Effective Lagrangians for linearized gravity in Randall-Sundrum model

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Abstract

We construct the second variation Lagrangian for Randall-Sundrum model with two branes, study its gauge invariance, find the corresponding equations of motion and decouple them. We also derive an effective Lagrangian for this model in the unitary gauge, which describes a massless graviton, massive gravitons and a massless scalar radion. It is shown that the mass spectra of these fields and their couplings to the matter are different on different branes.

1 Introduction

The Kaluza-Klein (KK) hypothesis has been discussed in theoretical physics for three quarters of a century. In accordance with this hypothesis, space-time may have extra dimensions, which are unobservable for certain reasons. The explanation of this unobservability, which was put forward in the original papers by Kaluza and Klein, implies that the extra dimensions are compactified and have a very small size of the order of the Planck length \( l_{Pl} = 1/M_{Pl} \).

In 1983 Rubakov and Shaposhnikov put forward a new scenario for Kaluza-Klein theories, which was based on the idea of localization of fields on a domain wall [1]. They have also proposed an ansatz for multidimensional metric, which is compatible with this hypothesis [2].

In the last years there appeared indications that scenarios of this type can arise in the theory of strings [3] (see Ref. [4] for a review and references). In this approach our three spatial dimensions are supposed to be realized as a three-dimensional hypersurface embedded into a multidimensional space-time. Such hypersurfaces are called 3-branes, or just branes. The main goal of such scenarios was to find a solution to the hierarchy problem. It was solved either due to the sufficiently large characteristic size of extra dimensions [5], or due to exponential warp factor appearing in the metric [6, 7]. In both approaches gravity in multidimensional space-time becomes "strong" not at the energies of the order of \( 10^{19} \) GeV, but at much lower energies, maybe of the order of \( 1 \div 10 \) TeV. An attractive feature of these models is that they predict new effects which can be observed at the coming collider experiments.
In paper [6] an exact solution for a system of two branes interacting with gravity in five-dimensional space-time was found, which allows one to study the effect of the gravitational field induced by the branes. This model is called Randall-Sundrum model, and it was widely discussed in the literature (see Refs. [4, 8] for reviews and references). However, there are some issues which are still to be clarified. In the present paper we present a detailed derivation of the Lagrangian of quadratic fluctuations around the Randall-Sundrum solution and discuss its properties. In particular, we carry out a diagonalization of this Lagrangian and of the corresponding equations of motion.

The paper is organized as follows. In Sect. 2 we derive an expression for the Lagrangian of quadratic fluctuations around an arbitrary background convenient for our purposes. In Sect. 3 this result is employed for the calculation of the Lagrangian of quadratic fluctuations in the Randall-Sundrum model. We also discuss here the gauge transformations in the model and identification of physical degrees of freedom. In Sect. 4 we calculate the couplings of the fields, arising from the five-dimensional metric, to the matter. We emphasize that for the proper interpretation of the interactions Galilean coordinates on the branes should be used. Sect. 5 contains some discussion of the effective theories on the branes.

2 Second variation Lagrangian

First, let us consider the standard gravitational action with cosmological constant in $4 + d$-dimensional space-time $E$ with coordinates $\{x^M\}, M = 0, 1, \ldots d + 3$:

$$S_g = \frac{1}{16\pi \hat{G}} \int_E (R - \Lambda) \sqrt{-g} d^{d+4}x,$$

where $\hat{G}$ is the multidimensional gravitational constant. The signature of the metric $g_{MN}$ is chosen to be $(-1, 1, \cdots 1)$. Let $\gamma_{MN}$ be a fixed background metric. We denote $\hat{\kappa} = \sqrt{16\pi \hat{G}}$ and parameterize the metric $g_{MN}$ as

$$g_{MN} = \gamma_{MN} + \hat{\kappa} h_{MN}.$$  \hspace{1cm} (2)

If we substitute this formula into (1) and retain only the terms of the zeroth order in $\hat{\kappa}$, we get the following Lagrangian, which is usually called the second variation Lagrangian:

$$L_g^{(2)}/\sqrt{-\gamma} = -\frac{1}{4} \left( \nabla_R h_{MN} \nabla^R h^{MN} - \nabla_R h \nabla^R h + 2 \nabla_M h^{MN} \nabla_N h - 2 \nabla^R h^{MN} \nabla_M h_{RN} \right) + \frac{1}{4} (R - \Lambda) \left( h_{MN} h^{MN} - \frac{1}{2} h h \right) + G^{MN} h_{MR} h^R_N - \frac{1}{2} G^{MN} h_{MN} h.$$ \hspace{1cm} (3)

Here $\gamma = \det \gamma_{MN}, h = h^M_N, \nabla$ denotes the covariant derivative with respect to metric $\gamma_{MN}$, and $G_{MN} = R_{MN} - \frac{1}{2} \gamma_{MN} (R - \Lambda)$. This formula in rather complicated notations can be found in [2].

