On Self-Adjointness of 3D–Dirac Operators with Singular Potentials

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Abstract. We consider the Dirac operator on \( \mathbb{R}^3 \),

\[
D_{A,\Phi,\delta_S} = \sum_{j=1}^{3} \alpha_j \left(i \partial_{x_j} + A_j(x)\right) + \alpha_0 m - \Phi(x) I_4 + \delta_S,
\]

with magnetic potential \( A(x) = (A_1(x), A_2(x), A_3(x)) \) and electrostatic potential \( \Phi(x) \), where \( \alpha_j, j = 0, 1, 2, 3 \) are the Dirac matrices, \( \delta_S \) is a singular potential where \( \delta_S \) is the Dirac \( \delta \)-function with support on an enough smooth surface \( S \subset \mathbb{R}^3 \) divided \( \mathbb{R}^3 \) on two open domains \( \Omega_+, \Omega_- \) with common unbounded boundary \( S \), and \( \Gamma \) is \( 4 \times 4 \) matrix. We associate with the formal Dirac operator \( D_{A,\Phi,\delta_S} \), the unbounded in the Hilbert space \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) operator \( \mathcal{D} \) with domain defined by some interaction conditions on the surface \( S \). The purpose of the paper is to give conditions of the self-adjointness of the operator \( \mathcal{D} \).

1. Introduction

The aim of the paper is the investigation of self-adjointness of 3 – \( D \) Dirac operators with singular potentials supported on unbounded hyper-surfaces in \( \mathbb{R}^3 \). It should be noted that the same problem for Schrödinger operators have attracted a lot of attention in the last time (see for instance [1–7]). We consider the operator

\[
D_{A,\Phi,\delta_S} u(x) = (D_{A,\Phi} + \delta_S) u(x), \quad x \in \mathbb{R}^3,
\]

where

\[
D_{A,\Phi} = \sum_{j=1}^{3} \alpha_j \left(i \partial_{x_j} + A_j(x)\right) + \alpha_0 m - \Phi(x) I_4
\]

is the Dirac operators defined on vector-valued distributions on \( \mathbb{R}^3 \) with values in \( \mathbb{C}^4 \). In formula (2),

\[
\alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3,
\]

are the \( 4 \times 4 \) Dirac matrices satisfying the relations

\[
\alpha_j \alpha_k + \alpha_k \alpha_j = \delta_{jk} I_4, \quad j, k = 0, 1, 2, 3,
\]

where \( I_n \) is the \( n \times n \) unit matrix,
and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the $2 \times 2$ Pauli matrices satisfying the relations

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} I_2, \quad j, k = 1, 2, 3. \quad (7)$$

In Eq. (2), $A = (A_1, A_2, A_3)$ is the vector-valued potential of the magnetic field $H = \nabla \times A$, $\Phi$ is the electrostatic potential of the electric field $E = \nabla \Phi$, and $m$ is the mass of the particle. We use the system of coordinates for which the Planck constant $\hbar = 1$, the light speed $c = 1$, and the charge of the particle $e = 1$.

The singular potential $Q_s$ is of the form $Q_s = \Gamma \delta_S$, where $\Gamma = (\Gamma_{ij})_{i,j=1}^4$ is a $4 \times 4$ matrix, and $\delta_S$ is the delta-function with support on an enough smooth hyper-surface $S \subset \mathbb{R}^3$, which divides $\mathbb{R}^3$ in two open domains $\Omega_{\pm}$, with the common boundary $S$. The distribution $\Gamma \delta_S$ acts on the test functions $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ as

$$\Gamma \delta_S(\varphi) = \int_S \Gamma(s) \varphi(s) ds,$$

where $ds$ is the Lebesgue measure on $S$. We assume that $S$ is an unbounded hyper-surface of the bounded geometry (see for instance, [8]).

The main aim of the paper is to give sufficient conditions of the self-adjointness of realization of the formal Dirac operator as an unbounded operator in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$. The free 3D–Dirac operators perturbed by singular potentials supported on compact hyper-surfaces in $\mathbb{R}^3$ considered in the recent papers [9–14]. In these papers the spectral properties of Dirac operators with $\delta$–shell interactions of constant strength supported on closed smooth hyper-surfaces in $\mathbb{R}^3$ have been studied. Applying the boundary triplet techniques and the Birman–Schwinger principle the authors investigated the self-adjointness of unbounded operators associated with the formal Dirac operators.

