ON BIPARTITE DISTANCE-REGULAR GRAPHS
WITH EXACTLY TWO IRREDUCIBLE
T-MODULES WITH ENDPOINT TWO

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Abstract

Let \( \Gamma \) denote a bipartite distance-regular graph with diameter \( D \geq 4 \) and valency \( k \geq 3 \). Let \( X \) denote the vertex set of \( \Gamma \), and let \( A \) denote the adjacency matrix of \( \Gamma \). For \( x \in X \) let \( T = T(x) \) denote the subalgebra of \( \text{Mat}_X(\mathbb{C}) \) generated by \( A, E_0^*, E_1^*, \ldots, E_D^* \), where for \( 0 \leq i \leq D \), \( E_i^* \) represents the projection onto the \( i \)th subconstituent of \( \Gamma \) with respect to \( x \). We refer to \( T \) as the Terwilliger algebra of \( \Gamma \) with respect to \( x \). An irreducible \( T \)-module \( W \) is said to be thin whenever \( \dim E^*_i W \leq 1 \) for \( 0 \leq i \leq D \). By the endpoint of \( W \) we mean \( \min \{ i \mid E^*_i W \neq 0 \} \). For \( 0 \leq i \leq D \), let \( \Gamma_i(z) \) denote the set of vertices in \( X \) that are distance \( i \) from vertex \( z \). Define a parameter \( \Delta_2 \) in terms of the intersection numbers by \( \Delta_2 = (k - 2)(c_3 - 1) - (c_2 - 1)\mu^2_{22} \). In this paper we prove the following are equivalent: (i) \( \Delta_2 > 0 \) and for \( 2 \leq i \leq D - 2 \) there exist complex scalars \( \alpha_i, \beta_i \) with the following property: for all \( x, y, z \in X \) such that \( \partial(x, y) = 2 \), \( \partial(x, z) = i \), \( \partial(y, z) = i \) we have \( \alpha_i + \beta_i |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)| \); (ii) For all \( x \in X \) there exist up to isomorphism exactly two irreducible modules for the Terwilliger algebra \( T(x) \) with endpoint two, and these modules are thin.

1 Introduction

In this paper we obtain a combinatorial characterization of bipartite distance-regular graphs with exactly two irreducible modules of the Terwilliger algebra of endpoint 2, both

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of which are thin (see Section 2 for formal definitions). Our combinatorial characterization is closely related to the 2-homogeneous property of Curtin [3] and Nomura [14].

Throughout this introduction let \( \Gamma \) denote a bipartite distance-regular graph with diameter \( D \geq 4 \) and valency \( k \geq 3 \). Let \( X \) denote the vertex set of \( \Gamma \). For \( x \in X \), let \( T = T(x) \) denote the Terwilliger algebra of \( \Gamma \) with respect to \( x \). It is known that there exists a unique irreducible \( T \)-module with endpoint 0, and this module is thin [8, Proposition 8.4]. Moreover, Curtin showed that up to isomorphism \( \Gamma \) has exactly one irreducible \( T \)-module with endpoint 1, and this module is thin [11, Corollary 7.7].

We now discuss the irreducible \( T \)-modules of endpoint 2. For \( 0 \leq i \leq D \), let \( \Gamma_i(z) \) denote the set of vertices in \( X \) that are distance \( i \) from vertex \( z \). In [7, Theorem 3.11], Curtin proved that the following are equivalent: (i) For all \( i \) \((2 \leq i \leq D - 2)\) and for all \( x, y, z \in X \) with \( \partial(x, y) = 2, \partial(x, z) = i, \partial(y, z) = i \), the number \( |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z)| \) is independent of \( x, y, z \); (ii) For all \( x \in X \) there exists a unique irreducible \( T \)-module for the Terwilliger algebra \( T(x) \) with endpoint 2, and this module is thin. When these equivalent conditions hold, \( \Gamma \) is said to be almost 2-homogeneous.

Now define a parameter \( \Delta_2 \) in terms of the intersection numbers by \( \Delta_2 = (k-2)(c_3-1) - (c_2-1)p_2^2 \). In this paper we prove the following are equivalent: (i) \( \Delta_2 > 0 \) and for \( 2 \leq i \leq D - 2 \) there exist complex scalars \( \alpha_i, \beta_i \) with the following property: for all \( x, y, z \in X \) such that \( \partial(x, y) = 2, \partial(x, z) = i, \partial(y, z) = i \) we have \( \alpha_i + \beta_i \Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_{i-1}(z) = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)| \); (ii) For all \( x \in X \) there exist up to isomorphism exactly two irreducible modules for the Terwilliger algebra \( T(x) \) with endpoint 2, and these modules are thin. We also compute \( \alpha_i, \beta_i \) in terms of the intersection numbers of \( \Gamma \).

We remark that this paper is part of a continuing effort to understand and classify the bipartite distance-regular graphs with at most two irreducible modules of the Terwilliger algebra with endpoint 2, both of which are thin. Please see [5]–[7], [10]–[12] for more work from this ongoing project.

## 2 Preliminaries

In this section we review some definitions and basic results concerning distance-regular graphs. See the book of Brouwer, Cohen and Neumaier [2] for more background information.

Let \( \mathbb{C} \) denote the complex number field and let \( X \) denote a nonempty finite set. Let \( \text{Mat}_X(\mathbb{C}) \) denote the \( \mathbb{C} \)-algebra consisting of all matrices whose rows and columns are indexed by \( X \) and whose entries are in \( \mathbb{C} \). Let \( V = \mathbb{C}^X \) denote the vector space over \( \mathbb{C} \) consisting of column vectors whose coordinates are indexed by \( X \) and whose entries are in \( \mathbb{C} \). We observe \( \text{Mat}_X(\mathbb{C}) \) acts on \( V \) by left multiplication. We call \( V \) the standard module. We endow \( V \) with the Hermitian inner product \( \langle , \rangle \) that satisfies \( \langle u, v \rangle = u^t \overline{v} \) for \( u, v \in V \), where \( t \) denotes transpose and \( \overline{\cdot} \) denotes complex conjugation. For \( y \in X \) let \( y \) denote the element of \( V \) with a 1 in the \( y \) coordinate and 0 in all other coordinates. We observe \( \{ y \mid y \in X \} \) is an orthonormal basis for \( V \).

Let \( \Gamma = (X, R) \) denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set \( X \) and edge set \( R \). Let \( \partial \) denote the path-length distance function for \( \Gamma \), and set \( D := \max\{ \partial(x, y) \mid x, y \in X \} \). We call \( D \) the diameter of \( \Gamma \). For a
vertex $x \in X$ and an integer $i$ let $\Gamma_i(x)$ denote the set of vertices at distance $i$ from $x$. We abbreviate $\Gamma(x) = \Gamma_0(x)$. For an integer $k \geq 0$ we say $\Gamma$ is regular with valency $k$ whenever $|\Gamma(x)| = k$ for all $x \in X$. We say $\Gamma$ is distance-regular whenever for all integers $h, i, j$ $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of $x$ and $y$. The $p^h_{ij}$ are called the intersection numbers of $\Gamma$.

For the rest of this paper we assume $\Gamma$ is distance-regular with diameter $D \geq 4$. Note that $p^h_{ij} = p^h_{ji}$ for $0 \leq h, i, j \leq D$. For convenience set $c_i := p^i_{i,i-1}$ $(1 \leq i \leq D)$, $a_i := p^i_{i,i}$ $(0 \leq i \leq D)$, $b_i := p^i_{i,i+1}$ $(0 \leq i \leq D - 1)$, $k_i := p^0_{i,i}$ $(0 \leq i \leq D)$, and $c_0 = b_D = 0$. By the triangle inequality the following hold for $0 \leq h, i, j \leq D$: (i) $p^0_{ij} = 0$ if one of $h, i, j$ is greater than the sum of the other two; (ii) $p^h_{ij} \neq 0$ if one of $h, i, j$ equals the sum of the other two. In particular $c_i \neq 0$ for $1 \leq i \leq D$ and $b_i \neq 0$ for $0 \leq i \leq D - 1$. We observe that $\Gamma$ is regular with valency $k = k_1 = b_0$ and that

$$c_i + a_i + b_i = k \quad (0 \leq i \leq D).$$

Note that $k_i = |\Gamma_i(x)|$ for $x \in X$ and $0 \leq i \leq D$. By [2] p. 127,

$$k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq D).$$

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq D$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $(x, y)$-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

For notational convenience, we define $A_{D+1}$ to be the zero matrix. We call $A_i$ the $i$th distance matrix of $\Gamma$. We abbreviate $A := A_1$ and call this the adjacency matrix of $\Gamma$. We observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^D A_i = J$; (aiii) $\overline{A}_i = A_i$ $(0 \leq i \leq D)$; (av) $A_i = A_i$ $(0 \leq i \leq D)$; (av) $A_i A_j = \sum_{h=0}^D p^h_{ij} A_h$ $(0 \leq i, j \leq D)$, where $I$ (resp. $J$) denotes the identity matrix (resp. all 1’s matrix) in $\text{Mat}_X(\mathbb{C})$. Using these facts we find $A_0, A_1, \ldots, A_D$ is a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{C})$. We call $M$ the Bose-Mesner algebra of $\Gamma$. It turns out that $A$ generates $M$ [1] p. 190).

