) be the image of \( B_{(h,x)} \) under the exponential map. As \( H \times Y_x \) is a slice at \((h,x)\) for the \( G_x \)-action on \( G_x \times Y_x \), \( V_{(h,x)} \) is an open neighborhood of \((h,x)\) in \( H \times Y_x \). By [Mic96, Sec. 6.1(4)], the exponential map \( B_x \rightarrow Y_x \) maps \( B_x^G \) onto \( Y_x^G \), and the exponential map \( B_{(h,x)} \rightarrow V_{(h,x)} \) maps \((B_{(h,x)})_H = B_{(h,x)}^H \) onto \((V_{(h,x)})_H = V_{(h,x)}^H \). As \( T_{(h,x)}^* \) is an open subset of the closed subgroup \( h^* \) by Proposition 6.10(4), we have
$T_h^* T_{(h,x)} = T_h h^*$. Hence, the preimage of $V_{(h,x)} \cap (T_{(h,x)}^* \times Y_{x}^{G_x})^H$ under the exponential map is given by $(T_h h^* \times B_{G_x}^*) \cap B_{(h,x)}^H$. As $T_h h^*$, $B_{G_x}^*$, and $B_{(h,x)}^H$ are open subsets of the origin in linear spaces, it follows that $S_{(h,x)}$ is the germ of a smooth manifold.

As $V_{(h,x)}$ is a slice at $(h,x)$ for the $G_x$-action on $G_x \times Y_x$, there is a $G_x$-equivariant diffeomorphism $G_x \times H V_{(h,x)} \to G_x V_{(h,x)} \subset G_x \times Y_x$ that restricts to a $G_x$-equivariant diffeomorphism

$$G_x / H \times V_{(h,x)} \cap (T_{(h,x)}^* \times Y_{x}^{G_x})^H \to G_x \left( V_{(h,x)} \cap (T_{(h,x)}^* \times Y_{x}^{G_x})^H \right)$$

and induces a homeomorphism on the quotients by $G_x$. The $G_x$-invariant functions on $G_x / H \times V_{(h,x)} \cap (T_{(h,x)}^* \times Y_{x}^{G_x})^H$ coincide with the smooth functions on $V_{(h,x)} \cap (T_{(h,x)}^* \times Y_{x}^{G_x})^H$ by [DS87] Prop. 5.2 and [NGStS03] Thm. 1.2.2(5) making this homeomorphism into an isomorphism of differentiable spaces from the smooth manifold $V_{(h,x)} \cap (T_{(h,x)}^* \times Y_{x}^{G_x})^H$ onto a subspace of the differentiable space $G_x \setminus (G_x \times Y_x)$. Finally, it is easy to see that the inertia space $|A(G_x \times Y_x)| \subset G_x \setminus (G_x \times Y_x)$ is isomorphic as a differentiable space to an open neighborhood of $G(h,x)$ in the inertia space $|A(G \times M)|$, see [FPS15] Prop. 3.6, and that the germ of the image of $V_{(h,x)} \cap (T_{(h,x)}^* \times Y_{x}^{G_x})^H$ in $|A(G \times M)|$ at $G(h,x)$ is $R_{(h,x)}$, completing the proof.

In order for the germs $S_{(h,x)}$ to define a stratification, one must verify that for each $(h,x) \in A_0(G \times M)$ there is a neighborhood $U$ in $A_0(G \times M)$ and a decomposition $Z$ of $U$ as defined in Appendix A.2 such that for all $(k,y) \in A_0(G \times M)$, the germ $S_{(k,y)}$ coincides with the germ of the piece of $Z$ containing $(k,y)$. For the remainder of this section, we fix $(h,x)$, a slice $Y_x$ at $x$ for the $G_x$-action on $M$, and a slice $V_{(h,x)}$ at $(h,x)$ for the $G_x$-action on $G_x \times Y_x$ given by an open neighborhood of $(h,x)$ in $H \times Y_x$ that is the exponential image of a ball in the normal space $B_{(h,x)}$ as in the proof of Proposition 6.11. Set $U := GV_{(h,x)} \cap A_0(G \times M)$. Note that $U$ is indeed an open neighborhood of $(h,x)$ in $A_0(G \times M)$, as $G_x V_{(h,x)}$ is an open $G_x$-invariant neighborhood of $(h,x)$ in $A_0(G_x \times Y_x)$ and so the $G$-saturation is as well by [FPS15] Prop. 3.6. We now define the decomposition $Z$ of $U$.

Given $(k,y) \in U$ there is a $\tilde{g} \in G$ such that $\tilde{g}(k,y) \in V_{(h,x)}$. Put $(k,y) = \tilde{g}(k,y)$ and $K = G_{(k,y)} \leq H$, and let $T_{(k,y)}$ be a Cartan subgroup in $K$ associated to $k$. Define $U_{\tilde{g}}^{T_{(k,y)}}(k,y)$ to be the $G$-saturation of the set of points $(l,z) \in (V_{(h,x)})_K \cap (T_{(k,y)} \times Y_x)_{G_y}$ such that $T_{(k,y)}$ is also a Cartan subgroup of $K$ associated to $l$ and such that the $\simeq$ class of $l$ at $z$ in $T_{(k,y)}$ is diffeomorphic to $T_{(k,y)}^*$. Define the piece $Z$ containing $(k,y)$ to be the connected component of $U_{\tilde{g}}^{T_{(k,y)}}(k,y)$ containing $(k,y)$.

By [Sch80] Proposition 1.2(3), the slice representations of points in the same orbit type are isomorphic. Hence, if $z \in (Y_x)_{G_y}$, then the slice representations $N_y$ and $N_z$ for the action of $G_x$ on $Y_x$ at $y$ and $z$, respectively, are isomorphic as $G_y$-representations. This isomorphism induces an isomorphism of $T_{(k,y)}$-representations, which induces a diffeomorphism of $\simeq$ classes at $y$ onto $\simeq$ classes at $z$. Then the $\simeq$ class of $l$ at $y$ is diffeomorphic to the $\simeq$ class of $l$ at $z$, and hence to the $\simeq$ class of $k$ at $y$. Therefore, the set $U_{\tilde{g}}^{T_{(k,y)}}(k,y)$ can be written as

$$U_{\tilde{g}}^{T_{(k,y)}}(k,y) = G \left( (V_{(h,x)})_K \cap (T_{(k,y)}^* \times (Y_x)_{G_y}) \right).$$
where $T^*_{(k,y)}$ denotes the union of $\simeq$ classes in $T_{(k,y)}$ that are diffeomorphic to $T^*_{(k,y)}$. Choosing representatives $k_0, \ldots, k_r$ from the collection of such $\simeq$ classes with $k_0 = k$ and noting that this collection is finite by Proposition 6.10, we can express

\begin{equation}
\mathcal{U}^T_{(k,y)}(\tilde{k}, \tilde{y}) = G\left(\bigcup_{i=0}^{r}(V(h,x))_K \cap (T^*_{(k_i,y)} \times (Y_x)_{G_y})\right).
\end{equation}

In particular, it will be clear below that for every $(k, y) \in V(h,x)$ with isotropy group $K$ and every Cartan subgroup $T_{(k,y)}$ of $K$ associated to $k$, the connected component of $T_{(k,y)}$ containing $k$ is the only connected component that intersects the projection of $V(h,x)$ onto $G_x$. Note that $\mathcal{U}^T_{(k,y)}(\tilde{k}, \tilde{y})$ is clearly a subset of $\Lambda_0(G \rtimes M)$ as $T_{(k,y)} \leq G_y$.

We now demonstrate that $\mathcal{U}^T_{(k,y)}(\tilde{k}, \tilde{y})$ depends only on the orbit $G(k, y)$. The proof follows those of [PFS15, Lemmata 4.18 & 4.19], though is simpler in the situation at hand.

**Lemma 6.12.** The set $\mathcal{U}^T_{(k,y)}(\tilde{k}, \tilde{y})$ does not depend on the choice of $(\tilde{k}, \tilde{y})$, $\tilde{g}$, nor the Cartan subgroup $T_{(k,y)}$.

**Proof.** Suppose $g \in G$ such that $g(\tilde{k}, \tilde{y}) =: (k′, y′) \in V(h,x)$. Then $g\tilde{g}^{-1}(k, y) = (k′, y′) \in V(h,x) \subset G_x \times Y_x$. This implies first that $\tilde{h} := g\tilde{g}^{-1} \in G_x$ as $\tilde{h}y = y′$ so that $\tilde{h}Y_x \cap Y_x \neq \emptyset$, and then that $\tilde{h} \in H$ as $\tilde{h}V(h,x) \cap V(h,x) \neq \emptyset$. Then $G_{y′} \leq G_x$ so that $K′ := G_{(k′, y′)} = \tilde{h}K\tilde{h}^{-1} \leq H$.

