ON THE REVERSE ISODIAMETRIC PROBLEM AND
DVORETZKY-ROGERS-TYPE VOLUME BOUNDS

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Abstract. The isodiametric inequality states that the Euclidean ball maximizes the volume among all convex bodies of a given diameter. We are motivated by a conjecture of Makai Jr. on the reverse question: Every convex body has a linear image whose isodiametric quotient is at least as large as that of a regular simplex. We relate this reverse isodiametric problem to minimal volume enclosing ellipsoids and to the Dvoretzky-Rogers-type problem of finding large volume simplices in any decomposition of the identity matrix.

As a result, we solve the reverse isodiametric problem for o-symmetric convex bodies and obtain a strong asymptotic bound in the general case. Using the Cauchy-Binet formula for minors of a product of matrices, we obtain Dvoretzky-Rogers-type volume bounds which are of independent interest.

1. Introduction

Let $K^n$ be the family of convex bodies in $\mathbb{R}^n$, that is, full-dimensional convex compact sets. If $K = -K$, we say that $K$ is o-symmetric, and we denote by $K^n_o$ the family of all such convex bodies. Further, we denote by $D(K) = \max\{\|x - y\| : x, y \in K\}$ the diameter and by $\text{vol}_n(K)$ the volume of $K \in K^n$. If the dimension is clear from the context, we just write $\text{vol}(K)$.

A classic result in convexity is the isodiametric inequality due to Bieberbach [8] (cf. [16, Sect. 8.2]) which states that the Euclidean unit ball $B^n_2$ maximizes the isodiametric quotient. More precisely,

$$\text{idq}(K) := \frac{\text{vol}(K)}{D(K)^n} \leq \frac{\text{vol}(B^n_2)}{D(B^n_2)^n}, \quad \text{for all } K \in K^n.$$  

Boxes of arbitrarily large diameter and constant volume show that $\text{idq}(K)$ is in general not bounded from below by any constant $c > 0$. In the spirit of the reverse isoperimetric inequality by Ball [1], one may wonder whether there is a suitable linear transformation $A$ such that the linear image $AK$ of $K$ has an isodiametric quotient that can be bounded away from zero.

Makai Jr. [21] posed the conjecture that there is always a linear image whose isodiametric quotient is at least as large as that of a regular simplex. He was motivated by an application to the minimal density of non-separable...
lattice arrangements of convex bodies, and based his conjecture on the solution of the reverse isodiametric problem in the plane, which was found by Behrend [5] already in 1937.

**Conjecture 1.1** (Makai Jr. [21]). For every $K \in \mathcal{K}^n$ there is a linear transformation $A \in \text{GL}_n(\mathbb{R})$ such that

\[ \text{idq}(AK) \geq \frac{\sqrt{n+1}}{n ! 2^n}, \]

with equality sign if and only if $AK$ is a regular simplex.

If we assume that $K \in \mathcal{K}^n_o$, then an $A \in \text{GL}_n(\mathbb{R})$ exists, such that

\[ \text{idq}(AK) \geq \frac{1}{n!}, \]

with equality sign if and only if $AK$ is a regular crosspolytope.

To be more precise, a regular simplex is a simplex all of whose edges have the same length. A regular crosspolytope in $\mathbb{R}^n$ is the convex hull of $\pm u_1, \ldots, \pm u_n$, where $\{u_1, \ldots, u_n\}$ is an orthonormal basis of $\mathbb{R}^n$. While Conjecture 1.1 is open for any dimension $n \geq 3$, Makai Jr. [21, Lem. 2] proved that there is always some $A \in \text{GL}_n(\mathbb{R})$ such that

\[ \text{idq}(AK) \geq \frac{\text{vol}(\text{conv}\{\mathbb{B}_2^n, \pm(\sqrt{n}, 0, \ldots, 0)\})}{\left(\binom{2n}{n} \frac{\pi^n}{n!}\right)} \approx \frac{\sqrt{n+1}}{n ! 8^n}. \]

In this work, we relate the reverse isodiametric problem to minimal volume enclosing ellipsoids and to the Dvoretzky-Rogers-type problem of finding large volume simplices in any decomposition of the identity matrix.

We begin our investigation with the central definition regarding isodiametric quotients.

**Definition 1.2.** A convex body $K \in \mathcal{K}^n$ is in isodiametric position (or Behrend position), if

\[ \text{idq}(K) = \max_{A \in \text{GL}_n(\mathbb{R})} \text{idq}(AK). \]

This definition is justified because standard arguments in convexity show that the supremum of the isodiametric quotient of linear images of a fixed $K \in \mathcal{K}^n$ is always attained (see Lemma 2.1). We prove in Theorem 2.7 that the Behrend position is unique up to rotations, scalings and translations.

In Section 2, we elaborate on the crucial observation that a convex body $K$ is in Behrend position if and only if its normalized difference body $\text{Di}(K - K)$ is in Löwner position, which means that its volume minimal enclosing ellipsoid is the Euclidean unit ball. This relationship allows to use a result of Barthe [3] on the minimal volume of an $o$-symmetric convex body in Löwner position, leading to the solution of Conjecture 1.1 in the $o$-symmetric case.

**Theorem 1.3.** Let $K \in \mathcal{K}^n_o$ be in Behrend position. Then

\[ \text{idq}(K) \geq \frac{1}{n!}. \]

Equality holds if and only if $K$ is a regular crosspolytope.
Behrend observed that the directions of the line segments attaining the diameter of a planar convex body in isodiametric position correspond to a well-distributed point configuration on the unit circle. In Section 3, we show how his ideas can be extended to arbitrary dimension and use this information to significantly improve the asymptotic bound (4) as follows.

**Theorem 1.4.** Let \( K \in K^n \) be in Behrend position. Then

\[
\text{idq}(K) \gtrsim \sqrt{\frac{n+1}{n!e^n}}.
\]

An extremely useful result in Convex Geometry is the characterization of the Löwner position in terms of the existence of a decomposition of the identity matrix as a non-negative linear combination of rank-one matrices. **Theorem 2.3** shows that, for convex bodies in Behrend position, such a decomposition is induced by the directions of line segments attaining the diameter. The proof of **Theorem 1.4** in Section 3 uses crucially that we can find such diametrical directions which span a simplex of large volume.

These observations motivate our studies in Section 4, where we are interested in the following problem: Given a decomposition of the \( n \times n \) identity matrix into a sum of \( m \) rank-one matrices of the form \( uu^\top \), find \( j \) of the decomposing vectors \( u \) that together with the origin span a \( j \)-dimensional simplex of large volume. The famous Dvoretzky-Rogers lemma from [11] gives an estimate for the case \( j = n \) which is however not sensitive to the parameter \( m \). Writing \( \text{DR}(m,n,j) \) for the largest possible volume that can always be guaranteed (see **Problem 4.1** for a precise definition), we use the Cauchy-Binet formula for the minors of a product of matrices and prove

\[
\text{DR}(m,n,j)^2 \geq \frac{\binom{n}{j} \left( \frac{m}{n} \right)^j}{\left( \frac{n}{j} \right)!^2}.
\]

(5)

This estimate turns out to be sharp for interesting triples \((m,n,j)\). For example, sharpness for the triple \( \left( \binom{n+1}{2}, n, 2 \right) \) is related to the existence of a set of many equiangular lines. In the particular case \((m,n,j) = (m,n,n)\), the bound (5) was recently obtained in [13] by probabilistic methods. As a corollary to these Dvoretzky-Rogers-type volume bounds, we get a second proof of the asymptotic estimate in **Theorem 1.4** (see **Corollary 4.7**).

We complement the discussion on the reverse isodiametric problem by studying its dual counterpart in Section 5. Replacing the diameter of \( K \) by the minimum width \( w(K) \), we define the **isominwidth quotient** by

\[
\text{iwq}(K) := \frac{\text{vol}(K)}{w(K)^n}.
\]

We then consider the reverse isominwidth problem, which asks for an upper bound on the minimum isominwidth quotient of a linear image of any given \( K \in K^n \). The strong duality between the diameter and the minimum width implies characterization results regarding \( \text{iwq}(K) \) that are analogous to those in Section 2.

