ON INITIAL VALUE AND TERMINAL VALUE PROBLEMS FOR SUBDIFFUSIVE STOCHASTIC RAYLEIGH-STOKES EQUATION

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Abstract. In this paper, we study two stochastic problems for time-fractional Rayleigh-Stokes equation including the initial value problem and the terminal value problem. Here, two problems are perturbed by Wiener process, the fractional derivative are taken in the sense of Riemann-Liouville, the source function and the time-spatial noise are nonlinear and satisfy the globally Lipschitz conditions. We attempt to give some existence results and regularity properties for the mild solution of each problem.

1. Introduction. Stochastic partial differential equations (SPDEs) play an important role in the modeling of many phenomena in various fields, such as fluid mechanics, physics, astrophysics, hydrodynamics, biology, etc. [8, 22, 24, 14, 15]. In addition, fractional differential equations (FDEs) have received much attention recently with successful applications in various sciences such as engineering, physics, biology, etc. [13, 23, 25]. For some impressive works in recent time on stochastic fractional differential equations (SFDEs), the readers can refer to [9, 16, 17, 21, 27, 36, 35, 30, 29].

Let \( \mathcal{X} \subset \mathbb{R}^N \), \( N \geq 1 \), be a bounded domain (with sufficiently smooth boundary \( \partial \mathcal{X} \) for \( N \geq 2 \)). In this paper, we study two following problems for an SFDE named Rayleigh-Stokes equation (RSE).
• The initial value problem: Consider the following time-fractional Rayleigh-Stokes stochastic equation
\[
\left( \frac{\partial}{\partial t} - \Delta - \kappa \frac{\partial}{\partial t^\alpha} \Delta \right) u = F(t, u) + \sigma(t, u) \dot{W}(t), \quad \text{on} \quad J \times \mathcal{X},
\]  
(1)
where \( J = (0, T) \), \( \kappa \) is a positive constant and \( 0 < \alpha < 1 \), \( \Delta \) is the Laplacian operator. This equation is subjected to the Dirichlet boundary condition
\[
u(t, x) = 0, \quad (t, x) \in J \times \partial \mathcal{X},
\]  
(2)
and the initial value condition
\[
u(0, x) = \psi(x), \quad x \in \mathcal{X}.
\]  
(3)
Here, \( \partial^\alpha / \partial t^\alpha \) stands for the Riemann-Liouville fractional derivative
\[
\frac{\partial^\alpha}{\partial t^\alpha} v(t) := \frac{\partial}{\partial t} \int_0^t \mu_{1-\alpha}(s) v(t-s, x) ds, \quad \mu_\beta(s) := \frac{1}{\Gamma(\beta)} s^{\beta-1}, \quad (\beta > 0),
\]  
(4)
whereupon \( \Gamma \) is the Gamma function, see [23]. \( \{W(t, \cdot)\} \) represents a standard Wiener process (see Subsection 2.2), the generalized derivative \( \dot{W}(t) = \frac{\partial}{\partial t} W(t) \) describes a white noise, the nonlinear term \( F \) and the time-spatial-noise \( \sigma \) will be specified later. The problem of finding \( u \) satisfying (1),(2),(3) is called an initial value problem.

• The terminal value problem: Let the initial condition (3) be replaced by the final value condition
\[
u(T, x) = \varphi(x), \quad x \in \mathcal{X}.
\]  
(5)
Then, the problem of finding \( u \) satisfying (1),(2),(5) is called a final value problem.

The Rayleigh-Stokes problem arises in modeling the behavior of some non-Newtonian fluids. The Riemann-Liouville derivative in this model has been found to be flexible in describing viscoelastic behaviors of the flow [1, 7, 26]. Let us propose some details on the history of two problems we are interested in and some related studies. We begin with the deterministic model of the initial value problem (when the term \( \sigma(t, u) \dot{W}(t) \) disappears). For the homogeneous problem, the Sobolev regularities for both smooth and nonsmooth initial data was established in [1]. For the linear case, Shen [26] used fractional Laplace transform to contribute the exact solution. In [34], the exact solutions was constructed by using the eigenfunction expansion on a rectangular domain. The readers can refer to some other studies on the exact solutions for the Rayleigh-Stokes problem in [10],[31]. In [33], Zaky extended and developed Legendre-tau algorithms for solving one-dimensional and two-dimensional fractional Rayleigh-Stokes problem for a heated generalized second grade fluid. Some other works [4, 5, 6, 12] used different numerical methods to study the Rayleigh-Stokes problem with a heated generalized second grade fluid. In contract to the initial value problem, the study of the terminal value problem is still limited. We can list here some papers concerned with the deterministic model of this kind of problem [19, 20, 28].

To the best of our knowledge, both the initial value problem and terminal value problem for the stochastic Rayleigh-Stokes equation driven by Wiener process have not been investigated in the literature. This motivated us to study the existence, uniqueness and regularity of each of two stochastic problems. It is the fact that the analysis technique used in the deterministic case cannot be applied in the stochastic case. The problems of finding suitable spaces for the solutions in the stochastic case...
is our challenges since the solutions are more complex and it is required to cleverly
improve the estimates for the solution operators.

The main contributions of this paper are as follows. In Section 2, we provide some
preliminary results including some solution spaces, fractional calculus and stochastic
analysis techniques. The existence, uniqueness result and some regularity properties
of the mild solution to the initial value problem (1),(2),(3) is investigated in Section
3. In Section 4, we continue to study the existence, uniqueness, regularity of the
mild solution to the terminal value problem (1),(2),(5).

2. Preliminaries.

2.1. Hilbert scale spaces. For convenience, we denote by $L^p := L^p(0, T)$, $L^p_r :=
L^p(0, T)$, $p \geq 1$. By $L^r := L^r(\mathcal{X})$ \((r \geq 1), H^\xi := H^\xi(\mathcal{X}) \ (\xi \geq 0), \) we denote
the Lebesgue and the Sobolev-Slobodeckij spaces respectively, and $H^\xi_0(x)$
the closure of $C^\infty(\mathcal{X})$ in $H^\xi_0$. Suppose that the operator $A = -\Delta$ is defined on the domain
$D(A) = H^1 \cap H^2$. Since $\mathcal{X}$ is a bounded domain with sufficiently smooth boundary
and $A$ is self-adjoint on $L^2$, it is well-known there exists a sequence of eigenfunctions
$\phi_j \in D(A)$ and a sequence of corresponding eigenvalues $\lambda_j$ such that
$A\phi_j(x) = \lambda_j \phi_j(x)$ on $\mathcal{X}$. Additionally, the sequence \{\lambda_j\}_{j=1,2,...} is positive, non-decreasing,
and tends to infinity as $j \to \infty$, \{\phi_j\}_{j \geq 1} forms an orthonormal basis of $L^2$.

Next, we introduce the Hilbert scale spaces which will be defined by basing on
$\{\phi_j\}_{j=1}$. For a non-negative number $\xi$, we define by $H^\xi$ the space of all functions
$v \in L^2$ such that

$$\|v\|_{H^\xi} := \left( \sum_{j \geq 1} \lambda_j^{2\xi} |\langle v, e_j \rangle|^2 \right)^{\frac{1}{2}} < \infty,$$

where $\langle .., .. \rangle$ is the usual inner product of $L^2$. If $\xi = 0$, one can see $H^0 = L^2$. Let $H^{-\xi}$
be the dual space of $H^\xi$. The fractional operator (see [3]) $A^\xi : H^{\xi/2} \to H^{-\xi/2}$
be defined as

$$A^\xi v := \sum_{j \geq 1} \lambda_j^{\xi} \langle v, e_j \rangle e_j, \ \ v \in H^{\xi/2}.$$

2.2. Stochastic processes on the Hilbert scale spaces. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in \mathcal{T}})$
be a filtered complete probability space, which satisfies that the filtration \{\mathcal{F}_t\}_{t \in \mathcal{T}}
is a right continuous increasing family and that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.
Throughout this paper, for each $\xi \geq 0$, notation $\{W^\xi(t, .)\}_{t \in \mathcal{T}}$ describes a $H^\xi$-valued
Wiener process which is defined on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in \mathcal{T}})$ with a finite trace covariance
operator $Q_\xi$ on $H^\xi$. Let us denote \{\beta_j^{(\xi)}\}_{k=1} the spectrum of $Q_\xi$, i.e. $Q_\xi^{\phi_j^{(\xi)}} = \beta_j^{(\xi)} \phi_j^{(\xi)}$, then the finite trace of $Q_\xi$ is $\text{Tr}(Q_\xi) = \sum_{j=1}^{\infty} \beta_j^{(\xi)} < \infty$. According to the
above setting, $\{W^\xi(t, .)\}_{t \in \mathcal{T}}$ can be expanded in the form

$$W^\xi(t, x) = \int_0^\infty \sum_{j=1}^{\infty} \beta_j^{(\xi)}(t) \phi_j^{(\xi)}(x),$$

whereupon $\{\phi_j^{(\xi)}(t)\}_{j=1}^{\infty}$ is a sequence of mutually independent one-dimension standard
Wiener processes.

Let us recall the definition of the expectation. For a random variable $\chi : \Omega \to H^\xi$
and a real number $q \geq 2$, we note that $\mathbb{E}\|\chi\|^q_{H^\xi} := \int_\Omega \|\chi(\omega)\|^q_{H^\xi} d\mathbb{P}(\omega)$. Notation
$L^q : H^\xi := L^q(\Omega; H^\xi)$ denotes the space of all $H^\xi$-valued random variables endowed
with the norm \( \|v\|_{L^2_2(H^s_2)}^q := \mathbb{E}\|v\|_{H^s_2}^q < \infty \), for all \( v \in L^2_2(H^s_2) \). Let us define by 
\( L_0^2(\hat{H}^s_2, \hat{H}^s_2') \) the space of all linear bounded operators \( \Psi \) from \( Q^{1/2}_{s/2} \hat{H}^s_2 \) to \( \hat{H}^s_2' \) such that \( \Psi Q^{1/2}_{s/2} \) is a Hilbert-Schmidt operator from \( \hat{H}^s_2 \) to \( \hat{H}^s_2' \) where its norm is given by

\[
\|\Psi\|_{L^2_0(\hat{H}^s_2, \hat{H}^s_2')} := \left\{ \sum_{j=1}^{\infty} \left\| \Psi Q^{1/2}_{s/2} \phi_j \right\|_{H^s_2'}^2 \right\}^{1/2} < \infty, \quad \text{for } \Psi \in L^2_0(\hat{H}^s_2, \hat{H}^s_2').
\]

In the case \( \xi = \xi' \), we will write \( L^2_0(\hat{H}^s_2) \) in stead of \( L^2_0(\hat{H}^s_2, \hat{H}^s_2') \), and just \( L^2_0 \) if \( \xi = 0 \).

The following lemma is useful to estimate the stochastic integrals, and is a straightforward consequence of the Burkholder-Davis-Gundy inequality proved in Da Prato and Zabczyk [2, Lemma 7.2, page 182].

**Lemma 2.1** (Burkholder-Davis-Gundy-type inequality). *Given \( q \geq 2 \), \( t_1, t_2 \in \mathcal{J} \), and let \( \{\Phi(t)\}_{t_1 \leq t \leq t_2} \) be a \( L^2_0(\hat{H}^s_2, \hat{H}^s_2') \)-valued predictable stochastic process satisfying*

\[
\mathbb{E}\left[ \left( \int_{t_1}^{t_2} \|\Phi(s)\|_{L^2_0(\hat{H}^s_2)}^2 ds \right)^{q/2} \right] < \infty.
\]

*Then, it holds that*

\[
\mathbb{E}\left[ \left\| \int_{t_1}^{t_2} \Phi(s)dW_\xi(s,.) \right\|_{\hat{H}^s_2}^q \right] \leq C(q) \mathbb{E}\left[ \left( \int_{t_1}^{t_2} \|\Phi(s)\|_{L^2_0(\hat{H}^s_2)}^2 ds \right)^{q/2} \right],
\]

*where the constant \( C(q) \) is equal to \( \left( \frac{q(q-1)}{2} \right)^{1/2} \left( \frac{q}{q-1} \right)^{(q-2)/2} \).

### 2.3. Solution spaces and assumptions on nonlinearities.

Let \( \xi \geq 0 \) and \( q \geq 2 \). By \( L^\infty_\mathcal{J} L^2_0 H^s_2 := L^\infty([0, T]; L^q(\Omega, H^s_2)) \), we denote the space of all \( \mathcal{F}_t \)-adapted measurable processes \( w \) from \( \mathcal{J} \) to \( L^2_0 H^s_2 \) such that

\[
\|w\|_{L^\infty_\mathcal{J} L^2_0 H^s_2} := \sup_{t \in \mathcal{J}} \|w(t)\|_{L^2_0 H^s_2} < \infty.
\]

For \( \rho > 0 \), we introduce the following space

\[
L^\infty_{\mathcal{J}, \rho} L^2_0 H^s_2 := \left\{ w \in L^\infty_\mathcal{J} L^2_0 H^s_2 : \quad \|w\|_{L^\infty_{\mathcal{J}, \rho} L^2_0 H^s_2} := \sup_{t \in \mathcal{J}} e^{-\rho t} \|w(t)\|_{L^2_0 H^s_2} < \infty \right\},
\]

where \( \mathcal{J}_\rho = \mathcal{J} \setminus \{0\} \).