In contrast to these works we consider the Dirac operators with variable electrostatic and magnetic potentials, and the variable strength $\Gamma(s)$ of the interaction on unbounded hyper-surfaces.

We use the following notations:

- $L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the Hilbert space of 4-dimensional vector-functions $u = (u^1, u^2, u^3, u^4)$ depending on $x \in \mathbb{R}^3$, with the scalar product

  $$\langle u, v \rangle = \int_{\mathbb{R}^3} (u(x), v(x))_{\mathbb{C}^4} dx.$$

- $H^s(\mathbb{R}^3, \mathbb{C}^4)$ is the Sobolev space of vector-functions on $\mathbb{R}^3$ with values in $\mathbb{C}^4$.

- Let $S$ be a non-compact hyper-surface in $\mathbb{R}^3$ of the class $C^m$, $m \geq 1$. We say that $S$ has a bounded geometry if there exists $r > 0$ such that:

  (i) There exists a covering of $S$ by the system $\{B(x_j, r)\}_{j=1}^\infty$ of open balls $B(x_j, r) = \{x \in \mathbb{R}^3 : |x - x_j| < r\}$ such that the covering $\{B(x_j, 2r)\}_{j=1}^\infty$ has a finite multiplicity. It means that there exists a number $N \in \mathbb{N}$ such that each point $s \in S$ belongs not more than $N$ balls $B(x_j, 2r)$.

  (ii) For every $j \in \mathbb{N}$ there exists a system of coordinates $y = (y', y_n) \in B(x_j, 2r)$ such that

    $$S \cap B(x_j, 2r) = \{ y = (y', y_3) : y_3 = \phi_j(y'), y' = (y_1, y_2) \in \mathbb{R}^2_{y'} \cap B(x_j, 2r) \},$$

    with $\sup_{n} \in \mathbb{N}$, $|\alpha| \leq m$, $y' \in \mathbb{R}^2_{y'} \cap B(x_j, 2r)$, $|\partial^\alpha \phi_l(y')| < \infty$.

- We denote by $C_0(S)$ the class of bounded continuous functions on $S$. 


2. Realization of the Dirac operators with singular potentials as unbounded operators in $L^2(\mathbb{R}^3, \mathbb{C}^4)$

We define the product $Q_s u$ of the singular potential $Q_s = \Gamma \delta_s$ with $u \in H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4)$ as the distribution in $\mathcal{D}'(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{D}'(\mathbb{R}^3) \otimes \mathbb{C}^4$ acting on the test functions $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$. Namely

$$
(\Gamma \delta_s u)(\varphi) = \frac{1}{2} \int_S (\Gamma(s) (u_+(s) + u_-(s)) \cdot \varphi(s))_{\mathbb{C}^4} ds.
$$

Let $H^1(\Omega_\pm, \mathbb{C}^4)$ be the restriction spaces on $\Omega_\pm$ of distributions in $H^1(\mathbb{R}^3, \mathbb{C}^4)$, and let $H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4) = H^1(\Omega_+, \mathbb{C}^4) \oplus H^1(\Omega_-, \mathbb{C}^4)$. We set $u_\pm = \gamma_s^\mp u$, where $\gamma_s^\pm : H^1(\Omega_\pm, \mathbb{C}^4) \to H^{1/2}(S, \mathbb{C}^4)$ are the trace operators. Let $u \in H^1(\Omega_\pm, \mathbb{C}^4)$ and $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$. Then integrating by parts we obtain that

$$
\langle D_{A,\Phi, Q_s} u, \varphi \rangle = \int_{\mathbb{R}^3} (D_{A,\Phi, Q_s} u(x), \varphi(x))_{\mathbb{C}^4} dx
= \int_{\Omega_+ \cup \Omega_-} (D_{A,\Phi} u(x), \varphi(x))_{\mathbb{C}^4} dx
- \int_S (i\alpha' \cdot \nu(s) (u_+(s) - u_-(s)), \varphi(s))_{\mathbb{C}^4} ds
+ \frac{1}{2} \left( \int_S \Gamma(s) (u_+(s) + u_-(s)), \varphi(s) \right)_{\mathbb{C}^4} ds,
$$

where $\nu(s)$ is the unit normal vector to $S$ at the point $s \in S$ directed to $\Omega_-$,

$$
\alpha' \nu(s) = \alpha_1 \nu_1(s) + \alpha_2 \nu_2(s) + \alpha_3 \nu_3(s).
$$