Finding good lower bounds on the quotient \( \text{iwq}(K) \) is an intricate and longstanding problem, most commonly known as Pál’s problem. However, based on the experiences we made concerning the Behrend position, we are
able to give a complete solution to the reverse isominwidth problem. In Theorem 5.4 we prove that for every convex body $K \in \mathcal{K}^n$ there is a linear transformation $A \in GL_n(\mathbb{R})$ such that
\[ \text{iwq}(AK) \leq 1, \]
and that equality holds if and only if $AK$ is a cube.

2. Convex bodies in Behrend position and the $\alpha$-symmetric reverse isodiametric inequality

In this section, we establish a close relationship between the Behrend position and the well-known Löwner position of a convex body. As a result we obtain the solution to the reverse isodiametric problem for $\alpha$-symmetric convex bodies.

Let us first justify the definition of the Behrend position by showing that the supremal isodiametric quotient among the linear images of a fixed convex body is always attained. We refer the reader to the textbook by Gruber [16, Ch. 6] for information on the set of convex bodies as a metric space.

**Lemma 2.1.** For every $K \in \mathcal{K}^n$, there exists an $A \in GL_n(\mathbb{R})$ such that
\[ \text{idq}(AK) = \sup_{B \in GL_n(\mathbb{R})} \text{idq}(BK). \]

**Proof.** First observe that by the scaling- and translation-invariance of the isodiametric quotient it suffices to consider $K \in \mathcal{K}^n$ containing the origin in their interior and linear maps $A$ that are volume-preserving. Therefore,
\[ \sup_{A \in GL_n(\mathbb{R})} \frac{\text{vol}(AK)}{D(\text{vol}(AK))^{n}} = \frac{\text{vol}(K)}{\inf_{A \in S_K} D(\text{vol}(AK))^{n}}, \]
where
\[ S_K = \left\{ A \in GL_n(\mathbb{R}) : \det(A) = 1 \text{ and } 2 \left( \frac{\text{vol}(K)}{\kappa_n} \right)^{\frac{1}{n}} \leq D(\text{vol}(AK)) \leq D(K) \right\}. \]

Note, that the lower bound on the diameter of $AK$ follows from (1).

Now, take a sequence of convex bodies $AK$, $A \in S_K$, whose diameters converge to the infimum in (6). As we have fixed the origin to be contained in $K$ and by the definition of $S_K$, this sequence is bounded in the sense that all of its members are contained in a ball of diameter $D(K)$. In view of Blaschke’s selection theorem (cf. [16, Thm. 6.3]), there exists a convergent subsequence with limit $\bar{K} = AK$, for $A \to \bar{A} \in S_K$. By the continuity of the diameter function with respect to the Hausdorff distance, we get that $D(\text{vol}(AK))^{n} = \inf_{A \in S_K} D(\text{vol}(AK))^{n}$, finishing the proof. \( \square \)

2.1. Behrend position versus Löwner position. In the sequel, we say that two points $x$ and $y$ in a convex body $K \in \mathcal{K}^n$ determine a diametrical segment of $K$ if $D(K) = \|x - y\|$, and in this case we say that $\frac{x - y}{\|x - y\|}$ is a diametrical direction. We denote by
\[ D_K = \left\{ u \in \mathbb{S}^{n-1} : u \text{ is a diametrical direction of } K \right\} \]
\[ = \left\{ u \in \mathbb{S}^{n-1} : \exists x \in K \text{ such that } x + D(K)[0, u] \subseteq K \right\}. \]
the set of all diametrical directions of $K$. Note that $D_K$ is $o$-symmetric.

A convex body $K \in \mathcal{K}^n$ is in Löwner position if $\mathbb{B}_o^n$ is a minimum volume ellipsoid containing $K$. For background information, references, and a discussion of the history regarding the Löwner (and John) position we refer the reader to the survey article by Henk [18].

It is well-known that for every $K \in \mathcal{K}^n$, there exists an $A \in \mathrm{GL}(\mathbb{R})$ and a translation $t \in \mathbb{R}^n$ such that $AK + t$ is in Löwner position, and that the minimal volume ellipsoid containing $K$ is unique. Moreover, the Löwner position of a convex body is characterized by the existence of contact points that decompose the $n \times n$ identity matrix $I_n$. More precisely,

**Theorem 2.2** (cf. [16, Ch. 11]). Let $K \in \mathcal{K}^n$ be such that $K \subseteq \mathbb{B}_o^n$. The following are equivalent:

(i) $K$ is in Löwner position.

(ii) There exists an $m \geq n$, contact points $u_1, \ldots, u_m \in \text{bd}(K) \cap S^{n-1}$, and scalars $\lambda_1, \ldots, \lambda_m \geq 0$ such that

$$I_n = \sum_{i=1}^m \lambda_i u_i u_i^\top \quad \text{and} \quad \sum_{i=1}^m \lambda_i u_i = 0.$$

Moreover, if $K$ is $o$-symmetric, then the condition $\sum_{i=1}^m \lambda_i u_i = 0$ can be dropped, and one can choose $m \leq \binom{n+1}{2}$.

We are now set in order to state the main result of this section.

**Theorem 2.3.** Let $K \in \mathcal{K}^n$. The following are equivalent:

(i) $K$ is in Behrend position.

(ii) $K - K$ is in Behrend position.

(iii) $\frac{1}{2}D(K - K)$ is in Löwner position.

(iv) $\text{conv}(D_K)$ is in Löwner position.

(v) There exists an $m \in \{n, \ldots, \binom{n+1}{2}\}$, scalars $\lambda_1, \ldots, \lambda_m \geq 0$, and diametrical directions $u_1, \ldots, u_m \in D_K$, such that

$$I_n = \sum_{i=1}^m \lambda_i u_i u_i^\top.$$

The proof of Theorem 2.3 rests on two key lemmas for which we introduce some notation. We write $\mathrm{NO}_1(n)$ for the set of non-orthogonal matrices $M \in \mathrm{GL}(\mathbb{R}) \setminus \mathcal{O}(n)$ with $|\det(M)| = 1$. Geometrically, $\mathrm{NO}_1(n)$ contains all volume-preserving linear maps that do not keep the unit ball $\mathbb{B}_o^n$ invariant.

**Lemma 2.4.** Let $K \in \mathcal{K}_o^n$ be such that $K \subseteq \mathbb{B}_o^n$. The following are equivalent:

(i) $K$ is in Löwner position.

(ii) For every $M \in \mathrm{NO}_1(n)$, we have $K \nsubseteq M(\mathbb{B}_o^n)$.

Proof. In order to show (i) $\Rightarrow$ (ii), we use the fact that $\mathbb{B}_o^n$ is the unique ellipsoid of volume $\text{vol}(\mathbb{B}_o^n)$ containing $K$. Therefore, if $M \in \mathrm{NO}_1(n)$, then $M(\mathbb{B}_o^n) \neq \mathbb{B}_o^n$ and $\text{vol}(M(\mathbb{B}_o^n)) = \text{vol}(\mathbb{B}_o^n)$, and hence we get $K \nsubseteq M(\mathbb{B}_o^n)$.

Now we show (ii) $\Rightarrow$ (i). Let us consider $M \in \mathrm{GL}(\mathbb{R})$ with $|\det(M)| \leq 1$. If $|\det(M)| = 1$, then either $M \in \mathcal{O}(n)$ (and then $M(\mathbb{B}_o^n) = \mathbb{B}_o^n$), or
Since
\[ M \notin \mathcal{O}(n), \] one then by (ii) implies that \( K \nsubseteq M(B^n_2). \) If, on the contrary, \(|\det(M)| < 1\), let us denote by \( N := \det(M)^{-1/2}M \) and observe that \(|\det(N)| = 1\). Again, if \( N \notin \mathcal{O}(n)\), then \( K \nsubseteq N(B^n_2) = \det(M)^{-1/2}M(B^n_2)\), and since \(|\det(M)| < 1\), we also have that \( K \nsubseteq M(B^n_2) \). Finally, we suppose that \( N \in \mathcal{O}(n) \). In order to verify that \( K \nsubseteq M(B^n_2) \), we make use of the fact that there exists a touching point \( u \in K \cap S^{n-1} \) (we verify it at the end of the proof). Indeed, under this assumption, in view of \( S^{n-1} = N(S^{n-1}) \), we have \( u \in K \cap N(S^{n-1}) = K \cap \det(M)^{-1/2}M(S^{n-1}) \), therefore \( u \notin M(B^n_2) \), and thus \( K \nsubseteq M(B^n_2) \), concluding the proof of (i).