We define by \( C_\rho L^2_0 H^s_2 := C(\mathcal{J}, L^2_0 H^s_2) \) the space of all continuous \( \mathcal{F}_t \)-adapted measurable processes \( w \) from \( \mathcal{J}_\rho \) to \( L^2_0 H^s_2 \) such that

\[
\|w\|_{C_\rho L^2_0 H^s_2} := \sup_{t \in \mathcal{J}_\rho} \|w(t)\|_{L^2_0 H^s_2} < \infty.
\]

Similarly, the notation \( C_{t, *, \rho} L^2_0 H^s_2 := C(\mathcal{J}_*, L^2_0 H^s_2) \), presents the set of all continuous \( \mathcal{F}_t \)-adapted measurable processes \( w \) from \( \mathcal{J}_* \) to \( L^2_0 H^s_2 \). By \( C_{t, * \rho} L^0_0 H^s_2 := C^\theta(\mathcal{J}_*; L^0_0 H^s_2) \), we introduce the following Hölder continuous space of exponent \( \theta \)

\[
C_{t, * \rho} L^0_0 H^s_2 = \left\{ w \in C_{t, * \rho} L^0_0 H^s_2 : \quad \|w\|_{C_{t, * \rho} L^0_0 H^s_2} := \sup_{r, t \in \mathcal{J}_*} \frac{\|w(r) - w(t, .)\|_{L^0_0 H^s_2}}{|r - t|^\theta} < \infty \right\}.
\]
By \( Z_{β}^0;L_{β}^{q} \mathcal{H}_{ξ}^0 := Z^{β}(\mathcal{J};L_{β}^{q} \mathcal{H}_{ξ}^0) \), we denote the space of all processes \( w \) in the space \( C_{t},L_{β}^{q} \mathcal{H}_{ξ}^0 \) such that \( \lim_{t \to 0^{+}}t^{β} E \|w(t,.)\|_{L_{β}^{q} \mathcal{H}_{ξ}^0}^{q} \) exists finitely and

\[
\|w\|_{Z_{β}^0;L_{β}^{q} \mathcal{H}_{ξ}^0} := \sup_{t \in \mathcal{T}} t^{β} \|w(t,.)\|_{L_{β}^{q} \mathcal{H}_{ξ}^0}^{q} = \left( \sup_{t \in \mathcal{T}} t^{β} \|w(t,.)\|_{L_{β}^{q} \mathcal{H}_{ξ}^0}^{q} \right)^{\frac{1}{q}} < \infty.
\]

Next, let us introduce the following definitions for the nonlinearity and the time-spatial noise.

**Definition 2.2** (Globally Lipschitz nonlinearity). Let \( q \geq 2 \) and \( ξ \geq 0 \). The nonlinearity \( F \) is said to be globally Lipschitz and written by \( F \in LG_{β}(L_{β}^{q} \mathcal{H}_{ξ}^0) \) if \( F : \mathcal{T} \times \mathcal{X} \times L_{β}^{q} \mathcal{H}_{ξ}^0 \to L_{β}^{q} \mathcal{H}_{ξ}^0 \) and there exists a positive function \( K_{1} := K_{1}(t) \) on \( \mathcal{T} \) such that

\[
E \|F(t,.,w^{\dagger}(t,.)) - F(t,.,w^{\ddagger}(t,.))\|_{L_{β}^{q} \mathcal{H}_{ξ}^0}^{q} \leq K_{1}(t) E \|w^{\dagger}(t,.) - w^{\ddagger}(t,.)\|_{L_{β}^{q} \mathcal{H}_{ξ}^0}^{q},
\]

for all \( w^{\dagger},w^{\ddagger} \) in \( L_{β}^{q} \mathcal{H}_{ξ}^0 \) and \( t \in \mathcal{T} \).

**Definition 2.3** (Globally Lipschitz time-spatial-noise). Let \( q \geq 2 \) and \( ξ \geq 0 \). The time-spatial-noise \( σ \) is said to be globally Lipschitz and written by \( σ \in LG_{β}(L_{β}^{q} \mathcal{H}_{ξ}^0; L_{β}^{q}(\mathcal{H}_{ξ}^0)) \) if \( σ : \mathcal{T} \times L_{β}^{q} \mathcal{H}_{ξ}^0 \to L_{β}^{q}(\mathcal{H}_{ξ}^0) \) and there exists a positive function \( K_{2} := K_{2}(t) \) on \( \mathcal{T} \) satisfying

\[
E \|σ(t,w^{\dagger}) - σ(t,w^{\ddagger})\|_{L_{β}^{q}(\mathcal{H}_{ξ}^0)}^{q} \leq K_{2}(t) E \|w^{\dagger} - w^{\ddagger}\|_{L_{β}^{q}(\mathcal{H}_{ξ}^0)}^{q},
\]

for all \( w^{\dagger},w^{\ddagger} \) in \( L_{β}^{q} \mathcal{H}_{ξ}^0 \) and \( t \in \mathcal{T} \).

3. Existence, regularity of the solution of the initial value problem.

3.1. Mild solution for initial value problem. In this subsection, we give the definition of mild solution to Problem (1)-(3). To do this, let us first consider the following deterministic problem

\[
\begin{cases}
\left( \frac{∂}{∂t} - \Delta - \kappa \frac{∂^{α}}{∂x^{α}} \Delta \right) u &= F(t,u), \quad (t,x) \in \mathcal{T} \times \mathcal{X}, \\
u(t,x) &= 0, \quad (t,x) \in \mathcal{T} \times \partial \mathcal{X}, \\
u(0,x) &= ψ(x), \quad x \in \mathcal{X},
\end{cases}
\]

(6)

We now express \( u(t,.) \) in the form \( u(t,x) = \sum_{j=1}^{∞} u_{j}(t)φ_{j}(ξ)(x) \), where we denote \( u_{j}(t) := \langle u(t,.),φ_{j}(ξ) \rangle_{\mathcal{H}_{ξ}^0} \). Due to the arguments in [1], the coefficients \( u_{j}(t) \) have the following expression

\[
u_{j}(t) = \mathcal{R}_{α,j}(t)ψ_{j} + \int_{0}^{t} \mathcal{R}_{α,j}(t-s)F_{j}(s,.,u(s,.,))ds,
\]

(7)

whereupon

\[
\mathcal{R}_{α,j}(t) = \int_{0}^{∞} e^{-st} \mathcal{K}_{α,j}(s)ds
\]

\[
\mathcal{K}_{α,j}(s) = \frac{\kappa λ_{j}s^{α} \sin απ}{π (-s + \kappa λ_{j}s^{α} \cos απ + λ_{j})^{2} + π (\kappa λ_{j}s^{α} \sin απ)^{2}}.
\]

(8)
For fixed $s$, $F_j(s, u(s, \cdot))$ denotes by the $j$-th coefficient of $x \mapsto F(s, x, u(s, x))$ in $H^2$. For the sake of convenience, for $t \in J$, we define

$$G_\alpha(t) := \sum_{j=1}^\infty \langle \cdot, \phi_j^{(\xi)} \rangle_{H^2} R_{\alpha,j}(t) \phi_j^{(\xi)}, \quad (9)$$

Then, we have the following representation for the solution to Problem (6)

$$u(t, x) = G_\alpha(t) \psi(x) + \int_0^t G_\alpha(t-s) F(s, x, u(s, x)) ds. \quad (10)$$

Motivated by (10), we state the definitions of mild solution to the initial value problem (1)-(3) as follows:

**Definition 3.1.** An $\mathcal{F}_t$-adapted process $\{u(t, \cdot)\}_{t \in J}$ in $C_t^0 L^q_t H^2_c$ with some $\theta \in (0, 1)$, $q \geq 2$, $\xi \geq 0$, is called a mild solution of Problem (1) if there holds that

$$u(t, x) = G_\alpha(t) \psi(x) + \int_0^t G_\alpha(t-s) F(s, x, u(s, x)) ds$$

$$+ \int_0^t G_\alpha(t-s) \sigma(s, u(s, x)) dW_\xi(s, x). \quad (11)$$

for almost all $\omega$ for fixed $t$.

3.2. Properties of solution operators. We begin with the following lemma which shows basic bounds for the functions $R_{\alpha,j}$, $j \geq 1$.

**Lemma 3.2.** Given $\alpha \in (0, 1)$, $\delta \in [0, 2]$, $\epsilon > 0$ and $j \in \mathbb{Z}^+$. Then

a) $K_{\alpha,j}(s) \leq c_{\alpha,1} \lambda_j^{-1} s^{-\alpha}$, for all $s > 0$. Furthermore, there exists $s_0 > 0$ such that

$$K_{\alpha,j}(s) \leq C_{\alpha,1} \lambda_j^{-1} s^{-\alpha-\delta}, \text{ for all } s \geq s_0, \quad (12)$$

where $C_{\alpha,1}$, $c_{\alpha,1}$ are two positive constants.

b) $R_{\alpha,j}(0) = 1$ and $0 < R_{\alpha,j}(t) \leq \frac{c_{\alpha,2}}{1 + \lambda_j t^{1-\alpha}}$, for $t \in J_*$, where $c_{\alpha,2}$ is a positive constant.

c) The classical derivative of $R_{\alpha,j}(t)$ finitely exists for $t \in J_*$ and is bounded as follows

$$\frac{d}{dt} R_{\alpha,j}(t) \leq c_{\alpha,3} (\lambda_j^{-1} + \lambda_j^{\delta-1} t^{-(2+\alpha-\delta+\epsilon)}), \quad t \in J_*, \quad (13)$$

where $c_{\alpha,3}$ is a positive constant.

**Proof.** a) Consider $K_{\alpha,j}(s)$ as defined in (8). Then,

$$(-s + \kappa \lambda_j s^\alpha \cos \alpha \pi + \lambda_j^2) \geq (\kappa \lambda_j s^\alpha \sin \alpha \pi)^2$$

$$s^2 + \kappa^2 \lambda_j^2 s^{2\alpha} \cos^2 \alpha \pi + s^2 \sin^2 \alpha \pi + \lambda_j^2 - 2 \lambda_j s - 2 \kappa \lambda_j s^{\alpha+1} \cos \alpha \pi + 2 \kappa \lambda_j^2 s^\alpha \cos \alpha \pi$$

$$= s^2 + \lambda_j^2 s^{2\alpha} (\kappa s^\alpha - 2 \cos \alpha \pi) + 2 \lambda_j s (\kappa s^\alpha \cos \alpha \pi - 1),$$

which show that there exists $s_0 > 0$ such that

$$(-s + \kappa \lambda_j s^\alpha \cos \alpha \pi + \lambda_j^2) \geq s^2 + \lambda_j^2 \quad \text{for all } s \geq s_0.$$
We now estimate $s^2 + \lambda_j^2$ by applying the Young inequality. For $\delta \in (0, 2)$, we set $p := p(\delta) = \frac{2}{\delta}$, $q := q(\delta) = \frac{\delta^2}{2}$, and $\frac{2}{p} + \frac{1}{q} = 1$. The Young inequality yields

$$\frac{sp}{p} + \frac{q}{q} \geq s\lambda_j.$$  

It follows that there exists a positive constant $C_0$ such that

$$s^2 + \lambda_j^2 \geq C_0 s^{\delta - 2}, \quad \text{for } \delta \in (0, 2). \quad (14)$$

On the other hand, it is clear that (14) holds for $\delta = 0$ or $\delta = 2$. Hence, there exists a positive constant $C_{\alpha,1}$ such that

$$K_{\alpha,j}(s) \leq \frac{\lambda_j s^\alpha \sin \alpha \pi}{\pi(s^2 + \lambda_j^2)} \leq C_{\alpha,1} \lambda_j^{\alpha - 1} s^{\alpha - \delta}, \quad \text{for all } s \geq s_0, \delta \in [0, 2]. \quad (15)$$

In addition, since $(-s + \kappa \lambda_j^\alpha \cos \alpha \pi + \lambda_j^2 + (\kappa \lambda_j^\alpha \sin \alpha \pi)^2 \geq (\kappa \lambda_j^\alpha \sin \alpha \pi)^2$, there exists $c_{\alpha,1} > 0$ such that

$$K_{\alpha,j}(s) \leq \frac{\lambda_j s^\alpha \sin \alpha \pi}{\pi(\lambda_j s^\alpha \sin \alpha \pi)^2} \leq c_{\alpha,1} \lambda_j \alpha s^{-\alpha}, \quad \text{for all } s > 0. \quad (16)$$

b) The proof of part b) can be found in [18].

