A simplified formalism of the algebra of partially transposed permutation operators with applications

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Abstract
Herein we continue the study of the representation theory of the algebra of permutation operators acting on the n-fold tensor product space, partially transposed on the last subsystem. We develop the concept of partially reduced irreducible representations, which allows us to significantly simplify previously proved theorems and, most importantly, derive new results for irreducible representations of the mentioned algebra. In our analysis we are able to reduce the complexity of the central expressions by getting rid of sums over all permutations from the symmetric group, obtaining equations which are much more handy in practical applications. We also find relatively simple matrix representations for the generators of the underlying algebra. The obtained simplifications and developments are applied to derive the characteristics of a deterministic port-based teleportation scheme written purely in terms of irreducible representations of the studied algebra. We solve an eigenproblem for the generators of the algebra, which is the first step towards a hybrid port-based teleportation scheme and gives us new proofs of the asymptotic behaviour of teleportation fidelity. We also show a connection between the density operator characterising port-based teleportation and a particular matrix composed of an irreducible representation of the symmetric group, which encodes properties of the investigated algebra.

Keywords: symmetric group, irreducible representation, port-based teleportation

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1. Introduction

In the classical theory of representation of the symmetric group $S(n)$, the swap or permutation representation, closely related to the famous Schur–Weyl duality [15], plays an important role not only in pure mathematics, but also has found a wide range of applications in quantum information theory. Here we mention only a few of them, i.e. qubit quantum cloning [11], the theory of quantum gates [1, 7, 8], quantum error correcting codes [21], distillation of quantum entanglement [12], task of optimal compression of identical particles [31] and some aspects of theory of reference frames [32]. The basic objects in the description are the operators $V_d(\sigma)$ representing the group elements $\sigma \in S(n)$ acting in the tensor product space $\mathbb{C}^d^\otimes n$ as permutations on the basis vectors in the algebra of tensor operators. It turns out that the above mentioned picture can be modified by introducing into it the notion of partial transposition, which has a well-established position in entanglement detection theory due to the famous Peres–Horodecki criterion [16, 26], investigating set of states with positive partial transposition (PPT) [17] and, from the view of representation theory, recent developments in port-based teleportation (PBT) protocol [23, 28] and the theory of universal qudit quantum cloning machines [27], where the representation theory of partially transposed operators $V_d(\sigma)$ was used. Because of this non-trivial connection between possible practical applications and pure mathematical theory, there is a need to study the much deeper properties of the operators $V_d(\sigma)$ affected by partial transposition. In the series of papers [24, 29], the first major step was taken and the theory of irreducible representations (irreps) of partially transposed permutation operators $V_d^n(\sigma)$ was formulated, where by $t_n$ we denote partial transposition with respect to the last subsystem. It is known that there is a connection between the algebra $A_d^n(d)$ and the walled Brauer algebra (WBA) [3, 9, 22], which is a sub-algebra of the Brauer algebra [5, 14, 25]. Namely, algebra $A_d^n(d)$ is a representation of WBA [33]. From [6] we know that whenever $d > n - 1$ the dimension of WBA is equal to $n!$, which is the same as for $A_d^n(d)$ [24, 29]. In this case these two algebras are isomorphic and we know the characterisation of the irreducible components. When condition $d > n - 1$ is not satisfied we have $\dim A_d^n(d) < n!$, while the dimension of WBA is still equal to $n!$. Because of this we do not have isomorphism between these two algebras and a full investigation in this case is needed. Namely, in both cases, from the point of view of possible applications, additional knowledge about the explicit orthogonal bases in every irreducible space and matrix representation of irreps is required. As we mentioned at the beginning of this section, such full characterisation of $A_d^n(d)$ including matrix representations of irreps for all possible relations between $n$ and $d$ was presented in [24, 29].

The formulas describing the representation theory of the algebra $A_d^n(d)$ in the original picture given in [24, 29] are complicated and difficult to use in practical applications. Fortunately, it turns out that if we use a particular kind of irrep of the symmetric group, namely a partially reduced irreducible representation (PRIR), the complicated expressions in the representation theory of the algebra $A_d^n(d)$ may be simplified significantly and may be written in a more explicit way, which allows much easier application. The biggest benefit is the reduction of the complexity of existing equations by reducing the number of sums over all permutations from the symmetric group, which significantly improves the time computation and allows us to prove new properties of $A_d^n(d)$. It should be mentioned that in applying PRIRs we, in fact, do not lose generality, because any irrep of $S(N)$ may be unitarily transformed into PRIR form. Because of the great importance of the described tools further analysis is required, so in this paper we develop the idea of PRIRs first introduced in [28] by presenting their new properties.

Our paper is organised as follows. In section 2 we recall briefly the structure of irreps of the algebra of partially transposed permutation operators $A_d^n(d)$. We present all of the most important theorems of this paper in their original form, taken from [24]. In section 3 we
For self-consistency of this paper, we present here all the most important facts about algebra operators. In particular, we simplify the expressions for two special matrices describing the properties of the underlying algebra. Section 5 is fully devoted to the application of the simplified representation formalism of the algebra \( \mathcal{A}_n^t(d) \) to deterministic PBT (dPBT). We show the deep connection between the PBT operator and the matrix which encodes the properties of the investigated algebra. We solve an eigenproblem for matrix generators of \( \mathcal{A}_n^t(d) \) which is one step toward a hybrid scheme of PBT [18]. This result also allows us to present an alternative proof of the lower bound of teleportation fidelity presented in [2]. In particular, we derive in a simpler way than previously expressions for the fidelity in a deterministic version of the protocol and we describe some of its asymptotic properties.

2. Representation theory of algebra of the partially transposed permutation operators

For self-consistency of this paper, we present here all the most important facts about algebra \( \mathcal{A}_n^t(d) \) of partially transposed permutation operators, preceded by an introduction to the notation, which is essential for proper understanding of further sections. Then, in section 2.2, we only briefly summarise in a possible simple way the structure of \( \mathcal{A}_n^t(d) \) and explain why the original picture is inconvenient for practical use. This should give the reader a flavour of the problem before section 2.3, where all the important technical details are presented. In both the following subsections, as well later, we keep the original notation taken from [24, 29].

2.1. Definitions and notations

Let us start here by considering a permutational representation \( V \) of the group \( S(n) \) in the space \( \mathcal{H} \equiv (\mathbb{C}^d)^{\otimes n} \) defined in the following way:

**Definition 1.** \( V : S(n) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \) and
\[
\forall \sigma \in S(n) \quad V(\sigma)e_1 \otimes e_2 \otimes \cdots \otimes e_n = e_{\sigma^{-1}(1)} \otimes e_{\sigma^{-1}(2)} \otimes \cdots \otimes e_{\sigma^{-1}(n)},
\]
where \( d \in \mathbb{N} \) and \( \{e_i\}_{i=1}^d \) is an orthonormal basis of the space \( \mathbb{C}^d \).

The representation \( V : S(n) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \) is defined in a given basis \( \{e_i\}_{i=1}^d \) of the space \( \mathbb{C}^d \) (and consequently in a given basis of \( H \)), so in fact it is a matrix representation.

**Remark 2.** The representation \( V : S(n) \to \text{Hom}((\mathbb{C}^d)^{\otimes n}) \), which we will denote more shortly as \( V(S(n)) \), depends explicitly on the dimension \( d \), so in fact we should write \( V(S(n)) \equiv V_d(S(n)) \), but for simplicity we will omit the index \( d \), unless it is necessary for clarity.

Let us assume that we are given the partition \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \) of some natural number \( n \) (we denote this by \( \mu \vdash n \)), then by \( h(\mu) \) we denote the height (equivalently the number of rows) of the corresponding Young diagram \( Y^\mu \). Since there is one-to-one correspondence between partitions of natural number \( n \) and inequivalent irreps of the symmetric group \( S(n) \), we use symbols \( \alpha, \mu \), etc interchangeably for partitions and irreps in the underlying algebra and projections onto irreducible subspaces of \( \mathcal{A}_n^t(d) \). We show the deep connection between the PBT operator and the matrix which encodes the properties of the investigated algebra. We solve an eigenproblem for matrix generators of \( \mathcal{A}_n^t(d) \) which is one step toward a hybrid scheme of PBT [18].
Proposition 3. The irrep $\psi^\mu$ of $S(n)$, indexed by the partition $\mu \vdash n$ is contained in $V(S(n))$ if $d \geq k \equiv h(\mu)$. In particular if $d \geq n$ then all irreps of $S(n)$ are included in the representation $V(S(n))$. When $d \geq k \equiv h(\mu)$ then the multiplicity of the irrep $\psi^\mu$ of $S(n)$ is equal to
\[
\frac{1}{n!} \sum_{\sigma \in S(n)} \chi^\mu(\sigma^{-1})d^{(\sigma)} = \frac{1}{n!} \sum_{\sigma \in S(n)} \chi^\mu(\sigma^{-1})\chi^V(\sigma),
\]
where $\chi^\mu(\cdot)$ is the character of $\psi^\mu$, $l(\sigma)$ is the number of cycles in the permutation $\sigma$ and $\chi^V(\sigma) = d^{(\sigma)}$ is the character of the representation $V(S(n))$.

The representation $V(S(n))$ extends in a natural way to the representation of the group algebra $\mathbb{C}[S(n)]$ and in this way we obtain the algebra
\[
\mathcal{A}_n(d) \equiv \text{span}_\mathbb{C}\{V(\sigma) : \sigma \in S(n)\} \subset \text{Hom}(\mathbb{C}^d \otimes \mathbb{C}^d)
\]
of operators representing the elements of the group algebra $\mathbb{C}[S(n)]$. Note that the algebra $\mathcal{A}_n(d)$ contains a natural subalgebra
\[
\mathcal{A}_{n-1}(d) \equiv \text{span}_\mathbb{C}\{V(\sigma_{n-1}) : \sigma_{n-1} \in S(n-1)\}.
\]

The algebra of partially transposed operators with respect to the last subsystem, the basic object of this paper, is defined in the following way:

Definition 4. For $\mathcal{A}_n(d) \equiv \text{span}_\mathbb{C}\{V(\sigma) : \sigma \in S(n)\}$ we define a new complex algebra
\[
\mathcal{A}_n^\sigma(d) \equiv \text{span}_\mathbb{C}\{V^\sigma(\sigma) : \sigma \in S(n)\} \subset \text{Hom}(\mathbb{C}^d \otimes \mathbb{C}^d),
\]
where the symbol $t_n$ denotes the partial transposition with respect to the last subsystem in the space $\text{Hom}(\mathbb{C}^d \otimes \mathbb{C}^d)$. The elements $V^\sigma(\sigma) : \sigma \in S(n)$ will be called natural generators of the algebra $\mathcal{A}_n^\sigma(d)$.

Notation 5. Further in the text whenever partial transposition $t_n$ changes the elements $V(\sigma_n) \in V(S(n)) \subset \mathcal{A}_n^\sigma(d)$, we will write $V'(\sigma)$ instead of $V^\sigma(\sigma)$. In particular, whenever $\sigma = (n-1, n)$ we will write simply $V'$. When the partial transposition does not change the elements $V(\sigma_n) \in V(S(n-1)) \subset \mathcal{A}_n^\sigma(d)$, therefore, in the following we will write $V(\sigma_{n-1})$ instead of $V^\sigma(\sigma_{n-1})$.

Remark 6. The algebra $\mathcal{A}_n^\sigma(d)$ is defined as the algebra of operators acting in the space $(\mathbb{C}^d)^\otimes$, so in this way we obtain a natural representation of the algebra $\mathcal{A}_n^\sigma(d)$ in the space $(\mathbb{C}^d)^\otimes$. The algebra $\mathcal{A}_n^\sigma(d)$ is semi-simple [24], so this natural representation is a direct sum of irreps of the algebra $\mathcal{A}_n^\sigma(d)$.

2.2. Introduction to the problem

The important feature of the algebra $\mathcal{A}_n^\sigma(d)$ is the fact that it contains a subalgebra $\mathcal{A}_{n-1}(d)$, generated by operators representing the subgroup $S(n-1) \subset S(n)$, which are not changed by the partial transposition (these operators will be denoted $V_d(\sigma) : \sigma \in S(n-1)$). In the papers [24, 29] it has been shown that algebra $\mathcal{A}_n^\sigma(d)$ splits into a direct sum of two left
ideals \( A_{\sigma}^d(d) = \mathcal{M} \oplus S \), which differ structurally and, in consequence, the structure of the irreps of the algebra \( A_{\sigma}^d(d) \) is of two kinds and it is strictly connected with the irreps of the groups \( S(n-2) \) and \( S(n-1) \). In particular, the matrix elements of the representations of the algebra \( A_{\sigma}^d(d) \) are expressed in terms of matrix elements of the representations of the groups \( S(n-2) \) and \( S(n-1) \). The irreps of the first type of \( A_{\sigma}^d(d) \) (later called non-trivial) are indexed by irreps of the group \( S(n-2) \) and they are strictly connected with the representations of the group \( S(n-1) \). Speaking more precisely in these representations, when the condition \( d > n-2 \) is satisfied, the elements \( V_d(\sigma) : \sigma \in S(n) \) of the algebra \( A_{\sigma}^d(d) \) are represented as in the representations of the group \( S(n-1) \) induced by irreps of \( S(n-2) \) and the dimension of such a representation of the first type of \( A_{\sigma}^d(d) \) is equal to the dimension the induced representation of \( S(n-1) \). When \( d \leq n-2 \) the situation is more complicated. In this case some of the irreps of the first type may be defined on some subspace of the representation space of induced representation of \( S(n-1) \). In both cases the non-trivially partially transposed generators are represented in these representations by complicated expressions. In particular, the equation for transposition generators \( V_d[(a,n)] \), where \( a = 1, \ldots, n-1 \), as well as being important expressions for the projectors onto the non-trivial irreducible subspaces of the algebra \( A_{\sigma}^d(d) \), are also very complicated and have high complexity since we have to deal with sums over all elements from the permutation group. These complicated formulas were derived for arbitrary forms of the irreps of the groups \( S(n-2) \) and \( S(n-1) \) in terms of which the representations of the algebra \( A_{\sigma}^d(d) \) are expressed, so they are not very practical in terms of some applications. The representations of the second type are indexed by some irreps of the group \( S(n-1) \). In this case the generators \( V(\sigma) : \sigma \in S(n-1) \) of the algebra \( A_{\sigma}^d(d) \) are represented naturally by operators of irreps of \( S(n-1) \), whereas the non-trivially partially transposed operators are represented by zero operators. So, in the representations of this type only the subalgebra \( A_{\sigma-1}(d) \) of \( A_{\sigma}^d(d) \) is represented non-trivially, therefore these irreps of the algebra \( A_{\sigma}^d(d) \) may be called semi-trivial and we will not consider them.

