High Dimensional Combinatorial Random Walks and Colorful Expansion

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Abstract

Random walks on expander graphs have been extensively studied. We define a high order combinatorial random walk on high dimensional simplicial complexes. This walk moves at random between neighboring i-dimensional faces (e.g. edges) of the complex, where two i-dimensional faces are considered neighbors if they share a common (i + 1)-dimensional face (e.g. a triangle). We prove that if the links of the complex are good spectral expanders, then the high order combinatorial random walk converges quickly.

We derive our result about the convergence of the high order random walk by defining a new notion of high dimensional combinatorial expansion of a complex, which we term colorful expansion. We show that spectral expansion of the links of the complex imply colorful expansion of the complex, which in turn implies fast convergence of the high order random walk. We further show the existence of bounded degree high dimensional simplicial complexes which satisfy our criterion, and thus form an explicit family of high dimensional simplicial complexes in which the high order random walk converges rapidly.

Our high order combinatorial random walk is different than the high order topological random walk that has recently been defined in [PR]. The topological random walk is associated with the concentration of the spectrum of the high order laplacian on the space that is orthogonal to the coboundaries, while we, in a sense, need to argue about the concentration of the spectrum on the space that is orthogonal to the constant functions, which is a much larger space.

1 Introduction

Random walks on expander graphs, and especially on those of a bounded degree, have played a pivotal role in Computer Science, Mathematics, Physics and more. It is well known that a random walk on an expander graph converges very quickly to its stationary distribution (which is the uniform distribution in the case of regular graphs).

The question we address in this work is the following:

Are there bounded degree high dimensional simplicial complexes (which are the high dimensional analogous of graphs) in which a high order random walk converges quickly to its stationary distribution.

In a nutshell, our answer is the following: We provide a local criterion on a complex that implies fast convergence of the high order random walk on it. Moreover, we show an explicit

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family of bounded degree complexes which meet that criterion, and hence are bounded degree complexes for which the high order random walk mixes quickly.

A simplicial complex $X$ of dimension $d$ can be viewed as a $(d+1)$-hypergraph with a closure property. Namely, if an $i$-hyperedge is in the hypergraph, then so are all of its subsets. An $(i+1)$-hyperedge is called an $i$-dimensional face of the complex. The vertices of $X$, which are the 0-dimensional faces, are denoted by $X(0)$, the edges by $X(1)$, the triangles by $X(2)$, and in general, the $i$-dimensional faces are denoted by $X(i)$. A complex is termed bounded degree if the number of faces (of any dimension) which are incident to every vertex is bounded by a constant, independent of the number of vertices in the complex.

An important component in simplicial complexes is the link of a face, which is, informally, its local view of the complex. These local views play a central role in our work.

A complex is said to be regular in its $i$ dimension, if every $i$-dimensional face is incident in the same number of $(i+1)$-dimensional faces, and every $(i+1)$-dimensional face is incident in the same number of $d$-dimensional faces. We consider a $d$-dimensional simplicial complex $X$, which is regular in its $d-1$ dimension. We define a high order combinatorial random walk on $X$ to be the walk that starts from any $(d-1)$-dimensional face and moves between neighboring $(d-1)$-dimensional faces at random, where two $(d-1)$-dimensional faces are considered neighbors if they share a common $d$-dimensional face. For example, in a 2-dimensional simplicial complex, which has vertices edges and triangles, we are interested in a walk that starts from an edge and moves between neighboring edges at random, where two edges are considered neighbors if they share a common triangle. If the complex is regular in the dimension of its edges, then the number of triangles on every edge is fixed for all edges, and thus the stationary distribution of the discussed walk is the uniform distribution on the edges of the complex. We are interested in finding a bounded degree $d$-dimensional complex, for every $d \in \mathbb{N}$, in which the described walk converges quickly to the uniform distribution.

We note that the high order combinatorial random walk is defined for every dimension of the complex which is regular. This walk is defined in an analogous way to the walk defined for $i = d-1$, and its stationary distribution is the uniform distribution on the $i$-dimensional faces of the complex.

For regular graphs, convergence of the described random walk is normally deduced from the concentration of the spectrum of the graph’s adjacency matrix (or laplacian). Analogously, for a $d$-dimensional simplicial complex $X$, we define the $i$-graph of $X$, $G_i = G_i(X)$, which reflects the neighboring relations in $X(i)$, as follows. The vertices of $G_i$ are the $i$-dimensional faces of $X$, and there is an edge between two vertices if the corresponding $i$-dimensional faces are neighbors. We are interested in the spectrum of the $\{0,1\}$-adjacency matrix of $G_{d-1}$, $A_{d-1} = A(G_{d-1})$, whose rows and columns are indexed by the vertices of $G_{d-1}$, (or analogously, by the $(d-1)$-dimensional faces of $X$), and $A_{d-1}(u,v) = 1$ if there is an edge $\{u,v\}$ in $G_{d-1}$, otherwise it equals 0.

The question we address here is how one could construct a bounded degree $d$-dimensional complex whose $A_{d-1}$ adjacency matrix has a good spectrum, and thus its associated high order combinatorial random walk mixes quickly.

**Colorful expansion** Our method in asserting the fast convergence of the high order combinatorial random walk is as follows. We define a new notion of high order combinatorial expansion of complexes which we term colorful expansion. This is a new notion of expansion for high dimensional simplicial complexes that is not implied by previously studied notions as coboundary expansion and cosystolic expansion. A $d$-dimensional simplicial complex is a colorful expander if for every subset $S$ of $i$-dimensional faces of the complex there is a large set of $(i+1)$-dimensional
faces of the complex that seem *colorful* by $S$, i.e., we count the $(i + 1)$-dimensional faces which are hit by $S$ but are not fully covered by $S$. The colorful expansion is defined in a way such that if a $d$-dimensional simplicial complex $X$ is a colorful expander, then the graph $G_{d-1}(X)$ is a combinatorial expander. We then use the relation between combinatorial expansion and spectral expansion of graphs in order to deduce that $A_{d-1}$ has a concentrated spectrum, and hence the high order combinatorial random walk converges quickly.

We provide a local-to-global criterion on the complex which implies fast mixing of its high order combinatorial random walk. Namely, if the underlying graphs of the links of all of its faces (including the link of the empty set, which is the underlying graph of the entire complex) are good spectral expanders, then the complex is a colorful expander, and hence the high order combinatorial random walk mixes fast.

We note that we actually prove a stronger claim. We show that if the complex satisfies this local expansion criterion, then for every $0 \leq i \leq d - 1$, the graph $G_i$ is a combinatorial expander. Thus, the high order combinatorial random walk mixes quickly on every dimension which is regular.

1.1 Related work

In a recent work [Opp] it was shown that if all of the links of a $d$-dimensional simplicial complex $X$ are good spectral expanders and the complex is connected, then the underlying graph of the complex (i.e., $X(0) \cup X(1)$) is a good expander graph, or in other words, that the graph $G_0$ is a combinatorial expander. In this paper we show that the spectral expansion of the links imply combinatorial expansion of $G_i$ for all $0 \leq i \leq d - 1$.

A recent work of [PR] has studied high order topological random walks on simplicial complexes. The topological walk is different than the combinatorial walk we present here, and it does not imply convergence of the combinatorial walk. The topological walk was designed to expose the topological properties of the complex. The topological random walk is associated with the concentration of the spectrum of the high order laplacian on the space that is orthogonal to the coboundaries; while the combinatorial random walk, in a sense, is related to the concentration of the spectrum on the space that is orthogonal to the constant functions, which is a much larger space.

We study the spectrum of the high order adjacency matrix of the complex, $A_{d-1}$, and derive some good bounds on it from the spectrum of the links of the complex. Garland [Gar] in a seminal work has studied the spectrum of high order laplacians associated with a $d$-dimensional simplicial complex. Garland has shown that if all the links of a complex are good spectral expanders, then the eigenspace of the weighted oriented laplacian that is orthogonal to the coboundaries has a good spectrum. This is somewhat in the spirit of what we get here. However, Garland could only obtain that eigenvectors of the laplacian which are orthogonal to the space of the coboundaries are small, while here we want to get a bound on all of the eigenvalues of $A_{d-1}$ besides the first trivial one corresponding to the constant functions. There is no known way to obtain our result from Garland argument. See also [GW] for more discussion on Garland’s work and the fact that it does not imply the required spectrum bound for $A_{d-1}$.

In a recent work of the first coauthor and Evra [EK], a different notion of high order expansion has been studied (we elaborate on that later). The studied notion is called cosystolic expansion. It was shown by [EK] that if all the links of a $d$-dimensional simplicial complex are good spectral expanders and good coboundary expanders, then the $(d - 1)$-skeleton of the complex is a cosystolic expander. Though the spirit of the proof there might resemble at first glance the method of the proof that we use here, the obstacles and the solutions are different. [EK] could only show cosystolic expansion of small sets. They then use a reduction of [KKL]
showing that co-systolic expansion of small sets implies co-systolic expansion of the \((d - 1)\)-
skeleton of a given \(d\)-dimensional complex. In our work it is crucial for us to obtain expansion of
large sets, whose norm is up to \(1/2\), and the reduction of [KKL] could not work here since it
does not imply colorful expansion. Thus, the expansion here is achieved in a method which is
different than the one used in [EK].

1.2 Expander graphs

We recall some basic properties of expanders. Let \(G = (V, E)\) be an undirected graph. For a
vertex \(v \in V\), denote by \(N(v) = \{u \in V \mid \{v, u\} \in E\}\) the set of neighbors of \(v\). The degree of
a vertex \(v \in V\), denoted by \(\deg(v)\), is the number of edges which contain \(v\). The graph is said
to be \(k\)-regular if for every vertex \(v \in V\), \(\deg(v) = k\). A family of graphs is called expanding if
the expansion of every graph in the family is a constant independent of its size.

1.2.1 Combinatorial expansion

From a combinatorial point of view, an expander graph is defined according to its edge expansion
as follows.

**Definition 1.1** (Edge expansion). Let \(G = (V, E)\) be a \(k\)-regular graph. The edge expansion
of \(G\), denoted by \(h(G)\), is defined as,

\[
 h(G) = \min_{S \subseteq V, 0 < |S| \leq \frac{|V|}{2}} \frac{|E(S, \overline{S})|}{|S|}
\]

where \(E(S, \overline{S})\) is the set of edges with one endpoint in \(S\) and one endpoint in \(\overline{S}\).

We just note here that while in graphs this is the common definition, when moving to higher
dimensions the normalized edge expansion, \(\tilde{h}(G) = h(G)/k\), is more convenient.

Intuitively, the edge expansion measures how well the edges are spread out in the graph.
Large edge expansion implies that every subset of vertices has relatively many outgoing edges.
In other words, every subset of vertices is expanding.

The combinatorial expansion criterion is that \(h(G) \geq \epsilon\) for some \(\epsilon > 0\). In this case, we
say that the graph is an \(\epsilon\)-combinatorial expander. A family of graphs is said to be a family
of \(\epsilon\)-combinatorial expanders if there exists an \(\epsilon > 0\) such that every graph in the family is an
\(\epsilon\)-combinatorial expander. As \(\epsilon\) is larger, the graph is considered to be a better expander.

1.2.2 Spectral expansion

The algebraic definition of expansion is related to the eigenvalues of the graph’s adjacency
matrix as follows.

Let \(G = (V, E)\) be a \(k\)-regular graph. The adjacency matrix of \(G\) is a square \(|V| \times |V|\) matrix,
denoted by \(A = A(G)\), indexed by the graph’s vertices, where \(A_{uv} = 1\) if there is an edge
\(\{u, v\} \in E\), and 0 otherwise. Since \(G\) is an undirected graph, its adjacency matrix is a symmetric
matrix, which means all of its eigenvalues have real values. Denote by \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\)
the eigenvalues of \(A\). The largest eigenvalue of every \(k\)-regular graph is \(\lambda_1 = k\). Denote
by \(\lambda = \lambda(G) = \max\{|\lambda_2|, |\lambda_n|\}\) the second largest eigenvalue of \(A\) in absolute value, and by
\(\tilde{\lambda} = \lambda/k\) its normalized value.

The expansion criterion from this point of view is that \(\tilde{\lambda}(G) \leq \epsilon\) for some \(0 < \epsilon < 1\). In
this case, the graph is said to be an \(\epsilon\)-spectral expander. Similar to the combinatorial expansion
case, a family of graphs is said to be a family of $\epsilon$-spectral expanders if there exists an $0 < \epsilon < 1$ such that every graph in the family is an $\epsilon$-spectral expander.

