Generalized $b$-Symbol Weights of Linear Codes and $b$-Symbol MDS Codes

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Abstract—Generalized pair weights of linear codes are generalizations of minimum symbol-pair weights, which were introduced by Liu and Pan (2022) recently. Generalized pair weights can be used to characterize the ability of protecting information in the symbol-pair read wire-tap channels of type II. In this paper, we introduce the notion of generalized $b$-symbol weights of linear codes over finite fields, which is a generalization of generalized Hamming weights and generalized pair weights. We obtain some basic properties and bounds of generalized $b$-symbol weights which are called Singleton-like bounds for generalized $b$-symbol weights. As examples, we calculate the generalized weight matrices for simplex codes and Hamming codes. We provide a necessary and sufficient condition for a linear code to be a $b$-symbol MDS code by using the generator matrix and the parity check matrix of this linear code. Finally, a necessary and sufficient condition of a linear isomorphism preserving $b$-symbol weights between two linear codes is obtained. As a corollary, we get the classical MacWilliams extension theorem when $b = 1$.

Index Terms—Generalized $b$-symbol weights, $b$-symbol MDS codes, linear isomorphisms preserving $b$-symbol weights, MacWilliams extension theorem.

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I. INTRODUCTION

In 2011, MOTIVATED by the limitations of the reading process in high density data storage systems, Cassuto and Blaum [3] introduced a new metric framework, named symbol-pair distance, to protect against pair errors in symbol-pair read channels, where the outputs are overlapping pairs of symbols. They also established relationships between the minimum Hamming distance and the minimum pair distance of the code, and obtained lower and upper bounds on the code sizes by using symbol-pair distance. In [4], the authors established a Singleton-like bound for symbol-pair codes and constructed MDS symbol-pair codes (meeting this Singleton-like bound). Several works have been done on the constructions of MDS symbol-pair codes (see, for example, [4], [5], [6], [13], [14], [15], [17], and [18]). In [8] and [9], the authors calculated the symbol-pair distances of repeated-root constacyclic codes of lengths $p^s$ and $2p^s$, respectively. In 2016, Yaakobi, Bruck and Siegel [24] generalized the notion of symbol-pair weight to $b$-symbol weight. Yang, Li and Feng [25] showed the Plotkin-like bound for the $b$-symbol weight and presented a construction on irreducible cyclic codes and constacyclic codes meeting the Plotkin-like bound.

On the other hand, the notion of generalized Hamming weights appeared in the 1970s and has become an important research object in coding theory after Wei’s work [23] in 1991. Wei [23] showed that the generalized Hamming weight hierarchy of linear codes has a close connection with cryptography. Since then, lots of works have been done in computing and describing the generalized Hamming weight hierarchies of certain linear codes (see, for example, [1], [12], [22] and [26]). In [16], we introduced the notion of generalized pair weights of linear codes, which is a generalization of minimum symbol-pair weights of linear codes. We obtained some bounds for generalized pair weights and gave an application of generalized pair weights of linear codes to symbol-pair read wire-tap channels of type II.

It is well known that the MacWilliams extension theorem plays a central role in coding theory. MacWilliams [19] and later Bogart, Goldberg, and Gordon [2] proved that, every linear isomorphism preserving Hamming weights between two linear codes over finite fields can be induced by a monomial matrix. It is interesting to ask how about the behavior of every linear isomorphism preserving the $b$-symbol weights between two linear codes. Unfortunately, we found that a linear isomorphism induced by a permutation matrix may not preserve the $b$-symbol weight between two linear codes. In [16], the authors provided a necessary and sufficient condition for a linear isomorphism preserving pair weights between two linear codes.

In 2018, Ding, Zhang and Ge [7] established the Singleton-like bound

$$d_b(C) \leq n + b - k$$

for an $[n,k]$-linear code $C$, where $d_b(C)$ is the minimum $b$-symbol weight of $C$ defined in Section 2. Since no linear code exists that reaches this bound in [7] when $b > k$, we give an improvement for the Singleton-like bound

$$d_b(C) \leq \min\{n + b - k, n\}$$

in Theorem 3.5, which is a small part of the Singleton-like bound for generalized weight matrices of linear codes in Theorem 3.11. Linear codes meeting this bound are called $b$-symbol MDS codes in this paper (see Def. 3.7). Then we show that the length of $b$-symbol MDS codes is as large as...
possible when \( b \geq k \) in Corollary 5.6, which is different from the MDS conjecture that the length of a 1-symbol MDS code is less than or equal to \( q + 1 \) or \( q + 2 \) (some special cases). Let \( n_{r,b} \) be the number of all the subspaces of dimension \( r \) of a vector space of dimension \( k \). It is interesting that when we study the \( b \)-symbol weights of linear codes and \( b \)-symbol MDS codes, we found that the length \( n \leq n_{1,b+1} \) for any \([n,k]\)-linear \( b \)-symbol MDS code over \( \mathbb{F}_q \) if \( b = k - 1 \) in Corollary 5.7. And the MDS conjecture is that if \( b = 1 \) then \( n \leq n_{1,b+1} \) for any nontrivial \([n,k]\)-linear \( b \)-symbol MDS code over \( \mathbb{F}_q \) except \( q \) is even and \( k = 3 \) or \( k = q - 1 \). Hence it is bold to conjecture that

\[
n \leq n_{1,b+1}
\]

for any \([n,k]\)-linear \( b \)-symbol MDS code over \( \mathbb{F}_q \) except some special cases, for example, \( q \) is even and \( k = 3 \) or \( k = q - 1 \).

In this paper, we unify the works of [16] and [23] to introduce generalized \( b \)-symbol weights of linear codes for \( 1 \leq b \leq n \). We define the generalized weight matrix \( D(C) \) of a linear code \( C \) in Section 2. The parameters about generalized \( b \)-symbol weights of the linear code \( C \) for \( 1 \leq b \leq n \) can be obtained from the generalized weight matrix \( D(C) \). Some properties on the generalized weight matrix \( D(C) \) are proved in Theorem 3.11. And we calculate the generalized weight matrices \( D(C) \) of simplex codes and two special Hamming codes in Section 4. In Section 6, we provide a necessary and sufficient condition of a linear isomorphism preserving \( b \)-symbol weights between two linear codes. As a corollary, when \( b = 1 \), we obtain the classical MacWilliams extension theorem. Using this result, an algorithm to determine whether an isomorphism between two linear codes preserves \( b \)-symbol weights is provided. And we explain why this algorithm is more efficiently than simply checking \( b \)-symbol weights of all the codewords of two codes in Remark 6.6.

This paper is organized as follows. Section 2 provides some preliminaries. We introduce the notion of generalized \( b \)-symbol weights of linear codes and give a characterization of the \( b \)-symbol weight of an arbitrary subspace of linear codes. In Section 3, we give a relationship between generalized Hamming weights and generalized \( b \)-symbol weights of linear codes and obtain a Singleton-like bound for generalized \( b \)-symbol weights. As examples, we calculate generalized weight matrices \( D(C) \) in Section 4, when \( C \) are simplex codes or two special Hamming codes. In Section 5, we provide a necessary and sufficient condition for a linear code to be a \( b \)-symbol MDS code by using a generator matrix and a parity check matrix of this linear code, and we give a new construction of an \([n,k]\)-linear \((k-1)\)-symbol MDS code over \( \mathbb{F}_q \) by using this result. In Section 6, we study linear isomorphisms preserving \( b \)-symbol weights of linear codes, and obtain a necessary and sufficient condition of a linear isomorphism preserving \( b \)-symbol weights.

II. PRELIMINARIES

Throughout this paper, let \( \mathbb{F}_q \) be the finite field of order \( q \), where \( q = p^r \) and \( p \) is a prime. And let \( \mathbb{N} = \{0,1,2,\cdots\} \) be the set of all natural numbers and \( \mathbb{N}^+ = \mathbb{N} \setminus \{0\} \).

An \( \mathbb{F}_p \)-subspace \( C \) of dimension \( k \) of \( \mathbb{F}_q^n \) is called an \([n,k]\)-linear code for \( k \leq n \in \mathbb{N}^+ \). The dual code \( C^\perp \) of \( C \) is defined as

\[
C^\perp = \{ \mathbf{x} \in \mathbb{F}_q^n | \mathbf{x} \cdot \mathbf{c} = 0, \forall \mathbf{c} \in C \},
\]

where \("^\perp"\) is the standard Euclidean inner product.

For \( n, b \in \mathbb{N}^+ \), we always assume \( 1 \leq b \leq n \) in this paper.

Definition 2.1: ([24]) For any \( \mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n \), the \( b \)-symbol distance between \( \mathbf{x} \) and \( \mathbf{y} \) is defined as

\[
d_b(\mathbf{x}, \mathbf{y}) = |\{0 \leq i \leq n - 1 | (x_i, x_{i+1}, \cdots, x_{i+b-1}) \neq (y_i, y_{i+1}, \cdots, y_{i+b-1}) \}|,
\]

where the indices are taken modulo \( n \). The \( b \)-symbol weight of \( \mathbf{x} \) is defined as \( w_b(\mathbf{x}) = d_b(\mathbf{x}, \mathbf{0}) \).

Definition 2.2: Let \( D \) be an \( \mathbb{F}_q \)-subspace of \( \mathbb{F}_q^n \), the \( b \)-symbol support of \( D \), is

\[
\chi_b(D) = \{0 \leq i \leq n - 1 | \exists \mathbf{x} = (x_0, x_1, \cdots, x_{n-1}) \in D, (x_i, x_{i+1}, \cdots, x_{i+b-1}) \neq (0, 0, \cdots, 0)\},
\]

where the indices are taken modulo \( n \). The \( b \)-symbol weight of \( D \) is defined as \( w_b(D) = |\chi_b(D)| \).

In particular, if \( C \) is an \([n,k]\)-linear code over \( \mathbb{F}_q \), the minimum \( b \)-symbol weight of \( C \) is defined as

\[
d_b(C) = \min_{ \mathbf{c} \neq \mathbf{0} \in C } d_b(\mathbf{c}, \mathbf{c}') = \min_{ \mathbf{c} \neq \mathbf{0} \in C } w_b(\mathbf{c}).
\]

We denote by \( \langle S \rangle \) the \( \mathbb{F}_q \)-subspace generated by the subset \( S \subseteq \mathbb{F}_q^n \). By Definition 2.2, we know that \( w_b(\mathbf{x}) = w_b(\langle \mathbf{x} \rangle) \) for any \( \mathbf{x} \in \mathbb{F}_q^n \). For convenience, we denote \( \chi_b(\mathbf{x}) = \chi_b(\langle \mathbf{x} \rangle) \) for any \( \mathbf{x} \in \mathbb{F}_q^n \).