As promised, we show that (ii) implies that \( K \cap S^{n-1} \neq \emptyset \). If, on the contrary, \( K \subseteq \rho B^n_2 \), for some \( \rho \in (0, 1) \), we can consider the matrix \( M_\rho := \text{diag}(\varepsilon, \varepsilon^{-1}, 1, \ldots, 1) \in \text{GL}_n(\mathbb{R}) \setminus \mathcal{O}(n) \), for \( \varepsilon \in (0, 1) \), which of course satisfies \( \det(M_\rho) = 1 \). Since \( \lim_{\varepsilon \to 1} M_\varepsilon = I_n \), there exists \( \varepsilon_0 \) close enough to 1 such that \( \rho B^n_2 \subseteq M_{\varepsilon_0}(B^n_2) \), therefore implying that \( K \subseteq \rho B^n_2 \subseteq M_{\varepsilon_0}(B^n_2) \), and thus contradicting (ii).

The second lemma has been shown by Behrend [5, Satz 7u & 11u] in the case of the plane \( n = 2 \).

**Lemma 2.5.** Let \( K \in \mathcal{K}^n_0 \) be such that \( D(K) = 2 \). The following are equivalent:

(i) \( K \) is in Behrend position.

(ii) \( K \) is in Löwner position.

**Proof.** Since \( D(K) = 2 \), we have \( K \subseteq B^n_2 \). Therefore, using Lemma 2.4,

\[ K \text{ is in Löwner position} \]

\[ \iff \text{for all } M \in \text{NO}_1(n) \exists u \in K \text{ such that } \|M(u)\| \geq 1 \]

\[ \iff \text{for all } M \in \text{NO}_1(n) : D(M(K)) \geq 2 = D(K) \]

\[ \iff K \text{ is in Behrend position}. \]

**Proof of Theorem 2.3.** (i) \( \iff \) (ii). Since \( M(K - K) = M(K) - M(K) \) for every \( M \in \text{GL}_n(\mathbb{R}) \), and \( D(K - K) = 2D(K) \), we can conclude that

\[ K \text{ is in Behrend position} \]

\[ \iff \text{for all } M \in \text{NO}_1(n) : D(M(K)) \geq D(K) \]

\[ \iff \text{for all } M \in \text{NO}_1(n) : D(M(K - K)) \geq D(K - K) \]

\[ \iff K - K \text{ is in Behrend position}. \]

(ii) \( \iff \) (iii). This follows from Lemma 2.5.

(iii) \( \iff \) (iv). Observe that \( u \in \frac{1}{D(K)}(K - K) \cap S^{n-1} \) if and only if \( u \in D_K \).

This means that the contact points \( u_i, i \in [m] \), in Theorem 2.2, belong to both \( \frac{1}{D(K)}(K - K) \) and \( D_K \), which shows the claimed equivalence.

(iv) \( \iff \) (v). Apply Theorem 2.2 to the body \( \text{conv}(D_K) \).

In contrast to the Behrend position, it is in general not true that \( K \in \mathcal{K}^n_0 \) is in Löwner position if and only if \( \frac{1}{D(K)}(K - K) \) is. The following proposition provides examples showing that in fact neither implication holds in general.
Proposition 2.6. 

(i) For $\frac{\sqrt{3}}{2} < r \leq 1$, the “sailing boat”

$$K_r = \text{conv} \left\{ \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}$$

is in Löwner position but $\frac{1}{\text{D}(K_r)}(K_r - K_r)$ is not.

(ii) For $r \in [0, 1)$, let $T_r = \text{conv} \left\{ (0, 1)^T, (\pm \sqrt{1 - r^2}, -r)^T \right\}$ be a triangle, and let $Q_2 = \text{conv} \{ \pm (1, 1)^T \}$ be the axis parallel square inscribed into the unit circle. Then, for $\varepsilon > 0$ small enough, the septagon

$$K_{\varepsilon} = \text{conv} \left\{ (1 - \varepsilon)Q_2, T_{1/2 - \varepsilon} \right\}$$

is not in Löwner position, but $\frac{1}{\text{D}(K_{\varepsilon})}(K_{\varepsilon} - K_{\varepsilon})$ is.

Proof. (i): Using Theorem 2.2, one checks that the equilateral triangle $T_2$ with vertices $(0, 0)^T, (\frac{\sqrt{3}}{2}, 1)^T$, and $(-\frac{\sqrt{3}}{2}, 1)^T$ is in Löwner position. Since $T_2 \subseteq K_r \subseteq B_2$, the pentagon $K_r$ is in Löwner position as well.

Now, $D(K_r) = \{(0, 1)^T - (\pm \sqrt{1 - r^2}, -r)^T\} = \sqrt{2(r + 1)}$, and moreover the segments $[(0, 1)^T, (\pm \sqrt{1 - r^2}, -r)^T]$ are the only diametrical segments of $K_r$. Hence, for $\frac{\sqrt{3}}{2} < r \leq 1$,

$$\frac{1}{\text{D}(K_r)}(K_r - K_r) \cap S^1 = \left\{ \frac{1}{\sqrt{2(r + 1)}} (\pm \sqrt{1 - r^2}, \mp (r + 1))^T \right\}.$$ 

Therefore, the arc with midpoint $e_2$ and angle $\pi/2$ of the circle $S^1$ contains all diametrical directions of $K_r$ in its interior. Thus, the diameter condition (Lemma 3.1) implies that $\frac{1}{\text{D}(K_r)}(K_r - K_r)$ is not in Behrend position. In view of Theorem 2.3 this shows that $\frac{1}{\text{D}(K_r)}(K_r - K_r)$ is not in Löwner position.

(ii): First of all, the unit circle $B_2$ is the smallest circle containing $T_{1/2 - \varepsilon}$, and hence it is the smallest circle containing $K_{\varepsilon}$. The contact points $\text{bd}(K_{\varepsilon}) \cap S^1$ are exactly the vertices of the triangle $T_{1/2 - \varepsilon}$. The reader quickly convinces herself that for $\varepsilon > 0$ these contact points do not provide a decomposition of the identity $I_2$ according to Theorem 2.2, so that $K_{\varepsilon}$ is not in Löwner position.

On the other hand, for $\varepsilon$ small enough, the diameter of $K_{\varepsilon}$ is attained by the opposite pairs of vertices of $(1 - \varepsilon)Q_2$. Therefore, $Q_2 \subseteq \frac{1}{\text{D}(K_{\varepsilon})}(K_{\varepsilon} - K_{\varepsilon}) \subseteq B_2$, and since $Q_2$ is in Löwner position, the difference body $\frac{1}{\text{D}(K_{\varepsilon})}(K_{\varepsilon} - K_{\varepsilon})$ is as well.

Based on Theorem 2.3, we can now give a succinct characterization of uniqueness of the Behrend position of a convex body.

Theorem 2.7. The Behrend position of a convex body is unique up to orthogonal transformations, scalings, and translations.

Proof. The isodiametric quotient of a convex body is clearly invariant under orthogonal transformations, scalings, and translations. Hence, the property of a convex body to be in Behrend position is invariant under these transformations as well.
In order to show that this is an exhaustive list of such transformations, it suffices to consider \( o \)-symmetric convex bodies. In fact in view of Theorem 2.3, \( K \) is in Behrend position if and only if its difference set \( K - K \) is. Therefore, let \( K \in \mathcal{K}_o^n \) be in Behrend position and furthermore let \( K \subseteq \mathbb{B}_2^n \) and \( D(K) = 2 \), which deals with the freedom of scalings. Now, let \( A \in \text{GL}_n(\mathbb{R}) \), with \( \left| \det(A) \right| = 1 \), be such that \( AK \) is in Behrend position as well. Note that this implies that \( D(AK) = 2 \). By Lemma 2.5, both \( K \) and \( AK \) are in Löwner position. In particular, \( K \subseteq A^{-1}B_n^2 \) and \( B_n^2 \) is the unique minimal volume ellipsoid containing \( K \). Hence, \( A^{-1}B_n^2 = B_n^2 \) and thus \( A \) is an orthogonal transformation. \( \square \)

2.2. The \( o \)-symmetric reverse isodiametric inequality. We conclude this section with a proof of Conjecture 1.1 for \( o \)-symmetric convex bodies.

