Effective field theories for disordered systems from the logarithmic derivative of the wave-function.

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We consider a spinless particle moving in a random potential on a d-dimensional torus. Introducing the gradient of the logarithm of the wave-function transforms the time independent Schrödinger equation into a stochastic differential equation with the random potential acting as the source. Using this as our starting point we write functional integral representations for the disorder averaged density of states, the two point correlator of the absolute value of the wave-function as well as the conductivity for a d-dimensional system. We use the well studied one dimensional system with Gaussian disorder to illustrate that these quantities can be computed reliably in the current formalism by using standard approximation techniques. We also indicate the possibility of including magnetic fields.

I. INTRODUCTION

The study of the localization of non-interacting electrons in disordered media has progressed considerably since the initial work of Anderson in 1958. Early work on one dimensional (1D) disordered systems included the study of the spectral densities by Halperin as well as a calculation by Berezinskii using diagramatic techniques showing that all states are localized in 1D disordered systems, although this is generally difficult to extend to higher dimensions. Abrahams et al. introduced a scaling theory of localization predicting that a metal-insulator transition occurs in dimensions greater than two, although there seems to be experimental evidence for a transition in two dimensions. Making use of the replica trick, the problem was mapped onto a non-linear σ-model, which gave quantitative confirmation of the scaling approach. Efetov’s supersymmetry approach introduced a mathematically more rigorous alternative to the replica trick, which he used to prove, amongst other things, a conjecture of Gor’kov and Eliashberg that random matrix theory can be applied to the energy level statistics of particles in disordered systems.

Notwithstanding the considerable amount of work that has gone into the investigation of the localization problem, there are still many outstanding problems, for instance the lack of a order parameter to describe the second order metal-insulator phase transition. Also, finding an analytically tractable description of the localization problem, of which there has been little progress, would lead to a better understanding of disordered based phenomena, such as the Quantum Hall Effect. For this reason, any additional approaches for studying the localization problem, possibly leading to new insights, are useful.

In general, we would like to calculate disordered averages of observables that depend on a random potential \( V(x) \). These disordered averages can be calculated when the exact dependence of the observable on the random potential is known. However, when this dependence is not known, for example, the density of states and correlators of the wave-function, other methods of averaging these observables over the disorder are needed. Usually, the disorder averages of advanced or retarded Green’s functions, \( G^{\pm}(E) = (E - H \pm i\epsilon)^{-1} \), are calculated since their dependence on \( V(x) \) is known. These averages are then related to the averages of the observable. Thus, one would calculate the average of the advanced Green’s function and then relate it to the density of states using

\[
\langle \rho(E) \rangle = -\lim_{\epsilon \to 0} \frac{1}{\pi L^d} \text{Im} \text{Tr}(G(E)),
\]

where the angle brackets denote averaging over the disorder. Both of the main field theoretic techniques for investigating disordered systems, the supersymmetry and replica methods, are based on calculating the averages of products of Green’s functions using a generating function and then extracting information from the result.

In this paper we would like to propose a complementary approach for calculating disorder averages. This approach entails a transformation where we change from the random potential \( V(x) \) to a new set of random variables, which can be related to the logarithmic derivative of the wave-function and energy of a particle moving in the random potential. Using this transformation allows us to calculate directly averages of the density of states and correlations of the wave-function and its absolute value.

In section II, we will introduce the formalism, both for one-dimensional systems and for higher dimensions, and show how disordered averaged observables are calculated within this framework. Since the one dimensional system with Gaussian disorder is probably the best studied disordered system, with a variety of well known results available, it is ideal for testing and developing approximation techniques within our formalism with the ultimate aim of extending these techniques to higher dimensions, and possibly also to the case of a magnetic field. Therefore we focus in section III on the one dimensional Gaussian disordered system in order to illustrate how the formalism can be applied, using standard approximation techniques, to recover known results for the
density of states, and to obtain results for the 2-point correlator of the absolute value of the wave function as well as the conductivity. In section II, we also give a realization of the model to show how the parameters describing the Gaussian disorder can be related to microscopic quantities. In the final section, section IV, we numerically calculate and generate plots of the main results obtained in section II in order to obtain a understanding of the results.

II. FORMALISM

We consider a particle moving on a d-dimensional torus, and in a periodic, random potential caused by impurities in a system. We wish to calculate observed quantities of this particle, which under the assumption of self-averaging implies the averaging of these observables over the different realizations of the random potential, i.e.,

$$\langle \hat{O} \rangle = Z^{-1} \int [dV] \hat{O}[V] P[V],$$

(2)

where $Z = \int [dV] P[V]$ and $P[V]$ is the probability distribution describing the random potential. If we assume that the impurities are quenched, then the movement of the particle is described by the time independent Schrödinger equation. We impose periodic boundary conditions and for the moment assume that time reversal symmetry is not broken, so that the wave function can always be chosen real.

To introduce our formalism, we consider the logarithmic derivative of the wave function instead of the wave function, and correspondingly change from the Schrödinger equation to the equation of motion of this new variable. There are several advantages to this transformation, particularly from a functional integration point of view. Firstly, in contrast to the Schrödinger equation in which the random potential multiplies the wave function, the equation of motion governing the new variable is a non-linear stochastic differential equation which under the assumption of localization the length cannot be extracted from the correlations of the wave function as these are always short ranged due to the random phase cancellations. Instead, the correlations of the absolute values of the wave function should be computed. The current formulation is ideal for this purpose, as will be illustrated later.

Although the strategy is identical, there are subtle differences in the introduction of the formalism for one-dimensional systems, where we transform from the scalar wave function to a scalar variable, and higher dimensions, where the transformation is from the scalar wave function to a vector variable. For this reason, we first introduce the formalism for one-dimensional systems and then afterwards consider the more general theory in higher dimensions.

A. One dimensional formalism

In one dimension, we introduce the following real valued field related to the logarithmic derivative of the wave function,

$$\phi = \frac{\psi'}{\psi},$$

(3)

where we use the notation that $\phi \equiv \phi(x)$, unless the argument is specifically stated.

The periodic boundary conditions on the wave function implies that $\phi$ is also periodic, but cannot be a constant function. Using $\phi$ in the Schrödinger equation, we obtain the first order Ricatti equation,

$$V(x) = E + \phi' + \phi^2.$$  

(4)

where we work in units of $\frac{\hbar^2}{2m}$. Note that in these units, $V$ and $E$ have the dimensions $(\text{length})^{-2}$, while $\phi$ has the dimensions $(\text{length})^{-1}$.

Let us now consider (4). In principle, if we knew the functional dependence of the observable $\hat{O}$ on $V$, we could compute the desired quantities directly from (4). However, except in extremely trivial cases, we do not know this dependence and, in particular, we do not know the functional dependence of eigenvalues and eigenfunctions on $V$, which are the averages we would like to compute. On the other hand, the functional dependence of these, and many other observables, on the variable $\phi$ is fairly easy to determine (see section I.D). It therefore seems like a good strategy to change integration variables in (4) from $V$ to $\phi$ using the simple relation (3). Doing so will shift the complexity of (2) from the observables to the action (probability distribution) for $\phi$, which will generally be highly non-linear, even though $P[V]$ may be simple, i.e. Gaussian. The latter problem is, however, more amenable to treatment through the arsenal of perturbative and non-perturbative field theoretic techniques than the original problem as stated in (2).

To facilitate the change of variables (3) in (2), we introduce the identity

$$1 = N^{-1} \int [d\phi] dE \delta \left( \int dx \phi(x) \right) \delta[V - \bar{E} - \phi' - \phi^2]|J|$$

$$\equiv N^{-1} \int [d\phi] dE \delta[V - \bar{E} - \phi' - \phi^2]|J|,$$

(5)

where the functional integral is over all possible non-constant periodic configurations of $\phi$, $N$ is the total number of states (dimension of the Hilbert space), and the Jacobian $|J| = |\det(-\frac{d}{dx} + 2\phi)|$. Note that the role of $\bar{E}$ under our change of variables is to replace the integration over the constant mode of $V$, which cannot be done with $\phi$. It is easy to see, using the conditions imposed on $\phi$,
that the operator $-\frac{d^2}{dx^2} + 2\phi$ can be transformed to $-\frac{d^2}{dx^2}$ through a similarity transformation, thus not affecting the determinant. Therefore, $|J|$ is simply a multiplicative constant, which can be combined with the normalization of the functional integral.

Inserting the identity (8) into the averages of (2), and then completing the integration over the disorder, allows us to obtain a field theory, formulated in terms of the variable $\phi$, for the disordered average of observables

$$
\langle \hat{O} \rangle = Z^{-1} \int [d\phi] d\bar{E} \hat{O}[\bar{E}, \bar{A}] |J| \delta[\bar{\nabla} \times \bar{A}] 
\times P[\bar{E} + \bar{\nabla} \cdot \bar{A} + \bar{A} \cdot \bar{A}],
$$

(11a)

with $f$ having the same meaning as in (8).

Using the identity (8) in (8) and integrating over the disorder gives the corresponding field theory for the disordered average of observables in higher dimensions,

$$
\langle \hat{O} \rangle = Z^{-1} \int [d\bar{A}]d\bar{E} \hat{O}[\bar{E}, \bar{A}] |J| \delta[\nabla \times \bar{A}]
\times P[\bar{E} + \nabla \cdot \bar{A} + \bar{A} \cdot \bar{A}],
$$

(11b)

where

$$
Z = \int [d\phi] d\bar{E} P[\bar{E} + \phi' + \phi^2] = N\bar{Z}/|J|.
$$

(6b)

It should be noted that although we considered only a random potential when constructing this field theory, it is also possible to obtain the field theory when there is both a random potential $V(x)$ and a deterministic potential $W(x)$. In this case the result is similar to (8), except the energy is now shifted by $W(x)$,

$$
\langle \hat{O} \rangle = Z^{-1} \int [d\phi] d\bar{E} \hat{O}[\bar{E}, \phi] P[\bar{E} - W + \phi' + \phi^2].
$$

(7)

### B. Higher dimensions

In higher dimensions, we introduce a real valued vector field related to the gradient of the logarithm of the wave function,

$$
\bar{A} = \nabla \log \psi.
$$

(8)

Since the wave functions are assumed to be of class $C^2$, we note by direct computation that $\nabla \times \bar{A} = 0$. Also, periodicity of $\psi$ demands that $\bar{A}$ is periodic and does not contain a constant mode.

Using $\bar{A}$ in the higher dimensional Schrödinger equation, we obtain

$$
V(\bar{x}) = E + \nabla \cdot \bar{A} + \bar{A} \cdot \bar{A}.
$$

(9)

As in the one-dimensional case, we can introduce an identity to implement a change of variables, based on (6), between $V(\bar{x})$ and the field $\bar{A}$, constrained as described above:

$$
1 = N^{-1} \int [d\bar{A}] d\bar{E} \delta[V - \bar{E} - \nabla \cdot \bar{A} - \bar{A} \cdot \bar{A}] \delta[\nabla \times \bar{A}] |J|,
$$

(10)

with $f$ having the same meaning as in (8).

