Bohr’s correspondence principle in quantum field theory and classical renormalization scheme: the Nelson model.

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(Dated: March 8, 2016)

Abstract: In the mid Sixties Edward Nelson proved the existence of a consistent quantum field theory that describes the Yukawa-like interaction of a non-relativistic nucleon field with a relativistic meson field. Since then it is thought, despite the renormalization procedure involved in the construction, that the quantum dynamics should be governed in the classical limit by a Schrödinger-Klein-Gordon system with Yukawa coupling. In the present paper we prove this fact in the form of a Bohr correspondence principle. Besides, our result enlighten the nature of the renormalization method employed in this model which we interpret as a strategy that allows to put the related classical Hamiltonian PDE in a normal form suitable for a canonical quantization.

Keywords: Nelson model, Renormalization, Schrödinger-Klein-Gordon system, Yukawa interaction, Classical limit.

MSC2010: 35Q40, 81T16, 81Q20

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Modern theoretical physics explains how matter interacts with radiation and proposes phenomenological models of quantum field theory that in principle describe such fundamental interaction. Giving a firm mathematical ground to these models is known to be a difficult task related to renormalization theory \([22, 43, 65, 66, 71, 81]\). Since the fifties there were spectacular advances in these problems culminating with the perturbative renormalization of quantum electrodynamics, the birth of the renormalization group method and the renormalizability of gauge field theories. Nevertheless, conceptual mathematical difficulties remain as well as outstanding open problems, see \([74, 90]\). The purpose of the present article is twofold: first to prove the quantum-classical correspondence for the renormalized Nelson model and second to point out a conceptually different point of view on renormalization through the analysis of an elementary example of quantum field theory and the study of the relationship with its classical formulation.

The so-called Nelson model is a system of Quantum Field Theory that has been widely studied from a mathematical standpoint [see e.g. \(1, 14–17, 24, 34, 41, 47, 51–53, 83, 88, 99\)]. It consists of non-relativistic spin zero particles interacting with a scalar boson field, and can be used to model various systems of physical interest, such as nucleons interacting...
with a meson field. In the mid sixties Edward Nelson rigorously constructed a quantum
dynamic for this model free of ultraviolet (high energy) cutoffs in the particle-field coupling,
see [84]. This is done by means of a renormalization procedure: roughly speaking, we need
to subtract a divergent quantity from the Hamiltonian, so the latter can be defined as a self-
adjoint operator in the limit of the ultraviolet cutoff. The quantum dynamics is rather singular
in this case (renormalization is necessary); and the resulting generator has no explicit form as
an operator though it is unitarily equivalent to an explicit one. Since the work of Gross [69]
and Nelson [84] it is believed, but never proved, that the renormalized dynamics is generated
by a canonical quantization of the Schrödinger-Klein-Gordon (S-KG) system with Yukawa
coupling. In other words, the quantum fluctuations of the particle-field system are centered
around the classical trajectories of the Schrödinger-Klein-Gordon system at certain scale and
the renormalization procedure preserves the suitable quantum-classical correspondence as
well as being necessary to define the quantum dynamics. We give a mathematical formulation
of such result in Theorem I.1 in the form of a Bohr correspondence principle. The proof of
the above theorem is rather technical and constitutes a large part of this article.

Recently, the authors of this paper have studied the classical limit of the Nelson model, in
its regularized version [7, 39]. We have proved that the quantum dynamic converges, when
an effective semiclassical parameter $\varepsilon \to 0$, towards a non-linear Hamiltonian flow on a
classical phase space. This flow is governed by a Schrödinger-Klein-Gordon system, with a
regularized Yukawa-type coupling. To extend the classical-quantum correspondence to the
system without ultraviolet cutoff, we rely on the recent techniques elaborated in the mean-
field approximation of many-body Schrödinger dynamics in [8–11] as well as the result with
cutoff [7]. As a matter of fact the renormalization procedure, implemented by a dressing
transform, generates a many-body Schrödinger dynamics in a mean-field scaling. So it was
convenient that the mean-field approximation was derived with the same general techniques
that allow to prove equally the classical approximation for QFT models. The result is further
discussed in Subsection I.2, and all the details and proofs are provided in Section V.

As mentioned before, the study of the quantum-classical correspondence led us to a reinter-
pretation of renormalization, for the Nelson model and maybe in general, as a procedure that
allows to put the related classical Hamiltonian PDE in a normal form suitable for a canoni-
cal quantization. This normal form is implemented by a near identity change of coordinates
close in spirit to the work of Jalal Shatah on nonlinear Klein-Gordon equation with quadratic
terms, see [98]. Usually renormalization in quantum electro-dynamics and QFT is addressed
through the renormalization group method, see e.g. [25, 81]. In the eighties and nineties
two deep ideas have found fruitful applications beyond their birthplace: the normal form
techniques are extended to PDEs and the renormalization group method is adapted to ODEs
and PDEs after the influential works of Shatah [98], Goldenfeld and al. [29] respectively
(see also Bricmont and al. [26]). These ideas resulted in a strong research activity, see e.g.
[2, 38, 76, 82, 86, 105]. The connection between the normal form and the renormalization
group has been highlighted for ODEs in [36]. So this suggests that renormalization can be carried out through normal forms prior to quantization in a Hamiltonian framework rather than using a renormalization group method and dealing with Lagrangians, actions and Feynman amplitudes. Actually such point of view may be linked in some sense to long standing ideas of Paul Dirac and Hendrik Kramers on "classical" renormalization of quantum electrodynamics, see [37]. Moreover, both normal form and group renormalization methods seem to be related techniques that deal with the problem of reparametrizing the dependence of a physical theory with respect to its high/low energy scales (or fast/slow motion).

The proposed approach (if successful) is in our opinion more mathematically founded due to the long history of normal form theory for ODEs and PDEs. It relies less in physical considerations and intuitions that have surrounded the subject of renormalization with an impenetrable air of mystery. Although our suggestion is based on a very simple model (no mass or charge renormalization is needed) we hope that it could be extended in the future to more realistic quantum field theories and it will be of interest for other colleagues. A further discussion of this point is continued in Subsection I.3, and more details are provided in Section IV for the Nelson model.

For the sake of presentation, we collected the notations and basic definitions—used throughout the paper—in the Subsection I.1 below. In subsection I.2 we present our main result on the classical-quantum correspondence principle and in Subsection I.3 we discuss our alternative point of view on renormalization. The rest of the paper is organized as follows: in Section II we review the basic properties of the quantum system and the usual procedure of renormalization; in Section III we analyze the classical S-KG dynamics and the classical dressing transformation; in Section IV we apply the renormalization via classical dressing to the Nelson model; in Section V we study in detail the classical limit of the renormalized Nelson model, and prove our main Theorem I.1.

1. Notations and general definitions.

* We fix once and for all $0 < \bar{\varepsilon}, m_0, M$. We also define the function $\omega(k) = \sqrt{k^2 + m_0^2}$.

* The effective (semiclassical) parameter will be denoted by $\varepsilon \in (0, \bar{\varepsilon})$.

* Let $\mathcal{Z}$ be a Hilbert space; then we denote by $\Gamma_s(\mathcal{Z})$ the symmetric Fock space over $\mathcal{Z}$. We have that

$$\Gamma_s(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} \mathcal{Z}^\otimes_n \text{ with } \mathcal{Z}^\otimes_n = \mathbb{C}^n.$$ 

* Let $X$ be an operator on a Hilbert space $\mathcal{Z}$. We will usually denote by $D(X) \subset \mathcal{Z}$ its domain of definition, and by $Q(X) \subset \mathcal{Z}$ the domain of definition of the corresponding quadratic form.
Let $S : Z \supseteq D(S) \to Z$ be a densely defined self-adjoint operator on $Z$. Its second quantization $d\Gamma(S)$ is the self-adjoint operator on $\Gamma_s(Z)$ defined by

$$d\Gamma(S)|_{D(S)^\otimes_n} = \varepsilon \sum_{k=1}^n 1 \otimes \cdots \otimes S \otimes \cdots \otimes 1 .$$

The operator $d\Gamma(1)$ is usually named the number operator and denoted by $N$.

We denote by $C_0^\infty(N)$ the subspace of finite particle vectors:

$$C_0^\infty(N) = \{ \psi \in \Gamma_s(Z) ; \exists \bar{n} \in \mathbb{N}, \psi|_{Z_{\bar{n},n}} = 0 \forall n > \bar{n} \} .$$

Let $U$ be a unitary operator on $Z$. We define $\Gamma(U)$ to be the unitary operator on $\Gamma_s(Z)$ given by

$$\Gamma(U)|_{Z^\otimes_n} = \bigotimes_{k=1}^n U .$$

If $U = e^{itS}$ is a one parameter group of unitary operators on $Z$, $\Gamma(e^{itS}) = e^{itd\Gamma(S)}$.

On $\Gamma_s(Z)$, we define the annihilation/creation operators $a^\#(g), g \in Z$, by their action on $f^\otimes_n \in Z^{\otimes_0,n}$ (with $a(g)f_0 = 0$ for any $f_0 \in Z^{\otimes_0,0} = \mathbb{C}$):

$$a(g)f^\otimes_n = \sqrt{\varepsilon n} \langle g, f \rangle_Z f^{\otimes(n-1)} ;$$
$$a^\#(g)f^\otimes_n = \sqrt{\varepsilon (n + 1)} g \otimes_s f^{\otimes_n} .$$

They satisfy the Canonical Commutation Relations (CCR), $[a(f), a^\#(g)] = \varepsilon \langle f, g \rangle_Z$.

If $Z = L^2(\mathbb{R}^d)$ it is useful to introduce the operator valued distributions $a^\#(x)$ defined by

$$a(g) = \int_{\mathbb{R}^d} \tilde{g}(x)a(x)dx , \quad a^\#(g) = \int_{\mathbb{R}^d} g(x)a^\#(x)dx .$$

$H = \Gamma_s(L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)) \simeq \Gamma_s(L^2(\mathbb{R}^3)) \otimes \Gamma_s(L^2(\mathbb{R}^3))$. We denote by $\psi^\#(x)$ and $N_1$ the annihilation/creation and number operators corresponding to the nucleons (conventionally taken to be the first Fock space), by $a^\#(k)$ and $N_2$ the annihilation/creation and number operators corresponding to the meson scalar field (second Fock space). In particular, we will always use the following $\varepsilon$-dependent representation of the CCR if not specified otherwise:

$$[\psi(x), \psi^\#(x')] = \varepsilon \delta(x - x') , \quad [a(k), a^\#(k')] = \varepsilon \delta(k - k') .$$
* We will sometimes use the following decomposition:

\[ \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \text{with} \quad \mathcal{H}_n = (L^2(\mathbb{R}^3))^\otimes_n \otimes \Gamma_s(L^2(\mathbb{R}^3)) . \]

We denote by \( T^{(n)} := T \big|_{\mathcal{H}_n} \) the restriction to \( \mathcal{H}_n \) of any operator \( T \) on \( \mathcal{H} \).

* On \( \mathcal{H} \), the Segal quantization of \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \ni \xi = \xi_1 \oplus \xi_2 \) is given by \( R(\xi) = (\psi^*(\xi_1) + \psi(\xi_1) + a^*(\xi_2) + a(\xi_2)) / \sqrt{2} \), and therefore the Weyl operator becomes \( W(\xi) = e \frac{i}{\sqrt{2}}(\psi^*(\xi_1) + \psi(\xi_1)) e \frac{i}{\sqrt{2}}(a^*(\xi_2) + a(\xi_2)) \).

* Given a Hilbert space \( \mathcal{Z} \), we denote by \( \mathcal{L}(\mathcal{Z}) \) the \( \mathcal{C}^* \)-algebra of bounded operators; by \( \mathcal{K}(\mathcal{Z}) \subset \mathcal{L}(\mathcal{Z}) \) the \( \mathcal{C}^* \)-algebra of compact operators; and by \( \mathcal{L}^1(\mathcal{Z}) \subset \mathcal{K}(\mathcal{Z}) \) the trace-class ideal.

* We denote classical Hamiltonian flows by boldface capital letters (e.g. \( \mathbf{E}(\cdot) \)); their corresponding energy functional by script capital letters (e.g. \( \mathbf{E} \)).

* Let \( f \in \mathcal{S}'(\mathbb{R}^d) \). We denote by \( F(f)(k) \) its Fourier transform

\[ F(f)(k) = \frac{1}{(2\pi)^d/2} \int_{\mathbb{R}^d} f(x) e^{-ik \cdot x} dx . \]

* We denote by \( C_0^\infty(\mathbb{R}^d) \) the infinitely differentiable functions of compact support. We denote by \( H^s(\mathbb{R}^d) \) the non-homogeneous Sobolev space:

\[ H^s(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) , \int_{\mathbb{R}^d} (1 + |k|^2)^s |F(f)(k)|^2 dk < +\infty \right\} ; \]

and its “Fourier transform”

\[ \mathcal{F} H^s(\mathbb{R}^d) = \left\{ f , \mathcal{F}^{-1} f \in H^s(\mathbb{R}^d) \right\} . \]

* Let \( \mathcal{Z} \) be a Hilbert space. We denote by \( \Psi(\mathcal{Z}) \) the set of Borel probability measures on \( \mathcal{Z} \).

2. The classical limit of the renormalized Nelson model.

The Schrödinger-Klein-Gordon equations with Yukawa-like coupling is a widely studied system of non-linear PDEs in three dimension [see e.g. 18, 19, 31, 48–50, 58, 87]. This system can be written as:

\[
\begin{align*}
    i\partial_t u &= -\frac{\Lambda}{2M} u + Vu + Au \\
    (\Box + m_0^2)A &= -|u|^2
\end{align*}
\]
where $m_0, M > 0$ are real parameters and $V$ is a non-negative potential that is confining or equal to zero. Using the complex field $\alpha$ as a dynamical variable instead of $A$ (see Equation (47) of Section III), the aforementioned dynamics can be seen as a Hamilton equation generated by the following energy functional, densely defined on $L^2 \oplus L^2$:

$$E(u, \alpha) := \langle u, (-\frac{\Delta}{2M} + V)u \rangle_2 + \langle \alpha, \omega \alpha \rangle_2 + \frac{1}{16\pi^2} \int_{\mathbb{R}^6} \frac{1}{\sqrt{2\omega(k)}} (\bar{a}(k)e^{-ik \cdot x} + \alpha(k)e^{ik \cdot x}) u(x)^2 \, dx \, dk.$$  

With suitable assumptions on the external potential $V$, one proves the global existence of the associated flow $E(t)$; a detailed discussion can be found in Subsection III.3 where the precise condition on $V$ is given by Assumption (A1). So there is a Hilbert space $\mathcal{D} = Q(-\Delta + V) \oplus \mathcal{F}H^\frac{1}{2}(\mathbb{R}^3)$ densely imbedded in $L^2 \oplus L^2$ such that there exists a classical flow $E : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ that solves the Schrödinger-Klein-Gordon equation (S-KG$_\alpha[Y]$) written using the complex field $\alpha$.

A question of significant interest, both mathematically and physically, is whether it is possible to quantize the Schrödinger-Klein-Gordon dynamics with Yukawa coupling as a consistent theory that describes quantum mechanically the particle-field interaction. As mentioned previously, E. Nelson rigorously constructed a self-adjoint operator satisfying in some sense the above requirement. Afterward the model is proved to satisfy some of the main properties that are familiar in the axiomatic approach to quantum fields, see [27]. Furthermore, asymptotic completeness is established in [5]. The problem of quantization of such infinite dimensional nonlinear dynamics is related to constructive quantum field theory. The general framework is as follows.

Let $\mathcal{Z}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We define the associated symplectic structure $\Sigma(\mathcal{Z})$ as the pair $\{\mathcal{Y}, B(\cdot, \cdot)\}$ where $\mathcal{Y}$ is $\mathcal{Z}$ considered as a real Hilbert space with inner product $\langle \cdot, \cdot \rangle_\mathcal{Y} = \text{Re}(\cdot, \cdot)$, and $B(\cdot, \cdot)$ is the symplectic form defined by $B(\cdot, \cdot) = \text{Im}(\cdot, \cdot)$. Following Segal [93], we define a (bosonic) quantization of the structure $\Sigma(\mathcal{Z})$ any linear map $R(\cdot)$ from $\mathcal{Y}$ to self-adjoint operators on a complex Hilbert space such that:

* The Weyl operator $W(z) = e^{iR(z)}$ is weakly continuous when restricted to any finite dimensional subspace of $\mathcal{Y}$;

* $W(z_1)W(z_2) = e^{-\frac{1}{2}B(z_1, z_2)}W(z_1 + z_2)$ for any $z_1, z_2 \in \mathcal{Y}$ (Weyl’s relations).

When the dimension of $\mathcal{Z}$ is not finite, there are uncountably many irreducible unitarily inequivalent Segal quantizations of $\Sigma(\mathcal{Z})$ (or representation of Weyl’s relations). A representation of particular relevance in physics is the so-called Fock representation [33, 42] on the symmetric Fock space $\Gamma_s(\mathcal{Z})$. Once this representation is considered there is a natural way to

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1 Sometimes the shorthand notation $L^2 \oplus L^2$ is used instead of $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, if no confusion arises.

2 If $V = 0$, $\mathcal{D}$ could be the whole space $L^2 \oplus L^2$. 
quantize polynomial functionals on $\mathcal{Z}$ into quadratic forms on $\Gamma_s(\mathcal{Z})$ according to the Wick or normal order (we briefly outline the essential features of Wick quantization on Section V.2, the reader may refer to [8, 23, 35] for a more detailed presentation).

Following this rules, the formal quantization of the classical energy $\mathcal{E}$ yields a quadratic form $h$ on the Fock space $\Gamma_s(L^2 \oplus L^2)$ which plays the role of a quantum energy. The difficulty now lies on the fact that the quadratic form $h$ do not define straightforwardly a dynamical system (i.e. $h$ is not related to a self-adjoint operator). Nevertheless, according to the work of Nelson it is possible in our case to define for any $\sigma_0 \in \mathbb{R}_+$, a renormalized self-adjoint operator $H_{ren}(\sigma_0)$ associated in some specific sense to $h$ (see Sections II and IV for details). However, the relationship between the classical and the quantum theory at hand is obscured by the renormalization procedure and it is unclear even formally if the quantum dynamics generated by $H_{ren}(\sigma_0)$ are still related to the original Schrödinger-Klein-Gordon equation. Therefore, we believe that it is mathematically interesting to verify Bohr’s correspondence principle for this model.

**Bohr’s principle:** The quantum system should reproduce, in the limit of large quantum numbers, the classical behavior.

This principle may be reformulated as follows. We make the quantization procedure dependent on an effective parameter $\varepsilon$, that would converge to zero in the limit. The physical interpretation of $\varepsilon$ is of a quantity of the same order of magnitude as the Planck’s constant, that becomes negligible when large energies and orbits are considered. In the Fock representation, we introduce the $\varepsilon$-dependence in the annihilation and creation operator valued distributions $\psi^#(x)$ and $a^#(k)$, whose commutation relations then become $[\psi(x), \psi^*(x')] = \varepsilon \delta(x - x')$ and $[a(k), \allowbreak a^*(k')] = \varepsilon \delta(k - k')$. If in the limit $\varepsilon \to 0$ the quantum unitary dynamics converges towards the Hamiltonian flow generated by the Schrödinger-Klein-Gordon equation with Yukawa interaction, Bohr’s principle is satisfied.

If the phase space $\mathcal{Z}$ is finite dimensional, the quantum-classical correspondence has been proved in the context of semiclassical or microlocal analysis, with the aid of pseudo-differential calculus, Wigner measures or coherent states [see e.g. 4, 30, 32, 54, 55, 70, 72, 73, 78, 80, 91]. If $\mathcal{Z}$ is infinite dimensional, the situation is more complicated, and there are fewer results for systems with unconserved number of particles [7, 8, 12, 44, 45, 59]. The approach we adopt here makes use of the infinite-dimensional Wigner measures introduced by Ammari and Nier [8, 9, 10, 11]. Given a family of normal quantum states $(\rho_\varepsilon)_{\varepsilon \in (0, \bar{\varepsilon})}$ on the Fock space, we say that a Borel probability measure $\mu$ on $\mathcal{Z}$ is a Wigner measure associated to it if there exists a sequence $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \bar{\varepsilon})$ such that $\varepsilon_k \to 0$ and

\begin{equation}
\lim_{k \to \infty} \text{Tr}[\rho_{\varepsilon_k} W(\xi)] = \int_{\mathcal{Z}} e^{i \sqrt{\mathbf{N}} \text{Re}(\xi, z)} d\mu(z), \forall \xi \in \mathcal{Z}.
\end{equation}

$^3$ $W(\xi)$ is the $\varepsilon_k$-dependent Weyl operator explicitly defined by (73).
Wigner measures are related to phase-space analysis and are in general an effective tool for the study of the classical limit. We denote by $M(Q_ε, \varepsilon \in (0, \bar{\varepsilon}))$ the set of Wigner measures associated to $(Q_ε)_{ε \in (0, \bar{\varepsilon})}$. Let $e^{-i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)}$ be the quantum dynamics on $Γ_ε(\mathcal{Z})$, $\mathcal{Z} = L^2 \oplus L^2$. Then the time-evolved quantum states can be written as $(e^{-i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)}Q_εe^{i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)})_{ε \in (0, \bar{\varepsilon})}$. Bohr’s principle is satisfied if Wigner measures of time evolved quantum states are exactly the pushed forward, by the classical flow $E(t)$, of the initial Wigner measures at time $t = 0$; i.e.

\[(2) \quad M(e^{-i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)}Q_εe^{i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)}, \varepsilon \in (0, \bar{\varepsilon})) = \left\{ E(t) # µ, \ µ \in M(Q_ε, \varepsilon \in (0, \bar{\varepsilon})) \right\}. \]

To ensure that $M(Q_ε, \varepsilon \in (0, \bar{\varepsilon})) \neq \emptyset$, it is sufficient to assume that there exist $\delta > 0$ and $C > 0$ such that, for any $ε \in (0, \bar{\varepsilon})$, $\text{Tr}[Q_εN^0] < C$; where $N$ is the number operator of the Fock space $Γ_ε(\mathcal{Z})$ with $\mathcal{Z} = L^2 \oplus L^2$. Actually, we make the following more restrictive assumptions: Let $(Q_ε)_{ε \in (0, \bar{\varepsilon})}$ be a family of normal states on $Γ_ε(\mathcal{L}^2(\mathbb{R}^3) \oplus \mathcal{L}^2(\mathbb{R}^3))$, then

\[(A_0) \quad \exists \xi > 0, \ ∀ ε \in (0, \bar{\varepsilon}), \ ∀ k \in \mathbb{N}, \ \text{Tr}[Q_εN^k_1] \leq C^k; \]

\[(A_ρ) \quad \exists C > 0, \ ∀ ε \in (0, \bar{\varepsilon}), \ \text{Tr}[Q_ε(N + U_∞ H_0 U_∞)] \leq C; \]

where $N_1$ is the nucleonic number operator, $N = N_1 + N_2$ the total number operator, $H_0$ is the free Hamiltonian defined by Equation (5) and $U_∞$ is the unitary quantum dressing defined in Lemma II.3. As a matter of fact, it is possible in principle to remove Assumption $(A_0)$, but it has an important role in connection with the parameter $σ_0$ related to the renormalization procedure. This condition restricts the considered states $Q_ε$ to be at most with $[\xi/ε]$ nucleons.

We are now in a position to state precisely our result: the Bohr’s correspondence principle holds between the renormalized quantum dynamics of the Nelson model generated by $H_{\text{ren}}(σ_0)$ and the Schrödinger-Klein-Gordon classical flow generated by $Ω$. The operator $H_{\text{ren}}(σ_0)$ is constructed in Subsection II.3 according to Definition II.13. Recall that $D = Q(-\Delta + V) \oplus \mathcal{F}H^\frac{1}{2}(\mathbb{R}^3)$.

**Theorem 1.1.** Let $E : \mathbb{R} \times D → D$ be the Schrödinger-Klein-Gordon flow provided by Theorem III.14 and solving the equation (S-KG$_ε$)[Y] with a potential $V$ satisfying Assumption $(A_V)$. Let $(Q_ε)_{ε \in (0, \bar{\varepsilon})}$ be a family of normal states in $Γ_ε(\mathcal{L}^2(\mathbb{R}^3) \oplus \mathcal{L}^2(\mathbb{R}^3))$ that satisfies Assumptions $(A_0)$ and $(A_ρ)$. Then:

(i) there exists a $σ_0 \in \mathbb{R}^+$ such that the dynamics $e^{-i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)}$ is non-trivial on the states $Q_ε$.

(ii) $M(Q_ε, \varepsilon \in (0, \bar{\varepsilon})) \neq \emptyset$

(iii) for any $ε \in \mathbb{R}$,

\[(3) \quad M\left(e^{-i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)}Q_εe^{i\frac{\sigma_0}{\varepsilon}H_{\text{ren}}(\varepsilon)} , \varepsilon \in (0, \bar{\varepsilon})\right) = \left\{ E(t) # µ, \ µ \in M(Q_ε, \varepsilon \in (0, \bar{\varepsilon})) \right\}. \]

Furthermore, let $(ε_k)_{k \in \mathbb{N}} \subset (0, \bar{\varepsilon})$ be a sequence such that $\lim_{k \to \infty} ε_k = 0$ and $M(Q_{ε_k}, k \in \mathbb{N}) = \{µ\}$, i.e.: for any $ξ \in L^2 \oplus L^2$,

$$\lim_{k \to \infty} \text{Tr}[Q_{ε_k} W(ξ)] = \int_{L^2 \oplus L^2} e^{i\sqrt{2Re(ξ, ζ)}} dµ(ζ).$$
Then for any \( t \in \mathbb{R} \), \( \mathcal{M}(e^{-i \frac{i}{\hbar^2} H_{\text{ren}}(\sigma_0)} \rho e^{i \frac{i}{\hbar^2} H_{\text{ren}}(\sigma_0)}, k \in \mathbb{N}) = \{ \mathbf{E}(t) \# \mu \} \), i.e.:

\[
\lim_{k \to \infty} \text{Tr}
\left[
\rho e^{i \frac{i}{\hbar^2} H_{\text{ren}}(\sigma_0)} \rho e^{i \frac{i}{\hbar^2} H_{\text{ren}}(\sigma_0)} W(\xi)\right] = \int_{L^2 \oplus L^2} e^{i \sqrt{\frac{2}{\hbar}} \text{Re}(\xi, z)} d(\mathbf{E}(t) \# \mu)(z), \quad \forall \xi \in L^2 \oplus L^2.
\]

Remark 1.2. * The choice of \( \sigma_0 \) is related to our Definition II.13 of the renormalized dynamics and the localization of states \( (\rho_k)_{t \in (0, \bar{\varepsilon})} \) satisfying Assumption \((A_0)\) (see Lemma V.2). Actually, one can take any \( \sigma_0 \geq 2K(\mathcal{C} + 1 + \bar{\varepsilon}) \) where \( K > 0 \) is a constant given in Theorem II.10.

* We remark that every Wigner measure \( \mu \in \mathcal{M}(\rho_k, \varepsilon \in (0, \bar{\varepsilon})) \), with \((\rho_k)_{t \in (0, \bar{\varepsilon})}\) satisfying Assumption \((A_\rho)\) is a Borel probability measure on \( \mathcal{D} \) equipped with its graph norm, hence the push-forward by means of the classical flow \( \mathbf{E} \) is well defined (see Section V.3).

* Adopting a shorthand notation, the last assertion of the above theorem can be written as:

\[
\rho_{\varepsilon_k} \to \mu \iff \forall t \in \mathbb{R}, \quad e^{-i \frac{i}{\hbar^2} H_{\text{ren}}(\sigma_0)} \rho_{\varepsilon_k} e^{i \frac{i}{\hbar^2} H_{\text{ren}}(\sigma_0)} \to \mathbf{E}(t) \# \mu.
\]

3. An alternative renormalization procedure.

The quantization of physically interesting classical Hamiltonian systems with infinite degrees of freedom often leads to nowhere defined energy operators, that at best make sense only as quadratic forms. In order to cure the ultraviolet and infinite volume divergencies typical of QFT, a renormalization procedure is needed [64]. On the non-perturbative level, this has been usually done by manipulating a regularization of the quadratic form which one obtains by a formal quantization. The goal is to define another, related, form that is closed and semi-bounded when the regularization is removed: hence one gets a unique self-adjoint operator and eventually this will lead to a construction of an interacting QFT verifying some of the Wightman axioms [102]. This “Hamiltonian approach” was considered in few situations [e.g. 46, 56, 60–63, 84]. We briefly review it in the following paragraphs.

Starting with a one particle Hilbert space \( \mathcal{Z} \) that has an associated symplectic structure \( \Sigma(\mathcal{Z}) \) and a Fock representation on \( \Gamma_s(\mathcal{Z}) \), one easily shows that the free dynamic is unitarily implemented. Actually, the free classical Hamiltonian is described by a positive closed densely defined quadratic form \( \mathcal{E}_0 : D(\mathcal{E}_0) \subset \mathcal{Z} \to \mathbb{R}_+ \). In addition, the Wick quantization procedure Wick(\( \cdot \)) maps \( \mathcal{E}_0 \) into a closed, positive, and densely defined quadratic form \( h_0 = \text{Wick}(\mathcal{E}_0) \) on \( \Gamma_s(\mathcal{Z}) \) which in turn is associated to a unique self-adjoint operator \( H_0 \).

The interacting classical system is described by a “perturbation” of \( \mathcal{E}_0 \), namely \( \mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 : D(\mathcal{E}) \subset \mathcal{Z} \to \mathbb{R} \), where \( D(\mathcal{E}) \cap D(\mathcal{E}_0) \) is dense in \( \mathcal{Z} \). Usually, one follows the same rules and maps \( \mathcal{E} \) by Wick onto a quadratic form \( h = \text{Wick}(\mathcal{E}) \) on \( \Gamma_s(\mathcal{Z}) \). The difficulty is that
This is done by means of a self-adjoint operator \( h \). Therefore it is not possible, a priori, to associate a self-adjoint operator to \( h \), and hence to define a quantum dynamics.

The non-perturbative renormalization of the Hamiltonian operator is usually done as follows. A family \( (h_\sigma)_{\sigma \geq 0} \) of closed, densely defined, and semi-bounded quadratic forms is introduced, such that for any \( \sigma \geq 0 \), \( Q(h) \subset Q(h_\sigma) \), and for any \( \Psi, \Phi \in Q(h) \), \( \lim_{\sigma \to \infty} h_\sigma(\Psi, \Phi) = h(\Psi, \Phi) \). The idea is, roughly speaking, to manipulate the form \( h_\sigma \), or the associated self-adjoint operator \( H_\sigma \), in a way such that in the limit \( \sigma \to \infty \) a self-adjoint operator is obtained. This is done by means of a \( \sigma \)-dependent transformation \( U_\sigma \) on \( \Gamma_s(\mathbb{Z}) \), that is called dressing. The purpose of the dressing is two-fold: single out the counter term that is causing the unboundedness from below of \( h \), and yield the correct domain where the renormalized (dressed) operator is densely defined and (hopefully) self-adjoint. As explained in detail in Section II, for the Nelson model the dressing \( U_\sigma \) is a unitary transformation, the divergent term causing the unboundedness is a scalar function \( E_\sigma \) (self-energy), and the domain of definition of the renormalized dressed operator \( \hat{H}_{ren} \) is a subset of the free form domain \( Q(h_0) \). In fact one shows that that \( h_\sigma(\cdot, \cdot) = h_\sigma(U_\sigma \cdot, U_\sigma \cdot) - E_\sigma(\cdot, \cdot) \) is a densely defined quadratic form that is closed and bounded from below uniformly with respect to \( \sigma \), and therefore defines a unique self-adjoint operator \( \hat{H}_{ren} \) in the limit. More precisely, \( \hat{H}_{\sigma} \) converges to \( \hat{H}_{ren} \) in the norm resolvent sense, see [27].

There are more complicated situations in which the above procedure is not sufficient, and a so-called wavefunction renormalization is required. This yields a change of Hilbert space for the renormalized dynamics. The idea is that we can define a new Hilbert space \( \mathcal{R} \) as the completion of the pre-Hilbert space defined by \( \langle \hat{\Psi}, \hat{\Phi} \rangle_{\mathcal{R}} = \lim_{\sigma \to \infty} \langle U_\sigma \Psi, U_\sigma \Phi \rangle_{\Gamma_s(\mathbb{Z})} \); where \( \Psi, \Phi \in D \), the latter being dense in \( \Gamma_s(\mathbb{Z}) \), and \( \Omega \in D \) is the Fock vacuum (obviously in this case \( U_\sigma \) is not unitary). If \( U_\sigma[D] \subset D(H_\sigma) \), the renormalized Hamiltonian \( \hat{H}_{ren} \) on \( \mathcal{R} \) is defined by \( \langle \hat{\Psi}, \hat{H}_{ren} \hat{\Phi} \rangle_{\mathcal{R}} = \lim_{\sigma \to \infty} \frac{h_\sigma(U_\sigma \Psi, U_\sigma \Phi) - s_\sigma(U_\sigma \Psi, U_\sigma \Phi)}{\|U_\sigma \Omega\|_{\Gamma_s(\mathbb{Z})}} \), where \( s_\sigma(U_\sigma \Psi, U_\sigma \Phi) \) is the singular part of \( h_\sigma \). Now suppose that: to the singular part \( s_\sigma \) is associated an operator \( S_\sigma \) defined on \( D(H_\sigma) \); and for any \( \Psi \in D \), \( \|(H_\sigma - S_\sigma)U_\sigma \Psi\|_{\Gamma_s(\mathbb{Z})}/\|U_\sigma \Omega\|_{\Gamma_s(\mathbb{Z})} \) is uniformly bounded with respect to \( \sigma \). Then it follows from Riesz’ representation theorem that \( \hat{H}_{ren} \) is a densely defined (symmetric) operator on \( \mathcal{R} \). The existence of eventual self-adjoint extensions of \( \hat{H}_{ren} \) has to be then proved by other means.

