Analytical Solution of Magnetically Dominated Astrophysical Jets/Winds

LIANG CHEN 1 and BING ZHANG 2

1 Key Laboratory for Research in Galaxies and Cosmology, Shanghai Astronomical Observatory, Chinese Academy of Sciences, 80 Nandan Road, Shanghai 200030, China
2 Department of Physics and Astronomy, University of Nevada, Las Vegas, Las Vegas, NV 89154, USA

(Received ...; Revised ...; Accepted ...)

Submitted to ...

ABSTRACT

We present an analytical solution of a highly magnetized jet/wind flow. The left-hand of the general force-free jet/wind equation (the “pulsar” equation) is separated into a rotating and a non-rotating term. The two equations with either term can be solved analytically and the two solutions match each other very well. Therefore, we obtain a general approximate solution of a magnetically dominated jet/wind, which covers from the non-relativistic to relativistic regimes, with the drift velocity well matching the cold plasma velocity. The acceleration of a jet includes three stages. 1. The jet flow is located within the Alfvén critical surface (i.e. the light cylinder), has a non-relativistic speed, and is dominated by toroidal motion. 2. The jet is beyond the Alfvén critical surface where the flow is dominated by poloidal motion and becomes relativistic. The total velocity in these two stages follows the same law $v\Gamma = \Omega R$. 3. The evolution law is replaced by $v\Gamma \approx 1/(\theta\sqrt{2-\nu})$, where $\theta$ is the half opening angle of the jet and $0 \leq \nu \leq 2$ is a free parameter determined by the magnetic field configuration. This is because the earlier efficient acceleration finally breaks the causality connection between different parts in the jet, preventing a global solution. The jet has to carry local charges and currents to support an electromagnetic balance. This approximate solution is consistent with known theoretical results and numerical simulations and it is more convenient to directly compare with observations. This theory may be used to constrain the spin of black holes in astrophysical jets.

Keywords: galaxies: jets - gamma-ray burst: general - magnetic fields - black hole physics - accretion, accretion disks

1. INTRODUCTION

Astrophysical jets/winds are a very common astronomical phenomenon. They have been observed in different types of sources, such as active galactic nuclei (AGNs, Urry & Padovani 1995; Blandford et al. 2019), γ-ray bursts (GRBs, Piran 2004; Mészáros 2005; Kumar & Zhang 2015; Zhang 2018), tidal disruption events (TDEs, Burrows et al. 2011; Zauderer et al. 2011), X-ray binaries (Mirabel & Rodríguez 1999), etc. Observationally, a collimated jet can be accelerated to a highly relativistic speed (e.g., Lithwick & Sari 2001; Ghisellini et al. 2014; Chen 2018; Blandford et al. 2019) and its scale ranges many orders of magnitude during its propagation outwards (Urry & Padovani 1995; Asada & Nakamura 2012; Blandford et al. 2019). Jets play an important role in many astrophysical phenomena, such as galaxy evolution (through feedback, see e.g., Fabian 2012), black hole (BH) growth (through shedding the angular momentum to facilitate accretion, see e.g., Blandford & Payne 1982; Marconi et al. 2004) and particle acceleration (related to ultra/very-high energy cosmic rays and neutrinos, see e.g., IceCube Collaboration et al. 2018a; Anchordoqui 2019; Blandford et al. 2019). They are also the laboratory to study electrodynamical processes in curved spacetime.

Corresponding author: Liang Chen

chenliang@shao.ac.cn
(e.g., extracting BH rotating energy, see Blandford & Znajek 1977; Thorne & MacDonald 1982; MacDonald & Thorne 1982).

The observations of astrophysical jets, especially AGN jets, have accumulated a huge amount of data and present a variety of observational phenomena. Very-long-baseline Interferometry (VLBI) observations of AGN jets have revealed a feature of collimation and acceleration during the propagation of the jet from the core (e.g., Homan et al. 2015, and references therein). Because of its proximity to Earth, the M87 jet has been extensively observed, with a large amount of observational data showing that the jet is continuously accelerated from non-relativistic to relativistic speed and collimated up to a distance over $\sim 300$ pc (see e.g., Kovalev et al. 2007; Asada et al. 2014; Hada et al. 2017). This source presents the typical feature of a magnetically dominated jet (Meier et al. 2001; Marscher et al. 2008). Some AGN jets present a limb-brightening phenomenon (i.e. the edge of jet is brighter than its spine, for example, M87 jet, Kovalev et al. 2007; Ly et al. 2007; Walker et al. 2018). The radio emission from AGN jets is usually polarized and its Faraday rotation measure (RM) sometimes presents a systematic gradient with respect to the jet axis and also along the jet axis (e.g., Asada et al. 2002; Gabuzda et al. 2004; Hovatta et al. 2012; Park et al. 2019a). The polarization angle of AGN optical emission presents a smoothly rotating feature by a large angle during the outburst (e.g., over $\sim 720^\circ$ for PKS 1510-089 Marscher et al. 2008, 2010; Abdo et al. 2010). The $\gamma$-ray emission from some AGN jets presents a periodic variability with a timescale of months to years (e.g., Ackermann et al. 2015; Zhou et al. 2018). Also the innermost position angle of some AGN jets shows evidence of an oscillatory behavior (e.g., Lister et al. 2013; Walker et al. 2018).

The long-standing issue about how a jet is launched and which mechanism determines its collimation and acceleration is one of the most fundamental questions in astrophysics. Theoretically, one needs to obtain a self-consistent global jet model that connects the central accreting system to the large distance where the observed radiation is produced. It is generally believed that jets are a magnetic phenomenon. This motivates the investigation of magnetohydrodynamic (MHD) models for jets. Through releasing the gravitational energy of an accretion disk (AD, see, Blandford & Payne 1982, for the BP process) and/or extracting the rotational energy of the central compact object (CO, see Blandford & Znajek 1977; Zamaninasab et al. 2014, for BH case, the BZ process), the central accreting system can launch an electromagnetically dominated jet\(^1\) (see e.g., McKinney et al. 2012, and references therein), of which the electromagnetic stress provides the principal torque acting on the AD/CO, and most of the energy is liberated electromagnetically. Generally speaking, an ordered rotating magnetic field is the indispensable ingredient to globally launch a collimated relativistic jet.

The MHD equations governing jet physics are highly non-linear, and therefore need to be generally solved numerically (see e.g., Vlahakis & Königl 2003a; McKinney et al. 2012; Chatterjee et al. 2019, and references therein). Compared with the complex numerical solutions, analytical solutions are only achieved for some special cases. Yet, such analytical studies may provide simple scalings, which could offer a qualitative understanding of the basic properties of relativistic MHD flows. Until now, a great deal of analytical work has been done on MHD outflows (see e.g., Blandford 1976; Begelman & Li 1994; Vlahakis & Königl 2003a,b; Beskin & Nokhrina 2006; Narayan et al. 2007; Lyubarsky 2009, and references therein). If the pressure and/or internal energy are negligible, the plasma only contributes to the dynamics. One can then make a cold gas assumption to simplify the MHD equations. If the gas inertia can be further ignored in a highly magnetized flow, the plasma plays a dynamically negligible role and the electromagnetic field can be solved self-consistently. In this case, the force-free condition becomes a good approximation (see e.g., Blandford & Znajek 1977; Spruit 1996, 2010; McKinney 2006a; Narayan et al. 2007, and references therein).

In the case of a highly magnetized flow, a magnetic stream function ($\Psi$) is usually employed to measure the poloidal magnetic field in an axisymmetric system, which is conserved along a magnetic field line (e.g., Narayan et al. 2007). To solve MHD equations, a self-similar distribution of the magnetic stream function (e.g., $\Psi \propto R_{0}^\nu$ holding on the foot-point of magnetic field lines threading the AD plane) is often used to separate variables. Under a further assumption of “flat rotation” on the AD plane (i.e. an angular velocity $\Omega \propto R_{0}^{-1}$, not a Keplerian profile, see e.g., Li et al. 1992; Narayan et al. 2007), one can derive a globally analytical solution for the following cases: 1) the $\nu = 0$ monopole solution (this case does not need the “flat rotation” assumption, see Michel 1973), 2) the $\nu = 1$ parabolic solution (Blandford 1976; Narayan et al. 2007), 3) the $\nu = 3/4$ approximate solution of Blandford & Payne (1982). In general, numerical methods are necessarily needed to solve the equation for an arbitrary $\nu$ value and an asymptotic behavior

---

\(^1\) Even a turbulent AD can form a simple power-law profile of the height-integrated toroidal current, and therefore can further launch a nearly force-free, stationary, collimated and ordered Poynting-flux-dominated jet in the polar region as shown in some MHD simulations (e.g., Hawley & Krolik 2006; McKinney 2006a; McKinney & Narayan 2007a).
may be obtained in the limit of the highly relativistic case (e.g., Vlahakis & Königl 2003a; Beskin et al. 2004; Beskin & Nokhrina 2006; Narayan et al. 2007; Tchekhovskoy et al. 2008; Komissarov et al. 2009; Pu & Takahashi 2020). In another aspect, some MHD simulations explore some parameter spaces and find that the poloidal configuration of magnetic fields changes little from the non-rotation to the rotation cases (e.g., Tchekhovskoy et al. 2008). In the case of a highly magnetized flow, given a magnetic field configuration, one can derive the velocity profile during the outward propagation of a flow. It has been found that the bulk Lorentz factor of the flow has an asymptotic power-law dependence on the distance from the central CO/AD in the highly relativistic regime (e.g., Narayan et al. 2007; Tchekhovskoy et al. 2008). However, the question regarding how the flow is accelerated from the non-relativistic case to the relativistic case can be only studied numerically (e.g., Tchekhovskoy et al. 2008; McKinney et al. 2012; Nakamura et al. 2018).

The status of the field may be summarized as follows. 1) Numerical simulations become the reliable and popular method to explore the underlying physics of magnetic field configuration, jet launching, collimating, and acceleration (e.g. Chatterjee et al. 2019 recently simulated a jet spanning over 5 orders of magnitude in distance). On the other hand, due to its complexity, some simulation results are not very intuitive for understanding the physical details. It is also challenging to directly test theoretical results against observations, although some efforts have been made recently (see e.g., Dexter et al. 2012; Mościbrodzka et al. 2016; Ceccobello et al. 2018; Davelaar et al. 2019; Chael et al. 2019; Event Horizon Telescope Collaboration et al. 2019; Tsunetoe et al. 2020, and references therein). 2) As a complementary method to numerical simulations, an analytical self-consistent model can presents a better understanding of jet physics and may be easier to confront with observations directly. However, analytic models proposed so far does not cover a large enough parameter space and cannot describe the transition regime from the non-relativistic to the relativistic regime. 3) An increasing amount of observational data have been collected, but direct testing models against these observational data is still lacking (Spruit 2010).

To achieve a comparison between observations and theory, a global relativistic jet solution is needed, which should satisfy the following conditions: 1) The solution should be physically (mathematically) reasonable. 2) The solution can describe the transition from the non-relativistic regime to the relativistic regime. 3) The solution for an ensemble of jets should have features/trends consistent with the previous theoretical results (analytic/semi-analytic, numerical simulation results) and observations. 4) The expression of solution should be explicitly analytical and comprehensive, so that it can be easily used for further developments (e.g., adding radiative processes) and comparisons with observations. 5) The solution can be an approximation (i.e. an approximately quantitative description of a global jet), but it should be accurate enough in the collimated jet region.

In this paper, we study the problem of a magnetically-dominated jet/wind under the force-free condition, and find an approximately analytical solution to the magnetic stream function that meets the above requirements. Based on this solution, the jet properties (velocity, current, charge, etc.) can be further explored in detail. In Section 2, starting from the first principle, we present the basic MHD equations to describe a highly magnetized jet flow. An approximate solution is provided in Section 3 in the case of negligible gravity. Based on the approximate solution, the electromagnetic field configuration is presented in Section 4, and a detailed flow velocity profile is obtained and drawn in Section 5. The questions about how much current, charge and power a jet carries are discussed in Section 6. Under the force-free condition, in principle, one cannot derive plasma fluid acceleration due to the omission of inertia. We discuss the flow dynamics and how a cold plasma velocity is related to the electromagnetic field drift velocity in Section 7 (and also the jet flow density). In general, we have ignored the effect of general relativity (GR) to derive the solutions. In Section 8, we apply the approximate solution on a BH system with the gravity effect explicitly included near the BH. As a boundary condition, the CO/AD offers gas, charges and currents to support the electromagnetic field in the jet flow, which is discussed in Section 9. A further note on jet stability is presented in Section 10, which is followed by a summary Section in 11. For simplicity, we employ the natural unit system throughout the paper, i.e. the light speed $c = 1$ and the gravitational constant $G = 1$. Some important formulae are also presented in Gaussian unit in Appendix I.

2. BASIC EQUATIONS

We start with a brief derivation of the well-known “pulsar equation” following, e.g. Narayan et al. (2007). We consider a steady-state ($\partial/\partial t = 0$) jet flow with an infinite conductivity,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0,$$ (1)
which means that the electric field vanishes in the plasma fluid comoving frame\(^2\).

A magnetic field has no divergence and therefore can be expressed as \( \mathbf{B} = \nabla \times \mathbf{A} \), where \( \mathbf{A} \) is the vector potential. In axisymmetric coordinates, one can define a magnetic stream function \( \Psi = r \sin \theta A_\phi \) and \( \mathbf{\Phi} = r \sin \theta B_\phi \) for convenience. Therefore, \( \mathbf{B} \) can be expressed as

\[
\mathbf{B} = \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta} \mathbf{\hat{r}} - \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r} \mathbf{\hat{\theta}} + \frac{\Phi}{r \sin \theta} \mathbf{\hat{\phi}} = -\frac{1}{R} \frac{\partial \Phi}{\partial z} \mathbf{\hat{R}} + \frac{\Phi}{R} \mathbf{\hat{\phi}} + \frac{1}{R} \frac{\partial \Psi}{\partial R} \mathbf{\hat{Z}} \tag{2}
\]

in spherical \((r, \theta, \phi)\) and cylindrical coordinates \((R, \phi, z)\), respectively. It is easy to prove that \( \mathbf{B} \cdot \nabla \Psi = 0 \), suggesting that \( \Psi \) is conserved along a magnetic field line. This also suggests conservation of the enclosed magnetic flux within radius \( R \), i.e.

\[
F_B = \iint \mathbf{B} \cdot d\mathbf{S} = 2\pi \Psi = \text{const}. \tag{3}
\]

This suggests that the interior of a jet has smaller \( \Psi \) values. For an axisymmetric system, rotation ensures that a magnetic field line stays in a magnetic stream surface \( \Psi = \text{const} \), which is also where the frozen plasma fluid streams. Therefore, the global structure of the magnetic field configuration can evolve in a self-similar form.

The frozen-in condition also implies that the magnetic stream surface would be equipotential \((\mathbf{E} \cdot \mathbf{B} = 0\), see Equation 1\). Therefore, \( \mathbf{E} \) can be written in a form,

\[
\mathbf{E} = -\Omega \nabla \Psi = -\Omega r \sin \theta \hat{\phi} \times \mathbf{B}, \tag{4}
\]

where \( \Omega \) is the angular velocity of magnetic field line, which in principle is not necessarily equal to any angular velocity of the fluid matter. One can imagine that the angular velocity of the entire field line all the way up to infinity is determined by the rotation of the CO/AD at the foot-point, which implies that \( \Omega \) would be conserved along a magnetic field line. This is indeed the case in a steady state, where the electric field is non-curl, i.e.

\[
0 = \nabla \times \mathbf{E} = -r \sin \theta \hat{\phi} (\mathbf{B} \cdot \nabla \Omega).
\]

This implies that \( \Omega \) is conserved along a magnetic field line and therefore is only a function of \( \Psi \). Since the electric field is perpendicular to the magnetic stream surface, the electric force would be crucial to maintain the force balance between different surfaces (see Section 7.1 for a discussion of force balance).

Generally speaking, the motion of a cold gas with negligible gravity can be written as

\[
\rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B}, \tag{6}
\]

where \( \rho_e \) refers to the charge density, \( \mathbf{j} \) the current density, \( \mathbf{u} = \Gamma \mathbf{v} \) the four-velocity (the spatial part), and \( \rho \) the plasma proper density. If a plasma is sufficiently rarefied so that it does not exert significant force on the magnetic field (i.e. the inertial term can be neglected) - but it is still sufficiently dense to support charges and currents to maintain the magnetic field, one then has the so called force-free approximation\(^3\), which is reduced from Equation 6 as

\[
\rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B} = 0, \tag{7}
\]

where the electric current density follows

\[
4\pi \mathbf{j} = \nabla \times \mathbf{B} = -\frac{1}{R} \frac{\partial \Phi}{\partial z} \mathbf{\hat{R}} - \frac{1}{R} \left( \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial R^2} - \frac{1}{R} \frac{\partial \Psi}{\partial R} \right) \hat{\phi} + \frac{1}{R} \frac{\partial \Phi}{\partial R} \mathbf{\hat{Z}}.
\]

Because the \( \hat{\phi} \) component of \( \mathbf{E} \) vanishes, one expects that the poloidal component \( \mathbf{j}_p \) and \( \mathbf{B}_p \) would be parallel to each other \((\mathbf{j}_p = \Phi^\prime \mathbf{B}_p / 4\pi \) with \( \Phi^\prime = d\Phi / d\Psi \), which implies that the electric current would also flow in the magnetic stream

\(^2\) The perfect/ideal MHD condition corresponds to flux freezing with a high magnetic Reynold’s number.

\(^3\) This is a reasonable approximation for highly magnetized flows (e.g., Blandford & Znajek 1977; McKinney 2006a). In terms of the standard magnetization parameter \( \sigma \), we assume \( \sigma \gg 1 \) (see discussion in Section 7.2 and Michel 1969; Goldreich & Julian 1970).
surface. One immediately has $\mathbf{B} \cdot \nabla \Phi = 0$, which implies that $\Phi$ is also conserved along the magnetic field line, and therefore is only a function of $\Psi$. This indicates that the enclosed current within $R$ is also conserved, i.e.

$$J = \iint \mathbf{j} \cdot d\mathbf{S} = \frac{\Phi}{2} = \text{const.}$$

(9)

Substituting relations $\rho_e = (\nabla \cdot \mathbf{E})/4\pi$, $\mathbf{j} = (\nabla \times \mathbf{B})/4\pi$, Equations 2 and 4 into Equation 7, one finally gets the “pulsar equation” (i.e. the cross-field equation, see e.g., Lovelace et al. 1986; Okamoto 1974; Narayan et al. 2007),

$$\Phi = \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} - \cot \theta \frac{\partial \Psi}{\partial \theta} + \left(\frac{\partial \Psi}{\partial \theta}\right)^2 = 0,$$

(10)

with $\Omega' \equiv d\Omega/d\Psi$, which can be also reduced from the Grad-Shafranov equation in the limit of force-free condition (see e.g., Lyubarsky 2009). The first line in Equation 10 corresponds to the non-rotation term, while the second line is related to the rotation induced term. This partial differential equation is clearly singular at $(1 - \Omega^2 r^2 \sin^2 \theta) \to 0$, which corresponds to the Alfvén critical surface (ACS) where the rotation velocity of the magnetic field lines reach the speed of light (the so-called “light cylinder” in the case of pulsars). Both $\Psi$ and $\Omega$ are conserved along magnetic field lines, and they can be determined by the properties of the CO/AD at the foot-point, which can be taken as boundary conditions. The enclosed $z$-direction magnetic flux would vanish when approaching the polar-axis, which implies $\Psi(r, \theta = 0) = 0$. On the AD plane or the CO “surface”, the value of $\Psi$ is finite. Given proper boundary conditions, the above 2D equation can be solved in principle, at least numerically. On the other hand, solving this equation analytically may give simple analytical scalings, which could offer qualitative understanding of the basic properties of relativistic MHD flows. Many analytical or semi-analytical studies have done this, but only for very specific cases (see Section 1 and e.g., Blandford 1976; Blandford & Zhukov 1977; Blandford & Payne 1982; Contopoulos 1995; Narayan et al. 2007, and references therein). Comparing with previous results, in this paper we consider whether there is a more general analytic solution for Equation 10, even an approximate one. To do this, the plasma fluid is assumed flowing outward from an accreting system consisting of a spherical CO surrounded by an infinitely thin AD (i.e., the foot-points of magnetic field lines, e.g., Narayan et al. 2007; Tchekhovskoy et al. 2010). In principle, at large distances from the central engine, the external environment inevitably plays a role to balance the Lorentz force in the jet flow (e.g., Lyubarsky 2009). From this perspective, the force-free approach can only apply where the magnetic pressure dominates over external pressure (e.g., McKinney 2006a). The jet/wind (our interested region) would have a finite magnetic flux. The boundary condition at the equator should be taken to balance with the external magnetic pressure. The problem tackled here, following Narayan et al. (2007), the boundary is treated with externally supplied parameters (for example $\nu$ and $\lambda$, see below), which would be determined by disk/external boundary properties.

3. AN APPROXIMATE SOLUTION

In the region of $\Omega r \sin \theta \gg 1$ (it will be shown in Section 5.1 that this corresponds to the relativistic case), the second line of Equation 10 dominates, i.e.

$$\frac{\Phi}{\Omega^2 r^2 \sin^2 \theta} - \left(\frac{\partial \Psi}{\partial r}\right)^2 - \frac{1}{r} \frac{\partial \Psi}{\partial r} - \cot \theta \frac{\partial \Psi}{\partial \theta} = 0.$$ (11)

We call it the rotation term Equation. Conversely, in the region of $\Omega r \sin \theta \ll 1$ (corresponding to the non-relativistic case), the first line of Equation 10 dominates, i.e.

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial \Psi}{\partial \theta} = 0.$$ (12)

We call it the non-rotation term Equation. In fact, in the non-relativistic region (Section 5.1), whether the rotation term can be ignored depends on whether the rotation velocity of electromagnetic field ($\Omega r \sin \theta$) is relativistic or not: if

---

4. $\Phi$ conserves approximately in the limit of a highly magnetized jet, see discussion in Section 7.

5. The $\dot{\phi}$ component of Equation 7 is automatically satisfied, while the $\dot{r}$ and $\dot{\theta}$ components yield the same equation.

6. The coefficients of the highest derivatives ($\frac{\partial^2 \Psi}{\partial r^2}$ and $\frac{\partial^2 \Psi}{\partial \theta^2}$) go to zero at the singularity.
the rotation velocity is close to the speed of light, one cannot ignore the rotation term; whereas in the non-relativistic case, one can safely ignore the rotation term. It is interesting that the solution of the rotation term Equation 11 can approximately satisfy the non-rotation term Equation 12 (under certain conditions, see below), in the regimes where one can safely ignore the rotation term. It is interesting that the solution of the rotation term Equation 11 can be close to the speed of light, one cannot ignore the rotation term; whereas in the non-relativistic case, one can safely ignore the rotation term.