We now recall the dual idempotents of $\Gamma$. To do this fix a vertex $x \in X$. We view $x$ as a “base vertex.” For $0 \leq i \leq D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y, y)$-entry

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X).$$

We call $E^*_i$ the $i$th dual idempotent of $\Gamma$ with respect to $x$ [13] p. 378). We observe (ei) $\sum_{i=0}^D E^*_i = I$; (eii) $E^*_i = E^*_i$ $(0 \leq i \leq D)$; (eiii) $E^*_i = E^*_i$ $(0 \leq i \leq D)$; (eiv) $E^*_i E^*_j = \delta_{ij} E^*_i$ $(0 \leq i, j \leq D)$. By these facts $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for a commutative subalgebra
\(M^* = M^*(x)\) of \(\text{Mat}_X(\mathbb{C})\). We call \(M^*\) the dual Bose-Mesner algebra of \(\Gamma\) with respect to \(x\) [15, p. 378]. For \(0 \leq i \leq D\) we have

\[
E_i^* V = \text{Span}\{\hat{y} \mid y \in X, \partial(x, y) = i\},
\]

so \(\dim E_i^* V = k_i\). We call \(E_i^* V\) the \(i\)th subconstituent of \(\Gamma\) with respect to \(x\). Note that

\[
V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum)}. \quad (4)
\]

Moreover \(E_i^*\) is the projection from \(V\) onto \(E_i^* V\) for \(0 \leq i \leq D\).

We recall the Terwilliger algebra of \(\Gamma\). Let \(T = T(x)\) denote the subalgebra of \(\text{Mat}_X(\mathbb{C})\) generated by \(M, M^*\). We call \(T\) the Terwilliger algebra of \(\Gamma\) with respect to \(x\) [15, Definition 3.3]. Recall \(M\) is generated by \(A\) so \(T\) is generated by \(A\) and the dual idempotents. We observe \(T\) has finite dimension. By construction \(T\) is closed under the conjugate-transpose map so \(T\) is semisimple [15, Lemma 3.4(i)].

By a \(T\)-module we mean a subspace \(W\) of \(V\) such that \(BW \subseteq W\) for all \(B \in T\). Let \(W\) denote a \(T\)-module. Then \(W\) is said to be irreducible whenever \(W\) is nonzero and \(W\) contains no \(T\)-modules other than 0 and \(W\).

By [9, Corollary 6.2] any \(T\)-module is an orthogonal direct sum of irreducible \(T\)-modules. In particular the standard module \(V\) is an orthogonal direct sum of irreducible \(T\)-modules. Let \(W, W'\) denote \(T\)-modules. By an isomorphism of \(T\)-modules from \(W\) to \(W'\) we mean an isomorphism of vector spaces \(\sigma : W \rightarrow W'\) such that \((\sigma B - B \sigma)W = 0\) for all \(B \in T\). The \(T\)-modules \(W, W'\) are said to be isomorphic whenever there exists an isomorphism of \(T\)-modules from \(W\) to \(W'\). By [4, Lemma 3.3] any two nonisomorphic irreducible \(T\)-modules are orthogonal. Let \(W\) denote an irreducible \(T\)-module. By [15, Lemma 3.4(iii)] \(W\) is an orthogonal direct sum of the nonvanishing spaces among \(E_0^* W, E_1^* W, \ldots, E_D^* W\). By the endpoint of \(W\) we mean \(\min\{i \mid 0 \leq i \leq D, E_i^* W \neq 0\}\).

By the diameter of \(W\) we mean \(\min\{i \mid 0 \leq i \leq D, E_i^* W \neq 0\}\) - 1. We say \(W\) is thin whenever the dimension of \(E_i^* W\) is at most 1 for \(0 \leq i \leq D\).

Fix a decomposition of the standard module \(V\) into an orthogonal direct sum of irreducible \(T\)-modules. For any irreducible \(T\)-module \(W\), the multiplicity of \(W\) is the number of irreducible modules in this decomposition which are isomorphic to \(W\). It is well-known that the multiplicity is independent of the decomposition of \(V\).

By [8, Proposition 8.4] \(\Gamma\) has a unique irreducible \(T\)-module with endpoint 0. We denote this \(T\)-module by \(V_0\). We call \(V_0\) the primary module. It appears in \(V\) with multiplicity 1 and it has basis \(\{s_i \mid 0 \leq i \leq D\}\), where

\[
s_i = \sum_{y \in \Gamma_i(x)} \hat{y}. \quad (5)
\]

Recall \(\Gamma\) is bipartite whenever \(a_i = 0\) for \(0 \leq i \leq D\). For the rest of this paper we assume \(\Gamma\) is bipartite. In order to avoid trivialities we assume the valency \(k \geq 3\). We now recall a basic formula. Setting \(a_i = 0\) in (11) we find

\[
b_i + c_i = k \quad (0 \leq i \leq D). \quad (6)
\]

In the rest of the paper we will consider the following situation.
**Notation 2.1** Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, intersection numbers $b_i, c_i$, and distance matrices $A_i$ $(0 \leq i \leq D)$. Let $V$ denote the standard module for $\Gamma$. We fix $x \in X$ and let $E_i^{\ast} = E_i^{\ast}(x)$ $(0 \leq i \leq D)$ and $T = T(x)$ denote the dual idempotents and the Terwilliger algebra of $\Gamma$ with respect to $x$, respectively. Let $V_0$ denote the irreducible $T$-module with endpoint 0, and let $V_1$ denote the subspace of $V$ spanned by the irreducible $T$-modules with endpoint 1. We define the set $U$ to be the orthogonal complement of $E_2^{\ast}V_0 + E_2^{\ast}V_1$ in $E_2^{\ast}V$.

With reference to Notation 2.1 we now define a certain partition of $X$ that we will find useful.

**Definition 2.2** With reference to Notation 2.1 fix vertex $y \in X$ such that $\partial(x, y) = 2$. For all integers $i, j$ define $D_j = D_j(x, y)$ by

$$D_j = \{ z \in X \mid \partial(x, z) = i \mbox{ and } \partial(y, z) = j \}.$$ 

We observe $D_j = \emptyset$ unless $0 \leq i, j \leq D$ and $i + j$ is even.

### 3 Local eigenvalues

Later in the paper we will consider the irreducible $T$-modules with endpoint 2. In order to discuss these we introduce some parameters we call local eigenvalues.

**Definition 3.1** With reference to Notation 2.1 let $\Gamma_2 = \Gamma_2^{\ast}(x)$ denote the graph with vertex set $X = \Gamma_2(x)$ and edge set $R = \{ yz \mid y, z \in X, \partial(y, z) = 2 \}$. The graph $\Gamma_2$ has exactly $k_2$ vertices and it is regular with valency $p_{22}$. Let $\tilde{A}$ denote the adjacency matrix of $\Gamma_2$. The matrix $\tilde{A}$ is symmetric with real entries. Therefore $\tilde{A}$ is diagonalizable with all eigenvalues real. Let $\eta_1, \eta_2, \ldots, \eta_{k_2}$ denote the eigenvalues of $\tilde{A}$. We call $\eta_1, \eta_2, \ldots, \eta_{k_2}$ the local eigenvalues of $\Gamma$ with respect to $x$.

With reference to Notation 2.1 consider the second subconstituent $E_2^{\ast}V$. We recall the dimension of $E_2^{\ast}V$ is $k_2$ and observe $E_2^{\ast}V$ is invariant under multiplication by $E_2^{\ast}A_2E_2^{\ast}$. Note that for an appropriate ordering of the vertices of $\Gamma$ we have

$$E_2^{\ast}A_2E_2^{\ast} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{A}$ is as in Definition 3.1. Apparently the action of $E_2^{\ast}A_2E_2^{\ast}$ on $E_2^{\ast}V$ is essentially the adjacency map for $\Gamma_2$. In particular, the action of $E_2^{\ast}A_2E_2^{\ast}$ on $E_2^{\ast}V$ is diagonalizable with eigenvalues $\eta_1, \eta_2, \ldots, \eta_{k_2}$. Note that the vector $s_2$ from (5) is in $E_2^{\ast}V$ and it is an eigenvector for $E_2^{\ast}A_2E_2^{\ast}$ with eigenvalue $p_{22}$. By [1] Corollary 7.7 there exists, up to isomorphism, a unique irreducible $T$-module with endpoint 1. It has diameter $D - 2$ and it appears in $V$ with multiplicity $k - 1$. Let $W_1$ denote an irreducible $T$-module with endpoint 1 and choose nonzero $v \in E_2^{\ast}W_1$. By [10] Corollary 9.1, $v$ is an eigenvector for $E_2^{\ast}A_2E_2^{\ast}$ with eigenvalue $b_3 - 1$. Reordering the local eigenvalues if necessary, we have $\eta_i = p_{22}$ and $\eta_i = b_3 - 1$ ($2 \leq i \leq k$). We define the set $\Phi_2 := \{ \eta_i \mid k + 1 \leq i \leq k_2 \}$. Note that the set $\Phi_2$ is the same as the set $\Phi_2$ from [6] Definition 4.8].
Lemma 3.2 With reference to Notation 2.1, the local eigenvalues \( \eta \in \Phi_2 \) are exactly the eigenvalues of \( E_2^* A_2 E_2^* \) on \( U \).

