Extending torsors under a quasi-finite flat group scheme - Preliminary Version

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Abstract

Let $R$ be a Henselian discrete valuation ring of field of fractions $K$ and of residue field $k$ of characteristic $p > 0$. In an earlier work, we studied the question of extending torsors on $K$-curves into torsors over $R$-regular models of the curves but we restricted ourselves to the case when the structural $K$-group scheme of the torsor admits a finite flat model over $R$. Here, we study the case where it has a quasi-finite flat model which is not necessarily finite. Under some additional assumptions, we prove that this reduces to the finite case. In a second part, combining our result with recent work by Holmes, Malcho and Wise on the relation between the log Picard functor and Néron models of Jacobians, we give an application of our result in the semi-stable case.

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1 Introduction

All over this paper, $R$ denotes a discrete valuation ring with field of fractions $K$ and residue field $k$ of characteristic $p > 0$. In sections 4.1 and 4.2, $R$ is assumed to be Henselian. In sections 4.3 and some results of section 5, $R$ is supposed to be Henselian, Japanese with $k$ perfect.

Let $X$ be a faithfully flat $R$-morphism of finite type, and let $Y$ be its generic fiber. Assume that we are given a finite commutative $K$-group scheme $G$ and an fppf $G$-torsor $Y \to X$. The problem of extending the $G$-torsor $Y \to X$ consists in finding a finite and flat $R$-group scheme $\mathcal{G}_f$ whose generic fiber is isomorphic to $G$, and an fppf $\mathcal{G}_f$-torsor $\mathcal{Y} \to \mathcal{X}$ whose generic fiber is isomorphic to $Y \to X$ as a $G$-torsor.

This question has been investigated in various settings, first by Grothendieck. For example, the case when $G$ is a constant group scheme of order coprime to $p$ and $X \to \text{Spec}(R)$ is smooth with geometrically connected fibers, does have a solution; see [2, X, §3.1 and §3.6].

A natural way to solve this problem is to start by searching for a finite flat $R$-model of $G$ (if any), and then study the question of extending the $G$-torsor. In [16], this has been done by Tossici in the case where $p$ divides $|G|$. More precisely, he studied the extension of torsors under commutative finite flat group schemes over local schemes under some extra assumptions. He also studied, using the so-called effective models, the extension of $\mathbb{Z}/p\mathbb{Z}$-torsors and $\mathbb{Z}/p^2\mathbb{Z}$-torsors imposing the normality of $\mathcal{Y}$.

However, an $R$-finite flat model of $G$ doesn’t always exist. In [3], Antei and Emsalem approached the issue differently. Assuming that $G$ is affine, instead of looking for a finite flat model of $G$, they choose to work with a more general model of $G$ that is flat but only quasi-finite, and, then, extend the torsor over some scheme $\mathcal{X}'$, which is obtained by modifying the special fiber of $\mathcal{X}$. By allowing such models of $G$, they solved the problem of extending any $G$-torsor up to a modification of $\mathcal{X}'$, without any assumptions on the residue characteristic. When $\mathcal{X}$ is a relative curve, this modification is obtained by performing a finite sequence of Néron blow-ups of $\mathcal{X}$ along closed subschemes of the special fiber.

In a previous paper (cf. [13]), we considered the problem of extending fppf $G$-torsors over a smooth projective $K$-curve $C$, endowed with a $K$-rational point $Q$, and sought an extension over some $R$-regular model $\mathcal{C}$ of $C$. We first noticed that the existence of an fppf extension is in general a strong requirement: if we assume that we have a finite flat model $\mathcal{G}_f$ of $G$, that is in addition étale, our extended torsor –if it exists– should be unramified, which is quite a strong condition. A way to relax this condition is to work inside a larger category, namely the category of logarithmic torsors. More precisely, we endow $\mathcal{C}$ with the logarithmic structure induced by its special fiber $\mathcal{C}_k$, seen as a divisor. Then logarithmic torsors over $\mathcal{C}$ are, roughly speaking, tamely ramified along $\mathcal{C}_k$. We proved that this problem of extending torsors reduces to the question of extending group functors and morphisms between them. Namely, we proved:

**Theorem 0.** Let $C$ be a smooth projective and geometrically connected curve over $K$, endowed with a $K$-rational point $Q$. Let $f : C \to \text{Spec}(R)$ be a regular model of $C$ and let $Q$ be the $R$-section that extends $Q$ over $C$. Let $\mathcal{G}_f$ be a commutative finite flat group scheme over $R$ and let $\mathcal{G}_f^D$ denote
its Cartier dual. We have a canonical isomorphism:

\[ H^1_{klf}(C, \mathcal{Q}, G_f) \xrightarrow{\sim} \text{Hom}(G^D_f, \text{Pic}^\log_{C/R}). \]

where \( H^1_{klf}(C, \mathcal{Q}, G_f) \) denotes the cohomology group that classifies logarithmic \( G_f \)-torsors over \( C \), pointed relatively to \( \mathcal{Q} \).

Keeping the assumptions of §6, it is now natural to ask what happens in the case where \( G \) does not have a finite flat model. In particular, we would like to study the case when \( G \) has a quasi-finite and flat model which is not necessarily finite. Note that the previous theorem uses the fact that \( \mathcal{G} \) is finite since its Cartier dual is not defined otherwise. Nevertheless, under some additional assumptions, the quasi-finite case reduces to the finite one. Indeed, in the case where \( R \) is supposed to be Henselian, any quasi-finite flat and separated \( R \)-group scheme can be written as an extension of an étale group scheme by a finite group scheme. More precisely, if \( G \) admits a quasi-finite flat and separated \( R \)-model \( \mathcal{G} \), we can write:

\[ 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{E} \to 0 \]

where \( \mathcal{F} \) is an \( R \)-finite flat group scheme and \( \mathcal{E} \) an \( R \)-étale group scheme with trivial special fiber.