Proof of Theorem 1.3. A crucial result of Barthe [3] states that for every \( L \in \mathcal{K}_o^n \) in Löwner position, we have

\[
\text{vol}(L) \geq \frac{2^n}{n!},
\]

and equality holds if and only if \( L \) is a regular crosspolytope.

Since \( K \) is in Behrend position, Theorem 2.3 implies that \( \frac{1}{D(K)}(K - K) = \frac{2}{D(K)}K \) is in Löwner position. Hence, we obtain

\[
\text{idq}(K) = \frac{\text{vol}(K)}{D(K)^n} = \frac{1}{2^n} \text{vol} \left( \frac{2}{D(K)}K \right) \geq \frac{1}{n!},
\]

and the equality case characterization follows from that of Barthe. \( \square \)

3. The diametrical directions of a body in Behrend position are well-distributed

In contrast to the \( o \)-symmetric situation, a complete solution of Makai Jr.’s conjecture for arbitrary convex bodies \( K \in \mathcal{K}^n \) still seems to be elusive. However, in the following we make significant progress on asymptotic bounds on the isodiametric quotient of a convex body in Behrend position.

As discussed in the introduction, Behrend obtained an optimal result in the plane. Based on his ideas, we show that in isodiametric optimal position, the diametrical directions of a convex body give rise to a well-distributed point set on the sphere. Once this distribution property is established, a strong asymptotic bound follows easily.

At the heart of Behrend’s arguments lies what he calls the diameter condition:

Lemma 3.1 ([5, p. 733, Satz 9u]). The diameters of any \( K \in \mathcal{K}^2 \) in Behrend position cannot be placed in the interior of a right angle.

It turns out that Behrend’s proof and therefore this property can be generalized to higher dimensions. In order to state the extension, we define the angle between a linear subspace \( L \) and a non-zero vector \( v \in \mathbb{R}^n \) as

\[
\angle(L, v) = \min_{z \in L \setminus \{0\}} \left( \arccos \frac{z^Tv}{\|z\|\|v\|} \right).
\]
Lemma 3.2. Let \( K \in \mathbb{K}^n \) be in Behrend position, let \( D_K \subseteq \mathbb{S}^{n-1} \) be the set of diametrical directions of \( K \), and let \( 1 \leq i \leq n - 1 \). Then, for every \( i \)-dimensional linear subspace \( L \),

(i) there is some \( v \in D_K \) such that \( \angle(L, v) \leq \arccos(\sqrt{i/n}) \), and
(ii) there is some \( w \in D_K \) such that \( \angle(L, w) \geq \arccos(\sqrt{i/n}) \).

Moreover, the cube \( C_n = [-1, 1]^n \) and the subspaces \( L_i = \text{lin}\{e_1, \ldots, e_i\} \), where \( e_i \) denotes the \( i \)th coordinate unit vector, show that the bounds cannot be improved.

Proof. (i): Let \( L \) be a fixed \( i \)-dimensional linear subspace. For the sake of contradiction, we assume that \( \angle(L, v) > \arccos(\sqrt{i/n}) \), for every \( v \in D_K \).

This means, that \( |u^Tv| < \sqrt{i/n} \), for every \( v \in D_K \) and every \( u \in L \cap \mathbb{S}^{n-1} \).

Hence, a given diametrical direction \( v \in D_K \) constitutes an angle
\[
\omega = \frac{\pi}{2} - \arccos(u^Tv) = \arcsin(u^Tv) < \arcsin\left(\sqrt{\frac{i}{n}}\right)
\]
with the hyperplane \( L^\perp \) orthogonal to \( L \). By \( \cos(\arcsin(x)) = \sqrt{1-x^2} \), this implies that \( \cos^2 \omega > \frac{n-i}{n}(1+\delta) \).

Via a suitable rotation of \( K \), we assume that \( L = \text{lin}\{e_1, \ldots, e_i\} \). For a small \( \varepsilon > 0 \), we consider the linear map \( A_\varepsilon = \text{diag}(1, \ldots, 1, 1-\varepsilon, \ldots, 1-\varepsilon) \in \text{GL}_n(\mathbb{R}) \) having its first \( i \) entries equal to 1. Using elementary trigonometry, we see that the length of a line segment \( \ell \) that constitutes an angle \( \omega \) with \( L^\perp \), changes under the transformation \( A_\varepsilon \) according to the formula
\[
\|A_\varepsilon \ell\| = \|\ell\| \sqrt{1 - 2\varepsilon \cos^2 \omega + \varepsilon^2 \cos^2 \omega} = \|\ell\| \left(1 - \varepsilon \cos^2 \omega + O(\varepsilon^2)\right).
\]
Let \( K' = A_\varepsilon K \). By compactness of \( K \), we can choose \( \varepsilon \) small enough such that \( \angle(L, v') > \arccos(\sqrt{i/n}) \) for every diametrical direction \( v' \in D_K \) of \( K' \) as well. Thus, if \( L \) is a line segment whose image under \( A_\varepsilon \) attains \( D(K') \) and which makes an angle of \( \omega \) with \( L^\perp \), we get by (8) that
\[
D(K') = \|\ell\| \left(1 - \varepsilon \cos^2 \omega + O(\varepsilon^2)\right) \leq D(K) \left(1 - \varepsilon \cos^2 \omega + O(\varepsilon^2)\right).
\]
Clearly, we have \( \text{vol}(K') = (1-\varepsilon)^n \text{vol}(K) \), and therefore for \( \varepsilon \) small enough
\[
\text{idq}(K') = \frac{\text{vol}(K')}{D(K')^n} \leq \frac{\text{vol}(K)}{D(K)^n} \left(1 - \varepsilon \cos^2 \omega + O(\varepsilon^2)\right)^n
\]
\[
= \text{idq}(K) \frac{1 - (n-i)\varepsilon + O(\varepsilon^2)}{1 - n\varepsilon \cos^2 \omega + O(\varepsilon^2)}
\]
\[
\geq \text{idq}(K) \frac{1 - (n-i)\varepsilon + O(\varepsilon^2)}{1 - (n-i)\varepsilon (1 + \delta) + O(\varepsilon^2)}
\]
\[
> \text{idq}(K).
\]
This is in contradiction that \( K \) is in Behrend position and hence proves our claim.
(ii): The statement (ii) holds for the $i$-dimensional linear subspace $L$ if and only if (i) holds for its orthogonal complement $L^\perp$. Indeed, by (i) there exists some $w \in D_K$ such that $\angle(L^\perp, w) \leq \arccos(\sqrt{(n-i)/n})$. Therefore,

$$
\angle(L, w) = \frac{\pi}{2} - \angle(L^\perp, w) \geq \arcsin\left(\sqrt{\frac{n-i}{n}}\right) = \arccos\left(\sqrt{\frac{i}{n}}\right),
$$

in view of the identity $\arcsin(x) = \arccos(\sqrt{1-x^2})$.

We conclude the proof by showing that the cube $C_n = [-1,1]^n$ does not allow for a smaller angle than $\arccos(\sqrt{i/n})$ in (i). First of all, $C_n$ is in Löwner position (cf. [18, Sect. 2]), and thus by Theorem 2.3, it is also in Behrend position. The diametrical directions of $C_n$ are precisely its vertex directions. For the linear subspace $L_i = \text{lin}\{e_1, \ldots, e_i\}$ and any vertex $v \in \{-1,1\}^n$ of $C_n$, we have

$$
\angle(L_i, v) = \min_{z \in L_i, \{0\}} \left( \frac{\arccos \left( \frac{z^\top v}{\|z\|\|v\|} \right)}{\|z\|\|v\|} \right) = \arccos\left( \sqrt{\frac{i}{n}} \right),
$$

where $z_v = (v_1, \ldots, v_i, 0, \ldots, 0)^\top$. Hence, the inequalities in (i) and (ii) cannot be improved in general.

\[\Box\]

**Remark 3.3.**

(i) Since $\arccos(\sqrt{1/2}) = \pi/4$, we retrieve Behrend’s diameter condition by Lemma 3.2 (ii), for $n = 2$.

