Metric Regularity of Quasidifferentiable Mappings, Limiting Quasidifferential Sum, and Optimality Conditions

Dolgopolik M.V.∗†

January 1, 2019

Abstract

We obtain new necessary and sufficient conditions for the local metric regularity of a multifunction in terms of Demyanov-Rubinov-Polyakova quasidifferentials of the distance function to this multifunction. We also propose a new MFCQ-type constraint qualification for a parametric system of quasidifferential equality and inequality constraints, and prove that it ensures the metric regularity of a multifunction associated with this system. To strengthen this result in the finite dimensional case, we introduce a limiting quasidifferential sum of a quasidifferentiable function, and prove that a limiting version of the constraint qualification introduced in this paper ensures the metric regularity of the parametric system as well. As an application, we utilise our constraint qualifications to strengthen existing optimality conditions for quasidifferentiable programming problems with equality and inequality constraints. In the end of the paper, we present a simple example in which the optimality conditions in terms of quasidifferentials detect the non-optimality of a given point, while optimality conditions in terms of various subdifferentials fail to disqualify this point as non-optimal.

1 Introduction

Metric regularity plays a very important role in various parts of optimization theory and numerical analysis, including stability analysis of perturbed optimization problems, subdifferential calculus, analysis of optimality conditions etc. [3, 22, 23, 29, 11, 1] Necessary and/or sufficient conditions for metric regularity are usually expressed in terms of various slopes, subdifferentials and coderivatives [6, 22, 23, 24]. However, if one studies nonsmooth problems with quasidifferentiable data and wants to utilise quasidifferential calculus [10, 11], these conditions for metric regularity become very inconvenient, since one has to compute and use subdifferentials/coderivatives and quasidifferentials simultaneously. In this case it seems more reasonable to apply necessary and/or sufficient for metric regularity in terms of quasidifferentials. Such conditions were first obtained by Uderzo [34, 35].

∗Saint Petersburg State University, Saint Petersburg, Russia
†Institute of Problems in Mechanical Engineering, Saint Petersburg, Russia
One of the main goals of this paper is to improve the main results of [34, 35] and obtain simple conditions for metric regularity in terms of quasidifferentials. With the use of general results on metric regularity [22] we obtain new necessary and sufficient conditions for metric regularity of multifunctions in terms of quasidifferential of the distance function to this multifunction (see [15] for some results on the quasidifferentiability of this function). These conditions significantly generalize and improve some results from [31]. For example, our conditions, unlike the ones in [35], are invariant under the choice of quasidifferentials. However, both our conditions and the one in [34, 35] have a significant drawback. Namely, one must verify the validity of certain inequalities in a neighbourhood of a given point to apply these conditions. To overcome this issue, we introduce a new MFCQ-type constraint qualification for a parametric system of quasidifferentiable equality and inequality constraint and demonstrate that this constraint qualification guarantees the local metric regularity of a multifunction associated with this system (see [27] for a discussion of constraint qualifications for quasidifferentiable optimization problems with inequality constraints). To improve this result in the finite dimensional case, we introduce a limiting quasidifferential sum of a quasidifferentiable function, and demonstrate that a limiting version of our MFCQ-type constraint qualification formulated in terms of limiting quasidifferential sum ensures the metric regularity of the parametric system as well.

As an application, we utilise our constraint qualifications to obtain new necessary optimality conditions for quasidifferentiable programming problems with equality and inequality constraints that strengthen existing optimality conditions for these problems in terms of quasidifferentials [32, 33, 31] (optimality conditions for such problems involving, e.g. the Demyanov difference of quasidifferentials, can be found in [17]). We also present a simple example in which our optimality conditions detect the non-optimality of a given point, while optimality conditions in terms of Clarke, Michel-Penot, Jeyakumar-Luc, Ioffe and Mordukhovich subdifferentials fail to disqualify this point as non-optimal.

The paper is organized as follows. Necessary and sufficient conditions for metric regularity of multifunctions in terms of quasidifferentials are obtained in Section 3. In this section, we also introduce two new MFCQ-type constraint qualifications for parametric systems of quasidifferentiable equalities and inequalities and study their connection with metric regularity. These constraint qualifications are applied to the derivation of new optimality conditions for quasidifferentiable programming problems in Section 4. Finally, some basic definitions and facts from quasidifferential calculus are collected in Section 2.

2 Quasidifferentiable Functions

From this point onwards, let $X$ be a real Banach space. Its topological dual space is denoted by $X^*$, whereas the canonical duality pairing between $X$ and $X^*$ is denoted by $\langle \cdot, \cdot \rangle$.

Let $U \subset X$ be an open set. Recall that a function $f: U \to \mathbb{R}$ is called Dini (Hadamard) directionally differentiable at a point $x \in U$, if for any $v \in X$ there
exists the finite limit

\[
\begin{align*}
    f'_\mathcal{D}(x,v) &= \lim_{\alpha \to 0^+} \frac{f(x + \alpha v) - f(x)}{\alpha} \\
    (f'_H(x,v) &= \lim_{[\alpha,v'] \to [0,v]} \frac{f(x + \alpha v') - f(x)}{\alpha})
\end{align*}
\]

(see [18] for a discussion about the limit in the definition of Hadamard directional derivative). Clearly, if \( f \) is Hadamard directionally differentiable at \( x \), then \( f'_H(x,\cdot) = f'_\mathcal{D}(x,\cdot) \). Therefore, it is natural to refer simply to the directional derivative of \( f \) at \( x \), and denote it by \( f'(x,\cdot) \).

**Definition 1.** A function \( f : U \to \mathbb{R} \) is called Dini (Hadamard) quasidifferentiable at a point \( x \in U \) if \( f \) is Dini (Hadamard) directionally differentiable at \( x \) and its directional derivative can be represented as the difference of two continuous sublinear functions or equivalently if there exists a pair \( \mathcal{D}f(x) = [\mathcal{D}_f(x), \overline{\mathcal{D}}_f(x)] \) of convex weak* compact sets \( \mathcal{D}_f(x), \overline{\mathcal{D}}_f(x) \subset X^* \) such that

\[
    f'(x,v) = \max_{x^* \in \mathcal{D}_f(x)} \langle x^*, v \rangle + \min_{y^* \in \overline{\mathcal{D}}_f(x)} \langle y^*, v \rangle \quad \forall v \in X.
\]

The pair \( \mathcal{D}_f(x) \) is called a Dini (Hadamard) quasidifferential of \( f \) at \( x \), while the sets \( \mathcal{D}_f(x) \) and \( \overline{\mathcal{D}}_f(x) \) are called the Dini (Hadamard) subdifferential and superf(di)fferential of \( f \) at \( x \) respectively.

**Remark 1.** Following the usual convention, we identify \( X^* \) with \( X \) in the case when \( X \) is either a finite dimensional or a Hilbert space. Therefore, in particular, if \( X = \mathbb{R}^n \), then a quasidifferential is a pair of convex compact subsets of \( \mathbb{R}^n \), while if \( X \) is a Hilbert space, then a quasidifferential is a pair of weakly compact convex subsets of \( X \).

A calculus of quasidifferentiable functions can be found in [10]. Here we only mention that any finite DC (difference-of-convex) function is Hadamard quasidifferentiable. Note also that a quasidifferential of a function \( f \) is not unique. In particular, for any quasidifferential \( \mathcal{D}_f(x) \) of \( f \) at \( x \) and any weak* compact convex set \( C \subset X^* \) the pair \( \mathcal{D}_f(x) + C, \overline{\mathcal{D}}_f(x) - C \) is a quasidifferential of \( f \) at \( x \) as well.

In the general case quasidifferential mapping \( \mathcal{D}_f(\cdot) \) might not possess any continuity properties; however, for many nonsmooth functions appearing in applications it is outer semicontinuous (o.s.c.). Recall that if a function \( f \) is quasidifferentiable in a neighbourhood \( U \) of a point \( x \in X \), then a quasidifferential mapping \( \mathcal{D}_f(\cdot) \) defined in this neighbourhood is said to be o.s.c. at \( x \), if the corresponding multifunctions \( \mathcal{D}_f : U \to X^* \) and \( \overline{\mathcal{D}}_f : U \to X^* \) are o.s.c. at \( x \), i.e. for any open sets \( V_1, V_2 \subset X^* \) such that \( \mathcal{D}_f(x) \subset V_1 \) and \( \overline{\mathcal{D}}_f(x) \subset V_2 \) there exists \( \delta > 0 \) such that \( \mathcal{D}_f(x') \subset V_1 \) and \( \overline{\mathcal{D}}_f(x') \subset V_2 \) for all \( x' \in U \) with \( \|x' - x\| < \delta \).

As it was pointed out in [26], a quasidifferential of a continuously codifferentiable function is outer semicontinuous (see [11] for the definition of continuously codifferentiable function). Hence, in particular, the class of functions for which there exists an o.s.c. quasidifferential mapping is closed under all standard algebraic operations, the pointwise maximum and minimum of finite families of function, and composition with smooth functions, since the class of continuously codifferentiable functions is closed under all these operations [10] [12] [14]. Furthermore, any DC function has an o.s.c. quasidifferential mapping. Indeed,
if \( f = f_1 - f_2 \), where \( f_1 \) and \( f_2 \) are finite closed convex functions, then one can define \( \partial f(z) = [\partial f_1(z), -\partial f_2(z)] \), where \( \partial f_i(z) \) is the subdifferential of \( f_i \) in the sense of convex analysis. Note that this quasidifferential is correctly defined and o.s.c. due to the fact that the subdifferential of a finite closed convex function defined on a Banach space is nonempty at every point (see, e.g., [16, Proposition I.5.2. and Corollary I.2.5]) and outer semicontinuous.

Let us also recall a certain extension of the definition of quasidifferentiability to the case of vector-valued functions that was utilised in [19, 35].

