THE PRODUCT OF SIMPLE MODULES OVER KLR ALGEBRAS AND QUIVER GRASSMANNIANS

YINGJIN BI

Abstract. In this paper we study the product of two simple modules over KLR algebras using the quiver Grassmannians for Dynkin quivers. More precisely, we establish a bridge between the Induction functor on the category of modules over KLR algebras and the irreducible components of quiver Grassmannians for Dynkin quivers via a sort of extension varieties, which is an analogue of the extension group in Hall algebras. As a result, we give a necessary condition when the product of two simple modules over a KLR algebra is simple using the set of irreducible components of quiver Grassmannians. In particular, in some special cases, we provide a proof for the conjecture recently proposed by Lapid and Minguez.

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1. Introduction

In this paper we will establish a bridge between the representations of a Dynkin quiver $Q$ and the Induction functor on the category of modules of the KLR algebra $R_Q$ associated with $Q$. The motivation of this paper is from the common quantum cluster structure on a subcategory of modules of $R_Q$ and on a subcategory of representations of the preprojective algebra $\Lambda_Q$ (see [KKKO18] and [GLS13]). The relation between modules of $R_Q$ and that of $\Lambda_Q$ is an important topic in cluster algebras, as the former category is a monoidal category, and the latter category is an additive category. However, there is no direct connection between these two fields.

Before stating this, let us review some previous works concerning this topic. We firstly introduce the main notations in this paper. Let us consider two elements $\alpha, \beta$ in root lattice $Q^+ := \mathbb{N}[\alpha_i]_{i \in I}$ and denote by $\text{KP}(\alpha)$ (resp: $\text{KP}(\beta)$) the set of Konstant partitions associated with the isomorphic class (up to shifted) of simple modules over the algebra $R_\alpha$ (resp: $R_\beta$): e.t. the $\alpha$ (resp: $\beta$)-part of $R_Q$. We remark that $\text{KP}(\alpha)$ (resp: $\text{KP}(\beta)$) can be thought of as the set of isomorphic class of representations of $Q$ with dimension $\alpha$ (resp: $\beta$). Denote by $E_\alpha$ (resp: $E_\beta$) the representation space of $Q$ with dimension $\alpha$ (resp: $\beta$). The set $\text{KP}(\alpha)$ coincides with the set of $G_\alpha$-orbits in $E_\alpha$ (see Section 2.1 for more details). Denote by $L(\lambda)$ the simple modules corresponding to the Konstant partition $\lambda \in \text{KP}(\alpha)$.

**Problem 1.1.** Let $\alpha, \beta$ be two elements in $Q^+$ and $(\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta)$ be a pair of Konstant partitions. Describe the pair $(\mu, \nu)$ such that the Induction $L(\mu) \circ L(\nu)$ of the simple modules $L(\mu)$ and $L(\nu)$ is also a simple module.

This question is proposed by Kleshchev and Ram in [KR11, Problem 7.6]. But they used the words of the set of vertices of $Q$ to describe this condition rather than Konstant partitions. This problem is quite important: Since the product $L(\mu) \circ L(\nu)$ can be regarded as the product of two dual canonical bases in the quantum coordinate ring $A_q(n)$ via the categorification of quantum groups, this problem means that describing the pair $(\mu, \nu)$ when $b^*_\mu b^*_\nu$ is also a dual canonical base (up to some $q$-power scaling), where $b^*_\mu$ (resp: $b^*_\nu$) is the dual canonical base associated with $\mu$ (resp: $\nu$). Namely, $b^*_\mu b^*_\nu \in q\mathbb{Z}B^*$. This problem is closely related to [BZ92, Conjecture 1.7]. That is: $b^*_\mu b^*_\nu \in q\mathbb{Z}B^*$ if and only if $b^*_\mu b^*_\nu = q^n b^*_\nu b^*_\mu$ for some integer $n$. But Leclerc shows that this conjecture is not true in general, (see [Lec03]), and then propose the notion of real dual canonical bases $b^*$: e.t. $b^*b^* \in q\mathbb{Z}B^*$. Therefore, it is quite difficult to give an equivalent condition for $b^*_\mu b^*_\nu \in q\mathbb{Z}B^*$. For instance, classification of real dual canonical bases is a difficult open problem until now. Recently, in type $A$, Lapid
and Minguez propose [LM22, Conjecture 3.1 C] to describe the real dual canonical bases in terms of the notion of rigid modules over the preprojective $\Lambda_Q$. There is no related results concerning the condition $b_\mu^* b_\nu^* \in q\mathbb{Z}^*$. In [KKKO18] Kang, Kashiwara, Kim and Oh proved this conjecture using the notion of $R$-matrix on the category of modules over KLR algebras (or, quiver Hecke algebras) and [McN14, Lemma 7.5] given by McNamara, (see [KKKO18, Corollary 4.1.2]). More precisely, they show that the self-dual simple module $L'$ (resp: $L''$) corresponding to $b_\mu^*$ (resp: $b_\nu^*$) is the head (resp: socle) of the module $L(\mu) \circ L(\nu)$.

Based on this result, Lapid and Minguez propose the [LM22, Conjecture 5.1]. That is: the socle $L''$ is the simple module with the Konstant partition $\mu \star \nu$. Here $\mu \star \nu$ refers to the generic extension of Lusztig’s nilpotent variety $\Lambda_\mu$ and $\Lambda_\nu$. In other words, they provide more information about Conjecture 1.2 and interpret $b''^*$ in terms of the modules of the preprojective algebra $\Lambda_Q$. They essentially provide a clue of the relation between modules over the KLR algebra $R_Q$ and modules over the preprojective algebra $\Lambda_Q$. We remark here that they use the language of representations of general linear group over non-Archimedean local field. Similarly, we can consider the modules over quantum affine groups, as they are both the categorification of quantum coordinate ring $A_q(n)$, (see [Fu20]).

As we pointed out before, there is no direct connection between modules over $R_Q$ and modules over $\Lambda_Q$. A naive idea is to use the representations of $Q$. Because Ringel has given an interpretation of modules of $\Lambda_Q$ in terms of representations of quiver $Q$ (see [Rin98]), and Varagnolo and Vasserot gave a geometric description of KLR algebras in terms of Lusztig’s category on representation spaces for $Q$ (see [VV11]). It seems like there is only the representations of $Q$ to connect the above two categories. In fact, in type $A$, Lapid and Minguez already considered this approach in a series of their works (see [LM18], [LM22], etc.). They use the language of representations of general linear groups over non-Archimedean fields and propose a lot of Conjectures on this topic (see also [LM22]). They call this approach as a down-to-earth approach.

The goal of the present paper is to establish a bridge between representations of $Q$ and modules of $R_Q$. The relations between representations of $Q$ and modules of $\Lambda_Q$ will be given in another paper [Bi21]. More precisely, we will answer Problem 1.1 in terms of the irreducible
components of quiver Grassmannians for the quiver $Q$. We will also give a sufficient condition when $q^nL(\mu \cdot \nu) = \text{soc} L(\mu) \circ L(\nu)$ for some integer $n$ in terms of the irreducible components of quiver Grassmannians. In other words, we give a proof for [LM22, Conjecture 5.1] in some special cases. Let me explain this more explicitly.

First, let $\gamma \in Q^+$ and a partition $\gamma = \alpha + \beta$, we have that the set $\text{KP}(\gamma)$ is identified with the set of $G_\gamma$-orbits in $E_\gamma$ in Dynkin cases. We define a partial ordering on $\text{KP}(\gamma)$ so that $\lambda' \leq \lambda$ if and only if $O_\lambda \subset O_{\lambda'}$ where $O_\lambda$ refers to the $G_\gamma$-orbit corresponding to $\lambda$ in $E_\gamma$ and $\overline{O}_\lambda$ refers to closure of the orbit $O_\lambda$ (see [CB93] for more details). For a pair $(\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta)$, one defines $(\mu', \nu') \leq (\mu, \nu)$ by $\mu' \leq \mu$ and $\nu' \leq \nu$.

Let us define the quiver Grassmannian of the representation $M_\lambda$ associated with $\lambda \in \text{KP}(\gamma)$. Roughly speaking, it is a variety classifying the subrepresentations of $M_\lambda$ with dimension vector $\beta$. Namely,

$$\text{Gr}_\beta(M_\lambda) = \{ W \in \text{Gr}(\beta, \gamma) \mid \text{such that } M_\lambda(W) \subset W \}$$

where $\text{Gr}(\beta, \gamma) = \prod_{i=1}^{n} \text{Gr}(\beta_i, \gamma_i)$ is the product of Grassmannians of the vector space $\mathbb{C}^{\gamma_i}$ with dimension $\beta_i$. Here we write $\beta = \sum_{i \in I} \beta_i \alpha_i$ and $\gamma = \sum_{i \in I} \gamma_i \alpha_i$. It is well-known that $\text{Gr}_\beta(M_\lambda)$ is a projective variety.

Let us consider the set of the irreducible components of $\text{Gr}_\beta(M_\lambda)$: Following [CER12, Lemma 2.4], we have

$$\text{Gr}_\beta(M_\lambda) = \bigcup_{(\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta)} \text{Gr}(\mu, \nu, \lambda)$$

where $\text{Gr}(\mu, \nu, \lambda)$ refers to the subvariety consisting of $W \in \text{Gr}_\beta(M_\lambda)$ such that the restriction of $M_\lambda$ to $W$ is isomorphic to $M_\nu$, e.t. $(M_\lambda)_W \cong M_\nu$ and the restriction of $M_\lambda$ to the quotient space $V/W$ is isomorphic to $M_\mu$, namely, $(M_\lambda)_{V/W} \cong M_\mu$, and $\text{Gr}(\mu, \nu, \lambda)$ refers to its Zarisky closure.

In order to describe the set of irreducible components of $\text{Gr}_\beta(M_\lambda)$, we introduce the notion of generic pair. Following [CFR13, Definition 7.3], we call $(\mu, \nu)$ a generic pair of $\lambda$ if there exists no $\nu' < \nu$ such that $(M_\lambda)_W \cong M_{\nu'}$ and $(M_\lambda)_{V/W'} \cong M_{\mu}$ for some subspace $W' \in \text{Gr}_\beta(M_\lambda)$ and there exists no $\mu' < \mu$ such that $(M_\lambda)_W \cong M_{\nu'}$ and $(M_\lambda)_{V/W'} \cong M_{\mu'}$ for some subspace $W'' \in \text{Gr}_\beta(M_\lambda)$. Thanks to Professor Cerulli Irelli and Reineke, we have the following Lemma

**Lemma 1.3.** [Cerulli Irelli and Reineke] Under the above assumption, the set of irreducible components of $\text{Gr}_\beta(M_\lambda)$ is identified with the following set

$$\text{ext}^\text{ger}_{\alpha, \beta}(\lambda) = \left\{ (\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta) \mid \text{such that } (\mu, \nu) \text{ is a generic pair of } \lambda \text{ and } \dim \text{Hom}_Q(M_\nu, M_\lambda) = \dim \text{Hom}_Q(M_\nu, M_\mu) + \dim \text{Hom}_Q(M_\nu, M_\mu) \right\}$$

We denote this set by $\text{ext}^\text{ger}_{\alpha, \beta}(\lambda)$. Here we introduce the main notion of this paper
Definition 1.4. We call a pair \((\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta)\) for a support pair if for any \(\lambda < \mu \oplus \nu\) we have \((\mu, \nu) \notin \text{ext}_{\alpha, \beta}^{\text{ger}}(\lambda)\). This notion is inspired by the support of the image of simple perverse sheaf IC(\(\lambda\)) for \(\lambda\) under the Restriction functor.

Here is the main result in this paper.

Theorem 1.5. Let \(L(\mu)\) and \(L(\nu)\) be two simple modules of \(R_Q\) where \(\mu, \nu\) are their Konstant partitions, respectively. If \(L(\mu) \circ L(\nu)\) is a simple module, then \((\mu, \nu)\) is a support pair. Namely, for any non-trivial extension \(\lambda \in \text{ext}(\mu, \nu)\), we have
\[
\dim \text{Hom}_Q(M_{\nu}, M_{\mu} \oplus M_{\nu}) > \dim \text{Hom}_Q(M_{\nu}, M_{\lambda})
\]
and
\[
\dim \text{Hom}_Q(M_{\mu}, M_{\mu} \oplus M_{\nu}) > \dim \text{Hom}_Q(M_{\mu}, M_{\lambda})
\]

Remark 1.6. This theorem reveals a relation between representations of \(Q\) and induction of two simple modules of \(R_Q\). We see that the condition \(L(\mu) \circ L(\nu)\) is a simple module is a quite subtle condition. It is difficult to describe this condition in general.

Let us go back to Conjecture 1.2. Based on [LM22, Conjecture 5.1] introduced by Lapid and Minguez, we consider the generic extension \(\mu \ast \nu\) of any pair \((\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta)\). Namely, the module \(M_{\mu \ast \nu}\) satisfies \(M_{\mu \ast \nu} \in \text{Ext}^1_Q(M_{\mu}, M_{\nu})\) and \(\dim \text{Ext}^1_Q(M_{\mu \ast \nu}, M_{\mu \ast \nu})\) is minimal with respect to elements in \(\text{Ext}^1_Q(M_{\mu}, M_{\nu})\). We remark that \(\mu \ast \nu\) always less than or equal to \(\mu \ast \nu\) in the sense of Lapid and Minguez, but \(\mu \ast \nu = \mu \ast \nu\) holds in the most of the examples considered in [LM22]. Therefore, the following theorem gives an answer to this conjecture in some special cases.

Theorem 1.7. Under the above assumption, if \((\mu, \nu) \in \text{ext}^\text{ger}_{\alpha, \beta}(\mu \ast \nu)\), then \(q^n L(\mu \ast \nu)\) is a submodule of \(L(\mu) \circ L(\nu)\) for some integer \(n \in \mathbb{Z}\).

Remark 1.8. The difficulty is to connect the representations of \(Q\) and modules of \(\Lambda_Q\). After obtaining this connection, it is possible to give an answer to that conjecture. Although this theorem is not good enough to prove [LM22, Conjecture 5.1], this is the first attempt to prove this conjecture. Please see Example 7.21.

The strategy used in this paper

1. Firstly, we transform the product \(L(\lambda) \circ L(\mu)\) into the Induction functor on the indecomposable projective modules over \(R_Q\): \(\text{ind}^{\alpha+\beta}_{\alpha, \beta} \mathbb{P} \nu\) via the bilinear paring [KL09, Section 2.5].

2. Secondly we transform \(\text{ind}^{\alpha+\beta}_{\alpha, \beta} \mathbb{P} \nu\) into the Induction functor on simple perverse sheaves on representation space: \(\text{Res}^{\alpha+\beta}_{\alpha, \beta} \text{IC}(\nu)\) via the results in [VV11] and [McN17].

3. Thirdly, we study the geometric properties of functor \(\text{Res}^{\alpha+\beta}_{\alpha, \beta} \text{IC}(\nu)\) given in [Sch09], [Lus91] and [Lus93] using the graded quiver varieties as in [LP13] and [VV03].

4. At last we connect the \(\text{Res}^{\alpha+\beta}_{\alpha, \beta} \text{IC}(\nu)\) with quiver Grassmannians given in [Rei01] and [CER12].
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2. Premise

In this section, we recall some basic facts on quiver representation theory for Dynkin quivers.

2.1. Notations. In this paper, a quiver $Q = (I, \Omega)$ consists of a set of vertices $I$ and a set of arrows $\Omega$. Denote by $n$ the number of vertices if there is no danger of confusion. For an arrow $h$, one denotes by $s(h)$ and $t(h)$ its source and target, respectively. For our purpose, we often assume that $Q$ is of Dynkin type. It follows that $I$ endows with an order such that if there exists an arrow from $i_k$ to $i_l$ then $k < l$. $\Omega$ gives rise to a symmetric matrix $C_Q$, simply write $C$, as follows.

\begin{equation}
(2.1) \quad a_{i,j} = \begin{cases} 
2 & \text{if } i = j \\
\#\{h : i \rightarrow j\} - \#\{h : j \rightarrow i\} & \text{otherwise}
\end{cases}
\end{equation}

We write $\alpha_i$ for its simple roots and let $Q = \mathbb{Z}[\alpha_i]_{i \in I}$ be its root lattice. The matrix $C$ induces a bilinear form on $Q$, we write $(-, -)$ for it. We denote $Q^+ \subset Q$ as the positive lattice. Set a bilinear form on $Q^+$ by

\begin{equation}
(2.2) \quad \langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i - \sum_{h \in \Omega} \alpha_{s(h)} \beta_{t(h)}
\end{equation}

It is well known that $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$. We write $W$ for its Weyl group generated by simple reflections $s_i$ associated with simple roots $\alpha_i$ for any $i \in I$. For any $w \in W$, we write $l(w)$ for the length of $w$. Denote by $R^+$ the set of positive roots. The ground field in this paper is the complex number field $\mathbb{C}$, we sometimes write $k$ for simplicity.

Let $w_0 \in W$ be the longest element in $W$. For a reduced expression of $w_0 = s_{i_m} \cdots s_{i_2} s_{i_1}$ which is adapted with $\Omega$, we obtain the roots as follows:

\begin{equation}
(2.3) \quad \alpha_{i_1}; s_{i_1}(\alpha_{i_2}); \cdots; s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}); \cdots; s_{i_1} s_{i_2} \cdots s_{i_{m-1}}(\alpha_{i_m})
\end{equation}

Let us denote them by $\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})$ for $1 \leq k \leq m$ and give them an order so that $\beta_l < \beta_k$ if and only if $l < k$. In other words, we give an ordering on $R^+$.

Representations of Dynkin quivers. Let $M = (V, x)$ be a representation of $Q$ where $V$ is a $I$-graded vector space and $x = (x_h)_{h \in \Omega}$ is a tuple of matrices $x_h : V_{s(h)} \rightarrow V_{t(h)}$. We sometimes write the matrices $x$ for the representation $M$ if there is no danger of confusion. For a representation $M = (V, x)$, we write $\dim M$ for its dimension $\sum_{i \in I} \dim(V_i) \alpha_i \in Q^+$. 
Given two representations \( M, N \) of \( Q \), we denote by \( \text{Hom}_Q(M, N) \) the vector space of \( Q \)-morphisms between \( M \) and \( N \) and write \([M, N]\) (resp: \([M, N]^1\)) for \( \dim(\text{Hom}_Q(M, N)) \) (resp: \( \dim(\text{Ext}^1_Q(M, N)) \)). One has

\[
\langle \dim M, \dim N \rangle = [M, N] - [M, N]^1
\]

For any vertex \( i \in I \), we associate it with the indecomposable projective (resp: injective) representation \( P_i \) (resp: \( I_i \)), and the head (resp: socle) of \( P_i \) (resp: \( I_i \)) is the simple module \( S_i \).

Denote by \( M_\beta \) the indecomposable module corresponding to the root \( \beta \). If the order of positive roots is given as before, then

\[
\text{Hom}_Q(M_{\beta_a}, M_{\beta_b}) = \text{Ext}_Q(M_{\beta_a}, M_{\beta_b}) = 0 \quad \text{for } \beta_a < \beta_b
\]

Let \( \alpha \in Q^+ \) and \( V \) be an \( I \)-graded space such that \( \text{dim} V = \alpha \), one denotes by \( G_\alpha \) the group \( \prod_{i \in I} \text{GL}(V_i) \) and denote by \( E_\alpha \) the representation space \( \oplus_{h \in \Omega} \text{Hom}_k(V_{s(h)}, V_{t(h)}) \) endowed with a \( G_\alpha \) action by \( g \cdot x = (g_{t(h)})^{-1}x(g_{s(h)}) \). For two \( I \)-graded vector spaces \( V, W \), we denote by \( \text{Hom}_I(W, V) \) the space \( \oplus_{i \in I} \text{Hom}_k(W_i, V_i) \) and by \( \Omega_k(W, V) \) the space \( \oplus_{h \in \Omega} \text{Hom}_k(W_{s(h)}, V_{t(h)}) \).

2.2. Konstant Partitions. Let us consider the set of the \( G_\alpha \)-orbits in \( E_\alpha \), which is denoted by \( \text{KP}(\alpha) \). For an element \( \lambda \in \text{KP}(\alpha) \), we denote by \( \mathcal{O}_\lambda \) the \( G_\alpha \)-orbit corresponding to \( \lambda \), and write \( \delta_\lambda \) for the dimension of the orbit \( \mathcal{O}_\lambda \).

One defines an ordering on \( \text{KP}(\alpha) \) so that \( \lambda' \leq \lambda \) if \( \mathcal{O}_\lambda \subset \mathcal{O}_{\lambda'} \). Let \( \text{KP}(\alpha) \) be the Grothendieck group of the category of \( Q \)-representations. Denote by \( K_0(Q) \) the representations with dimension vector \( \alpha \). It is well known that \( \text{KP}(\alpha) \cong K_0(Q) \), as any \( G_\alpha \)-orbit corresponds to an isomorphic class of \( Q \). In Dykin cases, By Gabriel’s Theorem one can write any element \( \lambda \) in \( \text{KP}(\alpha) \) as

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)
\]

Where \( \lambda_k \in R^+ \) for any \( k \in [1, s] \) such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \) and \( \sum_{i=1}^s \lambda_i = \alpha \). We call \( \lambda \) as in (2.6) a Konstant partition of \( \alpha \). Denote by \( M_\mu \) the module corresponding to the Konstant partition \( \mu \in \text{KP}(\alpha) \).

