LADDER COSTS FOR RANDOM WALKS
IN LÉVY RANDOM MEDIA

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Abstract. We consider a random walk $Y$ moving on a Lévy random medium, namely a one-dimensional renewal point process with inter-distances between points that are in the domain of attraction of a stable law. The focus is on the characterization of the law of the first-ladder height $Y_T$ and length $L_T(Y)$, where $T$ is the first-passage time of $Y$ in $\mathbb{R}^+$. The study relies on the construction of a broader class of processes, denoted Random Walks in Random Scenery on Bonds (RWRSB) that we briefly describe. The scenery is constructed by associating two random variables with each bond of $\mathbb{Z}$, corresponding to the two possible crossing directions of that bond. A random walk $S$ on $\mathbb{Z}$ with i.i.d increments collects the scenery values of the bond it traverses: we denote this composite process the RWRSB. Under suitable assumptions, we characterize the tail distribution of the sum of scenery values collected up to the first exit time $T$. This setting will be applied to obtain results for the laws of the first-ladder length and height of $Y$. The main tools of investigation are a generalized Spitzer-Baxter identity, that we derive along the proof, and a suitable representation of the RWRSB in terms of local times of the random walk $S$. All these results are easily generalized to the entire sequence of ladder variables.

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1. Introduction

An essential component of fluctuation theory in discrete time is the study of the first-ladder height and time of a one-dimensional (1D) random walk $S = (S_n)_{n \in \mathbb{N}_0}$, respectively given by the first maximal value reached by $S$, and by the corresponding time. In this context, the Wiener-Hopf techniques, introduced by Spitzer, Baxter and others, offer a main tool of investigation, and allow for the derivation of several fundamental identities that relate the distributions of these first-ladder random variables to that of the underlying random walk (see [24, 25, 26], and [8, 10] for reviews). These results, that have then been established in different formulations by many authors, open the way to a refined understanding of the first-ladder quantities of the ladder (ascending or descending) process, and of conditioned random walks ([11, 12, 1, 15, 9].

The aim of the present work is to generalize these kind of results to random walks moving on a one-dimensional random medium having i.i.d. heavy-tailed inter-distances. Specifically, the random medium that we consider is a renewal point process $\omega = (\omega_k)_{k \in \mathbb{Z}}$ with i.i.d. inter-distances in the domain of attraction of a stable variable. The random
walk on the random medium $\omega$ is then defined as $Y = (Y_n)_{n \in \mathbb{N}_0}$, where $Y_n := \omega_{S_n}$ and $S = (S_n)_{n \in \mathbb{N}_0}$ is an underlying random walk on $\mathbb{Z}$, independent of $\omega$.

The process $Y$ can be seen as a generalization of the \textit{Lévy flights}, that are random walks with i.i.d. heavy-tailed jumps, and as a discrete time version of the \textit{Lévy-Lorentz gas} (see \cite{5} and \cite{6,7} for some related extensions). All these are processes that have been receiving a surge of attention as they model phenomena of anomalous transport and anomalous diffusion (see e.g. \cite{16,28,3,20,29} for some general or recent references).

From its definition, it turns out that the process $Y$ performs the same jumps as $S$ but on the marked points of $\omega$ instead of $\mathbb{Z}$. Thus the first-ladder times of $S$ and $Y$ correspond, and we set

$$T := \min\{n > 0 : S_n > 0\} \equiv \min\{n > 0 : Y_n > 0\}.$$ 

A complete characterization of the law of $T$ can then be derived from the classical Spitzer-Baxter identities, and specifically from the so called Sparre-Andersen identity \cite{22,23} (see also \cite{10} for a general treatment, and \cite{19} for the specific lattice case). In particular, if the random walk $S$ has symmetric jumps as in our definition, then the law of $T$ is in the domain of attraction of a $1/2$-stable law.

The characterization of the distribution of $Y_T$, the first-ladder height of $Y$, is in general an open problem, as the double source of randomness creates a non-trivial dependence between the increments of the process and makes the analysis of the corresponding motion much harder than the classical independent case, studied for example in \cite{21,12}. For the same reason, the ascending ladder process $(Y_{T_k})_{k \in \mathbb{N}_0}$, where $T_k$ is the time corresponding to the $k$-th maximum value reached by $Y$ (so that $T_0 = 0$ and $T_1 \equiv T$), is not in general a renewal process.

We will approach the problem by considering a slightly generalized setting, appearing in several applications, in which a cost process $C$ is associated with a real (continuous or discrete) random walk $S$, that is assumed to be the control process. As a simple but paradigmatic example, suppose that each jump of the random walk takes a given and possibly random cost (e.g. time or energy) to be performed. We could then be interested in the total cost accumulated when the walk reaches its first maximum, that is the quantity $C_T$. As a first step of our analysis, we will derive a generalized Spitzer-Baxter identity for $(T, C_T)$ under the assumption that the process $C$ has i.i.d increments, eventually depending on the control process (Theorem 2.1). When the cost process is chosen to be exactly equal to $S$, we recover the classical Spitzer-Baxter identity. With different choices of $C$, we are able to derive information on different kinds of first-ladder random variables associated with the process, such as its first-ladder length.

We then move to \textit{Random Walks in Random Scenery on Bonds}. In this setting, as already mentioned, at each step the walker collects the scenery values of the bond it traverses. It is apparent that these quantities can be seen, in the same spirit as the previous part, as the increments of a cost process $C$ associated with $S$. On the other hand such increments are now not i.i.d. and thus Theorem 2.1 can not be directly applied. However, assuming that $S$ has symmetric increments and thanks to a representation of the process in terms of the local times of $S$, we will be able to express the generating function of $C_T$ in a simpler form, that allows for the implementation of the generalized Spitzer-Baxter identities. The explicit results are derived from Tauberian theorems under mild assumptions about the scenery process. Then we will adapt the techniques used to analyze the first-ladder quantities to obtain analogous results for the $k$-th ladder costs, $C_{T_k}$.

Finally, these results are applied in order to derive the tail distributions of the first-ladder length and height of the process $Y$. In turn, the latter can be used to infer on the
law of the first-passage time of a generalized Lévy-Lorentz gas.

The paper is organized as follows. Section 2 is devoted to the rigorous definition of cost and control processes, random walks in random scenery on bonds and the related first-ladder quantities. At the same time, we provide the statement of the associated main results. All the proofs of these theorems are presented in Section 3, together with some explicit applications to random walks in Lévy random media and Lévy-Lorentz gas.

2. Setup and Main Results

We start by defining a convention that will be used throughout the paper. If $X$ is a random variable in the domain of attraction of an $\alpha$-stable law, with $\alpha \in (0, 2]$, we define $\hat{\alpha} = \min\{1, \alpha\}$. If $X \equiv 0$, we set $\hat{\alpha} = +\infty$.

Let us consider a process $S := (S_n)_{n \in \mathbb{N}_0}$ taking value on $\mathbb{R}$ and, for a fixed $\ell \in \mathbb{N}$, an $\ell$-dimensional process $C := (C_n)_{n \in \mathbb{N}_0}$, that could depend non-trivially on $S$. We denote by $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ the increments of the joint process $(S, C)$. $C$ is referred to as the cost process while $S$ is the control process. We avoid to write explicitly the dependence on $S$ of the cost process, unless explicitly needed.

Example 1. As a simple but paradigmatic example to be used in next sections, consider the one-dimensional cost process obtained by choosing $\eta_k = |\xi_k|$, that is

$$C_n(S) = \sum_{k=1}^{n} |S_k - S_{k-1}| =: L_n(S), \quad \forall n \in \mathbb{N}_0.$$  

It is apparent that $L_n(S)$ measures the total length of the walker after $n$ steps.

We define the first-ladder time in $(0, \infty)$ of $S$ (or first-passage time of $S$) as

$$T := \min\{n > 0 : S_n > 0\}$$

and the corresponding first-ladder height (or leapover) as the control process stopped at $T$, i.e. $S_T$. In the same spirit, we can define the first-ladder cost as the value of the cost process $C$ stopped at $T$, i.e. $C_T$. With the choice (2.1), $L_T(S)$ is the first-ladder length of $S$, that is the length of the process $S$ up to its first-passage in $(0, \infty)$.

We now give an overview of our main results, characterizing the law of $C_T$ under different assumptions. The first result is an explicit expression for the joint generating function of $(T, S_T, C_T)$ under the assumption that the joint process $(S, C)$ has i.i.d. increments. This result will be instrumental for the analysis of first-ladder quantities related to the random walk in random media $Y$, though not directly applicable to it. We then introduce a general process, called Random Walk in Random Scenery on Bonds from its analogy to Random Walks in Random Scenery [17], so to obtain a suitable representation of $Y_T$ and of $L_T(Y)$ in this setting. This new process, for which we will state our main result, can be seen as a cost process coupled with $S$, having dependent increments also depending on a random scenery assigned to the bonds of $Z$. Finally, as explicit applications of the main result, we derive the law of the first-ladder quantities for the random walks in Lévy random media.

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1It is apparent that we can equivalently define an $(\ell + 1)$-dimensional process and choose an arbitrary coordinate to play the role of the control process and the remaining $\ell$’s as the cost process. We prefer to stick to a more explicit notation for the sake of clarity.
2.1. Cost process with i.i.d. increments. The investigation of first-ladder time and height of a 1D random walk is nowadays a well-established topic in fluctuation theory. Among well-known results, that are derived under the assumption of independent and identically distributed increments of the walk \( S \), the Spitzer-Baxter identity provides an explicit formula for the generating function of the first-ladder time \( T \) and height \( S_T \):

\[
E \left[ zn^{T}e^{itS_T} \right] = 1 - \exp \left( -\sum_{n=1}^{\infty} \frac{zn}{n} \int_{\{S_n>0\}} e^{itS_n}d\mathbb{P} \right).
\]

In the same spirit of these classical results, assuming that the process \((S,C)\) is the sum of i.i.d. random variables \(\xi_k, \eta_k \in \mathbb{N}\) we derive an identity akin to the Spitzer-Baxter identity for the joint control and cost processes:

**Theorem 2.1.** Suppose that the joint process \((S,C)\) has i.i.d. increments. Then, for any \(t \in \mathbb{R}, s \in \mathbb{R}^\ell\) and \(z \in (0,1)\),

\[
E \left[ z^{T}e^{itS_T}e^{isC_T} \right] = 1 - \exp \left( -\sum_{n=1}^{\infty} \frac{zn}{n} \int_{\{S_n>0\}} e^{itS_n}e^{isC_n}d\mathbb{P} \right),
\]

\[
E \left[ \sum_{n=0}^{T-1} zn^{S_n}e^{isC_n} \right] = \exp \left( +\sum_{n=1}^{\infty} \frac{zn}{n} \int_{\{S_n\leq0\}} e^{itS_n}e^{isC_n}d\mathbb{P} \right).
\]

As in the classical setting, the identities (2.4) and (2.5) allow to determine the laws of \(T, S_T, C_T\) from the knowledge of quantities that do not depend on the first-passage event (right-hand sides of the identities).