Formula (9) yields that $D_{A,\Phi, Q_s} u \in L^2(\mathbb{R}^3, \mathbb{C}^4)$, if and only if the following interaction (transmission) condition on the hyper-surfaces $S$ holds for almost all $s \in S$,

$$
-i\alpha' \cdot \nu(s) (u_+(s) - u_-(s)) + \frac{1}{2} \Gamma(s) (u_+(s) + u_-(s)) = 0.
$$

Condition (10) can be written of the form

$$
a_+(s)u_+(s) + a_-(s)u_-(s) = 0,
$$

where

$$
a_+(s) = \frac{1}{2} \Gamma(s) - i\alpha \cdot \nu(s),
\quad a_-(s) = \frac{1}{2} \Gamma(s) + i\alpha \cdot \nu(s).
$$

We associate with the formal Dirac operator $D_{A,\Phi, Q_s}$ an unbounded in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ operator $D_{A,\Phi, a_+, a_-}$, defined by the regular Dirac operator $D_{A,\Phi}$, with domain

$$
\text{dom}D_{A,\Phi, a_+, a_-} = H^1_{a_+, a_-}(\mathbb{R}^3 \setminus S, \mathbb{C}^4)
= \{ u \in H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4) : a_+(s)u_+(s) + a_-(s)u_-(s), s \in S \}.
$$

We also associate with the formal Dirac operator $D_{A,\Phi, Q_s}$ the operator of the transmission problem

$$
D_{A,\Phi, a_+, a_-} u(x) = \left\{ \begin{array}{ll} D_{A,\Phi} u(x), & x \in \mathbb{R}^3 \setminus S \\ a_+(s)u_+(s) + a_-(s)u_-(s) = 0, & s \in S \end{array} \right.,
$$

acting from $H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4)$ into $L^2(\mathbb{R}^3, \mathbb{C}^4)$. 


3. A priori estimates for the operators $\mathcal{D}_{A,\Phi,a,+,-}$

We consider the analogue of the Shapiro–Lopatinsky condition (see for instance [15]) for the operator $\mathcal{D}_{A,\Phi,a,+,-}$ at the points $s \in S$. Let

$$
\begin{align*}
\left\{ \begin{array}{l}
(\alpha_1 \xi_1 + \alpha_2 \xi_2 + i\alpha_3 \partial_z) \psi(\xi', z) = 0, z \in \mathbb{R} \setminus \{0\} \\
 a_+(s) \psi_+(\xi', 0) + a_-(s) \psi_-(\xi', 0) = 0, s \in S
\end{array} \right.
\end{align*}
$$

be the family of 1-dimensional transmission problems on $\mathbb{R}$ depending on the parameter $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2$, defined on the functions $\psi(\xi', z) \in H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}) = H^1(\mathbb{R}^+, \mathbb{C}^1) \oplus H^1(\mathbb{R}^-, \mathbb{C}^1)$, and depending on the parameter $\xi' \in \mathbb{R}^2$, where $\psi_\pm(\xi', 0) = \lim_{s \to +0} \psi(\xi', \pm \varepsilon)$,

$$
a_+(s) = \frac{1}{2} \Gamma(s) - i\alpha_3, a_-(s) = \frac{1}{2} \Gamma(s) + i\alpha_3.
$$

We say that the operator $\mathcal{D}_{A,\Phi,a,+,-}$ of the transmission problem satisfies the Shapiro–Lopatinsky condition at the point $s \in S$, if for every $\xi' \in S^1 = \{ \xi' \in \mathbb{R}^2 : |\xi'| = 1 \}$, the problem (15) has the trivial solution only in $H^1(\mathbb{R} \setminus \{0\}, \mathbb{C})$.