1.2.3 Random walks on expanders

A very useful property of expander graphs is their pseudorandom behavior. We explain it in relation to a random walk on a graph, which is defined as follows.

**Definition 1.2 (Random walk).** A random walk on a graph $G = (V, E)$ is a sequence of vertices $v_0, v_1, \ldots \in V$ such that,

1. $v_0$ is chosen from some initial probability distribution on the vertices.
2. For every $i$, the vertex $v_{i+1}$ is chosen uniformly at random from $N(v_i)$.

A random walk on every non-bipartite regular connected graph converges to the uniform distribution over the vertices at some point. It is known that as the graph is a better expander, the speed of convergence is faster. In particular, it means that for every vertex in the graph, when considering a random walk that starts from it, after a few steps the probability to be at any vertex in the graph is almost the same, i.e., it is no different to choosing a uniformly random vertex in the graph.

The above statement can be formalized as follows. For a graph $G = (V, E)$, denote by $u = (1/|V|, \ldots, 1/|V|)$ the uniform distribution on the vertices of $G$. Consider a random walk that starts with some probability distribution $p_0 \in \mathbb{R}^{|V|}$. Denote by $p_i$ the probability distribution after $i$ steps of the walk. A rapid mixing of the walk is defined as follows.

**Definition 1.3 (Rapid mixing of a random walk).** Let $G = (V, E)$ be a graph. The random walk on $G$ is said to be $\epsilon$-rapidly mixing for some $\epsilon > 0$, if for any initial probability distribution $p_0 \in \mathbb{R}^{|V|}$ and any $i \in \mathbb{N}$,

$$\|p_i - u\|_1 \leq \epsilon \sqrt{|V|}$$

Then the following Theorem holds.

**Theorem 1.4.** [HLW, Theorem 3.2] Let $G = (V, E)$ be an $\epsilon$-spectral expander. Then the random walk on $G$ is $\epsilon$-rapidly mixing.

As can be seen from this theorem, the speed of convergence towards the uniform distribution is controlled by the spectral expansion of the graph. Thus, as $\epsilon$ is smaller the graph is considered to be a better expander.

It has been proven that in the case of graphs, combinatorial expansion and spectral expansion are related. A spectral expander is also a combinatorial expander, and a combinatorial expander, in some sense, is also a spectral expander. We get into the details of this relationship later. Now, let us introduce the higher dimensional version of expanders.

1.3 High dimensional expanders

The generalization of expander graphs to higher dimensions comes from the study of simplicial complexes. We start by introducing some basic definitions of complexes.
1.3.1 Definitions of simplicial complexes

A *simplicial complex* $X$, with a set of vertices $V$, is a collection of subsets of $V$, i.e., $X \subseteq 2^V$, which is closed under inclusions. Namely, for any element $\sigma \in X$, all of its subsets $\tau \subset \sigma$ are also elements of $X$. Each such element is called a *face*. The *dimension* of a face $\sigma \in X$ is defined as $\dim(\sigma) = |\sigma| - 1$, i.e., one less than the number of vertices in it. Note that the empty set, which is a face of dimension $-1$, is a face in every simplicial complex. The dimension of the entire complex, $\dim(X)$, is defined as the maximal dimension of a face in it. The complex is said to be *pure* if for every face $\sigma \in X$ with $\dim(\sigma) < \dim(X)$ there is a face of maximal dimension $\tau \in X$, $\dim(\tau) = \dim(X)$, such that $\sigma \subset \tau$. We only deal with pure simplicial complexes, so for convenience sake, every time we mention a complex we mean a pure complex.

Let $X$ be a $d$-dimensional simplicial complex, i.e., the maximum dimension of a face in $X$ is $d$. For any $-1 \leq i \leq d$, denote by $X(i) = \{ \sigma \in X \mid \dim(\sigma) = i \}$ the collection of $i$-dimensional faces of $X$. An $\mathbb{F}_2$ valued function on $X(i)$, $f : X(i) \rightarrow \mathbb{F}_2$, is called an $i$-cochain of $X$. The space of all $i$-cochains of $X$ is denoted by $C^i(X) = \{ f : X(i) \rightarrow \mathbb{F}_2 \}$.

1.3.2 Graphs as simplicial complexes

Let us demonstrate the above definitions with the simple case of graphs. A 1-dimensional simplicial complex $X$ can be viewed as a graph $G = (V,E)$. The collection of 0-dimensional faces of $X$ is the set of vertices of $G$, i.e., $V = X(0)$. The collection of 1-dimensional faces of $X$ is the set of edges of $G$, i.e., $E = X(1)$. A 0-cochain $S : V \rightarrow \{0,1\}$ can be viewed as a subset of vertices, $S \subseteq V$, where $v \in S$ if and only if $S(v) = 1$. Similarly, a 1-cochain $F : E \rightarrow \{0,1\}$ can be viewed as a subset of edges, $F \subseteq E$, where $e \in F$ if and only if $F(e) = 1$.

For ease of notation, we treat cochains at every dimension as subsets of faces; for an $i$-cochain $A \in C^i(X)$ and an $i$-dimensional face $\sigma \in X(i)$, if $A(\sigma) = 1$ we say that $\sigma \in A$, and otherwise we say that $\sigma \notin A$.

1.3.3 Norms in simplicial complexes

Before giving the definitions of expansion of simplicial complexes we need to introduce the following definition of norms in the complex. Recall that in a graph $G = (V,E)$, the degree of a vertex $v \in V$ is defined as the number of edges which contain $v$. Similarly, in a $d$-dimensional simplicial complex $X$, define the degree of a face $\sigma \in X$ to be the number of $d$-dimensional faces which contain $\sigma$. Since we work with non-regular simplicial complexes, i.e., the degree of two faces of the same dimension might be different, we normalize the weight of every face according to its degree as follows. Denote by $w : X \rightarrow [0,1]$ the weight function on faces of the complex. For an $i$-dimensional face $\sigma \in X(i)$, the weight of $\sigma$ is defined as,

$$w(\sigma) = \frac{\deg(\sigma)}{\sum_{\tau \in X(i)} \deg(\tau)}$$

Then, for an $i$-cochain $A \in C^i(X)$, define the norm of $A$ to be $\|A\| = \sum_{\sigma \in A} w(\sigma)$. This norm satisfies the following basic properties.

- For any $i$-cochain $A \in C^i(X)$, $\|A\| = 0$ if and only if $A = \emptyset$, and $\|A\| = 1$ if and only if $A = X(i)$.
- For any two $i$-cochains $A,A' \in C^i(X)$, $\|A \cup A'\| \leq \|A\| + \|A'\|$ with equality if and only if $A \cap A' = \emptyset$, and $\|A \setminus A'\| \geq \|A\| - \|A'\|$ with equality if and only if $A' \subseteq A$.
Note that in the case of a 1-dimensional complex where the corresponding graph is \(k\)-regular, the weight of every vertex is exactly \(1/|V|\), and the norm of a subset of vertices \(S \subseteq V\) is its normalized support size, i.e., \(\|S\| = |S|/|V|\).

### 1.3.4 Expansion of simplicial complexes

High order combinatorial expansion has received much attention recently. There are mostly two variants of combinatorial expansion that were studied for high dimensional simplicial complexes. They both generalize the Cheeger constant (i.e., the edge expansion parameter) of a graph to higher dimensions. The exact definitions are pretty technical so here we just illustrate their meanings.

The first studied generalization is termed \emph{coboundary expansion} \cite{LM}. Given an \(i\)-cochain \(A \in C^i(X)\), the coboundary expansion measures the number of \((i + 1)\)-dimensional faces of \(X\) that are touched \emph{odd} many times by \(A\). For every complex there are \(i\)-cochains that are trivially not expanding (they are called the \(i\)-coboundaries, which generalize the role of the set of all the vertices in a graph that can never expand). The requirement of this expansion measure is that \(i\)-cochains that are far from the \(i\)-coboundaries will expand a lot, i.e., there will be many \((i + 1)\)-dimensional faces that touch them odd many times. The existence of bounded degree expanders according to this definition is not known! (either random or explicit).

The next studied combinatorial expansion in higher dimensions is termed \emph{cosystolic expansion} \cite{G, EK}. This definition is a relaxation of the previous definition. It requires that each \(i\)-cochain which do not expand is either a trivial non-expanding cochain (i.e., an \(i\)-coboundary), or that it is very large. The motivation for studying this relaxation is that it implies a property called the topological overlapping of a complex which was heavily studied \cite{G, DKW}. \cite{KKL, EK} constructed the first known explicit bounded degree high dimensional complexes that are cosystolic expanders.

**Remark 1.5.** In the 1-dimensional case, if \(X\) is a cosystolic expander, it means that it is a union of large disjoint coboundary expanders. Any connected component has to be a coboundary expander by itself, and the size of the smallest component has to be large.

### 1.4 Our contribution

In this work we introduce a new notion of a random walk on high dimensional simplicial complexes. Our main result is a local criterion on the complex which implies that the high order combinatorial random walk on it mixes quickly. In order to prove it we introduce a new definition of combinatorial expansion in higher dimensions, which we term colorful expansion. We prove that this local criterion on the complex implies that it is a colorful expander, which in turn implies the mixing behavior of the high order combinatorial random walk on it. Then we show an explicit construction of bounded degree complexes, such that the high order combinatorial random walk on them mixes quickly.

**1.4.1 Combinatorial random walks on high dimensional simplicial complexes**

We generalize the notion of a random walk on graphs to higher dimensions by letting the walk to be on any dimension of the complex. Namely, for any \(0 \leq i \leq d-1\), the \(i\)-dimensional random walk moves between \(i\)-dimensional faces of \(X\), where two faces are considered neighbors if they are contained in an \((i+1)\)-dimensional face, i.e., for any \(\sigma \in X(i)\), \(N(\sigma) = \{\sigma' \in X(i+1) \mid \sigma \cup \sigma' \in X(i+1)\}\). Before introducing the definition of the high order combinatorial random walk we define regularity of dimension \(i\) in the complex.
Definition 1.6 (Regularity of a dimension in a complex). Let $X$ be a $d$-dimensional simplicial complex. $X$ is called regular in dimension $i$ if the following two conditions hold.

- For every $\sigma, \sigma' \in X(i)$, $|N(\sigma)| = |N(\sigma')|$, i.e., each $i$-dimensional face has the same number of neighbors.
- For every $\tau, \tau' \in X(i+1)$, $\deg(\tau) = \deg(\tau')$, i.e., each $(i+1)$-dimensional face is contained in the same number of $d$-dimensional faces.

We introduce the following definition of a high order combinatorial random walk.

Definition 1.7 (High order combinatorial random walk). Let $X$ be a $d$-dimensional simplicial complex. For every $0 \leq i \leq d-1$, where $X$ is regular in dimension $i$, an $i$-dimensional combinatorial random walk on $X$ is a sequence of $i$-dimensional faces $\sigma_0, \sigma_1, \ldots \in X(i)$ such that,

1. $\sigma_0$ is chosen from some initial probability distribution on the $i$-dimensional faces.
2. For every $j$, the face $\sigma_{j+1}$ is chosen uniformly at random from $N(\sigma_j)$.

1.4.2 Local criterion for mixing of combinatorial random walks

In order to explain the local criterion which implies the mixing of the high order combinatorial random walk, we need to introduce the definition of links in the complex.

Definition 1.8 (Link). Let $X$ be a $d$-dimensional simplicial complex. For any face $\sigma \in X$, the link of $\sigma$, denoted by $X_\sigma$, is the $(d - |\sigma|)$-dimensional simplicial complex which is defined as $X_\sigma = \{\tau \in X \mid \sigma \cup \tau \in X\}$.

Intuitively, a link of a face is its neighborhood in the complex. Another way to look at it is by noting that the link of $\sigma \in X$ is obtained by taking all of the faces $\tau \in X$ which contain $\sigma$, and then removing $\sigma$ from all of them, yielding a smaller complex which corresponds to the local view of $\sigma$.

Since the link of every face $\sigma \in X$ is a simplicial complex, all of the definitions of simplicial complexes apply to it as well. For every $-1 \leq i \leq d - |\sigma|$, denote by $C^*_\sigma(X_\sigma)$ the space of $i$-cochains of $X_\sigma$, by $w_\sigma : X_\sigma \to [0, 1]$ the weight function of the link, and by $\| \cdot \| = \| \cdot \|_\sigma^i : C^*_\sigma \to [0, 1]$ the norm associated with $X_\sigma$.