2.2. Random Walk in Random Scenery on Bonds. Let \(\zeta^\pm = (\xi_k^\pm)_{k \in \mathbb{Z}}\) be two sequences of i.i.d. real-valued random variables defining the random scenery; \(\xi_k^+\) and \(\xi_k^-\) are the scenery values at bond \(k\) of \(\mathbb{Z}\). In what follows, the random walk \(S\) is considered to have i.i.d. symmetric increments \(\xi_k\) in \(\mathbb{Z}\), independent of \(\zeta^\pm\). We then consider the cost process \(C = (C_n)_{n \in \mathbb{N}_0}\), depending on \(S\) and \(\zeta^\pm\), such that \(C_0 = 0\) and, for \(n \in \mathbb{N}\),

\[
C_n := \sum_{k=1}^{n} \eta_k, \quad \text{with} \quad \eta_k = \begin{cases} \sum_{j=S_{k-1}}^{S_{k-1}-1} \zeta^+_j, & \text{if} \; \xi_k > 0, \\ 0, & \text{if} \; \xi_k = 0, \\ \sum_{j=S_k}^{S_k-1} \zeta^-_j, & \text{if} \; \xi_k < 0. \end{cases}
\]

Basically, each \(\eta_k\) collects all the scenery values corresponding to the bonds that have been crossed in the corresponding jump \(\xi_k\) of \(S\), whereas \(\zeta^+\) determines the weight associated with bond traversed to the right and \(\zeta^-\) with bond traversed to the left. In particular, the cost process \(C\) depends on both \(\zeta^\pm\) and \(S\) and is called Random Walk in Random Scenery on Bonds (RWRSB).\(^2\)

\(^2\)It is worth stressing that we are not assuming that \(\eta\) and \(\xi\) are independent.
The presence of the random scenery breaks down the i.i.d. assumption of the generalized Spitzer-Baxter identity stated in Theorem 2.1. We will show how to leverage the results for first-ladder quantities associated with the control process \( S \) to infer properties on the first-ladder quantity \( C_T \) in this setting. As it will be clear in the proof, to characterize \( C_T \) we need to consider also the even part of the scenery random variables, defined as 
\[
c_k := \frac{\gamma_k^+ + \gamma_k^-}{2}, \quad \forall k \in \mathbb{Z}.
\]
This analysis will lead to the following:

**Theorem 2.2.** Let \( S \) be a random walk on \( \mathbb{Z} \) with i.i.d. symmetric increments in the basin of attraction of a \( \beta \)-stable distribution. Consider a RWRSB with \( \xi_k^+ \) and \( \xi_k^0 \) that are non-negative (or non-positive) and belong to the domain of attraction of stable random variables with index \( \gamma_+ \) and \( \gamma_0 \) respectively. Let \( C \) be the RWRSB satisfying (2.6) and set \( \rho := \min\{\hat{\gamma}_+, \hat{\gamma}_0\} \). Then, if \( \hat{\gamma}_+ \beta < \hat{\gamma}_0 \beta \), or if \( \hat{\gamma}_+ \beta > \hat{\gamma}_0 \beta \) and \( \hat{\gamma}_0 = 1 \),
\[
P[C_T > x] \sim K(x) x^{-\rho/2}, \quad \text{as } x \to \infty,
\]
where \( K(x) \) is a slowly varying function.

Moreover, in all the other cases, the tail distribution of \( C_T \) satisfies the same upper bound given in (2.7), while the lower bound, under the further hypothesis of stability of the \( \xi_k^0 \)'s, decays polynomially with exponent \( \min\{\hat{\gamma}_0, \hat{\gamma}_+/2, \hat{\beta}/2\} \).

We anticipate that a slightly more general theorem is valid under suitable assumptions (see Thm. 3.13): for the ease of the reader we postpone the technical details and the complete statement to the dedicated section.

It is worth noting that, when \( \mathbb{E}(\xi_k^+) < \infty \), the exponent of the asymptotic tail of the first-ladder cost is ruled solely by the properties of the underlying random walk \( S \), as expected.

### 2.3. First-ladder quantities for the random walks in Lévy random media \( Y \).

Let \( \zeta := (\zeta_k)_{k \in \mathbb{Z}} \) be a sequence of i.i.d. positive random variables, whose common distribution belongs to the basin of attraction of a \( \gamma \)-stable distribution, with \( 0 < \gamma \leq 2, \gamma \neq 1 \). The recursive sequence of definitions
\[
\omega_0 := 0, \quad \omega_k - \omega_{k-1} := \zeta_k, \quad \text{for } k \in \mathbb{Z},
\]
determines a marked point process \( \omega := (\omega_k)_{k \in \mathbb{Z}} \) on \( \mathbb{R} \), which we call the random medium. For a fixed \( \omega \), we define the discrete time process \( Y := (Y_n)_{n \in \mathbb{N}_0} \) setting
\[
Y_n \equiv Y_n(\omega, S) := \omega_{S_n}, \quad \forall n \in \mathbb{N}_0.
\]
In simple terms, \( Y \) performs the same jumps as \( S \) but on the points of \( \omega \), thus it is called random walk on the random medium.

The presence of the random medium creates a dependence between the increments of the process and provides a more realistic model of motion in inhomogeneous media with respect to the classical hypothesis of i.i.d. jumps. However, as before, the double source of randomness makes the analysis of the model much harder than the classical independent case, and even standard results of the classical theory of random walks, such as transience and recurrence, or central limit theorems, have been only recently obtained under suitable hypotheses (\( \square \)). On the other hand it is easy to see that the first-ladder time \( T \) is the same for both \( Y \) and the underlying random walk \( S \). Our aim is thus to characterize the asymptotic law of the first-ladder length \( L_T(Y) \) and height \( Y_T \). By noting that \( Y \) can be seen as a RWRSB driven by \( S \) with scenery \( \zeta^+ = -\zeta^- \equiv \zeta \), and similarly for \((L_n(Y))_{n \in \mathbb{N}_0}\) with the choice \( \zeta^+ = \zeta^- \equiv \zeta \), we get from Theorem 2.2 the following results:
Corollary 2.3.
\[ \mathbb{P}(Y_T > x) \sim K x^{-\gamma / 2}, \quad \text{as } x \to \infty. \]
where \( K \) is an explicit constant (see Eqs. \[3.42\]/\[3.43\]).

Corollary 2.4.
\[ \mathbb{P}(L_T(Y) > x) \leq K_{up}(x) x^{-\gamma / 2}, \]
with \( K_{low}(x) \) and \( K_{up}(x) \) suitable slowly varying functions that are matching if \( \gamma, \beta \in (0, 1) \) (see Subsection 3.3.2).

Notice that, except for the case \( \gamma \in (0, 1) \) and \( \beta \in (1, 2] \), the exponents of the decay for the lower and upper bounds match.

We are also interested in the continuous-time process \( X := (X_t)_{t \geq 0} \), whose trajectories interpolate those of the walk \( Y \) and have unit speed. Formally it can be defined as follows: given a realization \( \omega \) of the medium and a realization \( S \) of the dynamics, we define the sequence of collision times \( T_n = T_n(\omega, S) \) via
\[ T_0 := 0, \quad T_n := \sum_{k=1}^{n} |\omega_{S_k} - \omega_{S_{k-1}}|, \quad \text{for } n \geq 1. \]
Since the length of the \( n \)th jump of the walk is given by \( |\omega_{S_n} - \omega_{S_{n-1}}| \), \( T_n \) represents the global length of the trajectory \( Y \) up to the \( n \)th collision. In other words, \( T_n = L_n(Y) \), and it can be seen as a RWRSB (see also [7]). Finally, \( X_t = X_t(\omega, S) \) is defined by the equations
\[ X_t := Y_n + \text{sgn}(\xi_{n+1})(t - T_n), \quad \text{for } t \in [T_n, T_{n+1}). \]
The process \( X \) is also important from the standpoint of applications as it is a generalization of the so-called Lévy-Lorentz gas [5], that is obtained under the further assumption that the underlying random walk is simple and symmetric.

Functional limit theorems for the processes \( Y \) and \( X \), with suitable scaling, have been derived in [6/7/27] under different set of hypotheses. In particular, when \( \gamma \in (0, 1) \) or when the underlying random walk performs heavy-tailed jumps, the processes \( Y \) and \( X \) are shown to exhibit an interesting super-diffusive behavior [7/27].

Let us define the first-passage time in \((0, \infty)\) by
\[ \mathcal{T}(X) := \inf \{ t > 0 : X_t > 0 \}. \]
Notice that in this continuous setting the notion of first-ladder height becomes trivial, while that of first-ladder length of \( X \) indeed corresponds to \( \mathcal{T}(X) \), being the speed of the process \( X \) set equal to 1. By construction, and using the previous notation, it can be seen that
\[ \mathcal{T}(X) = \sum_{k=1}^{\mathcal{T}} |Y_k - Y_{k-1}| - Y_{\mathcal{T}} = L_{\mathcal{T}}(Y) - Y_{\mathcal{T}}. \]
This relation shows that, beyond their intrinsic interest, the derivation of the law of the first-ladder height and length of \( Y \) will allow to infer information on the first-passage time of the process \( X \). Indeed, the continuous first-passage time \( L_{\mathcal{T}}(Y) - Y_{\mathcal{T}} \) can be seen as the value of the RWRSB with scenery on bonds \( \zeta^+ = 0 \) and \( \zeta^- = 2\zeta \) at time \( \mathcal{T} \). As a consequence, we have
Corollary 2.5.

\[(2.16) \quad K_{\text{low}}(x) \ t^{-\min\{\gamma, \beta/2\}} \leq \mathbb{P}[T(X) > t] \leq K_{\text{up}}(x) \ t^{-\gamma \beta/2},\]

where \(K_{\text{low}}(x)\) and \(K_{\text{up}}(x)\) are slowly varying functions that are matching if \(\gamma \in (1, 2]\) (see Subsection 3.3.3).

3. Proofs of results

We give the proof of the main results described in the previous section, together with some useful corollaries. We also discuss some applications.

3.1. Results in the case of \((S, C)\) with i.i.d. increments. In this section we assume that the process \((S, C)\) has i.i.d. increments and we introduce the characteristic functions

\[(3.1) \quad \phi_{\xi_1, \eta_1}(t, s) := \mathbb{E} \left[ e^{i(t \xi_1 + s \eta_1)} \right], \quad \phi_{\eta_1}(s) := \phi_{\xi_1, \eta_1}(0, s) \quad \text{with } t \in \mathbb{R}, \ s \in \mathbb{R}^k.\]

We start proving the generalized Spitzer-Baxter identities stated in Thm. 2.1. The proof follows the line of that for the classical Spitzer-Baxter identity as in [10], Paragraph 8.4.

**Proof of Thm. 2.1.** As \((\xi_k, \eta_k)_{k \in \mathbb{N}}\) are i.i.d. random variables, we have

\[(3.2) \quad \mathbb{E} \left[ \sum_{n=0}^{\infty} z^n e^{itS_n} e^{isC_n} \right] = \frac{1}{1 - z\phi_{\xi_1, \eta_1}(t, s)} = f_+^{-1}(z, t, s)f_-(z, t, s)\]

where

\[(3.3a) \quad f_+(z, t, s) := \exp \left( -\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{\{S_n > 0\}} e^{itS_n} e^{isC_n} d\mathbb{P} \right),\]

\[(3.3b) \quad f_-(z, t, s) := \exp \left( +\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{\{S_n \leq 0\}} e^{itS_n} e^{isC_n} d\mathbb{P} \right).\]

Split the sum in the left-hand side of (3.2) as

\[(3.4) \quad \mathbb{E} \left[ \sum_{n=0}^{\infty} z^n e^{itS_n} e^{isC_n} \right] = \mathbb{E} \left[ \sum_{n=0}^{T-1} z^n e^{itS_n} e^{isC_n} \right] + \mathbb{E} \left[ \sum_{n=T}^{\infty} z^n e^{itS_n} e^{isC_n} \right].\]

The second term on the right-hand side of (3.4) can be rewritten as

\[(3.5) \quad \mathbb{E} \left[ \sum_{n=T}^{\infty} z^n e^{itS_n} e^{isC_n} \right] = \mathbb{E} \left[ z^T e^{itS_T} e^{isC_T} \sum_{n=0}^{\infty} z^n e^{i(S_{n+\tau} - S_\tau)} e^{i(s(C_{n+\tau} - C_\tau))} \right] = \mathbb{E} \left[ z^T e^{itS_T} e^{isC_T} \right] / (1 - z\phi_{\xi_1, \eta_1}(t, s)),\]

where in the last passage we use the fact that \(S_{n+\tau} - S_\tau\) and \(S_n\) have the same distribution, independent of \(S_\tau\), and the equivalent property for \(C\). By using (3.2) and (3.5) in (3.4) we get

\[(3.6) \quad [1 - \mathbb{E} \left[ z^T e^{itS_T} e^{isC_T} \right]] f_-(z, t, s) = f_+(z, t, s) \mathbb{E} \left[ \sum_{n=0}^{T-1} z^n e^{itS_n} e^{isC_n} \right].\]

We now apply standard Wiener-Hopf argument: the convolution of two measures restricted to \((0, +\infty)\) remains restricted to \((0, +\infty)\) (and the same for \((-\infty, 0]\)); by expanding the exponential functions in (3.3), we can associate \(f_+\) and \(f_-\) with \(P^*\) and \(Q^*\) in Lemma A.1.
respectively, and similarly the remaining terms on both sides of (3.6) correspond to $P$ and $Q$. The results (2.4) and (2.5) immediately follow using the lemma.

Theorem 2.1 is particularly useful when the right-hand side of the identities (2.4) and (2.5) can be computed explicitly. This happens, for example, when the law of the joint process $(S, C)$ satisfies some symmetry property. In particular the following definitions will be helpful.