Example 2.1. Let us consider the type \( A_n \), we denote by \( [1, n] \) the set of integers \( k \) such that \( 1 \leq k \leq n \). We fix an orientation of \( A_n \) as follows:

\[
1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n
\]

It is well-known that the set \( R^+ \) of positive roots is identified with the set of segments \([a, b]\) such that \( a \leq b \). The ordering of \( R^+ \) is given by

\[
[i, j] < [k, l] \text{ if } i < k \text{ or } i = k \text{ and } j < l
\]

The Gabriel’s Theorem implies that any indecomposable representations of \( A_n \) is of the form \( M[a, b] \) such that \( \text{dim} M[a, b] = \sum_{a \leq i \leq b} \alpha_i \).
For each representation $M$ of $A_n$, one can decompose every representation as

$$M \cong M[a_1, b_1] \oplus M[a_2, b_2] \oplus M[a_m, b_m]$$

for some $m \geq 1$. We arrange this tuple $([a_1, b_1], [a_2, b_2], \cdots, [a_m, b_m])$ such that $[a_k, b_k] \geq [a_{k+1}, b_{k+1}]$ for any $k \in [1, m - 1]$, which is called multisegment. It follows that for any $\alpha \in Q^+$, the set of the Konstant partitions $\text{KP}(\alpha)$ is identified with the set of multisegments $([a_1, b_1], \cdots, [a_m, b_m])$ such that $\sum_{i=1}^{m}[a_k, b_m] = \alpha$. We list some properties of indecomposable representations.

$$[M[i, j], M[k, l]] = 1 \text{ if and only if } k \leq i \leq l \leq j$$

$$[M[k, l], M[i, j]] = 1 \text{ if and only if } k + 1 \leq i \leq l + 1 \leq j.$$  

Moreover, for each $[M[k, l], M[i, j]] = 1$, we have the following short exact sequence

$$0 \to M[i, j] \to M[i, l] \oplus M[k, j] \to M[k, l] \to 0$$

where we formally set $M[i, j] = 0$ if $i < 1$ or $j > n$ or $j < i$.

For a multisegment $\lambda = ([a_1, b_1], [a_2, b_2], \cdots, [a_m, b_m])$, one sets

$$r_{i,j}(\lambda) = \{|k| \in [1, m] \mid |i, j| \subset [a_k, b_k]\}$$

where $|A|$ refers to the number of the elements in the set $A$. It is well known that for any two multisegments $\lambda', \lambda$

$$\lambda' \leq \lambda \iff r_{i,j}(\lambda') \geq r_{i,j}(\lambda) \text{ for any } (i, j)$$

2.2.1. Extension of Konstant partitions. Let $\alpha, \beta, \gamma \in Q^+$ such that $\alpha + \beta = \gamma$, let $\mu \in \text{KP}(\alpha)$ and $\nu \in \text{KP}(\beta)$. We define $\text{ext}(\mu, \nu) \subset \text{KP}(\gamma)$ as the subset of $\text{KP}(\gamma)$ consisting of $\lambda$ satisfying $M_\lambda \in \text{Ext}_Q^1(M_\mu, M_\nu)$ and set

$$\text{ext}(\alpha, \beta) := \bigcup_{\mu \in \text{KP}(\alpha), \nu \in \text{KP}(\beta)} \text{ext}(\mu, \nu)$$

Definition 2.2. For any $\lambda \in \text{KP}(\gamma)$ and a decomposition $\gamma = \alpha + \beta$, we denote by $\text{ext}_{\alpha, \beta}(\lambda)$ the subset of $\text{KP}(\alpha) \times \text{KP}(\beta)$ consisting of pairs $(\mu, \nu)$ such that $\lambda \in \text{ext}(\mu, \nu)$. Let us define the $\text{ext}_{\alpha, \beta}^{\text{min}}(\lambda)$ by

$$\text{ext}_{\alpha, \beta}^{\text{min}}(\lambda) = \{(\mu, \nu) \in \text{ext}_{\alpha, \beta}(\lambda) \mid \text{there is no pair } (\mu', \nu') \in \text{ext}_{\alpha, \beta}(\lambda) \text{ satisfying } (\mu', \nu') < (\mu, \nu)\}$$

Here by $(\mu', \nu') < (\mu, \nu)$ we mean the condition $\mu' \leq \mu$ and $\nu' < \nu$ or $\mu' < \mu$ and $\nu' \leq \nu$.

Example 2.3. Let $\lambda \in \text{KP}(\gamma)$. If the representation $M_\lambda$ admits a decomposition $M_\lambda = M_\mu \oplus M_\nu$ such that $\mu \in \text{KP}(\alpha)$ and $\nu \in \text{KP}(\beta)$, then it is easy to see that

$$(\mu, \nu) \in \text{ext}_{\alpha, \beta}^{\text{min}}(\lambda)$$

However, the elements in $\text{ext}_{\alpha, \beta}^{\text{min}}(\lambda)$ don’t satisfy $M_\lambda = M_\mu \oplus M_\nu$ in general. For example: Let us consider the case $A_3$, let $\lambda = [1, 3] + [2, 2]$ and $\alpha = \alpha_1 + \alpha_2$ and $\beta = \alpha_2 + \alpha_3$. It is easy to see that $\text{ext}_{\alpha, \beta}^{\text{min}}(\lambda) = ([1, 2], [2, 3])$. 
3. Hall maps for representation spaces

We will next recall the Hall maps for representation spaces (see [Sch09] for more details). Fix $\alpha, \beta = m_i\alpha_i \in Q^+$ and let $\gamma = \alpha + \beta = \sum_{i \in I} n_i\alpha_i$. Given two $I$-graded vector spaces $W \subset V$ such that $\dim V = \gamma$ and $\dim W = \beta$. Set $\text{Gr}(\beta, \gamma) = \prod_{i \in I} \text{Gr}(m_i, n_i)$ where $\text{Gr}(m_i, n_i)$ are the Grassmannians of $n_i$ for $m_i$. There are two descriptions for it: First, we consider it as

$$\text{Gr}(\beta, \gamma) = \{ W' \subset V \mid \dim W' = \beta \}$$

Secondly, we interpret it as a quotient variety. Set

$$\text{Hom}_I(W, V)^0 = \{ f = (f_i)_{i \in I} \in \text{Hom}_I(W, V) \mid f \text{ is injective.} \}$$

which is an open subset of $\text{Hom}_I(W, V)$. In other words, $f_i \in \text{Mat}_{n_i \times m_i}$ with rank $m_i$ for each $i \in I$. The $G_\beta$ acts on $\text{Hom}_I(W, V)$ by $g \cdot f = (f_i g_i^{-1})$. There is a canonical map

$$\Pi : \text{Hom}_I(W, V)^0 \to \text{Gr}(\beta, \gamma)$$

$$f \mapsto f(W)$$

It is easy to see that $\Pi$ is a $G_\beta$-torsor of $\text{Gr}(\beta, \gamma)$. In other words, $\Pi^{-1}(W') \cong G_\beta$ for any $W' \in \text{Gr}(\beta, \gamma)$.

3.1. Induction maps. Let us define the variety $E_{\beta, \gamma}$ by

$$E_{\beta, \gamma} = \{(x, W') \in E_\gamma \times \text{Gr}(\beta, \gamma) \mid x(W') \subset W' \}$$

Let us write $W' = f(W)$ for some element $f$ of $\text{Hom}_I(W, V)^0$ via (3.3). The relation $xf(W) \subset f(W)$ gives rise to a unique $y \in E_\beta$ such that $xf = fy$, as $f$ is injective. More precisely, for each arrow $h : i \to j$, let us consider the following diagram

$$\begin{array}{ccc}
0 & \rightarrow & W_i \\
\uparrow y_h & & \downarrow x_h \\
0 & \rightarrow & W_j \\
\rightarrow & f_i & \rightarrow \\
& & V_i \\
\rightarrow & f_j & \rightarrow \\
& & V_j
\end{array}$$

each matrix $x_h$ gives rise to a unique matrix $y_h$ such that $x_h f_i = f_j y_h$. Therefore, in terms of the second definition of $\text{Gr}(\beta, \gamma)$, we define

$$E'_{\beta, \gamma} = \{(x, y, f) \in E_\gamma \times E_\beta \times \text{Hom}_I(W, V)^0 \mid xf - fy = 0 \}$$

The $G_\beta$-action on it is given by $g \cdot (x, y, f) = (x, g \cdot y, g \cdot f)$. It follows by (3.3) that it is a $G_\beta$-torsor of $E_{\beta, \gamma}$ under this action via the map $(x, y, f) \mapsto (x, f(W))$.

Let us consider the canonical map

$$q : E_{\beta, \gamma} \to E_\gamma$$

$$(x, W') \mapsto x$$
and

\[ q' : E_{\beta, \gamma} \rightarrow \text{Gr}(\beta, \gamma) \]

\[ (x, W') \mapsto W' \]

The map \( q \) is a projective map and \( q' \) is a vector bundle with trivial fiber

\[ P_{\Omega}(W, V) = \oplus_{h \in \Omega} P_{m_{s(h)}, m_{t(h)}} \]

where \( P_{m_{s(h)}, m_{t(h)}} \) is the parabolic subalgebra of \( \text{Hom}_k(V_{s(h)}, V_{t(h)}) \) which are of the form

\[
\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}
\]

where the block of 0 is in \( \text{Mat}_{(n - m)_{t(h)} \times m_{s(h)}} \). In order to see this, let us consider the tuple of matrices \( x = (x_h)_{h \in \Omega} \in E_\gamma \) such that \( x(W) \subset W \) for a fixed subspace \( W \). For any arrow \( h : i \rightarrow j \), the condition \( x(W) \subset W \) means that the matrix \( x_h : V_i \rightarrow V_j \) is of the following form

\[
\begin{pmatrix} y_h & u_h \\ 0 & z_h \end{pmatrix}
\]

where \( y_h \in \text{Hom}_k(W_i, W_j) \), \( u_h \in \text{Hom}_k(V_i/W_i, W_j) \), and \( z_h \in \text{Hom}_k(V_i/W_i, V_j/W_j) \). The corresponding parabolic subalgebra \( P_{m_i, m_j} \) consists of the elements with this form. To sum up all the arrows, we see that the condition \( x(W) \subset W \) for a fixed subspace \( W \) is equivalent to \( x \in P_{\Omega}(W, V) \). It implies that \( E_{\beta, \gamma} \) is a smooth variety and the fiber of \( p \) at \( M \) is a projective variety.

**Definition 3.1.** Consider the map (3.6) \( q : E_{\beta, \gamma} \rightarrow E_\gamma \). For a representation \( M \in E_\gamma \), one calls its fiber \( q^{-1}(M) \) the *quiver Grassmannian* of \( M \) with dimension \( \beta \), and denoted by \( \text{Gr}_\beta(M) \).

Next, one defines

\[ E_{\beta, \gamma}^{(1)} = \{ (x, W', \rho_\beta, \rho_\alpha) \in E_{\beta, \gamma} \times G_\beta \times G_\alpha \mid \rho_\beta : W' \sim W, \rho_\alpha : V/W' \sim V/W \} \]

We will give another form of this variety. For an element \( (x, y, f) \in E_{\beta, \gamma}' \) and an arrow \( h : i \rightarrow j \), let us consider the following diagram

\[
\begin{array}{ccc}
0 & \rightarrow & W_i \\
\downarrow y_h & & \downarrow x_h \\
V_i/f_iW_i & \rightarrow & V_i/0 \\
\downarrow z_h & & \\
W_j & \rightarrow & V_j/0 \\
0 & \rightarrow & f_jV_j \\
\end{array}
\]

where \( u_i, u_j \) are the cokernel of \( f_i, f_j \), respectively, and \( z_h \) is the unique map induced by \( y_h, x_h \). It follows that any element \( (x, y, f) \in E_{\beta, \gamma}' \) gives rise to a unique map \( (u, z) \in \text{Hom}_I(V, V/W) \times \text{Hom}_\Omega(V/W, V/W) \). Set

\[ \mathcal{E}xt_\gamma(\alpha, \beta) = \{ (x, y, z, f, u) \in E_{\beta, \gamma}' \times \text{Hom}_I(V, V/W) \times \text{Hom}_\Omega(V/W, V/W) \mid 0 \rightarrow y \xrightarrow{f} x \xrightarrow{u} z \rightarrow 0 \} \]

We will show that \( \mathcal{E}xt_\gamma(\alpha, \beta) \cong E_{\beta, \gamma}^{(1)} \). First we define the following map by

\[
f : \mathcal{E}xt_\gamma(\alpha, \beta) \rightarrow E_{\beta, \gamma}'
\]

\[
(x, y, z, f, u) \mapsto (x, y, f)
\]
This map is a $G_\alpha$-torsor over $E'_{\beta,\gamma}$. Let us take any element $(x, y, z', f, u') \in \mathcal{E}xt_\gamma(\alpha, \beta)$, the element $(x, y, f) \in E'_{\beta,\gamma}$ gives rise to a unique tuple $(x, y, z, f, u)$ as before. Since $u$ be the cokernel of $(x, y, f)$, by the above diagram we have that there exists a unique $g \in G_\alpha$ such that $u' = gu$. It implies that $z' = g \cdot z$. Therefore, the variety $\mathcal{E}xt_\gamma(\alpha, \beta)$ is a $G_\alpha \times G_\beta$-torsor over $E_{\beta,\gamma}$.

Let us consider $E_{\beta,\gamma}^{(1)}$: For any element $(x, f(W), \rho_\beta, \rho_\alpha)$, it is easy to see by (3.5) that $(f(W), \rho_\beta)$ gives rise to a unique $f \in \text{Hom}_I(W, V)$. The element $\rho_\alpha$ gives rise to a unique $u' = \rho_\alpha u$ using (3.8). The induced pair $(f, u')$ yields a short exact sequence

$$0 \to y \xrightarrow{f} x \xrightarrow{u'} z \to 0$$

Therefore, we obtain a bijection map

$$\mathcal{E}xt_\gamma(\alpha, \beta) \cong E_{\beta,\gamma}^{(1)}$$

Here we remark that we just check the bijection between the closed points of $E_{\beta,\gamma}^{(1)}$ and that of $\mathcal{E}xt_\gamma(\alpha, \beta)$. It is easy to construct a bijection map between $\mathcal{E}xt_\gamma(\alpha, \beta) \cong E_{\beta,\gamma}^{(1)}$ using $(f, u')$ and (3.5).

There is a map $p : E_{\beta,\gamma}^{(1)} \to E_\beta \times E_\alpha$ sending $(x, y, z, f, u)$ to $(y, z)$. Hence, we obtain

$$E_\beta \times E_\alpha \xleftarrow{p} E_{\beta,\gamma}^{(1)} \xrightarrow{r} E_{\beta,\gamma} \xrightarrow{q} E_\gamma$$

$$(y, z) \leftarrow (x, y, z, f, u) \mapsto (x, f(W)) \mapsto x$$

**Lemma 3.2.** Given two orbits $O_\mu$ and $O_\nu$ in $E_\alpha$ and $E_\beta$, respectively, we have

$$qrp^{-1}(O_\mu \times O_\nu) = \bigsqcup_{\lambda \in \text{Ext}_I^{(1)}(\mu, \nu)} O_\lambda$$

**Proof.** Let us consider the subset $p^{-1}(O_\mu \times O_\nu)$. By the definition $\mathcal{E}xt_\gamma(\alpha, \beta)$, one has $(x, y, z, f, u) \in p^{-1}(O_\mu \times O_\nu)$ if and only if $y \in O_\nu$, $z \in O_\mu$, and there exists a short exact sequence

$$0 \to y \xrightarrow{f} x \xrightarrow{u} z \to 0$$

Since $qr(x, y, z, f, u) = x$, we have that $x \in qrp^{-1}(O_\mu \times O_\nu)$ if and only if there exists a short exact sequence above. That means $x \in \text{Ext}_I^{(1)}(\mu, \nu)$. Therefore, this leads to our conclusion.

Let us consider the following fibre diagram

$$\text{Gr}(\mu, \nu, M_\lambda) \xrightarrow{q'} M_\lambda$$

$$\xrightarrow{d}$$

$$r^{-1}(O_\mu \times O_\nu) \xrightarrow{q} \mathcal{E}xt_\gamma(\mu, \nu)$$

(3.11)
where $M_\lambda$ is a point in $E_\gamma$, and $\text{Gr}(\mu, \nu, M_\lambda)$ is the $q^{-1}(M_\lambda) \cap r_p^{-1}(\mathcal{O}_\mu \times \mathcal{O}_\nu)$. Namely, by (3.10), we have

\begin{equation}
(3.12) \quad \text{Gr}(\mu, \nu, M_\lambda) = \{ fW \in \text{Gr}_\beta(M_\lambda) \mid (M_\lambda)_fW \cong M_\nu; (M_\lambda)_{V/fW} \cong M_\mu \}
\end{equation}

3.2. **Restriction maps.** Let $F_{\beta, \gamma}$ be the closed subset of $E_\gamma$ consisting of representations $y$ such that $y(W) \subset W$. That is equal to $P_\Omega(W, V)$. Let $P_t(W, V) \subset GL(\gamma)$ be the parabolic subgroup associated with $W$. We consider the following diagram

\begin{equation}
(3.13) \quad E_\alpha \times E_\beta \xleftarrow{\kappa} F_{\beta, \gamma} \xrightarrow{i} E_\gamma
\end{equation}

where $\kappa(y) = (y|_{V/W}, y|_W)$ and $i$ is the closed embedding. Note that $\kappa$ is a vector bundle of rank

\begin{equation}
(3.14) \quad \text{rank} \kappa = \sum_{i \in I} \alpha_i \beta_i - \langle \alpha, \beta \rangle
\end{equation}

3.2.1. **The orbits and restriction maps.** Let us consider the map $\kappa_\lambda : \overline{\mathcal{O}_\lambda} \cap F_{\beta, \gamma} \rightarrow \mathcal{O}_\mu \times \mathcal{O}_\nu$. The fiber of $\kappa_\lambda^{-1}(M, N)$ is identified with

\begin{equation}
(3.15) \quad \{ E \in F_{\beta, \gamma} \mid 0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0 \quad \text{and} \quad E \geq M_\lambda \}
\end{equation}

**Lemma 3.3.** If $\lambda' \geq \lambda$ then we have $\kappa_\lambda^{-1}(M, N) \subset \kappa_{\lambda'}^{-1}(M, N)$.

**Proof.** It is easy to see this lemma by $\overline{\mathcal{O}_\lambda} \subset \overline{\mathcal{O}_{\lambda'}}$. \hfill \Box

**Lemma 3.4.** For any $\lambda \in \text{ext}(\alpha, \beta)$, let $\overline{\mathcal{O}_\lambda}$ be the closure of the orbit $\mathcal{O}_\lambda \subset E_\gamma(Q)$. We have

\begin{equation}
\kappa_{\lambda'}^{-1}(\overline{\mathcal{O}_\lambda}) = \bigcup_{(\mu, \nu) \in \text{ext}_{\alpha, \beta}^\text{min}(\lambda)} \overline{\mathcal{O}_\mu \times \mathcal{O}_\nu}
\end{equation}

**Proof.** For simplicity we write $\kappa$ for the restriction of $\kappa$ to $\kappa_{\lambda'}^{-1}(\overline{\mathcal{O}_\lambda})$ in the proof. Note that it is not a smooth map in general.

We first show that $\kappa_{\lambda'}^{-1}(\overline{\mathcal{O}_\lambda})$ is a closed subset. Let $(\mu, \nu)$ be a pair such that $\overline{\mathcal{O}_\mu \times \mathcal{O}_\nu} \subset \kappa_{\lambda'}^{-1}(\overline{\mathcal{O}_\lambda})$. It is enough to check that for any point $(M \times N) \in \overline{\mathcal{O}_\mu \times \mathcal{O}_\nu}$ there exists $G \in \overline{\mathcal{O}_\lambda \cap F_{\beta, \gamma}}$ such that $\kappa(G) \equiv M \times N$. The fact $M \oplus N \geq M_\mu \oplus M_\nu \geq M_\lambda$ implies that $\mathcal{O}_{M \oplus N} \subset \overline{\mathcal{O}_\lambda}$. Let $G = M \oplus N \subset \overline{\mathcal{O}_\lambda \cap F_{\beta, \gamma}}$, it is obvious that $\kappa(M \oplus N) = (M, N)$. Therefore, we obtain

\begin{equation}
\kappa_{\lambda'}^{-1}(\overline{\mathcal{O}_\lambda}) = \bigcup_{(\mu, \nu) \in KP(\alpha) \times KP(\beta)} \overline{\mathcal{O}_\mu \times \mathcal{O}_\nu}
\end{equation}

We will see that here $(\mu, \nu) \in \text{ext}_{\alpha, \beta}^\text{min}(\lambda)$. Otherwise, there is a pair $(\mu', \nu') \in \text{ext}_{\alpha, \beta}(\lambda)$ with $\mu' < \mu$ and $\nu' \leq \nu$ or $\mu' \leq \mu$ and $\nu' < \nu$. The short exact sequence

\begin{equation}
0 \rightarrow M_{\mu'} \rightarrow M_\lambda \rightarrow M_{\mu'} \rightarrow 0
\end{equation}

implies that

\begin{equation}
\left( \begin{array}{cc} g_1 \cdot M_{\mu'} & u \\ 0 & g_2 \cdot M_{\mu'} \end{array} \right) \in \mathcal{O}_\lambda \cap F_{\beta, \gamma}
\end{equation}

for some $u \in \text{Hom}_Q(V/W, W)$ and any elements $g_1 \in G_{\beta}, g_2 \in G_{\alpha}$. This then implies that $\mathcal{O}_{\mu'} \times \mathcal{O}_{\nu'} \subset \kappa^{-1}(\mathcal{O}_\lambda)$. It follows that $\mathcal{O}_{\mu'} \times \mathcal{O}_{\nu'}$ is contained in an irreducible component.
of $\kappa^{-1}(\Omega_{\lambda})$, but $\overline{\Omega}_{\mu} \times \overline{\Omega}_{\nu} \not\subseteq \overline{\Omega}_{\mu'} \times \overline{\Omega}_{\nu'}$ contradicts the fact $\overline{\Omega}_{\mu} \times \overline{\Omega}_{\nu}$ is a component of $\kappa^{-1}(\Omega_{\lambda})$. Thus, we obtain a contradiction. □

**Lemma 3.5.** Under the assumptions as in Definition (2.2). If $(\mu, \nu) \in \text{ext}_{\alpha, \beta}^{\min}(\lambda)$, then for any $\lambda'$ such that $\lambda \leq \lambda' \leq \mu \oplus \nu$ we have $(\mu, \nu) \in \text{ext}_{\alpha, \beta}^{\min}(\lambda')$

**Proof.** By $\lambda' \leq \mu \oplus \nu$ we have $\overline{\Omega}_{\mu} \times \overline{\Omega}_{\nu}$ is contained in $\kappa^{-1}(\Omega_{\lambda'})$. Since $\overline{\Omega}_{\mu} \times \overline{\Omega}_{\nu}$ is an irreducible component of $\kappa^{-1}(\Omega_{\lambda})$ and $\kappa^{-1}(\Omega_{\lambda'}) \subset \kappa^{-1}(\Omega_{\lambda})$, we have that $\overline{\Omega}_{\mu} \times \overline{\Omega}_{\nu}$ is an irreducible component of $\kappa^{-1}(\Omega_{\lambda'})$. □

**Example 3.6.** We take $\mu$ and $\nu$ as the unique minimal element in $\text{KP}(\alpha)$ (resp: $\text{KP}(\beta)$). In other words, $M_\mu$ and $M_\nu$ are rigid modules. For any $\lambda \leq \mu \oplus \nu$, we have $\text{ext}_{\alpha, \beta}^{\min}(\lambda) = \{(\mu, \nu)\}$, as $\kappa^{-1}(\Omega_{\mu \oplus \nu}) \subset \overline{\Omega}_{\mu} \times \overline{\Omega}_{\nu} = E_\alpha \times E_\beta$.