We consider the homogeneous system

$$
\begin{align*}
(\alpha_1 \xi_1 + \alpha_2 \xi_2 + i\alpha_3 \frac{d}{dz}) \psi(\xi', z) = 0, z \in \mathbb{R}, \xi' = (\xi_1, \xi_2),
\end{align*}
$$

of ordinary differential equations where

$$
\psi(\xi', z) = \left( \begin{array}{l}
\psi^1(\xi', z) \\
\psi^2(\xi', z)
\end{array} \right),
$$

and $\psi^j(\xi', z) \in \mathbb{C}^2$, $j = 1, 2$. Then, the equation (16) splits into two equations

$$
\begin{align*}
(\sigma_1 \xi_1 + \sigma_2 \xi_2 + i\sigma_3 \frac{d}{dz}) \psi^j(\xi', z) = 0, \quad j = 1, 2.
\end{align*}
$$

System (17) has the exponential solutions $\psi^j(\xi', z) = h^j(\xi') e^{\pm |\xi'| z}$, where $h^j(\xi') \in \mathbb{C}^2$. Substituting $\psi^j(\xi', y_0)$ in (17) we obtain that the vectors $h^j(\xi') \in \mathbb{C}^2$ satisfy the equations

$$
\Lambda^j(\xi') h^j(\xi') = \left( \sigma_1 \xi_1 + \sigma_2 \xi_2 \pm i\sigma_3 |\xi'| \right) h^j(\xi') = 0.
$$

Note that $\det \Lambda_\pm(\xi') = 0$ for every $\xi' \neq 0$. Hence, $\dim \ker \Lambda_\pm(\xi') = 1$. Note also that the vectors

$$
g^\pm(\xi') = \left( \begin{array}{c}
\mp i |\xi'| \\
\pm i |\xi'|
\end{array} \right), \xi = \xi_1 + i \xi_2
$$

are linearly independent solutions of equation (18). One can see that the system (16) has 4 linearly independent exponential solutions

$$
\psi_{+,1}(\xi', z) = h_{+,1}(\xi') e^{\xi'|z}, \psi_{+,2}(\xi', z) = h_{+,2}(\xi') e^{-\xi'|z},
$$

$$
\psi_{-,1}(\xi', z) = h_{-,1}(\xi') e^{-\xi'|z}, \psi_{-,2}(\xi', z) = h_{-,2}(\xi') e^{\xi'|z},
$$

where

$$
h_{+,1}(\xi') = \left( \begin{array}{c}
g^+ \psi_{+,1}(\xi') \\
0
\end{array} \right), h_{+,2}(\xi') = \left( \begin{array}{c}
0 \\
g^+ \psi_{+,2}(\xi')
\end{array} \right), h_{-,1}(\xi') = \left( \begin{array}{c}
g^- \psi_{-,1}(\xi') \\
0
\end{array} \right), h_{-,2}(\xi') = \left( \begin{array}{c}
0 \\
g^- \psi_{-,2}(\xi')
\end{array} \right).
$$
Note that the system \( \{ h_{+1}(\xi'), h_{+2}(\xi'), h_{-1}(\xi'), h_{-2}(\xi') \} \) is orthogonal in \( \mathbb{C}^4 \) for every \( \xi' \neq 0 \).

The solutions of Eq. (16) in \( H^1(\mathbb{R} \setminus \{ 0 \}, \mathbb{C}^4) \) have the following form

\[
\psi(\xi', z) = \begin{cases} 
\psi_+(\xi', z), z > 0 \\
\psi_-(\xi', z), z < 0 
\end{cases},
\]

where

\[
\psi_+(\xi', z) = C_{+1} \psi_{-1}(\xi', z) + C_{+2} \psi_{-2}(\xi', z), z > 0,
\]

\[
\psi_-(\xi', z) = C_{-1} \psi_{+1}(\xi', z) + C_{-2} \psi_{+2}(\xi', z), z < 0,
\]

with arbitrary complex constants \( C_{+1}, C_{+2}, C_{-1}, C_{-2} \). Substituting \( \psi_{\pm}(\xi', z) \) in the transmission condition

