ON THE DUAL VARIABLE OF THE CAUCHY STRESS TENSOR IN ISOTROPIC FINITE HYPERELASTICITY

CLAUDE VALLEE, DANIELLE FORTUNE, AND CAMELIA LERINTIU

ABSTRACT. Elastic materials are governed by a constitutive law linking the second Piola-Kirchhoff stress tensor \( \Sigma \) and the right Cauchy-Green strain tensor \( C = F^TF \). Isotropic elastic materials are the special ones for which the Cauchy stress tensor \( \sigma \) depends solely of the left Cauchy-Green strain tensor \( B = FF^T \). In this paper we revisit the following property of isotropic hyperelastic materials: if the constitutive law linking \( \Sigma \) and \( C \) derives from a potential \( \alpha \), then \( \sigma \) and \( \ln B \) are linked by a constitutive law deriving from the potential \( \alpha \circ \exp \). We give a new and concise proof which is based on an explicit formula expressing the derivative of the exponential of a tensor.

1. INTRODUCTION

According to the mass conservation principle, the mass density per unit volume \( \rho \) and its initial value \( \rho_0 \) are in the ratio

\[
\frac{\rho_0}{\rho} = \det F = (\det C)^{\frac{1}{2}} = (\det B)^{\frac{1}{2}}.
\]

The relation

\[
(\det F)\sigma = F\Sigma F^T
\]

between the Cauchy stress tensor \( \sigma \) and the second Piola-Kirchhoff stress tensor \( \Sigma \) can be rewritten

\[
\frac{\sigma}{\rho} = F\frac{\Sigma}{\rho_0} F^T.
\]

Let us agree to formulate the elastic materials constitutive laws as:

\[
\frac{\Sigma}{\rho_0} = h(C).
\]

The polar decomposition \( F = RU \) of the deformation gradient \( F \) implies:

\[
B = FF^T = RU^2R^T = RCR^T \text{ or } C = R^T BR.
\]

This allows to translate the relation between \( \Sigma \) and \( C \) by a law satisfied by \( \sigma \):

\[
\frac{\sigma}{\rho} = RUh(R^T BR)UR^T = (RU R^T)Rh(R^T BR)R^T (RU R^T)
\]

where we have enlightened the tensor \( RUR^T \) which is nothing else than the square root \( B^{\frac{1}{2}} \) of the positive definite symmetric tensor \( B \). A priori, for elastics materials, the tensor \( \frac{\sigma}{\rho} \) is a function of \( B \) and \( R \):

\[
\frac{\sigma}{\rho} = B^{\frac{1}{2}} Rh(R^T BR)R^T B^{\frac{1}{2}}.
\]

It will depend solely of \( B \) in a single case: when the tensor \( Rh(R^T BR)R^T \) does not depend on the rotation \( R \). The rotations forming a group, the only possible
tensorial functions $h$ are those satisfying the relations of isotropy with respect to $B$:

$$Rh(R^T BR)R^T = h(B) \quad \text{or} \quad R^T h(B) R = h(R^T BR).$$

Because of the relation $C = R^T BR$, the isotropy of the function $h$ can alternatively be expressed with respect to $C$:

$$Rh(C)R^T = h(RCR^T) \quad \text{or} \quad R^T h(RCR^T) R = h(C).$$

To summarize: if the law $\frac{\Sigma}{\rho_0} = h(C)$ is isotropic, then $\frac{\sigma}{\rho}$ depends only of $B$, and it is the sole case: furthermore, under this isotropy condition

$$\frac{\sigma}{\rho} = B^{\frac{1}{2}} h(B) B^{\frac{1}{2}}.$$

In this paper, we revisit the property of isotropic hyperelastic materials for which the existence of a potential expressing the constitutive law between $\Sigma$ and $C$ implies the existence of a potential linking $\sigma$ and $\ln B$.

2. ISOTROPY OF THE CONSTITUTIVE LAW LINKING $\frac{\Sigma}{\rho_0}$ AND $B$

Let $\Omega$ be a rotation, if we change $B$ into $\Omega^T B \Omega$, then $B^{\frac{1}{2}}$ is changed into $\Omega^T B^{\frac{1}{2}} \Omega$ and $\frac{\Sigma}{\rho_0}$ is changed in:

$$\Omega^T B^{\frac{1}{2}} \Omega h(\Omega^T B \Omega) \Omega^T B^{\frac{1}{2}} \Omega = \Omega^T B^{\frac{1}{2}} h(B) B^{\frac{1}{2}} \Omega = \Omega^T \frac{\sigma}{\rho} \Omega.$$

The isotropy of the constitutive law linking $\frac{\Sigma}{\rho_0}$ and $C$ is thus transferred to the constitutive law linking $\frac{\sigma}{\rho}$ and $B$.

3. COAXIALITY OF $B$ AND $h(B)$

**Theorem 3.1.** Because $h$ is isotropic, the symmetric tensors $B$ and $h(B)$ are coaxial (i.e. they have the same eigenvectors).

