AN INTEGRAL FORMULA FOR THE $Q'$-PRIME CURVATURE IN 3-DIMENSIONAL CR GEOMETRY

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Abstract. We give an integral formula for the total $Q'$-curvature of a three-dimensional CR manifold with positive CR Yamabe constant and nonnegative Paneitz operator. Our derivation includes a relationship between the Green’s functions of the CR Laplacian and the $P'$-operator.

1. Introduction

The $Q'$-curvature, introduced to three-dimensional CR manifolds by the first- and third-named authors [2] and to higher-dimensional CR manifolds by Hirachi [9], has recently emerged as the natural CR counterpart to Branson’s $Q$-curvature in conformal geometry. The analogies are especially strong in dimension three, where it is known that the total $Q'$-curvature is a biholomorphic invariant — indeed, it is a multiple of the Burns–Epstein invariant [2, 3] — and gives rise to a CR invariant characterization of the standard CR three-sphere.

The above discussion is complicated by the fact that the $Q'$-curvature is most naturally defined only for pseudo-Einstein contact forms. A pseudohermitian manifold $(M^3, J, \theta)$ is pseudo-Einstein if the curvature $\mathcal{R}$ and torsion $A_{11}$ of the Tanaka–Webster connection satisfy $R_1 = iA_{11,1}$. It is known [8] that if $\theta$ is a pseudo-Einstein contact form, then $\hat{\theta} := e^{\Upsilon} \theta$ is pseudo-Einstein if and only if $\Upsilon$ is a CR pluriharmonic function. Moreover, if $M^3$ is embedded in $\mathbb{C}^2$, then pseudo-Einstein contact forms arise from solutions of Fefferman’s Monge–Ampère equation [5]. For a pseudo-Einstein manifold $(M^3, J, \theta)$, the $Q'$-curvature is defined by

$$Q' := -2\Delta R + R^2 - 4|A_{11}|^2.$$ 

The behavior of $Q'$ under the conformal transformation of $\theta$ to $\hat{\theta}$ is controlled by the $P'$-prime operator $P'$ and the Paneitz operator $P$, which have the local expressions

$$P'(u) := 4\Delta^2 u - 8 \text{Im}(A_{11}u_1)\overline{1} - 4 \text{Re}(Ru_1)\overline{1},$$

$$P(u) := \Delta^2 u + T^2 u - 4 \text{Im}(A_{11}u_1)\overline{1}.$$
More precisely, if $\hat{\theta} = e^Y \theta$ and $\theta$ are both pseudo-Einstein, then
\begin{equation}
\tag{1}
e^{2Y} \hat{Q}' = Q' + P'(Y) + \frac{1}{2} P(Y^2).
\end{equation}

From this formula, it is clear that the total $Q'$-curvature is independent of the choice of pseudo-Einstein contact form. A direct computation also shows that if the holomorphic tangent bundle of $M$ is trivial, then the total $Q'$-curvature is a multiple of the Burns–Epstein invariant [3].

The CR Yamabe constant of a CR manifold $(M^3, J)$ is the infimum of the total (Tanaka–Webster) scalar curvature over all contact forms of volume one. For CR manifolds $(M^3, J)$ with positive CR Yamabe constant and nonnegative Paneitz operator, the first- and third-named authors [2] showed that $\int Q' \leq 16\pi^2$ with equality if and only if $(M^3, J)$ is CR equivalent to the standard CR three-sphere. The main goal of this note is to refine this statement by giving an integral formula for the total $Q'$-curvature in terms of the Green’s function of the CR Laplacian:

**Theorem 1.** Let $(M^3, J, \hat{\theta})$ be a pseudo-Einstein manifold with positive CR Yamabe constant and nonnegative Paneitz operator. Given any $p \in M$, it holds that
\begin{equation}
\tag{2}
\int_M Q' = 16\pi^2 - 4 \int_M G_L^2 |A_{11}|^2 \hat{\theta} - 12 \int_M 3 \log(G_L) P_A \log(G_L)
\end{equation}

where $G_L$ is the Green’s function for the CR Laplacian with pole $p$ and $\hat{\theta} = G_L^2 \theta$. In particular,
$$\int_M Q' \leq 16\pi^2$$

with equality if and only if $(M^3, J)$ is CR equivalent to the standard CR three sphere.

