LONG-TIME ASYMPTOTICS FOR THE NONLOCAL MKDV EQUATION

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Abstract. In this paper, we study the Cauchy problem with decaying initial data for the nonlocal modified Korteweg-de Vries equation (nonlocal mKdV)

\[ q_t(x, t) + q_{xxx}(x, t) - 6q(x, t)q(-x, -t)q_x(x, t) = 0, \]

which can be viewed as a generalization of the local classical mKdV equation. We first formulate the Riemann-Hilbert problem associated with the Cauchy problem of the nonlocal mKdV equation. Then we apply the Deift-Zhou nonlinear steepest-descent method to analyze the long-time asymptotics for the solution of the nonlocal mKdV equation. In contrast with the classical mKdV equation, we find some new and different results on long-time asymptotics for the nonlocal mKdV equation and some additional assumptions about the scattering data are made in our main results.

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1. Introduction

The pioneering work for the nonlocal systems has been done by Ablowitz and Musslimani when they studied the nonlocal NLS equation with PT symmetry [3]. This research field has attracted much attention from both mathematics and the physical application of nonlinear optics and magnetics [14, 24, 25]. Since the nonlocal NLS was found, a number of other nonlocal integrable systems has been introduced from a mathematical viewpoint. For instance, from symmetry reduction of general AKNS system, some new reverse space-time and reverse time nonlocal nonlinear integrable version of the NLS, mKdV, sine-Gordon equation were found [5]. Recently, Yang constructed some new nonlocal integrable equations by simple variable transformations on local equations [32].

Like the local case, the nonlocal integrable systems also possess integrable properties, for example, nonlocal NLS equation admits infinite number of conservation laws and can be solved by using the inverse scattering transform (IST) [4]. Some exact solutions of nonlocal mKdV equation including soliton, kink, rogue-wave and breather were obtained through either Darboux transformation or IST. These solutions have displayed some new properties which are different from those of local equation [18, 19]. In physical application, the nonlocal mKdV possesses the shifted parity and delayed time reveal symmetry, and thus it can be related to the Alice-Bob system [23]. For instance, a special approximate solution of the nonlocal mKdV was applied to theoretically capture the salient features of two correlated dipole blocking events in atmospheric dynamical systems [27].

However, there has been still not much work on the Riemann-Hilbert method to the nonlocal systems except to the recent paper [26], where Rybalko and Shepelsky obtained the long-time asymptotics of the solution for the nonlocal Schrodinger equation via the nonlinear steepest-descent method. In this paper, we apply Riemann-Hilbert (RH) method and Deift-Zhou nonlinear steepest-descent method to analyze longtime asymptotics of the Cauchy problem of the nonlocal mKdV equation

\[ q_t(x, t) + q_{xxx}(x, t) - 6q(x, t)r(x, t)q_x(x, t) = 0, \]  \hspace{1cm} (1.1a)

\[ q(x, 0) = q_0(x), \]  \hspace{1cm} (1.1b)

where \( r(x, t) = q(-x, -t) \) is a symmetry reduction of an AKNS system, and the initial data \( q_0(x) \) decays rapidly to zero as \( x \to \pm \infty \).

In 1970's, the solutions of the Cauchy problem for many integrable nonlinear wave equations was obtained by solving an associated RH problem on the complex plane [2]. More precisely, starting with initial data, the direct scattering transform gives rise to certain spectral functions whose time evolution is simple. Then the solution of the original
Cauchy problem can be recovered via the IST characterized in terms of RH problem whose jump matrix depends on the given spectral functions.

In 1993, Deift and Zhou introduced the nonlinear steepest-descent method to analyze the asymptotics of the solutions of RH problems \([12]\). It involves a series of counter deformation aiming to reduce the original RH problem to the one whose jump matrix is decaying fast (as \(t \to \infty\)) to the identity matrix everywhere except near some stationary phase points; and it is the contour near these points that determine the leading order of the long time asymptotics which can be obtained explicitly after rescaling the RH problem. This method has been used to study rigorously the long–time asymptotics of a wide variety of integral systems, such as the mKdV equation \([12]\) and the non-focusing NLS equation \([11]\), the sine-Gordon equation \([10]\), the modified Schrödinger equation \([20, 21]\), the KdV equation \([15]\), the Cammasa–Holm equation \([9]\), Fokas-Lenells equation \([29]\), derivative NLS equation \([31]\), short pulse equation \([28, 30]\), Sine-Gordon equation \([16]\), Kundu-Eckhaus Equation \([33]\).

In \([12]\), Deift and Zhou obtained the explicit leading order long-time asymptotic behavior of the solution to the classical mKdV equation

\[
\begin{align*}
q_t(x, t) + q_{xxx}(x, t) - 6q^2(x, t)q(x, t) &= 0, \\
q(x, 0) &= q_0(x),
\end{align*}
\]  

(1.2a, 1.2b)

using the nonlinear steepest descent method. Here we extend above results to give the asymptotic behavior of solution of nonlocal mKdV equation \((1.1)\), but it will be much different from that on the classical mKdV equation \((1.2)\) in the following three aspects.

(i) For our nonlocal mKdV equation, the jump matrix of the RH problem involve two reflection coefficients \(r_1(k)\) and \(r_2(k)\), but there is only one reflection coefficient \(r(k)\) for the local mKdV equation, which is specified by \(r_1(k) = r(k)\), \(r_2(k) = \overline{r(k)}\), \(|r(k)| < 1\).

(ii) In the analysis of the local equations, the great difference from the nonlocal case is that \(1 - r_1(k)r_1(k)\) is complex-valued, which leads to \(\text{Im} \nu(\zeta) \neq 0\). We will find below that \(\text{Im} \nu(\zeta)\) contributes to both the leading order and the error terms in the asymptotics for the nonlocal mKdV equation. To obtain asymptotic behavior of solution of nonlocal mKdV equation, we have used Slightly different method from that in \([12]\).

(iii) At last, in contrast with the asymptotic of local mKdV equation, we obtain the long time asymptotic of nonlocal mKdV equation as follows

\[
q(x, t) = \frac{4\epsilon \text{Re} \beta(\zeta, t)}{\tau^{\frac{1+\alpha}{2}} - \text{Im} \nu(\zeta)} + O(\epsilon \tau^{-\frac{1+\alpha}{2}}|\text{Im} \nu(\zeta)| + |\text{Im} \nu(\zeta)|).
\]

Note that the decay rate of the leading term depends on \(\zeta = \frac{\tau}{t}\) through \(\text{Im} \nu(\zeta)\), while \(\text{Im} \nu(\zeta) = 0\) for all \(\zeta \in I\) in the local mKdV equation.
Organization of this paper is as follows. In Section 2 we present the IST and express the solution of nonlocal mKdV equation (1.1) in terms of a RH problem. In Section 3 we conduct several deformations to obtain a model RH problem convenient for consequent analysis. In Section 4 we derive the long-time behavior of nonlocal mKdV equation (1.1) in the similarity sector.

2. INVERSE SCATTERING TRANSFORM AND THE RIEMANN-HILBERT PROBLEM

Since (1.1a) is a member of AKNS systems, the standard method of IST was applied in [19]. We reformulate the IST to express the solution of (1.1) in terms of a RH problem for convenience of the consequent analysis.

The nonlocal mKdV equation (1.1a) admits the Lax pair

\[ \Phi_x + ik\sigma_3 \Phi = U \Phi, \]
\[ \Phi_t + 4ik^3\sigma_3 \Phi = V \Phi, \]

where \( \Phi(x,t,k) \) is a 2 \times 2 matrix valued eigenfunction, \( k \in \mathbb{C} \) is the spectral parameter, and

\[ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & q(x,t) \\ q(-x,-t) & 0 \end{pmatrix}, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \]

with

\[ A = -2iq(x,t)q(-x,-t)k + q(-x,-t)q_x(x,t) + q(x,t)q_x(-x,-t), \]
\[ B = 4k^2q(x,t) + 2iq_x(x,t)k + 2q^2(x,t)q(-x,-t) - q_{xx}(x,t), \]
\[ C = 4k^2q(-x,-t) + 2iq_x(-x,-t)k + 2q^2(x,t)q(-x,-t) - q_{xx}(-x,-t). \]

Let \( \Psi_j(x,t,k), \ j = 1,2, \) be the 2 \times 2 matrix valued solutions of the linear Volterra integral equations

\[ \Psi_1(x,t,k) = I + \int_{-\infty}^{x} e^{ik(y-x)\hat{\sigma}_3} (U(y,t)\Psi_1(y,t,k)) \, dy, \quad k \in (\mathbb{C}_+ , \mathbb{C}_-) , \]
\[ \Psi_2(x,t,k) = I + \int_{\infty}^{x} e^{ik(y-x)\hat{\sigma}_3} (U(y,t)\Psi_2(y,t,k)) \, dy, \quad k \in (\mathbb{C}_- , \mathbb{C}_+ ) , \]

where \( \hat{\sigma}_3 \) acts on a 2 \times 2 matrix \( A \) by \( \hat{\sigma}_3 A = [\hat{\sigma}_3, A] \), i.e. \( e^{\hat{\sigma}_3} \) is a matrix exponential. \( e^{\sigma_3}Ae^{-\sigma_3} = \mathbb{R}_k \) and the notation \( k \in (\mathbb{C}_+ , \mathbb{C}_-) \) indicates that the first and second columns are valid for \( k \in \mathbb{C}_+ \) and \( k \in \mathbb{C}_- \), respectively. From (2.2), we can prove that \( \Psi_1(x,t,\cdot) \) is continuous for \( k \in (\mathbb{C}_+ , \mathbb{C}_-) \) and analytic for \( k \in (\mathbb{C}_+ , \mathbb{C}_-) \). Moreover we can derive the large \( k \) asymptotics of \( \Psi_j \) (c.f. [7])

\[ \Psi_j(x,t,k) = I + O(k^{-1}), \quad k \to \infty, \]

where the error term is uniformly with respect to \( x, t \).
Then the Jost solutions $\Phi_j(x, t, k)$, $j = 1, 2$, of (2.1) are defined as follow
\[
\Phi_j(x, t, k) = \Psi_j(x, t, k)e^{(-ikx-4ik^3t)\sigma_3},
\] (2.4)

Since $U$ is traceless, $\det \Phi_j(x, t, k) \equiv 1$ for all $x$, $t$, and $k$. And for $k \in \mathbb{R}$, $\Phi_j(x, t, k)$ can be related by scattering matrix $S(k)$
\[
\Phi_1(x, t, k) = \Phi_2(x, t, k)S(k), \quad k \in \mathbb{R},
\] (2.5)

where
\[
S(k) = \begin{pmatrix}
s_{11}(k) & s_{12}(k) \\
s_{21}(k) & s_{22}(k)
\end{pmatrix}, \quad k \in \mathbb{R},
\] (2.6)
is independent of $x$ and $t$.

