Forward and Inverse Cascades in EMHD Turbulence

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Abstract. Electron magnetohydrodynamics (EMHD) provides a simple fluid-like description of physics below the proton gyro-scale in collisionless plasmas, such as the solar wind. In this paper, we discuss forward and inverse cascades in EMHD turbulence in the presence of a strong mean magnetic field. Similar to Alfvén waves, EMHD waves, or EMHD perturbations, propagate along magnetic field lines. Therefore, two types of EMHD waves can exist: waves moving parallel to and waves moving anti-parallel to the magnetic field lines. For energy cascade in EMHD turbulence, the relative amplitudes of opposite-traveling waves are important. When the amplitudes are balanced, we will see fully-developed forward cascade with a $k^{-7/3}$ energy spectrum and a scale-dependent anisotropy. On the other hand, when the amplitudes are imbalanced, we will see inverse cascade, as well as (presumably not fully developed) forward cascade. The underlying physics for the inverse cascade is magnetic helicity conservation.

1. Introduction
In the solar wind, it is well known that the observed magnetic energy spectrum shows a break near the ion gyro-scale [1]. Above the break, the spectrum is compatible with a $k^{-5/3}$ spectrum and, below the break, the spectrum is steeper than the $k^{-5/3}$ spectrum. In this paper, we focus on the nature of turbulence below the break.

Magnetohydrodynamics (MHD) is a powerful and relatively simple tool for physics on scales larger than the ion gyro-scale. Indeed, the observed $k^{-5/3}$ spectrum above the break is consistent with a theoretical prediction for MHD turbulence in a strongly magnetized medium [2]. However, it is obvious that MHD is invalid below the break because the fluid approximation, in which MHD is based on, is invalid there.

Although fluid approximation is violated below the break, it is still possible to construct a fluid-like model for the small scales. Electron MHD (EMHD) model [3] is the simplest model among such fluid-like models for the small scales. The most fundamental assumption of the EMHD model is that, below the ion gyro-scale, ions provide only smooth charge background and electron ‘fluid’ moves and carries current.

As in MHD, waves propagate along magnetic field lines in the presence of a strong mean magnetic field and their interactions are important for turbulence in EMHD. Since perturbations propagate along magnetic field lines in EMHD, two types of waves can exist: waves moving parallel to and waves moving anti-parallel to the magnetic field lines. Then two types of interactions can exist: interactions between counter-propagating waves and self-interactions between waves moving in one direction. Note that interactions of the latter type are absent in MHD, but they are present in EMHD.
Throughout the paper, let us call EMHD waves moving parallel to the magnetic field `positive' waves and those moving anti-parallel to it `negative' waves. Then, let us consider a collection of `positive' and `negative' EMHD wave packets with similar sizes. In this case, the relative amplitudes of the positive and the negative wave packets are important. In case two types of amplitudes are balanced, numerical studies have revealed that interactions between opposite-traveling wave packets are more important than interactions between wave packets moving in the same direction [4, 5]. On the other hand, if there is substantial imbalance in their amplitudes, then interactions between dominant wave packets, i.e. either positive wave packets or negative wave packets that have larger amplitudes, become important and result in inverse cascade [6, 7, 8].

In this paper, we review the scaling relations of forward cascade in balanced EMHD turbulence and the nature of inverse cascade in imbalanced EMHD turbulence. We describe numerical method in Section 2. We discuss properties of EMHD waves and their implications for turbulence cascade in Section 3. We present scaling relations and numerical results for forward cascade in Section 4 and for inverse cascade in Section 5. We give summary in Section 6.

2. Numerical Methods

2.1. Incompressible EMHD equation

In this subsection, we briefly describe derivation of the incompressible EMHD equation [3]. The EMHD model deals with physics below the ion gyro-scale and the most important assumption of the model is that the ions create only smooth motionless charge background. Therefore, in the EMHD model, we consider motions of electrons only (see Figure 1). Since motions of electrons create current density, we can write

\[ \mathbf{v}_e = -\frac{\mathbf{J}}{n_e e} = -\frac{c}{4\pi n_e e} \nabla \times \mathbf{B}, \]  

where \( \mathbf{v}_e \) is the electron velocity, \( \mathbf{J} \) is the electric current density, \( \mathbf{B} \) is the magnetic field, \( n_e \) is the electron number density, and \( e \) is the absolute value of the electric charge.

