APPROXIMATE QUARTIC LIE ∗-DERIVATIONS

HEEJEONG KOH

Abstract. We will show the general solution of the functional equation
\[ f(x + ay) + f(x - ay) + 2(a^2 - 1)f(x) = a^2f(x + y) + a^2f(x - y) + 2a^2(a^2 - 1)f(y) \]
and investigate the stability of quartic Lie ∗-derivations associated with the given functional equation.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms. Hyers [7] gave a first affirmative partial answer to the question of Ulam. Afterwards, the result of Hyers was generalized by Aoki [1] for additive mapping and by Rassias [14] for linear mappings by considering a unbounded Cauchy difference. Later, the result of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. For further information about the topic, we also refer the reader to [10], [8], [2] and [3].

Recall that a Banach ∗-algebra is a Banach algebra (complete normed algebra) which has an isometric involution. Jang and Park [9] investigated the stability of ∗-derivations and of quadratic ∗-derivations with Cauchy functional equation and the Jensen functional equation on Banach ∗-algebra. The stability of ∗-derivations on Banach ∗-algebra by using fixed point alternative was proved by Park and Bodaghi and also Yang et al.; see [12] and [19], respectively. Also, the stability of cubic Lie derivations was introduced by Fošner and Fošner; see [6].

Rassias [13] investigated stability properties of the following quartic functional equation
\[ f(x + 2y) + f(x - 2y) + 6f(x) = 4f(x + y) + 4f(x - y) + 24f(y). \]

Received by the editors October 15, 2015. Accepted October 23, 2015.

2010 Mathematics Subject Classification. 39B55, 39B72, 47B47, 47H10.

Key words and phrases. Hyers-Ulam-Rassias stability, quartic mapping, Lie ∗-derivation, Banach ∗-algebra, fixed point alternative.

© 2015 Korean Soc. Math. Educ.
It is easy to see that \( f(x) = x^4 \) is a solution of (1.1) by virtue of the identity
\[
(1.2) \quad (x + 2y)^4 + (x - 2y)^4 + x^4 = 4(x + y)^4 + 4(x - y)^4 + 24y^4.
\]
For this reason, (1.1) is called a quartic functional equation. Also Chung and Sa-hoo [4] determined the general solution of (1.1) without assuming any regularity conditions on the unknown function. In fact, they proved that the function \( f : \mathbb{R} \to \mathbb{R} \) is a solution of (1.1) if and only if \( f(x) = A(x, x, x, x) \), where the function \( A : \mathbb{R}^4 \to \mathbb{R} \) is symmetric and additive in each variable.

In this paper, we deal with the following functional equation:
\[
(1.3) \quad f(x + ay) + f(x - ay) + 2(a^2 - 1)f(x) = a^2 f(x + y) + a^2 f(x - y) + 2a^2(a^2 - 1)f(y)
\]
for all \( x, y \in X \) and an integer \( a(a \neq 0, \pm 1) \). We will show the general solution of the functional equation (1.3), define a quartic Lie \(*\)-derivation related to equation (1.3) and investigate the Hyers-Ulam stability of the quartic Lie \(*\)-derivations associated with the given functional equation.

2. A Quatric Functional Equation

In this section let \( X \) and \( Y \) be real vector spaces and we investigate the general solution of the functional equation (1.3). Before we proceed, we would like to introduce some basic definitions concerning \( n \)-additive symmetric mappings and key concepts which are found in [16] and [18]. A function \( A : X \to Y \) is said to be \( \textit{additive} \) if \( A(x + y) = A(x) + A(y) \) for all \( x, y \in X \). Let \( n \) be a positive integer. A function \( A_n : X^n \to Y \) is called \( n \)-\( \textit{additive} \) if it is additive in each of its variables. A function \( A_n \) is said to be \( \textit{symmetric} \) if \( A_n(x_1, \ldots, x_n) = A_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \) for every permutation \( \{\sigma(1), \ldots, \sigma(n)\} \) of \( \{1, 2, \ldots, n\} \). If \( A_n(x_1, x_2, \ldots, x_n) \) is an \( n \)-additive symmetric map, then \( A^n(x) \) will denote the diagonal \( A_n(x, x, \ldots, x) \) and \( A^n(rx) = r^n A^n(x) \) for all \( x \in X \) and all \( r \in \mathbb{Q} \), such a function \( A^n(x) \) will be called a \( \textit{monomial function} \) of degree \( n \) (assuming \( A^n \neq 0 \)). Furthermore the resulting function after substitution \( x_1 = x_2 = \cdots = x_s = x \) and \( x_{s+1} = x_{s+2} = \cdots = x_n = y \) in \( A_n(x_1, x_2, \ldots, x_n) \) will be denoted by \( A^{s,n-s}(x, y) \).

