Quantum theory can consistently describe the use of itself in Frauchiger-Renner’s Gedankenexperiment

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Theoretical physics has faced many challenges since the advent of quantum mechanics. Recently, Frauchiger and Renner have presented a no-go theorem, which makes quantum mechanics more controversial. However, from our perspective, the process of proving appears questionable. Therefore, we discuss the validity of their proof approach in this letter. Here, we propose a simple thought experiment that clarifies how correctly the attributed quantum state can be written in problems similar to Frauchiger and Renner’s Gedankenexperiment. In the next step, with the help of the correct form of the quantum state, it is demonstrated that a fallacy occurred in the proof of the no-go theorem, which means it cannot be valid because of the wrong proof. Ultimately, getting help from Hardy’s paradox, we investigate whether there is an approach to modify their proof in order to lend the no-go theorem validity.

I. INTRODUCTION

Results of experiments are not generally deterministic according to the mathematical framework of quantum mechanics[1, 2]. This indeterminism has become a contentious topic not only in physics but also in philosophy[3]. Determinism is acceptable in common sense coming from classical physics. This has led to the acceptance of theories that claim quantum mechanics is incomplete, in which one of the most prominent scientists, Albert Einstein, believed[4]. Additionally, to create an ontological challenge, Erwin Schrödinger proposed a thought experiment known as Schrödinger’s cat[3, 5]. Next, Eugene Wigner modified Schrödinger’s cat experiment and made it more challenging from the epistemological perspective[6, 7]. Recently, Frauchiger and Renner, inspired by Wigner’s friend experiment, have proposed a new thought experiment that leads to Frauchiger-Renner’s (FR) no-go theorem that can strengthen the foundations of many-worlds interpretation[8-11]. However, with the help of Bohmian interpretation[11], it is claimed that single world can satisfy the completeness of quantum mechanics is the many-worlds variable, we demonstrate one of the assumptions used in the proof of the no-go theorem cannot be used.

First, we found it necessary to review the implementation of FR Gedankenexperiment in which four observers, F, F, W and W, perform their measurements in turn. In each round of the Gedankenexperiment, observer F tosses the coin in her lab L, then in response to the result, provides a spin-1/2 particle to observer F. If the result is tails, she will provide a particle polarised in |→⟩ direction, and in case of heads, she will send a particle polarised in |↓⟩ direction. In the next step, F measures the particle in her lab L, which has been prepared by F, to find |↓⟩ or |↑⟩. After that, W and W measure contents of L and L on their pertinent particle with respect to their setups, respectively. The measurement setup of W has been set on the basis \( \{|\uparrow\rangle, |\downarrow\rangle\} \), where (h,t) stands for (heads, tails), and the one of W has been set on the basis \( \{|\uparrow\rangle, |\downarrow\rangle\} \).

In this step, we derive the pure quantum state governing the experiment. It is obvious that the results of these four experiments are correlated. The way one can mathematically write this correlation is challenging, although it does not seem difficult at first. We begin to illuminate the idea by showing the correlation between F and F, which gives

\[
|\psi\rangle^{(1)}_{FF} = \sqrt{\frac{1}{3}}|\uparrow\rangle_F|\downarrow\rangle_L + \sqrt{\frac{2}{3}}|t\rangle_F|\rightarrow\rangle_L.
\]
By using \(|\rightarrow\) = \(\sqrt{2}(|\uparrow\rangle + |\downarrow\rangle)\) and the symbol \(:\) := that means the item on the right-hand side is being defined to be what is on the left-hand side, the quantum state can be written in a new form
\[
|\psi\rangle_{FF}^{(1)} = \sqrt{\frac{1}{3}} |h\rangle_F |\downarrow\rangle_F + \sqrt{\frac{1}{3}} |t\rangle_F |\uparrow\rangle_F + \sqrt{\frac{1}{3}} |t\rangle_F |\downarrow\rangle_F = |\psi\rangle_{FF}^{(2)}.
\] (2)