In this paper, we would like to introduce a different point of view on the non-perturbative renormalization. It is founded not on manipulations at the quantized level, but on suitable transformations of the classical energy before quantization. These transformations are related to normal form techniques for Hamiltonian PDEs [67, 75]. In particular for the Nelson model, we exploit a classical counterpart of the dressing transformation that puts the S-KG equation into a normal form suitable for quantization, see Subsection III.1. We remark that we obtain the same renormalized Nelson Hamiltonian \( H_{ren} \) as the one obtained in Section II by standard techniques in [84]. The idea of using the properties of the classical field equations to
construct a quantum field theory dates to a series of works by I. E. Segal [94–97] and Streater [101]. Some years later, “Segal’s program” has been continued by Balaban [20, 21]. Local relativistic interacting quantum fields can be constructed, but not the interacting quantum dynamics. Our approach to non-perturbative renormalization by means of normal forms of Hamiltonian PDEs may be seen as a further continuation of Segal’s program, though we adopt a different point of view. While the cited works concentrated on the construction of the relativistic self-interacting fields, we focus on particle-field interacting Hamiltonians.

The procedure is described in detail in Section IV, but we will briefly outline the strategy here, for the idea is very simple. Instead of quantizing directly the energy functional $\mathcal{E}$, we make a “change of coordinates” in the classical phase space $\mathcal{Z}$. Let $D : \mathcal{Z} \to \mathcal{Z}$ be a symplectomorphism, such that there exists a dense subset $\mathcal{D} \subset D(\mathcal{E})$ of $\mathcal{Z}$ such that $D[\mathcal{D}] \subset \mathcal{D}$. Then we can calculate the energy in the new coordinates, i.e. $\mathcal{E}(D(z))$, for any $z \in \mathcal{D}$. It turns out that not only the quantization Wick($\mathcal{E} \circ D$) differs from Wick($\mathcal{E}$), but it has also different properties as a quadratic form. In particular, for a specific choice of $D$, Wick($\mathcal{E} \circ D$) is closed and bounded from below and the associated self-adjoint operator is the dressed renormalized Hamiltonian $\hat{H}_{\text{ren}}$ (while Wick($\mathcal{E}$) is not even bounded from below). It follows that, for the Nelson model, a suitable classical near identity change of coordinates is sufficient to renormalize the quantum dynamics, without any additional manipulation (only a self-energy renormalization is needed).

In addition, $D$ gives informations on the relation between quantum dressed dynamics $\hat{H}_{\text{ren}}$, and the undressed one $H_{\text{ren}}$. The canonical map $D$ can be associated to a group of symplectomorphisms $(D(\theta))_{\theta \in \mathbb{R}}$ with generator $\mathcal{D}$, by the relation $D = D(1)$. It turns out that Wick($\mathcal{D}$) is a well-defined self-adjoint operator $T_\omega$, and therefore it generates a strongly continuous one-parameter unitary group $e^{-i\theta T_\omega}$ that is the quantum analogous of $D(\theta)$. If we now denote $\hat{\mathcal{E}} = \mathcal{E} \circ D$, we define the corresponding classical evolution group by $\hat{E}(t)$; also, we denote by $E(t)$ the classical evolution group associated to $\mathcal{E}$. By definition, these groups are related by $E(t) = D(1) \circ \hat{E} \circ D(-1)$. At the quantum level, it is not possible to define directly the evolution corresponding to $E(t)$, for Wick($\mathcal{E}$) is not associated to a self-adjoint operator. Nevertheless, the right hand side of the previous relation has a quantum correspondent in $e^{-i\hat{H}_{\text{ren}} T_\omega} e^{-i\hat{H}_{\text{ren}} T_\omega} e^{-i\hat{E} T_\omega}$, and that provides an indirect definition of the undressed renormalized dynamics $H_{\text{ren}}$:

$$e^{-i\hat{H}_{\text{ren}}} := e^{-i\hat{H}_{\text{ren}} T_\omega} e^{-i\hat{E} T_\omega} e^{-i\hat{H}_{\text{ren}} T_\omega}.$$  

To sum up, we think that this new point of view has some advantages over the usual Hamiltonian approach, since the manipulations are done at the classical level where observables are commuting (in our case, an explicit near identity change of coordinates is sufficient and there is no need of introducing cut-offs, and divergent quantities like the quantum self-energy do not show up explicitly). More importantly, the approach draws the link with the topic of normal forms for Hamiltonian PDEs where substantial advances have been accomplished. Therefore, we believe that this point of view deserves further study and we hope that it will be
of interest in clarifying some of the mathematical methods and techniques of ”constructive” QFT.

II. THE QUANTUM SYSTEM: NELSON HAMILTONIAN.

In this section we define the quantum system of ”nucleons” interacting with a meson field, and review the standard renormalization procedure due to Nelson [84]. Since we are interested in the classical limit and our original and dressed Hamiltonians depend in an effective parameter $\varepsilon \in (0, \bar{\varepsilon})$, we need to derive several estimates that are uniform with respect to $\varepsilon$. So this explains why we go through the technical details related to the renormalization procedure of the Nelson model.

On $\mathcal{H} = \Gamma_s(L^2(\mathbb{R}^3)) \otimes \Gamma_s(L^2(\mathbb{R}^3))$ we define the following free Hamiltonian as a positive self-adjoint operator given by:

\[
H_0 = \int_{\mathbb{R}^3} \psi^*(x) \left(-\frac{\Delta}{2\mu} + V(x)\right)\psi(x) dx + \int_{\mathbb{R}^3} a^*(k)\omega(k)a(k) dk = d\Gamma\left(-\frac{\Delta}{2\mu} + V\right) + d\Gamma(\omega),
\]

where $V \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^+)$). We denote its domain of self-adjointness by $D(H_0)$. We denote by $d\Gamma$ the second quantization acting either on the first or second Fock space, when no confusion arises.

Now let $\chi \in C_0^\infty(\mathbb{R}^3); 0 \leq \chi \leq 1$ and $\chi \equiv 1$ if $|k| \leq 1, \chi \equiv 0$ if $|k| \geq 2$. Then, for all $\sigma > 0$ define $\chi_\sigma(k) = \chi(k/\sigma)$; it will play the role of an ultraviolet cutoff in the interaction. The Nelson Hamiltonian with cutoff has thus the form:

\[
H_\sigma = H_0 + \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \psi^*(x) \left( a^*\left(\frac{e^{ikx}}{\sqrt{2\mu}}\chi_\sigma\right) + a\left(\frac{e^{-ikx}}{\sqrt{2\mu}}\chi_\sigma\right)\right)\psi(x) dx.
\]

We will denote the interaction part by $H_\sigma = H_\sigma - H_0$.

**Remark II.1.** There is no loss of generality in the choice of $\chi$ as a radial function [see 5, Proposition 3.9].

The following proposition shows the self-adjointness of $H_\sigma$, see e.g. [7, Proposition 2.5] or [40].

**Proposition II.2.** For any $\sigma > 0$, $H_\sigma$ is essentially self-adjoint on $D(H_0) \cap C_0^\infty(N)$.

To obtain a meaningful limit when $\sigma \to \infty$, we use a dressing transformation, introduced in the physics literature by Greenberg and Schweber [68] following the work of van Hove [103, 104]. The dressing and the renormalization procedure are described in Sections II.1 and II.2 respectively. In Section II.3 we discuss a possible extension of the renormalized Hamiltonian on $\mathcal{H}_n$ to the whole Fock space $\mathcal{H}$. The extension we choose is not the only possible one, however the choice is motivated by two facts: other extensions should provide the same classical limit, and our choice $\hat{H}_{\text{ren}}(\sigma_0)$ is, in our opinion, more consistent with the quantization procedure of the classical energy functional.
1. Dressing.

The dressing transform was introduced as an alternative way of doing renormalization in the Hamiltonian formalism, and has been utilized in a rigorous fashion in various situations [see e.g. 56, 63, 71, 84]. For the Nelson Hamiltonian, it consists of a unitary transformation that singles out the singular self-energy.

From now on, let $0 < \sigma_0 < \sigma$, with $\sigma_0$ fixed. Then define:

$$g_\sigma(k) = -\frac{i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} \frac{k^2}{2M} + \omega(k) \chi_\sigma(k) - \chi_\sigma_0(k)$$;

$$E_\sigma = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\omega(k)} \frac{(\chi_\sigma - \chi_\sigma_0)^2(k)}{k^2 + \omega(k)} dk - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\chi_\sigma(k) (\chi_\sigma - \chi_\sigma_0)(k)}{\omega(k)} dk .$$

The dressing transformation is the unitary operator generated by (the dependence on $\sigma_0$ will be usually omitted):

$$T_\sigma = \int_{\mathbb{R}^3} \psi^*(x) \left( a^\dagger(g_\sigma e^{-ik\cdot x}) + a(g_\sigma e^{-ik\cdot x}) \right) \psi(x) dx .$$

The function $g_\sigma \in L^2(\mathbb{R}^3)$ for all $\sigma \leq \infty$; therefore it is possible to prove the following Lemma, e.g. utilizing the criterion of [40].

**Lemma II.3.** For any $\sigma \leq \infty$, $T_\sigma$ is essentially self-adjoint on $C^\infty_0(N)$. We denote by $U_\sigma(\theta)$ the corresponding one-parameter unitary group $U_\sigma(\theta) = e^{-i\theta T_\sigma}$.

For the sake of brevity, we will write $U_\sigma := U_\sigma(-1)$. We remark that $T_\sigma$ and $H_\sigma$ preserve the number of “nucleons”, i.e.: for any $\sigma \leq \infty$, $\sigma' < \infty$:

$$[T_\sigma, N_1] = 0 = [H_\sigma', N_1] .$$

The above operators also commute in the resolvent sense. We are now in position to define the dressed Hamiltonian

$$\hat{H}_\sigma := U_\sigma(H_\sigma - \varepsilon N_1 E_\sigma)U_\sigma^* .$$

The operator $\hat{H}_\sigma$ is self-adjoint for any $\sigma < \infty$, since $H_\sigma$ and $N_1$ are commuting self-adjoint operators and $U_\sigma$ is unitary. The purpose is to show that the quadratic form associated with $\hat{H}_\sigma|_{\mathcal{H}_0}$ satisfies the hypotheses of KLMN theorem, even when $\sigma = \infty$, so it is possible to define uniquely a self-adjoint operator $\hat{H}_\infty$. In order to do that, we need to study in detail the form associated with $\hat{H}_\sigma^{(n)}$.

By Equation (11), it follows immediately that

$$\hat{H}_\sigma^{(n)} = \varepsilon U_\sigma^{(n)} \left( \frac{H_\sigma^{(n)}}{\varepsilon} - (\varepsilon n) E_\sigma \right) (U_\sigma^{(n)})^* .$$
A suitable calculation [5, 84] yields:

\[ \hat{H}^{(n)}_\sigma = H^{(n)}_\sigma + \varepsilon^2 \sum_{i<j} V_\sigma(x_i - x_j) + \frac{\varepsilon}{2M} \sum_{j=1}^n \left( a^*(r_\sigma e^{-ik \cdot x_j})^2 + a(r_\sigma e^{-ik \cdot x_j})^2 \right) + 2a^*(r_\sigma e^{-ik \cdot x_j})a(r_\sigma e^{-ik \cdot x_j}) - 2 \left( D_{\sigma} a(r_\sigma e^{-ik \cdot x_j}) + a^*(r_\sigma e^{-ik \cdot x_j})D_{\sigma} \right) ; \]

where \( D_{\sigma} x_j = -i\nabla_{x_j} \) and

\[ r_\sigma(k) = -ik g_\sigma(k) , \]

\[ V_\sigma(x) = 2\text{Re} \int_{\mathbb{R}^3} \omega(k) |g_\sigma(k)|^2 e^{-ik \cdot x} dk - 4\text{Im} \int_{\mathbb{R}^3} \frac{\bar{g}_\sigma(k)}{2\pi} \frac{\chi_\sigma(k)}{\sqrt{2\omega(k)}} e^{-ik \cdot x} dk . \]

It is also possible to write \( \hat{H}_\sigma \) in its second quantized form as:

\[ \hat{H}_\sigma = H_0 + \hat{H}_I(\sigma) ; \]

\[ \hat{H}_I(\sigma) = H_I(\sigma_0) + \frac{1}{2} \int_{\mathbb{R}^6} \psi^*(x)\psi^*(y)V_\sigma(x - y)\psi(x)\psi(y) dx dy \]

\[ + \frac{1}{2M} \int_{\mathbb{R}^3} \psi^*(x) \left( a^*(r_\sigma e^{-ik \cdot x})^2 + a(r_\sigma e^{-ik \cdot x})^2 \right) + 2a^*(r_\sigma e^{-ik \cdot x})a(r_\sigma e^{-ik \cdot x}) - 2 \left( D_{\sigma} a(r_\sigma e^{-ik \cdot x}) + a^*(r_\sigma e^{-ik \cdot x})D_{\sigma} \right) \psi(x) dx . \]

**Remark II.4.** The dressed interaction Hamiltonian \( \hat{H}_I(\sigma) \) contains a first term analogous to the undressed interaction with cutoff, a second term of two-body interaction between nucleons, and a more singular term that can be only defined as a form when \( \sigma = \infty \).

2. **Renormalization.**

We will now define the renormalized self-adjoint operator \( \hat{H}^{(n)}_{\sigma_\infty} \). A simple calculation shows that \( E_\sigma \to -\infty \) when \( \sigma \to +\infty \); hence the subtraction of the self-energy in the definition (11) of \( \hat{H}_\sigma \) is necessary. It is actually the only renormalization necessary for this system. We prove that the quadratic form associated with \( \hat{H}^{(n)}_\sigma \) of Equation (13) has meaning for any \( \sigma \leq \infty \), and the KLMN theorem [see 89, Theorem X.17] can be applied, with a suitable choice of \( \sigma_0 \).

We start with some preparatory lemmas:

**Lemma II.5.** For any \( 0 \leq \sigma \leq \infty \), the symmetric function \( V_\sigma \) satisfies:

(i) \( V_\sigma(1 - \Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^3)) \);
(ii) \((1 - \Lambda)^{-1/2}V_{\sigma}(1 - \Lambda)^{-1/2} \in \mathcal{K}(L^2(\mathbb{R}^3))\).

In particular, \(V_{\sigma} \in L^s(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3)\), for any \(s \in [2, +\infty]\).

**Proof.** It is sufficient to show [11, Corollary D.6] that \(V_{\sigma} \in L^{3,\infty}(\mathbb{R}^3)\) (weak-\(L^p\) spaces). Write \(V_{\sigma} = V_{\sigma}^{(1)} + V_{\sigma}^{(2)}\),

\[
V_{\sigma}^{(1)}(x) = 2\text{Re} \int_{\mathbb{R}^3} \omega(k)|g_{\sigma}(k)|^2 e^{-ik \cdot x} dk = 2(2\pi)^{3/2} \text{Re} \mathcal{F}(\omega|g_{\sigma}|^2)(x);
\]

\[
V_{\sigma}^{(2)}(x) = -2 \sqrt{2} \text{Im} \int_{\mathbb{R}^3} \frac{\bar{g}_{\sigma}(k)}{\sqrt{\omega(k)}} \frac{X_{\sigma}(k)}{2\pi} e^{-ik \cdot x} dk = -2 \sqrt{2} \text{Im} \mathcal{F}(\bar{g}_{\sigma} \frac{X_{\sigma}}{\sqrt{\omega}})(x).
\]

• \([V_{\sigma}^{(1)}]\). For any \(\sigma \leq \infty, \omega|g_{\sigma}|^2 \in L^{s'}(\mathbb{R}^3), 1 \leq s' \leq 2\). Then \(V_{\sigma}^{(1)} \in L^s(\mathbb{R}^3)\) for any \(s \in [2, +\infty]\); furthermore \(V_{\sigma}^{(1)} \in C_0(\mathbb{R}^3)\) (the space of continuous functions converging to zero at infinity). Hence \(V_{\sigma}^{(1)} \in L^{3,\infty}(\mathbb{R}^3)\).

• \([V_{\sigma}^{(2)}]\). For any \(\sigma \leq \infty, \bar{g}_{\sigma} \frac{X_{\sigma}}{\sqrt{\omega}} \in L^{s'}(\mathbb{R}^3), 1 < s' \leq 2\). Therefore \(V_{\sigma}^{(2)} \in L^s(\mathbb{R}^3)\) for any \(s \in [2, +\infty]\). It remains to show that \(V_{\sigma}^{(2)} \in L^{3,\infty}(\mathbb{R}^3)\). Define \(f(k) \in L^2(\mathbb{R}^3)\):

\[
f(k) := \frac{X_{\sigma}(k)(\chi_{\sigma} - \chi_{\sigma_0})(k)}{\omega(k)} \left(\frac{k^2}{2M} + \omega(k)\right).
\]

Then there is a constant \(c > 0\) such that \(|V_{\sigma}^{(2)}(x)| \leq c|\mathcal{F}(f)(x)|\), where the Fourier transform is intended to be on \(L^2(\mathbb{R}^3)\). The function \(f\) is radial, so we introduce the spherical coordinates \((r, \theta, \phi) \equiv k \in \mathbb{R}^3\), such that the \(z\)-axis coincides with the vector \(x\). We then obtain:

\[
\lim_{R \to +\infty} \int_{B(0,R)} f(k)e^{-ik \cdot x} dk = \lim_{R \to +\infty} \int_0^R dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \ r^2 f(r)e^{-ir|x|\cos \theta} \sin \theta
\]

\[
= 2\pi \lim_{R \to +\infty} \int_0^R dr \int_{-1}^1 dy \ r^2 f(r)e^{-ir|x|y} = \frac{4\pi}{|x|} \lim_{R \to +\infty} \int_0^R f(r)r \sin(r|x|)dr.
\]

Since for any \(\sigma \leq +\infty, f(r)r \in L^1(\mathbb{R})\) we can take the limit \(R \to +\infty\) and conclude:

\[
\mathcal{F}(f)(x) = \frac{4\pi}{|x|} \int_0^{+\infty} f(r)r \sin(r|x|)dr.
\]

Therefore, for any \(x \in \mathbb{R}^3 \setminus \{0\}\), there exists a \(0 < \tilde{c} \leq 4\pi c||f(r)r||_{L^1(\mathbb{R})}\) such that:

\[
|V_{\sigma}^{(2)}(x)| \leq \frac{\tilde{c}}{|x|}.
\]

Let \(\lambda\) be the Lebesgue measure in \(\mathbb{R}^3\). Since \(\{x : |V_{\sigma}^{(2)}| > t\} \subset \{x : \frac{\tilde{c}}{|x|} > t\}\), there is a positive \(C\) such that:

\[
\lambda\{x : |V_{\sigma}^{(2)}(x)| > t\} \leq \lambda\{x : \frac{\tilde{c}}{|x|} > t\} \leq \frac{C}{t^2}.
\]

Finally (22) implies \(V_{\sigma}^{(2)} \in L^{3,\infty}(\mathbb{R}^3)\). 

\(\square\)
Lemma II.6. There exists $c > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma \leq +\infty$:  

\begin{align}
(23) & \quad \left\| (H_0 + 1)^{-1/2} D_{x_j} a(r_\sigma e^{-i k \cdot x_j}) (H_0 + 1)^{-1/2} \right\|_{L(\mathcal{H}_n)} \leq \frac{c}{\sqrt{\varepsilon}} \| \omega^{-1/2} r_\sigma \|_2 ; \\
(24) & \quad \left\| (H_0 + 1)^{-1/2} a^* (r_\sigma e^{-i k \cdot x_j}) D_{x_j} (H_0 + 1)^{-1/2} \right\|_{L(\mathcal{H}_n)} \leq \frac{c}{\sqrt{\varepsilon}} \| \omega^{-1/2} r_\sigma \|_2 .
\end{align}

Moreover, (23) holds if we replace the left $H_0$ by $d\Gamma(-\frac{A}{2M} + V)$ and the right $H_0$ by $d\Gamma(\omega)$ and similarly (24) holds if we replace the left $H_0$ by $d\Gamma(\omega)$ and the right $H_0$ by $d\Gamma(-\frac{A}{2M} + V)$.

Proof. Let $S_n \equiv S_n \otimes 1$ be the symmetrizer on $\mathcal{H}_n$ (acting only on the $\{x_1, \ldots, x_n\}$ variables) and $\Psi_n \in \mathcal{H}_n$ with $n > 0$. Then:

$$\langle \Psi_n, d\Gamma(-\Delta) \Psi_n \rangle = \langle \Psi_n, (n\varepsilon) S_n (D_{x_j})^2 \otimes 1^{n-1} \Psi_n \rangle = (n\varepsilon) \langle \Psi_n, (D_{x_j})^2 \Psi_n \rangle .$$

Hence $(n\varepsilon) \| D_{x_j} \Psi_n \|^2 \leq \| (d\Gamma(-\Delta) + 1)^{1/2} \Psi_n \|^2$. It follows that

$$\left\| D_{x_j} (d\Gamma(-\Delta) + 1)^{-1/2} \right\|_{L(\mathcal{H}_n)} \leq \frac{1}{\sqrt{n\varepsilon}} ; \left\| (d\Gamma(-\Delta) + 1)^{-1/2} D_{x_j} \right\|_{L(\mathcal{H}_n)} \leq \frac{1}{\sqrt{n\varepsilon}} .$$

Using (25) we obtain for any $\Psi_n \in \mathcal{H}_n$, with $\| \Psi_n \| = 1$:

$$\left\| (H_0 + 1)^{-1/2} D_{x_j} a(r_\sigma e^{-i k \cdot x_j}) (H_0 + 1)^{-1/2} \Psi_n \right\| \leq \frac{c}{\sqrt{n\varepsilon}} \| a(r_\sigma e^{-i k \cdot x_j}) (d\Gamma(\omega) + 1)^{-1/2} \Psi_n \| \leq \frac{c}{\sqrt{n\varepsilon}} \| \omega^{-1/2} r_\sigma \|_2 ;$$

where the last inequality follows from standard estimates on the Fock space [see 7, Lemma 2.1]. The bound (24) is obtained by adjunction. \hfill \blacksquare

Lemma II.7. There exists $c > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma \leq +\infty$:

\begin{align}
(26) & \quad \left\| (H_0 + 1)^{-1/2} a^* (r_\sigma e^{-i k \cdot x_j}) a(r_\sigma e^{-i k \cdot x_j}) (H_0 + 1)^{-1/2} \right\|_{L(\mathcal{H}_n)} \leq c \| \omega^{-1/2} r_\sigma \|^2 ; \\
(27) & \quad \left\| (H_0 + 1)^{-1/2} (a^* (r_\sigma e^{-i k \cdot x_j}))^2 (H_0 + 1)^{-1/2} \right\|_{L(\mathcal{H}_n)} \leq c \| \omega^{-1/4} r_\sigma \|^2 ; \\
(28) & \quad \left\| (H_0 + 1)^{-1/2} (a(r_\sigma e^{-i k \cdot x_j}))^2 (H_0 + 1)^{-1/2} \right\|_{L(\mathcal{H}_n)} \leq c \| \omega^{-1/4} r_\sigma \|^2 .
\end{align}

The same bounds hold if $H_0$ is replaced by $d\Gamma(\omega)$.

Proof. First of all observe that, since $m_0 > 0$, there exists $c > 0$ such that, uniformly in $\varepsilon \in (0, \bar{\varepsilon})$:

$$\left\| (H_0 + 1)^{-1/2} (d\Gamma(\omega) + 1)^{1/2} \right\|_{L(\mathcal{H})} \leq c ; \left\| (H_0 + 1)^{-1/2} (N_2 + 1)^{1/2} \right\|_{L(\mathcal{H})} \leq c .$$
Equation (26) is easy to prove:
\[
\left\| (H_0 + 1)^{-1/2} a^* (r_\sigma e^{-ik\cdot x}) a (r_\sigma e^{-ik\cdot x}) (H_0 + 1)^{-1/2} \right\|_{L(H_0)} \leq c \left\| (d\Gamma(\omega) + 1)^{-1/2} a^* (r_\sigma e^{-ik\cdot x}) a (r_\sigma e^{-ik\cdot x}) (d\Gamma(\omega) + 1)^{-1/2} \right\|_{L(H_0)} \leq c \|\omega^{-1/2} r_\sigma\|_2^2.
\]
For the proof of (27) the reader may refer to [5, Lemma 3.3 (iv)]. Finally (28) follows from (27) by adjunction. □

**Lemma II.8.** There exists \(c(\sigma_0) > 0\) such that for any \(\varepsilon \in (0, \bar{\varepsilon})\) and \(\lambda \geq 1:\)
\[
(H_0 + \lambda)^{-1/2} H_f(\sigma_0)(H_0 + \lambda)^{-1/2} \leq c(\sigma_0) \lambda^{-1/2}(n\varepsilon);
\]
\[
(H_0 + \lambda)^{-1/2} e^2 \sum_{i < j} V_\sigma(x_i - x_j)(H_0 + \lambda)^{-1/2} \leq c(\sigma_0) \lambda^{-1/2} \sqrt{n\varepsilon(1 + n\varepsilon)}.
\]

**Proof.** The inequality (29) can be proved by a standard argument on the Fock space [see e.g. 39, Proposition IV.1].

To prove (30) we proceed as follows. First of all, by means of (i), Lemma II.5 we can write:
\[
\left\| (-\Delta_n + \lambda)^{-1/2} V_\sigma(x_i - x_j)(-\Delta_n + \lambda)^{-1/2} \right\|_{L(H_0)} \leq \lambda^{-1/2} \left\| V_\sigma(x_i)(-\Delta_n + \lambda)^{-1/2} \right\|_{L(H_0)} \leq c(\sigma_0) \lambda^{-1/2}.
\]
Therefore \(V_\sigma(x_i - x_j) \leq c(\sigma_0) \lambda^{-1/2} (-\Delta_n + \lambda)\). Let \(\Psi_n \in H_0\); using its symmetry, and some algebraic manipulations we can write:
\[
\langle \Psi_n, \varepsilon^2 \sum_{i < j} V_\sigma(x_i - x_j) \Psi_n \rangle \leq c(\sigma_0) (n\varepsilon)^2 \langle \Psi_n, (\lambda^{-1/2}(D_{x_i})^2 + \lambda^{1/2}) \Psi_n \rangle
\]
\[
= c(\sigma_0) \langle \Psi_n, N_1(\lambda^{-1/2} d\Gamma(D_x^2) + \lambda^{1/2} N_1) \Psi_n \rangle
\]
\[
\leq c(\sigma_0) \lambda^{-1/2} \left[ \left\| N_1^{1/2}(d\Gamma(D_x^2) + \lambda)^{1/2} \Psi_n \right\|^2 + \left\| N_1(d\Gamma(D_x^2) + \lambda)^{1/2} \Psi_n \right\|^2 \right]
\]
\[
\leq c(\sigma_0) \lambda^{-1/2} \langle \Psi_n, (N_1 + N_1^2)(d\Gamma(D_x^2) + \lambda) \Psi_n \rangle;
\]
where the constant \(c(\sigma_0)\) is redefined in each inequality. The result follows since \(N_1\) commutes with \(d\Gamma(D_x^2)\). □

Combining Lemmas II.6, II.7 and II.8 together, we can prove easily the following proposition.

**Proposition II.9.** There exist \(c > 0\) and \(c(\sigma_0) > 0\) such that for any \(\varepsilon \in (0, \bar{\varepsilon})\), \(\lambda \geq 1\), \(\sigma_0 < \sigma \leq +\infty\) and for any \(\Psi \in D(N_1)\):
\[
\left\| (H_0 + \lambda)^{-1/2} \hat{H}_f(\sigma_0)(H_0 + \lambda)^{-1/2} \Psi \right\| \leq \left[ c(\lambda^{-1/2} r_\sigma)\right] \cdot \left( (N_1 + 1)\Psi \right) .
\]
Consider now $\hat{H}_I(\sigma)^{(n)}$. It follows easily from Equation (31) above that for any $\sigma_0 < \sigma \leq +\infty$, and $\Psi_n \in D(H_0^{1/2}) \cap \mathcal{H}_n$:

$$
\left| \langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n \rangle \right| \leq \left[ c(n\varepsilon + 1)\left( ||\omega^{-1/2}r_{\sigma}||_2^2 + ||\omega^{-1/4}r_{\sigma}||_2^2 + ||\omega^{-1/2}r_{\sigma}||_2 \right) + c(\sigma_0)(n\varepsilon + 1)\lambda^{-1/2} \right] \langle \Psi_n, H_0^{(n)}\Psi_n \rangle + c(\sigma_0)(n\varepsilon + 1)\lambda^{-1/2} \langle \Psi_n, \Psi_n \rangle.
$$

(32)

Consider now the term ($||\omega^{-1/2}r_{\sigma}||_2^2 + ||\omega^{-1/4}r_{\sigma}||_2^2 + ||\omega^{-1/2}r_{\sigma}||_2$); by definition of $r_{\sigma}$, there exists $c > 0$ such that, uniformly in $\sigma \leq +\infty$:

$$
||\omega^{-1/2}r_{\sigma}||_2^2 + ||\omega^{-1/4}r_{\sigma}||_2^2 + ||\omega^{-1/2}r_{\sigma}||_2 \leq c(\sigma_0^{-2} + \sigma_0^{-1}).
$$

(33)

Hence for any $\sigma_0 \geq 1$ there exist $K > 0$ ($K = 2c$), $c(\sigma_0) > 0$ and $C(n, \varepsilon, \lambda, \sigma_0) > 0$ such that (32) becomes:

$$
\left| \langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n \rangle \right| \leq \left[ \frac{K(n\varepsilon + 1)}{\sigma_0} + c(\sigma_0)(n\varepsilon + 1)\lambda^{-1/2} \right] \langle \Psi_n, H_0^{(n)}\Psi_n \rangle + c(\sigma_0)(n\varepsilon + 1)\lambda^{-1/2} \langle \Psi_n, \Psi_n \rangle.
$$

(34)

Therefore choosing

$$
\sigma_0 > 2K(n\varepsilon + 1)
$$

(35)

and then $\lambda > (2c(\sigma_0)(n\varepsilon + 1))^2$, we obtain the following bound for any $\Psi_n \in D(H_0^{1/2}) \cap \mathcal{H}_n$, with $a < 1$, $b > 0$ and uniformly in $\sigma_0 < \sigma \leq +\infty$:

$$
\left| \langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n \rangle \right| \leq a\langle \Psi_n, H_0^{(n)}\Psi_n \rangle + b\langle \Psi_n, \Psi_n \rangle.
$$

(36)

Applying KLMN theorem, (36) proves the following result [see e.g. 5, 84, for additional details].

**Theorem II.10.** There exists $K > 0$ such that, for any $n \in \mathbb{N}$, and $\varepsilon \in (0, \bar{\varepsilon})$ the following statements hold:

(i) For any $(2K(n\varepsilon + 1)) < \sigma_0 < \sigma \leq +\infty$, there exists a unique self-adjoint operator $\hat{H}_I^{(n)}$ with domain $\hat{D}_\sigma^{(n)} \subset D((H_0^{(n)})^{1/2}) \subset \mathcal{H}_n$ associated to the symmetric form $\hat{h}_\sigma^{(n)}(\cdot, \cdot)$, defined for any $\Psi, \Phi \in D((H_0^{(n)})^{1/2})$ as:

$$
\hat{h}_\sigma^{(n)}(\Psi, \Phi) = \langle \Psi, H_0^{(n)}\Phi \rangle + \langle \Psi, \hat{H}_I(\sigma)^{(n)}\Phi \rangle.
$$

(37)

The operator $\hat{H}_I^{(n)}$ is bounded from below, with bound $-b_{\sigma_0}(\sigma)$ (where $|b_{\sigma_0}(\sigma)|$ is a bounded increasing function of $\sigma$).
(ii) The following convergence holds in the norm topology of $L(H_n)$:

$$\lim_{\sigma \to +\infty} (z - \hat{H}_\sigma^{(n)})^{-1} = (z - \hat{H}_\infty^{(n)})^{-1}, \quad \text{for all} \quad z \in \mathbb{C} \setminus \mathbb{R}. $$

(iii) For any $t \in \mathbb{R}$, the following convergence holds in the strong topology of $L(H_n)$:

$$s - \lim_{\sigma \to +\infty} e^{-i\frac{t}{\sigma} \hat{H}_\sigma^{(n)}} = e^{-i\frac{t}{\sigma} \hat{H}_\infty^{(n)}}. $$

Remark II.11. The operator $\hat{H}_\infty^{(n)}$ can be decomposed only in the sense of forms, i.e.

$$\hat{H}_\infty^{(n)} = H_0^{(n)} + \hat{H}_I^{(n)}(\infty);$$

where $+$ has to be intended as the form sum.

3. Extension of $\hat{H}_\infty^{(n)}$ to $\mathcal{H}$.

We have defined the self-adjoint operator $\hat{H}_\infty^{(n)}$ which depends in $\sigma_0$ for each $n \in \mathbb{N}$. Now we are interested in extending it to the whole space $\mathcal{H}$. This can be done in at least two different ways, however we choose the one that is more suitable to interpret $\hat{H}_\infty$ as the Wick quantization of a classical symbol.

Let $K$ be defined by Theorem II.10. Then define $\mathcal{R}(\varepsilon, \sigma_0) \in \mathbb{N}$ by:

$$\mathcal{R}(\varepsilon, \sigma_0) = \left[ \frac{\sigma_0 - 2K}{2K \varepsilon} - 1 \right];$$

where the square brackets mean that we take the integer part if the number within is positive, zero otherwise.

**Definition II.12 ($\hat{H}_{\text{ren}}(\sigma_0)$).** Let $0 \leq \sigma_0 < +\infty$ be fixed. Then we define $\hat{H}_{\text{ren}}(\sigma_0)$ on $\mathcal{H}$ by:

$$\hat{H}_{\text{ren}}(\sigma_0) \mid_{\mathcal{H}_n} = \begin{cases} 
\hat{H}_\infty^{(n)} & \text{if } n \leq \mathcal{R}(\varepsilon, \sigma_0) \\
0 & \text{if } n > \mathcal{R}(\varepsilon, \sigma_0) 
\end{cases} $$

where $\mathcal{R}(\varepsilon, \sigma_0)$ is defined by (41). We may also write $\hat{H}_{\text{ren}}(\sigma_0) = H_0 + \hat{H}_{\text{ren},I}(\sigma_0)$ as a sum of quadratic forms.

The operator $\hat{H}_{\text{ren}}(\sigma_0)$ is self-adjoint on $\mathcal{H}$, with domain of self adjointness:

$$\hat{D}_{\text{ren}}(\sigma_0) = \left\{ \Psi \in \mathcal{H} \mid \Psi \mid_{\mathcal{H}_n} \in \hat{D}_\infty^{(n)} \text{ for any } n \leq \mathcal{R}(\varepsilon, \sigma_0) \right\}. $$

Acting with the dressing operator $U_\infty$ defined in Lemma II.3 (with the same fixed $\sigma_0$ as for $\hat{H}_{\text{ren}}(\sigma_0)$), we can also define the undressed extension $H_{\text{ren}}(\sigma_0)$. 