2.3. Technical summary of known results

As we will see later in the analysis of the algebra \( A_{\sigma}^d(d) \), as well in applications to PBT, the matrix \( Q \) plays a very important role, which appears naturally in the theory of representation of the algebra \( A_{\sigma}^d(d) \), namely we have (see [24, 29]).

**Definition 7.** For any irrep \( \varphi^{\alpha} \) of dimension \( d_{\alpha} \) of the group \( S(n-2) \) we define the block matrix

\[
Q_{\sigma-1}(\alpha) \equiv Q(\alpha) = (d_{\alpha} \varphi^{\alpha}_{\sigma}[(a,n-1)(a,b)(b,n-1)]) = (Q_{ij}(\alpha)) \in M((n-1)d_{\alpha}, \mathbb{C}),
\]

(6)

where \( a, b = 1, \ldots, n-1 \), \( i, j = 1, \ldots, d_{\alpha} \). The blocks of the matrix \( Q(\alpha) \) are labelled by indices \( (a,b) \), whereas the elements of the blocks are labelled by the indices of the irreducible representation \( \varphi^{\alpha} = (\varphi^{\alpha}_{ij}) \) of the group \( S(n-2) \).

Below we recall the most important spectral properties of the above defined matrices.

**Proposition 8.** The matrices \( Q(\alpha) \) are Hermitian, positive semi-definite. Eigenvalues \( \lambda_{\nu}(\alpha) \) of \( Q(\alpha) \) are labelled by the irreps \( \psi^{\nu} \in \text{ind}S^{(n-1)}(\varphi^{\alpha}) \), and the multiplicities of \( \lambda_{\nu}(\alpha) \) are equal to \( d_{\nu} \). Moreover at most one (up to the multiplicity) eigenvalue \( \lambda_{\nu}(\alpha) \) of the matrix \( Q(\alpha) \) may be equal to zero.
Remark 9. The matrix $Q_{y}^{a}(\alpha)$ in the representation space has the form:

$$Q_{y}^{a}(\alpha) = \begin{pmatrix}
1 & \varphi^{a}[(1, 2)] & \cdots & \varphi^{a}[(1, n-2)] & 1 \\
\varphi^{a}[(2, 1)] & 1 & \cdots & \varphi^{a}[(2, n-2)] & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cdots & \cdots & 1 & 1
\end{pmatrix},$$

where every $\varphi^{a}[(a, b)] = \{\varphi^{a}[(a, b)]\}$ is a representation matrix of permutation $(a, b)$ in an irrep of $S(n-2)$ labelled by $\alpha$. It is worth mentioning here that in the general case there is always the possibility to choose matrices $\varphi^{a}$ to be unitary, so we obtain $\varphi^{a}_{y}[(a, b)] = \varphi^{a}_{y}[(a, b)]$. In our paper constraints are even stronger because representations $\varphi^{a}[(a, b)]$ are in the form of symmetric and real matrices, so we have $\varphi^{a}_{y}[(a, b)] = \varphi^{a}_{y}[(a, b)]$.

Before we go further let us define a few additional concepts, in particular the so-called rank of the partition for the corresponding Young diagram.

Definition 10. By $\widehat{S}(N)$ we denote the set of all irreps of the symmetric group $S(N)$, and by $[\widehat{S}(N)]$ its cardinality. If $\psi^{\mu} \in \widehat{S}(N)$ is an irrep of the group $S(N)$ we write

$$\widehat{S}_{d}(N) \equiv \{ \psi^{\mu} \in \widehat{S}(N) : h(\mu) \leq d \} \Rightarrow \widehat{S}_{N}(N) = \widehat{S}(N).$$

The above set contains all irreps of $S(n)$ whose corresponding Young diagrams have no more than $d$ rows.

Definition 11. Let $\psi^{\mu}$ be any irrep of the group $S(n)$, $\mu = (\mu_{1}, \ldots, \mu_{k})$ its partition and $Y^{\mu}$ the corresponding Young diagram. The rank $r = r(\mu)$ (or $r(Y^{\mu})$) of the partition $\mu$ is the length of the diagonal of its Young diagram.

Now we are in a position to present the spectral theorem for the matrices $Q(\alpha)$ (see [24]).

Theorem 12.

(a) Let $\varphi^{a}$ be any irrep of the group $S(n-2)$, $\alpha = (\alpha_{1}, \ldots, \alpha_{k})$ its partition and $Y^{\alpha}$ the corresponding Young diagram. Suppose that for some index $1 \leq i \leq k$ the sequence $\nu = (\alpha_{1}, \ldots, \alpha_{i} + 1, \ldots, \alpha_{k})$ is a partition of $n-1$, so it defines an irrep $\psi^{\nu}$ of the group $S(n-1)$. For Young diagrams it means that the Young diagram $Y^{\nu}$ is obtained from the Young diagram $Y^{\alpha}$ by adding, in the $i$th row, one box (we denote this by $\nu \in \alpha$). Then the corresponding matrix $Q_{n-1}^{a}(\alpha)$ has the following eigenvalues:

(i) If $r(Y^{\alpha}) = r(Y^{\nu})$, then

$$\lambda_{\nu}(\alpha) = d + \alpha_{i} + 1 - i, \quad i = 1, \ldots, k + 1,$$

and if $i = k + 1$ we set $\alpha_{k+1} = 0$.

(ii) If $r(Y^{\alpha}) + 1 = r(Y^{\nu})$ which may occur only if $i = r + 1$, then

$$\lambda_{\nu}(\alpha) = d.$$

The case (ii) describes the situation when by adding, in a proper way, one box to Young diagram $Y^{\alpha}$ we extend its diagonal. The multiplicity of the eigenvalue $\lambda_{\nu}(\alpha)$ is equal to \text{dim } \psi^{\nu}, and the number of pairwise distinct eigenvalues of the matrix $Q_{n-1}^{a}(\alpha)$ is equal to the rank of Young diagrams $Y^{\gamma}$ that one can obtain from the Young diagram $Y^{\alpha}$ by adding, in a proper way, one box.
The unitary matrix $Z(\alpha) = (z(\alpha)_{\beta}^{\gamma})$ which reduces the induced representation $\Phi^{\alpha} = \text{ind}_{k(n-2)}^{k(n-1)}(\phi^{\alpha})$ into the irreducible components has a form

$$z(\alpha)_{\beta}^{\gamma} = \frac{d_{\nu}}{\sqrt{N_{\nu}^{\gamma}(n-1)!}} \sum_{\sigma \in S(n-1)} \psi_{\sigma(\gamma)}^{\nu} (\sigma^{-1}) \delta_{\sigma(\nu)\sigma} \varphi_{\nu}^{\alpha} [(an-1)\sigma(q'n-1)],$$

(11)

with

$$N_{\nu}^{\gamma} = \left( E_{\nu,\gamma}^d \right)_{\sigma(\gamma)}^{\nu(\sigma)} = \frac{d_{\nu}}{(n-1)!} \sum_{\sigma \in S(n-1)} \psi_{\sigma(\gamma)}^{\nu} (\sigma^{-1}) \delta_{\sigma(\nu)\sigma} \varphi_{\nu}^{\alpha} [(q'n-1)\sigma(q'n-1)],$$

(12)

where $\psi^{\nu}$ are representations of the group $S(n-1)$ whose Young diagrams are obtained from the Young diagram $Y^{\nu}$ by adding, in a proper way, one box and $(\psi_{\sigma(\gamma)}^{\nu}(\sigma))$ is a matrix form of $\sigma \in S(n-1)$ in the representation $\psi^{\nu}$. $E_{\nu,\gamma}^d$ and is a Hermitian projector of rank one in the representation space $\Phi^{\alpha}$ defined by $\psi^{\nu}$ (see definition 79 in appendix C of [24]). The double index $(q', r')$ is fixed and chosen in such a way that $N_{\nu}^{\gamma} > 0$, which is always possible because $E_{\nu,\gamma}^d$ is a positive semi-definite matrix. Moreover we have

$$\sum_{ak} \sum_{bl} z^{\dagger}(\alpha)_{k}^{\delta a} \Phi^{\alpha}(\sigma)_{k}^{\delta a} z(\alpha)_{l}^{\mu b} = \delta^{\delta a \mu b} \varphi_{l j}^{\nu}(\sigma).$$

(13)

In particular

$$\sum_{ak} \sum_{bl} z^{\dagger}(\alpha)_{k}^{\delta a} Q(\alpha)_{k}^{ab} z(\alpha)_{l}^{\mu b} = \delta^{\delta a \mu b} \lambda_{\nu}^{\sigma}(\alpha),$$

(14)

so the columns of the matrix $Z(\alpha) = (z(\alpha)_{\beta}^{\gamma})$ are eigenvectors of the matrix $Q(\alpha)$.

**Remark 13.** The indices $j', q', r', a'$ are in fact parameters. Expression (11) is complicated in practical applications because, although it looks quite explicit, the normalisation factor $N_{\nu}^{\gamma} = \left( E_{\nu,\gamma}^d \right)_{\sigma(\gamma)}^{\nu(\sigma)}$ cannot be given explicitly without specifying the form of the irreps $\varphi^{\nu}$ of $S(n-1)$ and irrep $\phi^{\alpha}$ of $S(n-2)$. Moreover equation (11) contains two sums over all permutations from $S(n-1)$, which causes high complexity during explicit calculations.

**Remark 14.** Part (a) of theorem 12 gives an explicit eigenvalues of the matrix $Q_{n-1}(\alpha)$ on the partition $\alpha = (\alpha_1, \ldots, \alpha_k)$, which defines the the irreducible representation $\varphi^{\alpha}$ on the dimension parameter $d$.

There is also another expression for eigenvalues $\lambda_{\nu}^{\sigma}(\alpha)$ of matrices $Q(\alpha)$.

**Lemma 15.** Let $\varphi^{\alpha}$ be any irrep of the group $S(n-1)$, $\alpha$ its partition, $\chi^{\nu}$ its character and let $\psi^{\nu}$ be all irreps of the group $S(n-1)$ whose Young diagrams are obtained from the Young diagram $Y^{\alpha}$ by adding, in a proper way, one box. By $\chi^{\nu}$ we denote their characteristics, where $\nu$ is the partition of $n-1$ which labels the representation $\psi^{\nu}$. Then the distinct eigenvalues of the matrix $Q(\alpha)$ generated by the irrep $\varphi^{\alpha}$ of $S(n-2)$ are labelled by the partitions $\nu$ and are of the form
\[ \lambda_\nu(\alpha) = d + \frac{(n-1)(n-2)}{2} \chi^\nu[(a, b)] - \frac{(n-2)(n-3)}{2} \chi^\nu[(c, d)] / d_\nu, \] (15)

where \((a, b)\) for \(a, b \leq n-2\) is an arbitrary transposition in \(S(n-2)\) and \((c, d)\) for \(c, d \leq n-1\) is transposition in \(S(n-1)\). The eigenvalue \(\lambda_\nu(\alpha)\) has multiplicity \(d_\nu\).

**Remark 16.** Since irreducible characters \(\chi^\nu[(a, b)], \chi^\nu[(c, d)]\) are constant on conjugacy classes it is enough to take \((a, b) = (c, d) = (1, 2)\) in equation (15) of lemma 15. The reader may notice that the quantity \(\lambda_\nu(\alpha)\) is of non-zero value only if \(\nu \in \alpha\). In this paper we assume that this assumption is always satisfied.

In the next and last part of this section we briefly recall the basic properties of irreps of the algebra \(A_\nu^n(d)\). The irreps of the algebra \(A_\nu^n(d)\) are of two kinds and we describe them in the matrix form. Let us start with the following.

**Proposition 17.** The first kind of irrep, denoted by \(\Phi^\alpha\), is determined by irreps \(\varphi^\alpha\) of the group \(S(n-2)\), such that \(\varphi^\alpha \in \hat{S}_d(n-2)\), and we have

\[ \Phi^\alpha : A_\nu^n(d) \to \mathbb{M}(\text{rank}Q(\alpha), \mathbb{C}). \] (16)

The representation space \(S(\Phi^\alpha)\) of \(\Phi^\alpha\) has the following structure

\[ S(\Phi^\alpha) = \bigoplus_{\nu \in I \mid h(\nu) \leq d} S(\psi^\nu), \] (17)

where

\[ \text{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha) = \bigoplus_{\nu \in I} \psi^\nu. \] (18)

In the above, by \(I\) we denote the set of irreps of the group \(S(n-1)\), which appear in the above decomposition of induced representation of \(S(n-1)\) into irreducible components \(\psi^\nu\). In the reduced matrix basis \(f \equiv \{f_\nu^\nu : h(\nu) \leq d, \ j_\nu = 1, \ldots, d_\nu\}\) of the representation \(\Phi^\alpha\) (see [24]), the natural generators \(V^\nu[(a, n)]\) and \(V(\sigma_{n-1})\) of \(A_\nu^n(d)\) are represented by the following matrices

\[ \sum_{k=1, \ldots, d_a} \sqrt{\lambda_\xi} \epsilon^\xi(\alpha)_{h\nu}^{\xi\nu}(\alpha)_{h\nu}^{\xi\nu} \sqrt{\lambda_\nu} : \xi, \nu \in I, \ a = 1, \ldots, n-1, \] (19)

\[ M_f^\nu \left[ \sum_{k=1, \ldots, d_a} \epsilon^\xi(\alpha)_{h\nu}^{\xi\nu}(\sigma_{n-1})_{h\nu}^{\xi\nu} \right] \] (20)

where the matrices \(Z(\alpha) = (z(\alpha)_{h\nu}^{\xi\nu})\) are defined in the theorem 12. Expression (20) shows that irrep \(\Phi^\alpha\) of algebra \(A_\nu^n(d)\) is \(\text{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)\) of subalgebra \(\mathbb{C}[S(n-1)]\).

