ON THE DEGENERATIONS OF SOLVABLE LEIBNIZ ALGEBRAS

J. M. CASAS, A. KH. KHUDOYBERDIYEV, M. LADRA, B. A. OMIROV

ABSTRACT. The present paper is devoted to the description of rigid solvable Leibniz algebras. In particular, we prove that solvable Leibniz algebras under some conditions on the nilradical are rigid and we describe four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradical. We show that the Grunewald-O’Halloran’s conjecture “any $n$-dimensional nilpotent Lie algebra is a degeneration of some algebra of the same dimension” holds for Lie algebras of dimensions less than six and for Leibniz algebras of dimensions less than four. The algebra of level one, which is omitted in the 1991 Gorbatsevich’s paper, is indicated.

1. Introduction

Largely because of their importance to string theory, quantum field theory and other branches of fundamental research in mathematical physics, noncommutative analogs of many classical constructions have received much attention in the past few years [8, 10].

The noncommutative analog of Lie algebras are Leibniz algebras, discovered by Loday when he handled periodicity phenomena in algebraic $K$-theory [15]. This algebraic structure found applications in several fields as Physics and Geometry [9, 14, 16, 17].

Important subjects playing a relevant role in Mathematics and Physics are degenerations, contractions and deformations of Lie and Leibniz algebras. Namely, in [20] the notion of contractions of Lie algebras on physical grounds was introduced: if two physical theories (like relativistic and classical mechanics) are related by a limiting process, then the associated invariance groups (like the Poincaré and Galilean groups) should also be related by some limiting process. If the velocity of light is assumed to go to infinity, relativistic mechanics “transforms” into classical mechanics. This also induces a singular transition from the Poincaré algebra to the Galilean one.

Another example is a limiting process from quantum mechanics to classical mechanics under $\hbar \to 0$ of the Planck constant, that corresponds to the contraction of the Heisenberg algebras to the abelian ones of the same dimensions [7].

Nevertheless, as it was proved in [21], the notions of deformations, contractions and degenerations are isomorphic over the fields $\mathbb{R}$ or $\mathbb{C}$. Degenerations of Lie and Leibniz
algebras were the subject of numerous papers, see for instance [3, 4, 5, 11, 18, 19, 21] and references given therein, and their research continues actively. These facts motivate that we focus our attention in the study of degenerations of solvable Leibniz algebras.

In order to do so, we know that an $n$-dimensional Leibniz algebra may be considered as an element $\lambda$ of the affine variety $\text{Hom}(V \otimes V, V)$ via the mapping $\lambda: V \otimes V \to V$ defining the Leibniz bracket on a vector space $V$ of dimension $n$. Since Leibniz algebras are defined via polynomial identities, the set of $n$-dimensional Leibniz algebra structures, $\mathcal{Leib}_n$, forms an algebraic subset of the variety $\text{Hom}(V \otimes V, V)$ and the linear reductive group $\text{GL}_n(F)$ acts on $\mathcal{Leib}_n$ via change of basis, i.e.,

$$(g \ast \lambda)(x, y) = g\left(\lambda(g^{-1}(x), g^{-1}(y))\right), \quad g \in \text{GL}_n(F), \ \lambda \in \mathcal{Leib}_n.$$ 

The orbits $\text{Orb}(-)$ under this action are the isomorphism classes of algebras. Note that solvable (respectively, nilpotent) Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action.

Let $V$ be an $n$-dimensional vector space over a field $F$. The bilinear maps $V \times V \to V$ form an $F^{n^2}$-dimensional affine space. We shall consider the Zariski topology on this space. Recall, a set is called irreducible if it cannot be represented as a union of two nontrivial closed subsets, otherwise it is called reducible. The maximal irreducible closed subset of a variety is called an irreducible component. From algebraic geometry we know that an algebraic variety is a union of irreducible components and that closures of open sets produce irreducible components. Therefore, for the description of a variety it is very important to find all open sets. Since under the above action the variety of Leibniz algebras consists of orbits of algebras, the description of the variety is reduced to find the open orbits. By Noetherian consideration there are a finite number of open orbits. In any variety of algebras there are algebras with open orbits (so-called rigid algebras). Thus, the closure of orbits of rigid algebras gives irreducible components of the variety. Hence, to describe the variety of algebras it is enough to describe all rigid algebras.

A powerful tool in the study of a variety of algebras is that for constructive subsets of algebraic varieties, the closures with respect to the Euclidean and the Zariski topologies coincide. In particular, for an algebraically closed field $F$, the limit in usual Euclidean topology leads to the same limit as in the Zariski topology. It has lead to consideration of such notions as deformations and degenerations of algebras. In fact, a rigid algebra is characterized by absence of any degeneration coming from any algebra, that is, the orbit of the rigid algebra does not belong to the closure of the orbit of any other algebra. Existence or absence of degeneration in a given variety of algebras is revealed
by construction or by using invariant arguments. This approach is very effective in case of nilpotent and solvable algebras.

The description of a variety of any class of algebras is a very difficult problem. Note that for the description of the variety of nilpotent Lie algebras with dimensions less than eight the works [4, 13, 19] are devoted. The complete description of orbits closure of four-dimensional Lie algebras is given in [5]. To the investigation of the variety of Leibniz algebras the work [1] is devoted. In particular, in [1] it is described all irreducible components of the varieties of complex nilpotent Leibniz algebras of dimensions less than 5.

On the other hand, Grunewald and O’Halloran in [13] proposed the following:

**Conjecture:** Any $n$-dimensional nilpotent Lie algebra is a degeneration of some algebra of the same dimension.

In other words there is not nilpotent rigid algebra in the variety of Lie algebras, although a rigid Lie algebra exists in the subvariety of nilpotent Lie algebras. The statement is based on the fact that second cohomology groups of rigid algebras are trivial, while for nilpotent Lie algebras, they are always nontrivial. Similarly to the case of Lie algebras, Balavoine proved the general principles for deformations and rigidity of Leibniz algebras [3].

In this paper we prove that solvable Leibniz algebras, whose nilradical is rigid in the variety of nilpotent Leibniz algebras, cannot be obtained as a degeneration of a solvable Leibniz algebra with different nilradical. In other words, any solvable Leibniz algebra with a given rigid nilradical, such that there is not other solvable Leibniz algebra with the same nilradical, is rigid. The description of solvable Leibniz algebras with three-dimensional rigid nilradical is obtained. Moreover, we prove that the Conjecture above is true for dimensions less than six and for the case of Leibniz algebras the Conjecture is true for dimension less than four. Finally, we find one algebra which was omitted in the work [11].

