Heat Dissipation Rate in a Nonequilibrium Viscoelastic Medium

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Living non-Newtonian matter like cell cortex and tissues are driven out-of-equilibrium at multiple spatial and temporal scales. The stochastic dynamics of a particle embedded in such a medium is non-Markovian, given by a generalized Langevin equation. Due to the non-Markovian nature of the dynamics, the heat dissipation and entropy production rate cannot be computed using the standard methods for Markovian processes. In this work, to calculate heat dissipation, we use an effective Markov description of the non-Markovian dynamics which includes the degrees-of-freedom of the medium. Specifically, we calculate entropy production and heat dissipation rate for a spherical colloid in a non-Newtonian medium whose rheology is given by a Maxwell viscoelastic element in parallel with a viscous fluid element, connected to different temperature baths. This problem is nonequilibrium for two reasons, the medium is nonequilibrium due to different effective temperatures of the bath, and the nonequilibrium driving of the particle by an external stochastic force. When the medium is nonequilibrium, the effective non-Markov dynamics of the particle may lead to negative value of heat dissipation and entropy production rate. The positivity is restored when the medium's degree-of-freedom is considered. When the medium is at equilibrium and the only nonequilibrium component is the external driving, the correct dissipation is obtained from the effective description of the particle.

I. INTRODUCTION

A non-equilibrium steady state (NESS) is maintained by a constant input of energy. The NESS is characterized by a positive entropy production rate (EPR) and heat dissipation rate (HDR) \[1–3\]. For a colloidal suspension, the NESS is also characterized by the violation of fluctuation dissipation relation. The dissipation in the linear response regime for a Newtonian fluid where the friction is constant can be computed using the Harada-Sasa relation \[4, 5\].

However, in general, the rheology of living and non-living active matter is non-Newtonian \[6, 7\]. Moreover, these systems are driven out-of-equilibrium by processes operating at multiple spatial and temporal scales. For instance, the cell cortex in specific contexts can be modeled as a Maxwell fluid with relaxation time depending on factors like the cross-linker turnover rate, actin polymerization, and depolarization rate \[8\]. At a larger scale, the fluidity of tissues is governed by T1 transitions regulated by the activity of actomyosin \[9, 10\]. Most if not all of these processes are non-equilibrium. This makes it different from the passive non-Newtonian matter where the complex rheology is a result of relaxation at multiple timescales that do not consume energy \[7, 11\].

Motivated by the examples of non-Newtonian active matter, in this work we compute EPR and HDR for a colloidal particle in a non-Newtonian fluid, driven out-of-equilibrium at different scales.

In ref. \[12\] a generalization of Harada-Sasa relation to calculate HDR of a particle in a non-Newtonian medium is proposed. In the following, through an example, we show that the proposed generalization does not give the correct HDR and EPR. We show that, to calculate HDR of a particle in a nonequilibrium medium it is necessary to include the medium’s dissipative degrees of freedom (DOF) along with that of the particles DOF. We find that, ignoring the medium’s DOF may lead to a negative EPR and when it is included in the dynamics the EPR is always positive. This work is also of interest in understanding the dissipation in non-Markovian systems in general \[13–16\].

In this work, we consider a spherical colloid embedded in a complex medium comprising of a Maxwell viscoelastic element \[17, 18\] in parallel to a viscous fluid element (see Fig. 1), with an external fluctuating force \(f\) acting on it. The stochastic dynamics of the colloid is given by a generalized Langevin equation (GLE). We represent the GLE which is non-Markovian by an equivalent Markovian dynamics and use Harada-Sasa relation to compute the HDR and EPR. The results thus obtained are compared with that of the expression proposed in ref. \[12\]. We find that the above result is only true when the non-Newtonian medium is passive.

In the following, first, we briefly outline the derivation of the Langevin dynamics of the particle from the stress equations and then calculate the HDR for the GLE thus obtained using the relation proposed in ref. \[12\]. We then obtain the HDR using the effective Markov description of the particle and summarize the similarity and differences in the two cases.

