COMPLEX DYNAMICS IN THE SEGMENTED DISC DYNAMO

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(Communicated by Miguel Sanjuan)

ABSTRACT. The present work is devoted to giving new insights into the segmented disc dynamo. The integrability of the system is studied. The paper provides its first integrals for the parameter \( r = 0 \). For \( r > 0 \), the system has neither polynomial first integrals nor exponential factors, and it is also further proved not to be Darboux integrable. In addition, by choosing an appropriate bifurcation parameter, the paper proves that Hopf bifurcations occur in the system and presents the formulae for determining the direction of the Hopf bifurcations and the stability of bifurcating periodic solutions.

1. Introduction. Since the discovery of the Lorenz chaotic system, chaos has been developed and intensively studied in the past four decades. This study about chaos has concentrated on not only proposing new and interesting chaotic systems, but also enhancing complex dynamics and topological structure based on the existing chaotic systems [5, 8, 13].

In order to understand magnetic field generation and reversals in astrophysical bodies, model dynamos have been extensively investigated during the past decades [1, 4, 6, 10, 11]. The self-exciting disc dynamo has frequently been invoked as a simple prototype of dynamo action, analogous to the dynamo process that is believed to operate in the liquid conducting core of the Earth and in the convective envelope of the Sun. Considering that the conventional treatment of the simplest such model, the self-exciting homopolar disc dynamo, was not self-consistent, Moffatt introduced a segmented disc dynamo in which the current associated with the radial diffusion of the magnetic field could be included in a simple way [10]. The dynamo is described by the following set of ordinary differential equations:

\[
\begin{align*}
\dot{x}(t) &= r(y - x) = P(x, y, z), \\
\dot{y}(t) &= mx - (1 + m)y + xz = Q(x, y, z), \\
\dot{z}(t) &= g(mx^2 + 1 - (1 + m)xy) = R(x, y, z),
\end{align*}
\]

where \( x(t) \) and \( y(t) \) denote the magnetic fluxes due to radial and azimuthal current distributions, \( z(t) \) is the angular velocity of the disc; \( g \) measures the applied torque, and \( r \) and \( m \) are positive constants that depend on the electrical properties of the circuit [6, 10].
The system (1.1) is not identical to the Lorenz system [10]. Knobloch added the term \( -vz' \) to the right side of the third equation of system (1.1) and system (1.1) becomes the Lorenz system [6].

The integrability of systems of differential equations is one of central topics in the theory of ordinary differential equations. The Darboux theory of integrability plays a central role in the integrability of the polynomial differential models. We can compute the Darboux first integrals by knowing a sufficient number of algebraic invariant surfaces (the Darboux polynomials) and of the exponential factors (see [9, 12] and the references therein).

Here, we shall use the Darboux theory of integrability to study the Darboux integrability of the segmented disc dynamo (1.1). We further contribute to the understanding of the complexity, or more precisely of the topological structure of the dynamics of system (1.1) by studying its integrability. Obviously, system (1.1) is integrable for \( r = 0 \), and we provide its first integrals. For \( r > 0 \), the system has neither polynomial first integrals nor exponential factors, and it is also further proved not to be Darboux integrable.

On the other hand, by choosing an appropriate bifurcation parameter, the paper proves that Hopf bifurcations occur in system (1.1) when the bifurcation parameter exceeds a critical value and presents the formulae for determining the direction of the Hopf bifurcations and the stability of bifurcating periodic solutions by applying the normal form theory [3, 7].

The paper is organized as follows. Sect. 2 investigates the dynamical behaviors of the segmented disc dynamo. Sect. 3 investigates the integrable of the system. In Sect. 4, by using the normal form theory, the direction of Hopf bifurcations and the stability of bifurcating periodic solutions are analyzed in detail. In Sect. 5, some numerical simulations are presented to illustrate the theoretical analysis. And Sect. 6 concludes the paper.

2. Dynamical behaviors of system (1.1). This system is invariant under the transformation

\[
(x, y, z) \rightarrow (-x, -y, z).
\]

Namely, the system has rotation symmetry around the z-axis. The divergence of system (1.1) is \( \nabla \cdot V = -(1 + m + r) \), and the system is dissipative.

2.1. Equilibria and stability. For \( r > 0, g > 0 \) and \( m > 0 \), system (1.1) always has two isolated equilibria \( P_+ [1, 1, 1] \) and \( P_- [-1, -1, 1] \). The characteristic equation about the equilibrium \( P_- \) is

\[
\lambda^3 + (m + r + 1) \lambda^2 + g (1 + m) \lambda + 2gr = 0. \tag{2.1}
\]

The characteristic equation about the equilibrium \( P_+ [1, 1, 1] \) is the same as that of \( P_- [-1, -1, 1] \).