We now establish important symmetry properties of the scattering matrix (2.6). It can be verified that if $\Psi(x, t, k)$ is the solution of (2.2a), then $\Lambda \bar{\Psi}(-x, -t, -k)\Lambda^{-1}$ is the solution of (2.2b) with $\Lambda = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}$. Notice (2.4) and the uniqueness of the solution of the Volterra equation (2.2), we arrives that
\[
\Phi_2^{(2)}(x, t, k) = \bar{\Lambda}\Phi_1^{(1)}(-x, -t, -\bar{k}), \quad \Phi_1^{(2)}(x, t, k) = \Lambda^{-1}\bar{\Phi}_1^{(2)}(-x, -t, -\bar{k}),
\] (2.7)

where $\Phi_i^{(j)}(x, t, k)$ denotes the $j$-th column of the matrix $\Phi_i(x, t, k)$. Rewrite the relation between the Jost solutions (2.5) as
\[
\Phi_1^{(1)}(x, t, k) = s_{11}(k)\Phi_2^{(1)}(x, t, k) + s_{21}(k)\Phi_2^{(2)}(x, t, k), \quad (2.8a)
\]
\[
\Phi_2^{(1)}(x, t, k) = s_{12}(k)\Phi_2^{(1)}(x, t, k) + s_{22}(k)\Phi_2^{(2)}(x, t, k), \quad (2.8b)
\]
the scattering data can be represented in terms of $\Phi_i^{(j)}$, and from (2.7), we reach the following symmetry
\[
s_{11}(k) = \det(\Phi_1^{(1)}(x, t, k), \Phi_2^{(2)}(x, t, k))
\]
\[
= \det(\Lambda^{-1}(\Phi_1^{(1)}(-x, -t, -\bar{k}), \Phi_2^{(2)}(-x, -t, -\bar{k}))\Lambda) = s_{11}(-\bar{k}),
\] (2.9a)
\[
s_{22}(k) = \det(\Phi_2^{(1)}(x, t, k), \Phi_1^{(2)}(x, t, k))
\]
\[
= \det(\Lambda^{-1}(\Phi_2^{(1)}(-x, -t, -\bar{k}), \Phi_1^{(2)}(-x, -t, -\bar{k}))\Lambda) = s_{22}(-\bar{k}),
\] (2.9b)
\[
s_{12}(k) = \det(\Phi_1^{(2)}(x, t, k), \Phi_2^{(2)}(x, t, k))
\]
\[
= \det(\Lambda(\Phi_2^{(1)}(-x, -t, -\bar{k}), \Phi_1^{(1)}(-x, -t, -\bar{k}))\Lambda) = s_{21}(-\bar{k}).
\] (2.9c)

Further more, we can also verify that if $\Psi(x, t, k)$ is the solution of (2.2a), then $\Lambda \bar{\Psi}(-x, -t, -k)\Lambda^{-1}$ is the solution of (2.2b). So following the above procedure, we obtain another symmetry property
\[
s_{12}(k) = s_{21}(k).
\] (2.10)

Finally, from (2.9) and (2.10), $S(k)$ can be written in the form
\[
S(k) = \begin{pmatrix}
a_1(k) & b(k) \\
b(k) & a_2(k)
\end{pmatrix},
\] (2.11)
Using scattering relation (2.5), we have the jump condition for
\[
\begin{aligned}
| a_1(k) | &= \frac{a_1(-k)}{a_1(k)}, \\
| a_2(k) | &= \frac{a_2(-k)}{a_2(k)}, \\
b(k) &= \overline{b(-k)}. 
\end{aligned}
\tag{2.12}
\]

In accordance with the case of local equations [7, 17], the scattering matrix $S(k)$ is uniquely determined by the initial data $q_0(x)$, and we can conclude that:

(1) $a_1(k)$ is analytic for $k \in \mathbb{C}_+$, and continuous for $k \in \mathbb{C}_-$; $a_2(k)$ is analytic for $k \in \mathbb{C}_-$, and continuous for $k \in \mathbb{C}_+$.

(2) $a_j(k) = 1 + O(k^{-1})$, $j = 1, 2$ and $b(k) = O(k^{-1})$ as $k \to \infty$.

(3) $a_1(k) = \frac{a_1(-k)}{a_1(k)}, k \in \mathbb{C}_+$; $a_2(k) = \frac{a_2(-k)}{a_2(k)}, k \in \mathbb{C}_-$; $b(k) = \overline{b(-k)}, k \in \mathbb{R}$.

(4) $a_1(k)a_2(k) - b(k)^2 = 1, k \in \mathbb{R}$, (follows from $\det S(k) = 1$).

Now we define the matrix valued function $M$ as
\[
M(x, t, k) = \begin{cases}
\Psi^{(1)}(x, t, k) & \text{Im } k > 0, \\
\Psi^{(2)}(x, t, k) & \text{Im } k < 0.
\end{cases}
\tag{2.13}
\]

Using scattering relation (2.5), we have the jump condition for $M(x, t, k)$ across $k \in \mathbb{R}$
\[
M_+(x, t, k) = M_-(x, t, k)J(x, t, k), \quad k \in \mathbb{R},
\tag{2.14}
\]
where $M_{\pm}$ is the limiting value of $M$ as $k$ approaches $\mathbb{R}$ from $\mathbb{C}_{\pm}$, and
\[
J(x, t, k) = e^{(-ikx-\frac{4ik^3}{3})\tau_3} \begin{pmatrix} 1 - r_1(k)r_2(k) & -r_2(k) \\ r_1(k) & 1 \end{pmatrix}, \quad k \in \mathbb{R},
\tag{2.15}
\]
and reflection coefficients are defined by
\[
r_1(k) = \frac{b(k)}{a_1(k)}, \quad r_2(k) = \frac{b(k)}{a_2(k)}. \tag{2.16}
\]

From the symmetry of scattering data (2.12), $r_1$ and $r_2$ also possess the symmetry property
\[
r_1(k) = \overline{r_1(-k)}, \quad r_2(k) = \overline{r_2(-k)}, \quad k \in \mathbb{R},
\tag{2.17}
\]
and the determinant property 4 implies that
\[
1 - r_1(k)r_2(k) = \frac{1}{a_1(k)a_2(k)}, \quad k \in \mathbb{R}. \tag{2.18}
\]

We assume that $a_1(k)$ and $a_2(k)$ have no zeros in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively so that one can assemble the above facts into the form of a Riemann-Hilbert problem
\[
\begin{cases}
M(x, t, k) \text{ analytic in } \mathbb{C} \setminus \mathbb{R}, \\
M_+(x, t, k) = M_-(x, t, k)J(x, t, k), \quad k \in \mathbb{R}, \\
M(x, t, k) \to I, \quad k \to \infty.
\end{cases}
\tag{2.19}
\]

**Remark 2.1.** RH problem (2.19) can be regard as a generalization of the RH problem associated with the mKdV equation. In the local case, the reflection coefficients are specified by
\[
r_1(k) = r(k), \quad r_2(k) = \overline{r(k)}, \quad k \in \mathbb{R},
\tag{2.20}
\]
with $|r(k)| < 1$. 

Inverse, if RH problem (2.19) has a unique solution for all \((x, t)\), the solution \(q(x, t)\) of (1.1) is given by (c.f. [1, 8, 13])
\[
q(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{12}.
\] (2.21)

3. Reduction to a model RH problem

The deformations of the RH problem (2.19) are similar to the local case [12, 22], where the original RH problem is deformed to the one whose jump matrix decays to the identity matrix everywhere as \(t \to \infty\) except near the stationary points. Then an explicitly solvable RH problem is introduced to obtain a model RH problem for which long time asymptotics can be conveniently performed.

Let \(I = (-N, 0)\) be the interval with \(N > 0\) and let \(\zeta = x/t\) be the variable with \(\zeta \in I\). Let \(M(x, t, \cdot)\) denote the unique solution of the RH problem (2.19), and the phase of the exponentials \(e^{\pm i\Phi(\zeta)}\) in the jump matrix (2.15) is defined by
\[
\Phi(\zeta, k) = 2ik\zeta + 8ik^3, \tag{3.1}
\]
which admits two stationary points
\[
k_0 = \sqrt{-\frac{\zeta}{12}}, \quad k_0 = -\sqrt{-\frac{\zeta}{12}}, \tag{3.2}
\]
such that \(\frac{d\Phi}{dk}(\pm k_0) = 0\).

Now we deform the RH problem (2.19) with the following steps.

**Step 1** The first step is to search for upper/lower and lower/upper triangular factorizations of the jump matrix. For this purpose, we introduce a scalar RH problem
\[
\begin{cases}
\delta \text{ analytic in } \mathbb{C} \setminus [-k_0, k_0], \\
\delta_+ = \delta_- (1 - r_1(k)r_2(k)), \quad |k| < k_0, \\
\delta \to 1 \quad k \to \infty.
\end{cases} \tag{3.3}
\]
Direct calculation shows that (3.3) admits a unique solution
\[
\delta(\zeta, k) = e^{\frac{i}{2\pi} \int_{-k_0}^{k_0} \ln(1 - r_1(s)r_2(s)) \frac{ds}{s}}, \quad k \in \mathbb{C} \setminus [-k_0, k_0]. \tag{3.4}
\]
The symmetry (2.17) implies that
\[
\delta(\zeta, k) = \overline{\delta(\zeta, -k)}, \tag{3.5}
\]
moreover, integrating by parts in formula (3.4) yields
\[
\delta(\zeta, k) = \frac{(k - k_0)^{i\nu(\zeta)}}{(k + k_0)^{i\nu(\zeta)}} e^{\tilde{\chi}(\zeta, k)}, \tag{3.6}
\]
where \(\tilde{\chi}(\zeta, k)\) is a uniformly bounded function with respect to \(\zeta \in I\) and \(k \in \mathbb{C} \setminus \mathbb{R}\), which is defined by
\[
\tilde{\chi}(\zeta, k) = -\frac{1}{2\pi i} \int_{-k_0}^{k_0} \ln(k - s) d\ln(1 - r_1(s)r_2(s)), \tag{3.7}
\]
and $\nu(\zeta)$ is a bounded function defined by

$$\nu(\zeta) = -\frac{1}{2\pi} \ln(1 - r_1(k_0)r_2(k_0)) = -\frac{1}{2\pi} \ln |1 - r_1(k_0)r_2(k_0)| - \frac{i}{2\pi} \Delta(\zeta)$$

(3.8)

with

$$\Delta(\zeta) = \int_{-\infty}^{k_0} d\arg(1 - r_1(s)r_2(s)).$$

We assume that

$$\Delta(\zeta) \in (-\pi, \pi), \quad \zeta \in \mathcal{I},$$

then $\nu(\zeta)$ is single valued and

$$|\text{Im} \nu(\zeta)| < \frac{1}{2}, \quad \zeta \in \mathcal{I}.$$

(3.9)

(3.10)

Consequently the singularity of $\delta(\zeta, k)$ at $k = \pm k_0$ is square integrable.