II. PARTICLE IN A VISCOELASTIC MEDIUM

Consider a viscoelastic medium schematic shown in Fig. 1 consisting of a Maxwell viscoelastic element and a viscous fluid element in parallel. The total stress in the
medium is
\[ \sigma_{ij} = -p \delta_{ij} + \sigma^1_{ij} + \sigma^2_{ij}, \] (1)
where \( p \) is the pressure and \( \sigma^1_{ij} \) and \( \sigma^2_{ij} \) are the symmetric traceless part of the stress tensor due to the viscous and Maxwell viscoelastic element respectively. The viscous stress is given by
\[ \sigma^1_{ij} = \eta (\partial_i v_j + \partial_j v_i) + \partial^2_{ij}, \] (2)
where \( \eta \) is the viscosity, \( v \) is the velocity, and \( \partial^2_{ij} \) is the stochastic component of the stress due to the temperature bath \( T_1 \). The Maxwell stress \( \sigma^2_{ij} \) is given by
\[ \sigma^2_{ij} = \gamma (\partial_i v_j + \partial_j v_i) + \partial^2_{ij} \] (3)
where \( \gamma = \gamma/B \) is the Maxwell relaxation time, \( B \) is the elastic modulus, \( \gamma \) is the viscosity, and \( \partial^2_{ij} \) is the stochastic component of the stress due to the temperature bath \( T_2 \). The variance of the stochastic component of the stresses is
\[ \langle \partial^2_{ij}(x,t)\partial^2_{kl}(x',t') \rangle = 2T_0 \eta \delta(x - x') \delta(t - t') \left[ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right]. \] (4)
where \( \alpha \in (1, 2) \). The incompressibility condition is \( \nabla \cdot v = 0 \) and the dynamics in the Stokes limit is \( \nabla \cdot \sigma = 0 \). In Fourier space Eq. 1 reads (through the text tilde denotes the Fourier transform defined as \( \tilde{\phi}(\omega) = \int_{-\infty}^{\infty} dt \phi(t) e^{-i\omega t} \))
\[ \tilde{\sigma}_{ij}(\omega) = -p \delta_{ij} + \left( \frac{\gamma}{-i\omega\tau + 1} + \eta \right) (\partial_i \tilde{v}_j + \partial_j \tilde{v}_i) + \tilde{\partial}^2_{ij} \] (5)
Stochastic dynamics of a spherical colloidal particle of radius \( a \) embedded in this medium is obtained by integrating the stress in Eq. 5 over the surface of the sphere and using no-slip boundary condition leading to the GLE (see appendix of ref. 21 for derivation)
\[ -\omega^2 m \ddot{x} - i\omega \gamma(\omega) \dot{x} = -k \ddot{x} + \ddot{f}_i + \dot{\xi}_1 + \dot{\xi}_2, \] (6)
where \( m \) is the mass of the particle, \( k \) is the stiffness of the harmonic potential, \( f \) is the external stochastic force applied on the particle, and \( \xi_1 \) is zero mean Gaussian white noise of variance \( \langle \xi_1(\omega) \xi_1(\omega') \rangle = 2T_1 \gamma \delta(\omega + \omega') \), and \( \xi_2 \) is zero mean Gaussian noise of variance \( \langle \xi_2(\omega) \xi_2(\omega') \rangle = 2T_2 \gamma \delta(\omega + \omega')/(\omega^2 \tau^2 + 1) \), and the frequency dependent friction coefficient
\[ \gamma(\omega) = \left( \frac{\gamma_1 + \frac{\gamma_2}{-i\omega\tau + 1}}{-i\omega\tau + 1} \right), \] (7)
where \( \gamma_1 = 6\pi \eta a \) and \( \gamma_2 = 6\pi \gamma a \). In the time domain Eq. 6 reads
\[ m \ddot{x} + \frac{\gamma_2}{\tau} \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \dot{x} + \gamma_1 \ddot{x} = -k \ddot{x} + f_i + \dot{\xi}_1 + \dot{\xi}_2, \] (8)
where \( \xi_1 \) and \( \xi_2 \) are noise sources either thermal or active with correlations are \( \langle \xi_1(t) \xi_2(t') \rangle = 2T_2 \gamma e^{-(t-t')/\tau}/\tau \) and \( \langle \xi_1(t) \xi_1(t') \rangle = 2T_1 \gamma \delta(t - t') \), and \( f \) is an external fluctuating force with correlation \( \langle f(t)f(t') \rangle = 2\Lambda_t e^{-(t-t')/\tau}/\tau \). The correlation function is defined as
\[ C_{\dot{x}\dot{x}}(t) = \langle (\dot{x}(t) - \langle \dot{x} \rangle)(\dot{x}(0) - \langle \dot{x} \rangle) \rangle, \] (9)
and the response function is defined by the relation
\[ \langle x(t) - \langle x \rangle \rangle = v_s + \epsilon \int_{-\infty}^{t} \chi(t - t') f_p(t') dt', \] (10)
where the averages with subscript \( \epsilon \) are over the steady state with perturbed force \( f_p(t) \) and averages without subscript are for \( \epsilon = 0 \). The dynamics given by Eq. 8 is out of equilibrium when the fluctuation dissipation relation is not satisfied, i.e.,
\[ \tilde{C}_{\dot{x}\dot{x}}(\omega) \neq 2T \tilde{\chi}(\omega). \] (11)
where \( T \) is the temperature. Through the text prime in superscript denotes the real component and double prime the imaginary component. The equilibrium limit is obtained when fluctuations \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \) are thermal, i.e., \( T_1 = T_2 = T \) where \( T \) is the temperature of the bath and \( \Lambda_\alpha = 0 \). In the following, we calculate the EPR and HDR at NNESS when the system is out of equilibrium.