c) For $t, r \in J_*$ such that $r > t$, we will estimate $|R_{\alpha,j}(r) - R_{\alpha,j}(t)|$. We have

$$|R_{\alpha,j}(r) - R_{\alpha,j}(t)| \leq \int_{s_0}^{s_0} |e^{-sr} - e^{-st}| |K_{\alpha,j}(s)| ds + \int_{s_0}^{\infty} |e^{-sr} - e^{-st}| |K_{\alpha,j}(s)| ds$$

$$= : I_1 + I_2. \quad (17)$$

Noting that $|e^{-sr} - e^{-st}| = |e^{-st} (1 - e^{-s(r-t)})| \leq se^{-st}(r-t)$. For $I_1$, one can see from (16) that

$$I_1 \leq \int_{s_0}^{s_0} e^{-st}(r-t)K_{\alpha,j}(s) ds \leq c_{\alpha,1}(r-t)\lambda_j^{-1} \int_{s_0}^{s_0} e^{-st}s^{\delta - \alpha} ds. \quad (18)$$

For $I_2$, it follows from (15) that

$$I_2 \leq C_{\alpha,1} \int_{s_0}^{\infty} e^{-st}(r-t)\lambda_j^{\delta - 1}s^{\delta - \alpha} ds \leq C_{\alpha,1}(r-t)\lambda_j^{\delta - 1} \int_{s_0}^{\infty} e^{-st}s^{\delta + \alpha - \delta} ds, \quad \delta \in [0, 2].$$

Noting that there always exists $C_{\alpha,2} > 0$ such that $e^{-st} \leq C_{\alpha,2}(st)^{-(2+\alpha - \delta + \epsilon)}$, for all $\epsilon > 0$, then

$$I_2 \leq C_{\alpha,1}C_{\alpha,2}(r-t)\lambda_j^{\delta - 1} \int_{s_0}^{\infty} t^{-(2+\alpha - \delta + \epsilon)} s^{-\gamma} ds$$

$$\leq C_{\alpha,1}C_{\alpha,2}(r-t)\lambda_j^{\delta - 1} t^{-(2+\alpha - \delta + \epsilon)} \int_{s_0}^{\infty} s^{-\gamma} ds. \quad (19)$$

From (17)-(19), we deduce that there exists $c_{\alpha,3} > 0$ such that

$$|R_{\alpha,j}(r) - R_{\alpha,j}(t)| \leq c_{\alpha,3}(r-t) \left(\lambda_j^{\delta - 1} t^{-(2+\alpha - \delta + \epsilon)} \right).$$

This implies that the classical derivative of $R_{\alpha,j}(t)$ finitely exists for $t \in J_*$ and is bounded as

$$\frac{d}{dt}R_{\alpha,j}(t) \leq c_{\alpha,3} \left(\lambda_j^{\delta - 1} t^{-(2+\alpha - \delta + \epsilon)} \right), \quad t \in J_*.$$  

This completes the proof. □
The following lemma gives the appropriate estimates for the solution operators. From now on, if an operator $\mathcal{G} : L^q_0 \rightarrow L^q_0$, for $\xi, \xi' \geq 0$, satisfies
\[
\|\mathcal{G} v\|_{L^q_0} \leq C \|v\|_{L^q_0}, \quad v \in L^q_0,
\]
we will write $\|\mathcal{G}\|_{L^q_0 \rightarrow L^q_0}$ for short.

**Lemma 3.3.** Given $t \in I$, $\xi \geq 0$ and $q \geq 2$. Then, there holds
\[
\|\mathcal{G}_\alpha(t)\|_{L^q_0 \rightarrow L^q_0} \leq c_{\alpha,2}.
\]

**Proof.** Let $v \in L^q_0$ and $v_j := \langle v, \phi_j \rangle$. By applying Lemma 3.2 and the fact that $1 + \lambda_j t^{1-\alpha} \geq 1$, it is obvious that $R_{\alpha,j}(t) \leq c_{\alpha,2}$, for $t \in I$. It follows that
\[
\|\mathcal{G}_\alpha(t) v\|_{L^q_0} = \left[ \mathbb{E} \left( \sum_{j=1}^{\infty} R_{\alpha,j}(t) \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \leq c_{\alpha,2} \|v\|_{L^q_0}.
\]
This completes the proof. \qed

**Lemma 3.4.** Consider $\xi \geq 0$ and $q \geq 2$. Let $\epsilon_1, \epsilon_2$ be two positive constants satisfying $\alpha + \epsilon_1 < 1$ and $\epsilon_2 < \min\{\alpha + \epsilon_1, 1 - (\alpha + \epsilon_1)\}$. There exists $c_{\alpha,4}, c_{\alpha,5} > 0$ such that, for $t, r \in I$, satisfying $0 < r - t < 1$, it holds
\[
\|\mathcal{G}_\alpha(r) - \mathcal{G}_\alpha(t)\|_{L^q_0} \leq c_{\alpha,4}(r - t)^{1-(\alpha + \epsilon_1)},
\]
\[
\|\mathcal{G}_\alpha(t) - \mathcal{G}_\alpha(t)\|_{L^q_0} \leq c_{\alpha,5}(r - t)^{\epsilon_2} r^{-(\alpha + \epsilon_1 + \epsilon_2)}.
\]

**Proof.** Let $v \in L^q_0$ and $\mathcal{R}_{\alpha,j}(t) := \frac{d}{dt} R_{\alpha,j}(t)$. Applying Part c of Lemma 3.2 implies that
\[
\|\mathcal{G}_\alpha(r) v - \mathcal{G}_\alpha(t) v\|_{L^q_0} = \left[ \mathbb{E} \left( \sum_{j=1}^{\infty} \left[ R_{\alpha,j}(r) - R_{\alpha,j}(t) \right] \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q} \leq c_{\alpha,3} \left[ \mathbb{E} \left( \sum_{j=1}^{\infty} \int_t^r \mathcal{R}_{\alpha,j}(s) ds \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q},
\]
where $\delta \in [0, 2]$ and $\epsilon > 0$.

- Firstly, we prove that (21) holds. By choosing $\delta = 2$ and $\epsilon = \epsilon_1$, one arrives at
\[
\|\mathcal{G}_\alpha(r) v - \mathcal{G}_\alpha(t) v\|_{L^q_0} \leq c_{\alpha,3} \left[ \mathbb{E} \left( \sum_{j=1}^{\infty} \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q}.
\]
It follows that there exists $c_{\alpha,4} > 0$ such that
\[
\|G_\alpha(r)v - G_\alpha(t)v\|_{L^q_\alpha H^2_x} \leq c_{\alpha,4}(r - t)^{1-(\alpha+\epsilon_1)}\left[\mathbb{E}\left(\sum_{j=1}^{\infty} \lambda_j^2(\xi+1)v_j^2\right)^{q/2}\right]^{1/q} \\
\leq c_{\alpha,4}(r - t)^{1-(\alpha+\epsilon_1)}\|v\|_{L^q_\alpha H^2_x+1}.
\]

- Next, we prove that (22) holds. By choosing $\delta = 1$, $\epsilon = \epsilon_1$, one also have
\[
\|G_\alpha(r)v - G_\alpha(t)v\|_{L^q_\alpha H^2_x} \leq c_{\alpha,3}\left[\sum_{j=1}^{\infty} \left(\frac{(r-t)^{\alpha+\epsilon_1} - (r-t)^{\alpha+\epsilon_1}}{\alpha+\epsilon_1}\right)^2 \lambda_j^2v_j^2\right]^{q/2}\ |
\leq c_{\alpha,3}\left[\sum_{j=1}^{\infty} \left(\frac{(r-t)^{\alpha+\epsilon_1} - (r-t)^{\alpha+\epsilon_1}}{\alpha+\epsilon_1}\right)^2 \lambda_j^2v_j^2\right]^{q/2}\ |
\leq c_{\alpha,3}\left[\sum_{j=1}^{\infty} \lambda_j^{-1}(r-t) + \frac{(r-t)^{\alpha+\epsilon_1}}{\alpha+\epsilon_1}\right]^{q/2}\ |
\]
Since $r^{\alpha+\epsilon_1} \geq (r-t)^{\alpha+\epsilon_1}t^{\rho}$, one can see
\[
\frac{(r-t)^{\alpha+\epsilon_1}}{\alpha+\epsilon_1} \leq \frac{(r-t)^{\alpha+\epsilon_1}}{\alpha+\epsilon_1}(r-t)^{\alpha+\epsilon_1}\|v\|_{L^q_\alpha H^2_x}.
\]
By two latter estimates, we deduce that there exists $c_{\alpha,5} > 0$ such that
\[
\|G_\alpha(r)v - G_\alpha(t)v\|_{L^q_\alpha H^2_x} \leq c_{\alpha,5}(r - t)^{\alpha+\epsilon_1}t^{\rho}\|v\|_{L^q_\alpha H^2_x}.
\]
The proof is completed.

3.3. The existence and uniqueness of mild solution to Problem (1),(2),(3). In this part, we state the main results for Problem (1),(2),(3). The first theorem states the existence and uniqueness of the solution to Problem (1)-(3) in the space $L^\infty_\rho L^q_\alpha H^2_x$. The second theorem investigates the regularity of the solution on the space $C^1_{\gamma,x} L^q_\alpha H^2_x$, for any positive constant $\gamma$ satisfying $\gamma < \min\{1 - \alpha, \frac{1}{2}\}$.

Theorem 3.5. Let $\xi \geq 0$ and $q \geq 2$. Assume that $\psi \in L^q_\omega H^2_x$, $F \in L_{glo}(L^q_\omega H^2_x)$, $F(\cdot, 0) = 0$, and $\sigma \in L_{glo}(L^q_\alpha H^2_x; L^q_0(H^2_x)), \sigma(\cdot, 0) = 0$. Assume further that $K_1, K_2 \in L^\infty$. Then, there exists $p > 0$ such that Problem (1)-(3) has a unique solution $u \in L^\infty_\rho L^q_\alpha H^2_x$.

Proof. Let $\rho$ be a positive constant and $\Phi : L^\infty_\rho L^q_\alpha H^2_x \to L^\infty_\rho L^q_\alpha H^2_x$ be defined as follows
\[
\Phi w(t,x) = G_\alpha(t)\psi(x) + \int_0^t G_\alpha(t-s)F(s, w(s,x))ds \\
+ \int_0^t G_\alpha(t-s)\sigma(s, w(s,x))d\mathcal{W}_x(t,x).
\]
Step 1. Firstly, it is required to show that $\Phi$ is well-defined on $L^\infty_t L^q_w \hat{H}^\xi_z$. The first term $G_\alpha(t)\psi(x)$ can be estimated by using Lemma 3.3 as $\|G_\alpha(t)\psi\|_{L^q_w \hat{H}^\xi_z} \leq c_{\alpha,2} \|\psi\|_{L^q_w \hat{H}^\xi_z}$ for $t \in \mathcal{J}$. This leads to

\[ e^{-\rho t} \|G_\alpha(t)\psi\|_{L^q_w \hat{H}^\xi_z} \leq c_{\alpha,2} e^{-\rho t} \|\psi\|_{L^q_w \hat{H}^\xi_z} \leq c_{\alpha,2} \|\psi\|_{L^q_w \hat{H}^\xi_z}, \]

which implies that $\|G_\alpha(t)\psi\|_{L^\infty_t L^q_w \hat{H}^\xi_z} \leq c_{\alpha,2} \|\psi\|_{L^q_w \hat{H}^\xi_z}$. For the second term, by applying Lemma 3.3, $F(\cdot,\cdot,0) = 0$, and $F \in L_{glo}(L^q_w \hat{H}^\xi_z)$, one can see

\[
\left\| \int_0^t G_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_w \hat{H}^\xi_z} \leq \int_0^t \left\| G_\alpha(t-s)F(s,\cdot,w(s,\cdot)) \right\|_{L^q_w \hat{H}^\xi_z} ds
\leq c_{\alpha,2} \int_0^t \left( E \left\| F(s,\cdot,w(s,\cdot)) \right\|_{H^\xi_z} \right)^{\frac{q}{2}} ds
\leq c_{\alpha,2} K_1 \int_0^t \left( E \left\| w(s,\cdot) \right\|_{H^\xi_z} \right)^{\frac{q}{2}} ds, \tag{23}
\]

where we have used $E \left\| F(s,\cdot,w(s,\cdot)) \right\|_{H^\xi_z} \leq K_1 \int_0^t E \left\| w(s,\cdot) \right\|_{H^\xi_z}^{\frac{q}{2}} ds$. It follows that

\[
\begin{align*}
& e^{-\rho t} \left\| \int_0^t G_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_w \hat{H}^\xi_z} \\
& \leq c_{\alpha,2} K_1 \int_0^t e^{-\rho(t-s)} \left\| w(s,\cdot) \right\|_{L^q_w \hat{H}^\xi_z} ds.
\end{align*}
\]

Using the fact that $\int_0^t e^{-\rho(t-s)} ds = \rho^{-1}(1-e^{-\rho t}) = \rho^{-1}$, we deduce that

\[
\left\| \int_0^t G_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^\infty_t L^q_w \hat{H}^\xi_z} \leq c_{\alpha,2} K_1 \rho^{-1} \left\| w \right\|_{L^\infty_t L^q_w \hat{H}^\xi_z}.
\]

We continue to estimate the last term by considering two following cases.

