Relativistic Particles on Quantum Space-time

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ABSTRACT

We discuss alternatives to the usual quantization of a relativistic particle which result in discrete spectra for position and time operators.

1 Introduction

Space-time noncommutativity can arise from the quantization of a point particle.[1],[2] For this one only needs to start with a reparametrization invariant action and then employ alternatives to the standard gauge fixing condition that identifies the particle’s world line parameter with the time-component of the position four-vector. Alternative gauge fixing conditions can lead to nontrivial Dirac brackets between different components of the position four-vector, and noncommutative space-time appears upon replacing the Dirac brackets with quantum commutators. The procedure has been employed in order to recover various interesting deformations of the Heisenberg algebra.[3],[4],[5],[6],[7],[8] In particular, for a suitable gauge choice one obtains the Snyder algebra,[9] which is a Lorentz covariant deformation and is characterized by a discrete spectrum for the position operators. With another gauge choice, a particle action written on a continuous space-time can lead to a discrete spectrum for the time operator in the quantum theory.[10] The quantum description of the particle in these gauges is thus distinct from what one obtains in the standard gauge. This gauge dependence for the quantum description of the particle is analogous to the presence of anomalies in quantum field theory.

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The previous derivations of Snyder’s algebra and its resulting discretized position spectra have involved either deforming the standard action for a relativistic free particle or introducing extra degrees of freedom to the system, which are later eliminated using gauge fixing conditions.\cite{3},\cite{7},\cite{11},\cite{12} By the standard action we are referring to

\begin{equation}
S = -m\int d\lambda \sqrt{-\dot{x}_\mu \dot{x}^\mu},
\end{equation}

or equivalent reparametrization invariant expressions describing a massive particle. Here $x^\mu(\lambda)$, $\mu = 0, 1, ... 3$, defines the trajectory of the particle, $m$ is the mass, $\lambda$ is an arbitrary evolution parameter and the dot indicates differentiation with respect to $\lambda$. We choose $c = 1$ and metric tensor $[g_{\mu\nu}] = \text{diag}(-1, 1, 1, 1)$. In this article we show that a discrete spectra for the position operators can be obtained directly from (1.1), \textit{without deforming the action or introducing additional degrees of freedom to the system}. This only requires finding a suitable gauge condition to fix $\lambda$. We do not recover the full Lorentz covariant algebra of Snyder in this case because, like with the standard gauge, the gauge condition breaks Lorentz covariance. Nevertheless, the algebra we obtain is sufficient for getting a discretized space in the quantum theory, and the full Poincaré algebra is realized by the Dirac brackets. The quantum theory carries the spin zero irreducible representation of the Poincaré group, and, like with the standard gauge, the zero-component of the four-momentum serves as a Hamiltonian for the theory, generating evolution along the trajectory.

Starting with yet another gauge fixing of the parameter $\lambda$ in (1.1), one can get a discrete spectrum for the time operator. Unlike in \cite{10}, we shall not introduce additional particle degrees of freedom for this purpose. Instead, we shall only require that one of the spatial coordinates $x^i$ in (1.1) be an angular variable, thereby implying the existence of a coordinate singularity. Alternatively, the latter can be promoted to a real singularity by replacing $g_{\mu\nu}$ by a black hole metric. The resulting quantum algebra agrees with what was found previously for the BTZ black hole from symmetry considerations,\cite{13} and aspects of the quantum theory were studied previously by several authors.\cite{14},\cite{15}

After first reviewing the standard gauge in section two, we give the gauge condition which leads to discretized space in section three, and discretized time in section four.

## 2 Standard gauge

Using the Dirac Hamiltonian formalism,\cite{16} we now review the standard gauge constraint. The four-momenta,

\begin{equation}
p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}_\nu \dot{x}^\nu}},
\end{equation}

obtained from (1.1) are canonically conjugate to the space-time coordinates,

\begin{equation}
\{x^\mu, p_\nu\} = \delta_\nu^\mu \quad \{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = 0,
\end{equation}

