ANYONS AS SPINNING PARTICLES

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Abstract

A model-independent formulation of anyons as spinning particles is presented. The general properties of the classical theory of (2+1)-dimensional relativistic fractional spin particles and some properties of their quantum theory are investigated. The relationship between all the known approaches to anyons as spinning particles is established. Some widespread misleading notions on the general properties of (2+1)-dimensional anyons are removed.

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1 Introduction

It is now well known that the case of (2+1)-dimensional space-time is a special one from the point of view of spinning particles: only in this case spin can take arbitrary values on the real line \([1, 2]\). Such particles with fractional (arbitrary) spin, called anyons, satisfy an exotic fractional statistics, realizing representations of the braid group \([3]–[5]\).

It is believed that anyons in the form of quasiparticles are related to the planar physical phenomena such as fractional quantum Hall effect and high-\(T_c\) superconductivity \([6]\).

The theory of anyons as spinning particles has specific features just from the point of view of classical mechanics. It is obvious that fractional spin particles cannot be described at the classical level by the standard pseudoclassical approach which uses Grassmann variables for taking into account spin degrees of freedom \([7]\). This is due to the fact that Grassmann variables lead after quantization to the finite-dimensional representations of the (2+1)-dimensional Lorentz group. Such representations correspond only to the cases of integer or half-integer spin \([8]\), and, therefore, the description of anyons requires some nonstandard approaches. Nevertheless, the first model of relativistic particle with fractional (arbitrary) spin was constructed in ref. \([9]\) on the basis of an analogy with the pseudoclassical approach \([10]\). In that model spin degrees of freedom were described by commuting translationally invariant variables and configuration space of the system had a nontrivial topology corresponding to the topology of the simplest nonrelativistic anyon system \([11]\).

Another attempt to construct a model of relativistic fractional spin particle was undertaken in ref. \([12]\) following the (3+1)-dimensional approach of Balachandran et al. \([13]\). The paper \([12]\) comprised the discussion of the classical theory and contained some general statements on the properties of (2+1)-dimensional relativistic spinning particles. First such a statement is that in the presence of spin the angular momentum/boost vector of a particle has a contribution proportional to its momentum. This contribution must, according to the paper \([12]\), satisfy the algebra of the (2+1)-dimensional Lorentz group \(SO(2, 1)\) at the level of Poisson brackets. These two statements being correct only in some special cases but misleading in the general case, became, unfortunately, popular notions and were reproduced noncritically in a series of papers on the subject. In particular, the parallelness of the spin contribution to the energy-momentum vector was declared in ref. \([14]\), devoted to the electromagnetic interaction of anyons, as to be a fundamental property for spinning particles in 2+1 dimensions. Moreover, the attempt to incorporate both general statements of ref. \([12]\) led to the construction of the self-contradicting model \([15]\). Subsequently, that model was criticized in the literature (see, e.g., ref. \([16]\)), but the above-mentioned basic misleading statements were not removed. In turn, the paper \([12]\) itself contained another misleading statement, that the second time derivative of the coordinates of a free relativistic spinning particle has to be equal to zero. This last statement was reproduced in the recent paper \([17]\), where it was also declared that it is not possible to keep simultaneously the free nature of the anyon and the translational invariance.

The purpose of the present paper is to investigate the general properties of the classical theory of relativistic fractional spin particles and some properties of the corresponding quantum theory. This, in particular, will allow us to demonstrate that the above-mentioned widespread notions on (2+1)-dimensional anyons are incorrect in the general case, and, therefore, can, as it has been stated above, be misleading. The analysis is based on the Poincaré group being the exact symmetry group of the theory, and, so, it will also
give us a possibility to demonstrate that the general statement of ref. [17] is incorrect.

Our consideration includes all the known approaches to the description of anyons as (2+1)-dimensional spinning particles. Such approaches do not use Chern-Simons U(1) gauge field constructions taking place in another, more developed approach to anyons [18, 19, 6], and they can be classified in the following way. The essential feature of the first above-mentioned model [9] is that in the case of quantizing the system by the reduced phase space quantization method, spin is present in it as an independent operator. As a result, fractional spin states are described there by means of multi-valued wave functions in the case of the standard realization of the spin operator or by single-valued functions in the case of its nonstandard realization (see below, section 6). Two analogous possibilities with single- and multi-valued wave functions take also place in the case of the covariant quantization of the model by the Dirac method [9]. Another model of such type was proposed in ref. [20] and subsequently analysed in ref. [21], where some very superficial speculations were given on the relation of the approach of the models [9, 20] to another possible approach to anyons which uses infinite-dimensional representations of the universal covering group SO(2,1) of the (2+1)-dimensional Lorentz group SO(2,1) (or SL(2,R), equivalent to it) [8,20–25]. This latter approach ascends in its origin to the (2+1)-dimensional model of the relativistic particle with torsion [22,26]. At last, fractional spin particles can be described proceeding from the nontrivial symplectic two-form corresponding to the monopole-like Poisson structure on the phase space of a relativistic particle [23]. For the first time such a possibility to describe anyons was, probably, pointed out (indirectly) in ref. [1] (see also ref. [27] and relevant papers [28]). Recently, it was exploited in refs. [14, 29] for introducing the interaction of anyons with electromagnetic and gravitational fields.

The present investigation will result in establishing the relationship between all the listed approaches to anyons as spinning particles.

The paper is organized as follows. In section 2 we “derive” two different but related possibilities for describing relativistic massive particles with an arbitrary fixed spin \( s \neq 0 \) within the framework of the classical canonical approach. It will be done proceeding from the general properties of the classical Poincaré group and exploiting an observation of a similarity, peculiar to the (2+1)-dimensional space-time, of the general case, characterized by \( s \neq 0 \), to the spinless case, \( s = 0 \). Here we take into account the Jacobi identities for independent phase space variables of the system, that allows us to generalize our preliminary analysis of ref. [30] to the most general case. One of the two possibilities, called by us as a minimal formulation (approach), consists in using for the purpose only the coordinate and momentum phase space variables and in fixing the value of nontrivial spin in a strong sense. Another possibility, called an extended formulation, uses auxiliary internal phase space variables and fixes spin in a weak sense, with the help of the corresponding spin constraint. The approach of refs. [14,23,27,29] with nontrivial symplectic two-form corresponds to the minimal formulation and is contained in it as a special, covariant, particular case. The extended formulation comprises two other above-mentioned approaches to anyons as spinning particles.

Section 3 is devoted to the investigation of the minimal approach. Here we shall find the general form of the brackets for the coordinates of the particle and the corresponding form for the total angular momentum vector, and, moreover, investigate the Lorentz properties of the coordinates. Then we shall consider two special cases in more detail. The first one comprises covariant coordinates being nonlocalizable in the sense of brackets,
whereas the second case is given by localizable coordinates, having trivial brackets between different components, but being noncovariant ones from the point of view of their Lorentz properties. We shall establish the connection between these two cases and use it for realizing the quantization procedure for the minimal canonical approach.

Section 4 is devoted to the consideration of the general properties of the extended canonical formulation and its relationship to the minimal one. The concrete models corresponding to the extended formulation are considered in two subsequent sections. In section 5 we analyse in detail the model proposed by us in paper [30]. We shall discuss this model within the framework of the lagrangian and hamiltonian approaches, and construct the general solutions to its equations of motion. This, in particular, will allow us to demonstrate that the classical analog of the quantum Zitterbewegung takes place in the general case here, and, as a consequence, the above-mentioned statement of refs. [15, 17] on the second derivative of the coordinates of a free relativistic spinning particle in 2+1 dimensions turns out generally to be incorrect. We shall conclude this section by discussing the quantization of the model. At the classical level, the model contains two cases with essentially different topology of the internal phase subspace of the system being a two-sheet or one-sheet hyperboloid. Correspondingly, the quantum theory comprises as an essential ingredient the infinite-dimensional representations of the \( \text{SL}(2,\mathbb{R}) \) group either of the so called discrete type series (half-bounded representations) or of the continuous series (unbounded representations).

In section 6, the model of refs. [20, 21] is considered. We shall get another form for its lagrangian and with the help of it analyse the classical theory of the model. Then we shall discuss the quantum theory revealing its analogy with that for the model from section 5. This observation together with the general results obtained in section 4 will help us to establish the one-to-one correspondence of the reduced phase space description of the model with the canonical description of the model from section 5 for the case corresponding to the continuous series of representations of \( \text{SL}(2,\mathbb{R}) \) at the quantum level. Thus, we shall find that these two extended models together with corresponding quantum schemes turn out to be closely related. In conclusion of this section we shall discuss a possible interpretation of the model ascending to the original analysis of Leinaas and Myrheim on the topology of configuration space of the systems of identical particles, which resulted in their discovery of fractional statistics [3]. Moreover, here we shall comment upon the relationship of the model to the approach of Balachandran et al. [13].

Section 7 comprises a summary of the paper and a list of the main results removing misleading notions on the general properties of anyons.

## 2 Spin in 2+1 dimensions

In classical mechanics, the algebra of (2+1)-dimensional Poincaré group ISO(2,1) is given by the Poisson brackets of its generators being the energy-momentum vector \( p_\mu \) and the total angular momentum vector \( J_\mu \) dual to the total angular momentum tensor \( J^{\nu\lambda} \),

\[
J_\mu = -\frac{1}{2} \epsilon_{\mu\nu\lambda} J^{\nu\lambda},
\]

\[
\{p_\mu, p_\nu\} = 0, \quad (2.1)
\]

\[
\{J_\mu, J_\nu\} = -\epsilon_{\mu\nu\lambda} J^\lambda, \quad (2.2)
\]

\[
\{J_\mu, p_\nu\} = -\epsilon_{\mu\nu\lambda} p^\lambda, \quad (2.3)
\]
where we have used the metric $\eta_{\mu\nu} = \text{diag}(-,+,+)$ and a totally antisymmetric tensor $\epsilon_{\mu\nu\lambda}$, $\epsilon^{012} = 1$. The components of the total angular momentum vector $J^\mu$ are the generators of the space rotations ($\mu = 0$) and Lorentz boosts ($\mu = 1, 2$), whereas the energy-momentum vector $p^\mu$ generates the space-time translations.

The quantities $p^\mu p_\mu$ and $p^\mu J_\mu$ lie in the center of algebra (2.1)–(2.3). So, in the quantum case their analogs are the Casimir operators of the quantum mechanical Poincaré group ISO(2,1) with eigenvalues characterizing its irreducible representations. In the massive case, $-p^2 = m^2 > 0$, the pseudoscalar quantity

$$S = \frac{p^\mu J_\mu}{\sqrt{-p^2}}$$

(2.4)

has a sense of spin and the eigenvalues of its quantum analog can take any values $s \in \mathbb{R}$ [2]. Moreover, we have here the following specific situation due to the (pseudo)-scalar nature of spin. In (2+1)-dimensional space-time a relativistic particle with fixed mass $m > 0$ and fixed spin $s \neq 0$ has the same number of degrees of freedom as the relativistic massive scalar particle with zero spin. Proceeding from this observation, we pass over to the investigation of possible approaches for describing relativistic massive particle of an arbitrary (fixed) spin.

Within the framework of the canonical approach such a system can be described in the following way. Introduce coordinates of the particle $x_\mu$, $\mu = 0, 1, 2$, and conjugate to the momenta $p_\mu$,

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu}. \quad (2.5)$$

These brackets mean that momenta $p^\mu$ simultaneously are the generators of the space-time translations. The brackets for the coordinates themselves we denote as

$$\{x_\mu, x_\nu\} = \epsilon_{\mu\nu\lambda} R^\lambda, \quad (2.6)$$

having in mind that $R^\mu$ can be some function of $x_\mu$ and $p_\mu$, and, possibly, of other (auxiliary) variables. Its admissible structure will be analysed below.