\[
a_+(s)\psi_+(\xi', 0) + a_-(s)\psi_-(\xi', 0) = 0, \quad |\xi'| = 1, \quad s \in S,
\]

we obtain a linear system of equations with respect to \( C_{+1}, C_{+2}, C_{-1}, C_{-2} \),

\[
a_+(x)h_{-1}(\xi')C_{+1} + a_+(x)h_{-2}(\xi')C_{+2} + a_-(x)h_{+1}(\xi')C_{-1} + a_-(x)h_{+2}(\xi')C_{-2} = 0.
\]

(21)

We denote by \( \mathcal{L}(x, \xi') \), \( x \in S, |\xi'| = 1 \) the 4 \times 4 matrix with columns

\[
a_+(x)h_{-1}(\xi'), a_+(x)h_{-2}(\xi'), a_-(x)h_{+1}(\xi'), a_-(x)h_{+2}(\xi').
\]

Note that the Shapiro–Lopatinsky condition holds at the point \( s \in S \) if and only if the system (21) has the trivial solution only for every \( \xi' \) such that \( |\xi'| = 1 \). Hence, the Shapiro–Lopatinsky condition for the point \( s \in S \) accepts the form

\[
\det \mathcal{L}(s, \xi') \neq 0
\]

for every \( \xi' \in \mathbb{R}^2 : |\xi'| = 1 \).

**Proposition 1.** Let \( S \subset \mathbb{R}^3 \) be an unbounded hyper-surface of bounded geometry, \( A = (A_1, A_2, A_3) \in L^\infty(\mathbb{R}^3, \mathbb{C}^3) \) the magnetic potential, \( \Phi \in L^\infty(\mathbb{R}) \) the electric potential, and \( \Gamma \in C_0(S, \mathbb{C}^{4 \times 4}) \). Then, the Shapiro–Lopatinsky condition is satisfied for every point \( x \in S \) uniformly. That is

\[
\inf_{s \in S, |\xi'| = 1} |\det \mathcal{L}(s, \xi')| > 0.
\]

(23)

Then there exists \( C > 0 \) such that for every \( u \in H^1_{a_+a_-}(\mathbb{R} \setminus S, \mathbb{C}^4) \) the a priori estimate holds

\[
\|u\|^2_{H^1(\mathbb{R} \setminus S, \mathbb{C}^4)} \leq C \left( \|D_{A,\Phi}u\|^2_{L^2(\mathbb{R}^3, \mathbb{C}^4)} + \|u\|^2_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \right).
\]

(24)

The scheme of the proof of Proposition 1 is standard (see for instance [15], p. 16).

1. The ellipticity of the operator \( D_{A,\Phi} \) implies the local a priory estimates at every point \( x \in \mathbb{R}^3 \setminus S \).

2. The ellipticity of the operator \( D_{A,\Phi} \) and the Shapiro–Lopatinsky condition implies the local a priory estimates at every points \( s \in S \).

3. If \( S \) is a hyper-surface of bounded geometry there exists for fix \( \varepsilon > 0 \) a countable covering \( \{ B_\varepsilon(x_j) \}_{j=1}^N \) \( B_\varepsilon(x_j) = \{ x \in \mathbb{R}^3 : |x - x_j| < \varepsilon \} \) of \( \mathbb{R}^3 \) with the finite multiplicity such that the local a priory estimates hold at every set \( B_\varepsilon(x_j) \). This allows us to glue together local a priori estimates and obtain a global estimate (24) by means of the partition of unity. If \( S \) is an compact hyper-surface it is sufficient a finite covering.
4. Parameter dependent Dirac operators with singular potentials

We consider the parameter dependent operator of the transmission problem

\[
\mathcal{D}_{A,\phi,a_+,a_-}(\mu) = \mathcal{D}_{A,\phi,a_+,a_-} - i\mu I_4
\]

acting from \(H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)\). The equality

\[
(\alpha \cdot D - i\mu I_4)(\alpha \cdot D + i\mu I_4) = (-\Delta + \mu^2) I_4
\]

yields that \(D_{A,\phi} - i\mu I_4\) is the elliptic operator with parameter (see for instance [15], p.22-27).