PROOF. By the definition of \( U \), we find that \( U \) is invariant under the action of \( E_2^* A_2 E_2^* \). Apparently the restriction of \( E_2^* A_2 E_2^* \) to \( U \) is diagonalizable with eigenvalues \( \eta_i \) \((k + 1 \leq i \leq k_2)\). The result follows. \( \blacksquare \)

We now define certain scalars \( \Delta_i \) \((2 \leq i \leq D - 1)\). To do this recall that, by \([2, \text{Lemma 4.1.7}]\), we have

\[
p_{2i}^i = c_i (b_{i-1} - 1) + b_i (c_{i+1} - 1)
\]

for \( 1 \leq i \leq D - 1 \).

Definition 3.3 With reference to Notation 2.1, for \( 2 \leq i \leq D - 1 \) we define

\[
\Delta_i = (b_{i-1} - 1) (c_{i+1} - 1) - (c_2 - 2) p_{2i}^i.
\]

Lemma 3.4 (\([3, \text{Theorem 12}], [6, \text{Corollary 4.13}]\)) With reference to Notation 2.1 and Definition 3.3, we have \( \Delta_i \geq 0 \) for \( 2 \leq i \leq D - 1 \). Moreover \( \Delta_2 = 0 \) if and only if \(|\Phi_2| \leq 1\).

Let \( W \) denote a thin irreducible \( T \)-module with endpoint 2. Then \( E_2^* W \) is a one-dimensional eigenspace for \( E_2^* A_2 E_2^* \); we call the corresponding eigenvalue the local eigenvalue of \( W \). Note this local eigenvalue of \( W \) is contained in the set \( \{\eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2}\} \).

We will need the following result.

Lemma 3.5 (\([5, \text{Lemma 10.10 and Theorem 11.7}]\)) With reference to Notation 2.1, let \( W, W' \) denote thin irreducible \( T \)-modules with endpoint 2. Then \( W \) and \( W' \) are isomorphic as \( T \)-modules if and only if they have the same local eigenvalue.

4 Lowering and raising matrices

With reference to Notation 2.1, in this section we recall the lowering matrix and the raising matrix of the algebra \( T \).

Definition 4.1 With reference to Notation 2.1 we define matrices \( L = L(x) \), \( R = R(x) \) by

\[
L = \sum_{h=1}^{D} E_{h-1}^* A E_h^*, \quad R = \sum_{h=0}^{D-1} E_{h+1}^* A E_h^*.
\]

Note that \( A = L + R \) \([4, \text{Lemma 4.4}]\). We call \( L \) and \( R \) the lowering matrix and the raising matrix of \( T \) with respect to \( x \), respectively.

Lemma 4.2 With reference to Notation 2.1 let \( y, z \in X \). Then the following (i), (ii) hold.
With reference to Notation 2.1, choose an integer

Lemma 4.4

With reference to Notation 2.1 let

\[ L_{zy} = \begin{cases} 1 & \text{if } \partial(z, y) = 1 \text{ and } \partial(x, z) = \partial(x, y) - 1, \text{ and } 0 \text{ otherwise.} \\ R_{zy} = \begin{cases} 1 & \text{if } \partial(z, y) = 1 \text{ and } \partial(x, z) = \partial(x, y) + 1, \text{ and } 0 \text{ otherwise.} \end{cases} \]

PROOF. Immediate from Definition [4.1] and elementary matrix multiplication.

With reference to Notation 2.1 let \( y, z \in X \). We display the \((z, y)\)-entry of certain products of the matrices \( L \) and \( R \). To do this we need another definition.

A sequence of vertices \([y_0, y_1, \ldots, y_t]\) of \( \Gamma \) is a walk in \( \Gamma \) if \( y_{i-1}y_i \) is an edge for \( 1 \leq i \leq t \).

Lemma 4.3 With reference to Notation 2.1 choose \( y, z \in X \) and let \( m \) denote a positive integer. Assume that \( y \in \Gamma_i(x) \). Then the following (i)-(iii) hold.

(i) The \((z, y)\)-entry of \( R^m \) is equal to the number of walks \([y = y_0, y_1, \ldots, y_m = z]\), such that \( y_j \in \Gamma_{i+j}(x) \) for \( 0 \leq j \leq m \).

(ii) The \((z, y)\)-entry of \( R^mL \) is equal to the number of walks \([y = y_0, y_1, \ldots, y_m+1 = z]\), such that \( y_i \in \Gamma_{i-1}(x) \) and \( y_j \in \Gamma_{i-2+j}(x) \) for \( 2 \leq j \leq m + 1 \).

(iii) The \((z, y)\)-entry of \( LR^m \) is equal to the number of walks \([y = y_0, y_1, \ldots, y_{m+1} = z]\), such that \( y_j \in \Gamma_{i+j}(x) \) for \( 0 \leq j \leq m \) and \( y_{m+1} \in \Gamma_{i+m-1}(x) \).

PROOF. Immediate from Lemma [4.2] and elementary matrix multiplication.

Lemma 4.4 With reference to Notation 2.1 choose an integer \( i \) \((2 \leq i \leq D - 2)\). Let \( y, z \in X \) such that \( y \in \Gamma_2(x), z \in \Gamma_i(x) \). Then the following (i)-(iii) hold:

(i) \((LR^{i-1})_{zy} = \begin{cases} b_i c_{i-1} c_{i-2} \cdots c_1 & \text{if } \partial(z, y) = i - 2, \\ 0 & \text{if } \partial(z, y) = i, \text{ otherwise.} \end{cases} \)

(ii) \((R^{i-1}L)_{zy} = \begin{cases} c_2 c_{i-1} c_{i-2} \cdots c_1 & \text{if } \partial(z, y) = i - 2, \\ 0 & \text{if } \partial(z, y) = i, \text{ otherwise.} \end{cases} \)

(iii) \((R^{i-2})_{zy} = \begin{cases} c_{i-2} c_{i-3} \cdots c_1 & \text{if } \partial(z, y) = i - 2, \\ 0 & \text{otherwise.} \end{cases} \)

PROOF. Since \( \partial(x, y) = 2, \partial(x, z) = i \), and \( \Gamma \) is bipartite, we find \( \partial(z, y) \in \{i - 2, i, i + 2\} \). Now by Lemma [4.3], the \((z, y)\)-entry of each of \( LR^{i-1}, R^{i-1}L, \) and \( R^{i-2} \) is 0 whenever \( \partial(z, y) = i + 2 \). Let \( D^j_{\ell} = D^j_{\ell}(x, y) \) \((0 \leq \ell, j \leq D)\) be as in Definition 2.2.

Assume \( \partial(z, y) = i \). By Lemma [4.3], \((LR^{i-1})_{zy} \) equals the number of walks \([y = y_0, y_1, \ldots, y_{i+1} = z]\), such that \( y_j \in \Gamma_{2+j}(x) \) for \( 0 \leq j \leq i - 1 \). Since \( \partial(z, y) = i \) each such walk is actually a path passing through \( D^j_{\ell+1} \). Note that \( z \) has \( c_i - |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)| \) neighbours in \( D^j_{\ell+1} \), and that there are precisely \( c_{i-1} c_{i-2} \cdots c_1 \) paths of length \( i - 1 \) from each vertex in \( D^j_{\ell+1} \) to \( y \). It follows from the above comments that

\[(LR^{i-1})_{zy} = (c_i - |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)|) c_{i-2} \cdots c_1.\]
By Lemma 4.3 $\text{(R}^i L)_{zy}$ equals the number of walks $[y = y_0, y_1, \ldots, y_i = z]$, such that $y_j \in \Gamma_j(x)$ for $1 \leq j \leq i$. Since $\partial(z, y) = i$ each such walk is actually a path passing through $D_1^1$. Note that there are precisely $c_{i-1}c_{i-2} \cdots c_1$ paths of length $i - 1$ from $z$ to each vertex in $D_1^1$ which is at distance $i - 1$ from $z$. It follows from the above comments that

$$\text{(R}^i L)_{zy} = |\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)|c_{i-1}c_{i-2} \cdots c_1.$$ 

By Lemma 4.3 $\text{(R}^i L)_{zy}$ equals the number of walks $[y = y_0, y_1, \ldots, y_i = z]$, such that $y_j \in \Gamma_{2+i}(x)$ for $0 \leq j \leq i - 2$. Since $\partial(z, y) = i$, this number equals $0$.

Assume finally that $\partial(z, y) = i - 2$. Using Lemma 4.3 and similar reasoning as above, we find that $\text{(R}^i L)_{zy} = b_i c_{i-1}c_{i-2} \cdots c_1$, $(\text{R}^i L)_{zy} = c_2 c_{i-1}c_{i-2} \cdots c_1$, and $(\text{R}^{i-2})_{zy} = c_{i-2}c_{i-3} \cdots c_1$. This completes the proof.

**Lemma 4.5** With reference to Notation 2.1, the following (i), (ii) hold for $2 \leq i \leq D$:

(i) $E_i^* A_{i-2} E_2^* = \frac{1}{c_1 c_2 \cdots c_{i-2}} E_i^* \text{(R}^{i-2}) \text{E}_2^*$.

(ii) $E_i^* A_i E_2^* = \frac{1}{c_1 c_2 \cdots c_i} E_i^* \left(\text{R}^{i-2} L + \text{R}^{i-2} LR + \cdots + LR^{i-1} - \sum_{j=1}^{i-1} b_j c_j \text{R}^{i-2}\right) E_2^*$.