Now, let us introduce the constraint

$$p^2 + m^2 \approx 0. \quad (2.7)$$

This constraint fixes the mass in a weak sense [31], and its quantum analog is nothing else as the Klein-Gordon equation. The total angular momentum for a spinning particle can be taken in the form generalizing that for the spinless (scalar) particle:

$$J_\mu = -\epsilon_{\mu\nu\lambda} x_\nu p^\lambda + J^\mu. \quad (2.8)$$

The second term $J^\mu$ in eq. (2.8) takes into account spin $s \neq 0$. This can be done in two different ways. One possibility consists in introducing some auxiliary internal phase space variables $z_n$, $n = 1, \ldots, 2N$, independent of external variables $x_\mu$ and $p_\mu$, $\{z_n, x_\mu\} = \{z_n, p_\mu\} = 0$, and in supplementing the mass shell constraint (2.7) with the constraint

$$pJ - sm \approx 0. \quad (2.9)$$

Here we suppose that $J^\mu$ depends on these internal variables. Relations (2.7), (2.8) and definition (2.4) mean that this constraint fixes the value of the particle spin. Due to
independence of $J_\mu$ on $x_\mu$ (see below), the constraints (2.9) and (2.7) form the trivial algebra of the first class constraints. Moreover, in order to have a pure spin system, without any other internal degrees of freedom, we must supplement constraint (2.9) with a corresponding number of first and/or second class constraints to eliminate the remaining $N-1$ internal phase space degrees of freedom ($2(N-1)$ variables). As a result, the system will have the same number of degrees of freedom as the relativistic spinless particle in 2+1 dimensions.

Another possibility consists in fixing the value of spin in a strong sense, without introducing any auxiliary internal degrees of freedom and corresponding constraints freezing these degrees of freedom. It can be done by choosing the second term in the total angular momentum (2.8) in the form

$$J_\mu = -se_\mu^{(0)} + J_\mu^\perp,$$

(2.10)

where $e_\mu^{(0)} = p_\mu/\sqrt{-p^2}$ and $J_\mu^\perp$ is transverse to $p_\mu$, $J_\mu^\perp p^\mu = 0$. Indeed, in this case the system will have the same number of degrees of freedom as a scalar massive particle has. Moreover, if we shall satisfy relations (2.2) and (2.3), then representation (2.10) together with eqs. (2.4) and (2.7) will guarantee that we have a relativistic massive particle with nontrivial spin $s$. We shall call this latter possibility for the description of fractional (arbitrary) spin particles the minimal approach (formulation), whereas the former one will be called the extended approach.

The vector $e_\mu^{(0)}$, introduced above, can be supplemented with the pair of momentum-dependent objects $e_\mu^{(i)} = e_\mu^{(i)}(p)$, $i = 1, 2$, so that the three $e_\mu^{(\alpha)}$, $\alpha = 0, 1, 2$, will form a complete triad:

$$e_\mu^{(\alpha)}\eta_{\alpha\beta}e_\nu^{(\beta)} = \eta_{\mu\nu}, \quad e_\mu^{(\alpha)}\eta^{\mu\nu}e_\nu^{(\beta)} = \eta^{\alpha\beta},$$

(2.11)

whose orientation can be chosen in different ways, in particular, we can fix it as

$$e_{\mu\nu\lambda}e^{(0)\mu}e^{(1)\nu}e^{(2)\lambda} = 1.$$

Such a triad will be used in a further analysis.

Before passing over to the analysis of the two described possibilities and their relationship, let us make some general observations, which will be necessary for a subsequent consideration. First of all, we note that the brackets (2.3) together with (2.1) and (2.5) prescribe $J_\mu$ to be independent of $x_\mu$ in both approaches. This means that in the minimal formulation the transverse part $J_\mu^\perp$ of the spin addition $J_\mu$ can depend only on $p_\mu$. Then in both cases the brackets (2.2) lead to the condition

$$\{J_\mu, J_\nu\} = -\epsilon_{\mu\nu\lambda} \left( J^\lambda + (p^\sigma \partial^\lambda - p^\lambda \partial^\sigma) J_\sigma - p^\lambda R_{\sigma\lambda} p_\sigma \right).$$

(2.12)

Here and below we use the notation $\partial^\mu = \partial/\partial p_\mu$. Moreover, the functions $R_\mu$ defining the brackets of the coordinates must be chosen in such a way, that the Poisson brackets for independent phase space variables $x_\mu$, $p_\mu$ (and $z_n$ in the extended case) would satisfy corresponding Jacobi identities. The Jacobi identity $\{\{x_\mu, x_\nu\}, p_\lambda\} + cycle = 0$ is reduced to the condition of independence of $R_\mu$ on $x_\mu$, whereas the identity $\{\{x_\mu, x_\nu\}, x_\lambda\} + cycle = 0$ is reduced to the condition

$$\partial^\mu R_\mu = 0.$$  

(2.13)

Relations (2.12) and (2.13) will play an important role in what follows.
3 Minimal canonical approach

In this section we shall analyse in detail the minimal canonical approach. First we shall find the most general admissible form for the quantities $R_\mu$ defining the brackets of coordinates of the particle, and, besides, we shall establish the form of the transverse part $J^\perp_\mu$ of the angular momentum addition $J_\mu$. Then we shall analyse the Lorentz properties of the coordinates $x_\mu$, and determine the form of the angular momentum addition and the form of the brackets $\{x_\mu, x_\nu\}$ compatible with the requirement of covariant (vector) behavior of the coordinates with respect to the Lorentz transformations. The form of the brackets will turn out to be a nontrivial one corresponding to the monopole-like symplectic two-form considered in refs. [14, 23, 27, 29]. After that we shall find the general form for the total angular momentum vector $J_\mu$ compatible with the trivial brackets $\{x_\mu, x_\nu\} = 0$, and shall determine the corresponding Lorentz properties of $x_\mu$. In conclusion of this section we shall comment on the quantum theory corresponding to the minimal canonical approach.

As we have pointed out in the end of the previous section, in the minimal approach $J_\mu = J_\mu(p)$, and, so, $\{J_\mu, J_\nu\} = 0$. Therefore, the transverse part of $J_\mu$ can be presented as

$$J^\perp_\mu = -\epsilon_{\mu\nu\lambda} A^\nu p^\lambda,$$

(3.1)

where $A_\mu$ is some function of $p_\mu$, being defined up to the arbitrary term of the form $p_\mu a(p)$ and having the dimensionality $[A] = [p]^{-1}$. Then we find the condition (2.12) is equivalent to the condition $p_\mu (R_\mu - \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda) = s/\sqrt{-p^2}$, and, therefore,

$$R_\mu = -s \frac{p_\mu}{(-p^2)^{3/2}} + \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda + R^\perp_\mu$$

(3.2)

solves eq. (2.12), where $R^\perp_\mu$ is an addition orthogonal to $p_\mu$, $R^\perp_\mu p_\mu = 0$. Now, using eq. (3.2), we find that eq. (2.13) is reduced to the condition $\partial_\mu R^\perp_\mu = 0$, whose general solution is $R^\perp_\mu = -\epsilon_{\mu\nu\lambda} p^\nu \partial^\lambda g$ with an arbitrary function $g = g(p)$, $[g] = [p]^{-2}$. Therefore, as a general solution to eqs. (2.12) and (2.13) we get finally the functions

$$R_\mu = -s \frac{p_\mu}{(-p^2)^{3/2}} + \epsilon_{\mu\nu\lambda} \partial^\nu \tilde{A}^\lambda$$

(3.3)

containing arbitrary functions $\tilde{A}_\mu = A_\mu + p_\mu g$.

Hence, a (2+1)-dimensional relativistic massive particle with arbitrary spin $s$ can be described by the coordinates $x_\mu$ and conjugate momenta $p_\mu$ with brackets given by eqs. (2.1), (2.9) and (2.10), where $R_\mu$ is given, in turn, by eq. (3.3). Constraint (2.7) prescribes the particle to have a fixed mass $m > 0$, whereas the nontrivial value of spin $s$ is coded in the form of the total angular momentum vector

$$J_\mu = -\epsilon_{\mu\nu\lambda} x^\nu p^\lambda - s c_\mu^{(0)} - \epsilon_{\mu\nu\lambda} A^\nu p^\lambda.$$

(3.4)

Its specific form together with nontrivial form of brackets (2.10), (3.3) guarantees the fulfillment of the Poincaré algebra (2.1)–(2.3).

Here a remark is in order. The difference between $\tilde{A}_\mu$ from eq. (3.3) and $A_\mu$ from eq. (3.4) coincides with the arbitrariness up to which $A_\mu$ has been defined itself when we have introduced it into consideration in eq. (3.1). This difference can be removed by redefining the coordinates: $x_\mu \to x'_\mu = x_\mu + p_\mu g$. Such a redefinition does not change the form of the
total angular momentum vector (3.4), but the functions $R_\mu$ giving the brackets for the redefined coordinates take the form

$$R_\mu = -s \frac{P_\mu}{(-p^2)^{3/2}} + \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda. \quad (3.5)$$

So, without loss of generality, within the minimal approach the brackets for the coordinates can be taken in the form (2.6) with the functions $R_\mu$ defined by eq. (3.5). Further on we shall use this form of the functions $R_\mu$ as giving the general case.

Let us pass over to the analysis of the Lorentz properties of the coordinates $x_\mu$. From the form of the brackets

$$\{J_\mu, x_\nu\} = -\epsilon_{\mu\nu\lambda} x^\lambda - \eta_{\mu\nu} \epsilon_{\lambda\rho\sigma} p^\lambda \partial^\rho A^\sigma + \epsilon_{\mu\rho\sigma} (p_\nu \partial^\rho - \partial_\nu p^\rho) A^\sigma \quad (3.6)$$

we conclude that in the general case the three $x_\mu, \mu = 0, 1, 2$, has transformation properties different from those of a Lorentz vector. Simple algebraic calculations show that the coordinates of the particle $x_\mu = x^c_\mu$ have covariant transformation properties given by the brackets

$$\{J_\mu, x^c_\nu\} = -\epsilon_{\mu\nu\lambda} x^c_\lambda \quad (3.7)$$

only in the case when $A^\mu$ has a special form $A^\mu = p^\mu a$, where $a = a(p^2), [a] = [p]^{-2}$, is an arbitrary function. In its turn, this means that the coordinates of the particle have a covariant nature when

$$J_\mu = -\epsilon_{\mu\nu\lambda} x^c_\nu p^\lambda - se_\mu^{(0)}, \quad (3.8)$$

and

$$\{x^c_\mu, x^c_\nu\} = -se_{\mu\nu\lambda} \frac{p^\lambda}{(-p^2)^{3/2}}. \quad (3.9)$$

Therefore, the coordinates of the particle have a covariant sense only in the case when their brackets as well as the total angular momentum vector have special fixed form, given by eqs. (3.3) and (3.8), respectively. The form of brackets (3.3) means that in the quantum case there is no representation in which the operators corresponding to the covariant coordinates $x^c_\mu$ would be diagonal. We shall return to this point in the end of the section. Another specific feature which we have here is that only in this covariant case the spin addition $J^c_\mu = -se_\mu^{(0)}$ in the total angular momentum vector (3.8) is parallel to the energy-momentum vector $P_\mu$. It is necessary to stress that due to the nontrivial brackets (3.3), neither the first “orbital” term from $J_\mu$ nor the second one do not satisfy the Lorentz algebra at the level of Poisson brackets, but only the total vector (3.8) does. Therefore, we conclude that the information on the spin of the system is coded simultaneously in the form of the total angular momentum vector and in the form of the brackets for the covariant coordinates, and only bearing in mind this fact one can call the second term $J^c_\mu = -se_\mu^{(0)}$ as the spin vector. Moreover, we see that the properties of the covariance (3.7) and localizability of the coordinates $x_\mu = x^l_\mu$,

$$\{x^l_\mu, x^l_\nu\} = 0, \quad (3.10)$$

are compatible iff $s = 0$. It is necessary also to note that the general case, given by eqs. (3.3) and (3.4), is connected with the special case (3.8), (3.9) with covariant coordinates $x^c_\mu$ through the relation

$$x^c_\mu = x_\mu + A_\mu. \quad (3.11)$$
As we shall see below, this relation turns out to be important in the construction of the corresponding quantum scheme for the minimal approach.

