We prove an analogue of Selberg’s explicit formula for Motohashi’s product (see arXiv:1104.1358v3 [math.NT]). We also provide a zero-density theorem for the product, which follows from Soundararajan’s theorem for moments of the Riemann zeta-function on the critical line.

Key words: Riemann zeta-function, Riemann hypothesis.

1. Introduction. In [1] we gave a proof of the following theorem.

Assume RH. Then there exists an infinite sequence of pairs of real numbers \((T_1, T_2)\), \(T_1 = T, T_2 = T + H\), with arbitrarily large values of \(T\) and \(H = c(\log \log T)^{-1}\), such that

\[
|\zeta(1 + iT_1)| |\zeta(1 + iT_2)| \ll (\log \log T)^{-2}
\]

and

\[
(\log T)^{\varepsilon} |\zeta(1 + iT_1)|^4 |\zeta(1 + iT_2)|^4 \\
+ (\log T)^{\varepsilon} \zeta(1 + iT_1 + H)^2 \zeta(1 - iT_1 - H)^2 \zeta(1 + iT_2 + H)^2 \zeta(1 - iT_2 - H)^2 \\
+ (\log T)^{\varepsilon} \zeta(1 + iT_1 - H)^2 \zeta(1 - iT_1 + H)^2 \zeta(1 + iT_2 - H)^2 \zeta(1 - iT_2 + H)^2 \\
\ll (\log T)^{-1}.
\]

We put

\[
z = \exp(A \log \log T \log \log \log T)
\]

with the same constant \(A\) as in [1], set

\[
\xi(d) = \lambda_d(z) = \begin{cases} 
\mu(d) & \text{if } d < z, \\
\mu(d) \frac{\log(z^2/d)}{\log z} & \text{if } z \leq d < z^2, \\
0 & \text{otherwise},
\end{cases}
\]

\[
P_d(s, T_1, T_2) = \prod_{p|d} \left(1 - \left(1 - \frac{1}{p^s}\right) \left(1 - \frac{1}{p^s+iT_1}\right) \left(1 - \frac{1}{p^s+iT_2}\right) \left(1 - \frac{1}{p^{2s-i(T_1-T_2)}}\right) \left(1 - \frac{1}{p^{2s-i(T_1-T_2)}}\right)^{-1}\right),
\]

and write

\[
J(s, T_1, T_2) = \frac{\zeta(s)\zeta(s-iT_1)\zeta(s+iT_2)\zeta(s-i(T_1-T_2))}{\zeta(2s-i(T_1-T_2))},
\]

\[
K(s, T_1, T_2) = \sum_{d \leq z^2} \lambda_d(z) P_d(s, T_1, T_2).
\]

The theorem below shows that on RH the absolute value of the product \(JK\) cannot exceed a quite small function \(((\log \log T)^{\varepsilon})\) in a quite large neighborhood (within the distance \(c(\log \log T)^{-1}\)) to the left of the line \(\Re s = 1\).

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**Theorem 1.** Assume the Riemann hypothesis. Let \( s_0 = \sigma_0 + it_0 \) be a point such that
\[
|J(s_0, T_1, T_2)K(s_0, T_1, T_2)| \geq (\log \log T)^\varepsilon
\]
with arbitrarily small fixed \( \varepsilon > 0 \), and
\[
\sigma_0 = 1 - \frac{E_0}{\log \log T} \geq 1 - \frac{E}{\log \log T}, \quad C \log \log \log T \leq |t_0| \leq T/2. \tag{1}
\]
Then \( E_0 \geq c_2(\varepsilon) > 0 \).

Here we call the product \( J(s_0, T_1, T_2)K(s_0, T_1, T_2) \) Motohashi’s product (see \([2]\)).

In this paper, we prove an analogue of Selberg’s formula for the logarithmic derivative of Motohashi’s product. We also provide a zero-density theorem for the product, which follows from Soundararajan’s theorem for moments of the Riemann zeta-function on the critical line \([3]\).

2. Lemmas and the results. The following lemmas are found in Ingham \([4]\), theorems 26 and 27. The proofs use the functional equation for the zeta-function.

