ENERGY DECAY AND GLOBAL SMOOTH SOLUTIONS FOR A FREE BOUNDARY FLUID-NONLINEAR ELASTIC STRUCTURE INTERFACE MODEL WITH BOUNDARY DISSIPATION

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ABSTRACT. We consider a nonlinear, free boundary fluid-structure interaction model in a bounded domain. The viscous incompressible fluid interacts with a nonlinear elastic body on the common boundary via the velocity and stress matching conditions. The motion of the fluid is governed by incompressible Navier-Stokes equations while the displacement of elastic structure is determined by a nonlinear elastodynamic system with boundary dissipation. The boundary dissipation is inserted in the velocity matching condition. We prove the global existence of the smooth solutions for small initial data and obtain the exponential decay of the energy of this system as well.

1. Introduction. We consider a free boundary fluid-structure interaction model which consists of the viscous incompressible fluid and the nonlinear elastic structure. Both fluid and elastic body are contained in $\Omega$ which is a smooth bounded domain in $\mathbb{R}^3$. This domain is divided into two parts by the common interface of fluid and structure where the interaction takes place. The inner part is taken over by the elastic body, denoted by $\Omega_e(t) \subset \Omega$ while the fluid occupied the exterior part $\Omega_f(t) = \Omega \backslash \Omega_e(t)$. We denote the fluid portion and the solid portion in the domain $\Omega$ at the start time by $\Omega_e$ and $\Omega_f$, respectively. Both $\Omega_e$ and $\Omega_f$ are also smooth bounded domains in $\mathbb{R}^3$. Moreover, we use the symbol $\Gamma_c = \partial \Omega_e \cap \partial \Omega_f$ to stand for the
common boundary of fluid and solid at $t = 0$ (For more details, see [2, 9, 10, 11]). The motion of the fluid is described by the incompressible Navier-Stokes equations (see [14]):

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } \Omega_f(t),$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega_f(t),$$

in which the vector field $u \in \mathbb{R}^3$ is the velocity of the fluid, while the displacement of elastic structure $w \in \mathbb{R}^3$ is dominated by the following nonlinear elastodynamic system:

$$w_{tt} = \text{div} \, DW(\nabla w) + w,$$

which is derived by variational methods from the action functional

$$I[w] = \int_0^T \int_{\Omega_w} \left[ \frac{1}{2} |w_t|^2 - W(\nabla w) + |w|^2 \right] dx \, dt,$$

where $W(F) : \mathbb{R}^{3 \times 3} \to \mathbb{R}$ is the stored-energy function of the elastic material. The term $|w|^2$ in (3) is needed for the energy estimate consideration, see [8]. The interaction occurs on the common boundary $\Gamma_c(t)$ via the natural transmission boundary conditions matching the velocity and the stress.

Fluid-structure interaction models have drawn considerable attention from both engineers and mathematical researchers. At beginning, these models were considered in a finite element framework (see ([4], [5], [6]) and reference therein). Recently, the topics about the mathematical theory of existence, uniqueness and stability of solutions for such models have been becoming quite attractive. For the linear elastic material, local in time well-posedness of the free boundary model was first established by Coutand and Shkoller in [2] and improved in [7, 10] and [11] where there are no dissipative mechanisms on the interface. Then the global-in-time existence for the fluid-structure system with damping is established in [8] and [9] for small initial data and linear isotropic elastic material. In [8], the authors consider the case that there are boundary dissipative term on the common interface $\Gamma_c(t)$ and internal dissipative term $\alpha w_t, \alpha \geq 0$ in the wave equations. And in [9], only internal dissipative term in wave equations are considered. Besides, [13], the global solutions and energy decay are also obtained for the linear wave equations with variable coefficients coupling with incompressible viscous fluid. For other topics on fluid-linear elastic structure system, see the short review in the introduction in [9].

As for the nonlinear elastic material, in [3], Coutand and Shkoller developed the short-time wellposedness theory for the system in which fluid couples with some specific quasilinear elastic material.

In this paper we assume that the elastic body is a general nonlinear material to consider the global smooth solutions and energy decay of the system where the fluid interacts the nonlinear elastic body determined by the action functional (3). The nonlinearity of the elasticity equations causes more difficulty and increase the complexity for us to establish our result and we need to do careful estimates to overcome this difficulty. For the fluid part of this system, we employ the method and the estimates obtained in [8]. For the elastic body, we use the multiplier methods and invoke the nonlinear estimates and compactness results derived in [16], [17], [1] and [18] to deal with our problem.

Let $\eta(x, t) : \Omega \to \Omega(t)$ be the position function with $\Omega(t) = \Omega$ being the different states of the system with respect to different time. With the help of position
function, the incompressible Navier-Stokes equations can be reformulated in the Lagrangian framework:

\[
\begin{aligned}
\partial_t v^i - \partial_j (a^{ki} a^{lj} \partial_k v^j) + \partial_k (a^{ki} q) &= 0 & \text{in} & & \Omega_f \times (0, T), \\
\partial_k a^{ki} v^i &= 0 & \text{in} & & \Omega_f \times (0, T),
\end{aligned}
\]

where \(v(x,t)\) and \(q(x,t)\) denote the Lagrangian velocity and the pressure of the fluid over the initial domain \(\Omega\). The matrix \(a\) is decomposed into three scalar parts \(a^{ij}\), \(a^i\), and \(a\). The nonlinear elastodynamic equations for the displacement function \(w(x,t) = \eta(x,t) - x\) are formulated as the following:

\[
w_{tt} - \text{div} \, DW(\nabla w + \mathbb{I}) + w = 0 \quad \text{in} \quad \Omega_e \times (0, T), \quad i = 1, 2, 3.
\]

We seek a solution \((v, w, q, a, \eta)\) to the system (4) and (5), where the matrix \(a = (a^{ij})(i,j = 1, 2, 3)\) and \(\eta \big|_{\Omega_f}\) are determined in the following way:

\[
\begin{aligned}
a_t &= -a : \nabla v : a, \quad \text{in} \quad \Omega_f \times (0, T), \\
\eta_t &= v \quad \text{in} \quad \Omega_f \times (0, T), \\
a(x,0) &= I, \quad \eta(x,0) = x, \quad \text{in} \quad \Omega_f,
\end{aligned}
\]

where the symbol “:” stands for the usual multiplication between matrices and \(\mathbb{I}\) represents the identity matrix in \(\mathbb{R}^{3 \times 3}\).

Making use of the notation \(a\), we rewrite (4) as

\[
\begin{aligned}
\partial_t v - \text{div} \, (a : a^T \nabla v) + \text{div} \, (aq) &= 0 & \text{in} & & \Omega_f \times (0, T), \\
\text{tr}(a : \nabla v) &= 0 & \text{in} & & \Omega_f \times (0, T).
\end{aligned}
\]

On the interface \(\Gamma_e\) between \(\Omega_f\) and \(\Omega_e\), we assume the transmission boundary condition

\[
w_t = v - \gamma w_{N_f} \quad \text{on} \quad \Gamma_e \times (0, T),
\]

where the constant \(\gamma > 0\) and

\[
w_{N_f} = DW(\nabla w + \mathbb{I})\nu,
\]

and the matching of stress

\[
w_{N_f} = (a : a^T \nabla v)\nu - q(a\nu) \quad \text{on} \quad \Gamma_e \times (0, T),
\]

where \(\nu = (\nu_1, \nu_2, \nu_3)\) is the unit outward normal with respect to \(\Omega_e\). On the outside fluid boundary \(\Gamma_f = \partial \Omega\), we impose the non-slip condition

\[
v = 0 \quad \text{on} \quad \Gamma_f \times (0, T).
\]

We supplement the system (4) and (5) with the initial data \(v(x,0) = v_0(x)\) and \((w(x,0), w_t(x,0)) = (w_0(x), w_t(x))\) in \(\Omega_f\) and \(\Omega_e\), respectively. Let \(H = \{u \in L^2(\Omega_f) : \text{div} \, u = 0, \quad u \cdot \nu|_{\Gamma_f} = 0\}\) and \(V = \{u \in H^1(\Omega_f) : \text{div} \, u = 0, \quad u|_{\Gamma_f} = 0\}\).

Based on the initial data \(v_0\), the initial pressure \(q_0\) is determined by solving the problem

\[
\begin{aligned}
\Delta q_0 &= -\partial_i v_0^j \partial_k v_0^j \quad \text{in} \quad \Omega_f, \\
\nabla q_0 \cdot \nu &= \Delta v_0 \cdot \nu \quad \text{on} \quad \Gamma_f, \\
-q_0 &= -\partial_i v_0^j \nu_j + w_{N_f}^i \nu_i, \quad \text{on} \quad \Gamma_e.
\end{aligned}
\]
Let the initial data $v_0 \in V \cap H^3(\Omega_f)$, $w_0 \in H^4(\Omega_e)$, and $w_1 \in H^3(\Omega_e)$ be provided. Moreover, $v_0, w_0$ and $w_1$ are supposed to satisfy the compatibility conditions as follows:

\begin{align}
(w_0)_{\mathcal{N}_e} &= \gamma^{-1}(v_0 - w_1) \text{ on } \Gamma_e, \\
(\nabla v_0 \cdot \nu, \tau) &= \langle (w_0)_{\mathcal{N}_e}, \tau \rangle \text{ on } \Gamma_c,
\end{align}

\begin{align}
\partial_t(l_{\nabla w}(\int_0^1 D^2W(\mathbb{I} + s\nabla w)))(0) \nu &= \gamma^{-1}(v'(0) - w''(0)) \text{ on } \Gamma_c, \quad (15) \\
\langle \partial_t(a : a^T : \nabla v)(0) \cdot \nu, \tau \rangle &= \langle \partial_t(l_{\nabla w}(\int_0^1 D^2W(\mathbb{I} + s\nabla w)))(0) \nu, \tau \rangle \quad (16)
\end{align}

with

\begin{align}
\partial_t(l_{\nabla w}(\int_0^1 D^2W(\mathbb{I} + s\nabla w)))(0) \nu &= \gamma^{-1}(v''(0) - w'''(0)) \text{ on } \Gamma_c, \quad (17) \\
\langle \partial_t(a : a^T : \nabla v)(0) \cdot \nu, \tau \rangle &= \langle \partial_t(l_{\nabla w}(\int_0^1 D^2W(\mathbb{I} + s\nabla w)))(0) \nu, \tau \rangle \quad (18)
\end{align}

and

\begin{align}
\partial_{ttt}(l_{\nabla w}(\int_0^1 D^2W(\mathbb{I} + s\nabla w)))(0) \nu &= \gamma^{-1}(v_{ttt}(0) - w_{tttt}(0)) \text{ on } \Gamma_c, \quad (19) \\
\langle \partial_{ttt}(a : a^T : \nabla v)(0) \cdot \nu, \tau \rangle &= \langle \partial_{ttt}(l_{\nabla w}(\int_0^1 D^2W(\mathbb{I} + s\nabla w)))(0) \nu, \tau \rangle \quad (20)
\end{align}

\begin{align}
v_{ttt}(0) = 0, \quad \text{ on } \Gamma_f. \quad (21)
\end{align}

We now consider the hypothesis on the stored-energy function.

**H1** The elastic body of the system is in equilibrium with $w = 0$, i.e.

\begin{equation}
DW(\mathbb{I}) = 0.
\end{equation}

By this assumption, we rewrite (5) as following:

\begin{equation}
w_{tt} - \mathcal{N}_w(t)w + w = 0, \quad (23)
\end{equation}

where the operator $\mathcal{N}_w(t)\phi$ is defined for $\phi \in H^2(\Omega_e)$ as

\begin{equation}
\mathcal{N}_w(t)\phi = \text{div} l_{\nabla \phi} N_w(t),
\end{equation}

and $N_w(t) = \int_0^1 D^2W(\mathbb{I} + s\nabla w)ds$. And in above definition, $l_XK$ is defined as a $k-1$ order tensor field in $\mathbb{R}^{3 \times 3}$ by

\begin{equation}
l_XK(X_1, ..., X_{k-1}) = K(X_1, ..., X_{k-1}, X), \quad \text{for } X_i, X \in \mathcal{X}(\mathbb{R}^{3 \times 3}), \quad 1 \leq i \leq k-1.
\end{equation}

We need to impose the strong ellipticity condition at the zero equilibrium for the material function $W$.

**H2** There exists a positive constant $\mu > 0$ such that

\begin{equation}
D^2W(\mathbb{I})(b_1 \otimes b_2, b_1 \otimes b_2) \geq \mu | b_1 |^2 | b_2 |^2, \quad \text{for all } b_1, b_2 \in \mathbb{R}^3. \quad (24)
\end{equation}

The fourth order tensor $D^2W(F) = \left( \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(F) \right)$ is the Hessian of $W$ with respect to the variable $F = (F_{ij}) \in \mathbb{R}^{3 \times 3}$.
Following the approach in [8] and [10], we construct the short-time solutions for the system (4) and (5) with boundary conditions (9)-(11) under the assumptions of Theorem 1.1 in Section 5, which is essential for the global existence of the solutions for the system we consider.

Our main results are given as follows.

**Theorem 1.1.** Let the assumptions (H1) and (H2) hold and $\Omega_e$ be star-shaped with respect to a fixed point $x_0 \in \Omega_e$. Suppose that initial data $v_0 \in V \cap H^4(\Omega_e), v_t(0) \in V \cap H^2(\Omega_e), v_{tt}(0) \in V, v_{ttt}(0) \in H, v_{t0} \in H^4(\Omega_e), v_{t1} \in H^3(\Omega_e)$ are given small such that the corresponding compatibility conditions (13)-(21) hold and $\gamma \geq 2C > 0$, where the constant $C$ depends on the value of initial data. Then, there exists a unique global smooth solution $(v, w, q, a, \eta)$ such that

$$v \in L^\infty([0, \infty); H^4(\Omega_f)); \quad v_t \in L^\infty([0, \infty); H^2(\Omega_f));$$

$$v_{tt} \in L^\infty([0, \infty); L^2(\Omega_f)), \quad \nabla v_{ttt} \in L^2([0, \infty); L^2(\Omega_f));$$

$$\partial^k_v w \in C([0, \infty); H^{4-k}(\Omega_e)), \quad k = 0, 1, 2, 3, 4,$$

with $\partial^j_q w \in L^\infty([0, \infty); H^{4-j}(\Omega_f));$ if $j = 0, 1, 2$; $a, a_t \in L^\infty([0, \infty); H^3(\Omega_f)), a_{tt} \in L^\infty([0, \infty); H^2(\Omega_f)), a_{ttt} \in L^\infty([0, \infty); L^2(\Omega_f)), \nabla a_{ttt} \in L^2([0, \infty); L^2(\Omega_f)), \partial^k_v a \in L^2([0, \infty); L^2(\Omega_f))$ and $\eta|_{\Omega_f} \in \mathcal{C}([0, \infty); H^4(\Omega_f))$.

**Remark 1.** Given initial data small the total energy of the system decays exponentially, where the total energy is defined in (124) later.

The proof of Theorem 1.1 will be given in Section 4. Next, we also list the result of short-time existence of this system and our result in Theorem 1.1 is based on this short-time existence results. We deal with the proof of the following assertion in Section 5.

**Theorem 1.2.** Let the assumptions (H1) and (H2) hold and $\Omega_e$ be star-shaped with respect to a fixed point $x_0 \in \Omega_e$. Suppose that initial data $v_0 \in V \cap H^4(\Omega_f), v_t(0) \in V \cap H^2(\Omega_f), v_{tt}(0) \in V, v_{ttt}(0) \in H, v_{t0} \in H^4(\Omega_e), v_{t1} \in H^3(\Omega_e)$ are given small such that the above listed compatibility conditions hold. Then, there exists a time $T > 0$ depending on initial data such that there is a unique solution $(v, w, q, a, \eta)$ on the interval $[0, T]$ with the following regularity

$$v \in L^\infty([0, T]; H^4(\Omega_f)); \quad v_t \in L^\infty([0, T]; H^2(\Omega_f));$$

$$v_{tt} \in L^\infty([0, T]; L^2(\Omega_f)), \quad \nabla v_{ttt} \in L^2([0, T]; L^2(\Omega_f));$$

$$\partial^k_v w \in C([0, T]; H^{4-k}(\Omega_e)), \quad k = 0, 1, 2, 3, 4,$$

with $\partial^j_q w \in L^\infty([0, T]; H^{4-j}(\Omega_f));$ if $j = 0, 1, 2$; $a, a_t \in L^\infty([0, T]; H^3(\Omega_f)), a_{tt} \in L^\infty([0, T]; H^2(\Omega_f)), a_{ttt} \in L^\infty([0, T]; L^2(\Omega_f)), \nabla a_{ttt} \in L^2([0, T]; L^2(\Omega_f)), \partial^k_v a \in L^2([0, T]; L^2(\Omega_f))$ and $\eta|_{\Omega_f} \in \mathcal{C}([0, T]; H^4(\Omega_f))$.

2. **Preliminaries.** We list some lemmas and estimates in the literature which are needed in the proof of Theorem 1.1. Constant $C$ may be different from line to line throughout this note.

Following Lemma 3.1 in [7], we derive a higher regularity version.
Lemma 2.1. Assume that \( \| \nabla v \|_{L^\infty([0,T];H^2(\Omega_f))} \leq M \). Let \( p \in [1, \infty] \) and \( 1 \leq i, j, k, l \leq 3 \). With \( T \in [0, \frac{1}{CM}] \) where \( C > 0 \) is large enough, the following statements hold:

(i) \( \| \nabla \eta \|_{H^2(\Omega_f)} \leq C \) for \( t \in [0,T] \);
(ii) \( \| a \|_{H^1(\Omega_f)} \leq C \) for \( t \in [0,T] \);
(iii) \( \| a \|_{L^2(\Omega_f)} \leq C \| \nabla v \|_{L^p(\Omega_f)} \) for \( t \in [0,T] \);
(iv) \( \| \partial a_i \|_{L^r(\Omega_f)} \leq C \| \nabla v \|_{L^r(\Omega_f)} \| \partial_t a \|_{L^2(\Omega_f)} + C \| \nabla \partial_t v \|_{L^p(\Omega_f)} \), \( t \in [0,T] \) where \( 1 \leq p, p_1, p_2 \leq \infty \) are given such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \);
(v) \( \| \partial_j a_i \|_{L^2(\Omega_f)} \leq C \| \nabla v \|_{H^1(\Omega_f)} \| \nabla v \|_{H^2(\Omega_f)} + C \| \nabla v \|_{H^2(\Omega_f)} \), \( t \in [0,T] \);
(vi) \( \| a_{il} \|_{L^2(\Omega_f)} \leq C \| \nabla v \|_{L^2(\Omega_f)} \| \nabla v \|_{L^2(\Omega_f)} + C \| \nabla v_t \|_{L^2(\Omega_f)} \) and \( \| a_{il} \|_{L^2(\Omega_f)} \leq C \| v \|_{L^2(\Omega_f)} + C \| \nabla v_t \|_{L^2(\Omega_f)} \) for \( t \in [0,T] \);
(vii) \( \| a_{ij} \|_{L^2(\Omega_f)} \leq C \| \nabla v \|_{L^2(\Omega_f)} + C \| \nabla v_t \|_{L^2(\Omega_f)} \) and \( \| a_{ij} \|_{L^2(\Omega_f)} \leq C \| v \|_{L^2(\Omega_f)} + C \| \nabla v_t \|_{L^2(\Omega_f)} \) for \( t \in [0,T] \);
(viii) for every \( \epsilon \in (0, \frac{1}{2}) \) and all \( t \leq T^* = \min \{ \frac{T}{1+M}, T \} \), we have
\[
\| \delta_{jk} - a^{jl} a^{kl} \|^2_{H^1(\Omega_f)} \leq \epsilon, \tag{25}
\]
and
\[
\| \delta_{jk} - a^{jl} \|^2_{H^2(\Omega_f)} \leq \epsilon, \tag{26}
\]
In particular, the form \( a^{jl} a^{kl} \xi_{ij} \xi_{ik} \) satisfies the ellipticity estimates
\[
a^{jl} a^{kl} \xi_{ij} \xi_{ik} \geq \frac{1}{C} |\xi|^2, \quad \xi \in \mathbb{R}^{3 \times 3}, \tag{27}
\]
for all \( t \in [0, T^*] \) and \( x \in \Omega_f \), provided \( \epsilon \leq \frac{1}{C} \) with \( C \) sufficiently large.

Using a similar method carried out in Lemma 2.1, we establish the following estimates for \( a \). As for the proof, we leave it out.

