Using overlap of sky localization probability maps for filtering potentially lensed pairs of gravitational-wave signals*

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ABSTRACT

When a gravitational wave passes by a massive object, its trajectory gets curved which can result in multiple images separated by a time delay, a phenomenon known as strong lensing. Although gravitational wave lensing has not been observed yet, it is currently predicted that it will be possible to detect with third-generation detectors. Moreover, the future detectors are expected to significantly increase the gravitational-wave detection rate up to $O(10^5 - 10^6) \text{ yr}^{-1}$ which could also increase the chance of observing gravitational wave lensing. Therefore, it is necessary to develop accurate and efficient tools to filter potentially lensed gravitational-wave signal pairs. In the study, we propose three statistics to analyze the overlap of two sky localization probability maps, which could provide a quick preliminary lensing analysis of two gravitational-wave signals within 3 seconds. We simulate 200 lensed pairs of gravitational-wave signals for 5 signal-to-noise ratios to investigate the performance of the search methodology. By setting up a threshold with a specific false positive rate $\text{FPR} = 10^{-2}$ for the three overlap statistics, we conclude that it is possible to filter out more than 99% of the non-lensed event pairs while keeping all lensed event pairs.

Key words: Gravitational wave – Gravitational lensing – Skymap overlap

1 INTRODUCTION

Gravitational wave (GW) detection in 2015 (Abbott et al. 2016), 93 confident GW events have been detected by the Advanced LIGO (Aasi et al. 2015) and the Advanced Virgo (Acernese et al. 2014) in the first (O1), the second (O2), and the third observing run (O3) (Abbott et al. 2019, 2021b,a; The LIGO Scientific Collaboration et al. 2021a). Although there is no compelling evidence in lensing of the observed GW events in O3a (The LIGO Scientific Collaboration et al. 2021b), theoretical predictions suggest roughly one strong GW lensing event per year at the Advanced LIGO-Advanced Virgo-KAGRA design sensitivity (Wierda et al. 2021; Xu et al. 2021). Therefore, it is expected that the first lensed GW event pair will be detected in the coming years (Ng et al. 2018; Wierda et al. 2021).

There are different methods to detect strongly lensed GW event pairs (The LIGO Scientific Collaboration et al. 2021b). One method (Haris et al. 2018; Hanuksele et al. 2019) uses the overlap of the posterior distributions of the source parameters to check for the consistency of the source properties between events. The overlap of the two posterior distributions is expected to be high when the two events come from the same source. Another method (Liu et al. 2021; Lo & Hernandez 2021) performs a full Bayesian analysis, namely the joint-parameter estimation (joint-PE), on the detected events.

Joint-PE provides a solid statistical framework to calculate the probability of a GW event pair being lensed images from the same source. The current number of confident GW events is 93 (Abbott et al. 2019, 2021b,a; The LIGO Scientific Collaboration et al. 2021a), which gives 4278 pairs of GW events. However, performing joint-PE on all event pairs is very time-consuming. Moreover, for a proposed Einstein Telescope (ET) survey, the predicted event rate of binary neutron star (BNS) mergers is as high as $O(10^5 - 10^6) \text{ yr}^{-1}$ and $O(10^4) \text{ yr}^{-1}$ for NS-BH mergers (Amaro-Seoane et al. 2009). This means that the detector sensitivity and the number of GW signals in the future will be far higher and will further increase the time for doing joint-PE on all possible event pairs. For the purpose

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of sustainability, it is necessary to establish a low latency filter to screen out event pairs that are definitely not lensed.

There are several fast algorithms for rapid identification of strongly lensed GW events developed to address the sustainability issue. One of the algorithms is aimed to search for strongly lensed multiple GW images - from the fact that the signals come from the same source, it is possible to split the joint PE into two easier PEs, that we can use the result from one event to perform the inference of the another event (Janquart et al. 2021). Such methodology enables relatively quick multiple-image analyses with high accuracy. Another approach proposed by Goyal et al. uses a machine learning model to predict the probability of lensing in event pairs from the time-frequency maps and posterior distributions of GW events (Goyal et al. 2021). The computational time of the lensing probability is estimated to be around 3 seconds with most of the time spent on loading necessary files. Both of the mentioned works have successfully reduced the computational expense of filtering potentially lensed GW event pairs.