**Proof.** It is well-known that $A_i = v_i(A)$, where $v_j$ ($0 \leq j \leq D$) are polynomials of degree $j$ defined recursively by

$$v_0(\lambda) = 1, \quad v_1(\lambda) = \lambda, \quad c_{j+1} v_{j+1}(\lambda) = \lambda v_j(\lambda) - b_{j-1} v_{j-1}(\lambda).$$

From (7) it is easy to see that for $0 \leq j \leq D$ we have

$$v_j(\lambda) = \frac{1}{c_1 c_2 \cdots c_j} \lambda^j - \sum_{\ell=1}^{j-1} \frac{b_{j-1} c_{\ell}}{c_1 c_2 \cdots c_j} \lambda^{j-1} + \text{(lower degree terms)}.$$ 

Since $A = R + L$ it follows from the above comments that

$$A_{i-2} = v_{i-2}(R + L) = \frac{1}{c_1 c_2 \cdots c_{i-2}} (R + L)^{i-2} - \sum_{\ell=1}^{i-1} \frac{b_{j-1} c_{\ell}}{c_1 c_2 \cdots c_{i-2}} (R + L)^{i-4} + \cdots$$

and

$$A_i = v_i(R + L) = \frac{1}{c_1 c_2 \cdots c_i} (R + L)^i - \sum_{\ell=1}^{i-1} \frac{b_{i-1} c_{\ell}}{c_1 c_2 \cdots c_i} (R + L)^{i-2} + \cdots$$

The result now follows from the facts that $RE_j^* V \subseteq E_{j+1}^* V$ and $LE_j^* V \subseteq E_{j-1}^* V$.

**Lemma 4.6** With reference to Notation 2.1, pick $v \in U$. Then the following (i), (ii) hold.

(i) The vector $v$ is an eigenvector of $E_2^* A_2 E_2^*$ if and only if $v$ is an eigenvector of $E_2^* LRE_2^*$.
Moreover, since $v$ is an eigenvector of $E_2^*A_2E_2^*$ with eigenvalue $\eta \in \Phi_2$, then the corresponding eigenvalue for $E_2^*RLR_2E_2^*$ is $c_2\eta + k$.

**Proof.** (i) By Lemma 4.5(ii) we have

$$E_2^*A_2E_2^* = \frac{1}{c_2}E_2^*(RL + LR - kI)E_2^*.$$ Moreover, since $v \in U$, we have $E_2^*RLE_2^*v = RLv = 0$. The result follows.

(ii) Immediate from comments in the proof of (i) above.

## 5 Algebraic condition implies combinatorial property

With reference to Notation 2.1 assume that up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. In this section we prove that for $2 \leq i \leq D - 2$ there exist complex scalars $\alpha_i, \beta_i, \gamma_i$, not all zero, with the following property: for all $y, z \in X$ such that $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$ we have

$$\alpha_i + \beta_i|\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)| + \gamma_i|\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)| = 0.$$

Since up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, we can split the irreducible $T$-modules with endpoint 2 into two isomorphism classes. Let $V_2^1$ denote the sum of $T$-modules from the first of these isomorphism classes, and let $V_2^2$ denote the sum of $T$-modules from the second of these isomorphism classes. Let $d_1$ denote the diameter of the irreducible $T$-modules from the first isomorphism class, and let $d_2$ denote the diameter of the irreducible $T$-modules from the second isomorphism class. Note that $d_1, d_2 \in \{D - 4, D - 3, D - 2\}$ by [3, Theorem 10.1].

**Lemma 5.1** With reference to Notation 2.1 assume that up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. Let $F_1, F_2, F_3, F_4, F_5 \in T$ and pick an integer $i$, $2 \leq i \leq D - 2$. Assume that $E_i^*F_5E_2^*v$ is nonzero for every nonzero $v \in E_2^*V_i$. Then the matrices

$$E_i^*F_1E_2^*, \ E_i^*F_2E_2^*, \ E_i^*F_3E_2^*, \ E_i^*F_4E_2^*, \ E_i^*F_5E_2^*$$

are linearly dependent.

**Proof.** Since $V_0$ is thin, there exist scalars $r_{0,j}$ ($1 \leq j \leq 4$) such that for $v \in E_2^*V_0$ we have $E_i^*F_jE_2^*v = r_{0,j}E_i^*F_5E_2^*v$. Since all irreducible $T$-modules with endpoint 1 are thin and mutually isomorphic, there exist scalars $r_{1,j}$ ($1 \leq j \leq 4$) such that for $v \in E_2^*V_1$ we have $E_i^*F_jE_2^*v = r_{1,j}E_i^*F_5E_2^*v$. Since all the summands of $V_2^1$ ($\ell \in \{1, 2\}$) are thin and mutually isomorphic, there exist scalars $r_{2,j}^\ell$ ($1 \leq j \leq 4$) such that for $v \in E_2^*V_2^\ell$ we have $E_i^*F_jE_2^*v = r_{2,j}^\ell E_i^*F_5E_2^*v$.

We will now show that there exist scalars $\lambda_i$ ($1 \leq i \leq 5$) such that $\lambda_i$ are not all zero and such that

$$\lambda_1E_i^*F_1E_2^* + \lambda_2E_i^*F_2E_2^* + \lambda_3E_i^*F_3E_2^* + \lambda_4E_i^*F_4E_2^* + \lambda_5E_i^*F_5E_2^* = 0.$$ (8)
Let $S$ denote the matrix
\[
\begin{pmatrix}
 r_{0,1} & r_{0,2} & r_{0,3} & r_{0,4} & 1 \\
 r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} & 1 \\
 r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & 1 \\
 r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & 1
\end{pmatrix}
\]
Since the rank of $S$ is less or equal to 4, the system of equations $S(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 0, 0)^t$ has a nontrivial solution. Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^t$ be such a nontrivial solution. Choose an arbitrary $v \in V$ and note that
\[
E_v^* = v_0 + v_1 + v_2^1 + v_2^2,
\]
where $v_0 \in V_0$, $v_1 \in V_1$, $v_2^1 \in V_2^1$ and $v_2^2 \in V_2^2$. Using the above comments we find
\[
E_i^* F_j E^*_v = r_{0,j} E_i^* F_3 E^*_v v_0 + r_{1,j} E_i^* F_5 E^*_v v_1 + r_{2,j} E_i^* F_2 E^*_v v_2 + r_{2,j} E_i^* F_2 E^*_v v_2 \quad (1 \leq j \leq 4).
\]
It follows that
\[
\lambda_1 E_i^* F_1 E^*_v + \lambda_2 E_i^* F_2 E^*_v + \lambda_3 E_i^* F_3 E^*_v + \lambda_4 E_i^* F_4 E^*_v + \lambda_5 E_i^* F_5 E^*_v = 0
\]
and hence (8) holds.

\begin{lemma}
With reference to Notation 2.1 assume that up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. For $2 \leq i \leq D - 2$, the matrices
\[
E_i^* R_i^{j-1} E_2^*, \quad E_i^* R_i^{j-1} L E_2^*, \quad E_i^* A_i E_2^*, \quad E_i^* R_i^{-2} E_2^*
\]
are linearly dependent.
\end{lemma}

\begin{proof}
By [0, Lemma 5.3(iii), Lemma 7.3(ii)] and [3, Theorem 9.4(i), Lemma 10.2(i)], $R_i^{-j} v \neq 0$ for each nonzero $v \in E_2^* v$. Therefore, by Lemma 5.1, there exist scalars $\lambda_j$ ($1 \leq j \leq 5$), not all zero, such that
\[
\lambda_1 E_i^* R_i^j L^2 E_2^* + \lambda_2 E_i^* R_i^{j-1} E_2^* + \lambda_3 E_i^* R_i^{j-1} L E_2^* + \lambda_4 E_i^* A_i E_2^* + \lambda_5 E_i^* R_i^{-2} E_2^* = 0.
\]
We will show that $\lambda_1 = 0$. Choose $y, z \in X$ such that $\partial(x, y) = 2$, $\partial(x, z) = i$ and $\partial(y, z) = i + 2$. Then by Lemma 4.3 and elementary matrix multiplication, the $(z, y)$-entry of $E_i^* R_i^j L^2 E_2^*$ is nonzero, while by Lemma 4.4 the $(z, y)$-entries of $E_i^* R_i^{j-1} E_2^*$, $E_i^* R_i^{j-1} L E_2^*$, and $E_i^* R_i^{-2} E_2^*$ are all zero. Similarly, the $(z, y)$-entry of $E_i^* A_i E_2^*$ is 0. Hence $\lambda_1 = 0$ and the result follows.
\end{proof}

\begin{theorem}
With reference to Notation 2.1 assume that up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. Then for $2 \leq i \leq D - 2$ there exist complex scalars $\alpha_i, \beta_i, \gamma_i$ which are not all zero and with the following property: for all $y, z \in X$ such that $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$ we have
\[
\alpha_i + \beta_i |\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)| + \gamma_i |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)| = 0.
\]
\end{theorem}
PROOF. Note that the theorem holds for $i = 2$ with $\alpha_2 = 0$, $\beta_2 = 1$ and $\gamma_2 = -1$. Assume $i \geq 3$. By Lemma 5.2 there exist scalars $\lambda_{1,i}, \lambda_{2,i}, \lambda_{3,i}$ and $\lambda_{4,i}$ which are not all zero and such that

$$
\lambda_{1,i}L_i R^{i-1} E_2^* + \lambda_{2,i} L_i A_i E_2^* + \lambda_{3,i} A_i E_2^* + \lambda_{4,i} A_i E_2^* = 0. 
$$

