CERTAIN RESULTS ON $q$-STARLIKE AND $q$-CONVEX ERROR FUNCTIONS

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Abstract. The error function occurs widely in multiple areas of mathematics, mathematical physics and natural sciences. There has been no work in this area for the past four decades. In this article, we estimate the coefficient bounds with $q$-difference operator for certain classes of the spiralike starlike and convex error function associated with convolution product using subordination as well as quasi-subordination. Though this concept is an untrodden path in the field of complex function theory, it will prove to be an encouraging future study for researchers on error function.

1. Introduction and Preliminaries

The error function $erf$ defined by [1] p. 297

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} \exp(-t^2) dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!},$$

is the subject of intensive studies and applications during the last years. Several properties and inequalities of error function can be found in [5, 10, 13, 14]. In [11, 12] the authors study the properties of complementary error function occurring widely in almost every branch of applied mathematics and mathematical physics, e.g., probability and statistics [9] and data analysis [16].

Its inverse, introduced by Carlitz [8], which we will denote by $inverf$, appears in multiple areas of mathematics and the natural sciences. A few examples include concentration-dependent diffusion problems [31], solutions to Einstein’s scalar-field equations and in heat conduction problem [9, 27].

Now, we recall the definitions of fundamental class of analytic functions. Let $A$ denote the class of functions of the form

$$f(z) = z + a_2z^2 + \cdots,$$

which are analytic in the unit disk $U := \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by $f(0) = 0$, $f'(0) = 1$.

Also, let $S$ be the subclass of $A$ consisting of functions univalent in $U$. Here and subsequently, $\Omega$ denotes the class of analytic functions of the form

$$w(z) = w_1z + w_2z^2 + w_3z^3 + \cdots,$$

analytic and satisfying a condition $|w(z)| < 1$ in $U$, known as a class of Schwarz functions. To recall the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $U$. Then we say that the function $f$ is subordinate to $g$, if there exists a Schwarz function $w$, such that $f(z) = g(w(z))$, $z \in U$. We denote this subordination by $f \prec g$ (or $f(z) \prec g(z)$, $z \in U$). In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to the conditions $f(0) = g(0)$, $f(U) \subset g(U)$.

An extension of the notion of the subordination is the quasi-subordination introduced by Robertson in [37]. We call a function $f$ quasi-subordinate to a function $g$ in $U$ if there exist the Schwarz function $\omega$ and an analytic functions $\varphi$ satisfying $|\varphi(z)| < 1$ such that $f(z) = \varphi(z)g(\omega(z))$ in $U$. We then write $f \prec_\omega g$. If $\varphi(z) \equiv 1$ then the quasi-subordination reduces to the subordination. If we set $\omega(z) = z$, then $f(z) = \varphi(z)g(z)$ and we say that $f$ is majorized by $g$ and it is written as $f(z) \ll g(z)$ in $U$. Therefore quasi-subordination is a generalization of the notion of the subordination as well

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as the majorization that underline its importance. Related works of quasi-subordination may be found in [15, 25].

For \( f \) given by (1.2) and \( g \) with the Taylor series \( g(z) = z + b_2 z^2 + \cdots \) their Hadamard product (or convolution), denoted by \( f \ast g \), is defined as

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]

We will use also the symbol of the Hadamard product to the product of the class as follows

\[
(1.4) \quad B \ast C = \{ \psi \ast \phi : \psi \in B, \phi \in C \}.
\]

Let \( E_r f \) be a normalized analytic function which is obtained from (1.1), and give \( n \) by

\[
(1.5) \quad E_r f(z) = \sqrt{\frac{\pi}{2}} \text{erf}(\sqrt{z}) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^n.
\]

Applying a notation (1.4) we define a family of an analytic function as follows

\[
(1.6) \quad E = A \ast E_r = \left\{ F : F(z) = (f \ast E_r f)(z) = z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} a_n}{(2n-1)(n-1)!} z^n, f \in A \right\},
\]

where we denote by \( E \) the class that consists of a single function \( E_r f \).

