CERTAIN SUBCLASSES OF SPIRALLIKE UNIVALENT FUNCTIONS RELATED WITH PASCAL DISTRIBUTION SERIES

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Abstract. The purpose of the present paper is to find the necessary and sufficient conditions for the subclasses of analytic functions associated with Pascal distribution to be in subclasses of spiral-like univalent functions and inclusion relations for such subclasses in the open unit disk $D$. Further, we consider the properties of integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

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1. Introduction and definitions

Denote by $\mathcal{A}$ the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be subclass of $\mathcal{A}$ whose members are given by (1) and are univalent in $\mathbb{D}$. For functions $f \in \mathcal{S}$ be given by (1) and $g \in \mathcal{S}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of $f$ and $g$ by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}.$$

The two well known subclass of $\mathcal{S}$, are namely the class of starlike and convex functions (for details see Robertson [17]). A function $f \in \mathcal{S}$ is said to be starlike of order $\gamma$ ($0 \leq \gamma < 1$), if and only if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma \quad (z \in \mathbb{D}).$$

This function class is denoted by $\mathcal{S}^*(\gamma)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{D})$ is starlike with respect to the origin.

A function $f \in \mathcal{S}$ is said to be convex of order $\gamma$ ($0 \leq \gamma < 1$) if and only if

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \gamma \quad (z \in \mathbb{D}).$$

This class is denoted by $\mathcal{K}(\gamma)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. By Alexander’s relation, it is a known fact that

$$f \in \mathcal{K} \iff zf'(z) \in \mathcal{S}^*.$$
A function \( f \in S \) is said to be spirallike if
\[
\Re \left( e^{-i\alpha} \frac{zf''(z)}{f'(z)} \right) > 0
\]
for some \( \alpha \) with \( |\alpha| < \frac{\pi}{2} \) and for all \( z \in \mathbb{D} \) this class of spiral-like function was introduced by [20]. Also \( f(z) \) is convex spiral-like if \( zf'(z) \) is spiral-like. Due to Murugusundramoorthy [8, 9], we consider subclasses of spiral-like functions as below:

**Definition 1.1.** For \( 0 \leq \rho < 1, 0 \leq \gamma < 1 \) then
\[
S(\xi, \gamma, \rho) := \left\{ f \in S : \Re \left( \frac{zf'(z)}{(1-\rho)f(z) + \rho zf'(z)} \right) > \gamma \cos \xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}.
\]

By virtue of Alexander’s relation, we define the following subclass:

**Definition 1.2.** For \( 0 \leq \rho < 1, 0 \leq \gamma < 1 \) then
\[
K(\xi, \gamma, \rho) := \left\{ f \in S : \Re \left( \frac{zf''(z) + f'(z)}{f'(z) + \rho zf'(z)} \right) > \gamma \cos \xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}.
\]

By specialising the parameter \( \rho = 0 \) we remark the following:

**Definition 1.3.** For \( 0 \leq \gamma < 1 \) then
\[
S(\xi, \gamma) := \left\{ f \in S : \Re \left( \frac{zf'(z)}{f(z)} \right) > \gamma \cos \xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}
\]
and
\[
K(\xi, \gamma) := \left\{ f \in S : \Re \left( \frac{zf''(z) + f'(z)}{f'(z)} \right) > \gamma \cos \xi, \ |\xi| < \frac{\pi}{2}, \ z \in \mathbb{D} \right\}.
\]

Now we state the necessary sufficient conditions for \( f \) in the above classes.

**Lemma 1.4.** [8, 9] A function \( f(z) \) of the form (1) is in \( S(\xi, \gamma, \rho) \) if
\[
\sum_{n=2}^{\infty} ((1-\rho)(n-1) \sec \xi + (1-\gamma)(1+n\rho-\rho)) |a_n| \leq 1 - \gamma,
\]
where \( |\xi| < \frac{\pi}{2} \), \( 0 \leq \rho < 1, 0 \leq \gamma < 1 \).