**Proof.** Let $n$ be an eigenvector of $B$ chosen unitary, and let us consider the rotation of angle $\pi$ around $n$:

$$S = (\cos \pi) I + (1 - \cos \pi) nn^T = 2nn^T - I$$

with $I$ as the identity tensor. Such a symmetry $S$ leaves $n$ unchanged and changes any orthogonal vector to $n$ in its opposite. The tensor $B$ being symmetric, its other two eigenvectors are orthogonal to $n$, as a consequence $S^T BS = B$.

The isotropy condition implies $S^T h(B) S = h(S^T BS)$ or $h(B) S = S h(B)$, therefore $h(B)[Sn] = S[h(B)n]$ or $S[h(B)n] = h(B)n$. Since the sole vectors unchanged by $S$ are the vectors parallel to $n$, the last equality is possible only when the vector $h(B)n$ remains parallel to the vector $n$, that is to say when $n$ is also an eigenvector for $h(B)$. \[ \square \]

We easily deduce from this coaxiality property the two next corollaries, which will reveal important in the following.

**Corollary 3.2.** $B$ and $h(B)$ commute.

**Corollary 3.3.** For every real number $s$, $h(B)$ commutes with the power $B^s$ of $B$.

The choice $s = \frac{1}{2}$ allows one to simplify the expression $\frac{\sigma}{\rho} = B^{\frac{1}{2}} h(B) B^{\frac{1}{2}}$ in

$$\frac{\sigma}{\rho} = h(B) B.$$
4. HYPERELASTIC MATERIALS

4.1. Existence of a potential between the second Piola-Kirchhoff stress tensor $\Sigma$ and the right Cauchy-Green strain tensor $C$. Let us consider a derivable function $\alpha$ of $C$, its derivative $D\alpha(C)$ is a linear mapping from the space of symmetric tensors to $\mathbb{R}$. Thus, there exists a symmetric tensor denoted $\frac{\partial \alpha}{\partial C}$ such that for every variation $\delta C$ of $C$:

$$D\alpha(C)\delta C = \text{tr}\left(\frac{\partial \alpha}{\partial C}\delta C\right).$$

Hyperelastic materials are those for which there exists a function $\alpha$ such that

$$\frac{\Sigma}{\rho_0} = \frac{\partial \alpha}{\partial C}$$

In this assumption, we will say that the constitutive law linking the tensors $\frac{\Sigma}{\rho_0}$ and $C$ derives from the potential $\alpha$.

4.2. Derivative of the exponential of a matrix. Let us consider a square matrix $A$ and a real number $t$, the exponential $\exp(tA)$ is the solution of the matricial ordinary differential equation

$$\frac{d}{dt} \exp(tA) = A \exp(tA)$$

which is equal to $I$ at $t = 0$. Let $\delta A$ be a variation of $A$, in the varied equation

$$\frac{d}{dt} D(\exp)(tA)(t\delta A) = \delta A[\exp(tA)] + AD(\exp)(tA)(t\delta A)$$

let us introduce the square matrix $M(t)$ defined by

$$D(\exp)(tA)(t\delta A) = [\exp(tA)]M(t)$$

The varied equation becomes

$$[\frac{d}{dt} \exp(tA)]M(t) + [\exp(tA)]\frac{dM}{dt} = \delta A[\exp(tA)] + A[\exp(tA)]M(t)$$

and simplifies itself into the ordinary differential equation

$$\frac{dM}{dt} = [\exp(-tA)]\delta A[\exp(tA)]$$

which can be integrated by quadrature. Because $M(0)$ vanishes, we easily deduce from it the value of $M(1)$ and thereafter the variation of the exponential of a matrix $[10]$:

$$D(\exp)(A)(\delta A) = [\exp(A)] \int_0^1 [\exp(-sA)]\delta A[\exp(sA)]ds.$$
4.3. Existence of a potential between the Cauchy stress tensor and the logarithm of the left Cauchy-Green strain tensor.

**Theorem 4.1.** If the tensor \( \Sigma \) derives from a potential \( \alpha \) of the tensor \( C \), then the tensor \( \sigma \) derives from the potential \( \alpha \circ \exp \) of the tensor \( \ln B \).

**Proof.** By deriving the compound function \( \alpha \circ \exp \), we find successively:

\[
D(\alpha \circ \exp)(\ln B)\delta B = D\alpha(B)(D(\exp)(\ln B)\delta B)
\]

\[
= \text{tr}(\frac{\partial \alpha}{\partial B}[D(\exp)(\ln B)\delta B]) = \text{tr}(h(B)B\int_0^1 B^{-s}\delta BB^sds) = \int_0^1 \text{tr}[h(B)BB^{-s}\delta BB^s]ds.
\]

