ON THE ELLIPTIC GENUS OF GENERALISED KUMMER VARIETIES

MARC A. NIEPER-WISSKIRCHEN

Abstract. Borisov and Libgober (1) recently proved a conjecture of Dijkgraaf, Moore, Verlinde, and Verlinde (see 4) on the elliptic genus of a Hilbert scheme of points on a surface. We show how their result can be used together with our work on complex genera of generalised Kummer varieties (9) to deduce the following formula on the elliptic genus of a generalised Kummer variety $A^{[[n]]}$ of dimension $2(n-1)$:
$$\text{Ell}(A^{[[n]]}) = n^4 \psi^{-2} \cdot \left( \psi^2 |_{-2} V(n) \right).$$
Here $\psi(z, \tau) := 2\pi i \vartheta(-z, \tau) \vartheta'(0, \tau)$ is the weak Jacobi form of weight $-1$ and index $\frac{1}{2}$, and $V(n)$ is the Hecke operator sending Jacobi forms of index $r$ to Jacobi forms of index $nr$ (see 7).

1. Introduction

The elliptic genus can be defined for every compact complex manifold as a function of a complex variable $\tau \in \mathbb{H}$ and a complex variable $z \in \mathbb{C}$. For a Calabi-Yau manifold (Calabi-Yau in the (very weak) sense that its first Chern class vanishes up to torsion) the elliptic genus turns out to be a weak Jacobi form of weight zero and index $d/2$, where $d$ is the dimension of the manifold. A proof of this fact can be found in 11, and also in 3.

The elliptic genus is a generalization of Hirzebruch’s $\chi_y$-genus. It was shown in 3 that in fact the elliptic genus of a Calabi-Yau generally encodes more information than the $\chi_y$-genus provided the dimension of the manifold is twelve or greater than thirteen.

Compact hyperkähler (see for example 14) manifolds are in particular Calabi-Yau manifolds in the sense given above. There are two main series of examples of compact hyperkähler manifolds, the Hilbert schemes of points on surfaces and the generalised Kummer varieties (see 1). The $\chi_y$-genera for these manifolds have been known for quite some time (see 11), since they depend only on the Hodge numbers of these manifolds. One is naturally led to the question of generalising these results to elliptic genera.

Dijkgraaf, Moore, Verlinde and Verlinde showed in 4 that the elliptic genus of a symmetric product of a manifold can be expressed in terms of the elliptic genus of the manifold itself. They conjectured that their formula also holds for the Hilbert scheme of points on surfaces since these Hilbert schemes can be seen as resolutions of symmetric powers of surfaces. Very recently, Borisov and Libgober managed to prove that conjecture (see 9) with methods of toric geometry.

In 11, we expressed the value of a genus on a generalised Kummer variety in terms of twisted versions of this genus evaluated on the Hilbert schemes of points on a surface $X$ with $c_1(X)^2 \neq 0$. In this work it is shown that the result of 2 on the elliptic genera of these Hilbert schemes can be used to calculate their twisted elliptic genera. We do this and are led to a formula for the elliptic genus of a
generalised Kummer variety. By specialising our result we reproduce Göttsche’s and Soergel’s formula for the $\chi_y$-genus of the generalised Kummer varieties.

One last word on the notation: The symbol $\sum_{a|n}$ stands for summing over the positive divisors of the integer $n$.

2. Complex genera

2.1. Some notes on complex genera and the elliptic genus. Let $\Omega := \Omega^1 \otimes \mathbb{Q}$ denote the (rational) complex cobordism ring. By a result of Milnor ([15]), it is generated by the cobordism classes $[X]$ of all complex manifolds $X$, and two complex manifolds $X$ and $Y$ lie in the same cobordism class if and only if they have the same Chern numbers, i.e. the Chern numbers determine the cobordism class and vice versa. Recall that the sum in the ring is induced by the disjoint union of manifolds, and the product by the cartesian product of manifolds.