According to Ref. [10], the two equilibria \( P_+ \) and \( P_- \) are both asymptotically stable when the following conditions are met.

\[
m < 1, \quad r < \frac{(m + 1)^2}{1 - m};
\]

or

\[
m > 1.
\]
For $r = \frac{(m+1)^2}{1-m}$, the Jacobian matrix has a pair of purely imaginary eigenvalues and a nonzero real eigenvalue. Therefore, $P_+$ and $P_-$ are both not hyperbolic but weak repelling focus.

2.2. Coexistence of chaotic attractors with different types of equilibria. Generally, it is difficult to analytically specify parametric regions of a chaotic system. Therefore, certain numerical indices for identifying chaotic properties of system orbits are verified. There are three kinds of cases in system (1.1) as follows.

Case 1. Chaotic attractor coexists with two stable node-foci.

When parameters $(r, m, g) = (4, 0.5, 11.6)$, the chaotic attractor can be observed (see Figure 1(a)). The Poincaré map also shows that this system is chaotic (see Figure 1(b)). The three Lyapunov exponents with initial values $(0.2094, 0.2547, -0.2240)$ of system (1.1) are $L_1 = 0.012596$, $L_2 = -0.034469$, and $L_3 = -5.478127$. The equilibria $P_+$ and $P_-$ are both stable, whose characteristic values are $-0.03086766497 \pm 4.130776455i, -5.438264670$. Therefore, system (1.1) has neither homoclinic orbits nor heteroclinic orbits joining $P_+$ and $P_-$, and can display a chaotic attractor, not satisfying the Sil’nikov theorem.

Case 2. Chaotic attractor coexists with two saddle-foci.

When parameters $(r, m, g) = (5, 0.5, 11.6)$, the chaotic attractor can be observed (see Figure 2(a)). The Poincaré map is shown in Figure 2(b). The three Lyapunov exponents with initial values $(0.2094, 0.2547, -0.2240)$ of system (1.1) are $L_1 = 0.387939$, $L_2 = -0.007317$, and $L_3 = -6.880622$. It is noted that the chaotic attractor is a bit different from Case 1, because the equilibria $P_+$ and $P_-$ are both unstable. Figure 1(b) and Figure 2(b) also show sheets of the chaotic attractors visualized on the Poincaré maps. It is clear that some sheets are folded. In addition, system (1.1) has rotation symmetry around the $z$-axis, so the scatter of points of the Poincaré maps on the $x - z$ plane and $y - z$ plane should be symmetrical, as shown in Figure 1(b) and Figure 2(b).

Case 3: Chaotic attractor coexists with two nonhyperbolic equilibria.

When parameters $(r, m, g) = (4.5, 0.5, 11.6)$, the chaotic attractor can be observed (see Figure 3(a)). The Poincaré map is shown in Figure 3(b). The three Lyapunov exponents with initial values $(0.2094, 0.2547, -0.2240)$ of system (1.1) are $L_1 = 0.310598$, $L_2 = -0.005019$, and $L_3 = -6.3055804$. The Lyapunov exponent spectrum for $r \in [5, 7]$ are shown in Figure 4.

3. Darboux integrability of system (1.1).

3.1. Preliminary results. Let $R[x, y, z]$ be the ring of the real polynomials in the variables $x, y$, and $z$. We say that $f(x, y, z) \in R[x, y, z]$ is a Darboux polynomial of system (1.1) if it satisfies

$$
\frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q + \frac{\partial f}{\partial z} R = fL_f, \tag{3.1}
$$

for some polynomial $L_f$, called the cofactor of $f(x, y, z)$. If $f(x, y, z)$ is a Darboux polynomial, then the surface $f(x, y, z) = 0$ is an invariant algebraic surface, which means that if an orbit of system (1.1) has a point on the surface $f(x, y, z) = 0$, then the whole orbit is contained in it.

