Energy conservation and Jarzynski equality are incompatible for quantum work

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Measuring the fluctuations of work in coherent quantum systems is notoriously problematic. Aiming to reveal the ultimate source of these problems, we demand of work measurement schemes the sheer minimum and see if those demands can be met at all. We require (A) energy conservation for arbitrary initial states of the system and (B) the Jarzynski equality for thermal initial states. By energy conservation we mean that the average work must be equal to the difference of initial and final average energies, and that untouched systems must exchange \textit{deterministically} zero work. Requirement (B) encapsulates the second law of thermodynamics and the quantum–classical correspondence principle. We prove that work measurement schemes that do not depend on the system’s initial state satisfy (B) if and only if they coincide with the famous two-point measurement scheme, thereby establishing that state-independent schemes cannot simultaneously satisfy (A) and (B). Expanding to the realm of state-dependent schemes allows for more compatibility between (A) and (B). However, merely requiring the state-dependence to be continuous still effectively excludes the coexistence of (A) and (B), leaving the theoretical possibility open for only a narrow class of exotic schemes.

I. INTRODUCTION

Work is a central notion in both mechanics and thermodynamics, being essentially the only touching point between the two theories. When information about the full statistics of work is required, which is especially relevant for small systems, classical mechanics answers by means of trajectories [1, 2], leaving no prescription for accessing the statistics of work in the quantum regime [3–11]. This uncertainty has bred a number of approaches towards measuring work in the quantum regime [3–7, 9, 11–22]. Nevertheless, they all become problematic in certain regimes [5–8, 10, 23].

In this paper, we show that the fundamental source of all these problems is the incompatibility of energy conservation and Jarzynski’s fluctuation relation for work [2]. We prove that essentially no physically meaningful scheme measuring quantum work can simultaneously satisfy (A) energy conservation for all states and (B) Jarzynski equality (JE) for all thermal states.

Since the JE necessarily holds in classical mechanics, requiring it also in the quantum regime is a way of imposing a weak form of quantum–classical correspondence principle on the distribution of work, without going into the specifics of the quantum–classical transition. Thus, elegantly incorporating both the second law of thermodynamics and the correspondence principle, the JE can serve as a high-level filter through which physically meaningful definitions of work fluctuations ought to be able to pass.

Undeniably, energy conservation is another such filter, and that virtually no definition of work fluctuations can pass through both, means that quantum work cannot be thought of as a classical random variable. In our definition, a measurement scheme obeys energy conservation if the work statistics measured by the scheme reflect energy conservation for the unmeasured system. Specifically, the average work performed on an unmeasured thermally isolated system is equal to the difference between the final and initial average energies of the system, and a proper measurement of work must output a random variable the first moment of which coincides with the average work on the unmeasured system. We call this condition (A\textsubscript{1}). Another simple consequence of energy conservation is that, if a system is untouched, then zero work with probability 1 is performed on it. Therefore, a correct measurement of work must output exactly that statistics; this is condition (A\textsubscript{2}). In this paper, we will consider only these two aspects of the broader energy conservation condition (A).

Our analysis is divided into two parts. First, we focus on work measuring schemes that do not depend on the initial state of the system. We prove that, if a state-independent scheme satisfies the JE for all thermal initial states (we call the set of all such schemes the JE class), then the average of the work measured by such a scheme cannot coincide with the unmeasured average work whenever the system’s initial state has coherences in the energy eigenbasis (i.e., there is a conflict with A\textsubscript{1}). One encounters the same kind of conflict with A\textsubscript{1} when using the famous two-point measurement (TPM) scheme [3, 4, 12, 24], where the first energy measurement erases the coherences in the state, thereby distorting the measured average [5, 7, 8]. Interestingly, the similarity between the TPM scheme and the JE class goes deeper: the schemes in the JE class that yield the correct average for initial thermal states produce the same statistics as the TPM scheme; this relation can be denoted as JE ⇒ TPM. The reverse, namely, that work statistics produced by the TPM scheme satisfy the JE (i.e., JE ⇔ TPM), has been known from the very conception of the scheme [3, 4]; which was in fact what motivated the experimental implementations of the TPM scheme [25–27]. Thus, JE ⇔ TPM for state-independent schemes.

In the second part, we ask whether (A) and (B) can be reconciled by using measurements of work that are allowed to depend on the system’s initial state. Considering physically meaningful only schemes that depend on
the state continuously, we show that $\mathfrak{A}_1$ and $\mathfrak{B}$ can be made compatible. However, we prove that, when requiring energy conservation to be satisfied to a fuller extent, namely, imposing $\mathfrak{A}_2$ alongside $\mathfrak{A}_1$, the compatibility with $\mathfrak{B}$ breaks down for almost all reasonable state-dependent schemes, leaving the theoretical possibility open for only an exotic class of schemes. Interestingly, simply requiring $\mathfrak{A}_1$ and $\mathfrak{A}_2$ to hold at the same time already severely restricts the class of acceptable schemes.

II. FORMAL SETUP

Throughout this paper, we will focus on thermally isolated systems. This is the most general setting, in the sense that any open system is a part of a larger closed system consisting of the system itself and all the external systems with which it is correlated, interacts, and will interact during the process under study [28, 29]; this picture seems to incorporate also the process of quantum measurement [30–33]. For such systems, a process is described by a time-dependent Hamiltonian $H(t)$ that at the beginning of the process ($t = t_{in}$) has some value $H$, and has some value $H'$ at the end ($t = t_{fin}$), generating the unitary evolution operator $U = \exp\[−i \int_{t_{in}}^{t_{fin}} dt H(t)\]$. 

Suppose the scheme one uses to access the fluctuations of work does not depend on the system’s initial state. Take two different ensembles of the same system, represented by density matrices $ρ_1$ and $ρ_2$, make them undergo the same process, and denote the measured probability distributions of work outputs, $W$, by $p_W^{(1)}$ and $p_W^{(2)}$ [34]. Then, since the scheme is state-independent, for any mixture $ρ_{λ} = λρ_1 + (1 − λ)ρ_2$, with $0 < λ < 1$, 

$$p_W^{(λ)} = λp_W^{(1)} + (1 − λ)p_W^{(2)},$$  

(1)

since, in the mixed ensemble, with probability $λ$ one lands at the sub-ensemble described by $p_W^{(1)}$, and likewise for $p_W^{(2)}$. It was proven in Ref. [8] that this condition is equivalent to the existence of a state-independent (yet necessarily process-dependent) positive-operator-valued measure (POVM [35]), $\{M_W\}_W$, such that 

$$p_W = \text{Tr}(M_W ρ).$$  

(2)

Saying, in other words, that any random variable representing work is nothing but an outcome of a quantum measurement, Eq. (2) not only describes the fluctuations of work, but also simultaneously constitutes a means to observing these fluctuations.

Accordingly, whenever we refer to a state-dependent scheme, we understand a POVM at least some of the elements of which depend on the initial state of the system.

Thus, $\mathfrak{A}_1$ means that 

$$\sum_W W \text{Tr}(M_W ρ) = \text{Tr}(UρU^\dagger H') − \text{Tr}(ρH) = \text{Tr}(\dot{W})$$  

(3)

is required to hold for all $ρ$’s. Here, $\text{Tr}(ρH)$ and $\text{Tr}(UρU^\dagger H')$ are, respectively, the initial and final average energies, and 

$$\dot{W} = U^\dagger H' U − H$$  

(4)

is the so-called operator of work (OW) [5].

Requirement $\mathfrak{B}$ reads as 

$$\sum_W e^{−βW} \text{Tr}(M_W τ_β) = e^{−β(F_β[H') − F_β[H]),}$$  

(5)

for any $β > 0$, where $τ_β = \frac{1}{Z_β[H]} e^{−βH}$ is the thermal state, with $Z_β[H] = \text{Tr} e^{−βH}$ and $F_β[H] = −β^{-1} \ln Z_β[H]$ being, respectively, the partition function and equilibrium free energy.

For the sake of the forthcoming discussion, let us briefly review the two paradigmatic work measurement schemes, TPM and OW, in the light of requirements $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{B}$.

In the TPM scheme [3, 4], one first measures the energy at the beginning, obtaining the outcome $E_a$ and post-measurement state $P_aρP_a/\text{Tr}(ρP_a)$ with probability $\text{Tr}(ρP_a)$, where $E_a$ and $P_a$ are, respectively, the eigenvalues and eigoprojectors of $H (H = \sum_a E_a P_a)$ and $ρ$ is the initial state of the system. Then, the unitary process is implemented and the energy of the system is measured again. This yields the outcome $E_{a}'$ with conditional probability $\text{Tr}\left[U \frac{P_aρP_a}{\text{Tr}(ρP_a)} U^\dagger P_{a}'\right]$, where $E_{a}'$ and $P_{a}'$ are the eigenvalues and eigoprojectors of $H'$. Thus, according to the scheme, the outcomes of work and their probabilities are 

$$W_{a k} = E_{a}' − E_a, \quad P_{a k}^{\text{TPM}} = \text{Tr}(ρM_{a k}^{\text{TPM}}),$$  

(6)

where 

$$M_{a k}^{\text{TPM}} = P_a U^\dagger P_{k}' U P_a.$$  

(7)

Noting that $[M_{a k}^{\text{TPM}}, H] = 0$ for all $a$ and $k$, we see that $\sum W_{a k} M_{a k}^{\text{TPM}}$ commutes with $H$ for any $H$, $H'$, and $U$. Whereas there exist such $H$, $H'$, and $U$ for which $[U^\dagger H' U, H] \neq 0$ and hence $\dot{W}$ does not commute with $H$, and therefore $\dot{W}$ cannot be equal to $\sum W_{a k} M_{a k}^{\text{TPM}}$.

