VIOLATION OF THE ENERGY CONSERVATION LAW IN
LORENTZ-DIRAC EQUATIONS FOR MORE THAN ONE CHARGE

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An exact solution of Lorentz-Dirac equations where the energy conservation law is violated, is described herein for the case of two charges.

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Lorentz-Dirac equations are currently accepted as the equations of motion for charged particles in classical electrodynamics. These equations were derived by Dirac in a classical paper in 1938 [1]; the same equations were also obtained by Rohrlich on the basis of an action principle in his well-known book [2]. In the case of two charges, the equations are:

\[
m_1 a^\mu_1 = \left( e_1/c \right) F^{\mu\alpha}_{\text{ext}} v_\alpha^1 + \left( e_1/c \right) F^{\mu\alpha}_{\text{ret}} v_\alpha^1 + \left( 2 e_1^2 / 3 c^3 \right) \left( \dot{a}^\mu_1 - \left( 1/c^2 \right) a^\lambda_1 a^\lambda_1 v^\mu_1 \right)
\]

where we use Rohrlich’s notation. The equation for charge \( e_2 \) is the same as (1), but with indexes 1 and 2 interchanged. The first term on the right-hand side (RHS) in (1), is Lorentz’s force due to the external field. The second term on the RHS, links charge \( e_2 \) charge \( e_2 \) by means of a purely retarded interaction, and represents the only mutual interaction of the charged particles in Lorentz-Dirac equations. Finally, the third term is the well known radiation reaction term, constituted by two parts: “Schott term”, involving the third derivative of the charge position, and “Larmor term”, involving the square of the acceleration. In the case of only one charge in an external field, the second term on the RHS of (1) disappears.

Equations in (1) are the final achievement of Dirac’s conception, according to which it is possible to formulate a consistent set of classical equations of motion for point charges. Dirac reached these equations starting from Maxwell equations and the principles of relativity, masking the divergences associated with the singularities of the field at the position of the charges, by means of a mass renormalization procedure. Extensive theoretical work, carried out especially by Rohrlich [3], led to consider that Lorentz-Dirac equa-

\[\text{(1)}\]
tions coherently describe the one-charge case in an external field. However, the situation does not remain the same when several charges are involved. In fact, a few years after Dirac derived his equations, Eliezer applied them to the case of two equal-mass particles with charges of equal magnitude but opposite signs, moving in a straight line and colliding head-on. This led to a controversy that still holds true. According to Eliezer, the charges stop before colliding, then turn back and move away from each other, with an acceleration that is always different from zero, and with a velocity that tends toward the speed of light. This behavior is contrary to what common understanding would suggest for two charges of opposite signs, and is also contradictory to the law of energy conservation. Eliezer’s conclusions were questioned by Clavier, Plass, and Rohrlich. However, more recent detailed numerical calculations by Baylis and Huschilt, and Kasher, came to support Eliezer’s conclusions.

The pathological behavior of Lorentz-Dirac equations in Eliezer head-on collision has received, in general, little attention in textbooks (one exception is Parrott’s book ). Several reasons may help to explain this lack of attention. In spite of the one-dimensional character of Eliezer’s collision, the retarded nature of the interaction between the charges gives rise to mathematical complications. In particular, the numerical treatment of & are rather elaborated, and questions about the convergence of the iterative process cannot be easily answered. This problem is related, in turn, to the issue of the existence and the uniqueness of the solution of Lorentz-Dirac equations, which still remains an open question in the case of several charges.
Another source of troubles comes from the fact that in Eliezer’s collision, the kinetic energy of the charges, the total rate of the radiation emitted by them, as well as the energy stored in the field of the charges, are all time-dependent. This makes it very difficult to analyze the energy conservation law at any given moment. Although the kinetic energy and the total rate of the radiation are well defined concepts, the same does not occur with the concept of “energy stored in the field of the charges”, which makes for even further troubles. We also want to point out that the catastrophic behavior of Eliezer’s solutions has been obtained merely as a mathematical result of Lorentz-Dirac equations, but without identifying any physical cause—an identification that would be convenient for a deeper understanding.

From our point of view, Lorentz-Dirac equations cannot be correct, because they do not take into account all the radiation emitted by the charges. In fact, in the case of two charges, radiation must be described by: a Larmor term for each charge, terms that are indeed present in (1); plus a term that considers the interference effect between the fields of both charges, which is not present in (1). The second term on the RHS of (1) is the only one that considers both particles, but this term has very little to do with the interference radiation, which must involve the product of the acceleration of two charges, and this is clearly not the case. So, in order to put this conjecture in a quantitative way, it would be ideal to find one exact solution of the Lorentz-Dirac equations for two charges where the total rate of radiation (including the interference term) can be calculated in a precise way. It is the purpose of this note to present one such solution, which is the first exact
solution for the Lorentz-Dirac equations for more than one charge present in
the literature related to the subject [9]. As it will be shown below, Lorentz-
Dirac equations, with appropriate external fields to be determined, allow for
the motion of two charges moving in a plane at opposite ends of a diameter
revolving at constant angular velocity $\omega$ in a fixed circular orbit of radius
$a$. This solution can be considered as a sort of generalization of the exact
solution found by Sokolov and Kolesnikova [10] for one charge in a circular
orbit.