\textbf{Definition II.13} ($H_{\text{ren}}(\sigma_0)$). Let $0 \leq \sigma_0 < +\infty$ be fixed. Then we define the following operator on $\mathcal{H}$:

\begin{equation}
H_{\text{ren}}(\sigma_0) = U_\infty^*(\sigma_0) \hat{H}_{\text{ren}}(\sigma_0) U_\infty(\sigma_0).
\end{equation}

The operator $H_{\text{ren}}(\sigma_0)$ is self-adjoint on $\mathcal{H}$, with domain of self adjointness:

\begin{equation}
D_{\text{ren}}(\sigma_0) = \{ \Psi \in \mathcal{H} : \Psi|_{\mathcal{H}_n} \in e^{-\frac{\epsilon}{2T_\infty} D^{(n)}_\infty} \text{ for any } n \leq \mathcal{N}(\epsilon, \sigma_0) \}.
\end{equation}

\textbf{Remark II.14.} Let $\sigma_0 \geq 0$ be fixed. Then the $\hat{h}_\infty$ given by (11) defines, in the limit $\sigma \to \infty$, a symmetric quadratic form $\hat{h}_\infty$ on $D(H^{1/2}_0) \subset \mathcal{H}$. Also $\hat{H}_{\text{ren}}(\sigma_0)$ defines a quadratic form $\hat{h}_{\text{ren}}$.

We have\footnote{$\mathbb{I}_{[0,3]}(N_1)$ is the orthogonal projector on $\bigoplus_{n=0}^{\mathbb{R}} \mathcal{H}_n$.}:

\begin{equation}
\hat{h}_\infty(\mathbb{I}_{[0,3]}(N_1) \cdot, \cdot) = \hat{h}_{\text{ren}}(\mathbb{I}_{[0,3]}(N_1) \cdot, \cdot).
\end{equation}

However, we are not able to prove that there is a self-adjoint operator on $\mathcal{H}$ associated to $\hat{h}_\infty$, and it is possible that there is none.

\section{THE CLASSICAL SYSTEM: S-KG EQUATIONS.}

In this section we define the S-KG system, with initial data in a suitable dense subset of $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, that describes the classical dynamics of a particle-field interaction. Then we introduce the classical dressing transformation (viewed itself as a dynamical system), and then study the transformation it induces on the Hamiltonian functional. Finally, we discuss the global existence of unique solutions of the classical equations, both in their original and dressed form.

\textit{The Yukawa coupling:} The S-KG[$Y$] system (Schrödinger-Klein-Gordon with Yukawa interaction), or undressed classical equations, is defined by:

\begin{equation}
(S-\text{KG}[Y])
\begin{cases}
i \partial_t u = -\Delta u + Vu + Au \\
(\Box + m_0^2)A = -|u|^2
\end{cases};
\end{equation}

where $V : \mathbb{R}^3 \to \mathbb{R}$ is an external potential. If we introduce the complex field $\alpha$, defined by

\begin{align}
A(x) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{1}{\sqrt{2\omega(k)}} (\bar{\alpha}(k)e^{-ik \cdot x} + \alpha(k)e^{ik \cdot x})dk , \\
\bar{A}(x) &= -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{1}{2\omega(k)} (\alpha(k)e^{ik \cdot x} - \bar{\alpha}(k)e^{-ik \cdot x})dk ,
\end{align}

where $\mathbb{I}_{[0,3]}(N_1)$ is the orthogonal projector on $\bigoplus_{n=0}^{\mathbb{R}} \mathcal{H}_n$. 
we can rewrite \((\text{S-KG[Y]})\) as the equivalent system\(^\text{5}\):

\[
(\text{S-KG}_\omega[Y]) \quad \begin{cases}
i\partial_t u = -\frac{\Lambda}{2M} u + Vu + Au \\
i\partial_t \alpha = \omega \alpha + \frac{1}{\sqrt{2\omega}} F(|u|^2)
\end{cases}
\]

**The “dressed” coupling:** The system that arises from the dressed interaction is quite complicated. We will denote it by \(\text{S-KG[D]}\), and it has the following form\(^\text{6}\):

\[
(\text{S-KG[D]}) \quad \begin{cases}
i\partial_t u = -\frac{\Lambda}{2M} u + Vu + (W * |u|^2)u + [(\varphi * A) + (\xi * \partial_i A)]u + \sum_{i=1}^{3} [(\rho(i) * A)\partial_i u + (\zeta(i) * A)^2]u \\
\quad + (\Box + m^2_0)A = -\varphi * |u|^2 + i \sum_{i=1}^{3} \rho(i) * ([u\partial_i u] - \sqrt{2M}(\zeta(i) * \partial_i A))
\end{cases}
\]

where: \(V, W, \varphi : \mathbb{R}^3 \to \mathbb{R}\) with \(W, \varphi\) even; \(\xi : \mathbb{R}^3 \to \mathbb{C}\), even; \(\rho : (\mathbb{R}^3)^3 \to \mathbb{C}\), odd; and \(\zeta : (\mathbb{R}^3)^3 \to \mathbb{R}\), odd. Obviously also \(\text{S-KG[D]}\) can be written as an equivalent system \(\text{S-KG}_\omega[D]\), with unknowns \(u\) and \(\alpha\) (omitted here). As discussed in detail in Section III.3, with a suitable choice of \(W, \varphi, \xi, \rho\) and \(\zeta\) the global well-posedness of \(\text{S-KG[D]}\) follows directly from the global well-posedness of \(\text{S-KG[Y]}\).

1. **Dressing.**

We look for a classical correspondent of the dressing transformation \(U_\infty(\theta)\). Since \(U_\infty(\theta)\) is a one-parameter group of unitary transformations on \(\mathcal{H}\), the classical counterpart of its generator is expected to induce a non-linear evolution on the phase-space \(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\), using the quantum-classical correspondence principle for systems with infinite degrees of freedom [see e.g. 8, 57, 72]. The resulting “classical dressing” \(D_{\theta_\infty}(\theta)\) plays a crucial role in proving our results: on one hand it is necessary to link the S-KG classical dynamics with the quantum dressed one; on the other it is at the heart of the ”classical” renormalization procedure.

Let \(g \in L^2(\mathbb{R}^3)\); define the following functional \(\mathcal{D}_g : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \mathbb{R}\),

\[
\mathcal{D}_g(u, \alpha) := \int_{\mathbb{R}^6} \big(g(k)\tilde{\alpha}(k)e^{-ik \cdot x} + \tilde{g}(k)\alpha(k)e^{ik \cdot x}\big)|u(x)|^2\ dx \, dk .
\]

\(^{5}\)The two systems are equivalent since \((1 + \omega^2)\text{Re} \alpha \in L^2(\mathbb{R}^3) \iff A \in H^{s+1/2}(\mathbb{R}^3), (1 + \omega^2)\text{Im} \alpha \in L^2(\mathbb{R}^3) \iff \partial_t A \in H^{s-1/2}(\mathbb{R}^3)\). In \(\text{S-KG}_\omega[Y]\) the unknowns are \(u\) and \(\alpha\).

\(^{6}\)We denote by \(\partial_i\) the derivative with respect to the \(i\)-th component of the variable \(x \in \mathbb{R}^3\). Analogously, we denote by \(\xi^{(i)}\) the \(i\)-th component of a 3-dimensional vector \(v\).
The functional $\mathcal{D}_g$ induces the following Hamiltonian equations of motion:

\begin{equation}
\begin{aligned}
\begin{cases}
i \partial_t u = A_g u \\
i \partial_t \alpha = gF(|u|^2)
\end{cases} ;
\end{aligned}
\end{equation}

where

\begin{equation}
A_g(x) = \int_{\mathbb{R}^3} \left( g(k) \tilde{\alpha}(k) e^{-ik \cdot x} + \tilde{g}(k) \alpha(k) e^{ik \cdot x} \right) dk ,
\end{equation}

\begin{equation}
F(|u|^2)(k) = \int_{\mathbb{R}^3} e^{-ik \cdot x} |u(x)|^2 dx .
\end{equation}

Observe that for any $g \in L^2(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$, $A_g(x) \in \mathbb{R}$. This will lead to an explicit form for the solutions of the Cauchy problem related to (50). The latter can be rewritten in integral form, for any $\theta \in \mathbb{R}$:

\begin{equation}
\begin{aligned}
\begin{cases}
u_{\theta}(x) = u_0(x) \exp \left\{-i \int_0^\theta \left( (A_g)_{\tau} (x) \right) d\tau \right\} \\
\alpha_{\theta}(k) = \alpha_0(k) - ig(k) \int_0^\theta F(|u|^2)(k) d\tau
\end{cases} ;
\end{aligned}
\end{equation}

where $(A_g)_\theta$ is defined by (51) with $\alpha$ replaced by $\alpha_\tau$; analogously we define $B_g$ by (51) with $\alpha$ replaced by $\beta$.

**Lemma III.1.** Let $s \geq 0$, $s - \frac{1}{2} < \zeta \leq s + \frac{1}{2}$, $(1 + \omega^\zeta) g \in L^2(\mathbb{R}^3)$. Also, let $u, v \in H^s(\mathbb{R}^3)$ and $(1 + \omega^\zeta) \alpha, (1 + \omega^\zeta) \beta \in L^2(\mathbb{R}^3)$. Then there exist constants $C_s, C_\zeta > 0$ such that:

\begin{equation}
|| (A_g - B_g) u ||_{H^s} \leq C_s \max_{w \in [u,v]} ||w||_{H^s} ||(1 + \omega^\frac{1}{2})g||_2 ||(1 + \omega^\zeta)(\alpha - \beta)||_2 ,
\end{equation}

\begin{equation}
|| A_g (u - v) ||_{H^s} \leq C_s \max_{\zeta \in [\alpha, \beta]} ||(1 + \omega^\zeta)\zeta||_2 ||(1 + \omega^\frac{1}{2})g||_2 ||u - v||_{H^s} ,
\end{equation}

\begin{equation}
\left\| (1 + \omega^\zeta) g \int_{\mathbb{R}^3} e^{-ik \cdot x} ((u - v)\tilde{v} + (\tilde{u} - v)u) dx \right\|_2 \leq C_\zeta \max_{w \in [u,v]} ||w||_{H^s} ||(1 + \omega^\frac{1}{2})g||_2 ||u - v||_{H^s} .
\end{equation}

**Proof.** If $s \in \mathbb{N}$, the results follow by standard estimates, keeping in mind that $|k| \leq \omega(k) \leq |k| + m_0$. The bounds for non-integer $s$ are then obtained by interpolation. ■

**Proposition III.2.** Let $\theta \in \mathbb{R}$, $(u_0, \alpha_0) \in L^2 \oplus L^2$. If $(u_\theta, \alpha_\theta) \in C^0(\mathbb{R}, L^2 \oplus L^2)$ is a solution of (53), then it is unique, i.e. any $(v_\theta, \beta_\theta) \in C^0(\mathbb{R}, L^2 \oplus L^2)$ that satisfies (53) is such that $(v_\theta, \beta_\theta) = (u_\theta, \alpha_\theta)$.

**Proof.** We have:

\begin{align*}
\frac{i}{2} \partial_\theta \left( ||u_\theta - v_\theta||^2_2 + ||\alpha_\theta - \beta_\theta||^2_2 \right) = \text{Im} \left( \left( u_\theta - v_\theta, ((A_g)_\theta - (B_g)_\theta) u_\theta + (B_g)_\theta (u_\theta - v_\theta) \right) \right) \\
+ \left( \alpha_\theta - \beta_\theta, g \int_{\mathbb{R}^3} e^{-ik \cdot x} ((u_\theta - v_\theta)\tilde{v} + (\tilde{u} - \tilde{v})u) dx \right) .
\end{align*}

The result hence is an application of the estimates of Lemma III.1 with $s = 0$ and Gronwall’s Lemma. ■
Now that we are assured that the solution of (53) is unique, we can construct it explicitly. Since \( A_g(x) \) is real, it follows that for any \( \theta \in \mathbb{R} \): \(|u_\theta| = |u_0|\). Therefore \( F(|u_\theta|^2) = F(|u_0|^2) \), and
\[
\alpha_\theta(k) = \alpha_0(k) - i\theta g(k)F(|u_0|^2)(k) \, .
\]

Substituting this explicit form in the expression for \( u_\theta \), we obtain the solution for any \((u_0, \alpha_0) \equiv (u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\):
\[
\begin{align*}
(u_\theta(x) & = u(x) \exp \left\{ -i\theta A_g(x) + i\theta^2 \text{Im} \int_{\mathbb{R}^3} F(|u|^2)(k)g(k)^2e^{ikx} dk \right\} \\
\alpha_\theta(k) & = \alpha(k) - i\theta g(k)F(|u|^2)(k) \, .
\end{align*}
\]

This system of equations defines a non-linear symplectomorphism: the “classical dressing map” on \( L^2 \oplus L^2 \).

**Definition III.3.** Let \( g \in L^2(\mathbb{R}^3) \). Then \( D_g(\cdot) : \mathbb{R} \times (L^2 \oplus L^2) \rightarrow L^2 \oplus L^2 \) is defined by (57) as:
\[
D_g(\theta)(u, \alpha) = (u_\theta, \alpha_\theta) \, .
\]

The map \( D_g(\cdot) \) is the Hamiltonian flow generated by \( \mathcal{D}_g \).

Using the explicit form (57) and Lemma III.1, it is straightforward to prove some interesting properties of the classical dressing map. The results are formulated in the following proposition, after the definition of useful classes of subspaces of \( L^2 \oplus L^2 \).

**Definition III.4.** Let \( s \geq 0, s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2} \). We define the spaces \( H^s(\mathbb{R}^3) \oplus \mathcal{F} H^s(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \):
\[
H^s(\mathbb{R}^3) \oplus \mathcal{F} H^s(\mathbb{R}^3) = \left\{ (u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3), u \in H^s(\mathbb{R}^3) \text{ and } \mathcal{F}^{-1}(\alpha) \in H^s(\mathbb{R}^3) \right\} .
\]

**Proposition III.5.** Let \( s \geq 0, s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2} \); and \( g \in \mathcal{F} H^s(\mathbb{R}^3) \). Then
\[
D_g : \mathbb{R} \times (H^s \oplus \mathcal{F} H^s) \rightarrow H^s \oplus \mathcal{F} H^s \text{ ;}
\]
i.e. the flow preserves the spaces \( H^s \oplus \mathcal{F} H^s \). Furthermore, it is a bijection with inverse \((D_g(\theta))^{-1} = D_g(-\theta)\). Hence the classical dressing is an Hamiltonian flow on \( H^s \oplus \mathcal{F} H^s \).

**Corollary III.6.** Let \( s \geq 0, s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2} \), \( \theta \in \mathbb{R} \), and \( g \in \mathcal{F} H^s(\mathbb{R}^3) \). Then there exists a constant \( C(g, \theta) > 0 \) and an \( \lambda(s) \in \mathbb{N}^+ \) such that for any \((u, \alpha) \in H^s \oplus \mathcal{F} H^s\):
\[
\|D_g(\theta)(u, \alpha)\|_{H^s \oplus \mathcal{F} H^s} \leq C(g, \theta)\|u, \alpha\|^{\lambda(s)}_{H^s \oplus \mathcal{F} H^s} .
\]

Using the positivity of both \(-\Delta\) and \( V \), and Corollary III.6 one also obtains the following result.

**Corollary III.7.** Let \( V \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}_+) \); and let \( Q(-\Delta + V) \subseteq L^2(\mathbb{R}^3) \) be the form domain of \(-\Delta + V\). Then for any \( \frac{1}{2} \leq \varsigma \leq \frac{3}{2} \), and \( g \in \mathcal{F} H^s(\mathbb{R}^3) \):
\[
D_g : \mathbb{R} \times (Q(-\Delta + V) \oplus \mathcal{F} H^s) \rightarrow Q(-\Delta + V) \oplus \mathcal{F} H^s .
\]
2. Classical Hamiltonians.

In this section we define the classical Hamiltonian functionals that generate the undressed and dressed dynamics on \( L^2 \oplus L^2 \). Then we show that they are related by a suitable classical dressing: the quantum procedure described in Section II.2 is reproduced, in simplified terms, on the classical level.

**Definition III.8 (\( \mathcal{E}, \hat{\mathcal{E}} \)).** The undressed Hamiltonian (or energy) \( \mathcal{E} \) is defined as the following real functional on \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \):

\[
\mathcal{E}(u, \alpha) := \left\langle u, \left(-\frac{\Delta}{2M} + V\right)u\right\rangle + \left\langle \alpha, \omega \alpha \right\rangle + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \frac{1}{\sqrt{2\omega(k)}} \left( \tilde{\alpha}(k) e^{-ik \cdot x} + \alpha(k) e^{ik \cdot x} \right) \left| u(x) \right|^2 \, dx \, dk .
\]

We denote by \( \mathcal{E}_0 \) the free part of the classical energy, namely

\[
\mathcal{E}_0(u, \alpha) := \left\langle u, \left(-\frac{\Delta}{2M} + V\right)u\right\rangle + \left\langle \alpha, \omega \alpha \right\rangle .
\]

Let \( \chi_{\pi} \in L^\infty(\mathbb{R}^3) \cap \mathcal{F} H^{-1/2}(\mathbb{R}^3) \) such that \( \chi_{\pi} (k) = \chi_{\pi} (-k) \) for any \( k \in \mathbb{R}^3 \). Then (again as a real functional on \( L^2 \oplus L^2 \)) the dressed Hamiltonian \( \hat{\mathcal{E}} \) is defined as \(^7\):

\[
\hat{\mathcal{E}}(u, \alpha) := \left\langle u, \left(-\frac{\Delta}{2M} + V\right)u\right\rangle + \left\langle \alpha, \omega \alpha \right\rangle + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \frac{\chi_{\pi}(k)}{\sqrt{2\omega(k)}} \left( \tilde{\alpha}(k) e^{-ik \cdot x} + \alpha(k) e^{ik \cdot x} \right) \left| u(x) \right|^2 \, dx \, dk \ldots
\]

\[ + \frac{1}{2M} \int_{\mathbb{R}^6} \left( r_{\omega}(k) \tilde{\alpha}(k) e^{-ik \cdot x} + \tilde{r}_{\omega}(k) \alpha(k) e^{ik \cdot x} \right) \left( r_{\omega}(l) \tilde{\alpha}(l) e^{-il \cdot x} + \tilde{r}_{\omega}(l) \alpha(l) e^{il \cdot x} \right) \left| u(x) \right|^2 \, dx \, dl \]

\[ - \frac{2}{M} \text{Re} \int_{\mathbb{R}^6} r_{\omega}(k) \tilde{\alpha}(k) e^{-ik \cdot x} \tilde{\alpha}(x) D_x u(x) \, dx \, dk \ldots + \frac{1}{2} \int_{\mathbb{R}^6} V_{\omega}(x-y) \left| u(x) \right|^2 \left| u(y) \right|^2 \, dx \, dy .
\]

**Remark III.9.** We denote by \( D(\mathcal{E}) \subset L^2 \oplus L^2 \) the domain of definition of \( \mathcal{E} \), and by \( D(\hat{\mathcal{E}}) \subset L^2 \oplus L^2 \) the domain of definition of \( \hat{\mathcal{E}} \). We have that \( D(\mathcal{E}) \supset C_0^\infty \subset C_0^\infty \) and \( D(\hat{\mathcal{E}}) \supset C_0^\infty \subset C_0^\infty \). Therefore both \( \mathcal{E} \) and \( \hat{\mathcal{E}} \) are densely defined, and \( D(\mathcal{E}) \cap D(\hat{\mathcal{E}}) \) is dense in \( L^2 \oplus L^2 \).

We are interested in the action of \( \mathcal{E} \) and \( \hat{\mathcal{E}} \) on \( H^1 \cap \mathcal{F} H^{1/2} \), since this emerges naturally as the energy space of the system, at least when \( V = 0 \).

**Lemma III.10.** Let \( \theta \in \mathbb{R} \), \( g \in \mathcal{F} H^{1/2}(\mathbb{R}^3) \). Then for any \( u \in Q(V) \cap H^1(\mathbb{R}^3) \), and \( \alpha \in \mathcal{F} H^{1/2}(\mathbb{R}^3) \): \( D_\theta(\theta)(u, \alpha) \subset D(\mathcal{E}) \).

**Proof.** Let \( u \in Q(V) \), and \( \alpha \in L^2(\mathbb{R}^3) \). Then

\[
\langle u_\theta, Vu_\theta \rangle_2 = \langle u, Vu \rangle_2 ;
\]

We recall that: \( g_{\omega}(k) = -i (2\pi)^{3/2} 1 - \chi_{\pi}(k) \sum_{\mathbb{R}^3} \omega(k) g_{\omega}(k)^2 e^{-ik \cdot x} dk - 4i \mathfrak{m} \int_{\mathbb{R}^3} \frac{\bar{g}_{\omega}(k)}{(2\pi)^{3/2} \sqrt{2\omega(k)}} e^{-ik \cdot x} dk \).

Also, \( D_x = -i \nabla_x ; r_{\omega}(k) = -ikg_{\omega}(k) \).
where \( u_0 \) is defined in Equation (57), and it is the first component of \( D_g(\theta)(u, \alpha) \). Also, for any \((u, \alpha) \in H^1 \oplus \mathcal{F}H^{\frac{1}{2}}\) we have that:

\[
\left| \int_{\mathbb{R}^6} |u(x)|^2 \frac{1}{\sqrt{\omega(k)}} \alpha(k) e^{i k \cdot x} \, dx \right| = C \left| \int_{\mathbb{R}^3} \frac{1}{|k|\omega(k)} \left( \omega^{1/2} \alpha(k) \left( \int_{\mathbb{R}^3} |D_\lambda u(x)|^2 e^{i k \cdot x} \, dx \right) \left( \int_{\mathbb{R}^3} |D_\lambda u(x)|^2 \, dx \right) \right) \right| 
\leq 2C \left\| \frac{1}{|k|\omega(k)} \right\|_2 \left[ \|\omega^{1/2} \alpha\|_2 \|u\|_H^1 \right] < +\infty.
\]

The result then follows since \( D_g(\theta) \) maps \( H^1 \oplus \mathcal{F}H^{\frac{1}{2}} \) into itself by Proposition III.5.  

The functional \( \mathcal{E} \) is independent of \( g_\infty \), while \( \hat{\mathcal{E}} \) depends on it. In addition, we know that \( g_\infty \) has been fixed, at the quantum level, to renormalize the Nelson Hamiltonian, and it is the function that appears in the generator of the dressing transformation \( U_\infty \). Hence, since we are establishing a correspondence between the classical and quantum theories, we expect it to be the function that appears in the classical dressing too. Two features of \( g_\infty \) are very important in the classical setting: the first is that \( g_\infty \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3) \) for any \( \chi_{\sigma_0} \in L^{\infty} \cap \mathcal{F}H^{-\frac{1}{2}} \); the second is that it is an even function, i.e. \( g_\infty(k) = g_\infty(-k) \) for any \( k \in \mathbb{R}^3 \). Using the first fact, one shows that \( D_{g_\infty}(\cdot) \) maps the energy space into itself (and that will be convenient when discussing global solutions); using the second property we can simplify the explicit form of \( D_{g_\infty}(\cdot) \).

**Lemma III.11.** Let \( \theta \in \mathbb{R} \), and \( g \in L^2(\mathbb{R}^3) \). If \( g \) is an even or odd function, then the map \( D_g(\theta) \) defined by (57) becomes:

\[
(59) \quad D_g(\theta)(u(x), \alpha(k)) = \left( u(x)e^{-i\theta\sigma_3(x)} , \alpha(k) - i\theta g(k)F(|u|^2)(k) \right).
\]

**Proof.** Consider \( I(x) := \int_{\mathbb{R}^3} F(|u|^2)(k)|g(k)|^2 e^{i k \cdot x} \, dk \). We will show that \( \tilde{I}(x) = I(x) \). We have that:

\[
\tilde{I}(x) = \int_{\mathbb{R}^6} |u(x')|^2 |g(k)|^2 e^{-i k \cdot (x-x')} \, dx' \, dk = \int_{\mathbb{R}^6} |u(x')|^2 |g(-k)|^2 e^{i k \cdot (x-x')} \, dx' \, dk.
\]

Now if \( g \) is either even or odd, \( |g(-k)| = |g(k)| \). Hence \( \tilde{I}(x) = I(x) \), therefore \( \text{Im} I(x) = 0 \).  

We conclude this section proving its main result: \( \mathcal{E} \) and \( \hat{\mathcal{E}} \) are related by the \( D_{g_\infty}(1) \) classical dressing\(^8\).

**Proposition III.12.** For any \( u \in Q(V) \cap H^1(\mathbb{R}^3), \alpha \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3) \), and for any \( \chi_{\sigma_0} \in L^{\infty}(\mathbb{R}^3) \cap \mathcal{F}H^{-\frac{1}{2}}(\mathbb{R}^3) \):

\[
(1) \quad (u, \alpha) \in D(\mathcal{E});
\]

\(^8\) We recall again that \( g_\infty = -i(2\pi)^{-3/2} \chi_{\sigma_0}(k) \chi_{\sigma_0}(k) \chi_{\sigma_0}(k) \).
(2) \((u, \alpha) \in D(\hat{\mathcal{E}})\);

(3) \(\hat{\mathcal{E}}(u, \alpha) = \mathcal{E} \circ D_{g_\alpha}(1)(u, \alpha)\).

Remark III.13. Relation (3) of Proposition III.12 actually holds for any \((u, \alpha) \in D_{g_\alpha}(-1)D(\hat{\mathcal{E}})\).

Proof of Proposition III.12. The statement (1) is just an application of Lemma III.10 when \(\theta = 0\). If (3) holds formally, than (2) follows directly, since by Lemma III.10 the right hand side of (3) is well defined. It remains to prove that the relation (3) holds formally. This is done by means of a direct calculation, that we will briefly outline here.

\[
\mathcal{E} \circ D_{g_\alpha}(1)(u, \alpha) = \left\langle u e^{-iA_{g_\alpha}}, \frac{D_x}{2M}D_x(u e^{-iA_{g_\alpha}}) \right\rangle_2 + \left\langle u, Vu \right\rangle_2 + \left\langle \alpha, \omega \alpha \right\rangle_2
\]

(a) \(+ 2\text{Im}\left\langle \alpha, \omega g_\alpha F_u \right\rangle_2 + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \frac{1}{\sqrt{2\omega(k)}} \bar{\alpha}(k)e^{-ik \cdot x}|u(x)|^2 \text{d}x\text{d}k \)

(b) \(+ \|\omega g_\alpha F_u\|_2^2 + \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \frac{1}{\sqrt{2\omega(k)}} g_\alpha(k)F_u(k)e^{ik \cdot x}|u(x)|^2 \text{d}x\text{d}k \).

After some manipulation, taking care of the ordering, the first term on the right hand side becomes:

\[
\left\langle u e^{-iA_{g_\alpha}}, \frac{D_x}{2M}D_x(u e^{-iA_{g_\alpha}}) \right\rangle_2 = \left\langle u, -\frac{\Delta}{2M}u \right\rangle_2
\]

(c) \(+ \frac{1}{2M} \left\langle A_{g_\alpha}u, A_{g_\alpha}u \right\rangle_2 \)

(d) \(- i(u, A_{g_\alpha}g_\alpha u)_2 \)

(e) \(- \frac{1}{M} \left\langle u, \int_{\mathbb{R}^3} dk(D_x\bar{r}_\alpha(k)\alpha(k)e^{ik \cdot x} + r_\alpha(k)\bar{\alpha}(k)e^{-ik \cdot x}D_x)u \right\rangle_2 \).

The proof is concluded making the following identifications (the other terms sum to the free part):

(a) \(+ (d) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \frac{\chi_{\sigma_0}}{\sqrt{2\omega(k)}} \left(\bar{\alpha}(k)e^{-ik \cdot x} + \alpha(k)e^{ik \cdot x}\right)|u(x)|^2 \text{d}x\text{d}k \)

(b) \(= \frac{1}{2} \int_{\mathbb{R}^6} V_\alpha(x-y)|u(x)|^2|u(y)|^2 \text{d}x\text{d}y \)

(c) \(= \frac{1}{2M} \int_{\mathbb{R}^3} \left(r_\alpha(k)\bar{\alpha}(k)e^{-ik \cdot x} + \bar{r}_\alpha(k)\alpha(k)e^{ik \cdot x}\right)\left(r_\alpha(l)\bar{\alpha}(l)e^{-il \cdot x} + \bar{r}_\alpha(l)\alpha(l)e^{il \cdot x}\right)|u(x)|^2 \text{d}x\text{d}k\text{d}l \)

(e) \(- \frac{2}{M} \text{Re} \int_{\mathbb{R}^6} r_\alpha(k)\bar{\alpha}(k)e^{-ik \cdot x}\bar{u}(x)D_xu(x) \text{d}x\text{d}k \).

\[\blacksquare\]
3. Global existence results.

In this section we discuss uniqueness and global existence of the classical dynamical system: using a well-known result on the undressed dynamics, we prove uniqueness and existence also for the dressed system.

The Cauchy problem associated to $\partial_t^\alpha$ by the Hamilton’s equations is\(^9\) (S-KG\(_a\)[Y]). Theorem III.14 below is a straightforward extension of [31, 87] that includes a (confining) potential on the NLS equation. As proved in [28, 85], the quadratic potential is the maximum we can afford to still have Strichartz estimates and global existence in the energy space. Therefore we make the following standard assumption on $V$:

**Assumption** $A_V$. $V \in C^\infty(\mathbb{R}^3, \mathbb{R}_+)$, and $\partial^\alpha V \in L^\infty(\mathbb{R}^3)$ for any $\alpha \in \mathbb{N}^3$, with $|\alpha| \geq 2$ (i.e. at most quadratic positive confining potential).

**Theorem III.14** (Undressed global existence). Assume $A_V$. Then there is a unique Hamiltonian flow solving (S-KG\(_a\)[Y]):

$$E : \mathbb{R} \times (Q(-\Delta + V) \oplus F H^{\frac{1}{2}}(\mathbb{R}^3)) \to Q(-\Delta + V) \oplus F H^{\frac{1}{2}}(\mathbb{R}^3).$$

If $V = 0$, then there is a unique Hamiltonian flow

$$E : \mathbb{R} \times (H^s(\mathbb{R}^3) \oplus F H^s(\mathbb{R}^3)) \to H^s(\mathbb{R}^3) \oplus F H^s(\mathbb{R}^3).$$

for any $0 \leq s \leq 1$, $s - \frac{1}{2} \leq \zeta \leq s + \frac{1}{2}$.

**Theorem III.15** (Dressed global existence). Assume $A_V$. Then for any $\chi_{\sigma_0} \in L^\infty(\mathbb{R}^3) \cap F H^{-\frac{1}{2}}(\mathbb{R}^3)$, there is a unique Hamiltonian flow:

$$\hat{E} : \mathbb{R} \times (Q(-\Delta + V) \oplus F H^{\frac{1}{2}}(\mathbb{R}^3)) \to Q(-\Delta + V) \oplus F H^{\frac{1}{2}}(\mathbb{R}^3).$$

If $V = 0$, then there is a unique Hamiltonian flow

$$\hat{E} : \mathbb{R} \times (H^s(\mathbb{R}^3) \oplus F H^s(\mathbb{R}^3)) \to H^s(\mathbb{R}^3) \oplus F H^s(\mathbb{R}^3).$$

for any $0 \leq s \leq 1$, $s - \frac{1}{2} \leq \zeta \leq s + \frac{1}{2}$. For any $V$ that satisfies $A_V$, the flows $\hat{E}$ and $E$ are related by:

$$\hat{E} = D_{g_0}(-1) \circ E \circ D_{g_0}(1), \quad E = D_{g_0}(1) \circ \hat{E} \circ D_{g_0}(-1).$$

**Proof of Theorem III.15.** The theorem is a direct consequence of the global well-posedness result of Theorem III.14, the relation $\hat{E} = \partial_t^\alpha \circ D_{g_0}(1)$ proved in Proposition III.12, and the regularity properties of the dressing proved in Proposition III.5. \(\blacksquare\)

---

\(^9\) The Cauchy problem associated to $\partial_t^\alpha$ is equivalent to (S-KG[D]), setting: $W = V_{\sigma_0}$, $\varphi = (2\pi)^{-3/2} \chi_{\sigma_0}$, $\xi = \frac{(2\pi)^{3/2}}{\sqrt{m}}(F(\frac{k^2}{\chi_0}g_{\sigma_0}) - F(i \frac{k^2}{\chi_0} g_{\sigma_0}))$, $\rho = \frac{\sqrt{m}}{\sqrt{\pi}} F(\sqrt{\omega} g_{\sigma_0})$, and $\zeta = \frac{i}{\sqrt{m}} F(\frac{\vec{k}}{\chi_0} g_{\sigma_0})$. 

IV. CLASSICAL RENORMALIZATION OF THE NELSON MODEL.

In this section we would like to describe in some detail the classical approach to renormalization outlined in Section I.3, for the quantization of the S-KG energy functional \( \mathcal{E} \) of Definition III.8.

The Wick quantization of \( \mathcal{E} \)—defined on \( L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \)—yields a formal operator on \( \Gamma_s(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)) \):

\[
(\mathcal{E})^{\text{Wick}} = \int_{\mathbb{R}^3} \psi^*(x) \left( -\frac{\Delta}{2M} + V(x) \right) \psi(x) \, dx + \int_{\mathbb{R}^3} a^*(k) \omega(k) a(k) \, dk
\]

\[
+ \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \psi^*(x) \frac{1}{\sqrt{2\omega(k)}} (a(k)e^{ikx} + a^*(k)e^{-ikx}) \psi(x) \, dx \, dk.
\]

Such quantity \((\mathcal{E})^{\text{Wick}}\) makes sense only as a densely defined quadratic form

\[
h(\cdot, \cdot) = \text{Wick}(\mathcal{E})(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{(\mathcal{E})^{\text{Wick}}},
\]

because the last term of the right hand side of (65) creates, roughly speaking, a Klein-Gordon particle with a wavefunction that is not square-integrable. In addition, the quadratic form \(h(\cdot, \cdot)\) is not bounded from below. We should therefore define a suitable transformation (the classical dressing), to modify the interaction part of \( \mathcal{E} \) in the spirit of normal forms. It turns out that a near identity change of coordinates in the phase space is sufficient (more precisely, a non-linear canonical transformation). For any function \( \chi_{\sigma_0} : \mathbb{R}^3 \rightarrow \mathbb{C} \) we can define \( D_{\chi_{\sigma_0}} := D_{g_{\sigma_0}}(1) \), where \( D_{g_{\sigma_0}}(1) \) is given by (59) with

\[
g_{\sigma_0}(k) = -i \frac{(2\pi)^{-3/2} 1 - \chi_{\sigma_0}(k)}{\sqrt{2\omega(k)}} \frac{k^2}{M} + \omega(k).
\]

The map \( D_{\chi_{\sigma_0}} \) is a candidate to be the classical dressing, since it cancels the singular term in \( \mathcal{E} \). As we will see, the choice of \( \chi_{\sigma_0} \) modifies the self-adjoint Hamiltonian that results from Wick quantization. The properties of \( D_{\chi_{\sigma_0}} \) have been discussed in Sections III.1 and III.2. Since it is suitable that \( D_{\chi_{\sigma_0}} \) map the domain of definition of \( \mathcal{E} \) into itself, we restrict \( \chi_{\sigma_0} \) to be in \( L^\infty(\mathbb{R}^3) \cap \mathcal{F}H^{-\frac{1}{2}}(\mathbb{R}^3) \).