Definition 2.3: Let \( C \) be an \([n,k]\)-linear code over \( \mathbb{F}_q \). For \( 1 \leq r \leq k \), and \( 1 \leq b \leq n \), the \( r \)th generalized \( b \)-symbol weight of \( C \) is defined as \( d_b^r(C) = \min\{ w_b(D) | D \leq C, \dim(D) = r \} \). The parameters \( d_1^1(C), d_2^2(C), \cdots, d_b^b(C) \) are called generalized \( b \)-symbol weights of \( C \).

Remark 2.4: When \( b = 1 \), the \( r \)th generalized 1-symbol weight \( d_1^r(C) \) of \( C \) is the \( r \)th generalized Hamming weight of an \([n,k]\)-linear code \( C \) over \( \mathbb{F}_q \) for \( 1 \leq r \leq k \) defined by Wei [23]. Also we know that \( d_1^1(C) \) and \( d_2^2(C) \) are the minimum Hamming weight and the minimum pair weight of a linear code \( C \), respectively.

When \( b = n \), \( w_b(\mathbf{c}) = n \) for any nonzero \( \mathbf{c} \in C \) and \( d_n^n(C) = n \) for any \( 1 \leq r \leq k \). If \( D \) is an \( \mathbb{F}_q \)-subspace of \( C \) with \( \dim(D) \geq 1 \), we have \( d_n^n(C) \leq d_n^r(D) \) for any \( 1 \leq r \leq \dim(D) \).

For convenience, we let \( d_b(C) = d_b^1(C) \) for any linear code \( C \) when \( r = 1 \). Since we want to study all the generalized \( b \)-symbol weights of the linear code \( C \) for \( 1 \leq b \leq n \), we introduce the following definition.

Definition 2.5: For an \([n,k]\)-linear code \( C \) over \( \mathbb{F}_q \), we define an \( n \times k \) matrix \( D(C) \) over the field of real numbers as follows:

\[
D(C) = (d_b^r(C))_{n \times k} = \begin{pmatrix}
d_1^1(C) & d_2^2(C) & \cdots & d_b^b(C) \\
\vdots & \vdots & \ddots & \vdots \\
d_n^1(C) & d_n^2(C) & \cdots & d_n^k(C)
\end{pmatrix}_{n \times k},
\]
where \(d_k^b(C)\) is the \(r\)th generalized \(b\)-symbol weight of \(C\), for \(1 \leq b \leq n\) and \(1 \leq r \leq k\). The matrix \(D(C)\) is called the generalized weight matrix of a linear code.

**Remark 2.6:** (a) The elements of the first row of the generalized weight matrix \(D(C)\) are generalized Hamming weights of \(C\), which are used in wire-tap channel of type II [23]. The elements of the second row of \(D(C)\) are generalized pair weights of \(C\), which are used in symbol-pair read wire-tap channels of type II [16]. The elements of the first column of \(D(C)\) are all the minimum \(b\)-symbol weights of \(C\) for \(1 \leq b \leq n\).

(b) The elements \(d_k^b(C)\) of \(D(C)\) satisfy some certain rules (see Theorems 3.5, 3.10 and 3.11). Then we can calculate the generalized weight matrices of simplex codes and Hamming codes by using those rules of the generalized weight matrix in Section 4. There is also a closed connection between \(D(C)\) and the interior structure of the linear code (see Theorem 3.10 (e)).

(c) We should mention that if \(C\) is a \(b\)-symbol MDS code (defined in Section 3), then \(D(C)\) must satisfy certain conditions in Theorem 3.11.

Let \(U\) be an \(F_q\)-vector space of dimension \(k\). We denote by \(U/W\) the quotient space modulo \(W\), where \(W\) is an \(F_q\)-subspace of \(U\). For any \(r, k \in \mathbb{N}\), let

\[
\begin{align*}
PG^<(U) &= \{V \subseteq U \mid \dim(V) = r\}, \\
PG^\leq r(U) &= \{V \subseteq U \mid \dim(V) \leq r\}.
\end{align*}
\]

It is trivial that \(\dim(\{0\}) = 0\) and \(PG^0(U) = \{0\}\). Let \(n_{r,k}\) be the number of all the subspaces of dimension \(r\) of a vector space of dimension \(k\). It is easy to see that

\[
n_{r,k} = \begin{cases} 
1, & \text{if } r = 0; \\ 
\prod_{i=0}^{r-1} \frac{q^k - q^i}{q^k - q^i}, & \text{if } 1 \leq r \leq k; \\ 
0, & \text{if } r > k.
\end{cases}
\]

Let \(C\) be an \([n, k]\)-linear code with a generator matrix \(G = (G_0, \cdots, G_{n-1})\), where \(G_i\) is the \(i\)th column vector of \(G\) for \(0 \leq i \leq n-1\). For any \(V \in PG^\leq k(F_q^n)\), the function \(m_G^b : PG^\leq k(F_q^n) \rightarrow \mathbb{N}\) is defined as follows:

\[
m_G^b(V) = |\{0 \leq i \leq n-1 \mid (G_i, G_{i+1}, \cdots, G_{i+b-1}) \subseteq V\}|
\]

where the indices are taken modulo \(n\). By using the function \(m_G^b\), we define the function \(\theta_G : PG^\leq k(F_q^n) \rightarrow \mathbb{N}\) to be

\[
\theta_G^b(U) = \sum_{V \subseteq PG^\leq k(U)} m_G^b(V)
\]

for any \(U \subseteq PG^\leq k(F_q^n)\).

For an \([n, k]\)-linear code \(C\) over \(F_q\) with a generator matrix \(G\), we know that for any \(1 \leq r \leq k\) and an \(F_q\)-subspace \(D\) of dimension \(r\) of \(C\), there exists a unique \(F_q\)-subspace \(\tilde{D}\) of dimension \(r\) of \(F_q^n\) such that \(D = \tilde{D}G\). Also we know that for any nonzero codeword \(c \in C\), there exists a unique nonzero vector \(y \in F_q^n\) such that \(c = yG = (yG_0, yG_1, \cdots, yG_{n-1})\), where \(G = (G_0, \cdots, G_{n-1})\).

**Lemma 3.1:** Assume the notation is given above. Let \(D\) be an \(F_q\)-subspace of \(F_q^n\). Then

\[
w_b(D) = w_1(D) + \sum_{H \in \mathbb{H}(\chi(D)), |H| \leq b-1} |H| + \sum_{H \in \mathbb{H}(\chi(D)), |H| \geq b} (b-1),
\]

where \(\mathbb{H}(\chi(D))\) is the set of all the holes of \(\chi(D)\).

**Proof:** Suppose \(i \in \chi(D)\), there exists \(x = (x_0, x_1, \cdots, x_{n-1}) \in D\) such that \(x_i \neq 0\). Then we know

\[
(x_{i+b+1}, x_{i+b+2}, \cdots, x_{i}),
\]

\[
(x_{i-b+2}, x_{i-b+3}, \cdots, x_{i+1}),
\]

\[
(x_i, x_{i+1}, \cdots, x_{i+b-1})
\]

are not \(0\). Hence \(i-b+1, i-b+2, \cdots, i \in \chi_b(D)\).

If \(H = \{a_0 + 1, a_0 + 2, \cdots, a_0 + |H|\}\) is an element of \(\mathbb{H}(\chi_1(D))\) and \(|H| \leq b-1\), we have \(H \subseteq \chi_b(D)\) since

\[
a_0 + |H| + 1 \in \chi_1(D).
\]
If $H = \{a_0 + 1, a_0 + 2, \cdots, a_0 + |H|\}$ is an element of $\mathbb{H}(\chi_1(D))$ and $|H| \geq b$, we have
\[
\{a_0 + |H| - b + 2, a_0 + |H| - b + 3, \cdots, a_0 + |H|\} \subseteq \chi_0(D)
\]
and $\{a_0 + 1, a_0 + 2, \cdots, a_0 + |H| - b + 1\} \subseteq \mathbb{Z}_n \setminus \chi_0(D)$. Hence
\[
w_b(D) = w_1(D) + \sum_{H \in \mathbb{H}(\chi_1(D)), |H| \leq b-1} |H| + \sum_{H \in \mathbb{H}(\chi_1(D)), |H| \geq b} (b - 1).
\]

**Theorem 3.2:** Assume the notation is given above. Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$. For $1 \leq r \leq k - 1$, we have
\[
\min\{d_1^r(C) + b - 1, n\} \leq d_b^r(C) \leq \min\{bd_1^r(C), n\}
\]

**Proof:** If $d_1^r(C) + b - 1 \geq n$, then
\[
|\chi_1(D)| \geq d_1^r(C) \geq n - b + 1
\]
and $n - |\chi_1(D)| \leq b - 1$ for any $\mathbb{F}_q$-subspace $D$ of dimension $r$ of $C$. Then we have $|H| \leq b - 1$ for any $H \in \mathbb{H}(\chi_1(D))$. By Lemma 3.1,
\[
w_b(D) = w_1(D) + \sum_{H \in \mathbb{H}(\chi_1(D)), |H| \leq b-1} |H| + \sum_{H \in \mathbb{H}(\chi_1(D)), |H| \geq b} (b - 1).
\]

For any $\mathbb{F}_q$-subspace $D$ of dimension $r$ of $C$. Hence
\[
\min\{d_1^r(C) + b - 1, n\} \leq d_b^r(C) = n.
\]

If $d_1^r(C) + b - 1 < n$, then there exists an $\mathbb{F}_q$-subspace $E$ of $C$ such that $\dim(E) = r$ and $w_b(E) = d_b^r(C)$. By Lemma 3.1, we have
\[
w_b(E) = w_1(E) + \sum_{H \in \mathbb{H}(\chi_1(E)), |H| \leq b-1} |H| + \sum_{H \in \mathbb{H}(\chi_1(E)), |H| \geq b} (b - 1).
\]

If $|H| \leq b - 1$ for any $H \in \mathbb{H}(\chi_1(E))$, then $d_b^r(C) = w_b(E) = n > d_1^r(C) + b - 1$.

If there exists $H \in \mathbb{H}(\chi_1(E))$ such that $|H| \geq b$, then
\[
d_b^r(C) = w_b(E) \geq w_1(E) + \sum_{H \in \mathbb{H}(\chi_1(E)), |H| \geq b} (b - 1)
\]
\[
\geq w_1(E) + b - 1
\]
\[
\geq d_b^r(C) + b - 1.
\]

Hence, we get $\min\{d_1^r(C) + b - 1, n\} \leq d_b^r(C)$.