Now, let us find the general form for the total angular momentum vector $\mathbf{J}_\mu$ compatible with the property of localizability of the coordinates (3.10). In correspondence with eq. (3.5), equality (3.10) takes place when $A_\mu$ is subject to the equation

$$\partial_\mu A_\mu - \partial_\nu A_\mu = -s \epsilon_{\mu\nu\lambda} \frac{p^\lambda}{(-p^2)^{3/2}}. \quad (3.12)$$

This equation means that the “gauge field” $A_\mu$ has the curvature of the SO(2,1) monopole and is defined up to the “gauge transformation”

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu f, \quad (3.13)$$

where $f = f(p)$ is an arbitrary function of dimensionality $[f] = [p]^0$. Up to the “gauge transformation” (3.13), the “gauge potential” can be chosen in the form:

$$A_\mu = -s \epsilon_{\mu\nu\lambda} \frac{p^\nu \xi^\lambda}{p^2 + \sqrt{-p^2}(p\xi)}, \quad (3.14)$$

where $\xi^\mu$ is an arbitrary fixed timelike unit vector, $\xi^2 = -1$. Since the denominator on the r.h.s. of eq. (3.14) contains the factor $(1 - e^{(0)}\xi)$, we can choose the vector $\xi_\mu$ in the form

$$\xi^\mu = \frac{p^0}{|p^0|} \cdot (1, 0, 0) \quad (3.15)$$

in order to avoid a singularity not only for $p^0 > 0$ but also for $p^0 < 0$. Such a choice of $\xi^\mu$ is possible since from the very beginning we have borne in mind that $p^2 < 0$. Hence, the sectors with $p^0 > 0$ and $p^0 < 0$ are separated (i.e. the phase space is supposed to be disconnected), and therefore, the sign factor in eq. (3.13) can be considered as a constant one in these two sectors. On the other hand, if we omit the sign factor in eq. (3.13), we will have the singularity for the case $p^0 < 0$: $A_i \propto \epsilon^{ij}p^j/p^k$ as $p^i \rightarrow 0$, $i = 1, 2$, where we have introduced the notation $\epsilon^{ij} = \epsilon^{(0)}ij$.

Let us note here that due to the form of the total angular momentum vector (3.4), the gauge transformation (3.13) can be “absorbed” into the first, orbital-like term by redefining the coordinates, $x_\mu \rightarrow x_\mu + \partial_\mu f$, since such a redefinition is a canonical transformation. Therefore, without loss of generality we can use the fixed form of the “gauge potential” (3.14) with $\xi_\mu$ given by eq. (3.13).

Thus, the specific form of the total angular momentum (3.4) with the SO(2,1) “monopole gauge potential” $A_\mu$ given by eq. (3.14) guarantees the fulfillment of algebra (2.2) in the case when the coordinates of the particle have the property of localizability (3.10). But as we have pointed out above, in this case the coordinates of the particle have noncovariant properties given by eq. (3.6), where $A_\mu$, in turn, is given by eqs. (3.14), (3.15). Indeed, in the case under consideration the total angular momentum is presented in the form

$$\mathbf{J}_\mu = -\epsilon_{\mu\nu\lambda} x^{\nu\lambda} + J^i_\mu, \quad (3.16)$$

where

$$J^i_0 = -s \frac{p_0}{|p^0|}, \quad J^i_l = -s \frac{p_i}{\sqrt{-p^2} + |p^0|}, \quad (3.17)$$
and we have the transformation properties of the coordinates $x^l_{ij}$ given by the brackets

$$\{J_\mu, x^l_{ij}\} = -\epsilon_{\mu \nu \lambda} x^{j \lambda} + \Delta_{\mu \nu}, \quad (3.18)$$

where

$$\Delta_{0 \nu} = 0, \quad \Delta_{\mu \nu} = \frac{s}{|p^0| + \sqrt{-p^2}} \cdot e^{(i)}_{\nu},$$

with

$$e^{(i)0} = \frac{p^0}{|p^0|} \cdot \frac{p^j}{\sqrt{-p^2}}, \quad e^{(i)j} = \delta^{ij} + \frac{p^j p^i}{\sqrt{-p^2} (|p^0| + \sqrt{-p^2})}. \quad (3.19)$$

The presence of the noncovariant term $\Delta_{\mu \nu}$ in eq. (3.18) means that the coordinates $x^l_{ij}$ have noncovariant transformation properties with respect to the Lorentz boosts, whose generators are $J_\mu$. The quantities $e^{(i)}_{\mu}$ appearing here are the components of the triad (2.11) satisfying another orientation condition: $e_{\mu \nu \lambda} e^{(0)\mu} e^{(1)\nu} e^{(2)\lambda} = -p^0/|p^0|$

Let us stress once more that though $J^j_{\mu} = s\sqrt{-p^2}$, nevertheless, due to the relation $J^j e^{(i)} \neq 0$, here the spin addition $J^j_{\mu}$ defined by eq. (3.17) is not parallel to the energy-momentum vector $p_\mu$ in correspondence with the general results declared after eq. (3.9).

Due to noncovariant properties of the localizable coordinates $x^l_{ij}$, one could forget about them and work only in terms of the covariant coordinates $x^c_{ij}$. But we are forced to recall about localizable coordinates as soon as we pass over to the consideration of the quantum theory. Indeed, due to nontrivial form of brackets (3.9), we have a problem of constructing the operators corresponding to the classical coordinates $x^c_{ij}$. This problem can be solved with the help of the localizable coordinates $x^l_{ij}$, which we redenote here as $q_\mu$, in the following way. According to eq. (3.11), we can realize the operators $\hat{x}^c_{ij}$ with the help of equality

$$\hat{x}^c_\mu = \hat{q}_\mu + \hat{A}_\mu, \quad (3.20)$$

where $\hat{A}_\mu$ is the quantum analog of the “gauge potential” given by eqs. (3.14) and (3.13). Therefore, we can choose the momentum representation with the operators $\hat{p}_\mu$ being diagonal, and in correspondence with classical relations (2.3) and (3.10), the operators $\hat{q}_\mu$ will be realized as $\hat{q}_\mu = i\partial/\partial p^\mu$. On the other hand, we can choose the coordinate representation with the operators $\hat{q}_\mu$ being diagonal and with $\hat{p}_\mu = -i\partial/\partial q^\mu$. In this case due to the specific form of the “gauge potential” $A_\mu$ being present on the r.h.s. of eq. (3.20), the operator $\hat{x}^c_\mu$ will have a nonlocal form. In both cases of the momentum and coordinate representations, noncovariant properties of the operator $\hat{q}_\mu$ are coded in the nontrivial form of the angular momentum operator

$$\hat{J}_\mu = -\epsilon_{\mu \nu \lambda} \hat{q}^\nu \hat{p}^\lambda + \hat{J}^j_{\mu},$$

where $\hat{J}^j_{\mu}$ is the quantum analog of the spin addition given by eq. (3.17). This operator being rewritten in terms of $\hat{x}^c_{ij}$ with the help of relation (3.20) takes, clearly, the quantum form corresponding to the classical equality (3.8).

Therefore, we conclude that the manifest covariance of the minimal approach being formulated in terms of the coordinates $x^c_{ij}$ is inevitably lost under transition to the quantum theory, where we are forced to use localizable, $[\hat{q}_\mu, \hat{q}_\nu] = 0$, but noncovariant operators $\hat{q}_\mu$. Moreover, in the coordinate representation with diagonal operators $\hat{q}_\mu$, the covariant operators $\hat{x}^c_{ij}$ turn out to be nonlocal operators. These shortcomings can be removed by using the extended canonical formulation for the relativistic fractional spin particles.
4 Extended canonical formulation

Let us turn to the general analysis of the extended formulation and its relation to the minimal one. As we have pointed out in sect. 2, in the extended case the spin addition \( J_\mu \) in the total angular momentum vector \( J_\mu \) depends on auxiliary internal variables \( z_n \) and generally it can also depend on \( p_\mu \) (see eq. (2.12)). It is reasonable to restrict this general analysis by adding the rather natural assumption that \( J_\mu \) depends only on the internal variables, \( J_\mu = J_\mu(z_n) \). Moreover, let us suppose that the coordinates of the particle \( x_\mu \) are localizable, i.e. they have the brackets

\[
\{x_\mu, x_\nu\} = 0. \tag{4.1}
\]

As a result, the general condition (2.12) takes the very simple form

\[
\{J_\mu, J_\nu\} = -\epsilon_{\mu\nu\lambda}J^\lambda. \tag{4.2}
\]

Therefore, under the assumptions made above, the components of \( J_\mu \) themselves must form the Lorentz algebra \( so(2,1) \), and in this case both \( J_\mu \) and \( x_\mu \) are Lorentz vectors, i.e. here the property of the covariance of the coordinates is compatible with that of localizability (4.1) thanks to the introduction of the auxiliary internal variables.

Now, let us reveal the relation of this formulation to the minimal one, whereas concrete examples of models realizing the extended formulation will be considered in the two subsequent sections. To this end, we decompose the vector \( J_\mu = J_\mu(z_n) \) into longitudinal and transverse parts:

\[
J_\mu = -J^{(0)}\epsilon_\mu^{(0)} + J_\mu^\perp. \tag{4.3}
\]

where \( J^{(0)} = J_\mu \epsilon_\mu^{(0)} \), \( J_\mu^\perp = (\eta_{\mu\nu} + \epsilon_\mu^{(0)}\epsilon_\nu^{(0)})J_\nu \), and taking into account the mass shell constraint (2.7), we present the spin constraint (2.9) in the equivalent form

\[
J^{(0)} - s \approx 0. \tag{4.4}
\]

All the quantities in the system can be classified as those being either gauge-invariant or gauge-noninvariant ones with respect to the gauge transformations generated by this constraint [31]. In particular, the brackets

\[
\{J^{(0)}, x_\mu\} = -\frac{1}{\sqrt{-p^2}}J_\mu^\perp \tag{4.5}
\]

mean that the coordinates \( x_\mu \) are not gauge-invariant quantities.

With the help of decomposition (4.3), we can present the total angular momentum (2.8) in the equivalent form

\[
J_\mu = -\epsilon_{\mu\nu\lambda}\tilde{x}^\nu p^\lambda - J^{(0)}\epsilon_\mu^{(0)} \tag{4.6}
\]

with

\[
\tilde{x}_\mu = x_\mu + \frac{1}{p^\nu}\epsilon_{\mu\nu\lambda}p^\nu J^\lambda. \tag{4.7}
\]

The redefined coordinates \( \tilde{x}_\mu \) form a Lorentz vector similar to the initial coordinates \( x_\mu \), but unlike the latter ones, they have the trivial brackets with \( J^{(0)} \),

\[
\{\tilde{x}_\mu, J^{(0)}\} = 0. \tag{4.8}
\]
Hence, $\tilde{x}_\mu$ is the gauge-invariant extension [31] of the initial vector $x_\mu$. The total angular momentum vector (4.6) written in terms of the gauge-invariant coordinates (4.7) has the same form (in the weak sense, under taking into account the spin constraint (4.4)) as the total angular momentum vector from the minimal canonical formulation had when it was presented in terms of the covariant coordinates $x^c_\mu$. It is necessary to note that the both terms in eq. (4.6) are gauge-invariant vectors. The brackets for the redefined coordinates,

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = -J^{(0)} e_{\mu \nu \lambda} \frac{p^\lambda}{(-p^2)^{3/2}},$$

(4.9)

have the same form (in the weak sense) as the covariant coordinates $x^c_\mu$ had. The gauge-invariant term $-J^{(0)} e^{(0)}_\mu$ in the angular momentum vector (4.6) carries alongside with the constraint (4.4) and brackets (4.9) the information on the nontrivial value of the spin, and, therefore, we arrive at a complete correspondence with the special case of the minimal formulation given in terms of the coordinates $x^c_\mu$.