**Definition 2.** Let \( Z \) be a real Banach space, and \( U \subseteq X \) be an open set. A function \( F: U \to Z \) is called scalarly quasidifferentiable at a point \( x \in U \), if \( F \) is Dini directionally differentiable at \( x \), i.e. for any \( v \in X \) there exists the limit

\[
F'(x, v) = \lim_{\alpha \to +0} \frac{1}{\alpha} (F(x + \alpha v) - F(x)),
\]

and for any \( z^* \in Z^* \) the function \( \langle z^*, F'(x, \cdot) \rangle \) can be represented as the difference of sublinear functions, i.e. there exists a pair convex weak* compact sets \( \overline{\partial} F(x; z^*), \overline{\partial} F(x; z^*) \subseteq X^* \) such that

\[
\langle z^*, F'(x, v) \rangle = \max_{x^* \in \overline{\partial} F(x; z^*)} \langle x^*, v \rangle + \min_{z^* \in \overline{\partial} F(x; z^*)} \langle y^*, v \rangle \quad \forall v \in X.
\]

For any \( z^* \in Z^* \) the pair \( \partial F(x; z^*) = [\overline{\partial} F(x; z^*), \overline{\partial} F(x; z^*)] \) is called a scalar quasidifferential of \( F \) at \( x \) (corresponding to \( z^* \)).

**Remark 2.** Below, as usual, we use the term “quasidifferential”, instead of “Dini quasidifferentiable”. Also, when we say that a function \( f \) is quasidifferentiable at a point \( x \), we suppose that a quasidifferential of \( f \) at \( x \) is given. Alternatively, one can define a quasidifferential as an equivalence class, and work with these equivalence classes instead; however, in author’s opinion, this approach leads to somewhat cumbersome formulations of the main results. That is why we do not adopt it in this article.

### 3 Metric Regularity of Quasidifferentiable Maps

In this section we obtain several sufficient conditions for the metric regularity of multifunctions in terms of quasidifferentials, and introduce a so-called limiting quasidifferential sum, which in some cases allows one to obtain stronger conditions for metric regularity than with the use of quasidifferentials.

#### 3.1 General Conditions for Metric Regularity

Let \( Y \) be a complete metric space, and \( F: X \rightrightarrows Y \) be a given set-valued mapping with closed values, whose graph is denoted by Graph \( F \). For any \( y \in Y \), \( r > 0 \) and any set \( C \subseteq Y \) denote \( B(y, r) = \{ z \in Y \mid d(y, z) \leq r \} \) and \( d(y, C) = \inf_{z \in C} d(y, z) \). As usual, we put \( d(y, \emptyset) = +\infty \).

Recall that \( F \) is called metrically regular near a point \( (\overline{\tau}, \overline{y}) \in \text{Graph} \ F \), if there exist \( K > 0 \) and \( r > 0 \) such that

\[
d(x, F^{-1}(y)) \leq K d(y, F(x)) \quad \forall (x, y) \in B(\overline{\tau}, r) \times B(\overline{y}, r).
\]
The greatest lower bound of all $K$ for which the inequality above is satisfied with some $r > 0$ is called the norm of metric regularity of $F$ near $(\overline{\tau}, \overline{\eta})$. For the general theory of metric regularity see [22, 23, 2].

At first, our aim is to obtain sufficient conditions for the metric regularity of the set-valued mapping $F$ in the case when the distance function $x \to d(y, F(x))$ is quasidifferentiable for any $(x, y)$ in a neighbourhood of $(\overline{\tau}, \overline{\eta})$. For any $y \in Y$ and $x \in X$ denote $\psi_y(x) = d(y, F(x))$, and define

$$|\nabla \psi_y|(x) = \limsup_{u \to x, \psi_y(u) \to \psi_y(x)} \frac{\max\{\psi_y(x) - \psi_y(u), 0\}}{\|x - u\|}.$$

Recall that $|\nabla \psi_y|(x)$ is called the strong slope of $\psi_y$ at $x$.

**Theorem 1.** Let for any $y \in Y$ the function $\psi_y(\cdot)$ be l.s.c., and let $(\overline{\tau}, \overline{\eta}) \in \text{Graph} \ F$ and $K > 0$ be given. Suppose that there exists $r > 0$ such that for any $(x, y) \in B(\overline{\tau}, r) \times B(\overline{\eta}, r)$ with $y \notin F(x)$ the function $\psi_y(\cdot)$ is quasidifferentiable at $x$, and there exists $y^* \in \partial \psi_y(x)$ for which

$$d(0, \partial \psi_y(x) + \{y^*\}) > \frac{1}{K}. \tag{1}$$

Then for any $(x, y) \in B(\overline{\tau}, r) \times B(\overline{\eta}, r)$ such that $Kd(y, F(x)) < r - d(x, \overline{\tau})$ one has $d(x, F^{-1}(y)) \leq Kd(y, F(x))$, which, in particular, implies that the set-valued mapping $F$ is metrically regular near $(\overline{\tau}, \overline{\eta})$ with the norm of metric regularity not exceeding $K$.

Moreover, suppose that $Y$ is a Banach space, $X$ is finite dimensional, and for any $y \in Y$ the functions $\psi_y(\cdot)$ are Hadamard quasidifferentiable on $B(\overline{\tau}, r)$ with some $r > 0$. Then for the metric regularity of $F$ near $(\overline{\tau}, \overline{\eta})$ with the norm of metric regularity not exceeding $K$ it is necessary and sufficient that for any $t > K$ there exists a neighbourhood $U$ of $(\overline{\tau}, \overline{\eta})$ such that for any $(x, y) \in U \setminus \text{Graph} \ F$ there exists $y^* \in \partial \psi_y(x)$ for which $d(0, \partial \psi_y(x) + \{y^*\}) \geq t^{-1}$.

**Proof.** Let us show that under the assumptions of the theorem one has

$$|\nabla \psi_y|(x) > K^{-1} \quad \forall (x, y) \in \left( B(\overline{\tau}, r) \times B(\overline{\eta}, r) \right) \setminus \text{Graph} \ F.$$

Then applying [22, Theorem 2b] one obtains the desired result.

Indeed, fix $(x, y) \in B(\overline{\tau}, r) \times B(\overline{\eta}, r)$ with $y \notin F(x)$. From (1) it follows that the convex compact subsets $B(0, K^{-1})$ and $\partial \psi_y(x) + \{y^*\}$ of the space $X^*$ endowed with the weak* topology are disjoint. Applying the separation theorem one obtains that there exists $v \in X$ with $\|v\| = 1$ such that

$$\langle x^*, v \rangle \leq \langle z^*, v \rangle \quad \forall x^* \in \partial \psi_y(x) + \{y^*\} \quad \forall z^* \in B(0, K^{-1})$$

or equivalently $\langle x^*, v \rangle \leq -K^{-1}$ for any $x^* \in \partial \psi_y(x) + \{y^*\}$. Hence, as it is easy to see, one has $\psi_y(x, v) \leq -K^{-1}$. Therefore there exists a sequence $\{\alpha_n\} \subset (0, +\infty)$ such that

$$\lim_{n \to \infty} \frac{\psi_y(x + \alpha_n v) - \psi_y(x)}{\alpha_n} \leq -\frac{1}{K}.$$ 

Consequently, $\psi_y(x) - \psi_y(x + \alpha_n v) > 0$ for any sufficiently large $n \in \mathbb{N}$, and

$$\limsup_{n \to \infty} \frac{\max\{\psi_y(x) - \psi_y(x + \alpha_n v), 0\}}{\alpha_n} \geq \frac{1}{K},$$


which yields $|\nabla\psi_y|(x) \geq K^{-1}$, since $\|v\| = 1$.

Let us now prove the second part of the theorem. Indeed, by [22, Theorem 2b] the multifunction $F$ is metrically regular near $(\overline{x}, \overline{y})$ with the norm of metric regularity not exceeding $K$ iff for any $t > K$ there exist a neighbourhood $U$ of $(\overline{x}, \overline{y})$ such that $|\nabla\psi_y|(x) \geq t^{-1}$ for any $(x, y) \in U \setminus G_F$.

Taking into account the facts that $X$ is finite dimensional, and the functions $x \rightarrow \psi_y(x)$ are Hadamard quasidifferentiable, and applying [2, Proposition 2.8] one obtains that $|\nabla\psi_y|(x) = -\min_{y \in \mathcal{B}} \psi_y(x, v)$. Hence with the use of the explicit expression for the rate of steepest descent of a quasidifferentiable function (see [10, section V.3.1]) one gets

$$|\nabla\psi_y|(x) = \max_{y^* \in \partial\psi_y(x)} \min_{x^* \in \overline{\partial}_\psi_y(x) + \{y^*\}} \|x^*\|.$$  

Consequently, $|\nabla\psi_y|(x) \geq t^{-1}$ iff $d(0, \partial\psi_y(x) + \{y^*\}) \geq t^{-1}$ for some $y^* \in \overline{\partial}_\psi_y(x)$, which implies the required result.

Remark 3. (i) It is easy to check that condition (1) is satisfied for some $y^* \in \overline{\partial}_\psi_y(x)$ iff there exists $v \in X$ with $\|v\| = 1$ such that $\psi_y(x, v) < -K^{-1}$. Therefore this condition is invariant with respect to the choice of quasidifferentials of the functions $\psi_y$.

(ii) Sufficient conditions for the metric regularity of a continuous single-valued mapping $F$ between Banach spaces in terms of quasidifferentials of the functions $\psi_y(x) = ||y - F(x)||$ were first obtained by Uderzo [34] (see also [35]). However, the conditions in [34] are more restrictive then the ones stated in the theorem above. Indeed, by [34, Theorem 4.3] for the metric regularity of $F$ near a point $(\overline{x}, F(\overline{x}))$ it is sufficient that there exist $m > 0$ and $r > 0$ such that for any $x \in B(\overline{x}, r)$ and $y \in B(F(\overline{x}), r)$ with $y \neq F(x)$ one has

$$d(0, \partial\psi_y(x) + y^*) > m \quad \forall y^* \in \overline{\partial}_\psi_y(x). \quad (2)$$

It is easy to see that this condition fails to hold true even for the very simple function $F(x_1, x_2) = |x_1| - |x_2|$, when $\overline{x} = 0$ and $\overline{y} = 0$ (here $X = \mathbb{R}^2$ and $Y = \mathbb{R}$). Indeed, for $x = 0$ and any $y > 0$ a quasidifferential of the function $\psi_y(x) = |y - F(x)|$ has the form

$$\partial\psi_y(0) = \text{co} \left\{ \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ -1 \end{array} \right) \right\}, \quad \overline{\partial}_\psi_y(0) = \text{co} \left\{ \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} -1 \\ 0 \end{array} \right) \right\},$$

and for $y^* = (0, 0) \in \overline{\partial}_\psi_y(0)$ one has $0 \in \partial\psi_y(0) + y^*$. Thus, condition (2) is not satisfied. On the other hand, one can easily check that sufficient conditions from [22, Theorem 4] are satisfied, and the function $F(x) = |x_1| - |x_2|$ is metrically regular near the point $(0, 0)$. Note also that condition (2), unlike condition (1), depends on the choice of quasidifferential. For instance, it is not valid for the identity function $F(x) = x$ (here $X = Y = \mathbb{R}$), if one chooses the pair

$$\partial\psi_y(x) = -\text{sign}(y - x) + [-1, 1], \quad \overline{\partial}_\psi_y(x) = [-1, 1],$$

as a quasidifferential of the function $\psi_y(x) = |y - F(x)| = |y - x|$ at every point $x$ such that $y \neq x$. 

