SCHUR-FINITENESS IN $\lambda$-RINGS

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Abstract. We introduce the notion of a Schur-finite element in a $\lambda$-ring.

Since the beginning of algebraic $K$-theory in [G57], the splitting principle has proven invaluable for working with $\lambda$-operations. Unfortunately, this principle does not seem to hold in some recent applications, such as the $K$-theory of motives. The main goal of this paper is to introduce the subring of Schur-finite elements of any $\lambda$-ring, and study its main properties, especially in connection with the virtual splitting principle.

A rich source of examples comes from Heinloth’s theorem [H1], that the Grothendieck group $K_0(A)$ of an idempotent-complete $\mathbb{Q}$-linear tensor category $A$ is a $\lambda$-ring. For the category $\mathcal{M}^{\text{eff}}$ of effective Chow motives, we show that $K_0(\text{Var}) \to K_0(\mathcal{M}^{\text{eff}})$ is not an injection, answering a question of Grothendieck.

When $A$ is the derived category of motives $\mathcal{D}m_{gm}$ over a field of characteristic 0, the notion of Schur-finiteness in $K_0(\mathcal{D}m_{gm})$ is compatible with the notion of a Schur-finite object in $\mathcal{D}m_{gm}$, introduced in [Mz].

We begin by briefly recalling the classical splitting principle in Section 1, and answering Grothendieck’s question in Section 2. In section 3 we recall the Schur polynomials, the Jacobi-Trudi identities and the Pieri rule from the theory of symmetric functions. Finally, in Section 4, we define Schur-finite elements and show that they form a subring of any $\lambda$-ring. We also state the conjecture that every Schur-finite element is a virtual sum of line elements.

Notation. We will use the term $\lambda$-ring in the sense of [Ber, 2.4]; we warn the reader that our $\lambda$-rings are called special $\lambda$-rings by Grothendieck, Atiyah and others; see [G57] [AT] [A].

A $\mathbb{Q}$-linear category $A$ is a category in which each hom-set is uniquely divisible (i.e., a $\mathbb{Q}$-module). By a $\mathbb{Q}$-linear tensor category (or $\mathbb{Q}$TC) we mean a $\mathbb{Q}$-linear category which is also symmetric monoidal and such that the tensor product is $\mathbb{Q}$-linear. We will be interested in $\mathbb{Q}$TC’s which are idempotent-complete.
1. Finite-dimensional $\lambda$-rings

Almost all $\lambda$-rings of historical interest are finite-dimensional. This includes the complex representation rings $R(G)$ and topological $K$-theory of compact spaces [AT 1.5] as well as the algebraic $K$-theory of algebraic varieties [G57]. In this section we present this theory from the viewpoint we are adopting. Little in this section is new.

Recall that an element $x$ in a $\lambda$-ring $R$ is said to be even of finite degree $n$ if $\lambda_t(x)$ is a polynomial of degree $n$, or equivalently that there is a $\lambda$-ring homomorphism from the ring $\Lambda_n$ defined in [12] to $R$, sending $a$ to $x$. We say that $x$ is a line element if it is even of degree 1, i.e., if $\lambda^n(x) = 0$ for all $n > 1$.

We say that $x$ is odd of degree $n$ if $\sigma_t(x) = \lambda_{-t}(x)^{-1}$ is a polynomial of finite degree $n$. Since $\sigma_{-t}(x) = \lambda_t(-x)$, we see that $x$ is odd just in case $-x$ is even. Therefore there is a $\lambda$-ring homomorphism from the ring $\Lambda_{-n}$ defined in [12] to $R$ sending $b$ to $x$.

We say that an element $x$ is finite-dimensional if it is the difference of two even elements, or equivalently if $x$ is the sum of an even and an odd element. The subset of even elements in $R$ is closed under addition and multiplication, and the subset of finite-dimensional elements forms a subring of $R$.

Example 1.1. If $R$ is a binomial $\lambda$-ring, then $r$ is even if and only if some $r(r - 1) \cdots (r - n) = 0$, and odd if and only if some $r(r + 1) \cdots (r + n) = 0$. The binomial rings $\prod_{k=1}^{n} \mathbb{Z}$ are finite dimensional. If $R$ is connected then the subring of finite-dimensional elements is just $\mathbb{Z}$.

There is a well known family of universal finite-dimensional $\lambda$-rings $\{\Lambda_n\}$.

Definition 1.2. Following [AT], let $\Lambda_n$ denote the free $\lambda$-ring generated by one element $a = a_1$ of finite degree $n$ (i.e., subject to the relations that $\lambda^k(a) = 0$ for all $k > n$). By [Ber 4.9], $\Lambda_n$ is just the polynomial ring $\mathbb{Z}[a_1, ..., a_n]$ with $a_i = \lambda^i(a_1)$.

Similarly, we write $\Lambda_{-n}$ for the free $\lambda$-ring generated by one element $b = b_1$, subject to the relations that $\sigma^k(b) = 0$ for all $k > n$. Using the antipode $S$, we see that there is a $\lambda$-ring isomorphism $\Lambda_{-n} \cong \Lambda_n$ sending $b$ to $-a$, and hence that $\Lambda_{-n} \cong \mathbb{Z}[b_1, ..., b_n]$ with $b_k = \sigma^k(b)$.

Consider finite-dimensional elements in $\lambda$-rings $R$ which are the difference of an even element of degree $m$ and an odd element of degree $n$. The maps $\Lambda_m \to R$ and $\Lambda_{-n} \to R$ induce a $\lambda$-ring map from $\Lambda_m \otimes \Lambda_{-n}$ to $R$.

Lemma 1.3. If an element $x$ is both even and odd in a $\lambda$-ring, then $x$ and all the $\lambda^i(x)$ are nilpotent. Thus $\lambda_t(x)$ is a unit of $R[\!\!\!\!t\!\!\!\!]$.

Proof. If $x$ is even and odd then $\lambda_t(x)$ and $\sigma_{-t}(x)$ are polynomials in $R[\!\!\!\!t\!\!\!\!]$ which are inverse to each other. It follows that the coefficients $\lambda^i(x)$ of the $t^i$ are nilpotent for all $i > 0$.

If $R$ is a graded $\lambda$-ring, an element $\sum r_i$ is even (resp., odd, resp., finite-dimensional) if and only if each homogeneous term $r_i$ is even (resp., odd, resp., finite-dimensional). This is because the operations $\lambda^n$ multiply the degree of an element by $n$.

The forgetful functor from $\lambda$-rings to commutative rings has a right adjoint; see [Kr] pp. 20–21. It follows that the category of $\lambda$-rings has all colimits. In particular, if $B \leftarrow A \to C$ is a diagram of $\lambda$-rings, the tensor product $B \otimes_A C$ has the structure of a $\lambda$-ring. Here is a typical, classical application of this construction, originally proven in [AT 6.1].

