Tests of analytical hadronisation models using event shape moments in $e^+e^-$ annihilation

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Abstract. Predictions of analytical models for hadronisation, namely the dispersive model, the shape function and the single dressed gluon approximation, are compared with moments of hadronic event shape distributions measured in $e^+e^-$ annihilation at centre-of-mass energies between 14 and 209 GeV. In contrast to Monte Carlo models for hadronisation, analytical models require to adjust only two universal parameters, the strong coupling and a second quantity parametrising non-perturbative corrections. The extracted values of $\alpha_S$ are consistent with the world average and competitive with previous measurements. The variance of event shape distributions is compared with predictions given by some of these models. Limitations of the models, probably due to unknown higher order corrections, are demonstrated and discussed.

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1 Introduction

In previous studies, moments of event shape distributions have been compared to perturbative predictions of quantum chromodynamics (QCD, [1–4]) in next-to-leading order, simulating the hadronisation process by Monte Carlo models [5, 6]. Alternatively there exist models describing hadronisation analytically. This paper aims to study these models qualitatively and quantitatively by measurements of the strong coupling and the model parameters.

The dispersive model [7] is based on the assumption of a nonperturbatively continued strong coupling. The shape function [8] additionally describes a modification of the shape of the perturbatively calculated distribution. The single dressed gluon approximation [9, 10] estimates the perturbative part more completely with reduced perturbative uncertainties of the prediction. The models test the predicted energy evolution of the strong coupling. Their parameters can be determined—one of them the value of $\alpha_S$ at some reference energy—and the assumption of universality of these parameters can be probed. To date, primarily the distributions themselves and the mean values (first moments) have been studied [11–14].

This analysis uses data measured by JADE [5] in the years 1979...1986 at six centre-of-mass (cms) energies in the energy range of $Q = 14...44$ GeV, and data measured by OPAL at 12 energy points over the whole LEP energy range of $91...209$ GeV and combined into 4 energy ranges [6]. The large energy range covered by the measurements allows to test the employed assumptions selectively.

The outline of the paper is as follows. In Sect. 2 we present the observables used in the analysis. In Sect. 3 we describe the perturbative QCD predictions and introduce the analytical models which describe the hadronisation process. In Sect. 4 we discuss predictions for event shape moments at hadron level and compare them with the measurements. The predictions for the event shape distribution variance following from the dispersive model and the shape function are tested as well. In Sect. 5 we summarize and give our conclusions.

2 Event shape moments

Event shape variables are a convenient way to characterise properties of hadronic final states. They are calculated from particle momenta and energies. For definitions of the variables we refer to [6].

In a hadronic event in $e^+e^-$ annihilation the virtual vector boson $Z^0/\gamma^*$ generated by annihilation of electron and positron decays into a quark pair $q\bar{q}$. The quarks may radiate gluons which radiate further gluons or decay into another quark pair. The final state of this parton shower is called parton level. By the process of hadronisation partons are transferred into hadrons. The predictions used in this work describe the hadron level. The variables measured in the experiments have to be corrected for the effects of limited detector acceptance and resolution to probe the hadron level.

The event shapes considered here are thrust $T$, C-parameter $C$, heavy jet mass $M_H$, jet broadening vari-
ables $B_T$ and $B_W$, and the transition value $y^{2D}_{23}$ between 2- and 3-jet final states defined using the Durham jet reconstruction scheme. The $\alpha_S$ determination in [5] is based on moments of these variables, in [6] on distributions and moments of these variables. Their theoretical description by perturbation theory is currently the most advanced [15–17].

A generic event shape variable is denoted by the symbol $y$. Regions dominated by multiple jets give large values of $y$, while two narrow jets give $y \approx 0$. Thrust $T$ is an exception to this rule. By using $y = 1 - T$ instead, the condition is fulfilled for all event shapes. $B_W$, $y^{2D}_{23}$ and $M_H$ are sensitive to only one suitably chosen hemisphere of the event (one-hemisphere variables), $1 - T$, $C$ and $B_T$ are sensitive to the whole hadronic event (two-hemisphere variables).

The $n$th, $n = 1, 2, \ldots$ moment of the distribution of an event shape variable $y$ is defined by

\[
\langle y^n \rangle = \int_0^{y_{\text{max}}} dy y^n \frac{1}{\sigma_{\text{tot.}}} \frac{d\sigma}{dy},
\]

where $y_{\text{max}}$ is the kinematically allowed upper limit of the variable and $\sigma_{\text{tot.}}$ denotes the total hadronic cross section.

Predictions have been made available for the moments of event shapes. Their evolution with cms energy allows direct tests of the predicted energy evolution of the strong coupling $\alpha_S$. Furthermore it enables the determination of a single value of $\alpha_S$ at a definite energy scale—for example $\alpha_S(M_{Z^0}^2)$ at the rest energy of the $Z^0$ boson. Theoretical calculations always involve a full integration over phase space, which implies that comparison with data always probes all of the available phase space. This is in contrast to QCD predictions for distributions: these are commonly only compared with data—e.g. in order to measure $\alpha_S$—in restricted regions, where the theory is able to describe the data well, see e.g. [18]. Comparisons of QCD predictions for moments of event shape distributions with data are thus complementary to tests of the theory using distributions.

3 Theory

The QCD prediction for $\langle y^n \rangle$ in next-to-leading-order (NLO) of the strong coupling $\alpha_S \equiv \alpha_S/(2\pi)$ has the form

\[
\langle y^n \rangle_{\text{NLO}} = A_n \hat{\alpha}_S + (B_n - 2A_n) \hat{\alpha}_S^2.
\]

Very recently, perturbative NNLO predictions for event shape moments became available [19]. The nonperturbative models discussed in the following sections need to be adapted to these new NNLO predictions before a meaningful study based on NNLO can be done. Therefore, here we restrict ourselves to consistently using NLO together with the currently available models.

The coefficient $A_n$ and $B_n$ are obtained by numerical integration of the QCD matrix elements using the program EVENT2 [20]. These predictions were also used in [5, 6].

The coupling $\hat{\alpha}_S$, and therefore the QCD prediction depends on the renormalisation scale $\mu_R$, see e.g. [21]. The prediction is changed by this dependence as shown in [5]. For clearer notation the renormalisation scale factor is defined as $x_\mu = \mu_R/Q$: a truncated fixed order QCD calculation such as [2] will then depend on $x_\mu$.

Infrared and collinear stability are essential for a perturbative description of the event shape variables $1 - T$, $C$ and $B_T$. However they do not suffice for a perturbative description of the hadron level as hadronisation takes place at low energy scales, where the perturbative description breaks down. The evolution of partons to hadrons can be approximated by analytical calculations. These analytical calculations are generally motivated from the transition from perturbative to non-perturbative regime. A perturbation series in quantum field theory is a divergent series, see e.g. [22]. To obtain finite results, regularisations have to be applied. From suitable prescriptions non-perturbative terms are found, which typically scale with inverse powers of the cms energy [7–9].

3.1 Dispersive model

This model [7, 23, 24] is based on the assumption of a non-perturbatively defined strong coupling $\alpha_S(Q^2)$ which remains finite at and below the Landau pole $A_{\text{QCD}}$. The Landau pole is the scale where the usual perturbative coupling diverges. The matching scale $Q = \mu_I$, marking the border between perturbative and non-perturbative region, is not uniquely defined. Usually it is taken as $\mu_I \approx 2 \text{GeV}$.

As the non-perturbative coupling cannot be calculated, it is parametrised universally in a simple way by the zeroth moment of the extended coupling over the non-perturbative region,

\[
\alpha_0(\mu_I) = \frac{1}{\mu_I} \int_0^{\mu_I} dQ \alpha_S(Q^2).
\]

In first approximation, non-perturbative corrections generate a simple shift of the perturbative differential distribution $d\sigma_{\text{NLO}}/dy$ of the event shape variables $1 - T$, $C$, $B_T$, $B_W$ and $M_H^2$ when relating parton to hadron level,

\[
\frac{d\sigma_{\text{had.}}}{dy} = \frac{d\sigma_{\text{NLO}}}{dy} (y - a_y \cdot P).
\]

This prediction is valid only if the value of the event shape variable $y$ is not too large ($y \ll 1$), and the cms energy $Q$ is not too small ($Q \gg A_{\text{QCD}}/y$). Only the numerical factor $a_y$ depends on the event shape variable $y$, see Table 1. However $P$ depends on the hard scale $Q$ but is universal.