- If $q = 2$, the Itô isometry yields

\[
\begin{align*}
& \left\| \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,\cdot) \right\|_{L^q_w \hat{H}^\xi_z} \\
& = \left( E \int_0^t \left\| G_\alpha(t-s)\sigma(s,w(s,\cdot)) \right\|_{L^q_w(\hat{H}^\xi_z)} ds \right)^{\frac{1}{2}} \\
& \leq \left( \int_0^t \left\| G_\alpha(t-s) \right\|_{L^q_w(\hat{H}^\xi_z)} E \left\| \sigma(s,w(s,\cdot)) \right\|_{L^q_w(\hat{H}^\xi_z)} ds \right)^{\frac{1}{2}}. \tag{24}
\end{align*}
\]

Lemma 3.3, $\psi(\cdot,0) = 0$, and $\psi \in L_{glo}(L^q_w \hat{H}^\xi_z; L^2_w(\hat{H}^\xi_z))$ allow that

\[
\begin{align*}
e^{-\rho t} \left\| \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,\cdot) \right\|_{L^q_w \hat{H}^\xi_z} \\
& \leq c_{\alpha,2} e^{-\rho t} \left( \int_0^t K_2^q(s)E \left\| w(s,\cdot) \right\|_{H^\xi_z}^{q} ds \right)^{\frac{1}{2}} \\
& \leq c_{\alpha,2} K_2 \int_0^t \left( e^{-\rho(t-s)}E \left\| w(s,\cdot) \right\|_{L^q_w \hat{H}^\xi_z}^{q} \right)^{\frac{1}{2}} ds \\
& \leq c_{\alpha,2} K_2 \int_0^t \left( e^{-\rho(t-s)} ds \right)^{\frac{1}{2}} \left\| w \right\|_{L^\infty_t L^q_w \hat{H}^\xi_z} \int_0^t \left( e^{-\rho(t-s)}ds \right)^{\frac{1}{2}}.
\end{align*}
\]
By the above arguments, one can see
\[ \leq c_{\alpha,2} \| K_2 \|_{L_t^\infty} \| w \|_{L_{t_0}^\infty L_x^2 H_x^\infty} q^{-\frac{1}{4}} \rho^{-\frac{1}{4}}. \] (25)
where we have used the fact that \( E \| \sigma(s, w(s, \cdot)) \|_{L_t^q(H_x^\infty)}^q \leq K_2^q(s) E \| w(s, \cdot) \|_{H_x^\infty}^q. \)

- If \( q > 2 \), the consequence of Burkholder-Davis-Gundy’s inequality yields
\[
\left\| \int_0^t G_\alpha(t-s) \sigma(s, w(s, \cdot)) dW_x(s, \cdot) \right\|_{L_x^2 H_x^\infty}
\leq C(q) \left[ \mathbb{E} \left( \int_0^t \| G_\alpha(t-s) \sigma(s, w(s, \cdot)) \|_{L_x^2(H_x^\infty)}^2 \, ds \right) \right]^{\frac{1}{2}}
\leq C(q) \left[ \mathbb{E} \left( \int_0^t \| G_\alpha(t-s) \|_{H_x^\infty \rightarrow H_x^\infty}^2 \| \sigma(s, w(s, \cdot)) \|_{L_t^q(H_x^\infty)}^2 \, ds \right) \right]^{\frac{1}{2}}
\leq c_{\alpha,2} C(q) \left[ \mathbb{E} \left( \int_0^t \| \sigma(s, w(s, \cdot)) \|_{L_t^q(H_x^\infty)}^2 \, ds \right) \right]^{\frac{1}{2}}. \] (26)

From the Hölder inequality we deduce
\[
\left\| \int_0^t G_\alpha(t-s) \sigma(s, w(s, \cdot)) dW_x(s, \cdot) \right\|_{L_x^2 H_x^\infty}
\leq c_{\alpha,2} C(q) \left( \int_0^t ds \right)^{(q-2)/(2q)} \left( \int_0^t \mathbb{E} \| \sigma(s, w(s, \cdot)) \|_{L_t^q(H_x^\infty)}^q \, ds \right)^{1/q}
\leq c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \| K_2 \|_{L_t^\infty} \left( \int_0^t \mathbb{E} \| w(s, \cdot) \|_{H_x^\infty}^q \, ds \right)^{1/q}, \tag{27}
\]
where we recall that \( \sigma(\cdot, 0) = 0, \sigma \in L_{x_0}(L_x^2 H_x^\infty; L_0^2(H_x^\infty)). \) Hence, we derive the estimate
\[ e^{-pt} \left\| \int_0^t G_\alpha(t-s) \sigma(s, w(s, \cdot)) dW_x(s, \cdot) \right\|_{L_x^2 H_x^\infty}
\leq c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \| K_2 \|_{L_t^\infty} \left( \int_0^t e^{-qs(t-s)} e^{-qs} \| w(s, \cdot) \|_{L_x^2 H_x^\infty}^q \, ds \right)^{1/q}
\leq c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \| K_2 \|_{L_t^\infty} \| w \|_{L_{t_0}^\infty L_x^2 H_x^\infty} q^{-\frac{1}{4}} \rho^{-\frac{1}{4}}. \tag{28}\]

By the above arguments, one can see
\[ \| \Phi w \|_{L_{t_0}^\infty L_x^2 H_x^\infty} \leq c_{\alpha,2} \| \psi \|_{L_x^2 H_x^\infty}
+ \left( c_{\alpha,2} T \| K_1 \|_{L_t^\infty} \rho^{-1} + c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \| K_2 \|_{L_t^\infty} q^{-\frac{1}{4}} \rho^{-\frac{1}{4}} \right) \| w \|_{L_{t_0}^\infty L_x^2 H_x^\infty}, \]
which implies that \( \Phi \) is well-defined on \( L_{t_0}^\infty L_x^2 H_x^\infty \).

**Step 2.** Next, we will apply Banach’s fixed point theorem to show that \( \Phi \) has a unique fixed point. For \( w \in L_{t_0}^\infty L_x^2 H_x^\infty \), we define
\[ \Phi_1 w(t, x) := \int_0^t G_\alpha(t-s) F(s, x, w(s, x)) \, ds, \]
\[ \Phi_2 w(t, x) := \int_0^t G_\alpha(t-s) \sigma(s, w(s, x)) dW_x(s, x). \]
For $w^1, w^\uparrow \in L^\infty_{t,p} L^q H^s_x$, by a similar way as in (23), it is obvious that
\[
e^{-\rho t} \|\Phi_1 w^1(t, \cdot) - \Phi_1 w^\uparrow(t, \cdot)\|_{L^q H^s} \\
\leq e^{-\rho t} \int_0^t \|G_\alpha(t-s) \left(F(s, \cdot, w^1(s, \cdot)) - F(s, \cdot, w^\uparrow(s, \cdot))\right)\|_{L^q H^s} \, ds \\
\leq c_{\alpha,2} \int_0^t e^{-\rho(t-s)} e^{-\rho s} \left(\mathbb{E}\|w^1(s, \cdot) - w^\uparrow(s, \cdot)\|_{H^s}^q\right)^{\frac{q}{2}} \, ds \\
\leq c_{\alpha,2} \|K_1\|_{L^\infty_p} \|w^1 - w^\uparrow\|_{L^\infty_{t,p} L^q H^s_x} \rho^{-1}. \tag{29}
\]

We continue with the second operator $\Phi_2$ in the two following cases.

• If $q = 2$, similarly to (24)-(25), one arrives at
\[
e^{-\rho t} \|\Phi_2 w^1(t, \cdot) - \Phi_2 w^\uparrow(t, \cdot)\|_{L^q H^s} \leq c_{\alpha,2} \|K_2\|_{L^\infty_p} \|w^1 - w^\uparrow\|_{L^\infty_{t,p} L^q H^s_x} q^{-\frac{1}{2}} \rho^{-\frac{1}{4}}. \tag{30}
\]

• If $q > 2$, by a similar way as in (26)-(28), one arrives at
\[
e^{-\rho t} \|\Phi_2 w^1(t, \cdot) - \Phi_2 w^\uparrow(t, \cdot)\|_{L^q H^s} \leq c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \|K_2\|_{L^\infty_p} \|w^1 - w^\uparrow\|_{L^\infty_{t,p} L^q H^s_x} q^{-\frac{1}{4}} \rho^{-\frac{1}{4}}. \tag{31}
\]

Combining (29), (30), (31), we deduce that
\[
\|\Phi w^1(t, \cdot) - \Phi w^\uparrow(t, \cdot)\|_{L^\infty_{t,p} L^q H^s} \\
\leq \|\Phi_1 w^1(t, \cdot) - \Phi_1 w^\uparrow(t, \cdot)\|_{L^\infty_{t,p} L^q H^s} + \|\Phi_2 w^1(t, \cdot) - \Phi_2 w^\uparrow(t, \cdot)\|_{L^\infty_{t,p} L^q H^s} \\
\leq \left(c_{\alpha,2} T \|K_1\|_{L^\infty_p} \rho^{-1} + c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \|K_2\|_{L^\infty_p} q^{-\frac{1}{4}} \rho^{-\frac{1}{4}}\right) \|w^1 - w^\uparrow\|_{L^\infty_{t,p} L^q H^s_x}. \tag{32}
\]

By choosing $\rho = \overline{p}$ with $\overline{p}$ is large enough such that
\[
c_{\alpha,2} T \|K_1\|_{L^\infty_p} \overline{p}^{-1} + c_{\alpha,2} C(q) T^{\frac{q-2}{2q}} \|K_2\|_{L^\infty_p} q^{-\frac{1}{4}} \overline{p}^{-\frac{1}{4}} < 1,
\]
the estimate (32) shows that $\Phi$ is a contraction on $L^\infty_{t,p} L^q H^s_x$. Hence, we conclude that Problem (1)-(3) has a unique solution $u \in L^\infty_{t,p} L^q H^s_x$. \qed

**Remark 1.** In Theorem 3.5, the function $\sigma$ is defined by Definition 2.3. Here, we can find a constant $C_\sigma$ such that $C_\sigma \rightarrow 0$ as $\rho \rightarrow \infty$, and $e^{-\rho t}\|\Phi_2 w^1(t, \cdot) - \Phi_2 w^\uparrow(t, \cdot)\|_{L^q H^s}$ is bounded by $C_\rho\|w^1 - w^\uparrow\|_{L^\infty_{t,p} L^q H^s_x}$ (see (30)-(31)). Therewith, the global existence has been obtained consequently. However, we note that the above result does not hold true if $\sigma$ is locally Lipschitz continuous. For example, let us consider $q = 2$ for simplicity and assume that $\sigma$ satisfies the following locally Lipschitz continuity
\[
\|\sigma(t, w^1) - \sigma(t, w^\uparrow)\|_{L^q H^s_x} \leq \left(\|w^1\|_{L^q H^s_x} + \|w^\uparrow\|_{L^q H^s_x}\right) \|w^1 - w^\uparrow\|_{L^q H^s_x}. \tag{33}
\]
Let $U_R$ be the closed ball of center at zero and radius $R$ in $L^\infty_{t,p} L^q H^s_x$. For $w^1, w^\uparrow$ taken in this ball, the following chain can be obtained by making uses of Lemma...
Consider the condition (33)

\[ e^{-\rho t} \left\| \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,\cdot) \right\|_{L^2_\omega H^\alpha_5} \leq e^{-\rho t} \left( \int_0^t \|G_\alpha(t-s)\|_{H^\alpha_5 \to H^\alpha_5}^2 E \|\sigma(s,w(s,\cdot))\|_{L^2_\omega H^\alpha_5}^2 \, ds \right)^{1/2} \]

\[ \leq e^{-\rho t} \left( \int_0^t \left( \|w^1(s,\cdot)\|_{L^2_\omega H^\alpha_5}^2 + \|w^4(s,\cdot)\|_{L^2_\omega H^\alpha_5}^2 \right)^2 \|w^1(s,\cdot) - w^4(s,\cdot)\|_{L^2_\omega H^\alpha_5}^2 \, ds \right)^{1/2} \]

\[ \leq c_{\alpha,2} e^{-\rho t} \left( \int_0^t 4R^2 e^{4\rho s - 2\rho t} ds \right)^{1/2} \|w^1 - w^4\|_{L^\infty_t L^2_\omega H^\alpha_5}. \]

By taking the essential supremum of \( e^{-\rho s} \|w^1(s,\cdot) - w^4(s,\cdot)\|_{L^2_\omega H^\alpha_5} \) on \((0,T)\), we imply from the above chain that

\[ e^{-\rho t} \left\| \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,\cdot) \right\|_{L^2_\omega H^\alpha_5} \leq 2Rc_{\alpha,2} \left( \int_0^t e^{4\rho s - 2\rho t} ds \right)^{1/2} \|w^1 - w^4\|_{L^\infty_t L^2_\omega H^\alpha_5}. \]

A direct computation shows that \( e^{-2\rho t} \int_0^t e^{4\rho s} ds \) equals \((e^{2\rho t} - e^{-2\rho t})/(4\rho)\), and so it tends to positive infinity as \( \rho \) approaches positive infinity. Therefore, we cannot derive the contraction of the mapping \( \Phi \) analogously Proof of Theorem 3.5.