To answer this question, we proceed as follows:

In sections 4.1 and 4.2, if \( F \) and \( E \) denote the generic fibers of \( \mathcal{F} \) and \( \mathcal{E} \) respectively, we observe that any fppf pointed \( G \)-torsor \( Y \to C \) can be written as a tower of an fppf pointed \( F \)-torsor and an étale pointed \( E \)-torsor. Then, we study the extension of these two torsors. In particular, we show that it is always possible to choose an \( R \)-model of \( C \) where the extension of the \( G \)-torsor is unique (cf. Corollary 4.8).

In section 4.3, after restricting our assumptions a bit, we study the existence of such an extension. To do so, we use a result in [9] that states that any torsor under a quasi-finite group scheme has a reduction into a torsor under a finite group scheme. In particular, we prove the following:

**Theorem 4.9.** We assume that \( R \) is Henselian Japanese and that \( k \) is perfect. Let \( C \) be a smooth projective and geometrically reduces \( K \)-curve with a point \( Q \). Let \( \mathcal{C} \) be a regular model of \( C \) over \( R \), which supposed to be irreducible with a geometrically reduced special fiber. Then, the extension of pointed fppf \( G \)-torsors over \( C \) into pointed \( \mathcal{G} \)-log torsors over \( C \) is equivalent to that of pointed fppf \( F \)-torsors over \( C \) into pointed \( F \)-log torsors over \( C \).

Finally, in section 5, we give an application in the semi-stable case. First, we observe that Theorem 0 above can be improved in the case where the curve \( C \) is semi-stable. Indeed, it was proved in [8] that the subsheaf of the log Picard functor of degree zero line bundles is the Néron model if the Jacobian of \( C \). In particular, if \( \mathcal{G}_f \) is finite as in Theorem 0, the morphism \( \mathcal{G}_f \to \text{Pic}^\log_{C/R} \) factors through the Néron model of the Jacobian of \( C \). Therefore, if the \( G \)-torsor \( Y \to C \) is geometrically connected, we prove that it extends into a log \( \mathcal{G}_f \)-torsor over \( C \) if and only if \( \mathcal{G}_f^D \) is the schematic closure of \( G^D \) in \( \mathcal{J} \). Now, going back to the case where \( G \) admits a quasi-finite flat separated model which is not finite, and using the previous work, we prove the following:

**Theorem 5.7.** We assume that \( R \) is Henselian, Japanese and that \( k \) is perfect. Let \( C \) be a smooth projective semi-stable and geometrically connected \( K \)-curve with a point. Let \( \mathcal{C} \) be some
R-regular model of $C$ which is assumed to be irreducible and which is endowed with its canonical log structure, $J$ the Jacobian of $C$ and $\mathcal{J}$ the Néron model of $J$. Let $G$ be a finite commutative $K$-group scheme of order $r$ and with quasi-finite flat and separated $R$-model $\mathcal{G}$ that is written as in §8. Then, a pointed fppf $G$-torsor $Y \to C$ extends into a log $G$-torsor over $C$ if and only if the morphism $h_Y : G^D \to J$ factors through $h'_Y : F^D \to J$ and the latter extends into an $R$-morphism $F^D \to \mathcal{J}$. In particular, if $h'_Y$ is injective, then it factors through $F\mathcal{J}[r]_K$ and $F^D$ is necessarily the schematic closure of $F^D$ in $\mathcal{J}$.

As for the fppf case, we have the following:

**Theorem 5.9.** With the same assumptions as in Theorem 5.7, an fppf pointed $G$-torsor $Y \to C$ extends into an fppf $G$-torsor over $C$ if and only if $h_Y : G^D \to J$ factors through $h'_Y : F^D \to J$ and the latter extends into an $R$-morphism $F^D \to \mathcal{J}^0$. In addition, if $h'_Y$ is injective, then it factors through $F\mathcal{J}^0[r]_K$ and $F^D$ is necessarily the schematic closure of $F^D$ in $\mathcal{J}^0$.

Throughout this paper, all schemes and log schemes are assumed to be locally noetherian.

2 Preliminaries

2.1 Log schemes and log torsors

Let $X$ be a noetherian and regular scheme and let $j : U \to X$ be an open subset whose complementary is a divisor $D$ over $X$. Then the inclusion

$$O_X \cap j_*O'_U \to O_X$$

defines a fine and saturated log structure on $X$, which we call the log structure defined by $D$. It is clear from the definition that $U$ is the open of triviality of this log structure.

Spec($R$) can be seen as a fine saturated log scheme with the log structure defined by Spec($k$) seen as a divisor. Spec($K$) is then the open of triviality of this log structure.

If $\mathcal{X}$ is an $R$-scheme, then it can be seen as a log scheme with the log structure induced by its special fiber $\mathcal{X}_k$ seen as a divisor. Then the log structure is trivial over the generic fiber $\mathcal{X}_K$. This log structure is called the canonical log structure on $\mathcal{X}$.

We consider here the category of fine and saturated log schemes, endowed with the Kummer log flat topology (we write sometimes klf to refer to this topology for simplicity). We refer to [10] or [7, §2.2] for the definition of this Grothendieck topology. A torsor in this category, defined with respect to the klf topology, is called a logarithmic torsor (or a log torsor). For $X$ an $R$-log scheme and $G$ an $R$-group scheme, we denote by $H^1_{klf}(X, G)$ the first cohomology group that classifies $G$-logarithmic torsors over $X$. Moreover, a Kummer log flat cover of a scheme endowed with the trivial log structure is just a cover for the fppf topology. So, in this paper, the category of schemes is endowed with the fppf topology.