The structure of the irreps of the second kind is much simpler,

**Proposition 18.** The irreps of the second kind, denoted as \(\Psi^\nu\), are determined by the irreps \(\psi^\nu\) of the group \(S(n-1)\), such that \(h(\nu) < d\). In this case we have

\[ \Psi^\nu : A_\nu^n(d) \to \mathbb{M}(d_\nu, \mathbb{C}), \] (21)

where the representation space \(S(\Psi^\nu)\) of \(\Psi^\nu\) is simply \(S(\psi^\nu) = S(\psi^\nu)\), and
\[ \Psi^\nu(a) = \begin{cases} 0 : a \notin S(n-1), \\ \psi^\nu(\sigma_{n-1}) : a = \sigma_{n-1} \in S(n-1). \end{cases} \] (22)

In this case only the elements of \( S(n-1) \) which are not changed by partial transpose are represented non-trivially. The remaining natural generators of the algebra \( A^d_n(d) \) are not invertible [24].

Using the properties of the irreps of \( A^d_n(d) \) described in [24] one can derive the following decomposition of the natural representation of the algebra \( A^d_n(d) \) (see remark 6) into irreducible components.

**Theorem 19.** The algebra \( A^d_n(d) \) in its natural representation in the space \( \mathbb{C}^d \otimes \mathbb{C}^d \) has the following decomposition into irreps:

\[ A^d_n(d) = \left( \bigoplus_{\nu : h(\nu) < d} m^\nu \Psi^\nu \right) \oplus \left( \bigoplus_{\alpha : h(\alpha) \leq d} m_\alpha \Phi^\alpha \right). \] (23)

The multiplicities \( m_\alpha \) are equal to the multiplicities of the irreps \( \varphi^\alpha \) of \( S(n-2) \) in the representation \( V_d(S(n-2)) \) (see proposition 3)

\[ m_\alpha = \frac{1}{(n-2)!} \sum_{\sigma \in S(n-2)} \chi^\alpha(\sigma^{-1}) d^l(\sigma), \] (24)

where \( l(\sigma) \) is the number of disjoint cycles in the permutation \( \sigma \) and

\[ m^\nu = d^n - \sum_{\alpha, \nu \in \text{ind}(S(n-2))} m_\alpha. \] (25)

**Remark 20.** Note that from theorem 19 it follows that when \( d \geq n \) all possible irreps of the first kind and second kind are included in the decomposition of \( A^d_n(d) \). When \( d < n \) then the conditions \( h(\alpha) \leq d \) and \( h(\nu) < d \) limit the variety of irreps appearing in the decomposition given through (23).

As was presented in [24] the orthogonal projectors \( F^\nu(\alpha) \) onto non-trivial irreducible subspaces of the algebra \( A^d_n(d) \) have in fact a very complicated form:

**Proposition 21.** Projectors onto non-trivial irreducible spaces of algebra \( A^d_n(d) \) are of the form

\[ F^\nu(\alpha) = \frac{1}{\lambda(\nu)} \sum_{\nu} \sum_{\text{odd}} (z^{-1}(\alpha))_{j^\nu}^{\nu} u_{tb}^{\nu}(\alpha) z^\nu(\alpha), \] (26)

where \( z(\alpha)^{\nu}_{t^\nu} \) is given in equation (11) in theorem 12, and

\[ u_{tb}^{\nu}(\alpha) = \frac{d^n}{(n-2)!} V^n([a,n]) \sum_{\sigma \in S(n-2)} \varphi^\alpha_{\nu}(\sigma^{-1}) V([an-1](\sigma)(bn-1)]. \] (27)

Indeed, equation (26) contains matrix elements of \( Z(\alpha) \) which are in general difficult to compute (see remark 13).
3. New results for partially reduced irreducible representations

In this section we recall the notion of PRIRs introduced in [28] and derive new properties for them. The concept of PRIRs plays a crucial role in the simplification of the representation of the algebra $A_{n}^{d}(d)$ (see section 4), the derivation of the fidelity in the deterministic version of the PBT, and the new proof of the lower bound on fidelity in the deterministic version of the PBT presented [2] (see section 5.3).

Let us consider an arbitrary unitary irrep $\psi^{\mu}$ of $S(n)$. It can always be unitarily transformed to reduced form $\psi_{R}^{\mu}$, such that

$$\forall \pi \in S(n-1) \quad \psi_{R}^{\mu}(\pi) = \bigoplus_{\alpha \in \mu} \varphi^{\alpha}(\pi), \quad (28)$$

where $\varphi^{\alpha}$ are irreps of $S(n-1)$. By $\alpha \in \mu$ we understand such Young diagrams $\alpha$ which can be obtained from $\mu$ by removing one box in the proper way. We see that the restriction of the irrep $\psi^{\mu}$ of $S(n)$ to the subgroup $S(n-1)$ has a block-diagonal form of completely reduced representation, which in matrix notation takes the form

$$\forall \pi \in S(n-1) \quad \psi_{R}^{\mu}(\pi) = (\delta^{\alpha\beta} \varphi_{i_{\alpha}j_{\beta}}^{\alpha}). \quad (29)$$

The block structure of this reduced representation allows us to introduce such a block indexation for $\psi_{R}^{\mu}$ of $S(n)$, which gives

$$\forall \sigma \in S(n) \quad \psi_{R}^{\mu}(\sigma) = (\psi_{k_{\alpha,l_{\beta}}}(\sigma) = (\psi_{i_{\alpha}j_{\beta}}^{\alpha\beta}(\sigma)). \quad (30)$$

where the indices $k_{\mu}, l_{\mu}$ are standard matrix indices, the matrices on the diagonal $(\psi_{R}^{\mu})^{\alpha\alpha}(\sigma) = (\psi_{i_{\alpha}j_{\alpha}}^{\alpha\alpha}(\sigma))$ are of the dimension of the corresponding irrep $\varphi^{\alpha}$ of $S(n-1)$. The reader should note that the off-diagonal blocks need not be square. From this it follows that we may introduce the idea of the PRIR which we define in the following way.

**Definition 22.** An irrep $\psi^{\mu}$ of the group $S(n)$ is the PRIR if it has a reduced form on the subgroup $S(n-1) \subset S(n)$, i.e. we have

$$\forall \sigma \in S(n-1) \quad \psi_{R}^{\mu}(\sigma) = \bigoplus_{\alpha \in \mu} \varphi^{\alpha}(\sigma). \quad (31)$$

For such representations, in general, we will use the block indexation described in equation (30).

**Remark 23.** Clearly for a given irrep $\psi^{\mu}$ of the group $S(n)$ its PRIR is not given uniquely.

**Remark 24.** It is obvious that any irrep of $S(n)$ can be unitarily transformed into the PRIR representation.

The first new result regarding PRIRs is summarised in the following proposition, which plays a similar role to the standard orthogonality relation for irreps.

**Proposition 25.** The PRIRs $\psi_{R}^{\mu}, \psi_{R}^{\nu}$ of $S(n)$ satisfy the following bilinear summation rule

$$\forall \alpha, \beta \in \mu \quad \forall \gamma, \delta \in \nu \quad \sum_{a=1}^{n} \sum_{k_{\beta}=1}^{d_{\beta}} (\psi_{R}^{\mu})^{k_{\beta} \alpha \beta} [(a,n)] (\psi_{R}^{\nu})^{\gamma \delta}_{k_{\beta} j_{\delta}} [(a,n)] = n \frac{d_{\beta}}{d_{\mu}} \delta^{\alpha \gamma} \delta^{\beta \delta} \delta_{i_{\alpha}j_{\beta}}, \quad (32)$$
where $\alpha, \beta, \gamma$ are irreps of $S(n-1)$ contained in the irreps $\mu, \nu$ of $S(n)$.

The proof of the above proposition follows similarly to the proof of proposition 17 in [28], but its generalisation is necessary for further applications in this paper. Next we prove one more summation rule, which is crucial in order to prove theorem 28, which is the main result of this section.

**Lemma 26.** Let $\psi^\alpha_R \in PRIR$ representations of the group $S(n-1)$ included in $\Phi^\alpha = \text{ind}_{S(n-2)}^{S(n-1)}(\varphi^\alpha)$, then

$$
\sum_{\nu \in \Phi^\alpha} d_{\nu} (\psi^\alpha_R)_{k_\nu, l_\nu} [(b, n - 1)] = (n - 1)d_{\alpha} \delta_{b,n-1} \delta_{l_\nu},
$$

(33)

where the summation is over partitions $\nu$ labelling irreps of $S(n-1)$ contained in $\Phi^\alpha$.

**Proof.** Let

$$
\sum_{\nu \in \Phi^\alpha} d_{\nu} (\psi^\alpha_R)_{k_\nu, l_\nu} [(b, n - 1)] = x_{j_\nu, k_\nu} (b), \quad b = 1, \ldots, n - 1.
$$

(34)

We define a Hermitian matrix

$$
X(b) \equiv (x_{j_\nu, k_\nu} (b)) \in \mathbb{M} (\dim \varphi^\alpha, \mathbb{C}),
$$

(35)

such that

$$
X(n - 1) = (n - 1)d_{\alpha} \mathbf{1}_{d_{\alpha}},
$$

(36)

where $\mathbf{1}_{d_{\alpha}}$ denotes the identity operator of dimension $d_{\alpha}$. Now from equation (34) we obtain

$$
\sum_{\nu \in \Phi^\alpha} d_{\nu} (\psi^\alpha_R)_{j_\nu, k_\nu} [(b, n - 1)] (\psi^\mu_R)_{k_\nu, l_\nu} [(b, n - 1)] = \sum_{k_\nu} x_{j_\nu, k_\nu} (b) (\psi^\mu_R)_{k_\nu, l_\nu} [(b, n - 1)]
$$

(37)

Making the summation over $b = 1, \ldots, n - 1$ and applying proposition 25 to the left-hand side of (37), we obtain

$$
(n - 1)d_{\alpha} \delta_{j_\nu, l_\nu} = \sum_{k_\nu} \sum_{b = 1}^{n-1} x_{j_\nu, k_\nu} (b) (\psi^\mu_R)_{k_\nu, l_\nu} [(b, n - 1)].
$$

(38)

Multiplying both sides of (38) by $d_{\mu}$ and making the summation over $\mu \in \Phi^\alpha$ we have

$$
(n - 1)^2 d_{\alpha}^2 \delta_{j_\nu, l_\nu} = \sum_{k_\nu} \sum_{b = 1}^{n-1} x_{j_\nu, k_\nu} (b) x_{k_\nu, l_\nu} (b) = \sum_{b = 1}^{n-1} x_{j_\nu, l_\nu}^2 (b) + x_{j_\nu, l_\nu} (n - 1) = \sum_{b = 1}^{n-2} x_{j_\nu, l_\nu}^2 (b) + (n - 1)^2 d_{\alpha}^2 \delta_{j_\nu, l_\nu},
$$

(39)

which means that

$$
\sum_{b = 1}^{n-2} x_{j_\nu, l_\nu}^2 (b) = \sum_{b = 1}^{n-2} (X^2(b))_{j_\nu, l_\nu} = 0.
$$

(40)
From the above it follows that
\[ \forall b = 1, \ldots, n - 2 \quad X^2(b) = 0 \iff X(b) = 0, \]   (41)
since the matrices \( X(b) \) are Hermitian, so the matrices \( X^2(b) \) are positive semi-definite.