**Lemma 1.5.** A function \( f(z) \) of the form (1) is in \( S(\xi, \gamma, \rho) \) if
\[
\sum_{n=2}^{\infty} n((1-\rho)(n-1) \sec \xi + (1-\gamma)(1+n\rho-\rho)) |a_n| \leq 1 - \gamma,
\]
where \( |\xi| < \frac{\pi}{2} \), \( 0 \leq \rho < 1, 0 \leq \gamma < 1 \).

**Proof.** Using Alexander type Theorem which states that, If \( f \in K(\xi, \gamma, \rho) \) if and only if \( zf' \in S(\xi, \gamma, \rho) \), Thus \( z + \sum_{n=2}^{\infty} na_n z^n \) is in \( K(\xi, \gamma, \rho) \). Hence by wringing \( a_n \) in Lemma [1.4] by \( na_n \) we get the desired result. 

**Lemma 1.6.** A function \( f(z) \) of the form (1) is in \( S(\xi, \gamma) \) if
\[
\sum_{n=2}^{\infty} ((n-1) \sec \xi + (1-\gamma)) |a_n| \leq 1 - \gamma,
\]
where \( |\xi| < \frac{\pi}{2} \), \( 0 \leq \gamma < 1 \).
Lemma 1.7. A function \( f(z) \) of the form (1) is in \( K(\xi, \gamma) \) if
\[
\sum_{n=2}^{\infty} n[(n-1)\sec\xi + (1-\gamma)]|a_n| \leq 1 - \gamma, \tag{5}
\]
where \(|\xi| < \frac{\pi}{2} \), \(0 \leq \gamma < 1\).

Definition 1.8. A function \( f \in S \) is said to be in the class \( R^\tau(\vartheta, \delta) \), \((\tau \in \mathbb{C}\setminus\{0\}, 0 < \vartheta \leq 1; \delta < 1) \), if it satisfies the inequality
\[
\left| (1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1 \right| < 1 \quad (z \in \mathbb{U}).
\]

The class \( R^\tau(\vartheta, \delta) \) was introduced earlier by Swaminathan [21](for special cases see the references cited there in).

Lemma 1.9. [21] If \( f \in R^\tau(\vartheta, \delta) \) is of form (1), then
\[
|a_n| \leq \frac{2|\tau| (1-\delta)}{1 + \vartheta(n-1)}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{6}
\]

The bounds given in (6) is sharp.

A variable \( x \) is said to be Pascal distribution if it takes the values 0, 1, 2, 3, \ldots with probabilities
\[
(1-q)^m, \quad \frac{qm(1-q)^m}{1!}, \quad \frac{q^2m(m+1)(1-q)^m}{2!}, \quad \frac{q^3m(m+1)(m+2)(1-q)^m}{3!}, \ldots \text{ respectively,}
\]
where \( q \) and \( m \) are called the parameters, and thus
\[
P(x = k) = \binom{k + m - 1}{m - 1} q^k (1-q)^m, \quad k = 0, 1, 2, 3, \ldots.
\]

Lately, El-Deeb [4](also see [1]) introduced a power series whose coefficients are probabilities of Pascal distribution
\[
\Theta^m_q(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}(1-q)^m z^n, \quad z \in \mathbb{D}
\]
where \( m \geq 1; 0 \leq q \leq 1 \) and one can easily verify that the radius of convergence of above series is infinity by ratio test. Now, we define the linear operator
\[
\Lambda^m_q(z) : \mathcal{A} \to \mathcal{A}
\]
defined by the convolution or Hadamard product
\[
\Lambda^m_q f(z) = \Theta^m_q(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}(1-q)^m a_n z^n, \quad z \in \mathbb{D}.
\]

