Transmission phase shift of phonon-assisted tunneling through a quantum dot

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I. INTRODUCTION

Electron transport through a mesoscopic system, e.g. a quantum dot (QD), has been extensively investigated in the last two decades. Due to the fact that the size of a mesoscopic device is within the phase coherent length, the phase of a wave function plays a key role on the electronic transport. So the transmission amplitude $t = |t|e^{i\theta}$, which describes the electron tunneling through a mesoscopic system, is a complex number. Its magnitude square $|t|^2$ is the transmission probability which is observable in the measurement of current or conductance. The transmission phase $\theta$ describes the phase change when an electron tunnels through a device. This phase $\theta$ is in general lost in the measurement of current or conductance, so that it is difficult to acquire in experiment. Using an AB interference ring device, Yacoby et al. tried, for the first time, to measure the transmission phase $\theta$ through a QD. Coupled years later, Schuster et al. utilized an open multi-terminal AB ring device to successfully measure the phase $\theta$. Since then, to investigate the transmission phase has generated a great deal of theoretical and experimental interest with a fare amount of efforts focusing in this field. On the experimental side, for example, Buks et al. reported that controlled decoherence could be achieved in a device with a QD that is capacitively coupled to a quantum point contact in coherence could be achieved in a device with a QD that was subsequently observed.

Since then, to investigate the transmission phase has generated a great deal of theoretical and experimental interest with a fare amount of efforts focusing in this field. On the experimental side, for example, Buks et al. reported that controlled decoherence could be achieved in a device with a QD that is capacitively coupled to a quantum point contact in close vicinity. The phase evolution in the Kondo regime was experimentally investigated a few years back and was found to be highly sensitive to the onset of Kondo correlation. Recently, Leturcq et al. investigated the magnetic field symmetry and the phase rigidity of the nonlinear conductance in an AB ring. On the other hand, the success of these experiments generates a number of theoretical studies. In early of 1980s, Buttiker found the phase rigidity in a two-terminal AB ring device due to the time-reversal symmetry and the current conservation. After the experiment by Schuster et al. many follow-up theoretical efforts focused on and tried to interpret the measured results of the transmission phase $\theta$ through a QD, in particular, the abrupt lapses of $\theta$ between two adjacent resonances and the similar behavior of $\theta$ for all resonant peaks. In addition, some works have also studied the transmission phase in the Kondo regime or with the photon-assisted tunneling process under a time-dependent external field etc.

Another subject, the electron-phonon (e-ph) interaction in a single-molecular QD, has also generated a great deal of interest in recent years. The phonon-assisted tunneling peaks or steps have been experimentally observed in various single-molecule transistor systems. Park et al. observed phonon-assisted tunneling sub-steps in the I-V curves in a single-C₆₀ transistor device, and those sub-steps are attributed to the coupling of electron and the C₆₀-surface vibration mode. In another experiment by Leroy et al. the current and the conductance of a suspended individual single-wall nanotube device are measured, and the phonon-assisted sub-peaks on the two sides of the main resonant peak are clearly visible in the differential conductance versus gate voltage, which is due to the radial breathing phonon mode. On the theoretical side, the influence of e-ph interaction on the mesoscopic transport is also studied by several groups. Many interesting results, e.g. the phonon-assisted sub-peaks, etc., are first theoretically predicted, and then experimentally observed.

In this paper, we investigate the transport behavior of a molecular QD system having an e-ph interaction by using the scattering matrix method. We focus mainly on the transmission phase of the phonon-assisted tunneling sub-peaks, as well as the phonon-induced dephasing process. The results exhibit that the transmission phase $\theta$ drops between two adjacent (sub)-peaks and $\theta$ rises again near the position of sub-peaks. In particular, the characteristic of phonon-assisted tunneling process in the transmission phase is much more pronounced and visible than these sub-peaks in the conductance. Afterwards we discuss the dephasing ratio. While at zero temperature and at low bias $V (V < \omega_0$ with $\omega_0$ being the phonon frequency), the electronic transport through the molecular QD is completely coherent because the electron can not absorb or emit phonons under this condition. How-
ever, if at non-zero temperature or at a high bias $V_{\text{bias}}$ ($V_{\text{bias}} > \omega_0$), the dephasing process occurs. In the limit of high bias $V_{\text{bias}}$ ($V_{\text{bias}} \gg \omega_0$), the dephasing ratio goes as square of the e-ph interaction strength $\lambda$ in the weak interaction region, but it is linearly dependent on $\lambda$ in the strong interaction region. In addition, we also consider an open AB ring device with a molecular QD embedded in one of its arms, and find that it is feasible to experimentally measure the influence of the e-ph interaction through the transmission phase.