Theorem 1 is motivated by similar work in conformal geometry: Gursky [6] used the total $Q$-curvature to characterize the standard four-sphere among all Riemannian manifolds with positive Yamabe constant and Hang–Yang [7] rederived this result by giving an integral formula for the total $Q$-curvature in terms of the Green’s function for the conformal Laplacian.

The key technical difficulty in the proof of Theorem 1 comes from the potential need to consider the $Q'$-curvature of a contact form which is not pseudo-Einstein. On the one hand, $\log G_L$ need not be CR pluri-harmonic, and hence $G_L^2 \theta$ need not be pseudo-Einstein; this problem is overcome by adapting ideas from [2]. On the other hand, estimates for $\log G_L$ are usually derived in CR normal coordinates (cf. [10]), but CR normal coordinates need not be specified in terms of a pseudo-Einstein contact form. We overcome the latter issue by using Moser’s contact form, which is necessarily pseudo-Einstein, as a replacement for CR normal coordinates.
Ignoring these technical difficulties, the idea of the proof of Theorem 1 is to observe that \( \hat{\theta} := G_L^2 \theta \) has vanishing scalar curvature away from the pole, and hence \( \hat{Q} \) has a particularly simple expression. Equation (1) relates \( Q \) and \( \hat{Q} \) in terms of \( P'(\log G_L) \) and \( P((\log G_L)^2) \). Using normal coordinates, we can compute these latter functions near the pole \( p \), at which point (2) follows from (1) by integration. As an upshot of this approach, we relate \( \log G_L \) and the Green’s function for \( P' \); we expect this relation to be useful for future studies of the \( Q' \)-curvature.

This note is organized as follows: In Section 2, we recall necessary facts about Moser’s contact form and use it to relate \( Q' \) and \( \hat{Q}' \). In Section 3, we integrate this relation to prove Theorem 1.

2. Moser’s contact form and normal coordinates

Moser’s normal form for a real hypersurface in \( \mathbb{C}^2 \) (see, e.g., [4]) reads
\[
v = |z|^2 - E(u, z, \bar{z})
\]
where \( (z, w) \in C^2, w = u + iv \), and
\[
E(u, z, \bar{z}) = -c_{42}(u)z^4\bar{z}^2 - c_{24}(u)z^2\bar{z}^4
- c_{33}(u)z^3\bar{z}^3 + O(7).
\]
Hereafter we use \( O(k) \) to denote \( O(\rho^k) \) where \( \rho := (|z|^4 + u^2)^{1/4} \).

Associated to the defining function
\[
r = \frac{1}{2i}(w - \bar{w}) - |z|^2 + E(u, z, \bar{z}),
\]
we have Moser’s contact form
\[
\theta = i\partial r = \frac{1}{2}dw - i\bar{z}dz + i(E_z dz + E_u \frac{1}{2}dw)
\]
in which we have used \( E_w = E_{u \frac{1}{2}} \) and \( E_z := \partial E/\partial z, E_u := \partial E/\partial u, \) etc.

We call coordinates \( (z, u) \) for real hypersurface \( \{ r = 0 \} \) Moser’s normal coordinates. We are going to compute pseudohermitian quantities with respect to Moser’s contact form in Moser’s normal coordinates.

Compute
\[
d\theta = ig_{1\bar{1}}dz \wedge d\bar{z} + \theta \wedge \phi
\]
where
\[
g_{1\bar{1}} = 1 - E_{zz} - \lambda E_{u\bar{z}} - \bar{\lambda} E_{uz} - |\lambda|^2E_{uu},
\]
\[
\phi = a_1 dz + a_{\bar{1}} d\bar{z}
\]
in which
\[
\lambda = \frac{\bar{z} - E_z}{-i + E_u} = i\bar{z} - iE_z + \bar{z}E_u + O(6)
\]
\[
a_1 = \frac{-E_{uz} - \lambda E_{uu}}{i + E_u}, a_{\bar{1}} = (a_1).
\]
The order counting follows the rule that \(z, \bar{z}\) are of order 1 and \(u\) is of order 2. Here we have also used the relation between \(dw\) and \(\theta\):
\[
dw = \frac{2}{1 + iE_u}(\theta + i(\bar{z} - E_z)dz).
\]

