Traveling waves for a model of the Belousov-Zhabotinsky reaction

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Abstract

Following J.D. Murray, we consider a system of two differential equations that models traveling fronts in the Noyes-Field theory of the Belousov-Zhabotinsky (BZ) chemical reaction. We are also interested in the situation when the system incorporates a delay $h \geq 0$. As we show, the BZ system has a dual character: it is monostable when its key parameter $r \in (0, 1]$ and it is bistable when $r > 1$. For $h = 0$, $r \neq 1$, and for each admissible wave speed, we prove the uniqueness of monotone wavefronts. Next, a concept of regular super-solutions is introduced as a main tool for generating new comparison solutions for the BZ system. This allows to improve all previously known upper estimations for the minimal speed of propagation in the BZ system, independently whether it is monostable, bistable, delayed or not. Special attention is given to the critical case $r = 1$ which to some extent resembles to the Zeldovich equation.

Keywords: Belousov-Zhabotinsky reaction; comparison solutions; minimal speed; sliding solution method; bistable; monostable.

2000 Mathematics Subject Classification: 34K12, 35K57, 92D25

March 4, 2013
1. Introduction and main results

One of useful objects associated with the famous Belousov-Zhabotinsky chemical reaction is the following dimensionless non-linear system \([21, 22]\)

\[
\begin{align*}
    u_t(t,x) &= \Delta u(t,x) + u(t,x)(1 - u(t,x) - rv(t,x)), \\
    v_t(t,x) &= \Delta v(t,x) - bu(t,x)v(t,x),
\end{align*}
\](1)
called the Belousov-Zhabotinsky (BZ for short) reaction-diffusion system. The coefficients \(r, b\) are positive and \(u, v\) correspond to the bromous acid and bromide ion concentrations respectively. The front solution \((u, v) = (\phi, \theta)(v \cdot x + ct)\) of system (1) provides an appropriate mathematical tool for the description of planar waves propagating in a thin layer of reactant solution filled in a Petri dish \([23]\). Due to the chemical interpretation of (1), only non-negative fronts are meaningful. Another requirement is the existence of the limits \((\phi, \theta)(-\infty) = (0, a), (\phi, \theta)(+\infty) = (1, 0)\) with \(a > 0\). The exact value of \(a\) is not relevant: after rescaling \(u, v\), we can take \(a = 1\). By the experimental data \([21, 22]\), \(r > 1\). Nevertheless, almost all previous analytical studies of wavefronts (with two exceptions given in Propositions \([11, 10]\) considered the case \(r \in (0, 1)\) which was proved to be of the monostable type. We observe that the standard definition \([26]\) of monostability/bistability needs an obvious modification in order to be applied to system (1) which has a continuum of non-negative equilibria. The degeneracy of the equilibrium \((0, 1)\) is a special feature of model (1) complicating its analysis. For example, the recent Liang-Zhao general theory \([14]\) of spreading speeds for abstract monostable evolution systems can not be employed here despite the fact that system (1) is formally monostable and monotone for \(0 < r \leq 1\). This obliged us in \([24]\) to present a complete proof of the existence of the minimal speed of front propagation in (1) when \(r \leq 1\). On the other hand, we show here that, for each \(r > 1\), the BZ system possesses a unique wavefront solution, in full accordance with its formal bistability.

Now, as it was argued in \([24]\), a better theoretical prediction for propagation speeds in model (1) can be also obtained by taking into account delayed effects during the generation of the bromous acid. For simplicity, and in order to connect with various analytical investigations, we will use here the following delayed version of (1) proposed by Wu and Zou in \([27]\):

\[
\begin{align*}
    u_t(t,x) &= \Delta u(t,x) + u(t,x)(1 - u(t,x) - rv(t,x)), \\
    v_t(t,x) &= \Delta v(t,x) - bu(t,x)v(t,x),
\end{align*}
\](2)

During the last decades considerable efforts have been made in studying the wave propagation in (1), (2). The attention was focused on the stability, numerical approximation \([19, 22, 23]\) and existence \([12, 13, 15, 16, 17, 21, 22, 25, 26, 27, 28]\) of fronts. After linear changes, systems (1), (2) acquire good monotonicity properties: they are quasi-monotone as partial differential equations \([20, 26]\) and they are monotone in the sense of Wu and Zou \([27]\). Hence, the front existence may be handled by the standard comparison technique well established for several decades \([24, 26]\). Thus the existence of fronts for the BZ system is no longer an issue, in difference with the determination or satisfactory approximation of the minimal speed of propagation in (1), (2). Precisely this problem is our main concern here. It is quite noteworthy that a similar question (formulated as linear versus non-linear determinacy of the minimal speed) for a Lotka-Volterra reaction-diffusion competition model has received a considerable attention during the last few years \([8, 9, 10]\). Finally, our secondary concern is the uniqueness of wavefronts (cf. (1)): since these have to be monotone, we prove their uniqueness in the non-delayed non-degenerate case (i.e. \(r \neq 1, h = 0\)) by applying the Berestycki-Nirenberg sliding solution argument \([2, 8]\).
1.1. Some previously known results

For the sake of completeness, we state the most relevant known existence results for (1), (2). First of them was proved in [21, 22], it gives a lower bound for the admissible front speeds. Set

$$c_1 := \max \{ 2R, \sqrt{1-r}, (\sqrt{r^2 + 2b/3} - r)/\sqrt{2b + 4r} \}. $$

**Proposition 1.** Let $r, b > 0$. If system (1) has a positive componentwise monotone wavefront (or, shortly, monotone wavefront) connecting $(0, 1)$ with $(1, 0)$ then $c \geq c_1$.

It is easy to see that the estimation of Proposition 1 has the form $c \geq 2 \sqrt{1+r}$ for positive $r \leq 11/12 = 0.917 \ldots$. The next assertion summarizes the main existence results from [12, 13, 28].

**Proposition 2.** System (1) has a positive monotone wavefront $(u,v)(x,t) = (\phi, \theta)(v \cdot x + ct)$, $|v| = 1$, connecting $(0, 1)$ with $(1, 0)$ for each velocity $c \geq c_k = \{ \begin{array}{ll} 2 \sqrt{1-r}, & \text{if } rb + r \leq 1; \\
2 \sqrt{b}, & \text{if either } b + r > 1, b < 1, r \in (0, 1) \text{ or } b = 1, r < 1; \\
2, & \text{if } b > 1, r \in (0, 1]. \end{array}$

**Proof.** The first condition was proved in [13, Theorem 3] under additional assumption $r + b > 1$. All wavefronts to (1) are monotone. On the other hand, the delayed response may imply the loss of wave’s monotonicity [7]. Therefore it is worthy to emphasize that the inclusion of delay as in (2) does not change the monotone shape of fronts, see [24, Theorem 6].

By [21 Section 8], all wavefronts to (1) are monotone. On the other hand, the delayed response may imply the loss of wave’s monotonicity [7]. Therefore it is worthy to emphasize that the inclusion of delay as in (2) does not change the monotone shape of fronts, see [24, Theorem 6].

**Proposition 4.** If, for some $r, b > 0$, system (2) has a wavefront $(u, v) = (\phi, \theta)(v \cdot x + ct)$, $|v| = 1$, connecting $(0, 1)$ with $(1, 0)$, then $\phi'(t), -\theta'(t) > 0$ and $\theta(t), \phi(t) \in (0, 1)$ for all $t \in \mathbb{R}$.

1.2. General remarks about our approach and some useful relations

The speed of front propagation in (1), (2) can be estimated by means of the truncation method [28], the shooting technique [12, 13] and the upper and lower solutions [16, 17, 27]. Here, we use the latter approach complementing it by a useful idea about how to generate new comparison solutions. The main working tool will be regular super-solutions defined in Section 5.1. Theorem 1 from the mentioned section is instrumental for the proofs of existence: its application with different regular super-solutions yields Theorems 7, 8. The same super-solutions are then used
in the bistable case, see Theorems5, 9. Conceptually, Theorem 4 is very close to highly non-trivial Theorem 1(iv) from 13 (see also 4). The proofs of Theorem 17 and the mentioned Chen and Guo result are, however, completely different.

Asymptotic expansions of the eventual fronts at infinity are another key ingredient of our approach. In combination with a sliding solution argument they lead to

**Theorem 5.** Let \( r \neq 1, b > 0 \). Then for each fixed admissible wave speed \( c \) the monotone wavefront \((u, v) = (\phi, \theta)(v \cdot x + ct), |v| = 1\), connecting equilibria \((0, 1)\) and \((1, 0)\) of system (17) is unique (up to a translation).

We also will need the following relations between the components of wavefront profile:

**Theorem 6.** Consider \( \phi, \theta \) as in Proposition 2 and set \( \psi(t) := 1 - \theta(t) \). We have

**A.** Let \( r \in (0, 1) \), \( K \geq 1 \), \( L \in (0, 1) \) satisfy \( K \geq b/(1 - r) \geq L \). Then

\[
L\phi(t - c\theta) < \psi(t) < K\phi(t), \quad t \in \mathbb{R}.
\]

If \( b + r = 1 \), then \( \phi(t - c\theta) \leq \psi(t) \leq \phi(t) \). Hence, if \( h = 0 \) then \( \phi \equiv \psi \).

**B.** Let \( r \geq 1 \). Then \( \psi(t) > \phi(t) \), \( t \in \mathbb{R} \).

**C.** Suppose that \( r \in (0, 1] \), then \( \psi^2(t) < M\phi(t), t \in \mathbb{R} \), \( M := \max[1, 2b] \).

By part [A], the BZ system with \( h = 0, b + r = 1 \) essentially reduces to the KPP-Fisher equation 7, 21. Part [B] has a clear chemical interpretation: the sum of the (normalized) concentrations of the bromous acid and bromide ion in the propagating wavefront is strictly less than the concentration of the bromide ion far ahead of the wavefront. Part [C] connects 2 with the delayed Zeldovich equation \( u_t(t, x) = \Delta u(t, x) + \beta u^2(t - h, x)(1 - u(t, x)) \), \( \beta = \min[b, 0.5] \). Actually this relation suggested the correct form of asymptotic expansions (5) below (see also 24, Lemma 26 and Corollary 27).

### 1.3. Main results: monostable case

For the non-delayed BZ reaction 21 and \( r \in (0, 1) \), the existence of the minimal speed of front propagation \( c_\ast(1) \) was proved in 26, p. 333. The speed \( c_\ast(1) \), however, is minimal only for the fronts taking values in special domains \( \Pi \) called the balance polyhedrons. Since the BZ system has a continuum of equilibria, none of these domains can cover the whole region admissible for wavefronts, see 26, Fig. 5.1, p. 334]. The existence of the positive minimal speed independent on \( \Pi \) was established in 24, Theorem 7], by means of regular super-solutions. By Theorem 8 below, \( c_\ast = 2\sqrt{1 - r} - rb\exp(-2h(1 - r)) + r \leq 1 \). However, due to Proposition 11 it may happen that \( c_\ast \) is not linearly determined (i.e. \( c_\ast > 2\sqrt{1 - r} \), cf. 5, 8, 10]. Even for the non-delayed BZ system, the exact value of \( c_\ast \) in the case \( rb + r > 1 \) is unknown and represents an interesting open problem. The next theorems show that the use of regular super-solutions in the Wu and Zou approach 27 yields important improvements of the estimations of \( c_\ast \) even for the non-delayed model. Set \( b' := be^{-c\psi(1)} \) and let \( c_\psi = c_\psi(r, b, h) \) be the unique positive root 24 of the equation

\[
c = 2\max \left\{ \sqrt{1 - r}, \frac{\sqrt{b'}}{\sqrt{1 + b'}} \right\} = \begin{cases} 2\sqrt{1 - r}, & \text{if } rb\exp(-2h(1 - r)) + r \leq 1; \\ 2\sqrt{b'/1 + b'}, & \text{if } rb\exp(-2h(1 - r)) + r \geq 1. \end{cases}
\]
Theorem 7. Let \( r \in (0, 1] \), \( c \geq c_k \). Then system (2) has a positive monotone front \((u, v) = (\phi, \theta)(v \cdot x + ct)\), \( |v| = 1 \), connecting \((0, 1)\) with \((1, 0)\) and such that (i) if \( r = 1 \), \( c > c_k \), then
\[
\phi(t) = \frac{2c^2}{t^2} - \frac{8c}{3b}(c^2(1 + h + \frac{1}{b}) - 4) \ln(-t) + O(\frac{1}{t^3}), \quad \theta(t) = 1 + \frac{2c}{t} - \frac{4}{3}(c^2(1 + h + \frac{1}{b}) - 4) \ln(-t) + O(\frac{1}{t^3}), \quad t \to -\infty;
\]
(ii) if \( r \in (0, 1) \), \( c > 2 \sqrt{1 - r} \), then, for some \( \epsilon > 0 \) and \( \lambda := 0.5(c - \sqrt{c^2 - 4(1 - r)}) \), it holds
\[
\phi(t) = e^{br} + O(e^{b(r+\epsilon)t}), \quad \theta(t) = 1 - \frac{be^{(b+\epsilon)t}}{1-r} + O(e^{b(r+\epsilon)t}), \quad t \to -\infty.
\]

Theorem 8. Assume that \( r \in (0, 1] \), \( c \in [2 \sqrt{1 - r}, c_k] \) and
\[
f(c^2, r, b, h) := c^2 \left( \frac{\omega}{8r} + \frac{h}{2} \right) + \ln \frac{c^2}{4br} - \frac{\omega h}{2} - \frac{1 - r}{r} > 0,
\]
where \( \omega_* = 8.21093 \ldots \) denotes the greatest positive root of the equation \( \omega = 4 + 2 \ln \omega \). Then system (2) has a positive monotone front connecting \((0, 1)\) with \((1, 0)\). Asymptotic formulas (6) (when \( c > 2 \sqrt{1 - r} \)) and (5) (when \( r = 1 \)) are fully applicable for this wavefront.