Inspired by earlier results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example, [2, 6, 7, 18, 19, 21]) and by the recent investigations related with distribution series (see for example, [1, 3, 4, 5, 11, 10, 13, 14, 16], we obtain necessary and sufficient condition for the function \( \Phi^m_q \) to be in the classes \( S(\xi, \gamma, \rho) \) and \( K(\xi, \gamma, \rho) \), and information regarding the images of functions belonging in \( R^\tau(\vartheta, \delta) \) by applying convolution operator. Finally, we provide conditions for the integral operator \( G^m_q(z) = \int^z_0 \Theta^m_q(t) \frac{dt}{t} \) belonging to the above classes.
2. The necessary and sufficient conditions

In order to prove our main results, we will use the following notations, for $m \geq 1$ and $0 \leq q < 1$:

$$\sum_{n=0}^{\infty} \binom{n + m - 1}{m - 1}q^n = \frac{1}{(1-q)^m}; \quad \sum_{n=0}^{\infty} \binom{n + m}{m}q^n = \frac{1}{(1-q)^{m+1}}$$

and

$$\sum_{n=0}^{\infty} \binom{n + m + 1}{m + 1}q^n = \frac{1}{(1-q)^{m+2}}. \quad (7)$$

By simple computation we get the following relations:

$$\sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1}q^{n-1} = \sum_{n=0}^{\infty} \binom{n + m - 1}{m - 1}q^n - 1 = \frac{1}{(1-q)^m} - 1 \quad (8)$$

$$\sum_{n=2}^{\infty} (n - 1) \binom{n + m - 2}{m - 1}q^{n-1} = qm \sum_{n=0}^{\infty} \binom{n + m}{m}q^n = \frac{qm}{(1-q)^{m+1}} \quad (9)$$

and

$$\sum_{n=2}^{\infty} (n - 1)(n - 2) \binom{n + m - 2}{m - 1}q^{n-1} = q^2 m(m + 1) \sum_{n=0}^{\infty} \binom{n + m + 1}{m + 1}q^n$$

$$= \frac{q^2 m(m + 1)}{(1-q)^{m+2}}. \quad (10)$$

**Theorem 2.1.** If $m > 0$, then $\Theta_q^m(z) \in S(\xi, \gamma, \rho)$ if

$$[(1-\rho) \sec \xi + \rho (1-\gamma)] \frac{qm}{(1-q)^{m+1}} \leq 1 - \gamma. \quad (11)$$

**Proof.** Since

$$\Theta_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1}q^{n-1}(1-q)^m z^n.$$ 

Using the Lemma 1.4 it suffices to show that

$$\sum_{n=2}^{\infty} [(1-\rho)(n - 1) \sec \xi + (1-\gamma)(1+n\rho-\rho)] \leq 1 - \gamma. \quad (12)$$
From (12) we let

\[ M_1(\xi, \gamma, \rho) = \sum_{n=2}^{\infty} [(1 - \rho) \sec \xi (n - 1) + (1 - \gamma)(1 + n\rho - \rho)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m \]

But \( M_1(\xi, \gamma, \rho) \) is bounded above by \( 1 - \gamma \) if and only if (11) holds. \( \square \)

**Theorem 2.2.** If \( m > 0 \), then \( \Theta_{q}^m(z) \in K(\xi, \gamma, \rho) \) if

\[
[(1 - \rho) \sec \xi + (1 - \gamma)] \frac{mq}{1 - q} + \frac{m(m + 1)q^2}{(1 - q)^2} + \frac{2(1 - \rho)sec\xi + (1 - \gamma)(4 - \rho)}{1 - q} \]

\[
\leq 1 - \gamma. \tag{13}
\]