**Lemma 1.** There exists a sequence of numbers \( T_2, T_3, \ldots \), such that
\[
m < T_m < m + 1 \quad (m = 2, 3, \ldots)
\]
and
\[
\left| \frac{\zeta'}{\zeta}(s) \right| < A \log^2 t \quad (-1 \leq \sigma \leq 2, t = T_m).
\]

**Lemma 2.** In the region obtained by removing from the half-plane \( \sigma \leq -1 \) the interiors of a set of circles of radius \( \frac{1}{2} \) with centres at \( s = -2, -4, -6, \ldots \), i.e. in the region defined by
\[
\sigma \leq -1, \quad |s - n| \geq \frac{1}{2} \quad (n = -2, -4, -6, \ldots),
\]
we have
\[
\left| \frac{\zeta'}{\zeta}(s) \right| < A \log(|s| + 1).
\]

The following lemma is a consequence of lemmas \([1]\) and \([2]\) (cf. Lemma 9 of \([3]\)).

**Lemma 3.** There exists a sequence of numbers \( T_2, T_3, \ldots \), such that
\[
m < T_m < m + 1 \quad (m = 2, 3, \ldots)
\]
and
\[
\left| \frac{\zeta'}{\zeta}(s) \right| < A \log^2 m,
\]
for \( \sigma \geq -m - 1/2, t = \pm T_m \), or \( \sigma = -m - 1/2, |t| < T_m \).

Consider the function \( Z(s) = J(s, T_1, T_2)K(s, T_1, T_2) \). By Lemma 3 of \([1]\), for \( \sigma > 1 \) \( Z(s) \) is given by the convergent Dirichlet series
\[
Z(s) = \sum_{n=1}^{\infty} \sigma_{iT_1}(n)\sigma_{-iT_2}(n) \left( \sum_{d|n} \xi(d) \right) n^{-s}.
\]
Recalling the definition of \( \xi(d) \) we see that for \( \sigma \) sufficiently large we can make the power series expansion for \( \log Z(s) \) as \( \log(1 + x) \) and represent this as the convergent Dirichlet series.
Alternatively, we can expand \( \log K_n(s) \) and use the known expansion for \( \log J(s) \). Differentiating term by term, we obtain the Dirichlet series for \( \frac{Z'}{Z}(s) \). Denote the coefficients of the series by \( \Sigma(n) \):

\[
\frac{Z'}{Z}(s) = \sum_{n=2}^{\infty} \frac{\Sigma(n)}{n^s}.
\]

The following formula is an analogue of Selberg’s formula for \( \zeta(s) \). We denote the complex zeros of \( \zeta(s) \) by \( \rho = \beta + i\gamma \), the zeros of \( K(s, T_1, T_2) \) by \( r \), and the poles of \( K(s, T_1, T_2) \) by \( \nu \).

**Theorem 2.** Let \( x > 1 \) and write

\[
\Sigma_x(n) = \begin{cases} 
\Sigma(n) & \text{for } 1 \leq n \leq x, \\
\Sigma(n) \frac{\log^2 \frac{x^3}{n} - 2 \log^2 x}{2 \log^2 x} & \text{for } x \leq n \leq x^2, \\
\Sigma(n) \frac{\log^2 \frac{x^3}{n}}{2 \log^2 x} & \text{for } x^2 \leq n \leq x^3.
\end{cases}
\]

Then for

\[
s \notin S_1 = \{1, 1 + iT_1, 1 - iT_2, 1 + i(T_1 - T_2)\}, \quad s \notin \overline{S}_1 = \left\{ \frac{1}{2} + \frac{i}{2}(T_1 - T_2) \right\},
\]

\[
s \notin S_\rho = \{\rho, \rho + iT_1, \rho - iT_2, \rho + i(T_1 - T_2)\}, \quad s \notin \overline{S}_\rho = \left\{ \frac{\rho}{2} + \frac{i}{2}(T_1 - T_2) \right\},
\]

\[
s \notin S_{-2q} = \{-2q, -2q + iT_1, -2q - iT_2, -2q + i(T_1 - T_2)\}, \quad s \notin \overline{S}_{-2q} = \left\{ -q + \frac{i}{2}(T_1 - T_2) \right\}
\]