$\delta(\zeta, k)$ can be written in another way:

$$\delta(\zeta, k) = (\frac{k - k_0}{k + k_0})^{\nu(\zeta)} e^{\chi(\zeta, k)},$$

(3.11)

where

$$\chi(\zeta, k) = \frac{1}{2\pi i} \int_{-k_0}^{k_0} \ln\left(\frac{1 - r_1(s)r_2(s)}{1 - r_1(k_0)r_2(k_0)}\right) \frac{ds}{s - k}$$

$$= \tilde{\chi}(\zeta, k) - \frac{1}{2\pi i} \ln\left(\frac{1 - r_1(k_0)r_2(k_0)}{1 - r_1(k_0)r_2(k_0)}\right) \ln(k + k_0).$$

(3.12)

In the local case, $\chi(\zeta, k)$ is equivalent to $\tilde{\chi}(\zeta, k)$ by symmetry (2.20), so $\chi(\zeta, k)$ is uniformly bounded. However $\chi(\zeta, k)$ is singular at $k = -k_0$ for nonlocal equation.

Lemma 3.1. Let $S = \{k' \in \mathbb{C}||k' + k_0| \geq \frac{k_0}{2}\}$ denote the complex plane minus a neighborhood of $-k_0$. Then $\chi(\zeta, k)$ is uniformly bounded with respect to $\zeta \in \mathcal{I}$ and $k \in S$, i.e.,

$$\sup_{\zeta \in \mathcal{I}} \sup_{k \in S} |\chi(\zeta, k)| \leq C$$

(3.13)

Proof. Since $\delta(\zeta, k) \to 1$ as $k \to \infty$, $\chi(\zeta, k)$ is uniformly bounded with respect to $\zeta \in \mathcal{I}$ and $k \in \{k' \in \mathbb{C}||k' + k_0| \geq G\}$ by (3.11), where $G$ is a large enough constant. Let $k = -k_0 + u e^{i\alpha}$, where $\frac{k_0}{2} \leq u < G$ and $\alpha \in (-\pi, \pi]$. By (3.12)

$$|\chi(\zeta, k)| \leq C + C \left|\ln\left(\frac{1 - r_1(k_0)r_2(k_0)}{1 - r_1(k_0)r_2(k_0)}\right)\right| |\ln(ue^{i\alpha})|$$

$$\leq C + C \left|\ln\left(\frac{1 - r_1(k_0)r_2(k_0)}{1 - r_1(k_0)r_2(k_0)}\right)\right| \left|\ln \left|\frac{k_0}{2}\right|\right|$$

(3.14)

Symmetry (2.17) implies that

$$r_j(0) = r_{-j}(0), \quad j = 1, 2,$$

(3.15)

thus $\chi(\zeta, k)$ is also uniformly bounded with respect to $\zeta \in \mathcal{I}$ and $k \in \{k' \in \mathbb{C}||k' + k_0| \leq \frac{k_0}{2}\}$ by (3.14) and (3.15). □
Figure 1. The jump contour $\Gamma$ and the open sets $\{V_j\}_{j=1}^6$

Remark 3.2. In the analysis of the local equations, the chief difference from the nonlocal case is that $1 - r_1(k)r_2(k)$ is complex-valued, that is, $\text{Im}\nu(\zeta) \neq 0$. We will find below that $\text{Im}\nu(\zeta)$ contributes to both the leading order and the error terms in the asymptotics for the nonlocal mKdV equation.

Conjugating the RH problem (2.19) by

$$\delta(\zeta,k)^{-\sigma_3} = \left(\begin{array}{cc}
\delta(\zeta,k)^{-1} & 0 \\
0 & \delta(\zeta,k)
\end{array}\right)$$

leads to the factorization problem for $\tilde{M}(x,t,k) = M(x,t,k)\delta(\zeta,k)^{-\sigma_3}$,

$$\begin{cases}
\tilde{M}_+(x,t,k) = \tilde{M}_-(x,t,k)\tilde{J}(x,t,k), & k \in \mathbb{R}, \\
\tilde{M}(x,t,k) \to I, & k \to \infty,
\end{cases}$$

where

$$\tilde{J} = \begin{cases}
1 - \delta(\zeta,k)^2r_2(k)e^{-t\Phi(\zeta,k)} & 0 & 1 \\
0 & 1 & -\delta(\zeta,k)^2r_1(k)e^{t\Phi(\zeta,k)} & 0 & 1 \\
1 & 0 & -\delta_+(\zeta,k)^2r_4(k)e^{-t\Phi(\zeta,k)} & 0 & 1 \\
(\delta_-(\zeta,k)^{-1}r_3(k)e^{t\Phi(\zeta,k)})^{-1} & 1 & 0 & 1 & 0
\end{cases}$$

with

$$r_3(k) = \frac{r_1(k)}{1 - r_1(k)r_2(k)}, \quad r_4(k) = \frac{r_2(k)}{1 - r_1(k)r_2(k)}.$$  

Step 2 In accordance with the local case, we introduce oriented counter $\Gamma$ and open sets $\{V_j\}_{j=1}^6$ as depicted in Figure 1 and define $m(x,t,k)$ by
Figure 2. The domains \( \{U_j\}_{j=1}^4 \) in the complex \( k \)-plane. \( \text{Re} \Phi = 0 \) on the curves.

\[
m(x, t, k) = \begin{cases} 
\tilde{M}(x, t, k) \begin{pmatrix} 1 & 0 \\ -\delta(\zeta, k) - 2r_{1,a}(x, t, k)e^{t\Phi(\zeta, k)} & 1 \end{pmatrix}, & k \in V_1, \\
\tilde{M}(x, t, k) \begin{pmatrix} 1 & \delta(\zeta, k)^2 r_{4,a}(x, t, k)e^{-t\Phi(\zeta, k)} \\ 0 & 1 \end{pmatrix}, & k \in V_3, \\
\tilde{M}(x, t, k) \begin{pmatrix} 1 & 0 \\ \delta(\zeta, k) - 2r_{3,a}(x, t, k)e^{t\Phi(\zeta, k)} & 1 \end{pmatrix}, & k \in V_4, \\
\tilde{M}(x, t, k) \begin{pmatrix} 1 & \delta(\zeta, k)^2 r_{2,a}(x, t, k)e^{-t\Phi(\zeta, k)} \\ 0 & 1 \end{pmatrix}, & k \in V_6, \\
\tilde{M}(x, t, k), & \text{elsewhere}, 
\end{cases}
\]

(3.20)

where \( r_{j,a} \) is the analytic approximation of \( r_j \) with small error \( r_{j,r} \), \( j = 1, \cdots, 4 \). More precisely, since \( \{r_j(k)\}_{j=1}^4 \) are sufficiently smooth and decaying, we can closely follow the proof of Lemma 4.8 in [22] to obtain similar decompositions:

Dividing the complex \( k \)-plane into four parts \( U_j \), \( j = 1, \cdots, 4 \) as in Figure 2 so that

\[
\{k|\text{Re} \Phi(\zeta, k) < 0\} = U_1 \cup U_3, \quad \{k|\text{Re} \Phi(\zeta, k) > 0\} = U_2 \cup U_4, 
\]

(3.21)

we can introduce decompositions

\[
r_j(k) = \begin{cases} 
\rho_{j,a}(x, t, k) + \rho_{j,r}(x, t, k), & j = 1, 2, \ |k| > k_0, k \in \mathbb{R}, \\
\rho_{j,a}(x, t, k) + \rho_{j,r}(x, t, k), & j = 3, 4, \ |k| < k_0, k \in \mathbb{R}, 
\end{cases}
\]

(3.22)

such that
(1) \( r_{j,a}(x,t,k) \) is defined and continuous for \( k \in \overline{U}_j \), analytic for \( k \in U_j \), and for each \( K > 0 \) satisfies

\[
|r_{j,a}(x,t,k) - r_j(k_0)| \leq C_k |k - k_0| e^{\frac{1}{4} |\text{Re} \Phi(\zeta,k)|},
\]

where the constant \( C \) is independent of \( \zeta,t,k \).

(2) \( r_{1,a} \) and \( r_{2,a} \) satisfy

\[
|r_{j,a}(x,t,k)| \leq C \frac{e^{\frac{1}{4} |\text{Re} \Phi(\zeta,k)|}}{1 + |k|}, \quad k \in \overline{U}_j, \quad \zeta \in I, \quad t > 0, \quad j = 1, 2,
\]

where the constant \( C \) is independent of \( \zeta,t,k \).

(3) \( r_{1,r} \) and \( r_{2,r} \) satisfy

\[
|r_{j,r}(x,t,k)| \leq C \frac{|k - k_0| t^{-3/2}}{1 + |k|^2}, \quad k \in (-\infty, -k_0) \cup (k_0, \infty), \\
\zeta \in I, \quad t > 0, \quad j = 1, 2,
\]

where the constant \( C \) is independent of \( \zeta,t,k \).

(4) \( r_{3,r} \) and \( r_{4,r} \) satisfy

\[
|r_{j,r}(x,t,k)| \leq C |k^2 - k_0^2| t^{-3/2}, \quad k \in (-k_0, k_0), \\
\zeta \in I, \quad t > 0, \quad j = 1, 2,
\]

where the constant \( C \) is independent of \( \zeta,t,k \).