The rest of this paper is organized as follows. We introduce the model and derive the formula of transmission amplitude in Sec. II. In Sec. III, we present the numerical results and their discussions. In Sec. IV, we study the phase measurement by using an open AB ring device. Finally, a brief summary is presented in Sec. IV. Some detailed derivation of the transmission amplitude is given in Appendix.

II. MODEL AND FORMULATION

The system under consideration is a molecular QD coupled to left and right leads in the presence of a local phonon mode, and it can be described by the following Hamiltonian:

$$H = H_0 + H_1,$$

where

$$H_0 = \sum_{\alpha,k} \epsilon_{\alpha k} c_{\alpha k}^{\dagger} c_{\alpha k} + \epsilon_0 b^\dagger b, \quad (1)$$

$$H_1 = \lambda (b^\dagger + b) d^\dagger d + \sum_{\alpha,k} [t_{\alpha k} c_{\alpha k}^{\dagger} d + H.c]. \quad (2)$$

Here $c_{\alpha k}^{\dagger}$ ($c_{\alpha k}$) and $d^\dagger$ ($d$) are the electron creation (annihilation) operators in the lead $\alpha = L, R$ and the QD, respectively. $b^\dagger$ ($b$) is the phonon creation (annihilation) operator in the QD. Due to large level spacing of the molecular QD, only one relevant quantum level $\epsilon_0$ is considered. The electron in the QD is coupled to a single phonon mode $\omega_0$, and $\lambda$ and $t_{\alpha k}$ describe the strength of the e-ph interaction and the coupling between the QD and the leads, respectively.

In the following, we apply the S-matrix scattering formalism to derive the transmission amplitude, the transmission phase and the current. From Hamiltonian (1), the S matrix can be written as:

$$S = 1 - i \int_{-\infty}^{\infty} dt_1 e^{iH_0 t_1} H_1 e^{-iH_0 t_1} e^{-\eta|t_1|}$$

$$-i \int_{-\infty}^{\infty} dt_1 dt_2 e^{iH_0 t_2} H_1 \tilde{G}_r (t_2 - t_1)$$

$$\times H_1 e^{-iH_0 t_1} e^{-\eta(|t_1| + |t_2|)}, \quad \eta \to 0^+ \quad (4)$$

The single-particle Green’s function operator $\tilde{G}_r(t)$ is, $\tilde{G}_r(t) = -i\theta(t)e^{-iHt}$. By using the S matrix, the final state $|f>$ can be obtained from the initial state $|i>$, with $|f> = S \times |i>$. Considering an initial state $|i> = |\epsilon_i, n, L>$, which denotes an electron with energy $\epsilon_i$ in the left lead and $n$ phonons in the QD, the final state $|f>$ can be expressed as:

$$|f> = S \times |i> = S \times |\epsilon_i, n, L>$$

$$= \sum_{m=-\infty}^{+\infty} [r_m(\epsilon_f, \epsilon_i)|\epsilon_f, m, L + t_m(\epsilon_f, \epsilon_i)|\epsilon_f, m, R>]$$

$$= \sum_{m=-\infty}^{+\infty} [r_m(\epsilon_i)\delta(\epsilon_i + n\omega_0 - \epsilon_f - m\omega_0)|\epsilon_f, m, L>$$

$$+ t_m(\epsilon_i)\delta(\epsilon_i + n\omega_0 - \epsilon_f - m\omega_0)|\epsilon_f, m, R>]. \quad (5)$$