By a complex genus $\phi$ with values in a $\mathbb{Q}$-algebra homomorphism $\phi : \Omega \to R$. Examples of complex genera can be constructed in the following way:

Example 1. Let $\mathcal{C}$ be the category of complex manifolds and $\mathfrak{A}$ the category of abelian groups. Let $R$ be any $\mathbb{Q}$-algebra and $\Phi$ a natural transformation from the functor $\mathcal{C} \to \mathfrak{A}, X \mapsto K^0(X) \otimes \mathbb{Q}$ to the functor $\mathcal{C} \to \mathfrak{A}, X \mapsto (K^0(X) \otimes R)^\times$ that associates to each $X$ in the category of compact complex manifolds a group homomorphism $\Phi_X : K^0(X) \to (K^0(X) \otimes R)^\times$. For every such natural transformation,

$$\phi : X \mapsto \chi(X, \Phi_X(T_X))$$

(1)

defines a complex genus.

Proof. It follows directly from the naturality of $\Phi_X$ that $\phi(X + Y) = \phi(X) + \phi(Y)$. Obviously, $\phi(\emptyset) = 0$, where “+” denotes the disjoint union of manifolds. Furthermore,

$$\phi(X \times Y) = \chi(X \times Y, \Phi_{X \times Y}(T_{X \times Y}))$$

$$= \chi(X \times Y, \Phi_{X \times Y}(T_X \oplus T_Y))$$

$$= \chi(X \times Y, \Phi_{X \times Y}(T_X) \otimes \Phi_{X \times Y}(T_Y))$$

$$= \chi(X \times Y, \Phi_X(T_X) \boxtimes \Phi_Y(T_Y))$$

$$= \chi(X, \Phi_X(T_X)) \cdot \chi(Y, \Phi_Y(T_Y))$$

$$= \phi(X) \cdot \phi(Y)$$

and

$$\phi(\text{point}) = \chi(\text{point}, \Phi_{\text{point}}(0)) = \chi(\text{point}, 1) = 1.$$

Finally, we have to show that $\phi$ depends only on the characteristic numbers of $X$. By the Hirzebruch-Riemann-Roch formula, we have $\phi(X) = \int_X \text{ch}(\Phi_X(T_X)) \text{td}(T_X)$, i.e. it suffices to show that $\text{ch}(\Phi_X(E))$ depends only on the Chern classes of $E$ for any bundle $E$ on $X$. This can be seen as follows: Obviously, $E \mapsto \text{ch}(\Phi_X(E))$ is a natural transformation from the functor $X \mapsto K^0(X) \otimes \mathbb{Q}$ to the functor $X \mapsto \mathbb{H}^0(X) \otimes R$. By the universality property of the Chern classes it follows that $\text{ch}(\Phi_X(E))$ can be given as an universal polynomial (depending only on $\Phi_X$) in the Chern classes of $E$ over $R$ (see for example [14]).

The definition of the elliptic genus is an application to the concept outlined in the example. From now on let use write $q := e^{2\pi i\tau}$ for $\tau \in \mathbb{H}$, and $y^z := e^{\pi iz}$ for $z \in \mathbb{C}$, i.e. $y = e^{2\pi iz}$.
ON THE ELLIPTIC GENUS OF GENERALISED KUMMER VARIETIES

Definition 1. For every complex manifold \( X \) of dimension \( d \), we define the following virtual bundle

\[
\mathcal{E}_X(z, \tau) := \Phi_X(T_X, z, \tau) \in K^0(X)[[y]]y^{1/2}
\]

on \( X \), where

\[
\Phi_X(E, z, \tau) = y^{-d/2} \bigotimes_{n=1}^{\infty} \left( \bigwedge_{y^n-1} E^* \otimes \bigwedge_{y^{-1}q^n} E \otimes S_{q^n} E^* \otimes S_{q^n} E \right)
\]

\[
\in K^0(X)[[y]]y^{1/2},
\]

for any bundle \( E \) on \( X \). Here, \( \bigwedge^i(E) \) is defined as the Chern roots of \( E \) as follows:

\[
\bigwedge^i(E) := \bigoplus_{i=0}^\infty \bigwedge^i(E)t^i,
\]

for any bundle \( E \) denote the generating functions for the exterior respective symmetric powers of \( E \).

