ROYDEN’S LEMMA IN INFINITE DIMENSIONS AND LOOP SPACES AS HILBERT-HARTOGS MANIFOLDS

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ABSTRACT. We prove the Royden’s Lemma for complex Hilbert manifolds, i.e., that a holomorphic imbedding of the closure of a finite dimensional, strictly pseudoconvex domain into a complex Hilbert manifold extends to a holomorphic imbedding of the product of this domain with the unit ball in Hilbert space. This reduces several problems concerning complex Hilbert manifolds to open subsets of a Hilbert space. As an illustration we prove a version of Behnke’s Continuity Principle for complex Hilbert manifolds and give some results on generalized loop spaces of complex manifolds.

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1. Introduction and statement of results

1.1. Royden’s Lemma. Our main result in this paper is the following Royden’s Lemma for complex Hilbert manifolds.

Theorem 1. Let \( f : \overline{D} \to X \) be a holomorphic imbedding of a closure of a finite dimensional bounded strictly pseudoconvex domain \( D \subseteq \mathbb{C}^q \) to a complex Hilbert manifold \( X \). Then there exists an extension of \( f \) to a holomorphic imbedding \( \tilde{f} : \overline{D} \times \overline{B}^\infty \to X \).

Here \( B^\infty(r) \) stands for the ball of radius \( r \) in \( l^2 \), \( B^\infty := B^\infty(1) \) is the unit ball. Hilbert manifolds in this paper are modeled over \( l^2 \), and are assumed to be second countable. It is worth to mention at this point that all holomorphic Hilbert bundles over such \( D \) are trivial, see [Bu]. This statement allows to reduce some questions, such as the proof of the Continuity Principle for example, from Hilbert manifolds to open subsets in \( l^2 \).

1.2. Continuity Principle and Hilbert-Hartogs manifolds. Analogously to the finite dimensional case we say that a complex Hilbert manifold \( X \) is \( q \)-Hartogs (or, simply Hartogs when \( q = 1 \)) if every holomorphic mapping \( f : H^1_q(r) \to X \) of the \( q \)-concave Hartogs figure \( H^1_q(r) := (\Delta^q \times \Delta(r)) \cup (A^q_{r_1} \times \Delta) \subseteq \mathbb{C}^{q+1} \) with values in \( X \) extends to a holomorphic map \( \tilde{f} : \Delta^{q+1} \to X \) of the unit polydisk to \( X \). Here \( A^q_{r_1,r_2} = \Delta^q(r_2) \setminus \Delta^q(r_1) \) is a ring domain, \( r_2 > r_1 \). We say that \( X \) is Hilbert-Hartogs when \( q \) is irrelevant or, clear...
from the context. Our next goal is to show that Hilbert-Hartogs manifolds possess much stronger extension properties than it is postulated in their definition.

**Definition 1.** A $q$-concave Hartogs figure in $l^2$ is the following open set

$$H_q^\infty(r) := (\Delta^q \times B^\infty(r)) \cup (A_{q-1}^1 \times B^\infty),$$

where $0 < r < 1$.

**Theorem 2.** Let $X$ be a $q$-Hartogs Hilbert manifold. Then for every $r > 0$ every holomorphic mapping $f : H_q^\infty(r) \to X$ extends to a holomorphic mapping $\tilde{f} : \Delta^q \times B^\infty \to X$.

As an immediate consequence we obtain the following statement.

**Corollary 1.** Let $D$ be a domain in a complex Hilbert manifold $X$ which is $q$-pseudoconcave at a boundary point $p$. Then every holomorphic map $f : D \to Y$ to a $q$-Hartogs Hilbert manifold $Y$ extends holomorphically to a neighborhood of $p$.

The proof of this corollary follows from Theorem 2 by appropriately placing the Hartogs figure $H_q^\infty(r)$ near $p \in \partial D$. Our next goal is the following infinite dimensional version of the classical Continuity Principle in the form of Behnke, see [Bh].

**Theorem 3.** (Continuity Principle) Let $D$ be an open subset in a complex Hilbert manifold $X$ and let $Y$ be a $q$-Hartogs Hilbert manifold. Suppose we are given a sequence of analytic $q$-disks $\{\varphi_n : \Delta^q \to X\}$ such that

i) $\varphi_n$ converge uniformly on $\Delta^q$ to an imbedded analytic $q$-disk $\varphi_0 : \Delta^q \to X$;

ii) $\varphi_n(\partial \Delta^q) \subset D$ and $\varphi_n(\partial \Delta^q) \subset D$.

Then every holomorphic mapping $f : D \to Y$ holomorphically extends to a neighborhood of $\varphi_0(\Delta)$ which doesn’t depend on $f$.

The main ingredient in the proofs of Theorems 2 and 3 is the Royden’s Lemma of Theorem 1. The same for Corollary 1 because for placing the Hartogs figure “near $p$” one should place the discolor $\Delta^q \times B^\infty$ first.

**1.3. Loop spaces as Hilbert-Hartogs manifolds.** One finds Hilbert-Hartogs manifolds more often than one could expect. In order to provide such examples we concentrate our attention in section 3 on loop spaces of complex manifolds. Fix a compact (with boundary or not) manifold $S$ and a finite dimensional complex manifold $X$. Then the manifold $W_{S,X}^{k,2} := W^{k,2}(S,X)$ of Sobolev $W^{k,2}$-maps carries a natural structure of a complex Hilbert manifold, see [L1] or, section 3. Here $k \geq n = \dim_{\mathbb{R}} S$ to ensure that mappings from this space are continuous. Manifold $W_{S,X}^{k,2}$ is usually called a generalized loop space of $X$. We prove the following statement.

