THE ERDŐS-RÉNYI LAW OF LARGE NUMBERS FOR BALLISTIC RANDOM WALK IN RANDOM ENVIRONMENT

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Abstract. We consider a one dimensional ballistic nearest-neighbor random walk in a random environment. We prove an Erdős-Rényi strong law for the increments.

1. Definitions and main results

The classical Erdős-Rényi strong law of large numbers asserts as follows.

Theorem 1.1 (Erdős-Rényi, 1970). Consider a random walk $S_n = \sum_{i=1}^{n} X_i$ with $X_i$ i.i.d., satisfying $EX_1 = 0$. Set $\phi(t) = E[e^{tX}]$ and let $D_\phi^+ = \{ t > 0 : \phi(t) < \infty \}$. Let $\alpha > 0$ be such that $\phi(t)e^{-\alpha t}$ achieves its minimum value for some $t$ in the interior of $D_\phi^+$. Set $1/A_\alpha := -\log \min_{t > 0} \phi(t)e^{-\alpha t}$. Then, $A_\alpha > 0$ and

$$\max_{0 \leq j \leq n - \lceil A_\alpha \log n \rceil} \frac{S_{j + \lceil A_\alpha \log n \rceil} - S_j}{A_\alpha \log n} \rightarrow \alpha, \ a.s.$$  

In the particular case of $X_i \in \{-1, 1\}$, the assumptions of the theorem are satisfied for any $\alpha \in (0, 1)$. The theorem also trivially generalizes to $EX_1 \neq 0$, by considering $Y_i = X_i - EX_1$.

Theorem 1.1 is closely related to the large deviation principle for $S_n/n$ given by Cramer’s theorem, see e.g. [3] for background. Indeed, with $I(x) = \sup_{t}(tx - \log \phi(t))$ denoting the rate function, one observes that $I(\alpha) = 1/A_\alpha$ and that

$$\alpha = \inf \{ x > 0 : I(x) > 1/A_\alpha \}.$$  

In this paper, we prove an analogous statement for standard one dimensional random walk in random environment (RWRE), in the case of positive velocity. We begin by introducing the model. Fix a realization $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ with $\omega_i \in (0, 1)$ of a collection of i.i.d. random variables, which we call the environment. With $p$ denoting the law of $\omega_0$ and $\sigma(p)$ its support, denote by $P = p^\mathbb{Z}$ the law of the environment on $\Sigma_p := \sigma(p)^\mathbb{Z}$. We make throughout the following assumption.

Condition 1.2 (Uniform Ellipticity). There exists $\kappa \in (0, 1)$ such that $\sigma(p) \subset [\kappa, 1 - \kappa]$ almost surely.

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Letting \( \rho_i := (1 - \omega_i)/\omega_i \), we note that the ellipticity assumption gives a deterministic uniform upper and lower bounds on \( \rho_i \).

It will be useful for us to consider also different laws of the environment \( \Sigma = [\kappa, 1 - \kappa]^2 \), not necessarily product laws. Such laws will be denoted \( \eta \). Equipping \( \Sigma \) with the standard shift, the space of measures (stationary/ergodic) on \( \Sigma \) are denoted \( M_1(\Sigma) \) (\( M_1^\star(\Sigma) \/ M_1^0(\Sigma) \)), respectively; similar definitions hold when \( \Sigma \) is replaced by \( \Sigma_p \).

On top of \( \omega \) we consider the RWRE, which is a nearest neighbor random walk \( \{X_t\}_{t \in \mathbb{Z}} \). Conditioned on the environment \( \omega \), \( \{X_t\} \) is a Markov chain with transition probabilities

\[
\pi(i, i + 1) = 1 - \pi(i, i - 1) = \omega_i.
\]

We denote the law of the random walk, started at \( i \in \mathbb{Z} \) and conditioned on a fixed realization of the environment \( \omega \), by \( \mathbb{P}_i^\omega \) (the so-called quenched law). For any measure \( \eta \in M_1(\Sigma) \), the measure \( \eta(d\omega) \otimes \mathbb{P}_i^\omega \) is referred to as the annealed law, and denoted by \( \mathbb{P}_i^\eta \); with some abuse of notation, we sometimes say annealed law for the restriction of \( \mathbb{P}_i^\eta \) to path space. If \( \eta = P \) then we write \( \mathbb{P}_i \) for \( \mathbb{P}_i^P \). We use similar conventions for expectations, e.g. \( \mathbb{E}_i^\eta \) for expectation with respect to \( \mathbb{P}_i^\eta \), etc.

### 1.1. The potential \( V \) and functional \( S \)

Introduce the potential function, which is defined as

\[
V_\omega(j) = \begin{cases} 
\sum_{i=1}^j \log \rho_i(\omega), & \text{if } j > 0; \\
0, & \text{if } j = 0; \\
-\sum_{i=j+1}^0 \log \rho_i(\omega), & \text{if } j < 0.
\end{cases}
\]

(1.3)

and the Lyapunov function, see \[2\],

\[
S(n, \omega) = \begin{cases} 
\sum_{i=0}^{n-1} e^{V_\omega(i)}, & \text{if } n > 0, \\
0, & \text{if } n = 0, \\
\sum_{i=n}^{1} e^{V_\omega(i)}, & \text{if } n < 0.
\end{cases}
\]

(1.4)

By definition, for \( n > m \geq 0 \) we can decompose \( S(n, \omega) \) as

\[
S(n, \omega) = S(m, \omega) + e^{V_\omega(m)} S(n - m, \theta^m \omega).
\]

(1.5)

Another important property of \( S(n, \omega) \) is its relation to hitting times. Let \( \tau_A = \inf\{t > 0 : X_t \in A\} \) and abbreviate \( \tau_i = \tau_1 \) for \( i \in \mathbb{Z} \). Then, see e.g. \[4\] (2.1.4),

\[
\mathbb{P}_x^\omega[\tau_0 > \tau_y] = \frac{S(x, \omega)}{S(y, \omega)}, \quad \text{for } y > x > 0.
\]

(1.6)

Also, for \( n > 0 \),

\[
\max_{0 \leq j \leq n-1} V_\omega(j) \leq S(n, \omega) \leq n \max_{0 \leq j \leq n-1} V_\omega(j).
\]

(1.7)
1.2. Rate functions and modified environments. We follow [1] in introducing the function
\[ \phi(\omega, \lambda) = \mathbb{E}_0^\omega [e^{\lambda \tau_1} \mathbb{1}_{\tau_1 < \infty}], \]
and the hitting time quenched rate function, defined for \( \eta \in M_1(\Sigma) \),
\[ I_{\eta}^{\tau,q}(u) = \sup_{\lambda \in \mathbb{R}} \{ \lambda u - \int \log \phi(\omega, \lambda) \eta(d\omega) \}. \]
We denote the empirical field \( R_n \in M_1(\Sigma) \) by
\[ R_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\theta^j \omega}. \]
It is well known, see e.g. [3], that under \( P \), the sequence \( R_n \) satisfies a large deviation principle in \( M_1(\Sigma) \), equipped with the topology of weak convergence, with rate function \( h(\cdot | \eta) \), the so-called specific relative entropy.

We need to consider the RWRE conditioned on not hitting the origin, i.e. conditioned on \( \tau_0 = \infty \). Using Doob’s h-transform, it is straightforward to check that such conditional law is equivalent to using a transformed environment, namely for all measurable \( A \) and \( i \in \mathbb{Z}^+ \),
\[ P_\omega^i[A | \tau_0 = \infty] = P_{\hat{\omega}^i}[A], \]
where
\[ \hat{\omega}^i = \frac{\omega_i S(i + 1, \omega)}{S(i, \omega)}. \]
Note that due to (1.4), we have that \( \hat{\omega}^i \in [0, 1] \).

For \( L \) a positive integer, consider the following ergodic (with respect to shifts, if the law of \( \omega \) is ergodic) environment obtained as a transformation of \( \omega \),
\[ \hat{\omega}^L_i = \frac{\omega_i S(L + 1, \theta^{i-L} \omega)}{S(L, \theta^{i-L} \omega)}. \]
Here again, \( \hat{\omega}^L_i \in [0, 1] \). Introduce the function
\[ I^F(x, \eta) = \lim_{L \to \infty} \sup_{\lambda \leq 0} \left\{ \lambda - x \int \log \phi(\hat{\omega}^L, \lambda) \eta(d\omega) \right\}, \quad \eta \in M_1^s(\Sigma). \]
The existence of the limit in (1.12) is due to the following lemma, whose proof appears in Section 5.

**Lemma 1.3.** For any fixed \( i \), the sequence \( \{\hat{\omega}^L_i\} \) is decreasing in \( L \in \mathbb{Z}^+ \). Moreover the limit in (1.12) exists for any \( \eta \in M_1^s(\Sigma) \).

For \( \eta \in M_1^s(\Sigma) \), \( I^F(x, \eta) \) has a natural interpretation as a rate function for the quenched LDP of the hitting times of the random walk in random environment, conditioned on never hitting the origin, see Appendix A.
1.3. Statement of main result. With all needed information gathered, we state the main result of the paper.

**Theorem 1.4.** Let \( P = p^Z \) satisfy Condition 1.2. Set
\[
s = \sup\{\theta > 0 : E_p \theta \leq 1\}.
\]
Assume that \( s \in (1, \infty] \). Fix \( A > 0 \). Then, for \( k = k(n) \) positive integer such that \( k(n)/\log n \to A \),
\[
\max_{1 \leq t \leq n-k} \frac{X_{t+k} - X_t}{k} \to_{n \to \infty} x^*(A), \quad \mathbb{P}_0^a - a.s.,
\]
where
\[
x^*(A) = \inf\{x > 0 : I^*(x) > 1/A\},
\]
and
\[
I^*(x) = \inf_{\eta \in M^*_1(\Sigma_p)} \left\{ I^F(x, \eta) + x h(\eta|P) \right\}.
\]
(Compare (1.14) and (1.15) with (1.1) and (1.2).)

Let
\[
v_p := \frac{1 - E_p(\rho_0)}{1 + E_p(\rho_0)}.
\]
We remark, see [5], that the condition \( s \in (1, \infty] \) is equivalent to \( E_p(\rho_0) < 1 \) and is also equivalent to the convergence
\[
\frac{X_n}{n} \to_{n \to \infty} v_p > 0, \quad \mathbb{P}_0^a - a.s..
\]
That is, we are dealing here with the transient ballistic case. It also implies that \( E_p \log \rho_0 < 0 \).

We further note that it follows from the definitions that \( x \mapsto I^*(x) \) is a convex increasing function on \( \mathbb{R}_+ \), with \( I^*(0) = 0 \) and \( I^*(\infty) \to_{x \to \infty} \infty \). Thus, \( I^* \) is continuous on its domain and strictly increasing in the set \( \{x : \infty > I^*(x) > 0\} \). Therefore, \( x^*(A) \) is well defined and satisfies \( A I^*(x^*(A)) = 1 \). It is also obvious from Theorem 1.4 that \( x^*(A) \leq 1 \).

1.4. Proof strategy. The standard proof of Theorem 1.1 and of its extensions to sums of weakly dependent random variables usually consists of an upper and of a lower bounds for increments within time intervals (which we refer to as temporal blocks) of length \( A_0 \log n \). The former relies only on the upper large deviations bound for such sums while the latter in addition to the lower large deviations bound requires also sufficiently weak dependence which enables to split the sum into weakly dependent disjoint blocks (this step is, of course, trivial in the independent case). In this way the corresponding random walk is split into weakly dependent temporal blocks. Such a temporal splitting is not possible in our case of random environment, since (under the annealed measure) increments of the random walk in disjoint time intervals are strongly correlated. So instead, in the proof of Theorem 1.4 we use a spatial decoupling of the walk in order to obtain both upper and lower bounds on maximal increments. This leads to several complications. First, the increments of the walk in different spatial blocks are not independent. Secondly, and more important, the walk may visit a
block many times, and the probability to do so depends not only on the environment in the block but also on adjacent blocks.

The first difficulty is relatively easily dispensed with by appealing to a standard non-backtracking estimate (Lemma 2.2). This allows us to consider only blocks of size $c \log n$ for some large $c$. To address the second issue, we use the environment $\hat{\omega}$, see (1.10), representing the environment under the condition of not backtracking at all, and use it to introduce the crucial quantity $\chi(k, x, c, \eta)$ which serves as a proxy for the probability of having a fast segment of the walk in a block of length $xk$ with $k = k(n)$ such that $k/\log n \to A$, under the ergodic measure $\eta$, see (3.2) for the definition and the crucial Lemma 3.1 for the representation of the maximal increment in terms of $\chi$ (by fast segment we mean a segment which crosses the block faster than typical, that is with speed $1/x$). The rest of the proof involves a study of $\chi$, which is an expectation (with respect to $P$) of functions of the environment (some of which represent quenched large deviations). As in [1], the latter can be represented in terms of a variational problem involving a change in the environment, and a function of quenched large deviations estimate for the RWRE, see (1.16) for the form of the variational principle.

We remark that the proof of Lemma 1.3 requires several approximation steps due to the fact that the environment $\hat{\omega}$ is not an ergodic environment under $\eta$. This is carried out in Section 5. On the other hand, the identification of the rate function requires a study of the variational principle, and it is in the latter study that we use the assumption that $s > 1$, see the statement of Theorem 1.4.

1.5. **Notation.** For two sequences (of possibly random variables) $a_n$ and $b_n$ we will say $a_n \sim b_n$ if it holds almost surely that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 1.$$ 

We say that $a_n = o(b_n)$ if $a_n/b_n \to 0$ almost surely (with respect to the measure under consideration) as $n \to \infty$. Constants, whose values are fixed throughout the paper, are denoted by $\alpha_i$ and we fix

$$(1.19) \quad C_\kappa := (1 - \kappa)/\kappa > 1.$$ 

The shift operator on $\Sigma$ is denoted by $\theta$, so $(\theta^i \omega)_j := \omega_{i+j}$. We also define the flipped and reversed environments $\bar{\omega}$ and $r(\omega)$ by

$$(1.20) \quad \bar{\omega}_i := 1 - \omega_i, \quad r(\omega)_i := \omega_{-i}.$$ 

Recall that $\tau_i := \inf\{t > 0 : X_t = i\}$. We denote the subsequent visits to a site by $\tau_i(j) := \inf\{t > \tau_i(j - 1) : X_t = i\}$, $j \geq 2$, with $\tau_i(1) = \tau_i$. We denote by $\ell(i, t)$ the partial local time of a site $i$ up to time $t$, i.e.

$$(1.21) \quad \ell(i, t) := \sum_{j=0}^{t} 1[X_j = i].$$

The local time at $i$ is defined as $\ell(i) := \lim_{t \to \infty} \ell(i, t)$. 


We also define some functionals that depend on $V$ and $S$ and will be useful later.

\begin{equation}
W(n, \omega) = \frac{e^{V_{\omega}(n)}}{S(n, \omega)},
\end{equation}

\begin{equation}
\xi_n(i, \omega) = \frac{W(i + 1, \theta^{n-i}\omega)}{1 + S(i, \theta^{n-i}\omega)W(n-i, \omega)},
\end{equation}

and

\begin{equation}
\bar{\xi}(i, \omega) = \frac{W(i + 1, \omega)}{1 + S(i, \omega)/S(-\infty, \omega)}.
\end{equation}

We will see in (5.20) below that $\xi_n(i, \omega) \leq \bar{\xi}(i, \theta^n i \omega)$.

2. Non-backtracking estimate

We provide in this section non-backtracking estimates which will be crucial in obtaining spatial decoupling of events.

**Lemma 2.1.** Assume $\eta \in M_1(\Sigma)$. Then, for every $n \geq 1$,

\begin{equation}
W(n, \omega) \leq \frac{1 - 2\kappa}{\kappa} \left(1 - \left(\frac{\kappa}{1 - \kappa}\right)^n\right) \leq \frac{1 - \kappa}{\kappa}, \eta \text{ - a.s.}
\end{equation}

**Proof.** First observe that $W(n, \omega)$ satisfies

\begin{equation}
\frac{1}{W(n, \omega)} = \frac{1}{\rho_n} \left(1 + \frac{1}{W(n-1, \omega)}\right),
\end{equation}

and hence

\begin{equation}
\frac{1}{W(n, \omega)} \geq \frac{\kappa}{1 - \kappa} \left(1 + \frac{1}{W(n-1, \omega)}\right).
\end{equation}

Inductively applying this relation we get

\begin{equation}
\frac{1}{W(n, \omega)} \geq \sum_{j=1}^{n-i} \left(\frac{\kappa}{1 - \kappa}\right)^j + \left(\frac{\kappa}{1 - \kappa}\right)^{n-i} \frac{1}{W(i, \omega)}.
\end{equation}

Using that $W(1, \omega) = \rho_0 \leq (1 - \kappa)/\kappa$ we conclude

\begin{equation}
\frac{1}{W(n, \omega)} \geq \sum_{j=1}^{n} \left(\frac{\kappa}{1 - \kappa}\right)^j = \left(\frac{\kappa}{1 - \kappa}\right) \frac{1 - \left(\frac{\kappa}{1 - \kappa}\right)^n}{1 - \left(\frac{\kappa}{1 - \kappa}\right)},
\end{equation}

yielding the first inequality in (2.1). The second inequality follows from monotonicity in $n$. \qed

For $a$ a positive integer, set $\tau_y^{BT}(a) = \inf\{t > \tau_y : X_t = y - a\}$ (possibly $\tau_y^{BT} = \infty$) as the first backtracking time of $a$ steps for the walk after hitting $y$, and introduce the backtracking event

\begin{equation}
B(n, a) = \bigcup_{y=1}^{n} \{\tau_y^{BT}(a) < \tau_{y+1}\}.
\end{equation}

The following standard lemma shows that large logarithmic in $n$ backtracks are not occurring before hitting position $n$. 