### III. Entropy Production Rate

For the stochastic dynamics given by
\[ \gamma_i \dot{x}_i = f_i + \xi_i, \] (12)
where \( i \in (1, ..., N) \), and \( \xi_i \) is a Gaussian white noise with correlation \( 2T \gamma_i \gamma_i \) is the friction coefficient, and \( f_i \) is force on the variable \( i \), using Harada-Sasa relation the heat dissipated by variable \( i \) is given by
\[ h_i = \gamma_i (\dot{x}_i)^2 + \gamma_i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( \tilde{C}_{\dot{x}_i\dot{x}_i}(\omega) - 2T_i \tilde{\chi}(\omega) \right). \] (13)
The total HDR \( h \) and the EPR \( s \) is
\[
h = \sum_{i=1}^{N} h_i \quad \text{and} \quad s = \sum_{i=1}^{N} \frac{h_i}{T_i},
\]
(14)
At equilibrium \( T_i = T \) and \( \tilde{C}_{i;\dot{x}_i} = 2T\chi_i'(\omega) \), for which \( h_i = 0 \). In ref. [12] an expression for computing heat dissipation rate for the equation
\[
\int_{-\infty}^{t} dt' \gamma(t-t')\dot{x}(t') = -\Gamma x(t) + \xi(t),
\]
(15)
where \( t \in (1, ..., N) \) was proposed by generalizing the Harada-Sasa relation to
\[
h_{\text{NM}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\gamma}'(\omega) \left( \tilde{C}_{x\dot{x}}(\omega) - 2T\tilde{\chi}_i'(\omega) \right),
\]
(16)
where \( \tilde{\gamma}'(\omega) \) is the real component of the friction and \( T \) is the temperature of the bath. We now calculate the heat dissipation using Eq. (16) for the dynamics given by Eq. (6).

From Eq. (7) we get
\[
\gamma'(\omega) = \gamma_1 + \frac{\gamma_2}{\omega^2 + 1}.
\]
(17)
Rearranging Eq. (8)\]
\[
\Gamma(\omega)\tilde{x}(\omega) = \dot{\bar{f}}(\omega) + \xi_1(\omega) + \frac{\xi_2}{i\omega\tau + 1},
\]
(18)
where
\[
\Gamma(\omega) = \left( k - \omega^2m - \frac{i\gamma_2\omega}{-i\omega\tau + 1} - i\omega\gamma_1 \right).
\]
(19)
The response function as defined by Eq. (10) for the dynamics in Eq. (18) is \( \tilde{x}_i(\omega) = -i\omega/\Gamma(\omega) \), from which we get its real component as
\[
\tilde{x}_i' = \frac{\omega^2}{\Gamma(\omega)} \left( \frac{\gamma_2}{\omega^2 + 1} + \gamma_1 \right).
\]
(20)

The spectrum of correlation function as defined in Eq. (9) for the dynamics in Eq. (18)\]
\[
\tilde{C}_{x\dot{x}}(\omega) = \frac{\omega^2}{\Gamma(\omega)} \left( 2T_1\gamma_1 + \frac{2T_2\gamma_2}{\omega^2 + 1} + \frac{2T_2\gamma_2}{\omega^2 + 1} \right).
\]
(21)
Substituting Eq. (20) and Eq. (21) in Eq. (16) and taking \( T = T_1 \) gives
\[
h_{\text{NM}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\omega^2\gamma'(\omega)}{\Gamma(\omega)} \left( \frac{\gamma_2}{\omega^2 + 1} + \frac{\gamma_2(T_2 - T_1)}{\omega^2 + 1} \right).
\]
(22)
The integrand can be negative for \( T_1 > T_2 \) which will lead to negative HDR and EPR. In this framework, it is not possible to explain the negative EPR.