**Proof.** In view of Lemma 1.5, we have to show that

\[
\sum_{n=2}^{\infty} n[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m \leq 1 - \gamma. \tag{14}
\]
Writing \( n = (n - 1) + 1 \) and \( n^2 = (n - 1)(n - 2) + 3(n - 1) + 1 \). From (14), consider the expression

\[
M_2(\xi, \gamma, \rho) = \sum_{n=2}^{\infty} n[\sec \xi + (1 - \gamma)(1 + n\rho - \rho)]
\]

\[
\times \binom{n + m - 2}{m - 1} q^{n-1}(1 - q)^m
\]

\[
= [(1 - \rho)\sec \xi + (1 - \gamma)](1 - q)^m \sum_{n=2}^{\infty} n^2 \binom{n + m - 2}{m - 1} q^{n-1}
\]

\[
- (1 - \rho)[\sec \xi - (1 - \gamma)(1 - q)\rho] \sum_{n=2}^{\infty} n \binom{n + m - 2}{m - 1} q^{n-1}
\]

\[
= [(1 - \rho)\sec \xi + (1 - \gamma)](1 - q)^m \sum_{n=2}^{\infty} (n - 1)(n - 2) \binom{n + m - 2}{m - 1} q^{n-1}
\]

\[
+ [2(1 - \rho)\sec \xi + (1 - \gamma)(4 - \rho)](1 - q)^m \sum_{n=2}^{\infty} (n - 1) \binom{n + m - 2}{m - 1} q^{n-1}
\]

\[
+ [(1 - \gamma)(2 - \rho)](1 - q)^m \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}
\]

\[
+ [(1 - \gamma)(2 - \rho)](1 - q)^m \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}
\]

Hence, \( M_2(\xi, \gamma, \rho) \) is bounded above by \( 1 - \gamma \) if (13) is satisfied.

3. Inclusion Properties

Making use of the Lemma 1.9, we will focus the influence of the Pascal distribution series on the classes \( S(\xi, \gamma, \rho) \) and \( K(\xi, \gamma, \rho) \).
Theorem 3.1. If $f \in \mathcal{R}^\gamma(\vartheta, \delta)$ then $\Lambda_n^m f(z)$ is in $\mathcal{S}(\xi, \gamma, \rho)$ if and only if

$$2\frac{|\tau|}{\vartheta} (1 - \delta) \left\{ \frac{[(1 - \rho)\sec\xi + \rho(1 - \gamma)]}{[(1 - q) - (1 - q)^m - q(m - 1)(1 - q)^m]} \right\} \leq 1 - \gamma. \quad (15)$$

Proof. In view of Lemma 1.4, it is required to show that

$$\sum_{n=2}^{\infty} [(1 - \rho)(n - 1) \sec\xi + (1 - \gamma)(1 + n\rho - \rho)] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m |a_n| \leq 1 - \gamma.$$ 

Let

$$M_3(\xi, \gamma, \rho) = \sum_{n=2}^{\infty} [(1 - \rho)(n - 1) \sec\xi + (1 - \gamma)(1 + n\rho - \rho)] \times \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m |a_n|.$$ 

Since $f \in \mathcal{R}^\gamma(\vartheta, \delta)$, then by Lemma 1.9, we have

$$|a_n| \leq \frac{2|\tau|}{1 + \vartheta(n - 1)} (1 - \delta), \quad n \in \mathbb{N} \setminus \{1\}$$

and $1 + \vartheta(n - 1) \geq \vartheta n$. Thus, we have

$$M_3(\xi, \gamma, \rho) \leq \frac{2|\tau|}{\vartheta} (1 - \delta) \left[ \sum_{n=2}^{\infty} \frac{1}{n} [(1 - \rho)(n - 1) \sec\xi + (1 - \gamma)(1 + n\rho - \rho)] \right] \times \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m $$

$$= \frac{2|\tau|}{\vartheta} (1 - \delta) (1 - q)^m \left[ \sum_{n=2}^{\infty} [(1 - \rho)\sec\xi + \rho(1 - \gamma)] \right]$$

$$+ (1 - \rho)(1 - \gamma - \sec\xi) \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}.$$ 

