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**Numerical simulations of stochastic conformable space–time fractional Korteweg-de Vries and Benjamin–Bona–Mahony equations**

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**Abstract:** In this paper, we investigate the effect of white noise on conformable time and space fractional KdV and BBM equations. For this purpose, we convert these equations with external noise to homogeneous conformable time and space fractional KdV and BBM equations with defined transformation and then we solve them by modified Kudryashov method. We bring our numerical results in some figures in the last section.

**Keywords:** The modified Kudryashov method, Partial differential equations, Conformable fractional derivative, Nonlinear PDEs, Benjamin–Bona–Mahony equation, Korteweg-de Vries equation.

**1 Introduction**

L'Hospital expressed the idea of the fractional derivative in his letter to Leibniz [23], since that time it has attracted the attention of many researchers who have tried to propose a definition on a fractional derivative like Liouville, Caputo, Hadamard, Riemann [10, 18, 31]. Mathematical models established by using fractional derivatives have better overlapping with experimental data rather than the models with integer order derivatives. The most use of these fractional derivatives is their applications as the modelling term in many fields of sciences such as chemistry, physics, biology, financial modelling, control theory and other fields [24, 27, 30]. A new idea of fractional derivative involving two singular kernels has been suggested in [6] and presented its properties and applications to fractional differential equations. In [14], it has been investigated the Riemann-Liouville and the Caputo fractional derivative of the Dirac delta function and its Laplace transform to explore the solution for fractional-order system. In [44], short memory fractional derivatives and a short memory fractional modelling approach are introduced. In [37], the q-homotopy analysis transform method is used to the mathematical model of the cancer chemotherapy effect in the sense of Caputo fractional. P. Veeresha et al. have used the q-homotopy analysis transform method for finding the solution for a fractional Richards equation describing the water transport in unsaturated porous media and time-fractional coupled Burgers equations [35, 39]. In [36], the approximated analytical solutions for nonlinear dispersive fractional Zakharov-Kuznetsov equations are obtained with the help of two novel techniques, called fractional natural decomposition method and q-homotopy analysis transform method.

Khalil et al. [19] introduced a definition for the fractional derivative which is satisfied most properties of classical derivative despite the other definition of the fractional derivative. Its applications can be referred to the quantum mechanics and the fluid dynamics [4, 5, 8, 20, 32, 46]. In [45] B. Xin et al. developed Bertrand duopoly game to that based on conformable fractional derivative. S. He et al. solved the conformable fractional memcapacitor system by using conformable differential transform method [16]. A new class of smooth solutions for the Newton's law of cooling with conformable fractional derivative was gained [28].

When one or more terms of a differential equation are a stochastic process, it is called stochastic differential equation (SDE). SDEs have many applications in physics, biology, chemistry, mechanics and economics [9, 25, 34].

The Korteweg-de Vries (KdV) equation was concluded by Korteweg and de Vries as a model nonlinear equation for the propagation of shallow water waves along a canal and wave motion in plasmas [21], so the KdV equation is useful for the modelling biological and physical phenomenon. The third order nonlinear partial differential KdV equation happens in many fields of physics such as in water waves, plasmas and fiber optics [13, 17]. When weakly dispersive long waves would be with moving sur-
2 Conformable fractional derivative and its some properties

In this section, we express the definition and some properties of the conformable fractional derivative.

Definition 1. [19] Let $f : [0, \infty) \to \mathbb{R}$ be a function. The conformable derivative of order $\alpha$ is determined by

$$\left( T^\alpha f\right)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad \text{for all } t > 0 \text{ and } \alpha \in (0, 1].$$

Theorem 2. [19] Assume $\alpha \in (0, 1)$ and $f, g$ be differentiable at a point $t > 0$. Then

(i) $T^\alpha (af + bg) = aT^\alpha(f) + bT^\alpha(g)$, for all $a, b \in \mathbb{R}$,
(ii) $T^\alpha(\theta) = \theta$, for all constant functions $\theta$,
(iii) $T^\alpha(\theta) = 0$, for all constant functions $f(t) = \theta$,
(iv) $T^\alpha(fg) = fT^\alpha(g) + gT^\alpha(f)$,
(v) $T^\alpha\left( \frac{f}{g} \right) = \frac{\alpha fT^\alpha \left( \frac{g}{f} \right) - T^\alpha(g)}{g^2}$,
(vi) If $f$ is differentiable, then $T^\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

Theorem 3. [1] (Integration by parts) Assume $f, g : [a, b] \to \mathbb{R}$ are the functions such that $fg$ is differentiable. Then

$$\int_a^b f(t) T^\alpha(g)(t) \, dt(a, t) = f g(\frac{b}{a}) - \int_a^b g(t) T^\alpha(f)(t) \, da(t, a).$$

Someone who wants to know more properties of the conformable fractional derivative like Laplace transform, exponential functions, the Gronwall’s inequality, etc. can find them in [1].