From lemma 26 one can easily deduce the following

**Corollary 27.** Let
\[ \sigma = \gamma(b, n - 1) \in S(n - 1) : \gamma \in S(n - 2), \quad b = 1, \ldots, n - 1, \]   (42)
then
\[ \sum_{\nu \in \Phi^\alpha} d_\nu(\psi^\alpha_R)_{\nu, j, k_n}[\gamma(b, n - 1)] = (n - 1)d_{\alpha} \delta_{b, n - 1} \varphi^\alpha_{j, k_n}(\gamma). \]   (43)

In particular if \( \sigma \in S(n - 1) \), and \( \sigma \notin S(n - 2) \subset S(n - 1) \), then
\[ \sum_{\nu \in \Phi^\alpha} d_\nu(\psi^\alpha_R)_{\nu, j, k_n}(\sigma) = 0. \]   (44)

If \( \sigma \in S(n - 2) \subset S(n - 1) \), then
\[ \sum_{\nu \in \Phi^\alpha} d_\nu(\psi^\alpha_R)_{\nu, j, k_n}(\sigma) = (n - 1)d_{\alpha} \varphi^\alpha_{j, k_n}(\sigma). \]   (45)

Now we are in the position to prove the main result of this section, namely we have the following

**Theorem 28.** Let \( \psi^\alpha_R \) are PRIR representations of the group \( S(n - 1) \) included in \( \Phi^\alpha = \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \) i.e.
\[ \Phi^\alpha = \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \simeq \bigoplus_{\nu \in \Phi^\alpha} \psi^\alpha_R, \]   (46)
then \( \forall \sigma \in S(n - 1) \) we have the following summation rule
\[ \sum_{\nu \in \Phi^\alpha} d_\nu(\psi^\alpha_R)_{\nu, j, k_n}[(a, n - 1)\sigma(b, n - 1)] = (n - 1)d_{\alpha} \delta_{\sigma(b)} \varphi^\alpha_{j, k_n}[(a, n - 1)\sigma(b, n - 1)]. \]   (47)

In particular, we have
\[ \sum_{\nu \in \Phi^\alpha} d_\nu(\psi^\alpha_R)_{\nu, j, k_n}[(a, n - 1)\sigma(b, n - 1)] = (n - 1)d_{\alpha} \delta_{b, n - 1} \varphi^\alpha_{j, k_n}. \]   (48)

**Proof.** From corollary 27 it follows that, in order to calculate
\[ \forall \sigma \in S(n - 1) \quad \sum_{\nu \in \Phi^\alpha} d_\nu(\psi^\alpha_R)_{\nu, j, k_n}[(a, n - 1)\sigma(b, n - 1)], \]   (49)
we need to establish when the permutation \((a, n - 1)\sigma(b, n - 1)\) belongs to the subgroup \(S(n - 2) \subset S(n - 1)\), otherwise the sum is equal to zero. It is easy to check that
(a, n - 1) σ(b, n - 1) ∈ S(n - 2) ⊂ S(n - 1)  

if and only if

\[ \sigma : b \mapsto a \]  

and from this follows the statement of the theorem.

\[ \square \]

**Proposition 29.** Suppose that \( \psi^\nu_R \) is a PRIR representation of the group \( S(n) \), then we have

\[
\sum_{a=1}^{n} (\psi^\nu_R)_{a \alpha a}^{\alpha \alpha} ([a, n]) = \lambda_\nu(\alpha) \delta_{\alpha a},
\]

where \( \alpha \in \nu \), \( j_\alpha \) are PRIR indices of the irrep \( \psi^\nu_R \) and \( \lambda_\nu(\alpha) \) is given by lemma 15, theorem 12 or equivalently in corollary 37.

The proof of proposition 29 follows from proposition A.2 of appendix and lemma 15.

**4. Application of PRIRs to the representation theory of the algebra \( \mathcal{A}_d^N(d) \)**

In the following subsections we derive simpler forms of the matrices \( Q(\alpha) \), \( Z(\alpha) \) and projectors \( F_\nu(\alpha) \), which where defined in [24] (or see section 2) by use of the concept of PRIRs introduced in section 3. Additionally as a second result we present explicit and relatively simple expressions for the matrix elements of the permutation operators \( V'([a, n]) \) for \( a = 1, \ldots, n - 1 \), which are now more convenient for practical application.

**4.1. Simplification of the matrices \( Q(\alpha) \) and \( Z(\alpha) \) and the matrix representation of \( V'([a, n]) \)**

Let us consider the induced representation of \( S(n - 1) \) \( \Phi^\alpha = \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \), where \( \varphi^\alpha \) is a given irrep of \( S(n - 2) \). It is known that the decomposition into irreps \( \psi^\nu \) of \( S(n - 1) \)

\[
\Phi^\alpha = \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \simeq \bigoplus_{\nu \in \Phi^\alpha} \psi^\nu
\]

is simple reducible. This means that there exists a unitary matrix \( Z(\alpha) \), which reduces the representation to an irreducible block-diagonal form. In [24] such a matrix \( Z(\alpha) \) was constructed for arbitrary forms of irreps \( \psi^\nu \) of \( S(n - 1) \) and irrep \( \varphi^\alpha \) of \( S(n - 2) \) (see theorem 12(b) in section 2). First let us observe that on the right-hand side of the main equality (49) of theorem 28 we obtain the matrix elements of induced representation \( \Phi^\alpha = \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \), i.e. we have

\[
\Phi^\alpha = \left( (\psi^\nu_R)^{ab}_{j_\alpha k_\alpha} \right) = \left( \delta_{\alpha \sigma(b)} \varphi^\alpha_{j_\alpha k_\alpha} ([a, n - 1] \sigma(b, n - 1)) \right),
\]

which is a standard matrix form of the induced representation \( \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \). Having this we are now in the position to formulate the following.

**Theorem 30.** Let \( \Phi^\alpha = \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \simeq \bigoplus_{\nu \in \Phi^\alpha} \psi^\nu_R \), where \( \psi^\nu_R \) are PRIR representations of the group \( S(n - 1) \), then the corresponding matrix \( Z_R(\alpha) = (z_R(\alpha)_{j_\alpha k_\alpha}) \), which reduces the induced representation \( \Phi^\alpha = \text{ind}^{S(n-1)}_{S(n-2)}(\varphi^\alpha) \) into the direct sum \( \bigoplus_{\nu \in \Phi^\alpha} \psi^\nu_R \), has the following form:
\[
\zeta_R(\alpha)_{\bar{k}\bar{n}\lambda\epsilon\nu} = \frac{1}{\sqrt{n - 1}} \sqrt{d_{\bar{\nu}}} (\psi_R^{\nu})^\alpha_{\bar{k}\bar{n}\lambda\epsilon\nu} \left[ (\alpha, n - 1) \right],
\]

(55)

where \((\alpha, k_a)\) and \((\xi_{\ell}, j_{\ell})\) are the PRIR indices in \(\psi_R^{\nu}\), corresponding to a reducible structure for the subgroup \(S(n - 2)\) (see equation (30)). The irrep \(\varphi^\nu\) of \(S(n - 2)\) is included in every \(\psi_R^{\nu}\ \in \Phi^\alpha\). The matrix \(Z_R(\alpha) = \left( \zeta_R(\alpha)_{\bar{k}\bar{n}\lambda\epsilon\nu} \right)\) is unitary and satisfies

\[
\sum_{\bar{k}} \sum_{\bar{n}} \bar{\epsilon} R(\bar{\alpha}, \bar{k}, \bar{n}) \bar{\lambda} R(\bar{\alpha}, \bar{k}, \bar{n}) \zeta_R(\alpha)_{\bar{k}\bar{n}\lambda\epsilon\nu} = \bar{\delta} R(\bar{\nu}, \bar{\mu}, \bar{\lambda}) \delta_R(\bar{\mu}, \bar{n}, \bar{\epsilon}, \nu), \quad \forall \sigma \in S(n - 1)
\]

(56)

and

\[
\sum_{\bar{k}} \sum_{\bar{n}} \bar{\epsilon} R(\bar{\alpha}, \bar{k}, \bar{n}) \bar{\lambda} R(\bar{\alpha}, \bar{k}, \bar{n}) \zeta_R(\alpha)_{\bar{k}\bar{n}\lambda\epsilon\nu} = \bar{\delta} R(\bar{\nu}, \bar{\mu}, \bar{\lambda}) \delta_R(\bar{\mu}, \bar{n}, \bar{\epsilon}, \nu),
\]

(57)

as well

\[
\sum_{\nu} \sum_{\nu} \bar{\epsilon} R(\bar{\alpha}, \bar{k}, \bar{n}) \bar{\lambda} R(\bar{\alpha}, \bar{k}, \bar{n}) \zeta_R(\alpha)_{\bar{k}\bar{n}\lambda\epsilon\nu} = \bar{\delta} R(\bar{\nu}, \bar{\mu}, \bar{\lambda}) \delta_R(\bar{\mu}, \bar{n}, \bar{\epsilon}, \nu).
\]

(58)

**Proof.** An application of equation (32) from proposition 25 and theorem 28 for PRIRs to the equation for the matrix \(Z(\alpha)\) given in theorem 12(b) leads directly to expressions from the statements of the theorem. \(\square\)

One can see that comparing the expression for \(Z_R(\alpha)\) given through equation (55) with the general formula for \(Z(\alpha) = (z(\alpha))_{\bar{k}\bar{n}\lambda\epsilon\nu}\) in theorem 12, we have substantial simplification, which vitally important in the practical application of our tools (see section 5). The main advantage over the previous expression is that there is no sum over all permutations, so we remove complexity of order \((n - 1)!\). This will allow us to produce expressions for matrix elements of the algebra of operators that are tractable (see proposition 33). Moreover, the results contained in theorem 30 also solve the problem of the eigenvectors of the matrix \(Q(\alpha)\). Namely, they are given by the columns of the matrix \(Z_R(\alpha) = \left( \zeta_R(\alpha)_{\bar{k}\bar{n}\lambda\epsilon\nu} \right)\), which are now relatively simple.

It is well known that the columns of any unitary matrix form a set of orthonormal vectors. Using this fact as a corollary from the properties of the matrix \(Z_R(\alpha)\) given in theorem 30, we obtain the following.

**Corollary 31.** The set of \((n - 1)d_\alpha\) vectors

\[
T^\nu (\xi_{\ell}, j_{\ell}) = \left( T^\nu (\xi_{\ell}, j_{\ell})_{\bar{k}\bar{n}\lambda\epsilon\nu} \right) \in \mathbb{C}^{(n - 1)d_\alpha}
\]

forms an orthonormal basis of the space \(\mathbb{C}^{(n - 1)d_\alpha}\).

Next from the properties of the matrix \(Z_R(\alpha)\) we obtain the following corollary, which is a direct consequence of corollary 31.

**Corollary 32.** The matrix \(Z_R(\alpha)\) diagonalises the matrix \(Q(\alpha)\):

\[
\sum_{\bar{k}} \sum_{\bar{n}} \bar{\epsilon} R(\bar{\alpha}, \bar{k}, \bar{n}) \bar{\lambda} R(\bar{\alpha}, \bar{k}, \bar{n}) \zeta_R(\alpha)_{\bar{k}\bar{n}\lambda\epsilon\nu} = \bar{\delta} R(\bar{\nu}, \bar{\mu}, \bar{\lambda}) \delta_R(\bar{\mu}, \bar{n}, \bar{\epsilon}, \nu).
\]

(60)
which means that the vectors $T^ν(ξ, J_ξ)$ from corollary 31 are eigenvectors of the matrix $Q(α)$. The numbers $λ_α(α)$ are given by lemma 15, theorem 12 or equivalently in corollary 37.

The next important consequence of the simplification of matrix $Z(α)$ by the PRIR approach is relatively handy expressions for the matrix representations of the generators of algebra $A_α^ν(d)$, in particular for $V'$. Namely, we have the following.

**Proposition 33.** In the irrep $Φ^ν$ of the algebra $A_α^ν(d)$ we have the following matrix representation of elements $V'[(a, n)]$

$$M_ν^ν[V'[(a, n)]]|ξ_ν, ξ_ν′⟩ = \frac{1}{n - 1} \sqrt{d_νd_ν′} \sum _{κ_ν} \sqrt{λ_ν(α)λ_ν(α′)}[[a, n - 1]|ψ_ν^αξ_ν⟩⟨ψ_ν′αξ_ν′|[(a, n - 1)]√λ_ν(α).$$

where $ν, ν′, Φ ≠ \emptyset$ and the subscript $f$ (see proposition 17) means that the matrix representation is calculated in reduced basis $f \equiv \{j_ν^ν : h(ν) ≤ d, \ j_ν = 1, \ldots , dν\}$ of the ideal $Φ^ν$. In particular, for $a = n - 1$ expression (61) reduces to

$$M_ν^ν[V'[(a, n)]]|ξ_ν, ξ_ν′⟩ = \frac{1}{n - 1} \sqrt{d_νd_ν′} \sqrt{λ_ν(α)λ_ν(α′)}[d_ν^νδ_ν′δ_ν′, Φ′_ν|ξ_ν⟩⟨ξ_ν′].$$

Later in this paper we use a simplified notation for the matrix elements $V'[|(an)|]$ in the reduced basis $f$:

$$M_ν^ν[V'[|(an)|]|ξ_ν, ξ_ν′⟩ ≡ M_ν^ν[V'[|(an)|]|ξ_ν⟩⟨ξ_ν′],$$

where

- $ξ_ν$ labels irreps of $S(n - 2)$ included in $ψ^ν ∈ Φ(α)$
- $j_ν$ labels indices in $ξ_ν$
- $ξ_ν′$ labels irreps of $S(n - 2)$ included in $ψ^ν′ ∈ Φ(α)$
- $j_ν′$ labels indices in $ξ_ν′$

Next we exploit the idea of PRIRs introduced in [28], with additional results presented in section 3 for the simplification of the set of orthogonal projections $F_ν(α)$ given in proposition 21 onto non-trivial irreducible spaces of the algebra $A_α^ν(d)$. The results proven below will allow us to use them in section 5.2, where the properties for the PBT scheme, and its connection with the matrix $Q(α)$, are provided. As we can see, equation (26) is not explicit since we have to compute coefficients $z(α)^ν$ and $N_ν^ν$ separately, which are given by highly complicated equations (see remark 13). Using PRIRs we have the following simplification.

**Proposition 34.** Using the PRIR representation we can simplify the form of the operators $F_ν(α)$ from proposition 21 to the explicit expression of type

$$F_ν(α) = \frac{1}{λ_ν(α)}(n - 1)^d_ν \sum _{h=1}^{n - 1} \sum _{k_ν} \sum _{γ∈S(n-1)} (ψ_ν^α|k_νk_ν′(γ^{-1})V′(b, n)V′[(b, n - 1)γ(b, n - 1)].$$

$$\quad (64)$$

Indeed, the reader may notice that we simplified the complicated expressions for $z(α)^ν$ and $N_ν^ν$, and now we have only one sum over all permutations from $S(n - 1)$ instead of two of them. Next, directly from the result contained in proposition 34 we obtain the following.