Then by employing \((\bar{\omega}F, |\text{fail}\rangle_F\) and \(|\text{ok}\rangle_F, |\text{fail}\rangle_F\) instead of \((|\chi\rangle_F, |\text{fail}\rangle_F\) and \(|\tau\rangle_F, |\text{fail}\rangle_F\), respectively, we can write
\[
|\psi\rangle_{FF}^{(1)} = 3\sqrt{\frac{1}{12}} |\text{fail}\rangle_F |\text{fail}\rangle_F + \sqrt{\frac{1}{12}} |\text{fail}\rangle_F |\text{ok}\rangle_F
- \sqrt{\frac{1}{12}} |\text{ok}\rangle_F |\text{fail}\rangle_F + \sqrt{\frac{1}{12}} |\text{ok}\rangle_F |\text{ok}\rangle_F
:= |\psi\rangle_{FF}^{(3)}.
\] (3)

It is clear that each of the quantum states \(|\psi\rangle_{FF}^{(1)}, |\psi\rangle_{FF}^{(2)}\) and \(|\psi\rangle_{FF}^{(3)}\) is equivalent to the other one. Nevertheless, this naming style plays a beneficial role in our argument. We further explain in the next few lines about using them.

Thus far, we have derived a quantum state based on the correlation between \(F\) and \(F\) measurement results. The next step is to extend the quantum state that is applied to predict all four measurements' results in each round of the Gedankenexperiment. To achieve this goal, we consider \(|\psi\rangle_{FF}^{(1)}\). Regardless of the contractual experiment setup, it is expected that \(|h\rangle_W |\downarrow\rangle_W\) corresponds to \(|h\rangle_F |\downarrow\rangle_F\), and \(|t\rangle_W |\rightarrow\rangle_W\) corresponds to \(|t\rangle_F |\rightarrow\rangle_F\). By this method, \(|\psi\rangle_{FF}^{(1)}\) expands to
\[
|\psi\rangle_{FFW}^{(1)} = \sqrt{\frac{1}{3}} |h\rangle_F |\downarrow\rangle_F |h\rangle_W |\downarrow\rangle_W
+ \sqrt{\frac{2}{3}} |t\rangle_F |\rightarrow\rangle_F |t\rangle_W |\rightarrow\rangle_W.
\] (4)

We rewrite \(|\psi\rangle_{FFW}^{(1)}\) in the basis states that the observers are supposed to measure, by using \(|\rightarrow\) = \(\sqrt{\frac{1}{2}}(|\uparrow\rangle + |\downarrow\rangle)\) and substituting \((\bar{\omega}F, |\text{fail}\rangle_W\) and \(|\text{ok}\rangle_W, |\text{fail}\rangle_W\) for \((|\chi\rangle_W, |\text{fail}\rangle_W\) and \(|\tau\rangle_W, |\text{fail}\rangle_W\), respectively\) \(^2\). This gives
\[
|\psi\rangle_{FFW}^{(1)} = \left[ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle + \sqrt{\frac{1}{6}} |t\rangle |\uparrow\rangle + \sqrt{\frac{1}{6}} |t\rangle |\downarrow\rangle \right]|\text{fail}\rangle |\text{fail}\rangle
+ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle |\text{fail}\rangle |\text{ok}\rangle
+ \left[ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle - \sqrt{\frac{1}{6}} |t\rangle |\uparrow\rangle - \sqrt{\frac{1}{6}} |t\rangle |\downarrow\rangle \right]|\text{ok}\rangle |\text{fail}\rangle
+ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle |\text{ok}\rangle |\text{ok}\rangle.
\] (5)

In the similar way, \(|\psi\rangle_{FFW}^{(2)}\) and \(|\psi\rangle_{FFW}^{(3)}\) can be written as
\[
|\psi\rangle_{FFW}^{(2)} = \left[ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle + \sqrt{\frac{1}{6}} |t\rangle |\uparrow\rangle + \sqrt{\frac{1}{6}} |t\rangle |\downarrow\rangle \right]|\text{fail}\rangle |\text{fail}\rangle
+ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle |\text{fail}\rangle |\text{ok}\rangle
+ \left[ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle - \sqrt{\frac{1}{6}} |t\rangle |\uparrow\rangle - \sqrt{\frac{1}{6}} |t\rangle |\downarrow\rangle \right]|\text{ok}\rangle |\text{fail}\rangle
+ \sqrt{\frac{1}{12}} |h\rangle |\downarrow\rangle |\text{ok}\rangle |\text{ok}\rangle.
\] (6)