We introduce a new variable
\[ \Lambda = \alpha \Psi^\lambda. \]  
(13)

In the region approaching the polar axis, one can expect that the magnetic flux vanishes and thus \( \Psi \to 0 \). Therefore, in the region near the polar axis, mathematically \( \lambda > 0 \) is required to guarantee a finite value of \( \Omega \) (notice that \( \lambda > 0 \) is likely unphysical). Magnetic field lines just around the polar axis would connect to the central CO, which is expected to rotate with a roughly constant angular velocity\(^8\) (see details below), implying \( \lambda = 0 \) for the case of magnetic field lines threading the CO. In the region where magnetic field lines threading the AD, one usually expects \( \lambda < 0 \). The choice of \( \Phi \), which determines the toroidal magnetic field, is important (e.g., Sulkanen & Lovelace 1990; Camenzind 1987). Physically, it is the rotation which develops a toroidal magnetic field and poloidal electric field in the lab frame (or the inertial frame), which seems to imply that \( \Phi \) would not be chosen arbitrarily and would be self-consistently determined by the MHD equations, given \( \Psi \) and \( \Omega \) specified (e.g., Beskin & Tchekhovskoy 2005; Contopoulos et al. 2013; Huang et al. 2019). To solve the rotation term Equation 11, one can see that in the case of

\[ \Phi = -\beta \Omega \Psi = -\beta \alpha \Psi^\lambda + 1, \]  
(14)

deeply understood due to the fact that it is the rotation of the poloidal magnetic field (related to the magnetic flux) that produces the toroidal magnetic field. In term of Equation 15, Equation 11 can be expressed as

\[ \lambda r^2 \left( \frac{H_t'}{H_t} \right)^2 + r_0 \frac{H_t''}{H_t} + 2r \frac{H_t'}{H_t} = \frac{\beta^2 (1 + \lambda)}{\sin^2 \theta} - \lambda \left( \frac{T_r'}{T_r} \right)^2 - \frac{T_t''}{T_t} - \cot \theta \frac{T_r'}{T_r}, \]  
(16)

where \( H_t' = dH_t/dr \), \( H_t'' = d^2H_t/dr^2 \), \( T_r'' = d^2T_r/d\theta^2 \) and \( T_r' = dT_r/d\theta \). The left-hand side of the above equation is only a function of \( r \) (the \( r \) component) and the right-hand side is only a function of \( \theta \) (the \( \theta \) component). Both equal the same constant. Let us set this constant as \( (1 + \lambda)^2 \nu \), a choice making the solution of the \( r \) component equation concise:

\[ H_t(r) = r^\nu. \]  
(17)

We introduce a new variable
\[ y = \sin^2 \theta, \]  
(18)

which makes the \( \theta \)-component equation become

\[ 4 (1 - y) y T_t'' + 4 \lambda (1 - y) y^2 T_t'^2 - 2 (3y - 2) y T_t T_r' - \beta^2 (\lambda + 1) T_r'^2 + \nu (\nu + 1) y T_r'^2 = 0, \]  
(19)

where \( T_r'' = dT_r/d\theta \) and \( T_t'' = d^2T_t/dy^2 \). Let us first consider its asymptotic properties. In the case of \( \theta \ll 1 \) (keeping in mind: \( T \to 0 \) when \( \theta \to 0 \)), the leading order terms give

\[ \lambda \left( \frac{dT_r}{d\theta} \right)^2 + T_r \frac{d^2T_r}{d\theta^2} + \cot \theta T_r \frac{dT_r}{d\theta} - \frac{\beta^2 (\lambda + 1) T_r'^2}{\sin^2 \theta} = 0. \]  
(20)

\(^7\) They may be constrained by proper boundary conditions (e.g., Contopoulos et al. 2013).

\(^8\) In case of threading a BH, magnetic field lines at different polar angles may not rotate with exactly the same frequency, see discussion in Section 8.
It is clear that this equation has a solution of the form

\[ T_r (\theta) \propto \theta^3. \]  

(21)

Therefore, a magnetic field line forms a “general-parabolic” configuration\(^9\) at \( \theta \ll 1 \), i.e. \( \Psi \propto r^\nu \theta^3 \). Now, let us consider what value \( \beta \) might take. In order to do this, we have to consider the higher order terms of Equation 19, which may be comparable to the non-rotation term. Therefore, we have to consider the original Equation 10. Let us substitute a general form of \( \Psi \propto r^\nu \theta^3 \) (1 + \( a_1 \theta + a_2 \theta^2 + a_3 \theta^3 + \ldots \)) (with coefficients \( a_1, a_2, a_3 \ldots \) to be determined) into the original Equation 10. One gets the first two leading order terms (note that \( a_1 = 0 \)),

\[
\begin{align*}
\beta (\beta - 2) - \left[ \nu (1 + \nu + \lambda \nu) - \frac{\beta^2}{3} (1 + \lambda) - \frac{\beta}{3} + 4a_2 (1 + \beta + \beta \lambda) \right] \Omega^2 \Psi^2/\nu \theta^4 - 2\beta/\nu \\
= \beta (\beta - 2) - \left[ \nu (1 + \nu + \lambda \nu) - \frac{\beta^2}{3} (1 + \lambda) - \frac{\beta}{3} + 4a_2 (1 + \beta + \beta \lambda) \right] \Omega^2 \nu^2 \theta^4 \approx 0.
\end{align*}
\]

(22)

It can be seen that, for any values of \( \beta, \lambda, \nu \), one always has an \( a_2 \) to make the second term vanish, whereas the first term yields \( \beta \approx 2 \) (\( \beta = 0 \) corresponds to the non-rotation case). We note that this choice cannot guarantee that the higher order terms vanish. So this solution is only an approximation. In the case of \( \beta < 2\nu \) (i.e. \( \nu \gtrsim 1 \)), the first term dominates over the second term in Equation 22, whereas in the case of \( \beta > 2\nu \) (i.e. \( \nu \lesssim 1 \)), the second term dominates over the first one. We therefore expect that the approximation \( \beta \approx 2 \) would be more efficient in the former case (i.e. \( \nu \gtrsim 1 \)). In another aspect, from Equation 2, one has \( B_r \propto \theta^{\beta-2} \), which implies that the choice of \( \beta = 2 \) can avoid singularity or vanishing magnetic fields on the magnetic polar axis (either \( B_\theta \) or \( B_\phi \) vanishes). This choice is also supported by some numerical simulations, which showed that this choice corresponds to a minimum torque (i.e. the least amount of toroidal magnetic fields) and is the one picked by a “real” system (see, e.g., Michel 1969; Contopoulos 1995; Narayan et al. 2007; Tchekhovskoy et al. 2008). Although the coefficient \( \beta \approx 2 \) is derived asymptotically (\( \theta \ll 1 \)), which must hold throughout the jet region due to the fact that \( \Phi, \Omega \) and \( \Psi \) are each conserved along magnetic field lines. In Appendix A, we come back to this question and present another proof of the relation \( \Psi \approx -2\Omega \Psi \) based on a more physical consideration, showing that this relation is only valid in the limit of a highly magnetized jet flow (e.g., Lyubarsky 2009).

Equation 19 can be solved analytically, with a general solution,

\[ T_r (y) = A_2 e^{\frac{\pi}{2\nu}} \int_0^\gamma \frac{G_1(\nu + 1, 2; \frac{1}{\nu}; t^\nu) G_2(\nu + 1, 2; t^\nu)}{G_1(\nu + 1, 2; \frac{1}{\nu}; t^\nu) G_2(\nu + 1, 2; t^\nu)} dt, \]

(23)

where

\[
\begin{align*}
a_1 &= b - s - \beta s - \frac{s}{\nu}; \ b_1 &= \frac{1}{2} - s + \frac{s}{\nu}; \ c_1 = 1 + \beta + \frac{s\beta}{\nu}, \\
a_2 &= -b - s + \frac{s}{\nu}; \ b_2 &= \frac{1}{2} - s - \frac{s}{\nu}; \ c_2 = 1 - b - \frac{s\beta}{\nu}, \\
G_1 &= \frac{\beta (s + \nu)}{2\nu} 2F_1 (a_1, b_1, c_1, t) t^{-1 + \frac{\beta (s + \nu)}{2\nu}} + \frac{a_1 b_1}{c_1} 2F_1 (a_2, b_2, c_2, t) t^{-1 - \frac{\beta (s + \nu)}{2\nu}}, \\
G_2 &= \frac{\beta (s + \nu)}{2\nu} 2F_1 (a_2, b_2, c_2, t) t^{-1 - \frac{\beta (s + \nu)}{2\nu}} + \frac{a_2 b_2}{c_2} 2F_1 (a_1, b_1, c_1, t) t^{-1 + \frac{\beta (s + \nu)}{2\nu}}, \\
G_3 &= 2F_1 (a_2, b_2, c_2, t) t^{-\frac{\beta (s + \nu)}{2\nu}}, \\
G_4 &= 2F_1 (a_1, b_1, c_1, t) t^{\frac{\beta (s + \nu)}{2\nu}},
\end{align*}
\]

(24)

where \( A_1, a_2 \) are integration constants, \( 2F_1 (a, b, c, x) \) are the Hypergeometric functions, and the constant

\[ s = \nu \lambda \]

(25)

measures the slope of the radial profile of the angular velocity on the AD plane. The constant \( A_2 \) measures the amplitude of \( T_r \) and therefore the strength of the magnetic field. Here, we just normalize \( T_r (\theta = \pi/2) = 1 \) for

---

\(^9\) In the case of \( \nu = 0 \), \( \Psi \) is only a function of \( \theta \), which presents a monopole solution. The case of \( \nu = \beta \) leads to \( \Psi \propto R \), which gives a cylindrical solution.
simplicity, but in reality $T_r$ can be multiplied by an arbitrary constant and still satisfy the original equation (see below and Appendix B). Therefore, one can set $A_2 = 1$ to make $T_r (y = 1) = 1$ and $T_r (y = 0) = 0$. However, $A_1$ is still unconstrained, which means that there are infinite solutions with different values of $A_1$ that can all satisfy $\Psi (r, \theta = 0) = 0$. Before further discussing how to determine $A_1$, let us consider the case near the CO/AD, where the non-rotation term would dominate. The non-rotation term Equation 12 can be easily solved (see also Tchekhovskoy et al. 2008), in terms of variable separated $\Psi = H_{nr} (r) T_{nr} (\theta)$ (where the subscript ‘nr’ denotes ‘non-rotation’), i.e.

$$\Psi = r^n T_{nr} (\theta),$$

$$T_{nr} (\theta) = C_2 y^2 F_1 \left( 1 - \frac{\nu}{2}, \frac{1}{2} + \frac{\nu}{2}, 2, y \right) = 2 F_1 \left( \frac{\nu}{2} - \frac{1}{2}, \frac{1}{2}, \frac{\nu^2}{\nu}, \mu_2^2 \right) - C_4 \mu_2 F_1 \left( \frac{\nu}{2} - \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \mu_2^2 \right),$$

where we define\(^{10}\),

$$\mu = \cos \theta,$$

$$C_1 = \frac{\nu \Gamma (3/2 - \nu/2) \Gamma (\nu/2)}{\Gamma (1 - \nu/2) \Gamma (\nu/2 + 1/2)}$$

$$C_2 = \frac{\Gamma (3/2 - \nu/2) \Gamma (1 + \nu/2)}{\sqrt{\pi}}.$$  

Notice that we have used the condition $\Psi (r, \theta = 0) = 0$ and the normalization $T_{nr} (\theta = \pi/2) = 1$, keeping in mind that the amplitude of $T_{nr}$ (or $\Psi$) can be multiplied by an arbitrary constant (see below and Appendix B). The same applies to the rotation term equation. The radial component also follows a power law distribution, i.e. $H_{r/nr} (r) = r^n$. This is the so-called self-similar solution employed in many papers (e.g. Narayan et al. 2007). The scaling relations in these solutions can capture key aspects of the jet problem as also found in MHD simulations (see e.g. Narayan et al. 2007; Tchekhovskoy et al. 2008).

As discussed above, there are infinite solutions with different values of $A_1$ that can satisfy the rotation term Equation 19. They are also expected to present different trends when approaching $\theta \to \pi/2$ even though they all satisfy $T_r (\theta = \pi/2) = 1$. One may wonder whether there is an $A_1$ that can make the rotation and non-rotation solutions have approximately the same trend when approaching $\theta \to \pi/2$, i.e. $T_r (y) \approx T_{nr} (y)$.

Differentiating Equation 23 with $y$, one gets

$$\frac{1}{T_{nr}} \frac{dT_r}{dy} = \frac{\nu}{s + \nu} A_1 G_3 (y) + G_4 (y)$$

which can be set to equal $\frac{1}{T_{nr}} \frac{dT_r}{dy}$ at $y = 1$ to guarantee that $T_{nr}$ and $T_r$ follow the same trend approaching $y \to 1$. Therefore, one can get a constraint on $A_1$:

$$A_1 = D \frac{D_1 + D_2}{D_3 + D_4}.$$  

where,

$$D = [\beta (s + \nu) - \nu] \frac{\Gamma (1 + \beta + s/\nu)}{\Gamma (1 - \nu/2 + (1 + \beta - s)/2 + s/\nu)} \frac{\Gamma (\nu/2 - \beta s/\nu + (2 - \beta + s)/2)}{\Gamma (\nu/2 + (2 + \beta + s)/2 + \beta s/\nu)},$$

$$D_1 = (2 - \nu + \nu^2) \frac{\Gamma (3/2 - \nu/2) \Gamma (1 + \nu/2)}{\Gamma (2 - \nu/2) \Gamma (3/2 + \nu/2)},$$

$$D_2 = (\beta - \nu) \frac{\Gamma (s + \nu)}{\Gamma (\nu + \nu)} \frac{\Gamma (2 + \beta + s/\nu)}{\Gamma (\nu + 1/2)} \frac{\Gamma (\nu/2 - \beta s/\nu + (2 - \beta + s)/2 + \beta s/\nu)}{\Gamma (\nu/2 + (3 + \beta + s)/2 + \beta s/\nu)},$$

$$D_3 = (2 + \nu - \nu^2) \frac{(\beta s + \beta s - \nu) \Gamma (3/2 - \nu/2) \Gamma (1 + \nu/2) \Gamma (1 - \beta - s/\nu)}{\Gamma (\nu - \nu/2) \Gamma (3/2 + \nu/2)},$$

$$D_4 = (\beta + \nu) \frac{\Gamma (s + \nu)}{\Gamma (\nu + \nu)} \frac{\Gamma (2 - \beta - s/\nu)}{\Gamma (\nu - \nu/2) \Gamma (3/2 - \nu/2)} \frac{\Gamma (\nu/2 - (\beta + s - 1)/2 - \beta s/\nu)}{\Gamma (\nu/2 + (3 - b + s)/2 - \beta s/\nu)}.$$  

\(^{10}\) Notice that both $C_1$ and $C_2$ are symmetric functions of $\nu$ centered at $\nu = 1/2$. The former increases from $C_{1, \nu=0.5} = 1$ to the maximum value at $C_{1, \nu=1} = [\Gamma (1/4)]^2 / 16 \pi^2 = 1.094$ and continuously decreases through $C_{1, \nu=1} = 1$ to $C_{1, \nu=2} = 0$. The later decreases from $C_{2, \nu=0.5} = 1/2$ to the minimum value $C_{2, \nu=1/2} = [\Gamma (1/4)]^2 / 16 \sqrt{\pi} = 0.4635$ and then continuously increases through $C_{2, \nu=1} = 1/2$ to $C_{2, \nu=2} = 1$.  

With this choice of $A_1$, both rotation and non-rotation terms reach $T_r (y = 1) = T_{nr} (y = 1) = 1$. Furthermore, they follow the same trend approaching $y \to 1$ with an error on the order of $\sqrt{1-y} = \mu$. In another limit in the region of $\Omega r \sin \theta \gg 1$, the rotation term dominates, so that $\Psi = r^{\nu} T_r$ can be considered as a good approximation for the solution of the original Equation 10. In the case of $\beta \approx 2$ in particular, at $\theta \ll 1$, even though the non-rotation term can be ignored compared with the rotation term, they follow the same trend, i.e. $T_r \propto T_{nr} \propto \theta^2$ (with the error on the order of $\sin^2 \theta \approx \theta^2$). It means that, for the choice of $\beta = 2$, either the form $\Psi = r^{\nu} T_{nr} (\theta)$ or $\Psi = r^{\nu} T_r (\theta)$ can be considered as an approximation for the solution of the original Equation 10. This is why some MHD simulations found that the poloidal structure of magnetic fields changes quite mildly from the non-rotation to the rotation cases (e.g., Tchekhovskoy et al. 2008). For the parameter $\nu$, we consider $0 \leq \nu \leq 2$ because we are interested in the case of a collimated jet whose 1) enclosed magnetic flux increases with increasing radius, and whose 2) strength of magnetic field decreases with increasing radius (e.g., Contopoulos 1995; Vlahakis & Königl 2003a; Narayan et al. 2007). The $\nu$ value is a free parameter in the model, which can be determined from observations (see Section 4). Throughout the paper, we take $\nu = 3/4$ as a typical value in later discussion (see more discussion on this choice in Section 9 and Blandford & Payne 1982; Narayan et al. 2007; McKinney & Narayan 2007a,b).

In the case of $\lambda = 0$ (threading a CO), the rotation term solution (Equation 23) can be reduced to the simple form

$$T_r (\theta) = \frac{1}{\nu} \Gamma \left( \frac{\nu}{2} - \frac{\beta}{2} + 1 \right) \Gamma \left( \frac{1}{2} - \frac{\nu}{2} - \frac{\beta}{2} \right) P_\nu^\beta (\mu)$$

$$= \frac{1}{\nu} \Gamma \left( 1 + \beta \right) \sqrt{\pi} \Gamma \left( \frac{1}{2} - \frac{\nu}{2} + \frac{\beta}{2} \right) \left( 1 + \frac{\nu}{2} + \frac{\beta}{2} \right) y^{\beta/2} _2 F_1 \left( \frac{\beta}{2} - \frac{\nu}{2} + \frac{\beta}{2} + 1, \frac{1}{2} + \frac{1}{2} + 1, y \right), \quad (34)$$

where $P_\nu^\beta (\mu)$ is the Legendre function with the order $\beta$ and the degree $\nu$. This formula can be also directly derived by solving Equation 19. In the following discussion, we will focus on the case of $\beta = 2$, i.e.

$$\Phi = -2 \Omega \Psi. \quad (35)$$

In Figure 1, we plot the comparison between $T_r (\theta)$ and $T_{nr} (\theta)$ for various cases: $s = -3/2$ corresponding to foot-points on a Keplerian AD and $s = 0$ corresponding to foot-points on a CO. In both cases, they match each other in the regions of $\theta \ll 1$ and $\theta \to \pi/2$.

The parameter $\Psi$ determines the poloidal magnetic field, while $\Phi = -2 \Omega \Psi$ determines the toroidal magnetic field. Given $\Psi$ and $\Omega$, other physical qualities can be estimated (as a reminder, this approximate solution applies only in the magnetically dominated region). Due to the fact that $T_{nr}$ and $T_r$ match each other with an error no larger than $\theta^2$ at $\theta \ll 1$ and no larger than $\cos \theta$ at $\theta \to \pi/2$, one therefore expects that 1) if a quantity relates to the second-order derivative of $\Psi$ (e.g., toroidal current or charge density), its estimation will bring an error in the region $\theta \to \pi/2$ (although not very large, see Section 6.2 for detail) but is still accurate enough in the region $\theta \ll 1$; 2) if a quantity is only related to the first-order derivative of $\Psi$ (e.g., electromagnetic fields, velocity, poloidal current and jet power), the estimation will be accurate enough for both regions $\theta \to \pi/2$ and $\theta \ll 1$ (see discussion below). Therefore, even though the approximation $\Psi = r^{\nu} T_r (\theta) = r^{\nu} T_{nr} (\theta)$ may not exactly guarantee a smooth transition through the singular point (this condition is usually employed to solve the Equation 10 numerically, e.g., Li et al. 1992; Contopoulos et al. 1999, 2013), it still presents enough precision for practical purposes.

Equations 11 (rotation term) and 12 (non-rotation term) are originally separated from Equation 10 based on $\Omega r \sin \theta \gg 1$ or $\Omega r \sin \theta \ll 1$, respectively, with the conditions corresponding to the jet being relativistic or non-relativistic (see Section 5). The fact that $T_{nr}$ matches $T_r$ in the regions $\theta \ll 1$ or $\theta \to \pi/2$ implies that our approximate solution applies to either $\theta \ll 1$ or $\theta \to \pi/2$, and either non-relativistic or relativistic. Therefore, the solution can describe the acceleration of collimated AGN jets ($\theta \ll 1$) continuously from the non-relativistic to the relativistic regimes (e.g., M87 jet, Kovalev et al. 2007; Hada et al. 2017). As a result, either $\theta \ll 1$ or $\theta \to \pi/2$, and either non-relativistic or relativistic regimes can be adapted to the approximate solution. Recall that this approximate solution does not assume $\Omega \propto R_0^{-1}$ (a “flat rotation” on the AD plane, see e.g., Li et al. 1992; Narayan et al. 2007) and can apply to the general $\Omega \propto R_0^s$, including $s = -3/2$ for a Keplerian AD. It is the ansatz Equation 13 that helps to solve the problem to get the approximated solution. The fact that $\Psi = r^{\nu} T_{nr} (\theta)$ is not a function of $\alpha$ or $\lambda$ also indicates that the configuration of the magnetic stream surface is independent of rotation (see next Section 4 for more discussion).

---

11 Either $T_r$ or $T_{nr}$ can be expanded to series $a_0 + a_1 \mu + a_2 \mu^2 + a_3 \mu^3 + \cdots$, and $dT/d\mu = (dy/d\mu)(dT/dy)$.
In this aspect, $\Omega$ could be any piecewise-defined function of $\Psi$ (each sub-function should follow the form $\alpha \Psi^\lambda$, but could have different $\alpha$ and $\lambda$ values). One can always derive the approximate ($\Omega$ independent) solution $\Psi = r^n T_{nr}(\theta)$.

In principle, $\Psi$ can be positive or negative, which corresponds to $B_p$ pointing in two opposite directions. The scalar angular velocity is defined as $\Omega = \mathbf{\Omega} \cdot \hat{z}$, which implies that $\Omega$ can be also positive or negative, corresponding to $\mathbf{\Omega}$ direction being along or opposite to the polar-axis. Whether $B_p$ rotates forward or backward is determined by the sign of $\Phi \approx -2\Omega \Psi$. In the following Sections, we just discuss the case that both $\Psi$ and $\Omega$ are positive, i.e. both $\mathbf{\Omega}$ rotating vector and $\mathbf{B}_p$ projection vector are along the polar-axis (one of the pair of opposite jets). For other choices of $\mathbf{\Omega}$ and $\mathbf{B}_p$, some formulae in following sections may need to change signs. The sign convention for different cases is presented in Appendix B. One can also multiply an arbitrary constant to $\Psi$, which still satisfies the original Equation 10. This constant determines the absolute value of the magnetic field strength, which further affects other quantities. The effect is discussed in Appendix B in detail.

4. ELECTROMAGNETIC FIELD CONFIGURATION

For the specific cases of $\nu = 0$, 1 and 2, the magnetic stream function $\Psi = r^n T$ expresses very simple functions: $\Psi = 1 - \cos \theta$ for $\nu = 0$ (monopole); $\Psi = r(1 - \cos \theta)$ for $\nu = 1$ (parabola); and $\Psi = r^2 \sin^2 \theta$ for $\nu = 2$ (cylinder), which are exact solutions of Equation 10 and have been studied in previous works under some special assumptions (see Appendix C and Michel 1973; Istomin & Pariev 1994; Blandford 1976; Narayan et al. 2007). General asymptotical properties of $T(\theta)$ at $\theta \ll 1$ and $\theta \rightarrow \pi/2$ can be found in Appendix D. $\Psi = \text{const}$ defines a magnetic stream surface, in which the magnetic field lines lie, plasma fluids stream, and the currents flow. Therefore, a magnetic stream surface can measure the configuration of magnetic fields and the jet flow. In the region of $\theta \ll 1$, given a magnetic stream surface specified, the jet half opening angle and half width are defined by

$$\theta = C_2^{-1/2} \Psi_0^{1/2} r^{-\nu/2},$$
$$R = C_2^{-1/2} \Psi_0^{1/2} z^{1-\nu/2},$$

(36)

which show a “general-parabolic” configuration (see Narayan et al. 2007). Generally, if the magnetic field line is rooted at the foot-point $r_0$ and $\theta_0$, the conservation of $\Psi$ implies $r_0^\nu T(\theta_0) = C_2 r^n \theta^2$, which indicates that the footpoint location and the parameter $\nu$ can be derived through measuring the jet configuration (i.e. the $r$-$\theta$ or $R$-$z$ relations, for the special case of magnetic field lines threading a BH\footnote{In principle, one needs to consider GR effects to deal with magnetic field lines threading a BH. As shown in McKinney & Narayan (2007a,b); Parfrey et al. (2019), GR effects do not qualitatively change the magnetic field configuration even close to the BH. See Section 8 for more information.}, see Section 8.).