Choose a Cartan subgroup $T_{(k′, y′)}$ of $K′$ associated to $k′$. Then $\tilde{h}T_{(k,y)}\tilde{h}^{-1}$ is as well a Cartan subgroup of $K′$ associated to $k′$, so by [BtD95, IV. Prop. 4.6], there is a $\tilde{k} \in K′$ such that $T_{(k,y)} = \tilde{k} \tilde{h}T_{(k,y)}\tilde{h}^{-1}$ $\tilde{k}^{-1}$. As conjugation by elements of $G_x$ is easily seen to map $\simeq$ classes to $\simeq$ classes and $\tilde{k}h(k,y) = \tilde{k}(k′, y′) = (k′, y′)$, we have $\tilde{k} \tilde{h}T_{(k,y)}\tilde{h}^{-1} = (T_{(k′, y′)})^*$ and hence $\tilde{k} \tilde{h}T_{(k,y)}\tilde{h}^{-1} = (T_{(k′, y′)})^{**}$. Then as $\tilde{k} \in K′$ so that $\tilde{k} \tilde{h}K\tilde{h}^{-1} \tilde{k}^{-1} = K′$ and, as $K′ \leq G_{y′}$, $\tilde{k} \tilde{h}G_y\tilde{h}^{-1} \tilde{k}^{-1} = G_{y′}$, it follows that

\[ \tilde{k} \tilde{h}\left((V(h,x))_K \cap (T^*_{(k,y)} \times (Y_x)_{G_y})\right) = (V(h,x))_{K′} \cap ((T_{(k′, y′)})^{**} \times (Y_x)_{G_{y′}}). \]

Hence, $\mathcal{U}^T_{(k,y)}(\tilde{k}, \tilde{y})$ does not depend on the choice of $(\tilde{k}, \tilde{y})$ nor $\tilde{g}$. Applying the same argument where $(k′, y′) = (k, y)$ so that $T_{(k′, y′)} = T_{(k,y)}$ is another choice of Cartan subgroup demonstrates that $\mathcal{U}^T_{(k,y)}(\tilde{k}, \tilde{y})$ does not depend on the choice of Cartan subgroup.

We may henceforth denote $\mathcal{U}^T_{(k,y)}(\tilde{k}, \tilde{y})$ simply as $\mathcal{U}(G(k, y))$ and let $\mathcal{U}_{(\tilde{k}, \tilde{y})}$ denote the connected component of $\mathcal{U}(k, y)$ containing $(\tilde{k}, \tilde{y})$. The partition $\mathcal{Z}$ of $U$ then can be written as

\begin{equation}
\mathcal{Z} = \{\mathcal{U}_{(\tilde{k}, \tilde{y})} \subset U \mid (\tilde{k}, \tilde{y}) \in U\}.
\end{equation}

In order to show that $\mathcal{Z}$ is a decomposition of $U$ as defined in Appendix A.2, we first demonstrate that it is finite. If $(k′, y′) \in U = GV(h,x)$ is chosen such that $y′$ has $G$-isotropy type $(G_y)$ (i.e. $G_{y′}$ is conjugate to $G_y$ in $G$), and $(k′, y′)$ has $G_y$-isotropy type $(K)$, then there is a $g \in G$ such that $gy′$ has $G$-isotropy group $G_y$ and $g(k′, y′)$ has $G_y$-isotropy group $K$. To see this, first choose $\tilde{g} \in G$ such that $\tilde{g}G_y\tilde{g}^{-1} = G_{y′} = G_y$. 


Then choose $\tilde{k} \in G_y$ such that $\tilde{k}gG(k',y')\tilde{g}^{-1}K = K$. As $\tilde{k} \in G_y$, $\tilde{k}g(k',y') = g(k',y')$ is the desired point. Then as the set $U(k,y)$ is determined by choosing one of the finitely many isotropy types $(G_y)$ for the $G$-action on $Y_x$, one of the finitely many isotropy types $(K)$ for the $G_y$-action on $G_x \times Y_y$, and one of the finitely many diffeomorphism classes of $\simeq$ classes in a choice of Cartan torus, there are finitely many set $U(k,y)$ in $U = GV(h,x)$. It remains to show that each $U(k,y)$ has finitely many connected components, which we will accomplish following the proof of [FPS15] Lemma 4.17 & 4.21, i.e. by giving a description of $U(k,y)$ as the $G$-saturation of a semialgebraic set.

Fix a $U(k,y)$ with $(k,y) \in V(h,x)$ and let $K = G(k,y)$. As $(N_x)_{G_y}$ is formed by removing from $(N_x)^{G_y}$ the finite collection of isotropy types it properly contains, and as each isotropy type is semialgebraic by [Bre80] Theorem A, it follows that $(N_x)_{G_y}$ and hence $(Y_x)_{G_y}$ is semialgebraic for the same reason. As the actions of $G_x$ on $N_x$ and $H$ on $N(h,x)$ are linear, and as $N(h,x)$ is a quotient of a product with $N_x$ so that the linear structures are compatible, both $(N_x)_{G_y}$ and $(N_h)_{G_y}$ are closed under multiplication by scalars $t \in (0,1]$. As $K^0 \leq H^0$, any maximal torus in $K^0$ is contained in a maximal torus in $H^0$. Moreover, as $V^K(h,x)$ is connected and closed under multiplication by scalars $t \in (0,1]$, taking the limit of $t(k,y)$ as $t \to 0$, we see that $h \in K$, and in particular is in the connected component of $K$ containing $k$. Following the proof of [BD95] IV. Prop. 4.2, we may then choose a Cartan subgroup of $H$ associated to $h$ by taking the group generated by $h$ as well as a maximal torus in $H^0$ that contains a maximal torus in $K^0$. That is, we may assume that $h \in T(k,y) \leq T(h,x)$ and $T(k,y) = T(h,x) \cap K$.

Now, a slice $Y_k$ at $y$ for the $G$-action on $M$ is isomorphic as a $G_y$-space to a $G_y$-invariant subset of $Y_x$ by [Bre72] II. Corollary 4.6. Hence, the $\simeq$ classes in $T(k,y)$ (defined in terms of the action on $Y_y$) are unions of $\simeq$ classes in $T(h,x)$ (defined in terms of the action on $Y_x$) intersected with $T(k,y)$. By Proposition 6.10 $T(h,x)$ contains a finite number of $\simeq$ classes, each an open subset of a closed subgroup of $T(h,x)$. Hence, we may assume that $V(h,x)$ is small enough to only intersect $\simeq$ classes of $T(h,x)$ whose closures contain $h$. Then the exponential map associated to the product metric on $G_x \times Y_x$ maps the subset

$$\text{(6.8)} \quad (N(h,x))_K \cap (T_h(k^*) \oplus (N_x)_{G_y}) \cap B(h,x)$$

onto $(V(h,x))_K \cap (T^{(k,y)} \times (Y_x)_{G_y})$. Then $\text{(6.8)}$ is a semialgebraic subset of $N(h,x)$, and is invariant under the action of $t \in (0,1]$. Because there are only finitely many $\simeq$ classes in $T(k,y)$, there are $l_1,\ldots,l_N \in T(k,y)$ such that each group $l_j^*$, $j = 1,\ldots,N$, has dimension less than dim $k^*$, and

$$T^{(k,y)} = \bigcup_{j=1}^N l_j^*.$$

Then the exponential function maps the semialgebraic set

$$\text{(6.9)} \quad (N(h,x))_K \cap \left( (T_h(k^*) \setminus \bigcup_{j=1}^N T_h(l_j^*)) \oplus (T_x Y_x)_{G_y} \right) \cap B(h,x)$$

onto $(V(h,x))_K \cap (T^{*}(k,y) \times (Y_x)_{G_y})$. Finally, $(V(h,x))_K \cap (T^{*}(k,y) \times (Y_x)_{G_y})$ is given by the finite union of such sets over $\simeq$ classes diffeomorphic to $T^{*}(k,y)$. This set, and hence its $G$-saturation $U(k,y)$, has finitely many connected components by [BCR98] Thrm. 2.4.5. We have hence demonstrated the following.
Lemma 6.13. The partition $Z$ of $U = GV_{(h,x)}$ given by Equation (6.9) is finite.

Restricting the inverse of the exponential map from $V_{(h,x)}$ to $V_{(h,x)} \cap A_0(G_x \ltimes Y_x)$ yields an $H$-equivariant embedding $\iota$ of a neighborhood of $(h,x)$ in $V_{(h,x)} \cap A_0(G_x \ltimes Y_x)$ into the normal space $N_{(h,x)}$, where the stratum of $(h,x)$ is mapped onto the subspace

$$N^H_{(h,x)} \cap \left( T_h h^* \times (T_x Y_x)^G_x \right).$$

Using the description of the image of each stratum given in Equation (6.9), one sees immediately that the homotopy defined as multiplication in $\iota$ follows that $g$ have further that to immediately that the homotopy defined as multiplication in $N_{(h,x)}$ by scalars $t \in [0,1]$ contracts the image of $\iota$ onto the origin preserving the image of $\iota$. The linearity of the $H$-action on $N_{(h,x)}$ ensures that this homotopy is $H$-equivariant, and scalars $t \in (0,1]$ preserve the images of strata as demonstrated above. We therefore have proven the following.

Proposition 6.14. The inertia groupoid $\Lambda G$ of a proper Lie groupoid $G$ satisfies the local contractibility condition (LC) of Definition 5.7.

The proof of the following is similar to that of [FPS15] Prop. 4.20.

Proposition 6.15. The germs of the $U(G(k,y))$ coincide with the stratification. That is, for $(\tilde{k}, \tilde{y}) \in U = GV_{(h,x)}$, the germs $[U(G(k,y))]((\tilde{k}, \tilde{y}))$, $[U(g,\tilde{y})]((\tilde{k}, \tilde{y}))$ and $S_{(\tilde{k}, \tilde{y})}$ coincide.