(5) The following symmetries are valid:

\[
r_{j,a}(\zeta,t,k) = \overline{r_{j,a}(\zeta,t,-k)}, \quad r_{j,r}(\zeta,t,k) = \overline{r_{j,r}(\zeta,t,-k)}, \quad j = 1, \cdots, 4.
\]

As a result, the function \( m(x,t,k) \) satisfies the RH problem

\[
\begin{cases}
m_+(x,t,k) = m_-(x,t,k)v(x,t,k), & k \in \Gamma, \\
m(x,t,k) \to I, & k \to \infty,
\end{cases}
\]

(3.28)
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LONG-TIME ASYMPTOTICS FOR THE NONLOCAL MKDV EQUATION

Figure 3. Contour \(X\) in the complex \(z\)-plane

where the jump matrix is

\[
v(x,t,k) = \begin{cases}
    \begin{pmatrix}
        1 & 0 \\
        \delta(\zeta,k)^{-2}r_{1,a}(x,t,k)e^{t\Phi(\zeta,k)} & 1
    \end{pmatrix}, & k \in \mathcal{V}_1 \cap \mathcal{V}_2, \\
    \begin{pmatrix}
        1 & 0 \\
        -\delta(\zeta,k)^2r_{4,a}(x,t,k)e^{-t\Phi(\zeta,k)} & 1
    \end{pmatrix}, & k \in \mathcal{V}_2 \cap \mathcal{V}_3, \\
    \begin{pmatrix}
        1 & 0 \\
        -\delta(\zeta,k)^{-2}r_{3,a}(x,t,k)e^{t\Phi(\zeta,k)} & 1
    \end{pmatrix}, & k \in \mathcal{V}_4 \cap \mathcal{V}_5, \\
    \begin{pmatrix}
        1 & 0 \\
        \delta(\zeta,k)^{-2}r_{2,a}(x,t,k)e^{-t\Phi(\zeta,k)} & 1
    \end{pmatrix}, & k \in \mathcal{V}_5 \cap \mathcal{V}_6, \\
    \begin{pmatrix}
        1 & 0 \\
        -\delta(\zeta,k)^2r_{3,r}(x,t,k)e^{-t\Phi(\zeta,k)} & 1
    \end{pmatrix}, & k \in \mathcal{V}_1 \cap \mathcal{V}_6, \\
    \begin{pmatrix}
        1 & 0 \\
        -\delta(\zeta,k)^{-2}r_{4,r}(x,t,k)e^{-t\Phi(\zeta,k)} & 1
    \end{pmatrix}, & k \in \mathcal{V}_3 \cap \mathcal{V}_4.
\end{cases}
\]

(3.29)

By (3.5) and (3.27), \(v(\zeta,t,k)\) satisfies the symmetry

\[
v(\zeta,t,k) = v(\zeta,t,-k), \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \in \Sigma.
\]

(3.30)

Step 3 Let \(m^X(\zeta,z)\) be the solution of the RH problem in the complex \(z\)-plane:

\[
\begin{cases}
    m^X(\zeta,z)_+ = m^X(\zeta,z)_-v^X(\zeta,z), & z \in X, \\
    m^X(\zeta,z) \to I, & z \to \infty,
\end{cases}
\]

(3.31)

where contour \(X = X_1 \cup \cdots \cup X_4\) is shown in Figure 3.
The jump matrix is
\[
v^X(\zeta, z) = \begin{cases} 
  \frac{1}{q_1(\zeta)} z^{-2i\nu(\zeta)} e^{\frac{i\pi^2}{2} z}, & z \in X_1, \\
  \frac{q_2(\zeta)}{1} z^{2i\nu(\zeta)} e^{-\frac{i\pi^2}{2} z}, & z \in X_2, \\
  1, & z \in X_3, \\
  \frac{1}{q_1(\zeta)q_2(\zeta)} z^{-2i\nu(\zeta)} e^{\frac{i\pi^2}{2} z}, & z \in X_4.
\end{cases}
\] (3.32)

We point out that in the local case (see [22]), \(q_1\) and \(q_2\) are defined as
\[
q_1(\zeta) = \delta(\zeta, k)^{-2} r_r(k) \left( \frac{k - k_0}{\sqrt{48k_0}} \right)^{2i\nu(\zeta)} \bigg|_{k=k_0} 
= e^{-\chi(\zeta,k_0)} r_r(k_0) e^{2i\nu(\zeta) \ln(2\sqrt{15k_0^{3/2})}},
\] (3.33a)
\[
q_2(\zeta) = \frac{1}{q_1(\zeta)}.
\] (3.33b)

By contrast, to keep the blondness of the function \(q_j(\zeta), \zeta \in \mathcal{I}, j = 1, 2\) for the nonlocal case, we let
\[
q_1(\zeta) = \delta(\zeta, k)^{-2} r_1(k) \left( \frac{2(k - k_0)}{k_0} \right)^{2i\nu(\zeta)} \bigg|_{k=k_0} 
= e^{-\chi(\zeta,k_0)} r_1(k_0) e^{2i\nu(\zeta) \ln 4},
\] (3.34a)
\[
q_2(\zeta) = \delta(\zeta, k)^2 r_2(k) \left( \frac{2(k - k_0)}{k_0} \right)^{-2i\nu(\zeta)} \bigg|_{k=k_0} 
= e^{\chi(\zeta,k_0)} r_2(k_0) e^{-2i\nu(\zeta) \ln 4}.
\] (3.34b)

From Lemma [A.1] the unique solution \(m^X(\zeta, z)\) of the RH problem (3.31) can be explicitly expressed in terms of parabolic-cylinder function. Together with \(D(\zeta, t)\) defined by
\[
D(\zeta, t) = e^{-\frac{t\delta(\zeta,k_0)}{2} \sigma_3} \tau^{-\frac{i\nu(\zeta)}{2} \sigma_3},
\] (3.35)
where
\[
\epsilon = \frac{k_0}{2}, \quad \rho = \epsilon \sqrt{48k_0}, \quad \tau = t\rho^2 = 12k_0^3 t,
\]
\[
\phi(\zeta, z) = \Phi \left( \zeta, k_0 + \frac{\epsilon}{\rho} z \right) = -16i k_0^3 + \frac{iz^2}{2} + \frac{iz^3}{12\rho},
\] (3.36)
we use \(m^X(\zeta, z)\) to introduce \(m_0(\zeta, t, k)\) for \(k\) near \(k_0\):
\[
m_0(\zeta, t, k) = D(\zeta, t)m^X(\zeta, \frac{\sqrt{\tau}}{\epsilon}(k - k_0)) D(\zeta, t)^{-1}, \quad |k - k_0| \leq \epsilon,
\] (3.37)
and extend it to a neighborhood of \(-k_0\) by symmetry:
\[
m_0(\zeta, t, k) = \overline{m_0(\zeta, t, -k)} = \overline{m_0(\zeta, t, k)}, \quad |k + k_0| \leq \epsilon.
\] (3.38)
Remark 3.3. The method introduced in [22] cannot be imitated indiscriminately to deal with the situation where $\text{Im} \nu(\zeta) \neq 0$. In [22] $D(\zeta, t)$ is defined by

$$D(\zeta, t) = e^{-\frac{t\phi(\zeta, 0)}{2} \sigma_3} t^{-\frac{\nu(\zeta)}{2} \sigma_3}. \quad (3.39)$$

Notice (3.35), we replace $t$ by $\tau$ in (3.39) to define $D(\zeta, t)$. Actually, our adjustment including that to the function $q_j(\zeta), j = 1, 2$ (see (3.33) and (3.34)) is also valid for the study of the local case.

Then we use $m_0(\zeta, t, k)$ to introduce function $\hat{m}(\zeta, t, k)$:

$$\hat{m}(\zeta, t, k) = \begin{cases} m(\zeta, t, k)m_0(\zeta, t, k)^{-1}, & |k \pm k_0| < \epsilon, \\ m(\zeta, t, k), & \text{elsewhere}. \end{cases} \quad (3.40)$$

By the RH problems (3.20) and the definition of $m_0$, $\hat{m}(\zeta, t, k)$ satisfies the following RH problem

$$\begin{cases} \hat{m}(\zeta, t, k)_+ = \hat{m}(\zeta, t, k)_-\hat{v}(\zeta, t, k), & k \in \hat{\Gamma}, \\ \hat{m}(\zeta, t, k) \to I, & k \to \infty, \end{cases} \quad (3.41)$$

where $\hat{\Gamma} = \Gamma \cup \{k \mid |k \pm k_0| = \epsilon\}$ is oriented as in Figure 4 and the jump matrix is

$$\hat{v}(\zeta, t, k) = \begin{cases} m_0(\zeta, t, k)v(\zeta, t, k)m_0+(\zeta, t, k)^{-1}, & |k_0 \pm k_0| < \epsilon, \\ m_0(\zeta, t, k), & |k_0 \pm k_0| = \epsilon, \\ v(\zeta, t, k), & \text{elsewhere}. \end{cases} \quad (3.42)$$

The model RH problem (3.41) is finally obtained.
4. Long time asymptotics

We use the model RH problem (3.41) to derive the asymptotics of the nonlocal mKdV equation in the similarity sector.

Let $\Sigma$ denote the counter $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_6 \subset \mathbb{C}$, where

$$
\begin{align*}
\Sigma_1 &= \{ se^{\frac{iu}{2}} \mid 0 \leq s < \infty \}, \\
\Sigma_2 &= \{ se^{-\frac{iu}{2}} \mid 0 \leq s < \infty \}, \\
\Sigma_3 &= \{ se^{\frac{3iu}{4}} \mid 0 \leq s < \infty \}, \\
\Sigma_4 &= \{ se^{\frac{ui}{4}} \mid 0 \leq s < \infty \}, \\
\Sigma_5 &= \{ s \mid 0 \leq s < \infty \}, \\
\Sigma_6 &= \{ -s \mid 0 \leq s < \infty \},
\end{align*}
$$

are oriented as in Figure 5. For $r > 0$, we denote $\Sigma_r = \Sigma_1 \cup \cdots \cup \Sigma_6$, where $\Sigma_r = \Sigma_j \cap D(0, r)$, $j = 1, \cdots, 6$ and $D(k, r)$ is the disk of radius $r$ centered at $k$.

**Lemma 4.1.** Let $\Gamma_{\Sigma} = \pm k_0 + \Sigma^\epsilon$ and let $\Gamma' = \Gamma \setminus \Gamma_{\Sigma}$ as shown in Figure 6. Let $\hat{w}(\zeta, t, k) = \hat{v}(\zeta, t, k) - I$, then $\hat{w}(\zeta, t, k)$ satisfies

$$
\begin{align*}
\|\hat{w}(\zeta, t, \cdot)\|_{L^p(\Gamma')} &= O(\epsilon^{\frac{1}{p}}\tau^{-1}), \quad p = 1, 2, \\
\|\hat{w}(\zeta, t, \cdot)\|_{L^\infty(\Gamma')} &= O(\tau^{-1}),
\end{align*}
$$

uniformly with respect to $\zeta \in \mathcal{I}$, as $\tau \to \infty$.