The starting point for deriving the EMHD equation is the magnetic induction equation:

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_e \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \]  

where we consider motions of electrons only and \( \eta \) is magnetic diffusivity. Substituting equation (1) into equation (2), we obtain the incompressible EMHD equation

\[ \frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi n_e e} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta \nabla^2 \mathbf{B}. \]  

2.2. Numerical method

We use a pseudospectral code to solve the normalized 3-dimensional incompressible EMHD equation in a periodic box of size \( 2\pi \):

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta' \nabla^2 \mathbf{B}, \]  

where the magnetic field, time, and length are normalized by a mean field \( B_0 \), the whistler time \( t_w = L^2(\omega_{pe}/c)^2/\Omega_e \) (\( \omega_{pe} \) = electron plasma frequency; \( \Omega_e = \) electron gyrofrequency), and a characteristic length scale \( L \) [9]. The resistivity \( \eta' \) in equation (4) is dimensionless. The magnetic field consists of a uniform background field and a fluctuating field: \( \mathbf{B} = B_0 + \mathbf{b} \). The strength of the uniform background field, \( B_0 \), is set to 1. We use either 256\(^3\) or 512\(^3\) grid points.
In EMHD, ions are assumed to provide uniform charge background below the ion gyro-scale. Therefore, only electrons move and carry current, which results in $J \propto -v_e$.

Hyperdiffusivity is used for the diffusion term. The power of hyperdiffusivity is set to 3, so that the dissipation term in the above equation is replaced with $\eta_3 (\nabla^2)^3 \mathbf{B}$, where the value of $\eta_3$ is approximately $3 \times 10^{-10}$ for $256^3$ and $1 \times 10^{-11}$ for $512^3$.

Throughout the paper, we use the following convention: the field $\mathbf{B}$ actually means the magnetic field divided by $(4\pi \rho)^{1/2}$, where $\rho$ is density. Thus, the field $\mathbf{B}$ is, in fact, the Alfvén velocity, although we refer to it `magnetic field’ in some places.

3. Properties of EMHD waves and turbulence cascade

3.1. Properties of EMHD waves

In some respects, EMHD waves are similar to Alfvén waves. For example, they propagate along magnetic field lines. But, in many respects, EMHD waves are different from Alfvén waves. In this subsection, we briefly discuss properties of EMHD waves (see Ref. [8] for further discussions).

First, EMHD waves are dispersive. From equation (4), we can show that the dispersion relation for an EMHD wave is $\omega \propto k k_\parallel B_0$, where $k$ is the wavenumber, $k_\parallel$ is the wavenumber parallel to the mean magnetic field, and $B_0$ is the Alfvén speed of the background magnetic field [11, 9, 7]. The dispersion relation indicates that an EMHD wave packet of size $l$ ($\sim 1/k$) propagates along the mean magnetic field at the speed of

$$V_w \propto k B_0.$$  

(5)

If we consider a group of different-size wave packets traveling in one direction along the mean magnetic field, the dispersion relation implies that smaller wave packets are faster than larger ones. Therefore, EMHD wave packets moving in the same direction can collide each other and produce small-scale structures through nonlinear interactions during collisions [12, 7].

Second, an EMHD wave is circularly polarized [13, 14] and it has a non-zero magnetic helicity. The sign of magnetic helicity depends on the direction of propagation. A ‘positive’ wave, which moves in the positive direction along the magnetic field, has a positive magnetic helicity. In fact, the magnetic helicity of a positive wave is $b^2 / k$, where $k$ is the wave number and $b$ is the amplitude of the wave, which allows us to write

$$H(k) = E_b(k) / k,$$  

(6)
where $H(k)$ is the magnetic helicity spectrum and $E_b(k)$ is the magnetic energy spectrum. On the other hand, a ‘negative’ wave, which moves in the negative direction along the magnetic field, has a negative magnetic helicity. Therefore, if there is imbalance in the amplitudes of positive and negative EMHD wave packets with similar sizes, then the net magnetic helicity of the system is not zero. For example, if positive wave packets have larger amplitudes than negative wave packets, then the net magnetic helicity is positive.