Proof. As $S_{(\tilde{k}, \tilde{y})}$ and $U(G(k,y))$ depend only on the orbit of $(\tilde{k}, \tilde{y})$ we assume $(\tilde{k}, \tilde{y}) = (k, y) \in V_{(h,x)}$ and let $K = G_{(k,y)} \leq H$. Fix a slice $Y_g$ at $y$ for the $G_x$-action on $Y_x$, and then $Y_g$ is a slice at $y$ for the $G$-action on $M$ by [Bre72, II. Corollary 4.6]. Note that $Y_g$ may not be the image of $N_y$ under the exponential map, though it is diffeomorphic to an open subset $B_y \subset N_y$ as a $G_y$-space. Then a slice $V_{(k,y)}$ for the $G_y$-action on $G_y \times Y_y$ is given by a neighborhood of $(k, y)$ in $K \times Y_y$ by [DK00] Proposition (3.1.1)(i)]. As $Y_y \subset Y_x$, we may assume by shrinking $V_{(k,y)}$ if necessary that $V_{(k,y)} \subset V_{(h,x)}$, and as well that $V_{(k,y)}$ does not intersect the product of $M$ with any connected component of $K$ except $kK^o$. Choose a Cartan subgroup $T_{(k,y)}$ of $K$ associated to $k$, and then we may further shrink $V_{(k,y)}$ to assume by Proposition 6.14 that $V_{(k,y)}$ does not intersect (the product of $M$ with) the closure of any of the finitely many $\simeq$ classes of $T_{(k,y)}$ diffeomorphic but not equal to $T_{(k,y)}$. Let $Q := GV_{(k,y)}$; we claim that

$$U(G(k,y)) \cap Q = G \left( V^K_{(k,y)} \cap (T^*_{(k,y)} \times Y^G_{(k,y)}) \right).$$

Let $(\tilde{l}, \tilde{z}) \in U(G(k,y)) \cap Q$. Then there is a $g' \in G$ such that $(l, z) := g'(\tilde{l}, \tilde{z}) \in (V_{(h,x)})_K \cap (T^*_{(k,y)} \times (Y_x)_G)$, and in particular $T_{(k,y)}$ is a Cartan subgroup of $K$ associated to $l$. As $(l, z) \in Q$, there is a $g \in G$ such that $g(l, z) \in V_{(k,y)}$. Because $z \in Y_x$ and $gz \in Y_y \subset Y_x$, we have $g \in G_x$. Then as $(l, z) \in V_{(h,x)}$ and $g(l, z) \in V_{(k,y)} \subset V_{(h,x)}$, we have further that $g \in H$. Moreover, as $G_{(l,z)} = K$ and $g(l, z) \in V_{(k,y)}$, so that $G_{g(l,z)} \leq K$, it follows that $G_{g(l,z)} = K$, i.e. $g(l, z) \in V^K_{(k,y)}$. In the same way, as $G_z = G_y$, and as $V_{(k,y)} \subset G_y \times Y_y$ so that $gz \in Y_y$, we have $G_{gz} = G_y$, i.e. $g(l, z) \in V^K_{(k,y)} \cap (K \times Y^G_{(k,y)})$. Finally, as $l \in T^*_{(k,y)}$ and as $g^{-1}g$ is in $V_{(k,y)}$ and hence the same connected component of $K$ as $k$, we may by Lemma 6.7 conjugate further by a $\tilde{k} \in K \leq G_y$, which fixes $V^K_{(k,y)}$ and $Y^G_{(k,y)}$, so that $\tilde{k}g(l, z) \in V^K_{(k,y)} \cap (T^*_{(k,y)} \times Y^G_{(k,y)})$. But as $V_{(k,y)} \cap T^*_{(k,y)} \subset T^*_{(k,y)}$, $\tilde{k}g(l, z) \in V^K_{(k,y)} \cap (T^*_{(k,y)} \times Y^G_{(k,y)})$. 

Let $\tilde{k}g(l, z) \in V^K_{(k,y)} \cap (T^*_{(k,y)} \times Y^G_{(k,y)})$. 


Conversely, if \((\tilde{j}, \tilde{w}) \in G \left( V^K_{(k, y)} \cap (T^*_{(k, y)} \times Y^G_y) \right)\), then there is a \(\tilde{g} \in G\) such that 
\((j, w) := \tilde{g}(\tilde{j}, \tilde{w}) \in V^K_{(k, y)} \cap (T^*_{(k, y)} \times Y^G_y)\). Then as \(V_{(k, y)} \subseteq V_{(h, x)}\) and \(Y_y \subseteq Y_x\), we have 
\((j, w) \in (V_{(h, x)})_K \cap (T^*_{(k, y)} \times (Y_x)_G),\) and so \((j, w) \in \mathcal{U}(G(k, y))\).

Finally, restricting to a neighborhood of \((k, y)\) that only intersects one connected component of \(\mathcal{U}(G(k, y))\), we see that the germ of \(\mathcal{U}(k, y)\) at \((k, y)\) coincides with \(S_{(k, y)}\).

Since the \(S_{(h, x)}\) are germs of smooth \(G\)-submanifolds of \(G \times M\), and the piece associated to a point \((\tilde{k}, \tilde{y}) \in U\) has the same set germ as \(S_{(l, z)}\) at \((l, z) \in \mathcal{U}(\tilde{k}, \tilde{y})\), it follows that the pieces of \(\mathcal{Z}\) are smooth submanifolds of \(G \times M\) invariant under the \(G\)-action. We now verify that \(\mathcal{Z}\) is a decomposition indeed. The proof is similar to that of \[FPS15\] Prop. 4.22.

**Proposition 6.16.** The decomposition \(\mathcal{Z}\) satisfies the condition of frontier.

**Proof.** Suppose there are points \((h, x)\) and \((k, y)\) with \(\mathcal{U}(G(h, x)) \cap \overline{\mathcal{U}(G(k, y))} \neq \emptyset\). As the pieces of \(\mathcal{Z}\) are defined to be connected components, it is sufficient to show that \(\mathcal{U}(G(h, x)) \cap \overline{\mathcal{U}(G(k, y))}\), which is obviously closed in \(\mathcal{U}(G(h, x))\), is also open in \(\mathcal{U}(G(h, x))\). Note that we can assume with no loss of generality that one of the points in question is \((h, x)\), the point used to define \(U\), as we may restrict consideration to a neighborhood of that point. Moreover, as the piece \(\mathcal{U}(G(h, x))\) may be defined in terms of any point it contains, we may assume that \(G(h, x) \subset U\) in \(\mathcal{U}(G(h, x)) \cap \overline{\mathcal{U}(G(k, y))}\). Similarly, we assume by choosing another representative of the orbit if necessary that \((k, y) \subset V_{(h, x)}\). By Proposition 6.13 an open neighborhood of \((h, x)\) in \(\mathcal{U}(G(h, x))\) is given by \(G(V^H_{(h, x)} \cap (T^*_{(h, x)} \times Y^G_x))\) for a sufficiently small slice \(V_{(h, x)}\) at \((h, x)\). We will show that \(G(V^H_{(h, x)} \cap (T^*_{(h, x)} \times Y^G_x))\) is contained in \(\mathcal{U}(G(k, y))\).

Let \(K := G(k, y) \leq H\). As explained in the proof of Lemma 6.13 (before the statement of the Lemma), \(h\) is in the connected component of \(K\) containing \(k\), and we may choose Cartan subgroups \(T_{(h, x)}\) and \(T_{(k, y)}\) such that \(h \in T_{(k, y)} \leq T_{(h, x)}\) and \(T_{(k, y)} = T_{(h, x)} \cap K\). Then we have \(h \in T^*_{(k', y)}\) for some \(k'\) whose \(\simeq\) class at \(y\) is diffeomorphic to \(T^*_{(k, y)}\).

Similarly, as \(G_y \leq G_x\), it follows that \(Y^G_x \subseteq (Y_x)_G\). From these observations, we have

\[(h, x) \in V^H_{(h, x)} \cap (T^*_{(k', y)} \times Y^G_x) \subset (V_{(h, x)})_K \cap (T^*_{(k', y)} \times (Y_x)_G)\]

Now, let \(l \in T^*_{(h, x)}\) so that \(Y^l_x = Y^h_x\). In particular, as \(h \in K \leq G_y\) and \(y \in Y_x\), it follows that \(l \in G_y\). Similarly, as \(l \in T_{(h, x)}\), \(k \in T_{(k, y)} \leq T_{(h, x)}\), and as \(T_{(h, x)}\) is abelian, we have \(l(k, y) = (k, y)\) so that \(l \in K\). In particular, \(l \in T_{(h, x)} \cap K = T_{(k, y)}\). This demonstrates \(T^*_{(h, x)} \subset T_{(k, y)}\). As \(Y_y \subset Y_x\), we have that the relation \(\simeq\) at \(x\) implies \(\simeq\) at \(y\), so that the \(\simeq\) classes at \(y\) are the intersection with \(T_{(k, y)}\) of a (finite) union of \(\simeq\) classes at \(y\). That is, using Proposition 6.10 (2), \(T^*_{(h, x)} \subset T^*_{(k', y)}\). Then

\[V^H_{(h, x)} \cap (T^*_{(h, x)} \times Y^G_x) \subset (V_{(h, x)})_K \cap (T^*_{(k', y)} \times (Y_x)_G) \subset \mathcal{U}(G(k, y))\]

Considering the \(G\)-saturations of both sides of this inclusion, it follows that an open neighborhood of \((h, x)\) in \(\mathcal{U}(G(k, x))\) is contained in \(\mathcal{U}(G(h, x)) \cap \mathcal{U}(G(k, y))\). This completes the proof. \(\square\)
6.4. Proof of Theorem 6.6 Let $G$ be a proper Lie groupoid. We now demonstrate that the stratification given by Equation \( 6.4 \) is well-defined, i.e. that it does not depend on the choice of a point in an orbit, a slice at that point, nor the riemannian metric.

**Proposition 6.17.** Let $G$ be a proper Lie groupoid, and let $x, y \in G_0$ be points in the same orbit. Let $g \in G_1$ such that $s(g) = t(g) = x$, and let $h \in G_1$ such that $s(h) = x$ and $t(h) = y$. Put $g' := hgh^{-1}$. Then $S_g = S_{g'}$. In particular, $S_g$ does not depend on the choice of a slice $Y_x$ and hence does not depend on the choice of transversally invariant metric.