### 4. Quiver Grassmannians and Extension varieties

In this section, we will give some new interpretation of quiver Grassmannians and extension groups. First, let us recall the notion of generic extension of two representations of $Q$. Following [Rei01], we have

**Definition 4.1.** [Rei01, Definition 2.2] Let $M$ and $N$ be two representations of $Q$. We say $M \ast N$ is a generic extension of $M, N$ if $M \ast N \in \text{Ext}^1_Q(M, N)$ such that $\dim \text{Ext}^1_Q(M \ast N, M \ast N)$ is minimal with respect to any element $E$ in $\text{Ext}^1_Q(M, N)$. If we write $M$ and $N$ for matrices $(V, x)$ and $(W, y)$ respectively, then any extension of $E \in \text{Ext}^1_Q(M, N)$ can be expressed as the matrix block

\[(1) \quad M_u = \begin{pmatrix} y & u \\ 0 & x \end{pmatrix}\]

where $u \in \text{Hom}_Q(V, W)$. Let us set

\[(2) \quad \text{Hom}^1_Q(M, N) = \{(u, f) \in \text{Hom}_Q(V, W) \times \text{Hom}_F(V, V) \mid f \in \text{Hom}_Q(M_u, M_u)\}\]

Here we regard $\text{Hom}_Q(M_u, M_u)$ as the solution of the equation

\[M_{u; h} f_i = f_j M_{u; h} \text{ for each arrow } h : i \to j\]

where $M_u = (M_{u; h})_{h \in \Omega}$.

It yields a map $g : \text{Hom}^1_Q(M, N) \to \text{Hom}_Q(V, W)$ by sending $(u, f)$ to $u$. It is easy to see that $g^{-1}(u) = \text{Hom}_Q(M_u, M_u)$. There is a unique open subvariety of $\text{Hom}_Q(V, W)$ such that $\text{Hom}_Q(M_u, M_u)$ with minimal dimension. The notion of generic extension of $M, N$ exactly means that the element $u$ for $M \ast N$ in (1) is in this open locus.

#### 4.1. Quiver Grassmannians

Let us consider the following set

\[(3) \quad G_{\lambda} = \{g \in G_{\gamma} \mid g \cdot x_{\lambda}(W) \subset W\} = \left\{g \in G_{\gamma} \mid g \cdot \begin{pmatrix} y & u \\ 0 & x \end{pmatrix} \in F_{\beta, \gamma}\right\}\]

where $y = N$, $x = M$ and $x_{\lambda} = \begin{pmatrix} y & u \\ 0 & x \end{pmatrix}$ for $u \in \text{Hom}_Q(W, V/W)$. It is obvious to see that $F_{\beta, \gamma} \subset G_{\lambda}$. 
Lemma 4.2. Under the above assumption, we have \( G_\lambda/P_{\beta,\gamma} \cong \text{Gr}_\beta(M_\lambda) \).

Proof. By Definition (3.1), we see that \( \text{Gr}_\beta(M_\lambda) = \{ W' \in \text{Gr}(\beta,\gamma) \mid x_\lambda(W') \subset W' \} \). Recall that (3.1) any elements in \( \text{Gr}(\beta,\gamma) \) are of the form \( g \cdot W \) for some \( g \in G_\gamma \), we have \( x_\lambda(g \cdot W) \subset g \cdot W \). In other words, \( g^{-1} \cdot x_\lambda(W) \subset W \). It follows that

\[
(4.4) \quad g \cdot W \in \text{Gr}_\beta(M_\lambda) \iff g^{-1} \in G_\lambda
\]

It implies that there is a surjective map \( G_\lambda \to \text{Gr}_\beta(M_\lambda) \) sending \( g \) to \( g^{-1}W \). The stabilizer of \( W \) is exactly \( P_{\beta,\gamma} \). It leads to that \( G_\lambda/P_{\beta,\gamma} \cong \text{Gr}_\beta(M_\lambda) \). \( \square \)

We will give a decomposition of quiver Grassmannians. In what follows, we fix a decomposition \( V = W \oplus U \) such that \( \dim W = \beta \) and \( \dim U = \alpha \), and we write

\[
x_\lambda = \begin{pmatrix} y & u \\ 0 & x \end{pmatrix}
\]

where \( y \in E_\beta, x \in E_\alpha \) and \( u \in \text{Hom}_\Omega(U, V) \).

Let us set

\[
(4.5) \quad G_\lambda(\mu, \nu) = \left\{ g \in G_\gamma \mid g \cdot \begin{pmatrix} y & u \\ 0 & x \end{pmatrix} = \begin{pmatrix} y' & u' \\ 0 & x' \end{pmatrix} \text{ for } u' \in \text{Hom}_\Omega(U, W) \right\}
\]

It is easy to see that \( P_\lambda(\mu, \nu) := P_{\beta,\gamma} \cap G_\lambda(\mu, \nu) = \begin{pmatrix} \text{Aut}_Q(N) & \text{Hom}_\Omega(U, W) \\ 0 & \text{Aut}_Q(M) \end{pmatrix} \)

Theorem 4.3. Under the above assumption, we have

\[
G_\lambda(\mu, \nu)/P_\lambda(\mu, \nu) = \{ g \cdot W \in \text{Gr}_\beta(M_\lambda) \mid x_{\lambda|gW} = y \text{ and } x_{\lambda|/gW} = x \}
\]

\( G_\lambda(\mu, \nu) \) is an irreducible variety and

\[
\dim G_\lambda(\mu, \nu) = \dim \text{Hom}_\Omega(U, W) + [M_\lambda, M_\mu * M_\nu]
\]

In particular, we have

\[
\dim G_\lambda(\mu, \nu)/P_\lambda(\mu, \nu) = [M_\lambda, M_\mu * M_\nu] - [M_\mu, M_\mu] - [M_\nu, M_\nu]
\]

Proof. Suppose \( g \cdot \begin{pmatrix} y & u \\ 0 & x \end{pmatrix} = \begin{pmatrix} y' & u' \\ 0 & x' \end{pmatrix} \) and write \( x'_\lambda = g \cdot x_\lambda \). Since \( g \cdot x_{\lambda|W} = x_{\lambda|g^{-1}W} \), we obtain that \( x_{\lambda|W} = x_{\lambda|g^{-1}W} = y' \) and \( x_{\lambda|/gW} = x_{\lambda|/g^{-1}W} = x' \). Therefore, the corresponding (4.4) implies the first formula.

For any \( u \in \text{Hom}_\Omega(U, W) \), one defines \( M_u = \begin{pmatrix} y & v \\ 0 & x \end{pmatrix} \). Set

\[
(4.6) \quad T = \{(v, g) \in \text{Hom}_\Omega(U, W) \times \text{Hom}_I(V, V) \mid g \in \text{Hom}_Q(M_\lambda, M_u)\}
\]

It is easy to see that there are two obvious maps

\[
(4.7) \quad \text{Hom}_I(V, V) \xleftarrow{\pi} T \xrightarrow{\psi} \text{Hom}_\Omega(U, W)
\]

\[
g \mapsto (v, g) \mapsto v
\]
It is easy to see that \( \psi^{-1}(v) = \text{Hom}_Q(M_\lambda, M_\mu) \). We will show that \( \psi \) is surjective. We just check that for any \( v \in \text{Hom}_Q(U, W) \) the homomorphism space \( \text{Hom}_Q(M_\lambda, M_\mu) \neq 0 \). Otherwise, it implies by equation 2.4 that

\[
\langle \gamma, \gamma \rangle = -[M_\lambda, M_\mu]^1 \leq 0
\]

and then leads to \( \langle \gamma, \gamma \rangle \leq 0 \), which is a contradiction, as \((-,-)\) is a positive-definite bilinear form. Let \( M \ast N \) be the generic extension of \( M, N \). Using [Har13, Exercise 3.22 in Chapter 2] and the fact that the generic extension consists of an open dense subset \( \text{Hom}_{\overline{\Omega}}^\text{ger}(U, W) \) of \( \text{Hom}_Q(U, W) \), we see that

\[
\dim T - \dim \text{Hom}_Q(U, W) = [M_\lambda, M_\mu * M_\nu]
\]

Meanwhile, we obtain that \( T \) is an irreducible variety, as the restriction \( \psi \) to the open generic locus \( \text{Hom}_{\overline{\Omega}}^\text{ger}(U, W) \) is a vector bundle and \( \psi^{-1}(\text{Hom}_{\overline{\Omega}}^\text{ger}(U, W)) \) is a dense open subspace of \( T \).

Let us consider the map \( \pi \): We restrict it to the open subset \( G_\gamma \subset \text{Hom}_I(V, V) \). For any \( g \in G_\gamma \), we have \( \pi^{-1}(g) = \{v\} \) or \( \emptyset \), as \( M_\nu \) is uniquely determined by \( g \) via the equation \( g x_\lambda = x_\nu g \). Therefore, we obtain \( \dim \text{Im} \pi \cap G_\gamma = \dim T \). On the other hand, it is easy to see that \( G_\gamma(M_\mu, M_\nu) = \text{Im} \pi \cap G_\gamma \). Therefore, we see that \( G_\gamma(M_\mu, M_\nu) \) is irreducible and its dimension is \( \dim \text{Hom}_Q(U, W) + [M_\lambda, M_\mu * M_\nu] \), as \( \text{Im} \pi \) is irreducible.

**Proposition 4.4.** Under the above assumption, for any \( \mu \oplus \nu \geq \lambda \geq \mu * \nu \), we have

\[
\dim \kappa^{-1}_\lambda(M_\mu, M_\nu) = \dim \text{Hom}_Q(U, W) + [M_\lambda, M_\mu * M_\nu] - [M_\lambda, M_\lambda]
\]

Moreover, it is an affine irreducible variety. From then on, we denote by \( e_\lambda(\mu, \nu) \) the dimension of \( \kappa^{-1}_\lambda(M_\mu, M_\nu) \).

**Proof.** It is easy to see that the \( G_\gamma \cdot M_\lambda \cap F_{\beta, \gamma} \) is an open dense subvariety of \( \overline{\Omega}_\lambda \cap F_{\beta, \gamma} \). It implies that \( \overline{\Omega}_\lambda \cap \kappa^{-1}_\lambda(M_\mu, M_\nu) \) is an open dense subvariety of \( \kappa^{-1}_\lambda(M_\mu, M_\nu) \). It is straightforward to see that

\[
\kappa^{-1}_\lambda(M_\mu, M_\nu) = \{g \cdot M_\lambda \mid g \in G_\gamma(\mu, \nu)\}
\]

Since \( \text{Aut}_Q(M_\lambda) \subset G_\gamma(\mu, \nu) \), then

\[
\text{Aut}_Q(M_\lambda) \cap \kappa^{-1}_\lambda(M_\mu, M_\nu) \cong G_\gamma(\mu, \nu) / \text{Aut}_Q(M_\lambda)
\]

Since \( G_\gamma(\mu, \nu) \) is an irreducible variety, it implies that \( \text{Aut}_Q(M_\lambda) \cap \kappa^{-1}_\lambda(M_\mu, M_\nu) \) is an irreducible variety and then \( \kappa^{-1}_\lambda(M_\mu, M_\nu) \) is irreducible. As it is a closed subset of \( \text{Hom}_Q(U, W) \), we have \( \kappa^{-1}_\lambda(M_\mu, M_\nu) \) is affine.

**Lemma 4.5.** Let \( \lambda' > \lambda \) for some \( \lambda' \leq \mu \oplus \nu \), we have \( e_{\lambda'}(\mu, \nu) \leq e_\lambda(\mu, \nu) \).

**Proof.** Since \( \overline{\Omega}_{\lambda'} \subset \overline{\Omega}_\lambda \), it implies that \( \kappa^{-1}_{\lambda'}(M_\mu, M_\nu) \subset \kappa^{-1}_\lambda(M_\mu, M_\nu) \).

### 4.2. Irreducible components of quiver Grassmannians

In this section, we will consider the set of irreducible components of quiver Grassmannians. Let us fix a partition of \( \gamma = \alpha + \beta \) and a Konstant partition \( \lambda \in \text{KP}(\gamma) \). Consider the quiver Grassmannian \( \text{Gr}_\beta(M_\lambda) \). By [CER12], the irreducible components of \( \text{Gr}_\beta(M_\lambda) \) are the closure of the following subvarieties

\[
S_N = \{W' \in \text{Gr}_\beta(M_\lambda) \mid x_{\lambda|W'} \cong N\}
\]
for some $N \in E_\beta$. Similarly, we have the irreducible components are the closure of the subverities

$$T_M = \{ W' \in \text{Gr}_\beta(M_\lambda) \mid x_{\lambda|W'/W' \cong M} \}$$

for some $M \in E_\alpha$. Therefore, for such a $S_N$, there exists a $M \in E_\alpha$ such that $S_N = T_M$

In other words, the subvariety $S_N \cap T_M$ is a open subvariety of the component $\overline{S}_N$.

**Definition 4.6.** We define a pair $(\mu, \nu)$ is a *component pair* of $\lambda$ if $\overline{S}_\mu = T_\nu$ is a component of $\text{Gr}_\beta(M_\lambda)$. Therefore, the set of irreducible components of $\text{Gr}_\beta(M_\lambda)$ is equal to the set of component pairs $(\mu, \nu)$ of $\lambda$. We denote it by $\text{ext}_{\alpha, \beta}^\text{ger}(\lambda)$.

**Remark 4.7.** Note that for any component pair $(\mu, \nu)$, we have that $M_\mu$ is the generic quotient of $M_\lambda$ by $M_\nu$ e.t. there exists no $\mu' < \mu$ such that

$$0 \to M_\mu \to M_\lambda \to M_{\mu'} \to 0$$

and $M_\nu$ is the generic subrepresentation of $M_\lambda$ with quotient $M_{\mu'}$ e.t. there exists no $\nu' < \nu$ such that

$$0 \to M_{\nu'} \to M_\lambda \to M_\mu \to 0$$

We call $(\mu, \nu)$ a *generic pair* if $M_\mu$ is the generic quotient of $M_\lambda$ by $M_\nu$ and $M_\nu$ is the generic subrepresentation of $M_\lambda$ with quotient $M_{\mu'}$. Therefore, a component pair is a generic pair. But a generic pair is not a component pair in general.

Thanks to Cerulli Irelli and Reineke, the following lemma gives a description of component pairs.

**Lemma 4.8.** [Cerulli Irelli and Reineke] Under the above assumption, the subvariety $\overline{S}_N$ is an irreducible component of $\text{Gr}_\beta(M_\lambda)$ if and only if

$$[N, M_\lambda] - [N, N] = [N, T]$$

here $T$ is the generic quotient of $M_\lambda$ by $N$, (see Remark 4.7).

**Proof.** Let us assume that $[N, M_\lambda] - [N, N] = [N, T]$. It is well known that $\dim \overline{S}_N = [N, M_\lambda] - [N, N]$ and tangent space at $N$ is $T_N \text{Gr}_\beta(M_\lambda) = \text{Hom}_Q(N, T)$. Since

$$[N, T] = \dim T_N \text{Gr}_\beta(M_\lambda) \geq \dim T_N \overline{S}_N \geq \dim \overline{S}_N = [N, M_\lambda] - [N, N]$$

then the equation (4.8) implies that $N$ is a nonsingular point in $\overline{S}_N$. Meanwhile, by [Hub17, Lemma 3.1], the identity $\dim T_N \text{Gr}_\beta(M_\lambda) = \dim T_N \overline{S}_N$ implies that $\overline{S}_N$ is an irreducible component of $\text{Gr}_\beta(M_\lambda)$.

If $\overline{S}_N$ is an irreducible component of $\text{Gr}_\beta(M_\lambda)$, let us take a nonsingular point $g \cdot N$ in $\overline{S}_N$. This means $\dim T_{g \cdot N} \overline{S}_N = \dim T_{g \cdot N} \text{Gr}_\beta(M_\lambda) = [N, T]$. It follows by the fact $g \cdot N$ is nonsingular that $\dim T_{g \cdot N} \overline{S}_N = \dim \overline{S}_N = [N, M_\lambda] - [N, N]$. Hence, we obtain

$$[N, M_\lambda] - [N, N] = [N, T]$$

$\square$
Therefore, we see that

**Theorem 4.9.** [Reineke] Under the above assumption, we have the set of irreducible components of $\text{Gr}_\beta(M_\lambda)$ is identified with the following set

$$\text{ext}^\text{ger}_{\alpha,\beta}(\lambda) = \left\{ (\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta) \mid \text{Such that } (\mu, \nu) \text{ is a generic pair of } \lambda \text{ and} \right. [M_\mu, M_\lambda] = [M_\nu, M_\lambda] + [M_\nu, M_\mu]$$

The notion of generic pair is given in Remark 4.7.

*Proof.* Following the above lemma.

**Example 4.10.** In the case $A_3$, let us consider $\lambda = [1, 2] + [2, 3]$. If $\beta = \alpha_2 + \alpha_3$, let $V = V_1 \oplus V_2 \oplus V_3$ such that $\dim V_1 = \dim V_3 = 1$ with base $e_1, e_3$ and $\dim V_2 = 2$ with bases $e_2, e'_2$. Set $x_1 : e_1 \to e_2$ and $x_2 : e'_2 \to e_3$. Fix $W = \langle e_1, e_2 \rangle$, we have

$$\text{Gr}_\beta(M_\lambda) = \{ gW \in \text{Gr}(1, 2) \mid x_2(ge_2') \subset W_3 = V_3 \}$$

It is easy to see that $\text{Gr}_\beta(M_\lambda) = \text{Gr}(1, 2) \cong \mathbb{P}^1$.

If $\beta = \alpha_1 + \alpha_2$ and $W = \langle e_1, e_2 \rangle$, then

$$\text{Gr}_\beta(M_\lambda) = \{ gW \in \text{Gr}(1, 2) \mid x_1(e_1) \subset gW_2 = \langle ge_1 \rangle \}$$

This means $\text{Gr}_\beta(M_\lambda) = \{ pt \}$.

Let us consider $\text{Gr}_\beta(M_\lambda)$ with $\beta = \alpha_2 + \alpha_3$. There are two generic pairs

$$([2, 2] + [3, 3], [1, 1] + [2, 2]) \quad \text{and} \quad ([2, 3], [1, 2])$$

by

$$0 \to S_2 \oplus S_3 \to M_\lambda \to S_1 \oplus S_2 \to 0$$

$$0 \to M[2, 3] \to M_\lambda \to M[1, 2] \to 0$$

But $\text{ext}^\text{ger}_{\alpha,\beta}(\lambda) = ([2, 3], [1, 2])$. This follows from the equation

$$(4.9) \quad [S_2 \oplus S_3, S_2 \oplus S_3] + [S_2 \oplus S_3, S_1 \oplus S_2] = 2 + 1 > [S_2 \oplus S_3, M[2, 3] \oplus M[1, 2]] = 2$$

In the case $A_2$, following [CFR13, Section 8.3], for any dimension vector $\gamma = (d_1, d_2)$ the Konstant partitions in $\text{KP}(\gamma)$ is given by the rank $r$ of the matrix $x_1 : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$. Let $\beta = (e_1, e_2) < (d_1, d_2)$ be a sub dimension vector of $\gamma$. If $r < e_1 - e_2 + d_2$, the set of irreducible components of $\text{Gr}_\beta(M_r)$ is identified with

$$\text{ext}^\text{ger}_{\alpha,\beta}(M_r) = \{ a \in \mathbb{N} \mid \max\{0, r + e_1 - d_1, r - d_2 + e_2\} \leq a \leq \min\{e_1, e_2, r\} \}$$

5. **Graded quiver varieties**

In this section, we will recall the notion of graded quiver varieties and relations between representation space for $Q$ and the graded quiver varieties.
5.1. **Repetition quivers.** We will briefly discuss the notion of repetition algebras. For a quiver $Q$ we define a height function $\xi : I \to \mathbb{Z}$ such that
\[
\xi_i = \xi_j + 1 \quad \text{if there exists an arrow } h : i \to j
\]
It is easy to see that the minimal $\xi_k$ appears at the sink points and the maximal $\xi_l$ appears at the source points. The values of points in any path in $Q$ are uniquely determined by the value of the sink of this path. Therefore, for a connected $Q$, any two height functions differ by a constant. We fix a height function $\xi$ and define
\[
\hat{I} = \{(i, p) \in I \times \mathbb{Z} | p - \xi_i \in 2\mathbb{Z}\}
\]
We attach to $Q$ the infinite repetition quiver $\widehat{Q}$, defined as the quiver with vertex set $\hat{I}$ and two type arrows:
\[
(h, p) : (i, p) \to (j, p + 1) \quad \text{if } h : i \to j
\]
\[
(\bar{h}, q) : (j, q) \to (i, q + 1) \quad \text{if } h : i \to j
\]
for all $(i, p), (i, q) \in \hat{I}$. In fact, it is well known that $\widehat{Q}$ is the quiver of a $\mathbb{Z}$-cover of the preprojective algebra associated with $Q$.

