MULTIPLICATIVE FUNCTIONS RESEMBLING THE MÖBIUS FUNCTION

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Abstract. A multiplicative function $f$ is said to be resembling the Möbius function if $f$ is supported on the square-free integers, and $f(p) = \pm 1$ for each prime $p$. We prove $O$- and $\Omega$-results for the summatory function $\sum_{n \leq x} f(n)$ for a class of these $f$ studied by Aymone, and the point is that these $O$-results demonstrate cancellations better than the square-root saving. It is proved in particular that the summatory function is $O(x^{1/3+\varepsilon})$ under the Riemann Hypothesis. On the other hand it is proved to be $\Omega(x^{1/4})$ unconditionally. It is interesting to compare these with the corresponding results for the Möbius function.

1. Introduction

The Möbius function $\mu$ is an arithmetic multiplicative function supported on square-free integers with $\mu(p) = -1$ for each prime $p$. Thus $\mu(1) = 1$, $\mu(n) = 0$ if $n$ is divided by a square of prime, and $\mu(n) = (-1)^k$ if $n$ is a product of $k$ distinct prime factors. The role of $\mu$ is central in the theory of prime numbers, and an evidence is that the Riemann Hypothesis is equivalent to the estimate $M(x) \ll x^{1/2+\varepsilon}$ for arbitrary $\varepsilon > 0$, where $M(x)$ denotes the summatory function of $\mu$, that is

$$M(x) = \sum_{n \leq x} \mu(n).$$

It is also known, unconditionally, that $M(x) = \Omega(\sqrt{x})$ with the $\Omega$-symbol being defined as the negation of the $o$-symbol. All these are classical, and the reader is referred to Titchmarsh [11, Chap. 14] for details. The best-known upper bound estimate for $M(x)$ is obtained in Soundararajan [10].

A random model of the Möbius function was proposed by Wintner [12], and studied by many authors from various perspectives. An arithmetic function $f$ is said to be a random multiplicative function if

(i) $f$ is multiplicative, and supported on the square-free integers;

Date: June 10, 2022.

2000 Mathematics Subject Classification. 11M26, 11N37.

Key words and phrases. Möbius function, random multiplicative function, zeta-function.
(ii) \((f(p))_p\) is an independent sequence of random variables with distribution

\[
P(f(p) = 1) = P(f(p) = -1) = \frac{1}{2},
\]

where \(p\) runs over the set of primes.

Denote by \(M_f(x)\) the summatory function

\[
M_f(x) = \sum_{n \leq x} f(n).
\]

Then a theorem of Wintner states that \(M_f(x) \ll x^{1/2+\varepsilon}\) almost surely, and also \(M_f(x) = \Omega(x^{1/2-\varepsilon})\) almost surely. For developments concerning the above \(\ll\) and \(\Omega\)-results of \(M_f\), see Halász [2], Lau-Tenenbaum-Wu [7], and Harper [3] and the references therein.

An arithmetic function \(f\) is said to be resembling the Möbius function if

(i) \(f\) is multiplicative, and supported on the square-free integers,

(ii) \(f(p) = \pm 1\) for each prime \(p\).

It is therefore an interesting problem whether there is an \(f\) resembling the Möbius function, such that its summatory function satisfies \(M_f(x) = o(\sqrt{x})\). Aymone [1] has studied this problem and provided a class of examples of these \(f\). Let \(\chi\) be a real non-principal Dirichlet character modulo \(q \geq 3\), and define

\[
g_\chi(p) = \begin{cases} 
\chi(p), & \text{if } p \nmid q; \\
1, & \text{if } p | q.
\end{cases}
\]

This \(g_\chi\) extends to a completely multiplicative function \(g_\chi : \mathbb{N} \to \{-1, 1\}\). The example discovered in [1] is \(f = \mu^2 g_\chi\), where clearly \(f\) is resembling the Möbius function. It is proved in [1] that, under the condition

\[
\sum_{p \leq x} |1 - f(p)\chi(p)| \ll \sqrt{x} \exp(-c\sqrt{\log x}),
\]

one has

\[
M_f(x) \ll \sqrt{x} \exp(-c(\log x)^{1/4})
\]

where here and throughout \(c\) stands for positive constants not necessarily the same at each occurrence; while under the Riemann Hypothesis for the zeta-function,

\[
M_f(x) \ll x^{2/5+\varepsilon}
\]

for arbitrary \(\varepsilon > 0\).