Now, we exploit the \( U(1) \) symmetry of \( \mathcal{E} \) with respect to the first variable. This symmetry leads to the conservation of the “mass” of \( u \), i.e. of \( \|u\|^2 \). The \( U(1) \) symmetry of \( \mathcal{E} \) induces a conservation also at the quantum level: the Wick quantization of \( \mathcal{E} \) “commutes” with the Wick quantization of \( \|u\|^2 \), and the latter is the self-adjoint number operator \( N_1 \otimes 1 \) for the nucleons. More precisely, we have that for any \( \Psi, \Phi \in \mathcal{Q}(h) \), for any \( n_1 \leq n_2 \in \mathbb{N} \):

\[
h(\Psi, 1_{[n_1,n_2]}(N_1)\Phi) = h(1_{[n_1,n_2]}(N_1)\Psi, \Phi) = h(1_{[n_1,n_2]}(N_1)\Psi, 1_{[n_1,n_2]}(N_1)\Phi);
\]

where the quadratic form \( h \) is defined by (66), and \( 1_{[0,n]}(N_1) \) is a spectral projection of \( N_1 \otimes 1 \). Let \( \mathcal{H} = \Gamma_s(L^2 \otimes L^2) \), it is possible to rewrite \( \mathcal{H} = \bigoplus \mathcal{H}_n \) with \( \mathcal{H}_n = (L^2(\mathbb{R}^3))^{\otimes n} \otimes \)
\( \Gamma_t(L^2(\mathbb{R}^3)) \). Equation (67) implies that \( h \) factorizes into a direct sum of forms on each \( \mathcal{H}_n \), i.e. for any \( n \in \mathbb{N} \), there exists a quadratic form \( h_n \) densely defined on \( \mathcal{H}_n \) such that \( h = \bigoplus_n h_n \). Since \( D_{\chi_{\sigma_0}} \) preserves the \( U(1) \) invariance\(^{10} \), also the form \( \hat{h}(\chi_{\sigma_0}) = \text{Wick}(\mathcal{E} \circ D_{\chi_{\sigma_0}}) \) factorizes as a direct sum of forms \( \hat{h}_n(\chi_{\sigma_0}) \) on \( \mathcal{H}_n \). However, even if we may hope\(^{11} \) that each \( \hat{h}_n(\chi_{\sigma_0}) \) is bounded from below, this cannot be expected for \( \hat{h}(\chi_{\sigma_0}) \) due to asymptotic instability of bosonic systems.

Fix \( n \in \mathbb{N} \); we can prove that, for a suitable choice of \( \chi_{\sigma_0} \), \( \hat{h}_n(\chi_{\sigma_0}) \) is closed and bounded from below by the KLMN theorem (it is exactly the quadratic form associated to \( \hat{h}_n(\chi_{\sigma_0}) \), and the bounds are proved in Section II.2) The usual choice of the function \( \chi_{\sigma_0} \) is described at the beginning of Section II (it is, roughly speaking, a smooth characteristic function of \( 0 \leq |k| \leq \sigma_0 \)). We need also \( 2K(1 + n\varepsilon) < \sigma_0 < \infty \), where \( K \) is defined in Theorem II.10, to apply the KLMN theorem. The choice of \( \sigma_0 \) depends on \( n \), and thus we cannot define a unique self-adjoint operator for any \( n \in \mathbb{N} \). Nevertheless, for any \( \sigma_0 \in \mathbb{R}^+ \), we can define a unique self-adjoint operator \( \hat{H}_{\text{ren}}(\sigma_0) \) on \( \Gamma_t(L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)) \) by:

\[
\hat{H}_{\text{ren}}(\sigma_0) = \begin{cases} \text{Wick}(\mathcal{E} \circ D_{\chi_{\sigma_0}})|_{\mathcal{H}_n} & \text{if } n \leq \mathcal{R} \\ 0 & \text{if } n > \mathcal{R} \end{cases},
\]

where \( \mathcal{R} \) is defined by (41), and \( \text{Wick}(\mathcal{E} \circ D_{\chi_{\sigma_0}})|_{\mathcal{H}_n} \) is the unique self-adjoint operator defined by \( \hat{h}_n \).

It remains to consider the quantization, if any, of \( D_{\chi_{\sigma_0}} \). The classical dressing is the Hamiltonian flow, at time 1, generated by \( \mathcal{D}_{g_{\sigma_0}} \). As stated in Lemma II.3, \( (\mathcal{D}_{g_{\sigma_0}})^{\text{Wick}} \) is a densely defined self-adjoint operator on \( \Gamma_t(L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)) \). Therefore the quantization of \( D_{\chi_{\sigma_0}} \) would be the unitary operator \( e^{-\frac{i}{\hbar}(\mathcal{D}_{g_{\sigma_0}})^{\text{Wick}}} \). Analogously, for any \( \sigma_0 \in \mathbb{R}^+ \) the quantization of the dressed flow\(^{12} \) \( \hat{F}_{\sigma_0}(t) \) would be the renormalized unitary evolution \( e^{-\frac{i}{\hbar}\hat{H}_{\text{ren}}(\sigma_0)} \). So quantizing (64) we can define the undressed unitary dynamics for any \( \sigma_0 \in \mathbb{R}^+ \):

\[
e^{-i\frac{\varepsilon}{\hbar}H_{\text{ren}}(\sigma_0)} = e^{-\frac{i}{\hbar}(\mathcal{D}_{g_{\sigma_0}})^{\text{Wick}}} e^{-i\frac{\varepsilon}{\hbar}\hat{H}_{\text{ren}}(\sigma_0)} e^{\frac{i}{\hbar}(\mathcal{D}_{g_{\sigma_0}})^{\text{Wick}}};
\]

and the undressed renormalized Hamiltonian \( H_{\text{ren}}(\sigma_0) \) as its generator.

1. The relation between \( \sigma_0 \) and \( ||\varepsilon||_2 \).

The parameter \( \sigma_0 \) is necessary to make the renormalized Hamiltonian operator self-adjoint; and for each \( \sigma_0 \in \mathbb{R}^+ \) the quantum dynamics may be defined on the subspace of \( \mathcal{H} \) with at

\(^{10} \mathcal{E} \circ D_{\chi_{\sigma_0}}(\mathcal{E}^{\phi}u, \alpha) = \mathcal{E} \circ D_{\chi_{\sigma_0}}(u, \alpha) \) for any \( \phi \in \mathbb{R} \).

\(^{11} \) This is due to the fact that \( \mathcal{E} \) is unbounded from below on \( D(\mathcal{E}) \), but bounded from below when restricted to any cylinder with bounded \( u \)-norm \( B_u(0, C) \cap D(\mathcal{E}) \); see Section IV.1 below.

\(^{12} \) In Section III.3 above the dressed flow was defined as \( \mathcal{E}(\varepsilon) \).
most \( \Re(\sigma_0) \) nucleons. However, the choice of \( \sigma_0 \) remains in some sense arbitrary. In this section we argue that the number of nucleons is related to the boundedness from below of the classical energy \( E \).

The classical energy \( E \), when defined on the whole \( D(E) \), is unbounded from below:

\[
\inf_{(u, \alpha) \in D(E)} E(u, \alpha) = -\infty.
\]

However, we can take advantage of the conservation of the \( L^2 \)-norm of \( u \). Indeed it can be easily seen that on each hollow cylinder\(^\text{13}\) \( S_u(0, C) \cap D(E) \) the energy becomes bounded from below. One can also prove that that the energy is bounded from below on the cylinder \( B_u(0, \sqrt{C}) \cap D(E) \):

\[
\inf_{(u, \alpha) \in D(E)} \|u\|_2 \geq \sqrt{C} E(u, \alpha) > -\infty,
\]

and both \( B_u(0, \sqrt{C}) \) and \( D(E) \) are conserved by the Hamiltonian flow \( E \). Therefore it seems quite natural to consider the restriction of the S-KG theory to \( B_u(0, \sqrt{C}) \); and in particular to consider classical probability distributions concentrated in \( B_u(0, \sqrt{C}) \). Since we would like to quantize our theory in the Fock space \( \Gamma_s(\mathbb{L}^2 \oplus \mathbb{L}^2) \), we may ask which are the families of normal quantum states on the Fock space that have Wigner measures concentrated on \( B_u(0, \sqrt{C}) \). The families that satisfy Assumption \( (A_0) \), or equivalently that have at most \([E/\epsilon]\) nucleons (see Lemma \( \text{V.2} \)), have measures concentrated on \( B_u(0, \sqrt{C}) \) according to [10, Lemma 2.14] or [13, Theorem 3.1]. If in addition we take into account that to a classical theory with \( U(1) \) invariance in \( u \), corresponds a quantum theory that preserves the number of nucleons; we we are led to believe that a quantization of \( E \) should be meaningful, as an operator bounded from below, on the subspace \( \bigoplus_{n \leq [E/\epsilon]} \mathcal{H}_n \) of the Fock representation. As discussed above, we are able to define the quantization \( H_{\text{ren}} \) of \( E \) thanks to the classical dressing renormalization. It is remarkable that we have also the freedom, by choosing \( \sigma_0 \), to make \( H_{\text{ren}} \) self-adjoint and different from zero precisely on the aforementioned relevant subspace \( \bigoplus_{n \leq [E/\epsilon]} \mathcal{H}_n \).

We remark that there are families of quantum states with Wigner measures concentrated on \( B_u(0, \sqrt{C}) \) that have non-zero components in every subspace with fixed nucleons. In this case however, the Wick quantization procedure yields an unboundedness from below of the quantum energy, due to the fact that the number of nucleons is conserved by the dynamics, but unbounded on the state. This asymptotic instability or unboundedness from below of the energy, that is characteristic of the bosonic quantum system, seems to prevent the definition of a suitable non-trivial dynamics for these states.

---

\(^{13}\) We define a hollow cylinder as \( S_u(0, C) = \{(u, \alpha) \in L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d), ||u||_2 = C\} \).
2. Symplectic character of $D_{\chi_{V_0}}$.

To complete our description of the classical renormalization scheme, we explicitly prove that the classical dressing is a (non-linear) symplectic map for the real symplectic structure $\{(L^2 \oplus L^2)_R, \text{Im}(\cdot, \cdot)_{L^2 \oplus L^2}\}$. We denote by $dD_g(\theta)_{(u,\alpha)} \in \mathcal{L}(L^2 \oplus L^2)$ the (Fréchet) derivative of $D_g(\theta)$ at the point $(u, \alpha) \in L^2 \oplus L^2$.

**Proposition IV.1.** Let $g \in L^2(\mathbb{R}^3)$ be an even or odd function. Then for any $\theta \in \mathbb{R}$, $D_g(\theta)$ is differentiable at any point $(u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. In addition, it satisfies for any $(v_1, \beta_1), (v_2, \beta_2) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:

\[
\text{Im}(dD_g(\theta)_{(u,\alpha)}(v_1, \beta_1), dD_g(\theta)_{(u,\alpha)}(v_2, \beta_2))_{L^2 \oplus L^2} = \text{Im}((v_1, \beta_1), (v_2, \beta_2))_{L^2 \oplus L^2}.
\]

**Proof.** We recall that with the assumptions on $g$, $D_g(\theta)$ has the explicit form:

\[
D_g(\theta)(u(x), \alpha(k)) = \left(u(x)e^{-i\theta A_g(x)}, \alpha(k) - i\theta g(k)F(|u|^2)(k)\right);
\]

where $A_g$ and $F$ are defined by Equations (51) and (52) respectively. The Fréchet derivative of $D_g(\theta)$ is easily computed, and yields

\[
dD_g(\theta)_{(u,\alpha)}(v(x), \beta(k)) = \left((v(x) - i\theta B_g(x)u(x))e^{-i\theta A_g(x)}, \beta(k) - 2i\theta g(k)\text{Re}(F(\bar{u}v)(k))\right)
= \left(i(v, \beta), ii(v, \beta)\right);
\]

where we recall that $B_g(x)$ is $A_g(x)$ with $\alpha$ substituted by $\beta$. Then we have:

\[
\text{Im}(i(v_1, \beta_1), i(v_2, \beta_2))_{L^2} = \text{Im}(v_1, v_2)_{L^2} + 2\theta\text{Re}\left(\langle B_g^{(1)}u, v_2 \rangle_{L^2} - \langle v_1, B_g^{(2)}u \rangle_{L^2}\right),
\]

\[
\text{Im}(ii(v_1, \beta_1), ii(v_2, \beta_2))_{L^2} = \text{Im}(\beta_1, \beta_2)_{L^2} + 2\theta\text{Re}\left(\langle g \text{Re}(\bar{u}v_1), \beta_2 \rangle_{L^2} - \langle \beta_1, g \text{Re}(\bar{u}v_2) \rangle_{L^2}\right).
\]

The result then follows, noting that $\langle g \text{Re}(\bar{u}v_1), \beta_2 \rangle_{L^2} = \langle v_1, B_g^{(2)}u \rangle_{L^2}$ and $\langle \beta_1, g \text{Re}(\bar{u}v_2) \rangle_{L^2} = \langle B_g^{(1)}u, v_2 \rangle_{L^2}$. 

V. THE CLASSICAL LIMIT OF THE RENORMALIZED NELSON MODEL.

In this section we discuss in detail the classical limit of the renormalized Nelson model, both dressed and undressed. The Subsections from V.1 to V.5 are dedicated to prove the convergence of the dressed dynamics. The obtained results are summarized by Theorem V.26. In Subsection V.6 we study the classical limit of the dressing transformation. Finally, in Section V.7 we prove Theorem I.1 and overview the results.
1. The integral formula for the dressed Hamiltonian.

The results of this and the next subsection are similar in spirit to the ones previously obtained in [7, Section 3] for the Nelson model with cutoff and in [11, Section 3] for the mean field problem. However, some additional care has to be taken, for in this more singular situation the manipulations below are allowed only in the sense of quadratic forms. We start with a couple of preparatory lemmas. The proof of the first can be essentially obtained following [6, Lemma 6.1]; the second is an equivalent reformulation of Assumption (A0):

\[ \exists \varepsilon > 0, \forall \varepsilon \in (0, \varepsilon), \forall k \in \mathbb{N}, \quad \text{Tr}[Q_\varepsilon N_k^k] \leq \mathbb{C}^k. \]

We recall that the Weyl operator \( W(\xi), L^2 \oplus L^2 \ni \xi = \xi_1 \oplus \xi_2 \), is defined as:

\[ W(\xi) = e^{\frac{i}{\hbar}(\psi(\xi_1) + \phi(\xi_1))} e^{\frac{i}{\hbar}(\omega(\xi_2) + a(\xi_2))}. \]

**Lemma V.1.** For any \( \xi = \xi_1 \oplus \xi_2 \) such that \( \xi_1 \in Q(-\Delta + V) \subset H^1 \) and \( \xi_2 \in D(\omega^{1/2}) \equiv \mathcal{F} H^{1/2} \), there exists \( C(\xi) > 0 \) that depends only on \( \|\xi_1\|_{H^1} \) and \( \|\xi_2\|_{\mathcal{F} H^{1/2}} \), such that for any \( \varepsilon \in (0, \varepsilon) \):

\[ \|H_0^{1/2} W(\xi)\Psi\| \leq C(\xi)\|(H_0 + \varepsilon)^{1/2}\Psi\|, \quad \forall \Psi \in Q(H_0); \]

\[ \|(H_0 + 1)^{1/2}(N_1 + 1)^{1/2} W(\xi)\Psi\| \leq C(\xi)\|(H_0 + \varepsilon)^{1/2}(N_1 + \varepsilon)^{1/2}\Psi\|, \quad \forall \Psi \in Q(H_0) \cap Q(N_1). \]

In an analogous fashion, for any \( \xi \in L^2 \oplus L^2 \), \( r > 0 \), there exist \( C(\xi) > 0 \) that depends only on \( \|\xi_1\|_2 \) and \( \|\xi_2\|_2 \), such that for any \( \varepsilon \in (0, \varepsilon) \):

\[ \|(N_1 + N_2)^{r/2} W(\xi)\Psi\| \leq C(\xi)\|(N_1 + N_2 + \varepsilon)^{r/2}\Psi\|, \quad \forall \Psi \in Q(N_1^r). \]

**Lemma V.2.** Let \( (Q_\varepsilon)_{\varepsilon \in (0, \varepsilon)} \) be a family of normal states on \( \mathcal{H} \). Then \( (Q_\varepsilon)_{\varepsilon \in (0, \varepsilon)} \) satisfies Assumption (A0) if and only if for any \( \varepsilon \in (0, \varepsilon) \) there exists a sequence \( (\Psi_i(\varepsilon))_{\varepsilon \in \mathbb{N}} \) of orthonormal vectors in \( \mathcal{H} \) with non-zero components only in \( \bigoplus_{n=0}^{|\xi_1|} \mathcal{H}_n \) and a sequence \( (\lambda_i(\varepsilon))_{\varepsilon \in \mathbb{N}} \in \ell^1 \), with each \( \lambda_i(\varepsilon) > 0 \), such that:

\[ Q_\varepsilon = \sum_{\varepsilon \in \mathbb{N}} \lambda_i(\varepsilon)\langle \Psi_i(\varepsilon) \rangle\langle \Psi_i(\varepsilon) \rangle. \]

The explicit \( \varepsilon \)-dependence of \( \Psi_i \) and \( \lambda_i \) will be often omitted.

**Proof.** We start assuming (A0). Let \( Q_\varepsilon = \sum_{\varepsilon \in \mathbb{N}} \lambda_i(\varepsilon)\langle \Psi_i(\varepsilon) \rangle\langle \Psi_i(\varepsilon) \rangle \) be the spectral decomposition of \( Q_\varepsilon \). Then

\[ \text{Tr}[Q_\varepsilon N_k^k] = \sum_{\varepsilon \in \mathbb{N}} \lambda_i(\langle \Psi_i(\varepsilon) \rangle\langle \Psi_i(\varepsilon) \rangle) \leq \mathbb{C}^k \Rightarrow \sum_{\varepsilon \in \mathbb{N}} \lambda_i(\langle \Psi_i, (N_1/\mathbb{C})^k\Psi_i \rangle) \leq 1. \]

Let \( \mathbb{1}_{(L, +\infty)}(N_1) \) be the spectral projection of \( N_1 \) on the interval \( (L, +\infty) \); and choose \( L > \mathbb{C} \). Then it follows that:

\[ 1 \geq \text{Tr}[Q_\varepsilon \mathbb{1}_{(L, +\infty)}(N_1)(N_1/\mathbb{C})^k] = \sum_{\varepsilon \in \mathbb{N}} \lambda_i(\langle \Psi_i, \mathbb{1}_{(L, +\infty)}(N_1)(N_1/\mathbb{C})^k\Psi_i \rangle) \geq \sum_{\varepsilon \in \mathbb{N}} \lambda_i(L/\mathbb{C})^k\langle \Psi_i, \mathbb{1}_{(L, +\infty)}(N_1)\Psi_i \rangle. \]
Therefore \((L/\mathbb{C})^k\langle \Psi_i, I_{[L+\infty]}(N_1)\Psi_i \rangle \leq 1\) for any \(k \in \mathbb{N}\) and for any \(\Psi_i\). Now \((L/\mathbb{C})^k\) diverges when \(k \to \infty\), while \(\langle \Psi_i, I_{[L+\infty]}(N_1)\Psi_i \rangle\) does not depend on \(k\), so their product is uniformly bounded if and only if \(I_{[L+\infty]}(N_1)\Psi_i = 0\) for any \(L > \mathbb{C}\). The result follows immediately, recalling that the eigenvalues of \(N_1\) are of the form \(\epsilon n_1\), with \(n_1 \in \mathbb{N}\).

The converse statement that Assumption \((A_0)\) follows if \(\varrho_e = \sum_{i \in \mathbb{N}} A_i|\Psi_i\rangle\langle \Psi_i|\), with each \(\Psi_i\) with at most \([\mathbb{C}/\epsilon]\) particles is trivial to prove.

In this subsection, we will consider only families of states \((\varrho_{e}(t))_{t \in (0, \bar{\epsilon})}\) that satisfy Assumption \((A_0)\) and the following assumption:

\((A'_0)\) \quad \exists C > 0, \forall \epsilon \in (0, \bar{\epsilon}) \quad \text{Tr}[\varrho_{e}(N_1 + H_0)] \leq C .

**Definition V.3** \((\varrho_{e}(t), \tilde{\varrho}_{e}(t))\). We define the dressed time evolution of a state \(\varrho_{e}\) to be

\[\tilde{\varrho}_{e}(t) = e^{-i\hat{H}_{ren}}\varrho_{e}(t) e^{i\hat{H}_{ren}},\]

where the dependence on \(\sigma_0\) of \(\hat{H}_{ren}\) is omitted, and the \(\sigma_0\) is chosen such that the dynamics is non-trivial on the whole subspace with at most \([\mathbb{C}/\epsilon]\) nucleons (see Lemma V.2 and the discussion in Section I.2). We also define the dressed evolution in the interaction picture to be

\[\tilde{\varrho}_{e}(t) = e^{i\hat{H}_0}\varrho_{e}(t) e^{-i\hat{H}_0}.\]

To characterize the evolved Wigner measures corresponding to \(\tilde{\varrho}_{e}(t)\), it is sufficient to study its Fourier transform; this is done studying the evolution of \(\text{Tr}[\tilde{\varrho}_{e}(t)W(\xi)]\) by means of an integral equation.

**Proposition V.4.** Let \((\varrho_{e})_{t \in (0, \bar{\epsilon})}\) be a family of normal states on \(\mathcal{H}\) satisfying Assumptions \((A_0)\) and \((A'_0)\). Then for any \(t \in \mathbb{R}\), \(Q(-\Delta + V) \oplus D(\omega^{1/2}) \ni \xi = \xi_1 \oplus \xi_2:\)

\[
\text{Tr}[\tilde{\varrho}_{e}(t)W(\xi)] = \text{Tr}[\varrho_{e}W(\xi)] + \frac{i}{\epsilon} \int_0^\epsilon \text{Tr}[\varrho_{e}(s)[\hat{H}_{ren,\xi}, W(\hat{\xi}_s)]] ds, \tag{74}
\]

where \(\hat{\xi}_s = e^{is(-\Delta + V)}\xi_1 \oplus e^{-is}\xi_2\). The commutator \([\hat{H}_{ren,\xi}, W(\hat{\xi}_s)]\) has to be intended as a densely defined quadratic form with domain \(Q(H_0)\), or equivalently as an operator from \(Q(H_0)\) to \(Q(H_0)^*\).

**Proof.** The family \((\varrho_{e})_{t \in (0, \bar{\epsilon})}\) satisfies Assumption \((A_0)\), therefore by Lemma V.2:

\[
\text{Tr}[\tilde{\varrho}_{e}(t)W(\xi)] = \sum_{i \in \mathbb{N}} A_i \langle e^{i\hat{H}_0} e^{-i\hat{H}_{ren}} \Psi_i, W(\xi) e^{i\hat{H}_0} e^{-i\hat{H}_{ren}} \Psi_i \rangle .
\]

By Assumption \((A'_0)\), it follows that \(\Psi_i \in Q(H_0)\) for any \(i \in \mathbb{N}\). Hence the right hand side is differentiable in \(t\) by Lemma V.1, since \(Q(H_0)\) is the form domain of both \(H_0\) and \(\hat{H}_{ren}\). Using the Duhamel formula and the fact that \(e^{-i\hat{H}_0} W(\xi) e^{i\hat{H}_0} = W(\hat{\xi}_s)\), we then obtain:

\[
\text{Tr}[\tilde{\varrho}_{e}(t)W(\xi)] = \sum_{i \in \mathbb{N}} A_i \left( \langle \Psi_i, W(\xi) \Psi_i \rangle + \frac{i}{\epsilon} \int_0^\epsilon \langle e^{-i\hat{H}_{ren}} \Psi_i, [\hat{H}_{ren,\xi}, W(\hat{\xi}_s)] e^{-i\hat{H}_{ren}} \Psi_i \rangle ds \right); \tag{75}
\]
where $[\hat{H}_{ren,I}, W(\xi_e)]$ makes sense as a quadratic form on $Q(H_0)$. The result is then obtained using Lebesgue’s dominated convergence theorem on the right hand side, by virtue of Assumption $(A'_p)$ and Lemma V.1.

### 2. The commutator $[\hat{H}_{ren,I}, W(\tilde{\xi}_e)]$.

In this subsection we deal with the commutator $[\hat{H}_{ren,I}, W(\tilde{\xi}_e)]$. The goal is to show that each of its terms converges in the limit $\varepsilon \to 0$, either to zero or to a suitable phase space symbol.

For convenience, we recall some terminology related to quantization procedures in infinite dimensional phase spaces (see [8] for additional informations). Let $Z$ be a Hilbert space (the classical phase space). In the language of quantization, we call a densely defined functional $\mathcal{A} : D \subset Z \to \mathbb{C}$ a (classical) symbol. We say that $A$ is a polynomial symbol if there are densely defined bilinear forms $b_{p,q}$ on $Z^{\otimes p} \times Z^{\otimes q}$, $0 \leq p \leq \tilde{p}$, $0 \leq q \leq \tilde{q}$ (with $p, \tilde{p}, q, \tilde{q} \in \mathbb{N}$) such that

$$
\mathcal{A}(z) = \sum_{0 \leq p \leq \tilde{p}, 0 \leq q \leq \tilde{q}} b_{p,q}(z^\otimes p, z^\otimes q).
$$

(75)

The Wick quantized quadratic form $(\mathcal{A})^{\text{Wick}}$ on $\Gamma_s(Z)$ is then obtained, roughly speaking, replacing each $z(\cdot)$ with the annihilation operator valued distribution $a(\cdot)$; each $\bar{z}(\cdot)$ with the creation operator valued distribution $a^*(\cdot)$; and putting all the $a^*(\cdot)$ to the left of the $a(\cdot)$. We denote, with a straightforward notation, the class of all polynomial symbols on $Z$ by $\bigoplus_{(p,q) \in \mathbb{N}^2}^{alg} Q_{p,q}(Z)$. If $\mathcal{A} : Z \to \mathbb{C}$ and the bilinear forms $b_{p,q}(z^\otimes p, z^\otimes q)$ in (75) can all be written as $\langle \zeta^\otimes p, \tilde{b}_{p,q} \zeta^\otimes q \rangle_{Z^\otimes q}$ for some bounded (resp. compact) operator $\tilde{b}_{p,q} : Z^{\otimes p} \to Z^{\otimes q}$, we say that $\mathcal{A}$ is a bounded (resp. compact) polynomial symbol. We denote the class of all bounded (resp. compact) polynomial symbols by $\bigoplus_{(p,q) \in \mathbb{N}^2}^{alg} \mathcal{P}_{p,q}(Z)$ (resp. $\bigoplus_{(p,q) \in \mathbb{N}^2}^{alg} \mathcal{P}_{p,q}^\infty(Z)$). We remark that $\mathcal{E}'$, $\mathcal{E}$ and $\mathcal{D}_q$ defined in Section III are all polynomial symbols\footnote{In $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$, we adopt the notation $z = (u, a)$; and to each $u(x)$ it corresponds the operator valued distribution $\psi(x)$, to each $a(k)$ the distribution $a(k)$. The Wick quantization is again obtained by substituting each $(u^\#(x), a^\#(k))$ with $(\psi^\#(x), a^\#(k))$, and using the normal ordering of creators to the left of annihilators.} on $L^2 \otimes L^2$. 

**Lemma V.5.** Let $(\xi_e)_{e \in (0, \tilde{e})}$ satisfy the same assumptions as in Proposition V.4. Then there exist maps $\mathcal{B}_j(\cdot) : Q(-\Delta + V) \oplus D(\omega^{1/2}) \to \bigoplus_{(p,q) \in \mathbb{N}^2}^{alg} Q_{p,q}(L^2 \otimes L^2)$, $j = 0, \ldots, 3$, such that
for any \( t \in \mathbb{R}, \xi \in Q(\Delta + V) \oplus D(\omega^{1/2}) \):

\[
\text{Tr}[\hat{\rho}_\varepsilon(t)W(\xi)] = \text{Tr}[\hat{\rho}_\varepsilon W(\xi)] + \sum_{j=0}^3 \varepsilon^j \int_0^t \text{Tr}[\hat{\rho}_\varepsilon(s)W(\xi)(\mathcal{B}_j(\xi))]^{\text{Wick}} ds
\]

(76)

\[
= \text{Tr}[\hat{\rho}_\varepsilon W(\xi)] + \sum_{j=0}^3 \varepsilon^j \int_0^t \text{Tr}[\hat{\rho}_\varepsilon(s)W(\xi)B_j(\xi)] ds ;
\]

where the \((\mathcal{B}_j(\xi))^{\text{Wick}}\) make sense as densely defined quadratic forms. To simplify the notation, we have set \( B_j(\cdot) := (\mathcal{B}_j(\cdot))^{\text{Wick}} \).

**Proof.** We only sketch the proof here since it follows the same lines as in [11, 77]. By (74), the only thing we have to prove is that, in the sense of quadratic forms, \( \frac{1}{\varepsilon} [\hat{H}_{\text{ren,l}}, W(\xi)] = \sum_{j=0}^3 \mathcal{B}_j(\xi) B_j(\xi) \). First of all, we remark that \( \hat{\rho}_\varepsilon \) is the Wick quantization of a polynomial symbol\(^\text{15}\); i.e. \( \hat{H}_{\text{ren,l}} = (\hat{\rho}_\varepsilon)^{\text{Wick}} \), with

\[
\hat{\rho}_\varepsilon(u, \alpha) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \frac{\chi_{\varepsilon_0}}{\sqrt{2\omega(k)}} (\tilde{a}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}) |u(x)|^2 \text{d}x \text{d}k
\]

(77)

\[
+ \frac{1}{2M} \int_{\mathbb{R}^6} (r_\infty(k)\tilde{a}(k)e^{-ik\cdot x} + r_\infty(k)\alpha(k)e^{ik\cdot x})(r_\infty(l)\tilde{a}(l)e^{-il\cdot x} + r_\infty(l)\alpha(l)e^{il\cdot x}) |u(x)|^2 \text{d}x \text{d}k \text{d}l
\]

\[-\frac{2}{M} \text{Re} \int_{\mathbb{R}^6} r_\infty(k)\tilde{a}(k)e^{-ik\cdot x}u^\ast(x)D_x u(x) \text{d}x \text{d}k + \frac{1}{2} \int_{\mathbb{R}^6} V_\infty(x-y) |u(x)|^2 |u(y)|^2 \text{d}x \text{d}y .
\]

We also recall, according to [8, Proposition 2.10 for bounded polynomial symbols] and [77, Proposition 2.1.30 for the general case], that essentially for any \( \mathcal{A} \in \bigoplus_{(p,q) \in \mathbb{N}_2} Q_{p,q}(L^2 \oplus L^2) \) the following formula is true, in the sense of forms, for any suitably regular \( \xi \in L^2 \oplus L^2 \):

\[
W^\ast(\xi)(\mathcal{A})^{\text{Wick}} W(\xi) = (\mathcal{A}(\cdot) + \frac{i\varepsilon}{\sqrt{2}} \hat{\xi})^{\text{Wick}} .
\]

(78)

Roughly speaking, the Weyl operators \( W(\xi) \) translate each creation/annihilation operator by \( \mp \frac{in}{\sqrt{2}} \hat{\xi} \). The result then follows immediately on the states \( \hat{\rho}_\varepsilon(s) \):

\[
[\hat{H}_{\text{ren,l}}, W(\xi)] = W(\xi)^\ast(\hat{H}_{\text{ren,l}})W(\xi) - \hat{H}_{\text{ren,l}} = W(\xi)(\hat{\rho}_\varepsilon(\cdot) + \frac{i\varepsilon}{\sqrt{2}} \hat{\xi})^{\text{Wick}} - \hat{\rho}_\varepsilon(\cdot) ;
\]

finally we define \( \sum_{j=0}^3 \varepsilon^j \mathcal{B}_j(\xi)(z) = \frac{1}{\varepsilon} (\hat{\rho}_\varepsilon(z) + \frac{i\varepsilon}{\sqrt{2}} \hat{\xi})^{\text{Wick}} - \hat{\rho}_\varepsilon(\cdot) \), to factor out the \( \varepsilon \)-dependence.

We state the next lemma without giving the tedious proof, that is based on the same type of estimates given in Section II.2 for the full operator \( \hat{H}_{\text{ren,l}} \).

\(^{15}\)To be precise, we are considering here the quadratic form \( h_{\text{ren,l}} \), defined and different from zero on the whole space \( \mathcal{H} \), since it agrees with \( \langle \cdot, \hat{H}_{\text{ren,l}} \cdot \rangle \) when restricted to vectors that belong to \( \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n \) (being here the case by Lemma V.2).
Lemma V.6. For any $j = 0, 1, 2, 3, \xi \in Q(-\Delta + V) \cap D(\omega^{1/2})$ and $\mathbb{C} > 0$, there exists $C_j(\xi) > 0$ such that for any $\Phi, \Psi \in D(H_0^{1/2}) \cap D(N_1)$, with $\Phi$ or $\Psi$ in $\bigoplus_{n \in [\xi/\varepsilon]} \mathcal{H}_n$ and for any $s \in \mathbb{R}$ and $\varepsilon \in (0, \bar{\varepsilon})$:

\[
|\langle \Phi, B_j(\xi \Psi) \rangle| \leq C_j(\xi) \|(N_1 + H_0 + \varepsilon)^{1/2} \Phi\| \cdot \|(N_1 + H_0 + \varepsilon)^{1/2} \Psi\|.
\]

(79)

Thanks to this lemma we are now in a position to prove that the higher order terms in $\varepsilon$ of Equation (76) (namely those with $j > 0$) vanish in the limit $\varepsilon \to 0$.