where $t_m(\epsilon)$ and $r_m(\epsilon)$ are the transmission amplitude and the reflection amplitude with accompanying absorption or emission of $|m - n|$ phonons. At zero temperature, the phonon number $n$ in the initial state $|i>$ must be zero and then $t_m(\epsilon_i)$ can be written as (the detailed derivation is shown in the Appendix):

$$t_m(\epsilon_i) = \int d\epsilon_f t_m(\epsilon_f, \epsilon_i)$$

$$= -i \frac{\Gamma e^{-\epsilon^2}}{\sqrt{m!}} \sum_{l=0}^{m} \frac{(-1)^{m-l} m!}{l!(m-l)!}$$

$$\times \sum_{n=0}^{\infty} \frac{\lambda^{2n+m}}{n!} G(\epsilon_i - n\omega_0 - l\omega_0) \quad (6)$$

Obviously, $t_0(\epsilon)$ describes the amplitude of an elastic tunneling process which is coherent. While $t_m(\epsilon)$ ($m \neq 0$) is the amplitude of an inelastic tunneling for emitting $m$ phonons. Due to emission of phonons, thus, leaving a trace in the QD for the inelastic tunneling process, an inelastically tunnelled electron loses its phase coherence. So at zero temperature the transmission phase shift through the QD is $2\lambda$.

$$\theta = \arg \{t_0(0)\}. \quad (7)$$

From $t_m(\epsilon)$, the total transmission probability (including the coherent and the non-coherent parts) through the QD is $T_{\text{tot}}(\epsilon) = \sum_{m=0}^{\infty} |t_m(\epsilon)|^2$, and the transmission probability of the non-coherent part is $T_d(\epsilon) = \sum_{m=1}^{\infty} |t_m(\epsilon)|^2$.

III. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we numerically study the transmission phase $\theta$ and the dephasing ratio $T_d/T_{\text{tot}}$. In our numerical calculations, the phonon frequency $\omega_0$ is set as the energy unity ($\omega_0 = 1$). Notice that the main result of this paper, Eq. (6), is obtained at zero temperature, and thus so are the numerical results and their discussions.
However, results at low temperature should be similar to that at zero temperature. Fig. 1 shows the coherent transmission probability $|t_0|^2$ [namely, $|t_0(0)|^2$] and the phase $\theta$, as a function of renormalized level $\pi_0$, which can be tuned by the gate voltage in an experiment. Notice that $|t_0|^2$ is proportional to the linear conductance $G$ through the QD, $G = (e^2/h)|t_0|^2$. Due to the e-ph interaction, several interesting features are manifested. In addition to the main peak related to the single level, new satellite sub-peaks appear in the curve of $|t_0|^2 - \pi_0$ at $-\pi_0 = n\omega_0$ ($n = 1, 2, \ldots$). The sub-peaks only exist on the right hand side of the main peak, and their heights increase with the increase of the e-ph coupling strength $\lambda$. The sub-peak at $\pi_0 = -\omega_0$ corresponds to the phonon-assisted tunneling process as shown in the inset of Fig. 1(a), in which an incident electron from the left lead first emits a phonon and tunnels to the level $\pi_0$, and subsequently reabsorbs a phonon and tunnels forward to the right lead. Since this process does not leave a trace in the QD, it maintains the phase coherence. Meanwhile at zero temperature, there is no phonon in the QD in the initial state and the absorption process can not occur, so that the satellite sub-peaks only exist on the negative $\pi_0$ side [see Fig. 1(a)].

Next, we study the transmission phase $\theta$ that exhibits a non-monotonic behavior. Across the main resonant peak, $\theta$ continuously rises by a value of $\pi$. This result is consistent with the previous theoretical and experimental findings. Because of the e-ph interaction, $\theta$ drops between the main peak and the next sub-peak or between two adjacent sub-peaks, and rises again across a sub-peak, such that a dip appears around $-\pi_0 = n\omega_0$. These dips are much more pronounced than those sub-peaks in the transmission probability $|t_0|^2$. For example, for $\lambda = 0.8$ the 2-nd phonon-assisted sub-peak is so small that it is hardly visible [see Fig. 1(a)], however even the 4-th dip can be clearly seen [see Fig. 1(b)]. The sensitivity of the transmission phase $\theta$ to the e-ph interaction provides a new way to detect the strength of the e-ph interaction.