**Theorem 4.** A generalized loop space of a $q$-Hartogs complex manifold is a $q$-Hartogs Hilbert manifold.

This provides us a lot of interesting examples of infinite dimensional Hilbert-Hartogs manifolds. For example let $S = \mathbb{S}^1$ be the circle and $X$ a connected Riemann surface. Remark that $X$ is Hartogs if and only if $X$ is different from the Riemann sphere $\mathbb{P}^1$. Therefore all loop spaces $W_{S^1,X}^{k,2}$ for $X \neq \mathbb{P}^1$ are Hilbert-Hartogs. Or, let $G$ be a complex Lie group. Then by [ASY] $G$ is Hartogs. Therefore the loop space, which is classically denoted as $LG$, is also Hartogs (1-Hartogs to be precise). Moreover, some “a priori unknown” compact complex manifolds (like surfaces from the class $VII_0^+$ of $\mathbb{S}^3$) if they
Royden’s Lemma and Continuity Principle

2. ROYDEN’S LEMMA AND CONTINUITY PRINCIPLE

2.1. Proof of the Royden’s Lemma. Let \( \mathcal{X} \) be a complex Hilbert manifold modeled over \( \mathbb{P}^2 \). Let \( f : \bar{D} \to \mathcal{X} \) be a holomorphic imbedding and choose \( r > 0 \) such that \( f \) extends to a \( r \)-neighborhood of \( D \), i.e \( D^r = \{ z \in \mathbb{C}^4 : d(z, \partial D) < r \} \). Set \( f(\bar{D}) = M \) and \( f(D^r) = M^r \). Since \( M \) is compact there exists a finite covering \((\Omega_\alpha, \varphi_\alpha)_{\alpha \in A}\) of \( M \) by coordinate balls.

**Step 1.** One can choose this covering in such a way that:

- \( \forall \alpha \in A, U_\alpha := \varphi_\alpha(\Omega_\alpha) \subset D^r \times B^\infty(\delta) \), where \( B^\infty(\delta) \) is the ball of radius \( \delta > 0 \) centered at zero in \( \mathbb{P}^2 \). Moreover, \( \varphi_\alpha(\Omega_\alpha \cap M^r) \subset D^r \times \{ 0 \} \).
- \( \forall z \in f^{-1}(\Omega_\alpha \cap \Omega_\beta \cap M^r) \) one has \( \varphi_\alpha \circ \varphi_\beta^{-1}(z, 0) = (z, 0) \).
- \( \forall \alpha \) and \( \forall z \in f^{-1}(\Omega_\alpha \cap M^r) \) one has \( \varphi_\alpha(f(z)) = (z, 0) \).
- \( \forall \alpha, \beta \in A, \forall z \in f^{-1}(\Omega_\alpha \cap \Omega_\beta \cap M^r) \) one has \( (d\varphi_\alpha^{-1})_{f(z)} = (d\varphi_\beta^{-1})_{f(z)} \) and, moreover, \( (d^2\varphi_\alpha^{-1})_{f(z)} = (d^2\varphi_\beta^{-1})_{f(z)} \).

The first two items are proved in Lemma 2.1 of [AZ], the third and fourth in subsections 2.3 and 1.4 of [AZ]. This means that the normal bundle to \( M \) is trivialized to second order.

Set of \( \Omega = \bigcup_{\alpha \in A} \Omega_\alpha \). Assuming that \( \mathcal{X} \) is second countable we can refine our covering to get a (countable this time) covering by coordinate balls, still denoted as \( \{ \Omega_\alpha \}_{\alpha \in A} \), and construct a partition of unity \( \{ \eta_\alpha \}_{\alpha \in A} \) on \( \Omega \) subordinated to this refined \( \{ \Omega_\alpha \}_{\alpha \in A} \) such that \( \operatorname{supp} \eta_\alpha \subset \Omega_\alpha \) for \( \alpha \in A \). See [AZ] for details and remark that in notations of [AZ] \( \{(B_n, g_n)\} \) does the job. Consider functions \( \theta_\alpha : \Omega \to \mathbb{P}^2 \) defined as follows

\[
\theta_\alpha(m) = \begin{cases} 
\eta_\alpha(m)\varphi_\alpha(m) & \text{if } m \in \Omega_\alpha, \\
0 & \text{otherwise.}
\end{cases} \tag{2.1}
\]

Function \( \theta_\alpha \) is of class \( C^\infty \) on \( \Omega \) supported in \( \Omega_\alpha \). Let \( < e_j > \) be the canonical base of \( \mathbb{P}^2 \), i.e., \( e_1 = (1, 0, ...) \) and so on. Decompose \( \varphi_\alpha(m) = (\varphi_\alpha^1(m), \varphi_\alpha^2(m)) \) with respect to this base, i.e., \( \varphi_\alpha^1(m) \leq e_1 \) and \( \varphi_\alpha^2(m) \leq e_1 > _1 = < e_2, e_3, ... > \). Remark that the following \( \mathbb{P}^2 \)-valued function

\[
u_\alpha(m) := \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{(\partial_\theta_\alpha)_{\varphi_\alpha^{-1}(\xi, \varphi_\alpha^2(m))} \circ (d\varphi_\alpha^{-1})_{(\xi, \varphi_\alpha^2(m))}(e_1)}{\xi - \varphi_\alpha^1(m)} d\xi d\bar{\xi} \tag{2.2}
\]

is well defined and smooth on \( \Omega_\alpha \). Indeed, if \( (\xi, \varphi_\alpha^2(m)) \) leaves \( \varphi_\alpha(\operatorname{supp} \eta_\alpha) \) the form \( \partial_\theta_\alpha |_{\varphi_\alpha^{-1}(\xi, \varphi_\alpha^2(m))} \) vanishes.