Take a pseudohermitian coframe
\[
\tag{6}
\theta^1 := dz - ia^1\theta,
\]
\[
a^1 := g^{1\bar{1}}a_{\bar{1}}
\]
where \(g^{1\bar{1}} := (g_{1\bar{1}})^{-1}\), such that
\[
\tag{7}
d\theta = ig_{1\bar{1}}\theta^1 \wedge \theta^\bar{1}.
\]
The dual frame \(Z_1\) (such that \(\theta(Z_1) = 0, \theta^1(Z_1) = 1,\) and \(\theta^\bar{1}(Z_1) = 0\)) reads
\[
\tag{8}
Z_1 = \frac{\partial}{\partial z} + \lambda \frac{\partial}{\partial u} = \hat{Z}_1 + O(5) \frac{\partial}{\partial u}
\]
where \(\hat{Z}_1 := \partial_z + i\bar{z}\partial_u\).

Differentiating \(\theta^1\) from (6) gives
\[
\tag{9}
d\theta^1 = \theta^1 \wedge \tilde{\omega}_1^1 + iZ_1(a^1)\theta \wedge \theta^\bar{1}
\]
by (7), where
\[
\tag{10}
\tilde{\omega}_1^1 = a_1\theta^\bar{1} - iZ_1(a^1)\theta.
\]
Differentiating (7) gives no \(\theta\) component of
\[
dg_{1\bar{1}} - g_{1\bar{1}}\tilde{\omega}_1^1 - g_{1\bar{1}}\tilde{\omega}_1^\bar{1}
\]
\[
= [Z_1g_{1\bar{1}} - g_{1\bar{1}}a_1]\theta^1 + \text{conjugate}
\]
where \(\tilde{\omega}_1^\bar{1}\) is the complex conjugate of \(\tilde{\omega}_1^1\). It follows that the pseudohermitian connection form \(\omega_1^1\) reads
\[
\tag{11}
\omega_1^1 = \tilde{\omega}_1^1 + (g^{1\bar{1}}Z_1g_{1\bar{1}} - a_1)\theta^1.
\]
We also reads from (9) that
\[
\tag{12}
A_1^1 = iZ_1(a^1).
\]

Substituting (11) into the structure equation \(d\omega_1^1 = Rg_{1\bar{1}}\theta^1 \wedge \theta^\bar{1}\) mod \(\theta\), we obtain the Tanaka-Webster (scalar) curvature
\[
\tag{13}
R = Z_1^\prime a_1 + Z_1a_1^1 + Z_1^1a_1 - Z_1^\prime Z_1^\prime g_{1\bar{1}} + a_1(Z_1^\prime g_{1\bar{1}} - a_1^1) - a_1^1 a_1
\]
where we have used \(g^{1\bar{1}}\) to raise the indices, e.g., \(Z_1 := g^{1\bar{1}}Z_1 = (g^{1\bar{1}}Z_1) = (Z_1^\dagger), a_1^1 := g^{1\bar{1}}a_1^1\). We then compute the lowest order terms of \(Z_1^\prime a_1^1\),
By (14) and alike formulas, we can compute
\[ Z^1 g_{1\bar{1}} = -E_{uu} - iE_{uz} - zE_{uuu} + \bar{z}E_{uu\bar{z}} - i|z|^2 E_{uuu} + O(3), \]
\[ Z^1 g_{1\bar{1}} = -E_{zz}^1 - 2i\bar{z}E_{z\bar{z}} + izE_{zzu} + \bar{z}^2 E_{uuu} - iE_{uz} \]
\[-2|z|^2 E_{uuu} - \bar{z}E_{uu} - i\bar{z}|z|^2 E_{uuu} + O(4). \]

Here we have counted \( z, \bar{z} \) of order 1, \( u \) of order 2, and used \( g^{1\bar{1}} = 1 + O(4), \lambda = i\bar{z} - iE_z + \bar{z}E_u + \text{h.o.t.}, a_1 = iE_{uz} - \bar{z}E_{uu} + \text{h.o.t.} \), \( Z_1 = \partial_z + i\bar{z}\partial_u + \text{h.o.t.} \). From (12) we compute
\[ (15) \quad A^1_1 = E_{uu\bar{z}} - 2izE_{uu\bar{z}} + z^2 E_{uuuu} + O(3). \]