6
3.2 Parametric Systems of Equalities and Inequalities

Note that in order to verify the metric regularity of a multifunction with the use of the theorem above, one must check that condition (1) holds true at all points in a neighbourhood of a given point \((\overline{\tau}, \overline{\gamma})\), which is a common drawback of general results on metric regularity (cf. [22]). However, as in the case of sufficient conditions in terms of various subdifferentials and coderivatives, in some particular cases one can obtain sufficient conditions for the metric regularity that involve only quasidifferentials of certain functions at the point \((\overline{\tau}, \overline{\gamma})\) itself.

Our next goal is to obtain such conditions for a set-valued mapping associated with a parametric system of nonlinear equality and inequality constraints.

Let \(Y\) be a real Banach space, \(P\) be a metric space of parameters, while \(F: X \times P \to Y\) and \(g_i: X \times P \to \mathbb{R}, i \in I = \{1, \ldots, m\}\), be given functions. For any \(y \in Y\) and \(z_i \in \mathbb{R}, i \in I\), consider the following parametric system

\[
F(x, p) = y, \quad g_i(x, p) \leq z_i, \quad i \in I. \tag{3}
\]

Denote by \(S(p, y, z) = \{x \in X \mid F(x, p) = y, g_i(x, p) \leq z_i, i \in I\}\) the solution set of this system, where \(z = (z_1, \ldots, z_m) \in \mathbb{R}^m\). We also denote \(S(p) = S(p, 0, 0)\), and sometimes use the notation \(F_y(x) = F(x, p)\).

Let us introduce a constraint qualification that ensures the metric regularity of a multifunction associated with system (3). For the sake of shortness we consider the case \(y = 0\) and \(z = 0\) only; since the general case can be easily reduced to this one by replacing \(F(x, p)\) with \(F(x, p) - y\), and \(g_i(x, p)\) with \(g_i(x, p) - z_i\). Suppose that the functions \(g_i(\cdot, \overline{\gamma}), i \in I\), are quasidifferentiable at a point \(\overline{x}\) such that \(\overline{x} \in S(\overline{\gamma})\), and the mapping \(F(\cdot, \overline{\gamma})\) is scalarly quasidifferentiable at this point, and denote their quasidifferentials at this point by \(\partial F_y(\overline{x}, \overline{\gamma})\) and \(\partial F_y(\overline{x}, \overline{\gamma})\), \(y^* \in Y^*\), respectively. Introduce the sets

\[
\partial F_y(\overline{x}, \overline{\gamma}) = \partial F_y(\overline{x}, \overline{\gamma}) + \partial F_y(\overline{x}, \overline{\gamma}),
\]

\[
\partial F_y(\overline{x}, \overline{\gamma}) = \partial F_y(\overline{x}, \overline{\gamma}) + \partial F_y(\overline{x}, \overline{\gamma}).
\]

It should be noted that these sets are sometimes called \textit{quasidifferential sums} and were considered e.g. in [25], and they are not invariant with respect to the choice of the corresponding quasidifferentials. For any \(x \in X\) and \(p \in P\) define \(I(x, p) = \{i \in I \mid g_i(x, p) = 0\}\), and denote \(S_X = \{x \in X \mid \|x\| = 1\}\).

**Definition 3.** One says that the Mangasarian-Fromovitz constraint qualification \(\text{in terms of quasidifferentials (q.d.-MFCQ)}\) holds at \((\overline{\tau}, \overline{\gamma})\), if

\[
\inf_{y^* \in S_X} \inf \{\|x^*\| : x^* \in [\partial F_y(\overline{x}, \overline{\gamma})]^+\} > 0, \tag{4}
\]

and there exists \(v \in X \setminus \{0\}\) such that \(\langle x^*, v \rangle = 0\) for all \(x^* \in [\partial F_y(\overline{x}, \overline{\gamma})]^+\) and \(y^* \in Y^*\), while \(\langle x^*, v \rangle < 0\) for all \(x^* \in [\partial F_y(\overline{x}, \overline{\gamma})]^+\) and \(i \in I(\overline{\tau}, \overline{\gamma})\).

Let \(A_1, \ldots, A_s\) be nonempty subsets of a linear space \(E\). Recall that these sets are said to be \textit{linearly independent} (or to have \textit{full rank}), if the inclusion \(0 \in \lambda_1 A_1 + \ldots + \lambda_s A_s\) with \(\lambda_i \in \mathbb{R}\) is valid only for \(\lambda_i = 0, i \in \{1, \ldots, s\}\). Clearly, the sets \(A_i, i \in \{1, \ldots, s\}\) are linearly independent if for all \(x_i \in A_i, i \in \{1, \ldots, s\}\) the vectors \(x_1, \ldots, x_s\) are linearly independent. The following proposition explains how q.d.-MFCQ is connected with the standard MFCQ.

\[7\]
Proposition 1. Let $Y$ be the space $\mathbb{R}^l$ equipped with Euclidean norm $|\cdot|$, and $F(\cdot) = (f_1(\cdot), \ldots, f_l(\cdot))$, where the functions $f_j : X \times P \to \mathbb{R}$ are quasidifferentiable in $x$ at $(x,\bar{p})$. Then the mapping $F(\cdot,\bar{p})$ is scalarly quasidifferentiable at $\bar{x}$, and q.d.-MFCQ holds at $(\bar{x},\bar{p})$ iff the sets $[\partial_x f_j(\bar{x},\bar{p})]_+^+$, $1 \leq j \leq l$, are linearly independent, and there exists $\bar{v} \in X \setminus \{0\}$ such that $\langle \bar{x}^*, \bar{v} \rangle = 0$ for all $x^* \in [\partial_x f_j(\bar{x},\bar{p})]_+^+$ and $1 \leq j \leq l$, while $\langle \bar{x}^*, \bar{v} \rangle < 0$ for all $x^* \in [\partial_x g_i(\bar{x},\bar{p})]_+^+$ and $i \in I(\bar{x},\bar{p})$.

Proof. From the fact that the functions $f_i(\cdot,\bar{p})$ are quasidifferentiable at $\bar{x}$ it follows that the mapping $F(\cdot,\bar{p})$ is directionally differentiable at this point, and

$$F'(\cdot,\bar{p})(\bar{x},v) = \left( f'_1(\cdot,\bar{p})(\bar{x},v), \ldots, f'_l(\cdot,\bar{p})(\bar{x},v) \right)$$

for any $v \in X$. Therefore, for any $z^* = (z_1, \ldots, z_l) \in \mathbb{R}^l$ one has

$$\langle z^*, F'(\cdot,\bar{p})(\bar{x},v) \rangle = \sum_{j=1}^l z_j \left( \max_{x^* \in \partial_x f_j(\bar{x},\bar{p})} \langle x^*, v \rangle + \min_{y^* \in \partial_y f_j(\bar{x},\bar{p})} \langle y^*, v \rangle \right),$$

which implies that $F'(\cdot,\bar{p})$ is scalarly quasidifferentiable at $\bar{x}$, and for any $z^*$ one can define

$$\partial F(\bar{x}, z^*) = \sum_{j=1}^l \left( \partial_x f_j(\bar{x},\bar{p}) - [z_j]_+ \partial_x f_j(\bar{x},\bar{p}) \right),$$

$$\partial F(\bar{x}, z^*) = \sum_{j=1}^l \left( \partial_x f_j(\bar{x},\bar{p}) - [z_j]_+ \partial_x f_j(\bar{x},\bar{p}) \right),$$

where $[t]_+ = \max\{t,0\}$ for any $t \in \mathbb{R}$. Hence for any $z^*$ one has

$$[\partial F(\bar{x}, z^*)]_+^+ = \sum_{j=1}^l z_j [\partial_x f_j(\bar{x},\bar{p})]_+^+.$$  \hfill (5)

Consequently, if (3) holds true, then the sets $[\partial_x f_j(\bar{x},\bar{p})]_+^+$, $1 \leq j \leq l$, are linearly independent, since otherwise $0 \in [\partial F(\bar{x}, z^*)]$ for $z^* = \lambda/|\lambda|$, where $\lambda \in \mathbb{R}^l \setminus \{0\}$ is such that $0 \in \lambda_1 [\partial_x f_1(\bar{x},\bar{p})]_+^+ + \ldots + \lambda_l [\partial_x f_l(\bar{x},\bar{p})]_+^+$, which is impossible. Conversely, if the sets $[\partial_x f_j(\bar{x},\bar{p})]_+^+$, $1 \leq j \leq l$, are linearly independent, then $0 \notin [\partial F(\bar{x}, z^*)]_+^+$ for any $z^* \neq 0$. Applying the separation theorem and the fact that the set $[\partial F(\bar{x}, z^*)]_+^+$ is weak$^*$ compact one obtains that there exists $v \in X$ and $\delta > 0$ such that $\langle \bar{x}^*, v \rangle \geq \delta$ for all $x^* \in [\partial F(\bar{x}, z^*)]_+^+$. Therefore, $\inf\{|\bar{x}^*| \mid x^* \in [\partial F(\bar{x}, z^*)]_+^+\} > 0$ for any $z^* \neq 0$. Hence taking into account the facts that this infimum is obviously continuous with respect to $z^*$ (see (5)), and the unit sphere in $\mathbb{R}^l$ is compact one gets that (4) holds true. It remains to note that the equivalence between the second conditions from q.d.-MFCQ and the proposition (the existence of $v$) follows from (5).