Action (1) is invariant under general coordinate transformations $y^M = y^M(x)$, the corresponding transformation of the metric being

$$g'_{RS}(y) \frac{\partial y^R}{\partial x^M} \frac{\partial y^S}{\partial x^N} = g_{MN}(x).$$ \hspace{1cm} (4)

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Let us consider infinitesimal coordinate transformations

$$y^M(x) = x^M + \hat{\kappa} \xi^M(x).$$

(5)

Representing the initial and the transformed metrics as

$$g_{MN}(x) = \gamma_{MN}(x) + \hat{\kappa} h_{MN}(x),$$
$$g'_{MN}(y) = \gamma_{MN}(y) + \hat{\kappa} h'_{MN}(y),$$

substituting these formulas into Eq. (4) and keeping the terms up to the first order in $\hat{\kappa}$, we get the transformation law for the variation $h_{MN}$,

$$h'_{MN}(x) = h_{MN}(x) - \gamma_{MR} \partial_N \xi^R - \gamma_{NR} \partial_M \xi^R - \xi^R \partial_R \gamma_{MN},$$

which can be equivalently rewritten as

$$h'_{MN}(x) = h_{MN}(x) - (\nabla_M \xi_N + \nabla_N \xi_M).$$

(6)

It is not difficult to check that the action built with the second variation Lagrangian (3) is invariant under transformations (6). Therefore, the latter can be interpreted as the gauge transformations of the field $h_{MN}$.

Usually the background metric $\gamma_{MN}$ is a solution of the Einstein equations

$$R_{MN} - \frac{1}{2} \gamma_{MN} (R - \Lambda) = 8\pi \hat{G} T_{MN}$$

(7)

with a certain energy-momentum tensor of the matter $T_{MN}$. We express $G_{MN}$ and $(R - \Lambda)$ in terms of the energy-momentum tensor $T_{MN}$ and substitute it into (3), which gives the second variation Lagrangian in the form

$$L_g^{(2)} / \sqrt{-\gamma} = -\frac{1}{4} \left( \nabla_R h_{MN} \nabla^R h^{MN} - \nabla_R h \nabla^R h + 2 \nabla_M h^{MN} \nabla_N h - 2 \nabla^R h^MN \nabla_M h_RN \right) + \frac{\Lambda}{2(d+2)} \left( h_{MN} h^{MN} - \frac{1}{2} h h \right) -$$

$$- \frac{4\pi \hat{G}}{d+2} T^R_R \left( h_{MN} h^{MN} - \frac{1}{2} h h \right) + \left( 8\pi \hat{G} T^{MN} h_{MR} h^R_N - 4\pi \hat{G} T^{MN} h_{MN} h \right).$$

In the next section this expression will be calculated for the Randall-Sundrum model.

### 3 Diagonalization of the Lagrangian in the Randall-Sundrum model

The Randall-Sundrum model [6] describes the gravity in a five-dimensional space $E$ with two branes embedded in it (it is often referred to as the RS1 model). We denote the coordinates as $\{x^\mu, x^4\}$, $\mu = 0, 1, 2, 3$, the coordinate $x^4$ parameterizing the fifth dimension. It forms the orbifold $S^1/Z_2$, which is the circle of the circumference $2R$ with points $(x^\mu, x^4)$ and $(x^\mu, -x^4)$ identified. Correspondingly, the metric $g_{MN}$ satisfies the orbifold symmetry conditions

$$g_{\mu\nu}(x^\rho, -x^4) = g_{\mu\nu}(x^\rho, x^4), \quad g_{\mu4}(x^\rho, -x^4) = -g_{\mu4}(x^\rho, x^4), \quad g_{44}(x^\rho, -x^4) = g_{44}(x^\rho, x^4).$$

(9)
The branes are located at the fixed points of the orbifold, \( x^4 = 0 \) and \( x^4 = R \). In addition we have the usual periodicity condition which identifies points \((x^\mu, x^4)\) and \((x^\mu, x^4 + 2nR)\).

The action of the model is equal to

\[
S = S_g + S_1 + S_2,
\] (10)

where \( S_g \) is given by (1) with \( d = 1 \) and

\[
S_1 = V_1 \int_E \sqrt{\tilde{g}} \delta(x^4) d^5x,
\] (11)

\[
S_2 = V_2 \int_E \sqrt{\tilde{g}} \delta(x^4 - R) d^5x.
\] (12)

The subscripts 1 and 2 label the branes.

The Randall-Sundrum solution for the metric is given by

\[
\gamma_{MN} dx^M dx^N = e^{2\sigma} \eta_{\mu\nu} dx^\mu dx^\nu + dx^4 dx^4,
\] (13)

where \( \eta_{\mu\nu} \) is the Minkowski metric and the function \( \sigma(x^4) \) in the interval \(-R \leq x^4 \leq R\) is equal to \( \sigma(x^4) = -k|x^4| \). The parameter \( k \) has the dimension of mass, and \( \Lambda, V_{1,2} \) are related to it as follows:

\[
\Lambda = -12k^2, \quad V_1 = -V_2 = -\frac{3k}{4\pi G}.
\]

We see that brane 1 has positive energy density, whereas brane 2 has negative one. The function \( \sigma \) has the properties

\[
\partial_4 \sigma = -k \text{sign}(x^4), \quad \frac{\partial^2 \sigma}{\partial x^4^2} = -2k(\delta(x^4) - \delta(x^4 - R)) \equiv -2k \tilde{\delta}.
\] (14)

Now we construct the second variation Lagrangian for the Randall-Sundrum model. To this end we parameterize the metric in accordance with (2), substitute it into action (10) and keep the terms of the zeroth order in \( \hat{\kappa} \). The contribution of the term \( S_g \) is given by (8) with

\[
T_{MN} = -\frac{3k}{4\pi G} \sqrt{\gamma} \gamma_{\mu\nu} \delta_{\mu}^M \delta_{\nu}^N \tilde{\delta},
\] (15)

\( \tilde{\gamma} \) denoting the determinant of \( \gamma_{\mu\nu} \). The contribution of the branes is given by

\[
\Delta L_{1,2}^{(2)} = 3k(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2} \tilde{h} \tilde{h}) \delta \sqrt{-\gamma},
\] (16)

where \( \tilde{h} = \gamma^{\mu\nu}h_{\mu\nu} \).

