UPPER AND LOWER BOUNDS FOR EIGENVALUES OF THE CLAMPED PLATE PROBLEM*

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Abstract. In this paper, we study estimates for eigenvalues of the clamped plate problem. A sharp upper bound for eigenvalues is given and the lower bound for eigenvalues in [10] is improved.

1. Introduction

A membrane has its transverse vibration governed by equation
\[ \Delta u = -\lambda u, \quad \text{in } \Omega \]
with the boundary condition
\[ u = 0, \quad \text{on } \partial \Omega, \]
where \( \Delta \) is the Laplacian in \( \mathbb{R}^n \) and \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). It is classical that there is a countable sequence of eigenvalues
\[ 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty, \]
and a sequence of corresponding eigenfunctions \( u_1, u_2, \cdots, u_k, \cdots \) such that
\[ \Delta u_k = -\lambda_k u_k, \quad \text{in } \Omega. \]
The eigenfunctions form an orthonormal basis of \( L^2(\Omega) \).

On the other hand, the vibration of a stiff plate differs from that of a membrane not only in the equation which governs its motion but also in the way the plate is fastened to its boundary. A plate spanning a domain \( \Omega \) in \( \mathbb{R}^n \) has its transverse vibrations governed by
\[
\begin{cases}
\Delta^2 u = \Gamma u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
where \( \nu \) denotes the outward unit normal to the boundary \( \partial \Omega \). Namely, not only is the rim of the plate firmly fastened to the boundary, but the plate is clamped so that lateral motion can occur at the edge. One calls it a clamped plate problem. It is known that this problem has a real and discrete spectrum
\[ 0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_k \leq \cdots \to +\infty, \]
where each \( \Gamma_i \) has finite multiplicity which is repeated according to its multiplicity.

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For the eigenvalues of the clamped plate problem (1.1), Agmon [1] and Pleijel [22] gave the following asymptotic formula,

\[ \Gamma_k \sim \frac{16\pi^4}{(B_n \text{vol}(\Omega))^\frac{2}{n}} k^\frac{4}{n}, \quad k \to \infty. \]

This implies that

\[ \frac{1}{k} \sum_{j=1}^{k} \Gamma_j \sim \frac{n}{n + 4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^\frac{2}{n}} k^\frac{4}{n}, \quad k \to \infty, \]

where \( B_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). Furthermore, Levine and Protter [17] proved that the eigenvalues of the clamped plate problem (1.1) satisfy

\[ \frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n + 4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^\frac{2}{n}} k^\frac{4}{n}. \]

The formula (1.2) shows that the coefficient of \( k^\frac{4}{n} \) is the best possible constant. Thus, it will be interesting and very important to find the second term on \( k \) of the asymptotic expansion formula of \( \Gamma_k \). The authors [10] have made effort for this problem. We have improved the result due to Levine and Protter [17] by adding to its right hand side two terms of lower order in \( k \):

\[ \frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n + 4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^\frac{2}{n}} k^\frac{4}{n} \]

\[ + \left( \frac{n + 2}{12n(n + 4)} - \frac{1}{1152n^2(n + 4)} \right) \frac{\text{vol}(\Omega)}{I(\Omega)} \frac{n}{n + 2} \frac{4\pi^2}{(B_n \text{vol}(\Omega))^\frac{2}{n}} k^\frac{2}{n} \]

\[ + \left( \frac{1}{576n(n + 4)} - \frac{1}{27648n^2(n + 2)(n + 4)} \right) \left( \frac{\text{vol}(\Omega)}{I(\Omega)} \right)^2, \]

where

\[ I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 \, dx \]

is called the moment of inertia of \( \Omega \). On the other hand, if one can obtain an upper bound with optimal order of \( k \) for eigenvalue \( \Gamma_k \), then one can know the exact second term on \( k \). From our knowledge, there is no any result on upper bounds for eigenvalue \( \Gamma_k \) with optimal order of \( k \). In [12], Cheng and Yang have established a recursion formula in order to obtain upper bounds for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian. Hence, if one can get a sharper universal inequality for eigenvalues of the clamped plate problem, we can also derive an upper bound for eigenvalue \( \Gamma_k \) by making use of the recursion formula due to Cheng and Yang [12]. On the investigation of universal inequalities for eigenvalues of the clamped plate problem, Payne, Pólya and Weinberger [21] proved

\[ \Gamma_{k+1} - \Gamma_k \leq \frac{8(n + 2)}{n^2 k} \sum_{i=1}^{k} \Gamma_i. \]
Chen and Qian [7] and Hook [14], independently, extended the above inequality to
\[
\frac{n^2k^2}{8(n + 2)} \leq \sum_{i=1}^{k} \frac{\Gamma_{i}^{1/2}}{\Gamma_{k+1} - \Gamma_{i}} \sum_{i=1}^{k} \Gamma_{i}^{1/2}.
\]

Recently, answering a question of Ashbaugh [2], Cheng and Yang [12] have proved the following remarkable estimate:
\[
\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_{i}) \leq \left(\frac{8(n + 2)}{n^2}\right)^{1/2} \sum_{i=1}^{k} (\Gamma_{i}(\Gamma_{k+1} - \Gamma_{i}))^{1/2}.
\]

Furthermore, Wang and Xia [25] (cf. Cheng, Ichikawa and Mametsuka [8], [9]) have proved
\[
\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_{i})^2 \leq \frac{8(n + 2)}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_{i})\Gamma_{i}.
\]

The first author has conjectured the following:

**Conjecture.** Eigenvalue \(\Gamma_j\)'s of the clamped plate problem (1.1) satisfy
\[
\sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_{j})^2 \leq \frac{8(n + 2)}{n^2} \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_{j})\Gamma_{j}.
\]

If one can solve the above conjecture, then from the recursion formula of Cheng and Yang [12], we can derive an upper bound for the eigenvalue \(\Gamma_k\) with the optimal order of \(k\). But it seems to be hard to solve this conjecture.