Proposition V.7. Let $(\mathcal{Q}_\varepsilon)_{\varepsilon \in (0, \bar{\varepsilon})}$ satisfy Assumptions (A_0) and (A'_0); let $\xi \in Q(-\Delta + V) \cap D(\omega^{1/2})$. Then the following limit holds for any $t \in \mathbb{R}$:

\[
\lim_{\varepsilon \to 0} \sum_{j=1}^{3} \varepsilon^j \int_{0}^{t} \text{Tr}[\mathcal{Q}_\varepsilon(s) W(\xi \Psi_j) B_j(\xi \Psi_j)] ds = 0.
\]

(80)

Proof. By Lemma V.2 we can write $\mathcal{Q}_\varepsilon = \sum_i \lambda_i |\Psi_i\rangle \langle \Psi_i|$, where each $\Psi_i$ has non-zero components only in the subspace $\bigoplus_{n \in [\xi/\varepsilon]} \mathcal{H}_n$, and each $\lambda_i > 0$. Assumption (A'_0) then translates on the fact that each $\Psi_i$ is on the domain $Q(H_0) \cap Q(N_1)$, and in addition $\sum_i \lambda_i |\Psi_i, (N_1 + H_0)\Psi_i\rangle \leq C$, uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon})$. Therefore we can write

\[
\left| \sum_{j=1}^{3} \varepsilon^j \int_{0}^{t} \text{Tr}[\mathcal{Q}_\varepsilon(s) W(\xi \Psi_j) B_j(\xi \Psi_j)] ds \right| \leq \sum_{j=1}^{3} \varepsilon^j \sum_i \lambda_i \int_{0}^{t} \left| \langle W^*(\xi \Psi_j) e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i, B_j(\xi \Psi_j) e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i \rangle \right| ds.
\]

Using now Lemma V.6 and then Lemma V.1 and the fact that $N_1$ commutes with $\hat{H}_\text{ren}$ we obtain

\[
\left| \sum_{j=1}^{3} \varepsilon^j \int_{0}^{t} \text{Tr}[\mathcal{Q}_\varepsilon(s) W(\xi \Psi_j) B_j(\xi \Psi_j)] ds \right| \leq \sum_{j=1}^{3} \varepsilon^j C_j(\xi) \sum_i \lambda_i \int_{0}^{t} \left| (N_1 + H_0 + \varepsilon)^{1/2} W^*(\xi \Psi_j) e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i \right| ds

\cdot \left| (N_1 + H_0 + \varepsilon)^{1/2} e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i \right| ds

\leq \sum_{j=1}^{3} \varepsilon^j C(\xi) C_j(\xi) \sum_i \lambda_i \int_{0}^{t} \left| e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i, (N_1 + H_0 + \varepsilon) e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i \right| ds

\leq \sum_{j=1}^{3} \varepsilon^j C(\xi) C_j(\xi) \sum_i \lambda_i \left( \langle \Psi_i, (N_1 + \varepsilon) \Psi_i \rangle + \int_{0}^{t} \left| e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i, H_0 e^{-i\varepsilon \hat{H}_\text{ren}} \Psi_i \right| ds \right).
\]
Now we consider the term $\langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i, H_0 e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i \rangle$. First of all we write it as

$$
\langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i, H_0 e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i \rangle = \langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i, (\hat{H}_{\text{ren}} - \hat{H}_{\text{ren},1}) e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i \rangle
$$

$$
= \sum_{n=0}^{[\mathcal{E}/\varepsilon]} \langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i^{(n)}, (\hat{H}_{\text{ren}}^{(n)} - \hat{H}_{\text{ren}}(\infty)) e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i^{(n)} \rangle
$$

(81)

$$
\leq \sum_{n=0}^{[\mathcal{E}/\varepsilon]} \left( |\langle \Psi_i^{(n)}, \hat{H}_{\text{ren}}^{(n)} \Psi_i^{(n)} \rangle| + |\langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i^{(n)}, \hat{H}_{\text{ren}}(\infty) e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i^{(n)} \rangle| \right).
$$

The idea now is to use the bound of Equation (36) on $|\langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i^{(n)}, \hat{H}_{\text{ren}}(\infty) e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i^{(n)} \rangle|$. The crucial point is that since we have chosen $\sigma_0$ such that the dynamics is non-trivial for any $n \leq [\mathcal{E}/\varepsilon]$, it follows that there exist an $a < 1$ and a $b < \infty$ both independent of $\varepsilon$ and $n$ such that the bound (36) holds for any $n \leq [\mathcal{E}/\varepsilon]$. Therefore we obtain

$$
\langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i, H_0 e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i \rangle \leq a \langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i, H_0 e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i \rangle + b \langle \Psi_i, \Psi_i \rangle
$$

(82)

$$
+ \sum_{n=0}^{[\mathcal{E}/\varepsilon]} \left| \langle \Psi_i^{(n)}, \hat{H}_{\text{ren}}^{(n)} \Psi_i^{(n)} \rangle \right|
$$

Now since $a < 1$, we may take it to the left hand side and use again (36) on $|\langle \Psi_i^{(n)}, \hat{H}_{\text{ren}}^{(n)} \Psi_i^{(n)} \rangle|$:

$$
\langle e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i, H_0 e^{-i\frac{\hat{H}_{\text{ren}}}{\varepsilon}} \Psi_i \rangle \leq \frac{1}{1-a} \langle \Psi_i, H_0 \Psi_i \rangle + \frac{2b}{1-a} \langle \Psi_i, \Psi_i \rangle
$$

(83)

Finally, since the state is normalized (i.e. $\sum_i \alpha_i \langle \Psi_i, \Psi_i \rangle = 1$), we conclude:

$$
\left| \sum_{j=1}^{3} e^{i \int_0^t \text{Tr}[Q_\varepsilon(s) W(\xi_s) B_j(\xi_s)] ds} \right| \leq t \sum_{j=1}^{3} e^{i \int_0^t C(\xi) C_j(\xi)} \sum_i \alpha_i \left( \langle \Psi_i, N_1 \Psi_i \rangle + \frac{1}{1-a} \langle \Psi_i, H_0 \Psi_i \rangle \right)
$$

$$
+ \left( \frac{2b}{1-a} + \bar{\varepsilon} \right) \langle \Psi_i, \Psi_i \rangle
$$

$$
\leq t \sum_{j=1}^{3} e^{i \int_0^t C(\xi) C_j(\xi)} \left( 1 + \frac{1}{1-a} \right) \sum_i \alpha_i \langle \Psi_i, (N_1 + H_0) \Psi_i \rangle + \frac{2b}{1-a} + \bar{\varepsilon}
$$

$$
\leq t \sum_{j=1}^{3} e^{i \int_0^t C(\xi) C_j(\xi)} \left( 1 + \frac{1}{1-a} \right) C + \frac{2b}{1-a} + \bar{\varepsilon}.
$$

The right hand side has no implicit dependence on $\varepsilon$, so it converges to zero when $\varepsilon \to 0$. ■

By the same argument used from (81) to (83) above, we can prove the following useful lemma.
Lemma V.8. If a family of states \((q_\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})}\) satisfies Assumptions \((A_0)\) and \((A'_0)\), then for any \(t \in \mathbb{R}\), \((q_\varepsilon(t))_{\varepsilon \in (0,\bar{\varepsilon})}\) and \((\tilde{q}_\varepsilon(t))_{\varepsilon \in (0,\bar{\varepsilon})}\) satisfy Assumptions \((A_0)\) and \((A'_0)\). In particular, there exist \(a(\mathbb{C}) < 1\) and \(b(\mathbb{C}) > 0\) such that uniformly on \(\varepsilon \in (0,\bar{\varepsilon})\):

\[
\begin{align*}
\text{(84)} & \quad \text{Tr}[q_\varepsilon(t)N_1^k] \leq C^k, \quad \forall k \in \mathbb{N}; \\
\text{(85)} & \quad \text{Tr}[q_\varepsilon(t)(N_1 + H_0)] \leq \frac{1}{1 - a(\mathbb{C})} C + \frac{2b(\mathbb{C})}{1 - a(\mathbb{C})};
\end{align*}
\]

and the same bounds hold for \((\tilde{q}_\varepsilon(t))_{\varepsilon \in (0,\bar{\varepsilon})}\).

It remains to study the limit of the \(B_0(\cdot)-\text{term}\) in (76). As already pointed out in Lemma V.5, we know that \(B_0\) is a Wick quantization. More precisely, there exist a densely defined map from the one-particle space to polynomial symbols in \(\bigoplus_{(p,q) \in \{(i,j) \mid 0 \leq i, j \leq 1, 2 \leq i+j \leq 3\}} Q_{p,q}(L^2 \oplus L^2)\). In order to apply the convergence results of Ammari and Nier [8], we need to show that the symbol of \(B_0\) may be approximated by a compact one, with an error that vanishes in the limit \(\varepsilon \to 0\).

To improve readability, we will write \(B_0(\tilde{\xi})\) in a schematic fashion. The precise structure of each term will be discussed and analyzed in the proof of the sequent proposition. In addition, as seen in Equation (16), the dressed interaction quadratic form \(\hat{H}_t(\infty)\) can be split in three terms: the first is just the interaction term \(H_t(\sigma_0)\) of the Nelson model with cutoff (with \(\sigma\) replaced by \(\sigma_0\)), whose classical limit has been analyzed by the authors in [7]; the second is a “mean-field” term for the nucleons, of the same type as the ones analyzed by Ammari and Nier in [11]; the last one has a structure similar to the interaction part of the Pauli-Fierz model [see e.g. 15–17, 100], and thus will be called of “Pauli-Fierz type”. We will concentrate on the analysis of the Pauli-Fierz type terms of \(B_0\), while for a precise treatment of the others the reader may refer to [7, 11]. In order to highlight the different parts of \(B_0(\tilde{\xi}) = B_0(\tilde{\xi}_1, \tilde{\xi}_2)\), we will underline with different style and color the Nelson, mean-field and Pauli-Fierz type terms:

\[
(86) \quad B_0(\tilde{\xi}_1, \tilde{\xi}_2) = (B_0(\tilde{\xi}_1, \tilde{\xi}_2))^\text{Wick} = \underbrace{(a^* + a)(\tilde{\xi}_1 \psi^* - \tilde{\xi}_1 \psi) + \text{Im}(\hat{\xi}_2) \psi^* \psi + \tilde{\xi}_1 \psi^* \psi \psi - \tilde{\xi}_1 \psi^* \psi^* \psi}_{\text{mean-field}}
\]

\[
\quad \quad + \underbrace{(a^* a^* + aa + a^* a)(\tilde{\xi}_1 \psi^* - \tilde{\xi}_1 \psi) + (\tilde{\xi}_2 a^* - \tilde{\xi}_2 a) \psi^* \psi + (a^* D_x + D_x a)(\psi^* \tilde{\xi}_1 - \tilde{\xi}_1 \psi)}_{\text{Pauli-Fierz}}
\]

\[
\quad \quad \quad + \underbrace{\psi^* D_x \tilde{\xi}_2 \psi - \psi^* \tilde{\xi}_2 D_x \psi}_{\text{Pauli-Fierz}}.
\]

Proposition V.9. There exists a family of maps \((B_0^{(m)})_{m \in \mathbb{N}}\) such that:

\[* \text{ For any } m \in \mathbb{N} \]

\[
B_0^{(m)}(\cdot) : Q(-\Delta + V) \oplus D(\omega^{3/4}) \rightarrow \bigoplus_{(p,q) \in \{(i,j) \mid 0 \leq i, j \leq 1, 2 \leq i+j \leq 3\}} P^{\infty}_{p,q}(L^2 \oplus L^2); \]
* For any \( \xi \in Q(-\Delta + V) \oplus D(\omega^{3/4}) \), there exist a sequence \((C^{(m)}(\xi))_{m \in \mathbb{N}}\) that depends only on \( \|\xi\|_{Q(-\Delta + V) \oplus D(\omega^{3/4})} \) such that \( \lim_{m \to \infty} C^{(m)} = 0 \); and such that for any two vectors \( \Phi, \Psi \in \mathcal{H} \cap D(N_1) \), and for any \( \varepsilon \in (0, \varepsilon) \):

\[
(87) \quad \left| \langle (H_0 + 1)^{-1/2} \Phi, (\mathcal{B}_0(\xi) - \mathcal{B}_0^{(m)}(\xi))^\text{Wick} (H_0 + 1)^{-1/2} \Psi \rangle \right| \leq C^{(m)}(\xi) \| (N_1 + \varepsilon)^{1/2} \Phi \| : \| (N_1 + \varepsilon)^{1/2} \Psi \| .
\]

Remark V.10. Contrarily to what it was previously assumed throughout Section V, in this proposition we need additional regularity on \( \xi_2 \), namely \( \xi_2 \in D(\omega^{3/4}) \subset D(\omega^{1/2}) \). This will not be a problem in the following, since we will extend our results to any \( \xi \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \) by a density argument, and \( D(\omega^{3/4}) \) is still dense in \( L^2(\mathbb{R}^3) \).

**Proof of Proposition V.9.** To prove the proposition, we need to analyze each term of Equation (86), and prove that either it has a compact symbol or it can be approximated by one, in a way that (87) holds. The analysis for Nelson terms has been carried out in [7, Proposition 3.11 and Lemma 3.15]. In addition, using Lemma II.5 we see that \( V_\infty \) satisfies the hypotheses of the mean field potentials in [11], therefore (87) holds also for the mean-field terms [see in particular Section 3.2 of 11]. For the sake of completeness, we explicitly write the Nelson and mean field part of Equation (86):

\[
(a^* + a)(\xi_1 \psi^* - \bar{\xi}_1 \psi) = -\frac{1}{\sqrt{2(2\pi)^3}} \int_{\mathbb{R}^3} \left( a^* \left( e^{ik \cdot x} \psi(y) - \bar{\xi}_1(x) \psi(x) \right) - a \left( e^{ik \cdot x} \psi(y) - \bar{\xi}_1(x) \psi(x) \right) \right) dx ;
\]

\[
\text{Im}(\xi_2) \psi \psi^* = -\frac{1}{\sqrt{2(2\pi)^3}} \int_{\mathbb{R}^6} \left( \frac{k \cdot x}{\sqrt{2\omega(\kappa)}} \right) \left( \xi_2(k) e^{ik \cdot x} - \bar{\xi}_2(k) e^{-ik \cdot x} \right) \psi^*(x) \psi(x) dx ;
\]

\[
\bar{\xi}_1 \psi \psi^* - \xi_1 \psi^* \psi^* = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^6} V_\infty(x - y) \left( \bar{\xi}_1(y) \psi^*(x) \psi(y) - \xi_1(y) \psi^*(x) \psi(y) \psi(x) \right) dx dy .
\]

Now we start to study in detail terms of Pauli-Fierz type.

1) \( \bar{\xi}_1 a \psi, \xi_1 a^* \psi^* \).

\[
\bar{\xi}_1 a \psi = -\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^3} \left( a(r_\infty e^{-ik \cdot x}) \right) \bar{\xi}_1(x) \psi(x) dx .
\]

We recall that \( r_\infty \sim \kappa g_\infty \), where \( g_\sigma \) is defined by (7) for any \( \sigma \leq \infty \). Let \( \bar{\xi}_1 a \psi \) be the symbol associated to \( \bar{\xi}_1 a \psi \), i.e. \( \bar{\xi}_1 a \psi = (\bar{\xi}_1 a \psi)^\text{Wick} \) Now, since \( r_\infty \notin L^2(\mathbb{R}^3) \), we cannot

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16 We recall that for the Nelson model \( \mathcal{Z} = L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \), thus we denote the variable \( z \) by \( u \oplus \alpha \).
expect that \( \tilde{\xi}_1 \alpha \alpha u \) is defined for any \( u, \alpha \in L^2(\mathbb{R}^3) \), and therefore that it is a compact symbol. We introduce the approximate symbol \( \tilde{\xi}_1 \alpha \alpha u^{(m)} \) defined by

\[
\tilde{\xi}_1 \alpha \alpha u^{(m)} = (\tilde{\xi}_1 \alpha \alpha u^{(m)})^{\text{Wick}} = -\frac{1}{2\sqrt{2}M} \int_{\mathbb{R}^3} \left( a(r_{\sigma_m} e^{-ik \cdot x}) \right)^2 \tilde{\xi}_1(x) \psi(x) dx ,
\]

with \((\sigma_m)_{m \in \mathbb{N}} \subset \mathbb{R}\) such that \( \lim_{m \to \infty} \sigma_m = \infty \). First of all, we prove that (87) holds for \( \tilde{\xi}_1 \alpha \alpha u - \tilde{\xi}_1 \alpha \alpha u^{(m)} \):

\[
\left| \left\langle (H_0 + 1)^{-1/2} \Phi, (\tilde{\xi}_1 \alpha \alpha u - \tilde{\xi}_1 \alpha \alpha u^{(m)})(H_0 + 1)^{-1/2} \Phi \right\rangle \right| \leq \frac{1}{2\sqrt{2}M} \left\| \tilde{\xi}_1 \right\|_2 \left\| (d \Gamma(\omega) + 1)^{1/2} \right\| (H_0 + 1)^{-1/2} \Phi \cdot \sup_{x \in \mathbb{R}^3} \left| (d \Gamma(\omega) + 1)^{-1/2} \left( a((r_\omega - r_{\sigma_m}) e^{-ik \cdot x}) \right)^2 (d \Gamma(\omega) + 1)^{-1/2} \right| .
\]

We use (28) of Lemma II.7 and the fact that \((d \Gamma(\omega) + 1)^{1/2}(H_0 + 1)^{-1/2}\) is bounded with norm smaller than one to obtain

\[
\left| \left\langle (H_0 + 1)^{-1/2} \Phi, (\tilde{\xi}_1 \alpha \alpha u - \tilde{\xi}_1 \alpha \alpha u^{(m)})(H_0 + 1)^{-1/2} \Phi \right\rangle \right| \leq \frac{c}{2\sqrt{2}M} \left\| \tilde{\xi}_1 \right\|_2 \left\| \alpha^{-1/4}(r_\omega - r_{\sigma_m}) \right\|_2 \Phi \cdot \left\| (N_1 + \tilde{\epsilon})^{1/2} \Phi \right\| \cdot \left\| (N_1 + \tilde{\epsilon})^{1/2} \Phi \right\| ,
\]

with \( C(m)(\tilde{\xi}_1) = C(\tilde{\epsilon}, \tilde{\xi}_1) \left\| \omega^{-1/4}(r_\omega - r_{\sigma_m}) \right\|_2 \) for some \( C(\tilde{\epsilon}, \tilde{\xi}_1) > 0 \). The sequence \((C(m)(\tilde{\xi}_1))_{m \in \mathbb{N}}\) converges to zero since by our choice of \((\sigma_m)_{m \in \mathbb{N}}\):

\[
\lim_{m \to \infty} \left\| \omega^{-1/4}(r_\omega - r_{\sigma_m}) \right\|_2 = 0 .
\]

It remains to show that \( \tilde{\xi}_1 \alpha \alpha u^{(m)} \) is a compact symbol. Such symbol can be written as

\[
\tilde{\xi}_1 \alpha \alpha u^{(m)} = -\frac{1}{2\sqrt{2}M} \int_{\mathbb{R}^3} \tilde{\xi}_1(x) \tilde{r}_{\sigma_m}(k) \tilde{r}_{\sigma_m}(k') e^{ik(k+k') \cdot x} \alpha(k) \alpha(k') u(x) dx .
\]

Now we can define an operator \( \tilde{b}_{\alpha \alpha u} : (L^2 \oplus L^2)^{\otimes 3} \to \mathbb{C} \) in the following way. Let the maps \( \pi_1, \pi_2 : L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) be the projections on the first and second space respectively. Then we define the operator \( \tilde{b}_{\alpha \alpha u} \) as:

\[
\tilde{b}_{\alpha \alpha u} : (u, \alpha) \otimes 3 \mapsto \alpha(k) \alpha(k') u(x) \in L^2(\mathbb{R}^9) \to \langle f, \alpha \alpha u \rangle_{L^2(\mathbb{R}^9)} \in \mathbb{C} ;
\]

where \( f(k, k', x) = -\frac{1}{2\sqrt{2}M} \tilde{\xi}_1(x) \tilde{r}_{\sigma_m}(k) \tilde{r}_{\sigma_m}(k') e^{ik(k+k') \cdot x} \in L^2(\mathbb{R}^9) \). Therefore \( \tilde{b}_{\alpha \alpha u} \) is bounded and finite rank, and therefore compact. The proof for the corresponding adjoint term

\[
\tilde{\xi}_1 \alpha^* \alpha^* \psi^* = -\frac{1}{2\sqrt{2}M} \int_{\mathbb{R}^3} \left( a^*(r_\omega e^{-ik \cdot x}) \right)^2 \tilde{\xi}_1(x) \psi^*(x) dx
\]

can be obtained directly from the above, using the following approximation with compact symbol:

\[
\tilde{\xi}_1 \alpha^* \alpha^* \psi^{(m)} = (\tilde{\xi}_1 \alpha \alpha u^{(m)})^{\text{Wick}} = -\frac{1}{2\sqrt{2}M} \int_{\mathbb{R}^3} \left( a^*(r_{\sigma_m} e^{-ik \cdot x}) \right)^2 \tilde{\xi}_1(x) \psi^*(x) dx .
\]
Again we approximate this term by
\[
\xi_1aa\psi* = -\frac{1}{2\sqrt{2}M} \int_{\mathbb{R}^3} \left( a(r_{oo} e^{-ik\cdot x}) \right)^2 \xi_1(x)\psi^*(x) dx .
\]

Therefore, since $\tilde{\xi}_1\alpha\alpha\bar{u}(m)$ is a compact symbol. We define an operator $\tilde{b}_{\alpha\alpha\bar{u}} : (L^2 \oplus L^2)^{\otimes^2} \rightarrow L^2 \oplus L^2$ by
\[
\tilde{b}_{\alpha\alpha\bar{u}} : (u, \alpha)^{\otimes^2} \in (L^2 \oplus L^2)^{\otimes^2} \quad \longrightarrow \quad \alpha(k)\alpha(k') \in L^2(\mathbb{R}^n)
\]
\[
\tilde{b}_{\alpha\alpha\bar{u}} \left( \int_{\mathbb{R}^6} \tilde{f}(k, k', \cdot)\alpha(k)\alpha(k')dkdK' \oplus 0 \right) \in L^2 \oplus L^2 ;
\]
where $f(k, k', x) = -\frac{1}{2\sqrt{2}M} \xi_1(x)r_{\sigma_m}(k) r_{\sigma_m}(k') e^{-ik\cdot k'\cdot x}$. By definition, we have that
\[
\xi_1aa\bar{u}(m) = \langle (u, \alpha), \tilde{b}_{\alpha\alpha\bar{u}}(u, \alpha)^{\otimes^2} \rangle_{L^2 \oplus L^2} .
\]

It is easily seen that the operator $\tilde{c}_{\alpha\alpha\bar{u}}$ is bounded. It is in fact compact: let $\beta_j \rightarrow \beta$ in $L^2(\mathbb{R}^n)$ be a weakly convergent (bounded) sequence such that $\max\{\sup_j ||\beta_j||_{L^2(\mathbb{R}^n)}, ||\beta||_{L^2(\mathbb{R}^n)} \} = X < \infty$; then
\[
\left\| \tilde{c}_{\alpha\alpha\bar{u}}(\beta - \beta_j) \right\|_{L^2 \oplus L^2} = \left\| \langle f(k, k', x), (\beta - \beta_j)(k, k') \rangle_{L^2_{\bar{u}}(\mathbb{R}^n)} \right\|_{L^2_{\bar{u}}(\mathbb{R}^n)} \longrightarrow 0 ,
\]
by Lebesgue’s dominated convergence theorem, using the uniform bound
\[
\left| \langle f(k, k', x), (\beta - \beta_j)(k, k') \rangle_{L^2_{\bar{u}}(\mathbb{R}^n)} \right|^2 \leq \left\| f(k, k', x) \right\|^2_{L^2_{\bar{u}}(\mathbb{R}^n)} \left( \left\| \beta \right\|^2_{L^2(\mathbb{R}^n)} + \left\| \beta_j \right\|^2_{L^2(\mathbb{R}^n)} \right) \leq \frac{8X}{3M^2} \left\| r_{\sigma_m} \right\|^2_{L^2} \left\| \xi_1(x) \right\|^2 \in L^1(\mathbb{R}^3) .
\]

Therefore, since $\tilde{c}_{\alpha\alpha\bar{u}}$ is compact and $\pi_2 \otimes \pi_2$ is bounded, it follows that $\tilde{b}_{\alpha\alpha\bar{u}}$ is compact. Again, that implies the result holds also for the adjoint term
\[
\tilde{\xi}_1aa\psi = -\frac{1}{2\sqrt{2}M} \int_{\mathbb{R}^3} \left( a^*(r_{oo} e^{-ik\cdot x}) \right)^2 \tilde{\xi}_1(x)\psi(x) dx .
\]

3) $\tilde{\xi}_1aa\psi$, $\tilde{\xi}_1aa\psi*$. 
\[
\tilde{\xi}_1aa\psi = -\frac{1}{\sqrt{2}M} \int_{\mathbb{R}^3} a^*(r_{oo} e^{-ik\cdot x}) a(r_{oo} e^{-ik\cdot x}) \tilde{\xi}_1(x)\psi(x) dx ,
\]
\[
\tilde{\xi}_1aa\psi* = -\frac{1}{\sqrt{2}M} \int_{\mathbb{R}^3} a^*(r_{oo} e^{-ik\cdot x}) a(r_{oo} e^{-ik\cdot x}) \tilde{\xi}_1(x)\psi^*(x) dx .
\]

The proof for this couple of terms goes on exactly like the previous one, i.e. approximating $r_{oo}$ with $r_{\sigma_{m}}$ and showing that the corresponding operator $\tilde{c}_{\alpha\alpha\bar{u}}$ is compact, for it maps weakly convergent sequences into strongly convergent ones.
4) $\tilde{\xi}_2\alpha^*\psi, \xi_2\alpha^*\psi$.

$$\tilde{\xi}_2\alpha^*\psi = -\frac{\sqrt{2}}{M} \int_{\mathbb{R}^6} \text{Im}(\xi_2(k')\tilde{r}_\infty(k')e^{ik'x})a(r_\infty e^{-ikx})\psi^*(x)\psi(x)dxdk'.$$

We approximate it by the symbol $\tilde{\xi}_2\alpha\bar{u}^{(m)}$ defined by:

$$\tilde{\xi}_2\alpha^*\psi^{(m)} = (\tilde{\xi}_2\alpha\bar{u}^{(m)})^{\text{Wick}} = -\frac{\sqrt{2}}{M} \int_{\mathbb{R}^6} \psi^*(x)\chi_m(D_x)\text{Im}(\xi_2(k')\tilde{r}_\infty(k')e^{ik'x})a(r_\sigma e^{-ikx})\psi(x)dxdk'$$

where $\chi_m$ is the smooth cut-off function defined at the beginning of Section II, while $r_\sigma$ is the usual regularization of $r_\infty$ defined above. First of all we check that the approximation satisfies (87). By the chain rule, two parts have to be checked:

$$\left|\langle (H_0 + 1)^{-1/2} \Phi, (\tilde{\xi}_2\alpha^*\psi - \tilde{\xi}_2\alpha^*\psi^{(m)}) (H_0 + 1)^{-1/2} \Psi \rangle \right| \leq \frac{\sqrt{2}\pi\sqrt{2}}{M} \left(\int_{\mathbb{R}^3} dx \psi^*(x)(1 - \chi_m(D_x))\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a(r_\infty e^{-ikx})\psi(x)(H_0 + 1)^{-1/2}\Psi\right)$$

$$+ \left|\langle (H_0 + 1)^{-1/2} \Phi, \int_{\mathbb{R}^3} dx \psi^*(x)\chi_m(D_x)\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a((r_\infty - r_\sigma)e^{-ikx})\psi(x)(H_0 + 1)^{-1/2}\Psi\rangle \right|;$$

and we will consider them separately. For the first part we have:

$$\left|\langle (H_0 + 1)^{-1/2} \Phi, \int_{\mathbb{R}^3} dx \psi^*(x)(1 - \chi_m(D_x))\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a(r_\infty e^{-ikx})\psi(x)(H_0 + 1)^{-1/2}\Psi\rangle \right|$$

$$\leq \sum_{n=0}^{\infty} n\epsilon\left|\langle (H_0^{(n)} + 1)^{-1/2} \Phi, (1 - \chi_m(D_\lambda))\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a((r_\infty - r_\sigma)e^{-ikx})(H_0^{(n)} + 1)^{-1/2}\Psi \rangle \right|_{H^2}$$

$$\leq \sum_{n=0}^{\infty} n\epsilon\left|\langle (1 - D_\lambda^{1/2} (1 - \chi_m(D_\lambda)))\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a((r_\infty - r_\sigma)e^{-ikx})(H_0^{(n)} + 1)^{-1/2}\Psi \rangle \right|_{\mathcal{F}^{H^{1/2}}}$$

$$\leq (1 + \epsilon)\left\|\xi_2\right\|_{\mathcal{F}^{H^{1/2}}} \cdot \left\|\omega^{-1/2}r_\infty\right\|_2 \cdot \left\|\langle (1 - D_\lambda^{1/2} (1 - \chi_m(D_\lambda)))\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a((r_\infty - r_\sigma)e^{-ikx})(H_0^{(n)} + 1)^{-1/2}\Psi \rangle \right|_{\mathcal{F}^{L^2(\mathbb{R}^3)}}$$

where in the last inequality we have utilized the following bound:

$$n\epsilon\left|\langle (1 - D_\lambda)(H_0^{(n)} + 1)^{-1/2} \Phi \rangle_{\mathcal{F}^{H^{1/2}}} \cdot \left\|\omega^{-1/2}r_\infty\right\|_2 \cdot \left\|\langle (1 - D_\lambda^{1/2} (1 - \chi_m(D_\lambda)))\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a((r_\infty - r_\sigma)e^{-ikx})(H_0^{(n)} + 1)^{-1/2}\Psi \rangle \right|_{\mathcal{F}^{L^2(\mathbb{R}^3)}} \cdot \left\|\langle (N_1 + \epsilon)\right\|_{\mathcal{F}^{H^{1/2}}} \cdot \left\|\left\|\langle (1 - D_\lambda^{1/2} (1 - \chi_m(D_\lambda)))\text{Im}\mathcal{F}^{-1}(\xi_2\tilde{r}_\infty(x))a((r_\infty - r_\sigma)e^{-ikx})(H_0^{(n)} + 1)^{-1/2}\Psi \rangle \right|_{\mathcal{F}^{L^2(\mathbb{R}^3)}} \right.$$
i.e. it satisfies (87), for \( \lim_{m \to \infty} ||\omega^{-1/2}(r_{\infty} - r_{\sigma_m})||_2 = 0 \). Now it remains to show that \( \tilde{\xi}_2 \alpha \tilde{u}(m) \) is a compact symbol:

\[
\tilde{\xi}_2 \alpha \tilde{u}(m) = -\frac{(2\pi)^{3/2}}{M} \int_{\mathbb{R}^6} \tilde{u}(x) \chi_m(D_x) \text{Im} F^{-1}(\tilde{\xi}_2 \tilde{r}_{\infty})(x) \tilde{r}_{\sigma_m}(k) e^{i k \cdot x} \alpha(k) u(x) dx dk .
\]

As for the previous terms, we define an operator \( b_{\alpha \tilde{u}} : (L^2 \oplus L^2)^{\otimes 2} \to L^2 \oplus L^2 \) by

\[
b_{\alpha \tilde{u}} : (u, \alpha) \otimes (u, \alpha) \mapsto \alpha(k) u(x) \in L^2(\mathbb{R}^6) \mapsto \left( f'(x, D_x) u(x) \oplus 0 \right) \in L^2 \oplus L^2 ;
\]

where \( f'(x, D_x) = -\frac{(2\pi)^{3/2}}{M} \chi_m(D_x) F^{-1}(\tilde{r}_{\sigma_m} \alpha)(x) \text{Im} F^{-1}(\tilde{\xi}_2 \tilde{r}_{\infty})(x) \). We can easily prove that \( f' : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) is a compact operator. The cutoff function \( \chi_m \in L_0^{\infty}(\mathbb{R}^3) \) by hypothesis\(^\text{17}\). Now both \( \tilde{r}_{\sigma_m} \alpha \) and \( \tilde{\xi}_2 \tilde{r}_{\infty} \) belong to \( L^1(\mathbb{R}^3) \), since \( r_{\sigma_m}, \omega, \omega_{1/2} \tilde{\xi}_2 \omega_{-1/2} \tilde{r}_{\infty} \in L^2(\mathbb{R}^3) \). Therefore \( F^{-1}(\tilde{r}_{\sigma_m} \alpha) \text{Im} F^{-1}(\tilde{\xi}_2 \tilde{r}_{\infty}) \in L_0^{\infty}(\mathbb{R}^3) \), hence \( f'(x, D_x) \in \mathcal{K}(L^2(\mathbb{R}^3)) \). It immediately follows that \( b_{\alpha \tilde{u}} \) is compact, and the proof is complete. As usual, this result implies the one for the adjoint term

\[
\tilde{\xi}_2 \alpha^* \psi^* \psi = -\frac{\sqrt{3}}{M} \int_{\mathbb{R}^6} \text{Im}(\tilde{\xi}_2(k') \tilde{r}_{\infty}(k') e^{i k' \cdot x}) \alpha^*(r_{\infty} e^{-i k' \cdot x}) \psi^*(x) \psi(x) dx dk'.
\]

5) \( D_x a \tilde{\xi}_1 \psi, \ a^* D_x \psi^* \tilde{\xi}_1, D_x a \psi^* \tilde{\xi}_1, \ a^* D_x \tilde{\xi}_1 \psi \).

\[
D_x a \tilde{\xi}_1 \psi = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \tilde{\xi}_1(x) D_x a(r_{\infty} e^{-i k \cdot x}) \psi(x) dx .
\]

The approximated symbol \( D_x a \tilde{\xi}_1 \psi(m) \) is given by

\[
D_x a \tilde{\xi}_1 \psi(m) = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \tilde{\xi}_1(x) D_x a(r_{\sigma_m} e^{-i k \cdot x}) \psi(x) dx .
\]