Remark 4. With the use of the separation theorem one can easily check that under the assumptions of the proposition above the vector $v$ from q.d.-MFCQ exists iff

$$\text{co} \left\{ [\partial_x g_i(\bar{x},\bar{p})]_+^+ \mid i \in I(\bar{x},\bar{p}) \right\} \cap \text{cl span} \left\{ [\partial_x f_j(\bar{x},\bar{p})]_+^+ \mid 1 \leq j \leq l \right\} = \emptyset, \hfill (6)$$
where the closure is taken in the weak$^\ast$ topology. Furthermore, if this span is weak$^\ast$ closed (in particular, if it is finite dimensional), then (6) is equivalent to the following condition: for any $x_i^* \in [\mathcal{D}_i g_i(x,\overline{p})]^\ast$, $i \in I(x,\overline{p})$, and $y_j^* \in [\mathcal{D}_j f_j(x,\overline{p})]^\ast$, $1 \leq j \leq l$, there exists $v \in X$ such that
\[
\langle x_i^*, v \rangle < 0 \quad \forall i \in I(x,\overline{p}), \quad \langle y_j^*, v \rangle = 0 \quad \forall j \in \{1, \ldots, l\}.
\]
The implication (4) $\Rightarrow$ (7) follows from the separation theorem, while the opposite implication follows from the fact that if the intersection in (6) is not empty, then it is impossible to find $v$ satisfying (7) for those $x_i^*$ and $y_j^*$ that correspond to a vector from the intersection. Note that condition (7) is a “pointwise” version of the second condition from q.d.-MFCQ. Let us finally point out that in the case when $l = 1$ the “linear independence condition” from q.d.-MFCQ is reduced to $0 \notin [\mathcal{D}_1 f_1(x,\overline{p})]^\ast$.

Likewise the standard Mangasarian-Fromowitz constraint qualification (see, e.g., [33]), q.d.-MFCQ can be used to obtain sufficient conditions for metric regularity. For the sake of simplicity we consider only the case when the functions $F$ and $g_i$ are continuous on $X \times P$, although the theorem below holds true under weaker assumptions. Note also that in the theorem below, unlike in the main results of [35], we do not assume that there exists a Fréchet smooth renorming of $Y$.

**Theorem 2.** Suppose that the functions $F$ and $g_i$, $i \in I$, are continuous. Let also a point $(x,\overline{p}) \in X \times P$ be such that $\overline{x} \in S(\overline{p})$, and there exist a neighbourhood $U$ of $(x,\overline{p})$ such that

1. for any $(x,p) \in U$ the mapping $F(\cdot,p)$ is scalarly quasidifferentiable at $x$, and the functions $g_i(\cdot,p)$, $i \in I(x,\overline{p})$ are quasidifferentiable at $x$,

2. the multifunctions $\mathcal{D}_i g_i(\cdot)$, $i \in I(x,\overline{p})$ are o.s.c. at $(x,\overline{p})$, while the multifunction $(x,p) \mapsto \mathcal{D}_p F_p(x,y^*)$ is o.s.c. at $(x,\overline{p})$ uniformly with respect to $y^* \in S_{Y^*}$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{D}_p F_p(x,y^*) \subseteq \mathcal{D}_p F_p(x,y^*) + B(0,\varepsilon)$ for all $y^* \in S_{Y^*}$ and $(x,p) \in B(x,\delta) \times B(\overline{p},\delta)$;

3. the set $D(y) = \{(\mathcal{D}_p F_p(x,y^*), y^*) \in S_{Y^*} \times (y^*, y) : \|y^*\| = \|y\|\}$ is weak$^\ast$ closed for any $y \in S_{Y^*}$.

Suppose, finally, that q.d.-MFCQ holds at $(x,\overline{p})$. Then there exist $K > 0$, a neighbourhood $V$ of $(x,\overline{p})$, and a neighbourhood $W$ of zero in $Y \times \mathbb{R}^l$ such that:

\[
d(x, S(p,y,z)) \leq K \left(\|F(x,p) - y\| + \sum_{i=1}^m \max\{g_i(x,p) - z_i, 0\}\right)
\]

for all $(x,p) \in V$ and $(y,z) \in W$, i.e. the multifunction $\Phi_p(x) = \{F(x,p)\} \times \prod_{i=1}^m \{g_i(x,p) + \infty\}$ is metrically regular near $(x,0,0)$ with the norm of metric regularity not exceeding $K$ for all $p$ in a neighbourhood of $\overline{p}$.

**Proof.** Let $p > 0$ be such that $B(x,\overline{p}) \times B(\overline{p},\overline{r}) \subseteq U$. Our aim is to check that there exist $r \in (0,\overline{r})$ and $a > 0$ such that for any $p \in B(\overline{p},\overline{r})$ one has $|\nabla|\psi(x,y,z)| > a$ for all $(y,z) \in B((0,0),r)$ and $x \in B(x,\overline{p})$ such that $(y,z) \notin \Phi_p(x)$, where $\psi(x,y,z) = d(y,z,\Phi_p(x))$, and the space $Y \times \mathbb{R}^l$ is equipped with the norm $\|(y,z)\| = \|y\| + \sum_{i=1}^l |z_i|$. Then applying [22] Theorem 2b one obtains
that \(d(x, \Phi^{-1}(y, z)) \leq a^{-1}d((y, z), \Phi_p(x))\) for all \(p \in B(\overline{p}, r)\), \((y, z) \in B((0, 0), r)\) and \(x \in B(\overline{p}, r)\) such that \(a^{-1}d((y, z), \Phi_p(x)) < r - \|x - \overline{x}\|\), i.e. \(\text{K} = a^{-1}\) holds true for all such \(x, p, y,\) and \(z\). Hence with the use of the continuity of the functions \(F\) and \(g\), one can find \(\delta < r\) such that \(\text{K}\) with \(K = a^{-1}\) holds true for all \(p \in B(\overline{p}, \delta)\), \((y, z) \in B((0, 0), \delta)\) and \(x \in B(\overline{p}, \delta)\), and the proof is complete.

Let us compute the directional derivative of the mapping \(||F(\cdot, p) - y||\). Denote \(\omega(y) = \|y\|\). Recall that \(\partial \omega(y) = \{y^* \in S_{Y^*} \mid \|y\| = \langle y^*, y \rangle\}\) for any \(y \neq 0\), where \(\partial \omega(y)\) is the subdifferential of \(\omega\) at \(y\) in the sense of convex analysis. Fix \((x, p) \in U\) and \(y \in Y\). From the definition of scalar quasidifferentiability it follows that for any \(h \in X\) one has

\[
F_p(x + \alpha h) - F_p(x) = \alpha F'_p(x, h) + o(\alpha) \quad \forall \alpha \geq 0,
\]

where \(\|o(\alpha)\|/\alpha \to 0\) as \(\alpha \to +0\). Hence

\[
\left| \|F_p(x + \alpha h) - y\| - \|F_p(x) - y\| - \alpha \omega(F_p(x) - y, F'_p(x, h)) \right| \\
\leq \left| \|F_p(x) - y + \alpha F'_p(x, h) - \alpha F'_p(x, h)\| - \|F_p(x) - y\| - \alpha \omega(F_p(x) - y, F'_p(x, h)) \right| \\
\leq \|F_p(x) - y + \alpha F'_p(x, h)\| - \|F_p(x) - y\| - \alpha \omega(F_p(x) - y, F'_p(x, h)) + \|o(\alpha)\|.
\]

Dividing this inequality by \(\alpha\) and passing to the limit as \(\alpha \to +0\) one gets that the function \(||F_p(\cdot) - y||\) is directionally differentiable at \(x\), and for any \(h \in X\) and \(y \in Y\) one has

\[
\left| \|F_p(\cdot) - y\|'(x, h) - \omega'(F_p(x) - y, F'_p(x, h)) \right| = \sup_{y^* \in \partial \omega(F_p(x) - y)} \left( \max_{x^*_1 \in \partial F_p(x, y^*)} \langle x^*_1, h \rangle + \min_{x^*_2 \in \partial F_p(x, y^*)} \langle x^*_2, h \rangle \right) \\
\leq \sup_{y^* \in \partial \omega(F_p(x) - y)} \max_{x^* \in [\partial F_p(x, y^*)]^+} \langle x^*, h \rangle, \tag{9}
\]

if \(F(x, p) \neq y\), while

\[
\left| \|F_p(\cdot) - y\|'(x, h) \right| \leq \sup_{y^* \in S_{Y^*}} \max_{x^* \in [\partial F_p(x, y^*)]^+} \langle x^*, h \rangle, \tag{10}
\]

in the case when \(F(x, p) = y\), since \(\|y\| = \sup_{y^* \in S_{Y^*}} \langle y^*, y \rangle\).

Now, we can utilise q.d.-MFCQ, and the outer semicontinuity of the quasidifferential mappings to prove the inequality \(\|\nu\|_{(y, z, p)}(x) > 0\). Denote by \(\nu\) the infimum in \(\|\nu\|_{(y, z, p)}(x) > 0\). From assumption\(\text{K}\) the fact that the set \(D(y)\) is convex, and the separation theorem it follows that for any \(y \in S_Y\) there exists \(h_y\) with \(\|h_y\| = 1\) such that \(\langle x^*, h_y \rangle \leq -\infty\) for all \(x^* \in D(y)\). With the use of the second condition in q.d.-MFCQ one obtains that \(\langle x^*, h_y + tv \rangle \leq -\infty\) for all \(x^* \in D(y)\) and \(t \geq 0\). Hence applying the fact that the quasidifferential mapping \((x, p) \mapsto \partial F_p(x; y^*)\) is o.s.c. at \((\overline{p}, \overline{p})\) uniformly with respect to \(y^* \in S_{Y^*}\), one gets that for any \(t \geq 0\) there exists \(r_1(t) \in (0, \overline{r})\) such that for any \(y \in S_Y\) one has

\[
\langle x^*, h_y + tv \rangle \leq -\frac{\overline{r}}{2} \quad \forall x^* \in [\partial F_p(x; y^*)]^+ \quad \forall y^* \in \partial \| \cdot \| \tag{11}
\]

for all \((x, p) \in B(\overline{p}, r_1(t)) \times B(\overline{p}, r_1(t))\). Furthermore, from the second condition in q.d.-MFCQ, and assumption\(\text{K}\) it follows that for any \(t \geq 0\) there exists \(r_2(t)\) such that

\[
\langle x^*, tv \rangle \leq \frac{\overline{r}}{4} \quad \forall x^* \in [\partial F_p(x; y^*)]^+ \quad \forall y^* \in S_{Y^*}. \tag{12}
\]
for all \((x, p) \in B(\mathcal{T}, r_3(t)) \times B(\mathcal{P}, r_2(t))\).