Definition 2 (see [3]). Let \( X \) be a compact complex manifold of dimension \( d \). The elliptic genus \( \text{Ell}(X, z, \tau) \) of \( X \) is defined to be the Euler characteristic \( \chi(X, \mathcal{E}_X(z, \tau)) \in \mathbb{Q}[y]y^{1/2} \) of the bundle \( \mathcal{E}_X(z, \tau) \) on \( X \).

The natural transformation \( X \mapsto \Phi_X \) of Definition 1 is in fact a natural transformation as in Example 1. Therefore, \( X \mapsto \text{Ell}(X) \) induces in fact a complex genus.

Remark 1. The elliptic genus is a generalization of the \( \chi_y \)-genus of Hirzebruch. More precisely,

\[
\chi_y(X) = y^{d/2} \lim_{\tau \to i\infty} \text{Ell}(X, z, \tau) = y^{d/2} \text{Ell}(X, z, \tau)_{|y=0} \in \mathbb{Q}[y],
\]

which follows directly from the definitions.

By Hirzebruch’s theory of genera and multiplicative sequences ([12]), the \( R \)-valued complex genera \( \phi \) are in one-to-one correspondence with the formal power series \( f_\phi \in R[[x]] \) over \( R \) with constant coefficient one. The correspondence is given as follows:

\[
\phi(X) = \int_X \prod_{i=1}^d f_\phi(x_i)
\]

for all complex manifolds \( X \), where \( d \) is the dimension of \( X \), and \( x_1, \ldots, x_d \) are the Chern roots of its tangent bundle.

The representation of the elliptic genus in terms of power series is as follows:

Proposition 1. Let \( x_1, \ldots, x_d \) be the Chern roots of \( X \). We have

\[
\text{Ell}(X, z, \tau) = \prod_{i=1}^d \left( x_i \frac{\vartheta \left( \frac{x_i}{2\pi i}, z, \tau \right)}{\vartheta(0, z, \tau)} \right),
\]

where Jacobi’s theta function is defined as

\[
\vartheta : \mathbb{C} \times \mathbb{H} \to \mathbb{C}, (z, \tau) \mapsto 2q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1-q^n)(1-q^n y)(1-q^n y^{-1})
\]

\[
\vartheta(z, \tau) := \lim_{x \to 0} x \frac{\vartheta \left( \frac{x}{2\pi i}, z, \tau \right)}{\vartheta(0, z, \tau)} = 2\pi i \frac{\vartheta(-z, \tau)}{\vartheta(0, \tau)} \in \mathbb{Q}[[y]]y^{1/2},
\]

Proof. See [3].

However, the power series in \( x_i \) on the right hand side of the product sign has not constant coefficient one. Nevertheless, we can achieve this by following an idea proposed in [13], where it has been worked out on the \( \chi_y \)-genus: Let us define

\[
\psi(z, \tau) := \lim_{x \to 0} x \frac{\vartheta \left( \frac{x}{2\pi i}, z, \tau \right)}{\vartheta(0, z, \tau)} = 2\pi i \frac{\vartheta(-z, \tau)}{\vartheta(0, \tau)} \in \mathbb{Q}[[y]]y^{1/2},
\]
where \( \vartheta'(0, \tau) := \left. \frac{\partial}{\partial z} \right|_{z=0} \vartheta(z, \tau) \). Furthermore, we set
\[
f(z, \tau)(x) := \frac{x \vartheta\left(\frac{xz}{2\pi i}, -z, \tau\right)}{\vartheta\left(\frac{xz}{2\pi i}, \tau\right)} \in \mathbb{Q}[\![y^{\pm 1}]]\![x].
\] (9)
This is a power series in \( x \) with constant coefficient one. With this definition, we have
\[
\text{Ell}(X, z, \tau) = \int_X \prod_{i=1}^d f(z, \tau)(x_i),
\] (10)
which can be seen by expanding \( f(z, \tau) \) as a Taylor series in \( x \) and looking at the degree \( d \) term.