By (14) and alike formulas, we can compute \( R \) through (13):
\[ (16) \quad R = -2E_{uu} + E_{uuuu} - 2i\bar{z}E_{z\bar{z}u} - 2i\bar{z}E_{z\bar{z}\bar{z}u} \]
\[ + 4|z|^2 E_{z\bar{z}uu} - z^2 E_{uuzz} - \bar{z}^2 E_{uuuu} + 2i\bar{z}|z|^2 E_{uuuu} \]
\[-2i|z|^2 E_{uuuu} + |z|^4 E_{uuuu} + O(3) \]

We can then compute \( R_{1,1} = Z_1 R, A^1_{1,1}, \) and obtain the pseudo-Einstein tensor as follows:
\[ (17) \quad R_{1,1} - iA^1_{1,1} = E_{zzzz} - 4izE_{uu} + 3i\bar{z}E_{z\bar{z}uu} \]
\[ -3iE_{zz\bar{z}u} - 2izE_{zzz\bar{z}u} + 6|z|^2 E_{zz\bar{z}uu} \]
\[ -3\bar{z}^2 E_{z\bar{z}uu} + 6izE_{z\bar{z}z\bar{z}u} + 6i|z|^2 E_{zz\bar{z}uu} \]
\[ -3izE_{zzzz} - z^2 E_{uuzzzz} - 3iz|z|^2 E_{uuuu} \]
\[ -i\bar{z}^3 E_{uuuu} + iz^2 E_{uumu} - 2\bar{z}^2 |z|^2 E_{uuumu} \]
\[-6iz|z|^2 E_{uumu} + 3|z|^4 E_{uumuu} + \bar{z}|z|^2 E_{uumu} \]
\[ + i\bar{z}|z|^4 E_{uumuu} + O(2). \]

From (17) along the \( u \)-curve (a chain) where \( z = 0 \), we conclude that \( R_{1,1} - iA^1_{1,1} = 0 \) (terms in \( O(2) \) all vanish because of special structure of Moser’s normal form) and does not vanish identically in general. The reason is that the coefficient of \( z \) in \( E_{zzzz} \) is \( c_{42}(u) \) which is proportional to the Cartan tensor.

In general a pseudo-Einstein contact form may not be a “normalized” contact form that gives CR normal coordinates. So we take the contact form associated to the solution \( \psi \) to the complex Monge–Ampère equation:
\[ (18) \quad J[\psi] := \det \begin{bmatrix} \psi & \psi_\bar{z} & \psi_{\bar{w}} \\ \psi_z & \psi_{zz} & \psi_{z\bar{w}} \\ \psi_w & \psi_{w\bar{z}} & \psi_{ww} \end{bmatrix} = 1 \]
in \( \Omega \) and \( \psi = 0 \) on \( \partial\Omega \). The contact form \( \theta := i\partial\bar{\psi} \) is pseudo-Einstein. We want to compute \( \Delta_b, P, P \) w.r.t. this \( \theta \), but in Moser’s normal
coordinates \((z, u)\). For \(r\) having a form of \((3)\) multiplied by \(4^{1/3}\), we have

\[
J[r] = 1 + O(\rho^4) \tag{19}
\]

Lee-Melrose’s asymptotic expansion reads

\[
\psi \sim r \sum_{k \geq 0} \eta_k (r^3 \log r)^k \quad \text{near } \partial \Omega = \{ r = 0 \} \subset C^2
\]

with \(\eta_k \in C^\infty(\bar{\Omega})\). This means that for large \(N\), \(\psi - r \sum_{k=0}^{N} \eta_k (r^3 \log r)^k\) has many continuous derivatives on \(\bar{\Omega}\) and vanishes to high order at \(\partial \Omega\). It follows from \((18), (19),\) and \((20)\) that

\[
J[r \eta_0] = 1 + O(\rho^4) \quad \text{and} \quad \eta_0 = 1 + O(\rho^4). 
\]