In the following, we show that, in general, Eq. (16) does not lead to the correct expression for HDR and EPR. As shown in the following text, it leads to the correct result only when the viscoelastic medium is passive, i.e., \( T_1 = T_2 \).

The GLE in Eq. (8) is non-Markovian, to analyze the HDR and EPR we need an equivalent Markov representation. T A Markovian dynamics corresponding to Eq. (8) can be obtained by defining \( p \)
\[
p = \frac{\gamma}{\tau} \int_{-\infty}^{t} dt' e^{-\frac{(t-t')}{\tau}} \dot{x},
\]
(23)
using this substitution we get the following Markov dynamics corresponding to Eq. (8):
\[
m\dot{v} = -p - \gamma_1v - kx + f + \xi_1,
\]
(24)
\[
\dot{x} = v,
\]
(25)
\[
\tau\dot{p} = -p + \gamma_2v + \xi_2,
\]
(26)
\[
\tau_n\dot{f} = -f + \xi_n,
\]
(27)
where \( \xi_1, \xi_2, \) and \( \xi_n \) are zero mean Gaussian white noise of variance \( 2T_1\gamma_1, 2T_2\gamma_2, \) and \( 2\Lambda_n \) respectively. We emphasize that the variable \( p \) is not just a convenient representation but has a physical interpretation as the force on the particle due to the Maxwell stress in Eq. (8), i.e.,
\[
p(t) = \int_{\partial V} dS \sigma_2(x, t),
\]
(28)
where \( \partial V \) is the surface of the particle. Hence, this representation is unique and justifies the noise source \( \xi_2 \) in Eq. (20).

For a Markovian dynamics, the HDR and EPR can be obtained using the standard methods [3, 23, 24]. Here, we use the Harda-Sasa relation as given by Eq. (13). The total dissipation is the sum of dissipation due to variables \( v \) and \( p \). Using Eq. (13) the dissipation corresponding to \( v \) is
\[
h_v = \gamma_1 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( \tilde{C}_{vv}(\omega) - 2T_1\tilde{\chi}_v'(\omega) \right),
\]
(29)
where \( T_1 \) is the temperature of the bath corresponding to friction coefficient \( \gamma_1 \). The dissipation corresponding to variable \( p \) is
\[
h_p = \frac{1}{\gamma_2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( \tilde{C}_{pp}(\omega) - 2T_2\tilde{\chi}_p'(\omega) \right),
\]
(30)
where \( \tilde{C}_{pp} = \langle p^2(\omega) \rangle - \langle p(\omega) \rangle^2 \) is the correlation function, \( \tilde{\chi}_p = (\delta p)/\delta f_p \) is the response function, the averages are over the steady state distribution, and \( T_2 \) is the temperature of the bath corresponding to friction coefficient \( \gamma_2 \). Notice that \( \gamma_2 \) is in the denominator, \( \gamma_2 \) is the mobility corresponding to variable \( p \). The entropy production rate is
\[
s = \frac{h_v}{T_1} + \frac{h_p}{T_2},
\]
(31)
Substituting Eq. (21) and Eq. (20) in Eq. (29) we get
\[
h_v = h_b + h_{db},
\]
(32)
where the heat flow between the temperature bath \( T_1 \) and \( T_2 \) is
\[
h_b = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2}{| \Gamma(\omega) |^2} \left( \frac{2\gamma_1 \gamma_2 (T_2 - T_1)}{\omega^2 \tau^2 + 1} \right),
\]
(33)
and the heat flow from the driving force \( f \) to the bath \( T_1 \) is
\[
h_{df} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2}{| \Gamma(\omega) |^2} \left( \frac{2\Lambda_n \gamma_1}{\omega^2 \tau^2 + 1} \right),
\]
(34)
From Eq. 24 to 27 we get
\[
\tilde{p}(\omega) = -i\omega \gamma_2 \left( \frac{\tilde{p}(\omega) + \xi_1(\omega)}{\Gamma(\omega)(-i\omega\tau + 1)} \right) + \left( \frac{k - \omega^2 m - i\omega \gamma_1}{\Gamma(\omega)(-i\omega\tau + 1)} \right) \xi_2.
\]
(35)
The corresponding correlation spectrum reads
\[
\tilde{C}_{pp} = \frac{1}{| \Gamma(\omega) |^2 \left( \omega^2 \tau^2 + 1 \right)} \left[ \omega^2 \gamma_2^2 \left( 2T_1 \gamma_1 + \frac{2\Lambda_n}{\omega^2 \tau^2 + 1} \right) \right.
\]
\[+ \left. (k - \omega^2 m)^2 + \omega^2 \gamma_2^2 \right] 2T_2 \gamma_2]
\]
(36)
and the response function reads
\[
\chi_p = \frac{\gamma_2 (k - \omega^2 m - i\omega \gamma_1)}{\Gamma(\omega)(-i\omega\tau + 1)}.
\]
(37)
The real component of this response function is
\[
\chi'_p = \frac{\gamma_2 \left( (k - \omega^2 m)^2 + \omega^2 \gamma_1 (\gamma_2 + \gamma_1) \right)}{| \Gamma(\omega) |^2 \left( \omega^2 \tau^2 + 1 \right)}.
\]
(38)
Substituting Eq. 38 and Eq. 36 into Eq. 30 we get
\[
h_p = -h_b + h_{dp},
\]
(39)
where \( h_b \) is given by Eq. 33 and the heat flow from the driving force \( f \) to the bath \( T_2 \) is
\[
h_{dp} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{| \Gamma(\omega) |^2 \left( \omega^2 \tau^2 + 1 \right)} \left( \frac{2\Lambda_n \gamma_2 \omega^2}{\omega^2 \tau^2 + 1} \right).
\]
(40)
Fig. 2 show the direction of heat flow between different baths. The heat flow is from the driving force \( f \) to the two baths \( T_1 \) and \( T_2 \) although \( T_2 \) and \( f \) are not directly connected. The heat flow between the baths \( T_1 \) and \( T_2 \) is from “hotter” to “colder”.

