An Alternative to Exact Renormalization and Cosmological Solutions in String Theory

Jean Alexandre and Nick E. Mavromatos,
Department of Physics, King’s College London,
The Strand, London WC2R 2LS, U.K.

Abstract

In this work we review the application of a functional method, serving as an alternative to the Wilsonian Exact Renormalization approach, to stringy bosonic $\sigma$-models with metric and dilaton backgrounds on a spherical world sheet [1]. We derive an exact evolution equation for the dilaton with the amplitude of quantum fluctuations, driven by the kinetic term of the two-dimensional world-sheet theory. The linear dilaton conformal field theory, corresponding to a linearly (in cosmic Einstein-frame time) expanding Universe, appears as a trivial fixed point of this equation. With the help of conformal-invariance conditions, we find a logarithmic dilaton as another, exact and non trivial, fixed-point solution, and determine the corresponding target-space physical metric. Cosmological implications of our solutions are briefly discussed, in particular the transition (exit) from the expanding Universe of the linear dilaton to the Minkowski vacuum, corresponding to the non-trivial fixed point of our generalised flow. This novel renormalization-group method may therefore offer new insights into exact properties of string theories of physical significance.

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1 Introduction

Infinities in quantum field theory have been well understood today, and in fact acquire an important physical significance, linked with the phase structure of the theory. They can be “absorbed” into appropriately renormalized operators and coupling constants, which then “run” with the scale, in a way characteristic of the theory or, better, class of theories. The Renormalization Group (RG) [2] is therefore a very powerful method, which provides insight into the scale dependence of the various coupling constants of a quantum field theory, and through this, important information on the underlying phase structure. In this way several important “universality” properties of physical theories, that is common behaviour of apparently different physical systems, have been understood by means of specific perturbations of “fixed-point” (scale invariant) theories.

The so-called “exact” renormalization program [3], with the set up of appropriate flow equations for the coupling constants, helps compute the effective action of the theory in the
limit where the cutoff goes to zero. In order to avoid the introduction of a running world sheet cut off, we present here an alternative approach which has been proposed in [4]. The method dealt with the computation of the effective action of a theory by introducing a parameter, which controls quantum fluctuations of the dynamical variables rather than their Fourier modes, and looking at the corresponding evolution of the effective action, constructed as usual by means of a Legendre transform of the connected-graph generator functional. This exact functional method was extended successfully to incorporate QED [5].

The idea is to control quantum fluctuations of a system by varying the bare mass of a theory, for a fixed cut off (or any other regulator). When this mass is large, quantum fluctuations are frozen and the system is classical. As this bare mass decreases, quantum fluctuations gradually appear and the system gets dressed. The interesting point is that an exact evolution equation for the effective action with the bare mass can be obtained, and it was shown that the flows in this bare mass are equivalent to the usual renormalization flows at one loop. Beyond one loop, the non-perturbative aspect of this method leads to new insights into renormalization flows. A review of this method, as well as applications to different models, can be found in [6].

The Exact renormalization group method lies at the heart of the first quantization of string theory [7]. Indeed, the latter is formulated as a two-dimensional world-sheet quantum field theory, propagating in various target-space background fields, which from a two-dimensional viewpoint appear as (an infinite set of) “coupling constants” $g^i$, with the index $i$ running in both field-theory-species space and target space time, e.g. $g^i = \{G_{\mu\nu}(y), \Phi(y)\ldots\}$, with $y$ coordinates of a $D$-dimensional target-space time, $G_{\mu\nu}$ a metric tensor, $\Phi$ a dilaton field etc. In conventional RG approaches, the world-sheet infinities of the underlying (renormalizable) two-dimensional quantum field theory are absorbed in appropriately renormalized background fields $g^i_R$, characterized by some $\beta$-functions $\beta^i = dg^i_R/d\ln(L/a)^2$, with $\ln(L/a)^2$ a Wilsonian world-sheet renormalization scale, with $L$ the size of the world-sheet of the string, and $a$ an appropriate short-distance cutoff. Removal of the (length) cutoff $a \to 0$, therefore, corresponds to infinite-size world sheets.

In general, as in any other renormalized theory, there are relevant, marginal and irrelevant world-sheet deformations, which perturb a fixed-point theory away from the local scale invariant (conformal) point. Consistency of the underlying string theory requires conformal (local scale, Weyl) invariance, which in turn implies certain conditions on the coupling constants, characterizing the fixed-point theory. These conditions may be summarized by the vanishing of the so-called Weyl-anomaly coefficients of the $\sigma$-model, related to the $\beta^i$-functions by $\tilde{\beta}^i = \beta^i + \delta g^i$, where $\delta g^i$ are appropriate functionals having the form of target-space diffeomorphism variations of the corresponding couplings/fields $g^i$. In this way, the fixed points in the theory space of stringy $\sigma$-models are not simply given by the vanishing of the $\beta^i$, but by the vanishing of

$$\tilde{\beta}^i = \beta^i + \delta g^i = 0 \quad (1)$$

The extra terms $\delta g^i_R$ express the underlying target-space diffeomorphism invariance of the string theory, whose couplings $g^i_R$ are fields in target space. The reader should notice that this latter property is not a feature shared by the coupling constants of ordinary field theories. The physical importance of (1) lies, of course, on the fact that it determines the consistent space-time backgrounds of strings, thereby providing important insight into the low-energy physics phenomenology [7].