Applying the second condition in q.d.-MFCQ, and the facts that \(\|h_y\| = 1\) for any \(y \in S_Y\), and the sets \([\mathcal{P}_y g_i(\mathcal{T}, \mathcal{P})]^+\) are obviously weak* compact (and thus bounded) one can find \(t_0 > 0\) such that \(\langle x^*, h_y + t_0v \rangle \leq -\kappa\) for all \(x^* \in [\mathcal{P}_y g_i(\mathcal{T}, \mathcal{P})]^+, i \in I(\mathcal{T}, \mathcal{P}),\) and \(y \in S_Y\). Hence with the use of the outer semicontinuity of the mappings \(\mathcal{P}_y g_i(\cdot)\) at \((\mathcal{T}, \mathcal{P})\) one obtains that there exists \(r_3 > 0\) such that

\[
\langle x^*, h_y + t_0v \rangle \leq -\frac{\kappa}{2} \quad \forall x^* \in [\mathcal{P}_y g_i(x, p)]^+ \forall i \in I(\mathcal{T}, \mathcal{P}) \forall y \in S_Y.
\]

(13)

for all \((x, p) \in B(\mathcal{T}, r_3) \times B(\mathcal{P}, r_3)\). Finally, since \(g_i\) are continuous, there exists \(r_4 > 0\) and \(\varepsilon > 0\) such that \(g_i(x, p) < -\varepsilon\) for any \((x, p) \in B(\mathcal{T}, r_4) \times B(\mathcal{P}, r_4)\) and \(i \notin I(\mathcal{T}, \mathcal{P})\).

Define \(r = \min\{r_1(t_0), r_2(t_0), r_3, r_4/2, \varepsilon\}\), and fix any \((x, p) \in B(\mathcal{T}, r) \times B(\mathcal{P}, r)\) and \((y, z) \in B((0,0), r)\) such that \((y, z) \notin \Phi_p(x)\). Note that \(g_i(\cdot) - z_i < 0\) in a neighbourhood of \((x, p)\) for any \(i \notin I(\mathcal{T}, \mathcal{P})\), since \(r < \min\{r_4, \varepsilon\}\). Hence

\[
d((y, z), \Phi(\cdot)) = \|F(\cdot) - y\| + \sum_{i \in I(\mathcal{T}, \mathcal{P})} \max\{g_i(\cdot) - z_i, 0\}
\]

in a neighbourhood of \((x, p)\), i.e. the indices \(i \notin I(\mathcal{T}, \mathcal{P})\) can be discarded from consideration. Observe also that

\[
\max\{g_i(\cdot, p) - z_i, 0\}'(x, h) = \begin{cases} [g_i(\cdot, p)]'(x, h), & \text{if } g_i(x, p) > z_i, \\
\max\{[g_i(\cdot, p)]'(x, h), 0\}, & \text{if } g_i(x, p) = z_i, \\
0, & \text{otherwise},
\end{cases}
\]

and \([g_i(\cdot, p)]'(x, h) \leq \max_{x^* \in [\mathcal{P}_y g_i(\cdot, p)]^+} (x^*, h)\).

If \(F(x, p) \neq y\), then with the use of (9), (11), (13), and (14) one obtains that

\[
\psi'(y, z, p)(x, h) = \|F(\cdot, p) - y\|'(x, h) + \sum_{i \in I(\mathcal{T}, \mathcal{P})} \max\{g_i(\cdot, p) - z_i, 0\}'(x, h) \leq -\frac{\kappa}{2}
\]

where \(h = h_w + t_0v\) and \(w = (F(x, p) - y)/\|F(x, p) - y\|\) (note that \(\|h\| \leq 1 + t_0\|v\|\), since \(\|h_w\| = 1\)). On the other hand, if \(F(x, p) = y\), then there exists \(k \in I(\mathcal{T}, \mathcal{P})\) such that \(g_k(x, p) > z_k\). Consequently, applying (10), (12), (13), and (14) one gets that

\[
\psi'(y, z, p)(x, h) = \|F(\cdot, p) - y\|'(x, h) + \max\{g_i(\cdot, p) - z_k, 0\}'(x, h) + \sum_{i \in I(\mathcal{T}, \mathcal{P}) \setminus \{k\}} \max\{g_i(\cdot, p) - z_i, 0\}'(x, h) \leq \frac{\kappa}{4} + \frac{\kappa}{4} = -\frac{\kappa}{4},
\]

where \(h = t_0v\). Therefore, for any \((x, p) \in B(\mathcal{T}, r) \times B(\mathcal{P}, r)\) and \((y, z) \in B((0,0), r)\) such that \((y, z) \notin \Phi_p(x)\) one has

\[
|\nabla| \psi'(y, z, p)(x) \geq -\psi'(y, z, p) \left(x, \frac{h}{\|h\|}\right) \geq \frac{\kappa}{4(1 + t_0\|v\|)}
\]

and the proof is complete.

\[\square\]
Remark 5. Let \( F \) be as in Proposition 1 and \( X = \mathbb{R}^n \). In this case one can reformulate the sufficient conditions for the metric regularity of the mapping \( F \) from the theorem above in a different way. Namely, let the set \( \partial_+ F(\tau, p) \) consists of all \( l \times n \) matrices whose \( j \)th row is a vector from \( \partial_{x_j} f_j(\tau, p) \). The set \( \overline{\partial}_+ F(\tau, p) \) is defined in a similar way. Then the pair \( \partial_+ F(\tau, p) = [\partial_+ F(\tau, p), \overline{\partial}_+ F(\tau, p)] \) is, in fact, a quasidifferential of the mapping \( F(\cdot, p) \) at \( \tau \) (see [13 Appendix III]). From Theorem 2 it follows that for the mapping \( F(\cdot, p) \) to be metrically regular near \( (\tau, F(\tau, p)) \) with the norm of metric regularity \( K > 0 \) for all \( p \) in a neighbourhood of \( \overline{p} \) it is sufficient that \( l \leq n \), and all matrices from the set \([\partial_+ F(\tau, p)]^+ = [\partial_+ F(\tau, p), \overline{\partial}_+ F(\tau, p)]^+ \) have full rank. Note that a similar condition on the set \([\partial_+ F(\tau, p)]^+ \) was introduced by Demyanov in [7] for the analysis of nonsmooth implicit functions and a nonsmooth Newton method for codifferentiable vector-valued functions.

Remark 6. If a function \( f : X \to \mathbb{R} \) is quasidifferentiable at a point \( x \), then \( \min_{v \in [\partial f(x)]^+} \langle x^*, v \rangle \leq f'(x, v) \leq \max_{v \in [\partial f(x)]^+} \langle x^*, v \rangle \) for any \( v \in X \), i.e. the quasidifferential sum \([\partial f(x)]^+ \) is a convexificator of \( f \) at \( x \) (see [9, 23, 38]). With the use of the separation theorem and the inequalities above one can easily check that if \( f \) is Gâteaux differentiable at \( x \), then \( f'(x) \in [\partial f(x)]^+ \) regardless of the choice of quasidifferential. Consequently, if \( X = \mathbb{R}^n \), \( f \) is Lipschitz quasidifferentiable near \( x \), and a quasidifferential mapping \( \partial f \) is o.s.c. at \( x \), then \( \partial_{C_{\mathbb{R}^1}} f(x) \subseteq [\partial f(x)]^+ \), where \( \partial_{C_{\mathbb{R}^1}} f(x) \) is the Clarke subdifferential of \( f \) at \( x \) [5]. By [14] Corollary 2 under the assumptions of Theorem 2 the functions \( F(\cdot, p) \) and \( g_i(\cdot, p) \) are Lipschitz continuous near \( \tau \) with the same Lipschitz constant for all \( p \) in a neighbourhood of \( \overline{p} \), provided \( F \) has the same form as in Proposition [11]. Therefore, if \( X = \mathbb{R}^n \), then \( \partial_{C_{\mathbb{R}^1}} f(\cdot, p)(\tau) \subseteq [\partial f(\cdot, p)]^+ \), and the same inclusion holds true for \( f_i(x, p) \). Thus, if \( X = \mathbb{R}^n \) and \( Y = \mathbb{R}^l \), then Theorem 2 is a simple corollary to the sufficient conditions for metric regularity in terms of the Clarke subdifferential [11] Theorem 1.1 (see also [11]). On the other hand, if either \( X \) or \( Y \) is infinite dimensional, then Theorem 2 does not follow from the main results of [11, 4].

3.3 Limiting Quasidifferential Sum and Metric Regularity

In some cases q.d.-MFCQ fails to hold true for metrically regular quasidifferentiable mappings, which makes Theorem 2 inapplicable. For example, for the function \( F(x) = |x_1| - |x_2| \) one can define

\[
\partial F(0) = \text{co} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}, \quad \overline{\partial} F(0) = \text{co} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\},
\]

which implies that \([\partial F(0)]^+ = \{ x \in \mathbb{R}^2 \mid \max \{|x_1|, |x_2|\} \leq 1 \} \), and q.d.-MFCQ is not satisfied at the origin, since \( 0 \in [\partial F(0)]^+ \), despite the fact that the function \( F \) is metrically regular near the point \((0,0)\) (see Remark 3). To overcome this issue in some cases one can utilise a standard limiting construction from the theory of subdifferentials (see, e.g. [23, 30]).