The purpose of this paper is to improve on these, and also prove an \(\Omega\)-result. Our main results are stated in the following two theorems.
Theorem 1.1. Let $\chi$ be a real non-principal Dirichlet character modulo $q \geq 3$, and $g_\chi$ the completely multiplicative function extended from $\chi$ as in (1.1). Let $f = \mu^2 g_\chi$. Then

(i) there exists a positive constant $c$ such that

$$M_f(x) \ll x^{1/2} \exp \left( -c \frac{(\log x)^{3/5}}{((\log \log x)^{1/5}} \right);$$

(ii) under the Riemann Hypothesis for the zeta-function,

$$M_f(x) \ll x^{1/3+\varepsilon}.$$ 

Theorem 1.1 improves (1.3) without applying the additional condition (1.2), and improves (1.4) under the same condition. Theorem 1.1 can be compared with the $\Omega$-result below.

Theorem 1.2. Let $\chi$ be a real non-principal Dirichlet character modulo $q \geq 3$, and $g_\chi$ the completely multiplicative function extended from $\chi$ as in (1.1). Let $f = \mu^2 g_\chi$. Then

$$M_f(x) = \Omega(x^{1/4}).$$

Note that Theorem 1.2 is unconditional. The same result has been proved by Aymone [1] under the Riemann Hypothesis for $L(s, \chi)$. Unconditionally Klurman et al. [6] have proved that $M_f(x) = \Omega(x^{1/4-\varepsilon})$ for arbitrary $\varepsilon > 0$.

We use standard notations in number theory. The Riemann Hypothesis for the zeta-function means that all the non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. The conductor $q$ of $\chi$ is considered as fixed, and therefore some $\ll$ or $O$-constants may depends on $q$, but we do not make these dependences explicit. The expression $f(x) = \Omega(g(x))$ means the negation of the estimate $f(x) = o(g(x))$, that is the inequality $|f(x)| \geq cg(x)$ is satisfied for some arbitrarily large values of $x$. The letter $c$ stands for positive constants not necessarily the same at each occurrence, and $s = \sigma + it$ denotes a complex variable. The letter $\varepsilon > 0$ stands for a real number arbitrarily small, not necessarily the same at each occurrence.

2. Preparations

To prove Theorems 1.1 and 1.2, the first step is to compute the generating function for $f$. Recall that $f = \mu^2 g_\chi$ and $g_\chi$ is the completely multiplicative function extended from
\( \chi \) as in (1.1). Write \( s = \sigma + it \) as usual. Then in the half-plane \( \sigma > 1 \) we have
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left( 1 + \frac{f(p)}{p^s} \right) = \prod_p \left( 1 + \frac{g_{\chi}(p)}{p^s} \right) \\
= \prod_{p \nmid q} \left( 1 + \frac{1}{p^s} \right) \prod_{p \mid q} \left( 1 + \frac{\chi(p)}{p^s} \right).
\]
Since \( \chi \) is real, we have \( \chi(p)^2 = 1 \) for \( p \nmid q \), and hence
\[
\left( 1 + \frac{\chi(p)}{p^s} \right) = \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1} \left( 1 - \frac{1}{p^{2s}} \right).
\]
It follows that
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{L(s, \chi)P(s)}{\zeta(2s)}
\]
where
\[
P(s) = \prod_{p \nmid q} \left( 1 - \frac{1}{p^s} \right)^{-1}.
\]
The function \( P \) above has no zero, and has infinitely many poles at
\[
\rho = i \frac{2\pi j}{\log p}, \quad p\mid q, \quad j = 0, \pm 1, \pm 2, \ldots \quad (2.3)
\]
which are lying on the imaginary axis. The pole with \( j = 0 \) is of order \( \omega(q) \) where \( \omega(q) \) denotes the number of distinct prime divisors of \( q \), while other poles are simple. The formula (2.1) holds in the half-plane \( \sigma > 1 \), while the function on the right-hand side of (2.1) is meaningful in the whole complex plane by the functional equations of \( \zeta(s) \) and \( L(s, \chi) \).

This section is devoted to lemmas that are needed for Theorem 1.1.

**Lemma 2.1.** Let \( P \) be as in (2.2), and \( h(n) \) defined as the coefficients in the Dirichlet series expression
\[
P(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}
\]
which holds in the half-plane \( \sigma > \frac{1}{2} \). Denote by \( M_h(x) \) the summatory function \( \sum_{n \leq x} h(n) \). Then there exists a positive constant \( c \) such that
\[
M_h(x) \ll x^{1/2} \exp \left( -c \left( \log x \right)^{3/5} \left( \log \log x \right)^{1/5} \right).
\]
Proof. By definition (2.4),
\[ h(n) = \sum_{d \leq n, \text{gcd}(d, n) = 1} \mu(d) \]
where the condition "\( p | d \Rightarrow p | q \)" means that all the prime divisors of \( d \) are divisors of \( q \). It follows that
\[ M_h(x) = \sum_{d \leq x, \text{gcd}(d, x) = 1} \sum_{m^2 \leq x/d} \mu(m). \]

