How many colors to color a random graph?
Cavity, Complexity, Stability and all that

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We review recent progress on the statistical physics study of the problem of coloring random graphs with $q$ colors. We discuss the existence of a threshold at connectivity $c_q = 2q \log q - \log q - 1 + o(1)$ separating two phases which are respectively COL(orable) and UNCOL(orable) with $q$ colors. We also argue that the so-called one-step replica symmetry breaking ansatz used to derive these results give exact threshold values, and draw a general phase diagram of the problem.

Since the first observation in 1852 by Francis Guthrie that any planar map could be colored with only 4 colors, graph coloring has grown to become an important problem both in combinatorial mathematics and in statistical physics. Given a graph, or a lattice, and given a number $q$ of colors, it consists in assigning a color to each vertex such that no edge has two equally colored end vertices. When defined on random graphs, the problem turns out to be NP-Complete and to display an interesting phase transitions at the so-called $q$-COL/UNCOL connectivity $c_q$: graphs of average connectivity $c < c_q$ do have proper $q$-colorings with high probability (approaching one for graph size $N \to \infty$), whereas graphs of higher connectivity require more than $q$ colors. Here we review the recent progress on the statistical physics approach to the characterization of the phase diagram of this problem.

A very first approximation for physicists working in disordered systems is the so-called annealed computation (the first moment method in computer science). Take two connected vertices: the probability that they share the same color for a random assignment is $1/q$, hence they have different colors with probability $1 - 1/q$. A crude estimate of the probability that a random configuration colors a graph of average connectivity $c$ is easily obtained: there are $cN/2$ links and each of them has a probability $1 - 1/q$ to be satisfied, therefore the number of COL configurations is

$$N(c) \propto q^N \left(1 - \frac{1}{q}\right)^{cN/2} \propto e^{N\Sigma(c)}, \quad \text{where} \quad \Sigma(c) = \log q + \frac{c}{2} \log (1 - 1/q). \quad (0.1)$$

It is straightforward to deduce from the preceding formula the existence of a critical connectivity $c_q \approx 2q \ln q - \ln q$. For $c > c_q$, $\Sigma(c) < 0$ and the number of COL assignments is vanishing exponentially with the size of the graph while for $c < c_q$, $\Sigma(c) > 0$ and therefore there is an exponentially huge number of COL assignments: the COL/UNCOL transition is easily seen already at the annealed level. Such considerations are far from being only hand-waving; in fact, similar computations, using the first and second moment methods, allow to rigorously show that $2q \ln q - \ln q - 1 + o(1) \geq c_q \geq 2q \ln q - 2 \ln q + o(1)$. To go beyond these inequalities, we turn toward the use of more complex statistical physics tools.
It is quite immediate, for a statistical physicist, to realize that the coloring problem is equivalent to knowing if the energy of the ground-state of an anti-ferromagnetic Potts model on a random graph is zero (COL) or not (UNCOL). Inspired by spin glass theory, one can use the replica/cavity approach to analytically study the problem, following the seminal work of \[7\]. In this particular context of constraint satisfaction problems, the zero temperature cavity method is particularly well suited and thanks to many recent developments, precise and detailed studies have been made possible. Although the method is not fully rigorous, the self-consistency of the main underlying hypothesis has been checked in different optimization problems and therefore all the features of the solution derived by the statistical physics approach we will review here are conjectured to be exact results, not approximations. Of course, it is of first importance to develop rigorous mathematical approaches in order to confirm them.

The analytical results on the phase diagram are summarized on Fig. 1; it is very similar to the one first observed for other optimization problems such as the K-Satisfiability\[8\]. Let us discuss it for the particular case of the 3-coloring (we refer the readers to \[11\],\[12\],\[15\] for more details). When one varies the connectivity \(c\), there actually exist many distinct phases, separated by thresholds connectivity \(c_d\), \(c_m\), \(c_q\) and \(c_{SP}\). The most important point is of course the critical COL/UNCOL transition that happens at \(c_q = 4.69\). It separates the COL phases at \(c < c_q\) from the UNCOL phase at \(c > c_q\). But in the COL region, there actually exist distinct phases that differ by the structure of their phase space. First, For \(c < c_d \simeq 4.42\), the set of COL assignments builds one cluster which is basically connected and from one single valley in the phase space. This phase is called the EASY-COL phase, for it is generally quite easy for any algorithm to find a COL assignment in this phase.
On the other hand, for \( c > c_d \), this phase space becomes disconnected and the COL assignments are grouped into many clusters: this is the phenomenon of Replica Symmetry Breaking (RSB), familiar to anyone who had practiced mean field spin glass theory. This phase space is characterized by the presence of many non-ergodic valleys and it is now impossible for any physical dynamics to get from one cluster to another. Moreover, the phase space develops many metastable states at higher energies (corresponding to UNCOL configurations) and any too simple algorithm, such as steepest descent or simulated annealing, would get trapped into some these metastable states in a same way aging systems undergo a glass transition; the COL region in this phase is thus said to be HARD-COL. For \( c > c_q \), there are no more COL solutions and all valleys are UNCOL. In the whole RSB phase, the number of such valleys is growing exponentially with the size of the problem and the cavity method allows the computation of the logarithm of this number in a very similar way one computed the annealed logarithm number of COL solutions \( \Sigma(c) \) in eq. (0.1). This quantity, which is called the complexity is a central element within the cavity method. All these results where first obtained by Zecchina and collaborators.

This complex structure of the phase space is however very hard to study. In fact it is only when it is not too complex, i.e. when configurations simply group into valleys —the so-called one-step Replica Symmetry Breaking (1RSB) ansatz— that one is able to solve the equations. However the situation is generically much more complex: we do have valleys within valleys within valleys etc. . . which obviously make the whole approach very difficult! In fact, most computations are treating the phase space as if it was only 1RSB, neglecting the effect of valleys within valleys. Recently, by studying more precisely the structure of this phase space, we show that, luckily enough, there is a zone in this complex RSB phase which is indeed 1RSB; for the 3-coloring, it happens for \( c_m \approx 4.51 < c < c_G \approx 5.08 \). Even more luckily, it turns out that \( c_q \), the COL/UNCOL threshold connectivity, is precisely inside this zone, and that we are thus able to compute it without doing any approximation. In other word, the original computation of was made in a stable 1RSB zone, and is therefore valid. This knowledge of the phase space allows us to draw a more complete and quite generic phase diagram (fig.1) where we show all the different phases and transitions (COL/UNCOL, RSB/1RSB...) that the system undergoes while varying connectivity.

It turns out that the feature of this phase diagram are the same for every number of colors \( q \geq 3 \) (as well as for other satisfaction problem). For instance, in the specific case of random graph with \( \text{fixed connectivity} \) (where graphs are still random but constructed in such a way that each vertexes have the same connectivity) one can derived analytically all critical connectivities for any number \( q \) of colors. We illustrate these computations in Fig.2, where the different phases and transition discussed here are clearly seen. It can be checked directly that the critical COL/UNCOL transition always happens inside the 1RSB stable zone. Within this approach, we also show that the value of the COL/UNCOL threshold is asymptotically \( c_q = 2q \log q - \log q + 1 + o(1) \) which agrees perfectly with mathematical bounds mentioned before.

To conclude, the statistical physics approach, via the cavity method, of the coloring problem turns out to be rather fruitful. Not only it is consistent with indepen-
Fig. 2. Phase diagram of the $q$–coloring problem on regular fixed connectivity random graphs.

dently established rigorous mathematical results, but it also allows for calculation and determination of the phase diagram and for a sharper, though not rigorous, determination of threshold values.

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