Abstract

Motivated by a path planning problem we consider the following procedure. Assume that we have two points \( s \) and \( t \) in the plane and take \( K = \emptyset \). At each step we add to \( K \) a compact convex set that does not contain \( s \) nor \( t \). The procedure terminates when the sets in \( K \) separate \( s \) and \( t \). We show how to add one set to \( K \) in \( O(1 + k\alpha(n)) \) amortized time plus the time needed to find all sets of \( K \) intersecting the newly added set, where \( n \) is the cardinality of \( K \), \( k \) is the number of sets in \( K \) intersecting the newly added set, and \( \alpha(\cdot) \) is the inverse of the Ackermann function.

1 Introduction

Consider the path planning problem from robotics, also known as the piano mover’s problem [8] [2, Ch.13]: Given an initial and a target configuration of a robot, the task is to decide whether the robot can move from the initial to the target configuration without colliding with itself or a surrounding object (and to find such a transformation if it exists). The problem is typically tackled by setting up a configuration space \( \mathbb{X} \) where every robot position is encoded as a single point. Then \( \mathbb{X} \) is partitioned into a free space \( \mathbb{F} \subseteq \mathbb{X} \) of allowed configurations and its complement \( \overline{\mathbb{F}} = \mathbb{X} \setminus \mathbb{F} \) denoting configurations that collide with obstacles. The initial and final state are denoted by two points \( s \) and \( t \) in \( \mathbb{F} \), and the task is to decide whether \( s \) and \( t \) are in the same path-connected component of \( \mathbb{F} \).

The following approach to solve the path planning problem is discussed by Wang, Chiang and Yap [12]. Assume for simplicity that the configuration space \( \mathbb{X} \) is a unit cube in \( \mathbb{R}^d \). For any given subcube, which we call box from now, we can decide whether the box is entirely contained in \( \mathbb{F} \), entirely contained in \( \overline{\mathbb{F}} \), or both contains points of \( \mathbb{F} \) and \( \overline{\mathbb{F}} \). We color a box green, red, or yellow, respectively, depending on the predicates outcome. Now, starting with the entire \( \mathbb{X} \), we build a quadtree structure and keep subdividing yellow boxes into \( 2^d \) boxes of equal size until one of the following events occur:

1. Points \( s \) and \( t \) lie in green boxes and are connected by a path that lies entirely in green boxes. Such a path is a solution to the path planning problem. See Figure I left, for an illustration.

2. Each path from \( s \) to \( t \) intersects some red square. In this case, no collision-free path from \( s \) to \( t \) can exist, and we say that the red boxes separate \( s \) and \( t \). See Figure I right, for an illustration.
The described subdivision strategy is also used for the task of segmentation of digital images; see [1] and references therein. In that situation, the approach would decide whether the pixels \( s \) and \( t \) belong to the same connected component of the image.

How quickly can we decide whether one of the two conditions is satisfied? Condition (1) can be easily checked by union-find [11]: just create a new element for each new green box and make unions to keep together adjacent green boxes, always checking whether the boxes containing \( s \) and \( t \) fall into the same set. That means that the amortized complexity of checking condition (1) is almost linear in the number of green boxes produced. For condition (2), the case seems less clear – an alternative way of phrasing the condition is to check whether the union of green and yellow boxes contains \( s \) and \( t \) in the same connected component. The union-find approach cannot directly be applied because yellow regions might turn into red and, therefore, the area covered by the boxes may shrink. In this paper, we discuss how to test the second condition in the planar case (\( d = 2 \)).

Figure 1: Left: Configuration space with two (convex) holes. When subdividing the marked yellow box according to the dashed lines, \( s \) and \( t \) become connected. Right: Configuration space with an annulus-shaped obstacle. When subdividing the marked yellow box, the union of red boxes separates \( s \) and \( t \), so no path can exist.

We consider the following generalization of the problem. We have two points \( s \) and \( t \) in the plane. We get a set \( K \) of compact, convex sets in the plane iteratively, adding the sets one by one. Each of the sets added to \( K \) is disjoint from \( s \) and \( t \). In the motivating problem, the red boxes would be the elements of \( K \). At the end of the insertion of a new compact convex set into \( K \), we want to know whether \( K \) separates \( s \) and \( t \). That is, we want to know whether each path from \( s \) to \( t \) has to intersect some element of \( K \). Thus, we want a semi-dynamic data structure to store \( K \) that allows the insertion of new elements to \( K \) and decides whether \( K \) separates \( s \) and \( t \).

We show that we can maintain \( K \) under insertions using a slightly more sophisticated union-find approach. The time to insert a new set \( K_u \) into \( K \) is the time we need to find all the \( k \) elements of \( K \) intersecting \( K_u \), plus \( O(k) \) union-find operations. The idea is based on a classical parity argument saying that \( s \) and \( t \) are separated if and only if we can find a closed curve contained in the union of the elements of \( K \) that is crossed an odd number of times by the line segment \( \ell \) from \( s \) to \( t \). We maintain a union-find data structure for the sets of \( K \) and augment it by storing additional information about the parity of crossings with the line segment \( \ell \). Using this additional knowledge, we can quickly decide whether adding a new set to \( K \) forms a cycle that separates \( s \) and \( t \), and the information can be maintained under unions and path compressions without asymptotic overhead.