Observe that inequality (7) can be written as \( c > c_\omega = c_\omega(r, b, h) \) where \( c_\omega \) is the unique positive root of the equation \( f(c^2, r, b, h) = 0 \) considered with fixed \( r, b, h \).

1.4. Main results: bistable case

The next assertion can be considered as a dual to Theorems 7, 8. Indeed, it essentially amounts to the non-existence of bistable waves for \( c > c_k(r, b, h) \) and \( c > c_\omega(r, b, h) \):

Theorem 9. Let \( r > 1 \), \( b > 0 \). Then system (2) has at most one (a unique, if \( h = 0 \)) positive monotone wavefront \((u, v) = (\phi, \theta)(v \cdot x + c_* t), \phi > 0, |v| = 1\), connecting \((0, 1)\) with \((1, 0)\). The (unique) velocity of propagation \( c_* \) satisfies the inequality \( c_*(r, b, h) \leq \min(c_\omega(r, b, h), c_\omega(r, b, h)) \). In addition, \( c_\omega(r, b, h) \) is non-increasing in \( h \).

The wave existence problem for the bistable BZ delayed system requires a different approach and it is not considered here. In the non-delayed case, the wavefront existence was established by Kanel in [13, Theorem 4]. In view of Theorem 9, Kanel’s result can be reformulated as

Proposition 10. Let \( r = 0 < r \). Then system (1) has a positive monotone wavefront for the speed \( c_* \) such that \( c_k := b/(2 \sqrt{(r+b) \min(1,b)(r+b) - 0.5b}) \leq c_* < 2 \sqrt{\min(1,b)} \).

| \((r,b)\) | Propositions (9,10) | Theorems (9,10) | Theorems (9,10) | Numerical \( c_* \) | Propositions (9,10) |
|---------|-----------------|----------------|----------------|----------------|-----------------|
| (0.5,3) | \( c \geq 2 \) | \( c > 1.82 \ldots \) | \( c > 1.62 \ldots \) | \( c_* = 1.46 \ldots \) | \( c_* \geq c_\omega = 1.414 \ldots \) |
| (0.5,10) | \( c \geq 2 \) | \( c > 1.90 \ldots \) | \( c > 1.71 \ldots \) | \( c_* = 1.50 \ldots \) | \( c_* \geq c_\omega = 1.444 \ldots \) |
| (1,5) | \( c \geq 2 \) | \( c > 1.82 \ldots \) | \( c > 1.47 \ldots \) | \( c_* = 1.13 \ldots \) | \( c_* \geq c_\omega = 0.289 \ldots \) |
| (5,0.5) | \( c_* \leq 1.41 \ldots \) | \( c_* \leq c_\omega = 1.15 \ldots \) | \( c_* \leq c_\omega = 0.59 \ldots \) | \( c_* = 0.12 \ldots \) | \( c_K = 0.067 \ldots \) |
Example In Table 1, for \( h = 0 \), we compare results of Theorems [7][8][9] with previously known ones (Propositions [2][10]). Notation like \( c > 1.82 \ldots \) means that system (1) has a positive front for each velocity \( c > 1.82 \ldots \). Numerical estimations of the minimal speed \( c_* \) are taken from [19, Table 3] and [23, Table 1]. Lower bounds for \( c_* \) are computed from Propositions [4][10].

Finally, the organization of the paper is as follows. Sections 2, 3 contain the proofs of Theorems 6, 5, 7, 8 and 9, respectively. Asymptotic behavior of profiles at infinity is analyzed in Section 5. Our main technical result (Theorem 17) is proved in Section 5.1.

2. Proof of Theorem 6

Let \((u, v) = (\phi, \theta)(v \cdot x + ct)\) be a wavefront to (2). After introducing \( \psi(t) = 1 - \theta(t - c\tau) \), we obtain the following boundary value problem for the determination of fronts in the BZ system:

\[
\begin{align*}
\phi''(t) - c\phi'(t) + \phi(t)(1 - r - \phi(t) + r\psi(t)) &= 0, \\
\psi''(t) - c\psi'(t) + b\psi(t - ch)(1 - \psi(t)) &= 0, \\
\phi > 0, \psi < 1, \phi(-\infty) &= \phi(-\infty) = 0, \phi(+\infty) = \psi(+\infty) = 1.
\end{align*}
\]

(A) Set \( z(t) := K\phi(t) - \psi(t) \). It is easy to see that

\[
z''(t) - c\phi'(t) + [K(1 - r)\phi(t) - b\psi(t - ch) + K\phi(t)\psi(t - \phi(t)) + b\psi(t)\phi(t - ch)] = 0.
\]

Since \( z(-\infty) = 0, z(+\infty) = K - 1 \geq 0 \), the non-positivity of \( z \) at some points implies the existence of some \( \tau \) such that \( z(\tau) \leq 0, z'(\tau) = 0, z''(\tau) \geq 0 \). But \( z(\tau) \leq 0 \) implies \( K\phi(\tau) \leq \psi(\tau) \) and therefore

\[
0 = z''(\tau) + [K(1 - r)\phi(\tau) + K\phi(\tau)\psi(\tau - \phi(\tau)) + b(\psi(\tau) - 1)\psi(\tau - ch)]
\]

\[
> [K(1 - r)\phi(\tau) + K\phi(\tau)\psi(\tau - \phi(\tau)) + b(\psi(\tau) - 1)\phi(\tau)]
\]

\[
\geq \phi(\tau)[K(1 - r) - b] + K\phi(\tau)(rK - 1 + b) \geq 0,
\]

a contradiction. The latter inequality holds obviously if \( rK - 1 + b \geq 0 \). If \( rK - 1 + b < 0 \), then

\[
K(1 - r) - b + K\phi(\tau)(rK - 1 + b) > K(1 - r) - b + K(rK - 1 + b) = (rK + b)(1 - K) \geq 0.
\]

Next, set \( z(t) := L\phi(t - ch) - \psi(t) \). We have \( z(-\infty) = 0, z(+\infty) = L - 1 \leq 0 \), so that the non-positivity of \( z \) at some points would imply the existence of some \( \tau \) such that \( z(\tau) \leq 0, z'(\tau) = 0, z''(\tau) \leq 0 \). But then \( L\phi(\tau - ch) \geq \psi(\tau) \) and therefore

\[
0 = z''(\tau) + (L(1 - r) - b)\phi(\tau - ch) + L\phi(\tau - ch)\psi(\tau - ch) - \phi(\tau - ch) + b\phi(\tau - ch)\psi(\tau)
\]

\[
< (L(1 - r) - b)\phi(\tau - ch) + L\phi(\tau - ch)\psi(\tau - ch) - \phi(\tau - ch) + b\phi(\tau - ch)\psi(\tau)
\]

\[
\leq L_\ast := (L(1 - r) - b)\phi(\tau - ch) + L\phi(\tau - ch)(rL - 1 + b) \leq 0,
\]

a contradiction. The latter inequality is obvious if \( rL - 1 + b \leq 0 \). If \( rL - 1 + b > 0 \) then

\[
L_\ast \leq \phi(\tau - ch)((L(1 - r) - b) + L(rL - 1 + b)) = \phi(\tau - ch)(rL + b)(L - 1) \leq 0.
\]

(B) Consider \( z(t) := \psi(t) - \phi(t) \). We have that \( z(\pm\infty) = 0 \),

\[
z''(t) - c\psi'(t) + b\phi(t - ch)(1 - \psi(t)) - \phi(t)(1 - r - \phi(t) + r\psi(t)) = 0, \quad t \in \mathbb{R}.
\]
If \( z(s) \leq 0 \) at some \( s \) then there exists \( \tau \) such that \( 0 \geq z(\tau) = \min_{s \in \mathbb{R}} z(t) \). We have that \( z''(\tau) \geq 0 \), 
\[ -\phi(\tau)(1 - r - \phi(\tau)) \geq -\phi(\tau)(1 - r - \phi(\tau)) = \phi(\tau)(r - 1)(1 - \phi(\tau)) \geq 0, \]
contradicting to (9).

(C) Consider \( z(t) = M\phi(t) - \phi^2(t) \). Since \( M \geq 1 \), we have \( z(-\infty) = 0, \ z(+\infty) = M - 1 \geq 0 \).
Thus the non-positivity of \( z \) implies that \( z(\tau) \leq 0, \ z''(\tau) \geq 0 \) for some \( \tau \in \mathbb{R} \). Hence,
\[ M\phi(\tau) \leq \phi^2(\tau), \quad M\phi'(\tau) = 2\phi(\tau)\phi'(\tau), \quad M\phi''(\tau) \geq 2\phi(\tau)\phi'(\tau) + 2(\phi'(\tau))^2, \]
\[ 0 \geq 2\phi(\tau)\phi''(\tau) + 2(\phi'(\tau))^2 - 2c\phi(\tau)\phi'(\tau) + M\phi(\tau)(1 - r + \phi(\tau) - \phi(\tau)) \]
\[ = 2\phi(\tau)\phi''(\tau) - 2c\phi(\tau)\phi'(\tau) + 2b\phi(\tau)(\phi(\tau - ch)(1 - \phi(\tau))), \]
so that
\[ 0 \geq 2(\phi'(\tau))^2 + M\phi(\tau)(1 - r + \phi(\tau) - \phi(\tau)) - 2b\phi(\tau)(\phi(\tau - ch)(1 - \phi(\tau))) \]
\[ \geq \phi(\tau)[M(1 - r) + \phi(\tau)(M - 2b) + \psi^2(\tau)(2b - 1)] \geq 0, \]
a contradiction. Here we observe that the polynomial \( p(z) := M(1 - r) + z(Mr - 2b) + z^2(2b - 1) \), 
\( z := \phi(\tau) \in (0, 1) \), satisfies \( p(0) = M(1 - r) \geq 0, \ p(1) = M - 1 \geq 0 \), so that \( p(\phi(\tau)) \geq 0 \) if 
\( 2b - 1 \leq 0 \). If \( 2b - 1 \leq 0 \) then we choose \( M := 2b > 1 \) to obtain
\[ p(z) = 2b(1 - r) - 2b(z - 1) + z^2(2b - 1) = 2b(1 - r)(1 - z) + z^2(2b - 1) > 0. \]

3. Asymptotics of wavefront profiles

First, we observe that the derivatives \( \phi', \phi'' \) of wavefront components are bounded and uniformly continuous on \( \mathbb{R} \) so that \( \phi^{(i)}(\pm \infty) = \phi^{(i)}(\pm \infty) = 0 \). This fact is well known (cf. [27], Section 2)) and its proof is omitted. Incidentally, the relation \( \phi'(\pm \infty) = 0 \) implies the positivity of each admissible speed (i.e. \( c > 0 \)) it suffices to integrate the second equation of (8) on \( \mathbb{R} \).

Next, assume that \( r \in (0, 1] \). Using Theorem 6.3.11 if \( r = 1 \) and integrating (8) on \((0, t_0], \ t \leq t_0 \), we get, for sufficiently large negative \( t_0 \),
\[ \phi'(t) < \phi'(t) + \int_{t_0}^t \phi(s)(1 - r + \psi(s) - \phi(s))ds = c\phi(t), \ \psi'(t) + b \int_{t_0}^t \phi(s-ch)(1-\psi(s))ds = c\psi(t), \]
and therefore \( z(t) := \phi(t)/\phi(t) < c, \ t \leq t_0, \ \phi \in L_1(\mathbb{R}). \). Furthermore, \( z \) satisfies the equation
\[ z' + z^2 - cz + (1 - r) = f(t), \]
where \( f(t) := \phi(t) - \phi(t) \).

Let \( \lambda = \lambda(c) \leq \mu = \mu(c) \) denote the roots of the characteristic equation \( x^2 - cx + (1 - r) = 0 \).

Lemma 11. Let \((\phi, \psi)\) be a traveling front of (8) and \( r \in (0, 1) \). Then (a) \( c \geq 2 \sqrt{1-r} \), (b) there exists finite limit \( \lim_{t \to -\infty} \phi'(t)/\phi(t) = [\lambda, \mu] \) as \( t \to -\infty \).

proof. Recall that \( f(t) \to 0 \) as \( t \to -\infty \). (a) Suppose that \( c < 2 \sqrt{1-r} \). Then, for some \( t_h \leq t_0 \), it holds \( f(t) - z^2 + cz - (1 - r) < 0 \) for \((t, z) \in (-\infty, t_h) \times [0, 2c]. \). However, as a simple analysis of the direction field for equation (10) shows, this contradicts to the property \( z(t) \in (0, c), \ t \leq t_h \).