To establish a low latency filter, we propose the use of sky localization probability maps (skymaps) to analyze the possibility of an event pair coming from the same source. The deflection angle is usually a few hundred arc seconds for black holes and galaxy clusters (Schneider et al. 1992), which is negligibly small as it is far less than the uncertainty of the localization of the source (Aasi et al. 2016). Therefore, the skymaps of the images from the same source are expected to be highly overlapped. In other words, event pairs with non-overlapping skymaps are definitely not lensed pairs. We use this fact to analyze the possibility of whether two GW signals are a lensed pair and filter potentially lensed pairs by comparing their skymaps. With the use of overlap, we can reject a large fraction of event pairs that are definitely not image pairs. However, one must note that having a high overlap in the skymaps does not necessarily imply the events are a lensed pair. In contrast to the machine learning approach in Goyal et al. (2021), we investigate using measures of the overlap of skymaps without involving sophisticated learning models, which can be easily incorporated into search pipelines such as the subthreshold search (Li et al. 2021) to send alerts for performing follow-up analyses. The goal of the low latency filter is to reduce time cost by rapidly filtering out most of the non-lensed GW event pairs for further lensing analysis. Therefore, we would like to quantitatively investigate the efficiency of using skymap overlap to filter the non-lensed pairs while not rejecting the lensed pairs. With a low latency skymap generated by a rapid sky localization algorithm such as BAYESTAR (Singer & Price 2016), a quick preliminary lensing analysis on two GW signals can be done within a minute.

In this paper, we present three overlap statistics: normalized posterior overlap, 90% credible region overlap and cross-HPD statistic, to quantitatively investigate the efficiency of using skymap overlap to filter the non-lensed pairs while not rejecting the lensed pairs. The details of the statistics are described in Sec. 3. Using simulated GW events, we study the distribution of skymap overlap of lensed GW event pairs under different signal-to-noise ratios (SNRs). In order to filter potentially lensed event pairs, a reasonable threshold of skymap overlap that can filter out non-lensed event pairs while keeping all of the lensed event pairs should be set with a specific false positive rate (FPR). Based on the results, we compare the effectiveness of the three statistics and discuss the feasibility of using skymap overlap to filter potentially lensed pairs of GW signals.

This paper is organized as follows: In Sec. 2 and 3, we establish the methodology of the search by introducing posterior distribution and the three statistics mentioned. In Sec. 4, we investigate the feasibility of using skymap overlap for filtering potentially lensed pairs of GW signals with the receiver operating characteristic (ROC) curves. In Sec. 5, we conclude our findings on the feasibility of skymap overlap in this paper and its potential work in the future.

2 SKYMAP

2.1 Posterior Distribution

To generate a skymap, we use the observed data to estimate the source location of a GW projected on the celestial sphere under a fixed waveform model. This process relies on parameter estimation, which allows us to quantify the uncertainty of the GW location on the celestial sphere. We first define a hypothesis $H$ that the GW signal fits the fixed waveform model. Then for each GW source parameter, we set up a prior probability distribution $p(\theta|H)$, which represents the a priori knowledge of the model parameters before observing the data. Using Bayes’ theorem, the prior probability distribution $p(\theta|H)$ and the data $d$ of the GW signals are used to compute the posterior distribution $p(\theta|d,H)$ of the waveform parameters (Veitch et al. 2015) defined as

$$p(\theta|d,H) \equiv \frac{p(\theta|H)p(d|\theta,H)}{p(d|H)}.$$  (1)

2.2 Sky Localization Probability Map

A waveform model typically has many parameters that we can denote as $\theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$. The posterior distribution $p(\theta|d,H)$ represents the joint posterior distribution of all parameters. To determine the posterior distribution of a specific parameter, we marginalize the nuisance parameters, where the posterior distribution of a specific parameter $p(\theta_1|d,H)$ is obtained by

$$p(\theta_1|d,H) \equiv \int d\theta_2 \ldots d\theta_N p(\theta|d,H).$$  (2)

After obtaining the posterior distribution of the right ascension and the declination of the source location, which is referred as skymap in this paper, we compute the three overlap statistics of a skymap pair described in the following section.