(ii) For $u \in S^{n-1}$ and $\varphi \geq 0$, let $C(u, \varphi) = \{v \in S^{n-1} : \angle(u, v) \leq \varphi\}$ be the spherical cap with center $u$ and angle $\varphi$. The case $i = 1$ of Lemma 3.2 (i) then says that the caps of radius $\arccos(\sqrt{1/n})$ and with centers at the diametrical directions of $K$ induce a spherical covering, that is,

$$
S^{n-1} = \bigcup_{u \in D_K} C(u, \arccos(\sqrt{1/n})).
$$

A consequence of Lemma 3.2 is an asymptotic lower bound on the isodiametric quotient of a convex body in Behrend position that improves dramatically upon Makai Jr.’s original estimate (4).

**Theorem 3.4.** Let $K \in \mathcal{K}^n$ be in Behrend position. Then

$$
\text{idq}(K) \geq \frac{1}{\sqrt{n!n^2}} \approx \frac{\sqrt{n+1}}{n!e^2}.
$$

*Proof.* The idea of the proof is to use Lemma 3.2 in order to guarantee the existence of diametrical directions of $K$ that span a simplex of large volume.

More precisely, let $v_1 \in D_K$ be chosen arbitrarily. In view of Lemma 3.2 ii), for every $1 \leq i \leq n-1$, there exists a diametrical direction $v_{i+1} \in D_K$ such that $\angle(L_i, v_{i+1}) \geq \arccos(\sqrt{i/n})$, where $L_i = \text{lin}\{v_1, \ldots, v_i\}$. By definition of $D_K$, there are translation vectors $t_1, \ldots, t_n \in \mathbb{R}^n$ such that the segment $S_i = t_i + D(K)[0, v_i]$ is contained in $K$, for $1 \leq i \leq n$. Clearly, the volume of $K$ is then lower bounded by the volume of $\text{conv}\{S_1, \ldots, S_n\}$. A result of Groemer [15] (cf. [6, Thm. 2]) says that this volume is minimal if the line
Theorem 2.3 is actually a generalization of the famous Dvoretzky-Rogers lemma (cf. [9]). It shows that the Dvoretzky-Rogers lemma implies (11). The proof of (10) for every decomposition \( I_n = \sum_{i=1}^{m} \lambda_i u_i u_i^\top \) of the identity \( I_n = \sum_{i=1}^{m} \lambda_i u_i u_i^\top \) with the usual conditions \( u_1, \ldots, u_m \in S^{n-1} \) and \( \lambda_1, \ldots, \lambda_m \geq 0 \), we have proven that for any \( i \)-dimensional linear subspace \( L \), there is an index \( 1 \leq j \leq m \) such that \( \angle(L, u_j) \geq \arccos(\sqrt{n}/n) \). The Dvoretzky-Rogers lemma merely provides a choice \( \{u_{j_1}, \ldots, u_{j_n}\} \subseteq \{u_1, \ldots, u_n\} \) of \( n \) of the decomposing vectors such that, for \( 1 \leq i \leq n - 1 \), the angle \( \angle(L_i, u_{j_{i+1}}) \geq \arccos(\sqrt{n}/n) \), where \( L_i = \text{lin}\{u_{j_1}, \ldots, u_{j_i}\} \).

However, with regard to the reverse isodiametric problem, we are actually interested in finding a simplex \( S = \text{conv}\{0, u_{j_1}, \ldots, u_{j_n}\} \) that is spanned by a choice of the decomposing vectors \( u_i \), and which has a large volume. This motivates the following more general Dvoretzky-Rogers-type problem, which asks to find \( j \)-dimensional simplices of large volume in any decomposition of the identity.

**Problem 4.1.** For \( 1 \leq j \leq n \leq m \), let \( \text{DR}(m, n, j) \) be the largest number \( \nu \geq 0 \) such that, for every \( u_1, \ldots, u_m \in S^{n-1} \) and \( \lambda_1, \ldots, \lambda_m \geq 0 \) with \( I_n = \sum_{i=1}^{m} \lambda_i u_i u_i^\top \), there always exist indices \( 1 \leq i_1 < \ldots < i_j \leq m \) fulfilling

\[
\text{vol}(j(\text{conv}\{0, u_{i_1}, \ldots, u_{i_j}\})) \geq \nu.
\]

A couple of remarks regarding the constants \( \text{DR}(m, n, j) \) are in order:

- The constant \( \text{DR}(m, n, j) \) is non-increasing in \( m \), because every decomposition of \( I_n \) into \( m \) summands can be turned into one with \( m + 1 \) summands.

- For every decomposition \( I_n = \sum_{i=1}^{m} \lambda_i u_i u_i^\top \) of the identity with \( m \geq \binom{n+1}{2} \) vectors, there is a subset \( u_{i_1}, \ldots, u_{i_{\ell}} \), for some \( \ell \leq \binom{n+1}{2} \), that also decomposes the identity. Hence \( \text{DR}(m, n, j) = \text{DR}(\binom{n+1}{2}, n, j) \), for \( m \geq \binom{n+1}{2} \).

- The proof of Theorem 3.4 shows that the Dvoretzky-Rogers lemma implies the estimate

\[
\text{DR}(m, n, n) \geq \frac{1}{\sqrt{n!} n^2},
\]

which is however not sensitive to the value of \( m \).
In the following, we use the classical Cauchy-Binet formula for the minors of a product of two matrices in order to provide estimates on $\text{DR}(m, n, j)$ in terms of $m, n$ and $j$. The obtained bounds improve in particular the Dvoretzky-Rogers bound $(10)$ on $\text{DR}(m, n, n)$ and they turn out to be sharp for interesting families of triples $(m, n, j)$. For the sake of notation, we write $[n] = \{1, \ldots, n\}$ for the set of the first $n$ natural numbers, and $\binom{[n]}{i}$ for the family of all $i$-element subsets of $[n]$. Given a matrix $M \in \mathbb{R}^{n \times m}$ and index sets $I \in \binom{[n]}{i}$ and $J \in \binom{[n]}{j}$, we denote by $M_{I,J}$ the submatrix of $M$ which remains after we delete all rows of $M$ with indices not in $I$, and all columns of $M$ with indices not in $J$.

**Theorem 4.2** (Cauchy-Binet formula, cf. [10, Ch. 4]). Let $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{m \times n}$, and let $I, J \in \binom{[n]}{i}$. Then

$$
\det((AB)_{I,J}) = \sum_{K \in \binom{[m]}{i}} \det(A_{I,K}) \det(B_{K,J}).
$$

**Corollary 4.3.** Let $u_1, \ldots, u_m \in \mathbb{S}^{n-1}$ and $\lambda_1, \ldots, \lambda_m \geq 0$ be such that $\sum_{i=1}^{m} \lambda_i u_i u_i^\top = I_n$. Then, for every $1 \leq i \leq n$, we have

$$
\binom{n}{i} = \sum_{J \in \binom{[i]}{i}} \lambda_J \det(\langle U_J \rangle^\top U_J),
$$

where $\lambda_J = \prod_{j \in J} \lambda_j$ and $U_J = (u_j : j \in J) \in \mathbb{R}^{n \times i}$.