\((q = 1, 2, 3, \ldots),
\]

\[
s \notin S_\nu = \left\{ \nu = i \left( \frac{\pi k}{\log p} + \frac{1}{2}(T_1 - T_2) \right) \quad (k \in \mathbb{Z} \subset \mathbb{Z}, \quad p < z^2) \right\},
\]

\[
s \notin S_r = \{r\},
\]

we have

\[
\frac{Z'}{Z}(s) = \sum_{n<x^3} \frac{\Sigma_x(n)}{n^s} + \sum_{u \in S_1 \cup S_\rho \cup S_{-2q} \cup S_\nu} \frac{x^u - s(1 - x^{u-s})^2}{\log^2 x (u - s)^3}
\]

\[ - \sum_{u \in S_1 \cup S_\rho \cup S_{-2q} \cup S_\nu} \frac{x^u - s(1 - x^{u-s})^2}{\log^2 x (u - s)^3}.
\]

(2)

where the zeros and the poles are counted according to their multiplicities.

For the proof, we follow the argument of Selberg in Lemma 10 of [5]. Note that the analogue of Lemma 3 for the function \( K(s, T_1, T_2) \) should be with the bound

\[
\left| K'(s, T_1, T_2, z) \right| < A_{T_1, T_2, z} m,
\]

(3)

for a sequence of contours \( C_m : \sigma_m \leq \sigma \leq \alpha, t = \pm T_m, \text{ or } \sigma = \sigma_m, |t| < T_m (\sigma_m \to -\infty, T_m \to +\infty) \). The sequence \( \sigma_m, T_m \) is chosen properly so as to avoid zeros and poles of \( K \) (and zeros of \( J \)). This can be done using Jensen’s theorem for \( K \). In view of the bound (3) we need degree 3 in the denominator for the integral to tend to zero.
Next, we give a zero-density theorem for Motohashi’s product. Recall that
\[ K_X(s) = K(s, T_1, T_2) = \sum_{d \leq X^2} \lambda_d(X) P_d(s, T_1, T_2), \]
and
\[ J(s) K_X(s) = 1 + \sum_{n=X}^{\infty} \frac{\sigma_{T_1}(n)\sigma_{-iT_2}(n)}{n} \left( \sum_{d|n} \lambda_d(X) \right) n^{-s}. \]

Let
\[ f_X(s) = J(s) K_X(s) - 1, \]
and define \( N_K(\sigma, T) \) to be the number of zeros \( r \) of the function \( K_X(s) \) such that \( \Re r > \sigma \geq \frac{1}{2}, \ 0 < \Im r \leq T \).

**Lemma 4.** If for some \( X = X(\sigma, T) \), \( X < T^{A_1} \),
\[ \int_0^T |f_X(\sigma + it)|^{1/\kappa} dt = O \left( T^{l(\sigma)} \log^m T \right) \]
as \( T \to \infty \), uniformly for \( \alpha \leq \sigma \leq \beta \), where \( l(\sigma) \) is a positive non-increasing function with a bounded derivative, \( \kappa \) lies between \( \frac{1}{2} \) and \( \frac{3}{4} \), and \( m \) is a constant \( \geq 0 \), then for any fixed \( A_2 \geq 1 \)
\[ N_K(\sigma, T) = O \left( T^{l(\sigma)} \log^{m+1} T \right) + O \left( \frac{T \log T}{X^{A_2}} \right) \]
uniformly for \( \alpha + 1/\log T \leq \sigma \leq \beta \).

This lemma is a modified form of Theorem 9.16 of [6].

**Lemma 5.** Assume RH. For every positive real number \( k \) and every \( \varepsilon > 0 \) we have
\[ \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} dt \ll_k,\varepsilon T \left( \log T \right)^{k^2+\varepsilon}. \]

This is a theorem of K. Soundararajan [3].

The following lemma is an analogue of Theorem 9.19 (B) of [6] (Ingham’s zero density theorem) and follows from Soundararajan’s theorem and the upper bound for \( K_X(s) \) with \( X = z \) in [1].

**Lemma 6.** Assume RH.
\[ \int_0^T |f_X(1/2 + d \log \log T \log \log \log T/ \log T + it)|^{1/\kappa} dt = O \left( T^{1-c_d \log \log T \log \log \log T/ \log T} \right). \]
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