Abstract

In this paper we investigate whether the use of a noiseless, classical feedback channel will increase the capacity of a quantum discrete memoryless channel to transmit classical information. This problem has been previously analyzed by Bowen and Nagarajan \cite{2} for the case of protocols restricted to product input states. They showed that feedback did not increase the information capacity. In this paper we introduce a quantum analogue of classical causality \cite{11,15} and prove a capacity theorem (in regularized form) for the transmission of classical information.

1 Introduction:

In classical information theory a noiseless feedback channel between sender and receiver will not increase the Shannon capacity of a channel. In the quantum case, the situation is more complex because there are number of possible feedback capacities corresponding to any channel: the simplest of these is the product-state input or HSW capacity of a channel aided by
feedback. For this case, Bowen and Nagarajan have shown that there is no
capacity increase over the no-feedback case. On the other hand, for the case of
quantum capacities, it has been shown by Bowen [3] that the use of a classical
feedback channel may increase the value of the channel capacity from $Q$ to $Q_E$
where $Q$ is the quantum capacity of the channel, and $Q_E$ is entanglement-
assisted capacity of the channel (see [5] for rigorous definitions). In this work,
we will concentrate on classical feedback and the effect on the ”full” classical
capacity of the channel (in which case entangled input states are allowed).
In this regard it is worth referring to an important conjecture of quantum
information theory, that the unassisted capacity, called $C$, is in fact additive:

$$C(\Phi \otimes \Psi) = C(\Phi) + C(\Psi)$$

for any two quantum channels $\Phi$ and $\Psi$ [9]. It is natural to conjecture
whether a similar additivity property holds in the case of (suitably defined)
classical feedback capacity (which in the sequel, will be denoted $C_F$). If such
a conjecture were true, then by the result of Bowen [2] it would follow that
unconstrained classical feedback would not increase the channel capacity.
However, this turns out not to be the case, as Devetak and co-workers [16]
have produced an example of a discrete memoryless quantum channel for
which the feedback capacity exceeds the Holevo product state capacity. In
this paper we will give a coding theorem for quantum discrete memoryless
channels with classical feedback and demonstrate that, as in the classical
case, the definitions and proofs are highly dependent on an analog of the
classical notion of causality introduced in [11].

In what follows we make use of standard notation for information transfer
through quantum channels: a quantum channel $\Phi$ with input space $\mathcal{H}$ is
modelled as a trace-preserving completely positive map on density matrices
$\rho \in \mathcal{B}(\mathcal{H})$ and will be represented by a Kraus decomposition \{${E_j}$\} with
$\sum_j E_j^* E_j = I$ as $\Phi(\rho) = \sum_j E_j \rho E_j^*$. In the direct part of the coding theorem
proof below we will consider encoding using density operators picked from
ensembles of the form \{$p_x, \rho_x$\}, where $p_x$ is the probability of picking density
matrix $\rho_x$.

We will work with a quantum analogue of classical mutual information, which
we define for a bipartite quantum system $\rho_{AB}$, by

$$I(\mathcal{A} : \mathcal{B}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

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where $S(\sigma)$ is the von Neumann entropy of the density matrix $\sigma$. This quantity was introduced by Adami and Cerf [12].

In particular we will consider states of the form

$$\rho_{AB} = \sum_i p_i |i\rangle \langle i| \otimes \rho_i^B,$$

where $\{|i\rangle\}$ is an orthonormal basis for subspace $A$ (for clarity we will generally represent quantum subsystems with calligraphic letters as opposed to ordinary capitals for classical registers). Such a state is said to exhibit “classical-quantum” correlations and we have the following form for $I(A : B)$:

$$I(A : B) = S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i) = \chi(\mathcal{E}),$$

where $\mathcal{E} = \{p_i, \rho_i\}$ and $\chi(\mathcal{E})$ is the Holevo quantity of ensemble $\mathcal{E}$.

The quantum mutual information allows us to more neatly express the quantities of interest in classical information transfer.

## 2 Feedback Code Definition and feedback capacity upper bound

What follows is a formulation of the feedback communication protocols similar to that introduced by Bowen and Nagarajan in their paper [2]. As before, the message source is a finite alphabet stochastic process satisfying the asymptotic equipartition property (AEP), see [1].

Given a rate $R > 0$, we define an $n$-block feedback code of size $N = 2^{nR}$ for channel $\Phi$ acting on states in the input Hilbert space $\mathcal{H}$ as a quadruple $\mathcal{C}_n = (\mathcal{E}_F, \mathcal{M}_F, \mathcal{N}_F, f)$ consisting of

1) strings $i_1^n(l) \in \mathcal{A}^n$ forming a classical code $\mathcal{C}_N$ of size $N$. These strings should be viewed as the elements of the image of a mapping from the message space $\mathcal{M}$ to the space of input strings.

2) an input ensemble $\mathcal{E} = \{p_{i_1^n(l)}, \rho_{i_1^n(l)}\}$ where each $\rho_{i_1^n(l)}$ is a density matrix in $\mathcal{H}^{\otimes n}$. For convenience we will denote this Hilbert space as $\otimes_{k=1}^n \mathcal{H}_k$, with
the index $k$ referring to the space on which the $k$-th sequential channel action occurs.

3) a collection of measurements $\mathcal{M}_F$ given by measurements $\mathcal{M}_1, \ldots, \mathcal{M}_n$, where $\mathcal{M}_j$ acts in the space $\mathcal{L}(\mathcal{H}_j) \otimes \cdots \otimes \mathcal{L}(\mathcal{H}_j)$, $1 \leq j \leq n$, and an array $N_F$ of "associated" trace preserving completely positive maps $\{N_2^{(k_1)}, \ldots, N_n^{(k_{n-1})}\}$, where $k_i$ runs through the outcomes of measurement $M_i$. Also $N_j^{(k_j-1)}$ acts on the space $\otimes_{k=j+1}^n \mathcal{H}_k$, $1 \leq j \leq n-1$. We denote the elements of $N_j^{(k_j-1)}$ by $\{N_j^{(k_j-1)}\}$ and those of $\mathcal{M}_i$ by $\{F_i\}$. For clarity, assume that the outcomes $k_n$ of $\mathcal{M}_n$ are strings $i^n_1(l) \in \mathcal{C}_n$ and a spodge (error) denoted "er".