Using (8), we get

$$M_3(\xi, \gamma, \rho) = \frac{2|\tau|}{\vartheta} (1 - \delta)(1 - q)^m \left\{ [(1 - \rho)\sec\xi + \rho(1 - \gamma)] \left[ \sum_{n=0}^{\infty} \left( \frac{n + m - 1}{m - 1} \right) q^n \right] - 1 \right\}$$

$$+ \frac{(1 - \rho)(1 - \gamma)}{q(m - 1)} \left[ \sum_{n=0}^{\infty} \left( \frac{n + m - 2}{m - 2} \right) q^n - 1 - (m - 1)q \right] \}$$

$$= \frac{2|\tau|}{\vartheta} (1 - \delta) \left\{ [(1 - \rho)\sec\xi + \rho(1 - \gamma)] [1 - (1 - q)^m]$$

$$+ \frac{(1 - \rho)(1 - \gamma - \sec\xi)}{q(m - 1)} [(1 - q) - (1 - q)^m - q(m - 1)(1 - q)^m] \right\}.$$

But $M_3(\xi, \gamma, \rho)$ is bounded by $1 - \gamma$, if (15) holds. This completes the proof of Theorem 3.1. □
Applying Lemma 1.5 and using the same technique as in the proof of Theorem 2.2, we have the following result.

**Theorem 3.2.** If \( f \in \mathcal{R}^\tau(\vartheta, \delta) \), then \( \Lambda^m_q f(z) \) is in \( \mathcal{K}(\xi, \gamma, \rho) \) if and only if
\[
\frac{2|\tau|}{\vartheta} (1 - \delta) \left[ \frac{[(1 - \rho) \sec \xi + (1 - \gamma)] m(m + 1)q^2}{(1 - q)^2} + \frac{2(1 - \rho) \sec \xi + (1 - \gamma)(4 - \rho)}{1 - q} \right] \leq 1 - \gamma.
\]

4. **An integral operator**

**Theorem 4.1.** If the function \( G^m_q(z) \) is given by
\[
G^m_q(z) = \int_0^z \frac{\Theta^m(t)}{t} dt, \quad z \in \mathbb{D}
\]
then \( G^m_q(z) \in \mathcal{K}(\xi, \gamma, \rho) \) if and only if
\[
[(1 - \rho) \sec \xi + \rho(1 - \gamma)] \frac{qm}{(1 - q)^{m+1}} \leq 1 - \gamma.
\]

**Proof.** Since
\[
G^m_q(z) = z + \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \frac{z^n}{n}
\]
then by Lemma 1.5 we need only to verify that
\[
\sum_{n=2}^{\infty} \frac{1}{n} \left[ (1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \leq 1 - \gamma,
\]
or, equivalently
\[
\sum_{n=2}^{\infty} \left[ (1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \leq 1 - \gamma.
\]
The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 2.1, and so we omit the details. \( \square \)

**Theorem 4.2.** If \( m > 0 \), then the integral operator \( G^m_q \) given by (16) is in \( \mathcal{S}(\xi, \gamma, \rho) \) if and only if
\[
[(1 - \rho)\sec \xi + \rho(1 - \gamma)] [1 - (1 - q)^m]
+ \frac{(1 - \rho)(1 - \gamma - \sec \xi)}{q(m - 1)} [(1 - q) - (1 - q)^m - q(m - 1)(1 - q)^m] \leq 1 - \gamma.
\]

**Proof.** Since
\[
G^m_q(z) = z + \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \frac{z^n}{n}
\]
then by Lemma 1.4 we need only to verify that
\[
\sum_{n=2}^{\infty} \frac{1}{n} \left[ (1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho) \right] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \leq 1 - \gamma.
\]
Thus, we have

\[ M_4(\xi, \gamma, \rho) = \sum_{n=2}^{\infty} \frac{1}{n}[(1 - \rho)(n - 1) \sec \xi + (1 - \gamma)(1 + n\rho - \rho)] \times \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m \]

\[ = (1 - q)^m \left[ \sum_{n=2}^{\infty}[(1 - \rho)\sec \xi + \rho(1 - \gamma)] + (1 - \rho)(1 - \gamma - \sec \xi) \frac{1}{n} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1} \right]. \]