If in the motivating subdivision procedure we always subdivide a largest yellow box, we
obtain $O(1)$ time per yellow box and $O(\alpha(n))$ amortized time per red box, where $n$ is the number of red boxes and $\alpha(\cdot)$ is the inverse of the Ackermann function. Thus, we obtain the same asymptotic behavior for testing conditions (1) and (2).

Roadmap. In Section 2 we discuss a criterion to decide when $\mathcal{K}$ separates $s$ and $t$ in the static case. In Section 3 we extend this to the semi-dynamic case, where sets get added to $\mathcal{K}$. In Section 4 we discuss the application to the motivating subdivision procedure.

Our aim is to provide a self-contained exposition. Some of the arguments are an adaptation of Cabello and Giannopoulos [3] to this simpler setting, others can be shorten substantially using machinery from Algebraic Topology.

2 Static connectivity

Let $\mathcal{K}$ denote a finite family of compact convex sets in the plane, and let $\mathbb{K}$ denote their union. We use the notation $\mathbb{K} = \mathbb{R}^2 \setminus \mathcal{K}$. Let $s$ and $t$ be points in $\mathbb{K}$.

The set $\mathbb{K}$ separates $s$ and $t$ if they are in different path-connected components of $\mathbb{K}$. Equivalently, $\mathbb{K}$ separates $s$ and $t$ if each path in the plane from $s$ to $t$ intersects $\mathbb{K}$. We also say that $\mathcal{K}$ separates $s$ and $t$.

In the next subsection we discuss a criterion to decide when $\mathbb{K}$ separates $s$ and $t$. The criterion is based on considering all polygonal paths contained in $\mathbb{K}$, and thus is computationally unfeasible. In Subsection 2.2 we discuss how this criterion can be checked in the intersection graph of $\mathcal{K}$, and thus obtain a discrete version suitable for computations.

We will consistently use Greek letters $\pi, \gamma, \tau, \ldots$ only for (polygonal) curves.

2.1 Topological criterion for separation

A polygonal curve $\pi$ is generic (with respect to $s$ and $t$) if $\pi$ does not contain $s$ nor $t$ and the line segment from $s$ to $t$ does neither contain an endpoint of $\pi$ nor a self-intersection of $\pi$. We will assume in our discussion that all the polygonal curves are generic. We can enforce this assumption making a rotation, so that $\ell$ is horizontal, and replacing the point $s$ by $s' = s + (0, \varepsilon)$, for an infinitesimal $\varepsilon > 0$. We always use the same perturbed point $s'$. Since $\mathcal{K}$ is finite, separation of $s$ and $t$ with $\mathbb{K}$ is equivalent to separation of $s'$ and $t$ with $\mathbb{K}$. The computations can then be made using simulation of simplicity [6].

We fix $\ell$ as the line segment joining $s'$ and $t$. The crossing number of $\ell$ with a polygonal curve $\pi$ is the number of intersections of $\ell$ and $\pi$. We denote by $\text{cr}_2(\ell, \pi)$ the modulo 2 value of the crossing number of $\ell$ and $\pi$. Thus, $\text{cr}_2(\ell, \pi) = 1$ if and only if the crossing number is odd. For the whole paper, any arithmetic involving $\text{cr}_2(\cdot, \cdot)$ is done modulo 2.

It is important to use always the same perturbed point $s'$. Then, if a polygonal curve $\pi$ is the concatenation of $\pi'$ and $\pi''$, we have $\text{cr}_2(\ell, \pi) = \text{cr}_2(\ell, \pi') + \text{cr}_2(\ell, \pi'')$. If we would use different perturbed points and the common endpoint of $\pi'$ and $\pi''$ lies in the line segment $st$, then the inequality does not necessarily hold.

A polygonal curve $\pi$ is closed if its endpoints coincide. It is simple if it does not have any self-intersections, except for the common endpoint in the case of closed polygonal paths.

Note that in the following lemma we do not require simple curves.

Lemma 1. The set $\mathbb{K}$ separates $s$ and $t$ if and only if there exists a closed polygonal curve $\pi$ contained in $\mathbb{K}$ such that $\text{cr}_2(\ell, \pi) = 1$.

Proof. We use the following classical argument, which sometimes is an intermediary step towards a proof of the Jordan’s curve theorem: A simple closed polygonal curve $\pi$ separates $s'$ and $t$ if and only if $\ell$ and $\pi$ have an odd crossing number. See, for example, Mohar and Thomassen [9, Section 2.1] for a formal proof.
Assume that $\mathcal{K}$ contains a closed polygonal curve $\pi$ such that $\ell$ and $\pi$ have an odd crossing number. The curve $\pi$ may have self-intersections. If $\pi$ is not simple, we can split it at self-intersections arbitrarily to obtain simple, closed polygonal paths $\pi_1, \ldots, \pi_k$ that have, all together, the same image as $\pi$. Since we have $1 = \text{cr}_2(\ell, \pi) = \sum_i \text{cr}_2(\ell, \pi_i)$, at least one of the curves $\pi_i$ has $1 = \text{cr}_2(\ell, \pi_i)$. Such a curve $\pi_i$ separates the endpoints of $\ell$, and thus separates $s$ and $t$. It follows that there is no path in $\mathbb{R}^2 \setminus \pi_i$ from $s$ to $t$. Since $\bar{\mathcal{K}} \subset \mathbb{R}^2 \setminus \pi_i$, there is no path in $\bar{\mathcal{K}}$ from $s$ to $t$.