Now we invoke the estimate (see for example [5, Theorem 12.7])
\[ \sum_{m \leq y} \mu(m) \ll y \exp \left( -c \frac{(\log y)^{3/5}}{(\log \log y)^{1/5}} \right) \] (2.5)
which follows from the Vinogradov-Korobov zero-free region of the Riemann zeta-function. Writing \( \Phi(y) \) for the function on the right-hand side of (2.5), we see that \( \Phi(y) \) is increasing for \( y \geq 100 \), say. Thus, for \( \sqrt{x/d} \geq 100 \),
\[ \sum_{m^2 \leq x/d} \mu(m) \ll \Phi \left( \sqrt{\frac{x}{d}} \right) \ll \Phi(\sqrt{x}). \]
The above sum is bounded for \( \sqrt{x/d} \leq 100 \), and therefore
\[ M_h(x) \ll \Phi(\sqrt{x}) \sum_{d \leq x, \text{gcd}(d, x) = 1} 1. \] (2.6)

We are going to prove that
\[ \sum_{d \leq x, \text{gcd}(d, x) = 1} 1 \leq (\log x)^{\omega(q)} \] (2.7)
where \( \omega(q) \) is the number of distinct prime divisors of \( q \). We let \( \omega(q) = k \) and let \( q = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) be the canonical decomposition of \( q \). The condition "\( p | d \Rightarrow p | q \)" means that \( d \) must be of the form \( d = p_1^{\beta_1} \cdots p_k^{\beta_k} \) where \( \beta_1, \ldots, \beta_k \) are nonnegative integers. Note that \( \beta_j \) may exceed \( \alpha_j \), but they must be bounded from above as
\[ \beta_j \leq \frac{\log x}{\log p_j} \leq \log x, \]
since \( p_j^{\beta_j} \leq x \) for each \( j \). This proves (2.7). The assertion of the lemma is a consequence of (2.6) and (2.7). \( \square \)
Lemma 2.2. Let \( h(n) \) be as in Lemma 2.1. Then
\[
\sum_{n \leq x} |h(n)| \ll x^{1/2} (\log x)^{\omega(q)},
\]
where \( \omega(q) \) denotes the number of different prime divisors of \( q \).

Proof. By (2.4),
\[
|h(n)| = \left| \sum_{dm^2 = n \atop p|d \Rightarrow p|q} \mu(m) \right| \leq \sum_{dm^2 = n \atop p|d \Rightarrow p|q} |\mu(m)|.
\]
(2.8)

Taking summation over \( n \), we have
\[
\sum_{n \leq x} |h(n)| \ll \sum_{n \leq x} \sum_{dm^2 = n \atop p|d \Rightarrow p|q} |\mu(m)| \ll \sum_{m \leq \sqrt{x}} \sum_{d \leq x/m^2 \atop p|d \Rightarrow p|q} 1.
\]
The assertion of the lemma now follows from this and (2.7).

3. Proof of Theorem 1.1(i)

Proof of Theorem 1.1(i). We start from (2.1) and (2.4) to get
\[
f(n) = \sum_{dm = n} \chi(d)h(m),
\]
and hence
\[
M_f(x) = \sum_{n \leq x} f(n) = \sum_{dm \leq x} \chi(d)h(m)
\]
\[
= \left( \sum_{dm \leq x \atop d \leq D} + \sum_{dm \leq x \atop m \leq M} - \sum_{dm \leq x \atop d \leq D} \right) \chi(d)h(m)
\]
\[
= S_1 + S_2 - S_3,
\]
say. Here \( D \) and \( M \) are parameters satisfying \( DM = x \) but to be decided later. We start from
\[
S_1 \ll \sum_{d \leq D} \left| \sum_{m \leq x/d} h(m) \right|,
\]
and then apply Lemma 2.1 to the inner sum. Note that the function
\[
\exp \left( -c \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right)
\]
is decreasing. By Lemma 2.1,

\[ S_1 \ll \sum_{d \leq D} \left( \frac{x}{d} \right)^{1/2} \exp \left( -c \frac{(\log \frac{x}{D})^{3/5}}{(\log \log \frac{x}{D})^{1/5}} \right) \]

\[ \ll D^{1/2} x^{1/2} \exp \left( -c_1 \frac{(\log D)^{3/5}}{(\log \log D)^{1/5}} \right) \] (3.2)

for some positive constant \( c_1 \). Now we invoke the trivial bound \(|\sum_{d \leq x} \chi(d)| \leq q\) as well as Lemma 2.2 to get

\[ S_2 = \sum_{m \leq M} h(m) \sum_{d \leq x/m} \chi(d) \ll \sum_{m \leq M} |h(m)| \ll M^{1/2} \log^{\omega(q)} x, \] (3.3)

where we recall that in this paper \( q \) is treated as a constant. Furthermore, by Lemma 2.1

\[ S_3 = \left( \sum_{m \leq M} h(m) \right) \left( \sum_{d \leq D} \chi(d) \right) \ll \left| \sum_{m \leq M} h(m) \right| \ll M^{1/2}. \] (3.4)

It turns out that the optimal choice is

\[ M = x \exp \left( -\frac{c_1}{2} \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right) \]

and \( D = x/M \). This proves Theorem 1.1(i).

\[ \square \]

4. Further preparations

This section is devoted to lemmas that are needed for Theorem 1.1(ii). The first of these is Perron’s formula in the following form, which is a modification of Titchmarsh [11, Lemma 3.12].