In such a case, there will exist a $ρ$ for which Eq. (3) is violated, which shows that the TPM scheme is incompatible with $\mathfrak{A}_1$. Nonetheless, we note that the TPM scheme always satisfies $\mathfrak{A}_2$. Moreover, in the classical limit, where all the commutators go to zero, the discrepancy with $\mathfrak{A}_1$ also vanishes (see Refs. [36, 37] for more details about the TPM scheme’s classical limit in continuous-variable systems).

In the OW scheme [5], the statistics of work is identified with the measurement statistics of the operator $\dot{W}$. Therefore, by construction, the scheme satisfies both requirements $\mathfrak{A}_1$ and $\mathfrak{A}_2$. However, $\mathfrak{B}$ is not satisfied as, in general, 

$$\langle e^{−β\dot{W}} \rangle_{\text{OW}} := \text{Tr}(τ_β e^{−β\dot{W}}) \geq e^{−β(F_β[H'] − F_β[H])}.$$
When the system's Hilbert space is finite-dimensional, the inequality is strict when (and only when) $\beta[W, H] = \beta[U^1H'U, H] \neq 0$, which straightforwardly follows from the Golden–Thompson inequality [38]. However, the OW scheme satisfies $\mathfrak{B}$ in the classical limit. In finite-dimensional spaces, this is a simple consequence of all the commutators vanishing in the classical limit [5]. In continuous-variable systems, issues with continuity may arise, so is not mathematically guaranteed anymore (see discussion in Appendix A). Nonetheless, for a harmonic oscillator, by performing an explicit calculation in Appendix A, we show that the OW scheme satisfies $\mathfrak{B}$ in the $h \to 0$ limit.

That $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{B}$ become compatible in the classical limit for the TPM and OW schemes (and in fact for some other schemes too [11, 39]), is yet another indication that the classical approach to determining the statistics of work is unlikely to provide useful guidance for the quantum regime.

III. JARZYNSKI EQUATION CLASS

To be able to characterize the JE class, namely, the set of state-independent schemes each of which satisfies the JE for all thermal initial states, let us rewrite Eq. (5) as

$$\sum_W \text{Tr}(M_W e^{-\beta H'} e^{-\beta W}) = \text{Tr} e^{-\beta H'} \forall \beta, \quad (8)$$

where $\{M_W\}_W$ is a POVM that does not depend on the state of the system, but may depend on the process (i.e., $H, H',$ and $U$).

Recalling the eigenresolutions $H = \sum_{a=1}^{N} E_a P_a$ and $H' = \sum_{k=1}^{K} E_k' P_k'$ and introducing the degeneracies

$$g_a := \text{Tr} P_a \quad \text{and} \quad g_k' = \text{Tr} P_k', \quad (9)$$

we obtain from Eq. (8) that

$$\sum_{W,a} \text{Tr}(M_W P_a) e^{-\beta(E_a + W)} = \sum_{k=1}^{K} (g_k' e^{-\beta E_k'}) \quad (10)$$

must hold for all $\beta$'s. Whence, using the expansion $e^{x} = \sum_{N=0}^{\infty} \frac{x^{N}}{N!}$, we find that

$$\sum_{W,a} \text{Tr}(M_W P_a) (E_a + W)^N = \sum_{k=1}^{K} g_k' (E_k')^N \quad (11)$$

for all $N \in \mathbb{N}$.

Keeping in mind that everything above is invariant under a global constant energy shift, we choose the ground state of $H'$ to have zero energy. Moreover, since the eigenresolutions $H' = \sum_{k=1}^{K} E_k' P_k'$ already accounts for possible degeneracies, all values of $E_k'$ are distinct. Therefore, the order of the eigenvectors of $H'$ can be chosen such that $0 = E_1' < E_2' \cdots < E_K'$. Let us now take $N = 2L, L \in \mathbb{N}$, and divide Eq. (11) by $(E_K')^N$. Then, as $L \to \infty$, the right-hand side will converge to $g_k'$. Now, if, for all $a$ and $W$, $|E_a + W| < E_K'$, then the left-hand side will converge to 0, which cannot be. Likewise, there cannot exist such $a$ and $W$ for which $|E_a + W| > E_K'$, because otherwise the left-hand side would diverge. Therefore, there have to exist at least one pair of $a$ and $W$ such that $|E_a + W| = E_K'$, and $|E_b + W'| \leq |E_a + W|$ for all other pairs $b$ and $W'$. And since the equality between the left- and right-hand sides of Eq. (11) is maintained in the $L \to \infty$ limit, it also holds that

$$\sum_{|E_a + W| = E_K'} \text{Tr}(M_W P_a) = g_k'. \quad (12)$$

Moreover, if there exist such $b$ and $W'$ that $E_b + W' = -E_K'$, then necessarily $\text{Tr}(M_W P_b) = 0$. Indeed, taking $N = 2L + 1$, dividing Eq. (11) by $(E_K')^N$, and taking the $L \to \infty$ limit, we will obtain the same $g_k'$ on the right-hand side, but $\text{Tr}(M_W P_b)$ will enter the left-hand side with a negative sign, which means that equality between the left and right-hand sides is possible only if $\text{Tr}(M_W P_b) = 0$. In other words, for all $N \in \mathbb{N}$, $g_k' (E_k')^N$ is equal to the sum of all the terms on the left-hand side with maximal $E_a + W$. Thus, for any $N$, we can eliminate those terms from both sides of Eq. (11) and reiterate the arguments above. Doing so $K$ times, we will conclude that the set of outcomes of work coincides with energy differences,

$$\{W\}_W = \{E_k' - E_a\}_{a=1}^{K} \quad (12)$$

so that we can simply label the outcomes as $W_{ab} = E_k' - E_a$ and the corresponding POVM elements as $M_{ab}$.

Furthermore,

$$\sum_{a,b,m} \text{Tr}(M_{bm} P_a) = g_k' \quad (13)$$

and

$$\text{Tr}(M_{bm} P_a) = 0 \text{ if } E_m' - E_a + E_a \notin \{E_k'\}_{k=1}^{K}. \quad (14)$$

A. JE class does not satisfy $\mathfrak{A}_1$

For a state-independent scheme to satisfy $\mathfrak{A}_1$, Eq. (3) must be satisfied for any state $\rho$, which means that $\sum_W W \text{Tr}(M_W) = W$ must hold.

On the other hand, the conditions in Eq. (14) rather strongly restrict the set of eigensubspaces of $H$ among which the matrices $M_{ab}$ are allowed to have coherences. So much so that, for any nontrivial $H$ and $H'$, there exists a $U$ such that $\hat{W}$ has coherences where $\sum_W W \text{Tr}(M_W)$ cannot. This means that the schemes in the JE class are incompatible with $\mathfrak{A}_1$.

For brevity of the presentation, let us illustrate this fact for the case of nondegenerate set of work outcomes, namely, when $E_k' - E_a = E_m' - E_a$ necessitates $k = m$ and $a = b$. Note that this case is the generic one—those
$H$ and $H'$ for which the set of energy differences has degeneracies constitute a $0$-measure subset in the space of all $H$’s and $H'$’s. For this configuration, Eq. (14) simply states that $\text{Tr}(M_{ak}P_k) = 0$ when $b \neq a$, which means that, for any $k$, $M_{ak}$ belongs to the $a$’th eigensubspace of $H$; in other words,

$$M_{ak} = P_a M_{ak} P_k.$$  \hfill (15)

With this taken into account, Eq. (13) simplifies to

$$\sum_a \text{Tr}(M_{ak}) = g_k.'$$  \hfill (16)

Now, in view of Eq. (15), $[M_{ak}, H] = 0$ for any $a$ and $k$, and therefore $\sum W W \text{Tr}(M_W)$ commutes with $H$. Hence, $\sum W W \text{Tr}(M_W)$ cannot be equal to $\bar{W}$ whenever $[U \dagger H' U, H] \neq 0$. Note the similarity of this argumentation to that for the TPM scheme below Eq. (7).