**Lemma 35.** The matrix form of the projector $F_ν(α)$ on non-trivial irreducible spaces of the algebra $A_α^ν(d)$ in the reduced basis $f$ has the following form:
\[ M^\nu_{\mu}[F_\nu(\alpha)]_{\xi_1\xi_2\xi_3\xi_4}^\eta = \delta^{\eta\mu} \delta^{\nu\xi_1} \delta_{\xi_2\xi_3} \delta_{\xi_4,\xi_4}, \]  

i.e. in the irrep \( \Phi^\alpha \) of the algebra \( \mathcal{A}_n^\nu(d) \), in the reduced basis \( f \) the projector \( F_\nu(\alpha) \) takes its canonical form with ones on the diagonal in the position of the irrep \( \psi^\nu \) of the group \( S(n - 1) \) only.

This result is obtained by a direct calculation using PRIRs. From the statement of lemma 35, using a decomposition of the natural representation of the algebra \( \mathcal{A}_n^\nu(d) \) into its irreps, we easily deduce the following:

**Corollary 36.**

\[ \text{Tr} M^\nu_{\mu}[F_\nu(\alpha)] = d_\nu, \]  

and from this we obtain

\[ \text{Tr}_H F_\nu(\alpha) = m_\alpha d_\nu, \]  

where \( H = (\mathbb{C}^d)^{\oplus n} \) and \( m_\alpha \) is the multiplicity the irreps \( \varphi^\alpha \) of \( S(n - 2) \) in the representation \( V_\nu(S(n - 2)) \) (see proposition 3).

The trace \( \text{Tr}_H F_\nu(\alpha) \) can also be computed in another way, directly from expression (64)

\[ \text{Tr}_H F_\nu(\alpha) = \frac{d_\nu}{\lambda_\nu(\alpha)} \frac{1}{(n - 1)!} \sum_{k=1}^{n-1} \sum_{k_1, k_2} (\psi^\nu)^{\alpha\alpha}_{k_1, k_2} (\gamma^{-1}) \text{Tr}_H (V'([b, n])V([b, n - 1] \gamma(b, n - 1))), \]  

where

\[ \text{Tr}_H (V'([b, n])V([b, n - 1] \gamma(b, n - 1))) = \text{Tr}_H V([n - 1, n] \gamma). \]

Using the well-known fact that \( \text{Tr}_H V_\nu(\sigma) = d_\nu(\sigma) = \chi^V(\nu)(\sigma) \) is the character of the permutation representation of \( S(n) \), where \( I(\sigma) \) is the number of cycles in the permutation \( \sigma \in S(n) \) we obtain

\[ \text{Tr}_H F_\nu(\alpha) = \frac{1}{\lambda_\nu(\alpha)} \frac{d_\nu}{(n - 1)!} \sum_{k=1}^{n-1} \sum_{k_1, k_2} (\psi^\nu)^{\alpha\alpha}_{k_1, k_2} (\gamma^{-1}) d_\nu(I(\sigma) - I(\gamma)), \]  

and it is easy to check that \( I([n - 1, n] \gamma) = I(\gamma), \quad \gamma \in S(n - 1) \). Further, using the orthogonality relations for irreps we obtain one more,

\[ \text{Tr}_H F_\nu(\alpha) = \frac{1}{\lambda_\nu(\alpha)} \frac{d_\nu}{(n - 1)!} \sum_{k=1}^{n-1} \sum_{k_1, k_2} (\psi^\nu)^{\alpha\alpha}_{k_1, k_2} (\gamma^{-1}) d_\nu(I(\gamma)) = \frac{(n - 1)}{\lambda_\nu(\alpha)} d_\nu m_\alpha. \]

As a corollary from corollary 36 we obtain the following.

**Corollary 37.** The eigenvalues \( \lambda_\nu(\alpha) \) of the matrix \( Q(\alpha) \) (see definition 7) are of the form

\[ \lambda_\nu(\alpha) = (n - 1) \frac{m_\alpha d_\alpha}{m_\mu d_\mu}. \]

This an equivalent expression for the eigenvalues of \( Q(\alpha) \) given in lemma 15 and theorem 12(a). This formula was obtained in [28] using a different method.
4.2. New matrix operators in algebra $A^I_n(d)$

In this paragraph we define a new set of matrix operators which gives a useful description of the generator $V'$ of underlying algebra $A^I_n(d)$ in the matrix form. The derived expressions are similar to those which can be obtained for groups. Descriptions of the latter can be found in the classical textbooks [10] and [13] or in appendix F of [28]. As we have shown in section 4, in the irrep of the ideal $\mathcal{M}$ labelled by partition $\alpha$ we have a basis labelled by $\mu, \xi, \iota, \phi$, where $\mu$ is a partition of $n - 1$, $\xi$ is a partition of $n - 2$ differing from $\mu$ by one block and $\iota, \phi$ labels a basis in irrep $\xi$. We thus have a vector basis $\{|\phi^\mu_{\xi,\iota}(\alpha, r)\rangle\}$, where $r$ labels multiplicity in our representation. With this basis we can associate flip operators (or, matrix basis operators):

$$E^\mu_{\xi,\iota}(\alpha) \equiv \sum_r |\phi^\mu_{\xi,\iota}(\alpha, r)\rangle \langle \phi^\mu_{\xi,\iota}(\alpha, r)|. \quad (73)$$

In the reduced matrix basis $f$ given in proposition 17 the matrix elements of the above operators are of the form

$$M^f_{\alpha} \left[ E^\mu_{\xi,\iota}(\alpha) \right]^\rho = \delta^\mu_\rho \delta^\gamma_\alpha \delta^\iota_\xi \delta^\iota_\phi \delta^\gamma_\alpha \delta^\iota_\xi \delta^\iota_\phi. \quad (74)$$

Below we will show the explicit form of some of these flip operators in terms of the elements of our algebra $A^I_n(d)$.

**Theorem 38.** Let us define the following set of operators in the algebra $A^I_n(d)$:

$$E^\mu_{\iota,\alpha}(\alpha) \equiv \frac{m_\alpha}{\sqrt{m_\mu m_\nu}} P_\mu E^\alpha_{\iota,\nu} V' P_\nu, \quad (75)$$

where

$$P_\mu = \frac{d_\mu}{(n - 1)!} \sum_{\sigma \in S(n - 1)} \chi^\mu((\sigma^{-1}) V(\sigma), \quad E^\alpha_{\iota,\alpha}(\alpha) = \frac{d_\alpha}{(n - 2)!} \sum_{\pi \in S(n - 2)} \varphi^\alpha_{\iota,\alpha}(\pi^{-1}) V(\pi). \quad (76)$$

Then operators in (75) form a subset in the set of operators given through (73). Clearly the operators $E^\mu_{\iota,\alpha}(\alpha)$ belong to the ideal $\mathcal{M}$.

**Proof.** To prove the statement of the theorem we have to show that the operators given in (75) form a subset contained in the set composed of operators given through (73). To do so we compute matrix elements of $E^\mu_{\iota,\alpha}(\alpha)$ in reduced basis $f$ and compare them to expression (74). In order to compute the desired matrix elements, we first have to calculate in the PRIR representation matrix elements $\psi^\alpha_R$ of $S(n - 1)$:

$$\psi^\alpha_R[E^\alpha_{\iota,\alpha}]_{\iota,\alpha} = \delta^\alpha_{\iota,\alpha} \varphi^\alpha_{\iota,\alpha} = \delta^\alpha_{\iota,\alpha} \frac{d_\alpha}{(n - 2)!} \sum_{\pi \in S(n - 2)} \varphi^\alpha_{\iota,\alpha}(\pi^{-1}) \varphi^\alpha_{\iota,\alpha}(\pi) \quad (77)$$

The expression above means that in the PRIR representation $\psi^\alpha_R$ of $S(n - 1)$ the operator $E^\alpha_{\iota,\alpha}$ is represented in such a way that among the diagonal blocks $\langle \phi^\alpha_R| \varphi^\alpha_{\iota,\alpha} \sim \varphi^\alpha_{\iota,\alpha}$, the block $\langle \phi^\alpha_R| \varphi^\alpha_{\iota,\alpha} \sim \varphi^\alpha_{\iota,\alpha}$ is non-zero and in this block the operator $E^\alpha_{\iota,\alpha}$ is represented by the standard matrix basis $e_{\iota,\alpha}$. From this it follows that in the irrep $M^f_{\alpha}$ of the algebra $A^I_n(d)$ we have...
Next using equation (62) from proposition 33 and
\[ M^\gamma_j [P^\gamma \mu \nu \alpha \beta \gamma, \xi, \xi] = \delta^\alpha_{\mu^\prime} \delta^\beta_{\nu^\prime} \delta_{\xi, \xi} \delta_{\gamma, \gamma} = \delta^\alpha_{\mu^\prime} \delta^\beta_{\nu^\prime} \delta_{\xi, \xi} \delta_{\gamma, \gamma}, \]  
(89)

we calculate
\[ M^\gamma_j [P^\gamma \mu \nu \alpha \beta \gamma, \xi, \xi] = \sum_{\gamma} \sum_{\mu^\prime} \sum_{\nu^\prime} \sum_{\alpha} \sum_{\beta} \delta^\alpha_{\mu^\prime} \delta^\beta_{\nu^\prime} \delta_{\xi, \xi} \delta_{\gamma, \gamma} \]
\[ \times \sqrt{m_\mu m_\nu} \delta_{\xi, \xi} \delta_{\gamma, \gamma} = \frac{\sqrt{m_\mu m_\nu}}{m_\alpha} \delta_{\xi, \xi} \delta_{\gamma, \gamma} \]
(89)

Taking these all together we write
\[ M^\gamma_j [P^\gamma \mu \nu \alpha \beta \gamma, \xi, \xi] = \frac{\sqrt{m_\mu m_\nu}}{m_\alpha} \delta_{\xi, \xi} \delta_{\gamma, \gamma} \]
(89)

Finally, the matrix elements of the operators given in (75) are of the form
\[ M^\gamma_j [E^\mu \nu \alpha \beta \gamma, \xi, \xi] = \frac{\sqrt{m_\mu m_\nu}}{m_\alpha} \delta_{\xi, \xi} \delta_{\gamma, \gamma} \]
(89)

We see that in the matrix \( M^\mu_j [E^\mu \nu \alpha \beta \gamma] \) the only non-zero block is the block with indices \( (\mu, \nu) \) and inside this block the only non-zero sub-block has indices \( (\alpha, \alpha) \), and this sub-block is equal to the standard matrix basis element \( e_{\alpha, \beta} \). Comparing equation (82) to the expression for matrix elements in (74), we see that operators from (75) indeed form a subset in the set of operators given in (73). This completes the proof.

If we define in standard way:

**Definition 39.**

\[ P_\alpha \equiv \sum_{\alpha} E^\alpha_{\alpha \alpha} = \frac{d_\alpha}{n-2} \sum_{\pi \in S(n-2)} \chi^\alpha(\pi^{-1})V(\pi), \]
(83)

then we obtain the following corollary from equation (82)

**Corollary 40.**

\[ M^\mu_j (P^\mu \mu \nu \nu \gamma) = M^\mu_j (P^\mu \mu \nu \nu \gamma) M^\mu_j (P^\mu \nu \nu \gamma), \]
(84)

where
\[ M^\mu_j (P^\mu \mu) = (\delta^\mu_{\mu^\prime} \delta^\nu_{\nu^\prime} \delta_{\xi, \xi} \delta_{\gamma, \gamma}), \]
(85)

so it is a block-diagonal matrix with only one non-zero diagonal block with indices \( (\mu, \mu) \). The operator \( P_\alpha \) vanishes on the right-hand side because, from the structure of \( M^\mu_j (V^\gamma) \), we have
\[ M^\mu_j (P^\mu_\alpha) M^\mu_j (V^\gamma) = M^\mu_j (V^\gamma). \]
(86)
Remark 41. The operator $E_{i_{\alpha},\beta}^{\mu\nu} (\alpha)$ is non-zero if and only if the irreps labelled by partitions $\alpha$ and $\mu, \nu$ are in the relation
\[ \mu = \alpha + \square \quad \land \quad \nu = \alpha + \square. \] (87)

The total number of non-zero operators, for a given $\alpha$, is equal to $(d, \mu; N(M^\alpha; S(n-1)))^2$, where $N(M^\alpha; S(n-1))$ is the number of irreps of $S(n-1)$ in the irrep $M^\alpha$ of the algebra $A_d (d)$.