Assume that there is no path in $\bar{\mathcal{K}}$ from $s$ to $t$. Consider the path-connected component $A$ of $\bar{\mathcal{K}}$ that contains $s$. Since $t$ is in a different cell of $\bar{\mathcal{K}}$ and $\mathcal{K}$ is a finite collection of compact, convex bodies, there exists a simple closed curve $\pi$ contained in the boundary of $A$ that separates $s$ and $t$. We can make shortcuts in $\pi$ to obtain a simple closed polygonal curve $\pi'$ contained in $\mathcal{K}$ that separates $s$ and $t$. (This can be shown formally using the convexity of the elements of $\mathcal{K}$ and the compactness of $\mathcal{K}$.) The resulting simple polygonal path $\pi'$ separates $s$ and $t$, and thus the crossing number of $\ell$ and $\pi'$ is odd. \hfill $\square$

**Lemma 2.** Let $K_u$ and $K_v$ be two compact convex sets of $\mathcal{K}$. For any two generic polygonal curves $\pi$ and $\pi'$ contained in $K_u \cup K_v$ with the same endpoints, we have $\text{cr}_2(\ell, \pi) = \text{cr}_2(\ell, \pi')$.

**Proof.** First note that $K_u \cup K_v$ does not separate $s$ and $t$. This can be seen as follows. Let $\mathbb{S}^1$ be the set of directions. Consider the set of directions of the vectors $\bar{\mathcal{K}}x$, for all $x \in K_u$. Since $K_u$ is convex and $s \not\in K_u$, this directions cover less than half of $\mathbb{S}^1$. A similar statement holds for $K_v$. It follows that there exists some ray from $s$ to infinity in $\mathbb{R}^2 \setminus (K_u \cup K_v)$. Similarly, there exists a ray from $t$ to infinity in $\mathbb{R}^2 \setminus (K_u \cup K_v)$. Those two rays and an extra path far enough can be combined to obtain a path from $s$ to $t$ in $\mathbb{R}^2 \setminus (K_u \cup K_v)$. Thus, $K_u \cup K_v$ does not separate $s$ and $t$.

Since $K_u \cup K_v$ does not separate $s$ and $t$, Lemma 1 implies that any closed path $\gamma$ contained in $K_u \cup K_v$ has $\text{cr}_2(\ell, \gamma) = 0$. The concatenation of $\pi$ and the reverse of $\pi'$ is a closed path contained in $K_u \cup K_v$, and therefore $\text{cr}_2(\ell, \pi) + \text{cr}_2(\ell, \pi') = 0$. \hfill $\square$

### 2.2 Criterion on the Intersection Graph

Consider the **intersection graph** of $\mathcal{K}$ and denote it by $G$. Each element $K_v \in \mathcal{K}$ is a node of $G$; we will denote the node by $v$ to match standard graph theory notation. There is an edge $uv$ in $G$ if and only if $K_u$ and $K_v$ intersect. The graph $G$ is an abstract graph. Next we provide a geometric representation.

For each node $v$ of $G$ choose a point $p_v$ in $K_v$. For each edge $uv$ of $G$, let $\gamma(uv)$ be a polygonal path from $p_u$ to $p_v$ contained in the union $K_u \cup K_v$. Since $K_u$ and $K_v$ are convex and intersect, we can always choose $\gamma(uv)$ with at most 2 edges. The pair

$$\{\{p_v \mid v \in V(G)\}, \{\gamma(uv) \mid uv \in E(G)\}\}$$

is a drawing of $G$. (It is not necessarily an embedding because drawings of edges may cross, for example when four axis-parallel squares have disjoint interiors but share a vertex.) For each walk $W = e_1 \ldots e_k$ in $G$, let $\gamma(W)$ be the polygonal path obtained by concatenating $\gamma(e_1), \ldots, \gamma(e_k)$. If $W$ is a closed walk, then $\gamma(W)$ is a closed polygonal curve.

**Lemma 3.** The set $\mathcal{K}$ separates $s$ and $t$ if and only if there exists a closed walk $W$ in $G$ such that $\text{cr}_2(\ell, \gamma(W)) = 1$.

**Proof.** Assume that $\mathcal{K}$ separates $s$ and $t$. Because of Lemma 1, there is some polygonal curve $\pi$ contained in $\mathcal{K}$ such that $\text{cr}_2(\ell, \pi) = 1$. We break the path $\pi$ into pieces such that each piece is contained in the union of 2 sets from $\mathcal{K}$. Let $\pi_1, \ldots, \pi_k$ be the resulting pieces, each of them a polygonal curve. For each piece $\pi_i$, let $x_i$ and $y_i$ be the endpoints of $\pi_i$, and let $K_{x_i}$ and $K_{y_i}$...
be the elements of \( K \) that contain \( x_i \) and \( y_i \), respectively, so that \( \pi_i \) is contained in \( K_{u_i} \cup K_{v_i} \). Note that \( u_i v_i \) is an edge of \( G \). Let \( W \) be the closed walk with edges \( u_1 v_1, \ldots, u_k v_k \).

