Acceleration of colliding shells around a black hole
– Validity of the test particle approximation in the Banados-Silk-West process –

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(Dated: February 8, 2011)

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Abstract

Recently, Banados, Silk and West (BSW) showed that the total energy of two colliding test particles has no upper limit in their center of mass frame in the neighborhood of an extreme Kerr black hole, even if these particles were at rest at infinity in the infinite past. We call this mechanism the BSW mechanism or BSW process. The large energy of such particles would generate strong gravity, although this has not been taken into account in the BSW analysis. A similar mechanism is seen in the collision of two spherical test shells in the neighborhood of an extreme Reissner-Nordström black hole. In this paper, in order to draw some implications concerning the effects of gravity generated by colliding particles in the BSW process, we study a collision of two spherical dust shells, since their gravity can be exactly treated. We show that the energy of two colliding shells in the center of mass frame observable from infinity has an upper limit due to their own gravity. Our result suggests that an upper limit also exists for the total energy of colliding particles in the center of mass frame in the observable domain in the BSW process due the gravity of the particles.

PACS numbers: 04.70.Bw
I. INTRODUCTION

Recently, Banados, Silk and West (BSW) showed that two test particles can collide with arbitrarily high energy in the center of mass frame near an extremal Kerr black hole, even though these particles were at rest at infinity in the infinite past \cite{1}. We call this mechanism the BSW mechanism or the BSW process, and several aspects of this mechanism have been reported in Refs. \cite{2–13}. If this mechanism was really workable, it might be possible to probe Planck-scale physics by observing the neighborhood of an extreme or almost extreme Kerr black hole. However, it is not yet clear whether particles can really be accelerated with sufficient efficiency to produce collisions with Planckian energies. To answer this question, it is necessary to consider, among other things, the effect of particle size, the effect of gravitational radiation on the trajectories of the particles, and the effect of the gravity generated by the particles themselves at the event horizon. In this paper, we focus on the third effect.

The BSW mechanism is also interesting from a purely relativistic point of view. The result obtained in Ref. \cite{1} seems to imply that the energy of the colliding particles in the center of mass frame can be as large as the mass of the Kerr black hole in the background spacetime, and hence the gravity generated by the particles is so strong that another black hole forms. Such intense gravity cannot be described using a linear perturbation approximation of the Kerr spacetime. Thus, even if the energy of the particles is initially small enough that the gravity due to these particles is well described by the linear perturbation approximation, the gravity generated by the energy of these particles might finally become too strong to invoke linear perturbation analysis. If this inference is true, the Kerr black hole is unstable against minor perturbations induced by dropping small particles into the black hole under finely tuned initial conditions. This conclusion is paradoxical, because it is well known that a Kerr black hole is stable against small perturbations. Therefore, it is important to investigate how large the energy of colliding particles in the center of mass frame can be, by taking into account the gravity due to the particles.

However, it is difficult to treat the effects of gravity generated by particles around a Kerr black hole. One of the reasons for this difficulty is the fact that Kerr spacetime is not very symmetric. Hence, in the present paper, we consider charged particles or charged spherical shells around a spherically symmetric charged black hole known as a Reissner-Nordström
black hole, and then show that a process similar to the BSW process is also possible in this system. Next, in order to draw some implications concerning the effects of the gravity generated by colliding objects in this process, we study a collision of spherical dust shells whose gravity is exactly treatable by the Israel formalism [14].

The organization of this paper is as follows. In Sec. II, we briefly review the BSW mechanism. In Sec. III, we study a collision between a charged test particle and a neutral test particle around an extreme Reissner-Nordström black hole, in order to show that a BSW-like process is also possible in this spacetime. We study a spherical charged dust shell in Sec. IV and a collision between the charged dust shell and a neutral dust shell in Sec. V. The final section is devoted to the discussion.

In this paper, we adopt geometric units for which Newton’s gravitational constant $G$ and the speed of light $c$ are unity, and we adopt an abstract index notation in which all Latin indices except for $t$, $r$, and $i$ indicate a type of a tensor, whereas all Greek indices except for $\phi$ represent components with respect to the coordinate basis. The exceptional indices $t$, $r$, $\phi$ respectively denote the components of time, radial and azimuthal coordinates in the spherical polar coordinate system, and $i$ specifies the $i$-th particle, shell or region. The signature of the metric is $\text{diag}[-,+,+,+]$.

II. PARTICLE COLLISIONS AROUND EXTREME KERR BLACK HOLES

In this section, we briefly review the results described in Ref. [1].

We consider a collision of two test particles with an identical inertial mass $m$. We denote the 4-velocities of these particles by $u^a_{(i)}$ ($i = 1, 2$). Then, the total 4-momentum of these particles at the collision event is given by $p^a = m(u^a_{(1)} + u^a_{(2)})$, and their total energy in the center of mass frame, which hereafter will be called simply the CM energy, is given by

$$ E_{\text{cm}} = \sqrt{-g_{ab} p^a p^b} = \sqrt{2m} \sqrt{1 - g_{ab} u^a_{(1)} u^b_{(2)}}, $$

where $g_{ab}$ is the metric tensor.