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1Since $C$ is semi-stable, $\mathcal{J}[r]$ is quasi-finite, hence can be written as an extension of an $R$-étale group scheme, by an $R$-finite one that we denote by $F\mathcal{J}[r]$. We write $F\mathcal{J}[r]_K$ for its generic fiber.
2.2 Extension of torsors under a finite flat group scheme

In this section, we fix a $K$-group scheme $G$, a smooth projective and geometrically connected $K$-curve $C$, endowed with a $K$-rational point $Q$, and $C$ a regular $R$-model of $C$. In an earlier word, we considered the problem of extending fppf pointed $G$-torsors over $C$ and sought an extension over $C$. We first emphasize that the existence of an fppf extension is in general a strong requirement: if we assume that we have a finite flat model $G$ of $G$, that is in addition ´etale, our extended torsor –if it exists– should be unramified. But this is quite a strong condition. In order to relax this condition, the idea was to work inside a larger category, namely the category of logarithmic torsors. More precisely, we endow $C$ with the logarithmic structure induced by its special fiber $C_k$ seen as a divisor. The results mentioned in this section come from [13].

The data of a pointed torsor over $C$ (relative to $Q$) is equivalent to that of a torsor which is trivial when restricted to $Q$. Furthermore, we have the following bijective correspondence:

$$\{\text{pointed fppf } G\text{-torsors over } C, \text{ relative to } Q\} \to \text{Hom}(G^D, \text{Pic}_{C/K})$$

$$Y \mapsto h_Y.$$  \hspace{1cm} (1)

where $G^D$ is the Cartier dual of $G$ and $\text{Pic}_{C/K}$ is the relative Picard functor of $C$ over $K$. In addition, since $G$ is assumed to be finite, each morphism $G^D \to \text{Pic}_{C/K}$ factors through $\text{Pic}_{C/K}^0 = J$, the Jacobian of the curve $C$.

In fact, the bijective correspondence in (1) also holds over $R$. Indeed, if $Q$ is the $R$-section over $C$ that extends $Q$, if $G$ is a finite flat $R$-group scheme and if we assume that $f_*\mathcal{O}_C = \mathcal{O}_R$, we have an isomorphism:

$$H^1_{\text{fppf}}(C, Q, G) \simeq \text{Hom}(G^D, \text{Pic}_{C/R}).$$ \hspace{1cm} (2)

where $H^1_{\text{fppf}}(C, Q, G)$ is the cohomology group that classifies fppf $G$-pointed torsors over $C$, relatively to $Q$, and $\text{Pic}_{C/R}$ the relative Picard functor of $C$ over $R$.

**Definition 2.1.** We recall that $\text{Spec}(R)$ is seen as a log scheme via the log structure induced by $\text{Spec}(k)$. Let $(\text{Sch}/R)$ denote the category of schemes over $R$, and $(\text{fs}/R)$ the category of fine and saturated log schemes over $R$, endowed with klf topology. Then $(\text{Sch}/R)$ can be seen as a (full) subcategory of $(\text{fs}/R)$ as follows: given a morphism of schemes $T \to \text{Spec}(R)$, we endow $T$ with the inverse log structure of that on $\text{Spec}(R)$. The topology on $(\text{Sch}/R)$ induced by the klf topology is the fppf one.

1. Recall the definition of the following functor:

$$\mathbb{G}_{m,\log,R} : (\text{fs}/R) \to (\text{Ab})$$

$$T \mapsto \Gamma(T, M_{\text{fpp}}^R)$$

This is a sheaf for the Kummer log flat topology [10, Theorem 3.2].

2. Consider the following functor

$$(\text{Sh}/R) \to (\text{Sets})$$

$$T \mapsto \{\mathbb{G}_{m,\log,C} - \text{log torsors on } C_T\}.$$
The log Picard functor, denoted by \( \text{Pic}_C^{\log} \), is defined to be the fppf sheafification on \((\text{Sch}/R)\) of the previous functor. Furthermore, it is clear that the generic fiber of the log Picard functor is \( \text{Pic}_{C/K} \), the usual relative Picard functor of \( C \) over \( K \).

**Proposition 2.2.** If \( C \) has a point, then we have an isomorphism

\[
\text{Pic}_C^{\log}(T) \simeq H^1_{dR}(C, \mathbb{G}_{m, \log, C})/H^1_{dR}(T, \mathbb{G}_{m, \log, R})
\]

We investigated the correspondence between torsors and homomorphisms of group schemes in the logarithmic setting. We proved the following:

**Theorem 2.3.** Let \( C \) be a smooth projective and geometrically connected curve over \( K \), endowed with a rational point \( Q \). Let \( f : C \to \text{Spec}(R) \) be a regular model of \( C \) and let \( Q \) be the \( R \)-section that extends \( Q \) over \( C \). Let \( G \) be a commutative finite flat group scheme over \( R \) and let \( G^D \) denote its Cartier dual. We have a canonical isomorphism:

\[
H^1_{dR}(C, Q, G) \xrightarrow{\sim} \text{Hom}(G^D, \text{Pic}_C^{\log}).
\]

where \( H^1_{dR}(C, Q, G) \) denotes the cohomology group that classifies logarithmic \( G \)-torsors over \( C \), pointed relatively to \( Q \).

**Proposition 2.4.** With the assumptions of the previous theorem, if \( J \) denotes the Jacobian of \( C \) and \( J \) its Néron model over \( R \), then the closed immersion \( J \to \text{Pic}_{C/K} \) extends uniquely into an \( R \)-morphism \( J \to \text{Pic}_C^{\log} \). In addition, the log Picard functor coincides with the Néron model of \( \text{Pic}_{C/K} \) in the étale site.