In this paper, we will try to use a fact that eigenfunctions of the clamped plate problem (1.1) form an orthonormal basis of the Sobolev Space \(W^{2,2}(\Omega)\) to get an upper bound for eigenvalues of the clamped plate problem (1.1). A similar fact for the Dirichlet eigenvalue problem of the Laplacian is also used by Li and Yau [18] and Kröger [15]. Furthermore, we will give an improvement of the inequality (1.3).

Let \(\Omega\) be a bounded domain with a smooth boundary \(\partial \Omega\) in the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\). Let \(d(x) = \text{dist}(x, \partial \Omega)\) denote the distance function from the point \(x\) to the boundary \(\partial \Omega\) of \(\Omega\). We define
\[
\Omega_r = \left\{ x \in \Omega \mid d(x) < \frac{1}{r} \right\}.
\]

**Theorem 1.1.** Let \(\Omega\) be a bounded domain with a smooth boundary \(\partial \Omega\) in \(\mathbb{R}^n\). Then there exists a constant \(r_0 > 0\) such that eigenvalues of the clamped plate problem (1.1) satisfy
\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \leq \frac{1 + \frac{4(n + 4)(n^2 + 2n + 6)}{n + 2} \frac{\text{vol}(\Omega_{r_0})}{\text{vol}(\Omega)}}{(1 - \frac{\text{vol}(\Omega_{r_0})}{\text{vol}(\Omega)})^{\frac{n+4}{n}}} \frac{n}{n + 4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{4/3}},
\]
for \(k \geq \text{vol}(\Omega)r_0^n\).
Remark 1.1. Since \( \text{vol}(\Omega_{r_0}) \to 0 \) when \( r_0 \to \infty \), we know that the upper bound in the theorem 1.1 is sharp in the sense of the asymptotic formula due to Agmon and Pleijel.

Corollary 1.1. Let \( \Omega \) be a bounded domain with a smooth boundary \( \partial \Omega \) in \( \mathbb{R}^n \). If there exists a constant \( c_0 \) such that

\[
\text{vol}(\Omega_r) \leq c_0 \text{vol}(\Omega) \frac{n+1}{n} \frac{1}{r}
\]

for \( r > \text{vol}(\Omega)\frac{1}{n} \), then there exists a constant \( r_0 \) such that eigenvalues of the clamped plate problem \( (1.1) \) satisfy

\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \leq \frac{n}{n+4} \left( \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{\frac{4}{n}}} \left( k^{\frac{4}{n}} + c_0 c(n) k^{\frac{3}{n}} \right) \right),
\]

for \( k = \text{vol}(\Omega)r_0^n > c_n^n \), where \( c(n) \) is a constant depended only on \( n \).

Theorem 1.2. Let \( \Omega \) be a bounded domain with a piecewise smooth boundary \( \partial \Omega \) in \( \mathbb{R}^n \). Eigenvalue \( \Gamma_j \)'s of the clamped plate problem \( (1.1) \) satisfy

\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \left( \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}} \right) + \frac{n+2}{12n(n+4)} \frac{\text{vol}(\Omega)}{I(\Omega)}^{\frac{n}{n+4}} k^{\frac{4}{n}} + \frac{(n+2)^2}{1152n(n+4)^2} \left( \frac{\text{vol}(\Omega)}{I(\Omega)} \right)^2,
\]

where \( I(\Omega) \) is the moment of inertia of \( \Omega \).

2. Upper bounds for eigenvalues

In this section, we will study the upper bounds for eigenvalues of the clamped plate problem \( (1.1) \).

Proof of Theorem 1.1. Since \( d(x) \) is the distance function from the point \( x \) to the boundary \( \partial \Omega \) of \( \Omega \), we define a function \( f_r \) for any fixed \( r \) by

\[
f_r(x) = \begin{cases} 
1, & x \in \Omega, d(x) \geq \frac{1}{r}, \\
r^2 d^2(x), & x \in \Omega, d(x) < \frac{1}{r}, \\
0, & \text{the other}.
\end{cases}
\]

Let \( u_j \) be an orthonormal eigenfunction corresponding to the eigenvalue \( \Gamma_j \), that is, \( u_j \) satisfies

\[
\begin{align*}
\Delta^2 u_j &= \Gamma_j u_j, & \text{in } \Omega, \\
u_j &= \frac{\partial u_j}{\partial \nu} = 0, & \text{on } \partial \Omega, \\
\int_{\Omega} u_i(x) u_j(x) dx &= \delta_{ij}, & \text{for any } i, j.
\end{align*}
\]
Thus, \( \{u_j\} \) forms an orthonormal basis of the Sobolev Space \( W^{2,2}_0(\Omega) \). For an arbitrary fixed point \( z \in \mathbb{R}^n \) and \( r > 0 \), a function

\[
g_{r,z}(x) = e^{i\langle z, x \rangle} f_r(x),
\]

with \( i = \sqrt{-1} \), belongs to the Sobolev Space \( W^{2,2}_0(\Omega) \). Hence, we have

\[
g_{r,z}(x) = \sum_{j=1}^{\infty} a_{r,j}(z) u_j(x),
\]

where

\[
a_{r,j}(z) = \int_{\Omega} g_{r,z}(x) u_j(x) dx.
\]

