Compact complex manifolds whose complement of an analytic subset is Kähler

Hirokazu Shimobe

Abstract

In this paper, we consider a class of balanced manifolds and provide a proof of a problem proposed by Silva — that compact complex manifolds that are Kähler outside an analytic subset are balanced. Next, as a special case of this theorem, we prove that compact complex manifolds which are Kähler outside a Kähler submanifold are class C. Finally, we prove that a blow up along a submanifold of Hironaka’s examples are Kähler, and we establish that Hironaka’s examples are examples of this theorem.

1 Introduction

Compact complex manifolds can be approximately characterized by the Hermitian metric they admit. If the Hermitian metric is closed, the compact complex manifold is termed a Kähler manifold. There are many examples of compact Kähler manifolds. For example, any submanifold of complex projective spaces are compact Kähler manifolds.

There are compact complex manifolds with no Kähler metrics. If the Hermitian metric is co-closed, the compact complex manifold is termed a balanced manifold. It is trivial that any Kähler manifolds is balanced, but there are balanced manifolds which do not admit any Kähler metrics. For example, Hironaka’s examples, which are class C manifolds (see Definition 2.3), are balanced, but not Kähler. Hironaka’s examples are obtained by appropriately modifying compact Kähler manifolds.

Class C manifolds are compact complex manifolds which are Kähler outside an analytic subset. Therefore it is natural to consider if any compact complex manifolds that are Kähler outside an analytic subset are balanced. This problem was first conjectured by [14] (also see [2], [3]). In this paper we present a simple proof of this conjecture. Explicitly, we prove the following theorem.

Theorem 1.1. Let $X$ be a compact complex manifold of dimension $\geq 3$. Let $Z \subset X$ be an analytic subset of codimension $\geq 2$ such that $X \setminus Z$ is Kähler. Then $X$ is balanced.

Proof. See Theorem 3.1.
In this theorem, the condition of codimension $Z$ cannot be omitted. For example, the direct product of the Hopf surface and a compact Kähler manifold is Kähler outside a divisor. However, this product manifold does not admit any balanced metrics \(^2\).

Hironaka’s examples serve as evidence of manifolds that satisfy the conclusion of Theorem 1.1. However, we unfortunately do not know if the conclusion of Theorem 1.1 is the best possible one. Hence, we will prove the following theorem as a special case of Theorem 1.1.

**Theorem 1.2.** Let $X$ be a compact complex manifold of dimension $\geq 3$. Let $Z \subset X$ be a Kähler submanifold of codimension $\geq 2$ such that $X \setminus Z$ is Kähler. Then $X$ is class C (that is, $X$ is bimeromorphic to a compact Kähler manifold).

**Proof.** See Proposition 3.5.

In Theorem 1.2, the conclusion is the best possible one. That is, we can see that the Hironaka’s examples satisfy assumptions and the conclusion by proving the following proposition.

**Proposition 1.3.** Let $X$ be a manifold from Hironaka’s example and $Z$ the submanifold homologous to 0. Then a blow up $\tilde{X}$ along $Z$ is Kähler.

**Proof.** See Proposition 4.1.

It is trivial that Hironaka’s examples serve as examples of Theorem 1.2. Based on the above considerations, we believe that we can strengthen conclusion of Theorem 1.1. That is, we propose the following conjecture.

**Conjecture 1.4.** Let $X$ be a compact complex manifold of dimension $\geq 3$. Let $Z \subset X$ be an analytic subset of codimension $\geq 2$ such that $X \setminus Z$ is Kähler. Then $X$ is class C.

### 2 Preliminaries

Let $X$ be a compact complex manifold ($\dim X = n$) and $\omega$ the Hermitian form (metric). In this paper, the Hermitian metric will be identified with the associated positive-definite $C^\infty(1, 1)$-form $\omega > 0$ on $X$.

**Definition 2.1.** If $X$ admits a Hermitian metric $\omega$ such that $d\omega^{n-1} = 0$, $X$ is termed a balanced manifold.