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and are subject to the mass shell condition
\[ \Psi_1 = p^\mu p_\mu + m^2 \approx 0 , \] (2.3)
where \( \approx \) indicates equality in weak sense. \( \Psi_1 \) generates gauge motion on the phase space associated with reparametrizations of \( \lambda \). The gauge symmetry is fixed after imposing an additional constraint \( \Psi_2 \approx 0 \). The standard choice for \( \Psi_2 \) identifies \( \lambda \) with the time coordinate \( x^0 \),
\[ \Psi_2 = x^0 - \lambda \approx 0 , \] (2.4)
and, as a result, (2.3) and (2.4) form a second class set. Dirac brackets\(^\text{[16]}\)
\[ \{ A, B \}_\text{DB} = \{ A, B \} + \frac{1}{\{ \Psi_1, \Psi_2 \}} \left( \{ A, \Psi_1 \} \{ \Psi_2, B \} - \{ A, \Psi_2 \} \{ \Psi_1, B \} \right) , \] (2.5)
are then employed to write down a consistent algebra on phase space. The result is
\[ \{ x^\mu, p_\nu \}_\text{DB} = \delta^\mu_\nu - \delta^0_\nu p^\mu_p^0 \quad \{ x^\mu, x^\nu \}_\text{DB} = \{ p_\mu, p_\nu \}_\text{DB} = 0 \] (2.6)
It is not Lorentz covariant, since neither is (2.4). The Poincaré algebra easily follows from the Dirac brackets (2.6),
\[ \{ j^{\mu \nu}, p^\rho \}_\text{DB} = \eta^{\mu \rho} p^\nu - \eta^{\nu \rho} p^\mu \] (2.7)
\[ \{ j^{\mu \nu}, j^{\rho \sigma} \}_\text{DB} = \eta^{\mu \rho} j^{\nu \sigma} - \eta^{\nu \rho} j^{\mu \sigma} - \eta^{\mu \sigma} j^{\nu \rho} + \eta^{\nu \sigma} j^{\mu \rho} , \] (2.8)
where
\[ j^{\mu \nu} = x^\mu p^\nu - x^\nu p^\mu , \] (2.9)
and so the quantum theory carries the spinless irreducible representation of the Poincaré group.

The quantum operators \( \hat{x}^i \), associated with the position coordinates \( x^i \), have continuous spectra, while the time coordinate \( x^0 \) remains a commuting parameter in the quantum theory. \( p^0 \) serves as the Hamiltonian for the system, generating evolution in \( x^0 \), i.e., for any function \( \mathcal{F}(x, p, \lambda) \) on the phase space,
\[ \frac{d}{dx^0} \mathcal{F}(x, p) = \{ \mathcal{F}(x, p), \mathcal{H} \}_\text{DB} + \frac{\partial}{\partial x^0} \mathcal{F}(x, p) , \quad \mathcal{H} = p^0 \] (2.10)
The Hamilton equation (2.10) gets replaced by the corresponding Heisenberg equation in the quantum theory.

3 Discrete space gauge

We now introduce an alternative to the gauge fixing condition (2.4), which deforms the Dirac brackets (2.6), while preserving the Poincaré algebra (2.7),(2.8). It leads to a different quantum
description of the relativistic free particle. The alternative gauge condition is†

\[
\Psi_2 = x^0 + \frac{p_0 \vec{x} \cdot \vec{p}}{\Lambda^2 + \vec{p}^2} - \lambda \approx 0 ,
\]

(3.1)

where \( \vec{x} \) and \( \vec{p} \) denote three-vectors and Λ is some energy scale (Λ ≠ 0). The new term in \( \Psi_2 \) vanishes in the limit Λ → ∞ and so we recover the standard gauge fixing condition in the limit. The gauge condition (3.1) says that the evolution parameter \( \lambda \) is a momentum dependent rescaling of the time coordinate \( x^0 \) along any free particle world line,

\[
\left. \frac{d\lambda}{dx^0} \right|_{p_\mu=\text{const}} = \frac{\Lambda^2}{\Lambda^2 + \vec{p}^2} \tag{3.2}
\]

[To obtain this result set (3.1) strongly equal to zero and use \( dx^i / dx^0 = \dot{x}^i / \dot{x}^0 = p^i / p^0 \), which is valid along the particle world line.] Then \( \lambda \) increases monotonically as the particle evolves in the time-like direction, and (3.1) a valid gauge condition.