**Remark 2.1**. Remark that \( u_\alpha \) vanishes near \( \partial \Omega_\alpha \cap \varphi_\alpha^{-1}(\{z_1 = 0\}) \). Indeed, \( \partial_\theta_\alpha |_{\varphi_\alpha^{-1}(\xi, \varphi_\alpha^2(m))} \) vanishes identically in \( \xi \) for such \( m \).

**Step 2.** Function \( u_\alpha \) satisfies

\[
\bar{\partial} u_\alpha = \partial \theta_\alpha \tag{2.3}
\]

on \( \Omega_\alpha \) and consequently \( u_\alpha \) extends by zero to the whole of \( \Omega \) still satisfying (2.3).
Note that due to the holomorphicity of \( \varphi_\alpha \) one obviously has
\[
(\bar{\partial} \varphi_\alpha \circ d\varphi_\alpha^{-1})(\xi, \varphi_\alpha(m)) = \bar{\partial}(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi, \varphi_\alpha(m)),
\]
and therefore
\[
u_\alpha(m) = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi, \varphi_\alpha(m))}{\xi - \varphi_\alpha'(m)} \, d\xi \wedge d\overline{\xi} = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi, \varphi_\alpha(m))}{\xi - \varphi_\alpha'(m)} \, d\xi \wedge d\overline{\xi}.
\]
(2.4)
Denote coordinates in the chart \((\Omega_\alpha, \varphi_\alpha)\) as \(z = (z_1, z')\). Setting \(m = \varphi_{\beta}^{-1}(z)\) in (2.4) we get after the coordinate change \(\xi - \varphi_\alpha'(m) \rightarrow \xi\) the following
\[
u_\alpha \circ \varphi_\alpha^{-1}(z) = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi, z')}{\xi - z_1} \, d\xi \wedge d\overline{\xi} = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi + z_1, z')}{\xi} \, d\xi \wedge d\overline{\xi}.
\]
Then for any \(j \in \mathbb{N}^*\) we have
\[
\frac{\partial(u_\alpha \circ \varphi_\alpha^{-1})}{\partial z_j}(z) = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi, z')}{\xi} \, d\xi \wedge d\overline{\xi} = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi + z_1, z')}{\xi} \, d\xi \wedge d\overline{\xi} = \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})}{\partial z_j}(z),
\]
and this means that \(\partial u_\alpha = \bar{\partial} \varphi_\alpha\) on \(\Omega_\alpha\).

Since \(\theta_\alpha\) vanishes near \(\partial \Omega_\alpha\) we see that \(u_\alpha\) is holomorphic there. By the Remark 2.1 above and uniqueness theorem for holomorphic functions this implies that \(u_\alpha\) vanishes near \(\partial \Omega_\alpha\) and therefore extends by zero to the whole of \(\Omega\) still satisfying (2.3) there. **Step 3.** Set \(\theta := \sum_{\alpha \in A} \theta_\alpha\) and \(u := \sum_{\alpha \in A} u_\alpha\). We claim that \(\forall z \in \mathring{D}\) one has
\[
\theta(f(z)) - u(f(z)) = (z, 0).
\]
(2.5)
First remark that \(\forall z \in \mathring{D}\) we have
\[
\theta(f(z)) = \sum_{\alpha \in A} \eta_\alpha(f(z)) \varphi_\alpha(f(z)) = \sum_{\alpha \in A} \eta_\alpha(f(z))(z, 0) = (z, 0).
\]
(2.6)
Therefore all we need is to prove that
\[
u(f(z)) = 0.
\]
(2.7)
We have that
\[
(d\varphi_\alpha^{-1})(z, 0) = (d\varphi_\beta^{-1})(z, 0)
\]
for all \(\alpha, \beta\). Denote therefore this operator simply as \(d\varphi_\alpha^{-1}\). Write now
\[
u(f(z)) = \sum_{\alpha \in A} \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi, \varphi_\alpha(f(z)))}{\xi} \, d\xi \wedge d\overline{\xi} = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} \frac{\sum_{\alpha \in A} \partial(\varphi_\alpha \circ \varphi_\alpha^{-1})(\xi, \varphi_\alpha(f(z)))}{\xi} \, d\xi \wedge d\overline{\xi}.
\]
Since
\[ \sum_{\alpha \in A} (\tilde{\partial}_\alpha)_{\varphi^{-1}_\alpha}(\xi, z', 0) = \sum_{\alpha \in A} \tilde{\partial}(\eta_{\alpha} \varphi_{\alpha}) f(\xi, z') = \left( \sum_{\alpha \in A} (\tilde{\partial}_\alpha)_{f(\xi, z')} \right) (\xi, z', 0) = \]
\[ = \tilde{\partial} \left( \sum_{\alpha \in A} \eta_{\alpha} \right)_{f(\xi, z')} (\xi, z', 0) = 0 \]
we get (2.7) and therefore (2.5).

