Bose-Einstein Condensation in the Framework of $\kappa$-Statistics

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In the present work we study the main physical properties of a gas of $\kappa$-deformed bosons described through the statistical distribution function $f_\kappa = Z^{-1} \exp_{(\kappa)}(\beta(\frac{1}{2}mv^2 - \mu)) - 1^{-1}$. The deformed $\kappa$-exponential $\exp_{(\kappa)}(x)$, recently proposed in Ref. [G.Kaniadakis, Physica A 296, 405, (2001)], reduces to the standard exponential as the deformation parameter $\kappa \to 0$, so that $f_0$ reproduces the Bose-Einstein distribution. The condensation temperature $T_c^\kappa$ of this gas decreases with increasing $\kappa$ value, and approaches the $^4$He(I) - $^4$He(II) transition temperature $T_\lambda = 2.17 K$, improving the result obtained in the standard case ($\kappa = 0$). The heat capacity $C(T) = 2 \beta T^3$ is a continuous function and behaves as $B_n T^{3/2}$ for $T < T_c^\kappa$, while for $T > T_c^\kappa$, in contrast with the standard case $\kappa = 0$, it is always increasing.

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I. INTRODUCTION

The liquid $^4$He behaviour at low temperature and pressure has been studied in the approximation of the standard ideal boson gas, and described by the Bose-Einstein (BE) statistical distribution:

$$f = \frac{1}{\exp(\epsilon) - 1}, \quad (1)$$

where $\epsilon = \beta \left(\frac{1}{2}mv^2 - \mu\right)$. The condensation temperature, calculated within this model, has the value $T_c = 3.07 K$, while the fluid-superfluid transition temperature for the $^4$He, measured at the saturation pressure, is $T_\lambda = 2.17 K$. Therefore the difference between the two results is about forty percent. Moreover the heat capacity behavior presents in the two cases some qualitative and quantitative differences. Some models depending on a free parameter based on the group theory, describing the liquid $^4$He behavior, have been proposed in Refs. [8, 9, 10, 11]. Recently the liquid $^4$He is considered in the framework of the nonextensive statistical mechanics [12, 13, 14, 15]. In Ref. [8], a model is developed for the liquid $^4$He, starting from the nonextensive quantum statistical distribution of Refs. [16, 17, 18, 19], given by:

$$f = \frac{1}{[\exp(\epsilon)]^{-1} - 1}, \quad (2)$$

where $\exp_q(x) = [1 + (1 - q)x]^{1/q}$. The condensation temperature $T_c^\kappa$ obtained in Ref. [8] by using the Eq. (2), decreases with increasing $q$ value, becoming quite similar to the experimental result.

In the present effort we propose a study using a new theoretical model, in which the liquid $^4$He is viewed as a particles gas described by a deformed Bose-Einstein statistical distribution, recently proposed in Ref. [12, 13], depending on one continuous parameter $\kappa$, given by:

$$f = \frac{1}{\exp_{(\kappa)}(\epsilon) - 1}, \quad (3)$$

being

$$\exp_{(\kappa)}(x) = \left(\sqrt{1 + \kappa^2 x^2 + \kappa x}\right)^{1/\kappa}. \quad (4)$$

The paper is organized as follows: in the Sect. II, after recalling the main mathematical properties of $\exp_{(\kappa)}(x)$, we define the entropy which gives the distribution given by Eq. (3) using the maximum entropy principle. In Sect. III we introduce a family of integrals for defining the physical properties of the gas and subsequently we calculate the particle number and energy. In Sect. IV the condensation temperature and heat capacity of the system are analyzed, while in the Sect. V some conclusions are reported.

II. THE $\kappa$-STATISTICS

In Ref. [11] it has been shown that the deformed exponential $\exp_{(\kappa)}(x)$ given by Eq. (3) satisfies the equation:

$$\exp_{(\kappa)}(x) \exp_{(\kappa)}(-x) = 1. \quad (5)$$

It is easy to verify that the most general solution of the Eq. (5) is given by $\exp_{(\kappa,g)}(x) = \exp_{(\kappa)}(g(\kappa)x)/\kappa$, where the generator $g(x)$ of the $\exp_{(\kappa,g)}(x)$ is an arbitrary, odd and increasing function, behaving as $g(x) \sim x$ as $x \to 0$. The choice $g(x) = x$ reduces to the $\exp_{(\kappa)}(x)$ while if we set $g(x) = \sinh(x)$ we obtain the standard exponential $\exp(x)$.