Any Kähler manifold is balanced, but there exist many balanced manifolds with no Kähler metrics. We introduce examples of balanced manifolds with no Kähler metrics, which are based on Hironaka’s examples \(^1\). \(^1\)

**Example 2.2.** \(^1\) Let $M$ be a compact Kähler manifold ($\dim M \geq 3$). Let $\pi: Y \to M$ be a blow up at a point. Then $\mathbb{P}^2 \subset Y$. Let $C \subset \mathbb{P}^2 \subset Y$ be a curve that is smooth except for one double point $b$ with normal crossing. There exists an open neighbourhood $U$ of $b$ in $Y$ such that $C \cap U = C_1 \cup C_2$. Let $\pi_1: U_1 \to U$
be a blow up of $C$. Let $\pi_2: U_2 \rightarrow U_1$ be a blow up of the strict transform of $C_2$. Let $\psi: Z \rightarrow Y \backslash b$ be a blow up of $C \backslash b$. Then, there exists a smooth modification $\rho: X \rightarrow Y$ that is obtained by gluing $\psi$ and $\pi_2 \circ \pi_1$. Then $X$ is non-Kähler, because $\rho^{-1}(b) =: Z_1 + Z_2$ includes a submanifold $Z_1$ that is homologous to zero. $X$ is known as Hironaka’s example. By [6], since the class of compact balanced manifolds is invariant under modifications, Hironaka’s examples are balanced.

Hironaka’s examples are also known as class C manifolds.

**Definition 2.3.** If $X$ is bimeromorphic to a compact Kähler manifold, $X$ is termed a class C manifold.

In other words, class C manifolds are compact complex manifolds that are Kähler outside an analytic subset. Therefore class C manifolds can also be characterized by the existence of a Kähler current, which is a smooth Kähler form outside an analytic subset. It is trivial that any class C manifold admits a Kähler current. Conversely, if $X$ admits a Kähler current, it is known that $X$ is a class C manifold [10]. By definition, class C manifolds admit many properties of compact Kähler manifolds. For example, it is important that $\partial \bar{\partial}$ lemma holds for class C manifolds. That is, for any $d$-closed pure type form $u$ on any class C manifold, the following exactness properties are equivalent,

$$u \text{ is } d\text{-exact} \iff u \text{ is } \partial\text{-exact} \iff u \text{ is } \overline{\partial}\text{-exact} \iff u \text{ is } \partial\overline{\partial}\text{-exact}.$$

Next, we define the real (1,1) Bott-Chern cohomology and the real (1,1) Aeppli cohomology, which are important in this paper.

**Definition 2.4.** Let $A^{p,q}(X)$ be the space of smooth $(p,q)$ forms on $X$. The real (1,1) Bott-Chern cohomology $H^{1,1}_{BC}(X)$ and the real (1,1) Aeppli cohomology $H^{1,1}_{A}(X)$ of $X$ are defined as follows.

$$H^{1,1}_{BC}(X) := \left\{ \phi \in A^{1,1}_R(X) | d\phi = 0 \right\} \cap \left\{ i\partial \overline{\partial} \psi | \psi \in A^{0,0}(X) \right\}$$

$$H^{1,1}_{A}(X) := \left\{ \phi \in A^{1,1}_R(X) | \partial \overline{\partial} \phi = 0 \right\} \cap \left\{ \partial \overline{\psi} + \overline{\partial} \psi | \psi \in A^{p,q-1}(X) \right\}$$

In general, the Bott-Chern cohomology and the Aeppli cohomology are not isomorphic, but they are isomorphic on any class C manifold.

### 3 Proof of Theorem 1.1 and Theorem 1.2

The following theorem is a conjecture of Silva [14].