**Lemma 4.1.** Let \( F(s) = \sum a_n n^{-s} \) be the Dirichlet series expression for \( F \) in the half-plane \( \sigma > 1 \), where \( a_n \ll \psi(n) \), \( \psi \) being non-decreasing, and

\[ \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} \ll \frac{1}{\sigma - 1} \] (4.1)

as \( \sigma \to 1^+ \). If \( b > 0, b + \sigma > 1, x \) is half an odd integer, then

\[ \sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s + w) \frac{x^w}{w} dw + O \left( \frac{x^b}{T(\sigma + b - 1)} \right) \]

\[ + O \left( \frac{\psi(2x)x^{1-\sigma} \log x}{T} \right). \] (4.2)
Lemma 4.2. Assume the Riemann Hypothesis for the zeta-function. Let \( m \geq 0 \) be a fixed integer. Then in the half-plane \( \sigma \geq \frac{1}{2} + \delta \) we have
\[
\zeta^{(m)}(s) \ll (|t| + 2)^\varepsilon
\]
and
\[
\frac{d^m}{ds^m} \left( \frac{1}{\zeta(s)} \right) \ll (|t| + 2)^\varepsilon
\]
where \( \delta > 0 \) and \( \varepsilon > 0 \) are arbitrarily small. We may require that \( \delta \leq \varepsilon \).

Proof. This follows from \([11, \text{Theorem 14.14(B)}]\) and \([11, \text{Theorem 14.16}]\). \(\square\)

Lemma 4.3. Unconditionally we have, uniformly for \( \frac{1}{2} \leq \sigma \leq 2 \),
\[
\int_{-T}^{T} \frac{|L(\sigma + it, \chi)|}{|\sigma + it|} dt \ll \log T.
\]

Proof. By Cauchy’s inequality,
\[
\int_{-T}^{T} \frac{|L(\sigma + it, \chi)|}{|\sigma + it|} dt \ll \left( \int_{-T}^{T} \frac{1}{|\sigma + it|} dt \right)^{1/2} \left( \int_{-T}^{T} \frac{|L(\sigma + it, \chi)|^2}{|\sigma + it|} dt \right)^{1/2}.
\]
The first integral in braces is \( \ll \log T \). To bound the second integral in braces, we note that, uniformly for \( \frac{1}{2} \leq \sigma \leq 2 \),
\[
\int_{-T}^{T} |L(\sigma + it, \chi)|^2 dt \ll T \log T.
\]
It follows by partial integration that
\[
\int_{-T}^{T} \frac{|L(\sigma + it, \chi)|^2}{|\sigma + it|} dt \ll \log^2 T.
\]
This proves the lemma. \(\square\)

The above lemmas will be applied to estimate the tail in (2.4), that is
\[
Z_M(s) = \frac{P(s)}{\zeta(2s)} - \sum_{m \leq M} \frac{h(m)}{m^s}
\]
with \( M \geq 1 \) being a parameter, in the half-plane \( \sigma \geq \frac{1}{2} + \delta \). We observe that
\[
Z_M(s) = \sum_{m > M} \frac{h(m)}{m^s}
\]
in the half-plane \( \sigma \geq 1 + \delta \). In the above \( \delta > 0 \) is arbitrarily small.
Proposition 4.4. Assume the Riemann Hypothesis for the zeta-function. Let \( Z_M(s) \) be as in (4.3). Then, in the half-plane \( \sigma \geq \frac{1}{2} + \delta \) with \( \delta > 0 \) arbitrarily small, \( Z_M(s) \ll M^{1/4-\sigma} M^\varepsilon (|t| + 2)^\varepsilon \).

Proof. By definition (2.2),
\[
P(s) = \sum_{d \mid m^2 = n} \frac{1}{d^s}
\]
where the condition “\( p \mid d \Rightarrow p \mid q \)” means that all the prime divisors of \( d \) are divisors of \( q \). Hence by (2.4),
\[
|h(n)| = \left| \sum_{d \mid m^2 = n} \mu(m) \right| \leq \sum_{d \mid m^2 = n} |\mu(m)|.
\]
(4.5)
If we relax the constraint on \( d \), we have
\[
|h(n)| \leq \sum_{m^2 \mid n} |\mu(m)| \leq 2^{\omega(n)},
\]
where \( \omega \) is the number of distinct prime divisors of \( n \). A rough bound for the above is \( |h(n)| \ll n^\varepsilon \). It follows from (4.5) that
\[
\sum_{n=1}^{\infty} \frac{|h(n)|}{n^\sigma} \leq P(\sigma) \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{2\sigma}} \ll \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^\sigma} = \frac{\zeta(\sigma)}{\zeta(2\sigma)} \ll \frac{1}{\sigma - 1}
\]
(4.6)
as \( \sigma \to 1 \) from the right.