Remark 2. The power series \( \psi \) has the following representation as an infinite product:
\[
\psi(z, \tau) = \left( y^{-\frac{1}{2}} - y^{\frac{1}{2}} \right) \prod_{n=1}^{\infty} \frac{(1 - q^n y)(1 - q^n y^{-1})}{(1 - q^n)^2}.
\] (11)
This can be seen for example by considering the product representations of \( \vartheta(\tau, z) \) and \( \vartheta'(0, \tau) \) given in [1].

2.2. Twisted genera and the twisted elliptic genus. Following our work on complex genera of generalised Kummer varieties (16), we define twisted genera of compact complex manifolds as follows:

Definition 3. Let \( \phi \) be a complex genus with values in the \( \mathbb{Q} \)-algebra \( R \). By \( \phi_t \) we denote the genus with values in \( R[t] \) that is given by
\[
\phi_t(X) := \int_X \prod_{i=1}^d (f_{\phi}(x_i)e^{tx_i})
\] (12)
for any complex manifold \( X \) of dimension \( d \), and with Chern roots \( x_1, \ldots, x_d \).

Remark 3. Obviously, if \( \phi \) a the genus \( \phi \) is given by a natural transformation \( \Phi \) as in Example 1, we have
\[
\phi_t(X) = \chi(X, \Phi_X(T_X) \otimes K_X^{-t}).
\] (13)
for any integer \( t \). Of course, this can also be written as
\[
\phi_t(X) = \chi(X, \Phi_{t,X}(T_X))
\] (14)
with the natural transformation \( \Phi_t \) defined by \( \Phi_{t,X}(E) := \Phi_X(E) \otimes (\det E)^t \) for any bundle \( E \) on \( X \).

In general, the twisted version of a complex genus holds more information than the untwisted genus itself. In the case of the elliptic genus, however, this is not true, i.e. the elliptic genus encodes all the information of its twisted version.

As we are interested only in manifolds of even dimension for the purpose of proving our main theorem, let \( X \) be of even dimension \( d = 2n \) from now on.

Proposition 2. The elliptic genus twisted by \( t \psi \), \( t \) an integer, of the compact complex manifold \( X \) can be expressed by the usual elliptic genus via
\[
\text{Ell}_{t\psi}(X, z, \tau) = y^{2nt} q^{nt^2} \text{Ell}(X, z + t\tau, \tau).
\] (15)
Proof. We will make use of the following transformation property of the theta function (see for example [3]):
\[
\vartheta(z + \tau, \tau) = q^{-\frac{t}{2}} y^{-1} \vartheta(z, \tau)
\] (16)
where $q^{-\frac{1}{2}}$ stands for $\exp(-\pi i \tau)$. It follows by definition that
\[ f(z, \tau, x) = -q^{\frac{1}{2}}ye^{-x\psi}f(x, z + \tau, \tau). \tag{17} \]
Now we use (10) and find that
\[ \text{Ell}_{t\psi}(X, z, \tau) = q^{nt^2}y^{2nt}\text{Ell}_{(t-1)\psi}(X, z + \tau, \tau) \tag{18} \]
by the definition of twisted genera. The proposition then follows by induction on $t$. \qed

From this we deduce the following proposition on the Fourier coefficients of twisted elliptic genera:

**Proposition 3.** Let $t$ be an integer and $c_t(m, l)$ for $m, 2l \in \mathbb{Z}$ be such that
\[ \sum_{m, 2l \in \mathbb{Z}} c_t(m, l)q^m y^l = \text{Ell}_{t\psi}(X, z, \tau). \tag{19} \]
These coefficients are connected to the coefficients $c(m, l) := c_0(m, l)$ of the untwisted elliptic genus via
\[ c_t(m, l) = c(m - lt + nt^2, l - 2nt). \tag{20} \]