Thus, the quadratic Lagrangian for the variation \( h_{MN} \) can be written as follows:

\[
L/\sqrt{-\gamma} = -\frac{1}{4} \left( \nabla_R h_{MN} \nabla^R h^{MN} - \nabla_R h \nabla^R h + 2\nabla_M h^{MN} \nabla_N h - 2\nabla_R h^{MN} \nabla_M h_{RN} \right) + 2k^2(h_{MN}h^{MN} - \frac{1}{2} h h) + 4k(h_{MN}h^{MN} - \frac{1}{2} h h) + 3k \tilde{h} \tilde{h} - 6k h_{M\nu} h^{M\nu} + 3k(h_{\mu\nu}h^{\mu\nu} - \frac{1}{2} \tilde{h} \tilde{h}) \right) \tilde{\delta}.
\] (17)
It turns out to be convenient to transform the term \( \frac{1}{2} \nabla^R h^{MN} \nabla_M h_{RN} \) to the standard Fierz-Pauli form

\[
\frac{1}{2} \nabla^R h^{MN} \nabla_M h_{RN} = \frac{1}{2} \nabla_M h^{MN} \nabla_R h_{MN} + \frac{1}{2} h^{MN} h^{PQ} R_{MPNQ} - \frac{1}{2} h^{MN} h_{NP} R^P_M, \tag{18}
\]

where \( R_{MPNQ} \) and \( R^P_M \) are the curvature and the Ricci tensors of the metric \( \gamma_{MN} \). If we calculate these terms explicitly, the Lagrangian becomes

\[
L/\sqrt{-\gamma} = -\frac{1}{4} \left( \nabla_R h_{MN} \nabla^R h^{MN} - \nabla_R h \nabla^R h + 2 \nabla_M h^{MN} \nabla_N h - 2 \nabla_M h^{MN} \nabla_R h_{RN} + \frac{k^2}{2} (h_{MN} h^{MN} + h h) + \left[ -k h_{MN} h^{MN} + k h - k h_{\mu\nu} h^{\mu\nu} + 3 k (h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} \bar{h} \bar{h}) \right] \tilde{\delta}. \tag{19}
\]

Now let us discuss the gauge invariance of this Lagrangian. The gauge transformations (6) in the case under consideration can be found explicitly and turn out to be

\[
h_{\mu\nu}'(x) = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu + 2 \gamma_{\mu\nu} \partial_4 \xi_4) \tag{20}
\]
\[
h_{\mu4}'(x) = h_{\mu4}(x) - (\partial_\mu \xi_4 + \partial_4 \xi_\mu - 2 \partial_4 \sigma \xi_\mu),
\]
\[
h_{44}'(x) = h_{44}(x) - 2 \partial_4 \xi_4,
\]

where the functions \( \xi^M(x) \) satisfy the orbifold symmetry conditions

\[
\xi^\mu(x^\nu, -x^4) = \xi^\mu(x^\nu, x^4), \tag{21}
\]
\[
\xi^4(x^\nu, -x^4) = -\xi^4(x^\nu, x^4).
\]

These gauge transformations in other parameterizations were discussed in papers \[10\] and \[11\]. We will use them to remove the gauge degrees of freedom of the field \( h_{MN} \). To this end we first make a gauge transformation with

\[
\xi_4(x^\nu, x^4) = \frac{1}{4} \int_{-x^4}^{x^4} h_{44}(x^\nu, y) dy - \frac{x^4}{4R} \int_{-R}^{R} h_{44}(x^\nu, y) dy. \tag{22}
\]

One can easily see that \( \xi_4 \) satisfies the orbifold symmetry condition. After this transformation \( h_{44} \) takes the form

\[
h_{44}'(x^\nu) = \frac{1}{2R} \int_{-R}^{R} h_{44}(x^\nu, y) dy \tag{23}
\]

and therefore it does not depend on \( x^4 \). Moreover, there are no remaining gauge transformations with \( \xi_4 \).

Now let us consider the components \( h_{\mu4} \). Due to orbifold symmetry

\[
h_{\mu4}(x, -x^4) = -h_{\mu4}(x, x^4), \tag{24}
\]

and the gauge transformations for \( h_{\mu4} \) read

\[
h_{\mu4}'(x, x^4) = h_{\mu4}(x, x^4) - (\partial_4 \xi_\mu - 2 \partial_4 \sigma \xi_\mu). \tag{25}
\]
Let us show that we can impose the condition $h'_\mu 4 = 0$. One can easily find that the corresponding equation for $\xi_\mu$ is

$$\partial_4 \left( e^{-2\sigma} \xi_\mu \right) = e^{-2\sigma} h_{\mu 4}.$$  

Thus, we have

$$\xi_\mu (x', x^4) = e^{2\sigma} \int_0^{x^4} e^{-2\sigma} h_{\mu 4} (x', y) \, dy,$$  \hspace{1cm} (26)

and $\xi_\mu$ satisfies the orbifold symmetry condition (21). We will call the gauge with

$$h_{\mu 4} = 0, \ h_{44} = h_{44}(x') \equiv \phi(x')$$  \hspace{1cm} (27)

the unitary gauge. After we have imposed this gauge, there remain gauge transformations satisfying

$$\partial_4 \left( e^{-2\sigma} \xi_\mu \right) = 0.$$  \hspace{1cm} (28)

They will be important for removing the gauge degrees of freedom of the massless mode of the gravitational field.