(b1) Let \( c = 2 \sqrt{1-r} \) and take some small \( e > 0 \). By analyzing the direction field again, we can see that there exists \( t_e \) such that \((t, z(t)) \in (-\infty, t_e) \times (-e + c/2, c/2 + e) \) for \( t \leq t_e \). Hence,
\[ z(t) \to c/2 = \lambda = \mu \text{ as } t \to -\infty. \] (b2) The situation when \( c > 2 \sqrt{1-r} \) is similar to (b1).
Corollary 12. Let \( r \in (0, 1) \). Then there are \( t_1, m \in [0, 1] \) and \( r(c) \in [\lambda(c), \mu(c)] \) such that \((\psi(t + t_1), \phi(t + t_1), \psi'(t + t_1)) = (-) \alpha \psi^{(r(c))(1/2)}(1 - r) \), \( r(s) \in (1 + o(1)), \ t \to -\infty \).

Proof. By Lemma 11, \( \phi(t), \phi'(t) \) decay exponentially at \(-\infty\). Then \( \psi(t) \) has the same property due to Theorem 6. Therefore we can apply Proposition 7.2 from 18 together with Theorem 6A to system 5 in order to obtain the above asymptotic formulas for \( \phi, \phi', \psi \). Note that \( m = 1 \) only when \( c = 2 \sqrt{1 - r} \).

Lemma 13. Let \( \phi, \psi \) be a wavefront for 5 and \( r > 1 \). Then, for some \( A > 0, t_2 \in \mathbb{R} \) and small \( \sigma > 0 \), it holds \( \phi(t + t_2) = e^{\xi(t)} + O(e^{2c + \sigma |y|}), \psi(t + t_2) = Ae^\xi + O(e^{(\mu(c)+\sigma |y|)}, \ t \to -\infty \).

Proof. Integrating the first equation of 5 from \(-\infty \) to \( t \), and using the inequality \( 1 - r - \phi(t) + \rho(t) < 0 \) for all large negative \( t \) (say, for \( t \leq T \) where, simplifying, we can take \( T = 0 \), we obtain that \( \phi'(t) - c \phi(t) > 0 \) for \( t \leq 0 \). Thus \( \phi(t) < \phi(0)e^\rho, \ t \leq 0 \). Similarly, from the second equation of 5, we deduce \( \psi(t) > \psi(0)e^\xi, \ t \leq 0 \). The latter equation can be written as \( \psi'(t) - c \psi(t) = F(t), \) where \( F(t) := 2b(t)\phi(t) - \phi(t) = O(e^\xi), \psi(t), \psi'(t) = O(1), \ t \to -\infty \). But then Proposition 7.1 guarantees that \( \psi(t), \psi'(t) = O(e^{\xi - \sigma |y|}), \ t \to -\infty \), for each small \( \sigma > 0 \). Now, writing the first equation of 5 as \( \phi''(t) - c \phi'(t) + (1 - r) \phi(t) = G(t), \) where \( G(t) = O(e^{2c + \sigma |y|}), \ t \to -\infty \), we find analogously that \( \phi(t) = B e^{\xi(t)} + O(e^{2c + \sigma |y|}), \ t \to -\infty, \) where \( \sigma > 0, B \geq 0 \). To prove that \( B > 0 \), it suffices to repeat the proof of Lemma 11 (note that \( \psi(t) \) is bounded on \( \mathbb{R} \), because otherwise it blows up in a finite time). Hence, \( F(t) = O(e^{\mu(c)\xi}), \ t \to -\infty, \ \psi(t) > O(\phi(t)), \ t \leq 0, \mu(c) > c \). By Proposition 7.1, this yields the required asymptotic formula for \( \phi \).

Next, we consider the case when \( t \to +\infty \). In order to linearize system 5 along the positive steady state \((1, 1)\), we use the change of variables \( \phi(t) = 1 - \xi(t), \psi(t) = 1 - \theta(t - ch) \), which leads to

\[
\begin{align*}
\xi''(t) - c \xi'(t) - \xi(t)(1 - \xi(t) + r \theta(t - ch)) + r \theta(t - ch) &= 0, \\
\theta''(t) - c \theta'(t) - b \theta(t)(1 - \xi(t)) &= 0.
\end{align*}
\]

(11)

The characteristic equation \((c^2 - cz - 1)(c^2 - cz - b) = 0\) for this system at the zero equilibrium has two positive \((\tilde{\xi}_2, \tilde{\xi}_3 = 0.5(c + \sqrt{c^2 + 4b}))\) and two negative eigenvalues \((\tilde{\xi}_1 = 0.5(c - \sqrt{c^2 + 4b}))\), respectively.

Lemma 14. Let \( r > 0 \). Then for some appropriate \( A \geq 0, t_0, d, d_1 \) and small \( \sigma > 0 \), we have that \((\phi(t + t_0), \phi'(t + t_0)) = -A e^{\xi(t)}(1, \tilde{\xi}_1) + \)

\[
\begin{align*}
&\left\{(1 - r e^{(t - ch)/(b - 1)}, -r \xi^1 e^{(t - ch)/(b - 1)} + O(e^{(\mu(c)+\sigma |y|)}), \ b \neq 1, \\
&\left(1 - r(e + d_1)e^{(t - ch)/(c - 2 \xi_1)}, -r \xi_1(t + d_1)e^{(t - ch)/(c - 2 \xi_1)} + O(e^{(\mu(c)+\sigma |y|)}), \ b = 1, \\
&\right.\right.
\end{align*}
\]

\[
(\phi(t + t_0), \psi(t + t_0)) = (1 - e^{(t - ch)}, -e^{(t - ch)} + O(e^{\xi(t - \sigma |y|)}), \ t \to +\infty.
\]

Proof. Since \( \theta(\infty) = \xi(\infty) = 0 \) and the linear system \( y''(t) - cy'(t) - y(t) + rz(t - ch) = 0, \)

\(y''(t) - cy'(t) - y(t) - rz(t - ch) = 0\) possesses an exponentially dichotomy on \( \mathbb{R}^+ \), the perturbed system \( y''(t) - cy'(t) - y(t)(1 - \xi(t) + r \theta(t - ch)) + r \theta(t - ch) = 0, \)

\(z''(t) - cz'(t) - bz(t)(1 - \xi(t)) = 0\) is also exponentially dichotomic on \( \mathbb{R}^+ \). As a consequence, we obtain that \( \theta(t), \xi'(t), \xi(t) = O(e^\theta), \ t \to +\infty, \) for some negative \( l \). Moreover, by applying the Levinson asymptotic integration theorem 6 to the second equation of 11, we find (cf. 17 Lemma 19) that, for some \( t_0 \),

\[
(\theta(t + t_0), \xi'(t + t_0)) = (c^{1/2}(1 + o(1)), -e^{\xi(t)(1 + o(1))), \ t \to +\infty.
\]
Then [18, Proposition 7.2] applied to the second equation of (11) yields the required estimation

\[ (θ(t + t₀), θ'(t + t₀)) = (e^{ıt}, 0) + O(e^{t(1−c)/2}) , \quad t \to +∞ . \]  

(12)

Let simplify (12) by assuming \( t₀ = 0 \). If \( b ≠ 1 \) then \( ζ₁ ≠ ζ₂ \) and \( y = ξ(t) + re^{ıt(1−c)/2} / (b+1) = O(e^b) \) satisfies

\[ y''(t) − cy'(t) − y(t)(1 + m(t)) = O(e^{ıt(1−c)}) , \quad t \to +∞ , \]  

(13)

where \( m(t) = O(e^c) \). Applying again Proposition 7.2 from [18], we conclude that if \( ζ₁ > ζ₂ \) (equivalently, \( b ∈ (0, 1) \)) then \( y(t), y'(t) = O(e^{ıt(1−c)}) \) with \( c’ ∈ (0, c) \) so that

\[ (ξ(t), ξ'(t)) = \frac{re^{ıt(1−c)}}{1−b}(1, ζ₁) + O(e^{ıt(1−c)}) , \quad t \to +∞ . \]

When \( ζ₁ < ζ₂ \) (that is \( b > 1 \)), we find similarly that, for some \( A > 0 \) and \( t \to +∞ , \)

\[ 0 < ξ(t) = Ae^{ıt} + re^{ıt(1−c)/2} / (b+1) + O(e^{ıt(1−c)}), \quad ξ'(t) = Aζ₁e^{ıt} + rζ₁e^{ıt(1−c)/2} / (b+1) + O(e^{ıt(1−c)}). \]

Finally, if \( b = 1 \) then \( ζ₁ = ζ₂ \) and therefore

\[ y = ξ(t) + \frac{re^{ıt(1−c)}}{2ζ₁} = O(e^b) \]

satisfies (13). As a consequence, we obtain (once more invoking [18, Proposition 7.2]) that, for some real \( d, d₂ \), it holds \((ξ, ξ')\) = \( r(t + d, ζ₁t + d₂)e^{ıt(1−c)/2} / (c - 2ζ₁) + O(e^{ıt(1−c)}) \), \( t \to +∞ . \)

4. Proof of Theorem 5

The proof is based on the Berestycki-Nirenberg sliding solution argument. Let \((φ₁, ψ₁), (φ₂, ψ₂)\) be two different (modulo translation) traveling fronts of (8) considered with \( h = 0 \). By Lemma 14 without restricting the generality, we may assume that \( φ₁ \) and \( ψ₂ \) have the same first terms of their asymptotic expansions at \(+∞\). In addition, due to Lemma 13 (employed when \( r > 1 \)) and Corollary 2 (for \( r < 1 \)), we can index \( φ₁ \) in such a way that either \( φ₂(t) > ψ₂(t) \) on some infinite interval \((−∞, T] \) or \( φ₁, ψ₂ \) also have the same first asymptotic exponential terms at \( −∞ \) (recall that \( r ≠ 1 \)). In each case, the closed set \( S = \{ s : ψ₁(t + s) ≥ ψ₂(t), t ∈ \mathbb{R} \} \) is non-empty and contains finite \( s_* := \inf S \). Similarly, there exists the leftmost \( t_* \) such that \( ψ₁(t_*+s*) ≥ ψ₂(t_*), t ∈ \mathbb{R} \).

Let us show that actually \( s_* = 0 \). Indeed, if \( s_* > 0 \) then, due to the chosen asymptotic behavior of \( ψ₂ \) at ±∞, we find that, for each \( s ∈ (0, s_*), \) it holds \( ψ₁(t + s) − c > ψ₂(t) \) for all \( t ∈ \mathbb{R} \) except for some compact interval. This implies the existence of finite \( i \) such that \( δ(i) = 0, δ''(i) ≥ 0, \)

\( δ(i) := ψ₁(t + s) − ψ₂(t) ≥ 0. \) If we suppose additionally that \( s_* ≥ t_* \) then \( ψ₁(i + t) − ψ₂(i) > 0, t ∈ \mathbb{R}. \) (Note that \( ψ₁(i + t) − ψ₂(i) = 0 \) implies that \( s_* = t_* \) and \( φ₁(i + s) − φ₂(i) = 0. \) Since also \( δ(i) = 0, δ''(i) = 0, \) the solution uniqueness theorem for (8) assures that \( (φ₁, ψ₁)(t + s) = (φ₂, ψ₂)(i) \). But then we get from (8) the following contradiction:

\[ 0 = δ''(i) − cδ(i) + bφ₂(i + s) − ψ₂(i)(1 - ψ₂(i)) > 0. \]  

(14)

Hence, we have to consider the case when \( t*-s* > 0 \) and \( φ₁(i + t) − ψ₂(i) ≤ 0. \) Note that \( δ(±∞) = 0, δ(i) ≥ 0, \) and therefore \( δ(t) \) has at least two local maxima at some \( t_j: t_j < i < t_j. \) Since \( δ''(t_j) ≤ 0, δ'(t_j) = 0, \) estimations similar to (14) shows that \( φ₁(t_j + s) − ψ₂(t_j) ≥ 0, j = 1, 2. \)
Next, set $S_\alpha(t) := \phi_1(t + s_\alpha + a) - \phi_2(t)$. Functions $S_\alpha(t)$ are increasing in $a$ and strictly positive on $[t_1, t_2]$ for all large $a > 0$. On the other hand, $S_\alpha(t)$ has at least one zero on $(t_1, t_2)$. This means that for some $a_\ast \geq 0$ and $t_\ast \in (t_1, t_2)$ function $S_\ast(t) := S_\ast(t)$ reaches at $t_\ast$ its zero global minimum on $[t_1, t_2]$. Therefore $S_\ast''(t_\ast) \geq 0, S_\ast'(t_\ast) = 0, S_\ast(t_\ast) = 0$, so that, due to (3),

$$0 = S_\ast''(t_\ast) - cS_\ast'(t_\ast) + r\phi_2(t_\ast)(\psi_1(t_\ast + s_\ast + a_\ast) - \psi_2(t_\ast)) \geq 0.$$  

(15)

This shows that $a_\ast = 0$ and that

$$\psi_1'(t + s_\ast) - \psi_2'(t) = \psi_1(t + s_\ast) - \psi_2(t) = \psi_1'(t + s_\ast) - \psi_2'(t) = \phi_1(t + s_\ast) - \phi_2(t) = 0.$$  

But then, by the uniqueness theorem for (5), $(\phi_1, \psi_1)(t + s_\ast) \equiv (\phi_2, \psi_2)(t), t \in \mathbb{R}$ contradicting to our choice of $(\phi_\ast, \psi_\ast)$. Therefore we conclude that $s_\ast = 0$ and $\delta(t) > 0, t \in \mathbb{R}$. In the remainder of the proof we will analyze three possible mutual positions of $t_\ast$ and $0$.