Lemma 2.2. Assume that $P = p^Z$ satisfies the conditions of Theorem 1.4. Then there exists a constant $a_0 > 0$ so that, for any $A > 0, c > 0$ and $k = k(n) \sim A \log n$ so that $ck$ is an integer, and all $n$ large,

\[ \mathbb{P}_0^a[B(n, ck)] \leq n^{1 - a_0 c A}. \]

Proof. Observe that

\[ \mathbb{P}_0^a[B(n, ck)] = \mathbb{P}_0^a \left[ \bigcup_{y=1}^n \{ \tau_y^{BT}(ck) < \tau_{y+1} \} \right] \leq \sum_{y=1}^n \mathbb{P}_0^a[\tau_y^{BT}(ck) < \tau_{y+1}], \]

and therefore,

\[ \mathbb{P}_0^a[B(n, ck)] \leq \sum_{y=1}^n \mathbb{P}_0^a[\tau_y^{BT}(ck) < \tau_{y+1}], \]

From (2.4) and we obtain that, for all large $n$,

\[ \mathbb{P}_0^a[B(n, ck)] \leq n \int e^{V(ck)} \min(1, e^{V(ck)}) \eta(d\omega). \]

Let $\mu = \int \log \rho_0 dp$, which is negative by assumption. From (2.4) and

\[ \frac{W(ck, \omega)}{W(ck, \omega) + 1} \leq \min(1, e^{V(ck)}) \]

we obtain that, for all large $n$,

\[ \mathbb{P}_0^a[B(n, ck)] \leq n \int e^{V(ck)} \eta(d\omega), \]

with $\alpha, \alpha_0$ depending on $p$ only, where the second inequality is due to Hoeffding’s inequality. Recalling that $k \sim A \log n$ concludes the proof. \qed

3. A reduction to block estimates, large deviations, and proof of Theorem 1.4

In this section we reduce the Erdős-Renyi problem to a block estimate, and state a large deviations estimate for the block. The proof of both these estimates is technical and will be provided in subsequent sections. We then show how the block estimates yield the proof of Theorem 1.4.

For $k$ integer and $c, x \in \mathbb{R}_+$, set

\[ f(\omega, x, k) = \mathbb{P}_0^x[\tau_x > k] \quad \text{and} \quad g(\omega, x, c, k) = S(xk, \omega)/S(ck, \omega), \]

where $\hat{\omega}$ is as in Lemma 1.10. (Here and in the sequel, we abuse notation by writing $xk$ and $ck$ instead of $\lfloor xk \rfloor$ or $\lfloor ck \rfloor$, as appropriate.) For $\eta \in M_+(\Sigma)$, set

\[ \chi(k, x, c, \eta) := \int \left( \frac{1 - f(\omega, x, k)}{1 - f(\omega, x, k)(1 - g(\omega, x, c, k))} \right) \eta(d\omega). \]

When $\eta = P$, we omit $\eta$ from the notation and write $\chi(k, x, c)$ for $\chi(k, x, c, \eta)$. 
3.1. A block estimate. Introduce the notation
\[ \dot{X}(k, n) := \max_{1 \leq t \leq n-k} \frac{X_{t+k} - X_t}{k}. \]

The main result of this subsection, whose proof is postponed to Section 4, is the following lemma. Recall the asymptotic velocity \( v_p \), see (1.17).

**Lemma 3.1.** Fix \( A > 0 \) and set \( k = k(n) \sim A \log n \) integer. Then, there exists a constant \( \alpha_1 > 0 \) so that for any \( c > x > 0 \), and all \( n \) large enough,
\begin{equation}
\mathbb{P}_0^a \left[ \dot{X}(k; \tau_{[v_p n]}) \geq x \right] \leq n \chi(k, x, c) + n^{1-\alpha_1 Ac},
\end{equation}
and
\begin{equation}
\mathbb{P}_0^a \left[ \dot{X}(k; \tau_{[v_p n]} < x \right] \leq \exp \left( - \left( \frac{v_p n - k}{ck} \right) - 1 \right) \chi(k, x, c) + n^{1-\alpha_1 Ac}.
\end{equation}

Note that the statement of the lemma is trivial if \( x > 1 \), for then \( f(\omega, x, k) = 1 \) and \( \chi(k, x, c, \eta) = 0 \), as expected.

In the rest of the paper, we always take \( c > c_0 \) where \( \alpha_i Ac_0 > 2 \), \( i = 0, 1 \). This ensures that the error terms in the right hand side of (3.4) and (3.5), and also of (2.3), are summable. We also recall our convention to write for brevity \( ck, xk \) instead of \( \lfloor ck \rfloor, \lfloor xk \rfloor \).

3.2. A logarithmic estimate for \( \chi \). The following result, which gives a representation of \( \chi \) from Lemma 3.1 in terms of the function \( I^* \), is a crucial ingredient in the proof of Theorem 1.4. Its proof is technically involved and postponed to Section 6.

**Proposition 3.2.** Under the assumptions of Theorem 1.4, and \( c > c_0 \), we have that
\[ \lim_{k \to \infty} \frac{1}{k \log \chi(k, x, c)} = I^*(x), \]
where \( I^* \) is as in (1.16).

3.3. Proof of Theorem 1.4. We now combine Lemma 3.1 with Proposition 3.2 to prove Theorem 1.4. Throughout, \( k = k(n) \) is as in the statement of the theorem.

**Proof of Theorem 1.4.** Fix \( \gamma > 0 \) small. Let \( E_0 = E_0(\gamma) = \{ n \leq \tau_{[v_p n/(1-\gamma)]} \} \) and \( E_1 = E_1(\gamma) = \{ n \geq \tau_{[v_p n/(1+\gamma)]} \} \). By (1.18),
\begin{equation}
\frac{\tau_{[v_p n]}}{n} \to 1, \quad \mathbb{P}_0^a - a.s.,
\end{equation}
implying that \( E_0, E_1 \) occur for all large enough \( n \), almost surely under \( \mathbb{P}_0^a \).

From Lemma 3.1 (applied with \( n/(1-\gamma) \) and \( n/(1+\gamma) \)) and the fact that \( k(n) \sim k(n/(1-\gamma)) \sim k(n/(1+\gamma)) \) we get the following bounds
\[ \mathbb{P}_0^a \left[ E_0, \dot{X}(k, n) \geq x \right] \leq \frac{n}{1 - \gamma} \chi(k, x, c) + n^{1-\alpha_1 Ac}, \]
and
\[ \mathbb{P}_0^a \left[ E_1, \dot{X}(k, n) < x \right] \leq \exp \left( - \left( \frac{v_p n/(1+\gamma) - k}{ck} \right) - 1 \right) \chi(k, x, c) + n^{1-\alpha_1 Ac}. \]
From Proposition \[ \text{3.2} \] we obtain that
\[
\lim_{n \to \infty} \frac{\log(n \chi(k, x, c))}{\log n} = 1 - AI^*(x),
\]
and therefore for every \( \varepsilon > 0 \) there is a constant \( \alpha_\varepsilon > 0 \) such that
\[
\alpha_\varepsilon^{-1} n^{1 - AI^*(x) - \varepsilon} \leq n \chi(k, x, c) \leq \alpha_\varepsilon n^{1 - AI^*(x) + \varepsilon},
\]
hence, for some constant \( c'_\varepsilon > 0 \),
\[
\text{(3.8)} \quad \P_0^a [E_1, \hat{X}(k, n) < x] \leq \exp \left(- \frac{c'_\varepsilon}{(1 + \gamma)ck} n^{1 - AI^*(x) - \varepsilon} \right) + n^{1 - \alpha_0 Ac}.
\]
Fix now \( x < x^*(A) \) and set \( \varepsilon = A(I^*(x^*(A)) - I^*(x))/2 \). Because \( I^* \) is strictly increasing in a neighborhood of \( x^*(A) \), we have with \( AI^*(x^*(A)) = 1 \) that \( \varepsilon > 0 \) and \( 1 - AI^*(x) - \varepsilon > 0 \), which together with the choice \( c > c_0 \), imply that the right hand side of (3.8) is summable. Together with (3.6), it follows from the Borel-Cantelli lemma that \( \lim \inf_{n \to \infty} \hat{X}(k, n) \geq x^*(A), \P^a\text{-a.s.} \)

For the other bound let \( n_j = \max\{n : k(n) = j\} \). Since \( k \sim A \log n \), there exist constants \( \alpha_0, \alpha_{10} > 0 \) such that for all values of \( n \) with \( k(n) = j \), we have that \( n \geq \alpha_0 e^{\alpha_{10} j} \). Therefore by (3.7) and (3.4), we have for any \( x > x^*(A) \) and \( \varepsilon > 0 \) that
\[
\sum_{j=1}^{\infty} \P_0^a [E_0, \hat{X}(j, n_j) \geq x] \leq \sum_{j=1}^{\infty} \alpha_\varepsilon n_j^{1 - AI^*(x) + \varepsilon} + C
\]
\[
\leq \sum_{j=1}^{\infty} c_j^\varepsilon e^{\varepsilon / 2} n_j^{1 - AI^*(x) + \varepsilon} + C,
\]
where \( C \) is some constant coming from the summation of the error term in (3.4). Taking \( \varepsilon = A(I^*(x) - I^*(x^*(A)))/2 \) and using that \( I^*(x) > I^*(x^*(A)) = 1/A \) makes the sum in (3.9) finite, therefore by the Borel-Cantelli lemma for all but a finite number of \( j \)'s we have that \( \hat{X}(k, n_j) \leq x \), almost surely. For every \( n \) there is a \( j \) such that \( n \leq n_j \) and \( k(n) = k(n_j) \), therefore \( \hat{X}(k, n) \leq \hat{X}(j, n_j) \), and hence for all but a finite number of \( n \) we have \( \hat{X}(k, n) \leq x \). Since \( x > x^*(A) \) is arbitrary, we obtain together with (3.6) that
\[
\limsup_{n \to \infty} \hat{X}(k, n) \leq x^*(A), \quad \P^a - a.s.,
\]
concluding the proof. \( \square \)

4. Proof of Lemma \[ \text{3.1} \]

We provide in this section the proof of Lemma \[ \text{3.1} \] which was used in the proof of Theorem \[ \text{1.4} \].

Proof of Lemma \[ \text{3.1} \]. Fix \( x, c \) as in the lemma. It follows from Lemma \[ \text{2.2} \] that
\[
\text{(4.1)} \quad \P_0^a[B(n, ck)] \leq n^{1 - \alpha_0 Ar}. \quad \sum_{j=1}^{\infty} \P_0^a [E_0, \hat{X}(j, n_j) \geq x] \leq \sum_{j=1}^{\infty} \alpha_\varepsilon n_j^{1 - AI^*(x) + \varepsilon} + C
\]
\[
\leq \sum_{j=1}^{\infty} c_j^\varepsilon e^{\varepsilon / 2} n_j^{1 - AI^*(x) + \varepsilon} + C,
\]
where \( C \) is some constant coming from the summation of the error term in (3.4). Taking \( \varepsilon = A(I^*(x) - I^*(x^*(A)))/2 \) and using that \( I^*(x) > I^*(x^*(A)) = 1/A \) makes the sum in (3.9) finite, therefore by the Borel-Cantelli lemma for all but a finite number of \( j \)'s we have that \( \hat{X}(k, n_j) \leq x \), almost surely. For every \( n \) there is a \( j \) such that \( n \leq n_j \) and \( k(n) = k(n_j) \), therefore \( \hat{X}(k, n) \leq \hat{X}(j, n_j) \), and hence for all but a finite number of \( n \) we have \( \hat{X}(k, n) \leq x \). Since \( x > x^*(A) \) is arbitrary, we obtain together with (3.6) that
\[
\limsup_{n \to \infty} \hat{X}(k, n) \leq x^*(A), \quad \P^a - a.s.,
\]
concluding the proof. \( \square \)
we write in the proof $v_{pn}$ for $[v_{pn}]$. We bound
\[
\mathbb{P}_0^\omega \left[ X(k, \tau_{v_{pn}}) \geq x \right] = \mathbb{P}_0^\omega \left[ \bigcup_{t=1}^{\tau_{v_{pn}}-k} \{X_{t+k} - X_t \geq xk\} \right] \\
\leq \mathbb{P}_0^\omega \left[ B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{y=-ck}^{v_{pn}-xk} \bigcup_{t=1}^{\tau_{v_{pn}}-k} \{X_{t+k} - y \geq xk, X_t = y\} \right\} \right] \\
+ \mathbb{P}_0^\omega [B(v_{pn}, ck)].
\] (4.2)

Turning to the first term in the right hand side of (4.2), recalling the local time $\ell(\cdot, \cdot)$, see (1.21), we have that
\[
\mathbb{P}_0^\omega \left[ B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{y=-ck}^{v_{pn}-xk} \bigcup_{t=1}^{\tau_{v_{pn}}-k} \{X_{t+k} - y \geq xk, X_t = y\} \right\} \right] \\
\leq \sum_{y=-ck}^{v_{pn}-xk} \mathbb{P}_0^\omega \left[ B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{t=1}^{\tau_{v_{pn}}-k} \{X_{t+k} - y \geq xk, X_t = y\} \right\} \right] \\
\leq \sum_{y=-ck}^{v_{pn}-xk} \mathbb{P}_0^\omega \left[ B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{j=1}^{\ell(y, \tau_{v_{pn}}-k)} \{X_{\tau_y(j)+k} - y \geq xk\} \right\} \right] \\
(4.3)
\]
\[
= \sum_{y=-ck}^{v_{pn}-xk} \mathbb{P}_0^\omega [\tau_y \leq \tau_{v_{pn}}] \mathbb{P}_0^{\theta_y^\omega} \left[ B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{j=1}^{\ell(0, \tau_{v_{pn}}-y-k)} \{X_{\tau_0(j)}+k \geq xk\} \right\} \right],
\]
where $\{X_{\tau_0(j)+k} \geq xk\} = \emptyset$ if $\tau_0(j) = \infty$. Set $\tilde{\tau}_y(0) = 0$ and, for $j \geq 0$, define recursively $\tilde{\tau}_y(j) = \inf\{t > t_y(j) : X_t = y \text{ or } X_t = y + xk\}$ and $t_y(j) = \inf\{t \geq \tilde{\tau}_y(j-1) : X_t = y\}$. These are the consecutive attempts for the walk to cross the interval $[y, y+xk]$. We represent the event
\[
B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{j=1}^{\infty} \{X_{\tilde{\tau}_0(j)+k} \geq xk\} \right\} \text{ in the right hand side of (4.3)}
\]
as
\[
B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{i=1}^{\infty} \{\tilde{\tau}_0(i) - t_0(i) \leq k, X_{\tilde{\tau}_0(i)} = xk\} \right\} \\
\bigcap \left\{ \bigcup_{j=1}^{\infty} \{X_{\tilde{\tau}_0(j)} = 0\} \bigcup \{\tilde{\tau}_0(j) - t_0(j) > k, X_{\tilde{\tau}_0(j)} = xk, t_0(j+1) < \tau_{v_{pn} - y}\} \right\}
\]
which is a subset of
\[
B(v_{pn}, ck)^C \bigcap \left\{ \bigcup_{i=1}^{\infty} \{\tilde{\tau}_0(i) - t_0(i) \leq k, X_{\tilde{\tau}_0(i)} = xk\} \right\} \\
\bigcap \left\{ \bigcup_{j=1}^{\infty} \{X_{\tilde{\tau}_0(j)} = 0\} \bigcup \{\tilde{\tau}_0(j) - t_0(j) > k, X_{\tilde{\tau}_0(j)} = xk, t_0(j+1) < \tau_{ck}\} \right\}
\]
and therefore, using the Markov property,

\begin{equation}
\mathbb{P}_0^\omega \left[ B(v_p n, ck) \cap \bigcap_{i=1}^{\ell(0,\tau_{p n-y-k})} \{X_{\tau_0(i)+k} \geq \omega k \} \right] \leq \sum_{i=1}^{\infty} \mathbb{P}_0^\omega[\tau_0(i) - t_0(i) \leq k, X_{\tau_0(i)} = \omega k] \\
\times (\mathbb{P}_0^\omega[\{X_{\tau_0(1)} = 0\} \cup \{\tau_0(1) - t_0(1) > k, X_{\tau_0(1)} = \omega k, t_0(2) < \tau_0(1)\})^{i-1} \\
= 1 - \mathbb{P}_0^\omega[\tau_0(1) > k, X_{\tau_0(1)} = \omega k] = 1 - \mathbb{P}_0^\omega[\tau_0(1) > k, X_{\tau_0(1)} = \omega k, t_0(2) < \tau_0(1)] \\
= 1 - \mathbb{P}_0^\omega[\tau_0(1) = 0] - \mathbb{P}_0^\omega[\tau_0(1) > k, X_{\tau_0(1)} = \omega k] \mathbb{P}_{x_k}^\omega[\tau_0 < \tau_0(1)].
\end{equation}

We calculate the probabilities in the right hand side of (4.4) separately.

\[\mathbb{P}_0^\omega[\tau_0(1) > k, X_{\tau_0(1)} = \omega k] = \mathbb{P}_0^\omega[\tau_{xk} > k, \tau_0 > \tau_{xk}] = \omega_0 \mathbb{P}_1^\omega[\tau_{xk} > k - 1, \tau_0 > \tau_{xk}] = \omega_0 \mathbb{P}_1^\omega[\tau_{xk} \geq k | \tau_0 > \tau_{xk}] \mathbb{P}_1^\omega[\tau_0 > \tau_{xk}].\]

Recall, see (1.10), that \(\mathbb{P}_0^\omega\) is the law of the random walk in the environment \(\omega\), conditioned on not hitting the origin. With this it holds by the Markov property

\[\mathbb{P}_1^\omega[\tau_{xk} \geq k | \tau_0 = \omega k = \infty] = \lim_{N \to \infty} \mathbb{P}_1^\omega[\tau_{xk} \geq k | \tau_0 > \tau_N] = \lim_{N \to \infty} \frac{\mathbb{P}_1^\omega[\tau_{xk} \geq k \cap \tau_0 > \tau_N]}{\mathbb{P}_1^\omega[\tau_0 > \tau_N]} = \lim_{N \to \infty} \frac{\mathbb{P}_1^\omega[\tau_0 > \tau_{xk} \cap \tau_0 > \tau_N]}{\mathbb{P}_1^\omega[\tau_0 > \tau_{xk}] \mathbb{P}_1^\omega[\tau_0 > \tau_N]} = \mathbb{P}_1^\omega[\tau_{xk} \geq k | \tau_0 > \tau_{xk}],\]

and thus,

\[\mathbb{P}_0^\omega[\tau_0(1) > k, X_{\tau_0(1)} = \omega k] = \omega_0 \mathbb{P}_1^\omega[\tau_{xk} \geq k] \mathbb{P}_1^\omega[\tau_0 > \tau_{xk}].\]

We also have

\[\mathbb{P}_0^\omega[\tau_0(1) = 0] = \mathbb{P}_0^\omega[\tau_0 < \tau_{xk}] = (1 - \omega_0) + \omega_0 \mathbb{P}_1^\omega[\tau_0 < \tau_{xk}] = (1 - \omega_0) + \omega_0 \left(1 - \frac{1}{S(xk, \omega)}\right) = 1 - \frac{\omega_0}{S(xk, \omega)},\]

(4.5)
Using (4.5) in (4.4) we get
\[
\begin{align*}
1 - \mathbb{P}_0^\omega(X_{t_0(1)} = k, X_{t_0(1)} = x) & = 1 - \mathbb{P}_0^\omega(\tau_{t_0(1)} > k, X_{t_0(1)} = x) \mathbb{P}_0^\omega(\tau_{t_0(1)} < \tau) \\
& = \omega_0 \mathbb{P}_1^\omega(\tau_{x_k} < k) \mathbb{P}_1^\omega(\tau_{t_0(1)} > \tau) \\
& \leq \omega_0 \mathbb{P}_1^\omega(\tau_{x_k} < k)(S(x_k, \omega))^{-1} - \omega_0 \mathbb{P}_1^\omega(\tau_{x_k} \geq k)(S(x_k, \omega))^{-1} \mathbb{P}_1^\omega(\tau_{t_0(1)} < \tau) \\
& = \frac{1 - \mathbb{P}_1^\omega(\tau_{x_k} \geq k) \mathbb{P}_1^\omega(\tau_{t_0(1)} < \tau)}{1 - \mathbb{P}_1^\omega(\tau_{x_k} < k) \mathbb{P}_1^\omega(\tau_{t_0(1)} < \tau)}.
\end{align*}
\] (4.6)

Using this in (4.2)–(4.4) we get
\[
\begin{align*}
P_{p, n}^\omega(\hat{X}(k, \tau_{v_p n})) \geq x & \leq \sum_{y = -ck}^{vpn - ck} \mathbb{P}_0^\omega(\tau_{x_k} < k) - \mathbb{P}_1^\omega[\tau_{x_k} \geq k] \mathbb{P}_1^\omega[\tau_{t_0} < \tau] + \mathbb{P}_0^\omega[B(v_p n, c)].
\end{align*}
\] (4.7)

Taking expectations with respect to \( P = p^n \) and using stationarity, we obtain
\[
\begin{align*}
P_0^\omega[\hat{X}(k, \tau_{v_p n}) \geq x] & \leq n \int \left( 1 - f(\omega, x, k) \right) P(\omega) + \mathbb{P}_0^\omega[B(v_p n, c)]
\end{align*}
\] (4.8)

Together with (4.1), this concludes the proof of (4.4).