Corollary 1. Under the assumptions demanded in Lemma 2.1, it follows that for \( t \in [0,T] \) with some \( T > 0 \) and \( 1 \leq i, j, k \leq 3 \),

1. \( \| \partial_t a_{il} \|_{L^2} \leq C \| v \|_{H^2}^2 + C \| v_t \|_{H^2}^2 \| v \|_{H^2} + C \| v_t \|_{H^2} \);  
2. \( \| \partial_t^2 a \|_{L^2} \leq C \| v \|_{H^2}^2 \| v_t \|_{H^2} + C \| v_t \|_{H^2} \| v \|_{H^2} + C \| v_t \|_{H^2} + C \| \nabla v_{tt} \|_{L^2} \);  
3. \( \| \partial_{ij} a_{il} \|_{L^2} \leq C \| \nabla v \|_{H^2}^2 + C \| \nabla v_t \|_{H^2} \| v_t \|_{H^2} + C \| v_t \|_{H^2} \| v_t \|_{H^2} \);  
4. \( \| \partial_{ij} a_{il} \|_{L^2} \leq C \| v \|_{H^2}^2 \| v_t \|_{H^2} + C \| v_t \|_{H^2} \| v_t \|_{H^2} + C \| v_t \|_{H^2} \);  
5. \( \| \partial_{ij} a_{il} \|_{L^2} \leq C \| v \|_{H^2}^2 + C \| v_t \|_{H^2} + C \| v_t \|_{H^2} + C \| v_t \|_{H^2} \);  

Similar to Lemma 3.2 in [7], we obtain a pointwise a-priori estimates with higher regularity for variable coefficient Stokes system.

Lemma 2.2. Assume that \( v \) and \( q \) are solutions of the system
\[
\begin{align*}
\partial_t v - \partial_j (a^{jl} a^{ki} \partial_k v^i) + \partial_k (a^{ki} q) &= 0 & \text{in} & & \Omega_f \times (0,T), \\
a^{ki} \partial_k v^i &= 0 & \text{in} & & \Omega_f \times (0,T), & i = 1, 2, 3 \\
v^i &= 0 & \text{on} & & \Gamma_f, \\
a^{jl} a^{kl} \partial_k v^i v_j - a^{ki} q v_k &= (\omega^i) v^i, & \text{on} & & \Gamma_c,
\end{align*}
\]
where \( a^{ij} \in L^\infty(\Omega_f) \) satisfies Lemma 2.1 with \( \epsilon = \frac{1}{C} \) sufficiently small. Then we have
\[
\| v \|_{H^{s+2}(\Omega_f)} + \| q \|_{H^{s+1}(\Omega_f)} \leq C \| v_t \|_{H^s(\Omega_f)} + C \| w_{tv} \|_{H^{s+\frac{1}{2}}(\Gamma_c)}, \quad s = 0, 1, 2
\]
for \( t \in (0, T) \). Moreover, we obtain for \( t \in (0, T) \), (1)
\[
\| v_t \|_{H^3(\Omega_f)} + \| q_t \|_{H^2(\Omega_f)} \leq C \| v_t \|_{L^2(\Omega_f)} + C \| (w_t)_{tv} \|_{H^\frac{1}{2}(\Gamma_c)} + C L(t)
\]
\[
+ C(\| v \|_{H^3(\Omega_f)} + \| q \|_{H^2(\Omega_f)}) (\| v \|_{H^2(\Omega_f)} + \| q \|_{H^1(\Omega_f)})
\]
where \( T \leq \frac{1}{C M} \) and \( C \) is sufficiently large; and (2)
\[
\| v_{tt} \|_{H^2(\Omega_f)} + \| q_{tt} \|_{H^1(\Omega_f)} \leq C \| v_{tt} \|_{L^2(\Omega_f)} + C \| (w_{tt})_{tv} \|_{H^\frac{1}{2}(\Gamma_c)} + C L(t)
\]
\[
+ C(\| v \|_{H^3(\Omega_f)} + \| q \|_{H^2(\Omega_f)}) (\| v \|_{H^2(\Omega_f)} + \| q \|_{H^1(\Omega_f)})
\]
\[
+ \| v \|_{H^2(\Omega_f)} (\| v \|_{H^3(\Omega_f)} + \| v \|_{H^2(\Omega_f)})
\]
From now on, for simplicity, we omit specifying the domains \( \Omega_f \) and \( \Omega_e \) in the norms involving the velocity \( v \) and the displacement \( w \). But we still emphasize the boundary domains \( \Gamma_c \) and \( \Gamma_f \).

Now we turn to the nonlinear elastodynamic system. Let \( w \) be a solution of (5) on \([0, T]\) for some \( T > 0 \). Set
\[
C(t) = DW(\nabla w + I).
\]
We differentiate \( C(t) \) in time and have
\[
\dot{C}(t) = \sum_{i,j=1}^{3} \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}} (Dw + \mathbb{I}) \frac{\partial w^i}{\partial x^j} _{kl} = l_{Dw} D^2 W.
\]
Then, the derivatives of order \( j \) of \( C(t) \) for \( j = 2, 3, 4 \) with respect to \( t \) are listed as follows:
\[
\dot{C}(t) = l_{\nabla w_t} D^2 W + l_{\nabla w_t} (l_{\nabla w_t} D^3 W),
\]
\[
C^{(3)}(t) = l_{\nabla w_{tt}} D^2 W + 3l_{\nabla w_t} (l_{\nabla w_t} D^3 W) + l_{\nabla w_t} l_{\nabla w_t} l_{\nabla w_t} D^4 W,
\]
\[
C^{(4)}(t) = l_{\nabla w_{ttt}} D^2 W + 4l_{\nabla w_t} (l_{\nabla w_t} D^3 W) + 6l_{\nabla w_t} l_{\nabla w_t} l_{\nabla w_t} l_{\nabla w_t} D^5 W + l_{\nabla w_t} l_{\nabla w_{tt}} l_{\nabla w_t} D^4 W + l_{\nabla w_t} l_{\nabla w_t} l_{\nabla w_t} l_{\nabla w_t} l_{\nabla w_t} D^5 W.
\]
Define
\[
B_w(t) \phi = \text{div} l_{\nabla w_t} [D^2 W(\nabla w + \mathbb{I})], \quad \text{for } \phi \in H^2(\Omega_e).
\]
Let
\[
\phi_{\nu R} = (l_{\nabla w} D^2 W) \nu, \quad \text{for } \phi \in H^2(\Omega_e)
\]
be a vector field on the interface \( \Gamma_c \), given by the formulas
\[
\langle \phi_{\nu R}, X \rangle = D^2 W(\nabla \phi, X \otimes \nu), \quad \text{for } \phi \in H^2(\Omega_e), \quad X \in \mathcal{X}(\Gamma_c).
\]
With the help of the above notations we introduce, we differentiate (5) or (23) in time for \( j \geq 1 \) times and obtain
\[
w^{(j)}_{tt} - B_w(t) w^{(j)} + w^{(j)} - r_{j-1}(t) = 0,
\]
where the remainder term caused by the nonlinearity of \( W \) is
\[
r_{j-1}(t) = \text{div} (C^{(j)}(t)) - B_w(t) w^{(j)} \quad \text{as } \quad j \geq 2; \quad r_0(t) = 0.
\]
Therefore, with above preparations, we introduce the following various types of energy: the energy for (5) or (23)

\[ V_0^e(t) = \frac{1}{2} \|w\|_{L_2}^2 + \|v\|_{L_2}^2 + \int_{\Omega_e} (|\nabla w_1 N_1(w(t), \nabla w)_{R_{3 \times 3}} dx), \]

the first level energy of the whole system

\[ V_0(t) = V_0^e(t) + \frac{1}{2} \|v\|_{L_2}^2, \]

for \(j \geq 1\), the energy for (36)

\[ V_j^e(t) = \frac{1}{2} \|w_j\|_{L_2}^2 + \|v_j\|_{L_2}^2 + \int_{\Omega_e} (|\nabla w_j D^2 W, \nabla w_j)_{R_{3 \times 3}} dx, \]

the j-th level energy of the fluid-structure interaction model

\[ V_j(t) = V_j^e(t) + \frac{1}{2} \|v_j\|_{L_2}^2, \]

and the total energy of the whole system and the elastic body

\[ \mathcal{Q}(t) = \sum_{j=0}^{3} V_j(t), \quad \text{and} \quad \mathcal{E}^e(t) = \sum_{k=0}^{4} \|w^{(k)}(t)\|_{H^{4-k}(\Omega_e)}^2. \]

Moreover, we denote a remainder term which will appear by \( L(t) = \sum_{k=3}^{4} (\mathcal{E}^e(t))^\frac{k}{2}. \)

Now, with necessary notation introduced above, we give some lemmas which will be frequently utilized.

Suppose that the assumption (H2) holds. Hence, there exists a constant \( \gamma_0 > 0 \) such that if \( w \in H^1(\Omega_e) \) with

\[ \sup_{x \in \Omega_e} |\nabla w| \leq \gamma_0, \quad (38) \]

then

\[ D^2W(\nabla w + \mathbb{I})(b_1 \otimes b_2) \geq |b_1|^2 |b_2|^2, \quad (39) \]

for all \( b_1, b_2 \in \mathbb{R}^3 \). Thus, it yields the following lemma, by Gårding’s inequality.

**Lemma 2.3.** [18, Lemma 2.1] Let (H2) hold. Then there is \( \gamma_0 > 0 \) such that, if \( w \in H^1(\Omega_e) \) is such that the condition (38) holds, then

\[ \int_{\Omega_e} D^2W(\nabla w + \mathbb{I})(\nabla \phi, \nabla \phi) dx \geq c_{\gamma_0} \|\nabla \phi\|_{L_2}^2, \quad \phi \in H^1(\Omega_e), \quad (40) \]

for some \( c_{\gamma_0} > 0 \).

By slightly modifying the proofs of Lemma 2.1-2.3 and Theorem 2.1 in [16, 17], we obtain the following lemmas. Before the statement of these lemmas, we collect a few basic properties of Sobolev space which we’ll use often first.

(i) Let \( s_1 > s_2 \geq 0 \) and \( \mathcal{D} \) be a bounded, open set with smooth boundary in \( \mathbb{R}^n (n \geq 2) \). For any \( \epsilon > 0 \) there is \( c_{\epsilon} > 0 \) such that

\[ \|\phi\|_{H^{s_2}(\mathcal{D})} \leq \epsilon \|\phi\|_{H^{s_1}(\mathcal{D})} + c_{\epsilon} \|\phi\|_{H^{s_2}(\mathcal{D})}, \quad \forall \phi \in H^{s_1}(\mathcal{D}). \quad (41) \]

(ii) If \( s > \frac{n}{2} \), then for each \( k = 0, \ldots \), we have \( H^{s+k}(\mathcal{D}) \subset C^k(\mathcal{D}) \) with continuous inclusion.

(iii) If \( r \triangleq \min\{s_1, s_2, s_1 + s_2 - \lceil \frac{n}{2} \rceil - 1\} \geq 0 \), then there is a constant \( c > 0 \) such that

\[ \|f_1 f_2\|_{H^r(\mathcal{D})} \leq c \|f_1\|_{H^{s_1}(\mathcal{D})} \|f_2\|_{H^{s_2}(\mathcal{D})} \quad \forall f_1 \in H^{s_1}(\mathcal{D}), f_2 \in H^{s_2}(\mathcal{D}). \quad (42) \]
(iv) Let $s_j \geq 0, j = 1, \ldots, k$ and $r = \min_{1 \leq i \leq k} \min_{j_i \leq \cdots \leq j_k} \{s_j, + \cdots + s_j, -(i-1) \times \left(\frac{3}{2} + 1\right)\} \geq 0$. Then there is a constant $c > 0$ such that
\[
\|f_1 \cdots f_k\|_{H^r(\mathcal{D})} \leq c \|f_1\|_{H^{s_1}(\mathcal{D})} \cdots \|f_k\|_{H^{s_k}(\mathcal{D})} \quad \forall f_j \in H^{s_j}(\mathcal{D}), 1 \leq j \leq k.
\] (43)

**Lemma 2.4.** Let $\gamma_0 > 0$ be given and $\phi \in H^1(\mathcal{D})$ be such that the condition (38) hold. Let $f(\cdot, \cdot)$ be a smooth function on $\overline{\Omega} \times \mathbb{R}^n$ and $F(x) = f(x, \nabla \phi)$. Then there is $c_{\gamma_0} > 0$, depending on $\gamma_0$, such that
\[
\|F\|_{H^{k}(\mathcal{D})} \leq c_{\gamma_0} \sum_{j=1}^{k} (1 + \|\phi\|_{H^{m}(\mathcal{D})})^j,
\] (44)
for $0 \leq k \leq m - 1$ and $m \geq \left[\frac{3}{2}\right] + 3$. Moreover, if $f(x, 0) = 0$ for $x \in \mathcal{D}$ and $\|\phi\|_{H^{m}(\mathcal{D})} \leq \gamma_0$, then there is $c_{\gamma_0} > 0$ such that
\[
\|F\|_{H^{k}(\mathcal{D})} \leq c_{\gamma_0} \|\phi\|_{H^{k+1}(\mathcal{D})}, \quad 0 \leq k \leq m - 1.
\] (45)

**Lemma 2.5.** Let $\gamma_0 > 0$ be given and $w$ satisfy the problem (5) on the interval $[0, T]$ for some $T > 0$ such that
\[
\sup_{0 \leq t \leq T} \|w(t)\|_{H^{m}(\Omega_\gamma)} \leq \gamma_0.
\] (46)

For $1 \leq j \leq 2$, let
\[
\gamma_j(t) = \text{div} C^{(j+1)}(t) - B_w(t)w^{(j+1)}, \quad r_{j, \Gamma_s} = |C^{(j+1)}(t) - B_w(t)w^{(j+1)}|^2, \quad \nu,
\] where the operator $B_w(t)$ is given by (34). Then,
\[
\|\gamma_j(t)\|_{H^{2-j}(\Omega_\gamma)}^2 \|r_{j, \Gamma_s}(t)\|_{H^{3-j}(\Gamma_s)}^2 \leq c_{\gamma_0} \sum_{k=2}^{3} \mathcal{E}_k(t), \quad j = 1, 2.
\] (47)

**Lemma 2.6.** Let $\gamma_0 > 0$ be given and $w$ satisfy the problem (5) on the interval $[0, T]$ for some $T > 0$ such that (46) is true. Then there is $c_{\gamma_0} > 0$, which depends on $\gamma_0$, such that
\[
\|\phi\|_{H^{k+1}(\Omega_\gamma)}^2 \leq c_{\gamma_0} (\|B_w(t)\|_{H^{k-1}(\Omega_\gamma)}^2 + \|\nu\|_{H^{k-1}(\Omega_\gamma)}^2 + \|\phi\|_{H^k(\Omega_\gamma)}^2), \quad 1 \leq k \leq 3,
\] (48)
for $\phi \in H^{k+1}(\Omega_\gamma)$ and $t \in [0, T]$.

Combining all the lemmas from 2.4 to 2.6, we arrive at the following theorem for $Q^c(t) = \sum_{j=0}^{3} V_j^c(t)$ and $\mathcal{E}^c(t)$.

**Theorem 2.7.** Let $\gamma_0 > 0$ be given and $w$ satisfy the problem (5) on the interval $[0, T]$ for some $T > 0$ such that (46) holds. Then there are constants $c_{0, \gamma_0} > 0$ and $c_{\gamma_0} > 0$, which depend on $\gamma_0$, such that
\[
c_{0, \gamma_0} Q^c(t) \leq \mathcal{E}^c(t) \leq c_{\gamma_0} Q^c(t) + c_{\gamma_0} L(t) + c_{\gamma_0} \sum_{j=0}^{2} \|w^{(j)}\|_{H^{3-j}(\Gamma_s)}^2,
\] (49)
\[
-c_{\gamma_0} L(t) - c_{\gamma_0} \mathcal{L}^c(t) + \sum_{j=0}^{3} \int_{\Gamma_s} \{(l_{\nabla w^{(j)}} D^2 W) \cdot \nu, w^{(j+1)}\} d\sigma \leq \mathcal{Q}^c(t),
\] (50)
\[
\mathcal{Q}^c(t) \leq c_{\gamma_0} L(t) + c_{\gamma_0} \mathcal{L}^c(t) + \sum_{j=0}^{3} \int_{\Gamma_s} \{(l_{\nabla w^{(j)}} D^2 W) \cdot \nu, w^{(j+1)}\} d\sigma,
\] (51)
and
\[ V_0^c(t) \leq c_\gamma \left[ \int_0^t L(\tau) + \|w_{\nu\nu}\|_{L^2(\Gamma_c)}^2d\tau + V_0^c(0) \right]e^t. \]  
(52)

Thus, all the preparations for the proof of Theorem 1.1 have been made.

3. A-priori estimates. We derive a priori estimates for the global existence of solutions to the dissipative fluid-nonlinear structure system when the initial data are sufficiently small.

Assume that
\[ \|v_0\|_{H^4}, \|v_t(0)\|_{H^2}^2, \|v_{tt}(0)\|_{H^1}^2, \|v_{ttt}(0)\|_{H^2}^2, \|w_0\|_{H^4}, \|w_1\|_{H^3}^2 \leq \epsilon, \]
where \( \epsilon > 0 \) is given small enough, i.e. \( \epsilon < \gamma_0 \).

We demand several auxiliary estimates involving different levels of energy.

3.1. First level estimates. As defined above, \( V_0(t) \) is the first level energy. We deal with this energy in this subsection.

**Lemma 3.1.** Suppose that all the assumptions of Theorem 1.1 are true, the following inequality holds for \( t \in [0, T] \)
\[ V_0(t) + \int_0^t D_0(\tau)d\tau \leq V_0(0) + C\gamma_0 \int_0^t L(\tau)d\tau, \]
(53)
where
\[ D_0(t) = \frac{1}{C} \|\nabla v(t)\|_{L^2}^2 + \gamma \|w_{\nu\nu}(t)\|_{L^2(\Gamma_c)}^2 \]
(54)
is a dissipative term.

**Proof.** Take the \( L^2 \)-inner product of (4) with \( v \) and (5) with \( w_t \), respectively.

Hence, by (27), we have
\[ \frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \frac{1}{C} \|Dv\|_{L^2}^2 \leq -\int_{\Gamma_c} \langle (a : a^T : Dv)\nu - q(aw), v \rangle d\sigma \]
(55)
and
\[ \frac{1}{2} \frac{d}{dt} \|w_t\|_{L^2}^2 + \frac{1}{2} \|w\|_{L^2}^2 \leq \int_{\Omega_e} \langle \nabla w, N_w(t), \nabla w \rangle dx \]
\[ -\frac{1}{2} \int_{\Omega_e} N_w'(t)(\nabla w_t, \nabla w, \nabla w) dx = \int_{\Gamma_e} \langle w_{\nu\nu}, w_t \rangle d\sigma, \]
(56)
where \( N_w'(t)(\nabla w_t, \nabla w, \nabla w) = \int_0^1 s \frac{\partial^3 W}{\partial F_{ij} \partial F_{kl} \partial F_{\mu\nu}}(\nabla w + I)ds \frac{\partial w_i}{\partial x^j} \frac{\partial w_k}{\partial x^l} \frac{\partial w_{\mu\nu}}{\partial x^\sigma} \) as \( i, j, k, l, \alpha, \beta = 1, 2, 3. \)

Note that
\[ \int_0^t \int_{\Omega_e} N_w'(t)(\nabla w_t, \nabla w, \nabla w) dx d\tau \leq C\gamma_0 \int_0^t L(\tau)d\tau. \]

Add (55) and (56) together and integrate in time from 0 to \( t \). With the help of boundary condition (9) and (10), we arrive at the resulted inequality.

In order to proceed further, we have to derive the multiplier identities which will play a central role in our calculations.

Denote \( \hat{w} = w^{(j)} \) for \( 0 \leq j \leq 3 \). From (23) and (36), \( \hat{w} \) satisfies the following equations:
\[ \hat{w}_{tt} - A_w(t)\hat{w} + \hat{w} - r(t) = 0, \]
(57)
where \( r(t) = r_{j-1}(t) \) as \( j = 2, 3; \) \( r(t) = 0 \) when \( j = 0, 1 \) and the operator \( A_w(t) = B_w(t) \) if \( j \geq 1 \), or else \( A_w(t) = N_w(t) \) if \( j = 0 \).