and \(^2\)
\[
|\psi\rangle_{FFW}^{(3)} = \left[ \frac{\sqrt{3}}{16} |h\rangle + |t\rangle \right]|\uparrow\rangle |\downarrow\rangle |\text{fail}\rangle |\text{fail}\rangle
+ \sqrt{\frac{1}{48}} |h\rangle + |t\rangle \left[- |\uparrow\rangle + |\downarrow\rangle \right]|\text{fail}\rangle |\text{ok}\rangle
- \sqrt{\frac{1}{48}} |h\rangle - |t\rangle \left[ |\uparrow\rangle + |\downarrow\rangle \right]|\text{ok}\rangle |\text{fail}\rangle
+ \sqrt{\frac{1}{48}} |h\rangle - |t\rangle \left[- |\uparrow\rangle + |\downarrow\rangle \right]|\text{ok}\rangle |\text{ok}\rangle.
\] (7)

It should be noted that some may try to increase the dimensions of the Hilbert space by taking into account the memory of the observers or the data recorded in the registers of the measuring devices or anything else like that\(^3\), but employing them only leads to a more complicated quantum state without any significant effect. We neglect them for clean handwriting.

As it can be seen from equations (5-7), the quantum states \(|\psi\rangle_{FFW}^{(1)}, |\psi\rangle_{FFW}^{(2)}\) and \(|\psi\rangle_{FFW}^{(3)}\) are not equivalent to one another while as shown in the previous part, the quantum states \(|\psi\rangle_{FF}^{(1)}, |\psi\rangle_{FF}^{(2)}\) and \(|\psi\rangle_{FF}^{(3)}\) are equivalent. Therefore, when one applies Born's rule to these

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1 Due to the beauty of the text, indexing has been avoided on the right side of the equation\(^2\), and this method has been also adopted for the following two formulas.

2 Lazarovici and Hubert used \(|\psi\rangle_{FFW}^{(3)}\) in their work\(^3\).
III. THE SIMPLE THOUGHT EXPERIMENT

Now, the question arises as to which of these quantum states are appropriate. Before answering, it has to be considered that we just made three quantum states while we could make more by choosing other kinds of basis states. In fact, we can attribute countless quantum states to FR Gedankenexperiment, each of which makes a different probability distribution to predict the results of a unique experimental setup. In addition, these quantum states are observer-independent, which means one has full discretion in choosing them. Due to this observer-independency, relativistic interpretations are not helpful here. Thus just one of the quantum states is suitable. Then the new question is: what is the only acceptable quantum state?

We designed a simple experiment and analyzed it to answer the last question. In this experiment, we consider a spin-1/2 particle provided in |0⟩ direction, which has been given to Wigner’s friend, and she wants to determine whether the spin of the particle is |+⟩ or |−⟩. After her determination of the result, Wigner does his experiment to find out whether the spin of the particle is |0⟩ or |1⟩. Following the procedure outlined above, there are two types of quantum states in this case:

\[ |φ^{(1)}_F⟩ = |0⟩_F \text{ expanding} |φ^{(1)}_{FW}⟩ = |0⟩_F |0⟩_W \] (8)

and with the use of |0⟩ = \( \sqrt{\frac{1}{2}}(|+⟩ + |−⟩) \), we obtain

\[ |φ^{(2)}_F⟩ = \sqrt{\frac{1}{2}}(|+⟩_F + |−⟩_F) \text{ expanding} \]
\[ |φ^{(2)}_{FW}⟩ = \sqrt{\frac{1}{2}}(|+⟩_F |+⟩_W + |−⟩_F |−⟩_W). \] (9)