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field $\mathbb{C}$. Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not noticed we shall consider non-nilpotent solvable algebras.

2. PRELIMINARIES

In this section we give necessary definitions and results for understanding main parts of the work.
Definition 2.1 ([15]). A vector space $L$ over a field $F$ with a binary operation $[-,-]$ is called a Leibniz algebra, if for any $x, y, z \in L$ the so-called Leibniz identity holds

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

Every Lie algebra is a Leibniz algebra, but the bracket in a Leibniz algebra needs not be skew-symmetric.

For a Leibniz algebra $L$ consider the following lower central and derived series:

$$L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1,$$

$$L^{[1]} = 1, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1.$$ 

Definition 2.2. A Leibniz algebra $L$ is said to be nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $L^p = 0$ (respectively, $L^{[q]} = 0$).

It is well known [2] that in Leibniz algebras case, in each dimension, there exists a unique (up to isomorphism) algebra with maximal index of nilpotency whose multiplication table is:

$$\text{NF}_n : [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1.$$ 

Denote by $\mathcal{Leib}_n$ (respectively, by $\mathcal{LN}_n$ and $\mathcal{LR}_n$) the set of all $n$-dimensional (respectively, nilpotent and solvable) Leibniz algebras.

Remark 2.3. Null-filiform Leibniz algebras of dimension $n$ can be characterized as $n$-dimensional nilpotent Leibniz algebras such that the $n$-th term in the lower central series is nontrivial. This means that their orbits are open sets in the variety of $n$-dimensional nilpotent Leibniz algebras with respect to the Zariski topology, hence null-filiform Leibniz algebras of dimension $n$ are rigid.

Let $\lambda$ and $\mu$ be Leibniz algebras of the same dimension over a field $F$.

Definition 2.4. It is said that an algebra $\lambda$ degenerates to an algebra $\mu$, if $\text{Orb}(\mu)$ lies in the Zariski closure of $\text{Orb}(\lambda), \overline{\text{Orb}(\lambda)}$. We denote this by $\lambda \to \mu$.

The degeneration $\lambda \to \mu$ is called a direct degeneration if there is not a chain of nontrivial degenerations of the form: $\lambda \to \nu \to \mu$.

The level of an algebra $\lambda$, denoted by $\text{lev}_n(\lambda)$, is the maximum length of a chain of direct degenerations, which, of course, ends with the algebra $a_n$ (the algebra with zero multiplication).

Remark 2.5. Recall that any $n$-dimensional algebra degenerates to the algebra $a_n$.

Further we shall use the fact from the algebraic groups theory on constructive subsets of algebraic varieties that their closures relative to the Euclidean and the Zariski
topologies coincide. It is not difficult to see that the $GL_n(\mathbb{C})$-orbits are constructive sets. Therefore, the usual Euclidean topology on $\mathbb{C}^n$ leads to the same degenerations as does the Zariski topology, that is, the following condition $\lambda \rightarrow \mu$ implies that

there exists $g_t \in GL_n(\mathbb{C}(t))$ such that $\lim_{t \to 0} g_t \ast \lambda = \mu$,

where $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$.

**Remark 2.6.** It is easy to note that a rigid nilpotent (solvable) algebra cannot be obtained by degeneration of any other nilpotent (solvable) algebra.

Further we shall need the following results.

**Proposition 2.7 ([13]).** Let $G$ be a reductive algebraic group over $\mathbb{C}$ with Borel subgroup $B$ and let $X$ be an algebraic set on which $G$ acts rationally. Then

$$G \ast x = G \ast (B \ast x) \text{ for all } x \in X.$$ 

Note that for the classification of solvable Leibniz algebras with given nilradical, it is important the number of nil-independent derivations of the nilradical. Namely, for a solvable Leibniz algebra with nilradical $N$, the dimension of the complementary vector space to $N$ is not greater than the maximal number of nil-independent derivations of $N$.

**Theorem 2.8 ([6]).** Let $R$ be a solvable Leibniz algebra whose nilradical is $NF_n$. Then there exists a basis $\{e_1, e_2, \ldots, e_n, x\}$ of the algebra $R$ such that the multiplication table of $R$ with respect to this basis has the following form:

$$RNF_n : \begin{cases}
[e_i, e_1] = e_{i+1}, & 1 \leq i \leq n - 1, \\
[x, e_1] = e_1, \\
[e_i, x] = -ie_i, & 1 \leq i \leq n.
\end{cases}$$

In [13] it was shown that the rigid nilpotent Lie algebras in dimensions less than six are the following:

$$n_3 : [e_1, e_2] = -[e_2, e_1] = e_3;$$
$$n_4 : [e_1, e_2] = -[e_2, e_1] = e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_4;$$
$$n_5 : [e_1, e_2] = -[e_2, e_1] = e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_4, \quad [e_1, e_4] = -[e_4, e_1] = e_5, \quad [e_2, e_3] = -[e_3, e_2] = e_5.$$

Due to [1] we can present the list of three-dimensional nilpotent rigid Leibniz algebras:

$$\lambda_4(\alpha) : [e_1, e_1] = e_3, \quad [e_2, e_2] = \alpha e_3, \quad [e_1, e_2] = e_3, \quad \alpha \neq 0;$$
$$\lambda_5 : [e_2, e_1] = e_3, \quad [e_1, e_2] = e_3;$$
$$\lambda_6 : [e_1, e_1] = e_2, \quad [e_2, e_1] = e_3.$$
Proposition 2.9 (I). Let $\lambda$ be a complex non Lie algebra of $\text{Leib}_n$. Then $\lambda \to n_2 \oplus \mathbb{C}^{n-2}$, where $n_2 : [e_1, e_1] = e_2$ is a two-dimensional non-abelian nilpotent Leibniz algebra.

Consider the following algebras:

\[ p^\pm_n : [e_1, e_i] = e_i, \quad [e_i, e_1] = \pm e_i, \quad i \geq 2, \]

\[ n^\pm_3 : [e_1, e_2] = e_3, \quad [e_2, e_1] = \pm e_3. \]

Theorem 2.10 (I). Let $\lambda$ be an $n$-dimensional algebra. Then

1. If the algebra $\lambda$ is skew-commutative, then $\text{lev}_n(\lambda) = 1$ if and only if it is isomorphic to $p_n^-$ or to the algebra $n_3^- \oplus a_{n-3}$ ($n \geq 3$). In particular, the algebra $\lambda$ is a Lie algebra.

