Periodic solutions of a phase-field model with hysteresis *

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Abstract

In the present paper we consider a partial differential system describing a phase-field model with temperature dependent constraint for the order parameter. The system consists of an energy balance equation with a fairly general nonlinear heat source term and a phase dynamics equation which takes into account the hysteretic character of the process. The existence of a periodic solution for this system is proved under a minimal set of assumptions on the curves defining the corresponding hysteresis region.

Keywords: evolution system, hysteresis, phase transitions, periodic solutions.

1 Introduction

In the space-time cylinder $Q := [0, T] \times \Omega$, where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$ and $T > 0$ is a fixed final time, consider the system

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\[ u_t - \Delta u = h(u,v) \quad \text{in } Q, \]  
\[ v_t - \kappa \Delta v + \partial I(u;v) \ni g(u,v) \quad \text{in } Q, \]  
\[ u = v = 0 \quad \text{on } (0,T) \times \partial \Omega, \]  
\[ u(x,0) = u(x,T), \quad v(x,0) = v(x,T) \quad \text{on } \Omega. \]  

Here, \( I(u;\cdot) \) is the indicator function of the interval \([f_*(u),f^*(u)]\), \( \partial I(u;\cdot) \) is its subdifferential in the sense of convex analysis, \( h, g, f_*, f^* \) are given functions with the properties specified in the next section, \( \kappa > 0 \) is a given constant.

For convenience, denote system (1.1)–(1.4) by \((P)\). System \((P)\) can be regarded as a dynamical model of a phase transition process between two distinct phases (such as solid-liquid) placed in the container \( \Omega \). The state variables \( u = u(t,x) \) and \( v = v(t,x) \) are then interpreted as the relative temperature and the order parameter (phase fraction of an individual phase), respectively. Eq. (1.2) with \( g \equiv 0 \) and \( \kappa = 0 \) models a continuous hysteresis operator of generalized play type generated by the curves \( v = f_*(u) \) and \( v = f^*(u) \), see [1–3] for details. The introduction of the latter operator to the model accounts for hysteretic relationship between \( u \) and \( v \), playing in this case the roles of the input and output functions, respectively.

Recent years have seen a considerable amount of works on partial differential equations with hysteresis. In particular, the questions on existence, uniqueness and large time behaviour of solutions to Cauchy problems for systems with state-dependent constraints, relevant to (1.1)–(1.3) were addressed in a number of papers (see, e.g., [4–11], and references therein). Periodic problems for systems with hysteresis describing phase transitions have also received a keen attention. Among related contributions, we mention the works [12–14] studying periodic processes in elastoplastic bodies, [15] dealing with the Stefan problem in a one-dimensional spatial domain, that involves a relay hysteresis operator, and [16] which considers the system (1.1)–(1.4) with \( h(u,v) = v, \ g \equiv 0, \) and \( \kappa = 0 \). We note, however, that the requirements on the functions \( f_* \) and \( f^* \) in [10]: \( f_*, f^* \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) are non-decreasing Lipschitz continuous, \( f_*(u) = f^*(u) \) for \( u \in (-\infty,a] \cup [b,\infty) \) with \( a < 0 < b \), \( f_* \) is convex on \((-\infty,b)\) and \( f^* \) is concave on \((a,\infty)\), which are indispensable for the proof in [10], appear to be too demanding. In our paper, we dispense with these assumptions on the functions \( f_* \) and \( f^* \) retaining only the Lipschitz continuity. In this respect, as a byproduct of our analysis, we also improve the results on the existence of a solution to the Cauchy problem from [7] by removing the assumptions of smoothness, monotonicity and boundedness on the functions \( f_* \) and \( f^* \). Moreover, the convergences of approximate solutions which the authors obtain in [10] The proof of Theorem 2.1] would not be sufficient to treat general nonlinear right-hand sides as in (1.1), (1.2). Another advance of our paper is that we allow a diffusion effect for \( v \) assuming the coefficient of the interfacial energy \( \kappa \) to be nonzero.

Here, we would like to mention that hysteresis curves encountered in the practice of physical measurements, stress-strain hysteresis loops in shape memory alloy
wires, load-displacement hysteresis curves in composite structures, magnetic hysteresis curves of nano-minerals may genuinely occur to lack the smoothness. Also, we add that for the completed relay operator and the truncated play operators employed to approximate the former (see [3]) the curves describing the corresponding hysteresis regions are piecewise linear but nonsmooth.

The purpose of the present paper is to prove the existence of a solution to system (P) with sufficiently general \( h, g, f_\ast, \) and \( f^\ast \). In our approach to establish the existence for problem (P), first, we construct a family of suitable approximate problems based on the Yosida regularization \( \partial I^\lambda(u; \cdot), \lambda > 0, \) of the subdifferential \( \partial I(u; \cdot) \). We then further regularize the nonsmooth functions \( f_\ast \) and \( f^\ast \) describing \( \partial I^\lambda(u; \cdot) \) by sequences of mollifiers depending on a regularizing parameter \( \varepsilon > 0 \). Next, we rewrite thus obtained approximate problem \( (P)_{\lambda, \varepsilon} \) as a single abstract differential equation with a general nonlinearity subject to periodic condition. Applying a standard for such equations technique invoking a fixed point argument we find a solution of the abstract equation which provides us with a solution to the approximate problem \( (P)_{\lambda, \varepsilon} \). After that, we establish a priori estimates independent of \( \varepsilon \) for solutions of the approximate problems and performing a limiting procedure as \( \varepsilon \to 0 \) we obtain an intermediate approximate problem \( (P)_\lambda \) depending now on the parameter \( \lambda \) only. Deriving uniform estimates with respect to \( \lambda \) for the latter system, we finally prove the existence of a solution to problem (P) through the passage-to-the-limit procedure when \( \lambda \to 0 \). We note that in order to get suitable compactness properties and, thus, to legitimate this passage-to-the-limit we exploit essentially the properties derived from the specific structure of the approximate equations for (1.2). This also allows us, inter alia, to treat general nonlinearities in Eqs. (1.1), (1.2).

2 Preliminaries and hypotheses on the data

In this section, we recall some notions which we use in the paper and posit assumptions on the data describing Problem (P).