The theoretical calculations are based on the variable $M_H^2$ because of the problems with the NLO description of $M_H$ as discussed in [5, 6, 25].
for the event shape variables $1 - T, C, B_T, B_W$ and $M^2_H$, and has the form [23, 24]

$$\mathcal{P} = \frac{4C_F}{\pi^2} \cdot \mathcal{M} \cdot \{a_0 - \left[\alpha_S(\mu_R^2)\right] + 2\alpha_0^2(\mu_R^2) \left[\ln \frac{\mu_R}{\mu_I} + 1 + \frac{K}{4\pi\beta_0}\right] + \mathcal{O}(\alpha_S^3)\} \times \frac{\mu_I}{Q}$$  

(5)

In the $\overline{\text{MS}}$ renormalisation scheme the constant $K$ has the value

$$K = C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5}{9} \frac{\mu_I}{\mu_F}$$  

(6)

with $n_f = 5$ at the studied energies, and the beta function coefficient is $\beta_0 = 23/(12\pi)$. The color factors have the values $C_F = 4/3, C_A = 3$ [26], the Milan factor $\mathcal{M}$ is known in two loops, $\mathcal{M} = 1.49 \pm 20\%$ (for flavour number $n_f = 3$ at the relevant low scales). The cited uncertainty results [24] from the estimation of the next order contribution, $\mathcal{M}_{\text{NNLO}} = \mathcal{M}_{\text{NLO}} \cdot (1 + \mathcal{O}(\alpha_S/\pi))$.

Table 1. Coefficients $a_n$ of power correction $\propto 1/Q$ of event shape variables in the dispersive model [27, 28]

| event shape variable $y$ | $1 - T$ | $C$ | $B_T$ | $B_W$ | $y_{23}$ | $M^2_H$ |
|--------------------------|--------|-----|-------|-------|--------|---------|
| $a_n$                    | 2      | $3\pi$ | 1     | 1/2   | 0      | 1       |

Applying the dispersive model calculation [24] of the normalized event shape distribution in definition (1) of the unpysical range of negative variable values, gives

$$\langle y^n \rangle = \int_0^1 dy y^n \cdot \frac{d\sigma}{dy}(y) \approx \int_0^1 dy (y + a_y P)^n \cdot \frac{d\sigma_{\text{NLO}}}{dy}(y).$$

The predictions for the moments on hadron level become:

$$\langle y^1 \rangle = \langle y^1 \rangle_{\text{NLO}} + a_y P$$  

(7)

$$\langle y^2 \rangle = \langle y^2 \rangle_{\text{NLO}} + 2\langle y^1 \rangle_{\text{NLO}} \cdot a_y P + (a_y P)^2$$  

(8)

$$\langle y^3 \rangle = \langle y^3 \rangle_{\text{NLO}} + 3\langle y^2 \rangle_{\text{NLO}} \cdot a_y P + 3\langle y^1 \rangle_{\text{NLO}} \cdot (a_y P)^2 + (a_y P)^3$$  

(9)

$$\langle y^4 \rangle = \langle y^4 \rangle_{\text{NLO}} + 4\langle y^3 \rangle_{\text{NLO}} \cdot a_y P + 6\langle y^2 \rangle_{\text{NLO}} \cdot (a_y P)^2 + 4\langle y^1 \rangle_{\text{NLO}} \cdot (a_y P)^3 + (a_y P)^4$$  

(10)

$$\langle y^5 \rangle = \langle y^5 \rangle_{\text{NLO}} + 5\langle y^4 \rangle_{\text{NLO}} \cdot a_y P + 10\langle y^3 \rangle_{\text{NLO}} \cdot (a_y P)^2 + 10\langle y^2 \rangle_{\text{NLO}} \cdot (a_y P)^3 + 5\langle y^1 \rangle_{\text{NLO}} \cdot (a_y P)^4 + (a_y P)^5.$$  

(11)

Previous studies [29] indicate that the parameters $\alpha_S(M^2_{Z'})$ and $\alpha_0(\mu_I)$ when fitted to $B_T$ and $B_W$ distributions via [24] are not compatible with values derived from $1 - T$ and $C$. Therefore improved predictions for these distributions were given. They also describe a compression of the distribution peak, i.e. a narrowing in the two jet region [24, 30]. The non-perturbative factor $\mathcal{P}$ in the case of $\langle B_T^+ \rangle$ and $\langle B_W^+ \rangle$ is replaced by [31]

$$\mathcal{P}\langle B_T^+ \rangle = \mathcal{P} \cdot \left(\frac{\pi}{2\sqrt{2}C_F \hat{\alpha}_s(1 + K \hat{\alpha}_s/(2\pi))} \right)$$

$$\left(\frac{3}{4} - \frac{2\pi\beta_0}{3C_F} + \eta_0\right),$$  

(12)

and

$$\mathcal{P}\langle B_W^+ \rangle = \mathcal{P} \cdot \left(\frac{\pi}{2\sqrt{2}C_F \hat{\alpha}_s(1 + K \hat{\alpha}_s/(2\pi))} \right)$$

$$\left(\frac{3}{4} - \frac{\pi\beta_0}{3C_F} + \eta_0\right),$$  

(13)

with a rescaled coupling $\hat{\alpha}_s(Q^2) \equiv \alpha_S(e^{-3/2}Q^2)$ and a constant $\eta_0 \simeq -0.6137$.

No power correction coefficient has been calculated in the dispersive model for the variable $y_{23}^2$. The purely perturbative prediction describes the first moments of $y_{23}^2$ well [14, 31]—therefore we also compare it with the higher moments.

The dispersive model gives predictions for several observables and contains only universal free parameters $\alpha_S(M^2_{Z'})$ and $\alpha_0(\mu_I)$.

### 3.2 Shape function

Korchemsky and Tafat [8] describe properties of the event shape variables $1 - T, C$ and $M^2_H$ not included in NLO perturbation theory by a so called shape function, which does not depend on the variable nor the cms energy. This is more general than the dispersive model, as not only a shift of the perturbative prediction is predicted but also a compression of the distribution peak.

The prediction is deduced from studying the two jet region (i.e. $y \ll 1$) in the distribution of the event shape variable $y$. The prediction for the differential distribution is

$$\int_0^Q \frac{1}{\sigma} \frac{d\sigma}{dy} = \int_0^Q y \cdot dy f_y(\varepsilon) \cdot \frac{d\sigma_{\text{NLO}}}{dy}(y - \varepsilon/Q),$$  

(14)

with a non-perturbative function $f_y(\varepsilon)$, dependent on one scale parameter $\varepsilon$. This function is derived from the shape function $f(\varepsilon_L, \varepsilon_R)$ [8], which depends on two scale parameters $\varepsilon_L, \varepsilon_R$ for the two hemispheres of the event. By the compression of the distribution the validity of the prediction is extended compared to the dispersive model to $y \simeq 1 / \Lambda_{\text{QCD}}$. The shape function can be parametrised [8] by its “first moment”:

$$\lambda_1 = \int d\varepsilon_R \int d\varepsilon_L (\varepsilon_R + \varepsilon_L) f(\varepsilon_R, \varepsilon_L) \equiv \langle \varepsilon_R + \varepsilon_L \rangle,$$  

(15)

its “second moment”:

$$\lambda_2 = \langle (\varepsilon_R + \varepsilon_L)^2 \rangle.$$  

(16)
and a $Q$ dependent function $\delta \lambda_2(Q)$. Predictions for the moments of event shape variables can be derived, and $\lambda_1$ and $\lambda_2$ can be fitted to the data.