Now we prove that $d_b^r(C) \leq bd_1^r(C)$. There exists an $\mathbb{F}_q$-subspace $D$ of $C$ such that $\dim(D) = r$ and $\dim(D) = d_b^r(C)$. By Lemma 3.1, we have
\[
w_b(D) = w_1(D) + \sum_{H \in \mathbb{H}(\chi_1(D)), |H| \leq b-1} |H| + \sum_{H \in \mathbb{H}(\chi_1(D)), |H| \geq b} (b - 1)
\]
\[
\leq w_1(D) + (b - 1)(|\mathbb{H}(\chi_1(D))|)
\]
\[
\leq bw_1(D).
\]

Hence $d_b^r(C) \leq w_b(D) \leq bw_1(D) = bd_1^r(C)$.

**Remark 3.3:** When $r = 1$, Proposition 3 of [24] is a corollary of Theorem 3.2. When $r = 2$, Theorem 3.2 of [16] is a corollary of Theorem 3.2.

Before we give the next theorem, we need the following lemma.

**Lemma 3.4:** Let $C$ be an $[n, k]$-linear code over $\mathbb{F}_q$. Then
\[(a)\] For any $i \in \mathbb{Z}_n$, there exists an $\mathbb{F}_q$-subspace $D$ of $C$ such that $\dim(D) = k - 1$ and $i \in \mathbb{Z}_n \setminus \chi_1(D)$.
\[(b)\] For any $i \in \mathbb{Z}_n$ and $1 \leq b < k$, there exists an $\mathbb{F}_q$-subspace $D$ of $C$ such that $\dim(D) = k - b$ and $\{i, i + 1, \ldots, i + b - 1\} \subseteq \mathbb{Z}_n \setminus \chi_1(D)$.

**Proof:** (a) If $i \in \chi_1(C)$, we know that every $\mathbb{F}_q$-subspace $D$ of $C$ with $\dim(D) = k - 1$ satisfies $i \in \mathbb{Z}_n \setminus \chi_1(D)$.

Assume $i \in \chi_1(C)$. Let
\[
D = \{e = (c_0, c_1, \ldots, c_{n-1}) \in C \mid c_i = 0\},
\]
then $D$ is an $\mathbb{F}_q$-subspace $D$ of $C$ such that $\dim(D) = k - 1$ and $i \in \mathbb{Z}_n \setminus \chi_1(D)$.

(b) We prove (b) by induction on $b$. When $b = 1$, we have proved the statement in (a).

Assume $b \geq 2$. By (a), there exists an $\mathbb{F}_q$-subspace $E$ of $C$ such that $\dim(E) = k - 1$ and $i \in \mathbb{Z}_n \setminus \chi_1(D)$.

By induction, there exists an $\mathbb{F}_q$-subspace $D$ of $E$ such that $\dim(D) = k - b$ and
\[
\{i + 1, i + 2, \ldots, i + b - 1\} \subseteq \mathbb{Z}_n \setminus \chi_1(D).
\]
Since $i \in \mathbb{Z}_n \setminus \chi_1(D)$, we have
\[
\{i, i + 1, i + 2, \ldots, i + b - 1\} \subseteq \mathbb{Z}_n \setminus \chi_1(D).
\]

Hence we obtain the $\mathbb{F}_q$-subspace $D$, which is what we expect.

For any $[n, k]$-linear code $C$ over $\mathbb{F}_q$, it is easy to know that
\[
b \leq d_b^1(C) \leq d_b^2(C) \leq \cdots \leq d_b^{k-1}(C) \leq d_b^k(C) \leq n.
\]
We give an improvement of these inequalities in the next theorem.

**Theorem 3.5:** Assume the notation is given above. Then
\[(a)\] For $1 \leq r \leq k - 1$, if $d_b^{r+1}(C) < n$, then $d_b^r(C) < d_b^{r+1}(C)$.
\[(b)\] If $k > b$, then $d_b^{k-b}(C) < n$.
\[(c)\] If $k > b$, then $b \leq d_b^r(C) < d_b^2(C) \leq \cdots < d_b^{k-b}(C) < d_b^{k-b+1}(C) \leq \cdots \leq d_b^k(C) \leq n$.
\[(d)\] For $1 \leq r \leq k - 1$, $d_b^r(C) \leq \min\{n - k + b + r - 1, n\}$, which is called the Singleton-like bound for generalized $b$-symbol weights. In particular, when $r = 1$, $d_b^1(C) \leq \min\{n - k + b, n\}$, which is called the Singleton-like bound for $b$-symbol weights.

**Proof:** (a) There exists an $\mathbb{F}_q$-subspace $E$ of $C$ such that $\dim(E) = r + 1$ and $w_b(E) = d_b^{r+1}(C)$. Then there exists
\[
H = \{a_0 + 1, a_0 + 2, \cdots, a_0 + |H|\} \in \mathbb{H}(\chi_1(E))
\]
such that \(|H| \geq b\) otherwise \(n = w_b(E) = d_b^{r+1}(C)\) by Lemma 3.1. Suppose
\[
\tilde{E} = \{x = (x_0, x_1, \ldots, x_{n-1}) \in E \mid x_{a_0} = 0\}.
\]
Then we get \(\tilde{E} \in E\) and \(\dim(\tilde{E}) = r\) since \(a_0 \in \chi_1(E)\).
Then \(\chi_1(\tilde{E}) \subseteq \chi_1(E) \setminus \{a_0\}\) and \(a_0 \in b(E) \setminus \chi_0(\tilde{E})\) since
\(\{a_0, a_0 + 2, \ldots, a_0 + |H|\} \subseteq \chi_0(\tilde{E})\) and \(|H| \geq b\).
Hence \(d_b^r(C) \leq w_b(E) = d_b^{r+1}(C)\).
(b) By Lemma 3.4 (b), there exists an \(\mathbb{F}_q\)-subspace \(D\) of \(C\) such that
\[
\dim(D) = k - b\quad \text{and}
\]
\[
\{0, 1, \ldots, b - 1\} \subseteq \mathbb{Z}_n \setminus \chi_1(D).
\]
By Lemma 3.1, we have \(d_b^{k-b}(C) \leq w_b(D) < n\).
(c) By (a) and (b), we have
\[
b \leq d_b^1(C) < d_b^2(C) < \cdots < d_b^{k-b}(C)
\]
\[
\leq d_b^{k-b+1}(C) \leq \cdots \leq d_b^k(C) \leq n.
\]
Suppose \(d_b^{k-b}(C) = d_b^{k-b+1}(C) < n\), then
\[
d_b^{k-b}(C) < d_b^{k-b+1}(C),
\]
and by (a) which is a contradiction. Hence
\[
b \leq d_b^1(C) < d_b^2(C) < \cdots < d_b^{k-b}(C)
\]
\[
< d_b^{k-b+1}(C) \leq \cdots \leq d_b^k(C) \leq n.
\]
(d) If \(1 \leq r \leq k - b\), then
\[
d_b^r(C) \leq d_b^{r+1}(C) - 1 \leq \cdots \leq d_b^{k-b}(C) - (k - b - r)
\]
\[
\leq n - (k - b - r) = n + b + r - k - 1 \leq n.
\]
Since \(d_b^r(C) \leq n \leq n + b + r - k - 1\) when \(k - b + 1 \leq r \leq k\), we have
\[
d_b^r(C) \leq \min\{n + b + r - k - 1, n\}
\]
for \(1 \leq r \leq k\).

Remark 3.6: When \(b = 1\), the statement (c) of Theorem 3.5 is
\[
1 \leq d_1^1(C) < d_1^2(C) < \cdots < d_1^{k-1}(C) < d_1^k(C) \leq n
\]
which was proved by Wei [23]. When \(b = 2\), the statement (c) of Theorem 3.5 is
\[
2 \leq d_2^1(C) < d_2^2(C) < \cdots < d_2^{k-1}(C) \leq d_2^k(C) \leq n
\]
which was proved by Liu and Pan [16].

From Theorem 3.5, we have the following definition.

Definition 3.7: An \([n, k]\)-linear code \(C\) over \(\mathbb{F}_q\) with \(d_b(C) = \min\{n - k + b, n\}\) is called a \(b\)-symbol maximum
distance separable (\(b\)-symbol MDS) code.

Remark 3.8: Theorem 2.4 of [7] gives \(d_b^1(C) \leq n - k + b\),
but there is no \([n, k]\)-linear code \(C\) over \(\mathbb{F}_q\) such that \(d_b^1(C) = n - k + b\) when \(k < b\).
Hence our bound in statement (d) of
Theorem 3.5 is an improvement of Theorem 2.4 of [7].
Also there is an \([n, k]\)-linear code \(C\) over \(\mathbb{F}_q\) such that \(d_b^1(C) = \min\{n + b - k, n\}\) when \(k < b\),
for example \(1\)-MDS codes by using Theorem 3.11.

For any subset \(J \subseteq \mathbb{Z}_n\), let \(J[b] = \bigcup_{i=0}^{b-1} (J + i)\) and \(J[-b] = \bigcup_{i=0}^{b-1} (J - i)\).

Lemma 3.9: For any \(\mathbb{F}_q\)-subspace \(D\) of \(\mathbb{F}_q^n\), we have
(a) \(j \in \mathbb{Z}_n \setminus \chi_0(D)\) if and only if \(\{j\}[b] \subseteq \mathbb{Z}_n \setminus \chi_1(D)\).
(b) \(\chi_1(D)[-b] = \chi_b(D)\).
(c) For any \(1 \leq b \leq n - 1\), \(\chi_b(D)[-2] = \chi_{b+1}(D)\).
Proof: (a) It is easy to prove that by the definitions of \(\chi_1(D)\) and \(\chi_b(D)\).
(b) For any \(j \in \chi_1(D)[-b]\), there exists \(0 \leq i \leq b - 1\) such that \(j \in \chi_1(D) - i\). Then \(j + i \in \chi_1(D)\) and \(j \in \chi_b(D)\) by (a). Hence \(\chi_1(D)[-b] \subseteq \chi_b(D)\).
For any \(j_1 \in \chi_b(D)\), there exists \(x = (x_0, x_1, \ldots, x_{n-1}) \in D\) such that
\[
(x_j, x_{j+1}, \ldots, x_{j+b-1}) \neq 0.
\]
Then there exists \(0 \leq i_1 \leq b - 1\) such that \(x_{j_1 + i_1} \neq 0\) and \(j_1 + i_1 \in \chi_1(D)\).
Hence \(j_1 \in \chi_1(D)[-b]\) and \(\chi_1(D)[-b] \geq \chi_b(D)\).
(c) We have
\[
\chi_b(D)[-2] = \chi_b(D) \cup \chi_b(D)[-1)
\]
\[
= \chi_b(D)[-1] \cup \bigcup_{i=0}^{b-1} (\chi_1(D) - i)
\]
\[
= \bigcup_{i=0}^{b-1} (\chi_1(D) - i) \cup (\bigcup_{i=1}^{b} (\chi_1(D) - i))
\]
\[
= \bigcup_{i=0}^{b} (\chi_1(D) - i) = \chi_{b+1}(D),
\]
where the last equality holds by using (b).