By theorem 38, the operators $E_{i_{\alpha},\beta}^{\mu\nu} (\alpha)$ satisfy the following multiplication rule:
\[ E_{i_{\alpha},\beta}^{\mu\nu} (\alpha)E_{k_{\gamma},\delta}^{\sigma\pi} (\beta) = \delta_{\alpha\beta}\delta_{\delta\gamma} E_{i_{\alpha},k_{\gamma}}^{\mu\nu} (\alpha). \] (88)

We shall now also show this directly, by using their expression in terms of algebra elements (75). We have
\[
E_{i_{\alpha},\beta}^{\mu\nu} (\alpha)E_{k_{\gamma},\delta}^{\sigma\pi} (\beta) = \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} \frac{m_{\beta}}{\sqrt{m_{\pi}m_{\sigma}}} P_{\mu} E_{i_{\alpha},k_{\gamma}}^{\pi\sigma} V^\nu \delta_{\beta\gamma} P_{\pi} E_{i_{\alpha},k_{\delta}}^{\nu\sigma} V^\mu P_{\sigma} \\
= \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} \frac{m_{\beta}}{\sqrt{m_{\pi}m_{\sigma}}} \delta_{\alpha\beta}\delta_{\delta\gamma} E_{i_{\alpha},k_{\gamma}}^{\mu\nu} V^\nu P_{\pi} V^\mu P_{\sigma}. \] (89)

To obtain the desired result the most important step is to calculate $V^\nu P_{\mu} V^\mu$:
\[
V^\nu P_{\mu} V^\mu = \frac{d_{\mu}}{(n-1)!} \sum_{\sigma \in S(n-1)} \chi^{\mu}(\sigma^{-1}) V^\nu \chi(\sigma) V^\mu \\
= \frac{d_{\mu}}{(n-1)!} \sum_{\sigma \in S(n-2)} \sum_{a=1}^{n-1} \chi^{\mu}[(a, n-1) \pi] V^\nu \chi[(\pi^{-1}) (a, n-1)] V^\mu. \] (90)

Now using the PRIR structure we expand the expression on the right-hand side of (90)
\[
\chi^{\mu}[(a, n-1) \pi] = \sum_{\xi,\eta,\kappa} \sum_{\gamma,\delta,\rho} (\psi^{\eta\rho}_{\pi})^{\xi_{\kappa}} \gamma \delta_{\kappa\delta}^{\rho} \delta_{\gamma\rho}^{\eta} \varphi^{\gamma}_{\kappa\rho} (\pi) \] (91)

and together with the identity $V^\nu V^{[(a, n-1)]} V^\mu = d^{n-1} V^\mu$ we obtain
\[
V^\nu P_{\mu} V^\mu = \frac{d_{\mu}}{(n-1)!} \sum_{\sigma \in S(n-2)} \sum_{a=1}^{n-1} \sum_{\xi,\eta,\kappa} \sum_{\gamma,\delta,\rho} \psi^{\eta\rho}_{\pi} \delta_{\kappa\delta}^{\rho} \delta_{\gamma\rho}^{\eta} \varphi^{\gamma}_{\kappa\rho} (\pi) d^{n-1} V^\mu. \] (92)

Further using proposition 29 and the definition of $P_{\xi,\rho}$ we find
\[
V^\nu P_{\mu} V^\mu = \sum_{\xi,\rho} \frac{m_{\rho}}{m_{\xi}} P_{\xi,\rho} V^\mu. \] (93)

Finally, all together this implies that
\[
E_{i_{\alpha},\beta}^{\mu\nu} (\alpha)E_{k_{\gamma},\delta}^{\sigma\pi} (\beta) = \frac{m_{\alpha}}{\sqrt{m_{\mu}m_{\nu}}} \frac{m_{\beta}}{\sqrt{m_{\pi}m_{\sigma}}} P_{\mu} E_{i_{\alpha},k_{\gamma}}^{\pi\sigma} V^\nu \delta_{\beta\gamma} P_{\pi} E_{i_{\alpha},k_{\delta}}^{\nu\sigma} V^\mu P_{\sigma} = \delta_{\alpha\beta}\delta_{\delta\gamma} E_{i_{\alpha},k_{\gamma}}^{\mu\nu} (\alpha). \] (94)

Of course, since operators in (75) form a subset in the set of operators given by (73), we can also prove the composition rule (88) using the definition from (73) directly. Using the
properties of \( P_\mu, P_\alpha \) and \( E_{i\alpha}^{\alpha} \), one can deduce the decomposition of \( V' \) in terms of matrix operators given in definition 38, namely we have

**Proposition 42.** Operator \( V' \) has the following decomposition in terms of operators \( E_{i\alpha}^{\alpha} \) given in theorem 38:

\[
V' = \sum_\alpha \sum_{\mu, \nu \in \alpha} \sum_{i \in \alpha} \frac{\sqrt{m_\mu m_\nu}}{m_\alpha} E_{i\alpha}^{\mu\nu}(\alpha),
\]

where summations are over partitions for which \( \varphi_\alpha \in \hat{S}_d(n-2) \) and \( \psi_\mu, \psi_\nu \in \hat{S}_d(n-1) \).

Now if we define a new set of operators as

\[
P_{\mu\nu}^{\alpha}(\alpha) \equiv \sum_{i \in \alpha} E_{i\alpha}^{\mu\nu}(\alpha) = \frac{m_\alpha}{\sqrt{m_\mu m_\nu}} P_\mu \left( \sum_{i \in \alpha} E_{i\alpha}^{\alpha} \right) V' P_\nu = \frac{m_\alpha}{\sqrt{m_\mu m_\nu}} P_\mu P_\alpha V' P_\nu,
\]

then together with expression (88) we can formulate the following.

**Corollary 43.** The operators \( P_{\mu\nu}^{\alpha}(\alpha) \) given in (96) satisfy

\[
P_{\mu\nu}^{\alpha}(\alpha) P_{\rho\sigma}^{\beta}(\beta) = \delta_{\sigma\rho} \delta_{\alpha\beta} P_{\mu\nu}^{\alpha}(\beta).
\]

The operators \( E_{i\alpha}^{\mu\nu}(\alpha) \) for a given \( \alpha \) form an algebra isomorphic with the matrix algebra \( M(d_\alpha N[M_\alpha^\beta : S(n-1)], \mathbb{C}) \subset M(d^n, \mathbb{C}) \). Directly from the multiplication rule for the operators \( E_{i\alpha}^{\mu\nu}(\alpha) \) we also have the following.

**Corollary 44.** The subset of \( d_\alpha^2 \) operators of the form \( E_{i\alpha}^{\mu\nu}(\alpha) \) satisfy

\[
E_{i\alpha}^{\mu\nu}(\alpha) E_{i'\alpha}^{\rho\sigma}(\alpha) = \delta_{\alpha\beta} \delta_{\mu\sigma} E_{i\alpha}^{\mu\nu}(\alpha),
\]

so it forms a subalgebra isomorphic with the matrix algebra \( M(d_\alpha, \mathbb{C}) \), but again as matrices the operators \( E_{i\alpha}^{\mu\nu}(\alpha) \) belong to matrix algebra \( M(d^n, \mathbb{C}) \).

From equation (88) it follows that operators \( E_{i\alpha}^{\mu\nu}(\alpha) \) may be represented by the standard elementary matrices \( e_{ij} \in M(d_\alpha N[M_\alpha^\beta : S(n-1)], \mathbb{C}) \), but this matrix representation is not equivalent to the matrix representation in the irreps \( M_\alpha^\beta \) of the algebra \( \mathcal{A}_d^n \).

5. Application to deterministic port-based teleportation

As we mentioned at the beginning of this paper, the algebraic tools described in previous sections have an explicit connection with a novel PBT protocol introduced and analysed for the qubit case in the series of papers [18–20] and extended to the qudit case, partially using a graphical representation of Temperley–Lieb algebra in [30] and fully in [23, 28]. It can be shown that the description of the probabilistic and dPBT can be written purely in terms of characteristics of the algebra \( \mathcal{A}_d^n \), and it is not only a different description of the problem. Namely, only by using the representation theory of \( \mathcal{A}_d^n \) are we able to describe PBT in dimensions higher than two in an efficient way.
5.1. Port-based teleportation protocol and algebra \( \mathcal{A}_d^n(d) \)

Here we give a brief description of the dPBT protocol and its connection with the algebra of partially transposed permutation operators with respect to the last subsystem. We present the connection between the PBT operator encoding the performance of the teleportation protocol and matrices \( \mathcal{O}(\alpha) \), which encode properties of the underlying algebra \( \mathcal{A}_d^n(d) \). Further, we apply a simplified formalism of the algebra \( \mathcal{A}_d^n(d) \) to present alternative proofs of theorems for entanglement fidelity \( F \) in the case of dPBT. Using the PRIR basis we solve an eigenproblem for the generators \( V'[\beta \alpha, n] \), where \( a = 1, \ldots, n - 1 \), which allows us to present a discussion about the asymptotic behaviour of \( F \) as well as re-derive the lower bound on \( F \) found previously in [2]. It is worth mentioning that having a full spectral analysis of \( V'[\beta \alpha, n] \) gives us possibilities for further investigations of hybrid schemes of PBT for qudits [18].

In the standard version of the PBT protocol Alice and Bob share a large resource state composed of \( N \) copies of the maximally entangled state \(|\psi^+\rangle\). Each copy is a two-qubit state called a port. Alice wishes to teleport to Bob an unknown state \( \rho \). To do so she performs a joint measurement from the set of positive-operator valued measure (POVM) \( \{ \Pi_i \}_{i=1}^{N} \) on her half of the resource state and the unknown state \( \rho \), and obtains an outcome \( i \in \{1, \ldots, N\} \) and communicates it to Bob. When Bob receives the information from Alice he discards all the ports except the \( i \)th, which is the teleported state. The most important features of this kind of protocol are the lack of unitary correction as in the ordinary teleportation scheme [4] and the fact that the teleported state is always successfully teleported but it is distorted. This means that the fidelity \( F \) between the unknown state \( \rho \) and teleported state is smaller than one and is a function of \( N \) as well as local Hilbert space dimension \( d \). Namely, we have the following expression for the fidelity in the mentioned scenario:

\[
F = \frac{1}{d^2} \sum_{i=1}^{N} \text{Tr} \left[ \sigma_i \tilde{\rho}^{-1/2} \sigma_i \tilde{\rho}^{-1/2} \right],
\]

where \( N = n + 1 \). In the above \( \tilde{\rho} \) is called the PBT operator and together with operators \( \sigma_i \), has the following representation in terms of partially transposed permutation operators \( V'[\beta \alpha, n] \) for \( i = 1, \ldots, n - 1 \):

\[
\sigma_i = \frac{1}{d^2} V'[\beta \alpha, n], \quad \tilde{\rho} = \sum_{i=1}^{N} \sigma_i = \frac{1}{d^2} \sum_{i=1}^{N} V'[\beta \alpha, n].
\]

From this construction it follows clearly that PBT operator \( \rho \) is an element of the previously studied algebra of the partially transposed permutation operators with respect to the last subsystem \( \mathcal{A}_d^n(d) \). We know that algebra \( \mathcal{A}_d^n(d) \) decomposes into a direct sum of two ideals, i.e. \( \mathcal{A}_d^n(d) = \mathcal{M} \oplus S \), and for analysis of the dPBT scheme only knowledge about ideal \( \mathcal{M} \) is crucial. What is most important (see [28]) is that due to symmetries in the system operator \( \rho \) is diagonal in the blocks represented by projectors \( F_{\nu}(\alpha) \) spanning the irreps contained in \( \mathcal{M} \) of the algebra \( \mathcal{A}_d^n(d) \) and can be written as

\[
\tilde{\rho} = \frac{1}{d^2} \rho = \frac{1}{d^2} \sum_{\alpha, d(\alpha) \leq d} \sum_{\mu \in \Phi^\alpha} \lambda_{\nu}(\alpha) F_{\nu}(\alpha),
\]

where the numbers

\[
\lambda_{\nu}(\alpha) = \frac{N m_{\nu} d_{\alpha}}{m_{\alpha} d_{\nu}}
\]

are eigenvalues of the sum \( \rho = \sum_{i=1}^{N} V'[\beta \alpha, n] \).
5.2. Spectrum of the PBT operator and its connection with matrix $Q(\alpha)$

In this section we focus on spectral analysis of the operator $\rho$ given in equation (101). The first spectral property of the operator $\rho$, which one can easily calculate using proposition 3, is the following.

Proposition 45.

$$\text{Tr} \rho = (n - 1)d^{n-1},$$

where the trace is taken in the space $(\mathbb{C}^d)^{\otimes n}$.

The next step to describe the spectrum of $\rho$ is to find the matrix form $M^{R}(\rho)$, where $R \in \{\Phi^\alpha, \Psi^\nu\}$, of this the operator $\rho$ in the irreps $\Phi^\alpha, \Psi^\nu$ of the algebra $A^{\otimes n}_t(d)$. This of course depends on the choice of the basis in the irreps $\Phi^\alpha, \Psi^\nu$. For example, in the irrep $\Phi^\alpha$ in the basis $E(\alpha) = \{e_{ij}(\alpha) : a, b = 1, \ldots, n - 1, i, j = 1, \ldots, d_\alpha\}$ (see [24]) we have

$$M^{\Phi^\alpha}_{E(\alpha)}(\rho) = Q(\alpha).$$

The above result holds only when $d \geq n - 1$, nevertheless it shows a connection between the spectra of matrices $Q(\alpha)$ and $\rho$. Using the so-called reduced basis $f \equiv \{f^\nu_j : h(\nu) \leq d, j_\nu = 1, \ldots, d_\nu\}$ in the irreps $\Phi^\alpha$ and the arbitrary basis in the irreps $\Psi^\nu$, one can prove a much stronger result.

Proposition 46. For any $n, d \geq 2$ we have

$$M^{\Phi^\alpha}_{B}(\rho) = \text{diag}(\lambda^\nu(\alpha)) \in \mathbb{M}(\text{rank}Q(\alpha), \mathbb{C}),$$

where $\lambda^\nu(\alpha)$ are all non-zero eigenvalues of the matrix $Q(\alpha)$ including their multiplicities. For the irreps of second kind $\Psi^\nu$ we have

$$M^{\Psi^\nu}_{B}(\rho) = 0 \in \mathbb{M}(d_\nu, \mathbb{C}),$$

for any basis $B$ in the irrep $\Psi^\nu$.

From the above proposition and from theorem 19 we deduce one of the main results of this paper, the structure of the spectrum of the operator $\rho$.

Theorem 47. Let $A^{\otimes n}_t(d)$ be the algebra of partially transposed operators in $(\mathbb{C}^d)^{\otimes n}$. Then for any $n, d \geq 2$ the non-zero eigenvalues of the operator $\rho$ are the non-zero eigenvalues $\lambda^\nu(\alpha)$, including multiplicity, of the matrices $Q(\alpha)$, where irreps $\alpha$ of $S(n - 2)$ are those which appear in the decomposition of $A^{\otimes n}_t(d)$ in theorem 19, i.e. when $h(\alpha) \leq d$. The multiplicity $m^\nu_{\nu, \alpha}$ of the eigenvalue $\lambda^\nu(\alpha)$ of operator $\rho$ is equal to

$$m^\nu_{\nu, \alpha} = m_\alpha d_\nu,$$

where $m_\alpha$ is given in theorem 19 and the multiplicity $m_0$ of the eigenvalue 0 of operator $\rho$ is equal to

$$m_0 = d^{n-1}(d - n + 1).$$

The explicit equation for the eigenvalues $\lambda^\nu(\alpha)$ of the operator $\rho$ are given in the lemma 15, theorem 12(b) and corollary 37. From the properties of the eigenvalues of the matrices $Q(\alpha)$
one can deduce several properties of the spectrum of the operator $\rho$. The first such a result concerns the spectral radius of the operator $\rho$.

