Non-standard quantum (1+1) Poincaré group: a $T$–matrix approach

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Abstract

The Hopf algebra dual form for the non–standard uniparametric deformation of the (1+1) Poincaré algebra $iso(1,1)$ is deduced. In this framework, the quantum coordinates that generate $Fun_{qs}(ISO(1,1))$ define an infinite dimensional Lie algebra. A change in the basis of the dual form is obtained in order to compare this deformation to the standard one. Finally, a non–standard quantum Heisenberg group acting on a quantum Galilean plane is obtained.
1 Introduction

Quantum deformations were introduced as quantizations of some integrable models characterized by quadratic Poisson brackets [1]. The essential relation between these brackets and Lie bialgebras was established by Drinfel’d [2]. This link provides a framework to understand the quantization of such Poisson–Lie structures as a dual process to the (bialgebra) deformation of universal enveloping Lie algebras [3]. More recently, the $T$–matrix approach has been introduced in such a way that the Hopf algebra dual form $T$ summarizes duality in a universal (representation independent) setting [4, 5, 6]. The transfer matrices of certain integrable systems can be seen as particular realizations of the dual form $T$ and, conversely, the construction of new integrable models could be guided by the obtention of new $T$–matrices. In general, this approach provides a “canonical” formalism in which quantum objects are defined in a completely similar language to their classical counterparts. This “proper” setting provides a generalization of many group theoretical results to the non–commutative cases in a straightforward –although possibly cumbersome– way (for instance, see [7] for an example in the context of $q$–special function theory).

From an algebraic point of view, the $T$–matrix approach emphasizes the equivalent role that both quantum algebras and quantum groups play as Hopf algebra deformations and reveals the importance of solvable Lie algebras in this context, a fact that has been also pointed out in [8].

The Lie bialgebra structures compatible with a determined Lie algebra give a primary characterization for its quantum algebra deformations. So far, the deformations coming from the (non–degenerate) coboundary Lie bialgebra structures classified by Belavin and Drinfel’d [9] have been deeply studied (the so–called “standard” deformations [10, 11]). In some cases, their corresponding $T$–matrices have been deduced and, by using contraction methods, these results have been extended to some quantum non–semisimple groups [4, 5, 6, 12, 13].

In this paper we deal with the non–semisimple inhomogeneous algebra $iso(1, 1) \simeq t_2 \otimes so(1, 1)$ with classical commutation rules

$$[K, P_{\pm}] = \pm 2P_{\pm}, \quad [P_+, P_-] = 0.$$ (1.1)

As a real form, the algebra $iso(1, 1)$ is isomorphic to the (1+1) dimensional Poincaré algebra: $K$ and $P_\pm$ generate, respectively, the boosts and the translations along the light-cone. A non–standard coboundary Lie bialgebra $(iso(1, 1), \delta^{(n)})$ is generated by the classical $r$–matrix

$$r^{(n)} = K \wedge P_+.$$ (1.2)

As usual, $\delta^{(n)}(X) = [1 \otimes X + X \otimes 1, r^{(n)}]$ and $r^{(n)}$ verifies the Classical Yang–Baxter Equation (CYBE). The set of coboundary structures for this algebra is completed by the Lie bialgebra $(iso(1, 1), \delta^{(s)})$ given by

$$r^{(s)} = K \wedge (P_- + P_+),$$ (1.3)

that generates the standard deformation [14, 15] (this $r^{(s)}$ fulfills the modified CYBE). As a particular feature of the (1+1) Poincaré algebra [16, 17], there also
exists a non–coboundary \( \text{iso}(1,1) \) bialgebra with cocommutator
\[
\delta^{(nc)}(K) = 0, \quad \delta^{(nc)}(P_{\pm}) = P_{\pm} \wedge K. \tag{1.4}
\]
It can be easily checked that the quantum Poincaré algebra of \([18]\) is a deformation of \( \text{iso}(1,1) \) “in the direction” of \( \delta^{(nc)} \).

Starting from the quantization \( U_{w,\text{iso}}(1,1) \) (reviewed in Section 2) of the Lie bialgebra \( (\text{iso}(1,1), \delta^{(n)}) \), the construction of its Hopf algebra dual form is developed in Section 3. The main result of this section is that the non–standard quantum \( T \)–matrix is written as a product of usual exponentials. We recall that the \((1+1)\) Poincaré dual form given in \([12]\) (that corresponds to the quantum algebra \([18]\)) does contain \( q \)–exponential factors.

In Section 4, the non–standard quantum \( \text{ISO}(1,1) \) group is deduced. We emphasize a relevant difference of the result so obtained with respect to the \( T \)–matrices already known: in \([1,4,12,13]\) the classical dynamical variables (i.e., the group coordinates under a certain factorization of the group elements) generate a (solvable) finite dimensional Lie bialgebra \( (g_x, \delta) \), and the quantum group is constructed as a deformation \( U_q(g_x) \). In our case, the Poisson algebra \( g_x \) defined by the Sklyanin bracket coming from (1.2) is an infinite dimensional Lie algebra, and its quantization is obtained by applying the Weyl prescription onto this infinite–dimensional object. In particular, this enhances the differences between the \( T \)–matrix of \([12]\) and the one presented here.

On the other hand, the symmetric character of the \( T \)–matrix is used to transform in a consistent way the light–cone quantum coordinates into the quantum space–time cartesian coordinates. This allows us to compare the results here obtained to those of \([13]\) that correspond to a “standard” quantum \( \text{ISO}(1,1) \) group dual to the deformation generated by (1.3). By following the scheme developed in \([13]\), it is possible to understand \( U_{w,\text{iso}}(1,1) \) as a quantization of the Poisson–Lie similarity group \( SB(2) \).