Let $y, z \in X$ such that $\partial(x, y) = 2$, $\partial(x, z) = i$ and $\partial(y, z) = i$. Observe that since $\partial(x, y) = 2$ and $\partial(x, z) = i$ we have $(E_i^* L_i R^{i-1} E_2^*)_{zy} = (L_i R^{i-1})_{zy}$, $(E_i^* L_i R^{i-1} E_2^*)_{zy} = (R^{i-1} L)_zy$, $(E_i^* A_i E_2^*)_{zy} = (A_i)_{zy}$, and $(E_i^* A_i E_2^*)_{zy} = (R^{i-2} E_2^*)_{zy}$. Now observe that $(E_i^* A_i E_2^*)_{sy} = 1$ by (3). By Lemma 4.4 we find

$$
(E_i^* L_i R^{i-1} E_2^*)_{zy} = (c_i - |\Gamma_i-1(x) \cap \Gamma_i-1(y) \cap \Gamma(z)|)c_{i-1}c_{i-2} \cdots c_1,
$$

$$
(E_i^* R^{i-1} L E_2^*)_{zy} = |\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)|c_{i-1}c_{i-2} \cdots c_1,
$$

$$
(E_i^* R^{i-2} E_2^*)_{zy} = 0.
$$

Set $\alpha_i = \lambda_{1,i} c_1 \cdots c_1 + \lambda_{3,i}$, $\beta_i = \lambda_{2,i} c_{i-1} \cdots c_1$ and $\gamma_i = -\lambda_{1,i} c_{i-1} \cdots c_1$. If $\alpha_i = \beta_i = \gamma_i = 0$, then $\lambda_{1,i} = \lambda_{2,i} = \lambda_{3,i} = 0$. It follows from (11) that $\lambda_{4,i} E_i^* R^{i-2} E_2^* = 0$. Since $\lambda_{1,i}, \lambda_{2,i}, \lambda_{3,i}, \lambda_{4,i}$ are not all zero, we have $E_i^* R^{i-2} E_2^* = 0$. But this is a contradiction, since by Lemma 4.3(i), $(E_i^* R^{i-2} E_2^*)_{wy} \neq 0$ for $w \in \Gamma_i(x) \cap \Gamma_{i-2}(y)$. Therefore $\alpha_i, \beta_i$ and $\gamma_i$ are not all zero. Taking the $(z, y)$-entry of both sides of (11) and using the above information, we obtain the desired result.

6 The $\gamma_i = 0$ case

With reference to Notation 2.4, assume that up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. In the last section we showed that for $2 \leq i \leq D - 2$ there exist complex scalars $\alpha_i, \beta_i, \gamma_i$ which are not all zero and with the following property: for all $y, z \in X$ such that $\partial(x, y) = 2$, $\partial(x, z) = i$, $\partial(y, z) = i$ we have

$$
\alpha_i + \beta_i |\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)| + \gamma_i |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)| = 0.
$$

In this section, we will show that $\gamma_i \neq 0$ for $2 \leq i \leq D - 2$. We will need the following lemma.

Lemma 6.1 ([5] Section 10]) With reference to Notation 2.4, let $W$ denote a thin irreducible $T$-module of endpoint 2. Let $v \in E_2^* W$ be an eigenvector for $E_2^* A_2 E_2^*$ with eigenvalue $\eta$. For all integers $2 \leq i \leq D - 1$, define vectors $v_i^+ = E_i^* A_{i-2} v$, $v_i^- = E_i^* A_{i+2} v$. There exist unique real scalars $\varphi_i = \varphi_i(W)$, $\omega_i = \omega_i(W)$ ($2 \leq i \leq D - 2$) such that

$$
v_i^- = \varphi_i v_i^+, \quadLv_i^+ = \omega_i v_i^+.
$$

The scalars $\varphi_i$ and $\omega_i$ satisfy the following recurrence:

$$
\varphi_2 = -(\eta + 1), \quad \omega_2 = b_2 - c_2 \varphi_2, 
$$

$$
\varphi_i = \frac{b_{i+1}}{\omega_{i-1}} \varphi_{i-1}, \quad \omega_i = c_{i-2} \varphi_{i-1} + b_i - c_i \varphi_i \quad (3 \leq i \leq D - 2).
$$

Moreover, the scalars $\omega_2, \omega_3, \ldots, \omega_{D-3}$ are all positive.
For the rest of this section, we will use the following notation.

**Notation 6.2** With reference to Notation 2.1, assume that up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. Let $\alpha_i, \beta_i, \gamma_i$ ($2 \leq i \leq D - 2$) denote the scalars from Theorem 5.3. Let $W, W'$ denote non-isomorphic thin irreducible $T$-modules of endpoint 2 with corresponding local eigenvalues $\eta, \eta'$, respectively. We define scalars $\psi, \psi'$ by

$$
\psi = \begin{cases} 
\frac{b_2 - \frac{b_2}{1+\eta}}{\infty}, & \text{if } \eta \neq -1 \\
\frac{b_2}{1+\eta}, & \text{if } \eta = -1
\end{cases} \quad \psi' = \begin{cases} 
\frac{b_2 - \frac{b_2}{1+\eta'}}{\infty}, & \text{if } \eta' \neq -1 \\
\frac{b_2}{1+\eta'}, & \text{if } \eta' = -1.
\end{cases}
$$

(13)

Fix nonzero vectors $v \in E_2^W, v' \in E_2^W$. We note $v, v'$ are eigenvectors for $E_2^i A_2 E_2^i$ with eigenvalues $\eta, \eta'$, respectively. We define scalars $\varphi_i(W), \varphi_i(W'), \omega_i(W), \omega_i(W')$ ($2 \leq i \leq D - 2$) as in Lemma 6.1.

With reference to Notation 6.2, recall that we wish to show $\gamma_i \neq 0$ ($2 \leq i \leq D - 2$). We will proceed by contradiction. As our proof is a bit long, we will need some intermediary results. In the following two lemmas, we will assume there exists an integer $i$ ($2 \leq i \leq D - 2$) such that $\gamma_i = 0$, and we obtain some preliminary results that ultimately will lead to a contradiction.

**Lemma 6.3** With reference to Notation 6.2, assume there exists an integer $i$ ($2 \leq i \leq D - 2$) such that $\gamma_i = 0$. Then $\varphi_i(W) = \varphi_i(W')$. Moreover, these scalars are both nonnegative.

**Proof.** Setting $\gamma_i = 0$ in (9), we find that $|\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)|$ is constant for all $y, z \in X$ such that $\partial(x, y) = 2, \partial(x, z) = i, \partial(y, z) = i$. Let $\tau$ denote this constant. Now let $z, y$ denote arbitrary vertices in $X$. By computing the $(z, y)$-entry of both sides, we shall now show that

$$
E_2^i R^{-1} E_2^i - c_{i-1} c_2 E_2^i R^{-2} E_2^i - \tau c_{i-1} c_{i-2} \ldots c_1 E_2^i A_i E_2^i = 0.
$$

(14)

Indeed, it is clear that the $(z, y)$-entry of both sides of (14) is zero unless $z \in \Gamma_i(x)$ and $y \in \Gamma_2(x)$. So let $z \in \Gamma_i(x), y \in \Gamma_2(x).$ Observe that $(E_2^i R^{-1} E_2^i)_{zy} = (R^{-1} L)_{zy},$

$(E_2^i R^{-2} E_2^i)_{zy} = (R^{-2})_{zy},$ and $(E_2^i A_i E_2^i)_{zy} = (A_i)_{zy}$. Using Lemma 1.3 and (3), one verifies the $(z, y)$-entries of both sides of (14) are equal, and hence (14) holds.

Since the $T$-module $W$ has endpoint 2, we observe $E_2^i R^{-1} E_2^i v = 0$ and thus

$$
\tau c_{i-1} c_{i-2} \ldots c_1 E_2^i A_i E_2^i v = -c_{i-1} c_2 E_2^i R^{-2} E_2^i v.
$$

(15)

By [5] Corollary 9.3, Theorem 9.4], we find

$$
E_2^i A_i E_2^i v = -v_i^+ - v_i^-,
E_2^i R^{-2} E_2^i v = c_{i-2} c_{i-3} \ldots c_2 v_i^+,
$$

(16)

where $v_i^+, v_i^-$ are as in Lemma 6.1. Combining the information in (15), (16), we find $\tau^{-1} (c_2 - \tau) v_i^+ = v_i^-$. Thus by Lemma 6.1 $\varphi_i(W) = \tau^{-1} (c_2 - \tau)$. We note that $\tau^{-1} (c_2 - \tau) \geq 0$ since $|\Gamma(x) \cap \Gamma(y)| = c_2$. By a similar argument, $\varphi_i(W') = \tau^{-1} (c_2 - \tau)$, and the result follows. 

\[12\]
Lemma 6.4 With reference to Notation 6.2, assume there exists an integer \( i \) (2 ≤ \( i \) ≤ \( D - 2 \)) such that \( \gamma_i = 0 \). Then \( \eta, \eta' \neq -1 \).

Proof. Suppose, to the contrary, that \( \eta = -1 \). Then by (11), (12), we find \( \varphi_2(W) = 0 \), \( \varphi_1(W) = 0 \). By Lemma 6.3 we find \( \varphi_i(W') = 0 \), and hence \( \varphi_2(W') = 0 \) by (12). So \( \eta' = -1 \) by (11). Thus \( W, W' \) have the same local eigenvalue, and hence are isomorphic by Lemma 3.5 a contradiction.