For the sake of simplicity in this subsection we consider only the finite dimensional case, i.e. we suppose that both \( X \) and \( Y \) are finite dimensional. The infinite dimensional case can be treated in essentially the same way it is done in the theory of limiting subdifferentials [23, 30].
**Definition 4.** Let a function \( f : X \times P \to \mathbb{R} \) be quasidifferentiable in \( x \) in a neighbourhood of a point \((\overline{\mathcal{X}}, \overline{\mathcal{P}}) \in X \times P\), and let there exist a quasidifferential mapping \( \mathcal{D}_x f(\cdot) \) that is o.s.c. at \((\overline{\mathcal{X}}, \overline{\mathcal{P}})\). The set
\[
[\mathcal{D}_x f(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+ = \limsup_{(x,p) \to (\overline{\mathcal{X}}, \overline{\mathcal{P}}), (x,p) \neq (\overline{\mathcal{X}}, \overline{\mathcal{P}})} [\mathcal{D}_x f(x,p)]^+,
\]
where \( \limsup \) is the outer limit, is called a limiting quasidifferential sum in \( x \) of \( f \) at \((\overline{\mathcal{X}}, \overline{\mathcal{P}})\) (associated with the quasidifferential mapping \( \mathcal{D} f(\cdot) \)).

The limiting quasidifferential sum \([\mathcal{D}_x f(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+\) is a nonempty compact but not necessarily convex set, since from the outer semicontinuity of the quasidifferential mapping \( \mathcal{D}_x f(\cdot) \) it follows that the sets \([\mathcal{D}_x f(x,p)]^+\) lie within a bound set for all \((x,p)\) in a sufficiently small neighbourhood of \((\overline{\mathcal{X}}, \overline{\mathcal{P}})\). Furthermore, it is clear that \([\mathcal{D}_x f(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+\) is a limiting quasidifferential sum. In particular, from the fact that \( \mathcal{D}_x (f_1 + f_2)(\cdot) = \mathcal{D}_x f_1(\cdot) + \mathcal{D}_x f_2(\cdot) \) it follows that
\[
[\mathcal{D}_x (f_1 + f_2)(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+ \subseteq [\mathcal{D}_x f_1(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+ + [\mathcal{D}_x f_2(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+.
\]

We leave the derivation of other similar results to the interested reader. Let us finally note that a limiting quasidifferential sum of \( f \) in \( x \) is obviously not invariant under the choice of quasidifferential.

**Remark 7.** One can define limiting quasidifferential instead of limiting quasidifferential sum; however, since subdifferential \( \partial_x f(x,p) \) and superdifferential \( \overline{\mathcal{D}}_x f(x,p) \) are not independent, in this case one must consider the outer limit of \( \mathcal{D}_x f(x,p) \) in \( X^* \times X^* \), which is no longer a pair of sets, but a subset of \( X^* \times X^* \) (i.e. it is fruitless to consider the outer limits of \( \partial_x f(x,p) \) and \( \overline{\mathcal{D}}_x f(x,p) \) as \((x,p) \to (\overline{\mathcal{X}}, \overline{\mathcal{P}})\) separately). That is why it seems more straightforward to define limiting quasidifferential sum directly, rather than define it via limiting quasidifferential.

With the use of limiting quasidifferential sum we can significantly strengthen Theorem 2 in the finite dimensional case.

**Theorem 3.** Let \( Y \) be the space \( \mathbb{R}^l \) equipped with the Euclidean norm, \( F(\cdot) = (f_1(\cdot), \ldots, f_l(\cdot)) \), where \( f_j : X \times P \to \mathbb{R} \), and let the functions \( f_j \), \( 1 \leq j \leq l \), and \( g_i \), \( i \in I \), be continuous. Let also a point \((\overline{\mathcal{X}}, \overline{\mathcal{P}}) \in X \times P \) be such that \( \overline{\mathcal{P}} \in S(\overline{\mathcal{P}}) \), and there exists a neighbourhood \( U \) of \((\overline{\mathcal{X}}, \overline{\mathcal{P}}) \) such that:

1. for any \((x,p) \in U\) the functions \( f_j(\cdot, p) \), \( 1 \leq j \leq l \), and \( g_i(\cdot, p) \), \( i \in I \), are quasidifferentiable at \( x \);
2. there exist quasidifferential mappings \( \mathcal{D}_x f_j(\cdot) \), \( 1 \leq j \leq l \), and \( \mathcal{D}_x g_i(\cdot) \), \( i \in I \), defined on \( U \) and o.s.c. at \((\overline{\mathcal{X}}, \overline{\mathcal{P}})\);
3. limiting q.d.-MFCQ holds at \((\overline{\mathcal{X}}, \overline{\mathcal{P}})\), i.e. the sets \([\mathcal{D}_x f_j(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+ \), \( 1 \leq j \leq l \), are linearly independent, and there exists \( v \in X \setminus \{0\} \) such that \((x^*, v) = 0\) for any \( x^* \in [\mathcal{D}_x f_j(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+ \), \( 1 \leq j \leq l \), and \((x^*, v) < 0\) for any \( x^* \in [\mathcal{D}_x g_i(\overline{\mathcal{X}}, \overline{\mathcal{P}})]^+ \), \( i \in I \).
Then there exist $K > 0$, a neighbourhood $V$ of $(\overline{x}, \overline{p})$, and a neighbourhood $W$ of zero in $Y \times \mathbb{R}^l$ such that for all $(x, p) \in V$ and $(y, z) \in W$ one has

$$d(x, S(p, y, z)) \leq K \left( \|F(x, p) - y\| + \sum_{i=1}^{m} \max\{g_i(x, p) - z_i, 0\} \right).$$

**Proof.** From the definition of limiting quasidifferential sum and the first condition in limiting q.d.-MFCQ it follows that there exists a neighbourhood $V_1$ of $(\overline{x}, \overline{p})$ and $\varepsilon > 0$ such that for any $(x, p) \in V_1 \setminus \{(\overline{x}, \overline{p})\}$ one has

$$\inf_{y^* \in S_Y} \inf \{\|x^*\| : x^* \in [\mathcal{D}F_p(x; y^*)]^+\} \geq \varepsilon > 0. \quad (15)$$

Indeed, arguing by reductio ad absurdum, suppose that for any $n \in \mathbb{N}$ there exists $(x_n, p_n) \in U$, $y_n^* \in S_Y^*$ and $x_n^* \in [\mathcal{D}F_{p_n}(x_n; y_n^*)]^+$ such that $\|x_n - \overline{x}\| + d(p_n, \overline{p}) < 1/n$ and $\|y_n^*\| < 1/n$. Without loss of generality one can suppose that $y_n^*$ converges to some $y^* = (y_1, \ldots, y_l) \in S_Y^*$. Passing to the limit as $n \to \infty$, and applying $(15)$ one obtains that

$$0 \in \limsup_{n \to \infty} [\mathcal{D}F_{p_n}(x_n; y_n^*)]^+ = \sum_{j=1}^{l} y_j [\mathcal{D}f_j(\overline{x}, \overline{p})]^+_\infty,$$

i.e. the sets $[\mathcal{D}f_j(\overline{x}, \overline{p})]^+_\infty$, $1 \leq j \leq l$, are linearly dependent, which is impossible. Thus, $(15)$ holds true. Furthermore, arguing in a similar way one can easily check that for any $\varepsilon > 0$ and $t > 0$ there exists a neighbourhood $V_2(\varepsilon, t)$ of $(\overline{x}, \overline{p})$ such that for any $(x, p) \in V_2(\varepsilon, t) \setminus \{(\overline{x}, \overline{p})\}$ one has $(x^*, tv) \leq \varepsilon$ for all $x^* \in [\mathcal{D}F_p(x; y^*)]^+$ and $y^* \in S_Y^*$, where $v$ is from the limiting q.d.-MFCQ.

With the use of $(15)$ and the separation theorem one gets that for any $(x, p) \in V_1 \setminus \{(\overline{x}, \overline{p})\}$ and $y^* \in S_Y^*$ there exists $h \in X$ with $\|h\| = 1$ such that $(x^*, h) \leq -\varepsilon$ for all $x^* \in [\mathcal{D}F_p(x; y^*)]^+$. Hence for any $(x, p) \in (V_1 \cap V_2(\varepsilon/4, t))$, $(x, p) \neq (\overline{x}, \overline{p})$ and any $y^* \in S_Y^*$ there exists $h \in X$ with $\|h\| = 1$ such that

$$(x^*, h + tv) \leq -\frac{\varepsilon}{2} \quad \forall x^* \in [\mathcal{D}F_p(x; y^*)]^+. \quad (16)$$

Taking into account the fact that limiting quasidifferential sum is a compact set one obtains that there exists $t_0 > 0$ such that

$$\max \{\langle x^*, h + t_0v \rangle : x^* \in [\mathcal{D}g_i(\overline{x}, \overline{p})]^+_{\infty}, i \in I(\overline{x}, \overline{p})\} \leq -\varepsilon,$$

for any $h \in B(0, 1)$. Hence and from the definition of limiting quasidifferential sum it follows that there exists a neighbourhood $V_3$ of $(\overline{x}, \overline{p})$ such that for any $(x, p) \in V_3 \setminus \{(\overline{x}, \overline{p})\}$ and $h \in B(0, 1)$ one has

$$\max \{\langle x^*, h + t_0v \rangle : x^* \in [\mathcal{D}g_i(x, p)]^+_{\infty}, i \in I(\overline{x}, \overline{p})\} \leq -\frac{\varepsilon}{2}.$$

Moreover, one can suppose that $\sup_{(x, p) \in V_3} g_i(x, p) < 0$ for any $i \notin I(\overline{x}, \overline{p})$.

Now, arguing in the same way as in the proof of Theorem $2$ one can easily check that for any $(x, p) \in V_1 \cap V_2(\varepsilon/4, t_0) \cap V_3$, $(x, p) \neq (x', p')$ and for all $(y, z)$ lying in a sufficiently small neighbourhood of zero and such that $(y, z) \notin \Phi(x, p)$ one has

$$\psi_{(y, z, p)}(x, h + t_0v) \leq \begin{cases} -\frac{\varepsilon}{2}, & \text{if } F(x, p) \neq y, \\ -\frac{\varepsilon}{4}, & \text{if } F(x, p) = y, \end{cases}$$
where $h$ is from (16) in the case $F(x, p) \neq y$, and $h = 0$ otherwise (note that $(x, p) \in V_2(x/4, t_0)$). Therefore, $|\nabla \psi(x, z, p)(x) | \geq x/4(1 + t_0||v||)$ for any such $x$, $p$, $y$, and $z$, and applying [22, Theorem 2b] we arrive at the desired result.

Let us consider a simple example demonstrating that Theorem 3 is significantly sharper than Theorem 2.