Let us consider the CM energy at a collision event of two particles in a Kerr spacetime
whose metric is given by

\[ ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4aMr\sin^2 \theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2 \theta}{\rho^2}\right) \sin^2 \theta d\phi^2, \]  

(2)

where \( M \) and \( a \) are the mass parameter and the Kerr parameter which represents the angular momentum of this system, respectively, and

\[ \Delta = r^2 + a^2 - 2Mr, \]  

(3)

\[ \rho^2 = r^2 + a^2 \cos \theta. \]  

(4)

In the case of \( M^2 > a^2 \), the equation \( \Delta = 0 \) has two real roots \( r = r_\pm := M \pm \sqrt{M^2 - a^2} \): \( r = r_+ \) corresponding to the black hole (BH) horizon, whereas \( r = r_- \) corresponds to the Cauchy horizon. In the case of \( M^2 = a^2 \), these two horizons degenerate into one horizon \( r = r_+ = r_- = M \). In the case of \( M^2 < a^2 \), there is no root of \( \Delta = 0 \), and hence this case is naked singular. In this section, we assume \( M \geq |a| \).

For simplicity, we focus on two particles whose orbits are confined to the equatorial plane \( \theta = \pi/2 \). By integrating the geodesic equations and using the normalization condition of the 4-velocity \( g_{ab}u^a(\mathbf{i})u^b(\mathbf{i}) = -1 \) \((i = 1, 2)\), we have

\[ u^r(\mathbf{i}) = \pm \frac{1}{r^2} \sqrt{T_i^2 - \Delta[r^2 + (\ell_i - aE_i)^2]}, \]  

(5)

\[ u^\phi(\mathbf{i}) = -\frac{1}{r^2} \left[ (aE_i - \ell_i) - \frac{aT_i}{\Delta} \right], \]  

(6)

\[ u^t(\mathbf{i}) = -\frac{1}{r^2} \left[ a(aE_i - \ell_i) - \frac{(r^2 + a^2)T_i}{\Delta} \right], \]  

(7)

where \( E_i, \ell_i \) are constants of integration, and \( T_i = E_i(r^2 + a^2) - \ell_i a \). Note that \( E_i \) and \( \ell_i \) correspond to the specific energy and angular momentum of the \( i \)-th particle.

We assume that each particle is marginally bound : \( E_i = 1 \). In this case, \( u^r(\mathbf{i}) \) and \( u^\phi(\mathbf{i}) \) vanish in the limit of \( r \to \infty \), or in other words, these particles were at rest at infinity in the infinite past. By substituting Eqs. (5)–(7) into Eq. (1), we obtain the square of the CM energy at the collision event as

\[ E_{cm}^2 = \frac{2m^2}{r(r^2 - 2Mr + a^2)} \left( 2a^2(r + M) - 2Ma(\ell_1 + \ell_2) - \ell_2 \ell_1 (r - 2M) + 2(r - M)r^2 \right. \]

\[ - \sqrt{2M(a - \ell_1)^2 - \ell_1^2 r + 2Mr^2} \sqrt{2M(a - \ell_2)^2 - \ell_2^2 r + 2Mr^2}. \]  

(8)
In the extreme case $a = M$, the CM energy at the degenerate horizon $r = r_\pm = M$ is given in the simple form

$$E_{cm}(r \to r_+) = \sqrt{2m} \sqrt{\frac{\ell_2 - 2M}{\ell_1 - 2M} + \frac{\ell_1 - 2M}{\ell_2 - 2M}}.$$  \hspace{1cm} (9)

We see from the above equation that if either $\ell_1$ or $\ell_2$ is equal to $2M$ and the other is not, the CM energy at the collision event does not have an upper limit.  \(^1\)

This result can be understood as follows. We denote the world line of the 1st particle by $x^\mu = x^\mu(\tau)$, where $\tau$ is its proper time. Then, in the extreme case $a = M$, the square of (5) gives

$$\left(\frac{dr}{d\tau}\right)^2 - \frac{2M(r - M)^2}{r^3} = 0,$$ \hspace{1cm} (10)

where we have taken $\mathcal{E}_1 = 1$ and $\ell_1 = 2M$. We can see from this equation that if this particle is falling toward the black hole, its asymptote is the degenerate horizon $r = M$. Since the only possible causal line on the future horizon is outward null, this particle asymptotically becomes outward null, even though it is falling toward the black hole\(^2\). Hence, the closer the relative velocity between this 1st particle and a 2nd particle with $\ell_2 \neq 2M$ approaches to the speed of light, the closer the collision event approaches the BH horizon. As a result, the collision of these particles can lead to an indefinitely large CM energy near the horizon. Here, note that the 1st particle cannot reach the horizon within a finite time span, and hence the CM energy never diverges.