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An important remark is in order at this point. In a perturbative $\alpha'$ (Regge slope) expansion, the leading order graviton $\beta$-functions contains the Ricci tensor of the string. Ignoring non-constant dilatons for the moment, this would imply that a consistent background in strings is the Ricci flat space time, which is a vacuum solution of a target-space (Einstein) effective theory. In the presence of dilatons, the effective action becomes more complicated, of the form of a Brans-Dicke scalar/gravity theory [7].

A non-trivial time dependence of the dilaton, in fact linear time dependence, has been shown in [8] to be possible, satisfying the conditions [11], leading to string theory backgrounds with cosmological significance. The resulting space-time metrics where actually found to describe Universes with a linearly expanding (or contracting) scale factor, in the so-called Einstein cosmic frame. The latter is a system of target-space-time coordinates and field configurations, in which the metric field was redefined by appropriate exponential dilaton factors so as to ensure a canonical form for the Einstein curvature term in the effective action, with no Brans-Dicke scale factors in front. The cosmic time on the other hand, was defined in such a way that the metric in the Einstein frame had the standard Robertson-Walker form

$$ds^2 = g_{\mu\nu}^E dx^\mu dx^\nu = -dt_E^2 + a^2(t_E) d\vec{x}^2,$$

where $\vec{x} = \{x^1, \ldots x^{D-1}\}$ (2)

For future use we remind the reader that the Einstein-frame metric $g_{\mu\nu}^E$ is related to the metric $G_{\mu\nu}$ of the original $\sigma$-model ($\sigma$-model-frame metric) by [8]

$$g_{\mu\nu} = e^{-4\Phi/(D-2)} G_{\mu\nu}$$

and the time $t_E$ is related to the time coordinate $X^0$ of the $\sigma$-model by:

$$dt_E = e^{-2\Phi/(D-2)} dX^0$$

The appropriate dilaton configuration leading to (2) was linear in $X^0$:

$$\Phi(X^0) = -QX^0, \quad Q^2 = \frac{D - 26}{3}$$

while the $\sigma$-model metric $G_{\mu\nu} = \eta_{\mu\nu}$ was Minkowski flat, corresponding, in the Einstein frame, to a Robertson-Walker metric with a scale factor depending linearly on the Einstein time-$t_E$, which can be expanding or contracting, depending on the algebraic sign of $Q$. In fact, the solution of [8] allowed for a consistent formulation of strings in other than $D = 26$ dimensions, and it was the first example of a non-critical (supercritical) string [9], with the time $X^0$ playing the rôle of the Liouville mode, which has time-like signature.

Liouville strings correspond to $\sigma$-models which are perturbed by deformations away from their conformal points in string theory space $\{g^i\}$. The conformal invariance is restored, provided the theory is “dressed” by the Liouville mode, $\varphi$, which appears as an extra dynamical field in the $\sigma$-model, coupled to the world-sheet curvature term in the action, hence contributing to the dilaton field, which thereby acquires special significance [9, 8]. If the non-critical stringy $\sigma$-model has a central charge deficit, i.e. the central charge of the theory is less than that necessary for the $\sigma$-model to be critical, then the Liouville mode has space-like signature, in the sense that the kinetic term of the Liouville field in the two-dimensional action appears with the signature of the kinetic term of the target spatial coordinate fields in a stringy $\sigma$-model.
On the other hand, if there is a central charge surplus (supercritical string [8]), the Liouville field has time-like signature, and may play the rôle of the target time [8, 10].

The restoration of conformal invariance by the presence of this extra \( \sigma \)-model field means that a non-critical string in \( D \) target-space dimensions can become a critical one in \((D+1)\) dimensions. However, in [10] an alternative to the traditional Liouville dressing procedure has been suggested, according to which the dimensionality of the target space of the non-critical string remains the same as that before the dressing. This can be achieved as follows: one starts with a non-critical \( \sigma \)-model formulated on a \( D \)-dimensional target space time \((X^0, \vec{x})\). The initial non-criticality may be due to catastrophic cosmic events, for instance the collision of two brane worlds [11]. The Liouville dressing procedure initially leads to a \((D+1)\)-dimensional target space, but, eventually, dynamical reasons [11], such as minimization of appropriate effective potentials, impose the identification of the world-sheet zero mode of the Liouville field itself with (a function of) the target time \( X^0 \), thereby keeping the target space \( D \)-dimensional. Only certain backgrounds are consistent with such an identification, but fortunately among those there are some with cosmological interest, including inflationary models [12], and in general expanding Universes in string theory [13], which for large cosmic times asymptote to the linear-dilaton solutions of [8].