Varying the action corresponding to the Lagrangian (19) we get the following equations of motion:

$$\frac{1}{2} \nabla_R \nabla^R h_{MN} - \frac{1}{2} \gamma_{MN} \nabla_R \nabla^R h - \frac{1}{2} \nabla_M \nabla_N h + \frac{1}{2} \gamma_{MN} \nabla_R \nabla^S h_{RS} -$$

$$- \frac{1}{2} \nabla_M \nabla^R h_{RN} - \frac{1}{2} \nabla_N \nabla^R h_{RM} + k^2 (h_{MN} + \gamma_{MN} h) + \left[-4kh_{MN} -$$

$$- k (h_{M\nu} \delta_\nu^\mu + h_{\mu N} \delta_\nu^\nu) + k \gamma_{MN} \ddot{h} + k \delta_\mu^\mu \delta_\nu^\nu \gamma_{\mu \nu} + 6 k \delta_\mu^\mu \delta_\nu^\nu (h_{\mu \nu} - \frac{1}{2} \gamma_{\mu \nu} \ddot{h}) \right] \ddot{\delta} = 0.$$  \hspace{1cm} (29)

Contracting these equations, we get a very simple equation

$$\frac{3}{2} \nabla^R \nabla^S h_{RS} - \frac{3}{2} \nabla_R \nabla^R h + 6 k^2 h - 3 k \ddot{h} = 0.$$  \hspace{1cm} (30)

which turns out to be very useful. The equations of motions for different components are derived from (29) and in the unitary gauge (27) take the form:

1) $\mu \nu$-component

$$\frac{1}{2} \left( \partial_\rho \partial^\rho h_{\mu \nu} - \partial_\mu \partial^\rho h_{\rho \nu} - \partial_\nu \partial^\rho h_{\rho \mu} + \frac{\partial^2 h_{\mu \nu}}{\partial x^4^2} \right) - 2 k^2 h_{\mu \nu} + \frac{1}{2} \partial_\mu \partial_\nu \ddot{h} + \frac{1}{2} \partial_\mu \partial_\nu \phi +$$

$$+ \frac{1}{2} \gamma_{\mu \nu} \left( \partial_\rho \partial^\rho h_{\rho \sigma} - \partial_\rho \partial^\rho \ddot{h} - \frac{\partial^2 \ddot{h}}{\partial x^4^2} - 4 \partial_4 \sigma \partial_4 \ddot{h} - \partial_\rho \partial^\rho \phi + 12 k^2 \phi \right)$$

$$+ \left[ 2 k h_{\mu \nu} - 3 k \gamma_{\mu \nu} \phi \right] \ddot{\delta} = 0.$$  \hspace{1cm} (31)

2) $\mu 4$-component,

$$\partial_4 \left( \partial_\mu \ddot{h} - \partial^\sigma h_{\mu \sigma} \right) - 3 \partial_4 \sigma \partial_\mu \phi = 0,$$  \hspace{1cm} (32)

which plays the role of a constraint,

3) 44-component

$$\frac{1}{2} \left( \partial^\mu \partial^\nu h_{\mu \nu} - \partial_\mu \partial^\nu \ddot{h} \right) - \frac{3}{2} \partial_4 \sigma \partial_4 \ddot{h} + 6 k^2 \phi = 0.$$  \hspace{1cm} (33)
The contracted equation can also be rewritten in this gauge to be

$$\frac{3}{2}(\partial^\mu\partial^\nu h_{\mu\nu} - \partial_\mu \partial^\mu \tilde{h}) - \frac{15}{2} \partial_4 \sigma \partial_4 \tilde{h} - \frac{3}{2} \partial^2 \tilde{h} - \frac{3}{2} \partial_\mu \partial^\mu \phi + 30 k^2 \phi - 12 k \phi \tilde{\delta} = 0. \quad (34)$$

Let us first consider the 44-component and the contracted equation. Multiplying Eq. (33) by 3 and subtracting it from Eq. (34), we obtain the following equation, containing $\tilde{h}$ and $\phi$ only:

$$\frac{\partial^2 \tilde{h}}{\partial x^4^2} + 2 \partial_4 \sigma \partial_4 \tilde{h} - 8 k^2 \phi + 8 k \phi \tilde{\delta} + \partial_\mu \partial^\mu \phi = 0. \quad (35)$$