2. If the algebra $\lambda$ is commutative, then $\text{lev}_n(\lambda) = 1$ if and only if it is isomorphic to $p_n^+$ or to the algebra $n_3^+ \oplus a_{n-3}$ ($n \geq 3$). In particular, the algebra $\lambda$ is a Jordan algebra.

Remark 2.11. We note that the algebra $p_n^+$ is not a Jordan algebra.

3. Main results

We divide the main section into three subsections where we study the rigidness of solvable Leibniz algebras with rigid nilradical, describe such four-dimensional algebras with three-dimensional radical and present one algebra of level one, which was omitted in the work [I].

3.1. Rigidity of solvable Leibniz algebras with rigid nilradical.

In this subsection we investigate the rigidity of solvable Leibniz algebras with rigid nilradical.

Definition 3.1. The algebras whose orbits are open sets in the variety $\text{Leib}_n$ with respect to the Zariski topology are said to be rigid.

Remark 3.2. The notion of rigidity is characterized by absence of any degeneration coming from any algebra, that is, the orbit of the rigid algebra does not belong to the closure of the orbit of any other algebra.

Let $N$ be a nilpotent Leibniz algebra. Denote by $LR_n(N)$ the set of all $n$-dimensional solvable Leibniz algebras whose nilradical is $N$. 

For any \( m \) \((1 \leq m \leq n)\) define the subset \( \wedge_m \subset \mathcal{L}R_n \) such that \( \wedge_m = \{ \lambda = (c_{i,j}^k) \} \) with the properties:

\[
\sum_{k_1=n-m+1}^{n} \sum_{k_2=n-m+1}^{n} \cdots \sum_{k_{s-1}=n-m+1}^{n} c_{i_1,i_2,k_3}^{k_1} c_{i_2,i_4,k_5}^{k_2} \cdots c_{i_{s-1},i_s}^{k_{s-1},i_s} = 0, \quad n - m + 1 \leq i_1, i_2, \ldots, i_s \leq n,
\]

\[
c_{i,j}^k = 0, \quad 1 \leq i, j \leq n, \quad 1 \leq k \leq n - m,
\]

where \( c_{i,j}^k \) are structural constants and \( s \) any fixed number.

Let us observe that \( R \in \wedge_m \) if and only if \( R \) contains the nilpotent ideal \( N = \langle \{ e_{n-m+1}, e_{n-m+2}, \ldots, e_n \} \rangle \) satisfying \( R^2 \subseteq N \).

It is not difficult to see that \( \wedge_m \) is a Zariski closed subset of \( \mathcal{L}R_n \), but it is not \( \text{GL}_n(\mathbb{C}) \)-stable. However, the set \( \wedge_m \) is \( B \)-stable, where \( B \) is the Borel subgroup of \( \text{GL}_n(\mathbb{C}) \) consisting of upper triangular matrices.

**Proposition 3.3.** Let \( R_1, R_2 \in \mathcal{L}R_n \) and let \( R_1 \in \mathcal{L}R_n(N_1) \), \( R_2 \in \mathcal{L}R_n(N_2) \). If \( R_1 \to R_2 \), then \( \dim N_2 \geq \dim N_1 \).

**Proof.** Let \( \dim N_1 = m \), then choose \( g \in \text{GL}_n(\mathbb{C}) \) such that \( R' = g \ast R_1 \in \wedge_m \). Since \( B \ast R' \in \wedge_m \) and \( \wedge_m \) is a closed set, then \( \overline{B \ast R'} \in \wedge_m \). By Proposition 2.7 and by condition \( R_1 \to R_2 \) we conclude that \( R_2 \in \text{GL}_n(\mathbb{C}) \ast \wedge_m \). Therefore, the algebra \( R_2 \) contains a nilpotent ideal of dimension \( m \). Since \( N_2 \) is the nilradical of \( R_2 \), we get \( \dim N_2 \geq m \).

**Corollary 3.4.** Let \( R_1 \in \mathcal{L}R_n(N_1) \) and \( R_2 \in \mathcal{L}R_n(N_2) \). If \( \dim N_1 = \dim N_2 \) and \( R_1 \to R_2 \), then \( N_1 \to N_2 \).

**Proof.** Let \( g_t \) be a family such that \( \lim_{t \to 0} g_t \ast R_1 = R_2 \). By Proposition 3.3 we have that \( \lim_{t \to 0} g_t \ast N_1 \) is a nilpotent ideal of \( R_2 \). Therefore, we get \( \dim N_1 = \dim (\lim_{t \to 0} g_t \ast N_1) = \dim N_2 \). Since \( N_2 \) is the nilradical of \( R_2 \), then \( \lim_{t \to 0} g_t \ast N_1 = N_2 \), i.e., \( N_1 \to N_2 \).

Consider now a solvable Leibniz algebra \( R \) with rigid nilradical \( N \).

**Proposition 3.5.** Let \( R^2 = N \) and suppose that there exists a solvable Leibniz algebra \( R_1 \) such that \( R_1 \to R \). Then \( R_1 \in \mathcal{L}R_n(N) \).

**Proof.** Let \( N_1 \) be the nilradical of the algebra \( R_1 \). Note that by the Proposition 3.3 \( \dim N_1 \leq \dim N \).

If \( \dim N_1 < \dim N \), then we have \( \dim R_1^2 \leq \dim N_1 < \dim N = \dim R^2 \), which is a contradiction to the condition \( R_1 \to R \) by a consequence of [13 Theorem 1.4] (see also [11 Corollary]).

If \( \dim N_1 = \dim N \), then by Corollary 3.4 we conclude that \( N_1 \to N \). Since \( N \) is a rigid algebra, then we get \( N_1 \cong \mathbb{N} \).
Corollary 3.6. The algebra $\text{RNF}_n$ is a rigid algebra of the variety $LR_{n+1}$.

From the above, we conclude that for a rigid nilpotent Leibniz algebra $N$ in the variety $LN_s$ and for $R \in LR_n(N)$ there are only two possibilities: $R$ is rigid in $LR_n$ or there exists a rigid algebra $R_1 \in LR_n(N)$ such that $R_1 \to R$.