**Theorem 2.1.** A function \( f : X \to Y \) is a solution of the functional equation (1.3) if and only if \( f \) is of the form \( f(x) = A^4(x) \) for all \( x \in X \), where \( A^4(x) \) is the diagonal of the 4-additive symmetric mapping \( A_4 : X^4 \to Y \).
Proof. Assume that $f$ satisfies the functional equation (1.3). Letting $x = y = 0$ in the equation (1.3), we have

$$f(0) = 2a^2(a^2 - 1)f(0),$$

that is, $f(0) = 0$. Putting $x = 0$ in the equation (1.3), we get

$$f(ax) + f(-ay) = a^2f(y) + a^2f(-y) + 2a^2(a^2 - 1)f(y)$$

(2.1) for all $y \in X$. Replacing $y$ by $-y$ in the equation (2.1), we obtain

$$f(ax) + f(-ay) = a^2f(y) + a^2f(-y) + 2a^2(a^2 - 1)f(-y)$$

(2.2) for all $y \in X$. Combining two equations (2.1) and (2.2), we have $f(y) = f(-y)$, for all $y \in X$. That is, $f$ is even. We can rewrite the functional equation (1.3) in the form

$$f(x) + \frac{1}{2(a^2 - 1)}f(x + ay) + \frac{1}{2(a^2 - 1)}f(x - ay)$$

$$- \frac{a^2}{2(a^2 - 1)}f(x + y) - \frac{a^2}{2(a^2 - 1)}f(x - y) - a^2f(y) = 0$$

for all $x, y \in X$ and an integer $a(a \neq 0, \pm 1)$. By Theorem 3.5 and 3.6 in [18], $f$ is a generalized polynomial function of degree at most 4, that is, $f$ is of the form

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A(x) + A^0(x)$$

(2.3) for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of $Y$, and $A^i(x)$ is the diagonal of the $i$-additive symmetric mapping $A_i : X^i \to Y$ for $i = 1, 2, 3, 4$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$. Substituting (2.3) into the equation (1.3) we have

$$A^4(x + ay) + A^3(x + ay) + A^2(x + ay) + A^1(x + ay)$$

$$+ A^4(x - ay) + A^3(x - ay) + A^2(x - ay) + A^1(x - ay)$$

$$+ 2(a^2 - 1)[A^4(x) + A^3(x) + A^2(x) + A^1(x)]$$

$$= a^2[A^4(x + y) + A^3(x + y) + A^2(x + y) + A^1(x + y)$$

$$+ A^4(x - y) + A^3(x - y) + A^2(x - y) + A^1(x - y)]$$

$$+ 2a^2(a^2 - 1)[A^4(y) + A^3(y) + A^2(y) + A^1(y)]$$
for all $x, y \in X$. Note that
\[
\begin{align*}
A^4(x + ry) + A^4(x - ry) &= 2A^4(x) + 12r^2A^{2,2}(x, y) + 2r^4A^4(y), \\
A^3(x + ry) + A^3(x - ry) &= 2A^3(x) + 6r^2A^{1,2}(x, y), \\
A^2(x + ry) + A^2(x - ry) &= 2A^2(x) + 2r^2A^2(y), \\
A^1(x + ry) + A^1(x - ry) &= 2A^1(x).
\end{align*}
\]

Since $a \neq 0, \pm 1$, we have
\[
A^3(y) + A^2(y) + A^1(y) = 0
\]
for all $y \in X$. Thus
\[
f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x) = A^4(x)
\]
for all $x \in X$.