**Definition 3.1.** The joint process $(S, C)$ is called $\tau$-symmetric if, for all $x \in \mathbb{R}$, $y \in \mathbb{R}^\ell$ and $n \in \mathbb{N}$,

$$
\mathbb{P}(S_n \in dx, C_n \in dy) = \mathbb{P}(-S_n \in dx, C_n \in dy).
$$

The joint process $(S, C)$ is called $\sigma$-symmetric if, for all $x \in \mathbb{R}$, $y \in \mathbb{R}^\ell$ and $n \in \mathbb{N}$,

$$
\mathbb{P}(S_n \in dx, C_n \in dy) = \mathbb{P}(-S_n \in dx, -C_n \in dy).
$$

To simplify notations we also define the function

$$
\Phi(z, s) := \exp\left(-\frac{1}{2\pi} \int_0^\pi \ln[1 - z\phi_{\xi_1, \eta_1}(t, s)] dt\right).
$$

It is easy to see that the generalized Spitzer-Baxter can be directly used to give the following explicit relation involving both $\tau$-symmetric and $\sigma$-symmetric processes, with i.i.d. increments.

**Corollary 3.2.** Consider a cost process $C'$ such that $(S, C')$ is $\tau$-symmetric, together with a cost process $C^\circ$ such that $(S, C^\circ)$ is $\sigma$-symmetric, all with i.i.d. increments. Then, for all $z \in (0, 1)$ and $s \in \mathbb{R}^\ell$,

$$
(1 - \mathbb{E}\left[z^T e^{is(C'_T + C^{\circ}_T)}\right]) \left(1 - \mathbb{E}\left[z^T e^{is(C'_T - C^{\circ}_T)}\right]\right) = (1 - z\phi_{\eta_1}(s)) \Phi^2(z, s),
$$

where $\eta = (\eta_k)_{k \in \mathbb{N}}$ are the cost increments associated with $C' + C^\circ$.

**Proof.** Choosing $C = C' + C^\circ$ in (2.4), we get for $t = 0$

$$
1 - \mathbb{E}\left[z^T e^{is(C'_T + C^{\circ}_T)}\right] = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{\{S_n > 0\}} e^{is(C_n + C^\circ_n)} d\mathbb{P}\right),
$$

On the other hand, using the definitions (3.7) and (3.8), we also have

$$
\int_{\{S_n > 0\}} e^{is(C_n - C^\circ_n)} d\mathbb{P} = \int_{\{S_n < 0\}} e^{is(C_n + C^\circ_n)} d\mathbb{P}.
$$

Putting together Eqs. (3.11) and (3.12), and using the i.i.d. assumption about the increments $\eta$ of $C' + C^\circ$, we have

$$
\left(1 - \mathbb{E}\left[z^T e^{is(C'_T + C^{\circ}_T)}\right]\right) \left(1 - \mathbb{E}\left[z^T e^{is(C'_T - C^{\circ}_T)}\right]\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{\{S_n = 0\}} e^{is(C_n + C^\circ_n)} d\mathbb{P}\right).
$$

The integral at denominator in the r.h.s. of the above equation is the anti-transform with respect to $S_n$, evaluated at $S_n = 0$, of the joint transform of a $n^{th}$ convolution of $(\xi, \eta)$. 


Hence, by using again the i.i.d. assumption about the increments \((\xi_k, \eta_k)_{k \in \mathbb{N}}\), and the fact that the convolution becomes a product in the transform domain, we get

\[
\exp \left( + \sum_{n=1}^{\infty} \frac{z^n}{n} \int_{\{S_n=0\}} e^{is(C_n+c_n)} \ d\mathbb{P} \right) = \exp \left( + \sum_{n=1}^{\infty} \frac{z^n}{n \pi} \int_0^\pi \phi_{\xi_1,\eta_1}(t,s) \ dt \right)
\]

\[
= \exp \left( - \frac{1}{\pi} \int_0^\pi \ln[1-z\phi_{\xi_1,\eta_1}(t,s)] \ dt \right)
\]

(3.13)

from which the result follows. \(\square\)

In the \(\gamma\)-symmetric case, Eq. (3.10) allows to obtain an explicit representation of the generating function of the first-ladder cost:

**Corollary 3.3.** If \((S,C)\) is \(\gamma\)-symmetric with i.i.d. increments, then

\[
\mathbb{E} \left[ z^T e^{i s C_T} \right] = 1 - \sqrt{1-z\phi_{\eta_1}(s)} \Phi(z,s).
\]

(3.14)

**Proof.** Directly follows from (3.10) with \(C^* \equiv 0\). \(\square\)

**Remark 3.4.** If \((S,C)\) is \(\gamma\)-symmetric with i.i.d. increments, and the control process \(S\) has continuous distribution, then \(\Phi(z,s) = 1\) and thus

\[
\mathbb{E} \left[ z^T e^{i s C_T} \right] = 1 - \sqrt{1-z\phi_{\eta_1}(s)},
\]

(3.15)

leading to the behavior of \(C_T\) stated in [2], where a combinatorial proof of this result has been provided. Notice that the dependence of this joint generating function on the random walk distribution comes only through the costs, that in general depend on \(S\). The presence of a discrete jump distribution yields a correction term \(\Phi(z,s)\) that instead explicitly depends on the random walk, as already underlined in [19]. Observe also that the identity (3.15) generalizes the classical Sparre-Anderson identity, which is recovered for \(s = 0\).

The generalized Spitzer-Baxter identities stated in Theorem 2.1 together with Corollaries 3.2 and 3.3 provide the key element to identify the law of the first-ladder quantities involved in it (see e.g. [10] for the classical treatment).

While the laws of \(T\) and \(S_T\) are well known under quite general hypotheses on the random walk \(S\) with i.i.d. increments (see [12] and references therein), the focus will rather be given to the first-ladder cost. We now derive the asymptotic distribution of \(C_T\), under the assumption that the joint process is \(\gamma\)-symmetric or \(\omega\)-symmetric.

**Proposition 3.5.** Assume that \((S,C)\) is \(\gamma\)-symmetric with i.i.d. increments \((\xi_k, \eta_k)_{k \in \mathbb{N}}\). If \(\eta_1\) is in the basin of attraction of a \(\gamma\)-stable law, then \(C_T\) is in the basin of attraction of a \(\gamma/2\)-stable law. More precisely

(A) If \(\phi_{\eta_1}(s) = 1 + i \nu s + o(s)\) for \(s \to 0^+\), with \(\nu > 0\) real and finite (similarly if \(\nu < 0\)), then as \(x \to +\infty\)

\[
\mathbb{P}[C_T > x] \sim \sqrt{\frac{z}{\pi}} \Phi(1,0) x^{-1/2}, \quad \mathbb{P}[C_T < -x] = o(x^{-1/2}),
\]

(3.16)

(B) If \(\phi_{\eta_1}(s) = 1 - c_1 s^\gamma + o(s^\gamma)\), for \(s \to 0^+, \gamma \in (0,1]\) and \(c_1 \in \mathbb{C}\) a complex constant with \(\Re(c_1) > 0\), then as \(x \to \infty\)

\[
\mathbb{P}[C_T > x] \sim \frac{Cp_+}{\Gamma(1-\gamma/2)} x^{-\gamma/2}, \quad \mathbb{P}[C_T < -x] \sim \frac{Cp_-}{\Gamma(1-\gamma/2)} x^{-\gamma/2},
\]

(3.17)
where
\[ C = \frac{\Phi(1,0)}{\cos(\pi \gamma/4)} \left[ \Re(c_1)^2 + \Im(c_1)^2 \right]^{1/4} \cos \left( \frac{1}{2} \arctan \left( \frac{\Im(c_1)}{\Re(c_1)} \right) \right) , \]
and
\[ p_+ = 1 - p_- = \frac{1}{2} \left( 1 - \frac{\sin \left( \frac{1}{2} \arctan \left( \frac{\Im(c_1)}{\Re(c_1)} \right) \right)}{\cos \left( \frac{1}{2} \arctan \left( \frac{\Im(c_1)}{\Re(c_1)} \right) \right) \tan \left( \frac{\pi \gamma}{4} \right) } \right) \in [0,1]. \]

If \( p_+ = 0 \) or \( p_- = 0 \), then we interpret (3.17) as \( o(x^{-\gamma/2}) \).

(C) If \( \phi_{\eta_1}(s) = 1 + ic_2 s \log(1/s) + o(s \log(1/s)) \), with \( s \to 0^+ \) and \( c_2 \in \mathbb{R} \) a positive constant (similarly for \( c_2 < 0 \), then as \( x \to \infty \)

\[ \mathbb{P}[C_T > x] \sim \frac{c_2 \log(x)}{\pi} \Phi(1,0)x^{-1/2} , \quad \mathbb{P}(C_T(Z) < -x) = o \left( \frac{\sqrt{\log(x)}}{x^{1/2}} \right) , \]

(D) If \( \phi_{\eta_1}(s) = 1 - c_3 s^\gamma + o(s^\gamma) \), with \( s \to 0^+ \), \( \gamma \in (1,2] \) and \( c_3 \in \mathbb{R}^+ \) a positive constant, then as \( x \to \infty \)

\[ \mathbb{P}[C_T > +x] \sim \mathbb{P}[C_T < -x] \sim \frac{C}{2 \Gamma(1-\gamma/2)} x^{-\gamma/2} , \]

where
\[ C = \Phi(1,0)\sqrt{c_3} \begin{cases} 1/\cos(\pi \gamma/4) , & \text{if } \gamma \in (1,2) , \\ 2/\pi , & \text{if } \gamma = 2 . \end{cases} \]

Proof. From Corollary 3.3, we have
\[ \mathbb{E} \left[ e^{isC_T} \right] = 1 - \sqrt{1 - \phi_{\eta_1}(s)\Phi(1,s)} . \]
Since the result depends solely on the behaviour of the characteristic function \( \phi_{\eta_1} \) around 0 (see e.g. [14]), or equivalently on the tail distributions of \( \eta_1 \), the tail asymptotic of \( C_T \)
can be readily determined.

Similarly, in the \( \sigma \)-symmetric case we have the following:

**Proposition 3.6.** Assume that \((S,C)\) is \( \sigma \)-symmetric with i.i.d. increments \((\xi_k,\eta_k)_{k \in \mathbb{N}}\),
and let \( \gamma \in (0,2] \) such that \( \phi_{\eta_1}(s) = 1 - c_4 s^\gamma + o(s^\gamma) \) for some \( c_4 \in \mathbb{R}^+ \). In the above notation, and for all \( \gamma \neq 2 \), it holds that

\[ \mathbb{P}[\left| C_T \right| > x] \sim K \cdot x^{-\gamma/2} , \]

where the constant is explicit \( K = \Phi(1,0)\sqrt{c_3}/\Gamma(1-\gamma/2) \) whenever \( C_T \) is non-negative (or non-positive).

Proof. The proof follows by setting \( z = 1 \) in Eq. (3.10), and performing a series expansion around \( s = 0 \) on both sides. More specifically, the ansatz \( \hat{\phi}_{\pm C_T}(s) = 1 - c_\pm s^\alpha + o(s^\alpha) \),
with \( c_\pm \) complex conjugate constants, provides \( |c_+| = |c_-| = \sqrt{c_3}\Phi(1,0) \) and \( \alpha = \gamma/2 \).

If \( \gamma \in (0,2) \), then \( \alpha \in (0,1) \) and it turns out that \( \Re(c_\pm) \neq 0 \). Furthermore, for a non-negative cost process we know that \( c_\pm = ce^{\pm i\theta} \), which concludes the proof. Notice
that if \( \gamma = 2 \) (and thus \( \alpha = 1 \)) we do not know if \( \Re(c_\pm) \neq 0 \), and hence we can not draw
any conclusions about the tail distribution of \( C_T \).
Applications. The \( \ast \)-symmetric (\( \ast \)-symmetric) condition is fulfilled in the following situations. Consider a joint process \((S, C)\) with i.i.d. increments \((\xi_k, \eta_k)_{k \in \mathbb{N}}\) such that, for a given function \(g: \mathbb{R} \mapsto \mathbb{R}^\ell\),

\[
\eta_k = g(\xi_k) \quad \forall k \in \mathbb{N}.
\]

It is apparent that if the function \(g\) is even (odd) the joint process \((S, C)\) is \(\ast\)-symmetric (\(\ast\)-symmetric). As a main example, let us consider the one-dimensional cost process \(C \equiv L\) defined in (2.1), corresponding to the length of the process \(S\), obtained by choosing \(g(\xi_k) = |\xi_k|\). Applying the above result we will obtain a complete characterization of the asymptotic law of the first-ladder length \(L_T(S)\). Similarly, by choosing \(g(\xi_k) = \xi_k\) we will fully characterize the asymptotic behavior of the first-ladder height (or leapover) \(S_T\). Both results will be of great use in the next section, we thus state them explicitly for the ease of later reference.