Next proposition establishes a relationship between a solvable algebra and its nilradical.

Proposition 3.7. For any solvable algebra $R$ with nilradical $N$ there exists a degeneration: $R \to N \oplus \mathbb{C}^k$, where $k = \dim R/N$.

Proof. We choose a basis $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ of $R$ such that $N = \langle\{e_{k+1}, \ldots, e_n\}\rangle$. A degeneration is given by the family $g_t$ defined as follows:

$$g_t(e_i) = \begin{cases} t^{-1}e_i & \text{if } 1 \leq i \leq k, \\ e_i & \text{if } k+1 \leq i \leq n. \end{cases}$$

Indeed,

$$g_t[e_i, e_j] = g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = t^2 g_t([e_i, e_j]) = t^2 [e_i, e_j] \to 0, \quad 1 \leq i, j \leq k,$$

$$g_t[e_i, e_j] = g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = t g_t([e_i, e_j]) = t [e_i, e_j] \to 0, \quad 1 \leq i \leq k, \quad k+1 \leq j \leq n,$$

$$g_t[e_i, e_j] = g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = g_t([e_i, e_j]) = [e_i, e_j], \quad k+1 \leq i, j \leq n.$$

Now we present a family, which will be useful in the sequel,

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_i) = t^{-i+1}e_i, \quad 3 \leq i \leq n,$$

that degenerates the algebra $\text{NF}_n$ to the so-called filiform algebra $\mathbb{F}_n$:

$$\mathbb{F}_n : [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n - 1.$$

3.2. Classification of four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradicals.

In this subsection we classify four-dimensional solvable Leibniz algebras whose nilradical is rigid and three-dimensional.

First at all, in the following proposition we describe the derivations of the three-dimensional nilpotent rigid Leibniz algebras $\lambda_4(\alpha)$, $\lambda_5$ and $\lambda_6$. Recall that a derivation of a Leibniz algebra $(L, [-, -])$ is a $F$-linear map $d : L \to L$ such that $d[x, y] = [d(x), y] + [x, d(y)]$, for all $x, y \in L$. 
Proposition 3.8. In the algebras $\lambda_4(\alpha)$, $\lambda_5$ and $\lambda_6$ there exist bases such that their derivations have the following forms:

\[
\begin{align*}
\text{Der } (\lambda_4(\alpha)) & = 
\begin{pmatrix}
  a_1 & 0 & a_3 \\
  0 & b_2 & b_3 \\
  0 & 0 & a_1 + b_2 \\
\end{pmatrix}, \quad \alpha \neq \frac{1}{4};
\quad \text{Der } (\lambda_4(\frac{1}{4})) = 
\begin{pmatrix}
  a_1 & a_2 & a_3 \\
  0 & a_1 & b_3 \\
  0 & 0 & 2a_1 \\
\end{pmatrix}, \\
\text{Der } (\lambda_5) & = 
\begin{pmatrix}
  a_1 & 0 & a_3 \\
  0 & b_2 & b_3 \\
  0 & 0 & a_1 + b_2 \\
\end{pmatrix}, \quad \text{Der } (\lambda_6) = 
\begin{pmatrix}
  a_1 & a_2 & a_3 \\
  0 & 2a_1 & a_2 \\
  0 & 0 & 3a_1 \\
\end{pmatrix}.
\end{align*}
\]

Proof. Taking the following change of basis in the algebra $\lambda_4(\alpha)$:

\[f_1 = e_1, \quad f_2 = e_2 + \beta e_1, \quad f_3 = e_3,\]

with $\beta = -\frac{1 + \sqrt{1-4\alpha}}{2}$, we deduce that the multiplications of $\lambda_4(\alpha)$ becomes of the form:

\[[f_1, f_1] = f_3, \quad [f_2, f_1] = \beta f_3, \quad [f_1, f_2] = (1 + \beta) f_3.\]

If $\beta \neq -\frac{1}{2}$ (i.e., $\alpha \neq \frac{1}{4}$), then setting $f'_1 = f_1 - \frac{1}{2\beta + 1} f_2$, $f'_2 = \frac{1}{\beta} f_2$, we derive

\[[f_2, f_1] = f_3, \quad [f_1, f_2] = \beta' f_3, \quad (3.1)\]

where $\beta' = \frac{\sqrt{1-4\alpha}-1}{\sqrt{1-4\alpha}+1}$.

If $\beta = -\frac{1}{2}$ (i.e., $\alpha = \frac{1}{4}$), then putting $f'_2 = -2 f_2$, we get

\[\lambda_4(\frac{1}{4}) : [f_1, f_1] = f_3, \quad [f_2, f_1] = f_3, \quad [f_1, f_2] = -f_3. \quad (3.2)\]

By checking the derivation property for algebras (3.1) and (3.2) we obtain

\[
\begin{align*}
\text{Der } (\lambda_4(\alpha)) & = 
\begin{pmatrix}
  a_1 & 0 & a_3 \\
  0 & b_2 & b_3 \\
  0 & 0 & a_1 + b_2 \\
\end{pmatrix}, \quad \alpha \neq \frac{1}{4};
\quad \text{Der } (\lambda_4(\frac{1}{4})) = 
\begin{pmatrix}
  a_1 & a_2 & a_3 \\
  0 & a_1 & b_3 \\
  0 & 0 & 2a_1 \\
\end{pmatrix}.
\end{align*}
\]

The derivations of the algebras $\lambda_5$ and $\lambda_6$ are obtained directly applying the derivation property. \(\square\)

Below, we prove that there do not exist four-dimensional solvable Leibniz algebras with nilradical $\lambda_4(\frac{1}{4})$.

Proposition 3.9. There are not four-dimensional solvable Leibniz algebras with three-dimensional nilradical $\lambda_4(\frac{1}{4})$.