Before discussing the algebraic treatment of our solution, it may be con-
venient to make some general comments on Lorentz-Dirac equa-
tions, since
treatment of this subject is almost absent in current literature. Besides, they
may help those readers not very familiar with radiation reaction effects, and,
on the other hand, they will allow us to set the framework of our discussion.

It may seem rather natural to study the validity of Lorentz-Dirac equa-
tions by means of experiments, instead of resorting to theoretical disquisition
as we are doing here. However, due to other effects that appear in experi-
ments that make it almost impossible to reach a clear conclusion, the troubles
with Lorentz-Dirac equations cannot be clarified with the help of practical
measurements. The motion of an electron in a storage ring is a good ex-
ample for illustrating this situation, since in these machines radiation reaction
plays an important role. The starting point for the description of the elec-
tron motion in a storage ring is Lorentz-Dirac equations for one electron
[11, 12]. Now, in a machine of a few Gev, the magnetic external force is
much greater than the Larmor term, which in turn is much greater than the Schott term. These differences make it difficult to test Lorentz-Dirac equations; in addition, there are several other complications. For example, in the vacuum chamber of the machine there is not only one electron, but many of them interacting with each other. In particular, these interactions lead to a loss of electrons from the beam. Furthermore, the particle trajectory is influenced by the interaction of the electron with the wall of the vacuum chamber, as well as by the inhomogeneity of the magnetic field. But the most striking phenomenon that rules out the use of Lorentz-Dirac equations for an accurate description of the motion are well-established quantum effects on electron motion [11, 12].

An experimental set-up of Eliezer’s problem considering a collision between an electron and a positron, will pose difficulties of the same kind as those observed in the case of a storage ring. So, even though Eliezer’s problem has a clear meaning in a classical context, where position and momentum have precise values at any time, quantum effects are unavoidable in the experiment and, therefore, Lorentz-Dirac equations do not apply.

Why then should we insist on looking for exact solutions of Lorentz-Dirac equations if they are of little help for an accurate practical description of the motion of the charge? In our opinion, the study of these equations in their own context, regardless of experiments or practical applications, is interesting because their coherence is deeply related to the mass renormalization problem, which still remains somewhat obscure in classical as well as quantum
theory. It was precisely in Lorentz-Dirac equations where, for the first time, Dirac introduced the concept of mass renormalization, in order to deal with the divergent self-energy of a point charge. Thus, it would not be surprising to find that the violation of the energy conservation law in these equations is related to the mass renormalization problem. Now, if the Lorentz-Dirac equations are incorrect, as we clearly show below, the natural question that arises is: What are the correct equations of motion for point charges? This question assumes that the divergence due to the point nature of the charges can be consistently renormalized with the energy conservation law; but, is it possible that this program could not be carried out? In order to answer these and other related questions, Lorentz-Dirac equations must be studied in their own classical context. The analysis of exact solutions becomes a powerful tool for this purpose.

Let us now describe our solution. We assume that the motion occurs on the $X - Y$ plane, clockwise in a circle of radius $a$ centered at the origin. The two identical particles of charges $e < 0$ and mass $m$, are rotating with constant angular velocity $\omega$, at the end of a diameter. At time $t$, let charge 1 be the one located at $x = -a \cos \omega t$, $y = a \sin \omega t$. Then, the retarded position of charge 2, which determines the field over charge 1 at time $t$, is defined by the retarded time $t'$, time that also defines the retarded position of charge 1 respect to charge 2 at time $t$. This property is due to the symmetry of the motion. Now, let $2\varphi = \omega t - \omega t'$ be the angle between the diameters of the actual and the retarded positions. Then, we have $\varphi = \beta \cos \varphi$, where $\beta = a\omega/c$. In particular, we obtain $d\beta/d\varphi > 0$. Thus, $\beta$ is a strictly
monotonous increasing function of $\varphi$. Therefore, since there exists a one to
one correspondence between $\beta$ and $\varphi$, we will use either of them to represent
the velocity of the charges. Moreover, as $\beta < 1$, then, $\varphi < \varphi_{\text{cri}} = 0.739$, and
since $\beta > 0$, the angle $\varphi$ is in the interval $0 < \varphi < \varphi_{\text{cri}}$.