**Case A:** $t_\ast < 0$. Recall that $\psi_1(t) > \psi_2(t), \phi_1(t + t_\ast) \geq \phi_2(t)$. Due to the coincidence of the principal terms of asymptotic representations for $\psi_1, \psi_2$ at $+\infty$, we see that, for every small $\delta \in (0, |t_\ast|)$ the graphs of functions $\psi_1(t - \delta)$ and $\psi_2(t)$ have at least one intersection on some interval $(T, +\infty)$. In fact, we may assume that $\psi(t - \delta) > \psi_2(T)$ and $\psi_1(t - \delta) < \psi_2(t), t \in [T_1, +\infty)$, for some $T_1 > T$. It is clear also that $\phi_1(t - \delta) > \phi_2(t)$ for all $t \in \mathbb{R}$. Next, we consider the family of functions $\phi_1(t - \delta) + a$ and the following non-empty and closed set

$$\mathfrak{A} := \{a \geq 0 : \phi_1(t - \delta) + a \geq \psi_2(t), t \in [T, +\infty)\}.$$  

Set $a_\ast = \inf \mathfrak{A}$, it is evident that $a_\ast > 0$ and that $w(t) := \psi_1(t - \delta) + a_\ast - \psi_2(t)$ has at least one zero $t_\rho \in (T, +\infty)$, where, in addition, $w'(t_\rho) = 0, w''(t_\rho) \geq 0$. But then, due to equations (8),

$$0 = w''(t_\rho) - cw'(t_\rho) + b\left(a_\ast \phi_2(t_\rho) + [\phi_1(t_\rho - \delta) - \phi_2(t_\rho)](1 - \psi_1(t_\rho - \delta))\right) > 0,$$

a contradiction proving that $t_\ast \geq 0$. In fact, we have established a stronger result: for every $\delta > 0$, the inequality $\phi_1(t - \delta) > \phi_2(t)$ does not hold on any infinite interval $[T, +\infty)$.

As a consequence, there exists a minimal $\rho \in (0, t_\ast]$ such that $\phi_1(t + \rho) \geq \phi_2(t)$ for all $t \in [T, +\infty)$. That is, for every small $\delta > 0$, equation $\phi_1(t + \rho - \delta) = \phi_2(t)$ has at least one root on $(T, +\infty)$ (otherwise, $\phi_1(t + \rho - \delta) < \phi_2(t), t > T$, for some $\delta \to 0$ and therefore $\phi_1(t + \rho) \leq \phi_2(t), t \geq T$, implying a contradiction: $\phi_1(t + \rho) \equiv \phi_2(t), \psi_1(t + \rho) > \psi_2(t), t \geq T$).

**Case B:** $t_\ast = 0$, so that $\psi_1(t) > \psi_2(t), \phi_1(t) \geq \phi_2(t), t \in \mathbb{R}$, and, for each $\delta > 0$, the inequalities $\phi_1(t - \delta) > \phi_2(t), \psi_1(t - \delta) > \psi_2(t)$ do not hold on any interval $[T, +\infty)$. Now, it is easy to see that, in fact, $S_\ast(t) := \phi_1(t) - \phi_2(t) > 0, t \in \mathbb{R}$. Indeed, otherwise $S_\ast(t) = 0$ for some $t_\ast$ and thus we get a contradiction as in (15), where $s_\ast = a_\ast = 0$ should be taken. Hence, for a fixed $T$ and small $\delta > 0$ in such a way that

$$\phi_2(T) - 2\alpha(t - \psi_1(T - \delta)) > 0,
\psi_1(T - \delta) > \psi_2(T), \phi_1(T - \delta) - \phi_2(T) > 2/3.$$  

(16)

In the next stage of the proof, we apply the sliding solution argument to the families $\epsilon + \psi_1(t - \delta)$ and $2\epsilon + \phi_1(t - \delta)$. It is clear that the sets

$$\mathcal{E}_1 := \{\epsilon \geq 0 : \epsilon + \psi_1(t - \delta) \geq \psi_2(t), t \in [T, +\infty)\},$$

$$\mathcal{E}_2 := \{\epsilon \geq 0 : 2\epsilon + \psi_1(t - \delta) \geq \phi_2(t), t \in [T, +\infty)\}$$
are closed and non-empty, and that \( e_j = \inf \mathcal{E}_j \) are positive. Suppose first that \( e_1 \geq e_2 \). The difference \( \gamma(t) := e_1 + \psi_1(t) - \psi_2(t) \) reaches its global minimum at some point \( t_m > T \) where \( \gamma(t_m) = \gamma'(t_m) = 0 \) and \( \gamma''(t_m) \geq 0 \). We also have that

\[
2e_1r + \phi_1(t_m - \delta) \geq 2e_2r + \phi_1(t_m - \delta) \geq \phi_2(t_m).
\]

Therefore, using (8) again, we find that

\[
0 = \gamma''(t_m) - c\gamma'(t_m) + b[e_1\phi_2(t_m) + (\phi_1(t_m - \delta) - \phi_2(t_m))(1 - \psi_1(t_m - \delta))] \geq
\]

\[
be_1 [\phi_2(t_m) - 2r(1 - \psi_1(t_m - \delta))] > be_1 [\phi_2(T) - 2r(1 - \psi_1(T - \delta))] \geq 0,
\]

a contradiction. So \( e_1 < e_2 \) and the difference \( \alpha(t) := 2e_2r + \phi_1(t - \delta) - \phi_2(t) \) reaches its global minimum at some point \( t_n > T \) where \( \alpha(t_n) = \alpha'(t_n) = 0 \) and \( \alpha''(t_n) \geq 0 \). We also have that

\[
e_2 + \phi_1(t_n - \delta) > e_1 + \phi_1(t_n - \delta) \geq \phi_2(t_n).
\]

But then, after invoking (8), we get a contradiction:

\[
0 = \alpha''(t_n) - cc\alpha'(t_n) + r\phi_1(t_n - \delta)[2e_2 + \phi_1(t_n - \delta) - \phi_2(t_n)] +
\]

\[
2e_2r(1 + \phi_1(t_n - \delta) + 2e_2r - r\phi_2(t_n)) > 2e_2r(1 - \phi_2(t_n)) - 1 + 1.5\phi_1(t_n - \delta) + 2e_2r > 0.
\]

**Case C:** \( t > 0 \). For a fixed large \( T > 0 \), we consider \( \phi_1(t + \rho) \) where \( \rho \in [0, t] \) was defined in the last lines of subsection ‘Case A’. Then \( \psi_1(t + \rho) > \psi_2(t), t \in \mathbb{R} \), and, for each small \( \delta > 0 \), the equation \( \phi_1(t + \rho - \delta) = \phi_2(t) \) has at least one root on \( (T, +\infty) \). From this point we can follow the proof given in Case B (beginning from (16)). Actually, it will be literally the same proof if \( \rho = 0 \). If \( \rho > 0 \) we have to replace, starting from (16), \( \psi_1(t - \delta), \psi_1(t - \delta) \) with \( \phi_1(t + \rho - \delta), \psi_1(t + \rho - \delta) \). Note also that \( e_1 = 0, e_2 > 0 \) if \( \delta \in (0, \rho) \).

5. Regular super-solutions and proof of Theorems 7, 8

Assume that \( r > 0 \) and \( c^2 > 4(1 - r) \). Recall that \( \lambda = \lambda(c) < \mu = \mu(c) \) denote the real roots of the characteristic equation \( \chi(z, c) := z^2 - cz + (1 - r) = 0 \). Fix some positive \( v \in (\lambda, \mu) \). If \( r \in (0, 1) \) then we define \( k \) as the maximal positive integer such that \( k\lambda \leq v \) and \( (k + 1)\lambda > v \). Obviously, if \( k > 1 \) then we have \( \chi(j\lambda, c) \leq 0 \) for all \( j = 2, \ldots, k \).

5.1. Regular super-solutions and a preparatory theorem

To prove the existence of monostable fronts, we will use Wu and Zou version [27] of the upper and lower solutions method. Below, we propose a trick which increases the effectiveness of this approach for the BZ system. We will show that it suffices to find only two solutions (instead of four ones which must agree amongst themselves) of a system of differential inequalities.

**Definition 15.** Assume that continuous and piece-wise \( C^1 \)-smooth functions \( \psi_+, \phi_+ \) are positive and have positive derivatives in some neighborhoods \( O_1, O_2 \) of the sets \( (-\infty, t_1], (-\infty, t_2] \), respectively. We admit here that \( \psi'_+, \phi'_+ \) has a finite set \( D = \{d_1 < d_2 < \ldots < d_M, d_M < \min(t_1, t_2) \} \) of the discontinuity points and one-sided derivatives of \( \psi_+, \phi_+ \) satisfy \( \psi'_+(d_i^-) > \psi'_+(d_i^+) \), \( \phi'_+(d_i^-) > \phi'_+(d_i^+) \). Suppose also that \( \psi_+(\infty) = \phi_+(\infty) = 0, \psi_+(t_1) = \phi_+(t_2) = 1, \) and that \( \psi_+, \phi_+ \) are \( C^2 \)-smooth in some vicinities of \( t_1, t_2 \) and that
Lemma 19. For a fixed positive \( \nu \in (\lambda, \mu) \), \( m \in (0, 1) \), and some positive constants \( C_1, \epsilon \), it holds

\[
\psi_\epsilon(t) = O(t^\nu), \quad (\phi_\epsilon(t), \phi_\epsilon'(t), \phi_\epsilon''(t)) = C_1(-t)^m e^{\nu t}(1, 1, 1 + o(1)), \quad t \to -\infty.
\]

D2. If \( t \leq \min[t_1, t_2] \), \( t \not\in D \), then

\[
\begin{align*}
&\text{D1.} \quad \text{If } t_1 < t_2 \text{ then } \psi_\epsilon''(t) = c_\epsilon \phi_\epsilon'(t) + \phi_\epsilon(t)(1 - r - \phi_\epsilon(t) + r \psi_\epsilon(t)) \leq 0, \\
&\text{D2.} \quad \text{If } t_1 < t_2 \text{ then } \psi_\epsilon''(t) = c_\epsilon \phi_\epsilon'(t) + b \phi_\epsilon(t)(1 - \psi_\epsilon(t)) < 0.
\end{align*}
\]

\[
(17)
\]

D3. If \( t_1 < t_2 \) then \( \psi_\epsilon''(t) = c_\epsilon \phi_\epsilon'(t) + \phi_\epsilon(t)(1 - \phi_\epsilon(t)) < 0 \), \( t \in [t_1, t_2] \).

D4. If \( t_1 > t_2 \) then \( \psi_\epsilon''(t) = c_\epsilon \phi_\epsilon'(t) + b \min[1, \phi_\epsilon(t) - c h](1 - \psi_\epsilon(t)) < 0 \), \( t \in [t_2, t_1] \).

We will call such \( (\psi_\epsilon, \phi_\epsilon) \) a regular super-solution for \( \mathcal{B} \). Observe that we may suppose that \( \phi_\epsilon \) is defined, strictly increasing and smooth on \( [t_2, +\infty) \), this fact is implicitly used in D4.

Remark 16. Suppose that \( \psi_\epsilon, \phi_\epsilon \) are increasing and that inequalities \( \mathcal{D}_3, \mathcal{D}_4 \) hold for all \( t \leq \max[t_1, t_2] \). Then conditions \( \mathcal{D}_3, \mathcal{D}_4 \) are satisfied automatically. Indeed, in case \( \mathcal{D}_3 \), we have

\[
(18)
\]

\[
\psi_\epsilon''(t) = c_\epsilon \phi_\epsilon'(t) + b \min[1, \phi_\epsilon(t) - c h](1 - \psi_\epsilon(t)) \leq \Lambda_2(\phi_\epsilon, \psi_\epsilon(t)) < 0, \quad t \in [t_2, t_1].
\]

Note that the upper solutions for the BZ system proposed in earlier works (e.g. see \( \mathcal{D}_1, \mathcal{D}_2 \)) have ‘correct’ behavior at \(-\infty\) and therefore do not satisfy condition \( \mathcal{D}_1 \). ‘Correct’ here means ‘asymptotically similar to the true wavefront’ (i.e. satisfying \( \mathcal{D}_5, \mathcal{D}_6 \)).

Theorem 17. Suppose that for given parameters \( b, c > 2 \sqrt{1 - r}, \ r \in (0, 1), \ h \geq 0 \), system \( \mathcal{B} \) has a regular super-solution \( (\psi_\epsilon, \phi_\epsilon) \). Then there exists a monotone wavefront for \( \mathcal{B} \) moving at the velocity \( c \) and satisfying \( \mathcal{D}_5, \mathcal{D}_6 \).

To prove Theorem 17 we will need several auxiliary statements. The first of them can be viewed as a variant of the Perron theorem for piece-wise continuous solutions, cf. \( \mathcal{D}_5, \mathcal{D}_7 \).