We apply Perron’s formula in the form of Lemma 4.1, in which we take \( \psi(n) = n^\varepsilon \) and \( b = 1 + \frac{1}{\log M} \). Then (4.2) becomes, for \( M \) being half an odd integer,
\[
\sum_{m \leq M} \frac{h(m)}{m^s} = \frac{1}{2\pi i} \int_{b-iV}^{b+iV} P(s+w) M^w \frac{M^w}{w} dw + O\left( \frac{M^{1+\varepsilon}}{V} \right)
\]
(4.7)
Write \( w = u + iv \) where \( u \) and \( v \) are the real and imaginary parts of \( w \) respectively. We shift the contour in (4.7) to the left till the vertical line \( u = a \) where
\[
a = \frac{1}{4} - \sigma + \delta
\]
(4.8)
with \( \delta > 0 \) arbitrarily small, passing the simple pole of the integrand at \( w = 0 \), and also poles at \( w = \rho - s \) with \( \rho \) as in (2.3) satisfying \( |v| \leq V \). Hence the integral in (4.7) can be written as
\[
\int_{b-iV}^{b+iV} = -\int_{C_1} + \int_{C_2} + \int_{a-iV}^{a+iV} + R
\]
(4.9)
where \( C_1 \) means the segment \( \{ w = u - iV : a \leq u \leq b \} \), \( C_2 \) means \( \{ w = u + iV : a \leq u \leq b \} \), and \( R \) denotes the summation of all the residues arising from the poles mentioned above.

We want to estimate the three integrals in (4.9), and to this end we need to estimate their integrands in the half-plane \( u \geq a \). Note that we are assuming that \( \sigma \geq \frac{1}{2} + \delta \). Thus, in the half-plane \( u \geq a \) under consideration, we have \( \Re(w + s) = u + \sigma \geq a + \sigma = \frac{1}{4} + \delta \), and therefore

\[
0 \leq \Re(w + s) - \sigma = \frac{1}{4} + \delta - \sigma = \frac{1}{4} + \delta - \delta = \frac{1}{4} < \infty.
\]

The definition (4.8) of \( a \) guarantees that, in the half-plane \( u \geq a \), we have \( \Re(2s + 2w) = 2\sigma + 2u \geq \frac{1}{2} + \delta \), and Lemma 4.2 applies. Hence the integrals on \( C_1 \) and \( C_2 \) can be estimated as

\[
\left( - \int_{C_1} + \int_{C_2} \right) \frac{P(s + w) M^w}{\zeta(2s + 2w)} dw \ll \int_a^b (|t| + V + 2)^\epsilon \frac{M^u}{V} du \ll MV^{\epsilon-1}(|t| + 2)^\epsilon.
\]

On the vertical segment, we have

\[
\int_{a-iV}^{a+iV} \frac{P(s + w) M^w}{\zeta(2s + 2w)} \frac{M^w}{w} dw \ll M^a \int_{-V}^V \frac{(|t| + |v| + 2)^\epsilon}{\sqrt{a^2 + v^2}} dv \ll M^{1/4 - \sigma}(MV)^\epsilon(|t| + 2)^\epsilon.
\]

It follows that

\[
- \int_{C_1} + \int_{C_2} + \int_{a-iV}^{a+iV} \ll (M^{1/4 - \sigma} + MV^{-1})(MV)^\epsilon(|t| + 2)^\epsilon. \tag{4.10}
\]

This finishes the estimation of the three integrals.

It remains to compute \( R \), the summation of the residues. The residue of the simple pole at \( w = 0 \) is

\[
P(s) \frac{1}{\zeta(2s)}. \tag{4.11}
\]

For \( \rho \neq 0 \), the pole at \( w = \rho - s \) is simple with residue

\[
c_q \frac{M^{\rho-s}}{\zeta(2\rho)(\rho - s)}. \tag{4.12}
\]

where \( c_q \) is a constant depending on \( q \). Before analyzing (4.12), we need a rough bound