III. PARTICLE COLLISION AROUND EXTREME REISSNER-NORDSTRÖM BLACK HOLES

Since the large CM energy of the particles produces strong gravity, we must take this into account when evaluating how large the CM energy can become in the BSW mechanism. However, it is difficult to treat the effects of gravity due to the particles, partly because Kerr spacetime is not very symmetric.

\(^1\) More generally, the case of the Kerr Newmann black holes is discussed in [7].

\(^2\) One might expect that an unstable circular orbit for a massive particle is possible at the horizon $r = M$ from Eq. (10). However, we can show that this is not true. The detailed discussions are given in [11].
However, a mechanism similar to the BSW mechanism has been reported for Reissner-Nordström spacetime [8]. In this case, we can see the effects of the gravity generated by the colliding objects, since Reissner-Nordström spacetime is more symmetric than Kerr spacetime. This will be carried out in the next section. For now, we will ignore the effects of gravity.

The metric and gauge 1-form of Reissner-Nordström spacetime is

\[
\begin{align*}
\text{ds}^2 &= -f \text{d}t^2 + f^{-1} \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2), \\
A_a &= \frac{Q}{r} (\text{d}t)_a,
\end{align*}
\]

(11)

where the function \( f \) is defined by

\[
f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},
\]

(13)

\( M \) and \( Q \) being the mass parameter and the charge, respectively. In the case of \( M^2 > Q^2 \), the equation \( f = 0 \) has two positive roots, \( r = r_\pm := M \pm \sqrt{M^2 - Q^2} \); \( r = r_+ \) corresponds to the BH horizon, whereas \( r = r_- \) corresponds to the Cauchy horizon. In the case of \( M^2 = Q^2 \), the BH and Cauchy horizons degenerate into one horizon \( r = r_\pm = M \). In the case of \( M^2 < Q^2 \), there is no real root of \( f = 0 \), i.e., no horizon. In this section, we assume \( M \geq |Q| \).

The action of a charged test particle subjected to the Lorentz force is given by

\[
S = -m \int \text{d}\tau - q \int \sum_{\mu=0}^3 A_\mu \frac{dx^\mu}{d\tau} d\tau,
\]

(14)

where \( \tau, m \) and \( q \) are the proper time, inertial mass and charge of the test particle, respectively. From the minimal action principle, we obtain its equation of motion. Without loss of generality, we may assume that the orbit of the particle is confined to the equatorial plane \( \theta = \pi/2 \). We can easily integrate the time and azimuthal components of the equation of motion and obtain

\[
\frac{dt}{d\tau} = \frac{1}{f} \left( \mathcal{E}_c - \frac{q}{m} A_t \right) \quad \text{and} \quad \frac{d\phi}{d\tau} = \frac{\ell_c}{r^2},
\]

(15)

where \( \mathcal{E}_c \) and \( \ell_c \) are integration constants which correspond to the specific energy and angular momentum of the particle, respectively. We assume that \( \mathcal{E}_c \) is positive. By using the normalization condition of the 4-velocity and the above results, we obtain the energy equation

\[
\left( \frac{dr}{d\tau} \right)^2 + V = 0,
\]

(16)
where $V$ is the effective potential defined by

$$V = - \left( E_c - \frac{qQ}{mr} \right)^2 + \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \left( 1 + \frac{\ell^2}{r^2} \right).$$

(17)

For simplicity, hereafter, we consider a particle radially falling toward the black hole, i.e., the case of $\ell_c = 0$. If the background spacetime is extreme $Q = M$ and the charge of the particle is $q = E_c m$, the effective potential becomes

$$V = - \frac{(E_c^2 - 1)(r - M)^2}{r^2}.$$ 

(18)

From the above equation, we can see that this charged test particle asymptotically approaches the future degenerate horizon $r = r_\pm = M$, i.e., the outward null if $E_c > 1$.

We consider another particle with an identical inertial mass $m$ to the charged particle but with a vanishing charge, which is also radially falling toward the black hole. Then, let us consider the collision between this neutral particle and the charged particle with the effective potential (18). We assume that the specific energy of the neutral particle is equal to that of the charged particle, $E_c$. We can easily see that, by this assumption, the absolute value of the velocity of the neutral particle is larger than that of the charged particle at the same radial coordinate. Hence, the charged particle corresponds to the 1st particle, whereas the neutral particle corresponds to the 2nd particle in Eq. (1). The square of the CM energy at the collision event is obtained as

$$E_{cm}^2 = \frac{2m^2 r}{r - M} \left[ 1 - \frac{M}{r} + E_c^2 - \sqrt{E_c^2 - 1} \sqrt{E_c^2 - \left( 1 - \frac{M}{r} \right)^2} \right].$$

(19)

We can easily see that the CM energy diverges in the limit $r \to r_\pm = M$. This is similar to the BSW mechanism.