We are now in position to introduce the local criterion on the complex which implies that the high order combinatorial random walk on it mixes well. Recall that in graphs we have the Expander mixing lemma, which states that in a good spectral expander graph, the number of edges that lie inside of every subset of vertices is approximately the same as we expect in a random graph. In high dimensional simplicial complexes, our criterion is that the underlying graph of every local view of the complex would have a similar mixing behavior. This property is termed in [EK] as skeleton expansion, which is defined as follows.

For any simplicial complex $X$, consider its 1-dimensional skeleton $X(0) \cup X(1)$. Note that this 1-dimensional skeleton is essentially a graph, so we relate to it as the underlying graph of the complex. For every subset of vertices $S \subseteq X(0)$, denote by $E(S) = E(S, S) \subseteq X(1)$ the set of edges with two endpoints in $S$.

Definition 1.9 (Skeleton expander). Let $X$ be a $d$-dimensional simplicial complex. $X$ is said to be an $\alpha$-skeleton expander if for every face $\sigma \in X$ (including $\sigma = \emptyset$), and for every subset of vertices $S \in X_\sigma(0)$ the following holds,

$$\|E(S)\|_\sigma \leq \|S\|_\alpha (\|S\|_\sigma + \alpha)$$
In words, a simplicial complex is a skeleton expander if the 1-dimensional skeletons of all of its links behave like spectral expanders. We prove the following theorem.

**Theorem 1.10** (Main Theorem, informal, for formal see Theorem 3.3). If the complex is an $\alpha$-skeleton expander for small enough $\alpha$, then the high order combinatorial random walk on it mixes rapidly.

### 1.4.3 Colorful expansion of high dimensional simplicial complexes

Our new definition of combinatorial expansion in higher dimensions is the main building block in proving the mixing behavior of the high order combinatorial random walk. As mentioned, both of the current generalizations of combinatorial expansion to higher dimensions define the expansion measure according to the coboundaries of the complex.

We want to suggest a different approach. In the case of graphs, the expanding edges of a subset of vertices were the outgoing edges of the subset, i.e., the edges with one endpoint in the subset and one endpoint outside of it. In a similar essence we generalize this notion to higher dimensions. For a $d$-dimensional simplicial complex $X$ and an $i$-cochain $A \in C^i(X)$, we say that an $(i + 1)$-dimensional face is expanding if it contains at least one $i$-dimensional face in $A$ and at least one $i$-dimensional face outside of $A$. Intuitively, an expanding $(i + 1)$-dimensional face looks like an outgoing face of the cochain; it is not fully inside of it neither fully outside of it. The following definition formalizes this notion.

**Definition 1.11** (Expanding faces). Let $X$ be a $d$-dimensional simplicial complex. For any $i$-cochain $A \in C^i(X)$, the $(i + 1)$-cochain of expanding faces of $A$, denoted by $\psi(A) \in C^{i+1}(X)$, is defined as,

$$\psi(A) = \{\tau \in X(i + 1) \mid \exists \sigma, \sigma' \subset \tau, \dim(\sigma) = \dim(\sigma') = i \land \sigma \in A \land \sigma' \notin A\}$$

Since every face which is fully inside of $A$ or fully outside of $A$ is not expanding, then the trivial non-expanding sets are either $\emptyset$ or $X(i)$. Thus, the distance of $A$ from the trivial non-expanding sets is $\min\{\|A\|, \|\bar{A}\|\}$. Then, the definition of colorful expansion follows.

**Definition 1.12** (High order colorful expansion). Let $X$ be a $d$-dimensional simplicial complex. $X$ is said to be an $\epsilon$-colorful expander if for every $0 \leq i \leq d - 1$, and every $i$-cochain $A \in C^i(X)$ the following holds: If $\|\psi(A)\| = 0$ then $A \in \{\emptyset, X(i)\}$, otherwise, $\|\psi(A)\| \geq \epsilon \min\{\|A\|, \|\bar{A}\|\}$.

The following is the main technical theorem of our work.

**Theorem 1.13** (Colorful expansion Theorem, informal, for formal see Theorem 3.1). If the complex is an $\alpha$-skeleton expander for small enough $\alpha$, then it is a high order colorful expander.

We then prove the following theorem.

**Theorem 1.14** (Colorful expansion implies spectral expansion Theorem, informal, for formal see Theorem 3.2). If the complex is an $\epsilon$-colorful expander, then its high order adjacency matrix has a concentrated spectrum.

where the combination of Theorems 1.13 and 1.14 implies Theorem 1.10.

### 1.4.4 Explicit family of bounded degree colorful expanders of every dimension

As proven in [EK], Ramanujan complexes are excellent skeleton expanders. Thus, using the explicit construction of Ramanujan complexes from [LSV2] yields an explicit construction of bounded degree high dimensional colorful expanders, and hence the high order combinatorial random walk on them mixes well.
Theorem 1.15 (Explicit bounded degree colorful expanders Theorem, informal, for formal see 
Theorem 3.4). There exist explicit bounded degree complexes in which the high order combi-
atorial random walk on them mixes quickly.

1.5 Colorful expansion proof sketch

Before getting into the technical details, let us give a high level overview of the proof that
skeleton expansion of a complex implies colorful expansion. Recall that we consider an $i$-
cochain in the complex which is not a trivial non-expanding cochain, and we want to show that
it has relatively many expanding $(i + 1)$-dimensional faces. Since we need to show that every
cochain is expanding, this cochain can be of any size, i.e., its norm can be up to $1/2$.

In order to get many expanding faces we use the following two notions. First, we define
fat faces, which informally mean they carry many fat faces of the dimension above. Then at
Lemma 4.4 we prove that these fat faces actually carry a lot of the weight of the original cochain.
This Lemma is derived from the definition of fat faces, and thus up to now we did not use any
expansion property of the complex. Next we define the second notion of degenerate faces. These
are non-fat faces which carry many fat faces of the dimension above. They play the main role of
our proof. The way we use them is as follows. We show at Lemma 4.6 that a non-fat face cannot
carry many full faces of the original cochain, where a full face of dimension $i + 1$ means that all
of its $i$-dimensional faces are in $A$. In order to prove this Lemma we use the skeleton expansion
property of the complex. Therefore, we have received two opposite phenomena occurring on
the same face. On the one hand, this face has many fat faces of the dimension above, and
thus carries a lot of faces of the original cochain. On the other hand, it is a non-fat face, and
thus many faces in its link are not full. This is where we conclude that there must be many
expanding faces.

We want to emphasize the novelty of this approach. Up to now, there existed no way to
use these degenerate faces. It was known how to achieve a good expansion by them for one
dimension above, but past there it got stuck. That is why in the work of [EK] they could
achieve expansion only for cochains of small size. These degenerate faces contributed to their
error term, and thus it was necessary to bound their size and ignore them. In our work, we
show a way to climb with the degenerate faces from any dimension of the complex up to the
top. This is where we utilize the expansion of the links of the complex in order conclude that at
the highest dimension there are many expanding faces. That is how we are able to claim that
every cochain is expanding, and not only small sets.

Thus, the proof strategy is to step down the dimensions one by one. If at any dimension we
have many degenerate faces, i.e., many fat faces of the dimension above are sitting on non-fat
faces, then we can use these faces to conclude that there are many expanding faces at the top of
the complex. If not, we know that at the current dimension the contribution of the degenerate
faces is negligible. Thus, we step down to the next dimension and repeat the same process. If
we reach the lowest dimension of the complex, we know that at every dimension almost all of
the weight of the cochain is carried just by fat faces. Now, since that by the definition of fat
faces there cannot be many fat faces at any dimension (as we prove in Corollary 4.5) we know
that almost all of the weight of the cochain is carried by a small subset of vertices. But by
the skeleton expansion, there are not many faces of any dimension living on a small subset of
vertices. Thus, the rest of the faces must be distributed on non-fat faces, which again gives us
many expanding faces.
1.6 Organization of this paper

The paper is organized as follows. In Section 2 we present some preliminaries. In section 3 we provide the exact formulations of our theorems that were described in the introduction in an informal way. In Section 4 we prove the Colorful expansion Theorem (Theorem 3.1). In Section 5 we prove the local criterion for fast convergence of the high order combinatorial random walk (Theorems 3.2 and 3.3). In Section 6 we describe an explicit family of bounded degree complexes for which the high order combinatorial random walk mixes quickly (Theorem 3.4).

2 Preliminaries

We provide here some definitions and lemmas that will be useful for us in the following sections.

2.1 Combinatorial and spectral expansion in graphs

In the proof that colorful expansion implies rapid mixing of the high order combinatorial random walk we are going to use the relation between the combinatorial expansion and the spectral expansion of graphs. For that we use the following two lemmas.

Lemma 2.1 (Cheeger’s inequality). Let \( G = (V,E) \) be a \( k \)-regular graph, and let \( \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \ldots \geq \lambda_n \) be the normalized eigenvalues of its adjacency matrix. Then,

\[
\tilde{\lambda}_2 \leq 1 - \frac{h(G)^2}{2}
\]

On the other side of the spectrum, Trevisan [T] has given a Cheeger-type inequality for the smallest eigenvalue of a graph, which we describe now. A \( k \)-regular graph \( G = (V,E) \) is said to have a bipartite component if there is a non-empty subset of vertices \( S \subseteq V \), and a partition of \( S \) into two disjoint subsets, \( S_1 \cup S_2 = S \) such that \( |E(S,\bar{S})| = 0 \), \( |E(S_1)| = 0 \), and \( |E(S_2)| = 0 \). In this case the normalized smallest eigenvalue of the graph’s adjacency matrix equals \(-1\). The graph is said to be close to having a bipartite component if there is such a partition with a small number of violating edges compared to the number of edges incident in \( S \), where an edge \( e \in E \) is said to be violating if \( e \in E(S,\bar{S}) \cup E(S_1) \cup E(S_2) \). In order to measure this, for each partition as above we define the following vector \( y \in \{-1,0,1\}^{|V|} \) where \( y_v = 1 \) if \( v \in S_1 \), \( y_v = -1 \) if \( v \in S_2 \), and otherwise it equals 0. This vector is a convenient way to package the information about each partition. Now we can define the bipartiteness ratio of a graph.

Definition 2.2 (Bipartiteness ratio). Let \( G = (V,E) \) be a \( k \)-regular graph. The bipartiteness ratio of \( G \), denoted by \( \beta(G) \), is defined as,

\[
\beta(G) = \min_{y \in \{-1,0,1\}^{|V|}} \frac{\sum_{\{u,v\} \in E} |y_u + y_v|}{\sum_{v \in V} |y_v|}
\]

Intuitively, this measures the minimal proportion between the number of violating edges and the size of \( S \), over all possible partitions. We just note that in this measure, there is a difference between different violating edges. A violating edge \( e \in E(S,\bar{S}) \) contributes 1 to the sum, while a violating edge \( e \in E(S_1) \cup E(S_2) \) contributes \( 2 \). This measure helps us bound the smallest eigenvalue of the graph’s adjacency matrix, with the following lemma which is proven in [T].
Lemma 2.3 (Bipartiteness inequality). Let $G = (V, E)$ be a $k$-regular graph, and let $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \ldots \geq \tilde{\lambda}_n$ be the normalized eigenvalues of its adjacency matrix. Then,

$$\tilde{\lambda}_n \geq -1 + \frac{\tilde{\beta}(G)^2}{2}$$

where $\tilde{\beta}(G) = \beta(G)/k$ is the normalized bipartiteness ratio.

2.2 Norms and links in simplicial complexes

In order to prove the colorful expansion of a complex, we need the following properties of norms and links.

It is often very useful to move between a global cochain in the complex and a localized cochain of a face in it. Intuitively, this process can be described as follows. In order to localize a cochain to a face $\sigma \in X$, we restrict it to $\sigma$ and then we remove $\sigma$ from every face in the cochain. In the other direction, we take a cochain in the link of $\sigma$ and add $\sigma$ to every face in it, obtaining a cochain in the global complex. Let us give the formal definitions of it.