Proposition 1.4 (Splitting Principle). If $x$ is any even element of finite degree $n$ in a $\lambda$-ring $R$, there exists an inclusion $R \subseteq R'$ of $\lambda$-rings and line elements $\ell_1, ..., \ell_n$ in $R'$ so that $x = \sum \ell_i$. 

Proof. Let $\Omega_n$ denote the tensor product of $n$ copies of the $\lambda$-ring $\Lambda_1 = \mathbb{Z}[\ell]$; this is a $\lambda$-ring whose underlying ring is the polynomial ring $\mathbb{Z}[\ell_1, \ldots, \ell_n]$, and the $\lambda$-ring $\Lambda_n$ of Definition 1.2 is the subring of symmetric polynomials in $\Omega_n$; see [AT] §2.

Let $R'$ be the pushout of the diagram $\Omega_n \in\Lambda_n \to R$. Since the image of $x$ is $1 \otimes x = a \otimes 1 = (\sum \ell_i) \otimes 1$, it suffices to show that $R \to R'$ is an injection. This follows from the fact that $\Omega_n$ is free as a $\Lambda_n$-module.

Corollary 1.5. If $x$ is any finite-dimensional element of a $\lambda$-ring $R$, there is an inclusion $R \subseteq R'$ of $\lambda$-rings and line elements $\ell_i, \ell'_i$ in $R'$ so that

$$x = (\sum \ell_i) - (\sum \ell'_i).$$

Scholium 1.6. For later use, we record an observation, whose proof is implicit in the proof of Proposition 4.2 of [AT]: $\lambda^m(\lambda^n x) = P_{m,n}(\lambda^1 x, \ldots, \lambda^m x)$ is a sum of monomials, each containing a term $\lambda^i x$ for $i \geq n$. For example, $\lambda^2(\lambda^3 x) = \lambda^6 x - x \lambda^5 x + \lambda^4 x^2 \lambda^2 x^2$ (see [Kn] p. 11).

2. $K_0$ of tensor categories

The Grothendieck group of a $\mathbb{Q}$-linear tensor category provides numerous examples of $\lambda$-rings, and forms the original motivation for introducing the notion of Schur-finite elements in a $\lambda$-ring.

A $\mathbb{Q}$-linear tensor category is exact if it has a distinguished family of sequences, called short exact sequences and satisfying the axioms of $\mathbb{Q}$, and such that each $A \otimes -$ is an exact functor. In many applications $A$ is split exact: the only short exact sequences are those which split. By $K_0(A)$ we mean the Grothendieck group of an exact category, i.e., the quotient of the free abelian group on the objects $[A]$ by the relation that $[B] = [A] + [C]$ for every short exact sequence $0 \to A \to B \to C \to 0$.

Let $A$ be an idempotent-complete exact category which is a $\mathbb{Q}$TC for $\otimes$. For any object $A$ in $\mathcal{A}$, the symmetric group $\Sigma_n$ (and hence the group ring $\mathbb{Q}[\Sigma_n]$) acts on the $n$-fold tensor product $A^\otimes n$. If $A$ is idempotent-complete, we define $\Lambda^n A$ to be the direct summand of $A^\otimes n$ corresponding to the alternating idempotent $\sum (-1)^{\sigma(n)!}$ of $\mathbb{Q}[\Sigma_n]$. Similarly, we can define the symmetric powers $\text{Sym}^n(A)$. It turns out that $\lambda^n(A)$ only depends upon the element $[A]$ in $K_0(A)$, and that $\lambda^n$ extends to a well defined operation on $K_0(A)$.

The following result was proven by F. Heinloth in [Hl, Lemma 4.1], but the result seems to have been in the air; see [May] p. 486, [LLO] 5.1 and [B1, B2]. A special case of this result was proven long ago by Swan in [Sw].

Theorem 2.1. If $\mathcal{A}$ is any idempotent-complete exact $\mathbb{Q}$TC, $K_0(\mathcal{A})$ has the structure of a $\lambda$-ring. If $A$ is any object of $\mathcal{A}$ then $\lambda^n([A]) = [\Lambda^n A]$.

Kimura [Kim] and O’Sullivan have introduced the notion of an object $C$ being finite-dimensional in any $\mathbb{Q}$TC $\mathcal{A}$: $C$ is the direct sum of an even object $A$ (one for which some $\Lambda^n A \cong 0$) and an odd object $B$ (one for which some $\text{Sym}^n(B) \cong 0$). It is immediate that $[C]$ is a finite-dimensional element in the $\lambda$-ring $K_0(\mathcal{A})$. Thus the two notions of finite dimensionality are related.

Example 2.2. Let $\mathcal{M}^{\text{eff}}$ denote the category of $\mathbb{Q}$-linear pure effective Chow motives with respect to rational equivalence over a field $k$. Its objects are summands of smooth projective varieties over a field $k$ and morphisms are given by Chow groups. Thus $K_0(\mathcal{M}^{\text{eff}})$ is the group generated by the classes of objects, modulo the relation $[M_1 \oplus M_2] = [M_1] + [M_2]$. Since $\mathcal{M}^{\text{eff}}$ is a $\mathbb{Q}$TC, $K_0(\mathcal{M}^{\text{eff}})$ is a $\lambda$-ring.

By adjoining an inverse to the Lefschetz motive to $\mathcal{M}^{\text{eff}}$, we obtain the category $\mathcal{M}$ of Chow motives (with respect to rational equivalence). This is also a $\mathbb{Q}$TC, so $K_0(\mathcal{M})$ is a $\lambda$-ring.
The category $\mathcal{M}^{\text{eff}}$ embeds into the triangulated category $\text{DM}_{\text{gm}}^{\text{eff}}$ of effective geometric motives; see [MVW 20.1]. Similarly, the category $\mathcal{M}$ embeds in the triangulated category $\text{DM}_{\text{gm}}$ of geometric motives [MVW 20.2]. Bondarko proved in [Bo] 6.4.3 that $K_0(\text{DM}_{\text{gm}}^{\text{eff}}) \cong K_0(\mathcal{M}^{\text{eff}})$ and $K_0(\text{DM}_{\text{gm}}) \cong K_0(\mathcal{M})$. Thus we may investigate $\lambda$-ring questions in these triangulated settings. As far as we know, it is possible that every element of $K_0(\text{DM}_{\text{gm}})$ is finite-dimensional.

Recall that a motive $M$ in $\mathcal{M}^{\text{eff}}$ is a phantom motive if $H^*(M) = 0$ for every Weil cohomology $H$.

**Proposition 2.3.** Let $M$ be an object of $\mathcal{M}^{\text{eff}}$. Then if $[M] = 0$ in $K_0(\mathcal{M}^{\text{eff}})$, then $M$ is a phantom motive.