**Corollary 3.7.** Let \(S\) have i.i.d. symmetric increments in the domain of attraction of a \(\beta\)-stable law. Then the first-ladder length \(L_T(S)\) is in the domain of attraction of a \(\hat{\beta}/2\)-stable law. More precisely, writing \(\phi_{\xi_1}(s) = 1 - \nu s^\beta + o(s^\beta)\), we have

\[
P(L_T(S) > x) \sim \frac{\sqrt{C}}{\Gamma(1 - \hat{\beta}/2)} \Phi(1, 0) x^{-\hat{\beta}/2} \quad \text{as } x \to \infty,
\]

where

\[
C := \begin{cases} 
\nu / \cos(\pi \hat{\beta}/2) & \text{if } \beta \in (0, 1) \\
2\nu / \pi \log(x) & \text{if } \beta = 1 \\
\mathbb{E}[|\xi_1|] & \text{if } \beta \in (1, 2)
\end{cases}
\]

**Remark 3.8.** As a notable application, consider the situation in which a random time is needed to perform a jump for the random walker \(S\). In this case, the total time can be considered (per our notation) as a cost associated with the random walk. In particular (but see [2] for details and physical motivations) suppose that the time taken to perform a jump is correlated with its length. This is indeed the case for 1D Lévy walks, which are a continuous-time interpolation (with unit speed) of 1D RW with i.i.d. and heavy-tailed jumps, a.k.a. Lévy flights [28]. Thus, \(L_T(S)\) corresponds to the first-passage time for the wait-then-jump model associated with a Lévy walk, as mentioned in [2].

**Remark 3.9.** It is worthwhile to point out that Corollary 3.7 extends and completes a previous result by Sinai (see [21, Theorem 3]). One can easily retrace his proof in the presence of an appropriate cost, still fulfilling necessary hypotheses, in order to get the basin of attraction of \(L_T(S)\) rather than the leapover, but under the assumption that the random variables \(\xi_k\)'s have stable distribution.

The domain of attraction of the leapover \(S_T\), instead, stems from standard results of fluctuation theory (see [12] and references therein). Here is obtained by applying Proposition 3.6.

**Corollary 3.10.** Let \(S\) have i.i.d. symmetric increments in the domain of attraction of a \(\beta\)-stable law with \(\phi_{\xi_1}(s) = 1 - \nu s^\beta + o(s^\beta)\). Then the first-ladder length \(S_T(S)\) is in the domain of attraction of a \(\beta/2\)-stable law. More precisely

\[
P(S_T > x) \sim \frac{\sqrt{\nu}}{\Gamma(1 - \beta/2)} \Phi(1, 0) x^{-\beta/2} \quad \text{as } x \to +\infty.
\]
Notice that the limiting case \( \beta = 2 \) is included in \cite[Theorem 4]{2}, hence the tail asymptotic \( \mathbb{P}(S_T > x) \propto x^{-1} \) still holds true.

Another helpful tool, that will be used repeatedly throughout the proof of our main result, concerns linear combinations of the random variables \( L_T(S) \) and \( S_T \):

**Lemma 3.11.** Consider a cost process \( C_T \) with characteristic function

\[
\mathbb{E} [e^{isC_T}] = \mathbb{E} \left[ e^{ia(s+o(s))L_T(S)} \cdot e^{ib(s+o(s))S_T} \right],
\]

where \( a, b \) are given constants satisfying \( a, b \in \mathbb{R}, a \neq 0 \). The basin of attraction of the random variable \( C_T(S) \) is the same as for \( L_T(S) \).

**Proof.** The proof has to be split according to the fact that \( b \) is of the same sign as \( a \) or of opposite sign. Without loss of generality, we can assume that \( a > 0 \): when \( a < 0 \), inequalities are intended to be reversed. Firstly, note that the case \( b = 0 \) is trivial. If \( b \neq 0 \) we have the following cases.

(i) If \( b > 0 \), it is enough to observe that for sufficiently small \( s \)

\[
0 < (a + o(1))L_T(S) \leq C_T(S) \leq (a + b + o(1))L_T(S),
\]

given that \( 0 < S_T \leq L_T(S) \). In fact, by applying Corollary 3.7, we can immediately conclude that \( \mathbb{P}(C_T(S) > x) \propto x^{-\beta/2} \).

(ii) If \( b < 0 \), as long as \( a + b > 0 \) the above argument is still valid by reversing the inequality sign. When \( a + b \leq 0 \), instead, we have to exploit \cite{1} together with Corollary 3.2. More specifically, we can write

\[
(1 - \mathbb{E} \left[ e^{is(aL_T \pm bS_T)} \right]) \left( 1 - \mathbb{E} \left[ e^{is(aL_T \mp bS_T)} \right] \right) = (1 - \mathbb{E}[e^{is(a|x_1| \pm b\xi_1)}]) \phi^2(1, s),
\]

where we have omitted terms of order \( o(s) \) to lighten the notation. Notice that higher order corrections are irrelevant to the final result, since we are only interested in the leading term of the asymptotic expansion. It is obvious that the right-hand side of Eq. (3.24) is proportional to \( s^\beta \), since \( \mathbb{E}[a|x_1| \pm b\xi_1] = a\mathbb{E}[|\xi_1|] \neq 0 \), and \( 1 - \phi(aL_T - bS_T(s)) \propto s^\beta/2 \) thanks to (i). The desired conclusion immediately follows by comparing both sides of Eq. (3.24).

\( \square \)

It is worth noting that, as long as \( \beta \in [1, 2] \), or if \( \beta \in (0, 1) \) and \( |a|^\beta \neq |b|^\beta \cos(\pi \beta/2) \), the statement can be seen alternatively as a direct consequence of Lemma 3.1.

**Remark 3.12.** In the presence of spatio-temporal correlations, as explained in Remark 3.8, notice that the cost process \( C_T \) defined in Lemma 3.11 with \( b = -a \) corresponds to the first-passage time \( L_T(S) - S_T \) for the Lévy Walk.

### 3.2. Results for ladder costs associated with RWRSB

The focus of the present section is the cost process \( C = C(S, \xi^\pm) \) defined in Section 2 and called RWRSB. We remind that the process \( C \) collects all the scenery values \( \xi^\pm_k \) corresponding to the bonds that have been crossed in every jump of \( S \), taking into account also the travel direction. In particular, the random scenery creates a dependence between the increments of \( C \), and breaks down the i.i.d. assumption of the generalized Spitzer-Baxter identity stated in Theorem 2.1.
In this subsection, we will study the first-ladder costs associated with RWRSB, and extend the results derived in the previous subsection to this general context. This analysis will lead to Theorem 2.2 that provides the asymptotic distribution of $C_T$ under the assumption that the underlying random walk has i.i.d. symmetric increments. As stressed just after Theorem 2.2, our main result is now stated and proved emphasizing all the different scenarios arising as the parameters of the problem vary.

As a final observation, we underline that the extension to ladder costs $(C_{T_k})_{k \in \mathbb{N}_0}$, where $T_k$ is the ladder time corresponding to the $k$-th maximum value reached by $Y$, will be directly dealt with along the proof of the main theorem.

First of all, let us explicitly fix some notations. We consider a symmetric underlying random walk $S$ in $\mathbb{Z}$ with i.i.d. discrete increments $(\xi_k)_{k \in \mathbb{N}}$, whose corresponding characteristic function is, for $s \to 0^+$,

$$\phi_{\xi_1}(s) = 1 - \nu s^\beta + o(s^\beta)$$

with $\beta \in (0, 2]$ and $\nu \in \mathbb{R}^+$. The related characteristic function of $|\xi_1|$ is, for $s \to 0^+$, of the form

$$\phi_{|\xi_1|}(s) = 1 + \nu s^\beta + o(s^\beta),$$

where

$$\nu = \begin{cases} -\nu[1 - i \tan(\pi \beta/2)], & \text{for } \beta \in (0, 1), \\ -\nu[1 - i \frac{\beta}{\pi} \log(1/s)], & \text{for } \beta = 1, \\ i\mathbb{E}[|\xi_1|], & \text{for } \beta \in (1, 2]. \end{cases}$$

We also explicitly write the common characteristic function of the random variables $c_k^\pm$’s that we suppose being in the basin of attraction of a $\gamma^\pm$-stable law respectively, with $\gamma^\pm \in (0, 2]$, $\gamma^\pm \neq 1$: for $\theta \to 0^+$

$$\phi_{c_k^\pm}(\theta) = \begin{cases} 1 - c_\pm \theta^{\gamma^\pm} + o(\theta^{\gamma^\pm}), & \gamma^\pm = \hat{\gamma}_0 \in (0, 1); c_\pm \in \mathbb{C}, \Re(c_\pm) > 0, \\ 1 + i\mu_\pm \theta - c_\pm \theta^{\gamma^\pm} + o(\theta^{\gamma^\pm}), & \gamma^\pm = \hat{\gamma}_0 \in (1, 2], \hat{\gamma}_0 = 1; \mu_\pm \in \mathbb{R}, c_\pm \in \mathbb{C}, \\ 1, & \gamma^\pm = \hat{\gamma}_0 = +\infty \implies c_k^\pm \equiv 0. \end{cases}$$

We will also need to refer to the even part of the scenery values $\zeta_k^0 = \frac{\zeta_k^+ + \zeta_k^-}{2}$, for all $k \in \mathbb{Z}$, and assume that their common characteristic function is given by

$$\phi_{\zeta_k^0}(\theta) = \begin{cases} 1 - c_0 \theta^{\gamma_0} + o(\theta^{\gamma_0}), & \gamma_0 = \hat{\gamma}_0 \in (0, 1); c_0 \in \mathbb{C}, \Re(c_0) > 0, \\ 1 + i\mu_0 \theta - c_0 \theta^{\gamma_0} + o(\theta^{\gamma_0}), & \gamma_0 \in (1, 2], \hat{\gamma}_0 = 1; \mu_0 \in \mathbb{R}, c_0 \in \mathbb{C}, \\ 1, & \gamma_0 = \hat{\gamma}_0 = +\infty \implies \zeta_k^0 \equiv 0. \end{cases}$$

Sometimes we will ask for the extra-assumption that the $\zeta_k^0$’s have stable distribution with index $\gamma_0$. Along the proof, we will explicitly mention where and when this hypothesis enters in a fundamental way.

Let us discuss the relationship between the cost exponents $\hat{\gamma}_\pm$ and $\hat{\gamma}_0$, which will be crucial for the structure of the proof. By applying Lemma 4.1, it is easy to verify that

- if $\hat{\gamma}_+ \neq \hat{\gamma}_-$, we have $\hat{\gamma}_0 = \min\{\hat{\gamma}_+, \hat{\gamma}_-\}$
- if $\hat{\gamma}_+ = \hat{\gamma}_-$:
  - $\hat{\gamma}_0 = \hat{\gamma}_+ = \hat{\gamma}_-$
  - $\hat{\gamma}_0 > \hat{\gamma}_+ = \hat{\gamma}_-$, including two possible cases:
(a) \((0,1) \ni \hat{\gamma}_0 > \hat{\gamma}_+ = \hat{\gamma}_- \in (0,1) \implies \zeta_1 = -\zeta_1 + h(\zeta_1) \text{ with } h(\zeta_1) \neq 0\) and \(\hat{\gamma}_0 = \hat{\gamma}_h(\zeta_1);\)

(b) \(\hat{\gamma}_0 = +\infty \implies \zeta_0^0 \equiv 0.\)

Finally, let \((T_n)_{n \geq 0}\) be the sequence of ladder times of the process \(Y\), namely the consecutive times when the random walk attains a new maximum value. Formally, they are recursively defined by

\[
T_0 = 0, \quad T_n := \min\{k > T_{n-1} : Y_k > Y_{T_{n-1}}\} \quad \forall n \in \mathbb{N},
\]
so that \(T_1 \equiv T\). Notice that by construction they are equivalent to the ladder times of the underlying random walk \(S\) and in particular, by the Markov property, they give rise to a renewal process.

