On non-symmetric axial corner-layer flow

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Abstract. The problem of asymmetric incompressible axial flow in a corner formed of two intersecting plates at a right angle is considered. The asymptotic behaviour of the flow far away from the corner is analysed. Two types of asymptotic behaviour are found. It is shown that the flow is very sensitive to the asymmetry parameter. A comparison of the results with computations of full Navier–Stokes equations was performed.

1. Introduction

The axial corner-layer flows serve to mimic different flows at joints of aerodynamic bodies. The prevalence of such flows is the cause of their experimental and theoretical study for decades [1]. However, due to complexity of laboratory and numerical modelling, the relevant information, except some simple cases, is rather limited. For example, there are difficulties in implementing the symmetric flow with zero stream-wise pressure gradient in laboratory experiments [2]. Even relatively small inaccuracies in installing the model in a wind tunnel give rise to an asymmetry of the flow and to a stream-wise pressure gradient. The only theoretical study visiting the non-symmetric corner-layer flows [3] suggests using a flow asymmetry parameter to categorize the flows, but presents neither relevant boundary conditions, nor examples of such flows.

In this paper, self-similar symmetric and asymmetric axial laminar flows at right corner are considered. For the first time, a system of quite general asymptotic boundary conditions that may be useful for numerical solution to reduce the dimension of the problem in the asymmetric case is obtained. A numerical analysis of the boundary conditions is given. Computations of the asymmetric flows are presented.

2. Self-similar incompressible corner-layer flow with stream-wise pressure gradient

Let us consider in Cartesian coordinates \((x^*, y^*, z^*)\) a semi-infinite right dihedral corner with faces at coordinates \((x^*, 0, z^*)\) and \((x^*, y^*, 0)\), with the borders at coordinates \((0, 0, z^*)\) and \((0, y^*, 0)\), so that the corner edge has coordinates \((x^*, 0, 0)\), where \(x^* \geq 0\), \(y^* \geq 0\), \(z^* \geq 0\). The corner is exposed to an axial (along the edge) steady laminar flow of Newtonian fluid with kinematic viscosity \(\nu\) and velocity \(u^*(x^*, y^*, z^*)\), so that \(u^*(x^*, k z^*, z^*) \to (U_\infty \cdot (x^*/L)^m, 0, 0)\) when \(z^* \to \infty (k > 0)\), where \(U_\infty > 0\) and \(L > 0\) are the characteristic velocity and the length, respectively. The value \(m\) describes the stream-wise pressure gradient (when \(m = 0\), the gradient is equal to zero [3]).
Let us define the Reynolds number as \( \text{Re} = U_\infty L/\nu \) and introduce new variables \([4]\) with
\[
\eta = c \frac{y^*}{x^*}, \quad \zeta = c \frac{z^*}{x^*}
\]
and
\[
u(\eta, \zeta) = \frac{v^*(x^*, y^*, z^*)}{U_\infty(x^*L)^m}, \quad w(\eta, \zeta) = \frac{w^*(x^*, y^*, z^*)}{U_\infty(x^*L)^m},
\]
where \( c = \sqrt{\text{Re}(1 + m)/2} \). In accordance with the theoretical analyses performed in previous studies cited above the dependent variables can be expressed as the velocity potentials
\[
\phi = (1 - m) \eta u - \frac{v}{1 + m}, \quad \psi = (1 - m) \zeta u - \frac{w}{1 + m}
\]
and the modified vorticity \( \theta = \psi_\zeta - \phi_\eta \). The advantage of them in further analysis and computations is that two of four resulting similarity equations are linear:
\[
u_{\eta\eta} + \phi_{\eta\zeta} + \phi u_\eta + \psi u_\zeta + \frac{2m}{1 + m} (1 - u^2) = 0, \quad (1)
\]
\[
\phi_\eta + \psi_\zeta = \frac{2u}{1 + m}, \quad (2)
\]
\[
\psi_\eta - \phi_\zeta = \theta, \quad (3)
\]
\[
\theta_{\eta\eta} + \theta_{\zeta\zeta} + \phi \theta_\eta + \psi \theta_\zeta + \frac{2u}{1 + m} [\theta + (1 - m)\eta u_\zeta - \zeta u_\eta] = 0, \quad (4)
\]
with the boundary conditions at the walls
\[
u = \phi = \psi = 0,
\]
and at infinity
\[
u(k \zeta, \zeta) \rightarrow 1, \quad \theta(k \zeta, \zeta) \rightarrow 0, \quad \zeta \rightarrow \infty.
\]