Let us demonstrate now how the most general case from the minimal formulation, given by eqs. (3.4), (3.5), is related to the extended formulation. This will also allow us to get some relations useful for subsequent considerations.

First of all, from the brackets

$$\{J^{(0)}, J^2\} = 0$$

(4.10)

and

$$\{J^{(0)}, J^\perp_\mu\} = e_{\mu \nu \lambda} e^{(0)\nu} J^\perp_\lambda,$$

(4.11)

it follows that the constraint (4.4) generates the rotations of the two-dimensional vector $J$ with components $J^{(i)} = e^{(i)}_\mu J^\mu$, $i = 1, 2$, where $e^{(i)}_\mu$ are the components of the triad (2.11). Therefore, generally the gauge orbits generated by the action of the constraint (4.4) are one-dimensional spheres $S^1$ in the three-dimensional space with coordinates $J^{(\alpha)}$, $\alpha = 0, 1, 2$. These orbits are the sections of the set of two-dimensional hyperboloids $J^2 = -J^{(0)2} + J^{(i)2} = \text{const}$ by the hyperplane $J^{(0)} = s$.

We can pass over from the initial set of auxiliary internal coordinates $z_n, n = 1, \ldots, 2N$, to the set of variables $J^{(0)}, \varphi$ and $\tilde{z}_a, a = 1, \ldots, 2(N - 1)$, such that $0 < \varphi \leq 2\pi$,

$$\{\varphi, J^{(0)}\} = 1,$$

(4.12)

and $\{\tilde{z}_a, J^{(0)}\} = \{\tilde{z}_a, \varphi\} = \{\tilde{z}_a, \tilde{x}_\mu\} = \{\tilde{z}_a, p_\mu\} = 0$. The concrete form of $\tilde{z}_a = \tilde{z}_a(z_n, p^\mu)$ is not important for us here. The variables $J^{(0)}$ and $\varphi$ form a pair of action-angle type variables. As a result, we arrive at the parametrization for the components $J^{(i)}$ of the two-dimensional vector $J$:

$$J^{(i)} = r \cdot n^i, \quad r = \sqrt{J^{(0)2} + C(\tilde{z}_a)}, \quad n^i = (\sin \varphi, \cos \varphi).$$

(4.13)

We assume here that $C(\tilde{z}_a)$ and corresponding region of possible values of $J^{(0)}$ are defined by concrete model. Definition (4.7) and eq. (4.13) lead to the brackets

$$\{\tilde{x}_\mu, \varphi\} = -e^{(1)\nu} \partial_\mu e^{(2)\nu}.$$

(4.14)

The angle variable $\varphi$, obviously, parametrizes the points on the above-mentioned gauge orbits $S^1$'s. Therefore, proceeding from the initial set of the phase space variables $x_\mu, p_\mu$
and $z_n$, we can pass over to the set of the variables $\tilde{x}_\mu, p_\mu, J^{(0)}, \tilde{z}_a$ and $\varphi$. The variables $\tilde{x}_\mu, p_\mu, J^{(0)}$ and $\tilde{z}_a$ form the complete set of the variables being gauge-invariant with respect to the gauge transformations generated by the spin constraint (4.4), whereas $\varphi$ is the only gauge-noninvariant variable. For completeness we stress once more that in accordance with the general discussion from section 2, it is necessary to supplement the spin constraint (4.4) and the mass shell constraint (2.7) with a corresponding number of first and/or second class constraints which would “freeze” internal degrees of freedom described by the variables $\tilde{z}_a$.

Now, it is obvious that the reduction of the system to the surface of the constraint (4.4) can be realized by introducing a (local) gauge condition

$$\varphi - \varphi_0 \approx 0,$$

(4.15)

where $\varphi_0$ is some fixed point on a one-dimensional gauge orbit $S^1$. The Dirac brackets must be calculated with the help of the pair of second class constraints (4.4) and (4.15). Due to the gauge invariance of the quantities $\tilde{x}_\mu, p_\mu$ and $J_\mu$ with respect to the gauge transformations generated by the constraint (4.4), all the Dirac brackets between them will coincide with the corresponding initial brackets. At the same time, we find that the Dirac brackets $\{x_\mu, x_\nu\}^*$ have the form given by eqs. (2.6) and (3.5) with the functions $A_\mu$ given by

$$A_\mu = \frac{1}{p^2} \epsilon_{\mu\nu\lambda\rho} p^\nu J^\lambda, \quad J_\mu = -s e_\mu^{(0)} + e_\mu^{(i)} \gamma^i,$$

(4.16)

where quantities $\gamma^i$ are defined, in turn, by eq. (4.13) at $J^{(0)} = s$ and $\varphi = \varphi_0$.

Therefore, we conclude that the gauge-invariant coordinates (4.7) have the same sense here as the covariant “nonlocalizable” coordinates $x_\mu^c$ from the minimal canonical formulation. On the other hand, the initial gauge-noninvariant variables $x_\mu$ in the following sense are analogous to the coordinates corresponding to the general case of the minimal formulation, given by the total angular momentum vector (4.4) and by the functions $R_\mu$ (3.5); here the Dirac brackets $\{x_\mu, x_\nu\}^*$, appearing after reduction of the system to the surface defined by eqs. (4.4) and (1.13), correspond to the brackets of the coordinates $x_\mu$ from the minimal formulation.

It is necessary to note that after reduction of the system, in correspondence with the general results obtained for the minimal formulation, the initial gauge-noninvariant coordinates $x_\mu$ will not generally have covariant properties with respect to the Lorentz transformations due to a noncovariant character of eq. (4.13) giving the reduced subspace. Only in one case, when $-J^2 = s^2$, and, so, $\gamma^{(i)} = 0$ and $A_\mu = 0$ in eq. (4.16), gauge-invariant coordinates $\tilde{x}_\mu$ coincide with $x_\mu$. From the point of view of the reduction, such a case can be understood as a limit one, in which the above-mentioned gauge orbits shrink into just a point. We shall consider the concrete model comprising such a limit case in the next section, where, in particular, we shall demonstrate that this case will also be a special one from the lagrangian point of view.

Thus, we have revealed the relation of the extended canonical formulation to the minimal one through the reduction of the former formulation. But as we have pointed out in the beginning of the section, the advantage of the former formulation is that it allows us to describe the system in terms of the covariant coordinates $x_\mu$ having commuting components in the sense of the Poisson brackets (4.4) when fixing the spin is realized in the theory in the weak sense via the constraint (2.9). So, we pass over to the consideration of concrete models within the framework of the extended formulation.
5 Minimal extended model

In this section we shall consider the extended model with a minimal number of $2N = 2$ internal auxiliary phase space variables. At the quantum level, as we shall see, this model will correspond to the approach with infinite-dimensional representations of $\text{SL}(2, \mathbb{R})$ group \[22, 23\].

The minimal extended model can be constructed in the following way. First we note that the scalar $J^2$ lies in the center of the algebra (4.2), \[\{J^2, J_\mu\} = 0\]. Therefore, it can be fixed by introducing the condition

$$J^2 = C,$$

where $C$ is some real constant which corresponds here to $C(\bar{z}_a)$ from eq. (4.13). So, if we consider the dependent variables $J_\mu$, subject to the condition (5.1), as the internal variables themselves, we shall have the internal phase subspace with $2N = 2$ independent variables. This subspace has the structure defined by the constant $C$: for $C = -\alpha^2$, $\alpha > 0$, eq. (5.1) sets two disconnected sheets of the hyperboloid:

$$J_0 = \varepsilon \sqrt{\alpha^2 + J_i^2}, \quad \varepsilon = \pm, \quad i = 1, 2,$$

whereas in the case $C = \beta^2$, $\beta > 0$, it defines a one-sheet hyperboloid degenerating into the cone at $\beta = 0$.

Thus, the minimal extended model can be given by the variables $x_\mu$ and $p_\mu$ with canonical brackets (2.1), (2.5), (4.1), and by the spin variables $J_\mu$ subject to the condition (5.1) and forming the $\text{so}(2, 1)$ algebra (4.2). Moreover, the model has the set of two first class constraints (2.7) and (2.9). It is necessary to stress that for the case $J^2 = -\alpha^2 < 0$, the spin constraint has nontrivial solutions only when $s^2 \geq \alpha^2$, and, therefore, the model is defined only for $s^2 \geq -J^2$.

Let us construct now the lagrangian corresponding to the model. The brackets (4.2) for the internal variables $J_\mu$ can be derived from a kinetic lagrangian

$$L_{\text{kin}} = -\frac{J^2}{J^2 + (J_\mu)_{\nu}} \varepsilon_{\mu\nu\lambda} \zeta^\mu J^\nu \cdot J^\lambda$$

containing an arbitrary fixed unit timelike vector $\zeta^\mu$, $\zeta^2 = -1$. The simplest way to be convinced that it is so consists in checking the fact that under a Lorentz transformation of $J_\mu$, the kinetic term (5.3) is changed by a total derivative, and, therefore, it corresponds to a Lorentz invariant term in the action. Then, choosing $\zeta^\mu = (1, 0, 0)$, and parametrizing the variables $J_\mu$ as

$$J_\mu = \left( J_0, \sqrt{J_0^2 + C \cdot n_i} \right), \quad n_i = (\cos \varphi, \sin \varphi),$$

where $0 \leq \varphi < 2\pi$ and $-\infty < J_0 < \infty$ in the case $C = \beta^2 \geq 0$, whereas $J_0$ can take values in the region $[\alpha, +\infty)$ or $(-\infty, -\alpha]$ when $J^2 = -\alpha^2 < 0$, one gets

$$L_{\text{kin}} = J_0 \cdot \dot{\varphi}.$$ 

From here we find that the brackets for independent variables $J_0$ and $\varphi$ have the form

$$\{\varphi, J_0\} = 1.$$
Using these brackets and parametrization (5.4), one can get convinced that the lagrangian (5.3) indeed leads to the brackets (4.2).

The total lagrangian of the model can be obtained by the inverse Legendre transformation from the requirement that it would lead to the constraints (2.7) and (2.9). This gives

\[
L = \frac{1}{2\epsilon}(\dot{x}_\mu - vJ_\mu)^2 - \frac{1}{2}\epsilon m^2 + smv + L_{\text{kin}}. \tag{5.6}
\]

Lagrangian (5.6), with \(\epsilon\) and \(v\) being the Lagrange multipliers, leads to the primary constraints

\[
p_\epsilon \approx 0, \quad p_v \approx 0, \tag{5.7}
\]

and to the mass shell and spin conditions (2.7) and (2.9) as the secondary constraints, where \(p_\epsilon\) and \(p_v\) are the momenta conjugate to \(e\) and \(v\), \(\{e, p_\epsilon\} = 1\), \(\{v, p_v\} = 1\).

The corresponding action \(A = \int L d\tau\) is invariant with respect to the reparametrizations:

\[
\delta x_\mu = \gamma \dot{x}_\mu, \quad \delta J_\mu = \gamma \dot{J}_\mu, \quad \delta e = \frac{d}{d\tau}(\gamma e), \quad \delta v = \frac{d}{d\tau}(\gamma v),
\]

\[
\delta L = \frac{d}{d\tau}(\gamma L), \tag{5.8}
\]

whose generator is the mass shell constraint, and with respect to the transformations generated by the spin constraint (2.9):

\[
\delta e = 0, \quad \delta v = \dot{\rho}, \quad \delta x_\mu = \rho J_\mu, \quad \delta J_\mu = -\rho e^{-1}\epsilon_{\mu\nu\lambda}\dot{x}^\nu J^\lambda,
\]

\[
\delta L = \frac{d}{d\tau}(\rho (sm + e^{-1}\dot{x}J - J^2 e^{-1}(\dot{x}J + (\dot{x}\zeta)(\zeta J)) \cdot (J^2 + (J\zeta)^2)^{-1})). \tag{5.9}
\]

Here \(\gamma = \gamma(\tau)\) and \(\rho = \rho(\tau)\) are infinitesimal parameters of the transformations.