We turn to the proof of (3.5). Recall that \( k \sim A \log n \) and that we write \( ck \) for \( \lfloor ck \rfloor \). Split the interval \([1, v_p n]\) into blocks of size approximately \( ck \).

\[
\begin{align*}
\mathbb{P}_0^\omega[\hat{X}(k, \tau_{v_p n}) < x] & = \mathbb{P}_0^\omega \bigcap_{t=1}^{\lfloor v_p n - ck \rfloor} \{ X_{t+k} - X_t < k \} \\
& \leq \mathbb{P}_0^\omega \bigcap_{t \in T \cap [0, v_p n - ck]} \{ X_{t+k} - X_t < k \} \\
& = \mathbb{P}_0^\omega \bigcap_{t=1}^{\lfloor v_p n - ck \rfloor} \{ x \in [z_i - ck, z_i + k] : X_t = z_i \} \\
& = \mathbb{P}_0^\omega \bigcap_{t=1}^{\lfloor v_p n - ck \rfloor} \{ \hat{t}_i - 1 > \tau_{z_i} \}.
\end{align*}
\] (4.9)

The event \( \{ X_{\tau_{z_i}(j)+k} - z_i < x \} \) only depends on the environment in the sites \([z_i - k, z_i + k]\). Even though we are considering disjoint blocks of the environment, there is still dependence on the local times. To induce the independence we will make use of another event, first define \( \hat{t}_i = \inf\{ t > \tau_{z_i+1} : X_t = z_i \} \) and
\[
\hat{B}_n(c, k) = \bigcap_{i=1}^{\lfloor v_p n - ck \rfloor} \{ \hat{t}_i > \tau_{z_i+1} \}.
\]
Then from (4.9) and the Markov property we have

\[
\mathbb{P}_0^\omega \left[ \bigcap_{i=1}^{\lfloor vn/k \rfloor} \bigcap_{j=1}^{\ell(z_i, \tau_{z_i} + 1) - k} \{ X_{\tau_{z_i}(j) + k} - z_i < xk \} \right] = \prod_{i=1}^{\lfloor vn/k \rfloor - 1} \mathbb{P}_0^\omega \left[ \bigcap_{j=1}^{\ell(z_i, \tau_{z_i} + 1) - k} \{ X_{\tau_{z_i}(j) + k} - z_i < xk \} \right] + \mathbb{P}_0^\omega [ B_n(c, k)^\ell ]
\]

(4.10)

where we used that \( B_n(c, k) \supset B(v_p n, ck) \). We next control the main term in (4.10):

\[
\mathbb{P}_0^\omega \left[ \bigcap_{j=1}^{\ell(z_i, \tau_{z_i} + 1) - k} \{ X_{\tau_{z_i}(j) + k} - z_i < xk \} \right] = \mathbb{P}_{0}^{\theta_{z_i} \omega} \left[ \bigcap_{j=1}^{\ell(0, \tau_{x_i} - 1) - k} \{ X_{\tau_{x_i}(j) + k} < xk \} \right].
\]

Using analogous computations as in (4.4) we obtain

\[
\mathbb{P}_0^\omega \left[ \bigcap_{j=1}^{\ell(0, \tau_{x_i} - 1) - k} \{ X_{\tau_{x_i}(j) + k} < xk \} \right] = 1 - \frac{\mathbb{P}_0^\omega [ \bar{\tau}_0(1) \leq k, X_{\tau_0(1)} = xk ]}{1 - \mathbb{P}_0^\omega [ \bar{\tau}_0(1) = 0 ] - \mathbb{P}_0^\omega [ \bar{\tau}_0(1) > k, X_{\tau_0(1)} = xk ] \mathbb{P}_x^\omega [ \bar{\tau}_0 < \tau_{z_1} ]}
\]

\[
= 1 - \frac{\mathbb{P}_1^{\omega}[\tau_{xk} \leq k]}{\mathbb{P}_1^{\omega}[\bar{\tau}_{xk} > k](1 - S(ck, \omega)/S(ck, \omega))}
\]

\[
= 1 - \frac{\mathbb{P}_1^{\omega}[\bar{\tau}_{xk} > k]}{\mathbb{P}_1^{\omega}[\bar{\tau}_{xk} > k](1 - S(ck, \omega)/S(ck, \omega))}
\]

Now going back to the product in (4.10) we have

\[
\prod_{i=1}^{\lfloor vn/k \rfloor - 1} \mathbb{P}_0^\omega \left[ \bigcap_{j=1}^{\ell(z_i, \tau_{z_i} + 1) - k} \{ X_{\tau_{z_i}(j) + k} - z_i < xk \} \right]
\]

\[
= \prod_{i=1}^{\lfloor vn/k \rfloor - 1} \mathbb{P}_0^{\theta_{z_i} \omega} \left[ \bigcap_{j=1}^{\ell(0, \tau_{x_i} - 1) - k} \{ X_{\tau_{x_i}(j) + k} < xk \} \right]
\]

\[
= \prod_{i=1}^{\lfloor vn/k \rfloor - 1} \mathbb{P}_1^{\omega}[\bar{\tau}_{xk} > k] S(ck, \omega)/S(ck, \omega)
\]

\[
= \prod_{i=1}^{\lfloor vn/k \rfloor - 1} \frac{\mathbb{P}_1^{\omega}[\bar{\tau}_{xk} > k]}{\mathbb{P}_1^{\omega}[\bar{\tau}_{xk} > k](1 - S(ck, \omega)/S(ck, \omega))}
\]

(4.11)

The terms in the last product of (4.11) are independent since they depend on disjoint subsets of the environment. Moreover, by stationarity they are
identically distributed. Integrating (4.11) with respect to $P$ we obtain
\[
\int \left( \prod_{i=1}^{\lfloor vn - k \rfloor - 1} \frac{f(\theta^z \omega, x, k)g(\theta^z \omega, x, c, k)}{1 - f(\theta^z \omega, x, k)(1 - g(\theta^z \omega, x, c, k))} \right) P(d\omega)
\]
\[
= \left( \int \frac{f(\omega, x, k)g(\omega, x, c, k)}{1 - f(\omega, x, k)(1 - g(\omega, x, c, k))} P(d\omega) \right)^{\lfloor vn - k \rfloor - 1}
\]
\[
= (1 - \int \frac{1 - f(\omega, x, k)}{1 - f(\omega, x, k)(1 - g(\omega, x, c, k))} P(d\omega))^{\lfloor vn - k \rfloor - 1}
\]
\[
\leq \exp \left( - (\lfloor \frac{vn - k}{ck} \rfloor - 1) \int \frac{1 - f(\omega, x, k)}{1 - f(\omega, x, k)(1 - g(\omega, x, c, k))} P(d\omega) \right).
\]
Taking expectations in (4.10) and using the last estimate together with (4.1), this yields (3.5).

\section{Environment partitioning, approximations, and proof of Lemma 1.3}

We introduce a partition of the interval $[0, ck - 1]$ that will be useful when controlling maxima of the potential using empirical fields. We then introduce approximations of various rate functions, and then provide the proof of Lemma 1.3

\subsection{Environment partitioning, basic LDP, and reverse environment}

We begin by introducing a partition of the environment into blocks.

\textbf{Definition 5.1 (ε-partitioning).} Choose $\varepsilon > 0$ small enough so that $x/\varepsilon$ is an integer. Divide the interval $[0, xk - 1] \cap \mathbb{Z}$ into disjoint intervals $I_1, I_2, \ldots, I_{x/\varepsilon}$ of approximate length $\varepsilon k$ in the most even way possible, so for every $i, j$ we have $||I_i| - |I_j|| \leq 1$. Define the intervals $\bar{I}_1, \bar{I}_2, \ldots, \bar{I}_{(c-x)/\varepsilon}$ in the same way as a partitioning for the interval $[xk,cx - 1] \cap \mathbb{Z}$ (observe that since we are using the same value of $\varepsilon$ we cannot assure that $(c-x)/\varepsilon$ is an integer). For every interval we define its empirical field
\[
R_{i,\varepsilon} := \frac{1}{|I_i|} \sum_{j \in I_i} \delta_{\theta^j \omega} \quad \text{and} \quad \bar{R}_{i,\varepsilon} := \frac{1}{|\bar{I}_i|} \sum_{j \in \bar{I}_i} \delta_{\theta^j \omega}.
\]

Define for $m < n$
\[
R_m^n := \frac{1}{n - m} \sum_{j=m}^{n-1} \delta_{\theta^j \omega}.
\]
The next standard lemma exploits the product structure of $P = p^\mathbb{Z}$ to show a joint LDP for appropriate vectors of empirical processes $(R_{i,m})_{i=1}^B$. We omit the straightforward proof.

\textbf{Lemma 5.2.} For any constants $0 = s_0 < s_1 < s_2 < \ldots < s_B = 1$ in $[0,1]$, the vector of empirical fields $(R_{s_1 n}, R_{s_2 n}, \ldots, R_{s_B n})$ satisfies, under $P = p^\mathbb{Z}$, a large deviation principle in $M_1(\Sigma)^B$, equipped with the product topology, with the rate function
\[
I_R(\eta_1, \ldots, \eta_B) := \sum_{i=1}^B (s_i - s_{i-1}) h(\eta_i|P).
\]
Now we make use of the blocks to estimate the value of $g$, as defined in (3.1), in terms of empirical fields. Define

$$
\delta_\varepsilon = \delta_\varepsilon (\{|R_{i,\varepsilon}|_{i=1,\ldots,x/\varepsilon}, \{|\bar{R}_{i,\varepsilon}|_{i=1,\ldots,[(c-x)/\varepsilon]| \}
$$

as

$$
\delta_\varepsilon =: \max_{1 \leq j \leq [(c-x)/\varepsilon]} j \sum_{i=1}^{x/\varepsilon} \int \log \rho_0(\omega) \bar{R}_{i,\varepsilon} (d\omega) - \max_{1 \leq j \leq x/\varepsilon} j \sum_{i=1}^{x/\varepsilon} \int \log \rho_0(\omega) R_{i,\varepsilon} (d\omega).
$$

and

(5.2)

$$
\Delta_\varepsilon = \Delta_\varepsilon (\{|R_{i,\varepsilon}|_{i=1,\ldots,x/\varepsilon}, \{|\bar{R}_{i,\varepsilon}|_{i=1,\ldots,[(c-x)/\varepsilon] |}\} := \varepsilon \delta_\varepsilon - \varepsilon \sum_{i=1}^{x/\varepsilon} \int \log \rho_0(\omega) R_{i,\varepsilon} (d\omega).
$$

Recall the constant $C_\kappa$, see (1.19).

**Lemma 5.3.** Suppose $\omega_x \in [\kappa, 1 - \kappa]$ for all $x$. Then, it holds that

$$
-\frac{1}{k} \log \frac{S(xk, \omega)}{S(ck, \omega)} \leq \left( \Delta_\varepsilon + \varepsilon \log C_\kappa \right)^{+} + \frac{(c + x) \log C_\kappa}{\varepsilon k} + \frac{\log (xk)}{k},
$$

and for large enough $k$,

$$
-\frac{1}{k} \log \frac{S(xk, \omega)}{S(ck, \omega)} \geq \left( \Delta_\varepsilon - \varepsilon \log C_\kappa \right)^{+} - \frac{(c + x) \log C_\kappa}{\varepsilon k} - \frac{\log (ck)}{k}.
$$

**Proof.** Using (1.7) we get

(5.3) \quad \log \frac{S(xk, \omega)}{S(ck, \omega)} \leq \max_{0 \leq j \leq xk-1} V_\omega(j) - \max_{0 \leq j \leq ck-1} V_\omega(j) + \log (xk),

and

(5.4) \quad \log \frac{S(xk, \omega)}{S(ck, \omega)} \geq \max_{0 \leq j \leq xk-1} V_\omega(j) - \max_{0 \leq j \leq ck-1} V_\omega(j) - \log (ck),

Consider the partitioning in Definition 5.1. The size of each interval satisfies $ek \leq |I_i|, |ar{I}_i| \leq ek + 1$ for every $i$ possible, hence

$$
V_\omega(xk) = \sum_{j=0}^{xk-1} \log \rho_j(\omega) = \sum_{i=1}^{x/\varepsilon} \sum_{m \in I_i} \log \rho_m(\omega)
$$

$$
= \sum_{i=1}^{x/\varepsilon} |I_i| \int \log \rho_0(\omega) R_{i,\varepsilon}
$$

$$
= \varepsilon k \sum_{i=1}^{x/\varepsilon} \int \log \rho_0(\omega) R_{i,\varepsilon} + a_\varepsilon(x, k, \omega),
$$

where $a_\varepsilon(x, k, \omega)$ is the error by difference in interval lengths and have the following deterministic bounds

$$
-\frac{x}{\varepsilon} \log C_\kappa \leq a_\varepsilon(x, k, \omega) \leq \frac{x}{\varepsilon} \log C_\kappa.
$$
Hence,

\[ V_\omega(j) = V_\omega(i_m) + \sum_{i=i_m}^j \log \rho_i(\omega) \]

\[ \leq \sum_{i=1}^m \sum_{q \in i} \log \rho_q(\omega) + (\varepsilon k + 1) \log C_\kappa \]

\[ \leq (\varepsilon k + 1) \max_{i=1}^m \int \log \rho_0(\omega) R_{i,\varepsilon}(d\omega) + (\varepsilon k + 1) \log C_\kappa. \]

Hence,

\[ \frac{1}{k} \max_{0 \leq j \leq xk} V_\omega(j) = \frac{1}{k} \max_{1 \leq m \leq x/\varepsilon} \max_{j \in I_m} V_\omega(j) \]

\[ \leq \frac{(\varepsilon k + 1)}{k} \max_{1 \leq m \leq x/\varepsilon} \sum_{i=1}^m \int \log \rho_0(\omega) R_{i,\varepsilon}(d\omega) + \frac{(\varepsilon k + 1)}{k} \log C_\kappa, \]

and

\[ \frac{1}{k} \max_{z k \leq j \leq xk} V_\omega(j) \geq \frac{(\varepsilon k + 1)}{k} \max_{1 \leq m \leq [z(x)/\varepsilon]} \sum_{i=1}^m \int \log \rho_0(\omega) R_{i,\varepsilon}(d\omega) - \frac{(\varepsilon k + 1)}{k} \log C_\kappa. \]

Analogous calculations hold for the empirical fields \( \bar{R}_{i,\varepsilon} \):

\[ \frac{1}{k} \max_{z k \leq j \leq xk} V_\omega(j) - V_\omega(xk) \]

\[ \leq (\varepsilon + \frac{1}{k}) \max_{1 \leq m \leq [z(x)/\varepsilon]} \sum_{i=1}^m \int \log \rho_0(\omega) \bar{R}_{i,\varepsilon}(d\omega) + (\varepsilon + \frac{1}{k}) \log C_\kappa, \]

and

\[ \frac{1}{k} \max_{z k \leq j \leq xk} V_\omega(j) - V_\omega(xk) \]

\[ \geq (\varepsilon + \frac{1}{k}) \max_{1 \leq m \leq [z(x)/\varepsilon]} \sum_{i=1}^m \int \log \rho_0(\omega) \bar{R}_{i,\varepsilon}(d\omega) - (\varepsilon + \frac{1}{k}) \log C_\kappa. \]

By the definition it holds that \( |\delta_\varepsilon| \leq c \log C_\kappa / \varepsilon \). Also for positive \( b \) we have \((a + b)^+ \leq a^+ + b\) and hence

\[ (\Delta_\varepsilon + \frac{\delta_\varepsilon}{k} + (\varepsilon + \frac{x}{\varepsilon k}) \log C_\kappa)^+ \leq (\Delta_\varepsilon + \varepsilon \log C_\kappa)^+ + \frac{(c + x) \log C_\kappa}{\varepsilon k}, \]

and if \( b < |a| \) it holds \((a - b)^+ \geq a^+ - b\). Therefore for large enough \( k \)

\[ (\Delta_\varepsilon + \frac{\delta_\varepsilon}{k} - (\varepsilon + \frac{x}{\varepsilon k}) \log C_\kappa)^+ \geq (\Delta_\varepsilon - \varepsilon \log C_\kappa)^+ - \frac{(c + x) \log C_\kappa}{\varepsilon k}, \]

Applying those bounds to (5.3) and (5.4) yields the lemma. \( \Box \)

Recall the notation \( \hat{\omega} \) and \( \hat{\omega}^L \) (see (1.10) and (1.11)).