Throughout the paper, we denote by \( H \) the Hilbert space \( L^2(\Omega) \) with the standard inner product \( \langle \cdot, \cdot \rangle_H \), and by \( V \) the Sobolev space \( H^1(\Omega) \).

Given a Hilbert space \( X \) with the inner product \( \langle \cdot, \cdot \rangle_X \) and a convex, lower semicontinuous function \( \varphi : X \to \mathbb{R} \cup \{+\infty\} \) which is not identically \(+\infty\), the subdifferential \( \partial \varphi(x) \) of \( \varphi \) at a point \( x \in X \) is, in general, a set defined by the rule

\[
\partial \varphi(x) = \{ h \in X; \langle h, y - x \rangle_X \leq \varphi(y) - \varphi(x) \ \forall y \in X \}.
\]

Let \( f_\ast, f^\ast \) be two Lipschitz continuous functions defined on \( \mathbb{R} \). Then, the subdifferential of the indicator function \( I(u; \cdot), u \in \mathbb{R}, \)

\[
I(u; v) := \begin{cases} 
0 & \text{if } f_\ast(u) \leq v \leq f^\ast(u), \\
+\infty & \text{otherwise},
\end{cases}
\]

of the interval \([f_\ast(u), f^\ast(u)]\) has the form:
the functions

For $\lambda > 0$, the Yosida regularization of $\partial I(u; v)$ is the function

$$
\partial I^\lambda(u; v) = \begin{cases} 
0 & \text{if } v \notin [f_*(u), f^*(u)], \\
[0, +\infty) & \text{if } v = f^*(u) > f_*(u), \\
\{0\} & \text{if } f_*(u) < v < f^*(u), \\
(-\infty, 0) & \text{if } v = f_*(u) < f^*(u), \\
(-\infty, +\infty) & \text{if } v = f_*(u) = f^*(u).
\end{cases}
$$

(2.1)

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. For $\varepsilon > 0$, denote by $f_\varepsilon(u)$, $u \in \mathbb{R}$, the following regularization of the function $f(u)$:

$$
f_\varepsilon(u) := \int_{\mathbb{R}} f(s) \rho_\varepsilon(u - s) \, ds = \int_{\mathbb{R}} f(u - \varepsilon s) \rho(s) \, ds,
$$

(2.3)

where $\rho \in C^\infty(\mathbb{R})$ is such that $\rho \geq 0$, $\rho(0) = 0$ when $|s| \geq 1$, $\rho(s) = \rho(-s)$, $\int_{\mathbb{R}} \rho(s) \, ds = 1$, $\rho_\varepsilon(s) := \varepsilon^{-1} \rho\left(\frac{s}{\varepsilon}\right)$.

The lemma below follows directly from the definition of $f_\varepsilon(u)$ and the properties of $\rho_\varepsilon(s)$.

**Lemma 2.1.** The function $f_\varepsilon(u)$ possesses the following properties:

1. $f_\varepsilon(u) \in C^\infty(\mathbb{R})$;
2. $f_\varepsilon(u)$ is Lipschitz continuous with the same Lipschitz constant as $f(u)$;
3. $f_\varepsilon(u) \to f(u)$ as $\varepsilon \to 0$ uniformly on $\mathbb{R}$;
4. $|f_\varepsilon| \lesssim |f|$.

Problem (1.1)–(1.4) is considered under the following hypotheses:

**H1** the functions $h, g : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz continuous, the function $h$ is bounded, and the Lipschitz constant in the variable $v$ of $g$: $L_\varepsilon < \kappa/C_P$, where $C_P$ is the best constant of the Poincaré inequality on $\Omega$;

**H2** the functions $f_*, f^* : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous, $f_*(u) \leq f^*(u)$ for all $u \in \mathbb{R}$, and there exist constants $a, b$, $a < b$, such that $f_*(u) = f^*(u)$ for $u \in \mathbb{R} \setminus (a, b)$.

Next, we define a notion of solution to our Problem (P).

**Definition 2.1.** A pair $\{u, v\}$ is called a solution of system (1.1)–(1.4) if

(i) $u, v \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$;

(ii) $u' - \Delta u = h(u, v)$ in $H$ a.e. on $[0, T]$;

(iii) $v' - \kappa \Delta v + \partial I(u; v) \ni g(u, v)$ in $H$ a.e. on $[0, T]$;
(iv) \( u = v = 0 \) on \( \partial \Omega \) (in the sense of traces) a.e. on \([0, T]\);

(v) \( u(0) = u(T), \, v(0) = v(T) \) in \( H \),

where the prime denotes the derivative with respect to \( t \).

We note that in view of the variational characterization of subdifferential, the part (iii) of the above definition implies that

\[
f_*(u) \leq v \leq f^*(u) \text{ a.e. in } Q(T)
\]

(2.4)

and

\[
\langle v' - \kappa \Delta v(t) - g(u(t), v(t)), v(t) - z \rangle_H \leq 0
\]

(2.5)

for all \( z \in H \) with \( f_*(u(t)) \leq z \leq f^*(u(t)) \) a.e. in \( \Omega \) for a.e. \( t \in [0, T] \).

3 Approximate problems

First, we note that if \( (u, v) \) is a solution to Problem \((P)\), then \([1.1], [1.3], [1.4]\), and Hypothesis \((H1)\) imply that \( |u|_{L^\infty(Q)} \leq M_1 \), where \( M_1 > 0 \) is a constant depending on \(|h|_{L^\infty(Q)}\) and \( T \) only (see, e.g., [17, Proposition 10]). By virtue of this estimate, from Hypothesis \((H2)\) and \([2.4]\) we see that also \( |v|_{L^\infty(Q)} \leq M_2 \) for a constant \( M_2 > 0 \) depending on \( M_1 \) and the Lipschitz constants of \( f_*, f^* \). Therefore, we may now assume (cutting off outside the set where \( u \) and \( v \) are bounded, if necessary) that the functions \( h, g, f_*, f^* \) are all bounded and globally Lipschitz continuous.

In order to prove the existence of a solution to our Problem \((P)\), we approximate the latter by a family of suitable problems depending on two approximation parameters which we introduce next.

Let \( \varepsilon > 0 \) and \( f_{\varepsilon}(u) \) and \( f_{\varepsilon}^*(u) \) be the regularizations as in \([2.3]\) of the functions \( f_*(u) \) and \( f^*(u) \), respectively.