As presented in the MHD simulations by e.g., Tchekhovskoy et al. (2008), the poloidal magnetic field configuration in the final rotating state is nearly the same as in the initial non-rotating state, despite the fact that the final steady solution has a strong axisymmetric toroidal filed $B_\phi$. The same result is also shown in our solutions, i.e. either $B_z$ or $B_\theta$ is independent of angular velocity (see Equations 2 and 27), which indicates that the poloidal magnetic structure is independent of rotation, and collimation seems to be achieved by the poloidal field itself (see Equation 37). Rotation twists the poloidal magnetic field to make a toroidal component and induce a poloidal current. The force that this poloidal current received in the magnetic field is balanced by the force exerted by the appeared poloidal electric field to guarantee a force-free condition (see Section 7.1 for a detailed analysis). Therefore, the conservation of magnetic field flux makes the poloidal field configuration almost change free. Roughly, the collimating hoop stress associated with the toroidal field is canceled by the decollimating effect of the pressure gradient associated with the same field (balance with centrifugal forces for plasma rotation, see discussion in Section 7.1 and e.g., Ostriker 1997; McKinney & Narayan 2007b; Narayan et al. 2007; Tchekhovskoy et al. 2008). In fact, each field line is collimated by the pressure associated with the field line itself further out. This result applies for the case of $\Omega \propto R_0^{-1}$ (on the AD plane), compared with the case of a “flat rotation” $\Omega \propto R_0$ studied by Narayan et al. (2007). This type of “general parabolic” collimating jets have been observed in many AGNs, for example, the continuously collimation of the M87 jet indicates $R \propto z^{0.56}$ (e.g., Asada & Nakamura 2012; Hada et al. 2013), which is consistent with Equation 37 with $\nu = 0.88$.

Now, let’s consider the magnetic topology, which can be described by the ratio of toroidal to poloidal magnetic field strength:

$$\frac{-B_\phi}{B_p} = -\frac{\Phi}{|\nabla \Psi|} = \frac{2\Omega \Psi}{|\nabla \Psi|} = \frac{\Omega r}{\sqrt{\nu^2 + (\frac{1}{\nu} \frac{d\Psi}{d\theta})^2}}.$$

(37)
It can be easily proved that, in the region of $\theta \ll 1$, one has $-B_\phi/B_p = \Omega r \sin \theta$, which implies that the toroidal magnetic field vanishes on the polar-axis (so do the velocity and Poynting flux, see below). In the region of $\theta \rightarrow \pi/2$, one can also approximate\footnote{The maximum error of the last equality/approximation is 50% when $\nu \rightarrow 0$. A more accurate approximation is $-B_\phi/B_p \approx \Omega \sqrt{2(1 - \cos \theta)}$.} $-B_\phi/B_p = 2\Omega r/\sqrt{\nu^2 + C_1^2} \approx \Omega r \sin \theta$. Therefore, the approximation $-B_\phi/B_p \approx \Omega R$ roughly applies throughout the entire jet region (ignoring the GR effects, see GRMHD simulations by, e.g., McKinney 2006a; Pu & Takahashi 2020), which implies that the helical magnetic field is approximately shaped as an “Archimedean spiral (the arithmetic spiral)” on the magnetic stream surface. Such a configuration also describes the structure of interplanetary magnetic fields being twisted by the Sun’s rotation (often called Parker’s spiral, Parker 1958). See Appendix E for the details of calculating the 3D morphology of helical magnetic field lines.

At the foot-point, the rotation velocity of magnetic fields may not be very relativistic, which indicates that the magnetic field is approximately shaped as an “Archimedean spiral (the arithmetic spiral)” on the magnetic stream surface. Such a configuration also describes the structure of interplanetary magnetic fields being twisted by the Sun’s rotation (often called Parker’s spiral, Parker 1958). See Appendix E for the details of calculating the 3D morphology of helical magnetic field lines.

At the foot-point, the rotation velocity of magnetic fields may not be very relativistic, which indicates that the magnetic field is approximately shaped as an “Archimedean spiral (the arithmetic spiral)” on the magnetic stream surface. Such a configuration also describes the structure of interplanetary magnetic fields being twisted by the Sun’s rotation (often called Parker’s spiral, Parker 1958). See Appendix E for the details of calculating the 3D morphology of helical magnetic field lines.

At the foot-point, the rotation velocity of magnetic fields may not be very relativistic, which indicates that the magnetic field is approximately shaped as an “Archimedean spiral (the arithmetic spiral)” on the magnetic stream surface. Such a configuration also describes the structure of interplanetary magnetic fields being twisted by the Sun’s rotation (often called Parker’s spiral, Parker 1958). See Appendix E for the details of calculating the 3D morphology of helical magnetic field lines.

At the foot-point, the rotation velocity of magnetic fields may not be very relativistic, which indicates that the magnetic field is approximately shaped as an “Archimedean spiral (the arithmetic spiral)” on the magnetic stream surface. Such a configuration also describes the structure of interplanetary magnetic fields being twisted by the Sun’s rotation (often called Parker’s spiral, Parker 1958). See Appendix E for the details of calculating the 3D morphology of helical magnetic field lines.

At the foot-point, the rotation velocity of magnetic fields may not be very relativistic, which indicates that the magnetic field is approximately shaped as an “Archimedean spiral (the arithmetic spiral)” on the magnetic stream surface. Such a configuration also describes the structure of interplanetary magnetic fields being twisted by the Sun’s rotation (often called Parker’s spiral, Parker 1958). See Appendix E for the details of calculating the 3D morphology of helical magnetic field lines.
Keplerian AD, with $\nu = 3/4$ indicates $B_\nu \propto R^{-3}$. The helical magnetic field implies that jet synchrotron emission would be polarized and its RMs\textsuperscript{18} would present a systematic gradient with respect to the jet axis, a phenomenon that is exactly observed in AGN jets (see e.g., Asada et al. 2002; Gabuzda et al. 2004; Hovatta et al. 2012). Furthermore, the RM is also expected to decrease along the jet axis, which is also observed in the M87 jet recently (Park et al. 2019a). When projected on the sky plane with a viewing angle to the jet axis, one roughly expects a transverse magnetic field at the spine and an orthogonal magnetic field at the edge of the jet. Therefore, the polarization angle (measured by the polarization $\mathbf{E}$ vector) would be aligned with the jet along the spine and the orthogonal at the edge, which is the spine/sheath polarization angle pattern shown in some AGN jets (e.g., Attridge et al. 1999; Pushkarev et al. 2005; Kravchenko et al. 2017).

From Equation 4, one always has $E = \Omega r \sin \theta B_\rho \approx -B_\phi$. Near the disk plane,

$$
\mathbf{E} = -\Omega r^{-\nu - 1} \left\{ \nu (1 - C_1 \cos \theta) \hat{r} + [C_1 + \nu (\nu - 1) \cos \theta] \hat{\theta} \right\}.
$$

(42)

Since $\mathbf{E}$ is always perpendicular to $\mathbf{B}_p$, the inclined angle of $\mathbf{E}$ with respect to the polar-axis would be equal to the angle of $\mathbf{B}_p$ with respect to the mid-plane of the AD, $\theta_{B_p,AD}$. In the region of $\theta \ll 1$, one has

$$
E_r = -\nu \Omega \Psi r^{-1} = -\nu \Omega C_2^{1/\nu} \Psi^{1-1/\nu} \theta^{2/\nu} = -\alpha \nu C_2^{1+1/\nu} \lambda^{\nu + \nu + 2\lambda - 3} R^{2\lambda + 2},
$$

$$
E_\theta = -2C_2^{1/2} \Omega \Psi^{1/2} \nu^{2-1} = -2C_2^{1/2} \Omega \Psi^{1-1/\nu} \theta^{1-2/\nu} = -2\alpha C_2^{1+1/\nu} \lambda^{\nu + \nu - 2\lambda - 2} R^{2\lambda + 1},
$$

$$
E_\phi = E_R = E_\theta = B_\phi,
$$

$$
E_z = -\frac{2 - \nu}{2} \theta E_\theta.
$$

(43)

It can be seen that $E_r$ decreases faster than $E_\theta$ along the magnetic field line, and therefore $E_\theta$ would dominate eventually. Both $E_r$ and $E_\theta$ present stratified configurations.

5. JET VELOCITY AND ACCELERATION

By assumption, the ideal MHD condition implies that the electric field vanishes in the fluid rest frame. Magnetic field lines lie and the fluid streams in the magnetic stream surfaces. Combining Equations 1 and 4 yields

$$
\mathbf{v} = \Omega r \sin \theta \hat{\phi} + \kappa \mathbf{B},
$$

(44)

where $\kappa$ is a function of position. Therefore, one has the poloidal velocity being parallel to the poloidal magnetic field and $v_\phi B_p - B_\phi v_\rho = \Omega r \sin \theta B_p$. This implies that the fluid elements behave like beads on a rigid wire (magnetic field line), which rotates with angular velocity $\Omega$, i.e. the plasma fluid element slides along the rotating magnetic field lines (the plasma are actually frozen on the magnetic field line as seen in the "co-rotating frame"). A simple analysis can present a trend of velocity profile in the region far from the ACS. For example, in the region far below the ACS where $B_p$ dominates over $B_\phi$ while the toroidal velocity may be larger than the poloidal velocity, one immediately has $v_\phi \approx \Omega R$ (i.e., the plasma fluid almost co-rotates with the magnetic field). On the other hand, in the region far above the ACS, conservation of angular momentum implies $v_\phi \approx 1/\Omega R$ (see Equation 90 below in Section 7.2 and also Blandford & Payne 1982).

In principle, one has to consider the inertia to calculate the plasma velocity, which will be discussed in Section 7.2. Here we consider a magnetically dominated jet with a Poynting flux $\mathbf{S} = \mathbf{E} \times \mathbf{B}/4\pi = v_\perp B^2/4\pi$, with $v_\perp = \mathbf{E} \times \mathbf{B}/\mathbf{B}^2$ being the velocity component perpendicular to $\mathbf{B}$. This relation can be interpreted as that the magnetic energy is advected with the plasma fluid along a direction perpendicular to $\mathbf{B}$, and therefore the Poynting flux is converted to the kinetic energy of the plasma, just like the conversion of enthalpy into kinetic energy in a hydrodynamic flow (Spruit 2010). One therefore expects that, for a highly magnetized jet, the plasma fluid may roughly move perpendicular to $\mathbf{B}$ with a velocity $\mathbf{v} = v_\perp$. Generally, the velocity can be also decomposed into two mutually perpendicular components $\mathbf{v} = \mathbf{E} \times \mathbf{B}/\mathbf{B}^2 + \zeta \mathbf{B}$, with $\zeta = \kappa + \Omega r \sin \theta B_\phi/B^2$. In the force-free limit, $\kappa$ or $\zeta$ cannot be constrained due to the omission of the fluid inertia. In the following discussion, we choose $\zeta = 0$ (following, e.g., Landau & Lifshitz 1975; MacDonald & Thorne 1982; Narayan et al. 2007), which corresponds to a net velocity of

\textsuperscript{18} When polarized emission passes through a magnetized medium, the electric vector position angle would rotate with a wavelength ($\lambda_\nu$)-dependent law $\Delta \chi = RM \lambda_\nu^2$. One roughly has the rotation measure $RM \propto \int n_e B_{\parallel} dl$, where $n_e$ is the electron density, $B_{\parallel}$ the line of sight component of the magnetic field and the integral should be taken over the entire path (Gardner & Whiteoak 1966).
the plasma being at the minimum, i.e. the so called “drift velocity”;

\[ v = v_d = \frac{E \times B}{B^2} = \Omega r \sin \theta \left( \frac{\rho - B_\phi}{B^2} \right) = \Omega r \sin \theta \left( \frac{B_p^2}{B^2} \rho - \frac{B_\phi B_p}{B^2} \right), \]

(45)

where \( B_p \) is the poloidal magnetic field vector. In fact, any cold plasma fluid which is carried along with a highly magnetized flow only has a slightly modified velocity relative to the drift velocity, which will be discussed in Section 7.2. In terms of charged particles inside a plasma fluid, the drift motion of the center of particle gyration in the magnetic field induces a magnetic force, which almost balances the force exerted by the electric field. Therefore, the drift velocity is independent of the particle properties but is determined by the electromagnetic field configuration, which implies that every particle follows the same drift velocity and forms an overall motion of the plasma fluid. For practical purposes, we assume that the fluid has the same velocity as the drift velocity throughout the paper. The fact of \( v_p/v_\phi = -(\rho B_\phi/B_p) \) implies that velocity also forms a helical structure, which is always perpendicular to the magnetic field direction. Therefore, contrary to the case of the magnetic field, one expects that the toroidal velocity dominates in the region of \( \Omega R \lesssim 1 \) (near the CO/AD), while the poloidal velocity dominates in the region \( \Omega R \gtrsim 1 \) (see Equation 46). This is clearly seen in Figure 2. In this case, the velocity field can be roughly expressed as

\[ v_\phi = \Omega r \sin \theta \frac{B_p^2}{B^2} \approx \frac{\Omega R}{1 + (\Omega R)^2}, \]

\[ v_p = -\Omega r \sin \theta \frac{B_\phi B_p}{B^2} \approx \frac{1}{1 + (\Omega R)^2}, \]

\[ v = \Omega r \sin \theta \frac{B_p}{B} \approx \frac{\Omega r \sin \theta}{\sqrt{1 + \frac{4(\Omega R)^2}{\nu^2 + (\Omega R)^2}}} \approx \frac{\Omega R}{\sqrt{1 + (\Omega R)^2}}. \]

(46)

In the limit of \( \Omega R \to +\infty \), the velocity reaches the speed of light\(^{19}\) (for the radiation conditions at infinity, see e.g., Pan & Yu 2016). In this case, the fast critical surface is located at infinity, so the magnetic field lines are entirely inside the fast critical surface (see e.g., Vlahakis & Königl 2003a; Narayan et al. 2007). Roughly speaking, at the ACS, the jet reaches about a fraction \( \approx 1/\sqrt{2} \) of the final speed. This expression also presents an appropriate asymptotic trend of the toroidal velocity, i.e. \( v_\phi \approx \Omega R \) in the region \( \Omega R < 1 \) and \( v_\phi \approx 1/\Omega R \) in the region \( \Omega R \gg 1 \) as discussed above. In the following subsections, we will present detailed velocity distributions during jet propagation outward.

5.1. Total Velocity

From Equations 45, one has a four-velocity,

\[ \frac{1}{(\nu \Gamma)^2} = \frac{1}{(\Omega r \sin \theta)^2} + \frac{B_\phi^2}{(\Omega r \sin \theta B_p)^2} - 1 = \frac{1}{(\nu \Gamma)^2} + \frac{1}{(\nu \Gamma)^2}, \]

(47)

where one can define mathematical functions \( V_1 \) and \( V_2 \) (no physical meaning but will be frequently used in the following discussion),

\[ V_1 = (\nu \Gamma)_1 = \frac{E}{B_p} = \Omega r \sin \theta = \Omega R_0 \sin \theta T^{-1/\nu}, \]

\[ V_2 = (\nu \Gamma)_2 = \frac{\sqrt{B_\phi^2 - E^2}}{\sqrt{(\Omega r \sin \theta B_p)^2 - E^2}} - 1, \]

(48)

\(^{19}\) In a real jet system, one has to consider the loaded gas to calculate the fluid velocity. During jet propagation outwards, the magnetically dominated condition would finally be broken. As a result, the jet cannot be accelerated further to reach the speed of light. We will discuss this in Section 7.2.
Under the assumption that the rotation velocity of electromagnetic field is not very relativistic on the AD plane, the above velocity profiles (Equation 50) apply from non-relativistic to relativistic regimes. In the case of either efficient acceleration (dominated by \( V \propto \text{curvature of the poloidal field line} \) (Beskin et al. 2004; Lyubarsky 2009; Komissarov et al. 2009). The transition points \( \theta \) does not explicitly depend on the magnetic field angular velocity, but is determined purely by the local curvature of the poloidal field line. As discussed above, the condition \( \theta \ll 1 \) does not mean that the plasma fluid has to be relativistic. Actually, the above velocity profiles (Equation 50) apply from non-relativistic to relativistic regimes. In the case of \( \nu \gg 1 \), the first term always dominates since \( V_2 \) increases faster than \( V_1 \) with increasing \( r \) (see Chatterjee et al. 2019, for GRMHD simulation of this case). In the case of \( \nu \ll 1 \), \( V_2 \) increases slower than \( V_1 \), which implies that the second term dominates the first one beyond a certain distance where the four velocity reaches \( (v\Gamma)_1 = (v\Gamma)_2 = (2/\sqrt{2-\nu}) \theta^{-1} \) and remains valid during jet propagation. A larger velocity with \( (v\Gamma) \theta > 2/\sqrt{2-\nu} \) cannot guarantee the existence of a global solution, since different parts of the jet are not causally connected with each other (see e.g., Zakamska et al. 2008; Lü et al. 2009; Komissarov et al. 2009). Such a feature also meets the constraints from the VLBA observations of AGN jets (with \( \Gamma \theta \sim 0.1 \sim 0.3 \), see e.g., Jorstad et al. 2005; Clausen-Brown et al. 2013). It can be seen that \( V_2 \) does not explicitly depend on the magnetic field angular velocity, but is determined purely by the local curvature of the poloidal field line (Beskin et al. 2004; Lü et al. 2009; Komissarov et al. 2009). The transition points from efficient acceleration (dominated by \( V_1 \)) to inefficient acceleration (dominated by \( V_2 \)) forms a (causal) critical surface (CCS, e.g., Beskin \\& Nokhrina 2006), where,

\[
\begin{align*}
\Omega R^2 z^{-1} & = C_2^{-1} \Omega z^{-\nu} \left( \left( \frac{2}{\sqrt{2-\nu}} \right) C_2^{1/2} \Psi^{1/2} - \frac{1}{2} |B_\theta| R z^{-\nu} C_2^{-1} \right) = \alpha C_2^{2} z^{-\lambda (2-\nu) - 1} R^{2 \lambda + 2} = \frac{2}{\sqrt{2-\nu}}, \\
(v\Gamma)_{\text{CCS}} & = \frac{1}{\sqrt{2}} \left( - \left( \frac{2}{\sqrt{2-\nu}} \right) C_2^{1/2} \Psi^{1/2} \Omega^{1/2} - \frac{1}{2} \right) \left( \frac{2}{\sqrt{2-\nu}} \right) \Psi^{1/2} z^{-\nu} / \Omega^{1/2} \left( 2 \nu + \frac{1}{\sqrt{2-\nu}} \right)^{1/2} C_2^{-1/2} \Psi^{1/2} z^{-\nu} / \Omega^{1/2} \right)^{1/2} \cdot
\end{align*}
\]

(51)

It can be seen that both \( V_1 \) and \( V_2 \) depend on \( R \) at a given height \( z \), so the jet should be structured. In the case of \( s = -3/2 \) (magnetic field lines threading an AD and \( \nu = 3/4 \)), one has \( V_{1,\theta} \propto R^{-3} \) and \( V_{2,\theta} \propto R^{-1} \), which always presents a faster spine surrounded by a slower layer. In the case of \( s = 0 \) (magnetic field lines threading a CO), \( V_{1,\theta} \propto R \) (Blandford \\& Znajek 1977) and \( V_{2,\theta} \propto R^{-1} \) indicate a slower spine surrounded by a faster interlayer, which is further surrounded by a slower outerlayer. The “outmost” magnetic field lines threading the equator of CO form a stream surface \( R = C_2^{-1/2} \Psi^{1/2} z^{-\nu} / \Omega^{1/2} \), which intersects with the CCS at \( \Omega C_2^{-1/2} \Psi^{1/2} z^{-\nu} / \Omega^{1/2} = 2/\sqrt{2-\nu} \). Therefore, in the region with a height less than \( z_{\text{ccs}} \), the jet is always dominated by \( V_{1,\theta} \), i.e. a slower spine surrounded by a faster layer, with a maximum velocity located at the “outmost” surface \( (v\Gamma)_{\text{max}} = \Omega R = \Omega C_2^{-1/2} \Psi^{1/2} z^{-\nu} / \Omega^{1/2} \). In the region with a height larger than \( z_{\text{ccs}} \), the jet would be separated into an inner part (dominated by \( V_{1,\theta} \)) and an outer part (dominated by \( V_{2,\theta} \)), i.e. a slower spine surrounded by a faster interlayer, which is further surrounded by a slower outerlayer. In this case, the maximum velocity happens at the CCS \( (v\Gamma)_{\text{CCS}} = (\Omega z / \sqrt{2-\nu})^{1/2} \) at a given \( z \). This “anomalous” configuration (slower spine + faster interlayer + slower outerlayer) is also seen in some simulations (e.g., Komissarov et al. 2007; Tchekhovskoy et al. 2008; Nakamura et al. 2018). The formed spine/layer jet structure

---

20 On the AD plane \( (\theta = \pi/2) \), \( V_2 \) is an increasing function of \( \nu \), which has a minimum value at \( \nu = 0 \), \( V_{2,\nu=0,\theta=1} = 1/\sqrt{3} \), increasing through \( V_{2,\nu=1} = 1 \), and reaching \( V_{2,\nu=2} \rightarrow +\infty \).

21 This definition of \( \mathcal{R} \) is valid in the limit of \( \theta \ll 1 \), which guarantees a positive value for a concave surface. One always has the form \( V_{2,\theta} \propto \theta^{-1} \), although its coefficient \( (2/\sqrt{2-\nu}) \) may depend on the fourth order coefficient \( a_4 \) of the series \( T(\theta) = C_2 \theta^2 + a_4 \theta^4 + \cdots \).
plays an important role in the unified model of radio loud (RL) AGNs (see Chiaberge et al. 2000), explaining the γ-ray emission from misaligned radio galaxies (see Ghisellini et al. 2005; Chen 2017) and possibly also γ-ray prompt and afterglow emission of GRBs (see, e.g., Berger et al. 2003). This velocity map on the $R - z$ plane is plotted in Figure 4 (the case $s = 0$, see Section 8 for the parameter setting).

Generally speaking, one expects a global, continuous acceleration and collimation of the jet during its propagation onwards until the magnetic domination condition is violated (i.e. $\sigma$ drops below 1). This exactly matches the observations in M87 (Kovalev et al. 2007; Asada et al. 2014; Hada et al. 2017; Walker et al. 2018; Park et al. 2019b), 1H 0323+342 (Hada et al. 2018) and other AGNs (e.g., Homan et al. 2015). From Equations 37 and 50, the collimation and acceleration of the jet follow the same law as a function of increasing distance, $v \Gamma \propto R \propto z^{1-\nu/2}$ (in the case of the $V_1$ term dominating, see also Beskin & Nokhrina 2006), which is also observed in M87 as presented by e.g., Mertens et al. (2016) with $\nu \approx 0.88$. As discussed above, a jet should be structured and with helical motion. This may explain why there is a slightly complicated velocity profile in M87: having various values of proper motion velocities even at a given distance from the AGN jet core (e.g., apparent speeds range from $\lesssim 0.5c$ to $\gtrsim 2c$ in the inner $\sim 2$ mas during various epochs of observations, see e.g., Kovalev et al. 2007; Asada et al. 2014; Hada et al. 2017; Walker et al. 2018; Park et al. 2019b). In the $V_1$ dominated regime ($v \Gamma = \Omega R$), the angular velocity can be derived through measuring the jet width and velocity, which, in turn, may be employed to constrain the spin of the BH (see Section 8 for the discussion of a special case of magnetic field lines threading a BH).

In a force-free treatment and for an axisymmetric system, both the Poynting flux and the Lorentz forces (either magnetic or electric ones) vanish on the axis, where the motion of (cold) plasma flow is only controlled by gravity. The plasma near the polar axis therefore inevitably falls towards the CO, provided a not very large initial velocity. If one ignores the gravity pull, the plasma velocity there would vanish as shown in some simulations (see e.g., Penna et al. 2013; White & Chrystal 2020), which is consistent with the expected drift velocity discussed here (see also Narayan et al. 2007). It is worth noting again that the force-free treatment only applies to a high-magnetization region, while a cold plasma MHD approach can be employed to study a low-magnetization region (see e.g., Beskin et al. 1992; Hirotani & Okamoto 1998; Beskin & Malyshekin 2000; Narayan et al. 2007; Lyubarsky 2009, for an interesting case study on a low-magnetization axis surrounded by a high-magnetization region).

5.2. Toroidal Velocity

From Equation 45, the toroidal velocity can be expressed as

$$\frac{1}{v_\phi} = \frac{1}{\Omega r \sin \theta} + \frac{1}{\Omega r \sin \theta B_p^2} = \frac{1}{(v_\phi)_1} + \frac{1}{(v_\phi)_2},$$

with

$$(v_\phi)_1 = \Omega r \sin \theta = V_1,$n

$$(v_\phi)_2 = \Omega r \sin \theta \frac{B_p^2}{B_\phi^2} \approx \frac{1}{\Omega r \sin \theta} \approx \frac{1}{V_1}.$$ (53)

On the AD plane, one has $(v_\phi)_{2,\theta=\pi/2} = (v^2 + C_1^2)/4\Omega r_0$, which is usually larger than $(v_\phi)_{1,\theta=\pi/2}$. Therefore, the toroidal velocity is dominated by $(v_\phi)_1$ initially and increases as the jet propagates outwards. This is followed by a decreasing phase dominated by $(v_\phi)_2$ roughly beyond the ACS ($\Omega r \sin \theta \approx 1$), where the toroidal velocity reaches the maximum value $v_\phi \approx 1/2$. Within the ACS, the plasma fluid roughly co-rotates with the magnetic field lines since the magnetic fields are roughly poloidal dominated. After the jet propagates outside of the ACS, the toroidal magnetic field becomes dominated. The rotation of the plasma lags behind magnetic fields since the toroidal component of the plasma (the “slide motion”) along magnetic field line would dominate.