Set \( \psi := \theta - u \). Remark that by step 2 \( \psi \) is holomorphic in \( \Omega \) and by step 3
\[ \psi(f(z)) = (z, 0). \] (2.8)

**Step 4. Differential \( d\psi f(z) \) is bijective \( \forall z \in \tilde{D} \).** As above we write \( d\varphi f(z) \) for \( (d\varphi_\alpha) f(z) \) since it does not depend on the chart. We have
\[ (d\theta)_{f(z)} = \sum_{\alpha} \eta_{\alpha}(f(z)) (d\varphi_\alpha)_{f(z)} + \sum_{\alpha} (d\eta_{\alpha})_{f(z)} \varphi_{\alpha}(f(z)) = \sum_{\alpha} \eta_{\alpha}(f(z)) (d\varphi_\alpha)_{f(z)} + \]
\[ + \sum_{\alpha} (d\eta_{\alpha})_{f(z)}(z, 0) = 1 \cdot (d\varphi)_{f(z)} + d(1)_{f(z)}(z, 0) = (d\varphi)_{f(z)}. \]

Now we are going to prove that \( du f(z) = 0 \). Fix \( \alpha_0 \) such that \( \Omega_{\alpha_0} \ni f(z) \) and for any \( \alpha \in A \) write
\[ u_{\alpha} \circ \varphi_{\alpha_0}^{-1}(z, w) = \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} (\tilde{\partial}_\alpha)_{\varphi^{-1}_\alpha(\xi_1 + \varphi_{\alpha_0} \varphi_{\alpha_0}^{-1}(z, w))} (d\varphi_{\alpha}^{-1})_{\varphi_{\alpha_0}^{-1}(\xi_1 + \varphi_{\alpha_0} \varphi_{\alpha_0}^{-1}(z, w))}(\xi_1) \frac{d\xi \wedge d\bar{\xi}}{\xi}, \] (2.9)
here \( (z, w) \) are natural coordinates in \( D^r \times D^l \) and \( m = \varphi_{\alpha_0}^{-1}(z, w) \) as usual.

Let us compute differentials under the integral in (2.9). First
\[ (\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha_0} \varphi_{\alpha_0}^{-1}(z, w))} = (\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha_0} \varphi_{\alpha_0}^{-1}(z, w))} = (\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w))} = (\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w))} = (\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w))} \]
\[ = (\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w))} + d(\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w))}(d\varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)}(0, w) + o(||w||). \] (2.10)

Since \( \varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)) = f(\xi_1 + z) \) we have that
\[ (\tilde{\partial}_\alpha)_{\varphi_{\alpha}^{-1}(\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w))} = (\tilde{\partial}_\alpha)_{f(\xi_1 + z)} + d(\tilde{\partial}_\alpha)_{f(\xi_1 + z)}(d\varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)}(0, w) + o(||w||) \]

Compute furthermore
\[ (d\varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)} = (d\varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)} + d(\varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)) + o(||w||) = \]
\[ = (d\varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)} + d(\varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)) + o(||w||) \] (2.11)
then
\[ (d\varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)} = (d\varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)} + (d^2 \varphi_{\alpha}^{-1})_{\xi_1 + \varphi_{\alpha} \varphi_{\alpha_0}^{-1}(z, w)}(0, w) + o(||w||) \]

Substituting (2.10) and (2.11) to (2.9) and taking a sum on \( \alpha \) we get
\[ u \circ \varphi_{\alpha_0}^{-1}(z, w) = \sum_{\alpha} \frac{1}{2\pi i} \int_{\xi \in \mathbb{C}} (\tilde{\partial}_\alpha)_{\varphi_{\alpha_0}^{-1}(\xi_1 + \varphi_{\alpha_0} \varphi_{\alpha_0}^{-1}(z, w))} (d\varphi_{\alpha}^{-1})_{\varphi_{\alpha_0}^{-1}(\xi_1 + \varphi_{\alpha_0} \varphi_{\alpha_0}^{-1}(z, w))}(\xi_1) \frac{d\xi \wedge d\bar{\xi}}{\xi} \]
\[ \frac{1}{2\pi i} \sum_{\alpha \in A} \int \left[ (\bar{\partial} \theta_{\alpha})_{f(\xi_{1} + z)} + d(\bar{\partial} \theta_{\alpha})_{f(\xi_{1} + z)} (d\varphi^{-1})_{\xi_{1} + (z,0)} (0,w) + o(||w||) \right] \times \]
\[ \times \left[ (d\varphi^{-1})_{\xi_{1} + (z,0)} + (d^2 \varphi^{-1})_{\xi_{1} + (z,0)} (0,w) + o(||w||) \right] \frac{d\xi \wedge d\bar{\xi}}{\xi} = u \circ \varphi^{-1}_{\alpha}(z,0) + \]
\[ + \frac{1}{2\pi i} \sum_{\alpha \in A \in C} \int d(\bar{\partial} \theta_{\alpha})_{f(\xi_{1} + z)} \left[ (d\varphi^{-1})_{\xi_{1} + (z,0)} (0,w), (d\varphi^{-1})_{\xi_{1} + (z,0)} (e_1) \right] \frac{d\xi \wedge d\bar{\xi}}{\xi} + \]
\[ + \frac{1}{2\pi i} \sum_{\alpha \in A} \int (\bar{\partial} \theta_{\alpha})_{f(\xi_{1} + z)} (d^2 \varphi^{-1})_{\xi_{1} + (z,0)} [(0,w), e_1] \frac{d\xi \wedge d\bar{\xi}}{\xi} + o(||w||) \]

(2.12)

Taking into account the fact that \( d\varphi^{-1}(z,0) \) and \( d^2 \varphi^{-1}(z,0) \) do not depend on \( \alpha \) and that
\[ \bar{\partial} \theta_{f(z)} = \sum_{\alpha \in A} (\bar{\partial} \eta_{\alpha})_{f(z)} \cdot \varphi_{\alpha}(f(z)) = \sum_{\alpha \in A} \bar{\partial}_f(z) (0,0) = 0 \]
we conclude from (2.12) that \( d(u \circ \varphi^{-1}_{\alpha}(z,0)) = 0 \). So \( d\psi = d\varphi \) and \( d\psi \) is invertible on a neighborhood \( V \subset \Omega \) of \( M \). Then one can define the map \( F : \tilde{D} \times B^{\infty}(\epsilon) \rightarrow X \) by \( F(z,w) = \psi^{-1}(z,w) \) for \( \epsilon > 0 \) small enough. Theorem \( \square \) is proved.