In order to account for the retarded interaction between the charges
(the second term on the RHS of (1)), we introduce the radial and tan-
gential unitary vectors to the orbit at the location of charge 1, which are
$\hat{r} = -\hat{i} \cos \omega t + \hat{j} \sin \omega t$ and $\hat{t} = \hat{i} \sin \omega t + \hat{j} \cos \omega t$, respectively. Then, the
radial component $E_r = \hat{r} \cdot E$ and the tangential component $E_t = \hat{t} \cdot E$ of
the retarded electric field of charge 2 at the location of charge 1 at time $t$,
are obtained from Liénard-Wiechert’s well-known formulas for the field of a
point charge. We get:

\[
(4a^2/e)E_r = (\cos \varphi + \varphi \sin \varphi)^{-3} \left\{ \left( \varphi \sin 2\varphi + \cos^2 \varphi \right) \left( 1 + \varphi^2 \cos 2\varphi \sec^2 \varphi \right) - 2\varphi^2 \left( 1 + \varphi \tan \varphi \right) \cos 2\varphi \right\},
\]

\[
(4a^2/e)E_t = (\cos \varphi + \varphi \sin \varphi)^{-3} \left\{ \left( \varphi \cos 2\varphi - \sin \varphi \cos \varphi \right) \left( 1 + \varphi^2 \cos 2\varphi \sec^2 \varphi \right) + 2\varphi^2 \left( 1 + \varphi \tan \varphi \right) \sin 2\varphi \right\}.
\]

For this type of motion, the magnetic field at the position of each charge
has only one component along axis $Z$, given by:

\[
B_z = -\sin \varphi E_r - \cos \varphi E_t.
\]

The motion under consideration cannot be obtained without external fields.
We will assume here the existence of a time-independent external electric
field tangent to the orbit, and with a fixed magnitude around the orbit circle. In order to show that this electric field arise from Maxwell’s equations, we consider the ideal sources: a charge density \( \rho(x,t) \) that vanishes identically everywhere, and a charge current vector proportional to time \( t \) given by 

\[
J(x,t) = -At \delta(r - b) \hat{\phi},
\]

where we use cylindrical coordinates \((r, \phi, z)\); \( \delta \) is the usual Dirac delta function, and \( \hat{\phi} \) is the unit vector tangent to the circle \( r = \text{ fixed} \), pointing in the direction of increasing \( \phi \). Parameters \( A \) and \( b \) are positive, with radius \( b \) of the infinitely long solenoid smaller than orbit radius \( a \). Then, it is easy to show that for these sources the fields:

\[
E(x,t) = \left( \frac{\mu_0}{2} \right) A \left\{ r - \theta (r - b) \left( r - \frac{b^2}{r} \right) \right\} \hat{\phi}
\]

\[
B(x,t) = -\mu_0 At \left\{ 1 - \theta(r-b) \right\} \hat{k}
\]

(5)

where \( \theta(r) \) represents the step function, are solutions of Maxwell’s equations everywhere. The proof comes easier in cylindrical coordinates. This proof uses either standard rigor with step and delta functions, or the distribution theory. We need the fields (5) only in region \( r > b \), in which case they reduce to \( E = (\mu_0 Ab^2 / 2r) \hat{\phi}; \ B = 0 \). The tangential electric field can take any value at a given orbit circle. In addition to the external electric field, we will also consider an homogeneous time-independent external magnetic field pointing in the negative direction of axis \( Z \). The fourth component of Lorentz-Dirac equation (1) is then identically satisfied by \( z = 0 \) at any time. Equation (1) for the component along the \( X \) axis, \( x = -a \cos \omega t \), of charge 1, can be combined with the component along the \( Y \) axis, \( y = a \sin \omega t \), of the same charge, in order to write them in terms of radial and tangential component
as follows:

\[
\hat{t} \left\{ eE_t + eE_{t}^{\text{ext}} - \left( \frac{2e^2}{3a^2} \right) \beta^3 \gamma^4 \right\} = \dot{\hat{r}} \left\{ -ma \omega^2 \gamma - eE_r + e\beta B_z + e\beta B_{z}^{\text{ext}} \right\},
\]

(6)

where \( E_{t}^{\text{ext}} \) is the tangential component of the external electrical field, \( B_{z}^{\text{ext}} \) is the external magnetic field along the negative axis \( Z \), and \( \gamma \) is the usual relativistic parameter given by \((1 - \beta^2)^{-1/2}\). It is easy to show that charge 2 also satisfies equation (6). From this equation, it follows that must have the following value:

\[
eE_{t}^{\text{ext}} = \left( \frac{e^2}{a^2} \right) f(\varphi),
\]

(7)

where function \( f(\varphi) \) is given by:

\[
f(\varphi) = (2/3)\varphi^3 \cos \varphi (\cos^2 \varphi - \varphi^2)^{-2} - (a^2/e)E_t.
\]

(8)

A detailed analysis shows that \( f(\varphi) \) is a positive strictly monotonous increasing function of \( \varphi \). For this reason, for a given radius \( a \) and \( a \) given velocity \( v = a\omega \), equation (7) determines a perfectly well-defined external field \( E_{t}^{\text{ext}} \), which in turn imposes a restriction on parameters \( b \) and \( A \) in (5).