It is well-known that, constructing global existence of problems that including locally Lipschitz continuous nonlinearity and noise is not an easy task. In the future, we will study global existence of RSE with more a general assumption than (33).

Now, we study the regularity of the solution in the space \( L^\infty_t L^2_\omega H^\alpha_5 \). From now on, we use \( a \lesssim b \) to describe that there exists a positive constant \( C \) such that \( a \leq C b \).

**Theorem 3.6.** Consider \( \xi \geq 0 \) and \( q \geq 2 \). Assume that \( \psi, F, \sigma \) satisfy the conditions of Theorem 3.5. Then, the following regularity estimate holds

\[ \|u\|_{L^\infty_t L^2_\omega H^\alpha_5} \lesssim \|\psi\|_{L^\infty_t \omega}. \]

**Proof.** We aim to estimate \( \|u(t,\cdot)\|_{L^2_\omega H^\alpha_5} \) firstly by using some materials which have been proved in the proof of Theorem 3.5. Recall that \( \|G_\alpha(t)\psi\|_{L^2_\omega H^\alpha_5} \leq c_{\alpha,2} \|\psi\|_{L^2_\omega H^\alpha_5} \) for \( t \in \mathcal{J} \). From (23), we have

\[ \left\| \int_0^t G_\alpha(t-s)F(s,\cdot,u(s,\cdot))ds \right\|_{L^2_\omega H^\alpha_5} \leq c_{\alpha,2} \|K_1\|_{L^\infty_\omega} \int_0^t \|u(s,\cdot)\|_{L^2_\omega H^\alpha_5} \, ds \]

\[ \leq c_{\alpha,2} \|K_1\|_{L^\infty_\omega} T^{\frac{q-1}{q}} \left( \int_0^t \|u(s,\cdot)\|_{L^2_\omega H^\alpha_5}^q \, ds \right)^{\frac{1}{q}}. \]

- If \( q = 2 \), estimate (24) leads to

\[ \left\| \int_0^t G_\alpha(t-s)\sigma(s,u(s,\cdot))dW_\xi(s,\cdot) \right\|_{L^2_\omega H^\alpha_5} \leq c_{\alpha,2} \left( \int_0^t K^2_\alpha(s)E \|u(s,\cdot)\|_{H^\alpha_5}^q \, ds \right)^{\frac{1}{q}} \]

\[ \leq c_{\alpha,2} \|K_2\|_{L^\infty_\omega} \left( \int_0^t \|u(s,\cdot)\|_{L^2_\omega H^\alpha_5}^q \, ds \right)^{\frac{1}{q}}. \]

(34)
If $q > 2$, estimate (27) implies
\[
\left\| \int_0^t \mathcal{G}_\alpha(t-s)\sigma(s,u(s,\cdot))dW_\xi(s,\cdot) \right\|_{L^q_tH^s_L} \leq c_{a,2}C(q)T^{\frac{2q-2}{2q}} \|K_2\|_{L^{\infty}} \left( \int_0^t \|u(s,\cdot)\|_{L^q_tH^s_L}^q ds \right)^{\frac{1}{q}}.
\] (35)

By the above arguments, one can see there exists $M_{q,\alpha,1}, M_{q,\alpha,2}(T) > 0$ such that
\[
\|u(t,\cdot)\|_{L^q_tH^s_L}^q \leq M_{q,\alpha,1} \|\psi\|_{L^q_tH^s_L}^q + M_{q,\alpha,2}(T) \int_0^t \|u(s,\cdot)\|_{L^q_tH^s_L}^q ds.
\]

By applying the Gronwall inequality (see [32]),
\[
\|u(t,\cdot)\|_{L^q_tH^s_L}^q \leq M_{q,\alpha,1} \|\psi\|_{L^q_tH^s_L}^q \exp(tM_{q,\alpha,2}(T)) \leq M_{q,\alpha,1} \|\psi\|_{L^q_tH^s_L}^q \exp(TM_{q,\alpha,2}(T)),
\]
which implies that $\|u\|_{L^q_tH^s_L} \lesssim \|\psi\|_{L^q_tH^s_L}$.

Next, we prove a regularity result for the solution in the space $C_{t,s}^{1,0} L^q_t H^s_L$. For $b \in (0,1)$ and $q \geq 1$, we define
\[
\mathcal{K}^q_{t,s} := \left\{ f : \sup_{t \in T_s} \left( \int_0^t (t-s)^{-b} |f(s)|^q ds < \infty \right) \right\}.
\] (36)

**Theorem 3.7.** Consider $\xi \geq 0$ and $\epsilon_1, \epsilon_2 > 0$ such that $\alpha + \epsilon_1 < 1$ and $\epsilon_2 < \min\{\alpha + 1, 1 - (\alpha + \epsilon_1)\}$. Let $q$ be a positive constant satisfying $q(\alpha + \epsilon_1 + \epsilon_2) < 1$. Assume that $\psi \in L^q_t H^{s+1}_L$, $F, \sigma$ satisfy conditions in Theorem 3.5. Assume further that $K_1 \in \mathbb{K}_{t,s}^{1,\alpha+\epsilon_1+\epsilon_2}$ and $K_2 \in \mathbb{K}_{t,s}^{0,\alpha+\epsilon_1+\epsilon_2}$. Then, $u \in C_{t,s}^{1,0} L^q_t H^s_L$ and satisfies
\[
\|u(r,\cdot) - u(t,\cdot)\|_{L^q_tH^s_L} \lesssim (r-t)^{\epsilon_2} \left( \|\psi\|_{L^q_tH^{s+1}_L} + \|u\|_{L^{\infty}_tL^q_tH^s_L} \right), \quad \text{for } t, r \in T_s.
\] (37)

**Corollary 1.** Consider the case $\xi = 0$. If all assumptions in Theorem 3.7 are satisfied, then $u$ belongs to $\in C_{t,s}^{1,0} L^q_t L^2_x$ and satisfies
\[
\|u(r,\cdot) - u(t,\cdot)\|_{L^q_tL^2_x} \lesssim (r-t)^{\epsilon_2} \left( \|\psi\|_{L^q_tH^s_L} + \|u\|_{L^{\infty}_tL^q_tL^2_x} \right), \quad \text{for } t, r \in T_s.
\]

**Corollary 2.** If $K_1, K_2$ are two positive constants (independent of $t$), then $K_1 \in \mathbb{K}_{t,s}^{1,\alpha+\epsilon_1+\epsilon_2}$ and $K_2 \in \mathbb{K}_{t,s}^{1,\alpha+\epsilon_1+\epsilon_2}$. Hence, the result (37) holds in this case.

**Proof.** From the integral equation (11), it is obvious that
\[
\|u(r,\cdot) - u(t,\cdot)\|_{L^q_tH^s_L} \leq \| (\mathcal{G}_\alpha(r) - \mathcal{G}_\alpha(t)) \psi \|_{L^q_tH^s_L} + \left\| \int_t^r \mathcal{G}_\alpha(r-s) F(s,\cdot,u(s,\cdot)) ds \right\|_{L^q_tH^s_L} \\
+ \left\| \int_0^t (\mathcal{G}_\alpha(r-s) - \mathcal{G}_\alpha(t-s)) F(s,\cdot,u(s,\cdot)) ds \right\|_{L^q_tH^s_L} \\
+ \left\| \int_t^r \mathcal{G}_\alpha(r-s) \sigma(s,u(s,\cdot)) dW_\xi(s,\cdot) \right\|_{L^q_tH^s_L} \\
+ \left\| \int_0^t (\mathcal{G}_\alpha(r-s) - \mathcal{G}_\alpha(t-s)) \sigma(s,u(s,\cdot)) dW_\xi(s,\cdot) \right\|_{L^q_tH^s_L} \\
=: \mathcal{E}_1(r,t) + \mathcal{E}_2(r,t) + \mathcal{E}_3(r,t) + \mathcal{E}_4(r,t) + \mathcal{E}_5(r,t).
\] (38)
By Lemma 3.4, we directly obtain an upper bound for $\mathcal{E}_1(r, t)$

$$
\mathcal{E}_1(r, t) \leq \|G_\alpha(r) - G_\alpha(t)\|_{L^2_0 \dot{H}^{s_1+\epsilon} \to L^2_0 \dot{H}^s} \|v\|_{L^2_0 \dot{H}^{s_1+\epsilon}} \lesssim (r - t)^{1-(\alpha+\epsilon_1)} \|v\|_{L^2_0 \dot{H}^{s_1+\epsilon}}.
$$

(39)

For $\mathcal{E}_2(r, t)$, we can bound it by using Lemma 3.3 and the condition $F \in L_{glo}(L^q_0 \dot{H}^s)$, $F(\cdot, \cdot, 0) = 0$ as

$$
\mathcal{E}_2(r, t) \leq \int_t^r \|G_\alpha(r - s)F(s, \cdot, u(s, \cdot))\|_{L^2_0 \dot{H}^s} ds
\lesssim (r - t)^{\epsilon_2} \int_0^t (t - s)^{-(\alpha+\epsilon_1+\epsilon_2)} \|F(s, \cdot, u(s, \cdot))\|_{L^2_0 \dot{H}^s} ds
\lesssim (r - t)^{\epsilon_2} \int_0^t (t - s)^{-(\alpha+\epsilon_1+\epsilon_2)} K_1(s) \left( E \|u(s, \cdot)\|_{L^q_0 \dot{H}^s} \right)^{\frac{1}{q}} ds.
$$

It follows that

$$
\mathcal{E}_3(r, t) \lesssim (r - t)^{\epsilon_2} \left( \sup_{t \in J} \int_0^t (t - s)^{-(\alpha+\epsilon_1+\epsilon_2)} K_1(s) ds \right) \|u\|_{L^\infty_t L^2_0 \dot{H}^s}
\lesssim (r - t)^{\epsilon_2} \|u\|_{L^\infty_t L^2_0 \dot{H}^s}.
$$

(40)

We continue with the last two terms $\mathcal{E}_4(r, t)$, $\mathcal{E}_5(r, t)$ by considering them in two cases of $q$.

- If $q = 2$, we can bound $\mathcal{E}_4(r, t)$ by using a similar way as in (24)-(25)

$$
\mathcal{E}_4(r, t) = \left( E \int_t^r \|G_\alpha(r - s)\sigma(s, u(s, \cdot))\|_{L^2_0 \dot{H}^s}^q ds \right)^{\frac{1}{q}} \lesssim \left( \int_t^r K_2^q(s) E \|u(s, \cdot)\|_{L^q_0 \dot{H}^s} ds \right)^{\frac{1}{q}}.
$$

It follows that

$$
\mathcal{E}_4(r, t) \lesssim (r - t)^{\frac{1}{q}} \|u\|_{L^\infty_t L^2_0 \dot{H}^s}.
$$

(41)

For $\mathcal{E}_5(r, t)$, the Itô isometry and Lemma 3.4 allow that

$$
\mathcal{E}_5(r, t) = \left( E \int_0^t \|G_\alpha(r - s) - G_\alpha(t - s)\|_{L^q_0 \dot{H}^s}^q ds \right)^{\frac{1}{q}}
\leq \left( E \int_0^t \|G_\alpha(r - s) - G_\alpha(t - s)\|_{H^s \to \dot{H}^s}^q \|\sigma(s, u(s, \cdot))\|_{L^q_0 \dot{H}^s} ds \right)^{\frac{1}{q}}
\lesssim (r - t)^{\epsilon_2} \left( E \int_0^t (t - s)^{-q(\alpha+\epsilon_1+\epsilon_2)} \|\sigma(s, u(s, \cdot))\|_{L^q_0 \dot{H}^s} ds \right)^{\frac{1}{q}}.
$$

Since $\sigma(\cdot, 0) = 0$, $\sigma \in L_{glo}(L^2_0 \dot{H}^s; L^q_0 \dot{H}^s)$, we deduce that

$$
\mathcal{E}_5(r, t) \lesssim (r - t)^{\epsilon_2} \left( \int_0^t (t - s)^{-q(\alpha+\epsilon_1+\epsilon_2)} K_2^q(s) E \|u(s, \cdot)\|_{L^q_0 \dot{H}^s} ds \right)^{\frac{1}{q}}.
$$
Remark 2. Now, let us explain the reason why we need the estimate in part a) of Lemma 3.2 instead of the overestimate
\[ K_{\alpha,j}(s) \leq c_{\alpha,j} \lambda_j^{-1} s^{-a}, \quad \text{for all } s > 0, \]  
which was used in [18]. Recall that the estimate in part a) of Lemma 3.2 helps us to obtain the result \( u \in C_t^\epsilon \mathcal{L}_q^2 \dot{H}^\xi_2 \). In contrast to this, by using the overestimate

\[
\begin{aligned}
&\lesssim (r-t)_{2}^2 \left( \sup_{t \in J_*} \int_t^r (t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)} K_2^q(s) ds \right)^{\frac{1}{q}} \|u\|_{L_t^\infty \mathcal{L}_q^2 \dot{H}^\xi_2} \\
&\lesssim (r-t)^2 \|u\|_{L_t^\infty \mathcal{L}_q^2 \dot{H}^\xi_2}.
\end{aligned}
\]  