We remark that $\exp_{(\kappa)}(x)$ reduces to $\exp(x)$ as $\kappa$ approaches to zero. Namely $\exp_{(0)}(x) = \exp(x)$ where $\exp_{(0)}(x) \equiv \lim_{\kappa \to 0} \exp_{(\kappa)}(x)$. It is interesting to note

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that \( \exp_{(\kappa)}(x) \) satisfies several properties of \( \exp(x) \), some of these properly deformed. For instance \( \exp_{(\kappa)}(x) \) is a positive monotonic increasing function \( \forall x \in \mathbb{R} \), symmetric with respect to the parameter \( \kappa \), being \( \exp_{(-\kappa)}(x) = \exp_{(\kappa)}(x) \). Asymptotically the \( \exp_{(\kappa)}(x) \) follows the power law given by

\[
\exp_{(\kappa)}(x) \sim \frac{(2|\kappa|x)^{\pm 1/|\kappa|}}{2\kappa} . \tag{6}
\]

Others useful relations obtained in Ref. [12] are: for \( \forall a \in \mathbb{R} \) it is verified that \( \exp_{(\kappa)}(ax) = [\exp_{(\kappa)}(x)]^a \) and holds the relation:

\[
\exp_{(\kappa)}(x) \exp_{(\kappa)}(y) = \exp_{(\kappa)}(x \oplus y) , \tag{7}
\]

where the \( \kappa \)-sum \( x \oplus y \), is defined through \( x \oplus y = x \sqrt{1 + \kappa^2 y^2} + y \sqrt{1 + \kappa^2 x^2} \) [12] [3] [4].

Moreover it is remarkable that the inverse function of \( \kappa \)-exponential, namely \( \kappa \)-logarithm, is defined by

\[
\ln_{(\kappa)}(x) = \frac{x^{\kappa} - x^{-\kappa}}{2\kappa} . \tag{8}
\]

It is a monotonic increasing function \( \forall x \in \mathbb{R}^+ \) and reduces to the standard logarithm when \( \kappa \to 0 \). The asymptotic behaviour of the \( \ln_{(\kappa)}(x) \) follows again a power law given by

\[
\ln_{(\kappa)}(x) \sim \frac{1}{2|\kappa|} x^{2|\kappa|} ; \quad \ln_{(\kappa)}(x) \sim -\frac{1}{2|\kappa|} x^{2|\kappa|} . \tag{9}
\]

Others properties of the \( \ln_{(\kappa)}(x) \) can be found in Ref. [12]. It is well known that the maximal entropy principle asserts that for a given entropy

\[
S_{\kappa} = \int_\mathbb{R} d^nv \sigma_{\kappa}(f) , \tag{10}
\]

the most probable distribution for the system (formed by boson particles interacting with a bath) is obtained as solution of the following variational equation:

\[
\delta \left[ S_{\kappa} - \beta \int_\mathbb{R} d^nv \frac{1}{2} mv^2 f + \beta \mu \int_\mathbb{R} d^nv f \right] = 0 , \tag{11}
\]

where the constants \( \beta \) and \( \beta \mu \) are the Lagrange multipliers.

In Ref. [13] it has been postulated for the local entropy \( \sigma_{\kappa}(f) \) the following expression:

\[
\sigma_{\kappa}(f) = -\int d f \ln_{(\kappa)}(f) , \tag{12}
\]

where \( \ln_{(\kappa)}(f) \) is defined through Eq. (8). Then, from Eq. (11) the classical distribution follows

\[
f_{\kappa} = \exp_{(\kappa)} \left( -\beta \left( \frac{1}{2} mv^2 - \mu \right) \right) . \tag{13}
\]

We note that for \( \kappa \to 0 \) the entropy \( S_{\kappa} \) reduces to the well known Shannon-Boltzmann-Gibbs entropy and the distribution of Eq. (13) becomes the MB one.