Using (8), we get

\[ M_4(\xi, \gamma, \rho) = (1 - q)^m \left\{ [(1 - \rho)\sec \xi + \rho(1 - \gamma)] \left[ \sum_{n=0}^{\infty} \left(\frac{n + m - 1}{m - 1}\right) q^n - 1 \right] \right. \]

\[ + \frac{(1 - \rho)(1 - \gamma - \sec \xi)}{q(m - 1)} \left[ \sum_{n=0}^{\infty} \left(\frac{n + m - 2}{m - 2}\right) q^n - 1 - (m - 1)q \right] \left\} \right. \]

\[ = \left\{ [(1 - \rho)\sec \xi + \rho(1 - \gamma)] [1 - (1 - q)^m] \right. \]

\[ + \frac{(1 - \rho)(1 - \gamma - \sec \xi)}{q(m - 1)} [(1 - q) - (1 - q)^m - q(m - 1)(1 - q)^m] \right\}. \]

But \( M_4(\xi, \gamma, \rho) \) is bounded by \( 1 - \gamma \), if (15) holds. This completes the proof of Theorem 3.1. \( \square \)

The proof of Theorem 4.2 is similar to that of Theorem 4.1, so we omitted the proof of Theorem 4.2.

5. Corollaries and consequences

By taking \( \rho = 0 \) in Theorems 2.1, 3.1, 4.1, 4.2, we obtain the necessary and sufficient condition for Pascal distribution series be in the classes \( S(\xi, \gamma) \) and \( K(\xi, \gamma) \) from the following corollaries.

**Corollary 5.1.** If \( m > 0 \), then \( \Theta^m_q \) is in \( S(\xi, \gamma) \) if and only if

\[ \frac{qm \sec \xi}{(1 - q)^{m+1}} \leq 1 - \gamma. \]

**Corollary 5.2.** If \( m > 0 \), then \( \Theta^m_q \) is in \( K(\xi, \gamma) \) if and only if

\[ \frac{[\sec \xi(1 - \gamma)]m(m + 1)q^2}{(1 - q)^2} + \frac{[2\sec \xi + 4(1 - \gamma)]mq}{1 - q} \]

\[ + \frac{[2(1 - \gamma)](1 - (1 - q)^m)}{1 - q} \leq 1 - \gamma. \]
Corollary 5.3. If $f \in R^\tau(\vartheta, \delta)$ then $\Lambda^m_q$ is in $S(\xi, \gamma)$ if and only if
\[
\frac{2|\tau|(1-\delta)}{\vartheta} \left\{ \sec \xi \left[ 1 - (1-q)^m \right] \right. \\
+ \frac{(1-\gamma - \sec \xi)}{q(m-1)} \left[ (1-q) - (1-q)^m - q(m-1)(1-q)^m \right] \right\} \leq 1 - \gamma.
\]

Corollary 5.4. If $f \in R^\tau(\vartheta, \delta)$, then $\Lambda^m_q$ is in $K(\xi, \gamma)$ if and only if
\[
\frac{2|\tau|(1-\delta)}{\vartheta} \left[ \sec \xi + (1-\gamma) \right] \frac{m(m+1)q^2}{(1-q)^2} + \frac{2\sec \xi + 4(1-\gamma)}{1-q} \cdot \frac{mq}{1-q} \\
+ \frac{2(1-\gamma)}{1-(1-q)^m} \leq 1 - \gamma.
\]

Corollary 5.5. If $m > 0$, then the integral operator $G^m_q(z)$ given by \((16)\) is in $K(\xi, \gamma)$ if and only if
\[
\frac{qm \sec \xi}{(1-q)^{m+1}} \leq 1 - \gamma.
\]

Corollary 5.6. If $m > 0$, then the integral operator $G^m_q(z)$ given by \((16)\) is in $S(\xi, \gamma)$ if and only if
\[
\sec \xi \left[ 1 - (1-q)^m \right] \\
+ \frac{(1-\gamma - \sec \xi)}{q(m-1)} \left[ (1-q) - (1-q)^m - q(m-1)(1-q)^m \right] \leq 1 - \gamma.
\]

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