**Proof.** Since $\mathcal{M}^{\text{eff}}$ is an additive category, $[M] = 0$ implies that there is another object $N$ of $\mathcal{M}^{\text{eff}}$ such that $M \oplus N \cong N$. But every effective motive is a summand of the motive of a scheme, hence we may assume $N = M(X)$. If $M$ is not a phantom motive, there is a Weil cohomology and an $i$ such that $H^i(M) \neq 0$. But then $H^i(M) \oplus H^i(X) \cong H^i(X)$; since these are finite-dimensional vector spaces, this implies $H^i(M) = 0$, a contradiction. □

Here is an application of these ideas. Recall that any quasi-projective scheme $X$ has a motive with compact supports in $\text{DM}^{\text{eff}}$, $M^c(X)$. If $k$ has characteristic 0, this is an effective geometric motive, and if $U$ is open in $X$ with complement $Z$ there is a triangle $M^c(Z) \to M^c(X) \to M^c(U)$; see [MVW] 16.15. It follows that $[M^c(X)] = [M^c(U)] + [M^c(Z)]$ in $K_0(\mathcal{M}^{\text{eff}})$. (This was originally proven by Gillet and Soulé in [GS, Thm. 4] before the introduction of $\text{DM}$, but see [GS] 3.2.4).

**Definition 2.4.** Let $K_0(\text{Var})$ be the Grothendieck ring of varieties obtained by imposing the relation $[U] + [X \setminus U] = [X]$ for any open $U$ in a variety $X$. By the above remarks, there is a well defined ring homomorphism $K_0(\text{Var}) \to K_0(\mathcal{M}^{\text{eff}})$. Grothendieck asked in [G53, p.174] if this morphism was far from being an isomorphism. We can now answer his question.

**Theorem 2.5.** The homomorphism $K_0(\text{Var}) \to K_0(\mathcal{M}^{\text{eff}})$ is not an injection.

**Remark 2.5.1.** After this paper was posted in 2010, we were informed by J. Sebag that Grothendieck’s question had also been answered in [LS, Remark 14].

For the proof, we need to introduce Kapranov’s zeta-function. If $X$ is any quasi-projective variety, its symmetric power $S^nX$ is the quotient of $X^n$ by the action of the symmetric group. We define $\zeta_t(X) = \sum [S^nX]t^n$ as a power series with coefficients in $K_0(\text{Var})$.

**Lemma 2.6.** ([Gul]) The following diagram is commutative:

\[
\begin{array}{ccc}
K_0(\text{Var}) & \xrightarrow{\zeta_t} & 1 + K_0(\text{Var})[[t]] \\
M^c & \downarrow & M^c \\
K_0(\mathcal{M}^{\text{eff}}) & \xrightarrow{\sigma_t} & 1 + K_0(\mathcal{M}^{\text{eff}})[[t]].
\end{array}
\]

**Proof.** It suffices to show that $[M^c(S^nX)] = \text{Sym}^n[M^c(X)]$ in $K_0(\mathcal{M}^{\text{eff}})$ for any $X$. This is proven by del Baño and Navarro in [BN] 5.3. □

**Definition 2.7.** Following [LL04, 2.2], we say that a power series $f(t) = \sum r_nt^n \in R[[t]]$ is determinantly rational over a ring $R$ if there exists an $m, n_0 > 0$ such that the $m \times m$ Hankel matrices $(r_{n+i+j})_{i,j=1}^m$ have determinant 0 for all $n > n_0$. 


The name comes from the classical fact (Émile Borel [1894]) that when $R$ is a field (or a domain) a power series is determinantly rational if and only if it is a rational function $p(t)/q(t)$. For later use, we observe that $\deg(q) < m$ and $\deg(p) < n_0$. (This is relation (a) in [1894].)

Clearly, if $f(t)$ is not determinantly rational over $R$ and $R \subset R'$ then $f(t)$ cannot be determinantly rational over $R'$.

As observed in [LL04 2.4], if $f$ is a rational function in the sense that $gf = h$ for polynomials $g(t), h(t)$ with $g(0) = 1$ then $f$ is determinantly rational. For example, if $x = a_i$ is a finite-dimensional element of a $\lambda$-ring $R$, with $a$ even and $b$ odd, then $\lambda(x)$ and $\lambda_i(-b) = \lambda_i(b)^{-1}$ are polynomials so $\lambda(x) = \lambda_i(a)\lambda_i(b)$ and $\sigma_i(x) = \lambda_i(x)^{-1}$ are rational functions. This was observed by André in [A05].

**Proof of Theorem 2.5.** Let $X$ be the product $C \times D$ of two smooth projective curves of genus $> 0$, so that $p_0(X) > 0$. Larsen and Lunts showed in [LL04 2.4, 3.9] that $\zeta_1(X)$ is not determinantly rational over $R = K_0(\text{Var})$. On the other hand, Kimura proved in [Kim] that $X$ is a finite-dimensional object in $\mathcal{M}_\text{eff}$, so $\sigma_1(X) = \lambda_i(X)^{-1}$ is a determinantly rational function in $R' = K_0(\mathcal{M}_\text{eff})$. It follows that $R \to R'$ cannot be an injection. \hfill $\Box$

### 3. Symmetric functions

We devote this section to a quick study of the ring $\Lambda$ of symmetric functions, and especially the Schur polynomials $s_\pi$, referring the reader to [Macd] for more information. In the next section, we will use these polynomials to define the notion of Schur-finite elements in a $\lambda$-ring.

The ring $\Lambda$ is defined as the ring of symmetric “polynomials” in variables $\xi_i$. More precisely, it is the subring of the power series ring in $\{\xi_n\}$ generated by $e_1 = \sum \xi_n$ and the other elementary symmetric power sums $e_i \in \Lambda$; if we put $\xi_r = 0$ for $r > n$ then $e_i$ is the $i^{th}$ elementary symmetric polynomial in $\xi_1, \ldots, \xi_n$; see [AT]. A major role is also played by the homogeneous power sums $h_n = \sum \xi_1 \cdots \xi_n$ (where the sum being taken over $i_1 \leq \cdots \leq i_n$). Their generating functions $E(t) = \sum e_n t^n$ and $H(t) = \sum h_n t^n$ are $\prod(1 + \xi_i t)$ and $\prod(1 - \xi_i t)^{-1}$, so that $H(t)E(-t) = 1$. In fact, $\Lambda$ is a graded polynomial ring in two relevant ways (with $e_n$ and $h_n$ in degree $n$):

$$\Lambda = \mathbb{Z}[e_1, \ldots, e_n, \ldots] = \mathbb{Z}[h_1, \ldots, h_n, \ldots].$$

Given a partition $\pi = (n_1, \ldots, n_r)$ of $n$ (so that $\sum n_i = n$), we let $s_\pi \in \Lambda_n$ denote the Schur polynomial of $\pi$. The elements $e_n$ and $h_n$ of $\Lambda$ are identified with $s_{(1, \ldots, 1)}$ and $s_{(n)}$, respectively. The Schur polynomials also form a $\mathbb{Z}$-basis of $\Lambda$ by [Macd 3.3].

