Initial State Causes the Structural Balance of Complex Networks With Dynamical Models

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ABSTRACT What initial conditions may lead the dynamic complex network into structural balance? In the sense of social network, the solution to this problem explains that the initial connection relations between individuals are one of the reasons to achieve social stability. In order to obtain the solution, the special Riccati differential equation is chosen as the mathematic model of the dynamical complex network in this paper. Different from the existing results, we focus on the exact solution to the Riccati differential equation and draw the mathematic initial conditions from the exact solution. By using the existing results about the real logarithm of the matrix, the unique exact solution is obtained under the given initial conditions. This solution is a real continuous and differentiable symmetric matrix, and it is seen from which that how the initial state can lead to approximate asymptotically the structural balance more clearly. It turns out that the plus or minus sign of eigenvalues of the initial link matrix affects the asymptotic behavior of the solution, and the maximum positive eigenvalue and its eigenvector with nonzero entries play a key role in approximating asymptotically the structural balance. Finally, the numerical simulations show the validity of methods in this paper.

INDEX TERMS Social network, structural balance, Riccati differential matrix equation, time-varying link matrix.

I. INTRODUCTION

Relations among individuals are activated and changed over time in a social system, and thus the dynamic complex network with time-varying links is employed to describe the dynamical behaviors of the social system. In the dynamic complex network, the nodes denote the individuals and the links denote the relations among individuals. The overall links, which represent the relation strengths between nodes, may be represented by means of a square matrix whose entries are time-varying. We call this matrix as a link matrix, and the dynamic characteristics of the complex network can be mathematically investigated.

From the angle of large-scale system theory [1], dynamical complex network consists of nodes subsystem and links subsystem. Most of the existing research results about complex networks focus on the dynamic characteristics of the nodes, and the links are regarded as the auxiliary part. For example, the dynamic characteristics of the nodes are investigated with constant links [2]–[4], time-varying links [5]–[8], and unknown constant links [9], [10] respectively. Ref. [11], [12] discussed the consensus criteria for the linear multi-agent systems with sampled-data and sampled-data control for a class of linear time-varying system respectively. The above research show that the nodes are the main body. In fact, as a part of the complex network, the states of the links subsystem will be affected by the states of nodes subsystem, due to the coupling relationships between these two subsystems. For instance, in the biological neural network, Gamma oscillations in neurons (nodes) may cause the synaptic facilitation (links) [13], [14]. In the web winding system, how to regulate web (links) tension by controlling the speed of the motors (nodes) is a hot topic [15]. These examples illustrate that it is of great significance to study the dynamic characteristics of links. Hence, in this paper, different from the above results concern the dynamical behavior of the nodes subsystem, we focus on the characteristics of the links subsystem.

One of the dynamic characteristics of the links subsystem is the asymptotic structural balance, which means that the link relations may asymptotically reach a certain stability, which
is called structural balance. The concept of structural balance of social network was originally introduced in [16], [17]. Heider [16] proposed the structural balance based on the triangle in social psychology firstly. This concept was generalized to the complex network with positive and negative links, and a theorem that all nodes are partitioned into two factions, such that all links between nodes of the same faction are positive and all links of the two different factions are negative is proved subsequently [17], [18].

The early works focused on the static properties of structural balance. For the dynamic complex network, the link matrix is regarded with time-varying entries. A noteworthy issue is to understand how a complex network reaches a structural balanced state based on the given mathematic model. The first exploration with discrete-time models for structural balance was conducted by Antal et al. [19]. Local triad dynamics and constrained triad dynamics were two typical discrete-time models in [20], [21]. Furthermore, in constrained triad dynamics, the final states sometimes were not structural balanced, which is called as “jammed state” in [19], [22]. The case with a simple continuous time model for structural balance was firstly proposed by Kulakowski et al. [23]. The further investigation were given in [24], [25]. The main concern of them is to explore the mathematical conditions that leads the network to the structural balance asymptotically. The paper [24] shows that for a random initial matrix whose entries satisfied an absolutely continuous distribution with bounded support, the system reaches a balanced matrix in finite time with a probability converging to 1. The asymptotic structural balance with a non-symmetric link matrix is considered in [25].

