Research Article

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A pair of equations in unlike powers of primes and powers of 2

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Abstract: In this article, we show that every pair of large even integers satisfying some necessary conditions can be represented in the form of a pair of one prime, one prime squares, two prime cubes, and 187 powers of 2.

Keywords: circle method, Linnik problem, powers of 2

MSC 2010: 11P32, 11P05, 11P55

1 Introduction

As an approximation to Goldbach’s problem, Linnik proved in 1951 [1] under the assumption of the Generalized Riemann Hypothesis (GRH), and later in 1953 [2] unconditionally, that each large even integer \( N \) is a sum of two primes \( p_1, p_2 \) and a bounded number of powers of 2, namely

\[
N = p_1 + p_2 + 2^\nu + \cdots + 2^\nu. \tag{1.1}
\]

In 2002, Heath-Brown and Puchta [3] applied a rather different approach to this problem and showed that \( k = 13 \) and, on the GRH, \( k = 7 \). In 2003, Pintz and Ruzsa [4] established this latter result and announced that \( k = 8 \) is acceptable unconditionally. Elsholtz, in an unpublished manuscript, which is yet to appear in print, showed that \( k = 12 \); this was proved independently by Liu and Liu [5].

In 1999, Liu et al. [6] proved that every large even integer \( N \) can be written as a sum of four squares of primes and a bounded number of powers of 2, namely

\[
N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^\nu + \cdots + 2^\nu. \tag{1.2}
\]

And Platt and Trudgian [7] got that \( k = 45 \) suffices.

In 2001, Liu and Liu [8] proved that every large even integer \( N \) can be written as a sum of eight cubes of primes and a bounded number of powers of 2, namely

\[
N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^\nu + \cdots + 2^\nu. \tag{1.3}
\]

The acceptable value \( k = 330 \) was determined by Platt and Trudgian [7].

In 2011, Liu and Lü [9] considered a hybrid problem of (1.1)–(1.3),

\[
N = p_1 + p_2^2 + p_3^3 + p_4^4 + 2^\nu + \cdots + 2^\nu. \tag{1.4}
\]

They showed that \( k = 161 \) is acceptable and Platt and Trudgian [7] revised it to 156.

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As a generalization, recently, Hu and Yang [10] first considered the simultaneous representation of pairs of positive even integers \( N_1 \gg N_2 \), in the form

\[
\begin{align*}
N_1 &= p_1 + p_2^2 + p_3^3 + p_4^4 + \cdots + p_k^k, \\
N_2 &= p_5 + p_6^2 + p_7^3 + p_8^4 + \cdots + p_k^k,
\end{align*}
\]

where \( k \) is a positive integer. They proved that the simultaneous Eq. (1.5) is solvable for \( k = 455 \).

The primary purpose of this article is to sharpen this result considerably by establishing the following theorem.

**Theorem 1.1.** For \( k = 187 \), Eq. (1.5) is solvable for every pair of sufficiently large positive even integers \( N_1 \) and \( N_2 \) satisfying \( N_2 \gg N_1 \).

Our proof of Theorem 1.1 uses the Hardy-Littlewood circle method. We make a new estimate of minor arcs and draw on some strategies adopted in the works of Hu and Yang [10] and Kong and Liu [11].

Throughout this article, the letter \( \epsilon \) denotes a positive constant, which is arbitrarily small but may not the same at different occurrences.

## 2 The proof of Theorem 1.1

Throughout this article, we assume that \( N_i, i = 1, 2 \) are sufficiently large even integers satisfying \( N_2 \gg N_1 \). Then, we set

\[
P_i = N_i^{1/9 - 2\epsilon}, \quad Q_i = N_i^{8/9 + \epsilon}, \quad L = \log_2 N_i
\]

for \( i = 1, 2 \).