Let us consider $\widehat{\Delta} = \Delta^+ \times \mathbb{Z}$. We now describe a natural labelling of the vertices of $\widehat{Q}$ by $\widehat{\Delta}$. Let $\gamma_i$ be the root associated with indecomposable injective representation $I_i$ for each $i \in I$. There exists a unique bijection $\phi : \hat{I} \to \widehat{\Delta}$ defined inductively as follows.

1. $\phi(i, \xi_i) = (\gamma_i, 0)$ for $i \in I$.
2. Suppose $\phi(i, p) = (\beta, m)$ then
\[
\phi(i, p - 2) = (\tau \beta, m) \quad \text{if } \tau \beta \in \Delta^+
\]
\[
\phi(i, p - 2) = (-\tau \beta, m - 1) \quad \text{if } \tau \beta \in \Delta^-
\]
\[
\phi(i, p + 2) = (\tau^{-1} \beta, m) \quad \text{if } \tau^{-1} \beta \in \Delta^+
\]
\[
\phi(i, p + 2) = (-\tau^{-1} \beta, m + 1) \quad \text{if } \tau^{-1} \beta \in \Delta^-
\]
We remark that the Auslander-Reiten quiver $\Gamma_Q$ can be considered as a subquiver of $\widehat{Q}$ via a map sending $M\beta \to \phi^{-1}(\beta, 0)$. Define $\widehat{\Gamma}_Q = \phi^{-1}(\Delta^+ \times \{0\})$. It is easy to see that $\widehat{\Gamma}_Q$ is the vertex set of $\Gamma_Q$.

We define the $q$-analog of the Cartan matrix $C_q$ on $\mathbb{N}[I \times \mathbb{Z}]$ by
\[
C_q : \mathbb{N}[I \times \mathbb{Z}] \to \mathbb{N}[I \times \mathbb{Z}]
\]
\[
V(i, p) \mapsto V(i, p - 1) + V(i, p + 1) - \sum_{j \sim i} a_{i,j} V(j, p)
\]
Here, $i \sim j$ means that $i \to j$ or $j \to i$ in $\Omega$.

5.2. **Graded quiver varieties.** Let
\[
W = \bigoplus_{(i, p) \in \hat{I}} W_i(p)
\]
be a finite dimensional $\tilde{J}$-graded vector space. Nakajima [Nak04] has associated with $W$ an affine variety $\mathfrak{M}_0(W)$. More precisely, let
\[
\tilde{J} = \{(i, p) \in I \times \mathbb{Z} \mid (i, p - 1) \in \tilde{I}\}
\]
and let
\[
V = \bigoplus_{(i, p) \in \tilde{J}} V_i(p)
\]
be a finite dimensional $\tilde{J}$-graded vector space. Define
\begin{align*}
L^\bullet(V, W) &= \bigoplus_{(i, p) \in \tilde{J}} \text{Hom}(V_i(p), W_i(p - 1)) \\
L^\bullet(W, V) &= \bigoplus_{(i, p) \in \tilde{I}} \text{Hom}(W_i(p), V_i(p - 1)) \\
E^\bullet(V) &= \bigoplus_{(i, p) \in \tilde{J}, j \sim i} \text{Hom}(V_i(p), V_j(p - 1))
\end{align*}
Here $i \sim j$ means that $i \rightarrow j$ or $j \rightarrow i$ in $\Omega$. Put $M^\bullet(V, W) = E^\bullet(V) \oplus L^\bullet(W, V) \oplus L^\bullet(V, W)$. An element of $M^\bullet(V, W)$ is written by $(B, \alpha, \beta)$, and its components are denoted by
\begin{align*}
B_{ij}(p) &\in \text{Hom}(V_i(p), V_j(p - 1)) \\
\alpha_i(p) &\in \text{Hom}(W_i(p), V_i(p - 1)) \\
\beta_i(p) &\in \text{Hom}(V_i(p), W_i(p - 1))
\end{align*}
Let $\Lambda^\bullet(V, W)$ be the subvariety of $M^\bullet(V, W)$ defined by the equations
\[
\alpha_i(p - 1)\beta_i(p) + \sum_{i \rightarrow j} B_{ji}(p - 1)B_{ij}(p) - \sum_{k \rightarrow i} B_{ki}(p - 1)B_{ik}(p) = 0
\]
The group $G_V = \prod_{(i, p) \in \tilde{J}} \text{GL}(V_i(p))$ acts on $M^\bullet(V, W)$ by
\[
g \cdot (B, \alpha, \beta) = (g_j(p - 1)B_{ij}(p)g_i^{-1}(p), g_i(p - 1)\alpha_i(p), \beta_i(p)g_i^{-1}(p))
\]
This action preserve $\Lambda^\bullet(V, W)$. One defines the affine variety
\[
\mathfrak{M}_0^\bullet((V, W)) = \Lambda^\bullet(V, W) \sslash G_V
\]
By definition, the coordinate ring of $\mathfrak{M}_0^\bullet(V, W)$ is the ring of $G_V$-invariant functions on $\Lambda^\bullet(V, W)$. If there is a graded vector space $V'$ such that $V_i(p) \leq V'_i(p)$ for all $(i, p) \in \tilde{J}$, then we have a natural closed embedding $\mathfrak{M}_0^\bullet(V, W) \subset \mathfrak{M}_0^\bullet(V', W)$. Finally, one can define
\[
\mathfrak{M}_0^\bullet(W) = \bigcup_V \mathfrak{M}_0^\bullet(V, W)
\]
Let $\mathfrak{M}_0^\bullet_{\text{reg}}(V, W)$ be the open subset of $\mathfrak{M}_0^\bullet(V, W)$ parametrizing the closed free $G_V$-orbits. For a given $W$, we have $\mathfrak{M}_0^\bullet_{\text{reg}}(V, W) \neq \emptyset$ for only a finite number of $V$'s. Nakajima has shown that these subvarieties $\mathfrak{M}_0^\bullet_{\text{reg}}(V, W)$ give a stratification of $\mathfrak{M}_0^\bullet(V, W)$ as follows.
\[
\mathfrak{M}_0^\bullet(V, W) = \bigsqcup_V \mathfrak{M}_0^\bullet_{\text{reg}}(V, W)
\]
A necessary condition for \( \mathcal{M}_0^\text{reg}(V, W) \neq \emptyset \) is that
\[
\dim W_i(p) - \dim V_i(p + 1) - \dim V_i(p - 1) + \sum_{i \sim j} V_j(p) \geq 0
\]
for each \((i, p) \in \tilde{I}\). In other words, \( W - C_q(V) \geq 0 \). In this case, we say that \((V, W)\) is a dominate pair. We denote by \( \text{IC}_W(V) \) the intersection cohomology complex of the closure of the stratum \( \mathcal{M}_0^\text{reg}(V, W) \).

5.2.1. An isomorphism. Let \( \gamma = \sum_{i=1}^r \gamma_i \alpha_i \). We define an \( \tilde{I} \)-graded vector space \( W^\gamma \) by taking
\[
W_i(p) = C_{\gamma_i} \text{ if } \phi(i, p) = (\alpha_i, 0)
\]
and \( W_i(p) = 0 \) for others \((i, p) \in \tilde{I}\). Following [HL15, Theorem 9.11]

**Theorem 5.1** ([HL15, Theorem 9.11]). We have a \( G_\gamma \)-equivariant isomorphism
\[
\Psi : \mathcal{M}_0^\bullet(W^\gamma) \cong E_\gamma
\]
such that

1. The stratification given by \( G_\gamma \)-orbit on \( E_\gamma \) coincides with the stratification (5.13).
   Namely, \( \Psi(\mathcal{M}_0^\text{reg}(V, W)) = \bigodot_\lambda \) for some \( \lambda \in \text{KP}(\gamma) \);
2. As a corollary, we have \( \text{IC}(\lambda) = \Psi_*(\text{IC}_W(V)) \).

Let us set
\[
\langle - , - \rangle : \mathbb{N}[I \times \mathbb{Z}] \times \mathbb{N}[I \times \mathbb{Z}] \to \mathbb{Z}
\]
and \( q^\pm V = (V'(i, p)) \) such that \( V'(i, p) = V(i, p \pm 1) \). For a pair \((V_1, W_1; V_2, W_2)\) of graded vector spaces, one defines \( d(V_1, W_1; V_2, W_2) \) by
\[
d(V_1, W_1; V_2, W_2) = \langle \dim V_1, q^{-1}(\dim W_2 - C_q \dim V_2) \rangle + \langle \dim V_2, q \dim W_1 \rangle
\]
and set
\[
e(V_1, W_1; V_2, W_2) = d(V_1, W_1; V_2, W_2) - d(V_2, W_2; V_1, W_1)
\]

5.3. Representation varieties. In this section we will consider \( \mathcal{M}^\bullet(V, W) \) as the quotient of representation varieties for an algebra. Based on the tuples \((B, \alpha, \beta)\), we will give an algebra \( \Lambda \) as follows.

Let \( \Lambda \) be the algebra associated with \( \Lambda^\bullet(V, W) \). The vertex set of the quiver \( \tilde{\Gamma}_Q \) of \( \Lambda \) is \( I \times \mathbb{Z} \) and the set of arrows of this quiver is given by
\[
(h, p) : (i, p) \to (j, p - 1) \text{ if } (i, p) \in \tilde{J} \text{ and } h : i \to j
\]
\[
(\tilde{h}, p) : (j, p) \to (i, p - 1) \text{ if } (i, p) \in \tilde{J} \text{ and } h : i \to j
\]
\[
(a, p) : (i, p) \to (i, p - 1) \text{ if } (i, p) \in \tilde{I}
\]
\[
(b, p) : (i, p) \to (i, p - 1) \text{ if } (i, p) \in \tilde{J}
\]
We denote by $R$ the idea generated by the equations (5.10). Therefore, we have $\Lambda = k\Gamma_Q/R$. Here we can think of $\Lambda$ as the algebra $R$ in the sense of [KS16].

Following [CFR14], if $W$ subject to (5.15), then the algebra $\Lambda$ is induced from the category $\mathcal{H}_Q$ whose objects are given by $0 \to P \to Q$ where $P, Q$ are the projective representation of $Q$, and whose morphisms are given by $(f, g)$ such that

$$
\begin{array}{c}
0 \\
\downarrow f \\
P' \\
\downarrow g \\
Q'
\end{array}
\quad
\begin{array}{c}
0 \\
\downarrow f' \\
P'' \\
\downarrow g' \\
Q''
\end{array}
$$

for any pair $(P \to Q), (P' \to Q')$. There is a finite set of generators given by the projective cover of nonprojective indecomposable representations $M_\beta$ of $Q$ and projective representations. Namely, we take the projective cover of the indecomposable representation $M_\beta$.

$$0 \to P_\beta \to Q_\beta \to M_\beta \to 0$$

if $M_\beta$ is not a projective representation of $Q$ and $(M_\beta = M_\beta)$ if $M_\beta$ is a projective representation. For simplicity, we also write $(P_\alpha \to Q_\alpha)$ for $(M_\beta = M_\beta)$ if $M_\beta$ is a projective representation of $Q$. Set

$$(5.19) \quad B_Q \overset{\text{def}}{=} \text{End}_{\mathcal{H}_Q}(\bigoplus_{\beta \in R^+} (P_\beta \to Q_\beta))^\text{op}$$

Following [CFR14, Theorem 4.11], we have

$$(5.20) \quad B_Q \cong \Lambda$$

We will define the Res functor from $\Lambda - \mod$ to $kQ - \mod$. Let $\mathcal{M}$ be a representation of $\Lambda$, we write it for $(V, W, x)$, where $V$ is the graded vector space such that $V_\beta \neq 0$ only if $M_\beta$ is not a projective representation, and $W_\beta \neq 0$ only if $M_\beta$ is a projective representation. It follows from the definition of $B_Q$ (5.20) that $x_h : V_\beta \to V_{\beta'}$ for each irreducible morphism $h : M_{\beta'} \to M_\beta$. One defines

$$(5.21) \quad \text{Res} : \Lambda - \mod \to kQ - \mod$$

by sending $\mathcal{M} = (V, W, x)$ to $(W, y)$ where $y : W_i \to W_j$ is given by the composition $x_{h_1}x_{h_2} \cdots x_{h_r}$ of a following path in the Auslander-Reiten Quiver $\Gamma_Q$ of $Q$.

$$(5.22) \quad P_i \overset{h_r}{\to} S_i \overset{h_{r-1}}{\to} \cdots \overset{h_2}{\to} \tau^{-1}S_j \overset{h_1}{\to} P_j$$

Where $\tau$ is the Auslander-Reiten translation.

We will define the functor $\hat{\mathcal{M}}$ from $kQ - \mod$ to $\Lambda - \mod$: Let $M$ be a representation of $Q$, for each root $\beta \in R^+$, we define

$$(5.23) \quad \hat{M}_\beta \overset{\text{def}}{=} \text{Im}(\text{Hom}_Q(Q_\beta, M) \to \text{Hom}_Q(P_\beta, M))$$
and the matrices $x_h : \widehat{M}_\beta \to \widehat{M}_{\beta'}$ by

\[
\begin{align*}
\xymatrix{
\text{Hom}_Q(Q_\beta, M) \ar[r] & \widehat{M}_{\beta'} \
\text{Hom}_Q(f, M) \ar[u] \ar[r] & \text{Hom}_Q(P_\beta, M) & \text{Hom}_Q(g, M) \ar[u] \ar[r] & \text{Hom}_Q(Q_{\beta'}, M) \\
\text{Hom}_Q(Q_{\beta'}, M) \ar[r] & \widehat{M}_{\beta'} & \text{Hom}_Q(P_{\beta'}, M) \ar[uu] &}
\end{align*}
\]

Where $f, g$ are induced by $M_{\beta'} \to M_\beta$. Therefore, we obtain the functor

(5.25) \[\hat{-} : kQ - \text{mod} \to \Lambda - \text{mod}\]

In what follows, we write $i$ for the root of indecomposable representation $P_i$, and always consider $\beta$ as the roots of nonprojective indecomposable representation of $Q$. For a dimension vector $d$ for $\Lambda$, we express $d = d_V + d_W$ as a partition of $d$ such that $d_V$ with support in $R^+/\{i\}_{i \in I}$ and $d_W$ with support in $\{i\}_{i \in I}$. For the corresponding graded vector $(V, W)$, we have that $\Lambda^\bullet(V, W)$ as in (5.10) is regarded as the representation variety $\text{Rep}_{d}(\Lambda)$. If $d_W = \gamma$, then Theorem 5.1 implies that

(5.26) \[\Phi : \mathfrak{M}_0^\bullet(W) \cong E_\gamma\]

In order to define the graded quiver varieties via this algebraic approach, we will give the meaning of stable points of $\text{Rep}_{d}(\Lambda)$.

We call a module $M$ over $\Lambda$ for a stable module if it subjects to $\text{Hom}_\Lambda(S_\beta, M) = 0$ for all simple modules $S_\beta$ with support on the nonprojective roots. We say that $M$ is costable if it subjects to $\text{Hom}_\Lambda(M, S_\beta) = 0$ for all simple modules $S_\beta$ with support on the nonprojective roots. Denote by $\Lambda^{st}(V, W)$ the subvariety consisting of the stable modules in $\text{Rep}_{d}(\Lambda)$. It is easy to see that the stable modules coincide with the stable points in the sense of [Nak04], as any submodule $S_\beta$ of $M$ is stable under the matrices $B_h$ and $\beta(S_\beta) = 0$ for (5.9).

Following [KS14, Section 3.4], we define graded quiver variety by the $G_V$-quotient

(5.27) \[\mathfrak{M}_0^\bullet(V, W) \overset{\text{def.}}{=} \Lambda^{st}(V, W)/G_V\]

As $\text{Res} g \cdot \mathcal{M} = \text{Res} \mathcal{M}$ for any $g \in G_V$, the Res functor (5.21) gives rise to a proper map as follows.

(5.28) \[\pi : \mathfrak{M}(V, W) \to \mathfrak{M}_0^\bullet(W)\]

In [KS14, Section 3.4], we have

(5.29) \[\mathfrak{M}_0^{\text{reg}}(V, W) \cong \{[\mathcal{M}] \in \mathfrak{M}(V, W) | \mathcal{M} \text{ is a costable module.}\}\]

For any element $M \in \mathfrak{M}_0^\bullet(W)$, we see that $\widehat{M}$ is a stable and costable module of $\Lambda$. It leads to that there exists a unique graded vector space $V$ such that $[\widehat{M}] \in \mathfrak{M}_0^{\text{reg}}(V, W)$. 

Following [LP13, Section 3.2], the functor $\hat{M}$ is a functor on the split Grothendieck group $K_0^{\text{split}}(kQ)$ generated by the isomorphic class of representations of $Q$, subject to the relation

$$[M] + [N] - [M \oplus N] = 0$$

They show that $\dim\hat{M}$ is uniquely determined by the representation $M$ up to isomorphism. That is: the following map

$$\dim\hat{-} : K^{\text{split}}_0(kQ) \rightarrow \mathbb{N}[\beta]_{\beta \in R^+}$$

$$M \mapsto \dim\hat{M}$$

is bijective.

**Definition 5.2.** We assume that $d_W = \gamma$ for some $\gamma \in Q^+$ and denote by $V(\gamma)$ the set of $v \in \mathbb{N}\Gamma_Q$ such that $\dim\hat{M} = v + \gamma$ for a $M \in E_\gamma$. By Theorem 5.1, there is a bijection $\Pi$ given by

$$\Pi : K\mathbb{P}(\gamma) \rightarrow V(\gamma)$$

$$\lambda \mapsto \dim\hat{M}_\lambda - \gamma$$

Suppose that $\lambda' \leq \lambda$, this means $\Omega_\lambda \subset \overline{\Omega}_{\lambda'}$. Using Theorem 5.1, we see that

$$\mathfrak{M}_{\lambda'}^{\text{reg}}(\Pi(\lambda), W) \subset \mathfrak{M}_\lambda^{\text{reg}}(\Pi(\lambda'), W)$$

It follows by (5.13) that $\Pi(\lambda) \leq \Pi(\lambda')$. Therefore, we obtain $\Pi(\lambda') \geq \Pi(\lambda)$ for any $\lambda' \leq \lambda$ in $K\mathbb{P}(\gamma)$. It is easy to see by Theorem 5.1 that $V(\gamma)$ is the set of dimension vector $v \in \mathbb{N}\Gamma_Q$ such that there is a stable and costable module $M$ of $\Lambda$ such that $\dim\hat{M} = v + \gamma$. For simplicity, we denote by $v_\lambda$ the image $\Pi(\lambda)$ of $\lambda \in K\mathbb{P}(\gamma)$. Following [CFR13] or [LP13, Corollary 3.15], $v_\lambda = (v_U)_{U \in \Gamma_Q}$ where

$$v_U = \dim \text{Im}(\text{Hom}_Q(Q_U, M) \rightarrow \text{Hom}_Q(P_U, M))$$

for the projective resolution of indecomposable non-projective representations $U$. It is easy to see that $\Pi(\mu \oplus \nu) = \Pi(\mu) \oplus \Pi(\nu)$.

Following [KS14, Section 4.5], we will define the following functor.

**Lemma 5.3.** We define the functor $CK$ by

$$CK : kQ - \text{mod} \rightarrow k\Gamma_Q - \text{mod}$$

$$M \mapsto (\text{Ext}_Q^1(M_\beta, M))_\beta$$

where $\beta$ runs over the non-projective roots. This functor is identified with the definition given by [KS14, Section 4.5]

**Proof.** For a representation $M$ of $Q$, the restriction of the functor $K_R$ as in [KS14] to the category $\mathcal{H}_Q$ equals

$$K_R(M)(\iota : P \rightarrow Q) = \text{Hom}_Q(Q, M) \rightarrow \text{Hom}_Q(P, M)$$
for any \( \iota : P \to Q \), and the restriction of \( K_{LR} \) as in [KS14] to the category \( \mathcal{H}_Q \) equals \( \hat{M} \). Following [KS14, Section 4.5], the functor \( CK(M) \) is the cokernel of the natural map \( \hat{M} \to K_R(M) \), namely,

\[
CK(M)(\iota : P \to Q) = \text{Ext}^1_Q(\text{coker} \, \iota, M)
\]

By [CER12], we have that the support of \( CK(M) \) is in the set of non-projective indecomposable modules \( U \) over \( kQ \). Since \( CK(M)(\iota : P_U \to Q_U) = \text{Ext}^1_Q(U, M) = D\text{Hom}_Q(M, \tau U) \) for the projective resolution of \( U \) and \( CK(M)(\text{Id} : P \to P) = 0 \), then \( CK(M) \) is a module of the Auslander algebra \( k\Gamma_Q \) by [CFR13, Section 6.3].

Here is an important theorem.

**Theorem 5.4 ([KS16])**. Under the above assumption, we have

1. \( \mathfrak{M}^{reg}_0(V, W) = \{ M \in \mathfrak{M}_0(W) \mid \text{dim} \, \hat{M} = (w, v) \} \);
2. [KS16, Lemma 4.2] The fiber of \( \pi : \mathfrak{M}^*(V, W) \to \mathfrak{M}^*_0(W) \) at the point \( M \) such that \( \text{dim} \, \hat{M} = (v_0, w) \) is homeomorphic to the quiver Grassmannian \( \text{Gr}_{v-v_0}(CK(M)) \).