**Proof.** Let $\gamma$ denote the intersection of $\Gamma'$ and the line $k_0 + Re^{i\pi}$, i.e.

$$
\gamma = \{ k_0 + u e^{i\frac{\pi}{4}} \mid u \in (-\sqrt{2}k_0, -\frac{k_0}{2}) \cup \left[ \frac{k_0}{2}, \infty \right) \}.
$$

**Figure 5.** The contour $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_6$. 

![Diagram](image-url)
Figure 6. The contour $\Gamma' = \Gamma \setminus \Gamma_\Sigma$.

Let $k = k_0 + u e^{i\pi/4}$. By (3.42) and (3.29), $\hat{w}$ has the following form on $\gamma$:

$$
\hat{w}(\zeta, t, k) = \begin{cases} 
(0 \ 0 \\
\delta(\zeta, k)^{-2} r_{1,a}(x, t, k) e^{t\Phi(\zeta, k)} \ 0 \\
0 \\
-\delta(\zeta, k)^{-2} r_{3,a}(x, t, k) e^{t\Phi(\zeta, k)} \ 0 
\end{cases},
$$

$$
0 \leq u < \infty,
$$

$$
0 \leq u < -k_0^2.
$$

(4.4)

It’s enough to prove that $\delta(\zeta, k)^{\pm 1}$ is uniformly bounded on $\gamma$ with respect to $\zeta \in I$, i.e.

$$
\sup_{\zeta \in I} \sup_{k \in \Gamma'} |\delta(\zeta, k)^{\pm 1}| \leq C
$$

(4.5)

From Lemma 3.1 $\chi(\zeta, k)$ is uniformly bounded on $\gamma$ with respect to $\zeta \in I$. Thus

$$
|\delta(\zeta, k)^{\pm 1}| = \left| \left( \frac{k - k_0}{k + k_0} \right)^{\pm i\nu} e^{\pm \chi(\zeta, k)} \right| \leq C \left| \left( \frac{ue^{i\pi/4}}{ue^{i\pi/4} + 2k_0} \right)^{\pm i\nu} \right|
$$

$$
\leq C \left| \left( 1 + \frac{2k_0}{ue^{-i\pi/4}} \right)^{\pm i\nu} \right|,
$$

(4.6)

where $1 + \frac{2k_0}{ue^{-i\pi/4}}$ satisfies the following inequalities

$$
\begin{cases} 
1 < \left| 1 + \frac{2k_0}{ue^{-i\pi/4}} \right| \leq (17 + 4\sqrt{2})^{\frac{1}{2}}, & u \in \left[ \frac{k_0}{2}, \infty \right), \\
1 < \left| 1 + \frac{2k_0}{ue^{-i\pi/4}} \right| \leq (17 - 4\sqrt{2})^{\frac{1}{2}}, & u \in \left( -\sqrt{2}k_0, -\frac{k_0}{2} \right].
\end{cases}
$$

(4.7)

By (4.6) and (4.7), $\delta(\zeta, k)^{\pm 1}$ is uniformly bounded on $\gamma$ with respect to $\zeta \in I$. Since similar arguments apply to the remaining parts of $\Gamma'$, this prove (4.5).
Since the decompositions of $r_j$ ($r_j = r_{j,a} + r_{j,r}, j = 1, \cdots, 4$) is similar to the local case, we can follow [22] to accomplish the rest of the proof by (3.23), (3.24), (3.25) and (3.26).

We normalize the jump matrix $v(\zeta, t, k)$ on $\Gamma_{\Sigma}$

$$v_0(\zeta, t, z) = v\left(\zeta, t, k_0 + \frac{\epsilon z}{\rho}\right), \quad z \in \Sigma^\rho, \quad (4.8)$$

which has the form of

$$v_0(\zeta, t, z) = \begin{cases}
\begin{pmatrix}
1 & 0 \\
R_1(\zeta, t, z)(\bar{z}_\rho)^{-2i\nu(\zeta)e^{t\phi(\zeta,z)}} & 1 \\
1 & R_2(\zeta, t, z)(\bar{z}_\rho)^{2i\nu(\zeta)e^{-t\phi(\zeta,z)}} \\
0 & 1 \\
-R_3(\zeta, t, z)(\bar{z}_\rho)^{-2i\nu(\zeta)e^{t\phi(\zeta,z)}} & 1 \\
1 & -R_4(\zeta, t, z)(\bar{z}_\rho)^{2i\nu(\zeta)e^{-t\phi(\zeta,z)}}
\end{pmatrix}, & z \in \Sigma^\rho_1, \\
\begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
1 - S_1(\zeta, t, z)S_2(\zeta, t, z) & -S_2(\zeta, t, z)(\bar{z}_\rho)^{2i\nu(\zeta)e^{-t\phi(\zeta,z)}} \\
S_1(\zeta, t, z)(\bar{z}_\rho)^{-2i\nu(\zeta)e^{t\phi(\zeta,z)}} & 1 \\
S_3(\zeta, t, z)(\bar{z}_\rho)^{-2i\nu(\zeta)e^{t\phi(\zeta,z)}} & -S_4(\zeta, t, z)(\bar{z}_\rho)^{2i\nu(\zeta)e^{-t\phi(\zeta,z)}} \\
0 & 1 - S_3(\zeta, t, z)S_4(\zeta, t, z)
\end{pmatrix}, & z \in \Sigma^\rho_5,
\end{cases} \quad (4.9)$$

where $\epsilon$, $\rho$ and $\phi$ are defined by (3.36). The phase $\phi(\zeta, z)$ is identical to the local case, which is a smooth function of $(\zeta, z) \in \mathcal{I} \times \mathbb{C}$ satisfying condition (2.10) and (2.11) of Theorem 2.1 in [22]. Moreover, $\{R_j(\zeta, t, k)\}_1^4$ and $\{S_j(\zeta, t, k)\}_1^4$ satisfy the following Lemma.

**Lemma 4.2.** There exist constants $(\alpha, L) \in [1/2, 1) \times (0, \infty)$ such that the functions $\{R_j(\zeta, t, z)\}_1^4$ and $\{S_j(\zeta, t, k)\}_1^4$ satisfy the inequalities:

$$\begin{align*}
|R_1(\zeta, t, z) - q_1(\zeta)| & \leq L \frac{\epsilon z}{\rho} e^{\frac{t |a|}{2}}, & z \in \Sigma^\rho_1, \\
|R_2(\zeta, t, z) - q_2(\zeta)| & \leq L \frac{\epsilon z}{\rho} e^{\frac{t |a|}{2}}, & z \in \Sigma^\rho_2, \\
|R_3(\zeta, t, z) - \frac{q_3(\zeta)}{1 - q_1(\zeta)q_2(\zeta)}| & \leq L \frac{\epsilon z}{\rho} e^{\frac{t |a|}{2}}, & z \in \Sigma^\rho_3, \\
|R_4(\zeta, t, z) - \frac{q_4(\zeta)}{1 - q_1(\zeta)q_2(\zeta)}| & \leq L \frac{\epsilon z}{\rho} e^{\frac{t |a|}{2}}, & z \in \Sigma^\rho_4, \\
|S_j(\zeta, t, z)| & \leq L \frac{\epsilon z}{\rho} t^{-\frac{3}{2}}, & j = 1, 2, & z \in \Sigma^\rho_5, \\
|S_j(\zeta, t, z)| & \leq L \frac{\epsilon z}{\rho} t^{-\frac{3}{2}}, & j = 3, 4, & z \in \Sigma^\rho_6,
\end{align*} \quad (4.10)$$

where $q_1(\zeta)$ and $q_2(\zeta)$ are defined by [3.34].
Proof. Equations (3.29) and (4.9) imply that
\[
\begin{align*}
R_1(\zeta, t, z) &= \delta(\zeta, k)^2 r_{1,a}(x, t, k) (\frac{z}{\rho})^{2i\nu(\zeta)}, \\
R_2(\zeta, t, z) &= \delta(\zeta, k)^2 r_{2,a}(x, t, k) (\frac{z}{\rho})^{-2i\nu(\zeta)}, \\
R_3(\zeta, t, z) &= \delta(\zeta, k)^2 r_{3,a}(x, t, k) (\frac{z}{\rho})^{2i\nu(\zeta)}, \\
R_4(\zeta, t, z) &= \delta(\zeta, k)^2 r_{4,a}(x, t, k) (\frac{z}{\rho})^{-2i\nu(\zeta)}, \\
S_1(\zeta, t, z) &= \delta(\zeta, k)^2 r_{1,r}(x, t, k) (\frac{z}{\rho})^{2i\nu(\zeta)}, \\
S_2(\zeta, t, z) &= \delta(\zeta, k)^2 r_{2,r}(x, t, k) (\frac{z}{\rho})^{-2i\nu(\zeta)}, \\
S_3(\zeta, t, z) &= \delta(\zeta, k)^2 r_{3,r}(x, t, k) (\frac{z}{\rho})^{2i\nu(\zeta)}, \\
S_4(\zeta, t, z) &= \delta(\zeta, k)^2 r_{4,r}(x, t, k) (\frac{z}{\rho})^{-2i\nu(\zeta)}.
\end{align*}
\]

Let \( k = k_0 + \frac{1}{\rho} z \). Using the expression (3.11), we can write
\[
R_1(\zeta, t, z) = e^{-2\chi(\zeta, k)} r_{1,a}(x, t, k) e^{2i\nu(\zeta) \ln \left( \frac{2(k+k_0)}{k_0} \right)}, \quad z \in \Sigma_1^\rho.
\]

By (3.23), we find
\[
R_1(\zeta, t, 0) = e^{-2\chi(\zeta, k_0)} r_{1}(k_0) e^{2i\nu(\zeta) \ln 4} = q_1(\zeta).
\]

Similar arguments imply that
\[
R_2(\zeta, t, 0) = q_2(\zeta), \quad R_3(\zeta, t, 0) = \frac{q_1(\zeta)}{1 - q_1(\zeta) q_2(\zeta)}, \quad R_4(\zeta, t, 0) = \frac{q_2(\zeta)}{1 - q_1(\zeta) q_2(\zeta)}.
\]

Note that \( z \in \Sigma_1^\rho \) is equivalent to \( k \in k_0 + \Sigma_1^\rho \). For \( z \in \Sigma_1^\rho \), we have
\[
|R_1(\zeta, t, z) - q_1(\zeta)| \leq \left| e^{-2\chi(\zeta, k)} - e^{-2\chi(\zeta, k_0)} \right| \left| r_{1,a}(x, t, k) e^{2i\nu(\zeta) \ln \left( \frac{2(k+k_0)}{k_0} \right)} \right| \\
+ \left| e^{-2\chi(\zeta, k_0)} \right| \left| r_{1,a}(x, t, k) - r_1(k_0) \right| \left| e^{2i\nu(\zeta) \ln \left( \frac{2(k+k_0)}{k_0} \right)} \right| \\
+ \left| e^{-2\chi(\zeta, k_0)} r_1(k_0) \right| \left| 1 - e^{-2i\nu(\zeta) \ln \left( \frac{k+k_0}{2k_0} \right)} \right| \left| e^{2i\nu(\zeta) \ln \left( \frac{2(k+k_0)}{k_0} \right)} \right|.
\] (4.13)

From Lemma 3.1 \( e^{2\chi(\zeta, k_0)} \) is bounded for \( \zeta \in \mathcal{I} \). Moreover, let \( k = k_0 + u e^{i\pi/4} \), we have
\[
\left| e^{2i\nu(\zeta) \ln \left( \frac{2(k+k_0)}{k_0} \right)} \right| = \left| \left( 4 + \frac{2u}{k_0} e^{i\pi/4} \right)^{2i\nu(\zeta)} \right|,
\]
where \( 4 + \frac{2u}{k_0} e^{i\pi/4} \) satisfies
\[
(17 - 4\sqrt{2})^2 < \left| 4 + \frac{2u}{k_0} e^{i\pi/4} \right| \leq 4, \quad 0 \leq u < \epsilon.
\]
Thus \( e^{2i\nu(\zeta) \ln \left( \frac{2(k+k_0)}{k_0} \right)} \) is uniformly bounded with respect to \( \zeta \in \mathcal{I} \) and \( k \in k_0 + \Sigma_1^\rho \).