In the literature listed above, the mathematic models play an important role in the investigation on asymptotic structural balance for the dynamic complex networks. One of these mathematic models is the matrix Riccati differential equation (abbreviated as RDE). For example, the special matrix Riccati differential equation $\dot{X}(t) = X(t)^2$ is employed in [23]–[25], where $X(t)$ is the link matrix between nodes. For a given matrix Riccati differential equation, the noteworthy issue is to investigate what initial conditions of links states can result in the structural balance in a dynamical complex network. The investigated results will help to understand how the initial imbalance society becomes the structural balanced via social dynamics [21].

In this paper, we also consider the special matrix Riccati differential equation $\dot{X}(t) = X(t)^2$. The link matrix $X(t)$ is supposed to be symmetric, which means that the complex network is undirected in which the relation strengths between two nodes are peer-to-peer equal. Similar to [24], [25], we are concerned about the link matrix whose entries are best understood as an outgrowth of a theory from social psychology known as structural balance. Different from [24], [25], if $X(t)$ is the real continuous and differentiable symmetric matrix (abbreviated as RCDMS), the unique exact solution $X(t)$ satisfying the initial condition $X(t)|_{t=0} = X(0)$ is given in this paper. Compare to [25], [26], the advantage of the exact RCDMS solution given in this paper is being able to more clearly show the mechanism by which the initial state can drive the solution to approximate asymptotically the structural balance. Moreover, by using the exact RCDMS solution given in this paper, we find out that the plus or minus sign of eigenvalues of the initial link matrix affects the asymptotic behavior of the solution, and the maximum positive eigenvalue and its eigenvector with nonzero entries play a key role in approximating asymptotically the structural balance.

This paper is organized as follows. In section 2, we introduce the related works of the structural balance and compare it with our work in detail. In section 3, we introduce the matrix Riccati differential equation as the model of the complex network, and by using the results about the real logarithm of matrix introduced in [26], [27], we get the mathematic conditions by which the unique RCDMS solution exists. The exact RCDMS solution is given based on the mathematic conditions. In section 4, we discuss the initial conditions that make the network lead to structural balance with two cases. In section 5, we use numerical simulations to show the validity of the results in this paper. The conclusion is given in section 6.

II. RELATED WORK
There exist many results about structural balance for dynamical signed network.

References [28], [29] regarded the complex network as a large-scale system, which was composed of nodes subsystem and links subsystem and proposed two different adaptive control schemes for the complex network so that the link subsystem could asymptotically reach a structural balance. References [30] designed a state observer for the links subsystem and proposed a control scheme based on the information of the observer so that the state of the links subsystem could track a structural balance asymptotically. References [31], [32] discussed the same problem for the discrete-time complex dynamical network. References [24], [25] considered a nonlinear matrix Riccati differential equation as a model and investigated what initial conditions of links states could result in the structural balance in the dynamical complex networks.

However, the above existing results [28]–[32] considered a linear matrix Riccati differential equation as a model of the links subsystem. In this paper, different from the above literature, we choose a nonlinear matrix Riccati differential equation as a model. Different from [24], [25], the unique exact solution $X(t)$ satisfying the initial condition $X(t)|_{t=0} = X(0)$ is given in this paper. Hence, it can show the mechanism by which the initial state can drive the solution to approximate asymptotically the structural balance more clearly.

III. THE MODEL OF THE LINKS OF THE DYNAMICAL COMPLEX NETWORK
Consider the following Riccati matrix differential equation which governs the evolution of the links over time:

$$\dot{X}(t) = X(t)^2$$  \hspace{1cm} (1)
where $X(t) = (x_{ij}(t)) \in \mathbb{R}^{N \times N}$ is the real continuous and differentiable symmetric matrix (abbreviated as RCDSM) for $t \in [0, b) \subseteq [0, +\infty)$, where $b$ denotes the positive real number or $+\infty$.