3. Boundary conditions
The given boundary conditions are insufficient for solving equations (1)–(4) numerically. Instead, they can be replaced with asymptotic boundary conditions derived by analysing the behaviour of the flow far away from the corner edge \([5]\). Without loss of generality let us consider the limit as \( \zeta \rightarrow \infty \) and \( \eta = \text{const} \). Assuming the Reynolds number large, the boundary-layer approximation can be used in the limit yielding the following asymptotic solutions \([5]\):
\[
u(\eta, \zeta) = U_0(\eta) + \frac{U_1(\eta)}{\zeta} + O \left( \frac{1}{\zeta^2} \right), \quad \phi(\eta, \zeta) = \Phi_0(\eta) + \frac{\Phi_1(\eta)}{\zeta} + O \left( \frac{1}{\zeta^2} \right),
\]
\[
\psi(\eta, \zeta) = \zeta \Psi_0(\eta) + \Psi_1(\eta) + O \left( \frac{1}{\zeta} \right), \quad \theta(\eta, \zeta) = \zeta \Theta_0(\eta) + \Theta_1(\eta) + O \left( \frac{1}{\zeta} \right).
\]
Let us introduce the flow asymmetry parameter \( \gamma \) \([3]\) such that the condition \( \psi = (1 + \gamma)/(1 + m)\zeta + \lambda_1 + O(1/\zeta) \), where \( \lambda_1 \) is a constant to be defined, is satisfied at the bisecting line. Then,
the resulting system of governing equations can be reduced to the following system of three ordinary differential equations:

\[ U_{0,\eta\eta} + \Phi_{0,\eta} U_{0,\eta} + \frac{2m}{1+m} (1 - U_{0}^{2}) = 0, \quad (5) \]

\[ \Phi_{0,\eta} + \Psi_{0} = \frac{2U_{0}}{1+m}, \quad (6) \]

\[ \Psi_{0,\eta\eta} + \Phi_{0,\eta} \Psi_{0,\eta} + \left[ \Psi_{0}^{2} - \left( \frac{1}{1+m} + \gamma \right)^{2} \right] + \frac{1-m}{1+m} (1 - U_{0}^{2}) = 0, \quad (7) \]

with the boundary conditions

\[ U_{0}(0) = \Phi_{0}(0) = \Psi_{0}(0) = 0, \quad (8) \]

\[ U_{0}(\infty) = 1, \quad \Psi_{0}(\infty) = \frac{1}{1+m} + \gamma. \quad (9) \]

Leaving only the principal terms in the expansion makes the asymptotic boundary conditions for equations (1)–(4) incompatible when \( \eta \to \infty \) and \( \zeta \to \infty \), so the next terms must be taken into account. The corresponding equations are linear and after some manipulations can be reduced to the following system:

\[ U_{1} \equiv 0, \]

\[ \Phi_{1} \equiv 0 \]

\[ \Theta_{1} = \Psi_{1,\eta}, \]

\[ \Psi_{1,\eta\eta} + \Phi_{0} \Psi_{1,\eta} + \Psi_{0} \Psi_{1} = \left( \frac{1}{1+m} + \gamma \right) \lambda_{1}, \]

with the boundary conditions \( \Psi_{1}(0) = 0 \) and \( \Psi_{1}(\infty) = \lambda_{1} \), where

\[ \lambda_{1} = \lim_{\eta \to \infty} \Psi_{1}^{(+\gamma)}(\eta) = \lim_{\zeta \to \infty} \left[ \Psi_{0}^{(-\gamma)}(\zeta) - \left( \frac{1}{1+m} + \gamma \right) \zeta \right], \]

and the superscripts denote the asymmetry parameters.