**Lemma 5.4.** For any \( x > L \) and \( \eta \in M^*_1(\Sigma) \),

\[ \int \log \rho_0(\hat{\omega}^L) \eta(d\omega) \leq \int \log \rho_0(\hat{\omega}) \eta(d\omega) \leq - \left| \int \log \rho_0(\omega) \eta(d\omega) \right|, \]
Proof. Since \( \hat{\omega}_x^L \geq \hat{\omega}_x \) for all \( x > L \), we have that \( \rho_x(\hat{\omega}) \geq \rho_x(\hat{\omega}_x^L) \), and the first inequality follows. To get the second one observe that by the definition of the transformed probabilities it holds that

\[
\int \log \rho_x(\hat{\omega}) \eta(d\omega) = \int \log \frac{\rho_x(\omega)S(x-1, \omega)}{S(x+1, \omega)} \eta(d\omega)
\]

(5.5)

\[
= \int \log \rho_0(\omega) \eta(d\omega) + \int \log \frac{S(x-1, \omega)}{S(x+1, \omega)} \eta(d\omega).
\]

The second term satisfies

\[
\frac{S(x-1, \omega)}{S(x+1, \omega)} = \sum_{j=0}^{x-2} e^{V_\omega(j)} e^{V_\omega(x-2) - \sum_{j=0}^{x-2} e^{V_\omega(j) - V_\omega(x-2)}}
\]

but \( V_\omega(j) - V_\omega(x) = -\sum_{i=j+1}^{x} \log \rho_i(\omega) = V_\tau(\theta^x \omega)(x-j) \) and therefore, recalling the notation \( r(\cdot) \) and \( \hat{\omega} \) for the reversed and flipped environment, we have

\[
\frac{S(x-1, \omega)}{S(x+1, \omega)} = \frac{S(x-1, r(\theta^x \hat{\omega}))}{\rho_x \rho_{x-1} S(x+1, r(\theta^x \hat{\omega}))}.
\]

Integrating with respect to \( \eta \) we obtain that

\[
\int \log \frac{S(x-1, \omega)}{S(x+1, \omega)} \eta(d\omega) = \int \log \frac{S(x-1, r(\theta^x \hat{\omega}))}{\rho_x \rho_{x-1} S(x+1, r(\theta^x \hat{\omega}))} \eta(d\omega)
\]

\[
= -2 \int \log \rho_0(\omega) \eta(d\omega) + \int \log S(x-1, \hat{\omega}) \eta(d\omega) - \int \log S(x+1, \hat{\omega}) \eta(d\omega)
\]

\[
\leq -2 \int \log \rho_0(\omega) \eta(d\omega),
\]

where the second equality used the stationarity of the measure, while the inequality used that \( S(x-1, \hat{\omega}) \leq S(x+1, \hat{\omega}) \), as the latter is a sum of positive terms. Substituting the last display in (5.5), we obtain that

\[
\int \log \rho_x(\hat{\omega}) \eta(d\omega) \leq - \int \log \rho_0(\omega) \eta(d\omega).
\]

To conclude the proof one just needs to notice that \( \hat{\omega}_x \geq \omega_x \) for all \( x \). \( \square \)

5.2. Approximate rate functions and LDP for conditioned environment. We introduce the following functions

\[
\phi_i(\hat{\omega}, \lambda) = \mathbb{E}_i^{\hat{\omega}}[e^{\lambda \tau_{i+1}}],
\]

\[
\phi_{i,M}(\hat{\omega}, \lambda) = \mathbb{E}_i^{\hat{\omega}}[e^{\lambda \tau_{i+1} \mathbb{1}[\tau_{i+1} < M]}],
\]

\[
\hat{\phi}^L_M(\lambda, \omega) = \phi^M(\lambda, \hat{\omega}^L) = \mathbb{E}_0^{\hat{\omega}^L}[e^{\lambda \tau_1 \mathbb{1}[\tau_1 < M]}],
\]

\[
\hat{\phi}_L(\lambda, \omega) = \phi(\lambda, \hat{\omega}^L) = \mathbb{E}_0^{\hat{\omega}^L}[e^{\lambda \tau_1}].
\]
Observe that by Lemma 5.4 there are no concerns about $\tau_{i+1}$ being finite on those functions. Fix $1 \leq J < xk$ integer and define

$$\hat{I}_J(x, k, \omega) = \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{xk-1} \log \phi_i(\hat{\omega}, \lambda) \right\},$$

$$\hat{I}_{J,M}(x, k, \omega) = \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{xk-1} \log \phi_{i,M}(\hat{\omega}, \lambda) \right\},$$

$$\hat{I}_L^J(x, k, \omega) = \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{xk-1} \log \phi_L^M(\lambda, \theta^i \omega) \right\},$$

$$\hat{I}_{L,M}^J(x, k, \omega) = \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{xk-1} \log \phi_L^M(\lambda, \theta^i \omega) \right\}.$$

In general, we suppress the notations $k$ and $\omega$ from these functions when no confusion is possible, we also suppress $J$ in case it is 0.

**Lemma 5.5.** Suppose $\omega_i \in [\kappa, 1 - \kappa]$ for all $i$. Then, for any $J < xk$ (possibly depending on $k$), it holds that

$$-\frac{1}{k} \log \mathbb{P}^{\hat{\omega}}_J[\tau_{xk} \leq k] \geq \hat{I}_J(x, k, \omega).$$

Further, there is a constant $\alpha_3 = \alpha_3(\kappa, x) > 0$ so that for any $M > 0$ and $l \in (0, 1)$,

$$-\frac{1}{k} \log \mathbb{P}^{\hat{\omega}}_J[\tau_{xk} \leq k(1 + l)] \leq \hat{I}_{J,M}(x, k, \omega) + \alpha_3 l - \frac{1}{k} \log \left( 1 - 2e^{-\frac{2M^2}{xk^2}} \right).$$

**Proof.** Let $\tau_+(i) = \inf\{t > 0 : X_{\tau_i + t} = i + 1\}$ denote the time to hit $i + 1$ from $i$. We can decompose $\tau_{xk}$ as the sum of such variables and therefore for $\lambda \leq 0$ we have from Chebyshev’s inequality and the strong Markov property that

$$\mathbb{P}^{\hat{\omega}}_J[\tau_{xk} \leq k] = \mathbb{P}^{\hat{\omega}}_J \left[ \sum_{i=J}^{xk-1} \tau_+(i) \leq k \right]$$

$$\leq \exp \left( \inf_{\lambda \leq 0} \left\{ \sum_{i=J}^{xk-1} \log \mathbb{E}^{\hat{\omega}}_i [e^{\lambda \tau_{i+1}} 1[\tau_{i+1} < \infty]] - \lambda k \right\} \right)$$

$$\leq \exp \left( -k \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{xk-1} \log \mathbb{E}^{\hat{\omega}}_i [e^{\lambda \tau_{i+1}}] \right\} \right),$$

which implies (5.6).

We turn to the proof of (5.7). Let

$$F_M(\lambda, x, k, \hat{\omega}) = \lambda - \frac{1}{k} \sum_{i=J}^{xk-1} \log \phi_{i,M}(\hat{\omega}, \lambda), \quad \lambda_M^*(x) = \arg \max_{\lambda \leq 0} F_M(\lambda, x, k, \hat{\omega}).$$
Since
\[
F_M(0, x, k, \hat{\omega}) = -\frac{1}{k} \sum_{i=J}^{xM-1} \log \mathbb{P}_{j}^{\hat{\omega}}[\tau_{i+1} < M] < -\frac{1}{k} \sum_{i=J}^{xM-1} \log \mathbb{P}_{j}^{\hat{\omega}}[\tau_{i+1} = 1] = \lim_{\lambda \to -\infty} F_M(\lambda, x, k, \hat{\omega}),
\]
and \(E_{\hat{\omega}}^j(\tau_{i+1} | \tau_{i+1} < M) \geq 1\) with strict inequality when \(i > 1\), it follows that \(\lambda_M(x) \neq 0\) is well defined. Now let \(A_{\omega, M} = \{\tau_+ (i) \leq M, \text{ for all } i \in [J, xM - 1]\}\) and denote by \(\tilde{\mathbb{P}}_{\omega, M}\) the quenched law of \(\{\tau_+ (i)\}_{J \leq i < xM}\) conditioned on the event \(A_{\omega, M}\). Fix \(l > 0\) and write
\[
\mathbb{P}_{j}^{\hat{\omega}}[\tau_{xk} < k(1 + l)] \geq \mathbb{P}_{j}^{\hat{\omega}}[\tau_{xk} < k(1 + l) | A_{\omega, M}] \mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M}]
\]
\[
= \tilde{\mathbb{P}}_{j}^{\hat{\omega}, M}[k(1 - l) < \tau_{xk} < k(1 + l)] \mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M}]
\]
\[
= \tilde{\mathbb{P}}_{j}^{\hat{\omega}, M}[k(1 - l) < \sum_{i=J}^{xM-1} \tau_+ (i) < k(1 + l)] \mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M}]
\]
(5.9)
\[
= \int \mathbb{1}([z_i])_i : k(1 - l) < \sum_{i} z_i < k(1 + l)] \tilde{\mathbb{P}}_{\omega, M}(dz) \mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M}].
\]
Since \(\tilde{\mathbb{P}}_{\omega, M}\) is a product measure we can write
(5.10)
\[
\tilde{\mathbb{P}}_{\omega, M}(dz) = \prod_{i=J}^{xM-1} \mathbb{P}_{i}^{\hat{\omega}, M}(dz_i),
\]
where \(\mathbb{P}_{i}^{\hat{\omega}, M}\) corresponds to the quenched law of \(\tau_+ (i)\) under \(A_{\omega, M}\). Define the tilted measure \(\mathbb{Q}_{i}^{\hat{\omega}, M}\) by
\[
\mathbb{Q}_{i}^{\hat{\omega}, M}(dz) = \frac{\mathbb{P}_{j}^{\hat{\omega}}[\tau_+ (i) < M | e^{\lambda_M(x)} z]}{\phi_i(M) \lambda_M(x)} \mathbb{P}_{i}^{\hat{\omega}, M}(dz).
\]
Since
\[
\int e^{\lambda_M(x)} z \mathbb{P}_{i}^{\hat{\omega}, M}(dz) = \frac{\mathbb{E}_{j}^{\hat{\omega}}[e^{\lambda_M(x)} \tau_+ (i) \mathbb{1}[\tau_+ (i) < M]]}{\mathbb{P}_{j}^{\hat{\omega}}[\tau_+ (i) < M]} = \frac{\phi_i(M) \lambda_M(x)}{\mathbb{P}_{j}^{\hat{\omega}}[\tau_+ (i) < M]},
\]
then \(\mathbb{Q}_{i}^{\hat{\omega}, M}\) is indeed a probability measure. We also consider the joint product measure \(\mathbb{Q}_{\hat{\omega}, M}\) in a way analogous to \(5.10\):
\[
\mathbb{Q}_{\hat{\omega}, M}(dz) = \prod_{i=J}^{xM-1} \mathbb{Q}_{i}^{\hat{\omega}, M}(dz_i)
\]
\[
= \frac{\mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M} e^{\lambda_M(x)} \sum_{i} z_i]}{\prod_{i=J}^{xM-1} \phi_i(M) \lambda_M(x)} \tilde{\mathbb{P}}_{\omega, M}(dz),
\]
\[
= \frac{\mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M} e^{\lambda_M(x)} \sum_{i} z_i]}{\prod_{i=J}^{xM-1} \phi_i(M) \lambda_M(x)} \tilde{\mathbb{P}}_{\omega, M}(dz),
\]
\[
= \frac{\mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M} e^{\lambda_M(x)} \sum_{i} z_i]}{\prod_{i=J}^{xM-1} \phi_i(M) \lambda_M(x)} \tilde{\mathbb{P}}_{\omega, M}(dz) = \frac{\mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M} e^{\lambda_M(x)} \sum_{i} z_i]}{\prod_{i=J}^{xM-1} \phi_i(M) \lambda_M(x)} \tilde{\mathbb{P}}_{\omega, M}(dz),
\]
\[
= \frac{\mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M} e^{\lambda_M(x)} \sum_{i} z_i]}{\prod_{i=J}^{xM-1} \phi_i(M) \lambda_M(x)} \tilde{\mathbb{P}}_{\omega, M}(dz) = \frac{\mathbb{P}_{j}^{\hat{\omega}}[A_{\omega, M} e^{\lambda_M(x)} \sum_{i} z_i]}{\prod_{i=J}^{xM-1} \phi_i(M) \lambda_M(x)} \tilde{\mathbb{P}}_{\omega, M}(dz),
\]
Now we prove that see (5.8) for the definition of $\lambda_M^*(x)$. Introduce the set $C(z) = \{k(1-l) < z < k(1+l)\}$. From (5.9) we get
\[
\mathbb{P}_{\hat{\lambda}}[\tau_k < k(1+l)] \geq \int 1[C(\sum_i z_i)] \tilde{\mathbb{P}}^{\hat{\lambda}}_{\hat{\lambda} M}(dz) \mathbb{P}_{\hat{\lambda}}^{\hat{\lambda} M}[A_{\hat{\lambda}, M}]
\]
\[
= \prod_{i=1}^{k-1} \phi_{i,M}(\hat{\lambda}, \lambda^*_M(x)) \int \frac{\tilde{\mathbb{P}}^{\hat{\lambda}}_{\hat{\lambda} M}[A_{\hat{\lambda}, M}] e^{\lambda^*_M(x)(k-1)}}{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x))} 1[C(\sum_i z_i)] \tilde{\mathbb{P}}^{\hat{\lambda} M}(dz)
\]
\[
\geq \prod_{i=1}^{k-1} \phi_{i,M}(\hat{\lambda}, \lambda^*_M(x)) \int \frac{\tilde{\mathbb{P}}^{\hat{\lambda}}_{\hat{\lambda} M}[A_{\hat{\lambda}, M}] e^{\lambda^*_M(x) \sum_i z_i}}{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x))} 1[C(\sum_i z_i)] \tilde{\mathbb{P}}^{\hat{\lambda} M}(dz)
\]
\[
\geq \prod_{i=1}^{k-1} \phi_{i,M}(\hat{\lambda}, \lambda^*_M(x)) \int 1[C(\sum_i z_i)] Q^{\hat{\lambda} M}_{ij} (dz)
\]
(5.11)
\[
= e^{-kI_{x, M}(k, \omega)} e^{\lambda^*_M(x)k} Q^{\hat{\lambda} M}_{ij} \left[ k(1-l) < \sum_{i=1}^{k-1} \tau_+ (i) < k(1+l) \right].
\]

Now we prove that
\[
Q^{\hat{\lambda} M}_{ij} \left[ k(1-l) < \sum_{i=1}^{k-1} \tau_+ (i) < k(1+l) \right] \geq 1 - 2e^{-\frac{2k^2}{s M^2}}.
\]
(5.12)

First observe that the moment generating function of $\tau_+ (i)$ under $Q^{\hat{\lambda} M}_{ij}$ is
\[
\phi_{i,M}(\hat{\lambda}, \lambda) = \int e^{\lambda z} Q^{\hat{\lambda} M}_{ij} (dz)
\]
\[
= \mathbb{P}_{\hat{\lambda}}[\tau_+ (i) < M] \int \frac{e^{(\lambda + \lambda^*_M(x))z}}{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x))} Q^{\hat{\lambda} M}_{ij} (dz)
\]
\[
= \frac{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x) + \lambda)}{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x))}.
\]

Therefore
\[
\int \tau_+ (i) d Q^{\hat{\lambda} M}_{ij} = \frac{\phi_{i,M}^\prime(\hat{\lambda}, \lambda^*_M(x))}{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x))},
\]
and thus
\[
\int \sum_{i=1}^{k-1} \tau_+ (i) d Q^{\hat{\lambda} M}_{ij} = \sum_{i=1}^{k-1} \frac{\phi_{i,M}^\prime(\hat{\lambda}, \lambda^*_M(x))}{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x))}.
\]

Since $\lambda^*_M(x) \in (-\infty, 0)$, it is a critical point of the function $F_{M}$, i.e.
\[
\frac{\partial}{\partial \lambda} \left( \lambda - \frac{1}{k} \sum_{i=1}^{k-1} \log \phi_{i,M}(\hat{\lambda}, \lambda) \right) \bigg|_{\lambda = \lambda^*_M(x)} = 0 \Leftrightarrow \sum_{i=1}^{k-1} \frac{\phi_{i,M}^\prime(\hat{\lambda}, \lambda^*_M(x))}{\phi_{i,M}(\hat{\lambda}, \lambda^*_M(x))} = k,
\]
implying that
\[ \int \sum_{i=J}^{x-1} \tau_+(i) dQ_{\hat{\omega},M} = k. \]

Since under \( Q_{\hat{\omega},M} \) we have \( \tau_+(i) < M \) for all \( i \), and \( Q_{\hat{\omega},M} \) is (quenched) a product measure, we can use Hoeffding’s inequality (see e.g. [3, Corollary 2.4.7]) to conclude (5.12).