For \( \lambda, \varepsilon > 0 \), we consider the following approximate periodic problem denoted by \((P)_{\lambda, \varepsilon}\):

\[
u' - \Delta u = h(u, v) \quad \text{in } H \text{ a.e. on } [0, T],
\]

(3.1)

\[
v' - \kappa \Delta v + \partial I^\varepsilon_*(u; v) = g(u, v) \quad \text{in } H \text{ a.e. on } [0, T],
\]

(3.2)

\[
u = v = 0 \quad \text{on } \partial \Omega \text{ a.e. on } [0, T],
\]

(3.3)

\[
u(0) = u(T), \quad v(0) = v(T) \quad \text{in } H,
\]

(3.4)

where \( \partial I^\varepsilon_*(u; v) \) is defined as \( \partial I^\varepsilon_*(u; v) \) in \([2.2]\) with \( f_* \) and \( f^* \) replaced by \( f_\varepsilon \) and \( f_\varepsilon^* \), respectively.

A pair of functions \( \{u, v\} \) is called a solution to \((P)_{\lambda, \varepsilon}\) if \( u, v \in W^{1, 2}(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \) and \((3.1)-(3.3)\) hold.

In this section, we prove the existence of solutions for problems \((P)_{\lambda, \varepsilon}, \lambda, \varepsilon > 0\). To this aim, first, for convenience, we rewrite \((P)_{\lambda, \varepsilon}\) as the following equivalent to \((P)_{\lambda, \varepsilon}\) periodic problem for a single abstract nonlinear differential equation in the Hilbert space \( H := H \times H \):

\[
z'(t) + \partial \varphi(z(t)) = F(z(t)) \quad \text{in } H \text{ for a.e. } t \in [0, T],
\]

(3.5)
Proof. Let \( z := (u, v) \in \mathbf{H} \),
\[
\varphi(z) := \begin{cases} 
\frac{1}{2} |\nabla u|_H^2 + \frac{\kappa}{2} |\nabla v|_H^2 + \frac{1}{2\lambda} |v|_H^2 & \text{if } z \in H_1^1(\Omega) \times H_0^1(\Omega), \\
+\infty & \text{otherwise},
\end{cases}
\] (3.7)
and the notion of a solution to (3.6) naturally extends from Definition 2.1.

It is a standard matter to show that the function \( \varphi \) is a strictly monotone operator, \( \text{(see, e.g., [18, Theorem 1.4])} \) (see, e.g., [18, §2.3]) implies that there exists a unique solution \( z = z(f) \) of (3.11) below) naturally extends from Definition 2.1.

Given a function \( f \in L^2(0, T; \mathbf{H}) \) consider the following periodic problem associated with (3.5), (3.6):
\[
z'(t) + \partial \varphi(z(t)) = f(t) \quad \text{in } \mathbf{H} \quad \text{for a.e. } t \in [0, T],
\] (3.10)
\[
z(0) = z(T) \quad \text{in } \mathbf{H}.
\] (3.11)
It is a standard matter to show that the function \( \varphi \) defined by (3.7) satisfies the assumptions (A.1)–(A.3) of [18]. Moreover, since the subdifferential \( \partial \varphi \) (see (3.9)) of this function is obviously strictly monotone operator, \( \text{[18, Theorem 1.4]} \) (see, also, [19, §2.3]) implies that there exists a unique solution \( z = z(f) \) of (3.10), (3.11).

Next, define the solution operator \( \mathcal{T} : L^2(0, T; \mathbf{H}) \to C([0, T]; \mathbf{H}) \) which with each \( f \in L^2(0, T; \mathbf{H}) \) associates the unique solution \( z = z(f) \) of Problem (3.10), (3.11).

**Proposition 3.1.** The operator \( \mathcal{T} : L^2(0, T; \mathbf{H}) \to C([0, T]; \mathbf{H}) \) is weak-strong continuous in the sense that if \( f_n \to f \) weakly in \( L^2(0, T; \mathbf{H}) \) for \( f, f_n \in L^2(0, T; \mathbf{H}), n \geq 1 \), then \( \mathcal{T}(f_n) \to \mathcal{T}(f) \) strongly in \( C([0, T]; \mathbf{H}) \).

**Proof.** Let \( f_n \to f \) weakly in \( L^2(0, T; \mathbf{H}) \) and \( z_n := \mathcal{T}(f_n), n \geq 1 \). Then, we have
\[
z'(n) + \partial \varphi(z_n(t)) = f_n(t) \quad \text{in } \mathbf{H} \quad \text{for a.e. } t \in [0, T],
\] (3.12)
\[
z_n(0) = z_n(T) \quad \text{in } \mathbf{H}.
\] (3.13)
Testing Eq. (3.12) by \( z_n'(t) \) and applying Young’s inequality we obtain (cf. (3.7))
\[
|z_n'|_H^2 + \frac{d}{dt} \langle \partial \varphi(z_n), z_n \rangle_\mathbf{H} \leq C_1 \quad \text{a.e. on } [0, T],
\] (3.14)
where $C_1 := \sup_{n\geq 1} \| f_n \|_{H^1}^2$, $\cdot \|_{H^1}$ and $\langle \cdot, \cdot \rangle_{H^1}$ are the norm and inner product in $H^1$, respectively. Similarly, testing Eq. (3.12) by $\partial \varphi(z_n)$ we have
\[
\frac{d}{dt} \langle \partial \varphi(z_n), z_n \rangle_{H^1} + |\partial \varphi(z_n)|_{H^1}^2 \leq C_1 \quad \text{a.e. on } [0, T].
\] (3.15)

Now, we test (3.12) by $z_n$ and invoke the Poincaré inequality to obtain
\[
\frac{d}{dt} |z_n|_{H^1}^2 + \langle \partial \varphi(z_n), z_n \rangle_{H^1} \leq C_2 \quad \text{a.e. on } [0, T],
\]
where $C_2 > 0$ is a constant depending on $C_1$, $\lambda$ and the constant of the Poincaré inequality. Integrating this inequality from 0 to $T$ and taking account of the periodicity condition (3.13) we deduce that
\[
\int_0^T \langle \partial \varphi(z_n(\tau)), z_n(\tau) \rangle_{H^1} d\tau \leq C_2 T.
\] (3.16)