**Definition 2.5.** If \( G \) is a finite flat commutative group scheme over \( R \), we call a pointed \( G \)-log torsor over \( C \) whose associated morphism \( G^D \to \text{Pic}_C^{\log} \) (from Theorem 2.3) factors through \( J \), a Néron-log torsor. Given the separatedness of \( J \), the factorization is unique when it exists.

**Corollary 2.6.** Let \( G \) be a commutative finite \( K \)-group scheme and let \( Y \to C \) be an fppf pointed \( G \)-torsor. If the associated morphism \( h_Y : G^D \to J \) extends to a morphism \( G^D \to J \) for some \( R \)-finite flat model \( G \) of \( G \), then the pointed \( G \)-torsor \( Y \to C \) extends uniquely to a Néron-log (pointed) \( G \)-torsor over \( C \).

**Proposition 2.7.** Let \( G \) be a commutative finite \( K \)-group scheme and let \( Y \to C \) be an fppf pointed \( G \)-torsor. If the associated morphism \( h_Y : G^D \to J \) extends to a morphism \( G^D \to J \) for some \( R \)-finite flat model \( G \) of \( G \), then the extended Néron-log torsor (cf. Corollary 2.6) lifts into an fppf one if and only if the \( R \)-morphism \( G^D \to J \) factors through \( J^0 \), the identity component of \( J \).

### 3 Existence of a finite flat model

Let \( G \) be a finite commutative group scheme over \( K \), killed by an integer \( r \). Let \( C \) be a smooth projective and geometrically connected \( K \)-curve with Jacobian curve \( J \). Assume that we have an injective morphism of \( K \)-group schemes \( G \to J \) which hence factors through \( J[r] \). If \( J \) is the Néron model of \( J \), we let \( G \) be the schematic closure of \( G \) inside \( J[r] \).

If \( C \) is semistable, then \( J[r] \) is flat and quasi-finite (cf. [4, §7.3, Lemma 2]). According to [12, Lemma 1.1], we have an exact sequence:

\[
0 \to \mathcal{F} \mathcal{J}[r] \to \mathcal{J}[r] \to \mathcal{E} \mathcal{J}[r] \to 0
\]

\[\text{an } R\text{-model of } G \text{ is an } R\text{-group scheme whose generic fiber is isomorphic to } G.\]
where $FJ[r]$ is a finite flat group scheme over $R$ and $EJ[r]$ is an étale group scheme over $R$ with trivial special fiber. In particular, it follows from [1, §IX, Lemma 2.2.3] that $FJ[r]$ is the largest finite subgroup scheme in $J[r]$.

The schematic closure $G$ of $G$ in $J[r]$ is flat and quasi-finite. Hence we have as previously an exact sequence:

$$0 \to F \to G \to E \to 0$$

where $F$ is a finite flat group scheme over $R$ and $E$ is an étale group scheme over $R$ with trivial special fiber. We would like to find a necessary and sufficient condition for $G$ to be finite.

We denote by $FJ[r]_K$ the generic fiber of $FJ[r]$.

**Lemma 3.1.** $G$ is finite if and only if $G \to J[r]$ factors through $FJ[r]$.

**Proof.** If $G$ is finite, since $FJ[r]$ is the largest finite subgroup scheme inside $J[r]$, then $G \to J[r]$ factors through $FJ[r]$, hence $G \to J[r]$ factors through $FJ[r]_K$.

On the other hand, if $G \to J[r]$ factors through $FJ[r]$ since $FJ[r]$ is closed inside $J[r]$ (it is a kernel), $G$ is the schematic closure of $G$ in $FJ[r]$, hence it is finite (closed immersions are finite and the composition of two finite morphisms is finite). $\square$

4 Extension of torsors under a quasi-finite flat group scheme

We assume here $R$ to be Henselian. Let $G$ be a finite commutative group scheme over $K$ and let $G$ be a quasi-finite flat and separated model of $G$ over $R$. We have an exact sequence:

$$0 \to F \to G \to E \to 0.$$ (4)

We fix for this section a smooth projective and geometrically connected $K$-curve $C$ with a point $Q$, and a regular $R$-model $\mathcal{C}$ of $C$, which is seen as a log scheme with its canonical log structure. We denote by $Q$ the $R$-section that extends $Q$. Let’s give an fppf pointed $G$-torsor $Y \to C$.

By viewing $G$ as the extension of $E$ (the generic fiber of $E$) by $F$ (the generic fiber of $F$), we first decompose the $G$-torsor $Y \to C$ into an fppf $F$-torsor and an étale $E$-torsor. Then, we will study the extension of these two components and whether they imply the extension of the initial torsor.

4.1 Components of a quasi-finite torsor

Let $Y_0 \to C$ be a logarithmic pointed $G$-torsor. If $G$ acts on $Y_0$ on the right, we define an action $\sim$ of $G$ on $Y_0 \times E$ in the following way:

$$\sim : (G, Y_0 \times E) \to Y_0 \times_{\text{Spec}(R)} E \quad (g, (y, f)) \mapsto (yg^{-1}, gf)$$

Define the $R$-logarithmic scheme $Y'_0 := Y_0 \times^G E := (Y_0 \times E)/\sim$. It is called the contracted product of $Y_0$ and $E$.