Proof. Let us assume the contrary. Let $R \in \mathcal{L}R_4(\lambda_4(\frac{1}{4}))$. We choose a basis $\{x, f_1, f_2, f_3\}$ of $R$ such that $\{f_1, f_2, f_3\}$ is the basis of $\lambda_4(\frac{1}{4})$ chosen in the proof of
Proposition 3.8. Since the algebra $R$ is non-nilpotent, the restriction of the right multiplication operator $\mathcal{R}_x$ to $\mathbf{L}_4(\frac{1}{4})$ is a non-nilpotent derivation of $\mathbf{L}_4(\frac{1}{4})$. Then using the form of this derivation from Proposition 3.8 we have

\[
[f_1, x] = a_1f_1 + a_2f_2 + a_3f_3, \quad [f_2, x] = a_1f_2 + b_3f_3, \quad [f_3, x] = 2a_1f_3, \\
[f_1, f_1] = f_3, \quad [f_2, f_1] = f_3, \quad [f_1, f_2] = -f_3.
\]

Since $\mathcal{R}_x|\mathbf{L}_4$ is non-nilpotent, we can suppose $a_1 = 1$. It is easy to see that the right annihilator of the algebra $R$ only consists of $\{f_3\}$. Therefore,

\[
[f_1, x] = f_1 + a_2f_2 + a_3f_3, \quad [f_2, x] = f_2 + b_3f_3, \quad [f_3, x] = 2f_3, \\
[x, f_1] = -f_1 - a_2f_2 + \alpha_3f_3, \quad [x, f_2] = -f_2 + \beta_3f_3, \quad [x, x] = \gamma_3f_3, \\
[f_1, f_1] = f_3, \quad [f_2, f_1] = f_3, \quad [f_1, f_2] = -f_3.
\]

Considering the Leibniz identity

\[
0 = [x, [f_2, f_1]] = [[x, f_2], f_1] - [[x, f_1], f_2] = [-f_2 + \beta_3f_3, f_1] - [-f_1 - a_2f_2 + \alpha_3f_3, f_2] = -f_3 - f_3 = -2f_3,
\]

we have a contradiction with the assumption.

The following theorem gives the classification of four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradicals.

**Theorem 3.10.** Up to isomorphism, there exist three four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradicals. Namely,

- **$R_1^4$**: \[
\begin{align*}
[e_2, e_1] &= e_3, \quad [e_1, e_2] = \beta e_3, \quad [x, e_1] = -e_1, \quad [x, e_2] = -\beta e_2, \\
[e_1, x] &= e_1, \quad [e_2, x] = \beta e_2, \quad [e_3, x] = (\beta + 1)e_3,
\end{align*}
\]

where $\beta = \frac{\sqrt{1 - 4\alpha - 1}}{1 - 4\alpha + 1}$ for $\alpha \neq 0, \frac{1}{4}$;

- **$R_2^4$**: \[
\begin{align*}
[e_2, e_1] &= e_3, \quad [e_1, e_2] = e_3, \quad [x, e_1] = -e_1, \quad [x, e_2] = -e_2, \\
[e_1, x] &= e_1, \quad [e_2, x] = e_2, \quad [e_3, x] = 2e_3;
\end{align*}
\]

- **$R_3^4$**: \[
\begin{align*}
[e_1, e_1] &= e_2, \quad [e_2, e_1] = e_3, \quad [x, e_1] = -e_1, \\
[e_1, x] &= e_1, \quad [e_2, x] = 2e_2, \quad [e_3, x] = 3e_3.
\end{align*}
\]

**Proof.** Here we shall use the form of the algebra $\mathbf{L}_4(\alpha)$ as in the proof of Proposition 3.8 after the change of basis, i.e., the form $\mathbf{L}_4(\beta)$. Consider the class $\mathcal{L}R_4(\mathbf{L}_4(\beta))$. Due to Proposition 3.8 we can choose a basis $\{x, f_1, f_2, f_3\}$ of the algebra of $\mathcal{L}R_4(\mathbf{L}_4(\beta))$ such that $\mathcal{R}_x|\mathbf{L}_4(\beta)$ is a non-nilpotent derivation of $\mathbf{L}_4(\beta)$. Therefore, in the algebra of $\mathcal{L}R_4(\mathbf{L}_4(\beta))$ we have the following products:

\[
[f_2, f_1] = f_3, \quad [f_1, f_2] = \beta f_3, \\
[f_1, x] = a_1f_1 + a_3f_3, \quad [f_2, x] = b_2f_2 + b_3f_3, \quad [f_3, x] = (a_1 + b_2)f_3.
\]
It is easy to see that the right annihilator of the algebra consists of \( \{f_3\} \). Hence we get

\[
[1, f_1] = a_1 f_1 + a_3 f_3, \quad [f_2, x] = b_2 f_2 + b_3 f_3, \quad [f_3, x] = (a_1 + b_2) f_3,
\]

we can suppose that

\[
[x, f_1] = -a_1 f_1 + a_3 f_3, \quad [x, f_2] = -b_2 f_2 + \beta_3 f_3, \quad [x, x] = \gamma_3 f_3,
\]

we derive \( b_2 = a_1 \beta \).

Similarly, from \( R_{x|\lambda x} \) we have \( a_1 = b_2 \neq 0 \). Consequently, we can assume \( a_1 = 1, \ b_2 = \beta \).

Taking the change of basis:

\[
e_1 = f_1 - \frac{a_3}{\beta} f_3, \quad e_2 = f_2 - b_3 f_3, \quad e_3 = f_3, \quad x' = x - \frac{\gamma_3}{\beta + 1} f_3,
\]

we can suppose that \( a_3 = b_3 = \gamma_3 = 0 \) and the multiplication table has the form

\[
[e_1, x] = e_1, \quad [e_2, x] = \beta e_2, \quad [e_3, x] = (1 + \beta) e_3,
[x, e_1] = -e_1 + a_3 e_3, \quad [x, e_2] = -\beta e_2 + \beta_3 e_3,
[e_2, e_1] = e_3, \quad [e_1, e_2] = \beta e_3.
\]

Consider the chain of equalities

\[
[x, [e_1, x]] = [[x, e_1], x] - [[x, x], e_1] = [-e_1 + a_3 e_3, x] = -e_1 + a_3 (1 + \beta) e_3.
\]

On the other hand, \([x, [e_1, x]] = [x, e_1] = -e_1 + a_3 e_3\).

Comparing the coefficients at the basis elements, we obtain \( a_3 \beta = 0 \) which implies \( a_3 = 0 \).

Similarly, from

\[
[x, [e_2, x]] = [[x, e_2], x] - [[x, x], e_2] = [-\beta e_2 + \beta_3 e_3, x] = -\beta^2 e_2 + \beta_3 (1 + \beta) e_3,
[x, [e_2, x]] = [x, \beta e_2] = -\beta^2 e_2 + \beta \beta_3 e_3,
\]

we deduce \( \beta_3 = 0 \). Thus the algebra \( R_4^1 \) is obtained.