The Liouville dressing procedure may be viewed in some sense as a special renormalization of the two-dimensional world-sheet theory, in which the RG scale is itself a dynamical field, responsible for the restoration of conformal invariance. The zero mode of the Liouville field may be related to the logarithm of the world-sheet area, which thus provides the running scale. The restoration of conformal invariance implies the independence of the target-space dynamics from world-sheet physics.

An exact RG approach to Liouville strings was applied in [17], where the standard quantum Liouville field theory was derived as a solution of a flow equation. However, the two-dimensional theory suffers from the same inconsistencies and ambiguities of the higher-dimensional analogue, mentioned in the beginning of the section. For instance, a truncation of the space of action functionals is necessary in the analysis of [17], which although consistent with Weyl invariance, nevertheless is not unambiguous and moreover its physical significance is not entirely clear.

From such a point of view, it then seems necessary to explore the possibility of applying alternative approaches, such as the functional method of [4], to stringy \( \sigma \)-models. In our framework both critical and non-critical \( \sigma \)-models appear formally equivalent, since, as mentioned above, a Liouville-dressed string is a critical one in one dimension higher, or sometimes [10] even in the same dimension. In particular, we shall be interested in finding novel cosmological backgrounds in string theory by following such alternative “renormalization” methods. As we shall see in this work, this novel method is closer in spirit to the Liouville approach of [10, 12], in which although one starts with an independent scale, the Liouville mode, eventually the latter is identified with a function of the cosmic time. A similar thing will characterize our approach: the control parameter, which plays the rôle of a running scale in the approach of [4], becomes a function of the time coordinate of the string after quantum dressing, and this will lead to interesting cosmological backgrounds of strings, corresponding to fixed points of the alternative flow equations. As we shall demonstrate [11], though, such backgrounds will necessarily characterize only non-critical dimension strings.

Moreover, our new renormalization method will lead to results of physical significance that one could not obtain in conventional world-sheet renormalization-group approaches. In par-
ticular, in the expanding Universe solution of [8] it was not clear how the Universe exits from that phase into the static Minkowski space time, which could serve as an equilibrium point. As we shall show in this work, our new approach offers an interesting solution to this puzzle. The Minkowski space time corresponds to a non-trivial fixed point of the generalized flow of our control parameter, with a non-trivial dilaton field, while the linear dilaton solution, with a linearly (in Einstein-frame cosmic time) expanding scale factor, is simply the trivial fixed point of the flow. Thus, a flow of the control parameter from the trivial to the non-trivial fixed point, standard in any renormalization approach, provides such an exit from the expanding Universe to the static Minkowski phase, in view of the control parameter becoming a function of cosmic time after quantum dressing.

However, from a technical point of view it should be stressed that, if one wishes to apply the alternative method of [4, 5] to the world-sheet field theory of a stringy $\sigma$-model, one needs a modification. In the case of bosonic or supersymmetric strings propagating in the background of the massless string-multiplet, which we shall be dealing with in this work, i.e. in the absence of tachyons, no mass term is present in the two-dimensional world-sheet action. Nevertheless, the same method as in [4] can be used, but now one should be looking at the evolution of the quantum theory with the amplitude of the kinetic term, controlled by a dimensionless parameter $\lambda$.

The structure of the article is as follows: the evolution equation for the dilaton with respect to the control parameter $\lambda$ is derived in Section 2. We study the fixed point solutions of this exact equation, taking into account also the Weyl invariance constraints of the $\sigma$-model. We first demonstrate, as a consistency check, that the linear dilaton configuration of [8] is an exact solution of our evolution equation, corresponding to a trivial fixed point. Then we proceed, in section 3, to find another exact solution, pertaining to a non-trivial fixed point of the evolution with $\lambda$. This solution is characterized, in four large (uncompactified) dimensions, by a dilaton configuration, which is logarithmic in the $\sigma$-model-frame time, $X^0$, and a target-space metric, which, in the Einstein physical frame, turns out to be the (static) Minkowski space time. Cosmological implications of the non trivial solution, in particular its role as providing the exit phase of the linearly expanding Universe of [8], are discussed briefly in Section 4, emphasizing the fact that only non-critical-dimension string models can admit such time-dependent vacua. The non trivial solution in dimensions other than four can accommodate accelerating Universes without a cosmic horizon. Finally, in section 5 we present our Conclusions and Outlook.

## 2 Exact evolution equation

We consider a spherical world sheet with a curvature scalar $R$. The bare action of the Bosonic $\sigma$-model in Graviton and Dilaton backgrounds reads [7]

$$S = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \left\{ \gamma^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + R \phi_B(X^0) \right\},$$

where $\eta_{\mu\nu}$ is the flat Minkowski target-space metric. The parameter $\lambda$ plays the role of $1/\alpha'$ and controls the amplitude of the kinetic term: for $\lambda \gg 1$, the latter dominates over the bare dilaton $\phi_B$ term, and the theory is classical. As $\lambda$ decreases, the effects of the interactions in $\phi_B$ gradually appear and the quantum theory settles in.