As in the local case, using (3.23), (3.24) and the estimate
\[
|\text{Re} \phi(\zeta, ve^{i\pi/4})| \leq \frac{2v^2}{3}, \quad 0 \leq v < \rho,
\]
we can prove that for \( z \in \Sigma_1^\rho \), the following inequalities holds:
\[
|r_{1,a}(x, t, k) - r_1(k_0)| \leq C \frac{\epsilon |z|}{\rho} e^{\frac{\epsilon}{\rho} |z|^2}, \quad (4.14)
\]
\[
|r_{1,a}(x, t, k)| \leq C e^{\frac{\epsilon}{\rho} |z|^2}. \quad (4.15)
\]
Thus,
\[ |R_1(\zeta, t, z) - q_1(\zeta)| \leq C e^{\frac{|s|}{\rho}} e^{-2\chi(\zeta,k)} - e^{-2\chi(\zeta,k_0)} + C e^{\frac{|s|}{\rho}} e^{\frac{|s|}{\rho}} \]
\[ + C \left| 1 - e^{-2i\nu(\zeta) \ln \left( \frac{k+k_0}{2k_0} \right)} \right|. \] (4.16)

Employing (3.12), we have
\[ |\chi(\zeta, k) - \chi(\zeta, k_0)| \leq C \left| \ln \left( \frac{k_0 + k}{2k_0} \right) \right| + C \left| \int_{-k_0}^{k_0} \ln \left( \frac{s-k}{s-k_0} \right) d\ln(1 - r_1(s)r_2(s)) \right|. \] (4.17)

Then we can follow the proof of Lemma 4.9 in [22] to derive the following inequalities
\[ \left| 1 - e^{-2i\nu(\zeta) \ln \left( \frac{k+k_0}{2k_0} \right)} \right| \leq C k_0^{-1} |k - k_0|, \] (4.18)
\[ |e^{-2\chi(\zeta,k)} - e^{-2\chi(\zeta,k_0)}| \leq C |k - k_0| \ln |k - k_0|. \] (4.19)

Notice that (4.19) is slightly different from Lemma 4.9 in [22], which is caused by the first term of right-hand side of (4.17).

Using (4.16), (4.18) and (4.19), We can verify (4.10) in the case of \( z \in \Sigma^\rho \); the case of \( z \in \Sigma^\rho_j, \) \( j = 2, 3, 4 \) are similar. Using (3.25), (3.26) and the uniform boundness of \( e^{-2\chi(\zeta,k)} \) and \( e^{2i\ln \left( \frac{2(k+k_0)}{k_0} \right)} \), it’s clear that \( S_j(\zeta,t,z) \), \( j = 1, \cdots, 4 \) satisfy the inequalities in (4.10). \( \Box \)

**Lemma 4.3.** On \( \Gamma_\Sigma \), function \( \hat{w} \) satisfies
\[ \hat{w}(\zeta, t, k) = O \left( \tau^{-\frac{3}{2} + |\Im \nu(\zeta)|} e^{-\frac{\tau}{24\nu^2 |k+k_0|^2}} \right), \]
\[ \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad k \in \pm k_0 + \bigcup_{j=1}^{4} \Sigma^\rho_j, \] (4.20)
and
\[ \hat{w}(\zeta, t, k) = O \left( \tau^{-\frac{3}{2} + 2|\Im \nu(\zeta)|} \right), \]
\[ \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad k \in \pm k_0 + \bigcup_{j=5}^{6} \Sigma^\rho_j, \] (4.21)
where the error term is uniform with respect to \( \zeta \in \mathcal{I} \) and \( k \in \Gamma_\Sigma \).

**Proof.** We prove the case of \( k \in k_0 + \Sigma^\rho \). Symmetries (3.30) and (3.38) implies that
\[ \hat{w}(\zeta, t, k) = \overline{\hat{w}(\zeta, t, -k)}, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad k \in \hat{\Gamma}. \] (4.22)
Thus the case of \( k \in -k_0 + \Sigma^\rho \) follows by the above symmetry. We let \( k = k_0 + \frac{z}{\rho} \), \( z \in \Sigma^\rho \). Then
\[ \hat{w}(\zeta, t, k) = m_{0-}(\zeta, t) v(\zeta, t, k) m_{0+}(\zeta, t, k)^{-1} - I \]
\[ = D(\zeta, t)m_\Sigma(\zeta, \sqrt{t}z) D(\zeta, t)^{-1} v_0(\zeta, t, z) D(\zeta, t) m_{\Sigma}^X(\zeta, \sqrt{t}z)^{-1} D(\zeta, t)^{-1} - I \]
By Lemma A.1, there exists a constant $G$ such that $m^\pm_\pm(\zeta, \sqrt{t}z)$ is uniformly bounded with respect to $\zeta \in \mathcal{I}$ and $|\sqrt{t}z| \geq G$; $m^\pm_\pm(\zeta, \sqrt{t}z)(\sqrt{t}z)^{i\nu(c)\sigma_3}$ is uniformly bounded with respect to $\zeta \in \mathcal{I}$ and $|\sqrt{t}z| < G$. Thus we write $\hat{w}$ as

$$\hat{w}(\zeta, t, k) = \begin{cases} D(\zeta, t)m^X(\zeta, \sqrt{t}z)u_1(\zeta, t, z)m^X_+(\zeta, \sqrt{t}z)^{-1}D(\zeta, t)^{-1}, & |\sqrt{t}z| \geq G, \\ D(\zeta, t)m^X(\zeta, \sqrt{t}z)(\sqrt{t}z)^{i\nu(c)\sigma_3}u_2(\zeta, t, z)m^X_+(\zeta, \sqrt{t}z)^{-1}D(\zeta, t)^{-1}, & |\sqrt{t}z| < G. \end{cases}$$

(4.23)

where

$$u_1(\zeta, t, z) = D(\zeta, t)^{-1}v_0(\zeta, t, z)D(\zeta, t) - v^X(\zeta, \sqrt{t}z)$$

(4.24)

$$u_2(\zeta, t, z) = (\sqrt{t}z)^{-i\nu(c)\sigma_3}u_1(\zeta, t, z)$$

(4.25)

By (3.35), $|D(\zeta, t)| = O(\tau^{\frac{|\text{Im}\nu(c)|}{2}})$. Consequently, it’s enough to prove that

$$u_1(\zeta, t, z) = O(\tau^{-\frac{\alpha}{2}} e^{-\frac{|t|z^2}{2\tau}}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in \bigcup_{j=1}^{4} \Sigma^0_j, \quad |\sqrt{t}z| \geq G,$$

(4.26)

$$u_1(\zeta, t, z) = O(\tau^{-\frac{\alpha}{2}} |\text{Im}\nu(c)|), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in \bigcup_{j=5}^{6} \Sigma^0_j, \quad |\sqrt{t}z| \geq G,$$

(4.27)

$$u_2(\zeta, t, z) = O(\tau^{-\frac{\alpha}{2}} e^{-\frac{|t|z^2}{2\tau}}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in \bigcup_{j=1}^{4} \Sigma^0_j, \quad |\sqrt{t}z| < G,$$

(4.28)

$$u_2(\zeta, t, z) = O\left(\tau^{-\frac{3}{2}}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in \bigcup_{j=5}^{6} \Sigma^0_j, \quad |\sqrt{t}z| < G$$

(4.29)

uniformly with respect to $(\zeta, z)$ in the given ranges.

For the case of $z \in \Sigma^0_1$, we have

$$u_1(\zeta, t, z) = \begin{pmatrix} 0 \\ \left(R_1(\zeta, t, z)e^{i(\phi(\zeta, z) - \phi(\zeta, 0))} - q_1(\zeta)e^{\frac{|t|z^2}{2}}\right)(\sqrt{t}z)^{-2i\nu(c)} \end{pmatrix}. $$

(4.30)

So only the $(21)$ entry of $u_1(\zeta, t, z)$ is nonzero, and for $|\sqrt{t}z| \geq G$ we find that

$$|u_1(\zeta, t, z)_{21}| = \left|R_1(\zeta, t, z)e^{i(\phi(\zeta, z) - \phi(\zeta, 0))} - q_1(\zeta)e^{\frac{|t|z^2}{2}}\right| \left|\left(\sqrt{t}z\right)^{-2i\nu(c)}\right|$$

$$\leq C\left(\left|R_1(\zeta, t, z) - q_1(\zeta)e^{\text{Re}\hat{\phi}(\zeta, z)}\right| e^{\frac{|t|z^2}{2}} \left|\sqrt{t}z\right|^{2\text{Im}\nu(c)}e^{\frac{\text{Re}(\nu)}{2}}} \right)$$

$$\leq C\left(\left|R_1(\zeta, t, z) - q_1(\zeta)e^{\text{Re}\hat{\phi}(\zeta, z)}\right| e^{\frac{|t|z^2}{2}} \left|\sqrt{t}z\right|^{2\text{Im}\nu(c)}\right),$$