To describe correctly the physical degrees of freedom in the model we write the multidimensional gravitational field as

$$h_{\mu\nu} = b_{\mu\nu} + \gamma_{\mu\nu}(\sigma - c) \phi + \frac{1}{2k^2} \left( \sigma - c + \frac{1}{2} + \frac{c}{2} e^{-2\sigma} \right) \partial_\mu \partial_\nu \phi, \quad (36)$$

with $\sigma = \sigma(x^4)$ and $c$ being a constant. We will see that the field $b_{\mu\nu}(x^\mu, x^4)$ describes a massless graviton [6, 7] and massive Kaluza-Klein spin-2 fields, whereas $\phi(x^\mu)$ describes a scalar field called radion. Apparently, the radion field was first identified in Ref. [10]. As it will be shown, relation (36) with an appropriately chosen constant $c$ diagonalizes the second variation Lagrangian and decouples the equations of motion (31)-(33). The form of the substitution (36) is suggested by the form of the gauge transformation that transforms the theory from the local Gaussian normal coordinates (i.e. coordinates corresponding to $g_{44} = 1$, $g_{\mu 4} = 0$) [10], where the second variation Lagrangian is diagonal in the bulk by construction, to our coordinates. We would like to note that in fact the equations of motion derived in Ref. [10] remain coupled at the fixed points of the orbifold, i.e. at the points where the branes are located. To decouple them the term $\sim e^{-2\sigma}$ in Eq. (36) is needed. It follows from this expression that

$$\tilde{h} = \tilde{b} + 4(\sigma - c) \phi + \frac{1}{2k^2} \left( \sigma - c + \frac{1}{2} + \frac{c}{2} e^{-2\sigma} \right) \partial_\mu \partial_\nu \phi. \quad (37)$$

Substituting (37) into Eq. (35) we get the equation

$$\partial_4(e^{2\sigma} \partial_4 \tilde{b}) + 2c \partial_\mu \partial^\mu \phi + \frac{2}{k^2}(\sigma - c + c e^{-2\sigma}) \partial_\mu \partial^\mu \phi \tilde{\delta} = 0.$$

Choosing the constant $c$ so that the boundary term, proportional to $\tilde{\delta}(x^4)$, vanishes, i.e.

$$c = \frac{kR}{e^{2kR} - 1}, \quad (38)$$

we reduce the equation to the following one:

$$\partial_4(e^{2\sigma} \partial_4 \tilde{b}) + 2c \partial_\mu \partial^\mu \phi = 0. \quad (39)$$

Next, taking into account that $\gamma_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu}$, we pass to the flat metric in this equation and denote $b = \eta^{\mu\nu} b_{\mu\nu}$. Then we get

$$\partial_4(e^{2\sigma} \partial_4 (e^{-2\sigma} b)) + 2ce^{-2\sigma} \partial_\mu \partial^\mu \phi = 0. \quad (40)$$

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Let us consider Fourier expansions of both terms in $x^4$. Since the term with the derivative $\partial_4$ has no zero modes, this equation implies that the radion field is a free massless field \[ \partial_\mu \partial^\mu \phi = 0, \] (41) and $\partial_4 (e^{-2\sigma} b) = e^{-2\sigma} f(x^\mu)$. Applying the same reasoning to the last relation, we get $\partial_4 (e^{-2\sigma} b) = 0$.

Recall that we have at our disposal the gauge transformations, satisfying (28). With the help of these transformations, we can impose the gauge

$\tilde{b} = b = 0$, \hspace{1cm} (42)

It is easy to see that there remain gauge transformations parameterized by $\xi_\mu = e^{2\sigma} \epsilon_\mu (x^\nu)$ with $\epsilon_\mu (x^\nu)$ satisfying $\partial^\mu \epsilon_\mu = 0$. Substituting expression (36) into Eqs. (32), (33) and passing to gauge (42), we arrive at the following system of relations:

$$
\partial_\mu \partial^\mu b_{\mu \nu} = 0, \hspace{1cm} (43)
$$

$$
\partial_4 (e^{-2\sigma} \partial^\mu b_{\mu \nu}) = 0. \hspace{1cm} (44)
$$

The remaining gauge transformations are sufficient to impose the condition

$$
\partial^\mu b_{\mu \nu} = 0. \hspace{1cm} (45)
$$

The conditions (42) and (45) define the gauge often called the transverse-traceless (TT) gauge. Having imposed this gauge, we are still left with residual gauge transformations

$$
\xi_\mu = e^{2\sigma} \epsilon_\mu (x^\nu), \hspace{0.5cm} \partial_\rho \partial^\rho \epsilon_\mu = 0, \hspace{0.5cm} \partial^\mu \epsilon_\mu = 0, \hspace{1cm} (46)
$$

which, as we will see shortly, are important for determining the number of degrees of freedom of the massless mode of $b_{\mu \nu}$.

In the gauge (42), (45) the constraint (32) and the 44-equation (33) are trivially satisfied. Let us turn to Eq. (51). Substituting expressions (36), (37) into it and passing to TT gauge, we transform this equation to the following form

$$
\frac{1}{2} \partial_\mu \partial^\rho b_{\mu \nu} + \frac{1}{2} \frac{\partial^2 b_{\mu \nu}}{\partial x^4 \partial x^4} - 2k^2 b_{\mu \nu} + 2kb_{\mu \nu} \delta = 0. \hspace{1cm} (47)
$$

Thus, we have decoupled all the equations.

To summarize, fluctuations around the Randall-Sundrum solution are described by the spin-2 field $b_{\mu \nu}(x^\mu, x^4)$ and the massless radion field $\phi(x^\mu)$. Their classical equations of motion are given by Eq. (47) and Eq. (41) respectively.

