The Noether–Lefschetz theorem

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Outline

1. Introduction
2. Quartic surfaces in $\mathbb{P}^3$
3. Proof
The Noether–Lefschetz theorem

Theorem (Noether–Lefschetz Theorem)

For a very general surface $S_d \subset \mathbb{P}_C^3$ of degree $d \geq 4$, the restriction map $\text{Pic}(\mathbb{P}_C^3) \to \text{Pic}(S_d)$ is an isomorphism.
M. Noether (1882): The only curves on a “general” surface $F_\mu \subset \mathbb{P}^3_C$ of degree $\mu \geq 4$ are complete intersections of $F_\mu$ with another surface
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The Noether–Lefschetz theorem

**Theorem (Noether–Lefschetz Theorem)**

*For a very general surface \( S_d \subset \mathbb{P}^3_\mathbb{C} \) of degree \( d \geq 4 \), the restriction map \( \text{Pic}(\mathbb{P}^3_\mathbb{C}) \to \text{Pic}(S_d) \) is an isomorphism.*

Very general means away from a countable union

- 1882: stated by M. Noether
- 1920s: proved by Lefschetz using topological methods for complex surfaces

**Generalizations?**

- Replace \( \mathbb{P}^3_\mathbb{C} \) by \( X \)
- Replace \( \mathbb{C} \) by \( k \)

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S. Lefschetz. On certain numerical invariants of algebraic varieties with application to abelian varieties (1921), p. 359.
The Noether–Lefschetz theorem

$X \subset \mathbb{P}^N_k$ smooth subvariety, $H \subset \mathbb{P}^N_k$ very general hypersurface

$$\text{Pic}(X) \longrightarrow \text{Pic}(X \cap H)$$

- $X$ smooth 3-fold + char $k = 0$:
  Moishezon 1960s, Carlson–Green–Griffiths–Harris 1983 (Hodge theory), Griffiths–Harris 1985 (degeneration + monodromy), Joshi 1995...

- $X \subset \mathbb{P}^N_k$ complete intersection + char $k \geq 0$:
  Deligne 1960s ($l$-adic cohomology)
The Noether–Lefschetz theorem

\( X \subset \mathbb{P}^N_k \) normal subvariety, \( H \subset \mathbb{P}^N_k \) very general hypersurface

\[ \text{Cl}(X) \longrightarrow \text{Cl}(X \cap H) \]

- \( X \) smooth 3-fold + char \( k = 0 \):
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- \( X \subset \mathbb{P}^N_k \) complete intersection + char \( k \geq 0 \):
  Deligne 1960s (\( l \)-adic cohomology)

- \( X \) normal 3-fold + char \( k = 0 \): \( \text{Cl}(X) \rightarrow \text{Cl}(X \cap H) \)
  Ravindra–Srinivas 2008 (formal completion)

- Today: \( X \) normal 3-fold + char \( k = p \geq 0 \)
Theorem

Let \( X \subset \mathbb{P}^N_k \) be a normal 3-fold and \( H \subset \mathbb{P}^N_k \) a very general hypersurface of degree 4 or \( \geq 6 \). Then \( \text{Cl}(X) \to \text{Cl}(X \cap H) \) is an isomorphism up to torsion.

- No cohomology, Hodge theory, or monodromy
- Works for algebraically closed \( k \) of infinite transcendence degree and in any characteristic
  - e.g. \( \mathbb{C}, \mathbb{Q}(t_1, t_2, t_3, \ldots), \mathbb{F}_p(t_1, t_2, t_3, \ldots) \)
Quartic surfaces in $\mathbb{P}^3$

**Set-up**

- $C \subset \mathbb{P}^3_K$ degree 4 elliptic curve of rank $\geq 18$
- $q_1, q_2, p_1, \ldots, p_{15} \in C(K)$ independent in $\text{Pic}(C)/\langle H \rangle$ where $H = \mathcal{O}_{\mathbb{P}^3}(1)|_C$

