Proposal for a Correction to the Temporal Correlation Coefficient Calculation for Temporal Networks

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Abstract

Measuring the topological overlap of two graphs becomes important when assessing the changes between temporally adjacent graphs in a time-evolving network. Current methods depend on the fraction of nodes that have persisting edges. This breaks down when there are nodes with no edges, persisting or otherwise. The following outlines a proposed correction to ensure that correlation metrics have the expected behavior.

1 Previous Proposal

[1] defines the topological overlap in the neighborhood of $i$ between two consecutive time steps $[t_m, t_{m+1}]$ as:

$$C_i(t_m, t_{m+1}) = \frac{\sum_j a_{ij}(t_m)a_{ij}(t_{m+1})}{\sqrt{[\sum_j a_{ij}(t_m)][\sum_j a_{ij}(t_{m+1})]}}$$

(1)

where $a_{ij}$ represents an entry in the unweighted adjacency matrix of the graph, so summing over $a_{ij}$ gives the interactions between $i$ and every other node.

Then the average topological overlap of the neighborhood of node $i$ as the average of $C_i(t_m, t_{m+1})$ over all possible subsequent temporal snapshots:

$$C_i = \frac{1}{M-1} \sum_{m=1}^{M-1} C_i(t_m, t_{m+1})$$

(2)

The average topological overlap of all $C_i$ can be said to represent the temporal clustering of all of the edges in the network, which we will call the temporal clustering coefficient (TCC).

$$C = \frac{1}{N} \sum_{i=1}^{N} C_i$$

(3)

Identically, the order of summations in this formulation can be reversed (simply to make the following more obvious), to give the average topological overlap of the graph at $t_m$ with the subsequent graph at $t_{m+1}$:

$$C_m = \frac{1}{N} \sum_{i=1}^{N} C_i(t_m, t_{m+1})$$

(4)
and the same $C$ is then given by:

$$C = \frac{1}{M-1} \sum_{m=1}^{M-1} C_m \quad (5)$$

According to [1] this formulation gives $C_m = 1$ if and only if the graphs at $t_m$ and $t_{m+1}$ have exactly the same configuration of edges, and $C_m = 0$ if the graph at $t_m$ and $t_{m+1}$ do not share any edges. This claim is only true if all of the $N$ nodes considered in the calculation have at least one edge.

Equation (1) results in an undefined $\frac{0}{0}$ in the case where the node $i$ has no edges in one or both time steps $t_m$ or $t_{m+1}$. If that undefined value is simply set to zero (choose zero because the node $i_m$ shares no edges with the node $i_{m+1}$), $C$ between identical unconnected graphs is equal to the fraction of connected nodes in the graph. The formula by [1] provides no method for dealing with networks where $N(t_m)$ (the number of nodes participating in the network) changes in time.

This presents a significant problem, because for small time snapshots, many temporal graphs are often unconnected (contain unconnected nodes) [1] and $N(t_m)$ changes constantly with time. For unconnected graphs, $C_m$ describing the relationship between two identical graphs gives $C_m \neq 1$, and can significantly underestimate the correlation between two graphs.

## 2 Proposed Correction

To ensure that $C_m$ has the expected behavior when the graph is unconnected, a constant $n$ is replaced by the maximum number of connected nodes in the network for the two time steps being compared. Using $\max[N(t_m), N(t_{m+1})]$ rather than simply $N(t_m)$ or $N(t_{m+1})$ ensures that $C < 1$ for all non-identical graphs, using $\min[N(t_m), N(t_{m+1})]$ would give $C = 1$ for disconnected graphs where the only change is an edge appearing or disappearing. Be aware that this method will still give a $\frac{0}{0}$ for correlation between two graphs both with zero edges.

$$C_m = \frac{1}{\max[N(t_m), N(t_{m+1})]} \sum_{i=1}^{N} C_i(t_m, t_{m+1}) \quad (6)$$

$$C = \frac{1}{M-1} \sum_{m=1}^{M-1} \left( \frac{1}{\max[N(t_m), N(t_{m+1})]} \sum_{i=1}^{N} C_i(t_m, t_{m+1}) \right) \quad (7)$$
3 Examples

Below are some simple examples to show the difference between the two methods.