Here is another description of $\Lambda$, taken from [Kn]: $\Lambda$ is isomorphic to the direct sum $R_{\text{eff}}$ of the representation rings $R(\Sigma_n)$, made into a ring via the outer product $R(\Sigma_m) \otimes R(\Sigma_n) \to R(\Sigma_{m+n})$. Under this identification, $e_n \in \Lambda_n$ is identified with the class of the trivial simple representation $V^*_{n}$ of $\Sigma_n$. More generally, $s_\pi$ corresponds to the class $[V_{\pi}]$ in $R(\Sigma_n)$ of the irreducible representation corresponding to $\pi$. (See [Kn III.3].)

**Proposition 3.1.** $\Lambda$ is a graded Hopf algebra, with coproduct $\Delta$ and antipode $S$ determined by the formulas

$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j, \quad S(e_n) = h_n \quad \text{and} \quad S(h_n) = e_n.$$
We prove the assertions about \( \pi \) at most \( p \) plying the antipode \( \Pi \).

The fact that there is a ring involution \( \lambda \) is an ideal.

Remark 3.2. Atiyah shows in [A, 1.2] that \( \Lambda \) is isomorphic to the graded dual \( R^* = \oplus \text{Hom}(R(\Sigma_n), \mathbb{Z}) \). That is, if \( \{v_\pi\} \) is the dual basis in \( R^n \) to the basis \( \{[V_\pi]\} \) of simple representations in \( R^n \), and the restriction of \( V_\pi \) to the basis \( \{V_\mu\} \) then \( v_\pi v_\mu = \sum e^{\mu\nu}_e v_\nu \) in \( R^* \). Thus the product studied by Atiyah on the graded dual \( R^* \) is exactly the algebra structure dual to the coproduct \( \Delta \).

Let \( \pi' \) denote the conjugate partition to \( \pi \). The Jacobi-Trudi identities \( s_\pi = \det |h_{\pi_i+j-1}| = \det |e_{\pi_i+j-1}| \) show that the antipode \( S \) interchanges \( s_\pi \) and \( s_{\pi'} \). (Jacobi conjectured the identities, and his student Nicolò Trudi verified them in 1864; they were rediscovered by Giovanni Giambelli in 1903 and are sometimes called the Giambelli identities).

Let \( I_{e,n} \) denote the ideal of \( \Lambda \) generated by the \( e_i \) with \( i \geq n \). The quotient \( \Lambda/I_{e,n} \) is the polynomial ring \( \Lambda_{n-1} = \mathbb{Z}[e_1, \ldots, e_{n-1}] \). Let \( I_{h,n} \) denote \( S(I_{e,n}) \), i.e., the ideal of \( \Lambda \) generated by the \( h_i \) with \( i \geq n \).

Proposition 3.3. The Schur polynomials \( s_\pi \) for partitions \( \pi \) containing \( (1^n) \) (i.e., with at least \( n \) rows) form a \( \mathbb{Z} \)-basis for the ideal \( I_{e,n} \). The Schur polynomials with at most \( n \) rows form a \( \mathbb{Z} \)-basis of \( \Lambda_n \).

Similarly, the Schur polynomials \( s_\pi \) for partitions \( \pi \) containing \( (n) \) (i.e., with \( \pi_1 \geq n \)) form a \( \mathbb{Z} \)-basis for the ideal \( I_{h,n} \).

Proof. We prove the assertions about \( I_{e,n} \); the assertion about \( I_{h,n} \) follows by applying the antipode \( S \). By [Macd I.3.2], the \( s_\pi \) which have fewer than \( n \) rows project onto a \( \mathbb{Z} \)-basis of \( \Lambda_{n-1} = \Lambda/I_{e,n} \). Since the \( s_\pi \) form a \( \mathbb{Z} \)-basis of \( \Lambda \), it suffices to show that every partition \( \pi = (\pi_1, \ldots, \pi_r) \) with \( r > n \) is in \( I_{e,n} \). Expansion along the first row of the Jacobi-Trudi identity \( s_\pi = \det |e_{\pi_i+j-1}| \) shows that \( s_\pi \) is in the ideal \( I_{e,r} \).

Corollary 3.4. The ideal \( I_{h,m} \cap I_{e,n} \) of \( \Lambda \) has a \( \mathbb{Z} \)-basis consisting of the Schur polynomials \( s_\pi \) for partitions \( \pi \) containing the hook \( (m, 1^{n-1}) = (m, 1, \ldots, 1) \).

Definition 3.5. For any partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \), let \( I_\lambda \) denote the subgroup of \( \Lambda \) generated by the Schur polynomials \( s_\pi \) for which \( \pi \) contains \( \lambda \), i.e., \( \pi_i \geq \lambda_i \) for \( i = 1, \ldots, r \). We have already encountered the special cases \( I_{e,n} = I_{(1_{1,1})} \) and \( I_{h,n} = I_{(n)} \) in Proposition 3.3 and \( I_{(m,1,\ldots,1)} = I_{h,m} \cap I_{e,n} \) in Corollary 3.4.

Example 3.6. Consider the partition \( \lambda = (2,1) \). Then \( I_\lambda = I_{h,2} \cap I_{e,2} \) by Corollary 3.3. \( \Lambda_\lambda \) is the pullback of \( \mathbb{Z}[a] \) and \( \mathbb{Z}[b] \) along the common quotient \( \mathbb{Z}[a]/(a^2) = \Lambda/(I_{(1,1)} + I_{(2)}) \). The universal element of \( \Lambda_\lambda \) is \( x = (a,b) \) and if we set \( y = (0,b^2) \) then \( \Lambda_{(2,1)} \cong \mathbb{Z}[x,y]/(y^2 - x^2) \). Since \( \lambda^0(b) = b^n \) for all \( n \), it is easy to check that \( \lambda^2(x) = y \) and \( \lambda^{2+1}(x) = xy^2 \).

Lemma 3.7. The \( I_\lambda \) are ideals of \( \Lambda \), and \( \{I_\lambda\} \) is closed under intersection.

Proof. The Pieri rule writes \( h_\mu s_\pi \) as a sum of \( s_\nu \), where \( \mu \) runs over partitions consisting of \( \pi \) and \( \mu \) other elements, no two in the same column. Thus \( I_\lambda \) is closed under multiplication by the \( h_\mu \). As every element of \( \Lambda \) is a polynomial in the \( h_\mu \), \( I_\lambda \) is an ideal.

If \( \mu = (\mu_1, \ldots, \mu_s) \) is another partition, then \( s_\mu \) is in \( I_\lambda \cap I_\mu \) if and only if \( \mu_i \geq \max(\lambda_i, \mu_i) \). Thus \( I_\lambda \cap I_\mu = I_{\lambda \cap \mu} \). \( \square \)
Consider the set $\Lambda$

Proof. Since $\Delta$ is a ring homomorphism, $\Lambda$ corresponds to a natural operation on $\lambda$. The formula $f(x) = u_\lambda(f)$ defines a natural operation. The operation $\lambda_\ast$ corresponds to $e_\lambda$. The operation $\sigma^n$, defined by $\sigma^n(x) = (-1)^n \lambda^n(-x)$, corresponds to $h_n$; this may be seen by comparing the generating functions $H(t) = E(-t)^{-1}$ and $\sigma_1(x) = \lambda^{-1}(x)^{-1}$.