3.2. Magnetic helicity and turbulence cascade

Magnetic helicity is a more strictly conserved quantity than magnetic energy: it is very difficult to destroy magnetic helicity even in the presence of turbulence, while it is relatively easy to destroy magnetic energy through turbulence energy cascade. The reason for this is that magnetic helicity spectrum is steeper than magnetic energy spectrum.

Magnetic helicity conservation can play an important role in energy cascade in EMHD turbulence. Let us first consider two interacting wave packets moving in the same direction. We assume that their sizes are similar and of order $l$ ($\sim 1/k$). Let the magnetic energy of each packet be $\sim b^2$. Then, as we discussed in the previous subsection, each wave packet has magnetic helicity of order $lb^2$. The sign of magnetic helicity of the packets is same since they are moving in the same direction. Nonlinear interactions occur when two wave packets collide. Due to magnetic helicity conservation, the net magnetic helicity before and after the collision is unchanged and is of order

$$\sim 2lb^2.$$ (7)

When two wave packets interact, their energy should ultimately decrease. If this indeed happens, then the scale $l$ should increase in order to conserve magnetic helicity. Therefore we expect inverse cascade, i.e. increase of the typical scale of wave packets. Note that the underlying physics for inverse cascade is magnetic helicity conservation.

In the previous paragraph, we considered interactions between wave packets moving in the same direction. Such interactions are important in imbalanced EMHD turbulence. In imbalanced turbulence, waves packets moving in one direction are stronger than those moving in the other direction. Therefore, imbalanced EMHD turbulence has two important properties. First, in imbalanced EMHD turbulence, it is possible that interactions between dominant wave packets are more efficient than interactions between dominant and sub-dominant wave packets. Second, in imbalanced EMHD turbulence, the net magnetic helicity is not zero. From the discussion in the previous paragraph, we expect inverse cascade in imbalanced EMHD turbulence due to conservation of magnetic helicity. We will discuss scaling relations of inverse cascade in imbalanced EMHD turbulence in Section 5.

Now let us consider balanced EMHD turbulence, in which opposite-traveling wave packets with similar sizes have similar amplitudes. In this case, interactions between two opposite-traveling wave packets with similar sizes should be more efficient than interactions between wave packets moving in the same direction. If we have two balanced opposite-traveling wave

1 If magnetic energy spectrum $E_b(k)$ is proportional to $k^{-m}$ ($k_p \leq k \leq k_d$), then we have $b^2/(\bar{b}^2) \propto k_pE_b(k_p)/(k_dE_b(k_d)k_d^2) \propto (k_d/k_p)^{m-1}$, where we use the fact $b^2 \sim kE(k)$ (with $l \sim 1/k$). Note that $m \geq 1$ for most cases (e.g. $m=5/3$ for a Kolmorogov spectrum). Similarly, if magnetic helicity spectrum $H(k)$ is proportional to $k^{-m_h}$ ($k_p \leq k \leq k_d$), we can show that $h^2/(\bar{h}^2) \propto (k_d/k_p)^{m_h-1}$. Since magnetic helicity spectrum is steeper than magnetic energy spectrum (see, for example, equation (6)), we have $m_h > m$ and $h^2/(\bar{h}^2) > b^2/(\bar{b}^2)$. Therefore, dissipation timescale for magnetic helicity is relatively larger than that for magnetic energy.

2 Note that nonlinear interactions between wave packets occur when they collide and, hence, same-size wave packets moving in the same direction do not interact efficiently because their propagation speed is same. Therefore, interactions between wave packets moving in the same direction happen between wave packets with slightly different sizes.

3 Note that, in the latter, wave packets with slightly different sizes interacts. Therefore, if locality of interaction
packets with size \( \sim l \) and energy \( \sim b^2 \), the net magnetic helicity is close to zero: \( lb^2 - lb^2 = 0 \). When they collide, either forward cascade or inverse cascade is allowed because it does not change the net magnetic helicity. Note that the net magnetic helicity after collision will still be close to zero as long as wave packets remain balanced. We know that collision of two balanced opposite-traveling wave packets results in forward cascade in MHD turbulence. Therefore, we expect that interactions between opposite-traveling wave packets in EMHD turbulence are likely to produce forward cascade. We will discuss scaling relations of forward cascade in balanced EMHD turbulence in Section 4.

4. Forward cascade in EMHD

4.1. Scaling Relations

In this subsection, we derive scaling relations of forward cascade in EMHD turbulence. As we mentioned in the previous section, the following scaling relations are based on interactions between opposite-traveling wave packets and will be valid for balanced EMHD turbulence.