Applying the above arguments for the class \( LR_4(\lambda_5) \) we derive the multiplication table:

\[
[f_1, x] = a_1 f_1 + a_3 f_3, \quad [f_2, x] = b_2 f_2 + b_3 f_3, \quad [f_3, x] = (a_1 + b_2) f_3,
[x, f_1] = -a_1 f_1 + a_3 f_3, \quad [x, f_2] = -b_2 f_2 + \beta_3 f_3, \quad [x, x] = \gamma_3 f_3,
[f_2, f_1] = f_3, \quad [f_1, f_2] = f_3.
\]
From the chain of equalities

\[ 0 = [x, [f_2, f_1]] = [[x, f_2], f_1] - [[x, f_1], f_2] = -b_2 f_2 + \beta_3 f_3, \alpha_3 f_3, f_2 = -b_2 f_2 + \alpha_3 f_3, f_2, \]

we have \( b_2 = a_1. \)

Since the restriction of the right multiplication operator on the element \( x \) to \( \mathbf{L}_5 \) is non-nilpotent, we have \( a_1 = b_2 \neq 0 \) and without loss of generality we can suppose \( a_1 = b_2 = 1. \)

Taking the change of basis

\[ e_1 = f_1 - a_3 f_3, \quad e_2 = f_2 - b_3 f_3, \quad e_3 = f_3, \quad x' = x - \frac{\gamma_3}{2} f_3, \]

we can suppose that \( a_3 = b_3 = \gamma_3 = 0 \) and the multiplication table has the form

\[
\begin{align*}
[e_1, x] &= e_1, & [e_2, x] &= e_2, & [e_3, x] &= 2e_3, \\
[x, e_1] &= -e_1 + a_3 e_3, & [x, e_2] &= -e_2 + \beta_3 e_3, \\
[e_2, e_1] &= e_3, & [e_1, e_2] &= e_3.
\end{align*}
\]

Applying the Leibniz identity to the brackets \([x, [x, e_1]]\) and \([x, [e_1, x]]\) with respect to the above multiplication, we derive that \( \alpha_3 = \beta_3 = 0. \) Thus, we obtain the algebra \( R_4^4. \)

Since an algebra of \( LR_4(\mathbf{L}_6) \) is nothing else but the algebra \( \mathbf{RNF}_3, \) the algebra \( R_3^4 \) is directly followed from Theorem 2.8.

It should be noted that thanks to Proposition 3.3 and Corollary 3.6 the algebras \( R_4^4, R_4^2 \) and \( R_3^4 \) are rigid in the variety \( LR_4. \)

The following theorem assert that the Conjecture is true for dimensions less than six.

**Theorem 3.11.** Any complex nilpotent Lie algebra of dimension less than six is not rigid in \( LR_n. \)

**Proof.** All solvable Lie algebras of dimension less than six have the following multiplication tables [12]:

\[
\begin{align*}
\mathbf{r}_3 &: [e_1, e_2] = -[e_2, e_1] = e_1 + e_3, \quad [e_3, e_2] = -[e_2, e_3] = e_3, \\
\mathbf{r}_4 &: [e_1, e_2] = -[e_2, e_1] = e_1 + e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_4, \quad [e_2, e_3] = -[e_2, e_3] = e_3, \\
\mathbf{r}_5 &: [e_1, e_2] = -[e_2, e_1] = e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_2, \quad [e_1, e_4] = -[e_4, e_1] = e_5, \quad [e_2, e_3] = -[e_3, e_2] = e_5.
\end{align*}
\]

It is easy to check that
Consider the following degenerates the algebra \( R \) where \( \beta \) is a constant. From [1] we have a unique two-dimensional rigid nilpotent Leibniz algebra \( R \).

**Remark 3.12.** Consider the following \( n \)-dimensional solvable Leibniz algebra

\[
R_n : [e_1, e_1] = e_2, \quad [e_i, e_1] = e_i + e_{i+1}, \quad 2 \leq i \leq n - 1.
\]

It is known that the algebra \( \mathcal{N}F_n \) is rigid in the variety of nilpotent Leibniz algebras [2]. However, this algebra it is not rigid in the variety of solvable Leibniz algebras. Indeed, the family of basis transformations

\[
g_t(e_i) = t^{-i}e_i, \quad 1 \leq i \leq n,
\]

degenerates the algebra \( R_n \) to \( \mathcal{N}F_n \).

Now we present a result which asserts that the Conjecture is true for the case of Leibniz algebras of dimensions less than four.

**Theorem 3.13.** Any nilpotent Leibniz algebra of dimension less than four is not rigid.

**Proof.** From [1] we have a unique two-dimensional rigid nilpotent Leibniz algebra \( n_2 : [e_1, e_1] = e_2 \). It is easy to check that the algebra \( r_2 \) with the multiplication table

\[
[e_2, e_1] = e_2
\]
degenerates to \( n_2 \) via the family of transformations:

\[
g_t : \quad g_t(e_1) = t^{-1}e_1 - t^{-2}e_2, \quad g_t(e_2) = t^{-2}e_2.
\]

For the three-dimensional case we have the rigid nilpotent algebras \( \lambda_4(\alpha), \lambda_5 \) and \( \lambda_6 \).

Let us consider the solvable Leibniz algebra

\[
r_{3,2}(\alpha) : \begin{cases}
[e_1, e_1] = e_3, & [e_1, e_2] = -(2 + \beta)\alpha e_1 + e_2 + e_3, \\
[e_2, e_1] = (2 + \beta)\alpha e_1 - e_2, & [e_2, e_2] = \alpha e_3, \quad \alpha \neq 0, \\
[e_3, e_1] = \beta e_3, & [e_3, e_2] = (2 + \beta)\beta e_3,
\end{cases}
\]

where \( \beta = \frac{1 - 4\alpha + \sqrt{1 - 4\alpha}}{2\alpha} \).

Then \( g_t \) defined as

\[
g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_3) = t^{-2}e_3
\]
degenerates the algebra $r_{3,2}(\alpha)$ to the algebra $\lambda_4(\alpha)$.