In the calculations above, the tunneling coupling strength $\Gamma$ ($\Gamma = \Gamma_L = \Gamma_R$) between the leads and the QD is set to be quite weak, $\Gamma = 0.1 \ll \omega_0$. With an increase of $\Gamma$, the phonon-assisted sub-peaks and the main peak in the curve of $|t_0|^2 - \pi_0$ gradually merge together and become indistinguishable, and the dips in the curve of $\theta - \pi_0$ are also gradually getting smaller (see Fig. 2). While $\Gamma \approx \omega_0$ (e.g. $\Gamma = 0.7$), all sub-peaks and all dips are almost invisible. Consequently, in order to experimentally detect the phonon-induced dips of the phase $\theta$ or the phonon-assisted sub-peaks, the coupling strength $\Gamma$ should be tuned to be less than $\omega_0/2$. In fact, the condition $\Gamma < \omega_0/2$ is normally satisfied in the experiments.

Let us study the amplitude $t_m(\epsilon)$ ($m = 1, 2, \ldots$) of the inelastic tunneling process. In this inelastic tunneling process, an incident electron emits $m$ phonons while tunneling through the QD. However it is prohibited when $V_{bias} < m\omega_0$. At small bias case $V_{bias} < \omega_0$ and at zero temperature, all inelastic tunneling processes are prohibited and the tunneling through the QD is coherent. On the other hand, with $V_{bias} > \omega_0$, inelastic tunneling processes occur and the tunneling through the QD is partly non-coherent. In the limit of larger bias voltage, $V_{bias} \gg \omega_0$, the total dephasing transmission probability $T_d(\epsilon)$ is: $T_d(\epsilon) = \sum_{m=1}^{\infty} |t_m|^2$. Fig. 3 shows the dephasing transmission probability $T_d$ (or $T_d/T_{tot}$) versus renormalized level $\pi_0$ and the e-ph interaction strength $\lambda$. While without the e-ph interaction (namely $\lambda = 0$), no inelastic tunneling process happens and thus $T_d = 0$. When $\lambda \neq 0$, the inelastic tunneling process occurs and $T_d$ is no longer zero. A serial of peaks are exhibited in the curve of $T_d$ versus $\pi_0$ and the interval between the two adjacent peaks is $\omega_0$ [see Fig. 3(a)]. As is seen from Fig. 3(a), a higher peak must correspond to a larger value of $\lambda$ (including the peak at $\pi_0 = 0$). This means that the dephasing probability $T_d$ monotonously increases with $\lambda$ regardless of the position of the renormalized level $\pi_0$. Next, in Fig. 3(b) we show the relative dephasing transmission probability $T_d/T_{tot}$ versus the e-ph coupling strength $\lambda$ in the resonant tunneling region (namely $\pi_0 = 0$). When $\lambda$ is small ($\lambda < 0.2\omega_0$), the relative dephasing transmission probability $T_d/T_{tot}$ increases parabolically with $\lambda$, but the dephasing probability $T_d/T_{tot}$ is found to increase linearly with increasing $\lambda$ between the range of $1 > \lambda > 0.4$. For large $\lambda$ case ($\lambda > 1$), $T_d/T_{tot} > 0.6$ and the dephasing inelastic tunneling processes dominate. In an experiment the parameter $g = (\lambda/\omega_0)^2$ is generally in the range from 0.1 to 1, though some special devices show a big variable range of $g$. In this $\lambda$ region, the degree of dephasing is linearly dependent on the e-ph coupling strength.