Conversely, assume that $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of a 4-additive symmetric mapping $A_4: X^4 \to Y$. Note that
\[
\begin{align*}
A^4(qx + ry) &= q^4A^4(x) + 4q^3rA^{3,1}(x, y) + 6q^2r^2A^{2,2}(x, y) + 4qr^3A^{1,3}(x, y) + r^4A^4(y) \\
c^sA^{s,t}(x, y) &= A^{s,t}(cx, y), \quad c^tA^{s,t}(x, y) = A^{s,t}(x, cy)
\end{align*}
\]
where $1 \leq s, t \leq 3$ and $c \in \mathbb{Q}$. Thus we may conclude that $f$ satisfies the equation (1.3). $\square$

3. QUARTIC LIE ∗-DERIVATIONS

Throughout this section, we assume that $A$ is a complex normed ∗-algebra and $M$ is a Banach $A$-bimodule. We will use the same symbol $|| \cdot ||$ as norms on a normed algebra $A$ and a normed $A$-bimodule $M$. A mapping $f : A \to M$ is a quartic homogeneous mapping if $f(\mu a) = \mu^4f(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$. A quartic homogeneous mapping $f : A \to M$ is called a quartic derivation if
\[
f(xy) = f(x)y^4 + x^4f(y)
\]
holds for all $x, y \in A$. For all $x, y \in A$, the symbol $[x, y]$ will denote the commutator $xy - yx$. We say that a quartic homogeneous mapping $f : A \to M$ is a quartic Lie derivation if
\[
f([x, y]) = [f(x), y^4] + [x^4, f(y)]
\]
for all $x, y \in A$. In addition, if $f$ satisfies in condition $f(x^*) = f(x)^*$ for all $x \in A$, then it is called the **quartic Lie $*$-derivation**.

**Example 3.1.** Let $A = \mathbb{C}$ be a complex field endowed with the map $z \mapsto z^* = \bar{z}$ (where $\bar{z}$ is the complex conjugate of $z$). We define $f : A \to A$ by $f(a) = a^4$ for all $a \in A$. Then $f$ is quartic and

$$f([a, b]) = [f(a), b^4] + [a^4, f(b)] = 0$$

for all $a \in A$. Also,

$$f(a^*) = f(\bar{a}) = \bar{a}^4 = \bar{f(a)} = f(a)^*$$

for all $a \in A$. Thus $f$ is a quartic Lie $*$-derivation.

In the following, $T^1$ will stand for the set of all complex units, that is,

$$T^1 = \{ \mu \in \mathbb{C} \mid |\mu| = 1 \}.$$

For the given mapping $f : A \to M$, we consider

$$\Delta f(x, y) := f([x, y]) - [f(x), y^4] - [x^4, f(y)]$$

for all $x, y \in A, \mu \in \mathbb{C}$ and $s \in \mathbb{Z}$ ($s \neq 0, \pm 1$).

**Theorem 3.2.** Suppose that $f : A \to M$ is an even mapping with $f(0) = 0$ for which there exists a function $\phi : A^5 \to [0, \infty)$ such that

$$\tilde{\phi}(a, b, x, y, z) := \sum_{j=0}^{\infty} \frac{1}{|s|^j} \phi(s^j a, s^j b, s^j x, s^j y, s^j z) < \infty$$

$$||\Delta f(a, b)|| \leq \phi(a, b, 0, 0, 0)$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \leq \phi(0, 0, x, y, z)$$

for all $\mu \in T^1_{\frac{1}{n_0}} = \{e^{i\theta} \mid 0 \leq \theta \leq \frac{2\pi}{n_0} \}$ and all $a, b, x, y, z \in A$ in which $n_0 \in \mathbb{N}$. Also, if for each fixed $b \in A$ the mapping $r \mapsto f(rb)$ from $\mathbb{R}$ to $M$ is continuous, then there exists a unique quartic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(b) - L(b)|| \leq \frac{1}{2|s|^4} \tilde{\phi}(0, b, 0, 0, 0).$$
Proof. Let $a = 0$ and $\mu = 1$ in the inequality (3.3), we have

$$\|f(b) - \frac{1}{s^4} f(sb)\| \leq \frac{1}{2|s|^4} \phi(0, b, 0, 0, 0)$$

for all $b \in A$. Using the induction, it is easy to show that

$$\|\frac{1}{s^{4t}} f(s^t b) - \frac{1}{s^{4k}} f(s^k b)\| \leq \frac{1}{2|s|^4} \sum_{j=k}^{t-1} \phi(0, s^j b, 0, 0, 0)$$