We note the following points:
• The variables $X^k, k \neq 0$, do not play a dynamical role in our study since we consider a dilaton depending on $X^0$ only. This is because we are interested, in the spirit of [8], in constructing cosmological backgrounds only.

• The expression for the classical dilaton $\phi_B$ is not explicitly given here, but it should in principle contain interactions (at least cubic in $X^0$) so as to generate quantum fluctuations.

The quantum theory is described by the effective action $\Gamma$, i.e. the proper-graph-generating functional, for which the exact evolution equation with respect to the control parameter $\lambda$, which reads [1]:

\[
\dot{\Gamma}_\lambda = \frac{1}{4\pi} \int d^2 \xi \sqrt{\gamma} \gamma^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \\
+ \frac{\eta_{\mu\nu}}{4\pi} \text{Tr} \left\{ \gamma^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \zeta^b} \left( \frac{\delta^2 \Gamma_\lambda}{\delta X^\mu(\zeta) \delta X^\nu(\xi)} \right)^{-1} \right\},
\]

where a dot over a letter denotes a derivative with respect to $\lambda$. In eq.(7), the symbol of the trace contains the quantum corrections to $S$. In order to obtain physical information on the system, eq.(7) should in principle be integrated from $\lambda = \infty$ to $\lambda = 1$, which is the appropriate regime of the full quantum theory. In the present article, though, we shall be dealing only with fixed-point solutions of eq.(7), and hence we shall not follow the $\lambda$-dependence. This is left for a future work.

We now derive the evolution equation for the quantum dilaton with $\lambda$. For this one must have knowledge of the functional dependence of $\Gamma$ on the quantum fields. This can be achieved by means of a gradient-expansion approximation, which assumes that, for any value of $\lambda$, $\Gamma$ takes the form:

\[
\Gamma = \frac{1}{4\pi} \int d^2 \xi \sqrt{\gamma} \left\{ \gamma^{ab} \kappa_\lambda(X^0) \partial_a X^0 \partial_b X^0 \\
+ \gamma^{ab} \tau_\lambda(X^0) \partial_a X^k \partial_b X^k + R \phi_\lambda(X^0) \right\},
\]

where $\kappa, \tau$ are $\lambda$-dependent functions of $X^0$. The latter are different for the time ($X^0$) and space ($X^k$) coordinates, since the respective quantum fluctuations are different.

The approximation (8), when plugged in the exact evolution equation (7), leads to, in the limit of a flat world-sheet metric [1]:

\[
\dot{\phi} = - \frac{\Lambda^2}{2 R^{(2)}} \left( \frac{1}{\kappa} + \frac{D-1}{\tau} \right) + \frac{\phi''}{4\kappa^2} \ln \left( 1 + \frac{2\Lambda^2\kappa}{R^{(2)}\phi'} \right),
\]

where $\Lambda$ is the world sheet UV cut off and a prime denotes a derivative with respect to $X^0$.

For consistency of eq.(9), we can check that the linear dilaton configuration (10) of [8], corresponding to a Minkowski flat $\sigma$-model-frame metric, or, equivalently, to a power-law expanding Robertson-Walker Universe in the Einstein-cosmic-time-$t_E$ frame, is an exact solution. After the redefinition $X^j \rightarrow \sqrt{D-X}X^j$ of the space coordinates of the string, one can see that the evolution equation (9) is satisfied by the well-known flat metric/linear dilaton configuration [8]:

\[
\kappa = 1 = -\tau, \quad \phi(X^0) = QX^0,
\]

which shows that the latter solution is exactly marginal with respect to the flows in $\lambda$. 


3 Non trivial fixed-point solution

Besides the expected flat metric/linear dilaton configuration, the evolution equation (9) has another non-trivial solution.

We consider a configuration with \( \kappa(X^0) = F \phi''(X^0) \), where \( F \) is a constant. For such a configuration, the evolution equation (9) reads

\[
\kappa \dot{\phi} = -\frac{\Lambda^2}{2R(2)} \left( 1 + (D-1)\frac{\kappa}{\tau} \right) + \frac{1}{4F} \ln \left( 1 + \frac{2\Lambda^2 F}{R(2)} \right),
\]

and one can see that it is possible to have a \( \lambda \)-independent solution: \( \dot{\phi} = 0 \), if

\[
\frac{\kappa}{\tau} = -\frac{1}{D-1} + \frac{R(2)}{2(D-1)\Lambda^2 F} \ln \left( 1 + \frac{2\Lambda^2 F}{R(2)} \right) = -c^2,
\]

where we have taken the negative sign corresponding to Minkowski signature, as is appropriate for large cut-off \( \Lambda \), in which case the ratio \( \kappa/\tau \) is necessarily negative, and \( c \) is a positive constant. After the redefinition \( X^i \rightarrow cX^i \) of the space coordinates of the string, the condition (12) shows that the target-space metric is conformally flat, and the non-trivial \( \lambda \)-independent solution of eq. (9) is such that

\[
g_{\mu\nu}(X^0) \propto \phi''(X^0) \eta_{\mu\nu}.
\]

In the stringy \( \sigma \)-model framework, any equilibrium solution, i.e., one that satisfies the equations of motion of a target-space effective action, must also satisfy the conformal-invariance conditions \(^1\) on the world-sheet. \( A \) priori, it is not clear that the configuration (13), (12) satisfies these conditions. However, in the next Section we display a more precise functional dependence for \( \phi \), which we conjecture satisfies the Weyl invariance conditions. This configuration has the form:

\[
ds^2 = \frac{\alpha'A}{(X^0)^2} \left( -(dX^0)^2 + dX_idX_i \right), \quad \phi = \phi_0 \ln X^0
\]

where \( \phi_0 \) and \( A \) are to be determined.