3 OVERLAP STATISTICS

3.1 Posterior Overlap

After performing the parameter estimation on two GW events separately, we obtain two posterior distributions $p(\hat{\Omega})$ and $q(\hat{\Omega})$ of their skymap coordinate $\hat{\Omega}$. We determine whether a GW signal pair is potentially lensed by the inner product of $p(\hat{\Omega})$ and $q(\hat{\Omega})$, which quantifies the overlap between two probability densities. The posterior overlap $D_{PO}$ between two skymaps is defined similarly to Harris et al. (2018) as

$$D_{PO} \equiv \int p(\hat{\Omega}')q(\hat{\Omega}')d\hat{\Omega}'$$  (3)

which is not normalized. In order for the posterior overlap of different GW signal pairs to be comparable to each other, we normalize the
posterior overlap with skymap of the two GW events, which is defined as
\[
D_{\text{NPO}} \equiv \frac{\int p(\hat{\Omega}') q(\hat{\Omega}') d\Omega'}{\sqrt{\int p(\hat{\Omega}') p(\hat{\Omega}') d\Omega'} \sqrt{\int q(\hat{\Omega}') q(\hat{\Omega}') d\Omega'}} \in [0, 1].
\]  

(4)

3.2 90% Credible Region Overlap

For a GW signal with an extremely large signal-to-noise ratio (SNR), the posterior distribution of the source location could be narrowly peaked. This may lead to an extremely small value of posterior overlap and normalized posterior overlap with another GW signal. As a result, the two GW signals may be recognized as a non-lensed pair and filtered out. Therefore, we propose to use a 90% credible region overlap.

Instead of finding the overlap probability of the source position on the sky, we directly compute the overlapping area of the 90% credible regions. The minimum value used to keep a high 90% credible region overlap when the difference between two event SNRs is high for lensed pairs. Therefore, it reduces the chance of lensed pairs being filtered out. The 90% credible region overlap \(D_{\text{90\%CR}}(p, q)\) of two GW signals is computed by masking their skymaps, which is defined as

\[
D_{\text{90\%CR}}(p, q) = \frac{\int \mathbb{1}_{\text{90\%CR}}[p(\hat{\Omega}')] \mathbb{1}_{\text{90\%CR}}[q(\hat{\Omega}')] d\Omega'}{\min(\int \mathbb{1}_{\text{90\%CR}}[p(\hat{\Omega}')] d\Omega', \int \mathbb{1}_{\text{90\%CR}}[q(\hat{\Omega}')] d\Omega')},
\]  

(5)

where \(\mathbb{1}_{\text{90\%CR}}[p(\hat{\Omega})]\) and \(\mathbb{1}_{\text{90\%CR}}[q(\hat{\Omega})]\) are the indicator function. This indicator function is 1 when the sky location is in the 90% credible region of the corresponding skymap, and 0 otherwise.

3.3 Cross-HPD Statistic

We introduce another statistic called the cross-highest posterior density (HPD). The purpose of this statistic is similar to that of the 90% credible region overlap. We use the maximum a posteriori (MAP), which is the mode of the posterior distribution, as an alternative of the 90% confidence level to decide the credible region used for computing the overlap of the skymap between two GW signals. We first compute the probability of the MAP of one skymap on another skymap, which is referred as the search probability \(\mathcal{H}\). It is defined as

\[
\mathcal{H}(p, q) = \int \mathbb{1}[q(\hat{\Omega}') > q(\hat{\Omega}_{p, \text{MAP}})] q(\hat{\Omega}') d\Omega',
\]  

(6)

where \(\mathbb{1}\) is the indicator function, \(q(\hat{\Omega})\) is the posterior distribution of sky position of an event, and \(\hat{\Omega}_{p, \text{MAP}}\) is the MAP estimate of \(\hat{\Omega}_p\), also known as the coordinates of the point at highest probability density.

\(\mathcal{H}(p, q)\) quantifies the overlap of the skymap between the event pairs. The lower the value of \(\mathcal{H}(p, q)\), the higher the overlap between the skymaps. The cross-HPD statistic \(D_{\text{HPD}}(p, q)\) is defined as

\[
D_{\text{HPD}}(p, q) \equiv \max \{1 - \mathcal{H}(p, q), 1 - \mathcal{H}(q, p)\},
\]  

(7)

which is symmetric in \(p\) and \(q\).

4 RESULTS & DISCUSSION

We simulate 200 lensed pairs of GW signals for 5 different SNRs to investigate the performance of the search methodology. We first generate the simulated data of 200 lensed pairs of GW signals with PyCBC (Nitz et al. 2021). Primary mass follows a power-law profile with slope -2.35, while the mass ratio and sky position are uniformly distributed. With the simulated data, we re-scale all the events to 5 different SNRs by adjusting the luminosity distance of each event. In total, there are 1,000 event pairs in the injection campaign. For details, see Sec. 4.1. We use a Bayesian inference library Bilby (Ashton et al. 2019; Romero-Shaw et al. 2020) to perform parameter estimation on the GW events. After obtaining the posterior distribution of the sky location of each event, the skymaps of all events are generated using ligo.skymap (Singer et al. 2016). We pair up the skymaps for every possible combination and separate them into lensed-pair group and non-lensed-pair group. The three overlap statistics of all event pairs are then computed. Details of the overlap statistics can be found in Sec. 4.1. Finally, we plot the receiver operating characteristic (ROC) curve to compare the performance of the overlap statistics. Details of the ROC curve can be found in Sec. 4.3.