We consider the parameter dependent 1 - D problem on \(\mathbb{R} \setminus \{0\}\),

\[
\mathcal{D}_{a_+,a_-}(s, \xi', \mu)\psi(\xi', \mu, z) = \begin{cases} 
(\alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} - i\mu I_4) \psi(\xi', \mu, z) = 0, z \in \mathbb{R} \setminus \{0\}, \\
\alpha_+(s)\psi_+(\xi', \mu, 0) + \alpha_-(s)\psi_-(\xi', \mu, 0) = 0, s \in S.
\end{cases}
\]

The homogeneous equation

\[
\left(\alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} - i\mu I_4\right)\psi(\xi', \mu, y_3) = 0
\]

has the exponential solutions

\[
\psi_{\pm}(\xi', \mu, z) = h^{\pm} e^{\pm \rho z}, \quad \rho = \sqrt{1|\xi'|^2 + \mu^2},
\]

where \(h^{\pm} \in \mathbb{C}^4\). Substituting (28) in equation (27) we obtain that the vectors \(h^{\pm}\) satisfy the equation

\[
\Lambda_{\pm}(\xi', \mu)h^{\pm} = (\alpha' \cdot \xi' \pm i\rho \alpha_3 - i\mu I_4)h^{\pm} = 0.
\]

One can see that the system (29) has four linearly independent solutions

\[
\begin{align*}
\mathbf{h}_{1,+}(\xi', \mu) &= \begin{pmatrix} i\mu e \\ f_+ \end{pmatrix}, \\
\mathbf{h}_{2,+}(\xi', \mu) &= \begin{pmatrix} f_+ \\ i\mu e \end{pmatrix}, \\
\mathbf{h}_{1,-}(\xi', \mu) &= \begin{pmatrix} i\mu e \\ f_- \end{pmatrix}, \\
\mathbf{h}_{2,-}(\xi', \mu) &= \begin{pmatrix} f_- \\ i\mu e \end{pmatrix},
\end{align*}
\]

where \(f_{\pm}(\xi', \mu) = \Lambda_{\pm}(\xi', \mu)e = \left(\frac{\pm i\rho}{\rho}\right)\). The exponentially decreasing solutions of the equation (27) are of the form

\[
\psi(z, \mu) = \begin{cases} 
(C^+_1 \mathbf{h}_{+,1}(\xi', \mu) + C^+_2 \mathbf{h}_{+,2}(\xi', \mu)) e^{\rho z}, & z < 0, \\
(C^-_1 \mathbf{h}_{-,1}(\xi', \mu) + C^-_2 \mathbf{h}_{-,2}(\xi', \mu)) e^{-\rho z}, & z > 0,
\end{cases}
\]

where \(C^+_1, C^+_2, C^-_1, C^-_2\) are arbitrary constants. Substituting \(\psi(z, \xi', \mu)\) in the transmission conditions

\[
a_+(s)\psi_+(\xi', \mu, 0) + a_-(s)\psi_-(\xi', \mu, 0) = 0, \quad s \in S,
\]