Consider the solvable Leibniz algebra

$$r_{3,1} : [e_2, e_1] = -e_2 + e_3, \quad [e_3, e_1] = -2e_3, \quad [e_1, e_2] = e_2 + e_3, \quad [e_2, e_2] = e_3.$$  

Then $r_{3,1} \to \lambda_5$ via $g_t$, which is given by

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-2}e_2, \quad g_t(e_3) = t^{-3}e_3.$$  

Due to Remark 3.12 we get $R_3 \to \lambda_6$. $\Box$

### 3.3. On the algebra of level one.

In this subsection we show that the result of Theorem 2.10 is not complete. Namely, the algebra $n_2 \oplus a_{n-2}$ is also an algebra of level one and it is not isomorphic to the algebras $p_n^\pm$ and $n_3^\pm \oplus a_{n-3}$.

**Theorem 3.14.** The $n$-dimensional commutative algebra $n_2 \oplus a_{n-2}$ is of level one.

**Proof.** Firstly, we shall prove that the algebra $n_2 \oplus a_{n-2}$ does not degenerate to $p_n^\pm$ and $n_3^\pm \oplus a_{n-3}$. Since $n_2 \oplus a_{n-2}$ is commutative, it is enough to prove it for $p_n^\pm$ and $n_3^\pm \oplus a_{n-3}$.

Let us assume the contrary, that is there exists a family $g_t \in \text{GL}_n(\mathbb{C})$ such that $n_2 \oplus a_{n-2} \to p_n^\pm$. Let $g_t$ be of the form

$$g_t(e_i) = \sum_{s=1}^{n} \alpha_{i,s}(t)e_s, \quad g_t^{-1}(e_i) = \sum_{s=1}^{n} \beta_{i,s}(t)e_s.$$  

Consider $g_t(e_2) = \sum_{i=1}^{n} \alpha_{2,i}(t)e_i$. We choose numbers $p, q \neq q$ such that

$$\lim_{t \to 0} \frac{\alpha_{2,p}(t)}{\alpha_{2,q}(t)} < \infty.$$  

Consider

$$g_t([g_t^{-1}(e_1), g_t^{-1}(e_p)]) = \beta_{1,1}(t)\beta_{p,1}(t)g_t(e_2) = \beta_{1,1}(t)\beta_{p,1}(t)\sum_{i=1}^{n} \alpha_{2,i}(t)e_i. \quad (3.3)$$  

Since in the algebra $p_n^\pm$ we have $[e_1, e_p] = e_p$, then $\lim_{t \to 0} g_t([g_t^{-1}(e_1), g_t^{-1}(e_p)]) = e_p$.

Therefore, we obtain

$$\lim_{t \to 0} \beta_{1,1}(t)\beta_{p,1}(t)\alpha_{2,p}(t) = 1, \quad \lim_{t \to 0} \beta_{1,1}(t)\beta_{p,1}(t)\alpha_{2,q}(t) = 0.$$  

On the other hand, 
\[ \lim_{t \to 0} \beta_{1,1}(t)\beta_{p,1}(t)\alpha_{2,p}(t) = \lim_{t \to 0} \beta_{1,1}(t)\beta_{p,1}(t)\alpha_{2,q}(t) \frac{\alpha_{2,p}(t)}{\alpha_{2,q}(t)} = \lim_{t \to 0} \beta_{1,1}(t)\beta_{p,1}(t)\alpha_{2,q}(t) \cdot \lim_{t \to 0} \frac{\alpha_{2,p}(t)}{\alpha_{2,q}(t)} = 0. \]

This is a contradiction with the assumption of the existence of \( g_t \), i.e., \( n_2 \oplus a_{n-2} \) does not degenerate to \( p_n^+ \).

Let us show that \( n_2 \oplus a_{n-2} \) does not degenerate to the algebra \( n_3^+ \oplus a_{n-3} \). Similarly as above we can assume the existence of a family \( g_t \).

From (3.3) we get 
\[ g_t([g_t^{-1}(e_1), g_t^{-1}(e_1)]) = \beta_{1,1}(t)\beta_{1,1}(t) \sum_{i=1}^{n} \alpha_{2,i}(t)e_i. \]

Since in the algebra \( n_3^+ \oplus a_{n-3} \) we have the product \([e_1, e_1]=0\), then 
\[ \lim_{t \to 0} g_t([g_t^{-1}(e_1), g_t^{-1}(e_1)]) = 0. \]

Consequently, \( \lim_{t \to 0} \beta_{1,1}(t)\beta_{1,1}(t)\alpha_{2,3}(t) = 0 \).

Similarly, from (3.3) with \( p=2 \), i.e., 
\[ g_t([g_t^{-1}(e_1), g_t^{-1}(e_2)]) = \beta_{1,1}(t)\beta_{2,1}(t) \sum_{i=1}^{n} \alpha_{2,i}(t)e_i \]

and of the product \([e_1, e_2]=e_3 \) in \( n_3^+ \oplus a_{n-3} \), we conclude 
\[ \lim_{t \to 0} \beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 1. \]

Using the equalities
\[ g_t([g_t^{-1}(e_2), g_t^{-1}(e_2)]) = g_t([\sum_{s=1}^{n} \beta_{2,s}(t)e_s, \sum_{s=1}^{n} \beta_{2,s}(t)e_s]) \]
\[ = \beta_{2,1}(t)\beta_{2,1}(t)g_t(e_2) = \beta_{2,1}(t)\beta_{2,1}(t) \sum_{i=1}^{n} \alpha_{2,i}(t)e_i \]

and \([e_2, e_2]=0\), we derive \( \lim_{t \to 0} \beta_{2,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 0 \).

Thus, we summarize
\[ \lim_{t \to 0} \beta_{1,1}(t)\beta_{1,1}(t)\alpha_{2,3}(t) = \lim_{t \to 0} \beta_{2,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 0, \quad \lim_{t \to 0} \beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 1. \]

However,
\[ \lim_{t \to 0} \left( \beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) \right)^2 = \lim_{t \to 0} \beta_{1,1}(t)\beta_{1,1}(t)\alpha_{2,3}(t) \cdot \lim_{t \to 0} \beta_{2,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 0. \]

Thus, the algebra \( n_2 \oplus a_{n-2} \) does not degenerate to \( n_3^+ \oplus a_{n-3} \).
Now we shall prove that \( \text{lev}_n(n_2 \oplus a_{n-2}) = 1 \). Assume that there exists a Leibniz algebra \( \lambda \) such that \( n_2 \oplus a_{n-2} \to \lambda \) is a direct degeneration. Then \( \dim \text{Orb}(\lambda) < \dim \text{Orb}(n_2 \oplus a_{n-2}) \) (see [11]).