3.1 Connected graph

![Connected Graph](image)

**Figure 1:** A connected graph. Here Methods 1 and 2 give the same result.

*topological overlap* in the neighborhood of $i$ between two consecutive time steps:

\[
(\mathbb{I}): C_i(t_m, t_{m+1}) : C_{i=1}(t_m, t_{m+1}) = \frac{1}{\sqrt{2}}, \ C_{i=2}(t_m, t_{m+1}) = \frac{1}{\sqrt{2}}, \ C_{i=3}(t_m, t_{m+1}) = 0
\]

**Method 1:**

\[
C = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{M-1} \sum_{m=1}^{M-1} C_i(t_m, t_{m+1}) \right) = \frac{1}{3} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 \right) = \frac{2\sqrt{2}}{3}
\]

**Method 2:**

\[
C = \frac{1}{M-1} \sum_{m=1}^{M-1} \left( \max\{N(t_m), N(t_{m+1})\} \sum_{i=1}^{N} C_i(t_m, t_{m+1}) \right) = \frac{1}{3} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 \right) = \frac{2\sqrt{2}}{3}
\]

3.2 Unconnected graph

![Unconnected Graph](image)

**Figure 2:** An unconnected graph. As the graphs are identical the expected value is one, but method 1 for calculating $C$ gives $\frac{2}{3}$, or the fraction of nodes that are participating in the network at time $t_m$.

*topological overlap* in the neighborhood of $i$ between two consecutive time steps:

\[
(\mathbb{I}): C_i(t_m, t_{m+1}) : C_{i=1}(t_m, t_{m+1}) = 1, \ C_{i=2}(t_m, t_{m+1}) = 1, \ C_{i=3}(t_m, t_{m+1}) = 0 = 0
\]

**Method 1:**

\[
C = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{1}{M-1} \sum_{m=1}^{M-1} C_i(t_m, t_{m+1}) \right) = \frac{1}{3} \left( \frac{1}{2} (1 + 1 + 0) \right) = \frac{2}{3}
\]

**Method 2:**

\[
C = \frac{1}{M-1} \sum_{m=1}^{M-1} \left( \max\{N(t_m), N(t_{m+1})\} \sum_{i=1}^{N} C_i(t_m, t_{m+1}) \right) = \frac{1}{3} \left( \frac{1}{2} (1 + 1 + 0) \right) = 1
\]
3.3 Time series with unconnected graphs

![Diagram of a network containing four nodes](image)

Figure 3: A network containing four nodes \(N = 4\) which becomes disconnected during the observed time window \(N(t_m) = 4\), however, \(N(t_{m+1}) = 3\) and \(N(t_{m+2}) = 2\). Note three time steps, so \(M = 3\).

| \(i\) | \(C_i(t_m, t_{m+1})\) | \(C_i(t_{m+1}, t_{m+2})\) |
|------|---------------------|---------------------|
| 1    | \(\frac{1}{\sqrt{6}}\) | \(\frac{\sqrt{2}}{2}\) |
| 2    | 1                   | 1                   |
| 3    | 1                   | 0                   |
| 4    | 0 \(\approx 0\)     | 0 \(\approx 0\)     |

Table 1: topological overlap in the neighborhood of \(i\) between two consecutive time steps

Method 1:

\[
C_m = \frac{1}{N} \sum_{i=1}^{N} C_i(t_m, t_{m+1})
\]

\[
C = \frac{1}{M-1} \sum_{m=1}^{M-1} C_m
\]

\(C_m = \frac{1}{4} \sum_{i=1}^{4} C_i(t_m, t_{m+1}) \approx 0.70\)