From prediction $[14]$ for the distribution of the variables $1 - T$, $C$ and $M_H^2$, predictions for the mean values follow $[8]$ by integration,

$$\langle (1 - T) \rangle = \langle (1 - T) \rangle_{NLO} + \frac{\lambda_1}{Q}$$

(17)

$$\langle C \rangle = \langle C \rangle_{NLO} + \frac{3\pi}{2Q} \left[ 1 - 5.73 \alpha_s(Q^2) \right]$$

(18)

$$\langle M_H^2 \rangle = \langle M_H^2 \rangle_{NLO} + \frac{\lambda_1}{2Q}$$

(19)

Analogously for the second moments one finds $[8],$

$$\langle (1 - T)^2 \rangle = \langle (1 - T)^2 \rangle_{NLO} + 2\frac{\lambda_1}{Q} \langle (1 - T) \rangle_{NLO} + \frac{\lambda_2}{Q^2}$$

(20)

$$\langle C^2 \rangle = \langle C^2 \rangle_{NLO} + \frac{9\pi^2}{4 Q^2} \left[ 1 - 10.76 \alpha_s(Q^2) \right]$$

$$+ \frac{1}{2Q} \left[ 2 \langle C \rangle_{NLO} - 4.30 \alpha_s(Q^2) \right]$$

(21)

$$\langle M_H^4 \rangle = \langle M_H^4 \rangle_{NLO} + \frac{\lambda_1}{Q} \langle M_H^2 \rangle_{NLO} + \frac{\lambda_2 + \delta \lambda_2(Q)}{2Q^2}$$

(22)

In this model the more strongly suppressed power corrections have an independent coefficient. The coefficient $\lambda_1$ is interpreted as the first “moment” of the shape function, $\lambda_2$ the second, and $\delta \lambda_2$ as a contribution accounting for the one-hemisphere character of $M_H^2$. Therefore these are universal scales.

### 3.3 Variance of event shape distributions

In $[32]$ the variance of event shape distributions on hadron level was found to be described in the dispersive model perturbatively without significant power corrections. This would open the possibility of an accurate $\alpha_s(M_Z^0)$ determination.

A theory predicting first and second moments of an event shape variable $y$ also gives a prediction for the variance $\text{Var}(y) = \langle y^2 \rangle - \langle y \rangle^2$, and so a simple prediction for the variance of event shape variables on hadron level is deduced $[32]$ by employing the predictions $[7]$ and $[8]$.

$$\text{Var}(y) = \langle y^2 \rangle_{NLO} - \langle y \rangle^2_{NLO}.$$  

(23)

In the dispersive model one obtains a purely perturbative expression for the variance, up to strongly suppressed corrections $O(\alpha_s/Q^2)$.

Predictions for the event shape variance can also be derived from the shape function. The first and second moment predictions $[17]$ and $[20]$ give the identical prediction in case of thrust,

$$\text{Var}(1 - T) = \langle (1 - T)^2 \rangle_{NLO} - \langle 1 - T \rangle^2_{NLO}.$$  

(24)

Analogously this follows from $[19]$ and $[22]$ for $\text{Var}(M_H^2)$. Var($C$) follows from $[18]$, $[21]$ and the perturbative LO coefficient $[20]$ up to $O(1/Q^2)$

$$\text{Var}(C) = \langle C^2 \rangle_{NLO} - \langle C \rangle^2_{NLO} - 3.23 \frac{\lambda_1}{Q} \alpha_s(Q^2),$$

(25)

showing an additional term $\propto \alpha_s/Q$ with coefficient $\lambda_1$ as in Subsect. $[52]$.

### 3.4 Single dressed gluon approximation

Gardi et al. $[9, 10]$ assume the existence of a reordering of the perturbative series, the so called skeleton expansion $[33]$—its existence is proven only for abelian field theory like QED. The first contribution to this expansion is a single dressed gluon (SDG) which resums running coupling effects of any order in $\alpha_s$. These are renormalons in the dominant contributions $\propto \beta_0^\nu$. The single dressed gluon graphs can be calculated completely. In this way the perturbative prediction $\langle (1 - T)^n \rangle_{pt.}$ for a moment of thrust can be approximated. This approximated series already diverges and various regularisations differ by definite powers of the cms energy $Q$. These predictions follow:

$$\langle (1 - T)^1 \rangle = \langle 1 - T \rangle_{pt.} + \frac{\nu_1}{Q}$$

$$\langle (1 - T)^2 \rangle = \langle (1 - T)^2 \rangle_{pt.} + \frac{\nu_2}{Q^2} + \frac{\nu_2}{Q^3}$$

$$\langle (1 - T)^3 \rangle = \langle (1 - T)^3 \rangle_{pt.} + \frac{\nu_3}{Q^3} + \frac{\nu_3}{Q^5}$$

$$\langle (1 - T)^4 \rangle = \langle (1 - T)^4 \rangle_{pt.} + \frac{\nu_4}{Q^5} + \frac{\nu_4}{Q^7}$$

In general, the non-perturbative correction is predicted as a sum of two terms with different powers in $Q$ and coefficients $\nu_n, \kappa_n$.

The thrust is used in the massless limit,

$$T = \sum_i \frac{|p_i \cdot n_T|}{\sum_i E_i} = \sum_i \frac{|p_i \cdot n_T|}{Q}.$$  

(26)

Here the denominator is changed with respect to the standard definition $[6]$ by $\sum_i |p_i| \rightarrow \sum_i E_i$, which does not change the thrust value as long as massless partons are considered. Under this change of definition the thrust value calculated with a “massive” virtual gluon is correct, as long as all (massless) partons generated in the hadronisation end up in the same event hemisphere w.r.t. to the thrust axis. This so called “inclusive” calculation is valid only if fragmentation of the gluon is approximately collinear. The perturbative SDG predictions for the moments of thrust are given as perturbation series in the coupling $\bar{\alpha}$ $[10]$, which relates to $\alpha_s(\mu_R^2)$ in the $\overline{\text{MS}}$ scheme by a simple shift of the Landau Pole $[14]$.

$$\bar{\alpha}(\mu_R^2) = \frac{\alpha_s(\mu_R^2)/\pi}{1 - \frac{5}{3} \beta_0 \alpha_s(\mu_R^2)}.$$  

(27)

$^4$ In particular the beta function coefficients $\beta_0$ have the same values as in the $\overline{\text{MS}}$ scheme.
The perturbative coefficients are given in so called log moments \( d_i \) of the characteristic thrust function. The first six log moments for \( \langle (1 - T)^1 \rangle \ldots \langle (1 - T)^4 \rangle \) were calculated by numerical integration [10]. The perturbative prediction of \( \mathcal{O}(\bar{a}_S^6) \) in the (obviously incomplete) SDG approximation has the form [34]

\[
\langle (1 - T)\rangle_{\text{SDG}}^{n} = A_n \cdot \bar{a} + B_n \cdot \bar{a}^2 + C_n \cdot \bar{a}^3 + D_n \cdot \bar{a}^4 + E_n \cdot \bar{a}^5 + F_n \cdot \bar{a}^6
\]

\[
= d_0 \cdot \bar{a} + \beta_0 d_1 \pi \cdot \bar{a}^2
\]

\[
+ \left( -\frac{1}{3} d_0 \pi^2 + d_2 \right) \beta_0^2 + \beta_1 d_1 \pi^3 \cdot \bar{a}^3
\]

\[
+ \left( -\pi^2 d_1 + d_3 \right) \beta_0^3 + \frac{5}{2} \beta_1 \beta_0 d_2 + \beta_2 d_1 \pi^4 \cdot \bar{a}^4
\]

\[
+ \left( d_4 + \frac{1}{5} d_0 \pi^4 - 2 \pi^2 d_2 \right) \beta_0^4 + \left( \frac{13}{3} \beta_1 d_3 - \beta_1 \pi^2 d_1 \right) \beta_0^2
\]

\[
+ 3 \beta_2 d_2 \beta_0 + \frac{3}{2} \beta_1^2 d_2 + \beta_3 d_1 \pi^5 \cdot \bar{a}^5
\]

\[
+ \left( -\frac{10}{3} \pi^2 d_3 + \pi^4 d_1 + d_5 \right) \beta_0^5 + \left( \frac{77}{12} d_4 - \frac{9}{2} \pi^2 d_2 \right) \beta_1 \beta_0^3
\]

\[
+ (6 d_3 - \pi^2 d_1) \beta_2 \beta_0^2 + \left( \frac{25}{6} \beta_1^2 d_3 + \frac{7}{2} \beta_3 d_2 \right) \beta_0
\]

\[
+ \beta_4 d_1 + \frac{7}{2} \beta_1 \beta_2 d_2 \right) \pi^5 \cdot \bar{a}^5; \quad (27)
\]

where the \( n \)-dependent \( d_i \) are taken from [10]. Higher order contributions from the running of the coupling are accounted for up to sixth order. Reference [10] gives the prediction explicitly in \( \mathcal{O}(\bar{a}^3) \) without the term \( \beta_1 d_1 \pi^2 \cdot \bar{a}^3 \). This term with marked effect on the \( \bar{a}^3 \)-coefficient results from considering the usual QCD beta function [34]. The symbols \( \beta_0 \ldots \beta_3 \) denote the usual beta function coefficients [35,36], \( \beta_4 \) is unknown. We set \( \beta_4 \) to zero.