For any \([n, k]\)-linear code \(C\) over \(\mathbb{F}_q\), it is easy to know that
\[
1 \leq d_1^r(C) \leq d_2^r(C) \leq \cdots \leq d_n^r(C) = n.
\]
Then we give an improvement of this inequalities in the next theorem.

Theorem 3.10: Let \(C\) be an \([n, k]\)-linear code over \(\mathbb{F}_q\). Let \(1 \leq r \leq k\). Then
(a) For \(1 \leq b \leq n - 1\), if \(d_b^{k+1}(C) < n\), then \(d_b^r(C) < d_b^{k+1}(C)\).
(b) \(1 \leq d_b^1(C) < \cdots < d_b^{k-r}(C) \leq d_b^{k-r+1}(C) \leq \cdots \leq d_b^k(C) = n\).
(c) For \(1 \leq b \leq n - 1\), \(d_b^{k+1}(C) = d_b^r(C) + 1\) if and only if there exists an \(\mathbb{F}_q\)-subspace \(E\) of \(C\) such that
\(\dim(E) = r\), \(d_b^r(C) = w_b(E) < n\) and \(\chi_b(E)\) is successive.
(d) If there exists an \(\mathbb{F}_q\)-subspace \(E\) of \(C\) such that \(\dim(E) = r\), \(d_b^r(C) = w_b(E) < n\) and \(\chi_b(E)\) is successive,
then \(d_b^r(C) = \min\{d_b^1(C) + b - 1, n\}\).
by induction. If \( d_{b-1}^r(C) = d_1^r(C) + b - 2 < n \), we know \( \chi_{b-1}(E) \) is successive and

\[
\omega_{b-1}(E) = w_1(E) + b - 2 = d_1^r(C) + b - 2 = d_{b-1}^r(C) < n
\]

since \( \chi_1(E) \) is successive and \( \chi_{b-1}(E) = \chi_1(E) + (\{b-1\}) \) by Lemma 3.9 (b). By (e), we get

\[
d_{b-1}^r(C) = d_{b-1}^r(C) + 1 = \min\{d_1^r(C) + b - 1, n\}.
\]

For two real number \( n \times k \) matrices \( A = (a_{ij})_{n \times k} \) and \( B = (b_{ij})_{n \times k} \), we call \( A \leq B \) when \( a_{ij} \leq b_{ij} \) for any \( 1 \leq i \leq n \) and \( 1 \leq j \leq k \). Before we give the Singleton-like bound for the generalized weight matrix, we need the following matrix

\[
D(n, k) = (a_{ij})_{n \times k} = \begin{pmatrix}
n - k + 1 & n - k + 2 & \cdots & n - 1 & n \\
n - k + 3 & n - k + 4 & \cdots & n & n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
n - 1 & n & \cdots & n & n \\
n & n & \cdots & n & n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
& & & \ddots & \vdots \\
& & & & n
\end{pmatrix}_{n \times k},
\]

where

\[
a_{ij} = \min\{n - k + i + j - 1, n\} \in \mathbb{N}
\]

for \( 1 \leq i \leq n \) and \( 1 \leq j \leq k \). Also we know that the generalized weight matrix of \([n, k]\)-linear 1-symbol MDS codes (Hamming MDS codes) is \( D(n, k) \). Then we have the following theorem.

**Theorem 3.11:** Assume the notation is given above. For any \([n, k]\)-linear code \( C \) over \( \mathbb{F}_q \),

(a) Every row of the generalized weight matrix \( D(C) \) of \( C \) is increasing from left to right. And every column of the matrix \( D(C) \) is increasing from up to down.

(b) \( D(C) \leq D(n, k) \), which is called the Singleton-like bound for the generalized weight matrix.

(c) For \( 1 \leq b \leq n \), \( C \) is a \( b \)-symbol MDS code if and only if the \((b, 1)\)-element of \( D(C) \) is the same as the \((b, 1)\)-element of \( D(n, k) \) if and only if the \( b \)th row of \( D(C) \) is the same as the \( b \)th row of \( D(n, k) \).

(d) Let \( b_0 = \min\{1 \leq b \leq n \mid C \text{ is a } b \text{-symbol MDS code}\} \), then \( C \) is a \( b \)-symbol MDS code for any \( b_0 \leq b \leq n \).

(e) In particular, \( C \) is a 1-symbol MDS code if and only if \( D(C) = D(n, k) \).

**Proof:** (a) It is easy to see.

Statements (b) and (c) have been proved in Theorem 3.5 (c).

(d) First \( \{1 \leq b \leq n \mid C \text{ is a } b \text{-symbol MDS code}\} \) is not empty set since \( C \) is an \( n \)-symbol MDS code. Then we only need to prove (d) when \( b_0 < n \) and \( b = b_0 + 1 \).

If \( d_1^b(C) = n \), then \( C \) is a \( b \)-symbol MDS code. If \( d_1^b(C) < n \), then

\[
n + b_0 - k = d_1^{b_0}(C) < d_1^b(C) \leq n + b - k
\]

by Theorem 3.10 (a). By \( b = b_0 + 1 \), we have \( d_1^b(C) = n + b - k \) and \( C \) is a \( b \)-symbol MDS code.

(e) It is easy to prove by (c) and (d).
IV. GENERALIZED WEIGHT MATRICES OF TWO CLASSES OF CODES

In this section, we calculate the generalized weight matrix $D(C)$ defined in Section 2, when $C$ are simplex codes or two special Hamming codes. First we assume

$$\mathbb{F}_q = \{\alpha_0 = 0, \alpha_1 = 1, \alpha_2, \ldots, \alpha_{q-1}\}$$

and define a partially order “$\leq$” on $\mathbb{F}_q$ which is

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{q-1}.$$ 

Let $x, y \in \mathbb{F}_q^k$, we define an order on $\mathbb{F}_q^k$ by using lexicographical order as follows: Two vectors $x, y$ are called ordered, denoted by $x \preceq y$, if and only if

$$x = (x_0, x_1, \ldots, x_{k-1}), y = (y_0, y_1, \ldots, y_{k-1})$$

such that there exists $0 \leq i_0 \leq k - 1$ with $x_j = y_j$ for any $0 \leq j \leq i_0 - 1$ and $x_{i_0} < y_{i_0}$ (which means $x_{i_0} \neq y_{i_0}$ and $x_{i_0} < y_{i_0}$). And $x < y$ means $x \preceq y$ and $x \neq y$.

Recall that $PG^j_q(1) = \{V_1, V_2, \ldots, V_n\}$ is the set of all subspaces of dimension 1 of $\mathbb{F}_q^k$. Then there exists a unique $v_i \in V_i$ such that the first nonzero component of $v_i$ is 1 for any $1 \leq i \leq n_1.k$.

For $k \geq 1$, let $H_{q,k} = (x_1^T, x_2^T, \cdots, x_{n_1,k}^T)$ be the $k \times n_1.k$ matrix over $\mathbb{F}_q$ such that $x_i \in V_i [1 \leq i \leq n_1.k]$ for $1 \leq i \leq n_1.k$ and $x_i, x_{i,s+1}$ for $1 \leq i \leq n_1,k, r, s$, then the linear code over $\mathbb{F}_q$ with the generator matrix $H_{q,k}$ is called simplex code denoted by $S_{q,k}$ and the linear code over $\mathbb{F}_q$ with the parity check matrix $H_{q,k}$ is called Hamming code denoted by $H_{q,k}$.

We need the following lemma to calculate generalized Hamming weights of simplex codes. And we know that every nonzero codeword in a simplex code has the same Hamming weights of simplex codes. And we know that every

$\text{Lemma 4.1: [Lemma 1 of [10]] Let } C \text{ be an } [n, k]\text{-linear code over } \mathbb{F}_q \text{ and } 1 \leq j \leq k - 1. \text{ The following are equivalent:}$

(a) $d_j^1(C) = w_1(c)$ for every nonzero codeword $c \in C$.

(b) $d_j^1(C) = w_1(E)$ for every subspace $E$ of $C$ of dimension $j$ with $1 \leq j < k$.

By Lemma 4.1, we know that if every nonzero codeword in an $[n, k]$-linear code $C$ has the same Hamming weight, then every $\mathbb{F}_q$-subspace of $C$ with a fixed dimension has the same Hamming weight. And we will use this result to obtain the generalized weight matrices of simplex codes in the following.

For $k \geq 1$, let $F_{q,k} = (x_1^T, x_2^T, \cdots, x_{n_1,k}^T)$ be the $k \times q^k$ matrix over $\mathbb{F}_q$ such that $x_i \in \mathbb{F}_q^k$ for $1 \leq i \leq n_1,k$ and $x_i, x_{i,s+1}$ for $1 \leq i \leq q^k$. By using this notion, we have the following theorem.

$\text{Theorem 4.2: Assume the notation is given above. Then}$

(a) $H_{q,k} = \begin{pmatrix} 0 & 1 \\ H_{q,k-1} & F_{q,k-1} \end{pmatrix}$ for $k \geq 2$.

(b) Let $S_{q,k}$ be the simplex code over $\mathbb{F}_q$ with the generator matrix $H_{q,k}$ for $k \geq 1$, then

$$d_j^1(S_{q,k}) = \min\{q^k - q^{k-j}, q - 1\} + i - 1, n\}$$

for $1 \leq j \leq k$ and $1 \leq i \leq n$.

$\text{Proof: (a) It is easy to prove by the definition of } H_{q,k}$.