Now, we refer to a notion of \( q \)-operators i.e. \( q \)-difference operator and \( q \)-integral operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of \( q \)-calculus was initiated by Jackson [17, 18]. He was the first mathematician who developed \( q \)-derivative and \( q \)-integral in a systematic way. Mohammed and Darus [30] studied approximation and geometric properties of \( q \)-operators in some subclasses of analytic functions in compact disk, Purohit and Raina [33], Kanas and R˘aducanu [26] have used the fractional \( q \)-calculus operators in investigations of certain classes of functions which are analytic in the open disk. Purohit [32] also studied these \( q \)-operators defined by using convolution of normalized analytic functions and \( q \)-hypergeometric functions. A comprehensive study on applications of \( q \)-calculus in operator theory may be found in [7]. Both operators play crucial role in the theory of relativity, usually encompasses two theories by Einstein, one in special relativity and the other in general relativity. Special relativity applies to the elementary particles and their interactions, whereas general relativity applies to the cosmological and astrophysical realm, including astronomy. Special relativity theory rapidly became a significant and necessary tool for theorists and experimentalists in the new fields of atomic physics, nuclear physics and quantum mechanics.

For \( 0 < q < 1 \) the Jackson’s \( q \)-derivative of a function \( f \in A \) is, by definition, given as follows

\[
(1.7) \quad D_q f(z) = \left\{ \begin{array}{ll}
\frac{f(z) - f(qz)}{(1-q)z} & \text{for} \quad z \neq 0, \\
f'(0) & \text{for} \quad z = 0,
\end{array} \right.
\]

and \( D_q^2 f(z) = D_q(D_q f(z)) \). From (1.7), we have

\[
D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},
\]

where

\[
(1.8) \quad [n]_q = \frac{1 - q^n}{1 - q},
\]

is sometimes called the basic number \( n \). If \( q \to 1^- \), \([n]_q \to n\). For a function \( h(z) = z^m \), we obtain

\[
D_q h(z) = D_q z^m = \frac{1 - q^m}{1 - q} z^{m-1} = [m]_q z^{m-1},
\]
and
\[
\lim_{q \to 1} D_q h(z) = \lim_{q \to 1} \left( \sum_{n=0}^{\infty} a_n q^n z^n \right) = mz^{n-1} = h'(z),
\]
where \( h' \) is the ordinary derivative. Jackson q-derivative satisfy known rules of differentiation, for example a q-analogue of Leibniz’ rule. As a right inverse, Jackson [18] introduced the \( q \)-integral of a function \( f \)
\[
\int_0^z f(t) dt = z(1-q) \sum_{n=0}^{\infty} a_n q^n z^n,
\]
provided that the series converges. For a function \( h(z) = z^n \), we have
\[
\int_0^z h(t) dt = \int_0^z t^m dt = \frac{z^{m+1}}{m+1} \quad (m \neq -1),
\]
and
\[
\lim_{q \to 1} \int_0^z h(t) dt = \int_0^z \frac{z^{m+1}}{m+1} = \frac{z^{m+1}}{m+1} \int_0^z h(t) dt,
\]
where \( \int_0^z h(t) dt \) is the ordinary integral.
As a consequence of (1.7), for \( F \in E \) we obtain
\[
(1.9) \quad D_q F(z) = 1 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(n-1)!} z^{n-1}.
\]
In the sequel we will use q-operators to the functions related to the conic sections, that were introduced and studied by Kanas et al. [19] – [25] and examined by several mathematicians in a series of papers, see for example Ramachandran et al. [34], Kanas and Răducanu [26], Sim et al. [38], etc. Kharasani [4] extended original definition to the p-valent functions generalizing the domains \( \Omega_k \) to \( \Omega_{k,\alpha} (0 \leq k < \infty, 0 \leq \alpha < 1) \) as follows:
\[
\Omega_{k,\alpha} = \{ w = u + iv : (u - \alpha)^2 > k^2(u - 1)^2 + k^2v^2 \}, \quad \Omega_{k,0} = \Omega_k.
\]
Various classes of functions were defined by the fact of the membership to the domain \( \Omega_{k,\alpha} \), for instance by setting \( w = p(z) = \frac{z^3}{f(z)} \) or \( p(z) = 1 + \frac{z^3}{f(z)} \). We note that the explicit form of function \( p_{k,\alpha} \) that maps the unit disk onto the domains bounded by \( \Omega_{k,\alpha} \) and such that 1 \( \in \Omega_{k,\alpha} \) is as follows
\[
p_{0,\alpha}(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad p_{1,\alpha}(z) = 1 + \frac{2(1 - \alpha)}{\pi^2} \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}},
\]
\[
p_{k,\alpha}(z) = \frac{(1 - \alpha)}{1 - k^2} \cos \left( A(k)i \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) - \frac{k^2 - \alpha}{1 - k^2} \quad (0 < k < 1),
\]
and
\[
p_{k,\alpha}(z) = \frac{(1 - \alpha)}{k^2 - 1} \sin^2 \left( \frac{\pi}{2\kappa(t)} K\left( \frac{\sqrt{z}}{\sqrt{1}}, t \right) \right) + \frac{k^2 - \alpha}{k^2 - 1} \quad (k > 1),
\]
where \( z \in \mathbb{U}, A(k) = \frac{2}{\pi} \arccos k \) and \( K(\omega, t) \) is the Legendre elliptic integral of the first kind
\[
K(\omega, t) = \int_0^\omega \frac{dx}{\sqrt{1 - x^2} \sqrt{1 - t^2x^2}} \quad (\kappa(t) = K(1, t)),
\]
with \( t \in (0, 1) \) chosen such that \( k = \cos \frac{\pi t}{2\kappa(t)} \).
By virtue of the properties of the domains, for \( p < p_{k,\alpha} \), we have
\[
(1.10) \quad \Re p(z) \geq \Re (p_{k,\alpha}(z)) > (k + \alpha)/(k + 1).
\]
Note that Kanas and Sugawa [23] proved the positivity of coefficients of the functions \( p_{k,0} \) that implies positivity of \( p_{k,\alpha} \) for \( 0 \leq \alpha < 1 \). Also, we note that the domains \( \Omega_{k,\alpha} \) are symmetric about real axis and starlike with respect to 1.