The EPR as obtained by substituting Eq. 33, 34, and 40 in Eq. 31 is
\[
s = \left( \frac{T_2 - T_1}{T_1 T_2} \right) h_b + \frac{h_{de}}{T_1} + \frac{h_{dp}}{T_2}.
\]
(41)
The first term on the right is quadratic in the temperature difference, hence, as expected, the EPR is always positive. The total heat dissipated
\[
h = h_v + h_p = h_{dv} + h_{dp},
\]
(42)
which is always positive for \( \Lambda_n \neq 0 \). We now compare this with Eq. 22. For \( T_1 = T_2 \), i.e., when the viscoelastic medium is passive, the total HDR obtained from Eq. 22 is equal to that given by Eq. 42 and the corresponding EPR is equal to that in Eq. 41. When \( T_1 \neq T_2 \) the two expressions may lead to very different values. As mentioned before it for large enough temperature difference \( (T_1 - T_2) \) the HDR as obtained in Eq. 22 and the corresponding EPR can be negative, whereas the HDR and EPR as given by Eq. 42 and Eq. 41 respectively are always positive.

IV. OVERDAMPED LIMIT

We now calculate the EPR and HDR in the overdamped limit of Eq. 8. This is obtained by simply setting \( m \to 0 \). In this limit Eq. 19 reduces to
\[
\Gamma(\omega) = \frac{\gamma_1 \tau (i\omega - \omega_1) (i\omega - \omega_2)}{(-i\omega\tau + 1)},
\]
(43)
where
\[
\omega_{1,2} = \frac{\gamma_2 + \gamma_1 + k\tau \pm \sqrt{(\gamma_2 + \gamma_1 + k\tau)^2 - 4k\gamma_1 \tau}}{2\gamma_1 \tau}.
\]
Substituting Eq. 43 in Eq. 33 we get
\[
h_b = \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\omega^2 \gamma_2 (T_2 - T_1)}{\tau^2 \gamma_1 (\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)},
\]
(44)
which upon integration gives
\[
h_b = \frac{\gamma_2 (T_2 - T_1)}{\tau (\gamma_2 + \gamma_1 + k\tau)}.
\]
(45)
Substituting Eq. 43 in Eq. 40 gives
\[
h_{dp} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\Lambda_n \gamma_2 \omega^2}{\tau^2 \gamma_1^2 (\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)(\omega^2 \tau^2 + 1)}.
\]
(46)
upon integration this reduces to
\[
h_{dp} = \frac{\gamma_2 \Lambda_n}{(\gamma_1 \tau + \tau_n^2 \gamma_1 + \tau_n \gamma_2 + \gamma_1 + k\tau)}.
\]
(47)
Similarly, substituting Eq. 43 in 44 we get

\[ h_{dv} = \frac{\Lambda_n(\gamma_1 \tau_n + \tau_n k + \tau(\gamma_2 + \gamma_1 + k \tau))}{\tau_n(\gamma_1 \tau + \tau_n^2 k + \tau_n(\gamma_2 + \gamma_1 + k \tau))(\gamma_2 + \gamma_1 + k \tau)}. \]  