Now let us consider the second variation action of the theory. Substituting (36), (37) with $c$ given by (38) into the Lagrangian (19) and taking into account the TT gauge conditions (42), (45) for $b_{\mu \nu}$, we obtain

$$
L/\sqrt{-\gamma} = -\frac{1}{4} \partial_\mu b_{\mu \nu} \partial^\rho b^{\nu \rho} - \frac{1}{4} (\partial_4 b_{\mu \nu} - 2\partial_4 \sigma b_{\mu \nu})(\partial_4 b^{\mu \nu} + 2\partial_4 \sigma b^{\mu \nu}) - \frac{3}{4} ce^{-2\sigma} \partial_\mu \phi \partial^\mu \phi. \hspace{1cm} (48)
$$
To get an effective Lagrangian for the system, it remains to decompose the field $b_{\mu\nu}$ into the modes with definite masses and to integrate $L$, Eq. (48), over $x^4$. After integration over $x^4$ we obtain the following four-dimensional effective Lagrangian for the field $\phi$:

$$L_\phi = -\frac{3}{2} e^{2kR} \frac{kR^2}{1} \partial_\mu \phi \partial^\mu \phi,$$

where the index is raised with the flat metric. To bring the kinetic term to the canonical form we have to rescale the field according to

$$\phi \rightarrow \sqrt{e^{2kR} - \frac{1}{3kR^2}} \phi.$$

Let us turn to the mode expansion of the field $b_{\mu\nu}$. In what follows we denote $x^4 = y$. Following Refs. [7, 12, 13], to perform the expansion we first find eigenfunctions $\psi_n(y)$ and eigenvalues $m_n$ of the problem

$$\frac{1}{2} d^2 + 2k(\delta(y) - \delta(y - R)) - 2k^2 \psi_n(y) = \frac{m_n^2}{2} e^{2kR} \psi_n(y).$$

(51)

The operator in the l.h.s. is a part of the equation (47). Eq. (51) can be solved exactly. To this end we first introduce the variable

$$\xi = \frac{m_n}{k} e^{k|y|}.$$

Note that the eigenvalue $m_n$ enters into the definition of $\xi$. Making the change of variables in Eq. (51) we transform it to

$$\left[ \frac{d^2}{d\xi^2} + \frac{1}{\xi} \frac{d}{d\xi} + 1 - \frac{4}{\xi^2} + \frac{4}{\xi} (\delta(\xi - \xi_1) - \delta(\xi - \xi_2)) \right] f(\xi) = 0,$$

(52)

where $\xi_1 = m_n/k$ and $\xi_2 = m_n/k \exp(kR)$. This equation without the $\delta$-function terms is just the Bessel equation, the general solution being a linear combination of Bessel and Neumann functions $J_2(\xi)$ and $Y_2(\xi)$:

$$Z_2(\xi) = aJ_2(\xi) + bY_2(\xi).$$

The term with $\delta$-functions can be taken into account by imposing the boundary condition

$$Z_2(\xi) + \frac{2}{\xi} Z_2(\xi) = 0$$

at $\xi = \xi_1 = m_n/k$ and $\xi = \xi_2 = m_n/k \exp(kR)$. The first boundary condition, at $\xi = \xi_1$, can be satisfied by an appropriate choice of the coefficients $a$ and $b$:

$$Z_2(\xi) = Y_1 \left( \frac{m_n}{k} \right) J_2(\xi) - J_1 \left( \frac{m_n}{k} \right) Y_2(\xi).$$

(53)

The second boundary condition, at $\xi = \xi_2$, defines the mass spectrum of the theory and can be rewritten as

$$J_1(\xi_1)Y_1(t\xi_1) - J_1(t\xi_1)Y_1(\xi_1) = 0,$$

(54)
where \( t = \exp(kR) \). There exists a theorem about such combinations of products of Bessel and Neumann functions, which asserts that for \( t > 1 \) this combination is an even function of \( x \) and its zeros are real and simple \(^{14}\). Thus, the non-zero masses of tensor gravitons are defined by the positive zeros \( \gamma_n, n = 1, 2, \cdots \) of this combination and explicitly given by the formula

\[
m_n = \gamma_n k \equiv \beta_n \exp(-kR).
\]  

As it was discussed in many papers (see Refs. \([6, 7, 12, 13]\)), the product \( kR \) is chosen to be of the order 30 ÷ 35 in order to solve the hierarchy problem. Since \( \exp(-kR) \) is a tiny factor, the ratio \( m_n/k \ll 1 \) for eigenvalues with small \( n \), such that \( \beta_n \sim \mathcal{O}(1) \). Using the expansions for the Bessel functions for the small argument it is easy to see that such \( \beta_n \) are approximately given by the roots of the equation

\[
J_1(\beta_n) = 0.
\]

Let us describe the complete orthonormal system \( \{\psi_n(y), n = 0, 1, \cdots\} \) of solutions of equation (51), satisfying

\[
\int_{-R}^{R} e^{2k|y|}\psi_n'(y)\psi_n(y)dy = \delta_{mn}.
\]

The normalized functions \( \psi_n(y) \) corresponding to the mass eigenvalue \( m_0 = 0 \) and non-zero masses \( m_n = \gamma_n k > 0 \) are given by

\[
\psi_0(y) = N_0 e^{-2k|y|}, \quad N_0 = \frac{k^{\frac{3}{2}}}{(1 - \exp(-2kR))^{\frac{3}{2}}}, \\
\psi_n(y) = N_n \left( Y_1(\gamma_n) J_2(\gamma_n e^{k|y|}) - J_1(\gamma_n) Y_2(\gamma_n e^{k|y|}) \right), \\
N_n = \frac{k^{\frac{3}{2}}}{(\exp(2kR)Z_2(\gamma_n \exp(kR)) - Z_2(\gamma_n))^{\frac{3}{2}}}, \quad \gamma_n = \frac{m_n}{k} = \beta_n e^{-kR},
\]

where \( Z_2 \) was defined in (53). Now we calculate the values of the eigenfunctions for small \( n \) at \( y = 0 \) and \( y = R \), which will be used later. Taking into account that \( \exp(kR) \gg 1 \) we obtain the following expressions:

\[
\psi_0(0) = N_0 \approx \sqrt{k}; \quad \psi_n(0) \approx \frac{\sqrt{k}}{|J_2(\beta_n)|} e^{-kR}, \\
\psi_0(R) = N_0 e^{-2kR} \approx \sqrt{k} e^{-2kR}; \quad \psi_n(R) \approx -\sqrt{k} e^{-kR}.
\]