Lemma 18. Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a bounded classical solution of the impulsive equation

\[
\psi'' + A \psi' + B \psi = f(t), \quad \Delta \psi|_{t_i} = \alpha_i, \quad \Delta \psi'|_{t_j} = \beta_j,
\]

where \( \{t_j\} \) is a finite increasing sequence, \( f : \mathbb{R} \to \mathbb{R} \) is bounded and continuous at every \( t \neq t_j \) and \( \Delta \psi|_{t_i} := \psi(t_i+) - \psi(t_i-) \). Assume that \( \xi_1 < 0 < \xi_2 \) are real roots of \( 2 \lambda + \alpha z + B = 0 \). Then

\[
\psi(t) = \frac{1}{\xi_1 - \xi_2} \left( \int_{t}^{\xi_1} e^{\xi_1(t-s)} f(s) ds + \int_{t}^{\xi_2} e^{\xi_2(t-s)} f(s) ds \right) + \frac{1}{\xi_2 - \xi_1} \left[ \sum_{i \neq j} e^{\xi_j(t-t_j)(\xi_j \alpha_j - \beta_j)} + \sum_{i = t_j} e^{\xi_j(t-t_j)(\xi_j \alpha_j - \beta_j)} \right], \quad t \neq t_j.
\]

\[
(19)
\]

Proof. It is straightforward to check that \( \psi \) defined by \( (19) \) verifies equation \( (18) \). \( \square \)

Lemma 19. For \( r \in (0, 1) \), set \( \alpha_1 := 1, \ b_1 := be^{-\lambda h}(1 - r) \). There are functions

\[
\phi_A(t) := A(a_1 e^{\lambda t} + a_2 e^{2\lambda t} + \ldots + a_k e^{k\lambda t}),
\]

\[
\psi_A(t) := A(b_1 e^{\lambda t} + b_2 e^{2\lambda t} + \ldots + b_k e^{k\lambda t}),
\]

and a polynomial \( P(x, y) \) such that, for all \( t \in \mathbb{R} \),

\[
\begin{align*}
\phi_A''(t) - c \phi_A'(t) + \phi_A(t)(1 - r - \phi_A(t) + r \psi_A(t)) &= A^2 \psi_A(t) = A^2 \psi_A(\epsilon)(1 - r - \psi_A(\epsilon) + r \phi_A(t)), \\
\psi_A''(t) - c \psi_A'(t) + b \phi_A(t - c \epsilon) &= 0.
\end{align*}
\]

\[
(20)
\]
Proof. Indeed, for a suitable polynomial $P(x, y)$, we have that

$$\phi''_1(t) - c\phi'_1(t) + \phi_A(t)(1 - r - \phi_A(t) + r\psi_A(t))$$

$$= A \sum_{j=1}^k (\chi(j, \lambda, c)a_j - \sum_{p=0}^j A\phi(a_p - rb_j))e^{i\lambda t} + A^2 P(e^{i\lambda t}, A)e^{i(j+1)t},$$

and

$$\psi''_1(t) - c\psi'_1(t) + b\phi_A(t - ch) = A \sum_{j=1}^k [\chi(j, \lambda, c)b_j + (bae^{-j\lambda ch})]e^{i\lambda t}. $$

In order to obtain (20), we define recursively $(j = 2, \ldots, k)$

$$a_1 = 1, b_1 = b \frac{e^{-\lambda ch}}{1 - r}, a_j = A \sum_{p=0}^j a_p (a_q - rb_b) \chi(j, \lambda, c), b_j = b \frac{ae^{-j\lambda ch}}{1 - r - \lambda (j, \lambda, c)}. $$

□

**Remark 20.** It is easy to see that, for some rational functions $a_j(b, r, \lambda, c, \tau)$ and $b_j(b, r, \lambda, c, \tau)$, it holds

$$a_j = A^{j-1} a_j(b, r, \lambda, c, e^{-\lambda ch}), b_j = A^{j-1} b_j(b, r, \lambda, c, e^{-\lambda ch}).$$

Therefore $A^{-1}(\phi_1(t), \psi_1(t)) = (1, be^{-\lambda ch}(1 - r^{-1})e^{i\lambda t} + A\Omega(e^{i\lambda t}), t \to -\infty$. This implies that for every $\tau_0 \in \mathbb{R}$ there exists $A_0 > 0$ such that $\phi_A, \psi_A, \phi'_A, \psi'_A$ are positive for all $t \leq \tau_0$, $A \in (0, A_0]$. In addition, the derivatives of $\phi_A, \psi_A$ have the property $\lim_{A \to 0} (\phi'_A(t), \psi'_A(t)) = (0, 0)$, $k = 0, 1, 2$, uniformly on $(-\infty, \tau_0]$.

Next, in order to get an analog of $\phi_A, \psi_A$ when $r = 1$, we consider the functions

$$\phi(t) := \frac{2c^2/b t^2 + A\ln(-t)}{t^3} + \frac{T \ln^2(-t)}{t^4},$$

$$\psi(t) := \frac{2c}{t} + \frac{C \ln(-t)}{t^2} + \frac{F \ln^2(-t)}{t^3}, \quad t < -e,$$

which coefficients $A, C, F$ depend only on $c, b, h$ and are defined explicitly by

$$A := -\frac{8c}{3b}(c^2(1 + h + \frac{1}{b} - 4), C := \frac{4}{3}(c^2(1 + h + \frac{1}{b} - 4), \text{ (so that } bA + 2cC = 0);$$

$$F := \frac{2}{3}(c^2(1 - 2h - 1) = \frac{2c^2}{b} - 6 + \frac{b}{2c}A = 2(1 - c^2 - c^2h) + \frac{1}{2}C).$$

Since

$$\frac{1}{(t - ch)^m} = \frac{1}{t^m} + \frac{mch}{tm + 1} + O\left(\frac{1}{tm + 2}\right), \quad \frac{\ln^k(ch - t)}{(t - ch)^m} = \frac{\ln^k(-t)}{t^m} + \frac{mch}{t} + O\left(\frac{1}{tm + 1}\right)$$

at $t = -\infty$, we find that

$$\phi(t) = \frac{2c^2/b t^2 + A\ln(-t)}{t^3} + \frac{4c^3h/b t^2 + T \ln^2(-t)}{t^4} + O\left(\frac{1}{t}\right) + O\left(\frac{\ln(-t)}{t^3}\right).$$

Then a straightforward computation shows that

$$R_2(t) := \phi''_1(t) - c\phi'_1(t) + \phi'_2(t)(\psi(t) - \phi_1(t)) = r_{11} \frac{\ln^2(-t)}{t^2} + r_{12} \frac{\ln(-t)}{t^3} + O(1)$$

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where
\[ r_{11} := 2cT + 2c^2Q/b + A \alpha, \quad r_{12} := 12A + 2AF - 4A\alpha^2/b - 2cT; \]

and
\[ R_2(t) := \psi'_{Q}(t) - c\psi'_{Q}(t) + b\phi_{T}(t - ch)(1 - \psi_{Q}(t)) = r_{21} \frac{\ln^2(-t)}{t^4} + r_{22} \frac{\ln(-t)}{t^4} + O(1), \]

with \( r_{21} := bT + 3cQ, \quad r_{22} := 6C(1 - c^2) + 3bch\alpha - 2cQ. \)

**Lemma 21.** There exist \( T, Q \) and \( \sigma = \sigma(T, Q, c, b, h) > e \) such that \( \phi_{T}(t) > 0, \psi_{Q}(t) > 0 \) and \( R_{1}(t) < 0, R_{2}(t) < 0 \) for all \( t \leq -\sigma. \)

**Proof.** Take \( T, Q \) such that \( r_{11} > 0 \) and \( r_{21} < 0. \) Then it is easy to see that there is \( \sigma = \sigma(T, Q, c, b, h) > e \) such that \( t^{2}\phi_{T}(t) > 0, \quad t^{2}R_{1}(t) > 0 \) and \( \psi_{Q}(t) < 0, \quad t^{2}R_{2}(t) < 0 \) for \( t < -\sigma. \) \( \square \)

**Lemma 22.** For every \( \epsilon > 0 \) there are \( T_{n}, Q_{n} \) sufficiently large in absolute value and \( \sigma_{n} = \sigma(T_{n}, Q_{n}, c, b, h) < -e, \) such that the above defined functions \( \phi_{T_{n}}, \psi_{Q_{n}} \)

I. are positive and strictly increasing on the interval \(( -\infty, \sigma_{n}) \);

II. are strictly decreasing on the interval \(( \sigma_{n}, \sigma_{n} + ch) \);

III. \( \phi'_{T_{n}}(\sigma_{n}) = \psi'_{Q_{n}}(\sigma_{n}) = 0 \) and \( \phi_{T_{n}}(\sigma_{n}) + \psi_{Q_{n}}(\sigma_{n}) < \epsilon; \)

IV. \( R_{1}(t) > 0 \) and \( R_{2}(t) > 0 \) for all \( t \leq \sigma_{n}. \)

**Proof.** Take \( \kappa_{n} \in [0.34, 0.98] \subset (1/3, 1) \) and consider the sequences \( T_{n} \to -\infty, \quad Q_{n} = -\kappa_{n}bT_{n}/c \to +\infty. \) It is easy to see that \( r_{11} < 0 \) and \( r_{21} < 0 \) for all sufficiently large \( n. \)

Now, it is clear that, for a given fixed interval \([\epsilon, -e] \), we have that \( \phi_{T}(t) < 0, \quad t \in [\epsilon, -e] \) for all sufficiently large negative \( T. \) On the other hand, for each \( T, \) function \( \phi_{T} \) is positive and strictly increasing on some interval \(( -\infty, v) \). These simple observations show that to every positive \( \epsilon \) we can indicate \( T_{0} < 0 \) such that, for each \( T \leq T_{0}, \) the functions \( \phi_{T}, \phi'_{T} \) are positive on some maximal interval \(( -\infty, \gamma_{1}(T)) \) and \( \phi_{T}(\sigma_{1}) < \epsilon/2, \quad \phi'_{T}(\sigma_{1}) = 0. \) The equation \( \phi'_{T}(\sigma_{1}) = 0 \) can be written as \( T = \Gamma_{1}(\sigma) \) with \( \Gamma_{1} \) satisfying

\[ \Gamma_{1}(\sigma_{1}) := -\frac{c^2}{b} \frac{\sigma_{1}^2}{\ln(-\sigma_{1})} + O(1), \quad \sigma_{1} \to -\infty, \] (21)

and strictly increasing on some maximal interval \(( -\infty, \gamma_{1}(c, b, h)) \). Hence, we see that \( \sigma_{1} = \sigma_{1}(T) \) depends continuously on \( T \) and monotonically converges to \( -\infty \) as \( T \to -\infty. \)

Furthermore, since the equation \( T = \Gamma_{1}(\sigma_{1}) \) has only one root \( \sigma_{1} \in ( -\infty, \gamma_{1}(c, b, h)), \) we may suppose that \( \phi'_{T}(\sigma) < 0 \) for all \( \sigma \in (\sigma_{1}, \sigma_{1} + ch). \)

Using (21) and the monotonicity of \( \Gamma_{1}, \) one can readily establish that

\[ \sigma_{1}(T) = -\frac{\sqrt{-Tb}\ln(-T)}{2c} + O(1), \quad T \to -\infty. \]

Similarly, there is \( Q_{0} > 0 \) such that, for each \( Q \geq Q_{0}, \) the functions \( \psi_{Q}, \psi'_{Q} \) are positive on some maximal interval \(( -\infty, \sigma_{2}(Q)) \) and \( \psi_{Q}(\sigma_{2}) < \epsilon/2, \quad \psi'_{Q}(\sigma_{2}) = 0. \) Equation \( \psi'_{Q}(\sigma_{2}) = 0 \) can be written as \( Q = \Gamma_{2}(\sigma_{2}) \) where

\[ \Gamma_{2}(\sigma_{2}) = \frac{2c}{3} \frac{\sigma_{2}^2}{\ln(-\sigma_{2})} + O(1), \quad \sigma_{2} \to -\infty, \]
strictly decreases on some maximal interval \((-\infty, \gamma_2(c, b, h))\). From this we deduce that \(\sigma_2 = \sigma_2(Q)\) depends continuously on \(Q\) and monotonically converges to \(-\infty\) as \(Q \to +\infty\). Also we may suppose that \(\psi'_2(\sigma) < 0\) on \((\sigma_2, \sigma_2 + ch)\). Next, we have that

\[
\sigma_2(Q_n) = -\sqrt{\frac{3Q_n}{8c}}(\ln Q_n)(1 + o(1)) = \sigma_1(T_n) \sqrt{\frac{3\kappa_n}{2}}(1 + o(1)), \quad n \to +\infty,
\]

and since \(3\kappa_n/2 \in [0.51, 1.47]\), it is always possible to choose \(\kappa_n\) in such a way that \(\sigma_1(T_n) = \sigma_2(Q_n) := \sigma_n\) for all large \(n\). Obviously, \(\kappa_n \to 2/3\).

Next, taking \(Q = Q_n, T = T_n\), we find that for some functions \(\alpha_j, \beta_j\), uniformly on \(n\) satisfying \(\alpha_j(t) = o(1), \beta_j(t) = O(1), t \to -\infty\), it holds

\[
R_1(t) = (2cT_n(1 - \kappa_n + \alpha_j(t)) + \beta_j(t) \frac{\ln^2(-t)}{t^3} - \frac{\kappa_nbc^{-1}T_n^2}{t^3} \ln^4(-t)(1 + \sigma_2(t)).
\]

In this way, we prove the existence of \(\delta_1 < -e\) which does not depend on \(n\) and such that \(R_1(t) < 0\) for all \(t\) from some fixed interval \((-\infty, \delta_1]\). Thus we may assume in the sequel that \(\sigma_n < \delta_1\).