- $\exists!$ quadric $Q_j = (f_j = 0)$ such that
  $$\text{Im}(\text{Pic}(Q_j) \to \text{Pic}(C)) = \langle 2q_j, H - 2q_j \rangle$$

- $\exists!$ $p_{16}$ such that
  $$p_1 + \cdots + p_{15} + p_{16} = C \cap T$$
  where $T = (g = 0) \subset \mathbb{P}^3$ is a quartic surface

**Claim**

$$\rho(f_1f_2 + \lambda g = 0) = 1$$ for any $\lambda \notin \overline{K}$
Claim: \( \rho(T_\lambda := (f_1 f_2 + \lambda g = 0)) = 1 \) for any \( \lambda \notin \overline{K} \)

\( Q_j = (f_j = 0) \) degree 2, \( T = (g = 0) \) degree 4

Proof of claim:
- \( \tilde{X} \to \mathbb{P}^1 \) pencil spanned by \( Q_1 + Q_2 \) and \( T \)
- \( D_\lambda \) curve on \( T_\lambda \)
- Can find \( \Gamma \to \mathbb{P}^1 \) and divisor \( \mathcal{D} \) on \( \tilde{X} \times \mathbb{P}^1 \Gamma \) that restricts to \( D_\lambda \)
- \( \text{Pic}(Q_1) \times_{\text{Pic}(C)} \text{Pic}(Q_2) \cong \mathbb{Z} \), so want to restrict \( \mathcal{D} \) to \( Q_1 + Q_2 \)
- But \( \mathcal{D} \) need not be \( \mathbb{Q} \)-Cartier along \( C = Q_1 \cap Q_2 \)
- \( (\mathcal{D}|_{Q_1})|_C - (\mathcal{D}|_{Q_2})|_C \) is supported over \( \{p_1, \ldots, p_{16}\} = C \cap T \)

\[
(\mathcal{D}|_{Q_1})|_C - (\mathcal{D}|_{Q_2})|_C = a_1 p_1 + \cdots + a_{16} p_{16} \in \langle H, 2q_1, 2q_2 \rangle
\]

- \( q_1, q_2, p_1, \ldots, p_{15} \in C(K) \) were chosen independent in \( \text{Pic}(C)/\langle H \rangle \)
- \( a_i \)'s are all the same \( a \)
Claim: \( \rho(T_\lambda := (f_1 f_2 + \lambda g = 0)) = 1 \) for any \( \lambda \notin \overline{K} \)

Proof of claim (continued):

1. \((D|_{Q_1})|_C - (D|_{Q_2})|_C = a(p_1 + \cdots + p_{16})\)
2. Local class group over \( p_i \) is \( \mathbb{Z} = \langle Q_{1p_i} \rangle = \langle -Q_{2p_i} \rangle \)
3. \( D - aQ_1 \) is \( \mathbb{Q} \)-Cartier along \( C \)
4. \( \text{Pic}(Q_1) \times \text{Pic}(C) \text{Pic}(Q_2) \cong \mathbb{Z} \) is generated by restriction of \( \mathcal{O}_{\mathbb{P}^3}(1) \)
5. \( (D - aQ_1)|_{Q_1+Q_2} \) comes from a divisor \( D \) on \( \mathbb{P}^3 \)
6. \( D|_{T_\lambda} \sim D_\lambda \)
Set-up

For simplicity: Assume $X$ smooth 3-fold

Set-up

$L$ very ample line bundle such that for a general net in $|L|$

\[ X \longrightarrow \mathbb{P}^2 \]

- all but finitely many fibers are irreducible curves, and
- generic fiber $C_\eta$ has genus $> \dim \text{Alb}(X)$.