Let us consider now the Lagrange equations of motion for \(e\) and \(v\),

\[
(\dot{x}_\mu - vJ_\mu)^2 + e^2 m^2 = 0, \quad \dot{x}J - vJ^2 - sme = 0. \tag{5.10}
\]

From the second equation we get the equality \(e = s^{-1}m^{-1}(\dot{x}J - vJ^2)\), and we can rewrite the first equation in the form

\[
\dot{x}^2 + s^{-2}(\dot{x}J)^2 = -v(1 + s^{-2}J^2) \cdot (vJ^2 - 2\dot{x}J).
\]

From here we conclude that iff

\[
-J^2 = \alpha^2 = s^2, \tag{5.11}
\]

there is the Lagrange constraint:

\[
\dot{x}^2 + s^{-2}(\dot{x}J)^2 = \dot{x}^2 - (\dot{x}J)^2 \cdot (J^2)^{-1} = 0, \tag{5.12}
\]

which means that the particle velocity vector \(\dot{x}_\mu\) is parallel to the spin vector \(J_\mu\). Note, that the analogous property of parallelness between \(p_\mu\) and \(J_\mu\) in the case (5.11) takes place in this model at the hamiltonian level. It is important to stress here that such a property of parallelness of the spin vector \(J_\mu\) to \(p_\mu\) is valid only in the weak sense, on the surface of the spin constraint. Therefore, in the case (5.11) the particle velocity turns out also to be parallel to its energy-momentum vector \(p_\mu\). Below we shall return to this point.

One can rewrite lagrangian (5.6) in a form revealing the speciality of the case (5.11) in a more explicit way. To this end, we find the multiplier \(v\) from the second equation...
assuming that $J^2 \neq 0$. We get $v = (J^3)^{-1}(\dot{x}J - sme)$, and after substituting this value into lagrangian (5.9), we arrive at the following form for the total lagrangian:

$$L = \frac{1}{2e} \left( \dot{x}^2 - (J^2)^{-1}(\dot{x}J)^2 \right) + \text{sm}(J^2)^{-1}(\dot{x}J) - \frac{1}{2} em^2 \left( 1 + s^2 (J^2)^{-1} \right) + L_{\text{kin}}. \quad (5.13)$$

The term linear in $e$ disappears from (5.13) only when eq. (5.11) takes place. Therefore, in this case the variation of the corresponding action over $e$ gives directly the Lagrange constraint (5.12). The spin constraint (2.9) appears from lagrangian (5.13) for this special case as the primary constraint, whereas the mass-shell constraint (2.7) is a secondary one.

Let us note that the general procedure of the reduction of the system, described in the previous section, can be applied also to the special case (5.11) if to consider it as a limit, e.g., as $J^2 = -\alpha^2$, $s^2 = \alpha^2 + \epsilon^2$, $\epsilon \to 0$. After such a reduction we get the strong equality $J_{\mu} = -se_{\mu}^{(0)}$. Due to the gauge-noninvariance of the vector $J_{\mu}$ (see eq. (4.11)), instead of the initial $so(2,1)$ algebra (4.2) we arrive as a result of the reduction at the trivial Dirac brackets for the components of the vector $J_{\mu}$, which turns out to be parallel to the energy-momentum vector $p_{\mu}$ in the strong sense on the reduced phase space.

Let us investigate now the classical motion of the particle in the model under consideration. It can be done in the simplest way within the framework of the hamiltonian approach proceeding from the initial form of lagrangian (5.6). The corresponding total hamiltonian has here the form

$$H = \frac{e}{2} (p^2 + m^2) + v(pJ - ms) + w_1 p_e + w_2 p_v, \quad (5.14)$$

where $w_{1,2} = w_{1,2}(\tau)$ are arbitrary functions of the evolution parameter, which are associated with the primary first class constraints (5.7) [31]. The equations of motion generated by the total hamiltonian via the Poisson brackets have the form:

$$\dot{p}_{\mu} = 0, \quad \dot{x}_{\mu} = ep_{\mu} + vJ_{\mu}, \quad \dot{J}_{\mu} = -ve_{\mu\lambda}p^\lambda J, \quad \dot{e} = w_1, \quad \dot{v} = w_2. \quad (5.15)$$

From here we see once again that since generally $J_{\mu}$ is not parallel to the conserved energy-momentum vector $p_{\mu}$, the velocity $\dot{x}_{\mu}$ is not parallel to $J_{\mu}$. Note, that the coordinates $\tilde{x}_{\mu}$, constructed according to eq. (4.7), have the evolution law given by the equation

$$\dot{\tilde{x}}_{\mu} = (e - vsm^{-1}) \cdot p_{\mu}, \quad (5.17)$$

where we have taken into account the mass shell and spin constraints. As we shall see below, the coordinates of the particle $x_{\mu}$, being gauge-noninvariant quantities with respect to the gauge transformations generated by the constraint (4.4), have the evolution law revealing, unlike the gauge-invariant coordinates $\tilde{x}_{\mu}$, the classical analog of the relativistic quantum Zitterbewegung which takes place in the general case for models of spinning particles [10].

Contracting $x_{\mu}$ and $J_{\mu}$ with the triad components $e_{\mu}^{(a)}$ given by eq. (3.19) and taking again into account the mass shell and spin constraints, we get

$$\dot{x}^{(0)} = -me + vs, \quad \dot{x}^{(i)} = vJ^{(i)}, \quad (5.18)$$

$$\dot{J}^{(0)} = s, \quad \dot{J}^{(i)} = -\epsilon^{ij}mv \epsilon^{ij} J^{(j)}, \quad (5.19)$$
where through \( \varepsilon^0 \) we have denoted the sign of the energy \( p^0 \). Now let us fix the variables \( v \) and \( e \) as

\[
v - v_0 = 0, \quad e - m^{-1}(1 + v_0 s) = 0,
\]

(5.20)

where \( v_0 \) is a constant, restricted by the condition

\[
v_0^2 \leq r^{-2}, \quad r = \sqrt{C + s^2},
\]

(5.21)

(see below). Conditions (5.20) are really the gauge conditions conjugate to the primary constraints (5.7), and the requirement of their stationarity leads, according to eq. (5.16), to fixing the multipliers \( w_{1,2} \) in the total Hamiltonian: \( w_1 = w_2 = 0 \). From the Lagrangian point of view the introduction of conditions (5.20) corresponds, obviously, to fixing the gauge freedom given by eqs. (5.8) and (5.9). Such a choice leads to a simple form for the general solutions to eqs. (5.18) and (5.19):

\[
x^{(0)} = \varepsilon^0 (\tau + \tau_0), \quad x^{(i)} = m^{-1} r k^{(i)} + x_0^{(i)}, \quad j^{(i)} = r n^{(i)}.
\]

(5.22)

Here \( k^{(1)} = n^{(2)} = \sin \varphi(\tau), \ k^{(2)} = n^{(1)} = \cos \varphi(\tau), \ \varphi(\tau) = -\varepsilon^0 m v_0 \tau + \varphi_0, \) and \( \tau_0, \ x_0^{(i)} \) and \( \varphi_0 \) are some constants of integration. The first equality means that in the rest frame system, \( p^i = 0, \ i = 1, 2 \), we have in fact the laboratory temporal gauge: \( x^0 = \varepsilon^0 (\tau + \tau_0) \). According to the second relation and eq. (3.19) defining the explicit form of the triad components \( e^{(i)}_\mu \), in this system the particle performs a circular motion. The radius of the circle, \( m^{-1} r \), is defined by the choice of the model’s constant \( C \), whereas the velocity is a gauge-dependent quantity, \( \dot{x}^{(i)} \dot{x}^{(i)} = v_0^2 r^2 \leq 1 \). In the case of arbitrary momentum \( p \neq 0 \) we have here a relativistic superposition of the circular motion and the rectilinear motion with a constant velocity (in the laboratory time) equal to \( p/p^0 \). Indeed, as follows from eqs. (5.17) and (5.20), here

\[
\dot{x}^0 = \frac{p^0}{m}
\]

(5.23)

and \( \dot{x}^i = p^i/m \). Therefore, the gauge-invariant coordinates \( \tilde{x}_\mu \) do not reveal any gauge-dependent circular motion, and, moreover,

\[
\frac{d\tilde{x}^i}{d\tilde{x}^0} = \frac{p^i}{p^0}
\]

(5.24)

in a complete correspondence with the ordinary evolution law for relativistic free particle. Using eqs. (5.23) and (5.24), the definition of the gauge-invariant coordinates (4.7) and the form of the triad (3.19), one can get convinced that in the general case, when \( p \neq 0 \), the coordinates \( x^i \) evolves in the way that has been described above.

Here it is necessary to make an important remark. Using eq. (5.23) and definition (4.7), one can find that the inequality \( \dot{x}^0 \neq 0 \) takes place only in the case when the velocity of the particle is not greater than the velocity of light. It means, in turn, that only when \( \dot{x}^2 \leq 0 \), one can pass over to the laboratory temporal gauge with \( |\dot{x}^0| = 1 \) for any value of the momentum \( p \), and, as consequence, the coordinate \( x^0 \) can be interpreted as a time only in this case. It is due to this reason we have introduced the restriction on the modulus of the constant \( v_0 \) in eq. (5.21) (for a more detailed discussion of this point see ref. [10]).

Thus, we conclude that generally the coordinates of the particle \( x_i \) are subject to the gauge-dependent circular motion being the classical analog of the relativistic quantum
Zitterbewegung [32]. Therefore, the equality to zero of the second time derivative of the coordinates of the particle is not a necessary condition for the relativistic spinning particle to be a free particle in 2+1 dimensions. Only in one case, when \( C = J^2 = -s^2 \), there is no such a circular motion in the system and here \( \dddot{x}_\mu = 0 \). At the same, the space components of the gauge-invariant coordinate \( \tilde{x}_\mu \) do not reveal a Zitterbewegung and evolves according to the ordinary law for the coordinates of the free relativistic spinless particle. Note, that in this respect the latter coordinates are analogous to the Foldy-Wouthuysen coordinates of the Dirac particle [33].

Now let us pass over to the discussion of the quantization of the model. We have here covariant variables \( x_\mu \) with classical brackets (4.1), and, so, can choose the coordinate representation with the operators \( \hat{x}_\mu \) being diagonal and with \( \hat{p}_\mu = -i\partial/\partial x^\mu \). Then the only problem consists in realizing the quantum analogs of the variables \( J_\mu \) subject to the condition (5.1) and forming with respect to the brackets the algebra \( so(2,1) \) (4.2). This problem was considered in the paper [34] in connection with the quantization of the models of relativistic particles with curvature and torsion [22, 35], and here it is reasonable to sketch the results necessary for the present consideration.