\[
\frac{1}{|\zeta(2\rho)|} \ll_q 1. \tag{4.13}
\]
This is trivially true if $|\rho|$ is bounded, say $|\rho| \leq 2$. For $|\rho| > 2$ we apply the functional equation $\zeta(s) = \gamma(s)\zeta(1-s)$ and that
\[ |\gamma(s)| \sim \left( \frac{|t|}{\pi} \right)^{1/2-\sigma} \]
as $|t| \to \infty$ in any strip $c_2 \leq \sigma \leq c_3$ of finite width. Recall that all $\rho$ are lying on the imaginary axis and of the form of (2.3), so that
\[ \frac{1}{|\zeta(2\rho)|} \ll \frac{1}{|\gamma(2\rho)||\zeta(1-2\rho)|} \ll |\rho|^{\varepsilon-1/2}, \]
where we have also applied Lemma 4.2. This proves the claim (4.13). Hence the summation of all the residues in (4.12) with $|\Im \rho - t| \leq V$ is
\[ \sum_{|\Im \rho - t| \leq V} c_q \frac{M^{\rho-s}}{\zeta(2\rho)(\rho-s)} \ll M^{-\sigma}(|t| + V + 2)^\varepsilon \sum_{|\Im \rho - t| \leq V} \frac{1}{|\rho-s|}. \]
To estimate the last sum over $\rho$, we note that, for $p|q$,
\[ \# \left\{ j : \frac{H}{2} < \left| \frac{2\pi j}{\log p} - t \right| \leq H \right\} \leq H \log p. \]
It follows that
\[ \sum_{|\Im \rho - t| \leq V} \frac{1}{|\rho-s|} \leq 1 + \sum_{1 \leq |\Im \rho - t| \leq V} \frac{1}{|\Im \rho - t|} \ll \sum_{p|q} \sum_{1 \leq m \leq \log V/\log 2} \frac{2^{m+1} \log p}{2^m} \ll (\log V) \sum_{p|q} (\log p) \ll (\log V)(\log q), \]
and hence
\[ \sum_{|\Im \rho - t| \leq V} c_q \frac{M^{\rho-s}}{\zeta(2\rho)(\rho-s)} \ll M^{-\sigma}(|t| + V + 2)^\varepsilon \quad (4.14) \]
on recalling that $q$ is a constant. The pole at $w = -s$ is of order $k$ where $k = \omega(q)$, and therefore the residue is

$$
\lim_{w \to -s} \frac{1}{(k-1)!} \frac{d^{k-1} \left( (w + s)^k \frac{P(s + w)}{\zeta(2s + 2w)} \frac{M^w}{w} \right)}{dw^{k-1}} \ll q \sum_{0 \leq l + m + n \leq k} \left| c_m \frac{M^{-s} \log^l M}{s^n} \right| \ll M^{-\sigma + \varepsilon},
$$

where

$$
c_m = \left. \frac{d^m}{ds^m} \left( \frac{1}{\zeta(s)} \right) \right|_{s=0}.
$$

It follows from (4.15), (4.14), and (4.11) that

$$
R - \frac{P(s)}{\zeta(2s)} \ll M^{-\sigma + \varepsilon}(|t| + V + 2)^\varepsilon,
$$

and this finishes the treatment of $R$.

Inserting (4.16) and (4.10) back to (4.9) and (4.7), we get

$$
\frac{P(s)}{\zeta(2s)} \sum_{m \leq M} h(m) \frac{M^{1+\varepsilon}}{m^s} \ll \frac{M^{1+\varepsilon}}{V} + \{MV^{-1} + M^{1/4-\sigma} + M^{-\sigma}\}(MV)^\varepsilon(|t| + V + 2)^\varepsilon
$$

which is, on taking $V = M^2$,

$$
\ll M^{1/4-\sigma} M^\varepsilon(|t| + 2)^\varepsilon.
$$

This proves the lemma for $M$ being half an odd integer. For general $M$ the above formula should be corrected by an error term of order $O(M^{-\sigma + \varepsilon})$ by [11, Lemma 3.19], and this correction can be absorbed. This completes the proof. \hfill \Box

5. **Proof of Theorem 1.1(ii)**

Now we are ready to prove the conditional part of Theorem 1.1 applying the idea of Montgomery and Vaughan [9].

**Proof of Theorem 1.1(ii).** Similarly to the unconditional case, we start from the formula

$$
M_f(x) = \sum_{n \leq x} f(n) = \sum_{dm \leq x} \chi(d)h(m)
$$

$$
= \left( \sum_{dm \leq x \atop m \leq M} + \sum_{dm \leq x \atop m > M} \right) \chi(d)h(m)
$$

$$
= S_1 + S_2.
$$
say. We assume in addition that $M < x$. For $S_1$, we invoke the trivial bound $|\sum_{d \leq x} \chi(d)| \leq q$ as well as Lemma 2.2 to get

$$S_1 = \sum_{m \leq M} h(m) \sum_{d \leq x/m} \chi(d) \ll M^{1/2} \log^c x,$$

where $c = \omega(q)$ and $q$ is treated as a constant.

For $S_2$, we are going to apply analytic methods to prove in the following that

$$S_2 \ll M^{-1/4} x^{1/2+\varepsilon}.$$  \hfill (5.2)

Once (5.2) is established, one sees immediately from this and (5.1) that the optimal choice of $M$ is $M = x^{2/3}$, and this proves the Theorem 1.1(ii).