To state the main theorem, it is also convenient to define the following constants:

\[
\rho_+ := \hat{\gamma}_+ \beta, \quad \rho_0 := \hat{\gamma}_0 \hat{\beta}.
\]

Note that \(\rho_+\) as well as \(\rho_0\) involves the stability indexes of both the scenery values and the underlying random walk. Indeed we heuristically expect that the asymptotic tail of the ladder cost should receive contributions from both elements of randomness, as the RWRSB interlaces the two (otherwise independent) processes. The two exponent will enters in the asymptotic tail of the cost, as substantiated in the proof of the following theorem.

**Theorem 3.13.** Let \(C\) be a cost process satisfying (2.6), \((C_{T_n})_{n \geq 0}\) the corresponding ladder cost process. Suppose that \(\zeta^+\) and \(\zeta^0\) are i.i.d. sequences of non-negative (similarly for non-positive) random variables. Then

- If \(\rho_+ < \rho_0\), then \(\exists K \in \mathbb{R}^+\) explicit constant such that
  \[
  \mathbb{P}[C_{T_n} > x] \sim K \cdot n \cdot x^{-\rho_+ / 2}.
  \]

  Moreover, if \(\hat{\gamma}_0 = +\infty\), we can relax the hypothesis about the sign of the \(\zeta_k\)'s, and refer to \(\mathbb{P}[|C_{T_n}| > x]\) in the statement.
- If \(\rho_+ > \rho_0\), \(\hat{\gamma}_0 = 1\), then \(\exists K(x)\) slowly varying function such that
  \[
  \mathbb{P}[C_{T_n} > x] \sim K(x) \cdot n \cdot x^{-\rho_0 / 2},
  \]

  where \(K(x) = K \sqrt{\log(x)}\) with \(K \in \mathbb{R}^+\) if \(\beta = 1\) and a real positive constant otherwise.

- If \(\rho_+ \geq \rho_0\), \(\hat{\gamma}_0 \in (0,1)\) and the \(\zeta^0\)'s have stable distribution, or if \(\rho_+ = \rho_0\) and \(\hat{\gamma}_0 = 1\), then \(\exists K_{low}(x), K_{up}(x)\) slowly varying functions such that
  \[
  K_{low}(x) \cdot n \cdot x^{-\min\{\hat{\gamma}_0, \hat{\beta}/2\}} \leq \mathbb{P}[C_{T_n} > x] \leq K_{up}(x) \cdot n \cdot x^{-\rho_0 / 2}.
  \]

Moreover: The upper bound holds true even without the stability assumption on \(\zeta^0\); If \(\hat{\gamma}_+ = +\infty\) we can remove this assumption also for the lower bound.

Finally, \(K_{up}(x) \equiv K_{up} \in \mathbb{R}^+\) and \(K_{low}(x) \equiv K_{low} \in \mathbb{R}^+\) unless \(\hat{\gamma}_0 = 1\) and \(\beta = 1\), for which \(K_{low}(x) = K_{up}(x) = K \sqrt{\log(x)}\) with \(K \in \mathbb{R}^+\), or \(\hat{\gamma}_0 \in (0,1)\) and \(\beta = 1\), for which \(K_{up}(x) = K_{up} \sqrt{\log(x)}\) with \(K_{up} \in \mathbb{R}^+\) and similarly for \(K_{low}(x)\) if the lower tail exponent is given by \(\beta/2\).
Remark 3.14. We note explicitly that when the bounds on the asymptotic law are not matching, they do not permit to infer the basin of attraction of the corresponding quantities. However, we anticipate that while the lower bounds are crude estimates, the upper bounds need a more refined argument. For this reason, we expect that the upper bounds provide the correct asymptotic behavior.

Preliminary tools. Following [7], it is convenient to introduce the family of random variables $N_n(k)$, for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, called local times on the bonds of the random walk $S$, and given by

$$N_n(k) := \# \{ j \in \{1, \ldots, n\} : [k-1, k] \subseteq [S_{j-1}, S_j]\},$$

where the notation $[a, b]$ denotes the closed interval between the real numbers $a$ and $b$, irrespective of their order. In other words, $N_n(k)$ is the number of times that the walk $S$ travels the bond $[k-1, k]$ and in turn can be split into $N_n^-(k) + N_n^+(k)$ where

$$N_n^-(k) := \# \{ j \in \{1, \ldots, n-1\} : S_j \geq k, S_{j+1} \leq k - 1\},$$
$$N_n^+(k) := \# \{ j \in \{1, \ldots, n-1\} : S_{j+1} \geq k, S_j \leq k - 1\},$$

denote the number of crossings of $[k-1, k]$, respectively, from right to left and from left to right. In the following, it will be useful to express the first-ladder height and length of the process $S$ in terms of local times. An easy check shows that

\begin{equation}
\begin{aligned}
(3.30a) \quad & \text{if } k \leq 0, \quad N_T^+(k) = N_T^-(k) = \frac{N_T(k)}{2}, \quad \sum_{k \leq 0} N_T(k) = L_T(S) - S_T, \\
(3.30b) \quad & \text{if } k > 0, \quad N_T(k) = \begin{cases} 1 & k \leq S_T \\
0 & k > S_T 
\end{cases}, \quad \sum_{k > 0} N_T(k) = S_T.
\end{aligned}
\end{equation}

Moreover, since the local times are non-negative integers, we can point out the following estimates. Without loss of generality, assume that $\gamma \in (0, 1)$ (when $\gamma \in (1, 2]$ it is sufficient to reverse the inequality sign). Then, $\forall \ k \in \mathbb{Z}$, we have $(N_T(k))^{\gamma} \leq N_T(k)$, and summing over bonds on both sides we get

$$\sum_{k \leq 0} (N_T(k))^{\gamma} \leq L_T(S) - S_T.\tag{3.31}$$

On the other hand, considering the variable $N_T(k)/(L_T(S) - S_T) \leq 1$ for $k \leq 0$ and summing over all bonds, we obtain

$$\sum_{k \leq 0} (N_T(k))^{\gamma} \geq (L_T(S) - S_T)^{\gamma}.\tag{3.32}$$

These inequalities will be used to provide upper and lower bounds for the generating function of the cost process. Other useful probabilistic results are postponed to Appendix B.

Proof of Theorem 2.2. As underlined in [7], the introduction of the local times $N_n(k)$’s provides an interpretation of the collision times $(T_n)_{n \in \mathbb{N}_0}$ (defined in (2.12)) as a random walk in random scenery on bonds. More generally, the cost process of the form (2.6)
satisfies the following identity
\begin{align}
C_T &= \sum_{k \in \mathbb{Z}} \left[ \mathcal{N}^+_T(k) \zeta^+_k + \mathcal{N}^-_T(k) \zeta^-_k \right] \\
&= \sum_{k \leq 0} \mathcal{N}_T(k) \zeta^0_k + \sum_{k > 0} \mathcal{N}_T(k) \zeta^+_k.
\end{align}

Since the local times are functions of $S$ only, it turns out that, given $S$, $C_T$ is a sum of independent random variables.

Using (3.34), we can rearrange the terms inside the characteristic function of $C_T$ and get, for $s \in \mathbb{R}$,
\begin{align}
\mathbb{E} \left[ e^{i s C_T} \right] &= \mathbb{E} \left[ \exp \left( i s \sum_{k=1}^T \eta_k \right) \right] = \mathbb{E} \left[ e^{i s [\sum_{k \leq 0} \mathcal{N}_T(k) \zeta^0_k + \sum_{k > 0} \mathcal{N}_T(k) \zeta^+_k]} \right] \\
&= \mathbb{E} \left[ \mathbb{E} \left[ \exp \left( i s \sum_{k \leq 0} \mathcal{N}_T(k) \zeta^0_k \right) \exp \left( i s \sum_{k > 0} \mathcal{N}_T(k) \zeta^+_k \right) \right] | S \right] \\
&= \mathbb{E} \left[ \prod_{k \leq 0} \phi_{\mathcal{N}^0_T}(s \mathcal{N}_T(k)) \cdot \left( \phi_{\mathcal{N}^+_T}(s) \right)^{S_T} \right],
\end{align}
where in the last line we used the conditional independence mentioned above.

**Case** $\hat{\gamma}_0 = +\infty \implies \rho_+ < \rho_0$. Here we get a direct result, since local times disappear. Explicitly, we have
\begin{align*}
\phi_{C_T}(s) &= \mathbb{E} \left[ \left( \phi_{\mathcal{N}^+_T}(s) \right)^{S_T} \right] = \begin{cases}
\mathbb{E} \left[ e^{-s^{\hat{\gamma}_0} \left[ |\Re(c_+) + o(1)| \right] S_T} \cdot e^{i s^{\hat{\gamma}_0} \left[ -|\Im(c_+) + o(1)| \right] S_T} \right] & \text{if } \hat{\gamma}_+ \in (0, 1), \\
\mathbb{E} \left[ e^{i s \cdot (c_+ + o(1)) S_T} \cdot e^{-s^{\hat{\gamma}_0} \left[ |\Re(c_+) + o(1)| \right] S_T} \right] & \text{if } \hat{\gamma}_+ = 1,
\end{cases}
\end{align*}
hence by applying Lemma [B.1] we immediately obtain
\begin{align*}
K_{low} \cdot x^{-\hat{\gamma}_0 \cdot \beta / 2} \leq \mathbb{P} \left[ |C_T(Y)| > x \right] \leq K_{up} \cdot x^{-\hat{\gamma}_0 \cdot \beta / 2},
\end{align*}
with $K_{low} = K_{up}$ if $\hat{\gamma}_+ = 1$ or $\Im(c_+) = 0$.

In the following two paragraphs we will move to the generating function formalism, which is justified by the additional hypothesis $\zeta^+_1, \zeta^0_1 \geq 0$. In particular, we will provide upper and lower bounds on the generating function 
\begin{align*}
\mathcal{G}_{C_T}(s) := \mathbb{E} [e^{-s C_T}],
\end{align*}
that will allow us to determine, respectively, lower and upper bounds for the tails of the random variable $C_T(Y)$.

We anticipate that this analysis requires, in some cases, the stability of $\zeta^0_1$. In the presence of a generic domain of attraction, the management of a random error can be technically difficult to control.

**Case** $\hat{\gamma}_0 = 1$ (equiv. $\gamma_0 \in (1, 2)$). We start from Eq. (3.35), that is
\begin{align}
\mathcal{G}_{C_T}(s) &= \mathbb{E} [e^{-s C_T}] = \mathbb{E} \left[ \prod_{k \leq 0} \mathcal{G}_{\zeta^0_1}(s \mathcal{N}_T(k)) \cdot \left( \mathcal{G}_{\zeta^+_1}(s) \right)^{S_T} \right], \quad s \geq 0.
\end{align}
Observing that \( sN_T(k) \to 0 \) pointwise as \( s \to 0 \), we can replace the generating functions with their expansions around 0

\[
\begin{align*}
(3.37) \quad G_{\zeta_0}(\theta) &= 1 - \mu_0 \theta + o(\theta), \quad \text{as } \theta \to 0^+, \\
(3.38) \quad G_{\zeta_1}(\theta) &= 1 - \hat{\mu}_+ \theta^{\hat{\gamma}_+} + o(\theta^{\hat{\gamma}_+}) \quad \text{as } \theta \to 0^+,
\end{align*}
\]

where

\[
\hat{\mu}_+ = \begin{cases} 
\frac{\Re(c_+)}{\cos(\pi \hat{\gamma}_+/2)} & \text{if } \hat{\gamma}_+ \in (0, 1), \\
\mu_+ & \text{if } \hat{\gamma}_+ = 1,
\end{cases}
\]

according to (3.28) and (3.27). Hence we obtain

\[
G_{C_T}(s) = \mathbb{E} \left[ e^{-\mu_0 s(L_T - S_T) + \sum_{k \leq 0} o(sN_T(k))} \cdot e^{-\hat{\mu}_+ (s^{\hat{\gamma}_+} + o(s^{\hat{\gamma}_+})) S_T} \right],
\]

and we can apply Lemma B.1 by defining

\[
Z_1 := e^{-\mu_0 s(L_T - S_T) + \sum_{k \leq 0} o(sN_T(k))} \quad \text{and} \quad Z_2 := e^{-s^{\hat{\gamma}_+} (\hat{\mu}_+ + o(1)) S_T}.
\]