As we have pointed out in the beginning of the section, there are two cases being essentially different from the point of view of the topology of the subspace described by the variables \( J_\mu \). This topology is defined by the value of the constant \( C \). First, let us consider the case when \( C = -\alpha^2, \alpha > 0 \). Here we have two disconnected sheets of the hyperboloid (5.2). In the quantum case the operators \( \hat{J}_\mu \) can be realized in the form

\[
\hat{J}_0 = z \frac{\partial}{\partial z} + \alpha, \quad \hat{J}_1 = -\frac{1 + z^2}{2} \frac{\partial}{\partial z} - \alpha z, \quad \hat{J}_2 = -i \frac{1 - z^2}{2} \frac{\partial}{\partial z} + i\alpha z. \tag{5.25}
\]

These operators act on the space of functions \( \psi(z) \), holomorphic in the unit disc \( |z| < 1 \) on the complex plane. They satisfy the commutation relations

\[
[\hat{J}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\lambda} \hat{J}_\lambda \tag{5.26}
\]

corresponding to the classical relations (4.2). Moreover, the Casimir operator of the \( so(2,1) \) algebra (5.20) takes here the value

\[
\hat{J}^2 = -\alpha(\alpha - 1) \tag{5.27}
\]

substituting the classical value \( C = -\alpha^2 \). The scalar product in the internal subspace,

\[
(\psi_1, \psi_2) = \frac{2\alpha - 1}{2} \int \int_{|z|<1} \overline{\psi}_1(z)\psi_2(z)(1 - |z|^2)^{2\alpha-2}d^2z, \tag{5.28}
\]

where bar means a complex conjugation, is defined so that the operators (5.25) are hermitian. The infinite set of functions

\[
\psi^n(z) = \sqrt{\Gamma(2\alpha + n)/\Gamma(n+1)\Gamma(2\alpha)} \cdot z^n, \quad n = 0, 1, 2, \ldots, \tag{5.29}
\]

represents a complete set of orthonormal functions on the space of the holomorphic functions, which are the eigenfunctions of the operator \( \hat{J}_0 \):

\[
\hat{J}_0 \psi^n = j^n_0 \psi^n, \quad j^n_0 = \alpha + n. \tag{5.30}
\]
So, relations (5.27) and (5.30) mean that the operators \( \hat{J}_\mu \) (5.25) correspond to the realization of the infinite-dimensional unitary discrete-type series of representations \( D^+_\alpha \) of the group \( \text{SL}(2, \mathbb{R}) \) on the space of holomorphic functions in the unit disc \( |z| < 1 \) [30]. Such a realization of the operators \( \hat{J}_\mu \) corresponds, in turn, to the classical variables \( J_\mu \) setting the upper sheet (\( \varepsilon = + \)) of the classical two-sheet hyperboloid (5.2). The realization corresponding to the lower sheet with \( \varepsilon = - \) can be obtained from the realization (5.25) through the obvious substitution:

\[
\hat{J}_0 \rightarrow -\hat{J}_0, \quad \hat{J}_1 \rightarrow -\hat{J}_1, \quad \hat{J}_2 \rightarrow \hat{J}_2.
\] (5.31)

In this case operators \( \hat{J}_\mu \) satisfy relations (5.26) and (5.27), whereas functions (5.29) are the eigenfunctions of the operator \( \hat{J}_0 \) with the eigenvalues \( j^n_0 = -(\alpha + n) \), i.e. for \( \varepsilon = - \) in eq. (5.2) we arrive at the unitary irreducible representations \( D^-_\alpha \) of the group \( \text{SL}(2, \mathbb{R}) \).

In the general case (5.2) we have a direct sum of the representations (5.25) and (5.31):

\[
D^+_\alpha \oplus D^-_\alpha.
\]

In the case \( C = \beta^2 \geq 0, \beta \geq 0 \), the quantization of the subsystem described by the vector \( J_\mu \) results in the irreducible unitary infinite-dimensional representations of the principal continuous series \( C^\varrho_{\sigma} \), characterized by the value of the Casimir operator

\[
\hat{J}^2 = \sigma = \beta^2 + 1/4
\] (5.32)

and by the eigenvalues

\[
j^n_0 = \varrho + n, \quad \varrho \in [0, 1), \quad n = 0, \pm 1, \pm 2, \ldots,
\] (5.33)

of the operator \( \hat{J}_0 \). In this case the operators \( \hat{J}_\mu \) are realized in the form of the linear differential operators

\[
\hat{J}_0 = -i \frac{\partial}{\partial \varphi} + \varrho, \quad \hat{J}_\pm = J_1 \pm iJ_2 = e^{\pm i\varphi} \left( -i \frac{\partial}{\partial \varphi} + \varrho \pm i \left( \beta - \frac{i}{2} \right) \right)
\] (5.34)

acting on the space of functions \( \psi(\varphi) \) being \( 2\pi \)-periodic in the angle variable \( \varphi \). These operators are hermitian ones with respect to the natural internal scalar product

\[
(\psi_1, \psi_2) = \frac{1}{2\pi} \int_0^{2\pi} \psi_1(\varphi)\psi_2(\varphi) d\varphi;
\] (5.35)

and here the set of the functions

\[
\psi^n(\varphi) = e^{in\varphi}
\] (5.36)

forms the complete orthonormal set of eigenfunctions of the operator \( \hat{J}_0 \) corresponding to the eigenvalues (5.33):

\[
\hat{J}_0 \psi^n = j^n_0 \psi^n.
\]

Here the following remark is in order. In the cases of the half-bounded representations \( D^+_\alpha \) and unbounded representations of the principal continuous series \( C^\varrho_{\sigma} \) the operators \( \hat{J}_\mu \) act on the spaces of single-valued functions being the holomorphic or \( 2\pi \)-periodic

---

\(^3\)With the help of the formal substitution \( \beta \rightarrow i\tilde{\beta} \), representation (5.34) can be transformed into the unitary irreducible representation of the supplementary continuous series for the case when \( 0 < \tilde{\beta} < 1/2 \). In this case the scalar product can be changed in such a way that the modified operators \( \hat{J}_\mu \) will be hermitian ones (see refs. [34, 36, 37]).
functions, respectively. The differential operator $\hat{J}_0$ contains in both cases a constant number addition being equal to $\pm \alpha$ or $\vartheta$. One can remove this constant addition via the appropriate transformation of the operators $\hat{J}_\mu$, not violating either the commutation relations or the value of the Casimir operator and eigenvalues of $\hat{J}_0$. In the case of representation (5.23) the corresponding transformation has the form

$$\hat{\tilde{J}}_\mu = z^\alpha \hat{J}_\mu z^{-\alpha},$$

(5.37)

whereas in the case (5.34) it is represented as

$$\hat{\tilde{J}}_\mu = e^{i\vartheta \varphi} \hat{J}_\mu e^{-i\vartheta \varphi}.$$  

(5.38)

In the first case the eigenfunctions (5.29) are substituted by the functions $\tilde{\psi}_n(z) = z^\alpha \psi_n(z)$, whereas in the second case the eigenfunctions (5.30) are substituted by $\tilde{\psi}_n(\varphi) = e^{i\vartheta \varphi} \psi_n(\varphi)$. In both cases, generally, these transformed functions are multi-valued ones, and we note that such multi-valuedness of the functions taking place, as will be shown just below, for the one-particle relativistic massive states with fractional spin here corresponds to the multi-valuedness of the wave functions describing two-particle systems of nonrelativistic anyons in one and two space dimensions [38]. Moreover, it is also interesting to note that the half-bounded representations $D^\pm_\alpha$ take place as representations of the algebra of observable operators for the system of two nonrelativistic anyons in one-dimensional space [39].

Let us consider now a field $\Psi(x, \cdot) = \sum_n \Psi^n(x) \psi^n(\cdot)$, being transformed according to one of the described unitary infinite-dimensional representations, which satisfies the equations:

$$(\hat{p}^2 + m^2)\Psi = 0, \quad (\hat{p} \hat{J} - sm)\Psi = 0.$$  

(5.39)

Here we suppose that dot means $z$ or $\varphi$, and $n = 0, 1, 2, \ldots$, in the first case, whereas in the second case $n = 0, \pm 1, \pm 2, \ldots$. These equations are the quantum analogs of the classical constraints (2.7) and (2.9) fixing the values of the corresponding Casimir operators $-\hat{p}^2$ and $\tilde{S}$ of the Poincaré group $ISO(2,1)$. The equations have nontrivial solutions under the appropriate choice of the parameter $s$ coordinated with the choice of the corresponding representation of $SL(2,\mathbb{R})$ characterized by the parameters $\alpha$ or $\vartheta$. The simplest way to get convinced in the validity of this statement consists in passing over to the momentum representation and choosing the rest frame system with $\mathbf{p} = 0$. In particular, in the case $C = -\alpha^2$, $s = \alpha$, we have the discrete type representations $D^\pm_\alpha$, and the states $\Psi \propto \psi^0(z)$ will describe the physical states with positive ($p^0 = m$) and negative ($p^0 = -m$) energies in the rest frame system for the cases of representations $D^+_\alpha$ and $D^-_\alpha$, respectively. So, we conclude that the quantized minimal extended model corresponds to the approach using infinite-dimensional unitary representations of the $SL(2,\mathbb{R})$ group [22, 23].

6 Anyon as a coupled two-particle system

This section is devoted to the consideration of the model proposed in ref. [20]. Here we shall get another form for its lagrangian, that will allow us to simplify considerably the analysis of the system and reveal a nontrivial relationship of the model to the minimal extended model from the preceding section. We shall also comment on the relationship
of the approach of Balachandran et al. [13] to the present one and we shall point out an interesting possible interpretation of the model explaining the title of the section.

Thus, let us consider the system, given by the lagrangian [20, 21]

\[ L = m\ddot{x} - \sigma e_{\mu\lambda}n^{\nu}n^{\lambda} - \frac{\sigma}{2}(e^2 + 1) - \frac{\rho}{2}(n^2 - 1) - \omega(en). \]  

Here \( n^{\mu} \) is translationally invariant vector, which together with corresponding conjugate momenta will play the role of the variables \( z_n \), whereas \( \sigma, \rho, \omega \) and \( e_{\mu} \) are Lagrange multipliers. The variation of the action over these multipliers gives the Lagrange constraints

\[ e^2 + 1 = 0, \quad n^2 - 1 = 0, \quad en = 0, \]  

\[ \sigma e_{\mu} + \omega n_{\mu} + s\epsilon_{\mu\nu\lambda}n^{\nu}n^{\lambda} - m\dot{x}_{\mu} = 0. \]  

Using constraints \( (6.3) \) alongside with the first and third ones from the set \( (6.2) \), we can express the Lagrange multipliers \( \sigma, \omega \) and \( e_{\mu} \) via \( \dot{x}_{\mu}, n_{\mu} \) and \( \dot{n}_{\mu} \). Then, putting their expressions into lagrangian \( (6.1) \), we get for it the following form, equivalent from the physical point of view to the initial one:

\[ L = -m\sqrt{-(\dot{x}_{\mu} - n_{\mu}(\dot{x}n) - sm^{-1}\epsilon_{\mu\nu\lambda}n^{\nu}n^{\lambda})^2} - \frac{\rho}{2}(n^2 - 1). \]  

This lagrangian leads to the mass shell constraint \( (2.7) \), the spin constraint \( (2.9) \) (see below) and the constraint \( \Pi \approx 0 \) as the first class constraints. Moreover, there is a set of second class constraints in the system:

\[ pn \approx 0, \quad p\pi \approx 0, \]  

\[ \pi n \approx 0, \quad n^2 - 1 \approx 0. \]  

Here by \( \Pi \) and \( \pi_{\mu} \) we denote the momenta conjugate to \( \rho \) and \( n^{\mu} \),

\[ \{\rho, \Pi\} = 1, \quad \{n_{\mu}, \pi_{\nu}\} = \eta_{\mu\nu}. \]

The constraint \( \Pi \approx 0 \) means that \( \rho \) and \( \Pi \) are pure gauge variables. They can be eliminated, e.g., by introducing the gauge \( \rho - \rho_0 \approx 0 \) to this constraint, where \( \rho_0 \) is some constant. At the same time, the second class constraints \( (6.3) \) and \( (6.6) \) have the following sense: the first pair serves for removing the nonphysical degree of freedom described by the timelike conjugate variables \( n^{(0)} = ne^{(0)} \) and \( \pi^{(0)} = \pi e^{(0)} \), \( \{n^{(0)}, \pi^{(0)}\} = -1 \), whereas constraints \( (6.6) \) remove (or freeze according to our terminology from sections 2 and 4) the radial oscillator degree of freedom described by the conjugate variables \( n_r = \sqrt{n^{12}} \) and \( \pi_r = \pi n, \{n_r, \pi_r\} = 1 \), being independent in the sense of brackets from \( n^{(0)} \) and \( \pi^{(0)} \). These four variables play, obviously, the role of the variables \( \tilde{z}_a \) discussed in section 4.

