Composite Skyrme Model with Vector Mesons

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ABSTRACT

We study the composite Skyrme model, proposed by Cheung and Gürsey, introducing vector mesons in a chiral Lagrangian. We calculate the static properties of baryons and compare with results obtained from models without vector mesons.

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More than thirty years ago, Skyrme[1] pointed out that the non-linear sigma model[2] with a quartic derivative term admits stable soliton solutions which he suggested to identify as the baryons. Starting from this consideration, the phenomenological aspects of the Skyrme soliton have been widely explored in the past years. It was shown the computed static properties of the $SU(2) \times SU(2)$ Skyrme model agree moderately well with experiments.[3] An example was the predicted pion decay constant which is 30% lower than the experimental value. Various attempts such as adding higher-order terms to the Lagrangian[4], including the effect of the pion mass[5], and extending the symmetry to $SU(3) \times SU(3)$[6] have failed to increase significantly the value of the pion decay constant.

The work of Cheung and Gürsey generalizes the Skyrme model by introducing $U$ fields, for which $U^n$ not $U$ transforms linearly under $SU(2) \times SU(2)$.[7] Due to this invariance of the generalized model is not modified the current algebra result with the new parameter $n$. They found this modification without any tricks gives for $n=3$ almost experimentally observed value for the pion decay constant $F_\pi$ to be 185MeV. The fact that $n=3$ is singled out led these authors to consider the possibility that this composite Skyrme model describes effective quarks in a nucleon. Moreover some other predicted static properties were shown to agree better with the experiment than the original model($n=1$), while a disagreement persists for the magnetic properties.

After the work of Ref.[7], Nam and Workman study the effects of chiral symmetry breaking in the composite model.[8] But the predictions for the magnetic properties were not improved noticeably and the fit to the pion decay constant even got worse.

In this paper we introduce the $\omega$ vector mesons in the composite Skyrme model and calculate the static properties of a nucleon including the maganetic properties. This problem is interesting in light of the fact that the incorporation of additional low-lying mesons into the effective Lagrangian is the most important modification to construct a realistic theory and make predictions more accurate. In fact the static properties of the nucleon in the “$\omega$-stabilized” Skyrme model($n=1$) of Adkins and Nappi[4] constituted an improvement over the unadulterated Skyrme model. Especially the prediction for the magnetic moment of the proton(neutron) was improved from 1.97(-1.24) to 2.34(-1.46) while the experimental value is 2.79(-1.91). There has been much work to extend the nonlinear $\sigma$ model to the low-lying vector meson resonances such as the $\omega$, $\rho$, and $A_1$.[9][10] In this respect the models in this paper are certainly incomplete insofar as they include the $\omega$ meson only but this clearly has the advantage that one is able to obtain in a simpler way guiding results for a full-scale calculation.

The object of this paper is then to have a global view of the “$\omega$-stabilized” composite Skyrme model, with and without the pion mass term. The Lagrangian for our purpose is described in terms of composite chiral field $U^n$ coupled to an $\omega$ vector meson field:

$$
\mathcal{L} = -\frac{1}{4}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)(\partial^\mu \omega^\nu - \partial^\nu \omega^\mu) + \frac{1}{2}m_\omega^2 \omega_\mu \omega^\mu + \beta \omega_\mu B^\mu \\
+ \frac{1}{16n^2}F_\pi^2 \text{tr}(\partial_\mu U^n \partial^\mu U^{-n}) + \frac{1}{8n^2}F_\pi^2 m_\pi^2 (\text{tr} U^n - 2),
$$

(1)
where \( n \) is an arbitrary positive integer. Here \( m_\omega \) is the mass of the omega, \( m_\pi \) the mass of the pion and \( B^\mu \) the normalized baryonic current

\[
B^\mu = \frac{1}{24\pi^2n} \epsilon^{\mu\nu\alpha\beta} \text{tr}[(U^{-n}\partial_\nu U^n)(U^{-n}\partial_\alpha U^n)(U^{-n}\partial_\beta U^n)].
\] (2)

The model still retains the simplicity of the effective theory and the original model of Adkins and Nappi\[4\] is contained as a special case of \( n=1 \). Note that the choice of the coefficients in the Lagrangian (1) guarantees that the composite model has the same chiral Lagrangian description of the pion-pion interaction, etc. as in conventional model. In fact when we expand the kinematic part of the Lagrangian, we can obtain the usual kinematic term using

\[
U = \exp\left(2i\vec{\tau} \cdot \vec{\phi}/F_\pi\right) \approx 1 + 2i\vec{\tau} \cdot \vec{\phi}/F_\pi.
\] (3)

Similarly when we consider the coupling of the pions to the W gauged bosons, the coefficient of the term \( \partial_\mu \vec{\phi} \cdot W^\mu \) in \( (\partial_\mu + ig\vec{\tau} \cdot W_\mu)U^n \) which gives the transition amplitude for the pions on the W’s remains not modified with the parameter \( n \). The constant \( \beta \) is related to the strength of the coupling of the omega to three pions, but we will take it as a free parameter and determine it by fitting the properties of the nucleons following the treatment of Ref.\[4\].