The transmission protocol is then to sequentially transmit the codeword by uses of the channel, at each stage measuring the state received so far and using a noiseless classical channel to transmit the measurement outcome to the sender. Formally: the first round of communication starts with the mapping: $\rho_{i^n_1(l)} \mapsto \omega_{i^n_1(l)}^0 := (\Phi \otimes I \otimes \cdots \otimes I)(\rho_{i^n_1(l)})$. The feedback measurement $\mathcal{M}_1$ is made and the outcome $k_1$ transmitted to the sender who then applies map $N_2^{(k_1)}$ to the state

$$\frac{F_{k_1}^1 \omega_{i^n_1(l)}^0 F_{k_1}^{1*}}{\text{tr}(F_{k_1}^1 \omega_{i^n_1(l)}^0 F_{k_1}^{1*})}.$$

Here we use the notation $F_{k_1}$ for $F_{k_1} \otimes I \otimes \cdots \otimes I$ and a similar agreement holds in what follows. The result of these operations is the state:

$$\omega^1(i^n_1(l), k_1) = \sum_j \frac{(N_2^{(k_1)}(F_{k_1}^1 \omega_{i^n_1(l)}^0 F_{k_1}^{1*}) N_2^{(k_1)*})}{\text{tr}(F_{k_1}^1 \omega_{i^n_1(l)}^0 F_{k_1}^{1*})}.$$

We proceed inductively, with $\omega^{m-1}(i^n_1(l), k^{m-1})$ obtained from $\omega^{m-2}(i^n_1(l), k^{m-2})$ by

$$\omega^{m-1}(i^n_1(l), k^{m-1}) = \sum_j \frac{(N_{m-1}^{(k_{m-1})}(F_{k_{m-1}}^m \otimes \cdots \otimes I)(k_{m-2})) F_{k_{m-1}}^{m-1} N_{m,j}^{(k_{m-1})*})}{\text{tr}(F_{k}^{m-1}(I \otimes \cdots \otimes I)(k_{1}^{m-2})) F_{k}^{(m-1)*})}.$$

For the fidelity of this procedure we consider random outcomes $K_j$ for every measurement $\mathcal{M}_j$, $j = 1, \ldots, n$. The final (random) estimate of the original classical string is then a fixed function $f(K_1, \ldots, K_n)$ taking values in $\mathcal{C}_N$. 
Define the error probability for this code as:

\[ P_{E, M_f, N_F} = 1 - \max_l [p(i_1^n(l))P(f(K_1, \ldots, K_n) = i_1^n(l)|i_1^n(l))] \]

then take the minimum over all codes \( (E, M_F, N_F) \):

\[ P_e(n, N) = \min P_{E, M_F, N_F}. \]

The rate \( R \) is achievable if \( \lim_{n \to \infty} P_e(n, N) = 0 \). The feedback capacity \( C_F \) is then defined as the supremum of all achievable rates.

These operations can be summarised in the quantum mutual information formalism by defining a sequence of extended Hilbert space quantum states. We begin by defining

\[ \rho_{A_1^n, X_1^{n-1}, Z_1^n}^0 = \sum_l p_{i_1^n(l)}|i_1^n(l)\rangle\langle i_1^n(l)| \otimes |i_1^n(l)\rangle\langle i_1^n(l)| \otimes |0\rangle\langle 0| \cdots |0\rangle\langle 0| \otimes \rho_{i_1^n(l)}. \]

The non italicised systems \( X_1^{n-1} \) (each in an initial state \( |0\rangle\langle 0| \)) and \( A_1^n \) are classical registers recording, respectively, the classical codewords and the outcomes of the feedback measurements. To achieve this, the POVM elements of a given feedback measurement are augmented to the form \( U_{k_i} \otimes M_{k_i} \), where \( U_{k_i} \) is a unitary operator acting on the register system \( X_i \). By applying the sequence of operations outlined above we obtain the states:

\[ \rho_{A_1^n, X_1^{n-1}, Z_1^n}^t = \sum_l p_{i_1^n(2)}(l)p(k_1|i_1^n(l)) \cdots p(k_i|i_1^n(l), k_1, \ldots, k_{i-1}) \otimes |i_1^n(l)\rangle\langle i_1^n(l)| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes \omega(i_1^n(l), k_1^{i-1})^t, \]

for \( 1 \leq t \leq n - 1 \).

Now after \( k \) rounds of communication, the state held by the receiver can be written in the EHS form as \( \text{tr}_{A_1^{n-1}}(\rho_{A_1^n, X_1^{n-1}, Z_1^n}^k) \). This reflects the fact that the ensemble held by the receiver at this point contains states (of the form \( \text{tr}_{A_1^{n-1}}(\rho_{A_1^n, X_1^{n-1}, Z_1^n}^k) \)) that can be labelled by strings of length \( k \) from the alphabet \( A \), or equivalently, indexed by the ”register” space \( A_1^n \). Taking the partial trace with respect to \( A_1^{k+1} \) leaves us with an EHS state.
indexed by this register. This can be viewed in an analogous way to the classical causal systems (introduced by Massey [11]): A feedback protocol such as that defined in this chapter implies the existence of a classical-quantum Markov chain $M_i^n \rightarrow A_i^n \rightarrow Z_i^n$. This in turn implies that

$$P(K_n = k_n | K_1^{n-1} = k_1^{n-1}, X_1^n = x_1^n, M_1^k = m_1^k) = P(K_n = k_n | K_1^{n-1} = k_1^{n-1}, X_1^n = x_1^n)$$

(2.2)

The operational interpretation, as in the classical case, is that the message is specified before the initial transmission encoding and the channel is only aware of the message identity via its past inputs, measurement outputs and current input.