### C. Translational Invariance

Central to our analysis will be the translational invariance of the effective action, which stems from the assumed translational invariance of the probability distribution, $P[V]$, and implies that translational invariance is restored after averaging over the disorder. The translational invariance of the action leads to the appearance of an implied integration over a collective coordinate in the functional integral, corresponding to integration over the moduli space associated with the translational symmetry. It is appropriate to make this integration over the collective coordinate explicit to ensure that is correctly taken into account. To do this, we use a method inspired by the Faddeev-Popov quantization method of gauge theories. For simplicity we consider here the one-dimensional case, and indicate below how to extend to higher dimensions.

We introduce the identity

$$
1 = c \int_{-L/2}^{L/2} dx_0 F'[\phi^{x_0}] \delta(F[\phi^{x_0}] - \nu),
$$

(12)

where $\phi^{x_0} \equiv \phi(x + x_0)$, $F$ is an arbitrary functional of $\phi$ which is not translationally invariant, and $F'[\phi^{x_0}]$ denotes the derivative with respect to $x_0$. Unless explicitly stated, the integration is over the interval $[-\frac{L}{2}, \frac{L}{2}]$. The proportionality constant $c$ can in some cases be divergent, due to the existence of the Gribov ambiguity, for certain choices of $F$. Under these conditions, one needs to be careful to extract the spurious divergent term after completing the functional integration, as this term cancels with a similar term in the normalization.
As is usually done in gauge theories, it is more convenient to implement the identity \( \frac{1}{c} \int d\nu f(\nu) \) after integrating both sides with \( c^{-1} \int d\nu f(\nu) \), where \( f \) is an arbitrary function, so that

\[
c^{-1} \int d\nu f(\nu) = \int dx_0 F'[\phi(x_0)] f(F[\phi(x_0)]),
\]

where the left hand side is independent of \( \phi \). Multiplying both the numerator and denominator of \( \frac{E}{c} \) by the left hand side of (1) gives

\[
\langle \hat{O} \rangle = Z_F^{-1} \int [d\phi] dE dx_0 F'[\phi(x_0)] f(F[\phi(x_0)]) \\
\times \hat{O}[E, \phi] P[E + \phi' + \phi^2],
\]

where the partition function is given by

\[
Z_F = \int [d\phi] dE dx_0 F'[\phi(x_0)] f(F[\phi(x_0)]) \\
\times P[E + \phi' + \phi^2].
\]

Note that the functional \( F[\phi] \) acts in a similar fashion to gauge fixing terms in conventional gauge field theories. Since the choice of \( F[\phi] \) is arbitrary, we can choose \( F[\phi] \) so that our calculations can be simplified. As we shall see later, this choice will depend on what observables we wish to average. Also, any proportionality constants that appear due to the use of the Faddeev-Popov quantization method cancels out since we use the identity in both the numerator and the denominator (although, as mentioned earlier, extra care is needed if Gribov copying occurs).

To extend to higher dimensions, we introduce a vector-valued functional \( \vec{F}[\vec{A}] \) and the associated Faddeev-Popov determinant \( \Delta[\vec{A}] = \det \frac{\partial F[\vec{A}]}{\partial \vec{A}} \). The results obtained above can then be generalized to higher dimensions through the replacement \( F[\phi(x)] \rightarrow \vec{F}[\vec{A}] \) and \( F'[\phi] \rightarrow \Delta[\vec{A}] \), the latter denoting the corresponding Jacobian.

D. Disordered averaged observables.

Instead of writing the observables as a functional of \( V \), we would like to obtain them directly as functionals of the fields \( \phi \) or \( \vec{A} \) and the energy, \( E \). It is possible to do this for observables like the density of states, correlators of the wave function, and the conductivity.

1. Density of States

The density of states at energy \( E \) is defined as

\[
\rho(E) = \frac{1}{Z_L} \sum_m \delta(E - E_m),
\]

where \( E_m \) are eigenvalues of the Schrödinger equation, and \( d \) denotes the dimension of a system of size \( L \). However, if we consider the identities, \( \rho(E) \), we see that the functional integral can be considered as a sum over all possible solutions of the Ricatti equations \( \frac{E}{c} \), i.e., all possible eigenstates with all possible eigenvalues for the corresponding Schrödinger equation, and is thus the total number of states. However, when fixing the integral over the energy at \( E \), the functional integral yields only the number of eigenstates at \( E \), and is therefore proportional to the density of states, \( \rho(E) \) for a particular configuration of the disorder. After integrating over \( V \), this yields the disordered averaged density of states. Thus, we obtain via inspection the formula for the average density of states, normalised by the total number of states, for one dimensional systems,

\[
\langle \rho(E) \rangle = \frac{1}{Z_L} \int [d\phi] P[E + \phi' + \phi^2],
\]

and for higher dimensional systems,

\[
\langle \rho(E) \rangle = \frac{1}{Z_L^d} \int [d\vec{A}] P[E + \vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{A}] \delta(\vec{\nabla} \times \vec{A}) |J|.
\]

2. Correlators of the wave function

To obtain the observable related to correlations of the wave function, \( \psi(x) \psi(y) \), we use the definition of the field \( \phi \) to write the unnormalised wave function as

\[
\psi(x) = \exp \left[ \int_0^x dx' \phi(x') \right].
\]

Up to a global phase factor, all information that can be obtained from the wave function can also be obtained from (17), including information about the phase, which we need to consider carefully when computing the localization length so as to avoid obtaining incorrect results due to random phase cancellation.

The phase of the wave function changes as the wave function changes sign. From \( \phi \), we see that \( \phi \) must diverge at these points. We thus need a prescription to calculate the integral in the exponent of (17) at these points, since the result must be finite. The prescription we use is to integrate over a contour from \( 0 \) to \( x \), where the contour avoids the positions on the real axis where there are singularities in \( \phi(x) \) by moving around them in the upper complex plane with a semi-circle of radius \( \epsilon \).

This contour integral can be written in terms of the principle value of the integral plus a phase which depends on the number of times a singularity occurs in the interval \( [0, x] \). Using this prescription, we are able to separate the phase from the integral over \( \phi \). Thus we obtain for the normalised wave function (the notation \( \oint \) is used for the
contour as described above)
\[
\psi(x) = \sqrt{N[\phi]} \exp \left[ \int_0^x dx' \phi(x') \right]
\]
\[
= \sqrt{N} \exp \left[ P \int_0^x dx' \phi(x') - i \pi \sum_j \text{Res} \phi(x_j) \right] \tag{18}
\]
where \(x_j\) are the positions of the singularities and
\[
N[\phi]^{-1} = \int dx \exp \left[ 2P \int_0^x dx' \phi(x') \right]. \tag{19}
\]
To avoid the problems associated with the random phase cancellation when computing the localization length, we need to calculate correlators between the absolute values of the normalised wave functions. Using (13) with (14), we obtain an equation for calculating the disordered average of the 2-point correlator of the wave function at fixed energy \(E\),
\[
\langle |\psi_E(x)\rangle |\psi_E(y)\rangle \rangle = Z_{F}^{-1} \int [d\phi] dx_0 F[\phi^{-0}] [F[\phi^{-0}]] N[\phi(x)] P[E + \phi' + \phi^2]
\times \exp \left[ P \int_0^x dx' \phi(x') \right] \exp \left[ P \int_0^y dx' \phi(x') \right]. \tag{20}
\]
It is easy to check that this correlator is translationally invariant and that it only depends on \(|x-y|\).
Since \(F[\phi^{-0}]\) and \(f\) are arbitrary, we make a choice which simplifies the calculation of the correlator by cancelling out the normalisation factor, \(N[\phi]\).

To do this, we choose \(f = 1\), and \(F[\phi^{-0}] = -\int dz \exp (2P \int_0^zd\phi(x' + x_0))\), so that
\[
F[\phi^{-0}] = 2\phi(x_0) \int dz \exp \left[ 2P \int_0^z dx' \phi(x' + x_0) \right]
= 2\phi(x_0) \exp \left[ 2P \int_{x_0}^0 dx' \phi(x') \right] N[\phi(x)]^{-1}. \tag{21}
\]
Using (21) in (20) we have
\[
\langle |\psi_E(x)\rangle |\psi_E(y)\rangle \rangle = Z_{F}^{-1} \int [d\phi] dx_0 \phi(x_0) P[E + \phi' + \phi^2]
\times \exp \left[ P \int_0^x dx' \phi(x') \right] + P \int_{x_0}^y dx' \phi(x'). \tag{22}
\]
Note that the integrand in (22) can be written as a total derivative to \(x_0\), and thus the integral over \(x_0\) naively gives a result of zero. This is, however, an artifact of the choice of the gauge in the Faddeev-Popov identity, which is zero due to Gribov copying.
To obtain the correct result for the averaged observable, it is necessary, after completing the functional integral, to extract the terms that give zero using some form of regularisation, and cancel them out with similar terms that occur in the normalization. What remains is the correct result for the disordered average of the observable.
Using the periodic boundary conditions of \(\phi\), we see that we can transform (22) into a form where the symmetries of the system are more explicit, meaning that in all possible paths connecting \(\tilde{0}\) and \(\tilde{x}\), it must do so along any other path, which implies that the associated singularity in \(\tilde{A}\) must appear in all possible paths connecting \(\tilde{0}\) and \(\tilde{x}\). This in turn implies that the singularity in \(\tilde{A}\) occurs on a surface separating \(\tilde{0}\) and \(\tilde{x}\) into disconnected regions. Any path connecting \(\tilde{0}\) and \(\tilde{x}\) may therefore cross a singularity and a prescription to handle this singularity is required. We can do this in the same way as the one-dimensional case: if \(t\) parameterizes the path, we can avoid the singularity by a detour in the complex plane. In this way the absolute value and random phase of the wave function can again be separated, with the principle value of the line integral determining the absolute value.