Defining a function

\[
\varphi_k(x) = g_{r,z}(x) - \sum_{j=1}^{k} a_{r,j}(z) u_j(x),
\]

we have \( \varphi_k = \frac{\partial \varphi_k}{\partial \nu} = 0 \) on \( \partial \Omega \) and

\[
\int_{\Omega} \varphi_k(x) u_j(x) dx = 0, \quad \text{for } j = 1, 2, \ldots, k.
\]

Therefore, \( \varphi_k \) is a trial function. From Rayleigh-Ritz formula, we have

\[
\Gamma_{k+1} \int_{\Omega} |\varphi_k(x)|^2 dx \leq \int_{\Omega} |\Delta \varphi_k(x)|^2 dx.
\]

From the definition of \( \varphi_k \) and (2.1), we have

\[
\int_{\Omega} |\varphi_k(x)|^2 dx = \int_{\Omega} |g_{r,z}(x) - \sum_{j=1}^{k} a_{r,j}(z) u_j(x)|^2
\]

\[
= \int_{\Omega} |f_r(x)|^2 dx - \sum_{j=1}^{k} |a_{r,j}(z)|^2
\]

\[
\geq \text{vol}(\Omega) - \text{vol}(\Omega_r) - \sum_{j=1}^{k} |a_{r,j}(z)|^2.
\]
From (2.5) and Stokes' formula, we infer
\[
\int_{\Omega} |\Delta \varphi_k(x)|^2 dx = \int_{\Omega} |\Delta g_{r,z}(x) - \sum_{j=1}^{k} a_{r,j}(z) \Delta u_j(x)|^2 dx
\]
\[
= \int_{\Omega} \left( |\Delta g_{r,z}(x)|^2 + \left| \sum_{j=1}^{k} a_{r,j}(z) \Delta u_j(x) \right|^2 \right) dx
\]
\[
- \int_{\Omega} \left( \Delta g_{r,z}(x) \sum_{j=1}^{k} a_{r,j}(z) \Delta u_j(x) + \Delta g_{r,z}(x) \sum_{j=1}^{k} a_{r,j}(z) \Delta u_j(x) \right) dx
\]
(2.8)
\[
= \int_{\Omega} |\Delta g_{r,z}(x)|^2 dx - \sum_{j=1}^{k} \Gamma_j|a_{r,j}(z)|^2
\]
\[
= \int_{\Omega} \left| -|z|^2 f_r(x) + 2i \langle z, \nabla f_r(x) \rangle + \Delta f_r(x) \right|^2 dx - \sum_{j=1}^{k} \Gamma_j|a_{r,j}(z)|^2
\]
\[
= \int_{\Omega} \left\{ \left( -|z|^2 f_r(x) + \Delta f_r(x) \right)^2 + 4 \langle z, \nabla f_r(x) \rangle^2 \right\} dx - \sum_{j=1}^{k} \Gamma_j|a_{r,j}(z)|^2
\]
since
\[
\Delta g_{r,z}(x) = e^{i(z,x)} \left( -|z|^2 f_r(x) + 2i \langle z, \nabla f_r(x) \rangle + \Delta f_r(x) \right).
\]
According to the definition of the function \( f_r \), we have
\[
\Delta f_r(x) = \begin{cases}
0, & x \in \Omega, \ d(x) \geq \frac{1}{r}, \\
r^2 \Delta d^2(x), & x \in \Omega, \ d(x) < \frac{1}{r}, \\
0, & \text{the other}.
\end{cases}
\]
Hence, we obtain, from the Schwarz inequality and \(|\nabla d(x)|^2 = 1,\)
\[
\int_{\Omega} \left\{ \left( -|z|^2 f_r(x) + \Delta f_r(x) \right)^2 + 4 \langle z, \nabla f_r(x) \rangle^2 \right\} dx
\]
(2.9)
\[
\leq |z|^4 \text{vol}(\Omega) + 24 r^2 |z|^2 \text{vol}(\Omega_r) + \int_{\Omega_r} \left( \Delta f_r(x) \right)^2 dx.
\]
For a point \( x \in \Omega \), there is a point \( y = y(x) \in \partial \Omega \) such that \( d(x) = \text{dist}(x,y) \), then we know that
\[
\Delta d^2(x) = 2n - \sum_{j=1}^{n-1} \frac{2}{1 - \kappa_j d(x)},
\]
where \( \kappa_1, \kappa_2, \ldots, \kappa_{n-1} \) are the principal curvatures of \( \partial \Omega \) at the point \( y \). Since the boundary \( \partial \Omega \) of the domain \( \Omega \) is smooth and a compact hypersurface, one has that all of \( \kappa_j \) are bounded. Without loss of generality, we can assume that \( |\kappa_j(y)| \leq \kappa \) for any \( y \in \partial \Omega, 1 \leq j \leq n-1 \), then it follows that if \( r \geq r_0 > n \kappa \), then we see from
\[
0 < \Delta d^2(x) < 2n, \ x \in \Omega_r
\]
and
\[ \int_{\Omega_r} \left( \Delta f_r(x) \right)^2 \, dx \leq 4n^2 r^4 \text{vol}(\Omega_r). \]
Hence, if \( r > r_0 \), then we obtain
\[ \int_{\Omega} |\Delta \varphi_k(x)|^2 \, dx \]
(2.11)
\[ \leq |z|^4 \text{vol}(\Omega) + 24 r^2 |z|^2 \text{vol}(\Omega_r) + 4n^2 r^4 \text{vol}(\Omega_r) - \sum_{j=1}^{k} \Gamma_j |a_{r,j}(z)|^2. \]
From (2.6), (2.7) and (2.11), we have
\[ \Gamma_{k+1} \left( \text{vol}(\Omega) - \text{vol}(\Omega_r) \right) \]
(2.12)
\[ \leq |z|^4 \text{vol}(\Omega) + 24 r^2 |z|^2 \text{vol}(\Omega_r) + 4n^2 r^4 \text{vol}(\Omega_r) + \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j) |a_{r,j}(z)|^2, \]
here \( r > r_0 \).
Let \( B_n(r) \) denote the ball with a radius \( r \) and the origin \( o \) in \( \mathbb{R}^n \). By integrating the above inequality on the variable \( z \) on the ball \( B_n(r) \), we derive
\[ r^n B_n \left( \text{vol}(\Omega) - \text{vol}(\Omega_r) \right) \Gamma_{k+1} \]
(2.13)
\[ \leq r^{n+4} B_n \left( \frac{n}{n+4} \text{vol}(\Omega) + 24 \frac{n}{n+2} \text{vol}(\Omega_r) + 4n^2 \text{vol}(\Omega_r) \right) \]
\[ + \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j) \int_{B_n(r)} |a_{r,j}(z)|^2 \, dz, \quad r > r_0. \]
From Parseval’s identity for Fourier transform, we have
\[ \int_{B_n(r)} |a_{r,j}(z)|^2 \, dz \leq \int_{\mathbb{R}^n} |a_{r,j}(z)|^2 \, dz \]
(2.14)
\[ = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i(z,x)} f_r(x) u_j(x) \, dx \right|^2 \, dz \]
\[ = (2\pi)^n \int_{\mathbb{R}^n} |f_r u_j(z)|^2 \, dz = (2\pi)^n \int_{\mathbb{R}^n} |f_r(x) u_j(x)|^2 \, dx \]
\[ \leq (2\pi)^n. \]
We obtain
\[ r^n B_n \left( \text{vol}(\Omega) - \text{vol}(\Omega_r) \right) \Gamma_{k+1} \]
(2.15)
\[ \leq r^{n+4} B_n \left( \frac{n}{n+4} \text{vol}(\Omega) + 24 \frac{n}{n+2} \text{vol}(\Omega_r) + 4n^2 \text{vol}(\Omega_r) \right) \]
\[ + (2\pi)^n \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j), \quad r > r_0. \]
Taking \( r = 2\pi \left( \frac{1 + k}{B_n(\text{vol}(\Omega) - \text{vol}(\Omega_{r_0}))} \right)^{\frac{1}{n}} \), noting \( k \geq \text{vol}(\Omega) r_0^n \) and \( \frac{2\pi}{(B_n)^n} > 1 \), then we can obtain \( r > r_0 \) and
\[
\frac{1}{1 + k} \sum_{j=1}^{k+1} \Gamma_j
\leq 16\pi^4 \frac{n}{n + 4} \text{vol}(\Omega) + \frac{24n}{n + 2} \text{vol}(\Omega_r) + 4n^2 \text{vol}(\Omega_r)
\leq \frac{1}{B_n^n} \frac{1}{(\text{vol}(\Omega) - \text{vol}(\Omega_{r_0}))^{\frac{n+4}{n}}} \leq \frac{1}{B_n^n} (1 + k)^{\frac{4}{n}}.
\]
(2.16)