To simplify the last integral, it is necessary to pay attention on the switchings because the matrix \( \delta B \) does not commute with the others. However, under the trace, we can make cross at the beginning the last term of the product of 5 matrices. Then from Corollary 2, we can switch this term \( B^s \) with \( h(B) \) and afterwards with \( B \), it ends up just before \( B^{-s} \). The product of the two matrices \( B^s \) and \( B^{-s} \) reduces to the identity tensor \( I \), and the integral simplifies itself into

\[
\text{tr}(h(B)B\delta B) = \text{tr}(\sigma\delta B).
\]

The final value of the integral allows to conclude to the constitutive law:

\[
\sigma = \frac{\partial (\alpha \circ \exp)}{\partial (\ln B)}
\]

\[\Box\]

5. Conclusion

Without resorting to the Taylor expansion of the logarithm [11] or of the exponential [8] of a symmetric tensor, nor to its spectral decomposition [5], we have given an intrinsic proof of the existence of the potential \( \alpha \circ \exp \) between \( \sigma \) and \( \ln B \).

Numerous isotropic hyperelastic constitutive laws expressing directly \( \sigma \) in term of \( \ln B \) have been proposed ([2], [6], [7], [9], [12]) and numerically implemented [4]. When the potential \( \alpha \circ \exp \) is convex, the consideration of its Legendre-Fenchel-Moreau transform is a tool to perform the inversion of the constitutive law ([1], [13], [14]), ie to express the Hencky logarithmic strain tensor \( \ln B \) in term of the Cauchy stress tensor \( \sigma \).

**References**

[1] Blume, J.A.: On the form of the inverted stress-strain law for isotropic hyperelastic solids. International Journal of Non-Linear Mechanics. 27(3), 413–421 (1992)

[2] Bruhns, O.T., Xiao, H., Meyers, A.: Constitutive inequalities for an isotropic elastic strain-energy function based on Hencky’s logarithmic strain tensor. Proceedings of the Royal Society of London, Series A-Mathematical Physical and Engineering Sciences. 457(2013), 2207–2226 (2001)

[3] Ciarlet, P.G.: Mathematical Elasticity, vol.1. 3D Elasticity, North-Holland, Amsterdam (1988)

[4] Feng, Z.-Q., Vallée, C., Fortuné, D., Peyraut, F.: The 3e hyperelastic model applied to the modeling of 3D impact problems. Finite Elements in Analysis and Design. 43(1), 51–58 (2006)

[5] Hoger, A.: The stress conjugate to logarithmic strain. International Journal of Solids and Structures. 23(12), 1645–1656 (1987)

[6] Ogden, R.W., Saccomandi, G., Sgura, I.: Fitting hyperelastic models to experimental data. Computational Mechanics. 34(6), 484–502 (2004)
[7] Peric, D., Owen, D.R.J., Honnor, M.E.: A model for Finite Strain elastoplasticity based on logarithmic strains - computational issues. Computer Methods in Applied Mechanics and Engineering. 94(1), 35–61 (1992)

[8] Sansour C.: On the dual variable of the logarithmic strain tensor, the dual variable of the Cauchy stress tensor, and related issues. International Journal of Solids and Structures. 38(50-51), 9221–9232 (2001)

[9] Sendova, T., Walton, J.R.: On strong ellipticity for isotropic hyperelastic materials based upon logarithmic strain. International Journal of Non-Linear Mechanics. 40(2-3), 195–212 (2005)

[10] Souriau, J.M.: Calcul linéaire. P.U.F, Paris (1959)

[11] Vallée, C.: Laws of isotropic hyperelastic behaviour (in french). International Journal of Engineering Science. 16(7), 451–457 (1978)

[12] Xiao, H., Chen, L.S.: Hencky’s elasticity model and linear stress-strain relations in isotropic finite hyperelasticity. Acta Mechanica. 157(1-4), 51–60 (2002)

[13] Xiao, H., Chen, L.S.: Hencky’s logarithmic strain and dual stress-strain and strain-stress relations in isotropic finite hyperelasticity. International Journal of Solids and Structures. 40(6), 1455–1463 (2003)

[14] Xiao, H., Bruhns, O.T., Meyers, A: Explicit dual stress-strain and strain-stress relations of incompressible isotropic hyperelastic solids via deviatoric Hencky strain and Cauchy stress. Acta Mechanica. 168(1-2), 21–33 (2004)

Laboratoire de Mécanique des Solides, UMR CNRS 6610, Université de Poitiers, SP2MI, Téléport 2, Boulevard Marie et Pierre Curie, B.P. 30179, 86962, Futuroscope-Chasseneuil Cedex, France, Tel.: 0033-(0)549-496798, Fax: 0033-(0)549-496791

E-mail address: vallée@lms.univ-poitiers.fr

Laboratoire de Mécanique des Solides, UMR CNRS 6610, Université de Poitiers, SP2MI, Téléport 2, Boulevard Marie et Pierre Curie, B.P. 30179, 86962, Futuroscope-Chasseneuil Cedex, France

Laboratoire de Mécanique des Solides, UMR CNRS 6610, Université de Poitiers, SP2MI, Téléport 2, Boulevard Marie et Pierre Curie, B.P. 30179, 86962, Futuroscope-Chasseneuil Cedex, France