**Lemma 3.2.** Let \( \hat{w} \) be a solution of (57) and \( H \) be a vector field on \( \overline{\Omega}_e \) with \( H(\hat{w}) = \nabla H \hat{w} \). And let the scalar function \( \xi(x) \in C^1(\overline{\Omega}_e) \). Then we have for \( \varrho > 0 \),

\[
\int_s^t \int_{\Gamma_e} (|\hat{w}_t|^2 - A_w(t)(\nabla \hat{w}, \nabla \hat{w}) - |\hat{w}|^2)(H, \nu)d\sigma d\tau \\
+ \int_s^t \int_{\Gamma_e} \langle \hat{w}_{\nu A}, 2H(\hat{w}) + \varrho \hat{w} \rangle d\sigma d\tau = \langle \hat{w}_t, 2H(\hat{w}) + \varrho \hat{w} \rangle_{L^2} |^t_s \\
+ \int_s^t \int_{\Omega_e} (|\hat{w}_t|^2 - A_w(t)(\nabla \hat{w}, \nabla \hat{w}) - |\hat{w}|^2)(\text{div} H - \varrho)dxd\tau \\
+ 2\int_s^t \int_{\Omega_e} A_w(t)(\nabla \hat{w}, \nabla \hat{w} : \nabla H)dxd\tau \\
- \int_s^t \int_{\Omega_e} DA_w(t)(\nabla \hat{w}, \nabla H \nabla w)dxd\tau \\
- \int_s^t \int_{\Omega_e} (r(t), 2H(\hat{w}) + \varrho \hat{w})dxd\tau
\]

(58)

and

\[
(\hat{w}_t, \xi \hat{w})_{L^2} |^t_s - \int_s^t \int_{\Omega_e} \langle r(t), \xi \hat{w} \rangle dxd\tau + \int_s^t \int_{\Omega_e} \xi(\hat{w}_t^2 - |\hat{w}_t|^2)dxd\tau \\
+ \int_s^t \int_{\Omega_e} \xi A_w(t)(\nabla \hat{w}, \nabla \hat{w})dxd\tau + \int_s^t \int_{\Omega_e} A_w(t)(\nabla \hat{w}, \nabla \xi \hat{w})dxd\tau \\
= \int_s^t \int_{\Gamma_e} \xi(\hat{w}_{\nu A}, \hat{w})d\sigma d\tau,
\]

(59)

where \( A_w(t) = D^2W(\nabla w + \mathbb{1}), DA_w(t) = D^3W \) if \( j \geq 1 \) or else \( A_w(t) = N_w(t) \), \( DA_w(t) = N'_w(t) \) if \( j = 0 \).

**Proof.** For (58), we take \( 2H(\hat{w}) + \varrho \hat{w} \) as the multiplier. Multiply (57) by \( 2H(\hat{w}) + \varrho \hat{w} \) and integrate by parts over \( (s, t) \times \Omega_e \). Thus, we have

\[
\int_s^t \int_{\Omega_e} \langle \hat{w}_{tt}, 2H(\hat{w}) + \varrho \hat{w} \rangle dxd\tau = \langle \hat{w}_t, 2H(\hat{w}) + \varrho \hat{w} \rangle_{L^2} |^t_s \\
+ \int_s^t \int_{\Omega_e} |\hat{w}_t|^2 (\text{div} H - \varrho)dxd\tau - \int_s^t \int_{\Gamma_e} |\hat{w}_t|^2 (H, \nu)d\sigma d\tau; \quad (60)
\]

\[
- \int_s^t \int_{\Omega_e} \langle A_w(t)\hat{w}, 2H(\hat{w}) + \varrho \hat{w} \rangle dxd\tau \\
= \int_s^t \int_{\Omega_e} \langle l \nabla A_w(t), 2\nabla H(\hat{w}) + \varrho \nabla \hat{w} \rangle dxd\tau \\
- \int_s^t \int_{\Gamma_e} \langle \hat{w}_{\nu A}, 2H(\hat{w}) + \varrho \hat{w} \rangle d\sigma d\tau = 2 \int_s^t \int_{\Omega_e} A_w(t)(\nabla \hat{w}, \nabla \hat{w} : \nabla H)dxd\tau \\
+ \int_s^t \int_{\Omega_e} (\varrho - \text{div} H)A_w(t)(\nabla \hat{w}, \nabla \hat{w})dxd\tau
\]

(61)
Lemma. Moreover, we intend to deal with the similar estimates in subsection 3.1-3.4.

\[- \int_s^t \int_{\Omega_c} DA_w(t)(\nabla \hat{w}, \nabla \hat{w}, \nabla_{H} \nabla w) \, dx \, dt \]

\[+ \int_s^t \int_{\Gamma_c} A_w(t)(\nabla \hat{w}, \nabla \hat{w})(H, \nu) \, d\sigma \, dt - \int_s^t \int_{\Gamma_c} \langle \hat{w}_{\nu A}, 2H(\hat{w}) + \varrho \hat{w} \rangle d\sigma dt, \]

with the help of

\[\nabla [H(\hat{w})] = \nabla \hat{w} : \nabla_{H} \nabla \hat{w}\]

and

\[\text{div} \ [A_w(t)(\nabla \hat{w}, \nabla \hat{w})H] = A_w(t)(\nabla \hat{w}, \nabla \hat{w}) \text{div} H + 2A_w(t)(\nabla \hat{w}, \nabla_{H} \nabla \hat{w})
\]

\[+ DA_w(t)(\nabla \hat{w}, \nabla \hat{w}, \nabla_{H} \nabla w); \]

and

\[\int_s^t \int_{\Omega_c} \langle \hat{w}, 2H(\hat{w}) + \varrho \hat{w} \rangle \, dx \, dt = \int_s^t \int_{\Omega_c} \varrho |\hat{w}|^2 \, dx \, dt - \int_s^t \int_{\Gamma_c} \text{div} H \, |\hat{w}|^2 \, d\sigma \, dt
\]

\[+ \int_s^t \int_{\Gamma_c} |\hat{w}|^2 \langle H, \nu \rangle d\sigma dt \quad (62)\]

Combining (60)-(62), we acquire (58).

As for (59), we regard \(\xi(x)\hat{w}\) as the multiplier. Carry out similar calculations and we’ll arrive at the multiplier identity (59).

Taking advantage of the above multiplier identities, we establish the following Lemma. Moreover, we intend to deal with the similar estimates in subsection 3.1-3.4 in a unified way.

The related boundary conditions of (57) are the following:

\[\hat{w}_t = \hat{v} - \gamma (\hat{w}_{\nu A} + \hat{r}(t)), \quad (63)\]

where \(\hat{v} = v^{(j)}\) for the related \(0 \leq j \leq 3\), and

\[\hat{w}_{\nu A} = \partial_t^{(j)}(a : a^T : \nabla \nu) + \partial_t^{(j)}(qa) \nu - \hat{r}(t), \quad (64)\]

where \(0 \leq j \leq 3\) and \(\hat{r}(t) = 0\) if \(j = 0, 1\) or else \(\hat{r}(t) = r_{j-1, \Gamma_c} = [C^{(j)}(t) - B_w(t)w(t)]\nu\) if \(j = 2, 3\). Besides, in order to simplify the notations, we denote \(V(t) = V_j(t)\) as \(0 \leq j \leq 3\) and \(V^c(t) = V_j^c(t)\).

**Lemma 3.3.** Under the assumptions of Theorem 1.1, it holds for \(0 \leq s < t \leq T\) with some \(T > 0\) that there exists a positive constant \(C\) such that

\[\int_s^t V(\tau) \, d\tau \leq CV(s) + CV(t) + C \int_s^t L(\tau) \, d\tau + C \int_s^t \int_{\Gamma_c} |\hat{w}_{\nu A}|^2 \, d\sigma \, d\tau
\]

\[+ C \int_s^t \|D\hat{w}\|^2_{L^2} \, d\tau. \quad (65)\]

**Proof.** Submit \(H = x - x_0\) and \(\varrho = 3 - \varepsilon\), where \(0 < \varepsilon < 1\) is to be determined, into (58). Note that \(\text{div} H = 3\) and \(DH = I\). Hence, we have

\[\int_s^t \int_{\Gamma_c} |\hat{w}_t|^2 - A_w(t)(\nabla \hat{w}, \nabla \hat{w}) - |\hat{w}|^2 \langle H, \nu \rangle d\sigma dt\]
\[ + \int_{\tau}^{t} \langle \hat{w}_{\nu_{\sigma}}, 2H(\hat{w}) + \varrho \hat{w} \rangle d\sigma d\tau \]
\[ = (\hat{w}_{t}, 2H(\hat{w}) + \varrho \hat{w})_{L^2} \bigg|_{s}^{t} + \varepsilon \int_{\tau}^{t} \int_{\Omega_{\epsilon}} (| \hat{w}_{t} |^2 - A_{w}(t)(\nabla \hat{w}, \nabla \hat{w}) - | \hat{w} |^2) dx d\tau \]
\[ + 2 \int_{s}^{t} \int_{\Omega_{\epsilon}} A_{w}(t)(\nabla \hat{w}, \nabla \hat{w}) dx d\tau - \int_{s}^{t} \int_{\Omega_{\epsilon}} DA_{w}(t)(\nabla \hat{w}, \nabla \hat{w}, \nabla H \nabla w) dx d\tau \]
\[ - \int_{s}^{t} \int_{\Omega_{\epsilon}} (r(t), 2H(\hat{w}) + \varrho \hat{w}) dx d\tau. \tag{66} \]

Now we estimate the terms in (66).

By Cauchy-Schwartz inequality and (39) or Lemma 2.3, we have
\[ | (\hat{w}_{t}, 2H(\hat{w}) + \varrho \hat{w})_{L^2} | \leq CV(s) + CV(t). \tag{67} \]

With the help of Lemma 2.5 and Young inequality, we derive that
\[ | \int_{s}^{t} \int_{\Omega_{\epsilon}} (r(t), 2H(\hat{w}) + \varrho \hat{w}) dx d\tau | \leq \varepsilon_{1} \int_{s}^{t} V^{\epsilon}(\tau) d\tau + C_{\epsilon_{1}} \int_{s}^{t} L(\tau) d\tau, \tag{68} \]
where \( \varepsilon_{1} > 0 \) is sufficiently small and to be determined later on.

Applying the property of Stored-function \( W \) and the Sobolev inequality \( ||\phi||_{L^{\infty}} \leq C||\phi||_{H^{2}} \), it follows that
\[ | \int_{s}^{t} \int_{\Omega_{\epsilon}} DA_{w}(t)(\nabla \hat{w}, \nabla \hat{w}, \nabla H \nabla w) dx d\tau | \leq C \int_{s}^{t} L(\tau) d\tau. \tag{69} \]

Using the Cauchy-Schwartz inequality, Young inequality and Lemma 2.3 again, we have
\[ | \int_{s}^{t} \int_{\Gamma_{\epsilon}} \langle \hat{w}_{\nu_{\sigma}}, 2H(\hat{w}) + \varrho \hat{w} \rangle d\sigma d\tau | \leq \varepsilon_{2} \int_{s}^{t} \int_{\Gamma_{\epsilon}} [A_{w}(t)(\nabla \hat{w}, \nabla \hat{w}) + | \hat{w} |^2] d\sigma d\tau \]
\[ + C_{\epsilon_{2}} \int_{s}^{t} \int_{\Gamma_{\epsilon}} | \hat{w}_{\nu_{\sigma}} |^2 d\sigma d\tau, \tag{70} \]
in which \( \varepsilon_{2} > 0 \) is also to be determined.

By the star-shaped condition for the domain \( \Omega_{\epsilon} \), it implies that there exists a constant \( \rho_{0} > 0 \) such that
\[ \langle x - x_{0}, \nu \rangle \geq \rho_{0}. \tag{71} \]

Therefore, we take \( \varepsilon_{2} \leq \frac{\rho_{0}}{2} \).

Insert (67)-(71) into (66) and thus we deduce that
\[ \varepsilon \int_{s}^{t} \int_{\Omega_{\epsilon}} | \hat{w}_{t} |^2 dx d\tau + (2 - \varepsilon) \int_{s}^{t} \int_{\Omega_{\epsilon}} A_{w}(t)(\nabla \hat{w}, \nabla \hat{w}) dx d\tau \]
\[ + \frac{\rho_{0}}{2} \int_{s}^{t} \int_{\Gamma_{\epsilon}} [A_{w}(t)(\nabla \hat{w}, \nabla \hat{w}) + | \hat{w} |^2] d\sigma d\tau \leq \varepsilon \int_{s}^{t} \int_{\Omega_{\epsilon}} | \hat{w} |^2 dx d\tau \]
\[ + \int_{s}^{t} \int_{\Gamma_{\epsilon}} | \hat{w}_{t} |^2 (H, \nu) d\sigma d\tau + C_{\epsilon_{2}} \int_{s}^{t} \int_{\Gamma_{\epsilon}} | \hat{w}_{\nu_{\sigma}} |^2 d\sigma d\tau \]
\[ + CV(s) + CV(t) + \epsilon_{1} \int_{s}^{t} V^{\epsilon}(\tau) d\tau + C_{\epsilon_{1}} \int_{s}^{t} L(\tau) d\tau. \tag{72} \]

Note that the Poincaré inequality
\[ ||\phi||_{L^{2}}^2 \leq C ||\nabla \phi||_{L^{2}}^2 + C \int_{\Gamma_{\epsilon}} | \phi |^2 d\sigma \]
and set $\varepsilon$ small enough so that $2 - \varepsilon(1 + 2CC_{\gamma_0}) \geq 1$ and $\frac{\rho_0}{2} - 2\varepsilon C \geq \frac{\rho_0}{2}$ hold. Then it leads to
\[
\varepsilon \int_s^t \int_{\Omega_e} |\dot{w}_t|^2 \, dx \, d\tau + \int_s^t \int_{\Omega_e} A_w(t)(\nabla \dot{w}, \nabla \tilde{w}) \, dx \, d\tau + \varepsilon \int_s^t \int_{\Omega_e} |\tilde{w}|^2 \, dx \, d\tau
\]
\[+ \frac{\rho_0}{4} \int_s^t \int_{\Gamma_e} [A_w(t)(\nabla \tilde{w}, \nabla \tilde{w}) + |\tilde{w}|^2] \, d\sigma \, d\tau \leq \int_s^t \int_{\Gamma_e} |\dot{w}_t|^2 \, d\sigma \, d\tau + CV(s) + CV(t) + \varepsilon_1 \int_s^t V^c(\tau) \, d\tau
\]
\[+ C\varepsilon_2 \int_s^t \int_{\Gamma_e} |\tilde{w}_{\nu,t}|^2 \, d\sigma \, d\tau + CV(s) + CV(t) + \frac{C\varepsilon_1}{\varepsilon} \int_s^t L(\tau) \, d\tau. \tag{73}
\]
Let the constant $\varepsilon_1 < \frac{\varepsilon}{2}$. Thus we have
\[
\int_s^t \int_{\Omega_e} |\dot{w}_t|^2 \, dx \, d\tau + \int_s^t \int_{\Omega_e} A_w(t)(\nabla \dot{w}, \nabla \tilde{w}) \, dx \, d\tau + \int_s^t \int_{\Omega_e} |\tilde{w}|^2 \, dx \, d\tau
\]
\[+ \frac{\rho_0}{4\varepsilon} \int_s^t \int_{\Gamma_e} [A_w(t)(\nabla \tilde{w}, \nabla \tilde{w}) + |\tilde{w}|^2] \, d\sigma \, d\tau \leq \int_s^t \int_{\Gamma_e} |\dot{w}_t|^2 \, d\sigma \, d\tau + CV(s) + CV(t) + \frac{C\varepsilon_1}{\varepsilon} \int_s^t L(\tau) \, d\tau. \tag{74}
\]
Add $\frac{1}{2} \int_s^t \int_{\Omega_e} |\tilde{v}|^2 \, dx \, d\tau$ to both sides of (74) and submit the boundary condition (63) into (74). Therefore, thanks to the Poincaré inequality $\|\phi\|_{L^2} \leq C\|\nabla \phi\|_{L^2}$ for $\phi \in H^1_{\gamma_j}(\Omega_j)$ and (47) in Lemma 2.5, (65) can be obtained directly from (74).

According to Lemma 3.3, for the first order energy, it implies that
\[
\int_s^t V_0(\tau) \, d\tau \leq CV_0(s) + CV(t) + C \int_s^t L(\tau) \, d\tau + C \int_s^t \int_{\Gamma_e} |w_{\nu,t}|^2 \, d\sigma \, d\tau
\]
\[+ C \int_s^t \|\nabla v\|^2_{L^2} \, d\tau. \tag{75}
\]
Next, we multiply (75) by sufficiently small $\varepsilon_3 > 0$ and add the resulted inequality to (53). Thus, we get the following lemma.

**Lemma 3.4.** Assume that all of the hypotheses in Theorem 1.1 hold. Then it’s true for $t \in [0,T]$ with some $T > 0$ that
\[
V_0(t) + \int_0^t V_0(\tau) \, d\tau \leq CV_0(0) + C \int_0^t L(\tau) \, d\tau. \tag{76}
\]

**Remark 2.** If the solution of this fluid-nonlinear structure interaction system can exist for all time and all the assumptions in Theorem 1.1 hold, then we may infer from the above lemma that the first level energy $V_0(t)$ decays exponentially.

### 3.2. Second level estimates

As we have defined in Section 2, the second order energy
\[
V_1(t) = V_1^*(t) + \frac{1}{2}\|v_t\|^2_{L^2}, \quad \text{with}
\]
\[
V_1^*(t) = \frac{1}{2}\|w_{tt}\|^2_{L^2} + \|w_t\|^2_{L^2} + \int_{\Omega_e} \langle l\nabla w, D^2W, \nabla w_t \rangle \, dx,
\]
and the corresponding dissipation term

\[ D_1(t) = \frac{1}{C} \| \nabla v_t(t) \|^2_{L^2} + \gamma \| (w_t)_{\nu_B} (t) \|^2_{L^2(\Gamma_c)}. \]

To start with deriving the similar estimates with (53) and (76) in above subsection, we differentiate the whole system in time. It follows that

\begin{align*}
\partial_t v_t - \partial_t \text{div} (a \cdot a^T : \nabla v) + \partial_t \text{div} (aq) &= 0 \quad \text{in} \quad \Omega_f \times (0, T), \quad (77) \\
\text{tr}(a_t : \nabla v) + \text{tr}(a : \nabla v_t) &= 0 \quad \text{in} \quad \Omega_f \times (0, T), \quad (78) \\
w_{ttt} - B_w(t) w_t + w_t &= 0 \quad \text{in} \quad \Omega_e \times (0, T). \quad (79)
\end{align*}

Besides, the boundary conditions with respect to (77)-(79) are

\begin{align*}
w_{tt} &= v_t - \gamma (w_t)_{\nu_B} \quad \text{on} \quad \Gamma_c \times (0, T), \quad (80) \\
(w_t)_{\nu_B} &= \partial_t (a : a^T : \nabla v) - \partial_t (qa)_{\nu} \quad \text{on} \quad \Gamma_c \times (0, T), \quad (81) \\
v_t &= 0 \quad \text{on} \quad \Gamma_f \times (0, T). \quad (82)
\end{align*}

**Lemma 3.5.** Let the assumptions in Theorem 1.1 hold. The following energy inequality holds for \( t \in [0, T] \)

\[ V_1(t) + \int_0^t D_1(\tau) d\tau \leq V_1(0) + \left| \int_0^t \langle R_1(\tau), v_t(\tau) \rangle d\tau \right| + C \int_0^t L(\tau) d\tau, \quad (83) \]

where

\[ \int_0^t \langle R_1(\tau), v_t(\tau) \rangle d\tau = - \int_0^t \int_{\Omega_f} \langle \partial_t (a : a^T) : \nabla v_t \rangle dx d\tau \]

\[ + \int_0^t \int_{\Omega_f} \langle \partial_t (a q) : \nabla v_t \rangle dx d\tau - \int_0^t \int_{\Omega_c} \langle \partial_t a \partial_t q, \nabla v_t \rangle dx d\tau. \quad (84) \]

**Proof.** Take \( L^2 \) inner product with \( v_t \) and \( w_{tt} \) to (77) and (79), respectively. Utilizing the boundary conditions (80)-(82), we attain that

\[ \frac{1}{2} \frac{d}{dt} \| v_t \|^2_{L^2} + \int_{\Omega_f} \langle a : a^T : \nabla v_t, \nabla v_t \rangle dx + \int_{\Omega_f} \langle \partial_t (a : a^T) : \nabla v, \nabla v_t \rangle dx \]

\[ - \int_{\Omega_f} \langle \partial_t (a q), \nabla v_t \rangle dx + \int_{\Gamma_c} \langle (\omega_t)_{\nu_B}, v_t \rangle d\sigma = 0 \quad (85) \]

and

\[ \frac{1}{2} \frac{d}{dt} \| w_{tt} \|^2_{L^2} + \| w_t \|^2_{L^2} + \int_{\Omega_e} \langle \| \nabla w_t, D^2 w, \nabla w_t \rangle \rangle dx \]

\[ - \frac{1}{2} \int_{\Omega_e} D^3 W(\nabla w_t, \nabla w_t, \nabla w_t) dx - \int_{\Gamma_c} \langle (w_t)_{\nu_B}, v_t - \gamma (w_t)_{\nu_B} \rangle d\sigma = 0. \quad (86) \]

Add (85) and (86) together and integrate in time from 0 to \( t \). Note that

\[ \int_{\Omega_e} D^3 W(\nabla w_t, \nabla w_t, \nabla w_t) dx \leq C L(t). \]

Due to the ellipticity of \( a(x, t) \), we get

\[ V_1(t) + \int_0^t D_1(\tau) d\tau \leq V_1(0) + C \int_0^t L(\tau) d\tau \]

\[ - \int_0^t \int_{\Omega_f} \langle \partial_t (a : a^T) : \nabla v, \nabla v_t \rangle dx d\tau + \int_0^t \int_{\Omega_f} \langle \partial_t (a q), \nabla v_t \rangle dx d\tau \quad (87) \]
Taking advantage of (78), we have
\[
\int_0^t \int_{\Omega_f} \langle \partial_t (a q), \nabla v_t \rangle \, dx \, dt = \int_0^t \int_{\Omega_f} \langle \partial_t a q, \nabla v_t \rangle \, dx \, dt + \int_0^t \int_{\Omega_f} \langle a \partial_t q, \nabla v_t \rangle \, dx \, dt \tag{88}
\]
\[
= \int_0^t \int_{\Omega_f} \langle \partial_t a q, \nabla v_t \rangle \, dx \, dt - \int_0^t \int_{\Omega_f} \langle \partial_t a \partial_t q, \nabla v \rangle \, dx \, dt
\]
We submit (88) into (87) and obtain (83).