Side lemma: If $\mathcal{M}$ is very ample, then $L = \mathcal{M}^{\geq 2}$ has these properties

Want to show

\[ \text{Pic}(X) \xrightarrow{\cong} \text{Pic}(T) \text{ (up to torsion)} \text{ for very general divisor } T \]

Recover statement for $\text{Pic}(X) \to \text{Pic}(X \cap H)$ by embedding $X$
Outline of proof

5 main steps

Want to show

\[ \text{Pic}(X) \xrightarrow{\sim} \text{Pic}(T) \text{ up to torsion} \]

**Injectivity**: Fiber \( X \) by curves to study \( \text{Pic}(X) \)

1. \( \text{Pic}(X) \rightarrow \text{Jac}(\text{generic fiber}) \)
2. \( \text{Jac}(\text{generic fiber}) \rightarrow \text{Jac}(\text{very general fiber}) \)

**Surjectivity**: Degeneration argument

3. Surjectivity mod torsion for \( X \) to reducible member \( S_0 + S_1 \in |L^2| \)
4. Specialize from very general \( T \in |L^2| \) to \( S_0 + S_1 \)
5. Go back from \( S_0 + S_1 \) to \( T \)

(\( +\epsilon \)) Degeneration argument for odd degrees
Step 1: \( \text{Pic}(X) \hookrightarrow \text{Jac}(C_\eta)(K) \)

- \( X \) smooth 3-fold, \( \phi: X \rightarrow \mathbb{P}^2 \) general net

\[
\begin{array}{c}
X' \xrightarrow{\text{bir}} X \\
\phi' \downarrow \downarrow \\
\phi' & \phi \\
\downarrow & \downarrow \\
\mathbb{P}^2 & \mathbb{P}^2
\end{array}
\]

- \( C_\eta \) = generic fiber of \( \phi' \) is a curve over \( K = k(\mathbb{P}^2) \)

Fix a base point of the net, \( E \subset X' \) corresponding exceptional divisor. Define

\[
\begin{array}{c}
\text{Pic}(X') \xrightarrow{\text{Pic}(X)} \text{Jac}(C_\eta)(K) \\
D' \xrightarrow{} (D' - \deg(D'|_{C_\eta})E)|_{C_\eta}
\end{array}
\]

- Restriction to \( \text{Pic}(X) \subset \text{Pic}(X') \) is injective
- Can show using assumptions on \(|\mathcal{L}|\)
Step 2: $\text{Jac}(C_\eta)(K) \hookrightarrow \text{Jac}(C_s)(k)$ for very general $s \in \mathbb{P}^2$

- $\text{Jac}(C_\eta)$ is an abelian variety over $K = k(\mathbb{P}^2)$
- Spreads out to family of abelian varieties $\mathcal{J} \to U \subset \mathbb{P}^2$ (relative Jacobian)

Fact: $\mathcal{J} \xrightarrow{\text{isogeny}} (\text{constant family } A \times U) \times (Q \text{ with no constant subfamilies})$
- $A \times U \to U$ has only constant sections (no nontrivial map $\mathbb{P}^2 \to A$)
- $Q \to U$ has only countably many sections (Lang–Néron theorem)
- Specialization of sections for $Q \to U$ to very general fiber is injective

Steps 1+2 $\implies$ injectivity
If $C_s$ is very general, then $\text{Pic}(X) \hookrightarrow \text{Pic}(Y)$ for $Y \supset C_s$
Constant part of $\mathcal{J} \to U$

- $L \subset \mathbb{P}^2$ line corresponds to $S \in |\mathcal{L}|$

Turns out that $A = \text{Pic}^0(X)$

- $\text{Pic}^0(X) \subset \text{Pic}(X)$ subgroup of algebraically trivial divisors
- $\text{Pic}(X)/\text{Pic}^0(X)$ is a finitely-generated group

By Chow's theory of the $K/k$-trace (whose definition is recalled below), one knows that $(A, t_A)$ is a $k(u)/k$-trace of $\mathcal{J} = J$ ([2], Ch. 8, Th. 12). Consequently, to prove the theorem of the base, it will suffice to prove the following result.

