HITTING DIAMONDS AND GROWING CACTI

SAMUEL FIORINI, GWENAËL JORET, AND UGO PIETROPAOLI

Abstract. We consider the following NP-hard problem: in a weighted graph, find a minimum cost set of vertices whose removal leaves a graph in which no two cycles share an edge. We obtain a constant-factor approximation algorithm, based on the primal-dual method. Moreover, we show that the integrality gap of the natural LP relaxation of the problem is $\Theta(\log n)$, where $n$ denotes the number of vertices in the graph.

Keywords: Approximation algorithm, primal-dual method, covering problem.

1. Introduction

Graphs in this paper are finite, undirected, and may contain parallel edges but no loops. We study the following combinatorial optimization problem: given a vertex-weighted graph, remove a minimum cost subset of vertices so that all the cycles in the resulting graph are edge-disjoint. We call this problem the diamond hitting set problem, because it is equivalent to covering all subgraphs which are diamonds with a minimum cost subset of vertices, where a diamond is any subdivision of the graph consisting of three parallel edges.

The diamond hitting set problem can be thought of as a generalization of the vertex cover and feedback vertex set problems: Suppose you wish to remove a minimum cost subset of vertices so that the resulting graph has no pair of vertices linked by $k$ internally disjoint paths. Then, for $k = 1$ and $k = 2$, this is respectively the vertex cover problem and feedback vertex set problem, while for $k = 3$ this corresponds to the diamond hitting set problem.

It is well-known that both the vertex cover and feedback vertex set problems admit constant-factor approximation algorithms. Hence, it is natural to ask whether the same is true for the diamond hitting set problem. The main contribution of this paper is a positive answer to this question.

Date: March 23, 2010.

A preliminary version of the paper will appear in the proceedings of the 14th Conference on Integer Programming and Combinatorial Optimization (IPCO 2010). This work was supported by the “Actions de Recherche Concertées” (ARC) fund of the “Communauté française de Belgique”. G.J. is a Postdoctoral Researcher of the “Fonds National de la Recherche Scientifique” (F.R.S.-FNRS). This work was done while U.P. was at Département de Mathématique - Université Libre de Bruxelles as a Postdoctoral Researcher of the F.R.S.-FNRS.

A $\rho$-approximation algorithm for a minimization problem is an algorithm that runs in polynomial time and outputs a feasible solution whose cost is no more than $\rho$ times the cost of the optimal solution. The number $\rho$ is called the approximation factor.
1.1. **Background and Related Work.** Although there exists a simple 2-approximation algorithm for the vertex cover problem, there is strong evidence that approximating the problem with a factor of $2 - \varepsilon$ might be hard, for every $\varepsilon > 0$ [11]. It should be noted that the feedback vertex set and diamond hitting set problems are at least as hard to approximate as the vertex cover problem, in the sense that the existence of a $\rho$-approximation algorithm for one of these two problems implies the existence of a $\rho$-approximation algorithm for the vertex cover problem, where $\rho$ is a constant.

Concerning the feedback vertex set problem, the first approximation algorithm is due to Bar-Yehuda, Geiger, Naor, and Roth [2] and its approximation factor is $O(\log n)$. Later, 2-approximation algorithms have been proposed by Bafna, Berman, and Fujito [1], and Becker and Geiger [3]. Chudak, Goemans, Hochbaum and Williamson [5] showed that these algorithms can be seen as deriving from the primal-dual method (see for instance [14, 8]). Starting with an integer programming formulation of the problem, these algorithms simultaneously construct a feasible integral solution and a feasible dual solution of the linear programming relaxation, such that the values of these two solutions are within a constant factor of each other.

These algorithms also lead to a characterization of the integrality gap\(^2\) of two different integer programming formulations of the problem, as we now explain. Let $C(G)$ denote the collection of all the cycles $C$ of $G$. A natural integer programming formulation for the feedback vertex set problem is as follows:

\[
\begin{align*}
\text{Min} & \quad \sum_{v \in V(G)} c_v x_v \\
\text{s.t.} & \quad \sum_{v \in V(C)} x_v \geq 1 \quad \forall C \in C(G) \\
& \quad x_v \in \{0, 1\} \quad \forall v \in V(G).
\end{align*}
\]

(Throughout, $c_v$ denotes the (non-negative) cost of vertex $v$.) The algorithm of Bar-Yehuda et al. [2] implies that the integrality gap of this integer program is $O(\log n)$. Later, Even, Naor, Schieber, and Zosin [7] proved that its integrality gap is also $\Omega(\log n)$.

A better formulation has been introduced by Chudak et al. [5]. For $S \subseteq V(G)$, denote by $E(S)$ the set of the edges of $G$ having both ends in $S$, by $G[S]$ the subgraph of $G$ induced by $S$, and by $d_S(v)$ the degree of $v$ in $G[S]$. Then, the following is a formulation for the feedback vertex set problem:

\[
\begin{align*}
\text{Min} & \quad \sum_{v \in V(G)} c_v x_v \\
\text{s.t.} & \quad \sum_{v \in S} (d_S(v) - 1) x_v \geq |E(S)| - |S| + 1 \quad \forall S \subseteq V(G) : E(S) \neq \emptyset \\
& \quad x_v \in \{0, 1\} \quad \forall v \in V(G).
\end{align*}
\]

\(^2\)The integrality gap of an integer programming formulation is the worst-case ratio between the optimum value of the integer program and the optimum value of its linear relaxation.
Chudak et al. [5] showed that the integrality gap of this integer program asymptotically equals 2. Constraints (2) derive from the simple observation that the removal of a feedback vertex set $X$ from $G$ generates a forest having at most $|G| - |X| - 1$ edges. Notice that the covering inequalities (1) are implied by (2).

1.2. Contribution and Key Ideas. First, we obtain a $O(\log n)$-approximation algorithm for the diamond hitting set problem, leading to a proof that the integrality gap of the natural LP formulation is $\Theta(\log n)$. Then, we develop a 9-approximation algorithm. Both the $O(\log n)$- and 9-approximation algorithm are based on the primal-dual method.

Our first key idea is contained in the following observation: every simple graph of order $n$ and minimum degree at least 3 contains a $O(\log n)$-size diamond. This directly yields a $O(\log n)$-approximation algorithm for the diamond hitting set problem, in the unweighted case. However, the weighted case requires more work.

Our second key idea is to generalize constraints (2) by introducing ‘sparsity inequalities’, that enable us to derive a constant-factor approximation algorithm for the diamond hitting set problem: First, by using reduction operations, we ensure that every vertex of $G$ has at least three neighbors. Then, if $G$ contains a diamond with at most 9 edges, we raise the dual variable of the corresponding covering constraint. Otherwise, no such small diamond exists in $G$, and we can use this information to select the right sparsity inequality, and raise its dual variable. This inequality would not be valid in case $G$ contained a small diamond.

The way we use the non-existence of small diamonds is perhaps best explained via an analogy with planar graphs: An $n$-vertex planar simple graph $G$ has at most $3n - 6$ edges. However, if we know that $G$ has no small cycle, then this upper bound can be much strengthened. (For instance, if $G$ is triangle-free then $G$ has at most $2n - 4$ edges.)

We remark that this kind of local/global trade-off did not appear in the work of Chudak et al. [5] on the feedback vertex set problem, because the cycle covering inequalities are implied by their more general inequalities. In our case, the covering inequalities and the sparsity inequalities form two incomparable classes of inequalities, and examples show that the sparsity inequalities alone are not enough to derive a constant-factor approximation algorithm.

The paper is organized as follows. Preliminaries are given in Section 2. Then, in Section 3, we define some reduction operations that allow us to obtain graphs having some desirable properties. Next, in Section 4, we deal with the unweighted version of the diamond hitting set problem and provide a simple $O(\log n)$-approximation algorithm. In Section 5, we turn to the weighted version of the problem. We present a $O(\log n)$-approximation algorithm and we prove that the integrality gap of the natural formulation of the problem is $\Theta(\log n)$. Finally, in Section 6, we introduce the sparsity inequalities, analyze their strength and obtain a 9-approximation algorithm.
2. Preliminaries

We refer the reader to Diestel [6] for undefined terms and notations concerning graphs. A cactus is a connected graph where each edge belongs to at most one cycle. Equivalently, a connected graph is a cactus if and only if each of its blocks is isomorphic to either $K_1$, $K_2$, or a cycle. Thus, a connected graph is a cactus if and only if it does not contain a diamond as a subgraph. A graph without diamonds is called a forest of cacti (see Figure 1 for an illustration).

A diamond hitting set, or simply hitting set, of a graph is a subset of vertices that hits every diamond of the graph. A minimum hitting set of a weighted graph is a hitting set of minimum total cost, and its cost is denoted by $OPT$.

Let $\mathcal{D}(G)$ denote the collection of all diamonds contained in $G$. From the standard IP formulation for a covering problem, we obtain the following LP relaxation for the diamond hitting set problem:

$$\text{Min} \sum_{v \in V(G)} c_v x_v$$

s.t. $\sum_{v \in V(D)} x_v \geq 1 \quad \forall D \in \mathcal{D}(G)$

$$x_v \geq 0 \quad \forall v \in V(G).$$

We call inequalities (3) diamond inequalities.

3. Reductions

In this section, we define two reduction operations on graphs: First, we define the ‘shaving’ of an arbitrary graph, and then introduce a ‘bond reduction’ operation for shaved graphs.

The aim of these two operations is to modify a given graph so that the following useful property holds: each vertex either has at least three distinct neighbors, or is incident to at least three parallel edges.

3.1. Shaving a graph. Let $G$ be a graph. Every block of $G$ is either isomorphic to $K_1$, $K_2$, a cycle, or contains a diamond. Mark every vertex of $G$ that is included in a block containing a diamond. The shaving of $G$ is the graph obtained by removing every unmarked vertex from $G$. A graph is shaved if all its vertices belong to a block containing
a diamond. Observe that, in particular, every endblock$^3$ of a shaved graph contains a diamond. See Figure 2 for an illustration.

![Figure 2](image.png)

**Figure 2.** (a) A graph $G$. (b) The graph obtained by shaving $G$.

### 3.2. Reducing a bond.

A *bond* of a graph $G$ is a connected subgraph $Q \subseteq G$ equipped with two distinguished vertices $v, w$ (called *ends*) satisfying the following requirements:

- $Q$ is a cactus with at least two blocks;
- the block-graph of $Q$ is a path;
- $v$ and $w$ belong to distinct endblocks of $Q$;
- $v$ and $w$ are not adjacent in $Q$;
- $Q - \{v, w\}$ is a non-empty component of $G - \{v, w\}$, and
- $Q$ contains all the edges in $G$ between $\{v, w\}$ and $V(Q) - \{v, w\}$.

Observe that $Q$ is “almost” an induced subgraph of $G$, since $Q$ includes every edge of $G$ between vertices of $Q$, except those between $v$ and $w$ (if any). The vertices in $V(Q) - \{v, w\}$ are said to be the *internal vertices* of $Q$. The bond $Q$ is *simple* if $Q$ is isomorphic to a path, *double* otherwise.

Let $G$ be a shaved graph. A vertex $u$ of $G$ is *reducible* if $u$ has exactly two neighbors in $G$, and there are at most two parallel edges connecting $u$ to each of its neighbors. The *bond reduction* operation is defined as follows. Let $u$ be a reducible vertex and let $Q_u$ be an inclusion-wise maximal bond of $G$ containing $u$, with ends $v$ and $w$. (Observe that such a bond exists by our hypothesis on $u$; moreover, it might not be unique.) Then, remove from $G$ every internal vertex of $Q_u$, and add one or two edges between $v$ and $w$, depending on whether $Q_u$ is simple or double. In the latter case, the two new parallel edges are said to be *twins*. See Figure 3 for an illustration of the operation. Observe that the resulting graph is also a shaved graph.

A crucial property of the bond reduction operation is that, when applying it iteratively, we never include in the bond to be reduced any edge coming from previous bond reductions. This is proved in the following lemma.

---

$^3$We recall that the block-graph of $G$ has the blocks of $G$ and the cutvertices of $G$ as vertices, a block and a cutvertex are adjacent if the former contains the latter. This graph is always acyclic. An endblock of $G$ is a vertex of the block-graph with degree at most one.
Figure 3. (a) A shaved graph $G$ with two maximal bonds (in grey). (b) Reduction of the first bond. (c) Reduction of the second bond. The graph is now reduced.