Next, to show that the sequence
\[
\{ \langle \partial \varphi(z_n), z_n \rangle \}_{n \geq 1}
\]
is bounded in $L^\infty(0, T; H^1)$
\[
(3.17)
\]
we note that $z_n$, $n \geq 1$, are $T$-periodic and we can thus consider their periodic extensions onto the whole real axis. Then, we take $t \in [T, 2T]$, $\tau \in (0, T)$, and integrate the inequality
\[
\frac{d}{dt} \langle \partial \varphi(z_n), z_n \rangle_{H^1} \leq C_1 \quad \text{a.e. on } [0, T],
\]
obtained from (3.15), from $t$ to $t$ to get
\[
\langle \partial \varphi(z_n(t)), z_n(t) \rangle_{H^1} \leq \langle \partial \varphi(z_n(\tau)), z_n(\tau) \rangle_{H^1} + 2C_1 T.
\]

Integrating this inequality over $\tau$ from 0 to $T$ and using (3.16) we obtain (3.17).

Integrating (3.14), (3.15) from 0 to $T$ and taking account of the periodicity condition (3.13) we see in view of (3.7) that the following uniform with respect to $n \geq 1$ estimates hold for the components $u_n, v_n$ of the solution $z_n$ of (3.12), (3.13):
\[
|u_n'|_{L^2(0,T;H^1)} + |\Delta u_n|_{L^2(0,T;H^1)} + |\nabla u_n|_{L^\infty(0,T;H^1)} + |v_n'|_{L^2(0,T;H^1)} + |\Delta v_n|_{L^2(0,T;H^1)} + |\nabla v_n|_{L^\infty(0,T;H^1)} \leq C_3,
\] (3.18)
for a positive constant $C_3$ independent of $n \geq 1$. On account of these uniform estimates, by weak and weak-star compactness results, there exists a subsequence (still indexed by $n$) of the sequence $z_n$, $n \geq 1$, and a function $z \in W^{1,2}(0, T; H^1) \cap L^\infty(0, T; V \times V) \cap L^2(0, T; H^2(\Omega) \times H^2(\Omega))$ such that
\[
z_n \to z \quad \text{weakly in } W^{1,2}(0, T; H^1) \cap L^2(0, T; H^2(\Omega) \times H^2(\Omega))
\] and weakly-star in $L^\infty(0, T; V \times V)$.
\] (3.19)
In particular, we also have

\[ z_n \to z \quad \text{in} \quad C([0, T]; \mathbf{H}). \]  

(3.20)

The convergences (3.19), (3.20) imply that \( z \) satisfies (3.10), (3.11). Therefore, \( z = T(f) \) and the claim of the proposition follows.

From Hypotheses \((H1), (H2), \) Lemma 2.1 (2), (4), and (3.8) we see that the mapping \( F : C([0, T]; \mathbf{H}) \to L^2(0, T; \mathbf{H}) \) is continuous and there exists a constant \( R > 0 \) such that

\[ |F(z)|_H \leq R \quad \text{for all} \quad z \in \mathbf{H}. \]  

(3.21)

We now introduce the set

\[ S_R = \{ f \in L^2(0, T; \mathbf{H}) : |f(t)|_H \leq R \quad \text{for a.e.} \quad t \in [0, T] \}, \]

and define the superposition \( F \circ T : S_R \to L^2(0, T; \mathbf{H}) \) of the solution operator \( T \) and \( F \):

\[ F \circ T(f) = F(T(f)). \]

From Proposition 3.1 and (3.21) it follows that \( F \circ T : S_R \to S_R \) is weak-weak continuous. Since the set \( S_R \) is obviously convex and compact in the weak topology of the space \( L^2(0, T; \mathbf{H}) \), from the Schauder fixed point theorem we conclude that there exists a fixed point \( f* \in S_R \) of the operator \( F \circ T : \)

\[ f* = F(T(f*)). \]

Setting \( z* := T(f*) \) we see that \( z* \) is a solution to the periodic problem (3.5), (3.6), which provides the desired solution of Problem \((P)_{\lambda, \varepsilon}, \lambda, \varepsilon > 0.\)

4 Well-posedness of problem \((P)\)

In this section, first we derive uniform a priori estimates independent of the parameter \( \varepsilon > 0 \) for solutions \( \{u_{\lambda \varepsilon}, v_{\lambda \varepsilon}\} \) of the approximate periodic Problem \((P)_{\lambda, \varepsilon}, \) which will allow us to derive the convergence of \( \{u_{\lambda \varepsilon}, v_{\lambda \varepsilon}\} \) as \( \varepsilon \to 0 \) to a solution \( \{u_{\lambda}, v_{\lambda}\} \) of an intermediate approximate problem depending on the parameter \( \lambda \) only. Then, we establish uniform bounds independent of the parameter \( \lambda \) for solutions \( \{u_{\lambda}, v_{\lambda}\} \) of the latter system and finally pass to the limit as \( \lambda \to 0 \) to obtain a solution to our original periodic problem \((P).\)

4.1 Passage-to-the-limit: \( \varepsilon \to 0 \)

Fix \( \lambda > 0 \). Repeating the reasoning in derivation of (3.13) from (3.12), (3.13), using Hypothesis \((H1)\) we obtain from (3.11), (3.3) the following uniform with respect to the parameter \( \varepsilon > 0 \) estimates for the first component of solutions \( \{u_{\lambda \varepsilon}, v_{\lambda \varepsilon}\} \) of the approximate periodic problems \((P)_{\lambda \varepsilon}:\)

\[ |u_{\lambda \varepsilon}'|_{L^2(0, T; H)} + |\Delta u_{\lambda \varepsilon}|_{L^2(0, T; H)} + |\nabla u_{\lambda \varepsilon}|_{L^\infty(0, T; H)} \leq R_1, \]  

(4.1)
for a positive constant $R_1$ depending on $|h|_{L^\infty(\mathbb{R}^2)}$ and the Lebesgue measure of $\Omega$, but independent of $\varepsilon$. On account of these uniform estimates, by weak and weak-star compactness results, there exists a null sequence $\varepsilon_n$, $n \geq 1$, in $(0,1]$ and a function $u_\lambda$ such that

$$u_{\lambda\varepsilon_n} \rightharpoonup u_\lambda \quad \text{weakly in } W^{1,2}(0,T;H) \cap L^2(0,T;H^2(\Omega))$$

and weakly-star in $L^\infty(0,T;V)$. (4.2)

In particular, we also have

$$u_{\lambda\varepsilon_n} \rightarrow u_\lambda \quad \text{in } C([0,T];H). \quad (4.3)$$

Invoking the Poincaré inequality from (4.1) we obtain

$$\int_0^T |u_{\lambda\varepsilon}^{\varepsilon}(\tau)|_H^2 d\tau \leq C_3^2 R_1.$$ 

(4.4)

In order to derive uniform estimates for $v_{\lambda\varepsilon}$, we recall first the following result.