To create standard forms, which describe quantum states in a more comprehensible manner, we rewrite |φ^{(1)}_{FW}⟩ and |φ^{(2)}_{FW}⟩ by using |0⟩ = \( \sqrt{\frac{1}{2}}(|+⟩ + |−⟩) \), |+⟩ = \( \sqrt{\frac{1}{2}}(|0⟩ + |1⟩) \) and |−⟩ = \( \sqrt{\frac{1}{2}}(|0⟩ - |1⟩) \). This gives

\[ |φ^{(1)}_{FW}⟩ = |0⟩_F |0⟩_W \text{ expanding} \]
\[ |φ^{(1)}_{FW}⟩ = |0⟩_F |0⟩_W + |−⟩_F |0⟩_W \] (10)

and

\[ |φ^{(2)}_{FW}⟩ = \sqrt{\frac{1}{2}}(|+⟩_F |+⟩_W + |−⟩_F |−⟩_W) \]
\[ = \frac{1}{2}(|+⟩_F |0⟩_W + |−⟩_F |0⟩_W + |+⟩_F |1⟩_W - |−⟩_F |1⟩_W). \] (11)

As we all know, a probability distribution of experiment’s results could be generally derived if the experiment is repeated by observers a sufficient number of times. In this simple experiment, we can use the probability distribution already obtained by the experimental data[21], which is written as

\[ P(F = x, W = y) = \langle |η⟩ |x⟩⟨x| \otimes |y⟩⟨y| |η⟩, \] (13)

where the variables x and y can be substituted with (+ or -) and (0 or 1), respectively, and |η⟩ is an arbitrary quantum state. By Substituting |φ^{(1)}_{FW}⟩ for |η⟩ in equation (13), we have

\[ P(F = +, W = 0) = \frac{1}{4} \] (14a)
\[ P(F = +, W = 1) = \frac{1}{4} \] (14b)
\[ P(F = −, W = 0) = \frac{1}{4} \] (14c)
\[ P(F = −, W = 1) = \frac{1}{4} \] (14d)

According to equations (14a-14d), the probability distribution differs from what is shown in equations (12a-12d). Moreover, they indicate that occurrence of (F=+, W=1) and (F=−, W=1) is impossible, which is extremely anti-empirical. In contrast, by Substituting |φ^{(2)}_{FW}⟩ for |η⟩ all of the probabilities satisfy equations (12a-12d), which shows |φ^{(2)}_{FW}⟩ is the desired answer. The quantum state
\(|\phi\rangle_{FW}^{(2)}\) is acceptable because it is extended under the fact that Wigner’s friend, who is the observer measuring the spin at first, changes the quantum state of the particle. As a consequence, the sequence of observations is prominent in case of systems that are measured more than once, and it should be considered when describing quantum states. In summary, the acceptable quantum state in FR Gedankenexperiment is \(|\psi\rangle_{FWW}^{(2)}\). Finding the correct form of the quantum state allows us to examine the proof process in the next part.

IV. THE FALLACY

FR no-go theorem is based on three assumptions, Q, C, and S, but we focused on assumption Q, which states that if the probability of an event is one, then an agent is certain that the event occurs. We point out three cases in which the no-go theorem used assumption Q and examine the correctness of its use. First, \(\overline{F}\) observes "tails" and becomes certain that \(W\) will observe "fail". In fact, when \(\overline{F}\) observes "tails", she knows that the operation of using assumption Q has failed, and thus their chain of reasoning is broken. Consequently, Frauchiger and Renner’s no-go theorem cannot be proved with the method they used, and their claimed results should not be accepted until one adopts a correct method of proof.

\[
P(F = \downarrow, W = \overline{ok}) = \langle \psi\rangle_{FWW}^{(2)}(I \otimes \downarrow \otimes |\overline{ok}\rangle \otimes I) |\psi\rangle_{FWW}^{(2)} = \frac{1}{3} \neq 0. \tag{15c}
\]

This indicates that two of the desired probabilities are not zero, although just one violation is enough to show that the operation of using assumption Q has failed, and thus their chain of reasoning is broken. Consequently, Frauchiger and Renner’s no-go theorem cannot be proved with the method they used, and their claimed results should not be accepted until one adopts a correct method of proof.