**Definition 1** For a sequence of EHS states corresponding to a $n$-block feedback code, as defined above, we have the quantum directed information, given by the formula

$$I(A_1^n \rightarrow Z_1^n) = \sum_{t=1}^{n} I_t(A_t^t : Z_t | Z_t^{t-1}),$$

where the notation $I_t$ refers to the mutual information calculated with respect to the EHS state $\rho^{t}_{A_t^n, X_t^{n-1}, Z_t^n}, \leq t \leq n$. Furthermore, we will make use of a related quantity

$$I_n(A_1^n \rightarrow Z_1^n) = \sum_{t=1}^{n} I_n(A_t^t : Z_t | Z_t^{t-1}).$$

We have the following lemma:

**Lemma 1** Directed Data Processing Inequality With the above definitions, we have the following inequality

$$I(M_1^n : Z_1^n) \leq I(A_1^n \rightarrow Z_1^n).$$

**Proof** We first show that

$$I(M_1^n : Z_1^n) \leq I_n(A_1^n \rightarrow Z_1^n).$$

To do this, we imitate the methods of Massey [11]:

$$S(Z_1^n | M_1^n) = \sum_{k=1}^{n} S(Z_k | Z_1^{k-1} M_1^n) \geq \sum_{k=1}^{n} S(Z_k | Z_1^{k-1} A_1^k M_1^n) = \sum_{k=1}^{n} S(Z_k | Z_1^{k-1} A_1^k),$$

(2.3)
where we have used the fact that conditioning reduces the conditional von Neumann entropy (a direct consequence of strong subadditivity) and that $M_1^n \rightarrow A_1^n \rightarrow Z_1^n$ is a classical-quantum Markov chain. It then follows that

$$I(M_1^n : Z_1^n) \leq \sum_{k=1}^n I_n(A_k^n : Z_k | Z_k^{k-1}) = I_n(A_1^n \rightarrow Z_1^n).$$

(2.4)

Furthermore, we have $I_n(A_k^n : Z_k | Z_k^{k-1}) \leq I_k(A_k^n : Z_k | Z_k^{k-1})$, $1 \leq k \leq n$, a consequence of the conditional version of the data-processing inequality (see appendix), from which the result follows.

Theorem 2  Under the conditions described above, we have

$$C_F = \lim \sup_{n \rightarrow \infty} \frac{1}{n} I(A_1^n \rightarrow Z_1^n).$$

Here the supremum is over all classical-quantum states corresponding to $n$-block feedback codes, subject to the additional constraint that the above limit exists.

The theorem will be proved in 2 parts: Below we will the demonstrate the converse

$$C \leq \lim \sup_{n \rightarrow \infty} \frac{1}{n} I(A_1^n \rightarrow Z_1^n),$$

and provide a code that asymptotically (with respect to $n \rightarrow \infty$) achieves this upper bound, hence showing that

$$C \geq \lim \sup_{n \rightarrow \infty} \frac{1}{n} I(A_1^n \rightarrow Z_1^n).$$

Proof  Fix $n$ and consider an $n$-block product state feedback code

$$C_n = \{\mathcal{E}, \mathcal{M}, \mathcal{N}, f\}.$$

Define the message as $M_1^n$ and $A_1^n$ as the input codeword random variable with probability distribution $P(A_1^n = i_1^n(l)) = p_{i_1^n}(l)$. Our proof will involve calculating the classical mutual information between this random variable and the random vector $K_1^n = (K_1, \ldots, K_n)$. The probability distribution of this random vector is determined by the following set of relations:

$$P(K_1 = k_1) = \sum_l p_{i_1^n(l)} \text{tr} (\rho_{i_1^n(l)} F_{k_1}^1 F_{k_1}^{1*})$$
and
\[ P(K_m = k_m | K_1^{m-1} = k_1^{m-1}) = \text{tr}((I \otimes \cdots \otimes \Phi \otimes \cdots \otimes I)\omega(i_1^n(l), k_1, \ldots, k_{m-1}) F_{k_m}^m F_{k_m}^{m*}) \]

for \( 1 \leq m \leq n \).

It is sufficient [18] to provide an upper bound on \( H(M) \), the single-letter entropy of the message source, when \( P_e(n, 2^{nR}) \to 0 \). This is done by using the Fano inequality and the classical data-processing inequality [?]:
\[
\frac{H(M)}{n} \leq \frac{H(M|f(K^n_1)) + I(M:f(K_1^n))}{1 + P_e(n, 2^{nR})nR + I(M:K_1^n)} \leq \epsilon_n + \frac{I(M:K_1^n)}{n}
\]  
(2.6)

where \( \epsilon_n \to 0 \) as \( n \to \infty \).