### B. JE class and the TPM scheme

As was noted in Sec. I, the JE class contains the TPM scheme, which is expressed in the fact that Eqs. (13) and (14) [and even the more specific Eqs. (15) and (16)] allow for more general POVMs. However, for the (generic) case of nondegenerate $H$ and nondegenerate set of outcomes described in Sec. IIIA, the schemes in the JE class that satisfy Eq. (3) for thermal states [40] are equivalent to the TPM scheme.

To see that, let us, taking into account Eqs. (12) and (15), rewrite Eq. (3) with $\rho = \tau_\beta$ as

$$\sum_a e^{-\beta E_a} \left[ \sum_k E_k' \text{Tr}(P_a U \dagger P_k' U) - g_a E_a \right]$$

$$= \sum_{a,k} e^{-\beta E_a} (E_k' - E_a) \text{Tr} M_{ak},$$

which has to hold for any $\beta$. Keeping in mind that no two $E_a$’s are the same, and setting $\beta = 0, 1, \ldots, A - 1$, we will arrive at a linear system of equations with the coefficient matrix being a Vandermonde matrix [41] with a non-zero determinant. Hence, for any $a$,

$$\sum_k (E_k' - E_a) \text{Tr}(P_a U \dagger P_k' U) = \sum_k (E_k' - E_a) \text{Tr} M_{ak},$$

where we have noted that $\sum_k \text{Tr}(P_a U \dagger P_k' U) = g_a$.

At this point, we assume that the operators $M_{ak}$ do not depend on $E_k'$’s within a small neighborhood. This assumption is natural for two reasons. First, the unitary evolution operator, $U = \exp[-i \int_{t_n} \text{d}H(t)]$ does not change when values of $H(s)$ are changed on a measure-zero set, and since changing the eigenvalues of $H' = H(t_n)$ is such an operation, $U$ remains intact. Second, since $0 = E_1' < E_2' < \cdots < E'_K$ and $K$ is finite, there exists an $\epsilon > 0$ such that the $\epsilon$-neighborhoods of $E_k'$’s are non-overlapping; we will assume $E_k'$-independence in these neighborhoods. Now, before the last-moment change in the eigenvalues of $H'$, the process runs without “knowing” about the change, so if $M_{ak}$’s were going to measure probabilities of work outcomes for $E_k'$’s, then a small variation that preserves the order of $E_k'$’s should not impact the measurement probabilities and hence the operators.

With this assumption, we can differentiate Eq. (17) with respect to $E_k'$’s, with all other parameters fixed, to find that

$$\text{Tr} M_{ak} = \text{Tr}(P_a U \dagger P_k' U)$$

(18)

for all $a$’s and $k$’s.

When the initial Hamiltonian has no degeneracies, i.e., $P_a = |a\rangle\langle a|$ for all $a$’s, then Eq. (15) implies that $M_{ak} \propto |a\rangle\langle a|$, and Eq. (18) shows that

$$M_{ak} = |a\rangle\langle a| U \dagger P_k' U |a\rangle\langle a| = M^{\text{TPM}}_{ak},$$

(19)

where the second equality is due to Eq. (7). Hence, for nondegenerate $H$’s, the JE class consists of the TPM scheme only.

If $H$ is degenerate, the schemes in the JE class are guaranteed to coincide with the TPM scheme only on diagonal initial states of the form $\rho = \sum a \rho_a P_a$:

$$\text{Tr}(P_a M_{ak}) = \text{Tr}(P_a P_b M_{ak}) = \rho_a \text{Tr}(M_{ak})$$

$$= \rho_a \text{Tr}(P_a U \dagger P_k' U) = \text{Tr}(P_a P_b U \dagger P_k' U) = P_k' \text{TPM},$$

where we used Eqs. (15), (18), and (6). However, whenever $\rho$ is not proportional to the identity operator in one of the eigensubspaces of $H$, there will exist a POVM in the JE class [i.e., one satisfying Eqs. (15) and (18)] for which $\text{Tr}(P_a M_{ak}) \neq P_k' \text{TPM}$. So, for degenerate $H$’s, the JE class contains schemes other than the TPM scheme.

We emphasize that the $H$’s and $H'$’s for which the JE class is larger than the TPM scheme comprise a 0-measure subset of the set of all possible $H$’s and $H'$’s. Which in particular means that, for any $H$ and $H'$ yielding a degenerate set of energy difference, there exist infinitesimal perturbations of $H$ and $H'$ that remove the degeneracy. Therefore, if we ask which schemes satisfy the JE and average work condition for thermal initial states for all processes at least in an arbitrarily small neighborhood of a given process, then the TPM scheme will be the only such scheme. In this sense, the JE class is equivalent to the TPM scheme.

### IV. STATE-DEPENDENT SCHEMES

Having excluded the possibility of simultaneously satisfying the conditions $\mathfrak{A}_1$ and $\mathfrak{B}$ with work measuring schemes that are limited to being state-independent, let us explore the opportunities when no such limitation is posed. Namely, when the work-measuring POVM is allowed to depend not only on the process, but also on the system’s initial state. On the one hand, this considerably increases the set of POVMs we are now allowed
to choose from. On the other hand, such schemes are inconvenient from the practical standpoint, as, in order to implement such a scheme, one has to acquire additional knowledge about incoming states and readjust the measurement setup accordingly.

Expectedly, this additional freedom increases the degree of compatibility of $\mathfrak{A}_1$ and $\mathfrak{B}$, and even $\mathfrak{A}_2$. In fact, by allowing for completely arbitrary state-dependent schemes, we can resolve the inconsistency problem altogether, for example, by implementing the following protocol: when $[\rho, H] = 0$, apply the TPM scheme; when $[\rho, H] \neq 0$, apply the OW scheme. Formally, this satisfies $\mathfrak{A}_1$, $\mathfrak{B}$, and $\mathfrak{A}_2$ simultaneously. However, this scheme is highly discontinuous: even for an infinitesimal deviation from an incoherent state, the statistics experiences a dramatic jump from having AK outcomes to d outcomes ($d$ is the Hilbert space dimension of the system), accompanied with a radical change of the probability distribution itself. We deem such discontinuous situations pathological, especially because any measurement has a finite resolution (e.g., of how diagonal a state is).

Therefore, we require the measurement schemes to depend on the state continuously, at least within a certain convex subset of all states. More specifically, let us consider a closed $\epsilon$-ball in the state space around the infinite-temperature state $\tau_0$ (which coincides with the maximally mixed state, $1/d$) as $B_\epsilon := \{\sigma : \|\sigma - \tau_0\| \leq \epsilon\}$, where $\epsilon > 0$ and $\|\cdot\|$ is the standard operator norm (in $\ell_2$ space) [41], and choose $\epsilon > 0$ such that all $\rho$’s in $B_\epsilon$ are positive-definite (such an $\epsilon$ always exists [42]). Our main continuity assumption is that all outcomes $W$ and POVM elements $M_W$ (see Sec. II for the definitions) are continuous functions of $\rho$ in $B_\epsilon$. As we will prove below, this requirement is restrictive enough to virtually exclude the simultaneous compatibility of $\mathfrak{A}_1$, $\mathfrak{B}$, and $\mathfrak{A}_2$. In various parts of the proof, we will also require the $W$’s and $M_W$’s to be once or twice differentiable with respect to $U$; these assumptions will be explicitly stipulated at relevant places.

Importantly, while not being able to provide full compatibility of $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{B}$, continuously state-dependent schemes do make those conditions more compatible. Indeed, with state-independent schemes, we could simultaneously satisfy $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (the OW scheme) as well as $\mathfrak{A}_2$ and $\mathfrak{B}$ (the TPM scheme), but not $\mathfrak{A}_1$ and $\mathfrak{B}$. Now, with the additional freedom, in Appendix C, by combining the ordinary (“forward”) TPM scheme with what we call “backward” TPM scheme, we construct a state-dependent measurement that simultaneously satisfies $\mathfrak{A}_1$ and $\mathfrak{B}$ (but not $\mathfrak{A}_2$). Thus, all the requirements are pairwise compatible when continuous state-dependent POVMs are allowed.

### A. Proof of incompatibility

Since we are going to invoke $\mathfrak{A}_2$, let us consider cyclic Hamiltonian processes which are close to being trivial, i.e., the unitary evolution operator they generate can be written as $U = e^{-i\tau H}$, where $\tau \ll 1$ and $H$ is some Hermitian operator with finite operator norm. For each state $\rho$, let us define $O_{\rho}$ to be the set of outcomes $W$ that are either 0 or tend to 0 as $x \to 0$. The rest of the outcomes will tend to non-zero values in the same limit; we denote the set of those outcomes by $N_{\rho}$. Then, the requirement $\mathfrak{A}_2$ means that, for any $\rho$, $\sum_{W \in O_{\rho}} p_W \to 1$, and thus $\sum_{W \in N_{\rho}} p_W \to 0$, in the $x \to 0$ limit. When all the eigenvalues of $\rho$ are strictly positive, which is the case for all the states in $B_\epsilon$, this entails

$$\sum_{W \in N_\rho} M_W \to 0 \quad \text{and} \quad \sum_{W \in O_\rho} M_W \to 1. \quad (20)$$

We emphasize that, like $M_W$’s and the sets $O_\rho$ and $N_\rho$, also the outcomes $W$ are in general functions of $H$, $H'$, $h$, $x$, and $\rho$.