To end with, Section 5 introduces a new quantum Heisenberg group obtained from the previous results by means of a formal transformation involving Clifford dual units. This quantum group acts naturally on a quantum Galilei plane, and can be easily embedded within the classification given in \([19]\).

2 Non–standard quantum \( \text{iso}(1,1) \) algebra

It is well known that in \( sl(2, \mathbb{R}) \) there exist two non trivial (coboundary) Lie bialgebra structures \([20]\). The first one is generated by \( r^{(s)} = J_+ \wedge J_- \) and underlies the (standard) Drinfel’d–Jimbo deformation for this algebra. The quantization of the non–standard bialgebra (with \( r^{(n)} = J_3 \wedge J_+ \)) was developed in \([21]\).

At a classical level, the involutive automorphism of \( sl(2, \mathbb{R}) \) given by \( S(J_3, J_\pm) = (J_3, -J_\pm) \) induces an Inönü–Wigner contraction of this algebra that is obtained as the limit \( \varepsilon \to 0 \) of the transformation \( (K, P_{\pm}) := \Gamma(J_3, J_\pm) = (J_3, \varepsilon J_\pm) \). The algebra
that arises under such a contraction is just (1.1). This procedure can be extended to the quantum case by taking into account the following “quantum” automorphism of $U_{z}sl(2, \mathbb{R})$

$$S_q(J_3, J_{\pm}; z) = (J_3, -J_{\pm}; -z).$$

(2.1)

Involution $S_q$ gives rise to a generalized Inönü–Wigner contraction:

$$(K, P_{\pm}; w) := \Gamma_q(J_3, J_{\pm}; z) = (J_3, \varepsilon J_{\pm}; z/\varepsilon),$$

(2.2)

where $K$ and $P_{\pm}$ and $w$ are the Lie generators and the deformation parameter of the contracted quantum algebra $U_{w}iso(1, 1)$. By applying the transformation $\Gamma_q$ onto the deformation given in [21] and making the limit $\varepsilon \to 0$, the quantum algebra $U_{w}iso(1, 1)$ is obtained (see [22]). It is worth recalling that this quantum algebra was firstly discovered in [23] with no reference to contraction procedures.

**Proposition 2.1.** Let $K$ and $P_{\pm}$ the infinitesimal generators of the Lie algebra $iso(1, 1)$. The Hopf algebra $U_{w}iso(1, 1)$ is given by the coproduct, counit, antipode

$$\Delta P_+ = 1 \otimes P_+ + P_+ \otimes 1,$$

$$\Delta P_- = e^{-wP_+} \otimes P_- + P_- \otimes e^{wP_+},$$

$$\Delta K = e^{-wP_+} \otimes K + K \otimes e^{wP_+},$$

$$\epsilon(X) = 0; \quad \gamma(X) = -e^{wP_+} X e^{-wP_+}, \quad \text{for } X \in \{K, P_{\pm}\},$$

$$\gamma(P_+) = -P_+, \quad \gamma(P_-) = -P_-, \quad \gamma(K) = -K + 2 \sinh(wP_+);$$

(2.3)

and the commutation relations

$$[K, P_+] = 2 \frac{\sinh(wP_+)}{w}, \quad [K, P_-] = -2 P_- \cosh(wP_+), \quad [P_+, P_-] = 0.$$  

(2.4)

(2.5)

This structure is a quantization of the non–standard Lie bialgebra of $iso(1, 1)$ generated by $r^{(n)} = K \wedge P_+$.

The center of $U_{w}iso(1, 1)$ is generated by

$$C_w = 2P_- \frac{\sinh(wP_+)}{w}.$$  

(2.6)

A classical $3 \times 3$ matrix representation of $iso(1, 1)$ is given by:

$$D(K) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad D(P_+) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad D(P_-) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

(2.7)

and a quantum matrix realization of $U_{w}iso(1, 1)$ coincides with the classical one:

$$D_q(X) = D(X), \quad X \in \{K, P_{\pm}\}.$$  

(2.8)

(2.9)
3 The universal $T$–matrix

Following \[4,5\] we consider the Hopf algebra dual form

$$T = \sum_{abc} p_{abc} \otimes X^{abc},$$

(3.1)

where $X^{abc}$ is a basis for $U_w iso(1,1)$ and its dual basis $p_{abc}$ generates $Fun_w(ISO(1,1))$ (sumation over repeated indices will be omitted from now on).

The $T$–matrix is constructed here starting from $U_w iso(1,1)$. A suitable basis to work is

$$A_+ = P_+, \quad A_- = e^{-wP_+} P_-, \quad H = e^{wP_+} K.$$  

(3.2)

The following coproduct and commutation rules are deduced from (2.3) and (2.6):

$$\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1,$$

$$\Delta(A_-) = e^{-2wA_+} \otimes A_- + A_- \otimes 1,$$

$$\Delta(H) = 1 \otimes H + H \otimes e^{2wA_+};$$

(3.3)

$$[H, A_+] = \frac{e^{2wA_+} - 1}{w}, \quad [H, A_-] = -2A_- e^{2wA_+}, \quad [A_+, A_-] = 0.$$  

(3.4)

The dual basis to $X^{abc}$ is defined by

$$\langle p_{abc}, X^{lmn} \rangle = \delta_a^l \delta_b^m \delta_c^n.$$  

(3.5)

and we set $\hat{a}_- = p_{100}, \hat{a}_+ = p_{010}$ and $\hat{\chi} = p_{001}$.