for $t > k \geq 0$ and $b \in A$. The inequalities (3.2) and (3.7) imply that the sequence \(\{\frac{1}{s^{4n}} f(s^n b)\}_{n=0}^\infty\) is a Cauchy sequence. Since $M$ is complete, the sequence is convergent. Hence we can define a mapping $L : A \to M$ as

$$L(b) = \lim_{n \to \infty} \frac{1}{s^{4n}} f(s^n b)$$

for $b \in A$. By letting $t = n$ and $k = 0$ in the inequality (3.7), we have

$$\|\frac{1}{s^{4n}} f(s^n b) - f(b)\| \leq \frac{1}{2|s|^4} \sum_{j=0}^{n-1} \phi(0, s^j b, 0, 0, 0)$$

for $n > 0$ and $b \in A$. By taking $n \to \infty$ in the inequality (3.9), the inequalities (3.2) implies that the inequality (3.5) holds.

Now, we will show that the mapping $L$ is a unique quartic Lie ∗-derivation such that the inequality (3.5) holds for all $b \in A$. We note that

$$\|\Delta_\mu L(a, b)\| = \lim_{n \to \infty} \frac{1}{|s|^{4n}} |\Delta_\mu f(s^n a, s^n b)|$$

$$\leq \lim_{n \to \infty} \frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{4n}} = 0,$$

for all $a, b \in A$ and $\mu \in T_{\frac{1}{n_0}}$. By taking $\mu = 1$ in the inequality (3.10), it follows that the mapping $L$ is a quartic mapping. Also, the inequality (3.10) implies that $\Delta_\mu L(0, b) = 0$. Hence

$$L(\mu b) = \mu^4 L(b)$$

for all $b \in A$ and $\mu \in T_{\frac{1}{n_0}}$. Let $\mu \in T^1 = \{\lambda \in C \mid |\lambda| = 1\}$. Then $\mu = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Let $\mu_1 = \mu^\frac{i\theta}{\pi} = e^\frac{i\theta}{\pi}$. Hence we have $\mu_1 \in T_{\frac{1}{\pi n_0}}$. Then

$$L(\mu b) = L(\mu_1^{n_0} b) = \mu_1^{4n_0} L(b) = \mu^4 L(b)$$

for all $\mu \in T^1$ and $a \in A$. Suppose that $\rho$ is any continuous linear functional on $A$ and $b$ is a fixed element in $A$. Then we can define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(r) = \rho(L(rb))$$
for all \( r \in \mathbb{R} \). It is easy to check that \( g \) is cubic. Let

\[
g_k(r) = \rho \left( \frac{f(s^k r b)}{s^{4k}} \right)
\]

for all \( k \in \mathbb{N} \) and \( r \in \mathbb{R} \).

Note that \( g \) as the pointwise limit of the sequence of measurable functions \( g_k \) is measurable. Hence \( g \) as a measurable quartic function is continuous (see [5]) and

\[
g(r) = r^4 g(1)
\]

for all \( r \in \mathbb{R} \). Thus

\[
\rho(L(rb)) = g(r) = r^4 g(1) = r^4 \rho(L(b)) = \rho(r^4 L(b))
\]

for all \( r \in \mathbb{R} \). Since \( \rho \) was an arbitrary continuous linear functional on \( A \) we may conclude that \( L(rb) = r^4 L(b) \) for all \( r \in \mathbb{R} \). Let \( \mu \in \mathbb{C} (\mu \neq 0) \). Then \( \frac{|\mu|}{|\mu|} \in \mathbb{T}^1 \). Hence

\[
L(\mu a) = L \left( \frac{\mu}{|\mu|} |\mu| b \right) = \left( \frac{\mu}{|\mu|} \right)^4 L(|\mu| b) = \left( \frac{\mu}{|\mu|} \right)^4 |\mu|^4 L(b) = \mu^4 L(b)
\]

for all \( b \in A \) and \( \mu \in \mathbb{C} (\mu \neq 0) \). Since \( b \) was an arbitrary element in \( A \), we may conclude that \( L \) is quartic homogeneous.