Theorem 6.5 With reference to Notation 6.2, \( \gamma_i \neq 0 \) (2 ≤ \( i \) ≤ \( D - 2 \)).

Proof. Suppose there exists an integer \( i \) such that \( \gamma_i = 0 \). By Lemma 6.3, \( \varphi_i(W) = \varphi_i(W') \). By Lemma 6.4, \( \eta, \eta' \neq -1 \). We now define polynomials \( p_j, P_j \) (0 ≤ \( j \) ≤ \( D - 2 \)) in \( \mathbb{R}[\lambda] \) as in (11), Definition 6.2. Using (11) Lemma 6.3, (11) Lemma 6.4, and treating separately the cases where \( j \) is odd and even, one uses induction on \( j \) to routinely prove that

\[
\varphi_j(W) = \frac{b_j b_j+1}{c_j-1c_j} P_j(\psi), \quad \varphi_j(W') = \frac{b_j b_j+1}{c_j-1c_j} P_j(\psi') \quad (2 \leq j \leq D - 2).
\]

Now let \( \theta_1 \) denote the second-largest eigenvalue of \( \Gamma \), and let \( \theta_d \) denote the smallest nonnegative eigenvalue of \( \Gamma \). In (11) Theorem 11.4 it is shown that \( \theta_1 \leq \eta, \eta' \leq \theta_d \), where \( \theta_1 = -1 - b_2 b_3 (\theta_d^2 - b_2)^{-1} \) and \( \theta_d = -1 - b_2 b_3 (\theta_d^2 - b_2)^{-1} \). By (11) Lemma 3.5, \( \theta_d > b_2 > \theta_d \), so \( \theta_1 < -1 < \theta_d \). We claim that \( \theta_1 \leq \eta, \eta' < -1 \). To the contrary, without loss of generality, suppose \( -1 < \eta \leq \theta_d \). By (11) Lemmas 5.6(i), 6.6(iii), 15.3(ii)), we find that \( P_1(\psi), P_1(\psi') \) have opposite signs. Thus in view of (17), we find \( \varphi_1(W) \) is negative, contradicting Lemma 6.3. Thus \( \theta_1 \leq \eta, \eta' < -1 \), and by (11) Lemmas 5.5(i), 15.3(ii)), we find \( P_1(\psi), P_1(\psi') \) are positive for \( 0 \leq j \leq D - 2 \). In view of the fact that \( \eta, \eta' < -1 \) and using (13), we find \( \psi, \psi' > 0 \).

By (17) and the fact that \( \varphi_i(W) = \varphi_i(W') \), we find

\[
P_{i-2}(\psi)P_i(\psi') - P_{i-2}(\psi')P_i(\psi) = 0.
\]

Now using (11) Definition 6.2, (10) Lemma 5.3, and treating the cases of \( i \) odd and even separately, we find

\[
c_j^{-1}c_j^{-1}(\psi' - \psi) \sum_{h=0}^{i-2} P_h(\psi)P_h(\psi') \frac{k_{i-2}b_i b_{i-1}}{k_h b_h b_{h+1}} = 0.
\]

Since all terms in the sum above are positive, we find \( \psi = \psi' \). Thus \( \eta = \eta' \), so \( W, W' \) are isomorphic by Lemma 3.5 a contradiction.

Corollary 6.6 With reference to Notation 2.1 assume that up to isomorphism \( \Gamma \) has exactly two irreducible \( T \)-modules with endpoint 2, and they are both thin. Then \( \Delta_2 > 0 \) and for \( 2 \leq i \leq D - 2 \) there exist complex scalars \( \alpha_i, \beta_i \) with the following property: for all \( y, z \in X \) such that \( \partial(x, y) = 2, \partial(x, z) = i, \partial(y, z) = i \) we have

\[
\alpha_i + \beta_i |\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)|.
\]

Proof. Since \( \Gamma \) has exactly two thin irreducible \( T \)-modules with endpoint 2, we see \( |\Phi_2| = 2 \) by Lemma 3.5 Thus \( \Delta_2 > 0 \) by Lemma 3.4. Equation (20) follows immediately from Theorem 5.3 and Theorem 6.5.
7 Some equations involving lowering and raising matrices

In this section we assume the following notation.

**Notation 7.1** With reference to Notation 2.1 assume that for $2 \leq i \leq D - 2$ there exist complex scalars $\alpha_i, \beta_i$ with the following property: for all $y, z \in X$ such that $\partial(x, y) = 2, \partial(x, z) = i, \partial(y, z) = i$ we have

$$\alpha_i + \beta_i |\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)|. \tag{21}$$

Assume the situation from Notation 7.1. Our goal is to prove that if $\Delta_2 > 0$, then up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. To do this, we will need some equations involving raising and lowering matrices.

**Lemma 7.2** With reference to Notation 7.1, for $2 \leq i \leq D - 2$ we have

$$LR^{i-1}E^*_2 = c_{i-1} \cdots c_1 (c_i - \alpha_i) E_i^* A_i E_2^* - \beta_i R^{i-1} LE^*_2 + c_{i-1} (b_i + \beta_i c_2) R^{i-2} E^*_2. \tag{22}$$

**Proof.** Let $y, z \in X$. We show that the $(z, y)$-entry of both sides of the above equation agree. First note that the $(z, y)$-entry of both sides is 0 if $\partial(x, y) \neq 2$. It follows from Lemma 4.3 that the $(z, y)$-entry of both sides is 0 if $\partial(x, z) \neq i$.

Assume now $\partial(x, y) = 2$ and $\partial(x, z) = i$. Observe that $(LR^{i-1}E^*_2)_{zy} = (LR^{i-1})_{zy}, (E^*_i A_i E_2^*)_{zy} = (A_i)_{zy}, (R^{i-1} LE^*_2)_{zy} = (R^{i-1} L)_{zy},$ and $(R^{i-2} E^*_2)_{zy} = (R^{i-2})_{zy}$. By the triangle inequality and since $\Gamma$ is bipartite we have $\partial(y, z) \in \{i - 2, i, i + 2\}$. In each of these three cases, we may use Lemma 4.3 (3), and (21) to verify that the $(z, y)$-entries of both sides of (22) agree. This completes the proof.

**Corollary 7.3** With reference to Notation 7.1, for $2 \leq i \leq D - 2$ we have

$$\alpha_i LR^{i-1}E^*_2 = (c_i - \alpha_i) \left(R^{i-1} LE^*_2 + R^{i-2} LE^*_2 + \cdots + RLR^{i-2} E^*_2 \right) +$$

$$- c_i \beta_i R^{i-1} LE^*_2 + \left(c_i c_{i-1} (\beta_i c_2 + b_i) + (\alpha_i - c_i) \sum_{j=1}^{i-1} b_{j-1} c_j \right) R^{i-2} E^*_2. \tag{23}$$

**Proof.** From Lemma 4.3 (ii) we find

$$LR^{i-1}E^*_2 = c_i c_{i-1} \cdots c_1 E_i^* A_i E_2^* - R^{i-1} LE^*_2 - R^{i-2} LE^*_2 - \cdots - RLR^{i-2} E^*_2 +$$

$$\sum_{j=1}^{i-1} b_{j-1} c_j R^{i-2} E^*_2. \tag{24}$$

Multiplying (23) with $c_i - \alpha_i$ and (22) with $c_i$ and then subtracting the resulting equations, we find (24).

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8 The $\alpha_i = 0$ case

With reference to Notation 7.1 in this section we assume that $\alpha_i = 0$ for some $i$ ($3 \leq i \leq D - 2$). We will show that in this case $\Delta_2 = 0$, where $\Delta_2$ is from Definition 3.3.

**Definition 8.1** With reference to Notation 7.1 assume that $\alpha_i = 0$ for some $i$ ($3 \leq i \leq D - 2$), and let $\ell \geq 3$ be the minimal integer such that $\alpha_\ell = 0$. For $3 \leq i \leq D - 2$ define $t_i$ as

$$t_i = c_i c_{i-1} (\beta c_2 + b_i) + (\alpha_i - c_i) \sum_{j=1}^{i-1} b_{j-1} c_j.$$ 

(25)

Moreover, let $u \in U$ be an eigenvector for $LR$, and let $\lambda$ be the corresponding eigenvalue. Define scalars $\lambda_i$ ($1 \leq i \leq \ell - 2$) by

$$\lambda_1 = \lambda, \quad \lambda_i = \frac{c_{i+1} - \alpha_{i+1}}{\alpha_{i+1}} (\lambda_1 + \lambda_2 + \ldots + \lambda_{i-1}) + \frac{t_{i+1}}{\alpha_{i+1}}.$$ 

(26)

Note that for $1 \leq i \leq \ell - 2$ we have $\lambda_i = \sigma_i \lambda + \rho_i$, where $\sigma_1 = 1$, $\rho_1 = 0$ and

$$\sigma_i = \frac{c_{i+1} - \alpha_{i+1}}{\alpha_{i+1}} (\sigma_1 + \sigma_2 + \ldots + \sigma_{i-1}) \quad (2 \leq i \leq \ell - 2),$$ 

(27)

$$\rho_i = \frac{c_{i+1} - \alpha_{i+1}}{\alpha_{i+1}} (\rho_1 + \rho_2 + \ldots + \rho_{i-1}) + \frac{t_{i+1}}{\alpha_{i+1}} \quad (2 \leq i \leq \ell - 2).$$ 

(28)

We now show that $\sigma_1 + \sigma_2 + \ldots + \sigma_i \neq 0$ for $1 \leq i \leq \ell - 2$.