**Definition 1.1.** Let \( 0 \leq k < \infty \), \( 0 \leq \alpha < 1 \), \(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\), \( 0 < q < 1 \), \( b \neq 0 \), and let \( p_{k,\alpha}(z) \) be defined as above. A function \( f \in \mathcal{A} \) is in the class \( \mathcal{ES}^p_{q,b}(p_{k,\alpha}) \) if

\[
(1.11) \quad 1 + \frac{1}{b} \left( (1 + i \tan \beta) \left( \frac{zD_qF(z)}{F(z)} \right) - i \tan \beta - 1 \right) \prec p_{k,\alpha}(z) \quad (z \in \mathbb{U}).
\]

A function \( f \in \mathcal{A} \) is in the class \( \mathcal{EC}^p_{q,b}(p_{k,\alpha}) \) if

\[
1 + \frac{1}{b} \left( (1 + i \tan \beta) \left( \frac{zD_qF(z)}{F(z)} \right) - i \tan \beta - 1 \right) \prec p_{k,\alpha}(z) \quad (z \in \mathbb{U}).
\]

Let \( \phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \) \((c_1 > 0)\) be an analytic function with positive real part on \( \mathbb{U} \) which maps the open unit disk \( \mathbb{U} \) onto a region starlike with respect to 1 and symmetric with respect to 1 and symmetric with respect to the real axis.

**Definition 1.2.** Let \( 0 \leq k < \infty \), \( 0 \leq \alpha < 1 \), \(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\), \( 0 < q < 1 \), \( b \neq 0 \). By \( \mathcal{ES}^p_{q,b}(\phi) \) we mean a family that consist of the functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
1 + \frac{1}{b} \left( (1 + i \tan \beta) \left( \frac{zD_qF(z)}{F(z)} \right) - i \tan \beta - 1 \right) \prec_q \phi(z) - 1 \quad (z \in \mathbb{U}),
\]

and let the class \( \mathcal{EC}^p_{q,b}(\phi) \) consist of the functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
1 + \frac{1}{b} \left( (1 + i \tan \beta) \left( \frac{zD_qF(z)}{F(z)} \right) - i \tan \beta - 1 \right) \prec_q \phi(z) - 1 \quad (z \in \mathbb{U}).
\]

The principal significance of the sharp bounds of the coefficients is the information about geometric properties of the functions. For instance, the sharp bounds of the second coefficient of normalized univalent functions readily yields the growth and distortion bounds. Also, sharp bounds of the coefficient functional \( |a_3 - \mu a_2^2| \) obviously help in the investigation of univalence of analytic functions. Apart from these \( n \)-th coefficient bounds were used to determine the extreme points of the classes of analytic functions. Estimates of Fekete-Szegő functional for various subclasses of univalent and multivalent functions were given, among other, in [26, 35, 36].