Next, replacing \( x, y \) by \( s^k x, s^k y \), respectively, and \( z = 0 \) in the inequality (3.4), we have

\[
||\Delta L(x, y)|| = \lim_{n \to \infty} \left| \frac{\Delta f(s^n x, s^n y)}{s^{4n}} \right|
\]

\[
\leq \lim_{n \to \infty} \frac{1}{|s|^{4n}} \phi(0, 0, s^n x, s^n y, 0) = 0
\]

for all \( x, y \in A \). Hence we have \( \Delta L(x, y) = 0 \) for all \( x, y \in A \). That is, \( L \) is a quartic Lie derivation. Letting \( x = y = 0 \) and replacing \( z \) by \( s^k z \) in the inequality (3.4), we get

\[
\left| \left| \frac{f(s^n z^*)}{s^{4n}} - \frac{f(s^n z)^*}{s^{4n}} \right| \right| \leq \frac{\phi(0, 0, 0, 0, s^n z)}{|s|^{4n}}
\]

for all \( z \in A \). As \( n \to \infty \) in the inequality (3.11), we have

\[
L(z^*) = L(z)^*
\]
for all $z \in A$. This means that $L$ is a quartic Lie $\ast$-derivation. Now, assume $L' : A \to A$ is another quartic $\ast$-derivation satisfying the inequality (3.5). Then

\[
\|L(b) - L'(b)\| \leq \frac{1}{|s|^{4n}} \|L(s^n b) - L'(s^n b)\| \leq \frac{1}{|s|^{4n}} \left( \|L(s^n b) - f(s^n b)\| + \|f(s^n b) - L'(s^n b)\| \right) \leq \frac{1}{|s|^{4n}} \sum_{j=0}^{\infty} \frac{1}{|s|^{4j}} \phi(0, s^{j+n} b, 0, 0, 0)
\]

which tends to zero as $k \to \infty$, for all $b \in A$. Thus $L(b) = L'(b)$ for all $b \in A$. This proves the uniqueness of $L$. □

**Corollary 3.3.** Let $\theta, r$ be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that

\[
\|\Delta_{\mu} f(a, b)\| \leq \theta(||a||^r + ||b||^r) \quad \|\Delta f(x, y) + f(z^*) - f(z)^*\| \leq \theta(||x||^r + ||y||^r + ||z||^r)
\]

for all $\mu \in T^1_{m_0}$ and $a, b, x, y, z \in A$. Then there exists a unique quartic Lie $\ast$-derivation $L : A \to M$ satisfying

\[
\|f(b) - L(b)\| \leq \frac{\theta ||b||^r}{2(||s||^r - |s|^r)}
\]

for all $b \in A$.

**Proof.** The proof follows from Theorem 3.2 by taking $\phi(a, b, x, y, z) = \theta(||a||^r + ||b||^r + ||x||^r + ||y||^r + ||z||^r)$ for all $a, b, x, y, z \in A$. □

In the following corollaries, we show the hyperstability for the quartic Lie $\ast$-derivations.

**Corollary 3.4.** Let $r$ be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that

\[
\|\Delta_{\mu} f(a, b)\| \leq ||a||^r ||b||^r \quad \|\Delta f(x, y) + f(z^*) - f(z)^*\| \leq ||x||^r ||y||^r ||z||^r
\]

for all $\mu \in T^1_{m_0}$ and $a, b, x, y, z \in A$. Then $f$ is a quartic Lie $\ast$-derivation on $A$. 

\[
\text{396} \quad \text{Heejeong Koh}
\]
Proof. By taking \( \phi(a, b, x, y, z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r||z||^r) \) in Theorem 3.2 for all \( a, b, x, y, z \in A \), we have \( \tilde{\phi}(0, b, 0, 0, 0) = 0 \). Hence the inequality (3.5) implies that \( f = L \), that is, \( f \) is a quartic Lie \( * \)-derivation on \( A \). \( \square \)

**Corollary 3.5.** Let \( r \) be positive real numbers with \( r < 4 \) and let \( f : A \to M \) be an even mapping with \( f(0) = 0 \) such that
\[
||\Delta f(x, y) + f(z^*) - f(z)\| \leq ||x||^r(||y||^r + ||z||^r)
\]
for all \( \mu \in T_1^1 \) and \( a, b, x, y, z \in A \). Then \( f \) is a quartic Lie \( * \)-derivation on \( A \).