**Remark 1:** (1) Eq. (1) was firstly proposed as a continuous-time model for structural balance by Mukherjee [1]. In social psychology, this model is proposed based on the outcome of a particular gossiping process. For instance, John wants to revise his opinion about Tom, then he asks everybody in this network what they think of Tom. If the opinion he thinks of Tom is the same as others’, then he will increase his opinion about Tom, otherwise, he will decrease his opinion about Tom [2]. Therefore, this model is widely used when discussing structural balance.

(2) **Model Differences:** There are many results about structural balance, such as [28]--[32]. However, these papers discussed the tracking problem of the network with structural balance based on the linear matrix Riccati differential equation. The model used in our paper is a special nonlinear balance based on the linear matrix Riccati differential equation. The model used in our paper is a special nonlinear matrix Riccati differential equation, which is different from the existing literature.

Consider $t \in [0, b)$ and the initial symmetric matrix $X(0)$. Next, the results in this paper show that under certain mathematical conditions for $X(0)$, the unique RCDSM solution $X(t)$ satisfying $X(t)|_{t=0} = X(0)$ may be exactly obtained.

Let the symmetric matrix function $S(t) = e^{-\int_0^t X(\tau)d\tau}$. It follows that:

$$\dot{S}(t) = -[\dot{X}(t) - X(t)^2] e^{-\int_0^t X(\tau)d\tau}.$$  

Taking Eq. (1) into Eq. (2), we have:

$$\dot{S}(t) = 0.$$  

It follows that:

$$\dot{S}(t) = \dot{S}(0) = -X(0)$$

By using $S(0) = I$, which denotes the identity matrix, we obtain the following result:

$$S(t) = I - tX(0)$$

That is:

$$e^{-\int_0^t X(\tau)d\tau} = I - tX(0)$$

By using Eq. (6), we obtain the following lemma.

**Lemma 1:** For $t \in [0, b)$, the Riccati differential equation (1) has the unique RCDSM-solution $X(t)$ satisfying $X(t)|_{t=0} = X(0)$ if and only if there exists one RCDSM-solution $X(t)$ such that Eq. (6) is true.

If $t \in [0, b)$, it is easily seen that the Eq. (6) is true if and only if the real logarithm of the matrix $I - tX(0)$ exists for all $t \in [0, b)$. By using the results in [26], [27], we obtain the following result.

**Lemma 2:** Consider the Riccati differential Eq. (1) for $t \in [0, b)$, there exists a unique RCDSM solution $X(t)$ satisfying $X(t)|_{t=0} = X(0)$ if and only if the matrix Eq. (6) has a unique real logarithm $-\int_0^t X(\tau)d\tau$ of $I - tX(0)$ for $t \in [0, b)$. This implies that the matrix $I - tX(0)$ is nonderogatory and all the eigenvalues of $I - tX(0)$ are positive real for $t \in [0, b)$.

Noticing that $X(0)$ is a symmetric matrix, and thus its eigenvalues are real numbers. Let $\lambda_i = \lambda_i(X(0))$, $i = 1, 2, \ldots, N$ denote the real eigenvalues of $X(0)$. Therefore, it is easily known that the eigenvalues of $I - tX(0)$ are $1 - t\lambda_i$, $i = 1, 2, \ldots, N$, $t \in [0, b)$.

From the result in [33], it is seen that for $t \in [0, b)$ the matrix $I - tX(0)$ is nonderogatory if and only if its Jordan blocks have distinct eigenvalues. Moreover, if the Jordan form of $X(0)$ is supposed as $\text{diag}[J_1(\lambda_1), J_2(\lambda_2), \ldots, J_r(\lambda_r)]$, where $J_k(\lambda_k)$ is the Jordan block for the eigenvalue $\lambda_k$, $k = 1, 2, \ldots, r$, then the Jordan form of $I - tX(0)$ is obtained as $\text{diag}[J_1(1 - t\lambda_1), J_2(1 - t\lambda_2), \ldots, J_r(1 - t\lambda_r)]$. Therefore, by using Lemma 2, we obtain the following result.