It is still difficult to treat the gravity of particles in Reissner-Nordström spacetime. Here, it is worthwhile to note that the world line of a radially moving test particle is equivalent to the trajectory of an infinitesimally thin spherical test shell by virtue of the symmetry of Reissner-Nordström spacetime. Since the CM energy at the collision event between two shells is the same as that given by Eq. (11) (see Eq. (38) in Sec. V), an indefinitely large CM energy is realizable also in this case. The gravity generated by an infinitesimally thin spherical shell can be treated analytically by the Israel formalism [14]. In the next section, we study the effects of the gravity of a thin spherical charged shell.
IV. INFINITESIMALLY THIN SPHERICAL CHARGED DUST SHELL

In this section, we study the gravity generated by a thin spherical shell with a non-vanishing charge. An infinitesimally thin shell is equivalent to a singular timelike hypersurface $\Sigma$ which divides spacetime into two regions $V_1$ and $V_2$ (see Fig. 1).

![Schematic spacetime diagram of a spherical timelike shell. The shell divides spacetime into two regions $V_1$ and $V_2$.](image)

The metric tensor $g_{ab}$ should be everywhere continuous, even though $\Sigma$ is singular. Thus the unit vector $n_a$ normal to $\Sigma$ is uniquely determined, and we can introduce the intrinsic metric $h_{ab} = g_{ab} - n_a n_a$ on $\Sigma$. The extrinsic curvature of $\Sigma$ is defined by

$$K_{\alpha\beta}^{(i)} := -h^c_a h^d_b \nabla^{(i)}_c n_d,$$  \hspace{1cm} (20)

which determines how $\Sigma$ is embedded in $V_i$, where $\nabla^{(i)}_a$ is the covariant derivative within $V_i$. Since the infinitesimally thin shell is a distributional source for the Einstein equations, $K_{ab}^{(1)}$ and $K_{ab}^{(2)}$ may not be identical to each other. Following Israel [14], we define a tensor field $S_{ab}$ on $\Sigma$ by

$$K_{ab}^{(2)} - K_{ab}^{(1)} = 8\pi \left( S_{ab} - \frac{1}{2} h_{ab} \text{tr } S \right).$$  \hspace{1cm} (21)

Through the Einstein equation, $S_{ab}$ is identified with the surface stress-energy tensor of the
shell as

$$S_{\mu\nu} = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} T_{\mu\nu} d\lambda, \quad (22)$$

where $\lambda$ is the Gaussian normal coordinate in which $\Sigma$ is located at $\lambda = 0$ [14]. The above relation implies that the stress-energy tensor of the shell can be written in the form

$$T_{ab}^{\text{shell}} = S_{ab}^{\delta}(\lambda). \quad (23)$$

Further, the Einstein equations lead to

$$D_a S^{ab} = [T_{cd} n^c h^{db}], \quad (24)$$

where $D_a$ is the covariant derivative within $\Sigma$, and the brackets on the right hand side of the equation represent the difference in a quantity $\Psi$ evaluated on both sides of $\Sigma$: $[\Psi] := \Psi^{(2)} - \Psi^{(1)}$.

We consider a spherically symmetric dust shell whose surface stress-energy tensor is given by

$$S_{ab} = \sigma u^a u^b, \quad (25)$$

where $\sigma$ is the energy density per unit area, which is assumed to be non-negative, and $u^a$ can be regarded as the 4-velocity of the shell. We assume that this dust shell may have a non-vanishing charge. According to the Birkhoff’s theorem, the spacetime except on the shell itself is Reissner-Nordström spacetime, and hence the metric in the region $V_i$ is given as

$$ds^2 = -f_{(i)} dt_{(i)}^2 + f_{(i)}^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (26)$$

where $f_{(i)}$ is

$$f_{(i)} = 1 - \frac{2M_i}{r} + \frac{Q_i^2}{r^2}. \quad (27)$$

Here, note that all coordinates except for the time $t$ are common to both $V_1$ and $V_2$. We assume $M_i \geq |Q_i|$ and denote the roots of $f_{(i)} = 0$ by $r_{(i)}^\pm := M_i \pm \sqrt{M_i^2 - Q_i^2}$. In this coordinate system, the components of the 4-velocity $u^a$ are

$$u^\mu_{(i)} = \left( \frac{dt_{(i)}}{d\tau}, \frac{dr}{d\tau}, 0, 0 \right), \quad (28)$$
where $\tau$ is chosen so that $u^a u_a = -1$.

Using the normalization condition of the 4-velocity, Eq. (24) leads to

$$\mu := 4\pi \sigma r^2 = \text{const.} \quad (29)$$

The above equation means that the proper mass $\mu$ of the shell is conserved.