3 Description of the modified Kudryashov method

We use the modified Kudryashov method to obtain the exact solutions of the nonlinear conformable derivative partial differential equations. In this section, we bring a brief review of the modified Kudryashov method [12].

Let $F$ be a nonlinear equation $u = u(x, t)$ and it has partial derivatives in the following form

$$F \left( u, T^\alpha_0(u), T^\alpha_1(u) \right) = 0,$$

where $\alpha \in (0, 1]$ be the derivative order and $T^\alpha_0(u)$ and $T^\alpha_1(u)$ are the conformable derivatives of $u$ with respect to $x$ and $t$.

Here, we bring the fundamental three steps of the modified Kudryashov method:
Step 1. We convert (3) from PDE to ODE. For that, we use the transformation (4)

\[ u(x, t) = U(\eta), \quad \eta = \frac{k_1}{a} x + \frac{k_2}{a_3} t, \]  

(4)

Now, we have this nonlinear ODE for new variable \( \eta \):

\[ R(U, U', U'', \ldots) = 0, \]  

(5)

where \( R \) is a function of \( u(\eta) \) and the ordinary derivatives are respect to \( \eta \).

Step 2. We suppose that the solution of (5) be shown as

\[ U(\eta) = \sum_{n=0}^{N} a_n P^n(\eta), \]  

(6)

here, \( a_i, 0 \leq i \leq N \) are constants such that \( a_N \neq 0 \). The \( N \) is determined by considering equation (5) such that we investigate homogeneous balancing between the highest order of derivatives and the nonlinear terms of this equation. Furthermore, \( P(\eta) = \frac{1}{1 - a_1} \) is a solution of the auxiliary equation (7)

\[ P'(\eta) = P(\eta)(P(\eta) - 1) \ln \alpha, \]  

(7)

where \( d \neq 0, a > 0, \alpha \neq 1 \).

Step 3. We substitute (6) into (5) along with (7). Finally, \( a_0, a_1, \ldots, a_N \) are gained by equating all the coefficients of the powers of \( P^i(\eta), (i = 0, 1, 2, \ldots) \) to zero. We solve this algebraic equations system by Maple and we can find amount of \( a_0, a_1, \ldots, a_N \) and constants and coefficients which are used in (4).

4 The stochastic CFDKdV

In this section, we consider the stochastic CFDKdV (1). First of all, we describe some of the concepts of the stochastic calculus.

Definition 4. [33] For \( t \in T, X_t \), a collection of random variables, is called a Stochastic process.

Brownian motion was discovered by Robert Brown in 1827 [33].

Definition 5. [33] A Brownian motion is a Stochastic process \( X_t \) if it has three following conditions for \( t \geq 0 \)

1. \( X_0 = 0 \).
2. \( X_t \) increases independently and steadily.
3. \( X_t \) has Gaussian increments i.e. it is normal with mean 0 and variance \( \sigma^2 t \).

When \( \sigma = 1 \), it is called standard Brownian motion [33].

Definition 6. [33] Let for \( t \geq 0 \), \( X_t \) defines a standard Brownian motion process, the white noise is \( \{dX(t), 0 \leq t < \infty \} \).

Wadati in [41] studied the stochastic KdV equation with classic derivative where \( \zeta(t) \) is the external noise which is dependent on time as follows

\[ \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x} = \zeta(t). \]  

We use his scheme for solving the stochastic CFDKdV equation (1). The stochastic CFDKdV equation (1) can be transformed into an unperturbed CFDKdV equation by the transformation

\[ u(x, t) = U(X, T) + W(T), \]  

(8)

\[ X = x - \int_0^t \frac{W(s)}{s^{1-\alpha}} \, ds, \quad T = t. \]  

(9)

We have for a function of \( X \) and \( T \):

\[ \frac{\partial^a}{\partial T^a} = \frac{\partial^a}{\partial X^a} \frac{\partial X^a}{\partial T^a} + \frac{\partial^a}{\partial T^a} \frac{\partial T^a}{\partial X^a} = \frac{\partial^a}{\partial X^a}, \]  

and

\[ \frac{\partial^a}{\partial T^a} = \frac{\partial^a}{\partial X^a} \frac{\partial X^a}{\partial T^a} - \frac{\partial^a}{\partial T^a} \frac{\partial T^a}{\partial X^a} = -6W(T) \frac{\partial^a}{\partial X^a} + \frac{\partial^a}{\partial T^a}. \]  