V. HARDY’S PARADOX AND FR NO-GO THEOREM

Let us now examine whether there is an escape route for rescuing FR no-go theorem. The no-go theorem needs to use logic such as Hardy’s paradox to show that at least one of the three assumptions is not valid. Hardy’s paradox considers some assumptions that lead to separability of joint probabilities in \(\mathcal{H}^{\otimes 2}\). Finally, it assumes four joint probabilities that the value of three of them is manipulated into being zero, and by using the assumptions, the fourth probability should be zero. The paradox occurs when there are some quantum states whose fourth probability is not zero\[^{[22, 23]}\]. In a similar manner, Frauchiger and Renner intended to consider a group of assumptions and three probabilities equal zero such that

\[
P(F = \downarrow, W = |ok\rangle) = 0 \tag{16a}
\]
\[
P(F = \downarrow, W = \overline{ok}) = 0, \tag{16b}
\]

and

\[
P(W = |\overline{ok}\rangle, W = |ok\rangle) = 0. \tag{17}
\]

However, quantum mechanics formalism makes it possible to show that

\[
P(W = |\overline{ok}\rangle, W = |ok\rangle) \neq 0. \tag{18}
\]

As a matter of fact, Hardy’s paradox has been written in \(\mathcal{H}^{\otimes 2}\) while FR no-go theorem is discussed in \(\mathcal{H}^{\otimes 4}\). In Hardy’s paradox, regardless of the assumptions, the fourth probability might be nonzero. However, as we have depicted in FIG.2, the joint probability \(P(W = |\overline{ok}\rangle, W = |ok\rangle)\) cannot be nonzero in FR Gedankenexperiment. Moreover, the correct approach to use Hardy’s paradox in \(\mathcal{H}^{\otimes 4}\) is introducing sixteen joint probabilities instead of four, each containing four events and not two. Now, consider four events, one of which happened; because the occurrence probability for
the rest of the events is not one (or zero), an observer cannot be certain about what is happening. As a result, assumption Q is not helpful here. Therefore, the FR no-go theorem cannot be valid, even with modifying the method adopted by Frauchiger and Renner. Nevertheless, it can be a new challenge to prove FR no-go theorem by changing the proof method.

VI. CONCLUSION

In this work, our adherence to the empirical aspect of quantum mechanics guided our progress. In fact, we based our work on this belief that before considering any ontological or epistemological interpretation, we should use an approach that satisfies empirical concerns. We demonstrated how to write the probability distribution of the results and, accordingly, the quantum state governing the experiment in such experiments where the measurement of agents causes entanglement between the results, which satisfied our empirical concerns. Based on this method, we have shown the correct form of writing the quantum states at each stage of FR Gedankenexperiment, which can be found in the appendices section for details. The use of the quantum state’s correct form led to the conclusion that assumption Q could not be used in FR no-go theorem. It means that the no-go theorem claimed by Frauchiger and Renner has not been proved correctly. It should be emphasized that we did not use any non-empirical assumptions to advance our goal. In other words, none of the ontological and epistemological interpretations were used in our work, and we only moved within the framework of empirical rules that have been proven so far.

In the following, we investigated whether it is possible to prove the no-go theorem by modifying the Gedankenexperiment protocol without changing the dimension of the governing Hilbert space. According to the assumptions of the Gedankenexperiment, which lead to the zeroing of several specific joint (or equivalently conditional) probabilities, we came to the conclusion that the no-go theorem cannot be proven under any circumstances. However, changing the dimension of the Hilbert space can be a solution to demonstrate the no-go theorem for future works.

ACKNOWLEDGEMENTS

Thanks are due to the management and staff of Khayyam University for providing a conducive environment for research and study during the Corona pandemic. We are grateful to Dustin Lazarovici for his helpful email responses. It is our pleasure to thank Kimia Mokaram Dori and Nayereh Saberian for their assistance in improving the final editing of this work. This work was supported by Ferdowsi University of Mashhad under Grant No. 3/56175 (1400/10/15).