**Lemma 8.2** With reference to Definition 8.1 we have $\sigma_1 + \sigma_2 + \ldots + \sigma_i \neq 0$ for $1 \leq i \leq \ell - 2$.

**Proof.** When $i = 1$ the result is clear. Assume now that $i \geq 2$ and that on the contrary we have $\sigma_1 + \sigma_2 + \ldots + \sigma_i = 0$. We claim that in this case also $\sigma_1 + \sigma_2 + \ldots + \sigma_{i-1} = 0$.

If $c_{i+1} - \alpha_{i+1} = 0$, then, by (27), we have $\sigma_i = 0$ and so $\sigma_1 + \sigma_2 + \ldots + \sigma_{i-1} = 0$.

If $c_{i+1} - \alpha_{i+1} \neq 0$, then, by (27), we have

$$\sigma_1 + \sigma_2 + \ldots + \sigma_{i-1} = \frac{\alpha_{i+1} \sigma_i}{c_{i+1} - \alpha_{i+1}}.$$ 

Therefore

$$0 = \sigma_1 + \sigma_2 + \ldots + \sigma_{i-1} + \sigma_i = \frac{\alpha_{i+1} \sigma_i}{c_{i+1} - \alpha_{i+1}} + \sigma_i = \frac{c_{i+1} \sigma_i}{c_{i+1} - \alpha_{i+1}}.$$ 

It follows that $\sigma_i = 0$, and so $\sigma_1 + \sigma_2 + \ldots + \sigma_{i-1} = 0$. But now it follows that $\sigma_1 = 0$, a contradiction.

**Lemma 8.3** With reference to Definition 8.1 for $1 \leq i \leq \ell - 2$ we have

$$LR^i u = \lambda_i R^{i-1} u.$$ 

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The result follows.

**Corollary 8.6** With reference to Definition 8.4 we have

\[
\ell(\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell+1}) + t_{\ell+1} = 0.
\]

**Proof.** Immediate from Lemma 8.4 and Lemma 8.5. 

**Theorem 8.7** With reference to Definition 8.7 we have \(|\Phi_2| \leq 1\). Furthermore, \(\Delta_2 = 0\).

**Proof.** By Lemma 8.2 and Lemma 4.6, \(\Phi_2\) consists of values \(\eta\) for which \(c_2 \eta + k\) is an eigenvalue of \(LRE_2\) on \(U\). Let \(\eta \in \Phi_2\) such that \(\lambda = c_2 \eta + k\). By Corollary 8.6 and since \(\lambda = \sigma_i \lambda + \rho_i\) for \(1 \leq i \leq \ell - 2\) we have

\[
\lambda(\sigma_1 + \sigma_2 + \cdots + \sigma_{\ell-2}) = -\frac{t_\ell}{c_\ell} \cdot (\rho_1 + \rho_2 + \cdots + \rho_{\ell-2}).
\]
As $\sigma_1 + \sigma_2 + \cdots + \sigma_{\ell-2} \neq 0$ by Lemma 3.2, we have 

$$\lambda = \frac{\ell u + (\rho_1 + \rho_2 + \cdots + \rho_{\ell-2})}{\sigma_1 + \sigma_2 + \cdots + \sigma_{\ell-2}}.$$ 

By definition the numbers $\rho_i, \sigma_i$ ($1 \leq i \leq \ell - 2$) and $t_\ell$ depend only on the intersection numbers of $\Gamma$ and on numbers $\alpha_i, \beta_i$ ($3 \leq i \leq \ell - 1$). Therefore $\lambda$ (and hence also $\eta$) is uniquely determined, so $|\Phi_2| \leq 1$. It follows from Lemma 3.4 that $\Delta_2 = 0$. 

9 Combinatorial property implies algebraic condition

With reference to Notation 7.1, assume $\Delta_2 > 0$. In this section we prove that up to isomorphism $\Gamma$ has exactly two irreducible $T$-modules with endpoint 2, and they are both thin. Observe that by Theorem 8.7 we have $\alpha_i \neq 0$ for $3 \leq i \leq D - 2$.

**Lemma 9.1** With reference to Notation 7.1, assume $\Delta_2 > 0$. Let $W$ denote an irreducible $T$-module with endpoint 2, and choose $u \in E_2^2 W$ which is an eigenvector for $LR$. Then $LR^{2-1} u \in \text{Span}\{R^{i-2}u\}$ for $2 \leq i \leq D - 2$.

**Proof.** We will prove the lemma by induction on $i$. Note that the statement of the lemma is true for $i = 2$ since $u$ is an eigenvector for $LR$. Assume $i \geq 3$. The result now follows from (23) using $E_2^2 u = u$, $Lu = 0$, and the induction hypothesis.

**Corollary 9.2** With reference to Notation 7.1, assume $\Delta_2 > 0$. Let $W$ denote an irreducible $T$-module with endpoint 2, and choose $u \in E_2^2 W$ which is an eigenvector for $LR$. Then the following (i)–(iii) hold.

(i) $E_i^* A_{i-2} E_2^* u = 1/(c_{i-2}c_{i-3} \cdots c_1) R^{i-2} u$ $(2 \leq i \leq D)$.

(ii) $E_i^* A_i E_2^* u \in \text{Span}\{R^{i-2}u\}$ $(2 \leq i \leq D - 2)$.

(iii) $E_i^* A_{i+2} E_2^* u \in \text{Span}\{R^{i-2}u\}$ $(2 \leq i \leq D - 2)$.

**Proof.** (i) Immediate from Lemma 4.5(i) using $E_2^* u = u$.

(ii) Immediate from Lemma 4.5(ii) using $E_2^* u = u$, $Lu = 0$ and Lemma 9.1.

(iii) Note that by [5, Corollary 9.3] we have $E_i^* A_{i+2} E_2^* u = -E_i^* A_{i-2} E_2^* u - E_i^* A_i E_2^* u$. The result follows from (i), (ii) above.

**Corollary 9.3** With reference to Notation 7.1, assume $\Delta_2 > 0$. Let $W$ denote an irreducible $T$-module with endpoint 2, and choose $u \in E_2^2 W$ which is an eigenvector for $LR$. Then $LR^{D-2} u \in \text{Span}\{R^{D-3}u\}$.

**Proof.** By Lemma 4.5(i) and since $E_2^* u = u$, we have $R^{D-2} u = c_1 \cdots c_{D-2} E_2^* D_{D-2} E_2^* u$. By [5, Theorem 9.4(iii)] we have $L E_2^* A_{D-2} u = b_{D-1} E_{D-1}^* A_{D-3} u + R E_{D-2}^* A_{D} u$. Combining these facts, the result now follows from Corollary 9.2(i), (iii).
Lemma 9.4 With reference to Notation 7.1 assume \( \Delta_2 > 0 \). Let \( W \) denote an irreducible \( T \)-module with endpoint 2 and diameter \( d \). Choose \( u \in E_2 W \) which is an eigenvector for \( LR \). Then the following is a basis for \( W \):

\[
R^i u \quad (0 \leq i \leq d).
\]

(29)

In particular, \( W \) is thin.

Proof. We first show that \( W \) is spanned by vectors (29). Let \( W' \) denote the subspace of \( V \) spanned by vectors (29) and note that \( W' \subseteq W \). We claim that \( W' \) is \( T \)-invariant. Observe that since \( RE_j^* V \subseteq E_j^* V \) for \( 0 \leq j \leq D - 1 \) and \( RE_D^* V = 0 \), \( W' \) is invariant under the action of \( E_j^* \) for \( 0 \leq j \leq D \), and so \( W' \) is \( M^* \)-invariant. By definition \( W' \) is invariant under \( R \). Note that by Lemma 9.1 and Corollary 9.3 \( W' \) is invariant under \( L \). Since \( A = R + L \) and since \( A \) generates \( M \), \( W' \) is \( M \)-invariant. The claim follows.

Hence \( W' \) is a \( T \)-module and it is nonzero since \( u \in W' \). By the irreducibility of \( W \) we have \( W' = W \). Since \( E_i^* W \neq 0 \) for \( 2 \leq i \leq d + 2 \) we have \( R^i u \neq 0 \) for \( 0 \leq i \leq d \).

Observe also that since \( RE_j^* V \subseteq E_j^* V \) for \( 0 \leq j \leq D - 1 \) and \( RE_D^* V = 0 \), we have that \( R^i u (0 \leq i \leq d) \) are linearly independent.

Theorem 9.5 With reference to Notation 7.1 assume \( \Delta_2 > 0 \). Then \( \Gamma \) has up to isomorphism exactly two irreducible \( T \)-modules with endpoint 2, and they are both thin.

Proof. Note that every irreducible \( T \)-module with endpoint 2 is thin by Lemma 9.4. We will now show that up to isomorphism \( \Gamma \) has exactly two irreducible \( T \)-modules with endpoint 2. By \[12\] Theorem 4.2, Lemma 3.7, Theorem 3.8, the set \( \Phi_2 = \{ \eta_{k+1}, \eta_{k+2}, \ldots, \eta_{k_2} \} \) has at most two elements.