**Theorem 3.1.** The dual basis $p_{qrs}$ can be expressed in terms of the dual coordinates $\hat{a}_-, \hat{a}_+, \hat{\chi}$ in the form

$$p_{qrs} = \frac{\hat{a}_q \hat{a}_r \hat{\chi}_s}{q! \ r! \ s!},$$

(3.6)

and the Hopf algebra dual form $T$ reads

$$T = e^{\hat{a}_- A_-} e^{\hat{a}_+ A_+} e^{\hat{\chi} H}.$$  

(3.7)

The proof of this statement follows by considering the structure tensor $F$ that gives us the coproduct of an arbitrary element of $U_w iso(1,1)$

$$\Delta(X^{abc}) := F^{abc}_{lmn,qrs} X^{lmn} \otimes X^{qrs}$$

(3.8)

and, by means of duality, the product in $Fun_w(ISO(1,1))$

$$p_{lmn} p_{qrs} = F^{abc}_{lmn,qrs} p_{abc}.$$  

(3.9)
In particular, it is easy to see that
\[
F^{abc}_{000;000} = \delta^a_q \delta^b_r \delta^c_s,
\]
\[
F^{abc}_{lmn;000} = \delta^a_l \delta^b_m \delta^c_n,
\]
\[
F^{000}_{lmn;qrs} = \delta^0_l \delta^0_m \delta^0_n \delta^0_q \delta^0_r \delta^0_s.
\]

Two recurrence relations for \(F\) can be deduced by observing that
\[
\Delta(X^{abc}) = \Delta(A_-)\Delta(X^{(a-1)bc}),
\]
\[
\Delta(X^{abc}) = \Delta(A_+)\Delta(X^{a(b-1)c}),
\]
and using (3.3) and (3.4). In this way we find:
\[
F^{abc}_{lmn;qrs} = \sum_{k=0}^{m} F^{(a-1)bc}_{lkn;m,qrs} \left(-2w\right)^{m-k} \frac{(m-k)!}{(m-k)!}, \quad a, q, l \geq 1,
\]
\[
F^{abc}_{lmn;qrs} = F^{a(b-1)c}_{l(m-1)n,qrs} + F^{a(b-1)c}_{l(m-1)n,qrs}, \quad b, m, r \geq 1.
\]
The recurrence relation corresponding to \(\Delta(X^{abc}) = \Delta(X^{abc-1})\Delta(H)\) is much harder

to find in general. However, for our purposes, we shall only need some particular cases of it.

The relations (3.13) and (3.14), together with (3.10), lead to
\[
F^{abc}_{100;qrs} = a \delta^a_q \delta^b_r \delta^c_s, \quad a \geq 1,
\]
\[
F^{abc}_{010;qrs} = b \delta^a_q \delta^b_r \delta^c_s, \quad b \geq 1,
\]
and by simple considerations we can also find that
\[
F^{abc}_{lmn;001} = c \delta^a_l \delta^b_m \delta^c_{n+1}, \quad c \geq 1.
\]
These are the elements of \(F\) that are relevant in order to compute the dual basis. Now, with the aid of (3.8) and (3.13), the following relation holds
\[
p_{100} p_{(q-1)rs} = \hat{p}^{abc}_{100,(q-1)rs} p_{abc} = a \delta^a_q \delta^b_r \delta^c_s p_{abc} = q \hat{p}_{qrs},
\]
so
\[
p_{qrs} = \frac{\hat{a}_-}{q} p_{(q-1)rs} = \ldots = \frac{\hat{a}_q}{q!} p_{0rs}.
\]

Straightforward calculations based on (3.16) and (3.17) together with the fact that \(p_{000} = 1\) complete the proof of the theorem.

4 The quantum group \(\text{Fun}_w(ISO(1, 1))\)

The structure tensor \(F\) also allows us to deduce the commutation rules between the generators of \(\text{Fun}_w(ISO(1, 1))\). As a result, we enunciate the following proposition:
**Proposition 4.1.** The dual coordinates $\hat{a}_-, \hat{a}_+$ and $\hat{\chi}$ satisfy

$$[\hat{\chi}, \hat{a}_+] = w(e^{2\hat{\chi}} - 1), \quad [\hat{\chi}, \hat{a}_-] = 0, \quad [\hat{a}_+, \hat{a}_-] = -2w\hat{a}_-. \quad (4.1)$$

**Proof:** The commutation relation of any two elements of $\text{Fun}_w(ISO(1,1))$ is

$$[p_{lmn}, p_{qrs}] = (F_{lmn, qrs}^{abc} - F_{qrs, lmn}^{abc})p_{abc}. \quad (4.2)$$

The explicit expressions (4.1) for $[\hat{\chi}, \hat{a}_-] \equiv [p_{001}, p_{100}]$ and $[\hat{a}_+, \hat{a}_-] \equiv [p_{010}, p_{101}]$ are straightforwardly derived from the relations involving $F$ in the previous section. In spite of the absence of a third general recurrence relation, we can again find the particular values of $F$ involved in

$$[\hat{\chi}, \hat{a}_+] = (F_{001, 010}^{abc} - F_{010, 001}^{abc})p_{abc}. \quad (4.3)$$

From (3.16) we obtain that $F_{010, 001}^{abc} = b \delta_0^c \delta_0^b \delta_0^a$. On the other hand, $F_{001, 010}^{abc}$ gives the coefficient of the term $H \otimes A_+ \in \Delta(X^{abc})$. Such a term appears either when $b = c = 1$ (this contribution annihilates the previous $p_{011}$ term) or in the cases $a = b = 0$ and $c$ arbitrary (note that, since $[H, A_+]$ does not produce $H$, this generator cannot appear as a byproduct of reordering processes). As a consequence,

$$F_{001, 010}^{abc} p_{abc} = p_{011} + 2w p_{001} + 4w p_{002} + \ldots + 2^k w p_{00k} + \ldots \quad (4.4)$$

Therefore,

$$[\hat{\chi}, \hat{a}_+] = w \sum_{k=1}^{\infty} \frac{1}{k!} 2^k \chi^k = w(e^{2\hat{\chi}} - 1). \quad (4.5)$$

Note that two algebras (4.1) with parameters $w$ and $w'$ (both of them different from zero) are isomorphic.