(42)

- If \( q \geq 2 \), we can bound \( \mathcal{E}_4(r,t) \) by using a similar way as in (26)-(28)

\[
\mathcal{E}_4(r,t) \leq C(q) \left[ \mathbb{E} \left( \int_t^r \|G_\alpha(r-s)\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^2 \|\sigma(s,u(s,\cdot))\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^q ds \right)^{\frac{2}{q}} \right]^\frac{1}{2}.
\]

This leads to

\[
\mathcal{E}_4(r,t) \lesssim (r-t)^{\frac{1}{2}} \|u\|_{L_t^\infty \mathcal{L}_q^2 \dot{H}^\xi_2}.
\]  

(43)

For \( \mathcal{E}_5(r,t) \), the consequence of Burkholder-Davis-Gundy’s inequality and Lemma 3.4 yield

\[
\begin{aligned}
\mathcal{E}_5(r,t) &\leq C(q) \left[ \mathbb{E} \left( \int_0^t \|G_\alpha(r-s) - G_\alpha(t-s)\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^2 \|\sigma(s,u(s,\cdot))\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^q ds \right)^{\frac{2}{q}} \right]^\frac{1}{2} \\
&\lesssim \left[ \mathbb{E} \left( \int_0^t \|G_\alpha(r-s) - G_\alpha(t-s)\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^2 \|\sigma(s,u(s,\cdot))\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^q ds \right)^{\frac{2}{q}} \right]^\frac{1}{2} \\
&\lesssim (r-t)^{\epsilon_2} \left[ \mathbb{E} \left( \int_0^t (t-s)^{-2(\alpha+\epsilon_1+\epsilon_2)} \|\sigma(s,u(s,\cdot))\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^q ds \right)^{\frac{2}{q}} \right]^\frac{1}{2}.
\end{aligned}
\]

(44)

Applying the Hölder inequality, \( \mathcal{E}_5(r,t) \)

\[
\begin{aligned}
&\lesssim (r-t)^{\epsilon_2} \left( \int_0^t ds \right)^{(q-2)/(2q)} \left[ \int_0^t (t-s)^{-q(\alpha+\epsilon_1+\epsilon_2)} \mathbb{E} \|\sigma(s,u(s,\cdot))\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^q ds \right]^{1/2} \\
&\lesssim (r-t)^{\epsilon_2} \left[ \mathbb{E} \left( \int_0^t (t-s)\|G_\alpha(s)\|_{\mathcal{L}_q^2(\dot{H}^\xi_2)}^q ds \right)^{\frac{2}{q}} \right]^\frac{1}{2} \\
&\lesssim (r-t)^{\epsilon_2} \|u\|_{L_t^\infty \mathcal{L}_q^2 \dot{H}^\xi_2}.
\end{aligned}
\]

Now, combining (38)-(44), we conclude that \( u \in C_t^\epsilon \mathcal{L}_q^2 \dot{H}^\xi_2 \) and satisfies

\[
\|u(r,\cdot) - u(t, \cdot)\|_{\mathcal{L}_q^2 \dot{H}^\xi_2} \lesssim (r-t)^{\epsilon_2} \left( \|\psi\|_{L^\infty \mathcal{L}_q^2 \dot{H}^\xi_{2+1}} + \|u\|_{L^\infty \mathcal{L}_q^2 \dot{H}^\xi_2} \right), \quad \text{for } t, r \in J_*,
\]

where we recall that \( \epsilon_2 < 1 - (\alpha + \epsilon_1) \). This completes the proof. \( \square \)
we could not obtain \( u \in C_{t,s}^0L_\alpha^q\tilde{H}_2^\xi \), with some \( \theta > 0 \). We can explain the reason as follows. Look at the terms \( \int_0^t (t-s)^{-(\alpha+\epsilon_1+\epsilon_2)}K_1(s)ds \) in (40) and \( \int_0^t (t-s)^{-(2-\alpha)}K_2(s)ds \) in (44). If the overestimate (45) is used instead of the estimate in part a) of Lemma 3.2, there would be a trouble because the two terms we have mentioned would be changed by \( \int_0^t (t-s)^{-(\alpha+\epsilon_1+\epsilon_2)}K_1(s)ds \) and \( \int_0^t (t-s)^{-(2-\alpha)}K_2(s)ds \) respectively. Since two latter integrals are not convergent, we could not obtain \( u \in C_{t,s}^0L_\alpha^q\tilde{H}_2^\xi \), with some \( \theta > 0 \). Hence, it is required to improve the overestimate (45) by a new one as in part a) of Lemma 3.2.

4. Existence, regularity of the solution of the terminal value problem.

4.1. Mild solution for terminal value problem. Consider the following deterministic problem

\[
\begin{cases}
(\frac{\partial}{\partial t} - \Delta - \kappa \frac{\partial^\alpha}{\partial |\xi|^\alpha} \Delta) u = F(t,u), & (t,x) \in J \times \mathcal{X}, \\
u(t,x) = 0, & (t,x) \in J \times \partial \mathcal{X}, \\
u(T,x) = \varphi(x), & x \in \mathcal{X},
\end{cases}
\tag{46}
\]

From equation (7), by substituting \( t = T \), we have

\[
u_j(0) = \mathcal{R}_{\alpha,j}^{-1}(T)\varphi_j - \int_0^T \mathcal{R}_{\alpha,j}^{-1}(T)\mathcal{R}_{\alpha,j}(T-s)F_j(s,.,u(s,.))ds,
\tag{47}
\]

where \( \varphi_j := \langle \varphi, \phi_j^{(\xi)} \rangle_{H_2^\xi} \), \( \mathcal{R}_{\alpha,j} \) is given by (8). Combining (7) and (47), we obtain

\[
u_j(t) = \mathcal{R}_{\alpha,j}^{-1}(T)\mathcal{R}_{\alpha,j}(t)\varphi_j + \int_0^t \mathcal{R}_{\alpha,j}(t-s)F_j(s,.,u(s,.))ds
\]

\[
- \int_0^T \mathcal{R}_{\alpha,j}^{-1}(T)\mathcal{R}_{\alpha,j}(T-s)F_j(s,.,u(s,.))ds.
\]

Defining the operator \( \mathcal{G}_\alpha(t) \) as in (9) and

\[
\mathcal{G}_{\alpha,1}^T(t) := \mathcal{G}_\alpha^{-1}(T)\mathcal{G}_\alpha(t), \quad \mathcal{G}_{\alpha,2}^T(t,r) := \mathcal{G}_{\alpha,1}^T(t)\mathcal{G}_\alpha(r) = \mathcal{G}_\alpha^{-1}(T)\mathcal{G}_\alpha(t)\mathcal{G}_\alpha(r),
\]

we have the following representation for the solution to Problem (46)

\[
u(t,x) = \mathcal{G}_{\alpha,1}^T(t)\varphi(x) + \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,u(s,x))ds
\]

\[
- \int_0^T \mathcal{G}_{\alpha,2}^T(t,T-s)F(s,x,u(s,x))ds.
\tag{48}
\]

Motivated by (48), we give the definitions of mild solution to the terminal value problem (1),(2),(5) as follows:

**Definition 4.1.** An \( \mathcal{F}_T \)-adapted process \( \{u(t,.)\}_{t \in \mathcal{T}} \) in \( Z_{t,s}^\theta L_\alpha^q\tilde{H}_2^\xi \) with some \( \theta \in (0,1) \), \( q \geq 2, \xi \geq 0 \), is called a mild solution of Problem (1) if there holds that

\[
u(t,x) = \mathcal{G}_{\alpha,1}^T(t)\varphi(x) + \int_0^t \mathcal{G}_\alpha(t-s)F(s,x,u(s,x))ds
\]

\[
- \int_0^T \mathcal{G}_{\alpha,2}^T(t,T-s)F(s,x,u(s,x))ds.
\]
\[ + \int_0^T G_{\alpha}(t - s) \sigma(s, u(s, x))dW_\xi(s, x) \]
\[ - \int_0^T G_{\alpha,2}(t, T - s) \sigma(s, u(s, x))dW_\xi(s, x), \]
for almost all \( \omega \) for fixed \( t \).

4.2. Properties of solution operators.

**Lemma 4.2.** Given \( t, r \in \mathcal{J}_s, q \geq 2 \) and \( \xi \geq 0 \). Then
\[
\|G_{\alpha,1}^T(t)\|_{L^q_\xi \hat{H}^2_\xi \rightarrow L^q_\xi \hat{H}^2_\xi} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} t^{-\alpha - 1}. 
\]

**Proof.** Let \( v \in L^q_\xi \hat{H}^2_\xi \). By using Lemma 3.2 and the fact that \( R_{\alpha,2}(T) \geq c_{\alpha,1}\lambda_j^{-1} \) (see [18]), we deduce that \( 0 \leq \frac{R_{\alpha,1}(t)}{R_{\alpha,2}(T)} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} \lambda_j^{-1} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} t^{-\alpha - 1} \). It follows that
\[
\|G_{\alpha,1}^T(t)v\|_{L^q_\xi \hat{H}^2_\xi} = \|G_{\alpha,1}^{-1}(T)G_{\alpha}(v)\|_{L^q_\xi \hat{H}^2_\xi} \leq \left[ E \left( \sum_{j=1}^{\infty} \left( \frac{R_{\alpha,2}(T)}{R_{\alpha,2}(T)} \right)^2 \lambda_j^{2q} v_j^2 \right)^{q/2} \right]^{1/q} 
\]
\[ \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} t^{-\alpha - 1} \|v\|_{L^q_\xi \hat{H}^2_\xi} . \]

This completes the proof. \( \square \)

**Lemma 4.3.** Let \( r, s, \tau \in \mathcal{J}_s \) such that \( r > t \). For any real number \( \xi \), there holds
\[
\left\| \frac{d}{dt} G_{\alpha}(t) \right\|_{L^q_\xi \hat{H}^2_\xi \rightarrow L^q_\xi \hat{H}^2_\xi} + \left\| \frac{d}{dt} G_{\alpha,1}^T(t) \right\|_{L^q_\xi \hat{H}^2_\xi \rightarrow L^q_\xi \hat{H}^2_\xi} \lesssim t^{-(\alpha - 1)}. \]

**Proof.** Since \( K_{\alpha,2}(s) \leq \lambda_j^{-1}s^{-\alpha} \), thanks to the fundamental theorem of calculus, one deduces
\[
|R_{\alpha,2}(r) - R_{\alpha,2}(t)| = \int_0^\infty |e^{-sr} - e^{-st}| K_{\alpha,2}(s) ds 
\]
\[ \lesssim \int_0^\infty \frac{|e^{-sr} - e^{-st}|}{\lambda_j s^\alpha} ds \leq \frac{r - t}{\lambda_j t^{2-\alpha}}. \]

Hence, the classical derivative of \( R_{\alpha,2}(T) \) finitely exists for \( t \in \mathcal{J}_s \), and is bounded as follows
\[
P_{\alpha,2}(t) = \frac{d}{dt} R_{\alpha,2}(t) \lesssim \lambda_j^{-1} t^{-(\alpha - 1)} . 
\]

Let \( v \in L^q_\xi \hat{H}^2_\xi \). Applying the latter estimate,
\[
\|G_{\alpha}(r)v - G_{\alpha}(t)v\|_{L^q_\xi \hat{H}^2_\xi} \]
\[ \lesssim \left[ E \left( \sum_{j=1}^{\infty} \left| P_{\alpha,2}(s) \right| \lambda_j^{2q} v_j^2 \right)^{q/2} \right]^{1/q} \]
\[ \lesssim \left( r - t \right) t^{-(\alpha - 1)} \left[ E \left( \sum_{j=1}^{\infty} \lambda_j^{2q} v_j^2 \right)^{q/2} \right]^{1/q} \lesssim (r - t) t^{-(\alpha - 1)} \|v\|_{L^q_\xi \hat{H}^2_\xi} , \]
where we have used $\lambda_j^{-1} \leq \lambda_1^{-1}$. Similarly, we have

$$
\|G_{\alpha,1}^T(t)\|_{L_2^qH_x^\xi} = \left[ \mathbb{E} \left( \sum_{j=1}^{\infty} R_{\alpha,j}^{-2}(T) |R_{\alpha,j}(r) - R_{\alpha,j}(t)|^2 \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q}
$$

$$
\leq \left[ \mathbb{E} \left( \sum_{j=1}^{\infty} R_{\alpha,j}^{-2}(T) \int_t^T Q_{\alpha,j}(s) ds \right)^{q/2} \right]^{1/q}
$$

$$
\leq (r-t)t^{-(2-\alpha)} \left[ \mathbb{E} \left( \sum_{j=1}^{\infty} \lambda_j^{2\xi} v_j^2 \right)^{q/2} \right]^{1/q}
$$

$$
\leq (r-t)t^{-(2-\alpha)} \|v\|_{L_2^qH_x^\xi}.
$$

From two latter estimates, we conclude that (50) holds.