**Proof.** Choose slices $Y_x$ and $Y_y$ for $G$ at $x$ and $y$, respectively. Then there are identifications $G|_{Y_x} \cong G_x \ltimes Y_x$ and $G|_{Y_y} \cong G_y \ltimes Y_y$ by \[\text{Proposition 3.1}.\] Under these identifications let $g = (h, x)$ and $g' = (k, y)$ for some $h \in G_x$ and $k \in G_y$. Choose a local bisection $\sigma: U \to G_1$ defined on an open neighborhood of $x$ such that $\sigma(x) = h$ and such that $t \circ \sigma|_{Y_x}$ induces a diffeomorphism from $Y_x$ to $Y_y$. The existence of such a bisection, after possibly shrinking $Y_x$ and $Y_y$, is guaranteed by \[\text{Proposition 3.9 \& proofs of Lemmata 5.1 \& 5.2}.\] Then we obtain an isomorphism $\Psi: G_x \ltimes Y_x \to G_y \ltimes Y_y$ which is given by the composition of the diffeomorphism

$$G|_{Y_x} \to G|_{Y_y}, \quad k \mapsto (\sigma(t(k))) k(\sigma(s(k)))^{-1}$$

with the identifications $G|_{Y_x} \cong G_x \ltimes Y_x$ and $G|_{Y_y} \cong G_y \ltimes Y_y$. Note that $\Psi$ obviously restricts to a diffeomorphism from $A_0(G_x \ltimes Y_x)$ onto $A_0(G_y \ltimes Y_y)$. Moreover, by construction, $\Psi(h, x)$ is the image of

$$(\sigma(x)) g(\sigma(x))^{-1} = hgh^{-1} = g'$$

under the identification $G|_{Y_y} \cong G_y \ltimes Y_y$, hence $\Psi(h, x) = (k, y)$. Now we choose a Cartan subgroup $T_{(h, x)}$ of $Z_G(h)$ associated to $h$. Then as $\Psi$ restricts to an isomorphism of the isotropy group in $G$ of $x$ onto the isotropy group of $y$, $\Psi(T_{(h, x)} \times \{x\})$ a Cartan subgroup of $Z_G(k)$ associated to $k$. Moreover, we claim that $\Psi(T_{(h, x)} \times \{x\}) = T_{(k, y)} \times \{y\}$. To see this, note that $\Psi_0: Y_x \to Y_y$ is a diffeomorphism that is $T_{(h, x)}$-equivariant with respect to the isomorphism $\tau: T_{(h, x)} \to T_{(k, y)}$ given by $\tau(s) = \pi_1 \circ \Psi_0(s, x)$, where $\pi_1$ denotes the projection $\pi_1: G \to G_y \cong G_y$. Then for $z \in Y_y$ and $s \in T_{(h, x)}$, $\tau(s)z = \tau(s)(\Psi_0 \circ \Psi_0^{-1}(z)) = \Psi_0(s(\Psi_0^{-1}(z)))$. Hence $\tau(s)z = z$ if and only if $s(\Psi_0^{-1}(z)) = \Psi_0^{-1}(z)$, from which it follows that $\Psi_0$ maps the set of points of $Y_x$ fixed by $s \in T_{(h, x)}$ onto the fixed set of $\tau(s)$ in $Y_y$. The isomorphism $\Psi$ then maps $S_{(h, x)}$ onto $S_{(k, y)}$. This proves the claim. If $x = y$, this argument shows that the stratification does not depend on the choice of the slice $Y_x$, hence is also independent on the choice of a transversally invariant riemannian metric.

Given Theorem 6.4 and Proposition 6.17, it remains only to verify Whitney (b)-regularity; this will be done in the next section. Recall that Whitney (b)-regular stratified spaces are topologically locally trivial, see \[\text{Proposition 3.11}.\] That $G$ is a sliceable differentiable groupoid follows from \[\text{Proposition 3.9 \& Corollary 3.11}.\] Similarly, by the definition of the stratification of $G_0\ltimes Y_0$ in terms of stratifications of the inertia spaces of slices $G_x \ltimes Y_x$, $x \in G_0$, the inertia groupoid $AG$ is even a sliceable differentiable stratified groupoid. Moreover, since the elements of a stratum of a slice $Y_x$ all have the same $G_x$-orbit type,
the inertia space is a differentiable stratified space by Proposition 6.32, and the orbit map is a stratified surjective submersion.

6.5. Whitney (b)-regularity. Here, we complete the proof of Theorem 6.6 by demonstrating that the stratifications of \( A_0 \mathcal{G} \) and \(|A \mathcal{G}|\) are Whitney (b)-regular. The proof follows [HP01] Thm. 4.3.7 and [FPS15] Prop. 4.23. Roughly, the proof involves giving a parameterization of a neighborhood of a point in \( A \mathcal{G} \) and its tangent space sufficient to describe the secants of points in neighboring strata. Note that in our argument we use that the pieces satisfy the condition of frontier, which was shown above.

Proposition 6.18. Let \( \mathcal{G} \) be a proper Lie groupoid. The orbit Cartan type stratifications of the loop space \( A \mathcal{G} \) and the inertia space \(|A \mathcal{G}|\) are both Whitney (b)-regular.

Proof. Because the claim is local, we may assume that the groupoid \( \mathcal{G} \) is given by the product of \( O \times O \rightrightarrows O \) and \( G \times Y \) where \( G \) is the isotropy group \( \mathcal{G}_x \) of some point \( x \in \mathcal{G}_0 \). \( O \) is an open neighborhood of \( x \) in its orbit, and \( Y \) is a slice through \( x \). Let \((h,x) \in A \mathcal{G}, H = Z_G(h), \) and \( V(h,x) \) a slice at \((h,x)\) for the \( G \)-action on \( G \times Y \) of the form \( \exp(B(h,x)) \), where \( B(h,x) \) is a ball around the origin in the normal space \( N(h,x) \); recall that \( V(h,x) \) is then an open neighborhood of \((h,x)\) in \( H \times V \). Now let us denote by \( \mathcal{Z} \) the decomposition of \( A_0 \mathcal{G} \) obtained by taking the saturations of the sets defined through Eq. (6.6), which amounts to taking their products with \( O \). Let \( R \) be the piece of \( \mathcal{Z} \) containing \((h,x)\), i.e. the set of points of the form \(((o,o),(l,z))\) where \( o \in O \) and \((l,z) \in V(h,x) \cap (T(h,x) \times Y^G) \). We show that for any stratum \( S \in \mathcal{Z} \) with \((h,x) \in S\), the Whitney condition (b) is satisfied at \((h,x)\) for the pair of strata \((R,S)\). To describe the stratum \( S \) in some more detail, consider an orbit \( G(k,y) \) for \((k,y) \in S\). As in the proof of Proposition 6.16 we may choose the representative \((k,y)\) of the orbit \( G(k,y) \) such that \((k,y) \in V(h,x) \), \( h \in T(k,y) \leq T(h,x) \), and \( h \in T(k',y) \) for some \( k' \). In particular, we then we have \( K \leq H \) for the isotropy group \( K := Z_{G_y}(k) \) of \((k,y)\) and \( G_y \leq G \). As shown above, \( S \) coincides with the connected component of \( \mathcal{U}(G(k,y)) \) containing \((k,y)\).

Suppose now that \(((u_i,u_i),(h_i,x_i))_{i \in \mathbb{N}} \) is a sequence in \( R \) and \(((o_i,o_i),(k_i,y_i))_{i \in \mathbb{N}} \) a sequence in \( S \), and that both sequences converge to \(((x,x),(h,x))\). Assume in addition that in a smooth chart around \(((x,x),(h,x))\) the secant lines \( \ell_i = \left(((u_i,u_i),(h_i,x_i)),((o_i,o_i),(k_i,y_i))\right) \) converge to a straight line \( \ell \), and the tangent spaces \( T((o_i,o_i),(k_i,y_i))S \) converge to a subspace \( \tau \). Then we must show that \( \ell \subset \tau \).

Note that the hypotheses imply that \(((x,x),(h,x))\in \mathcal{U}(G(h,x)) \cap \mathcal{U}(G(k,y)) \). By the proof of Proposition 6.16 and the choices of \((k,y)\) and \( T(k,y) \subset K \) we obtain the relation

\[
V^H_{(h,x)} \cap \left(T^+_y(x,y) \times Y^G\right) \subset \left(V_{(h,x)} \cap \left(T^+_y(x,y) \times (Y)^G\right)\right).
\]

Denote by \( \mathfrak{g}_x \) the Lie algebra of \( G \), by \( \mathfrak{h} \) the Lie algebra of \( H \), and let \( \mathfrak{m} \) denote the orthogonal complement of \( \mathfrak{h} \) in \( \mathfrak{g}_x \) with respect to the initially chosen bi-invariant metric on \( G \). Then there is a neighborhood \( U \subset \mathcal{G}_0 \cong O \times H \) \( V(h,x) \) of \((h,x)\) such that

\[
\Psi : U \to O \times \mathfrak{m} \times N(h,x), \quad [o,\exp|_{\mathfrak{m}}(\xi),\exp(h,x)(v)] \mapsto (o,\xi,v)
\]

is a smooth chart at \(((x,x),(h,x))\), where \( \exp|_{\mathfrak{m}} \) denotes the restriction of the exponential map of the Lie group \( G \) to \( \mathfrak{m} \), and \( \exp(h,x) \) the exponential function restricted to the open
ball \( B_{(h,x)} \subset N_{(h,x)} \). After possibly shrinking \( U \) there is an open neighborhood \( Q \) of \( H \) in \( G \) such that
\[
\Psi \left( O \times Q \left( V^H_{(h,x)} \cap (T^*_{(h,x)} \times Y^G) \right) \right) \subset O \times m \times \left( N^H_{(h,x)} \cap T_{(h,x)}(T^*_{(h,x)} \times Y^G) \right).
\]
We may assume that the sequences \((u_i, x_i), (h_i, x_i))_{i \in \mathbb{N}}\) and \((o_i, o_i), (k_i, y_i))_{i \in \mathbb{N}}\) are contained in \( U \). Since \((o_i, o_i), (k_i, y_i)) \in U(G(k, y))\), one knows that
\[
\Psi((o_i, o_i), (k_i, y_i)) \in O \times m \times H \left( (V_{(h,x)})_{K} \cap \left( T^*_{(k,y)} \times Y^G \right) \right).
\]
Recall that \( T^*_{(k,y)} \) consists of a finite collection of diffeomorphic \( \simeq \) classes in \( T_{(k,y)} \). Moreover, by Lemma [4.1][4], each such \( \simeq \) class is disjoint from the closures of the other classes. By passing to a subsequence, we may assume without loss of generality that each \( k_i \) is in one fixed class, i.e.
\[
((o_i, o_i), (k_i, y_i)) \in O \times G \left( (V_{(h,x)})_{K} \cap \left( T^*_{(k,y)} \times Y^G \right) \right)
\]
for all \( i \) and some fixed \( k' \in T_{(k,y)} \).