Lemma 3.1. Let $G_i$ be a graph obtained from $G$ after applying $i$ bond reductions. Let $E_i := E(G_i) - E(G)$. Let $v$ be a reducible vertex of $G_i$. Let $Q_v$ be a maximal bond of $G_i$ containing $v$. Then, $E(Q_v) \cap E_i = \emptyset$.

Proof. Arguing by contradiction, assume $E(Q_v) \cap E_i \neq \emptyset$. Let $j$ be the maximum index such that $j < i$ and $Q_v$ contains an edge $e$ produced during the $j$th bond reduction. Let $w$ be the vertex that has been reduced at iteration $j$, and denote by $Q_w$ the corresponding bond. Note that, if $Q_w$ is double, then $e$ and its twin edge $e'$ are both included in $Q_v$ (by construction). Now, replacing $e$ (and $e'$ if it exists) by $Q_w$ in $Q_v$ produces a bond of $G_j$ that includes $w$ and is strictly larger than $Q_w$. This contradicts the maximality of $Q_w$. □

A reduced graph $\tilde{G}$ of $G$ is any graph obtained from $G$ by iteratively applying a bond reduction, as long as there is a reducible vertex (see Figure 3). We remark that there is not necessarily a unique reduced graph of $G$ (consider for instance $K_3$ where two edges are doubled).

As mentioned earlier, every reduced graph has the following desirable property: every vertex either has at least three distinct neighbors or is incident to at least three parallel edges.

4. A $O(\log n)$-approximation algorithm in the unweighted case

In this section, we deal with the unweighted version of the diamond hitting set problem. We first show that every reduced graph with $n$ vertices contains a diamond of size $O(\log n)$, and then provide a $O(\log n)$-approximation algorithm.

4.1. Small diamonds in reduced graphs. As a first step, we show that every simple graph with minimum degree at least 3 contains a diamond of size $O(\log n)$.

Lemma 4.1. Every simple $n$-vertex graph with minimum degree at least 3 contains a diamond of size at most $6\log_3/2 n + 8$. Moreover, such a diamond can be found in polynomial time.
Hence, \( N \) least two neighbors in \( m \) other hand, we also have \( N \) in \( w \) The number of edges between \( N \) white. Since \( G \) respectively; so, \( q \) and \( p \) respectively. By Claim 4.2, there is no red vertex in \( C \) contains \( D \). Let \( P_i \) \( (i = 1, 2, 3) \) be the \( u-v_i \) path in \( T \) augmented with the edge \( v v_i \). Observe that \( C_2 \cup C_3 \subseteq P_1 \cup P_2 \cup P_3 \). Therefore, the size of \( D \) is
\[
||D|| \leq ||C_2 \cup C_3|| \leq ||P_1|| + ||P_2|| + ||P_3|| \leq d + (d + 1) + (d + 1) = 3d + 2.
\]

\[ \square \]

Claim 4.3. Let \( d \geq 2 \). If \( G \) has no diamond of size at most \( 3d + 2 \), then \( |N_d| \geq \frac{3}{2}|N_{d-2}| \).

Proof. Assume that \( G \) has no diamond of size at most \( 3d + 2 \). The claim is easily seen to hold if \( d = 2 \), so we assume \( d \geq 3 \).

For \( i \in \{1, \ldots, d\} \), let \( w_i \) and \( b_i \) be the number of white and black vertices in \( N_i \), respectively. By Claim 4.2, there is no red vertex in \( N_i \); thus, \( |N_i| = w_i + b_i \). Let \( p_i \) and \( q_i \) be the number of black vertices in \( N_i \) having one and two neighbors in \( N_{i-1} \), respectively; so, \( b_i = p_i + q_i \).

Now, fix \( i \in \{2, \ldots, d\} \). (Thus, vertices in both \( N_i \) and \( N_{i-1} \) are colored in black and white.) Since \( G \) has minimum degree at least \( 3 \), every vertex in \( N_{i-1} \) has at least one neighbor in \( N_i \). By Claim 4.2, every black vertex in \( N_{i-1} \) has at least one white neighbor in \( N_i \), showing
\[
(4) \quad w_i \geq b_{i-1}.
\]

The number of edges between \( N_{i-1} \) and \( N_i \) in \( G \) is exactly \( w_i + p_i + 2q_i =: m_i \). On the other hand, we also have \( m_i \geq 2w_{i-1} + b_{i-1} \), since every white vertex in \( N_{i-1} \) has at least two neighbors in \( N_i \) and every black vertex in \( N_{i-1} \) has at least one neighbor in \( N_i \). Hence,
\[
(5) \quad w_i + p_i + 2q_i \geq 2w_{i-1} + b_{i-1}.
\]
By Claim 4.2, if a white vertex in $N_{i-1}$ is adjacent to two black vertices in $N_i$, then the latter vertices are adjacent. Also, black vertices in $N_i$ can only be adjacent to white vertices in $N_{i-1}$, again by Claim 4.2.

These two observations imply

\[(6) \quad \frac{1}{2} w_{i-1} \geq q_i.\]

Using Eq. (5) and (6), we obtain

\[|N_i| = w_i + p_i + q_i \geq 2w_{i-1} + b_{i-1} - q_i \geq \frac{3}{2} w_{i-1} + b_{i-1}.\]

It follows

\[|N_d| \geq \frac{3}{2} w_{d-1} + b_{d-1}\]
\[= |N_{d-1}| + \frac{1}{2} w_{d-1}\]
\[\geq \frac{3}{2} w_{d-2} + b_{d-2} + \frac{1}{2} w_{d-1}\]
\[= |N_{d-2}| + \frac{1}{2} w_{d-2} + \frac{1}{2} w_{d-1}\]
\[\geq |N_{d-2}| + \frac{1}{2} w_{d-2} + \frac{1}{2} b_{d-2}\]
\[= \frac{3}{2} |N_{d-2}|,\]

as claimed. (The last inequality follows from Eq. (4).) \[\square\]

Now, we may prove Lemma 4.1. Let $d$ be the largest even integer such that $G$ has no diamond of size at most $3d + 2$. If $d \leq 2$, then $G$ has a diamond with size at most $3(d + 2) + 2 \leq 14 \leq 6 \log_{3/2} n + 8$ (since $n \geq 4$). Thus, assume $d \geq 4$. By Claim 4.3,

\[n \geq \sum_{i=0}^{d} |N_i|\]
\[\geq |N_0| + |N_2| + \cdots + |N_{d-2}| + |N_d|\]
\[\geq 1 + \frac{3}{2} + \cdots + \left(\frac{3}{2}\right)^{d/2-1} + \left(\frac{3}{2}\right)^{d/2}\]
\[\geq \left(\frac{3}{2}\right)^{d/2},\]

that is,

\[d \leq 2 \log_{3/2} n.\]

Therefore, $G$ has a diamond of size at most $3(d + 2) + 2 \leq 6 \log_{3/2} n + 8$.

To conclude, we note that the above proof is easily turned into a polynomial-time algorithm finding the desired diamond. \[\square\]

The same result holds for reduced graphs:

**Lemma 4.4.** Every reduced graph $\tilde{G}$ contains a diamond of size at most $6 \log_{3/2} |\tilde{G}| + 8$. 
Proof. We can assume that any two adjacent vertices of $\tilde{G}$ are adjacent through at most 2 parallel edges, since otherwise there is a diamond with three edges and the statement trivially holds. Thus, each vertex of $\tilde{G}$ has at least three distinct neighbors. Let $\tilde{G}'$ be the subgraph of $\tilde{G}$ obtained by replacing every double edge by a simple edge. Clearly, $\tilde{G}'$ is simple and has minimum degree at least 3. Therefore, by Lemma 4.1, $\tilde{G}'$ has a diamond (which can be found in polynomial time) of size at most $6\log_{3/2}|\tilde{G}'| + 8$, thus at most $6\log_{3/2}|\tilde{G}| + 8$. The result follows. \qed

4.2. The algorithm. Our algorithm for the diamond hitting set problem on unweighted graphs is described in Algorithm 1.

Algorithm 1 A $O(\log n)$-approximation algorithm for unweighted graphs.

- $X \leftarrow \emptyset$
- While $X$ is not a hitting set of $G$, repeat the following steps:
  - Compute a reduced graph $\tilde{G}$ of $G - X$
  - Find a diamond $\tilde{D}$ in $\tilde{G}$ of size at most $6\log_{3/2}|\tilde{G}| + 8$ (using Lemma 4.4)
  - Include in $X$ all vertices of $\tilde{D}$

The algorithm relies on the simple fact that every hitting set of a reduced graph $\tilde{G}$ of a graph $G$ is also a hitting set of $G$ itself. The set of diamonds computed by the algorithm yields a collection $\mathcal{D}$ of pairwise vertex-disjoint diamonds in $G$. In particular, the size of a minimum hitting set is at least $|\mathcal{D}|$. For each diamond in $\mathcal{D}$, at most $6\log_{3/2} n + 8$ vertices were added to the hitting set $X$. Hence, the approximation factor of the algorithm is $O(\log n)$.

5. A $O(\log n)$-approximation algorithm

The present section is devoted to a $O(\log n)$-approximation algorithm for the diamond hitting set problem in the weighted case, which is based on the primal-dual method. We start by defining, in Section 5.1, the actual LP relaxation of the problem used by the algorithm, together with its dual. Then, in Section 5.2, we describe the approximation algorithm and, in Section 5.3, we prove that it provides a $O(\log n)$-approximation for the diamond hitting set problem. Finally, in Section 5.4, we show that the integrality gap of the natural LP relaxation for the problem (see (3), page 4) is $\Theta(\log n)$. This last result is obtained using expander graphs with large girth.

5.1. The working LP and its dual. Our approximation algorithm is based on the natural LP relaxation for the diamond hitting set problem, given on page 4. To simplify the presentation, we do not directly resort to that LP relaxation but to a possibly weaker relaxation that is constructed during the execution of the algorithm, that we call the working LP. At each iteration, an inequality is added to the working LP. These inequalities, that we name blended diamond inequalities, are all implied by diamond inequalities
The final working LP reads:

\[
\text{(LP)} \quad \text{Min} \quad \sum_{v \in V(G)} c_v x_v \\
\text{s.t.} \quad \sum_{v \in V(G)} a_{i,v} x_v \geq \beta_i \quad \forall i \in \{1, \ldots, k\} \\
x_v \geq 0 \quad \forall v \in V(G),
\]

where \(k\) is the total number of iterations of the algorithm. The dual of (LP) is:

\[
\text{(D)} \quad \text{Max} \quad \sum_{i=1}^{k} \beta_i y_i \\
\text{s.t.} \quad \sum_{i=1}^{k} a_{i,v} y_i \leq c_v \quad \forall v \in V(G) \\
y_i \geq 0 \quad \forall i \in \{1, \ldots, k\}.
\]

The algorithm is based on the primal-dual method. It maintains a boolean primal solution \(x\) and a feasible dual solution \(y\). Initially, all variables are set to 0. Then the algorithm enters its main loop, that ends when \(x\) satisfies all diamond inequalities. At the \(i\)th iteration, a violated inequality \(\sum_{v \in V} a_{i,v} x_v \geq \beta_i\) is added to the working LP and the corresponding dual variable \(y_i\) is increased. In order to preserve the feasibility of the dual solution, we stop increasing \(y_i\) whenever some dual inequality becomes tight. That is, we stop increasing when \(\sum_{j=1}^{i} a_{j,v} y_j = c_v\) for some vertex \(v\), that is said to be tight. Furthermore, we also stop increasing \(y_i\) in case a ‘collision’ occurs (see Section 5.2.4). All tight vertices \(v\) (if any) are then added to the primal solution. That is, the corresponding variables \(x_v\) are increased from 0 to 1. The current iteration then ends and we check whether \(x\) satisfies all diamond inequalities. If so, then we exit the loop, perform a reverse delete step, and output the current primal solution.

The precise way the violated blended diamond inequality is chosen is defined in Sections 5.2.1 and 5.2.3. It depends among other things on the residual cost (or slack) of the vertices. The residual cost of vertex \(v\) at the \(i\)th iteration is the number \(c_v - \sum_{j=1}^{i-1} a_{j,v} y_j\). Note that the residual cost of a vertex is always nonnegative, and zero if and only if the vertex is tight.