Definition 2.4 (Localization). Let $X$ be a $d$-dimensional simplicial complex, and let $A \in C_i(X)$ be an $i$-cochain in the complex. For every $\sigma \in X$, the localization of $A$ to the link of $\sigma$ is defined as

$$A_{\sigma} = \{ \tau \in X | \sigma \cup \tau \in A \}.$$

Definition 2.5 (Lifting). Let $X$ be a $d$-dimensional simplicial complex, let $\sigma \in X$ be a face in the complex, and let $A \in C_i(\sigma)$ be an $i$-cochain in the link of $\sigma$. The lifting of $A$ to the global complex is defined as

$$A = \{ \sigma \cup \tau | \tau \in A \}.$$

The relation between the norm of the global cochain and the norm of the localized cochain is given by the following lemma.

Lemma 2.6 (Global-to-local Lemma). \[EK, Lemma 2.6\] Let $X$ be a $d$-dimensional simplicial complex. For every $0 \leq i < j \leq d$, every $j$-cochain $A \in C_j(X)$, and every $\sigma \in X(i)$,

$$\|A_{\sigma}\| = \left(\frac{j+1}{i+1}\right) w(\sigma) \|A_\sigma\|_\sigma$$

The following lemma is super useful as it lets us calculate the norm of a global cochain by dividing it into many small local views.

Lemma 2.7 (Sum of local views). \[EK, Lemma 2.7\] Let $X$ be a $d$-dimensional simplicial complex. For every $0 \leq i < j \leq d$ and every $j$-cochain $A \in C_j(X)$,

$$\|A\| = \sum_{\sigma \in X(i)} w(\sigma) \|A_\sigma\|_\sigma$$

Another thing we will use is a container of a cochain. For any $i$-cochain $A \in C_i(X)$, its container is an $(i+1)$-cochain of all the of the $(i+1)$-dimensional faces which contain a face of $A$.

Definition 2.8 (Container). Let $X$ be a $d$-dimensional simplicial complex. For every $0 \leq i \leq d-1$ and every $i$-cochain $A \in C_i(X)$, the container of $A$, denoted by $\Gamma(A) \in C_{i+1}(X)$, is defined as

$$\Gamma(A) = \{ \tau \in X(i+1) | \exists \sigma \in A, \sigma \subset \tau \}$$
The following lemma shows that the size of a container must be at least like the size of the cochain that it contains.

**Lemma 2.9 (Big container Lemma).** [EK, Lemma 2.3] Let $X$ be a $d$-dimensional simplicial complex. For every $0 \leq i \leq d-1$ and every $i$-cochain $A \in C^i(X)$,

$$\|\Gamma(A)\| \geq \|A\|$$

### 3 Main Theorems

In this section we present formally the main theorems of this work.

**Theorem 3.1 (Colorful expansion criterion).** Let $X$ be a $d$-dimensional $\alpha$-skeleton expander. If $\alpha$ satisfies,

$$\frac{1 - \log(\frac{1+2\alpha}{1-2\alpha})}{\log(1 + 3\alpha(\frac{2(1-2\alpha)}{1+2\alpha})^{\frac{1}{4}})} > 2^{d}$$

then $X$ is an $\epsilon$-colorful expander for,

$$\epsilon = \alpha \left( \frac{\alpha}{(d+1)^{4}(\frac{1+2\alpha}{1-2\alpha})^{\frac{2-d}{4}} + \alpha} \right)^{2^{d-2}}$$

**Theorem 3.2 (Colorful expansion implies spectral expansion).** Let $X$ be a $d$-dimensional $\epsilon$-colorful expander. For any $0 < i \leq d-1$ where $X$ is regular in dimension $i$, $G_i(X)$ is an $\left(1 - \frac{\epsilon^2}{2(i+2)^2}\right)$-spectral expander.

**Theorem 3.3 (Rapid mixing of the high order combinatorial random walk).** Let $X$ be a $d$-dimensional $\alpha$-skeleton expander. If $\alpha$ is as in Theorem 3.1 then for any $0 < i \leq d-1$ where $X$ is regular in dimension $i$, the $i$-dimensional combinatorial random walk on $X$ is $\left(1 - \frac{\epsilon^2}{2(i+2)^2}\right)$-rapidly mixing for $\epsilon$ as in Theorem 3.1.

**Theorem 3.4 (Explicit bounded degree colorful expanders).** For any $d \in \mathbb{N}$ there exists a constant $q_0 = q_0(d)$, such that if $X$ is a $d$-dimensional $q$-thick Ramanujan complex with $q \geq q_0$, then the $(d-1)$-dimensional combinatorial random walk on $X$ is $\epsilon$-rapidly mixing for $0 \leq \epsilon = \epsilon(q,d) < 1$.

### 4 Proof of colorful expansion criterion (Theorem 3.1)

#### 4.1 Proof sketch

Before getting into the technical details, let us give a high level overview of the proof. Recall that we consider an $i$-cochain in the complex which is not a trivial non-expanding cochain, and we want to show that it has relatively many expanding $(i+1)$-dimensional faces. Since we need to show that every cochain is expanding, this cochain can be of any size, i.e., its norm can be up to $1/2$.

In order to get many expanding faces we use the following two notions. First, we define *fat faces*, which informally mean they carry many fat faces of the dimension above. Then at Lemma 4.4 we prove that these fat faces actually carry a lot of the weight of the original cochain. This Lemma is derived from the definition of fat faces, and thus up to now we did not use any
expansion property of the complex. Next we define the second notion of degenerate faces. These are non-fat faces which carry many fat faces of the dimension above. They play the main role of our proof. The way we use them is as follows. We show at Lemma 4.6 that a non-fat face cannot carry many full faces of the original cochain, where a full face of dimension \(i+1\) means that all of its \(i\)-dimensional faces are in \(A\). In order to prove this Lemma we use the skeleton expansion property of the complex. Therefore, we have received two opposite phenomena occurring on the same face. On the one hand, this face has many fat faces of the dimension above, and thus carries a lot of faces of the original cochain. On the other hand, it is a non-fat face, and thus many faces in its link are not full. This is where we conclude that there must be many expanding faces.

We want to emphasize the novelty of this approach. Up to now, there existed no way to use these degenerate faces. It was known how to achieve a good expansion by them for one dimension above, but past there it got stuck. That is why in the work of [EK] they could achieve expansion only for cochains of small size. These degenerate faces contributed to their error term, and thus it was necessary to bound their size and ignore them. In our work, we show a way to climb with the degenerate faces from any dimension of the complex up to the top. This is where we utilize the expansion of the links of the complex in order conclude that at the highest dimension there are many expanding faces. That is how we are able to claim that every cochain is expanding, and not only small sets.

Thus, the proof strategy is to step down the dimensions one by one. If at any dimension we have many degenerate faces, i.e., many fat faces of the dimension above are sitting on non-fat faces, then we can use these faces to conclude that there are many expanding faces at the top of the complex. If not, we know that at the current dimension the contribution of the degenerate faces is negligible. Thus, we step down to the next dimension and repeat the same process. If we reach the lowest dimension of the complex, we know that at every dimension almost all of the weight of the cochain is carried just by fat faces. Now, since that by the definition of fat faces there cannot be many fat faces at any dimension (as we prove in Corollary 4.5) we know that almost all of the weight of the cochain is carried by a small subset of vertices. But by the skeleton expansion, there are not many faces of any dimension living on a small subset of vertices. Thus, the rest of the faces must be distributed on non-fat faces, which again gives us many expanding faces.

### 4.2 Helpful definitions and lemmas

We are now ready to get into the proof in details. We start by defining the notion of fat faces.

**Definition 4.1** (Fat faces). Let \(X\) be a \(d\)-dimensional simplicial complex, let \(A \in C^i(X)\), and let \(0 < \eta < 1\) be a fatness constant. The \(i\)-cochain of fat faces is defined as \(S^i(A) = A\), and for every \(0 \leq j < i\), define the \(j\)-cochain of fat faces as,

\[
S^j(A) = \{ \sigma \in X(j) \mid \|S^{j+1}(A)\|_\sigma \geq \eta^{2^{i-j-1}} \}
\]

Intuitively, a face is considered fat if it is contained in many fat faces of one dimension above. The reason for this definition is that it lets us capture how much of the weight of the cochain is carried by every face in the complex. If the cochain is concentrated on a few faces of lower dimensions, then almost all of them will be fat. Otherwise, there would be many fat faces sitting on non-fat faces. In order to capture this phenomenon we define the following cochains of degenerate faces.

**Definition 4.2** (Degenerate faces). Let \(X\) be a \(d\)-dimensional simplicial complex, let \(A \in C^i(X)\), and let \(0 < \eta < 1\) be the fatness constant. For any \(2 \leq j \leq i + 1\), a \(j\)-dimensional face
σ ∈ X(j) is said to be degenerate if it contains two (j − 1)-dimensional fat faces τ, τ′ ⊂ σ such that their intersection is a non-fat (j − 2)-dimensional face. The j-cochain of j-dimensional degenerate faces, denoted by T^j(A), is defined as,

\[ T^j(A) = \{ \sigma ∈ X(j) | \exists \tau, \tau' ⊂ \sigma, \tau, \tau' ∈ S^{j-1}(A) ∧ \tau' ∈ S^{j-1}(A) ∧ \tau' ∩ \tau' \notin S^{j-2}(A) \} \]

Another definition we need is the cochain of full faces. An i-dimensional face is said to be full if all of its (i − 1)-dimensional faces are fat. We define two notions of full faces. The first notion of full faces seen by σ, the face \( F^j(A, \sigma) = \{ \tau ∈ X(j) | ∀ \sigma ⊂ \tau ⊂ \tau, \dim(\tau') = j − 1, \tau' ∈ S^{j-1}(A) \} \)

Define the j-cochain of globally full faces in the complex as,

\[ F^j(A) = \{ \tau ∈ X(j) | ∀ \tau' ⊂ \tau, \dim(\tau') = j − 1, \tau' ∈ S^{j-1}(A) \} \]

Note that for every \( \sigma ∈ X \) and any \( \dim(\sigma) + 2 ≤ j ≤ i + 1 \), define the j-cochain of full faces seen by \( \sigma \) as,

\[ F^j(\sigma) = \{ \tau ∈ X(j) | ∀ \sigma ⊂ \tau' ⊂ \tau, \dim(\tau') = j − 1, \tau' ∈ S^{j-1}(A) \} \]

The following lemma shows that fat faces carry a large amount of the original cochain. Meaning, a face in the complex which has many fat faces in its link, must have many faces from the original cochain.

**Lemma 4.4 (Fat link Lemma).** Let \( X \) be a d-dimensional simplicial complex, let \( A ∈ C^i(X) \), and let \( 0 < \eta < 1 \) be the fatness constant. For any face \( \sigma ∈ X(j) \) and any \( \dim(\sigma) + 2 ≤ j ≤ i + 1 \), and any \( j \)-dimensional face \( \sigma ∈ X(j) \),

\[ \|A_\sigma\|_\sigma ≥ \eta^{2^{i−j−1}} \|S^{j+1}(A)\|_\sigma \]

**Proof.** Let \( 0 ≤ j < i \), and let \( \sigma ∈ X(j) \). Note that for every \( j < k < i \), and every fat face \( \tau ∈ S^k(A)_\sigma \), the face \( \sigma ∪ \tau ∈ S^k(A) \). Thus, by the definition of fat faces, the following holds.

\[ \|S^{k+1}(A)_{\sigma ∪ \tau}\|_{\sigma ∪ \tau} ≥ \eta^{2^{j−i−1}} \]

Now, considering the simplicial complex obtained by the link of \( \sigma \), we can use Lemma 2.7 in order to calculate \( \|S^{k+1}(A)_{\sigma}\|_\sigma \) by the sum of local views of one dimension lower.

\[ \|S^{k+1}(A)_{\sigma}\|_\sigma = \sum_{\tau ∈ X(k)_{\sigma}} w_\sigma(\tau)\|S^{k+1}(A)_{\sigma ∪ \tau}\|_{\sigma ∪ \tau} ≥ \sum_{\tau ∈ S^k(A)_{\sigma}} w_\sigma(\tau)\|S^{k+1}(A)_{\sigma ∪ \tau}\|_{\sigma ∪ \tau} \]

\[ \geq \sum_{\tau ∈ S^k(A)_{\sigma}} w_\sigma(\tau)\eta^{2^{j−k−1}} = \eta^{2^{j−k−1}} \sum_{\tau ∈ S^k(A)_{\sigma}} w_\sigma(\tau) = \eta^{2^{j−k−1}} \|S^k(A)_{\sigma}\|_\sigma \tag{4.1} \]

Applying (4.1) iteratively finishes the proof.