Proposition 3.9. If $\phi$ is an element of $\Lambda$, and $\Delta(\phi) = \sum \phi'_i \otimes \phi''_i$ then the corresponding natural operation on $\lambda$-rings satisfies $\phi(x + y) = \sum \phi'_i(x) \phi''_i(y)$.

Proof. Consider the set $\Lambda'$ of all operations in $\Lambda$ satisfying the condition of the proposition. Since $\Delta$ is a ring homomorphism, $\Lambda'$ is a subring of $\Lambda$. Since $\Delta(e_\lambda) = \sum c_i \otimes e_{n-i}$ and $\lambda^n(x + y) = \sum \lambda^i(x) \lambda^{n-i}(y)$, $\Lambda'$ contains the generators $e_\lambda$ of $\Lambda$, and hence $\Lambda' = \Lambda$.

The Littlewood-Richardson rule states that $\Delta([V_\pi])$ is a sum $\sum c_{\mu \nu}^\pi [V_\mu] \otimes [V_\nu]$, where $\mu \subseteq \pi$ and $\pi$ is obtained from $\mu$ by concatenating $\nu$ in a certain way; see [Macd, §1.9]. By Proposition 3.9, we then have

Corollary 3.10. $s_\pi(x + y) = \sum c_{\mu \nu}^\pi s_\mu(x) s_\nu(y)$.

4. SCHUR-FINITENESS IN $\lambda$-RINGS

In this section we introduce the notion of a Schur-finite element in a $\lambda$-ring $R$, and show that these elements form a subring of $R$ containing the subring of finite-dimensional elements. We conjecture that they are the elements for which the virtual splitting principle holds.

Definition 4.1. We say that an element $x$ in a $\lambda$-ring $R$ is Schur-finite if there exists a partition $\lambda$ such that $s_\lambda(x) = 0$ for every partition $\mu$ containing $\lambda$. That is, $I_\lambda$ annihilates $x$. We call such a $\lambda$ a bound for $x$.

By Remark 3.8, $x \in R$ may have no unique minimal bound $\lambda$. By Example 4.4 below, $s_\lambda(x) = 0$ does not imply that $\lambda$ is a bound for $x$.

Proposition 4.2. Each $I_\lambda$ is a radical $\lambda$-ideal, and $\Lambda_\lambda = \Lambda/I_\lambda$ is a reduced $\lambda$-ring. Thus every Schur-finite $x \in R$ with bound $\lambda$ determines a $\lambda$-ring map $f : \Lambda_\lambda \rightarrow R$ with $f(a) = x$.

Moreover, if $\lambda$ is a rectangular partition then $I_\lambda$ is a prime ideal, and $\Lambda_\lambda$ is a subring of a polynomial ring in which a becomes finite-dimensional.

In general, $\Lambda_\lambda$ is a subring of $\prod \Lambda_{\beta_i}$ and hence of a product of polynomial rings in which $a$ becomes finite-dimensional.

Proposition 4.2 verifies Conjecture 3.9 of [KKT].

Proof. Fix a rectangular partition $\beta = ((m + 1)^{n+1}) = (m + 1, ..., m + 1)$, and set $a = \sum a_i, b = \sum b_j$. Consider the universal $\lambda$-ring map $f : \Lambda \rightarrow \Lambda_m \otimes \Lambda_{-n} \cong Z[a_1, ..., a_m, b_1, ..., b_n]$
sending $e_1$ to the finite-dimensional element $a + b$ (see Definition 1.2). We claim that the kernel of $f$ is $I_3$. Since $\text{Ker}(f)$ is a $\lambda$-ideal, this proves that $I_3$ is a $\lambda$-ideal and that $\Lambda/I_3$ embeds into the polynomial ring $\mathbb{Z}[a_1, \ldots, a_m, b_1, \ldots, b_n]$. Since any partition $\lambda$ can be written as a union of rectangular partitions $\beta_i$, Lemma 3.7 implies that $I_3 = \cap I_{\beta_i}$ is also a $\lambda$-ideal.

By the Littlewood-Richardson rule 3.10, $f(s_{\pi}) = s_{\tau}(a + b) = \sum c_{\pi\nu}^\mu s_{\mu}(a)s_{\nu}(b)$, where $\mu$ and $\nu$ run over all partitions such that $\pi$ is obtained from $\mu$ by concatenating $\nu$ in a certain way. We may additionally restrict the sum to $\mu$ with at most $m$ rows and $\nu$ with $\nu_1 \leq n$, since otherwise $s_{\mu}(a) = 0$ or $s_{\nu}(b) = 0$. By Proposition 3.3, the $s_{\mu}(a)$ run over a basis of $\Lambda_m$ and the $s_{\nu}(b)$ run over a basis of $\Lambda_n$.

If $\pi$ contains $\beta$ then $f(s_{\pi}) = s_{\tau}(a + b) = 0$, because in every term of the above expansion, either the length of $\mu$ is $> m$ or else $\nu_1 > n$. Thus $I_3 \subseteq \text{Ker}(f)$.

For the converse, we use the reverse lexicographical ordering of partitions [Macd. p. 5]. For each $\pi$ not containing $\beta$, set $\mu_\pi = (\pi_1, \ldots, \pi_m)$; this is the maximal $\mu$ (for this ordering) such that $c_{\pi\nu}^\mu \neq 0$ (with $\nu_\pi = \pi - \mu_\pi$). Given $t = \sum_{\beta \subseteq \pi} d_\pi s_{\beta}$, choose $\mu$ maximal subject to $\mu = \mu_\pi$ for some $\pi$ with $d_\pi \neq 0$; choose $\mu$ maximal with $\mu = \mu_\pi$ and $d_\pi \neq 0$, and set $\nu = \nu_\pi$. Then the coefficient of $s_\mu(a)s_\nu(b)$ in $f(t)$ is $d_\pi \neq 0$. Thus $\text{Ker}(f) \subseteq I_3$. □

**Corollary 4.3.** $\Lambda_{(2,2)}$ is the subring $\mathbb{Z} + x\mathbb{Z}[a, b]$ of $\mathbb{Z}[a, b]$, where $x = a + b$.

**Proof.** By Proposition 4.2, $\Lambda_{(2,2)}$ is the subring of $\mathbb{Z}[a, b]$ generated $x = a + b$ and the $\lambda^n(x)$.

\[\lambda^{n+1}(x) = a\lambda^n(b) + \lambda^n(a) = ab^n + b^{n+1} = xb^n,\]

we have $\Lambda_{(2,2)} = \mathbb{Z}[x, xb, xb^2, \ldots, xb^n, \ldots] = \mathbb{Z} + x\mathbb{Z}[a, b]$. □

**Remark 4.3.1.** The ring $\Lambda_{(2,2)}$ was studied in [KKT 3.8], where it was shown that $\Lambda_{(2,2)}$ embeds into $\mathbb{Z}[x, y]$ sending $e_0$ to $xy^{n-1}$. This is the same as the embedding in Corollary 4.3, up to the change of coordinates $(x, y) = (a + b, b)$.