\[ \square \]

2.2. Infinite dimensional Hartogs figures. Now we shall prove Theorem 2 from Introduction. We identify \( \mathbb{C}^q \) with \( i^2_q := \text{span} \{ e_1, \ldots, e_q \} \subset i^2 \). For a unit vector \( v \in i^2 \) orthogonal to \( \mathbb{C}^q \) set \( L_v := \text{span} \{ e_1, \ldots, e_q, v \} \). Remark that \( L_v \cap H^{\infty}_q (r) = H^1_q (r) \) and therefore given a holomorphic mapping \( f : H^1_q (r) \rightarrow X \) its restriction to \( L_v \cap H^{\infty}_q (r) \) holomorphically extends to \( L_v \cap (\Delta^q \times B^{\infty}) \). We conclude that for every line \( < v > \subset \mathbb{C}^q \) the restriction \( f|_{L_v} \) holomorphically extends onto \( L_v \cap (\Delta^q \times B^{\infty}) \), giving us an extension \( \tilde{f} \) of \( f \) to \( \Delta^q \times B^{\infty} \). This extension is correctly defined because for unit vectors \( v \neq w \) orthogonal to \( \mathbb{C}^q \) the spaces \( L_v, L_w \) intersect only by \( \mathbb{C}^q \times \{ 0 \} \).

Let us prove the continuity of \( \tilde{f} \). Consider the sequence \( (Z_n)_{n \geq 1} \) defined by \( Z_n = (z^n, w^n) \) such that \( Z_n \rightarrow Z_0 = (z^0, w^0) \). Here \( z \in \Delta^q \) and \( w \in B^{\infty} \). Take \( R \) such that \( 1 - r < R < 1 \) with \( ||z^n||, ||w^n|| < R \) for all \( n \in \mathbb{N} \). Let \( \varphi_n : \Delta^q (R) \times \Delta \rightarrow X \) be an analytic disk defined by \( \varphi_n (z, \eta) = \tilde{f} (z, \eta w^n) \). Theorem 1 gives a neighborhood \( V \cong \Delta^{q+1} \times B^{\infty} \) of the graph of \( \varphi_0 \). Let \( v_n \in \mathbb{C}^q \) be such that \( L_{v_n} \) contains \( Z_n \). For \( w^n \) close enough to \( w^0 \) the graph of \( \varphi_n \) over \( (L_{v_n} \cap H^{\infty}_q (r) \subseteq V \), because \( (L_{v_n} \cap H^{\infty}_q (r) \subseteq H^{\infty}_q (r) \) where \( \tilde{f} \) is holomorphic. By the classical Hartogs extension theorem for holomorphic functions the graph of \( \varphi_n \) over the whole set \( \Delta^q (R) \times \Delta \) is contained in the neighborhood \( V \). Then, by maximum principle \( \varphi_n \) converges uniformly to \( \varphi_0 \). Therefore \( \tilde{f} \) is continuous.

What is left to prove is that this extension is Gâteaux differentiable. Take some \( z^0 \in \Delta^q \times B^{\infty} \) and fix some direction \( v \) at \( z^0 \). Let \( l := \{ z^0 + tv : t \in \mathbb{C} \} \) be the line through \( z^0 \) in the direction \( v \). Find (at most) two vectors \( v_1, v_2 \) such that \( e_1, \ldots, e_q, v_1, v_2 \) is the orthonormal basis of the subspace \( L \) containing \( \mathbb{C}^q, z^0 \) and \( l \). Indeed, it is sufficient to prove this in a \( L \)-neighborhood of \( \{ z \} \times \subset \mathbb{R} \) for every \( z \in \Delta^q \) and every \( v \in \text{span} \{ v_1, v_2 \} \). But the graph of \( f|_{\{ z \} \times \subset \mathbb{R} \} \) admits a neighborhood \( V \cong \Delta \times B^{\infty} \) and by continuity of \( f \) the mapping to the graph \( F = (\text{Id}, f) \) sends a \( L \)-neighborhood \( W \) of \( \{ z \} \times \subset \mathbb{R} \) to \( V \). Therefore by the Hartogs separate analyticity theorem for holomorphic functions our extended map \( \tilde{F} \) is holomorphic on every such \( W \) and therefore on \( L \cap (\Delta^q \times B^{\infty}) = \)
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$\Delta^q \times B^2$. In particular it is differentiable in the direction of $v$ at $z^0$, i.e., is Gâteaux differentiable. Every continuous Gâteaux differentiable map is holomorphic, see Theorem 8.7 in [Mu], therefore Theorem 2 is proved.

\[\square\]

**Remark 2.2.** One can also consider the following version of an infinite dimensional Hartogs figure:

\[H(\infty) = (B(\infty) \times B(\infty)) \cup (\overline{B(\infty)}(1-r)) \times B(\infty)\]  

(2.13)

Analogously to the proof of theorem above one can prove the following.

**Proposition 2.1.** Holomorphic maps from $H(\infty)$ to $q$-Hartogs Hilbert manifolds extend onto $B(\infty) \times B(\infty)$ (whatever $q \geq 1$ is).