By \[5\] Theorem 11.7, the set \( \Phi_2 \) coincides with the set of local eigenvalues of the thin irreducible \( T \)-modules with endpoint 2. By Lemma 3.5, thin irreducible \( T \)-modules with endpoint 2 are isomorphic if and only if they have the same local eigenvalue. By Lemma 3.4, \( \Phi_2 \) has exactly two elements, and the result follows.

We now present our main result.

Theorem 9.6 Let \( \Gamma \) denote a bipartite distance-regular graph with vertex set \( X \), valency \( k \geq 3 \), and diameter \( D \geq 4 \). Then the following are equivalent.

(i) \( \Delta_2 > 0 \) and for \( 2 \leq i \leq D - 2 \) there exist complex scalars \( \alpha_i, \beta_i \) with the following property: for all \( x, y, z \in X \) such that \( \partial(x, y) = 2 \), \( \partial(x, z) = i \), \( \partial(y, z) = i \) we have

\[
\alpha_i + \beta_i |\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)| = |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma(z)|.
\]

(ii) For all \( x \in X \) there exist up to isomorphism exactly two irreducible modules for the Terwilliger algebra \( T(x) \) with endpoint two, and these modules are thin.

Proof. Immediate from Corollary 6.6 and Theorem 9.5.
10  Scalars $\alpha_i$ and $\beta_i$ in terms of intersection numbers

With reference to Notation 7.1 assume that the equivalent conditions (i), (ii) of Theorem 9.6 hold. In this section we express scalars $\alpha_i$ and $\beta_i$ in terms of the intersection numbers of $\Gamma$. To do this, we first need the following definition.

Definition 10.1 With reference to Notation 2.1 and Definition 2.2, for all integers $i, j$

\[ w_{ij} = \sum_{z \in D_i^j} \hat{z}. \]

Observe that $w_{ij} = 0$ unless $0 \leq i, j \leq D$ and $i + j$ is even, and that $\|w_{ij}\|^2 = p^2_{ij}$. For all integers $i$ define vectors $w^+_i = w^+_i(x, y)$ and $w^-_i = w^-_i(x, y)$ by

\[ w^+_i = \sum_{z \in D_i^i} |\Gamma_{i-1}(z) \cap D_1^i| \hat{z}, \quad w^-_i = \sum_{z \in D_i^i} |\Gamma(z) \cap D_{i-1}^i| \hat{z}. \]

We observe $w^+_i = w^-_i = 0$ unless $1 \leq i \leq D$. Furthermore, $w^+_{11} = w_{11}$, $w^-_{11} = 0$, $w^+_{DD} = c_2w_{DD}$, $w^-_{DD} = kw_{DD}$, and $w^+_{22} = w^-_{22}$.

Theorem 9.6 can now be reformulated as follows.

Theorem 10.2 Let $\Gamma$ denote a bipartite distance-regular graph with vertex set $X$, valency $k \geq 3$, and diameter $D \geq 4$. Then the following are equivalent.

(i) $\Delta^2 > 0$ and for $2 \leq i \leq D - 2$ there exist complex scalars $\alpha_i, \beta_i$ with the following property: for all $x, y \in X$ such that $\partial(x, y) = 2$, we have

\[ \alpha_i w_i^+ + \beta_i w_i^- = w_i^-, \]

where vectors $w_i, w_i^+, w_i^-$ are from Definition 10.1.

(ii) For all $x \in X$ there exist up to isomorphism exactly two irreducible modules for the Terwilliger algebra $T(x)$ with endpoint two, and these modules are thin.

Lemma 10.3 ([13, Lemma 4.1]) With reference to Notation 2.1 the following (i)–(iii) hold.

(i) $p^2_{i-2,i} = p^2_{i,i-2} = k_i c_i c_{i-1} / (k(k - 1))$ (2 \leq i \leq D);

(ii) $p^2_i = k_i (c_i b_{i-1} - 1) + b_i (c_{i+1} - 1) / (k(k - 1))$ (1 \leq i \leq D - 1);

(iii) $p^2_{DD} = kD(b_{D-1} - 1) / (k - 1)$.

Lemma 10.4 ([13, Lemma 7.1]) With reference to Definition 10.1 the following (i), (ii) hold for 2 \leq i \leq D - 1.

(i) $\langle w^+_i, w_i \rangle = k_i c_i (b_{i-1} - 1) / k_2$;
(ii) \( \| w_{ii}^+ \|^2 = k_i c_i (c_2 (b_i - 1) - (c_2 - 1) b_i) / k_2. \)

**Lemma 10.5** With reference to Definition [10.3](#) the following (i), (ii) hold for \( 2 \leq i \leq D - 1 \).

(i) \( \langle w_{ii}^-, w_{ii} \rangle = c_i k_i (c_i b_i + c_i - 1 - k) / (k (k - 1)) \);

(ii) \( \langle w_{ii}^-, w_{ii}^+ \rangle = k_i c_i (b_i - 1) + c_i (b_i - 1) / k_2. \)

**PROOF.** (i) We first claim that \( \langle c_i w_{ii} - w_{ii}^-, w_{ii} \rangle = p_{ii+1, i+1}^2 (c_{i+1} - c_{i-1}). \) To show this observe that \( \langle c_i w_{ii} - w_{ii}^-, w_{ii} \rangle = \sum_{z \in D_i} |\Gamma(z) \cap D_{i+1}^1| \). Hence \( \langle c_i w_{ii} - w_{ii}^-, w_{ii} \rangle \) is equal to the number of ordered pairs \( (v, z) \), where \( v \in D_{i+1}^1 \), \( z \in D_i \), and \( \partial(v, z) = 1 \). In order to find this number, we fix \( v \in D_{i+1}^1 \) and observe \( |\Gamma(v) \cap D_i| = c_i + 1 - c_i - 1 \). The claim follows since there are exactly \( p_{ii+1, i+1}^2 \) vertices in \( D_{i+1}^1 \). The result now follows from the above claim, Lemma [10.3](#) and since \( \| w_{ii} \|^2 = p_{ii}^2. \)

(ii) We first claim that \( \langle (b_i - c_i) w_{ii} + w_{ii}^+, w_{ii} \rangle = c_2 c_i (p_{ii+1, i}^1 - p_{ii+1, i-1}^2). \) To show this observe that \( \langle (b_i - c_i) w_{ii} + w_{ii}^+, w_{ii} \rangle = \sum_{z \in D_i} |\Gamma(z) \cap D_{i+1}^1| |\Gamma_i(z) \cap D_{i+1}^1|. \) Hence \( \langle (b_i - c_i) w_{ii} + w_{ii}^+, w_{ii} \rangle \) is equal to the number of ordered triples \( (v, z, u) \), where \( v \in D_1 \), \( z \in D_i \), \( u \in D_{i+1}^1 \), \( \partial(v, z) = i - 1 \) and \( \partial(v, u) = 1 \). Therefore, \( \partial(v, u) = i \). Now pick \( v \in D_1 \) and note that we have \( c_2 \) choices for \( v \). Observe that \( \Gamma_i+1(x) \cap \Gamma_i(v) \subseteq D_{i+1}^1 \cup D_{i+1}^1. \) Moreover, every vertex in \( D_{i+1}^1 \) is at distance \( i \) from \( v \). Therefore, since \( u \) is in \( D_{i+1}^1 \) and \( \partial(v, u) = i \), we have \( p_{ii+1, i}^1 - p_{ii+1, i-1}^2 \) choices for \( u \). Finally, for every such pair \( (v, u) \) there are exactly \( c_i \) vertices \( z \) which are at distance \( i - 1 \) from \( v \) and at distance \( 1 \) from \( u \). Observe that all these vertices must be in \( D_i \). This proves the claim. The result now follows from the claim, Lemma [10.3](#)(i), Lemma [10.3](#) and since \( p_{ii+1, i}^1 = b_j k_i / k. \)

**Theorem 10.6** Let \( \Gamma \) denote a bipartite distance-regular graph with vertex set \( X \), valency \( k \geq 3 \), and diameter \( D \geq 4 \). Assume that the equivalent conditions (i), (ii) of Theorem [10.3](#) hold. Then for \( 2 \leq i \leq D - 2 \), \( \Delta_i > 0 \) and the following hold:

\[
\alpha_i = \frac{c_i (c_i - 1) (b_i - 1) - c_i (b_i - 1) (c_2 - 1)}{c_2 \Delta_i}
\]

and

\[
\beta_i = \frac{c_i (c_i + 1) (b_i - 1) - b_i (c_i + 1) (c_i - 1)}{c_2 \Delta_i},
\]

where scalars \( \Delta_i \) are from Definition [3.3](#).

**PROOF.** By Lemma [3.3](#), \( \Delta_i \geq 0. \) Suppose \( \Delta_i = 0. \) Then by [3](#) Theorem 13], for all \( x, y, z \in X \) with \( \partial(x, y) = 2 \), \( \partial(x, z) = i \), \( \partial(y, z) = i \), the number \( |\Gamma(x) \cap \Gamma(y) \cap \Gamma_i(x)| \) is independent of \( x, y, z \). By Theorem [5.3](#) we find [10] is satisfied with \( \gamma_i = 0 \), which contradicts Theorem [6.3](#). Hence \( \Delta_i > 0. \) To obtain the formulae for \( \alpha_i \) and \( \beta_i \), take the inner product of the vector equation in Theorem [10.2](#) with \( w_{ii} \) and \( w_{ii}^+ \), and then solve thus obtained system of linear equations using Lemma [10.4](#) and Lemma [10.5](#).
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