Note that \( \lim o_i = \lim u_i = x \). Moreover, each piece of \( Z \) is a product of a piece in \( GV_{(h,x)} \subset G \times Y \). We may project onto \( G \times \) and ignore the \( O \)-factor. Choose \( \bar{l}_i \in G \) such that \((k_i, \bar{y}_i) := \bar{l}_i(k_i, y_i) \in (V_{(h,x)})_{K} \) for all \( i \in \mathbb{N} \). Put \((h_i, \bar{x}_i) := l_i(h_i, x_i)\). After possibly passing to a subsequence, \((\bar{l}_i)_{i \in \mathbb{N}}\) converges to some \( \bar{l} \in H \), the secant lines \( \bar{l}_i = (h_i, \bar{x}_i), (k_i, \bar{y}_i) \) converge to a straight line \( \bar{l} \), and the tangent spaces \( T_{(k_i, \bar{y}_i)} S \) converge to a subspace \( \bar{\tau} \). By definition, and since \( \bar{l}_iT_{(k_i, \bar{y}_i)} S = T_{(k_i, \bar{y}_i)} S \) for all \( i \), one obtains \( \bar{l} = \bar{l} \ell \) and \( \bar{\tau} = \bar{l} \tau \). Hence, the first claim is shown, if \( \bar{\ell} \subset \bar{\tau} \).

Without loss of generality we may therefore assume that for all \( i \in \mathbb{N} \)
\[
(6.11) \quad (k_i, y_i) \in (V_{(h,x)})_{K} \cap \left( T^*_{(k',y)} \times Y^G \right),
\]
and then show \( \bar{\ell} \subset \tau \) for the sequences \((k_i, y_i)_{i \in \mathbb{N}}\) and \((h_i, x_i)_{i \in \mathbb{N}}\).

Eq. (6.11) now means in particular that
\[
\Psi(k_i, y_i) \in \{0\} \times \left( (N_{(h,x)})_{K} \cap \exp^{-1}(h) \left( T^*_{(k',y)} \times Y^G \right) \right).
\]

Since \( \overline{T^*_{(k',y)}} \) is an open and closed subset of a closed subgroup of \( G \) and also contains \( h \), the set
\[
V := N_{(h,x)} \cap T_{(h,x)} \left( \overline{T^*_{(k',y)}} \right) \times Y^G
\]
is a subspace of \( N_{(h,x)} \). Let \( W \) be the orthogonal complement of the invariant space \( V^H \) in \( V \) with respect to the \( H \)-invariant scalar product induced from \( V_{(h,x)} \). Then the image under the chart \( \Psi \) of every element of \( G \left( V^H_{(h,x)} \cap (T^*_{(h,x)} \times Y^G) \right) \) \( \cup \) \( U \) and every \((k_i, y_i)\) is contained in
\[
m \times (W_K \cup \{0\}) \times V^H.
\]
With respect to this decomposition, \((h, x)\) has coordinates \((0, 0, 0)\), each element of \( G \left( V^H_{(h,x)} \cap (T^*_{(h,x)} \times Y^G) \right) \) has coordinates contained in \( m \times 0 \times V^H \), and each sequence element \((k_i, y_i)\) has coordinates contained in \( \{0\} \times W_K \times V^H \). In particular, let
\[
\Psi(k_i, y_i) = (0, w_i, v_i)
\]
for every $i$. Since $W_K$ is invariant under multiplication by non-vanishing scalars, we have

$$\lim_{i \to \infty} \frac{\Psi(k_i, y_i) - \Psi(h_i, x_i)}{\|\Psi(k_i, y_i) - \Psi(h_i, x_i)\|} \in m \times W_K \times V^H.$$ 

By compactness of the unit sphere in $W$, the sequence $w_i$ converges to some $w \in SW$ after possibly passing to a subsequence. Then $w = \|w\| \hat{w}$. Since $W_K$ is invariant by non-vanishing scalars, we have

$$m \times \text{span} \ w \times V^H \subset \tau,$$

and

$$\ell = \text{span} (\xi, \hat{w}, v) \subset \tau,$$

proving the first claim.

Now let us show that the orbit Cartan type stratification of $|AG|$ is Whitney (b)-regular as well. To this end let us first choose a Hilbert basis of $H$-invariant polynomials $p_1, \ldots, p_k: \left(N^H_{(h,x)}\right)^\perp \to \mathbb{R}$ of the orthogonal complement of the invariant space $N^H_{(h,x)}$ in $N_{(h,x)}$. Next let $p_{k+1}, \ldots, p_N: N^H_{(h,x)} \to \mathbb{R}$ with $N = k + \dim N^H_{(h,x)}$ be a linear coordinate system of the invariant space. We can even choose these $p_i$ in such a way that $p_{k+1, \ldots, p_{N+\dim V^H}$ is a linear coordinate system of $V^H$. By construction, $p_1, \ldots, p_N$ then is a Hilbert basis of the normal space $N_{(h,x)}$. Denote by $p: N_{(h,x)} \to \mathbb{R}^N$ the corresponding Hilbert map. Recall that $p$ induces a chart of $|AG|$ over $G \setminus U$ by

$$\hat{\Psi}: G \setminus U \to \mathbb{R}^N, \ G \exp_{h,x}(v) \mapsto p(v).$$

Note that by $H$-invariance of $p$ and since for every orbit in $U$ there is a representative in $V_{(h,x)}$, the chart $\hat{\Psi}$ is well-defined indeed. A decomposition of $\hat{U} := \hat{\Psi}(G \setminus U)$ inducing the orbit Cartan type stratification on $G \setminus U$ is given by

$$\hat{\mathcal{Z}} := \{ \hat{\Psi}(G \setminus (S \cap (G \times Y)) \mid S \in \mathcal{Z}) \}.$$

Let $\hat{S} \in \hat{\mathcal{Z}}$ denote the stratum containing the orbit $G(h,x)$, and $\hat{S} \in \mathcal{Z}$ a stratum $\neq \hat{R}$ such that $G(h,x)$ lies in the closure of $\hat{S}$. Now consider sequences of orbits $G(h_i, x_i)_{i \in \mathbb{N}}$ in $\hat{R}$ and $G(k_i, y_i)_{i \in \mathbb{N}}$ in $\hat{S}$ such that both sequences converge to $G(h,x)$. Moreover, assume that the sequence of secants $\overline{\hat{\Psi}(G(h_i, x_i)), \hat{\Psi}(G(k_i, y_i))}$ converges to a line $\hat{l}$, and that the sequence of tangent spaces $T_{\hat{\Psi}(G(k_i, y_i))} \hat{S}$ converges to some subspace $\hat{\tau} \subset \mathbb{R}^N$. Using notation from before, we can choose representatives $(h_i, x_i)$ and $(k_i, y_i)$ having coordinates in $m \times (W_K \cup \{0\}) \times V^H \subset N_{(h,x)}$ such that

$$\Psi(h_i, x_i) = (0, 0, v_i) \in \{0\} \times \{0\} \times V^H \quad \text{and}$$

$$\Psi(k_i, y_i) = (0, w_i, v_i) \in \{0\} \times W_K \times V^H.$$ 

Next observe that by the Tarski–Seidenberg Theorem, the stratum $\hat{S}$ is semialgebraic as the image of the semialgebraic set $(W_K \times V^H) \cap B_{(h,x)}$ under the Hilbert map $p$. By the same argument, $p(W_K)$ is semialgebraic, too, and an analytic manifold, since $p(W_K) \cong N_H(K) \setminus W_K \cong H \setminus W_K$. Moreover, the equality

$$\hat{S} = (p(W_K) \times V^H) \cap p(B_{(h,x)})$$

proving the second claim.