According to Lemma 3.3, we find that
\[
\int_0^t V_1(\tau) d\tau \leq CV_1(0) + CV_1(t) + C \int_0^t L(\tau) d\tau + C \int_0^t \int_{\Gamma_c} |(w_t)_{\nu_\delta}|^2 \, d\sigma d\tau
\]
\[
+ C \int_0^t \|\nabla v_t\|^2_{L^2} \, dt.
\]
(89)

After a similar procedure with subsection 3.1, we conclude that

**Lemma 3.6.** Under the same hypotheses as in Lemma 57, we have for all \( t \in [0, T] \)
that
\[
V_1(t) + \int_0^t V_1(\tau) d\tau \leq CV_1(0) + C \int_0^t L(\tau) d\tau + C \int_0^t \int_{\Gamma_c}(R_1(\tau), v_t(\tau)) d\tau.
\]
(90)

### 3.3. Third level estimates

Here, we go further for the third level energy estimates. The third level energy is defined by
\[
V_2(t) = V_2^\tau(t) + \frac{1}{2} \|v_{tt}\|_{L^2}^2,
\]
with
\[
V_2^\tau(t) = \frac{1}{2}(\|w_{ttt}\|^2_{L^2} + \|w_{tt}\|^2_{L^2} + \int_{\Omega_c} (l\nabla w_{tt}, D^2W, \nabla w_{tt}) \, dx)
\]
and the dissipative term
\[
D_2(t) = \frac{1}{C} \| \nabla v_{tt}(t) \|^2_{L^2} + \gamma (|w_{ttt}|_{\nu_\delta}(t) \|w_{tt}\|^2_{L^2(\Gamma_c)}).
\]

Before the derivation of our estimates, we give the equations satisfied by \( v_{tt} \) and \( w_{tt} \).
\[
\partial_t v_{tt} - \partial_t \text{div} (a : a^T : \nabla v) + \partial_t \text{div} (a q) = 0 \quad \text{in} \quad \Omega_f \times (0, T),
\]
(91)
\[
tr(a_{tt} : \nabla v) + 2tr(a_{tt} : \nabla v_{tt}) + tr(a : \nabla v_{tt}) = 0 \quad \text{in} \quad \Omega_f \times (0, T),
\]
(92)
\[
w_{tttt} - B_w(t)w_{tt} + w_{tt} - r_1(t) = 0 \quad \text{in} \quad \Omega_c \times (0, T).
\]
(93)

Moreover, the boundary conditions with respect to (91)-(93) are the following:
\[
w_{ttt} = v_{tt} - \gamma (|w_{ttt}|_{\nu_\delta} + r_{1, \Gamma_c}) \quad \text{on} \quad \Gamma_c \times (0, T),
\]
(94)
\[
(w_{tt})_{\nu_\delta} = \partial_{tt}(a : a^T : \nabla v)\nu - \partial_{tt}(qa)\nu - r_{1, \Gamma_c} \quad \text{on} \quad \Gamma_c \times (0, T),
\]
(95)
\[
v_{tt} = 0 \quad \text{on} \quad \Gamma_f \times (0, T).
\]
(96)

**Lemma 3.7.** Suppose that the assumptions in Theorem 1.1 hold, then the following energy inequality holds for \( t \in [0, T] \)
\[
V_2(t) + \int_0^t D_2(\tau) d\tau \leq CV_2(0) + \int_0^t (R_2(\tau), v_{tt}) d\tau + C \int_0^t L(\tau) d\tau
\]
\[
+ 2\bar{\epsilon} \int_0^t \int_{\Omega_c} |w_{ttt}|^2 \, dx d\tau,
\]
(97)
where $0 < \varepsilon < 1$ sufficiently small and

$$
\int_0^t (R_2(\tau), v_{tt}) d\tau = 2 \int_0^t \int_{\Omega_f} \langle \partial_t (a : a^T) : \nabla v_t, \nabla v_{tt} \rangle dx d\tau
- \int_0^t \int_{\Omega_f} \langle \partial_t (a q), \nabla v_t \rangle dx d\tau + \int_0^t \int_{\Omega_f} \langle \partial_t (a : a^T) : \nabla v, \nabla v_{tt} \rangle dx d\tau.
$$

(98)

\textbf{Proof.} Take Euclidean dot product to (91) with $v_{tt}$ and integrate over $\Omega_f$. After integrating by parts, we obtain

$$
\int_0^t \frac{d}{dt} \| v_t \|^2_{L^2} + \int_{\Omega_f} \langle a : a^T : \nabla v_t, \nabla v_{tt} \rangle dx + \int_{\Gamma_c} \langle (w_{tt})_{\nu a} + r_1, v_{tt} \rangle d\sigma
+ \int_{\Omega_f} \langle \partial_t (a : a^T) : \nabla v, \nabla v_{tt} \rangle dx - \int_0^t \int_{\Omega_f} \langle w_{tt}, v_{tt} \rangle dx = 0.
$$

(99)

Next, we do the same operation to (93) with $w_{tt}$ as above and also integrate by parts over $\Omega_c$. Thus it follows that

$$
\int_0^t \frac{d}{dt} \| w_{tt} \|^2_{L^2} + \int_{\Omega_c} \langle \nabla w_{tt}, D^2 W, \nabla w_{tt} \rangle dx - \int_{\Gamma_c} \langle r_1(t), w_{tt} \rangle dx - \frac{1}{2} \int_{\Omega_c} D^3 W (\nabla w_{tt}, \nabla w_{tt}, \nabla w_t) dx - \int_{\Gamma_c} \langle (w_{tt})_{\nu a} - \gamma (w_{tt})_{\nu b} + r_1, v_{tt} \rangle d\sigma = 0.
$$

(100)

Adding (99) to (100) and integrating in time from 0 to $t$, it leads to

$$
V_2(t) + \int_0^t \int_{\Omega_f} \langle a : a^T : \nabla v_t, \nabla v_{tt} \rangle dx d\tau + \gamma \int_0^t \int_{\Gamma_c} \langle v_{tt} \rangle_{\nu b} \| w_{tt} \|_{L^2(\Gamma_c)}^2 d\tau
+ \int_0^t \int_{\Gamma_c} \langle r_1, v_{tt} \rangle d\sigma d\tau + \gamma \int_0^t \int_{\Gamma_c} \langle (w_{tt})_{\nu a}, r_1 \rangle d\sigma d\tau
+ \int_0^t \int_{\Omega_c} \langle R_2(\tau), v_{tt} \rangle d\tau = V_2(0) + \int_0^t \int_{\Omega_c} \langle r_1(t), w_{tt} \rangle dx d\tau
+ \frac{1}{2} \int_0^t \int_{\Omega_c} D^3 W (\nabla w_{tt}, \nabla w_{tt}, \nabla w_t) dx d\tau.
$$

(101)

By Lemma 2.5 and the Poincaré inequality, we have

$$
| \int_0^t \int_{\Gamma_c} \langle r_1, v_{tt} \rangle d\sigma d\tau | \leq C_\varepsilon \int_0^t \int_{\Gamma_c} | r_1 |^2 d\sigma d\tau + \varepsilon \int_0^t \int_{\Gamma_c} | v_{tt} |^2 d\sigma d\tau
\leq C_\varepsilon \int_0^t L(\tau) d\tau + \varepsilon \int_0^t | \nabla v_t |^2 d\tau,
$$

(102)

$$
\gamma | \int_0^t \int_{\Gamma_c} \langle (w_{tt})_{\nu a}, r_1 \rangle d\sigma d\tau | \leq \varepsilon \int_0^t \int_{\Gamma_c} | (w_{tt})_{\nu b} |^2 d\sigma d\tau + C_\varepsilon \gamma \int_0^t L(\tau) d\tau,
$$

(103)

where $0 < \varepsilon < 1$ is small enough and to be determined.

Similarly, also by using Lemma 2.5, we obtain

$$
| \int_0^t \int_{\Omega_c} \langle r_1(t), w_{tt} \rangle dx d\tau | \leq \varepsilon \int_0^t \int_{\Omega_c} | w_{tt} |^2 dx d\tau + C_\varepsilon \int_0^t L(\tau) d\tau.
$$

(104)
Note that
\[ \int_0^t \int_{\Omega_e} D^3W(\nabla w_{tt}, \nabla w_{tt}, \nabla w_t) dx \, dt \leq C \int_0^t L(\tau) \, d\tau. \tag{105} \]

Substitute (102)-(105) into (101) and set \( 1 - C\bar{\epsilon} \geq \frac{1}{2} \) and \( 1 - \frac{\xi}{4} \geq \frac{1}{2} \). Thus, via the uniformly ellipticity, we finally get (97).

From Lemma 3.3, we deduce that
\[ \int_0^t V_2(\tau) \, d\tau \leq CV_2(0) + CV_2(t) + C \int_0^t L(\tau) \, d\tau + C \int_0^t |(w_{tt})^{\nu}_s|^2 \, d\sigma \, d\tau \]
\[ + C \int_0^t \|\nabla v_{tt}\|_{L^2}^2 \, d\tau. \tag{106} \]

Multiply (106) by \( \epsilon' > 0 \) with \( \epsilon < \epsilon' < \min\left\{ \frac{1}{C}, \frac{1}{2} \right\} \) and add the resulted inequality to (97). Hence, it turns out that we arrive at the following lemma.

Lemma 3.8. Let the hypotheses in Theorem 1.1 be true. Then we have for \( t \in [0, T] \),
\[ V_2(t) + \int_0^t V_2(\tau) \, d\tau \leq CV_2(0) + C \int_0^t (R_2(\tau), v_{tt}) \, d\tau + C \int_0^t L(\tau) \, d\tau. \tag{107} \]

3.4. Fourth level estimates. We move on to the Fourth level energy estimates and repeat what we do as above. The fourth level energy is defined by
\[ V_3(t) = V_3(t) + \frac{1}{2} \|v_{ttt}\|_{L^2}^2 \quad \text{and} \]
\[ V_3(t) = \frac{1}{2} \|w_{tttt}\|_{L^2}^2 + \|w_{ttt}\|_{L^2}^2 + \int_{\Omega_e} \langle \nabla w_{ttt}, D^2W, \nabla w_{ttt} \rangle dx \]
as in the previous section. Besides, the dissipative term for the fourth energy is
\[ D_3(\tau) = \frac{1}{C} \|\nabla v_{ttt}(t)\|_{L^2}^2 + \gamma \|w_{ttt}(t)\|_{L^2(\Gamma_c)}. \]

First of all, as before, we differentiate the whole system three times in time and obtain
\[ \partial_t v_{ttt} - \partial_{tttt} \, \text{div} (a : a^T : \nabla v) + \partial_{tttt} \, \text{div} (a^q) = 0 \quad \text{in} \quad \Omega_f \times (0, T), \tag{108} \]
\[ \text{tr} [\partial_{tttt} (a : \nabla v)] = 0 \quad \text{in} \quad \Omega_f \times (0, T), \tag{109} \]
\[ w_{ttt}(t) - B_w(t) w_{ttt} + w_{ttt} - r_2(t) = 0 \quad \text{in} \quad \Omega_e \times (0, T). \tag{110} \]

And, the boundary conditions satisfied by the system (108)-(110) are as follows:
\[ w_{ttt} = v_{ttt} - \gamma [(w_{ttt})^{\nu}_s + r_{2, \Gamma_c}] \quad \text{on} \quad \Gamma_c \times (0, T), \tag{111} \]
\[ (w_{ttt})^{\nu}_s = \partial_{tttt} (a : a^T : \nabla v) - \partial_{tttt} (qa^q) + \gamma \nu - r_{2, \Gamma_c} \quad \text{on} \quad \Gamma_c \times (0, T), \tag{112} \]
\[ v_{ttt} = 0 \quad \text{on} \quad \Gamma_f \times (0, T). \tag{113} \]

Hence, we are ready to derive the energy estimates for the fourth order energy.

Lemma 3.9. Assume that the hypotheses of Theorem 1.1 hold, then the following energy estimate is true for \( t \in [0, T] \)
\[ V_3(t) + \int_0^t D_3(\tau) \, d\tau \leq CV_3(0) + \int_0^t (R_3(\tau), v_{ttt}) \, d\tau + C \int_0^t L(\tau) \, d\tau \]
\[ + 2\epsilon \int_0^t V_3(\tau) \, d\tau. \tag{114} \]
where $0 < \epsilon < 1$ is sufficiently small and

\[
\int_{0}^{t} (R_3(\tau), \nu_{tt}) d\tau = \int_{0}^{t} \int_{\Omega_f} \langle \partial_{ttt}(a : a^T) : \nabla v, \nabla \nu_{ttt} \rangle dx d\tau \\
- \int_{0}^{t} \int_{\Omega_f} \langle \partial_{ttt}(aq), \nabla \nu_{ttt} \rangle dx d\tau + 3 \int_{0}^{t} \int_{\Omega_f} \langle \partial_t(a : a^T) : \nabla v, \nabla \nu_{ttt} \rangle dx d\tau \\
+ 3 \int_{0}^{t} \int_{\Omega_f} \langle \partial_t(a : a^T) : \nabla v_t, \nabla \nu_{ttt} \rangle dx d\tau.
\]  

(115)

Proof. Take $L^2$ inner product with $v_{ttt}$ and $w^{(4)}$ to (108) and (110), respectively. From (111) and (112), we attain that

\[
\frac{1}{2} \frac{d}{dt} \| v_{ttt} \|^2_{L^2} + \int_{\Omega_f} \langle a : a^T : \nabla v, \nabla \nu_{ttt} \rangle dx + \int_{\Gamma_v} \langle (w_{ttt})_{\nu_B} + r_{2.\Gamma_v}, \nu_{ttt} \rangle d\sigma \\
+ \int_{\Omega_f} \langle \partial_{ttt}(a : a^T) : \nabla v, \nabla \nu_{ttt} \rangle dx + 3 \int_{\Omega_f} \langle \partial_{ttt}(a : a^T) : \nabla v_t, \nabla \nu_{ttt} \rangle dx \\
+ 3 \int_{\Omega_f} \langle \partial_t(a : a^T) : \nabla v_t, \nabla \nu_{ttt} \rangle dx - \int_{\Omega_f} \langle \partial_{ttt}(aq), \nabla \nu_{ttt} \rangle dx = 0
\]

(116)

and

\[
\frac{1}{2} \frac{d}{dt} \| w_{ttt} \|^2_{L^2} + \| w_{ttt} \|^2_{L^2} + \int_{\Omega_v} \langle k \nabla w_{ttt} : D^2 W, \nabla w_{ttt} \rangle dx \\
- \int_{\Omega_v} \langle r_2(t), w_{tttt} \rangle dx - \frac{1}{2} \int_{\Omega_v} D^3 W(\nabla w_{ttt}, \nabla w_{tt}, \nabla w_t) dx \\
- \int_{\Gamma_{\Gamma_v}} \langle (w_{ttt})_{\nu_B}, \nu_{tt} - \gamma [(w_{ttt})_{\nu_B} + r_{2.\Gamma_v}] \rangle d\sigma = 0.
\]

(117)

Add (116) and (117) together and we have analogous estimates to (102)-(105) as well. Using the similar method with that in Lemma 3.7, we may arrive at (114) and conclude the proof. \qed

As a consequence of Lemma 3.3, we have

\[
\int_{0}^{t} V_3(\tau) d\tau \leq C V_3(0) + CV_3(t) + C \int_{0}^{t} L(\tau) d\tau + C \int_{0}^{t} \int_{\Gamma_v} \| (w_{ttt})_{\nu_B} \|^2 d\sigma d\tau \\
+ C \int_{0}^{t} \| \nabla v_{ttt} \|^2_{L^2} d\tau.
\]

(118)

After the same procedure as that in Subsection 3.3, we acquire the following lemma.

Lemma 3.10. Suppose that the hypotheses in Theorem 1.1 are true. Then for $t \in [0,T]$,

\[
V_3(t) + \int_{0}^{t} V_3(\tau) d\tau \leq CV_3(0) + C \int_{0}^{t} (R_3(\tau), \nu_{ttt}) d\tau + C \int_{0}^{t} L(\tau) d\tau.
\]

(119)

3.5. Superlinear estimates. Our aim of this subsection is to deal with the perturbation terms in the second, third and fourth level energy estimates. They are

\[
\int_{0}^{t} (R_1(\tau), \nu_t(\tau)) d\tau, \quad \int_{0}^{t} (R_2(\tau), \nu_{tt}(\tau)) d\tau
\]
and

$$\int_0^t (R_3(\tau), v_H(\tau)) d\tau$$

The concrete presentation of the above three perturbation terms can be found in (84), (98) and (115), respectively. For the estimates of (84) and (98), we only list the results. For detail, refer to [8].

Lemma 3.11. [8, Lemma 4.10] We have

$$| \langle R_1(t), v_t \rangle | \leq C \| v \|_{H^1} \| v \|_{H^2} \| v_t \|_{H^1} (\| v \|_{H^2} + \| q \|_{H^1})$$

$$+ C \| v \|_{H^1} \| v \|_{H^2} \| q_t \|_{H^1},$$

for all $t \in [0, T]$.

Lemma 3.12. [8, Lemma 4.11] For $\epsilon_0 \in (0, \frac{1}{2}]$, we have

$$| \int_0^t \langle R_2(s), v_t(s) \rangle ds | \leq \epsilon_0 \int_0^t \| \nabla v_t \|_{L^2}^2 ds$$

$$+ C_{\epsilon_0} \int_0^t \| v \|_{H^3}^3 \| v \|_{H^3}^3 \| v_t \|_{H^1}^2 ds$$

$$+ C_{\epsilon_0} \int_0^t (\| v \|_{H^3}^3 + \| q \|_{H^3}^3)(\| v \|_{H^3}^3 \| v \|_{H^3}^3 + \| v_t \|_{H^1}^2) ds$$

$$+ \epsilon_0 \| v(t) \|_{H^3}^2 + C_{\epsilon_0} \| q(t) \|_{H^1} + \epsilon_0 \| v_t(t) \|_{H^2}$$

$$+ C_{\epsilon_0} \| v(t) \|_{H^2}^2 \| v(t) \|_{H^1} + C_{\epsilon_0} \| v(t) \|_{H^1} \| v(t) \|_{H^2} \| v_t(t) \|_{L^2}^2$$

$$+ C \int_0^t (\| v \|_{H^2} + \| v_t \|_{H}^2 + \| v_t \|_{H}^2 \| v \|_{H^1} \| v \|_{H^2} \| q_t \|_{H^1} \| v \|_{H^2}) ds$$

$$+ C \| v(0) \|_{H^3}^3 + C \| v_t(0) \|_{H^1} + C \| q_t(0) \|_{H^1}^2,$$

for all $t \in [0, T]$.