We next find an lower bound for \( \lambda^*_M(x) \). Toward this end, recall that \( \lambda^*_M(x) \) is the solution of the equation
\[ \frac{1}{k} \sum_{i=J}^{x-1} \phi'_{i,M}(\hat{\omega}, \lambda) = \phi_{i,M}(\hat{\omega}, \lambda), \]
but since \( \lambda \leq 0 \),
\[ \frac{\phi'_{i,M}(\hat{\omega}, \lambda)}{\phi_{i,M}(\hat{\omega}, \lambda)} = \frac{E_\lambda^{\pi}[\tau_{i+1} e^{\lambda \tau_{i+1}} \mathbb{1}[\tau_{i+1} < M]]}{E_\lambda^{\pi}[e^{\lambda \tau_{i+1}} \mathbb{1}[\tau_{i+1} < M]]} \leq \frac{\omega_i e^\lambda}{\omega_i e^\lambda} \leq 1 + \frac{e^\lambda}{\kappa}, \]
thus
\[ k = \sum_{i=J}^{x-1} \phi'_{i,M}(\hat{\omega}, \lambda^*_M(x)) \leq (xk - J) \left( 1 + \frac{e^{\lambda^*_M(x)}}{\kappa} \right) \]
\[ \leq xk \left( 1 + \frac{e^{\lambda^*_M(x)}}{\kappa} \right) \]
(5.13)
\[ \Rightarrow e^{\lambda^*_M(x)} \geq \kappa \left( \frac{1-x}{x} \right) \Rightarrow \lambda^*_M(x) \geq \log \left( \kappa \left( \frac{1-x}{x} \right) \right). \]

Combining (5.13), (5.12) with (5.11) yields (5.7). \( \square \)

Lemma 5.6. For any \( a, b \in \mathbb{Z}^+ \) it holds that
\[ S(a, \theta^b \hat{\omega}) = \frac{S(b+1, \omega)S(b, \omega)}{e^{V_\omega(b)}} \left( \frac{1}{S(b, \omega)} - \frac{1}{S(a+b, \omega)} \right). \]

Proof. By the definition of \( S \) we have
\[ S(a, \theta^b \hat{\omega}) = \sum_{i=0}^{a-1} e^{V_{\phi(i)}}. \]
Thus we have a telescopic sum and hence

\[ V_{\theta^b_\omega}(i) = \sum_{j=1}^i \log \rho_j(\theta^b_\omega) = \sum_{j=1}^i \log \rho_{j+b}(\omega) \]

\[ = \sum_{j=1}^i \left( \log \rho_{j+b}(\omega) + \log \frac{S(j+b-1,\omega)}{S(j+b+1,\omega)} \right) \]

\[ = \sum_{j=b+1}^{i+b} \log \rho_j(\omega) + \log \frac{S(b+1,\omega)S(b,\omega)}{S(i+b-1,\omega)S(i+b,\omega)} \]

\[ (5.14) \quad = V_\omega(i+b) - V_\omega(b) + \log \frac{S(b+1,\omega)S(b,\omega)}{S(i+b+1,\omega)S(i+b,\omega)}. \]

Hence

\[ S(a, \theta^b_\omega) = \frac{S(b+1,\omega)S(b,\omega)}{e^{V_\omega(b)}} \sum_{i=0}^{a-1} \frac{e^{V_\omega(i+b)}}{S(i+b+1,\omega)S(i+b,\omega)}, \]

but observe that

\[ \frac{e^{V_\omega(i+b)}}{S(i+b+1,\omega)S(i+b,\omega)} = \frac{1}{S(i+b,\omega)} - \frac{1}{S(i+b+1,\omega)}. \]

Thus we have a telescopic sum and

\[ S(a, \theta^b_\omega) = \frac{S(b+1,\omega)S(b,\omega)}{e^{V_\omega(b)}} \left( \frac{1}{S(b,\omega)} - \frac{1}{S(a,\omega)} \right), \]

concluding the proof. \(\square\)

**Lemma 5.7.** Suppose \(\omega_i \in [\kappa, 1 - \kappa]\) for all \(i\). Then, for any \(a > 0\) so that \(a \log M\) is an integer,

\[ \frac{1}{k} \sum_{i=1}^{xk-1} \mathbb{P}_{\tilde{\omega}}[\tau_{i+1} \geq M] \leq xe^{-M^{1+\log a/(a \log M)}} + C_k \sum_{i=1}^{xk-1} \xi_i(a \log M, \omega). \]

Moreover, if \(\eta \in M_1(\Sigma)\) then

\[ \lim_{M \to \infty} \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{xk-1} \mathbb{P}_{\tilde{\omega}}[\tau_{i+1} \geq M] = 0, \quad \eta - a.s. \]

**Proof.** Fix \(a > 0\) and observe that

\[ \mathbb{P}_{\tilde{\omega}}[\tau_{i+1} \geq M] = \mathbb{P}_{\tilde{\omega}}[\tau_{i+1} \geq M, \tau_{i+1} < \tau_{i-\log M}] + \mathbb{P}_{\tilde{\omega}}[\tau_{i+1} \geq M, \tau_{i+1} > \tau_{i-\log M}] \]

\[ (5.17) \quad \leq \mathbb{P}_{\tilde{\omega}}[\tau_{i+1,\log M} \geq M] + \mathbb{P}_{\tilde{\omega}}[\tau_{i+1} > \tau_{i-\log M}]. \]

We deal with each term in the right hand side of (5.17) separately. Concerning the first one, note that from any point \(m\) inside the interval \([i - a \log M, i + 1]\) we can exit in \(a \log M\) steps to the right. Recalling that \(\tilde{\omega}_i \geq \omega_i\), see (1.10) and using the ellipticity bound, it holds that

\[ \mathbb{P}_m[\tau_{i+1,\log M} \leq a \log M] \geq \kappa^a \log M, \]
and therefore, by the Markov property,
(5.18)
\[ P_\omega^\tau [\tau_{i+1, i-a \log M} \geq M] \leq \left(1 - \kappa \alpha \log M\right)^{M/(\alpha \log M)} \leq e^{-M^{1+\alpha \log \kappa / (\alpha \log M)}.} \]

Turning to the second term in the right hand side of (5.17), we have
\[
P_\omega^\tau \left[ \tau_{i+1} > \tau_i - a \log M \right] = P_\omega^{\theta_i - a \log M \omega} \left[ \tau_{a \log M + 1} > \tau_0 \right] = 1 - \frac{S(a \log M, \theta_i - a \log M \omega)}{S(1 + a \log M, \theta_i - a \log M \omega)}.
\]

By Lemma 5.6 we have
\[
S(1 + a \log M, \theta_i - a \log M \omega) = \frac{S(i - a \log M + 1, \omega)S(i - a \log M, \omega)}{e^{V_{\omega}(i - a \log M)}}.
\]

Also by (5.14) we have
\[
e^{V_{\omega}(i - a \log M)} = e^{V_{\omega}(i - a \log M)} + S(i - a \log M + 1, \omega)S(i - a \log M, \omega).
\]

Hence,
\[
P_\omega^\tau \left[ \tau_{i+1} > \tau_i - a \log M \right] = e^{V_{\omega}(i - a \log M)} \frac{S(i - a \log M + 1, \omega)S(i - a \log M, \omega)}{S(i + 1, \omega)S(i, \omega)}.
\]

Using (1.5), we have that
\[
S(i + 1, \omega) = S(i - a \log M, \omega) + e^{V_{\omega}(i - a \log M)} S(a \log M + 1, \theta_i - a \log M \omega),
\]
we get

\[
e^{V_\omega(i)} \frac{S(i - a \log M, \omega)}{S(i, \omega)(S(i + 1, \omega) - S(i - a \log M, \omega))} = \frac{S(i - a \log M, \omega) + e^{V_\omega(i-a \log M)} S(a \log M, \theta^{i-a \log M} \omega)}{S(a \log M + 1, \theta^{i-a \log M} \omega)}
\]

\[
\times \left(1 - \frac{1}{\kappa}\right) W(a \log M + 1, \theta^{i-a \log M} \omega) \frac{1}{1 + W(i - a \log M, \omega) S(a \log M, \theta^{i-a \log M} \omega)}
\]

\[
= C_\kappa \xi_i(a \log M, \omega),
\]

where we recall the constant \( C_\kappa \) and the random variable \( \xi_i(a \log M, \omega) \), see (1.19) and (1.23). Substituting this and (5.18) in (5.17) yields (5.15).

Turning to the proof of (5.16), suppose that \( \mu = \int \log \rho \, d\eta > 0 \). According to Lemma 2.1, \( \xi_i(a \log M, \omega) \leq C_\kappa \frac{1}{1 + W(i - a \log M, \omega) S(a \log M, \theta^{i-a \log M} \omega)} \).

Abbreviate \( W_i(a \log M) = W(i - a \log M, \omega), \ S_i(a \log M) = S(a \log M, \theta^{i-a \log M} \omega). \)

For any \( J \) we have

\[
\frac{1}{k} \sum_{i=J}^{x_k-1} \xi_i(a \log M, \omega)
\]

\[
= \frac{1}{k} \sum_{i=J}^{x_k-1} \xi_i(a \log M, \omega) \mathbb{1}[W_i(a \log M) \geq M^{-\mu/2}]
\]

\[
+ \frac{1}{k} \sum_{i=J}^{x_k-1} \xi_i(a \log M, \omega) \mathbb{1}[W_i(a \log M) < M^{-\mu/2}]
\]

\[
\leq \frac{C_\kappa}{k} \sum_{i=J}^{x_k-1} \left(1 + M^{-\mu/2} S_i(a \log M)\right)^{-1} + \frac{C_\kappa}{k} \sum_{i=J}^{x_k-1} \mathbb{1}[W_i(a \log M) < M^{-\mu/2}].
\]

While the first term in the right hand side of (5.19) is a Cesàro average to which the ergodic theorem can be applied, the second term needs more work. Observe that

\[
\frac{1}{W_i(a \log M)} = \sum_{j=0}^{i-a \log M-1} e^{V_\omega(j)} = \sum_{j=0}^{i-a \log M-1} e^{V_\omega(j) - V_\omega(i-a \log M)}
\]

\[
\leq \sum_{j=-\infty}^{-1} e^{V_{\theta^{i-a \log M} \omega}(j)} = S(-\infty, \theta^{i-a \log M} \omega).
\]
Thus,

\[(5.21) \quad \frac{1}{k} \sum_{i=1}^{x_k-1} \mathbb{1}[W_i(a \log M) < M^{-\mu/2}] \leq \frac{1}{k} \sum_{i=1}^{x_k-1} \mathbb{1}[S(-\infty, \theta^{i-a \log M} \omega) > M^{\mu/2}],\]

and now we can apply the ergodic theorem to the right hand side of \(5.21\). Taking the limit in \(k\) we obtain

\[
\lim_{k \to \infty} \frac{1}{xC_k^2k} \sum_{i=1}^{x_k-1} \mathbb{P}_i[\tau_i+1 > \tau_i-a \log M]
\leq \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{x_k-1} \left( (1 + M^{-\mu/2}S_i(a \log M))^{-1} + \mathbb{1}[S(-\infty, \theta^{i-a \log M} \omega) > M^{\mu/2}] \right)
\leq \int \left( 1 + \varepsilon M^{-\mu/2}S(a \log M, \omega) \right)^{-1} \eta(d\omega) + \eta(S(-\infty, \cdot) > M^{\mu/2}).
\]

(5.22)

To take the limit in \(M\) we deal with the terms separately. For the integral term in the right hand side of \(5.22\), define \(N_1(\omega) = \inf\{n : V_{\omega}(i)/i \geq (3/4)\mu \text{ for } i \geq n\} \) and observe that according to the ergodic theorem, \(\eta(N_1(\cdot) > K) \to 0 \) as \(K \to \infty\). Now,

\[
\int \left( 1 + M^{-\mu/2}S(a \log M, \omega) \right)^{-1} \eta(d\omega) \leq \int \left( 1 + M^{-\mu/2}eV_{\omega}(a \log M-1) \right)^{-1} \eta(d\omega)
= \int \frac{\mathbb{1}[N_1(\omega) < a \log M]}{1 + M^{-\mu/2}e\omega(a \log M-1)} \eta(d\omega) + \int \frac{\mathbb{1}[N_1(\omega) > a \log M]}{1 + M^{-\mu/2}e\omega(a \log M-1)} \eta(d\omega)
\leq \frac{1}{1 + M^{\mu/4}} + \eta(N_1(\cdot) > a \log M - 1),
\]

(5.23)

which tends to zero as \(M \to \infty\).

For the second term in the right hand side of \(5.22\), define \(N_2(\omega) = \inf\{n : V_{\omega}(i)/i \leq -(3/4)\mu \text{ for } i \geq n\} \) and observe that for \(\alpha_4 = (1 - e^{-(3/4)\mu})^{-1}\) it holds that

\[
S(-\infty, \omega) = \sum_{j=1}^{\infty} eV_{\omega}(-j) = \sum_{j=1}^{N_2(\omega)-1} eV_{\omega}(-j) + \sum_{j=N_2(\omega)}^{\infty} eV_{\omega}(-j) \leq \frac{C_{N_2(\omega)}^N}{C_\kappa - 1} \leq \frac{C_{N_2(\omega)}^N}{C_\kappa - 1} + \alpha_4,
\]

and for some constant \(\alpha_5\) depending only on \(\kappa\) we have

\[
\eta(S(-\infty, \omega) > M^{\mu/2}) \leq \eta\left( \frac{C_{N_2(\omega)}^N}{C_\kappa - 1} + \alpha_4 > M^{\mu/2} \right)
= \eta\left( N_2(\cdot) > \alpha_5 + \frac{\log(M^{\mu/2} - \alpha_4)}{\log C_\kappa} \right),
\]

(5.24)
which tends to zero as $M \to \infty$ due to the ergodic theorem. We conclude that if $\mu = \mu(\eta) > 0$ then
\[
\lim_{M \to \infty} \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{x_k-1} \mathbb{P}[\tau_{i+1} > \tau_i - a \log M] = 0, \quad \eta - \text{a.s.,}
\]
completing the proof of (5.16) in that case.

Now consider $\mu = \int \log \rho_0 d\eta \leq 0$ and observe that
\begin{equation}
(5.25)
\xi_i(a \log M, \omega) \leq W(a \log M + 1, \theta^i - a \log M) \omega).
\end{equation}

By the ergodic theorem and (5.25) we get that, $\eta$-a.s.,
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{x_k-1} \mathbb{P}[\tau_{i+1} > \tau_i - a \log M] = 0, \quad \eta - \text{a.s.,}
\]
completing the proof of (5.16) in that case.

Now observe that for any $K \geq 1$,
\[
W(K+1, \omega) = \frac{e^{V_\omega(K+1)}}{\sum_{j=0}^{K} e^{V_\omega(j)}} = \left( \sum_{j=0}^{K} e^{V_\omega(j) - V_\omega(K+1)} \right)^{-1} = \left( S(-K - 1, \theta^K + 1) \right)^{-1}.
\]

By the stationarity of the environment, for each $K$, $S(-K - 1, \omega)$ has the same distribution as $S(K+2, r(\omega)) - 1$ where $r(\omega)$ was defined in (1.20), and therefore we obtain that
\begin{equation}
(5.27)
\int W(a \log M + 1, \omega) \eta(d\omega) = \int \left( -S(a \log M - 1, \theta^a \log M + 1 \omega) \right)^{-1} \eta(d\omega)
\end{equation}
and
\begin{equation}
(5.28)
\int S(a \log M + 2, r(\omega)) \eta(d\omega) = 1.
\end{equation}

From the proof of Theorem 2.1.1. from [5] (originally [4, Theorem (1.7)]) we have that $\mu \leq 0$ implies that $S(n, \omega) \to \infty \eta$-a.s. as $n \to \infty$. Observe that $\omega$ is an environment with
\[
\int \log \rho_0(\omega) \eta(d\omega) = - \int \log \rho_0 \eta(d\omega) \geq 0,
\]
hence $S(a \log M + 2, r(\omega)) \to M \to \infty$, $\eta$-a.s. Since $S(n, \omega) \geq 1 + C_\kappa^{-1}$, then the function integrated in (5.27) is bounded by $C_\kappa$ and therefore by the dominated convergence theorem we get
\[
\lim_{M \to \infty} \int W(a \log M + 1, \omega) \eta(d\omega) = \lim_{M \to \infty} \int \frac{1}{S(a \log M + 2, r(\omega)) - 1} \eta(d\omega) = 0,
\]
so for $\mu \leq 0$ it holds that
\[
\lim_{M \to \infty} \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{x_k-1} \mathbb{P}[\tau_{i+1} > \tau_i - a \log M] = 0.
\]
Having proved (5.16) also in the case $\mu = \mu(\eta) \leq 0$, the proof of the lemma is complete. □
Lemma 5.8. Let \( \omega_i \in [\kappa, 1 - \kappa] \) for all \( i \). Then there exist \( \alpha_6, \alpha_7 > 0 \) depending only on \( \kappa, x \) so that for any \( M + L < J \),

\[
\begin{align*}
(5.28) & \quad |\hat{I}_J(x, k, \omega) - \hat{I}_{J,M}(x, k, \omega)| \leq \frac{\alpha_6}{k} \sum_{i=J}^{x^k-1} \mathbb{P}_i^{\omega}[\tau_{i+1} \geq M], \\
(5.29) & \quad |\hat{I}_{J,M}(x, k, \omega) - \hat{I}_J(x, k, \omega)| \leq \frac{\alpha_6}{k} \sum_{i=J}^{x^k-1} \mathbb{P}_i^{\omega}[\tau_{i+1} \geq M], \\
(5.30) & \quad |\hat{I}_{J,M}(x, k, \omega) - \hat{I}_{J,M}(x, k, \omega)| \leq \frac{\alpha_7M}{k} \sum_{i=J}^{x^k-1} \sum_{i=J-M}^{J+M} \xi(L - 1, \omega).
\end{align*}
\]

Moreover if \( J > L \) then

\[
\frac{1}{k} \sum_{i=J}^{x^k-1} \mathbb{P}_i^{\omega}[\tau_{i+1} \geq M] \leq \frac{1}{k} \sum_{i=J}^{x^k-1} \mathbb{P}_i^{\omega}[\tau_{i+1} \geq M].
\]

Proof. For clarity, we suppress the arguments \( k \) and \( \omega \) in \( \hat{I} \)'s. We start by proving (5.29). Consider

\[
\lambda_{M,L}^*(x) = \arg \max_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x^k-1} \log \hat{\phi}_L^M(\lambda, \theta^i \omega) \right\}.
\]

As in the proof of Lemma 5.5, we have that \( \lambda_{M,L}^* \in (-\infty, 0) \) is well defined. By definition, \( \hat{I}_J(x) \leq \hat{I}_{J,M}(x) \) for every \( x \). For the other inequality observe that

\[
\hat{I}_J(x) = \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x^k-1} \log \hat{\phi}_L^M(\lambda, \theta^i \omega) \right\}
\]

\[
= \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x^k-1} \log \hat{\phi}_L^M(\lambda, \theta^i \omega) + E_i \right\}
\]

\[
\geq \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x^k-1} \log \hat{\phi}_L^M(\lambda, \theta^i \omega) + e^{\lambda M} \mathbb{P}_i^{\omega}[\tau_{i+1} \geq M] \right\}
\]

\[
= \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x^k-1} \log \hat{\phi}_L^M(\lambda, \theta^i \omega) - \frac{1}{k} \sum_{i=J}^{x^k-1} \log \left[ 1 + \frac{e^{\lambda M} \mathbb{P}_i^{\omega}[\tau_{i+1} \geq M]}{\hat{\phi}_L^M(\lambda, \theta^i \omega)} \right] \right\}
\]

\[
\geq \lambda_{M,L}^*(x) - \frac{1}{k} \sum_{i=J}^{x^k-1} \log \hat{\phi}_L^M(\lambda_{M,L}^*(x), \theta^i \omega)
\]

\[
- \frac{1}{k} \sum_{i=J}^{x^k-1} \log \left[ 1 + \frac{e^{\lambda_{M,L}^*(x) M} \mathbb{P}_i^{\omega}[\tau_{i+1} \geq M]}{\hat{\phi}_L^M(\lambda_{M,L}^*(x), \theta^i \omega)} \right].
\]
and therefore

\[
\hat{I}_J^L(x) \geq \hat{I}_J^{L,M}(x) - \frac{1}{k} \sum_{i,j} \log \left[ 1 + \frac{e^{\lambda_{M,L}(x)M\mathbb{P}_i[\tau_{i+1} \geq M]}}{\phi_L^M(\lambda_{M,L}(x), \theta_i \omega)} \right] \\
\geq \hat{I}_J^{L,M}(x) - \frac{1}{k} \sum_{i,j} \frac{e^{\lambda_{M,L}(x)M\mathbb{P}_i[\tau_{i+1} \geq M]}}{\phi_L^M(\lambda_{M,L}(x), \theta_i \omega)} \\
\geq \hat{I}_J^{L,M}(x) - \frac{1}{k} e^{\lambda_{M,L}(x)(M-1)} \sum_{i,j} \mathbb{P}_i[\tau_{i+1} \geq M],
\]

where in (5.31) we used that $\phi_L^M(\lambda, \theta_i \omega) \geq \omega_i e^\lambda$. Therefore there is a constant $\alpha_0$ such that (5.29) holds. The proof of (5.28) is analogous.