IV. THE AB RING DEVICE

In Sections II and III, we only consider a simple device consisting of a QD coupled to two leads. However, in a real experiment to measure the transmission phase $\theta$, the device is an open AB ring with a QD embedded in one of the arms. Therefore, it is of experimental relevance to study the open AB ring device in this section. A QD is embedded in one arm of the ring, and the other arm is the reference arm with the transmission amplitude $t_{ref}$. Due to openness of the open AB ring device, the processes of multi-time circling around the ring is negligible. Note that only elastic tunneling process $t_0(\epsilon_i)$ is in interference with the reference arm. $T_{AB}(\epsilon_f, \epsilon_i)$, defined as the probability that an electron of energy $\epsilon_i$ incident from the left lead will be transmitted with energy $\epsilon_f$ into the right lead, can therefore be written as:

$$T_{AB}(\epsilon_f, \epsilon_i) = \sum_{m=0}^{\infty} \delta(\epsilon_i - \epsilon_f - m\omega_0)|t_m(\epsilon_i) + \delta_{m,0} e^{i\theta_{ref}}|^2.$$ (8)
where $\phi$ is the magnetic flux inside the ring. In the absence of the reference arm (namely, $t_{ref} = 0$), $T_{AB}(\varepsilon_f, \varepsilon_i)$ is reduced to $T(\varepsilon_f, \varepsilon_i) = \sum_{m=0}^{\infty} \delta(\varepsilon_f - \varepsilon_i - m\omega_0) |t_m(\varepsilon_i)|^2$, and this result is the same as that in the work by Wingreen et al. Using the transmission probability $T_{AB}(\varepsilon_f, \varepsilon_i)$, the current flowing through the AB ring is

$$J_{AB} = \frac{2e}{h} \int d\varepsilon_i \int d\varepsilon_f T_{AB}(\varepsilon_f, \varepsilon_i) f_L(\varepsilon_i)[1 - f_R(\varepsilon_f)]$$

and

$$\frac{2e}{h} \int d\varepsilon_i \int d\varepsilon_f T_{AB}(\varepsilon_f, \varepsilon_i) f_R(\varepsilon_i)[1 - f_L(\varepsilon_f)],$$

where $f_L(\varepsilon_i) = f(\varepsilon_i - \mu_L)$ and $f_R(\varepsilon_f) = f(\varepsilon_f - \mu_R)$ with the chemical potential $\mu_{L(R)} = \pm eV_{bias}/2$ and $f(\varepsilon)$ is the Fermi distribution function. Finally, the differential conductance $G_{AB}$ can be obtained from $G_{AB} = dJ_{AB}/dV_{bias}$.

Based on Eq. (10), we show the numerical results of the current $J_{AB}$ flowing through the open AB ring in Fig. 4. Figs. 4(a) and 4(b) correspond to small and large bias voltage cases, respectively. In both cases, the phonon-assisted sub-peaks can be observed on the right hand side of the main peak, and the sub-peak height increases with increasing e-ph coupling strength $\lambda$. These results are similar to those in the previous paper.