Now, we turn to (115), even though the computation is quite involved.

Lemma 3.13. For $\epsilon_0 \in (0, \frac{1}{2}]$, it follows that for all $t \in [0, T]$

$$\int_0^t (R_3(\tau), v_H(\tau)) d\tau \leq \epsilon_0 \int_0^t \| \nabla v_H \|_{L^2}^2 d\tau$$

$$+ C_{\epsilon_0} \int_0^t \| v \|_{H^3}^3 (\| v \|_{H^1} \cdot \| v \|_{H^3} + \| v_t \|_{H^1})^2 d\tau$$

$$+ C_{\epsilon_0} \int_0^t (\| v \|_{H^3}^3 + \| q \|_{H^3}^3)(\| v \|_{H^3}^3 + \| v_t \|_{H^1} \| v \|_{H^3} + \| v_H \|_{H^1})^2 d\tau$$

$$+ C_{\epsilon_0} \int_0^t (\| v_t \|_{H^2} + \| q_H \|_{H^1})(\| v \|_{H^2} + \| v_t \|_{H^1} \| v \|_{H^2}^\gamma) d\tau$$

$$+ C_{\epsilon_0} \int_0^t \| v \|_{H^3}^3 (\| v_t \|_{H^2}^\gamma + \| q_t \|_{L^2}^\gamma) d\tau + C_{\epsilon_0} \int_0^t \| q_t \|_{H^1} \| v \|_{H^2}^\gamma \| v_t \|_{H^3}^\gamma d\tau$$

$$+ C \int_0^t (\| v \|_{H^2}^\gamma + \| v \|_{H^2}^\gamma + \| v \|_{H^2}^\gamma \| v_H \|_{H^1} + \| v_t \|_{H^3}) \| q_t \|_{H^1} \| v \|_{H^2}^\gamma d\tau$$

$$+ C \| v(t) \|_{H^1}^2 + C \| v_t(t) \|_{H^1} + C \| q_t(t) \|_{H^1}^2,$$
+C \int_0^t (\|v\|_{H^3}^2 + \|v\|_{L^2}^2 \|v\|_{H^1} + \|v_t\|_{H^1} + \|v_{tt}\|_{H^1}) \|q_{tt}\|_{H^1} \|v_t\|_{H^3} \|v\|_{H^3}^2 d\tau \\
+ C \int_0^t \|v_{tt}\|_{H^1} \|q_{tt}\|_{H^1} (\|v\|_{H^2}^2 + \|v_t\|_{H^1} \|v\|_{H^1}) d\tau + \epsilon_0 \|q_{tt}\|_{H^1}^2 + \epsilon_0 \|v_{tt}\|_{H^3} \\
+ C_{\epsilon_0} \|v_t\|_{H^3}^2 + C \|v\|_{H^3} + C_{\epsilon_0} \|v_t\|_{H^1} \|v\|_{H^3}^2 + C_{\epsilon_0} \|v\|_{H^2}^2 \|v_t\|_{H^1} \\
+ C_{\epsilon_0} \|v_t\|_{H^3}^2 \|v_t\|_{H^1} \|v\|_{H^2} + C_{\epsilon_0} \|v_t\|_{H^1} \|v\|_{H^2} + C \|q_{tt}(0)\|_{H^1}^2 \\
+ C \|v_t(0)\|_{H^3}^2 + C \|v(0)\|_{H^2}^2 + C \|v_t(0)\|_{H^1}^2 + C \|v(0)\|_{H^3}^2 + C \|v_t(0)\|_{H^1}^2)

Proof. From (115), we have

\begin{align*}
| \int_0^t (R_3(\tau), v_{tt}) d\tau | & \leq \int_0^t \int_{\Omega_f} (\delta_{tt}(a : a^T) : \nabla v, \nabla v_{tt}) dx d\tau \\
& + 3 | \int_0^t \int_{\Omega_f} \langle \delta_t(a : a^T) : \nabla v_t, \nabla v_{tt} \rangle dx d\tau | \\
& + 3 | \int_0^t \int_{\Omega_f} \langle \delta_{ttt}(a : a^T) : \nabla v_t, \nabla v_{tt} \rangle dx d\tau | \\
& + | \int_0^t \int_{\Omega_f} \langle \delta_{tttt} a q, \nabla v_{tt} \rangle dx d\tau | + | \int_0^t \int_{\Omega_f} \langle a_{ttt}, \nabla v_{tt} \rangle dx d\tau | \\
& + 3 | \int_0^t \int_{\Omega_f} \langle \delta_{tt} a q_t, \nabla v_{tt} \rangle dx d\tau | + 3 | \int_0^t \int_{\Omega_f} \langle \delta_t a_{ttt}, \nabla v_{tt} \rangle dx d\tau | \\
& = R_{31} + R_{32} + R_{33} + R_{34} + R_{35} + R_{36} + R_{37}
\end{align*}

(120)

By Hölder inequality and Lemma 3.1, we get

\begin{align*}
\sum_{j=1}^4 R_{3j} & \leq C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|\nabla v\|_{L^\infty} (\|\nabla v\|_{H^1}^2 + \|\nabla v_t\|_{L^2} \|\nabla v\|_{L^\infty} + \|\nabla v_{tt}\|_{L^2}) d\tau \\
& + C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|\nabla v\|_{L^\infty} (\|\nabla v\|_{L^2} \|\nabla v\|_{L^\infty} + \|\nabla v_t\|_{L^2}) d\tau \\
& + C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|\nabla v_t\|_{L^2} \|\nabla v\|_{L^\infty} d\tau + C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|\nabla v_t\|_{L^2} \|\nabla v\|_{L^\infty} d\tau \\
& + C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|\nabla v_t\|_{L^6} (\|v\|_{H^2}^2 + \|\nabla v_t\|_{L^3}) d\tau \\
& + C \int_0^t \|q\|_{L^\infty} \|\nabla v_{ttt}\|_{L^2} (\|v\|_{H^3}^2 + \|v_t\|_{H^1} \|v\|_{H^1} + \|v_{tt}\|_{H^1}) d\tau \\
& \leq C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|v\|_{L^2} (\|v\|_{H^3}^2 + \|v_t\|_{H^1} \|v\|_{H^1} + \|v_{tt}\|_{H^1}) d\tau \\
& + C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|v_t\|_{H^1} \|v\|_{H^3} d\tau + C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|v_t\|_{H^1} \|v\|_{H^3}^2 d\tau \\
& + C \int_0^t \|\nabla v_{ttt}\|_{L^2} \|v_t\|_{H^1} \|v_{tt}\|_{H^2} d\tau \\
& + C \int_0^t \|q\|_{H^2} \|\nabla v_{ttt}\|_{L^2} (\|v\|_{H^3}^2 + \|v_t\|_{H^1} \|v\|_{H^3} + \|v_{tt}\|_{H^1}) d\tau
\end{align*}

(121)
and
\begin{align}
R_{35} + R_{36} & \leq C \int_0^t \| \nabla v_{ttt} \|_{L^2} \| q_t \|_{H^3} \left( \| v \|_{H^\frac{3}{2}}^2 + \| v \|_{H^1} \| v \|_{H^2}^\frac{3}{2} \right) d\tau \\
& + C \int_0^t \| D v_{ttt} \|_{L^2} \| q_t \|_{L^2} \| v \|_{H^3} d\tau,
\end{align}

where the Sobolev and interpolation inequalities are employed. Now we begin to treat \( R_{37} \). By the differentiated divergence-free condition (109), we deduce that
\begin{align}
R_{37} &= - \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_{ttt}, \nabla v \rangle dx d\tau - 3 \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_{ttt}, \nabla v_t \rangle dx d\tau \\
& - 3 \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_t, \nabla v_t \rangle dx d\tau \\
& = - \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_{ttt}, \nabla v \rangle dx \mid_0^t - 3 \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_{ttt}, \nabla v_t \rangle dx \mid_0^t - 3 \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_t, \nabla v_t \rangle dx \mid_0^t \\
& + \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_t, \nabla v \rangle dx d\tau + 4 \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_{ttt}, \nabla v_t \rangle dx d\tau \\
& + 6 \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_t, \nabla v_t \rangle dx d\tau + 3 \int_0^t \int_{\Omega_f} \langle q_{ttt} \mathbf{a}_t, \nabla v \rangle dx d\tau.
\end{align}

Applying Lemma 2.1 and Corollary 1 along with Hölder’s, Sobolev and interpolation inequalities, we have
\begin{align}
R_{37} & \leq \| \mathbf{a}_{ttt} (t) \|_{L^2} \| \nabla v (t) \|_{L^3} \| q_{ttt} (t) \|_{L^6} + 3 \| \mathbf{a}_{ttt} (t) \|_{L^2} \| \nabla v_t (t) \|_{L^3} \| q_{ttt} (t) \|_{L^6} \\
& + 3 \| \mathbf{a}_{tt} (t) \|_{L^3} \| \nabla v_{ttt} (t) \|_{L^2} \| q_{tt} (t) \|_{L^6} + 3 \| \mathbf{a}_{tt} (t) \|_{L^2} \| \nabla v_t (t) \|_{L^3} \| q_{tt} (t) \|_{L^6} \\
& + 3 \| \mathbf{a}_{tt} (0) \|_{L^2} \| \nabla v (0) \|_{L^3} \| q_{tt} (0) \|_{L^6} + 3 \| \mathbf{a}_{tt} (0) \|_{L^2} \| \nabla v_t (0) \|_{L^3} \| q_{tt} (0) \|_{L^6} \\
& + \int_0^t \| \partial_t^2 \mathbf{a} \|_{L^2} \| \nabla v \|_{L^3} \| q_{tt} \|_{L^6} d\tau + 4 \int_0^t \| \mathbf{a}_{ttt} \|_{L^2} \| \nabla v_t \|_{L^3} \| q_{tt} \|_{L^6} d\tau \\
& + 6 \int_0^t \| \mathbf{a}_t (0) \|_{L^2} \| \nabla v_{ttt} \|_{L^2} \| q_{tt} \|_{L^6} d\tau + 3 \int_0^t \| \mathbf{a}_t \|_{L^2} \| \nabla v_{ttt} \|_{L^2} \| q_{tt} \|_{L^6} d\tau.
\end{align}

The sum of the first three terms on the right hand side of the above estimate is bounded by
\begin{align}
C (\| \nabla v \|_{H^\frac{3}{2}}^2 + \| \nabla v \|_{L^2} \| \nabla v \|_{L^\infty} + \| \nabla v_{ttt} \|_{L^2}) \| q_{ttt} \|_{H^3} \| v \|_{H^\frac{3}{2}} \| v \|_{H^2} \\frac{3}{2} \\
& + C (\| \nabla v \|_{L^2} \| \nabla v \|_{L^\infty} + \| \nabla v_t \|_{L^2}) \| q_{tt} \|_{H^1} \| v \|_{H^\frac{3}{2}} \| v \|_{H^2} \\frac{3}{2} \\
& + C (\| \nabla v \|_{L^2} \| v \|_{H^1} \| q_{tt} \|_{H^1} \leq \epsilon_0 \| q_{tt} \|_{H^1}^2 + \epsilon_0 \| v_t \|_{H^1}^2 + C_0 \| v \|_{H^1}^6 \\
& + C \| v \|_{H^1}^2 + C_0 \| v \|_{H^1}^2 \| v \|_{H^2}^\frac{3}{2} + C_0 \| v \|_{H^2} \| v \|_{H^1} \\
& + C_0 \| v \|_{H^1}^2 \| v \|_{H^2}^\frac{3}{2} + C_0 \| v \|_{H^2} \| v \|_{H^1} \| v \|_{H^2} \\
& \| \mathbf{a}_{tt} (0) \|_{L^2} \| \nabla v_{ttt} \|_{L^2} \| q_{tt} \|_{L^6} d\tau + 3 \int_0^t \| \mathbf{a}_t \|_{L^2} \| \nabla v_{ttt} \|_{L^2} \| q_{tt} \|_{L^6} d\tau,
\end{align}

with the help of Lemma 3.1. Thus, thanks to the Agmon’s inequality \( \| \phi \|_{L^\infty} \leq C \| \phi \|_{H^\frac{3}{2}} \| \phi \|_{H^\frac{3}{2}} \), in particular, we obtain
\begin{align}
R_{37} & \leq \epsilon_0 \| q_{tt} \|_{H^1}^2 + \epsilon_0 \| v_t \|_{H^1}^2 + C_0 \| v \|_{H^1}^6 + C \| v \|_{H^1}^4 \\
& + C_0 \| v \|_{H^1}^2 \| v \|_{H^1}^2 + C_0 \| v \|_{H^1} \| v \|_{H^2} + C_0 \| v \|_{H^1} \| v \|_{H^1} \| v \|_{H^2} \\
& + C_0 \| v \|_{H^1} \| v \|_{H^2} + C \| q_{tt} (0) \|_{H^1}^2 + C \| v_{tt} (0) \|_{H^1}^2.
\end{align}
4. Energy decay and global existence of the system. We aim at the global existence of solutions and the energy decay estimates in this section.

Let the total energy of the whole system

\[ X(t) = \sum_{i=0}^{3} V_i(t) + \epsilon_1 (\| \nabla v \|^2_{L^2} + \| \nabla v_t \|^2_{L^2} + \| \nabla v_{tt} \|^2_{L^2}) \]  \hspace{1cm} (124)

and its equivalent version

\[ \mathcal{X}(t) = \frac{1}{2} \sum_{i=0}^{3} \| v^{(i)} \|^2_{L^2} + \mathcal{E}^e(t) + \epsilon_1 (\| \nabla v \|^2_{L^2} + \| \nabla v_t \|^2_{L^2} + \| \nabla v_{tt} \|^2_{L^2}) \]

where \( \epsilon_1 > 0 \) is given sufficiently small and to be determined later.

We make some preparations for the proof of Theorem 1.1.

We have

\[
\| \nabla v(t) \|^2_{L^2} = \| \nabla v(0) \|^2_{L^2} + \int_0^t \frac{d}{d\tau} \| \nabla v(\tau) \|^2_{L^2} d\tau \\
= \| \nabla v(0) \|^2_{L^2} + 2 \int_0^t \| \nabla v \|_{L^2} \| \nabla v_t \|_{L^2} d\tau \\
\leq \| \nabla v(0) \|^2_{L^2} + C \int_0^t (D_0(\tau) + D_1(\tau))d\tau,
\]  \hspace{1cm} (125)

Similarly, we obtain

\[
\| \nabla v_t(t) \|^2_{L^2} \leq \| \nabla v_t(0) \|^2_{L^2} + C \int_0^t (D_1(\tau) + D_2(\tau))d\tau
\]  \hspace{1cm} (126)

and

\[
\| \nabla v_{tt}(t) \|^2_{L^2} \leq \| \nabla v_{tt}(0) \|^2_{L^2} + C \int_0^t (D_2(\tau) + D_3(\tau))d\tau.
\]  \hspace{1cm} (127)

In addition, it follows from Lemmas 3.1 and 3.4 that

\[
V_0(t) + \int_0^t V_0(\tau)d\tau + \int_0^t D_0(\tau)d\tau \leq CV_0(0) + C \int_0^t L(\tau)d\tau.
\]  \hspace{1cm} (128)

Combining Lemmas 3.5, 3.6 and 3.11, we have

\[
V_1(t) + \int_0^t V_1(\tau)d\tau + \int_0^t D_1(\tau)d\tau \leq CV_1(0) + C \int_0^t L(\tau)d\tau \\
+ C \int_0^t P_1(\| v \|_{H^2}, \| q \|_{H^1}, \| v_t \|_{H^1}, \| q_t \|_{H^1})d\tau.
\]  \hspace{1cm} (129)
From Lemmas 3.7, 3.8 and 3.12,
\begin{align}
V_2(t) &+ \int_0^t V_2(\tau) d\tau + \int_0^t D_2(\tau) d\tau \leq CV_2(0) + C \int_0^t L(\tau) d\tau + \epsilon_0 \| v(t) \|_{H^3}^2 \\
&+ \epsilon_0 \| q(t) \|_{H^1}^2 + \epsilon_0 \| v_t(t) \|_{H^2}^2 + \epsilon_0 \int_0^t \| Dv_{tt} \|_{L^2}^2 \, d\tau \\
&+ P_2(\| v \|_{H^3}, \| v_t \|_{L^2}) \\
&+ \int_0^t P_3(\| v \|_{H^3}, \| q \|_{H^2}, \| v_t \|_{H^2}, \| q_t \|_{H^1}) d\tau \\
&+ P_4(\| v(0) \|_{H^3}, \| v_t(0) \|_{H^1}, \| q_t(0) \|_{H^1}).
\end{align}

Moreover, according to Lemma 3.9, 3.10 and 3.13, we attain
\begin{align}
V_3(t) &+ \int_0^t V_3(\tau) d\tau + \int_0^t D_3(\tau) d\tau \leq CV_3(0) + C \int_0^t L(\tau) d\tau + \epsilon_0 \| q_{tt} \|_{H^1}^2 \\
&+ \epsilon_0 \| v_t \|_{H^2}^2 + \epsilon_0 \int_0^t \| \nabla v_{tt} \|_{L^2}^2 \, d\tau + P_5(\| v \|_{H^3}, \| v_t \|_{H^1}, \| v_{tt} \|_{H^1}) \\
&+ \int_0^t P_6(\| v \|_{H^3}, \| q \|_{H^2}, \| v_t \|_{H^2}, \| q_t \|_{H^2}, \| v_{tt} \|_{H^2}, \| q_{tt} \|_{H^1}) d\tau \\
&+ P_7(\| v(0) \|_{H^3}, \| v_t(0) \|_{H^1}, \| q_{tt}(0) \|_{H^1}, \| v_{tt}(0) \|_{H^1}),
\end{align}

where the symbols $P_i$, $1 \leq i \leq 7$ denote the superlinear polynomials of their arguments, which are allowed to depend on $\epsilon_0$ from Lemmas 3.12 and 3.13. Now multiply (125)-(127) by sufficiently small $\epsilon_1$, sum up (128)-(131) and then add them together to obtain
\begin{align}
X(t) &+ \int_0^t X(\tau) d\tau \leq CX(0) + \epsilon_0 \| v(t) \|_{H^3}^2 + \epsilon_0 \| q_t(t) \|_{H^1}^2 + \epsilon_0 \| v_t(t) \|_{H^2}^2 \\
&+ \epsilon_0 \| q_{tt} \|_{H^1}^2 + C \int_0^t L(\tau) d\tau + P_1(\| v \|_{H^3}, \| v_t \|_{H^1}, \| v_{tt} \|_{H^1}) \\
&+ \int_0^t P_2(\| v \|_{H^3}, \| q \|_{H^2}, \| v_t \|_{H^2}, \| q_t \|_{H^2}, \| v_{tt} \|_{H^2}, \| q_{tt} \|_{H^1}) d\tau \\
&+ P_3(\| v(0) \|_{H^3}, \| v_t(0) \|_{H^1}, \| q_{tt}(0) \|_{H^1}, \| v_{tt}(0) \|_{H^1}), \| q_t(0) \|_{H^1})
\end{align}

Because of (49) in Theorem 2.7, we find that
\begin{align}
X(t) &+ \int_0^t X(\tau) d\tau \leq CX(0) + C\epsilon_0 \| v(t) \|_{H^3}^2 + C\epsilon_0 \| q_t(t) \|_{H^1}^2 + C\epsilon_0 \| v_t(t) \|_{H^2}^2 \\
&+ C\epsilon_0 \| q_{tt} \|_{H^1}^2 + C \int_0^t L(\tau) d\tau + P_1(\| v \|_{H^3}, \| v_t \|_{H^1}, \| v_{tt} \|_{H^1}) \\
&+ \int_0^t P_2(\| v \|_{H^3}, \| q \|_{H^2}, \| v_t \|_{H^2}, \| q_t \|_{H^2}, \| v_{tt} \|_{H^2}, \| q_{tt} \|_{H^1}) d\tau \\
&+ P_3(\| v(0) \|_{H^3}, \| v_t(0) \|_{H^1}, \| q_{tt}(0) \|_{H^1}, \| v_{tt}(0) \|_{H^1}), \| q_t(0) \|_{H^1}) \\
&+ CL(t) + C \sum_{j=0}^2 \| w_{\nu\beta}^{(j)} \|_{H^3}^2 d\tau + C \sum_{j=0}^2 \int_0^t \| w_{\nu\beta}^{(j)} \|_{H^3}^2 d\tau.
\end{align}

From (28), Lemma 2.4 and the properties of Sobolev space we list, we have
\begin{align}
\| v \|_{H^3}^2 + \| q \|_{H^2}^2 \leq CX(t).
\end{align}
Thanks to (29) and (134), similarly we obtain
\[ \| v_t \|_{H^3}^2 + \| q_t \|_{H^3}^2 \leq Cx(t) + Cx^2(t). \]  \( \text{(135)} \)

For (28) in case of \( s = 2 \), by (135) and Lemma 2.4,
\[ \| v \|_{H^4}^2 + \| q \|_{H^3}^2 \leq Cx(t) + Cx^2(t). \]  \( \text{(136)} \)

From (2) in Lemma 2.2, (134) and (136), via a similar way, we deduce that
\[ \| v_{tt} \|_{H^2}^2 + \| q_{tt} \|_{H^1}^2 \leq Cx(t) + C\tilde{P}(x(t)), \]  \( \text{(137)} \)

where \( \tilde{P} \) is a polynomial with the degree of each term of it is at least 2.