It remains to show (5.2). Again, we apply Perron’s formula in the form of Lemma 4.1 with $\psi(n) = n^\varepsilon$ and $b = 1 + \frac{1}{\log x}$. Then (4.2) becomes, for $x$ being half an odd integer,

$$S_2 = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L(s, \chi) Z_M(s) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right).$$

We shift the contour in (5.3) to the left till the vertical line $\sigma = a$ where

$$a = \frac{1}{2} + \delta$$

and $\delta > 0$ is arbitrarily small, without passing any singularities of the integrand in (5.3). The integral in (5.3) can be written as

$$\int_{b-iT}^{b+iT} = -\int_{C_1} + \int_{C_2} + \int_{a-iT}^{a+iT}$$

where $C_1$ means the segment $\{s = \sigma - iT : a \leq \sigma \leq b\}$, and $C_2$ means $\{s = \sigma + iT : a \leq \sigma \leq b\}$. The integrals on $C_1$ and $C_2$ can be estimated to be small. Since we are not assuming the Riemann Hypothesis for $L(s, \chi)$, we invoke the convexity bound

$$L(s, \chi) \ll \{q(|t| + 2)\}^{1/4}$$

in the half-plane $\sigma \geq a$, and we can put $q$ into the implied constant. From this and Proposition 4.4, we arrive at that

$$\left( -\int_{C_1} + \int_{C_2} \right) L(s, \chi) Z_M(s) \frac{x^s}{s} ds \ll \int_a^b M^{1/4} \left( \frac{x^\sigma}{M} \right)^T (MT)^{\varepsilon} \frac{x^\sigma}{s} ds.$$

Note that $M < x$, and therefore the last quantity is

$$\ll (MT)^{-3/4} x (MT)^\varepsilon.$$
On the vertical segment, we apply Lemma 4.3 so that
\[
\int_{a-iT}^{a+iT} L(s, \chi) Z_M(s) \frac{x^s}{s} ds \ll x^a M^{1/4-a+\varepsilon} \int_{-T}^{T} |L(a+it, \chi)| \frac{(|t|+2)^{\varepsilon}}{\sqrt{a^2+t^2}} dt \\
\ll M^{-1/4} x^{1/2} (MT)^{\varepsilon}.
\]
Inserting these back to (5.5) and (5.3), we get
\[
S_2 \ll \left( M^{-1/4} x^{1/2} + (MT)^{-3/4} x + \frac{x}{T} \right) (xT)^{\varepsilon}.
\]
Taking \( T = x \) proves (5.2) for \( x \) being half an odd integer, from which (5.2) for general \( x \) follows easily. This completes the proof of Theorem 1.1(ii). \( \square \)

6. Proof of Theorem 1.2

For simplicity we write
\[
F(s) = \frac{L(s, \chi) P(s)}{\zeta(2s)};
\] (6.1)
the proof of Theorem 1.2 depends on the location of poles of \( F \). Obviously poles of \( F(s) \) must be zeros of \( \zeta(2s) \). But we cannot exclude the possibility that some zeros of \( \zeta(2s) \) may be cancelled by zeros of \( L(s, \chi) \). It is this possibility that produces extra subtlety to the proof. What we indeed need to know is whether there are any simple zeros of \( \zeta(2s) \), on the vertical line \( \sigma = \frac{1}{4} \), that are not cancelled by the zeros of \( L(s, \chi) \). The next lemma says that there are infinitely many of them.

**Lemma 6.1.** On the segment
\[
\{ s = 1/4 + it : 0 \leq t \leq T \},
\] (6.2)
there are \( \gg T \log T \) simple zeros of \( \zeta(2s) \) that are not zeros of \( L(s, \chi) \). These zeros are simple poles of \( F(s) \).

Note that Lemma 6.1 does not depend on any unproved hypothesis.

**Proof.** It is known that a positive proportion of zeros of \( \zeta(s) \) are simple and on the critical line \( \sigma = \frac{1}{2} \), and Levinson [8] has proved that this proportion is more than \( \frac{1}{3} \). Thus \( \zeta(2s) \) has \( \gg T \log T \) simple zeros on the segment (6.2).