This is still a conjecture, since the new fixed-point configuration (14) of the \( \lambda \)-low is non perturbative in \( \alpha' \), so the corresponding Weyl anomaly coefficients are not known in a closed form. We base our conjecture that it is indeed conformally invariant on a heuristic inductive argument that there exists in principle a world-sheet renormalization scheme, reached from the standard \( \sigma \)-model scheme \(^1\) by certain field redefinitions, in which the Weyl anomaly coefficients vanish. We now demonstrate this explicitly to order \( \alpha' \) and then use inductive arguments to argue that this is true to all orders in \( \alpha' \).

To first order in \( \alpha' \), the beta functions for the bosonic world-sheet \( \sigma \)-model theory in graviton and dilaton backgrounds are \(^1\):

\[
\beta^g_{\mu\nu} = R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi + \frac{\alpha'}{2} R_{\mu\lambda\sigma} R_{\nu}{}^{\lambda\rho\sigma} + O(\alpha')^2,
\]

\[
\beta^\phi = \frac{D - 26}{6\alpha'} - \frac{1}{2} \nabla^2 \phi + \partial^\rho \phi \partial_\rho \phi + \frac{\alpha'}{16} R_{\mu\rho\sigma} R^{\mu\rho\sigma} + O(\alpha')^2.
\]

\(^1\)Strictly, the Weyl-invariance conditions, that take into account target-space diffeomorphisms \(^1\).
We consider first the tree-level beta functions for a configuration satisfying the condition

\[ \kappa(X^0) \propto \phi''(X^0), \]

with the power-law dependence

\[ \phi'(X^0) = \phi_0(X^0)^n, \]
\[ \kappa(X^0) = \kappa_0(X^0)^{n-1}. \]  

(16)

We observe that such an ansatz does not satisfy perturbative Weyl invariance conditions, but
the important point is that, for \( n = -1 \), all the other orders in \( \alpha' \) are, as functions of \( X^0 \),

homogenous to the tree level: we can write

\[ \beta_{\phi_00} = \frac{1}{(X^0)^2} \sum_{m=0}^{\infty} \xi_m \left( \frac{\alpha'}{\kappa_0} \right)^m, \]
\[ \beta_{\phi_jk} = \frac{\delta_{jk}}{(X^0)^2} \sum_{m=0}^{\infty} \zeta_m \left( \frac{\alpha'}{\kappa_0} \right)^m, \]
\[ \beta_{\phi} = \frac{1}{\alpha'} \sum_{m=0}^{\infty} \eta_m \left( \frac{\alpha'}{\kappa_0} \right)^m, \]  

(17)

where \( \xi_m, \zeta_m, \eta_m \) are \( \alpha' \)-independent coefficients. As a consequence, for \( \kappa_0 \) of the same order as
\( \alpha' \), the expansion of the beta functions in \( \alpha' \) is no longer valid.

The next step is to argue that the configuration

\[ \phi(X^0) = \phi_0 \ln(X^0), \]
\[ \kappa(X^0) = \frac{\alpha' A}{(X^0)^2}, \]  

(18)

where we write \( \kappa_0 = \alpha' A \), may satisfy conformal invariance at a non-perturbative level. We exploit the fact, well-known in string theory, that at higher orders in \( \alpha' \) the beta functions are not fixed uniquely, but can be changed by making local field redefinitions [16]: \( g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} \) and \( \phi \rightarrow \tilde{\phi} \), which leave the (perturbative) string S-matrix amplitudes invariant. This possibility of
field redefinition enables us to maintain conformal invariance to all orders in \( \alpha' \).

We illustrate this possibility with an explicit calculation to first order in \( \alpha' \). In our case, since we keep the target-space metric conformally flat, the redefinition of the metric must be such that \( \tilde{g}_{\mu\nu} \) is proportional to \( g_{\mu\nu} \), and thus we can consider the following redefinitions:

\[ \tilde{g}_{\mu\nu} = g_{\mu\nu} + \alpha' g_{\mu\nu} \left( b_1 R + b_2 \partial^\rho \phi \partial_\rho \phi + b_3 \nabla^2 \phi \right), \]
\[ \tilde{\phi} = \phi + \alpha' \left( c_1 R + c_2 \partial^\rho \phi \partial_\rho \phi + c_3 \nabla^2 \phi \right), \]  