As per $\mathfrak{A}_1$, we have that $\langle W \rangle = \text{Tr}(\hat{W}\rho)$, which, due to the Baker–Hausdorff lemma [43] stating that $e^{A \cdot B} e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \cdots$, amounts to

$$\langle W \rangle = x \text{Tr}(\rho \hat{w}_0) + i x^2 \text{Tr}(\rho [h, \hat{w}_0]) + O(x^3) \quad (21)$$

for $x \ll 1$, where we have introduced the operator

$$\hat{w}_0 := [i[h, H]]. \quad (22)$$

Now, let us assume that, for any $h$ and $\forall \rho \in B_\epsilon$, $M_W$’s with $W \in N_\rho$ are differentiable, at least once, with respect to $x$, in the vicinity of $x = 0$. In such a case, for $W \in N_\rho$, $M_W$ can be Taylor-expanded up to the first order: $M_W = x m_W + o(x)$, where the last term is the little-o of $x$, as per the standard asymptotic notation. We immediately see that $m_W = 0$ must hold, because otherwise, for sufficiently small $x$’s, $M_W$ would change its sign when $x \to -x$, whereas $M_W$ must always remain non-negative. Therefore, while maintaining non-negativity,

$$M_W = o(x) \quad (23)$$

whenever $W \in N_\rho$ [44].

Furthermore, let us assume that, for all $\rho$’s in $B_\epsilon$, all $W$’s in $O_\rho$ are differentiable at least once around $x = 0$. By the definition of $O_{\rho}$, this means that

$$W = x \hat{W} + o(x). \quad (24)$$

Lastly, for each $\rho$, let us remove from $O_{\rho}$ (and add to $N_{\rho}$) all those $W \in O_{\rho}$ for which $M_W \to 0$ as $x \to 0$; in other words, introduce the set

$$D_{\rho} = \{W : W \in O_{\rho} \text{ AND } \lim_{x \to 0} M_W = +0\}$$

and form $O'_{\rho} = O_{\rho}/D_{\rho}$ and $N'_{\rho} = N_{\rho} \cup D_{\rho}$. Note that the argument leading to Eq. (23) is universal, therefore, all POVM elements in $N'_{\rho}$ are $o(x)$.

Picking a $\rho$ and an $h$ such that $\text{Tr}(\rho [h, H]) \neq 0$ (which, as it is easy to see, can be done for any $\rho$ in $B_\epsilon$ such that $[\rho, H] \neq 0$), we can categorize all possible POVMs satisfying $\mathfrak{A}_2$ into the two classes: (i) the set $O'_\rho$ is either empty or each $\hat{W}$ in it is zero and (ii) the set $O'_\rho$ is not empty and at least one of the $\hat{W}$’s in it is not zero.
1. $\mathcal{O}_\rho'$ is either empty or each $\tilde{W}$ in it is zero

In this case, taking Eqs. (23) and (24) into account, we find that $A_2$ necessitates

$$
\mathbb{E}[W] = \sum_{W \in \mathcal{O}_\rho'} W \text{Tr}(w M_W) + \sum_{W \in \mathcal{N}_\rho'} W \text{Tr}(w M_W) = o(x).
$$

Whereas, with our choice of $h$, according to Eq. (21), $A_1$ enforces $\langle W \rangle \propto x$. We thus encounter a clear contradiction between $A_1$ and $A_2$.

Note that $\text{Tr}(\rho[h, H]) \neq 0$ holds only when $[\rho, H] \neq 0$, i.e., when the initial state has coherences in the eigenbasis of the initial Hamiltonian. Incidentally, $\text{Tr}(\rho[h, H]) \neq 0$ necessitates $[\rho, h] \neq 0$ and $[h, H] \neq 0$, meaning that the unitary has to act nontrivially on both the state and the Hamiltonian.

This conflict between the two manifestations of the energy conservation law, $A_1$ and $A_2$, opens up a completely new perspective on the inadequacy, for coherent states, of a large class of schemes, that includes the TPM scheme, the scheme we introduced in Appendix C, and the scheme proposed in Ref. [21] ($\mathcal{O}_\rho'$ does not depend on $\rho$ and is empty for all these schemes).

2. $\mathcal{O}_\rho'$ is not empty and at least one of the $\tilde{W}$‘s in it is not zero

Here we cannot use the same argument as above, that relied solely on $A_1$ and $A_2$, because both $\sum W \text{Tr}(\rho M_W)$ and $\text{Tr}(\rho \tilde{W})$ are $\propto x$. Therefore, we will also have to invoke $B$.

In order to proceed, we will need to assume that, for all $\rho$‘s in $B$, all $W$‘s and $M_W$‘s are double differentiable with respect to $x$, so that the Taylor decompositions

$$
\begin{align*}
\mathcal{O}_\rho' \ni W &= x V_W + x^2 v_W + o(x^2), \\
M_W &= m_W + x \mu_W + x^2 \nu_W + o(x^2), \\
\mathcal{N}_\rho' \ni W &= U_W + O(x), \\
M_W &= u_W + o(x^2),
\end{align*}
$$

hold. By the definition of $O_\rho'$, all $m_W > 0$ and at least one of $V_W$‘s is nonzero. All the quantities—$V_W$, $v_W$, $m_W$, $\mu_W$, $\nu_W$, $U_W$, and $u_W$—may depend on $\rho$, $H$, and $h$.

Due to the fact that $\sum W M_W = 1$ must hold for any $x$, we find that

$$
\begin{align*}
\sum_{W \in \mathcal{O}_\rho'} m_W &= 1, \\
\sum_{W \in \mathcal{O}_\rho'} \mu_W &= 0, \\
\sum_{W \in \mathcal{O}_\rho'} \nu_W + \sum_{W \in \mathcal{N}_\rho'} u_W &= 0.
\end{align*}
$$

Next, since on the one hand $\langle W \rangle = \sum W \text{Tr}(\rho M_W)$, while on the other hand $A_1$ requires that Eq. (21) to be true for all $x$, by comparing the first-order (in $x$) terms, we find that

$$
\sum_{W \in \mathcal{O}_\rho'} V_W \langle m_W \rangle = \langle \tilde{w}_0 \rangle = 0,
$$

where the operator $\tilde{w}_0$ is as defined in Eq. (22), and $\langle X \rangle = \text{Tr}(\rho X)$. By comparing the second-order terms, we further obtain

$$
\sum_{W \in \mathcal{O}_\rho'} \left[ V_W \langle m_W \rangle + v_W \langle m_W \rangle \right]
+ \sum_{W \in \mathcal{N}_\rho'} U_W \langle u_W \rangle = \frac{1}{2} \langle \tilde{w}_0 \rangle.
$$

Now, let us introduce the Hermitian operator

$$
Y(\rho) := \sum_{W \in \mathcal{O}_\rho'} V_W m_W - \tilde{w}_0,
$$

which, due to Eq. (28), has to satisfy

$$
\text{Tr}(\rho Y(\rho)) = 0 \quad \forall \rho \in B.
$$

As a traceless Hermitian matrix, $\rho^{1/2} Y(\rho) \rho^{1/2}$ is most generally represented as a commutator [45]: there always exist operators $F(\rho)$ and $G(\rho)$ (also possibly depending on $H$ and $h$) such that $\rho^{1/2} Y(\rho) \rho^{1/2} = i[F(\rho), G(\rho)]$, where $F$ and $G$ can be moreover chosen to be Hermitian and are otherwise arbitrary. Since $B$ was chosen so that all $\rho$‘s in it are positive-definite, we have that, in its most general form,

$$
Y(\rho) = i \rho^{-1/2}[F(\rho), G(\rho)] \rho^{-1/2}.
$$

At this point, we will have to make two “structural” assumptions:

Assumption $S_1$: Both $F(\rho)$ and $G(\rho)$ are continuous functions of $\rho$ in $B$. Moreover, one of them commutes with $\rho$. For definiteness, and without further loss of generality, let us choose $F(\rho)$ to be the one that commutes with $\rho$:

$$
[F(\rho), \rho] = 0 \quad \forall \rho.
$$

Denoting $\tilde{G}(\rho) = \rho^{-1/2} G(\rho) \rho^{-1/2}$, we thus have that

$$
Y(\rho) = i [F(\rho), \tilde{G}(\rho)].
$$

Now, let us come back to $A_2$ and note that, when $h \propto H$, the system is untouched, for arbitrary values of $x$. Therefore, work should be zero with probability one. In the present case, this means that, since $\sum_{W \in \mathcal{O}_\rho'} m_W = 1$, then all $V_W$‘s will have to go to zero in the limit where $h$ becomes proportional to $H$. In turn, this means that the operator $\sum_{W \in \mathcal{O}_\rho'} V_W m_W$ has to go to zero in the said limit, for arbitrary $\rho$. Given the explicit dependence of $Y(\rho)$ on $\rho$ necessitated by Eq. (31), and the fact that $\tilde{w}_0$ does not depend on $\rho$, if $\tilde{G}(\rho)$ does not commute with $\rho$ in the limit where $h$ becomes proportional to $H$, it cannot be arranged that $\sum_{W \in \mathcal{O}_\rho'} V_W m_W = \tilde{w}_0 + Y(\rho)$ goes to zero in that limit. Therefore, in the said limit, it must
hold that $\tilde{G}(\rho) = 0$. Since $\tilde{G}$ appears only in the commutator with $F(\rho)$, we can without loss of generality say that, as $h$ becomes proportional to $H$, Assumption $S_2$ necessitates that

$$\tilde{G}(\rho, h) \to 0 \quad \forall \rho.$$  \tag{35}$$

Moreover, keeping in mind that we are in the first-order level in terms of $x$, $\tilde{G}(\rho, h)$ can depend on $h$ only linearly.