Now from the Holevo bound [9], or alternatively, the data processing inequality [10], we have
\[ I(M : K_1^n) \leq I(M : Z_1^n). \]

Then applying the directed data-processing inequality, Lemma 2, we have
\[ I(M : Z_1^n) \leq I(A_1^n \to Z_1^n). \]

Substituting and letting \( n \to \infty \) then gives
\[ C_F \leq \lim \sup_{n \to \infty} \frac{I(A_1^n \to Z_1^n)}{n} \]

\[ \square \]

### 3 Achievability proof

In this section we complete the proof of Theorem 3 for discrete memoryless channels with feedback, by showing the direct part. We have already proved an upper bound for the feedback capacity using the Fano inequality. We will show that this bound is achievable by providing a code which asymptotically achieves this bound (as the number of channel uses \( \to \infty \)). The proof proceeds along similar lines to the HSW theorem proof, using a generalized version of the square-root measurements used in the direct part of the proof of that theorem.
We will consider the case of discrete memoryless channels. Our coding procedure is to use a "double-blocked" code— we will construct an $nl$-block feedback code from the simultaneous use of $l$ independent instances of an $n$-block feedback code. Our approach is essentially to perform an entangling measurement on the $l$-fold tensor product of the $k$-th round outputs of each $n$-block instance, with the purpose of correctly obtaining the classical strings labelling these states, with high probability. At the same time, these measurements will be shown to (in a sense defined later) not disturb the states they act on very much. Finally, we allow the block parameters $n, l \to \infty$ and obtain the asymptotic rate achievable using such an encoding and show that this rate is the upper bound obtained in the converse proof. The technical details of the approach are described below:

Given a rate $R > 0$ and the block-length $nl$, we define our code of size $N = 2^{nlR}$ in terms of $l$ copies of $n$-block feedback code $C_n$ of size $2^{nR}$: the code is random, and each codeword of length $nl$ is built up as a concatenation of the $l$ words $i_1^n(j)$ $1 \leq j \leq l$ chosen at random from the code $C_n$. The corresponding quantum codeword is the $l$-fold tensor product of the codeword states chosen. The communicating parties then use the following order of transmission and feedback operations: transmission rounds $kl + 1$ to $(k + 1)l$ consist of following the $(k + 1)$-th round of each copy of the protocol $C_n$, independently of the other copies for $1 \leq k \leq n - 1$. In addition at round $jl$, $1 \leq j \leq n$, the receiver performs a measurement $R_j$ with outcome denoted $R_j$, on the state in his possession at that point. The round $jl$ will, for convenience, be referred to as the $j$-th “global round”. The measurement will identify which state the receiver has transmitted up to that point. The specific form of these measurements $R_j$ will be discussed in the next section. The result of this measurement is returned to the sender and if it does not agree with the classical data sent, the protocol terminates with an error. The nature of the protocol implies that an error can occur on global round $j 1 \leq j \leq n$.

If no error occurs, the final outcome of the feedback protocol, is a function $f(R_1, \ldots, R_l)$ of $(R_1, \ldots, R_l)$, taking values in the set of input message strings of length $nl$.

For any choice of message $m$ will denote the input classical words of this protocol by $i_1^n(1, \ldots, l)$ or in shortened form $i^n(l)$, where this refers to the concatenation $(i_1^n(1), \ldots, i^n_l(l))$ of the classical codewords of the $l$ individual original copies of $C_n$. We suppress the dependence on $m$ here, because in the sequel, the message source will be assumed to have a uniform distribution over all possible messages, since, by the classical theory, this will yield the maximal average error probability [15].
The initial ensemble held by the sender is then of the form

\[ \{ p^n_i(1) \ldots p^n_i(l), \rho^n_i(1) \otimes \ldots \otimes \rho^n_i(l) \}, \]

reflecting the fact we consider, a priori, \( l \) tensor-product/independent input protocols. Equivalently, the input states for such a protocol can also be expressed in the EHS form discussed earlier as \( \rho^A_{n1} x^{n-1}_i z^n_i(1) \otimes \ldots \otimes \rho^A_{n1} x^{n-1}_i z^n_i(l) = \rho^A_{n1} x^{n-1}_i z^n_i(l) \), say, where each term in the tensor product is the EHS representation of the input protocol for one of the \( l \) copies of \( C_n \). Now the state transmitted in transmission rounds \((k-1)l+1\) to \( kl \) can be labelled with length \( l \) strings \( i_k(1, \ldots, l) = i_k(l) \) (in the notation introduced above). In the EHS representation the state held by the receiver at this point is given by \( \text{tr} \left( \rho^A_{n1} x^{n-1}_i z^n_i(l) \right) \), tracing out all except the first \( k \) rounds in each \( C_n \) sub-protocol. Thus the state held by the receiver at this time is labelled by the “classical” register \( A^k \), or alternatively, by strings \( i^k(l) \) of length \( l \). Similarly, the state received in rounds \((k-1)l\) to \((k+1)l\) can be labelled by length \( l \) strings \( i^k(l) \). In our notation, the same strings will label the measurement \( R_k \).

We will give a specification of the code, described generally in the first section, which achieves the capacity upper bound.

The decoding procedure is defined as follows: For \( \delta > 0 \), let \( \epsilon_k = 2^{-tko\delta^2} \), for some \( c > 0 \) and \( 1 \leq k \leq n \). We begin by constructing \( R_1 \) as a set of (Hermitian) operators \( R_r \) that satisfy sub-POVM condition:

\[ \sum_{r_1} R_{r_1} \leq I, \]

where \( r_1 = j^n_1 \) is the measurement outcome. For full generality, we will define \( R_0 = I - \sum_{j^n_1} R_{j^n_1} \) as the POVM element pertaining to a faulty output. The form of the measurements, described below, will depend explicitly on \( k_1(l) \), so the measurement made is in fact conditional on the feedback measurement outcomes for the individual copies of \( C_n \). The measurements \( R_j, 2 \leq j \leq n \) are constructed in a similar fashion and are conditional on the outcomes of \( R_1, \ldots, R_{j-1} \) and the feedback measurements of the individual ”copies” of \( C_n \), which will be represented by concatenated strings, in a slight abuse of notation, as \( i^{j-1}_1(l) = (i^j_1(l), \ldots, i^{j-1}_1(l)) \).