From Eq. (21) and the normalization condition of the 4-velocity, we obtain the energy equation for the shell as

$$\left(\frac{dr}{d\tau}\right)^2 + V_{\text{shell}} = 0, \quad (30)$$

where the effective potential $V_{\text{shell}}$ is written in the form

$$V_{\text{shell}} = -\left(\mathcal{E} - \frac{q\langle Q \rangle}{\mu r}\right)^2 + 1 - \frac{2\langle M \rangle}{r} + \frac{\langle Q \rangle^2}{r^2} - \left(\frac{\mu}{2r}\right)^2 + \left(\frac{q}{2r}\right)^2, \quad (31)$$

where

$$\mathcal{E} := \frac{M_2 - M_1}{\mu}, \quad \langle M \rangle := \frac{M_2 + M_1}{2}, \quad \langle Q \rangle := \frac{Q_2 + Q_1}{2} \quad \text{and} \quad q := Q_2 - Q_1. \quad (32)$$

Here note that $\mu\mathcal{E} = M_2 - M_1$ is equal to the Misner-Sharp energy concentrated on the shell [15, 16]. Thus, we may call $\mathcal{E}$ the specific energy of the shell, and we assume that it is positive. We can see that the effective potential (31) has almost the same form as that for a charged test particle (17) with $\ell_c = 0$, or equivalently, that for a spherical charged test shell. The differences between Eqs. (17) with $\ell_c = 0$ and (31) are regarded as the self-gravity and self-electric interaction terms.

Let us investigate whether the charged dust shell can asymptotically approach an outward null hypersurface as in the case of the charged test shell. The outside of the shell is the region $V_2$, and the BH horizon in this region is $r = r_+(2) := M_2 + \sqrt{M_2^2 - Q_2^2}$ which is the outward null hypersurface. Hence, we search for the condition which guarantees $r = r_+(2)$ to be the asymptote of the singular hypersurface $\Sigma$. This task is equivalent to solving $V(r_+(2)) = 0$ and $dV(r)/dr|_{r = r_+(2)} = 0$ in terms of the parameters, $Q_2$, $M_2$, $\mu$, $q$ and $\mathcal{E}$. We have

$$Q_2 = M_2, \quad (33)$$

$$q^2 - 2M_2q + 2M_2\mathcal{E}\mu - \mu^2 = 0. \quad (34)$$

Condition (33) implies that region $V_2$ is an extreme Reissner-Nordström spacetime. If the above conditions hold, the effective potential becomes

$$V_{\text{shell}} = -\frac{(\mathcal{E}^2 - 1)(r - M_2)^2}{r^2}. \quad (35)$$
Namely, if this charged dust shell is contracting, it asymptotically approaches the outward null hypersurface \( r = r_{\pm}^{(2)} = M_2 \) as long as \( \mathcal{E} > 1 \). Hence, it is likely that a BSW-type mechanism can occur in this system. It is worthwhile to note that this result is correct even if there is no central black hole, i.e., \( M_1 = Q_1 = 0 \).

V. EFFECT OF GRAVITY GENERATED BY COLLIDING SHELLS IN BSW PROCESS

FIG. 2: Schematic spacetime diagram of two spherical timelike shells. The two shells divide the spacetime into three regions \( V_1, V_2 \) and \( V_3 \).

In this section, we consider collisions of two spherical dust shells and discuss the CM energy at the collision event. These two shells divide the spacetime into three regions, \( V_1, V_2 \) and \( V_3 \), before the collision (see Fig. 2). We assume that the inner shell is the same as that considered in the preceding section, whose parameters satisfy conditions (33) and (34). We also assume that, as in Sec. III, the outer shell is composed of neutral dust and has the same specific energy \( \mathcal{E} \) as the inner charged shell. Because the outer shell is neutral, we have \( Q_2 = Q_3 \). Further, both shells are assumed to have an identical proper mass, \( \mu_{\text{out}} = \mu_{\text{in}} = \mu \).

The total stress-energy tensor of the shells is written in the form

\[
T_{ab}^\Sigma = \sigma_{\text{in}} u^a_{\text{in}} u^b_{\text{in}} \delta(\lambda_{\text{in}}) + \sigma_{\text{out}} u^a_{\text{out}} u^b_{\text{out}} \delta(\lambda_{\text{out}})
\]  

(36)
where \( \lambda_{\text{in}} \) and \( \lambda_{\text{out}} \) are the Gaussian normal coordinates of the inner and outer shells, respectively. Since the proper masses \( \mu \) of these shells are identical to each other, \( \sigma_{\text{in}} \) is equal to \( \sigma_{\text{out}} \) at the collision event.

Here, we introduce the “center of mass frame” at the collision event of the shells. It is composed of the orthonormal basis \((\bar{u}^a, \bar{n}^a, e^a_\theta, e^a_\phi)\) which satisfies the following condition

\[
T^{ab}_{\Sigma} \bar{u}_a \bar{n}_b = 0 = T^{ab}_{\Sigma} \bar{u}_a e^a_\theta = T^{ab}_{\Sigma} \bar{u}_a e^a_\phi.
\]  

(37)

The above condition means that the spatial components of the energy flux vanish in this frame. Then, the energy of the colliding shells in this frame, which is also called the CM energy \( E_{\text{cm}} \), is given by

\[
E_{\text{cm}} = \int T^{ab}_{\Sigma} \bar{u}_a \bar{u}_b r^2 \sin \theta d\bar{\lambda} d\theta d\phi = \sqrt{2\mu} \sqrt{1 - g_{ab} u^a_{\text{in}} u^b_{\text{out}}}, \]

(38)

where \( \bar{\lambda} \) is the proper length in the direction of \( \bar{n}^a \). As expected, the CM energy of the shells takes the same form as that of the particles. Details of this derivation are given in the Appendix.