Now to prove (5.30) we couple the random walks $X_t$ in the environment $\hat{\omega}$ and $Y_t^L$ in the environment $\hat{\omega}^L$ as follows. The walks start together ($X_0 = Y_0^L = J$) and move independently if they are on different sites. If $X_t = Y_t^L$ and $X_{t+1} = X_t + 1$ then $Y_{t+1}^L = X_{t+1}$. The coupling is possible because the transformed environment satisfies $\hat{\omega}^L \geq \hat{\omega}$ for every $x \geq L$. Let the joint measure of the coupling with the random walks initially at $J$ be $\hat{\hat{\omega}}^L$ and consider the event $B^L_M = \{X_t = Y_t^L, \text{ for all } t \in [0, M]\}$. From the definition of $\hat{\omega}_i^L$, using (1.5) we have that if $i > L$

\[
\hat{\omega}_i^L \leq \frac{\omega(S(L, i+L \omega) - S(i, \omega) - S(i, \omega) - S(i, \omega))}{S(L, \omega^L)} = \omega_i \left( \frac{S(i+1, \omega) - S(i, \omega) - S(i, \omega) - S(i, \omega)}{S(i, \omega) - S(i, \omega) - S(i, \omega)} \right).
\]

Observe that if $X_t = Y_t^L = i$, then the probability that they jump to different sites is $\hat{\omega}_i^L - \hat{\omega}_i$, hence it holds for $M + L < J$ that

\[
\hat{\hat{\omega}}^L(B^L_M)^c \leq M \max_{J-M \leq i \leq J+M} (\hat{\omega}_i^L - \hat{\omega}_i) \\
\leq M \sum_{i=J-M}^{J+M} (\hat{\omega}_i^L - \hat{\omega}_i) \\
= M \sum_{i=J-M}^{J+M} \omega_i \left( \frac{S(i-L, \omega) - S(i, \omega) - S(i, \omega) - S(i, \omega)}{S(i, \omega) - S(i, \omega) - S(i, \omega)} \right).
\]

By definition we have that $S(i, \omega) = S(i-L, \omega) + e^{\lambda_\omega(i-L)} S(L, \theta_i \omega)$, and therefore the right hand side of (5.33) equals

\[
M \sum_{i=J-M}^{J+M} \omega_i \left( \frac{S(i-L, \omega)}{S(i, \omega)} \frac{e^{\lambda_\omega(i)}}{e^{\lambda_\omega(i-L)}} \right) \\
\leq M \sum_{i=J-M}^{J+M} \frac{1}{W(L, \theta_i \omega)} \frac{W(L, \theta_i \omega)}{1 + W(L, \theta_i \omega) S(L, \theta_i \omega)} \\
= M \sum_{i=J-M}^{J+M} \xi_i (L-1, \omega) = H^J_y(L, M).
\]
Due to the coupling we have \( \hat{\phi}_L^M(\lambda, \theta^i \omega) \geq \phi_i,M(\lambda, \omega) \). To get an inequality in the other direction we use that
\[
\hat{\phi}_L^M(\lambda, \theta^i \omega) = \int e^{\lambda \tau_{i+1}} \mathbb{1}[\tau_{i+1} < M] d\hat{\mu}_L^i \geq L
\]
\[
= \int e^{\lambda \tau_{i+1}} \mathbb{1}[\tau_{i+1} < M] d\hat{\mu}_L^i \omega
\]
\[
= \int e^{\lambda \tau_{i+1}} \mathbb{1}[\tau_{i+1} < M, B_{M,t}^i] d\hat{\mu}_L^i \omega
\]
\[
+ \int e^{\lambda \tau_{i+1}} \mathbb{1}[\tau_{i+1} < M, (B_{M,t}^i)\mathbb{C}] d\hat{\mu}_L^i \omega
\]
\[
\leq \int e^{\lambda \tau_{i+1}} \mathbb{1}[\tau_{i+1} < M] d\hat{\mu}_L^i \omega + \hat{\mu}_i \omega [(B_{M,t}^i)\mathbb{C}]
\]
and thus
\[
(5.34) \quad 0 \leq \hat{\phi}_L^M(\lambda, \theta^i \omega) - \phi_i,M(\lambda, \omega) \leq H_1^\omega(L, M).
\]
It is easy to check that \( \hat{I}_{J,M}(x) \geq \hat{I}_{J,M}^L(x) \). Using (5.34) we get
\[
\hat{I}_{J,M}^L(x) \geq \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x_k-1} \log[\phi_i,M(\lambda, \theta^i \omega) + H_1^\omega(L, M)] \right\}
\]
\[
= \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x_k-1} \log \phi_i,M(\lambda, \theta^i \omega) - \frac{1}{k} \sum_{i=J}^{x_k-1} \log \left[ 1 + \frac{H_1^\omega(L, M)}{\phi_i,M(\lambda, \theta^i \omega)} \right] \right\}
\]
\[
\geq \lambda^*_M(x) - \frac{1}{k} \sum_{i=J}^{x_k-1} \log \phi_i,M(\lambda^*_M(x), \theta^i \omega)
\]
\[
- \frac{1}{k} \sum_{i=J}^{x_k-1} \log \left[ 1 + \frac{H_1^\omega(L, M)}{\phi_i,M(\lambda^*_M(x), \theta^i \omega)} \right]
\]
\[
\geq \hat{I}_{J,M}(x) - \frac{1}{k} \sum_{i=J}^{x_k-1} \frac{H_1^\omega(L, M)}{\phi_i,M(\lambda^*_M(x), \theta^i \omega)}
\]
\[
\geq \hat{I}_{J,M}(x) - \frac{1}{k \kappa} e^{-\lambda^*_M(x)} \sum_{i=J}^{x_k-1} H_1^\omega(L, M).
\]
Using (5.13), we conclude that for some constant \( \alpha_7 \),
\[
(5.35) \quad |\hat{I}_{J,M}^L(x) - \hat{I}_{J,M}(x)| \leq \frac{\alpha_7}{k} \sum_{i=J}^{x_k-1} H_1^\omega(L, M).
\]
The last statement of the lemma is a consequence of the inequality \( \hat{\omega}_L^i \geq \hat{\omega}_i \) for all \( i > L \). \( \square \)

**Lemma 5.9.** For \( J > L \) we have
\[
|\hat{I}_{J}(x, k, \omega) - \hat{I}_{L}^0(x, k, \omega)| \leq \frac{J}{k} \log \left( \frac{x}{\kappa^2(1 - x)} \right).
\]
Proof. Since the variational problems only involve \( \lambda \leq 0 \), it is straightforward that \( I^L_J(x, k, \omega) \geq I^L_0(x, k, \omega) \) if \( J > L \). Now define
\[
\lambda_L^\ast(x) = \arg \max_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=0}^{x-1} \log \hat{\phi}_L(\lambda, \theta^i \omega) \right\},
\]
then
\[
I^L_J(x, k, \omega) = \sup_{\lambda \leq 0} \left\{ \lambda - \frac{1}{k} \sum_{i=J}^{x-1} \log \hat{\phi}_L(\lambda^\ast_L(x), \theta^i \omega) \right\}
\]
\[
\geq I^L_0(x, k, \omega) + \frac{1}{k} \sum_{i=0}^{J-1} \log(\hat{\phi}_L(\lambda^\ast_L(x), \theta^i \omega))
\]
\[
\geq I^L_0(x, k, \omega) + \frac{J-1}{k} \log \left( \frac{\kappa^2(1-x)}{x} \right),
\]
where in the last line it was used that \( e^{\lambda^\ast_L(x)} \geq \kappa(1-x)/x \), which holds due to the same computation as (5.13). \( \square \)

Recall the notation \( \bar{\xi}(i, \omega) \), see (1.24).

**Lemma 5.10.** For any measure \( \eta \in M^\ast_1(\Sigma) \) it holds that
\[
\lim_{L \to \infty} \int \bar{\xi}(L, \omega) \eta(d\omega) = 0.
\]

**Proof.** We first prove the result for \( \eta \in M^\ast_1(\Sigma) \) and then extend it to all stationary measures. Consider first the case \( \int \log \rho_0 \eta(d\omega) \leq 0 \). Then,
\[
\int \bar{\xi}(L, \omega) \eta(d\omega) \leq \int W(L + 1, \omega) \eta(d\omega)
\]
\[
= \int W(L + 1, \omega) \eta(d\omega) \to 0, \text{ as } L \to \infty,
\]
where the limit is due to (5.27).

Consider next the case \( \mu = \int \log \rho_0 \eta(d\omega) > 0 \). We have that
\[
\int \bar{\xi}(L, \omega) \eta(d\omega) \leq C_\kappa \left( 1 + \frac{S(L, \omega)}{S(-\infty, \omega)} \right)^{-1} \eta(d\omega)
\]
\[
\leq C_\kappa \left( 1 + e^{-\mu L/2} e^{V_\omega(L-1)} \right)^{-1} \eta(d\omega) + \eta(S(-\infty, \cdot) > e^{\mu L/2}).
\]
Now (5.24) states that
\[
\eta(S(-\infty, \cdot) > e^{\mu L/2}) \leq \eta \left( N_2(\omega) > \alpha_5 + \frac{\log(e^{\mu L/2} - \alpha_4)}{\log C_\kappa} \right).
\]
(5.36)
Using (5.23) with $L$ instead of $a \log M$ we get
\begin{equation}
(5.37) \quad \int (1 + e^{-\nu/2L} e^{\lambda(L-1)})^{-1} \eta(d\omega) \leq \frac{1}{1 + e^{\nu L/2}} + \eta(N_1(\cdot) > L - 1).
\end{equation}
Since the right hand sides of both (5.36) and (5.37) tend to 0 as $L \to \infty$, the result for ergodic measures is proved.

Now consider $\eta \in M_1^e(\Sigma)$, then there is a family of measures $\{\eta_{\theta}\}_{\theta \in \mathbb{R}} \subset M^e_1(\Sigma)$ and a measure $\nu \in M_1(\mathbb{R})$ such that
\begin{equation}
(5.38) \quad \eta = \int \eta_{\theta} \nu(d\theta).
\end{equation}
Using (5.38) we obtain
\begin{equation}
(5.39) \quad \lim_{L \to \infty} \int \xi(L, \omega) \eta(d\omega) = \lim_{L \to \infty} \int \int \xi(L, \omega) \nu(d\theta) d\eta_{\theta}(\omega),
\end{equation}
and since $0 \leq \xi(L, \omega) \leq C_\nu$ due to Lemma 2.1, we can apply Fubini’s Theorem and dominated convergence theorem in (5.39) to obtain
\begin{align*}
\lim_{L \to \infty} \int \xi(L, \omega) \eta(d\omega) &= \lim_{L \to \infty} \int \int \xi(L, \omega) d\eta_{\theta}(\omega) \nu(d\theta) \\
&= \int \left( \lim_{L \to \infty} \int \xi(L, \omega) d\eta_{\theta}(\omega) \right) \nu(d\theta) \\
&= 0,
\end{align*}
which extends the result to stationary measures. \hfill \Box

We next prove Lemma 1.3.

**Proof of Lemma 1.3.** We first prove that for any fixed $i$ and $L$ we have that $\hat{\omega}_i^{L+1} < \hat{\omega}_i^L$. From the definition of $\hat{\omega}_i^L$,
\begin{equation}
(5.40) \quad \hat{\omega}_i^{L+1} - \hat{\omega}_i^L = \frac{\omega_i S(L + 2, \theta^{i-L-1} \omega)}{S(L + 1, \theta^{i-L-1} \omega)} - \frac{\omega_i S(L + 1, \theta^{i-L} \omega)}{S(L, \theta^{i-L} \omega)},
\end{equation}
while from (1.50) we have that
\begin{equation}
S(L + 2, \theta^{i-L-1} \omega) = S(L + 1, \theta^{i-L-1} \omega) + \rho_1(\theta^{i-L-1} \omega) S(L + 1, \theta^{i-L} \omega)
\end{equation}
\begin{equation}
(5.41) \quad = 1 + \rho_{i-L} S(L + 1, \theta^{i-L} \omega),
\end{equation}
and analogously
\begin{equation}
S(L + 1, \theta^{i-L-1} \omega) = 1 + \rho_{i-L} S(L, \theta^{i-L} \omega).
\end{equation}
Using (5.41) and (5.42) in (5.40) we obtain
\begin{align*}
\hat{\omega}_i^{L+1} - \hat{\omega}_i^L &= \omega_i \left( \frac{1 + \rho_{i-L} S(L + 1, \theta^{i-L} \omega)}{1 + \rho_{i-L} S(L, \theta^{i-L} \omega)} - \frac{S(L + 1, \theta^{i-L} \omega)}{S(L, \theta^{i-L} \omega)} \right) \\
&= \omega_i \left( \frac{S(L, \theta^{i-L} \omega) - S(L + 1, \theta^{i-L} \omega)}{(1 + \rho_{i-L} S(L, \theta^{i-L} \omega)) S(L, \theta^{i-L} \omega)} \right) < 0,
\end{align*}
which proves the claimed monotonicity. With it we can define a coupling analogous to the one defined in the proof of Lemma 5.8. Take $L' > L$, so for every $i$ we have that $\hat{\omega}_i^{L'} < \hat{\omega}_i^L$, hence we can define two random walks $Y_i^{L'}$.
and $Y^L_t$ such that for all $t$ we have that $Y^L_{t'} \leq Y^L_t$. This in turns implies that $\dot{\phi}_L(\lambda, \omega)$ is non-increasing in $L$ for $\lambda \leq 0$, which in turns gives that 

$$I^\phi_L(x, \eta) := \sup_{\lambda \leq 0} \left\{ \lambda - x \int \log \phi(\hat{\omega}^L, \lambda) \eta(d\omega) \right\}$$

is increasing in $L$. This yields the claimed existence of the limit in (1.12). \(\square\)

6. **Identification of the rate function, study of the variational problem, and proof of Proposition 3.2**

In this section, we prove Proposition 3.2. We begin with a preliminary computation.

**Lemma 6.1.** With $\Lambda(\lambda) = \log \int \rho_0^\lambda d\eta$, set

$$I_m(x) = \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \Lambda(\lambda) \},$$

then, with $\eta$ such that $s \in (1, \infty]$ as in the statement of Theorem 1.4,

$$\inf_{z > 0} \frac{I_m(z)}{z} = s.$$

**Proof.** The claim is trivial if $s = \infty$, since then $I_m(z) = \infty$ for any $z > 0$. So we restrict attention to $s \in (1, \infty)$. Since $\Lambda(\cdot)$ is differentiable, we can compute the critical point $\tilde{\lambda}(x)$ of $G_x(\lambda) = \lambda x - \Lambda(\lambda)$ to get

$$G'_x(\tilde{\lambda}(x)) = x - \Lambda'(\tilde{\lambda}(x)) = 0,$$

and then

$$\inf_{z > 0} \frac{I_m(z)}{z} = \inf_{z > 0} \frac{\tilde{\lambda}(z) - \Lambda(\tilde{\lambda}(z))}{z} = \inf_{z > 0} \left( \frac{\tilde{\lambda}(z) - \Lambda(\tilde{\lambda}(z))}{z} \right) =: \inf_{z > 0} H(z).$$

Since $\Lambda(\cdot)$ is analytic and (strictly) convex, its second derivative is strictly positive and so, by (6.2) and the implicit function theorem, the map $z \mapsto \tilde{\lambda}(z)$ is differentiable. Thus, the function $z \mapsto H(z)$ is differentiable on $\mathbb{R}_+$ and its derivative is

$$H'(z) = \lambda'(z) - \frac{\Lambda'(\tilde{\lambda}(z))\lambda'(z)}{z} + \frac{\Lambda(\tilde{\lambda}(z))}{z^2} = \Lambda(\tilde{\lambda}(z)) - \frac{\Lambda(\tilde{\lambda}(z))}{z^2}.$$

Therefore, the only positive solution to $H'(z) = 0$ is the solution to $\Lambda(\tilde{\lambda}(z)) = 0$. Thus, for $z^*$ satisfying the latter equality we have

$$\inf_{z > 0} H(z) = \tilde{\lambda}(z^*) - \frac{\Lambda(\tilde{\lambda}(z^*))}{z^*} = \tilde{\lambda}(z^*).$$

This means that the value of $\inf_{z > 0} I_m(z)/z$ is the solution to the equation

$$\log \int \rho_0^\lambda d\eta = 0,$$

concluding the proof. \(\square\)

As a second step toward the proof of Proposition 3.2, we derive upper and lower bounds on the scaled logarithmic limit of $\chi$. Recall the definition of the quantities $f$ and $g$ in (3.1) and $\chi$ in (3.2). From now on for simplicity we denote the vectors of measures $\{\eta^*_x = (\eta_1, \ldots, \eta_{k/x})$ and
follows that for some positive constant \( \alpha \) we deduce that \( \hat{I}_J > L \) for \( \eta \). We have that \( \hat{I}_J = \sum_{i=1}^{\eta} h(\eta_i) + \sum_{j=1}^{\eta} h(\bar{\eta}_j) \), and the constant \( C_\kappa \), see \eqref{eq:5.2}, is then invoked to compute the re-scaled logarithmic limit in terms of a variational problem (see \eqref{eq:6.23}).