Proof. By taking \( \phi(a, b, x, y, z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r||z||^r) \) in Theorem 3.2 for all \( a, b, x, y, z \in A \), we have \( \tilde{\phi}(0, b, 0, 0, 0) = 0 \). Hence the inequality (3.5) implies that \( f = L \), that is, \( f \) is a quartic Lie \( * \)-derivation on \( A \). \( \square \)

Now, we will investigate the stability of the given functional equation (3.1) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see [11] and [15].

**Definition 3.6.** Let \( X \) be a set. A function \( d : X \times X \to [0, \infty] \) is called a **generalized metric** on \( X \) if \( d \) satisfies
\begin{align*}
(1) & \quad d(x, y) = 0 \text{ if and only if } x = y; \\
(2) & \quad d(x, y) = d(y, x) \text{ for all } x, y \in X; \\
(3) & \quad d(x, z) \leq d(x, y) + d(y, z) \text{ for all } x, y, z \in X.
\end{align*}

**Theorem 3.7** (The alternative of fixed point [11], [15]). Suppose that we are given a complete generalized metric space \( (\Omega, d) \) and a strictly contractive mapping \( T : \Omega \to \Omega \) with Lipschitz constant \( l \). Then for each given \( x \in \Omega \), either
\[
d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,
\]
or there exists a natural number \( n_0 \) such that
\begin{align*}
(1) & \quad d(T^n x, T^{n+1} x) < \infty \text{ for all } n \geq n_0; \\
(2) & \quad \text{The sequence } (T^n x) \text{ is convergent to a fixed point } y^* \text{ of } T; \\
(3) & \quad y^* \text{ is the unique fixed point of } T \text{ in the set } \\
\Delta = \{ y \in \Omega | d(T^{n_0} x, y) < \infty \}; \\
(4) & \quad d(y, y^*) \leq \frac{1}{1-l} d(y, Ty) \text{ for all } y \in \Delta.
\end{align*}
Theorem 3.8. Let $f : A \to M$ be a continuous even mapping with $f(0) = 0$ and let $\phi : A^5 \to [0, \infty)$ be a continuous mapping such that
\begin{align}
(3.12) & \quad ||\Delta_\mu f(a, b)|| \leq \phi(a, b, 0, 0, 0) \\
(3.13) & \quad ||\Delta f(x, y) + f(z^*) - f(z)|| \leq \phi(0, 0, x, y, z)
\end{align}
for all $\mu \in T^1_{\mu_0}$ and $a, b, x, y, z \in A$. If there exists a constant $l \in (0, 1)$ such that
\begin{align}
(3.14) & \quad \phi(sa, sb, sx, sy, sz) \leq |s|^4 \phi(a, b, x, y, z)
\end{align}
for all $a, b, x, y, z \in A$, then there exists a quartic Lie $\ast$-derivation $L : A \to M$ satisfying
\begin{align}
(3.15) & \quad ||f(b) - L(b)|| \leq \frac{1}{2|s|^4(1 - l)} \phi(0, b, 0, 0, 0)
\end{align}
for all $b \in A$.

Proof. Consider the set
\[
\Omega = \{ g \mid g : A \to A, g(0) = 0 \}
\]
and introduce the generalized metric on $\Omega$,
\[
d(g, h) = \inf \{ c \in (0, \infty) \mid \| g(b) - h(b) \| \leq c\phi(0, b, 0, 0, 0), \text{for all } b \in A \}.
\]
It is easy to show that $(\Omega, d)$ is complete. Now we define a function $T : \Omega \to \Omega$ by
\[
T(g)(b) = \frac{1}{s^4} g(sb)
\]
for all $b \in A$. Note that for all $g, h \in \Omega$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) \leq c$. Then
\begin{align}
(3.16) & \quad \|g(b) - h(b)\| \leq c\phi(0, b, 0, 0, 0)
\end{align}
for all $b \in A$. Letting $b = sb$ in the inequality (3.17) and using (3.14) and (3.16), we have
\[
\|T(g)(b) - T(h)(b)\| = \frac{1}{|s|^4} \|g(sb) - h(sb)\|
\leq \frac{1}{|s|^4} c\phi(0, sb, 0, 0, 0) \leq cl \phi(0, b, 0, 0, 0),
\]
that is,
\[
d(Tg, Th) \leq cl.
\]
Hence we have that
\[
d(Tg, Th) \leq ld(g, h),
\]
for all \( g, h \in \Omega \), that is, \( T \) is a strictly self-mapping of \( \Omega \) with the Lipschitz constant \( l \). Letting \( \mu = 1, a = 0 \) in the inequality \((3.12)\), we get