Up to this point, except for Assumption $S_1$, all arguments have been general. From this point on, we introduce our second structural assumption:

Assumption $S_2$: Whenever $h$ is a real matrix in the eigenbasis of $H$, $\tilde{G}(\tau_β, h)$ is a purely complex matrix in that basis, for all thermal states $\tau_β \in B_\tau$.

Albeit sounding somewhat nonintuitive, this assumption has an intuitive rationale. Indeed, since the linear functional $\tilde{G}(h)$ has to go to zero as $h$ becomes proportional to $H$, it is natural to assume that $\tilde{G}(h)$ is a linear matrix function of $i[h, H]$ ($= \tilde{w}_0$). The most general form of such a function compatible with Eq. (35) is $\tilde{G} = \{J, \tilde{w}_0\} + i\{Q, \tilde{w}_0\} + T^\dagger \tilde{w}_0 T$, where $J$ and $Q$ are Hermitian operators and $T$ is an arbitrary operator; all three may depend on $\rho$ and $H$. Now, when the matrix $h$ is real in $H$’s eigenbasis, $\tilde{w}_0$ is purely complex. Therefore, Assumption $S_2$ subsumes the slightly more special Assumption $S'_2$:

$$\tilde{G}(\rho, h) = \{J(\rho), \tilde{w}_0\} + i\{Q(\rho), \tilde{w}_0\} + T(\rho)^\dagger \tilde{w}_0 T(\rho)$$ \tag{36}$$

where, in the eigenbasis of $H$, $J(\tau_β)$ is an arbitrary real Hermitian matrix, $Q(\tau_β)$ is an arbitrary purely complex Hermitian matrix, and $T(\tau_β)$ is an arbitrary real matrix, for all thermal states $\tau_β \in B_\tau$.

Finally, let us fix a $\tau_β \in B_\tau$, call it simply $\tau$, and invoke $\mathfrak{B}$: $\sum_W e^{-\beta W} \text{Tr}(\tau M_W) = 1$. As before, noting that this must hold for any $x$, and collecting all powers of $x$, we will see that, on the zeroth- and first-order levels, $\mathfrak{B}$ is secured by Eqs. (26) and (28). Whereas the second-order level will produce

$$\sum_{W \in \mathcal{O}_Γ} \left[ \left( \frac{\beta^2 V_W^2}{2} - \beta V_W \right) \langle m_W \rangle_\tau - \beta V_W \langle \mu_W \rangle_\tau \right] + \langle \mu_W \rangle_\tau + \sum_{W \in \mathcal{N}_Γ} e^{-\beta U_W} \langle u_W \rangle_\tau = 0,$$

which, using Eqs. (27) and (29), we can rewrite as

$$\sum_{W \in \mathcal{O}_Γ} \beta^2 V_W^2 \langle m_W \rangle_\tau = \beta \langle \{i[h, \tilde{w}_0] \} \rangle_\tau - 2 \sum_{W \in \mathcal{N}_Γ} (e^{-\beta U_W} + \beta U_W - 1) \langle u_W \rangle_\tau,$$

from where, by noting that $e^{-z} + z - 1 \geq 0$ for all $z \in \mathbb{R}$ (with equality only when $z = 0$), we arrive at

$$\sum_{W \in \mathcal{O}_Γ} \beta^2 V_W^2 \langle m_W \rangle_\tau \leq \beta \langle \{i[h, \tilde{w}_0] \} \rangle_\tau.$$ \tag{37}$$

On the other hand, using Eq. (26) and invoking the operator-valued Cauchy–Schwarz inequality proven in Refs. [46, 47], we observe that

$$\sum_{W \in \mathcal{O}_Γ} \beta^2 V_W^2 m_W = \left\| \sum_{W \in \mathcal{O}_Γ} (\sqrt{m_W})^2 \right\| \left( \sum_{W \in \mathcal{O}_Γ} (\beta |V_W| \sqrt{m_W})^2 \right) \geq \left( \sum_{W \in \mathcal{O}_Γ} \beta |V_W| m_W \right)^2 \geq \left( \sum_{W \in \mathcal{O}_Γ} \beta V_W m_W \right)^2 = \beta^2 \langle \tilde{w}_0 + Y(\tau) \rangle^2,$$

which leads to

$$\sum_{W \in \mathcal{O}_Γ} \beta^2 V_W^2 \langle m_W \rangle_\tau \geq \beta \langle \{i[h, \tilde{w}_0] \} \rangle_\tau.$$ \tag{38}$$

Now, as we prove Appendix D, as long as $\{h, H\} \neq 0$, \n
$$\langle \tilde{w}_0^2 \rangle_\tau > \langle \{i[h, \tilde{w}_0] \} \rangle_\tau.$$ \tag{39}$$

Furthermore, a simple calculation that takes Eq. (33) into account, leads us to

$$\langle \{i[h, Y(\tau)] \} \rangle_\tau = 2 \sum_{k,j} \text{Re}[h_{kj} \tilde{G}(\tau, h)_{jk} \text{p}_k (E_k - E_j) (f_j - f_k)],$$

where $f_k$ are the eigenvalues of $F(\tau)$. Let us now choose a $h$ such that $\{h, H\} \neq 0$ but all $h_{kj}$ are real; we can always do that as long as $H \neq 0$. For such a choice of $h$, according to Assumption $S_2$, all $\tilde{G}(\tau, h)_{kj}$’s will be purely complex, and therefore $\langle \{i[h, Y(\tau)] \} \rangle_\tau = 0$. Hence, in view of Eq. (39), Eq. (38) will yield

$$\sum_{W \in \mathcal{O}_Γ} \beta^2 V_W^2 \langle m_W \rangle_\tau > \beta \langle \{i[h, \tilde{w}_0] \} \rangle_\tau,$$ \tag{40}$$

which contradicts Eq. (37). Hence, also in this case, we see that $A_1$, $A_2$, and $B$ cannot hold all at the same time.

Notably, the setting of this subsection is met in the OW approach: it satisfies $A_1$ and $A_2$ by design [for it, $Y(\rho) = 0 \forall \rho$]. In a sense, the proof strategy in this subsection is inspired by this example in that we start with requiring $A_1$ and $A_2$, and then show that the structure enforced by them cannot be made compatible with $B$. While this “directionality” of reasoning does not affect the generality of the result, it can affect how efficiently one can check whether the result applies to a given scheme. To illustrate this point, assume we are given an “operator scheme,” namely, work is defined as a state-dependent operator. Say, we know about it only that it is continuous in state and that, on thermal states, it is given by

$$\tilde{w}_β = -\beta^{-1} \ln (e^{\beta H/2} e^{-\beta U^1 H^U} e^{\beta H/2}).$$ \tag{41}$$

This scheme is obviously designed to satisfy $B$ and, when $[U^1 H^U, H] = 0$, $\tilde{w}_β$ coincides with the operator of work $W$ [defined in Eq. (4)], which also means that $\tilde{w}_β$ has a
proper classical limit. However, while we immediately see that this scheme does not fall into the class of Sec. IV A 1, we cannot check whether it satisfies A 1 or A 2, let alone Assumptions S 1 and S 2, since we do not know how ˆω depends on general nonthermal states. Although we show in Appendix B that ˆω violates A 1 even for thermal states, and thereby our result applies to it, this example demonstrates that certifying whether a given scheme passes through the filter of our no-go result will in general be an ad hoc procedure which can be nontrivial in and of itself.