**Theorem 1:** Consider the Riccati differential equation (1) on $t \in [0, b)$. $X(0)$ is a given initial symmetric matrix. Then there exists the unique RCDSM solution $X(t)$ satisfying $X(t)|_{t=0} = X(0)$ if and only if the following two conditions are satisfied:

1. $1 - t\lambda_i > 0$ for $t \in [0, b)$, $i = 1, 2, \ldots, N$.
2. Jordan blocks of $X(0)$ have distinct eigenvalues.

Theorem 1 shows the mathematical conditions to ensure the existence and uniqueness of the RCDSM solution of the Riccati Eq. (1) on $t \in [0, b)$. If the conditions in Theorem 1 are true, the following results may be obtained by using Eq. (4) and Eq. (6).

$$X(t)[I - tX(0)] = X(0), \quad t \in [0, b)$$

$$X(t) = X(0)[I - tX(0)]^{-1}, \quad t \in [0, b)$$

The solution (7b) is also proposed in [9], [10]. However, the proof we provide is different, it is based on the usage of the equation (6) and the existence of real matrix logarithm in [26], [27].

Noticing that $X(0)$ is a symmetric matrix, and thus there exists the orthogonal matrix $P$ such that $P^TX(0)P = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Multiplying both sides of the equality (7a) by using $P^T$, $P$, respectively, and noticing that $PP^T = I$, we obtain that:

$$P^TX(t)PP^T[I - tX(0)]P = P^TX(0)P$$

$$= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$$

Let $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. The Eq. (8) may be simplified as:

$$P^TX(t)P[I - t\Lambda] = \Lambda$$

Therefore, it follows that:

$$X(t) = P\text{diag}(\frac{\lambda_1}{1 - t\lambda_1}, \frac{\lambda_2}{1 - t\lambda_2}, \ldots, \frac{\lambda_N}{1 - t\lambda_N})P^T, \quad t \in [0, b)$$

Let $p_i$, $i = 1, 2, \ldots, N$ denote the $i$-th column vector of orthogonal matrix $P$, which is also the eigenvector of the
eigenvalue $\lambda_i$ of $X(0)$ such that $p_i^T p_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$ Therefore, the solution (10) can be represented as:

$$X(t) = \sum_{i=1}^{N} \frac{\lambda_i}{1 - t\lambda_i} p_i p_i^T, \quad t \in [0, b] \quad (11)$$

The Eq.(11) shows that $X(t)p_i = \frac{\lambda_i}{1 - t\lambda_i} p_i$ for $t \in [0, b)$, $i = 1, 2, \ldots, N$, that is, $\frac{\lambda_i}{1 - t\lambda_i}$ is the $t$-th eigenvalue of solution matrix $X(0)$ with the eigenvector $p_i$.

**Remark 2:** Method Differences: In this paper, we firstly choose a symmetric function and combine with the existing results about the real logarithm of the matrix, we solve the exact solution of the nonlinear matrix Riccati differential equation (1) and draw the initial condition from the solution, which is different from the existing literature [24]. Then, we will analyze the dynamical behavior of the link of the complex network based on the exact solution and initial condition.

IV. THE DYNAMICAL BEHAVIOR OF THE LINKS

By using Theorem 1 and Eq.(11), we discuss the dynamical behavior of the RCDSM solution $X(0)$ for $t \in [0, b)$. Remarkably, the following results show that essentially for the initial matrix $X(0)$ with at least one positive eigenvalue, the system (1) may reach asymptotically the structure balance over time. This conclusion coincides essentially with the result in [24], [25]. Our advantage is that the RCDSM solution $X(0)$ is represented exactly as the function of the eigenvalues and eigenvectors of the initial matrix $X(0)$, and thus it is easier to find the dynamical characteristics of the structure of the balanced state in Eq.(1).

The following two cases are discussed.