**Proof.** By Proposition 3, we have
\[
\sum_{m, 2l \in \mathbb{Z}} c_t(m, l)q^m y^l = \text{Ell}_{t\psi}(X, z, \tau) = y^{2nt}q^{nt^2}\text{Ell}(X, z + t\tau, \tau)
= y^{2nt}q^{nt^2} \sum_{m, 2l \in \mathbb{Z}} c(m, l)q^m(yq^t)^l = \sum_{m, 2l \in \mathbb{Z}} c(m, l)q^{m+lt+nt^2} y^{l+2nt}
= \sum_{m, 2l \in \mathbb{Z}} c(m - lt + nt^2, l - 2nt)q^m y^l.
\]
Now compare coefficients. \qed

3. The generalised Kummer varieties

3.1. Definition of generalised Kummer varieties. Let $X$ be a smooth projective surface over the field of complex numbers. For every nonnegative integer we denote by $X^{[n]}$ the Hilbert scheme of zero-dimensional subschemes of $X$ of length $n$. By a result of Fogarty \[4\], this scheme is smooth and projective of dimension $2n$.

It can be viewed as a resolution $\rho : \mathcal{X}^{[n]} \rightarrow \mathcal{X}^{(n)}$ of the $n$-fold symmetric product $\mathcal{X}^{(n)} := X^n/\mathfrak{S}_n$ of $X$. The morphism $\rho$, sending closed points, i.e. subschemes of $X$, to their support counting multiplicities, is called the Hilbert-Chow morphism.

Let us briefly recall the construction of the generalised Kummer varieties introduced by Beauville \[1\]. Let $A$ be an abelian surface and $n > 0$. There is an obvious summation morphism $A^{(n)} \rightarrow A$. We denote its composition with the Hilbert-Chow morphism $\rho : A^{[n]} \rightarrow A^{(n)}$ by $\sigma : A^{[n]} \rightarrow A$.

**Definition 4.** The $n^{th}$ generalised Kummer variety $A^{[[n]]}$ is the fibre of $\sigma$ over $0 \in A$.

**Remark 4.** It was Beauville who showed among other things in \[1\] that the $n^{th}$ generalised Kummer variety is smooth projective, and irreducible holomorphic symplectic of dimension $2(n - 1)$.
3.2. Complex genera of generalised Kummer varieties. Let

\[ K(p) := \sum_{n=1}^{\infty} A^{[n]} \cdot p^n \in \Omega[[p]] \]  

(21)

be the generating series of the complex cobordism classes of the generalised Kummer varieties and similarly

\[ H_X(p) := \sum_{n=0}^{\infty} X^{[n]} \cdot p^n \in \Omega[[p]] \]  

(22)

the generating series to the Hilbert schemes of points on a surface \( X \).

In [16] we proved the following theorem:

**Theorem 1.** Let \( \phi : \Omega \to R \) be a complex genus with values in the \( \mathbb{Q} \)-algebra \( R \). For every smooth projective surface \( X \) with \( \int_X c_1(X)^2 \neq 0 \),

\[ \phi(K(p)) = \frac{1}{c_1(X)^2} \left( p \frac{d}{dp} \right)^2 \ln \frac{\phi_1(H_X(p)) \phi^{-1}(H_X(p))}{\phi(H_X(p))^2}. \]  

(23)

**Proof.** See [16]. \( \square \)

Taking a look at the proofs of [14] or at the results of [8], we see that \( \ln \phi_t(H_X(p)) \) is a polynomial in \( t \) of degree 2, and therefore

\[ \ln \frac{\phi_1(H_X(p)) \phi^{-1}(H_X(p))}{\phi(H_X(p))^2} = \frac{d^2}{dt^2} \ln \phi_t(H_X(p)) \bigg|_{t=0} \]  

(24)

So, the genus \( \phi \) of \( K(p) \) can be expressed as

\[ \phi(K(p)) = 2 \left( p \frac{d}{dp} \right)^2 A_{\phi}(p) \]  

(25)

for any smooth projective surface \( X \) with \( c_1(X)^2 \neq 0 \). Here,

\[ A_{\phi}(p) := \frac{1}{2c_1(X)^2} \left. \frac{d^2}{dt^2} \ln \phi_t(H_X(p)) \right|_{t=0}, \]  

(26)

which is independent of \( X \). We shall use the formula (25) to calculate the elliptic genera of the generalised Kummer varieties.