**Proof.** For $1 \leq i \leq n$, let $w_i = \sqrt{\lambda_i} u_i$, and write $W = (w_1, \ldots, w_m) \in \mathbb{R}^{n \times m}$. First, we show that $I_n = \sum_{i=1}^{m} w_i w_i^\top = WW^\top$. The first identity follows from the definition of the $w_i$, whereas the second follows from

$$
\langle e_k, \left( \sum_{i=1}^{m} w_i w_i^\top \right) e_l \rangle = \sum_{i=1}^{m} \langle e_k, w_i w_i^\top e_l \rangle = \sum_{i=1}^{m} \langle e_k, w_i e_l \rangle \langle w_i, e_l \rangle = \sum_{i=1}^{m} w_i w_{ik} = \langle e_k, WW^\top e_l \rangle,
$$

where we have used the alternative notation $\langle x, y \rangle = x^\top y$ for the standard scalar product to improve readability. Now, for every $I, J \in \binom{[n]}{i}$, let $\delta_{I,J} = 1$, if $I = J$, and $\delta_{I,J} = 0$, otherwise. **Theorem 4.2** then implies

$$
\delta_{I,J} = \det((I_n)_{I,J}) = \det((WW^\top)_{I,J}) = \sum_{K \in \binom{[m]}{i}} \det(W_{I,K}) \det((W^\top)_{K,J}).
$$
Therefore, using Theorem 4.2 once again, we arrive at

\[
\binom{n}{i} = \sum_{I \in \binom{[n]}{i}} \sum_{J \in \binom{[n]}{j}} \det(W_{I,J}) \det((W^T)_{J,I})
\]

\[
= \sum_{J \in \binom{[n]}{i}} \sum_{I \in \binom{[n]}{j}} \det((W^T)_{J,I}) \det(W_{I,J}) = \sum_{J \in \binom{[n]}{i}} \det((W^T W)_{J,J})
\]

\[
= \sum_{J \in \binom{[n]}{i}} \det((W_J)^TW_J) = \sum_{J \in \binom{[n]}{i}} \lambda_J \det((U_J)^TU_J).
\]

\[\square\]

**Lemma 4.4.** For \(m \in \mathbb{N}\) and \(c \geq 0\), let \(\Delta^c_m = \left\{ \lambda \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = c \right\}\). For \(d \in \mathbb{N}\) with \(d \leq m\), let \(\sigma_d(\lambda_1, \ldots, \lambda_m) = \sum_{I \subseteq \binom{[m]}{d}} \prod_{i \in I} \lambda_i\) be the \(d\)-th elementary symmetric polynomial. Then

\[
\max_{\lambda \in \Delta^c_m} \sigma_d(\lambda) = \sigma_d\left(\frac{c}{m}, \ldots, \frac{c}{m}\right) = \left(\frac{m}{d}\right)^d \frac{c^d}{m^d}.
\]

**Proof.** Let us first assume that if \(\lambda \in \Delta^c_m\) attains the maximum value \(\sigma_d(\lambda)\), then \(\lambda_i \neq 0\), for all \(1 \leq i \leq m\). In this case we use an indirect argument and suppose that the maximum is attained at some \(\lambda \in \Delta^c_m\) with \(\lambda_i > \lambda_j\), for some \(1 \leq i < j \leq m\). Let \(\varepsilon > 0\) be small enough such that \(\lambda_i - \lambda_j > \varepsilon\), and let \(\lambda = \lambda - \varepsilon e_i + \varepsilon e_j\). For a subset \(J \subseteq [m]\), we write \(\lambda_J = \prod_{j \in J} \lambda_j\). From the definition it follows that \(\bar{\lambda} \in \Delta^c_m\), and moreover we have

\[
\sigma_d(\bar{\lambda}) = \sum_{i \in I \subseteq \binom{[m]}{d-1}} \bar{\lambda}_I + \sum_{J \subseteq \binom{[m]}{d-2}} \bar{\lambda}_I + \sum_{J \subseteq \binom{[m]}{d-1}} \bar{\lambda}_I
\]

\[
= (\lambda_i - \varepsilon) \sum_{J \subseteq \binom{[m]}{d-1}} \lambda_J + (\lambda_j + \varepsilon) \sum_{J \subseteq \binom{[m]}{d-1}} \lambda_J
\]

\[
+ (\lambda_i - \varepsilon)(\lambda_j + \varepsilon) \sum_{J \subseteq \binom{[m]}{d-2}} \lambda_J + \sum_{I \subseteq \binom{[m]}{d-2}} \lambda_J
\]

\[
= \sigma_d(\lambda) + \varepsilon(\lambda_i - \lambda_j - \varepsilon) \sum_{J \subseteq \binom{[m]}{d-2}} \lambda_J,
\]

contradicting the maximality of \(\lambda\). Note, that the inequality in the last line above is strict, because all \(\lambda_i\) are assumed to be positive.

In order to finish the proof, we now show by induction on \(m\), that in every optimal solution \(\lambda \in \Delta^c_m\) we have \(\lambda_i \neq 0\), for all \(1 \leq i \leq m\). For \(m = d\), we have \(\sigma_d(\lambda) = \prod_{i=1}^d \lambda_i\). So, if one of the \(\lambda_i\) would vanish, then clearly the point is not a maximum. If \(m > d\), and without loss of generality \(\lambda_1 \geq \ldots \geq \lambda_k > \lambda_{k+1} = \ldots = \lambda_m = 0\), then in view of what was shown above \(\sigma_d(\lambda_1, \ldots, \lambda_m) = \sigma_d(\lambda_1, \ldots, \lambda_k) \leq \binom{k}{d} \frac{c^d}{k^d} < \binom{m}{d} \frac{c^d}{m^d}\). Hence, \(\lambda\) could not have been an optimum. \(\square\)

We are now prepared to give our estimates on the Dvoretzky-Rogers-type constants \(\text{DR}(m, n, j)\).
Theorem 4.5. Let $u_1, \ldots, u_n \in S^{n-1}$ and $\lambda_1, \ldots, \lambda_m \geq 0$ be such that

\[ I_n = \sum_{i=1}^m \lambda_i u_i u_i^\top, \]

for some $n \leq m \leq \left(\frac{n+1}{2}\right)$. Then, for every $1 \leq j \leq n$, there exist indices $1 \leq i_1 < \ldots < i_j \leq m$ such that

\[ \text{vol}_j(\text{conv}\{0, u_{i_1}, \ldots, u_{i_j}\})^2 \geq \binom{n}{j} \left(\frac{m}{n}\right)^j \left(\frac{j}{n}\right)^2. \]

In particular, we obtain

\[ \text{DR}(m, n, j)^2 \geq \binom{n}{j} \left(\frac{m}{n}\right)^j \left(\frac{j}{n}\right)^2. \]

These inequalities are best possible for the triples

- $(n, n, j)$, for $1 \leq j \leq n$,
- $(n+1, n, j)$, for $1 \leq j \leq n$, and
- $\left(\left\lceil \frac{n+1}{2}\right\rceil, n, 2\right)$, for $n \in \{2, 3, 7, 23\}$, but not for any other $n \leq 118$.

Proof. Let the elements of a subset $J \in \binom{[m]}{j}$ be indexed by $J = \{i_1, \ldots, i_j\}$, and let $S_J = \text{conv}\{0, u_{i_1}, \ldots, u_{i_j}\}$ be the corresponding simplex. With this notation, Corollary 4.3 gives us

\[
\binom{n}{j} = \sum_{J \in \binom{[m]}{j}} \lambda_J \det((U_J)^\top U_J) = \sum_{J \in \binom{[m]}{j}} \lambda_J (j! \text{vol}_j(S_J))^2 \leq (j!)^2 \sum_{J \in \binom{[m]}{j}} \lambda_J \left(\max_{J \in \binom{[m]}{j}} \text{vol}_j(S_J)\right)^2.
\]

By taking traces in $I_n = \sum_{i=1}^m \lambda_i u_i u_i^\top$ and using $\|u_i\| = 1$, we see that $n = \sum_{i=1}^m \lambda_i$. Thus, we can apply Lemma 4.4, and obtain that

\[
\sum_{J \in \binom{[m]}{j}} \lambda_J \leq \sum_{J \in \binom{[m]}{j}} \left(\frac{n}{m}\right)^j = \binom{m}{j} \left(\frac{n}{m}\right)^j.
\]

Continuing the previous estimate we therefore arrive at

\[
\binom{n}{j} \leq (j!)^2 \binom{m}{j} \left(\frac{n}{m}\right)^j \left(\max_{J \in \binom{[m]}{j}} \text{vol}_j(S_J)\right)^2,
\]

as desired.

Let us now discuss equality cases for certain triples of parameters. From the proof of the inequalities above we see that the bound on $\text{DR}(m, n, j)$ is tight if and only if there is a decomposition $I_n = \sum_{i=1}^m \lambda_i u_i u_i^\top$ such that

- (i) $\text{vol}_j(S_J) = \text{vol}_j(S_{J'})$, for every $J, J' \in \binom{[m]}{j}$, and
- (ii) $\lambda_1 = \ldots = \lambda_m = \frac{m}{n}$ (see Lemma 4.4).