V. DISCUSSION

In this paper, we examined the problem of measuring quantum work from an abstract point of view. Our goal was to understand why, despite many attempts in the literature, no satisfactory measurement scheme has been found so far. Our approach was to maximally distance from the specifics of the system and the process, and ask whether any measurement scheme, defined as broadly as possible, can produce work statistics that is consistent with energy conservation for the unmeasured system and Jarzynski’s fluctuation relation. The important aspect of these two conditions is that they aim to ensure the measurement results reflect the reality of the unmeasured system. Say, nothing happens to the system—there are no systems with which it could change energy. Even without measuring the system, energy conservation rule already tells us that, with probability 1, exactly 0 work is performed on the system. Therefore, any meaningful measurement of work, when applied to the system, should output 0, with probability 1, even if the measurement process disrupts the system. This was our condition A 2.

By the same logic, when we take a system in a known state and drive it unitarily according to a protocol we control, then, even without measuring the system, energy conservation tells us that the average work performed on the system is equal to the difference between the final and initial average energies of the system. Thus, we imposed the condition A 1 that a proper measurement of work should output a random variable the first moment of which coincides with that average, no matter how disruptive the measurement may be.

Note that A 1 and A 2 are not conditions about the measurement process itself. Namely, we do not ask whether energy is conserved during the joint evolution of the system, the apparatus, and the external driving agents.

We found that no reasonable measurement scheme can output such a random variable that would respect the above two manifestations of energy conservation and, at the same time, satisfy the Jarzynski equality whenever the system is in a thermal state (condition B).

If a scheme does not depend on the state and has to satisfy B, then it must coincide with the TPM scheme. This shows that, in the realm of state-independent schemes, the TPM scheme and Jarzynski equality are equivalent. For state-dependent schemes, we have proven that, assuming continuity in a small neighborhood of the infinite-temperature state, and adopting the structural assumptions S 1 and S 2 in the same neighborhood, one rules out the possibility of finding a quantum measurement that would measure work while simultaneously satisfying the three conditions (A 1, A 2, and B).

While our continuity assumptions can hardly constitute a loophole in any conceivable physical situation, breaking our structural assumptions, could in principle provide a loophole through which a meaningful work measurement scheme can emerge. That said, one cannot of course exclude the possibility of proving the incompatibility under milder assumptions (or none at all). Studying those possibilities is an interesting problem for future research.

More broadly, exploring the extent to which state-dependent schemes can be helpful in the related problems of quantum back-action evasion [48, 49], measuring fluctuations of heat [50, 51] and current [52–54], and estimating energetic cost of measurements [32, 55–57] is an interesting research avenue.

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APPENDIX A: CLASSICAL LIMIT OF THE OPERATOR OF WORK

Here, we will explore the question of whether the operator of work starts obeying the Jarzynski equality in the classical limit of a quantum continuous-variable system. We will study the simplest such example—the driven harmonic oscillator.

From the Baker–Cambell–Hausdorff formula,

\[
\langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} = \frac{1}{Z_{\beta}[\hat{H}]} \text{Tr} \left( e^{-\beta \hat{H}} e^{-\beta \hat{W}} \right) \\
= \frac{1}{Z_{\beta}[\hat{H}]} \text{Tr} e^{-\beta (\hat{H} + \hat{W})} + \frac{1}{2} \beta^2 [\hat{H}, \hat{W}] + \cdots \\
= \frac{1}{Z_{\beta}[\hat{H}]} \text{Tr} e^{-\beta \hat{U}^\dagger \hat{U} + \frac{1}{2} \beta^2 [\hat{H}, \hat{U}^\dagger \hat{U}]} + \cdots
\]

(A1)

one could naïvely expect that, since the commutator terms go to zero as \( \hbar \to 0 \), one would have that \( \langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} \) will tend to

\[
\frac{1}{Z_{\beta}[\hat{H}]} \text{Tr} e^{-\beta \hat{U}^\dagger \hat{U}} = e^{-\beta (F_\beta[\hat{H}^\dagger] - F_\beta[\hat{H}])}.
\]

However, due to the presence of infinite-norm operators, the \( \hbar \to 0 \) limit may not necessarily be continuous in such
systems. In order to see this, let us take an oscillator with the Hamiltonian
\[ H = \frac{1}{2}(q^2 + p^2), \quad (A2) \]
that is, the mass and the frequency are equal to 1, so that \( \beta \) is dimensionless.

Now, performing a process as a result of which the frequency of the oscillator changes, \( \beta^2[H, U^1 H'U] \) will necessarily contain a term proportional to \( \beta^2[p^2, q^2] = -2i\beta^2\hbar \{p, q\} \), where \( \{\cdot, \cdot\} \) is the anticommutator. Introducing the creation and annihilation operators \( a \) and \( a^\dagger \), we will have \( \beta^2[p^2, q^2] = (a^\dagger a)^2 - a^2 \). On the one hand, if \( \{|n\}\), are the eigenvectors of \( a^\dagger a \), then
\[ \langle n + 2 | \beta^2[p^2, q^2]|n \rangle = 2\beta^2\sqrt{(n + 1)(n + 2)} \], meaning that the norm of \( \beta^2[p^2, q^2] \) is infinity for any, no matter how small, non-zero value of \( \hbar \beta \).

On the other hand, if \( \beta \hbar = 0 \), \( \|\beta^2[p^2, q^2]\| = 0 \), which means that the \( \beta \hbar \to 0 \) limit is not continuous for \( \beta^2[H, U^1 H'U] \), which, in turn, implies that the naïve \( \beta \hbar \to 0 \) limit in Eq. (A1) has to be taken with extra care. We do this calculation for a specific illustrative example below and show that indeed \( \langle e^{-\beta W} \rangle \) converges to \( e^{-\beta(F_z H' - F_z[H])} \) as \( \beta \hbar \to 0 \).

Our goal will thus be to explicitly calculate
\[ \hat{W} = U^1 H'U - H \quad (A3) \]
and then calculate
\[ \langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} = \sum_n \langle \psi_n | \tau_\beta | \psi_n \rangle e^{-\beta W_n}, \quad (A4) \]
where \( W_n \) are the eigenvalues of \( \hat{W} \) and \( |\psi_n\rangle \) are its eigenvectors. Then, we are going to take the \( \beta \hbar \to 0 \) limit.

We know that, for any finite \( \beta \hbar \), \( \langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} \) will be larger than \( e^{-\beta(F_z H' - F_z[H])} \) [5].

Given the infinite-dimensional Hilbert space, one should of course distinguish between \( \hat{W} \)’s with spectra bounded and unbounded from below. In this section, we consider only \( \hat{W} \)’s whose spectra are bounded from below.

Since during the driving the Hamiltonian of the oscillator remains quadratic in \( q \) and \( p \), the generated unitary evolution operator \( U \) is Gaussian. Therefore, the transformation of \( q \) and \( p \) in the Heisenberg picture will be symplectic [58], which means, that introducing the column vectors \( z = (q, p) \) and \( z' = (U^1 qU, U^1 pU) \), we can write
\[ z' = S z, \quad (A5) \]
where \( S \) is a \( 2 \times 2 \) symplectic matrix that depends only on \( U \).

Now, if
\[ H' = z^T \Lambda z, \quad (A6) \]
where \( \Lambda > 0 \) is a \( 2 \times 2 \) real, symmetric matrix, then
\[ U^1 H'U = (z')^T \Lambda z' = z^T S^T \Lambda S z, \quad (A7) \]
and therefore, recalling Eq. (A2),
\[ \hat{W} = z^T V z \quad (A8) \]
with
\[ V = S^T \Lambda S - \frac{1}{2} I_2, \quad (A9) \]
where \( I_2 \) is the \( 2 \times 2 \) identity matrix. As mentioned above, we will consider the only the processes for which
\[ V > 0. \quad (A10) \]

In which case, by the Williamson’s theorem [58, 59], there exists a symplectic transformation, \( \Sigma \), that takes \( z \) to new coordinates \( \tilde{z} = (\tilde{q}, \tilde{p}) \), i.e.,
\[ \tilde{z} = \Sigma z, \quad (A11) \]
in which \( V \) is proportional to the identity. Namely, if \( w > 0 \) is the symplectic eigenvalue of \( V \), then
\[ \hat{W} = \tilde{z}^T (\Sigma^{-1})^T V \Sigma^{-1} \tilde{z} = w(\tilde{q}^2 + \tilde{p}^2). \quad (A12) \]
Introducing the symplectic form \( \Omega = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \), one can calculate \( w \) by noting that the eigenvalues of \( V \Omega \) are \( \pm iw \). Using that, it is easy to show that
\[ w = \det(V^{1/2}) = \sqrt{\det V}. \quad (A13) \]
Moreover, the symplectic transformation in Eq. (A12) is given by
\[ \Sigma = w^{-1/2} V^{1/2}. \quad (A14) \]

In order to calculate the trace in Eq. (A4), we will use some ready formulas for Gaussian states, presented in Ref. [60]. The results Ref. [60] are given for states, therefore we will introduce density operator
\[ \kappa_\beta = \frac{1}{Z_W} e^{-\beta \hat{W}}, \quad (A15) \]
where
\[ Z_W = \text{Tr} e^{-\beta \hat{W}} = (e^{\beta h w} - e^{-\beta h w})^{-1}, \quad (A16) \]
so that
\[ \langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} = Z_W \text{Tr} (\kappa_\beta \tau_\beta). \quad (A17) \]