\( \mathbb{E}[Z_2] \) is trivially asymptotically equivalent to \( G_{S_T}(\hat{\mu}_+ s^{\hat{\gamma}_+}) = 1 - c_2 s^{\gamma_2} + o(s^{\gamma_2}) \), with \( \gamma_2 = \hat{\gamma}_+ \beta / 2 \) and \( c_2 \) given by Corollary 3.10, so we have to focus our efforts on the random variable \( Z_1 \). First of all, notice that by definition \( \forall \epsilon > 0, \exists \bar{s}_k > 0 \) such that \( \forall s \leq \bar{s}_k \) we have \( |o(sN_T(k))| \leq \epsilon s N_T(k) \). As a consequence, we can write

\[
\forall s \leq \min \{ (\bar{s}_k)_{k \leq 0} \}, \quad \left| \sum_{k \leq 0} o(sN_T(k)) \right| \leq \sum_{k \leq 0} |o(sN_T(k))| \leq \epsilon s (L_T - S_T),
\]

and conclude that \( \sum_{k \leq 0} o(sN_T(k)) = o(s(L_T - S_T)) \). Using this observation, we can infer that the expansion with respect to \( s \) of \( \mathbb{E}[Z_1] \) has the same leading order of

\[
(3.39) \quad \mathbb{E}[G_V(s(L_T - S_T))] = G_{V_{(L_T - S_T)}}(s),
\]

where \( V \) is a random variable independent of \( L_T - S_T \) with \( \mathbb{E}[V] = \mu_0 \) and tail behavior determined by the little-o. The equality in Eq. (3.39) is due to the fact that if \( V, W \) are independent random variables, by means of the law of total expectation we can write

\[
G_{V \cdot W}(s) = \mathbb{E}[e^{-sV - W}] = \mathbb{E}[\mathbb{E}[e^{-sV - W}|W]] = \mathbb{E}[G_V(sW)] = \mathbb{E}[G_W(sV)].
\]

Starting from Eq. (3.39), we can therefore affirm that the asymptotic behavior of \( Z_1 \) is ruled by \( L_T - S_T \), that is the random variable with slower tail decay. Then

\[
\mathbb{E}[Z_1] = 1 - c_1 s^{\gamma_1} + o(s^{\gamma_1}),
\]

with \( \gamma_1 = \bar{\beta} / 2 \), except for the case \( \beta = \bar{\beta} = 1 \) where the constant \( c_1 \) is replaced by a slowly varying function \( c_1 \sqrt{\log(1/s)} \). Formally, this conclusion stems from the application of Lemma B.2

Upper bound: Using the estimates in Lemma B.1 we can finally write

\[
G_{C_T}(s) \leq \mathbb{E}[Z_1] \mathbb{E}[Z_2] + \sqrt{\text{Var}(Z_1) \text{Var}(Z_2)} = 1 - c_{up}s^{\min\{\rho_0, \rho_+\}/2} + o(s^{\min\{\rho_0, \rho_+\}/2}),
\]

given that \( \sqrt{\text{Var}(Z_1) \text{Var}(Z_2)} = \sqrt{(2 - 2\rho_0/2)(2 - 2\rho_+ /2)c_1 c_2 s^{\rho_0 + \rho_+ /2}} \), with an additional logarithmic factor if \( \rho_0 \leq \rho_+ \) and \( \beta = \bar{\beta} = 1 \).
Lower bound: Similarly, by means of the reversed inequality, we can state that
\[ G_C(s) \geq \mathbb{E}[Z_1] \mathbb{E}[Z_2] - \sqrt{\text{Var}(Z_1) \text{Var}(Z_2)} = 1 - c_{\text{low}} s^{\min(p_0,p_1)/2} + o(s^{\min(p_0,p_1)/2}), \]
with the same logarithmic correction as before.

In terms of the tail asymptotic for the cost process, we have therefore obtained
\[ K_{\text{low}}(x) \cdot x^{-\rho/2} \leq \mathbb{P}[C_T(Y) > x] \leq K_{\text{up}}(x) \cdot x^{-\rho/2}, \]
with matching slowly varying functions $K_{\text{low}}(x) = K_{\text{up}}(x)$ when $\rho_+ \neq p_0$ or $\rho_+ = p_0$ and $\beta \equiv \hat{\beta} = 1$. Notice that in most of the cases $K_{\text{low}}, K_{\text{up}}$ are simply real positive constants: it is enough to avoid the limiting case $\beta \equiv \hat{\beta} = 1$.

Case $\hat{\gamma}_0 \in (0,1)$. We proceed as before, substituting Eq. (3.38) and
\[ G_{C_0}(\theta) = 1 - \tilde{c}_0 \theta^{\hat{\gamma}_0} + o(\theta^{\hat{\gamma}_0}), \quad \text{with} \quad \tilde{c}_0 := \frac{\Re(c_0)}{\cos(\pi \hat{\gamma}_0/2)}, \quad \hat{\gamma}_0 \in (0,1), \quad \text{as} \quad \theta \to 0^+, \]
in Eq. (3.36), and we get
\[ G_C(s) = \mathbb{E} \left[ e^{-\tilde{c}_0 s^{\hat{\gamma}_0} \sum_{k \leq 0} (N_T(k))^{\hat{\gamma}_0} + o(s^{\hat{\gamma}_0} \sum_{k \leq 0} (N_T(k))^{\hat{\gamma}_0})} \cdot e^{-\tilde{\mu}_+ (s^{\hat{\gamma}_0} + o(s^{\hat{\gamma}_0}))s_T} \right] =: \mathbb{E}[Z_1 \cdot Z_2], \]
where $Z_2$ is the same random variable as in the previous paragraph. In order to provide an asymptotic expansion for $\mathbb{E}[Z_1]$ and $\text{Var}(Z_1)$, we can equivalently study $\mathbb{E}[G_V(sW)] = G_{V \cdot W}(s)$, with $V = \zeta_1^0$ and $W = \left( \sum_{k \leq 0} (N_T(k))^{\hat{\gamma}_0} \right)^{1/\hat{\gamma}_0}$. We have not direct information about the random variable $W$, but using the estimates in (3.31) and (3.32) on local times, Lemma 3.11 and Appendix B.1, we get
\[ w^{-\hat{\beta}/2} \propto \mathbb{P}[L_{\hat{T}} - S_{\hat{T}} > w] \leq \mathbb{P}[W > w] \leq \mathbb{P}[(L_{\hat{T}} - S_{\hat{T}})^{1/\hat{\gamma}_0} > w] \propto w^{-\hat{\gamma}_0 \hat{\beta}/2}, \]
with an additional term $\sqrt{\log(w)}$ on both sides when $\beta \equiv \hat{\beta} = 1$. Consequently, by applying Lemma 3.2 to the products $V \cdot (L_{\hat{T}} - S_{\hat{T}})$ and $V \cdot (L_{\hat{T}} - S_{\hat{T}})^{1/\hat{\gamma}_0}$, we obtain the bounds
\begin{align}
(3.40a) \quad & \mathbb{E}[Z_1] \leq 1 - c_1^+ s^{\min(\hat{\gamma}_0,\hat{\beta}/2)} + o(s^{\min(\hat{\gamma}_0,\hat{\beta}/2)}), \\
(3.40b) \quad & \mathbb{E}[Z_1] \geq 1 - c_1^- s^{\hat{\gamma}_0 \hat{\beta}/2} + o(s^{\hat{\gamma}_0 \hat{\beta}/2}),
\end{align}
that imply also
\[ \text{Var}(Z_1) = \mathbb{E}[Z_1^2] - \mathbb{E}[Z_1]^2 \leq 2c_1^- s^{\hat{\gamma}_0 \hat{\beta}/2} + o(s^{\hat{\gamma}_0 \hat{\beta}/2}). \]
More precisely, if $\hat{\gamma}_0 = \hat{\beta}/2$, we have to introduce a logarithmic correction in the upper bound for the expectation of $Z_1$, which means $\mathbb{E}[Z_1] \leq 1 - c_1^+ s^{\hat{\gamma}_0} \log(1/s) + o(s^{\hat{\gamma}_0} \log(1/s))$. If $\hat{\gamma}_0 = \beta/2$ and $\beta \equiv \hat{\beta} = 1$, the slowly varying function is $c_1^+ \log^{3/2}(1/s)$, whereas if $\hat{\gamma}_0 > 1/2$ and $\beta \equiv \hat{\beta} = 1$ we have $c_1^+ \sqrt{\log(1/s)}$. With regard to the lower bound and the estimate on the variance, the square root of the logarithm appears whenever $\beta \equiv \hat{\beta} = 1$. 

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Lower bound: By means of Lemma 3.11, we find that
\[ G_C(s) \geq \mathbb{E}[Z_1]\mathbb{E}[Z_2] - \sqrt{\text{Var}(Z_1)\text{Var}(Z_2)} \]
\[ \geq 1 - c_1^{-\beta} \hat{s}^{\beta/2} - c_2 s^{\hat{\gamma} + \beta/2} + \mathcal{O}(s^{\hat{\gamma} + \beta/2}) + \ldots, \]
with \(c_1^{-\beta}\) replaced by \(c_1^{\log(1/s)}\) if \(\beta = 1\), which results in an upper bound on the tail of the cost process
\[ \mathbb{P}[C_T(Y) > x] \leq K_{up}(x) \cdot x^{-\rho/2}, \]
with \(K_{up}(x)\) slowly varying function at infinity. Note that \(K_{up}(x) = K_{up}\sqrt{\log(x)}\) if \(\beta = 1\) and \(\rho_0 \leq \rho_+\), whereas it is simply a real positive constant in all the other cases.

Upper bound: This time the estimates (3.40), combined with the Cauchy-Schwarz inequality, are not always adequate to provide a meaningful bound. In fact, bearing in mind the previous clarification concerning the limiting cases, for which one needs to make some small changes in terms of slowly varying functions, we have
\[ G_C(s) \leq \mathbb{E}[Z_1]\mathbb{E}[Z_2] + \sqrt{\text{Var}(Z_1)\text{Var}(Z_2)} \]
\[ \leq \min \left\{ 1, 1 - c_1^{\min\{\hat{\gamma}_0, \hat{\beta}/2\}} - c_2 s^{\hat{\gamma} + \beta/2} + 2(2 - 2\hat{\gamma} + \beta/2)c_1^{-\beta}c_2 s^{\hat{\gamma}_0 + \beta/2} + \ldots \right\}, \]
which turns into
\[ \mathbb{P}[C_T(Y) > x] \geq \begin{cases} K_{low} \cdot x^{-\rho/2} & \text{if } \rho \equiv \rho_+ < \rho_0, \\ K_{low} \cdot x^{-\min\{\hat{\gamma}_0, \hat{\beta}/2\}} & \text{if } \rho \equiv \rho_0 < \rho_+ \text{ and } \min\{\hat{\gamma}_0, \hat{\beta}/2\} < (\rho_0 + \rho_+)/4, \\ K_{low} \cdot x^{-\rho/2} & \text{if } \rho \equiv \rho_+ = \rho_0, c_2 - \sqrt{2c_1c_2} > 0, \\ 0 & \text{otherwise}. \end{cases} \]
In particular, notice that this analysis substantiates Remark 3.14.

In order to guarantee the global existence of a substantial lower bound, we have to impose the stability condition on the law of \(c_k^{0}\)'s. Let us assume that \(G_k(\theta) = e^{-\tilde{a}_0^\theta - \mu_0\theta}\), with \(\mu_0 \geq 0\). Then the proof for the upper bound simply becomes
\[ G_C(s) \leq \mathbb{E}[e^{-\tilde{a}_0 s^{\gamma_0}(L_T-\mathbb{S})^{\gamma_0} \cdot e^{-s^{\hat{\gamma}_+ + o(1))S_T}}] \implies \mathbb{P}[C_T(Y) > x] \geq K_{low}(x) \cdot x^{-\min\{\hat{\gamma}_0, \hat{\beta}/2\}}, \]
making the tail asymptotic always non-trivial. More precisely, \(K_{low}(x) = K_{low}\sqrt{\log(x)}\) when the exponent is \(\hat{\beta}/2\) with \(\beta = \hat{\beta} = 1\), and \(K_{low}(x) = K_{low} \in \mathbb{R}^+\) otherwise (it is explicit only for the tail exponent \(\rho_+/2\)).