Given an initial ensemble as defined above,

\[ \mathcal{E} = \{ p^n_i(1) \ldots p^n_i(l), \rho^n_i(1) \otimes \ldots \otimes \rho^n_i(l) \}, \]

after \( tl \) rounds of communication, conditional on no error in measurements \( R_1, \ldots, R_{t-1} \) and given sequence of feedback outcomes

\[ (k^t_1(1), \ldots, k^t_1(l)) = k^t_1(l) \]
(in our notation), the state held by the receiver is in the ensemble denoted

$$\mathcal{E}(k_1^t(l), i_1^{t-1}(l)) = \{ p (i_1^n(l)|i_1^{t-1}(l), k_1^t(l)), \text{tr}_{(t+1-n)}(\omega_{i_1^n(l)}(k_1^t(l), i_1^{t-1}(l))) \},$$

where the notation $\text{tr}_{(t+1-n)}$ implies that we are tracing out all but the first $t$ rounds of communication for each copy of $C_n$ considered. To avoid confusion, note that the $i_1^{t-1}(l)$ upon which we condition, refers, as noted above, to the string that is the concatenation of the outcomes of the measurements $\mathcal{R}_1, \ldots, \mathcal{R}_t$. This reflects the fact that, with probability $p (i_1^n(l)|i_1^{t-1}(l), k_1^t(l))$, the joint state held by the sender and receiver is denoted $\omega_{i_1^n(l)}(k_1^t(l), i_1^{t-1}(l))$.

Here we define $\omega_{i_1^n(l)}(k_1^t(l), i_1^{t-1}(l))$ in terms of $\omega_{i_1^n(l)}(k_1^t(l), i_1^{t-2}(l))$ via the relation

$$\omega_{i_1^n(l)}(k_1^t(l), i_1^{t-1}(l)) = \otimes_{j=1}^t \mathcal{N}^{k_{t}(j)} \circ \otimes_{j=1}^{t-1} \mathcal{M}_{t, k_t(j)} \circ \left( \otimes_{j=1}^t \Phi^{(t)}(j) \right)$$

$$\mathcal{R}_{t-1, it-1}(l)(i_1^{t-2}(l), k_1^{t-1}(l)) \left( \omega_{i_1^n(l)}(k_1^{t-1}(l), i_1^{t-2}(l)) \right),$$

where $\mathcal{R}_{t-1, it-1}(l)(i_1^{t-2}(l), k_1^{t-1}(l))$ is the realisation of the single element POVM given by

$$\mathcal{R}_{t-1, it-1}(l)(k_1^{t-1}(l), i_1^{t-2}(l))(\rho) = \frac{\sqrt{R_{t-1, it-1}(l)(k_1^{t-1}(l), i_1^{t-2}(l))}}{\text{tr}(\rho R_{t-1, it-1}(l)(k_1^{t-1}(l), i_1^{t-2}(l)))} \frac{\rho R_{t-1, it-1}(l)(k_1^{t-1}(l), i_1^{t-2}(l))}{\text{tr}(\rho R_{t-1, it-1}(l)(k_1^{t-1}(l), i_1^{t-2}(l)))}$$

(3.7)

for POVM elements $R_{t-1, it-1}(l)(k_1^{t-1}(l), i_1^{t-2}(l))$ as specified below, corresponding to measurement outcome $i_t(l)$. $\mathcal{M}_{t, k_t(j)}$ is a realisation of the single-element POVM corresponding to the feedback measurement outcome $k_t(j)$ (on $j$-th copy of $C_n$), defined, for input density matrix $\rho$ as:

$$\mathcal{M}_{t, k_t(j)}(\rho) = \frac{F_{k_t(j)}^t \rho F_{k_t(j)}^{t*}}{\text{tr}(\rho F_{k_t(j)}^t F_{k_t(j)}^{t*})},$$

Also $\Phi^{(t)}(1) \otimes \ldots \otimes \Phi^{(t)}(l)$ represents the channel action on the $tl + 1 \rightarrow (t + 1)l$ rounds and $\mathcal{N}^{k_{t1}(1)} \otimes \ldots \otimes \mathcal{N}^{k_{tl}(l)}$, the feedback post-processing done by the sender before transmission on those rounds. Also, we have inductively

$$p (k_t(l)|i_1^n(l), k_1^{t-1}(l), i_1^{t-1}(l)) = \text{tr} \left( \otimes_{j=1}^t \Phi^{(t)}(j) \right)$$

$$\circ \left( \mathcal{R}_{t-1, it-1}(l)(i_1^{t-2}(l), k_1^{t-1}(l)) \left( \omega_{i_1^n(l)}(k_1^{t-1}(l), i_1^{t-2}(l)) \right) \right) \otimes_{j=1}^t F_{k_t(j)}^{t} F_{k_t(j)}^{t*}$$

(3.8)
where
\[
p(\rho_{i-1} | \rho_1, k_{i-1}, i_{i-1}^2) = \text{tr} \left( R_{i-1,i-1} (\rho_{i-2} | k_{1-i-1}, i_{i-1}^2) \right)
\]
and
\[
p \left( k_t | \rho_1, k_{i-1}, i_{i-1}^2 \right) = \text{tr} \left( \otimes_{j=1}^t \Phi^{(i)} (j) \right) \circ R_{i-1,i-1} (\rho_{i-2} | k_{1-i-1}, i_{i-1}^2) \right) \otimes_{j=1}^t F_{k_t(j)}^j F_{t_t(j)}^{*}
\]

(3.9)