Now, since both \( \hat{W} \) and \( H \) are purely quadratic, we can write
\[ \langle q | \kappa_\beta | \bar{q} \rangle = e^{-A_W q^2 - A^*_W \bar{q}^2 - 2B_W q \bar{q} + C_W}, \quad (A18) \]
\[ \langle q | \tau_\beta | \bar{q} \rangle = e^{-A_H q^2 - A^*_H \bar{q}^2 - 2B_H q \bar{q} + C_H}, \quad (A19) \]
and in Ref. [60] it is shown that
\[ \langle q | \kappa_\beta \tau_\beta | \bar{q} \rangle = e^{-A_q q^2 - D q^2 - 2B q \bar{q} + C}, \quad (A20) \]
\[ A = A_W - \frac{B_W^2}{A_W + A_H}, \]
\[ D = A_H^* - \frac{B_W^2}{A_W + A_H}, \]
\[ B = -\frac{B_W B_H}{A_W + A_H}, \]
\[ C = C_W + C_H + \frac{1}{2} \ln \frac{\pi}{A_W + A_H}. \]

Along with Eq. (A21), Eq. (A20) immediately yields
\[ \text{Tr}(\kappa_\beta \tau_\beta) = \int_{-\infty}^{\infty} dq \langle q | \kappa_\beta \tau_\beta | q \rangle \]
\[ = \frac{\pi e^{C_W + C_H}}{\sqrt{|A_W + A_H|^2 - (B_W + B_H)^2}}. \]

All that is left to do now is to determine how \( A_W, B_W, \) and \( C_W \) are expressed in terms of \( V \). We again use formulas from Ref. [60]: If
\[ x^W = \begin{pmatrix} x_{qq}^W & x_{qp}^W \\ x_{pq}^W & x_{pp}^W \end{pmatrix}, \]
with the elements defined as
\[ x_{qq}^W = 2\langle q^2 \rangle_{\kappa_\beta}, \quad x_{pp}^W = 2\langle p^2 \rangle_{\kappa_\beta}, \quad x_{qp}^W = \langle \{ q, p \}\rangle_{\kappa_\beta}, \]
then
\[ A_W = \frac{1 + h^{-2} \text{det} x^W}{4h^{-1}x_{qq}^W} - i \frac{x_{pp}^W}{2x_{qq}^W}, \]
\[ B_W = \frac{1 - h^{-2} \text{det} x^W}{4h^{-1}x_{qq}^W}, \]
\[ C_W = -\ln \sqrt{\pi h^{-1} x_{qq}^W}. \]

Obviously, identical formulas apply for \( H \).

Let us now calculate the correlators in Eq. (A24). In view of Eq. (A11),
\[ z = \Sigma^{-1} z. \]

So, keeping in mind Eq. (A12), by introducing creation-annihilation operators, and doing the simple algebra, we arrive at
\[ x^W = \hbar \coth(\hbar \beta \omega)(\Sigma^{-1})^T \Sigma^{-1}. \]

When taking Eqs. (A13) and (A14) into account, this formula amounts to
\[ x^W = \hbar \sqrt{\text{det} V} \coth \left( \frac{\beta \hbar}{2} \right) V^{-1}. \]

Analogously, reading the \( x \) matrix for \( H \) off Eq. (A28), we get
\[ x^H = \hbar \coth \left( \frac{\beta \hbar}{2} \right) \mathbb{1}_2. \]

It is easy to check that, in the \( \beta \hbar \to 0 \) limit,
\[ x^W = \beta^{-1} \frac{1}{\text{det} V} \begin{pmatrix} V_{22} & -V_{12} \\ -V_{12} & V_{11} \end{pmatrix} + O(\beta \hbar), \]
\[ x^H = 2\beta^{-1} \begin{pmatrix} \hbar & 0 \\ 0 & \hbar \end{pmatrix} + O(\beta \hbar). \]

Using these in Eq. (A25) and substituting the obtained quantities into Eq. (A23), through a somewhat tedious algebra, we find that
\[ \text{Tr}(\kappa_\beta \tau_\beta) = \beta \hbar \sqrt{\frac{\det V}{\det(V + \mathbb{1}/2)}} + O(\beta \hbar). \]

In order to finally calculate \( \langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} \), let us first notice that, due to Eq. (A9), \( \det(V + \mathbb{1}/2) = \det \Lambda \). Next, from Eq. (A16), and keeping in mind Eq. (A13), we see that
\[ Z_W = \beta \hbar^{-1} + O(1). \]

Substituting all these into Eq. (A17), we finally obtain
\[ \langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} = \frac{1}{2 \sqrt{\det \Lambda}} + O(\beta \hbar). \]

Finally, calculating \( Z_\beta[H'] \) and \( Z_\beta[H] \) in the same way as we calculated \( Z_W \), we find that
\[ e^{-\beta(F_\beta[H'] - F_\beta[H])} = \frac{Z_\beta[H']}{Z_\beta[H]} = \frac{1}{2 \sqrt{\det \Lambda}} + O(\beta \hbar), \]
which means
\[ \langle e^{-\beta \hat{W}} \rangle_{\tau_\beta} = e^{-\beta(F_\beta[H'] - F_\beta[H])} + O(\beta \hbar). \]

This expression shows that the OW scheme for continuous-variable systems is consistent in the classical limit, at least in the somewhat special case of \( \hat{W} > 0 \). The case of \( \hat{W} < 0 \) remains as an interesting open problem.

**APPENDIX B: THE OTHER OPERATOR OF WORK**

To see that \( \tilde{\omega}_\beta \) violates \( \mathfrak{A}_1 \), let us again consider the close-to-trivial cyclic processes, namely, those where \( H' = H \) and \( U = e^{-i \chi \hbar} \), with \( x \ll 1 \). First, we rewrite Eq. (41) as
\[ \tilde{\omega}_\beta = -\beta^{-1} \ln \left( \tau^{-1/2} U^1 \tau U \tau^{-1/2} \right), \]
where we have omitted \( \tau_\beta \)'s subscript for the sake of brevity of presentation. Through a simple calculation employing the Baker–Hausdorff lemma and the Taylor
expansion of \( \ln(1 + x) \) around \( x = 0 \), Eq. \((B1)\) will lead us to
\[
-\beta \dot{\omega}_\beta = i x \tau^{-1/2}[h, \tau] \tau^{-1/2} + O(x^2) - \frac{x^2}{2} \tau^{-1/2}[h, [h, \tau]] \tau^{-1/2} + \frac{x^2}{2} \tau^{-1/2}[h, \tau] \tau^{-1/2} + O(x^3).
\]
Therefore,
\[
\beta(\dot{\omega}_\beta)_\tau = -\frac{x^2}{2} \text{Tr}(\tau^{-1}[h, \tau]^2) + O(x^3).
\]

Whereas, reading from Eq. \((21)\), we have
\[
\beta(\dot{\omega})_\tau = -\frac{x^2}{2} \text{Tr}([h, \tau][h, \beta H]) + O(x^3).
\]

Neglecting the \( O(x^3) \) terms and working in the eigenbasis of \( H \), we can rewrite these equations as
\[
\beta(\dot{\omega}_\beta)_\tau = x^2 \sum_{k < j} |h_{kj}|^2 \left( p_k - p_j \right)^2 \frac{p_k + p_j}{2 np_k p_j} \beta \dot{W} = x^2 \sum_{k < j} |h_{kj}|^2 \left( p_k - p_j \right) \Delta_{kj} (\beta E_{kj} - 1) \sinh \Delta_{kj} \geq 0,
\]

where \( p_k = e^{-\beta E_k}/Z \) are the eigenvalues of \( \tau \), and \( \Delta_{kj} = \beta(E_j - E_k) \), and, when transitioning from equations \((B5)\) to \((B6)\) and \((B7)\) to \((B8)\), we used that \( p_k = p_j e^{-\Delta_{kj}} \).

Now, whenever \([h, H] \neq 0\), there exist such \( k_0 \) and \( j_0 \) for which \( \Delta_{k_0 j_0} > 0 \) and \( \Delta_{k_0 j_0} = 0 \). Since \( \Delta_{kj} \geq 0 \) for all \( k < j \), Eqs. \((B6)\) and \((B8)\) thus imply that
\[
\langle \dot{\omega}_\beta \rangle_\tau \geq \langle \dot{\omega} \rangle_\tau,
\]
provided the \( x \) is small enough for the \( O(x^3) \) terms to be irrelevant. The inequality will be strict. This inequality simply means that \( \dot{\omega}_\beta \) violates \( \mathfrak{A}_1 \) for all thermal states as long as \([h, H] \neq 0\).

1. Two curious properties of \( \dot{\omega} \)

Below, we will prove two results about \( \dot{\omega} \) that are not related to the goal of this appendix. However, we feel compelled to present them to more completely characterize the yet another “definition of work” that \( \dot{\omega} \) is.