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The normalisation of the
wave function can again be cancelled by an appropriate (non-unique) choice of $\vec{F}$ so that the correlation in higher dimensions is given by
\[
\langle \psi_E(\vec{x})|\psi_E(\vec{y})\rangle = \int [d\vec{A}]d\vec{x}_0\Delta[\vec{A}^{\alpha\beta}f(\vec{F}[\vec{A}^{\alpha\beta}])
\times P[E + \vec{\nabla} \cdot \vec{A} + \vec{A} \cdot \vec{A}]|\delta[\vec{\nabla} \times \vec{A}]
\times \exp \left[ P \int_{\vec{x}_0}^{\vec{x}} \vec{A}(\vec{x}') \cdot d\vec{x}' + P \int_{\vec{x}_0}^{\vec{y}} \vec{A}(\vec{x}') \cdot d\vec{x}' \right].
\] (24)

where
\[
\Phi(E, \omega) = \frac{4e^2}{\hbar L} \sum_{\alpha,\beta} \left[ \int dx \frac{d\phi_\alpha}{dx} \frac{d\phi_\beta}{dx} \right]^2 \times \delta(E - E_\alpha) \delta(E + \hbar \omega - E_\beta).
\] (26)

3. Conductivity

The conductivity of a system of non-interacting fermions is given by the Kubo formula. For our purposes it is convenient to integrate the Kubo formula by parts and use the periodic boundary conditions on the wave functions, so that the real part of the conductivity is then given by
\[
\text{Re}\sigma(\omega) = -\int dE \frac{\partial f(E)}{\partial E} \Phi(E, \omega),
\] (25)

\[
\langle \Phi(E, \omega) \rangle = \frac{4e^2}{\hbar L} Z^{-1} \int [d\phi_\alpha][d\phi_\beta] \delta[\hbar \omega - \phi_\alpha^2 - \phi_\beta^2 + \phi_\alpha^2 + \phi_\beta^2]P[E + \phi_\alpha + \phi_\beta]
\times N[\phi_\alpha]N[\phi_\beta] \left( \int dx \phi_\alpha(x) \exp \left[ \int_0^x dx' \phi_\alpha(x') + \int_0^x dx' \phi_\beta(x') \right] \right)^2
\] (27a)

where
\[
Z = \int dE d\bar{E} \left( \int [d\phi_\alpha][d\phi_\beta] \delta[\bar{E} - \phi_\alpha^2 - \phi_\beta^2 + \phi_\alpha^2 + \phi_\beta^2]P[E + \phi_\alpha + \phi_\beta] \right).
\] (27b)

Completing the integral over $\phi_\beta$, we have
\[
\langle \Phi(E, \omega) \rangle = \frac{4e^2}{\hbar L} Z^{-1} \int [d\phi_\alpha] dx d\bar{x} P[E + \phi_\alpha + \phi_\alpha^2]N[\phi_\alpha]N[\bar{\phi}_\beta] \phi_\alpha(\bar{x}) \phi_\alpha(x)
\times \exp \left( \int_0^x dx' [\bar{\phi}_\alpha(x') + \bar{\phi}_\beta(x')] \right) \exp \left( \int_0^x dx' [\phi_\alpha(x') + \bar{\phi}_\beta(x')] \right),
\] (28)

where $\bar{\phi}_\beta(x)$ is a functional of $\phi_\alpha$ and is determined by
\[
\hbar \omega - \phi_\alpha^2 - \phi_\beta^2 + \phi_\alpha^2 + \phi_\beta^2 = 0.
\] (29)

We can now use the Faddeev-Popov method, where we introduce the same choice of gauge for the $\phi_\alpha$ and $\bar{\phi}_\beta$ fields as in the previous section, which allow us to cancel out the $N[\phi_\alpha]$ and $N[\phi_\beta]$ normalisation factors respectively, so that

\[
\langle \Phi(E, \omega) \rangle = \frac{4e^2}{\hbar L} Z_F^{-1} \int [d\phi_\alpha] dx d\bar{x} dx_0 d\bar{x}_0 \phi_\alpha(\bar{x}) \phi_\alpha(x)(\bar{x}_0) \phi_\alpha(x_0) \bar{\phi}_\beta(\bar{x}_0) P[E + \phi_\alpha + \phi_\alpha^2]
\times \exp \left[ \int_0^x dx' \phi_\alpha(x') + \int_{\bar{x}_0}^{\bar{x}} dx' \bar{\phi}_\beta(x') + \int_{x_0}^{x} dx' \phi_\alpha(x') + \int_{\bar{x}_0}^{\bar{x}} dx' \bar{\phi}_\beta(x') \right],
\] (30a)
with

\[ Z_F = \int dE dE' \int [d\phi_0] dx_0 d\tilde{x}_0 P[E + \phi'_0 + \phi^2_0] F'\{\phi^0_0\} F'\{\phi^0_\beta\}. \] (30b)

In higher dimensions we use the same strategy as above to obtain

\[ \langle \Phi(E, \omega) \rangle = \frac{4e^2}{\hbar L^2} Z_F^{-1} \int [d\tilde{A}_\alpha] d\tilde{\omega} d\tilde{y}_0 d\tilde{x}_0 A_\alpha(\tilde{x}_0) \tilde{A}_\alpha(\tilde{y}_0) \Delta[\tilde{A}_\alpha] P[E + \nabla \cdot \tilde{A}_\alpha + \tilde{A}_\alpha \cdot \tilde{A}_\beta] |J_\alpha| \]

\[ \times \tilde{A}_\alpha(\tilde{x}_0) \tilde{A}_\alpha(\tilde{y}_0) \exp \left[ \oint_{\tilde{x}_0} \tilde{A}_\alpha(\tilde{x'}) \cdot d\tilde{x'} + \oint_{\tilde{y}_0} \tilde{A}_\beta(\tilde{x'}) \cdot d\tilde{x'} + \oint_{\tilde{y}_0} \tilde{A}_\alpha(\tilde{x'}) \cdot d\tilde{x'} + \oint_{\tilde{y}_0} \tilde{A}_\beta(\tilde{x'}) \cdot d\tilde{x'} \right]. \] (31)

where \( \tilde{A}_\beta \equiv \tilde{A}_\beta[\tilde{A}_\alpha] \) is the solution of

\[ \hbar \omega - \nabla \cdot \tilde{A}_\alpha - \tilde{A}_\alpha + \nabla \cdot \tilde{A}_\beta + \tilde{A}_\beta = 0. \] (32)

### III. ONE DIMENSIONAL SYSTEMS WITH GAUSSIAN DISORDER

In this section we consider one dimensional Gaussian disordered systems. We do so to illustrate how the formalism as described in the previous section can be applied, using standard approximation schemes, to recover known results for the density of states and the conductivity.

If we have Gaussian disorder, then \( \langle V(x)V(y) \rangle = t^{-1} \delta(x - y) \) and the probability distribution \( P[V] \) is given by

\[ P[V] = \exp \left( -l \int_{-L/2}^{L/2} dx V(x)^2 \right) \] (33)

where the dimension of \( l \) is (length). Normally, as in the previous section, we are interested in observables at fixed energy, \( E \). In this section we concentrate on these and therefore set \( \langle \tilde{O} E, \phi \rangle = \langle \tilde{O} E, \phi \rangle \delta(E - \tilde{E}) \) in what follows. Using (29) in (8), or its equivalent form (14), we obtain a \( \phi^4 \) field theory for calculating the disorder averages of fixed energy observables:

\[ \langle \tilde{O} \rangle = Z^{-1} \int [d\phi] \tilde{O}[E, \phi] \exp \left( -S[\phi, E] \right), \] (34a)

where the action is given by

\[ S[\phi, E] = l \int_{-L/2}^{L/2} dx (E + \phi' + \phi^2)^2. \] (34b)

Here \( Z \) and \( \tilde{O} \) denotes the normalization and observable in a generic gauge and we omit the subscript \( F \) of (14).

It is not possible to calculate the functional integral in (34) exactly, so we use perturbative approximations in order to calculate the disorder averages. If \( l \) is large, equivalent to a weak disorder system, we expand around a saddle point in (34). For small \( l \), or a strong disorder system, we use a Hubbard-Stratonovitch transformation to obtain a functional integral which can be approximated well in this regime.

### A. Weak disorder limit

For large \( l \), we calculate (34) perturbatively to one loop order using a saddle point approximation. Following a procedure similar to the one used in Zinn-Justin, we find approximate saddle point solutions which satisfy the saddle point equation to leading order in \( L^{-1} \), and thus become exact in the thermodynamic limit. These saddle point solutions for positive energies are

\[ \phi^m_c(x) = - \frac{m\pi}{L} \tan \left( \frac{m\pi}{L} x \right), \forall E \geq 0 \] (35a)

where \( m \) is the nearest even integer to \( \sqrt{E L}/\pi \), and for negative energies

\[ \phi^\pm_c(x) = \pm \sqrt{|E|} \tanh \sqrt{|E|}(x - \tilde{x}_0) \times \tanh \sqrt{|E|}(x + \tilde{x}_0), \forall E < 0 \] (35b)

where we assume a periodic continuation of \( \phi^\pm_c(x) \) outside \( [-\frac{L}{2}, \frac{L}{2}] \), and where the relative separation \( 2\tilde{x}_0 \) between the instanton and the anti-instanton pair is large. We shall see below that the constraint that \( \phi \) contains no constant mode implies \( 2\tilde{x}_0 = \frac{L}{2} \) so that this condition is automatically fulfilled. Under these conditions of well separated instanton and anti-instanton pairs the dilute gas approximation is valid.

We wish to emphasize that the two solutions obtained for \( E \geq 0 \) and \( E < 0 \) have different physical behaviour, which can be explained by noting that the potential changes from parabolic for positive energies to a double well potential for negative energies.

For positive energies, the requirement that the solution must have periodic boundary conditions, as well as the constraint that there cannot be a constant solution, implies that the energies are quantized, leading to (35a).

Also, the condition that \( m \) is an even integer is due to the periodic boundary conditions of the Schrödinger wave function.

We note that the solutions (35a) are topologically different for different \( m \), as each solution has a different number of singularities. Since the number of singularities are related to the number of nodes of the wave function, each saddle-point solution corresponds to wave functions with a different number of nodes. To be more precise,
when compactifying the real line to $S^1$ by identifying $+\infty$ and $-\infty$, we note that $\phi$ is a mapping from $S^1$ to $S^1$, where the mappings are classified according to winding numbers. It is simple to see that the solution $\phi^m_c$ has winding number $m$. In order to obtain a reliable approximation to the functional integral, it is necessary to sum over all the topologically different sectors. Additionally, since the fluctuations, $\eta$, are smoothly varying functions around the saddle point solutions, we see that all the information about the phase (which is determined by the number of singularities in $\phi$) is contained in the classical solutions, so that the localization length (which is extracted from the absolute value of the wave function) is purely determined by the fluctuations, $\eta$.