As shown in the preceding section, the outward null hypersurface \( r = r^{(2)}_{\pm} = M_2 \) is the asymptote of the inner shell. Thus, as in the case of the test shells, the closer the relative velocity between the inner and outer shells approaches to the speed of light, the closer the collision event approaches to the BH horizon in \( V_2 \), i.e., \( r = r^{(2)}_{\pm} \). As a result, the CM energy at the collision event can be indefinitely large, even if the gravity of the colliding shells is taken into account. However, we should note that if the two shells collide inside the BH horizon in \( V_3 \), i.e., \( r \leq r^{(3)}_+ = M_3 + \sqrt{M_3^2 - Q_3^2} \), distant observers like us could not see the collision of these shells. We should also note that \( r^{(3)}_+ \) is larger than \( r^{(2)}_{\pm} \) by virtue of the gravity generated by the outer shell. Thus, the observable CM energy is less than that of the collision at \( r = r^{(3)}_+ \) given in accordance with Eq. (38) as

\[
E_{\text{cm}}(r = r^{(3)}_+) = \sqrt{2\mu} \sqrt{\frac{1}{\sqrt{\mu}} \sqrt{\frac{Y}{2\sqrt{E(3E\mu + 2M_1) + 2E\sqrt{\mu}}}}},
\]  

(39)

where the function \( Y \) is given by

\[
Y = \sqrt{\sqrt{\frac{\mu^2 - 1}{2(\mu E + M_1) + 2(\mu - 1) \left( \sqrt{\mu E(3\mu + 2M_1) + 2E\mu + M_1} \right)} + \mu
		\times \sqrt{-2(\mu E + M_1) + 2(\mu + 1) \left( \sqrt{\mu E(3\mu + 2M_1) + 2E\mu + M_1} \right)} + \mu
		- E \left( 2E \left( \sqrt{\mu E(3\mu + 2M_1) + 2E\mu + M_1} \right) + \mu \right)}.
\]

(40)
We can see from the above result that the CM energy at the collision event between these shells has an upper limit in the observable domain \( r > r_{+}^{(3)} \), and this limit is determined by the mass \( M_1 \) of the “central black hole” and the proper mass of the two shells. Here, we should note that in order that the outer shell overtakes the inner shell, \((u_{\text{out}}^r)^2 - (u_{\text{in}}^r)^2\) should be positive at \( r = r_{+}^{(3)} \). We can see that this condition holds from

\[
[(u_{\text{out}}^r)^2 - (u_{\text{in}}^r)^2]_{r=r_{+}^{(3)}} = [2\mathcal{E}(\mathcal{E} \mu + M_1) + \mu] \\
\times \left[ 6\mathcal{E}^2 \mu + 4\mathcal{E} \sqrt{\mathcal{E} \mu (3\mathcal{E} \mu + 2M_1)} + 2\mathcal{E} M_1 + \mu \right] \\
\times \left[ 4(\sqrt{\mathcal{E} \mu (3\mathcal{E} \mu + 2M_1)} + 2\mathcal{E} \mu + M_1)^2 \right]^{-1} \\
> 0. \tag{41}
\]

In the case that the mass \( M_1 \) of the central black hole is much larger than the proper masses of the shells \( \mu \), the observable CM energy becomes

\[
E_{\text{cm}} \simeq 2^{1/4} \mathcal{E}^{1/4} \sqrt{\mathcal{E} - \sqrt{\mathcal{E}^2 - 1}} M_1^{1/4} \mu^{3/4}. \tag{42}
\]

We can see from the above equation that, also in this case, the observable CM energy is not indefinitely large. Thus, when estimating the size of the observable CM energy, the gravity caused by the colliding objects must not be ignored, even if their initial energy is very small.

VI. SUMMARY AND DISCUSSION

In this paper, we studied a collision of two spherical dust shells in order to determine the effects of the gravity generated by the colliding objects in the BSW process. We have shown that a contracting charged dust shell can asymptotically approach the outward null hypersurface, even if its gravity is taken into account. If such a shell collides with another contracting shell, the relative velocity between them can be arbitrarily close to the speed of light. However, we found that the CM energy of two colliding shells has an upper limit in the observable domain, since the event horizon moves outward due to the gravity of the outer shell. Our results suggest that two particles can not collide with an arbitrarily high energy in the center of mass frame in the observable domain, if we take into account the effects of the gravity of the colliding particles.