In this paper, we obtain coefficient estimates for the functions in the above defined class for \( q \)-difference operator associated with subordination and quasi subordination.

The following lemma is needed to prove our main results. Lemma [14] is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [29].

**Lemma 1.1.** \[8\] If \( w \in \Omega \), then

\[
|w_2 - tw_1^2| \leq \begin{cases} 
-t & \text{if } t < -1, \\
1 & \text{if } -1 \leq t \leq 1, \\
t & \text{if } t > 1.
\end{cases}
\]

When \( t < -1 \) or \( t > 1 \), the equality holds if and only if \( w(z) = z \) or one of its rotations. If \(-1 < t < 1 \), then equality holds if and only if \( w(z) = z^2 \) or one of its rotations. Equality holds for \( t = -1 \) if and only if

\[
w(z) = z \frac{\lambda + z}{1 + \lambda z} \quad (0 \leq \lambda \leq 1)
\]

or one of its rotations, while for \( t = 1 \) the equality holds if and only if

\[
w(z) = -z \frac{\lambda + z}{1 + \lambda z} \quad (0 \leq \lambda \leq 1)
\]

or one of its rotations.
Although the above upper bound is sharp, it can be improved in the case, when $-1 < t < 1$
\[
|w_2 - tw_1^2| + (1 + t)|w_1|^2 \leq 1 \quad (-1 < t \leq 0),
\]
\[
|w_2 - tw_1^2| + (1 - t)|w_1|^2 \leq 1 \quad (0 < t < 1).
\]

2. The Fekete-Szegő functional associated with conical sections

In this section we will consider the behavior of the Fekete-Szegő functional defined on the classes related to the conical domains

**Theorem 2.1.** Let $0 \leq k < \infty$, $0 \leq \alpha < 1$, and let $p_{k,\alpha}(z) = 1 + p_1z + p_2z^2 + \cdots$. Set
\[
\sigma_1 = \frac{10(p_1 + p_2)\vartheta_2^2 + 10\vartheta_2 bp_1^2}{9\vartheta_3 bp_1^2}, \quad \sigma_2 = \frac{10(p_1 - p_2)\vartheta_2^2 - 10\vartheta_2 bp_1^2}{9\vartheta_3 bp_1^2},
\]
and
\[
\sigma_3 = \frac{10\vartheta_2 p_2 + 10\vartheta_2 bp_1^2}{9\vartheta_3 bp_1^2}.
\]
If $f$ given by (1.2) belongs to $\mathcal{S}^q_{\vartheta_1,\vartheta_2}(p_{k,\alpha})$, then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 10bp_1 \vartheta_3 \left( \frac{p_2}{p_1} + \frac{9\mu \vartheta_3 - 10\vartheta_2}{10\vartheta_2^2} bp_1 \right) & \text{if } \mu < \sigma_1, \\ 10bp_1 \vartheta_3 \left( \frac{p_2}{p_1} + \frac{10\vartheta_2 - 9\mu \vartheta_3}{10\vartheta_2^2} bp_1 \right) & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ 10bp_1 \vartheta_3 \left( \frac{p_2}{p_1} + \frac{10\vartheta_2 - 9\mu \vartheta_3}{10\vartheta_2^2} bp_1 \right) & \text{if } \mu > \sigma_2, \end{cases}
\]
where $\vartheta = 1 + i \tan \beta$, $\vartheta_2 = [2]_q - 1$ and $\vartheta_3 = [3]_q - 1$. Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then
\[
|a_3 - \mu a_2^2| + \frac{10\vartheta_2^2}{9\vartheta_3 bp_1} \left( p_1 + p_2 + \frac{10\vartheta_2 - 9\mu \vartheta_3}{10\vartheta_2^2} bp_1^2 \right) |a_2|^2 \leq \frac{10bp_1}{\vartheta_3},
\]
and, if $\sigma_3 \leq \mu \leq \sigma_2$, then
\[
|a_3 - \mu a_2^2| + \frac{10\vartheta_2^2}{9\vartheta_3 bp_1} \left( p_1 - p_2 - \frac{10\vartheta_2 - 9\mu \vartheta_3}{10\vartheta_2^2} bp_1^2 \right) |a_2|^2 \leq \frac{10bp_1}{\vartheta_3}.
\]