Let $f, g \in R[x, y, z]$ be coprime. We say that a nonconstant function $E = e^{\frac{f}{g}}$ is an exponential factor of system (1.1) if $E$ satisfies
\begin{align}
\frac{\partial E}{\partial x} P + \frac{\partial E}{\partial y} Q + \frac{\partial E}{\partial z} R &= E L_e, \tag{3.2}
\end{align}

**Figure 1.** Parameters $r=4$, $m=0.5$, $g=11.6$, initial values $(0.2094, 0.2547, -0.2240)$; (a) chaotic attractor of system (1.1); (b) Poincaré map of system (1.1) on the $y$-$z$ plane

**Figure 2.** Parameters $r=5$, $m=0.5$, $g=11.6$, initial values $(0.2094, 0.2547, -0.2240)$; (a) chaotic attractor of system (1.1); (b) Poincaré map of system (1.1) on the $x$-$z$ plane

**Figure 3.** Parameters $r=4.5$, $m=0.5$, $g=11.6$, initial values $(0.2094, 0.2547, -0.2240)$; (a) chaotic attractor of system (1.1); (b) Poincaré map of system (1.1) on the $x$-$y$ plane
for some polynomial $L_c \in \mathbb{R}[x,y,z]$, called the cofactor of $E$ and having degree at most 1. Note (3.2) is equivalent to
\[
\frac{\partial g}{\partial x} P + \frac{\partial g}{\partial y} Q + \frac{\partial g}{\partial z} R = g L_f + f L_c.
\] (3.3)

For a geometrical and algebraic meaning of the exponential factors see [2].

A first integral $G$ of system (1.1) is of Darboux type if it has the form
\[
G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q},
\] (3.4)
where $f_1, \cdots, f_p$ are Darboux polynomials, $E_1, \cdots, E_q$ are exponential factors and $\lambda_j, \mu_k \in \mathbb{R}$ for $j = 1, \cdots, p$ and $k = 1, \cdots, q$.

3.2. **Darboux integrability.** We claim that the degree $k$ of the cofactor $L_f$ is less than or equal to 1. The claim follows from the fact that in (1.1)
\[
\deg(L_f) + \deg(f)
\]
\[
= \max \{ \deg(f) - 1 + \deg(P), \deg(f) - 1 + \deg(Q), \deg(f) - 1 + \deg(R) \}
\]
\[
\leq \deg(f) + 1.
\]

Therefore, without loss of generality, we can assume that the cofactor is of the form
\[
L_f = b_0 + b_1 x + b_2 y + b_3 z,
\]
and then (3.1) becomes
\[
\frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q + \frac{\partial f}{\partial z} R = f (b_0 + b_1 x + b_2 y + b_3 z).
\] (3.5)

**Theorem 3.1.** If $r = 0$, system (1.1) is integrable with the first integrals
\[
H_1 = x \quad \text{and} \quad H_2 = \frac{1}{2} \ln \left( (1 + m) x^2 g_1(x,y,z) + \frac{\sqrt{1 + m}}{\sqrt{4gx^2 - m - 1}} g_2(x,y,z), \right)
\]
where
\[
g_1(x,y,z) = (1 + m)^2 \left( gx^3 y - x^2 z + 1 \right) y + gx (mx^2 + 1)^2
\]
\[
- (1 + m) \left( 2gmx^3 y - x^2 z^2 - mx^2 z + 2gxy + z + m \right) x,
\]
\[
g_2(x,y,z) = \arctan \left( \frac{\sqrt{1 + m} \left( 2x^2 z + mx^2 - (m + 1)xy - 1 \right)}{\sqrt{4gx^2 - m - 1} \left( mx^2 - (m + 1)xy + 1 \right)} \right).
\]
It is straightforward to verify that $H_1$ and $H_2$ in the statement of the theorem are first integrals of system (1.1). Therefore the proof of Theorem 3.1 will be omitted and from now on we consider the cases in which $r \neq 0$.

**Theorem 3.2.** The following statements hold for system (1.1) with $r \neq 0$.

1. It has no Darboux polynomials with non-zero cofactor.
2. It has no polynomial first integrals.
3. It has no exponential factors.
4. It admits no Darboux first integrals.

**Proof.** (1) Let

$$f(x, y, z) = \sum_{i=0}^{n} f_i, \tag{3.6}$$

where each $f_i = f_i(x, y, z)$ is a homogeneous polynomial of degree $i$. Without loss of generality we can assume that $f_n \neq 0$ and $n \neq 0$.

Computing the terms of degree $n + 1$ in (3.5) yields

$$xz \frac{\partial f_n}{\partial y} + g (mx^2 - (1 + m) xy) \frac{\partial f_n}{\partial z} = f_n (b_1 x + b_2 y + b_3 z). \tag{3.7}$$

Solving the equation gets

$$f_n (x, y, z) = F (x, -2gmxy + g(m + 1)y^2 + z^2) \exp \left(b_3 \frac{y}{x} - \frac{b_1}{\sqrt{g(1 + m)}} k(x, y, z) - b_2 \frac{z \sqrt{g(1 + m)} + xgm k(x, y, z)}{\sqrt{g(1 + m)}} \right), \tag{3.8}$$

where $F$ is an arbitrary continuously differentiable function and

$$k(x, y, z) = \arctan \left(\frac{\sqrt{g(1 + m)} (mx - (1 + m)y)}{(1 + m) z} \right).$$

Since $f_n(x, y, z)$ is a polynomial of degree $n$, it forces that $b_1 = 0, b_2 = 0$, and $b_3 = 0$.