First of all, for every $1 \leq j \leq n$, we have

\[ \text{DR}(n, n, j) = \frac{1}{j!}. \]

In fact, if $\pm u_1, \ldots, \pm u_n$ are the vertices of a regular crosspolytope, then $\text{vol}_j(S_J) = 1/j!$, for every $J \in \binom{[n]}{j}$, and $I_n = \sum_{i=1}^n u_i u_i^\top$. 

Secondly, for every $1 \leq j \leq n$, we have

$$\text{DR}(n+1,n,j)^2 = \frac{(n-j+1)(n+1)^{j-1}}{n!(j!)^2}.$$  

Indeed, if $u_1, \ldots, u_{n+1}$ are the vertices of a regular simplex, then every $j$ of these vertices give rise to a $j$-dimensional simplex with the same volume. Moreover, the reader quickly convinces herself that the coefficients $\lambda_i$ in the corresponding decomposition of the identity matrix are all equal to $n/(n+1)$.

Finally, we consider the case $j = 2$ and $m = \binom{n+1}{2}$. Writing $J = \{\ell, k\}$, we get

$$\det((U_J)^\top U_J) = \det\left(\begin{array}{cc} 1 & u_\ell^\top u_k \\ u_k^\top u_\ell & 1 \end{array}\right) = 1 - (u_\ell^\top u_k)^2.$$  

Hence, the triangles $\text{vol}_2(S_J)$, $J \in \binom{[m]}{2}$, all have the same volume if

$$\cos(\alpha(u_\ell, u_k)) = |u_\ell^\top u_k| = \sqrt{1 - \binom{n}{2} \left(\frac{m}{2}\right)^2} = \frac{1}{\sqrt{n+2}},$$  

for every $1 \leq \ell < k \leq m = \binom{n+1}{2}$. In other words, the vectors $u_i$ are the directions of a set of $\binom{n+1}{2}$ equiangular lines.

To date, these special configurations are known to exist in only four different dimensions: For $n = 2$, the directions of the vertices of an equilateral triangle with barycenter at the origin give a system of three equiangular lines. For $n = 3$, we may take the vertex directions of a regular icosahedron. In dimensions $n = 7$ and $n = 23$, there exist sets of 28 and 276 equiangular lines with an angle of $\arccos(1/3)$ and $\arccos(1/5)$, respectively. These configurations are constructed in [19]. Moreover, a result of Neumann (cf. [19, Thm. 3.2]) states that, if there is a set of $\binom{n+1}{2}$ equiangular lines, then $\sqrt{n+2}$ is an odd integer. Bannai et al. [2] proved that this is neither possible for $n = 47$ nor $n = 79$, and thus the first unknown candidate is $n = 119$.

\[\square\]

**Remark 4.6.**

(i) The bound on the special case $\text{DR}(m,n,n)$ in Theorem 4.5 was obtained recently with probabilistic methods by Fodor, Naszódi & Zárnócz [13]. They also illustrate that the bound on $\text{DR}(n+1,n,n)$ is tight because of the regular simplex.

(ii) Theorem 2.3 (v) allows us to think of $\text{DR}(m,n,n)$ as the solution of a polynomial optimization problem. Using the scip solver [20], we have obtained numerical evidence that

$$\text{DR}(5,3,3) = \frac{1}{8} \quad \text{and} \quad \text{DR}(6,3,3) = \frac{1}{6\sqrt{2}}.$$  

Together with the proven values $\text{DR}(3,3,3) = 1/6$ and $\text{DR}(4,3,3) = 2/(9\sqrt{3})$ (cf. Theorem 4.5), this would solve Problem 4.1 in the case $n = j = 3$ completely. The experiments with scip furthermore suggest
Theorem 4.5

That DR(5,3,3) is attained by

\[
(u_1, \ldots, u_5) = \left( \begin{array}{ccccc}
0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\
1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}
\end{array} \right)
\]

and \((\lambda_1, \ldots, \lambda_5) = \left( \frac{1}{3}, \frac{2}{3}, \ldots, \frac{2}{3} \right)\).

As a corollary to Theorem 4.5, we get an alternative proof of the asymptotic bound in Theorem 1.4.

**Corollary 4.7.** Let \(K \in \mathcal{K}^n\) be in Behrend position. Then,

\[
\text{idq}(K) \geq \text{DR}(\binom{n+1}{2}, n, n) \geq \frac{(\frac{n+1}{2})^2}{\left(\frac{n(n+1)}{2}\right)^{\frac{1}{4}}} \approx \frac{\sqrt{n+1}}{n! e^{\frac{3}{2}}}
\]

For \(n = 2\), this is an alternative to Behrend’s solution of the reverse isodiametric problem in the plane. For 3-dimensional bodies \(K \in \mathcal{K}^3\) in Behrend position, it gives a lower bound of \(\text{idq}(K) \geq \frac{\sqrt{10}}{60} \approx 0.10541\), which is very close to the conjectured optimal value \(\frac{1}{6\sqrt{2}} \approx 0.11785\) in Conjecture 1.1.

**Proof.** Since \(K\) is in Behrend position, Theorem 2.3 (v) provides us with diametrical directions \(u_1, \ldots, u_m \in D_K\) and scalars \(\lambda_1, \ldots, \lambda_m \geq 0\), for some \(n \leq m \leq \binom{n+1}{2}\), such that \(I_n = \sum_{i=1}^m \lambda_i u_i u_i^\top\).

By the same reasoning as in the proof of Theorem 3.4, for any choice of indices \(1 \leq i_1 < \ldots < i_n \leq m\), we have

\[
\text{idq}(K) \geq \text{vol}(\text{conv}\{0, u_{i_1}, \ldots, u_{i_n}\})
\]

The definition of DR\((m,n,j)\) then implies that \(\text{idq}(K) \geq \text{DR}(\binom{n+1}{2}, n, n)\) and so we can employ the lower bound from Theorem 4.5. The asymptotics follow from Stirling’s approximation of the factorial function.

Therefore, the triple \((\binom{n+1}{2}, n, n)\) is the most interesting concerning the reverse isodiametric problem. In fact, the proof of Corollary 4.7 shows that the following claim would imply Makai Jr.’s Conjecture 1.1 (2).

**Conjecture 4.8.** For every \(n \in \mathbb{N}\), we have \(\text{DR}(\binom{n+1}{2}, n, n) = \frac{\sqrt{n+1}}{n! 2^{n+1}}\).

5. Analogies to the reverse isominwidth problem

For a convex body \(K \in \mathcal{K}^n\) and a direction \(u \in \mathbb{R}^n \setminus \{0\}\), the support function of \(K\) with respect to \(u\) is defined as \(h(K,u) = \max \{x^\top u : x \in K\}\). The width of \(K\) in direction \(u \in S^{n-1}\) is given by \(w(K,u) = h(K,u) + h(K,-u)\). Finally, the minimum width of \(K\) is defined as

\[
w(K) = \min_{u \in S^{n-1}} w(K,u).
\]

For an \(o\)-symmetric convex body \(K \in \mathcal{K}^n_o\), the minimum width and the diameter are dual to each other in the sense that

\[
w(K)D(K^*) = 4
\]

(cf. [14, (1.2)]). Here, \(K^* = \{x \in \mathbb{R}^n : x^\top y \leq 1, \forall y \in K\}\) denotes the polar body of \(K\). In the following, we elaborate on this duality and investigate the dual of the reverse isodiametric problem.
In analogy to the isodiametric quotient, we define the \textit{isominwidth quotient} of a convex body \( K \in \mathcal{K}^n \) as
\[
\text{iwq}(K) := \frac{\text{vol}(K)}{w(K)^n}
\]
and we may ask for upper and lower bounds on this magnitude. The question on optimal lower bounds is classical in Convex Geometry. Pál [22] proved that, for every planar \( K \in \mathcal{K}^2 \), we have
\[
\text{iwq}(K) \geq \frac{1}{\sqrt{3}},
\]
and that equality holds if and only if \( K \) is an equilateral triangle. In arbitrary dimension, the following bound is due to Firey [12] (see also Bezdek [7] for a slightly improved yet much more involved bound):
\[
\text{iwq}(K) \geq \frac{2}{\sqrt{3n!}}, \quad \text{for } K \in \mathcal{K}^n.
\]
However, the optimal bound (often called the \textit{convex Kakeya problem} or \textit{Pál problem}) is not known. Already in \( \mathbb{R}^3 \), one can slice a small neighbourhood of a vertex of a regular tetrahedron \( T_3 \), obtaining a new polytope \( T'_3 \), without reducing its minimum width. Hence, one gets \( \text{iwq}(T'_3) < \text{iwq}(T_3) \), so that \( T_3 \) is not a minimizer in Pál’s problem. If one continues slicing \( T'_3 \) in a certain way until no more slicing is possible without reducing the minimum width, Heil [17] conjectures that the resulting body is the solution to Pál’s question.