**Case I:** All eigenvalues of $X(0)$ are non-positive, that is, $\lambda_i = \lambda_s \leq 0$, $i = 1, 2, \ldots, N$.

In this case, $1 - t\lambda_i > 0$, $i = 1, 2, \ldots, N$ are true for $t \in [0, +\infty)$, and thus the Riccati differential equation (1) has the RCDSM solution satisfying $X(t)|_{t=0} = X(0)$ for $t \in [0, +\infty)$, which is given by Eq.(11). Invoking that $\lim_{t \to +\infty} \frac{\lambda_i}{1 - t\lambda_i} = 0$, $i = 1, 2, \ldots, N$, we notice that:

$$\lim_{t \to +\infty} X(t) = O \quad (ZeroMatrix) \quad (12)$$

**Case II:** The symmetric matrix has at least one positive eigenvalue.

In this case, suppose that there exist $s$ positive eigenvalues, $1 \leq s \leq N$, and all positive eigenvalues are ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$, if $s < N$, the other eigenvalues are non-positive, which are denoted by $\lambda_{s+1}, \lambda_{s+2}, \ldots, \lambda_N$.

Let $b = \frac{1}{\lambda_1}$. Consider the time interval $D \in [0, b)$. This implies that the RCDSM solution $X(0)$ of Eq.(1) satisfying $X(t)|_{t=0} = X(0)$ exists only on $D \in [0, b)$, which is given by Eq.(11). The Eq.(11) can be rewritten as:

$$X(t) = \frac{\lambda_1}{1 - t\lambda_1} p_1 p_1^T + \sum_{r=2}^{N} \frac{\lambda_r}{1 - t\lambda_r} p_r p_r^T \quad (13)$$

Let $p_i = (p_i^1, p_i^2, \ldots, p_i^N)^T$, $i = 1, 2, \ldots, N$. The following assumption is proposed.

**Assumption 1:** Assume that $p_j^1 \neq 0$ for $j = 1, 2, \ldots, N$. This means that all components of $p_1$ are nonzero.

**Lemma 3:** If Assumption 1 is satisfied, there exists a positive real number $\delta$ such that $x_{jk}(t) = |x_{jk}(t)| \text{sgn}(p_j^1 p_k^1)$ when $t \in [0, b)$ and $t > b - \delta$ where $j, k = 1, 2, \ldots, N$, $b = \lambda_1^{-1}$.

**Proof:** It is noticed that Assumption 1 implies that $p_j^1 p_k^1 \neq 0$ for $j, k = 1, 2, \ldots, N$. Therefore, we introduce the two positive real numbers $M_{jk} = \left| \frac{p_j^1 p_k^1}{p_j^1 p_k^1} \right|$ and $M = \max_{1 \leq j, k \leq N} \{ \frac{M_{jk}}{2} \}$. Since $\lim_{t \to b, t < b} \frac{\lambda_1}{1 - t\lambda_1} = +\infty$, so there exists a positive real number $\delta$ such that if $|t - b| < \delta$ (that is, $t > b - \delta$), $\frac{\lambda_1}{1 - t\lambda_1} > M$. By using Eq.(13), we obtain that:

$$x_{jk}(t) = \frac{\lambda_1}{1 - t\lambda_1} p_j^1 p_k^1 + \sum_{r=2}^{N} \frac{\lambda_r}{1 - t\lambda_r} p_r^1 p_r^1, \quad j, k = 1, 2, \ldots, N \quad (14)$$

Therefore, multiplying both sides of the Eq.(14) by using $p_k^1 p_j^1$, we notice that:

$$x_{jk}(t)p_j^1 p_k^1$$

$$= \frac{\lambda_1}{1 - t\lambda_1} (p_j^1 p_k^1)^2 + \sum_{r=2}^{N} \frac{\lambda_r}{1 - t\lambda_r} p_r^1 p_r^1$$

$$> M (p_j^1 p_k^1)^2 + \sum_{r=2}^{N} \frac{\lambda_r}{1 - t\lambda_r} p_r^1 p_r^1$$

$$\geq M_{jk} (p_j^1 p_k^1)^2 \quad (15)$$

This completes the proof of Lemma 3.