Let

$$f_n (x, y, z) = x^{n-2p} (g(1 + m)y^2 - 2gmxy + z^2)^p, \tag{3.9}$$

where $p$ is a nonnegative integer and $p \leq \frac{n}{2}$.

Computing the terms of degree $n$ in (3.5) yields

$$x \frac{\partial f_{n-1}}{\partial y} + g (m(x - y) - y) \frac{\partial f_{n-1}}{\partial z} + r (y - x) \frac{\partial f_n}{\partial x} + (m(x - y) - y) \frac{\partial f_n}{\partial y} = b_0 f_n. \tag{3.9}$$

**Case 1.** $p = 0$. Solving the equation (3.9) yields

$$f_{n-1} (x, y, z) = \frac{F (x, -2gmxy + g(1 + m)y^2 + z^2) - x^{n-1} k(x, y, z) b_0}{\sqrt{g(1 + m)}},$$

$$- \frac{nr x^{n-2} \left(gx k(x, y, z) - \sqrt{g(1 + m)} z\right)}{\sqrt{g(1 + m)}}.$$
where $F$ is an arbitrary continuously differentiable function. Since $f_{n-1}(x, y, z)$ is a polynomial, it forces that $b_0 = 0$. And the case is impossible.

**Case 2.** $p > 0$. Solving the equation (3.9) yields

$$f_{n-1}(x, y, z) = F(x, -2gmx + g(1 + m)y^2 + z^2) - b_0 \frac{k(x, y, z)}{x\sqrt{g(1 + m)}} f_n$$

$$+ \frac{\sqrt{g(1 + m)}}{x\sqrt{g(1 + m)}} g_4(x, y, z) - k(x, y, z)g_5(x, y, z) g_5(x, y, z),$$

(3.10)

where $F$ is an arbitrary continuously differentiable function and

$$g_3(x, y, z) = g(mx - (m + 1)y)^2 + (m + 1)z^2,$$

$$g_4(x, y, z) = (mx - (m + 1)y),$$

$$g_5(x, y, z) = px^{n-2p-1}(-2gmx + g(m + 1)y^2 + z^2)^{p-1}.$$

For the solution (3.10), the coefficient of $b_0$ is

$$- \frac{k(x, y, z)}{x\sqrt{g(1 + m)}} f_n.$$

Since $f_{n-1}(x, y, z)$ is a polynomial, it forces that $b_0 = 0$. And the case is impossible.

(2) As long as we let $b_0 = b_1 = b_2 = b_3 = 0$ in (3.5), the proof is similar to (1).

(3) Considering that system (1.1) has no Darboux polynomials with non-zero cofactor, we assume that $E = e^h$ with $h \in R[x, y, z]$ is an exponential factor of system (1.1) and we shall reach a contradiction. Without loss of generality, we assume that the cofactor $L$ is of the form $L = b_0 + b_1 x + b_2 y + b_3 z$, with $b_i \in R$ for $i = 0, 1, 2, 3$.

Clearly, by (3.3), $h$ satisfies

$$\frac{\partial h}{\partial x} P + \frac{\partial h}{\partial y} Q + \frac{\partial h}{\partial z} R = b_0 + b_1 x + b_2 y + b_3 z.$$

(3.11)

We write $h$ in the form

$$h = \sum_{i=0}^{n} h_i(x, y, z),$$

(3.12)

where each $h_i = h_i(x, y, z)$ is a homogeneous polynomial of degree $i$. Without loss of generality we assume that $h_n \neq 0$ and $n \neq 0$.

Computing the terms of degree $n + 1$ in (3.11) yields

$$xz \frac{\partial h_n}{\partial y} + g(mx^2 - (1 + m)xy) \frac{\partial h_n}{\partial z} = 0.$$

Solving the equation gets

$$f_n(x, y, z) = F(x, -2gmx + gmy^2 + gy^2 + z^2),$$

where $F$ is an arbitrary continuously differentiable function.

Let

$$f_n(x, y, z) = x^{n-2p}(-2gmx + gmy^2 + gy^2 + z^2)^p,$$

where $p$ is a nonnegative integer and $p \leq \frac{\nu}{2}$.