\[
\| \frac{1}{s^4} f(sb) - f(b) \| \leq \frac{1}{2|s|^4} \phi(0, b, 0, 0, 0)
\]

for all \( b \in A \). This means that

\[
d(Tf, f) \leq \frac{1}{2|s|^4}.
\]

We can apply the alternative of fixed point and since \( \lim_{n \to \infty} d(T^n f, L) = 0 \), there exists a fixed point \( L \) of \( T \) in \( \Omega \) such that

\[
(3.18) \quad L(b) = \lim_{n \to \infty} \frac{f(s^n b)}{s^{4n}},
\]

for all \( b \in A \). Hence

\[
d(f, L) \leq \frac{1}{1 - l} d(Tf, f) \leq \frac{1}{2|s|^4} \frac{1}{1 - l}.
\]

This implies that the inequality \((3.15)\) holds for all \( b \in A \). Since \( l \in (0, 1) \), the inequality \((3.14)\) shows that

\[
(3.19) \quad \lim_{n \to \infty} \frac{\phi(s^n a, s^n b, s^n x, s^n y, s^n z)}{|s|^{4n}} = 0.
\]

Replacing \( a, b \) by \( s^n a, s^n b \), respectively, in the inequality \((3.12)\), we have

\[
\frac{1}{|s|^{4n}} \| \Delta \mu f(s^n a, s^n b) \| \leq \frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{4n}}.
\]

Taking the limit as \( k \) tend to infinity, we have \( \Delta \mu f(a, b) = 0 \) for all \( a, b \in A \) and all \( \mu \in T^1_{\frac{1}{|a|}} \). The remains are similar to the proof of Theorem 3.2. \( \square \)

**Corollary 3.9.** Let \( \theta, r \) be positive real numbers with \( r < 4 \) and let \( f : A \to M \) be a mapping with \( f(0) = 0 \) such that

\[
\| \Delta f(x, y) + f(z^*) - f(z)^* \| \leq \theta(|x|^r + |y|^r + |z|^r)
\]

for all \( \mu \in T^1_{\frac{1}{2|a|}} \) and \( a, b, x, y, z \in A \). Then there exists a unique quartic Lie \(*\)-derivation \( L : A \to M \) satisfying

\[
\| f(b) - L(b) \| \leq \frac{\theta|b|^r}{2|s|^4(1 - l)}
\]

for all \( b \in A \).

**Proof.** The proof follows from Theorem 3.8 by taking \( \phi(a, b, x, y, z) = \theta(|a|^r + |b|^r + |x|^r + |y|^r + |z|^r) \) for all \( a, b, x, y, z \in A \). \( \square \)
In the following corollaries, we show the hyperstability for the quartic Lie $\ast$-derivations.

**Corollary 3.10.** Let $r$ be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that

$$
||\Delta_{\mu}f(a,b)|| \leq ||a||^r||b||^r
$$

$$
||\Delta f(x,y) + f(z^\ast) - f(z)\ast|| \leq ||x||^r||y||^r||z||^r
$$

for all $\mu \in \mathbb{T}_{1\frac{1}{20}}$ and $a, b, x, y, z \in A$. Then $f$ is a quartic Lie $\ast$-derivation on $A$.

**Proof.** By taking $\phi(a, b, x, y, z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r + ||z||^r)$ in Theorem 3.8 for all $a, b, x, y, z \in A$, we have $\tilde{\phi}(0, b, 0, 0, 0) = 0$. Hence the inequality (3.15) implies that $f = L$, that is, $f$ is a quartic Lie $\ast$-derivation on $A$.