**Remark 3:** Lemma 3 shows that in Case II, if Assumption 1 is satisfied, the sign of the entry $x_{jk}(t)$ of $X(0)$ in Eq.(13) is the same as the sign of $p_j^1 p_k^1$ when the time $t \in [0, b)$ and $t$ approximates to $b = \lambda_1^{-1}$. This implies that if $t > b - \delta$, $X(t)$ is a structure balance matrix, which is verified as follows.

$$x_{ik}(t)x_{jk}(t)$$

$$= |x_{ik}(t)| \cdot |x_{jk}(t)| \cdot \text{sgn}(p_k^1 p_j^1) \cdot \text{sgn}(p_j^1 p_k^1)$$

$$= |x_{ik}(t)| \cdot |x_{jk}(t)| \cdot \text{sgn}(p_j^1 p_k^1) \cdot \left( p_j^1 p_k^1 \right) \cdot \left( p_j^1 p_k^1 \right)$$

$$= |x_{ik}(t)| \cdot |x_{jk}(t)| \cdot \left( p_j^2 \cdot p_j^2 \right) \cdot \left( p_k^2 \right) > 0$$

$$i, j, k = 1, 2, \ldots, N \quad (16)$$

Summarize the above results as follows.

**Theorem 2:** Consider the Riccati differential Eq.(1). Suppose that the eigenvalues of the initial symmetric matrix $X(t)|_{t=0} = X(0)$ are $\lambda_i, i = 1, 2, \ldots, N$. If $X(t)$ is the RCDSM solution satisfying $X(t)|_{t=0} = X(0)$ on $[0, b)$, the following results are true.
(i) If the eigenvalues $\lambda_i \leq 0$, $i = 1, 2, \cdots, N$, then $b = +\infty$ and $\lim_{t \to +\infty} X(t) = O$ (Zero Matrix), which means that $\lim_{t \to +\infty} X(t)$ is not a structure balance matrix.

(ii) If $X(0)$ has at least one positive eigenvalue and Assumption 1 is true, then $b = \lambda_1^{-1}$ and there exists a real number $\delta > 0$ such that $X(t)$ is a structure balance matrix for $t > b - \delta$, where $\lambda_1 = \max\{ \lambda_i | \lambda_i > 0, 1 \leq i \leq N \}$.

**Remark 4:** (1) The result (ii) in Theorem 2 implies that $X(t)$ has the sign pattern of $\frac{\lambda_i}{1 - \lambda_i} p_i p_i^T$ as $t$ approaches $b = \lambda_1^{-1}$, but $X(t)$ may not be ensured to converge to $\frac{\lambda_i}{1 - \lambda_i} p_i p_i^T$.

In fact, let $E(t) = X(t) - \frac{\lambda_i}{1 - \lambda_i} p_i p_i^T$, the following equality is obtained by using Eq.(14) and the orthogonal properties of matrix $P$.

$$\|E(t)\|^2 = \text{trace}[E^2(t)] = \sum_{r=2}^{N} \left( \frac{\lambda_r}{1 - \lambda_r} \right)^2 \text{trace}(p_r p_r^T)$$

$$= \sum_{r=2}^{N} \left( \frac{\lambda_r}{1 - \lambda_r} \right)^2 p_r p_r^T = \sum_{r=2}^{N} \left( \frac{\lambda_r}{1 - \lambda_r} \right)^2$$

(17)

It is easily seen from Eq.(17) that $\lim_{t \to b, t < b} \|E(t)\| = 0$ if and only if $\sum_{r=2}^{N} \left( \frac{\lambda_r}{1 - \lambda_r} \right)^2 = 0$. This implies that $\lambda_r = 0, r = 2, \cdots, N$, which shows that $X(0)$ has only one positive eigenvalue and the others are zero.

(2) Some exceptions maybe occur if Assumption 1 is not satisfied. For example, consider the initial matrix $X(0)$ given as follows.

$$X(0) = \begin{pmatrix}
\begin{array}{cccc}
-1 & 1 & 0 & 9 \\
4 & \sqrt{2} & -1 & 1 \\
& \sqrt{3} & \sqrt{3} & -1 \\
& 0 & 1 & 0 \\
& 9 & -1 & 0 \\
& 4 & \sqrt{6} & 0 & 15 \\
& & & & 12
\end{array}
\end{pmatrix}$$

(18)