4.1 Injection Campaign

To study the distribution of skymap overlap of lensed-pair candidates under different SNRs and the efficiency of overlap statistics on filtering potentially lensed pairs of GW signals, we set up an injection campaign with 200 lensed GW event pairs. The localization uncertainty of a GW signal depends on its SNR, therefore we are interested in the effect of different SNRs on the skymaps of lensed GW signal pairs. The uncertainty of the localization is smaller when the SNR is higher. Therefore, we expect the overlap to be generally lower for GW event pairs with a higher SNR than those with a lower SNR. We use 5 different SNRs to study the difference in the filtering threshold for different SNR pairs by skymap overlap.

The largest SNR of the event pairs is chosen to be 50, it is a sufficiently high SNR upper limit due to the rarity of detecting a signal with such SNR. After setting up the upper limit and the lower limit of the event SNR, we uniformly divide the range between 8 and 50 to obtain 5 different SNRs and rescale the luminosity distances to attain the desired SNR.
Overlap for non-image pairs in the injection campaign to study the distribution of skymap coverage ensures that there is a sufficient number of lensed event pairs with relatively high overlap values are the false positives of YPE credible regions of non-image pairs, those non-image event bleregions covered about half of the sky, notethat there are overlaps RPP events come from different parts of the sky and their YPE credible regions are significantly larger than that of the events with SNR = 8.

Table 1. The parameters of the waveform simulated and the set-up of the injection campaign. 200 event pairs are generated. For simplicity, the spins of all events are set to zero. All events are rescaled to 5 different SNRs by adjusting the luminosity distance of each event. Finally, there are 2,000 events in total.

| SNR | Mean area of 90% credible region (deg$^2$) |
|-----|---------------------------------|
| 8   | 186.8                           |
| 18.5 | 34.8                           |
| 29  | 12.0                           |
| 39.5 | 6.4                            |
| 50  | 4.1                            |

Table 2. Mean of the 90% credible region area of the skymaps of 200 events simulated using HLV network for 5 SNRs. Details of the injection campaign refer to Table 1.

After performing the parameter estimation, we generate the skymaps for all events. The prior of chirp mass is set to be uniform from 4 $M_\odot$ to 150 $M_\odot$, and the two component masses are constrained between 5 $M_\odot$ to 150 $M_\odot$. The prior of geocent time is chosen to be a delta function that peaks at the injected parameter. The sampler used for the parameter estimation is dynesty (Speagle 2020). Finally, the three overlap statistics of each event pair are computed. The lensed event pairs consist of 1,000 samples which are 200 lensed events pairs for each of the 5 SNRs. For the non-lensed event pairs, the overlap of all the possible event pairs from the original 2,000 events is computed, which results in about 700,000 samples. By plotting the ROC curve of each statistic, we study the performance of the three statistics on filtering lensed candidate event pairs.

4.3 Performance of Statistics

With the overlap statistics of all the event pairs, the performance of the statistics is shown in a ROC curve. In a ROC curve, which shows the performance of a classification model at all classification thresholds by plotting the true positive rate (TPR) against the false positive rate (FPR).

The y-axis of the ROC curves is TPR, which is defined as

$$TPR = \frac{TP}{TP + FN},$$

where TP is the number of lensed event pairs which overlap value is higher than the filtering threshold, while FN is the number of lensed event pairs which overlap values are smaller than the filtering threshold.

The x-axis of the ROC curves is FPR, which is defined as

$$FPR = \frac{FP}{FP + TN},$$

where FP is the number of non-lensed event pairs which overlap value is higher than the filtering threshold, while TN is the number of non-lensed event pairs which overlap value is smaller than the filtering threshold.