Since the classical fundamental representation of $iso(1,1)$ (2.8) is a fundamental one for the quantum algebra $U_w iso(1,1)$, the specialization of the $T$–matrix (3.7) to this realization is just the classical $ISO(1,1)$ group element with non–commutative entries:

$$T^{D_q} = e^{\hat{a}_- D_q(A_-)} e^{\hat{a}_+ D_q(A_+)} e^{\hat{\chi} D_q(H)} = \begin{pmatrix} 1 & 0 & 0 \\ \hat{a}_- + \hat{a}_+ & \cosh 2\hat{\chi} & -\sinh 2\hat{\chi} \\ \hat{a}_- - \hat{a}_+ & -\sinh 2\hat{\chi} & \cosh 2\hat{\chi} \end{pmatrix}. \quad (4.6)$$

Thus, the multiplicative property of $T$ provides the coproduct for $\text{Fun}_w(ISO(1,1))$ (note that $D_q(P_\pm) = D_q(A_\pm), D_q(K) = D_q(H)$).

**Theorem 4.2.** The Hopf algebra $\text{Fun}_w(ISO(1,1))$ is given by the commutation rules (4.7), coproduct

$$\Delta(\hat{\chi}) = \hat{\chi} \otimes 1 + 1 \otimes \hat{\chi}, \quad \Delta(\hat{a}_-) = \hat{a}_- \otimes 1 + \cosh 2\hat{\chi} \otimes \hat{a}_+ + \sinh 2\hat{\chi} \otimes \hat{a}_+, \quad (4.7)$$

$$\Delta(\hat{a}_+) = \hat{a}_+ \otimes 1 + \cosh 2\hat{\chi} \otimes \hat{a}_- - \sinh 2\hat{\chi} \otimes \hat{a}_-;$$
counit and antipode

\[ \epsilon(X) = 0, \quad X \in \{ \hat{a}_+, \hat{a}_-, \hat{\chi} \}; \quad (4.8) \]

\[ \gamma(\hat{\chi}) = -\hat{\chi}, \quad \gamma(\hat{a}_+) = -e^{-2\hat{\chi}} \hat{a}_+, \quad \gamma(\hat{a}_-) = -e^{2\hat{\chi}} \hat{a}_-. \quad (4.9) \]

The “coincidence” between classical and quantum coordinates can be also extracted from the Poisson–Lie structure underlying this quantization. A Poisson–Hopf algebra of smooth functions on the group \( ISO(1,1) \) can be constructed by means of the classical non–standard \( r \)–matrix \( r^{(\alpha)} = K \wedge P_+ \). The Sklyanin bracket induced from this \( r \)–matrix is

\[ \{ f, g \} = r^{\alpha\beta} \left( X^L_\alpha f X^L_\beta g - X^R_\alpha f X^R_\beta g \right) \]

\[ = X^L_K f X^L_P g - X^L_P f X^L_K g - X^R_K f X^R_P g + X^R_P f X^R_K g. \quad (4.10) \]

We use the classical matrix representation similar to (4.6) to obtain the left and right invariant vector fields:

\[ X^L_K = \partial_\chi, \quad X^L_P = e^{2\chi} \partial_{a_+}, \quad X^L_P = e^{-2\chi} \partial_{a_-}, \quad (4.11) \]

\[ X^R_K = \partial_\chi + 2a_+ \partial_{a_+} - 2a_- \partial_{a_-}, \quad X^R_P = \partial_{a_+}, \quad X^R_P = \partial_{a_-}, \quad (4.12) \]

and we get the following result:

**Proposition 4.3.** The Poisson bracket

\[ \{ f, g \} = m \circ \left( (e^{2\chi} - 1) \partial_\chi \wedge \partial_{a_+} - 2a_- \partial_{a_+} \wedge \partial_{a_-} \right) (f \otimes g), \quad (4.13) \]

gives the structure of a Poisson–Hopf algebra to \( Fun(ISO(1,1)) \) (\( m(a \otimes b) = ab \)).

In particular for the coordinates \( a_+, a_- \) and \( \chi \) we get

\[ \{ \hat{\chi}, \hat{a}_+ \} = w(e^{2\chi} - 1), \quad \{ \hat{\chi}, \hat{a}_- \} = 0, \quad \{ \hat{a}_+, \hat{a}_- \} = -2w\hat{a}_-. \quad (4.14) \]

This means that the commutation rules (4.1) can be seen as a Weyl quantization \( \{ , \} \rightarrow w^{-1}[ , ] \) of the fundamental Poisson brackets given by (4.14) and that the classical coproduct does not change under quantization (compare to [4, 12]).