5.2. The algorithm. A formal definition of the algorithm is given in Algorithm 2. All the steps are explicit, except those labeled ⋆, which will be specified later.

Above, \(L\) is a collection of triples \((T, B, \{v, w\})\) used to guide the choice of subgraph \(S\), where \(T\) and \(B\) are internally disjoint \(v\)–\(w\) paths (see Section 5.2.2).

We remark that the set \(X\) naturally corresponds to a primal solution \(x\), obtained by setting \(x_v\) to 1 if \(v \in X\), to 0 otherwise, for every \(v \in V(G)\). This vector \(x\) satisfies the diamond inequalities (3) exactly when we exit the while loop of the algorithm, that is, when \(X\) becomes a hitting set.

The reverse delete step consists in considering the vertices of \(X\) in the reverse order in which they were added to \(X\) and deleting those vertices \(v\) such that \(X - \{v\}\) is still a
HITTING DIAMONDS AND GROWING CACTI

Algorithm 2 A $O(\log n)$-approximation algorithm for weighted graphs.

- $X \leftarrow \emptyset$; $y \leftarrow 0$; $i \leftarrow 0$; $\mathcal{L} \leftarrow \emptyset$

- While $X$ is not a hitting set of $G = (V,E)$, repeat the following steps:
  - $i \leftarrow i + 1$
  - Let $H$ be the graph obtained by shaving $G - X$
  - Find a reduced graph $\tilde{H}$ of $H$
  - Find a diamond $\tilde{D}$ in $\tilde{H}$ of size $\leq 6 \log 3/2 |\tilde{H}| + 8$ (using Lemma 4.4)
  - Find an induced subgraph $S$ of $H$, based on $\tilde{D}$ (see Section 5.2.2)
  - If $S$ is not consistent with $\mathcal{L}$, modify $S$ (see Section 5.2.1)
  - Compute a violated blended diamond inequality $\sum_{v \in V} a_{i,v} x_v \geq \beta_i$, based on $S$, $\mathcal{L}$ and the residual costs; add it to (LP) (see Section 5.2.3)
  - Increase $y_i$ until some vertex becomes tight, or a collision occurs (see Section 5.2.4)
  - Update $\mathcal{L}$ (see Section 5.2.5)
  - Add all tight vertices $v$ to $X$, in a certain order (see Section 5.2.6)
  - Re-update $\mathcal{L}$ (see Section 5.2.7)

- $k \leftarrow i$

- Perform a reverse delete step on $X$

hitting set. Observe that, because of this step, the hitting set $X$ output by the algorithm is inclusion-wise minimal.

The remainder of this section is organized as follows. In Section 5.2.1, we define the ‘support graph’ $S$ of the inequalities and then explain, in Section 5.2.2, how to modify it when it is not consistent with $\mathcal{L}$. After that, we define the blended diamond inequalities in Section 5.2.3, we define collisions and explain how to take care of them in Section 5.2.4, we specify the insertion order of vertices in the solution in Section 5.2.6, and we explain the way list $\mathcal{L}$ is updated in Sections 5.2.5 and 5.2.7.

5.2.1. The support graph of the inequalities. First, we need some definitions. Let $H$ be a shaved graph and let $\tilde{H}$ be a reduced graph of $H$ (as defined in Section 3). Vertices in $V(\tilde{H})$ and $V(H) - V(\tilde{H})$ are called branch vertices and internal vertices of $H$, respectively.

We will consider graphs that are equipped with a collection of specific subgraphs, called ‘pieces’, which have a structure similar to that of bonds. The first kind of such graphs are simply diamonds: Consider a diamond $D$, and replace each edge $vw \in E(D)$ with a path between $v$ and $w$. Each such path is a piece of the resulting diamond; the vertices $v,w$ are the ends of the piece, the others are the internal vertices of the piece.

A second kind of graph equipped with pieces is a necklace, defined as any graph obtained as follows. First, choose a cycle $C$ (cycles consisting of two parallel edges are allowed). Then, select a non-empty subset $Z$ of edges of $C$. Next, replace each edge $vw \in E(C) - Z$ of $C$ with a path between $v$ and $w$. Finally, replace each edge $vw \in Z$ either by two internally disjoint $v$–$w$ paths, or by a cactus whose block-graph is a path, in such a way that $v$ and $w$ lie in distinct endblocks of the cactus and in no other block.
In both cases, the subgraph by which an edge of $C$ has been replaced is called a piece of the necklace; ends and internal vertices of the piece are defined as expected. See Figure 4 for an illustration.

A piece $Q$ of a diamond or necklace is simple if $Q$ is isomorphic to a path, double otherwise. Let us point out that, while pieces and bonds look very similar at first sight, there are some differences between the two notions. (Notice for instance that a piece of a necklace could consist of a single edge or a cycle.)

A diamond or necklace $S$ together with its pieces is rooted in $H$ if

- $S$ is an induced subgraph of $H$;
- every end of a piece of $S$ is a branch vertex of $H$;
- every internal vertex of a piece of $S$ is an internal vertex of $H$, and
- there is no edge in $H$ between an internal vertex of a piece of $S$ and a vertex in $V(H) - V(S)$.

(Observe that this definition also depends on $\tilde{H}$, since the latter graph determines which vertices of $H$ are branch vertices.)

![Figure 4. A cycle $C$, where edges in $Z$ are thicker (left) and a necklace built from $C$ (right).](image)

Lemma 5.1. Let $H$ be a shaved graph. Let $\tilde{H}$ be a reduced graph of $H$. Then, given a diamond $\tilde{D} \subseteq \tilde{H}$, one can find in polynomial time a subgraph $S$ of $H$ such that one of the following conditions is satisfied:

(i) $S$ is a diamond rooted in $H$, with at most $|\tilde{D}|$ pieces;
(ii) $S$ is a necklace rooted in $H$, with at most $|\tilde{D}|$ pieces;
(iii) $S$ consists of two vertices with at least four parallel edges between them;
(iv) $S$ is isomorphic to $K_4$.

Proof. First, we associate to each edge of $\tilde{H}$ a corresponding primitive subgraph in $H$, defined as follows. Consider an edge $e \in E(\tilde{H})$. If $e$ was already present in $H$, then its primitive subgraph is the edge itself and its two ends. Otherwise, the primitive subgraph of $e$ is the bond whose reduction produced $e$. In particular, if $e$ has a twin edge $e'$, then the primitive subgraphs of $e$ and $e'$ coincide. The primitive subgraph $J$ of a subgraph $\tilde{J} \subseteq \tilde{H}$ is defined simply as the union of the primitive subgraphs of every edge in $E(\tilde{J})$. Note that primitive subgraphs are well-defined, thanks to Lemma 3.1. Also, notice that the primitive subgraph of a subgraph of $\tilde{H}$ is not defined per se, but with respect to the bond reductions which produced $\tilde{H}$ from $H$.
Let $K$ denote the subgraph of $\tilde{H}$ induced by $V(\tilde{D})$. Consider an induced subgraph $K'$ of $K$ that contains a diamond and is vertex-minimal with that property, that is, $K' - v$ is a forest of cacti for every $v \in V(K')$. Let $\mu$ be the maximum number of parallel edges between pairs of adjacent vertices in $K'$.

First, suppose $\mu = 1$. As the reader will easily check, the minimality of $K'$ implies that either $K'$ is a simple diamond, or $K'$ is isomorphic to $K_4$.

In the first case ($K'$ is a simple diamond), the primitive subgraph $S$ of $K'$ in $H$ can be seen as a diamond with $||K'||$ pieces. It follows from the definition of primitive subgraphs and the fact that $H$ is shaved, that $S$ is rooted in $H$. Also, we trivially have $||K'|| \leq ||\tilde{D}||$; thus, $S$ has at most $||\tilde{D}||$ pieces. Hence, $S$ satisfies (i).

In the second case ($K' \simeq K_4$), we may assume that the primitive subgraph of $K'$ is not isomorphic to $K'$ (otherwise, (iv) holds). Then, there is an edge $e \in E(K')$ such that the primitive subgraph of $e$ is a path of length at least 2. Let then $S$ be the primitive subgraph of $K' - e$. The subgraph $S$ is induced in $H$ and is a diamond with $||K' - e||$ pieces. Since, similarly as before, $S$ is rooted in $H$ and $||K' - e|| \leq ||\tilde{D}||$, it follows that $S$ satisfies (i).

Next, assume $\mu = 2$. Let $v, w$ be two vertices of $K'$ that are connected by two parallel edges $e_1, e_2$ in $K'$. Let $P$ be a shortest $v$-$w$ path in the graph $K' - \{e_1, e_2\}$. (Observe that such a path exists, since $K'$ contains a spanning diamond.) Then, all vertices of $K'$ are included in $P$. Thus, $K'$ is a simple cycle with some (and at least one) of its edges replaced by pairs of parallel edges. Let $S$ be the primitive subgraph of $K'$ in $H$. Then, $S$ is a necklace rooted in $H$ and has at most $|K'| \leq ||K'|| \leq ||\tilde{D}||$ pieces. Hence, $S$ satisfies (i).

Now, suppose $\mu \geq 3$. Then, $K'$ consists of two vertices $v$ and $w$ connected by $\mu$ parallel edges. If $H$ contains a pair of vertices with at least four parallel edges between them, then (iii) holds for the obvious choice of $S$, and we are done. Thus, assume there are at most three edges between any two vertices in $H$.

Let $S$ be the primitive subgraph of $K'$. If $K'$ contains no pair of twin edges, then $S$ consists of at most three edges between $v$ and $w$ and at least $\mu - 3$ longer paths between $v$ and $w$. By removing vertices from $S$ if necessary, we can ensure that $S$ satisfies (i). Otherwise, $K'$ contains at least one pair of twin edges. In this case, by removing vertices from $S$ if necessary, we can ensure that $S$ satisfies (ii), or (i) if $S$ has three parallel edges between $v$ and $w$.

Finally, we note that the above proof can be turned without difficulty into a polynomial-time algorithm computing $S$, thus concluding the proof.

We say that an induced subgraph $S \subseteq H$ is of type 1 (type 2, 3, 4, resp.) if it satisfies condition (i) (condition (ii), (iii), (iv), resp.) of the above lemma.

5.2.2. **Modifying the graph $S$.** Consider some iteration $i$ of the algorithm, the corresponding shaved graph $H$ and the induced subgraph $S$ of $H$ produced at that iteration. Here, we explain how to modify $S$ when it is not consistent with $\mathcal{L}$. If $S$ is of type 3 or 4, there
is no need to modify it. Hence, we assume that $S$ is of type 1 or 2 for the rest of this section. First, we need to introduce more terminology.

Consider a piece $Q$ of $S$ containing a cycle $C$. Then $C$ is a block of $Q$. A vertex $v$ of $C$ is said to be an end of the cycle $C$ if $v$ is an end of the piece $Q$ or $v$ belongs to a block of $Q$ distinct from $C$. Observe that $C$ has always two distinct ends. The cycle $C$ has also two handles, defined as the two $v$–$w$ paths in $C$, where $v$ and $w$ are the two ends of $C$.

A handle is trivial if it has no internal vertex, non-trivial otherwise.

The two handles of $C$ are labelled top and bottom as follows. Suppose $C$ has two non-trivial handles, and compute the minimum residual cost of an internal vertex in each handle. If this minimum is achieved in exactly one handle, then this handle is the top handle. If, on the other hand, the minimum is achieved in both, then the tie is broken arbitrarily, unless the cycle $C$ was considered in a previous iteration. In this case, we ensure that the tie is always broken in the same way (actually, we can use the list $L$ to determine this, see below). Now, if $C$ has only one non-trivial handle, then it is defined to be the top handle. Finally, if both handles of $C$ are trivial (that is, $C$ is a cycle of length 2), then one of them is chosen arbitrarily and called the top handle. In each of these three cases, the bottom handle is the other handle, as expected.

Now, we may give a precise definition of $L$: it is a collection of triples $(T,B,\{v,w\})$ satisfying all of the following conditions:

- $v,w$ are two distinct vertices of $H$,
- $T$ and $B$ are two $v$–$w$ paths in $H$ that are internally disjoint,
- $T$ has at least one internal vertex,
- internal vertices of $T$ and $B$ have exactly two neighbors in $H$.