Example 1. Let, as above, $X = \mathbb{R}^2$, $l = 1$, and $F(x) = |x_1| - |x_2|$. For any $x \in \mathbb{R}^2$ one can define

$$D F(x) = \left\{ \left( \begin{array}{c} \text{Sign}(x_1) \\ 0 \end{array} \right) \right\}, \quad D^2 F(x) = \left\{ \left( \begin{array}{c} 0 \\ -\text{Sign}(x_2) \end{array} \right) \right\},$$

where $\text{Sign}(t) = 1$, if $t > 0$, $\text{Sign}(t) = -1$, if $t < 0$, and $\text{Sign}(0) = [-1, 1]$. It is easily seen that this quasidifferential mapping is outer semicontinuous. Moreover, as it is easy to check, $|D^2 F(0)|_{\infty} = \{ x \in \mathbb{R}^2 \mid \max\{|x_1|, |x_2|\} = 1 \}$. Thus, $0 \not\in |D^2 F(0)|_{\infty}$, and the mapping $F$ is metrically regular near $(0, 0)$ by Theorem 3.

4 Optimality Conditions

Let us utilise (limiting) q.d.-MFCQ as a new constraint qualification for quasidifferential programming problems with equality and inequality constraints in order to obtain necessary optimality conditions for these problems. To this end, consider the following optimization problem:

$$\min u(x) \quad \text{subject to} \quad f_j(x) = 0, \quad j \in J, \quad g_i(x) \leq 0, \quad i \in I. \quad (P)$$

Here $u, f_j, g_i : X \to \mathbb{R}$ are given functions, $J = \{1, \ldots, l\}$ and $I = \{1, \ldots, m\}$. Denote by $\Phi_\lambda(x) = u(x) + \lambda \varphi(x)$ with

$$\varphi(x) = \sum_{j=1}^l |f_j(x)| + \sum_{i=1}^m \max\{g_i(x), 0\},$$

the $\ell_1$ penalty function for the problem $(P)$, where $\lambda \geq 0$ is the penalty parameter. Note that if the functions $u$, $f_j$ and $g_i$ are quasidifferentiable, then this penalty function is quasidifferentiable as well (see [10]).

Theorem 4. Let the following assumptions be valid:

1. $\bar{x}$ is a locally optimal solution of the problem $(P)$;
2. $u$ is quasidifferentiable at $\bar{x}$ and Lipschitz continuous near this point;
3. $f_j$, $j \in J$, and $g_i$, $i \in I$, are quasidifferentiable in a neighbourhood of $\bar{x}$, and there exist quasidifferential mappings $D f_j(\cdot)$, $j \in J$, and $D g_i(\cdot)$, $i \in I$ defined in a neighbourhood of $\bar{x}$ and o.s.c. at this point;
4. either q.d.-MFCQ holds at $\bar{x}$ or $X$ is finite dimensional, and limiting q.d.-MFCQ holds at this point.
Then there exists $\lambda^* \geq 0$ such that for any $\lambda \geq \lambda^*$ one has
\[ 0 \in \partial \Phi_\lambda(\bar{x}) + y^* \quad \forall y^* \in \overline{\partial} \Phi_\lambda(\bar{x}), \]
(17)
where $\partial \Phi_\lambda(\bar{x}) = \partial \Phi_\lambda(\bar{x}), \overline{\partial} \Phi_\lambda(\bar{x})$ is a quasidifferential of $\Phi_\lambda$ at $\bar{x}$. Moreover, for any $y^*_0 \in \overline{\partial} u(\bar{x}), y^*_i \in \overline{\partial} f_j(\bar{x})$ $\forall j$, and $z^*_i \in \overline{\partial} g_i(\bar{x})$ there exists $\mu_j, \lambda_i \geq 0$ such that $\lambda_i g_i(\bar{x}) = 0$ for all $i \in I$, and
\[ 0 \in \partial u(\bar{x}) + y^*_0 - \sum_{j=1}^i \mu_j (y^*_j + \overline{\partial} f_j(\bar{x})) + \sum_{j=1}^m \lambda_i (\overline{\partial} g_i(\bar{x}) + z^*_i). \]
(18)

Proof. With the use of either Theorem 2 or Theorem 3 one obtains that there exist a neighbourhood $U$ of $\bar{x}$ and $\tau \geq 0$ such that $\varphi(x) \geq \tau d(x, \Omega)$ for any $x \in U$, where $\Omega$ is the feasible region of the problem $(P)$. Hence by [13, Theorem 2.4 and Proposition 2.7] there exists $\lambda^* \geq 0$ such that for any $\lambda \geq \lambda^*$ the point $\bar{x}$ is a point of local minimum of the penalty function $\Phi_\lambda$. Consequently, applying the necessary conditions for a minimum in terms of quasidifferentials [10] to $\Phi_\lambda$, one gets that $0 \in \partial \Phi_\lambda(\bar{x}) + y^*$ for all $y^* \in \overline{\partial} \Phi_\lambda(\bar{x})$.

To prove the validity of (18), note that by the necessary condition for a minimum in terms of directional derivative one has
\[ \Phi'_\lambda(x, v) = u'(x, v) + \lambda \left( \sum_{j=1}^m |f_j(x, v)| + \sum_{i \in I(\bar{x})} \max \{ g'_i(x, v), 0 \} \right) \geq 0 \]
for any $v \in X$, where $I(\bar{x}) = \{ i \in I \mid g_i(\bar{x}) = 0 \}$ (here we used standard calculus rules for directional derivatives; see [10]). Let $y^*_0$, $y^*_i$, $\overline{\partial} f_j$ and $z^*_i$ be as in the formulation of the theorem. Define
\[ \eta(v) = s(\partial u(\bar{x}) + y^*_0, v) + \lambda \sum_{j=1}^m \max \left\{ s(\overline{\partial} f_j(\bar{x}) + \overline{\partial} f_j(\bar{x}), v), s(-y^*_j - \overline{\partial} f_j(\bar{x}), v) \right\} \]
\[ + \lambda \sum_{i \in I(\bar{x})} \max \left\{ s(\partial g_i(\bar{x}) + z^*_i, v), 0 \right\}, \]
where $s(C, v) = \sup_{x^* \in C} (x^*, v)$ for any $C \subset X^*$. Applying the definition of quasidifferential it is easy to see that $\eta(v) \geq \Phi'_\lambda(x, v) \geq 0$ for any $v \in X$. Therefore, 0 is a point of global minimum of the function $\eta$, which implies that $0 \in \partial \eta(0)$, where $\partial \eta(0)$ is the subdifferential of $\eta$ at 0 in the sense of convex analysis. Consequently, taking into account the fact that
\[ \partial \eta(0) = \partial u(\bar{x}) + y^*_0 + \lambda \sum_{j=1}^m \co \left\{ \overline{\partial} f_j(\bar{x}) + \overline{\partial} f_j(\bar{x}), -y^*_j - \overline{\partial} f_j(\bar{x}) \right\} \]
\[ + \lambda \sum_{i \in I(\bar{x})} \co \left\{ \partial g_i(\bar{x}) + z^*_i, 0 \right\} \]
we arrive at the required result. \[\square\]
Remark 8. Optimality conditions similar to but weaker than \[17\] were obtained in \[32, 33\] in the finite dimensional case under a different constraint qualification that involves some assumptions on so-called contact points of the sets \(\partial f_j(\mathbf{t})\) and \(\overline{\partial f_j(\mathbf{t})}\), i.e. such points \(x^*\) of a convex set \(C \subset X^*\) that \(s(C, v) = \langle x^*, v \rangle\) for a given direction \(v\). Note that one has to compute contact points of the sets \(\partial f_j(\mathbf{t})\) and \(\overline{\partial f_j(\mathbf{t})}\) for all feasible directions in order to check the validity of the constraint qualification from \[32, 33\], which is impossible in nontrivial cases. In contrast, q.d.-MFCQ is formulated in terms of problem data directly. In turn, optimality conditions similar to but weaker than \[15\] were derived in \[31\] under yet another constraint qualification in the case when \(X\) is finite dimensional, there are no inequality constraints, and there is only one equality constraint. Furthermore, note that sufficient conditions for the validity of this constraint qualification \[31\] Theorem 2 coincide with q.d.-MFCQ with \(I = \emptyset\) and \(l = 1\).

Let us also give a simple example demonstrating that in some cases the optimality conditions from the theorem above are much sharper than optimality conditions in terms of various subdifferentials.

**Example 2.** Let \(X = \mathbb{R}^2\), and consider the following optimization problem:

\[
\min u(x) = -x_1 + x_2 \quad \text{subject to} \quad f_1(x) = |x_1| - |x_2| = 0. \tag{19}
\]

Put \(\mathbf{t} = (0, 0)\). Observe that \(\mathbf{t}\) is not a locally optimal solution of problem \[19\], since for any \(t > 0\) the point \(x(t) = (t, -t)\) is feasible for this problem and \(u(x(t)) = -2t < 0 = u(\mathbf{t})\). Nevertheless, let us verify that several subdifferential-based optimality conditions fail to disqualify \(\mathbf{t}\) as a non-optimal solution.

We start with necessary optimality conditions in terms of the subdifferential of Michel-Penot \[20\], which we denote by \(\partial MP\). Let \(L(x, \lambda) = u(x) + \lambda f_1(x)\) be the Lagrangian function for problem \[19\]. For any \(h \in \mathbb{R}^2\) the Michel-Penot directional derivative of \(L(\cdot, \lambda)\) at \(\mathbf{t}\) has the form

\[
d_{MP}L(\cdot, \lambda)[\mathbf{t}, h] = \sup_{e \in \mathbb{R}^2} \limsup_{t \to +0} \frac{L(x + t(h + e)) - L(x + te)}{t} = \sup_{e \in \mathbb{R}^2} \left\{ -h_1 + h_2 + \lambda \left( |h_1 + e_1| - |h_1| - h_2 + e_2 + |e_2| \right) \right\} = -h_1 + h_2 + |\lambda| (|h_1| + |h_2|).
\]

Hence the Michel-Penot subdifferential of \(L(\cdot, \lambda)\) at \(\mathbf{t}\) has the form

\[
\partial MP L(\cdot, \lambda)(\mathbf{t}) = \partial \varphi(0) = \text{co} \left\{ \left( \frac{|\lambda| - 1}{|\lambda| + 1} \right), \left( \frac{|\lambda| - 1}{|\lambda| + 1} \right), \left( \frac{|\lambda| - 1}{|\lambda| + 1} \right), \left( \frac{|\lambda| - 1}{|\lambda| + 1} \right) \right\}
\]

where \(\varphi(h) = d_{MP}L(\cdot, \lambda)[\mathbf{t}, h]\). Consequently, for any \(\lambda \in \mathbb{R}\) such that \(|\lambda| \geq 1\) one has \(0 \in \partial MP L(\cdot, \lambda)(\mathbf{t})\), which implies that the optimality conditions from \[20\] are satisfied at \(\mathbf{t}\). Furthermore, note that \(\partial MP L(\cdot, \lambda)(\mathbf{t}) = \partial_{CL} L(\cdot, \lambda)(\mathbf{t})\), which implies that optimality conditions in terms of the Clarke subdifferential \[5\] Theorem 6.1.1] are satisfied at \(\mathbf{t}\) as well.