**Theorem 3.1.** Let $X$ be a compact complex manifold. Suppose that $\dim X \geq 3$. Let $Z \subset X$ be an analytic subset. Suppose that $\text{codim} Z \geq 2$. Assume that $X \backslash Z$ is Kähler. Then $X$ is balanced.
Suppose that \( X \) is not balanced. Then there exists a non-zero positive, bidegree \((1,1)\)-current \( T \) on \( X \) which is \((1,1)\)-component of a boundary i.e, \( T = \partial S + \overline{\partial} S \), where \( S \) is a \((1,0)\)-current (see [1]). Let \( \omega \) be a Kähler form on \( X \setminus Z \). By the hypothesis, we can extend \( \omega \) to a closed positive current on \( X \), since \( \text{codim} Z \geq 2 \) (see [12]), also called \( \omega \). By Demailly's regularization theorem (see [9]), we can find a non-zero, positive \((1,1)\)-form \( \eta \) such that \( \text{Supp} \eta \subset X \setminus Z \) and \( \omega_k \geq \eta - \epsilon_k \gamma \), where \( \gamma \) is a Hermitian metric on \( X \) and \( \omega_k \xrightarrow{w} \omega \) and \( (\epsilon_k) \) is a sequence of positive real numbers such that \( \epsilon_k \to 0 \) (\( k \to \infty \)). Moreover, we choose a proper modification \( p_k \) such that \( p_k^* \omega_k = \theta_k + [D_k] \), where \( \theta_k \) is a closed \((1,1)\)-form and \( D_k \) is a divisor (see [10]). Note \( \theta_k \geq p_k^* (\eta - \epsilon_k \gamma) \).

Assume that \( \text{Supp} T \not\subset Z \). By Theorem 5.6 of [6], there is the pull back \( T_k := p_k^* T \). The positive current \( T_k \) is the \((1,1)\)-component of a boundary. We may assume that \( T_k \) is non zero. Therefore,

\[
\int_{X_k} \theta_k^{n-1} \wedge T_k = 0.
\]

On the other hand,

\[
\int_{X_k} \theta_k^{n-1} \wedge T_k \geq \int_{X_k} p_k^* (\eta - \epsilon_k \gamma)^{(n-1)} \wedge T_k = \int_X (\eta - \epsilon_k \gamma)^{(n-1)} \wedge T > 0, \quad (k \gg 1).
\]

This is a contradiction. Hence \( \text{Supp} T \subset Z \). Since \( \text{codim} Z \geq 2 \), by the support theorem ([1]), \( T = 0 \). Therefore \( X \) is balanced.

\[ \square \]

Remark 3.2. In Theorem 3.1, the condition of \( \text{codim} Z \geq 2 \) cannot be omitted ([2]).

Remark 3.3. If \( \dim X = 3 \) and \( Z \) is a smooth curve, Theorem 3.1 was proven by [2].

Next, we will prove a key lemma in this paper.

Lemma 3.4. Let \( X \) be a compact complex manifold and \( Z \subset X \) a Kähler submanifold of codimension \( Z \geq 2 \). Let \( \pi : \hat{X} \to X \) be a blow up along \( Z \). Then the Kähler form \( \omega \) on the exceptional divisor \( E := \pi^{-1}(Z) \) can be extended to \( \hat{X} \) as a closed \((1,1)\) form.

Proof. Let \( i : E \to \hat{X} \) be an embedding. Suppose that there does not exist \( [u] \in \hat{H}^{1,1}_{BC}(\hat{X}) \) such that \( [i^* u] = [\omega] \). By [15], there exists the following isomorphism.

\[
i : \hat{H}^{1,1}_{BC}(\hat{X})/\pi^* H^{1,1}_{BC}(X) \cong H^{1,1}_{BC}(E)/(\pi|_E)^* H^{1,1}_{BC}(Z).
\]

Therefore, since \( [\omega] \in (\pi|_E)^* H^{1,1}_{BC}(Z) \), there exists \( \beta \in H^{1,1}_{BC}(Z) \) such that \( \omega = (\pi|_E)^* \beta \), but \( \beta^{n-1} = 0 \), contradiction \( (\omega^{n-1} \neq 0) \). Hence \( \omega \) can be extended to \( \hat{X} \) as a closed \((1,1)\) form. \[ \square \]
Class C manifolds are compact complex manifolds that are Kähler outside an analytic subset. Therefore, it is natural to consider whether compact complex manifolds obtained from Theorem 3.1 are close to being class C manifolds. In this context, we prove the following Proposition as a special case of Theorem 3.1.