Analogously, we can use the representation

\[
R_2(t) = (bT_n(1 - 3\kappa_n) + \alpha_j(t)) \frac{\ln^2(-t)}{t^3} + \frac{\kappa_n b^2 c^{-1} T_n^2}{t^3} \ln^4(-t)(1 + \sigma_2(t)),
\]

to establish that \(R_2\) is positive on some maximal interval \((-\infty, \delta_2)\), where \(\delta_2 = \delta_2(n)\) depends on \(n\), \(\lim \delta_2(n) = -\infty\) and \(R_2(\delta_2(n)) = 0\). Analyzing the latter equation, we find that there is a sequence \(b_n \to b\) such that

\[
\frac{3c}{2b_n} = \frac{T_n \ln^2(-\delta_2(n))}{\delta_2^2(n)}, \quad \text{and therefore} \quad \delta_2(n) = \frac{\sqrt{2bT_n}}{2c} \ln^{2/3}(-T_n)(1 + o(1)).
\]

Again, we have that \(\sigma_n < \delta_2(n)\) for all large \(n\), so that, without restricting the generality, we may suppose that both \(R_1(t), R_2(t)\) are positive on \((-\infty, \sigma_n]\).

**Remark 23.** For \(r = 1\) and small positive \(A\), we will define \(\phi_A, \psi_A\) by

\[
\phi_A(t) = \phi_T(t - A^{-1}), \quad \psi_A(t) = \psi_Q(t - A^{-1}),
\]

where sufficiently large \(T, Q\) are chosen as in Lemma 21. It is clear that for every \(\tau_0 \in \mathbb{R}\) there exists \(A_0 > 0\) such that \(\phi_A(t) > 0, \psi_A(t) > 0, \phi'_A(t) > 0, \psi'_A(t) > 0\) for all \(t \leq \tau_0, A \in (0, A_0]\). In addition, \(\lim_{A \to 0^+} (\phi_A(\tau), \psi_A(\tau)) = (0, 0), k = 0, 1, 2,\) uniformly on \((-\infty, \tau_0]\).

Now we are in the position to prove Theorem 17.

**Proof.** Consider the functions

\[
\Phi_r(t, A) = \min[1, \phi_A(t) + \psi_r(t)], \quad \Psi_r(t, A) = \min[1, \psi_A(t) + \psi_r(t)],
\]

where \(\phi, \psi\) are defined in Remark 23 if \(r = 1\) and in Lemma 19 for \(r \in (0, 1)\). By Remarks 20, 23 and the implicit function theorem, there exist smooth functions \(t_1(A), t_2(A)\), such that \(\lim_{A \to 0^+} t_1(A) = t_1, \lim_{A \to 0^+} t_2(A) = t_2\) and

\[
\Phi_r(t, A), \Psi'_r(t, A) > 0, \quad t < t_2(A), \quad \text{with} \quad \Phi_r(t, A) = 1, \quad t \geq t_2(A).
\]
\(\Psi_s(t, A), \Psi_s'(t, A) > 0, \ t < t_1(A), \) with \(\Psi_s(t, A) = 1, \ t \geq t_1(A),\)

\(\Phi'_s(t_2(A)) -\ A > \Phi'_s(t_2(A)) + A,\) \(\Psi'_s(t_1(A)) - A > \Psi'_s(t_1(A)) + A = 0.\)

We claim that for \(t \neq t_1(A), t_2(A), d_1, \ldots, d_M,\) and for sufficiently small positive \(A,\) the functions \(\Phi_s(t) := \Phi_s(t, A), \ \Psi_s(t) := \Phi_s(t, A)\) satisfy the system

\[
\begin{align*}
\Phi''_s(t) - c\Phi'_s(t) + \Phi_s(t)(1 - \Phi_s(t) + r\Psi_s(t)) & \leq 0, \\
\Psi''_s(t) - c\Psi'_s(t) + b\Phi_s(t - ch)(1 - \Psi_s(t)) & \leq 0, \\
\Phi'_s(d_j) > \Phi'_s'(d_j), \quad \Psi'_s(d_j) > \Psi'_s'(d_j).
\end{align*}
\]  

(22)

Since differential inequalities (22) hold trivially for all \(t \geq t' := \max\{t_1(A), t_2(A)\}\) (when \(\Phi_s(t) = \Psi_s(t) = 1\)), it suffices to prove (22) for \(t \in (-\infty, t')\). We will consider the following three cases.

**Case I.** Let \(t < t_* := \min\{t_1(A), t_2(A)\}\), then by Lemmas [19, 21] for all small \(A > 0, \)

\[
\Psi''_s(t) - c\Psi'_s(t) + b\Phi_s(t - ch)(1 - \Psi_s(t)) \leq \phi''_+(t) - c\phi'_+(t) + b\phi_+(t - ch)(1 - \phi_+(t)) - b(\phi_+(t - ch)\psi_+(t) + \phi_+(t - ch)\psi_+(t)) < 0,
\]  

(23)

due to assumption D2 of Definition [15] and the positivity of \(\phi_+, \psi_+, \phi_A, \psi_A.\) In a similar way (but this time using assumption D1) we can evaluate \(\Gamma\) defined by

\[
\Gamma := \Phi''_s(t) - c\Phi'_s(t) + \Phi_s(t)(1 - \Phi_s(t) + r\Psi_s(t)) = \phi''_+(t) - c\phi'_+(t) + \phi_+(t)(1 - \phi_+(t) + r\psi_+(t)) + \phi_A(t)(1 - \phi_A(t) + r\psi_A(t)) - 2\phi_A(t)\phi_+(t) + r\phi_A(t)\psi_A(t) < 0,
\]

If \(r \in (0, 1)\) then we obtain

\[
\Gamma = A\epsilon^{1+\epsilon}\rho + A^2\epsilon^{\rho+\delta}\phi_+(t) - c\phi'_+(t) + \phi_+(t)(1 - \phi_+(t) + r\psi_+(t)),
\]

and if \(r = 1\) then

\[
\Gamma = \frac{(-t)^\rho e^{\rho\rho}}{t - A^{-1}} + \phi''_+(t) - c\phi'_+(t) + \phi_+(t)(1 - \phi_+(t) + r\psi_+(t)).
\]

In each of these two cases, for some small \(A_0,\) we obtain \(\Gamma \leq C(-t)^\rho e^{\rho\rho}(t, c) + o(1)), A \in (0, A_0], t \to -\infty.\) Thus there exists \(r_* < i_*\) such that \(\Gamma\) is negative for all \(t \leq t_*\) uniformly on \(A \in (0, A_0].\)

On the other hand, since \(\lim_{A \to 0} \min\{t_1(A), t_2(A)\} = \min\{t_1, t_2\},\) we deduce from D2 and the above asymptotic representation of \(\Gamma\) the existence of \(A_1 \in (0, A_0]\) such that \(\Gamma\) is negative for all \(t \in [r_*, i_*], A \in (0, A_1].\) Thus (22) holds for \(t \in (-\infty, t_*)\) and sufficiently small \(A \in (0, A_1].\)

**Case II.** Suppose now that \(t_1(A) > t_2(A)\) and let \(t \in [t_*, t'_*] = [t_2(A), t_1(A)].\) We have

\[
\Phi''_s(t) - c\Phi'_s(t) + \Phi_s(t)(1 - \Phi_s(t) + r\Psi_s(t)) = -r(1 - \Psi_s(t)) \leq 0,
\]

and, for sufficiently small \(A,\) \(\Psi''_s(t) - c\Psi'_s(t) + b\Phi_s(t - ch)(1 - \Psi_s(t)) =

\[
\begin{align*}
\forall t & \in [t_*, t_* + ch]; \\
\psi''_+(t) - c\psi'_+(t) + b(1 - \psi_+(t)) + \psi'_+(t) - c\psi'_+(t) - b\psi_+(t) & \leq 0, \quad t \in [t_*, t_* + ch, t_*].
\end{align*}
\]

Here we recall that \(\psi''_+(t) - c\psi'_+(t) + b(1 - \psi_+(t))\) is negative on \([t_2 + ch, t_*]\) due to assumption D4.

On the other hand, by the same assumption, we have that, for \(t \in [t_*, t_* + ch]\) and all small \(A,
\]

\[
\forall t \in [t_*, t_* + ch]; \\
\psi''_+(t) - c\psi'_+(t) + b\psi_+(t - ch)(1 - \psi_+(t)) & \leq 0.
\]

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Case III. Similarly, if $t_{1}(A) < t_{2}(A)$ then for $t \in [t_{1}(A), t_{2}(A)]$, we obtain
\[
\Phi''(t) - c\Phi'(t) + \Phi(t)(1 - r - \Phi(t) + r\Psi(t)) = \Phi''(t) - c\Phi'(t) + \Phi(t)(1 - \Phi(t))
\]
for all small $A$. Additionally, $\Psi''(t) - c\Psi'(t) + b\Phi(t)(1 - \Psi(t)) = 0$. Since $\Phi'(d_{1} - A) > \Phi'(d_{1} + A)$, $\Psi'(d_{1} - A) > \Psi'(d_{1} + A)$ are obviously true for all small positive $A$, inequalities \((22)\) are proved for small $A > 0$.

So let us fix some small $A' > 0$ such that $\Phi(t) := \Phi_{0}(t, A')$, $\Psi(t) := \Phi_{0}(t, A')$ satisfy \((22)\). In the continuation, we will prove the existence of lower solutions $\Psi_{-}, \Phi_{-} : \mathbb{R} \rightarrow [0, 1)$, which are defined as smooth non-decreasing functions satisfying the following system:
\[
\begin{align*}
\Phi''(t) - c\Phi'(t) + \Phi(t)(1 - r - \Phi(t) + r\Psi(t)) & \geq 0, \\
\Psi''(t) - c\Psi'(t) + b\Phi(t)(1 - \Psi(t)) & \geq 0,
\end{align*}
\]
\[
\Phi_{0}(t, A') > \Phi_{-}(t) \quad \text{and} \quad \Psi_{0}(t, A') > \Psi_{-}(t), \quad t \in \mathbb{R}. \tag{24}
\]
We will treat separately each of the following cases: $r \in (0, 1)$ and $r = 1$.
Suppose that $r \in (0, 1)$. It follows from the definition of $\Phi_{0}(t, A')$, $\Psi_{0}(t, A')$ that for some positive $k_1 = k_1(A')$, $k_2 = k_2(A')$,
\[
\Phi_{0}(t, A') \geq k_1e^{ut}, \quad \Psi_{0}(t, A') \geq k_2e^{vt}, \quad t \leq 0.
\]
Set now $\Psi_{-}(t) \equiv 0$, and define $\Phi_{-}(t)$, $t \in \mathbb{R}$, as a unique (up to a translation) traveling front solution of the KPP-Fisher equation
\[
\phi''(t) - c\phi'(t) + \phi(t)(1 - r - \phi(t)) = 0, \quad \phi(-\infty) = 0, \quad \phi(+\infty) = 1 - r > 0.
\]
It is well known [7] that $\Phi_{-}(t)$ is strictly increasing and that $\Phi_{-}(t + s_{0}) = 0.5k_1(A')e^{ut} + O(e^{(u+s_{0})t})$, $t \to -\infty$, for some small positive $\delta$ and for an appropriate shift $s_{0} = s_{0}(A')$ which can be supposed to be zero. Hence, as a consequence of all mentioned properties of $\Phi_{-}, \Psi_{-}$, $r \in (0, 1)$, without restricting the generality, we may further assume that the third inequality in \((24)\) is also satisfied.

Let now $r = 1$. For sufficiently large $n$ such that $\phi_{T_{n}}(\sigma_{n}) < 1$, $\psi_{Q_{n}}(\sigma_{n}) < 1$, we consider the following $C^{1}$-smooth increasing functions
\[
\Phi_{-}(t, n) := \begin{cases} \phi_{T_{n}}(t + \sigma_{n}), & t \leq 0, \\
\phi_{T_{n}}(\sigma_{n}), & t \geq 0; \end{cases}
\]
\[
\Psi_{-}(t, n) := \begin{cases} \psi_{Q_{n}}(t + \sigma_{n}), & t \leq 0, \\
\psi_{Q_{n}}(\sigma_{n}), & t \geq 0. \end{cases}
\]

Lemma \([22]\) then implies that, for all $t \in \mathbb{R}$,
\[
\begin{align*}
\Phi''(t, n) - c\Phi'(t, n) + \Phi_{-}(t, n)(\Psi_{-}(t, n) - \Phi_{-}(t, n)) & > 0, \\
\Psi''(t, n) - c\Psi'(t, n) + b\Phi_{-}(t, n)(1 - \Psi_{-}(t, n)) & > 0.
\end{align*}
\]
Take now $n$ sufficiently large to have $T_{n} < T$, $Q_{n} > Q$ and $\sigma_{n} < -(A')^{-1}$. Then
\[
\phi_{A'}(t) = \phi_{T}(t - (A')^{-1}) > \phi_{T}(t + \sigma_{n}), \quad \psi_{A'}(t) = \psi_{Q}(t - (A')^{-1}) > \phi_{Q}(t + \sigma_{n}), t \leq 0.
\]
As a consequence,
\[
\begin{align*}
\Phi_{+}(t, A') & = \min\{1, \phi_{A'}(t) + \phi_{s}(t)\} > \phi_{T}(t + \sigma_{n}) = \Phi_{-}(t, n), \\
\Psi_{+}(t, A') & = \min\{1, \psi_{A'}(t) + \psi_{s}(t)\} > \psi_{Q}(t + \sigma_{n}) = \Psi_{-}(t, n), \quad t \in \mathbb{R}.
\end{align*}
\]
In order to finalize the proof of Theorem [17] for a fixed negative number \( B \leq -(1 + r + b) \), we consider nonlinear operators

\[
\mathcal{F}_1(\phi, \psi)(t) = \phi(t)(1 - r - B - \phi(t) + r\psi(t)), \quad \mathcal{F}_2(\phi, \psi)(t) = b\psi(t - c\nu)(1 - \psi(t)) - B\psi(t).
\]