**Lemma 6.2.** Under the assumptions of Theorem 1.4 we have that

\[
(6.3) \quad I^*_0(x, c) \leq \lim_{k \to \infty} \frac{1}{k} \log\chi(k, x, c) \leq I^*_u(x, c),
\]

where

\[
(6.4) \quad I^*_u(x, c) = \inf_{\{\eta\}_x, \{\bar{\eta}\}_x, c \in M^{(1)}_u(\Sigma_p)} \left\{ \left[ \left( \Delta_x(\{\eta\}_x, \{\bar{\eta}\}_x) - \varepsilon \log C_\kappa \right)^+ \right. \right.
\]

\[
- I^f \left( x, \frac{\varepsilon}{x} \sum_{i=1}^{\eta} \eta_i \right) + \varepsilon \sum_{i=1}^{\eta} h(\eta_i) + \varepsilon \sum_{j=1}^{\eta} h(\bar{\eta}_j),
\]

and

\[
(6.5) \quad I^*_1(x, c) = \inf_{\{\eta\}_x, \{\bar{\eta}\}_x, c \in M^{(1)}_u(\Sigma_p)} \left\{ \left[ \left( \Delta_x(\{\eta\}_x, \{\bar{\eta}\}_x) + \varepsilon \log C_\kappa \right)^+ \right. \right.
\]

\[
- I^f \left( x, \frac{\varepsilon}{x} \sum_{i=1}^{\eta} \eta_i \right) + \varepsilon \sum_{i=1}^{\eta} h(\eta_i) + \varepsilon \sum_{j=1}^{\eta} h(\bar{\eta}_j).
\]

**Proof.** The strategy of the proof is as follows. We first use Lemmas 5.5 and 6.9 to express the functions \( f \) and \( g \) as exponentials of functionals of empirical fields plus deterministic error terms, and use these expressions in upper and lower bounds on \( \chi \) (see \eqref{eq:6.17} and \eqref{eq:6.19}). Varadhan’s Lemma is then invoked to compute the re-scaled logarithmic limit in terms of a variational problem (see \eqref{eq:6.22}).

From now on, we denote \( \hat{I}_J^L(x, k, \omega) = I^L(x, R_{x,k}) \), where \( R_{x,k} \) is as in \eqref{eq:1.9} with \( n = xk \), and throughout \( L, M, J \) are positive integers as in Section 5. In what follows, we will assume that \( L \leq xk \) and that \( J \geq L + M \). (Eventually, we will take limits in \( k \to \infty \), followed by \( J \to \infty \) and then by \( M, L \to \infty \).)

We begin by estimating \( f(\omega, x, k) \):

\[
(6.6) \quad - \frac{1}{k} \log(1 - f(\omega, x, k)) = - \frac{1}{k} \log \mathbb{P}_x^\omega [\tau_{x,k} \leq k] \geq - \frac{1}{k} \log \mathbb{P}_x^\omega [\tau_{x,k} \leq k] \geq \hat{I}_J(x, k, \omega),
\]

where in \eqref{eq:6.6} we used Lemma 5.5. Since for \( i > L \) we have that \( \hat{\omega}_L^\kappa \geq \hat{\omega}_i \), for \( J > L \) we deduce that \( \hat{I}_J(x, k, \omega) \geq \hat{I}_J^L(x, k, \omega) \). Using Lemma 5.9 it follows that for some positive constant \( \alpha_8 \) independent of \( k \) and \( J \),

\[
(6.7) \quad - \frac{1}{k} \log(1 - f(\omega, x, k)) = - \frac{1}{k} \log \mathbb{P}_x^\omega [\tau_{x,k} \leq k] \geq \left( \hat{I}_L(x, R_{x,k}) - \frac{\alpha_8 J}{k} \right)^+.
\]
For the upper bound, let \( x'_l = x(1 + l) \) and \( k'_l = k/(1 + l) \). Then,
\[
\frac{1}{k} \log(1 - f(\omega, x, k)) = -\frac{1}{k} \log \mathbb{P}_x^k[\tau_{xk} \leq k] = -\frac{1}{k} \log \mathbb{P}_x^k[\tau_{xk} \leq k'](1 + l)]
\]
(6.8)
\[
\leq \hat{I}_{1,M}(x'_l, k'_l, \omega) + \alpha_3 l - \frac{1}{k'} \log \left( 1 - 2e^{-\frac{2k'/k^2}{x'/M^2}} \right),
\]
where in (6.8) again we used Lemma 5.5. By Lemma 5.8 we have
\[
\hat{I}_{1,M}(x'_l, k'_l, \omega) \leq \hat{I}_L(x'_l, k'_l, \omega) + \frac{\alpha_7 M}{k} \sum_{i=J}^{J+M} \xi_i(L - 1, \omega)
\]
\[
+ \frac{\alpha_6}{k} \sum_{i=J}^{x_k-1} \mathbb{P}^x_i[\tau_{i+1} \geq M].
\]
Using Lemma 5.7 and (7.3), we obtain
\[
\hat{I}_{1,M}(x'_l, k'_l, \omega) \leq \hat{I}_L(x'_l, k'_l, \omega) + \alpha_6 x'_l \Phi_1(M, R_{xk})
\]
\[
+ \alpha_7 x'_l \Phi_2(M, R_{xk}) + \alpha_6 x'_l e^{-M^{1+a log \kappa}/(a log M)},
\]
(6.9)
where we used the abbreviations
\[
\Phi_1(M, R_{xk}) = \int \bar{\xi}(a log M, \omega) R_{xk}(d\omega),
\]
\[
\Phi_2(M, R_{xk}) = M \sum_{j=-M}^{M} \int \bar{\xi}(L - 1, \theta^{j-M-L+1} \omega) R_{xk}(d\omega).
\]
Using (6.9) in (6.8) we obtain
\[
\frac{1}{k} \log(1 - f(\omega, x, k)) \leq \hat{I}_L(x'_l, R_{xk}) + \alpha_6 x'_l \Phi_1(M, R_{xk}) + \alpha_7 x'_l \Phi_2(M, R_{xk}) + \beta(k'_l, l, M, L)
\]
\[
= : \hat{I}_{l,M}^{L}(x, R_{xk}) + \beta(k'_l, l, M, L),
\]
where again for clarity we denoted
\[
\beta(k, l, M, L) = \alpha_3 l - \log \left( 1 - 2e^{-\frac{2k'/k^2}{x'/M^2}} \right) + \alpha_6 x'_l e^{-M^{1+a log \kappa}/(a log M)} + \alpha_8 J/k.
\]
Turning to bound \( g(\omega, x, c) \) in (3.1), we define
\[
I_{L}^{x,c}(\omega, x, c) = \left( \Delta_x - \epsilon \log C_{\kappa} \right)^+, \quad I_{u}^{x,c}(\omega, x, c) = \left( \Delta_x + \epsilon \log C_{\kappa} \right)^+.
\]
Lemma 5.3 and the definition of \( g \), see (3.1), imply that
\[
g(\omega, x, c, k) \geq \exp\{-kI_{u}^{x,c}(\omega, x, c) - (c + x) \log C_{\kappa}/\epsilon - \log(xk)\},
\]
and
\[
g(\omega, x, c, k) \leq \exp\{-kI_{L}^{x,c}(\omega, x, c) + (c + x) \log C_{\kappa}/\epsilon + \log(ck)\}.
\]
We now use the bounds on \( f, g \) to estimate \( \chi \). Denote 
\[
q^{c,e}_{1,k} = (c + x) \log C_e/\varepsilon + \log(xk).
\]
We use (6.14) and (6.15) to get the lower bound.

\[
(6.16)
\]
\[
\chi(k, x, c) = \int \left( \frac{1 - f(\omega, x, k)}{1 - f(\omega, x, k)(1 - g(\omega, x, c, k))} \right) P(d\omega)
\]
\[
\leq \int \frac{1}{1 - (1 - e^{-(kI^e(xR)) - \alpha_8J^+})} e^{-kI^{e+}(\omega, x, c) - q^{c,e}_{1, k}} P(d\omega)
\]
\[
= \int \left( 1 + \exp \left\{ (kI^e(x, R_k) - \alpha_8J)^+ - kI^{e+}(\omega, x, c) - q^{c,e}_{1, k} \right\} \right.
\]
\[
- \exp \left\{ -kI^{e+}(\omega, x, c) - q^{c,e}_{1, k} \right\}^{-1} P(d\omega)
\]
\[
\leq \int \exp \left\{ - (kI^{e+}(\omega, x, c) + q^{c,e}_{1, k} - (kI^e(x, R_k) - \alpha_8J)^+)^- \right\} P(d\omega)
\]
\[
\leq \int \exp \left\{ - (kI^{e+}(\omega, x) + q^{c,e}_{1, k} + \alpha_8J - kI^e(x, R_k))^- \right\} P(d\omega)
\]
\[
\leq e^{q^{c,e}_{1, k} + \alpha_8J} \int \exp \left\{ - \left( kI^{e+}(\omega, x, c) - kI^e(x, R_k) \right)^- \right\} P(d\omega),
\]

where in \((a)\) we used that for \( a, b \geq 0 \) we have that \((1 + e^{a-b} - e^{-b})^{-1} \leq e^{-(b-a)^-}\). Denote

\[
(6.18)
\]
\[
q^{c,e}_{2, k} = (c + x) \log C_e/\varepsilon + \log(ck).
\]
Similarly we use (6.12) and (6.14) to get the lower bound.

\[
(6.19)
\]
\[
\chi(k, x, c) = \int \left( \frac{1 - f(\omega, x, k)}{1 - f(\omega, x, k)(1 - g(\omega, x, c, k))} \right) P(d\omega)
\]
\[
\geq \int \frac{1}{1 - (1 - e^{-(kI^e_{I,M,L}(xR_k) - k\beta(k', l, M, L)))} (1 - e^{-kI^{e+}(\omega, x, c) + q^{c,e}_{2, k}}) P(d\omega)
\]
\[
= \int \left( 1 + \exp \left\{ kI^e_{I,M,L}(x, R_k) + \beta(k', l, M, L) - kI^{e+}(\omega, x, c) + q^{c,e}_{2, k} \right\} \right.
\]
\[
- \exp \left\{ - kI^{e+}(\omega, x, c) + q^{c,e}_{2, k} \right\}^{-1} P(d\omega)
\]
\[
\geq \int \left( 1 + \exp \left\{ - (kI^{e+}(\omega, x, c) + q^{c,e}_{2, k} - kI^e_{I,M,L}(x, R_k) - k\beta(k', l, M, L) \right\}^- \right\} P(d\omega)
\]
\[
\geq \frac{1}{2} \int \exp \left\{ - \left( kI^{e+}(\omega, x, c) - kI^e_{I,M,L}(x, R_k) \right)^- \right\} P(d\omega)
\]
\[
\geq e^{-q^{c,e}_{2, k} - k\beta(k', l, M, L)} \int \exp \left\{ - \left( kI^{e+}(\omega, x, c) - kI^e_{I,M,L}(x, R_k) \right)^- \right\} P(d\omega),
\]

where in \((b)\) we used that \((1 + e^a)^- \leq e^{-a^+}/2\) and \((a - b)^+ = (b - a)^-\).

We next take the rescaled logarithmic limit of \( \chi \). This has to be done in both the bounds of (6.17) and (6.19), but since the proofs are similar we
only consider the later, i.e. (6.19), which is slightly more complex. Taking the rescaled logarithm, (6.19) becomes

\[
- \frac{1}{k} \log \chi \leq - \frac{1}{k} \log \int \exp \left\{ - \left( kI_{l}^{g,\varepsilon}(\omega, x, c) - kI_{l,M,L}^{\Phi}(x, R_{xk}) \right) - q_{c,\varepsilon}^{c,\varepsilon} + k\beta(k_{l}, l, M, L) \right\} P(d\omega)
\]

(6.20)

\[
+ \frac{d_{c,\varepsilon}}{k} + k\frac{\log 2}{k}.
\]

From (6.13),

\[
\lim_{k \to \infty} \beta(k_{l}', l, M, L) = \alpha_{3}l + \alpha_{6}x_{l}^{'\varepsilon}e^{-M^{1+a\log \kappa/(a \log M)}},
\]

(6.21)

while

\[
\lim_{k \to \infty} \frac{d_{c,\varepsilon}^{c,\varepsilon}}{k} + \frac{\log 2}{k} = 0.
\]

This controls the last term in the right-hand side (6.20). The exponent in the integral in the right-hand side of (6.20) is non-positive, moreover the empirical fields emerge only in two functions in the exponent, viz.

\[
F_{1}(\eta) = \int \log \rho_{0}(\omega)\eta(d\omega), \quad F_{2}(\eta) = \int \log \phi(\lambda, \omega)\eta(d\omega).
\]

Such functions are measurable on \( M_{1}(\Sigma^{-}) \). It is straightforward that the mapping \( \eta \to F_{1}(\eta) \) is continuous in the weak topology. Moreover from [1, Lemma 6] we have that \( F_{2}(\eta) \) is also continuous in the weak topology. Recall, see Lemma 5.2, that the empirical fields \((\{R_{i,\varepsilon}\}_{i=1}^{x/\varepsilon}, \{\bar{R}_{i,\varepsilon}\}_{i=1}^{\lfloor(c-x)/\varepsilon\rfloor})\) satisfy a LDP with rate function

\[
I_{B}(\{\eta\}_{x}^{x/\varepsilon}, \{\bar{\eta}\}_{x,c}^{x/\varepsilon}) = \varepsilon \sum_{i=1}^{x/\varepsilon} h(\eta_{i}|p) + \varepsilon \sum_{j=1}^{\lfloor(c-x)/\varepsilon\rfloor} h(\bar{\eta}_{j}|p).
\]

We apply Varadhan's lemma to the limit of the integral term in (6.20) and get

\[
I_{u}(x, c) \triangleq \lim_{k \to \infty} - \frac{1}{k} \log \int \exp \left\{ - \left( kI_{l}^{g,\varepsilon}(\omega, x, c) - kI_{l,M,L}^{\Phi}(x, R_{xk}) \right) - q_{c,\varepsilon}^{c,\varepsilon} \right\} P(d\omega)
\]

(6.23)

\[
= \inf_{\{\eta\}_{x}^{x/\varepsilon}, \{\bar{\eta}\}_{x,c}^{x/\varepsilon} \in M_{1}^{+}(\Sigma_{p})} \mathcal{W}(\{\eta\}_{x}^{x/\varepsilon}, \{\bar{\eta}\}_{x,c}^{x/\varepsilon}),
\]

where

\[
\mathcal{W}(\{\eta\}_{x}^{x/\varepsilon}, \{\bar{\eta}\}_{x,c}^{x/\varepsilon}) = \left[ \left( \Delta_{e}(\{\eta\}_{x}^{x/\varepsilon}, \{\bar{\eta}\}_{x,c}^{x/\varepsilon}) - \varepsilon \log C_{e} \right) + I_{l,M,L}^{\Phi}(x, \frac{x}{e} \sum_{i=1}^{x/\varepsilon} \eta_{i}) \right]^{+}
\]

\[
+ \varepsilon \sum_{i=1}^{x/\varepsilon} h(\eta_{i}|p) + \varepsilon \sum_{j=1}^{\lfloor(c-x)/\varepsilon\rfloor} h(\bar{\eta}_{j}|p).
\]

The next step is to replace \( I_{l,M,L}^{\Phi} \) with \( I^{f} \) in the variational problem. First recall the definition of \( I_{u}^{f}(x, c) \) in (6.4). Fixing \( \varepsilon' > 0 \), there exist a vector
which concludes the proof. □

Arguing similarly for the lower bound, we also have

\[
\begin{align}
I^*_u(x, c) &> \left( \Delta_c(\{ \tilde{\eta}^c, \tilde{\eta}^s, \tilde{\eta}^t \}) - \varepsilon \log C_{\kappa} \right)^+ - I^f \left( x, \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \tilde{\eta}^c_i \right) \\
&\quad + \varepsilon \sum_{i=1}^{x/\varepsilon} h(\tilde{\eta}^c_i | p) + \varepsilon \sum_{j=1}^{(c-x)/\varepsilon} h(\tilde{\eta}^s_j | p) - \varepsilon'.
\end{align}
\]

From (6.24),

\[
(6.25) \quad I'_u(x, c) \leq \left[ \left( \Delta_c(\{ \tilde{\eta}^c, \tilde{\eta}^s, \tilde{\eta}^t \}) - \varepsilon \log C_{\kappa} \right)^+ \\
- \hat{I}^\Phi_{l, M, L}(x, \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \tilde{\eta}^c_i) \right] - \varepsilon \sum_{i=1}^{x/\varepsilon} h(\tilde{\eta}^c_i | p) + \varepsilon \sum_{j=1}^{(c-x)/\varepsilon} h(\tilde{\eta}^s_j | p).
\]

Now recall, see (6.12), that for a fixed measure \( \eta \),

\[
\hat{I}^\Phi_{l, M, L}(x, \eta) = \hat{I}^L(x', \eta) + \alpha_6 x' \Phi_1(M, \eta) + \alpha_7 x' \Phi_2(M, \eta).
\]

