Magnetostatics and the electric impact model

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Abstract The action of certain static magnetic fields on charged test particles is interpreted as a consequence of the interaction of the particles with electric dipole distributions emitted by other charged particles in relative motion. The dipole model of electric forces was initially conceived to emulate Coulomb’s law, but is applied here to a wide class of phenomena, such as forces between parallel conductors, a relativistic correction of Biot–Savart’s law, and the magnetic moment of a current loop.

1 Introduction

An impact model for electrostatic forces has been proposed by Wilhelm et al. (2013) that allows a description of the attraction and repulsion of electrically charged particles, in line with Coulomb’s law, by the interaction of massless electric dipoles travelling at the speed of light in vacuum, $c_0$. We now want to consider some special cases that would be treated under the heading “magnetostatic fields” in the framework of classical concepts.

2 Relative motion of charges

In preparation for the following sections, we study a body $B$ with a charge $q$ at rest in an inertial system $S’$ (at $x’ = 0$). It moves relative to another body $A$ with charge $Q$ ($|Q| \gg |q|$) at $x = 0$ of a coordinate system in $S$ with constant velocity, $v_S = |v_S|$, parallel to the $x$ axis and an impact parameter $b > 0$ on the $y$ axis. The discussion is first limited to the point of closest approach, where the classical electric field component of the charge $Q$ along $b = b \hat{b}$ (unit vector: $\hat{b}$) in $S$ is

$$E_b = \frac{Q}{4 \pi \varepsilon_0 b^2}$$

(1)

with the electric constant in vacuum $\varepsilon_0$. In system $S’$, the Lorentz transformations give (cf., v. Weizsäcker 1934; Greiner 1982; Jackson 2006):

$$E’_b = \gamma E_b$$

(2)

as well as

$$B’ = \frac{\gamma \mu_0}{c_0^2} E_b (v \times \hat{b}) = \gamma Q \frac{\mu_0}{4 \pi} \frac{v \times \hat{b}}{b^2},$$

(3)

with $\gamma = (1 - \beta^2)^{-1/2}$ the Lorentz factor ($\beta = v_S/c_0 < 1$), $B’$ the magnetic field and $\mu_0$ the permeability of the vacuum.

Without motion between $S$ and $S’$, Coulomb’s law could be described by the dipole model through the exchange of electric dipoles with momentum $p_D$ as:

$$p_D \frac{\Delta N_{q,Q}}{\Delta t'} = \frac{|Q| |q|}{4 \pi \varepsilon_0 b^2} = p_D \frac{\Delta N_{Q,q}}{\Delta t},$$

(4)

with the interaction rates $\Delta N_{q,Q}/\Delta t'$, $\Delta N_{Q,q}/\Delta t$ and $t’ = t$ in this case.

A relative motion of the bodies $A$ and $B$ leads to a distinction between the eigentimes in $S$ and $S’$, according to the special theory of relativity (STR, Einstein).
As seen from \( x = 0 \), the eigentime at \( x' = 0 \) then is according to the Lorentz transformations
\[
t' = \gamma t
\]
and the separation distance of \( B \) from \( A \) is
\[
r' = (b^2 + \frac{c^2 \beta^2 \gamma^2}{t^2})^{1/2} .
\]
Integration of the \( y \) component of the momentum transfer to \( B \) along the total path of \( q \) from \( t = -\infty \) to \( +\infty \) then yields
\[
(P^E)_y = \frac{\gamma Q q b}{4 \pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{dt}{(b^2 + \frac{c^2 \beta^2 \gamma^2}{t^2})^{3/2}} .
\]
With the substitution
\[
X = b^2 + \frac{c^2 \beta^2 \gamma^2}{t^2} ,
\]
the integral is listed in Bronstein and Semendjajew (1985) and can be evaluated as
\[
\int_{-\infty}^{\infty} \frac{dt}{X^{3/2}} = \int_{-\infty}^{\infty} \frac{dt}{X \sqrt{X}} = \frac{2}{\gamma c_0 \beta b^2} .
\]
Finally, with the help of Eq. (11), we get
\[
(P^E)_y = \frac{Q q}{2 \pi \varepsilon_0 b c_0 \beta} = \frac{Q q}{2 \pi \varepsilon_0 b v_S} .
\]
The assumption of a constant \( v_S \) requires special situations (large mass-to-charge ratios, but see also next section). In general, the charge \( q \) will be affected by the presence of \( Q \) leading to an elastic scattering on hyperbolic trajectories (cf., Rutherford’s formula).