First of all we prove that (87) is satisfied. Given \( \Phi \in \mathcal{H} \), we denote by \( \Phi_{n,p} \) its restriction to the subspace \( \mathcal{H}_{n,p} = \left( L^2(\mathbb{R}^3) \right)^{\otimes n} \otimes \left( L^2(\mathbb{R}^3) \right)^\otimes_p \) with \( n \) nucleons and \( p \) mesons. We also denote by \( X_n = \{ x_1, \ldots, x_n \} \) a set of variables, \( dX_n = dx_1 \cdots dx_n \) the corresponding Lebesgue measure (and analogously for \( K_p, dK_p \)). The proof is obtained by a direct calculation on the

---

\(^{17}\) We denote by \( L_0^{\infty}(\mathbb{R}^3) \) the set of bounded functions on \( \mathbb{R}^3 \) that vanish at infinity.
Fock space as follows:

\[
\left\| \left( (H_0 + 1)^{-1/2} \Phi, \int_{\mathbb{R}^3} \tilde{\xi}_1(x) D_x a((r_{\infty} - r_{\sigma_m}) e^{-ik \cdot x}) \psi(x) dx (H_0 + 1)^{-1/2} \Psi \right) \right\|
\]

\[= \left| \sum_{n, p=0}^{\infty} \sqrt{\varepsilon(n + 1)(p + 1)} \int_{\mathbb{R}^{n+p+2d}} \left( (H_0 + 1)^{-1/2} \Phi \right)_{n, p} (X_n; K_p) \tilde{\xi}_1(x) D_x (r_{\infty} - \tilde{r}_{\sigma_m})(k) e^{ik \cdot x} \right| \]

\[\leq \sum_{n, p=0}^{\infty} \sqrt{\varepsilon(n + 1)} \left\| \left( (H_0 + 1)^{-1/2} \Phi \right)_{n, p} (X_n; K_p) D_x \tilde{\xi}_1(x) \right\| \left\| (\omega(1))(H_0 + 1)^{-1/2} \Phi \right\| \]

\[\leq \sum_{n, p=0}^{\infty} \sqrt{\varepsilon(n + 1)} \left\| (-\Delta + V)^{1/2} \tilde{\xi}_1 \right\|_2 \cdot \left\| \omega^{-1/2}(r_{\infty} - r_{\sigma_m}) \right\|_2 \cdot \left\| (H_0 + 1)^{-1/2} \Phi \right\| \]

where in the last bound we have used Schwartz’s inequality and the fact that \( p \omega(k_j) \equiv \sum_{j=1}^{\tilde{p}} \omega(k_j) \) when acting on vectors of \( \mathcal{H}_{n,p} \). Now, since \( \lim_{m \to \infty} \left\| \omega^{-1/2}(r_{\infty} - r_{\sigma_m}) \right\|_2 = 0 \), Equation (87) holds with \( \mathcal{C}^{(m)}(\tilde{\xi}_1) = \frac{1}{\sqrt{2M}} \left\| (-\Delta + V)^{1/2} \tilde{\xi}_1 \right\|_2 \cdot \left\| \omega^{-1/2}(r_{\infty} - r_{\sigma_m}) \right\|_2 \). It remains to show that the classical symbol

\[D_x a \tilde{\xi}_1 u^{(m)} = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^6} \tilde{\xi}_1(x) D_x a(k) \tilde{r}_{\sigma_m}(k) e^{ik \cdot x} u(x) dx \]

is compact. Here we have written \( D_x a \tilde{\xi}_1 u^{(m)} = \langle \tilde{\xi}_1, D_x v \rangle \), with \( v_x(x) = \frac{(2\pi)^{3/2}}{\sqrt{2M}} F^{-1}(a \tilde{r}_{\sigma_m})(x) u(x) \); and that is defined for any \( u \in H^1(\mathbb{R}^3) \). However, since \( \tilde{\xi}_1 \in Q(-\Delta + V) \subset H^1(\mathbb{R}^3) \) and \( D_x \) is self-adjoint, we can write \( D_x a \tilde{\xi}_1 u^{(m)} = \langle D_x \tilde{\xi}_1, v \rangle \) for any \( v \in L^2(\mathbb{R}^3) \). It follows that \( D_x a \tilde{\xi}_1 u^{(m)} \) is defined for any \( u, a \in L^2(\mathbb{R}^3) \), since \( a, r_{\sigma_m} \in L^2 \) implies \( a \tilde{r}_{\sigma_m} \in L^1 \), and therefore \( F^{-1}(a \tilde{r}_{\sigma_m}) \in L^\infty \). It follows that the operator \( \tilde{b}_{D_x a u} : (L^2(\mathbb{R}^3))^2 \to \mathbb{C} \) defined as

\[\tilde{b}_{D_x a u} : (u, a) \xrightarrow{\otimes^2} \alpha(k) u(x) \in L^2(\mathbb{R}^6) \longrightarrow \langle f^\prime, \alpha u \rangle_{L^2(\mathbb{R}^6)} \in \mathbb{C} \]

is bounded and finite rank, and therefore compact.

\[a^* D_x \tilde{\xi}_1 \psi = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \tilde{\xi}_1(x) a^*(r_{\infty} e^{-ik \cdot x}) D_x \psi(x) dx \]

Again, the approximated symbol \( a^* D_x \tilde{\xi}_1 \psi \) is given by

\[a^* D_x \tilde{\xi}_1 \psi^{(m)} = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \tilde{\xi}_1(x) a^*(r_{\sigma_m} e^{-ik \cdot x}) D_x \psi(x) dx \]
Equation (87) is satisfied, and the proof follows the same guidelines as the one for the previous term \( D_x a \xi_1 \psi \). We give the compactness proof for the symbol

\[
\tilde{a} D_x \xi_1 u^{(m)} = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \tilde{\xi}_1(x) \tilde{a}(k) R_{\sigma_m}(k) e^{-ik \cdot x} D_x u(x) dx dk .
\]

We rewrite it as \( \tilde{a} D_x \xi_1 u^{(m)} = \langle (u, \alpha), \tilde{b}_0 D_{x,u}(u, \alpha) \rangle_{L^2 \otimes \mathbb{C}^2} \), with \( \tilde{b}_0 D_{x,u} : L^2 \oplus L^2 \rightarrow L^2 \oplus L^2 \) defined as

\[
\tilde{b}_0 D_{x,u} : (u, \alpha) \in L^2 \oplus L^2 \rightarrow u(x) \in L^2(\mathbb{R}^3) \rightarrow \begin{pmatrix} 0 & f'''(k) \end{pmatrix} \in L^2 \oplus L^2 ,
\]

where \( f'''(k) = \frac{1}{\sqrt{2M}} R_{\sigma_m}(k) \left( k \langle e^{ik \cdot x} \xi_1, u \rangle_{L^2_1} + \langle e^{ik \cdot x} D_x \xi_1, u \rangle_{L^2_1} \right) \). Now suppose that \( u_j \rightharpoonup u \) is a weakly convergent (bounded) sequence with bound \( X \). It follows that, uniformly in \( j \),

\[
|f'''(k_j)|^2 = \left| \frac{1}{\sqrt{2M}} R_{\sigma_m}(k) \left( k \langle e^{ik \cdot x} \xi_1, u_j \rangle_{L^2_1} + \langle e^{ik \cdot x} D_x \xi_1, u_j \rangle_{L^2_1} \right) \right|^2 \leq \frac{1}{2M^2} X^2 |r_{\sigma_m}(k)|^2 (k^2 + 1) ||\xi_1||^2_{L^2_1} \in L^1(\mathbb{R}^3) .
\]

In addition, \( \lim_{j \to \infty} |f'''(k_j) - f'''(k)|^2 = 0 \); therefore \( \tilde{c}_{a D_{x,u}} \) is a compact operator by Lebesgue’s dominated convergence theorem. So \( \tilde{b}_{0 D_{x,u}} \) is compact. The proofs above extend immediately to the adjoint terms

\[
\begin{align*}
\tilde{a}^* D_x \psi^* \xi_1 & = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) a^* (r_{\infty} e^{-ik \cdot x}) D_x \xi_1(x) dx ; \\
D_x a^* \psi^* \xi_1 & = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) D_x a (r_{\infty} e^{-ik \cdot x}) \xi_1(x) dx .
\end{align*}
\]

6) \( \psi^* D_x \xi_2 \psi, \psi^* \tilde{\xi}_2 D_x \psi \).

\[
\psi^* D_x \xi_2 \psi = \frac{(2\pi)^{3/2}}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) D_x F^{-1} (\xi_2 \tilde{\xi}) (x) \psi(x) dx .
\]

The approximated symbol, as for the terms of point 4, contains \( \chi_m(D_x) \):

\[
\psi^* D_x \xi_2 \psi^{(m)} = \frac{(2\pi)^{3/2}}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) \chi_m(D_x) D_x F^{-1} (\xi_2 \tilde{\xi}) (x) \psi(x) dx .
\]

As usual, we start proving that (87) holds. We remark that this is the only term where we
need $\xi_2 \in D(\omega^{3/4})$ instead of $D(\omega^{1/2})$.

\[
\left\langle (H_0 + 1)^{-1/2} \Phi, \int_{\mathbb{R}^3} \psi^*(x)(1 - \chi_m(D_x))D_x \mathcal{F}^{-1}(\xi_2 \tilde{r}_\infty)(x)\psi(x)dx \right\rangle \\
\leq \sum_{n=0}^{\infty} n\epsilon \left\| (H_0 + 1)^{-1/2} \Phi_n, (1 - \chi_m(D_{x_1}))D_{x_1} \mathcal{F}^{-1}(\xi_2 \tilde{r}_\infty)(x_1)(H_0 + 1)^{-1/2} \Psi_n \right\|
\leq \sum_{n=0}^{\infty} n\epsilon \left\| (1 - \Delta)^{-1/2}(1 - \chi_m(D_x)) \right\|_{L^2(\mathbb{R}^3)} \cdot \left( \left\| \mathcal{F}^{-1}(\xi_2 \tilde{r}_\infty) \right\|_{\infty} + \left\| \mathcal{F}^{-1}(k\xi_2 \tilde{r}_\infty) \right\|_{\infty} \right)
\cdot \left\| (1 - \Delta_{x_1})^{-1/2}(H_0 + 1)^{-1/2} \Phi_n \right\|_{\mathcal{H}_n} \cdot \left( \left\| D_{x_1}(H_0 + 1)^{-1/2} \Psi_n \right\|_{\mathcal{H}_n} + \left\| (H_0 + 1)^{-1/2} \Psi_n \right\|_{\mathcal{H}_n} \right)
\leq 2 \left\| (1 - D_x^2)^{-1/2}(1 - \chi_m(D_x)) \right\|_{L^2(\mathbb{R}^3)} \cdot \left\| \omega^{3/4} \xi_2 \right\|_2 \cdot \left\| \omega^{-1/4} r_\infty \right\|_2 \cdot \left\| \Phi \right\|
\cdot \left( \left\| (N_1 + \tilde{\epsilon})^{1/2} \Psi \right\| + \left\| \Psi \right\| \right);
\]

hence the result follows with

\[
C^{(m)}(\xi_2) = \frac{2\sqrt{2(2\pi)^3}}{M} \left\| (1 - D_x^2)^{-1/2}(1 - \chi_m(D_x)) \right\|_{L^2(\mathbb{R}^3)} \cdot \left\| \omega^{3/4} \xi_2 \right\|_2 \cdot \left\| \omega^{-1/4} r_\infty \right\|_2 ,
\]

since $\lim_{m \to \infty} \left\| (1 - D_x^2)^{-1/2}(1 - \chi_m(D_x)) \right\|_{L^2(\mathbb{R}^3)} = 0$. It remains to show that the symbol $\tilde{u}D_x \xi_2 u^{(m)}$ is compact. We introduce the operator $\tilde{b}_{\tilde{u}D_x u} : L^2 \oplus L^2 \to L^2 \oplus L^2$ such that $\tilde{u}D_x \xi_2 u^{(m)} = \langle (u, \alpha), \tilde{b}_{\tilde{u}D_x u}(u, \alpha) \rangle_{L^2 \oplus L^2}$:

\[
\tilde{b}_{\tilde{u}D_x u} : (u, \alpha) \in L^2 \oplus L^2 \mapsto u(x) \in L^2(\mathbb{R}^3) \mapsto \left( f'''(x, D_x)u(x) \oplus 0 \right) \in L^2 \oplus L^2 ,
\]

where $f'''(x, D_x) = \frac{2(2\pi)^{3/2}}{\sqrt{2M}} D_x \chi_m(D_x) \mathcal{F}^{-1}(\xi_2 \tilde{r}_\infty)(x)$. Now $f'''(x, D_x)$ is a compact operator: both $\chi_m(x)(x)$ and $\mathcal{F}^{-1}(\xi_2 \tilde{r}_\infty)(x)$ are in $L^2_0(\mathbb{R}^3)$. Therefore $\tilde{b}_{\tilde{u}D_x u}$ is compact. The proof extends immediately to the adjoint term

\[
\psi^* \tilde{\xi}_2 D_x \psi = \frac{2(2\pi)^{3/2}}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) \mathcal{F}(\tilde{\xi}_2 \tilde{r}_\infty)(x)D_x \psi(x)dx .
\]

3. Defining the time-dependent family of Wigner measures.

The last tool we need in order to take the limit $\epsilon \to 0$ of the integral formula (76) are Wigner measures. Throughout this section, we will leave some statements unproven; the reader may refer to [8, Section 6] for the proofs, and a detailed discussion of Wigner measures properties. We recall the definition of a Wigner measure associated with a family of states on $\mathcal{H} = \Gamma_\mathbb{R}(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3))$. 


**Definition V.11.** Let \((\varrho_\varepsilon)_{\varepsilon \in (0, \bar{\varepsilon})} \subset L^1(\mathcal{H})\) be a family of normal states; \(\mu \in \mathfrak{P}(L^2 \oplus L^2)\) a Borel probability measure. We say that \(\mu\) is a Wigner (or semiclassical) measure associated to \((\varrho_\varepsilon)_{\varepsilon \in (0, \bar{\varepsilon})}\), or in symbols \(\mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0, \bar{\varepsilon}))\), if there exist a sequence \((\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \bar{\varepsilon})\) such that \(\lim_{k \to \infty} \varepsilon_k = 0\) and

\[
\lim_{k \to \infty} \text{Tr} [\varrho_{\varepsilon_k} W(\xi)] = \int_{L^2 \oplus L^2} e^{i \sqrt{2} \text{Re}(\xi, z) t^2} d\mu(z), \quad \forall \xi \in L^2 \oplus L^2.
\]

We remark that the right-hand side is essentially the Fourier transform of the measure \(\mu\), so considering the sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) there is at most one probability measure that could satisfy (88). If (88) is satisfied, we say that to the sequence \((\varrho_{\varepsilon_k})_{k \in \mathbb{N}}\) corresponds a single Wigner (or semiclassical) measure \(\mu\), or simply \(\varrho_{\varepsilon_k} \to \mu\).

First of all, it is necessary to ensure that such a definition of Wigner measures is meaningful, i.e. that under suitable conditions the set of Wigner measures \(\mathcal{M}\) associated to a family of states is not empty. Since \(m_0 > 0\), it turns out that Assumption \((A'_p)\) is sufficient. Assumption \((A_p)\) would be sufficient as well, even if we will not use it for the moment.

**Lemma V.12.** Let \((\varrho_\varepsilon)_{\varepsilon \in (0, \bar{\varepsilon})}\) be a family of normal states on \(\mathcal{H}\), that satisfies Assumptions \((A'_p)\) and \((A_0)\). Then for any \(t \in \mathbb{R}\):

\begin{enumerate}[label=(\roman*)]
\item \(\mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0, \bar{\varepsilon})) \neq \emptyset\); \(\mathcal{M}(\tilde{\varrho}_\varepsilon(t), \varepsilon \in (0, \bar{\varepsilon})) \neq \emptyset\).
\item Any \(\mu \in \mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0, \bar{\varepsilon}))\) or in \(\mathcal{M}(\tilde{\varrho}_\varepsilon(t), \varepsilon \in (0, \bar{\varepsilon}))\)\footnote{In this section, we have used mostly the notation \(D(\omega^{1/2})\); however \(D(\omega^{1/2}) = \mathcal{F}H^{1/2}\), where the latter is defined in Definition III.4.} satisfies:
\[
\mu(B_u(0, \sqrt{\varepsilon}) \cap Q(-\Delta + V) \oplus D(\omega^{1/2})) = 1
\]
\item Moreover
\[
\int_{z=(u, \alpha) \in L^2 \oplus L^2} \|(-\Delta + V)^{1/2} u\|_2^2 + \|\alpha\|_{H^{1/2}}^2 d\mu(z) < \infty.
\]
\end{enumerate}

We recall that \(B_u(0, \sqrt{\varepsilon}) = \{(u, \alpha) \in L^2 \oplus L^2, \|u\|_2 \leq \sqrt{\varepsilon}\} \).

**Proof.** By (85) of Lemma V.8, we see that \(\varrho_\varepsilon(t)\) and \(\tilde{\varrho}_\varepsilon(t)\) satisfy \((A_0)\) and \((A'_p)\) at any time. Now \((i)\) follows by [8, Theorem 6.2] and \((ii)\) by \((iii)\) and [10, Lemma 2.14]. The third point is essentially a consequence of [11, Lemma 3.12]. However the latter result requires more regularity on the states \(\varrho_\varepsilon\). So we indicate here how to adapt the argument to our case. It is enough to assume \(t = 0\) and \(|\mu| = \mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0, \bar{\varepsilon}))\). The operators \(\frac{A}{2M} + V\) and \(\omega\) are

\[
\text{Tr} [\varrho_{\varepsilon_k} W(\xi)] = \int_{L^2 \oplus L^2} e^{i \sqrt{2} \text{Re}(\xi, z) t^2} d\mu(z), \quad \forall \xi \in L^2 \oplus L^2.
\]
positive (self-adjoint). So one can found non-decreasing sequences of finite rank operators $A_k$ and $B_k$ that converge weakly to $-\frac{\Delta}{2M} + V$ and $\omega$ respectively. In particular

$$b_k^{\text{wick}} = d\Gamma(A_k) \otimes 1 + 1 \otimes d\Gamma(B_k) \leq d\Gamma(-\frac{\Delta}{2M} + V) \otimes 1 + 1 \otimes d\Gamma(\omega) = H_0,$$

where $b_k(u, \alpha) = \langle u, A_k u \rangle + \langle \alpha, B_k \alpha \rangle \in \mathcal{P}^{\infty}_{1,1}(L^2 \oplus L^2)$. Let $P_k$ and $Q_k$ be the orthogonal projections on Ran($A_k$) and Ran($B_k$) respectively. Using the Fock space decomposition $\Gamma_s(L^2 \oplus L^2) \equiv \Gamma_s(P_k L^2 \oplus Q_k L^2) \otimes \Gamma_s(P_k^\perp L^2 \oplus Q_k^\perp L^2)$ where $P_k^\perp = 1 - P_k$ and $Q_k^\perp = 1 - Q_k$; one can writes $b_k^{\text{wick}} = (b_k)^{\text{Wick}}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)} \otimes 1_{\Gamma_s(P_k^\perp L^2 \oplus Q_k^\perp L^2)}$ and $Q_\epsilon \equiv \tilde{Q}_\epsilon$. Hence

$$\text{Tr}[Q_\epsilon b_k^{\text{wick}}] = \text{Tr}[\tilde{Q}_\epsilon b_k^{\text{Wick}}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)} \otimes 1_{\Gamma_s(P_k^\perp L^2 \oplus Q_k^\perp L^2)}] = \text{Tr}[\tilde{Q}_\epsilon b_k^{\text{Wick}}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}],$$

where $\tilde{Q}_\epsilon$ is a given reduced density matrix which is trace-class in $\Gamma_s(P_k L^2 \oplus Q_k L^2)$. So the problem is in some sense reduced to finite dimension. Now using Wick calculus (in finite dimension) $b_k^{\text{Wick}}$ can be written as an Anti-Wick operator by moving all the $a^\dagger$ to the right of $a$. So, one obtains $b_k^{\text{Wick}} = b_k^{A-\text{Wick}} + \epsilon T$ with $T(d\Gamma(P_k \oplus Q_k) + 1)^{-1}$ is bounded uniformly with respect to $\epsilon \in (0, \bar{\epsilon})$. Hence

$$\lim_{\epsilon \to 0} \text{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}[\epsilon^k e^{b_k^{A-\text{Wick}}}] = \lim_{\epsilon \to 0} \text{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}[\epsilon^k b_k^{\text{Wick}}] \leq \lim_{\epsilon \to 0} \text{Tr}_{\Gamma_s}(Q_\epsilon H_0) \leq C.$$

For details on the Anti-Wick quantization we refer the reader to [8]; in particular it is a positive quantization (see e.g. [8, Proposition 3.6]). Hence, we see that

$$\text{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}[\epsilon^k e^{b_k^{A-\text{Wick}}}] \leq \text{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}[\epsilon^k b_k^{A-\text{Wick}}]$$

where $b_k^{\epsilon,\chi}(u, \alpha) = \chi(u)\langle u, A_k u \rangle + \chi(\alpha)\langle \alpha, B_k \alpha \rangle$ for any cutoff function $\chi \in C^\infty_0(\mathbb{R}^3)$, $0 \leq \chi \leq 1$. Finally [8, Theorem 6.2] gives

$$\int_{z = (u, \alpha) \in L^2 \oplus L^2} b_k^{\epsilon,\chi}(u, \alpha) \, d\mu(z) = \lim_{\epsilon \to 0} \text{Tr}[Q_\epsilon b_k^{A-\text{Wick}}] = \lim_{\epsilon \to 0} \text{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}[\epsilon^k b_k^{A-\text{Wick}}] \leq C,$$

and the monotone convergence theorem proves $(iii)$. 

As we said above, our aim is to take the limit $\epsilon_k \to 0$ on the integral equation (76), for a suitable sequence contained in $(0, \bar{\epsilon})$. We may suppose that the sequence $(\epsilon_k)_{k \in \mathbb{N}}$ is chosen in such a way that there exist $\mu_0 \in \mathcal{M}(Q_\epsilon, \epsilon \in (0, \bar{\epsilon}))$ such that $(88)$ holds, i.e. $\mathcal{M}(Q_{\epsilon_k}, k \in \mathbb{N}) = \{\mu_0\}$. However, nothing a priori ensures that the sequence, or one of its subsequences $(\epsilon_k)_{k \in \mathbb{N}} \subset (\epsilon_k)_{k \in \mathbb{N}}$, is such that for any $t \in \mathbb{R}$:

$$\lim_{t \to \infty} \text{Tr}_{\Gamma_s}(Q_\epsilon(t) W(\xi)) = \int_{L^2 \oplus L^2} e^{i\sqrt{\Re(e, \xi)} d\tilde{\mu}_0(z)} , \forall \xi \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3);$$

where $\tilde{\mu}_0 : \mathbb{R} \to \Psi(L^2 \oplus L^2)$ is a map such that $\tilde{\mu}_0 = \mu_0$. The possibility of extracting such a common subsequence is crucial, since the integral equation involves all measures from zero to an arbitrary time $t$. To prove it is possible, we exploit the uniform continuity properties of $\text{Tr}(Q_\epsilon(t) W(\xi))$ in both $t$ and $\xi$, proved in the following lemma.
Lemma V.13. Let \((Q_e)_{e \in (0, \varepsilon)}\) be a family of quantum states on \(\mathcal{H}\) that satisfies Assumptions \((A_0)\) and \((A'_p)\). Then the family of functions \((t, \xi) \mapsto \tilde{G}_e(t, \xi) := \text{Tr}[\tilde{\xi}(t)W(\xi)]\) is uniformly equicontinuous on bounded subsets of \(\mathbb{R} \times (Q(-\Delta + V) \oplus D(\omega^{1/2}))\).

Proof. Let \((t, \xi), (s, \eta) \in \mathbb{R} \times (Q(-\Delta + V) \oplus D(\omega^{1/2}))\). Without loss of generality, we may suppose that \(s \leq t\). We write

\[
\left| \tilde{G}_e(t, \xi) - \tilde{G}_e(s, \eta) \right| \leq \left| \tilde{G}_e(t, \eta) - \tilde{G}_e(s, \eta) \right| + \left| \tilde{G}_e(t, \xi) - \tilde{G}_e(t, \eta) \right|;
\]

and define \(X_1 := \left| \tilde{G}_e(t, \eta) - \tilde{G}_e(s, \eta) \right|, X_2 := \left| \tilde{G}_e(t, \xi) - \tilde{G}_e(t, \eta) \right|\). Consider \(X_1\); we get by standard manipulations and Lemma V.2:

\[
X_1 \leq \sum_{j=0}^{3} \varepsilon^j \mathcal{L}(\eta) \sum_{i \in \mathbb{N}} \lambda_i \int_s^\infty \left| \langle e^{-i\tilde{\xi}B \text{ren} \Psi r}, W((\tilde{\eta})_s B (\tilde{\eta})_s) e^{-i\tilde{\xi}B \text{ren} \Psi r} \rangle \right| ds.
\]

Now using Lemma V.6 we obtain

\[
X_1 \leq \sum_{j=0}^{3} \varepsilon^j \mathcal{L}(\eta) \sum_{i \in \mathbb{N}} \lambda_i \int_s^\infty \left| (N_1 + H_0 + \tilde{\varepsilon})^{1/2} W^*((\tilde{\eta})_s) e^{-i\tilde{\xi}B \text{ren} \Psi r} \right| \left| (N_1 + H_0 + \tilde{\varepsilon})^{1/2} e^{-i\tilde{\xi}B \text{ren} \Psi r} \right| ds;
\]

then using Lemma V.1, and the fact that \(\|(\tilde{\eta})_s\|_{H^1} = \|\eta_1\|_{H^1}, \|(\tilde{\eta})_s\|_{H^{1/2}} = \|\eta_2\|_{H^{1/2}}\) we get

\[
X_1 \leq C(\eta) \sum_{j=0}^{3} \varepsilon^j \mathcal{L}(\eta) \int_s^\infty \text{Tr}[\tilde{\xi}(s) (N_1 + H_0 + \tilde{\varepsilon})] ds \leq \lambda - s [C(\eta) \sum_{j=0}^{3} \varepsilon^j \mathcal{L}(\eta) \left( \frac{C}{1 - a(\xi)} + \frac{2b(\xi)}{1 - a(\xi)} + \tilde{\varepsilon} \right) ;
\]

where in the last inequality we used Equation (85) of Lemma V.8. Now let’s consider \(X_2\); a standard manipulation using Weyl’s relation yields

\[
X_2 \leq \left\| \langle e^{i\tilde{\xi} \text{Im}(\xi, \eta)t}_{L^2 \oplus L^2} W(\xi - \eta) - 1 \rangle (N_1 + N_2 + 1)^{-1} \right\|_{\mathcal{L}(\Gamma_1, \mathcal{L}^2 \oplus \mathcal{L}^2)} \text{Tr} \left[ \tilde{\xi}(t) (N_1 + N_2 + 1) \right].
\]

Now we use the estimate in \([8, \text{Lemma 3.1}]\) and obtain

\[
X_2 \leq \|\xi - \eta\|_{L^2 \oplus L^2} \left( \varepsilon \|\eta\|_{L^2 \oplus L^2} + 1 \right) \text{Tr} \left[ \tilde{\xi}(t) (N_1 + N_2 + 1) \right] \leq \|\xi - \eta\|_{L^2 \oplus L^2} \left( \varepsilon \|\eta\|_{L^2 \oplus L^2} + 1 \right) \left( \frac{C}{1 - a(\xi)} + \frac{2b(\xi)}{1 - a(\xi)} + 1 \right),
\]

where in the last inequality we used again Equation (85) of Lemma V.8, keeping in mind that \(N_2 \leq d\Gamma(\omega) \leq H_0\).
Now using Lemma V.13 with the estimates on $X_1, X_2$ above and a diagonal extraction argument, we prove the following proposition. We omit the proof since it is similar to [11, Proposition 3.9].

**Proposition V.14.** Let $(\varrho_{\varepsilon_j})_{j \in \mathbb{N}}$ be a family of quantum states on $\mathcal{H}$ that satisfies Assumptions (A_0) and (A' p). Then for any sequence $(\varepsilon_j)_{j \in \mathbb{N}} \subseteq (0, \bar{\varepsilon})$ with $\lim_{k \to \infty} \varepsilon_k = 0$, there exists a subsequence $(\varepsilon_{k_j})_{j \in \mathbb{N}}$ such that there exists a map $\mu_t : \mathbb{R} \to \Psi(L^2 \oplus L^2)$ verifying the following statements:

\[(89) \quad \varrho_{\varepsilon_{k_j}}(t) \to \mu_t, \forall t \in \mathbb{R} ;
\]

\[(90) \quad \tilde{\varrho}_{\varepsilon_{k_j}}(t) \to \tilde{\mu}_t, \forall t \in \mathbb{R}, \text{ with } \tilde{\mu}_t = \mathbb{E}_0(-t) \# \mu_t ;
\]

\[(91) \quad \varrho_{\varepsilon_{k_j}}(t) W(\bar{\xi}_i) \to \mu_{\xi,t}, \forall t \in \mathbb{R} \text{ and } \forall \xi \in L^2 \oplus L^2, \text{ with } d\mu_{\xi,t}(z) = e^{i \sqrt{2} \text{Re}(\bar{\xi}_i z)} d\mu_t(z) ;
\]

where $\mathbb{E}_0(t) z = e^{-it(-\Delta + V)} u \oplus e^{-it\alpha} \varphi_0$ is the Hamiltonian flow associated with the free classical energy $\mathfrak{E}_0$, and $\bar{\xi}_i = \mathbb{E}_0(-t) \xi$. Moreover, $\mu_t$ and $\tilde{\mu}_t$ are both Borel probability measures on $Q(-\Delta + V) \oplus D(\omega^{1/2})$.

4. **The classical limit of the integral formula.**

We are finally ready to discuss the limit $\varepsilon \to 0$ of the integral formula (76). As a final preparation, we state a couple of preliminary lemmas. The first is a slight improvement of [8, Theorem 6.13]. The second can be easily proved by standard estimates on the symbol $\mathcal{B}_0^{(m)}(\xi)$ which we recall for convenience:

\[(92) \quad \mathcal{B}_0^{(m)}(\xi)(u, \alpha) = 2i \sqrt{2} \left( \text{Re} \mathcal{F} \left( \frac{\chi_m}{\sqrt{2}\alpha} \right)(x), \text{Im} \left( \bar{\xi}_1 u \right)(x) \right)_2 + i \sqrt{2} \left( u(x), \chi_m(D_x) \text{Im} \left( \mathcal{F} \left( \frac{\chi_m}{\sqrt{2}\alpha} \right) \right)(x) u(x) \right)_2
\]

\[\quad + i \sqrt{2} \text{Im} \left( u(x), (\chi_m(D_x) V_{\infty} \ast \bar{\xi}_1 u)(x) u(x) \right)_2 + \frac{i}{\sqrt{2}M^{1/2}} \text{Im} \left( \bar{\xi}_1(x), (\mathcal{F}^{-1}(\bar{r}_\sigma, \alpha)^2 + \mathcal{F}(r_\sigma, \bar{\alpha})^2 + \mathcal{F}^{-1}(\bar{r}_\sigma, \alpha)\mathcal{F}(r_\sigma, \bar{\alpha})(x) u(x) \right)_2
\]

\[\quad - \frac{2\sqrt{2}\pi}{M^{1/2}} \left( u(x), \chi_m(D_x) \text{Im} \left( \mathcal{F}^{-1}(\bar{r}_\sigma, \alpha)(x) \right) \mathcal{F}^{-1}(\bar{r}_\sigma, \alpha)(x) u(x) \right)_2
\]

\[\quad - \frac{i}{\sqrt{2}M^{1/2}} \left( \bar{\xi}_1(x), D_x \mathcal{F}^{-1}(\bar{r}_\sigma, \alpha)(x) u(x) \right)_2 = \frac{i}{\sqrt{2}M^{1/2}} \left( \bar{\xi}_1(x), D_x \mathcal{F}^{-1}(\bar{r}_\sigma, \alpha)(x) D_x u(x) \right)_2
\]

\[\quad + \frac{i}{\sqrt{2}M^{1/2}} \left( u(x), \chi_m(D_x) D_x \mathcal{F}^{-1}(\bar{r}_\sigma, \alpha)(x) u(x) \right)_2.
\]

**Lemma V.15.** Let $(\varepsilon_j)_{j \in \mathbb{N}} \subseteq (0, \bar{\varepsilon}), \lim_{j \to \infty} \varepsilon_j = 0$, and $\delta > 0$. Furthermore, let $(\varrho_{\varepsilon_j})_{j \in \mathbb{N}}$ be a sequence of normal states in $\mathcal{H}$ such that for some $C(\delta) > 0$,

\[(93) \quad \left\| (N_1 + N_2)^{\delta/2} \varrho_{\varepsilon_j}(N_1 + N_2)^{\delta/2} \right\|_{L^1(L^2 \oplus L^2)} \leq C(\delta),
\]
uniformly in $\varepsilon \in (0, \tilde{\varepsilon})$. Suppose that $\varrho_{e_j} \to \mu \in \mathfrak{P}(L^2 \oplus L^2)$ then the following statement is true:

$$\left( \forall \mathcal{A} \in \bigoplus_{(p,q) \in \mathbb{N}^2} \mathcal{P}^{\infty}_{p,q}(L^2 \oplus L^2) \right), \lim_{j \to \infty} \text{Tr} \left[ \varrho_{e_j}(\mathcal{A})^{\text{Wick}} \right] = \int_{L^2 \oplus L^2} \mathcal{A}(z)d\mu(z).$$

**Proof.** By linearity it is enough to assume $\mathcal{A} \in \mathcal{P}^{\infty}_{p,q}(L^2 \oplus L^2)$ for $(p, q) \in \mathbb{N}^2$ with $p + q < 2\delta$. Let $(P_R)_{R>0}$ be an increasing family of finite rank orthogonal projections on $L^2$ such that the strong limit $s - \lim_{R \to +\infty} P_R = 1$ holds. Let $\mathcal{A}_R(z) := \mathcal{A}(P_R \oplus P_R \chi)$ for any $z \in L^2 \oplus L^2$. One writes

$$\left| \text{Tr} \left[ \varrho_{e_j}(\mathcal{A})^{\text{Wick}} \right] - \int_{L^2 \oplus L^2} \mathcal{A}(z)d\mu(z) \right| \leq \left| \text{Tr} \left[ \varrho_{e_j}(\mathcal{A})^{\text{Wick}} \right] - \text{Tr} \left[ \varrho_{e_j}(\mathcal{A}_R)^{\text{Wick}} \right] \right|$$

$$\left| \text{Tr} \left[ \varrho_{e_j}(\mathcal{A}_R)^{\text{Wick}} \right] - \int_{L^2 \oplus L^2} \mathcal{A}_R(z)d\mu(z) \right|$$

$$\left| \int_{L^2 \oplus L^2} \mathcal{A}_R(z)d\mu(z) - \int_{L^2 \oplus L^2} \mathcal{A}(z)d\mu(z) \right|.$$

Using standard number estimates and the regularity of the states $(\varrho_{e_j})_j$, one shows

$$\left| \text{Tr} \left[ \varrho_{e_j}(\mathcal{A} - \mathcal{A}_R)^{\text{Wick}} \right] \right| \leq ||(N_1 + N_2)^{\delta/2} \varrho_{e_j}(N_1 + N_2)^{\delta/2}||_{L^1(L^2 \oplus L^2)} \|\mathcal{A} - \mathcal{A}_R\|,$$

where $\mathcal{A}$ and $\mathcal{A}_R$ denote the compact operators satisfying $\mathcal{A}(z) = (z^{\otimes q}, \mathcal{A}_R^{\otimes p})$ and $\mathcal{A}_R(z) = (z^{\otimes q}, \mathcal{A}_R^{\otimes p})$ respectively. Since $\mathcal{A}_R = (P_R \oplus P_R)^{\otimes q}, \mathcal{A}_R^{\otimes p}$ and $\mathcal{A}_R$ is compact, one shows that $\lim_{R \to +\infty} \|\mathcal{A} - \mathcal{A}_R\| = 0$. So the right hand side of (94) can be made arbitrary small by choosing $R$ large enough.