**Theorem 1.** Let $K$ be a finitely generated regular extension of a field $k$. Let $A$ be an abelian variety defined over $K$, and let $(B, \tau)$ be its $K/k$-trace. Then $A_K/\tau B_K$ is of finite type.

S. Lang and A. Néron. Rational points of abelian varieties over function fields (1959), p. 97

- $\text{Pic}^0(X) \overset{\sim}{\to} \text{Pic}^0(S)$ for general $S \in |\mathcal{L}|$
Step 3: $\text{Jac}(C_\eta)(K) \cong \{\text{sections over very general line pair in } U\}$

First step toward surjectivity

**Theorem (Graber–Starr 2013)**

If $\mathcal{J} \to B$ is a family of abelian varieties, then sections extend from a very general pair of incident lines in $B$.

- Split $\mathcal{J}$ into (constant family) $\times$ (family with no constant subfamilies)
- Simpler proof when the base is $U \subset \mathbb{P}^2$
  - Over very general line $L$ have \{sections of $\mathcal{J}|_L \to L$\} $\hookrightarrow \mathcal{J}_s(k)$

$\implies$ Surjectivity for reducible $S_0 + S_1 \in |\mathcal{L}^2|$: for $S_0, S_1 \in |\mathcal{L}|$ very general

$$\text{Pic}(X) \twoheadrightarrow \text{Pic}(S_0) \times_{\text{Pic}(S_0 \cap S_1)} \text{Pic}(S_1)$$
Step 4: From very general \( T \in |L^2| \) to reducible \( S_0 + S_1 \)

Let \( T \in |L^2| \) be very general, \( D^T \in \text{Pic}(T) \)

- \( \tilde{X} = \text{Bl}_{B_s|\Lambda|}(X) \to |\Lambda| = \mathbb{P}^1 \) pencil containing \( S_0 + S_1 \) and \( T \)
- Chow variety argument: after a base change \( \Gamma \to |\Lambda| \) can find a divisor \( D \) on \( \tilde{X} \times |\Lambda| \Gamma \) that restricts to \( D^T \) on \( T \)

- Look at restriction of \( D \) to \( S_0 + S_1 \), want to apply Step 3
- Problem: Singularities of \( \tilde{X} \times |\Lambda| \Gamma \) (blew up singular curve \( B_s|\Lambda| \))
- \( D \) may not be \( \mathbb{Q} \)-Cartier over \( S_0 \cap S_1 \cap T = \{ p_1, \ldots, p_m \} \)
- Singularities of \( \tilde{X} \times |\Lambda| \Gamma \) over \( \{ p_i \} \) locally look like

\[
\hat{O}_{\tilde{X} \times |\Lambda| \Gamma} \cong k[[y_1, y_2, y_3, t]]/(y_1 y_2 - t^r y_3)
\]
Step 4 cont.: From very general $T \in |L^2|$ to reducible $S_0 + S_1$

- If $\mathcal{D}$ not $\mathbb{Q}$-Cartier along $S_0 \cap S_1$ then $(\mathcal{D}|_{S_0})|_{S_0 \cap S_1}$ and $(\mathcal{D}|_{S_1})|_{S_0 \cap S_1}$ may differ
- $(\mathcal{D}|_{S_0})|_{S_0 \cap S_1} - (\mathcal{D}|_{S_1})|_{S_0 \cap S_1}$ is supported on $\{p_1, \ldots, p_m\}$
- Conjecture of Kollár (proved by Voisin when $k$ uncountable):
  If $T$ is very general, then $p_1, \ldots, p_m$ are linearly independent in $\text{Pic}(S_0 \cap S_1)$.
  