4. Elliptic genera of generalised Kummer varieties

In the following proposition, we shall calculate twisted elliptic genera of Hilbert schemes of points on surfaces. We shall make use of the following recent theorem of Borisov and Libgober ([2]):

**Theorem 2.** Let \( X \) be a complex projective surface. Let \( c(m, l) \) for \( m, 2l \in \mathbb{Z} \) be the Fourier coefficients of the elliptic genus of \( X \), i.e. \( Ell(X, z, \tau) = \sum_{m, 2l \in \mathbb{Z}} c(m, l) q^m y^l \).

Then the elliptic genera of the Hilbert schemes \( X^{[n]} \) of zero-dimensional subschemes of length \( n \) on \( X \) are given by:

\[ Ell(H_X(p), z, \tau) := \sum_{n=0}^{\infty} p^n Ell(X^{[n]}, z, \tau) = \prod_{i=1}^{\infty} \prod_{m, 2l \in \mathbb{Z}} (1 - p^i q^m y^l)^{-c(m, l)}. \]  

(27)

**Proof.** See [2]. \( \square \)
Proposition 4. As a corollary to the previous theorem, we have

\[
\text{Ell}_\psi(H_X(p), z, \tau) := \sum_{n=0}^{\infty} p^n \text{Ell}_\psi(X^{[n]}, z, \tau) = \prod_{i=1}^{\infty} \prod_{m, 2l \in \mathbb{Z}} (1 - p^i y^i q^m - c_{it}(m, l)),
\]

for every integer \( t \), i.e.

\[
\ln \text{Ell}_\psi(H_X(p), z, \tau) = \sum_{i, k=1}^{\infty} \frac{1}{k} c_{it}(m, l) p^{ik} y^{lk} q^{mk}.
\]

Proof. This proof consists just of a straight-forward calculation:

\[
\sum_{n=0}^{\infty} p^n \text{Ell}_\psi(X^{[n]}, z, \tau) = \sum_{n=0}^{\infty} p^n y^{2n t} q^{nt} \text{Ell}(X^{[n]}, z + t \tau, \tau)
= \prod_{i=1}^{\infty} \prod_{m, 2l \in \mathbb{Z}} (1 - (py^{2t} q^2)y^m(yq^l) - c_{(m, l)}
= \prod_{i=1}^{\infty} \prod_{m, 2l \in \mathbb{Z}} (1 - p^i q^{m+lt+it^2} y^{l+2it} - c_{(m, l)}
= \prod_{i=1}^{\infty} \prod_{m, 2l \in \mathbb{Z}} (1 - p^i y^l q^{m+it^2} y^{l+it} - c_{(m, l)}
= \prod_{i=1}^{\infty} \prod_{m, 2l \in \mathbb{Z}} (1 - p^i y^l q^{m-it^2} y^{l-it} - c_{(m, l)}
= \prod_{i=1}^{\infty} \prod_{m, 2l \in \mathbb{Z}} (1 - p^i y^l q^m - c_{(m, l)}

\]

\[\square\]

Lemma 1. Let \( X \) be a smooth projective surface. Let \( c_t(m, l) \) for \( m, 2l \in \mathbb{Z} \) be the Fourier coefficients of the twisted elliptic genus \( \text{Ell}_\psi \) of \( X \). We have

\[
\frac{d^2}{dt^2} \bigg|_{t=0} c_t(m, l) = u(m, l) \cdot c_1(X)^2,
\]

where \( u(m, l) \in \mathbb{Q} \) for \( m, 2l \in \mathbb{Z} \) is such that

\[
\sum_{m, 2l \in \mathbb{Z}} u(m, l) q^m y^l = \psi^2(z, \tau).
\]