4.3. The existence and uniqueness of mild solution to Problem (1), (2), (5).

For $q \geq 2$, we denote $K_b(L_1^q) := \{ f : \int_0^1 f(t) \in L_1^q \}$. This subsection is aimed to contribute the existence and uniqueness of mild solution to Problem (1), (2), (5). To this end, we prepare some useful estimates for the terms appearing in equation (48).

Firstly, the following lemma shows upper bounds for two first terms in $\mathcal{Z}_{1,1}^{1-\alpha} L_2^q H_x^\xi$.

**Lemma 4.4.** Given $q \geq 2$ and $w \in \mathcal{Z}_{1,1}^{1-\alpha} L_2^q H_x^\xi$ with $\xi \geq 0$.

a) Assume that $\varphi \in L_2^q H_x^\xi$. Then, $G_{\alpha,1}^T(t)\varphi \in \mathcal{Z}_{1,1}^{1-\alpha} L_2^q H_x^\xi$ and

$$
\|G_{\alpha,1}^T(t)\varphi\|_{\mathcal{Z}_{1,1}^{1-\alpha} L_2^q H_x^\xi} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} \|\varphi\|_{L_2^q H_x^\xi}.
$$

b) Assume that $F$ is continuous in time, $F(\cdot, \cdot, 0) = 0$, $F \in L_{glo}(L_2^q H_x^\xi)$ such that $K_1 \in K_\alpha(L_1^q)$, then $\int_0^t G_\alpha(t-s)F(s,x,w(s,x))ds \in \mathcal{Z}_{1,1}^{1-\alpha} L_2^q H_x^\xi$, and

$$
t^{1-\alpha} \left\| \int_0^t G_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L_2^q H_x^\xi} \leq c_{\alpha,2} \tilde{K}_{1,\xi \geq 2} \|w\|_{\mathcal{Z}_{1,1}^{1-\alpha} L_2^q H_x^\xi},
$$

where $\tilde{K}_{1,\xi \geq 2} := \sup_{t \in \mathcal{J}} t^{1-\alpha} \int_0^t s^{-(1-\alpha)} K_1(s)ds$.

**Proof.** a) By Lemma 4.3, we can see that $\frac{d}{dt} G_{\alpha,1}^T(t)\varphi$ finitely exists in $L_2^q H_x^\xi$, and so that $G_{\alpha,1}^T(t)\varphi$ belongs to $C_{t,1} L_2^q H_x^\xi$. In addition, by Lemma 4.2, one can see that

$$
t^{(1-\alpha)} \left( \mathbb{E} \left\| G_{\alpha,1}^T(t)\varphi \right\|_{H_x^\xi}^q \right)^{1/q} \leq c_{\alpha,2} \left( \mathbb{E} \left\| \varphi \right\|_{H_x^\xi}^q \right)^{1/q},
$$

which shows $G_{\alpha,1}^T(t)\varphi \in \mathcal{Z}_{1,1}^{1-\alpha} L_2^q H_x^\xi$.

b) According to Lemma 3.3, Lemma 4.3 and noting that $F$ is continuous with respect to the first variable, it is obvious that the quantity $\int_0^t G_\alpha(t-s)F(s,x,w(s,x))ds$ belongs to the space $C_{t,1} L_2^q H_x^\xi$. Since $F \in L_{glo}(L_2^q H_x^\xi)$ and $F(\cdot, \cdot, 0) = 0$, we have the following estimate

$$
\mathbb{E} \left\| F(s,\cdot,w(s,\cdot)) \right\|_{H_x^\xi}^q \leq K_1^q(s) \mathbb{E} \left\| w(s,\cdot) \right\|_{H_x^\xi}^q.
$$
Therefore,
\[
\left\| \int_0^t G_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega H^{\xi}_x} \leq \int_0^t \left\| G_\alpha(t-s)F(s,\cdot,w(s,\cdot)) \right\|_{L^q_\omega H^{\xi}_x} ds
\]
\[
\leq c_{\alpha,2} \int_0^t K_1(s) \left( \mathbb{E} \|w(s,\cdot)\|^q \right)^{1/q} ds,
\]
which associates with \( \left( \mathbb{E} \|w(s,\cdot)\|^q \right)^{1/q} \leq s^{-(1-\alpha)}\|w\|_{Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x} \) (since \( w \in Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x \)) to obtain that
\[
(t^{1-\alpha}) \left\| \int_0^t G_\alpha(t-s)F(s,\cdot,w(s,\cdot))ds \right\|_{L^q_\omega H^{\xi}_x} \leq c_{\alpha,2} \left( \sup_{t \in J_*} t^{1-\alpha} \int_0^t s^{-(1-\alpha)}K_1(s)ds \right) \|w\|_{Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x}.
\]

This implies that \( \int_0^t G_\alpha(t-s)F(s,x,w(s,x))ds \in Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x \). □

Next, the following lemma states an upper bound for the forth term in \( Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x \).

**Lemma 4.5.** Consider \( q \geq 2 \) satisfying \( q(1-\alpha) < 1 \) and \( w \in Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x \) with \( \xi \geq 0 \). Assume that \( \sigma \) is continuous in time, \( \sigma(\cdot,0) = 0 \), \( \sigma \in L_{\text{glo}}(L^q_\omega H^{\xi}_x; L^q_\omega(\langle H^{\xi}_x \rangle)) \) such that \( K_2 \in K_\alpha(L^q_\omega) \).

a) If \( q = 2 \), then
\[
\int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,x) \in Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x,
\]
and
\[
t^{1-\alpha} \left\| \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,x) \right\|_{L^q_\omega H^{\xi}_x} \leq c_{\alpha,2} \bar{K}_{2,q=2} \|w\|_{Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x}.
\]

b) If \( q > 2 \), then
\[
\int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,x) \in Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x,
\]
and
\[
t^{1-\alpha} \left\| \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,x) \right\|_{L^q_\omega H^{\xi}_x} \leq c_{\alpha,2} C(q) \bar{K}_{2,q=2} \|w\|_{Z_{t,s}^{1-\alpha} L^q_\omega H^{\xi}_x}.
\]

Here, \( \bar{K}_{2,q=2} := \sup_{t \in J_*} t^{1-\alpha} \int_0^t s^{-(1-\alpha)}K^q_2(s)ds \), \( \bar{K}_{2,q>2} := \sup_{t \in J_*} t^{1-\alpha} \int_0^t K^q_2(s)s^{-(1-\alpha)q}ds \).

**Proof.** Lemma 3.3, Lemma 4.3 combined with the continuity of \( \sigma \) with respect to the first variable allow that the integral \( \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,x) \in C_{t,s} L^q_\omega H^{\xi}_x \). Now, let us consider the two following cases for \( q \).

- If \( q = 2 \), then the Itô isometry yields that
  \[
  \mathbb{E} \left\| \int_0^t G_\alpha(t-s)\sigma(s,w(s,\cdot))dW_\xi(s,x) \right\|_{H^{\xi}_x}^q
  \]
  \[
  = \mathbb{E} \int_0^t \left\| G_\alpha(t-s)\sigma(s,w(s,\cdot)) \right\|_{L^q_\omega H^{\xi}_x}^q ds
  \]
\[
\begin{align*}
&\leq \int_0^t \|G_\alpha(t-s)\|_{\dot{H}_x^1}^{q/2} E \|\sigma(s, w(s, \cdot))\|_{L^q_{\omega}(\dot{H}_x^1)}^{q/2} ds \\
&\leq c_{\alpha,2} \int_0^t E \|\sigma(s, w(s, \cdot))\|_{L^q_{\omega}(\dot{H}_x^1)}^{q} ds.
\end{align*}
\]

Since \( \sigma \in L_{\text{glo}}(L^q_0\dot{H}_x^1; L^q_0(\dot{H}_x^1)) \) and \( \sigma(\cdot, 0) = 0 \), we immediately deduce that
\[
E \|\sigma(s, w(s, \cdot))\|_{L^q_{\omega}(\dot{H}_x^1)}^{q} \leq K^q_2(s) E \|w(s, \cdot)\|_{\dot{H}_x^1}^{q}.
\]
Furthermore, it follows from \( w \in Z^{-\alpha}_{1,\omega} L^q_0\dot{H}_x^1 \) that \( E \|w(s, \cdot)\|_{\dot{H}_x^1}^{q} \leq s^{-(1-\alpha)q} \|w\|_{Z^{-\alpha}_{1,\omega} L^q_0\dot{H}_x^1} \). Henceforth,
\[
E \left\| \int_0^t G_\alpha(t-s)\sigma(s, w(s, \cdot))dW_\xi(s, \cdot) \right\|_{H_x^1}^{q} \leq c_{\alpha,2} \int_0^t s^{-(1-\alpha)q} K^q_2(s) ds \|w\|_{Z^{-\alpha}_{1,\omega} L^q_0\dot{H}_x^1}^{q},
\]
which consequently implies
\[
t^{1-\alpha} \left\| \int_0^t G_\alpha(t-s)\sigma(s, w(s, \cdot))dW_\xi(s, \cdot) \right\|_{L^q_0\dot{H}_x^1}^{1/q} \leq c_{\alpha,2} \left( \sup_{t \in T} t^{(1-\alpha)q} \int_0^t s^{-(1-\alpha)q} K^q_2(s) ds \right)^{1/q} \|w\|_{Z^{-\alpha}_{1,\omega} L^q_0\dot{H}_x^1}.
\]

- If \( q > 2 \), then using again the consequence of the Burkholder-Davis-Gundy inequality and the same techniques as the previous case,
\[
E \left\| \int_0^t G_\alpha(t-s)\sigma(s, w(s, \cdot))dW_\xi(s, \cdot) \right\|_{H_x^1}^{q} \leq C^q(q) E \left[ \left( \int_0^t \|G_\alpha(t-s)\sigma(s, w(s, \cdot))\|_{L^q_{\omega}(\dot{H}_x^1)}^{2} ds \right)^{q/2} \right] \leq C^q(q)c_{\alpha,2} E \left[ \left( \int_0^t \|\sigma(s, w(s, \cdot))\|_{L^q_{\omega}(\dot{H}_x^1)}^{2} ds \right)^{q/2} \right].
\]

Applying the Hölder inequality,
\[
\left[ E \left\| \int_0^t G_\alpha(t-s)\sigma(s, w(s, \cdot))dW_\xi(s, \cdot) \right\|_{H_x^1}^{q} \right]^{1/q} \leq c_{\alpha,2} C(q) \left( \int_0^t ds \right)^{(q-2)/2q} \left( \int_0^t E \|\sigma(s, w(s, \cdot))\|_{L^q_{\omega}(\dot{H}_x^1)}^{q} ds \right)^{1/q} \leq c_{\alpha,2} C(q) t^{\frac{q-2}{2q}} \left( \int_0^t K^q_2(s) E \|w(s, \cdot)\|_{\dot{H}_x^1}^{q} ds \right)^{1/q},
\]
where we recall that \( \sigma(\cdot, 0) = 0, \sigma \in L_{\text{glo}}(L^q_0\dot{H}_x^1; L^q_0(\dot{H}_x^1)) \). Multiplying both sides of the latter estimates with \( t^{1-\alpha} \),
\[
t^{1-\alpha} \left\| \int_0^t G_\alpha(t-s)\sigma(s, w(s, \cdot))dW_\xi(s, \cdot) \right\|_{L^q_0\dot{H}_x^1} \leq c_{\alpha,2} C(q) \left( \sup_{t \in T} t^{\frac{q-2}{2q}} \int_0^t K^q_2(s) s^{-(1-\alpha)q} ds \right)^{1/q} \|w\|_{Z^{-\alpha}_{1,\omega} L^q_0\dot{H}_x^1}.
This completes the proof. \[\square\]

Using two previous lemmas, we now aim to establish the existence and uniqueness of the mild solution of the terminal value problem \((1),(3),(5)\). We denote the constants

\[\Pi_1 := c_{\alpha,2} \left( 1 + \frac{c_{\alpha,2}}{c_{\alpha,1}} T^{\alpha-1} \right) \left( \overline{K}_{1,q=2} + \overline{K}_{2,q=2}^{1/q} \right),\]

\[\Pi_2 := c_{\alpha,2} \left( 1 + \frac{c_{\alpha,2}}{c_{\alpha,1}} T^{\alpha-1} \right) \left( \overline{K}_{1,q=2} + C(q) \overline{K}_{2,q=2}^{1/q} \right).\]

**Theorem 4.6.** Assume that \(\varphi, F, K_1\) satisfy hypotheses in Lemma 4.4, and \(\sigma, K_2\) satisfy the ones in Lemma 4.5. If the condition \(\Pi_1 \leq 2 + \Pi_2 < 1\) holds, then Problem \((1),(3),(5)\) possesses a unique solution

\[u \in Z_{l,s}^{\alpha-1} L_0^q \dot{H}_x^q.\]  

Furthermore, there holds that

\[\|u(t,\cdot)\|_{L_0^q \dot{H}_x^q} \lesssim t^{\alpha-1}, \quad t \in \overline{J}_s.\]  

**Corollary 3.** Consider the case \(\xi = 0\). If all assumptions in Theorem 4.6 are satisfied, then \(u\) belongs to \(Z_{l,s}^{\alpha-1} L_0^q L_x^2\).