Submitting (134)-(137) and the boundary condition (63) into (133) that follows from (133) that
\[ \mathcal{X}(t) + \int_0^t \mathcal{X}(\tau)d\tau \leq Cx(0) + \mathcal{P}(x(t)) + \int_0^t \mathcal{P}(x(\tau))d\tau + \mathcal{P}(x(0)), \]  \( \text{(138)} \)

where \( \mathcal{P} \) is a superlinear polynomial as well. We rewrite (138) as
\[ \mathcal{X}(t) + \int_0^t \mathcal{X}(\tau)d\tau \leq C_0 \sum_{j=1}^m \int_0^t \mathcal{X}(\tau)\alpha_j d\tau + C_0 \sum_{k=1}^n \mathcal{X}(t)^{\beta_k} + C_0 \sum_{k=1}^n \mathcal{X}(0)^{\beta_k} \]
\[ + C_0 \mathcal{X}(0), \]  \( \text{(139)} \)

where \( C_0 \geq 1, \alpha_1, ..., \alpha_m > 1 \) and \( \beta_1, ..., \beta_n > 1 \).

Following [8, Lemma 5.1], we have

**Lemma 4.1.** Suppose that \( \mathcal{X} : [0, \infty) \rightarrow [0, \infty) \) is continuous for all \( t \) such that \( \mathcal{X}(t) \) is finite and assume that it satisfies
\[ \mathcal{X}(t) + \int_0^t \mathcal{X}(s)ds \leq C_0 \sum_{j=1}^m \int_0^t \mathcal{X}(s)^{\alpha_j} ds + C_0 \sum_{k=1}^n \mathcal{X}(t)^{\beta_k} + C_0 \sum_{k=1}^n \mathcal{X}(0)^{\beta_k} \]
\[ + C_0 \mathcal{X}(0), \]  \( \text{where } \alpha_1, ..., \alpha_m > 1 \) and \( \beta_1, ..., \beta_n > 1. \) Also, assume that \( \mathcal{X}(0) \leq \epsilon. \) If \( \epsilon \leq \frac{1}{C}, \) where the constant \( C \) depends on \( C_0, m, \alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n, \) we have \( \mathcal{X}(t) \leq C\epsilon e^{-\frac{t}{\epsilon}}. \)

**Proof.** Utilizing Lemma 4.1 and following the proof of [8, Theorem 2.1] and [18, Theorem 1.1], the proof of Theorem 2.7 is finished. \( \square \)

5. **Construction of solutions.** In this section, we turn to the proof of Theorem 1.2. Following the method for linear wave equations coupled in fluid-structure interaction system in [8], we first construct solutions in Lemma 5.1 to the linear problem for the given matrix \( a = I, \) and for given nonzero forcing \( f \) in liquid equations and \( \tilde{f} \) in elasticity equations, nonzero divergence condition \( g \) and nonzero difference of stresses \( h \) and velocity \( \tilde{h} \) on the common boundary \( \Gamma_c. \) Then, in the general case of given smooth elliptic matrix \( a(x, t) \), we apply a fixed point technique to the perturbed linear system, where
\[ f_i = -\partial_j((\delta_{jk} - a^{kl}a^{kl})\partial_k v_i) + \partial_k((\delta_{jk} - a^{jk}) q), \]  \( \text{(140)} \)
\[ g = (\delta_{jk} - a^{jk})\partial_k v_j, \]
Consider the linear coupled Stokes-elasticity system

\[ \tilde{f} = \text{div} l_{\nabla w} \left( \int_0^1 D^2W(1 + s\nabla w) - D^2W(1) ds \right), \]

\[ h_i = (\delta_{jk} - a^{ij}a^{kl})\partial_{k}v_{j} - (\delta_{jk} - a^{ij})q_{k} + \tilde{h}_i, \]

\[ \tilde{h} = (l_{\nabla w} \int_0^1 D^2W(1 + s\nabla w) - D^2W(1) ds) v, \]

Finally, we employ the retarded mollification technique for \( a \) to obtain the short-time existence results for the system considered in Theorem 1.1 as that in [10].

Before we state the lemma for the linear system, we give the compatibility conditions satisfied by the initial data of the linear equations

\[ (w_0)_{A_{\nu}} = \gamma^{-1}(v_0 - w_1) - \tilde{h}(0) \quad \text{on} \quad \Gamma_c \]  
\[ \langle \nabla v_0 \cdot \nu, \tau \rangle = \langle (w_0)_{A_{\nu}}, \tau \rangle + \langle h(0), \tau \rangle, \quad \text{on} \quad \Gamma_c \]  
\[ \partial_t (l_{\nabla w} A(x, t))(0) v = \gamma^{-1}(v'(0) - w''(0)) - \tilde{h}_t(0) \quad \text{on} \quad \Gamma_c \]  
\[ \langle \nabla v_t(0) \cdot \nu, \tau \rangle = \langle \partial_t (l_{\nabla w} A(x, t))(0)v, \tau \rangle + \langle h_t(0), \tau \rangle, \quad \text{on} \quad \Gamma_c \]

with

\[ \partial_{ttt}(l_{\nabla w} A(x, t))(0) v = \gamma^{-1}(v''(0) - w'''(0)) - \tilde{h}_{ttt}(0) \quad \text{on} \quad \Gamma_c \]  
\[ \langle \nabla v_{ttt}(0) \cdot \nu, \tau \rangle = \langle \partial_{ttt}(l_{\nabla w} A(x, t))(0)v, \tau \rangle + \langle h_{ttt}(0), \tau \rangle, \quad \text{on} \quad \Gamma_c \]

and

\[ \partial_{ttt}(l_{\nabla w} A(x, t))(0) v = \gamma^{-1}(v''(0) - w'''(0)) - \tilde{h}_{ttt}(0) \quad \text{on} \quad \Gamma_c \]  
\[ \langle \nabla v_{ttt}(0) \cdot \nu, \tau \rangle = \langle \partial_{ttt}(l_{\nabla w} A(x, t))(0)v, \tau \rangle + \langle h_{ttt}(0), \tau \rangle, \quad \text{on} \quad \Gamma_c \]

\[ v_{ttt}(0) = 0, \quad \text{on} \quad \Gamma_f, \]

and the regularity hypotheses of \( f, g, \tilde{f}, \tilde{h}, \tilde{\tilde{h}} \)

\[ f \in L^\infty([0, T]; H^2(\Omega_f)), f_t \in L^\infty([0, T]; H^1(\Omega_f)), f_{ttt} \in L^\infty([0, T]; L^2(\Omega_f)), \]  
\[ f_{ttt} \in L^2([0, T]; H^{-1}(\Omega_f)); \]  
\[ g \in C([0, T]; H^3(\Omega_f)), g_t \in L^\infty([0, T]; H^2(\Omega_f)), g_{ttt} \in L^\infty([0, T]; H^1(\Omega_f)), \]  
\[ g_{ttt} \in L^\infty([0, T]; H^{-1}(\Omega_f)) \cap L^2([0, T]; L^2(\Omega_f)); \]  
\[ \tilde{f} \in C([0, T]; H^2(\Omega_f)), \tilde{f}_t \in C([0, T]; H^1(\Omega_f)), \tilde{f}_{ttt} \in C([0, T]; L^2(\Omega_f)), \]  
\[ \tilde{f}_{ttt} \in L^2([0, T]; H^{-1}(\Omega_f)), \]  
\[ h, \tilde{h} \in L^\infty([0, T]; H^{\frac{3}{2}}(\Gamma_c)), h_t, \tilde{h}_t \in L^\infty([0, T]; H^{\frac{3}{2}}(\Gamma_c)), \]  
\[ h_{ttt}, \tilde{h}_{ttt} \in L^\infty([0, T]; H^{\frac{3}{2}}(\Gamma_c)), h_{ttt}, \tilde{h}_{ttt} \in L^2([0, T]; H^{-\frac{3}{2}}(\Gamma_c)). \]

Lemma 5.1. Consider the linear coupled Stokes-elasticity system

\[ v_t - \Delta v + \nabla q = f \quad \text{in} \quad \Omega_f \times (0, T), \]  
\[ \text{div} v = g \quad \text{in} \quad \Omega_f \times (0, T), \]  
\[ w_{tt} - \text{div} l_{\nabla w} A(x, t) + w = \tilde{f}, \quad \text{in} \quad \Omega_e \times (0, T) \]

with the boundary conditions

\[ w_{A_{\nu}} = \gamma^{-1}(v - w_t) - \tilde{h} \quad \text{on} \quad \Gamma_c \times (0, T), \]  
\[ \nabla v \cdot \nu - q = w_{A_{\nu}} + \tilde{h} \quad \text{on} \quad \Gamma_c \times (0, T), \]  
\[ v = 0, \quad \text{in} \quad \Gamma_f \times (0, T), \]

where \( w_{A_{\nu}} = (l_{\nabla w} A(x, t)) \nu). \]
Assume that $v_0 \in \mathbf{V} \cap H^4(\Omega_f), v_t(0) \in \mathbf{V} \cap H^2(\Omega_f), v_{tt}(0) \in \mathbf{H}, w_0 \in H^4(\Omega_c), w_1 \in H^3(\Omega_c)$ is subject to the compatibility conditions (141)-(145) and $A(x, t) \in C^\infty(\Omega_c \times (0, T]; T^2(\mathbb{R}^3))$ satisfying

$$A(x, t)(\xi \otimes \eta, \xi \otimes \eta) \geq \varsigma |\xi|^2|\eta|^2, \quad \text{for some} \quad \varsigma > 0, \forall \xi, \eta \in \mathbb{R}^3,$$

$f$ with the regularity conditions (145), $g$ with $g(0) = 0$ and (146), $\hat{f}$ satisfying (147), $\bar{h}, \hat{h}$ with regularity conditions (148) for some $T > 0$. Then there exists a unique solution $(v, w, q)$ satisfying

$$v \in L^\infty([0, T]; H^4(\Omega_f)); \quad v_t \in L^\infty([0, T]; H^3(\Omega_f));$$
$$v_{tt} \in L^\infty([0, T]; H^2(\Omega_f)); \quad$$
$$v_{ttt} \in L^\infty([0, T]; L^2(\Omega_f)), \quad \nabla v_{ttt} \in L^2([0, T]; L^2(\Omega_f));$$
$$\partial_t^k w \in C([0, T]; H^{4-k}(\Omega_c), \quad k = 0, 1, 2, 3, 4;$$
$$\partial_t^j q \in L^\infty([0, T]; H^{3-j}(\Omega_f))(j = 0, 1, 2),$$

Note that the initial pressure $q_0$ solves the elliptic system:

$$\begin{cases}
\Delta q_0 = -\partial_t v_0^h \partial_k v_0 + \text{div } f(0) & \text{in } \Omega_f, \\
\nabla q_0 \cdot \nu = \Delta v_0 \cdot \nu + f(0)\nu & \text{on } \Gamma_f, \\
-q_0 = -\partial_t v_0^h \nu_t + w_0^h \nu_t + h(0)\nu & \text{on } \Gamma_c.
\end{cases}$$

The proof of Lemma 5.1 is similar to that of Lemma 6.1 in [8], so we omit the proof of this lemma. But we still give the sketch of it.

First, we decompose the fluid part of the system in Lemma 5.1. Let $v = u + z$ and $z = z_1 + z_2 + z_3$. Following Lemma 6.1 in [8], $z_1$ satisfies

$$\begin{cases}
-\Delta z_1 + \nabla q_1 = 0 & \text{in } \mathbb{R}^3 \times (0, T), \\
\text{div } z_1 = E\text{gin} & \mathbb{R}^3 \times (0, T),
\end{cases}$$

$z_2$ satisfies the system

$$\begin{cases}
-\Delta z_2 + \nabla q_2 = 0 & \text{in } \Omega_f \times (0, T), \\
\text{div } z_2 = 0 & \text{in } \Omega_f \times (0, T), \\
z_2 = -z_1 + \lambda(t)\psi & \text{on } \Gamma_c \times (0, T), \\
z_2 = -z_1 & \text{on } \Gamma_f \times (0, T),
\end{cases}$$

and $z_3$ satisfies

$$\begin{cases}
(z_3)_t - \Delta z_3 + \nabla q_3 = -(z_1)_t - (z_2)_t & \text{in } \Omega_f \times (0, T), \\
\text{div } z_3 = 0 & \text{in } \Omega_f \times (0, T), \\
z_3 = 0 & \text{on } \Gamma_c \cup \Omega_f \times (0, T), \\
z_3(0) = 0 & \text{in } \Omega_f,
\end{cases}$$

where $E : H^s(\Omega_f) \to H^s(\mathbb{R}^3)$ is the continuous extension operator for $s = -1, 0, ..., 4$, $\lambda(t) = \int_{\Gamma_c} z_1(., t) \cdot \nu d\sigma - \int_{\Gamma_f} z_1(., t) \cdot \nu d\sigma$ for $t \in (0, T)$, and $\psi$ is a smooth compactly supported function on $\Gamma_c$ such that $\psi \geq 0$ with $\int_{\Gamma_c} \psi \cdot \nu d\sigma = 1$.

Thus, $(z, \bar{q})$ satisfies the Stokes system

$$\begin{cases}
z_t - \Delta z + \nabla \bar{q} = 0 & \text{in } \Omega_f \times (0, T), \\
\text{div } z = g & \text{in } \Omega_f \times (0, T), \\
z = \lambda(t)\psi & \text{on } \Gamma_c \times (0, T), \\
z = 0 & \text{on } \Gamma_f \times (0, T), \\
z(0) = 0 & \text{in } \Omega_f.
\end{cases}$$
By the regularity of all above equations, we have that
\[ z \in L^\infty([0, T]; H^4(\Omega_f)), \quad z_t \in L^\infty([0, T]; H^3(\Omega_f)), \quad z_{tt} \in L^\infty([0, T]; H^2(\Omega_f)), \quad z_{ttt} \in L^2([0, T]; H^1(\Omega_f)) \]

Then we turn to the divergence-free linear Stokes-elasticity system satisfied by \((u, q, w)\)
\[
\begin{align*}
    u_t - \Delta u + \nabla q &= f \quad \text{in} \quad \Omega_f \times (0, T), \\
    \text{div} \, u &= 0 \quad \text{in} \quad \Omega_f \times (0, T), \\
    w_{tt} - \text{div} \, v w A(x, t) + w &= \tilde{f}, \quad \text{in} \quad \Omega_c \times (0, T)
\end{align*}
\]
with the boundary conditions
\[
\begin{align*}
    w_{\nu} &= \gamma^{-1}(u + z - w_t) - \tilde{h} \quad \text{on} \quad \Gamma_c \times (0, T), \\
    \nabla u \cdot \nu - q \nu &= w_{\nu} + \tilde{h} \quad \text{on} \quad \Gamma_c \times (0, T), \\
    u &= 0, \quad \text{in} \quad \Gamma_f \times (0, T),
\end{align*}
\]
where \(\tilde{f} = f + \nabla \bar{q}\) and \(\tilde{h} = h - \nabla z \cdot \nu\). We apply the Galerkin’s method to the above system and use the estimates we obtain in Lemma 2.2 and elliptic estimates for wave equations as in [8]. Then we establish the wellposedness for the system in Lemma 5.1 and attain the solutions with the desired regularity.

Next, we consider the case of given time-dependent matrix \(a(x, t)\) with smooth coefficients \(a^{ik} \in C^\infty(\Omega_c \times [0, T])\) and nonlinear elasticity equations. We assume that \(a(x, t)\) is a small perturbation of the identity matrix satisfying
\[
\begin{align*}
    \|a - I\|^2_H \leq \epsilon, \quad \|\partial_t (a - I)\|^2_H \leq \epsilon, \quad \|\partial_{tt} (a - I)\|^2_H \leq \epsilon, \quad \|\partial_{ttt}(a - I)\|^2_H \leq \epsilon, \\
    \|\partial_{tt}(a - I)\|^2_H \leq \epsilon, \quad \|\partial_{ttt}(a - I)\|^2_H \leq \epsilon,
\end{align*}
\]
and
\[
\begin{align*}
    \|a : a^T - I\|^2_H \leq \epsilon, \quad \|\partial_t (a : a^T - I)\|^2_H \leq \epsilon, \quad \|\partial_{tt} (a : a^T - I)\|^2_H \leq \epsilon, \quad \|\partial_{ttt}(a : a^T - I)\|^2_H \leq \epsilon, \\
    \|\partial_{tt}(a : a^T - I)\|^2_H \leq \epsilon, \quad \|\partial_{ttt}(a : a^T - I)\|^2_H \leq \epsilon,
\end{align*}
\]
for all \(t \in [0, T]\) with \(T\) sufficiently small. In particular, the ellipticity condition \(a^{ij} a^{kl} \xi_{ij} \xi_{lk} \geq \frac{1}{c} |\xi|^2\) holds for \(\xi \in \mathbb{R}^{3 \times 3}\) and \(t \in [0, T]\).