**Lemma 4.1** ([7, Lemma 4.1]). Let $(u,v)$ be a solution of (3.1), (3.2). Then, the function

$$t \mapsto I_\lambda^{\varepsilon}(u;v)(t) = \frac{1}{2\lambda} \left| |v(t) - f^{\ast}_\varepsilon(u(t))|^+\right|^2_H + \frac{1}{2\lambda} \left| |f^{\ast}_\varepsilon(u(t)) - v(t)|^+\right|^2_H$$

is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt} I_\lambda^{\varepsilon}(u;v) \leq \langle \partial I_\lambda^{\varepsilon}(u;v), v' \rangle_H + L_0 |u'|_H |\partial I_\lambda^{\varepsilon}(u;v)|_H$$

a.e. in $(0,T)$, where $L_0$ is a common Lipschitz constant of $f_*$ and $f^*$ (see Lemma 2.1(2)). □

Now, testing Eq. (3.2) by $v'_{\lambda\varepsilon}$ we see in view of Lemma 4.1 that

$$\frac{1}{2} |v'_{\lambda\varepsilon}|_H^2 + \frac{d}{dt} \left( \frac{\kappa}{2} |\nabla v_{\lambda\varepsilon}|_H^2 + I_\varepsilon^{\lambda}(u_{\lambda\varepsilon};v_{\lambda\varepsilon}) \right)$$

$$\leq L_0^2 |u_{\lambda\varepsilon}'|_H^2 + \frac{1}{4} |\partial I_\varepsilon^{\lambda}(u_{\lambda\varepsilon};v_{\lambda\varepsilon})|_H^2 + R_2$$

a.e. on $(0,T)$, where $R_2 > 0$ is a constant independent of $\varepsilon$. Then, testing Eq. (3.2) by $-\kappa \Delta v_{\lambda\varepsilon}$ yields

$$\kappa^2 |\Delta v_{\lambda\varepsilon}|_H^2 + \frac{d}{dt} \left( \frac{\kappa}{2} |\nabla v_{\lambda\varepsilon}|_H^2 \right) \leq \langle \kappa \Delta v_{\lambda\varepsilon}, \partial I_\varepsilon^{\lambda}(u_{\lambda\varepsilon};v_{\lambda\varepsilon}) \rangle_H$$

$$+ \frac{1}{2} \kappa^2 |\Delta v_{\lambda\varepsilon}|_H^2 + 4R_2$$

a.e. in $(0,T)$.
a.e. on \((0, T)\). We evaluate the first term on the right-hand side of this inequality as follows

\[
\langle \partial I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon), \kappa \Delta v_\varepsilon \rangle_H = \\
\left( \frac{\kappa}{\lambda} [v_\varepsilon - f_\varepsilon^*(u_\varepsilon)]^+, \Delta (v_\varepsilon - f_\varepsilon^*(u_\varepsilon)) \right)_H + \left( \frac{\kappa}{\lambda} [v_\varepsilon - f_\varepsilon^*(u_\varepsilon)]^+, \Delta f_\varepsilon^*(u_\varepsilon) \right)_H + \\
\left( \frac{\kappa}{\lambda} [f_\varepsilon(u_\varepsilon) - v_\varepsilon]^+, \Delta (f_\varepsilon(u_\varepsilon) - v_\varepsilon) \right)_H + \left( \frac{\kappa}{\lambda} [f_\varepsilon(u_\varepsilon) - v_\varepsilon]^+, -\Delta f_\varepsilon(u_\varepsilon) \right)_H \\
= -\frac{\kappa}{\lambda} \left| \nabla [v_\varepsilon - f_\varepsilon^*(u_\varepsilon)]^+ \right|^2_H + \left( \frac{\kappa}{\lambda} [v_\varepsilon - f_\varepsilon^*(u_\varepsilon)]^+, \Delta f_\varepsilon^*(u_\varepsilon) \right)_H \\
- \frac{\kappa}{\lambda} \left| \nabla [f_\varepsilon(u_\varepsilon) - v_\varepsilon]^+ \right|^2_H + \left( \frac{\kappa}{\lambda} [f_\varepsilon(u_\varepsilon) - v_\varepsilon]^+, -\Delta f_\varepsilon(u_\varepsilon) \right)_H \\
\leq \frac{1}{8\lambda^2} \left( \left| [v_\varepsilon - f_\varepsilon^*(u_\varepsilon)]^+ \right|^2_H + \left| [f_\varepsilon(u_\varepsilon) - v_\varepsilon]^+ \right|^2_H \right) \\
+ 2\kappa^2 \left( |\Delta f_\varepsilon^*(u_\varepsilon)|^2_H + |\Delta f_\varepsilon(u_\varepsilon)|^2_H \right).
\]

Observing that \(\Delta f(u) = f'''(u)|\nabla u|^2_H + f''(u)\Delta u\) and invoking the Gagliardo-Nirenberg inequality from the last inequality we obtain

\[
\langle \partial I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon), \kappa \Delta v_\varepsilon \rangle_H \leq \frac{1}{8} \left| \partial I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon) \right|^2_H + 2\kappa^2 R_3 (|u_\varepsilon|^2_H + |\Delta u_\varepsilon|^2_H)
\]

for a constant \(R_3 > 0\) independent of \(\varepsilon\), so that Eq. (4.7) implies that

\[
\kappa^2 |\Delta v_\varepsilon|^2_H + \frac{d}{dt} \left( \frac{\kappa}{2} |\nabla v_\varepsilon|^2_H \right) \leq \frac{1}{8} \left| \partial I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon) \right|^2_H \\
+ 2\kappa^2 R_3 (|u_\varepsilon|^2_H + |\Delta u_\varepsilon|^2_H) \tag{4.8}
\]

a.e. on \((0, T)\). Similarly, testing Eq. (3.2) by \(\partial I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon)\) and using Lemma 3.1 we see that

\[
\frac{1}{2} \left| \partial I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon) \right|^2_H + \frac{d}{dt} I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon) \leq L_0^2 |u_\varepsilon|^2_H \\
+ 2\kappa^2 R_3 (|u_\varepsilon|^2_H + |\Delta u_\varepsilon|^2_H) + 4R_2 \tag{4.9}
\]

a.e. on \((0, T)\).