Computing the terms of degree $n$ in (3.11) yields

$$xz \frac{\partial f_{n-1}}{\partial y} + g(mx^2 - (1 + m)xy) \frac{\partial f_{n-1}}{\partial z} + r(y - x) \frac{\partial f_n}{\partial x} + (mx - my - y) \frac{\partial f_n}{\partial y} = 0.$$

(3.13)
Case 1. $p = 0$. Solving the equation (3.13) yields

$$f_{n-1}(x, y, z) = F(x, g(1 + m) y^2 - 2gmx + z^2)$$

$$- k(x, y, z) \frac{z^{n-2}\left(g(1 + m) y^2 - 2gmx + z^2\right)^{p-1}}{\sqrt{g(1 + m)^2}} g_0(x, y, z)$$

$$+ \frac{x^{n-2p-1}\left(g(1 + m) y^2 - 2gmx + z^2\right)^{p-1}}{(1 + m)^2} g_\tau(x, y, z),$$

where $F$ is an arbitrary continuously differentiable function, and

$$g_0(x, y, z) = (1 + m)^2 \left( (1 + m)^2 p + r (mp + n - 2p) \right) g(1 + m) y^2 - 2gmx + z^2 \right)$$

$$+ gm^2 p \left( (1 + m)^2 + (m - 2) r \right) x^2,$$

$$g_\tau(x, y, z) = gmp \left( (m + 1)^2 - (m - 2)r \right) x^2 z$$

$$- g(1 + m) \left( (m + 1)^2 p + mr (2n - 3p) \right) xyz$$

$$+ r (n - 2p) (m + 1) g(1 + m)y^2 + z^2 \right).$$

Since $f_{n-1}(x, y, z)$ is a polynomial, it forces that $p = 0$ and $n = 0$, which contradicts with the assumption $n \neq 0$. And this case is impossible.

Case 2. $p > 0$. Solving the equation (3.13) yields

$$f_{n-1}(x, y, z) = F(x, g(1 + m) y^2 - 2gmx + z^2)$$

$$- k(x, y, z) \frac{z^{n-2p-1}\left(g(1 + m) y^2 - 2gmx + z^2\right)^{p-1}}{\sqrt{g(1 + m)^2}} g_0(x, y, z)$$

$$+ \frac{x^{n-2p-2}\left(g(1 + m) y^2 - 2gmx + z^2\right)^{p-1}}{(1 + m)^2 g} g_\tau(x, y, z),$$

where $F$ is an arbitrary continuously differentiable function, and

$$g_0(x, y, z) = (1 + m)^2 \left( (1 + m)^2 p + r (mp + n - 2p) \right) g(1 + m) y^2 - 2gmx + z^2 \right)$$

$$+ gm^2 p \left( (1 + m)^2 + (m - 2) r \right) x^2,$$

$$g_\tau(x, y, z) = gmp \left( (m + 1)^2 - (m - 2)r \right) x^2 z$$

$$- g(1 + m) \left( (m + 1)^2 p + mr (2n - 3p) \right) xyz$$

$$+ r (n - 2p) (m + 1) g(1 + m)y^2 + z^2 \right).$$

Since $f_{n-1}(x, y, z)$ is a polynomial, it forces that $p = 0$ and $n = 0$, which contradicts with the assumption $n \neq 0$. And this case is impossible.

(4) By (3), it is obvious. \[\square\]

4. Hopf bifurcations in system (1.1). In this section, we apply the normal form theory [3, 7] to study the direction, stability and period of bifurcating periodic solutions for system (1.1).

We first consider the Hopf bifurcation of system (1.1) at $P_+ [1, 1, 1]$. When $m < 1$ and $r = r_0 = \frac{(1+m)^2}{1-m},$ Eq. (2.1) possesses a negative real root $\frac{2(m+1)}{m-1}$ and a pair of conjugate purely imaginary roots $\pm \sqrt{g(m+1)} \; i$. Under this condition, the transversality condition

$$\text{Re} \left( \lambda'(r_0) \right) \bigg|_{\lambda = \sqrt{g(m+1)}} = - \frac{g(m-1)^3}{2 \left[ g(m-1)^2 + 4 (m+1) \right] (m+1)} > 0,$$

is also satisfied. Accordingly, Hopf bifurcation at $P_+$ occurs. The above analysis is summarized as follows:

**Theorem 4.1.** (Existence of Hopf Bifurcation) Let $m < 1$. Then, as $r$ passes through the critical $r_0 = \frac{(1+m)^2}{1-m},$ system (1.1) undergoes a Hopf bifurcation at the equilibrium $P_+ [1, 1, 1]$. \[\Box\]
The direction, stability and period of bifurcating periodic solutions for system (1.1) are as follows.