(19)

where \( b_1, b_2, b_3, c_1, c_2, c_3 \) are constants and \( R, \nabla \) corresponds to the metric \( g_{\mu\nu} \). In the case of the configuration (18), we have (see Appendix B for details):

\[ \tilde{g}_{\mu\nu} = g_{\mu\nu} + \frac{g_{\mu\nu}}{A} \left( -b_1 (D-1)^2 + b_2 \phi_0^2 - b_3 (D-1) \phi_0 \right) = (1 + B) g_{\mu\nu}, \]
\[ \tilde{\phi} = \phi + \frac{1}{A} \left( -c_1 (D-1)^2 + c_2 \phi_0^2 - c_3 (D-1) \phi_0 \right) = \phi + C, \]  

(20)

where \( B, C \) are constants linear in \( b_1, b_2, b_3, c_1, c_2, c_3 \). Therefore, the redefinitions (19) consist of adding a constant to the dilaton and rescaling the metric, thus not changing the functional dependence of the configuration (18).
The new beta functions \( \beta_{\mu\nu}^g \to \tilde{\beta}_{\mu\nu}^g \) and \( \beta^\phi \to \tilde{\beta}^\phi \) are obtained via the appropriate Lie derivatives in theory space [16], as appropriate to the vector nature of the \( \beta^i \) functions in this space:

\[
\tilde{\beta}_{\mu\nu}^g - \beta_{\mu\nu}^g = \int (\tilde{g}_{\rho\sigma} - g_{\rho\sigma}) \frac{\delta \beta_{\mu\nu}^g}{\delta g_{\rho\sigma}} + \int (\tilde{\phi} - \phi) \frac{\delta \beta_{\mu\nu}^g}{\delta \phi} - \int \beta^\phi \frac{\delta (\tilde{g}_{\mu\nu} - g_{\mu\nu})}{\delta \phi}, \tag{21}
\]

and

\[
\tilde{\beta}^\phi - \beta^\phi = \int (\tilde{g}_{\rho\sigma} - g_{\rho\sigma}) \frac{\delta \beta^\phi}{\delta g_{\rho\sigma}} + \int (\tilde{\phi} - \phi) \frac{\delta \beta^\phi}{\delta \phi} - \int \beta^\phi \frac{\delta (\tilde{g}_{\mu\nu} - g_{\mu\nu})}{\delta \phi}. \tag{22}
\]

It is shown in [1], at one loop, that the beta functions obtained after this change of string parametrization can indeed be cancelled.

This analysis takes into account only the first non-trivial order in \( \alpha' \), whereas all the higher orders should also be taken into account. However, this first-order analysis provides the basis for an inductive argument. If conformal invariance is satisfied at order \( n \) in \( \alpha' \), as in the first-order case worked out above, there are always enough parameters in the redefinitions of the metric and dilaton at the next order, leaving the string configuration unchanged, which enable the beta functions to vanish and hence conformal invariance to be satisfied at the next order \( n + 1 \) in \( \alpha' \).

We stress again that these arguments are only heuristic at this stage, since the \( \beta \)-functions and Weyl anomaly coefficients are not exactly known to all orders in \( \alpha' \), and hence the world-sheet renormalization scheme in which they vanish is abstract. We mention at this point that it is known from standard analyses of \( \sigma \) models [16] that there is a formal scheme in which the dilaton dependence is simply that of one \( \sigma \)-model loop. For the graviton Weyl-anomaly coefficient this means that the dilaton dependence has the form of a target-space diffeomorphism: \( \nabla_\mu \partial_\nu \phi \). This is not the scheme we use in this work, and thus the reader should bear in mind that the background [13] proposed here does not correspond to a standard conformal field theory that is perturbative in \( \alpha' \). This reflects the non-perturbative nature of the novel renormalization-group method employed in our approach.

### 4 Cosmological implications

We now examine the physical significance of the new non-trivial fixed-point solution (18), and discuss briefly its cosmological implications. This leads to a value of the constant \( \phi_0 \) that appears in the configuration (18).

The relation between the physical metric in the Einstein frame and the string metric is given by [8]

\[
ds^2 = dt^2 - a^2(t)dx^k dx^k = \kappa(x^0) \exp \left\{ - \frac{4\phi(x^0)}{D-2} \right\} (dx^0 dx^0 - dx^k dx^k), \tag{23}
\]
where \( a(t) \) is the scale factor of a spatially flat Robertson-Walker-Friedmann Universe, and the \( x^\mu \) are the zero modes of \( X^\mu \). From the configuration (18), we have

\[
\frac{dt}{dx^0} = \varepsilon \sqrt{|\kappa_0|} (x^0)^{-1 - \frac{2\phi_0}{D-2}},
\]  

(24)

where \( \varepsilon = \pm 1 \), such that

\[
t = T + \sqrt{|\kappa_0|} \frac{(D-2)}{2|\phi_0|} (x^0)^{-\frac{2\phi_0}{D-2}},
\]  

(25)

where \( T \) is a constant. We find then a power law for the evolution of the scale factor:

\[
a(t) = a_0 |t - T|^\frac{D-2}{2\phi_0} + 1,
\]  

(26)

which is in general singular as \( t \to T \).