So we have

\[
\psi \sim r \eta_0 + \eta_1 r^4 \log r + h.o.t. \\
\sim r + O(\rho^6).
\]

Similar argument as for \(r\) before works for \(\psi\). Therefore, with respect to the pseudo-Einstein contact form defined by \(\psi\), we still have

\[
\theta = (1 + O(\rho^4)) \hat{\theta} + O(\rho^5) dz + O(\rho^5) d\bar{z},
\]

\[
\theta^1 = O(\rho^3) \hat{\theta} + (1 + O(\rho^8)) dz + O(\rho^8) d\bar{z},
\]

\[
Z_1 = \hat{Z}_1 + O(\rho^5) \frac{\partial}{\partial u},
\]

\[
\omega_1^1 = O(\rho^2) \hat{\theta} + O(\rho^3) dz + O(\rho^7) d\bar{z},
\]

\[
A_1^1 = O(\rho^2), \quad R = O(\rho^2),
\]

\[
g_{11} = 1 + O(\rho^4), \quad g_{1\bar{1}} = 1 + O(\rho^4)
\]

in view of \((3), (5), (10), (11), (15), (16),\) and \((4)\). Now let \(L\) denote the \(CR\) Laplacian:

\[
L := -4 \triangle_b + R
\]

where \(\triangle_b\) is the (positive) sublaplacian given by

\[
\triangle_b = \hat{Z}^1 \hat{Z}_1 - \omega_1^1 (\hat{Z}^1) \hat{Z}_1 + \text{conjugate}. \tag{22}
\]

Let \(G_L\) denote the Green’s function of \(L\), i.e.,

\[
LG_L = -4 \triangle_b G_L + RG_L = 16 \delta_p. \tag{23}
\]

Let \(\hat{P}' := 4 \hat{\triangle}_b^2, \hat{L} := -4 \hat{\triangle}_b\) denote the \(P'\) operator, the \(CR\) Laplacian for the Heisenberg group \(\mathbb{H}^1\), respectively. Observe that (cf. \([1]\))

\[
P'(\log G_L) = \hat{P}'(\log \frac{1}{2\pi \rho^2}) = 8\pi^2 S_p
\]

where \(S_p = S(p, \cdot)\) for \(S(p, \cdot)\) the kernel of the orthogonal projection \(\pi: L^2(\mathbb{H}^1) \to \mathcal{P}(\mathbb{H}^1)\) onto the space of \(CR\) pluriharmonic functions.
where we have used $G_L = \frac{1}{2\pi \rho^2}$. From (21) and (22) we obtain

$$\Delta_b = (1 + O(\rho^4))\hat{\Delta}_b + O(\rho^{10})\frac{\partial^2}{\partial u^2} + O(\rho^4)\frac{\partial}{\partial u}$$

$$+ O(\rho^5)\frac{\partial}{\partial u} + O(\rho^7)\hat{Z}_1$$

$$+ O(\rho^5)\frac{\partial}{\partial u} + O(\rho^7)\hat{Z}_1.$$

Write

$$G_L = \frac{1}{2\pi \rho^2} + \omega.$$

From (23), (24), and (21) we obtain $L\omega = a$ bounded function near $p$.

Therefore from subelliptic regularity theory of $L$, we see that $\omega$ is in the Folland–Stein space $S^{2,q}$ for any $q > 1$, and hence $w \in C^{1,\gamma}$. In fact, $\omega$ is $C^\infty$ smooth. Recall that

$$P' = 4\Delta_b - 8Im\nabla^1(A_1^\dagger \nabla_1) - 4Re\nabla^1(R\nabla_1)$$

$$\hat{P}' + 4(\Delta_b - \hat{\Delta}_b)$$

$$- 8Im\nabla^1(A_1^\dagger \nabla_1) - 4Re\nabla^1(R\nabla_1).$$

Write

$$\log G_L = \log(\frac{1}{2\pi \rho^2} + \omega)$$

$$= \log(\frac{1}{2\pi \rho^2}) + \log(1 + 2\pi \rho^2 \omega).$$

We can now compute

$$P'(\log G_L) = \hat{P}'(\log(\frac{1}{2\pi \rho^2})) + (P' - \hat{P}')(\log(\frac{1}{2\pi \rho^2}))$$