It is easy to check that \( \mathcal{F}_1, \mathcal{F}_2 \) are monotone in the sense that \( \mathcal{F}_1(\phi_1, \psi_1)(t) \leq \mathcal{F}_1(\phi_2, \psi_2)(t) \), \( t \in \mathbb{R} \), if \( 0 \leq \Phi_1(t) \leq \Phi_2(t) \leq 1 \), \( 0 \leq \Psi_1(t) \leq \Psi_2(t) \leq 1 \), \( t \in \mathbb{R} \). Let \( z_1 < 0 < z_2 \) be the real roots of the equation \( z^2 - cz + B = 0 \). Then every bounded solution \((\phi, \psi)\) of differential equations in (8) should satisfy the system of integral equations

\[
\phi(t) = \mathcal{N}_1(\phi, \psi)(t), \quad \psi(t) = \mathcal{N}_2(\phi, \psi)(t), \quad \text{where}
\]

\[
\mathcal{N}_1(\phi, \psi)(t) := \frac{1}{z_2 - z_1} \left( \int_{-\infty}^{t} e^{z_2(t-s)} \mathcal{F}_1(\phi, \psi)(s) ds + \int_{t}^{+\infty} e^{z_1(t-s)} \mathcal{F}_1(\phi, \psi)(s) ds \right).
\]

Conversely, each positive strictly monotone bounded solution \((\phi, \psi)\) of (25) yields a wavefront for (8). It is clear that the operators \( \mathcal{N}_1 \) are also monotone. Additionally, it is easy to see that \( \mathcal{N}_1(\phi, \psi)(t) \) is increasing if both \( \phi, \psi : \mathbb{R} \to [0, 1] \) are increasing functions.

Hence, taking into account (22), (24) and Lemma [18] we conclude that

\[
\Phi_-(t) \leq \Phi^{(1)}_-(t) := \mathcal{N}_1(\Phi_-, \Psi_-(t) \leq \mathcal{N}_1(\Phi_+, \Psi_+(t) := \Phi^{(1)}_+(t) \leq \Phi_+(t), \quad \Psi_-(t) < \Psi^{(1)}_-(t) := \mathcal{N}_2(\Phi_-, \Psi_-(t) \leq \mathcal{N}_2(\Phi_+, \Psi_+(t) := \Psi^{(1)}_+(t) \leq \Psi_+(t).
\]

Therefore the sequences of positive uniformly bounded (by 0 from below and by 1 from above) monotone continuous functions

\[
\Psi^{(n+1)}(t) = \mathcal{N}_2(\Phi^{(n)}, \Psi^{(n)})(t), \quad \Phi^{(n+1)}(t) = \mathcal{N}_1(\Phi^{(n)}, \Psi^{(n)})(t), \quad n = 1, 2, \ldots, \quad (26)
\]

and

\[
\Psi^{(n+1)}(t) = \mathcal{N}_2(\Phi^{(n)}, \Psi^{(n)})(t), \quad \Phi^{(n+1)}(t) = \mathcal{N}_1(\Phi^{(n)}, \Psi^{(n)})(t), \quad n = 1, 2, \ldots, \quad (26)
\]

are strictly increasing and decreasing, respectively. Set \( \Phi = \lim \Phi^{(n)}, \Psi = \lim \Psi^{(n)} \), then

\[
\Phi_-. \leq \Phi \leq \Phi_+, \quad \Psi_-. \leq \Psi \leq \Psi_+.
\]

Furthermore, a direct application of the Lebesgue’s dominated convergence theorem to (26) shows that the pair \((\Phi, \Psi)\) solves system (25). Since \( \Phi(t) > 0 \) for all \( t \), we may conclude from (25) that \( \Psi(t) > 0, t \in \mathbb{R} \). Note also that \( \Phi(-\infty) = \Psi(-\infty) = 0 \) in virtue of (27). Now, since \( \Phi, \Psi \) are positive, increasing and bounded functions, the values of \( \Phi(\infty), \Psi(\infty) \) are finite and positive. A standard argument based on the Barbalat lemma (cf. [27]) shows that \( \Phi(\infty) = \Psi(\infty) = 1 \).

Finally, the validity of asymptotic formula (5) follows from (27). To prove (6), we first observe that, due to (27) and Theorem [6] there exists \( T_b < 0 \) such that \( \Phi(t) \) and \( \Psi(t) \) are bounded (below and from above) by \( c_1 e^{ct} \) for some \( c_1 > 0 \) and all \( t \leq T_b \). Then we can apply Proposition 7.2 from [18] to the first equation of (8) in order to obtain the desired formula for \( \Phi(t) \). Using this formula and the change of variables \( y = \psi - be^{k(t-ch)}/(1 - r) \), we then get easily the second formula of (6), cf. the proof of Lemma [14] and Corollary [12].

5.2. Proof of Theorem [7]

The simplest form of regular super-solutions \( \phi_+, \psi_+ \) is exponential, we can write them as

\[
\phi_+(t) = e^{ct}, \quad \psi_+(t) = De^{ct}, \quad D = be^{-rch}/(cy - v^2), \quad (28)
\]

where \( D \) is chosen in such a way that the second inequality in D2 as well as D4 were satisfied (details are given below). In order to simplify the notation, in the sequel we will write \( b' = be^{-rch} \).
Lemma 24. Suppose that \( v \neq j \lambda \) is close to \( c/2, r > 0, \) and
\[
(cv - v^2)(1 + 1/b') > 1, \quad cv - v^2 \neq b'.
\] (29)

Then (28) determines a regular super-solution for (8).

Proof. Clearly, D1 is satisfied with \( C_1 = 1, t_1 = v^{-1} \ln((cv - v^2)/b') \neq 0, t_2 = 0. \) Still we have to check hypotheses D2, D3, D4. Depending on the sign of \( t_1, \) we will analyze the next two cases:

Case I. \( t_1 > 0 = t_2 \) or, equivalently, \( 0 < b'/(cv - v^2) < 1. \) If \( t \in [0, t_1], \) then D4 holds because
\[
\psi''(t) - cv'(t) + b \min(1, \phi_s(t - ch))(1 - \psi_s(t)) = \begin{cases} -b'e^{rt}\psi_s(t) < 0, & t \in [0, ch]; \\ b \left(1 + \frac{b}{c(v-c)}e^{rt-ch}\right) < 0, & t \in [ch, t_1]. 
\end{cases}
\]

If \( t \leq 0, \) we have that
\[
\psi''(t) - cv'(t) + b\phi_s(t - ch)(1 - \psi_s(t)) = -b'De^{2rt} < 0;
\]
\[
\phi''(t) - c\phi'(t) + \phi_s(t)(1 - r - \phi_s(t) + r\phi_s(t)) = e^{rt}[\chi(v,c) - \phi_s(t) + r\phi_s(t)] < 0. \quad (30)
\]

Case II. Next, let \( t_1 < t_2 = 0 \) so that \( b'/(cv - v^2) > 1. \) If \( t \in [t_1, 0] \) then
\[
\phi''(t) - c\phi'(t) + \phi_s(t)(1 - \phi_s(t)) = e^{rt}(1 - e^{rt} + v^2 - cv) 
\leq e^{rt}(1 - e^{rt} + v^2 - cv) = e^{rt}(1 - (cv - v^2)(1 + 1/b')) < 0,
\]
and D3 holds. Now, for \( t \leq t_1, \) condition D2 is true since
\[
\phi''(t) - c\phi'(t) + \phi_s(t)(1 - r - \phi_s(t) + r\phi_s(t)) = e^{rt}(\chi(v,c) + e^{rt}(-1 + rb'(cv - v^2)) \leq e^{rt}\max[\chi(v,c), 1 - (cv - v^2)(1 + 1/b')] < 0;
\]
\[
\phi''(t) - c\phi'(t) + b\phi_s(t - ch)(1 - \psi_s(t)) = -b'De^{2rt} < 0.
\]

This completes the proof of the lemma. \( \square \)

Corollary 25. The existence statement of Theorem 7 holds true.

Proof. First, we assume that \( c > c_\phi. \) Then clearly there is a positive \( u \) meeting all requirements of Lemma 24. This assures the existence of a regular super-solution for (8). By Theorem 17 system (8) has a positive monotone wavefront.

Next, we consider the case when \( c = c_\phi, \) \( r \in (0, 1). \) Let \( c_j \downarrow c_\phi \) be a strictly decreasing sequence of velocities and \( (\phi_j, \psi_j) \) be a sequence of corresponding traveling fronts (existing in virtue of the first part of the proof). Since
\[
0 = \psi_j(-\infty) + \psi_j(-\infty) < \phi_j(t) + \psi_j(t) < \phi_j(+\infty) + \psi_j(+\infty) = 2
\]
and the function \( \phi_j(t) + \psi_j(t) \) is increasing in \( t \) for each fixed \( j, \) we may assume that \( \phi_j(0) + \psi_j(0) = 3/2, \quad j = 1, 2, 3, \ldots. \) Using the standard compactness arguments and then applying the Lebesgue’s dominated convergence theorem to the system of integral equations (25):
\[
\phi_j(t) = N_1(\phi_j, \psi_j, c_j)(t), \quad \psi_j(t) = N_2(\phi_j, \psi_j, c_j)(t),
\]
we may assume, without restricting the generality, that \( \lim_j(\phi_j, \psi_j) = (\hat{\phi}, \hat{\psi}) \) uniformly on bounded intervals, where \((\hat{\phi}, \hat{\psi}) \) is a monotone solution of (8) with \( c = c_\phi. \) Since \((\hat{\phi}, \hat{\psi})(\pm\infty) \) are steady...
state solutions of \( \psi \) and \( \dot{\psi}(-\infty) + \dot{\psi}(\infty) \leq \dot{\psi}(0) + \dot{\psi}(0) = 3/2 \leq \dot{\psi}(0) + \dot{\psi}(\infty) \), we find that necessarily
\[
\dot{\psi}(-\infty) = 0, \ \dot{\psi}(\infty) \in [0, 1], \ \dot{\psi}(0) + \dot{\psi}(\infty) = 1.
\]
(if \( \dot{\psi}(-\infty) > 0 \), then \( \dot{\psi}(\infty) = -\dot{\psi}(-\infty) = 1 \) and thus \( \dot{\psi}(0) + \dot{\psi}(0) = 2 \), a contradiction). To finish the proof of the corollary, we have to establish that \( \ddot{\psi}(\infty) = 0 \). In order to prove this, we can apply the part [A] (for \( r \in (0, 1) \)) and the part [C] (when \( r = 1 \)) of Theorem 6 to find that either \( \psi_{1}(t) < K\psi_{1}(t), \ t \in \mathbb{R} \) (for \( r \in (0, 1) \)) or \( \psi_{1}(t) < \sqrt{M\psi_{1}(t)}, \ t \in \mathbb{R} \) (for \( r = 1 \)). Therefore either \( \psi_{1}(t) \leq K\psi_{1}(t), \ t \in \mathbb{R} \), or \( \psi_{1}(t) \leq \sqrt{M\psi_{1}(t)}, \ t \in \mathbb{R} \), so that \( \psi_{1}(\infty) = 0 \).

5.3. Proof of Theorem 8

Let now \( c < c_{b} \) so that for some \( v \) close to \( c/2 \)
\[
0 < (cv - v^2)(1 + \frac{1}{b}) \leq 1, \ cv - v^2 \neq b', \ v \neq j\lambda.
\]

Analyzing the proof of Lemma 24 under these assumptions, we see that \( t_{1} < 0 = t_{2} \) and the main obstacle to develop successfully the proof of Case II appears when we want to estimate expression (31) near \( t_{1} < 0 \). Therefore we may expect that, after an appropriate modification of super-solutions (30) in some neighborhood of \( t_{1} \), the result of Theorem 8 can be improved.

Below, we develop this idea by considering \( \psi_{s}(t) = e^{\nu t}, \ t \in \mathbb{R} \), and \( C^{1} \)-smooth function
\[
\psi_{s}(t) = \begin{cases} 
De^{\nu t}, & \text{if } t \leq t_{s}; \\
p + qt, & \text{if } t > t_{s}.
\end{cases}
\] (33)

Here \( p, q, t_{s} \), will be chosen to satisfy the first inequality in \( D_{2} \) for all \( t \in \mathbb{R} \). The mentioned inequality can be written as
\[
\psi_{s}(t) < \gamma(t) := r^{-1}(cv - v^2 + r - 1 + e^{rt}).
\] (34)

Now, assuming (33) and analyzing the mutual positions of convex graphs of the functions \( \gamma(t) \) and \( De^{\nu t} \), we deduce that these graphs should have exactly one point of intersection (or tangency) below the level \( y = 1 \). Indeed, otherwise \( cv - v^2 + r - 1 > 0 \) implies that \( \gamma(t) > De^{\nu t} \) for all \( t \) where \( \gamma(t) \leq 1 \). As a consequence, \( 1 = \gamma(s_{0}) > De^{\nu s_{0}} \) at some \( s_{0} \) which implies (30), a contradiction.