(b) Since $d_j^1(S_{q,k}) = w_1(c)$ for every nonzero codeword $c \in S_{q,k}$, which satisfies the statement (a) of Lemma 4.1, we know that

$$d_j^1(S_{q,k}) = w_1(E)$$

for every $E \in PG^j_q(S_{q,k})$ by Lemma 4.1. Let $V^j$ be the $\mathbb{F}_q$-subspace of $C$ generated by first $j$ rows of the matrix $H_{q,k}$, then

$$d_j^1(S_{q,k}) = w_1(V^j) = q^{k-1} + q^{k-2} + \cdots + q^{k-j} = \frac{q^k - q^{k-j}}{q - 1}$$

for $1 \leq j \leq k$ by (a). By Theorem 3.10 (d), we have

$$d_j^1(S_{q,k}) = \min\{q^k - q^{k-j} + 1 - 1, n\}$$

for $1 \leq j \leq k$ and $1 \leq i \leq n$.

Since any Hamming code is the dual code of some simplex code, we need the following lemma to calculate generalized Hamming weights of Hamming codes by generalized Hamming weights of simplex codes.

$\text{Lemma 4.3: [Theorem 3 of [23]] Let } C \text{ be an } [n, k]\text{-linear code over } \mathbb{F}_q \text{. Then}$

$$d_j^1(C) = \{1, 2, \ldots, n\} \setminus \{n + 1 - d_j^1(C^\perp) | 1 \leq j \leq n - k\}.$$
Example 4.5: Let $H_{2,4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}_{4 \times 15}$.

We denote the $i$th row vector of $H_{2,4}$ by $\alpha_i$ for $1 \leq i \leq 4$.

Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}_{13 \times 15}$.

and we denote the $i$th row vector of $A$ by $\beta_i$ for $1 \leq i \leq 13$.

Let $C_{2,4}$ be the $[15, 11]$-Hamming code over $\mathbb{F}_2$ with the parity check matrix $H_{2,4}$, then $\beta_i \in C_{2,4}$ for $1 \leq i \leq 13$. The first row of $D(C_{2,4})$ is

\[ \begin{pmatrix} 8 & 12 & 14 & 15 \\ 9 & 13 & 15 & 15 \\ 10 & 14 & 15 & 15 \\ 11 & 15 & 15 & 15 \\ 12 & 13 & 15 & 15 \\ 13 & 15 & 15 & 15 \\ 14 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \end{pmatrix}_{\times 4} \]

by Theorem 4.2, and the first row of $D(C_{2,4})$ is

\[ \begin{pmatrix} 35679101112131415 \end{pmatrix} \]

by Lemma 4.3. By Theorems 3.5 (a), 3.10 (a) and (d), and 3.11 (a), we have

\[ D(C_{2,4}) = \begin{pmatrix} 8 & 12 & 14 & 15 \\ 9 & 13 & 15 & 15 \\ 10 & 14 & 15 & 15 \\ 11 & 15 & 15 & 15 \\ 12 & 13 & 15 & 15 \\ 13 & 15 & 15 & 15 \\ 14 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \\ 15 & 15 & 15 & 15 \end{pmatrix}_{15 \times 4} \]

In fact, we know that the first column of $D(C_{2,4}^{T})$ is

\[ \begin{pmatrix} 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{pmatrix} \]

since there exists an $\mathbb{F}_q$-subspace $E_1$ of $C_{2,4}^\perp$ such that $\dim(E_1) = 1$, $d_b^1(C) = w_b(E_1) < n$ and $\chi_1(E_1)$ is successive, where $E_1 = (\alpha_1)$. Analogously, we can calculate the second column of $D(C_{2,4}^{T})$ by $E_2 = (\alpha_1, \alpha_2)$ and Theorem 3.10 (a) and (d).

And analogously calculate the first column of $D(C_{2,4})$ by $D_1 = (\beta_1)$.

Calculate the second column of $D(C_{2,4})$ by $D_2 = (\beta_1, \beta_2)$.

Calculate the third column of $D(C_{2,4})$ by $D_3 = (\beta_1, \beta_2, \beta_3)$.

Calculate the fourth column of $D(C_{2,4})$ by $D_4 = (\beta_1, \beta_2, \beta_3, \beta_4)$.

Using Theorems 3.5 (a) and 3.10 (a), we can calculate the rest columns of $D(C_{2,4})$.

Remark 4.6: In Theorem 4.2 and Examples 4.4 and 4.5, we calculate the generalized weight matrices of simplex codes and two special Hamming codes. By [16, Corollary A.2], it is easy to get the utility performance of simplex codes and those two special Hamming codes in symbol-pair read-tap channels of type II.

V. $b$-Symbol MDS Codes

In [7], the authors gave a sufficient condition for the existence of $b$-symbol MDS codes by using parity check matrices of linear codes. And they [7] constructed $b$-symbol MDS codes by using this condition. In this section, we give a necessary and sufficient condition for a linear code to be a $b$-symbol MDS code by using the generator matrix and the parity check matrix of this linear code, respectively.

Recall that we have assumed that $G = (G_0, \cdots, G_{n-1})$ is a generator matrix of an $[n, k]$-linear code $C$ over $\mathbb{F}_q$. Recall also that $J[b] = \bigcup_{i=0}^{b-1}(J + i)$ and $J[-b] = \bigcup_{i=b}^{n-1}(J + i)$ for any $J \subseteq \mathbb{Z}_n$. Now we take all the $i$th columns of $G$ such that $i \in J[b]$ and put them together to form a submatrix of $G$, which is denoted by $[G_j | j \in J[b]]$.

Theorem 5.1: Assume the notation is given above. An $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a $b$-symbol MDS code if and only if $\text{rank}([G_j | j \in J[b]]) = k$ for any $J \subseteq \mathbb{Z}_n$ such that $|J| = \max\{k - b, 0\} + 1$.

**Proof:** It is enough to prove that an $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a $b$-symbol MDS code if and only if $\text{rank}([G_j | j \in J[b]]) = k$ for any $J \subseteq \mathbb{Z}_n$ such that $|J| \geq \max\{k - b, 0\} + 1$.

Suppose $C$ is not a $b$-symbol MDS code. There exists a nonzero codeword $c_0 = (c_0, c_1, \cdots, c_{n-1}) \in C$ such that $w_b(c_0) \leq \min\{n - k + b, n\} - 1$ and a nonzero vector $y_0 \in \mathbb{F}_q^k$ such that $c_0 = y_0G$.

Note that $G = (G_0, \cdots, G_{n-1})$. Let $J_0 = \mathbb{Z}_n \setminus \{b\}(c_0)$, then $|J_0| = n - w_b(c_0) \geq \max\{k - b, 0\} + 1$.

By Lemma 3.9 (a), we have $J_0[b] \subseteq \mathbb{Z}_n \setminus \chi_1(c_0)$. Since $c_0 = (c_0, c_1, \cdots, c_{n-1}) = y_0G$, it follows that $c_0 \notin C$.
we have $c_i = y_0 \cdot G_i^T$, i.e., $c_i$ is equal to the Euclidean inner product between the vectors $y_0$ and $G_i^T$ for $0 \leq i \leq n - 1$. If

$$i \in J_0[b] \subseteq \mathbb{Z}_n \setminus \chi_1(c_0),$$

then $i \notin \chi_1(c_0)$ and $y_0 \cdot G_i^T = c_i = 0$. Hence

$$y_0[G_j \mid j \in J_0[b]] = 0.$$

Therefore

$$\text{rank}([G_j \mid j \in J_0[b]]) \leq k - 1,$$

which is a contradiction.

Conversely, assume there exists a subset $J_1 \subseteq \mathbb{Z}_n$ such that $|J_1| \geq \max\{k - b, 0\} + 1$ and

$$\text{rank}([G_j \mid j \in J_1]) \leq k - 1.$$

Then there exists a nonzero vector $y_1 \in \mathbb{F}_q^n$ such that $y_1[G_j \mid j \in J_1[b]] = 0$. Assume $c_i = y_1G_i$, then $J_1[b] \subseteq \mathbb{Z}_n \setminus \chi_1(c_1)$ and $J_1 \subseteq \mathbb{Z}_n \setminus \chi_0(c_1)$ by Lemma 3.9 (a). Hence

$$n - w_b(c_1) = |\mathbb{Z}_n \setminus \chi_b(c_1)| \geq |J_1| \geq \max\{k - b, 0\} + 1$$

and

$$w_b(c_1) \leq \min\{n - k + b, n\} - 1.$$

That is a contradiction, since $C$ is a $b$-symbol MDS code. ■

When $b = k - 1$, we give a construction of $(k - 1)$-symbol MDS code by using Theorem 5.1.

**Corollary 5.2:** Assume the notation is given above. There exists a $[q(k - 1), k]$-linear code $C$ over $\mathbb{F}_q$, which is a $(k - 1)$-symbol MDS code.

**Proof:** Let $W$ be an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^k$ such that $\dim(W) = k - 1$ and

$$\{w_0, w_1, \cdots, w_{k-2}\}$$

be a basis of $W$. Assume

$$\mathbb{F}_q = \{\alpha_0 = 0, \alpha_1 = 1, \alpha_2, \cdots, \alpha_{q-1}\}.$$ 

Since the quotient space $\mathbb{F}_q^k/W$ is of dimension one, we have

$$\mathbb{F}_q^k/W = \{W + \alpha_i x \mid 0 \leq i \leq q - 1\},$$

where $x \in \mathbb{F}_q/W$.

Let $G = (G_0, \cdots, G_{q(k-1)-1})$ be the $k \times q(k-1)$ matrix

$$\begin{pmatrix}
w_0 + \alpha_0 x, w_1 + \alpha_0 x, \cdots, w_{k-2} + \alpha_0 x, \\
w_0 + \alpha_1 x, w_1 + \alpha_1 x, \cdots, w_{k-2} + \alpha_1 x, \\
\vdots \\
w_0 + \alpha_{q-1-1} x, w_1 + \alpha_{q-1-1} x, \cdots, w_{k-2} + \alpha_{q-1-1} x,
\end{pmatrix}$$

where the column vector $w_a + \alpha_i x = G_{a+t(k-1)}$ for $0 \leq s \leq k - 2$ and $0 \leq t \leq q - 1$. Let $C$ be a $[q(k - 1), k]$-linear code over $\mathbb{F}_q$ such that $G$ is a generator matrix of $C$. Then we can show that $C$ satisfies the condition in Theorem 5.1, so that $C$ is a $(k - 1)$-symbol MDS code.