If \( \lambda \) is a non-Lie Leibniz algebra, then by Proposition 2.9 we have that \( \lambda \to n_2 \oplus a_{n-2} \). Then there exists a chain of direct degenerations \( \lambda \to \lambda_1 \to \cdots \to \lambda_k \to n_2 \oplus a_{n-2} \). Again by [11], we have that \( \dim \text{Orb}(n_2 \oplus a_{n-2}) < \dim \text{Orb}(\lambda_k) < \cdots < \dim \text{Orb}(\lambda_1) < \dim \text{Orb}(\lambda) \). This is a contradiction with \( \dim \text{Orb}(\lambda) < \dim \text{Orb}(n_2 \oplus a_{n-2}) \).

Let \( \lambda \) be a Lie algebra, then by assumption there exists a family \( g_t \) such that \( \lim_{t \to 0} g_t^* (n_2 \oplus a_{n-2}) = \lambda \).

Then from the following equalities
\[
g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = g_t\left(\sum_{s=1}^{n} \beta_{i,s}(t)e_s, \sum_{s=1}^{n} \beta_{j,s}(t)e_s\right) = \beta_{i,1}(t)\beta_{j,1}(t)g_t(e_2),
\]
we deduce \( \lambda(e_i, e_j) = \lambda(e_j, e_i) \). Since \( \lambda \) is a Lie algebra, it follows that it is abelian. Consequently, the algebra \( n_2 \oplus a_{n-2} \) is of level one.

\[\square\]

Acknowledgements

The first and third authors were supported by Ministerio de Ciencia e Innovación (European FEDER support included), grant MTM2009-14464-C02, and by Xunta de Galicia, grant Incite09 207 215 PR. The fourth author was partially supported by the Grant (RGA) No:11-018 RG/Math/ASI–UNESCO FR: 3240262715.

References

[1] Albeverio, S., Omirov, B.A., Rakhimov, I.S. Varieties of nilpotent complex Leibniz algebras of dimension less than five, Comm. Algebra 33(5) (2005), 1575–1585.
[2] Ayupov, Sh.A., Omirov, B.A. On some classes of nilpotent Leibniz algebras, Sib. Math. J. 42(1) (2001), 15–24.
[3] Balavoine, D. Déformations et rigidité géométrique des algèbres de Leibniz, Comm. Algebra 24(3) (1996), 1017–1034.
[4] Burde, D. Degenerations of 7-dimensional nilpotent Lie algebras, Comm. Algebra 33(4) (2005), 1259–1277.
[5] Burde, D., Steinhoff, C. Classification of orbit closures of 4-dimensional complex Lie algebras, J. Algebra 214(2) (1999), 729–739.
[6] Casas, J. M., Ladra, M., Omirov, B.A., Karimjanov, I.K. Classification of solvable Leibniz algebras with null-nilpotent nilradical, Linear Multilinear Algebra, DOI: 10.1080/03081080.
[7] Drühl, K. A theory of classical limit for quantum theories which are defined by real Lie algebras, J. Math. Phys. 19(7) (1978), 1600–1606.
[8] Ebrahimi-Fard, K. Loday-type algebras and the Rota-Baxter relation, Lett. Math. Phys. 61(2) (2002), 139–147.
ON THE DEGENERATIONS OF SOLVABLE LEIBNIZ ALGEBRAS

[9] Felipe, R., López-Reyes, N., Ongay, F. *R-Matrices for Leibniz Algebras*, Lett. Math. Phys. 63(2) (2003), 157–164.

[10] Golubchik, I.Z., Sokolov, V.V. *Generalized operator Yang-Baxter equations, integrable ODEs and nonassociative algebras*, J. Nonlinear Math. Phys. 7(2) (2000), 184–197.

[11] Gorbatsevich, V.V. *Contractions and degenerations of finite-dimensional algebras*, Soviet Math. (Iz. VUZ) 35 (1991), 17–24.

[12] Goze, M., Khakimdjanov, Y. *Nilpotent Lie algebras*, Mathematics and its Applications 361, Kluwer Acad. Publish., Dordrecht, 1996.

[13] Grunewald, F., O’Halloran, J. *Varieties of nilpotent Lie algebras of dimension less than six*, J. Algebra 112(2) (1998), 315–325.

[14] Kinyon, M.K., Weinstein, A. *Leibniz algebras, Courant algebroids and multiplications on reductive homogeneous spaces*, Amer. J. Math. 123(3) (2001), 525–550.

[15] Loday, J.-L. *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. (2) 39(3–4) (1993), 269–293.

[16] Lodder, J.M. *Leibniz homology and the James model*, Math. Nachr. 175 (1995), 209–229.

[17] Lodder, J.M. *Leibniz cohomology for differentiable manifolds*, Ann. Inst. Fourier (Grenoble) 48(1) (1998), 73–95.

[18] Rakhimov, I.S., Atan, K.A.M. *On contractions and invariants of Leibniz algebras*, Bull. Malays. Math. Sci. Soc. (2) 35(2A) (2012), 557–565.

[19] Seeley, C. *Degenerations of 6-dimensional nilpotent Lie algebras over C*, Comm. Algebra 18(10) (1990), 3493–3505.

[20] Segal, I.E. *A class of operator algebras which are determined by groups*, Duke Math. J. 18 (1951), 221–265.

[21] Weimar-Woods, E. *Contractions, generalized Inönü-Wigner contractions and deformations of finite-dimensional Lie algebras*, Rev. Math. Phys. 12(11) (2000), 1505–1529.

[J. M. Casas] Department of Applied Mathematics I, E. E. Forestal, University of Vigo, 36005 Pontevedra, Spain.

E-mail address: jmcasas@uvigo.es

[M. Ladra] Department of Algebra, University of Santiago de Compostela, 15782, Spain.

E-mail address: manuel.ladra@usc.es

[A. Kh. Khudoyberdiyev and B. A. Omirov] Institute of Mathematics, National University of Uzbekistan, Tashkent, 100125, Uzbekistan.

E-mail address: khabor@mail.ru, omirovb@mail.ru