In this work we are interested in obtaining \( \kappa \)-modified BE statistical distribution. Therefore, we can consider the local entropy having the form

\[
\sigma_{\kappa}(f) = -\int d f \ln_{(\kappa)} \left( \frac{f}{f + 1} \right) , \tag{14}
\]

that reproduces for \( \kappa = 0 \) the standard boson local entropy. By solving Eq. (11) the stationary distribution derived from Eq. (14) assumes the form:

\[
f_{\kappa} = \frac{1}{\exp_{(\kappa)} \left( \beta \left( \frac{1}{2} mv^2 - \mu \right) \right) - 1} . \tag{15}
\]

The Eq. (15) is the BE \( \kappa \)-modified that we are looking for and reduces to the BE stationary distribution as \( \kappa \) approaches to 0.

### III. IDEAL GAS OF \( \kappa \)-DEFORMED BOSONS

Let us study an ideal gas formed by \( N \) identical particles governed by the statistical distribution of Eq. (13). For calculating the typical thermodynamic properties of this gas, we will use in the following the integrals \( J_{n}^{\kappa}(x) \) defined by

\[
J_{n}^{\kappa}(x) = \int_0^\infty \frac{t^n}{\exp_{(\kappa)}(t + x) - 1} dt , \tag{16}
\]

which converge when \(-1 < n < 1/|\kappa| - 1\).

It is easy to verify, taking into account Eq. (11), that

\[
\frac{dJ_{n}^{\kappa}(x)}{dx} = n J_{n-1}^{\kappa}(x) . \tag{17}
\]

The asymptotic behaviour of the integrals \( J_{n}^{\kappa}(x) \) for \( x \to \infty \) is obtained immediately using Eq. (11):

\[
J_{n}^{\kappa}(x) \sim \frac{\Gamma(n + 1) \Gamma(1/|\kappa| - n - 1)}{\Gamma(1/|\kappa|)} x^{n + 1/|\kappa|} . \tag{18}
\]

Since it will be useful later on, we introduce the following auxiliary function:

\[
F_{n}^{\kappa}(x) = \frac{J_{n}^{\kappa}(x)}{J_{n-1}^{\kappa}(x)} , \tag{19}
\]

for which the asymptotic behaviour for \( \kappa \neq 0 \) assumes the form

\[
F_{n}^{\kappa}(x) \sim \frac{n|\kappa|}{1 - (n + 1)|\kappa|} x . \tag{20}
\]

On the other hand, in the case \( \kappa = 0 \), we have

\[
F_{n}^{0}(x) \sim n . \tag{21}
\]
Let us now consider the physical quantities in the thermodynamic limit that one can immediately evaluate from the knowledge of the distribution function defined through Eq. (17): the total particle number \( N = N_0^\kappa + \int_{-\infty}^{+\infty} f(v)d^3v \) is given by:

\[
N = N_0^\kappa + \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \beta^{-\frac{3}{2}} J_{1/2}^\kappa(\nu) , \tag{22}
\]

where \( \nu = |\mu| \), while \( N_0^\kappa \) is the particle number on the ground state \( \epsilon = 0 \):

\[
N_0^\kappa = \frac{1}{\exp(\kappa(\nu)) - 1} . \tag{23}
\]

For the total kinetic energy \( U^\kappa = \int_{-\infty}^{+\infty} \frac{1}{2} m v^2 f(v)d^3v \) we obtain

\[
U^\kappa = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \beta^{-\frac{3}{2}} J_{1/2}^\kappa(\nu) . \tag{24}
\]

Finally from Eq.s (22) and (24) we find straightforwardly:

\[
U^\kappa = (N - N_0^\kappa) k_B T F_{3/2}(\nu) . \tag{25}
\]

IV. PHYSICAL PROPERTIES OF IDEAL \( \kappa \)-DEFORMED BOSON GAS

The condensation temperature \( T_c^\kappa \) for the \( \kappa \)-deformed boson gas can be calculated by observing that at this temperature the potential \( U^\kappa \) is equal to zero and all the \( N \) particle are in the excited states. Then from Eq. (22) we have:

\[
V \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} (\kappa_B T_c^\kappa)^{\frac{3}{2}} J_{1/2}(0) = N . \tag{26}
\]