Indeed, let $f : H(\infty) \rightarrow X$. Choose $e_1, \ldots, e_{q-1}$ vectors of $l^2$. For any $v \in e_1, \ldots, e_{q-1}$, one has:

\[\langle e_1, \ldots, e_{q-1}, v \rangle \times l^2 \cap H(\infty) = H_q(\infty)\]  

(2.14)

Then, by Theorem 2 $f$ extends along $L_v$ for all $v \in e_1, \ldots, e_{q-1}$ to $\tilde{f} : B(\infty) \times B(\infty) \rightarrow X$. The proof of continuity and Gateaux differentiability is the same as above.

### 2.3. Continuity Principle for Hilbert manifolds

Our goal now is to prove the Continuity Principle from the Introduction. In what follows convergence of analytic $q$-disks $\Phi_k = \varphi_k(\Delta_q^k)$ to an analytic $q$-disk $\Phi_0 = \varphi_0(\Delta_q)$ will be understood as uniform convergence of $\varphi_k$ to $\varphi_0$ on some neighborhood of $\overline{D}$.

**Remark 2.3.** We shall apply our Lemma of Royden for $\Delta^q$ which is not strictly pseudoconvex. But, since we suppose that all our analytic disks are actually defined in a neighborhoods of corresponding closures, we can replace $D = \Delta^q$ by some strictly pseudoconvex $\tilde{D} \supset D$ close to $D$ and get the statement of Theorem 1 for such $D$-s as $\Delta^q$ or $\Delta^q \times B^k$ and so on.

**Proof of Theorem 3.** Due to Royden’s Lemma of Theorem 1 there exists a biholomorphic mapping $h$ between a neighborhood $V$ of $\Phi_0$ and $\Delta^q \times B^\infty$ sending $\Phi_0$ to $\overline{\Delta^q} \times \{0\}$. For $k > 1$ we have that $\Phi_k \subset V$ and therefore $h(\Phi_k)$ is a graph of some holomorphic $\psi_k : \Delta^q \rightarrow B^\infty$ with $\psi_k$ converging uniformly to zero as $k \rightarrow \infty$.

Take $k_0$ sufficiently big and make a coordinate change $h_0$ in (a neighborhood of) $\tilde{\Delta^q} \times B^\infty$ as follows: $h_0 : (z, w) \rightarrow (z, w - \psi_k(z))$. Mapping $f \circ h^{-1} \circ h_0^{-1}$ is defined and holomorphic on the Hartogs figure $H_q(\infty)$ for an appropriate $r > 0$. Theorem follows now from Theorem 2.

\[\square\]

### 3. Loop spaces of Hartogs manifolds are Hilbert-Hartogs

#### 3.1. Loop spaces of complex manifolds

Fix a compact, connected, $n$-dimensional real manifold (with boundary or not) $S$ and let us following [L1] describe the natural complex Hilbert structure on the Sobolev manifold $W^{k,2}(S, X)$ of $W^{k,2}$-maps of $S$ to a complex manifold $X$. To speak about Sobolev spaces it is convenient to suppose that $X$ is imbedded to some $\mathbb{R}^N$. If $X$ is not compact, we suppose that this imbedding is proper. For the following basic facts about Sobolev spaces we refer to [L1].
i) \( f \in W^{k,2}(\mathbb{R}^n) \iff (1 + ||\xi||)^k \hat{f} \in L^2(\mathbb{R}^n) \), where \( \hat{f} \) is the Fourier transform of \( f \).

Moreover this correspondence is an isometry by the Plancherel identity. One defines then for any positive \( s \) the space \( W^s(\mathbb{R}^n) = \{ f : (1 + ||\xi||)^s \hat{f} \in L^2(\mathbb{R}^n) \} \).

ii) If \( s > \frac{n}{2} + \alpha \) with \( 0 < \alpha < 1 \) then \( W^s(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n) \) and this inclusion is a compact operator. In particular \( W^{n,2}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \).

iii) If \( s > \frac{n}{2} + k \) for a positive integer \( k \), then \( W^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \).

iv) If \( 0 < s < \frac{n}{2} \) then \( W^s(\mathbb{R}^n) \subset L^{\frac{n}{n-2s}}(\mathbb{R}^n) \).

From (ii) one easily derives that if \( f, g \in W^{n,2}(\mathbb{R}^n) \) then \( fg \in W^{n,2}(\mathbb{R}^n) \). This enables to define correctly a \( W^{k,2} \)-vector bundle over an \( n \)-dimensional real manifold provided \( k \geq n \). By that we mean that the transition functions of the bundle are in \( W^{k,2} \). Condition \( k \geq n \) will be always assumed from now on. Take now \( f \in W^{k,2}(S, X) \). Note that by (ii) such \( f \) is H"older continuous. Consider the pullback \( f^*TX \) as a complex Sobolev bundle over \( S \). A neighborhood \( V_f \) of the zero section of \( f^*TX \to S \) is an open set of the complex Hilbert space \( W^{k,2}(S, f^*TX) \) of Sobolev sections of the pullback bundle. This \( V_f \) can be naturally identified with a neighborhood of \( f \) in \( W^{k,2}(S, X) \) thus providing a structure of complex Hilbert manifold on \( W^{k,2}(S, X) \), see [L1] for more details on this construction.

Another way to understand this complex structure on \( W^{k,2}(S, X) \) is to describe what are analytic disks in \( W^{k,2}(S, X) \).

Lemma 3.1. Let \( D \) and \( X \) be finite dimensional complex manifolds and let \( S \) be an \( n \)-dimensional compact real manifold with boundary. A mapping \( F : D \times S \to X \) represents a holomorphic map from \( D \) to \( W^{k,2}(S, X) \) (denoted by the same letter \( F \)) if and only if the following holds:

i) for every \( s \in S \) the map \( F(\cdot, s) : D \to X \) is holomorphic;

ii) for every \( z \in D \) one has \( F(z, \cdot) \in W^{k,2}(S, X) \) and this correspondence \( D \ni z \to F(z, \cdot) \in W^{k,2}(S, X) \) is continuous with respect to the Sobolev topology on \( W^{k,2}(S, X) \) (and the standard topology on \( D \)).