we obtain the system of linear equations

\[
C^+_1 a_-(s)\mathbf{h}_{+,1} + C^+_2 a_-(s)\mathbf{h}_{+,2} + C^-_1 a_+(s)\mathbf{h}_{-,1} + C^-_2 a_+(s)\mathbf{h}_{-,2} = 0,
\]
with respect to \((C_1^+, C_2^+, C_1^-, C_2^-)\). System (32) has the trivial solution if and only if
\[
\det \mathcal{M}(s, \xi', \mu) \neq 0 \text{ for } \mu^2 + |\xi'|^2 = 1,
\]
where \(\mathcal{M}(s, \xi', \mu)\) is the matrix with columns
\[
\{a_-(s)h_{+,1}(\xi', \mu), a_-(-s)h_{+,2}(\xi', \mu), a_+(s)h_{-,1}(\xi', \mu), a_+(s)h_{-,1}(\xi', \mu)\}.
\]
We say that the operator \(\mathcal{D}_{A, \Phi, a_+, a_-}(\mu)\) satisfies the uniform parameter dependent Shapiro–Lopatinsky condition if
\[
\inf_{s \in S, (\mu, \xi') \in \mathbb{R} \times \mathbb{R}^2 : \mu^2 + |\xi'|^2 = 1} |\det \mathcal{M}(s, \xi', \mu)| > 0. \tag{33}
\]
This condition provides that the operator
\[
\mathcal{D}_{a_+, a_-}(s, \xi', \mu) : H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^4) \to L^2(\mathbb{R}, \mathbb{C}^4)
\]
is invertible for every \(s \in S\) and \((\xi', \mu) \in \mathbb{R}^2 \times \mathbb{R}\), and
\[
\sup_{s \in S, (\mu, \xi') \in \mathbb{R} \times \mathbb{R}^2 : \mu^2 + |\xi'|^2 = 1} \left\| \mathcal{D}_{a_+, a_-}^{-1}(s, \xi', \mu) \right\|_{B(L^2(\mathbb{R}^3, \mathbb{C}^4), H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^4))} < \infty. \tag{34}
\]

**Proposition 2.** Let \(S \subset \mathbb{R}^3\) be a \(C^2\)-unbounded hyper-surface of bounded geometry, the magnetic potential \(A = (A_1, A_2, A_3) \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)\), the electrostatic potential \(\Phi \in L^\infty(\mathbb{R})\), \(\Gamma \in C_b(S, \mathbb{C}^{4 \times 4})\), and the Shapiro–Lopatinsky condition (33) for parameter dependent operator \(\mathcal{D}_{A, \Phi, a_+, a_-}(\mu)\), \(\mu \in \mathbb{R}\) is satisfied. Then, there exists \(\mu_0 > 0\) such that the operator
\[
\mathcal{D}_{A, \Phi, a_+, a_-}(\mu) : H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)
\]
is invertible for every \(\mu \in \mathbb{R} : |\mu| > \mu_0\).

The proof of Proposition 1 has the following steps (see for instance [15], p.22-27):

1. The ellipticity with parameter of the operator \(D_{A, \Phi}\) implies the local invertibility of \(D_{A, \Phi}\) at every point \(x \in \mathbb{R}^3 \setminus S\).
2. The ellipticity with parameter of the operator \(D_{A, \Phi}\) and the Shapiro–Lopatinsky condition (33) implies the local invertibility of \(D_{A, \Phi}\) at the points \(x \in S\).
3. Since \(S\) is a hyper-surface of bounded geometry, for fixed \(\varepsilon > 0\), there exists a countable covering \(\{B_\varepsilon(x_j)\}_{j=1}^N\) \((B_\varepsilon(x_j) = \{x \in \mathbb{R}^3 : |x - x_j| < \varepsilon\})\) of \(\mathbb{R}^3\) with a finite multiplicity such that there exist locally inverse operators for \(D_{A, \Phi, a_+, a_-}(\mu)\) at every ball \(B_\varepsilon(x_j)\) (in the case of the compact hyper-surface it is enough the finite covering). Then, applying the partition of unity to obtain the globally inverse operator for \(D_{A, \Phi, a_+, a_-}(\mu)\), for |\mu| large enough.

5. Self-adjointness of the operators \(D_{A, \Phi, a_+, a_-}\)

**Theorem 1.** Let \(S \subset \mathbb{R}^3\) be a \(C^2\)-hyper-surface of unbounded geometry, the potentials \(A \in L^\infty(\mathbb{R}^3, \mathbb{R}^3), \Phi \in L^\infty(\mathbb{R}^3)\) be real-valued, and \(\Gamma(s) = (\Gamma_{ij}(s))_{i,j=1}^4\) be the Hermitian matrix with elements \(\Gamma_{ij}(s) \in C_b(S)\). Let the parameter dependent uniform Shapiro–Lopatinsky condition (33) holds. Then the operator \(D_{A, \Phi, a_+, a_-}\) is self-adjoint in \(L^2(\mathbb{R}^3, \mathbb{C}^4)\).