**Lemma 4.1.** $Y_0 \to Y'_0$ is a logarithmic pointed $F$-torsor and $Y'_0 \to C$ is an étale pointed $E$-torsor.

**Proof.** The action of $E$ on $Y_0 \times^G E$ is defined by the natural action of $E$ on itself. For the fact that $Y'_0 \to \mathcal{X}$ is an $E$-torsor, it is proved in [5, III, §4, n3]. Here, the morphism of log schemes $Y'_0 \to \mathcal{X}$ is an isomorphism on the special fiber. Since the log structure is supported on the special fiber, it
follows that the log structures over $\mathcal{Y}_0'$ and $C$ are the same. Hence the log étale torsor $\mathcal{Y}_0' \to \mathcal{X}$ is étale.

It remains to see that $\mathcal{Y}_0 \to \mathcal{Y}_0'$ is an $F$-torsor. $F$ clearly acts on $\mathcal{Y}_0$ since $F$ is a subgroup scheme of $\mathcal{G}$. We need to show the isomorphism:

$$\mathcal{Y}_0 \times_{\mathcal{Y}_0'} \mathcal{Y}_0 \simeq \mathcal{Y}_0 \times F$$

For any $R$-scheme $T$, let $y_1, y_2 \in \mathcal{Y}_0(T)$. Then, since $\mathcal{Y}_0 \to C$ is a $G$-torsor, $\exists! g \in \mathcal{G}(T)$ such that $y_2 = y_1 g$. If $y_1$ and $y_2$ have the same image in $\mathcal{Y}_0'(T)$, then $(y_1, 1) = (y_1 g, 1) = (y_1, g^{-1})$, hence $g^{-1} \in \mathcal{F}(T)$ and so $g \in \mathcal{F}(T)$.

In particular, the fppf pointed $G$-torsor $Y \to C$ decomposes into an fppf pointed $F$-torsor $Y \to Y \times^G E$ and an étale pointed $E$-torsor $Y \times^G E \to C$. We call these two torsors the components of the $G$-torsor $Y \to C$.

### 4.2 On the unicity of the extension

Let $Y \to Y'$ and $Y' \to C$ be the components of the pointed fppf $G$-torsor $Y \to C$.

**Proposition 4.2.** If the fppf pointed $G$-torsor $Y \to C$ extends into a logarithmic $G$-torsor $\mathcal{Y} \to C$, then its components both extend into logarithmic torsors.

**Proof.** According to Lemma 4.1, the extended $G$-torsor $\mathcal{Y} \to C$ decomposes into a logarithmic pointed $F$-torsor $\mathcal{Y} \to \mathcal{Y} \times^G \mathcal{E}$, which extends the fppf pointed $F$-torsor $Y \to Y \times^G E$, and an étale pointed $E$-torsor $\mathcal{Y} \times^G \mathcal{E} \to C$, which extends the étale pointed $E$-torsor $Y \times^G E \to C$. 

**Definition 4.3.** Let $j : \text{Spec}(K) \to \text{Spec}(R)$ denote the open immersion. Let $\text{Ab}(K)$ (resp. $\text{Ab}(R)$) denote the category of abelian groups over $K$ (resp. $R$). For a sheaf $\mathcal{H}$ on $\text{Spec}(K)$, we define the following presheaf on $\text{Spec}(R)$:

$$T \mapsto \begin{cases} 
\mathcal{H}(T_K) & \text{if } T \text{ is a } K \text{-scheme} \\
0 & \text{otherwise}.
\end{cases}$$

We denote by $j_! \mathcal{H}$ the étale sheaf associated to the previous presheaf. It is called the extension by zero sheaf.

**Proposition 4.4.** The sheaf $j_! E$ is represented in the small étale site by $\mathcal{E}$.

**Proof.** Let $i$ be the inclusion: $\text{Spec}(k) \hookrightarrow \text{Spec}(R)$. Since $\mathcal{E}$ is an abelian sheaf, it is well-know that we have an exact sequence in the small étale site:

$$0 \to j_!(j^{-1} \mathcal{E}) \to \mathcal{E} \to i_*(i^{-1} \mathcal{E}) \to 0,$$

hence

$$0 \to j_! E \to \mathcal{E} \to i_*(\mathcal{E}_k) \to 0.$$ 

But $\mathcal{E}_k = 0$, therefore

$$j_! E \simeq \mathcal{E}.$$
Proposition 4.5. The pointed étale $E$-torsor $Y' \to C$ extends uniquely into a pointed étale $\mathcal{E}$-torsor over $C$.

Proof. By exactness of $j_!$ (cf. [14, §3, Proposition 3.14]) and Proposition 4.4, we have

$$j_!H^1_{\text{ét}}(C, E) \simeq H^1_{\text{ét}}(C, j_!E) \simeq H^1_{\text{ét}}(C, \mathcal{E})$$

\[\square\]

Notation 4.6. We denote by $Y' \to C$ the unique pointed étale $E$-torsor that extends the $E$-torsor $Y' \to C$.

We notice that $Y'$ is a smooth projective $K$-curve. If $Y'$ is not a regular model of $Y'$, let $q_1, \ldots, q_n$ be its singular points, and let $p_1, \ldots, p_m$ be their images by the morphism $Y' \to C$. We denote by $Y'_b$ the blow-up of $Y'$ at $Y' \times p_1, \ldots, Y' \times p_m$. It follows from [11, 2, §8, Proposition 1.12 (c)] that $Y'_b$ is isomorphic to $\mathcal{Y}'$.

Corollary 4.8. Assume that the fppf pointed $G$-torsor extends into a log pointed $\mathcal{G}$-torsor over $C$, then this extension is unique.

Proof. This follows from [7, Proposition 3.6]. \[\square\]

Corollary 4.8. Assume that the fppf pointed $G$-torsor extends into a log pointed $\mathcal{G}$-torsor over the regular model $C_1$ of $C$, then this extension is unique.