In the present model the total angular momentum has the form given by eq. \( (2.8) \) with spin addition equal to

\[ J_{\mu} = -\epsilon_{\mu\nu\lambda}n^{\nu}\pi^{\lambda}. \]  

It is this vector which defines the spin constraint \( (2.9) \) here. Vector \( (6.8) \) satisfies the \( so(2, 1) \) algebra \( (1.2) \) with respect to the initial canonical Poisson brackets of \( x_{\mu}, p_{\mu}, n_{\mu} \) and \( \pi_{\mu} \), but it has nonzero Poisson brackets with the constraints \( (6.5) \). This means that after necessary taking into account second class constraints, the “spin vector” \( (6.8) \) will
satisfy another (trivial) algebra with respect to the corresponding Dirac brackets, and, moreover, as we shall see, it will turn out to be parallel to the energy-momentum vector \( p_\mu \) on the surface of the second class constraints (5.3).

Thus, let us take into account the second class constraints (6.3), (6.6), that will allow us to exclude from the consideration nonphysical degrees of freedom described by the variables \( n^{(0)} \) and \( n_r \) and by their conjugate momenta. The simplest way to do this consists in reducing the system to the surface \( \Gamma \) specified by these constraints. On \( \Gamma \), the variables \( n_\mu \) and \( \pi_\mu \) can be parametrized in the form:

\[
\begin{align*}
  n_\mu &= e^{(i)}_\mu n^{(i)}, \\
  \pi_\mu &= J^{(0)} e^{(i)}_\mu \epsilon^{ij} n^{(j)},
\end{align*}
\]

where \( e^{(i)}_\mu, i = 1, 2 \), are two components of the triad (2.11), \( n^{(i)} = (\sin \varphi, \cos \varphi), 0 \leq \varphi < 2\pi \), and the quantity \( J^{(0)} \), \( -\infty < J^{(0)} < \infty \), is understood as an independent variable on the reduced phase space of the system. Parametrization (6.9) leads to the equality \( J_\mu = -J^{(0)} e^{(0)}_\mu \). Therefore, we see that on the surface of the second class constraints, \( J_\mu \) is indeed parallel to the vector \( p_\mu \) and the quantity \( J^{(0)} \) has, according to the definition (2.4), a sense of the spin variable. Then, reducing the symplectic 2-form \( \Omega = dp_\mu \wedge dx_\mu + d\pi_\mu \wedge dn_\mu \) to \( \Gamma \), we get the symplectic 2-form

\[
\omega = dp_\mu \wedge dy_\mu + dJ^{(0)} \wedge d\varphi
\]

on the reduced phase space described by the variables \( J^{(0)}, \varphi, p_\mu \) and

\[
y_\mu = x_\mu - J^{(0)} e^{(1)}_\nu \partial_\mu e^{(2)}_\nu.
\]

From this 2-form we find that the variables of the reduced phase space have the following nonzero Dirac brackets:

\[
\{y_\mu, p_\nu\}^* = \eta_{\mu\nu}, \quad \{\varphi, J^{(0)}\}^* = 1.
\]

With the help of eq. (6.10) one can find nontrivial Dirac brackets for the initial coordinates \( x_\mu \): \( \{x_\mu, p_\nu\}^* = \eta_{\mu\nu}, \)

\[
\{x_\mu, x_\nu\}^* = -J^{(0)} \epsilon_{\mu\nu\lambda} \frac{p^\lambda}{(-p^2)^{3/2}},
\]

\[
\{x_\mu, \varphi\}^* = -\epsilon^{(1)\sigma} \partial_\mu e^{(2)}_\sigma.
\]

The Dirac brackets for the initial “internal” variables \( n_\mu \) and \( \pi_\mu \) between themselves and with the coordinates \( x_\mu \) can be found with the help of the parametrization (6.9) and corresponding Dirac brackets of the reduced phase space variables.

Therefore, we can describe the reduced system in terms of the variables \( J^{(0)}, \varphi, p_\mu \) and \( x_\mu \) (or \( y_\mu \) instead of \( x_\mu \)). The total angular momentum has here the form given by eq. (4.6) (with \( x_\mu \) instead of \( \tilde{x}_\mu \)), whereas the spin constraint (2.9) is presented now in the form (4.14). Comparing the form of the brackets (5.12), (6.13) and (6.14) with the brackets (4.12), (4.9) and (4.14), respectively, one can conclude that we have arrived exactly at the system described in section 4 within a framework of the general analysis of the extended canonical formulation in the case when \( 2N = 2 \). That corresponding system comprises the variables \( \tilde{x}_\mu, p_\mu, J^{(0)} \) and \( \varphi \), whereas the variables \( \tilde{z}_a \) have been excluded already by us.
In order to quantize the system, we have to realize the operators corresponding to the coordinates \( x_\mu \), nonlocalizable in a sense of the Dirac brackets (6.13). It can be done as in the case of the minimal approach with the help of the localizable but noncovariant coordinates \( y_\mu \), whose noncovariance is conditioned by the noncovariant second term in eq. (6.10). We can choose a representation with the operators \( \hat{y}_\mu \) being diagonal, and work on the space of the \( 2\pi \)-periodic wave functions

\[
\Psi(y, \varphi) = \sum_n e^{i\tilde{n}\varphi} \psi^n(y).
\]  

(6.15)

Here we suppose that \( \tilde{n} = n \) and summation is realized over \( n \in \mathbb{Z} \). In accordance with the classical equalities (6.11), (6.12), we realize other operators as differential operators:

\[
\hat{p}_\mu = -i\partial/\partial y^\mu \quad \text{and} \quad \hat{J}^{(0)} = -i\partial/\partial \varphi + \vartheta,
\]

where \( \vartheta \) is an arbitrary constant, whose value can be restricted to the region \( \vartheta \in [0, 2\pi) \) without loss of generality. Under a choice of the internal scalar product in the form (5.35), the operator \( \hat{J}^{(0)} \) is a hermitian operator for any value of the constant \( \vartheta \). As in the case of the minimal approach, in this representation the operators corresponding to the initial coordinates \( x_\mu \), realized with the help of the quantum analog of eq. (6.10), turn out to be nonlocal operators. The quantum analogs of the mass shell and spin constraints turn into equations which single out the physical subspace of the system, and we find that the spin equation, \( (\hat{J}^{(0)} - s)\Psi = 0 \), has nontrivial solutions when the parameter \( \vartheta \) is chosen to be equal to the fractional part of the spin parameter \( s \). As a result, we find that the physical states are described by the wave functions of the form

\[
\Psi_{\text{phys}}(y, \varphi) \propto \int d^3p e^{ipy} \delta(p^2 + m^2) e^{is\varphi},
\]

where \([s]\) is the integer part of the spin parameter \( s \). On the other hand, as in the case of the quantum scheme for the model considered in the previous section, we can realize the operator \( \hat{J}^{(0)} \) in the standard form, without constant shift \( \vartheta \), \( \hat{J}^{(0)} = -i\partial/\partial \varphi \). In this case the spin equation will have nontrivial solutions if we choose the space of functions as that which is defined by the decomposition of the form (6.15), but with \( \tilde{n} \) to be equal to \( n + (s - [s]) \). As a result, the physical wave functions will contain a corresponding factor of the form \( e^{is\varphi} \) being multi-valued in general case. Thus, the described scheme of quantization reveals a similarity with that for the minimal approach and with the quantum scheme for the minimal extended model from the previous section for the case \( C \geq 0 \). The latter similarity forces us to investigate in more detail the relationship between the two extended models. Below we shall show that the reduced phase space description of the present model can be put in one-to-one correspondence with the canonical description of the previous model specified by the value of the constant \( C \geq 0 \).

To reveal such a relationship, let us note that here the internal phase space described by the variables \( \varphi \) and \( J^{(0)} \) having the Dirac brackets (6.12) is a cotangent bundle \( T^*S^1 \) of the one-dimensional sphere \( S^1 \). Topologically, this space is a cylinder, which, in turn, is equivalent to the one-sheet two-dimensional hyperboloid. Thus, let us construct the following vector on the reduced phase space of the system:

\[
J^{*\mu} = -\epsilon_{\mu\nu\lambda} n^\nu \pi^\lambda + \sqrt{C} n_\mu + \pi_\mu.
\]  

(6.16)

It has the fixed square, \( J^{*\mu}J^{*\mu} = C \), characterized by the constant \( C \), which is supposed to be a nonnegative one, \( C \geq 0 \). In correspondence with eq. (5.9), \( J^{*\mu}e^{(0)\mu} = J^{(0)} \) and
$J^{(i)\mu} = (\sqrt{C}\delta^{ij} + J^{(0)}\delta^{ij})\eta^{(j)}$, and we conclude that eq. (6.10) gives a map of the cylinder to the one-sheet two-dimensional hyperboloid (5.14) with $C \geq 0$.

Now, let us note that the spin constraint (2.9) has here exactly the same form in terms of the quantities $J^{\mu}_*$. Moreover, with the help of the parametrization of the reduced phase space (6.9) and corresponding Dirac brackets, one can check that $J^{\mu}_*$ satisfies the $so(2,1)$ algebra of the form (4.2), whereas here the Dirac brackets of $x^{\mu}$ with $J^{\nu}_*$ have the form

$$\{x^{\mu}, J^{\nu}_*\}^* = \frac{1}{p^2} (\eta^{\mu\nu}(pJ^*) - J^{*\nu}_*p^{\nu})$$

coinciding with the form of the corresponding Poisson brackets of the gauge-invariant coordinates $\tilde{x}_\mu$, given by eq. (1.7), with $J^{\mu}_*$ (see section 4). Then, due to the coincidence of the form of the Dirac brackets (6.13) for the coordinates $x^{\mu}$ with the form of the Poisson brackets (4.9) for the coordinates $\tilde{x}_\mu$, we arrive at the conclusion that the new coordinates

$$x^{\mu}_* = x^{\mu} - \frac{1}{p^2} \epsilon^{\mu\nu\lambda}p^{\nu}J^{*\lambda}$$

(6.17)

have the trivial Dirac brackets with $J^{\nu}_*$, $\{x^{\mu}_*, J^{\nu}_*\}^* = 0$, and

$$\{x^{\mu}_*, x^{\nu}_*\}^* = 0,$$

(6.18)

whereas the form of the total angular momentum vector of the system under consideration is presented in terms of $x^{\mu}_*$, $p^\mu$ and $J^{\mu}_*$ in the same form (2.8) as the total angular momentum vector of the minimal extended model from section 5. Therefore, both $x^{\mu}_*$ and $J^{\mu}_*$ are Lorentz vectors, and we conclude that we indeed have established a one-to-one correspondence of the reduced phase space description of the present model with the minimal extended model for the case when $C \geq 0$. At the same time, it is necessary to note that the reduced phase space of the system under consideration is topologically different from the phase space of the minimal extended model in the case when constant $C$ is chosen to be negative.