Let $\hat{S} \in \hat{\mathcal{Z}}$ denote the stratum containing the orbit $G(h,x)$, and $\hat{S} \in \mathcal{Z}$ a stratum $\neq \hat{R}$ such that $G(h,x)$ lies in the closure of $\hat{S}$. Now consider sequences of orbits $G(h_i, x_i)_{i \in \mathbb{N}}$ in $\hat{R}$ and $G(k_i, y_i)_{i \in \mathbb{N}}$ in $\hat{S}$ such that both sequences converge to $G(h,x).$ Moreover, assume that the sequence of secants $\overline{\hat{\Psi}(G(h_i, x_i)), \hat{\Psi}(G(k_i, y_i))}$ converges to a line $\hat{l}$, and that the sequence of tangent spaces $T_{\hat{\Psi}(G(k_i, y_i))} \hat{S}$ converges to some subspace $\hat{\tau} \subset \mathbb{R}^N$. Using notation from before, we can choose representatives $(h_i, x_i)$ and $(k_i, y_i)$ having coordinates in $m \times (W_K \cup \{0\}) \times V^H \subset N_{(h,x)}$ such that

$$\Psi(h_i, x_i) = (0, 0, v_i) \in \{0\} \times \{0\} \times V^H \quad \text{and}$$

$$\Psi(k_i, y_i) = (0, w_i, v_i) \in \{0\} \times W_K \times V^H.$$ 

Next observe that by the Tarski–Seidenberg Theorem, the stratum $\hat{S}$ is semialgebraic as the image of the semialgebraic set $(W_K \times V^H) \cap B_{(h,x)}$ under the Hilbert map $p$. By the same argument, $p(W_K)$ is semialgebraic, too, and an analytic manifold, since $p(W_K) \cong N_H(K) \setminus W_K \cong H \setminus W_K$. Moreover, the equality

$$\hat{S} = (p(W_K) \times V^H) \cap p(B_{(h,x)})$$

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Let $\hat{S} \in \hat{\mathcal{Z}}$ denote the stratum containing the orbit $G(h,x)$, and $\hat{S} \in \mathcal{Z}$ a stratum $\neq \hat{R}$ such that $G(h,x)$ lies in the closure of $\hat{S}$. Now consider sequences of orbits $G(h_i, x_i)_{i \in \mathbb{N}}$ in $\hat{R}$ and $G(k_i, y_i)_{i \in \mathbb{N}}$ in $\hat{S}$ such that both sequences converge to $G(h,x).$ Moreover, assume that the sequence of secants $\overline{\hat{\Psi}(G(h_i, x_i)), \hat{\Psi}(G(k_i, y_i))}$ converges to a line $\hat{l}$, and that the sequence of tangent spaces $T_{\hat{\Psi}(G(k_i, y_i))} \hat{S}$ converges to some subspace $\hat{\tau} \subset \mathbb{R}^N$. Using notation from before, we can choose representatives $(h_i, x_i)$ and $(k_i, y_i)$ having coordinates in $m \times (W_K \cup \{0\}) \times V^H \subset N_{(h,x)}$ such that

$$\Psi(h_i, x_i) = (0, 0, v_i) \in \{0\} \times \{0\} \times V^H \quad \text{and}$$

$$\Psi(k_i, y_i) = (0, w_i, v_i) \in \{0\} \times W_K \times V^H.$$ 

Next observe that by the Tarski–Seidenberg Theorem, the stratum $\hat{S}$ is semialgebraic as the image of the semialgebraic set $(W_K \times V^H) \cap B_{(h,x)}$ under the Hilbert map $p$. By the same argument, $p(W_K)$ is semialgebraic, too, and an analytic manifold, since $p(W_K) \cong N_H(K) \setminus W_K \cong H \setminus W_K$. Moreover, the equality

$$\hat{S} = (p(W_K) \times V^H) \cap p(B_{(h,x)})$$

proving the second claim.
holds true, where we have canonically identified $V^H$ with its image under the Hilbert map $p$. By Eq. (6.12), this implies that

$$\tilde{\tau} = \lim_{i \to \infty} T_{\tilde{\varphi}(G(k_i,y_i))} \tilde{S} = \lim_{i \to \infty} T_{p(w_i)}p(W_K) \times V^H.$$  

Since $p(W_K)$ is semialgebraic and an analytic manifold, [Loj65] Prop. 3, p. 103] by Lojasiewicz entails that $p(W_K)$ satisfies the Whitney condition (b) over the origin. This means after possibly passing to subsequences, that $\ell_{W_K} \subset \tau_{W_K}$, where $\ell_{W_K}$ is the limit line of the secants $p(w_i), 0$, and $\tau_{W_K}$ the limit of the tangent spaces $T_{p(w_i)}p(W_K)$ for $i \to \infty$. By Eqs. (6.12) and (6.13) this implies that

$$\tilde{\tau} \subset \ell_{W_K} \times V^H \subset \tau_{W_K} \times V^H = \tilde{\tau}.$$  

This finishes the proof. \hfill \Box

For a translation groupoid $G = G \ltimes M$, if $S$ is a stratum of $\Lambda_0(G \ltimes M)$, then by Lemma 2.10 [2.11 and 2.3], the induced stratum of the arrow space is $G \times S$, and the corresponding stratum of $(AG)_1 \ltimes_1 (AG)_1$ is $G \times G \times S$. Hence, Whitney (b)-regularity of the stratification of $\Lambda_0(G \ltimes M)$ implies Whitney (b)-regularity of the stratifications of the arrow space and $(AG)_1 \ltimes_1 (AG)_1$.

Note that $\Lambda_0 G$ is clearly topologically locally trivial because of its description in slices. Therefore, by Proposition 3.1, the inertia groupoid $\Lambda G$ is a differentiable stratified groupoid. Now, recall that the loop space $\Lambda G$ is a differentiable subspace of the smooth manifold $G_1$ and that the space of arrows $\Lambda G_1 = G_1 \ltimes G \Lambda G$ is a differentiable subspace of the smooth manifold $G_1 \ltimes G_1$. Similarly, if $x, y \in G_0$ are in the same orbit, then the slices $Y_x$ and $Y_y$ for $G$ can be chosen such that $G_{0|Y_x}$ and $G_{0|Y_y}$ are isomorphic by [PPT14] Lemma 5.1. This defines a diffeomorphism between the arrow spaces of $G_{0|Y_x}$ and $G_{0|Y_y}$ (which are both smooth manifolds) whose restriction defines an isomorphism between $\Lambda G_{0|Y_x}$ and $\Lambda G_{0|Y_y}$. It follows that the inertia groupoid $\Lambda G$ satisfies conditions (L1) to (L4) in Definitions 2.7 and 2.31, hence is sliceable. Moreover, by Proposition 6.14, the inertia groupoid $\Lambda G$ satisfies the local contractibility condition of Definition 5.7.

It is straightforward to verify that a weak equivalence $f = (f_0, f_1): G \to H$ of proper Lie groupoids induces a weak equivalence $\Lambda G \to \Lambda H$ given by the restriction of $f_1$ to the loop spaces. In particular, because the stratification of $\Lambda_0 G$ is defined in terms of slices for $G$, and the representation of the isotropy group on a slice is Morita invariant, the stratification of $\Lambda_0 G$ is obviously the pullback of the stratification of $\Lambda_0 H$ via $f_1$. Moreover, as the stratification of the inertia space $|\Lambda G|$ can be defined locally in terms of the actions of isotropy groups on slices, it is as well Morita invariant. This means that the isomorphism between $|\Lambda G|$ and $|\Lambda H|$ from Proposition 2.26 is an isomorphism of differentiable stratified spaces. We summarize these observations in the following.

**Theorem 6.19.** Let $G$ be a proper Lie groupoid. Then the inertia groupoid $\Lambda G$ is a proper reduced structurally (b)-regular differentiable stratified groupoid. Moreover the inertia groupoid $\Lambda G$ is sliceable and satisfies the local contractibility condition. Finally, the inertia space $|\Lambda G|$ inherits from $\Lambda_0 G$ via the canonical projection $\pi: \Lambda_0 G \to |\Lambda G|$ a Whitney (b)-regular stratification.

**Appendix A. Differentiable stratified spaces**

In this appendix we describe the category of differentiable stratified spaces used throughout this paper. Our notion of differentiable spaces is that of [NGSdS03] to
which we refer the reader for more details. For the definition of stratified spaces we follow Mather [Mat73] and [PH01] Chap. 1, except that we relax the assumption that the spaces under consideration are Hausdorff and only require that they are locally Hausdorff. Hence, a stratified space with smooth structure as defined in [PH01] Chap. 1 or a differentiable stratified space as defined in [FPS15] corresponds to a Hausdorff differentiable stratified space as defined here. Note that in addition to [PH01] [FPS15] various other concepts of structure sheaves respectively structure algebras of smooth functions over stratified spaces have been introduced in the literature. See for example the work by Kreck [Kre10] on stratifolds, by Lusala–Śniatycki [LS11, Sec. 4] on stratified subcartesian spaces, by Watts [Wat12, Wat15] on differential spaces, and finally by Somberg–Ván Lé–Vanzura [SLV15] [LSV13] on smooth structures on locally conic stratified spaces.

A.1. Differentiable spaces.

**Definition A.1** ([NGSdS03, Chap. 3]). Let \((X, \mathcal{O})\) be a locally \(\mathbb{R}\)-ringed space which we always assume to be commutative. One says that \((X, \mathcal{O})\) is an affine differentiable space, if there is a closed ideal \(\mathfrak{a} \subset C^\infty(\mathbb{R}^n)\) such that \((X, \mathcal{O})\) is isomorphic as a ringed space to the real spectrum of \(C^\infty(\mathbb{R}^n)/\mathfrak{a}\) equipped with its structure sheaf, which associates to each open set its localization over that set. Here, we consider the unique topology with respect to which \(C^\infty(\mathbb{R}^n)\) is a Frechét algebra. A locally \(\mathbb{R}\)-ringed space \((X, \mathcal{O})\) is a differentiable space if, for each \(x \in X\), there is an open neighborhood \(U\) of \(x\) such that the restriction \((U, \mathcal{O}_U)\) is an affine differentiable space. A differentiable space is reduced if for each open subset \(U\) of \(X\) the map \(\mathcal{O}(U) \to C(U)\) defined by the evaluation map is injective.