APPENDIX A: FINDING THE QUANTUM STATE

The method of arriving at equations (5), (6) and (7) from equations (1), (2) and (3), respectively, may be confusing. For the purpose of clarifying our method, we refer to Brukner’s work[18]. He proposes a thought experiment (Fig 2 in his work) similar to Frauchiger-Renner’s Gedankenexperiment. Although the protocol is different, the arrangement of agents and labs is identical. Therefore, his method in reaching the quantum state can be a suitable and reliable pattern to find the quantum state governing FR’s Gedankenexperiment. Equations (1), (2) and (3) in our work are similar to equation (4) in Brukner’s work, with little difference, which is we write our initial state in $\mathcal{H}^{\otimes 2}$ while Brukner wrote his ini-
tial state in $\mathcal{H}^{\otimes 4}$. Now, we show how the quantum state governing FR Gedankenexperiment can be written based on Brukner's method and finally the results of using his approach are not different from ours.

FR Gedankenexperiment has 4 agents who each performs their experiments in chronological order. In each round of Gedankenexperiment, first, agent $\bar{F}$ tosses her coin and finds out the result at $t_F$, and, depending on her outcome, she polarizes agent $F$'s particle. Next, agents $F$, $W$ and $W$ find out their measurement’s result at $t_F$, $t_W$ and $t_W$, respectively. It is obvious from the FR’s protocol that $t_F < t_F < t_W < t_W$. These four specific moments divide time into five different spans for each round of Gedankenexperiment. The five mentioned time spans are:

1- $t < t_F$:
In this time span, the quantum state, for each agent is

$$|\psi\rangle^{(FFWW)}_{t < t_F} \text{ for agents } F, F, W, W =\sqrt{\frac{1}{3}} |h\rangle_F |0\rangle_F |0\rangle_W |0\rangle_W + |\frac{2}{3} |t\rangle_F |0\rangle_F |0\rangle_W |0\rangle_W \right) \tag{A1}$$

where $|0\rangle_F$ means that there is no polarized particle in agent F’s hands to measure. Because agent $\bar{F}$ has not measured on her coin yet. Also, $|0\rangle_W (|0\rangle_W)$ denotes that if agent $W(W)$ decides to investigate the result of agent $\bar{F} (\bar{F})$’s measurement he finds out that the measurement has not been performed yet.

2- $t_F \leq t < t_F$:
Measuring the coin and polarizing the particle lead to the evolution of the quantum state under a unitary transformation. The evolved quantum state for agents $F$, $F$, $W$ and $W$ is obtained

$$|\psi\rangle^{(FFWW)}_{t_P < t < t_F} \text{ for agents } F, F, W, W =$$

$$U_F |\psi\rangle^{(FFWW)}_{t < t_F} \text{ for agents } F, F, W, W =$$

$$\sqrt{\frac{1}{3}} |h\rangle_F |\downarrow\rangle_F |h\rangle_W |0\rangle_W$$

$$+ |\frac{2}{3} |t\rangle_F |\downarrow\rangle_F |t\rangle_W |0\rangle_W \right) \tag{A2}$$

where $U_F$ denotes the mentioned unitary evolution that changes the initial quantum state (A1). On the other hand, from agent $\bar{F}$’s perspective, the quantum state collapses due to her own measurement. If she observes "heads", the quantum state collapses to

$$|\downarrow\rangle_F |h\rangle_W |0\rangle_W \right) \tag{A3}$$

And if she observes "tails", the quantum state collapses to

$$|\rightarrow\rangle_F |t\rangle_W |0\rangle_W \right) \tag{A4}$$

3- $t_F \leq t < t_W$:
Equation (A2) is evolved from agents $\bar{W} W, W$ point of view. This evolution begins shortly before $t_F$, when agent $F$ starts her measurement, and finished at $t_F$. The correct form of the unitary matrix is the one that gives us an evolved quantum state that is compatible with the empirical law obtained from section "III. THE SIM-