Moreover, we require that

- for every two distinct triples $(T,B,\{v,w\}),(T',B',\{v',w'\})$ in $L$, no two of the four paths $T,T',B,B'$ have an internal vertex in common.

The graph $S$ is consistent with a triple $(T,B,\{v,w\})$ in $L$ if any of the following conditions is satisfied:

(C1) $S$ contains no internal vertex of any of the paths $T$, $B$,
(C2) $S$ contains $T$ in one of its simple pieces and no internal vertex of $B$,
(C3) $S$ is of type 1 and contains the cycle $T \cup B$,
(C4) $S$ is of type 2 and contains $T \cup B$ in one of its (double) pieces.

We say that $S$ is consistent with the collection $L$ if $S$ is consistent with every triple in $L$.

Next, we explain how to modify $S$ if it is not consistent with $L$. Roughly speaking, the purpose of this step is to ensure that no vertex in a bottom handle becomes tight before some vertex in the corresponding top handle becomes tight. This property is crucial for our analysis of the algorithm.

We iteratively modify $S$, until it becomes consistent with $L$: Let $(T,B,\{v,w\}) \in L$ be such that $S$ is not consistent with $(T,B,\{v,w\})$. (Recall that $S$ is of type 1 or 2, by assumption.) Because (C1) is not satisfied, $S$ contains an internal vertex of $T$ or $B$. 
Claim 5.2. If $P$ is any path in $\{T, B\}$ such that one of its internal vertices is contained in $S$, then the whole path $P$ is contained in $S$.

Proof. Let $u$ be an internal vertex of $P$ contained in $S$. Because $u$ is an internal vertex of $P$, it has exactly two neighbors in $H$. Because $u$ is a vertex of $S$ and $S$ is a diamond or a necklace, $u$ has at least two neighbors in $S$. Thus, the two neighbors of $u$ in $H$ are in $S$. By repeating this argument, it follows that the whole path $P$ is contained in $S$. □

By what precedes, we may assume that some path $P \in \{T, B\}$ has at least one internal vertex in $S$. By Claim 5.2, $P$ is entirely contained in $S$. Because every end of a piece of $S$ is a branch vertex, all vertices of $P$ are in the same piece of $S$, say $Q$. Let $P'$ denote the other path in $\{T, B\}$.

Claim 5.3. Assume that the cycle $P \cup P' = T \cup B$ is entirely contained in $S$. Then, $S$ is a necklace with two pieces, one piece is a cycle with ends $v$ and $w$, and the other piece is either a $v$–$w$ path or also a cycle with ends $v$ and $w$.

Proof. Because $S$ is not consistent with $(T, B, \{v, w\})$, neither (C3) nor (C4) is satisfied. It follows that $S$ is a necklace, and the $v$–$w$ paths $P$ and $P'$ are contained in different pieces of $S$. In particular, $S$ has two distinct pieces sharing two distinct vertices, namely $v$ and $w$. This implies that $S$ has exactly two pieces, and $v$ and $w$ are the ends of both pieces. One of the pieces contains $P$ (namely, $Q$) and the other contains $P'$. The rest of the claim follows easily. □

Case 1. No internal vertex of $P'$ is contained in $S$.

First, assume that the path $P'$ has no internal vertex. Then, the whole cycle $P \cup P'$ is contained in $S$ because $S$ is induced in $H$. By Claim 5.3, we can transform $S$ into a diamond, by deleting a (possibly empty) subset of the vertices of $S$.

Second, assume that $P'$ has at least one internal vertex. Because $v$ and $w$ are vertices of $S$ that have at least one neighbor outside $S$, they are branch vertices. Thus, $v$ and $w$ are the ends of the piece $Q$.

If $Q$ is simple, then we have $P = B$ and $P' = T$ because otherwise $S$ would be consistent with $(T, B, \{v, w\})$. In this case, we redefine $S$ as the subgraph of $H$ induced by $(V(S) - V(B)) \cup V(T)$. (Thus, we “replace” $B$ with $T$ in $S$.)

If $Q$ is double, then $Q$ is a cycle with ends $v$ and $w$, and $S$ is a necklace. Then, we transform $S$ into a diamond, by redefining $S$ as $Q \cup P'$.

Case 2. Some internal vertex of $P'$ is contained in $S$.

By Claim 5.2 it follows that all vertices of $P'$ are contained in $S$. Hence, $S$ contains the whole cycle $P \cup P'$. Again, by Claim 5.3, we can transform $S$ into a diamond, by deleting a subset of the vertices of $S$.

In all the cases above, we either transform $S$ into a diamond, or make $S$ consistent with $(T, B, \{v, w\})$ without creating a new inconsistency. It follows that the modification process is finite, and produces a new induced subgraph $S$ still satisfying the requirements.
of Lemma \[5.1\] that is moreover consistent with all triples in \(\mathcal{L}\). Clearly, modifying \(S\) can be done in polynomial time.

5.2.3. Computing the inequality. Let \(C\) be a cycle contained in a piece \(Q\) of \(S\). If at least one handle of \(C\) is non-trivial, then we denote by \(t(C)\) (resp. \(b(C)\)) the minimum residual cost of an internal vertex in the top handle (resp. bottom handle) of \(C\). The convention is that \(b(C) = \infty\) if the bottom handle is trivial. It will be convenient to say that a vertex belongs to a handle if it is an internal vertex of that handle.

To the graph \(S\) we associate a unique blended diamond inequality of the form

\[
\sum_{v \in V(G)} a_{i,v} x_v \geq 1,
\]

whose support is contained in the vertex set of \(S\). For convenience, we call \(S\) the support graph of the inequality. If \(S\) is of type 1, 3, or 4, we let

\[
a_{i,v} := \begin{cases} 
1 & \text{if } v \in V(S), \\
0 & \text{if } v \in V(G) - V(S).
\end{cases}
\]

If \(S\) is of type 2, we choose a cycle of \(S\) and declare it to be special, as follows: If there is a cycle \(C\) in a piece of \(S\) such that \(t(C) = b(C)\), then we select such a cycle. Otherwise, we select an arbitrary cycle contained in a piece of \(S\). Then, we let

\[
a_{i,v} := \begin{cases} 
1 & \text{if } v \in V(S) \text{ and } v \text{ belongs to no handle}, \\
0 & \text{if } v \text{ belongs to the top handle of a non-special cycle } C \text{ and } t(C) < b(C), \\
1 & \text{if } v \text{ belongs to the bottom handle of a non-special cycle } C \text{ and } t(C) < b(C), \\
\frac{1}{2} & \text{if } v \text{ belongs to a handle of a non-special cycle } C \text{ and } t(C) = b(C), \\
1 & \text{if } v \text{ belongs to a handle of the special cycle}, \\
0 & \text{if } v \in V(G) - V(S).
\end{cases}
\]

**Lemma 5.4.** Every blended diamond inequality \[7\] is implied by the diamond and non-negativity inequalities.

**Proof.** If \(S\) is of type 1, 3, or 4, Inequality \[7\] is a diamond inequality. Otherwise, \[7\] is easily seen to be a convex combination of two diamond inequalities. \(\Box\)

5.2.4. Taking care of collisions. When increasing the variable \(y_i\), the residual cost \(c_v - \sum_{j=1}^i a_{j,v} y_j\) of every vertex \(v\) in \(S\) decreases, at a speed given by the coefficient \(a_{i,v}\). Also, for every cycle \(C\) included in a piece of \(S\), we have that \(t(C)\) and \(b(C)\) decrease, possibly at different speeds. We could simply increase \(y_i\) until some vertex \(v\) becomes tight (that is, until its residual cost drops to 0). However, by doing so, it could be that \(t(C) \leq b(C)\) no longer holds for some piece \(Q\) and some cycle \(C\) in \(Q\). (For instance, this would happen if \(b(C)\) decreases much faster than \(t(C)\).) We will need that \(t(C) \leq b(C)\) remains true in future iterations, so that the top and bottom handles of \(C\) do not interchange (this is used in the proof of Lemma \[5.6\]). For this reason, we have to keep track of ‘collisions’, as we now explain.

---

\(4\)This is an abuse of notation, since the support of the inequality is not always equal to \(S\).
A collision occurs if, for some cycle \( C \) in a piece of \( S \), we had \( t(C) < b(C) \) at the beginning of the iteration, and \( t(C) \) and \( b(C) \) become equal when increasing \( y_i \). As mentioned in the algorithm, we stop increasing \( y_i \) when a collision occurs. If no vertex became tight during the current iteration, then the algorithm will simply keep the same graph \( S \) in the next iteration, but will change the coefficient of every vertex \( v \) belonging to a handle of \( C \). The new coefficients are equal to \( 1/2 \) (or to 1 if \( C \) becomes the special cycle); in particular, \( t(C) \) and \( b(C) \) will decrease at the same speed in the future.

Finally, let us point out that the number of consecutive iterations executed by the algorithm before some vertex becomes tight is bounded by the number of cycles contained in pieces of \( S \), since a cycle can be involved in at most one collision. Hence, the total number of iterations of the algorithm is polynomial.

5.2.5. Updating the list \( L \) (first pass). We update the collection \( L \) as follows. If \( S \) is of type 1, 3, or 4, we leave \( L \) unchanged. If, on the other hand, it is of type 2, we add to \( L \) all triples \( (T, B, \{v, w\}) \) that were not yet present in \( L \), such that \( T \) is the top handle, \( B \) is the bottom handle and \( \{v, w\} \) are the ends of some cycle contained in a piece of \( S \), and \( T \) has at least one internal vertex. Because \( S \) is consistent with the original list \( L \), the updated list \( L \) satisfies the required properties.

5.2.6. The insertion order. When several vertices are added to \( X \) in the same iteration, we pick an enumeration of the triples of \( L \), say \( (T_1, B_1, \{v_1, w_1\}), \ldots, (T_\ell, B_\ell, \{v_\ell, w_\ell\}) \) having the property that all triples \( (T_i, B_i, \{v_i, w_i\}) \) such that \( X \) contains an internal vertex of both \( T_i \) and \( B_i \) come first, but otherwise arbitrarily. Then, we insert last all vertices \( u \) such that \( u \) is an internal vertex of \( T_1 \), followed by all vertices \( u \) such that \( u \) is an internal vertex of \( B_1 \), followed by all vertices \( u \) such that \( u \) is an internal vertex of \( T_2 \), and so on, ending with all vertices \( u \) such that \( u \) is an internal vertex of \( B_\ell \).

5.2.7. Updating the list \( L \) (second pass). After vertices have been added to \( X \) (hence, deleted from \( H \)), we remove from \( L \) all triples \( (T, B, \{v, w\}) \) such that \( V(T) \) or \( V(B) \) has a non-empty intersection with the new solution \( X \). We also remove all triples \( (T, B, \{v, w\}) \) such that one of the vertices of \( V(T) \) or \( V(B) \) will be removed when shaving \( G - X \).

5.3. Analysis of the algorithm. Before proceeding with the analysis of the algorithm, we need a lemma.

**Lemma 5.5.** Consider a 2-connected graph, some of whose vertices and edges are marked. If no marked vertex is incident to a marked edge, then the total number of marked vertices and edges is at most the total number of edges.

**Proof.** Consider any feasible assignment of marks to vertices and edges of a 2-connected graph. First, suppose that all vertices are marked and, therefore, that no edge is marked. Since the graph is 2-connected, the number of vertices is at most the number of edges, thus the result follows.

Now, suppose that there exists an unmarked vertex. If no vertex of the graph is marked, we are done. Otherwise, by connectivity, there exists an edge \( uv \) such that \( u \)
is unmarked and \( v \) is marked, hence \( uv \) is unmarked. Unmark \( v \) and mark \( uv \). This operation does not change the total number of marked elements and, by applying it iteratively as long as there exists a marked vertex, we eventually obtain a graph without any marked vertex. The result follows. \( \square \)

Lemma 5.6. Let \( X \) be the hitting set output by the algorithm. Then

\[
\sum_{v \in X} a_{i,v} \leq \left( 12 \log_{3/2} n + 16 \right) \beta_i
\]

for all \( i \in \{1, \ldots, k\} \).

Proof. First, we recall that \( \beta_i = 1 \) for all the inequalities in the working LP relaxation.

Consider the \( i \)th iteration of the algorithm. In what follows, \( H \) stands for the graph \( H \) at the \( i \)th iteration, and \( S \) is the support graph of the \( i \)th inequality of the working LP.