The magnetic energy spectrum for forward cascade in EMHD turbulence has been known since 1970’s [15, 11, 16] and scaling relations based on ‘critical balance’ [2] were first derived and numerically tested by Cho & Lazarian [4]. We can re-derive the scaling relations in a more intuitive way as follows: Let us consider two EMHD wave packets in a medium threaded by a uniform magnetic field \( B_0 \). We assume that the packets are moving in opposite directions (see Figure 2). Note that, a strong (local) mean field naturally makes EMHD turbulence anisotropic: eddies are elongated along the (local) mean-field direction [17]. Therefore, in Figure 2, we draw that \( l_\parallel > l_\perp \), where \( l_\parallel \) and \( l_\perp \) are the parallel and the perpendicular sizes of an eddy, respectively. Here ‘parallel’ and ‘perpendicular’ refer to the directions with respect to the (local) mean magnetic field.

As in most turbulence theories, let us assume locality of interactions: only collisions between similar-size eddies are important for energy cascade (see, however, Refs. [18] and [19]). When two oppositely traveling wave packets (of size \( l_\perp \) and \( l_\parallel \)) collide, one wave packet provides shear and, thus, distortion to the other (Figure 2). The time scale for complete distortion is of the order \( t_{\text{eddy}} \sim l_\perp/v_\perp \), where \( v_\perp \) is the fluctuating velocity on the scale \( l = l_\perp \). If the strengths of the fluctuating magnetic field is \( b_\parallel \), the fluctuating velocity \( v_\perp \) is roughly

\[
\mathbf{v} \propto \nabla \times \mathbf{b} \rightarrow v_\perp \propto b_\parallel/l_\perp.
\]

Therefore, we have

\[
t_{\text{eddy}} \sim l_\perp/v_\perp \propto l_\perp^2/b_\parallel.
\]

On the other hand, the duration of the collision is

\[
t_{\text{coll}} \sim l_\parallel/V_w \propto l_\parallel/(kB_0).
\]

As in the MHD turbulence case (see Ref. [2]), we can argue that there is a regime of turbulence where

\[
t_{\text{eddy}} \sim t_{\text{coll}}, \quad \text{or} \quad l_\perp B_0/(l_\parallel b_\parallel) \sim 1,
\]

which is known as ‘critical balance’. This type of turbulence is called ‘strong’ turbulence. Note that, in strong EMHD turbulence, the time scale for energy cascade, \( t_{\text{cas}} \), can be either the collision time \( t_{\text{coll}} \) or the eddy turnover time \( t_{\text{eddy}} \). When we combine this condition and the constancy of energy cascade rate,

\[
b_\parallel^2/t_{\text{cas}} \sim \text{constant},
\]

is a valid assumption in EMHD, we expect interactions between wave packets moving in one direction are less efficient than interactions between opposite-traveling wave packets.
At \( t \approx 0 \), only small-wavenumber Fourier modes are excited. At later times, energy gradually cascades down to large \( k \) regions and a \( k^{-7/3} \) inertial range develops. From [5].

we get the scaling relations

\[
E_b(k) \propto k^{-7/3} \quad \text{and} \quad l_\parallel \propto l_\perp^{1/3}, \quad \text{(or, } k_\parallel \propto k_\perp^{1/3})
\]

where the latter implies scale-dependent anisotropy: smaller eddies are more elongated. Here, \( k_\perp \) is the wavenumber perpendicular to the (local) mean magnetic field. Here, we used \( b^2 \sim kE_b(k) \).

Compared with the Goldreich-Sridhar scaling relations [2] for MHD turbulence, the energy spectrum is steeper and the anisotropy is stronger in EMHD turbulence.