There is, however, a complication in the saddle point approximation, since the approximation contains a zero mode, which is a manifestation of the translational invariance of the action \((4b)\). To circumvent the problems associated with the zero mode, we make the translational symmetry explicit in the form of a collective coordinate \(c\), which we introduce by using the Faddeev-Popov method discussed earlier. Inserting the identity \((22)\) into \((23)\) with the choice $F'[\phi^m_c] = \int dx \phi_0(x) \phi(x+x_0)$ in order to project out the zero mode $\phi_0 \equiv \frac{d\phi^m_c}{dx}$, we obtain the positive energy saddle point approximation

\[
\langle \hat{O} \rangle \approx Z^{-1} \sum_m \int dx_0 \int [d\eta] \exp (-l L (\Delta E)^2) F'[\phi^m_c + \eta] \delta(F[\phi^m_c + \eta]) \times \hat{O}[E, \phi^m_c(x-x_0), \eta(x-x_0)] \exp (-l \int_{-L/2}^{L/2} dx \eta \Delta_m^{-1} \eta), \quad \forall E \geq 0
\]  

where $\Delta E = E - (\frac{m \pi}{L})^2$ and the propagator is given by

\[
\Delta^{-1} = (-\frac{d^2}{dx^2} + 2E + 6 \phi^m_c^2).
\]

Here we noted that the constraint $\delta(\int dx \phi^m_c)$ is trivially satisfied so that the constraint $\delta(\int dx \phi)$ in \((33)\) simply becomes the constraint that the $\eta$ integration is over all non-constant modes. Also note that in the pure limit $(l \to \infty)$ the value of $m$, and thus the topological sector is fixed by the energy.

Integrating over the zero mode and making the approximation that $F'[\phi^m_c + \eta] \approx F'[\phi^m_c]$, we obtain

\[
\langle \hat{O} \rangle \approx Z^{-1} \sum_m \int dx_0 \int [d\eta] \exp (-l L (\Delta E)^2) F'[\phi^m_c] \times \hat{O}[E, \phi^m_c(x-x_0), \eta(x-x_0)] \exp (-l \int_{-L/2}^{L/2} dx \eta \Delta_m^{-1} \eta), \quad \forall E \geq 0
\]

where the accent denotes that the zero mode is excluded when calculating the functional integral. Also note that $F'[\phi^m_c]$ is a divergent constant (the zero-mode is not normalizable), that may depend on $m$. However, requiring that the disorder average gives the correct result in the pure limit, we find that $F'[\phi^m_c]$ is a constant independent of $m$, which can thus be incorporated into the normalization.

In the negative energy region, the saddle point equation has a double well potential. The constraint that $\phi$ has no constant mode does not allow us to obtain constant saddle point solutions situated at the minima of the potential, thus the only other possible solutions are instanton solutions, which we introduce by using the Faddeev-Popov method discussed earlier. This quasi-symmetry is due to the fact that for large system sizes, local translations are possible that only changes the action by terms of order.

\[
\int dx \phi_0(x) \phi(x+x_0)
\]

from one minima to the other. Since we must also satisfy periodicity, there must also be tunnelling back to the original minima. This to and fro tunnelling can occur multiple times, corresponding to topologically different sectors over which one has to sum, but since there is an exponential decay associated with each tunnelling process we consider only solutions \((35b)\), where the tunnelling occurs once.

Once again the saddle point approximation contains a zero mode, which needs to be integrated out. Additionally, the saddle point solution \((35b)\) allows another quasi-symmetry to exist. This quasi-symmetry is due to the fact that for large system sizes, local translations are possible that only changes the action by terms of order.

\[
\int dx \phi^m_c(x) \phi(x+x_0)
\]
der exp\((-cL)\). To be specific, for large separations in \(\phi(x)\), a translation in \(\bar{x}_0\) has an exponentially small effect on the action, so that to leading order in \(\frac{1}{L}\) it is a symmetry of the action, which in the \(L \to \infty\) limit becomes an exact symmetry. Associated with this approximate symmetry there is again an approximate zero-mode.

To circumvent the problems associated with the zero modes, we make the translational symmetries explicit in the form of collective coordinates which we wish to introduce simultaneously, we need to modify the identity in (12) so that

\[
1 = c \int_{-L/2}^{L/2} dx_0 d\bar{x}_0 \Delta[\phi] \delta(F[\phi(x_0, \bar{x}_0)]) \delta(F[\phi(x_0, \bar{x}_0)])
\]

(38a)

where

\[
\Delta[\phi] = \left| \frac{\partial F}{\partial \phi_x} - \frac{\partial F}{\partial \phi_{x_0}} \right|.
\]

(38b)

In order to project out the zero modes, we choose

\[
F[\phi(x_0, \bar{x}_0)] = \int dx \phi_0(x, \bar{x}_0) \phi(x + x_0) \tag{39a}
\]

\[
\bar{F}[\phi(x, \bar{x}_0)] = \int dx \bar{\phi}_0(x, \bar{x}_0) \phi(x + x_0) \tag{39b}
\]

where the zero mode \(\phi_0\) and quasi zero-mode \(\bar{\phi}_0\) are given by

\[
\phi_0(x, \bar{x}_0) = \frac{\partial \phi_\pm}{\partial x}, \quad \bar{\phi}_0(x, \bar{x}_0) = \frac{\partial \phi_\pm}{\partial \bar{x}_0}.
\]

(39c)

Using (38) and (39) in (34), changing the variables \(\phi(x + x_0) \to \phi(x)\) in the functional integral and then writing \(\phi = \phi_\pm + \eta\), we have

\[
\langle \hat{O} \rangle = Z^{-1} \int d\eta \int dx_0 d\bar{x}_0 \exp \left( -\int dx_0 \phi_\pm \eta(x - x_0) \right) \exp \left( -\int dx_0 \phi_\pm \eta(x - x_0) \right)
\]

\[
\times \Delta[\phi_\pm + \eta] \delta(F[\phi_\pm + \eta]) \delta(F[\phi_\pm + \eta]) \left| \int dx \frac{\partial \phi_\pm}{\partial x_0} \right|^{-1} \delta(x_0 - L/4).
\]

(40)

Here we have handled the constraint \(\delta(\int dx \phi_\pm)\) by restricting the \(\eta\) integration to be over non-constant modes, leaving the constraint \(\delta(\int dx \phi_\pm)\), which can explicitly be written as \(\delta(\int dx \phi_\pm) = \delta(\bar{x}_0 - L/4)/\int dx \frac{\partial \phi_\pm}{\partial x_0}\). Note that the constraint on \(\bar{x}_0\) forces the instanton and anti-instanton pair of (35b) to be separated by \(\frac{L}{2}\), which for large \(L\) allows us to use the dilute gas approximation.

\[
\phi_\pm(x) \approx \pm \sqrt{|E|} \tanh \sqrt{|E|}(x - \frac{L}{4}) \Theta(x)
\]

\[
\pm \sqrt{|E|} \tanh \sqrt{|E|}(x + \frac{L}{4}) \Theta(-x) \quad \forall E < 0
\]

(41)

After integrating over \(\bar{x}_0\) in equation (41), using the dilute gas results one finds \(S[\phi_\pm, E] = \frac{16}{3} |E|^{3/2}\), \(\Delta[\phi_\pm] \propto |E|^{1/2}\) and \(\int dx \frac{\partial \phi_\pm}{\partial x_0} \propto |E|^{1/2}\), which gives the saddle-point approximation for the negative energies

\[
\langle \hat{O} \rangle \approx Z^{-1} |E| \exp \left( -\frac{16}{3} |E|^{3/2} \right)
\]

\[
\times \int [d\eta'] dx_0 \hat{O}[E, \phi_\pm(x - x_0), \eta(x - x_0)]
\]

\[
\times \exp \left( -\int_{-L/2}^{L/2} dx \eta \Delta^{-1}_\perp \eta \right) \quad \forall E < 0
\]

(42a)

where

\[
\Delta^{-1} = (-\frac{d^2}{dx^2} + 2E + 6\phi_\pm^2),
\]

(42b)

and the notation \([d\eta']\) denotes that the zero modes are excluded in the functional integral.

To calculate the functional integrals in (36) and (42), we need to be able to calculate the propagator, \(\Delta\), and the determinant of its inverse, \(\det(\Delta^{-1})\). This involves solving the eigenvalue equation

\[
\left( -\frac{d^2}{dx^2} + 2E + 6\phi_\pm^2(x) \right) \Psi_n = \lambda_n \Psi_n
\]

(43)

where \(\phi_\pm\) is given by (35a) in the positive energy region, or by (31) in the negative energy region. Although it is
possible to solve for the eigenvalues and eigenfunctions of (43) exactly for the different energy regions, it is not possible to obtain closed expressions which is necessary when calculating the propagator. We thus need a consistent approximation for calculating the eigenvalues and eigenfunctions of (43) for both positive and negative energies. (These approximations, as well as comparisons to the exact results, are discussed in detail in the Appendix.)

For positive energies, we note that the dominant property of the classical solution (45a) is that it contains singularities which appear periodically with period $\frac{2m}{L}$. Thus, the eigenfunctions of (43) must be zero at the points where these singularities occur. Additionally, the periodicity of the classical solution leads to Bloch characteristics of the eigenfunctions, so that the eigenfunctions are also periodic over the interval $\frac{2m}{L}$, which implies that we only need to solve (43) over the interval $[-\frac{2m}{L}, \frac{2m}{L}]$.

We wish to make an approximation for the propagator that captures the essential characteristics of the original eigenfunctions. As a first approximation, we treat the $\tan^2$ potential in (43) to lowest order in perturbation theory, where we capture the singularities of the potential by imposing vanishing boundary conditions at $\pm \frac{2m}{L}$. We thus need to solve the eigenvalue equation

$$(-\frac{d^2}{dx^2} + 2E)\Psi_n = \lambda_n \Psi_n,$$

(44)

with vanishing boundary conditions at $\pm \frac{L}{2m}$. We can now calculate the approximate propagator and determinant in the positive energy region to give

$$\Delta_n(x, y) \equiv \left( -\frac{d^2}{dx^2} + 2E \right)^{-1} \approx 2 \sum_{n=1}^{\infty} \left( \cos(2n-1)\frac{m\pi}{L}x \cos(2n-1)\frac{m\pi}{L}y \right) \left( \frac{2n\frac{m\pi}{L}y}{(2n\frac{m\pi}{L})^2 + 2E} \right)$$

$$+ \frac{\sin(2n\frac{m\pi}{L}x)\sin(2n\frac{m\pi}{L}y)}{(2n\frac{m\pi}{L})^2 + 2E}$$

(45a)

and

$$|\det(\Delta_n^{-1})| \propto \frac{|m|}{\sqrt{2EL}} \sinh \left( \frac{\sqrt{2EL}}{|m|} \right).$$

(45b)

In the negative energy region one finds from the exact solutions two zero modes (which we must eliminate) and a doubly degenerate bound state at $\lambda = 4|E|$. The rest of the spectrum consists of four fold degenerate scattering states starting at $\lambda = 4|E|$. From the exact solution it turns out (See the Appendix) that the eigenvalues and eigenfunctions of the scattering states can be well approximated by that of a free particle. This amounts to writing $\tanh^2(x \pm x_0) = 1 - \sech^2(x \pm x_0)$ in the classical solution appearing in (42a) and then treating the $\sech^2(x \pm x_0)$ terms to lowest order in perturbation theory. Thus the approximate eigenvalue problem that has to be solved to obtain the spectrum and eigenfunctions of the scattering states is

$$(-\frac{d^2}{dx^2} + 4|E|)\Psi_n = \lambda_n \Psi_n,$$

(46)

where $\Psi_n$ are periodic on a system of length $L$. Taking into account the bound states and scattering states, approximated as a free particle spectrum, we obtain the determinant in the negative energy region as

$$|\det(\Delta_n^{-1})| \propto \sinh^4 \left( \sqrt{|E|L} \right).$$

(47a)

The contribution of the bound states in the propagator can be neglected as it involves the product of two very well localized eigenfunctions evaluated at points which are well separated. Taking into account only the scattering states, again approximated as free particle states, we find for the propagator

$$\Delta_\pm(x, y) = \frac{2}{L} \sum_{n=1}^{\infty} \left( \frac{\cos(2n\frac{m\pi}{L}(x-y))}{(2n\frac{m\pi}{L})^2 + 4|E|} \right).$$

(47b)

Using these approximate results for the propagator and determinant for positive energies (43) and negative energies (47) with the respective formulations for disordered averages, (36) and (42), now allows us to calculate the average value of observables to leading order in $L$.