As usual, we make the hedgehog ansatz for the soliton solution

\[
U(x) = U_0(x) = \exp[i\tau \cdot \hat{r} F(r)],
\] (4)

with the boundary condition \( F(0) = \pi, F(\infty) = 0 \). When we rotate the soliton with a time dependent \( SU(2) \) matrix \( A(t) \):

\[
U(t,x) = A(t)U_0(x)A^{-1}(t),
\] (5)

we obtain the Lagrangian

\[
L = -\tilde{M} + I\text{tr}[\partial_0 A \partial_0 A^\dagger] = -(4\pi F_\pi^2/m_\omega)\tilde{M} + (2\pi F_\pi^2/3m_\omega^3)\tilde{I}\text{tr}[\partial_0 A \partial_0 A^\dagger].
\] (6)

Here \( \tilde{M} \) and \( \tilde{I} \) are the dimensionless numbers related to \( F(r) \) and \( \omega = F_\pi \tilde{\omega} \) as

\[
\tilde{M} = \int_0^\infty dt\left[-\frac{1}{2} t^2 \tilde{\omega}'^2 - \frac{1}{2} t^2 \tilde{\omega}^2 + \tilde{\beta}\tilde{\omega}(\sin^2 nF)F' + \frac{1}{8} t^2 F'^2 + \frac{1}{4n^2} \sin^2 nF \right.
\]

\[
+ \frac{1}{4n^2}(m_\pi^2/m_\omega^2)t^2(1 - \cos nF)]
\]

\[
\tilde{I} = \frac{1}{n^2} \int_0^\infty t^2 \sin^2 nF dt + 2\tilde{\beta}^2 \int_0^\infty dt dt'[\sin^2 nF(t)]F'(t)[\sin^2 nF(t')]F'(t')B(t,t'),
\] (7)

where

\[
B(t,t') = \{\exp[-(t + t')](1 + t + t' + tt') - \exp(-|t - t'|)(1 + |t - t'| - tt')\}/tt'
\] (8)

and \( \tilde{\beta} = \beta m_\omega/2\pi^2 F_\pi \), \( r = t/m_\omega \).
From $\tilde{M}$ in eq.(7) we can numerically find $F(t)$ and $\tilde{\omega}(t)$ by employing relaxation techniques described in Ref.[1]. In Figs.1 and 2 we plot the solutions $F(t)$ and $\tilde{\omega}(t)$ for the models with the pion mass term. In these figures $\beta$ is chosen to fit the experimental value of the mass ratio

$$\frac{\tilde{M}}{\tilde{I}} = \frac{(5m_N - m_\Delta)(m_\Delta - m_N)}{36m_\omega^2} = 0.04605,$$

which is from the formula for the energy $E = M + J^2/2I$. Some obtained $\beta$ values are 4.9, 7.5, 8.6 for $n=1$, 4, 7. The pion decay constant $F_\pi$ corresponding to these values are 124, 121 and 120 MeV, showing essentially no dependence on $n$. When we apply this procedure on models without the pion mass term we obtain $\beta=3.3$, 2.0, 1.5 for $n=1$, 4, 7 with the pion decay constant 145, 194 and 219 MeV.

In order to compute some integrals for physical quantities with arbitrary values of $n$, we use following formulae:

$$\int d\Omega V^{i,a} = \frac{4}{3} \pi i [F_\pi^2 \frac{\sin^2 nF(r)}{n^2} + 2(\frac{\beta}{2\pi^2})^2 \frac{\sin^2 nF(r)}{r^2}] F'(r) \times \int_0^{\infty} dr' \frac{\sin nF(r')F'(r')}{m_\omega} B(m_\omega r, m_\omega r') \left[ (\partial_0 A) A^\dagger \tau^a \right],$$

$$\int d\Omega \mathbf{q} \cdot \mathbf{x} V^{i,a} = \frac{2\pi}{3} \left( \frac{1}{2n^2} F_\pi^2 + \frac{\beta \omega}{\pi^2} F'(r) \right) (\sin^2 nF) \left[ \tau^i \mathbf{q}^\dagger \tau^a A^\dagger \right],$$

$$\int d\Omega A^{i,a} = \frac{4}{3} \pi \left[ \frac{1}{4} F_\pi^2 (\frac{F'}{n} + \frac{\sin 2nF}{n^2 r}) + \frac{\beta \omega}{2\pi^2} (\frac{\sin 2nF}{r} F' + \frac{\sin^2 nF}{nr^2}) \right] \left[ \tau^i A^\dagger \tau^a \right].$$