From the definitions of \( \Phi_1 \) and \( \Phi_2 \) in (6.10) and (6.11), using Lemma 5.10 we obtain

\[
\lim_{M \to \infty} \Phi_1(M, \eta) = \lim_{L \to \infty} \Phi_2(M, \eta) = 0,
\]

and since \( \{ \tilde{\eta}^c \} \) does not depend on the variables \( l, M, L \),

\[
\lim_{M \to \infty} \lim_{L \to \infty} \hat{I}^\Phi_{l, M, L}(x, \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \tilde{\eta}^c_i) = I^f \left( x', \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \tilde{\eta}^c_i \right).
\]

From (1.12) we have \( I^F \) is a point-wise limit of convex functions, hence it is convex and since it is defined on the interval \((0, 1)\), it is also continuous on it. Taking the limit on \( l \) we obtain

\[
(6.26) \quad \lim_{l \to 0} \lim_{M \to \infty} \lim_{L \to \infty} \hat{I}^\Phi_{l, M, L}(x, \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \tilde{\eta}^c_i) = I^f \left( x, \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \tilde{\eta}^c_i \right).
\]

Using (6.26) on (6.25) and comparing to (6.4) we obtain,

\[
\lim_{l \to 0} \lim_{M \to \infty} \lim_{L \to \infty} I'_u(x, c) \leq I^*_u(x, c) + \varepsilon'.
\]

Since this holds for every \( \varepsilon' > 0 \) and using

\[
\lim_{l \to 0} \lim_{M \to \infty} \alpha_3 l + \alpha_6 x'_1 e^{-M^{1+1/\log \varkappa}/(\alpha \log M)} = 0
\]

in (6.21), it follows that

\[
(6.27) \quad \lim_{n \to \infty} - \frac{1}{k} \log \chi \leq I^*_u(x, c).
\]

Arguing similarly for the lower bound, we also have

\[
(6.28) \quad \lim_{n \to \infty} - \frac{1}{k} \log \chi \geq I^*_l(x, c),
\]

which concludes the proof. □

As a last preparatory step, we compute the minimization over measures of certain functionals appearing in the expressions \( I^*_u \) and \( I^*_l \).
Lemma 6.3. Define

\[ F_\varepsilon(\{\eta\}_{x,c}) := \max_{1 \leq j \leq \lfloor(c-x)/\varepsilon\rfloor} \sum_{i=1}^{j} \int \log \rho_0(\omega)\tilde{\eta}_i(d\omega), \]

and let

\[ I^F_\varepsilon(y) := \inf \left\{ \sum_{j=1}^{\lfloor(c-x)/\varepsilon\rfloor} h(\tilde{\eta}_j|p) : F_\varepsilon(\{\tilde{\eta}\}_{x,c}) = y \right\}. \]

Then

\[ I^F_\varepsilon(y) \geq sy. \]

Proof. The result is trivial for \( y \leq 0 \) and hence we only consider \( y > 0 \). Also, if \( s = \infty \) then the support of \( \log \rho_0 \) under \( p \) is contained in \( \mathbb{R}_- \) and then \( I^F_\varepsilon(y) = \infty \) for \( y > 0 \). So we consider in the sequel \( s \in (1, \infty) \). Since the only appearance of the measures \( \{\tilde{\eta}\}_{x,c} \) in \( F_\varepsilon(\{\tilde{\eta}\}_{x,c}) \) is through integration against the test function \( F_1(\eta) = \int \log \rho_0(\omega)\eta(d\omega) \), we consider the auxiliary problem

\[ I^*_m(x) = \inf_{\eta \in M_p^1(\Sigma)} \left\{ h(\eta|p) : \int \log \rho_0(\omega)\eta(d\omega) = x \right\}. \]

By the contraction principle, see [3, Theorem 4.2.1], \( I^*_m(x) \) is the rate function of the LDP for the random variable \( \int \log \rho_0(\omega)R_\eta(d\omega) \), under \( P = p^\eta \), and thus by Cramer’s theorem, we have that the function \( I^*_m(\cdot) \) coincides with the function \( I_m(\cdot) \) defined in (6.1) when \( \eta = P \).

To be able to reduce the analysis of \( F_\varepsilon(\{\tilde{\eta}\}_{x,c}) \) to a single variable, we use (6.29) and (6.32) in (6.30), obtaining

\[ I^F_\varepsilon(y) = \inf \left\{ \sum_{i=1}^{\lfloor(c-x)/\varepsilon\rfloor} I_m(x_i) : \max_{1 \leq j \leq \lfloor(c-x)/\varepsilon\rfloor} x_1 + \ldots + x_j = y \right\}. \]

Now for a vector \( x \in \mathbb{R}^{\lfloor(c-x)/\varepsilon\rfloor} \), define

\[ N(x) = \inf \left\{ m \in \mathbb{Z}^+ : \max_{1 \leq j \leq \lfloor(c-x)/\varepsilon\rfloor} \sum_{i=1}^{j} x_i = \sum_{i=1}^{m} x_i \right\}. \]

Using (6.34) we have

\[ I^F_\varepsilon(y) = \inf \left\{ \sum_{i=1}^{\lfloor(c-x)/\varepsilon\rfloor} I_m(x_i) : \max_{1 \leq j \leq \lfloor(c-x)/\varepsilon\rfloor} x_1 + \ldots + x_j = y \right\} \geq \inf \left\{ \sum_{i=1}^{N(x)} I_m(x_i) : \max_{1 \leq j \leq \lfloor(c-x)/\varepsilon\rfloor} x_1 + \ldots + x_j = y \right\} \]

\[ \geq \inf \left\{ N(x)I_m\left(\frac{y}{N(x)}\right) : \max_{1 \leq j \leq \lfloor(c-x)/\varepsilon\rfloor} x_1 + \ldots + x_j = y \right\}, \]

where the last inequality uses the convexity of \( I_m \). By Lemma 6.1, for any positive \( z \) we have \( I_m(z) \geq sz \), and hence

\[ N(x)I_m\left(\frac{y}{N(x)}\right) \geq sy, \]
which together with \(6.33\) proves \(6.31\).

We are finally ready to prove Proposition \(3.2\).

**Proof of Proposition \(3.2\).** To prove the proposition we simplify the variational problems in Lemma \(6.2\) using the property of positive velocity and then take the limit on \(\varepsilon \to 0\) to match the lower and upper bounds.

As in Lemma \(6.2\) the calculations for \(I^*_x(x, c)\) and \(I^*_x(x, c)\) are similar, so we will only develop \(I^*_x(x, c)\) as it has more technical details to consider. Define

\[
F_4(\{\eta\}_x^\varepsilon) = \max_{1 \leq j \leq x/\varepsilon} \sum_{j=1}^j \int \log \rho_0(\omega)\eta_j(d\omega) - \frac{x/\varepsilon}{x/\varepsilon} \int \log \rho_0(\omega)\eta_j(d\omega) \geq 0.
\]

Using \(6.33\) and \(6.34\), \(\Delta_4(\{\eta\}_x^\varepsilon, \{\bar{\eta}\}_x^\varepsilon)\), defined in \(6.2\), can be written as

\[
\Delta_4(\{\eta\}_x^\varepsilon, \{\bar{\eta}\}_x^\varepsilon) = \varepsilon F_5((\bar{\eta})_x^\varepsilon) - \varepsilon F_4((\eta)_x^\varepsilon).
\]

Substituting \(6.38\) in \(6.40\) and taking the infimum over \(\varepsilon\), we obtain

\[
I^*_x(x, c) = \inf_{\{\eta\}_x^\varepsilon \in M^*_x(\Sigma_p), y \in \mathbb{R}} \left\{ \varepsilon \left( y - F_4((\eta)_x^\varepsilon) + \log C_\kappa \right) + I^F(x, \frac{\varepsilon}{x} \sum_{i=1}^x \eta_i) \right\} - I^F(x, \frac{\varepsilon}{x} \sum_{i=1}^x \eta_i) + \varepsilon I^F_c(y).
\]

Now observe that

\[
\left[ \varepsilon \left( y - F_4((\eta)_x^\varepsilon) + \log C_\kappa \right) + I^F(x, \frac{\varepsilon}{x} \sum_{i=1}^x \eta_i) \right] = I^F(x, \frac{\varepsilon}{x} \sum_{i=1}^x \eta_i) + \varepsilon \sum_{i=1}^x h(\eta_i|p) + \varepsilon I^F_c(y)
\]

\[
\geq I^F(x, \frac{\varepsilon}{x} \sum_{i=1}^x \eta_i) - \varepsilon \left( y - F_4((\eta)_x^\varepsilon) + \log C_\kappa \right) + \varepsilon \sum_{i=1}^x h(\eta_i|p) + \varepsilon I^F_c(y)
\]

\[(d)\]

\[
\geq I^F(x, \frac{\varepsilon}{x} \sum_{i=1}^x \eta_i) + \varepsilon \sum_{i=1}^x h(\eta_i|p) - \varepsilon \log C_\kappa + \varepsilon (I^F_c(y) - y^+),
\]

where in \((d)\) we used that \([a-b]^- \geq b-a\) and in \((e)\) we used that \((a+b)^+ \leq a^+ + b^+\) and that \(F_4((\eta)_x^\varepsilon) \geq 0\). By Lemma \(6.3\) we have that

\[
I^F_c(y) - y^+ \geq 0
\]

Substituting \(6.41\) in \(6.40\) and taking the infimum over \(\{\eta\}_x^\varepsilon \in M^*_x(\Sigma_p)\) and \(y \in \mathbb{R}\) we obtain

\[
I^*_x(x, c) \geq \inf_{\{\eta\}_x^\varepsilon \in M^*_x(\Sigma_p)} \left\{ I^F(x, \frac{\varepsilon}{x} \sum_{i=1}^x \eta_i) + \varepsilon \sum_{i=1}^x h(\eta_i|p) \right\} - \varepsilon \log C_\kappa.
\]
Next we deal with the upper bound. As with $I_r^*(x, c)$, we substitute (6.38) and (6.30) in (6.4) to obtain

\begin{equation}
I_u^*(x, c) = \inf_{\{\eta\}_i \in M^*_1(\Sigma_p), y \in \mathbb{R}} \left\{ \varepsilon \left( y - F_4(\{\eta\}_i^z) - \log C_\kappa \right) + I^F \left( x, \frac{\varepsilon}{x} \sum_{i=1}^{x/\varepsilon} \eta_i \right) \right\}^{-1}
\end{equation}

(6.43)

\begin{equation}
+ \varepsilon \sum_{i=1}^{x/\varepsilon} h(\eta_i | p) + \varepsilon I^F_c(y)
\end{equation}

(6.44)

For $s \in (1, \infty)$ set $y^* = 0$ while for $s = \infty$ set $y^* \leq 0$ so that $I^F_c(y^*) < \infty$. The infimum in (6.43) can be bounded above by substituting $y = y^* \leq 0$, hence

\begin{equation}
I_u^*(x, c) \leq \inf_{\{\eta\}_i \in M^*_1(\Sigma_p)} \left\{ \varepsilon \left( y^* - F_4(\{\eta\}_i^z) - \log C_\kappa \right) + I^F \left( x, \frac{\varepsilon}{x} \sum_{i=1}^{x/\varepsilon} \eta_i \right) \right\}^{-1}
\end{equation}

(6.46)

\begin{equation}
+ \varepsilon \sum_{i=1}^{x/\varepsilon} h(\eta_i | p) + \varepsilon I^F_c(y^*)
\end{equation}

(6.47)

So with (6.42) and (6.44) applied to (6.28) and (6.27), we obtain

\begin{equation}
-\varepsilon I^F_c(y^*) \leq \inf_{\{\eta\}_i \in M^*_1(\Sigma_p)} \left\{ I^F \left( x, \frac{\varepsilon}{x} \sum_{i=1}^{x/\varepsilon} \eta_i \right) + \varepsilon \sum_{i=1}^{x/\varepsilon} h(\eta_i | p) \right\} + \lim_{n \to \infty} \frac{1}{K} \log \chi \leq \varepsilon \log C_\kappa.
\end{equation}

(6.48)

To conclude the proof we show that

\begin{equation}
\inf_{\{\eta\}_i \in M^*_1(\Sigma_p)} \left\{ I^F \left( x, \frac{\varepsilon}{x} \sum_{i=1}^{x/\varepsilon} \eta_i \right) + \varepsilon \sum_{i=1}^{x/\varepsilon} h(\eta_i | p) \right\}
\end{equation}

(6.49)

does not depend on $\varepsilon$ so we can take the limit $\varepsilon \to 0$ in (6.49). First, since specific relative entropy is affine, we have

\begin{equation}
\varepsilon \sum_{i=1}^{x/\varepsilon} h(\eta_i | p) = x h \left( \frac{\varepsilon}{x} \sum_{i=1}^{x/\varepsilon} \eta_i \mid p \right),
\end{equation}

(6.50)

so (6.46) becomes

\begin{equation}
\inf_{\{\eta\}_i \in M^*_1(\Sigma_p)} \left\{ I^F \left( x, \frac{\varepsilon}{x} \sum_{i=1}^{x/\varepsilon} \eta_i \right) + x h \left( \frac{\varepsilon}{x} \sum_{i=1}^{x/\varepsilon} \eta_i \mid p \right) \right\}
\end{equation}

(6.51)
Now observe that \( \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \eta_i \in M_{s}(\Sigma_p) \) and thus
\[
\inf_{\{\eta\}_{i} \in M_{s}(\Sigma_p)} \left\{ I^F \left( x, \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \eta_i \right) + xh \left( \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \eta_i \left| p \right. \right) \right\} \geq \inf_{\eta \in M_{s}(\Sigma_p)} \left\{ I^F \left( x, \eta \right) + xh \left( \eta \left| p \right. \right) \right\}.
\]

At the same time when considering the infimum over the vectors \( \{\eta\}_{i} \varepsilon \) we could restrict ourselves to having all the measures being equal, i.e. \( \eta_i = \eta \) for \( i = 1, \ldots, x/\varepsilon \), and hence the reverse inequality also holds, therefore
\[
\inf_{\{\eta\}_{i} \in M_{s}(\Sigma_p)} \left\{ I^F \left( x, \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \eta_i \right) + xh \left( \frac{x}{\varepsilon} \sum_{i=1}^{x/\varepsilon} \eta_i \left| p \right. \right) \right\} (6.49) = \inf_{\eta \in M_{s}(\Sigma_p)} \left\{ I^F \left( x, \eta \right) + xh \left( \eta \left| p \right. \right) \right\}.
\]

Finally, observe that \( I^F_{c}(y^*) \) is finite: if \( s = \infty \) we chose \( y^* \) that way, while \( s \in (1, \infty) \) implies that there are positive and negative drifts in the support of \( p \), and hence 0 is in the domain of \( I^F_{c} \). With this, substituting (6.50) in (6.45), we take \( \varepsilon \to 0 \) to conclude the proof. \( \square \)

7. Appendix A: LDP for the conditional random walk on random environment

In this short appendix, we show that \( I^F_{c} \) in (1.12) has, for ergodic laws on the environment, a natural interpretation in terms of the rate function for the large deviations of hitting times in a conditioned environment.

Proposition 7.1. Fix \( \eta \in M_{s}(\Sigma) \). Then,
\[
\lim_{k \to \infty} -\frac{1}{k} \log P^\omega_{1}[\tau_{xk} \leq k] = \lim_{k \to \infty} I^F \left( x, R_{xk} \right) = I^F \left( x, \eta \right), \quad \eta - a.s..
\]

Proof. In Lemma 5.5 and Lemma 5.8 we were able to approximate \( P^\omega_{1}[\tau_{xk} \leq k] \) using \( I^L \) and error terms. Consider \( J = J(k) \) such that \( \lim_{k \to \infty} J(k) = \infty \), but \( J = o(k) \). In this proof we show that
\[
(1) \quad \lim_{L \to \infty} \lim_{k \to \infty} I^L_{f}(x, k, \omega) = I^F \left( x, \eta \right), \quad \eta \text{ almost surely.}
\]
\[
(2) \quad \lim_{L \to \infty} \lim_{k \to \infty} \left| -\frac{1}{k} \log P^\omega_{1}[\tau_{xk} \leq k] - I^L_{f}(x, k, \omega) \right| = 0, \quad \eta \text{ almost surely.}
\]

For the first statement observe that, see (5.43),
\[
I^L_{f}(x, k, \omega) = I^\phi_{f}(x, R_{xk}).
\]

Moreover by Lemma 5.5
\[
\lim_{k \to \infty} \left| I^L_{0}(x, k, \omega) - I^F_{f}(x, k, \omega) \right| = 0, \quad \eta - a.s.
\]

Since \( I^L_{f}(x, R_{xk}) \to I^\phi_{f}(x, \eta) \) almost surely as \( k \to \infty \), the statement is proved.
For the second statement, according to Lemma 5.8 we have

$$\left| \hat{I}_j(x, k, \omega) - \hat{I}_j^L(x, k, \omega) \right| \leq \frac{\alpha_6}{k} \sum_{i=J}^{x-1} \mathbb{P}_i^\omega [\tau_{i+1} \geq M] + \frac{\alpha_6}{k} \sum_{i=J}^{x-1} \mathbb{P}_i^\omega [\tau_{i+1} \geq M] + \frac{\alpha_7}{k} \sum_{i=J}^{x-1} H_i^\omega(L, M)$$

We prove now that $\eta$ almost surely, for every $M$,

$$\lim_{L \to \infty} \lim_{k \to \infty} \frac{1}{k} \sum_{i=J}^{x-1} H_i^\omega(L, M) = 0 \quad (7.2)$$

Indeed,

$$\frac{1}{k} \sum_{i=J}^{x-1} H_i^\omega(L, M) = \frac{M}{k} \sum_{i=J}^{x-1} \sum_{j=i-M}^{i+M} \xi_j(L-1, \omega)$$

$$\leq \frac{M}{k} \sum_{i=J}^{x-1} \sum_{j=i-M}^{i+M} \xi(L-1, \theta^{j-M-L+1} \omega)$$

$$= \frac{M}{k} \sum_{j=-M}^{M} \sum_{i=J}^{x-1} \xi(L-1, \theta^{i+j-M-L+1} \omega)$$

$$\to M x \int_{-M}^{M} \xi(L-1, \omega) \eta(d\omega), \quad \text{as } k \to \infty, \quad (7.3)$$

where the limit is due to the ergodic theorem. Lemma 5.10 then concludes the proof of (7.2). Recalling that $J = o(k)$, the proof of Proposition 7.1 is completed using (7.2) and Lemma 5.7.

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