We claim that \( \text{cr}_2(\ell, \gamma(W)) = \text{cr}_2(\ell, \pi) = 1 \). To see this, consider for each piece \( \pi_i \) the polygonal curve \( \hat{\gamma}_i \) from \( p_{u_i} \) to \( p_{v_i} \) obtained by concatenating the line segment from \( p_{u_i} \) to \( x_i \), followed by \( \pi_i \), and followed by the line segment from \( y_i \) to \( p_{v_i} \). See Figure 2 for an example. For each piece \( \pi_i \), the polygonal curves \( \hat{\gamma}_i \) and \( \gamma(u_i v_i) \) have the same endpoints and are contained in the union \( K_{u_i} \cup K_{v_i} \). Because of Lemma 2, we have \( \text{cr}_2(\ell, \hat{\gamma}_i) = \text{cr}_2(\ell, \gamma(u_i v_i)) \). It follows that, if we define \( \hat{\gamma} \) as the concatenation of \( \hat{\gamma}_1, \ldots, \hat{\gamma}_k \), we have \( \text{cr}_2(\ell, \gamma(W)) = \text{cr}_2(\ell, \hat{\gamma}) \). Moreover, \( \text{cr}_2(\ell, \hat{\gamma}) = \text{cr}_2(\ell, \pi) \) because \( \hat{\gamma} \) is essentially \( \pi \) with some spokes connecting \( x_i \) to \( p_{u_i} \), where the number of crossings evens out. We conclude that \( \text{cr}_2(\ell, \gamma(W)) = \text{cr}_2(\ell, \hat{\gamma}) = \text{cr}_2(\ell, \pi) = 1 \). This finishes one direction of the proof.

![Figure 2: Notation in the proof of Lemma 3](image)

For the other direction, assume that \( G \) has a closed walk \( W \) such that the crossing number of \( \ell \) and \( \gamma(W) \) is odd. Since the closed polygonal path \( \gamma(W) \) is contained in \( \mathbb{K} \) by construction, Lemma 3 implies that \( \mathbb{K} \) separates \( s \) and \( t \). This proves the other direction.

We extend Lemma 3 to a necessary and sufficient condition for \( s \) and \( t \) being disconnected that involves only a few cycles of \( G \). Let \( T \) be any maximal spanning forest of \( G \), that is, \( T \) contains a spanning tree of each connected component of \( G \). For each edge \( e \) of \( G - E(T) \), let \( \text{cycle}(T, e) \) be the unique cycle in \( T + e \), and let \( \tau(T, e) \) be the curve \( \gamma(\text{cycle}(T, e)) \). That is, \( \tau(T, e) \) is the polygonal curve describing \( \text{cycle}(T, e) \) in the drawing.

**Lemma 4.** Let \( T \) be a maximal spanning forest of \( G \). The set \( \mathbb{K} \) separates \( s \) and \( t \) if and only if there exists some edge \( e \in E(G) \setminus E(T) \) such that \( \text{cr}_2(\ell, \tau(T, e)) = 1 \).

**Proof.** The essential idea is to use the so-called cycle space of a graph and the fact that \( \{ \text{cycle}(T, e) \mid e \in E(G) \setminus E(T) \} \) is a basis. We next provide the details using no background.

Since we can treat each connected component of \( G \) (and thus \( \mathbb{K} \)) independently, we will just assume that \( G \) has one connected component. This means that \( T \) is a spanning tree of \( G \).

Fix any node \( r \in V(G) \) and take the point \( p_r \in K_r \) as a basepoint. For each node \( v \in V(T) \), let \( T[r, v] \) be the simple walk in \( T \) from \( r \) to \( v \). For each edge \( uv \) of \( G \) we define a closed polygonal curve \( \lambda(uv) \) as the concatenation of \( \gamma(T[r, u]) \), \( \gamma(uv) \), and the reverse of \( \gamma(T[r, v]) \). Note that \( \lambda(uv) \) is a closed polygonal path through \( p_r \).

When \( uv \notin E(T) \), \( \lambda(uv) \) is \( \tau(T, uv) \) with a spoke following \( \gamma(T[r, u]) \), where \( w \) is the last common node of \( T[r, u] \) and \( T[r, v] \). This implies that

\[
\forall uv \in E(G) \setminus E(T) : \quad \text{cr}_2(\ell, \tau(T, uv)) = \text{cr}_2(\ell, \lambda(uv)).
\]

When \( uv \in E(T) \), then \( \lambda(uv) \) walks twice the same polygonal curve, and therefore

\[
\forall uv \in E(T) : \quad \text{cr}_2(\ell, \lambda(uv)) = 0.
\]
Assume that the points $s$ and $t$ lie in different path-components of $\overline{K}$. Because of Lemma $3$, there exists some closed walk $W$ in $G$ with $cr_2(\ell, \gamma(W)) = 1$. Let $u_1v_1, \ldots, u_kv_k$ be the sequence of edges in $W$, where $u_1 = v_k$. Using arithmetic modulo 2 we have

$$1 = cr_2(\ell, \gamma(W))$$

$$= \sum_{i=1}^{k} cr_2(\ell, \gamma(u_iv_i))$$

$$= \sum_{i=1}^{k} \left( cr_2(\ell, \gamma(T[r, u_i]) + cr_2(\ell, \gamma(u_iv_i)) + cr_2(\ell, \gamma(T[u_i, v_i])) \right)$$

$$= \sum_{i=1}^{k} cr_2(\ell, \lambda(u_iv_i)),$$

where in the third equality we have used that each node of $W$ is the endpoint of two consecutive edges of $W$, which implies that the new terms cancel out. This means that, for some edge $u_iv_i$ of $W$, we have $cr_2(\ell, \lambda(u_iv_i)) = 1$. This edge $u_iv_i$ cannot be in $T$ because of $2$. Therefore we have some edge $u_iv_i$ in $E(W)$, where $u_iv_i \notin E(T)$, with $cr_2(\ell, \lambda(u_iv_i)) = 1$. Because of $1$ we have $cr_2(\ell, \tau(T, u_iv_i)) = cr_2(\ell, \lambda(u_iv_i)) = 1$. This finishes the proof of one direction of the statement.