5.3. Poloidal Velocity

The poloidal velocity can be expressed as

$$\frac{1}{(v_p \Gamma_p)^2} = \frac{B_p^2}{(\Omega r \sin \theta B_\phi^2)^2} + \frac{2}{(\Omega r \sin \theta)^2} + \frac{B_\phi^2}{(\Omega r \sin \theta B_p)^2} - 1 = \frac{1}{(v_p \Gamma_p)_1^2} + \frac{1}{(v_p \Gamma_p)_2^2} + \frac{1}{(v_p \Gamma_p)_3^2}$$ (54)
where

\[
(v_p \Gamma_p)_1 = \frac{\Omega r \sin \theta |B_\phi|}{B_p} \approx (\Omega r \sin \theta)^2 = V_1^2,
\]

\[
(v_p \Gamma_p)_2 = \Omega r \sin \theta / \sqrt{2} = V_1 / \sqrt{2},
\]

\[
(v_p \Gamma_p)_3 = \frac{1}{\sqrt{\left(\Omega r \sin \theta B_p\right)^2 - 1}} = V_2.
\]

In the region of \( \theta \to \pi/2 \) (near the AD plane), one has \((v_p \Gamma_p)_{1,\theta-\pi/2} = 2(\Omega r)^2/\sqrt{\nu^2 + C_1^2}
[1 - C_1 \cos \theta (\nu^2 + C_1^2 - \nu) / (\nu^2 + C_1^2)] \approx V_{1,\theta-\pi/2}^2\), which dominates the initial poloidal velocity near the disk plane (the square dependence on \( R \) is roughly similar to that of Li et al. 1992). Beyond roughly the ACS (\( \Omega R = 1 \)), \((v_p \Gamma_p)_2\) becomes dominated. As the jet continues propagating outward, the poloidal velocity would be always dominated by \((v_p \Gamma_p)_2\) for \( \nu \geq 1 \), and it would be dominated by \((v_p \Gamma_p)_3\) beyond the CCS for \( \nu < 1 \).

In summary, one has a 3-stage acceleration diagram, as presented in Figure 3 (see Section 8 for parameter setting).

- **The 1st acceleration regime:** Initially from the AD/CO, the poloidal magnetic field dominates and the fluid plasma almost co-rotates with the magnetic field lines. The total velocity is dominated by the toroidal component. The toroidal velocity follows \((v_\phi \Gamma_\phi) = V_1\), while the poloidal component follows \((v_p \Gamma_p) = V_1^2\). This stage ends at the radius where the toroidal magnetic field starts to dominate and the fluid fails to co-rotate, which is roughly located at the ACS \((V_1 = \Omega R \approx 1)\).

- **The 2nd acceleration regime:** The total velocity is dominated by the poloidal component, which follows \((v_p \Gamma_p) = V_1/\sqrt{2}\); and the toroidal component decreases as \((v_\phi \Gamma_\phi) = 1/V_1\) (the total velocity still follows \(v \Gamma = V_1\)). In the case of \( \nu > 1 \), this acceleration regime continues all the way to “infinity” or when magnetic dominance condition is broken.

- **The 3rd acceleration regime:** This regime applies in the case of \( \nu < 1 \), when the velocity reaches \((v \Gamma)_1 \theta = 2/\sqrt{2 - \nu} \). Causality limits efficient acceleration in the 2nd acceleration regime. In this regime, one still has \((v_\phi \Gamma_\phi) = 1/V_1\) and the total velocity is dominated by the poloidal velocity, which follows \((v_p \Gamma_p) = V_2\).

As shown by Equation 47, the total velocity, in term of \( v \Gamma \), has only two acceleration regimes. The physical meanings of these two regimes can also be clearly presented by separately plotting the velocity and Lorentz factor acceleration profiles, as shown in Narayan et al. (2007). In fact, the bulk Lorentz factor can be also separated into three parts to match the above three stages, i.e.,

\[
\Gamma^2 = 1 + \frac{1}{V_1^2 + V_2^2}.
\]

The first stage is non-relativistic (within the ACS), i.e. \( \Gamma \sim 1 \). In the second stage, the jet becomes relativistic, i.e. \( \Gamma \sim v \Gamma \approx V_1 \), which is followed by the third stage \( \Gamma \approx v \Gamma \approx V_2 \). In limit of \( \Gamma \gg 1 \), Equation 56 is reduced to \(1/\Gamma^2 \approx 1/V_1^2 + 1/V_2^2\), which is the two-stage acceleration scenario that has been discussed by many authors (e.g., Vlahakis & Königl 2003a; Beskin et al. 2004; Beskin & Nokhrina 2006; Narayan et al. 2007; Tchekhovskoy et al. 2008; Komissarov et al. 2009; Pu & Takahashi 2020). In the case of \( \nu > 1 \), \( V_1 \) always dominates, which yields \( \Gamma \approx \Omega R \) for the relativistic case, as discussed by e.g., Chatterjee et al. (2019, through GRMHD simulations) and Narayan et al. (2007, through a semi-analytical study).

### 5.4. Helical Jet

One can define a “cycle period” of motion for the helical structure of velocity

\[
P = \frac{2\pi r \sin \theta}{v_\phi} = \frac{2\pi r \sin \theta}{(v_\phi)_1} + \frac{2\pi r \sin \theta}{(v_\phi)_2} = P_1 + P_2,
\]

where

\[
P_1 = \frac{2\pi r \sin \theta}{(v_\phi)_1} = \frac{2\pi \Omega}{0},
\]

\[
P_2 = \frac{2\pi B_\phi^2}{\Omega B_p^2} \approx 2\pi \Omega (r \sin \theta)^2 = \frac{2\pi}{\Omega} V_1^2.
\]
It can be seen that $P_1$ is the rotation period of the electromagnetic field (see Section 8 for a special case of magnetic field lines threading a BH), which dominates over $P_2$ within the ACS. In the region outside the ACS, the second term $P_2$ dominates. At $\theta \ll 1$, one has

$$P_2 = P_1 (\Omega R)^2 = 2\pi C_2^{-2/\nu} \Omega \Psi^{2/\nu} \theta^{2-4\nu} = 2\pi C_2^{-1} \Omega \Psi \theta^{2-\nu} = 2\pi \alpha C_2^2 \lambda^{-(2-\nu)} R^{2\lambda+2}. \quad (59)$$

The case $s = -3/2$ (magnetic field line threading the AD and $\nu = 3/4$) implies $P_2 \propto R^{-2}$, i.e. the spine needing a longer time to finish a cycle motion compared with the layer. The case $s = 0$ (magnetic field line threading the CO) gives $P_2 \propto R^2$, i.e. the cycle period increases from spine to layer.

One can also define an inclination angle between the poloidal and total velocities, i.e.

$$\tan \theta_{\text{inc}} = \frac{v_{\phi}}{v_p} = \frac{1}{2\Omega r} \left[ v^2 + \left( \frac{1}{T} \frac{dT}{d\theta} \right)^2 \right] = \frac{B_p}{|B_\phi|} \approx \frac{1}{V_1}, \quad (60)$$

which measures the orderliness of the velocity. At $\theta \to \pi/2$, one has

$$\tan \theta_{\text{inc}, \theta \to \pi/2} = \frac{\sqrt{v^2 + C_1^2}}{2\Omega r} \left[ 1 + \frac{C_1 (v^2 + C_1^2 - \nu)}{\nu^2 + C_1^2} \cos \theta \right]. \quad (61)$$

At $\theta \ll 1$, this angle roughly represents the angle between the velocity direction and the polar axis, i.e.

$$\tan \theta_{\text{inc}, \theta \ll 1} = C_2^{1/2} \Omega^{-1} \Psi^{-1/2} \nu^{-1} \theta^{2-\nu/2} = C_2^{1/\nu} \Omega^{-1} \Psi^{-1/\nu} \theta^{2\nu-1} = \alpha^{-1} C_2^{-1} \lambda^2 R^{2-2\lambda} \approx \frac{1}{V_1},$$

which is equal to the reciprocal of $V_1$ (see Equation 50).

By multiplying $v_p$ by the cycle period, one gets a cycle distance that fluid moves along the poloidal direction within each cycle period,

$$D = v_p P = \frac{2\pi \nu \sin \theta}{v_\phi/v_p} \approx P_2. \quad (63)$$

At $\theta \to \pi/2$ (near the AD), one has

$$D = \frac{4\pi \Omega r^2}{\sqrt{v^2 + C_1^2}} \left[ 1 - \frac{C_1 (v^2 + C_1^2 - \nu)}{\nu^2 + C_1^2} \cos \theta \right], \quad (64)$$

and at $\theta \ll 1$, one has

$$D = P_2 = 2\pi C_2^{-2/\nu} \Omega \Psi^{2/\nu} \theta^{2-4\nu} = 2\pi C_2^{-1} \Omega \Psi \theta^{2-\nu} = 2\pi \alpha C_2^2 \lambda^{-(2-\nu)} R^{2\lambda+2}, \quad (65)$$

which shows the same formula as the cycle period $P_2$ outside the ACS (see Equation 59).

Both velocity and magnetic field lines fall in the magnetic stream surface and show helical structures, but they are perpendicular to each other ($v_{\phi}/v_p = -B_p/B_\phi$). Similar qualities such as the inclination angle, the cycle distance and the orderliness can be also defined for magnetic field lines, which would be a reciprocal of that of velocity. During the propagation of a helical motion of a plasma fluid element in a helical magnetic field configuration, its synchrotron emission would present a continuous rotation of the polarization angle, which can be traced in some AGN jets despite of difficulties (e.g., Marscher et al. 2008, 2010; Abdo et al. 2010). The spin of the jet may also drive a rotational motion of the surrounded molecular outflow, which seems to be detected by polarization observations of AGN PG 1700+518 (Young et al. 2007; Yang et al. 2012). Figure 2 presents the topology of magnetic field lines (green line) and velocities (color gradient line). The black dashed line represents the location of the ACS, and the colors of the velocity lines represent $vT$. For parameter setting, see Section 8.

6. CURRENT, CHARGE AND JET POWER

The plasma supplies currents and charges as needed to support the electromagnetic field. In the force-free limit, the output jet power is dominated by the Poynting flux. Therefore, the current, charge and power of the jet can be self-consistently derived. Electromagnetic fields, velocity, and poloidal current depend on the first-order derivative of $\Psi$, so one can use $T_{nr}$ to approximately measure their structures over the entire jet region. On the other hand, the toroidal current and charge densities are related to the second-order derivatives of $\Psi$, while $T_{nr}$ and $T_r$ match each other only on the order of $\cos \theta$ in the region of $\theta \to \pi/2$. This would lead to some errors in estimating the toroidal current and charge densities near the AD plane (see below).
6.1. Current

From Equation 8, the poloidal current density reads

$$\mathbf{j}_p = \frac{\Phi' \mathbf{B}_p}{4\pi} = -\frac{(\lambda + 1) \Omega \mathbf{B}_p}{2\pi} = (\lambda + 1) \frac{B_\phi}{4\pi r} \left( \frac{1}{T} \frac{dT}{d\theta} \hat{\nu} - \nu \hat{\theta} \right) = (\lambda + 1) \frac{\Omega}{2\pi \sin \theta} \left( -\frac{dT}{d\theta} \hat{r} + \nu T \hat{\theta} \right),$$

(66)

which is always parallel to the poloidal magnetic field lines. Therefore, the current streams on the magnetic stream surface. For toroidal current, the situation seems slightly more complicated. In the regions of either outside of the ACS ($\Omega R > 1$) or where the jet is collimated ($\theta \ll 1$), $T_{\text{tr}}$ matches $T_\nu$ with an error no larger than $\sim \theta^2$. The toroidal current approximately vanishes. In the region inside the ACS ($\Omega R < 1$) and where the jet is not collimated ($\theta \to \pi/2$), $T_{\text{tr}}$ matches $T_\nu$ with an error no larger than $\sim \cos \theta$. The toroidal current cannot vanish, which is a small quantity compared with the poloidal one, i.e.

$$j_\phi = -\frac{1}{4\pi R} \left( \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial R^2} - \frac{1}{R} \frac{\partial \Psi}{\partial R} \right) \approx 0 \quad \Omega R > 1 \text{ or } \theta \ll 1,$$

$$j_\phi \ll j_p \quad \Omega R < 1 \text{ and } \theta \to \pi/2.$$  

(67)

For the cases with exact solutions of the pulsar Equation 10 (monopole, cylinder and parabola with $\Omega \propto R_0^{-1}$, see Appendix C), the toroidal currents exactly vanish over the entire jet (see also Bogovalov 1992, for the slow rotation case). This is consistent with the above analysis for the general cases. Physically, rotation induces a toroidal magnetic field and a poloidal current. The magnetic force made by the so called z-pinch effect is approximately balanced by the induced electric force. This makes the toroidal current a small quantity (i.e. almost vanishes in the region of $\Omega R > 1$ or $\theta \ll 1$) as indicated by Equation 79 (see below), which is also shown in some GRMHD simulations (i.e. the toroidal current in the “funnel” region almost vanishes, e.g., McKinney & Narayan 2007). Similar to velocity (Equation 44), the force-free condition (Equation 7) makes the current to be generally separated into two parts: the moving of charges and a sliding current along magnetic field lines, i.e. $\mathbf{j} = \rho_v \mathbf{v}_d + j_\phi \hat{B}$, where the parallel current is

$$j_\parallel \approx -\rho_\nu \Omega r \sin \theta B_\phi^2 / (B B_\phi) \text{ when considering the smallness of the toroidal current} \text{ (note the condition Equation 67).}$$

One therefore has a poloidal current $j_p \approx -\rho_\nu \Omega r \sin \theta B_p / B_\phi \approx \rho_\nu$ (see also the discussion on charge density in Section 6.2). In a non-rotating system, the toroidal magnetic field, the electric current and the electric field would vanish. What might exist is only the poloidal magnetic field (may be supported by the CO/AD, see Section 9). It is the rotation that generates a toroidal magnetic field, which induces a poloidal current. The force that this poloidal current receives in the magnetic field is balanced by the force exerted by the appeared poloidal electric field to guarantee the force-free condition (see Section 7.1 for a detailed analysis).

At $\theta \to \pi/2$ (near the AD), one has

$$j_p = \frac{\Omega}{2\pi} (\lambda + 1) \nu^{\nu-2} \left\{ [-C_1 - (\nu - 1) \nu \cos \theta] \hat{r} + [\nu - C_1 \nu \cos \theta] \hat{\theta} \right\}.$$  

(68)

At $\theta \ll 1$, one has

$$j_r = (\lambda + 1) \frac{B_\phi}{2\pi R} = -\frac{C_2}{\pi} (\lambda + 1) \Omega \nu^{\nu-2} = -\frac{C_2^{1/\nu}}{\pi} (\lambda + 1) \Omega \Psi^{1-2/\nu} \theta^{-2+4/\nu} = -\frac{\alpha C_2^{1+1/\nu}}{\pi} (\lambda + 1) z^{\lambda \nu-2\lambda+\nu-2} R^{2\lambda},$$

$$j_\theta = \frac{\nu C_2^{1/\nu}}{2\pi} (\lambda + 1) \Psi^{1/2} \nu^{\nu/2-2} = \frac{\nu C_2^{1/\nu}}{2\pi} (\lambda + 1) \Omega \Psi^{1-2/\nu} \theta^{-1+4/\nu} = \frac{\alpha \nu C_2^{1+1/\nu}}{2\pi} (\lambda + 1) z^{\lambda \nu-2\lambda+\nu-3} R^{2\lambda+1},$$

$$j_R = \frac{2 - \nu}{2} j_r,$$

$$j_z = j_p = j_r.$$  

(69)

Similar to $\mathbf{B}_p$, $j_r$ dominates over $j_\theta$ in the region far away from the CO/AD. In the case of $s = 0$ (magnetic field lines threading a CO), $j_r$ carries a negative value and is almost constant at a given height; while in the case of $s = -3/2$ (magnetic field lines threading an AD) $j_r$ carries a positive value, which decreases with increasing $R$ at a given height $z$. This current distribution is clearly presented in Figure 5.

6.2. Charge

As discussed above, it is the rotation of the system that produces both the toroidal magnetic field and the poloidal electric field, which provide a balance between magnetic and electric forces and in turn, also provide the condition for
the presence of charges (Porth & Fendt 2010). Generally, one has two methods to calculate the charge density. The first method is from the force-free condition,

\[
\rho_{c,ff} = \frac{(j \times B) \cdot E}{E^2} \approx_0 \frac{(\lambda + 1) B_\phi}{2\pi r \sin \theta} = \frac{-B_\phi}{B_p \Omega r \sin \theta} j_p \approx j_p, \tag{70}
\]

which implies that the charge density is always equal to the poloidal electric current density as discussed above. Similar to the enclosed current (Equation 9), one can define a linear charge density along the jet \(\rho_c = \int_0^R \rho_c 2\pi R'dR' \approx J = \Phi/2\), which is also a roughly conserved quality. The second method to calculate the charge density is from the divergence of electric field, i.e.

\[
\rho_{c,em} = \frac{\nabla \cdot E}{4\pi} = -\frac{\Omega \cdot B}{2\pi} + \Omega r \sin \theta j_\phi - \frac{\Omega'}{4\pi} |\nabla \Psi|^2 = -\frac{\Omega}{4\pi} \left( \nabla^2 \Psi + \frac{\Omega'}{\Omega} |\nabla \Psi|^2 \right),
\]

\[
\approx_0 \frac{\Omega \nu - 2}{4\pi} \left[ \frac{\lambda}{T} \left( \frac{dT}{d\theta} \right)^2 + 2\cot \theta \frac{dT}{d\theta} + (2 + \lambda \nu) \nu T \right]. \tag{71}
\]

In the case of \(s = 0\) (magnetic field threading a CO, \(\lambda' = 0\)), one has \(\rho_c = -\Omega \cdot B/2\pi\), which is similar to that in a pulsar magnetosphere (the so called Goldreich-Julian charge density \(\rho_{GJ}\), see Goldreich & Julian 1969). In principle, these two methods should self-consistently present a unique charge density. However, as discussed in Section 3, the charge density is related to the second-order derivative of \(\Psi\), and therefore our approximate solution would only be accurate at \(\theta \ll 1\), and would have an error at \(\theta \to \pi/2\). This is clearly shown when comparing Equations 70 and 71. At \(\theta = \pi/2\) (with \(\nu = 3/4\)), the case of \(s = 0\) yields \(\rho_{c,em}/\rho_{c,ff} = 0.375\), and the case of \(s = -3/2\) gives \(\rho_{c,em}/\rho_{c,ff} = 0.480\).

It is the global rotation of the magnetic fields that guarantees the existence of a charge density in the entire plasma region (Narayan et al. 2007), including near the CO surface or on the AD plane (see Section 9). On the other hand, in the case of a non-relativistic plasma flow, the electric force is relatively unimportant compared with the magnetic force. Some MHD simulations often ignore the electric force, of which such an approximation cannot give a self-consistent result \(\rho_e = \nabla \cdot E/4\pi = 0\), but nevertheless a good enough approximation to solve the magnetic field configuration (e.g., Yang et al. 2019, and references therein; see Appendix F for a detailed discussion).

### 6.3. Potential Difference

The magnetic stream surface is equipotential, which implies that there is a potential difference between two magnetic stream surfaces (layers \(l_1\) and \(l_2\), in the case of \(\lambda \neq -1\), see Equation 4 and also Blandford et al. 2019), i.e.

\[
\Delta V = \int_{l_1}^{l_2} \frac{dE}{n} = -\int_{l_1}^{l_2} \Omega \nabla \Psi \cdot dl_E = -\frac{\Omega \Psi}{2(\lambda + 1)} \bigg|_{l_1}^{l_2} = -\frac{\Omega \Psi}{2(\lambda + 1)} \bigg|_{l_1}^{l_2} = \frac{RB_\phi}{2(\lambda + 1)} \bigg|_{l_1}^{l_2} = \frac{J}{\lambda + 1}, \tag{72}
\]

In the case of \(\lambda = -1\), the coefficient \(1/(\lambda + 1)\) in the above equation should be replaced by \(\ln(\Psi_2/\Psi_1)\). The fluid plasma supports charge to cancel the electric field induced by the motion of fluid plasma, which makes jet self-balanced to maintain the electromagnetic configuration (see Section 7.1). Therefore, one anticipates a shorting out of the parallel electric field \(E_\parallel\) (to magnetic field): \(E \cdot B = 0\). However, there may be cases that in some regions the charges stream out along the magnetic field line and no new charges replenish the region. In extreme situations the charges could be totally depleted, forming a “gap” with \(E \cdot B \neq 0\) (see e.g., Hirota & Okamoto 1998; Chen et al. 2018; Parfrey et al. 2019). The parallel electric field \(E_\parallel\) would bring a potential difference along magnetic field line, the maximum of which is limited by Equation 72 (see e.g., Ruderman & Sutherland 1975, for a gap in the pulsar magnetosphere). For the case of magnetic field lines threading a BH, see Section 8.

### 6.4. Jet Power

In the force-free limit, the output jet energy flow is almost equal to the Poynting flux, \(S = E \times B/4\pi\), which follows the direction of the particle drifting velocity, transporting within the plane of magnetic stream surface. The \(r\) and \(z\) direction of the Poynting fluxes can be written as

\[
S_r = \frac{\Omega R}{4\pi} B_\phi B_r = \frac{\Omega^2}{2\pi} \frac{\Psi}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta} = \frac{B_\phi^2}{8\pi} \frac{\sin \theta}{T} \frac{dT}{d\theta},
\]

\[
S_z = \frac{\Omega R}{4\pi} B_\phi B_z = -\frac{\Omega \Psi}{4\pi R} \frac{\partial \Psi}{\partial R} = \frac{B_\phi^2}{8\pi} \sin^2 \theta \left( \frac{1}{T} \frac{dT}{d\theta}, \cot \theta + \nu \right). \tag{73}
\]
The enclosed Poynting flux between two magnetic stream surfaces \( \Psi_1 \) and \( \Psi_2 \) can be then easily calculated (two-side jet power, in case of \( \lambda \neq -1 \)),

\[
P_{\text{jet}} = 2 \int_{\theta_1}^{\theta_2} S_z 2 \pi r^2 \sin \theta d\theta = \frac{\Omega^2 \Psi^2}{|\lambda + 1|_1} = \frac{\Omega^2 F_B^2}{4 \pi^2 |\lambda + 1|_1} = \frac{R^2 B_z^2}{4 |\lambda + 1|_1} = \frac{J^2}{|\lambda + 1|_1},
\]

(74)

In the case of \( \lambda = -1 \), the coefficient \( 1/|\lambda + 1| \) in the above equation should be replaced by \( 2 \ln (\Psi_2/\Psi_1) \). The last equality shows a very simple relation between the jet power and the current carried by the plasma. Radio galaxy 3C 303 is hitherto the only source that has the measured electric current of its jet, i.e., \( \sim 3.9 \times 10^{18} \) A (through mapping polarization and Faraday rotation, Kronberg et al. 2011). Employing Equation 74, assuming a magnetic field threading a BH (\( \lambda = 0 \)) or a Keplerian AD (\( \lambda = -2 \) with \( \nu = 3/4 \)), one can easily estimate a jet power \( \sim 4.6 \times 10^{45} \) erg s\(^{-1}\), which is only a factor of \( \sim 2 \) smaller than that estimated through modeling the broadband spectral energy distribution \( \sim 1.0 \times 10^{46} \) erg s\(^{-1}\), Zhang et al. 2018). Combining with Equation 72, one has a maximum potential difference \( \Delta V \approx 1.7 \times 10^{15} \sqrt{P_{\text{jet}}/10^{36} \text{erg s}^{-1}} \) volts in the case of magnetic field lines threading a CO. Taking the radiative luminosity of the Crab Nebula \( (\gtrsim 10^{37} \text{ erg s}^{-1}, \text{ Hester 2008}) \) as a lower limit of jet/wind power, the potential difference can reach up to \( \gtrsim 5 \times 10^{15} \) volts, which is large enough to (theoretically) accelerate charged particles to emit the highest energy photons ever detected in the Crab Nebula \( (\approx 0.45 \text{ PeV}, \text{ Amenomori et al. 2019}) \). For a special case of magnetic field threading a BH, see Section 8.