Once again, we emphasise that we are conditioning here both on the initial input state, indexed by \(i_1^0\), and the sequence of receiver measurements, indexed by the strings \(k_1^1, i_{i-1}^1\). Then we calculate
\[
p(i_1^0 | k_1^1, i_1^1) = \frac{p(\rho_1, k_1^1, i_1^1 | i_1^0) p(i_1^0)}{\sum_{i_1^0} p(\rho_1, k_1^1, i_1^1)}
\]
The form of the measurements \(R_j\) is then given by the POVM elements:
\[
R_{i-1} (i_{i-1}^1, k_1^1) = \left( \sum_{r_t \neq i_t} \Gamma_{r_t} (k_t^1, i_{i-1}^1) \right)^{-1/2}
\]
\[
\Gamma_{r_t} (k_t^1, i_{i-1}^1) = \left( \sum \Gamma_{r_t} (k_t^1, i_{i-1}^1) \right)^{-1/2}
\]

where the summation is over all \(k_t^1, i_{i-1}^1\) and \(r_t \neq i_t\). Furthermore
\[
\Gamma_{r_t} (k_t^1, i_{i-1}^1) = \Pi_{r_t} (k_t^1, i_{i-1}^1) \Pi_{r_t} (k_t^1, i_{i-1}^1) \times \Pi_{r_t} (k_t^1, i_{i-1}^1)
\]
where \(\rho_t\) is shorthand for \(\rho_t (k_t^1, i_{i-1}^1)\), the ensemble average of
\[
E_t (k_t^1, i_{i-1}^1)
\]
and the typical subspace projectors \(\Pi_{r_t} (k_t^1, i_{i-1}^1)\) (defined in the appendix) are calculated with respect to ensemble \(\rho_t (k_t^1, i_{i-1}^1)\) or the corresponding EHS quantum state. The expected error probability on or before global round \(t\) (for error probability up to round \(t\) denoted \(P_t\) for this protocol is evaluated inductively as follows, where we have denoted the event that no error occurs on round \(t\) by \(E_t\):

\[
E (P_t) = E (P_{t-1}) + E (P (E_t | E_1, \ldots, E_{t-1})(1 - P_{t-1})) = E (P_{t-1}(1 - P (E_t | E_1, \ldots, E_{t-1}))) + E (P (E_t | E_1, \ldots, E_{t-1}))
\]

(3.10)
where we write $P(R_t|R_1, \ldots, R_{t-1})$ for the probability that an error occurs on round $t$ conditional on no error before this point. The expectation is taken over all random codes as constructed above. For $R \leq C$, from the argument given in the appendix it follows that

$$E(P_t) \leq E(P_{t-1}) + \epsilon'_t,$$

from which it is clear that

$$E(P_t) \leq \sum_{k=1}^{t} \epsilon'_k.$$

Now the error probability for the whole protocol is given by

$$E(P_n) \leq \sum_{k=1}^{n} k \epsilon'_k \leq \left(\sqrt{24} + 6\right) \sum_{k=1}^{n} k \epsilon_k + \sum_{k=1}^{n} k 2^{-t} t K,$$

which $\to 0$ when we let $l, n \to \infty$. Then by the standard argument, there a (non-random) code with asymptotically 0 error probability.

4 Conclusion:

We have generalised the usual protocols for classical communication using quantum channels, to include the possibility of feedback. We proved a (regularised, asymptotic) formula for the capacity using such protocols and showed that the usual classical capacity formula [9] follows from our formula. The capacity formula is expressed in terms of the quantum directed information, in an analogous sense to the classical feedback capacity result [15]. However, in a significant departure from the classical case, it was noted that the feedback capacity of a discrete memoryless channel may, at least for some channels, exceed its unassisted capacity if one allows the use of entangled input states.

Acknowledgements: The author would like to thank Yeo Ye and Yurii Suhov for useful and interesting discussions. This publication is an output from project activity funded by the Cambridge-MIT Institute limited (“CMI”). CMI is funded in part by the United Kingdom government. The activity was carried for CMI by Cambridge University and Massachusetts Institute of Technology. CMI can accept no responsibility for any information provided or views expressed.
5 Technical Remarks

In this section we will prove the estimates for the error probabilities that were used in the preceding section. First we require two lemmas: firstly to accurately describe to what extent the measurements $R_t$, $1 \leq t \leq n$, disturb the states they act on, and a second lemma given estimates for quantities involving typical projectors.

**Lemma 3** Let $\epsilon_t = 2^{-tlc^2}$ for some $c > 0$. Then for $t$ sufficiently large, we have

$$||\omega_{i_1}(k_1(1)) \otimes \ldots \otimes \omega_{i_l(l)}(k_1(l)) - \omega_{i_1}(\ell_1^{-1}(l), k_1(l))||_1 \leq \sum_{s=1}^t (\sqrt{24\epsilon_s} + 6\epsilon_s),$$

for all $k$. The norm $||A||_1$ is defined for all Hermitian $A$ in a finite dimensional space as the sum of the absolute values of the eigenvalues of $A$. Here, the states $\omega_{i_1}(j)$, $1 \leq j \leq n$ are defined recursively as per the specifications of the $n$-block feedback code $C_n$.

**Proof** This is done by induction. For $m = 1$, denote the POVM elements of $R_1$ as $R_{i_1}(\ell)$, with the understanding that we have "padded" the POVM elements by taking the tensor product of the original elements with identity operators on those copies of the channel space on which the measurement does not act. A similar convention is assumed for $R_{i_2}(\ell), \ldots, R_{i_n}(\ell)$. Then we have

$$\text{tr} \left( \omega_{i_1}(k_1(1)) \otimes \ldots \otimes \omega_{i_l(l)}(k_1(l)) R_{i_1}(\ell) \right) \geq 1 - 3\epsilon_1,$$

a consequence of Lemma 6 of Hayashi and Nagaoka [13]. This implies

$$||\omega_{i_1}(k_1(1)) \otimes \ldots \otimes \omega_{i_l(l)}(k_1(l)) - \sqrt{R_{i_1}(\ell) \omega_{i_1}(k_1(1)) \otimes \ldots \otimes \omega_{i_l(l)}(k_1(l)) \sqrt{R_{i_1}(\ell)}}||_1$$