According to [8, Theorem 6.2], the regularity of $(\varrho_{e_j})_j$ insures the bound

$$\int_{L^2 \oplus L^2} \|z\|_{L^2 \oplus L^2}^{2\delta} d\mu(z) \leq C(\delta).$$

Hence by dominated convergence the right hand side of (96) can also be made arbitrary small when $R$ is large enough since $\mathcal{A}(z)$ and $\mathcal{A}_R(z)$ are both bounded by $c||z||_{L^2 \oplus L^2}$ and $\mathcal{A}_R(z)$ converges pointwise to $\mathcal{A}(z)$.

To handle the right hand side of (95), we use a further regularization. Let $\chi \in C^0_0(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ in a neighborhood of 0 and $\chi_m(x) = \chi(\frac{x}{m})$ for $m > 0$. Recall that the Fock space has the decomposition $\Gamma_s(L^2 \oplus L^2) = \Gamma_s(P_R L^2 \oplus P_R L^2) \oplus \Gamma_s(P_R^* L^2 \oplus P_R^* L^2)$ where $P_R^* = 1 - P_R$. In this representation $\mathcal{A}_R^{\text{Wick}} = (\mathcal{A}_R)^{\text{Wick}} \Gamma_s(P_R L^2 \oplus P_R L^2) \otimes 1_{\Gamma_s(P_R^* L^2 \oplus P_R^* L^2)}$ and $\mathcal{A}_R = \hat{\varrho}_{e_j}$.

Hence using reduced density matrices $\varrho_{e_j}^R$ that are normalized positive trace-class operators in $\Gamma_s(P_R L^2 \oplus P_R L^2)$, one writes

$$\text{Tr} \left[ \varrho_{e_j}(\mathcal{A}_R)^{\text{Wick}} \right] = \text{Tr} \left[ \varrho_{e_j}(\mathcal{A}_R)^{\text{Wick}} \Gamma_s(P_R L^2 \oplus P_R L^2) \otimes 1_{\Gamma_s(P_R^* L^2 \oplus P_R^* L^2)} \right] = \text{Tr} \left[ \Gamma_s(P_R L^2 \oplus P_R L^2) \right] [\varrho_{e_j}^R(\mathcal{A}_R)^{\text{Wick}}] .$$

As in the proof of Lemma V.12, the Wick calculus gives that $(\mathcal{A}_R)^{\text{Wick}}$ can be written as an Anti-Wick operator by moving all the $a^*$ to the right of $a$. So, one obtains $(\mathcal{A}_R)^{\text{Wick}} = \mathcal{A}_R^{\text{Wick}}$.
\((\mathcal{A})^{A-\text{Wick}} + \varepsilon T\) with \(T(d\Gamma(P_R \oplus P_R) + 1)^{-\frac{p+q}{2}}\) is bounded uniformly with respect to \(\varepsilon \in (0, \varepsilon_0)\). We refer the reader to [8] where Weyl and Anti-Wick quantization are explained for "cylindrical" symbols. Hence

\[
\lim_{j \to \infty} \text{Tr} \left[ \mathcal{Q}_{E_j} (\mathcal{A})^{A-\text{Wick}} \right] = \lim_{j \to \infty} \text{Tr}_{\gamma} (P_{L^2} \oplus P_{L^2}) \left[ \mathcal{Q}_{E_j}^{R} (\mathcal{A})^{A-\text{Wick}} \right] = \lim_{j \to \infty} \text{Tr}_{\gamma} (P_{L^2} \oplus P_{L^2}) \left[ \mathcal{Q}_{E_j}^{R} (\mathcal{A})^{A-\text{Wick}} \right].
\]

Now we define \(\chi_{m,R}(z) := \chi_m(|P_R \oplus P_R|^2)\) and \(\mathcal{Q}_{E_j}^{R,m} := \chi_{m,R}(z)^{\text{Weyl}}\mathcal{Q}_{E_j}^{R} \chi_{m,R}(z)^{\text{Weyl}}\). So one writes

\[
\begin{align*}
\left| \text{Tr} \left[ \mathcal{Q}_{E_j}^{R} (\mathcal{A})^{A-\text{Wick}} \right] - \int_{L^2 \oplus L^2} \mathcal{A}(z) d\mu(z) \right| &\leq \left| \text{Tr} \left[ (\mathcal{Q}_{E_j}^{R} - \mathcal{Q}_{E_j}^{R,m}) (\mathcal{A})^{A-\text{Wick}} \right] \right| \\
&+ \left| \text{Tr} \left[ \mathcal{Q}_{E_j}^{R,m} (\mathcal{A})^{A-\text{Wick}} \right] - \int \chi_{m,R}^2(z) \mathcal{A}(z) d\mu(z) \right| \\
&+ \left| \int \chi_{m,R}^2(z) \mathcal{A}(z) d\mu(z) - \int \mathcal{A}(z) d\mu(z) \right|,
\end{align*}
\]

where the traces are on the Fock space \(\Gamma_\gamma(P_L L^2 \oplus P_R L^2)\) and the integrals are over \(L^2 \oplus L^2\). By dominated convergence the right hand side of (99) tends to 0 when \(m \to \infty\) at fixed \(R\). The right hand side of (97) can be made arbitrary small when \(m \to \infty\) using the following decomposition

\[
(\mathcal{Q}_{E_j}^{R,m} - \mathcal{Q}_{E_j}^{R}) = (\chi_{m,R} - 1) \mathcal{Q}_{E_j}^{R} \chi_{m,R}^{\text{Weyl}} + \mathcal{Q}_{E_j}^{R} (\chi_{m,R} - 1),
\]

which gives \(\text{Tr}(A) (\mathcal{A})^{A-\text{Wick}} = \text{Tr}[T_1 T_2 T_3 T_4]\) and a similar expression for \((B)\) with

\[
\begin{align*}
T_1 &= (N_R + 1)^{\frac{p+q}{2}} (\chi_{m,R} - 1)(N_R + 1)^{-\frac{2}{2}}, & T_2 &= (N_R + 1)^{\frac{2}{2}} \mathcal{Q}_{E_j}^{R} (N_R + 1)^{\frac{2}{2}} \\
T_3 &= (N_R + 1)^{\frac{2}{2}} \chi_{m,R}^{\text{Weyl}} (N_R + 1)^{\frac{2}{2}}, & T_4 &= (N_R + 1)^{\frac{2}{2}} (\mathcal{A})^{A-\text{Wick}} (N_R + 1)^{-\frac{2}{2}},
\end{align*}
\]

where \(N_R = d\Gamma(P_R \oplus P_R)\). The Weyl-Hörmander Pseudo-differential calculus gives that \(T_1 \to_{m \to \infty} 0\) in norm (since \(\delta > p + q\)) and that \(T_i, i = 2, 3, 4,\) are uniformly bounded with respect \(j \in \mathbb{N}\) and \(m > 0\) at fixed \(R\) (see e.g. [8, Proposition 3.2 and 3.3]).

To complete the proof, we remark that \(\text{Tr} \left[ \mathcal{Q}_{E_j}^{R,m} (\mathcal{A})^{A-\text{Wick}} \right] = \text{Tr} \left[ \mathcal{Q}_{E_j}^{R} \chi_{m,R}^{\text{Weyl}} (\mathcal{A})^{A-\text{Wick}} \chi_{m,R}^{\text{Weyl}} \right]\). So again by pseudo-differential calculus we know that \((\mathcal{A})^{A-\text{Wick}} = (\mathcal{A})^{\text{Weyl}} + \varepsilon b(\varepsilon)^{\text{Weyl}}\) with \(b(\varepsilon)\) belonging to the Weyl–Hörmander class symbol \(S_{p,q}(\varepsilon)^{p+q-2}, \frac{d^2}{\varepsilon^2}\) uniformly in \(\varepsilon\) (see [8, Section 3.2 and 3.4]). Therefore

\[
\lim_{j \to \infty} \text{Tr} \left[ \mathcal{Q}_{E_j}^{R,m} (\mathcal{A})^{A-\text{Wick}} \right] = \lim_{j \to \infty} \text{Tr} \left[ \mathcal{Q}_{E_j}^{R} \chi_{m,R}^{\text{Weyl}} (\mathcal{A})^{\text{Weyl}} \chi_{m,R}^{\text{Weyl}} \right],
\]

since \((d\Gamma(P_R \oplus P_R) + 1)^{-(q+p)/2} b(\varepsilon)^{\text{Weyl}} (d\Gamma(P_R \oplus P_R) + 1)^{-(p+q)/2}\) is uniformly bounded with respect to \(\varepsilon\). The Weyl-Hörmander pseudo-differential calculus gives \(\chi_{m,R}^{\text{Weyl}} (\mathcal{A})^{\text{Weyl}} \chi_{m,R}^{\text{Weyl}} = \)
\((\chi_{m,R}^2)_{\text{Weyl}} + \varepsilon c(\varepsilon)_{\text{Weyl}}\) with \(c(\varepsilon) \in S_{P_0}B_{P_0}(1, dz^2)\) uniformly in \(\varepsilon\) (see e.g. [8, Proposition 3.2]). Hence, according to [8, Theorem 6.2] one obtains

\[
\lim_{j \to \infty} \text{Tr} \left[ \hat{Q}_{e_j} (\mathcal{A}_{\text{Weyl}}) \right] = \lim_{j \to \infty} \text{Tr} \left[ \mathcal{Q}_{e_j} \left( \chi_{m,R}^2 \right)_{\text{Weyl}} \right] = \int_{L^2 \oplus L^2} \chi_{m,R}^2(z) d\mu(z).
\]

This yields the intended bound on (95) and completes the proof. ■

**Lemma V.16.** There exists \(C(\sigma_0) > 0\) depending only on \(\sigma_0 \in \mathbb{R}_+\) such that the following bound holds for \(\mathcal{B}_{0}^{(m)}\) uniformly in \(m \in \mathbb{N}\):

\[
(100) \quad \left| \mathcal{B}_{0}^{(m)}(\xi)(u, \alpha) \right| \leq C(\sigma_0) ||\xi||_{L^2} + ||(\xi + V)_{\text{FHM}/2}||_{L^2} + ||u||_{L^2} + ||(\xi + V)_{\text{FHM}/2}||_{L^2} + ||u||_{L^2} + ||\alpha||_{L^2} + ||(\xi + V)_{\text{FHM}/2}||_{L^2} + ||u||_{L^2} + ||\alpha||_{L^2}.
\]

It follows that:

1. For any \(\xi \in L^2 \oplus L^2\), for any \((u, \alpha) \in Q(\xi + V) \oplus D(\omega_{1/2})\), \(\lim_{m \to \infty} \mathcal{B}_{0}^{(m)}(\xi)(u, \alpha) = \mathcal{B}_0(\xi)(u, \alpha)\) and therefore the bound (100) holds also for \(\mathcal{B}_0\).

2. For any \(m \in \mathbb{N}\), \(\mathcal{B}_{0}^{(m)}(\cdot), \mathcal{B}_0(\cdot)\) are jointly continuous with respect to \(\xi \in L^2 \oplus L^2\) and \((u, \alpha) \in Q(\xi + V) \oplus D(\omega_{1/2})\).

Recall that for any \(\sigma_0 \geq 2K(\mathcal{C} + 1 + \varepsilon)\) there exists \(b > 0\) such that the operator \(\hat{H}_{ren}(\sigma_0) + b\) is non-negative uniformly for \(\varepsilon \in (0, \varepsilon)\). Let \(\mathcal{Q}_{e_{\varepsilon}}\) be a family of normal states on \(\Gamma_s(L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2))\), we consider the additional assumption:

\[
(A'') \quad \exists C > 0, \forall \varepsilon \in (0, \varepsilon), \text{Tr}[\mathcal{Q}_{e_{\varepsilon}}(\hat{H}_{ren}(\sigma_0) + b)] \leq C;
\]

**Proposition V.17.** Let \(\mathcal{Q}_{e_{\varepsilon}} \in L^1(\mathcal{H})\) be a family of normal states that satisfy Assumptions \((A_0), (A'_0)\) and \((A''_0)\) such that \(\sigma_0 \geq 2K(\mathcal{C} + 1 + \varepsilon)\). Then:

(i) For any sequence \((e_{\varepsilon_k})_{k \in \mathbb{N}} \subset (0, \varepsilon)\) converging to zero, there exist a subsequence \((e_{\varepsilon_{k_l}})_{l \in \mathbb{N}}\) and a map \(\mu_t : \mathbb{R} \to \Psi(L^2 \oplus L^2)\) such that \(\varepsilon_{\varepsilon_{k_l}}(t) \to \mu_t\) and \(\tilde{\varepsilon}_{\varepsilon_{k_l}}(t) \to \tilde{\mu}_t = E_0(-t)\mu_t\), for any \(t \in \mathbb{R}\).

(ii) The action of \(e^{-i\hat{H}_{ren}(\sigma_0)}\) is non-trivial on the states \(\mathcal{Q}_{e_{\varepsilon}}\).

(iii) The Fourier transform of \(\tilde{\mu}_t\) satisfies the following transport equation \(\forall \xi \in L^2 \oplus L^2\):

\[
\int_{L^2 \oplus L^2} \varepsilon^{i\mathcal{A}\mathcal{R}(\xi, z)} d\tilde{\mu}_t(z) = \int_{L^2 \oplus L^2} \varepsilon^{i\mathcal{A}\mathcal{R}(\xi, z)} d\mu_0(z) + \int_0^t \left( \int_{L^2 \oplus L^2} \mathcal{B}_{0}(\tilde{\xi}_s)(z) e^{i\varepsilon^{\mathcal{A}\mathcal{R}(\xi, z)}} d\mu_s(z) \right) ds;
\]

where the right hand side makes sense since \(\mathcal{B}_{0}(\tilde{\xi}_s)(z) e^{i\mathcal{A}\mathcal{R}(\xi, z)} \in L^1_t(\mathbb{R}, L^1_z[L^2 \oplus L^2, d\mu_z])\) for any \(\xi \in L^2 \oplus L^2\).

---

19 We recall that \(\mathcal{C}\) appears in Assumption \((A_0)\) and \(\sigma_0\) in Definition II.12 of \(\hat{H}_{ren}(\sigma_0)\). The condition \(\sigma_0 \geq K(\mathcal{C} + 1)\) ensures that the dressed dynamics is non-trivial on \(\bigoplus_{m=0}^\infty \mathcal{H}_m\) and hence non-trivial on the state \(\mathcal{Q}_{e_{\varepsilon}}\) according to Lemma V.2.
Proof. The first part of the proposition (i)–(ii) is just a partial restatement of Proposition V.14. We discuss the last assertion in (iii) about $\mathcal{B}_0(\tilde{\xi}_t)(z)e^{i\sqrt{2}\text{Re}(\tilde{\xi}_t)}$, before proving (101). Recall the fact that for any $\xi \in L^2 \oplus L^2$ and for any $t \in \mathbb{R}$, $\|\tilde{\xi}_t\|_{L^2 \oplus L^2} = \|\tilde{\xi}\|_{L^2 \oplus L^2}$. Using bound (100) of Lemma V.16 we obtain, setting $Q(-\Delta + V) \oplus D(\omega^{1/2}) \ni z = (u, \alpha)$:
\[
\left| \mathcal{B}_0(\tilde{\xi}_t)(z)e^{i\sqrt{2}\text{Re}(\tilde{\xi}_t)} \right| \leq C(\sigma_0)\|\xi\|_{L^2 \oplus L^2} \left( \|u\|^2_2 + \|(-\Delta + V)^{1/2}u\|^2_2 + \|\alpha\|^2_{H^{1/2}} \right) \\
+ \|u\|^2 \cdot \|(-\Delta + V)^{1/2}u\|^2_2 + \|\alpha\|^2_{H^{1/2}} + \|u\|^2 \cdot \|(-\Delta + V)^{1/2}u\|^2_2 \cdot \|\alpha\|^2_{H^{1/2}}.
\]
Now $\mu_t \in \mathcal{M}(\mathcal{Q}_t(t), \varepsilon \in (0, \varepsilon))$, therefore by Lemma V.12, $\mu_t(\mathcal{B}_0(0, \sqrt{\mathcal{C}}) \cap Q(-\Delta + V) \oplus D(\omega^{1/2})) = 1$ for any $t \in \mathbb{R}$. Then it follows that there exists $C(\mathcal{C}) > 0$ such that
\[
\left| \int_{L^2 \oplus L^2} \mathcal{B}_0(\tilde{\xi}_t)(z)e^{i\sqrt{2}\text{Re}(\tilde{\xi}_t)} \, d\mu_t(z) \right| \leq C(\mathcal{C})\|\xi\|_{L^2 \oplus L^2} \int_{L^2 \oplus L^2} \left( \|(-\Delta + V)^{1/2}u\|^2_2 + \|\alpha\|^2_{H^{1/2}} \right) d\mu_t(z) \\
\leq C(\mathcal{C})\|\xi\|_{L^2 \oplus L^2} J(t);
\]
where $J(t) < \infty$ by Lemma V.12. Actually, using the fact that the bound (85) is independent of $t$, it is easily proved that $J(t)$ does not depend on $t$ as well, i.e. $J(t) \in L^1(\mathbb{R})$.

We prove (101) by successive approximations. Consider $\text{Tr}[\tilde{\mathcal{E}}_{\varepsilon_k}(t)W(\xi)]$, $\xi \in L^2 \oplus L^2$. We can approximate $\xi$ with $(\xi^{(l)})_{l \in \mathbb{N}} \subseteq Q(-\Delta + V) \oplus D(\omega^{3/4})$, since the latter is dense in $L^2 \oplus L^2$, and $\lim_{l \to \infty} \text{Tr}[\tilde{\mathcal{E}}_{\varepsilon_k}(t)W(\xi^{(l)})] = 0$ uniformly in $\varepsilon_k$ by Lemma V.13. Now, for $\text{Tr}[\tilde{\mathcal{E}}_{\varepsilon_k}(t)W(\xi^{(l)})]$ the integral equation (76) holds. Proposition V.14 implies that $\tilde{\mathcal{E}}_{\varepsilon_k}(t) \to \tilde{\mu}_t = \mathcal{E}_0(t) \# \mu_t$, for any $t \in \mathbb{R}$. Therefore the left-hand side of (76) converges when $t \to \infty$ to $\int_{L^2 \oplus L^2} e^{i\sqrt{2}\text{Re}(\xi^{(0)}, z)} \, d\tilde{\mu}_t(z)$; and that in turn converges when $l \to \infty$ to $\int_{L^2 \oplus L^2} e^{i\sqrt{2}\text{Re}(\xi^{(0)}, z)} \, d\tilde{\mu}_t(z)$ by dominated convergence theorem. In addition,
\[
\lim_{l \to \infty} \sum_{j=1}^{3} e^{ij} \int_0^\infty \text{Tr}[\tilde{\mathcal{E}}_{\varepsilon_k}(s)W(\xi^{(l)})B_j(\xi^{(l)})] \, ds = 0;
\]
by Proposition V.7. It remains to show the convergence of the $B_0$ term in (76). We approximate $\mathcal{B}_0$ by the compact $\mathcal{B}_0^{(m)}$, because using Lemma V.2 and (87) of Proposition V.9 we obtain
\[
\left| \text{Tr}[\tilde{\mathcal{E}}_{\varepsilon_k}(s)W(\xi^{(l)}) \left( B_0(\xi^{(l)}) - B_0^{(m)}(\xi^{(l)}) \right) ] \right| \leq \sum_{l \in \mathbb{N}} \lambda_l \left| \langle W^*(\xi^{(l)}) e^{-i\tilde{\mathcal{E}}_{\varepsilon_k}} \Psi_l, (B_0(\xi^{(l)}) - B_0^{(m)}(\xi^{(l)}) \rangle \right| \\
\leq \sum_{l \in \mathbb{N}} \lambda_l C^{(m)}(\xi^{(l)}) \left| \langle (H_0 + 1)^{1/2} (N_1 + \tilde{\varepsilon})^{1/2} W^*(\xi^{(l)}) e^{-i\tilde{\mathcal{E}}_{\varepsilon_k}} \Psi_l \langle \right| \\
\leq \sum_{l \in \mathbb{N}} \lambda_l C^{(m)}(\xi^{(l)}) \left| \langle (H_0 + 1)^{1/2} (N_1 + \tilde{\varepsilon})^{1/2} e^{-i\tilde{\mathcal{E}}_{\varepsilon_k}} \Psi_l \langle \right| .
\]
Now, using the fact that $C^{(m)}(\xi^{(l)})$ depends only on $\|\xi^{(l)}\|_{Q(-\Delta + V) \oplus D(\omega^{3/4})} = \|\xi^{(l)}\|_{Q(-\Delta + V) \oplus D(\omega^{3/4})}$.
and Lemma V.1 we obtain
\[
\left| \text{Tr}[\mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s)(B_0(\tilde{\xi}^{(l)}_s) - B_0^{(m)}(\tilde{\xi}^{(l)}_s))] \right| \leq \sum_{i \in \mathbb{N}} \lambda_i C^{(m)}(\xi^{(l)}_s)C(\xi^{(l)}_s) \left\| (H_0 + 1)^{1/2} e^{-i\varepsilon_k \hat{H}_m} \right\|_{\mathcal{L}(\mathcal{H})} (N_1 + \varepsilon)^{1/2} \Psi_l^2 .
\]

We then use Equation (85) of Lemma V.8:
\[
\left| \text{Tr}[\mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s)(B_0(\tilde{\xi}^{(l)}_s) - B_0^{(m)}(\tilde{\xi}^{(l)}_s))] \right| \leq \sum_{i \in \mathbb{N}} \lambda_i C^{(m)}(\xi^{(l)}_s)C(\xi^{(l)}_s)(\mathcal{C} + \varepsilon) \frac{1}{1-m(\varepsilon)} C + \frac{2b(\varepsilon)}{1-m(\varepsilon)} .
\]

The right hand side goes to zero when \( m \to \infty \) uniformly with respect to \( \varepsilon_k \) and \( s \) by Proposition V.9, and therefore
\[
\lim_{m \to \infty} \int_0^t \text{Tr}[\mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s)(B_0(\tilde{\xi}^{(l)}_s) - B_0^{(m)}(\tilde{\xi}^{(l)}_s))] ds = 0 .
\]

So the next step is to prove
\[
\lim_{t \to \infty} \text{Tr}[\mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s)(\mathcal{R}_0^{(m)}(\tilde{\xi}^{(l)}_s))^{\text{Wick}}] = \int_{L^2 \oplus L^2} \mathcal{R}_0^{(m)}(\tilde{\xi}^{(l)}_s)(z) e^{i\sqrt{2}\text{Re}(\tilde{\xi}^{(l)}_s)z} d\mu_s(z) .
\]

This statement follows by applying Lemma V.15 with \( \delta = 2 \) and by checking the assumption (102)
\[
\|(N_1 + N_2)\mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s) (N_1 + N_2)\|_{\mathcal{L}(L^2 \oplus L^2)} \leq C ,
\]

uniformly in \( k \) for some \( C > 0 \). In fact (102) holds true by Assumptions (A_0)-(A^{''}_p), the Higher order estimate of Proposition A.4 and Lemma V.1. Remark that while \( \mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s) \) is not a non-negative trace-class operator, one can still apply Lemma V.15. In fact, one can write
\[
\text{Tr}[\mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s)B_0^{(m)}(\tilde{\xi}^{(l)}_s)] = \text{Tr}[W(\eta)\mathcal{Q}_{\varepsilon_k}(s)W(\eta) \mathcal{R}^{\text{Wick}}],
\]

for some \( \mathcal{R} \in \bigoplus_{p+q<4} \mathcal{P}^{\infty}_{p,q} (L^2 \oplus L^2) \) and with \( \eta = \frac{1}{2} \tilde{\xi}^{(l)}_s \). Remark now that \( W(\eta)\mathcal{Q}_{\varepsilon_k}(s)W(\eta) \) decomposes explicitly into a linear combination of non-negative trace-class operators satisfying all the assumption (93) of Lemma V.15. Note that the Wigner measures of \( \mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s) \) are identified through (91). Hence the dominated convergence theorem yields:
\[
\lim_{t \to \infty} \int_0^t \text{Tr}[\mathcal{Q}_{\varepsilon_k}(s)W(\tilde{\xi}^{(l)}_s)B_0^{(m)}(\tilde{\xi}^{(l)}_s)] ds = \int_0^t \left( \int_{L^2 \oplus L^2} \mathcal{R}_0^{(m)}(\tilde{\xi}^{(l)}_s)(z) e^{i\sqrt{2}\text{Re}(\tilde{\xi}^{(l)}_s)z} d\mu_s(z) \right) ds .
\]

By Lemma V.16, \( \lim_{m \to \infty} \mathcal{R}_0^{(m)}(\tilde{\xi}^{(l)}_s)(z) = \mathcal{B}_0(\tilde{\xi}^{(l)}_s)(z) \), so by dominated convergence theorem
\[
\lim_{m \to \infty} \int_0^t \left( \int_{L^2 \oplus L^2} \mathcal{R}_0^{(m)}(\tilde{\xi}^{(l)}_s)(z) e^{i\sqrt{2}\text{Re}(\tilde{\xi}^{(l)}_s)z} d\mu_s(z) \right) ds = \int_0^t \left( \int_{L^2 \oplus L^2} \mathcal{B}_0(\tilde{\xi}^{(l)}_s)(z) e^{i\sqrt{2}\text{Re}(\tilde{\xi}^{(l)}_s)z} d\mu_s(z) \right) ds .
\]
Above it is possible to apply the dominated convergence theorem due to a reasoning analogous to the one done at the beginning of this proof: roughly speaking, we have that $\mathcal{B}_0^{(m)}(\xi) \in L^\infty([R, L]_2^2; \mu_0)$ uniformly with respect to $m \in \mathbb{N}$. In an analogous fashion we finally obtain
\[
\lim_{l \to \infty} \int_0^l \left( \int_{L^2 \oplus L^2} \mathcal{B}_0(\xi) \, \langle \mathcal{B}_0(\xi), \mu_s(z) \rangle \, ds \right) \, \langle \mathcal{B}_0(\xi), \mu_s(z) \rangle \, ds.
\]

**Corollary V.18.** The transport equation (101) may be rewritten as
\[
\int_{L^2 \oplus L^2} e^{i \sqrt{2} \text{Re} \langle \xi \rangle} \, d\mu_t(z) = \int_{L^2 \oplus L^2} e^{i \sqrt{2} \text{Re} \langle \xi \rangle} \, d\mu_0(z) + i \sqrt{2} \int_0^l \left( \int_{L^2 \oplus L^2} e^{i \sqrt{2} \text{Re} \langle \xi \rangle} \, \langle \mathcal{B}_0(\xi), \mathcal{V}(s) \rangle \, d\mu_s(z) \right) \, ds;
\]
with the velocity vector field $\mathcal{V}(t)(z) = -i \mathcal{E}_0(-t) \circ \partial_2 (\xi_0) \circ \mathcal{E}_0(t)(z)$. In addition $\tilde{\mu}_t = \mathcal{E}_0(-t) \tilde{\mathcal{E}}(t) \# \mu_0$ is a solution of Equation (103).

**Proof.** It is proved by direct calculation, since $\mu_t(Q(-\Delta + V) \oplus \mathcal{F} H^{1/2}) = 1$ for any $t \in \mathbb{R}$ by Lemma V.12; and $E_t(t), E_0(t)$ are globally well-defined on this space (for $E_t(t)$, it is proved in Theorem III.15; for $E_0(t)$ it is trivial). The second point is proved by differentiating with respect to time and using Lemma V.16 and Lemma V.12 (iii).

5. **Uniqueness of the solution to the transport equation.**

As discussed in Corollary V.18, the dressed flow yields in the classical limit a solution of the transport equation (103). The second part of the same corollary suggests that it is important to study uniqueness properties of (103): it is by means of uniqueness that we can close the argument and reach a satisfactory characterization of the dynamics of classical states (Wigner measures). This subsection is devoted to prove that the family of Wigner measures $\tilde{\mu}_t$ of Proposition V.17 satisfies sufficient conditions, induced by the properties of $(Q_\xi)_{\xi \in (0, \ell)}$, to be uniquely identified with $E_0(-t) \tilde{\mathcal{E}}(t) \# \mu_0$. We use an optimal transport technique initiated by Ambrosio et al. [3] then extended by Ammari and Nier [11] to propagation of Wigner measures; and under minimal assumptions proved to be sufficient by Liard [77] (see also [79]).

In order to do that, we need to introduce a suitable topology on $\psi(L^2 \oplus L^2)$. Let $(e_j)_{j \in \mathbb{N}} \subset L^2 \oplus L^2$ be an orthonormal basis. Then
\[
d_w(z_1, z_2) = \left( \sum_{j \in \mathbb{N}} \frac{|\langle z_1 - z_2, e_j \rangle| L^2 \oplus L^2}{1 + j^2} \right)^{1/2},
\]
where $z_1, z_2 \in L^2 \oplus L^2$, defines a distance on $L^2 \oplus L^2$. The topology induced by $(L^2 \oplus L^2, d_w)$ is homeomorphic to the weak topology on bounded sets.
**Definition V.19** (Weak narrow convergence of probability measures). Let \((\mu_i)_{i \in \mathbb{N}} \subset \Psi(L^2 \oplus L^2)\). Then \((\mu_i)_{i \in \mathbb{N}}\) weakly narrowly converges to \(\mu \in \Psi(L^2 \oplus L^2)\), in symbols \(\mu_i \overset{n}{\rightharpoonup} \mu\), if

\[
\forall f \in C_b((L^2 \oplus L^2), \mathbb{R}), \quad \lim_{i \to \infty} \int_{L^2 \oplus L^2} f(z) d\mu_i(z) = \int_{L^2 \oplus L^2} f(z) d\mu(z);
\]

where \(C_b((L^2 \oplus L^2), \mathbb{R})\) is the space of bounded continuous real-valued functions on \((L^2 \oplus L^2, d_w)\).

It is actually more convenient to use cylindrical functions to prove narrow continuity properties. Therefore we also define the cylindrical Schwartz and compact support space of functions on \(L^2 \oplus L^2\).

**Definition V.20** (Spaces of cylindrical functions). Let \(f : L^2 \oplus L^2 \rightarrow \mathbb{R}\). Then \(f \in S_{cyl}(L^2 \oplus L^2)\) if there exists an orthogonal projection \(p : L^2 \oplus L^2 \rightarrow L^2 \oplus L^2\), dim(Ran \(p\)) = \(d < \infty\), and a rapid decrease function \(g \in S(\text{Ran} \ p)\) such that

\[
\forall z \in L^2 \oplus L^2, \quad f(z) = g(pz).
\]

Analogously, if \(g \in C_0^\infty(\text{Ran} \ p)\), then \(f \in C_0^\infty(\text{Ran} \ p)\), the cylindrical smooth functions with compact support.

We remark that neither \(S_{cyl}(L^2 \oplus L^2)\) nor \(C_0^\infty(\text{Ran} \ p)\) possess a vector space structure. Finally, for cylindrical Schwartz functions we define the Fourier transform:

\[
\mathcal{F}[f](\eta) = \int_{\text{Ran} \ p} e^{-2\pi i \text{Re}(\eta, z) \frac{1}{2} \eta \cdot z} f(z) dL_p(z),
\]

where \(dL_p\) denotes integration with respect to the Lebesgue measure on Ran \(p\). The inversion formula is then

\[
f(z) = \int_{\text{Ran} \ p} e^{2\pi i \text{Re}(\eta, z) \frac{1}{2} \eta \cdot z} \mathcal{F}[f](\eta) dL_p(\eta).
\]

With these definitions in mind, we can prove the following lemma.