From (2.5), one now gets the following Dirac brackets

\[
\{x_i, x_j\}_\text{DB} = \frac{1}{\Lambda^2} \epsilon_{ijk} L_k \tag{3.3}
\]

\[
\{x_i, p_j\}_\text{DB} = \delta_{ij} + \frac{p_i p_j}{\Lambda^2} \tag{3.4}
\]

\[
\{p_i, p_j\}_\text{DB} = 0 \tag{3.5}
\]

where \( i, j, k = 1, 2, 3 \) and \( L_i = \epsilon_{ijk} x_j p_k \) is the angular momentum. This is the classical analogue of Snyder’s algebra restricted to the reduced phase space spanned by \( x^i \) and \( p_i \). We do not recover the full Lorentz covariant algebra of Snyder because, as was true in section two, the gauge constraint spoils Lorentz covariance. For the time-like components \( x^0 \) and \( p^0 \), one instead gets the Dirac brackets

\[
\{x^0, x_i\}_\text{DB} = \frac{1}{\Lambda^2} \left( p_0 x_i + \frac{\vec{x} \cdot \vec{p}(\Lambda^2 - p_0^2 - m^2)}{p_0(\Lambda^2 + \vec{p}^2)} p_i \right) \tag{3.6}
\]

\[
\{x_i, p_0\}_\text{DB} = \left( 1 + \frac{\vec{p}^2}{\Lambda^2} \right) \frac{p_i}{p_0} \tag{3.7}
\]

\[
\{x_0, p_0\}_\text{DB} = \frac{\vec{p}^2}{\Lambda^2} \tag{3.8}
\]

\[
\{x_0, p_i\}_\text{DB} = \frac{p_0 p_i}{\Lambda^2} \tag{3.9}
\]

along with \( \{p_0, p_j\}_\text{DB} = 0 \). The Dirac brackets (3.3-3.9) reduce to those of the standard gauge (2.6) in the limit Λ → ∞.

Using the definition (2.9) of the Lorentz generators and the Dirac brackets (3.3-3.9), it can be checked that (2.7) and (2.8) are satisfied for all Λ ≠ 0. The Poincaré algebra is

†It is similar to a gauge condition used in [11].
therefore recovered and, like in the standard gauge, the quantum theory carries the spinless irreducible representation of the Poincaré group. Also, as in the standard gauge, \( p^0 \) serves as the Hamiltonian for the system. Here, though, it generates evolution in \( \lambda \), and not \( x^0 \). From (3.2), (3.7) and (3.8), one gets
\[
\{x^\mu, p^0\}_\text{DB} + \frac{\partial x^\mu}{\partial \lambda} = \frac{dx^\mu}{dx^0} = \frac{dx^\mu}{d\lambda}, \tag{3.10}
\]
where \( x_i, p_i \) and \( \lambda \) are regarded as independent variables in the partial derivative, while \( x^0 \) is defined using the constraint (3.1), and so \( \frac{dx^0}{d\lambda} = \delta_0^i \). Then for any function \( F(x, p, \lambda) \) on the phase space, one has the Hamilton equation
\[
\frac{d}{d\lambda} F(x, p, \lambda) = \{F(x, p, \lambda), H\}_\text{DB} + \frac{\partial}{\partial \lambda} F(x, p, \lambda), \quad H = p^0, \tag{3.11}
\]
To obtain the classical evolution in \( x^0 \), one can first solve (3.11) and then apply (3.2).‡ The quantum analogue of (3.11) gives a meaningful Heisenberg equation, because \( \lambda \) remains a commuting parameter upon quantization. Concerning the other Poincaré generators, the action of the three-momenta and Lorentz boosts generators on space-time is nonlinear, while the action of the angular momentum is undeformed. The latter follows from
\[
\{x_i, L_j\}_\text{DB} = \epsilon_{ijk} x_k \quad \{x_0, L_i\}_\text{DB} = 0 \quad \tag{3.12}
\]
Unlike with the standard gauge, here the position operators have discrete spectra in the quantum theory. This follows since the subalgebra spanned by the spatial components of position and momenta coincides with that of Snyder.[9] More explicitly, define
\[
A_i = \frac{1}{2}(L_i + \Lambda x_i) \quad B_i = \frac{1}{2}(L_i - \Lambda x_i), \tag{3.13}
\]
which from (2.8), (3.3) and (3.12), satisfy two \( SU(2) \) algebras
\[
\{A_i, A_j\}_\text{DB} = \epsilon_{ijk} A_k \quad \{B_i, B_j\}_\text{DB} = \epsilon_{ijk} B_k \quad \{A_i, B_j\}_\text{DB} = 0 \quad \tag{3.14}
\]
From the \( x_i L_i = 0 \), it follows that \( A_i A_i = B_i B_i \). In the quantum theory, we replace \( A_i \) and \( B_i \) by operators \( \hat{A}_i \) and \( \hat{B}_i \), and Dirac brackets by commutators of the operators divided by
\[
\frac{\Lambda^2}{\sqrt{\Lambda^2 - m^2}} \tan^{-1} \frac{p^0}{\sqrt{\Lambda^2 - m^2}},
\]
which is valid for \( \Lambda > m \). \( \mathcal{P}_0 \) is what one normally thinks of as the Hamiltonian, and from which one determines the evolution in quantum theory. However, here we cannot easily utilize \( \mathcal{P}_0 \) for the latter purpose, because \( x^0 \) gets promoted to a noncommuting operator in the quantum theory, which in particular will not commute with \( \mathcal{P}_0 \).
\( i\hbar \). Then \( \hat{A}_i \hat{A}_i, \hat{A}_3 \) and \( \hat{B}_3 \) form a complete set of independent commuting operators. These operators, along with the coordinate and angular momentum operators, \( \hat{x}_i \) and \( \hat{L}_i \), respectively, have discrete spectra. Denoting the eigenvectors by \( |j, m_A, m_B\rangle\), \( m_A, m_B = -j, 1 - j, \ldots, j \), one has