**Example 4.4.** Let $I$ be the ideal of $\Lambda_{(2,2)}$ generated by the $\lambda^2(i)$ ($i > 0$) and set $R = \Lambda_{(2,2)}/I$. Then $R$ is a $\lambda$-ring and $x$ is a non-nilpotent element such that $\lambda^2(x) = 0$ but $\lambda^{n+1}(x) \neq 0$. In particular, $\lambda^2(x) = 0$ yet $\lambda^3(x) \neq 0$.

To see this, we use the embedding of Corollary 4.3 to see that $I$ contains $x(\lambda^2x-1)$ and $(xb)(\lambda^2b-1)$ and hence the ideal $J$ of $\mathbb{Z}[a, b]$ generated by $x^2b$. In fact, $I$ is additively generated by $J$ and the $\{xb^2-1\}$. It follows that $R$ has basis $\{1, x^n, xb^{2n} | n \geq 1\}$. Since $\lambda^n(\lambda^2(x))$ is equivalent to $\lambda^{2n}(x) = xb^{2n-1}$ modulo $J$ by 1.6, it lies in $I$. Hence $I$ is a $\lambda$-ideal of $\Lambda_{(2,2)}$.

There is no $\lambda$-ring extension $R \subseteq R'$ in which $x = \ell_1 - \ell_2$ for line elements $\ell_i$, because we would have $\lambda^3(x) = \lambda^3(x + \ell_2) = 0$. On the other hand, there is a $\lambda$-ring extension $R \subseteq R'$ in which $x = \ell_1 + \ell_2 - \ell_3 - \ell_4$ for line elements $\ell_i$.

**Lemma 4.5.** If $x$ and $y$ are Schur-finite, so is $x + y$.

**Proof.** Given a partition $\lambda$, there is a partition $\pi_0$ such that whenever $\pi$ contains $\pi_0$, one of the partitions $\mu$ and $\nu$ appearing in the Littlewood-Richardson rule 3.10 must contain $\lambda$. If $x$ and $y$ are both killed by all Schur polynomials indexed by partitions containing $\lambda$, we must therefore have $s_{\tau}(x + y) = 0$. □

**Corollary 4.6.** Finite-dimensional elements are Schur-finite.

**Proof.** Proposition 5.3 shows that even and odd elements are Schur-finite. □

**Example 4.7.** If $R$ is a binomial ring containing $\mathbb{Q}$, then every Schur-finite element is finite-dimensional. This follows from Example 1.4 and [Macd. Ex. 1.3.4], which says that $s_{\tau}(r)$ is a rational number times a product of terms $r - c(x)$, where the $c(x)$ are integers.
Example 4.8. The universal element $x$ of the Schur-finite element $\Lambda_{(2,1)}$ is Schur-finite but not finite-dimensional. To see this, recall from Example 4.5 that $\Lambda_{(2,1)} \cong \mathbb{Z}[x,y]/(y^2 - x^3)$. Because $\Lambda_{(2,1)}$ is graded, if $x$ were finite-dimensional it would be the sum of an even and odd element in the degree 1 part $\{nx\}$ of $\Lambda_{(2,1)}$. If $n \in \mathbb{N}$, $nx$ cannot be even because the second coordinate of $\lambda^k(nx)$ is $\binom{n}{k} b^k$ by 4.2. And $nx$ cannot be odd, because the first coordinate of $\sigma^k(nx)$ is $(-1)^k \binom{n}{k} a^k$.

Lemma 4.9. Let $R \subset R'$ be an inclusion of $\lambda$-rings. If $x \in R$ then $x$ is Schur-finite in $R'$, if and only if $x$ is Schur-finite in $R$. In particular, if $x$ is finite-dimensional in $R'$, then $x$ is Schur-finite in $R$.

Proof. Since $s_\pi(x)$ may be computed in either $R$ or $R'$, the set of partitions $\pi$ for which $s_\pi(x) = 0$ is the same for $R$ and $R'$. The final assertion follows from Lemma 4.6.

Lemma 4.10. If $\pi$ is a partition of $n$, $s_{\pi'}(-x) = (-1)^n s_\pi(x)$.

Proof. Write $s_\pi$ as the homogeneous polynomial $f(e_1, e_2, ...)$ of degree $n$. Applying the antipode $S$ in $\Lambda$, we have $s_{\pi'} = f(h_1, h_2, ...)$. It follows that $s_{\pi'}(-x) = f(\sigma^1, \sigma^2, ...)(-x)$. Since $\sigma^3(-x) = (-1)^n \lambda^2(x)$, and $f$ is homogeneous, we have $s_{\pi'}(-x) = (-1)^n f(\lambda^1, \lambda^2, ...)(x) = s_\pi(x)$. □

Remark 4.10.1. If $a$ is a line element then $s_\pi(ax) = a^\pi s_\pi(x)$. From Lemma 4.9, we have $s_\pi(-ax) = (-a)^\pi s_\pi(x)$.

Theorem 4.11. The Schur-finite elements form a subring of any $\lambda$-ring, containing the subring of finite-dimensional elements.

Proof. The Schur-finite elements are closed under addition by Lemma 4.5. Since $\pi$ contains $\lambda$ just in case $\pi'$ contains $\lambda'$, Lemma 4.10 implies that $-x$ is Schur-finite whenever $x$ is. Hence the Schur-finite elements form a subgroup of $R$. It suffices to show that if $x$ and $y$ are Schur-finite in $R$, then $xy$ and all $\lambda^k(x)$ are Schur-finite.

Let $x$ be Schur-finite with rectangular bound $\mu$, so there is a map from the $\lambda$-ring $\Lambda_\mu$ to $R$ sending the generator $e$ to $x$. Embed $\Lambda_\mu$ in $R' = \mathbb{Z}[a_1, \ldots, b_1, \ldots]$ using Proposition 4.2. Since every element of $R'$ is finite-dimensional, $\lambda^k(e)$ is finite-dimensional in $R'$, and hence Schur-finite in $\Lambda_\mu$ by Lemma 4.9. It follows that the image $\lambda^k(x)$ of $\lambda^k(e)$ in $R$ is also Schur-finite.

Let $x$ and $y$ be Schur-finite with rectangular bounds $\mu$ and $\nu$, and let $\Lambda_\mu \to R$ and $\Lambda_\nu \to R$ be the $\lambda$-ring maps sending the generators $e_\mu$ and $e_\nu$ to $x$ and $y$. Since the induced map $\Lambda_\mu \otimes \Lambda_\nu \to R$ sends $e_\mu \otimes e_\nu$ to $xy$, we only need to show that $e_\mu \otimes e_\nu$ is Schur-finite. But $\Lambda_\mu \otimes \Lambda_\nu \subset \mathbb{Z}[a_1, \ldots, b_1, \ldots] \otimes \mathbb{Z}[a_1, \ldots, b_1, \ldots]$, and in the larger ring every element is finite-dimensional, including the tensor product. By Lemma 4.9, $e_\mu \otimes e_\nu$ is Schur-finite in $\Lambda_\mu \otimes \Lambda_\nu$.