We apply the transformations in equations (8) and (9) to equation (1)

\[ \zeta(t) = \frac{\partial^a}{\partial T^a} U + \frac{\partial^a}{\partial X^a} W + \frac{\partial^a}{\partial T^a} U + \frac{\partial^a}{\partial T^a} W \]  

(10)

\[ = \frac{\partial^a}{\partial T^a} U + \frac{\partial^a}{\partial T^a} W - 6W \frac{\partial^a}{\partial X^a} U + \frac{\partial^a}{\partial T^a} U + \frac{\partial^a}{\partial T^a} W \]  

(11)

\[ = \frac{\partial^a}{\partial T^a} U + \frac{\partial^a}{\partial X^a} U + \frac{\partial^a}{\partial T^a} W + \frac{\partial^a}{\partial T^a} W + 6U \frac{\partial^a}{\partial X^a} U + \frac{\partial^a}{\partial T^a} W \]  

(12)

Defining \( \zeta(t) = \frac{\partial^a}{\partial T^a} U \), for \( \alpha \in (0, 1] \). Equation (10) becomes the unperturbed CFDKdV equation.

5 Unperturbed CFDKdV

We solve unperturbed CFDKdV equation (14) by the modified Kudryashov method,

\[ \frac{\partial^a}{\partial T^a} U + \frac{\partial^a}{\partial X^a} U + 6U \frac{\partial^a}{\partial X^a} U = 0, \]  

for \( \alpha \in (0, 1] \). (14)

The transformation \( U(x, t) = U(\eta) \) that \( \eta = k_1 \left( \frac{x^a}{t^a} \right) - k_2 \left( \frac{x^a}{t^a} \right) \), converts the CDKdV equation (14) to

\[ -k_2 U' + k_1^2 U'' + 6k_1 UU' = 0, \]  

(15)
where derivative is respect to $\eta$. We integrate equation (15), then we have

$$-k_2 U + k_1^3 U'' + 3k_1 U^2 = A_1, \quad (16)$$

where $A_1$ is a constant. We obtain $N = 2$, due to the homogeneous balance between $U^2$ and $U''$, the highest order of nonlinear terms and the highest order of derivatives, in (16). Then we assume the solution of equation (5) is

$$U(\eta) = a_0 + a_1 P(\eta) + a_2 P^2(\eta). \quad (17)$$

Substitution of equation (17) and its second derivative into equation (16) gives

$$P^4 \left( 6k_1^3 a_2 (\ln a)^2 + 3k_1 a_2^2 \right)
+ P^3 \left( 2k_1 a_1 (\ln a)^2 - 10k_1^3 a_2 (\ln a)^2 + 6k_1 a_1 a_2 \right)
+ P^2 \left( -3k_1 a_1 (\ln a)^2 + 4k_1^3 a_2 (\ln a)^2 + k_1^3 a_1 (\ln a)^2 - 2k_1 a_2 + 3k_1 a_1^2 + 6k_1 a_1 a_2 = 0, \right.

$$k_2 a_2 + 3k_1 a_1 + 6k_1 a_0 a_2 = 0,
\left. k_1 a_1 (\ln a)^2 - k_1 a_2 + 6k_1 a_0 a_1 = 0, \right)
 \right) + (-k_2 a_0 + 3k_1 a_0 - A_1) = 0. \quad (18)$$

We make all the coefficients of the powers of each $P(\eta)$ equal to zero. Hence

$$6k_1^3 a_2 (\ln a)^2 + 3k_1 a_2^2 = 0,$n
$$2k_1 a_1 (\ln a)^2 - 10k_1^3 a_2 (\ln a)^2 + 6k_1 a_1 a_2 = 0,$n
$$-3k_1 a_1 (\ln a)^2 + 4k_1^3 a_2 (\ln a)^2 = 0,$n
$$-k_2 a_2 + 3k_1 a_1 + 6k_1 a_0 a_2 = 0,$n
$$k_1 a_1 (\ln a)^2 - k_1 a_2 + 6k_1 a_0 a_1 = 0,$n
$$-k_2 a_0 + 3k_1 a_0 - A_1 = 0. \quad (18)$$

We have the above nonlinear system that contains five algebraic equations for $a_0$, $a_1$, $a_2$ and $k_2$. By solving this system,

$$a_0 = 0,$n
$$a_1 = 2k_1^2 (\ln a)^2,$n
$$a_2 = -2k_1^2 (\ln a)^2,$n
$$k_2 = k_1^2 (\ln a)^2. \quad (19)$$