\[ \|A_\sigma\|_\sigma = \|S^i(A)_{\sigma}\|_\sigma ≥ \eta\|S^{i-1}(A)_{\sigma}\|_\sigma ≥ \eta\eta^2\|S^{i-2}(A)_{\sigma}\|_\sigma ≥ \cdots \]

\[ ≥ \eta \eta^2 \cdots \eta^{2^{j−i−2}}\|S^{j+1}(A)_{\sigma}\|_\sigma = \eta^{2^{j−i−1}}\|S^{j+1}(A)_{\sigma}\|_\sigma \]

\[ \square \]
Corollary 4.5 (Bounded norm of fat faces). Let $X$ be a $d$-dimensional simplicial complex, let $A \in C^i(X)$, and let $0 < \eta < 1$ be the fatness constant. For any $0 \leq j \leq i−1$,

$$\|S^j(A)\| \leq \eta^{-2^{i-j}}\|A\|$$

Proof. Let $0 \leq j \leq i−1$. Calculating $A$ by the sum of local views of dimension $j$, together with Lemma 4.4 yields the desired result.

$$\|A\| = \sum_{\sigma \in X(j)} w(\sigma)\|A_\sigma\|\|\sigma\| \geq \sum_{\sigma \in S^j(A)} w(\sigma)\|A_\sigma\|\|\sigma\| \geq \sum_{\sigma \in S^j(A)} w(\sigma)\eta^{2^{i-j}-1}\|S^{j+1}(A)_\sigma\|\|\sigma\|$$

$$\geq \sum_{\sigma \in S^j(A)} w(\sigma)\eta^{2^{i-j}-1}\eta^{2^{i-j}-1} \geq \eta^{2^{i-j}}\sum_{\sigma \in S^j(A)} w(\sigma) = \eta^{2^{i-j}}\|S^j(A)\|$$

\[\Box\]

Since we want to show that there are many expanding faces, we need to bound the number of full faces. We show that the number of full faces sitting on a non-fat face of any dimension is bounded.

Lemma 4.6 (Full faces on a non-fat face). Let $X$ be a $d$-dimensional $\alpha$-skeleton expander, let $A \in C^i(X)$, and let $0 < \eta < 1$ be the fatness constant. For any $0 \leq j \leq i−1$, and any non-fat $j$-dimensional face $\sigma \in X(j) \setminus S^j(A)$,

$$\|F^{i+1}(A,\sigma)_\sigma\| \leq (\eta^{2^{i-j}-1} + \alpha)\|S^{j+1}(A)_\sigma\|\|\sigma\| + \sum_{k=j+3}^{i+1} \|T^k(A)_\sigma\|\|\sigma\|$$

Proof. Let $0 \leq j \leq i−1$, and let $\sigma \in X(j) \setminus S^j(A)$ be a non-fat face of dimension $j$. Note that each vertex in the link of $\sigma$ corresponds to a face of dimension $j+1$ in the complex, and every edge corresponds to a $(j+2)$-dimensional face of the complex. Thus, the full faces of dimension $j+2$ are seen in the link of $\sigma$ as an edge between two fat vertices. Since $X$ is an $\alpha$-skeleton expander, the following holds.

$$\|F^{j+2}(A,\sigma)_\sigma\| = \|E(S^{j+1}(A)_\sigma)\|\|\sigma\| \leq \|S^{j+1}(A)_\sigma\|\|\sigma\|\|S^{j+1}(A)_\sigma\|\|\sigma\| + \alpha)$$

$$\leq (\eta^{2^{i-j}-1} + \alpha)\|S^{j+1}(A)_\sigma\|\|\sigma\|$$

(4.2)

where the last inequality holds since $\sigma$ is a non-fat face. Now, for every $j+2 < k \leq i+1$, and every full face $\tau \in F^k(A,\sigma)$, either all of its $(k−1)$-dimensional faces as seen by $\sigma$ are full, or it has at least one non-full $(k−1)$-dimensional face which is seen by $\sigma$. Denote by $F' \subseteq F^k(A,\sigma)$ the cochain of full $k$-dimensional faces which are seen by $\sigma$ such that all of their $(k−1)$-dimensional faces are also seen full by $\sigma$, i.e.,

$$F' = \{\tau \in F^k(A,\sigma) \mid \forall \sigma \subset \tau', \dim(\tau') = k−1, \tau' \in F^{k−1}(A,\sigma)\}$$

and denote by $F'' = F^k(A,\sigma) \setminus F'$ the $k$-dimensional full faces seen by $\sigma$ which contain at least one $(k−1)$-dimensional face which is seen by $\sigma$ as a non-full face. In order to bound $\|F'_\sigma\|\|\sigma\|$, we can use Lemma 2.7 to split it to local views of full faces of dimension $k−1$.

$$\|F'_\sigma\|\|\sigma\| = \sum_{\tau \in F^{k−1}(A,\sigma)} w(\tau)\|F'_{\sigma \cup \tau}\|\|\sigma \cup \tau\| \leq \sum_{\tau \in F^{k−1}(A,\sigma)} w(\tau) = \|F^{k−1}(A,\sigma)\|\|\sigma\|$$

where the last inequality follows since $\|F'_{\sigma \cup \tau}\|\|\sigma \cup \tau\| \leq 1$. Now, we claim that $F'' \subseteq T^k(A)$. The reason is the following: Consider a face $\tau \in F''$ and say $\tau = \sigma \cup \{v_1, v_2, \ldots, v_{k−j}\}$. Since $\tau$
is seen full by $\sigma$, all of its $(k - 1)$-dimensional faces which contain $\sigma$ are fat, i.e., for every $1 \leq l \leq k - j$, $\tau \setminus \{v_l\}$ is fat. Since $\tau \in F^k$, at least one of its $(k - 1)$-dimensional faces contains a non-fat $(k - 2)$-dimensional face which is seen by $\sigma$. This non-fat face is of the form $\tau \setminus \{v_{l_1}, v_{l_2}\}$ where $1 \leq l_1 < l_2 \leq k - j$. But then $\tau$ contains two fat $(k - 1)$-dimensional faces, $\tau \setminus \{v_{l_1}\}$ and $\tau \setminus \{v_{l_2}\}$, which their intersection is a non-fat $(k - 2)$-dimensional face. Thus, $\tau \in \Upsilon^k(A)$.

Now we can bound the norm of $F^k(A, \sigma)_\sigma$.

\[
\|F^k(A, \sigma)_\sigma\|_\sigma = \|F'_A \cup F''_A\|_\sigma \leq \|F'_A\|_\sigma + \|F''_A\|_\sigma
\]

By applying (4.3) iteratively for $k = i + 1, i, \ldots, j + 3$, we get,

\[
\|F^{i+1}(A, \sigma)_\sigma\|_\sigma \leq \|F^i(A, \sigma)_\sigma\|_\sigma + \|\Upsilon^j(A)\|_\sigma \\
\leq \|F^{i-1}(A, \sigma)_\sigma\|_\sigma + \|\Upsilon^j(A)\|_\sigma \\
\leq \|F^{j+1}(A, \sigma)_\sigma\|_\sigma + \|\Upsilon^{j+3}(A)\|_\sigma + \cdots + \|\Upsilon^j(A)\|_\sigma + \|\Upsilon^{j+1}(A)\|_\sigma \\
\leq (\eta^{2j-i-1} + \alpha)\|S^{j+1}(A)\|_\sigma + \sum_{k=j+3}^{i+1} \|\Upsilon^k(A)\|_\sigma
\]

where in the last inequality we used (4.2).

In the case that at some dimension the weight of the cochain is distributed on many faces, we get many degenerate faces. These degenerate faces are good for us, since the situation of almost all of the weight is concentrated on a few faces, the contribution of the degenerate faces is negligible. Then, when we step down one dimension lower, the degenerate faces of higher dimensions become bad. This is the motivation behind the following lemma which lets us bound the weight carried by the degenerate faces of higher dimensions.

**Lemma 4.7** (Small size of bad degenerate faces). Let $X$ be a $d$-dimensional $\alpha$-skeleton expander, let $A \in C^i(X)$, and let $0 < \eta < 1$ be the fatness constant. For any $2 \leq j \leq i + 1$,

\[
\|\Upsilon^j(A)\| \leq (j + 1)^2(\eta^{2j-i-1} + \alpha) \sum_{\sigma \in X(J-2)\setminus S^{j-2}(A)} w(\sigma)\|S^{j-1}(A)\|_\sigma
\]

**Proof.** Let $2 \leq j \leq i + 1$. Recall that for every $j$-dimensional degenerate face $\tau \in \Upsilon^j(A)$ there exists two $(j - 1)$-dimensional fat faces $\sigma, \sigma' \subset \tau$, such that $\sigma \cap \sigma' \notin S^{j-2}(A)$. Therefore, when considering the link of $\sigma \cap \sigma'$, $\tau$ is seen as an edge between two fat vertices, i.e., $\tau \in E(S^{j-1}(A)_{\sigma \cap \sigma'})$. In other words, for every $j$-dimensional degenerate face $\tau \in \Upsilon^j(A)$ there exists a $(j - 2)$-dimensional non-fat face $\sigma \in X(J-2) \setminus S^{j-2}(A)$, such that $\tau \in E(S^{j-1}(A)_{\sigma})$. Thus, by the union bound and the triangle inequality we get,

\[
\|\Upsilon^j(A)\| = \left\| \bigcup_{\sigma \in X(J-2)\setminus S^{j-2}(A)} E(S^{j-1}(A)_{\sigma}) \right\| \leq \sum_{\sigma \in X(J-2)\setminus S^{j-2}(A)} \|E(S^{j-1}(A)_{\sigma})\| \\
\] (4.4)

By Lemma 2.6 we can translate the global norm to the local norm.

\[
\|E(S^{j-1}(A)_{\sigma})\| = \left(\frac{j + 1}{j - 1}\right) w(\sigma)\|E(S^{j-1}(A)_{\sigma})\|_\sigma
\] (4.5)
By the $\alpha$-skeleton expansion, we can bound the edges between fat vertices in the link of $\sigma$.

$$
\|E(S^{j-1}(A)_\sigma)\|_\sigma \leq \|S^{j-1}(A)_\sigma\|_\sigma(\|S^{j-1}(A)\|_\sigma + \alpha) \leq (\eta^{2^{j-1}+1} + \alpha)\|S^{j-1}(A)\|_\sigma
$$

(4.6)

where the last inequality holds since $\sigma$ is a non-fat $(j - 2)$-dimensional face. Combining (4.4), (4.5), and (4.6) finishes the proof.

$$
\|\Upsilon^j(A)\| \leq \sum_{\sigma \in X(j-2) \setminus S^{j-2}(A)} (j + 1)(\eta^{2^{j-1}+1} + \alpha)w(\sigma)\|S^{j-1}(A)\|_\sigma
$$

$$
\leq (j + 1)^2(\eta^{2^{j-1}+1} + \alpha) \sum_{\sigma \in X(j-2) \setminus S^{j-2}(A)} w(\sigma)\|S^{j-1}(A)\|_\sigma
$$

\[\blacksquare\]

4.3 Proof of the Theorem

We are now ready to prove the colorful expansion criterion. Let $X$ be a $d$-dimensional $\alpha$-skeleton expander. Let $0 \leq i \leq d - 1$, and let $A \in C^i(X)$ be an $i$-cochain which is not $\emptyset$ or $X(i)$. Note that the expanding faces of $A$ and $\bar{A}$ are the same, i.e., $\psi(A) = \psi(\bar{A})$. We want to show that $\|\psi(A)\| \geq \epsilon \min\{\|A\|, \|\bar{A}\|\}$. Thus, assume without loss of generality that $\|A\| \leq 1/2$, otherwise take $\|\bar{A}\|$ as the cochain. Let $0 < \eta < 1$ be a fatness constant, and let $0 < c < 1$ be a concentration constant, both will be defined later.

Recall that the container of $A$, $\Gamma(A)$, contains all of the $(i + 1)$-dimensional faces which contain a face of $A$. These faces are the potentially expanding faces. The faces of $\Gamma(A)$ which are not expanding are exactly the $(i + 1)$-cochain of fat faces, $F^{i+1}(A)$. So we can say that $\psi(A) = \Gamma(A) \setminus F^{i+1}(A)$. We split our proof to two cases.