Property 1:
\[
\langle \dot{W} \rangle_\tau - \langle \dot{\omega} \rangle_\tau \geq \beta^{-1} S(\tau||U^\dagger \tau U');
\]

Property 2:
\[
\text{Spec}(\dot{\omega}_\beta)^\dagger \geq \text{Spec}(\dot{W})^\dagger,
\]
Taking the limit $\epsilon \to 0$ in Eqs. (B20) and (B21), it is straightforward to see that $\sum_{m=1}^{k} W_m \leq \sum_{m=1}^{k} \omega_m$, for $k = 1,\ldots,d-1$ and $\text{Tr} \hat{W} = \text{Tr} \hat{\omega}$, which concludes the proof of Eq. (B11).

**APPENDIX C: A SCHEME SIMULTANEOUSLY SATISFYING $\mathfrak{A}_1$ AND $\mathfrak{B}$**

Let us construct a scheme that satisfies $\mathfrak{A}_1$ and $\mathfrak{B}$ and depends continuously on the system’s state. The example illustrates that $\mathfrak{A}_2$ is an independent and essential requirement for any meaningful measurement of work.

As a first step, we devise a measurement scheme in the specific class of processes in which the system starts out in a possibly coherent state and unitarily evolves into a diagonal state: $(H, \rho) \rightarrow (H', \rho_D)$, where $\rho_D = V \rho V^\dagger$ and $[\rho_D, H'] = 0$. Here, we notice that the time-reversed process, $(H', \rho_D) \rightarrow (H, \rho)$, with $\rho = V^\dagger \rho_D V$, is describable by the TPM scheme.

Now, if the work statistics of a unitary process is $\{W_{\alpha}, p_\alpha\}_\alpha$, then, to the time-reversed process, it is reasonable to prescribe the work statistics $\{-W_{\alpha}, p_\alpha\}_\alpha$.

With this prescription, thinking of the process as the reverse of $(H, \rho) \rightarrow (H', \rho_D)$, and describing the latter by the TPM scheme [see Eqs. (6) and (7)], for the work statistics of $\mathcal{P}$ we obtain

$$W_{ak} = -(E_a - E_k),$$

$$p_{ak} = \text{Tr} \left( P'_a \rho_D \right) \frac{V^\dagger P'_a \rho_D P'_k V}{\text{Tr} \left( P'_k \rho_D \right)},$$

$$= \text{Tr} \left( P'_a V^\dagger P'_k \rho_D P'_k V \right).$$

Put in other words:

$$W_{ak} = E'_k - E_a, \quad p_{ak} = \text{Tr} \left( M_{ak} \rho \right),$$

with

$$M_{ak} = V^\dagger P'_a V P_a V^\dagger P'_k V.$$  \hspace{1cm} (C5)

Note that $M_{ak}$ depends on the initial state through $V$. We call this measurement “backward” TPM scheme as opposed to the standard TPM scheme that directly measures the “forward” process. Interestingly, the work distribution produced by the backward TPM scheme coincides with that of the scheme proposed in Ref. [6].

Having at our disposal this scheme, we can now describe any coherent-to-coherent process by decomposing it into coherent-to-incoherent and incoherent-to-coherent processes. More specifically, consider an arbitrary Hamiltonian process $(H, \rho) \rightarrow (H', \rho')$, with $\rho' = U \rho U^\dagger$, for which both $[\rho, H] \neq 0$ and $[\rho', H'] \neq 0$, and decompose it into

$$(H, \rho) \xrightarrow{R} (H, \rho') \xrightarrow{UR^\dagger} (H', \rho'),$$

where $\rho = R \rho R^\dagger$ is diagonal in $H$. Moreover, $R$ is chosen such that the diagonal of $\rho$ is ordered in the same way as the diagonal of $\rho$. More specifically, say, $\pi$ is the permutation matrix that reorders $\rho_D$ into $\rho_D$: $\rho_D = \pi \rho_D \pi^\dagger$, where $\rho_D$ is a diagonal matrix whose diagonal coincides with that of $\rho$ in the eigenbasis of $H$. Then, if $r^\dagger$ is the vector composed of the eigenvalues of $\rho$, organized in the decreasing order, then $\rho = \pi r^\dagger \pi^\dagger$. Thus, $R$ is the unitary operator that rotates $\rho$ into $\rho$. The operator $R$ is chosen in this way to guarantee that, when $\rho$ is diagonal, then $R = I$.

Now, using Eqs. (C4) and (C5), for the first part of the process (C6), we obtain

$$W_{ab}^I = E_b - E_a, \quad p_{ab}^I = \text{Tr} \left( M_{ab}^I \rho \right),$$

where the POVM is given by

$$M_{ab}^I = R^\dagger P_b R^\dagger P_a R.$$  \hspace{1cm} (C8)

Following the standard TPM scheme, for the second part of the process, we find

$$W_{ck}^{II} = E'_c - E_c, \quad p_{ck}^{II} = \text{Tr} \left( M_{ck}^{II} \rho \right),$$

where

$$M_{ck}^{II} = R^\dagger P_c R^\dagger P_b R.$$  \hspace{1cm} (C10)

Finally, for the overall process (C6), let us write

$$W_{abck} = W_{ab}^I + W_{ck}^{II}, \quad p_{abck} = p_{ab}^I p_{ck}^{II} = \text{Tr} \left[ M_{ab}^I \otimes M_{ck}^{II} \rho \otimes \rho \right],$$

where $M_{abck} = M_{ab}^I \otimes M_{ck}^{II}$ is a POVM that depends on the initial state through $R$.

From the perspective of a single copy of the system, this scheme is simply a state-dependent POVM. Indeed, with the same outcomes $W_{abck}, p_{abck}$ can be written as

$$p_{abck} = \text{Tr} \left[ \tilde{M}(\rho)_{abck} \rho \right],$$

where

$$\tilde{M}(\rho)_{abck} = p_{ck}^{II} M_{ab}^I.$$  \hspace{1cm} (C13)

Obviously, the operators $\tilde{M}(\rho)_{abck}$ constitute a POVM.

By construction, this definition satisfies $\mathfrak{A}_1$ and, keeping in mind that $R = I$ for initially diagonal states, it coincides with the TPM scheme, and therefore satisfies $\mathfrak{B}$. However, for $[\rho, H] \neq 0$, this definition does not satisfy $\mathfrak{A}_2$. Indeed, in order to measure work for some $[\rho, H] \neq 0$ in the trivial case of $H' = H$ and $U = I$, the scheme will first apply a unitary to diagonalize $\rho$ with $H$, thereby producing some nontrivial statistics according to Eqs. (C7) and (C8); then, it will rotate that state back to $\rho$, again producing nontrivial statistics as per Eqs. (C9) and (C10). In total, for the trivial process, the scheme will produce rather complicated work statistics described by Eq. (C11).
APPENDIX D: PROOF OF INEQUALITY (39)

To prove Eq. (39), let us switch to the eigenbasis of $H$ ordered in such a way that its eigenvalues $E_k$ are ordered increasingly. By explicit calculation, we then see that

$$\beta^2\langle \hat{w}_0^2 \rangle_\tau = \sum_{k<j} (p_k + p_j) \Delta_{kj}^2 |h_{kj}|^2,$$  \hspace{2cm} (D1)

where, as in Appendix B, $p_k = e^{-\beta E_k}/Z$ are the eigenvalues of $\tau$, and $\Delta_{kj} = \beta(E_j - E_k)$.

In the same notation, another simple calculation leads to

$$\beta \langle i[h, \hat{w}_0] \rangle_\tau = 2 \sum_{k<j} (p_k - p_j) \Delta_{kj}^2 |h_{kj}|^2 \geq 0.$$  \hspace{2cm} (D2)

Note that this inequality holds for any passive state $\tau$ and simply expresses the fact that when $\rho = \tau$ in Eq. (21), the term $\propto x$ vanishes, and the second term has to be positive because $\langle W \rangle$ is the work performed on a passive state and thus has to be nonnegative.

All in all,

$$\beta^2 \langle \hat{w}_0^2 \rangle_\tau - \beta \langle i[h, \hat{w}_0] \rangle_\tau = 2 \sum_{k<j} |h_{kj}|^2 (p_k + p_j) \Delta_{kj}^2 \left[ \frac{\Delta_{kj}}{2} - \tanh \frac{\Delta_{kj}}{2} \right].$$ \hspace{2cm} (D3)

Now, $[h, H] \neq 0$ means that $h$ has nonzero nondiagonal elements between some eigensubspaces of $H$. In other words, there exist some $k_0$ and $j_0$ for which $\Delta_{k_0j_0} > 0$ and $h_{k_0j_0} \neq 0$. Moreover, since $x - \tanh x > 0$ whenever $x > 0$, at least one of the summands in Eq (D3) is $> 0$, which thereby proves Eq. (39).

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