$$\leq \sqrt{24\epsilon_1} + 6\epsilon_1,$$

by an extension of the Winter tender measurement lemma [14]. Thus the required inequality holds in this case, noting that

$$\omega_{i_1}(\ell,l) = \frac{\sqrt{R_{i_1}(\ell) \omega_{i_1}(k_1(1)) \otimes \ldots \otimes \omega_{i_l(l)}(k_1(l)) \sqrt{R_{i_1}(\ell)}}}{\text{tr} \left( \omega_{i_1}(k_1(1)) \otimes \ldots \otimes \omega_{i_l(l)}(k_1(l)) R_{i_1}(\ell) \right)}.$$
Assume that the statement holds for \( m = t - 1 \). Then for \( m = t \) we have the following inequalities:

\[
\| \omega_{i_1^t(1)}(k_1^t(1)) \otimes \ldots \otimes \omega_{i_1^n(l)}(k_1^l(l)) - \omega_{i_1^t(l)}(i_1^t(l), k_1^l(l)) \| \\
\leq \| \mathcal{G}_{k_t(l)}(\otimes_{j=1}^t \omega_{i_1^t(j)}(k_1^{t-1}(j))) - \mathcal{G}_{k_t(l)} \left( \frac{\sqrt{R_{i_1^t(l)}} \otimes_{j=1}^t \omega_{i_1^t(j)}(k_1^{t-1}(j)) \sqrt{R_{i_1^t(l)}}}{\operatorname{tr} \left( \otimes_{j=1}^t \omega_{i_1^t(j)}(k_1^{t-1}(j)) R_{i_1^t(l)} \right)} \right) \| \\
+ \| \mathcal{G}_{k_t(l)} \left( \frac{\sqrt{R_{i_1^t(l)}} \otimes_{j=1}^t \omega_{i_1^t(j)}(k_1^{t-1}(j)) \sqrt{R_{i_1^t(l)}}}{\operatorname{tr} \left( \otimes_{j=1}^t \omega_{i_1^t(j)}(k_1^{t-1}(j)) R_{i_1^t(l)} \right)} \right) - \mathcal{G}_{k_t(l)} \left( \omega_{i_1^t(l)}(i_1^{t-1}(l), k_1^{t-1}(l)) \right) \| \\
\leq \sum_{n=1}^t (\sqrt{24\epsilon_n} + 6\epsilon_n),
\]

using the triangle inequality for the trace norm, the H"{o}lder inequality \( \operatorname{tr}(|A\,B|) \leq \|A\|_1 \|B\|_1 \), the fact that the trace distance is non-decreasing under completely positive operations and the inductive hypothesis. Here the completely positive map \( G \) is given by

\[
\mathcal{G}_{k_t(l)} = \otimes_{j=1}^t N_{k_t-1(j)} \otimes_{j=1}^t M_{t,k_t(j)} \circ \left( \otimes_{j=1}^t \Phi(t)(j) \right) \circ \otimes_{j=1}^t N_{t,k_t-1(j)}
\]

What follows is an summary of some important definitions and results concerning typical projectors (see [19] for a more comprehensive discussion): Consider a general classical-quantum system \( UXQ \) of the form

\[
\rho_{UXQ} = \sum_{u,x} p_{u,x} |(u, x)\rangle \langle (u, x)| \otimes \rho_{u,x},
\]

where \( u \) is defined on set \( U \) and \( x \) is defined on set \( X \). The set of typical sequences is defined by

\[
\mathcal{T}_{p,\delta}^n = \{ x^n_1 : \forall x |N(x|x^n_1) - np(x)| \leq n\delta \},
\]

where \( N(x|x^n_1) \) counts the number of occurrences of \( x \) in the word \( x^n_1 = (x_1, \ldots, x_n) \).
For a density matrix $\rho = \sum_k \lambda_k |k\rangle \langle k|$, define the probability distribution $P(K = k) = \lambda_k$ and for $\delta > 0$ the typical projector

$$\Pi_{\rho,\delta} = \sum_{k\in T_{\rho,\delta}^n} |k_1^n\rangle \langle k_1^n|.$$ Here we use the shorthand $|k_1^n\rangle \langle k_1^n| = |k_1\rangle \langle k_1| \otimes \ldots \otimes |k_n\rangle \langle k_n|$. 

We also define the conditionally typical projector as:

$$\Pi_{\rho,\delta}^{n}(u_1^n) = \otimes_u \Pi_{\rho,\delta}^{I_u},$$

where $I_u = \{i : u_i = u\}$ and $\Pi_{\rho,\delta}^{I_u}$ is the typical projector of $\rho_u$ in those positions of the $n$-factor tensor product (representing $u_1^n$) indicated by $I_u$.

This notation is slightly abused in the main part of this text, where the conditional typical projector is written, for example, as

$$\Pi_{r_t}(i_{1}^{l-1}(\emptyset), k_{1}^{l}(\emptyset)),$$

where $r_t = i_{1}^{l}(\emptyset)$ for some length $l$ string $i_{1}^{l}(\emptyset)$. This refers to a projector (with respect to a $l$ factor tensor product) of the form

$$\otimes_{j=1}^{l} \Pi_{\rho_{i_{1}^{l}(\emptyset)},k_{1}^{l}(\emptyset),\delta},$$

with $\Pi_{\rho_{i_{1}^{l}(\emptyset)},k_{1}^{l}(\emptyset),\delta}$ the usual typical projector.