If \( S \) is not of type 2, then the left hand side of (8) is simply the number of vertices of \( X \) contained in \( S \). It follows that, if \( S \) is of type 3, then the left hand side is at most 2; and, if \( S \) is of type 4, then the left hand side is at most 4.

Next, we consider the case where the type of \( S \) is 1 or 2.

Claim 5.7. Let \( Q \) be a piece of \( S \). Exactly one of the following four cases occurs:

(a) \( X \) contains no internal vertex of \( Q \),
(b) \( X \) contains exactly one vertex of \( Q \), and this vertex is a cutvertex of \( Q \),
(c) \( X \) contains exactly two vertices of \( Q \), and they belong to opposite handles of a cycle of \( Q \),
(d) \( X \) contains exactly one vertex per cycle of \( Q \), each belonging to some handle of the corresponding cycle.

Proof. Let \( Z \) be the set of vertices that \( X \) contained at the beginning of the \( i \)th iteration. (Thus, \( Z \) is the set of vertices that became tight at some iteration \( j \) with \( j < i \).) Every vertex \( u \in V(Q) \cap X \) has a corresponding witness, namely, a diamond \( D_u \subseteq G \) such that \( V(D_u) \cap (X \cup Z) = \{u\} \). (Such a subgraph exists because \( u \) was kept during the reverse delete step.) This witness \( D_u \) cannot contain any vertex that was removed during the shaving of \( G - Z \) at the beginning of iteration \( i \) since, by definition, these vertices are not included in any diamond of \( G - Z \). Hence, the diamond \( D_u \) is a subgraph of \( H - (X - \{u\}) \).

Suppose \( X \) contains some internal vertex \( u \) of \( Q \) (otherwise, (a) trivially holds). Let \( w_1 \) and \( w_2 \) be the two ends of \( Q \). Since no internal vertex of \( Q \) has a neighbor in \( V(H) - V(Q) \) in \( H \), the diamond \( D_u \) must contain \( w_1, w_2 \), and every cutvertex of \( Q \). Thus, \( X \) contains none of \( w_1 \) and \( w_2 \).

If \( u \) is a cutvertex of \( Q \), then there cannot be another internal vertex \( v \) of \( Q \) in \( X \), for otherwise \( D_v \) contains \( u \). Hence, (b) holds in this case.

Now, assume that \( u \) is not a cutvertex of \( Q \). By the previous observation, we may also assume that no other vertex in \( X \) is a cutvertex of \( Q \). This implies that each vertex in
would contain $v_{1,2}$ belongs to a handle of a cycle in $Q$. Let $A_1, A_2$ be the two handles of the cycle in $Q$ containing $u$, with $u \in V(A_1)$.

If $A_2 \not\subseteq D_u$, then $X$ must contain some internal vertex $v$ of $A_2$: Otherwise, replacing the path $A_1$ with $A_2$ in $D_u$ gives a diamond in $G - X$, a contradiction. It follows that $u$ and $v$ are the only internal vertices of $Q$ included $X$, since $\{u, v\}$ separates $w_1$ from $w_2$ in $Q$. Hence, (c) holds.

Finally, suppose $A_2 \subseteq D_u$. In this case, $X$ contains no internal vertex of $A_2$. Moreover, $D_u$ contains exactly one handle of each cycle in $Q$ that is distinct from $A_1 \cup A_2$. Consider such a cycle $C$. The set $X$ must contain some internal vertex of the handle of $C$ that is not in $D_u$ (since otherwise we could again find a diamond in $G - X$ using that handle).

Furthermore, $X$ contains at most two internal vertices of $C$. Otherwise, either $X$ contains two vertices $v, v'$ belonging to opposite handles of $C$, and $\{v, v'\}$ would separate $u$ from $w_1$ or $w_2$; or $X$ contains two vertices $v, v'$ belonging to the same handle of $C$, and $D_u$ would contain $v'$. Therefore, (d) holds. \hfill \Box

If $S$ is of type 1, then by Claim 5.7 we can see the vertices in $X \cap V(S)$ as marks on the pieces of $S$ such that, if the interior of a piece is marked, then none of its ends are. Hence, by Lemma 5.5, $|X \cap V(S)|$ is at most the number of pieces in $S$, which in turn is at most $6 \log_{3/2} n + 8$.

From now on, we assume that $S$ is of type 2. We split the left hand side of (8) into two parts: the vertices that are internal vertices of some piece of $S$, and the branch vertices.

Claim 5.8. Consider a piece $Q$ of $S$. The contribution of the internal vertices of $Q$ to the left hand side of (8) is at most 1 if $Q$ does not contain the special cycle of $S$, and at most 2 if $Q$ contains the special cycle of $S$.

Proof. By our choice of coefficients for inequality (8), the claim holds in cases (a), (b) and (c) of Claim 5.7. Thus, we assume that case (d) holds. First, we show that $X$ contains exactly one vertex belonging to the top handle of each cycle contained in $Q$.

We use the following property: For all $(T, B, \{v, w\})$ in $\mathcal{L}$, the minimum residual cost of a vertex belonging $T$ is less than or equal to the minimum residual cost of a vertex belonging to $B$. This holds when $(T, B, \{v, w\})$ is added to $\mathcal{L}$ (by the definition of top and bottom handles). The inequality is maintained as long as the support graph $S$ is unchanged (see Section 5.2.4). Moreover, the inequality is also maintained when a new support graph $S$ is chosen, because we ensure that $S$ is always consistent with $\mathcal{L}$ (see Section 5.2.2). Consequently, the inequality is maintained through all subsequent iterations.

Suppose, by contradiction, that there is a cycle $C$ inside the piece $Q$, with top handle $T$ and bottom handle $B$, such that $X$ contains a vertex belonging to $B$, say $u$. Because case (d) holds, (the final set) $X$ contains no vertex belonging to $T$. Because of the property above, at the iteration in which $u$ was added to (the evolving set) $X$, at least one vertex belonging to $T$ was also added to $X$, say $u'$. Now, because of our particular insertion order, it must be the case that $u$ was added after $u'$ in $X$. Since $u$ survived the reverse delete step, there is a diamond $D_u$ in $G$ witnessing the fact that $u$ is in (the final set)
X. This diamond $D_u$ does not contain $u'$. Hence, $D_u$ can be easily transformed into a diamond $D_{u'}$ containing $u'$ and disjoint from $X$, a contradiction. It follows that $X$ contains exactly one vertex belonging to the top handle of each cycle in $Q$.

Now, we may assume that $Q$ contains at least three cycles, and at least one cycle $C$ such that $t(C) = b(C)$, because otherwise the claim trivially holds. Consider such a cycle $C$, with top handle $T$, bottom handle $B$, and ends $v$, $w$. In every subsequent iteration such that $(T, B, \{v, w\})$ survives in $L$, case (C1) or (C4) arises (with respect to the corresponding subgraph $S$). It follows that $t(C) = b(C)$ holds in every subsequent iteration. In particular, at the iteration in which a vertex of $T$ is added to $X$, a vertex of $B$ is also added to $X$. Therefore, $Q$ contains at most one cycle $C$ such that $t(C) = b(C)$. This is due to the fact that case (d) arises and to our insertion order.

If $Q$ does not contain the special cycle, then the contribution of the internal vertices of $Q$ to the left hand side of (8) is clearly at most $1/2 \leq 1$. If $Q$ contains the special cycle, then the contribution of the internal vertices of $Q$ to the left hand side of (8) is at most 1. (Recall that the special cycle is chosen among the cycles $C$ contained in a piece of $S$ and such that $t(C) = b(C)$, if any.)

Theorem 5.9. Algorithm 2 is a $O(\log n)$-approximation for the diamond hitting set problem.

Proof. Letting $\alpha = 12 \log_{3/2} n + 16$, we have

$$\sum_{v \in X} c_v = \sum_{v \in X} \left( \sum_{i=1}^{k} a_{i,v} y_i \right) = \sum_{i=1}^{k} \left( \sum_{v \in X} a_{i,v} \right) y_i \leq \sum_{i=1}^{k} \alpha \beta_i y_i \leq \alpha OPT,$$

where the first equality holds because all vertices in $X$ are tight, and the first inequality follows from Lemma 5.6. The result follows. □

5.4. Integrality gap.

Proposition 5.10. The integrality gap of the LP defined by non-negativity and diamond inequalities is $\Theta(\log n)$.

Proof. By Theorem 5.9, we know that the integrality gap is $O(\log n)$. Now, we show that the integrality gap is also $\Omega(\log n)$, using expander graphs with large girth.

Let us first recall some standard notions from the theory of expanders. Let $G$ be a $d$-regular graph with $|G| \geq 2$. The spectral gap of $G$ is $d - \lambda(G)$, where $\lambda(G)$ denotes the second largest eigenvalue of the adjacency matrix of $G$. The vertex-expansion $h_v(G)$ is defined as the minimum of

$$\frac{|N(S) - S|}{|S|}$$
over all subsets $S \subset V(G)$ with $|S| \leq |G|/2$. (Here, $N(S)$ denotes the set of vertices of $G$ having a neighbor in $S$.) It is well-known that the spectral gap of $G$ can be used to derive a lower-bound on its vertex-expansion:

$$h_v(G) \geq \frac{1}{2d} (d - \lambda(G));$$

see for instance the survey by Hoory, Linial, and Wigderson \[10\], p. 454].

Lubotzky, Phillips, and Sarnak \[12\] proved that, for $d = 6$ and infinitely many values of $n$, there exists a $d$-regular graph $G$ on $n$ vertices with $\lambda(G) \leq 2\sqrt{d-1}$ and girth at least $\frac{4}{3} \log_{d-1} n$ (see also Margulis \[13\], and Biggs and Boshier \[4\]). By (9), the vertex-expansion of $G$ satisfies

$$h_v(G) \geq \frac{1}{2d} \left(d - 2\sqrt{d-1}\right) = \frac{1}{12} \left(6 - 2\sqrt{5}\right) > 0.$$

It is also known that the treewidth \[7\] $\text{tw}(H)$ of a graph $H$ satisfies

$$\text{tw}(H) \geq h_v(H) \cdot \frac{n}{4} - 1,$$

see for instance Grohe and Marx \[9\], Proposition 1]. It follows that

$$\text{tw}(G) \geq \xi n - 1,$$

where $\xi := \frac{1}{48} \left(6 - 2\sqrt{5}\right)$. Since removing a vertex from a graph decreases its treewidth by at most 1 and forest of cacti have treewidth at most 2, this implies that the minimum size of a hitting set satisfies

$$\text{OPT} \geq \xi n - 3.$$

On the other hand, the minimum size of a diamond in $G$ is at least the girth, that is, at least $\frac{4}{3} \log_5 n$. Thus, setting

$$x_v^* := \frac{3}{4 \log_5 n}$$

yields a feasible solution of the linear relaxation. The value of the objective function for $x^*$ is $\frac{3n}{4 \log_5 n}$. Therefore, the integrality gap of the LP is at least

$$4 \log_5 n \cdot \frac{\text{OPT}}{3n} \geq 4 \log_5 n \cdot \frac{\xi n - 3}{3n} = \Omega(\log n).$$

6. A 9-approximation algorithm

In this section, we give a primal-dual 9-approximation algorithm for the diamond hitting set problem. We start with a description of the algorithm in Section 6.1. This algorithm makes use of the sparsity inequalities. In order to describe them, we first bound the number of edges in a forest of cacti in Section 6.2; using this bound, in Sections 6.3 and 6.4 we introduce the sparsity inequalities, prove their validity, and show that they satisfy a key inequality that we need in the analysis of the algorithm. Finally, in Section 6.5 we prove that our algorithm provides a 9-approximation for the diamond hitting set problem.

\[5\]See Diestel \[6\] for a definition.
6.1. **The algorithm.** Our 9-approximation algorithm for the diamond hitting set problem is very similar to the $O(\log n)$-approximation algorithm. The main difference is that we use a different set of inequalities to build the working LP relaxation. (The working LP relaxation and its dual are defined in Section 5.1 on page 9.) See Algorithm 3 for a description of the algorithm.