\[
\hat{A}_i \hat{A}_i |j, m_A, m_B\rangle = \hbar^2 j(j + 1) |j, m_A, m_B\rangle
\]

\[
\hat{A}_3 |j, m_A, m_B\rangle = \hbar m_A |j, m_A, m_B\rangle
\]

\[
\hat{B}_3 |j, m_A, m_B\rangle = \hbar m_B |j, m_A, m_B\rangle,
\]

(3.15)

and also

\[
\hat{x}_3 |j, m_A, m_B\rangle = \frac{\hbar}{\Lambda} (m_A - m_B) |j, m_A, m_B\rangle
\]

\[
\hat{L}_3 |j, m_A, m_B\rangle = \hbar (m_A + m_B) |j, m_A, m_B\rangle
\]

(3.16)

While \( j, m_A \) and \( m_B \) can be integers or half-integers, the eigenvalues of \( \hat{L}_i \) are integers (times \( \hbar \)). Representing \( \hat{L}_i \) as the differntial operators \(-i\hbar \epsilon_{ijk} \frac{\partial}{\partial p_k}\), one then gets singlevalued angular momentum eigenfunctions in momentum space. The eigenvalues of \( \hat{x}_i \) are integers times \( \hbar / \Lambda \).

Additional remarks are:

i) The eigenvectors \( |j, m_A, m_B\rangle \) are not stationary. Neither are those which simultaneously diagonalize \( \hat{L}_i \hat{L}_i, \hat{L}_3, \) and \( \hat{A}_i \hat{A}_i \), since the last operator does not commute with the Hamiltonian operator \( \hat{p}^0 \).

ii) Using the commutation relations

\[
[\hat{A}_i, \hat{p}_j] = \frac{i\hbar}{2} \left( \epsilon_{ijk} \hat{p}_k + \Lambda \delta_{ij} + \frac{\hat{p}_i \hat{p}_j}{\Lambda} \right)
\]

\[
[\hat{B}_i, \hat{p}_j] = \frac{i\hbar}{2} \left( \epsilon_{ijk} \hat{p}_k - \Lambda \delta_{ij} - \frac{\hat{p}_i \hat{p}_j}{\Lambda} \right),
\]

(3.17)

one can write down the following differential representation for the \( SU(2) \) generators on the space of square-integrable functions \( \{F(\vec{p})\} \)

\[
\hat{A}_i = \frac{i\hbar}{2} \left( \Lambda \frac{\partial}{\partial p_i} + \frac{p_i p_j}{\Lambda} \frac{\partial}{\partial p_j} - \epsilon_{ijk} p_j \frac{\partial}{\partial p_k} \right)
\]

\[
\hat{B}_i = -\frac{i\hbar}{2} \left( \Lambda \frac{\partial}{\partial p_i} + \frac{p_i p_j}{\Lambda} \frac{\partial}{\partial p_j} + \epsilon_{ijk} p_j \frac{\partial}{\partial p_k} \right)
\]

(3.18)

The operators are symmetric for the scalar product defined using the measure \( d^3p/(\Lambda^2 + \vec{p}^2)^2 \).