  \[
  (\mathcal{D}|_{S_0})|_{S_0 \cap S_1} - (\mathcal{D}|_{S_1})|_{S_0 \cap S_1} = a(p_1 + \cdots + p_m)
  \]
- Replace $\mathcal{D}$ by $\mathcal{D} - aS_0$: this is $\mathbb{Q}$-Cartier along $S_0 \cap S_1$

\[
\begin{array}{cccc}
\tilde{X} \times |\Lambda| \Gamma & \rightarrow & \tilde{X} & \rightarrow \ X \\
\downarrow & & \downarrow & \nearrow \\
\Gamma & \rightarrow & \text{Bs}|\Lambda| & \rightarrow \ X
\end{array}
\]

- Step 3 \implies can find $D^X \in \text{Pic}(X)$ such that $D^X|_{S_0 + S_1} \sim (\mathcal{D} - aS_0)|_{S_0 + S_1}$
Step 5: Surjectivity: from reducible to back to v general

Have $D^X \in \text{Pic}(X)$ such that $D^X|_{S_0+S_1} \sim (D - aS_0)|_{S_0+S_1}$

On $\tilde{X} \times |\Lambda| \Gamma$: Want $D - aS_0 \sim \psi^* D^X$, so that $D^X|_T \sim D^T$

$\psi^* D^X - (D - aS_0)$ is a divisor on $\tilde{X} \times |\Lambda| \Gamma$ that is trivial on $S_0 + S_1$

- For smooth families, specialization of Néron–Severi groups is injective up to torsion. Argument only needs divisors to be $\mathbb{Q}$-Cartier
- A multiple of $D^X|_T - D^T$ is in $\text{Pic}^0(T)$
- $\text{Pic}^0(X) \cong \text{Pic}^0(T)$
- $\text{Pic}(X) \rightarrow \text{Pic}(T)$ is surjective up to torsion
Other degrees?

- Step 5 works for even multiples of $\mathcal{L}$
  - Specialized from $T \in |\mathcal{L}^2|$ to (member of $|\mathcal{L}|$)+(member of $|\mathcal{L}|$)
  - Recall set-up: $\mathcal{L} = M^m$ for $m \geq 2$
- Can specialize to (member of $|M^4|$)+(member of $|M^{\geq 2}|$)
- Get result for very general members of $|M^{\geq 6}|$

Conclusion

$\text{Pic}(X) \xrightarrow{\text{/torsion}} \text{Pic}(T)$ for very general $T \in |M^4|$ or $|M^{\geq 6}|$
Generalizations

Singular varieties?

- Steps 1–4 work if $X$ is normal, replacing
  - $\text{Pic}$ with $\text{Cl}$
  - $\text{Pic}^0$ with $\text{Cl}^0$ = subgroup of algebraically trivial Weil divisors
- Step 5 in char $0$: OK after base changing to a resolution of singularities of $X$
- Step 5 in char $p > 0$: OK after base changing to a purely inseparable alteration (purely inseparable morphism + partial resolution)

Torsion?

- Expect no prime-to-$p$-torsion
- Examples with $p$-torsion?? (Please let me know!)
If $\dim X = n \geq 4$

- Steps 1–2 show $\text{Cl}(X) \to \text{Pic}(C)$ is injective for a very general complete intersection curve in $|\mathcal{L}|$
- Steps 3–5 show $\text{Cl}(X) \to \text{Cl}(T)$ is surjective up to torsion for a very general complete intersection surface in $|\mathcal{L}^2| \cap |\mathcal{L}| \cap \cdots \cap |\mathcal{L}|$

By factoring $\text{Cl}(X) \to \text{Cl}(T)$ through a divisor in $|\mathcal{L}| = |\mathcal{M}_{\geq 2}|$, get

**Theorem**

Let $X \subset \mathbb{P}_k^N$ be a normal variety of dimension $\geq 4$ and $H \subset \mathbb{P}_k^N$ a very general hypersurface of degree $\geq 2$. Then $\text{Cl}(X) \to \text{Cl}(X \cap H)$ is an isomorphism up to torsion.
Thank you!