**Theorem 4.2.** Let \( m < 1 \) and \( r_0 = \frac{(1+m)^2}{1-m} \). For system (1.1),

1. bifurcating periodic solutions exist for sufficient small \( r - r_0 < 0 \). Moreover, the period solutions of system (1.1) from Hopf bifurcation at \( P_+ \) are non-degenerate, subcritical and orbitally unstable.

2. the period and characteristic exponent of the bifurcating periodic solution are:

\[
T = \frac{2\pi}{\omega_0} \left(1 + \tau_2 \varepsilon^2 + o(\varepsilon^4)\right),
\]

\[
\beta = \beta_2 \varepsilon^2 + o(\varepsilon^4),
\]

where

\[
\omega_0 = \sqrt{g(m+1)},
\]

\[
\tau_2 = \frac{(m + 1)\left(3g^2(m-1)^4 + 31g(m+1)(m-1)^2 + 22(m+1)^2\right)}{24g\left(g(m-1)^2 + m + 1\right)\left(g(m-1)^2 + 4(m+1)\right)^2},
\]

\[
\beta_2 = -\frac{1}{4} \left(\frac{(m-1)(m+1)^3\left(3g(m-1)^2 + m + 1\right)}{\left(g(m-1)^2 + 4m + 4\right)^2\left(g(m-1)^2 + m + 1\right)}\right),
\]

\[
\varepsilon^2 = \frac{r - r_0}{\mu_2} + o\left[(r - r_0)^2\right],
\]

\[
\mu_2 = -\frac{1}{4} \left(\frac{(m + 1)^4\left(3g(m-1)^2 + m + 1\right)}{g(m-1)^2\left(g(m-1)^2 + m + 1\right)\left(g(m-1)^2 + 4(m+1)\right)}\right).
\]

**Proof.** Let \( \omega_0 = \sqrt{g(m+1)} \) and \( r = r_0 = \frac{(1+m)^2}{1-m} \). Then, we have

\[
v_1 = \begin{pmatrix}
\frac{m+1}{gm^2 - 2gm + g + 4m + 4} & \left(m - 1 - \frac{2\omega_0}{g}\right) \\
\frac{m+1}{gm^2 - 2gm + g + 4m + 4} & \left(m - 1 + \frac{2\omega_0}{g}\right)
\end{pmatrix},
\]

\[
v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},
\]

which satisfy

\[
Av_1 = i\omega_0 v_1, \quad Av_2 = \frac{2(m+1)}{m-1} v_2,
\]

where \( A \) is the Jacobian matrix of system (1.1) at \( P_+ \), and

\[
A = \begin{pmatrix}
-r & r & 0 \\
1 + m & -1 - m & 1 \\
g(m-1) & -g(1+m) & 0
\end{pmatrix}.
\]
From system (1.1), define

$$P = (Re v_1, -Im v_1, v_2) = \begin{bmatrix}
-\frac{2(m+1)^2}{m-g-2g+4m+4} & \frac{m^2-1}{m^2-2g+g+4m+4} & \frac{m+1}{m-1}
-\frac{m^2-1}{m^2-2g+g+4m+4} & \frac{m^2-2g+g+4m+4}{(m-1)^2}
0 & 1 & 0
\end{bmatrix},$$

and

$$\begin{pmatrix}
x
y
z
\end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + P \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}.$$

Thus

$$\begin{cases}
\dot{x}_1 = -\omega_0 y_1 + F_1(x_1, y_1, z_1), \\
\dot{y}_1 = \omega_0 x_1 + F_2(x_1, y_1, z_1), \\
\dot{z}_1 = \frac{2(m+1)}{m-1} z_1 + F_3(x_1, y_1, z_1),
\end{cases}$$

where

$$F_1 = -\frac{(gm^2-2gm+g+4m+4)\omega_0 z_1 + (m-1)^2\omega_0 y_1 - 2 (m-1)}{g (m-1) (gm^2-2gm+g+4m+4)} y_1 \omega_0^2,$$

$$F_2 = -\frac{2}{g^2(m^2-2gm+g+4m+4)} \omega_0^4 x_1^2 + \frac{(m-1)^2}{g(m^2-2gm+g+4m+4)} y_1^2 + \frac{(m-1)^3}{g^2(m^2-2gm+g+4m+4)} x_1^2 y_1 + \frac{2\omega_0^3}{g^2(m-1)} x_1^2 z_1,$$

$$F_3 = \frac{2}{(gm^2-2gm+g+4m+4)^2} \frac{2(m-1)^2 (m+1)^2 (gm^2-2gm+g+2m+2) y_1^2}{y_1^2} + \frac{(m-1)(m+1)}{g^2(m^2-2gm+g+4m+4)^3} y_2^2 + \frac{1}{g^2(m^2-2gm+g+4m+4)^3} y_3^2$$

$$- \frac{g(m+1)^2}{gm^2-2gm+g+4m+4} y_3^2 - \frac{\omega_0 g (m+1)(m-1)^2}{g^2(m^2-2gm+g+4m+4)} y_1 y_3$$

$$- \frac{\omega_0 (m^2-1) (m-1)^2 g^2 - 4 (m+1)}{g^2(m^2-2gm+g+4m+4)^3} y_1 y_2$$

$$- \frac{4(m+1)^2 (gm^2-2gm+g+2m+2)}{gm^2-2gm+g+4m+4)^2} y_3^2 + \frac{4 (m+1)}{m-1} y_3.$$