At \( \theta \rightarrow \pi/2 \) (near the AD), the \( z \)-direction Poynting flux reads,

\[
S_z = \frac{B_z^2}{8\pi} (\nu + C_1 \cos \theta).
\]

(75)

At \( \theta \ll 1 \), one has,

\[
S_z = \frac{B_z^2}{4\pi} \frac{C_2}{\pi^2} \Omega^2 \Psi \nu^{-2} = \frac{C_2^2/\nu}{\pi} \Omega^2 \Psi^{2-2/\nu} \theta^{-2+4/\nu} = \frac{C_2^2}{\pi} \frac{C_2^2}{z^{2\lambda\nu+2\nu-4\lambda-4}} \nu^{4+2}. \]

(76)

In the case of \( \lambda = 0 \) (magnetic field threading a CO), one has \( S_z \propto R^2 \) at a given height, implying a hollow jet (the Poynting flux vanishes on the polar-axis), which may account for a limb-brightening phenomenon in some AGN jets (for example, M87, see e.g., Kovalev et al. 2007; Ly et al. 2007; Walker et al. 2018) and the apparent darkness of the inner pulsar wind nebulae of the Crab and other pulsars (e.g., Weisskopf et al. 2000; Hester et al. 2002; Kirk et al. 2009; Lyubarsky 2009). The hollow jet structure is also seen in MHD simulations (e.g., Hawley & Krolik 2006; Tchekhovskoy et al. 2008).

7. JET DYNAMICS

Since we consider a magnetohydrodynamic or force-free jet in this paper, we are unable to study the effects of mass-loading of the jet (see Ogilvie & Livio 2001; Casse & Keppens 2004 for mass-loading discussion in AGN jets and Lei et al. 2013 for GRB jets). However, it is expected that the main properties of a highly magnetized jet carry over to a mass-loaded jet, provided that the latter is electromagnetically dominated.

7.1. Force Balance

In assumption of force-free, the Lorentz force vanishes. In a real plasma flow system, when plasma inertia is considered, the Lorentz force cannot be neglected and would be responsible for plasma acceleration, \( \mathbf{F}_L = \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} \). It can be seen that in an axisymmetric system, the Lorentz force along the magnetic field line always vanishes: \( \mathbf{F}_{L,B} = 0 \). Similar to the analysis in Section 2, it can be proved that \( \Psi \) and \( \Omega \) are still conserved along magnetic field lines. Since the poloidal magnetic field \( B_r \) and the current density \( j_p \) may not be parallel to each other, \( \Phi \) may not be conserved. Because the electric field is always perpendicular to the magnetic field, one can naturally separate the Lorentz force into two parts: one along the direction of \( \mathbf{E} \) (i.e. \( \mathbf{F}_{L,E} \)) and another along the direction of \( \mathbf{E} \times \mathbf{B} \) (i.e. the direction of the drift velocity, \( \mathbf{F}_{L,\hat{v}_d} \)), i.e.

\[
\mathbf{F}_L = \mathbf{F}_{L,\hat{v}_d} + \mathbf{F}_{L,E} = \left[ \mathbf{j}_p \times \mathbf{B}_p + \left( \mathbf{j}_p \times \mathbf{B}_p \right) B_p \right]_{1,\hat{\phi}} + \left( \mathbf{j}_p \times \mathbf{B}_p \right)_{3,\hat{E}} + \mathbf{j}_p \times \mathbf{B}_p + \rho \mathbf{E}.
\]

(77)
The first two terms of Equation 77 correspond to the Lorentz force along the direction of the drift velocity, i.e.

$$\mathbf{F}_{L,v_d} = \frac{1}{4\pi R^2} \left( \frac{\partial \Phi}{\partial z} \frac{\partial \Psi}{\partial z} - \frac{\partial \Phi}{\partial R} \frac{\partial \Psi}{\partial R} \right) \left[ \frac{\dot{\psi} + \left( \frac{B_\phi}{B_p} \right) \dot{B}_p}{R^2} \right].$$  \hspace{1cm} (78)

This force has two effects: one is to accelerate the plasma, another is to make the plasma velocity direction tend to align with the drift velocity direction. Furthermore, we have the following conditions equivalent: 1) The Lorentz force vanishes in the magnetic stream surface (i.e. the force-free condition applies in the surface, $\mathbf{F}_{L,v_d} = 0$); 2) The poloidal current density, velocity, and magnetic field are parallel to each other ($j = \hat{v} \cdot B$) to that of magnetic field ($\mathbf{E} = \mathbf{F}_L$, $\mathbf{F} = \mathbf{L}$, $\mathbf{L} = \hat{v} \times B$); 3) The current flows in the magnetic stream surface; 4) $\Phi$ conserves along a magnetic field line, and thus is a function of $\Psi$ only. Releasing the force-free assumption (considering the inertia of plasma) requires that the above conditions are broken in order to accelerate the plasma fluid. As a result, these conditions apply approximately in the limit of highly magnetized jet (see below and e.g., Li et al. 1992).

The last three terms of Equation 77 refer to the Lorentz force along the direction of the electric field $\hat{E}$,

$$\mathbf{F}_{L,E} = \frac{B_p}{R} \left\{ \frac{\partial^2 \Psi}{\partial R^2} + \frac{\partial^2 \Psi}{\partial z^2} - \frac{1}{R} \frac{\partial \Psi}{\partial R} + \Phi \frac{4\pi j_p \cdot \mathbf{B}_p}{B_p^2} \frac{\Omega^2 R^2}{\hat{v}} \left[ \frac{\partial^2 \Psi}{\partial R^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{1}{R} \frac{\partial \Psi}{\partial R} + \frac{\Omega}{\hat{v}} \left( \left( \frac{\partial \Phi}{\partial R} \right)^2 + \left( \frac{\partial \Psi}{\partial z} \right)^2 \right) \right] \right\}. \hspace{1cm} (79)$$

which partially offers the “centripetal force” for plasma rotation and self-collimation for the outflows. The term 3 is well known as the $z$-pinch in the plasma physics (see Meier et al. 2001). In the case of poloidal current density $j_p$ parallel to that of magnetic field $B_p$ (i.e. $F_{L,v_d} = 0$), the term in the brace is reduced to the left-hand side term in Equation 10. Therefore, Equation 10 refers to the force-free condition along $\mathbf{E}$ ($F_{L,E} = 0$, see e.g., Lyubarsky 2009). Making term 4 equal 0 leads to Equation 12, while making combined terms 3 and 5 equal 0 leads to Equation 11. Due to the fact that $\Omega R$ almost measures the four velocity of plasma fluid (see Section 5), it can be seen that the electric force part is only important in the relativistic case. As discussed in Section 3, Equations 12 and 11 have approximately the same solutions, which implies that 1) the $\dot{\phi}$ component current density approximately vanishes (the term 4 in Equation 77), and that 2) the force the poloidal current receives from the magnetic field is balanced by the force exerted by the poloidal electric field to guarantee the force-free condition (in both relativistic and non-relativistic cases, see terms 3 and 5 in Equation 77), for more discussion on force balance, see e.g., Porth & Fendt 2010). As discussed above, in a non-rotation system, the toroidal magnetic field, the electric current and the electric field would vanish. What might exist is the poloidal magnetic field. It is the rotation that generates the toroidal magnetic field and induces the poloidal current. The force that this poloidal current receives from the magnetic field will be (almost exactly) balanced by the force exerted by the appeared poloidal electric field to guarantee the force-free condition (Equation 11). This leaves the toroidal current almost vanishing and the magnetic stream function being almost change free with rotation (Equation 12). This may explain why some MHD simulations found that the poloidal configuration of magnetic fields had little change from non-rotation to rotation cases (e.g., Tchekhovskoy et al. 2008).

This rotating poloidal magnetic field coil then drives the plasma fluid trapped in it outward along the magnetic field lines as they try to uncoil. As this twist propagates outward, the induced toroidal field pinches the plasma fluid toward the rotation axis. Therefore, a magnetically dominated jet would be self-collimated and accelerated (self balanced, see e.g., Blandford & Payne 1982; Heyvaerts & Norman 1989; Lynden-Bell 1996; Ostriker 1997; Meier et al. 2001; McKinney 2006a; Colgate et al. 2014). There may be a very special case, a bunch of magnetic field lines which anchor on and rotate with the CO/AD can load a bunch of magnetically dominated plasma moving along the field line. This bunch of magnetically dominated plasma may be also self-balanced. Therefore one may also employ the magnetic stream function to describe the system. In this viewpoint, the CO/AD and its rotation can be considered as merely a boundary providing a special electromagnetic field configuration, which further determines the outflow. In pulsars, the magnetic axis is usually misaligned from the rotation axis, so that a bunch of open magnetic field lines threading the magnetic polar region may launch a magnetically dominated collimated outflow, while the magnetic field lines near the magnetic dipolar equator would span through a torus-like region (e.g., Romanova et al. 2005; P´etri 2012). This offers an alternative explanation to “jet-torus” feature frequently observed in pulsar wind nebulae (Hester et al.

\[ 22 \] The magnetic force is almost balanced by the electric force, which is why both acceleration and collimation proceed very slowly.
The toroidal component (Equation 84) gives another conserved quality written as C.1, which gives rise to a very high Lorentz factor for the pulsar winds. In this case, the velocity of the plasma flow follows \( \Gamma = \Omega R \) for all polar angles (see Appendix C.1), which gives rise to a very high Lorentz factor for the pulsar winds.

There is another way to understand the effect of the Lorentz force. Generally speaking, the Lorentz force can be written as

\[
\mathbf{F}_L = \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left( \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} (\nabla \cdot \mathbf{E}) \mathbf{E} = -\frac{B^2 \hat{R}_B}{4\pi R_B} - \nabla \left( \frac{B^2}{8\pi} \right) + \frac{1}{4\pi} (\nabla \cdot \mathbf{E}) \mathbf{E},
\]

where \( \hat{R}_B \) denotes the radial vector of a curved magnetic field line and \( R_B \) denotes the radius of curvature of the magnetic field line. Therefore, we have \( \hat{R}_B/R_B = -d\hat{B}/ds = -\left( \mathbf{B} \cdot \nabla \right) \hat{B} \), with \( \hat{B} = \mathbf{B}/B \) being the unit vector along the magnetic field line. The gradient operator can be decomposed into two parts: those parallel and perpendicular to the magnetic field line, i.e. \( \nabla = \nabla_{\parallel} + \nabla_{\perp} \). The first term refers to a magnetic tension force of the magnetic field line, which appears whenever the magnetic field lines are curved and can be represented as a restoring force \(^{23}\). The second term represents the magnetic pressure \( (p_B = B^2/8\pi) \) gradient force, which occurs when the field strength, \( B \), varies from position to position. In principle, the magnetic pressure is isotropic. However, the component parallel to the magnetic field line is exactly canceled out by the component of the tension force in the same direction. Concerning the balance between various magnetic stream surfaces in a jet flow, the magnetic tension force points toward the polar axis, while the magnetic pressure gradient force points toward the polar axis in the case of magnetic field threading a CO \( (s = 0) \) and points away from the polar axis in the case of magnetic field threading a AD \( (s = -3/2 \) and \( \nu = 3/4) \). The third term is the electric field force, which points toward an opposite direction with that of the magnetic pressure gradient force. The magnetic pressure gradient and electric forces are two large numbers in the region outside the ACS \(^{24}\) (e.g., Beskin 2010). The difference between the two almost balances the magnetic tension force.

7.2. Jet Flow Velocity

In this subsection, we discuss the maximum velocity a jet can be accelerated to, and the relationship between the jet fluid velocity and the drift velocity. Hereafter (only) in this subsection, \( v \) (and \( \Gamma \)) refers to plasma fluid velocity (and Lorentz factor), while \( v_{d} \) (and \( \Gamma_{d} \)) indicates the drift velocity (and Lorentz factor).

Combining the continuity equation

\[
\nabla \cdot (\rho \mathbf{u}) = 0,
\]

and the general velocity Equation 44, one obtains a conserved quality along a magnetic field line

\[
4\pi \rho \Gamma \kappa = \eta (\Psi),
\]

which represents the mass flux per unit magnetic flux.

Equation 6 can be divided into the poloidal and toroidal components,

\[
\frac{u_p}{\partial l} = \frac{\sin \vartheta}{R} u_{\vartheta}^2 - \frac{B_{\phi}}{4\pi \rho R} \frac{\partial (RB_{\phi})}{\partial l},
\]

\[
\frac{u_p}{\partial l} = \frac{B_p}{4\pi \rho} \frac{\partial (RB_{\phi})}{\partial l},
\]

where \( l \) is the coordinate of the poloidal direction, and \( \vartheta \) is the angle between the \( l \) and \( z \) directions, i.e. \( \sin \vartheta = \partial R/\partial l \). The toroidal component (Equation 84) gives another conserved quality

\[
R \Gamma v_{\varphi} - \frac{RB_{\phi}}{\eta} = \mathcal{L} (\Psi),
\]

\(^{23}\) The magnetic tension force provides a restoring force for the Alfvén waves (Alfvén 1942).

\(^{24}\) According to Equation 41, the electromagnetic pressure is measured by the field in the rest frame of the plasma, \( B' = \sqrt{B^2 - E^2} = B/\Gamma \), which is a small value in the region far outside the ACS, even though both \( B \) and \( E \) are large values.
where the first term (of the left-hand side) measures matter angular momentum, the second term gives the torque associated with the magnetic stresses, and $\mathcal{L}$ represents the total angular momentum flux per unit mass flux.

Differentiating Equation 85 with $l$ and substituting it into Equation 83, together with Equation 44, one obtains another conserved quality

$$\Gamma - \frac{B_\phi \Omega R}{\eta} = \mathcal{E} (\Psi),$$

(86)

where $\mathcal{E}$ represents the total energy flux per unit rest energy flux. The first term of the left-hand side measures matter energy flux and the second term is for the Poynting flux. The ratio between the two terms defines a magnetization parameter (e.g., Lyubarsky 2009),

$$\sigma = \frac{-B_\phi \Omega R}{\eta \Gamma} = \frac{\mathcal{E} - \Gamma}{\Gamma}.$$  

(87)

It should be noted that, only in the case of high magnetization ($\sigma \gg 1$), the properties of plasma flow almost would not affect the details of the acceleration process, which is instead controlled by the magnetic field topology (see Equation 47 below and also e.g., the GRMHD simulation in Pu & Takahashi 2020).

Therefore,

$$\Gamma (\sigma + 1) = \mathcal{E}$$  

(88)

is conserved along a magnetic field line, which implies that $\sigma$ decreases during jet acceleration, and the maximum Lorentz factor of the plasma cannot exceed $\mathcal{E}$ even when all the Poynting flux energy is converted to kinetic energy (e.g., Beskin & Nokhrina 2006; Toma & Takahara 2013). In principle, from Equation 86, the conservation of $\Phi = B_\phi \Omega R$ in the force-free condition has to be broken in a real jet system to prevent Lorentz factor $\Gamma$ from being conserved along a magnetic field line (see also the Equation 78). However, in the case of high magnetization, one has $B_\phi R = \Phi$ approximately conserved and $\Gamma \sigma = -\Phi \Omega / \eta \approx \mathcal{E} (\sigma \gg 1$, see Equation 87). To determine $\mathcal{E}$ at a foot-point, one needs to consider a detailed mass-loading mechanism, which is not well understood yet (see e.g., Blandford & Payne 1982; Ogilvie & Livio 2001; Casse & Keppens 2004; Bai 2016; Lei et al. 2013, and references therein). In principle, the value of $\mathcal{E}$ cannot be arbitrarily large, because the plasma energy density cannot be arbitrarily small due to the fact that the plasma has to carry a proper charge and current to support the electromagnetic field (see Appendix G for more discussion on the maximum Lorentz factor). One expects that MHD mass-loaded outflows starting with a high value of the initial magnetization parameter $\sigma$ would roughly follow the force-free solution out to a certain distance, beyond which the MHD flows likely feel the effect of the inertia. It simply means that the force-free solutions can only apply (physically reasonable) up to a modest distance from the central engine. Furthermore, for a highly magnetized jet flow, the plasma fluid velocity is very close to the drift velocity, which will be shown below.

From Equations 85 and 86, one has another conservation law

$$\Gamma (1 - \Omega R v_\phi) = \mathcal{E} - \Omega \mathcal{L} = \varepsilon.$$  

(89)

Near the jet base, let us assume that the flow is sub-Alfvénic. One expects that the fluid motion is dominated by the toroidal velocity which is not very relativistic, i.e. $\Omega R_0 v_\phi \ll 1$ and $\Gamma \approx 1$. This yields $\varepsilon \approx 1$. Expressing $v_\phi$ and $v_p$ in term of $\Gamma$, one has

$$v_\phi = \frac{1}{\Omega R} \left(1 - \frac{\varepsilon}{\Gamma} \right),$$

$$v_p = \frac{B_p}{B_\phi} \left[\frac{1}{\Omega R} \left(1 - \frac{\varepsilon}{\Gamma} \right) - \Omega R\right],$$

(90)

which implies

$$\frac{\Gamma^2 - 1}{\Gamma^2} = \left[\frac{1}{\Omega R} \left(1 - \frac{\varepsilon}{\Gamma} \right)\right]^2 + \left[\frac{B_p}{B_\phi} \left[\frac{1}{\Omega R} \left(1 - \frac{\varepsilon}{\Gamma} \right) - \Omega R\right]\right]^2.$$  

(91)

Let us define $r_B = B_\phi / B_p$ and $\varpi = \Omega R$ for simplicity of writing. The above equation has a physical solution:

$$\Gamma = \sqrt{r_B^2 \varpi^2 (1 + r_B^2 - \varpi^2)} (-1 + \varepsilon^2 + \varpi^2) - \varepsilon (1 + r_B^2 - \varpi^2) \varepsilon \left(1 + r_B^2 - \varpi^2\right).$$  

(92)

Even if the velocity reaches $v_\phi \approx \Omega R_0 \approx 1/2$ and $\Gamma \approx 1/\sqrt{1 - v_\phi^2}$, one still has the $\varepsilon$ value close to 1: $\varepsilon \approx 0.87$.\end{footnote}
It can be seen that once the magnetic field configuration is specified, the cold plasma velocity would be completely determined (see below and e.g., McKinney 2006c).

From Equation 46, the drift velocity can be expressed as

$$\Gamma_d = \sqrt{\frac{1 + r_B^2}{1 + r_B^2 - \varpi^2}}.$$  \hspace{1cm} (93)

We can then define a deviation of plasma fluid velocity from drift velocity, i.e.

$$D_{\delta \theta} \equiv \frac{(v\Gamma - (u_a \Gamma_a_d))^2}{(u_a \Gamma_a_d)^2} = \frac{(1 + r_B^2 - \varpi^2)}{(u_a \Gamma_a_d)^2} \left[ \frac{1 + r_B^2 - \varpi^2 - 1}{\varpi^2 - 1} - 2x\sqrt{r_B^2 \varpi^2 (1 + r_B^2 - \varpi^2) (-1 + \varpi^2 + \varpi^2) + \varepsilon^2 [1 - \varpi^2 + r_B^2 (1 + \varpi^2)]} \right].$$  \hspace{1cm} (94)

As argued above, one has $\varepsilon \approx 1$ and $|B_\phi|/B_p \approx \Omega R = \varpi$ (see Equation 38). Therefore, the above equation can be approximated as

$$D_{\delta \theta} \approx \frac{\varpi^4 - 2\varpi^3 + \varpi^2}{\varpi^2 (\varpi^2 - 1)^2},$$  \hspace{1cm} (95)

which is a decreasing function of $\varpi$. In the limit of $\varpi = \Omega R \gg 1$, one has $D_{\delta \theta} \approx 1/\varpi^2 \ll 1$, implying that the relativistic jet has almost the same fluid and drift Lorentz factors. In the limit of $\varpi = \Omega R \ll 1$, one has $v\Gamma \approx \sqrt{2} \varpi \Gamma_d \ll 1$, indicating that the fluid and drift velocities are still roughly equal to each other. More exactly, in the limit of $\varpi = \Omega R \ll 1$, the deviation of $\varepsilon$ from unity could not be safely ignored (see Equations 89 and 94). In this case, let us set $v\Gamma = x u_a \Gamma_a_d$, which also implies $v_\phi \approx x v_\phi, a$ since the $\phi$ component velocity dominates. In this case, $|B_\phi|/B_p$ may also slightly differ from $\Omega R$. We may simply set $|B_\phi|/B_p = c_3 \Omega R$ (see Equation 38). Combining these relations and $v_\phi, a = v/ (1 + \varpi^2)$ (see Equation 52), one has

$$\varepsilon = 1 - \left(x - \frac{x^2}{2}\right) \varpi^2 + O(\varpi^4).$$  \hspace{1cm} (96)

Substituting this relation into Equation 94, one can derive a solution $x = 1 + c_3^2 \varpi^2/2 + O(\varpi^3)$, which indicates that the fluid has a velocity very close to the drift velocity. We plot the comparison between this cold plasma velocity (Equation 92) and drift velocity (Equation 93) in Figure 6. It can be seen that the drift velocity matches the cold plasma velocity very well. This result is not significantly affected by the conserved quality in Equation 89, which is set to $\varepsilon = 1 - (\Omega R_0)^2/2$ here with $R_0$ being the radial distance of the foot-point of magnetic field (see Equation 96). We note that Narayan et al. (2007) and Tchekhovskoy et al. (2008) reached a similar result in the limit of the relativistic case. As for why the plasma fluid velocity can be represented by a pure electromagnetic field quality - drift velocity, one may understand it as follows: 1) The plasma flow is highly magnetized so that it is dynamically unimportant and almost cannot affect the electromagnetic field configuration; 2) The ideal MHD condition indicates that the motion of the plasma fluid is governed by the electromagnetic field configuration (freezing effect); 3) The electromagnetic field configuration is self-consistently determined because the plasma can supply currents and charges as needed to support the electromagnetic field.

### 7.3. Jet Flow Density

Within the force-free approach, the inertia is ignored, which prevents a calculation of the flow density. In another aspect, given an available mass flux per magnetic flux $\eta$ (see Equation 82) that still satisfies a highly magnetically dominated condition, an approximation of the jet flow density along the magnetic field line can be derived, with the aid of Equations 44, 82, 3 and 46, i.e.

$$\rho = \frac{\eta}{4\pi u_p} B_p \sim \frac{\eta \Psi}{2\pi \Omega^2} \frac{\sqrt{1 + (\Omega R)^2}}{R^4} \sim \frac{\eta}{4\pi \Omega^2} \frac{B}{R^2},$$

$$\rho_1 = \Gamma \rho \sim \frac{\eta}{8\pi \Omega^2} B^2,$$  \hspace{1cm} (97)

where $\rho_1 = \Gamma \rho$ refers the density in the lab frame (see e.g., Broderick & Loeb 2009, for $\rho_1 \propto B^2$). Note that the derivation of the approximations makes use of Equation 46, which can be further improved by using the velocity formula from Sections 5.1 and 5.3.
8. BLACK HOLE JETS

As discussed in Section 4, the foot-point location can be constrained through measuring the large scale jet configuration with $2\sigma = C_2 \nu^{-2} R^2$ (see Equations 27 and 37). Furthermore, the angular velocity can be also constrained through measuring the jet width and its velocity (in the $v_J$ dominated regime, $v \Gamma = \Omega R$), which presents a potential method to estimate the BH spin for a jet launched from a rotating BH.