Following [Nak11, Theorem 3.14], the fiber \( \pi^{-1}(M) \cong \pi^{-1}(0) \) where

\[
\pi^{-1}(v : \mathfrak{M}^*(V-W, W') \to \mathfrak{M}^*_0(W')
\]

where \( W' = W - C_q V^0 \)

### 5.4 Hall maps and graded quiver varieties

Let \( W \) be a graded vector space with dimension vector \( \text{dim} W = \gamma \) and fix a partition \( W = W_1 \oplus W_2 \) such that \( \text{dim} W_1 = \alpha \) and \( \text{dim} W_2 = \beta \). Recall the Hall maps (3.13)

\[
E_\alpha \times E_\beta \xleftarrow{\kappa} F_{\beta, \gamma} \xrightarrow{\iota} E_\gamma
\]

and the birational proper map \( \pi : \mathfrak{M}^*(V_\lambda, \gamma) \to \mathfrak{M}^*_0(V_\lambda, \gamma) = \overline{\mathcal{O}}_\lambda \subset E_\gamma \). It induces the following diagram:

\[
\begin{array}{ccc}
\mathfrak{M}^*(V_\lambda, \gamma) & \xleftarrow{\iota} & \iota^{-1}(\mathfrak{M}^*(V_\lambda, \gamma)) \\
\downarrow{\pi} & & \downarrow{\pi \times \pi} \\
\overline{\mathcal{O}}_\lambda & \xleftarrow{\iota} & F_{\beta, \gamma} \cap \overline{\mathcal{O}}_\lambda \\
& & \xrightarrow{\kappa} \bigcup_{(\mu, \nu) \in \text{ext}_{\alpha, \beta}^{\min}(\lambda)} \overline{\mathcal{O}}_\mu \times \overline{\mathcal{O}}_\nu
\end{array}
\]

Here

\[
\iota^{-1}(\mathfrak{M}^*(V_\lambda, \gamma)) = \{ F \in \mathfrak{M}^*(V_\lambda, \gamma) \mid \text{Res} F(W_2) \subset W_2 \}
\]

Denote by \( Y_{\beta, \gamma} \) the subvariety \( \iota^{-1}(\mathfrak{M}^*(V_\lambda, \gamma)) \). Recall that for a representation \( F \) of \( B_Q \), \( \text{Res} F \) is given by (5.22)

\[
x_{h_i} : W_{i,p} \to W_{(j,p-k-1)}, \text{ where } x_h = \beta(j,p-k-1) B_{h_1} \cdots B_{h_m} \alpha_{i,p}
\]
If \( x_h(W^1_{(i,p)}) \subset W^1_{(p-k)} \), then there exists \( V^1_{(i,p-k)} \) for any \((i, p - k)\) inductively such that \( \text{Im} \alpha_{(i,p)} \) and \( \text{Im} B_{h_k} \) are contained in \( V^1 \) for all \( k \). This gives rise to a subspace \((V^1, W^1)\) such that \( F(V^1, W^1) \subset (V^1, W^1) \). Therefore, we obtain

\[
Y_{\beta, \gamma} = \bigcup_{V_2 \subset V_\lambda} \{ F \in \mathcal{M}^*(V_\lambda, \gamma) \mid F(V_2, W_2) \subset (V_2, W_2) \}
\]

Since \( F \) is the \( G_v \)-orbit of the module over \( B_Q \), then the subspace \( V_2 \) is uniquely determined by its dimension vector \( v_2 \). In what follows, we always write \( v_2 \) for the subspace \( V_2 \).

**Lemma 5.5.** Under the above assumption, we have

\[
\text{Res}(F_{(\mathcal{V}_2, W_2)}) = (\text{Res} F)_{|W_2}
\]

**Proof.** Recall that the functor \( \text{Res} F \) is given by

\[
x_h : W^1_{\mathcal{V}_2} \to W^1_{\mathcal{V}_2} \quad \text{where} \quad x_h = \beta_j B_{h_1} \cdots B_{h_m} \alpha_i
\]

\( F(v_2, W_2) \subset (v_2, W_2) \) implies that the image of the restrictions of \( \alpha_i, B_{h_k} \) to \( W_2 \) is contained in \( V_2 \). Hence, we get the map \( x_h \) doesn’t change under the restriction \((V_2, W_2)\). \( \square \)

**Definition 5.6.** Let \( v \) be a dimension vector. We call a partition \((v_1, v_2)\) of \( v \), \( v_1 + v_2 = v \), a **maximal partition** if there is an element \( F \in Y_{\beta, \gamma} \) such that \( F(V_2, W_2) \subset (V_2, W_2) \) and there exists no \( V'_2 \) such that \( V_2 \subset V'_2 \) and \( F(V'_2, W_2) \subset (V'_2, W_2) \). That is, \( v_2 \) is the maximal dimension vector of the subspaces \( V'_2 \) satisfying \( F(V'_2, W_2) \subset (V'_2, W_2) \) for the element \( F \in Y_{\beta, \gamma} \). We denote by \( KQ(v) \) the set of the maximal partition of \( v \).

Considering (5.35), we see that

\[
Y_{\beta, \gamma} = \bigcup_{(v_1, v_2) \in KQ(v)} Y_{\beta, \gamma}(v_1, v_2)
\]

where \( Y_{\beta, \gamma}(v_1, v_2) \) is \( \{ F \in \mathcal{M}^*(V_\lambda, \gamma) \mid F(V_2, W_2) \subset (V_2, W_2) \} \). Note that for any two different pairs \((v_1, v_2)\) and \((v'_1, v'_2)\), we have

\[
Y_{\beta, \gamma}(v_1, v_2) \cap Y_{\beta, \gamma}(v'_1, v'_2) = \emptyset
\]

Otherwise, let \( F \) be an nonzero element of \( Y_{\beta, \gamma}(v_1, v_2) \cap Y_{\beta, \gamma}(v'_1, v'_2) \), it follows that \( F(V_2, W_2) \subset (V_2, W_2) \) and \( F(V'_2, W_2) \subset (V'_2, W_2) \). Let \( V' = V_2 \oplus V'_2 \), it is easy to see that

\[
F(V_2 \oplus V'_2, W_2) \subset (V_2 \oplus V'_2, W_2)
\]

This is contradicted with the assumption that \( v_1 \) is maximal.

Let \( \lambda' \) be the map

\[
\kappa_{v_1, v_2} : Y_{\beta, \gamma}(v_1, v_2) \to \mathcal{M}^*(V_1, \alpha) \times \mathcal{M}^*(V_2, \beta)
\]

\[
F \mapsto F_{(v_1, v_1)} \times F_{(v_2, v_2)}
\]

We will show that \( F_{((v_2, v_2))} \) is a stable point. In other words, there is no subspace \( 0 \neq V' \subset V_2 \) such that \( F_{((v_2', v_2'))} \) is a sub representation of \( F_{((v_2, v_2))} \). Otherwise, \( F_{((v_2', v_2'))} \) is a nonzero subrepresentation of \( F \), which is contradict to the stable property of \( F \). Let us consider the
case \( F|_{(V_1, W_1)} \), if there exists nonzero subspace \( V' \) such that the restriction of \( F|_{(V_1, W_1)} \) to \((V', 0)\) is a subrepresentation of \( F|_{(V_1, W_1)} \), then the subspace \((V_2 \oplus V', W)\) satisfies 

\[
F(V_2 \oplus V', W) \subset (V_2 \oplus V', W)
\]

This is impossible for \((V_2, W)\), as we assume that \( V_2 \) is maximal.

Next, we will show that \( Y_{\beta, \gamma}(v_1, v_2) \) is an irreducible component. Following [VV03], we set a cocharacter \( \chi = q \text{Id}_{W_2} \oplus \text{Id}_{W_1} \). We will show that the subset \( F_{\beta, \gamma} \) is identified with

\[
(5.40) \quad \mathfrak{M}_{\beta, \gamma}^0(W)^{+X} = \left\{ x \in \mathfrak{M}_{\beta, \gamma}^0(W) \mid \lim_{q \to 0} \chi(q)x \text{ exists} \right\}
\]

Let us consider an arrow \( h : i \to j \) and the decompositions \( W^i = W^i_2 \oplus W^i_1 \) and \( W^j = W^j_2 \oplus W^j_1 \). The action \( \chi(q) \) on \( x_h \) is given by

\[
(5.41) \quad \left( \begin{array}{ccc}
q \text{Id} & 0 \\
0 & \text{Id}
\end{array} \right) \left( \begin{array}{ccc}
a & b \\
c & d
\end{array} \right) \left( \begin{array}{ccc}
q^{-1} \text{Id} & 0 \\
0 & \text{Id}
\end{array} \right) = \left( \begin{array}{ccc}
a & bq \\
cq^{-1} & d
\end{array} \right)
\]

where \( x_h = \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) \). The condition \( \lim_{q \to 0} \chi(q)x \) exists if and only if for each arrow \( h \), \( x_h \) is of the form \( \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) \). This means that the condition \( \lim_{q \to 0} \chi(q)x \) exists if and only if \( x \in F_{\beta, \gamma} \), as \( \mathfrak{M}_{\beta, \gamma}^0(W) \cong E_{\gamma} \) by Theorem 5.1.

Therefore, By [VV03, Remark 3.4], we see that \( Y_{\beta, \gamma} = \mathfrak{M}_{\beta, \gamma}^0(W)^{+X} \) and then obtain

\[
(5.42) \quad \kappa_{v_1, v_2} : Y_{\beta, \gamma}(v_1, v_2) \to \mathfrak{M}_{\beta, \gamma}^0(v_1, \alpha) \times \mathfrak{M}_{\beta, \gamma}^0(v_2, \beta)
\]

is a vector bundle whose fiber with dimension \( d(v_1, \alpha, v_2, \beta) \) (see (5.17)). Therefore, we obtain \( Y_{\beta, \gamma}(v_1, v_2) \) is an irreducible variety, as \( \mathfrak{M}_{\beta, \gamma}^0(v_2, \alpha) \times \mathfrak{M}_{\beta, \gamma}^0(v_1, \beta) \) is an irreducible variety. That means (5.38) gives rise to a set of irreducible components of \( Y_{\beta, \gamma} \).

5.4.1. **Open subvariety.** It is easy to see that the restriction \( \pi_{\beta} \) to \( \mathcal{O}_\lambda \cap F_{\beta, \gamma} \) is birational and its restriction to \( \iota^{-1}\mathfrak{M}_{\beta, \gamma}^0(V_\lambda, \gamma) \) is isomorphic to \( \mathcal{O}_\lambda \cap F_{\beta, \gamma} \). The equation (4.3) implies that

\[
G_\lambda / \text{Aut}_Q(M_\lambda) \cong \mathcal{O}_\lambda \cap F_{\beta, \gamma} = \bigcup_{(\mu, \nu) \in \text{ext}_{\alpha, \beta}^\text{reg}(\lambda)} U(\mu, \nu)
\]

here \( U(\mu, \nu) \) is the subvariety consisting of \( M' = gM_\lambda \) such that \( M'_W \geq M_\nu \) and \( M'_W \geq M_\mu \). For the second equation, it follows from Lemma 4.2 and \( \text{Aut}_Q(M_\lambda) \), \( P_{\beta, \gamma} \) are irreducible affine varieties. More precisely, the condition (4.4) implies that for any element \( g \in G_\lambda \)

\[
(5.43) \quad (x_{\lambda})_{g^{-1}W} = x_{\nu} \iff (g \cdot x_{\lambda})_W = x_{\nu}
\]

\[
(5.44) \quad (x_{\lambda})_{W/g^{-1}W} = x_{\nu} \iff (g \cdot x_{\lambda})_{W/W} = x_{\nu}
\]

Hence, the irreducible component \( S_\nu \) corresponds to the irreducible component \( U(\mu, \nu) \) as above, where \( \mu \) is the Konstanz partition associated with the generic quotient \( M_\mu \) of \( M_\lambda \) by \( M_\mu \).

The formula \( \iota^{-1}(\mathfrak{M}_{\beta, \gamma}^0(V_\lambda, \gamma)) \cong F_{\beta, \gamma} \cap \mathcal{O}_\lambda \) implies that

\[
(5.44) \quad Y_{\beta, \gamma}^\text{reg} := \iota^{-1}(\mathfrak{M}_{\beta, \gamma}^0(V_\lambda, \gamma)) = \bigcup_{(\mu, \nu) \in \text{ext}_{\alpha, \beta}^\text{reg}(\lambda)} U(\mu, \nu)
\]
This leads to the set of components of \( Y_{\beta,\gamma} \) coincides with the set of components of \( \text{Gr}_{\mu}(M_{\lambda}) \). That means that for each \((\mu, \nu) \in \text{Ext}_{\alpha, \beta}^{ger}(\lambda)\), there exists a unique maximal pair \((v_1, v_2)\) such that

\[
(5.45) \quad Y_{\beta,\gamma}(v_1, v_2) = U(\mu, \nu)^Y
\]

Here \( U(\mu, \nu)^Y \) refers to the closure of \( U(\mu, \nu) \) in the \( Y_{\beta,\gamma} \).

**Lemma 5.7.** Under the above assumption, we have that

\[
(5.46) \quad \text{Ext}_{\alpha, \beta}^{ger}(\lambda) = \{(v_1, v_2) \mid (v_1, v_2) \text{ is a maximal partition of } (v_{\lambda}, \gamma) \text{ for } (\alpha, \beta)\}
\]

Let us review the diagram (5.33). For a maximal partition \( v_1 + v_2 = v_{\lambda} \), we will show that the following diagram is commutative,

\[
(5.47) \quad \begin{array}{c}
Y_{\beta,\gamma}(v_1, v_2) \\
\downarrow \pi_{\beta}
\end{array} \quad \begin{array}{c}
\mathfrak{M}^\bullet(v_2, \alpha) \times \mathfrak{M}^\bullet(v_1, \beta)
\end{array} \quad \begin{array}{c}
U(\mu, \nu)
\end{array} \quad \begin{array}{c}
\kappa_{\lambda}
\end{array} \quad \begin{array}{c}
\mathfrak{O}_\mu \times \mathfrak{O}_\nu
\end{array}
\]

Where \((\mu, \nu)\) is the pair corresponding to \((v_1, v_2)\) via (5.45).

Taking \( F \in Y_{\beta,\gamma}(v_1, v_2) \), we have

\[
\kappa_{\lambda}\pi_{\beta}(F) = \kappa_{\lambda}\text{Res } F
\]

\[
= ((\text{Res } F)|_{W_2}, (\text{Res } F)|_{W_1})
\]

\[
= (\text{Res } F|_{V_2,W_2}, \text{Res } F|_{V_1,W_1}) \quad \text{By Lemma 5.5}
\]

\[
= (\pi \times \pi)\kappa_{v_1,v_2}(F)
\]

We define \( \pi \times \pi : \bigcup_{v_1+v_2=v_{\lambda}} \mathfrak{M}^\bullet(v_1, \alpha) \times \mathfrak{M}^\bullet(v_2, \beta) \to \bigcup_{\mu, \nu \in \text{Ext}_{\alpha, \beta}^{min}(\lambda)} \mathfrak{O}_\mu \times \mathfrak{O}_\nu \) by

\[
(5.48) \quad (F|_{(V_1,W_1)}, F|_{(V_2,W_2)}) \mapsto (\text{Res } F|_{(V_1,W_1)}, \text{Res } F|_{(V_2,W_2)})
\]

**Remark 5.8.** Note that the map \( \pi \times \pi \) is not a birational map in general, as

\[
\widehat{\text{Res } F} \neq F
\]

in general, (see [CFR14, Theorem 3.4]). Indeed, the \( \dim \widehat{\text{Res } F} \leq \dim F \), the equation holds if and only if \( F \) is a bistable point. This mean the map \( \pi \times \pi \) is a birational map if and only if \((v_\mu, v_\nu) = (v_1, v_2)\). This follows that \( M_{\lambda} = M_{\nu} \oplus M_{\mu} \).

We remark that the decomposition (5.35) of \( Y_{\beta,\gamma} \) coincides with the decomposition in the sense of Varagnolo and Vasserot [VV03] or Nakajima [Nak11, Section 3.5]. Recall that the
diagram for $E_\gamma$ is given by [VV03, Section 3.1]

\[
\begin{array}{ccc}
\bigcup_{v_1+v_2=v} \mathfrak{M}^* (v_1, \alpha) \times \mathfrak{M}^* (v_2, \beta) & \overset{\iota^{-1}(\mathfrak{M}^* (v, \gamma))}{\longleftarrow} & \mathfrak{M}^* (v, \gamma) \\
\downarrow \pi \times \pi & & \downarrow \pi \\
\bigcup_{v_1+v_2=v} \mathfrak{M}^*_0 (v_1, \alpha) \times \mathfrak{M}^*_0 (v_2, \beta) & \overset{\iota}{\longrightarrow} & \mathfrak{M}^*_0 (v, \gamma)
\end{array}
\] (5.49)

Here

\[
\iota^{-1}(\mathfrak{M}^* (v, \gamma)) = \bigcup_{v_1+v_2=v} \mathfrak{M} (v_1, v_2, \beta)
\] (5.50)

such that $\mathfrak{M} (v_1, v_2, \beta)$ is a vector bundle over $\mathfrak{M}^* (V_1, \alpha) \times \mathfrak{M}^* (V_2, \beta)$ of rank $d(v_1, \alpha, v_2, \beta)$ (see (5.17)). That means $\mathfrak{M} (v_1, v_2, \beta) = Y_{\beta, \gamma}(v_1, v_2)$. 

5.4.2. A birational map. Recall that the extension variety $\mathcal{E}xt_\lambda (\mu, \nu) = \kappa^{-1}_\lambda (M_\mu, M_\nu)$. We will give a birational map on $\mathcal{E}xt_\lambda (\mu, \nu)$.

For a pair $(\mu, \nu)$, the subvariety $\kappa^{-1}_\lambda (M_\mu, M_\nu) \cap \mathbb{O}_\lambda$ is an open subvariety of $\kappa^{-1}_\lambda (M_\mu, M_\nu)$. By Proposition 4.4, we see that this subvariety is an irreducible variety. Recall the decomposition (5.44)

\[
\kappa^{-1}_\lambda (M_\mu, M_\nu) \cap \mathbb{O}_\lambda \subset \mathbb{O}_\lambda \cap F_{\beta, \gamma} = \bigcup_{(\mu', \nu') \in \text{ext}^\text{ger}_{\alpha, \beta}(\lambda)} U(\mu', \nu')
\]

The disjoint union follows from (5.38) and (5.45). This means that there exists a unique $(\mu', \nu') \in \text{ext}^\text{ger}_{\alpha, \beta}(\lambda)$ such that

\[
\kappa^{-1}_\lambda (M_\mu, M_\nu) \cap \mathbb{O}_\lambda \subset U(\mu', \nu')
\]

Therefore, we obtain

\[
\mathcal{E}xt_\lambda (\mu, \nu) = \kappa^{-1}_\lambda (M_\mu, M_\nu) \subset U(\mu', \nu')
\] (5.51)

That means it is enough to consider the diagram (5.47) for $(\mu', \nu')$ to know $\mathcal{E}xt_\lambda (\mu, \nu)$.

\[
\begin{array}{ccc}
Y_{\beta, \gamma}(v_1, v_2) & \overset{\kappa_{v_1,v_2}}{\longrightarrow} & \mathfrak{M}^* (v_2, \alpha) \times \mathfrak{M}^* (v_1, \beta) \\
\downarrow \pi \times \pi & & \downarrow \pi \\
U(\mu', \nu') & \overset{\kappa_\lambda}{\longrightarrow} & \mathbb{O}_{\mu} \times \mathbb{O}_{\nu'}
\end{array}
\] (5.52)

Where $(v_1, v_2)$ is given by (5.45). Let us define $\pi \mathcal{E}xt_\lambda (\mu, \nu)$ by

\[
\pi \mathcal{E}xt_\lambda (\mu, \nu) = \kappa_{v_1,v_2}^{-1} (\text{Gr}_{v_1-v_\mu} (CK(M_\mu)) \times \text{Gr}_{v_2-v_\nu} (CK(M_\nu)))
\] (5.53)

Where $\kappa_{v_1,v_2}$ is the map in (5.52), and $\text{Gr}_{v_1-v_\mu} (CK(M_\mu))$ and $\text{Gr}_{v_2-v_\nu} (CK(M_\nu))$ are quiver Grassmannians of $CK(M_\mu)$ and $CK(M_\nu)$, respectively. By Theorem 5.4, we have

\[
(\pi \times \pi)^{-1} (M_\mu, M_\nu) = \text{Gr}_{v_1-v_\mu} (CK(M_\mu)) \times \text{Gr}_{v_2-v_\nu} (CK(M_\nu))
\]

This means by (5.51) that

\[
\pi \mathcal{E}xt_\lambda (\mu, \nu) = \kappa_{v_1,v_2}^{-1} (\pi \times \pi)^{-1} (M_\mu, M_\nu) = \pi_\beta^{-1} (\mathcal{E}xt_\lambda (\mu, \nu))
\]
Since the restriction $\pi_\beta$ to $\bigcap_\lambda \cap \kappa^{-1}_\lambda(M_\mu, M_\nu)$ is an isomorphism, we obtain the following birational map

$$\pi_\beta : \pi \mathcal{E}xt(\mu, \nu) \to \mathcal{E}xt(\mu, \nu) \quad (5.54)$$

On the other hand, $\pi \mathcal{E}xt(\mu, \nu)$ is a vector bundle over

$$\text{Gr}_{v_1 - v_\nu}(CK(M_\mu)) \times \text{Gr}_{v_2 - v_\nu}(CK(M_\nu))$$

with dimension of the fibers $d(v_1, \alpha, v_2, \beta)$ by (5.42). Therefore, we have the following Lemma.

**Lemma 5.9.** Under the above assumption and given a pair $(\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta)$, let $\lambda \in \text{ext}(\mu, \nu)$. There is a unique maximal pair $(v_1, v_2) \in \text{KQ}(\lambda)$ such that

$$\pi_\beta : \pi \mathcal{E}xt(\mu, \nu) \to \mathcal{E}xt(\mu, \nu)$$

is a birational map. Here $\pi \mathcal{E}xt(\mu, \nu)$ is a vector bundle over

$$\text{Gr}_{v_1 - v_\nu}(CK(M_\mu)) \times \text{Gr}_{v_2 - v_\nu}(CK(M_\nu))$$

with dimension of the fibers $d(v_1, \alpha, v_2, \beta)$.

## 6. Restriction and Induction functors

In this section, we review the Lusztig’s category and the perverse sheaves on graded quiver varieties.