(i) Assume that at every dimension the cochain is concentrated almost entirely on fat faces. Meaning, assume that for every $0 \leq j \leq i - 1$ the following holds,

$$
\sum_{\sigma \in X(j) \setminus S^j(A)} w(\sigma)\|S^{j+1}(A)\|_\sigma < c^{2^j}\|A\|
$$

(4.7)

This equation means that the weight of fat faces which sit on non-fat faces of every dimension is negligible. Since it holds for every dimension, it means that almost all of the weight of $A$ is concentrated on fat faces along all the way down to the lowest level.

In this case, we can bound $\|F^{i+1}(A)\|$ as follows. Denote by $F' \subseteq F^{i+1}(A)$ the faces in $F^{i+1}(A)$ which all of their vertices are fat, i.e., $F' = \{\sigma \in F^{i+1}(A) \mid \forall v \in \sigma, v \in S^0(A)\}$, and denote by $F'' = F^{i+1}(A) \setminus F'$ the faces in $F^{i+1}(A)$ which have at least one non-fat vertex. We bound each of them separately.

In order to bound $F'$ we show that there are not many $(i + 1)$-dimensional faces living on a small set of vertices. Recall that $E(S^0(A))$ is the set of edges in the complex which both of their endpoints are fat vertices. By the skeleton expansion of $X$ we know that $\|E(S^0(A))\| \leq \|S^0(A)\|\|(S^0(A)) + \alpha\)$. By Corollary 4.5 we know that $\|S^0(A)\| \leq \eta^{-2}\|A\|$. Namely, the cochain of fat vertices is not much larger than the original cochain. So by Lemma 2.7 we can calculate $\|F'\|$ by the local views of the edges which are only between fat vertices.

$$
\|F'\| = \sum_{\sigma \in X(1)} w(\sigma)\|F'_\sigma\|_\sigma = \sum_{\sigma \in E(S^0(A))} w(\sigma)\|F'_\sigma\|_\sigma
$$

$$
\leq \sum_{\sigma \in E(S^0(A))} w(\sigma) = \|E(S^0(A))\| \leq \|S^0(A)\|\|(S^0(A)) + \alpha\)
$$

(4.8)

$$
\leq \eta^{-2}\|A\|\|(\eta^{-2}\|A\| + \alpha) < \eta^{-2^{i+1}}(\frac{1}{2} + \alpha)\|A\|
$$
where the first inequality follows since \( \|F'\|_\sigma \leq 1 \), and the last inequality follows since we assumed that \( \|A\| \leq 1/2 \).

Now, by the definition of \( F'' \), each \( \sigma \in F'' \) contains at least one non-fat vertex. Thus, for every \( \sigma \in F'' \), there exists a vertex \( v \in X(0) \setminus S^0(A) \), such that \( \sigma \in F^i(A,v) \). This implies that \( F'' \subseteq \bigcup_{v \in X(0) \setminus S^0(A)} F^i(A,v) \). By the union bound and the triangle inequality we get the following upper bound.

\[
\|F''\| \leq \sum_{v \in X(0) \setminus S^0(A)} \|F^i(A,v)\| = \sum_{v \in X(0) \setminus S^0(A)} \|F^i(A,v)\|,
\]

where the last equality follows since \( F^i(A,v) \) contains only faces which contain \( v \), thus restricting it to \( v \) and lifting back to the global complex yields the same cochain. By Lemma 2.6 we can calculate the global norm by the local norm.

\[
\|F^i(A,v)\| = (i + 2)w(v)\|F^i(A,v)\|.
\]

By Lemma 4.7 we know that on every non-fat vertex \( v \in X(0) \setminus S^0(A) \) there are only a few full faces.

\[
\|F^i(A,v)\| \leq (\eta^{2i-1} + \alpha)\|S^1(A)\| + \sum_{j=3}^{i+1} \|\mathcal{Y}^j(A)\|
\]

Using Lemma 2.7 in order to translate the sum of local views to the global norm, together with Lemma 4.7 we get that for every \( 3 \leq j \leq i + 1 \),

\[
\sum_{v \in X(0) \setminus S^0(A)} w(v)\|\mathcal{Y}^j(A)\| \leq \sum_{v \in X(0)} w(v)\|\mathcal{Y}^j(A)\| = \|\mathcal{Y}^j(A)\|
\]

\[
\leq (j + 1)^2(\eta^{2i-j+1} + \alpha) \sum_{\sigma \in X(j-2),S^j(A)} w(\sigma)\|S^{j-1}(A)\|_\sigma
\]

\[
< (j + 1)^2(\eta^{2i-j+1} + \alpha)c^{2j-2}\|A\|
\]

where the last inequality follows from our assumption of this case at (4.7). This assumption also gives us an effective bound on the weight of fat edges sitting on non-fat vertices, i.e.,

\[
\sum_{v \in X(0) \setminus S^0(A)} w(v)\|S^1(A)\| < c\|A\|
\]

Combining all of these steps together, we get the following bound on \( F'' \).

\[
\|F''\| \leq \sum_{v \in X(0) \setminus S^0(A)} \|F^i(A,v)\| = (i + 2)\sum_{v \in X(0) \setminus S^0(A)} w(v)\|F^i(A,v)\|.
\]

\[
\leq (i + 2) \left( \eta^{2i-1} + \alpha \sum_{v \in X(0) \setminus S^0(A)} w(v)\|S^1(A)\| + \sum_{j=3}^{i+1} \sum_{v \in X(0) \setminus S^0(A)} w(v)\|\mathcal{Y}^j(A)\| \right)
\]

\[
\leq (i + 2) \left( \eta^{2i-1} + \alpha c\|A\| + \sum_{j=3}^{i+1} (j + 1)^2(\eta^{2i-j+1} + \alpha)c^{2j-2}\|A\| \right)
\]

\[
< (i + 2) \left( (\eta + \alpha)c\|A\| + (i + 2)^3(\eta + \alpha)c\|A\| \right) < (i + 2)^4(\eta + \alpha)c\|A\|
\]

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Finally, (4.3) and (4.9) yields an upper bound on $\|F^{i+1}(A)\|$.

$$\|F^{i+1}(A)\| = \|F''\| + \|F''\| < \left(\eta^{-2^{i+1}}(1/2 + \alpha) + (i + 2)^4(\eta + \alpha)c\right)\|A\|$$

By Lemma 2.9 we know that $\|\Gamma(A)\| \geq \|A\|$, thus we get the following lower bound on the cochain of expanding faces.

$$\|\psi(A)\| \geq \|\Gamma(A)\| - \|F^{i+1}(A)\| \geq \left(1 - \eta^{-2^{i+1}}(1/2 + \alpha) - (i + 2)^4(\eta + \alpha)c\right)\|A\|$$

(4.10)

(ii) In this case, at some dimension there are many fat faces sitting on non-fat faces. Let $0 \leq j \leq i - 1$ be the maximal such that for every $k + 1 \leq k \leq i - 1$,

$$\sum_{\sigma \in X(k) \setminus S^k(A)} w(\sigma)\|S^{k+1}(A)\| < c^k\|A\|$$

and for $j$ itself,

$$\sum_{\sigma \in X(j) \setminus S^j(A)} w(\sigma)\|S^{j+1}(A)\| \geq c^j\|A\|$$

That means that down to dimension $j$ almost all of the fat faces sit on fat faces, and at dimension $j$ there are many fat faces of dimension $j + 1$ sitting on non-fat faces of dimension $j$, i.e., there are many degenerate faces at dimension $j$. We show that from the local views of these degenerate faces there are many expanding faces. Consider a non-fat face $\sigma \in X(j) \setminus S^j(A)$. From Lemmas 2.9 and 4.4 we know that,

$$\|\Gamma(\sigma)\| = \|A\| \geq \eta^{j^2-j-1}\|S^{j+1}(A)\|$$

These faces are potentially expanding by the local view of $\sigma$. Out of these faces, the non-expanding ones are the faces that are seen full by $\sigma$. Since $\sigma$ is a non-fat face, Lemma 4.10 gives us an upper bound on the number of full faces in its link. Thus, we get the following.

$$\|\Gamma(\sigma)\| \setminus F^{j+1}(A, \sigma)\| \geq \|\Gamma(\sigma)\| - \|F^{j+1}(A, \sigma)\| \geq \eta^{j^2-j-1}\|S^{j+1}(A)\| - \sum_{k=j+3}^{i+1} \|\Upsilon^k(A)\|$$

(4.13)

Similar to the previous case, we can use Lemmas 2.7 and 4.7 in order to bound the norm of the degenerate faces of every dimension $j + 3 \leq k \leq i + 1$.

$$\sum_{\sigma \in X(k) \setminus S^k(A)} w(\sigma)\|\Upsilon^k(A)\| \leq \sum_{\sigma \in X(k)} w(\sigma)\|\Upsilon^k(A)\| = \|\Upsilon^k(A)\|$$

$$\leq (k + 1)^2(\eta^{j^2-k-1} + \alpha)\sum_{\sigma \in X(k-2) \setminus S^{k-2}(A)} w(\sigma)\|S^{k-1}(A)\|$$

(4.14)

$$< (k + 1)^2(\eta^{j^2-k-1} + \alpha)c^k\|A\|$$

where the last inequality follows since $j + 1 \leq k - 2 \leq i - 1$, which implies that our assumption at (4.11) holds.
Before summing over all of the local views, we need to show that by restricting ourselves to the links we do not add any non-expanding faces to the global count. In order to see that, note first that for every $\sigma \in X(j)$, $F_{i+1}(A)_\sigma \subseteq F_{i+1}(A,\sigma)_\sigma$, since every globally full face is seen full by $\sigma$. And also, note that $\Gamma(A_\sigma) \subseteq \Gamma(A)$. The reason is the following. Consider a face $\tau \in \Gamma(A_\sigma)$. By the definition of the container, there exists a face $\tau' \in A_\sigma$ of dimension $\dim(\tau) - 1$ such that $\tau' \subset \tau$. By the definition of the link, $\tau' \cup \sigma \in A$. But then $\tau' \cup \sigma \subset \tau \cup \sigma$ and $\dim(\tau \cup \sigma) = \dim(\tau' \cup \sigma) + 1$, which implies $\tau \cup \sigma \in \Gamma(A)$. And again, by the definition of the link, we get that $\tau \in \Gamma(A)_\sigma$. Thus,
\[
\Gamma(A_\sigma) \setminus F_{i+1}(A,\sigma)_\sigma \subseteq \Gamma(A)_\sigma \setminus F_{i+1}(A)_\sigma = (\Gamma(A) \setminus F_{i+1}(A))_\sigma
\] (4.15)

Now, by Lemma 2.7, we can sum over all of the local views of faces of dimension $j$.
\[
\|\psi(A)\| = \|\Gamma(A) \setminus F_{i+1}(A)\| = \sum_{\sigma \in X(j)} w(\sigma)\|\Gamma(A) \setminus F_{i+1}(A)_\sigma\|_\sigma \\
\geq \sum_{\sigma \in X(j) \setminus S^i(A)} w(\sigma)\|\Gamma(A) \setminus F_{i+1}(A)_\sigma\|_\sigma \\
\geq \sum_{\sigma \in X(j) \setminus S^i(A)} w(\sigma)\|S^i(A)_\sigma\|_\sigma - \sum_{k=j+3}^{i+1} \sum_{\sigma \in X(j) \setminus S^i(A)} w(\sigma)\|\Gamma_k(A)_\sigma\|_\sigma \\
\geq \left((\eta^{-1} - 1) \eta^{2^i - j - 1} - \alpha\right) \sum_{\sigma \in X(j) \setminus S^i(A)} w(\sigma)\|S^i(A)_\sigma\|_\sigma - \sum_{k=j+1}^{i+1} (k + 3)^2(\eta^{2^i - k - 1} + \alpha)c^{2k}\|A\| \\
\geq \left((\eta^{-1} - 1) \eta^{2^i - j - 1} - \alpha - \sum_{k=j+1}^{i+1} (k + 3)^2(\eta^{2^i - k - 1} + \alpha)c^{2k}\|A\| \right) c^{2^i - 2}\|A\| \\
\geq \left((\eta^{-1} - 1) \eta^{2^i - 1} - \alpha - (i + 2)^3(\eta + \alpha)c\right)c^{2^i - 1}\|A\| \\
\]
where in the second inequality we used (4.15), in the third we used (4.13), and in the forth we used (4.12) together with the assumption at (4.12). It is only left to simplify, so we get the following lower bound on the expanding faces.
\[
\|\psi(A)\| \geq \left((\eta^{-1} - 1) \eta^{2^i - j - 1} - \alpha - \sum_{k=j+1}^{i+1} (k + 3)^2(\eta^{2^i - k - 1} + \alpha)c^{2k}\|A\| \right) c^{2^i - 1}\|A\| \\
\geq \left((\eta^{-1} - 1) \eta^{2^i - 1} - \alpha - (i + 2)^3(\eta + \alpha)c\right)c^{2^i - 1}\|A\| \\
\]
Now, we define the following constants,
\[
\eta = \left(\frac{1 + 2\alpha}{2(1 - 2\alpha)}\right)^{2^d}, \quad c = \frac{\alpha}{(d + 1)^4(\eta + \alpha)} \\
\]
Note that in case (i), the lower bound on the expanding faces (4.10) satisfies,
\[
\left(1 - (i + 2)^4(\eta + \alpha)c - \eta^{-2^i + 1}(\frac{1}{2} + \alpha)\right)\|A\| \geq \left(1 - (d + 1)^4(\eta + \alpha)c - \eta^{-2^i}(\frac{1}{2} + \alpha)\right)\|A\| \\
= \left(1 - \alpha - \frac{2(1 - 2\alpha)}{1 + 2\alpha}(\frac{1}{2} + \alpha)\right)\|A\| = \alpha\|A\| \geq \epsilon\|A\| \\
and in case (ii), the lower bound on the expanding faces (4.10) satisfies,
\[
\left((\eta^{-1} - 1) \eta^{2^i - 1} - \alpha - (i + 2)^3(\eta + \alpha)c\right)c^{2^i - 1}\|A\| \geq \\
\left((\eta^{-1} - 1) \eta^{2d - 2} - \alpha - (d + 1)^3(\eta + \alpha)c\right)c^{2d - 2}\|A\| \geq \\
\left(3\alpha - \alpha - \alpha\right)c^{2d - 2}\|A\| = \alpha c^{2d - 2}\|A\| = \epsilon\|A\| \\
\]
\[
\]
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where in both cases we get \( \| \psi(A) \| \geq \epsilon \| A \| \), which finishes the proof.