**Algorithm 3** A 9-approximation algorithm.

- $X \leftarrow \emptyset$; $y \leftarrow 0$; $i \leftarrow 0$; $\mathcal{L} \leftarrow \emptyset$
- While $X$ is not a hitting set of $G = (V,E)$, repeat the following steps:
  - $i \leftarrow i + 1$
  - Let $H$ be the graph obtained by shaving $G - X$
  - Find a reduced graph $\tilde{H}$ of $H$
  - If $\tilde{H}$ contains a diamond $\tilde{D}$ with at most $2q - 1$ edges, then let $\sum_{v \in V} a_{i,v} x_v \geq \beta_i$ be a blended diamond inequality deduced from $\tilde{D}$ as in Section 5.2.3
  - Otherwise, in $\tilde{H}$, no two cycles of size at most $q$ share an edge. In this case, let $\sum_{v \in V} a_{i,v} x_v \geq \beta_i$ be the extended sparsity inequality with support $V(H)$
  - Check the inequality $\sum_{v \in V} a_{i,v} x_v \geq \beta_i$ w.r.t. $\mathcal{L}$, modify it if necessary, add it to the working LP
  - Increase $y_i$ until some vertex becomes tight, or a collision occurs
  - Update $\mathcal{L}$
  - Add all tight vertices to $X$, in a certain order
  - Re-update $\mathcal{L}$
- $k \leftarrow i$
- Perform a reverse delete step on $X$

The following sections are devoted to the definition of the extended sparsity inequalities. In the next paragraph we comment the last five steps in the main loop.

First, we only modify the current inequality $\sum_{v \in V} a_{i,v} x_v \geq \beta_i$ if it is a blended diamond inequality. This is done as described in Section 5.2.2. Note that none of the modifications increases the number of pieces of the support graph $S$.

Second, the tracking of collisions is performed as before (see Section 5.2.4).

Third, $\mathcal{L}$ is updated similarly as before (see Section 5.2.5): if the support graph of the current inequality contains double pieces (in case the current inequality is an extended sparsity inequality, the support graph is the whole graph $H$), we add all triples $(T,B,\{v,w\})$ that were not yet present in $\mathcal{L}$, where $T$ is the top handle, $B$ is the bottom handle, and $v,w$ are the ends of a cycle contained in a double piece of the support graph.

Finally, the insertion order and the second update of $\mathcal{L}$ are done exactly as previously (see Sections 5.2.6 and 5.2.7).
6.2. **Bounding the number of edges in a forest of cacti.** The following lemma provides a bound on the number of edges in a forest of cacti. For $i \in \{2, \ldots, q\}$, we denote by $\gamma_i(G)$ the number of cycles of length $i$ of a graph $G$.

**Lemma 6.1.** Let $F$ be a forest of cacti with $k$ components and let $q \geq 2$. Then

$$||F|| \leq \frac{q+1}{q} (|F| - k) + \sum_{i=2}^{q} \frac{q-i+1}{q} \gamma_i(F).$$

**Proof.** Denote by $\gamma_{>q}(F)$ the number of cycles of $F$ whose length exceeds $q$. We have

$$||F|| = |F| - k + \sum_{i=2}^{q} \gamma_i(F) + \gamma_{>q}(F).$$

In the right hand side, the first two terms represent the number of edges in a spanning forest of $F$, while the last terms give the number of edges that should be added to obtain the forest of cacti $F$.

Because every two cycles in $F$ are edge disjoint, we have

$$||F|| \geq \sum_{i=2}^{q} i \gamma_i(F) + (q+1) \gamma_{>q}(F).$$

Combining this with (10), we get

$$\gamma_{>q}(F) \leq \frac{1}{q} \left( |F| - k - \sum_{i=2}^{q} (i-1) \gamma_i(F) \right).$$

From (10) and (11), we finally infer

$$||F|| \leq |F| - k + \sum_{i=2}^{q} \gamma_i(F) + \frac{1}{q} \left( |F| - k - \sum_{i=2}^{q} (i-1) \gamma_i(F) \right) \leq \frac{q+1}{q} (|F| - k) + \sum_{i=2}^{q} \frac{q-i+1}{q} \gamma_i(F).$$

6.3. **The sparsity inequalities.** We define the load of a vertex $v$ in a graph $G$ as

$$\ell_G(v) := d_G(v) - \sum_{i=2}^{q} \lambda_i \gamma_i(G, v),$$

where, for $i \in \{2, \ldots, q\}$, $\gamma_i(G, v)$ denotes the number of cycles of length $i$ incident to $v$ in $G$ and

$$\lambda_i := \frac{q-i+1}{\lfloor i/2 \rfloor q}.$$

For $q = 5$, we have

$$\lambda_2 = \frac{4}{5}, \quad \lambda_3 = \frac{3}{5}, \quad \lambda_4 = \frac{1}{5}, \quad \text{and} \quad \lambda_5 = \frac{1}{10}.$$

**Lemma 6.2.** Let $X$ be a hitting set of a graph $G$ where no two cycles of length at most $q$ share an edge. Then,

$$\sum_{v \in X} \left( \ell_G(v) - \frac{q+1}{q} \right) \geq ||G|| - \frac{q+1}{q} |G| - \sum_{i=2}^{q} \frac{q-i+1}{q} \gamma_i(G) + \frac{q+1}{q}.$$

We call Inequality (12) a sparsity inequality.
Proof of Lemma 6.2. For \( i \in \{2, \ldots, q\} \) and \( j \in \{0, \ldots, i\} \), we denote by \( \xi_i^j \) the number of cycles of \( G \) that have length \( i \) and exactly \( j \) vertices in \( X \).

Letting \( |X| \) and \( |\delta(X)| \) respectively denote the number of edges of \( G \) with both ends in \( X \) and the number of edges of \( G \) having an end in \( X \) and the other in \( V(G) - X \), we have

\[
\sum_{v \in X} \ell_G(v) = 2|X| + |\delta(X)| - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j
\]

\[
= |X| + |G| - |G - X| - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j
\]

\[
\geq |X| + |G| - \frac{q+1}{q} (|G - X| - 1) - \sum_{i=2}^{q} \frac{q - i + 1}{q} \xi_i^0 - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j,
\]

where the last inequality follows from Lemma 6.1 applied to the forest of cacti \( G - X \) (notice that \( \gamma_i(G - X) = \xi_i^0 \)).

Because no two cycles of length at most \( q \) share an edge and, in a cycle of length \( i \), each subset of size \( j \) induces a subgraph that contains at least \( 2j - i \) edges, we have

\[
|X| \geq \sum_{i=2}^{q} \sum_{j=1+\lfloor \frac{i}{2} \rfloor}^{i} (2j - i) \xi_i^j.
\]

Thus, we obtain

\[
\sum_{v \in X} \ell_G(v) \geq \sum_{i=2}^{q} \sum_{j=1+\lfloor \frac{i}{2} \rfloor}^{i} (2j - i) \xi_i^j + |G| - \frac{q+1}{q} (|G - X| - 1) - \sum_{i=2}^{q} \frac{q - i + 1}{q} \xi_i^0 - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j.
\]

We leave it to the reader to check that, in the right hand side of the last inequality, the total coefficient of \( \xi_i^j \) is at least \(-\frac{2 - i + 1}{q}\), for all \( i \in \{2, \ldots, q\} \) and \( j \in \{0, \ldots, i\} \). Hence,

\[
\sum_{v \in X} \ell_G(v) \geq |G| - \frac{q+1}{q} |G| + \frac{q+1}{q} |X| + \frac{q+1}{q} \sum_{i=2}^{q} \frac{q - i + 1}{q} \gamma_i(G).
\]

Inequality (12) follows. \( \square \)

6.4. The extended sparsity inequalities. Consider a shaved graph \( H \) and denote by \( \tilde{H} \) a reduced graph of \( H \). Suppose that, in \( \tilde{H} \), no two cycles of length at most \( q \) share an edge. Thus, the graph \( H \) can be uniquely decomposed into simple or double pieces corresponding to edges or pairs of parallel edges in \( \tilde{H} \). Here, the pieces of \( H \) are defined as follows: let \( v \) and \( w \) be two adjacent vertices of \( \tilde{H} \), and let \( \tilde{J} \) denote the subgraph of \( \tilde{H} \) induced by \( \{v, w\} \). The primitive subgraph of \( \tilde{J} \) in \( H \), say \( J \), is a piece of \( H \) with ends \( v \) and \( w \). The vertices of \( H \) are of two types: the branch vertices are those that survive in \( \tilde{H} \), and the other vertices are internal to some piece of \( H \).

Consider a double piece \( Q \) of \( H \) (if any) and a cycle \( C \) contained in \( Q \). As before, denote by \( t(C) \) (resp. \( b(C) \)) the minimum residual cost of a vertex in the top handle (resp. bottom handle) of \( C \). Also, we choose a cycle of \( Q \) (if any) and declare it to be
special. If possible, the special cycle is chosen among the cycles $C$ contained in $Q$ with $t(C) = b(C)$. (So every double piece of $H$ has a special cycle.)

The extended sparsity inequality for $H$ reads

$$
\sum_{v \in V(H)} a_v x_v \geq \beta,
$$

where

$$
a_v := \begin{cases} 
\ell_{\tilde{H}}(v) - \frac{q + 1}{q} & \text{if } v \text{ is a branch vertex,} \\
1 & \text{if } v \text{ is an internal vertex of a simple piece,} \\
2 & \text{if } v \text{ is an internal vertex of a double piece and does not belong to any handle,} \\
0 & \text{if } v \text{ belongs to the top handle of a cycle } C \text{ with } t(C) < b(C), \\
2 & \text{if } v \text{ belongs to the bottom handle of a cycle } C \text{ with } t(C) < b(C), \\
1 & \text{if } v \text{ belongs to a handle of a cycle } C \text{ with } t(C) = b(C), \text{ or } C \text{ is special,}
\end{cases}
$$

and

$$
\beta := ||\tilde{H}|| - \frac{q + 1}{q} |\tilde{H}| - \sum_{i=2}^q \frac{q - i + 1}{q} \gamma_i(\tilde{H}) + \frac{q + 1}{q}.
$$

By convention, the support graph of the extended sparsity inequality (13) is defined to be $H$.

**Lemma 6.3.** Let $H$ be a graph and let $\tilde{H}$ be a reduced graph of $H$ such that no two cycles of length at most $q$ share an edge. Then Inequality (13) is valid, that is,

$$
\sum_{v \in X} a_v \geq \beta
$$

whenever $X$ is a hitting set of $H$.

**Proof.** Let $Y$ denote the set of branch vertices that are included in $X$. So $Y = V(\tilde{H}) \cap X$. Then $Y$ is not necessarily a hitting set of $\tilde{H}$. Indeed, $\tilde{H} - Y$ is a forest of cacti $F$ to which a certain number of extra edges are added: the extra edges are those corresponding to the vertices of $X$ that are internal to some piece, that is, vertices of $X - Y$. By our choice of coefficients,

$$
\sum_{v \in X - Y} a_v \geq ||\tilde{H} - Y|| - ||F||.
$$

In other words, the left hand side is at least the number of extra edges.

The rest of the proof closely follows the proof of Lemma 6.2. This time, all computations are made within the reduced graph. Precisely, $\xi_i^j$ denotes the number of cycles of $\tilde{H}$ that have length $i$ and exactly $j$ vertices in $Y$, $||Y||$ denotes the number of edges of $\tilde{H}$ with both ends in $Y$ and $|\delta(Y)|$ denotes the number of edges of $\tilde{H}$ having one end in $Y$ and the other in $V(\tilde{H}) - Y$. Then, we have

$$
\sum_{v \in Y} \ell_{\tilde{H}}(v) + \sum_{v \in X - Y} a_v = 2 ||Y|| + |\delta(Y)| - \sum_{i=2}^q \sum_{j=1}^i j \lambda_i \xi_i^j + ||\tilde{H} - Y|| - ||F||
$$

$$
= ||\tilde{H}|| + ||Y|| - ||F|| - \sum_{i=2}^q \sum_{j=1}^i j \lambda_i \xi_i^j.
$$
By using Lemma 6.1 to bound $||F||$ and the inequality $\xi_0^i \geq \gamma_i(F)$, we obtain

$$\sum_{v \in Y} \ell_{\tilde{H}}(v) + \sum_{v \in X - Y} a_v \geq ||\tilde{H}|| + |Y| - q + 1 \frac{q + 1}{q} (|F| - 1) - \sum_{i=2}^{q} \frac{q - i + 1}{q} \gamma_i(F) - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j$$

$$\geq ||\tilde{H}|| + |Y| - q + 1 \frac{q + 1}{q} (|\tilde{H}| - |Y| - 1) - \sum_{i=2}^{q} \frac{q - i + 1}{q} \xi_0^i - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j.$$

After performing the same steps as in the proof of Lemma 6.2 in the graph $\tilde{H}$ this time, we find

$$\sum_{v \in Y} \ell_{\tilde{H}}(v) + \sum_{v \in X - Y} a_v \geq ||\tilde{H}|| - q + 1 \frac{q + 1}{q} |\tilde{H}| + \frac{q + 1}{q} |Y| + \frac{q + 1}{q} - \sum_{i=2}^{q} \frac{q - i + 1}{q} \gamma_i(\tilde{H}).$$

This proves the validity of the extended sparsity Inequality (13). □

6.5. Analysis of the algorithm. We now prove a key inequality that will be used in the analysis of the algorithm.

**Lemma 6.4.** Let $H$ be a shaved graph and let $\tilde{H}$ be a reduced graph of $H$. Suppose that, in $\tilde{H}$, every diamond has at least $2q = 10$ edges. Then,

$$\sum_{v \in X} a_v \leq 8 \beta$$

for every minimal hitting set $X$ of $H$.