We use a fixed point argument for the linear perturbed system and establish the iteration as follows:
\[
\begin{align*}
    v_t^{(n+1)} - \Delta v^{(n+1)} + \nabla q^{(n+1)} &= f^{(n)} \quad \text{in} \quad \Omega_f \times (0, T), \quad (155) \\
    \text{div} \, v^{(n+1)} &= 0 \quad \text{in} \quad \Omega_f \times (0, T), \quad (156) \\
    w_{tt}^{(n+1)} - \text{div} \, v w^{(n+1)} D^2 W(1) + w^{(n+1)} &= \tilde{f}^{(n)} \quad \text{in} \quad \Omega_c \times (0, T) \quad (157)
\end{align*}
\]
with the boundary conditions
\[
\begin{align*}
    w_{\nu}^{(n+1)} &= \gamma^{-1}(v^{(n+1)} - w_t^{(n+1)}) - \tilde{h}^{(n)} \quad \text{on} \quad \Gamma_c \times (0, T), \quad (158) \\
    \nabla v_t^{(n+1)} \cdot \nu - q^{(n+1)} \nu &= w_{\nu}^{(n+1)} + \tilde{h}^{(n)} \quad \text{on} \quad \Gamma_c \times (0, T), \quad (159) \\
    v^{(n+1)} &= 0, \quad \text{in} \quad \Gamma_f \times (0, T), \quad (160)
\end{align*}
\]
where

\[ f_i^{(n)} = -\partial_j((\delta_{jk} - a^{jk})\partial_k v_i^{(n)}) + \partial_k((\delta_{jk} - a^{jk})q^{(n)}), \]

\[ g_j^{(n)} = (\delta_{jk} - a^{jk})\partial_k v_j^{(n)}, \]

\[ \tilde{f}^{(n)} = \text{div} (\nabla w^{(n)}(\int_0^1 D^2W(\| + s \nabla w^{(n)}) - D^2W(\|)ds), \]

\[ h_i^{(n)} = (\delta_{jk} - a^{jk})\partial_k v_i^{(n)}v_j - (\delta_{jk} - a^{jk})q^{(n)}v_k + \tilde{h}_i^{(n)}, \]

\[ \tilde{h}_i^{(n)} = (I_{\nabla w^{(n)}}(\int_0^1 D^2W(\| + s \nabla w^{(n)}) - D^2W(\|)ds))\nu, \]

for \( i, j, k, l = 1, 2, 3. \)

As in the proof of Lemma 5.1, we change variables \( u^{(n+1)} = v^{(n+1)} - z^{(n+1)}, \)
where \((z^{(n+1)}, q^{(n+1)})\) satisfies the Stokes system

\[
\begin{cases}
  z_t^{(n+1)} - \Delta z^{(n+1)} + \nabla q^{(n+1)} = 0 & \text{in } \Omega_f \times (0, T), \\
  \text{div} z^{(n+1)} = g^{(n)} & \text{in } \Omega_f \times (0, T), \\
  z^{(n+1)} = \lambda(t)^{(n)}\psi^{(n)} & \text{on } \Gamma_c \times (0, T), \\
  z^{(n+1)} = 0 & \text{on } \Gamma_f \times (0, T), \\
  z^{(n+1)}(0) = 0 & \text{in } \Omega_f.
\end{cases}
\]

where \( z^{(n+1)} = z_1^{(n+1)} + z_2^{(n+1)} + z_3^{(n+1)}, \lambda(t)^{(n)}\) and \( \psi^{(n)} \) are defined as in the
Lemma 5.1. Therefore, \( u^{(n+1)} \) satisfies the linear divergence-free Stokes-elasticity system

\[
\begin{align*}
  u_t^{(n+1)} - \Delta u^{(n+1)} + \nabla q^{(n+1)} &= f^{(n)} + \nabla q^{(n+1)} & \text{in } \Omega_f \times (0, T), \\
  \text{div} u^{(n+1)} &= 0 & \text{in } \Omega_f \times (0, T), \\
  w_{tt}^{(n+1)} - \text{div} (\nabla w^{(n+1)}D^2W(\|) + \nu^{(n+1)}) &= \tilde{f}^{(n)} & \text{in } \Omega_c \times (0, T),
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
  w_{tt}^{(n+1)} &= \gamma^{-1}(u^{(n+1)} + z^{(n+1)} - w_t^{(n+1)}) - \tilde{h}^{(n)} & \text{on } \Gamma_c \times (0, T), \\
  \nabla u^{(n+1)} \cdot \nu - q^{(n+1)}\nu &= w_{tt}^{(n+1)} + h^{(n)} + \nabla z^{(n+1)} \cdot \nu & \text{on } \Gamma_c \times (0, T), \\
  u^{(n+1)} &= 0 & \text{in } \Gamma_f \times (0, T).
\end{align*}
\]

According to the proof of Lemma 5.1, we obtain the existence and uniqueness of
\( z^{(n+1)} \) with

\[
\begin{align*}
  z^{(n+1)} &\in L^\infty([0, T]; H^4(\Omega_f)), \quad z_t^{(n+1)} \in L^\infty([0, T]; H^3(\Omega_f)), \\
  z_{tt}^{(n+1)} &\in L^\infty([0, T]; H^2(\Omega_f)), \quad z_{ttt}^{(n+1)} \in L^2([0, T]; H^1(\Omega_f)).
\end{align*}
\]

By fixed point argument, we conclude the construction of solutions of system
coupled by nonlinear elasticity equations and Navier-Stokes equations in Lagaragian
form with given lagaragian matrix \( a(x, t) \) by the following auxiliary assertion.
Before we state the assertion, we define a function space:

\[
\begin{align*}
  X(T) &= \{(v, w, q) : v \in L^\infty([0, T]; H^4(\Omega_f)); \quad v_t \in L^\infty([0, T]; H^3(\Omega_f)); \\
  &v_{tt} \in L^\infty([0, T]; H^2(\Omega_f)); v_{ttt} \in L^\infty([0, T]; L^2(\Omega_f)), \\
  &\nabla v_{ttt} \in L^2([0, T]; L^2(\Omega_f)); \\
  &\partial_t^k w \in C([0, T]; H^{4-k}(\Omega_c)), \quad k = 0, 1, 2, 3, 4; \\
  &\partial_t^j q \in L^\infty([0, T]; H^{3-j}(\Omega_f))(j = 0, 1, 2). \}
\end{align*}
\]
The norm of \(X(T)\) is defined by \(\| (v, w, q) \|_{X(T)}^2 = \| v \|_{L^\infty H^1}^2 + \| v_t \|_{L^\infty H^3}^2 + \| v_{tt} \|_{L^2}^2 + \| v_{ttt} \|_{L^2}^2 + \| \nabla v_{ttt} \|_{L^2}^2 + \sum_{k=0}^4 \| \partial^k_t w \|_{L^2([0,T];H^{k-5}(\Omega_r))}^2 + \sum_{j=0}^2 \| \partial^j q \|_{L^2 H^{j-3}}^2\).

Moreover, by (155)-(166), we define map
\[
\Gamma: (v^{(n)}, w^{(n)}, q^{(n)}) \to (v^{(n+1)}, w^{(n+1)}, q^{(n+1)}),
\]
which will be proved to be contractive map in the function space \(X(T, R)\) in the following lemma.

**Lemma 5.2.** Assume the initial data \((v_0, w_0, w_1)\) is small, that is
\[
\| v_0 \|_{H^1}, \| v_t(0) \|_{H^2}, \| v(0) \|_{H^1}, \| v_{tt}(0) \|_{L^2}, \| w_0 \|_{H^1}, \| w_1 \|_{H^3} \leq \bar{\varepsilon},
\]
where \(\bar{\varepsilon} > 0\) is a small parameter. Then, the map
\[
\Gamma: (v^{(n)}, w^{(n)}, q^{(n)}) \to (v^{(n+1)}, w^{(n+1)}, q^{(n+1)})
\]
has fixed point in the Banach space \(X(T, R)\), where
\[
X(T, R) = \{ (v, w, q) \in X(T) : \| (v, w, q) \|_{X(T)}^2 \leq R v(0) = v_0, v_t(0) \in V \cap H^2(\Omega_T), \ v_{tt}(0) \in H, v_t(0) = w(0), w_t(0) = w_1 \}
\]
and \(T, R\) both are sufficiently small positive numbers, which depend on initial data and \(a(x, t)\).

**Remark 3.** By Proposition 3.2.1. and Theorem 1.7.4. in [1], we can infer that \(X(T, R) \neq \emptyset\) and \(X(T, R)\) is a Banach space.

**Proof.** We begin the iteration by arbitrarily select an element in the space \(X(T, R)\), and denote it by \((v^{(0)}, w^{(0)}, q^{(0)})\). To obtain the result of this lemma, we need to prove that the mapping \(\Gamma\) maps from \(X(T, R)\) to \(X(T, R)\) and is a contraction in the norm of \(X(T)\). By Lemma 5.1, we can infer that
\[
(v^{(n+1)}, w^{(n+1)}, q^{(n+1)}) = \Gamma(v^{(n)}, w^{(n)}, q^{(n)}) \in X(T),
\]
if \((v^{(n)}, w^{(n)}, q^{(n)}) \in X(T)\). To finish the proof, we need the following estimates.

First, we consider the four energy level estimates for the system (161)-(166). For the first level energy \(V_0^{(n+1)} = \frac{1}{2} (\| u^{(n+1)} \|_{L^2}^2 + \| u_t^{(n+1)} \|_{L^2}^2 + \| u_{tt}^{(n+1)} \|_{L^2}^2 + \int_{\Omega_r} (\nabla u^{(n+1)} \cdot \nabla w^{(n+1)}(t) + (Dw + 3Dw)_{\mathbb{R}^{3 \times 3}}) dx)\), we have
\[
\begin{align*}
V_0^{(n+1)} + \int_0^t D_0^{(n+1)}(\tau) d\tau = & \ V_0^{(n+1)}(0) + \int_0^t (f^{(n)} + \nabla q^{(n+1)}, u^{(n+1)}) d\tau \\
& + \int_0^t (\tilde{f}^{(n)}, w_t^{(n+1)}) d\tau - \int_0^t \int_{\Gamma_r} (h^{(n)} + \nabla z^{(n+1)} \cdot \nu, u^{(n+1)}) dxd\tau \\
& + \int_0^t \int_{\Gamma_r} (w^{(n+1)} \Delta w, z^{(n+1)} - \gamma \cdot \tilde{h}^{(n)}) d\sigma d\tau, \\
\end{align*}
\]
(167)
which gives
\[
V_0^{(n+1)} + \int_0^t D_0^{(n+1)}(\tau)d\tau \leq V_0^{(n+1)}(0) + C \int_0^t \| (a : a^T - I) \nabla v^{(n)} \|_{L^2}^2 d\tau \\
+ C \int_0^t \| |\tau| (a - I) \nabla v^{(n)} \|_{L^2}^2 d\tau + C \int_0^t \| (a - I) q^{(n)} \|_{L^2}^2 d\tau \\
+ C \int_0^t \| w^{(n)} \|_{H^3}^4 d\tau + \frac{1}{2} \int_0^t \| w^{(n)} \|_{H^3}^2 d\tau \\
\leq V_0^{(n+1)}(0) + C \epsilon \int_0^T \| \nabla v^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^2}^2 d\tau + C \int_0^T \| w^{(n)} \|_{H^3}^4 d\tau e^\epsilon \epsilon^\epsilon (\epsilon^2 - 1) dt. 
\] (168)

By Gronwall’s inequality, we obtain
\[
V_0^{(n+1)} \leq (V_0^{(n+1)}(0) + C \epsilon \int_0^T \| \nabla v^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^2}^2 d\tau + C \int_0^T \| w^{(n)} \|_{H^3}^4 d\tau) e^{\epsilon^\epsilon (\epsilon^2 - 1) dt}. 
\] (169)

Then, we submit (169) into (168), which leads to
\[
V_0^{(n+1)} + \int_0^t D_0^{(n+1)}(\tau)d\tau \leq e^\epsilon V_0^{(n+1)}(0) \\
+ C \epsilon \int_0^t \| \nabla v^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^2}^2 d\tau \\
+ C \int_0^t \| w^{(n)} \|_{H^3}^4 \cdot \| w^{(n)} \|_{H^2}^2 + \| w^{(n)} \|_{H^2}^2 \cdot \| w^{(n)} \|_{H^3}^2 (\sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4}^2)^2) d\tau \\
+ C (e^\epsilon^\epsilon - 1) I_2, 
\] (170)

where
\[
I_2 = \epsilon \int_0^T \| \nabla v^{(n)} \|_{L^2}^2 + \| \nabla v^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^2}^2 + \int_0^T \| w^{(n)} \|_{H^3}^4 \cdot \| w^{(n)} \|_{H^2}^2 + \| w^{(n)} \|_{H^2}^2 \cdot \| w^{(n)} \|_{H^3}^2 (\sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4}^2)^2) d\tau; \\
V_2^{(n+1)} + \int_0^t D_2^{(n+1)}(\tau)d\tau \leq e^\epsilon V_0^{(n+1)}(0) \\
+ C \epsilon \int_0^t \| \nabla v^{(n)} \|_{L^6}^6 + \| \nabla v^{(n)} \|_{L^6}^6 + \| \nabla v^{(n)} \|_{L^6}^6 + \| \nabla v^{(n)} \|_{L^6}^6 + \| q^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^2}^2 d\tau 
\] (172)
\[ + C \int_0^t \| w^{(n)} \|_{H^3}^2 \cdot \| w^{(n)} \|_{H^2}^2 + \| w^{(n)} \|_{H^3}^4 \cdot (1 + \| w^{(n)} \|_{H^4})^2 (1 + \| w^{(n)} \|_{H^3}^2) \]
\[ + \| w^{(n)} \|_{H^2}^2 \| w^{(n)} \|_{H^3}^2 \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4})^2 d\tau \]
\[ + C(e^t - 1)(I_{31} + C_1), \]
where \( I_{31} = \int_0^T \| \nabla v^{(n)} \|_{L^6}^2 + \| \nabla v^{(n)} \|_{L^2}^2 + \| \nabla v^{(n)} \|_{L^2}^2 + \| q^{(n)} \|_{L^6}^2 + \| q^{(n)} \|_{L^2}^2 + \| u^{(n)} \|_{L^2}^2 d\tau \)
and \( I_{32} = \int_0^T \| \nabla w^{(n)} \|_{H^3}^2 \cdot \| w^{(n)} \|_{H^2}^2 + \| w^{(n)} \|_{H^1}^4 \cdot (1 + \| w^{(n)} \|_{H^4})^2 (1 + \| w^{(n)} \|_{H^3}^2) + \| w^{(n)} \|_{H^2}^2 \| w^{(n)} \|_{H^3}^2 \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4})^2 d\tau \) and
\[ V_3^{(n+1)} + \int_0^t \| \nabla w^{(n+1)} \|_{L^2}^2 d\tau \leq V_3^{(n+1)}(0) \] (173)
\[ - \int_0^t \int_{\Omega} \langle (\nu^{(n+1)} \nu^{(n+1)})_{\nu^{(n+1)}} + \nu^{(n+1)} \cdot \nu^{(n+1)} \rangle d\nu^{(n+1)} d\sigma^{(n+1)} \]
\[ + \int_0^t \| \nabla w^{(n+1)} + \nu^{(n+1)} + \nabla v^{(n+1)} \|_{L^2}^2 d\tau + \int_0^t \gamma^{(n+1)} \| u^{(n+1)} + \nu^{(n+1)} \|_{L^2}^2 d\tau \]
\[ - \int_0^t \int_{\Omega} \langle \nu^{(n+1)} \rangle \int_1^1 D^2 W(\| + s \nabla w^{(n)}) - D^2 W(\nu^{(n)}) d\nu^{(n+1)} \rangle d\nu^{(n+1)} d\sigma^{(n+1)} \]
For (173), we use the divergence-free condition (109) as in the proof of Lemma 3.13 to deal with the third term on the right hand side. By integrating by parts in time, we have the estimates of the last term. Thus, we obtain
\[ V_3^{(n+1)} + \int_0^t \| \nabla w^{(n+1)} \|_{L^2}^2 d\tau \leq V_3^{(n+1)}(0) + C \int_0^t V_3^{(n+1)}(\tau) d\tau \]
\[ + C \int_0^t \| \nabla v^{(n)} \|_{L^2}^2 + \| \nabla v^{(n)} \|_{L^2}^2 + \| \nabla v^{(n)} \|_{L^2}^2 + \| \nabla v^{(n)} \|_{L^2}^2 d\tau \]
\[ + \| q^{(n)} \|_{L^6}^2 + \| q^{(n)} \|_{L^6}^2 + \| q^{(n)} \|_{L^6}^2 d\tau \]
\[ + C \| u^{(n+1)}(t) \|_{L^2}^2 + C \| \nabla w^{(n+1)}(t) \|_{L^2}^2 + \| \nabla v^{(n)} \|_{L^2}^2 + \| \nabla v^{(n)} \|_{L^2}^2 \]
\[ + C \| \nabla u^{(n+1)}(0) \|_{L^2}^2 + \| \nabla u^{(n+1)}(0) \|_{L^2}^2 + \| \nabla u^{(n+1)}(0) \|_{L^2}^2 d\tau \]
\[ + C \| \nabla u^{(n+1)}(0) \|_{L^2}^2 + \| \nabla u^{(n+1)}(0) \|_{L^2}^2 + \| \nabla u^{(n+1)}(0) \|_{L^2}^2 \]
\[ + C \int_0^t \| w^{(n)} \|_{H^3}^2 \cdot \| w^{(n)} \|_{H^2}^2 + \| w^{(n)} \|_{H^3}^2 \cdot \| w^{(n)} \|_{H^2}^2 \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4})^2 \]
\[ + \| w^{(n)} \|_{H^3}^6 (1 + \| w^{(n)} \|_{H^4})^2 + \| w^{(n)} \|_{H^3}^2 \cdot \| w^{(n)} \|_{H^2}^2 \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4})^2 \]
\[ + \| w^{(n)} \|_{H^2}^6 \| w^{(n)} \|_{H^3}^2 \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4})^2 \]
\[ + \| w^{(n)} \|_{H^3}^6 \| w^{(n)} \|_{H^3}^2 \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^4})^2 d\tau \]
\[ + \int_0^t \int_{\Omega} \langle \nu^{(n+1)} \rangle \int_1^1 D^2 W(\| + s \nabla w^{(n)}) - D^2 W(\nu^{(n)}) d\nu^{(n+1)} \rangle d\nu^{(n+1)} d\sigma^{(n+1)} \]
and the estimates of the last term of (174)

$$\left| \int_0^t \int_{\Omega_x} \langle l \nabla w^{(n)} \rangle \left( \int_0^1 D^2 W(\mathbb{I} + s \nabla w^{(n)}) - D^2 W(\mathbb{I}) ds \right), \nabla w^{(n+1)}_{ittt} \right| dx \, dt$$

$$(174)$$

$$\leq \int_0^t \int_{\Omega_x} \langle l \nabla w^{(n)} \rangle \left( \int_0^1 D^2 W(\mathbb{I} + s \nabla w^{(n)}) ds \right), \nabla w^{(n+1)}_{ittt} \right| dx \, dt$$

$$+6 \int_0^t \int_{\Omega_x} \langle l \nabla w^{(n)} \rangle \left( \int_0^1 sD^3 W(\mathbb{I} + s \nabla w^{(n)}) ds \right), \nabla w^{(n+1)}_{ittt} \right| dx \, dt$$

$$+3 \int_0^t \int_{\Omega_x} \langle l \nabla w^{(n)} \rangle \left( \int_0^1 s^2 D^4 W(\mathbb{I} + s \nabla w^{(n)}) ds \right), \nabla w^{(n+1)}_{ittt} \right| dx \, dt$$

$$+| \int_0^t \int_{\Omega_x} \langle l \nabla w^{(n)} \rangle \left( \int_0^1 s^3 D^5 W(\mathbb{I} + s \nabla w^{(n)}) ds \right), \nabla w^{(n+1)}_{ittt} \right| dx \, dt|$$

$$\leq C \| w^{(n)}(t) \|_{H^3} \| \nabla w^{(n+1)}_{ittt}(t) \|_{L^2}^2 + C \| w_0 \|_{H^3} \| \nabla w^{(n+1)}_{ittt}(0) \|_{L^2}^2$$

$$+C \int_0^t \| w_i^{(n)} \|_{H^3}^2 \left( \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^3}^2) \right) \| \nabla w^{(n+1)}_{ittt}(t) \|_{L^2}^2 dx \, dt$$

$$(175)$$

$$\leq C \| w_i^{(n)}(t) \|_{H^3} \left( \sum_{j=1}^2 (1 + \| w^{(n)}(t) \|_{H^3}^2) \right) \| \nabla w^{(n+1)}_{ittt}(t) \|_{L^2}^2 dx \, dt$$

$$+C \| w_1 \|_{H^3} \left( \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^3}^2) \right) \| \nabla w^{(n+1)}_{ittt}(0) \|_{L^2}^2 dx \, dt$$

$$+C \int_0^t \| w_i^{(n)} \|_{H^3}^2 \left( \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^3}^2) \right) \| \nabla w^{(n+1)}_{ittt} \|_{L^2}^2 dx \, dt$$

$$+C \int_0^t \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^3}^2) \| \nabla w^{(n+1)}_{ittt} \|_{L^2}^2 dx \, dt$$

$$(176)$$

$$\leq C \int_0^t \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^3}^2) \| w_i^{(n)} \|_{H^3} \| \nabla w^{(n+1)}_{ittt} \|_{L^2}^2 dx \, dt$$

$$+C \int_0^t \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^3}^2) \| w_i^{(n)} \|_{H^3} \| \nabla w^{(n+1)}_{ittt} \|_{L^2}^2 dx \, dt$$