The SDG approximation [24] is complete in leading order \( \mathcal{O}(\bar{a}) \) by construction [10]. In higher orders this approximation only gives the terms \( \propto \beta_0 \). Therefore we use the numerically calculated [20] NLO coefficients. For the third and higher orders we employ the SDG approximation,

\[
\langle (1 - T)^n \rangle_{\text{pt.}} = \langle (1 - T)^n \rangle_{\text{NLO}} + C_n \cdot \bar{a}^3 + D_n \cdot \bar{a}^4 + E_n \cdot \bar{a}^5 + F_n \cdot \bar{a}^6.
\]

As the perturbative expansion is an asymptotic (divergent) series [22], the terms are expected to become smaller up to a certain order, and then become larger again. It is of interest

- at which expansion order this happens,
- how the measured coupling depends on the maximum expansion order,
- how the leading power correction \( \nu_{\alpha}/Q^\ell \) depends on the maximum expansion order,
- how the minimum perturbative term relates to the power correction.

The best approximation of the theory by an asymptotic series is expected when truncating it near the minimal term—including additional terms does not necessarily result in a better approximation. Therefore we study the analysis as a function of the truncation order.
4 Tests of non-perturbative models

Our tests use event shape moments with statistical and experimental uncertainties measured by JADE and published in [5] and analogous OPAL data published in [6], covering in total the energy range of 14 to 209 GeV. In the JADE data $e^+e^- \rightarrow b\bar{b}$ events have been subtracted on a statistical basis. Table 2 gives an overview of the data used.

Table 2. Year of data taking, energy range, integrated luminosity, average cms energy and the numbers of selected data events for each JADE [5] or OPAL [6] data sample. The horizontal lines in the OPAL ranges separate the data into the four energy ranges used for fit and presentation purposes

| year | range of Q in GeV | mean Q in GeV | luminosity (pb^{-1}) | selected events |
|------|-------------------|--------------|----------------------|-----------------|
| 1981 | 13.0 ... 15.0     | 14.0         | 1.46                 | 1783            |
| 1981 | 21.0 ... 23.0     | 22.0         | 2.41                 | 1403            |
| 1981, 1982 | 33.8 ... 36.0 | 34.6         | 61.7                 | 14313           |
| 1986 | 34.0 ... 36.0     | 35.0         | 92.3                 | 20876           |
| 1985 | 37.3 ... 39.3     | 38.3         | 8.28                 | 1585            |
| 1984, 1985 | 43.4 ... 46.4 | 43.8         | 28.8                 | 4376            |
| 1996, 2000 | 91.0 ... 91.5 | 91.3         | 14.7                 | 395695          |
| 1995, 1997 | 129.9 ... 136.3 | 133.1        | 11.26                | 630             |
| 1996 | 161.2 ... 161.6   | 161.3        | 10.06                | 281             |
| 1997 | 170.2 ... 172.5   | 172.1        | 10.38                | 218             |
| 1997 | 180.8 ... 184.2   | 182.7        | 57.72                | 1077            |
| 1998 | 188.3 ... 189.1   | 188.6        | 185.2                | 3086            |
| 1999 | 191.4 ... 192.1   | 191.6        | 29.53                | 514             |
| 1999 | 195.4 ... 196.1   | 195.5        | 76.67                | 1137            |
| 1999, 2000 | 199.1 ... 200.2 | 199.5        | 79.27                | 1900            |
| 1999, 2000 | 201.3 ... 202.1 | 201.6        | 37.75                | 519             |
| 2000 | 202.3 ... 205.5   | 204.9        | 82.01                | 1130            |
| 2000 | 205.5 ... 208.9   | 206.6        | 138.8                | 1717            |

JADE and OPAL are similar in construction and many parameters [37]. Consistent measurements can be expected from the simultaneous use. The analysis procedures for both data sets were constructed to be similar. Systematic variations of the JADE and OPAL analyses concern detector event reconstruction, selection cuts, Monte Carlo generators and background.

4.1 Dispersive model

Our test uses the first five moments of $1 - T$, $B_T$, $B_W$, $C$ and $y_{ZT}$, and the second and fourth moment of $M_H$. We compare the prediction with the hadron level data, varying two parameters $\alpha_S(M_Z)$ and $\alpha_0(\mu_I)$. Figs. 11 and 12 show the comparison of data and predictions [7, 8] in the case of $1 - T$ and $B_W$. The $\chi^2$/d.o.f. values—calculated from the statistical errors only—vary between 2 and 10, but should be regarded as being indicative only. Experimental differences between JADE and OPAL contribute significantly to the $\chi^2$/d.o.f. values. At low cms energies the predictions drop off again—this unphysical behaviour (not always visible in the figures) substantially contributes to the high $\chi^2$/d.o.f. values. The moments of the one-hemisphere variables $B_W$ and in particular $M_H$ are described better than in the respective comparison with hadronisation correction by Monte Carlo models [5].

To estimate the experimental systematic uncertainties, the fits are repeated based on the minimum overlap assumption [5] for combining the OPAL and JADE systematic errors.

We study uncertainties of theory parameters by the following variations:

- The renormalisation scale uncertainty (i.e. the effect of higher perturbative orders) is estimated by setting $x_f = 0.5$ and $x_\mu = 2.0$. Our default is always $x_f = 1$.
- The hadronisation uncertainty is estimated by setting $M = 1.49 \pm 20\%$.
- $\mu_I = 1\text{ GeV}$ and $\mu_I = 3\text{ GeV}$. The resulting deviations are included into the $\alpha_0(M_Z)$ uncertainty only, as $\alpha_0(\mu_I)$ depends on $\mu_I$ directly by its definition [3].

Tables 3 and 4 and Fig. 3 contain the $\alpha_0(M_Z)$ and $\alpha_0(\mu_I)$ results from the standard measurement of the moments of the six event shape variables and from the $\mu_I$, $\mu_I$ and $M$ variations. The values of $\alpha_0(M_Z)$ and $\alpha_0(\mu_I)$ measured from $\langle (1 - T)^1 \rangle$, $\langle C^1 \rangle$, $\langle B_T^1 \rangle$, $\langle (y_{ZT})^1 \rangle$, $\langle (M_H^2) \rangle$ are in agreement within errors with previous analyses, see [31, 39] and references therein. The $\alpha_0(M_Z)$ value from $\langle B_W^1 \rangle$ is lower and the $\alpha_0(\mu_I)$ value higher than in previous analyses because of the incomplete description of $\langle B_W^1 \rangle$ in NLO discussed below. This incompleteness affects the predictions at low centre-of-mass energies, leading to high $\alpha_0(\mu_I)$ values and low $\alpha_0$ values. Low energy data have not yet been analysed with a statistics comparable to ours.

The $\alpha_0(M_Z)$ results are similar to those in the analysis employing Monte Carlo models for hadronisation correction [5, 6]. In particular the $\alpha_0(M_Z)$ values steeply rising with moment order for the variables $1 - T$, $C$ and $B_T$ show the incomplete description of the multi jet region in NLO as already seen and discussed in [5, 6]. This is also seen in the large renormalisation scale uncertainties. The $\alpha_0(\mu_I)$ and $\alpha_0(M_Z)$ values from the moments $\langle B_W^0 \rangle$ are not universal but decrease with moment order $n$. We can validate only the explicitly calculated $\langle B_W^1 \rangle$ prediction.