Next, we consider optimality conditions in terms of the Jeyakumar-Luc subdifferential \[36\], which we denote by \(\partial^*\). By \[36\] Example 2.1] one has \(\partial^* f_1(\mathbf{t}) = \{(1, -1)^T, (-1, 1)^T\}\), and obviously \(\partial^* u(\mathbf{t}) = \{(-1, 1)\}\). Hence for any \(\lambda \in \mathbb{R}\) such that \(|\lambda| \geq 1\) one has \(0 \in \partial^* u(\mathbf{t}) + \lambda \text{ co } \partial^* f_1(\mathbf{t})\), i.e. the optimality conditions in terms of the Jeyakumar-Luc subdifferential \[36\] Corollary 3.4] are satisfied at \(\mathbf{t}\).
Let us now consider optimality conditions in terms of approximate (graded, Ioffe) subdifferentials (see [23, 30]), which we denote by \(\partial_a\). Observe that for any \(x \in \mathbb{R}^2\) such that \(x_2 > 0\) one has \(L(x, 1) = -x_1 + |x_1|\), which obviously implies that \(0 \in \partial_a L(x, 1)\) for any such \(x\), where \(\partial_a L(x, 1)\) is the Dini subdifferential of \(L(\cdot, 1)\) at \(x\). Therefore, \(0 \in \partial_a L(x, 1) = \limsup_{x \to x} \partial_{-x} L(x, 1)\), i.e. the optimality conditions in terms of approximate subdifferential [21, Proposition 12] are satisfied at \(x\).

Let us also consider optimality conditions in terms of the Mordukhovich basic subdifferential [29], which we denote by \(\partial_M\). One can check (see [28, p. 92–93]) that

\[
\partial_M f_1(x) = \text{co} \left\{ \left( -1, 1 \right), \left( 1, -1 \right) \right\} \cup \text{co} \left\{ \left( -1, 1 \right), \left( 1, 1 \right) \right\}.
\]

Therefore, \(-\nabla u(x) \in \partial_M f_1(x)\), i.e. the optimality conditions in terms of the Mordukhovich basic subdifferential [29, Theorem 5.19] hold true at \(x\).

Finally, let us verify that optimality conditions (18) from Theorem 4 are not satisfied at \(x\), i.e. unlike optimality conditions in terms of various subdifferentials, optimality conditions based on quasidifferentials detect the non-optimality of \(x\).

Arguing by reductio ad absurdum, suppose that (18) holds true. Then for \(y^*_1 = (1, 0) \in \partial f_1(x)\) and \(y^*_1 = (0, 1) \in \partial f_1(x)\) (see Example 1) there exist \(\mu_1, \mu_1 \geq 0\) such that

\[
0 \in \left( -1, 1 \right) - \mu_1 \text{co} \left\{ \left( 1, -1 \right), \left( 1, 1 \right) \right\} + \mu_1 \text{co} \left\{ \left( -1, 1 \right), \left( 1, 1 \right) \right\},
\]

or equivalently

\[
-1 - \mu_1 - \mu_1 \leq 0 \leq -1 - \mu_1 + \mu_1, \quad 1 + \mu_1 - \mu_1 \leq 0 \leq 1 + \mu_1 + \mu_1.
\]

From the third inequality it follows that \(1 + \mu_1 \leq \mu_1\), while from the second inequality it follows that \(1 + \mu_1 \leq \mu_1\). Therefore \(2 + \mu_1 \leq \mu_1\), which is impossible. Thus, optimality conditions (18) do not hold true at \(x\).

As it was shown in Example 1, the limiting q.d.-MFCQ holds at \(x\). Thus, by Theorem 4 one can conclude that optimality conditions (18) are not satisfied at \(x\) due to the non-optimality of this point.

References

[1] A. Auslender. Stability in mathematical programming with non-differentiable data. SIAM J. Control Optim., 22:239–254, 1984.

[2] D. Azé. A unified theory for metric regularity of multifunctions. J. Convex Anal., 13:225–252, 2006.

[3] J. F. Bonnans and A. Shapiro. Perturbation analysis of optimization problems. Springer Science+Business Media, New York, 2000.

[4] J. M. Borwein. Stability and regular point of inequality systems. J. Optim. Theory Appl., 48:9–52, 1986.

[5] F. H. Clarke. Optimization and Nonsmooth Analysis. Wiley–Interscience, New York, 1983.
[6] R. Cominetti. Metric regularity, tangent sets, and second-order optimality conditions. *Appl. Math. Optim.*, 21:265–287, 1990.

[7] V. F. Demyanov. Fixed point theorem in nonsmooth analysis and its applications. *Numer. Funct. Anal. Optim.*, 16:53–109, 1995.

[8] V. F. Demyanov. Exhauster and convexificators — new tools in nonsmooth analysis. In V. Demyanov and A. Rubinov, editors, *Quasidifferentiability and Related Topics*, pages 85–137. Kluwer Academic Publishers, Dordrecht, 2000.

[9] V. F. Demyanov and V. Jeyakumar. Hunting for a smaller convex subdifferential. *J. Glob. Optim.*, 10:305–326, 1997.

[10] V. F. Demyanov and A. M. Rubinov. *Constructive Nonsmooth Analysis*. Peter Lang, Frankfurt am Main, 1995.

[11] V. F. Demyanov and A. M. Rubinov, editors. *Quasidifferentiability and Related Topics*. Kluwer Academic Publishers, Dordrecht, 2000.

[12] M. V. Dolgopolik. Abstract convex approximations of nonsmooth functions. *Optim.*, 64:1439–1469, 2015.

[13] M. V. Dolgopolik. A unifying theory of exactness of linear penalty functions. *Optim.*, pages 1167–1202, 2016.

[14] M. V. Dolgopolik. A convergence analysis of the method of codifferential descent. *Comput. Optim. Appl.*, 71:879–913, 2018.

[15] S. I. Dudov. Subdifferentiability and superdifferentiability of distance functions. *Math. Notes*, 4:440–450, 1997.

[16] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems*. SIAM, Philadelphia, 1999.

[17] Y. Gao. Demyanov difference of two sets and optimality conditions of Lagrange multiplier type for constrained quasidifferentiable optimization. *J. Optim. Theory Appl.*, 104:377–394, 2000.

[18] F. Giannessi. A common understanding or a common misunderstanding? *Numer. Funct. Anal. Optim.*, 16:1359–1363, 1995.

[19] B. M. Glover. On quasidifferentiable functions and non-differentiable programming. *Optim.*, 24:253–268, 1992.

[20] A. Ioffe. A Lagrange multiplier rule with small convex-valued subdifferentials for nonsmooth problems of mathematical programming involving equality and nonfunctional constraints. *Math. Program.*, 58:137–145, 1993.

[21] A. D. Ioffe. Approximate subdifferentials and applications I: the finite dimensional theory. *Trans. Am. Math. Soc.*, 281:389–416, 1984.

[22] A. D. Ioffe. Metric regularity and subdifferential calculus. *Russ. Math. Surv.*, 55:501–558, 2000.
[23] A. D. Ioffe. On the theory of subdifferentials. *Adv. Nonlinear Anal.*, 1:47–120, 2012.

[24] A. D. Ioffe. *Variational Analysis of Regular Mappings: Theory and Applications*. Springer International Publishing, Cham, 2017.

[25] V. Jeyakumar and D. T. Luc. Nonsmooth calculus, minimality, and monotonicity of convexificators. *J. Optim. Theory Appl.*, 101:599–621, 1999.

[26] L. Kuntz. A characterization of continuously codifferentiable function and some consequences. *Optim.*, 22:539–547, 1991.

[27] L. Kuntz and S. Scholtes. Constraint qualifications in quasidifferentiable optimization. *Math. Program.*, 60:339–347, 1993.

[28] B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation I: Basic Theory*. Springer-Verlag, Berling, Heidelberg, 2006.

[29] B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation II: Applications*. Springer-Verlag, Berling, Heidelberg, 2006.

[30] J.-P. Penot. *Calculus Without Derivatives*. Springer Science+Business Media, New York, 2013.

[31] L. N. Polyakova. On the minimization of a quasidifferentiable function subject to equality-type quasidifferentiable constraints. In V. F. Demyanov and L. C. W. Dixon, editors, *Quasidifferential Calculus*, pages 44–55. Springer, Berlin, Heidelberg, 1986.

[32] A. Shapiro. On optimality conditions in quasidifferentiable optimization. *SIAM J. Control Optim.*, 22:610–617, 1984.

[33] A. Shapiro. Quasidifferential calculus and first-order optimality conditions in nonsmooth optimization. *Math. Program. Study*, 29:56–68, 1986.

[34] A. Uderzo. Fréchet quasidifferential calculus with applications to metric regularity of continuous maps. *Optim.*, 54:469–493, 2005.

[35] A. Uderzo. Stability properties of quasidifferentiable systems. *Vestn. St. Petesb. Univ. Ser. 10. Appl. Math., Inform., Control Process.*, 3:70–84, 2006. [in Russian]. Available online at: https://cyberleninka.ru/article/v/svoystva-ustoychivyosti-dlya-kvazidifferentsiruemyh-sistem-1.

[36] X. Wang and V. Jeyakumar. A sharp Lagrange multiplier rule for nonsmooth mathematical programming problems involving equality constraints. *SIAM J. Optim.*, 10:1136–1148, 2000.