Next, taking the sum of the inequalities (4.6), (4.8), and (4.9), then integrating the result from 0 to \(T\) we see in view of (3.4), (4.1), and (4.3) that

\[
|v_\varepsilon'|_{L^2(0,T;H)} + \kappa |\Delta v_\varepsilon|_{L^2(0,T;H)} + \left| \partial I_\varepsilon^\lambda (u_\varepsilon; v_\varepsilon) \right|_{L^2(0,T;H)} \leq R_4 \tag{4.10}
\]

for a constant \(R_4 > 0\) independent of \(\varepsilon\).

In view of (4.10), similarly to (3.16) we obtain

\[
\frac{\kappa}{2} \int_0^T \left| \nabla v_\varepsilon(\tau) \right|^2_H d\tau \leq R_5,
\]
where $R_5 > 0$ is a constant independent of $\varepsilon$. Invoking the Poincaré inequality we further derive
\begin{equation}
\frac{\kappa}{2} \int_0^T |v_{\lambda\varepsilon}(\tau)|_H^2 d\tau \leq R_5 C_P.
\end{equation}
(4.11)
Then, from (4.5), (4.4), and (4.11) we see that
\begin{equation}
\int_0^T \mathcal{I}_\lambda(u_{\lambda\varepsilon}(\tau), v_{\lambda\varepsilon}(\tau)) d\tau \leq \frac{1}{\lambda} R_6,
\end{equation}
(4.12)
where $R_6 > 0$ is a constant independent of $\varepsilon$ and $\lambda$. Taking account of (4.6), (4.8), (4.9), (4.11), (4.12) and reasoning as in derivation of (3.17) we infer that
\begin{equation}
|\nabla v_{\lambda\varepsilon_n}|_{L^\infty(0,T;H)} \leq R(\lambda)
\end{equation}
(4.13)
for a positive constant $R(\lambda)$ dependent on $\lambda$, $\kappa$, $L_0$, $T$, $R_1$, $R_2$, $R_3$, but independent of $\varepsilon$.

As before, the estimates (4.10), (4.13) imply the existence of a null sequence $\{\varepsilon_n\}_{n \geq 1}$ and a function $v_\lambda$ such that
\begin{equation}
v_{\lambda\varepsilon_n} \to v_\lambda \text{ weakly in } W^{1,2}(0,T;H) \cap L^2(0,T;H^2(\Omega))
\end{equation}
and weakly-star in $L^\infty(0,T;V)$.
(4.14)
In particular, we also have
\begin{equation}
v_{\lambda\varepsilon_n} \to v_\lambda \text{ in } C([0,T];H).
\end{equation}
(4.15)
Now, from the convergences (4.2), (4.3), (4.14), (4.15), Lemma 2.1 (2), (3), and the Arzela–Ascoli theorem we see that the pair $\{u_\lambda, v_\lambda\}$, $\lambda > 0$, is a solution of the following system, which we denote by $(P)_\lambda$: \begin{equation}
u' - \Delta u = h(u,v) \text{ in } H \text{ a.e. on } [0,T],
\end{equation}
(4.16)
\begin{equation}
u' - \kappa \Delta v + \partial I^\lambda(u;v) = g(u,v) \text{ in } H \text{ a.e. on } [0,T],
\end{equation}
(4.17)
\begin{equation}u = v = 0 \text{ on } \partial \Omega \text{ a.e. on } [0,T],
\end{equation}
(4.18)
\begin{equation}u(0) = u(T), \quad v(0) = v(T) \text{ in } H,
\end{equation}
(4.19)
where $\partial I^\lambda(u;v)$ is defined in (2.2).
A solution to $(P)_\lambda$ is a pair of functions $\{u, v\}$ such that $u, v \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ and (4.16)–(4.19) hold.

We note that the validity of the periodic condition (4.19) follows from (3.14) and (4.3), (4.15).
4.2 Passage-to-the-limit: \( \lambda \to 0 \)

We now derive a priori estimates uniform with respect to the parameter \( \lambda > 0 \) for solutions \((u_\lambda, v_\lambda)\) of Problem \((P_\lambda)\).

To this aim, we note that the constants \(R_1, R_4,\) and \(R_5\) in the uniform estimates of the previous subsection do not depend on \(\lambda\). Hence, for a solution \((u_\lambda, v_\lambda)\) of Problem \((P_\lambda)\) from (4.1), (4.10), and (4.11) we have

\[
|u_\lambda'|_{L^2(0,T;H)} + |\Delta u_\lambda|_{L^2(0,T;H)} + |v_{\lambda}|_{L^\infty(0,T;H)}
\]

\[
+ |v_\lambda'|_{L^2(0,T;H)} + \kappa|\Delta v_\lambda|_{L^2(0,T;H)} + |\partial I^\lambda(u_\lambda; v_\lambda)|_{L^2(0,T;H)} \leq R_7
\]

and

\[
|v_\lambda|_{L^2(0,T;H)} \leq R_7, \tag{4.20}
\]

for a constant \(R_7 > 0\) independent of \(\lambda\). In particular, as above we conclude that there exists a null sequence \(\lambda_n, n \geq 1\), in \((0,1]\) and functions \(u, v\) such that

\[
uo \to u \quad \text{weakly in } W^{1,2}(0,T;H) \cap L^2(0,T;H^2(\Omega))
\]

and weakly-star in \(L^\infty(0,T;V)\) \(\tag{4.21}\)

and, thus, strongly in \(C([0,T];H)\),

\[
v \to v \quad \text{weakly in } W^{1,2}(0,T;H), \tag{4.22}
\]

\[
\kappa \Delta v_\lambda \to \kappa \Delta v \quad \text{weakly in } L^2(0,T;H), \tag{4.23}
\]

\[
\partial I^\lambda(u_n, v_n) \to \xi \quad \text{weakly in } L^2(0,T;H) \tag{4.24}
\]

for some function \(\xi \in L^2(0,T;H)\).