We define the major arcs \( \mathcal{M}_1 \), \( \mathcal{M}_2 \) and minor arcs \( C(\mathcal{M}_1), C(\mathcal{M}_2) \) as usual, namely

\[
\mathcal{M}_i = \bigcup_{q \in \mathcal{P}_i, 1 < q, q \leq N_i} \bigcup_{(a, q) = 1} \mathcal{M}_i(a, q), \quad C(\mathcal{M}_i) = \left[ \frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right] \setminus \mathcal{M}_i,
\]

where \( i = 1, 2 \) and

\[
\mathcal{M}_i(a, q) = \left\{ \alpha_i : \left| \alpha_i - \frac{a}{q} \right| \leq \frac{1}{qQ_i} \right\}.
\]

It follows from the definition of \( P_i \) and \( Q_i \) that the arcs \( \mathcal{M}_1(a, q) \) and \( \mathcal{M}_2(a, q) \) are mutually disjoint, respectively. We further define

\[
\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 = \{ (\alpha_1, \alpha_2) : \alpha_1 \in \mathcal{M}_1, \alpha_2 \in \mathcal{M}_2 \}, \quad C(\mathcal{M}) = \left[ \frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right] \setminus \mathcal{M}.
\]

As in [10], let \( \delta = 10^{-4} \) and

\[
U_i = \left( \frac{N_i}{16(1 + \delta)} \right)^{1/3}, \quad V_i = U_i^{5/16}
\]

for \( i = 1, 2 \). We set

\[
f(a_i, N_i) = \sum_{p \leq \sqrt{N_i}} (\log p) e(p a_i), \quad g(a_i, N_i) = \sum_{p \leq N_i^{1/2}} (\log p) e(p^2 a_i),
\]

(2.2)
\[ S(\alpha_i, U_i) = \sum_{p - U_i} (\log p) e(p^3\alpha_i), \quad T(\alpha_i, V_i) = \sum_{p - V_i} (\log p) e(p^3\alpha_i), \] (2.3)

and set
\[ G(\alpha_i) = \sum_{v < L} e(2^v\alpha_i), \quad \delta_\lambda = \{ \alpha_i \in [0, 1] : |G(\alpha_i)| \geq \lambda L \}, \] (2.4)

for \( i = 1, 2. \)

Let
\[ R(N_1, N_2) = \sum \log p_1 \log p_2 \ldots \log p_{8} \]
be the weighted number of solutions of (1.5) in \((p_1, \ldots, p_8, v_1, \ldots, v_8)\) with

\[ p_1 \leq N_1, \quad p_2 \leq N_1^{1/2}, \quad p_4 \sim U_1, \quad p_5 \sim V_1, \quad p_5 \leq N_2, \quad p_7 \sim U_2, \quad p_8 \sim V_2, \quad v_j \leq L, \]

for \( j = 1, 2, \ldots, k. \) Then, \( R(N_1, N_2) \) can be written as follows:

\[ \left\{ \iiint_{C(\alpha_i)} + \iiint_{C(\alpha_i) \backslash \delta_\lambda} \right\} f(\alpha_i, N_1) g(\alpha_i, N_1) S(\alpha_i, U_1) T(\alpha_i, V_1) f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) \]
\[ \times T(\alpha_2, V_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) da_1 da_2 = R_1(N_1, N_2) + R_2(N_1, N_2) + R_3(N_1, N_2). \]

We will establish Theorem 1.1 by estimating the term \( R_1(N_1, N_2), R_2(N_1, N_2), \) and \( R_3(N_1, N_2). \) We need to show that \( R(N_1, N_2) > 0 \) for every pair of sufficiently large positive even integers \( N_2 \gg N_1 > N_2. \)

Let
\[ \sum_{h=1}^{q} e \left( \frac{ah^i}{q} \right) \]
for \( i = 1, 2, 3, 4. \) Then, we write
\[ B(n, q) = \sum_{(a, q) = 1}^{q} C_1(q, a) C_2(q, a) C_3(q, a) C_4(q, a) e \left( -\frac{an}{q} \right), \quad A(n, q) = \frac{B(n, q)}{q^4(q)}, \quad \mathcal{S}(n) = \sum_{q = 1}^{\infty} A(n, q). \]

Now, we need to quote two lemmas from Hu and Yang [10, Lemmas 2.3 and 2.4] as follows:

**Lemma 2.1.** For all integers \( n \equiv 0 \pmod 2, \) we have \( \mathcal{S}(n) \geq 0.2448. \)

**Lemma 2.2.** Let \( \mathcal{S}(N_1, k) = \{ n_1 \geq 2 \cap n_1 = N_1 - 2^n \cdots - 2^n \} \) with \( k \geq 2. \) Then, for \( N_1 \equiv N_2 \equiv 0 \pmod 2, \) we have
\[ \sum_{n_1 \in \mathcal{S}(N_1, k), n_2 \in \mathcal{S}(N_2, k), \atop n_1 \equiv n_2 \pmod 2} J(n_1) J(n_2) \geq 5.4671 N_1^{10/9} N_2^{10/9} L^k, \]

where \( J(n) \) is defined as
\[ J(n) = \sum_{m_x, m_y, m_z, m_w, m_r, m_s, m_t, m_u, m_v, m_w, m_r, m_s, m_t, m_u, m_v} m_2^{-1/2} (m_1 m_2 m_3 m_4 m_5 m_6 m_7 m_8 m_9) ^{-2/3}. \]

**Lemma 2.3.** For every pair of sufficiently large positive even integers \( N_1 \) and \( N_2 \) satisfying \( N_2 \gg N_1 > N_2, \)
\[ R_3(N_1, N_2) \geq 0.00101 N_1^{10/9} N_2^{10/9} L^k. \]
Proof. By Hu and Yang [10, Lemma 2.2] we note that

\[
R_i(N_i, N_2) = \iiint_{\mathcal{D}} f(a_1, N_i) g(a_1, N_i) S(a_1, U_1) T(a_1, V_1) f(a_2, N_2) g(a_2, N_2) S(a_2, U_2) T(a_2, V_2)
\]
\[
\times G_k(a_1 + a_2) e(-a_1 N_1 - a_2 N_2) d\alpha_1 d\alpha_2
\]
\[
\geq \left(\frac{1}{2\pi}\right)^2 \sum_{n_1 \in \mathcal{D}(N_{i,k})} \sum_{n_2 \in \mathcal{D}(N_{p,k})} \mathbb{S}(n_1) \mathbb{S}(n_2) f(n_1) f(n_2).
\]

Then, using Lemmas 2.1 and 2.2 the lemma now follows. \qed

Lemma 2.4. We have

\[
\text{meas}(\delta_1) < N_1^{-E(L)}
\]

with \(E(0.9322) > 845/1008 + 10^{-10}\).

Proof. According to Hu and Yang [12, Lemma 5.1], the following lemma can be calculated by computer. \qed

Lemma 2.5. For every pair of sufficiently large positive even integers \(N_1\) and \(N_2\) satisfying \(N_2 \gg N_1 > N_2\),

\[
R_2(N_1, N_2) \ll N_1^{10/9} N_2^{10/9} L^{k-1}.
\]

Proof. By the definition of \(C(.\mathcal{D})\), we have

\[
C(.\mathcal{D}) \subset \{(a_1, a_2) : a_1 \in C(.\mathcal{D}_1), a_2 \in [0, 1]\} \cup \{(a_1, a_2) : a_1 \in [0, 1], a_2 \in C(.\mathcal{D}_2)\}.
\]

Then, \(R_3(N_1, N_2)\) is

\[
\mathbb{S}(a_2, U_2) T(a_2, V_2) G_k(a_1 + a_2) e(-a_1 N_1 - a_2 N_2) d\alpha_1 d\alpha_2
\]
\[
\ll L^k \left\{ \iiint_{C(\mathcal{D}_1) \times [0,1]} + \iiint_{C(\mathcal{D}_2) \times [0,1]} \right\} f(a_1, N_1) g(a_1, N_1)
\]
\[
\times S(a_2, U_2) T(a_2, V_2) f(a_2, N_2) g(a_2, N_2) S(a_2, U_2) T(a_2, V_2) d\alpha_1 d\alpha_2
\]
\[
= L^k \left\{ \iiint_{1} + \iiint_{2} \right\},
\]

where we use the trivial bound of \(G(a_1 + a_2)\). We note that

\[
\iiint_{1} \ll \iiint_{C(\mathcal{D}_1) \times [0,1]} [f(a_1, N_1) g(a_1, N_1) S(a_1, U_1) T(a_1, V_1) f(a_2, N_2) g(a_2, N_2) S(a_2, U_2) T(a_2, V_2)] d\alpha_1 d\alpha_2
\]
\[
\ll N_1^{1065}/1008 + e \iiint_{C(\mathcal{D}_2) \times [0,1]} [f(a_2, N_2) g(a_2, N_2) S(a_2, U_2) T(a_2, V_2)] d\alpha_1 d\alpha_2,
\]