(4.31)

where $\hat{\phi}(\zeta, z) = \phi(\zeta, z) - \phi(\zeta, 0) - \frac{|t|z^2}{2}$. As in the local case [22], we can use the inequalities

$$|e^w - 1| \leq |w| \max(1, e^{\text{Re}w}), \quad w \in \mathbb{C},$$

$$\text{Re}\hat{\phi}(\zeta, z) \leq \frac{|z|^2}{4}, \quad \zeta \in \mathcal{I}, \quad z \in \Sigma^0_1,$$
and the boundness of \( q_j(\zeta), j = 1, 2 \) to find that
\[
\left| (u_1(\zeta, t, z))_{21} \right| \leq C \left( |R_1(\zeta, t, z) - q_1(\zeta)| + C t |\hat{\phi}(\zeta, z)| \right) e^{-\frac{t|z|^2}{4}} |\sqrt{t}z|^{2\text{Im} \nu(\zeta)},
\]
\[
\zeta \in \mathcal{I}, \quad t > 0, \quad z \in \Sigma^\rho_i, \quad |\sqrt{t}z| \geq G.
\]

(4.32)

It’s easy to verify that
\[
|\hat{\phi}(\zeta, z)| \leq C \frac{|z|^3}{\rho}, \quad \zeta \in \mathcal{I}, \quad z \in \Sigma^\rho,
\]
so together with Lemma 4.2, the right-hand of (4.32) is of order
\[
O \left( \frac{L|z|^\alpha e^{\frac{t|z|^2}{\rho^\alpha}}}{\rho^\alpha} + \frac{C t|z|^3}{\rho} e^{-\frac{t|z|^2}{4}} (t|z|^2) \text{Im} \nu(\zeta) \right)
\]
\[
= O \left( \left( \frac{(t|z|^2)^{\alpha/2+\text{Im} \nu(\zeta)}}{\tau^{\alpha/2}} + \frac{(t|z|^2)^{3/2+\text{Im} \nu(\zeta)}}{t^{1/2}} \right) e^{-\frac{t|z|^2}{12}} \right)
\]
\[
= O \left( \frac{1}{\tau^{\alpha/2}} + \frac{1}{t^{1/2}} \right) e^{-\frac{t|z|^2}{12}}, \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in \Sigma^\rho_i, \quad |\sqrt{t}z| \geq G.
\]

(4.33)

uniformly with respect to \((\zeta, z)\) in the given ranges. This proves (4.26) for \( z \in \Sigma^\rho_3 \); the cases of \( z \in \Sigma^\rho_j, j = 2, 3, 4 \), are similar. For the case of \( z \in \Sigma^\rho_6 \), we have
\[
u_1(\zeta, t, z) = \begin{pmatrix}
-S_1(\zeta, t, z) S_2(\zeta, t, z) \\
S_1(\zeta, t, z) e^{t(\phi(\zeta, z) - \phi(\zeta, 0))} (\sqrt{t}z)^{-2\nu(\zeta)} \\
-S_2(\zeta, t, z) e^{-t(\phi(\zeta, z) - \phi(\zeta, 0))} (\sqrt{t}z)^{-2\nu(\zeta)} \\
0
\end{pmatrix}.
\]

(4.34)

By Lemma 4.2 the (11) entry satisfies
\[
\left| (u_1(\zeta, t, z))_{11} \right| = |S_1(\zeta, t, z) S_2(\zeta, t, z)| \leq C \frac{|z|^2}{\rho} t^{-3}
\leq C \frac{|z|^2}{\tau} t^{-2} \leq C \tau^{-3}, \quad \zeta \in \mathcal{I}, \quad t > 0, \quad z \in \Sigma^\rho_5,
\]

(4.35)

and the (12) entry satisfies
\[
\left| (u_1(\zeta, t, z))_{12} \right| = \left| S_2(\zeta, t, z) e^{-t(\phi(\zeta, z) - \phi(\zeta, 0))} (\sqrt{t}z)^{-2\nu(\zeta)} \right|
\leq C \frac{|z|^{1-2\text{Im} \nu(\zeta)}}{\rho} t^{-\frac{3}{2}-\text{Im} \nu(\zeta)} \leq C \frac{|z|^{1-2\text{Im} \nu(\zeta)}}{\tau^{\frac{3}{2}}} t^{-1-\text{Im} \nu(\zeta)}.
\]

Because \(-\frac{1}{2} < \text{Im} \nu(\zeta) < \frac{1}{2}\), \((u_1(\zeta, t, z))_{12}\) is of order
\[
(u_1(\zeta, t, z))_{12} = O \left( t^{-\frac{3}{2}-\text{Im} \nu(\zeta)} \right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in \Sigma^\rho_5.
\]

(4.36)

Similarly, \((u_1(\zeta, t, z))_{21}\) is of order
\[
(u_1(\zeta, t, z))_{21} = O \left( t^{-\frac{3}{2}+\text{Im} \nu(\zeta)} \right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad z \in \Sigma^\rho_5.
\]

(4.37)

Using (4.35), (4.36) and (4.37), we prove (4.27) for \( z \in \Sigma^\rho_6 \); the case of \( z \in \Sigma^\rho_5 \) is similar.

On the other hand, (4.28) and (4.29), the estimates of \( u_2(\zeta, t, z) \), can be proved in the same way. □
Following Lemma 2.6 in [22], we use Lemma 4.1 and Lemma 4.3 to obtain the estimates:

\[
\|\hat{w}(\zeta, t, \cdot)\|_{L^2(\Gamma)} = O\left(e^{\frac{i}{2} \tau - \frac{\alpha}{2} + |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]
\[
\|\hat{w}(\zeta, t, \cdot)\|_{L^\infty(\Gamma)} = O\left(\tau^{-\frac{\alpha}{2} + |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]
\[
\|\hat{w}(\zeta, t, \cdot)\|_{L^p(\pm \zeta_0 + \Sigma^\epsilon)} = O\left(e^{\frac{i}{2} \tau - \frac{\alpha}{2} + |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]

where \(p \in [1, \infty)\) and the error terms are uniform with respect to \(\zeta \in \mathcal{I}\). Moreover, if taking account of the first and second columns of \(\hat{w}(\zeta, t, k)\) respectively in Lemma 4.3 we have

\[
\|\hat{w}^{(j)}(\zeta, t, \cdot)\|_{L^2(\Gamma)} = O\left(e^{\frac{i}{2} \tau - \frac{\alpha}{2} + (-1)^j |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad j = 1, 2
\]
\[
\|\hat{w}^{(j)}(\zeta, t, \cdot)\|_{L^\infty(\Gamma)} = O\left(\tau^{-\frac{\alpha}{2} + (-1)^j |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad j = 1, 2
\]
\[
\|\hat{w}^{(j)}(\zeta, t, \cdot)\|_{L^p(\pm \zeta_0 + \Sigma^\epsilon)} = O\left(e^{\frac{i}{2} \tau - \frac{\alpha}{2} - \frac{2}{p} + (-1)^j |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad j = 1, 2
\]

**Lemma 4.4.** The RH problem (3.41) has a unique solution for all sufficiently large \(\tau\). And for any \(\alpha \in (\lambda, 1)\) this solution satisfies

\[
\lim_{k \to \infty} (k \hat{m}(\zeta, t, k))_{12} = -\frac{2i e\Re \beta(\zeta, t)}{\tau^{\frac{\alpha}{2}} - |\Im \nu(\zeta)|} + O\left(e^{-\frac{1+\alpha}{2} + |\Im \nu(\zeta)| + |\Im \nu(\zeta)|}\right)
\]
\[
\tau \to \infty, \quad \zeta \in \mathcal{I},
\]

where \(\lambda = \max\left(\frac{1}{2}, \sup_{\zeta \in \mathcal{I}} 2|\Im \nu(\zeta)|\right)\), and the error term is uniform with respect to \(\zeta \in \mathcal{I}\) and \(\beta(\zeta, t)\) is defined by

\[
\beta(\zeta, t) = \frac{\sqrt{2} e^\frac{i \pi}{4} e^{-\frac{\pi \nu(\zeta)}{2}}}{q_1(\zeta) \Gamma(-i \nu(\zeta))} e^{-t \phi(\zeta, 0)} e^{-i \Re \nu(z)}.
\]

**Proof.** We define the integral operator \(\hat{C}\hat{w} : L^2(\hat{\Gamma}) + L^\infty(\hat{\Gamma}) \to L^2(\hat{\Gamma})\) by \(\hat{C}\hat{w} f = \hat{C}_-(f \hat{w})\), where \(\hat{C}_-(f \hat{w})\) is the boundary value of \(\hat{C}(f \hat{w})\) from the right side of \(\hat{\Gamma}\), and \(\hat{C}\) is the Cauchy operator associated with \(\hat{\Gamma}\):

\[
(\hat{C} f)(z) = \frac{1}{2\pi i} \int_{\hat{\Gamma}} f(s) \frac{1}{s - z} ds, \quad z \in \mathbb{C} \setminus \hat{\Gamma}.
\]

By (4.39),

\[
\|\hat{C}\hat{w}\|_{B(L^2(\hat{\Gamma}))} \leq C \|\hat{w}\|_{L^\infty(\hat{\Gamma})} = O\left(\tau^{-\frac{\alpha}{2} + |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}.
\]

Since \(\alpha \in (\lambda, 1)\), \(\|\hat{C}\hat{w}\|_{B(L^2(\hat{\Gamma}))}\) decays to 0 as \(\tau \to \infty\). Thus, there exists a \(T > 0\) such that \(I - \hat{C}\hat{w}(\zeta, t, \cdot) \in B(L^2(\hat{\Gamma}))\) is invertible for all \((\zeta, t) \in (0, \infty)\) with \(\tau > T\).