Based on the statistical errors, the values of $\alpha_0(\mu_I)$ from the $\langle B_W^1 \rangle$, $\langle (M_H^2) \rangle$ and $\langle (M_H^4) \rangle$ fits are significantly lower than those from the fits employing the statistical errors.

5 The case of $y_{ZT}$ where we compared with the purely perturbative NLO prediction is different. The energy evolution of the data appears consistent with the prediction but the measured higher moments at 14 GeV are too low. A suitable power correction would scale with a higher power of cms energy, and indeed power terms in $Q/Q^2$ and $1/Q^2$ are expected [7] and describe $y_{ZT}$ distributions better [38].

6 The $\chi^2$/d.o.f. values from these fits are substantially lower than those from the fits employing the statistical errors.

7 We list only results from moments that do not show problems in the multi jet region discussed below.
higher than those from \(\langle (1 - T)^1 \rangle\), \(\langle C^1 \rangle\) and \(\langle B^1 \rangle\). The NLO description of \(B_W\) and \(M_H\) in the two jet region is rather incomplete compared to \(1 - T\), \(C\) and \(B_T\)—especially at low energy where the value of the coupling is high. At low \(Q\) the NLO predictions of the \(B_W\), \(y_{\gamma\gamma}\) and \(M_H\) distributions are (unphysically) negative in a large range of the two jet region \([25, 38]\). This incompleteness of the NLO prediction for the moments—in the case of \(\langle B^1 \rangle\) the \(\alpha_s^2\) coefficient is even negative—is compensated by a larger power correction. The perturbative description then contributes less as is seen by the low \(\alpha_S(M_Z^2)\) values. Fits to JADE data of \(\langle B^1 \rangle\) alone give \(\alpha_S(M_Z^2) = 0.1029 \pm 0.0016\), \(\alpha_S(\mu) = 0.728 \pm 0.017\); to OPAL alone they return \(\alpha_S(M_Z^2) = 0.1242 \pm 0.0025\), \(\alpha_S(\mu) = 0.234 \pm 0.103\) (statistical errors). So at low energies where the coupling is large, the compensation is stronger, and the power correction is not universal (JADE and OPAL agree better on every other moment).

For averaging \(\alpha_S(M_Z^2)\) and \(\alpha_S(\mu)\) over many variables and moment orders, we exclude moments that suffer from the problems discussed in the preceding two paragraphs: The deficiencies in the description of the multi jet region for the two-hemispheres variables have already been seen in \([5, 6]\). To select observables with an apparently converging perturbative prediction, we consider as in \([5, 6]\) only those results for which the NLO term in equation (2) is less than half the corresponding LO term (i.e. \(|K| < 25\) or \(|K\alpha_s/2\pi| < 0.5\), with \(K = \mathcal{B}_n/A_n\). The \(K\) values are shown in Refs. \([5, 6]\).

The higher moments of the jet broadenings are excluded because of the incomplete \([29]\) description of their distributions in general. As in \([5]\) the first moment of \(B_W\) is excluded because the universality of the fit parameters could not be confirmed.

Thus we combine the results of \(\alpha_S(M_Z^2)\) and \(\alpha_S(\mu)\) from the moments \(\langle (1 - T)^1 \rangle\), \(\langle C^1 \rangle\), \(\langle B^1 \rangle\), \(\langle y_{\gamma\gamma}^3 \rangle\)\(\times\langle y_{\gamma\gamma}^5 \rangle\), and \(\langle M_H^2 \rangle\), \(\langle M_H^4 \rangle\). The parameters are consistent within total errors, and the combination procedure follows that used in \([5, 6]\). The correlations between \(\alpha_S(M_Z^2)\) and \(\alpha_S(\mu)\) have rather constant values from -0.80 to -0.93 and thus the combination is done individually for \(\alpha_S(M_Z^2)\) and \(\alpha_S(\mu)\). The averages are:

\[
\alpha_S(M_Z^2) = 0.1183 \pm 0.0007(\text{stat.}) \pm 0.0016(\text{exp.})
\]

\[
\pm 0.0011(\text{had.})^{+0.0052}_{-0.0042}(x_{\mu})
\]

\[
= 0.1183 \pm 0.0056(\text{tot.})
\]

\[
\alpha_S(\mu) = 0.493 \pm 0.006(\text{stat.}) \pm 0.008(\text{exp.})
\]

\(\rightarrow\) The covariances of the statistical errors are estimated by PYTHIA at hadron level at 91.2 GeV, the covariances of the experimental systematic uncertainties are taken as \(E_{ij} = \min\{\sigma_{\exp,i}^2 + \sigma_{\exp,j}^2\}\), and hadronisation and renormalisation scale uncertainties are found by repeating the analysis with varied parameters.
\[ \pm 0.050 \text{(had.)} ^{+0.028}_{-0.014} (x_\mu) \]
\[ = 0.493 \pm 0.058 \text{(tot.)}. \]

4.2 Shape function

In [8] values for the shape function parameters were determined by comparing a distribution of \( M_H^2 \) at 91 GeV with the NLO+NLLA prediction combined with power terms from the shape function to be \( \lambda_1 = 1.22 \text{ GeV}, \)
\[ \lambda_2 = 1.70 \text{ GeV}^2 \]
and \( \delta \lambda_2 \) from \( \delta \lambda_2 (10 \text{ GeV}) = 1.4 \text{ (GeV)}^2 \)
to \( \delta \lambda_2 (100 \text{ GeV}) = 1.2 \text{ (GeV)}^2 \).

We perform fits to the moments \( \langle (1 - T)^1 \rangle, \langle C^1 \rangle \) and \( \langle M_H^4 \rangle \) with two free parameters \( \alpha_S(M_Z^2) \) and \( \lambda_1 \). Fits to the higher moments \( \langle (1 - T)^2 \rangle, \langle C^2 \rangle \) and \( \langle M_H^4 \rangle \) are done with an additional parameter. Because of the weak energy dependence of \( \delta \lambda_2 \) in [22] we substitute the numerator \( \lambda_2 + \delta \lambda_2 (Q) \) by the fit parameter \( \lambda_2 \). The fits to \( \langle C^2 \rangle \) and \( \langle M_H^4 \rangle \) are not sensitive to the parameter \( \lambda_2 \) while the fit to \( \langle (1 - T)^2 \rangle \) gives \( \lambda_2 = (1.2 \pm 1.3) \text{ (GeV)}^2 \). For further analysis we therefore set \( \lambda_2 = 0 \).

Fig. 3 shows the comparison of the data with the prediction. The data are fitted well with \( \chi^2 / \text{d.o.f.} \) —based on the statistical errors—in the order of one. As before, the experimental systematic uncertainties are estimated by the minimum overlap assumption, and the renormalisation scale uncertainty by setting \( x_\mu = 0.5 \) and \( x_\mu = 2.0 \).

Fig. 4 shows the comparison of the data with the prediction. The data are fitted well with \( \chi^2 / \text{d.o.f.} \) —based on the statistical errors, the outer error bars show the total errors. The dashed lines indicate the weighted averages, the shaded bands show their errors. Only the measurements indicated by solid symbols are used for the averages.

Table 3 and Fig. 3 contain the results for \( \alpha_S(M_Z^2) \) and \( \lambda_1 \) from the standard measurement and the systematic variations.

The values of \( \alpha_S(M_Z^2) \) show the typical increase for the higher moments of \( 1 - T \) and \( C \). The values of \( \lambda_1 \) are consistent for the first two moments of every event shape variable. Furthermore they are consistent for the two-hemisphere variables \( 1 - T \) and \( C \). However, the values from \( M_H \) are not compatible. As the respective predictions are similar, we refer to the high values of \( \alpha_S(M_Z^2) \) from the moments \( M_H^2 \) and \( M_H^4 \) in the dispersive model, see Subsect. 4.1. The large power corrections compensate for the incomplete perturbative NLO description. Since the values are not consistent and no systematic evaluation of the theory uncertainty is available, we do not combine any of the fit parameters.
The $\lambda_1$ values from $1 - T$, $C$ and, less pronounced, from $M_H$ are smaller than the results in [8]. A substantial difference of [8] with our work is the inclusion of the NLLA approximation in the distribution prediction, which is not available for the moments.