On the other hand, the lower bound merely verifies the above result of more general validity concerning generic domains of attraction:
\[ G_C(s) \geq \mathbb{E}[e^{-\tilde{a}_0 s^{\gamma_0}(L_T-\mathbb{S})-\mu_0 s(L_T-\mathbb{S})} \cdot e^{-s^{\hat{\gamma}_+ + o(1))S_T}}] = \mathbb{E}[e^{-s^{\gamma_0}(\tilde{a}_0 + o(1))(L_T-\mathbb{S})} \cdot e^{-s^{\hat{\gamma}_+ + o(1))S_T}}], \]
\[ \mathbb{P}[C_T(Y) > x] \leq K_{up}(x) \cdot x^{-\rho/2}. \]

As a final point, we generalize all these asymptotic results to the law of ladder costs. Let \((T_n)_{n \geq 0}\) be the sequence of ladder times of the random walk \(S\) defined in the introduction. Notice that, for all \(n \geq 1\), the equality (3.33) still holds by replacing \(T\) with \(T_n\), together with the identity \(L_{T_n}(S) = \sum_{k \in \mathbb{Z}} N_{T_n}(k)\). Moreover, \(S_{T_n} = \sum_{k > 0} 1_{(0, S_{T_n})(k)}\). In order
to recast the characteristic or generating function of the cost \( C_{T_n} \) in a convenient manner, we need to introduce a further definition concerning the local times. Let \( N_{(t_0,t_f)}(k) \) be the number of crossings of \([k-1,k]\) observed in a specified time window \((t_0,t_f)\), that is

\[
N_{(t_0,t_f)}(k) := \#\{j \in \{t_0 + 1, \ldots, t_f\} : [k-1,k] \subseteq [S_{j-1}, S_j]\}.
\]

Hence we get

\[
C_{T_n} = \sum_{k \in \mathbb{Z}} [N_{T_n}^+(k) \zeta_k^+ + N_{T_n}^-(k) \zeta_k^-]
= \sum_{k \leq 0} N_{(0,T_1)}(k) \zeta_k^0 + \sum_{k \leq S_{T_1}} N_{(T_1,T_2)}(k) \zeta_k^0 + \cdots + \sum_{k \leq S_{T_{n-1}}} N_{(T_{n-1},T_n)}(k) \zeta_k^0
+ \sum_{k > 0} N_{(0,T_1)}(k) \zeta_k^+ + \sum_{k > S_{T_1}} N_{(T_1,T_2)}(k) \zeta_k^+ + \cdots + \sum_{k > S_{T_{n-1}}} N_{(T_{n-1},T_n)}(k) \zeta_k^+
= \sum_{k \leq 0} N_{T_n}(k) \zeta_k^0 + \sum_{k \in (0,S_{T_1}]} N_{(T_1,T_n]}(k) \zeta_k^0 + \cdots + \sum_{k \in (S_{T_{n-2}},S_{T_{n-1}])] N_{(T_{n-1},T_n]}(k) \zeta_k^0
+ \sum_{k > 0} 1_{(0,S_{T_n})}(k) \zeta_k^+,
\]

and also

\[
G_{C_{T_n}}(s) := \mathbb{E} \left[ e^{-sC_{T_n}} \right] = \mathbb{E} \left[ \mathbb{E} \left[ e^{-s \sum_{k \in \mathbb{Z}} [N_{T_n}^+(k) \zeta_k^+ + N_{T_n}^-(k) \zeta_k^-]} \bigg| S \right] \right]
= \mathbb{E} \left[ \prod_{k \leq 0} G_{\zeta_k^0} \left( sN_{T_n}(k) \right) \prod_{k \in (0,S_{T_1}]} G_{\zeta_k^0} \left( sN_{(T_1,T_n]}(k) \right) \cdots \right.
\]

\[
\left. \prod_{k \in (S_{T_{n-2}},S_{T_{n-1}])] G_{\zeta_k^0} \left( sN_{(T_{n-1},T_n]}(k) \right) \cdot \left( G_{\zeta_k^+} \right)^{S_{T_n}} \right].
\]

Then, when we consider upper and lower bounds for \( G_{C_{T_n}}(s) \), the relevant quantities to deal with are \( S_{T_n} \) and

\[
\sum_{k \leq 0} N_{T_n}(k) + \sum_{k \in (0,S_{T_1}]} N_{(T_1,T_n]}(k) + \cdots + \sum_{k \in (S_{T_{n-2}},S_{T_{n-1}])} N_{(T_{n-1},T_n]}(k) = L_{T_n} - S_{T_n}.
\]

More precisely, we are interested in their generating functions. Due to the renewal structure of the processes \((S_{T_n})_{n \geq 0}\) and \((L_{T_n}(S))_{n \geq 0}\), the ladder random variables can be seen as the sum of \(n\) i.i.d. first-ladder quantities. Thus, the previous results can be immediately generalized: we just have to introduce a multiplicative factor \(n\), which stems from the factorization of the expectations, in front of the slowly varying functions \(K(x), K_{up}(x)\) and \(K_{low}(x)\).

### 3.3. Results for the random walks in random media \(Y\). Consider the random walk on random medium \(Y\) defined in Eq. (2.9), under the hypothesis that the underlying random walk \(S\) has symmetric i.i.d. increments \((\xi_k)_{k \in \mathbb{N}}\). As already mentioned, the first-ladder height \(Y_T\) and the length \(L_T(Y)\) can be equivalently interpreted as first-ladder costs expressed as RWRSB with appropriate scenery. We then use our main Theorem 3.13 together with some simplifications occurring in this setting, to prove the Corollaries 2.3, 2.4 and 2.5.
In the following, we will refer to the notation introduced in previous sections, except for Eq. (3.27) that can be slightly simplified by considering only
\[
\phi_{\zeta_i}(\theta) = \begin{cases} 
1 - ce^{-\frac{i\pi}{2} \gamma \theta} + o(\theta), & \gamma = \hat{\gamma} \in (0, 1); \ c \in \mathbb{R}^+, \\
1 + i\mu \theta + o(\theta), & \gamma \in (1, 2], \hat{\gamma} = 1; \ \mu \in \mathbb{R}^+.
\end{cases}
\]

3.3.1. First-ladder height \(Y_T\).

Proof of Corollary 2.4. Recall that \(Y_T\) can be seen as the value at time \(T\) of a RWRSB driven by \(S\) and with scenery \(\zeta^+ = -\zeta^- = \zeta\). Notice that with this choice \(\hat{\gamma} = \hat{\gamma}_+ = \hat{\gamma}_-\), and \(\gamma_0 = +\infty\) implying that \(\rho_+ < \rho_0\). Therefore Theorem 3.13 applies.

Moreover, since \(\zeta\) are positive random variables (and thus also \(Y_T\) is positive), we can replace the characteristic functions with the generating functions in the appropriate part of the proof of the main theorem. This enable us to find explicit constants and get:

- If \(\gamma \in (1, 2]\), which means \(\hat{\gamma} = 1\), then \(\mathbb{E}[e^{-sY_T}] = \mathbb{E}[e^{-s(\mu + o(1))S_T}]\) and \(Y_T\) is in the basin of attraction of a stable law with parameter \(\beta/2\). Explicitly, by applying Corollary 3.10 we get
  \[
  \mathbb{P}(Y_T > x) \sim \frac{\sqrt{Mr^{\beta/2}}}{\Gamma(1 - \beta/2)} \Phi(1, 0) x^{-\beta/2}, \quad \text{as } x \to \infty.
  \]
  As previously mentioned, the limiting case \(\beta = 2\) is ensured by [12, Theorem 4].

- If \(\gamma \in (0, 1)\), then \(\hat{\gamma} = \gamma\) and we have \(\mathbb{E}[e^{-sY_T}] = \mathbb{E}[e^{-s^{\gamma}(c + o(1))S_T}]\) from which
  \[
  \mathbb{P}(Y_T > x) \sim \frac{\sqrt{Mr^{\beta/2}}}{\Gamma(1 - \gamma \beta/2)} \Phi(1, 0) x^{-\gamma \beta/2}, \quad \text{as } x \to \infty.
  \]

\(\square\)

3.3.2. First-ladder length.

Proof of Corollary 2.4. Recall that \(L_T(Y)\) can be seen as the value at time \(T\) of a RWRSB driven by \(S\) and with scenery \(\zeta^+ = -\zeta^- = \zeta\). Notice that with this choice \(\hat{\gamma} = \hat{\gamma}_+ = \hat{\gamma}_- = \gamma_0\) and therefore \(\rho_+ \geq \rho_0\). Following Theorem 3.13 we derive the next distinct cases.

- If \(\gamma \in (1, 2]\), that is \(\gamma_0 = \hat{\gamma}_+ = 1\), we obtain:
  (i) If \(E[|\xi_1|] < \infty\), which means \(\beta \in (1, 2]\) and \(\rho_0 < \rho_+\), then \(L_T(Y)\) is in the basin of attraction of a stable law with parameter 1/2,
    \[
    \mathbb{P}(L_T(Y) > x) \sim K \cdot x^{-1/2}, \quad \text{as } x \to \infty;
    \]
  (ii) If \(\hat{\beta} \equiv \beta = 1\), that is \(\rho_0 = \rho_+ = 1\), then
    \[
    \mathbb{P}(L_T(Y) > x) \sim K \cdot \sqrt{\log(x)} x^{-1/2}, \quad \text{as } x \to \infty;
    \]
  (iii) If \(\hat{\beta} \equiv \beta \in (0, 1]\), namely \(\rho_0 = \rho_+ = \hat{\beta}\), then \(L_T(Y)\) is in the basin of attraction of a stable law with parameter \(\hat{\beta}/2\),
    \[
    K_{low} \cdot x^{-\hat{\beta}/2} \leq \mathbb{P}(L_T(Y) > x) \leq K_{up} \cdot x^{-\hat{\beta}/2}, \quad \text{as } x \to \infty.
    \]

- If \(\gamma = \gamma_0 = \gamma_+ \in (0, 1)\), instead, we have to introduce the additional assumption of stability for the random medium. If we assume that \(G_{\zeta_i}(\theta) = e^{-\zeta \theta - \mu \theta}\), with \(\mu \geq 0\), we can write
  \[
  \mathbb{E}[e^{-\tilde{S}^{\gamma}(1 + s^\gamma(L_T - \bar{S}_T))}] \leq \mathbb{E}[e^{-\tilde{S}^{\gamma}(L_T - \bar{S}_T)^\gamma} \cdot e^{-\tilde{S}^{\gamma}(1 + s^\gamma(L_T - \bar{S}_T))}].
  \]
  In conclusion, in terms of the tail asymptotic for the first-ladder length, we have
(i) If $E(|\xi_1|) < \infty$, that is $\beta \in (1, 2]$,

$$K_{\text{low}} \cdot x^{-\min\left\{ \frac{1}{\gamma}, \frac{1}{2} \right\}} \leq \mathbb{P}(L_T(Y) > x) \leq \frac{\sqrt{E(|\xi_1|)\gamma}}{\Gamma(1 - \gamma/2)} \Phi(1, 0) x^{-\gamma/2}, \quad \text{as } x \to \infty;$$

(ii) If $\hat{\beta} \equiv \beta = 1$, which means $\rho_0 = \rho_+ = \gamma$, then

$$\frac{\sqrt{\nu \gamma}}{\Gamma(1 - \gamma/2)} \Phi(1, 0) x^{-\gamma/2} \leq \mathbb{P}(L_T(Y) > x) \leq \frac{\sqrt{2\nu \gamma \log (x)}}{\pi \Gamma(1 - \gamma/2)} x^{-\gamma/2}, \quad \text{as } x \to \infty;$$

(iii) If $\hat{\beta} \equiv \beta \in (0, 1)$, with $\rho_0 = \rho_+$, as $x \to \infty$

$$\frac{\sqrt{\nu \beta} \gamma}{\Gamma(1 - \beta \gamma/2)} \Phi(1, 0) x^{-\gamma \beta/2} \leq \mathbb{P}(L_T(Y) > x) \leq \frac{\sqrt{\nu \gamma \log (x)}}{\cos(\pi \beta/2) \Gamma(1 - \beta \gamma/2)} \Phi(1, 0) x^{-\gamma \beta/2}. $$

Observe that with this particular choice of RWRSB, we can obtain some more explicit constants due to the fact that we have dependence on the random variable $L_T$ instead of the linear combination $L_T - S_T$ and as a consequence we can apply Corollary 3.7, in addition to Corollary 3.10.

\[ \square \]