For the other direction, assume that there exists some edge $e \in E(G) \setminus E(T)$ such that $cr_2(\ell, \tau(T, e)) = 1$. Taking $W = cycle(T, e)$ and using that $\tau(T, e) = \gamma(W)$ by definition, this means that $W$ is a closed walk in $G$ with $cr_2(\ell, \gamma(W)) = 1$. It follows from Lemma $3$ that $K$ separates $s$ and $t$.

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3 Semi-dynamic connectivity

In this section we discuss the separation of $s$ and $t$ under the addition of new sets to $K$. We first describe a standard union-find data structure because we will build on it. Then we describe the setting and the notation we will use. It follows a description of the extension of the union-find data structure for our setting. Finally, we describe the data structure, its maintenance, and its correctness.

3.1 Preliminaries: Union-find

Here we review a standard union-find data structure and some of its properties. See [4, Chapter 21], [5, Chapter 5] or [7] for a comprehensive exposition.

A union-find data structure represents a disjoint set system supporting the operations MAKESET (create a new disjoint set with a single element), UNION (merge two sets), and FIND (return a representative of a given set). We can test whether two elements belong two the same set by testing whether the output of FIND for those two elements is the same. A common realization is to represent each disjoint set by a rooted tree in which each node holds one element of the set. The root of the tree holds the representative of the set. Each node has a pointer to its parent, while the root points to itself. Then FIND simply follows the parent pointer until it finds the root of the tree. The union operation merges two trees by making the root of one subtree a child of the root of the other. Thus, given two elements, we first locate the roots of their corresponding trees calling FIND, and then we proceed with the union.

Two optimizations are commonly used to obtain an efficient realization. Union-by-rank determines which root gets merged in a union operation: each root has a rank associated to it, in an union we simply make the root of lower rank a child of the root with larger rank, and we increase the rank of the root if both roots had the same rank. Path compression makes all nodes
found on a search path from a node to its root direct children of the root. For later reference and modification, we include pseudocode in Figure 3. Combining these two optimizations, each operation has an amortized time complexity of $\alpha$, and modification, we include pseudocode in Figure 3. Combining these two optimizations, each operation has an amortized time complexity of $\alpha(n)$, where $n$ is the number of elements in the set system and $\alpha(\cdot)$ is the extremely slow growing inverse Ackermann function. See references [4, Chapter 21], [7] or [10] for an analysis of the time complexity.

$$\text{Algorithm FIND}(u)$$
1. if $u \neq \text{parent}(u)$ then
2. $\text{parent}(u) \leftarrow \text{FIND}($parent($u$))
3. return parent($u$)

Figure 3: The main two operations in the union-find data structure. $u$ and $v$ are nodes of the tree.

3.2 Setting

Let $s$ and $t$ be two points in the plane. We have a finite family of convex sets $K$, all of them disjoint from $s$ and $t$. Following the previous notation, we denote by $K$ the union of the sets in $K$, and by $G$ the intersection graph of $K$.

Consider the addition of a new compact convex set $K_u$ to $K$. We use $K_{\text{new}}$ for the resulting set, $K_{\text{new}}$ for the union of its sets, and $G_{\text{new}}$ for the intersection graph of $K_{\text{new}}$.

The analysis of our data structure is based on a maximal spanning forest of the intersection graph of the convex sets. The definition of such spanning forest is iterative, as follows. Let $uv_1, \ldots, uv_k$ be an enumeration of the edges incident to $u$ in $G_{\text{new}}$. That is, $K_{v_1}, \ldots, K_{v_k}$ are the sets of $K$ intersecting the new set $K_u$. We consider adding the edges $uv_1, \ldots, uv_k$ to $G$ one by one. We thus define $G_0$ as the union of $G$ and a new vertex $u$ for $K_u$. For each index $1 \leq j \leq k$, we define the graph $G_j = G_{j-1} + uv_j$. Note that $G_{\text{new}} = G_k$. The intermediate graphs $G_1, \ldots, G_{k-1}$ are not intersection graphs of $K$ or $K_{\text{new}}$, but something in between.

If at the time of adding $uv_j$ the vertices $u$ and $v_j$ are already connected in the graph $G_{j-1}$, then we call $uv_j$ a cycle edge. Otherwise, $uv_j$ merges two connected components of $G_{j-1}$ and we call it a merge edge. Note that whether an edge is a cycle edge or a merge edge depends on the order used in the addition of edges.