We evaluate various static properties of nucleons using these formulae and compare them with the previous predictions of the Skyrme models. Table 1 shows the results from the broken chiral composite models having vector mesons and the pion mass term(BCVM) for $n=1$, 2 and 3. Note that $n=1$ case corresponds to the calculation of Adkins and Nappi.[4] They are compared with the predictions of the original($n=1^*$) and composite($n=3^*$) Skyrme model having the Skyrme term and the pion mass term(BCST). (results of $n=1^*$ are from [5] and results of $n=3^*$ are from [8].) The fourth row shows $\beta$ values for BCVM models, while it shows the $\epsilon$ values in the coefficient of Skyrme term for BCST models.[5][8] Similarly Table 2 shows results from the chiral theories having no pion mass term. Here results from the models with vector mesons(CVM) for $n=1$, 2, 3 are displayed in the first three columns while results from the original model (results of $n=1^*$ are from [3]) and composite model(results of $n=3^*$ are from [8][8][12]) having the Skyrme term(CST) are displayed in the next two columns.

As observed in previous studies, the pion mass term perturbs the chiral Skyrme soliton such that the value of $F_\pi$ is less than that found in chiral models. The excellent agreement of predicted $F_\pi$ with the experimental value in the chiral composite model(CST) is still seen in CVM model when $n=3$, while the predictions from broken chiral theories(BCVM or BCST) lie below the experimental value. They show small dependence on $n$. The various mean
Figure 1: The shape function $F(r)$ for values of $n=1 \beta=4.9$, $n=3 \beta=6.9$, $n=5 \beta=8.0$.

Figure 2: The omega field $\tilde{\omega}(r)$ for values of $n=1 \beta=4.9$, $n=3 \beta=6.9$, $n=5 \beta=8.0$. 
radii increase with n in all models and the addition of pion mass term increase them further. The models with vector mesons (CVM or BCVM) show steeper increase of mean radii with n than the models without vector mesons (CST or BCST). Generally the predictions for mean radii lie above the experimental values in all models.

The predictions for the proton magnetic moment increase with n and approach the experimental value in all models. Especially the results from CVM and BCVM are substantial improvements on those from CST and BCST. But the results from CVM, which are 2.18, 2.37, 2.45, 2.56 for n=1, 4, 7, 10, still lie below the experimental value 2.79. The best fit is obtained when n=2 for BCVM model with the calculated value 2.71. In the case of the neutron, the predicted values decrease with n and the best result -1.53 is obtained with CVM model when n=1. Thus the addition of vector mesons to n=1 models improves the result substantially both for the proton and neutron magnetic moment, while that which improves the result for \( \mu_p \) degrades the result for \( \mu_n \) for CVM and BCVM models when n>1. This fact is related with the results for \( \mu_{N\Delta} \). The calculations are 2.5-2.7 in CVM or BCVM models and 2.3 in CST or BCST models, showing small dependence on n. Since \( \mu_{N\Delta} \) is related to \( \mu_p - \mu_n \), the improvement of \( \mu_p \) with n results in the deterioration of \( \mu_n \). The axial coupling \( g_A \), which is calculated by integrating the axial current (12), increases with n for all models. The best result is given when n=2 for CVM or BCVM models and when n=3 for CST or BCST models. Another quantity which is related to \( g_A \) by the Goldberger-Treiman relation is the pion nucleon coupling constant \( g_{\pi NN} \). For theories with massless pions we calculated them from the long-distance behaviour of the shape function \( F(r) \), with the results in Table 2. Li and Liu argued that they can not be determined by the asymptotic behaviour of \( F(r) \) for theories with massive pions. Instead we use the formula

\[
g_{\pi NN} = \frac{4\pi}{9n}m^2_N F_\pi \int r^3 \sin nF(r)dr, \tag{13}\]

with the results shown in Table 1. It is explicitly checked in both cases that they satisfy the Goldberger-Treiman relation within the numerical errors. The calculated values of \( \sigma \) slightly decrease with n in BCVM such as 54, 61, 59, 56, 53, 50, 47, 45 MeV for n=1, 8, and approach the experimental value 36±20 MeV. The results are reasonable but are slightly larger than those from BCST.

The overall results from CVM or BCVM models seem to be more or less better than those from CST or BCST models. Especially the result for \( \mu_p \) is improved quite a lot while \( \mu_n \) is not. It was already observed that when we are away from the chiral limit, the worse numerical behaviour is resulted in the composite model. The addition of vector mesons in the composite models does not change this behaviour and the best fit from the composite models having vector mesons is obtained with the chiral theory (CVM) when n=3. Another strange prediction of the composite models was the peculiar oscillating behaviour of the nucleon charge density which is not seen experimentally. This behaviour still persists in CVM and BCVM models which we show in Fig. 3.
Figure 3: The charge density $\rho(t)$ for the proton and the neutron plotted as a function of the dimensionless variable $t$ with $n=2$ $\bar{\beta} = 6.1$. The density functions are normalized to give a unit charge for the proton.