3 Parallel conductors

Another interesting case consists of two (infinitely) long parallel conductors, \( C_1 \) and \( C_2 \), separated by \( b \) and carrying currents, \( I_1 \) and \( I_2 \), respectively. The classical treatment gives a circular magnetic field around \( C_1 \) produced by the current \( I_1 \), which we assume to be much larger than \( I_2 \). Applying the law of Biot–Savart, we get
\[
B_1(b) = \frac{\mu_0 I_1}{2 \pi b}
\]
at the location of the conductor \( C_2 \), and a force per length \( L \) of
\[
K = \frac{\mu_0}{2 \pi} \frac{I_1 I_2}{b}
\]
there.

How would the dipole model description cope with this situation? Let us assume that both conductors and their positive charges, \( Q_1 \) and \( Q_2 \), are at rest in system \( S \), whereas the negative charges in the conduction band are all in system \( S' \) and moving with the same speed:
\[
\nu_S = \frac{1}{Z} \frac{\Delta L}{\Delta t} = \frac{1}{Z} \frac{\Delta L'}{\Delta t'} = \frac{1}{Z} \frac{\gamma \Delta L}{\Delta t'} > 0 .
\]
The number of positive and negative charges, \( Z \), per length, \( \Delta L \), will be assumed to be the same in both conductors. With
\[
I_{1,2} = -|q_{1,2}| Z \frac{\nu_S}{\Delta L} ,
\]
where \(-|q_{1,2}|\) are the moving charges, we get currents in the same direction. Substitution of \( \nu_S \) in Eq. (14) yields
\[
I_{1,2} = \frac{|q_{1,2}|}{\Delta t} = -\gamma |q_{1,2}| \frac{\Delta L}{\Delta t'} .
\]
One can imagine that the negative charges are fixed on massless rods. The “Lorentz contraction” then leads to a factor of \( \gamma \) in the linear number density of the charges in the moving frame.

We construct, to demonstrate the principle, our experiment in such a way that the charges in the conductors are separated by distances much larger than the separation of the conductors, \( b \), i.e., we select \( b \ll \Delta L/Z \ll \Delta L \). In addition, we assume that the pairs of negative or positive charges, \((-|q_1|, -|q_2|)\) or \((Q_1, Q_2)\), maintain a constant separation distance of \( b \). Four electrostatic forces have then to be evaluated between the following pairs:

(1) \((Q_1, Q_2)\)
(2) \((-|q_1|, Q_2)\)
(3) \((-|q_1|, -|q_2|)\)
(4) \((Q_1, -|q_2|)\).

It is clear that neither the pairs (1) and (2) together give a contribution, nor pairs (3) and (4), as long as there is no relative motion. If there is such a motion, the example in Sect. 2 can directly be applied. According to Eq. (10) each isolated pair, even if its charges are in relative motion, does not provide a contribution differing from the electrostatic attraction or repulsion. Together with Eq. (15), however, differential momentum transfers from \(-|q_1|\) to \( Q_2 \) and \( Q_1 \) to \(-|q_2|\) results:
\[
\Delta P = \frac{q_1 Q_2}{2 \pi \varepsilon_0 b v_S} (\gamma - 1) = \frac{q_2 Q_1}{2 \pi \varepsilon_0 b v_S} (\gamma - 1) .
\]
The evaluation of Eq. (16) will be carried out for the length \( \Delta L \) in three steps:

• The charges \(-|q_1|\) move with \( \nu_S \) in case (2). We consider (as an approximation under our geometric assumptions) the interaction with \( Z \) charges \( Q_2 \)
to be the sum of $Z$ individual interactions of pairs \((-|q_1|, Q_2)\) per length, $L$, and get

$$\frac{P_2}{\Delta L} = Z \frac{\Delta P_2}{\Delta L} = - \frac{(Z/\Delta L)|q_1|Q_2}{2 \pi \varepsilon_0 b v_S} (\gamma - 1).$$ \hspace{1cm} (17)$$