$$+C \int_0^t \sum_{j=1}^2 (1 + \| w^{(n)} \|_{H^3}^2) \| w_i^{(n)} \|_{H^3} \| \nabla w^{(n+1)}_{ittt} \|_{L^2}^2 dx \, dt$$
\[
\begin{align*}
&\leq C \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|w_i^{(n)}(t)\|_{H^3}^2 \|\nabla w_t^{(n)}(t)\|_{L^2} \|\nabla w_{ttt}^{(n+1)}(t)\|_{L^2} \\
&+ C \sum_{j=1}^{2} (1 + \|w_0\|_{H^3}^j) \|w_1\|_{H^3}^2 \|\nabla w_1\|_{L^2} \|\nabla w_{ttt}^{(n+1)}(0)\|_{L^2} \\
&+ C \int_0^t \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|w_t^{(n)}\|_{H^3}^3 \|\nabla w_t^{(n)}\|_{L^2} \|\nabla w_{ttt}^{(n+1)}\|_{L^2} dt \right) \\
&+ C \int_0^t \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|w_t^{(n)}\|_{H^3}^2 \|\nabla w_{ttt}^{(n+1)}\|_{L^2} dt, \right) \\
&\leq C \|w^{(n)}(t)\|_{H^3} \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|\nabla w_{ttt}^{(n+1)}(t)\|_{L^2} \|\nabla w_{ttt}^{(n+1)}(0)\|_{L^2} \\
&+ C \|w_0\|_{H^3} \left( \sum_{j=1}^{2} (1 + \|w_0\|_{H^4}^j) \|\nabla w_{ttt}^{(n+1)}(0)\|_{L^2} \right) \\
&+ C \int_0^t \|w_t^{(n)}\|_{H^3}^2 \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|\nabla w_{ttt}^{(n+1)}\|_{L^2} dt \right) \\
&+ C \int_0^t \|w_t^{(n)}\|_{H^3} \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|\nabla w_{ttt}^{(n+1)}\|_{L^2} dt \right) \\
&\leq C \|w^{(n)}(t)\|_{H^3} \|w_t^{(n)}(t)\|_{H^3} \left( \sum_{j=1}^{2} (1 \\
&+ \|w^{(n)}(t)\|_{H^4}^j) \|\nabla w_{ttt}^{(n+1)}(t)\|_{L^2} \right) \\
&+ C \|w_0\|_{H^3} \|w_t^{(n)}(t)\|_{H^3} \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|\nabla w_{ttt}^{(n+1)}(0)\|_{L^2} \\
&+ C \int_0^t \|w^{(n)}(t)\|_{H^3} + 1 \|w_t^{(n)}\|_{H^3} \left( \sum_{j=1}^{2} (1 \\
&+ \|w^{(n)}(t)\|_{H^4}^j) \|\nabla w_{ttt}^{(n+1)}\|_{L^2} d\tau \right) \\
&+ C \int_0^t \left( \sum_{j=1}^{2} (1 + \|w^{(n)}(t)\|_{H^4}^j) \|w_t^{(n)}\|_{H^3} \|\nabla w_{ttt}^{(n)}\|_{L^2} \\
&+ \|\nabla w_{ttt}^{(n)}\|_{L^2} \|\nabla w_{ttt}^{(n)}\|_{L^2} \right) \|\nabla w_{ttt}^{(n+1)}\|_{L^2} d\tau, \right) \\
\end{align*}
\]
\[ \nabla w_{ittt}^{(n+1)}(x, t) \leq C \| w^{(n)}(t) \|_{H^3} \| w_t^{(n)}(t) \|_{H^3}^2 \sum_{j=1}^{2} (1 + \| w^{(n)}(t) \|_{H^3})^j \| \nabla w_t^{(n+1)}(t) \|_{L^2} \]

\[ + C \int_0^t \| w_t^{(n)}(t) \|_{H^3}^2 (1 + \| w^{(n)}(t) \|_{H^3})^j \| \nabla w_t^{(n+1)}(t) \|_{L^2} \, dt \]

\[ + \int_0^t \| u_t^{(n+1)}(t) \|_{L^2}^2 + \| \nabla u_t^{(n+1)}(t) \|_{L^2}^2 + \| \nabla u_t^{(n+1)}(t) \|_{L^6}^2 + \| \nabla v_t^{(n)}(t) \|_{L^2}^2 + \| \nabla u_t^{(n)}(t) \|_{L^6}^2 + \| \nabla q_t^{(n)}(t) \|_{L^2}^2 + \| \nabla q_t^{(n)}(t) \|_{L^6}^2 \, dt \]

Thus, we combine (174)-(180) and use Gronwall’s inequality as above estimates. This leads to that

\[ V_3^{(n+1)} + \int_0^t \| \nabla u^{(n+1)}_{tttt}(t) \|_{L^2}^2 \, dt \leq e^{Ct} V_3^{(n+1)}(0) + C(e^{Ct} - 1) I_4 \]

\[ + \tilde{P}_1(\| u^{(n)}(t) \|_{H^4}, \| u_t^{(n)}(t) \|_{H^3}, \| u_{tt}^{(n)}(t) \|_{H^2}, \| u_{ttt}^{(n)}(t) \|_{H^1}) \]

\[ + \tilde{P}_1(\| u^{(n)}(t) \|_{H^4}, \| u_t^{(n)}(t) \|_{H^3}, \| u_{tt}^{(n)}(t) \|_{H^2}, \| u_{ttt}^{(n)}(t) \|_{H^1})(0) \]

\[ + I_4 \int_0^t \| \nabla u^{(n+1)}_{tttt}(t) \|_{L^2}^2 + \| \nabla u^{(n+1)}_{ttt}(t) \|_{L^2}^2 + \| \nabla u^{(n+1)}_{tt}(t) \|_{L^2}^2 + \| \nabla u^{(n+1)}_{t}(t) \|_{L^2}^2 + \| \nabla u^{(n+1)}(t) \|_{L^2}^2 \, dt \]

where \( I_4 \) is defined similarly as above and \( \tilde{P}_1, \tilde{P}_2 \) are polynomials with the degree of each term of it is at least 2. The last term of (181) can be absorbed in the dissipation term \( \int_0^t \| \nabla u^{(n+1)}_{tttt} \|_{L^2}^2 \, dt \), \( i = 0, 1, 2 \) and \( \int_0^t \| | \nabla u^{(n+1)}_{tttt} \|_{L^2}^2 \). Moreover, (126) and (127) are needed. Next, by Lemma 2.2, we have the following pointwise stokes estimates:

\[ \begin{align*}
\| u^{(n+1)} \|_{H^{s+2}} + \| q^{(n+1)} \|_{H^{s+1}} &\leq C \| f^{(n)} + \nabla q^{(n+1)} \|_{H^s} + C \| u_t^{(n+1)} \|_{H^s} \\
+ C \| w_t^{(n+1)} \|_{H^{s+\frac{1}{2}}} &+ C \| u_t^{(n+1)} \|_{H^{s+\frac{1}{2}}(\Gamma_\infty)} + C \| \nabla z^{(n+1)} \cdot \nu \|_{H^{s+\frac{1}{2}}(\Gamma_\infty)}
\end{align*} \]

\[ s = 0, 1, 2 \]
for $t \in (0, T)$,
\[
\|u^{(n+1)}_t\|_{H^{s+2}} + \|u^{(n+1)}_t\|_{H^{s+1}} \leq C \|u^{(n+1)}_{tt}\|_{H^s} + C\|f^{(n)}_t + \nabla \tilde{q}^{(n+1)}\|_{H^s} + C\|\tilde{z}^{(n+1)}_{tt} + \mu \|_{H^{s+\frac{1}{2}}} (\Gamma_e),
\]
\[
= 0, 1
\]

for $t \in (0, T)$, and
\[
\|u^{(n+1)}_{tt}\|_{H^1} + \|u^{(n+1)}_{tt}\|_{H^1} \leq C\|u^{(n+1)}_{tt}\|_{L^2} + C\|f^{(n)}_t + \nabla \tilde{q}^{(n+1)}\|_{L^2} + C\|\tilde{z}^{(n+1)}_{tt} + \mu \|_{H^{s+\frac{1}{2}}} (\Gamma_e),
\]
\[
\|h^{(n)}_{tt}\|_{H^{s+\frac{1}{2}}} (\Gamma_e) \leq C\|\mathbf{a} : \mathbf{a}^T - I : \nabla v^{(n)}\|_{H^{s+1}} + C\|\mathbf{a} - I\| q^{(n)}\|_{H^{s+1}} + C\|w^{(n)}\|_{H^{s+2}} + \|w^{(n)}\|_{H^2}.
\]

By similar arguments to above estimates, we obtain the first-order time differentiated estimates for $s = 0, 1, 2$
\[
\|f^{(n)}_t + \nabla \tilde{q}^{(n+1)}\|_{H^1} + \|\tilde{z}^{(n+1)}_{tt} + \mu \|_{H^{s+\frac{1}{2}}} (\Gamma_e) \leq C\|\nabla v^{(n)}\|_{H^{s+1}} + \|q^{(n)}\|_{H^{s+1}} + C\|w^{(n)}\|_{H^{s+2}} + \|w^{(n)}\|_{H^1}.
\]

and
\[
\|u^{(n+1)}_{tt}\|_{H^{s+\frac{1}{2}}} (\Gamma_e) \leq C\|u^{(n+1)}_{tt}\|_{H^{s+1}} + \|u^{(n+1)}_{tt}\|_{H^{s+1}} + C\|w^{(n)}\|_{H^{s+2}} + \|w^{(n)}\|_{H^1}.
\]

and second order time differentiated estimates
\[
\|f^{(n)}_{tt} + \nabla \tilde{q}^{(n+1)}_{tt}\|_{L^2} + \|\tilde{z}^{(n+1)}_{ttt} + \mu \|_{H^{s+\frac{1}{2}}} (\Gamma_e) \leq C\|\nabla v^{(n)}\|_{H^2} + \|q^{(n)}\|_{H^2} + C\|w^{(n)}\|_{H^1} + \|q^{(n)}\|_{H^1}.
\]
\[ \| H^{(n)}_{tt} \|_{H^1(\Gamma_\omega)} \leq C(\| \nabla v^{(n)} \|_{H^2} + \| q^{(n)} \|_{H^2} + \| \nabla \psi^{(n)}_t \|_{H^1} + \| \tilde{\psi}^{(n)}_t \|_{H^1} \\
 + \| \nabla v^{(n)}_{tt} \|_{H^1} + \| q^{(n)}_{tt} \|_{H^1}) + C\| w^{(n)}_{tt} \|_{H^2}w^{(n)}_{tt}[1 + \left( \sum_{j=1}^{2}(1 + \| w^{(n)} \|_{H^4})^2 \right)] \\
 + C\| w^{(n)}_t \|_{H^3}^2(1 + \| u^{(n)} \|_{H^4})(1 + \| u^{(n)} \|_{H^4}) \] (192)

and
\[ \| (w^{(n+1)}_{tt})_{x,t} \|_{H^{1/2}(\Gamma_\omega)} \leq C\gamma^{-1}(\| u^{(n+1)}_t \|_{H^1} + \| z^{(n+1)}_t \|_{H^1} + \| w^{(n+1)}_{tt} \|_{H^1}) \\
 + C\| w^{(n)} \|_{H^2}u^{(n)}_{tt}[1 + \left( \sum_{j=1}^{2}(1 + \| u^{(n)} \|_{H^4})^2 \right)] \\
 + C\| u^{(n)}_t \|_{H^3}^2(1 + \| u^{(n)} \|_{H^4})(1 + \| u^{(n)} \|_{H^4}) \] (193)

Here, we also employ the elliptic estimates of the elasticity system. We list them below.
\[ \| w^{(n+1)} \|_{H^{s+2}} \leq C\| \tilde{\psi}^{(n)}_t \|_{H^{s}} + C\| w^{(n+1)}_t \|_{H^s} + C\| w^{(n+1)}_{tt} \|_{H^s} \\
 + C\gamma^{-1}(\| u^{(n+1)}_t \|_{H^{s+1}} + \| z^{(n+1)}_t \|_{H^{s+1}} + \| u^{(n+1)}_{tt} \|_{H^{s+1}}) + C\| \tilde{\psi}^{(n)}_t \|_{H^{s+1/2}(\Gamma_\omega)}, \] for \( s = 0, 1, 2 \) (194)

for \( t \in (0, T) \),
\[ \| w^{(n+1)}_t \|_{H^{s+2}} \leq C\| \tilde{\psi}^{(n)}_t \|_{H^{s}} + C\| w^{(n+1)}_t \|_{H^s} + C\| w^{(n+1)}_{tt} \|_{H^s} \\
 + C\gamma^{-1}(\| u^{(n+1)}_t \|_{H^{s+1}} + \| z^{(n+1)}_t \|_{H^{s+1}} + \| u^{(n+1)}_{tt} \|_{H^{s+1}}) + C\| \tilde{\psi}^{(n)}_t \|_{H^{s+1/2}(\Gamma_\omega)}, \] for \( s = 0, 1 \) (195)

for \( t \in (0, T) \), and
\[ \| w^{(n+1)}_{tt} \|_{H^{s}} \leq C\| \tilde{\psi}^{(n)}_t \|_{L^2} + C\| w^{(n+1)}_t \|_{L^2} + C\| w^{(n+1)}_{tt} \|_{L^2} \\
 + C\gamma^{-1}(\| u^{(n+1)}_t \|_{H^{s}} + \| z^{(n+1)}_t \|_{H^{s}} + \| u^{(n+1)}_{tt} \|_{H^{s}}) + C\| \tilde{\psi}^{(n)}_t \|_{H^{s+1}(\Gamma_\omega)}, \] (196)

for \( t \in (0, T) \).

Therefore, combining all the estimates above, we can easily see that as \( T, R \) sufficiently small, in particular, \( R \leq \varepsilon^2 \), the map \( \Gamma : X(T, R) \rightarrow X(T, R) \) and is also a contractive map. Then, by the Banach fixed point theorem, we finish the proof of Lemma 5.2.

Hence, we have finished the construction of short-time solutions of the system coupled by nonlinear elasticity equations and Navier-Stokes equations in Lagrangian form with given Lagrangian matrix \( a(x, t) \).

Finally, following [10], we employ the retarded mollification technique to complete the construction of the short time solution. We define the retarded mollification as [10]:
\[ \psi_\delta(a^{(j)})(x, t) = \delta^{-4} \int_{\mathbb{R}^3} \psi\left( \frac{y}{\delta}, \frac{\tau}{\delta} \right) \tilde{a}^{(j)}(x - y, t - \tau) dy d\tau, \]
where \( \delta > 0 \) is a small parameter and \( \psi(x, t) \) is a smooth, compactly supported function with support inside \( \{ |x|^2 < t \} \times [1, 2] \) and \( \int_{\mathbb{R}^3} \psi dx dt = 1 \). The function \( \tilde{a} \) is the continuous extension of \( a \) by the identity for \( t < 0 \) and by any continuous extension into \( \Omega_x \) and the rest of \( \mathbb{R}^3 \). Note that the value of \( \psi_\delta(a^{(j)}) \) at time \( t \) depends on the values of \( a \) before \( t - \delta \).
For a fixed \( \delta > 0 \), we apply the same method with [10]. By Lemma 5.2, we solve this system on the interval \([0, \delta]\) with \( a = I \), which produces the linear Stokes-nonlinear elasticity problem with divergence-free condition. We then use the solution \( v^\delta \) to compute the corresponding \( \eta^\delta = \int_0^t v^\delta ds + x \) and the coefficient matrix \( a^\delta = (\nabla \eta^\delta)^{-1} \) on the interval. Thus we can define \( \psi_3(a^\delta) \) on the next interval \([\delta, 2\delta]\). In particular, we have that \( ||a^\delta - \mathbb{1}||_{H^3} \leq \epsilon \) by its construction and so is \( \psi_3(a^\delta) \). Hence the modified form \( \text{div}(a : a^T : \nabla \cdot) \) corresponding to the matrix \( \psi_3(a^\delta) \) is still uniformly elliptic with the same ellipticity constant as \( a^\delta \). Then, we apply Lemma 5.2 to extend \( v^\delta \) to the next interval \([\delta, 2\delta]\), using the predetermined modified matrix \( \psi_3(a^\delta) \). For the detail of this procedure, refer to [10].

By Lemma 5.2, we obtain a sequence of solutions \((v^\delta, w^\delta, q^\delta) \in X(T, R)\) for \( T, R \) independent of \( \delta \). Observe that the modified coefficient matrix \( \psi_3(a^\delta) \) satisfies the estimates of Lemma 2.1.

Therefore, by passing to an appropriate subsequence the a priori estimates imply that

\[
\frac{\partial_t}{\partial t} v^\delta \rightarrow \frac{\partial_t}{\partial t} v, \quad \text{weakly-star in } L^\infty([0, T]; H^{4-j}(\Omega_f)), \quad j = 0, 1, 2; \quad \\
\psi_{ttt} \rightarrow v_{ttt}, \quad \text{in } L^2([0, T]; H^1(\Omega_f)); \quad \\
\frac{\partial_t^k}{\partial t} w^\delta \rightarrow \frac{\partial_t^k}{\partial t} w, \quad \text{weakly-star in } C([0, T]; H^{4-j}(\Omega_r)), \quad k = 0, 1, 2, 3, 4; \quad \\
\frac{\partial_t^l}{\partial t} q^\delta \rightarrow \frac{\partial_t^l}{\partial t} q, \quad \text{weakly-star in } L^\infty([0, T]; H^{3-j}(\Omega_f)), \quad l = 0, 1, 2.\]

By Proposition 1.7.9 in [1], we have

\[
\frac{\partial_t^j}{\partial t} v^\delta \rightarrow \frac{\partial_t^j}{\partial t} v, \quad \text{in } C([0, T]; H^{3-j}(\Omega_f)), \quad j = 0, 1, 2; \quad \\
\frac{\partial_t^k}{\partial t} w^\delta \rightarrow \frac{\partial_t^k}{\partial t} w, \quad \text{in } C([0, T]; H^{3-j}(\Omega_r)), \quad k = 0, 1, 2; \quad \\
q^\delta \rightarrow q, \quad \text{in } C([0, T]; H^2(\Omega_f)).\]

According to the definition of \( \eta^\delta \) and \( \eta^\delta = v^\delta \), we have \( \eta^\delta \rightarrow \eta \) in \( C([0, T]; H^3(\Omega_f)) \) and \( \eta^\delta \rightarrow \eta_t \) in \( C([0, T]; H^3(\Omega_f)) \). Thus, via applying Proposition 1.7.7 in [1], we arrive at \( \eta^\delta \rightarrow \eta \) in \( C([0, T] \times \Omega_f) \). This leads to \( \text{det} \nabla \eta^\delta \rightarrow \text{det} \nabla \eta \) in \( C([0, T] \times \Omega_f) \). As \( T \) is sufficiently small, by the definition of \( \eta^\delta \), we have \( |\nabla \eta^\delta| \geq c_0, (c_0 \text{ is independent of } \delta). \) Then, \( a^\delta = \frac{1}{\text{det} \nabla \eta^\delta} \text{cof} \nabla \eta^\delta \rightarrow a \) weakly-star in \( L^\infty([0, T]; H^3(\Omega_f)) \). Since \( a^\delta_1 = -a^\delta : \nabla v^\delta : a^\delta \), we also have that \( a^\delta \) is also bounded in \( L^\infty([0, T]; H^3(\Omega_f)) \) and \( a^\delta \rightarrow a \), weakly-star in \( L^\infty([0, T]; H^3(\Omega_f)) \). From above, we have \( \psi_3(a^\delta) \rightarrow a \text{ weakly-star in } L^\infty([0, T]; H^3(\Omega_f)) \) and \( \psi_3(a^\delta) \rightarrow a \text{ weakly-star in } L^\infty([0, T]; H^3(\Omega_f)) \). Hence, we obtain that \( \psi_3(a^\delta) \rightarrow a \) in \( C([0, T]; H^1(\Omega_f)) \) by Proposition 1.7.7 in [1].

Based on these convergence results, we may pass through the limit in the system (155)-(160) satisfied by the sequence to show that the limit still satisfies the system. The convergence is straightforward in linear terms. From the arguments in [10], we can see that \( \text{div}(a^\delta : (a^\delta)^T : \nabla v^\delta) \rightarrow \text{div}(a : a^T : \nabla v) \) in \( C([0, T]; L^2(\Omega_f)) \) and \( \text{div}(a^\delta q^\delta) \rightarrow \text{div}(aq) \) in \( C([0, T]; L^2(\Omega_f)) \). By (42) and (44),

\[
\| \text{div}DW(\nabla w^\delta + \mathbb{1}) - \text{div}DW(\nabla w + \mathbb{1}) \|_{H^1} \leq C\|w^\delta - w\|_{H^3}(\sum_{j=1}^2 (1 + ||w||_{H^4} + ||w^\delta||_{H^4})^j) \rightarrow 0.
\]
This means that \( \text{div} \, DW(\nabla w^\delta + \mathbb{I}) \rightarrow \text{div} \, DW(\nabla w + \mathbb{I}) \) in \( C([0, T]; H^1(\Omega_e)) \). Therefore, we recover the equations (4) and (5). For the boundary term and divergence condition, similar arguments also show their convergence. Thus, we complete the construction of short-time solutions of the system (4), (5) and (9)-(11). By Lemma 2.1 and Corollary 1, we arrive at the desired regularity of \( a \) and \( \eta \). Hence, we finish the proof of Theorem 1.2.

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