Furthermore,

$$g_{11} = \frac{1}{4} \left[ \frac{\partial^2 F_1}{\partial x_1^2} + i \frac{\partial^2 F_1}{\partial y_1^2} \right] + i \left( \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial y_1^2} \right) = -\frac{1}{2} \omega_0 (m^2-1) + i(m+1)^2,$$

$$g_{02} = \frac{1}{4} \left[ \frac{\partial^2 F_1}{\partial x_1^2} - \frac{\partial^2 F_1}{\partial y_1^2} - 2 \frac{\partial^2 F_2}{\partial x_1 \partial y_1} \right] + i \left( \frac{\partial^2 F_2}{\partial x_1^2} - \frac{\partial^2 F_2}{\partial y_1^2} + 2 \frac{\partial^2 F_1}{\partial x_1 \partial y_1} \right).$$
Meanwhile, one has

\[
g_{20} = \frac{1}{4} \left[ \left( \frac{\partial^2 F_1}{\partial x_1} - \frac{\partial^2 F_1}{\partial y_1} + 2 \frac{\partial^2 F_2}{\partial x_1 \partial y_1} \right) + i \left( \frac{\partial^2 F_2}{\partial x_1} - \frac{\partial^2 F_2}{\partial y_1} - 2 \frac{\partial^2 F_1}{\partial x_1 \partial y_1} \right) \right]
\]

\[
= \frac{(m + 1) (m - 1)^3 g_\omega}{2(m^2 g - 2gm + g + 4m + 4)^2} - \frac{(m + 1)^2 (5g^2 - 10gm + 5g + 12m + 12)}{2(m^2 g - 2gm + g + 4m + 4)^2} i,
\]

\[
G_{21} = \frac{1}{8} \left[ \frac{\partial^3 F_1}{\partial x_1^3} + \frac{\partial^3 F_1}{\partial x_1 \partial y_1^2} + \frac{\partial^3 F_2}{\partial x_1 \partial y_1} \right]
\]

\[
+ \frac{1}{8} i \left( \frac{\partial^3 F_2}{\partial x_1^2 \partial y_1} - \frac{\partial^3 F_2}{\partial x_1 \partial y_1^2} - \frac{\partial^3 F_1}{\partial x_1 \partial y_1} \right) = 0.
\]

By solving the following equations

\[
\lambda_3 w_{11} = -h_{11},
\]

\[
(\lambda_3 - 2i \omega_0) w_{20} = -h_{20},
\]

one obtains

\[
w_{11} = \frac{(m - 1)^3 (1 + m)}{4(m^2 g - 2gm + g + 4m + 4)^2},
\]

\[
w_{20} = \frac{(m - 1)^2 (m + 1) \left( (m - 1)^4 g^2 - 9 (m + 1) (m - 1)^2 g - 28 (1 + m)^2 \right)}{4(g^2 - 2gm + g + 4m + 4)^3 \left( gm^2 - 2gm + g + m + 1 \right)}
\]

\[
- \frac{(m - 1)^2 (m + 1) \left( 3(m - 1)^4 g^2 + 4(m + 1)(m - 1)^2 g - 8(1 + m)^2 \right)}{2g(gm^2 - 2gm + g + 4m + 4)^3 (gm^2 - 2gm + g + m + 1)} \omega_0.
\]