In order to have a Minkowski target space, one needs \( D - 2 + 2\phi_0 = 0 \). As was discussed earlier, the choices of \( D \) and \( \phi_0 \) are free, and lead to the determination of \( \kappa_0 \) (in a way which has not been determined yet). As a consequence, for a given dimension \( D \), it is always possible to choose \( \phi_0 \) so that the target space is static and flat. It may therefore find an application to the exit phase from the linearly expanding Universe associated with the linear dilaton of [8].

We note that, in terms of the Einstein time \( t \), the dilaton can be written, up to a constant, as:

\[
\phi = -\frac{D-2}{2} \ln |t - T|.
\]  

(27)

We observe that, like the scale factor (26), the dilaton has a singularity as \( t \to T \). It would be interesting to explore the applicability of this configuration to primordial cosmology. The sign of the expression (27) for the dilaton when \( D > 2 \) ensures that the string coupling is small at large times.

Finally, we mention for completeness that our solution (18) (or (26) (27) in the Einstein frame) should be compared and contrasted with the (isotropic-case) solution of [15], which describes a D-dimensional Universe whose constant-time slices are (D-1)-dimensional tori with time-dependent ‘rolling’ radii, whose flow is attributed to the existence of a rolling dilaton field. First, it should be remarked that the work of [15] is perturbative in \( \alpha' \), unlike our construction. Moreover, our main point in this work has been to associate the non-trivial fixed point solution (18) to a marginal configuration of flow with respect to a novel control parameter. In this way, we have provided arguments for the role of the solution, especially in the Minkowski case, as providing an exit phase from a linearly-expanding Universe. None of these aspects apply to the work of [15].

5 Conclusions

We have proposed here a new non-perturbative renormalization-group technique for the bosonic string, based on a functional method for controlling the quantum fluctuations, whose magnitudes are scaled by the value of \( \alpha' \). Using this technique, we have exhibited a new, non-perturbative time-dependent background solution. Using the field redefinition ambiguities of the target-space effective action, which leave the string S-matrix invariant, we have demonstrated that this solution is conformally invariant to \( \mathcal{O}(\alpha') \), and we have made a conjecture,
based on a heuristic and inductive argument, that conformal invariance can be maintained to all orders in $\alpha'$. We stress once again that our work, which is based on the flow equations of a non-perturbative Schwinger-Dyson-type effective action, is different in spirit from other work in the literature [14], where expressions for the $\beta$ functions to all orders in $\alpha'$ are obtained in a large-$N$-treatments, where $N$ is the number of target-space dimensions.

This new non-perturbative time-dependent background solution has related singularities in both the metric scale factor and the dilaton value at a specific value of the time in the Einstein frame. A full exploration of the possible cosmological applications of this solution lies beyond the scope of this paper, but we do note two interesting possibilities. One is that the temporal singularity might be relevant for primordial cosmology, i.e., the beginning of the Big Bang. The second possible application could be to describe the exit phase from the linearly expanding Universe associated with the linear dilaton of [8].

These are two phenomenological tasks for future work on this new non-perturbative time-dependent background solution. It is also desirable to explore in more detail the formal underpinnings of the solution. In particular, it is necessary to improve on our heuristic inductive argument that its conformal invariance may be maintained to all orders in $\alpha'$. We also note that the non-perturbative renormalization-group technique proposed here may have applications to other aspects of string theory.

There may be other non-trivial fixed point solutions of our flow equations, especially if more backgrounds, such as antisymmetric tensors and fields from the matter multiplet of strings, are included. In the spirit of what was discussed in this paper, such solutions may provide explanations on the phase transition of various stages of the string Universe, from grateful exit from inflation and reheating, to issues of possible pre-Big-Bang cosmologies [18], which could not be explained within the context of conventional world-sheet renormalization approaches to the low-energy limit of strings.

Before closing we would like to remark that it is also possible to apply the same method to a string with a tachyonic background. Such backgrounds may play an important rôle in early brane-Universe cosmology [13, 19], since they can provide the initial cosmological instability, decoupling relatively quickly from the spectrum. The tachyonic background involves a mass term for the world-sheet $\sigma$-model fields, and a usual Wilsonian Exact Renormalization method has been applied to this case [20], where the equation of motion of the quantum tachyon is found as a consequence of the independence of the theory from the world sheet cut off. This situation can be studied with the present method, where quantum fluctuations of the tachyon are controlled by the mass parameter in the world-sheet theory. The resulting evolution equation of the quantum theory would look like eq. (7), but without derivatives with respect to the world sheet coordinates inside the trace. A suitable derivative expansion for the effective action (together with the Weyl invariance conditions), plugged in this evolution equation, would then give coupled renormalization group equations for the tachyon and metric backgrounds. As in the case of the dilaton, studied in the present paper, a cut-off free evolution equation would then be obtained, enabling therefore one to study the quantum theory by means of renormalized physical quantities, in a way independent of unphysical cut off scales. We shall return to this issue in a future publication.
References