This completes the proof of Theorem 1.1.

\begin{proof}

Proof of the corollary 1.1. From (2.16) we have
\[
(2.17) \quad \frac{1}{1 + k} \sum_{j=1}^{k+1} \Gamma_j \leq \frac{1 + 4\left( \frac{6}{n+2} + n \right)(n + 4) \text{vol}(\Omega_r)}{\text{vol}(\Omega)} \frac{\text{vol}(\Omega)}{n + 4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{\frac{4}{n}}} (1 + k)^{\frac{4}{n}}.
\]

Since \( r = 2\pi \left( \frac{1 + k}{B_n(\text{vol}(\Omega) - \text{vol}(\Omega_{r_0}))} \right)^{\frac{1}{n}} \), we have
\[
\frac{\text{vol}(\Omega_r)}{\text{vol}(\Omega)} \leq c_0 \frac{B_n^\frac{4}{n}}{2\pi} \left( 1 - \frac{\text{vol}(\Omega_{r_0})}{\text{vol}(\Omega)} \right)^{\frac{1}{n}} (1 + k)^{-\frac{1}{n}}.
\]

Taking \( c_1 = 4\left( \frac{6}{n+2} + n \right)(n + 4) \frac{B_n^\frac{4}{n}}{2\pi} c_0 \), we have
\[
(2.18) \quad \frac{1}{1 + k} \sum_{j=1}^{k+1} \Gamma_j
\leq \frac{1 + c_1 \left( 1 - \frac{\text{vol}(\Omega_{r_0})}{\text{vol}(\Omega)} \right)^{\frac{1}{n}} (1 + k)^{-\frac{1}{n}}}{n + 4 \left( B_n \text{vol}(\Omega) \right)^{\frac{n}{4}}} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{\frac{4}{n}}} (1 + k)^{\frac{4}{n}}.
\]