### 6.1. Lusztig’s sheaves.

Let $\alpha \in Q^+$ and $(I)_\alpha$ be the set of words $[i_1i_2\cdots i_r]$ of $I$ such that $\sum_{k=1}^r \alpha_{i_k} = \alpha$. A word $i = [i_1i_2\cdots i_r] \in (I)_\alpha$, and a $I$-graded vector space $V$ such that $\dim V = \alpha$, let us define its flag variety as follows.

$$\mathcal{F}_1 = \{(0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = V) \mid \text{such that } \dim V_k/V_{k-1} = \alpha_{i_k}\} \quad (6.1)$$

Recall there is a vector bundle over it given by

$$\mathcal{F}_1 = \{(\phi, x) \mid \phi \in \mathcal{F}_1 \text{ and } x \in E_\alpha(Q) \text{ such that } x(V_k) \subset V_k\} \quad (6.2)$$

It yields a proper map as follows.

$$p : \mathcal{F}_1 \to E_\alpha(Q) \quad (6.3)$$

Let us recall the perverse sheaves on $E_\alpha(Q)$. Since $\mathcal{F}_1$, $E_\alpha(Q)$ are smooth varieties, there are constant perverse sheaves on them: $\mathcal{I}_{\mathcal{F}_1} = \mathbb{C}_{\mathcal{F}_1}[\dim \mathcal{F}_1]$ and $\mathbb{I}_{E_\alpha(Q)} = \mathbb{C}_{E_\alpha(Q)}[\dim E_\alpha(Q)]$.

The map $p$ gives rise to a complex $Q_1 = p^*\mathcal{I}_{\mathcal{F}_1}$ for a number $h_i$, which is called Lusztig’s sheaf. Because $h_i$ will not be mentioned in this paper, we always omit it for simplicity. By BBD Decomposition theorem, Lusztig’s sheaves are decomposed as a direct sum of simple perverse sheaves on $E_\alpha(Q)$. We denote $\mathcal{P}_\alpha$ by the set of simple perverse sheaves appearing in a summand of $\mathcal{F}_1$ for some word $i \in (I)_\alpha$. We write it as $\mathcal{P}$ if there is no danger of confusion. Since the perverse sheaves are of the form $\text{IC}(X, \lambda)$ where $X$ is local closed smooth subvarieties
of $E_\alpha(Q)$, $\lambda$ is a local system on $X$ and $\text{IC}(X, \lambda)$ is simple perverse sheaf associated with them. We always write $\lambda$ or $\text{IC}(\lambda)$ for $\text{IC}(X, \lambda)$ for the sake of simplicity. Hence, we get

$$(6.4) \quad L_i = \bigoplus_{\lambda \in P} L_i(\lambda) \boxtimes \text{IC}(\lambda)$$

where $L_i(\lambda)$ are $\mathbb{Z}$-graded vector spaces.

In particular, in Dynkin cases, all the simple perverse sheaves are of the form $\text{IC}(\mathbb{C}_\mathcal{O}[[\dim \mathcal{O}]]$ where $\mathcal{O}$ run over all $G_\alpha$-orbits.

**Borel-Moore Homology theory.** In this section, we will recall the notion of Borel-Moore homology and assume that $X$ is equivdimensional. Denote by $d$ the dimension of $X$.

**Definition 6.1.** For an algebraic variety $X$ over $\mathbb{C}$, let $\mathbb{D}_X$ be the dual complex of $X$, one defines its Borel-Moore Homology by

$$H_k(X) \overset{\text{def}}{=} H^{-k}(X, \mathbb{D}_X) = H^{-k}(p_* \mathbb{D}_X)$$

where $p : X \to \text{Spec}(\mathbb{C})$. We remark that $H^{-k}(p_* \mathbb{D}_X) \cong H^k(p_! \mathbb{C}_X)$, as $\mathbb{D}(p_! \mathbb{C}_X) = p_* \mathbb{D}_X$.

For $k = 2d$ we have that $H_{2d}(X) = \{ \text{components of } X \}$. If $i : Y \subset X$ is a closed subvariety of $X$ with dimension $r$, then the map $i_* i^! \mathbb{D}_X \to \mathbb{D}_X$ and equation $i^! \mathbb{D}_X = \mathbb{D}_Y$ induce a map $i_* : H_k(Y) \to H_k(X)$. It leads to $i_* [Y] \in H_{2r}(X)$. Thus, we can consider $[Y]$ as an element in $H_{2r}(X)$.

**Definition 6.2.** For a variety $X$ over $\mathbb{C}$, we define its Poincare polynomial by

$$P_q(X) \overset{\text{def}}{=} \sum_{k \in \mathbb{N}} \dim H_k(X) q^k$$

**Lemma 6.3.** Let $\pi : X \to Y$ be a proper map and $\mathbb{C}_X$ be the constant sheaf over $X$. For the complex $\pi_! \mathbb{C}_X$ and any point $y \in Y$, we have

$$(6.5) \quad (\pi_! \mathbb{C}_X)_y \cong H_{\bullet}(\pi^{-1}(y))$$

If $\pi$ has the isomorphic fiber at each point $y \in Y$, we have

$$\pi_! \mathbb{C}_X \cong P_q(\pi^{-1}(y)) \mathbb{C}_Y$$

6.2. **Restriction Functor.** We define the restriction functor as

$$\text{Res}^\gamma_{\alpha, \beta} : D^b_{G_\gamma}(E_\gamma(Q)) \to D^b_{G_\alpha \times G_\beta}(E_\alpha(Q) \times E_\beta(Q))$$

$$(6.6) \quad \mathbb{P} \mapsto \kappa_1 t^*(\mathbb{P})[-\langle \alpha, \beta \rangle]$$

Let $\lambda \in \text{KP}(\gamma)$ be a Konstant partition of $\gamma$, and $\text{IC}(\lambda)$ be the simple perverse sheaf associated with $\lambda$. It is well known that

$$(6.7) \quad \text{Res}^\gamma_{\alpha, \beta}(\text{IC}(\lambda)) = \sum_{(\mu, \nu) \in \text{KP}(\alpha) \times \text{KP}(\beta)} K_{\lambda}(\mu, \nu) \text{IC}(\mu) \boxtimes \text{IC}(\nu)$$

Where $K_{\lambda}(\mu, \nu) \in \mathbb{Z}[q, q^{-1}]$. 

Lemma 6.4. Let $\gamma = \alpha + \beta$ be a partition of $\gamma \in Q^+$, and $\lambda \in \text{KP}(\gamma)$, $\mu \in \text{KP}(\alpha)$ and $\nu \in \text{KP}(\beta)$. We set

$$d_\lambda(\mu, \nu) := \dim \mathcal{O}_\lambda - \langle \alpha, \beta \rangle - \dim \mathcal{O}_\mu - \dim \mathcal{O}_\nu$$

then we have

$$d_\lambda(\mu, \nu) = \alpha \cdot \beta + \beta \cdot \alpha + [M_\mu, M_\mu] + [M_\nu, M_\nu] - [M_\lambda, M_\lambda] - \langle \alpha, \beta \rangle$$

Proof. First, we have $\dim \mathcal{O}_\nu = \dim E_\eta - [M_\nu, M_\nu]$ where $\nu$ refers to $\lambda, \mu, \nu$ and $\eta$ refers to $\gamma, \alpha, \beta$ respectively. Using 2.4, we obtain

$$\dim \mathcal{O}_\nu = \dim E_\eta - [M_\nu, M_\nu]$$

$$= - \langle \eta, \eta \rangle + \eta \cdot \eta - [M_\nu, M_\nu]$$

$$= \eta \cdot \eta - [M_\nu, M_\nu]$$

where $\eta \cdot \eta = \sum_{i=1}^n \eta_i^2$. Thus, we have

$$\dim \mathcal{O}_\lambda - \dim \mathcal{O}_\mu - \dim \mathcal{O}_\nu - \langle \alpha, \beta \rangle$$

$$= \gamma \cdot \gamma - [M_\lambda, M_\lambda] - \alpha \cdot \alpha + [M_\mu, M_\mu] + \beta \cdot \beta + [M_\nu, M_\nu] - \langle \alpha, \beta \rangle$$

$$= \alpha \cdot \beta + \beta \cdot \alpha + [M_\mu, M_\mu] + [M_\nu, M_\nu] - [M_\lambda, M_\lambda] - \langle \alpha, \beta \rangle$$

It follows that $d_\lambda(\mu, \nu) = \alpha \cdot \beta + \beta \cdot \alpha + [M_\mu, M_\mu] + [M_\nu, M_\nu] - [M_\lambda, M_\lambda] - \langle \alpha, \beta \rangle$. \hfill \qed

Lemma 6.5. If $(\mu', \nu') < (\mu, \nu)$ then we have $d_{\mu, \nu}(\lambda) > d_{\mu', \nu'}(\lambda)$

Proof. If $(\mu', \nu') < (\mu, \nu)$ then we have $h(\lambda, \mu', \nu') < h(\lambda, \mu, \nu)$. It follows that $d_{\mu, \nu}(\lambda) > d_{\mu', \nu'}(\lambda)$ \hfill \qed

6.3. Induction functor. Let us recall the notion of Induction functor.

Definition 6.6. Recall the maps in equation (3.10). We define the Induction Functor by

$$\text{Ind} : D^b_{\lambda, \gamma}(E_\mu) \times D^b_{\lambda, \gamma}(E_\nu) \to D^b_{\lambda, \gamma}(E_\lambda)$$

$$(P, P') \mapsto qrp^*(P \boxtimes P')[\dim p]$$

Theorem 6.7. Let $\alpha, \beta$ be two elements in $Q^+$. For two Konstant partitions $\mu, \nu$ in $\text{KP}(\alpha)$ and $\text{KP}(\beta)$, respectively. We have

$$\text{IC}(\mu) * \text{IC}(\nu) = \bigoplus_{\lambda \in \text{KP}(\gamma)} J_\lambda(\mu, \nu) \text{IC}(\lambda)$$

where $J_\lambda(\mu, \nu) \neq 0$ only if there is a $\lambda' \in \text{ext}(\mu, \nu)$ such that $\lambda \geq \lambda'$.

Proof. Denote by $Y = rrp^{-1}((\mathcal{O}_\mu \times \mathcal{O}_\nu))$. By Lemma 3.2, the support of $\text{IC}(\mu) * \text{IC}(\nu)$ is contained in the set

$$\bigcup_{\lambda \in \text{ext}(\mu, \nu)} \mathcal{U}_\lambda$$

It follows that our conclusion. \hfill \qed
6.4. Perverse sheaves on graded quiver varieties. In this section, we will relate the perverse sheaves to the complex $\pi(V, W)$ induced from the graded quiver varieties, (see [Nak11]).

**Theorem 6.8.** Let $W$ be a $I$-graded vector space as in (5.15) such that $\dim W = \gamma$, and recall the map (5.28).

1. [Nak11, equation 3.15] For the proper map $\pi : M^\bullet(V, W) \to M^\bullet_0(V, W)$, we obtain

$$\pi_*(C_{\pi(M^\bullet(V, W))} \dim M^\bullet(V, W)) = \bigoplus_{V' \leq V} L_W(V, V') \boxtimes IC_W(V')$$

Here $V_v(w) = \{ v' \leq v \mid M^\bullet_0(V', W) \neq \emptyset \}$ and the polynomial of $L_W(V, V')$ is $a_{V, V', W}(t)$ in the sense of [Nak11, Formula 3.15]. Here $a_{V, V', W}(t) = 1$, $a_{V, V', W}(t) \neq 0$ unless $V' \leq V$, $a_{V, V', W}(t) = a_{V, V', W}(t^{-1})$, and

$$a_{V, V', W}(q) = a_{V, V', W}(q) \in q^{\dim M^\bullet(V - V', W)} [q, q^{-1}]$$

2. [SS16, Proposition 4.8] Suppose that $\dim W = \gamma$ and a decomposition $W = W_1 \oplus W_2$ such that $\dim W_1 = \alpha$ and $\dim W_2 = \beta$. The Restriction functor (6.6) on $\pi(v, w)$ is

$$Res_{\alpha, \beta} \pi(v, \gamma) = \bigoplus_{v_1 + v_2 = v} q^{\epsilon(v_2, \beta, v_1, \alpha)} \pi(v_1, \alpha) \boxtimes \pi(v_2, \beta)$$

Where $\epsilon(v_2, \beta, v_1, \alpha)$ is given in 5.18.

**Corollary 6.9.** Under the assumption in Theorem 5.1, for each $\lambda \in KP(\gamma)$ we have a resolution of orbit $\overline{\mathcal{O}}_\lambda$ with respect to the stratification of $G_{\gamma}$-orbits.

$$\pi_\lambda : M^\bullet(V, W) \to \overline{\mathcal{O}}_\lambda$$

for the $I$-graded vector space $V_\lambda$ given in Theorem 5.1. In particular, we have

$$\pi_*(C_{\pi(M^\bullet(V, W))} \dim M^\bullet(V, W)) = \bigoplus_{\lambda \geq \lambda} f_{V, W}(\lambda') IC(\lambda')$$

where $f_{V, W}(\lambda) \in \mathbb{Z}[q, q^{-1}]$.

**Proof.** Following the above theorem and Theorem 5.1. □

**Theorem 6.10.** Under the above assumption, let $(\mu, \nu) \in KP(\alpha) \times KP(\beta)$ and $\lambda \in \text{ext}(\mu, \nu)$. We have that the top degree of $K(\mu, \nu)$ is less than or equal to

$$2e_\lambda + d_\lambda(\mu, \nu)$$

If $(\mu, \nu) \in \text{ext}^{ger}(\alpha, \beta)$, then

$$K(\mu, \nu) = q^{\epsilon(v_1, \alpha, v_2, \beta)} P_q(Gr_{v_1 - v_\mu}(CK(M_\mu)) \times Gr_{v_2 - v_\nu}(CK(M_\nu)))$$

where $e_\lambda(\mu, \nu)$ is the dimension of $\text{Ext}_\lambda(\mu, \nu)$, (see 4.4).

**Proof.** Since $\pi : M^\bullet(V, W) \to M^\bullet_0(V, W)$ is a proper and birational map, we have

$$\pi(V, W)|_{\mathcal{O}_\lambda} \cong IC(\lambda)|_{\mathcal{O}_\lambda}$$
Denote by $Z$ the image $\kappa_\lambda(\Omega_\lambda \cap F_{\beta,\gamma})$ and by $j: Z \to E_\alpha \times E_\beta$ its embedding. Let us set

$$Z^0 \overset{\text{def.}}{=} \bigcup_{(\mu,\nu) \in \text{ext}^{ger}_{\alpha,\beta}(\lambda)} \Omega_\mu \times \Omega_\nu$$

It is easy to see by that $Z^0$ is an open subvariety of $Z$. Consider the following diagram,

$$\begin{array}{cccc}
\mathcal{M}^*(V_\lambda, \gamma) & \xleftarrow{\epsilon} & \epsilon^{-1}(\mathcal{M}^*(V_\lambda, \gamma)) & \xrightarrow{\kappa} & \bigcup_{v_1 + v_2 = v_\lambda} \mathcal{M}^*(v_1, \alpha) \times \mathcal{M}^*(v_2, \beta) \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \times \pi \\
\Omega_\lambda & \leftarrow & F_{\beta,\gamma} \cap \Omega_\lambda & \xrightarrow{\kappa_\lambda} & \bigcup_{(\mu,\nu) \in \text{ext}^{min}_{\alpha,\beta}(\lambda)} \Omega_\mu \times \Omega_\nu \\
\downarrow \epsilon & & \downarrow j & & \downarrow j' \\
\Omega_\lambda & \leftarrow & F_{\beta,\gamma} \cap \Omega_\lambda & \xrightarrow{\kappa_Z} & Z \\
\end{array}$$

(6.13)

It follows that

$$j^* \text{Res}_{\alpha,\beta} \pi(V_\lambda, \gamma) \cong j^* \kappa_\lambda \mu^* \pi[\langle \alpha, \beta \rangle]$$

$$= \kappa_{Z,1} j''^* \pi_1 \pi[\langle \alpha, \beta \rangle] \quad \text{By } j^* \kappa_\lambda = \kappa_{Z,1} j''^*$$

$$= \kappa_{Z,1} j''^* j^* \pi_1 \pi[\langle \alpha, \beta \rangle] \quad \text{By } j'' \cdot j' = j''$$

$$= \kappa_{Z,1} \epsilon''^* \pi(V_\lambda, \gamma) |_{\Omega_\lambda}$$

$$= \text{Res}_{\alpha,\beta} \pi(V_\lambda, \gamma) |_{\Omega_\lambda}$$

Similarly, we see that $j^* \text{Res}_{\alpha,\beta} \text{IC}(\lambda) \cong \text{Res}_{\alpha,\beta} \text{IC}(\lambda) |_{\Omega_\lambda}$. The identity (6.12) implies that

(6.14) $$j^* \text{Res}_{\alpha,\beta} \pi(V_\lambda, \gamma) \cong j^* \text{Res}_{\alpha,\beta} \text{IC}(\lambda)$$

By Theorem 6.8, we obtain

$$j^* \text{Res}_{\alpha,\beta} \pi(V_\lambda, \gamma) \cong \sum_{(v_1, v_2) \in \text{KQ}(v_\lambda)} q^{e(v_1, \alpha, v_2, \beta)} j^*(\pi(v_1, \alpha) \boxtimes \pi(v_2, \beta))$$

(6.15) $$\cong \sum_{(v_1, v_2) \in \text{KQ}(v_\lambda)} q^{e(v_1, \alpha, v_2, \beta)} f_{v_1, \alpha}(\mu) f_{v_2, \beta}(\nu) j^* \text{IC}(\mu) \boxtimes \text{IC}(\nu) + j^* P$$

Where $(\mu, \nu)$ is the unique element of $\text{ext}^{ger}_{\alpha,\beta}(\lambda)$ corresponding to $(v_1, v_2) \in \text{KQ}(v_\lambda)$ via Lemma 5.7, the support of $P$ is in the subvariety $Z^0/Z^0$, and $f_{v_1, \alpha}(\mu), f_{v_2, \beta}(\nu) \in \mathbb{Z}[q, q^{-1}]$.

The equation (6.14) implies that

(6.16) $$\text{Res}_{\alpha,\beta} \text{IC}(\lambda) = \sum_{(\mu,\nu) \in \text{ext}^{ger}_{\alpha,\beta}(\lambda)} q^{e(v_1, \alpha, v_2, \beta)} f_{v_1, \alpha}(\mu) f_{v_2, \beta}(\nu)(\text{IC}(\mu) \boxtimes \text{IC}(\nu)) + P + Q$$

where the support of $P$ is in the subvariety $Z/Z^0$, and the support of $Q$ is in the subvariety $E_\alpha \times E_\beta/Z$. 
We will prove

\[(6.17) \quad f_{v_1, \alpha}(\mu) = P_q(\Gr_{v_1-v_\mu}(CK(M_\mu))) \text{ and } f_{v_2, \beta}(\nu) = P_q(\Gr_{v_2-v_\nu}(CK(M_\nu)))\]

Let \(i_\mu\) be the embedding \(M_\mu \rightarrow E_\alpha\). Since \(\pi(v_1, \alpha) = \pi_!C[\dim E_\alpha], IC(\mu)|_{O_\mu} = C[\dim E_\alpha]\), and \(P_q(\Gr_{v_1-v_\mu}(CK(M_\mu))) = i_\mu^*\pi_!C\) by Lemma 6.3, we obtain

\[(6.18) \quad i_\mu^*\pi(v_1, \alpha) = P_q(\Gr_{v_1-v_\mu}(CK(M_\mu)))i_\mu^*IC(\mu)\]

Considering the equation (6.15), we see that

\[(6.19) \quad i_\mu^*j^*\pi(v_1, \alpha) = f_{v_1, \alpha}(\mu)i_\mu^*j^*IC(\mu) + i_\mu^*P'\]

where the support of \(P'\) is contained in \(\overline{\Omega}_\mu/\Omega_\mu\). Since \(M_\mu\) is not in the support of \(P'\), then we get \(i_\mu^*j^*\pi(v_1, \alpha) = f_{v_1, \alpha}(\mu)i_\mu^*j^*IC(\mu)\). The relation \(i_\mu^*j^* = i_\mu^*\) implies that \(P_q(\Gr_{v_1-v_\mu}(CK(M_\mu))) = f_{v_1, \alpha}(\mu)\). Similarly, we obtain \(f_{v_2, \beta}(\nu) = P_q(\Gr_{v_2-v_\nu}(CK(M_\nu)))\) in the same way.

Therefore, for any \((\mu, \nu) \in \ext_{\alpha, \beta}(\lambda)\), we get

\[
K_\lambda(\mu, \nu) = g^{\epsilon(v_1, \alpha; v_2, \beta)}f_{v_1, \alpha}(\mu)f_{v_2, \beta}(\nu)
\]

\[
= g^{\epsilon(v_1, \alpha; v_2, \beta)}P_q(\Gr_{v_1-v_\mu}(CK(M_\mu)) \times \Gr_{v_2-v_\nu}(CK(M_\nu)))
\]

For any \((\mu, \nu)\) such that \(K_\lambda(\mu, \nu) \neq 0\), we will show that the top degree of \(K_\lambda(\mu, \nu)\) is less than or equal to \(2e_\lambda + d_\lambda(\mu, \nu)\). The fact \(\pi(v_\lambda, \gamma) = IC(\lambda) + I\) for some sheaves with support on \(\overline{\Omega}_\lambda/\Omega_\lambda\) implies that \(K_\lambda(\mu, \nu) + f(q) = [\Res \pi(v_\lambda, \gamma), IC(\mu) \boxtimes IC(\nu)]\) for some polynomial \(f(q) \in \mathbb{Z}[q, q^{-1}]\). Let us consider

\[(6.20) \quad \Res \pi(v_\lambda, \gamma) = \sum_{(\mu', \nu') \leq (\mu, \nu)} H_\lambda(\mu', \nu') IC(\mu') \boxtimes IC(\nu') + K\]

where the support of \(K\) doesn't contain \((M_\mu, M_\nu)\).