A morphism of differentiable spaces \((f, \varphi): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)\) consists of a continuous map \(f: X \to Y\) and a morphism \(\varphi: \mathcal{O}_Y \to f_*\mathcal{O}_X\) of sheaves of \(\mathbb{R}\)-algebras such that for each \(x \in X\) the induced morphism on the stalks \(\varphi_x: \mathcal{O}_{Y, f(x)} \to \mathcal{O}_{X, x}\) is local, i.e., maps the maximal ideal \(m_{f(x)} \subset \mathcal{O}_{Y, f(x)}\) to the maximal ideal \(m_x \subset \mathcal{O}_{X, x}\).

Note that if \((X, C^\infty_X)\) and \((Y, C^\infty_Y)\) are reduced, a morphism of differentiable spaces \((f, \varphi): (X, C^\infty_X) \to (Y, C^\infty_Y)\) is fully determined by the map \(f: X \to Y\). The sheaf morphism \(\varphi\) is given in this case over each open \(V \subset Y\) by the pullback map \(f^*: C^\infty_Y(V) \to C^\infty_X(f^{-1}(V)), g \mapsto g \circ f|_V\). We therefore sometimes call a morphism between reduced differentiable spaces a smooth map, and just denote it by the underlying map \(f\).

By [NGSdS03, Thm. 3.23], a differentiable space \((X, \mathcal{O})\) is reduced if and only if each \(x \in X\) is contained in an open neighborhood \(V\) isomorphic as a differentiable space to a locally closed subset of the affine space \(\mathbb{R}^n\) with structure sheaf given by restrictions of smooth functions from \(\mathbb{R}^n\). We refer to such a \(V\) as an affine neighborhood of \(x\), and call an embedding \(\iota: V \hookrightarrow \mathbb{R}^n\) such that \((\iota, \iota^*): (V, \mathcal{O}_V) \to (\iota(V), C^\infty_{\mathbb{R}^n}(\iota(V)))\) is an isomorphism of locally ringed spaces a singular chart (of rank \(n\)) for \(X\).

We often denote the structure sheaf of a reduced differentiable space \(X\) by \(C^\infty_X\) or shortly by \(C^\infty\), if no confusion can arise. By a smooth submanifold of a differentiable space \(X\), we mean a differentiable subspace whose differentiable structure is that of a smooth manifold in the usual sense.

A.2. Stratified spaces. Let \(X\) be a paracompact separable locally Hausdorff topological space. A decomposition \(\mathcal{Z}\) of \(X\) is a locally finite partition of \(X\) into locally closed subspaces such that each \(S \in \mathcal{Z}\) is a countable union of smooth (not necessarily Hausdorff) manifolds such that the following condition of frontier is satisfied:
(CF) If $R \cap \overline{S} \neq \emptyset$ for $R, S \in Z$, then $R \subseteq \overline{S}$.

If $R \subseteq \overline{S}$, one writes $R \leq S$ and says that $R$ is incident to $S$. The incidence relation is an order relation on $Z$. The elements of a decomposition $Z$ are called its pieces.

In the following we provide a generalization of the definition of a stratification by Mather [Mat73] to the case of a locally Hausdorff space; cf. also [Pfl01, Sec. 1.2].

**Definition A.2.** Let $X$ be a paracompact separable and locally Hausdorff topological space. A *stratification* of $X$ is an assignment to each $x \in X$ of a germ $S_x$ of subsets of $X$ at $x$ such that for each $x \in X$ there is a Hausdorff neighborhood $U$ of $x$ in $X$ and a decomposition $Z$ of $U$ with the property that for each $y \in U$ the germ $S_y$ is equal to the germ at $y$ of the piece of $Z$ containing $y$. The set $X$ along with the stratification $S$ is called a *stratified space*. If, moreover, $X$ is a differentiable space, and for each $x \in X$, the germ $S_x$ is that of a smooth submanifold of $X$ as defined in the previous section, then we say $X$ is a *differentiable stratified space*.

A continuous function $f : (X, S) \to (Y, R)$ is a *morphism of stratified spaces* if, for each $x \in X$ with $f(x) = y$, there are open Hausdorff neighborhoods $U$ of $x$ and $V$ of $y$ with $U \subset f^{-1}(V)$ and decompositions of $U$ and $V$ inducing their respective stratifications such that for every $z \in U$ contained in the piece $S$ of $U$, there is an open neighborhood $O$ of $z$ in $U$ such that $f|_{S \cap O}$ maps into the piece of $V$ containing $f(z)$.

If $(X, O_X, S)$ and $(Y, O_Y, R)$ are differentiable stratified spaces, a function $f : X \to Y$ is a *differentiable stratified morphism* if it is simultaneously a morphism of differentiable spaces and a morphism of stratified spaces.

Note that the definition of a differentiable stratified space coincides, for a Hausdorff space $X$, with that of a stratified space with $C^\infty$ structure defined in [Pfl01 Section 1.3]; see also [FPS15 Section 2].

If $(X, Z)$ is a decomposed space, then the decomposition induces a stratification by assigning to $x \in X$ the germ at $x$ of the piece containing $x$; two decompositions of $X$ are *equivalent* if they induce the same stratification. The *depth* of $x \in X$ with respect to the decomposition $Z$ is the maximum $k$ such that $x \in S_0 \subset S_1 \subset \cdots \subset S_k$ for $S_i \in Z$.

Now, let $(X, S)$ be a stratified space and let $x \in X$. The proof of [Mat73 Lemma 2.1] (see also [Pfl01 Lem. 1.5.2]) is local, hence can be executed on a Hausdorff neighborhood of $x$. It therefore extends to our case and demonstrates that the depth of $x$ coincides for any decomposition of a Hausdorff neighborhood of $x$ inducing $S$. Hence, we may define the *depth of $x$ with respect to $S$* to be the depth with respect to any such decomposition. In the same way, the proofs of [Mat73 Lem. 2.2] and [Pfl01 Prop. 1.2.7]) extend to the situation of locally Hausdorff stratified space. So $X$ admits a decomposition $Z$ that induces $S$ and is maximal in the sense that for every Hausdorff open subset $U$ of $X$, the restriction of $Z$ to $U$ is coarser than any decomposition of $U$ that induces $S$. We will often refer to $Z$ simply as the *maximal decomposition of $X$*. Its pieces are called the *strata* of $X$. Note that if $X$ is a (Hausdorff) differentiable stratified space, the strata of $Z$ are obviously (Hausdorff) smooth manifolds.

We recall the following from [Pfl01 Section 1.4.1].

**Definition A.3.** A stratified space $(X, S)$ is *topologically locally trivial* if for every $x \in X$ in the stratum $S$ of $X$, there is a neighborhood $U$, a stratified space $(F, S^F)$, a point $o \in F$, and an isomorphism of stratified spaces $h : U \to (S \cap U) \times F$ such that $h^{-1}(y, o) = y$ for all $y \in S \cap U$, and such that $S^F_o$ is the germ of the set $\{o\}$. 
See [Pho01] Section 1.4.3 for the definitions of Whitney (a)- and Whitney (b)-regularity, which impose restrictions on how the tangent spaces of strata limit to one another. Note in particular that Whitney (b)-regularity implies topological local triviality; see [Pho01] Corollary 3.9.3.

A.3. Fibered products. By [NCSdS03] Theorem 7.6], the fibered product of differentiable spaces has a unique differentiable space structure with respect to which the projection maps are morphism of differentiable spaces. We now demonstrate that the same holds true for differentiable stratified spaces. Let \( X, Y, \) and \( Z \) be differentiable stratified spaces with respective stratifications \( S^X, S^Y, \) and \( S^Z. \) Suppose \( f: X \to Z \) and \( g: Y \to Z \) are differentiable stratified mappings. If \( f \) is in addition a stratified submersion, then we define a stratification of the fibered product \( X \times_g Y \) as follows. Let \( (x, y) \in X \times Y, \) let \( P \subset X \) be a subset whose germ \( [P]_x = S^X_x, \) and let \( R \subset Y \) such that \( [R]_y = S^Y_y \). Then we assign to \( (x, y) \in X \times Y \) the germ \( S_{(x, y)} := [P \times_g R]_{(x, y)} \). We refer to \( S \) as the induced stratification of \( X \times_g Y \) by the stratifications \( S^X \) and \( S^Y \).

**Lemma A.4.** Suppose \( X, Y, \) and \( Z \) are differentiable stratified spaces and \( f: X \to Z \) and \( g: Y \to Z \) are differentiable stratified mappings. If \( f \) is in addition a stratified submersion, then the induced stratification \( S \) is a stratification of \( X \times_g Y \).

**Proof.** For simplicity, we work with the maximal decompositions \( Z^X, Z^Y, \) and \( Z^Z \) of \( X, Y, \) and \( Z, \) respectively, which is clearly sufficient as the definition of the fibered product is local. Then the fact that \( Z^X \) and \( Z^Y \) are partitions of \( X \) and \( Y \) immediately implies that \( Z := \{P \times_g R | P \in Z^X, R \in Z^Y\} \) is a partition of \( X \times Y. \) Moreover, the fact that each \( P \in Z^X \) and \( R \in Z^Y \) is locally closed implies that \( P \times R \) is locally closed in \( X \times Y \). Hence \( P \times_g R = (P \times R) \cap (X \times Y) \) is locally closed in \( X \times Y \).

Given \( (x, y) \in X \times Y, \) let \( U_x \) and \( U_y \) be open neighborhoods of \( x \) in \( X \) and \( y \) in \( Y, \) respectively, such that each intersects finitely many elements of \( Z^X \) and \( Z^Y, \) and then \( (U_x \times U_y) \cap (X \times Y) \) is an open neighborhood of \( (x, y) \) in \( X \times Y \) that meets finitely many elements of \( Z. \) Therefore, \( Z \) is a locally finite partition of \( X \times Y \) into locally closed sets.