On the other hand, the FRT [24] prescription is consistent with Theorem 4.2. The (quadratic) commutation rules between the entries of the quantum matrix (4.6) that are derived from (4.1) coincide with the relations obtained via the FRT procedure and with the aid of the quantum \( R \)–matrix

\[ R = 1 \otimes 1 + wD_q(H) \wedge D_q(A_+). \quad (4.15) \]

Finally, it is worth recalling that in [4, 5, 12, 13] the \( T \)–matrix coordinates that generate \( Fun_q(G) \) close a solvable finite dimensional Lie (super) algebra (with the exception of the “esoteric” quantum \( GL(n) \) [5]). In particular, for the \( iso(1,1) \) case studied in [12] it is shown that

\[ [\pi, \pi_+] = -z \pi_+, \quad [\pi, \pi_-] = -z \pi_-, \quad [\pi_+, \pi_-] = 0. \quad (4.16) \]
This is no longer the case for the non–standard deformation (4.1). In fact, if we consider 
\[ \hat{a}_+, \hat{a}_- \text{ and } \hat{f}_m := e^{2m\hat{x}} (m \in \mathbb{Z}) \], we can identify 
\[ \text{Fun}_w(ISO(1,1)) \] with an infinite dimensional Lie algebra endowed with a Hopf algebra structure:
\[
\Delta(\hat{f}_m) = \hat{f}_m \otimes \hat{f}_m, \\
\Delta(\hat{a}_+) = \hat{a}_+ \otimes 1 + \hat{f}_1 \otimes \hat{a}_+, \tag{4.17}
\]
\[
\Delta(\hat{a}_-) = \hat{a}_- \otimes 1 + \hat{f}_{-1} \otimes \hat{a}_-. 
\]
Moreover, since
\[
\left[ \hat{f}_m, \hat{a}_+ \right] = 2w m (\hat{f}_{m+1} - \hat{f}_m), \tag{4.18}
\]
we have an infinite–dimensional Hopf subalgebra generated by \(\hat{f}_m\) and \(\hat{a}_+\) (note that \(H\) and \(A_+\) do generate a Hopf subalgebra aswell). It would be interesting to know whether this is a general feature of the non–standard deformations and to investigate the dependence of the dimensionality of the quantum algebra of coordinates with the parametrization. In principle, this infinite–dimensional aspect makes the approach formally closer to the realistic integrable models obtained without truncating the spectral parameter [5].

### 4.1 Standard versus non–standard deformations

In order to compare the results so far obtained to the ones given in [15] it is necessary to perform a “change of quantum coordinates” relating the “light cone” ones \((\hat{a}_+, \hat{a}_-)\) with the time and space quantum translations \((\hat{a}_1, \hat{a}_2)\). We recall the defining relations for the (1+1) quantum Poincaré group described in [15]. The group element was
\[
G = \begin{pmatrix}
1 & 0 & 0 \\
\hat{a}_1 & \cosh \hat{\theta} & \sinh \hat{\theta} \\
\hat{a}_2 & \sinh \hat{\theta} & \cosh \hat{\theta}
\end{pmatrix}, \tag{4.19}
\]
and the commutation rules between the coordinates read
\[
\left[ \hat{\theta}, \hat{a}_1 \right] = w' (\cosh \hat{\theta} - 1), \\
\left[ \hat{\theta}, \hat{a}_2 \right] = w' \sinh \hat{\theta}, \tag{4.20}
\]
\[
\left[ \hat{a}_1, \hat{a}_2 \right] = w' \hat{a}_1.
\]
This algebra can be also written in terms of generators \(\hat{g}_n = e^{n\hat{\theta}}\). The standard quantum (1+1) Poincaré plane of coordinates \((\hat{x}_1^{(s)}, \hat{x}_2^{(s)})\) characterized by \([\hat{x}_1^{(s)}, \hat{x}_2^{(s)}] = w' x_1^{(s)}\) is easily derived from these expressions.

Both standard and non–standard quantizations turn out to be Weyl quantizations of the classical coordinates preserving the multiplicative property of \(T^D_q\) and \(G\) (cfr. (L.13) and eq. (3.1) in [15]). This fact suggests the use of the classical relation between coordinates as an Ansatz for the change of quantum coordinates:
\[
\hat{\theta} = -2\hat{\chi}, \quad \hat{a}_1 = \hat{a}_+ + \hat{a}_-, \quad \hat{a}_2 = \hat{a}_- - \hat{a}_+ . \tag{4.21}
\]
This redefinition implies that $T^D_q$ and $G$ become identical. Hence, we have two different sets of commutation rules compatible with the same coproduct $\Delta(G) = G \otimes G$. In particular, the change (4.21) on (4.1) provides the non-standard brackets ($w' = -2w$)

$$
[\hat{\theta}, \hat{a}_1] = w' (\cosh \hat{\theta} - 1 - \sinh \theta),
$$

$$
[\hat{\theta}, \hat{a}_2] = w' (\sinh \hat{\theta} - (\cosh \hat{\theta} - 1)),
$$

$$
[\hat{a}_1, \hat{a}_2] = w' (\hat{a}_1 + \hat{a}_2). \tag{4.22}
$$

The non-standard quantum Poincaré plane (whose relations are invariant under the coaction defined by $G$ on a vector $(x_1^{(n)}, x_2^{(n)})$) is characterized by

$$
[\hat{x}_1^{(n)}, \hat{x}_2^{(n)}] = w' (x_1^{(n)} + x_2^{(n)}). \tag{4.23}
$$

A comparison between both quantizations shows that, in this quantum basis, the non-standard deformation seems to be constructed by adding some additional terms on the standard relations. It is also worth remarking the absolutely symmetrical role that both coordinates play in the non-standard case (see the quantum plane relation (4.23)). This kind of “symmetrical” quantization has been already related to the non-standard (2+1) deformations at a quantum algebra level [22]. In the last Section we shall analyze this symmetry from a more geometrical point of view.