### 4.3 Variance of event shape variables

From the first and second moments of event shape variables $y$ the variance $\text{Var}(y) = \langle y^2 \rangle - \langle y \rangle^2$ has been calculated [40]. The experimental uncertainties have been determined by systematic variations analogously to the moments at hadron level [5, 6]. The variance of $C$, $B_T$, $B_W$, $M_H$ and $M_H^2$ becomes larger with rising $Q$ while the variance of $1 - T$ becomes smaller, see Fig. 5 and [40]. Studying other observables defined in [6] we find that the variance of $T_{\text{maj.}}$ and $O$ becomes larger with rising $Q$ while the variance of $T_{\text{min.}}$, $S$, $M_L$ and $B_N$ becomes smaller, see [40]. This behaviour is determined by the multi jet region of the distribution [41]. The evolution is reproduced by Monte Carlo models qualitatively well; however data and models differ by typically up to five standard deviations.

Fig. 6 shows the comparison of the prediction (23) using the dispersive model and data for $1 - T$, $C$, $B_T$, $B_W$, $y_{23}$ and $M_H^2$. The fits of the free parameter $\alpha_S(M_Z)$ use central values of the measurements and the statistical errors. The energy evolution of prediction and data for $\text{Var}(B_W)$, $\text{Var}(1 - T)$, and $\text{Var}(y_{23}^D)$ coincides qualitatively (in case of $\text{Var}(y_{23}^D)$ except for the lowest energy points of 14 and 22 GeV), but not for $\text{Var}(C)$, $\text{Var}(B_T)$ and $\text{Var}(M_H^2)$. The $\chi^2$/d.o.f. values are large and the $\alpha_S(M_Z)$ values are not compatible with each other or with established QCD analyses.

The predictions using the shape function differ only for $\text{Var}(C)$ from the dispersive model. Fig. 7 shows the comparison of this prediction (24) and data. The fit gives $\chi^2$/d.o.f. = 50/8, $\alpha_S(M_Z) = 0.0963 \pm 0.0004$ and $\lambda_1 = (0.824 \pm 0.025)$ GeV (statistical errors). The correlation of $\alpha_S(M_Z)$ and $\lambda_1$ is 0.80. In the shape function description the $Q$ evolution of the perturbative description deviating from the data is corrected at low energies by the term $\propto \lambda_1$. Thus the steep energy evolution at low $Q$ is reproduced successfully. However, the fitted values $\alpha_S(M_Z)$ and $\lambda_1$ are not compatible with the corresponding values from $\langle C^1 \rangle$ or $\langle C^2 \rangle$.

### 4.4 Single dressed gluon approximation

The hadron level data are compared with the five different predictions which result from truncating [28] after order $O(\bar{a}^2)\ldots O(\bar{a}^6)$. The data are described by the predictions well. Fig. 8 shows the comparison of data and prediction for truncating after $\bar{a}^5$. The fit results for $\alpha_S$ and $\nu_t$ are given in Table 8. The coefficients $\kappa_2\ldots \kappa_4$ of the more strongly suppressed power correction in all instances turn out to be compatible with zero.

For every moment order we study specific properties to find the series describing the data best. The $\chi^2$/d.o.f. values from the various expansion orders do not differ significantly in a straight line fit.

\[9\] In case of $1 - T$ this is not seen very clearly in the plot, but significantly in a straight line fit.

\[10\] Comparisons for second to sixth expansion order, and more discussion, can be found in [41].
cantly for any moment order and thus do not discriminate between different expansion orders.

\((1 - T)^1\): For the first thrust moment at the order \(m_{\text{max}} = 5\) the power correction is compatible with zero, then it becomes negative. The values of \(\alpha_S(M_{Z\gamma})\) and the power correction reach their minimum at \(m_{\text{max}} = 5\), so the convergence appears best at \(m_{\text{max}} = 5\).

\((1 - T)^2\): Including higher perturbative orders the \(\alpha_S(M_{Z\gamma})\) and \(\nu_2\) values from the second thrust moment decrease. The purely perturbative description appears continually more accurate, we therefore choose \(m_{\text{max}} = 6\).

\((1 - T)^3\), \((1 - T)^4\): Fitting the third or fourth thrust moment, to compensate the negative perturbative terms the power correction increases when increasing the truncation order. Thus the NLO description appears more accurate than the SDG approximation, and we choose \(m_{\text{max}} = 2\).

Fig. 6 compares the perturbative terms and the power correction terms. In case of the first moment the power correction is larger for the third prediction from the SDG approximation already describes significantly for any moment order and thus do not discriminate between different expansion orders.

\((1 - T)^1\): For the first thrust moment at the order \(m_{\text{max}} = 5\) the power correction is compatible with zero, then it becomes negative. The values of \(\alpha_S(M_{Z\gamma})\) and the power correction reach their minimum at \(m_{\text{max}} = 5\), so the convergence appears best at \(m_{\text{max}} = 5\).

\((1 - T)^2\): Including higher perturbative orders the \(\alpha_S(M_{Z\gamma})\) and \(\nu_2\) values from the second thrust moment decrease. The purely perturbative description appears continually more accurate, we therefore choose \(m_{\text{max}} = 6\).

\((1 - T)^3\), \((1 - T)^4\): Fitting the third or fourth thrust moment, to compensate the negative perturbative terms the power correction increases when increasing the truncation order. Thus the NLO description appears more accurate than the SDG approximation, and we choose \(m_{\text{max}} = 2\).

Fig. 6 compares the perturbative terms and the power correction terms. In case of the first moment the power correction is compatible with zero; the purely perturbative prediction from the SDG approximation already describes the data well. The power correction is larger for the third and fourth moments, but is still small compared to the LO and NLO perturbative terms down to \(Q = 10\,\text{GeV}\). In case of the second moment the higher SDG terms up to \(O(\bar{a}^6)\) are comparable with the power correction.

4.5 Measuring \(\alpha_S(M_{Z\gamma})\) in the SDG approximation

The \(\alpha_S(M_{Z\gamma})\) values steeply rise with moment order in this model as observed before in Sect. 4.1. The problem of describing the higher thrust moments consistently—i.e. with compatible fit parameters—is not solved in this model. We measure the strong coupling in this model only from \(\langle (1 - T)^1 \rangle\).

The central value of the measurement and its statistical error are shown in table 6 as case \(m_{\text{max}} = 5\). To determine the experimental systematic uncertainty, the fit is repeated employing the minimum overlap assumption for the experimental uncertainties of the data. To estimate the theoretical uncertainty we study the following contributions:

- In order \(O(\bar{a}^6)\) of the perturbative prediction the unknown coefficient \(\beta_4\) appears. We set it to the simple Padé estimate \(\beta_4 = \beta_2^4/\beta_2\) instead of \(\beta_4 = 0\). This does not change any of the four cited digits of the \(\alpha_S(M_{Z\gamma})\) fit result.

- Instead of three loops we use the four known loops for calculating the \(Q\) evolution of \(\alpha_S\). This changes the fit result by less than 0.0001.

- The \(O(\bar{a}^2)\) coefficient \(B_1\) is approximated by SDG within 34\% [10], and the approximation is expected to be complete in high order [9]. Thus we vary the coefficient \(C_1\) by +34\% or −34\%. This results in changes of \(\alpha_S(M_{Z\gamma})\) by −0.0027 or +0.0031. This variation is consistent with truncating after expansion order \(m_{\text{max}} = 3\)

\[\text{There is no distinct sign of the power term. It can be positive or negative, or change its sign when using different regularisations [34]}.\]
or $m_{\text{max}} = 6$. A renormalisation scale variation is not studied and is expected to cause a small effect in a prediction of fifth order.

- As $\nu_1$ is one of the two fit parameters, the hadronisation error is part of the fit error.

The result is

$$\alpha_S(M_{Z^0}) = 0.1172 \pm 0.0007\,(\text{stat.}) \pm 0.0016\,\,(\text{exp.})$$

$$+0.0031\,-0.0027\,(\text{theo.})$$

$$= 0.1172 \pm 0.0036\,(\text{tot.}) .$$

It is our most precise value. The variation of the NNLO coefficient $C_1$ by $\pm 34\%$ is better motivated than the somewhat arbitrary variation of the renormalisation scale factor $x_\mu = 0.5, 2.0$. It leads to a smaller perturbative uncertainty than in the NLO analyses discussed above or in [5].