For any complex number $\mu$,
\[
|a_3 - \mu a_2^2| \leq \frac{10bp_1}{\vartheta_3} \max \left\{ 1, \frac{p_2}{p_1} + \frac{10\vartheta_2 - 9\mu \vartheta_3}{10\vartheta_2^2} bp_1 \right\}.
\]

**Proof.** If $f \in \mathcal{S}^q_{\vartheta_1,\vartheta_2}(p_{k,\alpha})$, then there is a Schwarz function $w \in \Omega$ of the form (1.3) such that
\[
1 + \frac{1}{b} \left( (1 + i \tan \beta) \left( \frac{zD_qF(z)}{F(z)} \right) - i \tan \beta - 1 \right) = p_{k,\alpha}(w(z)).
\]
We note that
\[
\frac{zD_qF(z)}{F(z)} = 1 + \frac{1 - [2]_q}{3} a_2 z + \left( \frac{[3]_q - 1}{10} a_3 + \frac{1 - [2]_q}{9} a_2^2 \right) z^2 + \ldots
\]
and
\[
p_{k,\alpha}(w(z)) = 1 + p_1 w_1 + (p_1 w_2 + p_2 w_1^2) z^2 + (p_1 w_3 + 2p_2 w_1 w_2 + p_3 w_1^3) z^3 + \cdots.
\]
Applying (2.5), (2.6) and (2.7), we obtain
\[
a_2 = \frac{3bp_1 w_1}{(1 + i \tan \beta)(1 - [2]_q)},
\]
and
\[ a_3 = \frac{10bp_1}{[3]q - 1} \left( w_2 - \left( \frac{p_2}{p_1} + \frac{p_1}{(1 + i \tan \beta)(1 - [2]q)} \right) w_2^2 \right). \]
Hence, by (2.8), (2.9), we get the following
\[ a_3 - \mu a_2^2 = \frac{10bp_1}{(1 + i \tan \beta)([3]q - 1)} (w_2 - tw_1^2), \]
where \( t = \frac{p_2}{p_1} + \frac{10(1 - [2]q) - 9\mu([3]q)b_p_1}{10(1 + i \tan \beta)(1 - [2]q)^2}. \)

The results (2.1) – (2.3) are established by an application of Lemma 1.1 and using the notation \( \varrho = 1 + i \tan \beta, \vartheta_2 = [2]q - 1 \) and \( \vartheta_3 = [3]q - 1. \) To show that the bounds in (2.1) – (2.4) are sharp, we define the function \( g_{\varphi n} \) (\( n = 2, 3, \ldots \)) by
\[ 1 + \frac{1}{b} \left( 1 + i \tan \beta \right) \left( \frac{zD_q g_{\varphi n}(z)}{g_{\varphi n}(z)} \right) = p_{k,\alpha} \left( z^{n-1} \right), \]
with \( g_{\varphi n}(0) = 0 = g_{\varphi n}'(0) - 1 \), and the function \( h_\lambda \) and \( k_\lambda \) (\( 0 \leq \lambda \leq 1 \)) by
\[ 1 + \frac{1}{b} \left( 1 + i \tan \beta \right) \left( \frac{zD_q h_\lambda(z)}{h_\lambda(z)} \right) = p_{k,\alpha} \left( -z^{n-1} \right), \]
\[ 1 + \frac{1}{b} \left( 1 + i \tan \beta \right) \left( \frac{zD_q k_\lambda(z)}{k_\lambda(z)} \right) = p_{k,\alpha} \left( -z^{n-1} \right), \]
with \( h_\lambda(0) = h_\lambda'(0) = 1 = k_\lambda(0) = k_\lambda'(0) = 1 = 0. \) Clearly \( g_{\varphi n}, h_\lambda, l_\lambda \in \mathcal{E}_{q,b}(k_{\alpha,\alpha}). \) Also we set \( g_\phi := g_{\varphi 2}. \) If \( \mu < \sigma_1 \) or \( \mu > \sigma_2, \) then the equality holds if and only if \( f \) is \( g_\phi \) or one of its rotations. When \( \sigma_1 < \mu < \sigma_2, \) the equality holds if and only if \( f \) is \( g_{\varphi 3} \) or one of its rotations. If \( \mu = \sigma_1, \) then the equality holds if and only if \( f \) is \( h_\lambda \) or one of its rotations. If \( \mu = \sigma_2, \) then the equality holds if and only if \( f \) is \( k_\lambda \) or one of its rotations.