In the limit \( \kappa \to 0 \), we obtain the standard condensation temperature \( T_c^0 \). Thus, by taking into account that \( J_{1/2}(0) = 2.612\sqrt{\pi}/2 \) and choosing \( m = m_{He^3} \), Eq. (26) yields the value \( T_c^0 = 3.07 K \). After these considerations it is easy to find from Eq. (26) the relation existing between the \( \kappa \)-modified \( T_c^\kappa \) and the standard \( T_c^0 \) one:

\[
\frac{T_c^\kappa}{T_c^0} = \left[ \frac{J_{1/2}(0)}{J_{1/2}(0)} \right]^{\frac{3}{2}} . \tag{27}
\]

From Eq. (24) we note that \( T_c^\kappa \) decreases with increasing \( \kappa \) value; so by increasing the \( \kappa \) value we could approach the condensation temperature \( T_c^\kappa \) to the \( ^4He \) experimental transition temperature, namely \( T_0 = 2.17 K \).

Now we consider the behaviour of the particle ratio \( N_0^\kappa /N \) in the ground state, as a function of the temperature for different \( \kappa \) values. This number is different from zero only for \( T < T_c^\kappa \) and after recalling Eq.s (22) and (24) we obtain:

\[
\frac{N_0^\kappa}{N} = \left[ 1 - \left( \frac{T}{T_c^\kappa} \right)^{\frac{3}{2}} \right] . \tag{28}
\]

This expression of \( N_0^\kappa/N \) can also be written as

\[
\frac{N_0^\kappa}{N} = \left[ 1 - \frac{J_{1/2}(0)}{J_{1/2}(0)} \left( \frac{T}{T_c^\kappa} \right)^{\frac{3}{2}} \right] . \tag{29}
\]

In Fig.1 the particle ratio is plotted at the ground state \( N_0^\kappa/N \) vs the temperature rate \( T/T_c^\kappa \) for different \( \kappa \) values. The shift of the curves \( N_0^\kappa/N \) towards left is due to the decreasing of \( T_c^\kappa \) as \( \kappa \) increases. This behaviour of \( T_c^\kappa \) is plotted in the insert of the fig.1. Besides it is remarkable that this decrease of \( T_c^\kappa \) allows us to approach the experimental liquid \( ^4He \) transition temperature \( T_\lambda = 2.17 K \). In fact for \( \kappa = 0.39 \) we obtain \( T_c^\kappa = 2.70 K \) (excess of 25%) while for \( \kappa = 0 \) (standard bosons) we have \( T_c^0 = 3.07 K \) (excess of 41%). Finally we calculate the heat capacity \( C_V^\kappa \) the \( \kappa \)-deformed bosons by using its definition

\[
C_V^\kappa = \left( \frac{\partial U^\kappa}{\partial T} \right)_V . \tag{30}
\]

For obtaining the expression of \( C_V^\kappa \) we distinguish the two following cases:

(I) \( T \leq T_c^\kappa \) \( \Rightarrow \) \( \nu = 0 \) , \( N_0^\kappa = 0 \)

(II) \( T > T_c^\kappa \) \( \Rightarrow \) \( \nu \neq 0 \) , \( N_0^\kappa = 0 \)

For case (I), after substituting Eq. (24) into Eq. (30) we have:

\[
\frac{C_V^\kappa}{k_B N} = \frac{5}{2} F_{3/2}(0) \left( \frac{T}{T_c^\kappa} \right)^{\frac{3}{2}} . \tag{33}
\]

On the other hand, for case (II), preliminary we differentiate Eq. (24) obtaining:

\[
\frac{dU^\kappa}{dT} = k_B N \left[ F_{3/2}(\nu) + T \frac{d\nu}{dT} \frac{dF_{3/2}(\nu)}{d\nu} \right] . \tag{34}
\]

We note that \( F_{3/2}(\nu) \) is given by Eq. (19), while \( dF_{3/2}(\nu)/d\nu \) can be calculated by combining Eq.s (19) and (17). Finally the quantity \( Td\nu/dT \) can be obtained by observing that the total particle number \( N \) given by Eq. (22) is a constant and then we have \( dN/dT = 0 \). This last relation yields:

\[
T \frac{d\nu}{dT} = -3 F_{3/2}(\nu) , \tag{35}
\]
thus we can write $C^\kappa_V$ in its definitive form in the case $T > T^\kappa_c$:

$$
\frac{C^\kappa_V}{k_B N} = \frac{5}{2} F_{3/2}^{\kappa}(\nu) - \frac{9}{2} F_{1/2}^{\kappa}(\nu) \quad .
$$