**Proposition 3.5.** Let \( X \) be a compact complex manifold of dimension \( X \geq 3 \) and \( Z \subset X \) be a Kähler submanifold of \( \text{codim}Z \geq 2 \). Assume that \( X \setminus Z \) is Kähler. Then \( X \) is class C.

**Proof.** Let \( \pi : \tilde{X} \to X \) be a blow up along \( Z \). We will prove that \( \tilde{X} \) is class C. Let \( E := \pi^{-1}(Z) \). Obviously, \( E \) is Kähler. Let \( \omega \) be the Kähler form on \( E \). Let \( i: E \hookrightarrow \tilde{X} \) be an embedding. By Lemma 3.4, there exists a closed (1,1) form \([u] \in H^{1,1}_{BC}(\tilde{X})\) such that \( i^*u = \omega \). By adding a boundary to \( u \), \( \tilde{\omega} := u + i\partial\overline{\partial}\psi \), we may assume that \( \tilde{\omega} > 0 \) on \( \bar{U} \), where \( U \) is an open neighborhood of \( E \) (see Proposition 3.3 (i) of [10]).

On the other hand, since \( \text{codim}Z \geq 2 \), any Kähler form on \( X \setminus Z \) has a trivial extension \( T^0 \) as a closed positive (1,1) current on \( X \) ([12]). Let \( \tilde{T}^0 := \pi^*T^0 \).

Let \( h \) be a Hermitian metric on \( \tilde{X} \). Then,

(i) There exists \( \epsilon > 0 \) such that \( \epsilon h < \tilde{\omega} \) on \( \bar{U} \).
(ii) There exists \( K > 0 \) such that \( K\tilde{T}^0 > \epsilon h - \tilde{\omega} \) on \( \tilde{X} \setminus \bar{U} \).

Hence there exists a \( K \in \mathbb{R}_{>0} \) such that \( K\tilde{T}^0 + \tilde{\omega} > \epsilon h \), i.e. \( \tilde{X} \) is class C, so \( X \) is class C.

We can provide many examples of non-Kähler, non-compact complex manifolds by using Proposition 3.5. Examples of non-Kähler, non-compact complex manifolds have been not well known.

**Proposition 3.6.** Let \( X \) be a non-Kähler twistor space and \( C \) the twistor line. Then, \( X \setminus C \) is non-Kähler.

**Proof.** First, we assume that \( X \) is non-class C. Then, by Proposition 3.5, \( X \setminus C \) is non-Kähler. This is because \( X \) must be class C if \( X \setminus C \) is Kähler. Contradiction.

Hence we assume that \( X \) is class C. Suppose that \( X \setminus C \) is Kähler. Then, by Theorem 5.5 of [3], since \( X \) is non-Kähler, one and only one of the following cases may occur:

(i) \( C \) is the (1,1)-component of a boundary,
(ii) \( C \) is part of the (1,1)-component of a boundary (see Definition 5.4 of [3]).

First, we assume (i). Let \( \pi: \tilde{X} \to X \) be a blow up along \( C \). Put \( E := \pi^{-1}(C) \).

Since \( E \) is Kähler, by Lemma 3.4 the Kähler form on \( E \) can be extended to \( \tilde{X} \) as a closed (1,1) form \( \alpha \). Since \( \text{codim}C = 2 \), the Kähler form on \( X \setminus C \) can be extended to \( X \) as a closed positive (1,1) current \( T \) ([12]). Then, similar to Proposition 3.5, for a sufficiently large \( K \in \mathbb{R}_{>0} \), \( \tilde{T} := K\pi^*T + \alpha \) is a Kähler current. The push forward \( \tilde{T} := \pi_\#\tilde{T} \) is a Kähler current which is smooth on \( X \setminus C \). Let \( \alpha \) be a representative element of the Bott-Chern cohomology group of \( \tilde{T} = \alpha + i\partial\overline{\partial}\phi \). Then, the integral \( \int_{C'} \alpha \) on the twistor fiber \( C' \) of \( X \setminus C \) is...
strictly positive. Since $C \sim C'$, the integral $\int_C \alpha$ is also strictly positive. On the other hand, by (i), since $C$ is the $(1,1)$-component of a boundary, $\int_C \alpha = 0$. This is a contradiction.