**Corollary 3.11.** Let $r$ be positive real numbers with $r < 4$ and let $f : A \to M$ be an even mapping with $f(0) = 0$ such that

$$
||\Delta_{\mu}f(a,b)|| \leq ||a||^r||b||^r
$$

$$
||\Delta f(x,y) + f(z^\ast) - f(z)\ast|| \leq ||x||^r(||y||^r + ||z||^r)
$$

for all $\mu \in \mathbb{T}_{1\frac{1}{20}}$ and $a, b, x, y, z \in A$. Then $f$ is a quartic Lie $\ast$-derivation on $A$.

**Proof.** By taking $\phi(a, b, x, y, z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r + ||z||^r)$ in Theorem 3.8 for all $a, b, x, y, z \in A$, we have $\tilde{\phi}(0, b, 0, 0, 0) = 0$. Hence the inequality (3.15) implies that $f = L$, that is, $f$ is a quartic Lie $\ast$-derivation on $A$.

**Acknowledgement**

The present research was conducted by the research fund of Dankook University in 2014.

**References**

1. T. Aoki: On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan* 2 (1950), 64-66.
2. N. Brillouët-Belluot, J. Brzdęk & K. Ciepliński: Fixed point theory and the Ulam stability. *Abstract and Applied Analysis* 2014, Article ID 829419, 16 pages (2014).
3. J. Brzdęk, L. Ćadariu & K. Ciepliński: On some recent developments in Ulam’s type stability. *Abstract and Applied Analysis* 2012, Article ID 716936, 41 pages (2012).
4. J.K. Chung & P.K. Sahoo: On the general solution of a quartic functional equation. *Bull. Korean Math. Soc.* 40 (2003), no. 4, 565-576.
5. St. Czerwik: On the stability of the quadratic mapping in normed spaces. *Abh. Math. Sem. Univ. Hamburg* 62 (1992), 59-64.
6. A. Fošner & M. Fošner: Approximate cubic Lie derivations. *Abstract and Applied Analysis* 2013, Article ID 425784, 5 pages (2013).
7. D.H. Hyers: On the stability of the linear equation. *Proc. Nat. Acad. Sci. U.S.A.* 27 (1941), 222-224.
8. D.H. Hyers, G. Isac & Th.M. Rassias: *Stability of Functional Equations in Several Variables*. Birkhäuser, Boston, USA, 1998.
9. S. Jang & C. Park: Approximate *-derivations and approximate quadratic *-derivations on C*-algebra. *J. Inequal. Appl.* 2011, Article ID 55 (2011).
10. S.-M. Jung: *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*. Vol. 48 of Springer Optimization and Its Applications, Springer, New York, USA, 2011.
11. B. Margolis & J.B. Diaz: A fixed point theorem of the alternative for contractions on a generalized complete metric space. *Bull. Amer. Math. Soc.* 126 (1968), no. 74, 305-309.
12. C. Park & A. Bodaghi: On the stability of *-derivations on Banach *-algebras. *Adv. Diff. Equat.* 2012, 2012:138 (2012).
13. J.M. Rassias: Solution of the Ulam stability problem for quartic mappings. *Glasnik Matematicki Series III* 34 (1999), no. 2, 243-252.
14. Th. M. Rassias: On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.* 72 (1978), 297-300.
15. I.A. Rus: *Principles and Applications of Fixed Point Theory*. Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).
16. P.K. Sahoo: A generalized cubic functional equation. *Acta Math. Sinica* 21 (2005), no. 5, 1159-1166.
17. S.M. Ulam: *Problems in Morden Mathematics*. Wiley, New York, USA, 1960.
18. T.Z. Xu, J.M. Rassias & W.X. Xu: A generalized mixed quadratic-quartic functional equation. *Bull. Malaysian Math. Sci. Soc.* 35 (2012), no. 3, 633-649.
19. S.Y. Yang, A. Bodaghi & K.A.M. Atan: Approximate cubic *-derivations on Banach *-algebra. *Abstract and Applied Analysis* 2012, Article ID 684179, 12 pages (2012).

**Department of Mathematical Education, Dankook University, 152, Jukjeon, Suji, Yonggin, Gyeonggi, Korea 448-701**

*Email address: khjmath@dankook.ac.kr (H. Koh)*