If we restrict to \( o \)-symmetric convex bodies \( K \in \mathcal{K}_o^n \) the situation gets much easier. Indeed, since \((w(K)/2)\mathbb{B}^n_2 \subseteq K\), one obtains
\[
\text{iwq}(K) \geq \frac{\text{vol}(\mathbb{B}^n_2)}{2^n},
\]
which holds with equality if and only if \( K \) is a Euclidean ball.

Analogously to the isodiametric quotient, there exists no upper bound on \( \text{iwq}(K) \) that is independent of the body \( K \in \mathcal{K}^n \). Hence, we may study whether the minimal isominwidth quotient among all linear images of \( K \) can be upper bounded by a constant only depending on the dimension \( n \).

First of all, an analogous argumentation as in Lemma 2.1 and Theorem 2.7 leads to

\textbf{Theorem 5.1.} \textit{For every } \( K \in \mathcal{K}^n \), \textit{there exists an } \( A \in \text{GL}_n(\mathbb{R}) \) \textit{such that}
\[
\text{iwq}(AK) = \inf_{B \in \text{GL}_n(\mathbb{R})} \text{iwq}(BK).
\]
\textit{Moreover, } \( A \) \textit{is unique up to orthogonal transformations and scalings.}

This of course allows to define yet another \textit{position} of a convex body, this time with respect to the minimum width.

\textbf{Definition 5.2.} A convex body \( K \in \mathcal{K}^n \) is in \textit{isominwidth position}, if
\[
\text{iwq}(K) = \min_{A \in \text{GL}_n(\mathbb{R})} \text{iwq}(AK).
\]
Now, we want to establish the analog of Theorem 2.3 for the isominwidth position. To this end, we need some further notation. Let
\[
W_K = \left\{ u \in S^{n-1} : h(K, u) + h(K, -u) = w(K) \right\}
\]
be the set of *minwidth directions*, that is, the directions in which the minimum width of $K$ is attained. It is well-known that if $u \in W_K$, then there exists an $x \in K$ such that $x + w(K)[0, u] \subseteq K$ (cf. [14]). Moreover, let $H(K, u) = \{x \in \mathbb{R}^n : x^T u = h(K, u)\}$ be the supporting hyperplane of $K$ in the direction $u$, and let $H^-(K, u)$ be the corresponding halfspace containing $K$.

Just as the Behrend position is strongly tied to the Löwner position, it turns out that the isominwidth position is linked to the so-called *John position* of a convex body. Dually to the Löwner position, $K \in \mathcal{K}_n$ is in John position if $B_n^{2} = \text{the maximum volume ellipsoid contained in } K$. The characterization of the John position by the existence of a certain decomposition of the identity is verbatim to Theorem 2.2, except for that we need to replace the condition $K \subseteq B_n^{2}$ by $B_n^{2} \subseteq K$ (cf. [18] and [16, Ch. 11]). We can now formulate the desired characterization of the isominwidth position.

**Theorem 5.3.** Let $K \in \mathcal{K}_n$. The following are equivalent:

(i) $K$ is in isominwidth position.

(ii) $K - K$ is in isominwidth position.

(iii) $\frac{1}{w(K)}(K - K)$ is in John position.

(iv) $\bigcap_{u \in W_K} H^-(B_n^{2}, u)$ is in John position.

(v) There exists an $m \in \{n, \ldots, \binom{n+1}{2}\}$, scalars $\lambda_1, \ldots, \lambda_m \geq 0$, and isominwidth directions $u_1, \ldots, u_m \in W_K$, such that

$$I_n = \sum_{i=1}^{m} \lambda_i u_i u_i^T.$$ 

The proof of this characterization is based on the same ideas as that for the Behrend position given in Section 2. For the sake of brevity, we do not give the details here and leave them to the reader.

As the main result of this section, we completely solve the reverse isominwidth problem. Curiously, it turns out that $o$-symmetric convex bodies have the worst minimum isominwidth quotient, which is in strong contrast to the Behrend position.

**Theorem 5.4.** Let $K \in \mathcal{K}_n$ be in isominwidth position. Then

$$\text{iwq}(K) \leq 1.$$ 

Moreover, equality holds if and only if $K$ is a cube.

The proof follows the ideas developed by Ball [1] for the volume ratio of a convex body in John position. We need the geometric version of Ball [1] of the geometric inequality of Brascamp & Lieb (cf. [4, Cor. 3]).

**Theorem 5.5** (Brascamp & Lieb 1976, Ball 1991). Let $\lambda_1, \ldots, \lambda_m \geq 0$ and $u_1, \ldots, u_m \in \mathbb{S}^{n-1}$, for some $n \leq m \leq \binom{n+1}{2}$, be such that $\sum_{i=1}^{m} \lambda_i u_i u_i^T = I_n$. Further, let $f_1, \ldots, f_m : \mathbb{R} \to [0, \infty)$ be measurable functions. Then

$$\int_{\mathbb{R}^n} \left( \prod_{i=1}^{m} f_i(x^T u_i)^{\lambda_i} \right) dx \leq \prod_{i=1}^{m} \left( \int_{\mathbb{R}} f_i(t) dt \right)^{\lambda_i}.$$ 

Moreover, if none of the $f_i$ is a Gaussian function, then equality holds if and only if $\{u_1, \ldots, u_m\}$ is an orthonormal basis of $\mathbb{R}^n$. 

Proof of Theorem 5.4. Applying a suitable scaling of $K$ we may suppose that $w(K) = 2$. Since $K \subseteq H^-(K, u)$, for every $u \in W_K$, we have

$$K \subseteq C = \bigcap_{u \in W_K} H^-(K, u).$$

We define $r_i = h(K, -u_i)$, for $1 \leq i \leq m$, and we observe that $h(K, u_i) = r_i + w(K, u_i) = r_i + 2$. Further, let

$$f_i(t) = \chi_{[r_i, r_i+2]}(t) = \begin{cases} 1 & \text{if } r_i \leq t \leq r_i + 2, \\ 0 & \text{otherwise}, \end{cases}$$

be the characteristic function of $[r_i, r_i + 2]$, and observe that

$$C = \left\{ x \in \mathbb{R}^n : \prod_{i=1}^m f_i(x^\top u_i)^{\lambda_i} = 1 \right\}.$$

Now, Theorem 5.5 yields that

$$\text{vol}(K) \leq \text{vol}(C) = \int_{\mathbb{R}^n} \chi_C(x) dx = \int_{\mathbb{R}^n} \left( \prod_{i=1}^m f_i(x^\top u_i)^{\lambda_i} \right) dx$$

$$\leq \prod_{i=1}^m \left( \int_{\mathbb{R}} f_i(t) dt \right)^{\lambda_i} = \prod_{i=1}^m 2^{\lambda_i} = 2^m \lambda_i = 2^n = w(K)^n,$$

as desired.

If we have equality, we need to have equality in each step of the estimate above. This means that $\text{vol}(K) = \text{vol}(C)$, and thus $K = C$. Moreover, since none of the characteristic functions $f_i$ is a Gaussian function, equality in Theorem 5.5 implies that $\{u_1, \ldots, u_m\}$ is an orthonormal basis. This means that $m = n$, and hence that $K = C$ is a cube. □

Acknowledgments. We thank Ambros Gleixner, Benjamin Müller, and Felipe Serrano from the Zuse Institute Berlin for discussions and help regarding the scip experiments described in Remark 4.6. Furthermore, we thank Peter Gritzmann and Martin Henk for the possibility of mutual research visits at TU Munich and TU Berlin. The first author also thanks TU Munich and the University Centre of Defence of San Javier, where part of this work has been done.

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