Proof. Such a statement is actually true for any genus on a surface \( X \): If \( \phi \) is any complex genus, then

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \phi_t(X) = c_1(X)^2,
\]

which follows from the fact that \( \phi_t(X) \) is polynomial of degree two in \( t \) with leading coefficient \( c_1(X)^2/2 \). From (\*) the lemma follows by the chain rule. \[\square\]

Proposition 5. For every smooth projective surface \( X \) and \( u(m, l) \) for \( m, 2l \in \mathbb{Z} \) as in Lemma [4] we have:

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \ln \text{Ell}_\psi(H_X(p), z, \tau) = c_1(X)^2 \sum_{i, k=1}^{\infty} \sum_{m, 2l \in \mathbb{Z}} i^2 \frac{1}{k} p^{ik} q^{mk} y^{lk} u(m, l).
\]

Proof. This proposition follows immediately from Proposition 4 and Lemma 1. \[\square\]
Now we can use that proposition together with the theorem on complex genera of generalised Kummer varieties to prove our main theorem:

**Theorem 3.** Let $J_{k,r}$ for $k, 2r \in \mathbb{Z}$ be the space of weak Jacobi forms of weight $k$ and index $r$. Following [7], we define for every $i \in \mathbb{N}$ a Hecke operator

$$V(i) : J_{k,r} \to J_{k,ir}, \phi \mapsto \phi|_{k} V(i)$$

by

$$(\phi|_{k} V(i))(z, \tau) := \sum_{a \mid i} \sum_{m, 2 \in \mathbb{Z}} c\left(\frac{mi}{a}, l\right) q^{am} y^{al},$$

where the $c(m, l)$ are the Fourier coefficients of $\phi$, i.e. $\phi(z, \tau) = \sum_{m, 2 \in \mathbb{Z}} c(m, l) q^{m} y^{l}$.

The following statements hold true: $\psi^2$ is a weak Jacobi form of weight $-2$ and index 1, $\psi^{-2} \left(\psi^2|_{-2} V(i)\right)$ is a weak Jacobi form of weight 0 and index $i - 1$, and the elliptic genera of the generalised Kummer varieties are given by

$$\text{Ell}(K(p)) = \sum_{i=1}^{\infty} \text{Ell}(A[ln]) p^n = \psi^{-2} \sum_{i=1}^{\infty} i^4 \left(\psi^2|_{-2} V(i)\right) p^i.$$

**Proof.** The holomorphic function $\psi^2$ is (up to a factor) the weak Jacobi form of weight $-2$ and weight index 1 (see [7]). Then note that the space of weak Jacobi forms of weight $-2$ lies inside the principal ideal spanned by $\psi^2$ in $J_{2\ast, \ast}$ which follows from the classification theorem on weak Jacobi forms in [7].

Let $X$ be any smooth projective surface with $c_1(X)^2 \neq 0$. By (28) and the previous proposition we have

$$\sum_{i=1}^{\infty} \text{Ell}(A[ln]) p^n$$

$$= \frac{1}{c_1(X)^2} \left(\frac{d}{dp}\right)^2 \left(\frac{d^2}{dp^2}\right)_{l=0} \ln \text{Ell}(H_X(p))$$

$$= \psi^{-2} \sum_{i=1}^{\infty} \sum_{m, 2 \in \mathbb{Z}} i^4 q^{mk} y^{lk} u(mi, l)$$

$$= \psi^{-2} \sum_{i=1}^{\infty} i^4 \sum_{a \mid i} \sum_{m, 2 \in \mathbb{Z}} a^{-3} \sum_{m, 2 \in \mathbb{Z}} u\left(\frac{mi}{a}, l\right) q^{am} y^{al},$$

which proves the rest of the theorem. \(\square\)