### 4.2 A realization of $T$ with entries in $U_w(iso(1, 1))$

In general, given a Lie group $G$ it is rather difficult to control how a given change of basis in $Fun_q(G)$ can be paired to a transformation of the generators of $U_q(g)$, $g=\text{Lie}(G)$, in such a way that duality is preserved. In the sequel we see how the solution to this problem is naturally contained within the $T$–matrix approach.

Let us consider the change of basis (4.21). We are interested in finding a set $\{A_1, A_2, A_{12}\}$ of generators of $U_w(iso(1, 1))$ such that the Hopf algebra dual form $T$ is preserved:

$$
T = e^{\hat{a}_-} A_- e^{\hat{A}_+} e^{\hat{X}H} = e^{\hat{a}_1 A_1} e^{\hat{a}_2 A_2} e^{\hat{A}_{12}}. \tag{4.24}
$$

Since “universal” computations to relate both $U_w(iso(1, 1))$ bases are extremely cumbersome, we can specialize the $T$ matrix to a (fundamental) representation $Q$ of $Fun_w(ISO(1, 1))$ given by

$$
Q(\hat{\chi}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q(\hat{a}_-) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q(\hat{a}_+) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -2w & 0 & 0 & 0 \\ 0 & 0 & 2w & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.25}
$$
A straightforward computation shows that
\[
T^Q := e^{Q(\hat{a}_-)A_-} e^{Q(\hat{a}_+)A_+} e^{Q(\hat{\chi})H} = \begin{pmatrix}
1 & 0 & H & A_+ \\
0 & e^{-2w A_+} & 0 & A_- \\
0 & 0 & e^{2w A_+} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
(4.26)

From this point of view, \( U_w(iso(1,1)) \) is a quantum similarity group (it is easy to check that \( Q(\hat{a}_+) \), \( Q(\hat{a}_-) \) and \( Q(\hat{\chi}) \) close a similarity algebra \( sb(2) \) with noncommuting coordinates \( A_+ \), \( A_- \) and \( H \). Moreover, the coproduct (3.3) is reproduced by the multiplicative property \( \Delta(T^Q) = T^Q \otimes T^Q \).

The new quantum coordinates admit a representation \( Q(\hat{a}_1) \), \( Q(\hat{a}_2) \) and \( Q(\hat{\theta}) \) derived from (4.21) and (4.25). By computing the corresponding exponentials we obtain a second expression for the dual form
\[
T^Q := e^{Q(\hat{a}_1)A_1} e^{Q(\hat{a}_2)A_2} e^{Q(\hat{\theta})A_{12}},
\]
\[
T^Q = \begin{pmatrix}
1 & 0 & -2A_{12} & A_1 - A_2 \\
0 & e^{-2w(\hat{A}_1 - \hat{A}_2)} & 0 & 1/2w(e^{-2w(\hat{A}_1 - \hat{A}_2)} - 2e^{-2w \hat{A}_1} + 1) \\
0 & 0 & e^{2w(\hat{A}_1 - \hat{A}_2)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
(4.27)

The relation between the two \( U_w(iso(1,1)) \) bases is now clear
\[
A_+ = A_1 - A_2, \\
A_- = \frac{1}{2w}(e^{-2w(\hat{A}_1 - \hat{A}_2)} - 2e^{-2w \hat{A}_1} + 1), \\
H = -2A_{12}.
\]
(4.28)

Note that \( \lim_{w \to 0} A_- = A_1 + A_2 \) and we recover the usual classical change of basis.

### 4.3 A Poisson–Lie realization of \( U_w(iso(1,1)) \)

The quantum algebra \( U_w(iso(1,1)) \) can be considered as a Lie bialgebra deformation of \( iso(1,1) \). The cocommutator \( \delta^{(n)} \) is a coboundary that gives the first order term in \( w \) of the antisymmetrized part of the coproduct (3.3):
\[
\delta^{(n)}(A_+) = 0, \\
\delta^{(n)}(A_-) = 2w A_- \wedge A_+, \\
\delta^{(n)}(H) = 2w H \wedge A_+.
\]
(4.29)

As we mentioned above, if we take into account the representation (4.21) of the Hopf algebra \( Fun_w(ISO(1,1)) \), the commutation rules (4.1) are linearized into the Lie algebra \( sb(2) \)
\[
[Q(\hat{\chi}), Q(\hat{a}_+)] = 2w Q(\hat{\chi}), \quad [\hat{\chi}, \hat{a}_-] = 0, \quad [Q(\hat{a}_+), Q(\hat{a}_-)] = -2w Q(\hat{a}_-),
\]
(4.30)
and the antisymmetric part of the coproduct (4.7) is reduced to
\[
\hat{\delta}(Q(\hat{a}_+)) = 2Q(\hat{\chi}) \wedge Q(\hat{a}_+), \\
\hat{\delta}(Q(\hat{a}_-)) = -2Q(\hat{\chi}) \wedge Q(\hat{a}_-), \\
\hat{\delta}(Q(\hat{\chi})) = 0.
\] (4.31)

It is easy to check that \((sb(2), \hat{\delta})\) is the dual Lie bialgebra of \((iso(1, 1), \delta)\). Moreover, \((sb(2), \hat{\delta})\) is a coboundary bialgebra generated by the classical \(r\) matrix \(\hat{r} = \frac{1}{w}Q(\hat{a}_+) \wedge Q(\hat{\chi})\) that verifies the CYBE.

This result can be related to the dual picture developed in the previous section. Provided that \(A_+, A_-\) and \(H\) are considered as classical (commutative) coordinates, the matrix \(T^Q (4.26)\) can be viewed as a \(4 \times 4\) realization of the group \(SB(2)\). Then, we obtain:

**Proposition 4.4.** The fundamental Poisson brackets
\[
\{H, A_+\} = \frac{e^{2wA_+} - 1}{w}, \quad \{H, A_-\} = -2A_- e^{2wA_+}, \quad \{A_+, A_-\} = 0, \quad (4.32)
\]
endow \(Fun(SB(2))\) with a Poisson–Lie structure.