## 5 Proof of rapid mixing of the high order combinatorial random walk (Theorems 3.2, 3.3)

In the most part of this section we prove that colorful expansion of high dimensional simplicial complexes implies the spectral expansion of every \( i \)-graph where the complex is regular in dimension \( i \). Then we prove the local criterion for rapid mixing of the \( i \)-dimensional combinatorial random walk on the complex.

### 5.1 Proof of colorful expansion implies spectral expansion (Theorem 3.2)

In order to prove that \( G_i = G_i(X) \) is a spectral expander we need to bound both \( \tilde{\lambda}_2(G_i) \) and \( \tilde{\lambda}_n(G_i) \). The bound on \( \tilde{\lambda}_2(G_i) \) is derived from the combinatorial expansion of \( G_i \), which is derived from the colorful expansion of \( X \). The bound on \( \tilde{\lambda}_n(G_i) \) is derived from the structure of the graph \( G_i \), which is far from having a bipartite component. We bound each of them in a separate lemma.

**Lemma 5.1.** Let \( X \) be a \( d \)-dimensional \( \epsilon \)-colorful expander. For any \( 0 < i \leq d - 1 \) where \( X \) is regular in dimension \( i \),

\[
\tilde{\lambda}_2(G_i(X)) \leq 1 - \frac{\epsilon^2}{2(i + 2)^2}
\]

**Proof.** Let \( 0 < i \leq d - 1 \) such that \( X \) is regular in dimension \( i \), and denote by \( G_i = (V_i, E_i) \) the \( i \)-graph of \( X \). Since \( X \) is regular in dimension \( i \), there exist \( k, l \in \mathbb{N} \) such that for every \( \sigma \in X(i), |N(\sigma)| = k \), and for every \( \tau \in X(i + 1), \deg(\tau) = l \).

Let \( S \subseteq V_i \) be a subset of vertices in \( G_i \) with \( 0 < |S| \leq \frac{|V_i|}{2} \). Denote by \( A \in C^i(X) \) the corresponding \( i \)-cochain in \( X \). Since \( X \) is regular in dimension \( i \), the following holds.

\[
\| \psi(A) \| = \frac{\sum_{\tau \in \psi(A)} \deg(\tau)}{\sum_{\tau \in X(i + 1)} \deg(\tau')} = \frac{|\psi(A)|}{|X(i + 1)|}
\]

Now, consider an \( i \)-dimensional face \( \sigma \in X(i) \). Note first that there are \( \frac{|N(\sigma)|}{i + 1} = \frac{k}{i + 1} \) faces of dimension \( i + 1 \) which contain \( \sigma \), since each \( (i + 1) \)-dimensional face which contains \( \sigma \) contributes \( i + 1 \) neighbors to \( \sigma \). Also note that for any \( d \)-dimensional face which contains \( \sigma \), \( \tau = \sigma \cup \{v_1, v_2, \ldots, v_{d-i}\} \in X(d) \), for every \( 1 \leq j \leq d - i \), the \( (i + 1) \)-dimensional face \( \sigma \cup \{v_j\} \) is contained in \( \tau \). In other words, for any \( \sigma \subseteq \tau \subseteq X(d) \) there are \( d-i \) faces of dimension \( i+1 \) which contain \( \sigma \) and are contained in \( \tau \). Therefore,

\[
\deg(\sigma) = \frac{1}{d-i} \sum_{|\tau| = |\sigma|+1} \deg(\tau) = \frac{1}{d-i} \sum_{|\tau| = |\sigma|+1} l = \frac{kl}{(d-i)(i+1)}
\]

where the first equality holds since each \( d \)-dimensional face which contains \( \sigma \) is counted \( d-i \) times, and the last equality holds since there are \( \frac{k}{i+1} \) faces of dimension \( i+1 \) which contain \( \sigma \). Thus, for every \( \sigma, \sigma' \in X(i) \), \( \deg(\sigma) = \deg(\sigma') \), which implies the following.

\[
\| A \| = \frac{\sum_{\sigma \in A} \deg(\sigma)}{\sum_{\sigma' \in X(i)} \deg(\sigma')} = \frac{|A|}{|X(i)|}
\]
Since each $(i+1)$-dimensional face contains $i+2$ faces of dimension $i$, we get the following relation between $|X(i)|$ and $|X(i+1)|$.

$$|X(i+1)| = \frac{1}{i+2} \sum_{\sigma \in X(i)} k = \frac{k|X(i)|}{(i+1)(i+2)} \quad (5.3)$$

We also claim that $|E(S, \bar{S})| \geq (i+1)\|\psi(A)\|$. The reason is the following. Consider an expanding face $\tau \in \psi(A)$, and denote by $|A_\tau|$ the number of faces of $\tau$ which are in $A$, and by $|\bar{A}_\tau|$ the number of faces of $\tau$ which are not in $A$. Then the number of edges $\tau$ contributes to $E(S, \bar{S})$ is,

$$|A_\tau|\ |\bar{A}_\tau| = |A_\tau|(i + 2 - |A_\tau|) \geq i + 1$$

where the inequality holds since $1 \leq |A_\tau| \leq i + 1$. Thus, by the colorful expansion of $X$, we can bound the number of outgoing edges of $S$.

$$|E(S, \bar{S})| \geq (i+1)\|\psi(A)\| = (i+1)|X(i+1)|\|\psi(A)\| \geq (i+1)|X(i+1)|\epsilon\|A\|$$

$$= \epsilon(i+1)\frac{|X(i+1)|}{|X(i)|}|A| = \frac{\epsilon k}{i+2}|A| = \frac{\epsilon k}{i+2}|S|$$

where the first equality is $(5.1)$, the following inequality follows from the colorful expansion of $X$, and the following equalities are $(5.2)$ and $(5.3)$. Since it holds for every subset of vertices $S \subseteq V_i$, we can bound the normalized edge expansion parameter of $G_i$.

$$\tilde{h}(G_i) = \min_{\substack{S \subseteq V_i \ni 0 < |S| \leq \frac{|V_i|}{2}}} \frac{|E(S, \bar{S})|}{k|S|} \geq \frac{\epsilon}{i + 2}$$

Using the Cheeger’s inequality (Lemma 2.1) we get the following upper bound on $\tilde{\lambda}_2(G_i)$.

$$\tilde{\lambda}_2(G_i) \leq 1 - \frac{\tilde{h}(G_i)^2}{2} \leq 1 - \frac{\epsilon^2}{2(i+2)^2}$$

which finishes the proof. \qed

**Lemma 5.2.** Let $X$ be a $d$-dimensional simplicial complex. For any $0 < i \leq d-1$ where $X$ is regular in dimension $i$,

$$\tilde{\lambda}_n(G_i(X)) \geq -1 + \frac{1}{2(i+2)^2}$$

*Proof.* Let $0 < i \leq d-1$ such that $X$ is regular in dimension $i$, and denote by $G_i = (V_i, E_i)$ the $i$-graph of $X$. Since $X$ is regular in dimension $i$, there exists a $k \in \mathbb{N}$ such that for every $\sigma \in X(i)$, $|N(\sigma)| = k$.

In order to bound $\tilde{\lambda}_n(G_i)$ we obtain a lower bound on the normalized bipartiteness parameter of the graph. Let $y \in \{-1,0,1\}^{|V_i|}$ be a partitioning vector on the vertices of $G_i$. Since the vertices of $G_i$ correspond to the $i$-dimensional faces of $X$, we can view $y$ as giving values to all the $i$-dimensional faces $\sigma \in X(i)$. Let $A = \{ \tau \in X(i+1) \mid \exists \sigma \subset \tau, \dim(\sigma) = i, y_\sigma \neq 0 \}$ be the $(i+1)$-cochain of faces which contain at least one $i$-dimensional face $\sigma \in X(i)$ for which $y_\sigma \neq 0$. We want to bound the following.

$$\frac{\sum_{\{u,v\} \in E_i} |y_u + y_v|}{\sum_{v \in V_i} k|y_v|}$$

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In the numerator we have a sum over the edges of $G_i$. Since every edge in $G_i$ corresponds to two neighboring $i$-dimensional faces, the following are equivalent.

$$\sum_{\{u,v\} \in E_i} |y_u + y_v| = \sum_{\tau \in X(i+1)} \sum_{\sigma,\sigma' \subset \tau \atop |\sigma| = |\sigma'| = |\tau|-1} |y_\sigma + y_{\sigma'}| = \sum_{\tau \in A} \sum_{\sigma,\sigma' \subset \tau \atop |\sigma| = |\sigma'| = |\tau|-1} |y_\sigma + y_{\sigma'}| \quad (5.4)$$

Now, consider a face $\tau \in A$. Denote by $j_1$, $j_0$, and $j_1$ the number of $i$-dimensional faces in $\tau$ which have the value $-1, 0,$ and $1$, correspondingly. Note that $j_0 = i + 2 - j_1 - j_1$. Also note that each pair of $i$-dimensional faces $\sigma, \sigma' \subset \tau$ with $y_\sigma = y_{\sigma'} \neq 0$ contribute 2 to the sum, and each pair of $i$-dimensional faces $\sigma, \sigma' \subset \tau$ with $f_{\sigma} = 0$ and $f_{\sigma'} \neq 0$ contribute 1 to the sum. Thus,

$$\sum_{\sigma,\sigma' \subset \tau \atop |\sigma| = |\sigma'| = |\tau|-1} |y_\sigma + y_{\sigma'}| = 2 \binom{j_1}{2} + 2 \binom{j_1}{2} + j_0j_1 - j_1 = j_1^2 - j_1 + j_1^2 + j_1 + (i + 2 - j_1 - j_1)(j_1 + j_1) = (i + 1)(j_1 + j_1) - 2j_1j_1 \geq (i + 1)(j_1 + j_1) - \frac{(j_1 + j_1)^2}{2} = (j_1 + j_1)(i + 1 - \frac{j_1 + j_1}{2})$$

where the inequality holds since $j_1 \leq j_1$ gets its maximum when both equal $\frac{j_1 + j_1}{2}$. Since we consider only faces which have at least one non-zero value, then $1 \leq j_1 \leq i + 2$. If $j_1 + j_1 = 1$, it means that there is an only one $i$-dimensional face in $\tau$ which is not 0, and thus $\tau$ contributes $i + 1$ to the sum, since either $j_0j_{j_1} = i + 1$ or $j_0j_{j_1} = i + 1$. Otherwise, $2 \leq j_1 + j_1 \leq i + 2$, and then by (5.5), $\tau$ contributes at least $i + 1$ to the sum.