Let $T$ be the maximal spanning forest of $G$. We define $T_0$ as the union of $T$ and a new vertex $u$ for $K_u$. For each index $1 \leq j \leq k$ we define

$$T_j = \begin{cases} T_{j-1} & \text{if } uv_j \text{ is a cycle edge}, \\ T_{j-1} + uv_j & \text{if } uv_j \text{ is a merge edge}. \end{cases}$$

It is easy to see by induction that, for each index $1 \leq j \leq k$, $T_j$ is a maximal spanning forest of $G_j$. We define $T_{\text{new}}$ as $T_k$. Thus $T_{\text{new}}$ is a maximal spanning forest of $G_{\text{new}} = G_k$.

As it was done in Section 2.2, for each $K_u$ we choose a point $p_u$ in $K_u$ and for each edge $uv$ we choose a polygonal curve $\gamma(uv)$. These choices are made in the first appearance of the node or edge, and remain invariant from there onwards.
3.3 Augmented union-find

We maintain a union-find data structure for the connected components of the graphs $G_j$. Recall that $T_j$ is a maximal spanning forest of $G_j$. For each node $v$ of $G_j$, we store a parity bit, denoted as $\text{parity}(v)$, with the following property:

- If $v$ is the root of a union-find tree, then $\text{parity}(v) = 0$.
- If $v$ has parent $w$ in a union-find tree, then $\text{parity}(v) = cr_2(\ell, T_j[w,v])$. That is, we look at the parity of the crossing number of $\ell$ with the polygonal curve from $p_v$ to $p_w$ defined by the drawing of $T_j$.

For the rest of the paper, any arithmetic involving parity bits is done modulo 2.

We next argue that the correct parity bits can be maintained in the same complexity as the union-find operations, assuming that only certain unions are made. That is clear for MAKESET by giving the new node parity 0.

Consider the FIND operation, which changes parent pointers due to path compression. Note that the graphs $G_j$ and $T_j$ do not change, but the union-find data structure does. Let $u, v, w$ be nodes such that, in the union-find data structure, $w$ is parent of $v$ and $v$ is parent of $u$. Note that

$$cr_2(\ell, \gamma(T_j[u,w])) = cr_2(\ell, \gamma(T_j[u,v])) + cr_2(\ell, \gamma(T_j[v,w])) = \text{parity}(u) + \text{parity}(v).$$

Therefore, when we update parent$(u) \leftarrow w$, we just have to set $\text{parity}(u) \leftarrow \text{parity}(u) + \text{parity}(v)$ to restore parity$(u)$ to its correct value.

We can now easily realize the augmented path compression. We define an extended function FINDEXT$(u)$ that, for all nodes $v$ from $u$ to the root $r$ of the tree containing $u$, sets parent$(u) = r$ and updates the value parity$(v)$ accordingly. Pseudocode is given in Figure 4.

**Algorithm** FINDEXT$(u)$

1. if $u \neq \text{parent}(u)$ then
2. \hspace{1em} $r \leftarrow \text{FIND}(\text{parent}(u))$
3. \hspace{1em} $\text{parity}(u) \leftarrow \text{parity}(u) + \text{parity}(\text{parent}(u))$
4. \hspace{1em} parent$(u) \leftarrow r$
5. return parent$(u)$

Figure 4: Extended find operation for an element $u$.

Finally, we discuss the extension UNION to UNIONEXT. Its arguments are two nodes $u$ and $v_j$ such that $uv_j$ is a merge edge and the union-find data structure stores the connectivity of $G_{j-1}$. Since $uv_j$ is a merge edge, we have $T_j = T_{j-1} + uv_j$. This means that the sets $K_u$ and $K_{v_j}$ intersect but $u$ and $v_j$ were in different connected components of $G_{j-1}$. Like before, we first find the roots $\bar{u}$ and $\bar{v}$ of their trees using FINDEXT$(\cdot)$. After this it holds that

$$\text{parity}(u) = cr_2(\ell, \gamma(T_{j-1}[\bar{u}, u])),$$

and similarly $\text{parity}(v) = cr_2(\ell, \gamma(T_{j-1}[\bar{v}, v]))$.

The walk $T_j[\bar{u}, \bar{v}]$ can be split into $T_{j-1}[\bar{u}, u]$ and $T_{j-1}[v, \bar{v}]$. Thus we have

$$cr_2(\ell, \gamma(T_j[\bar{u}, \bar{v}])) = cr_2(\ell, \gamma(T_{j-1}[\bar{u}, u])) + cr_2(\ell, \gamma(uv)) + cr_2(\ell, \gamma(T_{j-1}[v, \bar{v}])) = \text{parity}(u) + cr_2(\ell, \gamma(uv)) + \text{parity}(v).$$

The last values are either available through $\text{parity}(\cdot)$ or computable in constant time. If, for example, $\bar{u}$ gets $\bar{v}$ as its parent, then we have $\text{parity}(\bar{u}) = cr_2(\ell, \gamma(T_j[u, v]))$. The other case is similar. We provide the resulting pseudocode in Figure 5.

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The properties of union-find imply that each of the extended operations, \texttt{UnionExt} and \texttt{FindExt}, has an amortized complexity of $\alpha(n)$, where $n$ is the cardinality of $\mathcal{K}$.

### 3.4 Semi-dynamic data structure

We now describe the data structure to maintain $\mathcal{K}$. The data structure supports one operation: add a new compact convex set $K_u$ to $\mathcal{K}$ and then report whether $\mathcal{K} \cup \{K_u\}$ separates $s$ and $t$. We use the notation from Sections 3.2 and 3.3.