Proof. Let $Y \to C_1$ and $Y_0 \to C_1$ be two log $\mathcal{G}$-torsors that extend $Y \to C$. Hence, $Y \times^G \mathcal{E} \to C_1$ and $Y_0 \times^G \mathcal{E} \to C_1$ are both étale pointed $\mathcal{E}$-torsors that extend the étale pointed $E$-torsor $Y' \to C$. By Proposition 4.5 and the previous discussion, we conclude that $Y \times^G \mathcal{E} \to C_1 = Y_0 \times^G \mathcal{E} \to C_1 = Y_1$. On the other hand, $Y$ and $Y_0$ are both log pointed $\mathcal{F}$-torsors over the regular scheme $Y_1$ that extend the fppf $F$-torsor $Y \to Y'$. It follows from Proposition 4.7 that $Y = Y_0$. \[\square\]

4.3 On the existence of an extension

We saw in the previous section that it is always possible to choose the $R$-regular model of $C$ so that the extension of the initial $G$-torsor $Y \to C$ over it is unique. We now would like to study the existence of such an extension.

Theorem 4.9. [9, Theorem 12.1] We assume here that $R$ is Henselian Japanese and that $k$ is perfect. Let $\mathcal{X}$ be a normal, reduced, projective and flat $R$-scheme with geometrically reduced fibres, and let $Q$ be an $R$-point of $\mathcal{X}$. Let $\mathcal{G}$ be a quasi-finite $R$-group scheme of finite type and $\mathcal{Y} \to \mathcal{X}$ a pointed fppf $\mathcal{G}$-torsor. Then, there exists a finite flat group scheme $\mathcal{H}$ over $R$ and a pointed fppf $\mathcal{H}$-torsor $\mathcal{Y}_0 \to \mathcal{X}$ such that $\mathcal{Y}_0 \times^R \mathcal{G} \simeq \mathcal{Y}$ as pointed fppf $\mathcal{G}$-torsors.
Corollary 4.10. With the same assumptions of Theorem 4.9 and if $X$ is in addition given its canonical log structure, if $\mathcal{Y} \to X$ is a pointed $G$-log torsor, there exists a pointed $F$-log torsor $\mathcal{Y}_0 \to X$ such that $\mathcal{Y}_0 \times^F G \simeq \mathcal{Y}$ as pointed $G$-log torsors.

Proof. Theorem 4.9 can be rephrased in the log setting. Since $F$ is the largest finite subgroup of $G$, $H \to G$ factors through $F$. Therefore, the surjective map

$$H^1(X, Q, G) \to H^1(X, Q, \mathcal{G})$$

$\mathcal{T} \mapsto \mathcal{T} \times^H G$ factors through $H^1(X, F)$ in an obvious way. $
\square$

Theorem 4.11. We assume that $R$ is Henselian Japanese and that $k$ is perfect. Let $C$ be a smooth projective and geometrically reduces $K$-curve with a point $Q$. Let $\mathcal{C}$ be a regular model of $C$ over $R$, which supposed to be irreducible with a geometrically reduced special fiber and endowed with an $R$-section $Q$ that extends $Q$. Then, the extension of pointed fppf $G$-torsors over $C$ into pointed $G$-log torsors over $C$ is equivalent to that of pointed fppf $F$-torsors over $C$ into pointed $F$-log torsors over $C$.

Proof. It follows from the short exact sequence (4) the diagram:

$$
\begin{array}{ccc}
H^1_{klf}(C, Q, F) & \xrightarrow{f_1} & H^1_{klf}(C, Q, G) \\
g_1 & & \downarrow \downarrow f_2 \\
H^1_{fppf}(C, Q, F) & \xrightarrow{h_1} & H^1_{fppf}(C, Q, G) \\
g_2 & & \downarrow \downarrow h_2 \\
& & H^1_{\acute{e}t}(C, Q, E)
\end{array}
$$

By Corollary 4.10, $f_1$ is surjective, hence by exactness of the first line, $f_2 = 0$. If $g_1$ is surjective (hence an isomorphism by [6, Proposition 3.5]), by the commutativity of the second square, $h_2 \circ g_2 = 0$. On the other hand, the commutativity of the first square gives $h_1 = g_2 \circ f_1 \circ g_1^{-1}$. Hence, $h_2 \circ h_1 = 0$, and therefore, by exactness of the second line, $h_1$ is surjective. Finally, $g_2$ is surjective as well. Conversely, if $g_2$ is surjective, it is clear that $g_1$ is surjective as well. $
\square$

5 Application

5.1 Overview on the log Picard functor in the general case

Recently, the Picard log functor has been defined in a more general frame. Let $S$ be a log scheme and let $U$ be the open of triviality of the log structure on $S$. Let $X \to S$ be a family of logarithmic curves, i.e. a vertical integral log smooth morphism with geometric fibers that are reduced, connected and of dimension 1, smooth over $U$. In [15], following the ideas of Illusie and Kato, the authors constructed the analogue of the Picard functor in the logarithmic setting: the logarithmic Picard group that they denoted $\text{LogPic}_{X/S}$. It is the sheaf of isomorphism classes of the stack which parameterizes the logarithmic line bundles, i.e. torsors under the group scheme $\mathbb{G}_{m, \text{log}, S}$ which verify a certain condition called the condition of bounded monodromy.

Naturally, the logarithmic Picard group coincides with the ordinary Picard group over $X_U$, where the log structure is trivial. Furthermore, logarithmic line bundles have a natural notion of degree extending the notion of degree of classical line bundles (cf. [15]).