Let us consider the following diagram.

\[
\begin{array}{ccccccc}
\mathfrak{M}^\bullet(V_\lambda, \gamma) & \leftarrow & \ell^{-1}(\mathfrak{M}^\bullet(V_\lambda, \gamma)) & \rightarrow & \mathfrak{M}^\bullet(v_1, \alpha) \times \mathfrak{M}^\bullet(v_2, \beta) \\
\pi \downarrow & & \pi_\beta \downarrow & & \pi \times \pi \\
\overline{\Omega}_\lambda & \leftarrow & F_{\beta, \gamma} \cap \overline{\Omega}_\lambda & \rightarrow & \bigcup_{(\mu, \nu) \in \ext_{\alpha, \beta}(\lambda)} \overline{\Omega}_\mu \times \overline{\Omega}_\nu \\
\gamma' \downarrow & & i \downarrow & & i_{\mu, \nu} \uparrow \\
\mathcal{O}_\lambda & \leftarrow & \mathfrak{E}xt_\lambda(\mu, \nu) & \rightarrow & (M_\mu, M_\nu) \\
\end{array}
\]

It follows that

\[
i_{\mu, \nu}^* \Res \pi(v_\lambda, \gamma) = i^*_{\mu, \nu} \kappa_{\lambda}^! \pi_![-(\alpha, \beta)] = \epsilon_1 \pi^* \pi_![-(\alpha, \beta)] = \epsilon_1 \pi^* \pi_![-(\alpha, \beta)] = \epsilon_1 \pi^* \pi_![-(\alpha, \beta)] = \epsilon_1 \pi^* = \pi^*\]
Considering the diagram
\[
\begin{array}{ccc}
\mathbb{M}^*(V_\lambda, \gamma) & \xrightarrow{\nu} & \pi \mathcal{E}xt_\lambda(\mu, \nu) \\
\oplus & \downarrow \pi & \downarrow \pi_eta \\
\mathcal{O}_\lambda & \xrightarrow{i} & \mathcal{E}xt_\lambda(\mu, \nu) & \xrightarrow{\epsilon} (M_\mu, M_\nu)
\end{array}
\]

We see that
\[
\epsilon i^* \pi \| - \langle \alpha, \beta \rangle = \epsilon \pi \beta \| i^* \pi \| - \langle \alpha, \beta \rangle = \pi \beta \| i^* \pi \| = i^* \pi\]
\[
= \epsilon \pi \beta \| q \dim \mathcal{O}_\lambda - \langle \alpha, \beta \rangle = q^{-d_{\mu, \nu}} \mathcal{H}_\bullet(\pi \mathcal{E}xt_\lambda(\mu, \nu)) \text{ by Definition (6.1)}
\]

Therefore, we obtain \(i^* \| = q^{-d_{\mu, \nu}} \mathcal{H}_\bullet(\pi \mathcal{E}xt_\lambda(\mu, \nu)).\) The formula (6.20) implies that the top degree of \(H_\lambda(\mu, \nu)i^* \| \mathcal{H}_\bullet(\mu) \otimes \mathcal{H}_\bullet(\nu)\) is less than or equal to \(d_{\lambda} - \langle \alpha, \beta \rangle + 2e_\lambda(\mu, \nu)\). It follows that the top degree of \(H_\lambda(\mu, \nu)\) is less than or equal to
\[
2e_\lambda(\mu, \nu) + d_{\lambda} - \langle \alpha, \beta \rangle - d_{\mu} - d_{\nu}
\]

Since \(K_\lambda(\mu, \nu) = H_\lambda(\mu, \nu) + f(q),\) we obtain the top degree of \(K_\lambda(\mu, \nu)\) is less than or equal to
\[
2e_\lambda(\mu, \nu) + d_{\lambda} - \langle \alpha, \beta \rangle - d_{\mu} - d_{\nu}
\]

\[\square\]

7. Quiver Hecke algebras

Let us first recall the notion of KLR algebras. Set \(m_{i,j}\) as the number of arrows from \(i\) to \(j\) for \(i, j \in I,\) and
\[
q_{i,j}(u, v) = \begin{cases} 0 & i = j \\ (v - u)^{m_{i,j}}(u - v)^{m_{j,i}} & i \neq j \end{cases}
\]

**Definition 7.1.** Let \(\alpha = \sum_i c_i \alpha_i \in \mathbb{N}[I]\) and denote \(ht(\alpha) = \sum_i c_i = n.\) The **KLR algebra** (quiver Hecke algebra) \(R_\alpha\) is an associative \(\mathbb{Z}[q^\pm]\)-algebra generated by
\[
\{e_i | i \in \langle I \rangle_\alpha\} \cup \{x_1, \ldots, x_n\} \cup \{\tau_1, \ldots, \tau_{n-1}\}
\]

subject to the following relations
- \(x_k x_l = x_l x_k;\)
- the elements \(\{e_i | i \in \langle I \rangle_\alpha\}\) are mutually orthogonal idempotents whose sum is the identity \(e_\alpha \in R_\alpha;\)
- \(x_k e_i = e_i x_k \) and \(\tau_k e_1 = e_{k+1} \tau_k;\)
- \(e_i x_k - x_k e_i = \delta_{i,k+1} \tau_k e_i - \delta_{i,k} e_i;\)
- \(e_i x_k = q_{i,k+1} \tau_k e_i;\)
- \(\tau_k \tau_l = \tau_l \tau_k \text{ if } |k - l| > 1;\)
- \(e_i x_k = \frac{q_{i,k+1}(x_k x_{k+1}) - q_{i,k+1}(x_{k+2} x_{k+1})}{x_k - x_{k+2}} e_i\)
where \( t_k \) is the simple translation of \( n \) such that \( t_k(k) = k + 1, t_k(k + 1) = k \) and \( t_k(l) = l \) for \( l \neq k, k + 1 \).

We remark that \( R_\alpha \) is an associated \( \mathbb{Z} \)-graded algebra by setting \( \deg x_i = 2 \) and \( \deg \tau_k e_i = -(\alpha_{ik}, \alpha_{ik+1}) \). Therefore, the modules over \( R_\alpha \) endow with an \( \mathbb{Z} \)-graded structure. For a locally finite module \( M \), one defines

\[
\text{Dim } M = \sum_{i \in \mathbb{Z}} \dim M_i q^i
\]

And set \( M[1] \) so that \( M[1]_i = M_{i-1} \) for all \( i \in \mathbb{Z} \), which leads to the formula

\[
\text{Dim } M[1] = q \text{ Dim } M
\]

7.1. Geometrization of KLR algebras. Following [VV11], we consider the geometrization of KLR-algebras Let \( \alpha \) be a dimension vector in \( Q^+ \) and recall the Lusztig's sheaves (6.4), we set

\[
L_\alpha = \bigoplus_{i \in (I)_\alpha} L_i
\]

One defines \( R(\alpha) = \text{Ext}^\bullet_{GL(\alpha)}(L_\alpha, L_\alpha) \). For the notions here we refer to [VV11, Section 1.2]. It is well known that \( R(\alpha) \) is a KLR algebra associated with \( Q \). Since

\[
L_\alpha = \bigoplus_{\lambda \in P_\alpha} L(\lambda) \boxtimes IC(\lambda)
\]

for some \( \mathbb{Z} \)-graded vector spaces \( L(\lambda) \), then these \( L(\lambda) \) can be thought of as modules over \( R_\alpha \). The following theorem shows that all the simple modules over \( R_\alpha \) are of this form.

Remark 7.2. Note that all of \( L(\lambda) \) are self-dual simple modules, Since \( L_\alpha \) and IC(\( \lambda \)) are \( \mathbb{D} \)-invariant, where \( \mathbb{D} \) refers to Verdier duality.

Theorem 7.3 ([CG97, Theorem 8.6.12]). The non-zero numbers of collection \( \{L(\lambda)\} \) arising form (7.1) form a complete set of the isomorphism class of self-dual simple \( R(\alpha) \)-modules

Proof. By [CG97, Chapter 8] we have

\[
\text{Ext}^\bullet_{G(\alpha)}(L_\alpha, L_\alpha) = \bigoplus_{\lambda \in P_\alpha} \text{End}(L(\lambda)) \bigoplus_{\lambda, \gamma \in P_\alpha} \text{Hom}(L(\lambda), L(\gamma)) \text{Ext}^{\geq 1}_{G(\alpha)}(IC(\lambda), IC(\gamma))
\]

Thus, the simple modules are of the form \( L(\lambda) \), where \( \lambda \in P_\alpha = KP(\alpha) \).

7.2. Induction and Restriction functors. Let \( P_\lambda \) be the projective cover of \( L(\lambda) \) which are of the form \( \text{Ext}^\bullet_{G(\alpha)}(L_\alpha, IC(\lambda)) \). Let \( P_i = \text{Ext}^\bullet_{\mathbb{G}_m}(L_\alpha, L_i) \) be the projective module associated with word \( i \). Next we will define the product of projective module over \( R_Q = \bigoplus_{\alpha \in Q^+} R(\alpha) \)

Definition 7.4. Let \( \gamma = \alpha + \beta \) be a partition of \( \gamma \), and \( \mu, \nu \) be two Konstant partitions in KP(\( \alpha \)) and KP(\( \beta \)). Let \( P_\mu \) and \( P_\nu \) be the indecomposable projective modules associated with \( \mu, \nu \). One defines Induction Functor functor by

\[
P_\mu \star P_\nu = \text{Ext}^\bullet_{G(\gamma)}(L_\gamma, IC(\mu) \star IC(\nu))
\]
Let $\lambda \in \text{KP}(\gamma)$ and $P_\lambda$ be the indecomposable projective modules associated with $\lambda$. One defines the Restriction Functor by

$$\text{Res}^\gamma_{\alpha,\beta}(P_\lambda) = \text{Ext}^\bullet_{G_\alpha \times G_\beta}(L_\alpha \boxtimes L_\beta, \text{Res}^\gamma_{\alpha,\beta} IC(\lambda))$$

Remark 7.5. In [VV11, Section 4.6] and [McN17, Theorem 3.1], the above definition is equivalent to the Induction and Restriction functors given in [KL09, Section 3.1].

We denote by $R - \text{proj}$ the category of the graded projective modules over $R_Q$ and by $K^0(R)$ its Grothendieck group. We denote by $R - \text{gmod}$ the category of the graded finite dimensional modules over $R_Q$ and denote $K_0(R)$ as its Grothendieck group. It is easy to see that $K_0(R)$ is generated by simple module $L(\lambda)$ for all $\lambda \in \text{KP}(\alpha)$.

For a projective module $P$, one defines its dual by $P^\# = \text{Hom}_R(P, R(\alpha))$. In [KL09] they show that there exists an isomorphism $\gamma : K_0(R) \sim \rightarrow U_q(n)_A$, where $U_q(n)_A$ is the half-part of the quantum group induced from the underline graph of $Q$.

The functor $\text{Ext}^\bullet_{G_\alpha}(L_\alpha, -)$ gives the isomorphism

$$\text{Ext}^\bullet_{G_\alpha}(L_\alpha, -) : K_0(Q_\alpha) \sim \rightarrow K^0(R)$$

For any $L \in Q_\alpha$, we have

$$\text{Ext}^\bullet_{G_\alpha}(L_\alpha, D(L)) = \mathbb{P}_L^\#$$

where $\mathbb{P}_L$ refers to the projective module $\text{Ext}^\bullet_{G_\alpha}(L_\alpha, L)$.

There is a non-degenerate pairing as follows

$$\langle -, - \rangle : K^0(R) \times K_0(R) \rightarrow \mathbb{Z}[q, q^{-1}]$$

$$(P, M) \mapsto \dim \text{hom}_R(P, M)$$

where $\dim \text{hom}_R(P, M) = \sum_{n \in \mathbb{Z}} \dim \text{Hom}_R(P, M[n])q^n$

Lemma 7.6. Let $\alpha \in Q^+$, $L_\alpha \in \text{add}(\mathbb{P}_\alpha)$, and $P_\alpha = \text{Ext}^\bullet_{G_\alpha}(L_\alpha, L_\alpha)$. If we decompose $L_\alpha$ as

$$L_\alpha = \bigoplus_{\mu \in \mathbb{P}_\alpha} L_\alpha(\mu) \boxtimes IC(\mu)$$

then we have $\dim \text{hom}(P_\alpha, L(\mu)) = \dim L_\alpha(\mu)$ where $\dim L_\alpha(\mu) = \sum_{n \in \mathbb{Z}} \dim L_\alpha(\mu)nq^n$.

Proof. First we assume that $P_\alpha$ is indecomposable projective modules. For any $\mu' \in \text{KP}(\alpha)$, we have

$$\text{Hom}_R(P_{\mu'}, L(\mu)) = \delta_{\mu', \mu}$$

Since $P_\alpha \cong \bigoplus_{\mu \in \mathbb{P}} P_{\mu}^{\dim L_\alpha(\mu)}$, then

$$\text{Hom}_R(P_\alpha, L(\mu)) = \text{Hom}_R(P_{\mu}^{\dim L_\alpha(\mu)}, L(\mu)) = \dim L_\alpha(\mu)$$

$\square$
Next, we define the product on $K_0(R)$.

**Definition 7.7.** For two modules $M, N \in R - \text{gmod}$, we define their product $M \circ N$ so that

\[(7.7) \quad \langle P, M \circ N \rangle = \langle \text{Res} P, M \boxtimes N \rangle \]

for any projective modules $P \in R - \text{proj}$.

Since this pairing is non-degenerate, we get $M \circ N$ is well-defined.

**Remark 7.8.** The above definition coincides with the Induction Functor given in [KL09, Section 3.1] by [KL09, Proposition 3.3]. Let $\alpha + \beta = \gamma$ be a partition of $\gamma$. There exists a canonical embedding

\[R_\alpha \otimes R_\beta \to R_\gamma\]

We denote by $e_{\alpha, \beta}$ the image of $\sum_{i \in \langle I \rangle, j \in \langle I \rangle} e_{ij}$. The induction of two modules $M, N$ is given by

\[M \circ N = e_{\alpha, \beta} \otimes_{R_\alpha \otimes R_\beta} R_\gamma \boxtimes N.\]

For any $i \in \langle I \rangle$ we have

\[e_i R_\gamma e_{\alpha, \beta} = \text{hom}(P_i, \text{Res}_{\alpha, \beta} L) = \text{ext}^G(L_i, L_\alpha \star L_\beta)\]

by isomorphism (7.4)

Here $\text{hom}(A, B) = \oplus_{n \in \mathbb{Z}} \text{Hom}_{RQ}(A, B[1])$. Hence, we have

\[\text{hom}(P_1, M \circ N) = e_1 R_\gamma e_{\alpha, \beta} \otimes_{R_\alpha \otimes R_\beta} (M \boxtimes N)\]

\[= \text{hom}(\text{Res}_{\alpha, \beta} P_1, R_\alpha \otimes R_\beta) \otimes_{R_\alpha \otimes R_\beta} (M \boxtimes N)\]

Let us consider the functor $\text{Res}_{\alpha, \beta}(M) \overset{\text{def.}}{=} e_{\alpha, \beta} M$. Since

\[\text{hom}(P_j \boxtimes P_k, \text{Res}_{\alpha, \beta}(M)) = e_1 e_k M = \text{hom}(P_j \star P_k, M) = \text{hom}(P_{jk}, M)\]

then for any two projective modules $P_1, P_2$, we obtain

\[(7.8) \quad \text{hom}(P_1 \boxtimes P_2, \text{Res}_{\alpha, \beta}(M)) = \text{hom}(P_1 \star P_2, M)\]

**7.3. Two product of simple modules.** We will give an interpretation of the Jordan-Holder filtration of the Induction of two simple modules over $R_Q$ using the Decomposition (6.7).

**Lemma 7.9.** For $\gamma \in Q^+$ and a given $L_\alpha \in Q_\gamma$, let $P_\alpha$ be its corresponding projective module. Suppose that

\[\text{Res}_{\alpha, \beta}^\gamma(L_\alpha) = \sum_{\mu \in \text{KP}(\alpha), \nu \in \text{KP}(\beta)} J_\alpha(\mu, \nu)(IC_\mu \boxtimes IC_\nu)\]

Then we get

\[\langle P_\alpha, L(\mu) \circ L(\nu) \rangle = \text{Dim} J_\alpha(\mu, \nu)\]

**Proof.** Recall $\langle P_\alpha, L(\mu) \circ L(\nu) \rangle = \langle \text{Res} P_\alpha, L(\mu) \boxtimes L(\nu) \rangle$. By Definition (7.4), we get

\[\langle \text{Res} P_\alpha, L(\kappa) \boxtimes L(\nu) \rangle\]
\[\sum_{\mu' \in \text{KP}(\alpha), \nu' \in \text{KP}(\beta)} J_a(\mu', \nu') \otimes (P_{\mu'} \otimes P_{\nu'}), L(\mu) \boxtimes L(\nu)\]
\[= J_a(\mu, \nu)\]

Since \(L(\mu) \circ L(\nu)\) is finite dimensional, it admits a simple filtration. We will show that the multiplicity of \(L(\lambda)\) in this filtration, denoted by \([L(\mu) \circ L(\nu)) : L(\lambda)]\), is equal to \(\text{Dim} \hom_R(P_\lambda, L(\mu) \circ L(\nu))\). Namely,

\[(7.9) \quad \text{Dim} \hom_R(P_\lambda, L(\mu) \circ L(\nu)) = [L(\mu) \circ L(\nu)) : L(\lambda)]\]

We show this by induction on the length of this filtration. When \(L(\mu) \circ L(\nu)\) is a simple module, there is nothing to prove. Suppose that this holds when the length of the filtration of \(L(\mu) \circ L(\nu)\). Consider the following short exact sequence:

\[0 \rightarrow M \rightarrow L(\mu) \circ L(\nu) \rightarrow N \rightarrow 0\]

and applying the functor \(\text{hom}_R(P_\lambda, -)\), we obtain the following short exact sequence:

\[0 \rightarrow \text{hom}_R(P_\lambda, M) \rightarrow \text{hom}_R(P_\lambda, L(\mu) \circ L(\nu)) \rightarrow \text{hom}_R(P_\lambda, N) \rightarrow 0\]

This means

\[\text{Dim} \hom_R(P_\lambda, L(\mu) \circ L(\nu)) = \text{Dim} \hom_R(P_\lambda, M) + \text{Dim} \hom_R(P_\lambda, N)\]

On the other hand, the multiplicity of \(L(\lambda)\) in \(L(\mu) \circ L(\nu)\) is equal to

\[[L(\mu) \circ L(\nu) : L(\lambda)] = [M : L(\lambda)] + [N : L(\lambda)]\]

Our assumption implies that \([M : L(\lambda)] = \text{Dim} \hom_R(P_\lambda, M)\) and \([N : L(\lambda)] = \text{Dim} \hom_R(P_\lambda, N)\) and then \(\text{hom}_R(P_\lambda, L(\mu) \circ L(\nu)) = [L(\mu) \circ L(\nu) : L(\lambda)]\).

Using the above lemma, we get

**Theorem 7.10.** As above assumption, we have

\[[L(\kappa) \circ L(\nu) : L(\lambda)] = \text{Res}_{\alpha, \beta}^\gamma \text{IC}(\lambda) : \text{IC}(\kappa) \boxtimes \text{IC}(\nu)] = K_\lambda(\mu, \nu)\]

where \(\text{Res}_{\alpha, \beta}^\gamma \text{IC}(\lambda) : \text{IC}(\kappa) \boxtimes \text{IC}(\nu)\) refers to the multiplicity of \(\text{IC}(\kappa) \boxtimes \text{IC}(\nu)\) in \(\text{Res}_{\alpha, \beta}^\gamma \text{IC}(\lambda)\) as Lemma 7.9. In particular, we have \(K_\lambda(\mu, \nu) \neq 0\) only if \(\lambda \leq \mu \oplus \nu\), and the top degree of \(K_\lambda(\mu, \nu)\) is less than and equal to

\[2e_\lambda(\mu, \nu) + d_\lambda(\mu, \nu)\]

If \((\mu, \nu) \in \text{ext}_{\alpha, \beta}^{ger}(\lambda)\), then

\[K_\lambda(\mu, \nu) = q_{(v_1, \alpha, v_2, \beta)} \text{P}_q(\text{Gr}_{v_1 - v_\mu}(\text{CK}(M_\mu)) \times \text{Gr}_{v_2 - v_\nu}(\text{CK}(M_\nu)))\]

where \(e_\lambda(\mu, \nu)\) is the dimension of \(\text{Ext}_\lambda(\mu, \nu)\), (see 4.4).

**Proof.** Since \([L(\kappa) \circ L(\nu)) : L(\lambda)] = \text{Dim} \hom_R(P_\lambda, L(\kappa) \circ L(\nu))\). By the above Lemma and Theorem 6.10, we get required results \(\Box\)
Proposition 7.11. Suppose that $Q$ is a quiver without loops, (not necessary a Dynkin quiver). Let $\alpha + \beta = \gamma$ be a partition of $\gamma \in \mathbb{Q}^+$, $\mu, \nu \in \text{KP}(\alpha) \times \text{KP}(\beta)$, and $\lambda \in \text{KP}(\gamma)$. We have

$$[\text{Res}_{\alpha, \beta} L(\lambda) : L(\mu) \boxtimes L(\nu)] = [\text{IC}(\mu) \ast \text{IC}(\nu) : \text{IC}(\lambda)] = J_\lambda(\kappa, \nu)$$

where $[\text{IC}(\mu) \ast \text{IC}(\nu) : \text{IC}(\lambda)]$ refers to the multiplicity of $\text{IC}(\lambda)$ in $\text{IC}(\mu) \ast \text{IC}(\nu)$.

Proof. Recall $[\text{Res}_{\alpha, \beta} L(\lambda) : L(\mu) \boxtimes L(\nu)] = \dim \text{hom}(\mathbb{P}_\mu \boxtimes \mathbb{P}_\nu, \text{Res}_{\alpha, \beta} L(\lambda))$ and

$$\text{hom}(\mathbb{P}_\mu \boxtimes \mathbb{P}_\nu, \text{Res}_{\alpha, \beta} L(\lambda)) = \text{hom}(\mathbb{P}_\mu \boxtimes \mathbb{P}_\nu, L(\lambda))$$

Let us decompose $\mathbb{P}_\mu \ast \mathbb{P}_\nu = \bigoplus_{Y \subseteq \text{KP}(\gamma)} J_X(\mu, \nu) \mathbb{P}_Y$. It follows that $\text{hom}(\mathbb{P}_\mu \ast \mathbb{P}_\nu, L(\lambda)) = J_\lambda(\mu, \nu)$. The definition 7.4 implies that $[\text{IC}(\mu) \ast \text{IC}(\nu) : \text{IC}(\lambda)] = J_\lambda(\mu, \nu)$, we finish our proof.