Given a specific FPR or a specific filtering threshold, a higher TPR indicates a better performance of the statistic because less lensed event pairs are recognized as non-lensed event pairs by the statistic. We aim to set up a reasonable threshold with TPR $\approx$ 1 to filter non-lensed event pairs while keeping most of the lensed event pairs for further analysis. Ten ROC curves are shown in Figure 2 and Figure 3. The five ROC curves in Figure 2 indicate the performance of the three statistics for event pairs with one of the event SNRs is small (SNR = 8), while the five ROC curves in Figure 3 indicate the performance of the three statistics for event pairs with one of the event SNRs is large (SNR = 50).

Figure 2 shows the ROC curves of the three overlap statistics for event pairs with one event SNR = 8 while another is one of the 5 SNRs. In this case, the TPR of the three statistics shows nearly no difference when the FPR is fixed to a value between $10^{-2}$ and 1. The difference in the performance of the statistics is noticeable only when we fix FPR to a value smaller than around $10^{-2}$.

For a relatively small FPR value (FPR < $10^{-2}$), 90% credible region overlap shows the worst overall performance as it shows the
most significant drop at a lower FPR. However, all three statistics show a significant drop when the FPR is lower than $10^{-2}$. Within this range, the losses of lensed event pairs are significant for all three statistics. Therefore, the threshold should be fixed higher than $10^{-2}$ if we aim to keep most of the lensed event pairs. Thus, the performance of the three statistics is similar for a reasonable choice of threshold that can filter out non-lensed event pairs while keeping most of the lensed event pairs. The straight line of the TPR drop for 90% credible region overlap in FPR $< 10^{-3}$ is due to a lack of samples. The data points of the ROC curve are generated by using a threshold to separate the input data into FP, TP, FN, and TN.

Figure 3 shows the ROC curves of the three overlap statistics for event pairs with one event SNR = 50 while another is one of the 5 SNRs. In this case, the TPR of the three statistics shows nearly no difference when the FPR is fixed to a value between $10^{-2}$ and 1. The difference in the performance of the statistics is noticeable only when we fix FPR to a value smaller than around $10^{-3}$.

For a relatively small FPR value (FPR $< 10^{-3}$), 90% credible region overlap shows the worst overall performance. However, again, choosing a threshold with FPR lower than around $10^{-3}$ will lead to a significant loss in the number of lensed event pairs. Therefore, the filtering threshold should be fixed in an FPR that is higher than $10^{-3}$ in order to keep all of the lensed event pairs. Thus, the performance of the three statistics is similar for a reasonable choice of filtering threshold that can filter out non-lensed event pairs while keeping all of the lensed event pairs. Again, the straight line of the TPR drop for 90% credible region overlap in FPR $< 10^{-3}$ is due to the lack of samples.

For the purpose of filtering potentially lensed candidate event pairs, the three skymap overlap statistics demonstrated a similar performance while fixing FPR between $10^{-2}$ and 1. In the mock data study, we can set up a specific FPR = $10^{-2}$ for event pairs in all SNR to filter out 99% of the non-lensed candidate pairs, which demonstrated that a large fraction of non-image pairs could be rapidly filtered away using the skymap overlap.

Figure 1. The contours of 90% credible region of 200 skymaps with SNR = 8 are shown in the figure. Details of the injection campaign are referred to Table 1. The figure shows the sky coverage of the injections with SNR = 8.
Figure 2. Receiver operating characteristic (ROC) curves of event pairs with one event SNR = 8. The black solid line $x=y$ represents the ROC curve of a random classifier. ROC curves indicate the performance of the overlap statistics. The classifier is better if it can achieve a higher true positive rate given a false positive rate. The three statistics do not show a large difference in the region of FPR between $10^{-2}$ and 1.
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Figure 3. Receiver operating characteristic (ROC) curves of event pairs with one event SNR = 50. The black solid line $x = y$ represents the ROC curve of a random classifier. The classifier is better if it can achieve a higher true positive rate given a false positive rate. The three statistics do not show a large difference in the region of FPR between $10^{-2}$ and 1.
5 CONCLUSIONS

In this paper, we have demonstrated our proposed search methodology using sky map overlap to filter potentially lensed pairs of GW signals. By setting up a threshold with a specific FPR = 10^{-2} for the three overlap statistics, we can filter out more than 99% of the non-lensed event pairs while keeping most of the lensed event pairs, which shows the feasibility of using sky map overlap to filter potentially lensed event pairs of GW signals.

Since the computation of the three statistics for each event pair takes less than 3 seconds, a new trigger could be constructed to compute the sky map overlap of all existing events within a short period of time using the low latency skymaps generated by BAYESTAR (Singer & Price 2016) or other rapid sky localization algorithms.

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