Furthermore,

\[
G_{110} = \frac{1}{2} \left[ \left( \frac{\partial^2 F_1}{\partial x_1 \partial z_1} + \frac{\partial^2 F_2}{\partial y_1 \partial z_1} \right) + i \left( \frac{\partial^2 F_2}{\partial x_1 \partial z_1} - \frac{\partial^2 F_1}{\partial y_1 \partial z_1} \right) \right]
\]

\[
= \frac{(1 + m)^2 g}{gm^2 - 2gm + g + 4m + 4} + i \frac{(m + 1) \left( g^2 - 2gm + g + 2m + 2 \right)}{(gm^2 - 2gm + g + 4m + 4) (m - 1)} \omega_0,
\]

\[
G_{101} = \frac{1}{2} \left[ \left( \frac{\partial^2 F_1}{\partial x_1 \partial z_1} - \frac{\partial^2 F_2}{\partial y_1 \partial z_1} \right) + i \left( \frac{\partial^2 F_2}{\partial x_1 \partial z_1} + \frac{\partial^2 F_1}{\partial y_1 \partial z_1} \right) \right]
\]
Based the above calculation and analysis, one can compute the following quantities:

\[
g_{21} = G_{21} + (2G_{110}w_{11} + G_{101}w_{20})
\]

\[
C_1(0) = \frac{i}{2\omega_0} \left( g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3} |g_{20}|^2 \right) + \frac{1}{2} g_{21}
\]

\[
\mu_2 = - \frac{\Re C_1(0)}{\alpha'(0)}
\]

\[
\tau_2 = - \frac{\Im C_1(0) + \mu_2 \omega'(0)}{\omega_0}
\]

\[
\beta_2 = 2 \Re C_1(0) = - \frac{1}{4} \left( m - 1 \right) \left( m + 1 \right)^3 \left( 3g(m - 1)^2 + m + 1 \right)
\]

where

\[
\alpha'(0) = \Re (\lambda'(r_0)) = - \frac{g(m - 1)^3}{2(m^2 g - 2gm + g + 4m + 4)(m + 1)},
\]

\[
\omega_0 = \sqrt{g(m + 1)},
\]

\[
\omega'(0) = \frac{g(m - 1)^2}{\omega_0 (g(m^2 - 2gm + g + 4m + 4))},
\]

\[
T = \frac{2\pi}{\omega_0} \left( 1 + r_2 \varepsilon^2 + o(\varepsilon^4) \right),
\]

\[
\beta = \beta_2 \varepsilon^2 + o(\varepsilon^4),
\]

\[
\varepsilon^2 = \frac{r - r_0}{\mu_2} + o \left( (r - r_0)^2 \right).
\]
And the expression of the bifurcating periodic solution is (except for an arbitrary phase angle):

$$(x, y, z)^T = P(x_1, y_1, z_1)^T,$$

where the matrix $P$ is defined as in (4.1),

$$x_1 = \text{Re}u, \quad y_1 = \text{Im}u, \quad z_1 = w_{11}|u|^2 + \text{Re}\left(w_{20}u^2\right) + o\left(|u|^3\right),$$

and

$$u = \varepsilon e^{\frac{2\pi i}{T} t} + \frac{\varepsilon^2}{6\omega_0} \left(g_{02}e^{-\frac{4\pi i}{T}} - 3g_{20}e^{\frac{4\pi i}{T}} + 6g_{11}\right) + O\left(\varepsilon^3\right).$$

Because the system is invariant under the transformation $(x, y, z) \to (-x, -y, z)$, Theorems 4.1 and 4.2 are also true for $P_[-1, -1, 1].$

5. **Numerical simulations.** In this section, we present some numerical results of simulating system (1.1) at different values of $r, m$ and $g$. We can determine the direction of a Hopf bifurcation and the stability of the bifurcating periodic solution by the theorems in Section 4.

If $m = 0.1$ and $g = 11.6$, we can calculate

$$\mu_2 = -0.0078791696, \quad \beta_2 = 0.0043905618, \quad \tau_2 = 0.0013312230.$$

It is found that a stable periodic solution with initial values $(0.9929, 0.9692, 1.009)$ exists for $r = 1.3243$, as shown in Figure 5.

6. **Conclusions.** In this paper, we extensively investigate the dynamic properties of the segmented disc dynamo. The integrability of the system is studied. The author proves the system is integrable for the parameter $r = 0$. For $r > 0$, the system has neither polynomial first integrals nor exponential factors, and it is also further proved not to be Darboux integrable. In addition, by choosing an appropriate bifurcation parameter, the paper proves that Hopf bifurcations occur in the system when the bifurcation parameter exceeds a critical value and presents the formulae for determining the direction of the Hopf bifurcations and the stability of bifurcating periodic solutions. It is hoped that the investigation of the paper will shed some lights to more systematic studies of the segmented disc dynamo.

**Acknowledgments.** We express our sincere thanks to the anonymous referees for their rigorous comments and valuable suggestions helping to improve the original manuscript. The research is supported by the National Natural Science Foundation of China (No. 11671149), and the Science and Technology Planning Project of Guangdong Province, China (No. 2012B061800088).

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Figure 5. Waveform diagram and phase diagram for system (1.1) with $r = 1.3243$

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Received June 2015; revised July 2016.

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