It is obtained that the distinct eigenvalues of $X(0)$ are $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = -2$, $\lambda_4 = 0$. The corresponding eigenvectors are as follows.

$$p_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
0 \\
\frac{1}{\sqrt{3}}
\end{pmatrix}, \quad p_2 = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{6}} \\
\frac{1}{2\sqrt{3}}
\end{pmatrix}, \quad p_3 = \begin{pmatrix}
-\frac{1}{2} \\
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{6}} \\
\frac{1}{2\sqrt{3}}
\end{pmatrix}$$

$$p_4 = \begin{pmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} \\
-2\sqrt{3}
\end{pmatrix}$$

(19)

Obviously, Assumption 1 is not satisfied for $p_1, p_2$. By using Eq.(14), we notice that:

$$x_{32}(t) = \frac{\lambda_1}{1 - \lambda_1} p_1 p_1^T + \frac{\lambda_2}{1 - \lambda_2} p_2 p_2^T$$

$$+ \frac{\lambda_3}{1 - \lambda_3} p_3 p_3^T + \frac{\lambda_4}{1 - \lambda_4} p_4 p_4^T$$

$$= \frac{2}{1 - 2t} p_3 p_3^T$$

$$\leq \frac{2}{1 + 2t} p_3 p_3^T - \frac{2}{1 + 2t} p_3 p_3^T$$

(20a)

$$x_{22}(t) = \frac{2}{1 - 2t} p_2 p_2^T$$

$$- \frac{1}{1 + 2t} p_3 p_3^T$$

$$\leq \frac{2}{1 + 2t} p_3 p_3^T - \frac{2}{1 + 2t} p_3 p_3^T$$

(20b)

$$x_{23}(t) = \frac{2}{1 - 2t} p_3 p_3^T$$

$$- \frac{1}{1 + 2t} p_3 p_3^T$$

$$\leq \frac{2}{1 + 2t} p_3 p_3^T - \frac{2}{1 + 2t} p_3 p_3^T$$

(20c)

Therefore, by using Eq.(16) and $b = 0.5$, we notice that:

$$x_{32}(t)x_{22}(t)x_{23}(t) \leq \frac{1}{\sqrt{3}(1 + 2t)} \cdot 1 + 2t \cdot \frac{1}{\sqrt{3}(1 + 2t)}$$

$$\longrightarrow \frac{1}{24} < 0$$

(21)

The result Eq.(21) shows that is not a structure balance matrix.

(3) Result Differences: Based on the exact solution, we analyze the influence of the sign of the eigenvalue of the initial link matrix on the dynamical behavior of the complex network and this result is not mentioned in [24]. What is more, we prove that the maximum positive eigenvalue and its eigenvector with nonzero entries play a key role in tracking the structural balance.

**V. SIMULATION EXAMPLES**

In this section, we review our results in this paper and have a mathematical simulation to show the correctness of our results. Obviously, this simulation will be displayed in two parts according to the two different cases proposed in this paper.

In the following simulation, the two concepts of the positive triangle and the negative triangle are employed to explain the degree of approximating structural balance for the complete complex networks. The positive triangle and negative triangle are defined as $y_{jk}x_{ji}x_{kl} > 0$ and $x_{jk}x_{ji}x_{kl} \leq 0$ in the link matrix $X(t)$, respectively.

Consider the model proposed in Eq.(1), and we get the solution of Eq.(1) shown in Eq.(10). In this simulation, we assume that the complex network contains 50 nodes, that is, $N = 50$. The following two cases are considered.

**Case I:** All eigenvalues of $X(0)$ are non-positive, that is, $\lambda_i = \lambda_i \leq 0, i = 1, 2, \cdots, N$.

In this case, generate a symmetric matrix $X(0)$ whose all eigenvalues are non-positive with Matlab. The construction step for Matlab is as follow.
TABLE 1. The comparison between initial state and final state.

|            | The density of positive triangles | The density of negative triangles |
|------------|-----------------------------------|----------------------------------|
| The initial state | 0.3988                            | 0.6012                           |
| The final state  | 0.4335                            | 0.5665                           |