The value of the external magnetic field \( B_{z}^{\text{ext}} \) obtained from (6), is the following:

\[
B_{z}^{\text{ext}} = \left( e/4ar_0 \right) \left\{ 4\varphi \left( \cos^2 \varphi - \varphi^2 \right)^{-1/2} + (r_0/a) \varphi^{-1} \left[ \left( 4a^2/e \right) E_t \varphi \cos \varphi 
+ \left( 4a^2/e \right) E_r (\cos \varphi + \varphi \sin \varphi) \right] \right\},
\]

(9)

where \( r_0 = e^2/mc^2 \). For given radius and velocity, the above formula determines \( B_{z}^{\text{ext}} \) in a unique way. For \( \varphi \) tending to zero in (9), \( B_{z}^{\text{ext}} \) tends to
infinity, so that the magnetic fields compensate the repulsive electric field force between the charges.

The first component of (1), representing the energy conservation law, reduces to:

$$2e\mathbf{v} \cdot \mathbf{E}^{\text{ext}} = \left( \frac{4e^2c}{3a^2} \right) \beta^4 \gamma^4 - \left( \frac{4e^2c}{3a^2} \right) \beta^4 g(\varphi),$$  \hspace{1cm} (10)

where $g(\varphi)$ is given by:

$$g(\varphi) = \left( \frac{3a^2}{2e} \right) \varphi^{-3} \cos^3 \varphi E_t.$$  \hspace{1cm} (11)

Except for one factor, equation (10) coincides with equation (7). The left side of equation (10) corresponds to the rate at which the external field supplies energy to the system composed of the two charges. Now, the kinetic energy of the charges remains unchanged in this motion. Besides, the energy stored in the field of the charges remains also unchanged, since the position of the charges at two different times cannot be distinguished. We emphasize that we need not pay attention to the precise definition of the concept of energy stored in the field of the charges, since by the symmetry of the motion, this energy -whichever it may be- is time-independent. Therefore, according to the energy conservation law, all the power supplied to the system of two charges by the external electric field must be radiated away. This means that the RHS of (10) corresponds to the total rate of radiation emitted by the two charges. But for known motion of the charges, Maxwell’s equations determine the field generated by the charges, which in turn determines the total rate of radiation for this motion. We carried out this somewhat
elaborated calculation in a separate paper \[13\]. The total rate of radiation involves a Larmor term for each charge, plus an interference term due to the field of both charges. The Larmor terms for the two charges are precisely the first term on the RHS of (10), so its second term must coincide with the interference radiation term. If we put the interference radiation term in \[13\] in the form \(-(4e^2c/3a^2)\beta^4I(\varphi)\); then, function \(I(\varphi)\) must coincide with function \(g(\varphi)\) in (11). We plotted these functions in Fig. 1, showing that for velocities of the charges not close to the speed of light, function \(g(\varphi)\), which comes from Lorentz-Dirac equations, is qualitatively similar to \(I(\varphi)\). However, these functions are quantitatively different for any velocity of the charges, and therefore Lorentz-Dirac equations violate the energy conservation law. We remark that if we consider only one charge in circular motion, as in the Sokolov-Kolesnikova solution \[10\], the contradiction with the energy conservation law disappears, since in this case we have to take \(E_t = 0\) in (11).

Figure 1

The above discussion suggests that the source of troubles with the energy conservation law in Lorentz-Dirac equations for more than one charge, comes from an inappropriate consideration of the interference radiation between the charges. So, it seems natural to solve this problem by incorporating the interference radiation energy in the equations of motion. There are motions for which this program may succeed, as seems to be the case for the motion
considered herein. Nevertheless, this procedure may not work in all cases. Charges in close collision may present troubles, because the trick of oversimplifying the relevance of the fields in the neighborhood of the singularities of the charges by means of the mass renormalization procedure may critically disturb the exchange of energy momentum between the singularities. This disturbance may lead to a violation of the energy conservation law.

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FIGURE CAPTIONS

Figure 1: The solid line represents function $I(\varphi)$ of the interference rate of radiation given by $-\left(\frac{4e^2c}{3a^2}\right) \beta^4 I(\varphi)$. The dotted line represents the corresponding function $g(\varphi)$ given by (11), for the interference rate according to Lorentz-Dirac equations.
This figure "fig1-1.png" is available in "png" format from:

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