Using this notion of degree, they defined in [8] $\text{LogPic}_{X/\mathbb{R}}^0$, the subsheaf of $\text{LogPic}_{X/S}$ consisting of line bundle of degree zero, which they called the logarithmic Jacobian. In fact, this is a sheaf
in the category of log schemes and it provides the best possible extension of the Jacobian $\text{Pic}^0_{X/\mathcal{U}/U}$. In addition, it satisfies the Néron mapping property for log smooth morphisms.

Now, using the fact that the category of schemes over $S$ can be seen as a full subcategory of that of log schemes over $S$ (by endowing any $S$-scheme $T$ with the inverse log structure of that of $S$, i.e. the strict log structure), if we restrict the functor $\text{LogPic}^0_{X/S}$ to the category of schemes via this embedding, the resulting functor is called the the strict logarithmic Jacobian and denoted $\text{sLogPic}^0_{X/S}$. It clearly satisfies the Néron property. We have the following theorem:

**Theorem 5.1.** [8] If $S$ is log regular, $\text{sLogPic}^0_{X/S}$ is the Néron model of $\text{Pic}^0_{X/\mathcal{U}/U}$.

### 5.2 Case of a divisorial log structure

After a discussion with Poiret who also worked on the paper [8], it appeared that in the case where $S$ is the spectrum of a discrete valuation ring endowed with its divisorial log structure, the condition of monodromy boundedness is automatically satisfied, which means that the log line bundles consist of all the $\mathbb{G}_{m,\log,S}$-torsors. In particular, $\text{LogPic}_{X/S}$ when restricted to the category of schemes, via the embedding mentioned previously, coincides with the log Picard functor we defined in section 2.2.

**Example 5.2.** Let $C$ be a semi-stable curve and let $\mathcal{C}$ an $R$-model of $C$. Let $s = \text{Spec}(\overline{k})$ denote the unique geometric point over $\text{Spec}(R)$. If $\Gamma$ denotes the dual graph (which is assumed to be oriented) of $\mathcal{C}_s$ and if $M_R$ denotes the divisorial log structure over the base $\text{Spec}(R)$, one can define on $\Gamma$ a length map $l : \Gamma \to M_{R,s}$, where $M_R := M_R/\mathcal{O}_R^*$. Since, $M_R = \mathcal{O}_R \cap \mathcal{O}_K$, one computes easily that $M_{R,s} = \mathbb{N}$.

The data $\mathcal{X} = (\Gamma, l : \Gamma \to \mathbb{N})$ is called the tropical curve associated to $\mathcal{C}_s$ (cf. [15, §2.3]). In addition, on can define a topology on tropical curves (cf. [15, §3]), which allows to do homology on them. In particular, the length map $l$ can be extended to $H_1(\mathcal{X})$.

To any log line bundle over $\mathcal{C}$ is associated a class of morphisms $H_1(\mathcal{X}) \to \mathbb{N}^{gp} = \mathbb{Z}$, called the monodromy class (cf. [15, s 3.5 and §4.1]). A logarithmic line bundle is said to have bounded monodromy if for any $\gamma \in H_1(\mathcal{X})$, $\exists n \in \mathbb{N}$ such that $-nl(\gamma) \leq \alpha(\gamma) \leq nl(\gamma)$, where $\alpha : H_1(\mathcal{X}) \to \mathbb{Z}$ is some representative in the monodromy class of the line bundle (this condition does not depend of the choice a representative). It is clear that this condition is automatically satisfied in this setting. More generally, if the base $S$ is any log scheme with log structure denoted by $M_S$, the length map is defined on each geometric point $s$ and takes its values on $M_{S,s}$, which is assumed to be only partially ordered (in particular, the monodromy condition makes more sense than in $\mathbb{N}$).

**Proposition 5.3.** If $C$ is a smooth projective semi-stable $K$-curve with a point, $\mathcal{C}$ an $R$-regular model of $C$ endowed with its canonical log structure and $\mathcal{F}$ a finite flat $R$-group scheme, any $R$-morphism $\mathcal{F} \to \text{Pic}^0_{\mathcal{C}/R}$ factors through $\text{sLogPic}^0_{\mathcal{C}/R}$.

**Proof.** First, according to the previous example, we note that $\text{sLogPic}^0_{\mathcal{C}/R}$ is the subsheaf of $\text{Pic}^0_{\mathcal{C}/R}$ of degree zero log line bundles. On the other hand, according to Proposition 2.2, a morphism $\mathcal{F} \to \text{Pic}^0_{\mathcal{C}/R}$ gives rise to the class of log line bundles over $\mathcal{C} \times \mathcal{F}$. Since $\mathcal{F}$ is finite, these line bundles are torsion. Now, given the degree morphism recalled above, since $\mathbb{Z}$ has no torsion, these line bundles have degree zero.

**Corollary 5.4.** If $C$ is a smooth projective semi-stable and geometrically connected $K$-curve with a point, if $\mathcal{C}$ is an $R$-regular model of $C$ endowed with its canonical log structure and $\mathcal{F}$ a finite flat $R$-group scheme, any $\mathcal{F}$-log torsor over $\mathcal{C}$ is a Néron-$\mathcal{F}$-torsor.
Proof. It follows from Proposition 5.3 and Theorem 5.1.

**Proposition 5.5.** Let $C$ be a smooth projective semi-stable and geometrically connected $K$-curve with a point. Let $C$ be some $R$-regular model of $C$ endowed with its canonical log structure, $J$ the Jacobian of $C$ and $J$ the Néron model of $J$. If $F$ is a finite commutative $K$-group scheme with some finite flat $R$-model $\mathcal{F}$, then any pointed fppf $F$-torsor over $C$ which is geometrically connected extends into a logarithmic $\mathcal{F}$-torsor over $C$ if and only if $F^D$ is the schematic closure of $F^D$ in $\mathcal{J}$. Proof. Let $Y \to C$ be an fppf $F$-torsor which geometrically connected and let $h_Y : F^D \to J$ be its associated morphism. Since $Y$ is geometrically connected, $h_Y$ is injective. According to Corollary 5.4, it extends into a logarithmic torsor over $C$ if and only if the morphism $h_Y$ extends into an $R$-morphism $F^D \to \mathcal{J}$. Since $\mathcal{J}$ is separated, $F^D$ is necessarily the schematic closure of $F^D$ in $\mathcal{J}$.