**Proof.** Let $Y := V(\tilde{H}) \cap X$. We note that no two cycles of length at most 5 in $\tilde{H}$ have an edge in common, since $\tilde{H}$ has no diamond with at most 9 edges.

We have to prove that

$$\sum_{v \in Y} \left( \ell_{\tilde{H}}(v) - \frac{q + 1}{q} \right) + \sum_{v \in X - Y} a_v \leq 8 \cdot \left( ||\tilde{H}|| - \frac{q + 1}{q} |\tilde{H}| + \frac{q + 1}{q} |Y| + \sum_{i=2}^{q} \frac{q - i + 1}{q} \gamma_i(\tilde{H}) + \frac{q + 1}{q} \right).$$

We claim that

$$\sum_{v \in X - Y} a_v = ||\tilde{H} - Y|| - ||F||,$$

where the sum is taken over all vertices that are included in $X$ and that are internal to some piece. Indeed, since $X$ is minimal, there are four ways in which $X$ can intersect internal vertices of a piece $Q$ (see Claim 5.7 above):

- $X$ contains no internal vertex of $Q$,
- $X$ contains exactly one vertex of $Q$, and this vertex is a cutvertex of $Q$,
- $X$ contains exactly two vertices of $Q$, and they belong to opposite handles of a cycle of $Q$,
- $X$ contains exactly one vertex per cycle of $Q$, each belonging to some handle of the corresponding cycle.
In the three first cases, the choice for the coefficients ensures that the equality holds. In the fourth case, using the same arguments as in Claim 5.8 above, we prove that \( X \) contains one vertex in the top handle of each cycle of \( Q \). The left hand side can be rewritten as:

\[
\sum_{v \in Y} \left( \ell_{\tilde{H}}(v) - \frac{q+1}{q} \right) + \sum_{v \in X-Y} a_v = 2||Y|| + |\delta(Y)| - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j - \frac{q+1}{q} |Y| \\
+ ||\tilde{H} - Y|| - ||F||
\]

\[
\leq 2||\tilde{H}|| - |\delta(Y)| - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j - \frac{q+1}{q} |Y| \\
- 2||F|| - (||\tilde{H} - Y|| - ||F||).
\]

Let \( k \) denote the number of components of the forest of cacti \( F \). Since \( ||F|| = |F| - k + \sum_{i=2}^{q} \gamma_i(F) + \gamma_{>q}(F) \) and \( |F| = ||\tilde{H}|| - |Y| \), we have

\[
\sum_{v \in Y} \left( \ell_{\tilde{H}}(v) - \frac{q+1}{q} \right) + \sum_{v \in X-Y} a_v = 2||\tilde{H}|| - |\delta(Y)| - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j - \frac{q+1}{q} |Y| \\
- 2\left(||\tilde{H}|| - |Y| - k + \sum_{i=2}^{q} \gamma_i(F) + \gamma_{>q}(F)\right) \\
- (||\tilde{H} - Y|| - ||E(F)||) \\
\leq 2||\tilde{H}|| - 2||\tilde{H}|| - \sum_{i=2}^{q} \sum_{j=1}^{i} j \lambda_i \xi_i^j - 2 \sum_{i=2}^{q} \gamma_i(F) - 2 \gamma_{>q}(F) \\
- |\delta(Y)| - (||\tilde{H} - Y|| - ||F||) + \frac{q-1}{q} |Y| + 2k.
\]

Now, let \( \nu_i \ (i = \{2, \ldots, q\}) \) be defined as \( \nu_i := 0 \) for \( i \neq 3 \) and \( \nu_3 := \frac{3}{q} \). Notice that \( \nu_i \leq j \lambda_i \) for all \( i \in \{2, \ldots, q\} \) and \( j \in \{1, \ldots, i\} \). Also, \( \nu_i \leq \frac{q-i+1}{\lfloor i/2 \rfloor} q \) for all \( i \in \{2, \ldots, q\} \).
We proceed in two steps. We first show that

\[
\sum_{v \in Y} \left( \ell_{\tilde{H}}(v) - \frac{q+1}{q} \right) + \sum_{v \in X \setminus Y} a_v \leq 2||\tilde{H}|| - ||L|| - \sum_{i=2}^q \sum_{j=1}^{i} j \lambda_i \xi_i^j - 2 \sum_{i=2}^q \gamma_i(F) - 2 \gamma > q(F) \\
- \sum_{i=2}^q \nu_i(\xi^0_i - \gamma_i(F)) + \sum_{i=2}^q \nu_i(\xi^0_i - \gamma_i(F)) \\
- |\delta(Y)| - (||\tilde{H} - Y|| - ||F||) + \frac{q-1}{q}||Y|| + 2k
\]

\[
\leq 2||\tilde{H}|| - ||L|| - \sum_{i=2}^q \nu_i(\tilde{H}) - \sum_{i=2}^q (j \lambda_i - \nu_i) \xi_i^j \\
- (2 - \nu_i) \sum_{i=2}^q \gamma_i(F) - 2 \gamma > q(F) + \sum_{i=2}^q \nu_i(\xi^0_i - \gamma_i(F)) \\
- |\delta(Y)| - (||\tilde{H} - Y|| - ||F||) + \frac{q-1}{q}||Y|| + 2k
\]

\[
\leq 2||\tilde{H}|| - ||L|| - \sum_{i=2}^q \nu_i(\tilde{H}) + \sum_{i=2}^q \nu_i(\xi^0_i - \gamma_i(F)) \\
- |\delta(Y)| - (||\tilde{H} - Y|| - ||F||) + \frac{q-1}{q}||Y|| + 2k.
\]

Therefore, to prove Inequality (14), it suffices to show the following:

\[
6||\tilde{H}|| + \left(2 - 8 \cdot \frac{q+1}{q}\right)||\tilde{H}|| - \sum_{i=2}^q \left(8 \cdot \frac{q-i+1}{q} - \nu_i\right) \gamma_i(\tilde{H}) + 8 \cdot \frac{q+1}{q} \\
+ |\delta(Y)| - \frac{q-1}{q}||Y|| + ||\tilde{H} - Y|| - ||F|| - 2k - \sum_{i=2}^q \nu_i(\xi^0_i - \gamma_i(F)) \geq 0.
\]

We actually prove a slightly stronger inequality:

\[
6||\tilde{H}|| + \left(2 - 8 \cdot \frac{q+1}{q}\right)||\tilde{H}|| - \sum_{i=2}^q \left(8 \cdot \frac{q-i+1}{q} - \nu_i\right) \gamma_i(\tilde{H}) \\
+ |\delta(Y)| - \frac{q-1}{q}||Y|| + ||\tilde{H} - Y|| - ||F|| - 2k - \sum_{i=2}^q \nu_i(\xi^0_i - \gamma_i(F)) \geq 0.
\]

We proceed in two steps. We first show that

\[
(15) \quad 6||\tilde{H}|| + \left(2 - 8 \cdot \frac{q+1}{q}\right)||\tilde{H}|| - \sum_{i=2}^q \left(8 \cdot \frac{q-i+1}{q} - \nu_i\right) \gamma_i(\tilde{H}) \geq 0,
\]

then we show that

\[
(16) \quad |\delta(Y)| + ||\tilde{H} - Y|| - ||F|| \geq \frac{q-1}{q}||Y|| + 2k + \sum_{i=2}^q \nu_i(\xi^0_i - \gamma_i(F)).
\]
Let us start with Inequality (15). By multiplying by \( \frac{1}{3|H|} \), we rewrite it as:
\[
\frac{1}{|H|} \left( 2||\tilde{H}|| - \frac{1}{3} \sum_{i=2}^{q} \left( 8 \cdot \frac{q - i + 1}{i q} - \frac{\nu_i}{i} \right) i \gamma_i(\tilde{H}) \right) \geq \frac{8 \cdot 2q + 2 - 2}{3}.
\]
The above inequality has the following simple interpretation: the average load of a vertex should be at least the right hand side, where the load of vertex \( v \) is redefined as
\[
\ell'_H(v) := d_H(v) - \frac{1}{3} \sum_{i=2}^{q} \left( 8 \cdot \frac{q - i + 1}{i q} - \frac{\nu_i}{i} \right) i \gamma_i(\tilde{H}, v).
\]
Recalling that \( q = 5, \nu_i = 0 \) for \( i \neq 3 \) and \( \nu_3 = \frac{3}{5} \), we get that the right hand side is equal to \( \frac{28}{15} \) and the load is as follows:
\[
\ell'_H(v) = d_H(v) - \frac{1}{3} \left( 8 \cdot \frac{5 - 2 + 1}{2 \cdot 5} \right) \gamma_2(\tilde{H}, v) - \frac{1}{3} \left( 8 \cdot \frac{5 - 3 + 1}{3 \cdot 5} - \frac{5}{3} \right) \gamma_3(\tilde{H}, v) - \frac{1}{3} \left( 8 \cdot \frac{5 - 2 + 1}{4 \cdot 5} \right) \gamma_4(\tilde{H}, v) - \frac{1}{3} \left( 8 \cdot \frac{5 - 2 + 1}{5 \cdot 5} \right) \gamma_5(\tilde{H}, v) = d_H(v) - \frac{16}{15} \gamma_2(\tilde{H}, v) - \frac{7}{15} \gamma_3(\tilde{H}, v) - \frac{4}{15} \gamma_4(\tilde{H}, v) - \frac{8}{75} \gamma_5(\tilde{H}, v).
\]
As \( \tilde{H} \) is reduced, the minimum load of a vertex is \( 3 - \frac{7}{15} = \frac{28}{15} \) (think of a vertex incident to three edges, two of which determine a cycle of length 3). As a consequence, the average load of a vertex is at least the right hand side, and Inequality (15) follows.

Let us now prove Inequality (16). Since \( q = 5, \nu_i = 0 \) for \( i \neq 3 \) and \( \nu_3 = \frac{3}{5} \), the inequality reads \( |\delta(Y)| + ||\tilde{H} - Y|| - ||F|| \geq \frac{4}{5}|Y| + 2k + \frac{3}{5}(\xi^0_3 - \gamma_3(F)) \). We will prove the following slightly stronger inequality:
\[
|\delta(Y)| + ||\tilde{H} - Y|| - ||F|| \geq |Y| + 2k + \frac{3}{5}(\xi^0_3 - \gamma_3(F)). \tag{17}
\]
To prove it, we use arguments that are similar to those used in [5]. First observe that the quantities \( ||\tilde{H} - Y|| - ||F|| \) and \( \xi^0_3 - \gamma_3(F) \) respectively correspond to the number of extra edges and the number of extra triangles contained in \( \tilde{H} - Y \). Furthermore, each extra edge can generate at most one extra triangle, since \( \tilde{H} \) has no diamond with at most 9 edges.

We build a bipartite graph \( J \) as follows. Start with \( \tilde{H} \), and contract each of the \( k \) components of \( \tilde{H} - Y \) into a single vertex (as usual, we keep the newly created parallel edges, if any, and we remove the loops). Then, remove all edges having both endpoints in \( Y \).