So, the solution of equation (16) is

$$U(\eta) = 2k_1^2 (\ln a)^2 \frac{1}{1 + da^2} - 2k_1^2 (\ln a)^2 \frac{1}{(1 + da^2)^2}. \quad (19)$$

In the original variables, the solution for the equation (14) is

$$u(x, t) = 2k_1^2 (\ln a)^2 \frac{1}{1 + da} \left( \frac{k_1 (\ln a)^2}{1 + da} \right)^2. \quad (20)$$

6 The stochastic CFDBBM

The stochastic CFDBBM equation (2) can be transformed into an unperturbed CFDBBM equation by the transformation

$$u(x, t) = U(X, T) + W(T), \quad (21)$$

$$X = x - \int_0^t W(s) \frac{d\sigma}{s^{1-\alpha}} ds, \quad T = t. \quad (22)$$

We apply the transformations in equations (21) and (22) to equation (2)

$$\zeta(t) = \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^3 u}{\partial x^3} U U' + \frac{\partial^3 u}{\partial x^3} W W'. \quad (23)$$

Defining $\zeta(t) = \frac{\partial^\alpha W}{\partial T^\alpha}$, for $\alpha \in (0, 1]$. Equation (23) becomes the unperturbed CDBBM equation.

7 Unperturbed CFDBBM

We solve unperturbed CFDBBM equation (24) by the modified Kudryashov method,

$$\frac{\partial^\alpha U}{\partial T^\alpha} + \frac{\partial^3 U}{\partial x^3} U U' + \frac{\partial^3 U}{\partial x^3} W W' = 0, \quad (24)$$

The transformation $U(x, t) = U(\eta)$ that $\eta = k_1 \left( x - \frac{\alpha t}{\pi} \right) - k_2 \left( \frac{\alpha t}{\pi} \right)$, converts the CFDBBM equation (24) to

$$-k_2 U'' + k_1 U' + k_1^3 U''' + k_1 U U'' = 0, \quad (25)$$

where derivative is respect to $\eta$. We integrate equation (25), then we have

$$-k_2 U' + k_1 U + k_1^3 U''' + \frac{k_1}{2} U^2 = A_2, \quad (26)$$

where $A_2$ is a constant. We get $N = 2$, due to the homogeneous balance between $U^2$ and $U''$, the highest order of nonlinear terms and the highest order of derivatives, in (26). Then we assume the solution of equation (5) is

$$U(\eta) = a_0 + a_1 P(\eta) + a_2 P^2(\eta). \quad (27)$$

$\eta$ is the transformed variable.
Substitution of equation (27) and its second derivative into equation (26) gives

\[ p^6 \left( 6k_1^2a_2 \ln a \right)^2 + \frac{k_1}{2} a_2^2 \]
\[ + p^5 \left( 2k_1^2a_1 \ln a \right)^2 - 10k_1^2a_3 \ln a + k_1a_1a_2 \]
\[ + p^4 \left( -3k_1^2a_1 \ln a \right)^2 + 4k_1^2a_2 \ln a + k_1a_2 - k_2a_2 \]
\[ + \frac{k_1}{2}a_1^2 + k_1a_0a_2 \]
\[ + P \left( k_1^2a_1 \ln a \right)^2 + k_1a_1 - k_2a_1 + k_1a_0a_1 \]
\[ + (k_1a_0 - k_2a_0 + \frac{k_1}{2}a_0^2 - A_2) = 0. \]

We make all the coefficients of the powers of each \( P(\eta) \) equal to zero. Hence

\[ 6k_1^2a_2 \ln a + \frac{k_1}{2} a_2^2 = 0, \quad (28) \]
\[ 2k_1^2a_1 \ln a^2 - 10k_1^2a_3 \ln a + k_1a_1a_2 = 0, \quad (29) \]
\[ - 3k_1^2a_1 \ln a + 4k_1^2a_2 \ln a + k_1a_2 - k_2a_2 \]
\[ + \frac{k_1}{2}a_1^2 + k_1a_0a_2 = 0, \quad (30) \]
\[ k_1^2a_1 \ln a^2 + k_1a_1 - k_2a_1 + k_1a_0a_1 = 0, \quad (31) \]
\[ k_1a_0 - k_2a_0 + \frac{k_1}{2}a_0^2 - A_2 = 0. \quad (32) \]

We have the above nonlinear system that contains five algebraic equations for \( a_0, a_1, a_2 \) and \( k_2 \). By solving this system,

\[ a_0 = 0, \]
\[ a_1 = 12k_1^2 \ln a, \]
\[ a_2 = -12k_1^2 \ln a, \]
\[ k_2 = k_1 + k_1^2 \ln a. \]