**Corollary 5.6.** Assume that the assumptions of Proposition 5.5 are satisfied, that $F$ has order $r$ and that $\mathcal{J}[r]$ is written as in (3). Then, any pointed fppf $F$-torsor over $C$, which is geometrically connected, extends into a logarithmic torsor over $C$ if and only if $F^D$ is a subgroup of $\mathcal{F}J[r]_K$.

Proof. This follows from Proposition 5.5 and Proposition 3.1.

**Theorem 5.7.** We assume that $R$ is Henselian, Japanese and that $k$ is perfect. Let $C$ be a smooth projective semi-stable and geometrically connected $K$-curve with a point. Let $C$ be some $R$-regular model of $C$ which is assumed to be irreducible and which is endowed with its canonical log structure, $J$ the Jacobian of $C$ and $J$ the Néron model of $J$. Let $G$ be a finite commutative $K$-group scheme of order $r$ and quasi-finite flat separated $R$-model $\mathcal{G}$ that is written as in 4. Then, a pointed fppf $G$-torsor $Y \to C$ extends into a log $\mathcal{G}$-torsor over $C$ if and only if the morphism $h_Y : G^D \to J$ factors through $h_Y^r : F^D \to J$ and the latter extends into an $R$-morphism $F^D \to \mathcal{J}$. In particular, if $h_Y^r$ is injective, then it factors through $\mathcal{F}J[r]_K$ and $F^D$ is necessarily the schematic closure of $F^D$ in $\mathcal{J}$.

Proof. If the pointed fppf $G$-torsor $Y \to C$ extends into a log $\mathcal{G}$-torsor over $C$, according to Corollary 4.10, it comes from a logarithmic $\mathcal{F}$-torsor over $C$, which generically induces a morphism $F^D \to J$ and Corollary 5.4 allows to conclude. For the converse, if $h_Y$ factors through $F^D$, then there exists an fppf $F$-torsor $Y_0 \to C$ such that $Y \simeq Y_0 \times^F G$, which extends into a log $\mathcal{F}$-torsor $Y_0$ over $C$. Hence $Y_0 \times^F \mathcal{G}$ is a log $\mathcal{G}$-torsor over $C$ which extends the fppf $G$-torsor $Y \to C$.

The second part follows from Proposition 5.5 and Corollary 5.6.

**Example 5.8.** With the assumptions of Theorem 5.7, according to [13, §3], if the curve $C$ verifies Chiodo criterion, then $\mathcal{J}[r] = F\mathcal{J}[r]$. So if $h_Y^r$ is injective, the schematic closure of $F^D$ in $\mathcal{J}[r]$ is necessarily finite.

**Theorem 5.9.** With the same assumptions as in Theorem 5.7, an fppf pointed $G$-torsor $Y \to C$ extends into an fppf $\mathcal{G}$-torsor over $C$ if and only if $h_Y : G^D \to J$ factors through $h_Y^r : F^D \to J$ and the latter extends into an $R$-morphism $F^D \to \mathcal{J}^0$. In addition, if $h_Y^r$ is injective, then it factors through $F\mathcal{J}^0[r]_K$ and $F^D$ is necessarily the schematic closure of $F^D$ in $\mathcal{J}^0$.

Proof. The proof is similar to Theorem 5.7 but now uses Proposition 2.7.

**References**

[1] Groupes de monodromie en géométrie algébrique. I. Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim.
[2] Revêtements étales et groupe fondamental (SGA 1), volume 3 of Documents Mathématiques (Paris) [Mathematical Documents (Paris)]. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].

[3] Marco Antei and Michel Emsalem. Models of torsors and the fundamental group scheme. Nagoya Math. J., 230:18–34, 2018.

[4] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Néron models, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

[5] Michel Demazure and Pierre Gabriel. Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice par Corps de classes local par Michiel Hazewinkel.

[6] Jean Gillibert. Prolongement de biextensions et accouplements en cohomologie log plate. Int. Math. Res. Not. IMRN, (18):3417–3444, 2009.

[7] Jean Gillibert. Cohomologie log plate, actions modérées et structures galoisiennes. J. Reine Angew. Math., 666:1–33, 2012.

[8] David Holmes, Samouil Molcho, Orecchia Giulio, and Thibault Poiret. Models of jacobians of curves.

[9] Phung Hô Hai and Joao Pedro Dos Santos. Finite torsors on projective schemes defined over a discrete valutaion ring.

[10] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[11] Qing Liu. Algebraic geometry and arithmetic curves, volume 6 of Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[12] B. Mazur. Rational isogenies of prime degree (with an appendix by D. Goldfeld). Invent. Math., 44(2):129–162, 1978.

[13] Sara Mehidi. Extending torsors via regular models of curves. Manuscripta Mathematica, 2022.

[14] James S. Milne. Étale cohomology. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J., 1980.

[15] Samouil Molcho and Jonathan Wise. The logarithmic Picard group and its tropicalization. Compos. Math., 158(7):1477–1562, 2022.

[16] Dajano Tossici. Effective models and extension of torsors over a discrete valuation ring of unequal characteristic. Int. Math. Res. Not. IMRN, pages Art. ID rnm111, 68, 2008.
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