We can easily deduce Göttsche’s and Soergel’s formula on the $\chi_y$-genus of a generalised Kummer variety ([10]) from this theorem:

**Corollary 1** (Göttsche, Soergel ([10])). Let $n$ be a positive integer. The $\chi_y$-genus of the generalised Kummer variety $A[ln]$ is given by:

$$\chi_y(A[ln]) = y^{n-1} n^4 \sum_{a \mid n} a^{-3} \frac{y^a + y^{-a} - 2}{y + y^{-1} - 2}$$

**Proof.** We have

$$\chi_y(A[ln]) = y^{n-1} \text{Ell}(A[ln], \tau, z)\big|_{q=0} = y^{n-1} \psi^{-2}(\tau, z)|_{q=0} n^4 \left(\psi^2|_{-2} V(n)\right)(\tau, z)|_{q=0}.$$
Now we use that $\psi^2(\tau, z) \big|_{q=0} = (y + y^{-1} - 2)$ and the following representation of the Hecke operators (see [3])

$$
\left( \psi^2 \big|_{-2} V(n) \right)(\tau, z) = n^{-1} \sum_{a|n} \sum_{b=0}^{n/a-1} a^{-2} \psi^2 \left( \frac{a\tau + b}{n/a}, az \right),
$$

i.e.

$$
\left( \psi^2 \big|_{-2} V(n) \right)(\tau, z) \big|_{q=0} = n^{-1} \sum_{a|n} \sum_{b=0}^{n/a-1} a^{-2} \lim_{\tau \to i\infty} \psi^2 \left( \frac{a\tau + b}{n/a}, az \right)
$$

$$
= n^{-1} \sum_{a|n} \sum_{b=0}^{n/a-1} a^{-2} (y^a + y^{-a} + 2)
$$

$$
= \sum_{a|n} a^{-3} (y^a + y^{-a} + 2),
$$

to conclude the formula given in the corollary. □

**References**

[1] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782, 1983.

[2] Lev Borisov and Anatoly Libgober. McKay correspondence for elliptic genera. *arXiv:math.AG/0206241*.

[3] Lev A. Borisov and Anatoly Libgober. Elliptic genera of toric varieties and applications to mirror symmetry. *Invent. Math.*, 140(2):453–485, 2000.

[4] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New-York, 1982.

[5] Komaravolu Chandrasekharan. *Elliptic functions*, volume 281 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985.

[6] Robert Dijkgraaf, Gregory Moore, Erik Verlinde, and Herman Verlinde. Elliptic genera of symmetric products and second quantized strings. *Comm. Math. Phys.*, 185:197–201, 1997.

[7] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1985.

[8] Geir Ellingsrud, Lothar Göttsche, and Manfred Lehn. On the cobordism class of the Hilbert scheme of a surface. *J. Algebraic Geom.*, 10(1):81–100, 2001.

[9] John Fogarty. Algebraic families on an algebraic surface. *Amer. J. Math.*, 90:511–521, 1968.

[10] Lothar Göttsche and Wolfgang Soergel. Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. *Math. Ann.*, 296(2):235–245, 1993.

[11] Valery Gritsenko. Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms. *St. Petersburg Math. J.*, 11(5):781–804, 2000.

[12] Friedrich Hirzebruch. *Neue topologische Methoden in der Algebraischen Geometrie*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, Berlin, 1962.

[13] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung. *Manifolds and modular forms*, volume E20 of *Aspects of Mathematics*. Friedr. Vieweg and Sohn Verlagsgesellschaft mbH, Braunschweig, 1992.

[14] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.

[15] John Milnor. On the cobordism ring $\Omega^*$ and a complex analogue. *Amer. J. Math.*, 82:505–521, 1960.

[16] Marc Nieper-Wißkirchen. On the Chern numbers of Generalised Kummer Varieties. *To appear in: Math. Res. Lett.*

Mathematisches Institut der Univ. zu Köln, Weyertal 86–90, 50931 Köln, Germany

E-mail address: mnieper@mi.uni-koeln.de