Next we assume (ii). Since $C$ is not the $(1,1)$ component of a boundary, by Theorem 3.2 of [7], the Kähler form on $C$ can be extended to $X$ as a closed $(1,1)$ form. Similar to (i), we can define a Kähler current $T$ on $X$ which is smooth on $X \setminus C$. By the definition of part of the $(1,1)$-component of a boundary, we take the non-zero, positive, bidimension $(1,1)-\partial \bar{\partial}$-closed current $S$ such that $C + S$ is the $(1,1)$-component of a boundary and $\chi_{C,S} = 0$ (see Theorem 5.5 of [3]). We will calculate the intersection number $T.(C + S)$. Similarly to (i), $T.C = \alpha.C = \int_C \alpha > 0$, where $\alpha$ is a representative element of the Bott-Chern cohomology group of $T$. By Theorem 1.2 of [3], we take a sequence $\{\phi_k\}$ of closed $(1,1)$-forms on $X$ in the same Bott-Chern class of $T$ such that $\phi_k \geq \gamma - \lambda_k u$, where $\gamma$ is a Hermitian form such that $T > \gamma$, and $\{\lambda_k\}$ is a non-increasing sequence of continuous functions on $X$ such that $\lim_{k \to \infty} \lambda_k(x) = n(T, x)$ at every point $x \in M$ where $n(T, x)$ is the Lelong number of $T$ at $x \in X$. Hence $T.S = \phi_k.S \geq S(\gamma) - S(\lambda_k u)$. Similar to Theorem 5.5 of [3], by $k \to \infty$, we can prove

$$T.S \geq S(\gamma) > 0. \quad (3.1)$$

For the reader’s convenience, we will rewrite the proof of (3.1) according to Theorem 5.5 of [3]. We define a positive measure on $X$ as $\mu(A) := S(\chi_A u)$, where $A$ is a Borel subset of $X$. Then, since $0 \leq \lambda_k \leq \lambda_0$,

$$\lim_{k \to \infty} S(\lambda_k u) = \lim_{k \to \infty} \int_X \lambda_k d\mu = \int_X n(T, x) d\mu = 0.$$  

Hence we obtain (3.1). Therefore the intersection number $T.(C + S)$ is strictly positive. On the other hand, by (ii), since $C + S$ is the $(1,1)$-component of a boundary, $T.(C + S)$ must be zero. This is a contradiction. Therefore $X \setminus C$ is non-Kähler.

Remark 3.7. Except for the complex torus, nilmanifolds with a complex structure are all non-Kähler. Many of them have the structure of a holomorphic fiber bundle, and for the same reason as Proposition 3.6 the complement of the fiber is a non-Kähler, non-compact complex manifold. For example, the Iwasawa manifold admits the structure of $T^2$ torus bundle on $T^4$ torus, and the complement of the fiber is non-Kähler.

Remark 3.8. There are no nonconstant holomorphic functions on complex manifolds obtained by Proposition 3.6 and Remark 3.7.

4 Proof of Proposition 1.3

In this section, we present the proof of Proposition 1.3.

Proposition 4.1. We use the notation of Example 2.2. Let $X$ be any manifold from Hironaka’s example. Let $\pi: \tilde{X} \to X$ be a blow up along $Z_1$. Then $\tilde{X}$ is Kähler.
Remark 4.2. This Proposition gives non-trivial examples of Proposition 3.5. That is, we can take $Z_1$ as $Z$ in Proposition 3.5. Note that any manifold in Hironaka’s example is a non-Kähler class C manifold.