It follows from the fact that \( X \) is a minimal hitting set of \( H \) that, for each vertex \( v \in Y \), there exists a diamond \( D_v \) in \( \tilde{H} \) that is vertex-disjoint from \( Y - \{v\} \) and edge-disjoint from \( E(\tilde{H} - Y) - E(F) \). Moreover, for each extra edge \( e \in E(\tilde{H} - Y) - E(F) \), there exists a diamond \( D_e \) in \( \tilde{H} \) that is vertex-disjoint from \( Y \) and edge-disjoint from \( E(\tilde{H} - Y) - E(F) - \{e\} \). We call these diamonds a witness for \( v \) and \( e \), respectively.

In particular, for every \( v \in Y \), we can choose a component \( K \) of \( \tilde{H} - Y \) such that there are at least two edges between \( v \) and \( K \) in \( J \); we call the pair \((v, K)\) a primary pair. We remove from \( J \) one edge between \( v \) and \( K \) for each primary pair \((v, K)\). Noticed that we
removed exactly $|Y|$ edges from $J$; thus, to prove Inequality (17), it is enough to show that

\begin{equation}
||J|| + ||\tilde{H} - Y|| - ||F|| \geq 2k + \frac{3}{5}(\xi^0_3 - \gamma_3(F)).
\end{equation}

In the remainder of this proof, our aim is to show that, for each component $K$ of $\tilde{H} - Y$, the number of edges of $J$ incident to $K$ plus the number of extra edges in $K$ is at least 2 plus $3/5$ times the number of extra triangles in $K$, that is:

\begin{equation}
d_J(K) + ||K|| - ||K \cap F|| \geq 2 + \frac{3}{5}(\gamma_3(K) - \gamma_3(K \cap F)).
\end{equation}

Clearly, Inequality (19) implies Inequality (18).

First, we recall that, since $\tilde{H}$ is reduced, every vertex of $\tilde{H}$ has at least three neighbors. Consider a component $K$ of $\tilde{H} - Y$. Let $\eta = \eta(K) := ||K|| - ||K \cap F||$ denote the number of extra edges in $K$, and $\tau = \tau(K) := \gamma_3(K) - \gamma_3(K \cap F)$ denote the number of extra triangles in $K$. With these notations, Inequality (19) becomes

\begin{equation}
d_J(K) + \eta \geq 2 + \frac{3}{5}\tau.
\end{equation}

Note that $\eta \geq \tau$, since no two extra triangles have an edge in common. Therefore, we may assume that one of the two following cases occurs (otherwise (20) holds):

(i) $d_J(K) = 0$ and $\eta \leq 4$, or
(ii) $d_J(K) = 1$ and $\eta \leq 2$.

Both cases can be handled using similar arguments, hence we treat them in parallel and only highlight the differences when necessary.

In case (i), $K$ is a component of $\tilde{H}$, and hence every vertex of $K$ has at least three distinct neighbors in $K$. In case (ii), there are vertices in $V(K)$ having a neighbor in $Y$ in the graph $\tilde{H}$; these vertices are said to be special. Note that $K$ belongs to at most one primary pair $(v, K)$, because otherwise we would have $d_J(K) \geq 2$. It follows that there are at most two special vertices in $K$. Moreover, in $K$, every non-special vertex has at least three distinct neighbors, while every special vertex has at least two of them.

Let $L := K \cap F$. Each extra edge in $K$ has a witness in $\tilde{H}$, and this witness avoids all the other extra edges and the vertices in $Y$, thus $L$ is a component of $F$, and hence $L$ is a cactus.

First, we prove that every cycle in $L$ has at most 7 vertices. Arguing by contradiction, assume $C$ is a cycle of $L$ with $|C| \geq 8$. This cycle is a block of $L$, and each vertex of $C$ either has degree 2 in $L$, or is a cutvertex of $L$. Let $t$ be the number of vertices of $C$ having degree 2 in $L$. Now, for each cutvertex $v$ of $L$ included in $C$, there is at least one endblock $B_v$ of $L$ such that $v$ separates $B_v$ from $C$ in $L$. Observe that either $|B_v| = 2$, and thus one of its vertices has only one neighbor in $L$, or $|B_v| \geq 3$ and $B_v$ is a simple cycle, and thus at least two of its vertices have degree 2 in $L$. This directly gives the following lower bound on $\eta$:

\[
\eta \geq \begin{cases} 
t/2 + (|C| - t) & \text{in case (i)}, 
t/2 + (|C| - t) - 1 & \text{in case (ii)}. 
\end{cases}
\]
HITTING DIAMONDS AND GROWING CACTI

(The −1 in case (ii) comes from the existence of special vertices in K.) In case (i), since
η ≤ 4, we must have η = 4, |C| = t = 8, and L = C. However, by taking the union of C
with an extra edge of K we obtain a diamond in K having 9 edges, a contradiction. In
case (ii), we directly get a contradiction, because η ≤ 2 and t/2 + (|C| − t) − 1 ≥ 3. (The
last inequality is derived using t ≤ |C| and |C| ≥ 8.)

Consider an extra triangle in K. If there is only one extra edge e in that triangle, then
the unique cycle C in De − e is a cycle of L which has at least one edge in common with
the triangle. (Recall that De denotes the witness of e in K.) As we have seen, we must
have |C| ≤ 7. However, this implies ||De|| ≤ 9, contradicting the fact that every diamond
in H has at least 10 edges. Thus, every extra triangle contains at least two extra edges.
Since the extra triangles are edge disjoint, this implies

(21) η ≥ 2τ.

In case (ii), Inequality (21) allows us to easily show that (20) holds: First, suppose
τ ≥ 1. Then, by (21), we have
η ≥ 2τ ≥ 1 + \frac{3}{5}τ,

as desired. Now, assume τ = 0. Here, we only need to show η ≥ 1. Suppose the contrary,
that is, η = 0. Then K = L, and hence K is a cactus. Recall that K has at most two
special vertices, and that each of these vertices has at least two neighbors in K. Thus, if
some endblock B of K is isomorphic to K2 or to a cycle consisting of two parallel edges,
then some vertex of B has only one neighbor in K, a contradiction. This implies that all
the endblocks of K are simple cycles. Using this observation, it is easily seen that η ≥ 1
if there are more than one endblock in K. Thus, K consists of a single block, and hence
K is a simple cycle. However, since |K| ≥ 3, there is a non-special vertex in K with
degree 2, a contradiction. Therefore, we must have η ≥ 1, which concludes case (ii).

Now, consider case (i). Here, it is easily seen that η ≥ 2: First, |L| = |K| ≥ 4, since
every vertex of K has at least three distinct neighbors in K. Thus, if L has only one
endblock, then L must be a simple cycle on at least 4 vertices, implying η ≥ 2. Similarly,
if L has at least two endblocks, then by considering two such endblocks we deduce again
η ≥ 2. This implies that Inequality (20) holds if τ = 0. Hence, we may assume τ ≥ 1.

If η ≥ 3, then (21) gives

2 + \frac{3}{5}τ ≤ 2 + \frac{3}{10}η = η + 2 - \frac{7}{10}η = η - \frac{1}{10} ≤ η.

Thus, it remains to handle the case where η = 2 and τ = 1. We will show that this case
cannot happen, because it leads to a contradiction.

Let e1, e2, f denote the three edges forming the unique extra triangle in K, with e1
and e2 being the two extra edges of K. These two edges are incident to a common vertex,
which we denote v. Observe that this vertex must be contained in some endblock Bv of
L (otherwise, η ≥ 3).

Let Bf be the block of L including the edge f. Suppose Bf is a cycle. This cycle has
length at most 7. If v ∈ V(Bf), then each extra edge gives a diamond with at most 8
edges in K. Similarly, if v ∉ V(Bf), then using the two extra edges we obtain a diamond
in $K$ with at most 9 edges. Thus, in both cases we get a contradiction. This implies $B_f \cong K_2$. In particular, $B_f \neq B_v$, since otherwise we would have $|B_f| \geq 3$.

Now, $B_f$ cannot be an endblock of $L$, as otherwise some vertex of $B_f$ would have degree only 2 in $K$. This implies that there is an endblock of $L$ distinct from $B_v$ that contains a vertex having at most two neighbors in $K$, a contradiction. This concludes the proof. □

**Lemma 6.5.** For the minimal hitting set $X$ output by Algorithm 3, we have
\[ \sum_{v \in X} a_{i,v} \leq 9 \beta_i \]
for all $i \in \{1, \ldots, k\}$.

**Proof.** If the $i$th inequality of the working LP relaxation is an extended sparsity inequality, the result follows from the previous lemma. Thus, we may assume that the $i$th inequality is deduced from a diamond $\tilde{D}$ (thus $\beta_i = 1$). Hence, $\tilde{D}$ has at most $2q - 1 = 9$ edges. Let $S$ denote the support graph of the $i$th inequality. If $S$ is of type 1, then it has at most 9 pieces. If it is of type 2, it has at most 8 pieces. If it is of type 3 or 4, the left hand side is at most two. The result then follows from the arguments used in the proof of Lemma 5.6. □

Our main result directly follows from Lemma 6.5. The proof is identical to the proof of Theorem 5.9 and hence is omitted.

**Theorem 6.6.** Algorithm 3 is a 9-approximation for the diamond hitting set problem.

**Acknowledgements**

We thank Dirk Oliver Theis for his valuable input in the early stage of this research. We also thank Jean Cardinal and Marcin Kamiński for stimulating discussions.

**References**

[1] V. Bafna, P. Berman, and T. Fujito. A 2-approximation algorithm for the undirected feedback vertex set problem. *SIAM Journal on Discrete Mathematics*, 12(3):289–297, 1999.

[2] R. Bar-Yehuda, D. Geiger, J. Naor, and R. M. Roth. Approximation algorithms for the feedback vertex set problem with applications to constraint satisfaction and Bayesian inference. *SIAM Journal on Computing*, 27(4):942–959, 1998.

[3] A. Becker and D. Geiger. Optimization of Pearl’s method of conditioning and greedy-like approximation algorithms for the vertex feedback set problem. *Artificial Intelligence*, 83:167–188, 1996.

[4] N. L. Biggs and A. G. Boshier. Note on the girth of Ramanujan graphs. *Journal of Combinatorial Theory. Series B*, 49(2):190–194, 1990.

[5] F. A. Chudak, M. X. Goemans, D. S. Hochbaum, and D. P. Williamson. A primal-dual interpretation of two 2-approximation algorithms for the feedback vertex set problem in undirected graphs. *Operations Research Letters*, 22:111–118, 1998.

[6] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[7] G. Even, J. Naor, B. Schieber, and L. Zosin. Approximating minimum subset feedback sets in undirected graphs with applications. *SIAM Journal on Discrete Mathematics*, 13(2):255–267, 2000.
[8] M. X. Goemans and D. P. Williamson. The primal-dual method for approximation algorithms and its application to network design problems. In Approximation Algorithms for NP-Hard Problems, chapter 4, pages 144–191. PWS Publishing Company, 1997.

[9] M. Grohe and D. Marx. On tree width, bramble size, and expansion. Journal of Combinatorial Theory. Series B, 99(1):218–228, 2009.

[10] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. American Mathematical Society. Bulletin. New Series, 43(4):439–561, 2006.

[11] S. Khot and O. Regev. Vertex cover might be hard to approximate to within $2 - \varepsilon$. Journal of Computer and System Sciences, 74(3):334–349, 2008.

[12] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. Combinatorica, 8(3):261–277, 1988.

[13] G. A. Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. Problemy Peredachi Informatsii, 24(1):51–60, 1988.

[14] C. H. Papadimitriou and K. Steiglitz. Combinatorial Optimization: Algorithms and Complexity. Prentice-Hall, Englewood Cliffs, NJ, 1982.

DÉPARTEMENT DE MATHÉMATIQUE
UNIVERSITÉ LIBRE DE BRUXELLES
BRUSSELS, BELGIUM
E-mail address: sfiorini@ulb.ac.be

DÉPARTEMENT D’INFORMATIQUE
UNIVERSITÉ LIBRE DE BRUXELLES
BRUSSELS, BELGIUM
E-mail address: gjoret@ulb.ac.be

DIPARTIMENTO DI INGEGNERIA DELL’IMPRESA
UNIVERSITÀ DI ROMA “TOR VERGATA”
Rome, Italy
E-mail address: pietropaoli@disp.uniroma2.it