where we use the bound from [10, Lemma 2.5] and [13, Lemma 2.5] as follows:
By using the integral transformation of $\beta = a_1 + a_2$ and the periodicity of $G(\beta)$, we have

$$
\iint_{\{(a_1, a_2) \in [0, 1]^2 \mid G(a_1 + a_2) > ML\}} |f(a_1, N_1)| \, da_1 \, da_2 \\
= \frac{1}{2 \pi} \int_{-\pi}^{\pi} |f(a_1, N_1)| \, da_1 \, da_2.
$$

By the simple orthogonality and [14, Lemma 4.12], we have

$$
\int_0^1 |f^2(a_2, N_2)| \, da_2 \ll N_2^{1/2 + \epsilon}
$$

and

$$
\int_0^1 |g^2(a_2, N_2) S^2(a_2, U_2) T^2(a_2, V_2)| \, da_2 \ll N_2^{1/9 + \epsilon}.
$$

From Lemma 2.4 and Cauchy's inequality, we have

$$
\iint_{\{(a_1, a_2) \in [0, 1]^2 \mid G(a_1 + a_2) > ML\}} |f(a_1, N_1)| \, da_1 \, da_2 \ll N_1^{10/9 + \epsilon} N_2^{10/9} E(L).
$$

We choose $\lambda = 0.932$ and get

$$
\iint_1^2 \ll N_1^{1965/1008 - 845/1008 - \epsilon} N_2 \ll N_1^{10/9 - \epsilon} N_2^{10/9},
$$

since $N_2 \gg N_1$. Similarly,

$$
\iint_2^3 \ll N_2^{1965/1008 - 845/1008 - \epsilon} N_1 \ll N_2^{10/9 - \epsilon} N_1^{10/9}.
$$

Then,

$$
R_2(N_1, N_2) \ll N_2^{10/9 - \epsilon} N_2^{10/9} L^k + N_1^{10/9} N_2^{10/9 - \epsilon} L^k \ll N_1^{10/9} N_2^{10/9} L^{k-1}.
$$

Lemma 2.6. For every pair of sufficiently large even integers $N_1$ and $N_2$ satisfying $N_2 \gg N_1$, $N_2$,

$$
R_2(N_1, N_2) \leq 260.757 N_1^{10/9} N_2^{10/9} L^k.
$$

Proof. First, we need to estimate

$$
I = \iint_{\{(a_1, a_2) \in [0, 1]^2 \mid G(a_1 + a_2) > ML\}} |g^4(a_1, N_1) g^4(a_2, N_2) G^{20L}(a_1 + a_2)| \, da_1 \, da_2.
$$

Following the lines in [14], we note that
\[ I = \sum_{h \in \mathbb{Z}} r_{i4}(h) \sum_{M_k|p^i, p_j^k \leq N_k} \prod_{i=1}^{8} \log p_i, \]

where

\[ r_{i4}(h) = \sum_{4 \leq \nu \leq \mu \leq L} 1. \]

Let

\[ S(h) = \prod_{p > 2} \left( 1 + \frac{B(p, h)}{(p - 1)^6} \right), \]

where

\[ B(p, h) = \sum_{a=1}^{\varphi(p)} |C_2(p, a)|^6 e(ah/p). \]

As in the proof of [14, Lemma 3.2], we treat the case \( h \neq 0 \) and \( h = 0 \) separately and obtain

\[ I \leq 128^3 N_1 N_2 \sum_{h \neq 0} r_{i4}(h)S^2(h) + O(N_1 N_2 L^{20}). \]

Next, we estimate \( \sum_{h \neq 0} r_{i4}(h)S^2(h) \). Note that

\[ B(p, h) = \begin{cases} 
- (p + 1)^2, & \text{if } p \equiv 3 \mod 4 \text{ and } p|h, \\
- (p^2 + 6p + 1) - 4p(p + 1)\left(\frac{h}{p}\right), & \text{if } p \equiv 1 \mod 4 \text{ and } p|h, \\
(p - 1)(p + 1)^2, & \text{if } p \equiv 3 \mod 4 \text{ and } p|h, \\
(p - 1)(p^2 + 6p + 1), & \text{if } p \equiv 1 \mod 4 \text{ and } p|h. 
\end{cases} \]