[1] J. Alexandre, J. Ellis and N. E. Mavromatos, JHEP 0612 (2006) 071 [arXiv:hep-th/0610072].
[2] K. G. Wilson and J. B. Kogut, Phys. Rept. C12, 75 (1974). K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975); ibid. 55, 583 (1983).
[3] J. Polchinski, Nucl. Phys. B 231, 269 (1984); C. Wetterich, Phys. Lett. B 301, 90 (1993); M. Reuter and C. Wetterich, Nucl. Phys. B 391, 147 (1993); T. R. Morris, Int. J. Mod. Phys. A 9, 2411 (1994) [arXiv:hep-ph/9308265]; U. Ellwanger, Phys. Lett. B 335, 364 (1994) [arXiv:hep-th/9402077]; N. Tetradis and D. F. Litim, Nucl. Phys. B 464, 492 (1996) [arXiv:hep-th/9512073]; J. A. Adams, J. Berges, S. Bornholdt, F. Freire, N. Tetradis and C. Wetterich, Mod. Phys. Lett. A 10, 2367 (1995) [arXiv:hep-th/9507093].
[4] J. Alexandre and J. Polonyi, Annals Phys. 288, 37 (2001) [arXiv:hep-th/0010128].
[5] J. Alexandre, J. Polonyi and K. Sailer, Phys. Lett. B 531, 316 (2002) [arXiv:hep-th/0111152].
[6] J. Polonyi and K. Sailer, Phys. Rev. D 71, 025010 (2005) [arXiv:hep-th/0410271].
[7] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, Vol. I (Cambridge University Press, 1987); J.Polchinski, String Theory, Vol. 1 (Cambridge University Press, 1998).
[8] I. Antoniadis, C. Bachas, J. R. Ellis and D. V. Nanopoulos, Nucl. Phys. B 328, 117 (1989); Phys. Lett. B 257, 278 (1991).
[9] F. David, Mod. Phys. Lett. A 3, 1651 (1988); J. Distler and H. Kawai, Nucl. Phys. B 321, 509 (1989); J. Distler, Z. Hlousek and H. Kawai, Int. J. Mod. Phys. A 5, 391 (1990); see also: N. E. Mavromatos and J. L. Miramontes, Mod. Phys. Lett. A 4, 1847 (1989); E. D’Hoker and P. S. Kurzepa, Mod. Phys. Lett. A 5, 1411 (1990).
[10] J. R. Ellis, N. E. Mavromatos and D. V. Nanopoulos, Phys. Lett. B 293, 37 (1992) [arXiv:hep-th/9207103]; Invited review for the special Issue of J. Chaos Solitons Fractals, Vol. 10, (eds. C. Castro and M.S. El Naschie, Elsevier Science, Pergamon 1999) 345 [arXiv:hep-th/9805120]; Phys. Rev. D 63, 024024 (2001) [arXiv:gr-qc/0007044].
[11] E. Gravanis and N. E. Mavromatos, Phys. Lett. B 547, 117 (2002) [arXiv:hep-th/0205298]; N. E. Mavromatos, [arXiv:hep-th/0210079] (published in Oulu 2002 (Finland), Beyond the desert (ed. H.V. Klapdor-Kleingrothaus, IoP 2003)), 3.
[12] J. R. Ellis, N. E. Mavromatos and D. V. Nanopoulos, Phys. Lett. B 619, 17 (2005) [arXiv:hep-th/0412240]; J. Ellis, N. E. Mavromatos, D. V. Nanopoulos and A. Sakharov, [arXiv:gr-qc/0407089], New J. Phys. 6, 171 (2004).
[13] G. A. Diamandis, B. C. Georgalas, N. E. Mavromatos and E. Papantonopoulos, Int. J. Mod. Phys. A 17, 4567 (2002) [arXiv:hep-th/0203241]; G. A. Diamandis, B. C. Georgalas, N. E. Mavromatos, E. Papantonopoulos and I. Pappa, Int. J. Mod. Phys. A 17, 2241 (2002) [arXiv:hep-th/0107124].
[14] G. Michalogiorgakis and S. S. Gubser, Nucl. Phys. B 757, 146 (2006) \texttt{arXiv:hep-th/0605102}.

[15] M. T. Mueller, Nucl. Phys. B 337, 37 (1990).

[16] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B 293, 385 (1987).

[17] M. Reuter and C. Wetterich, Nucl. Phys. B 506, 483 (1997) \texttt{arXiv:hep-th/9605039}.

[18] see for instance: M. Gasperini and G. Veneziano, Phys. Rept. 373, 1 (2003) \texttt{arXiv:hep-th/0207130} and references therein; Phys. Rev. D 50, 2519 (1994) \texttt{arXiv:gr-qc/9403031}. E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. D 50, 4868 (1994) \texttt{arXiv:hep-th/9406216}.

[19] A. Sen, Phys. Scripta T117, 70 (2005) \texttt{arXiv:hep-th/0312153} and references therein.

[20] J. P. O’Dwyer, Mod. Phys. Lett. A 20, 807 (2005) \texttt{arXiv:hep-th/0502056}.