**Proposition 48.** For any $n, d \geq 2$ the biggest eigenvalue of the operator $\rho$ is of the form

$$\lambda_{\text{max}} = d + n - 2,$$

and has multiplicity $m_{\text{max}} = \frac{1}{(n-2)!} \sum_{\sigma \in S(n-2)} d(\sigma)$.

The above proposition easily follows from the form of $\lambda_{\nu}(\alpha)$ given in theorem 12(a). We can also calculate the minimal non-zero eigenvalue of the operator $\rho$.

**Proposition 49.** The minimal non-zero eigenvalue of the operator $\rho$ has the following value

$$\lambda_{\text{min}} = d - h(\alpha) : h(\alpha) < d.$$  

In particular we have:

(a) if $d \geq n - 1$, then $\lambda_{\text{min}} = d - h(\alpha) \geq 1$ with the multiplicity equal to

$$m_{\text{min}} = \frac{1}{(n-2)!} \sum_{\sigma \in S(n-2)} \text{sgn}(\sigma)d(\sigma),$$  

(b) if $d \leq n - 2$, then $\lambda_{\text{min}} = d - h(\alpha) = 1$ with the multiplicity given through expression (111).

### 5.3. Fidelity calculation in the case of a maximally entangled state as a resource state

Now that we have all the tools developed in the previous chapters, we are ready to apply them to the description of DPBT. In the first step, using proposition 33 we calculate the following quantity which appears in the expression for the fidelity $F$ given in equation (99) of the deterministic version of the protocol:

$$\text{Tr} \left[ M_{\nu}^\prime(\rho^{-1/2})M_{\nu}^\prime(\nu^\prime[(a,n)])M_{\nu}^\prime(\rho^{-1/2})M_{\nu}^\prime(\nu^\prime[(a,n)]) \right] \equiv \text{Tr}_{\Phi^\mu_n} \left[ \rho^{-1/2}(\nu)[a,n])\rho^{-1/2}(\nu^\prime)[(a,n)] \right].$$  

Namely, we have the following:

$$\text{Tr}_{\Phi^\mu_n} \left[ \rho^{-1/2}(\nu)[a,n])\rho^{-1/2}(\nu^\prime)[(a,n)] \right]$$

$$= \sum_{\nu,\lambda_\nu} \sum_{\mu,\lambda_\mu} \sum_{k_\nu,k_\mu} \frac{1}{(n-1)^2} \frac{d_\nu \alpha}{d_\lambda \alpha} \psi_{\lambda_\nu}(a,n-1) \sqrt{\lambda_\nu(\alpha)}(\psi^\mu_{\lambda_\mu})_{k_\nu,k_\mu} [(a,n-1)]$$

$$\times (\psi^\nu_{\lambda_\nu})_{k_\nu,k_\mu} [(a,n-1)] \sqrt{\lambda_\mu(\alpha)}(\psi^\mu_{\lambda_\mu})_{k_\nu,k_\mu} [(a,n-1)],$$  

(113)

which, after summation over $\xi_\nu, j_\xi_v$ and $\xi_\mu, j_\xi_\mu$ reduces to

$$\sum_{\nu,\lambda_\nu} \sum_{k_\nu} \frac{1}{(n-1)^2} \frac{d_\nu \alpha}{d_\lambda \alpha} \lambda_\nu(\alpha) \lambda_\mu(\alpha) \delta_{k_\nu,k_\mu} = \frac{1}{(n-1)^2} \sum_{\nu,\lambda_\nu} \frac{d_\nu \alpha}{d_\lambda \alpha} \lambda_\nu(\alpha) \lambda_\mu(\alpha).$$  

(114)

We can summarise the above calculations in the following proposition.
\textbf{Proposition 50.} In the irrep $\Phi^\alpha$ of the algebra $A^\alpha_n(d)$ we have

$$\text{Tr}_{\Phi^\alpha} \left[ \rho^{-1/2} V'(\ket{a,n}) \rho^{-1/2} V'(\ket{a,n}) \right] = \frac{1}{(n-1)^2} \sum_{\nu,\mu} d_\nu d_\mu \sqrt{\lambda_\nu(\alpha) \lambda_\mu(\alpha)}. \quad (115)$$

In order to calculate the trace of the operator $\rho^{-1/2} V'(\ket{a,n}) \rho^{-1/2} V'(\ket{a,n})$ in arbitrary representation $R(A^\alpha_n(d))$ it is enough to multiply $\text{Tr}_{\Phi^\alpha} \left[ \rho^{-1/2} V'(\ket{a,n}) \rho^{-1/2} V'(\ket{a,n}) \right]$ by the multiplicity of the irrep $\Phi^\alpha$ in the representation $R(A^\alpha_n(d))$, i.e. we have

\textbf{Corollary 51.} For any representation $R(A^\alpha_n(d))$ of the algebra $A^\alpha_n(d)$ we have

$$\text{Tr}_{R(A^\alpha_n(d))} \left[ \rho^{-1/2} V'(\ket{a,n}) \rho^{-1/2} V'(\ket{a,n}) \right] = \frac{1}{(n-1)^2} \sum_{\Phi^\alpha \epsilon R(A^\alpha_n(d))} \sum_{\nu,\mu \epsilon \Phi^\alpha} d_\nu d_\mu \sqrt{\lambda_\nu(\alpha) \lambda_\mu(\alpha)}. \quad (116)$$

In particular we have:

(a) for regular representation of the algebra $A^\alpha_n(d)$

$$\text{Tr}_{A^\alpha_n(d)} \left[ \rho^{-1/2} V'(\ket{a,n}) \rho^{-1/2} V'(\ket{a,n}) \right] = \frac{1}{n-1} \sum_{\alpha,\beta(\alpha) \subseteq d} \sum_{\nu,\mu \epsilon \Phi^\alpha} d_\nu d_\mu \sqrt{\lambda_\nu(\alpha) \lambda_\mu(\alpha)}. \quad (117)$$

(b) for the natural representation of the algebra $A^\alpha_n(d)$ in the space $\mathcal{H} = (\mathbb{C}^d) \otimes \mathbb{C}^{d^2}$

$$\text{Tr}_{\mathcal{H}} \left[ \rho^{-1/2} V'(\ket{a,n}) \rho^{-1/2} V'(\ket{a,n}) \right] = \frac{1}{(n-1)^2} \sum_{\alpha,\beta(\alpha) \subseteq d} \sum_{\nu,\mu \epsilon \Phi^\alpha} m_\alpha^\beta d_\nu d_\mu \sqrt{\lambda_\nu(\alpha) \lambda_\mu(\alpha)}, \quad (118)$$

where

$$m_\alpha = \frac{1}{(n-2)!} \sum_{\sigma \in S(n-2)} \chi^\alpha(\sigma^{-1}) d^\beta(\sigma). \quad (119)$$

Equation (118) leads to the following expression for the fidelity in the dPBT scheme when the resource state is a maximally entangled state, which was obtained independently in [28]:

$$F = \frac{1}{d^{N+2}} \sum_{\alpha,\beta(\alpha) \subseteq d} \left( \sum_{\nu,\mu \epsilon \Phi^\alpha} \sqrt{d_\nu d_\mu} m_\alpha^\beta \right)^2, \quad (120)$$

where sums over $\alpha$ and $\mu$ are taken whenever the number of rows in the corresponding Young diagrams is not greater than the dimension of the local Hilbert space $d$.

The reader may note that the formula for entanglement fidelity presented above is for the PBT operator $\hat{\rho}$, so the eigenvalues $\lambda_\nu(\alpha)$ used for calculations have to be rescaled by the factor $1/d^N$.

5.4. Properties of the fidelity

In this section we derive some basic properties of the standard fidelity given as
\[
F \equiv F_n(d) = \frac{n-1}{d^{n+1}} \text{Tr} \left[ \rho^{-\frac{1}{2}} V^\dagger [(a,n)] \rho^{-\frac{1}{2}} V [(a,n)] \right] = \frac{n-1}{d^{n+1}} \sum_{\alpha : \beta(\alpha) \leq d} \frac{d_\alpha}{m_\alpha} \left[ \sum_{\mu \neq 0} \lambda_\mu^{-\frac{1}{2}} (\alpha) m_\mu \right]^2.
\]

It appears that the power \(-\frac{1}{2}\) of \(\rho\) implies very particular properties of the fidelity \(F_n(d)\). First we prove the following.

**Theorem 52.** For any \(d \geq 2\) and \(n \geq 2\) we have
\[
F_n(d) \leq 1, \quad \lim_{n \to \infty} F_n(d) = 1.
\]

**Remark 53.** The first property of the fidelity \(F_n(d)\) is in fact the justification of the definition of the fidelity, but it is also a necessary statement in the proof of the second result in theorem 52.

In the proof of theorem 52 we need the spectral decomposition of the essential projectors \(V^\dagger [(a,n)]\) in the irrep \(M^\varphi_f\).

**Proposition 54.** The set of orthonormal vectors
\[
w_{i,\alpha} = w_{i,\alpha} [a, \alpha] = \left( \sqrt{\frac{m_{\alpha}}{d m_\alpha}} \otimes_{\beta(\alpha) \leq d} [a, n-1] \right) \in \mathbb{C}^{\text{dim} M^\alpha_f}, \quad i_\alpha = 1, \ldots, d_\alpha,
\]
where \(\rho = \alpha + \Box, \rho \neq \emptyset, \xi_{\rho} = \rho - \Box, j_{\xi_{\rho}} = 1, \ldots, \text{dim} \xi_{\rho}\) are PRIR indices, are eigenvectors of the matrix \(M^\rho_f (V^\dagger [(a,n)])\), i.e. we have
\[
M^\rho_f (V^\dagger [(a,n)]) w_{i,\alpha} [a, \alpha] = d w_{i,\alpha} [a, \alpha],
\]
and
\[
M^\rho_f (V^\dagger [(a,n)]) = d \sum_{i_\alpha=1}^{d_\alpha} w_{i,\alpha} [a, \alpha] w^\dagger_{i,\alpha} [a, \alpha].
\]

The remaining orthonormal eigenvectors of the matrix \(M^\rho_f (V^\dagger [(a,n)])\), corresponding to the eigenvalue 0, will be denoted as \(w_j [a, \alpha]\), where \(j = d_\alpha + 1, \ldots, \text{dim} M^\varphi_f\).

We define also

**Definition 55.** The rectangular matrix
\[
W \equiv W [a, \alpha] = [w_1 w_2 \ldots w_{d_\alpha}] \in \mathbb{M}(d_\alpha \times \text{dim} M^\rho_f, \mathbb{C})
\]
has the columns which are eigenvectors for eigenvalue \(d\) of the matrix \(M^\rho_f (V^\dagger [(a,n)])\), defined in proposition 54.

Next we will need also the dimension structure of the natural representation of \(A^n (d)\). Now we recall the following.

**Theorem 56.** The algebra \(A^n (d)\) in its natural representation in the space \((\mathbb{C}^d)^{\otimes n}\) has the following decomposition into irreps:

\[
\text{(121)}
\]

\[
\text{(122)}
\]

\[
\text{(123)}
\]

\[
\text{(124)}
\]

\[
\text{(125)}
\]
\[ V_d[A^\alpha_n(d)] = \bigoplus_{\alpha, h(\alpha) \leq d} m_\alpha \Phi^\alpha \oplus \bigoplus_{\nu, h(\nu) < d} M_\nu \Psi^\nu, \quad (127) \]

where the multiplicity \( m_\alpha \) is equal to the multiplicity of the irrep \( \varphi^\alpha \) of \( S(n-2) \) in the representation \( V_d[S(n-2)] \), i.e.

\[ m_\alpha = \frac{1}{(n-2)!} \sum_{\sigma \in S(n-2)} \chi_\alpha(\sigma^{-1}) d^\sigma, \quad (128) \]

and

\[ M_\nu = dm_\nu - \sum_{\alpha, \nu \in \text{ind} S(n-1)(\nu^n)} m_\alpha. \quad (129) \]

From the above theorem we deduce

**Corollary 57.** We have the following relation between the dimensions of the natural representation space and the dimensions and multiplicities of the irreps of the algebra \( A^\alpha_n(d) \)

\[ d^\alpha = \sum_{\alpha, h(\alpha) \leq d} m_\alpha \dim \Phi^\alpha + \sum_{\nu, h(\nu) < d} M_\nu \dim \psi^\nu, \quad (130) \]

or equivalently

\[ 1 = \frac{1}{d^\alpha} \sum_{\alpha, h(\alpha) \leq d} m_\alpha \dim \Phi^\alpha + \frac{1}{d^\nu} \sum_{\nu, h(\nu) < d} M_\nu \dim \psi^\nu. \quad (131) \]

Further, in the proof of the theorem 52 we will also need a very classical inequality. Namely, we have the following:

**Theorem 58.** Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) is convex in some subset \( D \subset \mathbb{R} \) of its domain, then for any numbers \( \lambda_1, \ldots, \lambda_m \in D \) and for any probability distribution \( \sum_{i=1}^m s_i = 1 \) we have

\[ f \left( \sum_{i=1}^m s_i \lambda_i \right) \leq \sum_{i=1}^m s_i f(\lambda_i). \quad (132) \]

Moreover, it is known that the function \( f : \mathbb{R} \to \mathbb{R} \), which is twice differentiable on a subset \( D \subset \mathbb{R} \) and

\[ f''(\lambda) \geq 0, \quad \lambda \in D, \quad (133) \]

is convex in subset \( D \subset \mathbb{R} \) of its domain.