Conjecture 4.12 (Virtual Splitting principle). Let $x$ be a Schur-finite element of a $\lambda$-ring $R$. Then $R$ is contained in a larger $\lambda$-ring $R'$ such that $x$ is finite-dimensional in $R'$, i.e., there are line elements $\ell_i, \ell'_j$ in $R'$ so that

$$x = \left( \sum \ell_i \right) - \left( \sum \ell'_j \right).$$

Example 4.13. The virtual splitting principle holds in the universal case, where $R_0 = \Lambda_\beta$. Indeed, we know that $x = \sum a_i + \sum b_j$ in $R'_0 = \mathbb{Z}[a_1, \ldots, b_1, \ldots]$. Since $\ell_j = -b_j$ is a line element, $x$ is a difference of sums of line elements in $R'_0$.

Unfortunately, although the induced map $f : R \to R \otimes_{R_0} R'_0$ sends a Schur-finite element $x$ to a difference of sums of line elements, the map $f$ need not be an injection.
Proposition 4.14. If a $\lambda$-ring $R$ is a domain, $R$ is contained in a $\lambda$-ring $R'$ such that every Schur-finite element of $R$ is a a difference of sums of line elements in $R'$.

Proof. Let $E$ denote the algebraic closure of the fraction field of $R$ and set $R' = W'(E)$; $R$ is contained in $R'$ by $R \overset{\lambda}{\to} W(R) \subset W(E)$. If $x \in R$ is Schur-finite then $\lambda_t(x)$ is determinantly rational in $E[[t]]$ and hence a rational function $p/q$ in $E(t)$ (see 24). Factoring $p$ and $q$ in $E[t]$, we have

$$\lambda_t(x) = \prod(1 - \alpha_i t)/\prod(1 - \beta_j t)$$

for suitable elements $\alpha_i, \beta_j$ of $E$. Since the underlying abelian group of $W(E)$ is $(1 + tE[[t]])$, $x$ and the $\ell_i = (1 - \alpha_i t)$ and $\ell'_j = (1 - \beta_j t)$ are line elements in $W(E)$, we are done.

The proof shows that a bound $\pi$ on $x$ determines a bound on the degrees of $p(t)$ and $q(t)$ and hence on the number of line elements $\ell_i$ and $\ell'_j$ in the virtual sum.

Corollary 4.15. The virtual splitting principle holds for reduced $\lambda$-rings.

Proof. Let $R$ be a reduced ring. If $P$ is a minimal prime of $R$ then the localization $R_P$ is a domain and $R$ embeds into the product $\prod E_P$ of the algebraic closures of the fields of fractions of the $R_P$. If in addition $R$ is a $\lambda$-ring then $R$ embeds into the $\lambda$-ring $R' = \prod W(E_P)$. If $x$ is Schur-finite in $R$ with bound $\pi$ then $\lambda_t(R)$ is determinantly rational and each factor of $\lambda_t(x)$ is a rational function in $E_P(t)$; the bound $\pi$ determines a bound $N$ on the degrees of the numerator and denominator in each component. By Theorem 4.14 there are line elements $\ell_1, \ldots, \ell_N, \ell'_1, \ldots, \ell'_N$ in each component so that $x = (\sum \ell_i) - (\sum \ell'_j)$ in $R'$.

As more partial evidence for Conjecture 4.12 we show that the virtual splitting principle holds for elements bounded by the hook $(2,1)$.

Theorem 4.16. Let $x$ be a Schur-finite element in a $\lambda$-ring $R$. If $x$ has bound $(2,1)$, then $R$ is contained in a $\lambda$-ring $R'$ in which $x$ is a virtual sum $\ell_1 + \ell_2 - a$ of line elements.

Proof. The polynomial ring $R[a]$ becomes a $\lambda$-ring once we declare $a$ to be a line element. Set $y = x + a$, and let $I$ be the ideal of $R[a]$ generated by $\lambda^3(y)$.

For all $n \geq 2$, the equation $s_{n,1}(x) = 0$ yields $\lambda^{n+1}(x) = x\lambda^{n}(x) = x^{n-1}\lambda^2(x)$ in $R$, and therefore $\lambda^{n+1}(y) = (a + x)x^{n-1}\lambda^2(x) = x^{n-2}\lambda^3(y)$. It follows from Scholium 4.4 that $\lambda^m(\lambda^3y) \in I$ for all $m \geq 1$ and hence that

$$\lambda^n(f \cdot \lambda^3y) = P_n(\lambda^3(f); 2, \lambda^3y, \ldots, \lambda^n(\lambda^3y))$$

is in $I$ for all $f \in R[a]$. Thus $I$ is a $\lambda$-ideal of $R[a]$, $A \overset{\lambda}{\to} R[a]/I$ is a $\lambda$-ring, and the image of $y$ in $A$ is even of degree 2. By the Splitting Principle 4.4 the image of $x = y - a$ is a virtual sum $\ell_1 + \ell_2 - a$ of line elements in some $\lambda$-ring containing $A$.

To conclude, it suffices to show that $R$ injects into $A = R[a]/I$. If $r \in R$ vanishes in $A$ then $r = f\lambda^3(y)$ for some $f = f(a)$ in $R[a]$. We may take $f$ to have minimal degree $d \geq 0$. Writing $f(a) = ca^d + g(a)$, with $c \in R$ and $\deg(g) < d$, the coefficient of $a^{d+1}$ in $f\lambda^3(y)$, namely $c\lambda^2(x)$, must be zero. But then $c\lambda^3y = 0$, and $r = g\lambda^3y$, contradicting the minimality of $f$.

Remark 4.17. The rank of a Schur-finite object with bound $\pi$ cannot be well defined unless $\pi$ is a rectangular partition. This is because any rectangular partition $\mu = (m + 1)^{n+1}$ contained in $\pi$ yields a map $R \to R'$ sending $x$ to an element of rank $m-n$. If $\pi$ is not rectangular there are different maximal rectangular sub-partitions with different values of $m - n$. 
For example, let $x$ be the element of Theorem 4.10. By Lemma 4.10, $-x$ also has bound $(2,1)$. Applying Theorem 4.10 to $-x$ shows that $R$ is also contained in a $\lambda$-ring $R''$ in which $x$ is a virtual sum $a - \ell_1 - \ell_2$ of line bundles. Therefore $x$ has rank 1 in $R'$, and has rank $-1$ in $R''$.

5. Rationality of $\lambda_t(x)$

Let $R$ be a $\lambda$-ring and $x \in R$. One central question is to determine when the power series $\lambda_t(x)$ is a rational function. (See [A05], [LL04], [HI], [Gui], [BI] [B2], [KKT] for example.) Following [LL04, 2.1], we make this rigorous by restricting to power series in $R[[t]]$ congruent to 1 modulo $t$ and define a (globally) rational function to be a power series $f(t)$ such that there exist polynomials $p, q \in R[t]$ with $p(0) = q(0) = 1$ such that $p(t) = f(t)q(t)$.