Let us comment on a possible interpretation of the model that has been considered. Such an interpretation will explain the title of the present section. For the purpose we note that the second term in the lagrangian (5.4) prescribes the spacelike vector $n^{\mu}$, $n^2 > 0$, to have a fixed (Minkowsky) length, $n^2 = 1$. One could omit this term from the lagrangian if one present $n^{\mu}$ in the first term as $n^{\mu} = q^{\mu}/\sqrt{q^2}$, where $q^{\mu}$ is supposed to be an arbitrary spacelike vector. The only difference which we have in this case is the absence of the second constraint from the pair of constraints (5.6), whereas other constraints from the set of constraints (5.3) and (5.6) are substituted here by $pq \approx 0$, $p\pi \approx 0$, and $\pi q \approx 0$, where now $\pi^{\mu}$ is the canonical momenta conjugate to $q^{\mu}$. In this case the last constraint means simply that the length of the spacelike vector $q^{\mu}$ is an unobservable quantity (and it can be fixed, e.g., by introducing the constraint $q^2 - 1 \approx 0$ as a gauge condition to the last constraint). This guarantees that the physical content of the model is not changed. Since $q^{\mu}$ is a translationally invariant vector, we can introduce the vectors $x^{\mu}_a$, $a = 1, 2$, via the definition

$$x^{\mu}_a = \frac{1}{2}(x^{\mu}_{1a} + x^{\mu}_{2a}), \quad q^{\mu}_a = x^{\mu}_{1a} - x^{\mu}_{2a}.$$

Then, in the rest frame of the system defined by the equality $p = 0$, the constraint $pq \approx 0$ is reduced to $x^{0}_1 = x^{0}_2$, and, therefore, here the condition that the vector $q^{\mu}$ is a spacelike
one means that $x_1 \neq x_2$. Thus, we can interpret the system as the system of two coupled relativistic particles described by the space-time coordinates $x_1^\mu$ and $x_2^\mu$, having a nontrivial configuration space defined by the condition $(x_1 - x_2)^2 > 0$, which prescribes the particles do not have coinciding space coordinates. In such an interpretation the system is related to the nonrelativistic system of two anyons discussed by Leinaas and Myrheim in their first original paper on the subject of fractional statistics \[3\]. Then the form of lagrangian of the model under consideration guarantees that such a system of coupled particles has only internal spin degree of freedom and the values of its corresponding relativistic mass and spin are fixed.

To conclude this section, we point out how the model considered here is related to the approach of Balachandran et al. \[13\] translated to the $(2+1)$-dimensional space-time. Proceeding from the original form of the lagrangian (6.1), one can supply the system of the two vectors $e_\mu$ and $n_\mu$ with a third one, $l_\mu = \epsilon_{\mu\nu\lambda}e_\nu n_\lambda$. Then, taking into account lagrangian constraints (6.2), we see that these vectors form the complete orthonormal set of vectors, $-e_\mu e_\nu + n_\mu n_\nu + l_\mu l_\nu = \eta_{\mu\nu}$. Now, from these vectors we can compose the matrix $\Lambda^{\mu\nu}$, $\Lambda^{\mu0} = e_\mu$, $\Lambda^{\mu1} = n_\mu$, $\Lambda^{\mu2} = l_\mu$, which, obviously, is the matrix of a Lorentz transformation: $\Lambda^{\mu\lambda}A_{\nu\lambda} = \eta^{\mu\nu}$. Such a matrix as a dynamical object is the main ingredient of the above-mentioned approach, where it is introduced for taking into account the spin degrees of freedom. Thus, the approach of the present model is related to the approach \[13\].

7 Summary and conclusions

We have shown that, within the framework of the minimal canonical approach, $(2+1)$-dimensional relativistic massive spinning particle can be described by the coordinates $x_\mu$ generally having nontrivial brackets given by eqs. (2.6) and (3.5). The form of the total angular momentum (3.4) of the particle with spin $s$ is correlated with the form of the brackets. Due to the presence of the term with specific dependence on the “gauge field” $A_\mu(p)$, the spin addition $J_\mu = -se_\mu^{(0)} - \epsilon_{\mu\nu\lambda}A_\nu p_\lambda$ in the total angular momentum vector (3.4) generally is not parallel to the energy-momentum vector of the particle $p_\mu$. Besides, in general case the coordinates $x_\mu$ have transformation properties different from those for a Lorentz vector. But there are two special cases here. One case is given by the trivial (up to the “gauge transformation” (3.13)) “gauge field” $A_\mu = 0$. Only in this case the corresponding coordinates $x_\mu^c$ form a Lorentz vector and the spin addition $J_\mu^c = -se_\mu^{(0)}$ turns out to be parallel to the vector $p_\mu$. But the price which we have to pay for the covariant properties of $x_\mu^c$ is the specific nontrivial form of their brackets (3.9), meaning that at the quantum level there is no representation where the corresponding operators $\hat{x}_\mu^c$ would be diagonal. It is necessary to stress that here either the first, orbital-like term in eq. (3.8), or the “spin vector” $J_\mu^c$ does not form the Lorentz algebra $so(2,1)$ with respect to the brackets, but only the total angular momentum vector $J_\mu$ does. Moreover, let us stress that within the framework of the minimal formulation we always have the equality $\{J_\mu, J_\nu\} = 0$. Another special case is characterized by the localizable coordinates $x_\mu^l$ having trivial brackets (3.10). This case is given by the $SO(2,1)$ “monopole gauge potential” defined by eq. (3.12), whose solution, in turn, is given by eqs. (3.14) and (3.13). Here the spin addition $J_\mu^l$ is not parallel to the energy-momentum vector $p_\mu$, and localizable coordinates $x_\mu^l$ have noncovariant transformation properties with respect to
the Lorentz boosts. We have demonstrated that the latter, noncovariant, special case turns out to be important for the construction of the quantum theory corresponding to the former case given by the covariant but nonlocalizable coordinates.

The properties of covariance and localizability of the coordinates can be simultaneously incorporated into the theory within the framework of the extended canonical formulation. Such a compatibility is achieved via introducing auxiliary internal phase space variables and by fixing the value of the particle spin in the weak sense, with the help of the spin constraint. Here the components of the spin addition $J_\mu$ form the $so(2,1)$ algebra with respect to the Poisson brackets, \( \{ J_\mu, J_\nu \} = -\epsilon_{\mu\nu\lambda} J^\lambda \), but, generally, this vector is not parallel to the energy-momentum vector $p_\mu$ in the weak sense, on the surface of the spin constraint. One can reformulate the extended approach in terms of variables which are gauge-invariant with respect to the gauge transformations generated by the spin constraint, presented in the dimensionless form (4.4). Such a reformulated extended canonical approach turns out to be equivalent to the minimal approach given in terms of the covariant nonlocalizable coordinates. On the other hand, the general case of the minimal formulation can be obtained from the extended formulation via reducing the latter one to the surface of the spin constraint. We have shown that there is only one special case, when the gauge-invariant extension $\tilde{x}_\mu$ of the coordinates of the particle, having the brackets corresponding to the brackets of the covariant coordinates $x^c_\mu$, coincides with the initial coordinates $x_\mu$. This case is characterized by the spin vector $J_\mu$ being parallel to the energy-momentum vector $p_\mu$ on the surface of the spin constraint. Here the gauge orbits generated by the spin constraint (4.4), which generally are one-dimensional spheres, shrink into just one point.

Such a special case is contained as a particular one in the concrete minimal extended model has been considered in section 5. We have shown that this case turns out also to be a special one from the point of view of the lagrangian approach. Only in this case there is the Lagrange constraint in the system prescribing the velocity of the particle to be parallel to the spin vector $J_\mu$, and, therefore, only in this case the energy-momentum vector of the particle, $p_\mu$, is parallel to its velocity $\dot{x}_\mu$. In the general case the coordinates of the particle are subject to the more complicated motion being a superposition of the circular and rectilinear motion along the momentum vector $p$. Such a motion represents by itself the classical analog of the quantum Zitterbewegung \([32]\), and, so, generally the second derivative of the coordinates of the particle (acceleration in the laboratory temporal gauge) is not equal to zero for a free relativistic spinning particle in 2+1 dimensions. At the same time, the above-mentioned gauge-invariant extension $\tilde{x}_\mu$ of the coordinates $x_\mu$ does not reveal any circular motion and its evolution law has the form (5.24) exactly coinciding with that for the evolution of the free relativistic spinless particle. Thus, the coordinates $\tilde{x}_\mu$ turn out to be analogous to the Foldy-Wouthuysen coordinates for the quantum massive Dirac particle \([33]\).

Let us stress once more that in the special case of the model with the spin parameter correlated with the value of the central element of the $so(2,1)$ algebra (1.2), \( s^2 = -J^2 \), the spin addition $J_\mu$ is parallel to $p_\mu$ on the surface of the spin constraint, the velocity of the particle is parallel to energy-momentum vector $p_\mu$, and so, as we have seen, the coordinates of the particle evolve in such a way that $\ddot{x}_\mu = 0$. At the same time, we have here the Poincaré group as the exact symmetry group of the system. Therefore, the general statement of the paper \([17]\), declaring the incompatibility of the free nature of the anyon (characterized there by the relation $\ddot{x}_\mu = 0$) and translation invariance of the
theory, turns out to be incorrect one.

We have pointed out that after reduction of the system to the surface of the spin constraint in the above-mentioned special case the spin addition \( J_\mu \) turns out to be parallel to \( p_\mu \) in the strong sense, but its components, in correspondence with the general results obtained for the minimal approach, form the trivial algebra with respect to the Dirac brackets. The quantization of the model in this special case leads to the representations of the discrete type series \( D_\alpha^\pm \) of the group \( SL(2, R) \), characterized by the parameter \( \alpha \) being correlated with the spin of the particle, \( s^2 = \alpha^2 \). Let us note here that different variants of linear differential equations for fractional spin fields, have been constructed up to now within a group-theoretical approach to anyons, are known only for the case of using such representations of the discrete type series \( [8, 20, 23, 25] \). Moreover, it is necessary to note that there is a vector set of linear differential equations, proposed in ref. \( [5] \), which itself fixes the choice of only these representations for the description of fractional spin fields. The quantum analogs of the mass shell (2.7) and spin constraints (2.9) appear there as a consequence of the corresponding basic linear differential equations.

We have shown that in the case when the parameter \( C \), defining the topology of the internal phase space of the minimal extended model, is nonnegative, the model turns out, in fact, to be equivalent to the model of refs. \( [20, 21] \). In turn, the latter model can be interpreted as the system of two coupled relativistic particles with nontrivial topology of configuration space, which prohibits the particles to have coinciding space coordinates. The form of the lagrangian of the model guarantees removing all the degrees of freedom corresponding to the relative motion of the particles being different from the spin degree of freedom, and, moreover, it prescribes the system to have fixed relativistic total mass and spin. We have also demonstrated that the model from section 6 is related to the approach of ref. \( [13] \), transferred to the case of 2+1 dimensions.

Finally, let us list the main results removing the misleading notions on the general properties of the (2+1)-dimensional anyons which have been cited in section 1.

A free relativistic particle with fractional (arbitrary) spin can be described in 2+1 dimensions in a Poincaré-invariant way. Under such a description, the classical analog of the relativistic quantum Zitterbewegung generally takes place, and, so, the condition \( \ddot{x}_\mu = 0 \) is not a necessary condition for a particle to be free.

The spin addition \( J_\mu \) of the total angular momentum vector cannot simultaneously satisfy the properties of parallelness to the energy-momentum vector \( p_\mu \) and \( so(2, 1) \) algebra \( \{J_\mu, J_\nu\} = -\epsilon_{\mu\nu\lambda} J^\lambda \). \( J_\mu \) can be (but is not necessarily) parallel to \( p_\mu \) in the strong sense within the framework of the minimal formulation, where its components satisfy the trivial algebra \( \{J_\mu, J_\nu\} = 0 \). On the other hand, spin addition satisfies \( so(2, 1) \) algebra within the framework of the extended formulation, where \( J_\mu \) can be (but is not necessarily) parallel to \( p_\mu \) only in the weak sense, on the surface of the spin constraint.

The general formulation introduced in this paper can give a new perspective on the remaining open questions in the study of (2+1)-dimensional anyons. The quantum field theory of fractional spin fields and the construction of consistent theory of electromagnetic interaction of anyons are two basic problems not yet solved, where the model-independent approach presented here can be important.
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