In principle, one should consider GR effects to study jet launching from a BH, which was not taken into account in this paper. In the following, we discuss how GR effects may play a significant role near the BH. The full GR effects have been studied in a growing number of GRMHD simulations recently, which are required to capture the complicity of outflows near the BH (e.g., McKinney 2006a,c; McKinney & Narayan 2007a; McKinney et al. 2012; Penna et al. 2013; Pu et al. 2015; Chatterjee et al. 2019). Theoretical analyses show that due to the GR frame dragging effect in the vicinity of the BH, a magnetic field line threading a BH does not co-rotate with the BH, but lags behind with an angular velocity approximately a half of that of the BH (e.g., Blundford & Znajek 1977; MacDonald & Thorne 1982). Numerical simulations show that the angular velocity is not constant and has $f_\Omega \approx 0.3 - 0.5$ for $\Omega = f_\Omega M_{BH}$ depending on the polar angles (e.g., McKinney & Narayan 2007b; Tchekhovskoy et al. 2010; Pan & Yu 2015). For our analytical treatment, we assume a constant $f_\Omega = 0.5$ throughout the following analysis. As presented by McKinney & Narayan (2007b), the impact of this polar angle dependent behavior on the jet properties is almost negligible. The electromagnetic energy would be always transported out from a BH so that the jet can be launched. This implies that as long as not very close to the BH, the Lorentz force would dominate gravity, so that the gravitational effect would not qualitatively change the magnetic field configuration, as shown in simulations (see e.g., McKinney & Narayan 2007a,b).

We employ a pseudo-Newtonian potential to measure GR effects around a rotating BH. The gravitational acceleration can be written as (Paczyński & Wiita 1980; Artemova et al. 1996)

$$g_{BH} = -\frac{M}{r^{2-\gamma}} (r - r_+)^\gamma,$$

with $\gamma = r_{ISCO}/r_+ - 1$, where $r_+ = (1 + \sqrt{1-a^2})r_g$ is the horizon radius of the BH, $r_g = M$ is the gravitational radius, $a = J_{\text{am}}/M^2$ is the BH spin parameter, and $J_{\text{am}}$ is the angular momentum. The inner-most stable circular orbit (ISCO) is $r_{ISCO} = \left\{ 3 + Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)} \right\} r_g$ with $Z_1 = 1 + (1 - a)^{1/3} \left[ (1 + a)^{1/3} + (1 - a)^{1/3} \right]$ and $Z_2 = \sqrt{3a^2 + Z_1^2}$. As presented by Artemova et al. (1996), this potential captures the essentials of the GR effects from the exact relativistic Kerr metric because the free-fall acceleration tends to be infinite when $r$ approaches the event horizon of the BH, and the position of the extremum of the boundary condition function is the same as that of the last stable circular orbit in the exact relativistic Kerr metric. Notice that the gravitational acceleration is not significantly dependent on the polar angle $\theta$ (e.g., MacDonald & Thorne 1982; Dhillingia et al. 2018).

With gravity, the motion equation now reads $\rho (u \cdot \nabla) u = \rho \dot{E} + j \times B + \rho g_{BH}$. In the case of the magnetic force dominating over gravity (the inertia is negligible) the force free condition (Equation 7) applies, so that the electromagnetic configuration would evolve self-consistently and our approximate solution applies. This implies that there is a critical surface where gravity balances magnetic force, outside of which our approximate solution would roughly apply. The magnetic force can be expressed in terms of acceleration, i.e.

$$|g_{BH}| = \sigma \left( \frac{B^2}{8\pi} \right)^{-1} |(j \times B)_r| = 8\nu \sigma \Omega^2 r \frac{1}{(1 + B^2/\hat{B}^2)} \approx 4\nu \sigma \Omega^2 r \frac{1 - \cos \theta}{(1 + B^2/\hat{B}^2)},$$

where $\sigma = U_B/\rho$ is the magnetization parameter for a non-relativistic flow. Note that the magnetic field would be poloidally dominated near the BH, i.e. $B^2_\phi/B^2_p < 1$. It can be seen that the magnetic force acceleration becomes smaller with a smaller $\theta$, which implies that this critical surface will extend to a larger distance from the BH at a smaller $\theta$. For $a = 0.998$, $\sigma = 10$, $\theta = \pi/4$ and $\nu = 3/4$, the combination of Equations 98 and 99 yields the critical balance point $r_{cr} \approx 1.4 r_g$.

To magnetically launch a jet from a BH, the rotation of the BH forces the magnetic field lines to rotate, forming a helical structure. This generates a Lorentz force exerting on the charged particles to force them move outwards. If

---

26 In some AGNs, the jet power may be even larger than the accretion power, which implies that the jet may be produced from a rotating BH (see e.g., Ghisellini et al. 2014; Chen 2018).
it is very close to the BH, strong gravity would force the particles to move inwards. One would therefore expect the existence of the so called stagnation surface, within which gravity dominates over the Lorentz force and particles move inwards, while beyond which the Lorentz force dominates over gravity and particles move outwards. This stagnation surface can be derived through the condition of gravity balancing the Lorentz force. Near the BH, the Lorentz force is dominated by the magnetic force, which implies that the stagnation surface is very close to the critical balance surface discussed above, but is a little bit more outside since the electric force would partially cancel the magnetic force. In principle, the force-free condition implies that the Lorentz force would vanish, and therefore, would prevent a further constraint on the stagnation surface. Our approximate solution can be used to make a very rough estimation through expressing the Lorentz force in terms of acceleration, i.e.,

$$|g_t| \approx \nu \sigma \Omega^2 r \left(1 - \cos \theta \right)^2 \left(\nu^2 - 5\nu + 6\right) \over \left(1 + B_\theta^2 / B_r^2 \right).$$

(100)

Similar to Equation 99, the stagnation surface is expected to extend to a larger distance from the BH at a smaller $\theta$. For $\nu = 3/4$, $a = 0.998$, $\sigma = 10$ and $\theta = \pi/4$, the combination of Equations 98 and 100 gives the stagnation point $r_{st} \approx 2.5 r_g$. This picture is roughly consistent with GRMHD simulations (see e.g., Takahashi et al. 1990; McKinney 2006a; Pu et al. 2015; Nakamura et al. 2018).

We now discuss how to constrain the BH spin and jet foot-points through measuring the large scale jet properties. The rotation angular velocity of a BH (the angular velocity of the zero angular momentum observer at BH’s horizon) is $\Omega_{BH} = a/2r_+$. The magnetic field line penetrating the BH rotates with

$$\Omega = f_\Omega \Omega_{BH} = \frac{a}{2 \left(1 + \sqrt{1 - a^2} \right) r_g}.$$  

(101)

The magnetic field line with the foot-point $r_0$ on the AD cannot rotate faster than the Keplerian velocity (in Boyer-Lindquist coordinates), i.e.,

$$\Omega \leq \Omega_K = \frac{1}{r_g \left( r_0 / r_g \right)^{3/2} + a} \leq \frac{1}{r_g \left( r_0 / r_g \right)^{3/2}},$$

(102)

which reaches the maximum value $\Omega_{K,\max}$ at the ISCO. Based on the measured angular velocity and given a BH mass, one can estimate the value (Equation 101) or the lower limit value ($\Omega \leq \Omega_{K,\max}$, Equation 102) of $a$. In another aspect, for any choices of $a$, the angular velocity of a magnetic field line cannot be larger than $1/r_g$ (corresponding $r_0 = r_g$ at ISCO for case $a = 1$ in Equation 102), which gives a constraint on the BH mass$^{27}$ $M \leq 1 / \Omega = R / (v \Gamma)$.

Assuming that the jet configuration and velocity profile represent the same magnetic stream surface, one can combine these relations with Equations 101 and/or 102 to obtain two equations of three quantities: BH mass, BH spin and foot-point location (measured by $\theta_0$ and $r_0$). In principle, given one of these three quantities, one can derive the remaining two. Generally, one can get a constraint on these three quantities based on the fact that $0 \leq a \leq 1$ and $0 \leq T(\theta) \leq 1$ (noticing that the velocity profile does not depend the absolute amplitude of $\Psi$, see Section 5 and Appendix B).

**Case I:** magnetic field line threading a BH,

$$T(\theta_0) \geq C_2 g^2 \left( 2v \Gamma / f_\Omega \right)^\nu,$$

$$a \geq a_{\min} = C_1^{1/\nu} g^{2\nu - 1} \left( 2v \Gamma / f_\Omega \right),$$

$$M \geq \frac{f_\Omega a_{\min} R}{2 \left(1 + \sqrt{1 - a_{\min}^2} \right) v \Gamma},$$

$$r_0 = r_+ = \left(1 + \sqrt{1 - a^2} \right) r_g.$$

$$^{27}$$The magnetic field threading the AD may rotate more slowly than the Keplerian velocity. However, it is still unclear how much slower it is. The magnetic field threading the BH has a maximum rotation angular velocity of $\Omega = f_\Omega / 2r_g$ (Equation 101 for $a = 1$). If this value is also assumed to be the maximum rotation velocity of the magnetic field threading the AD (McKinney & Narayan 2007a), one has a more severe constraint on the BH mass $M \leq f_\Omega R / (2v \Gamma)$. 


Case II: magnetic field line threading a AD,\[\theta_0 = \pi/2,\]
\[r_0 = C_2^{1/\nu} \theta_0^{2/\nu},\]
\[M \geq C_2^{3/\nu} (\nu \Gamma)^2 r \theta_0^{\nu-2},\]
\[\frac{1}{(r_{ISCO}/r_g)^{3/2}} \geq C_2^{3/\nu} (\nu \Gamma)^3 \theta_0^{\nu-3}.\] (104)

8.1. Properties of a Black Hole Jet

In this section, we consider a jet with magnetic field lines threading a BH. Let us further assume that the outermost magnetic stream surface is connected with the equator of a spinning BH and the innermost stream surface shrinks to the polar axis. For the outermost magnetic stream surface, one has ($\theta \ll 1$)
\[R_{\text{out}} = C_2^{-1/2} \left(1 + \sqrt{1 - a^2}\right)^{\nu/2} \left(\frac{z}{r_g}\right)^{1-\nu/2},\]
\[V_{1, \theta \ll 1, \text{out}} = C_2^{-1/2} \left(\frac{f_\Omega}{1/2}\right) \frac{a}{4 (1 + \sqrt{1 - a^2})^{1-\nu/2}} \left(\frac{z}{r_g}\right)^{1-\nu/2},\]
\[V_{2, \theta \ll 1, \text{out}} = \frac{2}{\sqrt{2-\nu}} C_2^{1/2} \left(1 + \sqrt{1 - a^2}\right)^{-\nu/2} \left(\frac{z}{r_g}\right),\]
\[P_1 = 3.44 \frac{1 + \sqrt{1 - a^2}}{a} \left(\frac{1/2}{f_\Omega}\right) \left(\frac{M}{10^8 M_\odot}\right) \text{ hr},\]
\[(v \Gamma)^2 B' = (\Omega R)^2 B_p = \frac{2}{\nu^2 + C_1^2} (\Omega R_0)^2 B_{p,0} = \frac{2}{\nu^2 + C_1^2} \left(\frac{f_\Omega}{1/2}\right)^2 \frac{a^2 B_{p,0}}{16},\] (105)

where $P_1$ is the cycle period of the magnetic field. It can be seen that the BH with a larger spin parameter will produce a faster jet at a given height $z$ from the central BH (see GRMHD simulation by Nakamura et al. 2018), which implies that one can constrain the BH spin through measuring the jet acceleration profile. For $f_\Omega = 1/2, \nu = 3/4$ and $a = 1$, at distance $z = 10^3 r_g$, one has $R_{\text{out}} = 109 r_g, V_{1, \theta \ll 1, \text{out}} = 27.3, V_{2, \theta \ll 1, \text{out}} = 16.4$ and $(v \Gamma)_{\theta \ll 1, \text{out}} = 14.1$ (the ACS $\Gamma$ out = 1 is located at $z = 4.99 r_g$ for $a = 1$, at $z = 67.4 r_g$ for $a = 0.3$, at $z = 800 r_g$ for $a = 0.1$). For BH spin parameters varying in the range $a \sim 0.1 - 1$ and with typical values of Lorentz factor $\Gamma \sim 10 - 30$ in AGN jets (e.g., Ghisellini et al. 2014; Chen 2018), the cycle period in the inertial frame would be on a timescale from sub-month to years ($P \approx (1 + \Gamma^2) P_1$, see Equation 58). These values are in the range of the observed periodical timescales of flux variability in AGN jets (e.g., Ackermann et al. 2015; Zhou et al. 2018; Peñil et al. 2020) and/or the oscillation timescales of the jet position angle (e.g., Lister et al. 2013; Walker et al. 2018). The last formula in above Equation 105 applies to the region where $\Omega R = v \Gamma$ is satisfied, where $B' = B/\Gamma = B_p$ is the strength of magnetic field measured in fluid rest frame, and $B_{p,0}$ is the strength of poloidal magnetic field near the CO equator.

For the case of $\nu < 1$, $V_{1, \theta \ll 1}$ crosses $V_{2, \theta \ll 1}$ at the CCS (see Equation 51): $\Omega R^2 z^{-1} = 2/\sqrt{2-\nu}$. The inside region of CCS is dominated by $V_{1, \theta \ll 1}$, while the outside region is dominated by $V_{2, \theta \ll 1}$. The CCS crosses the outermost magnetic stream surface at
\[\frac{z_{\text{crs}}}{r_g} = \left(1 + \sqrt{1 - a^2}\right) \left[\frac{2}{\sqrt{2-\nu}} \left(\frac{1/2}{f_\Omega}\right) \frac{4C_2}{a}\right]^{1/(1-\nu)} .\] (106)

For $f_\Omega = 1/2, \nu = 3/4$ and $a = 1$, one has $z_{\text{crs}} = 130 r_g$. Since $V_{1, \theta \ll 1}$ is an increasing function and since $V_{2, \theta \ll 1}$ is a decreasing function of the cylindrical radius $R$, one therefore expects that I) in the region of $z > z_{\text{crs}}$, the jet has a slower spine surrounded by a faster interlayer, which is further surrounded by a slower outlayer; and that II) in the region of $z < z_{\text{crs}}$, the jet only has a slower spine surrounded by a faster layer. For the former case, there is a maximum velocity at the CCS (Equation 51), i.e.
\[(v \Gamma)_{\text{CCS}} = V_{1, \theta \ll 1, \text{crs}}/\sqrt{2} = \frac{1}{(2-\nu)^{1/4}} \left[\frac{a}{4 (1 + \sqrt{1 - a^2})} \left(\frac{f_\Omega}{1/2}\right) \frac{z}{r_g}\right]^{1/2}.\] (107)

28 In other cases, it is an upper limit.
For \( f_\Omega = 1/2, \nu = 3/4 \) and \( a = 1 \), at \( z = 10^3 r_s \), one has \((v\Gamma)_{\text{CCS}} = 15.0\), which is larger than that at the outermost surface \((v\Gamma)_{\text{out}} = 14.1\). At \( z = 10^5 r_g \), one has \((v\Gamma)_{\text{CCS}} = 150\) and the jet open angle \( \theta_{op} = 2\sin^{-1}(R_{\text{out}}/z) = 2.22^\circ\), which are consistent with the typical values for GRBs\(^{29}\) (e.g., Lithwick & Sari 2001; Frail et al. 2001; Wang et al. 2018). At the critical point \( z = z_{\text{rms}} \), one has \((v\Gamma)_{\text{CCS, rms}} = 5.40\) for \( a = 1 \), and this value increases as the spin parameter \( a \) decreases. This means that for the jet region with \( v\Gamma \leq 5.4 \), \( V_{1, \theta < 1} \) always dominates (with \( f_\Omega = 1/2 \) and \( \nu = 3/4 \)). Notice that this maximum velocity \((v\Gamma)_{\text{CCS}}\) is over the entire jet region, as opposed to that along a magnetic field line (see Tchekhovskoy et al. 2008, for a similar discussion). Such overall jet velocity properties are clearly presented in Figure 4 (for \( \nu = 3/4 \)).

The total jet power is dominated by the Poynting flux, which can be estimated as (see Equation 74 and also Blandford & Znajek 1977; Tchekhovskoy et al. 2008),

\[
P_{\text{jet}} \sim 2 \times 10^{45} \frac{2}{\nu^2 + C_1^2} \left( 1 + \sqrt{1 - a^2} \right) \left( \frac{f_\Omega}{1/2} \right) \left( \frac{M}{10^8 M_\odot} \right)^2 \left( \frac{B_{p,0}}{10^4 G_\odot} \right)^2 \text{erg s}^{-1}
\]

\[
\sim 2 \times 10^{51} \frac{2}{\nu^2 + C_1^2} \left( 1 + \sqrt{1 - a^2} \right) \left( \frac{f_\Omega}{1/2} \right) \left( \frac{M}{10^8 M_\odot} \right)^2 \left( \frac{B_{p,0}}{10^4 G_\odot} \right)^2 \text{erg s}^{-1},
\]

which is large enough to provide the jet power in AGNs (e.g., Ghisellini et al. 2014; Chen 2018; Blandford et al. 2019) and GRBs (e.g., Piran 2004; Mészáros 2006; Kumar & Zhang 2015; Zhang 2018). The jet power increases with an increasing \( a \) and roughly follows a power-law dependence with a slope \( \sim 2 \). This is also consistent with what is observed in X-ray binaries (Narayan & McClintock 2012; Steiner et al. 2013).

The potential difference between the magnetic stream surfaces at polar angle \( \theta \) and at polar-axis near the BH can be estimated as (Equation 72)

\[
\Delta V \sim 8 \times 10^{19} \sqrt{\frac{2}{\nu^2 + C_1^2} \left( 1 + \sqrt{1 - a^2} \right) \left( \frac{f_\Omega}{1/2} \right) \left( \frac{M}{10^8 M_\odot} \right) \left( \frac{B_{p,0}}{10^4 G_\odot} \right) T(\theta) \text{ volts}}
\]

\[
\sim 8 \times 10^{22} \sqrt{\frac{2}{\nu^2 + C_1^2} \left( 1 + \sqrt{1 - a^2} \right) \left( \frac{f_\Omega}{1/2} \right) \left( \frac{M}{10^8 M_\odot} \right) \left( \frac{B_{p,0}}{10^4 G_\odot} \right) T(\theta) \text{ volts}}.
\]

In principle, any charged particle that crosses the magnetic stream surface would be accelerated by the potential difference. In another aspect, this provides a maximum potential difference reachable for a gap (if formed) near the polar cap region, which indicates that even making use of a small potential in the gap (corresponding to \( \theta \ll 1 \)) would be able to accelerate a charged particle to a very high energy (see the case of a pulsar magnetosphere, Ruderman & Sutherland 1975). Similar to the case in pulsar inner gaps, an induced strong electric field along the magnetic field line can serve to accelerate particles and further produce \( e^\pm \) pairs through a cascade of interaction between these high energy particles and background photons (see Hirotani & Okamoto 1998; Parfrey et al. 2019, for detailed simulations). This also offers a possible source of ultra high energy cosmic rays and neutrinos (Kotera & Olinto 2011; IceCube Collaboration 2013; IceCube Collaboration et al. 2018b,a; Anchordoqui 2019).

One needs to mention the caveat that the “pulsar” Equation 10 is written for a flat space. The application of its solution near a BH only yields approximate results.

\[
\text{9. THE CO/AD AS A BOUNDARY}
\]

In principle, a detailed jet launching mechanism is needed to produce the magnetic field configuration for launching a jet. This is beyond the scope of this paper. Here we consider the basic properties of the CO/AD (such as charges, currents, etc., as the boundary conditions of the plasma outflow) required to support a magnetic jet. Section 4 presents the electromagnetic field properties near the CO/AD, which can be used to constrain the current and charge properties of the CO/AD. When applying the Stokes’s theorem to \( j = \nabla \times B/4\pi \) and the Gauss’s theorem to \( \rho = \nabla \cdot E/4\pi \) on an infinite thin AD with an electromagnetic field described by Equations 39 and 42, one immediately gets the surface

\[^{29}\text{Note that the maximum velocity does not occur at the outermost magnetic stream surface and thus does not conflict with } (v\Gamma) \theta \leq 2/\sqrt{2 - \nu}.\]

\[^{29}\text{Also, for a stellar mass black hole, } z = 10^3 r_g \sim 10^{14} \text{ cm is compatible with the size of the progenitor star.}\]
current and charge densities

\[ J_\phi^e (R) = \frac{B_R (R)}{2\pi} = \frac{C_1}{2\pi} R^{\nu - 2}, \]

\[ J_{R}^e (R) = -\frac{B_\phi (R)}{2\pi} = \frac{\Omega R}{\pi} R^{\nu - 2}, \]

\[ \rho_c^e (R) = -\frac{1}{4\pi} \frac{C_1 B_\phi (R)}{R} = \frac{C_1 \Omega R}{2\pi} R^{\nu - 2}. \]  

(110)

The total charges carried by the CO can be also derived by a surface integration of electric fields over a sphere near the CO surface, which reads

\[ Q = \frac{1}{4\pi} \int \mathbf{E} \cdot d\mathbf{S} = r_0^2 \int_0^{\pi/2} E_{r, CO} (\theta) \sin \theta d\theta = -\frac{\nu C_1 \Omega R^3 B_{p, 0}}{(1 + \nu) (2 - \nu) \sqrt{\nu^2 + C_1^2}} = -S g (\mathbf{\Omega} \cdot \mathbf{B}) \frac{\nu C_1 r_0 \sqrt{P_{\text{jet}}}}{(1 + \nu) (2 - \nu)} = -S g (\mathbf{\Omega} \cdot \mathbf{B}) \frac{\nu C_1 f_{1/2} a F_B}{4\pi (1 + \nu) (2 - \nu)}, \]  

(111)

where \( r_0 \) is the radius of the CO, and \( E_{r, CO} (\theta) \) is the \( \hat{r} \) component of the electric field near the CO surface. For the typical value \( \nu = 3/4 \), one has \( |Q| = 0.281 \Omega r_0^3 B_{p, 0} \). Notice that in the case of a pulsar with a magnetosphere with an aligned dipolar magnetic field, the charge of the neutron star is \( |Q| = \Omega r_0^3 B_p/3 \) (see Ruderman & Sutherland 1975). Therefore, a BH/AD has to be charged in order to launch a magnetically dominated jet, which however cannot be produced by a neutral BH/AD even if it rotates rapidly or accretes lowly/highly. This is consistent with observations that 1) many AGNs host fast rotating BHs but are still radio quiet (RQ, no jet; see Reynolds 2014; Vasudevan et al. 2016; Xu et al. 2017; Walton et al. 2018; Sun et al. 2018); 2) RL and RQ AGNs present similar optical emission properties (Elvis et al. 1994; Shang et al. 2011), which are mainly determined by BH mass, spin and accretion rate; 3) jet can be produced from either a BH- advection-dominated accretion flow system (geometrically thick, radiatively inefficient, and with a low accretion rate, e.g., BL Lacertae objects, see Narayan & Yi 1995a,b; Yuan & Narayan 2014; Chen 2018) or a BH- standard Shakura-Sunyaev disk system (geometrically thin, radiatively efficient, and with a high accretion rate, e.g., flat spectrum radio quasars, see Shakura & Sunyaev 1973; Chen 2018), with the difference between the two cases being mainly determined by the accretion rate (e.g., Xu et al. 2009; Yuan & Narayan 2014). In this aspect, whether the BH/AD is charged may account for the RL/RQ dichotomy in AGNs (Kellermann et al. 1989; Antonucci 1993; Urry & Padovani 1995; Padovani et al. 2017; Blandford et al. 2019). This may be further related to the properties of the accreted plasma. Many theoretical works explore magnetized plasma flow accreted by a spinning BH and find that a relativistic jet can be launched via the BZ process in the funnel region, while the region outside the funnel is not preferred to launch a relativistic outflow (e.g., McKinney & Gammie 2004; Porth et al. 2019; Sądowski et al. 2013; Nakamura et al. 2018). This may suggest that a charged BH is more essential than a charged AD in launching a relativistic jet. For a relativistic jet considered here, which is powered by the rotation of the BH/AD, the BH charge is much smaller than the maximum value that is achievable theoretically (see Appendix H). The poloidal current of a BZ jet would exhaust the BH charge on timescale of \( \Delta t \sim Q/J \sim 10 M_8 \) min \((M_8 = M/10^8 M_\odot, \) see Equation 74 and 111). Such a small timescale indicates that a continuous accretion is necessary to maintain a “stable” BZ jet.