Generally the composite Skyrme models have better description of nucleon properties compared to the original $n=1$ model, but it still needs more improvements to increase the phenomenological power. One possibility is to use different integers $n$ in $U^n$ for each term in the effective Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\rho + \mathcal{L}_\sigma + \mathcal{L}_\omega$ of Ref. [10].

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Table 1: Results for broken chiral theories with(BCVM) and without vector mesons(BCST).

| Quantity | n=1 | n=2 | n=3 | n=1* | n=3* | Experiment |
|----------|-----|-----|-----|------|------|------------|
| $M_N$    | Input | Input | Input | Input | Input | 938.9 MeV |
| $M_\Delta$ | Input | Input | Input | Input | Input | 1232 MeV  |
| $F_\pi$  | 124 MeV | 124 MeV | 123 MeV | 108 MeV | 127 MeV | 186 MeV    |
| $\bar{\beta}$ or e | 4.9 | 6.1 | 6.9 | 4.84 | 1.30 | |
| $\langle r^2 \rangle_{E,I=0}$ | 0.74 fm | 1.01 fm | 1.19 fm | 0.68 fm | 0.99 fm | 0.72 fm |
| $\langle r^2 \rangle_{E,I=1}$ | 1.06 fm | 1.43 fm | 1.65 fm | 1.04 fm | 1.65 fm | 0.88 fm |
| $\langle r^2 \rangle_{M,I=0}$ | 0.94 fm | 1.41 fm | 1.76 fm | 0.95 fm | 1.29 fm | 0.81 fm |
| $\langle r^2 \rangle_{M,I=1}$ | 1.03 fm | 1.38 fm | 1.58 fm | 1.04 fm | 1.28 fm | 0.80 fm |
| $\mu_p$  | 2.33 | 2.71 | 3.03 | 1.97 | 2.37 | 2.79 |
| $\mu_n$  | -1.46 | -1.10 | -0.80 | -1.24 | -0.84 | -1.91 |
| $g_A$    | 0.83 | 1.28 | 1.64 | 0.65 | 1.4  | 1.23 |
| $g_{\pi NN}$ | 12.5 | 19.4 | 24.9 | 11.9 | 21  | 13.5 |
| $\sigma$ | 54 Mev | 61 MeV | 59 MeV | 49 MeV | 46 MeV | 36±20 MeV |
| $g_{\pi N\Delta}$ | 18.8 | 29.2 | 37.4 | 17.8 | 31.5 | 20.3 |
| $\mu_{N\Delta}$ | 2.67 | 2.70 | 2.71 | 2.3 | 2.27 | 3.29 |

$n=1^*$ and $n=3^*$ denote the results for broken chiral theories without vector mesons.

Table 2: Results for chiral theories with(CVM) and without vector mesons(CST).

| Quantity | n=1 | n=2 | n=3 | n=1* | n=3* | Experiment |
|----------|-----|-----|-----|------|------|------------|
| $M_N$    | Input | Input | Input | Input | Input | 938.9 MeV |
| $M_\Delta$ | Input | Input | Input | Input | Input | 1232 MeV  |
| $F_\pi$  | 145 MeV | 166 MeV | 182 MeV | 129 MeV | 185 MeV | 186 MeV    |
| $\bar{\beta}$ or e | 3.3 | 2.8 | 2.3 | 5.44 | 1.72 | |
| $\langle r^2 \rangle_{E,I=0}$ | 0.64 fm | 0.79 fm | 0.85 fm | 0.59 fm | 0.74 fm | 0.72 fm |
| $\langle r^2 \rangle_{M,I=0}$ | 0.91 fm | 1.32 fm | 1.61 fm | 0.92 fm | 1.18 fm | 0.81 fm |
| $\mu_p$  | 2.18 | 2.31 | 2.32 | 1.87 | 2.03 | 2.79 |
| $\mu_n$  | -1.53 | -1.33 | -1.17 | -1.31 | -1.17 | -1.91 |
| $g_A$    | 0.78 | 1.25 | 1.60 | 0.61 | 1.30 | 1.23 |
| $g_{\pi NN}$ | 10.1 | 14.1 | 16.4 | 8.9 | 13.2 | 13.5 |
| $g_{\pi N\Delta}$ | 15.2 | 21.1 | 24.7 | 13.2 | 19.8 | 20.3 |
| $\mu_{N\Delta}$ | 2.62 | 2.57 | 2.47 | 2.3 | 2.26 | 3.29 |

$n=1^*$ and $n=3^*$ denote the results for chiral theories without vector mesons.