Next we are going to show that the number of zeros of \( L(s, \chi) \) on the segment (6.2) is of lower order of magnitude. Let \( \psi \mod q \) denote a generic Dirichlet character, and \( N(\alpha, T, \psi) \) the number of zeros of the attached \( L \)-function \( L(s, \psi) \) lying in the rectangle
\[
\alpha \leq \sigma \leq 1, \quad 0 \leq t \leq T.
\]
Then Huxley’s classical zero-density estimate \[4\] states that
\[
\sum_{\psi \mod q} N(\alpha, T, \psi) \ll (qT)^{\frac{3}{5}(1-\alpha)+\varepsilon},
\]
where the summation is over all the characters \(\psi\) modulo \(q\). Now we take \(\alpha = \frac{3}{4}\), and pick up just the term with \(\psi = \chi\) on the left-hand side, to get
\[
N(3/4, T, \chi) \ll (qT)^{3/5+\varepsilon}.
\]
In particular on the segment \(\{s = \frac{3}{4} + it : 0 \leq t \leq T\}\) the number of zeros of \(L(s, \chi)\) is \(\ll (qT)^{3/5+\varepsilon}\). Note that our \(\chi\) is real. By symmetry, on the segment (6.2), the number of zeros of \(L(s, \chi)\) is also \(\ll (qT)^{3/5+\varepsilon}\).

Hence after possible cancellations, there are still \(\gg T \log T\) simple zeros of \(\zeta(2s)\) left on the same segment (6.2). This proves the lemma. \(\blacksquare\)

**Proof of Theorem 1.2.** The proof is naturally divided into two cases.
(i) The first case is that there exists a constant \(\theta > \frac{1}{4}\) such that
\[
\sup \{\beta : F(s) \text{ has a pole at } \beta + i\gamma\} = \theta;
\]
this implies that the Riemann Hypothesis for \(\zeta\) is false. Suppose \(M_f(x) \ll x^a\) for some \(a\) satisfying \(\frac{1}{4} < a < \theta\). For \(\sigma > 1\) we apply partial integration to get
\[
F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = s \int_{1}^{\infty} \frac{M_f(s)}{x^{s+1}} dx. \tag{6.3}
\]
The last integral is absolutely convergent in the half-plane \(\sigma \geq \frac{1}{2}(a + \theta)\), and hence represents an analytic function in the same region. This is a contradiction since \(\frac{1}{2}(a + \theta) < \theta\). This proves that
\[
M_f(x) = \Omega(x^a)
\]
for any \(a\) satisfying \(\frac{1}{4} < a < \theta\), which is actually stronger than the statement of the theorem.

(ii) It remains to treat the case that \(F(s)\) has no pole in the half-plane \(\sigma > \frac{1}{4}\). Note that this does not mean the Riemann Hypothesis holds for \(\zeta(2s)\), since its zeros off the line \(\sigma = \frac{1}{4}\) may possibly be cancelled by the zeros of \(L(s, \chi)\). In the present case (6.3) holds in the half-plane \(\sigma > \frac{1}{4}\). Thus
\[
F(s) = s \int_{1}^{\infty} \frac{M_f(s)}{x^{s+1}} dx, \tag{6.4}
\]
and finally we will make $\sigma \rightarrow \frac{1}{4}$ from the right. Suppose that

$$|M_f(x)| \leq \begin{cases} M_0, & \text{for } 1 \leq x \leq x_0, \\ \delta x^{1/4}, & \text{for } x \geq x_0. \end{cases}$$

Then (6.4) gives

$$F(s) \leq |s|M_0 \int_1^{x_0} \frac{dx}{x^{5/4}} + |s|\delta \int_{x_0}^{\infty} \frac{dx}{x^{\sigma+3/4}} \leq |s|M_0 \int_1^{\infty} \frac{dx}{x^{5/4}} + |s|\delta \int_{1}^{\infty} \frac{dx}{x^{\sigma+3/4}} = 4|s|M_0 + \frac{|s|\delta}{\sigma - \frac{1}{4}}.$$

Suppose that $\rho = \frac{1}{4} + i\gamma$ is a simple zero of $\zeta(2s)$ that is not cancelled by the zeros of $L(s,\chi)$; the existence of such a $\rho$ is guaranteed by Lemma 6.1. This $\rho$ is a simple pole of $F(s)$, and we have $L(\rho,\chi) \neq 0$. Of course $P(\rho) \neq 0$. Putting $s = \sigma + i\gamma$, and making $\sigma \rightarrow \frac{1}{4}$ horizontally from the right,

$$F(s) = \frac{L(s,\chi)P(s)}{\zeta(2s)} = \frac{L(s,\chi)P(s)}{\zeta(2s) - \zeta(2\rho)} \sim \frac{L(\rho,\chi)P(\rho)}{2(\sigma - \frac{1}{4})\zeta'(2\rho)}.$$

We therefore obtain a contradiction if

$$\delta < \left| \frac{L(\rho,\chi)P(\rho)}{4\rho\zeta'(2\rho)} \right|. \quad \Box$$

This proves the theorem.

Acknowledgments. The author would like to record her sincere thanks to Liming Ge, Fuzhou Gong, and Nanhua Xi for supports and encouragements, to Marco Aymone and Bingrong Huang for helpful suggestions leading to improvements in Theorem 1.1(ii), and to Oleksiy Klurman for bibliographical information.

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