Assume

$$J = \{s_1 + t_1(k-1), s_2 + t_2(k-1)\}$$

for $0 \leq s_1, s_2 \leq k - 2, 0 \leq t_1, t_2 \leq q - 1$ and $s_1 + t_1(k-1) \neq s_2 + t_2(k-1)$. Then the $\mathbb{F}_q$-subspace generated by all the columns of the matrix $[G_j \mid j \in J[k-1]]$ is

$$W_J = \langle w_{s_1} + \alpha_1 x, w_{s_1+1} + \alpha_1 x, \cdots, w_{k-2} + \alpha_1 x, \\
w_0 + \alpha_1 x, w_1 + \alpha_1 x, \cdots, w_{s_1-1} + \alpha_1 x, \\
w_{s_2} + \alpha_2 x, w_{s_2+1} + \alpha_2 x, \cdots, w_{k-2} + \alpha_2 x, \\
w_0 + \alpha_2 x, w_1 + \alpha_2 x, \cdots, w_{s_2-1} + \alpha_2 x \rangle,$$

where the indices of $w$ are taken modulo $k - 1$ and the indices of $\alpha$ are taken modulo $q$.

If $t_1 \neq t_2$, we have

$$(\alpha_{t_2} - \alpha_{t_1})x = w_{k-2} + \alpha_1 x - (w_{k-2} + \alpha_1 x) \in W_J$$

and $x \in W_J$. Hence $W_J = \langle x, w_0, w_1, \cdots, w_{k-2} \rangle$. If $t_1 = t_2$, we know $s_1 \neq s_2$ since $s_1 + t_1(k-1) \neq s_2 + t_2(k-1)$. We can assume $s_1 < s_2$, then

$$(\alpha_{t_2} - \alpha_{t_1})x = w_{s_1} + \alpha_1 x - (w_{s_1} + \alpha_1 x) \in W_J$$

and $x \in W_J$. Hence $W_J = \langle x, w_0, w_1, \cdots, w_{k-2} \rangle$. Hence $\mathbb{F}_q^k = \langle x, w_0, w_1, \cdots, w_{k-2} \rangle = W_J$ and $\text{rank}([G_j \mid j \in J[k-1]]) = k$. By Theorem 5.1, $C$ is a $(k - 1)$-symbol MDS code.

We give an example to illustrate Corollary 5.2.

**Example 5.3:** Assume the notation is given above. Let $k = 1 = b = 3$, $q = 2$ and $F_3 = \{0, 1, 2, 3, 4, 5\}$. Assume

$$w_0 = (1, 0, 0, 0), \ w_1 = (0, 1, 0, 0), \ w_2 = (0, 0, 1, 0), \ w_3 = (0, 0, 0, 1),$$

Then $G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \end{pmatrix}_{4 \times 15}$

and the [15, 4]-linear code $C$ with the generator matrix $G$ is a 3-symbol MDS code by Corollary 5.2.

When $b = 1$, we get the usually necessary and sufficient condition for a linear code to be a 1-symbol MDS code (Hamming MDS code) by using its generator matrix.

**Corollary 5.4:** Assume the notation is given above. An $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a 1-symbol MDS code if and only if $\text{rank}([G_j \mid j \in J]) = k$ for any $J \subseteq \mathbb{Z}_n$ such that $|J| = k$.

**Corollary 5.5:** Assume the notation is given above and $b \geq k \geq 1$. An $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a $b$-symbol MDS code if and only if $\text{rank}([G_j \mid i \leq j \leq i + b - 1]) = k$ for any $0 \leq i \leq n - 1$.

Given two positive integers $b$ and $k$ such that $b \geq k \geq 1$, we can construct an $[n, k]$-linear code $C$ over $\mathbb{F}_q$ such that $C$ is a $b$-symbol MDS code and $n$ is as large as possible in the following corollary.

**Corollary 5.6:** Assume the notation is given above and $b \geq k \geq 1$. For any $N \in \mathbb{N}^+$, there exists an $[n, k]$-linear code $C$ over $\mathbb{F}_q$ such that $C$ is a $b$-symbol MDS code and $n \geq N$.

**Proof:** Since $b \geq k$, there exists a $k \times b$ matrix $G_1$ over $\mathbb{F}_q$ such that $\text{rank}(G_1) = k$. Let $t \in \mathbb{N}^+$ such that $tb \geq N$,
we construct a $[tb, k]$-linear code $C_t$ over $\mathbb{F}_q$ with a generator matrix $\tilde{G}_t = [\tilde{G}_1, \tilde{G}_1, \ldots, \tilde{G}_1]$, where $\tilde{G}_1$ repeats $t$ times in $\tilde{G}_t$. By Corollary 5.5, we know that the linear code $C_t$ is a $b$-symbol MDS code.

Given two positive integers $b$ and $k$ such that $1 \leq b \leq k - 1$, we give an upper bound on $[n, k]$-linear codes which are $b$-symbol MDS codes in the following corollary.

**Corollary 5.7:** Assume the notation is given above and $1 \leq b \leq k - 1$. For any $[n, k]$-linear code $C$ over $\mathbb{F}_q$ which is a $b$-symbol MDS code, then $n \leq n_{1,k}$.

**Proof:** We only need to prove that $n \leq n_{1,k}$, for any $[n, k]$-linear code $C$ over $\mathbb{F}_q$ which is a $(k - 1)$-symbol MDS code, since any $[n, k]$-linear code $C$ over $\mathbb{F}_q$ which is a $b$-symbol MDS code for $1 \leq b \leq k - 1$ is a $(k - 1)$-symbol MDS code by Theorem 3.11 (d).

By Theorem 5.1, we know that an $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a $(k - 1)$-symbol MDS code if and only if $\text{rank}([G_j | j \in J[k - 1]]) = k$ for any $J \subseteq \mathbb{Z}_n$ such that $|J| = 2$. For any $0 \leq j \leq n - 1$, there is an $\mathbb{F}_q$-subspace $V_j$ of $\mathbb{F}_q^k$ such that $\text{dim}(V_j) = k - 1$ and

$$\{G_j, G_{j+1}, \ldots, G_{j+k-2}\} \subseteq V_j.$$

Suppose $n > n_{1,k} = n_{k-1,k}$ which is the number of all $\mathbb{F}_q$-subspaces of dimension $k - 1$ of $\mathbb{F}_q^k$, then there exist $j_1$ and $j_2$ such that

$$0 \leq j_1 < j_2 \leq n - 1$$

and $V_{j_1} = V_{j_2}$. Let $J_1 = \{j_1, j_2\}$, then

$$\text{rank}([G_j | j \in J_1[k - 1]]) < k$$

which is a contradiction.

We assume that $H = (H_0, \ldots, H_{n-1})$ is a parity check matrix of an $[n, k]$-linear code $C$ over $\mathbb{F}_q$. Then we take all the $i$th columns of $H$ such that $i \in J$ and put them together to form a submatrix of $H$, which is denoted by $[H_j | j \in J]$.

**Theorem 5.8:** Assume the notation is given above. An $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a $b$-symbol MDS code if and only if $\text{rank}([H_j | j \in J]) = |J|$ for any $J \subseteq \mathbb{Z}_n$ such that $|J| = 2$.

**Proof:** Suppose $C$ is not a $b$-symbol MDS code. There exists a nonzero codeword $c \in C$ such that

$$w_b(c) = \text{min}\{n - k + b, n\} - 1.$$

Let $J_0 = \chi_1(c)$. By Lemma 3.9 (b), we have

$$|J_0| = |\chi_1(c)| = |\chi_0(c)| = w_b(c) \leq \text{min}\{n - k + b, n\} - 1.$$

Since $He^T = 0$, we have $\sum_{j \in J_0} H_j c_j = 0$ where $c = (c_0, c_1, \ldots, c_{n-1})$. Then

$$\text{rank}([H_j | j \in J_0]) < |J_0|$$

which is a contradiction.

If there exists a subset $J_1 \subseteq \mathbb{Z}_n$ such that $|J_1| = |J_0| = n - k + b, n$ - 1 and

$$\text{rank}([H_j | j \in J_1]) < |J_1|,$$

then there exists a codeword $x = (x_0, x_1, \ldots, x_{n-1}) \in C$ such that $x_j = 0$ for any $j \in \mathbb{Z}_n \setminus J_1$ and

$$\sum_{j \in J_1} H_j x_j = 0.$$

By Lemma 3.9 (b), we have $\chi_1(x) \subseteq J_1$ and $\chi_0(x) \subseteq \chi_1(x) - b \subseteq J_1 - b$. Hence

$$w_b(x) = |\chi_0(x)| \leq |J_1 - b| \leq \text{min}\{n - k + b, n\} - 1.$$

This is a contradiction, since $C$ is a $b$-symbol MDS code.

When $b = 1$, we get the usually necessary and sufficient condition for a linear code to be a 1-symbol MDS code by using its parity check matrix.

**Corollary 5.9:** Assume the notation is given above. An $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a 1-symbol MDS code if and only if $\text{rank}([H_j | j \in J]) = n - k$ for any $J \subseteq \mathbb{Z}_n$ such that $|J| = n - k$.

By Corollaries 5.4 and 5.9, an $[n, k]$-linear code $C$ over $\mathbb{F}_q$ is a 1-symbol MDS code if and only if the dual code $C^\perp$ of $C$ is a 1-symbol MDS code. But the dual $C^\perp$ may not be a $b$-symbol MDS code, when $b \geq 2$ and $C$ is a $b$-symbol MDS code. And we know this by the following example.

**Example 5.10:** Let $C$ be the linear code over $\mathbb{F}_2$ with a generator matrix $( 1 \ 0 \ 1 )$. Then we know $d_2^2(C) = 3$ and $C$ is a 2-symbol MDS code. And $C^\perp$ is the linear code over $\mathbb{F}_2$ with a generator matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $d_2^2(C^\perp) = 2$ and $C^\perp$ is not a 2-symbol MDS code.

VI. LINEAR ISOMORPHISMS PRESERVING B-SYMBOL WEIGHTS

MacWilliams [19] and later Bogart, Goldberg, and Gordon [2] proved that every linear isomorphism preserving Hamming weights between two linear codes over finite fields can be induced by a monomial matrix. Unfortunately, a linear isomorphism induced by a permutation matrix may not preserve $b$-symbol weights of linear codes. In this section, we obtain a necessary and sufficient condition for a linear isomorphism preserving $b$-symbol weights between two linear codes over finite fields.