The system of equations (5)–(9) for the principal terms in the expansion is non-linear. Therefore, an additional analysis of its solutions depending on \( m \) and \( \gamma \) is required. For comparison of the results with the data of some previous studies, it is convenient also to introduce the Hartree pressure gradient parameter \( \beta = 2m/(1+m) \). The principal terms in the expansion of the boundary conditions were found on solving the system (5)–(9) as a boundary-value problem by setting \( U_{0}^{'}(0) \) as an additional boundary condition and finding \( \beta \) in the course of the solution [6]. To integrate the equations the ode15s solver of MATLAB was used [7].

Similar to the analysis of symmetric corner flow [8] it was found that the equations have two branches of solutions. The branch corresponding to the solutions with larger wall shear stress \( \sim U_{0}^{'}(0) \) is called the upper one, and the other branch is called the lower one. Examples of the obtained diagrams of \( U_{0}^{'}(0) \) and \( \Psi_{0}^{'}(0) \) vs. \( \beta \) for \( \gamma = \pm 0.25 \) and 0 are shown in Figure 1. It is seen that the change of \( \gamma \) leads to a change of the minimum of \( \beta \). Since the boundary conditions for the asymmetric flows are defined by a pair \( \pm \gamma \), then the self-similar flows in the corner are possible only for such \( \beta \), for which there are asymptotic solutions in both cases.
4. Corner-layer flow

For the numerical solution of (1)–(4) with the corresponding boundary conditions, the equations (2) and (3) were pre-transformed by differentiation and further algebraic manipulations to yield

\[
\begin{align*}
    u_{\eta\eta} + u_{\zeta\zeta} + \phi u_\eta + \psi u_\zeta + \frac{2m}{1 + m} (1 - u^2) &= 0, \\
    \phi_{\eta\eta} + \phi_{\zeta\zeta} &= \frac{2u_\eta}{1 + m} - \theta_\zeta, \\
    \psi_{\eta\eta} + \psi_{\zeta\zeta} &= \frac{2u_\zeta}{1 + m} + \theta_\eta, \\
    \theta_{\eta\eta} + \theta_{\zeta\zeta} + \phi\theta_\eta + \psi\theta_\zeta + \frac{2u}{1 + m} \left[ \theta + (1 - m)(\eta u_\zeta - \zeta u_\eta) \right] &= 0,
\end{align*}
\]

that is more convenient for numerical integration. The NITSOL package [9] used widely for solving this kind of non-linear problems showed, however, a very slow convergence. Analysis of the problem indicates that it is related apparently to nearly degenerate Jacobian of the considered set of equations. This is in accordance with findings of previous investigators of the symmetric problem. Therefore, an original method of solution, which will be described in a separate publication, was developed.

The computations show that the flow is very sensitive to the value of asymmetry parameter \(\gamma\). For example, Figure 2 demonstrates the solutions for both branches as absolute values of the normalized transverse velocity components \(q = \sqrt{v^2 + w^2}\) at \(\beta = 0\) with \(\gamma = 0\) and 0.01 (the range of \(\gamma\) is 0 to \(\pm 0.5\)). The results with \(\gamma = 0\) is the same as, e.g., obtained in [3]. Meanwhile, when \(\gamma = 0.01\), a significant distortion compared to the symmetric case takes place, the stream-wise velocity component experiencing much weaker change.

An important issue associated with the presence of multiple branches of asymptotic boundary conditions is the feasibility of resulting self-similar corner flows in practice. Its formal solution is
possible, e.g., by considering the transient problems by analogy with the symmetric case [8] that is beyond the scope of the present study. However, a qualitative comparison of the asymmetric self-similar flows with the results of computations of the full Navier–Stokes equations with ANSYS Fluent was performed. The details of both the numerical grid and the scheme will be described elsewhere. The obtained velocity distributions indicate in favour of the feasibility of the solution of the upper branch at least when $\beta$ is close to 0 that is also true for the symmetric flow [8].

5. Conclusions
The symmetric and asymmetric self-similar flows of viscous incompressible fluid along a semi-infinite dihedral corner with stream-wise pressure gradients were considered. The geometry of the problem was reduced to the consideration of the flow in quadrant on the plane with the origin coinciding with the vertex of the corner. For the region of the flow bounded by a rectangle with one vertex coinciding with the origin, the system of equations for compatible asymptotic boundary conditions was obtained and analysed numerically. Two types of the asymptotic behaviour was found. It is shown that the flow is very sensitive to the value of the asymmetry parameter.

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