In [9] the SDG prediction was fitted to $\langle 1 - T \rangle$ data, notably of the PETRA experiments TASSO und MARK J at cms energies down to 12 GeV. The results were $\alpha_S(M_{Z^0}) = 0.110$ und $\lambda_1 = 0.62$ GeV. In the old PETRA measurements $e^+e^- \rightarrow b\bar{b}$ events have not been subtracted. They enlarge $(1 - T)$ in an effect decreasing with energy faster than $1/Q$. This results in an increase of $\lambda_1$ and a decrease of $\alpha_S(M_{Z^0})$.

5 Summary

As an alternative to the description of hadronisation by Monte Carlo models we studied analytical models of hadronisation. These descriptions require no laborious tuning of a Monte Carlo model but fitting of $\alpha_S$ and an additional parameter. In this way, their consistency can be quantified.

We studied the dispersive model, the shape function and the single dressed gluon approximation, including the NLO predictions. All models describe the mean values of all studied event shape variables at cms energies $Q = 14...209$ GeV well. The dispersive model and the shape function also describe the higher moments of the one-hemisphere variables well. The measurements of $\alpha_S(M_{Z^0})$ and $\alpha_0(\mu_1)$ in the dispersive model from the mean values of $1 - T$, $C$, $B_T$, all studied moments of $y_{23}$ and $\langle M_H \rangle$, $\langle M_R \rangle$ confirm the universality of these parameters within total errors.

Higher moments of the two-hemisphere variables $1 - T$, $B_T$ and $C$ can only be described by higher values of $\alpha_S(M_{Z^0})$, moments of $B_W$ and $M_H$ only by higher values of the power correction coefficient. Both deviations apparently compensate for the deficiencies of the perturbative predictions.

Averaging the parameters from $\langle (1 - T)^1 \rangle$, $\langle C^1 \rangle$, $\langle B_1^T \rangle$, $\langle y_{23}^{D} \rangle$...$\langle y_{23}^{E} \rangle$, and $\langle M_H^1 \rangle$, $\langle M_R^1 \rangle$ in the dispersive
Event shape moments are barely sensitive to more strongly suppressed power corrections resulting from the shape function. In general, a detailed understanding of non-perturbative effects cannot reach any further than the reliability of the underlying perturbative structure which was studied in \cite{5, 6}.

For some variables the variance shows an unexpected energy evolution. Of the variables analysed here only the $1-T$ distributions are broader at low cms energy; the $M^2_B$, $C$, $B_T$ and $B_W$ distributions are narrower at low energy. The energy evolution of the variances is not given correctly by the pure NLO prediction. The description by an additional power correction from the shape function is not quantitatively correct.

The single dressed gluon approximation shows the same problems describing the higher thrust moments as the other models. Fitting the prediction for $\langle (1-T)^1 \rangle$ including terms up to fifth order gives power corrections compatible with zero and

$$\alpha_S(M_{Z0}) = 0.1172 \pm 0.0036 \text{(tot.)}.$$  

With a competitive total error of 3\% this is our most precise $\alpha_S$ measurement. It agrees with the world average of $\alpha_S(M_{Z0}) = 0.1189 \pm 0.0010$ \cite{42}. For this result a Monte Carlo model has only been used for correcting the data for $b\bar{b}$ background, taking into account its uncertainty in the experimental systematic error.

Moments are an interesting alternative to distributions: Specific parts of phase space are tested selectively and the energy evolution shows up clearly. This evolution can be studied thoroughly by the combination of the experiments JADE and OPAL. As in \cite{5} the most interesting energy range is provided by the JADE experiment, as the perturbative and non-perturbative effects both scale inversely with the energy.

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Table 3. Measurements of $\alpha_S(M_Z^0)$ and $\alpha_0(\mu_I)$ in the dispersive model from first moments of six event shape variables over the full analysed range of c.m. energy, 14...209 GeV. The $\chi^2$/d.o.f. values are based on the statistical errors only, see text for further discussion.

| $\langle(1 - T)^1\rangle$ | $\langle C^1\rangle$ | $\langle B_1^1\rangle$ | $\langle B^1_W\rangle$ | $\langle(y_{21})^1\rangle$ |
|---------------------------|------------------------|-------------------------|------------------------|-------------------------|
| $\alpha_S(M_Z^0)$         |                        |                         |                        |                        |
| Statistical error         | 0.0006                 | 0.0004                  | 0.0005                 | 0.0006                  | 0.0004                 |
| Experimental syst.        | 0.0015                 | 0.0012                  | 0.0011                 | 0.0015                  | 0.0016                 |
| $x_\mu$ variation: $x_\mu = 2.0$ | +0.0073               | +0.0073                 | +0.0047                | +0.0004                 | +0.0050                |
| $x_\mu = 0.5$            | -0.0062                | -0.0063                 | -0.0041                | +0.0009                 | -0.0038                |
| $\mu_I$ variation: $\mu_I = 1$ GeV | +0.0025               | +0.0030                 | +0.0017                | +0.0008                 | —                      |
| $\mu_I = 3$ GeV          | -0.0019                | -0.0022                 | -0.0013                | -0.0007                 | —                      |
| $M$ variation: $M - 20\%$ | +0.0011                | +0.0014                 | +0.0008                | +0.0004                 | —                      |
| $M + 20\%$              | -0.0011                | -0.0013                 | -0.0007                | -0.0004                 | —                      |
| Theoretical syst.        | 0.0078                 | 0.0080                  | 0.0051                 | 0.0012                  | 0.0050                 |
| $\alpha_0(\mu_I)$        | 0.490                  | 0.433                   | 0.514                  | 0.604                   | —                      |
| Statistical error         | 0.007                  | 0.003                   | 0.009                  | 0.015                   | —                      |
| Experimental syst.        | 0.010                  | 0.005                   | 0.009                  | 0.014                   | —                      |
| $x_\mu$ variation: $x_\mu = 2.0$ | -0.022                | -0.024                  | +0.013                 | -0.049                  | —                      |
| $x_\mu = 0.5$            | +0.031                 | +0.033                  | +0.011                 | +0.147                  | —                      |
| $M$ variation: $M - 20\%$ | +0.047                 | +0.034                  | +0.053                 | +0.086                  | —                      |
| $M + 20\%$              | -0.033                 | -0.025                  | -0.037                 | -0.057                  | —                      |
| Theoretical syst.        | 0.056                  | 0.048                   | 0.055                  | 0.170                   | —                      |
| Correlation $\alpha_S(M_Z^0), \alpha_0(\mu_I)$ | -0.86                   | -0.80                   | -0.93                  | -0.90                   | —                      |
| $\chi^2$/d.o.f.          | 47.8/10                | 66.3/10                 | 106/10                 | 75.3/10                 | 37.7/10                |
Table 4. Measurements of $\alpha_S(M_{Z^0})$ and $\alpha_0(\mu_I)$ in the dispersive model from second to fifth moments of six event shape variables over the full analysed range of c.m. energy, 14...209 GeV. The $\chi^2$/d.o.f. values are based on the statistical errors only, see text for further discussion.