Next, we focus on the differential conductance $G_{AB}$ ($G_{AB} = dJ_{AB}/dV_{bias}$) and its dependence on the renormalized level $\varepsilon_0$ or on the magnetic flux $\phi$. In fact, $\varepsilon_0$ and $\phi$ can be well controlled and are continuously tunable in an experiment. The differential conductance $G_{AB}$ is always a periodic function of the magnetic flux $\phi$ with a period of $2\pi$. Figs. 5(a) and 5(b) show the linear conductance $G_{AB}$ at zero bias voltage. Here a series of phonon-assisted sub-peaks exhibits in the curve of $G_{AB}$ versus $\varepsilon_0$, similar to that in the transmission probability $|t_0|^2$ [see Fig. 1(a)] because of small value of $t_{ref}$. Besides, the phonon-assisted tunneling processes can also be observed from the amplitude of the $G_{AB}$ oscillation versus the magnetic flux $\phi$ [see Fig. 5(b)]. While $\varepsilon_0 = 0$ or $-1$ (i.e. at the main peak or the 1-st sub-peak), the AB oscillation amplitudes are quite large since the phonon-assisted elastic tunneling processes play a role here. But at the position between two adjacent peaks (e.g. $\varepsilon_0 = -0.5$ or $-1.5$), the AB oscillation amplitude is quite weak. When a small bias voltage is applied between the left and right leads, all peaks in the curve of $G_{AB}$-$\varepsilon_0$, including the main peak and phonon-assisted sub-peaks, split into two [see Fig. 5(c)], and their positions are at $\varepsilon_0 = m\omega_0 \pm V_{bias}/2$. The reason is that at these values of $\varepsilon_0 = m\omega_0 \pm V_{bias}/2$, the renormalized level $\varepsilon_0$ is in line with the left or the right chemical potential $\mu_{L(R)} = \pm V_{bias}/2$, or the distance between $\varepsilon_0$ and $\mu_{L(R)}$ is just $m\omega_0$. The behavior of the conductance $G_{AB}$ versus the magnetic flux $\phi$ for small bias is similar with that of the linear conductance [see Figs. 5(b) and 5(d)]. Note that at zero bias or at small bias ($V_{bias} < \omega_0$) all tunnelings through the QD are completely coherent, and the small amplitude oscillation in $G_{AB}$ is due to the small transmission probability $|t_0|^2$. Finally, we investigate the large bias case ($V_{bias} > \omega_0$). At large bias $V_{bias}$ the peaks in the curve of $G_{AB}$ versus $\varepsilon_0$ clearly split into two with an interval of $V_{bias}$. Moreover, some extra sub-peaks emerge even on the left of the main peak. For example, the sub-peak marked by "B" in Fig. 5(e) stands on the left of the main peak at $\varepsilon_0 = -V_{bias}/2$, and their interval is $\omega_0$. In fact, this peak is from the inelastic tunneling process $t_1(\varepsilon)$ and a phonon is left in the QD with an electron tunneling through the dot. Fig. 5(f) shows the conductance $G_{AB}$ versus the magnetic flux $\phi$ while the level $\varepsilon_0$ is fixed on the peak positions of Fig. 5(e). The amplitude of the AB oscillation of the peak "B" is very weak, but the amplitudes are quite large for other three peaks. This gives a proof that the peak "B" is indeed from the inelastic tunneling process and the corresponding tunneling electron loses its phase coherent.

Since the differential conductance $G_{AB}$ is a periodic function of the magnetic flux $\phi$, one can make the Fourier expansion: $G_{AB}(\phi) = G_{AB0} + G'_{AB} \cos(\phi + \theta_0)$. Here the initial phase $\theta_0$ is a directly experimentally measurable quantity. Let us compare the measured phase $\theta_0$ with the transmission phase $\theta$. Fig. 6 shows the phase $\theta_0$ versus $\varepsilon_0$ for different bias $V_{bias}$. While at zero bias voltage the characteristics of the phase $\theta_0$, including the phonon-induced dips, are completely the same as the transmission phase $\theta$ (see Fig. 1). When a small bias voltage (e.g. $V_{bias} = 0.2 < \omega_0$) is applied between the two leads, the phase $\theta_0$ changes slightly but can still reflect the transmission phase $\theta$ quantitatively. So at zero or small bias the transmission phase $\theta$, including the intriguing characteristics due to the e-ph interaction, can be directly observed through the measurement of differential conductance versus the intra-dot level. For a molecular QD device, the phonon frequency $\omega_0$ is usually from 5mev to 35mev, so the condition $V_{bias} < \omega_0$ is easily reachable. On the other hand, at large bias case (e.g. $V_{bias} = 1.5\omega_0$), the phase $\theta_0$ deviates severely from the transmission phase $\theta$.

V. CONCLUSIONS

In summary, we study the influence of the electron-phonon (e-ph) interaction on the transmission phase and the dephasing while electron tunneling through a molecular quantum dot. It is found that the transmission phase versus the intra-dot level exhibits a non-monotonic behavior, and a pronounced dip emerges when the renormalized level locates at the position of the phonon-assisted sub-peaks. In particular, phonon-induced dips in the transmission phase are much more apparent than the phonon-assisted sub-peaks in the conductance. Besides, phonon-induce dephasing increases monotonically with the e-ph interaction strength $\lambda$. The dephasing probability $T_d$ is proportional to $\lambda^2$ at small $\lambda$, but $T_d \propto \lambda$ for large $\lambda$. In addition, the open AB ring device is investigated. At zero bias or small bias, the measurement phase
from the differential conductance versus the magnetic flux is found to have the same characteristics with the transmission phase, including the phonon-induced dips.