Now, let \( P \in Z^X \) and \( R \in Z^Y, \) and choose connected components \( P_0 \) of \( P \) and \( R_0 \) of \( R \). Then as \( f \) and \( g \) are stratified mappings, there is a piece \( S \in Z^Z \) with \( f(P_0), g(R_0) \subset S; \) see [Pho01] 1.2.10]. Moreover, as \( f \) is a stratified submersion, \( f \big| P_0 \) is by definition a submersion. Then [Lan02] Prop. 2.5 & 2.6] and the fact that \( f \big| P_0 \) is a submersion imply that \( f \big| P_0 \) is transversal to \( g \big| R_0 \). Hence, \( P_0 \times_g R_0 \) is a smooth submanifold of \( P_0 \times R_0 \) and hence of the differentiable space \( X \times_g Y. \) Therefore each connected component of \( P \times_g R \) is a smooth manifold.

Finally, suppose \( (P \times_g R) \cap (P' \times_g R') = \emptyset \) for \( P, P' \in Z^X \) and \( R, R' \in Z^Y. \) Choose \( (x, y) \in (P \times_g R) \cap (P' \times_g R'). \) Then for any open neighborhoods \( U_x \) and \( U_y \) of \( x \) and \( y \) in \( X \) and \( Y, \) respectively, \( U_x \times Y_y \) intersects \( P' \times_g R'. \) It follows that \( P \cap R' \cap R' \neq \emptyset \) so that \( P \subset R' \) and \( R \subset R' \), which imply \( P \times_g R \subset (P' \times_g R') \). That is, \( Z \) satisfies the condition of frontier and hence is a decomposition of \( X \times_g Y. \)

A.4. Tangent space. Assume that \( (X, C^\infty) \) is a differentiable space. Then, given a point \( x \in X, \) the maximal ideal \( m_x \subset C^\infty_x \) in the stalk at \( x \) is finitely generated, which implies that the quotient space \( m_x/m_x^2 \) is a finite dimensional real vector space. We will
call this space the Zariski cotangent space $T^*_x X$ of $(X,C^\infty)$ at $x$, and its dual $(m_x/m_x^2)^*$ the Zariski tangent space $T_x X$.

**Remark A.5.** There is another notion of a tangent bundle for a differentiable stratified space $(X,C^\infty)$, namely the stratified tangent space $(T^s X,C^\infty)$. If $(X,C^\infty)$ is Whitney (a)-regular, then $(T^s X,C^\infty)$ is also a differentiable stratified space. Indeed, $T^s X \subset TX$ with equality if and only if $(X,C^\infty)$ is a smooth manifold (without boundary). See [Pfl01 Sec. 2.1] for more details on stratified tangent bundles and [Pfl01 Rem. 2.2.4] for details on the relation between $T^s X$ and $TX$.

**A.5. Differential forms.** Let $(X,C^\infty)$ be a reduced differentiable stratified space. Let $U$ be an affine open subset of $X$ and $\iota : U \hookrightarrow \mathbb{R}^n$ be a singular chart of $X$. Denote by $I_\iota$ the sheaf of smooth functions vanishing on $\iota(U)$. Then define the sheaf $\Omega^k_i$ for $k = 0$ as $\iota^{-1}(C^\infty/I_\iota) \cong C^\infty_U$ and for $k \in \mathbb{N}^*$ as the following inverse image sheaf

$$\Omega^k_i := \iota^{-1}(\Omega^n_{R^n}/(I_\iota \Omega^k_{R^n} + dI_\iota \wedge \Omega^{k-1}_{R^n})).$$

Observe that by construction the exterior differential factors through the $\Omega^k$, hence we obtain a differential graded algebra $(\Omega^*_i,d)$. If $\kappa : V \hookrightarrow \mathbb{R}^m$ is another singular chart of $X$, there exists a unique sheaf isomorphism $\eta_{\kappa,\iota} : \Omega^*_i|_{\kappa(U \cap V)} \rightarrow \Omega^*_i|_{\iota(U \cap V)}$ extending the isomorphism of sheaves $\eta_{\kappa,\iota} : (C^\infty/I_\iota)|_{\kappa(U \cap V)} \rightarrow (C^\infty/I_\iota)|_{\iota(U \cap V)}$. We conclude that the cocycle condition

$$(A.1) \quad \eta_{\kappa,\iota} = \eta_{\kappa,\lambda} \circ \eta_{\lambda,\iota}$$

holds, where we denote by $\lambda : V \hookrightarrow \mathbb{R}^d$ a third singular chart of $X$. Hence the sheaves $\Omega^*_i$ glue to a globally defined sheaf $\Omega^*_X$ of so-called abstract $k$-forms on $X$ in such a way that the gluing maps preserve $d$. So we obtain a sheaf complex $(\Omega^*_X,d)$ of differential graded algebras. The complex of global sections $(\Omega^*_X,d)$ will be called the Grauert–Grothendieck complex of $X$.

**Remark A.6.** For $X \subset \mathbb{C}^n$ a complex space, the construction of the complex $\Omega^*_X(\mathcal{X})$ within the analytic category goes back to Grauert [GK61] and Grothendieck [Gro66].

Let us now describe how one can represent elements of $\Omega^k(X)$. To this end assume to be given an open covering $\mathcal{U}$ of $X$ by coordinate domains and a family $(\kappa_U)_{U \in \mathcal{U}}$ of singular charts $\kappa_U : U \hookrightarrow \tilde{U} \subset \mathbb{R}^{nu}$ such that $\tilde{U}$ is open and contains $\kappa_U(U)$ as a relatively closed subset. An element of $\Omega^k(X)$ can then be represented as a family $(\omega_U)|_{U \in \mathcal{U}}$, where $\omega_U \in \Omega^k(\tilde{U})$ and where, for any two $U,V \in \mathcal{U}$, one has

$$\eta_{\kappa_V,\kappa_U}(\omega_U) = [\omega_V]$$

on $U \cap V$. In the above, $[\omega_U]$ denotes the equivalence class of $\omega_U$ in $\Omega^k_{\kappa_U}$.

The above formula allows us to extend the forms $[\omega_U]$ globally as follows. Assume that $S$ is a stratum of $X$, and let $i_S : S \hookrightarrow X$ denote the canonical embedding. Given an element $\omega = (\omega_U)|_{U \in \mathcal{U}} \in \Omega^k(X)$ one observes that for any two $U,V \in \mathcal{U}$ the forms $i_{S \cap U}^* \kappa_U^*(\omega_U)$ and $i_{S \cap V}^* \kappa_V^*(\omega_V)$ coincide on the overlap $U \cap V$, hence glue together to an abstract global form on $S$ which we denote by $i_S^* \omega \in \Omega^k(S)$.

By construction, each of the sheaves $\Omega^*_X$ carries the structure of a $C^\infty$-module in a natural way. Therefore, we have the following.

**Proposition A.7.** The Grauert–Grothendieck complex $(\Omega^*_X,d)$ of a differentiable stratified space $(X,C^\infty)$ is a complex of fine sheaves.
Finally, we will define the pull-back morphism $f^* : \Omega^k_Y \to \Omega^k_X$ associated to a smooth map $f : X \to Y$ between reduced differentiable stratified spaces $(X, C^\infty_X)$ and $(Y, C^\infty_Y)$. By Proposition A.7, and the construction of the Grauert–Grothendieck complex it suffices to consider the case where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are affine. Choose open neighborhoods $U \subset \mathbb{R}^n$ of $X$ and $V \subset \mathbb{R}^m$ of $Y$ such that $X$ is closed in $U$ and $Y$ in $V$. Choose a smooth function $F : U \to V$ such that $F|_X = f$. For $\omega \in \Omega^k(V)$ representing an abstract $k$-form on $Y$ we put

$$f^*([\omega]) := [F^* \omega] \in \Omega^k(X).$$

Since $F^*$ maps the vanishing ideal $I_Y \subset C^\infty(V)$ to the vanishing ideal $I_X \subset C^\infty(U)$ and since $F^*$ commutes with $d$, $F^*$ maps the $I_Y \Omega^k(V) + dI_Y \wedge \Omega^k(V)$ to $I_X \Omega^k(U) + dI_X \wedge \Omega^k(U)$. Moreover, if $\tilde{F} : U \to V$ is another smooth function such that $\tilde{F}|_X = f$, then $F^* g - \tilde{F}^* g \in I_X$ and $F^* dg - \tilde{F}^* dg \in dI_X$ for all $g \in C^\infty(V)$, which entails that $F^* \omega - \tilde{F}^* \omega \in I_X \Omega^k(U) + dI_X \wedge \Omega^k(U)$. This proves that $[F^* \omega]$ neither depends on the particular choice of the representative of $[\omega]$ nor on the particular smooth $F$ extending $f$ to an open neighborhood of $X$. Hence $f^* : \Omega^k(Y) \to \Omega^k(X)$ is well-defined. Obviously, $d$ commutes with $f^*$, since it commutes with $F^*$.

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**Carla Farsi**, Department of Mathematics, University of Colorado at Boulder, Campus Box 395, Boulder, CO 80309-0395, USA

*Email address*: farsi@euclid.colorado.edu

**Markus J. Pflaum**, Department of Mathematics, University of Colorado at Boulder, Campus Box 395, Boulder, CO 80309-0395, USA and Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstr. 22, 04103 Leipzig, Germany

*Email address*: markus.pflaum@colorado.edu

**Christopher Seaton**, Department of Mathematics and Computer Science, Rhodes College, 2000 N. Parkway, Memphis, TN 38112, USA

*Email address*: seatonc@rhodes.edu