From theorem 58 we can deduce that

**Proposition 59.** The function \( f(\lambda) = \lambda^{-1/2} : \lambda \geq 0 \) is convex, because
\[ f''(\lambda) = \frac{1}{4} \lambda^{-5/2} \geq 0. \]

Now we are in a position to prove theorem 52. The proof is as follows.
Proof of theorem 52. First we prove the bound condition. Using the spectral decomposition

\[ M_j^\alpha(V'[(a,n)]) = d \sum_{i,n} w_{i,n} |a,\alpha\rangle |a,\alpha\rangle \]  

(134)

by a direct calculation one obtains

\[ \text{Tr} M_j^\alpha \left[ \rho^{-\frac{1}{2}} V'[(a,n)] \rho^{-\frac{1}{2}} V'[(a,n)] \right] = d^2 \sum_{i,n} \left( |w_{i,n}, M_j^\alpha (\rho^{-\frac{1}{2}}) w_{i,n} \rangle \right)^2. \]  

(135)

On the other hand, similarly we have

\[ \text{Tr} \left[ W^i M_j^\alpha (\rho^{-\frac{1}{2}}) W \right] = \sum_{p=1}^{\text{dim} M_j^\alpha} \sum_{i,n} \left( |w_{i,n}, M_j^\alpha (\rho^{-\frac{1}{2}}) w_{i,n} \rangle \right)^2 + \sum_{i,n} \left( |w_{i,n}, M_j^\alpha (\rho^{-\frac{1}{2}}) w_{i,n} \rangle \right)^2, \]  

(136)

where the rectangular matrix \( W \) is described in definition 55. From these equations we deduce

\[ \text{Tr} \left[ W^i M_j^\alpha (\rho^{-\frac{1}{2}}) W \right] \geq \text{Tr} M_j^\alpha \left[ \rho^{-\frac{1}{2}} V'[(a,n)] \rho^{-\frac{1}{2}} V'[(a,n)] \right]. \]  

(137)

so

\[ d^2 \text{Tr} \left[ W^i M_j^\alpha (\rho^{-1}) W \right] \geq \text{Tr} M_j^\alpha \left[ \rho^{-\frac{1}{2}} V'[(a,n)] \rho^{-\frac{1}{2}} V'[(a,n)] \right]. \]  

(138)

Now using the explicit form of the matrix \( W \) given in definition 55 and proposition 54 we obtain the following formula:

\[ \text{Tr} \left[ W^i M_j^\alpha (\rho^{-1}) W \right] = \frac{1}{(n-1)d} \text{dim} M_j^\alpha. \]  

(139)

Finally we obtain the following upper bound for the trace of the operator \( \rho^{-\frac{1}{2}} V'[(a,n)] \rho^{-\frac{1}{2}} V'[(a,n)] \) in the irrep \( M_j^\alpha \):

\[ \frac{d}{(n-1)} \text{dim} M_j^\alpha \geq \text{Tr} M_j^\alpha \left[ \rho^{-\frac{1}{2}} V'[(a,n)] \rho^{-\frac{1}{2}} V'[(a,n)] \right]. \]  

(140)

From this we deduce the upper bound for fidelity \( F_n(d) \) in the following way:

\[ F_n(d) = \frac{n-1}{d_{n+1}} \text{Tr} \left[ \rho^{-\frac{1}{2}} V'[(a,n)] \rho^{-\frac{1}{2}} V'[(a,n)] \right] \]

\[ = \frac{n-1}{d_{n+1}} \sum_{\alpha \in h(\alpha) \leq d} m_{\alpha} \text{Tr} M_j^\alpha \left[ \rho^{-\frac{1}{2}} V'[(a,n)] \rho^{-\frac{1}{2}} V'[(a,n)] \right] \]

\[ \leq \frac{n-1}{d_{n+1}} \sum_{\alpha \in h(\alpha) \leq d} m_{\alpha} \frac{d}{(n-1)} \text{dim} M_j^\alpha \]

(141)

Now from corollary 57 we have
\[
\frac{1}{d^n} \sum_{\alpha \in \Lambda(n)} m_\alpha \dim M_\alpha^\alpha = 1 - \frac{1}{d^n} \sum_{\nu \in \Lambda(n)} M_\nu \dim \psi_\nu < 1, \tag{142}
\]

so in this way we obtain the first statement of theorem 52

\[
F_n(d) \leq 1 - \frac{1}{d^n} \sum_{\nu \in \Lambda(n)} M_\nu \dim \psi_\nu < 1. \tag{143}
\]

In order to prove the remaining part of the theorem we consider the generalised fidelity

\[
F_n(d) = \frac{n-1}{d^{n+1}} \text{Tr}(V'(a,n)\rho - 1/2 V'(a,n)) = \frac{n-1}{d^{n+1}} \sum_{\alpha \in \Lambda(n)} \frac{d_\alpha}{d}\left[\sum_{\mu \neq \theta} \lambda_\mu^{-1}(\alpha) m_\mu \right]^2,
\]

which may be rewritten as follows

\[
F_n(d) = n-1 \frac{1}{(n-1)d^{n+1}} \sum_{\alpha \in \Lambda(n)} \frac{m_\alpha}{d}\left[\sum_{\mu \neq \theta} \lambda_\mu^{-1}(\alpha) d_\mu \right]^2. \tag{145}
\]

Further

\[
F_n(d) = \frac{1}{(n-1)d^{n+1}} \sum_{\alpha \in \Lambda(n)} m_\alpha \left[\sum_{\mu \neq \theta} s_\mu^{-\frac{1}{2}}(\alpha) \right]^2, \tag{146}
\]

where

\[
s_\mu = \frac{d_\mu \lambda_\mu(\alpha)}{d(n-1)d_\alpha} \quad \text{with} \quad \sum_{\mu \neq \theta} s_\mu = \sum_{\mu = \alpha + \Box} s_\mu = 1, \tag{147}
\]

because \(\sum_{\mu \neq \theta} d_\mu \lambda_\mu(\alpha) = \sum_{\mu = \alpha + \Box} d_\mu \lambda_\mu(\alpha) = \text{Tr} Q(\alpha) = d(n-1)d_\alpha\), thus the sum

\[
\sum_{\mu \neq \theta} \lambda_\mu^{-1}(\alpha) \tag{148}
\]

is a convex combination of the numbers \(\lambda_\mu^{-1/2}\), and we may use theorem 58 and proposition 59, which give

\[
\left(\sum_{\mu \neq \theta} s_\mu \lambda_\mu(\alpha)\right)^{-\frac{1}{2}} \leq \sum_{\mu \neq \theta} s_\mu \lambda_\mu^{-\frac{1}{2}}(\alpha). \tag{149}
\]

Using the above inequality we obtain the following lower bound for generalised fidelity:
\[
F_n(d) = \frac{n-1}{d^{n-1}} \sum_{\alpha \beta(\alpha) \leq d} m_\alpha d_\alpha \left( \sum_{\mu \neq \theta} s_\mu \lambda_\mu^{-1}(\alpha) \right)^2 \geq \frac{n-1}{d^{n-1}} \sum_{\alpha \beta(\alpha) \leq d} m_\alpha d_\alpha \left( \sum_{\mu \neq \theta} s_\mu \lambda_\mu(\alpha) \right)^{-1} = \frac{n-1}{d^{n-1}} \sum_{\alpha \beta(\alpha) \leq d} m_\alpha d_\alpha \left( \frac{\text{Tr} Q^2(\alpha)}{\text{Tr} Q(\alpha)} \right)^{-1}.
\]

(150)

Directly from definition 7 of the matrix \(Q(\alpha)\) we obtain that \(\text{Tr} Q^2(\alpha) = (n-1)d_n(d^2 + n - 2)\), and

\[
F_n(d) \geq \frac{n-1}{d^{n-1}} \sum_{\alpha \beta(\alpha) \leq d} m_\alpha d_\alpha \left( \frac{d^2 + n - 2}{d} \right)^{-1} = \frac{n-1}{d} \left( \frac{d^2 + n - 2}{d} \right)^{-1},
\]

(151)

so

\[
F_n(d) \geq \frac{n-1}{d^2 + n - 2},
\]

(152)

and thus recover the result of [2], obtained using a different method. Inequality (152) together with the upper bound \(F_n(d) < 1\) implies that

\[
\lim_{n \to \infty} F_n(d) = 1.
\]

(153)

6. Discussion and open problems

We found significant simplifications of the algebra \(A^p_n(d)\) of the partially transposed permutation operators with respect to the last subsystem by developing tools of PRIRs by proving a few new orthogonality theorems for them. We apply our successful approach to studying PRIRs to algebra \(A^p_n(d)\) by simplifying existing theorems. The main simplifications concern matrix \(Q\) given in definition 7 and matrix \(Z\) given in theorem 12 constructed from eigenvectors of \(Q\). We were able to reduce the complexity of the underlying expressions by reducing the number of sums over all permutations from \(S(n)\). Such a reduction allows us to perform any calculations, in particular devoted to practical applications, discussed later more efficiently. The second main result obtained thanks to the new approach is relatively simple equations for the matrix elements of operators \(V'[\{a,n\}]\) for \(a = 1, \ldots, n\) with a particular case when \(a = n - 1\).

Finally we applied derived simplifications to obtain characteristics of the dPBT scheme. First we gave an explicit connection between PBT operator \(\rho\) and matrix \(Q\) describing properties of the algebra \(A^p_n(d)\). We have shown that non-zero eigenvalues of \(Q\) are exactly eigenvalues of the operator \(\rho\). Later we presented derivation for the fidelity \(F\) of the teleported state and expressed the final result using parameters describing irreps of \(A^p_n(d)\), such as dimensions and multiplicities as well as global parameters \(d\) and \(N\). We presented asymptotic analysis of \(F\) showing that \(\lim_{N \to \infty} F = 1\) for fixed \(d\), which certifies our approach. Moreover, using a completely new method of computation based on analysis of the eigenproblem of \(V'[\{a,n\}]\) for \(a = 1, \ldots, n - 1\) we derived a known non-trivial lower bound for the fidelity \(F\) expressed only by global parameters as the number of ports \(N\) and dimension \(d\).
Despite the progress made in this paper and [23, 28], there are still a few open questions connected with the general theory of algebra $A_n^d$ and possible applications to PBT. The most interesting and important question, in the opinion of the authors, would be a full eigenanalysis of the PBT operator $\rho$ similarly as for $V'(a, n)$ in proposition 54. Namely, we would like to find its eigenvectors (since eigenvalues are known) in terms of parameters describing irreps of $A_n^d$ and analyse its entanglement with respect to some particular cuts. Such an analysis would be helpful in the extension of the hybrid PBT to higher dimensions. Additionally, the simplification presented in this paper should technically be easier from the perspective of a representation theory description of $1 \rightarrow N$ universal quantum cloning machines [27].

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Appendix. Summary of known fact about PRIRs

In this appendix, for self-consistency of our paper, we present known facts about PRIRs. For all proofs we refer the reader to [28].

**Proposition A.1.** Let $(\psi^\mu_\rho)^{\alpha\alpha}(\sigma) = \left(\psi^{\alpha\alpha}_i(\sigma)\right)$ be the matrices on the diagonal of the PRIR matrix $\psi^\mu_\rho(\sigma)$ where $\sigma \in S(n)$, then
\[
\forall \alpha \in \mu \quad \varphi^\alpha(\pi) (\psi^\mu_\rho)^{\alpha\alpha}[\langle a, n \rangle] \varphi^{-1} = (\psi^\mu_\rho)^{\alpha\alpha}[\langle \pi(a), n \rangle],
\]
and from this it follows that
\[
\forall \alpha \in \mu \quad \forall \pi \in S(n-1) \quad \forall a = 1, \ldots, n-1 \quad \text{Tr}[(\psi^\mu_\rho)^{\alpha\alpha}[\langle a, n \rangle]] = \text{Tr}[(\psi^\mu_\rho)^{\alpha\alpha}[\langle \pi(a), n \rangle]],
\]
so the trace in each diagonal block is constant on the transpositions, which naturally indexed the coset $S(n)/S(n-1)$.

**Proposition A.2.** The PRIR $\psi^\mu_\rho$ of $S(n)$ satisfies the following summation rules
\[
\sum_{a=1}^{n-1} (\psi^\mu_\rho)[\langle a, n \rangle] = \frac{n(n-1)}{2} \chi^\mu[\langle 1, 2 \rangle] d_\mu - \bigoplus_{\alpha \in \mu} \frac{(n-1)(n-2)}{2} \frac{\chi^\alpha[\langle 1, 2 \rangle]}{d_\alpha} 1_{\varphi^\alpha},
\]
which implies that for the diagonal blocks we have
\[
\forall \alpha \in \mu \quad \sum_{a=1}^{n-1} (\psi^\mu_\rho)^{\alpha\alpha}[\langle a, n \rangle] = \left[ \frac{n(n-1)}{2} \frac{\chi^\mu[\langle 1, 2 \rangle]}{d_\mu} - \frac{(n-1)(n-2)}{2} \frac{\chi^\alpha[\langle 1, 2 \rangle]}{d_\alpha} \right] 1_{\varphi^\alpha}.
\]
Remark A.1. Equation (A.3) in proposition A.2 may be written in a more explicit form as follows:

$$\forall \alpha \in \mu \sum_{a=1}^{n-1} (\psi^{\mu}_{[a]} \chi^{\alpha}_{[(a,n)]} = \left[ \frac{n(n-1)}{2} \chi^{\mu}_{[(1,2)]} - \frac{(n-1)(n-2)}{2} \chi^{\alpha}_{[(1,2)]} \right] \delta_{i_{\alpha}, j_{\alpha}}.$$  (A.5)

where $i_{\alpha}, j_{\alpha} = 1, \ldots, d_{\alpha}$.

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