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APPENDIX

In this appendix, we present a detailed derivation for the transmission amplitude $t_m(\varepsilon_i)$, Eq. (6). Because those states are normalized according to $\langle \varepsilon, n, \alpha \mid \varepsilon', n', \alpha' \rangle = \delta_{n,n'}\delta_{\alpha,\alpha'}\delta(\varepsilon - \varepsilon')$, only the last term in the S-matrix [Eq. (4)] contributes to the scattering matrix element $t_m(\varepsilon_f, \varepsilon_i)$ by $\frac{\omega}{\sqrt{m}}$. Therefore the scattering matrix element $t_m(\varepsilon_f, \varepsilon_i)$ is reduced into:

$$t_m(\varepsilon_f, \varepsilon_i) = \langle \varepsilon_f, m, R | S | \varepsilon_i, 0, L \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_1 dt_2 e^{-\eta(|t_1| + |t_2|)} \times e^{i(\varepsilon_f + m\omega_0)t_2 - i\varepsilon_1 t_1} \langle \varepsilon_f, m, R | H_1 e^{-iH(t_2 - t_1)} H_1 | \varepsilon_i, 0, L \rangle.$$

To take the change of variables: $t_1 = t_2$ and $t_2 = t_1$,

the integration over $t_1$ now yields a $\delta$ function of energies as $\eta \to 0^+$, and $t_m(\varepsilon_f, \varepsilon_i)$ changes into:

$$t_m(\varepsilon_f, \varepsilon_i) = \langle \varepsilon_f, m, R | S | \varepsilon_i, 0, L \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | H_1 e^{-iHt_2} H_1 | \varepsilon_i, 0, L \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | e^{i\lambda t_2} d^\dagger c_{Lk} | \varepsilon_i, 0, L \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | c_{Lk} e^{-iHt_2} d^\dagger | \varepsilon_i, 0, L \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | c_{Lk} e^{-iHt_2} d^\dagger | 0 \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | c_{Lk} e^{-iHt_2} d^\dagger | 0 \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | e^{-iHt_2} d^\dagger | 0 \rangle.$$

$$t_m(\varepsilon_f, \varepsilon_i) = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | e^{-iHt_2} d^\dagger | 0 \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | e^{-iHt_2} d^\dagger | 0 \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | e^{-iHt_2} d^\dagger | 0 \rangle = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | e^{-iHt_2} d^\dagger | 0 \rangle.$$

where $H_{Lk} = \sum_k |t_{Lk}(\varepsilon)|^2 \delta(\varepsilon - \varepsilon_{Lk}) H_{Lk}$. At zero temperature, the above equation can be rewritten as:

$$t_m(\varepsilon_f, \varepsilon_i) = -\theta(t_2 - t_1) \int_{-\infty}^{\infty} dt_2 e^{-\eta|t_2|} \times \langle \varepsilon_f, m, R | e^{-iHt_2} d^\dagger | 0 \rangle.$$

In order to calculate $Tr\{b^m(t)d^\dagger(t)d^\dagger\}$, we apply a canonical transformation with $\Pi = e^{sH} e^{-s}$ and $s = (\lambda/\omega_0)(b^\dagger - b)d^\dagger d$. Under this canonical transformation, Hamiltonian (1) becomes:

$$\Pi = \Pi_{el} + \Pi_{ph},$$

where

$$\Pi_{el} = \sum_{\alpha, k} \varepsilon_{ak} e^\dagger \varepsilon_{ak} + \sum_{\alpha, k} \varepsilon_{ak} e^\dagger \varepsilon_{ak} + H,$$

and

$$\Pi_{ph} = \omega_0 b^\dagger b.$$
where \( u = \lambda [1 - e^{-i\omega_0(t_2 - t_1)}] \). Substituting Eq. (18) into Eq. (17), we have:

\[
t_m(\varepsilon_f, \varepsilon_i) = -i2\pi \delta(\varepsilon_i - \varepsilon_f - m\omega_0) \frac{t_R(\varepsilon_f) G^*(\varepsilon_i)}{\sqrt{m!}}
\]

\[
\times \int_{-\infty}^{\infty} dt e^{i\varepsilon_i t} G^r(t) e^{-\lambda u e^{-im\omega_0 t}(u^*)^m} \times e^{-\lambda^2 m \frac{\Gamma}{m!}} \sum_{l=0}^{m} \frac{(-1)^{m-l} l!}{l!(m-l)!} \right]
\]

\[
\times \sum_{n=0}^{\infty} \lambda^{2n+m} \left( \varepsilon_i - n\omega_0 - l\omega_0 \right), \quad (18)
\]

where \( G^r(E) \) is the Fourier transform of \( G(t) \). Here we have assumed the symmetric coupling \( t_L(\varepsilon_i) = t_R(\varepsilon_i) \) and considered the wide-band limits case, so \( \Gamma = 2\pi t_L(\varepsilon_f) t^*_L(\varepsilon_i) \) is independent of the energy \( \varepsilon_i \) and \( \varepsilon_f \). In the wide-band limit the Green’s function \( G^r(\varepsilon) \) is easily calculated following the standard procedure,15,20,21

\[
G^r(\varepsilon) = \frac{1}{\varepsilon - \varepsilon_0 + i\Gamma}. \quad (19)
\]

From \( t_m(\varepsilon_f, \varepsilon_i) \), the transmission amplitude \( t_m(\varepsilon_i) \) can be obtained:

\[
t_m(\varepsilon_i) = \int d\varepsilon f t_m(\varepsilon_f, \varepsilon_i), \text{ and the result is given in Eq. (6) in the text.}
\]

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FIG. 1: (Color online) The transmission probability $|t_0|^2$ (a) and the transmission phase $\theta$ (b) vs. the renormalized level $\varepsilon_0$ for the different e-ph interaction strength $\lambda$ with $\Gamma = 0.1$. The inset in (a) is the schematic diagram for the phonon-assisted tunneling process.

FIG. 2: (Color online) The transmission probability $|t_0|^2$ (a) and the transmission phase $\theta$ (b) vs. the renormalized level $\varepsilon_0$ for the different $\Gamma$ with the e-ph interaction strength $\lambda = 0.7$.

FIG. 3: (Color online) (a) e-ph coupling strength $\lambda$ dependence of the dephasing probability, $T_d$. (b) The relative dephasing probability $T_d/T_{tot}$ vs. $\lambda$. The parameter $\Gamma = 0.1$ in (a) and (b). The dot line in (b) is guide to the eye.
FIG. 4: (Color online) The current $J_{AB}$ vs. the renormalized level $\bar{\varepsilon}_0$ for the different e-ph interaction strength $\lambda$ with $\Gamma = 0.1$, $\phi = 0$, and $t_{ref} = 0.1$.

FIG. 5: (Color online) (a, c, and e) are the differential conductance $G_{AB}$ vs. the level $\bar{\varepsilon}_0$ for the bias $V_{bias} = 0$ (a), 0.2 (c), and 1.5 (e) at $\phi = 0$. (b, d, and f) are the differential conductance $G_{AB}$ vs. the magnetic flux $\phi$ for the bias $V_{bias} = 0$ (b), 0.2 (d), and 1.5 (f). The other parameters are $\Gamma = 0.1$ and $t_{ref} = 0.1$.

FIG. 6: (Color online) The phase $\theta_0$ vs. the level $\bar{\varepsilon}_0$ for the different bias $V_{bias}$. The other parameters are $\lambda = 0.6$, $\Gamma = 0.1$, and $t_{ref} = 0.1$. 
(c) $V_{\text{bias}} = 0.2$

- $\lambda = 0$
- $\lambda = 0.3$
- $\lambda = 0.6$
- $\lambda = 0.8$
(e) \( V_{\text{bias}} = 1.5 \)

- \( \lambda = 0 \)
- \( \lambda = 0.3 \)
- \( \lambda = 0.6 \)
- \( \lambda = 0.8 \)