Since there exists a constant \( \alpha \) such that
\[
0 < v = \frac{\text{vol}(\Omega_{r_0})}{\text{vol}(\Omega)} \leq \frac{c_0}{(1 + k)^{\frac{1}{n}}} \leq \alpha < 1
\]
with \( r_0 = \left( \frac{1 + k}{\text{vol}(\Omega)} \right)^{\frac{1}{n}} \), we define a function

\[
G(v) = \frac{1 + c_1 (1 - v) \frac{1}{n} (1 + k)^{\frac{1}{n}}}{(1 - v)^{\frac{n+1}{n}}}
\]

with \( G(0) = 1 + c_1 (1 + k)^{-\frac{1}{n}} \). Since

\[
G'(v) = \frac{1 + \frac{4}{n} + c_1 (1 + \frac{3}{n})(1 - v) \frac{1}{n} (1 + k)^{\frac{1}{n}}}{(1 - v)^{\frac{2n+1}{n}}},
\]

by Lagrange mean value theorem, there exists \( 0 < \theta < 1 \) such that

\[
G(v) = G(0) + G'(\theta v)v.
\]

Hence, there exists a constant \( c(n) \) only depended on \( n \) such that

\[
G(v) = G(0) + G'(\theta v)v \
= 1 + c_1 (1 + k)^{-\frac{1}{n}} + \frac{1 + \frac{4}{n} + c_1 (1 + \frac{3}{n})(1 - \theta v) \frac{1}{n} (1 + k)^{-\frac{1}{n}}}{(1 - \theta v)^{\frac{2n+1}{n}}}v \
\leq 1 + c_1 (1 + k)^{-\frac{1}{n}} + \frac{1 + \frac{4}{n} + c_1 (1 + \frac{3}{n})}{(1 - \theta v)^{\frac{2n+1}{n}}}c_0 (1 + k)^{-\frac{1}{n}} \
\leq 1 + c_0 c(n) (1 + k)^{-\frac{1}{n}},
\]

that is,

\[
1 + c_1 \left( 1 - \frac{\text{vol}(\Omega r_0)}{\text{vol}(\Omega)} \right)^{\frac{1}{n}} (1 + k)^{-\frac{1}{n}} \leq 1 + c_0 c(n) (1 + k)^{-\frac{1}{n}}.
\]

Therefore, we obtain

\[
\frac{1}{1 + k} \sum_{j=1}^{k+1} \Gamma_j \leq \frac{n}{n + 4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{\frac{1}{n}}} \left( (1 + k)^{\frac{1}{n}} + c_0 c(n) (1 + k)^{\frac{1}{n}} \right).
\]

This finishes the proof of the corollary 1.1.

\[\square\]

### 3. Lower bounds for eigenvalues

In this section, we will give a proof of the theorem 1.2. The following lemma 3.1 will play an important role in the proof of theorem 1.2.

**Lemma 3.1.** For constants \( b \geq 2, \eta > 0 \), if \( \psi : [0, +\infty) \to [0, +\infty) \) is a decreasing function such that

\[-\eta \leq \psi'(s) \leq 0\]
and

$$A := \int_0^\infty s^{b-1}\psi(s)ds > 0,$$

then, we have

$$\int_0^\infty s^{b+3}\psi(s)ds \geq \frac{1}{b+4}(bA)^{\frac{b+4}{b+1}}\psi(0)^{-\frac{4}{b+1}} + \frac{1}{3b(b+4)\eta^2}(bA)^{\frac{b+2}{b+1}}\psi(0)^{\frac{2b+2}{b+1}}$$

$$+ \frac{(b+2)^2}{72b(b+4)^2}\eta^4 A\psi(0)^4 + \frac{q(b)}{\eta^6}(bA)^{\frac{b+2}{b+1}}\psi(0)^{\frac{6b+2}{b+1}}$$

$$\geq \frac{1}{b+4}(bA)^{\frac{b+4}{b+1}}\psi(0)^{-\frac{4}{b+1}} + \frac{1}{3b(b+4)\eta^2}(bA)^{\frac{b+2}{b+1}}\psi(0)^{\frac{2b+2}{b+1}}$$

$$+ \frac{(b+2)^2}{72b(b+4)^2}\eta^4 A\psi(0)^4.$$  \hspace{1cm} (3.1)

where

$$q(b) = \begin{cases} 
\frac{(13b^3 + 56b^2 - 52b - 32)(b+2)^3}{(12)^3b^4(b+4)^4}, & \text{for } b \geq 4 \text{ or } b = 2, \\
\frac{(4b^3 + 11b^2 - 16b + 4)(b+2)^3}{3 \times (12)^3b^3(b+4)^3}\eta^6, & \text{for } 2 < b < 4.
\end{cases}$$

Proof. By defining

$$\varphi(t) = \frac{\psi(\frac{\psi(0)}{\eta}t)}{\psi(0)},$$

we have $\varphi(0) = 1$ and $-1 \leq \varphi'(t) \leq 0$. Hence, without loss of generality, we can assume

$$\psi(0) = 1 \text{ and } \eta = 1.$$

Define

$$D := \int_0^\infty s^{b+3}\psi(s)ds.$$  \hspace{1cm} (3.2)