The coboundary structure of \((sb(2), \hat{\delta})\) is essential in order to construct the bracket (4.32) by using the Sklyanin procedure (4.10). The left and right invariant vector fields
\[
X^L_{Q(\hat{\chi})} = \partial_H, \quad X^L_{Q(\hat{a}_+)} = 2wH \partial_H + \partial_{A_+}, \quad X^L_{Q(\hat{a}_-)} = e^{-2wA_+} \partial_{A_-}, \quad (4.33)
\]
\[
X^R_{Q(\hat{\chi})} = e^{-2wA_+} \partial_H, \quad X^R_{Q(\hat{a}_+)} = -2w A_- \partial_{A_-} + \partial_{A_+}, \quad X^R_{Q(\hat{a}_-)} = \partial_{A_-}, \quad (4.34)
\]
are obtained from \(T^Q (4.26)\). The coproduct (3.3) is now the classical \(SB(2)\) group law. Moreover, the triangular nature of \(\hat{r}\) ensures the existence of a \(*_h\)–product quantizing (4.32). A similar construction has been fully developed in [15] for the standard deformation of the Heisenberg algebra.

## 5 A non–standard quantum Heisenberg group

It is well known that the points of the two dimensional Euclidean plane can be identified with the *complex numbers*, that is, each point with Cartesian coordinates \((x_1, x_2)\) corresponds to the complex number \(Z = x_1 + ix_2\). A similar description can be done for the \((1+1)\) Galilei and Poincaré planes by considering *dual numbers* and *double numbers*, respectively. Thus, points within the planes of these three geometries can be formally described in terms of numbers \(Z = x_1 + jx_2\), where the “imaginary” unit \(j\) is outside the real numbers, and can be equal either to the complex unit \(i\) \((i^2 = -1)\), to the dual unit \(\epsilon\) \((\epsilon^2 = 0)\) or to the double unit \(e\) \((e^2 = 1)\) [25, 26]. In this section we show how it is possible to extend formally the non–standard quantum Poincaré group studied in the previous Section to non–standard
quantum analogues of the (1+1) Euclidean and non–extended Galilei (Heisenberg) groups. This approach reveals also some traits of the non–standard Poincaré group which are somewhat hidden in the previous treatment.

If we perform the following formal change of basis (depending on the unit $j$) in (1.1):

$$
P_1 = \frac{1}{2}(P_- + P_+), \quad P_2 = \frac{1}{2}j(P_- - P_+), \quad J_{12} = -\frac{1}{2}jK,
$$

the commutation relations (1.1) turn into:

$$
[J_{12}, P_1] = P_2, \quad [J_{12}, P_2] = j^2P_1, \quad [P_1, P_2] = 0.
$$

The change (5.1) gives rise to a transformation of the group coordinates. By using the classical exponential factorization we have that

$$
e^{a - P_+}e^{a_+P_+}e^{\chi K} = e^{a - (P_+ + \frac{1}{j}P_2)}e^{a_+ (P_1 - \frac{1}{j}P_2)}e^{\chi (-\frac{2}{j}J_{12})},
$$

$$
e^{(a_+ - a_+)P_1\frac{1}{j}(a_+ - a_+)P_2}e^{-\frac{j}{2}\chi J_{12}} = e^{a_1P_1}e^{a_2P_2}e^{\theta J_{12}}.
$$

Hence, we define a quantum analogue of the expressions (5.3)

$$
\hat{a}_1 = \hat{a}_- + \hat{a}_+, \quad \hat{a}_2 = \frac{1}{j}(\hat{a}_- - \hat{a}_+), \quad \hat{\theta} = -\frac{2}{j}\hat{\chi},
$$

completed with a suitable defined transformation of the deformation parameter $w$ in the form

$$
v = -\frac{2}{j}w.
$$

Strictly speaking, expressions (5.4) and (5.5) are only defined for $j \neq \epsilon$, because the dual unit has no inverse. However, the Hopf algebras obtained by means of these substitutions are always well–defined:

**Proposition 5.1.** The Hopf algebra $Fun_v(ISO(1,1; j))$ has multiplication, coproduct, counit and antipode given by

$$
[\hat{\theta}, \hat{a}_1] = v\left((\cosh j\hat{\theta} - 1) - j \frac{\sinh j\hat{\theta}}{j}\right),
$$

$$
[\hat{\theta}, \hat{a}_2] = v\left(\frac{\sinh j\hat{\theta}}{j} - j \frac{\cosh j\hat{\theta} - 1}{j^2}\right),
$$

$$
[\hat{a}_1, \hat{a}_2] = v(\hat{a}_1 + j\hat{a}_2),
$$

$$
\Delta(\hat{\theta}) = \hat{\theta} \otimes 1 + 1 \otimes \hat{\theta},
$$

$$
\Delta(\hat{a}_1) = \hat{a}_1 \otimes 1 + \cosh j\hat{\theta} \otimes \hat{a}_1 + j \sinh j\hat{\theta} \otimes \hat{a}_2,
$$

$$
\Delta(\hat{a}_2) = \hat{a}_2 \otimes 1 + \cosh j\hat{\theta} \otimes \hat{a}_2 + \frac{\sinh j\hat{\theta}}{j} \otimes \hat{a}_1,
$$

$$
\epsilon(X) = 0, \quad X \in \{\hat{a}_1, \hat{a}_2, \hat{\theta}\};
$$
\[ \gamma(\hat{\theta}) = -\hat{\theta}, \]
\[ \gamma(\hat{a}_1) = -\cosh(j\hat{\theta})\hat{a}_1 + j\sinh(j\hat{\theta})\hat{a}_2, \quad (5.9) \]
\[ \gamma(\hat{a}_2) = -\cosh(j\hat{\theta})\hat{a}_2 + \frac{\sinh(j\hat{\theta})\hat{a}_1}{j}. \]