Since $\gamma$ can be approximated for small $\beta$ by the first two terms of the expansion

$$\gamma = 1 + \frac{1}{2} \beta^2 + \ldots,$$ \hspace{1cm} (18)

we find

$$\frac{P_2}{\Delta L} \approx - \frac{(Z/\Delta L)|q_1|Q_2 v_S}{4 \pi \varepsilon_0 b c_0^2}.$$ \hspace{1cm} (19)

Considering that the factor $\gamma$ in Eq. (18) has been taken care of in Eq. (16) and that $-(Z/\Delta L)Q_2 v_S$ can be expressed by $I_2$ in conductor $C_2$, half the specific force can be written as

$$K_2 = \frac{P_2}{2 \Delta L} \approx - \frac{\mu_0 I_1 I_2}{2 \pi b},$$ \hspace{1cm} (20)

noting that $c_0^2 = (\mu_0 \varepsilon_0)^{-1}$.

- The symmetry between cases (2) and (4) in Eq. (16) then yields for parallel currents

$$K_p = \frac{K_2 + K_4}{2 \Delta L} \approx - \frac{\mu_0 I_1 I_2}{2 \pi b}; \quad I_1 I_2 > 0.$$ \hspace{1cm} (21)

A full treatment without approximations will have to confirm that Eq. (21) is valid as an equality.

- A slight complication in our argumentation arises when we consider opposite currents in parallel conductors. The negative charges moving relative to the positive ones in the other conductor lead to the same attraction as in the previous example, namely $(K_2 + K_4)/\Delta L$. However, the negative charges in the conductors are now counter-moving with a relative speed of $2 v_S \ll c_0$. So, we have for these charges

$$\gamma' \approx 1 + \frac{2 v_S^2}{c_0^2} + \ldots$$ \hspace{1cm} (22)

and an additional specific force

$$K_4 \approx - \frac{\mu_0 I_1 I_2}{\pi b}; \quad I_1 I_2 < 0.$$ \hspace{1cm} (23)

This repulsion is twice as large as the attraction. Conceptually, this result could have been obtained by assuming that the negative charges are moving in one conductor and the positive charges in the other one.

Thus the dipole model gives an expression for the specific force equivalent to that of the classical magnetostatic field in Eq. (12) for both parallel and antiparallel currents.

### 4 Charge near a current

Assume moving negative charges causing a current, $I$, in a conductor, $C$, which is at rest in system $S$, together with its positive charges, and a negative test charge $-|q| = -|e|$ at a distance $b$ from $C$. Biot–Savart’s law gives a magnetic field according to Eq. (11) at the position of the test charge. If this is at “rest”, no Lorentz force would be expected. The dipole model provides a clear answer when this condition is fulfilled, namely, when the test charge is moving parallel to the conductor with $v_S/2$, half the speed of the negative charge carriers in the conductor, because then the effects of its positive and negative charges on the test charge cancel each other. This can be seen as a relativistic correction of the law of Biot–Savart (cf., Greiner 1982).

As an example, we consider a long conductor with a cross-section, $A$. In a certain volume, $V = A \Delta L$, with length $\Delta L$, there are $Z$ electrons as current carriers available. They are at rest in system $S'$ moving with

$$v_S = \frac{-\Delta L}{Z} \frac{I}{|e|}$$ \hspace{1cm} (24)

relative to the positive charges, cf., Eq. (14). If we place the test charge at $b \ll \Delta L$ from the conductor, the momentum transfer will not be significantly affected by charges outside $\Delta L$, and we can divide Eq. (16) by $\Delta t = \Delta L/(Z v_S)$ to find the force acting on the test charge (applying arguments similar to those of the previous section):

$$K = \frac{\Delta P}{\Delta t} \approx \frac{\mu_0 |e| I v_S}{4 \pi b}.$$ \hspace{1cm} (25)

Approximately $8.52 \times 10^{21}$ electrons are in the conduction band of copper for $A = 0.1 \text{ mm}^2$ and $L = 1 \text{ m}$ (Westphal 1956).