As mentioned earlier, this approximation is valid when $l$ is large in a sense determined by the other two scales, namely $E$ and $L$. To find the precise criterion one has to evaluate the higher order loop corrections to (37) and (42), e.g., in the case of the density of states, one has to evaluate higher order vacuum diagrams. Doing this, one finds that these contributions can be neglected under the condition that $\frac{2E}{L} \gg 1$.

There are two limits under which this condition can be fulfilled. Firstly, for a fixed energy $E$, we find that $l \gg \frac{4}{E}$, which is large for large system sizes. The approximation thus holds in the weak disorder limit. Secondly, for a fixed amount of disorder, we have that $E \gg \frac{4}{L}$. We thus find that the approximation also holds in the high energy limit. Note, however, that in the thermodynamic limit ($L \to \infty$), the condition cannot be satisfied, unless either $l \to \infty$ or $E \to \infty$, implying that using this approximation for disordered systems only holds for finite system sizes.

### 1. Density of states

We now apply the saddle-point approximation discussed above to the disorder averaged density of states given by (16). For positive energies, this amounts to computing (57) with the observable $O = 1$. Upon integrating over the functional integral to obtain the factor...
\begin{equation}
\langle \rho(E) \rangle = N_+ \sum_m \exp \left( -lL|E - (\frac{m\pi}{L})|^2 \right) \times \left[ \frac{\sqrt{2EL}}{|m|} \tanh \left( \frac{\sqrt{2EL}}{|m|} \right) \right]^{1/2}
\end{equation}

where $N_+$ is an unknown normalization factor, independent of the energy, that needs to be fixed in some manner.

For negative energies, the saddle-point approximation of (14) leads to (22) with $\tilde{O} = 1$. After integrating over the resulting functional integral, we obtain

\begin{equation}
\langle \rho(E) \rangle = N_- |E| \exp \left( -\frac{16}{3} |E|^{3/2} \right) \cosh \left( \sqrt{|E|}L \right),
\end{equation}

where $N_-$ is another energy independent normalization factor. The latter result agrees with the result of Halperin obtained by different means, although his result does not include the higher order corrections. It should be noted that in order to compare with the results of Lifshitz et al., which are given for a system of infinite size, one must take the $L \to \infty$ limit in the results above such that the ratio $El/L$ is fixed.

2. 2-point correlators

If we consider the disorder average of the 2-point correlator given by (22), we note the exponential terms can be written as

\begin{equation}
\exp \left[ P \int_x^y dx' \phi(x') + P \int_y^x dx' \phi(x') \right]
= \exp \left[ \int_x^y dx' \phi(x') + \int_y^x dx' \phi(x') \right] \times \exp \left[ \int_z^x dx' \eta(x') + \int_x^y dx' \eta(x') \right],
\end{equation}

where we have relabeled the the $x_0$ integration in (22) to $z$. The linear terms of $\eta$ in the exponential can be written as an integral over the interval $[\frac{x}{2}, \frac{x}{2}]$ by using a combination of step functions,

\begin{equation}
\int_x^y dx' \eta(x') + \int_y^x dx' \eta(x') = \int_{x/2}^{x/2} dx' S(x, y, z|x') \eta(x'),
\end{equation}

where $S(x, y, z|x')$ can be considered as a source term for the $\eta$ fields and is given by

\begin{equation}
S(x, y, z|x') = \theta(x - x') + \theta(y - x') - 2\theta(z - x').
\end{equation}

Furthermore, a periodic continuation is understood outside $[\frac{x}{2}, \frac{x}{2}]$.

Applying the positive energy saddle point approximation (27) to (22), and making the change of variables $z \to z + x_0$, we have

\begin{equation}
\langle |\psi_E(x)| |\psi_E(y)| \rangle = Z^{-1} \sum_m \int dx_0 dz \frac{\partial}{\partial z} \int [d\eta] \exp \left( -lL(\Delta \eta)^2 \right) \exp \left( -l \int_{-x/2}^{x/2} dx \eta \Delta_{-1} \eta \right)
\end{equation}

\begin{equation}
\times \exp \left[ 2P \int_x^0 dx' \phi(x') \right] \exp \left[ P \int_{x_0}^x dx' \phi_m'(x' - x_0 + \eta) + P \int_{x_0}^y dx' \phi_m'(x' - x_0 + \eta) \right].
\end{equation}

Note that as a result of the gauge that we chose to cancel the normalization of the wave functions giving rise to (23), equation (50) contains a total derivative with respect to $z$, which na"ively gives a result of zero when completing the integration over $z$. However, since we have a ratio of total derivatives, we should obtain a non-zero result if we use a consistent regularization method. With this in mind, we integrate over $z$ and cancel the result with a similar term in the denominator of (54). We can now integrate over the fluctuations to obtain

\begin{equation}
\frac{\langle |\psi_E(x)| |\psi_E(y)| \rangle}{\langle |\psi_E(0)| |\psi_E(0)| \rangle} \approx \frac{C(x, y)}{C(0)}
\end{equation}

where
\[ C(x, y) = \sum_m \int dx_0 \left| \cos \left( \frac{m\pi}{L} (x - x_0) \right) \right| \cos \left( \frac{m\pi}{L} (y - x_0) \right) \exp \left( -iL(\Delta E)^2 \right) \]
\[ \times (\det(\Delta_m^{-1}))^{-1/2} \exp \left( \frac{1}{4L} \int dx' dx'' S(x, y, x_0|x') \Delta_m(x', x'') S(x, y, x_0|x'') \right). \] (52)

After using (45) for the propagator and determinant, we have
\[ C(x, y) = \sum_m \int dx_0 \left| \cos \left( \frac{m\pi}{L} (x - x_0) \right) \right| \cos \left( \frac{m\pi}{L} (y - x_0) \right) \exp \left( -iL(\Delta E)^2 \right) \]
\[ \times \left[ \frac{\sqrt{2EL}}{|m|} \cosech \left( \frac{\sqrt{2EL}}{|m|} \right) \right]^{1/2} \exp \left( \frac{1}{4L} F(x, y, x_0) \right), \] (53a)

where
\[ F(x, y, x_0) = \frac{2}{L} \sum_{n=1}^{\infty} \frac{1}{D_{2n-1}} \left[ \cos \left( (2n - 1) \frac{m\pi}{L} (x - x_0) \right) + \cos \left( (2n - 1) \frac{m\pi}{L} (y - x_0) \right) - 2 \right]^2 \]
\[ + \frac{2}{L} \sum_{n=1}^{\infty} \frac{1}{D_{2n}} \left[ \sin \left( 2n \frac{m\pi}{L} (x - x_0) \right) + \sin \left( 2n \frac{m\pi}{L} (y - x_0) \right) \right]^2, \] (53b)

and
\[ D_n = \left[ \left( \frac{nm\pi}{L} \right)^2 + 2E \right] \left( \frac{nm\pi}{L} \right)^2. \] (54)

In the negative energy region, the saddle point approximation \((\ref{56b})\) is obtained using the dilute gas approximation. However, we find that the approximate saddle point solution \((\ref{55b})\) in a dilute gas approximation, \((\ref{11})\), breaks the symmetries of the system, thus we first need to write the exponential terms in \((\ref{22})\) in a form where the symmetries are explicit. We do this by using \((\ref{22})\) so that

\[ 2 \exp \left[ P \int_{x_0}^{x} dx' \phi(x') + P \int_{x_0}^{y} dx' \phi(x') \right] \]
\[ \rightarrow \exp \left[ P \int_{x_0}^{L-|x-y|} dx' \phi(x') + P \int_{x_0}^{0} dx' \phi(x') \right] + \exp \left[ P \int_{x_0}^{L+|x-y|} dx' \phi(x') + P \int_{x_0}^{0} dx' \phi(x') \right]. \] (55)

Using this form for the exponential insertions in \((\ref{22})\), along with saddle point approximation for negative energies \((\ref{22})\), and then integrating over the fluctuations, gives
\[ C(x, y) = \int dx_0 \exp \left[ P \int_{x_0}^{L-|x-y|} dx' \phi_c^\pm (x' - x_0) + P \int_{x_0}^{0} dx' \phi_c^\pm (x' - x_0) + \frac{1}{4L} F(L - |x-y|, x_0) \right] \]
\[ + \int dx_0 \exp \left[ P \int_{y}^{L-|x-y|} dx' \phi_c^\pm (x' - x_0) + P \int_{x_0}^{0} dx' \phi_c^\pm (x' - x_0) + \frac{1}{4L} F(|x-y|, x_0) \right], \] (56a)

where
\[ F(x, x_0) = \frac{4}{L} \sum_{n=1}^{\infty} \frac{1}{D_n} \left[ \cos \left( \frac{2n\pi}{L} x \right) - 2 \cos \left( \frac{2n\pi}{L} x - x_0 \right) - 2 \cos \left( \frac{2n\pi}{L} x_0 \right) + 3 \right] \] (56b)

and
\[ D_n = \left( \frac{2n\pi}{L} \right)^2 \left( \frac{2n\pi}{L} \right)^2. \] (57)

Note that in obtaining \((\ref{56b})\), we once again relabeled the
integration variable in (22) from \( x_0 \) to \( z \), translated \( z \to z + x_0 \), and then, as above, cancelled the \( z \) dependent terms giving rise to a total derivative with respect to \( z \). Note from the saddle point solution (11) that to leading order this correlation function decays or grows like \( \exp(\pm \sqrt{E}x) \), confirming the result of Lifshits et al.\(^4\) that the localization length is proportional to \( 1/\sqrt{E} \) for large negative energies.