$$+ P'(\log(1 + 2\pi \rho^2 \omega))$$

$$= 8\pi^2 S_p + \{4(\Delta_b - \hat{\Delta}_b) - 8Im\nabla^1(A_1^\dagger \nabla_1)$$

$$- 4Re\nabla^1(R\nabla_1)\}(\log(\frac{1}{2\pi \rho^2}))$$

$$+ P'(\log(1 + 2\pi \rho^2 \omega)).$$

Since $\omega$ is $C^\infty$ smooth, the third term is a bounded function near $p$. The second term is also bounded near $p$ in view of (21) and (24). So we conclude that

$$P'(\log G_L) = 8\pi^2 S_p + a$$

Similarly we can show

$$P((\log G_L)^2) = 8\pi^2 (\delta_p - S_p) + a.$$
On the other hand, we reduce computing the most singular term in $P_3(\log G_L)$ to computing $P_3(\log(\frac{1}{2\pi\rho^2}))$ by (21). In view of (21) we find that the most singular term in $P_3(\log(\frac{1}{2\pi\rho^2}))$ is a constant multiple of $\hat{P}_3(\log \rho)$ where $\hat{P}_3 = \hat{Z}_1\hat{Z}_1\hat{Z}_1$ is the $P_3$-operator w.r.t. the Heisenberg group $H_1$. Observe that $|z|^2 - iu$ is a CR function on $H_1$, i.e.,

$$\hat{Z}_1(|z|^2 - iu) = (\partial_z - iz\partial_u)(|z|^2 - iu) = 0.$$ 

It follows that the real part of $\log(|z|^2 - iu)$ is CR pluriharmonic. By (21) we have

$$\hat{P}_3 ((\log ||z|^2 - iu|) = \hat{Z}_1\hat{Z}_1\hat{Z}_1 (\log ||z|^2 - iu|) = 0.$$ 

Since $\log ||z|^2 - iu| = 2\log \rho$, we conclude that

$$\hat{P}_3(\log \rho) = 0.$$ 

It follows that

\begin{align*}
(29) \quad P_3(\log G_L) &= \hat{P}_3(\log(\frac{1}{2\pi\rho^2})) \\
&+ (P_3 - \hat{P}_3)(\log(\frac{1}{2\pi\rho^2})) \\
&+ P_3(\log(1 + 2\pi\rho^2\omega)) \\
&= 0 + (P_3 - \hat{P}_3)(\log(\frac{1}{2\pi\rho^2})) \\
&+ P_3(\log(1 + 2\pi\rho^2\omega)) \\
&= O(\rho).
\end{align*}

by (21). So $(\log G_L)P(\log G_L)$ has blow-up rate as $\log \rho$ near the pole $p$. Hence it is integrable with respect to the volume $\theta \wedge d\theta$ which has vanishing order $\rho^3$ near $p$.

3. A FORMULA FOR THE INTEGRAL OF $Q'$ CURVATURE

Let $\theta$ be a pseudo-Einstein contact form on $(M^3, J)$. By [2] Proposition 6.1], for any $\Upsilon \in C^\infty(M)$, it holds that $\hat{\theta} := e^{\Upsilon} \theta$ satisfies

\begin{align*}
(30) \quad e^{2\Upsilon} \hat{Q}' &= Q^{\Upsilon} + P'(\Upsilon) + \frac{1}{2}P(\Upsilon^2) \\
&- \Upsilon P(\Upsilon) - 16Re(\nabla^1\Upsilon)(P_3\Upsilon)_1
\end{align*}

where $P_3$ is the operator characterizing CR pluriharmonics. Recall that $P(\Upsilon) = 4\nabla^1(P_3\Upsilon)_1$.