Step 1. Generate a column vector \( \mathbf{b} = [b_1, b_2, \ldots, b_{50}]^T \) at random with the order \( \text{randn} \) in \((0, 1]\). Then, the column vector \( \mathbf{s} \) can be got as follow:

\[
\mathbf{s} = -50 \ast |\mathbf{b}|
\]  
(22)

Step 2. Generate a 50 by 50 matrix \( \mathbf{R} \) with random order as follow:

\[
\mathbf{R} = 2 \ast 10 \ast \text{randn}(50, 50) - 10 \ast \text{ones}(50, 50)
\]  
(23)

Step 3. Orthogonalize matrix \( \mathbf{R} \) to get matrix \( \mathbf{P}_0 \), whose column vectors are chosen as the eigenvectors of \( \mathbf{X}(0) \).

Step 4. Generate a diagonal matrix \( \mathbf{S} \), whose diagonal entries are the entries of \( \mathbf{s} \) respectively.

Step 5. \( \mathbf{X}(0) \) can be got as follow:

\[
\mathbf{X}(0) = \mathbf{P}_0 \ast \mathbf{S} \ast \mathbf{P}_0^T
\]  
(24)

By using the above Step1-Step5, the matrix \( \mathbf{X}(0) \) has the following properties:

a. The matrix \( \mathbf{X}(0) \) is symmetric;

b. All eigenvalues of \( \mathbf{X}(0) \) are non-positive.

The state of Riccati matrix differential Eq. (1) satisfying \( \mathbf{X}(t)|_{t=0}=\mathbf{X}(0) \) evolves as shown in Table 1 and Figure 1.

From (a) in Figure 1, it is observed that the link strengths of the network approximate to zero over time. That is, this network can not lead asymptotically to the structural balance. From another perspective, it can be seen from (b) in Figure 1 that the density of positive triangles in the network reach to 0.4335 asymptotically. Table 1 shows that the density of positive triangles increases from 0.3988 to 0.4335, but not 1 eventually. The above result indicates that there exist negative triangles in the complex network.

**Case II:** The symmetric matrix \( \mathbf{X}(0) \) has at least one positive eigenvalue.

In this case, generate a symmetric matrix \( \mathbf{X}(0) \) which has at least one positive eigenvalue with Matlab. The following formula is utilized to generate a column vector \( \mathbf{S} = [s_0, s_1, \ldots, s_{50}]^T \) as follow:

\[
\mathbf{S} = 0.5 \ast \text{rand}(1, 50) \ast \text{randint}(1, 50)
\]  
(25)

where the range of \( \mathbf{S} \) is in \([0, 0.5]\). Similar to the above Case I, the symmetric matrix \( \mathbf{X}(0) \) is given by using Eq.(24) with \( \mathbf{S} \) in Eq.(25).

The state of Riccati matrix differential Eq.(1) satisfying \( \mathbf{X}(t)|_{t=0}=\mathbf{X}(0) \) evolves as shown in Table 2 and Figure 2.

From (a) in Figure 2, it can be seen easily that the links of the network are partitioned into two parts in finite time. This implies by [15] that all nodes of the network Eq.(1) are partitioned into two antagonistic factions with positive links within a faction and negative links between them. That is, the complex network reach asymptotically to structural balance. What’s more, the form of solution shown in Eq.(11) lead to the result that this curve escape at \( \frac{1}{\lambda_1} \), where \( \lambda_1 \) is the maximal positive eigenvalue of \( \mathbf{X}(0) \). In another aspect, Table 2 shows that the density of positive triangles increase to 1 asymptotically. It means that all triangles in this complex network become asymptotically positive. Our simulation results also have practical significance in social network. If the matrix \( \mathbf{X}(t) \) is balanced, it means that all
individuals (representing people, companies or countries) in social network will segregate into two antagonistic but internally friendly factions. This results is similar with the viewpoint proposed in [16], [21], [24], [28].

VI. CONCLUSION
In this paper, we do some research about the dynamical characteristics of the complex networks, which is so called as asymptotically structural balance. We presented mathematically Riccati matrix differential equation as our continuous viewpoint proposed in [16], [21], [24], [28].

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