So, the solution of equation (26) is

\[ U(\eta) = 12k_1^2 \ln a \left( \frac{1}{1 + da^\eta} - 12k_1^2 \ln a \right)^2 \frac{1}{(1 + da^\eta)^2}. \]

In the original variables, the solution for the equation (24) is

\[ u(x, t) = 12k_1^2 \ln a \left( \frac{1}{1 + da^\left( \frac{k_1}{2} x^\eta - \frac{k_1}{3} \ln a^\eta \right)^2 \eta} \right) \]
\[ - 12k_1^2 \ln a \left( \frac{1}{1 + da^\left( \frac{k_1}{2} x^\eta - \frac{k_1}{3} \ln a^\eta \right)^2 \eta} \right)^2. \]

\[ \text{(a) } \alpha = 1 \]
\[ \text{(b) } \alpha = 1 \text{ with external noise} \]

**Figure 1**: Solution of CFDKdV when \( \alpha = 1 \) with and without external noise.

8 Numerical experiments the stochastic CFDKdV and CFDBBM

In this section, we illustrate the effect of noise on the solutions of CFDKdV and CFDBBM equations with different values of \( \alpha \). Figures 1 to 10 are related to CFDKdV equation (1) and Figures 11 to 20 are related to CFDBBM equation (2). We show both the solutions of the stochastic and unperturbed forms of equations. Numerical results show that the effect of noise is greater in the small order of conformable fractional derivative, \( \alpha \), and in \( \alpha \) close to one, we see less effect of the random noise in the solution of CFDKdV and CFDBBM equations.
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Figure 2: Solution of CFDKdV when \( \alpha = 0.9 \) with and without external noise

Figure 3: Solution of CFDKdV when \( \alpha = 0.8 \) with and without external noise
Figure 4: Solution of CFDKdV when $\alpha = 0.7$ with and without external noise

Figure 5: Solution of CFDKdV when $\alpha = 0.6$ with and without external noise

Figure 6: Solution of CFDKdV when $\alpha = 0.5$ with and without external noise
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(a) $\alpha = 0.4$
(b) $\alpha = 0.4$ with external noise

Figure 7: Solution of CFDKdV when $\alpha = 0.4$ with and without external noise

(a) $\alpha = 0.3$
(b) $\alpha = 0.3$ with external noise

Figure 8: Solution of CFDKdV when $\alpha = 0.3$ with and without external noise

(a) $\alpha = 0.2$
(b) $\alpha = 0.2$ with external noise

Figure 9: Solution of CFDKdV when $\alpha = 0.2$ with and without external noise
Figure 10: Solution of CFKdV when $\alpha = 0.1$ with and without external noise

Figure 11: Solution of CFDBBM when $\alpha = 1$ with and without external noise

Figure 12: Solution of CFDBBM when $\alpha = 0.9$ with and without external noise
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Figure 13: Solution of CFDBBM when \( \alpha = 0.8 \) with and without external noise

Figure 14: Solution of CFDBBM when \( \alpha = 0.7 \) with and without external noise

Figure 15: Solution of CFDBBM when \( \alpha = 0.6 \) with and without external noise
Figure 16: Solution of CFDBBM when $\alpha = 0.5$ with and without external noise

Figure 17: Solution of CFDBBM when $\alpha = 0.4$ with and without external noise

Figure 18: Solution of CFDBBM when $\alpha = 0.3$ with and without external noise
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Figure 19: Solution of CFDBBM when $\alpha = 0.2$ with and without external noise

Figure 20: Solution of CFDBBM when $\alpha = 0.1$ with and without external noise
9 Conclusion

In this paper, stochastic CFDKdV and CFDBBM equations are considered which are the equations that an external noise inter into them. First, we have converted stochastic CFDKdV and CFDBBM equations to unperturbed equations by defining a transformation. Then the conformable homogeneous equations are solved with modified Kudryashov method analytically. We have shown the solutions of the stochastic CFDKdV and CFDBBM equations in figures. It can be concluded that as α, the order of conformable fractional derivative, gets smaller, the noise effect will be greater on the solution of CFDKdV and CFDBBM equations.

Let the differential equation (1) or (2) be in form of

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^\beta u}{\partial x^\beta} + 6u\frac{\partial^\beta u}{\partial x^\beta} = \xi(t),
\]

for α, β ∈ (0, 1], in the case of α = β the solution is presented in this paper, but for the case of α ≠ β the defined transformations in this paper could not work out and we should find a new way to solve it.

Also, applying the numerical methods which are used in [38, 40] for solving the equations discussed in this paper can be an interesting topic for future work.

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