3. Conductivity

To calculate the disorder average of the conductivity given by (21) in a saddle point approximation, we first need to be able to solve for \( \tilde{\Phi} \) using (20). To do this, we make the ansatz that \( \phi_{\beta} \) consists of a classical and a fluctuating term, i.e. \( \phi_{\beta} = \phi_{\beta}^c + \chi \). Using this ansatz, as well as the expansion \( \phi_{\alpha} = \phi_{\alpha}^c + \eta \) in (23), and neglecting the coupling terms between the classical and fluctuating terms, we find that \( \chi = \eta \) and \( \phi_{\beta}^c \) must satisfy the saddle point equation with the energy shifted by \( \hbar \omega \),

\[
\hbar \omega - (\phi_{\alpha}^c)^2 - \phi_{\alpha}^{c'} + (\phi_{\beta}^c)^2 + \phi_{\beta}^{c'} = 0,
\]

with the positive energy solution given by

\[
\phi_{\beta}^c(x) = -\frac{q\pi}{L} \tan\left(\frac{q\pi}{L} x\right).
\]

To satisfy the periodic boundary conditions, we see that \( q \) must be the nearest integer to \((m^2 + \hbar^2 L^2/\pi^2)^{1/2}\).

Applying the saddle point approximation in the positive energy region (37) (where we relabel \( x_0 \) to \( z \)) and making the change of variables \( x \to x + z, \bar{x} \to \bar{x} + z \), \( x_0 \to x_0 + z \) and \( \bar{x}_0 \to \bar{x}_0 + z \) allows us to integrate over \( z \) (since the integrand is now independent of \( z \)) to obtain

\[
\frac{\langle \Phi(E, \omega) \rangle}{\langle \Phi(E, \omega_0) \rangle} = \frac{\sum_m \exp(-lL(\Delta E)^2)\Phi(\omega)}{\sum_m \exp(-lL(\Delta E)^2)\Phi(\omega_0)},
\]

with

\[
\Phi(\omega) = \int [d\eta] dx \bar{dx} dx_0 \bar{dx}_0 \frac{\partial}{\partial \eta_0} \exp \left[ \int_0^L dx' (\phi_{\alpha}^m(x') + \eta(x')) \right] \frac{\partial}{\partial \phi_{\alpha}^m(x')} \exp \left[ \int_0^L dx' (\phi_{\beta}^m(x') + \eta(x')) \right]
\]

\[
\times \left[ \phi_{\alpha}^m(x) + \eta(x) \right] \exp \left[ \int_0^L dx' \phi_{\alpha}^m(x') + \int_0^L dx' \phi_{\alpha}^m(x') + \int_0^L dx' \phi_{\beta}^m(x') + \int_0^L dx' \phi_{\beta}^m(x') \right]
\]

\[
\times \left[ \phi_{\beta}^m(\bar{x}) + \eta(\bar{x}) \right] \exp \left[ -\int_{-L/2}^{L/2} dx' \left( m^{-1} \eta(x') - S(x') \eta(x') \right) \right],
\]

where \( S(x') \equiv S(x, \bar{x}, 0|x') \) and the classical solutions in the positive energy region are now denoted by the superscript \( m \). Note that the total derivatives that appear are due to the original Faddeev-Popov method used to cancel out the normalizations of the wave functions. As before, we can cancel them with similar terms in the denominator of (60a).

Introducing source terms for the \( \eta(x) \) and \( \eta(\bar{x}) \) terms, integrating over the fluctuations and then integrating by parts, we have

\[
\tilde{\Phi}(\omega) = \int dx \bar{dx} \exp \left( \frac{i}{\hbar} F(x, \bar{x}) \right) \left[ \frac{\sqrt{2EL}}{m} \cosh \left( \frac{\sqrt{2EL}}{m} \right) \right]^{1/2}
\]

\[
\times \left[ \frac{q\pi}{L} \sin \left( \frac{q\pi}{L} x \right) \cos \left( \frac{m\pi}{L} x \right) - \frac{m\pi}{L} \sin \left( \frac{m\pi}{L} x \right) \cos \left( \frac{q\pi}{L} x \right) \right]
\]

\[
\times \left[ \frac{q\pi}{L} \sin \left( \frac{q\pi}{L} \bar{x} \right) \cos \left( \frac{m\pi}{L} \bar{x} \right) - \frac{m\pi}{L} \sin \left( \frac{m\pi}{L} \bar{x} \right) \cos \left( \frac{q\pi}{L} \bar{x} \right) \right],
\]

where

\[
F(x, \bar{x}) = \frac{2}{L} \sum_{n=1}^{\infty} \frac{1}{D_{2n}} \left[ \sin \left( 2n \frac{m\pi}{L} x \right) + \sin \left( 2n \frac{m\pi}{L} \bar{x} \right) \right]^2
\]

\[
+ \frac{2}{L} \sum_{n=1}^{\infty} \frac{1}{D_{2n-1}} \left[ \cos \left( 2n - 1 \frac{m\pi}{L} x \right) + \cos \left( 2n - 1 \frac{m\pi}{L} \bar{x} \right) - 2 \right]^2
\]
with $D_\alpha$ given in [54].

B. Strong disorder limit

In this limit, when $l$ is small, we use a Hubbard-Stratonovich transformation on (34), so that

$$\langle \hat{O} \rangle = Z^{-1} \int [d\Lambda] \hat{O}[E, \Lambda] \exp(-\frac{1}{l} \int dx \left[ \Lambda^2 + 2iE\Lambda \right]).$$

(62a)

The $\Lambda$ dependent observable is given by

$$\hat{O}[E, \Lambda] = \int [d\phi] \hat{O}[E, \phi] \times \exp(-\int dx \left[ t \phi'^2 + 2i\phi^2\Lambda \right])$$

(62b)

and

$$Z^{-1} = \int [d\Lambda] dE [\text{det}(\Lambda - t \frac{d^2}{dx^2} + 2i\Lambda)]^{-1/2} \times \exp(-\frac{1}{l} \int dx \left[ \Lambda^2 + 2iE\Lambda \right]).$$

(62c)

We now approximate (62c) by expanding (as described in Zinn-Justin[24]) up to first order in the loop corrections. We do this by first splitting the integral over $\Lambda$ into an integral over the constant mode, $\Lambda_0$, and non-
constant modes $\eta$. This is achieved by inserting the identity $\int d\Lambda_0 d\eta [\Lambda + \Lambda_0] \delta(\int d\eta)$ into the numerator and denominator of (62c). Integrating over $\Lambda$, we have

$$\langle \hat{O} \rangle = Z^{-1} \int d\Lambda_0 \int [d\eta] \hat{O}[E, \Lambda_0 + \eta] \exp \left[ -\frac{1}{l} \int dx \eta^2 \right] \times \exp \left[ -\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L \right],$$

(63)

where the observable $\hat{O}[E, \Lambda_0 + \eta]$ is given by (62b) with $\Lambda = \Lambda_0 + \eta$.

We wish to approximate (63) by neglecting the $\phi^2\eta$ coupling term that appears in $\hat{O}[E, \Lambda_0 + \eta]$. By examining the effective action for $\Lambda_0$, expanded around the equilibrium value for $\Lambda_0$, we find that one can neglect the coupling term if $\frac{\rho}{\rho_E} \ll 1$. As in the saddle point approximation, this condition can be fulfilled in two limits, namely the strong disorder limit ($l \ll \frac{\rho}{\rho_E}$), or the low energy limit ($E \ll \frac{\rho}{\rho_E}$). Also, in the thermodynamic limit, this condition always holds for fixed energy and disorder.

If we neglect the $\phi^2\eta$ coupling term, we can integrate out the $\eta$ integral, so that the disordered average is

$$\langle \hat{O} \rangle = Z^{-1} \int d\Lambda_0 \hat{O}(E, \Lambda_0) \exp \left[ -\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L \right]$$

(64a)

with the observable given by

$$\hat{O}(E, \Lambda_0) = \int [d\phi] \hat{O}_0[E, \phi] \exp \left[ -\int dx (t \phi'^2 + 2i\phi^2\Lambda_0) \right].$$

(64b)

Assuming that $\hat{O}[E, \phi]$ is real, we note that $\hat{O}^*(E, \Lambda_0) = \hat{O}(E, -\Lambda_0)$, which allows us to integrate over positive $\Lambda_0$ in (64a) if we take the real part of the integrand, thus

$$\langle \hat{O} \rangle = Z^{-1} \int_0^\infty d\Lambda_0 \Re \hat{O}(E, \Lambda_0)$$

$$\times \exp \left[ -\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L \right].$$

(65)

Here we considered only the lowest order approximation where we totally neglected the contribution from the $\phi^2\eta$ term, but it is easy to extend (64) to include the contribution of the quadratic terms in $\eta$ arising from the $\phi^2\eta$ coupling, upon which the $\eta$ integral can still be done to yield a determinant, which will give higher order corrections to (64).

1. Density of States

For the average density of states, the observable (64b) is

$$\hat{O}(E, \Lambda_0) = \left[ \text{det} \left( -\frac{d^2}{dx^2} + \frac{2i\Lambda_0}{l} \right) \right]^{-1/2}$$

$$\propto \frac{1}{\sqrt{l}} \frac{\sqrt{\Lambda_0} L}{2\sqrt{t}} \coth \left( \frac{1}{2} \sqrt{l} \frac{\Lambda_0}{2\sqrt{t}} L \right),$$

(66)

which we can use in (63) to obtain

$$\langle \rho(E) \rangle = \int_0^\infty d\Lambda_0 \Re \left[ (1 + i) \coth \left( \frac{1}{2} \sqrt{l} \frac{\Lambda_0}{2\sqrt{t}} L \right) \right]$$

$$\times \sqrt{\Lambda_0} \exp \left[ -\frac{\Lambda_0^2 L}{l} - 2iE\Lambda_0 L \right].$$

(67)

2. 2-point Correlations

The observable used to calculate the correlator in the dual region can be obtained from (22). Explicitly writing the total derivative and using the approximation in (64b), we integrate over the $\phi$ field to obtain
where $\Delta(x, x'') = (-\frac{d^2}{dx^2} + \frac{2i\Lambda}{L})^{-1}$.