**Corollary 4.** If \(K_1, K_2\) are two positive constants (independent of \(t\)), then conditions \(K_1 \in K_\alpha(L^1_t)\) and \(K_2 \in K_\alpha(L^1_t)\) hold. Hence, the results (53)-(54) hold in this case.

**Proof.** Let us first consider the case \(q = 2\) and \(\Pi_1 < 1\). To prove the existence of the mild solution in \(Z_{l,s}^{1-\alpha} L_0^q \dot{H}_x^q\), we will use the Banach contraction principle. After proving this mapping is well-defined, we will prove it has a unique fixed point in a ball of this space. For the sake of convenience, we naturally divide the proof into the following steps.

**Step 1. Constructing a well-defined mapping on** \(Z_{l,s}^{1-\alpha} L_0^q \dot{H}_x^q\). Let us denote

\[S\varphi(t,x) := G_{\alpha,1}^T(t) \varphi(x) + \int_0^t G_{\alpha}(t-s) F(s,x,w(s,x)) ds\]

\[+ \int_0^t G_{\alpha,2}(t-s) F(s,x,w(s,x)) ds + \int_0^t G_{\alpha}(t-s) \sigma(s,w(s,\cdot)) d\mathcal{W}_x(s,x)\]

\[+ \int_0^t G_{\alpha,2}(t-s) \sigma(s,w(s,\cdot)) d\mathcal{W}_x(s,x).\]

By applying Lemma 4.4 and Lemma 4.5, we deduce that the two first and forth terms of the right-hand-side belong to \(Z_{l,s}^{1-\alpha} L_0^q \dot{H}_x^q\). Let us consider the third and last terms. Recall that we have proved in Part b of Lemma 4.4 and to Part a of Lemma 4.5 that

\[t^{1-\alpha} \left\| \int_0^t G_{\alpha}(t-s) F(s,\cdot,w(s,\cdot)) ds \right\|_{L_0^q \dot{H}_x^q} \leq c_{\alpha,2} \overline{K}_{1,q=2} \|w\|_{Z_{l,s}^{1-\alpha} L_0^q \dot{H}_x^q},\]

\[t^{1-\alpha} \left\| \int_0^t G_{\alpha}(t-s) \sigma(s,w(s,\cdot)) d\mathcal{W}_x(s,\cdot) \right\|_{L_0^q \dot{H}_x^q} \leq c_{\alpha,2} \overline{K}_{2,q=2}^{1/q} \|w\|_{Z_{l,s}^{1-\alpha} L_0^q \dot{H}_x^q}.\]
for each $t \in J_*$. This together with Part a of Lemma 4.4 implies that
\[
\int_0^T G_{\alpha,2}^T(t, T - s) F(s, x, w(s, x)) ds = G_{\alpha,1}^T(t) \int_0^T G_{\alpha}(T - s) F(s, x, w(s, x)) ds \tag{57}
\]
and
\[
\int_0^T G_{\alpha,2}^T(t, T - s) \sigma(s, w(s, \cdot)) dW_t(s, x) = G_{\alpha,1}^T(t) \int_0^T G_{\alpha}(T - s) \sigma(s, w(s, \cdot)) dW_t(s, x) \tag{58}
\]
which belong to the space $Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi$. Thus, we conclude that $S w$ belongs to $Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi$.

**Step 2.** Finding a ball $\overline{B}(0; R) \subset Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi$ of center 0 and radius $R$ such that $S$ is a contraction mapping on. By taking supremum on $t \in J_*$, two inequalities yield that
\[
\left\| \int_0^T G_{\alpha}(t - s) F(s, \cdot, w(s, \cdot)) ds \right\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi} \leq c_{\alpha,2} \tilde{K}_{1, q \geq 2} \|w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi}.
\]
\[
\left\| \int_0^T G_{\alpha}(t - s) \sigma(s, w(s, \cdot)) dW_t(s, \cdot) \right\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi} \leq c_{\alpha,2} \tilde{K}_{2, q \geq 2} \|w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi}.
\]
By combining Part a and Part b of Lemma 4.4, one can obtain the following estimate
\[
\left\| \int_0^T G_{\alpha,2}^T(t, T - s) F(s, \cdot, w(s, \cdot)) ds \right\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} \left\| \int_0^T G_{\alpha}(T - s) F(s, \cdot, w(s, \cdot)) ds \right\|_{L_2^0 \dot{H}_t^\xi} \leq \frac{c_{\alpha,2}^2}{c_{\alpha,1}} \tilde{K}_{1, q \geq 2} \|w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi}.
\]
Similarly, combining Part a of Lemma 4.4 and Part a of Lemma 4.5 implies
\[
\left\| \int_0^T G_{\alpha,2}^T(t, T - s) \sigma(s, w(s, \cdot)) dW_t(s, \cdot) \right\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi} \leq \frac{c_{\alpha,2}}{c_{\alpha,1}} \left\| \int_0^T G_{\alpha}(T - s) \sigma(s, w(s, \cdot)) dW_t(s, \cdot) \right\|_{L_2^0 \dot{H}_t^\xi} \leq \frac{c_{\alpha,2}^2}{c_{\alpha,1}} \tilde{K}_{2, q \geq 2} \|w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi}. \tag{59}
\]
In the above arguments, we have used the estimates for the four last term of $S w$. Meanwhile, since $\varphi$ is defined by Lemma 4.4, the first term of $S w$ is estimated by $c_{\alpha,2}^2 c_{\alpha,1}^2 \|\varphi\|_{L_2^0 \dot{H}_t^\xi}$. All these estimates can be taken together to allow that
\[
\|S w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi} \leq c_{\alpha,2}^2 c_{\alpha,1}^2 \|\varphi\|_{L_2^0 \dot{H}_t^\xi} + \Pi_1 \|w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi}.
\]
Since $\Pi_1 < 1$, there always exists a positive number $R$ which satisfies the equation $c_{\alpha,2}^2 c_{\alpha,1}^2 \|\varphi\|_{L_2^0 \dot{H}_t^\xi} + \Pi_1 R = R$. Therefore, for all $w \in \overline{B}(0; R)$, we have $\|S w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi} \leq R$, i.e., $S w$ belongs to $Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi$. Moreover, the above arguments show that
\[
\|S w^\dagger - S w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi} \leq C_1 \|w^\dagger - w\|_{Z_{t,*,\alpha}^{1-\alpha} L_2^0 \dot{H}_t^\xi},
\]
where we recall that $F \in L_{glo}(L^q_x \dot{H}^\xi_x)$ and $\sigma \in L_{glo}(L^q_x \dot{H}^\xi_x; L^2_0(\dot{H}^\xi_x))$.

Finally, the existence and uniqueness of a mild solution $u$ in $Z_{1-\alpha}^{L^q_x \dot{H}^\xi_x}$ is obtained by the Banach contraction principle. For the case $q > 2$, $\Pi_2 < 1$, we note that the proof can be similarly based on Part a where it is necessary to estimate the Itô integral (59) in the case $q > 2$ (instead of $q = 2$) by using Part b of Lemma 4.5.

5. Conclusion. In this study, we consider the initial value problem and the terminal value problem for a stochastic time-fractional Rayleigh-Stokes equation, where the source function and the time-spatial noise are nonlinear. By using some useful stochastic analysis techniques and fractional calculus, we obtain the existence, uniqueness of the mild solution of each problem. Furthermore, some regularity properties and continuity results for the solutions are also proposed.

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REFERENCES

[1] E. Bazhlekova, B. Jin, R. Lazarov and Z. Zhou, An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid, Numer Math., 131 (2015), 1–31.
[2] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.
[3] L. Debri, Well-posedness of the multidimensional fractional stochastic Navier-Stokes equations on the torus and on bounded domains, J. Math. Fluid Mech., 18 (2016), 25–69.
[4] M. Dehghan, A computational study of the one-dimensional parabolic equation subject to non-classical boundary specifications. Numer. Methods Partial Differential Equations, 22 (2006), 220–257.
[5] M. Dehghan, The one-dimensional heat equation subject to a boundary integral specification, Chaos Solitons Fract., 32 (2007), 661–675.
[6] M. Dehghan and M. Abbaspadeh, A finite element method for the numerical solution of Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives, Eng Comput., 33 (2017), 587–605.
[7] C. Fetecau, M. Jamil, C. Fetecau and D. Vieru, The Rayleigh-Stokes problem for an edge in a generalized Oldroyd-B fluid, Z. Angew. Math. Phys., 60 (2009), 921–933.
[8] G. Hu, Y. Lou and P. D. Christofides, Dynamic output feedback covariance control of stochastic dissipative partial differential equations, Chem. Eng. Sci., 63 (2008), 4531–4542.
[9] Y. Jiang, T. Wei and X. Zhou, Stochastic generalized Burgers equations driven by fractional noises, J. Differ. Equ., 252 (2012), 1934–1961.
[10] M. Khan, The Rayleigh-Stokes problem for an edge in a viscoelastic fluid with a fractional derivative model, Nonlinear Anal. Real World Appl., 10 (2009), 3190–3195.
[11] R. Kruse, Strong and Weak Approximation of Semilinear Stochastic Evolution Equations, Springer, 2014.
[12] M. Lakestani and M. Dehghan, The use of Chebyshev cardinal functions for the solution of a partial differential equation with an unknown time-dependent coefficient subject to an extra measurement, J. Comput. Appl. Math., 235 (2010), 669–678.
[13] P. D. Lax, Functional Analysis, Wiley Interscience, New York, 2002.
[14] Y. Li and R. Wang, Regular measurable dynamics for reaction-diffusion equations on narrow domains with rough noise, Discrete Contin. Dyn. Syst., 38 (2018), 3663–3685.
[15] Y. Li and R. Wang, Limiting dynamics for stochastic reaction-diffusion equations on the Sobolev space with thin domains, Comput. Math. Appl., 79 (2020), 457–475.
[16] Y. Li and Y. Wang, The existence and asymptotic behavior of solutions to fractional stochastic evolution equations with infinite delay. J. Differential Equations, 266 (2019), 3514–3558.
[17] J. Liang, X. Qian, T. Shen and S. Song, Analysis of time fractional and space nonlocal stochastic nonlinear Schrödinger equation driven by multiplicative white noise, *J. Math. Anal. Appl.*, **466** (2018), 1525–1544.

[18] T. B. Ngoc, N. H. Luc, V. V. Au, N. H. Tuan and Z. Yong, Existence and regularity of inverse problem for the nonlinear fractional Rayleigh-Stokes equations, *Math. Meth. Appl. Sci.*, (2020), 1–27.

[19] H. L. Nguyen, H. T. Nguyen, K. Mokhtar and X. T. Duong Dang, Identifying initial condition of the Rayleigh-Stokes problem with random noise, *Math. Meth. Appl. Sci.*, **42** (2019), 1561–1571.

[20] H. L. Nguyen, H. T. Nguyen and Y. Zhou, Regularity of the solution for a final value problem for the Rayleigh-Stokes equation, *Math. Methods Appl. Sci.*, **42** (2019), 3481–3495.

[21] P. Niu, T. Helin and Z. Zhang, An inverse random source problem in a stochastic fractional diffusion equation, *Inverse Problems*, **36** (2020), 045002, 23 pp.

[22] M. Podlubny, *Fractional Differential Equations*, Academic Press, Inc., San Diego, CA, 1999.

[23] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Springer, 2007.

[24] N. Samko, A. Kilbas and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

[25] F. Shen, W. Tan, Y. Zhao and T. Masuoka, The Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative model, *Nonlinear Anal. Real World Appl.*, **7** (2006), 1072–1080.

[26] X. Su and M. Li, The regularity of fractional stochastic evolution equations in Hilbert space, *Stoch. Anal. Appl.*, **36** (2018), 639–653.

[27] N. H. Tuan, Y. Zhou, T. N. Thach and N. H. Can, Initial inverse problem for the nonlinear fractional Rayleigh-Stokes equation with random discrete data, *Commun. Nonlinear Sci. Numer. Simul.*, **78** (2019), 104873, 18 pp.

[28] H. Ye, J. Gao and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, **328** (2007), 1075–1081.

[29] C. Zhao and C. Yang, Exact solutions for electro-osmotic flow of viscoelastic fluids in rectangular micro-channels, *Appl. Math. Comput.*, **211** (2009), 502–509.

[30] G. Zou and B. Wang, Stochastic Burgers’ equation with fractional derivative driven by multiplicative noise, *Comput. Math. Appl.*, **74** (2017), 3195–3208.

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