If $D = \infty$, the conclusion is correct. Hence, one can assume that

$$D = \int_0^\infty s^{b+3}\psi(s)ds < \infty.$$ 

Thus, $\lim_{s \to \infty} s^{b+3}\psi(s) = 0$ holds. Putting $h(s) = -\psi'(s)$ for $s \geq 0$, we have

$$0 \leq h(s) \leq 1, \quad \int_0^\infty h(s)ds = \psi(0) = 1.$$

By making use of integration by parts, one has

$$\int_0^\infty s^b h(s)ds = b \int_0^\infty s^{b-1}\psi(s)ds = bA,$$  \hspace{1cm} (3.3)

$$\int_0^\infty s^{b+4}h(s)ds \leq (b+4)D.$$  \hspace{1cm} (3.4)
since $\psi(s) \geq 0$. By the same assertion as in [20], one can infer that there exists an $\epsilon \geq 0$ such that
\[
(3.5) \quad \int_{\epsilon}^{\epsilon+1} s^b ds = \int_{0}^{\infty} s^b h(s) ds = bA,
\]
\[
(3.6) \quad \int_{\epsilon}^{\epsilon+1} s^{b+4} ds \leq \int_{0}^{\infty} s^{b+4} h(s) ds \leq (b + 4)D.
\]
Since function $f(s)$ defined by
\[
(3.7) \quad f(s) = bs^{b+4} - (b + 4)\tau s^b + 4\tau s^{b+4} - 4\tau^2 s - \tau^2,
\]
only has two critical points, one is $s = \tau$, the other one is in the interval $(0, \tau)$, we have $f(s) \geq 0$. By integrating the function $f(s)$ from $\epsilon$ to $\epsilon + 1$, we deduce, from (3.3) and (3.4),
\[
(3.8) \quad b(b + 4)D - (b + 4)\tau^4 bA + 4\tau^{b+4} \geq \frac{1}{3} \tau^{b+2}, \quad \text{for any } \tau > 0.
\]
Hence, we have, for any $\tau > 0$,
\[
(3.9) \quad \int_{0}^{\infty} s^{b+3} \psi(s) ds = D \geq \frac{1}{b(b + 4)} \left\{ (b + 4)\tau^4 bA - 4\tau^{b+4} + \frac{1}{3} \tau^{b+2} \right\}.
\]
For $b \geq 4$ or $b = 2$, we have, from Taylor expansion formula,
\[
(1 + t)^{\frac{b+4}{b}} \geq 1 + \frac{4}{b} t + \frac{2(4 - b)}{b^2} t^2 + \frac{2(4 - b)(4 - 2b)}{3b^3} t^3
+ \frac{(4 - b)(2 - b)(4 - 3b)}{3b^4} t^4.
\]
\[
(1 + t)^{\frac{b+4}{b}} \geq 1 + \frac{b + 2}{b} t + \frac{(b + 2)}{b^2} t^2 + \frac{(b + 2)(2 - b)}{3b^3} t^3.
\]
Since it is not hard to prove
\[
(3.10) \quad \frac{1}{b + 1} = \int_{0}^{1} s^b ds \leq \int_{0}^{\infty} s^b h(s) ds = bA,
\]
by making use of the inequality $(s^b - 1)(h(s) - \chi(s)) \geq 0$ for $s \in [0, \infty)$, where $\chi$ is the characteristic function of the interval $[0, 1]$, we have
\[
(b + 1)bA \geq 1.
\]
Taking
\[
\tau = (bA)^{\frac{1}{b}} \left(1 + \frac{b + 2}{12(b + 4)}(bA)^{\frac{b+2}{b}}\right)^{\frac{1}{b}},
\]
we have
\[
(3.11) \quad (b + 4)\tau^4 bA - 4\tau^{b+4} + \frac{1}{3} \tau^{b+2}
= (bA)^{1+\frac{2}{b}} \left( b - \frac{b + 2}{3(b + 4)}(bA)^{\frac{b+2}{b}} \right) \left( 1 + \frac{b + 2}{12(b + 4)}(bA)^{\frac{b+2}{b}} \right)^{\frac{1}{b}}
+ \frac{1}{3} (bA)^{1+\frac{2}{b}} \left( 1 + \frac{b + 2}{12(b + 4)}(bA)^{\frac{b+2}{b}} \right)^{\frac{b+2}{b}}.
Putting
\[ t = \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} , \]
we derive, for \( b \geq 4 \) or \( b = 2 \),
\[ (b - \frac{b+2}{3(b+4)}(bA)^{\frac{2}{3}}) \left( 1 + \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} \right)^{\frac{1}{2}} \]
\[ = (b - 4t)(1 + t)^{\frac{1}{2}} \]
\[ \geq (b - 4t)(1 + \frac{4}{b} + \frac{2(4-b)}{b^2}t^2 + \frac{2(4-b)(4-2b)}{3b^3}t^3 \]
\[ + \frac{(4-b)(2-b)(4-3b)}{3b^4}t^4 \]
\[ = b - \frac{2(4+b)}{b}t^2 - \frac{4(4-b)(4+b)}{b^3}t^3 - \frac{(4-b)(2-b)(4+b)}{b^3}t^4 \]
\[ \geq b - \frac{2(4+b)}{b} \left( \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} \right)^2 - \frac{4(4-b)(4+b)}{b^3} \left( \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} \right)^3 \]
\[ - \frac{(4-b)(2-b)(4+b)}{b^3} \left( \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} \right)^4 \]
\[ \geq 1 + \frac{b+2}{b} + \frac{(b+2)(2-b)}{3b^3}t^3 \]
\[ = 1 + \frac{2+b}{b} \left( \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} \right)^2 + \frac{2+b}{b^2} \left( \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} \right)^3 \]
\[ + \frac{(b+2)(2-b)}{3b^3} \left( \frac{b+2}{12(b+4)}(bA)^{\frac{2}{3}} \right)^4 . \]