With the above definitions, we have the following lemma:

**Lemma 4** For typical projector $\Pi_{\rho,\delta}$, the following relations hold:

$$\text{tr}(\rho^{\otimes n} \Pi_{\rho,\delta}) \leq 2^{-n(S(\rho)+c\delta)}$$

and

$$\Pi_{\rho,\delta}^{\otimes n} \Pi_{\rho,\delta} \leq 2^{-n(S(\rho)+c\delta)} \Pi_{\rho,\delta}^{n}$$

**Proof**

We are now able to estimate the error probabilities, which we evaluate as:

$$P(E_k|E_1, \ldots, E_{k-1}) = 1 - \text{tr}(\omega_{i_{1}^{l}(1)}(k_{1}^{l}(1))) \otimes \ldots \otimes \omega_{i_{1}^{l}(l)}(k_{1}^{l}(l)) R_{r_t}(\emptyset) + \delta_{k-1},$$
where \( \delta_{t-1} = \sum_{m=1}^{t-1}(6\epsilon_m + \sqrt{24\epsilon_m}) \). This follows from the above Lemma. Then we have

\[
1 - \text{tr}(\otimes_{j=1}^{l} \omega_{i_{1}^{(j)}}(k_{1}^{(j)})) R_{i_{1}}(\underline{Q}) \leq 2\text{tr}\left(\otimes_{j=1}^{l} \omega_{i_{1}^{(j)}}(k_{1}^{(j)})(I - \Gamma_{i_{1}}(l', k_{1}^{(0)}, i_{1}^{t-1}(l)))\right) + 4\sum_{r_{t} \neq i_{1}} \text{tr}\left(\otimes_{j=1}^{l} \omega_{i_{1}^{(j)}}(k_{1}^{(j)}) \Gamma_{r_{t}}(k_{1}^{(l)}, i_{1}^{t-1}(l))\right).
\]

(5.11)

by Lemma 6 in Hayashi and Nagaoka [13]. Furthermore we have

\[
\text{tr}\left(\omega_{i_{1}^{(1)}}(k_{1}^{(1)}) \otimes \ldots \otimes \omega_{i_{1}^{(l)}}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l))\right) \geq 1 - 3\epsilon_{t}
\]

by Lemma 6 in Hayashi and Nagaoka [13]. In addition we have

\[
\sum \omega_{i_{1}^{(1)}}(k_{1}^{(1)})^2 \Pi_{i_{1}}^{l}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l)) = \text{tr}\left(\sum \omega_{i_{1}^{(1)}}(k_{1}^{(1)})^2 \Pi_{i_{1}}^{l}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l))\right).
\]

(5.12)

This in turn is equal to

\[
\text{tr}\left(\sum_{i_{1}^{(l)}} \Pi_{i_{1}}^{l}(k_{1}^{(l)}) \omega_{i_{1}^{(l)}}(k_{1}^{(l)})^{2} \Pi_{r_{t}}^{l}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l))\right) \leq 2^{-lH(\mathcal{Z}|\mathcal{Z}_{1}^{t-1}) - l\delta \epsilon} \text{tr}\left(\Pi_{i_{1}^{(l)}}^{l}(k_{1}^{(l)}) \omega_{i_{1}^{(l)}}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l))\right),
\]

for some constant \( c > 0 \). This follows from the inequality

\[
\Pi_{i_{1}^{(l)}}^{l}(k_{1}^{(l)}) \sum_{i_{1}, k_{1}} p_{i_{1}, k_{1}} \omega_{i_{1}, k_{1}}^{2} \Pi_{r_{t}}^{l}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l)) \leq 2^{-lH(\mathcal{Z}|\mathcal{Z}_{1}^{t-1}) - l\delta \epsilon} \Pi_{i_{1}^{(l)}}^{l}(k_{1}^{(l)}) \omega_{i_{1}^{(l)}}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l)).
\]

Then we have

\[
\text{tr}\left(\Pi_{i_{1}^{(l)}}^{l}(k_{1}^{(l)}) \omega_{i_{1}^{(l)}}(k_{1}^{(l)}) \Gamma_{i_{1}^{(l)}}(k_{1}^{(l)}, i_{1}^{t-1}(l))\right) \leq 2^{-lH(\mathcal{Z}|\mathcal{Z}_{1}^{t-1}) - l\delta \epsilon}.
\]

Putting these estimates together we get

\[
\mathbb{E}\left(1 - \text{tr}(\omega_{i_{1}^{(1)}}(k_{1}^{(1)}) \otimes \ldots \otimes \omega_{i_{1}^{(l)}}(k_{1}^{(l)}) R_{i_{1}}(l))\right) \leq 6\epsilon_{t} + 4.2^{n\left(H(\mathcal{Z}) - \frac{1}{n} I(\mathcal{A}_{1}; \mathcal{Z}|\mathcal{Z}_{1}^{t-1})\right)}.
\]
In exactly the same way we get

\[ \mathbb{E} \left( 1 - \text{tr}(\omega_{i_1(1)} \otimes \ldots \otimes \rho_{i_l(l)} R_{i_1(l)}) \right) \leq 6\epsilon_1 + 4.2^n(n^{R(1)} - n^{I(A_1:Z_1)}) , \]

where, for \( 1 \leq t \leq n \), \( R(t) = \log N_t \), for \( N_t \) defined as the number of strings \( i_t(l) \) that label the measurement \( R_t \). We necessarily have \( R(1) + \ldots + R(n) = R \). We have assumed in this paper that we consider only protocols for which the limit \( C = \lim_{n \to \infty} \frac{1}{n} I(A_1^n : Z_1^n) \) exists. For any positive numbers \( R(1), \ldots, R(n) \) as above satisfying \( R(t) \leq I(A_t^n : Z_t^n) \), we clearly have \( R \leq C \) and there exists \( K > 0 \) such that for \( n, l \) sufficiently large, we have

\[
P(E_t|E_1, \ldots, E_{t-1}) \leq \sum_{m=1}^{t-1} (6\epsilon_m + \sqrt{24\epsilon_m}) + 6\epsilon_t + 2^{-nlK} \\
\leq (\sum_{m=1}^t \epsilon_m)(6 + \sqrt{24}) + 2^{-nlK} = \epsilon'_t. \tag{5.13}
\]

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