The above result suggests that the azimuthal surface current density on the AD would follow a power-law distribution, the slope of which determines the configuration of the poloidal magnetic field in the jet. This simple boundary condition is consistent with some simulations, which show that although the fluid and electro-magnetic quantities in the AD are chaotic, the azimuthal surface current density (i.e. a vertically integrated toroidal current) exhibits a smooth and simple power-law behavior (e.g., McKinney & Narayan 2007a,b). Let us consider an AD with an azimuthal current but no plasma above/below it. Generally speaking, the circulating current \( I \) at \( (\theta', r') \) contributes to a magnetic stream function at \( (\theta, r) \) (see Jackson 1975; Lubow et al. 1994; Li & Cao 2019),

\[ \Psi (\theta, r) = 2I r \frac{1 + x^2 - 2x \cos \theta' \cos \theta}{\sqrt{1 + x^2 - 2x \cos \theta' + \theta}} K (k) - \left( 1 + x^2 - 2x \cos (\theta' + \theta) \right) E (k), \]  

(112)

where \( x = r'/r \) and \( K (k) \) and \( E (k) \) are the first and second kind of complete elliptic integrals with argument \( k = 4x \sin \theta' \sin \theta / \left( 1 + x^2 - 2x \cos (\theta' + \theta) \right) \). For a current following a power-law distribution on the AD \( (\theta' = \pi/2) \):

\[ I = J_{\phi}^e (R) dR = J_0 R^{\nu - 2} dR \]  

within the region \( (R_1, R_2) \), one has

\[ \Psi (r, \theta) = 2r^2 J_\phi^e (r) \int_{R_1/r}^{R_2/r} \frac{1 + x^2}{\sqrt{1 + x^2 + 2x \sin \theta}} K (k) - \left( 1 + x^2 + 2x \sin \theta \right) E (k) x^{\nu - 2} dx \]  

\[ \frac{x^{\nu - 2}}{(R_2/r) \gg 1} \frac{2\pi}{C_1 r^2 J_\phi^e (r)} T (\theta), \]  

(113)
with the argument reducing to \( k = 4x \sin \theta / (1 + x^2 + 2x \sin \theta) \), which surprisingly matches what we derived in the force-free model (Equation 27) in the limit of \((R_1/r) \ll 1 \ll (R_2/r)\). In fact, the condition \( j = \nabla \times \mathbf{B} = 0 \) above the AD can be expressed in the form of Equation 12, and therefore, they have the same solution. When the magnetic field starts rotation, nature chooses a magnetic stream function that is almost conserved, which determines the magnetic configuration.

From Equation 2, the components of the magnetic field can be expressed as (see also, Sakurai 1987),

\[
B_\phi (r, \theta) = 2J_\phi^s (r) \int_{R_1/r}^{R_2/r} \frac{(1 + x^2 \cos 2\theta) E(k) - (1 + x^2 - 2x \sin \theta) K(k)}{\sin (1 + x^2 - 2x \sin \theta) \sqrt{1 + x^2 + 2x \sin \theta}} \nu x^{-2} d\nu \left(\frac{(R_2/r) \sin \theta}{R_1/r} \right) - 2\pi J_\phi^s (r) \frac{\nu T(\theta)}{C_1 \sin \theta},
\]

\[
B_r (r, \theta) = 4J_\phi^s (r) \int_{R_1/r}^{R_2/r} \frac{\cos \theta \nu x^{s} E(k)}{(1 + x^2 - 2x \sin \theta) \sqrt{1 + x^2 + 2x \sin \theta}} \nu x^{-2} dx = \left(\frac{r - R_1}{R_2 - r} \right) \frac{\cos \theta}{\sin \theta} 2\pi J_\phi^s (r) \frac{1}{C_1 \sin \theta} \frac{dT}{d\theta}, \tag{114}
\]

The latter matches the poloidal magnetic fields we derived in Section 4. Notice that we did not consider GR effects when deriving this result. In principle, to calculate the BH charge one must consider a GR BH in the Einstein-Maxwell theory (e.g., Znajek 1977; Blandford & Znajek 1977; MacDonald & Thorne 1982) for a Kerr-Newman metric (Kerr 1963; Newman et al. 1965). This is left for future explorations.

Therefore, it seems that a simple disk model with a power-law scaling of azimuthal current may capture key properties of the jet such as jet configuration, collimation, Lorentz factor, etc. Theoretical analyses on the equipartition between magnetic and ram pressures suggest that the magnetic field and/or azimuthal current on an AD may follow a power-law distribution \( \propto R_0^{-5/4} \) (corresponding to \( \nu = 3/4 \), Blandford & Payne 1982; Narayan & Yi 1995b). This is confirmed by numerical simulations (McKinney & Narayan 2007a,b), where they found that despite the chaotic fluid and electro-magnetic quantities in the AD, the vertically integrated azimuthal current follows the above smooth and simple power-law behavior. One always has this result once the simulations reach a quasi-steady state regardless of the different initial conditions, e.g., multiple magnetic loops in the initial torus, a net vertical field, or loops of alternating poloidal directions. We, therefore, take \( \nu = 3/4 \) as the example value throughout the paper.

10. JET STABILITY

Since we assume an axial symmetry in a steady state in this paper, we cannot address the question of jet stability. However, jets in this paper have \( B_\phi \approx -\Omega RB_\rho \) (especially in the collimated region \( \theta \ll 1 \)), which marginally satisfies the stability criterion of Tomimatsu et al. (2001), suggesting that the jets are marginally stable to the kink instability, which can be avoided or mitigated through, e.g., the jet plasma being spinning faster than its internal Alfvén velocity (e.g., Nakamura & Meier 2004). Therefore, it is not expected that the jet would be (violently) kink unstable. This is also shown in some time-dependent simulations (e.g., Komissarov 2001; McKinney 2006a). Also, the spontaneous development of the spine-layer structure (case \( \nu < 1 \)) may be naturally stabilized (see e.g., Mizuno et al. 2007; Hardee 2007). Indeed, the jet in M87 is clearly Poynting-flux-dominated, accelerating and collimated over \(~ 300 \) pc (at a distance \( \lesssim 10^4 - 10^6 R_0 \) Meier et al. 2001; Marscher et al. 2008), and yet does not kink until well beyond HST-1.

In the case of magnetically dominated jet for GRBs, significant dissipation of the magnetic energy is needed to power the prompt \( \gamma \)-ray emission. One possibility is that kink instability may be induced via internal interactions between different parts of the jet (Zhang & Yan 2011) or due to the deceleration of jet as the jet encounters a large enough inertia from the ambient medium (Lazarian et al. 2019).

11. CONCLUSIONS & SUMMARY

In this paper, starting from the first principles, we study an analytical solution of a magnetized jet/wind flow. The explicit expression of the solution makes it easy for further developments (e.g. adding radiative processes) and to directly compare with the observational data. Our findings can be summarized as follows.

1. Through separating the force-free equation of a jet/wind plasma flow into the non-rotating and rotating parts, we found that each of the two equations can be solved analytically and that the two solutions match each other very well (in the regimes either \( \theta \ll 1 \) or \( \theta \to \pi/2 \), and either non-relativistic or relativistic). Therefore, we have a general approximate solution of a highly magnetized jet.

2. An ordered rotating magnetic field is the indispensable ingredient to globally launch, accelerate, and collimate a relativistic jet. The magnetic stream function \( \Psi = C_2 r^s \sin^2 \theta_2 F_1 \left(1 - \frac{\nu}{2} \frac{1}{\nu + 2}, \sin^2 \theta \right) \) (with \( 0 \leq \nu \leq 2 \) a
free parameter) controls the poloidal magnetic field configuration being a general parabola, while the toroidal magnetic field is determined by $\Phi = -2\Omega\Psi$ with an angular velocity $\Omega = a\Psi^\lambda$. The resulting helical magnetic field is dominated by the poloidal component within the ACS, and by the toroidal component outside. The large scale jet configuration can be used to constrain the foot-point locations of the magnetic fields.

3. For a highly magnetized jet/wind flow, the drift velocity matches the cold plasma velocity very well, which is almost always perpendicular to the magnetic field and therefore also forms a helical structure. Acceleration from non-relativistic through relativistic regimes is described, which is divided into three stages in the case of $\nu < 1$: 1) within the ACS, where the toroidal velocity dominates and the four velocity reads $v\Gamma = \Omega R$; 2) outside the ACS but within the CCS, where the poloidal velocity dominates and $v\Gamma = \Omega R$; and 3) outside the CCS, where the dominant poloidal velocity follow $v\Gamma \approx 2/(\theta\sqrt{2-\nu})$ due to a causality constraint. The acceleration in the case of $\nu \geq 1$ only has the former two stages and therefore always follows $v\Gamma = \Omega R$. The large scale jet velocity can be used to constrain angular velocity, and hence, the spin of the central BH.

4. The energy transportation of a magnetized jet/wind is dominated by the Poynting flux. The jet power is determined by the angular velocity and the magnetic flux, i.e. $P_{\text{jet}} = \Omega^2 F_{\text{B}}^2/(4\pi^2|\lambda+1|) \approx 2\times 10^{45}a^2M_8^2B_5^2$ erg s$^{-1}$ in the case of magnetic fields threading a BH ($\lambda = 0$ and $B_3 = B/10^6\text{Gs}$).

5. A jet/wind flow has to carry charges so that the electric field force can balance the magnetic field force. The charge density reads $\rho_e = -\Omega \cdot B/2\pi$ (in the case of magnetic fields threading a CO, $\lambda = 0$). The central rotating CO also has to be charged to globally launch a magnetized (BZ) jet with a total charge $|Q| \approx aF_B/8\pi \approx r_0\sqrt{P_{\text{jet}}} \approx 2.8 \times 10^{20}(1 + \sqrt{1-a^2}) M_8\sqrt{P_{\text{B}}^{44}} \text{C}$ (the BH case, the sign determined by $-Sg(\Omega \cdot B)$, $P_{44} = P_{\text{jet}}/10^{44}$ erg s$^{-1}$). Whether the BH (AD) is charged may account for the RL/RQ dichotomy in AGNs, with the aid of spin of the BH.

6. In a jet/wind flow, the toroidal current almost vanishes, while the poloidal current is about $j_p \approx \rho_e$. The total current carried by the jet reads $J = \sqrt{|\lambda+1|P_{\text{jet}}} \approx 5.8 \times 10^{17}\sqrt{P_{\text{B}}^{44}} \text{A}$ ($\lambda = 0$).

7. The magnetic stream surface is equipotential, within which the magnetic field lines lie, the current streams and the plasma flows. Crossing these surfaces can in principle lead to acceleration of the charged particles, which may be achieved through forming a gap in the polar region. The acceleration is limited by the potential difference between two magnetic stream surfaces $\Delta V = \sqrt{P_{\text{jet}}}/|\lambda+1| \approx 1.7 \times 10^{19}\sqrt{P_{\text{B}}^{44}} \text{volts}$ ($\lambda = 0$).

8. Given an available mass flux per magnetic flux ($\eta$) that still satisfies a highly magnetically dominated condition, one has approximations of the proper density $\rho = (\eta/4\pi\Omega^2) B/\sqrt{R^2}$ and the lab frame density $\rho_1 = \Gamma\rho \approx (\eta/8\pi\Psi\Omega^2) B^2$.

9. This approximate solution can (roughly) match known numerical simulation results and interpret most observations of AGN and GRB jets.

ACKNOWLEDGMENTS

Acknowledgments: We are grateful to the anonymous referee for useful comments and suggestions, and also appreciate the encouraging comments from Roger Blandford. We would like to thank Zhenhui Zhang, Defu Bu, Jiawen Li, Jiming Bai, Zhaoming Gan, Lei Huang, Hai Yang, Gang Li, Ming Chen, Hao Tong, Yunwei Yu, Henk Spruit, Xinwu Cao, Minfeng Gu, Shuangliang Li, Zhen Pan and Yuanping Hong for helpful discussions. LC thanks Tongxin Chen & Tongyun Chen. The work of LC is supported by the National Natural Science Foundation of China (grant U1831138). LC also acknowledges University of Nevada, Las Vegas, for hospitality when this work was carried out.
APPENDIX

A. THE RELATION $\Phi = -2\Omega \Psi$

In Section 2, we show that in the limit of force free condition, $\Phi = B_0 R$ is conserved along a magnetic field line. In Section 3, the relation $\Phi = -2\Omega \Psi$ is further employed to solve Equation 10, which is mainly based on a mathematical requirement (see Equation 22). Here, we present a brief discussion on a more physical consideration (see e.g., Lyubarsky 2009, for a similar discussion).

In principle, $\Phi = B_0 R = -\eta (E - \Gamma) / \Omega$ cannot be conserved along a magnetic field line (the fluid Lorentz factor $\Gamma$ would change, see Equation 86). In the limit of a highly magnetized jet ($\sigma \gg 1$, see Equation 87), there is a reduction $\Phi \approx -\eta E / \Omega$, which is therefore approximately conversed along the magnetic field line. The fluid velocity generally follows $v_\phi B_p - B_\phi v_p = \Omega R B_p$ (see Section 5). Let us consider the region $\Omega R \gg 1$, where the jet is already relativistic and $v_\phi \ll 1$ and $v_p \approx 1$ are satisfied. One immediately gets $-B_\phi / B_p \approx \Omega R$. In this collimated jet region (polar angle $\theta \ll 1$), the poloidal magnetic field can be assumed to be approximately uniform at a given height from the central CO/AD (being self-consistent with Equation 40, see also Lyubarsky 2009), which yields an enclosed magnetic flux $F_B = \pi R^2 B_p$. Comparing this with Equation 3 and with the aid of relation $-B_\phi / B_p \approx \Omega R$, one immediately gets $\Phi \approx -2\Omega \Psi$. Due to conservation of $\Psi$, $\Omega$, and $\Phi$ (in the limit of high magnetization), this relation would hold throughout the jet region, even though it was derived asymptotically (see Section 3).

B. THE MAGNETIC FIELD DIRECTION AND AMPLITUDE

The parameters $\Psi$ and $\Omega = \Omega \cdot \hat{z}$ can be both positive and negative, which correspond to the magnetic field lines pointing in two opposite directions. All the formulae in the main text are written assuming a positive $\Psi$ and $\Omega$. In the case of a negative $\Psi$ or a negative $\Omega$, one needs to multiply a sign factor to these formulae. The sign convention can be easily derived, which can be summarized as

$$
S_g (S_p) = S_g (v_p) = +1, \\
S_g (S_\phi) = S_g (v_\phi) = S_g (\Omega), \\
S_g (B_p) = S_g (j_\phi) = S_g (\Psi), \\
S_g (B_\phi) = S_g (E) = S_g (j_p) = S_g (\rho_e) = S_g (\Phi) = S_g (\Omega \cdot B) = S_g (\Omega) \cdot S_g (\Psi). 
$$

(B1)

It can be seen that $j_p$ and $B_\phi$ have the same sign convention ($E$ and $\rho_e$ also have the same sign convention), and therefore the Lorentz force does not change. This implies that the sign choice is dynamically not important. One therefore expects that the acceleration and velocity always point outwards from the CO/AD. This can be also illustrated by $S_g (S_p) = S_g (v_p) = +1$, which means that regardless of the sign of $\Psi$ or $\Omega$, the velocity and Poynting flux always point outwards for either of the pair of anti-parallel jets/winds. It can be also easily inferred that, in the case of $\Omega$ and $B$ projected in the same directions, the currents in both sides of the jet (having negative charges) flow inward towards the central CO ($s = 0$). In the case of $\Omega$ and $B$ projected in opposite directions (having a positive charge), on the other hand, both currents flow outwards from the central CO. The case of the AD may be different because there is a gradient of angular velocity (see Section 6.1).

For simplicity, the form $\Psi = r^u T (\theta)$ is taken in the main text. In fact, one can multiply $\Psi$ by an arbitrary constant $C_\Psi$, and the result should still be a solution of the original equations. In this case, $\Phi$, $B$, $E$, $j$ and $\rho_e$ should be amplified by $C_\Psi$, and the jet power (Poynting flux) should be multiplied by $C_\Psi^2$. The velocity does not change.

C. EXACT SOLUTIONS

In some special cases, the force free Equation 10 has exact solutions. These solutions are well known in the literature (e.g., Narayan et al. 2007), and we summarize them below for completeness. As shown below, these solutions can be also derived from our general solutions when the special conditions are met.
C.1. Monopole

In the case of a monopole magnetic configuration, the magnetic stream function is only a function of $\theta$. In this case, one has an exact solution for Equation 10 (with $\Omega = a \Psi^\lambda$, see Michel 1969),

\[
\Psi = 1 - \cos \theta, \\
\Phi = -\Omega \sin^2 \theta,
\]

which corresponds to our approximate solution for the case of $\nu = 0$, where we have the same $\Psi = 1 - \cos \theta$ but a different $\Phi = -2\Omega (1 - \cos \theta)$. This value of $\Phi$ and the exact value of $\Phi$ match each other at $\theta \ll 1$, but differ by 50% at $\theta = \pi/2$. This exact monopole solution shows a magnetic field that still follows $B_\phi/B_p = -\Omega r \sin \theta$ and

\[
B_\theta = 0, \\
B_r = \frac{1}{r^2} B_0, \\
B_\phi = -\frac{\Omega \sin \theta}{r} B_0.
\]

The drift velocity is also the same as our approximate solution (Equation 46):

\[
v_\theta = 0, \\
v_r = \frac{\Omega R^2}{1 + (\Omega R)^2}, \\
v_\phi = \frac{\Omega R}{1 + (\Omega R)^2}, \\
v = \frac{\Omega R}{\sqrt{1 + (\Omega R)^2}}, \\
v \Gamma = \Omega R.
\]

The Poynting flux and jet power read

\[
S = \frac{\Omega R \sqrt{1 + (\Omega R)^2}}{4\pi r^4} B_0^2, \\
S_r = \frac{\Omega^2 R^2}{4\pi r^4} B_0^2, \\
P_{\text{jet}} = \frac{2}{3} \Omega^2 B_0^3.
\]

C.2. Parabola

With an assumption of $\Omega = a \Psi^{-1}$, one can derive another exact solution of Equation 10:

\[
\Psi = r (1 - \cos \theta), \\
\Phi = -2a,
\]

which is the same as our approximate solution for the case $\nu = 1$, where the rotation curve is “flat” on the AD plane $\Omega \propto R_0^{-1}$ (see Blandford 1976; Narayan et al. 2007). The magnetic field configuration forms a parabola in this case.

C.3. Cylinder

Another exact solution of Equation 10 reads (see Istomin & Pariev 1994)

\[
\Psi = r^2 \sin^2 \theta, \\
\Phi = -2\Omega \Psi,
\]

which is the same as our approximate solution for the case of $\nu = 2$ (applies for $\Omega = a \Psi^\lambda$, see Section 3). The magnetic field configuration forms a cylinder in this case.
D. ASYMPTOTIC BEHAVIOR OF $T(\theta)$

Setting $\mu = \cos \theta$ and $\xi = \sin \theta$, one obtains the asymptotic behavior of $T(\theta)$ (see Equation 27) at $\theta \rightarrow \pi/2$ (i.e. $\mu \ll 1, \xi \rightarrow 1$)

$$T = 1 - C_1 \mu - \frac{\nu (\nu - 1)}{2} \mu^2,$$

$$\frac{dT}{d\theta} = \xi \left[ C_1 + \nu (\nu - 1) \mu - C_1 \frac{1}{2} \mu^2 \right],$$

$$\frac{d^2T}{d\theta^2} = -\nu (\nu - 1) \xi^2 + (1 + \xi^2) C_1 \mu + \nu (\nu - 1) \left[ 1 + \frac{1}{2} (\nu + 1) (\nu - 2) \xi^2 \right] \mu^2$$  \hspace{1cm} (D8)

and at $\theta \ll 1$ (i.e. $\xi \ll 1, \mu \rightarrow 1$)

$$T = C_2 \left\{ \xi^2 + \frac{1}{2} \left( 1 - \frac{\nu}{2} \right) \left( 1 + \frac{\nu}{2} \right) \xi^4 + \frac{1}{12} \left( 2 - \nu \right) \left( 2 \nu \right) \left( \frac{3}{2} + \frac{\nu}{2} \right) \xi^6 \right\},$$

$$\frac{dT}{d\theta} = C_2 \mu \left[ 2 \xi - \frac{1}{2} (\nu - 2) (\nu + 1) \xi^3 + \frac{1}{2} \left( 1 - \frac{\nu}{2} \right) \left[ 2 - \nu \right] \left( \frac{1}{2} + \nu \right) \left( \frac{3}{2} + \nu \right) \xi^5 \right],$$

$$\frac{d^2T}{d\theta^2} = C_2 \left\{ 2 \mu^2 + \left\{ -2 - \frac{3}{2} (\nu - 2) (\nu + 1) \right\} \xi^2 + \frac{1}{2} (\nu + 1) (\nu - 2) \left[ 1 + \frac{5}{16} (\nu - 4) (\nu + 3) \right] \mu^2 \right\} \xi^4. \hspace{1cm} (D9)$$

E. MAGNETIC FIELD 3D MORPHOLOGY

With the approximate solution $\Psi = r^\nu T(\theta)$ and $\Phi = -2\Omega \Psi$, one can derive 3D morphology of the magnetic field lines. It is known that the ratios of different components ($r, \theta, \phi$) of the magnetic field determine its 3D morphology. Combining with Equation 2, one can derive an equation that tracks a magnetic field line

$$dr : r d\theta : r \sin \theta d\phi = B_r : B_\theta : B_\phi = -\frac{1}{T} \frac{dT}{d\theta} : \nu : 2\Omega r. \hspace{1cm} (E10)$$

Supposing a magnetic field line anchored at $(r_0, \theta_0, \phi_0)$, one immediately has ($T_0 = T(\theta_0)$)

$$r = r_0 T_0^{1/\nu} T^{-1/\nu},$$

$$\phi - \phi_0 = \frac{2\Omega r_0 T_0^{1/\nu}}{\nu} \int_{\theta_0}^{\theta} T^{-1/\nu} \sin \theta' d\theta'. \hspace{1cm} (E11)$$

In the case of $\theta \ll \theta_0$, the angle rotation $(\phi - \phi_0)$ mainly happens in the region near $\theta$. The above Equation E11 is reduced to

$$r = r_0 T_0^{1/\nu} C_2^{-1/\nu} \theta^{-2/\nu},$$

$$\phi - \phi_0 = -\Omega r. \hspace{1cm} (E12)$$

Let us consider the number of rotation cycles, from foot-point to the ACS (assuming $\theta \ll 1$), of a magnetic field line anchored at the equatorial plane ($\theta_0 = \pi/2$)

$$N = \left| \frac{\phi_{ACS} - \phi_0}{2\pi} \right| \approx \frac{C_2^{1/(2-\nu)}}{2\pi (\Omega r_0)^{\nu/(2-\nu)} \frac{\nu=3/4}{2^{9/5} \pi (\Omega r_0)^{3/5}}. \hspace{1cm} (E13)$$

This indicates that a magnetic field line rotates even less than one cycle from its foot-point to the ACS, provided that the foot-point rotation velocity is not very slow ($\Omega r_0 \gtrsim 0.02$ for $\nu = 3/4$).

F. JET FLOW NEUTRALITY?

Some MHD simulations ignore the electric force due to its relative unimportance compared with the magnetic force. Therefore, the force-free condition reduces to $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$. Now, let us consider whether the solution of this equation can guarantee that charge vanishes, i.e. $\rho_e = \nabla \cdot \mathbf{E}/4\pi = 0$. Ignoring the electric force part, the force-free condition yields (steady and axisymmetric, see Equation 10)

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} \cot \theta \frac{\partial \Psi}{\partial \theta} + \Phi' \Phi = 0. \hspace{1cm} (F14)$$
Instead of appealing to possible numerical methods to solve this equation, here we try to seek analytical solutions. The \( \Psi \) function can be separated, \( \Psi = H(r)T(\theta) \), only in the case of \( \Phi = -\sqrt{\omega^2 \Psi^2 + \varrho^2} \) with \( \omega \) and \( \varrho \) being constants. In this case, we have a solution of \( T(\theta) \) the same as \( T_{nr}(\theta) \) (Equation 27). The function \( H(r) \) follows

\[
r^2 \frac{d^2 H}{dr^2} + \left[ \omega^2 r^2 - \nu (\nu - 1) \right] H = 0,
\]

with \( \nu (\nu - 1) \) being a separate constant. With transformations \( H = r^{1/2} F \) and \( x = \omega r \), one then has a standard Bessel equation

\[
x^2 \frac{d^2 F}{dx^2} + x \frac{dF}{dx} + \left[ x^2 - \left( \nu - \frac{1}{2} \right)^2 \right] F = 0.
\]

Considering the boundary condition: \( \Psi \) vanishes when \( \omega r \to 0 \), one can get a solution of Equation F15 which reads

\[
H(r) = C_h r^{\frac{1}{2}} J_{\nu - \frac{1}{2}}(\omega r),
\]

where \( C_h \) is the integration constant and \( J_{\nu - 1/2}(x) \) is the first kind of Bessel function with an order of \( \nu - 1/2 \).