From (3.11), (3.12) and (3.13), we obtain
\[ (b+4)^4bA - 4(b+4)^{b+4} + \frac{1}{3}(b+2)^{b+2} \]
\[ \geq b(bA)^{1+\frac{b}{3}} - \frac{2(4+b)}{b} \left( \frac{b+2}{12(b+4)}(bA) \right)^2 \]
\[ - \frac{4(4-b)(4+b)}{3b^2} \left( \frac{b+2}{12(b+4)} \right)^3 (bA)^{1-\frac{b}{3}} \]
\[
- \frac{(4 - b)(2 - b)(4 + b)}{b^3} \left( \frac{b + 2}{12(b + 4)} \right)^4 (bA)^{1 - \frac{4}{b}} \\
+ \frac{1}{3} (bA)^{1 + \frac{2}{b}} + \frac{2 + b}{3b} \left( \frac{b + 2}{12(b + 4)} \right) (bA) + \frac{2 + b}{3b^2} \left( \frac{b + 2}{12(b + 4)} \right)^2 (bA)^{1 - \frac{2}{b}} \\
+ \frac{(b + 2)(2 - b)}{9b^3} \left( \frac{b + 2}{12(b + 4)} \right)^3 (bA)^{1 - \frac{4}{b}}
\]

\[= b(bA)^{1 + \frac{2}{b}} + \frac{1}{3} (bA)^{1 + \frac{2}{b}} + \frac{1}{72} (b + 2)^2 (bA) + \frac{4(b - 1)(b + 4)}{3b^2} \left( \frac{b + 2}{12(b + 4)} \right)^3 (bA)^{1 - \frac{2}{b}} + \frac{(b^2 - 4)(8 - 3b)}{36b^3} \left( \frac{b + 2}{12(b + 4)} \right)^3 (bA)^{1 - \frac{4}{b}}.\]

From \( b \geq 4 \) or \( b = 2 \) and (3.11), we have

\[(bA)^{\frac{2}{b}} \geq \frac{1}{(b + 1)^{\frac{2}{b}}} \geq \frac{1}{3}\]

since \((b + 1)^{\frac{2}{b}} \leq 3\), and

\[(3.15)\quad \frac{4(b - 1)(b + 4)}{3b^2} + \frac{(b^2 - 4)(8 - 3b)}{36b^3} (bA)^{\frac{2}{b}} \geq \frac{13b^3 + 56b^2 - 52b - 32}{12b^3}.\]

According (3.9), (3.14) and (3.15), we obtain

\[
\int_0^\infty s^{b+3}\psi(s)ds = D \\
\geq \frac{1}{b(b + 4)} \left\{ (b + 4)\tau^4 bA - 4\tau^{b+4} + \frac{1}{3} \tau^{b+2} \right\} \\
\geq \frac{1}{b(b + 4)} \left\{ b(bA)^{1 + \frac{2}{b}} + \frac{1}{3} (bA)^{1 + \frac{2}{b}} + \frac{1}{72} (b + 2)^2 (bA) + \frac{13b^3 + 56b^2 - 52b - 32}{12b^3} \left( \frac{b + 2}{12(b + 4)} \right)^3 (bA)^{1 - \frac{2}{b}} \right\}.
\]

For \( 2 < b < 4 \), we can obtain the following inequality using the same method as the case of \( b \geq 4 \),

\[
\int_0^\infty s^{b+3}\psi(s)ds = D \\
\geq \frac{1}{b(b + 4)} \left\{ b(bA)^{1 + \frac{4}{b}} + \frac{1}{3} (bA)^{1 + \frac{4}{b}} + \frac{1}{72} (b + 2)^2 (bA) + \frac{12b^3 + 33b^2 - 48b + 12}{9b^3} \left( \frac{b + 2}{12(b + 4)} \right)^3 (bA)^{1 - \frac{2}{b}} \right\}.
\]

This finishes the proof of the lemma 3.1.
Proof of Theorem 1.2. Using the same notations as those of [10], we can obtain from Lemma 3.1 that
\[
\sum_{j=1}^{k} \Gamma_j \geq n B_n \int_0^{\infty} s^{n+3} g(s) ds
\]
(3.16)
\[
\geq \frac{n}{n+4} (B_n)^{-\frac{2}{n}} k^{\frac{n+4}{n}} g(0)^{-\frac{4}{n}} + \frac{1}{3(n+4)\eta^2} k^{\frac{n+2}{n}} (B_n)^{-\frac{2}{n}} g(0)^{\frac{2n-2}{n}}
\]
\[+ \frac{(n+2)^2}{72n(n+4)^2\eta^4} k g(0)^4,
\]
where \( g : [0, +\infty) \to [0, (2\pi)^{-n} \text{vol}(\Omega)] \) is a non-increasing function of \(|x|\) and \( g(x) \) is defined by \( g(|x|) := h^*(x) \). Here \( h^* \) is the symmetric decreasing rearrangement of \( h \), \( h \) is defined by \( h(z) := \sum_{j=1}^{k} |\hat{\varphi}_j(z)|^2 \), \( \hat{\varphi}_j(z) \) is the Fourier transform of the trial function \( \varphi_j(x) \),
\[
\varphi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}
\]
here \( u_j \) is an orthonormal eigenfunction corresponding to the eigenvalue \( \Gamma_j \).