Then, from the proof of [14, Lemma 4.3] we have

\[ S^2(h) \leq 8.54 \kappa^2(h) \prod_{p \geq 5} \frac{1 + \frac{b(p)}{(p - 1)^\gamma}}{1 + \frac{a(p)}{(p - 1)^\gamma}}. \]

where

\[ \kappa(h) = \begin{cases} 
\frac{25 + 15\left(\frac{h}{5}\right)}{32}, & \text{if } 3|h, 5|h, \\
3, & \text{if } 15|h, \\
\frac{3}{2}, & \text{if } 3|h, \\
0, & \text{if } 3|h. 
\end{cases} \]

\[ a(p) = \begin{cases} 
-(p + 1)^2, & \text{if } p \equiv 3 \mod 4, \\
3p^2 - 2p - 1, & \text{if } p \equiv 1 \mod 4, 
\end{cases} \]

\[ b(p) = \begin{cases} 
(p - 1)(p + 1)^2, & \text{if } p \equiv 3 \mod 4, \\
(p - 1)(p^2 + 6p + 1), & \text{if } p \equiv 1 \mod 4. 
\end{cases} \]
Define the multiplicative function $c(d)$ by

$$1 + \frac{1}{c(p)^2} = \left(1 + \frac{b(p)}{(p - 1)^a} \right)^2,$$

where $d$ is square-free and $(30, d) = 1$. Then,

$$\sum_{h \neq 0} r_{15}(h) S^2(h) \leq 8.54 \sum_{h \equiv 0 \mod 3} 2 \left( \frac{25 + 15b(h)}{32} \right)^2 \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)} + 8.54 \sum_{h \equiv 0 \mod 15} 2 \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)}.$$

$$= 8.54 \sum_{h \equiv 0 \mod 3} \left( \frac{25 + 15b(h)}{32} \right)^2 \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)} + 8.54 \sum_{h \equiv 0 \mod 15} 2 \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)} = 7.09 \sum_{h \equiv 0 \mod 3} \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)} + 14 \sum_{h \equiv 0 \mod 15} \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)}$$

$$= 7.09 \sum_{h \equiv 0 \mod 3} \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)} + 14 \sum_{h \equiv 0 \mod 15} \prod_{p \mid h, p > 5} 1 + \frac{1}{c(p)} = 7.09 \Sigma_1 + 14 \Sigma_2,$$

where the condition $(h)$ in $\sum_{h \equiv 0 \mod 3}$ denotes that the summation is taken over all $v_1, \ldots, v_{14}, \mu_1, \ldots, \mu_{14}$ satisfying $4 \leq v_j, \mu_j \leq L$ and $h = \sum_{j=1}^{14} (2^{v_j} - 1) \neq 0$.

Let us consider $\Sigma_1$. We have

$$\Sigma_1 \leq \sum_{d \mid N^e, \nu(d) \leq L} \frac{\nu^2(d)}{c(d)} \sum_{1 \leq \nu, \mu \leq L} 1 + O(N^e)$$

$$\leq L^{28} \sum_{d \mid N^e, \nu(d) \leq L} \frac{\nu^2(d)}{c(d)} \sum_{q(3d)} 1 + O(e)L^{28} + O(N^e)$$

$$\leq (c_1 + e)L^{28},$$

where $q(q)$ denotes the smallest positive integer $q$, such that $2^{q(q)} \equiv 1 \mod q$ and $c_1$ is a constant, which we will deal with later. Similarly, $\Sigma_2 \leq (c_2 + e)L^{28}$, where $c_2$ is a constant. Set

$$\beta(d) = \left( \frac{1}{q(3d)^{28}} \sum_{1 \leq \nu, \mu \leq L, q(3d)} 1 \right)^{-1}$$