Below, we show that along with the convergences (4.21)–(4.24) we also have

\[
v_\lambda \to v \quad \text{strongly in } C([0,T];H). \tag{4.25}
\]

To this end, take two arbitrary \(i,j \geq 1\) with \(i \neq j\) and denote \(u_i := u_\lambda^i, v_i := v_\lambda^i\). Then, from (4.17) it follows that

\[
v_j' - v_i' - \kappa(\Delta v_j - \Delta v_i) + \partial I^\lambda(u_j; v_j) - \partial I^\lambda(u_i; v_i) = g(u_j, v_j) - g(u_i, v_i),
\]

Testing this equality by \(v_j - v_i\), using the Lipschitz continuity of \(g\) and invoking Young’s inequality we have

\[
\frac{1}{2} \frac{d}{dt} |v_j - v_i|^2_H + \kappa|\nabla(v_j - v_i)|^2_H + \langle \partial I^\lambda(u_j; v_j) - \partial I^\lambda(u_i; v_i), v_j - v_i \rangle_H
\]

\[
\leq L_g |u_j - u_i|_H |v_j - v_i|_H + L_s |v_j - v_i|^2_H, \tag{4.26}
\]

where \(L_g\) and \(L_s\) are the Lipschitz constants of \(g\) in \(u\) and \(v\), respectively. Setting

\[
S^\lambda_{ij} = \langle \partial I^\lambda(u_j; v_j) - \partial I^\lambda(u_i; v_i), v_j - v_i \rangle_H.
\]

\[
(\partial I^\lambda(u_j; v_j) - \partial I^\lambda(u_i; v_i), v_j - v_i)_H.
\]

\[
S^\lambda_{ij} = \langle \partial I^\lambda(u_j; v_j) - \partial I^\lambda(u_i; v_i), v_j - v_i \rangle_H.
\]

\[
(\partial I^\lambda(u_j; v_j) - \partial I^\lambda(u_i; v_i), v_j - v_i)_H.
\]
we see from (2.2) that
\[ S^\lambda_{ij} = \left\langle \frac{1}{\lambda_j} [v_j - f^*(u_j)]^+ - \frac{1}{\lambda_i} [f_*(u_i) - v_i]^+, \frac{1}{\lambda_i} [f_*(u_i) - v_i]^+, v_j - v_i \right\rangle_H. \]

We have nine possible cases to estimate the value of \( S^\lambda_{ij} \) from below. First, assuming that \( v_j \geq f^*(u_j), v_i \geq f^*(u_i) \) we obtain
\[ S^\lambda_{ij} \geq \frac{1}{\lambda_j} (v_j - f^*(u_j)) - \frac{1}{\lambda_i} (v_i - f^*(u_i)), \]
\[
\frac{1}{\lambda_j} (v_j - f^*(u_j)) - \lambda_i \frac{1}{\lambda_i} (v_i - f^*(u_i)) + f^*(u_j) - f^*(u_i) \]
\[
\geq \lambda_j |\partial I^\lambda_j(u_j; v_j)|^2_H + \lambda_i |\partial I^\lambda_i(u_i; v_i)|^2_H - (\lambda_j + \lambda_i)|\partial I^\lambda_j(u_j; v_j)|_H |\partial I^\lambda_i(u_i; v_i)|_H \]
\[
- (|\partial I^\lambda_j(u_j; v_j)|_H + |\partial I^\lambda_i(u_i; v_i)|_H)|f^*(u_j) - f^*(u_i)|_H. \]

Second, when \( v_j \geq f^*(u_j), f_*(u_i) \leq v_i < f^*(u_i) \) we see that
\[ S^\lambda_{ij} \geq \frac{1}{\lambda_j} (v_j - f^*(u_j)), v_j - v_i, \]
\[
\geq -|\partial I^\lambda_j(u_j; v_j)|_H |f^*(u_j) - f^*(u_i)|_H. \]

Third, if \( v_j \geq f^*(u_j), v_i < f_*(u_i) \), then
\[ S^\lambda_{ij} \geq \frac{1}{\lambda_j} (v_j - f^*(u_j)) + \frac{1}{\lambda_i} (f_*(u_i) - v_i), v_j - v_i, \]
\[
\geq - (|\partial I^\lambda_j(u_j; v_j)|_H + |\partial I^\lambda_i(u_i; v_i)|_H) |f^*(u_j) - f^*(u_i)|_H. \]

The reasoning in the remaining cases:

\[
f_*(u_j) < v_j < f^*(u_j), v_i \geq f^*(u_i) \\
f_*(u_j) < v_j < f^*(u_j), f_*(u_i) \leq v_i < f^*(u_i) \\
f_*(u_j) < v_j < f^*(u_j), v_i < f_*(u_i) \\
\quad \quad v_j \leq f_*(u_j), v_i \geq f^*(u_i) \\
\quad \quad v_j \leq f_*(u_j), f_*(u_i) \leq v_i < f^*(u_i) \\
\quad \quad v_j \leq f_*(u_j), v_i < f_*(u_i) \]