PLE THOUGHT EXPERIMENT" of our work. So the evolved quantum state for agents $\bar{W} W, W$ is obtained

$$|\psi\rangle^{(FFWW)}_{t_F < t < t_W} \text{ for agents } W, W =$$

$$U_F |\psi\rangle^{(FFWW)}_{t_F < t < t_F} \text{ for agents } F, W, W =$$

$$\sqrt{\frac{1}{3}} |h\rangle_F |\downarrow\rangle_F |h\rangle_W |\downarrow\rangle_W$$

$$+ |\frac{2}{3} |t\rangle_F |\downarrow\rangle_F |t\rangle_W |\downarrow\rangle_W \right) \tag{A5}$$

Also, equations (A3) and (A4) are evolved due to agent $F$ measurement. The evolved form of equation (A3) is

$$|\downarrow\rangle_F |h\rangle_W |\downarrow\rangle_W \right) \tag{A6}$$

And, the evolved form of equation (A4) is

$$\sqrt{\frac{1}{2} |\downarrow\rangle_F |t\rangle_W |\downarrow\rangle_W \right) \tag{A7}$$

Moreover, the quantum state has collapsed for agent $F$ due to her own measurement. If she observes "up", the quantum state collapses to

$$|\uparrow\rangle_F |t\rangle_W |\uparrow\rangle_W \right) \tag{A8}$$

Also, if she observes "down", the quantum state collapses to

$$\sqrt{\frac{1}{2} |\downarrow\rangle_F |t\rangle_W |\downarrow\rangle_W \right) \tag{A9}$$

To facilitate the following analysis, we prefer to rewrite the quantum state of equation (A5) in the standard form. It gives

$$\sqrt{\frac{1}{12} \left[ \begin{array}{c} |h\rangle |\downarrow\rangle + |t\rangle |\uparrow\rangle + |t\rangle |\downarrow\rangle \end{array} \right] |fail\rangle |fail\rangle$$

$$+ \sqrt{\frac{1}{12} \left[ \begin{array}{c} |h\rangle |\downarrow\rangle - |t\rangle |\uparrow\rangle + |t\rangle |\downarrow\rangle \end{array} \right] |fail\rangle |ok\rangle$$

$$+ \sqrt{\frac{1}{12} \left[ \begin{array}{c} |h\rangle |\downarrow\rangle - |t\rangle |\uparrow\rangle - |t\rangle |\downarrow\rangle \end{array} \right] |ok\rangle |fail\rangle$$

$$+ \sqrt{\frac{1}{12} \left[ \begin{array}{c} |h\rangle |\downarrow\rangle + |t\rangle |\uparrow\rangle - |t\rangle |\downarrow\rangle \end{array} \right] |ok\rangle |ok\rangle \right). \tag{A10}$$

As can be seen, the standard form of the quantum state is exactly equal to equation (6).
4- $t_W \leq t < t_W$:

It should be noted that the measurements of agents $W$ and $F$ cannot change the predictions of other agents. So the quantum states for agents $F$ (equations (A6) or (A7)), $F$ (equations (A8) or (A9)) and $W$ (equation (A10)) do not change. For agent $W$, the quantum state collapses due to his own measurement. If he observes "fail", the collapsed quantum state is

$$\frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{fail}\rangle_W$$

$$+ \frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{ok}\rangle_W.$$  (A11)

Also, if he observes "ok", the collapsed quantum state is

$$\frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{fail}\rangle_W$$

$$+ \frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{ok}\rangle_W.$$  (A12)

5- $t_W \leq t$:

The quantum states of agents $\bar{F}$ and $F$ do not change. For agent $W$, also equations (A11) and (A12) do not change. For agent $W$, it collapses due to its own measurement. If he observes "fail", the collapsed quantum state is

$$\frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{fail}\rangle_W$$

$$+ \frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{ok}\rangle_W.$$  (A13)

Also, if he observes "ok", the collapsed quantum state is

$$\frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{fail}\rangle_W$$

$$+ \frac{1}{\sqrt{6}} \left[ |\downarrow \rangle_F + |\uparrow \rangle_F - |\downarrow \rangle_F + |\uparrow \rangle_F \right] |\text{ok}\rangle_W.$$  (A14)