Moreover, by (4.38) we have

\[
\|\hat{\mu} - I\|_{L^2(\hat{\Gamma})} = O\left(e^{\frac{i}{2} \tau - \frac{\alpha}{2} + |\Im \nu(\zeta)|}\right), \quad \tau \to \infty, \quad \zeta \in \mathcal{I}.
\]
where \( \hat{\mu} - I = (I - \hat{\mathcal{C}})^{-1}\hat{\mathcal{C}}I \in L^2(\hat{\Gamma}) \) is the solution of the integral equation
\[
(I - \hat{\mathcal{C}})(\mu - I) = \hat{\mathcal{C}}I.
\]

Consequently, by Lemma 2.9 in [22], there exists a unique solution \( \hat{m} \) of the RH problem (3.41) whenever \( \tau > T \), and in accordance with the local case, we can represent \( \hat{m} \) as
\[
\hat{m}(\zeta, t, k) = I + \hat{\mathcal{C}}(\hat{\mu}\hat{w}) = I + \frac{1}{2\pi i} \int_{\Gamma} \hat{\mu}(\zeta, t, s)\hat{w}(\zeta, t, s) \frac{ds}{s - k}.
\]

Finally, Lemma 2.10 in [22] and symmetry (4.22) imply that
\[
\lim_{k \to \infty} k(\hat{m}(\zeta, t, k) - I) = -\frac{1}{2\pi i} \int_{\Gamma} \hat{\mu}(\zeta, t, k)\hat{w}(\zeta, t, k) dk
\]
\[
= -\frac{1}{2\pi i} \left( \int_{|k-k_0|=\epsilon} + \int_{|k+k_0|=\epsilon} \right) \hat{\mu}(\zeta, t, k)\hat{w}(\zeta, t, k) dk - \frac{1}{2\pi i} \int_{\Gamma} \hat{\mu}(\zeta, t, k)\hat{w}(\zeta, t, k) dk
\]
\[
= -\frac{1}{\pi i} \text{Re} \left( \int_{|k-k_0|=\epsilon} \hat{\mu}(\zeta, t, k)(m_0(\zeta, t, k)^{-1} - I) dk \right) - \frac{1}{2\pi i} \int_{\Gamma} \hat{\mu}(\zeta, t, k)\hat{w}(\zeta, t, k) dk.
\]

By Lemma 4.1, we have
\[
\left( m_0(\zeta, t, k)^{-1} \right)^{(2)} = \left( D(\zeta, t)\epsilon X \left( \zeta, \sqrt{\frac{\tau}{\epsilon}} (k - k_0) \right) \right)^{-1} D(\zeta, t)^{-1}
\]
\[
= \begin{pmatrix} 0 & B^{(2)}(\zeta, t) \\ \sqrt{\tau}(k - k_0) & O(\tau^{-1 + \text{Im} \nu(\zeta)}) \end{pmatrix}, \quad \tau \to \infty, \quad \zeta \in \mathcal{I}, \quad |k - k_0| = \epsilon.
\]

where \( B(\zeta, t) \) is defined by
\[
B(\zeta, t) = -i \epsilon \begin{pmatrix} 0 & -\beta^X(\zeta)e^{-i\phi(\zeta, 0)\tau^{-i\nu(\zeta)}} \\ \gamma X(\zeta)e^{i\phi(\zeta, 0)\tau^{-i\nu(\zeta)}} & 0 \end{pmatrix}.
\]

Using (4.47) and (4.50) we find
\[
\int_{|k-k_0|=\epsilon} \left( \hat{\mu}(\zeta, t, k)(m_0(\zeta, t, k)^{-1} - I) \right)^{(2)} dk = \int_{|k-k_0|=\epsilon} (m_0(\zeta, t, k)^{-1} - I)^{(2)} dk
\]
\[
+ \int_{|k-k_0|=\epsilon} (\hat{\mu}(\zeta, t, k) - I)(m_0(\zeta, t, k)^{-1} - I)^{(2)} dk
\]
\[
= \frac{B^{(2)}(\zeta, t)}{\sqrt{\tau}} \int_{|k-k_0|=\epsilon} \frac{dk}{k - k_0} + O(\epsilon^{\tau^{-1 + \text{Im} \nu(\zeta)}} + O(\epsilon^{\frac{1}{2} \tau^{-\frac{1}{2} + \text{Im} \nu(\zeta)}})
\]
\[
= \frac{2\pi i B^{(2)}(\zeta, t)}{\sqrt{\tau}} + O(\epsilon^{\tau^{-1 + \text{Im} \nu(\zeta)}} + O(\epsilon^{\frac{1}{2} \tau^{\frac{1}{2} + \text{Im} \nu(\zeta)}})
\]
\[
= \frac{2\pi i B^{(2)}(\zeta, t)}{\sqrt{\tau}} + O(\epsilon^{\tau^{-\frac{1}{2} + \text{Im} \nu(\zeta)}} + O(\epsilon^{\frac{1}{2} \tau^{\frac{1}{2} + \text{Im} \nu(\zeta)}}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]
uniformly with respect to \( \zeta \in \mathcal{I} \). Notice that \( B^{(2)}(\zeta, t) \) contains \( \tau^{-i\nu(\zeta)} \), the order of the leading term of (4.51) is \( \tau^{-\frac{1}{2} + \text{Im} \nu(\zeta)} \). Since \( \alpha \in (\lambda, 1) \), the error term of (4.51) does make sense compared to the leading term.
On the other hand,
\[
\left|\int_{\Gamma} \left( \hat{\mu}(\zeta, t, k) \hat{w}(\zeta, t, k) \right)^{(2)} dk \right| = \left| \int_{\Gamma} \left( \hat{\mu}(\zeta, t, k) - I \right) \hat{w}^{(2)}(\zeta, t, k) dk + \int_{\Gamma} \hat{w}^{(2)}(\zeta, t, k) dk \right| \\
\leq \| \hat{\mu} - I \|_{L^2(\Gamma)} \| \hat{w}^{(2)} \|_{L^2(\Gamma)} + \| \hat{w}^{(2)} \|_{L^2(\Gamma)}.
\]
(4.2) and (4.43) implies that \( \| \hat{w}^{(2)} \|_{L^2(\Gamma)} = O(\epsilon \tau^{-1} + \epsilon \tau^{-\frac{1+\alpha}{2}+\text{Im}\nu(\zeta)}), \) and \( \| \hat{w}^{(2)} \|_{L^2(\Gamma)} = O(\epsilon \tau^{-1} + \epsilon \tau^{-\frac{1-\alpha}{2}+\text{Im}\nu(\zeta)}). \) Since \( \| \hat{\mu} - I \|_{L^2(\Gamma)} = O(\epsilon^2 \tau^{-\frac{1}{2}+\text{Im}\nu(\zeta)}) \) by (4.47) and \( \alpha \in (\lambda, 1) \), we find that
\[
\left|\int_{\Gamma} \left( \hat{\mu}(\zeta, t, k) \hat{w}(\zeta, t, k) \right)^{(2)} dk \right| = O(\epsilon \tau^{-1} + \epsilon \tau^{-\frac{1+\alpha}{2}+\text{Im}\nu(\zeta)} + \epsilon \tau^{-\frac{1-\alpha}{4}+\text{Im}\nu(\zeta)}), \quad \tau \to \infty, \quad \zeta \in \mathcal{I},
\]
(4.52)
uniformly with respect to \( \zeta \in \mathcal{I} \). Then (4.49) (4.50), (4.51) and (4.52) imply (4.44).

**Theorem 4.5.** Consider the Cauchy problem (1.1). We assume that the scattering data associated the initial data \( q_0(x) \) are such that:

1. \( a_1(k) \) and \( a_2(k) \) have no zeros in \( \mathbb{C}_+ \) and \( \mathbb{C}_- \) respectively;
2. For \( \zeta \in \mathcal{I} \), \( \Delta(\zeta) = \int_{-\infty}^{\infty} \text{arg}(1 - r_1(s)r_2(s)) \in (-\pi, \pi) \), where \( r_1(s) = \frac{b(s)}{a_1(s)} \) and \( r_2(s) = \frac{b(s)}{a_2(s)} \).

Then, for any \( \alpha \in (\lambda, 1) \) and \( N > 0 \), the solution \( q(x, t) \) defined by (2.21) satisfies
\[
q(x, t) = \frac{4\epsilon \text{Re} \beta(\zeta, t)}{\tau^{\frac{1}{2}-\text{Im}\nu(\zeta)}} + O(\epsilon \tau^{-\frac{1+\alpha}{2}+\text{Im}\nu(\zeta)}+\text{Im}\nu(\zeta)), \quad \tau \to \infty, \quad -Nt < x < 0,
\]
(4.53)
where the error term is uniform with respect to \( x \) in the given range.

**Proof.** Lemma 4.4 implies that (2.21) exists for all sufficiently large \( \tau \), and
\[
q(x, t) = 2i \lim_{k \to \infty} (kM(x, t, k))_{12} = 2i \lim_{k \to \infty} (k\hat{\mu}(x, t, k))_{12}
= \frac{4\epsilon \text{Re} \beta(\zeta, t)}{\tau^{\frac{1}{2}-\text{Im}\nu(\zeta)}} + O(\epsilon \tau^{-\frac{1+\alpha}{2}+\text{Im}\nu(\zeta)}+\text{Im}\nu(\zeta)).
\]
(4.54)

**Remark 4.6.** In contrast with the local mKdV equation, the decay rate of the leading term depends on \( \zeta = \frac{x}{t} \) through \( \text{Im}\nu(\zeta) \). Notice that in the local case, \( \text{Im}\nu(\zeta) = 0 \) for all \( \zeta \in \mathcal{I} \), and Theorem 4.5 regresses to the main result of [22].

**Remark 4.7.** In section 4 of [22], the conditions (1) and (2) in Theorem 4.5 were verified in the case of single box initial data, for which the scattering data can be calculated explicitly.
APPENDIX A.

Lemma A.1. The RH problem \[3.31\] has a unique solution \(m^X(\zeta, z)\) for each \(\zeta \in \mathcal{I}\). This solution satisfies
\[
m^X(\zeta, z) = I + \frac{i}{z} \left( \begin{array}{cc} 0 & -\beta^X(\zeta) \\ \gamma^X(\zeta) & 0 \end{array} \right) + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \zeta \in \mathcal{I},
\]
where the error term is uniform with respect to \(\arg z \in [0, 2\pi]\) and \(\zeta \in \mathcal{I}\). The functions \(\beta^X(\zeta)\) and \(\gamma^X(\zeta)\) are defined by
\[
\beta^X(\zeta) = \frac{\sqrt{2\pi} e^{i\pi/4} e^{-\pi\nu(\zeta)/2}}{q_1(\zeta) \Gamma(-i\nu(\zeta))},
\]
\[
\gamma^X(\zeta) = \frac{\sqrt{2\pi} e^{-i\pi/4} e^{-\pi\nu(\zeta)/2}}{q_2(\zeta) \Gamma(i\nu(\zeta))}.
\]
Moreover, for each closed disk \(K \in \mathbb{C}\) centered at the origin,
\[
\sup_{\zeta \in \mathcal{I}} \sup_{z \in K \setminus \mathcal{X}} |m^X(\zeta, z) z^{i\nu(\zeta)\sigma_3}| < \infty.
\]

Proof. The detailed proofs can be found in Appendix A in \[26\] and Appendix B in \[22\]. Notice that \(m^X(\zeta, z)\) is singular at the origin, which is different from the local case. Multiplying \(m^X(\zeta, z)\) by \(z^{i\nu(\zeta)\sigma_3}\) can remove the singularity. \(\square\)

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