iii) An alternative derivation of the Dirac brackets (3.3-3.9) may be possible starting from a gauge fixed Lagrangian. A first order formalism, analogous to [4],[7], should be convenient for this purpose since the gauge constraint involves momentum variables. The gauged fixed Lagrangian should then reduce to \(-m\sqrt{1-\vec{x}_i \vec{x}_i}\), or its equivalent, in the limit \( \Lambda \to \infty \).
4 Discrete time gauge

Now we introduce a gauge condition which from (1.1) leads to a discrete spectrum for the time. Here, we shall assume that one of the spatial coordinates $x^i$ can be identified with an angular variable $\phi$, $0 \leq \phi < 2\pi$. This implies the existence of either a coordinate singularity or a physical singularity on the spatial manifold. The latter can be associated with a black hole with axial symmetry. Let us assume this to be the case. Then we then need to replace the Minkowski metric of the previous two sections with the appropriate black hole metric tensor $g_{\mu\nu}$. $\phi$ corresponds to a Killing direction and its conjugate momenta $p_\phi$ is a constant along the particle geodesic. If, furthermore, we assume the metric tensor to be stationary, then there is an additional constant $p_0$ conjugate to the time $x^0$. Thus,

$$\{p_0, \Psi_1\} = \{p_\phi, \Psi_1\} = 0 \quad (4.1)$$

Other momenta may not have vanishing Poisson brackets with $\Psi_1$ since the constraint is a function of the background metric.

For the gauge constraint $\Psi_2$ we now choose

$$\Psi_2 = x^0 + \Theta p_\phi - \lambda \approx 0 , \quad (4.2)$$

where $\Theta$ is a constant. A similar choice was made in [1]. Substituting into (2.5) leads to nonvanishing Dirac brackets of the space-time coordinates with $\phi$,

$$\{x^\mu, \phi\}_{DB} = \Theta \frac{p_\mu}{p^0} \quad (4.3)$$

The Dirac brackets simplify if we replace $\phi$ by

$$\phi' = \phi + \Theta p_0 , \quad (4.4)$$

corresponding to a constant translation of the angular variable. In terms of $\phi'$, the only nonvanishing Dirac bracket between the space-time coordinates is

$$\{x^0, e^{i\phi'}\}_{DB} = i\Theta e^{i\phi'} \quad (4.5)$$

This agrees with the Poisson brackets of [14],[15], and with a special case ($c_2 = -c_3$) of the brackets found in [13] for the BTZ black hole, which preserves the isometry of the solution. For the Dirac brackets of the momenta, one finds

$$\{x^\mu, p_i\}_{DB} = \delta_i^\mu + \frac{\Theta}{2p^0} \delta_\phi g^{\rho\sigma} p_\rho p_\sigma \quad (4.6)$$

$$\{x^\mu, p_0\}_{DB} = \delta_0^\mu - \frac{p_\mu}{p^0} \quad (4.7)$$

$$\{p_i, p_0\}_{DB} = \frac{1}{2p^0} \partial_\iota g^{\mu\nu} p_\mu p_\nu \quad (4.8)$$

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\[ \{p_i, p_j\}_{DB} = 0 , \]  
\[ (4.6) \]

where \( g^{\mu\nu} \) denotes the inverse metric tensor and \( \partial_i = \partial / \partial x^i \).

In the quantum theory, the operator analogues \( \hat{x}^0 \) and \( e^{i\hat{\phi}^\prime} \) of \( x^0 \) and \( e^{i\phi^\prime} \), respectively, satisfy the commutation relation
\[
[\hat{x}^0, e^{i\hat{\phi}^\prime}] = -\hbar \Theta e^{i\hat{\phi}^\prime} 
\]
\[ (4.7) \]

It follows that \( \exp\left\{-\frac{2\pi i \hat{x}^0}{\hbar \Theta}\right\} \) is a central element and one can identify it with \( e^{i\chi} \mathbb{1} \) in an irreducible representation of the algebra. The spectrum for the time operator \( \hat{x}^0 \) is then discrete
\[
\hbar \Theta \left(n - \frac{\chi}{2\pi}\right), \quad n \in \mathbb{Z} 
\]
\[ (4.8) \]

Implications of such a discrete time spectrum have been discussed in [14],[15].

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\[ \text{§ For another derivation of a discrete time spectrum starting from point particle dynamics in 2+1 space-time, see [17].} \]
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