Proof. Put $\dim X = n$. Since $X$ is non-Kähler, there exists a non-zero positive current $T$ of bidegree $(n-1,n-1)$ such that $T = i\partial \bar{\partial} u$ (Theorem 2.4 of [8]). Let $S := T - \chi Z_1 + T$, where $\chi$ is the characteristic function. Then $S$ is the $\mathcal{d}$-closed positive current, and $\chi Z_1 + T = 0$. Since $X \setminus (Z_1 + Z_2)$ is isomorphic to the blow up along $C - \{b\}$ of $Y - \{b\}$, $X \setminus (Z_1 + Z_2)$ is Kähler. Let $\omega$ be the Kähler form. Since codim$(Z_1 + Z_2) \geq 2$, $\omega$ can be extended to $X$ as a positive, bidegree $(1,1)$ closed current (12). Choose a smooth $(1,1)$-form $\gamma$ on $X$ such that $\omega \geq \gamma \geq \alpha_k u$, where the notation of Theorem 1.2 of [3] has been used. By using the method of Theorem 5.5 of [3], we can choose a sequence $\{\omega_k\}$ of closed $(1,1)$-forms such that $\omega_k \geq \gamma - \lambda_k u$, where $\lambda_k \to 0$ as $k \to \infty$. Then, we can choose the smooth $(1,1)$-form $\omega_0$ as $\gamma$ which is positive definite on compact subset $K \subset X \setminus (Z_1 + Z_2)$. Since $K$ is arbitrary, we obtain $\omega, S = 0 \iff S = 0$. Since $T$ is a boundary,

$$0 = \omega, T = \omega_k (S + \chi Z_1 + T) = S(\omega_k) + \int_X \chi Z_1 T \wedge \omega_k + \int_X \chi Z_1 T \wedge \omega_k. \quad (4.1)$$

First, it is trivial that $S(\omega_k) \geq 0$. For the third term, since $Z_2$ is isomorphic to the projective space and $\int_{Z_2} \omega_k^n > 0$, there exists $\omega_2 \in [\omega_k]$ which is a Kähler form on a neighborhood $U_2 \supset Z_2$ (see Proposition 3.3 (i) of [10] and Theorem 3.1 (ii) of [7]). Therefore $\int_X \chi Z_1 T \wedge \omega_k \geq 0$. For the second term, since $Z_1$ is homologous to zero, $\omega_k|_{U_1}$ is $i\partial \bar{\partial}$-exact, so $\omega_k|_{U_1}$ is $d$-exact, where $U_1$ is a neighborhood of $Z_1$, that is, $\omega_k|_{U_1} = d\phi$. Therefore $\int_X \chi Z_1 T \wedge \omega_k = \int_X \chi Z_1 T \wedge d\rho_\epsilon \phi = 0$, where $\rho_\epsilon$ is a smooth function on $U_1 \supset \text{supp}\rho_\epsilon$ such that $\rho_\epsilon \equiv 1$ on a $\epsilon$-neighborhood of $Z_1$. Therefore, by (4.1), $\text{Supp} T \subset Z_1$.

Assume that $X$ is non-Kähler. Then, there exists a non-zero, positive, bidegree $(n-1,n-1)$-current $T$ such that $T = i\partial \bar{\partial} u$ (8). Put $E = \pi^{-1}(Z_1)$. If $\text{Supp} T \not\subset E$, then $\pi_* \overline{T} \not\subset Z_1$. Since $\pi_* \overline{T}$ is a non-zero, positive, bidegree $(n-1,n-1)$, $i\partial \bar{\partial}$-exact current, this is a contradiction, and therefore, $\text{Supp} T \subset E$. However, by Lemma 3.3, any Kähler form on $E$ can be extended to $X$ as a closed form $\alpha$. We may assume that $\alpha$ is positive definite on a neighborhood $U$ of $E$. (Proposition 3.3 (ii) of [10].) This contradicts Theorem 3.1 (ii) of [7]. Therefore $\overline{T} = 0$, $X$ is Kähler.

Remark 4.3. When $X$ is Moisheson and $\dim X = 3$, Proposition 4.1 was proved by [13] (Example 3).

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Osaka University, Toyonaka, Japan
E-mail address: rsc96831@gmail.com