Finally, we expand \( b_{\mu\nu}(x, y) \) as follows:

\[
b_{\mu\nu}(x, y) = \sum_n b_{\mu\nu}^n(x) \psi_n(y),
\]

\( b_{\mu\nu}^n(x) \) being ordinary spin-2 transverse traceless 4-dimensional fields, which can be quantized by the standard procedure. Substituting this decomposition into (48), passing to the corresponding action, integrating explicitly over the orbifold and rescaling the radion field \( \phi \) in accordance with (50), we get an effective action for the physical degrees of freedom of the 5-dimensional gravitational field

\[
S_{eff} = \int \left( -\frac{1}{4} \sum_n (\partial_{\mu} b_{\rho\sigma}^n \partial^{\mu} b_{\rho\sigma}^n + m_n^2 b_{\rho\sigma}^n b_{\rho\sigma}^n) - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \right) dx.
\]
Thus, we have a massless spin-2 tensor field, an infinite tower of massive spin-2 tensor fields and just one massless scalar radion. A very important point is that, due to the form of the solution with zero mass (56), the remaining gauge transformations (46) result in the following gauge transformations of the field $b_{\mu\nu}^0(x)$:

$$b_{\mu\nu}^0(x) = b_{\mu\nu}^0(x) - \left(\partial_\mu \zeta_\nu(x) + \partial_\nu \zeta_\mu(x)\right), \quad \partial_\mu \zeta_\mu(x) = 0, \quad \partial_\mu \partial^\mu \zeta_\nu(x) = 0. \quad (62)$$

This guarantees that the field $b_{\mu\nu}^0(x)$ has only two degrees of freedom, and, therefore, can be identified with the field of the massless graviton.

4 Interaction with matter on the branes

Now we have to find the interaction of these fields with the matter on the branes. The general form of this interaction is standard,

$$\frac{\hat{k}}{2} \int_{B_1} h^{\mu\nu}(x,0) T_{\mu\nu}^1 dx + \frac{\hat{k}}{2} \int_{B_2} h^{\mu\nu}(x,R) T_{\mu\nu}^2 \sqrt{-\tilde{\gamma}(R)} dx, \quad (63)$$

where $T_{\mu\nu}^1$ and $T_{\mu\nu}^2$ are energy-momentum tensors of the matter on brane 1 and brane 2 respectively:

$$T_{\mu\nu}^{1,2} = \frac{\delta L^{1,2}}{\delta \gamma_{\mu\nu}} - \gamma_{\mu\nu} L^{1,2}. \quad (64)$$

Substituting (36) into (63), decomposing $b_{\mu\nu}(x,y)$ according to (60) and rescaling the field $\phi$ according to (50), we find that the interaction of the graviton field $b_{\mu\nu}^n(x)$ and the massless canonically normalized radion field $\phi$ with the matter on brane 1 is given by

$$\frac{1}{2} \int_{B_1} \left( \kappa_1 b_{\mu\nu}^0(x) T^{\mu\nu} + \kappa_1 \sum_{n=1}^{\infty} \psi_n(0) \frac{1}{N_0} b_{\mu\nu}^n(x) T^{\mu\nu} - \frac{\kappa_2}{\sqrt{3}} \phi T_{\mu}^{\mu} \right) dx. \quad (65)$$

Here

$$\kappa_1 = \hat{k} N_0, \quad \kappa_2 = \kappa_1 e^{-kR} \quad (66)$$

(the normalization factor $N_0$ was defined in (56)). Since $\kappa_1$ is the coupling constant of the massless graviton, it can be expressed in terms of the 4-dimensional gravitational constant $G_1$ on brane 1 as $\kappa_1 = \sqrt{16\pi G_1}$, which gives the relation

$$G_1 = \frac{\hat{k} G_k}{1 - e^{-2kR}} \quad (67)$$

between the constants on brane 1 [6 8].

To find the interaction of the 5-dimensional gravity with the matter on brane 2 turns out to be a more complicated problem, because the coordinates $\{x^\mu\}$ in (13), in which we work, are not Galilean on this brane (coordinates are called Galilean, if $g_{00} = -1, g_{11} = g_{22} = g_{33} = 1$, see [15]).