Recall that $n_{r,k}$ is the number of all the subspaces of dimension $r$ of a vector space of dimension $k$. Let $\text{PG}^r(\mathbb{F}_q^k) = \{V_1^r, V_2^r, \ldots, V_{n_{r,k}}^r\}$ be the set of all subspaces of dimension $r$ of $\mathbb{F}_q^k$. There is a bijection between $\text{PG}^{k-r}(\mathbb{F}_q^k)$ and $\text{PG}^r(\mathbb{F}_q^k)$, which is defined by

$$\text{PG}^{k-r}(\mathbb{F}_q^k) \to \text{PG}^r(\mathbb{F}_q^k), V^{k-r} \to (V^{k-r})^\perp,$$

$$V^{k-r} \in \text{PG}^{k-r}(\mathbb{F}_q^k).$$

Hence $n_{r,k} = n_{k-r,k}$. For convenience, if $\frac{k}{2} < r \leq k$, we assume

$$\text{PG}^r(\mathbb{F}_q^k) = \{V_1^r = (V_1^{k-r})^\perp, V_2^r = (V_2^{k-r})^\perp, \ldots, V_{n_{r,k}}^r = (V_{n_{r,k}}^{k-r})^\perp\}.$$

Let $T_{r,s}$ be an $n_{r,k} \times n_{s,k}$ matrix over the rational number field $\mathbb{Q}$ such that

$$T_{r,s} = (t_{ij})_{n_{r,k} \times n_{s,k}}, \quad t_{ij} = \begin{cases} 1, & \text{if } V_i^r \subseteq V_j^s; \\ 0, & \text{if } V_i^r \not\subseteq V_j^s. \end{cases}$$
And let \( J_{m \times n} \) be the \( m \times n \) matrix with all entries being 1, i.e., \( J_{m \times n} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \). The following lemma can be found in [16].

**Lemma 6.1:** Assume the notation is given above, and \( 1 \leq r \leq s \leq z \leq k \). Then

(a) The sum of all rows of \( T_{r,s} \) is a constant row vector \( t = n_{r,s} 1 \).

(b) The matrix \( T_{1,k-1}^{-1} \) is an invertible matrix and

\[
T_{1,k-1}^{-1} = \frac{1}{q^k-2}(T_{1,k-1} - \frac{q^k-2}{q^k-1} J_{n_{1,k} \times n_{1,k}})
\]

for \( k \geq 2 \). The sum of all rows of \( T_{1,k-1}^{-1} \) is a constant row vector.

(c) \( T_{r,k-1} T_{1,k-1} = (q^{k-r-1}) T_{1,r} + \frac{k-r-1}{q^r-1} J_{n_{r,k} \times n_{1,k}} \)

and

\[
T_{r,k-1} T_{1,k-1}^{-1} = \frac{1}{q^r-1} T_{1,r} - \frac{q^r-1-1}{q^r-1} J_{n_{r,k} \times n_{1,k}}
\]

for \( k \geq r + 1 \).

(d) \( T_{r,s} T_{s,z} = n_{s-r,z-r} T_{r,s} \) for \( 1 \leq r \leq s \leq z \leq k \).

Let \( \varphi \) be an \( \mathbb{F}_q \)-linear isomorphism from \( C \) to \( \tilde{C} \), where \( C \) and \( \tilde{C} \) are two \([n,k]\)-linear codes over \( \mathbb{F}_q \). Let \( G = \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} \) be a generator matrix of \( C \) for some \( g_i \in \mathbb{F}_q^n \). Then \( \tilde{G} = \begin{pmatrix} \varphi(g_1) \\ \vdots \\ \varphi(g_k) \end{pmatrix} \) is a generator matrix of \( \tilde{C} \).

**Theorem 6.2:** Assume the notation is given above. Then \( w_b(\varphi(\psi)) \) is constant for any nonzero codeword \( c \in C \) if and only if \( \sum_{V \in \Omega_i \setminus \{1\}} (m^b_G(V) - m^b_G(V)) \) is constant for any \( 1 \leq i \leq n_{1,k} \), where \( s = \min\{b,k-1\} \) and \( \Omega_i = \{ V \in \text{PG}^{s-1}(\mathbb{F}_q^k) | V_1 \subseteq V \} \).

**Proof:** Let \( \psi \) be the \( \mathbb{F}_q \)-linear isomorphism from \( \mathbb{F}_q^k \) to \( C \) such that \( \psi(y) = yG \) for any \( y \in \mathbb{F}_q^k \). And let \( \tilde{\psi} \) be the \( \mathbb{F}_q \)-linear isomorphism from \( \mathbb{F}_q^k \) to \( \tilde{C} \) such that \( \tilde{\psi}(y) = y\tilde{G} \) for any \( y \in \mathbb{F}_q^k \). Then \( \tilde{\psi} = \psi \circ \varphi \) by the definition of \( \tilde{G} \). For any nonzero codeword \( c \in C \), there is a vector \( y \) such that \( c = \psi(y) \) and \( \tilde{c} = \varphi(c) = \varphi(y) \). By Lemma 2.7, we have

\[
w_b(c) = n - \theta^b_G((y)^\perp)
\]

and

\[
w_b(\tilde{c}) = n - \theta^b_G((\tilde{y})^\perp).
\]

Let \( \Delta_r = (m^b_G(V_1^r), m^b_G(V_2^r), \cdots, m^b_G(V_{n_{1,k}}^r)) \)

\[
\tilde{\Delta}_r = (m^b_G(V_1^r), m^b_G(V_2^r), \cdots, m^b_G(V_{n_{1,k}}^r))
\]

for \( 0 \leq r \leq \min\{b,k-1\} \) and

\[
\Gamma_{k-1} = (\theta^b_G(V_1^{k-1}), \theta^b_G(V_2^{k-1}), \cdots, \theta^b_G(V_{n_{1,k}}^{k-1}))
\]

and

\[
\tilde{\Gamma}_{k-1} = (\theta^b_G(V_1^{k-1}), \theta^b_G(V_2^{k-1}), \cdots, \theta^b_G(V_{n_{1,k}}^{k-1})).
\]

Assume \( s = \min\{b,k-1\} \), then we get

\[
\Gamma_{k-1} = \sum_{r=0}^{s} \Delta_r T_{r,k-1} = m^b_G(0) I + \sum_{r=1}^{s} \Delta_r T_{r,k-1} \quad \text{(VI.3)}
\]

and

\[
\tilde{\Gamma}_{k-1} = \sum_{r=0}^{s} \tilde{\Delta}_r T_{r,k-1} = m^b_G(0) I + \sum_{r=1}^{s} \tilde{\Delta}_r T_{r,k-1} \quad \text{(VI.4)}
\]

by the definition of \( \theta^b_G \).

Suppose \( a = w_b(c) - w_b(\tilde{c}) \) for any nonzero \( c \in C \). By Equations (VI.1) and (VI.2), we have \( \theta^b_G((y)^\perp) - \theta^b_G((\tilde{y})^\perp) = -a \) for any nonzero \( y \in \mathbb{F}_q^k \) and

\[
\Gamma_{k-1} - \tilde{\Gamma}_{k-1} = -a 1.
\]

By Equations (VI.3) and (VI.4), we have

\[
\sum_{r=1}^{s} (\Delta_r - \tilde{\Delta}_r) T_{r,k-1} = (m^b_G(0) - m^b_G(0) - a) 1
\]

and

\[
\sum_{r=1}^{s} (\Delta_r - \tilde{\Delta}_r) T_{r,k-1} T_{1,k-1}^{-1} = \frac{m^b_G(0) - m^b_G(0) - a}{n_{1,k-1}} 1.
\]

Then we have

\[
q \sum_{V \in \Omega_i \setminus \{1\}} \frac{1}{|V|} (m^b_G(V) - m^b_G(V))
\]

\[
- \sum_{r=2}^{s} \sum_{V \in \text{PG}^{r-1}(\mathbb{F}_q^k)} \frac{q^r-1-1}{q^r-1(q^k-1-1)} (m^b_G(V^r) - m^b_G(V^r))
\]

\[
= \frac{m^b_G(0) - m^b_G(0) - a}{n_{1,k-1}},
\]

since the element in the \( i \)th position of the vector

\[
\sum_{r=1}^{s} (\Delta_r - \tilde{\Delta}_r) T_{r,k-1} T_{1,k-1}^{-1}
\]

is

\[
q \sum_{V \in \Omega_i \setminus \{1\}} \frac{1}{|V|} (m^b_G(V) - m^b_G(V))
\]

\[
- \sum_{r=2}^{s} \sum_{V \in \text{PG}^{r-1}(\mathbb{F}_q^k)} \frac{q^r-1-1}{q^r-1(q^k-1-1)} (m^b_G(V^r) - m^b_G(V^r))
\]

by Lemma 6.1 (c), where \( \Omega_i = \{ V \in \text{PG}^{s-1}(\mathbb{F}_q^k) | V_1 \subseteq V \} \). Hence

\[
q \sum_{V \in \Omega_i \setminus \{1\}} \frac{1}{|V|} (m^b_G(V) - m^b_G(V))
\]

\[
= \sum_{r=2}^{s} \sum_{V \in \text{PG}^{r-1}(\mathbb{F}_q^k)} \frac{q^r-1-1}{q^r-1(q^k-1-1)} (m^b_G(V^r) - m^b_G(V^r))
\]

and

\[
\sum_{V \in \Omega_i \setminus \{1\}} (m^b_G(V) - m^b_G(V)) \text{ is constant for any } 1 \leq i \leq n_{1,k}.
\]

Suppose \( \sum_{V \in \Omega_i \setminus \{1\}} (m^b_G(V) - m^b_G(V)) = b \) for any \( 1 \leq i \leq n_{1,k} \). Then

\[
\sum_{r=1}^{s} (\Delta_r - \tilde{\Delta}_r) T_{r,k-1} T_{1,k-1}^{-1}
\]
and $\sum_{r=1}^s(\Delta_r - \tilde{\Delta}_r)T_{r,k-1}$ are constant vectors by Lemma 6.1 (b), since the element in the $i$th position of the vector $\sum_{r=1}^s(\Delta_r - \tilde{\Delta}_r)T_{r,k-1}$ is

$$q \sum_{V \in \Omega_i} \frac{1}{|V|}(m^b_G(V) - m^b_G(V))$$

$$- \sum_{r=2}^s \sum_{V' \in PG^s(P^*_q)} \frac{q^{r-1} - 1}{(q^{k-1} - 1)}(m^b_G(V') - m^b_G(V'))$$

$$= q \sum_{V' \in PG^s(P^*_q)} \frac{q^{r-1} - 1}{(q^{k-1} - 1)}(m^b_G(V') - m^b_G(V')).$$  

By Equations (VI.3) and (VI.4), again we know $\Gamma_{k-1} - \tilde{\Gamma}_{k-1} = \sum_{r=1}^s(\Delta_r - \tilde{\Delta}_r)T_{r,k-1} + (m^b_G(0) - m^b_G(0))I$ is a constant vector and $\theta^b_C((y^{1} \cdot \cdots \cdot y^{1})) - \theta^b_C((y^{1} \cdot \cdots \cdot y^{1}))$ is constant for any nonzero $y \in P^*_q$. Hence $w_b(c) - w_b(\varphi(c))$ is constant for any nonzero $c \in C$ by Equations (VI.1) and (VI.2).