(36)

Finally we can write the general expression of the specific heat as follows:

$$
C^\kappa_V = \begin{cases} 
\frac{5}{2} F_{3/2}^{\kappa}(0) \left( \frac{T}{T^\kappa_c} \right)^{\frac{3}{2}} & ; \quad T \leq T^\kappa_c; \\
\frac{5}{2} F_{3/2}^{\kappa}(\nu) - \frac{9}{2} F_{1/2}^{\kappa}(\nu) & ; \quad T > T^\kappa_c.
\end{cases}
$$

(37)

It is clear that for studying $C^\kappa_V$ as a function of $T$ one needs to know $\nu = \nu(T)$ for $T > T^\kappa_c$. In order to calculate this function we can combine Eqs. (22) and (26):

$$
\frac{T}{T^\kappa_c} = \left[ \frac{J^{\kappa}(0)}{J^{\kappa}(\nu)} \right]^{\frac{3}{2}} .
$$

(38)

Eq. (28) defines implicitly $\nu(T)$ and numerically one can see that $\nu(T)$ is an increasing function. This function can be obtained also directly from Eq. (28) by performing an integration.

In Fig.2 the specific heat $C^\kappa_V/Nk_B$ vs the temperature $T/T^\kappa_c$ for different values of $\kappa$ is plotted. All curves are continuous, while for $T = T^\kappa_c$ we obtain a discontinuity in the first derivative which is due to the phase transition.

$C^\kappa_V$, for $T < T^\kappa_c$, assumes the form $C^\kappa_V = B_\kappa T^{3/2}$ where $B_\kappa$ is a constant depending on $\kappa$ and its behaviour is the standard power law of the undeformed bosons. On the other hand, for $T > T^\kappa_c$ we have that $C^\kappa_V$ depends strongly on $\kappa$ and asymptotically it tends to the values $3/2Nk_B$ only when $\kappa = 0$. For $\kappa \neq 0$ we have that $C^\kappa_V \rightarrow \infty$ for $T \rightarrow \infty$. Moreover we can observe that the angular point, corresponding to the transition from the non condensate to condensate state, is shifted to the left as $\kappa$ increases, in accordance with the behaviour of $T^\kappa_c$ vs $\kappa$. Finally we consider the classical limit $T \rightarrow \infty$. This limit is equivalent to the limit $\nu \rightarrow \infty$ and for $\kappa = 0$ we obtain the well known result:

$$
C^\kappa_V = \frac{3}{2} Nk_B \quad .
$$

(39)

On the other hand, for $\kappa \neq 0$, after substituting into Eq. (37) the asymptotic behaviour of $F^{\kappa}(\nu)$ given by Eq. (20), we have

$$
C^\kappa_V \sim \frac{3}{2} Nk_B \frac{4|\kappa|}{(2-5|\kappa|)(2-3|\kappa|)} \nu(T) \quad .
$$

(40)

In conclusion we remark that the usual thermodynamic formalism of standard bosons, properly modified, can be adopted entirely to treat also the $\kappa$-deformed bosons. All the quantities $N^\kappa_0$, $U^\kappa$, $T^\kappa_c$, $C^\kappa_V$ of the 3-dimensional $\kappa$-bosons here considered have finite value when $0 < \kappa < 2/5$ (when $\kappa \geq 2/5$ the total kinetic energy $U^\kappa$ diverges).

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V. FIGURE CAPTIONS

Fig.1. Temperature dependence of the particle ratio in the ground state $N^\kappa_0/N$ for $\kappa$-deformed boson gas, with different $\kappa$ value. We remember that $T^\kappa_c$ is the standard condensation temperature obtained as $\kappa = 0$. In the insert is plotted the $\kappa$ dependence of the $\kappa$-deformed and standard condensation temperature ratio $T^\kappa_c/T^\kappa_0$ as $\kappa$ takes values in the range $0 < \kappa < 2/5$.

Fig.2. Temperature dependence of specific heat $C^\kappa_V/Nk_B$ of $\kappa$-deformed boson gas with different $\kappa$ values in the range $0 < \kappa < 2/5$. $T^\kappa_c$ is the standard condensation temperature obtained when $\kappa = 0$. 