A current of $I = 1 \text{ A}$ then requires $v_S = 0.73 \text{ mm s}^{-1}$, and an electron (at rest in $S$) at $b = 1 \text{ cm}$ experiences a repulsive force $K \approx 1.2 \times 10^{-27} \text{ N}$. A smaller cross-section of $A = 0.001 \text{ mm}^2$, for instance, would then lead to an increase of the quantities $v_S$ and $K$ by a factor of one hundred.

The significant feature of Eq. (25) is that the force $K$ depends on the speed of the charge carriers, and therefore on the characteristics of the conductor. This result might be used as an experimental test in a suitable arrangement.

### 5 Magnetic moment

A magnetostatic dipole field can also be treated in this context. A current, $I$, in a circular conductor with radius, $R$, will generate a magnetic moment
would result, where we have used a current $I = Q/T = Q v Q / (2 \pi R)$ generated by a charge, $Q$, moving around the dotted circle with a period, $T$, so that, for reasons of simplicity, its speed is $v Q = v = |v|$. We have split up the moving charge into $\pm |Q|/2$ as a trick to eliminate the averaged electrostatic force. We will first discuss two special cases in the framework of the dipole model: (a) Let $|q|$ move with the same velocity vector as indicated for $\pm |Q|/2$. Then there are no forces other than the electrostatic ones. (b) If $|q|$ moves antiparallel, we get $\beta' = 2 v / c_0$ and additional forces.

The equivalence of the force resulting from the dipole model with Eq. (27) can now be shown by integrating the movements of the charges $\pm |Q| / 2$ over a full revolution. With the help of Eqs. (11) and (23) (for which we have shown the equivalence with the dipole model), applied separately to $\pm |Q| / 2$, we find for the force averaged over a period $T$: 

$$\langle K_\pm \rangle = \frac{|q|}{T} \int_0^T \gamma E_\pm \, dt = \pm \frac{|Q|}{8 \pi \varepsilon_0 b^2} \frac{|q|}{T} b_\pm \int_0^T \gamma \, dt ~. \quad (28)$$

Substituting in Eq. (15)

$$\beta^2(\varphi) = \frac{4 v^2}{c_0^2} \cos^2 \frac{\varphi}{2} ~, \quad (29)$$

obtained from the vector sum of the velocities of $q$ and $Q$, and replacing $\, dt$ by the azimuthal angle $d \varphi = \pm v \, dt / R$ (30)

for the positive charge or the negative one, respectively, we get with $\cos \theta = R / b$ after a straightforward calculation, taking into account Eqs. (13) and (18), the total average force exerted by the electric dipoles

$$K_a = \langle (K_+) + (K_-) \rangle \cos \theta = - \frac{\mu_0}{4 \pi} \frac{R v^2}{b^3} |q| \, Q \, \hat{x} ~, \quad (31)$$

which is the same as $K_m$ in Eq. (27).

It should be noted that this integration involves accelerations of the charges in the conductor, therefore, we can only expect an estimate for small accelerations, i.e., for large $R$. How these considerations have to be applied to magnetic moments of charged particles is, of course, of great interest, but is beyond the scope of this conceptual presentation (for a recent discussion, see Hughes and Kinoshita [1993]).

6 Discussion and conclusions

The electric impact model not only describes electrostatic forces [Wilhelm et al. 2013], but also forces acting on moving charges seemingly caused by static magnetic fields. In Sect. 3 the speed of the negative charge carriers did not enter into the final result and thus the definition of the Système International d’Unités (SI; BIPM 2006) of the unit of the electric current is not affected:

The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 m apart in vacuum, would produce between these conductors a force equal to $2 \times 10^{-7}$ N per metre of length. This definition leads to $\mu_0 = 4 \pi \times 10^{-7}$ H m$^{-1}$ (exact).

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We again assume the masses of the bodies involved to be so large that their accelerations can be neglected.
On the other hand, it has been found in Sect. 4 that the application of Biot–Savart’s law in determining the Lorentz force acting on a charged particle depends on the speed of the charge carriers according to STR and, therefore, on the characteristics of the conductor.

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