\(C_{m+1} = \frac{1}{4} \sum_{i=1}^{4} C_i(t_{m+1}, t_{m+2}) \approx 0.45\)

\(\approx 0.57\)

Method 2:

\[
C_m = \frac{1}{\max\{N(t_m), N(t_{m+1})\}} \sum_{i=1}^{N} C_i(t_m, t_{m+1})
\]

\[
C = \frac{1}{M-1} \sum_{m=1}^{M-1} C_m
\]

\(C_m = \frac{1}{4} \sum_{i=1}^{4} C_i(t_m, t_{m+1}) \approx 0.70\)

\(C_{m+1} = \frac{1}{3} \sum_{i=1}^{4} C_i(t_{m+1}, t_{m+2}) \approx 0.57\)

\(\approx 0.64\)
3.4 Time series with identical unconnected graphs

Figure 4: A graph which becomes unconnected during the observed time window. Note that even though the second two time steps are identical, the $C$ between $t_m$ and $t_{m+1} \neq 1$ because the graph is disconnected. Because of this, $C$ for the time series of graphs is underestimated when performing the calculation using Method 1.

|         | $C_i(t_m, t_{m+1})$ | $C_i(t_{m+1}, t_{m+2})$ |
|---------|---------------------|------------------------|
| $C_{i=1}(t_m, t_{m+1})$ | $\frac{2}{\sqrt{6}}$ | $1$ |
| $C_{i=2}(t_m, t_{m+1})$ | $1$ | $1$ |
| $C_{i=3}(t_m, t_{m+1})$ | $1$ | $1$ |
| $C_{i=4}(t_m, t_{m+1})$ | $0$ | $0$ |

Table 2: topological overlap in the neighborhood of $i$ between two consecutive time steps

Method 1:

[1]: $C_m = \frac{1}{M-1} \sum_{m=1}^{M-1} C_m $

| $C_m$ | $C_{m+1}$ |
|-------|-----------|
| $\frac{1}{4} \sum_{i=1}^{4} C_i(t_m, t_{m+1}) \approx .70$ | $\frac{1}{4} \sum_{i=1}^{3} C_i(t_{m+1}, t_{m+2}) \approx 0.73$ |

Method 2:

[2]: $C_m = \frac{1}{\max[N(t_m), N(t_{m+1})]} \sum_{i=1}^{N} C_i(t_m, t_{m+1})$  

| $C_m$ | $C_{m+1}$ |
|-------|-----------|
| $\frac{1}{4} \sum_{i=1}^{4} C_i(t_m, t_{m+1}) \approx .70$ | $\frac{1}{3} \sum_{i=1}^{3} C_i(t_{m+1}, t_{m+2}) \approx 0.85$ |

If this time series of graphs were extended, with $t_{m+2}, t_{m+3}...$ identical to $t_{m+1}$, the time series would logically demonstrate a very high (asymptotically 1) temporal correlation, as the graphs hardly ever change. Calculated using the Method 2 formulation, $C \rightarrow 1$, but
using the Method 1 formulation, $C$ is asymptotically equal to the average fraction of nodes participating in the network.

![Convergence of TCC for time series with identical unconnected graphs](image)

Figure 5: Plot showing convergence behavior of TCC for Methods 1 and 2 for the time series described above, adding identical unconnected graphs to the series in figure (4).

4 Conclusion

Because Method 1 for calculating the temporal correlation coefficient relies on a fixed number of nodes in the network $N$, some modifications (presented as Method 2) need to be made to the formulation to avoid systematically underestimating the correlation between two unconnected graphs. In a time series of graphs which have, on average, unconnected nodes, the temporal correlation coefficient $C$ calculated using Method 1 will underestimate the correlation between two graphs by the fraction $\frac{\bar{N}_0}{N}$ where $\bar{N}_0$ is the average number of unconnected nodes over time and $N$ is the total number of nodes considered.

References

[1] V. Nicosia et al. ”Graph Metrics for Temporal Networks”. [arXiv:1306.0493] [physics.soc-ph]