Except for equation (A11), the rest of the quantum states can be divided into two general categories: evolved quantum states and collapsed quantum states. Equation (A11) is the initial state. Equations (A2), (A5) and (A10) belong to the category of the evolved quantum states, and the rest of the equations belong to the category of collapsed form. With the help of the evolved quantum states, we can find the joint probabilities, and with the help of the collapsed quantum states, we can find the conditional probabilities. However, it should be noted that only some of these probabilities can be suitable for the analysis of FR Gedankenexperiment. For example, if agent $F$ wants to find $P(W_{t_F \leq t < t_F} = \text{ok} \mid \bar{F}_{t_F} = \text{tails})$, she should use equation (A4) whereas to find $P(W_{t_W} = \text{ok} \mid \bar{F}_{t_F} = \text{tails})$, she should use equation (A7). For this example, it is obvious that our desired conditional probability is $P(W_{t = t_W} = \text{ok} \mid \bar{F}_{t_F} = \text{tails})$, which targets the event mentioned in Gedankenexperiment protocol. It can be easily shown that all desired probabilities can be obtained from equation (A10) (or equation (6) in our main text). Therefore, there is appropriate empirical justification for using equation (6). From another point of view, equation (6) is the evolved form of the initial state (A11). Exactly the same as Brukner’s work [18] in which he got from equation (4) to (5) and used equation (5) to predict the probabilities he wanted. Thus, for the purpose of simplifying the analysis in our main text, we only used equation (6) and named it “the quantum state”. In addition, as was done in our main text, for the sake of simplicity, we avoid putting time indices in appendix B.

APPENDIX B: FR GEDANKENEXPERIMENT

In this appendix, we demonstrate the fallacy with the quantum sates obtained in appendix A to illuminate all doubts.

In the first step, agent $\bar{F}$ obtains "tails" at $t = t_{\bar{F}}$. From her point of view, the quantum state is equal to $|\text{A4}\rangle$ at this moment. However, to predict the result of agent $W$’s measurement she should use equation (A7), as the evolution due to agent $F$’s measurement is considered in equation (A7). With the help of this quantum state, the probability of the occurrence of result "fail" for the measurement of agent $W$ at $t = t_W$ is equal to

$$P(W = \text{fail} \mid \bar{F} = \text{tails}) = \frac{1}{2} \left[ |\downarrow \rangle \langle \downarrow | (I \otimes I \otimes |\text{fail}\rangle \langle \text{fail}|) |\downarrow \rangle |\downarrow \rangle + |\downarrow \rangle \langle \downarrow | (I \otimes I \otimes |\text{fail}\rangle \langle \text{fail}|) |\uparrow \rangle |\uparrow \rangle + |\downarrow \rangle \langle \uparrow | (I \otimes I \otimes |\text{fail}\rangle \langle \text{fail}|) |\downarrow \rangle |\uparrow \rangle + |\downarrow \rangle \langle \uparrow | (I \otimes I \otimes |\text{fail}\rangle \langle \text{fail}|) |\uparrow \rangle |\uparrow \rangle \right] = \frac{1}{2} \neq 1.$$  (B1)

However, Frauchiger and Renner in equation (4) of their work [5] used $\rightarrow$ and $|\text{fail}\rangle \langle \text{fail}|$ instead of $\sqrt{\frac{1}{2}} |\downarrow \rangle_F \langle \downarrow | W_{t_W} |\downarrow \rangle + \sqrt{\frac{1}{2}} |\uparrow \rangle_F \langle \downarrow | W_{t_W} |\downarrow \rangle$ and $(I \otimes I \otimes |\text{fail}\rangle \langle \text{fail}|)$, respectively, which leads to $1$ for desired probability. Because of this wrong choice, their result for this step is wrong, and they could not use assumption Q there. Also, in the third step, with the help of equation (A12), the probability $P(F = |\uparrow \rangle \ W = |\text{ok}\rangle)$ becomes $\frac{1}{3}$, which is not equal to $1$. So, equation (6) in their work has a wrong value too.

As mentioned in appendix A, equation (6) can be used to find these probabilities. In our main text, we used equation (6) and the joint probabilities obtained from it, while we could have used the conditional probabilities obtained from it and reached the results in appendix B exactly.
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