Each dust shell can be regarded as an aggregation of many particles. Thus, it is worthwhile to study how a large CM energy of the constituent particles of the shells can be
achieved by a collision of two shells. We assume that the mass of the central black hole $M_1$ is much larger than that of a shell $\mu$, and the constituent particles have an identical rest mass $m$. Thus, the number $N$ of particles included in a shell is given by $N = \mu/m$. The energy $E$ of a particle in the center of mass frame is given by $E = E_{\text{cm}}/2N$. In this section, we denote the speed of light and the gravitational constant by $c$ and $G$, respectively. Then, from Eq. (42), we have

$$\frac{E}{mc^2} = \frac{E_{\text{cm}}}{2\mu c^2} \simeq \alpha \left( \frac{M_1}{\mu} \right)^{1/4},$$

where

$$\alpha = \frac{1}{2^{3/4}} \sqrt[4]{\mathcal{E}} - \sqrt{\mathcal{E}^2 - 1}. \tag{44}$$

We can easily see that $\alpha$ is a monotonically decreasing function of $\mathcal{E}$ for $\mathcal{E} \geq 1$ and $\alpha = 2^{-3/4} \simeq 0.59$ at $\mathcal{E} = 1$. We again note that we are interested in the case of $\mathcal{E} > 1$. Here, we introduce a parameter defined by

$$\beta = \frac{E}{m_{\text{pl}}c^2}, \tag{45}$$

where $m_{\text{pl}}$ is the Planck mass ($2.18 \times 10^{-5}$ g). When the shells collide with each other near the horizon $r = GM_1/c^2$, the mean separation between the constituent particles of the shells is given by

$$\ell \simeq \sqrt{\frac{4\pi(GM_1/c^2)^2}{2N}} \simeq 1.2 \left( \frac{\beta}{\alpha} \right)^2 \left( \frac{10^{-9}m_{\text{pl}}}{m} \right)^{3/2} \left( \frac{M_1}{M_{\odot}} \right)^{1/2} \text{cm}, \tag{46}$$

where $M_{\odot}$ is the solar mass, respectively. The above equation implies that the mean separation is much larger than the Compton wavelength if the mass $M_1$ of the central black hole is equal to $M_{\odot}$, if the mass $m$ of a constituent particle is equal to $10^{-9}$ times the Planck mass $m_{\text{pl}}$ and if the mean CM energy $E$ of a constituent particle is equal to or larger than the Planck energy, i.e., $\beta \geq 1$. This macroscopic separation implies that the dust approximation can be valid, even if $E$ is super-Planckian. Further, since the mean separation $\ell$ can be much smaller than $r = GM_1/c^2 \simeq 1.5 \times 10^5 (M_1/M_{\odot})$ cm, the continuum approximation can also be valid. Thus, although the mass of each constituent particle has to be rather large, we may say that the CM energy of the constituent particle can be super-Planckian within the dust shell approximation, even if it cannot be indefinitely large.

The CM energy of a particle, $E$, is rewritten from Eq. (43) as,

$$\beta = \frac{E}{m_{\text{pl}}c^2} = 1.4 \times 10^{-5} N^{-1/4} \alpha \left( \frac{m}{m_{\text{proton}}} \right)^{3/4} \left( \frac{M_1}{M_{\odot}} \right)^{1/4}, \tag{47}$$
where \( m_{\text{proton}} \) is the proton mass. Thus, within the dust shell approximation, since \( N \) is much larger than unity, the CM energy will be smaller than the Planck energy significantly if the mass of the constituent particles is equal to the proton mass. If we were allowed to extrapolate this equation to \( N \sim 1 \), we would obtain \( E \simeq 10^{-5}m_{\text{pl}} \) in the case \( m = m_{\text{proton}}, M_1 = 10M_\odot \) and \( \alpha = 0.59 \). Of course, since such an extrapolation might not be very accurate, much more detailed investigation will be necessary to evaluate an accurate value of the maximum CM energy.

**Acknowledgments**

We would like to thank Tomohiro Harada, Hideki Ishihara, Umpei Miyamoto and Takahiro Tanaka for useful discussions. H.T.’s work was supported in part by a Monbu Kagakusho Grant-in-aid for Scientific Research of Japan (No. 20540271).

**Appendix A: Center of mass frame for spherical shell**

The components of the orthonormal basis of the “center of mass frame” for two spherical shells considered in Sec.\( \text{V} \) are denoted as

\[
\begin{align*}
\vec{u}^a &= \left( \frac{dt}{d\pi}, \frac{dr}{d\pi}, 0, 0 \right), \\
\vec{n}^a &= \left( \frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, 0 \right), \\
e^{a}(\theta) &= \left( 0, 0, \frac{1}{r}, 0 \right), \\
e^{a}(\phi) &= \left( 0, 0, 0, \frac{1}{r \sin \theta} \right).
\end{align*}
\]