The following result may be proved in much the same way as Theorem 2.1 (we also use a notation \( \varrho = 1 + i \tan \beta \)).

**Theorem 2.2.** Let \( 0 \leq k < \infty, \ 0 \leq \alpha < 1, \) and let \( p_{k,\alpha}(z) = 1 + p_1 z + p_2 z^2 + \cdots. \) For \( f \in \mathcal{E}_{q,b}(k_{\alpha,\alpha}), \) it holds
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{5bp_1}{[3]q} \left( \frac{p_2}{p_1} + \frac{bp_1}{\varrho} - \frac{9\mu([3]q)b_p_1}{5[2]q^2 \varrho} \right) & \text{if } \mu < \sigma_1, \\ \frac{5bp_1}{[3]q} \left( \frac{p_2}{p_1} + \frac{bp_1}{\varrho} - \frac{9\mu([3]q)b_p_1}{5[2]q^2 \varrho} \right) & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{5bp_1}{[3]q} \left( \frac{p_2}{p_1} + \frac{bp_1}{\varrho} - \frac{9\mu([3]q)b_p_1}{5[2]q^2 \varrho} \right) & \text{if } \mu > \sigma_2, \end{cases} \]
where
\[ \sigma_1 = \frac{5[2]q^2 \varrho}{9[3]q b_p 1} \left( p_2 - p_1 + \frac{bp_1}{\varrho} \right), \quad \sigma_2 = \frac{5[2]q^2 \varrho}{9[3]q b_p 1} \left( p_1 + p_2 + \frac{bp_1}{\varrho} \right), \]
and
\[ \sigma_3 = \frac{5[2]q^2 \varrho}{9[3]q b_p 1} \left( p_2 + \frac{bp_1}{\varrho} \right). \]
Further, if \( \sigma_1 \leq \mu \leq \sigma_3, \) then
\[ |a_3 - \mu a_2^2| + \frac{5[2]q^2 \varrho}{9\mu([3]q)b_p 1} \left( p_1 - p_2 - \frac{bp_1}{\varrho} + \frac{9\mu([3]q)b_p_1}{5[2]q^2 \varrho} \right) |a_2| \leq \frac{5bp_1}{[3]q}. \]
and, if $\sigma_3 \leq \mu \leq \sigma_2$, then
\[ |a_3 - \mu a_2^2| + \frac{5|2|^2q}{2p_1 p_2 + \frac{b p_1^2}{q} - \frac{9\mu [3]_q b p_1}{2|2|^2q}} |a_2|^2 \leq \frac{5bp_1}{q[3]_q}. \]

For any complex number $\mu$
\[ |a_3 - \mu a_2^2| \leq \frac{5bp_1}{q[3]_q} \max \left\{ 1, \frac{p_2 + \frac{b p_1}{q} - \frac{9\mu [3]_q b p_1}{2|2|^2q}}{p_1} \right\}. \]

3. The Fekete-Szegő functional associated with quasi-subordination

We now direct our attention to the extension of the subordination idea. The principal difference is the assertion of quasi-subordination. We are thus led to the following strengthening of Theorem 2.1 and 2.2

**Theorem 3.1.** Let $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, $0 < q < 1$, $b \neq 0$ and let $\phi = 1 + i \tan \beta$. If $f$ of the form (1.2) belongs to $\mathcal{E}S_{q,b}^\beta (\phi)$, then
\[ |a_2| \leq \frac{3bc_1}{q(1 - [2]_q)}, \]

(3.1)
\[ |a_3| \leq \frac{10b}{q([3]_q - 1)} \left( c_1 + \max \left\{ |c_1|, \frac{|bc_1^2|}{q(2|2| - 1)} + |c_2| \right\} \right), \]

and for any complex number $\mu$,
\[ |a_3 - \mu a_2^2| \leq \frac{10b}{q([3]_q - 1)} \left( c_1 + \max \left\{ |c_1|, \frac{10(1 - [2]_q) + 9b([3]_q - 1)}{10q(1 - [2]_q)^2} \right\} |bc_1^2| + |c_2| \right\}. \]