For the proof we refer to [L1]. Now let us prove the Theorem 4 from the Introduction.

3.2. Proof of Theorem 4. Let \( F : H_q^1(r) \to W^{k,2}(S, X) \) be a holomorphic map. We represent this map as a map \( F : H_q^1(r) \times S \to X \) possessing properties (i) and (ii) of Lemma 3.1 above. From the fact that \( X \) is \( q \)-Hartogs we get that for every fixed \( s \in S \) mapping \( F(\cdot, s) : H_q^1(r) \to X \) extends holomorphically to \( F(\cdot, s) : \Delta^{q+1} \to X \) and we get a mapping \( F : \Delta^{q+1} \times S \to X \). It remains to prove that for every fixed \( z \in \Delta^{q+1} \) one has that \( F(z, \cdot) \in W^{k,2}(S, X) \) and that this correspondence is continuous.

Fix some \( z_0 \in \Delta^{q+1} \) and some \( s_0 \in S \). Take \( R < 1 \) such that \( (z_0, t_0) \in \Delta^{q+1}_R \). Let \( g_{s_0} \) be the map to the graph \( \Gamma_{F(s, s_0)} \) of \( F(\ast, s_0) \). I.e., \( g_{s_0} \) is defined by

\[
g_{s_0} : \Delta^{q+1} \ni z \mapsto (z, F(z, s_0)) \in \Delta^{q+1} \times X.
\]

(3.1)

By Royden’s Lemma for \( R < 1 \) there exists a holomorphic embedding \( G : \Delta^{q+1}_R \times \Delta^{m}_R \to \Delta^{q+1} \times X \) such that \( G(\ast, 0) = g_{s_0} \), here \( m = \text{dim}(X) \). Then \( V = G(\Delta^{q+1}_R \times \Delta^{m}_R) \) contains \( \Gamma_{F(s, s_0)} \) over \( \Delta^{q+1}_R \). Since on \( H_q^1(r) \) the map \( z \mapsto g_s(z) \in W^{k,2}(S, X) \) is continuous it exists \( \epsilon > 0 \) such that for \( s \in B(s_0, \epsilon) \) the graph \( \Gamma(\ast, s) \) over \( H_q^1(r) \) is contained in \( V \). Therefore by the Hartogs theorem for holomorphic functions the graph \( \Gamma(\ast, s) \) over \( \Delta^{q+1}_R \) is contained in \( V \) as well. By maximum principle one has for \( z \in \Delta^{q+1}_R \) and \( s \in B(s_0, \epsilon) \) that

\[
||G^{-1}(g_s(z, t))|| \leq \max_{z \in \partial \Delta^{q+1}_R} ||G^{-1}(g_s(z))|| \leq 
\]
there exists a constant, namely
\[ \psi \]
Set natural projection and consider its Fourier transform \( \hat{\psi} \) such that for all \( z \in \Delta_{q+1} \) one has
\[ ||(1 + ||*||)k\hat{\psi}(z)(*)||_{L^2(B(s_0,\epsilon),\mathbb{C}^m)} \leq M_{s_0}. \] (3.3)
Since \( S \) is compact one can cover it by a finite number of balls \( \{B(s_0,\epsilon)\}_{s_0 \in J} \) and by taking the maximum \( M = \max_{s_0 \in J} M_{s_0} \) one obtains the inequality
\[ ||(1 + ||*||)k\hat{\psi}(z)(*)||_{L^2(S,\mathbb{C}^m)} \leq M. \] (3.4)
Therefore for all \( z \in \Delta_{q+1} \) mapping \( \psi(z) \) is in \( W^{k,2}(S,\mathbb{C}^n) \) and consequently the map \( F(z,*) \) is in \( W^{k,2}(S,X) \). Now let us see that the correspondence \( z \mapsto F(z,*) \) is continuous on \( z \) in Sobolev topology. Indeed, the map \( z \mapsto (1 + ||*||)k\hat{\psi}(z)(*) \) is a holomorphic Hilbert space valued mapping that satisfies the maximum modulus principle, i.e., in particular it will continuously depend on \( z \). Lemma and theorem are proved.

The following statement gives us one more example of open sets \( U \subsetneq \hat{U} \) in Hilbert manifold such that holomorphic mappings extend from \( U \) to \( \hat{U} \), the previous one was \( H^\infty_q(r) \subsetneq \Delta^q \times B^\infty \) of Theorem 2. It shows that \( \hat{U} := W^{k,2}(S,\Delta^q \times \Delta^n) \) is the “envelope of holomorphy” of \( U := W^{k,2}(S,H^q_n(r)) \). Here \( H^q_n(r) := \Delta^q \times \Delta^n(r) \cup A^q_{r-1,1} \times \Delta^n \) stands for the \( q \)-concave Hartogs figure in \( \mathbb{C}^{g+n} \).

**Theorem 3.1.** Let \( X \) be a \( q \)-Hartogs Hilbert manifold. Then every holomorphic map \( F:W^{k,2}(S,H^q_n(r)) \rightarrow X \) extends to a holomorphic map \( \hat{F}:W^{k,2}(S,\Delta^q \times \Delta^n) \rightarrow X \).