Similarly, in the denominator the following are equivalent.

$$\sum_{v \in V_j} k|y_v| = \sum_{\{u,v\} \in E_i} (|y_u| + |y_v|) = \sum_{\tau \in X(i+1)} \sum_{\sigma,\sigma' \subset \tau \atop |\sigma| = |\sigma'| = |\tau|-1} (|y_\sigma| + |y_{\sigma'}|) = \sum_{\tau \in A} \sum_{\sigma,\sigma' \subset \tau \atop |\sigma| = |\sigma'| = |\tau|-1} (|y_\sigma| + |y_{\sigma'}|) \quad (5.6)$$

Every $\tau \in X(i + 1)$ contains $i + 2$ faces of dimension $i$, and for each pair of two $i$-dimensional faces $\sigma, \sigma' \subset \tau$, $|y_\sigma| + |y_{\sigma'}| \leq 2$. Thus, for every $\tau \in X(i + 1)$,

$$\sum_{\sigma,\sigma' \subset \tau \atop |\sigma| = |\sigma'| = |\tau|-1} (|y_\sigma| + |y_{\sigma'}|) \leq \sum_{\sigma,\sigma' \subset \tau \atop |\sigma| = |\sigma'| = |\tau|-1} 2 = 2 \binom{i + 2}{2} = (i + 2)(i + 1) \quad (5.7)$$

By (5.4), (5.5), (5.6), and (5.7), we obtain the following lower bound on the vector $y$.

$$\sum_{\{u,v\} \in E_i} \frac{|y_u + y_v|}{k|y_v|} \geq \frac{\sum_{\tau \in A} (i + 1)}{\sum_{\tau \in A} (i + 2)(i + 1)} = \frac{1}{i + 2}$$

Since it holds for any vector $y \in \{-1, 0, 1\}^{|V_i|}$, we get the following.

$$\tilde{\beta}(G_i) = \min_{y \in \{-1, 0, 1\}^{|V_i|}} \frac{\sum_{\{u,v\} \in E} |y_u + y_v|}{\sum_{v \in V} k|y_v|} \geq \frac{1}{i + 2}$$

Then by Lemma 2.3, we can bound $\tilde{\lambda}_n(G_i)$.

$$\tilde{\lambda}_n(G_i) \geq -1 + \frac{\tilde{\beta}(G_i)^2}{2} \geq -1 + \frac{1}{2(i + 2)^2}$$

which finishes the proof. □
Now the Theorem follows easily from Lemmas 5.1 and 5.2. Let \(0 < i \leq d - 1\) such that \(X\) is regular in dimension \(i\). By Lemma 5.1 we obtain the upper bound on \(\tilde{\lambda}_2(G_i(X))\),

\[
\tilde{\lambda}_2(G_i(X)) \leq 1 - \frac{\epsilon^2}{2(i+2)^2} 
\]

By Lemma 5.2 we obtain the lower bound on \(\tilde{\lambda}_n(G_i(X))\),

\[
\tilde{\lambda}_n(G_i(X)) \geq -1 + \frac{1}{2(i+2)^2} 
\]

Finally, since \(\epsilon^2 < 1\), we can conclude,

\[
\tilde{\lambda}(G_i(X)) = \max\{\tilde{\lambda}_2(G_i(X))|_i|,\tilde{\lambda}_n(G_i(X))|_i\} \leq 1 - \frac{\epsilon^2}{2(i+2)^2} 
\]

which finishes the proof.

5.2 Proof of local criterion for rapid mixing of the high order combinatorial random walk (Theorem 3.3)

Let \(0 < i \leq d - 1\) such that \(X\) is regular in dimension \(i\). By applying Theorem 3.1 we get that \(X\) is an \(\epsilon\)-colorful expander for \(\epsilon\) as in Theorem 3.1. Then, by applying Theorem 3.2 we get that \(G_i(X)\) is an \((1 - \frac{\epsilon^2}{2(i+2)^2})\)-spectral expander. Finally, by Theorem 1.4 we get that the \(i\)-dimensional combinatorial random walk on \(X\) is \((1 - \frac{\epsilon^2}{2(i+2)^2})\)-rapidly mixing.

6 Proof of explicit bounded degree colorful complexes (Theorem 3.4)

Ramanujan complexes were first defined in [LSV1], and were explicitly constructed in [LSV2]. For details on Ramanujan complexes we refer the readers to [L2].

As been proven in [EK], Ramanujan complexes are excellent skeleton expanders. Though, we need a stronger claim for the mixing of their 1-dimensional skeletons than the one appears in [EK]. We can get a better bound since we need a good mixing behavior only inside a subset of vertices and not between every two subsets. We start by defining a special type of complexes.

**Definition 6.1** (Partite regular complex). Let \(X\) be a \(d\)-dimensional simplicial complex. \(X\) is said to be partite regular if there exists a partition of its vertices to disjoint subsets \(V_0 \cup V_1 \cup \cdots \cup V_d = X(0)\), such that the following conditions hold.

- For every \(d\)-dimensional face \(\sigma \in X(d)\), \(\sigma \in V_0 \times V_1 \times \cdots \times V_d\).
- For any \(I \subset J \subset \{0,1,\ldots,d\}\) there exists \(k^I_j \in \mathbb{N}\), such that every face \(\sigma \in X \cap \prod_{i \in I} V_i\)
  is contained in \(k^I_j\) faces from \(X \cap \prod_{j \in J} V_j\).

For a partite regular complex \(X\), denote by \(X_{(i,j)} = (V_i \cup V_j, X(1) \cap (V_i \times V_j))\) the induced graph by partitions \(i\) and \(j\). Note that by the regularity of the complex, \(X_{(i,j)}\) is a bipartite bigraph, i.e., there exists \(k^I_j, k^J_j \in \mathbb{N}\), such that every vertex \(v \in V_i\) is contained in \(k^I_j\) edges and every vertex \(u \in V_j\) is contained in \(k^J_j\) edges. It is known (see [EGL] Lemma 3.1) for a proof) that \(\lambda_1(X_{(i,j)}) = (k^I_j k^J_j)^{\frac{1}{2}}\), where \(\lambda_1(X_{(i,j)})\) is the largest eigenvalue of the graph’s adjacency matrix. Denote by \(\tilde{\lambda}_2(X_{(i,j)}) = \lambda_2(X_{(i,j)})/(k^I_j k^J_j)^{\frac{1}{2}}\) the normalized second largest eigenvalue. The following mixing lemma for bipartite bigraphs is proven in [EGL].
Lemma 6.2. [EGL, Corollary 3.4] Let $G = (V_1 \cup V_2, E)$ be a bipartite biregular graph, and let $\tilde{\lambda}_2(G)$ be its normalized second largest eigenvalue. For any $S \subseteq V_1, T \subseteq V_2$,

$$\frac{|E(S,T)|}{|E|} \leq \sqrt{\frac{|S||T|}{|V_1||V_2|} (\sqrt{\frac{|S||T|}{|V_1||V_2|} + \tilde{\lambda}_2(G)})}$$

We use this lemma in order to prove the following proposition.

Proposition 6.3 (Skeleton mixing Lemma). Let $X$ be a partite regular $d$-dimensional complex, and let $\tilde{\lambda}_2(X) = \max_{0 \leq i < j \leq d} \tilde{\lambda}_2(X_{i,j})$. For any subset of vertices $S \subseteq X(0)$,

$$\|E(S)\| \leq \|S\| (\|S\| + \tilde{\lambda}_2(X))$$

Proof. Let $V_0 \cup V_1 \cup \cdots \cup V_d$ be the partition of $X(0)$. Note that since $X$ is a partite regular complex, for any $0 \leq i \leq d$ and every two vertices $u, v \in V_i$, $\deg(u) = \deg(v)$. It is achieved by the regularity property of the complex and by setting $I = \{i\}$ and $J = [d] = \{0, 1, \ldots, d\}$. This implies that for any $0 \leq i \leq d$,

$$\sum_{v \in X(0)} \deg(v) = (d + 1)|X(d)| = (d + 1) \sum_{v \in V_i} \deg(v) = (d + 1)|V_i||k_{i}^{[d]}|$$

Therefore, for every $S_i \subseteq V_i$,

$$\|S_i\| = \sum_{v \in S_i} w(v) = \frac{\sum_{v \in S_i} \deg(v)}{d + 1} = \frac{|S_i|}{(d + 1)|V_i|} \quad (6.1)$$

In a same way, for any $0 \leq i < j \leq d$,

$$\sum_{e \in X(1)} \deg(e) = \binom{d + 1}{2} |X(d)| = \frac{(d + 1)}{2} \sum_{e \in X(1) \cap (V_i \times V_j)} \deg(e) = \frac{(d + 1)}{2} |X(1) \cap (V_i \times V_j)||k_{i,j}^{[d]}|$$

And again for every $S_{ij} \subseteq X(1) \cap (V_i \times V_j)$,

$$\|S_{ij}\| = \sum_{e \in S_{ij}} w(e) = \frac{\sum_{e \in S_{ij}} \deg(e)}{d + 1} = \frac{2|S_{ij}|}{d(d + 1)|X(1) \cap (V_i \times V_j)|} \quad (6.2)$$

Now, let $S \subseteq X(0)$ be a subset of vertices in the complex. For every $0 \leq i \leq d$, denote by $S_i = S \cap V_i$ the vertices in partition $i$. For any $0 \leq i < j \leq d$, by (6.1) we know that,

$$\sqrt{\frac{|S_i||S_j|}{|V_i||V_j|}} = \sqrt{(d + 1)^2|S_i||S_j|} = (d + 1)\sqrt{|S_i||S_j|}$$

and by (6.2) we know that,

$$\frac{|E(S_i, S_j)|}{|X(1) \cap (V_i \times V_j)|} = \frac{d(d + 1)}{2} \|E(S_i, S_j)\|$$

Thus, by using Lemma 6.2 we get for any $0 \leq i < j \leq d$,

$$\|E(S_i, S_j)\| \leq \frac{2}{d} \sqrt{|S_i||S_j|} \left((d + 1)\sqrt{|S_i||S_j|} + \tilde{\lambda}_2(X)\right)$$

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Note that since $S = S_0 \cup S_1 \cup \cdots \cup S_d$, then $\|S\| = \sum_{i=0}^{d} \|S_i\|$. Thus, the sum $\sum_{i \neq j} \|S_i\|\|S_j\|$ is maximized when all of the subsets are equal, i.e., $\|S_i\| = \frac{|S|}{(d+1)}$ for all $0 \leq i \leq d$. That gives us the following bound on $\|E(S)\|$.

$$\|E(S)\| = \sum_{i \neq j} \|E(S_i, S_j)\| \leq \sum_{i \neq j} \frac{2}{d} \sqrt{\|S_i\|\|S_j\|} \left( (d+1) \sqrt{\|S_i\|\|S_j\|} + \tilde{\lambda}_2(X) \right)$$

$$\leq \sum_{i \neq j} \frac{2}{d} \sqrt{\frac{\|S\|^2}{(d+1)^2}} \left( (d+1) \sqrt{\frac{\|S\|^2}{(d+1)^2}} + \tilde{\lambda}_2(X) \right)$$

$$(6.3)$$

$$= \frac{d+1}{2} \frac{2}{d(d+1)} (\|S\| + \tilde{\lambda}_2(X)) = \|S\| (\|S\| + \tilde{\lambda}_2(X))$$

which finishes the proof. \qed

Now, as been proven in [EK], all of the links of a Ramanujan complex are partite regular and the normalized second largest eigenvalue of every induced bipartite biregular graph approaches 0 as a function of the dimension and the thickness of the complex. So we state the following lemma, which is proven in [EK].

**Lemma 6.4.** Let $X$ be a $d$-dimensional $q$-thick Ramanujan complex. There exists a constant $C = C(d)$ such that,

$$\max_{\sigma \in X, i \neq j} \tilde{\lambda}_2((X_\sigma)_{(i,j)}) \leq \frac{C}{\sqrt{q}}$$

Finally, by this lemma and by Proposition[6.3] we get that a $d$-dimensional $q$-thick Ramanujan complex is a $\left(\frac{C}{\sqrt{q}}\right)$-skeleton expander. Applying Theorem[3.3] to the Ramanujan complex yields a constant $q_0$, such that if $q \geq q_0$ then the $(d-1)$-dimensional combinatorial random walk on it is $\epsilon$-rapidly mixing for $0 < \epsilon = \epsilon(q,d) < 1$, which finishes the proof.

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