**Corollary 6.3:** Assume the notation is given above. Then $w_b(c) = w_b(\varphi(c))$ for any $c \in C$ if and only if there exists $c_0 \in C$ such that $w_b(c_0) = w_b(\varphi(c_0))$ and

$$\sum_{V \in \Omega_i} \frac{1}{|V|}(m^b_G(V) - m^b_G(V))$$

is constant for any $1 \leq i \leq n_{1,k}$, where $s = \min\{b,k-1\}$ and $\Omega_i = \{V \in PG^s(P^*_q) | V^1 \subseteq V\}$.

There is an example by using Corollary 6.3 to determine a linear isomorphism preserving $b$-symbol weights when $b = 2$ in [16, Section 5]. When $b = 1$, we obtain the classical MacWilliams extension theorem [2], [19] in the next corollary.

**Corollary 6.4:** Assume the notation is given above. Then $w_1(c) = w_1(\varphi(c))$ for any $c \in C$ if and only if there exists a monomial matrix $M$ such that $\varphi(x) = xM$ for any $x \in P^*_q$.

**Proof:** When $b = 1$, we have $w_1(c) = w_1(\varphi(c))$ for any $c \in C$ if and only if $m^b_1(V^1) = m^1_1(V^1)$ for any $1 \leq i \leq n_{1,k}$ by Corollary 6.3 and

$$\sum_{V \in \Omega_i} \frac{1}{|V|}(m^1_1(V) - m^1_1(V)) = m^1_1(V^1) - m^1_1(V^1)$$

for any $1 \leq i \leq n_{1,k}$. Hence $w_1(c) = w_1(\varphi(c))$ for any $c \in C$ if and only if there exists a monomial matrix $M$ such that $\varphi(x) = xM$ for any $x \in P^*_q$ by using the definitions of the functions $m^1_1$ and $m^b_1$.

From Theorem 6.2, we know that if we want to determine a linear isomorphism preserves $b$-symbol weights of linear codes or not, it is crucial to calculate the value $\sum_{V \in \Omega_i} \frac{1}{|V|}m^b_G(V)$ for an $[n,k]$-linear code $C$ with a generator matrix $G$, where $s = \min\{b,k-1\}$ and $\Omega_i = \{V \in PG^{S^b}(P^*_q) | V^1 \subseteq V\}$.

Recall that we have assumed that $G = (G_0, \ldots, G_{n-1})$ is a generator matrix of an $[n,k]$-linear code $C$ over $\mathbb{F}_q$. Then we assume $S_j = \mathbb{F}_qG_j + \mathbb{F}_qG_{j+1} + \cdots + \mathbb{F}_qG_{j+b-1}$ which is an $\mathbb{F}_q$-subspace of $\mathbb{F}^k_q$ and $\tilde{S}_j = (G_j, G_{j+1}, \ldots, G_{j+b-1})$ is a $k \times b$ submatrix of $G$ for $0 \leq j \leq n-1$. Also we know that $\dim(S_j) = \text{rank}(\tilde{S}_j)$.

**Theorem 6.5:** Assume $k_{ij} = \begin{cases} 1, & \text{if } V^1 \subseteq S_j; \\ 0, & \text{if } V^1 \not\subseteq S_j, \end{cases}$ for $1 \leq i \leq n_{1,k}$ and $1 \leq j \leq n$. We have

$$\sum_{V \in \Omega_i} \frac{1}{|V|}m^b_G(V) = \sum_{j=1}^n k_{ij} q^{-\text{rank}(S_j)}. \quad \Box$$

**Proof:** It is easy to prove this lemma by using the definition of the function $m^b_G$.

**Remark 6.6:** Let $C$ be an $[n,k]$-linear code over $\mathbb{F}_q$ with a generator matrix $G = (G_0, \ldots, G_{n-1})$, then we can calculate $f_i = \sum_{j=1}^n k_{ij} q^{-\text{rank}(S_j)}$ for $1 \leq i \leq n_{1,k}$ by the following steps. First we can calculate $\{S_0, S_1, \ldots, S_{n-1}\}$ and $|PG^1(S_i)| \leq q^{n-1}$. Assume $T = \bigcup_{i=1}^n PG^1(S_i)$, we have $|T| \leq n^{k+1}$. If $V^1 \not\subseteq T$, then $f_1 = 0$ by Theorem 6.5. So we only need to calculate $f_i$ for $|T|$ subspaces of dimension one of $\mathbb{F}^1_q$.

However, if we simply check $b$-symbol weights of all the codewords of $C$ and $\tilde{C}$, then we need to calculate $2 \cdot \frac{q^{n-1}}{n} - 1$ subspaces of dimension one of $\mathbb{F}^1_q$ for their $b$-symbol weights since $c$ and $\lambda$ have the same $b$-symbol weight for $c \in C$ and $\lambda \in F^*_q$. So the determination of a linear isomorphism preserving $b$-symbol weight by using Theorem 6.2 is more efficient, since $2 \cdot |T| \leq 2n \cdot \frac{q^n - 1}{q-1} < < 2 \cdot \frac{q^n - 1}{q-1}$ when $k >> b$. For example, when $C$ is a $[10,6]$-linear code $C$ over $\mathbb{F}^1_{31}$ and $b = 3$, then $2 \cdot |T| < 19860$ is much less than $2 \cdot \frac{31^6 - 31}{30} = 59166912$.

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**REFERENCES**

[1] P. Beelen, “A note on the generalized Hamming weights of Reed–Muller codes,” *Applicable Algebra Eng., Commun. Comput.*, vol. 30, no. 3, pp. 233–242, 2019.

[2] K. Bogart, D. Goldberg, and J. Gordon, “An elementary proof of the MacWilliams theorem on equivalence of codes,” *Inf. Control*, vol. 37, no. 1, pp. 19–22, Apr. 1978.

[3] Y. Cassuto and M. Blaum, “Codes for symbol-pair read channels,” *IEEE Trans. Inf. Theory*, vol. 57, no. 12, pp. 8011–8020, Dec. 2011.

[4] Y. M. Chee, L. Ji, H. M. Kiah, C. Wang, and J. Yin, “Maximum distance separable codes for symbol-pair read channels,” *IEEE Trans. Inf. Theory*, vol. 59, no. 11, pp. 7259–7267, Nov. 2013.

[5] B. Chen, L. Lin, and H. Liu, “Constacyclic symbol-pair codes: Lower bounds and optimal constructions,” *IEEE Trans. Inf. Theory*, vol. 63, no. 12, pp. 7661–7666, Dec. 2017.

[6] B. Ding, G. Ge, J. Zhang, T. Zhang, and Y. Zhang, “New constructions of MDS symbol-pair codes,” *Des., Codes Cryptogr.*, vol. 86, no. 4, pp. 841–859, 2018.

[7] B. Ding, T. Zhang, and G. Ge, “Maximum distance separable codes for $b$-symbol read channels,” *Finite Fields Their Appl.*, vol. 40, pp. 180–197, Jan. 2018.

[8] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and S. Sriboonchitta, “On the symbol-pair distance of repeated-root constacyclic codes of prime power lengths,” *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2417–2430, Apr. 2018.

[9] H. Q. Dinh, X. Wang, H. Liu, and S. Sriboonchitta, “On the symbol-pair distances of repeated-root constacyclic codes of length $2^i$,” *Discrete Math.*, vol. 342, no. 11, pp. 3062–3078, 2019.
[10] Y. Fan and H. Liu, “Generalized Hamming equiweight linear codes,” *Acta Electronica Sinica*, vol. 31, no. 10, pp. 1591–1593, 2003.
[11] C. W. Huffman and V. Pless, *Fundamentals of Error-Correcting Codes*. Cambridge, U.K.: Cambridge Univ. Press, 2003.
[12] G. Jian, R. Feng, and H. Wu, “Generalized Hamming weights of three classes of linear codes,” *Finite Fields Their Appl.*, vol. 45, pp. 341–354, May 2017.
[13] X. Kai, S. Zhu, and P. Li, “A construction of new MDS symbol-pair codes,” *IEEE Trans. Inf. Theory*, vol. 61, no. 11, pp. 5828–5834, Nov. 2015.
[14] X. Kai, S. Zhu, Y. Zhao, H. Luo, and Z. Chen, “New MDS symbol-pair codes from repeated-root codes,” *IEEE Commun. Lett.*, vol. 22, no. 3, pp. 462–465, Mar. 2018.
[15] S. Li and G. Ge, “Constructions of maximum distance separable symbol-pair codes using cyclic and constacyclic codes,” *Dess. Codes Cryptogr.*, vol. 84, no. 3, pp. 359–372, Sep. 2017.
[16] H. Liu and X. Pan, “Generalized pair weights of linear codes and linear isomorphisms preserving pair weights,” *IEEE Trans. Inf. Theory*, vol. 68, no. 1, pp. 105–117, Jan. 2022.
[17] J. Ma and J. Luo, “MDS symbol-pair codes from repeated-root cyclic codes,” *Dess. Codes Cryptogr.*, vol. 90, no. 1, pp. 121–137, Jan. 2022.
[18] J. Ma and J. Luo, “Constructions of MDS symbol-pair codes with minimum distance seven or eight,” *Dess. Codes Cryptogr.*, vol. 90, no. 10, pp. 2337–2359, Oct. 2022.
[19] F. MacWilliams, “A theorem on the distribution of weights in a systematic code,” *Bell Syst. Tech. J.*, vol. 42, no. 1, pp. 79–94, Jan. 1963.
[20] L. H. Ozarow and A. D. Wyner, “Wire-tap channel II,” *AT&T Bell Lab. Tech. J.*, vol. 63, no. 10, pp. 2135–2157, 1984.
[21] L. Storme and J. A. Thas, “M.D.S. Codes and arcs in PG(n, q) with q even: An improvement of the bounds of Bruen, Thas, and Blokhuis,” *J. Combinat. Theory, A*, vol. 62, no. 1, pp. 139–154, 1993.
[22] M. A. Tsfasman and S. G. Vladut, “Geometric approach to higher weights,” *IEEE Trans. Inf. Theory*, vol. 41, no. 6, pp. 1564–1588, Nov. 1995.

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