|          | $\langle(y_{23}^2)\rangle$ | $\langle(M_2^2)\rangle$ | $\langle(y_{23}^3)\rangle$ | $\langle(y_{23}^4)\rangle$ | $\langle(M_4^2)\rangle$ | $\langle(y_{23}^5)\rangle$ |
|----------|-----------------------------|--------------------------|-----------------------------|-----------------------------|--------------------------|-----------------------------|
| $\alpha_S(M_{Z^0})$ | 0.1182                      | 0.1158                   | 0.1173                      | 0.1160                      | 0.1154                   | 0.1144                      |
| Statistical error | 0.0009                      | 0.0007                   | 0.0013                      | 0.0017                      | 0.0010                   | 0.0020                      |
| Experimental syst. | 0.0021                      | 0.0024                   | 0.0020                      | 0.0025                      | 0.0034                   | 0.0033                      |
| $x_\mu$ variation: $x_\mu = 2.0$ | +0.0054                     | +0.0042                  | +0.0053                     | +0.0050                     | +0.0051                  | +0.0047                     |
| $x_\mu = 0.5$ | -0.0042                     | -0.0033                  | -0.0041                     | -0.0038                     | -0.0042                  | -0.0036                     |
| $\mu_I$ variation: $\mu_I = 1\text{GeV}$ | —                           | +0.0013                  | —                           | +0.0011                     | —                        | —                           |
| $\mu_I = 3\text{GeV}$ | —                           | -0.0010                  | —                           | -0.0009                     | —                        | —                           |
| $M$ variation: $M-20\%$ | —                           | +0.0006                  | —                           | +0.0005                     | —                        | —                           |
| $M+20\%$ | —                           | -0.0006                  | —                           | -0.0005                     | —                        | —                           |
| Theoretical syst. | 0.0054                      | 0.0045                   | 0.0053                      | 0.0050                     | 0.0053                  | 0.0047                      |
| $\alpha_0(\mu_I)$ | —                           | 0.601                    | —                           | 0.579                       | —                        | —                           |
| Statistical error | —                           | 0.011                    | —                           | 0.015                       | —                        | —                           |
| Experimental syst. | —                           | 0.012                    | —                           | 0.022                       | —                        | —                           |
| $x_\mu$ variation: $x_\mu = 2.0$ | —                           | -0.022                   | —                           | -0.022                     | —                        | —                           |
| $x_\mu = 0.5$ | —                           | +0.037                   | —                           | +0.034                     | —                        | —                           |
| $M$ variation: $M-20\%$ | —                           | +0.083                   | —                           | +0.079                     | —                        | —                           |
| $M+20\%$ | —                           | -0.056                   | —                           | -0.054                     | —                        | —                           |
| Theoretical syst. | —                           | 0.091                    | —                           | 0.086                       | —                        | —                           |
| Correlation $\alpha_S(M_{Z^0})$, $\alpha_0(\mu_I)$ | —                           | -0.88                    | —                           | -0.82                       | —                        | —                           |
| $\chi^2$/d.o.f. | 18.3/10                     | 25.2/10                  | 21.0/10                     | 18.7/10                     | 26.4/10                  | 15.2/10                     |
Table 5. Measurements of $\alpha_S(M_{Z^0})$ and $\lambda_1$ in the shape function model from the first and second moments of three event shape variables (second and fourth moment in case of $M_H$) over the full analysed range of c.m. energy, 14...209 GeV

|                  | $\langle (1 - T)^1 \rangle$ | $\langle C^1 \rangle$ | $\langle M_H^2 \rangle$ |
|------------------|-----------------------------|------------------------|--------------------------|
| $\alpha_S(M_{Z^0})$ | 0.1304                      | 0.1305                 | 0.1193                   |
| Statistical error | 0.0008                      | 0.0006                 | 0.0008                   |
| Experimental syst. | 0.0018                      | 0.0015                 | 0.0027                   |
| $x_\mu$ variation: $x_\mu = 2.0$ | +0.0069                    | +0.0064                | +0.0038                  |
| $x_\mu = 0.5$     | -0.0054                     | -0.0050                | -0.0025                  |
| $\lambda_1$ [GeV] | 0.499                       | 0.393                  | 1.010                    |
| Statistical error | 0.034                       | 0.024                  | 0.047                    |
| Experimental syst. | 0.065                       | 0.045                  | 0.075                    |
| $x_\mu$ variation: $x_\mu = 2.0$ | +0.053                      | +0.063                 | +0.044                   |
| $x_\mu = 0.5$     | -0.067                      | -0.075                 | -0.045                   |
| Correlation $\alpha_S(M_{Z^0}), \lambda_1$ | -0.96                      | -0.96                  | -0.94                    |
| $\chi^2$/d.o.f.   | 23.3/8                      | 22.6/8                 | 19.6/8                   |

|                  | $\langle (1 - T)^2 \rangle$ | $\langle C^2 \rangle$ | $\langle M_H^4 \rangle$ |
|------------------|-----------------------------|------------------------|--------------------------|
| $\alpha_S(M_{Z^0})$ | 0.1426                      | 0.1456                 | 0.1184                   |
| Statistical error | 0.0014                      | 0.0008                 | 0.0012                   |
| Experimental syst. | 0.0028                      | 0.0023                 | 0.0041                   |
| $x_\mu$ variation: $x_\mu = 2.0$ | +0.0111                    | +0.0117                | +0.0049                  |
| $x_\mu = 0.5$     | -0.0091                     | -0.0097                | -0.0037                  |
| $\lambda_1$ [GeV] | 0.516                       | 0.364                  | 0.964                    |
| Statistical error | 0.056                       | 0.023                  | 0.079                    |
| Experimental syst. | 0.096                       | 0.052                  | 0.133                    |
| $x_\mu$ variation: $x_\mu = 2.0$ | +0.044                      | +0.063                 | +0.031                   |
| $x_\mu = 0.5$     | -0.052                      | -0.054                 | -0.030                   |
| Correlation $\alpha_S(M_{Z^0}), \lambda_1$ | -0.95                      | -0.91                  | -0.92                    |
| $\chi^2$/d.o.f.   | 26.9/8                      | 18.3/8                 | 20.2/8                   |
Table 6. Measurements of $\alpha_S(M_Z)$ and $\nu_n$ from first to fourth moment of $1 - T$ over the full analysed range of cms energy, 14...209 GeV. The single dressed gluon approximation is used in maximum order $m_{\text{max}}$. The errors are statistical. Units of $\nu_n$ are GeV for $n = 1$ and $(\text{GeV})^2$ for $n = 2, 3, 4$

| $m_{\text{max}}$ | $\alpha_S(M_Z)$ | $\nu_n$ | correlation $\alpha_S(M_Z), \nu_n$ | $\chi^2$/d.o.f. |
|------------------|-----------------|---------|-----------------------------------|-----------------|
| $(1 - T)^1$      |                 |         |                                   |                 |
| 2                | 0.1314±0.0008   | 0.430±0.036 | -0.96                            | 29.1/10         |
| 3                | 0.1209±0.0007   | 0.352±0.039 | -0.97                            | 30.0/10         |
| 4                | 0.1178±0.0007   | 0.223±0.045 | -0.97                            | 29.3/10         |
| 5                | 0.1172±0.0007   | 0.035±0.058 | -0.98                            | 26.4/10         |
| 6                | 0.1182±0.0001   | -0.293±0.010 | 0.88                            | 24.0/10         |
| $(1 - T)^2$      |                 |         |                                   |                 |
| 2                | 0.1463±0.0009   | 1.855±0.455 | 0.73                             | 23.8/10         |
| 3                | 0.1426±0.0009   | 1.762±0.454 | 0.74                             | 24.0/10         |
| 4                | 0.1416±0.0008   | 1.644±0.458 | 0.73                             | 24.2/10         |
| 5                | 0.1412±0.0008   | 1.546±0.460 | 0.73                             | 24.3/10         |
| 6                | 0.1410±0.0008   | 1.465±0.470 | 0.73                             | 24.4/10         |
| $(1 - T)^3$      |                 |         |                                   |                 |
| 2                | 0.1523±0.0015   | 0.273±0.136 | 0.73                             | 20.7/10         |
| 3                | 0.1525±0.0015   | 0.274±0.137 | 0.73                             | 20.7/10         |
| 4                | 0.1531±0.0015   | 0.286±0.137 | 0.73                             | 20.7/10         |
| 5                | 0.1533±0.0015   | 0.298±0.136 | 0.73                             | 20.6/10         |
| 6                | 0.1534±0.0015   | 0.307±0.135 | 0.74                             | 20.6/10         |
| $(1 - T)^4$      |                 |         |                                   |                 |
| 2                | 0.1581±0.0017   | 0.059±0.024 | 0.55                             | 17.6/10         |
| 3                | 0.1625±0.0018   | 0.063±0.024 | 0.55                             | 17.5/10         |
| 4                | 0.1645±0.0019   | 0.075±0.023 | 0.56                             | 17.4/10         |
| 5                | 0.1652±0.0019   | 0.085±0.022 | 0.58                             | 17.3/10         |
| 6                | 0.1655±0.0019   | 0.095±0.021 | 0.60                             | 17.3/10         |
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