(48)

The total HDR and EPR is obtained upon substitution of Eq. 47-48 and Eq. 45 in Eq. 42 and Eq. 41 respectively. To check the validity of the results we take the viscous limit by taking \( \tau \to 0 \) and the total HDR to be passive \((T_1 = T_2 = T)\). In this limit the dynamics in Eq. 24 to 27 reduces to

\[ (\gamma_1 + \gamma_2) \dot{x} = -kx + f + \sqrt{2T(\gamma_1 + \gamma_2)} \xi, \]  

(49)

\[ \tau_n \dot{f} = -f + \xi_n. \]  

(50)

This is the dynamics of a particle in a Newtonian fluid of viscosity \( \eta \) driven by an Ornstein-Uhlenbeck process \( f \). The EPR given by Eq. 41 reduces to

\[ s = \frac{\Lambda_n}{T\tau_n(k\tau_n + (\gamma_1 + \gamma_2))}, \]  

(51)

which is same as that obtained for this dynamics directly in different contexts [25, 26].

In absence on an external driving \((\Lambda_n = 0)\) \( h_{dv} = h_{dp} = 0 \) and the total HDR \( h = h_b \), the EPR from Eq. 41 is

\[ s = \frac{(T_1 - T_2)^2}{k^2_{B} \gamma_{1}} \frac{k_2^2}{T_1 T_2} \]  

(52)

where we have defined \( k_2 = 6\pi \eta a B \) and substituted \( \tau = \gamma/B \). The EPR increases with the increase in the elasticity of the Maxwell element but decreases with the increase in the elasticity of the external potential. The increase in viscosity of both Maxwell and viscous elements leads to a decrease in the EPR. In the overdamped limit taking \( \tau \to 0 \) when \( T_1 \neq T_2 \) leads to \( h_b \to \infty \). To calculate this limit we need to include inertia which adds a high frequency cutoff to the correlation function. Similarly, for \( \tau_n \to 0 \) the dissipation \( h_{dv} \) and \( h_{dp} \to \infty \). Again, to calculate this limit we need to introduce a high frequency cutoff which for this case is provided by inertial relaxation.

V. DISCUSSION

In summary, we compute the heat dissipation and entropy production rate for a spherical particle suspended in a viscoelastic medium composed of a Maxwell fluid element and a viscous element in parallel driven by a stochastic force. The fluctuation corresponding to the viscosities of the fluid (\( \eta \)) and the Maxwell element (\( \gamma \)) act as two effective temperature baths \( T_1 \) and \( T_2 \) respectively. The dynamics of the particle is given by a generalized Langevin equation which is non-Markovian. This problem is nonequilibrium for two reasons, the effective temperature of the baths may be unequal and the particle is driven by an external stochastic force.

To calculate the heat dissipation and the entropy production rate for this case we write an effective Markov description of the non-Markovian dynamics. This is done by explicitly including the relaxation dynamics of the Maxwell fluid along with the particle dynamics. For this effective Markov description, we compute the dissipation using the Harada-Sasa relation. We find that the dissipation relation proposed in ref. [12] and that obtained in this work are different. The results match only when the medium is passive \((T_1 = T_2)\) and the only nonequilibrium input is the fluctuating force.

Particle in a viscoelastic medium driven by a fluctuating force is realized experimentally in ref. [27]. In this system the medium is passive and the only non-equilibrium component is the driving force on the particle. Hence, for this system, it is possible to calculate HDR using Eq. 16. However, for a similar experiment when the medium is active Eq. 16 cannot be used and a more detailed analysis of the kind proposed in this paper is required.

It has been shown that, in general, the non-Markovian dynamic can lead to a negative EPR [13–16]. We show that, indeed when the medium degree-of-freedom are not included the EPR can be negative for some values of the parameters, however, if all the relevant degrees of freedom are included the dynamics is Markovian and the EPR is always positive.

This approach is useful when the microscopic stress model is known. However, in general, the microscopic model for the medium is not experimentally accessible. For instance, using active and passive microrheology techniques, the correlation and response function of the embedded particle can be obtained. Form this inferring the equilibrium and nonequilibrium degrees of freedom of the medium may not be possible. One of the useful directions for the future will be to explore the limits in which the correct Markov description can be inferred from microrheology data.

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