**Proof.** Let \( f = (f^q,f^n):S \rightarrow \Delta^q \times \Delta^n \) be an element of \( W^{k,2}(S,\Delta^q \times \Delta^n) \). Consider the following analytic \( q \)-disk in \( W^{k,2}(S,\Delta^q \times \Delta^n) \)
\[ \varphi:(z,s) \in \Delta^q \times S \rightarrow (h_{f^n(s)}(z),f^n(s)), \] (3.5)
where \( h_u \) is an automorphism of \( \Delta^q \) interchanging \( a \) and \( 0 \). \( \Phi = \varphi(\Delta^q) \) is clearly an analytic disk in \( W^{k,2}(S,\Delta^q \times \Delta^n) \) possessing the following properties:
- \( \varphi(0,s) \) is our loop \( f \).
- For \( z \in \partial\Delta^q \) one has that \( \varphi(z,\cdot)(S) \subset A^q_{r-1,1+r} \times \Delta^n \), therefore \( \partial\Phi \subset W^{k,2}(S,H^q_n(r)) \).

Consider the following \((q+1)\)-disk in \( W^{k,2}(S,\Delta^q \times \Delta^n) \)
\[ \varphi_t(z,s) := \varphi(z,t,s) = (h_{f^n(s)}(z),tf^n(s)), \quad |t| < 1. \]
Then
- \( \varphi_0 \subset \Delta^q \times \{0\} \).
- \( \varphi_1 = \varphi \).
- For all \( t \in \Delta \) one has that \( \partial\Phi_t \subset W^{k,2}(S,H^q_n(r)) \).
Therefore for $\varepsilon > 0$ small enough the Hartogs figure

$$H^q_\varepsilon := \{ ||z|| < 1, |t| < \varepsilon \text{ or } 1 - \varepsilon < ||z|| < 1 + \varepsilon, |t| < 1 \}$$

is mapped by $\tilde{\varphi}$ to $W^{k,2}(S, H^q_\varepsilon(r))$.

We can now thicken it to an infinite dimensional Hartogs figure $H^\infty_q(\varepsilon)$ by multiplying it with $B^\infty(\varepsilon)$, where $B^\infty(\varepsilon)$ is a ball in $W^{k,2}(S, \mathbb{C}^{r+n})$. Taking $\varepsilon > 0$ smaller, if necessary, we can achieve that the map

$$\tilde{\varphi} : (z, s, f_2) \to \varphi(z, t, s) + f_2(s)$$

will send $H^\infty_q(\varepsilon)$ to $W^{k,2}(S, H^q_\varepsilon(r))$. Applying Theorem 2 we extend $\tilde{\varphi}$ to $\Delta^n \times B^\infty$ and therefore $F$ is extended to a neighborhood of $\varphi$. Finally, since $H^q_\varepsilon(r)$ and $\Delta^q \times \Delta^n$ are contractible the manifold $W^{k,2}(S, \Delta^q \times \Delta^n)$ is simply connected for any $S$. This insures that our extensions give a single valued holomorphic extension of $F$.

\[\square\]

Corollary 3.1. If $X$ is a $q$-Hartogs complex manifold then every holomorphic mapping $F : W^{k,2}(S, H^q_\varepsilon(r)) \to W^{k,2}(S, X)$ extends to a holomorphic mapping $\tilde{F} : W^{k,2}(S, \Delta^q \times \Delta^n) \to W^{k,2}(S, X)$.

This readily follows from Theorem 3.1 applied to $q$-Hartogs by Theorem 4 Hilbert manifold $X = W^{k,2}(S, X)$.

3.3. Loop spaces of compact complex manifolds are “almost Hartogs”. In \[Iv2\] we introduced the class $G_q$ of $q$-disk convex complex manifolds possessing a strictly positive $dd^c$-closed $(q, q)$-form. Sequence $\{G_q\}_{q=1}^\infty$ is rather exhaustive: $G_q$ contains all compact manifolds of dimension $q + 1$, see subsection 1.5 in \[Iv2\].

We think that the following statement should be true: Let $X$ be a compact manifold from the class $G_q$. Then

i) either $X$ contains a $(q + 1)$-dimensional spherical shell (remark that $X \in F_q$ implies that $\dim X \geq q + 1$);

ii) or, $X$ contains an uniruled compact subvariety of dimension $q$;

iii) or, $X$ is $q$-Hartogs.

Remark 3.1. a) This was proved in \[Iv2\] for $q = 1$ (in fact this particular statement was proved already in \[IV1\]), and in \[IS\] for $q = 2$. In the latter paper we proved almost the assertion stated above (for all $q$-s), but for holomorphic mappings with zero-dimensional fibers, see Proposition 12 there.

2. For $q = 1$ the item (ii) means just that $X$ contains a rational curve. For $q = 2$ we need to add few explanations to \[IS\]. We proved there that a meromorphic map from $H^2_\varepsilon(r)$ to such $X$ meromorphically extends to $\Delta^3 \setminus S$, where is a complete pluripolar set of Hausdorff dimension zero. If $S \neq \emptyset$ then $X$ contains a spherical shell of dimension 3. Otherwise $S$ is empty. If our map $f$ was in addition holomorphic on $H^2_\varepsilon(r)$ then the set $I_f$ of points of indeterminacy of the extension $\tilde{f}$ can be only discrete and then it is clear that for every $a \in I_f$ its full image $\tilde{f}[a]$ contains an uniruled analytic set of dimension two.

From the discussion above we obtain the following statement.

Corollary 3.2. Let $X$ be a compact complex manifold of dimension 2 (resp. of dimension 3). Then either $X$ is one of (i) or (ii) as above or, every generalized loop space $W^{k,2}(S, X)$ is Hartogs (resp. 2-Hartogs).
It might be interesting to think about $X$ from this Corollary as being (an unknown) surface of class $VI_{0}^{+}$ or as $S^{6}$.

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