The data structure has the following elements:

- an augmented union-find data structure as described in Section 3.3;
- for each element $K_v$ of $\mathcal{K}$, we store the point $p_v$;
- a semi-dynamic data structure $DS(\mathcal{K})$ that can find, for the new $K_u$, all the objects of $\mathcal{K}$ that intersect $K_u$.

The intersection graph $G$ and the maximal spanning forest $T$ are not kept. They are used only for the analysis.

We next describe how to insert $K_u$. We use the data structure $DS(\mathcal{K})$ to find the sets $K_{v_1}, \ldots, K_{v_k}$ of $\mathcal{K}$ that intersect $K_u$. We then insert $K_u$ in the data structure $DS(\mathcal{K})$ to obtain $DS(\mathcal{K}_{\text{new}})$. We choose a point $p_u$ in $K_u$ and create a new node $u$ in the extended union-find data structure.

We then iterate over the edges $uv_1, \ldots, uv_k$. We first decide whether the considered edge $uv_j$ is a merge edge or a cycle edge by checking whether $\texttt{FindExt}(u)$ and $\texttt{FindExt}(v_j)$ return the same representative. If $uv_j$ is a merge edge, we just call \texttt{UnionExt}(u, v_j) and continue with the next step of the filtration.

Otherwise, $uv_j$ is a cycle edge, and we proceed as follows. We want to check whether $\texttt{cr}_2(\ell, \tau(T_{j-1}, uv_j)) = \texttt{cr}_2(\ell, \tau(T_j, uv_j))$ is 1 or 0. For this, we use that $u$ and $v$ have already the same parent because of the calls \texttt{FindExt}(u) and \texttt{FindExt}(v). If we denote such a common parent by $r$, then

$$
\texttt{cr}_2(\ell, \tau(T_j, uv_j)) = \texttt{cr}_2(\ell, \gamma(T_{j-1}[u, v_j])) + \texttt{cr}_2(\ell, \gamma(uv_j)) \\
= \texttt{cr}_2(\ell, \gamma(T_{j-1}[u, r])) + \texttt{cr}_2(\ell, \gamma(T_{j-1}[v_j, r])) + \texttt{cr}_2(\ell, \gamma(uv_j)) \\
= \texttt{parity}(u) + \texttt{parity}(v_j) + \texttt{cr}_2(\ell, \gamma(uv_j)).
$$

---

\textbf{Algorithm} \texttt{UnionExt}(u, v)
1. $\bar{u} \leftarrow \texttt{FindExt}(u)$
2. $\bar{v} \leftarrow \texttt{FindExt}(v)$
3. $b \leftarrow \texttt{parity}(u) + \texttt{parity}(v) + \texttt{cr}_2(\ell, \gamma(uv))$
4. \textbf{if} $\text{rank}(\bar{u}) > \text{rank}(\bar{v})$ \textbf{then}
5. \hspace{0.5cm} $\text{parent}(\bar{v}) \leftarrow \bar{u}$
6. \hspace{0.5cm} $\text{parity}(\bar{v}) \leftarrow b$
7. \textbf{else} (* $\text{rank}(\bar{u}) \leq \text{rank}(\bar{v})$ *)
8. \hspace{0.5cm} $\text{parent}(\bar{u}) \leftarrow \bar{v}$
9. \hspace{0.5cm} $\text{parity}(\bar{u}) \leftarrow b$
10. \textbf{if} $\text{rank}(\bar{u}) = \text{rank}(\bar{v})$ \textbf{then}
11. \hspace{0.5cm} $\text{rank}(\bar{v}) = \text{rank}(\bar{v}) + 1$

Figure 5: Extended union for two nodes $u$ and $v$. 

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If \( cr_2(\ell, \tau(T_j, uv_j)) = 1 \), then we conclude that \( K_{\text{new}} \) separates \( s \) and \( t \) and we finish the algorithm. If \( cr_2(\ell, \tau(T_j, uv_j)) = 0 \), we proceed to the next edge \( uv_{j+1} \). Pseudocode for the insertion of \( K_u \) is given in Figure 6. This finishes the description of the algorithm.

\[
\begin{align*}
\text{Algorithm } & \text{ Adding } K_u \text{ to } K \\
1. & \quad \text{MAKESET}(u) \\
2. & \quad \text{parity}(u) \leftarrow 0 \\
3. & \quad \text{for } K_v \in K \text{ intersecting } K_u \text{ do} \\
4. & \quad \quad \bar{u} \leftarrow \text{FINDExt}(u) \\
5. & \quad \quad \bar{v} \leftarrow \text{FINDExt}(v) \\
6. & \quad \quad \text{if } \bar{u} \neq \bar{v} \text{ then} \\
7. & \quad \quad \quad \text{UNIONExt}(u, v) \\
8. & \quad \quad \text{else } (* uv \text{ a cycle edge} *) \\
9. & \quad \quad \quad b \leftarrow \text{parity}(u) + \text{parity}(v) + cr_2(\ell, \gamma(uv)) \\
10. & \quad \quad \quad \text{if } b = 1 \text{ then} \\
11. & \quad \quad \quad \quad \text{return } "K \cup \{K_u\} \text{ separates } s \text{ and } t!!" \\
\end{align*}
\]

Figure 6: Procedure for the addition of \( K_u \).