Using this in (65), extracting the terms that are $x_0$ independent and then cancelling the $x_0$ integral with a similar term in the normalization, we have

$$\frac{\langle \psi(x) | \psi(y) \rangle}{\langle \psi(0) | \psi(0) \rangle} = \frac{C(x, y)}{C(0, 0)} \tag{69a}$$

where

$$C(x, y) = \frac{\Lambda_0^2 L}{2iE \Lambda_0 L} \exp \left[ \frac{\Lambda_0^2 L}{2iE \Lambda_0 L} \right] \exp \left[ \frac{1}{4i} F(|x-y|) \right], \tag{69b}$$

with

$$F(x) = \frac{2}{L} \sum_{n=1}^\infty \frac{\cos (2\pi nx/L) - 1}{(2\pi/n)^2 + 2i\Lambda_0} (\frac{2\pi}{n})^2 \tag{69c}.$$

C. Microscopic realisation of the model

It is useful to have a microscopic realisation of the model presented in the previous sections that relates the parameters to microscopic quantities. For this purpose, we consider a one dimensional model of $N$ Dirac-delta scatterers placed randomly on a ring, with the potential given by

$$V(x) = a \sum_{i=1}^N \delta(x-x_i) - \frac{aN}{L}. \tag{70}$$

Here $a$ is a dimensionfull constant (units of $(\text{length})^{-1}$) that determines the strength of the scatters, and the subtracting term is chosen so that $\langle V \rangle = 0$. Applying our formulism, the disordered average of some observable $\hat{O}$ is now given by

$$\langle \hat{O} \rangle = Z^{-1} \int \frac{dx_i}{L} \prod_{i=1}^N \int [d\phi] d\tilde{E} \hat{O}(\tilde{E}, \phi)$$

$$\times \delta \left[ \tilde{E} + \phi' + \phi^2 - a \sum_{i=1}^N \delta(x-x_i) + \frac{aN}{L} \right] \tag{71}$$

where

$$Z = \int \frac{dx_i}{L} \prod_{i=1}^N \int [d\phi] d\tilde{E} \exp \left[ N \log \int \frac{dx}{L} e^{-ia\tilde{E}} \right]$$

$$\times \delta \left[ \tilde{E} + \phi' + \phi^2 - a \sum_{i=1}^N \delta(x-x_i) + \frac{aN}{L} \right]. \tag{72}$$

Introducing a Fourier representation for the functional Dirac-delta, we can write (71) as

$$\langle \hat{O} \rangle = Z^{-1} \int [d\phi] [d\Lambda] d\tilde{E} \hat{O}(\tilde{E}, \phi) \exp \left[ N \log \int \frac{dx}{L} e^{-ia\tilde{E}} \right]$$

$$\times \exp \left[ i \int dx \Lambda \left( \tilde{E} + \phi' + \phi^2 \right) \right]. \tag{73}$$

Performing the $\Lambda$ integration, we have:

$$\langle \hat{O} \rangle = Z^{-1} \int [d\phi] [d\Lambda] d\tilde{E} \hat{O}(\tilde{E}, \phi)$$

$$\times \exp \left[ -\frac{L}{2Na^2} \int dx (\tilde{E} + \phi' + \phi^2)^2 \right] \tag{75}$$

with

$$Z = \int [d\phi] [d\Lambda] \exp \left[ -\frac{L}{2Na^2} \int dx (\tilde{E} + \phi' + \phi^2)^2 \right]. \tag{76}$$

We can therefore identify the disorder parameter, $l$, of our Gaussian model with $l = L/2Na^2 = \bar{l}/2a^2$, with $\bar{l}$ the mean free path length.

IV. NUMERICAL RESULTS

In this section we numerically calculate the main results that we obtained in the previous sections and generate plots from these calculations in order to obtain a understanding of the results.
FIG. 1: Plot of the density of states (in arbitrary units) vs energy for \( L = 10 \). In (a), the result of the positive energy, weak disorder approximation for various values of \( l \) are shown. In (b), we plot the strong disorder approximation result, while (c) shows the result for the negative energy, weak disorder approximation.

**A. Density of States**

There are three different regions that we need to consider when calculating the disordered averaged density of states. Firstly, if \( E \gg \frac{L}{l} \), then the weak disorder saddle point approximation in the positive energy region, (48a), holds. For \( E \ll -\frac{L}{l} \), the weak disorder saddle point approximation for negative energies, (48b), is used. Finally, when \( -\frac{L}{l} \ll E \ll \frac{L}{l} \), we use the strong disorder approximation, (57). Thus, we find that the weak disorder saddle point approximation describes the high energy tails of the average density of states, while the strong disorder limit describes the low energy states. Note that for a fixed disorder, \( l \), the only approximation that holds in the thermodynamic limit, \( L \to \infty \), is the strong disorder approximation, as the regions described by the saddle point approximation tend to negative and positive infinity.

We calculate the disordered averaged density of states for finite \( L \), with various disorder values using the approximations in their respective energy regions. These results are shown in Fig. 1 in arbitrary units. Figure 1(a) shows the plot for the positive high energy tail for various disorder values. Note that at high energies, the density of states is peaked around the discrete values that one would get in the pure limit. Also, the width of the Gaussian distribution around these discrete values increase as the disorder in the system increases. Eventually, when the disorder is large enough (or the energy low enough, as seen in the plot) the Gaussian distributions start to overlap, which leads to a change in the density of states from a almost pure behaviour to a strong disorder behaviour. Note, however, that the criterion that \( E \gg \frac{L}{l} \) no longer holds in this region, and that the strong disorder approximation should be used instead. Figure 1(b) shows the density of states for the low energy states. As the disorder increases, the width of the strong disorder density of states increases. In Fig. 1(c), the result for the negative energy tail is shown. In this region the density of states falls off exponentially to zero. To describe the density of states over all energies, the arbitrary normalization factors appearing in the three different regions should be fixed by requiring a continuous matching at the transitional points \( E = \pm \frac{L}{l} \) and by imposing some global normalization condition. Here we have only imposed an arbitrary normalization within each region to exhibit the main features of the different regions.

From the plots in Fig. 1, we see that there is a crossover from the almost pure system behaviour at large positive energies to an exponential decay at negative energies. Of particular note is the crossover at small energies where the strong disorder approximation is valid leading to a non-zero result for the density of states at zero energies.

Also, we note that as the parameter \( l \) decreases, the width of the density of states in the negative energy region increases due to the creation of additional bound states in the more disordered system, whereas for large \( l \), that is a more pure system, there are less states at negative energies and the peak increases, leading to the \( E^{-1/2} \) singularity at zero energy for pure systems.

**B. 2 point correlators**

The correlation function in the weak disorder saddle point approximation is given by (51) using (53) for positive energies, and (54) for negative energies, while (69) gives the correlation function in the strong disorder approximation. Without loss of generality, we can set \( y = 0 \) and \( z = dL \), allowing us to calculate the correlation function in the appropriate energy regions where the various approximations hold. These results are shown in Fig. 2 and Fig. 3.

Figure 2 shows the result of the correlation function at a fixed value of \( l \) and \( L \) for various energies. Figure 2(a) is the correlation function calculated in the weak disorder saddle point approximation with a large positive energy \( (E \gg \frac{L}{l}) \). In this region the result is dominated by the pure solution, giving rise to the oscillations. The effect of
FIG. 2: Plots of the correlation function (normalized to one at \(d = 0\)) vs distance \(d\) showing the disorder dependence, where (a) is the positive energy, weak disorder saddle point approximation \((E = 1.5 \times 10^3)\), (b) is the strong disorder saddle point approximation \((E = -2.5)\) and (c) is the negative energy, weak disorder saddle point approximation \((E = -1.75 \times 10^3)\). The disorder value in all three plots is \(l = 1.0 \times 10^2\), and \(L = 10\).

As the energy is lowered even more, the region where the strong disorder approximation \((-\frac{L}{l} \ll E \ll \frac{L}{l})\) holds is reached as shown in Fig. 2(b). The correlation function decreases as the distance increases until \(d = 0.5L\) after which the correlation increases again. This is of course due to our ring topology. Since the bulk of the states occur in this region, we consider the behaviour of the strong disorder region in the next figure, Fig. 3.

Figure 2(c) shows the correlator function for large negative energies, where the weak disorder saddle approximation \(E \ll -\frac{L}{l}\) once again holds. Here the results show that the correlation function decays exponentially, where the decay length is determined by the energy and is largely disorder independent. This exponential behaviour is due to the formation of bound states in the disordered potential.

In Fig. 3, we consider the behaviour of the correlator in the strong disorder approximation, when the disorder and energy are varied. In Fig. 3(a), we keep the energy fixed and vary the disorder. As can be expected, there is a stronger decay when the disorder in the system is increased (smaller \(l\)). In Fig. 3(b), the disorder parameter is kept fixed, while the energy is varied. Once again, as is expected, the decay increases as the energy is lowered. Thus the strong disorder approximation gives a bridge from the almost pure behaviour at high positive energies to the strongly localized behaviour at strong negative energies due to the formation of bound states.

**V. CONCLUSION**

In this paper we introduced a functional integration formalism for studying disordered averaged observables that provides a complementary viewpoint to the standard field theoretic techniques used at present. The formalism
is based on changing variables from the random potential describing the disordered system to a new set of random variables related to the logarithmic derivative of the wave-function. This allows a more direct computation of certain disordered averages, such as the density of states or observables that explicitly depend on the wave function. In particular we showed how to calculate the disorder averages of the density of states (16), the 2-point correlators of the wave-function (22, 24), as well as the real part of the conductivity (30, 31).

As an illustration of how the formalism works, we considered one dimensional Gaussian disordered systems. We were able to obtain results for the weak disorder and strong disorder limits for the density of states (48, 67), and the 2-point correlators (53, 56, 69). Unfortunately we were only able obtain results of the conductivity in the weak disorder limit (61), as there is a complication in the perturbative expansion of the strong disorder limit when using the Hubbard-Stratonovitch transformation on (30), which we have as yet been unable to resolve. The formalism reproduced the results of Halperin for the density of states, and considering the 2-point correlator we showed that in the thermodynamic limit all states in one dimension are localised.

Future developments include the addition of a deterministic potential to the formalism, allowing magnetic interactions to be included. Also, the calculation in higher dimensions needs to be investigated further to see if signs of a metal insulator transition can be found.