and $m(x) = \prod_{2^e \leq x} 2^e - 1$. Then, we have

$$\sum_{p \mid d, p < 5 \atop \beta(d) \leq x} \frac{\nu^2(d)}{c(d)} \leq \prod_{p \mid m(x)} \left( 1 + \frac{1}{c(p)} \right) \leq \prod_{p \mid m(x)} \left( 1 + \frac{1}{c(p)} \right)^2 \prod_{p \mid m(x)} \left( 1 + \frac{1}{c(p)} \right)^2 \leq \left( \frac{8}{15} \right)^2 c_3 e^{2\gamma} \log^2 x,$$
where we use the fact  
$\frac{m(x)}{\phi(m(x))} \leq e^y \log x$ for $x \geq 9$. Here, $c_3 = \prod_{p \leq 5} \frac{1 - \frac{1}{p-1}}{\frac{1}{p-1}} \leq 1.3904^2$ can be found in the proof of [14, Lemma 4.1]. With $M = 35$, we have

$$c_1 = \left(1 + \sum_{\substack{d \leq 5 \leq \mu(d)}} \sum_{\substack{p \leq 5 \leq \mu(d)}} \frac{\mu^2(d)}{x^2} \frac{\phi(m(x))}{x^2} \leq \sum_{\substack{\beta(d) \leq 5 \leq \mu(d)}} \frac{\mu^2(d)}{x^2} \left(1 - \frac{1}{\beta(d)} - 1 \right) + \left( \frac{8}{15} \right)^2 c_3 e^{x_2} \frac{1 + \log M}{M} \leq 1.12031.$$

The constant $c_2 \leq 1.10302$ can be handled in the similar way. Then, the numerical computations provide

$$\sum_{h \neq 0} r_1(h)S^2(h) \leq 23.39L^8.$$

So we get

$$I \leq 3.83 \times 10^5 N_1 N_2 L^8 + O(N_1 N_2 L^6).$$

Next, by [11, Lemma 2.3] and [9, Lemma 2.5] we have

$$\iint_{(a_1, a_2) \in [0, 1]^2} |f^2(a_1, N_1) f^2(a_2, N_2) G^4(a_1 + a_2)| \, da_1 \, da_2 \leq 305.8869 N_1 N_2 L^4$$

and

$$\iint_{(a_1, a_2) \in [0, 1]^2} |S^4(a_1, U_1) T^4(a_1, V_1) S^4(a_2, U_2) T^4(a_2, V_2)| \, da_1 \, da_2 \leq 0.1289 N_1^{\frac{13}{9}} N_2^{\frac{13}{9}}$$

Thus,

$$R_1(N_1, N_2) \leq (AL)^{-9} \iint_{(a_1, a_2) \in [0, 1]^2} \left| f(a_1, N_1) g(a_1, N_1) S(a_1, U_1) T(a_1, V_1) f(a_2, N_2) g(a_2, N_2) \right| \, da_1 \, da_2$$

$$\leq (AL)^{-9} \left( \iint_{(a_1, a_2) \in [0, 1]^2} \left| f^2(a_1, N_1) f^2(a_2, N_2) G^4(a_1 + a_2) \right| \, da_1 \, da_2 \right)^{\frac{1}{2}}$$

$$\times \left( \iint_{(a_1, a_2) \in [0, 1]^2} \left| g^4(a_1, N_1) g^4(a_2, N_2) G^8(a_1 + a_2) \right| \, da_1 \, da_2 \right)^{\frac{1}{4}}$$

$$\times \left( \iint_{(a_1, a_2) \in [0, 1]^2} \left| S^4(a_1, U_1) T^4(a_1, V_1) S^4(a_2, U_2) T^4(a_2, V_2) \right| \, da_1 \, da_2 \right)^{\frac{1}{4}}$$

$$\leq 260.757 A^{k-9} N_1^{10/9} N_2^{10/9} L^k.$$

Combining Lemmas 2.3, 2.5, and 2.6, we obtain

$$R(N_1, N_2) > 0.00101 N_1^{10/9} N_2^{10/9} L^k - 260.757 A^{k-9} N_1^{10/9} N_2^{10/9} L^k.$$

We therefore solve the inequality

$$R(N_1, N_2) > 0$$
and get $k \geq 187$. Consequently, we deduce that every pair of large even integers $N_1$, $N_2$ satisfying $N_2 \gg N_1 > N_1$ can be written in the form of (1.5) for $k \geq 187$. Thus, Theorem 1.1 follows.

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