To solve this problem, we introduce the Galilean coordinates $\{z^\mu\}$ on brane 2 related to $\{x^\mu\}$ by $x^\mu = e^{kR} z^\mu$. When we do this in (63), we get for brane 2

$$\frac{\hat{k}}{2} \int_{B_2} h^{\prime \mu\nu}(z,R) T_{\mu\nu}^{\prime 2} dz, \quad (68)$$
where \( T^{a_{
u}, \mu} \) is the canonical energy-momentum tensor of the matter, \( h'_{\mu \nu}(z, R) = e^{2kR}h_{\mu \nu}(x, R) \), and the indices are raised with the flat metric. Thus, we see that in order to find the interaction of the 5-dimensional gravity with the matter on brane 2 we have to pass to the Galilean coordinates \( \{ z^\mu \} \) in the effective action (61) as well. This results in the following expression:

\[
S_{\text{eff}} = \int \left( -\frac{1}{4} \sum_n \left( e^{-2kR} \partial_\mu b^{n, \rho}_\sigma \partial^\mu b^{n, \rho}_\sigma + m_n^2 b^{n, \rho}_\sigma b^{n, \rho}_\sigma \right) - \frac{1}{2} e^{2kR} \partial_\mu \phi' \partial^\mu \phi' \right) dz, \tag{67}
\]

where the indices are also raised with the flat metric. To transform this expression to be a canonical action we have to rescale the fields \( b^{n, \rho}_\sigma, \phi' \) and the masses \( m_n \). We define the graviton fields \( u^{n, \rho}_\sigma = e^{-kR}b^{n, \rho}_\sigma \) and the radion field \( \varphi = e^{kR}\phi' \), as seen from brane 2, for which the expression (67) reduces to a canonical action in the Galilean coordinates \( \{ z^\mu \} \):

\[
S_{\text{eff}} = \int \left( -\frac{1}{4} \sum_n \left( \partial_\mu u^{n, \rho}_\sigma \partial^\mu u^{n, \rho}_\sigma + (m_n e^{kR})^2 u^{n, \rho}_\sigma u^{n, \rho}_\sigma \right) - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi \right) dz. \tag{68}
\]

Now we are able to write down the interaction of these fields with the matter on brane 2, which is found to be

\[
\frac{1}{2} \int_{B_2} \left( \kappa_2 u^{0}_{\mu \nu}(x) T^{\mu \nu} + \kappa_2 \sum_{n=1}^{\infty} \frac{\psi_n(R) e^{2kR}}{N_0} u^n_{\mu \nu}(x) T^{\mu \nu} - \frac{\kappa_1}{\sqrt{3}} \varphi T^n_\mu \right) \right) dz. \tag{69}
\]

The coupling constants \( \kappa_1 \) and \( \kappa_2 \) are defined by Eqs. (65). Since \( \kappa_2 \) is the coupling constant of the massless graviton, formally it gives the following relations between the 5-dimensional gravitational constant \( \hat{G} \) and the four-dimensional gravitational constant \( G_2 \) on brane 2

\[
G_2 = \frac{\hat{G}}{e^{2kR} - 1}, \tag{70}
\]

which coincides with the one found in [8].

5 Conclusions and discussion

In the present paper we have developed a Lagrangian description of linearized gravity in Randall-Sundrum model with two branes, which enabled us to find easily the physical degrees of freedom of this model and to construct an effective Lagrangian for them. If we take the limit \( R \to \infty \), the radion field \( \varphi \) drops from the Lagrangian, and we get the same degrees of freedom, as found in [11] for the case of infinite extra dimension.

A very important point of our study is the observation that the 5-dimensional gravity looks different, when viewed from different branes: the masses of the gravitational KK modes and the coupling constants of the massless fields differ in the exponential factor.

The case of finite extra dimension is more interesting from the point of view of possible phenomenology scenarios due to the presence of the radion field and of two coupling constants, \( \kappa_1 \) and \( \kappa_2 \). These scenarios will be discussed in a separate paper. Now we only mention two possibilities.

Let us assume that our brane is the brane with the positive energy density, in contrast to the standard RS1 scenario, so that the coupling of the massless mode \( \kappa_1 \sim 1/M_{Pl} \). If we
make a natural assumption that $kR \gg 1$, then $\kappa_1 \gg \kappa_2$, and we see from (58), (64) that the interactions of the massive KK modes and of the radion field with the matter on our brane are much weaker (exponentially suppressed) than that of the massless graviton. Therefore, the radion field does not affect Newton’s gravity on our brane, which arises from the interaction of the massless graviton in (64). The massive KK modes have masses of the order of $M_{pl}e^{-kR}$.

If one assumes that our brane is brane 2 with the negative energy density, then, as follows from (69), the interaction of massless radion field is not suppressed. In this case the constant $G_2$ can be identified with Newton’s constant, $G_2 = 1/M_{pl}^2$. The relation (70) gives the following relation between the Planck mass and the fundamental mass scale $M$ in the theory defined as $\hat{G} = 1/M^3$:

$$M_{pl}^2 = \frac{M^3}{k} \left( e^{2kR} - 1 \right) \approx \frac{M^3}{k} e^{2kR}.$$

(see [8]). By choosing, for example, $M \sim k \sim 1$ TeV we reproduce the correct value of the Planck mass, if the argument of the exponential factor satisfies $kR \approx 30 \div 35$. Then, as in the standard scenario, the couplings of the massive KK states and of the radion are of the order of 1 TeV$^{-1}$ [12, 13]. This can be easily seen from Eqs. (59), (69). The masses of the KK excitations, which can be read from Eqs. (55), (68), are of the order of 1 TeV for small $n$. The presence of the massless radion with such coupling leads to some predictions which are in contradiction with the available high energy physics data. To solve this problem, several mechanisms for generating a mass for the radion were proposed (see, for example, Ref. [16]).

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Note added

We decided to replace the text, because many misprints were found during these years. All of them have been corrected in the new version. We would like to thank M. Iofa, who pointed to us some of the misprints.

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