Now defining a function \( F \) by
\[
F(t) = \frac{n}{n+4} (B_n)^{-\frac{2}{n}} k^{\frac{n+4}{n}} t^{-\frac{4}{n}} + \frac{1}{3(n+4)\eta^2} k^{\frac{n+2}{n}} (B_n)^{-\frac{2}{n}} t^{\frac{2n-2}{n}}
\]
(3.17)
\[+ \frac{(n+2)^2}{72n(n+4)^2\eta^4} k t^4.\]

Since \( \eta \geq (2\pi)^{-n} B_n^{-\frac{2}{n}} \text{vol}(\Omega)^{-\frac{n+1}{n}} \), we obtain
\[
F'(t) \leq -\frac{4}{n+4} (B_n)^{-\frac{2}{n}} k^{\frac{n+4}{n}} t^{-1-\frac{4}{n}} + \frac{2(n-1)}{3n(n+4)} k^{\frac{n+2}{n}} (2\pi)^{2n} \text{vol}(\Omega)^{-\frac{2(n+1)}{n}} t^{-\frac{n-2}{n}}
\]
(3.18)
\[+ \frac{(n+2)^2}{18n(n+4)^2} k t^3 (2\pi)^{4n} (B_n)^{3\frac{n}{n}} \text{vol}(\Omega)^{-\frac{4(n+1)}{n}}
\]
\[= \frac{k}{n+4} t^{-\frac{4n+4}{n}} \times \left\{ \frac{2(n-1)}{3n} (2\pi)^{2n} k^{2\frac{n}{n}} \text{vol}(\Omega)^{-\frac{2(n+1)}{n}} t^{\frac{2n+2}{n}}
\]
\[+ \frac{(n+2)^2}{18n(n+4)^2} (2\pi)^{4n} (B_n)^{4\frac{n}{n}} \text{vol}(\Omega)^{-\frac{4(n+1)}{n}} t^{\frac{4n+4}{n}} \right\} \}
\]

Hence, we have
\[
\frac{n+4}{k} t^{\frac{n+4}{n}} F'(t)
\]
(3.19)
\[
\leq \frac{2(n-1)}{3n} (2\pi)^{2n} k^{2\frac{n}{n}} \text{vol}(\Omega)^{-\frac{2(n+1)}{n}} t^{\frac{2n+2}{n}}
\]
\[- 4(B_n)^{-\frac{4}{n}} k^{\frac{4}{n}} + \frac{(n+2)^2}{18n(n+4)} (2\pi)^{4n} (B_n)^{4\frac{n}{n}} \text{vol}(\Omega)^{-\frac{4(n+1)}{n}} t^{\frac{4n+4}{n}}.\]
Since the right hand side of (3.19) is an increasing function of $t$, if the right hand side of (3.19) is not larger than 0 at $t = (2\pi)^{-n}\text{vol}(\Omega)$, that is,

\[
\frac{2(n-1)}{3n} (2\pi)^{2n} k^2 \frac{4}{\text{vol}(\Omega)} \frac{2^{(n+1)}}{n} ((2\pi)^{-n}\text{vol}(\Omega))^{2n+2} \frac{4}{n} \]

\[
+ \frac{(n+2)^2}{18n(n+4)} (2\pi)^{4n} (B_n)^{\frac{4}{n}} \text{vol}(\Omega) - \frac{4}{n} ((2\pi)^{-n}\text{vol}(\Omega))^{\frac{4n+4}{n}} - 4(B_n)^{\frac{4}{n}} k^2 \leq 0,
\]

then one has from (3.19) that $F'(t) \leq 0$ on $(0, (2\pi)^{-n}\text{vol}(\Omega)]$. If $F'(t) \leq 0$, then $F(t)$ is a decreasing function on $(0, (2\pi)^{-n}\text{vol}(\Omega)]$. By a direct calculation, we have that (3.20) is equivalent to

\[
\frac{(n-1)}{6n} (2\pi)^{-2} k^2 + \frac{(n+2)^2}{72n(n+4)} (2\pi)^{-4} (B_n)^{\frac{4}{n}} \leq (B_n)^{-\frac{4}{n}} k^2.
\]

We now check the equation (3.21). Note that $(2\pi)^{-2} (B_n)^{\frac{4}{n}} < 1$, then one has

\[
\frac{(n-1)}{6n} (2\pi)^{-2} k^2 + \frac{(n+2)^2}{72n(n+4)} (2\pi)^{-4} (B_n)^{\frac{4}{n}} < (2\pi)^{-2} < \frac{1}{6} \frac{2\pi}{6} + \frac{1}{36} (2\pi)^{-2} < (2\pi)^{-2} \left( \frac{1}{6} \frac{2\pi}{6} + \frac{1}{36} \right)
\]

\[
< (2\pi)^{-2} k^2 < (B_n)^{-\frac{4}{n}} k^2.
\]

On the other hand, since $0 < g(0) \leq (2\pi)^{-n}\text{vol}(\Omega)$ and the right hand side of the formula (3.16) is $F(g(0))$, which is a decreasing function of $g(0)$ on $(0, (2\pi)^{-n}\text{vol}(\Omega)]$, then we can replace $g(0)$ by $(2\pi)^{-n}\text{vol}(\Omega)$ in (3.16) which gives inequality

\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n\text{vol}(\Omega))^{\frac{4}{n}}} k^2 \frac{4}{n} + \frac{n+2}{12n(n+4)} \frac{\text{vol}(\Omega)}{I(\Omega)} n + \frac{4\pi^2}{(B_n\text{vol}(\Omega))^{\frac{4}{n}}} k^2 \frac{4}{n} + \frac{(n+2)^2}{1152n(n+4)^2} \left( \frac{\text{vol}(\Omega)}{I(\Omega)} \right)^2.
\]

This completes the proof of Theorem 1.2.

\[\square\]

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