Corollary 7.12. Under above assumptions, $\text{Res}_{\alpha, \beta} L(\lambda) \neq 0$ only if there exists a pair $\mu, \nu$ in $\text{KP}(\alpha)$ and $\text{KP}(\beta)$ such that $\lambda \geq \mu \ast \nu$.

Proof. By the above Proposition and Theorem 6.7, we obtain our conclusion. Because we just consider the support of perverse sheaves $\text{IC}(\lambda)$, the conclusion of Theorem 6.7 holds for any quivers $Q$ without loops.

Remark 7.13. This Corollary is closely related to the notion semisimple modules $L(\alpha)$ where $\alpha$ is a positive root. Suppose $\text{Res}_{\mu, \nu} L(\alpha) \neq 0$, then there exists a pair $(\mu, \nu)$ such that $\alpha \geq \mu \ast \nu$. On the other hand, $\alpha$ is the minimal Konstant partition in $\text{KP}(\alpha)$. It follows that $\alpha = \mu \ast \nu$. Let $M_\alpha$ is indecomposable representation associated with $\alpha$. If there exists a pair $(\mu, \nu)$ such that

$$0 \to M_\nu \to M_\alpha \to M_\mu \to 0$$

It follows that $[M_\mu, M_\nu] \neq 0$. On the other hand, any summands in $M_\nu$ is greater than $\alpha$, as there is an injective map $M_\nu \to M_\alpha$ by (2.5). Similarly, we have any summands in $M_\mu$ is less than $\alpha$. Therefore, $\text{Res}_{\mu, \nu} L(\alpha) \neq 0$ only if $\mu_i < \alpha$ and $\nu_j > \alpha$ for any $\mu_i \in \mu$ and $\nu_j \in \nu$.

Corollary 7.14. Give two simple modules $L(\mu)$ and $L(\nu)$ such that $\mu \in \text{KP}(\alpha)$ and $\nu \in \text{KP}(\beta)$, let us assume that the head of $L(\mu) \circ L(\nu)$ is a simple module, which is denoted by $L(\mu) \triangledown L(\nu)$. Denote by $\mu \triangledown \nu$ the Konstant partition associated with this head. Denote by $L(\mu) \Delta L(\nu)$ the soc of $L(\mu) \circ L(\nu)$. We have

$$\mu \oplus \nu \geq \mu \triangledown \nu \geq \nu \ast \mu; \quad \mu \oplus \nu \geq \mu \Delta \nu \geq \mu \ast \nu$$

(7.10)

Proof. Let us consider the case $L(\mu \Delta \nu) \to L(\mu) \circ L(\nu)$. It follows by adjoint functor that $\text{Res}_{\beta, \alpha} L(\mu \Delta \nu) \to L(\nu) \boxtimes L(\mu)$. It implies that $J_{\mu \Delta \nu}(\nu, \mu) \neq 0$ and $K_{\lambda}(\mu, \nu) \neq 0$. By Corollary 7.12 and Theorem 7.10, we have $\mu \oplus \nu \geq \mu \Delta \nu \geq \mu \ast \nu$.

In the case $L(\mu) \circ L(\nu) \to L(\mu \triangledown \nu) \to 0$. Applying the functor $(\cdot)^\ast$, we obtain

$$0 \to L(\mu \triangledown \nu) \to q^{(\alpha, \beta)} L(\nu) \circ L(\mu)$$

By the above conclusion, we have

$$\mu \oplus \nu \geq \mu \triangledown \nu \geq \nu \ast \mu$$

□
7.3.1. Support pairs. In this section, we give the notion of support pair. This will give a necessary condition for some conjectures introduced in [LM22, Conjecture 1.3] and [KR11, Problem 7.6]. However, we should remark that this notion is not sufficient to answer these conjectures. Because this notion just focus on the supports of the perverse sheaves on representation spaces $E_\alpha$, (see Lemma 3.4). This will omit some information about the decomposition (6.7). One of the main reason is that the closure of orbits in $E_\alpha$ is singular in general. The singularity makes this approach more difficult than we expect. Therefore, we limit our attention to the support of perverse sheaves $IC(\lambda)$.

**Definition 7.15.** We call a pair $(\mu, \nu) \in KP(\alpha) \times KP(\beta)$ for a support pair if and only if for any $\lambda < \mu \oplus \nu$ we have $(\mu, \nu) \notin \text{ext}^{\text{ger}}_{\alpha, \beta}(\lambda)$. This notion is inspired by Proposition 6.10

For [KR11, Problem 7.6], we have the following theorem.

**Theorem 7.16.** Let $L(\mu)$ and $L(\nu)$ be two simple modules of $R_Q$ where $\mu, \nu$ are their Konstant partitions, respectively. If $L(\mu) \circ L(\nu)$ is a simple module, then $(\mu, \nu)$ is a support pair. Namely, for any non-trivial extension $\lambda \in \text{ext}(\mu, \nu)$, we have

$$[M_\nu, M_\mu \oplus M_\nu] > [M_\nu, M_\lambda]$$

and

$$[M_\mu, M_\mu \oplus M_\nu] > [M_\mu, M_\lambda]$$

**Proof.** It is easy to see that $L(\mu) \circ L(\nu)$ is a simple module if and only if $K_\lambda(\mu, \nu) = 0$ for all $\lambda < \mu \oplus \nu$. By Proposition 6.10, we have $(\mu, \nu) \notin \text{ext}^{\text{ger}}_{\alpha, \beta}(\lambda)$. Otherwise,

$$K_\lambda(\mu, \nu) = q^{(v_1, \alpha, v_2, \beta)} P_q(\text{Gr}_{v_1-v_\mu}(CK(M_\mu)) \times \text{Gr}_{v_2-v_\nu}(CK(M_\nu))) \neq 0$$

By the definition of support pair, we have that

$$[M_\nu, M_\mu \oplus M_\nu] > [M_\nu, M_\lambda]$$

Since $L(\mu) \circ L(\nu)$ is a simple module if and only if $L(\nu) \circ L(\mu)$ is a simple module, we see that

$$[M_\mu, M_\mu \oplus M_\nu] > [M_\mu, M_\lambda]$$

Another useful application of this theorem is related to [VV03, Conjecture 4.3].

**Remark 7.17.** Recall the [VV03, Conjecture 4.3] or [BZ92]: for a pair of dual canonical bases $(b^*(\mu), b^*(\nu))$

$$b^*(\mu) \ast b^*(\nu) \in q^{\beta^*}$$

if and only if $b^*(\mu) \ast b^*(\nu) = q^{(v_1, \alpha, v_2, \beta)} b^*(\mu \oplus \nu)$.

The above Theorem provides a proof of this conjecture. Because the simple modules $L(\mu), L(\nu)$ correspond to the dual canonical bases $b^*(\mu), b^*(\nu)$ under the isomorphism $K_0(R_Q) \cong U_q(n)^*$ (see [KL09]) and $K_{\mu \oplus \nu}(\mu, \nu) = q^{(v_1, \alpha, v_2, \beta)}$ by Theorem 7.10, since $v_1 = v_\mu$ and $v_2 = v_\nu$ when $M_\lambda = M_\mu \oplus M_\nu$ by Remark 5.8.

Here is another application of our results.
Theorem 7.18. Let $\alpha + \beta = \gamma$ be a partition of $\gamma \in Q^+$ and $(\mu, \nu) \in \text{KP} \alpha \text{KP} \beta$. If $(\mu, \nu) \in \text{ext}^{gr}_{\alpha, \beta} (\mu \ast \nu)$ then $q^n L(\mu \ast \nu)$ is the socle of $L(\mu) \circ L(\nu)$ for some integer $n$.

Proof. Let $\lambda = \mu \ast \nu$, we have $\text{Ext} \lambda (\mu, \nu) = \text{Hom}_Q (U, W)$. It implies that $2e_\lambda (\mu, \nu)$ is maximal with respect to any extension $M_\lambda \in \text{Ext}^1 \bigl( M_\mu, M_\nu \bigr)$. If $(\mu, \nu) \in \text{ext}^{gr}_{\alpha, \beta} (\mu \ast \nu)$, then $K_\lambda (\mu, \nu) \neq 0$ with the top degree $2e_\lambda (\mu, \nu) + d_\lambda (\mu, \nu)$ by Theorem 6.10.

For others $\lambda' > \mu \ast \nu$, we have the top degree of $K_{\lambda'} (\mu, \nu)$ is less than or equal to

$$2e_{\lambda'} (\mu, \nu) + d_{\lambda'} (\mu, \nu) < 2e_\lambda (\mu, \nu) + d_\lambda (\mu, \nu).$$

This means the top degree of $K_{\mu \ast \nu} (\mu, \nu)$ is maximal for all Knostant partitions $\lambda' \in \text{ext} (\mu, \nu)$. On the other hand, by Corollary 7.14 the Konstant partitions $\mu \triangle \nu$ of the socle of $L(\mu) \circ L(\nu)$ satisfy $\mu \triangle \nu \in \text{ext} (\mu, \nu)$. By [McN14, Lemma 7.5], $K_{\mu \triangle \nu} (\mu, \nu)$ admits the maximal top degree for all $K_{\lambda'} (\mu, \nu) \neq 0$. This leads to that $\mu \triangle \nu = \mu \ast \nu$.

Remark 7.19. This theorem gives a new viewpoint to [LM20, Conjecture 1.3]. We remark that this couldn’t answer that conjecture. Comparing the conditions in 7.18 and 7.16, it is easy to see that a support pair $(\mu, \nu)$ always satisfies the condition in 7.18.

Thanks to Lapid, he provides some useful examples as follows.

Example 7.20. We remark that if $L(\mu \ast \nu)$ is a submodule of $L(\mu) \circ L(\nu)$ the condition $(\mu, \nu) \in \text{ext}^\min \bigl( \mu \ast \nu \bigr)$ doesn’t hold in general. The counter-example is given by E. Lapid: In type $A_3$, let us consider the Konstant partitions $\mu = [2,1] = [2,2] + [1,1], \nu = [3,2] = [3,3] + [2,2]$. Namely, they are both semisimple representations of $A_3$. It is easy to see that $\text{Ch} L(\mu) = [2,1]$ and $\text{Ch} L(\nu) = [3,2]$. It implies by shuffle product that

$$(\text{Ch} L(\mu) \circ \text{Ch} L(\nu)) = [2,1,1,3,2] + [2,3,1,2] + [3,2,2,1]$$

By [LM22, Example 5.2], the socle of $L(\mu) \circ L(\nu)$ is $L([2,1,3,2])$. The representation associated with this segment is the direct sum of simple representation $S_2 \oplus S_1$ and $S_3 \oplus S_2$. On the other hand, $\text{ext}^{gr}([2,1] \ast [3,2]) = [2,3][1,2]$ by Example 4.10. Therefore, $[2,1][3,2]$ is not in $\text{ext}^{gr}_{\alpha, \beta}([2,1] \ast [3,2])$.

Meanwhile, following [LM22, Example 5.2], let us consider the following example.

Example 7.21. Let $\lambda = \sum_{i=1}^{r} [a_i, b_i]$ be a multisegment in $\text{KP} (\gamma)$ such that $a_i < a_{i+1}$ for all $i \in [1, r-1]$ and $b_i < b_{i+1}$ for all $i \in [1, r-1]$. For each $i$ let $t_i$ such that $a_i \leq t_i \leq b_i + 1$, $t_i < t_{i+1}$. Let

$$\mu = \sum_{i=1}^{r} [a_i, t_i] \quad \nu = \sum_{i=1}^{r} [t_i + 1, b_i]$$

We will show that $\lambda = \mu \ast \nu$ and $(\mu, \nu)$ is the generic pair of $\lambda$. However, the following identity

$$(7.11) \quad [M_\lambda, M_\mu \oplus M_\nu] = [M_\lambda, M_\lambda]$$

doesn’t hold in general. The counter example is given in 4.10. Therefore, the conclusion of Theorem 7.18 is not good enough. But this is the first attempt to prove [LM22, Conjecture 5.1].
Proof. Suppose that \( \mu \in KP(\alpha), \nu \in KP(\beta) \). We decompose \( V = W \oplus U \) and \( V^i = W^i \oplus U^i \) for \( i \in [1, r] \) such that \( \dim V_i = [a_i, b_i], \dim W = \mu, \) and \( \dim U = \nu. \)

We will show that \( \mu \ast \nu = \lambda \). Consider the following short exact sequence

\[
0 \to M_\nu \xrightarrow{f} M_\lambda \xrightarrow{g} M_\mu \to 0
\]

Since the head of \( M_\mu \) coincides with \( \oplus_{i=1}^r S_{a_i} \), we get \( \oplus_{i=1}^r S_{a_i} \subset \text{hd} M_\lambda \). Similarly, we have \( \oplus_{i=1}^r S_{b_i} \subset \text{soc} M_\lambda \). If there exists other \( c \in I \) such that \( S_c \in \text{hd} M_\lambda \), let us consider

\[
r_{c-1,c}(\lambda') = \sum_{[c-1,c] \subset [d,e]} d_{\lambda'}[d,e]
\]

where \( d_{\lambda'}[d,e] \) refers to the multiplicity of the segment \([d,e]\) in \( \lambda' \). It is easy to see that the rank \( r_{c-1,c}(\lambda') \) is less than \( \alpha_c \), as \( \sum_{[c] \subset [d,e]} d_{\lambda'}[d,e] = \alpha_c \) and there is a segment \([c,l]\) in \( \lambda' \) by our assumption. However, \( r_{c-1,c}(\lambda) = \alpha_c \). It implies that \( \lambda' > \lambda \) using (2.11).

Similarly, if there exists other \( c \in I \) such that \( S_c \in \text{soc} M_\lambda \), we obtain \( r_{c,c+1}(\lambda') < \alpha_c = r_{c,c+1}(\lambda) \). Thus, we have \( \lambda' = \sum_{k=1}^r [a_k, b'_k] \) such that \( \{b'_k\}_{k \in [1,r]} = \{b_k\}_{k \in [1,r]} \).

We will show that \( b_k = b'_k \) for any \( k \). Suppose that \( k \) is the minimal element with the condition \( b'_k \neq b_k \) segment. For simplicity, we assume that \( k = 1 \), otherwise, we subtract the segments \([a_i, b_i]\) for all \( i < k \). Since \( b_j = b'_j > b_1 \) for some \( j > 1 \), we have \([a_1, b_1] \subset [a_1, b_j]\). Denote by \( M' \) the representation corresponding to \([a_1, b_j]\). Let us consider the restriction \( M' \to U \) denoted by \( M'_U \), which is a subrepresentation of \( M_\nu \). As its soc is \( S_{b_j} \), it follows that \( M'_U = M[s + 1, b_j] \subset M[t_j + 1, b_j] \). On the other hand, \( M'_W = M[a_1, s] \subset M[a_1, t_1] \). It implies that \( t_j \leq s \leq t_1 \), which leads to a contradiction \( t_j > t_1 \). Therefore, we have \( \lambda = \mu \ast \nu \).

We will show that \( (\mu, \nu) \) is a generic pair of \( \lambda \). Suppose that \( (\mu', \nu) \) be a pair of multisegments such that \( \mu' < \mu \) and

\[
0 \to M_\nu \to M_\lambda \to M_{\mu'} \to 0
\]

We have \( \text{hd} M_{\mu'} \subset \text{hd} M_\lambda = \oplus_{i=1}^r S_{a_i} \). This means

\[
\mu' = ([a_i, v_{i}]_{i \in [1,s]})
\]

where \( s \leq r \). On the other hand, \( \mu' \) can be obtained from \( \mu \) by a sequence of extensions of 2.9. It is easy to see that the end points in the multisegments of the extension as in 2.9 are the same with that of the original multisegments. For instance, for the segments \(([a, b], [c, d])\) such that \( a \leq c \leq b \leq d \), the multisegments of their extension are \(([a, d], [c, b])\). This implies that the set

\[
\{v_{i}\}_{i \in [1,s]} \subset \{t_{j}\}_{j \in [1,r]}
\]

For simplicity, we assume that \( t_i + 1 \neq a_i \). Otherwise, we consider the sub multisegment \( \lambda' \) obtained by deleting those \([a_i, b_i]\) such that \( t_i + 1 = a_i \).

Let \( k \) be the minimal number \( i \) satisfying the relation \( v_i \neq t_i \). Namely, we have \( v_i = t_i \) for all \( i < k \) and \( v_k = t_i > t_k \) for some \( l > k \). For any pair \([a_k, v_k], [t_j + 1, b_j]\), the segments of their extension have the same end points with \([a_k, v_k], [t_j + 1, b_j]\) by 2.9 unless \( v_k = t_l \). But the segments in \( \lambda' \) implies that there exits no extension of \([a_k, v_k]\) and \([t_l, b_l]\). Otherwise, we get the segment \([a_k, b_l] \in \lambda \). The relation \( b_l \neq b_k \) leads to a contradiction. Therefore, the
segment \([a_k, b_k] \in \lambda\) is given by the unique extension of \([a_k, v_k], [t_k + 1, b_k]\). By a product, we obtain the segment \([t_k + 1, v_k]\).

Then the point \(t_k + 1\) and \(v_k = t_l\) will appear in the segments in \(\lambda\) by 2.9. This means \(t_k + 1, t_l \in \{a_i, b_i\}_{i \in [1, r]}\). Suppose that \(t_k + 1 = a_s\) for some \(s \in [1, r]\), then the segment containing the end point \(t_k + 1\) is of the form \([t_k + 1, b_s]\). The segment \([a_s, v_s]\) gives the head of \(M_\lambda\) with two copy of simple representation \(S_{a_s}\). It is a contradiction. If that \(t_k + 1 = b_s\) for some \(s \in [1, r]\), then the segment containing the end point \(t_k + 1\) is of the form \([a_s, t_k + 1]\).

We see that the socle of \(M_\lambda\) has two copy of simple representation \(S_{a_s}\), which leads to a contradiction. Therefore, the condition \(v_k > t_k\) leads to a contradiction. We prove that \(v_k = t_k\) for all \(k \in [1, r]\). That is \(\mu' = \mu\).

Similarly, one can show that \(\nu\) is the generic subrepresentation of \(M_\lambda\) by \(M_\mu\). Therefore, we have that \((\mu, \nu)\) is the generic extension of \(\lambda\).

If \([M_\mu, M_\nu]^1 = 0\), it is easy to see that \((\mu, \nu) \in \text{ext}_{\alpha, \beta}^{\text{min}}(\mu \ast \nu)\), as \(\mu \ast \nu = \mu \oplus \nu\). This give another proof of [LM22, Lemma 5.4].

7.3.2. In the case of rigid representations. In this section, we will consider the rigid representations of \(Q\).

**Corollary 7.22.** Suppose \(\mu\) and \(\nu\) are minimal elements in \(\text{KP}(\alpha)\) (resp: \(\text{KP}(\beta)\)). In other words, \(M_\mu\) and \(M_\nu\) are rigid modules. We have that \(L(\mu) \circ L(\nu)\) is a simple module if and only if \([M_\mu, M_\nu]^1 = [M_\nu, M_\mu]^1 = 0\) In other words \(M_\mu \oplus M_\nu\) is a rigid module.

**Proof.** If \(L(\mu) \circ L(\nu)\) is a simple module over \(R_Q\), then \((\mu, \nu)\) is a support pair by Theorem 7.16. That means for any \(\lambda < \mu \oplus \nu\) \((\mu, \nu) \notin \text{ext}_{\alpha, \beta}^{\text{ger}}(\lambda)\). But \((\mu, \nu)\) is the unique minimal pair in \(\text{KP}(\alpha) \times \text{KP}(\beta)\). It implies that there is no Konstant partition \(\lambda\) satisfying \(\lambda < \mu \oplus \nu\).

Namely, \(M_\mu \oplus M_\nu\) is a rigid representation of \(Q\).

Suppose that \(M_\mu \oplus M_\nu\) is a rigid representation of \(Q\). By Proposition 7.10, if simple module \(L(\lambda)\) in the Jordan-Holder filtration of \(L(\mu) \circ L(\nu)\), then \(\lambda \leq \mu \oplus \nu\). Since \(\mu \oplus \nu\) is the unique minimal Konstant partition in \(\text{KP}(\gamma)\), the only simple module in the Jordan Holder filtration of \(L(\mu) \circ L(\nu)\) is \(L(\mu \oplus \nu)\).

Next we will consider the case \([M_\mu, M_\nu]^1 = 1\).

**Proposition 7.23.** Let us assume that \([M_\mu, M_\nu]^1 = 1\). We have the following short exact sequence

\[0 \to q^{n_2}L(\mu \ast \nu) \to L(\mu) \circ L(\nu) \to q^{n_1}L(\mu \oplus \nu) \to 0\]

**Proof.** By Theorem 7.16, we have \(L(\mu) \circ L(\nu)\) is not a simple module. By Proposition 7.10, if \([L(\mu) \circ L(\nu) : L(\lambda)] \neq 0\) then \(M_\lambda \in \text{Ext}_{\alpha}^1(M_\mu, M_\nu)\). Since \([M_\mu, M_\nu]^1 = 1\), It implies that there are exactly two Konstant partitions \(\mu \oplus \nu\) and \(\mu \ast \nu\) satisfying \([L(\mu) \circ L(\nu) : L(\lambda)] \neq 0\). Since \((\mu, \nu) \in \text{ext}_{\alpha, \beta}^{\text{ger}}(\lambda)\), it follows by Theorem 7.18 that \(q^{n_2}L(\mu \ast \nu)\) is the socle of \(L(\mu) \circ L(\nu)\). The cokernel of this embedding gives rise to the simple module \(q^{n_1}L(\mu \oplus \nu)\).
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School of Mathematical Sciences, Beijing Normal University, Beijing 100875 P.R.China

Email address: yingjinbi@mail.bnu.edu.cn

URL: ORCID: orcid.org/0000-0003-0153-3274