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**APPENDIX A: PROPAGATORS AND DETERMINANTS**

We wish to calculate the propagator, ∆, and determinant, det(∆−1), where ∆−1 is given by (36b) or (42b). This involves solving the differential equation (43),

\[-\frac{d^2}{dx^2} + 2 \left( \frac{m\pi}{L} \right)^2 - 6 \left( \frac{m\pi}{L} \tan \left( \frac{m\pi}{L} x \right) \right)^2 \] \[\Psi_n = \left[ \lambda_n + 2 \left( \frac{m\pi}{L} \right)^2 - 2E \right] \Psi_n \] (A2)

The differential equation in (A2) can be solved exactly using the method of generalized ladder operators so that

\[\lambda_n = (n^2 + 6n + 3) \left( \frac{m\pi}{L} \right)^2 + 2E \quad \forall \ n \geq 0 \] (A3a)

and

\[\Psi_0 \propto \left( \cos \left( \frac{m\pi}{L} x \right) \right)^3 \]
\[\Psi_n \propto \prod_{i=0}^{n-1} \left( -\frac{d}{dx} + (i+3) \frac{m\pi}{L} \tan \left( \frac{m\pi}{L} x \right) \right) \left( \cos \left[ \frac{m\pi}{L} x \right] \right)^{n+3} \quad \forall \ n > 0. \] (A3b)

Unfortunately, although we have the exact solution, we do not have the eigenfunctions in a closed form. This makes the calculation of the propagator difficult. We thus wish to make an approximation to (A3) that allows us to calculate the propagator. Since the tan² potential in (A2) contains m singularities in the interval \([-\frac{L}{2}, \frac{L}{2}]\), the eigenfunctions must be zero where these singularities occur. The approximation that we make for (A2) must preserve this global property. The approximation that we make is to ignore the tan² term in (A2) and to change the boundary condition so that eigenvalues have zeros at the correct intervals. The eigenvalue equation that we must solve is thus

\[-\frac{d^2}{dx^2} + 2E \Psi_n = \lambda_n \Psi_n \] (A4)

with the boundary condition that \(\Psi(\pm \frac{L}{2}) = 0\).

Excluding the constant mode as required by (37), the solution of (A4) is

\[\lambda_n = \left( \frac{nm\pi}{L} \right)^2 + 2E \quad \forall \ n > 0 \] (A5a)
with
\[
\Psi_n = \sqrt{\frac{L}{2}} \cos \left( \frac{n\pi}{L}x \right) \quad \forall \text{ odd } n > 0
\]
\[
= \sqrt{\frac{L}{2}} \sin \left( \frac{n\pi}{L}x \right) \quad \forall \text{ even } n > 0.
\] (A5b)

As expected (A5a) and (A3a) are in good agreement for large \(n\).

The propagator can now be calculated using the energy representation to give
\[
\Delta_m(x, y) = \left( -\frac{d^2}{dx^2} + 2E \right)^{-1}
\]
\[
= \frac{2}{L} \sum_{n=1}^{\infty} \left( \frac{\cos((2n-1)\frac{\pi}{L}x)\cos((2n-1)\frac{\pi}{L}y)}{([2n-1]\frac{\pi}{L})^2 + 2E} + \frac{\sin((2n)\frac{\pi}{L}x)\sin((2n)\frac{\pi}{L}y)}{((2n)\frac{\pi}{L})^2 + 2E} \right)
\] (A6)

while the determinant can be calculated with various techniques, i.e. the identity 1.143.1 found in Gradsteyn and Ryzhik, so that
\[
|\det(\Delta_m^{-1})| = C \frac{|m|}{\sqrt{2EL}} \sinh \left( \frac{\sqrt{2EL}}{|m|} \right),
\] (A7)

where \(C\) is an energy independent constant, but not necessarily independent of \(m\). We determine \(C\) by requiring that the density of states \(48a\) be the correct solution in the pure limit. Obtaining the pure solution (up to a global normalization constant), requires that (A7) must be a constant independent of \(m\). However, since \(E = \left( \frac{m\pi}{L} \right)^2\) in this limit, and \(C\) is independent of \(E\), we find that \(C\) must also be independent of \(m\). Thus we see that \(C\) is constant that is independent of \(E\) and \(m\).

In the negative energy region, \(\phi_c\) is given by \([11]\) where we are using the dilute gas approximation. Thus the equation \([11]\) becomes
\[
\left( -\frac{d^2}{dx^2} + 4|E| + \tilde{\phi}_0 \right) \Psi_n = \lambda_n \Psi_n,
\] (A8a)

where the quasi zero-mode \(\tilde{\phi}_0\) is given by
\[
\tilde{\phi}_0 = \text{sech}^2 \left( \sqrt{|E|}(x - \frac{L}{4}) \right) \Theta(x) + \text{sech}^2 \left( \sqrt{|E|}(x + \frac{L}{4}) \right) \Theta(-x).
\] (A8b)

If we integrate over \([A8a]\), and use the condition that \(\tilde{\phi}_0\) is orthogonal to \(\Psi_n\), we find that the constraint that the eigenfunction cannot contain a zero mode is satisfied for all eigenfunctions except the one corresponding to \(\lambda_n = 4|E|\). The eigenfunction with this eigenvalue must be explicitly checked to see if the constraint is satisfied.

We first calculate the eigenfunctions and eigenvalues of \([A8]\) in the subinterval \([-\frac{L}{4}, \frac{L}{4}]\) or \([0, \frac{L}{4}]\) using the method of generalized operators \([12]\) or via the solution of a hypergeometric equation \([11]\), and find that the eigenvalues consists of two discrete eigenvalues, \(\lambda_0 = 0\) and \(\lambda_1 = 3|E|\), and a continuum of eigenvalues \(\lambda_k = (k^2 + 4)|E|\). The corresponding eigenfunctions are

\[
\Psi_0^+ \propto \text{sech}^2 \left( \sqrt{|E|}(x + \frac{L}{4}) \right) \Theta(\pm x),
\]
\[
\Psi_1^+ \propto \text{tanh} \left( \sqrt{|E|}(x + \frac{L}{4}) \right) \Theta(\pm x),
\]
\[
\Psi_k^+ \propto \left[ 3 \text{tanh}^2 \left( \sqrt{|E|}(x + \frac{L}{4}) \right) - 1 - k^2 \right] e^{\sqrt{|E|}kx} \Theta(\pm x) - 3ik \text{tanh} \left( \sqrt{|E|}(x + \frac{L}{4}) \right) e^{i\sqrt{|E|}kx} \Theta(\pm x).
\] (A9a)

We can now calculate the eigenfunctions for the full region \([-\frac{L}{4}, \frac{L}{4}]\) by matching the eigenfunctions in \([A9]\) at \(x = 0\), i.e. requiring that \(\Psi^+(0) = \Psi^-(0)\) and \(\frac{d\Psi^+}{dx}(0) = \frac{d\Psi^-}{dx}(0)\). The eigenfunctions are

\[
\Psi_0 \propto \pm \text{sech}^2 \left( \sqrt{|E|}(x + \frac{L}{4}) \right) \Theta(-x) + \text{sech}^2 \left( \sqrt{|E|}(x - \frac{L}{4}) \right) \Theta(x)
\] (A10a)
\[
\Psi_1 \propto \pm \frac{\text{tanh} \left( \sqrt{|E|}(x + \frac{L}{4}) \right)}{\text{cosh} \left( \sqrt{|E|}(x + \frac{L}{4}) \right)} \Theta(-x) + \frac{\text{tanh} \left( \sqrt{|E|}(x - \frac{L}{4}) \right)}{\text{cosh} \left( \sqrt{|E|}(x - \frac{L}{4}) \right)} \Theta(x)
\] (A10b)
\[
\Psi_k \propto \pm (2 - k^2) e^{\sqrt{|E|}k(x + \frac{L}{4})} \Theta(-x) \pm \frac{3\Theta(-x)}{\sqrt{|E|}} \frac{\partial}{\partial x} \left[ \text{tanh} \left( \sqrt{|E|}(x + \frac{L}{4}) \right) e^{\sqrt{|E|}k(x + \frac{L}{4})} \right]
\]
\[
\quad + (2 - k^2) e^{\sqrt{|E|}k(x - \frac{L}{4})} \Theta(x) + \frac{3\Theta(x)}{\sqrt{|E|}} \frac{\partial}{\partial x} \left[ \text{tanh} \left( \sqrt{|E|}(x - \frac{L}{4}) \right) e^{\sqrt{|E|}k(x - \frac{L}{4})} \right]
\] (A10c)
where \( k \neq 0 \) since the corresponding eigenfunction does not satisfy the constraint that there are no constant modes. Also, the periodicity requirements on the eigenfunctions imply that \( k \) must satisfy the equation

\[
e^i \sqrt{|E|} k L = \left( \frac{2 - k^2 + 3ik}{2 - k^2 - 3ik} \right)^2. \tag{A11}\]

Note that there are degenerate solutions for each eigenvalue, since the eigenfunctions can be constructed as a symmetric or an anti-symmetric solution.

As in the positive energy region, we do not have the eigenvalues and subsequently also not the eigenfunctions in a closed form, which makes calculation of the propagator and determinant difficult. We thus once again wish to make an approximation that will enable us to calculate the propagator and determinant. We note that the right hand side of (A1) is approximately unity, allowing us to obtain

\[
k = \frac{2n\pi}{L\sqrt{|E|}}, \quad \forall n \in \mathbb{Z}, n \neq 0 \tag{A12}\]

for large values of \( k \). The eigenfunctions \( \Psi_k \), can then be approximated by a plane wave, and is given by

\[
\Psi_k \propto e^{i \sqrt{|E|} k (x + \frac{1}{2})} \Theta(-x) + e^{i \sqrt{|E|} k (x - \frac{1}{2})} \Theta(x) \tag{A13}\]

with corresponding eigenvalue \( \lambda_n = (\frac{2n\pi}{L})^2 + 4|E| \). This implies, as is to be expected, that all the higher lying particle states can be very well approximated by free particle states and that this only breaks down for the lowest lying scattering states, where the potential is important. Note, however, that even the spectrum of the lower lying scattering states is well approximated by a free particle spectrum as the right hand side of (A11) is approximately unity also in this case. The eigenfunctions are, however, distorted away from plane waves due to the presence of the potential.

It is now possible to calculate the determinant using the above approximation so that

\[
|\det(\Delta^{-1})| = (3|E|)^2 \prod_{n=1}^{\infty} \left( \left( \frac{2n\pi}{L} \right)^2 + 4|E| \right)^4 \propto \sinh^4 \left( \sqrt{E} L \right), \tag{A14}\]

where we have once again used the identity 1.143.1 from Gradshteyn and Ryzhik. Note that the quadratic term in (A14) is due to the doubly degenerate bound state (from the symmetric and anti-symmetric eigenfunctions), while the product over the continuum eigenvalues is raised to the fourth power since there is a fourfold degeneracy in the continuum eigenfunctions (from the symmetric and anti-symmetric states, as well as the right moving and left moving plane waves).

In calculating the propagator, we neglect the contribution of the bound state \( \Psi_1 \), which gives only a small contribution to the propagator as it involves the product of two very well localized eigenfunctions evaluated at points which are well separated. Taking into account only the scattering states, (A13), we find for the propagator

\[
\Delta_{\pm}(x,y) = \frac{2}{L} \sum_{n=1}^{\infty} \left( \frac{\cos(2n\pi(x-y))}{(2n\pi)^2 + 4|E|} \right). \tag{A15}\]

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