By taking into account the power series expressions for the functions \( \sinh(jy)/j, \)
\( \cosh(jy) - 1 \) and \( \cosh(jy) - 1)/j^2 \), it is easy to check that all of them are always
real functions for real values of the argument \( y \in \mathbb{R} \), no matter of whether \( j \) is the
complex, dual or double unit. Therefore, note that each commutator (5.6) is written
as a pair formed by a “real” term and a “pure imaginary” term.

The above statement enables us to define the quantum Euclidean, Poincaré and
Galilean planes as follows:

**Proposition 5.2.** Consider the coordinates of the quantum space \((\hat{x}_1, \hat{x}_2)\) verifying
\[ [\hat{x}_1, \hat{x}_2] = v(\hat{x}_1 + j\hat{x}_2), \quad (5.10) \]
then, the co–action
\[ \left( \begin{array}{c} 1 \\ \hat{x}_1' \\ \hat{x}_2' \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ \hat{a}_1 & \cosh j\hat{\theta} & j\sinh j\hat{\theta} \\ \hat{a}_2 & \frac{1}{j}\sinh j\hat{\theta} & \cosh j\hat{\theta} \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ \hat{x}_1 \\ \hat{x}_2 \end{array} \right), \quad (5.11) \]
preserves the commutation rules (5.10) for the \((\hat{x}_1', \hat{x}_2')\) coordinates.

In the Euclidean case \((j = i)\), the resultant Hopf algebra can be compared to
the standard one obtained in [15] leading to similar comments to the ones done in
§4.1. For the Heisenberg case, the fact that \( j^2 = \epsilon^2 = 0 \) implies that commutation
rules (5.6) can be written as quadratic relations:
\[ [\theta, a_1] = -v \epsilon \theta, \]
\[ [\theta, a_2] = v (\theta - \epsilon \theta^2/2), \]
\[ [a_1, a_2] = v (a_1 + \epsilon a_2). \quad (5.12) \]

If we rename generators in the form \( \hat{\theta} = \delta, \hat{a}_1 = \alpha \) and \( \hat{a}_2 = \beta \), this algebra is
included within the classification of quantum Heisenberg groups given in [19] (see
also [27]) as a (Type I) two–parametric deformation with \( z = \frac{1}{2}v\epsilon \) and \( p = q = -v \).

It is interesting to note that the standard quantum Heisenberg group [15] is
obtained by supressing in (5.12) all “imaginary” terms that include the dual unit \( \epsilon \). The same is true for the Poincaré (resp. Euclidean) cases if we annihilate the
imaginary parts of the expressions. In this sense, it is worth remarking that an
approach with dual units is quite different from a strict contraction procedure. Thus,
if we perform the contraction [13]
\[ \hat{a}_1' = \hat{a}_1, \quad \hat{a}_2' = \lambda^{-1}\hat{a}_2, \quad \hat{\theta}' = \lambda^{-1}\hat{\theta}, \quad v' = \lambda^{-1}v, \quad (\lambda \to 0), \quad (5.13) \]
in either the non–standard quantum Poincaré or Euclidean groups we get exactly the standard quantum Heisenberg group.

The geometrical meaning of the relations (5.6) and (5.10) should deserve further study; in this non–standard quantum space the commutator of the coordinates $(\hat{x}_1, \hat{x}_2)$ of a point is, in some sense, the point itself.

6 Concluding remarks

Non–standard quantum deformations have received scant attention compared to the standard ones. However, they present interesting features: at a purely mathematical level, the existence of a $\ast_h$–product that quantizes the Poisson–Lie group is always guaranteed for them. From a physical point of view, they are naturally adapted to the null–plane basis of the Poincaré algebra (see [22, 28] for the construction of such null–plane deformations of the (2+1) and (3+1) dimensional cases). This paper can be seen as a first step in the study of these null–plane quantum groups. At this respect, the symmetric appearance of both spatial and time quantum translations, together with the coboundary nature of the underlying Lie bialgebras are worth emphasizing.

It is also interesting to note that the essential features of the $T$–matrix approach can be obtained without computing all the components of the dual tensors. We have also shown that the quantum group $Fun_w(ISO(1,1))$ is, in fact, an infinite dimensional Hopf–Lie algebra. This structure is rather different to the universal enveloping algebras encountered when computing $T$–matrices in the literature.

Since the Sklyanin bracket defines a quadratic algebra in terms of the group entries, it seems difficult, in general, to deduce from it the algebraic properties that characterize the dynamical algebras coming from different $r$–matrices. However, the results contained in the previous pages show that, if we take into account the specialization of the Sklyanin bracket to the group coordinates, the comparison between two Sklyanin algebras defined on the same group is much more easy. The $T$–matrix provides a canonical framework to consider the corresponding non–commutative problem and, for the cases so far explored, the main result is: at the algebra level, quantum groups are just the Weyl quantization of these “fundamental Sklyanin algebras”, perhaps endowed with a deformed group law.

An interesting question to be posed now is whether the study of the non–standard $sl(2, \mathbb{R})$ deformation from the $T$–matrix point of view would confirm the previous analysis. Work on this line is currently in progress.

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