(3.2)

Proof. If $f \in \mathcal{E}S_{q,b}^\beta (\phi)$, then there exist analytic functions $\varphi$ and $\omega$ with $|\varphi(z)| \leq 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that
\[ 1 + \frac{1}{b} \left( \varphi \left( \frac{zD_q F(z)}{F(z)} \right) - \varphi \right) = \varphi(z) (\phi(\omega(z)) - 1). \]

Since
\[ \frac{zD_q F(z)}{F(z)} = 1 + \frac{1 - [2]_q}{3} a_2 z + \left( \frac{[3]_q - 1}{10} a_3 + \frac{1 - [2]_q}{9} a_2 \right) z^2 + \ldots \]

(3.4)
\[ \varphi(z) (\phi(\omega(z)) - 1) = c_1 d_1 d_1 z + (c_1 d_1 d_1 + d_0 (c_1 d_1 + d_2 c_2)) z^2 + \ldots \]

From (3.3), (3.3) and (3.5), we get
\[ a_2 = \frac{3bc_1 d_0}{q(1 - [2]_q)} \]
\[ a_3 = \frac{10b}{q([3]_q - 1)} \left( c_1 d_1 d_1 + c_1 d_0 d_2 + d_0 \left( c_2 - \frac{b c_1 d_0}{q(1 - [2]_q)} \right) \omega_1^2 \right) \]
\[ |a_3 - \mu a_2^2| \leq \left\{ c_1 d_1 d_1 \right\} \left| \frac{10b}{q([3]_q - 1)} \left[ c_1 d_1 d_1 + c_1 d_0 \left( \omega_2 - \left( \frac{b c_1 d_0}{q(1 - [2]_q)} + \frac{9b c_1 d_0 [3]_q - 1}{10q(1 + i \tan \beta)(1 - [2]_q)^2} - \frac{c_2}{c_1} \right) \omega_1^2 \right) \right| \].

Since $\varphi$ is analytic in $U$, using the inequalities $|d_n| \leq 1$ and $|\omega_1| \leq 1$, we get
\[ |a_2| \leq \frac{3bc_1}{q(1 - [2]_q)}. \]

(3.6)
Applying Lemma [1.1] to
\[ \omega_2 = \left( \frac{c_2}{c_1} - \left[ \frac{bc_1}{\theta(1 - [2]q)} + \frac{9\mu b c_1 ([3]_q - 1)}{10\theta(1 - [2]q)^2} \right] \right) \omega_1^2, \]
which yields
\[ |a_3 - \mu a_2^2| \leq \frac{10b}{\theta([3]_q - 1)} \left( c_1 + \max \left\{ c_1, \left| \frac{10(1 - [2]q) + 9\mu b ([3]_q - 1)}{10\theta(1 - [2]q)^2} \right| bc_1^2 + |c_2| \right\} \right), \]
For \( \mu = 0 \), we get
\[ |a_3| \leq \frac{10b}{\theta([3]_q - 1)} \left( c_1 + \max \left\{ c_1, \left| \frac{bc_1^2}{\theta} \right| + |c_2| \right\} \right). \]

Analysis similar to that in the proof of the previous Theorem shows that

**Theorem 3.2.** Let \( -\frac{\pi}{2} < \beta < \frac{\pi}{2} \), \( 0 < q < 1 \) and \( b \neq 0 \). If \( f \) given by [1.2] belongs to \( \tilde{E}S_{q,b}(\phi) \), then
\[ |a_2| \leq \frac{3bc_1}{\theta([2]_q)}, \]
\[ |a_3| \leq \frac{5b}{\theta([3]_q)} \left( c_1 + \max \left\{ c_1, \frac{bc_1^2}{\theta} \right\} \right), \]
and for any complex number \( \mu \),
\[ |a_3 - \mu a_2^2| \leq \frac{5b}{\theta([3]_q)} \left( c_1 + \max \left\{ c_1, \frac{9\mu b c_1 ([3]_q - 1)}{10\theta(1 - [2]q)^2} \right\} bc_1^2 + |c_2| \right\} \).

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