### 4.2. Numerical results

Figure 3 shows magnetic energy spectrum in a decaying EMHD simulation with 512\(^3\) grid points (see Ref. [5]). At \( t=0 \), Fourier modes with \( 2 \leq k < 5 \) are excited. Here \( k = \sqrt{k_\parallel^2 + k_\perp^2} \) and, thus, the initial perturbation is isotropic (i.e. \( k_\parallel \sim k_\perp \)). The strength of the mean magnetic field (\( B_0 \)) is 1 and the rms value of the random magnetic field (\( b \)) is \( \sqrt{2} \). Therefore, the critical balance, \( bk_\perp/B_0k_\parallel = 1 \), is roughly satisfied at \( t=0 \). Figure 3 shows how the inertial range develops. The dashed curve in Figure 3 shows the initial spectrum. As the turbulence decays, the initial energy cascades down to smaller scales and, as a result, small-scale (i.e. large-\( k \)) modes are excited. When the energy reaches the dissipation scale at \( k \approx 100 \), the whole energy spectrum goes down without changing its slope (the dotted and the solid curves). The magnetic energy spectrum at this stage is very close to that of the predicted one:

\[
E_b(k) \propto k^{-7/3}.
\]

For the study of anisotropy, we use the data cube at \( t \approx 0.46 \). The solid curve in left panel of Figure 3 represents the spectrum at that time. Figure 4 shows the relation between \( k_\parallel \) and \( k_\perp \) (see [5] for the calculation method). The result is compatible with the expected anisotropy

\[
k_\parallel \propto k_\perp^{1/3}.
\]

### 5. Inverse cascade in EMHD

As we mentioned earlier, imbalanced EMHD turbulence shows inverse cascade. Let us consider two exemplary cases: freely decaying completely imbalanced EMHD turbulence and EMHD turbulence with completely imbalanced driving. Note that “completely imbalanced” in the former means that all EMHD waves are freely propagating in one direction at \( t=0 \) and “completely imbalanced driving” in the latter means ether positive waves or negative waves...
are driven. For the sake of simplicity, let us assume that only the positive waves are present at 
t=0 in the former case and only the positive waves are driven in the latter case.

In either case, the positive EMHD waves can interact with each other, generate opposite-traveling negative waves, develop small-scale structures, and dissipate magnetic energy. At the same time, they show inverse cascade due to magnetic helicity conservation. Note that magnetic helicity stays constant in the former case and it increases linearly in time in the latter case. In this section, we consider consequences of magnetic helicity conservation. Most discussions in the section are from Refs. [7, 8].

5.1. Freely decaying completely imbalanced EMHD turbulence

Let us first consider freely decaying completely imbalanced EMHD waves (i.e. EMHD waves moving in one direction at t=0). Suppose that they have a power-law energy spectrum between 
k = k_p and k_d:

$$E_b(k) \propto k^{-m}, \quad k_p \leq k \leq k_d,$$

where $k_d$ is the dissipation scale. Note that, even though they do not have a power-law spectrum at 
t=0, they develop a rough power-law-like spectrum at later times (see Figure 5). Then the magnetic energy and the magnetic helicity densities are approximately $\sim k_p E_b(k_p)$ and 
$\sim E_b(k_p)$, respectively. As the EMHD waves propagate, the helicity density stays the same, so that 
$E_b(k_p)$ is approximately constant. On the other hand, if there is no driving, the energy density, 
$\sim k_p E_b(k_p)$, decreases due to dissipation. Therefore $k_p$ should decrease, which means 
that the peak of the energy spectrum moves to smaller wavenumbers.

Numerical simulations confirm the theoretical expectations in the previous paragraph. Figure 5 
shows magnetic energy spectra of the dominant waves in a simulation with a completely 
imbalanced initial condition. At $t = 0$, Fourier modes between $k = 8$ and $k = 15$ are excited. 
The dotted curve in Figure 5 shows the initial spectrum. As time goes on, the initial energy 
cascades down to smaller scales and, as a result, a power-law-like spectrum forms for $k > 15$. 
But the magnetic energy spectrum for $k > 15$ is slightly steeper than $k^{-7/3}$. At the same time the 
peak of the energy spectrum moves to larger scales, so that the wavenumber at which the 
spectrum peaks, $k_p$, gets smaller. We clearly observe inverse cascade of magnetic energy.
5.2. EMHD turbulence with completely imbalanced driving

Now, let us consider EMHD turbulence with completely imbalanced driving. When driving injects magnetic energy into waves moving only in one direction, it also injects magnetic helicity into the system. Because the magnetic helicity is a well-conserved quantity, the net magnetic helicity density grows linearly in time in driven imbalanced EMHD turbulence:

\[ H_{\text{net}} = \epsilon_h t \approx (\epsilon_b/k_f) t, \]  

(17)

where \( k_f \) is the driving-scale wavenumber, \( \epsilon_h \) is the magnetic helicity injection rate, and \( \epsilon_b \) is the magnetic energy injection rate. On the other hand, the magnetic energy density follows

\[ b^2 = \epsilon_b - D_M < \epsilon_b \]  

\[ \rightarrow b^2 < \epsilon_b t, \]  

(18)

where \( D_M \) is the magnetic energy dissipation rate.