is fully symmetric and is left to the reader. Consequently, we always have
\[ S^\lambda_{ij} \geq -(\lambda_j + \lambda_i)|\partial I^\lambda_j(u_j; v_j)|_H |\partial I^\lambda_i(u_i; v_i)|_H \]
\[
- (|\partial I^\lambda_j(u_j; v_j)|_H + |\partial I^\lambda_i(u_i; v_i)|_H) (|f^*(u_j) - f^*(u_i)|_H + |f_*(u_j) - f_*(u_i)|_H) \]
\[
=: -\delta^\lambda_{ij} \quad \quad (4.28) \]
For convenience, denote \( \bar{u} := u_j - u_i, \bar{v} := v_j - v_i. \) Then, from (4.20)–(4.28) we infer that
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_H + \kappa \|\nabla \bar{v}\|^2_H \leq L_g |\bar{u}|_H |\bar{v}|_H + L_s |\bar{v}|^2_H + \delta^\lambda_{ij}.
\]
Invoking the Poincaré inequality, from this inequality we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\bar{v}\|^2_H \leq - \left( \frac{\kappa}{C_P} - L_s \right) \|\bar{v}\|^2_H + L_g |\bar{u}|_H |\bar{v}|_H + \delta^\lambda_{ij}.
\]
Denote \( c_0 := \frac{\kappa}{C_P} - L_s \) (\( c_0 > 0 \) by Hypothesis (H1)). We further have
\[
\frac{1}{2} \frac{d}{dt} (e^{2c_0 t} \|\bar{v}\|^2_H) \leq 2e^{2c_0 t} (L_g |\bar{u}|_H |\bar{v}|_H + \delta^\lambda_{ij}).
\]
Integrating this inequality from 0 to \( t \in (0, T] \) and using Hölder’s inequality we see in view of (4.20) that
\[
\|\bar{v}(t)\|_H^2 \leq e^{-2c_0 t} \|\bar{v}(0)\|_H^2 + 2 \int_0^t (L_g |\bar{u}(\tau)|_H |\bar{v}(\tau)|_H + \delta^\lambda_{ij}(\tau)) d\tau
\]
\[
\leq e^{-2c_0 t} \|\bar{v}(0)\|_H^2 + 4L_g R_T |\bar{u}|_{L^2(0,T;H)} + 2 \int_0^T \delta^\lambda_{ij}(\tau) d\tau. \tag{4.29}
\]
Taking \( t = T \) in (4.29) and substituting the resulting inequality, using the fact that \( \bar{v}(T) = \bar{v}(0) \), into (4.20) we obtain
\[
\|\bar{v}(T)\|_H^2 \leq \frac{2e^{2c_0 T}}{e^{2c_0 T} - 1} \left( 2L_g R_T |\bar{u}|_{L^2(0,T;H)} + \int_0^T \delta^\lambda_{ij}(\tau) d\tau \right).
\]
Applying Gronwall’s inequality to this inequality we conclude in view of the convergence (4.21) that \( u_i, i \geq 1 \), is a Cauchy sequence in the space \( C([0, T]; H) \). Hence, according to (4.22) we obtain the convergence (4.25).

To show that \( v \in L^\infty(0, T; V) \), we note that (4.17) implies that
\[
g(u(t), v(t)) - v'(t) - \partial I^\lambda(u(t), v(t)) + v(t) \in \partial \psi(v(t)) \tag{a.e. \( t \in [0, T] \)},
\]
where
\[
\psi(v) := \begin{cases} \frac{1}{2} |v|^2_H + \frac{1}{2} |\nabla v|^2_H & \text{if } v \in V, \\ +\infty & \text{if } v \in H \setminus V. \end{cases}
\]
\[\partial \psi(v) = v - \Delta v.\]

Since all the functions on the left-hand side of the inclusion above belong to \( L^2(0, T; H) \), from [20] Lemma 3.3 it follows that the function
\[
t \to \varphi(v(t)) = \frac{1}{2} |v(t)|^2_H + \frac{1}{2} |\nabla v(t)|^2_H.
\]
is absolutely continuous. Consequently, the function \( t \to |v(t)|_V = (|v(t)|_H^2 + |\nabla v(t)|_H^2)^{1/2} \) is continuous. Hence, the set \( \{v(t) : t \in [0, T]\} \) is bounded in the space \( V \). Since the embedding \( H \hookrightarrow V' \) is dense, the function \( t \to v(t) \) which is continuous from \([0, T]\) to \( H \) is also continuous from \([0, T]\) to \( \omega V \). Since in a reflexive Banach space the weak convergence coupled with the convergence of norms implies the norm convergence, we conclude that the function \( t \to v(t) \) is continuous from \([0, T]\) to \( V \). In particular, \( v \in L^\infty(0, T; V) \).

Given the convergences (4.21)–(4.25), to finish the proof that the pair \( \{u, v\} \) is a solution to Problem (P) it remains to show that

\[
\xi \in \partial I(u; v) \quad \text{a.e. on } (0, T). \tag{4.30}
\]

To this end, let \( z \) be an arbitrary function from \( L^2(0, T; H) \) such that \( z \in [f_*(u), f^*(u)] \) a.e. on \( Q \). For every \( n \geq 1 \), define \( z_n \) to be the pointwise projection of \( z \) onto the set \([f_*(u_n), f^*(u_n)]\). Then, \( z_n \in [f_*(u_n), f^*(u_n)] \) a.e. on \( Q \), \( n \geq 1 \), and \( z_n \to z \) in \( L^2(0, T; H) \) as \( n \to \infty \). Consequently, since the operator \( \partial I^{\lambda_n}(u_n; \cdot) \) is the subdifferential of the function \( I^{\lambda_n}(u_n; \cdot) \), from the definition of subdifferential we have

\[
(\partial I^{\lambda_n}(u_n; v_n), z_n - v_n) \leq I^{\lambda_n}(u_n; z_n) - I^{\lambda_n}(u_n; v_n) \leq -I^{\lambda_n}(u_n; v_n) \leq 0, \tag{4.31}
\]

\( n \geq 1 \). On the other hand, from the uniform boundedness of \( \{\partial I^{\lambda_n}(u_n; v_n)\} \), \( n \geq 1 \), in \( L^2(0, T; H) \) and (2.2) we see that

\[
|v_n - f^*(u_n)|^+ + |f_*(u_n) - v_n|^+ = \lambda_n \|\partial I^{\lambda_n}(u_n; v_n)\| \to 0
\]
in \( L^2(0, T; H) \) as \( n \to \infty \). Therefore, we infer that \( v \in [f_*(u), f^*(u)] \) a.e. on \( Q \). Passing now to the limit as \( n \to \infty \) in (4.31) we conclude that (4.30) holds and \( \{u, v\} \) is thus a solution to problem (P).

Finally, we note that the periodicity condition (iv) in Definition 2.1 follows from (4.19), (4.21), and (4.25).

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