The above consideration and a direct computation show that there exists a unique line \( y = p + qt \) which is tangent to the graphs of \( De^{\nu t} \) and \( \gamma(t) \) at the respective points \( t_{s} < t' \). From the tangency conditions \( q = Dve^{\nu t} = \gamma'(t'), \ p = De^{\nu t} - qt_{s} = \gamma(t') - qt' \), it follows easily that
\[
p = \frac{q}{v} \ln \frac{Dve^{\nu t}}{q}, \ q = \frac{(cv - v^2 + r - 1)v}{r \ln(eD)}, \ t_{s} = \frac{1}{v} \ln \frac{q}{De^{\nu t}}.
\]

It follows from the above construction that \( \psi_{s} \) defined by (33) is \( C^{1} \)-smooth and \( \psi_{s}(t) < \gamma(t) \) for all \( t \neq t' \). It is clear that, after making an arbitrarily small change of \( p, q, t_{s} \), we may assume that \( \psi_{s}(t) < \gamma(t) \) for all \( t \in \mathbb{R} \).

Hence, taking \( \psi_{s} \) as in (33) and \( \phi_{s}(t) = e^{rt} \), we have to check only the second inequality in \( D_{2} \) on the interval \([t_{s}, +\infty)\). This inequality can be written as \( b'(1 - p)/q < b' + ce^{rt} \). Since \( y = b't + ce^{rt} \) has a unique critical point (an absolute minimum) at \( t' = r^{-1}(cv/b') \), the latter inequality amounts to
\[
(1 - p)v < q \ln(ecv/b').
\] (35)
After recalling the definition of $p$ and $D$ and taking into account that $\nu$ can be chosen as close to $c/2$ as we want, we rewrite (5) as $\omega < 2(2 + \ln \omega)$, where

$$\omega = \frac{2r}{c^2/4 + r - 1} \ln \frac{4b'c}{c^2} \left(= \frac{c}{q}\right).$$

Notice here that the assumptions $c^2 > 4(1 - r)$ and $c^2 < 4/(1 + (b')^{-1})$ imply $r(b' + 1) > 1$ and $c^2 < 4b'r$ so that $\omega > 0$. Furthermore, since $\omega$ is decreasing in $c^2/4$, we find that

$$\omega > \frac{2r}{b'(1 + b') + r - 1} \ln r(b' + 1) = \frac{2r(1 + b')}{r(1 + b') - 1} \ln r(b' + 1) > 2.$$

A direct graphical analysis shows that the interval $\omega \in (0.14555 \ldots, 8.21093 \ldots)$ gives the solution of $\omega < 2(2 + \ln \omega)$. In consequence, since we additionally have $\omega > 2$, the latter inequality is equivalent to $\omega < \omega_* = 8.21 \ldots$ which can be written as (7). This proves Theorem 8. \hfill \Box

6. Proof of Theorem 9

The proof is divided into three claims.

Claim I: The propagation speed $c_*$ is unique. Indeed, suppose that $(\phi_1, \psi_1, c_1)$ and $(\phi_2, \psi_2, c_2)$, $c_1 < c_2$, solves the nonlinear eigenvalue problem (8). It follows from Lemmas 13 and 14 that there exist $p < q$ such that $\psi_1(t) > \psi_2(t)$ for all $t \in \mathbb{R} \setminus [p, q]$. As a consequence, the closed set

$$\mathcal{S} := \{s : \psi_1(t + s) \geq \psi_2(t), t \in \mathbb{R}\} \neq \mathbb{R}$$

is non-empty and has a finite $s_* := \inf \mathcal{S}$. It is clear that $\psi_1(t + s_*) \geq \psi_2(t)$, $t \in \mathbb{R}$, and since always $\psi_1(t + s_*) > \psi_2(t)$ for $t \ll -1$ and $t \gg 1$, we deduce that $\psi_1(t + s_*) = \psi_2(t)$ at some point $\tau$ (otherwise $s_* \neq \inf \mathcal{S}$). By a similar argument, there exists $t_*$ such that $\psi_1(t + t_*) \geq \psi_2(t)$, $t \in \mathbb{R}$, and $\phi_1(T + t_*) = \phi_2(T)$ for some $T$. Suppose first that $t_* \leq s_*$. Without restricting the generality, we may assume that $s_* = 0, \tau = 0$. Then $t_* \leq 0$ so that $\psi_1(t) \geq \psi_2(t)$, $t \in \mathbb{R}$, and thus we get

$$0 = (\psi_1' - \psi_2')'(0) - c_1(\psi_1' - \psi_2')(0) + (c_2 - c_1)(\psi_2'(0) + b(\phi_1(-c_1h) - \phi_2(-c_2h))(1 - \psi_1(0))) > 0,$$

a contradiction. Next, suppose that $t_* > s_*$. We may assume again that that $t_* = 0, T = 0$. Then $s_* < 0$ so that $\psi_1(t) > \psi_2(t)$, $t \in \mathbb{R}$, and thus we get

$$0 = (\phi_1 - \phi_2)''(0) - c_1(\phi_1 - \phi_2)'(0) + (c_2 - c_1)(\phi_2'(0) + r\phi_1(0)(\psi_1(0) - \psi_2(0))) > 0,$$

a contradiction. Hence $c_1 = c_2$ and Claim I is proved.

Claim II: $c_* \leq c_m := \min\{c_1, c_2\}$. Let $(\phi_*, \psi_*, c_*)$ be the solution of (8). On the contrary, suppose that $c_* > c_m$ and take an arbitrary $c' \in (c_m, c_*)$. Then $(\phi_*, \psi_*, c')$ is a lower solution:

$$\phi_*''(t) - c' \phi_*'(t) + \phi_1(t)(1 - \phi_1(t) + r\phi_1(t)) > 0, \quad \psi_*''(t) - c' \psi_*'(t) + b\phi_1(t - c'h)(1 - \psi_1(t)) > 0, \quad t \in \mathbb{R}.$$

For the same $c'$ we consider the upper solutions $\Phi_*(t) \equiv \min\{1, \phi_*(t)\}$, $\Psi_*(t) \equiv \min\{1, \psi_*(t)\}$, with $\phi_*, \psi_*$ defined in Subsections 5.2, 5.3. By Lemma 13, we may suppose (possibly, after a translation of $(\phi_*, \psi_*)$) that $\phi_1(t) < \Phi_*(t)$, $\psi_1(t) < \Psi_*(t)$, $t \in \mathbb{R}$. But then there exists (cf. the last part of Subsection 5.1 starting from formula (25)) a monotone traveling front propagating at the velocity $c' < c_*$. However, this contradicts to Claim I.
Claim III: Set $c_*(h) := c_*(r, b, h)$ for some fixed $r, b > 0$. Then $c_*(h)$ is a non-increasing function on its domain. Suppose that $c_*(h_1) > c_*(h_2)$ for some $h_1 > h_2$. Let $(\phi_j, \psi_j, c_*(h_j))$ be respective solutions of (8). Then, for a fixed $c \in (c_*(h_2), c_*(h_1))$, it holds

$$\phi_1'(t) - c\phi_1(t) + \phi_1(t)(1 - r - \phi_1(t) + r\phi_1(t)) > 0, \quad \psi_1'(t) - c\psi_1(t) + b\phi_1(t) - ch_1)(1 - \psi_1(t)) > 0, \quad t \in \mathbb{R},$$

$$\phi_2'(t) - c\phi_2(t) + \phi_2(t)(1 - r - \phi_2(t) + r\phi_2(t)) < 0, \quad \psi_2'(t) - c\psi_2(t) + b\phi_2(t) - ch_1)(1 - \psi_2(t)) < 0, \quad t \in \mathbb{R}.$$ 

Moreover, due to Lemmas 13 and 14, we may assume that $\phi_1(t) < \phi_2(t), \quad \psi_1(t) < \psi_2(t), \quad t \in \mathbb{R}$. Therefore $(\phi_1, \psi_1, c), \quad j = 1, 2$, forms a pair of upper and lower solutions for (8) considered with $c$ and $h_1$. As a consequence, system (8) with $h = h_1$ has two different propagation speeds: $c$ and $c_*(h_1) > c$. However, this is a contradiction with Claim I.

\[\square\]

Acknowledgments

The authors express their gratitude to the referee, whose critical comments and valuable suggestions helped to improve the original version of this paper. This research was supported by FONDECYT (Chile), projects 1080034 and 1110309, and by CONICYT (Chile) through PBCT program ACT-56.

References

[1] M. Aguerrea, C. Gomez, S. Trofimchuk, On uniqueness of semi-wavefronts (Diekmann-Kaper theory of a non-linear convolution equation revisited), Math. Ann., 354 (2012), 73-109.
[2] H. Berestycki, L. Nirenberg, Traveling waves in cylinders. Ann. Inst. H. Poincare Anal. Non. Lineaire 9, 497-572 (1992).
[3] X. Chen, J.-S. Guo, Uniqueness and existence of traveling waves for discrete quasilinear monostable dynamics. Math. Ann. 326 (2003), 123-146.
[4] X. Chen, S.-C. Fu, J.-S. Guo, Uniqueness and asymptotics of traveling waves of monostable dynamics on lattices, SIAM J. Math. Anal. 38 (2006) 233-258.
[5] A. Boumenir, V. M. Nguyen, Perron theorem in the monotone iteration method for traveling waves in delayed reaction-diffusion equations, J. Diff. Eqns, 244 (2008), 1551-1570.
[6] M.S.P. Eastham, The asymptotic solution of linear differential systems, London Math. Soc. Monogr. Ser., Clarendon Press, Oxford, 1989.
[7] A. Gomez, S. Trofimchuk, Monotone traveling wavefronts of the KPP-Fisher delayed equation, J. Diff. Eqns, 250 (2011), 1767-1787.
[8] J.-S. Guo, X. Liang, The minimal speed of traveling fronts for the Lotka-Volterra competition system, J. Dynam. Diff. Eqns, 23 (2011), 353-363.
[9] W. Huang, Problem on minimum wave speed for a Lotka-Volterra reaction-diffusion competition model, J. Dynam. Diff. Eqns, 22 (2010), 285-297.
[10] W. Huang, M. Han, Non-linear determinacy of minimum wave speed for a Lotka-Volterra competition model, J. Diff. Eqns, 251 (2011), 1549-1561.
[11] X.-J. Hou, Y. Li, K. R. Meyer, Traveling wave solutions for a reaction diffusion equation with double degenerate nonlinearities, Discrete Contin. Dyn. Syst., 26 (2010), 265-290.
[12] Ya. I. Kanel, Existence of a traveling-wave solution of the Belousov-Zhabotinskii system, Differential Equations, 26 (1990), 652-660.
[13] Ya. I. Kanel, Existence of a traveling-wave type solutions for the Belousov-Zhabotinskii system of equations II, Sib. Math. J., 32 (1991), 390-400.
[14] X. Liang, X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, J. Functional Anal., 259 (2010), 857-903.
[15] G. Liu, W.-T. Li, Traveling wavefronts of Belousov-Zhabotinskii system with diffusion and delay, Appl. Math. Letters, 22 (2009), 341-346.
[16] G. Lv, M. Wang, Traveling wave front in diffusive and competitive Lotka-Volterra systems, Nonlinear Analysis, RWA, 11 (2010), 1323-1329.
[17] S. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, J. Diff. Eqns, 171 (2001), 294–314.
[18] J. Mallet-Paret, The Fredholm alternative for functional differential equations of mixed type, J. Dynam. Diff. Eqns., 11 (1999), 1–48.
[19] V. S. Manoranj, A. R. Mitchell, A numerical study of the Belousov-Zhabotinskii reaction using Galerkin finite element methods, J. Math. Biology, 16 (1983), 251–260
[20] R. H. Martin, H. L. Smith, Abstract functional differential equations and reaction-diffusion systems, Trans. Amer. Math. Soc. 321 (1990), 1–44.
[21] J. D. Murray, On traveling wave solutions in a model for Belousov-Zhabotinskii reaction, J. Theor. Biol., 56 (1976), 329–353.
[22] J. D. Murray, Lectures on nonlinear differential equations. Models in biology, Clarendon Press, Oxford, 1977.
[23] D. A. Quinney, On computing travelling wave solutions in a model for the Belousov-Zhabotinskii reaction, J. Inst. Maths. Applics., 23 (1979), 193–201.
[24] E. Trofimchuk, M. Pinto, S. Trofimchuk, Traveling wavefronts for a model of the Belousov-Zhabotinskii reaction, preprint arXiv:1103.0176v2.
[25] W. C. Troy, The existence of traveling wave front solutions of a model of the Belousov-Zhabotinskii reaction, J. Diff. Eqns, 36 (1980), 89-98.
[26] A. I. Volpert, V. A. Volpert, and V. A. Volpert, Traveling Wave Solutions of Parabolic Systems, Translations of Mathematical Monographs, Vol. 140, Amer. Math. Soc., Providence, 1994.
[27] J. Wu, X. Zou, Traveling wave fronts of reaction-diffusion systems with delay, J. Dynam. Diff. Eqns., 13 (2001), 651–687. [Erratum in J. Dynam. Diff. Eqns, 20 (2008), 531–533].
[28] Q. Ye, M. Wang, Traveling wave front solutions of Noyes-Field System for Belousov-Zhabotinskii reaction, Nonlin. Anal. TMA, 11 (1987), 1289–1302.

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