If the magnetic spectrum has a peak at \( k_p \), then from equations (17) and (18) we have

\[ k_p E_b(k_p) \approx b^2 \approx \epsilon_b t \approx k_f H_{\text{net}} \approx k_f E_b(k_p), \]  

(19)

which implies

\[ k_p < k_f. \]  

(20)

Here we used \( H_{\text{net}} \approx E_b(k_p) \). We can show that \( k_p \) cannot be a fixed wavenumber: if we assume that \( k_p \) does not change in time, we will have a contradictory result. Recall that we are considering EMHD turbulence with completely imbalance driving. In this case, most of the energy near \( k = k_p \) resides in the dominant waves, which naturally produce sub-dominant waves. Suppose that an amount of magnetic energy is transferred from the dominant to the sub-dominant waves near \( k = k_p \). As a result of the energy transfer, the magnetic energy \( \sim k_p E_b(k_p) \) of the dominant waves near \( k = k_p \) decreases, while the magnetic helicity \( \sim E_b(k_p) \) of the dominant waves near \( k = k_p \) increases to compensate magnetic helicity transfer to the sub-dominant waves\(^4\). Therefore, \( k_p \) cannot be a constant and should decrease. In summary, when a continuous magnetic helicity injection is present, we expect that the peak of magnetic energy spectrum moves to smaller wavenumbers.

In numerical simulations of driven imbalanced EMHD turbulence, we can clearly see inverse cascade [8]. Figure 6 shows magnetic energy spectrum of the dominant waves in a simulation with completely imbalanced driving. The driving wavenumber \( k_f \) is approximately 40. The small-scale spectra of the dominant waves are very steep, partly due to our limited numerical resolution of 256\(^3\). The large-scale magnetic energy spectrum clearly shows inverse cascade: the peak of the spectrum gradually moves to larger scales. Although we do not observe a well-defined power-law spectrum at a given time, the location of the peaks follows \( E_b(k_p) \propto k^{-3/2} \). The fact that the magnetic energy spectrum for \( k < k_f \) does not follow a single power-law implies that inverse cascade in EMHD turbulence is not a self-similar process. Therefore, the nature of the inverse cascade in EMHD turbulence can be very different from cascades in other types of turbulence (see [8] for details).

\(^4\) When magnetic energy \( \Delta E \) is transferred from the positive waves near \( k = k_p \) to the negative waves near \( k = -k_p \), the magnetic helicity of the negative waves decreases: \( H^- \rightarrow H^- - \Delta E/k_p \). This is because the sign of magnetic helicity of the negative waves is negative and, hence, increase of magnetic energy in negative waves means decrease of magnetic helicity. The net magnetic helicity should not be changed by the energy transfer. Therefore, the magnetic helicity of the positive waves should increase: \( H^+ \rightarrow H^+ + \Delta E/k_p \).
6. Summary
Similar to Alfvén waves in MHD, EMHD waves propagate along magnetic field lines. Therefore, two types EMHD waves can exist: waves moving parallel to and waves moving anti-parallel to the magnetic field. The relative amplitudes of these two types of waves are important for turbulence cascade. In case their amplitudes are balanced, we observe efficient forward cascade. On the other hand, in case there is large imbalance in their amplitudes, inverse cascade becomes important.

Forward cascade in balanced EMHD turbulence has a steep magnetic energy spectrum, \( E_b(k) \propto k^{-7/3} \), and a strong anisotropic eddy shapes, \( l_\parallel \propto l_\perp^{1/3} \), where \( l_\parallel \) and \( l_\perp \) are the size of an eddy parallel and perpendicular to the (local) mean magnetic field, respectively. Inverse cascade in imbalanced EMHD turbulence does not show a well-defined power-law spectrum at a given time for the inverse cascade. However, the location of the peaks at different times follows \( E_b(k_p) \propto k_p^{-3/2} \), where \( k_p \) is the wavenumber at which the magnetic energy spectrum peaks.

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