Because \( \Psi \) vanishes at \( \theta \to 0 \), one expects that the constant \( \varrho \) should vanish (i.e., \( \varrho = 0 \)) to guarantee \( B_\varrho = -\sqrt{\omega^2 \Psi^2 + \varrho^2} / \sin \theta \) being not singular at \( \theta \to 0 \). Therefore one has \( \Phi = -\omega \Psi \), which implies that \( \omega \) represents the angular velocity of the electromagnetic field (although they may differ by a constant, see Section 3). In the case of \( \omega r \ll 1 \) (i.e., non-relativistic), \( H(r) \) reduces to \( H(r) = C_h \left[ \Gamma(\nu + 1/2) \right]^{-1} (\omega/2)^{\nu-1/2} r^\nu \). One can safely choose \( C_h = \Gamma(\nu + 1/2)(\omega/2)^{1/2-\nu} \) to guarantee that \( H(r) = r^\nu \) is the same as \( H_{nr}(r) \). The electric field can be derived through Equation 4, and the charge density can be then derived through \( \rho_e = \nabla \cdot \mathbf{E}/4\pi \), which would be the same as Equation 71. Therefore, the above solution (ignoring the electric force part) cannot generally guarantee that the charge density vanishes. However one still has, in the non-relativistic limit, the magnetic field configuration being approximately the same as that in the case of considering the electric force (see Equation 27) and the case with the electric force neglected (see more in Section 7.1).

G. THE MAXIMUM LORENTZ FACTOR

One has \( \Gamma(\sigma + 1) = \mathcal{E} \) conserved along a magnetic field line, which also approximates the theoretical maximum Lorentz factor that a jet can be accelerated to, if all the Poynting flux is converted to kinetic energy. In principle, the value of \( \mathcal{E} \) cannot be arbitrarily large, because the plasma energy density cannot be arbitrarily small, which is due to the fact that the plasma has to carry a proper charge and current density to support the electromagnetic field (see Section 6). Given the plasma being completely made up of charged particles, i.e. without any other neutral particles (pairs), one has a minimum plasma density \( \rho_{me} = \rho_e / r_{cm} \), where \( r_{cm} \) is the charge-to-mass ratio of the particle. While this minimum plasma density is still too small to maintain a poloidal current \( j_p \) (see Section 6) due to a small poloidal velocity \( v_p \approx (\Omega R)^2 \ll 1 \) (non-relativistic at the foot-point, see Section 5.3). Therefore, in order to sustain this poloidal current, additional plasma particles are required with a density roughly at least \( \rho_{min} \sim \rho_{me} / (\Omega R)^2 \). In this case, one has a maximum magnetization parameter at the foot-point, which is also the maximum Lorentz factor the plasma flow can reach, \( \gamma_{\text{max}} \approx \sigma_{\text{max}} \approx U_{B_0}/\rho_{\text{min}} \sim r_{cm}B_0\Omega R^2/4 \sim r_{cm}\Delta V/4 \) (see Equations 72 and 109 for the case of a BH engine).

H. ON BLACK HOLE CHARGE

In principle, a BH can be charged. Similar to the BH spin parameter, a dimensionless BH charge parameter can be defined in Kerr-Newman metric

\[
r_Q = \frac{Q}{M},
\]

which follows a constrain \( r_Q^2 + a^2 \leq 1 \) (see Kerr 1963; Newman et al. 1965). It is not easy for an astrophysical black hole to be charged unless there is a magnetosphere induced rotating magnetic field threading its horizon. For a charged CO that can launch a jet and that we are interested in here, the charge parameter roughly follows (see Equation 111)

\[
r_Q \approx \frac{r_0}{r_+} \sqrt{\frac{P_{\text{jet}}}{P_{\text{max}}}},
\]
where \( P_{\text{max}} = \frac{c^5}{G} = 3.63 \times 10^{59} \text{ erg s}^{-1} \) is the universal maximum power defined by fundamental constants. It can be seen that, for any astrophysical system, the charge parameter is a small value, provided that the astrophysical jet power being always very small compared with \( P_{\text{max}} \), even though such a “small” charge is essential to launch a relativistic powerful jet powered by BH rotation.

I. SOME FORMULAE IN GAUSSIAN UNITS

The formulae in the main text are written in natural units. For the convenience of readers, below we write some important formulae also in Gaussian units.

The ideal MHD condition Equation 1

\[
\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} = 0. \quad (I20)
\]

The force-free Equation 7

\[
\rho_e \mathbf{E} + \frac{\mathbf{j} \times \mathbf{B}}{c} = 0. \quad (I21)
\]

The electric field Equation 4

\[
\mathbf{E} = -\frac{1}{c} \Omega \nabla \Psi = -\frac{\Omega r \sin \theta}{c} \hat{\varphi} \times \mathbf{B}. \quad (I22)
\]

The enclosed current Equations 9 and 35

\[
\Phi = -2\Omega \Psi c = 2J. \quad (I23)
\]

The Poynting flux Equation 73

\[
S = c^4 \pi \mathbf{E} \times \mathbf{B}. \quad (I24)
\]

The drift velocity Equation 45

\[
\mathbf{v}_d = c \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \quad (I25)
\]

The flow density Equation 97

\[
\rho = \frac{\eta}{4\pi} \frac{B_p}{u_p} \approx \frac{c \eta \Psi}{2\pi \Omega^2} \sqrt{1 + \left(\Omega R/c\right)^2} \approx \frac{c \eta}{8\pi \Omega^2} \frac{B}{R^2} \approx \frac{\rho_i}{\Gamma}. \quad (I26)
\]

The current density Equations 8 and 66

\[
\mathbf{j}_p = \frac{c}{4\pi} \left( \nabla \times \mathbf{B} \right)_p = (\lambda + 1) c B_\phi \frac{1}{4\pi} \left[ \frac{d T}{d \theta} \hat{r} - \nu \hat{\theta} \right] = (\lambda + 1) \frac{\Omega}{2\pi \sin \theta} \left( \frac{r^{\nu-2}}{\Omega} \left( \frac{d T}{d \theta} \right)^2 + 2 \cot \theta \frac{d T}{d \theta} + (2 + \lambda \nu) \nu T \right) = \frac{j_p}{c}. \quad (I27)
\]

The charge density Equations 70 and 71

\[
\rho_e = \frac{\nabla \cdot \mathbf{E}}{4\pi} = \frac{\Omega \cdot \mathbf{B}}{2\pi c} + \frac{\Omega r \sin \theta \mathbf{j}_\phi}{c^2} = \frac{\nu T}{4\pi c} \left[ \nabla \Psi \right]^2 \frac{j_e = 0}{c} = -\nu \frac{\nu - 2}{4\pi c} \left[ \lambda T \left( \frac{d T}{d \theta} \right)^2 + 2 \cot \theta \frac{d T}{d \theta} + (2 + \lambda \nu) \nu T \right] = \frac{j_e}{c}. \quad (I28)
\]

The potential difference Equation 72

\[
\Delta V = -\frac{\Omega \Psi}{(\lambda + 1)c} \left|_1 \right|^2 = -\frac{\Omega F_B}{2\pi (\lambda + 1)c} \left|_1 \right|^2 = \frac{R B_\phi}{2(\lambda + 1)} \left|_1 \right|^2 = \frac{J}{(\lambda + 1)c} \left|_1 \right|^2. \quad (I29)
\]

The jet power Equation 74

\[
P_{\text{jet}} = \frac{\Omega^2 \Psi^2}{c |\lambda + 1|} \left|_1 \right|^2 = \frac{\Omega^2 F_B^2}{4\pi^2 c |\lambda + 1|} \left|_1 \right|^2 = \frac{c R^2 B_\phi^2}{4 |\lambda + 1|} \left|_1 \right|^2 = \frac{J^2}{c |\lambda + 1|} \left|_1 \right|^2. \quad (I30)
\]

The CO/BH total charge Equation 111

\[
Q = -Sg \left( \frac{\nu C_1 r_0 \sqrt{P_{\text{jet}}/c}}{(1 + \nu)(2 - \nu)} \right) = -\frac{Sg (\Omega \cdot \mathbf{B}) \nu C_1 f_{\Omega a} F_B}{4\pi (1 + \nu)(2 - \nu)}, \quad (I31)
\]
The BH gravitational radius
\[ r_g = \frac{GM}{c^2}. \] (I32)

The BH spin parameter
\[ a = \frac{cJ_{am}}{GM^2}. \] (I33)

The BH charge parameter
\[ r_Q = \frac{Q}{\sqrt{GM}} \approx \frac{r_0}{r_g} \sqrt{\frac{GP_{\text{jet}}}{c^5}}. \] (I34)

The BH rotating angular velocity
\[ \Omega_{\text{BH}} = \frac{ac}{2 \left(1 + \sqrt{1 - a^2}\right) r_g}. \] (I35)

The Keplerian angular velocity on a BH equator
\[ \Omega_K = \frac{c}{r_g \left[\left(r_0/r_g\right)^{3/2} + a\right]} . \] (I36)

REFERENCES

Abdo, A. A., Ackermann, M., Ajello, M., et al. 2010, Nature, 463, 919, doi: 10.1038/nature08841

Ackermann, M., Ajello, M., Albert, A., et al. 2015, ApJL, 813, L41, doi: 10.1088/2041-8205/813/2/L41

Alfvén, H. 1942, Nature, 150, 405, doi: 10.1038/150405d0

Amenomori, M., Bao, Y. W., Bi, X. J., et al. 2019, PhRvL, 123, 051101, doi: 10.1103/PhysRevLett.123.051101

Anchordoqui, L. A. 2019, PhR, 801, 1, doi: 10.1016/j.physrep.2019.01.002

Antonucci, R. 1993, ARA&A, 31, 473, doi: 10.1146/annurev.aa.31.090193.002353

Artymova, I. V., Björnsson, G., & Novikov, I. D. 1996, ApJ, 461, 565, doi: 10.1086/177084

Asada, K., Inoue, M., Uchida, Y., et al. 2002, PASJ, 54, L39, doi: 10.1093/pasj/54.3.L39

Asada, K., & Nakamura, M. 2012, ApJL, 745, L28, doi: 10.1088/2041-8205/745/2/L28

Asada, K., Nakamura, M., Doi, A., Nagai, H., & Inoue, M. 2014, ApJL, 781, L2, doi: 10.1088/2041-8205/781/1/L2

Attridge, J. M., Roberts, D. H., & Wardle, J. F. C. 1999, ApJL, 518, L87, doi: 10.1086/312078

Bai, X.-N. 2016, ApJ, 821, 80, doi: 10.3847/0004-637X/821/2/80

Bein, M. C., & Li, Z.-Y. 1994, ApJ, 426, 269, doi: 10.1086/174061

Berger, E., Kulkarni, S. R., Pooley, G., et al. 2003, Nature, 426, 154, doi: 10.1038/nature01998

Beskin, V., & Tschekhovskoy, A. 2005, A&A, 433, 619, doi: 10.1051/0004-6361:20041592

Beskin, V. S. 2010, MHD Flows in Compact Astrophysical Objects, doi: 10.1007/978-3-642-01290-7

Beskin, V. S., Istomin, Y. N., & Parev, V. I. 1992, Soviet Ast., 36, 642

Beskin, V. S., & Malyskin, L. M. 2000, Astronomy Letters, 26, 208, doi: 10.1134/1.20384

Beskin, V. S., & Nokhrina, E. E. 2006, MNRAS, 367, 375, doi: 10.1111/j.1365-2966.2006.09957.x

—. 2009, MNRAS, 397, 1486, doi: 10.1111/j.1365-2966.2009.14964.x

Beskin, V. S., Zakamska, N. L., & Sol, H. 2004, MNRAS, 347, 587, doi: 10.1111/j.1365-2966.2004.07229.x

Blandford, R., Meier, D., & Readhead, A. 2019, ARA&A, 57, 467, doi: 10.1146/annurev-astro-081817-051948

Blandford, R. D. 1976, MNRAS, 176, 465, doi: 10.1093/mnras/176.3.465

Blandford, R. D., & Payne, D. G. 1982, MNRAS, 199, 883, doi: 10.1093/mnras/199.4.883

Blandford, R. D., & Znajek, R. L. 1977, MNRAS, 179, 433, doi: 10.1093/mnras/179.3.433

Bogovalov, S. V. 1992, Soviet Astronomy Letters, 18, 337

Broderick, A. E., & Loeb, A. 2009, ApJ, 697, 1164, doi: 10.1088/0004-637X/697/2/1164

Burrows, D. N., Kennea, J. A., Ghisellini, G., et al. 2011, Nature, 476, 421, doi: 10.1038/nature10374

Camenzind, M. 1987, A&A, 184, 341

Cao, X. 2012, MNRAS, 426, 2813, doi: 10.1111/j.1365-2966.2012.21973.x

Casse, F., & Keppens, R. 2004, ApJ, 601, 90, doi: 10.1086/380441
Kirk, J. G., Lyubarsky, Y., & Petri, J. 2009, Astrophysics and Space Science Library, Vol. 357, The Theory of Pulsar Winds and Nebulae, ed. W. Becker, 421, doi: 10.1007/978-3-540-76965-1_16

Komissarov, S. S. 2009, MNRAS, 397, L41, doi: 10.1111/j.1745-3933.2009.00108.x

Komissarov, S. S., Barkov, M. V., Vlahakis, N., & Königl, A. 2007, MNRAS, 380, 51, doi: 10.1111/j.1745-3933.2007.00108.x

Komissarov, S. S., Vlahakis, N., Königl, A., & Barkov, M. V. 2009, MNRAS, 394, 1182, doi: 10.1111/j.1365-2966.2009.14410.x

Kotera, K., & Olinto, A. V. 2011, ARA&A, 49, 119, doi: 10.1146/annurev-astro-081710-102620

Krause, M., & Stöcker, H. 1985, A&A, 154, 21, doi: 10.1051/0004-6361:19851542101

Lazarian, A., Zhang, B., & Xu, S. 2019, ApJ, 882, 184, doi: 10.3847/1538-4357/ab2b38

Li, J., & Cao, X. 2019, ApJ, 872, 149, doi: 10.3847/1538-4357/ab2b38

Li, Z.-Y., Chiueh, T., & Begelman, M. C. 1992, ApJ, 394, 459, doi: 10.1086/171597

Lister, M. L., Aller, M. F., Aller, H. D., et al. 2013, AJ, 146, 120, doi: 10.1088/0004-6256/146/5/120

Lithwick, Y., & Sari, R. 2001, ApJ, 555, 540, doi: 10.1086/321455

Lyubarsky, Y. E. 2010, MNRAS, 402, 353, doi: 10.1111/j.1365-2966.2009.15877.x

MacDonald, D., & Thorne, K. S. 1982, MNRAS, 198, 345, doi: 10.1093/mnras/198.2.345

Marconi, A., Risaliti, G., Gilli, R., et al. 2004, MNRAS, 351, 169, doi: 10.1111/j.1365-2966.2004.07765.x

Martins, F., & LSlim, R. C. 2007, MNRAS, 377, 1403, doi: 10.1111/j.1365-2966.2007.11301.x

Marscher, A. P., Jorstad, S. G., D'Arcangelo, F. D., et al. 2008, Nature, 452, 966, doi: 10.1038/nature06895

Marscher, A. P., Jorstad, S. G., Lario, M. V., et al. 2010, ApJL, 710, L126, doi: 10.1088/2041-8205/710/2/L126

McKinney, J. C. 2006a, MNRAS, 368, 1561, doi: 10.1111/j.1365-2966.2006.10256.x

—. 2006b, MNRAS, 368, L30, doi: 10.1111/j.1745-3933.2006.00150.x

—. 2006c, MNRAS, 367, 1797, doi: 10.1111/j.1365-2966.2006.10087.x

McKinney, J. C., & Gammie, C. F. 2004, ApJ, 611, 977, doi: 10.1086/422244

McKinney, J. C., & Narayan, R. 2007a, MNRAS, 375, 513, doi: 10.1111/j.1365-2966.2006.11301.x

—. 2007b, MNRAS, 375, 531, doi: 10.1111/j.1365-2966.2006.11220.x

McKinney, J. C., Tchekhovskoy, A., & Bland ford, R. D. 2012, MNRAS, 423, 3083, doi: 10.1111/j.1365-2966.2012.21074.x

Meier, D. L., Koide, S., & Uchida, Y. 2001, Science, 291, 84, doi: 10.1126/science.291.5501.84

Mertens, F., Lobanov, A. P., Walker, R. C., & Hardee, P. E. 2016, A&A, 595, A54, doi: 10.1051/0004-6361/201628829

Mészáros, P. 2006, Reports on Progress in Physics, 69, 2259, doi: 10.1088/0034-4885/69/8/R01

Michel, F. C. 1969, ApJ, 158, 727, doi: 10.1086/150233

—. 1973, ApJL, 180, L133, doi: 10.1086/181169

Mirabel, I. F., & Rodríguez, L. F. 1999, ARA&A, 37, 409, doi: 10.1146/annurev.astro.37.1.409

Mizuno, Y., Hardee, P., & Nishikawa, K.-I. 2007, ApJ, 662, 835, doi: 10.1086/518106

Moll, R. 2009, A&A, 507, 1203, doi: 10.1051/0004-6361:200912266

Mościbrodzka, M., Falcke, H., & Shiokawa, H. 2016, A&A, 586, A38, doi: 10.1051/0004-6361/201526630

Nakamura, M., & Meier, D. L. 2004, ApJ, 617, 123, doi: 10.1086/425337

Nakamura, M., Asada, K., Hada, K., et al. 2018, ApJ, 868, 146, doi: 10.3847/1538-4357/aacb2d

Narayan, R., & McClintock, J. E. 2012, MNRAS, 423, 3083, doi: 10.1111/j.1365-2966.2012.21074.x

Narayan, R., McKinney, J. C., & Farmer, A. J. 2007, MNRAS, 375, 548, doi: 10.1111/j.1365-2966.2006.11272.x
Walton, D. J., Brightman, M., Risaliti, G., et al. 2018, MNRAS, 473, 4377, doi: 10.1093/mnras/stx2659
Wang, X.-G., Zhang, B., Liang, E.-W., et al. 2018, ApJ, 859, 160, doi: 10.3847/1538-4357/aabc13
Weisskopf, M. C., Hester, J. J., Tennant, A. F., et al. 2000, ApJL, 536, L81, doi: 10.1086/312733
White, C. J., & Chrystal, F. 2020, MNRAS, 498, 2428, doi: 10.1093/mnras/staa2423
Xu, Y., Baloković, M., Walton, D. J., et al. 2017, ApJ, 837, 21, doi: 10.3847/1538-4357/aa5df4
Xu, Y.-D., Cao, X., & Wu, Q. 2009, ApJL, 694, L107, doi: 10.1088/0004-637X/694/2/L107
Yang, J., Wu, F., Paragi, Z., & An, T. 2012, MNRAS, 419, L74, doi: 10.1111/j.1745-3933.2011.01182.x
Yang, X.-H., Bu, D.-F., & Li, Q.-X. 2019, ApJ, 881, 34, doi: 10.3847/1538-4357/ab2b47
Young, S., Axon, D. J., Robinson, A., Hough, J. H., & Smith, J. E. 2007, Nature, 450, 74, doi: 10.1038/nature06319
Yuan, F., & Narayan, R. 2014, ARA&A, 52, 529, doi: 10.1146/annurev-astro-082812-141003
Zakamska, N. L., Begelman, M. C., & Bland ford, R. D. 2008, ApJ, 679, 990, doi: 10.1086/587870
Zamaninasab, M., Clausen-Brown, E., Savolainen, T., & Tchekhovskoy, A. 2014, Nature, 510, 126, doi: 10.1038/nature13399
Zauderer, B. A., Berger, E., Soderberg, A. M., et al. 2011, Nature, 476, 226, doi: 10.1038/nature10366
Zhang, B. 2018, The Physics of Gamma-Ray Bursts, doi: 10.1017/9781139226530
Zhang, B., & Yan, H. 2011, ApJ, 726, 90, doi: 10.1088/0004-637X/726/2/90
Zhang, J., Du, S.-s., Guo, S.-C., et al. 2018, ApJ, 858, 27, doi: 10.3847/1538-4357/aaab22
Zhou, J., Wang, Z., Chen, L., et al. 2018, Nature Communications, 9, 4599, doi: 10.1038/s41467-018-07103-2
Znajek, R. L. 1977, MNRAS, 179, 457, doi: 10.1093/mnras/179.3.457
Figure 1. The comparison between the solutions of non-rotation and rotation terms: $T_{nr}(\theta)$ versus $T_r(\theta)$ (assuming $\nu = 3/4$). The upper panel shows the case of $s = \lambda\nu = 0$ (the magnetic field lines threading the CO), and the middle panel indicates the case of $s = -3/2$ (the magnetic field lines threading a Keplerian AD with $\Omega \propto R_0^{-3/2}$). In both cases, they match each other at $\theta \ll 1$ and $\theta \to \pi/2$. 
Figure 2. The configuration of magnetic field line and velocity on a magnetic stream surface (illustrated by the light blue semitransparent surface in the left panel) threading the equator of a BH with a spin parameter $a = 0.1$: the green line refers to the structure of a magnetic field line and the gradient colored line represents the velocity profile with the color measuring the value of the four velocity $v\Gamma$. The right panel is a 2D projection as seen edge on of the left 3D configuration panel. The black dashed line shows the ACS ($z = 396 \, r_g$).
Figure 3. The four velocity ($v\Gamma$) profile of a flow along a magnetic field line threading the equator of a BH with a spin parameter $a = 0.5$. The toroidal velocity presents two stages (within or outside of the ACS, the upper panel), while the poloidal velocity has three stages: the first turning point also happens at the ACS and the second one is determined by the causality (the CCS, the middle panel). The complementarity of the toroidal and poloidal velocities in the first two stages makes the total velocity following a uniform stage before the CCS, although the dominant term changes from the toroidal to poloidal one when crossing the ACS (the bottom panel, see Section 5 for detail). The vertical dashed black line shows the location of the ACS ($z = 26 r_g$), while the vertical dotted black line refers that of the CCS ($z = 3882 r_g$).
Figure 4. Left panel: The gradient colored map of velocity distribution on the poloidal plane for the case of magnetic field lines threading a BH with a spin parameter $a = 1$, where the color measures the value of four velocity $vT$. The thick black line shows the outermost magnetic field line threading the equator of the BH and the two thin black lines represent these with polar angles $\theta = 60^\circ$ and $30^\circ$, respectively. The four velocity increases as the jet propagating outwards along a magnetic field line. At a fixed height of $z \lesssim 130 \, r_g$ (depending on $a$), the velocity always increases with increasing radial distance from the polar axis (e.g., at $z = 100 \, r_g$, the right/bottom panel). At a fixed height of $z \gtrsim 130 \, r_g$, the velocity increases firstly and then decreases as increasing radial distance from the polar axis (e.g., at $z = 1000 \, r_g$ and $10000 \, r_g$ in the right/middle and right/upper panels, respectively). The red dashed line shows where the maximum velocity is reached.
Figure 5. The current and charge in a BH accretion/jet system (the AD rotating along the same direction of the BH with a spin parameter $a = 0.1$ and the case $\Omega \cdot B > 0$). The green lines show the current flows and their directions. The red symbols “+” on the flow from the AD refer to positive charges with their sizes indicating the relative charge densities, while the blue symbols “–” refer to that of negative charges of flow from the BH. The red symbols “+” on the AD show positive charges with their sizes indicating the relative charge surface densities. The radii of foot-points of the three current lines from the AD are $R_o = 10 \, r_g$, $15 \, r_g$ and $20 \, r_g$, respectively, while those of the current lines originate from the BH have polar angles $\theta = 30^\circ$, $60^\circ$ and $90^\circ$, respectively. Notice that the BH has a negative charge in this case.
Figure 6. The cold plasma velocity (blue) versus the drift velocity (red) of a highly magnetized jet. The bottom panel is for the magnetic field line threading the equator of a BH with the spin parameter \( a = 1 \) and \( a = 0.1 \) as presented by solid and dashed lines, respectively. The upper panel is for the magnetic field line threading the AD at radius \( R_0 = 10 \, r_g \), rotating with a Keplerian velocity, where the solid and dashed lines are the same as that in the bottom panel. It can be seen that the drift velocity matches the cold plasma velocity very well. This result does not significantly depend on the value of the conserved quality in Equation 89, which is set to \( \varepsilon = 1 - (\Omega R_0)^2 / 2 \) here (see Equation 96). See Section 7.2 for details.