It follows from the invariants of the extended union-find discussed in Section 3.3, that we are correctly computing the value \( cr_2(\ell, \tau(T_j, uv_j)) \). If \( cr_2(\ell, \tau(T_j, uv_j)) = 1 \), then Lemma 4 implies that \( K_{\text{new}} \) separates \( s \) and \( t \). From that point on, we only need to remember that \( s \) and \( t \) are separated.

If \( cr_2(\ell, \tau(T_j, uv_j)) = 0 \), then \( cr_2(\ell, \tau(T, uv_j)) \) will remain 0 for all future maximal spanning forests \( T \). This is so because the maximal spanning forest we maintain is monotone increasing: we only add vertices and edges, but never remove anything. Thus, we never need to check \( cr_2(\ell, \tau(T, uv_j)) \) again later. In particular, if \( K \) did not separate \( s \) and \( t \) and we have \( cr_2(\ell, \tau(T_j, uv_j)) = 0 \) for all \( j \), then

\[
\forall vv' \in E(G_{\text{new}}) \setminus E(T_{\text{new}}) : \quad cr_2(\ell, \tau(T_{\text{new}}, vv')) = 0.
\]

Since \( T_{\text{new}} \) is a maximal spanning forest of \( G_{\text{new}} \), Lemma 4 implies that \( K_{\text{new}} \) does not separate \( s \) and \( t \).

For each edge \( uv_j \) we make 2 calls to FINDExt, at most one call to UNIONExt, and additional \( O(1) \) work. This means that for each edge we spend \( O(\alpha(n)) \) amortized time, where \( n \) is the cardinality of \( K \). We also need the time needed to find the elements of \( K \) intersecting the new element \( K_u \). We conclude.

**Theorem 5.** Let \( s \) and \( t \) be two points in the plane. There is a semi-dynamic data structure to maintain a family \( K \) of \( n \) compact convex sets in the plane under insertions to decide whether \( K \) separates \( s \) can \( t \). The insertion of a new set \( K_u \) in \( K \) that intersects \( k \) sets of \( K \) takes \( O(1 + ko(n)) \) amortized time, plus the time needed to find the \( k \) elements of \( K \) intersecting \( K_u \).

Of course, once \( s \) and \( t \) are separated by \( K \), the insertion of each new set can be carried out in constant time, since we only need to remember that \( K \) separates \( s \) and \( t \).

4 Application to dynamic connectivity under subdivision

We consider now the motivating application discussed in the Introduction for \( d = 2 \).

We have two points \( s \) and \( t \) inside the unit square \( X \). Initially, the box \( X \) is colored yellow. In each iteration, we take a largest yellow box, subdivide it into 4 subboxes, and color each of
them as red, yellow, or green depending on the outcome of some oracle. The boxes containing \( s \) or \( t \) are always colored yellow or green. We want to know at which point the red boxes separate \( s \) and \( t \), meaning that each path from \( s \) to \( t \) contained in the unit square intersects some red box.

Boxes are assumed to contain their boundary, so that any two boxes intersect if their boundaries intersect, possibly only at a common vertex.

For our arguments it is convenient to surround \( X \) with 8 red boxes of the same size as \( X \). This reduces the problem to finding certain curves within the red region. Without those additional squares, we should also consider boundary-to-boundary curves.

We maintain through the algorithm the intersection graph \( H \) of the yellow and red boxes. This intersection graph \( H \) has one node for each box that is yellow or red, and an edge between two nodes whenever the corresponding boxes intersect. The graph \( H \) is stored using an adjacency list representation \[4\] Chapter 22. The adjacency list of each vertex is stored as a doubly linked list. Moreover, for the appearance of a node \( v \) in the adjacency list of \( u \), we keep a pointer to the appearance of \( u \) in the adjacency list of \( v \). With this, we can perform the deletion of a node \( v \) in time proportional to its degree.

When we want to subdivide a yellow box \( K_u \) represented by a node \( u \), we can locate its set of neighbors \( N = N_H(u) \) in the graph \( H \), delete \( u \) from the graph, subdivide \( K_u \) into four boxes, create the at most four new nodes representing the yellow and red boxes arising from the subdivision of \( K_u \), check for intersection each of them against each of the nodes in \( N \), and update the graph \( H \) accordingly. All this takes time \( O(1 + |N|) \) time.

If we always subdivide a largest yellow box, there are at most 12 other boxes intersecting it. This means that we can update the intersection graph \( H \) of yellow and red boxes in \( O(1) \) time. For choosing always a largest yellow box, we can use for example a queue for the yellow boxes. Thus, we spend \( O(1) \) time per subdivided yellow box and, for each red box, we get its neighboring red boxes in \( O(1) \) time. Using Theorem 5 for the red boxes, and a normal union-find for the green boxes, as discussed in the Introduction, we obtain the following result.

**Theorem 6.** Consider the subdivision procedure described in the Introduction where we always subdivide a largest yellow box. We can perform the subdivision until condition (1) or (2) occurs in \( O(n \alpha(n)) \) time, where \( n \) is the number of subdivisions performed.

Of course we can also perform the first \( n \) steps of the subdivision procedure in \( O(n \alpha(n)) \) time, and correctly report that neither condition (1) nor (2) hold.

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