Let $G_L$ be the Green’s function of the CR Laplacian (we assume $\Upsilon(J) > 0$). Set $\hat{\theta} = G_L^2 \theta$. Then $\hat{\theta}$ has vanishing scalar curvature away from the pole $p$. In particular, we have

$$\hat{Q}' = -4|\hat{A}_{11}|_\hat{\theta}^2.$$
away from the pole \( p \). Plugging this into (30), we see that away from \( p \),

\[
-4G_L^4|\hat{A}_{11}|_\theta^2 = Q' + 2P'(\log G_L) + 2P((\log G_L)^2) - 4(\log G_L)P(\log G_L) - 64Re(\nabla^1 \log G_L)(P_3(\log G_L))_1.
\]

Now assume \((M^3, J)\) is embedded in \( \mathbb{C}^2 \). Take \( \theta \) to be the pseudo-Einstein contact form associated to the solution to complex Monge–Ampère equation (18). We look at the order of \( G_L^4|\hat{A}_{11}|_\theta^2 \) near \( p \). The transformation law of torsion reads

\[
\hat{A}_{11} = G_L^{-2}(A_{11} + 2i(\log G_L)_{,11} - 4i(\log G_L)_{,1}(\log G_L),_1
\]

(see [11, p. 421]). Recall \( \dot{Z}_1 := \partial_z + iz\partial_u \). Observe that

\[
\dot{Z}_1 \log \rho^4 = \frac{2\bar{z}}{|z|^2 - iu},
\]

\[
\dot{Z}_1 \bar{Z}_1 \log \rho^4 = \frac{-4\bar{z}^2}{(|z|^2 - iu)^2} = -(\dot{Z}_1 \log \rho^4)^2.
\]

Therefore we have

\[
\dot{Z}_1 \bar{Z}_1 \log \frac{1}{2\pi \rho^2} - 2(\dot{Z}_1 \log \frac{1}{2\pi \rho^2})^2 = 0
\]

It follows from (21) and (33) that

\[
A_{11} = O(\rho^2)
\]

\[
2i(\log G_L)_{,11} - 4i(\log G_L)_{,1}(\log G_L),_1 = O(\rho^2)
\]

near \( p \). So from (32) and (34), we learn that

\[
G_L^4|\hat{A}_{11}|_\theta^2 = O(\rho^4)
\]

near \( p \). By (29), we obtain that the last two terms in (31) are \( L^1 \) and bounded near \( p \), respectively. In view of (27), (28), (35), and (31), we then have

\[
2P'(\log G_L) + 2P((\log G_L)^2)
\]

\[
= 16\pi^2 \delta_p - Q' - 4G_L^4|\hat{A}_{11}|_\theta^2
\]

\[
+ 4(\log G_L)P(\log G_L) + 64Re(\nabla^1 \log G_L)(P_3(\log G_L))_1.
\]

in the distribution sense. Integrating the last term in (36) gives

\[
-16 \int (\log G_L)P(\log G_L) + 64Re \int_{\text{around } p} (\log G_L)P_3(\log G_L)i\theta \wedge \theta^1.
\]

Here we have omitted the lower index “1” for the \( P_3 \) term. The boundary term in (37) vanishes by (29) and that \( \theta \wedge \theta^1 \) has vanishing order
of $\rho^3$ near $p$. Applying (36) to the constant function 1 yields

$$0 = 16\pi^2 - \int Q' - 4 \int G_L^4|\hat{A}_{11}|^2 - 12 \int (\log G_L) P(\log G_L)$$

by (37). Here notice that in the distribution sense,

$$2P'(\log G_L)(1) = 2 \int (\log G_L) P'(1) = 0$$

since $P'(1) = 0$. Similarly we get $2P((\log G_L)^2)(1) = 0$ since $P(1) = 0$. Assuming $P \geq 0$, we get that

$$\int Q' = 16\pi^2 - 4 \int G_L^4|\hat{A}_{11}|^2 - 12 \int (\log G_L) P(\log G_L)$$

$$\leq 16\pi^2.$$ 

Moreover, equality holds if and only if $\hat{A}_{11} \equiv 0$ and $\log G_L$ is pluriharmonic. Since also $\hat{R} \equiv 0$, we conclude that $(M \setminus \{p\}, \hat{\theta})$ is isometric to the Heisenberg group $\mathbb{H}^1$. Indeed, the developing map identifies the universal cover of $M \setminus \{p\}$ with $\mathbb{H}^1$, while the fact that a neighborhood of $p$ (equivalently, a neighborhood of infinity in $(M \setminus \{p\}, \hat{\theta})$) is simply connected implies that the covering map is trivial. By adding back the point $p$, we conclude that $(M, J)$ is CR equivalent to the standard CR three-sphere.

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