### 3.3.3. Continuous first-passage time for the generalized Lévy-Lorentz gas.

**Proof of Corollary 3.5.** Recall that the continuous first-passage time $T(X) = L_T(Y) - Y_T$ can be seen as the value at time $T$ of a RWRSB driven by $S$ and with scenery $\zeta^+ = 0$ and $\zeta^− = 2\zeta$. Notice that with this choice $\hat{\gamma} \equiv \hat{\gamma}_- = \hat{\gamma}_0 < \hat{\gamma}_+ = +\infty$, and hence $\rho = \min\{\rho_0, \rho_+\} = \rho_0 = \hat{\gamma}\hat{\beta}$. As a consequence, Eq. (3.36) simply becomes

$$G_{L_T(Y) - Y_T}(s) = \mathbb{E} \left[ \prod_{k \leq 0} G_{\zeta_k}(sN_T(k)) \right].$$

As before, following Theorem 3.13 we have to study different cases:

- If $\gamma \in (1, 2)$, that is $\hat{\gamma} = 1$, we obtain the following results.
  (i) If $E(|\xi_1|) < \infty$, with $\beta \in (1, 2]$ and $\hat{\beta} = 1$, then $T(X)$ is in the basin of attraction of a 1/2-stable law,

$$\mathbb{P}(T(X) > t) \sim K \cdot t^{-1/2}, \quad \text{as } t \to \infty;$$

(ii) If $\hat{\beta} \equiv \beta = 1$, then

$$\mathbb{P}(T(X) > t) \sim K \cdot \sqrt{\log(t)} t^{-1/2}, \quad \text{as } t \to \infty;$$

(iii) If $\hat{\beta} \equiv \beta \in (0, 1)$, similarly $T(X)$ is in the basin of attraction of a stable law with parameter $\hat{\beta}/2,

$$\mathbb{P}(T(X) > t) \sim K \cdot t^{-\hat{\beta}/2}, \quad \text{as } t \to \infty.$$  

- If $\gamma = \hat{\gamma} \in (0, 1)$, we get
  (i) If $E(|\xi_1|) < \infty$, that is $\beta \in (1, 2]$, we can conclude that

$$K_{\text{low}} \cdot t^{-\min\left\{ \frac{1}{\gamma}, \frac{1}{2} \right\}} \leq \mathbb{P}(T(X) > t) \leq K_{\text{up}}(t) \cdot t^{-\gamma/2}, \quad \text{as } t \to \infty,$$

with $K_{\text{up}}(t) = K_{\text{up}} \log(t)$ if $\gamma = 1/2$, constant otherwise.
(ii) If \( \hat{\beta} \equiv \beta = 1 \), we obtain
\[
K_{\text{low}} \cdot \sqrt{\log(t)} t^{-\min\{\gamma, \frac{1}{2}\}} \leq \mathbb{P}(\mathcal{T}(X) > t) \leq K_{\text{up}}(t) \cdot t^{-\gamma/2}, \quad \text{as } t \to \infty,
\]
with
\[
K_{\text{up}}(t) = \begin{cases} 
\sqrt{\log(t)} & \text{if } \gamma > 1/2, \\
\log^{3/2}(t) & \text{if } \gamma = 1/2, \\
1 & \text{if } \gamma < 1/2,
\end{cases}
\]

(iii) If \( \hat{\beta} \equiv \beta \in (0, 1) \), we get
\[
K_{\text{low}} \cdot t^{-\min\{\gamma, \frac{\hat{\beta}}{2}\}} \leq \mathbb{P}(\mathcal{T}(X) > t) \leq K_{\text{up}} \cdot t^{-\gamma\hat{\beta}/2}, \quad \text{as } t \to \infty.
\]
Notice that this specific choice of scenery does not require to restrict the hypothesis about the random medium to stable random variables, in contrast to the general statement of Theorem 3.13.

\[\square\]

**Appendix A. Lemma**

**Lemma A.1.** Let
\[
P(z, t) = \sum_{n=0}^{\infty} z^n p_n(t), \quad Q(z, t) = \sum_{n=0}^{\infty} z^n q_n(t),
\]
and
\[
P^*(z, t) = \sum_{n=0}^{\infty} z^n p^*_n(t), \quad Q^*(z, t) = \sum_{n=0}^{\infty} z^n q^*_n(t),
\]
where \( p_0(t) \equiv q_0(t) \equiv p^*_0(t) \equiv q^*_0(t) \equiv 1 \); and for \( n \geq 1 \), \( p_n \) and \( p^*_n \) as functions of \( t \) are Fourier transforms of measures with support in \( (0, \infty) \); \( q_n \) and \( q^*_n \) as functions of \( t \) are Fourier transforms of measures in \( (-\infty, 0] \). Suppose that for some \( z_0 > 0 \) the four power series converge for \( z \) in \( (0, z_0) \) and all real \( t \), and the identity
\[
P(z, t)Q^*(z, t) = P^*(z, t)Q(z, t)
\]
holds there. Then
\[
P = P^*, \quad Q = Q^*.
\]

Proof. see Section 8.4.1 of [10]. \[\square\]

**Appendix B. Probabilistic tools**

**B.1. Characteristic/generating function of a random variable raised to a power.**

Let \( X \) be a non-negative random variable and let us consider \( Y := X^\gamma, \gamma > 0 \). We want to determine the characteristic function of \( Y \), keeping in mind that the same reasoning also applies to the generating function. By means of Tauberian theorems, we know that
\[
\phi_X(s) = \mathbb{E}[e^{isX}] = 1 - ce^{-i \frac{\pi}{2} \beta s^\beta + o(s^\beta)} \implies \mathbb{P}(X > x) \sim \frac{c}{\Gamma(1 - \beta)} x^{-\beta}.
\]

Therefore, we immediately obtain
\[
\mathbb{P}(Y > y) = \mathbb{P}(X > y^{1/\gamma}) \sim \frac{c}{\Gamma(1 - \beta)} y^{-\beta/\gamma}.
\]
By observing that $Y$ has a finite mean if $\gamma < \beta$, then we can conclude that

$$
\phi_Y(s) = \begin{cases} 
1 + i \mathbb{E}[Y] s + o(s) & \text{if } \gamma < \beta, \\
1 - c \bar{\Gamma} e^{-\frac{\pi}{2} s^2 / \gamma} + o(s^{\beta / \gamma}) & \text{if } \gamma \geq \beta,
\end{cases}
$$

where $\bar{\Gamma} := \Gamma(1 - \beta / \gamma) / \Gamma(1 - \beta)$.

### B.2. Estimates on joint characteristic/generating functions.

**Lemma B.1.** Assume that, for $k \in \{1, 2\}$, $Z_k(s)$ is a complex random variable defined by $Z_k(s) := e^{isX_k}$ or $Z_k(s) := e^{-sX_k}$, whose average therefore corresponds to the characteristic or generating function of a real or non-negative random variable $X_k$.

If $|\mathbb{E}[Z_k(s)]| = 1 - c_k s^{\gamma_k} + o(s^{\gamma_k})$, with $\gamma_k \in (0, 1)$, $c_k \in \mathbb{R}^+$ and $s \to 0^+$, then by defining $\gamma := \min\{\gamma_1, \gamma_2\}$ and assuming $c_1 \neq c_2$ if $\gamma_1 = \gamma_2$ and $\Im(Z_1(s)) \neq 0$, we get

$$1 - k_+ s^{\gamma_1} + o(s^{\gamma_1}) \leq |\mathbb{E}[Z_1 Z_2(s)]| \leq 1 - k_- s^{\gamma_2} + o(s^{\gamma_2}),$$

where the positive constants $k_+ \geq k_-$, matching if $\gamma_1 \neq \gamma_2$, are functions of $c_1$, $c_2$.

**Proof.** To easy the notation, from now on we will drop the dependence on $s$ of $Z_k$’s. By definition

$$\mathbb{E}[Z_1 Z_2] = \mathbb{E}[Z_1] \mathbb{E}[Z_2] + \mathbb{Cov}(Z_1, Z_2),$$

where $\bar{Z}_2$ denotes the complex conjugate of $Z_2$ and

$$\mathbb{Cov}(Z_1, Z_2) := \mathbb{E}[(Z_1 - \mathbb{E}[Z_1])(Z_2 - \mathbb{E}[Z_2])].$$

From the Cauchy-Schwarz inequality, we have

$$|\mathbb{Cov}(Z_1, \bar{Z}_2)| \leq \sqrt{\mathbb{Var}(Z_1) \mathbb{Var}(Z_2)},$$

where

$$\mathbb{Var}(Z_k) := \mathbb{E}[|Z - \mathbb{E}[Z]|^2] = \mathbb{E}[|Z_k|^2] - |\mathbb{E}[Z_k]|^2.$$

In particular, it holds that

$$|\mathbb{E}[Z_1 Z_2]| \leq |\mathbb{E}[Z_1] |\mathbb{E}[Z_2]| + |\mathbb{Cov}(Z_1, \bar{Z}_2)| \leq |\mathbb{E}[Z_1]| |\mathbb{E}[Z_2]| + \sqrt{\mathbb{Var}(Z_1) \mathbb{Var}(Z_2)},$$

$$|\mathbb{E}[Z_1 Z_2]| \geq |\mathbb{E}[Z_1]| |\mathbb{E}[Z_2]| - |\mathbb{Cov}(Z_1, \bar{Z}_2)| \geq |\mathbb{E}[Z_1]| |\mathbb{E}[Z_2]| - \sqrt{\mathbb{Var}(Z_1) \mathbb{Var}(Z_2)}.$$

Since by assumptions

$$\mathbb{Var}(Z_k) = \mathbb{E}[|Z_k|^2] - [1 - c_k s^{\gamma_k} + o(s^{\gamma_k})]^2,$$

to determine the behavior of the variance, as $s \to 0^+$, we have to consider two possible cases:

- If $Z_k(s) = e^{-sX_k}$, then
  $$\mathbb{E}[|Z_k|^2] = \mathbb{E}[Z_k^2] = \mathbb{E}[e^{-2sX_k}] = 1 - c_k (2s)^{\gamma_k} + o(s^{\gamma_k}),$$
  and hence $\mathbb{Var}(Z_k) = (2 - 2^{\gamma_k}) c_k s^{\gamma_k} + o(s^{\gamma_k})$, with $2 - 2^{\gamma_k} \in (0, 1)$;
- If $Z_k(s) = e^{isX_k}$, then $\mathbb{E}[|Z_k|^2] = 1$, which implies $\mathbb{Var}(Z_k) = 2c_k s^{\gamma_k} + o(s^{\gamma_k})$.

Let us stress that even if $\mathbb{E}[Z_1]$, $\mathbb{E}[Z_2]$ are both characteristic functions, we get

$$k_- = c_1 + c_2 - 2 \sqrt{c_1 c_2} \geq 0,$$

with $k_- > 0$ whenever $c_1 \neq c_2$. The statement is therefore proved. \qed
Notice that Lemma B.1 can be easily generalized to a limiting case that is useful for the proof of Lemma 3.11 when $\beta = 2$. Indeed, assuming that $\gamma_1 < 1 \leq \gamma_2$ and $E[Z_2]$ is the characteristic function of a non-negative random variable $X_2$, then

$$|E[Z_1Z_2]| \sim |E[Z_1]| = 1 - c_1 s^{\gamma_1} + o(s^{\gamma_1})$$

still holds true.

Moreover, if $E[X_k] = 0$ and we focus on the characteristic functions, we can extend the result to the range $\gamma_k \in (0,2]$.

B.3. Tail asymptotic of the product of independent random variables.

Lemma B.2. Let $V, W$ be non-negative independent random variables characterized by the asymptotic tails

$$P[V > v] \sim c_V \cdot [\log(v)]^{k_V} v^{-\gamma_V}, \quad P[W > w] \sim c_W \cdot w^{-\gamma_W}, \quad \text{as } v, w \to +\infty,$$

with $\gamma_V, \gamma_W \in (0,2)$, $k_V \geq 0$ and $c_V, c_W > 0$. It holds that [13]

(A) If $\gamma_V < \gamma_W$, then

$$P[V \cdot W > z] \sim c_V \cdot c_W \cdot [\log(z)]^{k_V} z^{-\gamma_V}, \quad \text{as } z \to +\infty;$$

(B) If $\gamma_V = \gamma_W =: \gamma$, then

$$P[V \cdot W > z] \sim \frac{\gamma c_V c_W}{k_V + 1} \cdot [\log(z)]^{1+k_V} z^{-\gamma}, \quad \text{as } z \to +\infty.$$
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