**Proof:**

Let \(u, v \in H^1_{a_+, a_-}(\mathbb{R}^3 \setminus S, \mathbb{C}^4) = \text{Dom } D_{A, \Phi, a_+, a_-}\). Then integrating by parts and taking into account that \(u, v \in H^1_{a_+, a_-}(\mathbb{R}^3 \setminus S, \mathbb{C}^4)\) we obtain
\[
\langle D_{A, \Phi} u, v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} = \langle u, D_{A, \Phi} v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)}.
\]
Hence $\mathcal{D}_{A,\Phi,a_+,a_-}$ is a symmetric operator. The parameter dependent uniform ellipticity of $D_{A,\Phi}$ and the parameter dependent uniform Shapiro-Lopatinsky condition (33) imply the a priori estimate (24) and the invertibility $\mathcal{D}_{A,\Phi,a_+,a_-}(\mu) : H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ for $|\mu|$ large enough. A priori estimate (24) yields the closeness of $\mathcal{D}_{A,\Phi,a_+,a_-} = \mathcal{D}_{A,\Phi,a_+,a_-}(0)$ and the invertibility of $\mathcal{D}_{A,\Phi,a_+,a_-}(\mu)$ for $|\mu|$ large enough. It yields that the deficiency indices of $\mathcal{D}_{A,\Phi,a_+,a_-}$ are equal 0. Hence (see [16], page 100) the operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ with domain $H^1_{a_+,a_-}((\mathbb{R}^3 \setminus S, \mathbb{C}^4)$ is self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. □

5.1. Electrostatic $\delta$–shell interaction

As example of the application of Theorem 1 we consider the 3D–Dirac operator

$$D_{A,\Phi,2\eta} = \sum_{j=1}^{3} \alpha_j (i \partial_{x_j} + A_j) + \alpha_0 m - \Phi I_4 + 2\eta \delta_S$$

with electrostatic singular potential $2\eta \delta_S$. We assume that $A_j, \Phi \in L^\infty(\mathbb{R}^3), \eta \in C_b(S), S$ is a closed compact hypersurface or a hypersurface of bounded geometry. The operators $a_{\pm}$ for $D_{A,\Phi,2\eta}$ are of the form

$$a_+(s) = \eta(s)I_4 - i\alpha \cdot \nu(s), a_-(s) = \eta(s)I_4 + i\alpha \cdot \nu(s), s \in S$$

and we associate with the formal Dirac operator $D_{A,\Phi,2\eta}$ the unbounded operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $H^1_{a_+,a_-}(\mathbb{R}^3 \setminus S)$.

**Theorem 2.** Let $A_j, j = 1, 2, 3, \Phi, \eta$ be real-valued functions, and

$$\min \left\{ \inf_{s \in S} |\eta(s) - 1|, \inf_{s \in S} |\eta(s) + 1| \right\} > 0.$$  \hfill (37)

Then the operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ is self-adjoint.

**Proof:**

Theorem 2 follows from Theorem 1 since condition (37) yields the parameter dependent uniform Shapiro-Lopatinsky condition (33) for the transmission problem (14) with $a_{\pm}(s)$ given in (36). □

**Remark.** Condition (37) coincides with the condition of self-adjointness of $\mathcal{D}_{A,\Phi,a_+,a_-}$ obtained in the paper [10] where $A = 0, \Phi = 0, \eta \neq \pm 1$ is a constant.

6. Conclusion

The Dirac operator was introduced to give a quantum mechanical framework that takes relativistic properties of particles of spin $\frac{1}{2}$ into account. In the present paper we focus on a class of Dirac operators with singular potentials supported on surfaces in $\mathbb{R}^3$. Such operators are often used for idealization of problems of propagation of relativistic particles through out thin charged surfaces.

We obtained in the paper effective sufficient conditions for the unbounded operator $D_{A,\Phi,a_+,a_-}$ associated with the formal Dirac operator $D_{A,\Phi,Q}$ to be self-adjoint in the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^4)$. We give an application of general results for investigation of self-adjointness of the Dirac operator with the singular potential associated with the electrostatic $\delta$–shell interaction.

It should be noted that the self-adjointness of Dirac operators with $\delta$–interactions on unbounded surfaces is considered for the first time.
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