By using these basis vectors, the 4-velocities of the two shells are expressed in the form

\[
\begin{align*}
u^a_{\text{in}} &= \alpha_{\text{in}}\vec{u}^a + \beta_{\text{in}}\vec{n}^a, \\
u^a_{\text{out}} &= \alpha_{\text{out}}\vec{u}^a + \beta_{\text{out}}\vec{n}^a,
\end{align*}
\]

where \( \alpha_{\text{in}} \) and \( \alpha_{\text{out}} \) are assumed to be positive. The normalization of the 4-velocities leads to

\[
\alpha_{\text{in}}^2 - \beta_{\text{in}}^2 = 1 \quad \text{and} \quad \alpha_{\text{out}}^2 - \beta_{\text{out}}^2 = 1.
\]
By the normalization and orthogonal conditions, the normal vectors to the two shells are given by

\[ n^a_{\text{in}} = \beta_{\text{in}} \bar{u}^a + \alpha_{\text{in}} \bar{n}^a, \]  
(A8)

\[ n^a_{\text{out}} = \beta_{\text{out}} \bar{u}^a + \alpha_{\text{out}} \bar{n}^a. \]  
(A9)

The stress-energy tensor of the shells (36) at the collision event is expressed by using the basis of the CM frame as

\[ T^{ab}_{\Sigma} = \sigma \left[ \frac{\alpha_{\text{in}}^2}{\partial \lambda_{\text{in}} / \partial \lambda}^{-1} + \frac{\alpha_{\text{out}}^2}{\partial \lambda_{\text{out}} / \partial \lambda}^{-1} \right] \bar{u}^a \bar{u}^b + \left[ \frac{\alpha_{\text{in}} \beta_{\text{in}}}{\partial \lambda_{\text{in}} / \partial \lambda}^{-1} + \frac{\alpha_{\text{out}} \beta_{\text{out}}}{\partial \lambda_{\text{out}} / \partial \lambda}^{-1} \right] \left( \bar{u}^a \bar{n}^b + \bar{u}^b \bar{n}^a \right) \]

\[ + \left[ \frac{\beta_{\text{in}}^2}{\partial \lambda_{\text{in}} / \partial \lambda}^{-1} + \frac{\beta_{\text{out}}^2}{\partial \lambda_{\text{out}} / \partial \lambda}^{-1} \right] \bar{n}^a \bar{n}^b \delta(\bar{\lambda}), \]  
(A10)

where we have assumed that \( \sigma_{\text{in}} = \sigma_{\text{out}} = \sigma \) at the collision event.

The condition \( T^{ab}_{\Sigma} \bar{u}_a \bar{n}_b = 0 \) for the CM frame leads to

\[ \alpha_{\text{in}} \beta_{\text{in}} \left( \frac{\partial \lambda_{\text{in}}}{\partial \lambda} \right)_{\bar{\tau}}^{-1} + \alpha_{\text{out}} \beta_{\text{out}} \left( \frac{\partial \lambda_{\text{out}}}{\partial \lambda} \right)_{\bar{\tau}}^{-1} = 0. \]  
(A11)

From the definition of the orthonormal frame, we have

\[ n^a_{\text{in}} \frac{\partial \lambda_{\text{in}}}{\partial x^a} = 1 \]  
and

\[ u^a_{\text{in}} \frac{\partial \lambda_{\text{in}}}{\partial x^a} = 0, \]  
(A12)

and thus we have

\[ g^{ab} \frac{\partial \lambda_{\text{in}}}{\partial x^b} = n^a_{\text{in}}. \]  
(A13)

By using the above relation, Eqs. (A8) and (A9) lead to

\[ \left( \frac{\partial \lambda_{\text{in}}}{\partial \lambda} \right)_{\bar{\tau}} = n^a_{\text{in}} \bar{n}_a = \alpha_{\text{in}}. \]  
(A14)

In the same manner as above, we have

\[ \left( \frac{\partial \lambda_{\text{out}}}{\partial \lambda} \right)_{\bar{\tau}} = \alpha_{\text{out}}. \]  
(A15)

Thus, Eq. (A11) becomes

\[ \beta_{\text{in}} + \beta_{\text{out}} = 0. \]  
(A16)
By Eq. (A7), the above equation leads to
\[ \alpha_{\text{in}}^2 = \alpha_{\text{out}}^2. \] (A17)

Thus, we have \( \alpha_{\text{in}} = \alpha_{\text{out}} =: \alpha \). Further, by Eqs. (A5) and (A6) and one of the orthonormal conditions for \( u^a \) and \( n^a \), we obtain
\[ \alpha = \sqrt{\frac{1}{2}} \left( 1 - g_{ab} u_{\text{in}}^a u_{\text{out}}^b \right). \] (A18)

Hence we have
\[ T^{ab}_\Sigma \bar{u}_a \bar{u}_b = \sqrt{2} \sigma \sqrt{1 - g_{ab} u_{\text{in}}^a u_{\text{out}}^b} \delta(\bar{\lambda}). \] (A19)

The integration of the above quantity over \((\bar{\lambda}, \theta, \phi)\) gives “the CM energy” of the two shells.

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