**Lemma V.21.** Let \((\varphi_{\varepsilon})_{\varepsilon \in (0, \bar{\varepsilon})} \subset L^1(\mathcal{H})\) be a family of normal states that satisfies Assumptions \((A_0), (A'_0)\) and \((A''_0)\); \(\tilde{\mu}_t : \mathbb{R} \rightarrow \Psi(L^2 \oplus L^2)\) such that for any \(t \in \mathbb{R}\), \(\tilde{\mu}_t \in M(\tilde{\varphi}_{\varepsilon}(t), \varepsilon \in (0, \bar{\varepsilon}))\). If, in addition, \(\tilde{\mu}_t\) satisfies the integral equation (103), then the following statements are true:

* For any \(t \in \mathbb{R}\), and for any \((t_i)_{i \in \mathbb{N}} \subset \mathbb{R}\) such that \(\lim_{i \to \infty} t_i = t\),

\[
\tilde{\mu}_{t_i} \overset{n}{\rightharpoonup} \tilde{\mu}_t;
\]

i.e. \(\tilde{\mu}_t\) is a weakly narrowly continuous map.
The map $\tilde{\mu}_t$ solves the transport equation\footnote{Recall that $\nabla f = -iE_0(-t) \circ \partial_z(\hat{\delta} - \delta_0) \circ E_0(t)(z)$.} 

$$\partial_t \tilde{\mu}_t + \nabla^T \left( \nabla f(t) \tilde{\mu}_t \right) = 0$$

in the weak sense, i.e.

$$\forall f \in C_{0,cyl}^{\infty}(\mathbb{R} \times (L^2 \oplus L^2)),$$

\begin{equation}
\int_{\mathbb{R}} \int_{L^2 \oplus L^2} \left( \partial_t f + \text{Re} \langle \nabla f, \nabla t \rangle \right) d\tilde{\mu}_t dt = 0.
\end{equation}

**Proof.** Let $f \in S_{cyl}(L^2 \oplus L^2)$. Fubini’s theorem gives

$$\int_{L^2 \oplus L^2} f(z) d\tilde{\mu}_t(z) = \int_{\text{Ran} p} \mathcal{F}[f](\xi) \left( \int_{L^2 \oplus L^2} e^{2\pi i \text{Re} \langle \xi, z \rangle} d\tilde{\mu}_t(z) \right) dL_{\text{Ran} p}(\xi),$$

where $dL_{\text{Ran} p}$ is the Lebesgue measure on $\text{Ran} p$ and $\mathcal{F}(f)(\xi) = \int_{\text{Ran} p} f(z) e^{-2\pi i \text{Re} \langle \xi, z \rangle} dL_{\text{Ran} p}(z)$. Now we define $\tilde{G}_0(t, \xi) := \int_{L^2 \oplus L^2} e^{2\pi i \text{Re} \langle \xi, z \rangle} d\tilde{\mu}_t(z)$. Hence Equation (101) of Proposition V.17 gives

$$\tilde{G}_0(t, \xi) - \tilde{G}_0(s, \xi) = \int_s^t \left( \int_{L^2 \oplus L^2} \mathcal{F}^{-1}_0(\tilde{\xi}_s)(z) e^{2\pi i \text{Re} \langle \tilde{\xi}_s, z \rangle} d\mu_t(z) \right) d\tau;$$

and this proves that $t \mapsto \tilde{G}_0(t, \xi)$ is continuous for any $\xi \in L^2 \oplus L^2$ since the integrand in the right hand side of (106) is bounded with respect to $\tau$ by Proposition V.17. Remark that $\tilde{G}_0(t, \xi)$ is bounded by one for any $(t, \xi) \in \mathbb{R} \times (L^2 \oplus L^2)$. Therefore the map $t \mapsto \int_{L^2 \oplus L^2} f(z) d\tilde{\mu}_t(z)$ is continuous for any $f \in S_{cyl}(L^2 \oplus L^2)$. Finally, by an argument analogous to the one used at the beginning of the proof of Proposition V.17, it is easy to prove that $\int_{L^2 \oplus L^2} ||\xi||^2_{L^2 \oplus L^2} d\tilde{\mu}_t(z) \in L^1_t(\mathbb{R})$. In fact, we know that $\tilde{\mu}_t \left( B_0(0, \sqrt{\omega}) \cap Q(-\Delta + V) \oplus D(\omega^{1/2}) \right) = 1$ by Lemma V.12; and if $z = (u, \alpha)$ then the functions $\alpha \mapsto ||\alpha||_2^2 \leq ||\alpha||^2_{F^{1/2}}, \alpha \mapsto L^1_t \left[ L^2(\log L^2), d\mu_t(z) \right]$ uniformly in $t$ by Lemmas V.12 and V.8. Then it follows that $\tilde{\mu}_t$ is weakly narrowly continuous by [3, Lemma 5.1.12 - f], thus proving the first point.

Now we prove the second point by a similar argument as in [11] which we reproduce here for completeness. Let $g \in C_{0,cyl}^{\infty}(L^2 \oplus L^2)$; we integrate Equation (103) with respect to the measure $\mathcal{F}[g](\eta) dL_p(\eta)$ obtaining

$$\int_{L^2 \oplus L^2} g(z) d\tilde{\mu}_t(z) = \int_{L^2 \oplus L^2} g(z) d\mu_0(z) + 2\pi i \int_0^t \int_{\text{Ran} p} \left( \int_{L^2 \oplus L^2} \text{Re} \langle \eta, \nabla(s) \rangle d\tilde{\mu}_s(z) \right) dL_p(\eta) ds .$$

Let $\nabla g$ be the differential of $g : L^2 \oplus L^2 \rightarrow \mathbb{R}$, where here $L^2 \oplus L^2$ is considered as a real Hilbert space with scalar product $\text{Re} \langle \cdot, \cdot \rangle_{L^2 \oplus L^2}$. Then, by Fubini’s theorem and the properties of the Fourier transform, we get

$$\int_{L^2 \oplus L^2} g(z) d\tilde{\mu}_t(z) = \int_{L^2 \oplus L^2} g(z) d\mu_0(z) + \int_0^t \int_{L^2 \oplus L^2} \text{Re} \langle \nabla g(z), \nabla(s)(z) \rangle d\tilde{\mu}_s(z) ds .$$

\begin{equation}
\int_{L^2 \oplus L^2} g(z) d\tilde{\mu}_t(z) = \int_{L^2 \oplus L^2} g(z) d\mu_0(z) + \int_0^t \int_{L^2 \oplus L^2} \text{Re} \langle \nabla g(z), \nabla(s)(z) \rangle d\tilde{\mu}_s(z) ds .
\end{equation}
By Lebesgue’s differentiation theorem (with respect to $t$), we obtain

$$\partial_t \int_{L^2 \oplus L^2} g(z) d\tilde{\mu}_t(z) - \int_{L^2 \oplus L^2} \text{Re}(\nabla g(z), V(t)(z)) d\tilde{\mu}_t(z) = 0.$$  

Equation (105) is then obtained for $f(t, z) = \varphi(t)g(z)$, multiplying by $\varphi(t) \in C^0_0(\mathbb{R}, \mathbb{R})$, integrating with respect to $t$, and finally using integration by parts. The result for a generic $f \in C^\infty_{0,alg}(\mathbb{R} \times (L^2 \oplus L^2))$ follows immediately: $f(t, z) = g(t, p z)$ for some $g \in C^\infty_0(\mathbb{R} \times \text{Ran } p)$, and the latter can be approximated by a sequence $(g_j(t, p z))_{j \in \mathbb{N}} \subset C^\infty_0(\mathbb{R}) \otimes C^\infty_0(\text{Ran } p)$. ■

We need to check an hypothesis on the velocity vector field $V(t)$ (introduced in Corollary V.18) to prove the sought uniqueness result. This is done in the following lemma.

**Lemma V.22.** Let $(\tilde{\varrho}_e)_{\epsilon \in (0, \delta)} \subset L^1(\mathcal{H})$ be a family of normal states that satisfies Assumptions $(A_0)$ and $(A'_0)$; $\tilde{\mu}_t : \mathbb{R} \to \Psi(L^2 \oplus L^2)$ such that for any $t \in \mathbb{R}$, $\tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_e(t), \epsilon \in (0, \delta))$. Then $||V(t)||_{L^2 \oplus L^2} \in L^\infty_t(\mathbb{R}, L^1_t[L^2 \oplus L^2, d\mu_t(z)])$, i.e. the norm of the velocity vector field is integrable with respect to $\tilde{\mu}_t$, uniformly in $t \in \mathbb{R}$.

**Proof.** By Equation (100) of Lemma V.16 and the definition of $V(t)$ we have that for any $\xi \in L^2 \oplus L^2$:

$$\left| \text{Re} \langle \xi, V(t)(z) \rangle \right| \leq C(\sigma_0)||\xi||_{L^2 \oplus L^2} \left( ||u||_2^2 + ||(-\Delta + V)^{1/2} u||_2^2 + ||\alpha||_{F^H/t}^2 + ||u||_2 \cdot ||(-\Delta + V)^{1/2} u||_2^2 + ||u||_2 \cdot ||\alpha||_{F^H/t}^2 \right).$$

It is easy to prove an equivalent bound for the imaginary part, and hence obtain for any $\xi \in L^2 \oplus L^2$:

$$\left| \langle \xi, V(t)(z) \rangle \right| \leq C(\sigma_0)||\xi||_{L^2 \oplus L^2} \left( ||u||_2^2 + ||(-\Delta + V)^{1/2} u||_2^2 + ||\alpha||_{F^H/t}^2 + ||u||_2 \cdot ||(-\Delta + V)^{1/2} u||_2^2 + ||u||_2 \cdot ||\alpha||_{F^H/t}^2 \right).$$

Therefore it follows immediately that

$$||V(t)||_{L^2 \oplus L^2} \leq C(\sigma_0) \left( ||u||_2^2 + ||(-\Delta + V)^{1/2} u||_2^2 + ||\alpha||_{F^H/t}^2 + ||u||_2 \cdot ||(-\Delta + V)^{1/2} u||_2^2 + ||u||_2 \cdot ||\alpha||_{F^H/t}^2 \right).$$

The right hand side of the above equation is in $L^\infty_t(\mathbb{R}, L^1_t[L^2 \oplus L^2, d\mu_t(z)])$, as shown at the beginning of the proof of Proposition V.17. ■

As showed in [77, Theorem 3.1.1], Lemma V.21, Lemma V.22 and Lemma V.12 (ii)-(iii) are sufficient to prove uniqueness of the solution to the transport equation (105), as precisely formulated in the following proposition.

**Proposition V.23.** Let $(\varrho_k)_{k \in (0, \delta)} \subset L^1(\mathcal{H})$ be a family of normal states that satisfies Assumptions $(A_0),(A'_0)$ and $(A''_0)$. In addition, let $\tilde{\mu}_t : \mathbb{R} \to \Psi(L^2 \oplus L^2)$ such that for any $t \in \mathbb{R}$, $\tilde{\mu}_t \in \mathcal{M}(\varrho_k(t), k \in \mathbb{N})$ for some sequence $(\varrho_k)_{k \in \mathbb{N}}$ with $\varrho_k \to 0$ and $\tilde{\mu}_t$ satisfies the integral equation (103). Then $\tilde{\mu}_t = E_0(-t)\hat{E}(t)\mu_0$.  

6. The classical limit of the dressing transformation.

Let’s consider now the dressing transformation $U_{\kappa}(\theta) = e^{-i\frac{\theta}{\kappa}T_{\kappa}}$ on $\mathcal{H}$, with self-adjoint generator:

$$T_{\kappa} = (\mathcal{D}_{g_{\kappa}})^{Wick} = \int_{\mathbb{R}^3} \psi^*(x) \left( a^*(g_{\kappa}e^{-\kappa x}) + a(g_{\kappa}e^{-\kappa x}) \right) \psi(x) dx ;$$

$$g_{\kappa}(k) = -\frac{i}{2\pi} \frac{1}{2\omega(k)} \left( 1 - \chi_{\sigma}(k) \right) \in L^2(\mathbb{R}^3).$$

The family $(e^{-i\frac{\theta}{\kappa}T_{\kappa}})_{\theta \in \mathbb{R}}$ is a strongly continuous unitary group, and therefore can be seen as a dynamical system acting on quantum states. Therefore, given a family $(\mathcal{Q}_\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})}$ of normal quantum states on $\mathcal{H}$, we could determine the Wigner measures of

$$(107) \quad \hat{\mathcal{Q}}_\varepsilon(\theta) = e^{-i\frac{\theta}{\kappa}T_{\kappa}} \mathcal{Q}_\varepsilon e^{i\frac{\theta}{\kappa}T_{\kappa}}.$$

Since $T_{\kappa} = (\mathcal{D}_{g_{\kappa}})^{Wick}$, where $\mathcal{D}_{g_{\kappa}}$ is the classical dressing generator defined in Section III.1, we expect that under suitable assumptions, $(\mathcal{Q}_\varepsilon \rightarrow \mu \Rightarrow \hat{\mathcal{Q}}_\varepsilon(\theta) \rightarrow \mathcal{D}_{g_{\kappa}}(\theta) \# \mu)$, where $\mathcal{D}_{g_{\kappa}}(\theta)$ is the classical dressing transformation. The last assertion is indeed true, as explained in the following. Observe that the dressing generator $T_{\kappa}$ is equal to the interaction part $H_I(\sigma)$ of the Nelson model with cutoff, where $\frac{\chi_{\sigma}}{\sqrt{2\omega}}$ is replaced by $g_{\kappa}$, i.e. $T_{\kappa} = H_I(\sigma)|_{\chi_{\sigma}=g_{\kappa}}$. The classical limit of the Nelson model with cutoff has been treated by the authors in [7], thus the results below can be immediately deduced by the results in [7, d = 3, $H_0 = 0$ and $\frac{1}{\sqrt{\omega}} = g_{\kappa}]$. We recall also that $g_{\kappa}$, and therefore also $T_{\kappa}$ and $\mathcal{D}_{g_{\kappa}}$, depends on $\sigma_0 \in \mathbb{R}_+$.

**Lemma V.24.** Let $(\mathcal{Q}_\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})}$ be a family of normal (quantum) states on $\mathcal{H}$ that satisfies Assumptions (A0) and (A_p). Then for any $\sigma_0 \in \mathbb{R}_+$, $(\mathcal{Q}_\varepsilon(1))_{\varepsilon \in (0,\bar{\varepsilon})}$ satisfies Assumptions (A0) and (A_p).

**Proposition V.25.** Let $\mathcal{D}_{g_{\kappa}}: \mathbb{R} \times \mathcal{Q}(-\Delta + V) \oplus \mathcal{F}H^{1/2} \rightarrow \mathcal{Q}(-\Delta + V) \oplus \mathcal{F}H^{1/2}$ be the classical dressing transformation. Let $(\mathcal{Q}_\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})}$ be a family of normal quantum states on $\mathcal{H}$ that satisfies Assumption (A0) and Assumption (A_p) or $(A'_p)$. Then $\mathcal{M}(\mathcal{Q}_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) \neq \emptyset$ and for any $\sigma_0 \in \mathbb{R}_+$ and $\theta \in \mathbb{R}$,

$$(108) \quad \mathcal{M}(\hat{\mathcal{Q}}_\varepsilon(\theta), \varepsilon \in (0,\bar{\varepsilon})) = \left\{ \mathcal{D}_{g_{\kappa}}(\theta) \# \mu, \mu \in \mathcal{M}(\mathcal{Q}_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) \right\}.$$

Furthermore, let $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0,\bar{\varepsilon})$ be a sequence such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$. Then the following statement is true:

$$(109) \quad \mathcal{Q}_{\varepsilon_k} \rightarrow \mu \Leftrightarrow \left( \forall \theta \in \mathbb{R}, \forall \sigma_0 \in \mathbb{R}_+, \hat{\mathcal{Q}}_{\varepsilon_k}(\theta) \rightarrow \mathcal{D}_{g_{\kappa}}(\theta) \# \mu \right).$$
7. Overview of the results: linking the dressed and undressed systems.

Since as discussed in the previous subsection we can treat the dressing as a dynamical transformation with its own “time” parameter \( \theta \); we are able to link the classical limit of the dressed and undressed quantum dynamics via the classical dressing. In this way we are able to recover the expected classical S-KG dynamics for the undressed dynamics, and finally prove Theorem I.1.

First of all, we put together the results proved from Section V.1 to Section V.5 on the renormalized dressed dynamics and remove the Assumption \( (A_p') \) with the help of an approximation argument worked out in [10]. This is done in the following theorem.

**Theorem V.26.** Let \( \hat{\mathcal{E}} : \mathbb{R} \times Q(-\Delta + V) \oplus \mathcal{F} H^{1/2} \to Q(-\Delta + V) \oplus \mathcal{F} H^{1/2} \) be the dressed S-KG flow associated to \( \hat{\mathcal{E}} \). Let \( (\mathcal{Q}_e)_{e \in (0,\bar{\varepsilon})} \) be a family of normal states in \( \mathcal{H} \) that satisfies Assumptions \( (A_0) \) and \( (A_\varepsilon') \). Then for any \( \sigma_0 \geq 2K(\bar{\varepsilon} + 1 + \bar{\varepsilon}) \) the dynamics \( e^{-i\frac{\hbar}{\varepsilon} \hat{H}_{\text{ren}}(\sigma_0)} \) is non-trivial on every relevant sector with fixed nucleons of the state \( \mathcal{Q}_e ; \mathcal{M}(\mathcal{Q}_e, \varepsilon \in (0, \bar{\varepsilon})) \neq \emptyset \); and for any \( t \in \mathbb{R} \)

\[
\mathcal{M}\left(e^{-i\frac{\hbar}{\varepsilon} \hat{H}_{\text{ren}}(\sigma_0)} \mathcal{Q}_e e^{i\frac{\hbar}{\varepsilon} \hat{H}_{\text{ren}}(\varepsilon)} , \varepsilon \in (0, \bar{\varepsilon}) \right) = \left\{ \hat{\mathcal{E}}(t) \# \mu : \mu \in \mathcal{M}(\mathcal{Q}_e, \varepsilon \in (0, \bar{\varepsilon})) \right\} .
\]

Furthermore, let \( (\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \bar{\varepsilon}) \) be a sequence such that \( \lim_{k \to \infty} \varepsilon_k = 0 \). Then the following statement is true:

\[
\mathcal{Q}_{e_k} \to \mu \Leftrightarrow \left( \forall t \in \mathbb{R} , \ e^{-i\frac{\hbar}{\varepsilon_k} \hat{H}_{\text{ren}}(\varepsilon_k)} \mathcal{Q}_{e_k} e^{i\frac{\hbar}{\varepsilon_k} \hat{H}_{\text{ren}}(\varepsilon_k)} \to \hat{\mathcal{E}}(t) \# \mu \right) .
\]

**Proof.** Thanks to the argument briefly sketched below, we no longer need Assumption \( (A_p) \). Let \( \chi \in C^0_0(\mathbb{R}) \) such that \( 0 \leq \chi \leq 1 \), \( \chi \equiv 1 \) in a neighbourhood of 0 and \( \chi_{\mathbb{R}}(x) = \chi(x^\mathbb{R}) \). The approximation

\[
\mathcal{Q}_{e,R} = \frac{\chi_{\mathbb{R}}(\hat{H}_{\text{ren}}(\sigma_0)) \mathcal{Q}_e \chi_{\mathbb{R}}(\hat{H}_{\text{ren}}(\sigma_0))}{\text{Tr} \left[ \chi_{\mathbb{R}}(\hat{H}_{\text{ren}}(\sigma_0)) \mathcal{Q}_e \chi_{\mathbb{R}}(\hat{H}_{\text{ren}}(\sigma_0)) \right]} ,
\]

satisfies the Assumptions \( (A_0), (A_p'), (A_p'') \) and the property

\[
\|e^{-i\frac{\hbar}{\varepsilon} \hat{H}_{\text{ren}}(\sigma_0)} (\mathcal{Q}_e - \mathcal{Q}_{e,R}) e^{i\frac{\hbar}{\varepsilon} \hat{H}_{\text{ren}}(\sigma_0)}\|_{\mathcal{L}_1(\mathcal{H})} = \|\mathcal{Q}_e - \mathcal{Q}_{e,R}\|_{\mathcal{L}_1(\mathcal{H})} \leq \nu(R) ,
\]

where \( \nu(R) \) is independent of \( \varepsilon \) and \( \lim_{R \to \infty} \nu(R) = 0 \). The last claim follows by Assumption \( (A_p') \), Theorem II.10 and Definition II.12. Up to extracting a sequence which a priori depends on \( R \) and \( t \), we can suppose that \( \mathcal{M}(\mathcal{Q}_{e,R}, n \in \mathbb{N}) = \{\mu_{0,R} \} \), \( \mathcal{M}(\mathcal{Q}_{e,n}, n \in \mathbb{N}) = \{\mu_0\} \) and \( \mathcal{M}(\mathcal{Q}_{e_n}(t), n \in \mathbb{N}) = \{\mu_1\} \). In particular, applying Proposition V.23 we obtain

\[
\mathcal{M}(e^{-i\frac{\hbar}{\varepsilon} \hat{H}_{\text{ren}}(\sigma_0)} \mathcal{Q}_{e_n,R} e^{i\frac{\hbar}{\varepsilon} \hat{H}_{\text{ren}}(\sigma_0)}, n \in \mathbb{N}) = \{\hat{\mathcal{E}}(t) \# \mu_{0,R} \} .
\]
A general estimate proved in [10, Proposition 2.10] compares the total variation distance of Wigner (probability) measures with the trace distance of their associated quantum states. In our case, it implies

\[
\int_{L^2 @ L^2} |\mu_t - \hat{E}(t) \# \mu_{0,R}| \leq \liminf_{n \to \infty} \| e^{-i \frac{\sigma_0}{n}} H_{ren}^{(\sigma_0)} \left( Q_{e_n} - Q_{e_n,R} \right) e^{i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} \|_{L^1(\mathcal{H})} \leq \nu(R), \\
\int_{L^2 @ L^2} |\mu_0 - \mu_{0,R}| \leq \liminf_{n \to \infty} \| Q_{e_n} - Q_{e_n,R} \|_{L^1(\mathcal{H})} \leq \nu(R),
\]

where the left hand side denotes the total variation of the signed measures \( \mu_t - \hat{E}(t) \# \mu_{0,R} \) and \( \mu_0 - \mu_{0,R} \) respectively. Hence by the triangle inequality, we obtain

\[
\int_{L^2 @ L^2} |\mu_t - \hat{E}(t) \# \mu_0| \leq \int_{L^2 @ L^2} |\mu_t - \hat{E}(t) \# \mu_{0,R}| + \int_{L^2 @ L^2} |\mu_{0,R} - \mu_0| \leq 2\nu(R).
\]

This proves that

\[
\{ \hat{E}(t) \# \mu_0 \} \subset \mathcal{M}(e^{-i \frac{\sigma_0}{n}} H_{ren}^{(\sigma_0)} Q_{e_n} e^{i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}}, n \in \mathbb{N}).
\]

By reversing time and utilizing the analogue inclusion above, we prove (111). 

\[\blacksquare\]

**Proof of Theorem I.1**: Observe that using the definition of the renormalized dressed evolution \( Q_e(t) \) (Definition V.3) and the definition of the “dressing dynamics” \( \hat{q}_e(\theta) \) (Equation (107)), we obtain:

\[
e^{-i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} Q_e e^{i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} = e^{-i \frac{\theta}{T_n} - i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} e^{i \frac{\theta}{T_n} + i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} \hat{q}_e e^{i \frac{\theta}{T_n} + i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} e^{i \frac{\theta}{T_n} + i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} = \left((\hat{q}_e(-1))(t)\right)^{(1)}.
\]

Let \( (Q_e)_{e \in (0, \varepsilon)} \) be a family of normal states in \( \mathcal{H} \) that satisfies Assumptions (A0) and (A_\rho). In addition, as usual, let \( (\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \varepsilon) \) be a sequence such that \( \lim_{k \to \infty} \varepsilon_k = 0 \). Then we can use Lemma V.24, Proposition V.25 and Theorem V.26 to prove the following statement:

\[
Q_{e_k} \to \mu \iff \forall t \in \mathbb{R}, \ e^{-i \frac{\theta}{T_n} - i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} Q_{e_k} e^{i \frac{\theta}{T_n} + i \frac{\sigma_0}{n} H_{ren}^{(\sigma_0)}} \to D_{g_\omega}(1) \# \hat{E}(t) \# D_{g_\omega}(-1) \# \mu = [D_{g_\omega}(1) \circ \hat{E}(t) \circ D_{g_\omega}(-1)] \# \mu
\]

Therefore Theorem I.1 is proved, since by Equation (64) of Theorem III.15, \( D_{g_\omega}(1) \circ \hat{E}(t) \circ
The inference in the opposite sense is trivial.

As it has become evident with the above discussion, we do not prove Theorem I.1 directly; and it would be very difficult to do so, due to the fact that we do not know the explicit form of the generator $H_{\text{ren}}(\sigma_0)$ of the undressed dynamics. We know instead how the dressed generator $\hat{H}_{\text{ren}}(\sigma_0)$ acts as a quadratic form, and that is sufficient to characterize its dynamics in the classical limit, and obtain the results of Theorem V.26. The properties of the dressing transformation and of its classical counterpart are then crucial to translate the results on the dressed dynamics to the corresponding results on the undressed one.

### Appendix A: Uniform higher-order estimate

We prove in this section a higher-order estimate that bounds the meson number operator $N_2$ by the dressed Hamiltonian $\hat{H}_{\sigma}^{(n)}$ uniformly with respect to the effective (semiclassical) parameter $\epsilon$ and the cut-off parameter $\sigma$. Such type of estimates rely on the pull-through formula and they are known for the $P(\varphi)$ model [92] and for the Nelson model [5]. However, since the dependence of the dressed Hamiltonian $\hat{H}_{\sigma}^{(n)}$ on $\epsilon$ is somewhat nontrivial, we briefly indicate in this appendix how to obtain an uniform estimate.

**Lemma A.1.** For any $\epsilon \in (0, \bar{\epsilon})$ and any $\psi \in \mathcal{D}(N_2) \subset \mathcal{H}$,

$$||N_2\psi||^2 = \int_{\mathbb{R}^3} ||(N_2 + \epsilon)^{\frac{1}{2}}a(k)\psi||^2 \, dk .$$

**Proof.** Recall that $N_2$ and $a(k)$ depends in the parameter $\epsilon$ according to the notations of Subsection I.1. Taking care of domain issues as in [5, Lemma 2.1] one proves

$$||N_2\psi||^2 = \langle N_2^{\frac{1}{2}}\psi, \int_{\mathbb{R}^3} a^*(k)a(k) \, dk \, N_2^{\frac{1}{2}}\psi \rangle = \int_{\mathbb{R}^3} ||a(k)N_2^{\frac{1}{2}}\psi||^2 \, dk = \int_{\mathbb{R}^3} ||(N_2 + \epsilon)^{\frac{1}{2}}a(k)\psi||^2 \, dk .$$

$\blacksquare$
Recall that the interaction term $\hat{H}_I(\sigma)^{(n)}$ is given by (13). A simple computation yields

$$[a(k), \hat{H}_I(\sigma)^{(n)}] = e^2 \left[ \sum_{j=1}^{n} \frac{1}{2\sqrt{(2\pi)^3}} \frac{e^{ikx_j}}{\sqrt{\omega(k)}} - \frac{1}{M} \sum_{j=1}^{n} r_\sigma(k) e^{-ik\cdot x_j} a^*(r_\sigma e^{-ik\cdot x_j}) + r_\sigma(k) e^{-ik\cdot x_j} a(r_\sigma e^{-ik\cdot x_j}) - r_\sigma(k) e^{-ik\cdot x_j} D_{x_j} \right].$$

**Lemma A.2.** For any $\mathbb{C} > 0$ and $\sigma_0 \geq 2K(\mathbb{C} + 1 + \tilde{\varepsilon})$ there exist $c, b > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon})$, $\sigma_0 < \sigma \leq +\infty$ and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathbb{C}$, we have

$$\left\| (b + \varepsilon\omega(k) + \hat{H}_I(\sigma)^{(n)})^{-\frac{1}{2}} [a(k), \hat{H}_I(\sigma)^{(n)}](b + \hat{H}_I(\sigma)^{(n)})^{-\frac{1}{2}} \right\| \leq c \left( \frac{\varepsilon^{s}(k)}{\sqrt{\omega(k)}} + |r_\sigma(k)|\omega(k)^{-1/4} \right).$$

**Proof.** According to Proposition II.9 and Theorem II.10, $\hat{H}_I(\sigma)^{(n)}$ is $H_0^{(n)}$-form bounded with small bound that is uniform with respect to $\varepsilon \in (0, \tilde{\varepsilon})$, $\sigma_0 < \sigma \leq +\infty$ and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathbb{C}$. Hence $(H_0^{(n)})^{\pm}(b + \hat{H}_I(\sigma)^{(n)})^{-\frac{1}{2}}$ is uniformly bounded for some $b > 0$. So it is enough to prove the claimed bound with $H_0^{(n)}$ instead of $\hat{H}_I(\sigma)^{(n)}$. Now using similar estimates as in Lemma II.6 and the fact that $\sqrt{\varepsilon\omega(k)(b + \varepsilon\omega(k) + \hat{H}_I(\sigma)^{(n)})^{-\frac{1}{2}}}$ is uniformly bounded one correctly bounds all the terms of the commutator except the one with $a^*$. Remark that the commutator contains the power $e^2$ that controls the sum over $1 \leq j \leq n$ and the factor $1/\sqrt{\varepsilon\omega(k)}$. In order to bound the term with $a^*$, one uses the type of estimate in [5, Lemma 3.3 (ii)] with $s = 1/2$. Remark that one gets an $\varepsilon$-dependent estimate from [5, Lemma 3.3 (ii)] by noticing that $e^{1/4}(H_0^{(n)} + 1)^{-1/4}(d\Gamma_1(\omega) + 1)^{1/4}$ and $e^{1/4}(N_2 + 1)^{-1/4}(d\Gamma_1(1) + 1)^{1/4}$ are uniformly bounded\(^{21}\) and that $a^*$ contains $\sqrt{\varepsilon}$ that cancels the latter $e^{-1/4} \cdot \varepsilon^{-1/4}$. \hfill \blacksquare

Let $\mathbb{C} > 0$ and $\sigma_0 \geq 2K(\mathbb{C} + 1 + \tilde{\varepsilon})$ as in the above lemma. In particular $\hat{H}_I(\sigma)^{(n)}$ is a self-adjoint operator for any $\varepsilon \in (0, \tilde{\varepsilon})$, $\sigma_0 < \sigma \leq +\infty$ and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathbb{C}$.

**Lemma A.3 (The pull-through formula).** The following identity holds true for some $b < 0$, any $\phi \in D(N_2^{\frac{1}{2}}) \cap \mathcal{H}_n$ and $k$ almost everywhere in $\mathbb{R}^3$,

$$a(k)(b - \hat{H}_I(\sigma)^{(n)})^{-1}\phi = (b - \varepsilon\omega(k) - \hat{H}_I(\sigma)^{(n)})^{-1}a(k)\phi + (b - \varepsilon\omega(k) - \hat{H}_I(\sigma)^{(n)})^{-1}[a(k), \hat{H}_I(\sigma)^{(n)}](b - \hat{H}_I(\sigma)^{(n)})^{-1}\phi.$$

**Proof.** According to [5, Lemma 4.4] there exists $\psi \in (H_0^{(n)} + 1)^{-1}D(N_2^{\frac{1}{2}})$ such that $\phi = (b - \hat{H}_I(\sigma)^{(n)})\psi$ for some $b < 0$. So the claimed formula is equivalent to

$$(b - \varepsilon\omega(k) - \hat{H}_I(\sigma)^{(n)})a(k)\psi = a(k)(b - \hat{H}_I(\sigma)^{(n)})\psi + [a(k), \hat{H}_I(\sigma)^{(n)}]\psi.$$ 

The latter identity follows by a simple computation. \hfill \blacksquare

\(^{21}\) $d\Gamma_1(\cdot)$ is the $\varepsilon$-independent second quantization operator in [5]
Proposition A.4. For any $\mathcal{C} > 0$ and $\sigma_0 \geq 2K(\mathcal{C} + 1 + \bar{\varepsilon})$ there exist $c, b > 0$ such that the operator $\hat{H}_\sigma^{(n)}$ is self-adjoint and the following bound holds true:

$$\|N_2\psi\| \leq c\|(\hat{H}_\sigma^{(n)} + b)\psi\|, \quad \forall \psi \in D(\hat{H}_\sigma^{(n)}),$$

for any $\varepsilon \in (0, \bar{\varepsilon}), \sigma \in (\sigma_0, +\infty], n \in \mathbb{N}$ such that $n\varepsilon \leq \mathcal{C}$.

Proof. The operator $\hat{H}_\sigma^{(n)}$ is uniformly bounded from below. So by choosing $b > 0$ large enough one can take $\psi = (-b - \hat{H}_\sigma^{(n)})^{-1}\phi$. Now it is enough to prove the estimate for $\phi \in (H_0^{(n)} + 1)^{-1/2}D(N_2^{\frac{1}{2}})$. Using Lemma A.1 and Lemma A.3,

$$\|N_2\psi\|^2 = \int_{\mathbb{R}^3} \|(N_2 + \varepsilon)^{\frac{1}{2}}a(k)(b + \hat{H}_\sigma^{(n)})^{-1}\phi\|^2 \, dk \leq 2 \int_{\mathbb{R}^3} \|(N_2 + \varepsilon)^{\frac{1}{2}}(b + \varepsilon\sigma(k) + \hat{H}_\sigma^{(n)})^{-1}a(k)\phi\|^2 \, dk$$

(A1)

$$+ 2 \int_{\mathbb{R}^3} \|(N_2 + \varepsilon)^{\frac{1}{2}}(b + \varepsilon\sigma(k) + \hat{H}_\sigma^{(n)})^{-1}[a(k), \hat{H}_1(\sigma^{(n)})](b + \hat{H}_\sigma^{(n)})^{-1}\phi\|^2 \, dk.$$  

(A2)

Since $(N_2 + \varepsilon)^{\frac{1}{2}}(b + \varepsilon\sigma(k) + \hat{H}_\sigma^{(n)})^{-1/2}$ is uniformly bounded, by Lemma A.2 one shows

$$(A2) \leq c \int_{\mathbb{R}^3} |\chi_{\sigma(k)}| + |r_\sigma(k)|\omega(k)^{-1/4} \, dk \cdot \|(b + \hat{H}_\sigma^{(n)})^{-1/2}\phi\|^2$$

For simplicity we denote by $c$ any constant. In the same way, one also shows

$$(A1) \leq c \int_{\mathbb{R}^3} \|(b + \varepsilon\sigma(k) + \hat{H}_\sigma^{(n)})^{-1/2}a(k)\phi\|^2 \, dk \leq c \int_{\mathbb{R}^3} \|(b + \varepsilon\sigma(k) + H_0^{(n)})^{-1/2}a(k)\phi\|^2 \, dk = c\|N_2^{1/2}(b + H_0^{(n)})^{-1/2}\phi\|^2.$$

The last equality follows by a similar argument as in the proof of Lemma A.1. Hence, one obtains

$$\|N_2\psi\|^2 \leq c\left(\|\phi\|^2 + \|(b + H_0^{(n)})^{-1/2}\phi\|^2\right) = c\left(\|(b + \hat{H}_\sigma^{(n)})\psi\|^2 + \|(b + H_0^{(n)})^{1/2}\psi\|^2\right) \leq c\|(b + \hat{H}_\sigma^{(n)})\psi\|^2.$$

The last inequality is a consequence of the uniform boundedness of the operator $(b + H_0^{(n)})^{-1/2}(b + \hat{H}_\sigma^{(n)})^{-1/2}$ with respect to $\varepsilon, \sigma$ and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathcal{C}$. 

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