As noted in [2.7], it is well known that if $x$ is a finite-dimensional element then $\lambda_t(x)$ is a rational function. Larsen and Lunts observed in [LL04] that the property of being a rational function is not preserved by passing to subrings and proposed replacing ‘rational function’ by ‘determinantally rational function’ (see [2.7]). We propose an even weaker condition, which we now define.

Given a power series $f(t) = \sum r_n t^n \in R[[t]]$ and a partition $\pi$, we form the the Jacobi-Trudi matrix $(a_{ij})$ with $a_{i,j} = r_{\pi'_{i+j-1}}$ and define $s_\pi(f) \in R$ to be its determinant. (If $\pi$ has $m$ columns, $\pi'$ has $n$ rows and $(a_{ij})$ is an $m \times n$ matrix over $R$.) The terminology comes from the fact that the commutative ring homomorphism $\rho : \Lambda \rightarrow R$, defined by $\rho(x_n) = r_n$, satisfies $\rho(s_\pi) = \det(a_{ij})$ by the Jacobi-Trudi identities.

**Definition 5.1.** Let $R$ be a commutative ring. We say that a power series $f(t) = \sum r_n t^n \in R[[t]]$ is Schur-rational over $R$ if there exists a partition $\mu$ such that $s_\mu(f) = 0$ for every partition $\pi$ containing $\mu$.

If $\mu$ is a rectangular partition then $(a_{i,j})$ is the matrix $(r_{\pi'_{i+j-1}})$ in Definition 2.7 up to row permutation. It follows that if $f(t)$ is Schur-rational then it is determinantally rational. The converse fails, as we show in Example 5.2.

It is easy to see that a (globally) rational function is Schur-rational. Thus being Schur-rational is a property of $f$ intermediate between being rational and being determinantally rational.

**Example 5.2.** Let $R_m$ be the quotient of $\Lambda$ by the ideal generated by all $m$-fold products $x_{i_1} \cdots x_{i_m}$ where $|i_j - i_k| < 2m$ for all $j, k$. Then $f(t) = \sum x_n t^n$ is determinantally rational. On the other hand, $f(t)$ is not Schur-rational because for each $\lambda$ with $l$ rows there are lacunary partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$ (meaning that $\pi_1 \gg \pi_2 \gg \cdots \gg \pi_l \gg 0$) containing $\lambda$ which are nonzero in $R_m$, because $s_{\pi}(f)$ is an alternating sum of monomials and the diagonal monomial $\prod r_{\pi_i}$ is nonzero and occurs exactly once.

The notion of Schur-rationality is connected to Schur-finiteness.

**Proposition 5.3.** An element $x$ in a $\lambda$-ring is Schur-finite if and only if the power series $\lambda_t(x)$ is Schur-rational.

In particular, if $x$ is Schur-finite then $\lambda_t(x)$ is determinantally rational.

The “if” part of this proposition was proven in [KKT, 3.10] for $\lambda$-rings of the form $K_0(A)$, using categorical methods.

**Proof.** By definition, the power series $\lambda_t(x)$ is Schur-rational if and only if there is a partition $\mu$ so that for every $\pi$ containing $\mu$, the determinant $\det(\lambda^{\pi'_{i+j-1}}(x))$ is zero. Since this determinant is $s_\pi(x)$ by the Jacobi-Trudi identity, this is equivalent to $x$ being Schur-finite (definition 4.1). □
We conclude by connecting our notion of Schur-finiteness to the notion of a Schur-finite object in a $\mathbb{Q}$-linear tensor category $\mathcal{A}$, given in [Mz]. By definition, an object $A$ is Schur-finite if some $S_{\lambda}(A) \cong 0$ in $\mathcal{A}$. By [Mz 1.4], this implies that $S_{\pi}(A) = 0$ for all $\pi$ containing $\lambda$. It is evident that if $A$ is a Schur-finite object of $\mathcal{A}$ then $[A]$ is a Schur-finite element of $K_0(\mathcal{A})$. However, the converse need not hold. For example, if $\mathcal{A}$ contains infinite direct sums then $K_0(\mathcal{A}) = 0$ by the Eilenberg swindle, so $[A]$ is always Schur-finite.

Here are two examples of Schur-finite objects whose class in $K_0(\mathcal{A})$ is finite-dimensional even though they are not finite-dimensional objects.

**Example 5.4.** Let $\mathcal{A}$ denote the abelian category of positively graded modules over the graded ring $A = \mathbb{Q}[e]/(e^2 = 0)$. It is well known that $\mathcal{A}$ is a tensor category under $\otimes_{\mathbb{Q}}$, with the $\lambda$-ring $K_0(\mathcal{A}) \cong \Lambda_{-1} = \mathbb{Z}[b]; 1 = [\mathbb{Q}]$ and $b = [\mathbb{Q}[1]]$. The graded object $A$ is Schur-finite but not finite-dimensional in $\mathcal{A}$ by [Mz 1.12]. However, $[A]$ is a finite-dimensional element in $K_0(\mathcal{A})$ because $[A] = [\mathbb{Q}] + [\mathbb{Q}[1]]$.

**Example 5.5 (O’Sullivan).** Let $X$ be a Kummer surface; then there is an open subvariety $U$ of $X$, whose complement $Z$ is a finite set of points, such that $M(U)$ is Schur-finite but not finite-dimensional in the Kimura-O’Sullivan sense in the category $\mathcal{M}$ of motives [Mz 3.3]. However, it follows from the distinguished triangle

$$M(Z)(2)[3] \rightarrow M(U) \rightarrow M(X) \rightarrow M(Z)(2)[4]$$

that $[M(U)] = [M(Z)(2)[3]] + [M(X)]$ in $K_0(DM_{gm})$ and hence in $K_0(\mathcal{M})$. Since both $M(X)$ and $M(Z)(2)[3]$ are finite-dimensional, $[M(U)]$ is a finite-dimensional element of $K_0(\mathcal{M})$.

**Proposition 5.6.** Let $M$ be a classical motive. If $M$ is Schur-finite in $\mathcal{M}$, then $\lambda_t([M])$ is determinantly rational. If $\lambda_t([M])$ is determinantly rational, then there exists a partition $\lambda$ such that $S_\lambda(M)$ is a phantom motive.

**Proof.** If $M$ is Schur-finite, then there is a $\lambda$ such that $0 = [S_\pi M] = s_\pi([M])$ for all $\pi \supset \lambda$. Thus $[M]$ is Schur-finite in $K_0(\mathcal{M})$ or equivalently, by [Mz 5.3] $\lambda_t([M])$ is determinantly rational. If $\lambda_t([M])$ is Schur-finite, with bound $\lambda$, then $0 = s_\lambda([M]) = [S_\lambda M]$ in $K_0(DM_{eff})$. By Proposition 2.3, $S_\lambda(M)$ is a phantom motive.

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