A REMARK ABOUT 6J SYMBOLS AND YOUNG SEMI-NORMAL FORM

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1. Introduction

A semi-simple tensor category is determined up to equivalence by its Grothendieck ring and its 6j symbols with respect to a set of tree basis vectors. The 6j symbols are the coordinate representation of the associator. Despite their importance, we only know explicit formulas for 6j symbols in a few special cases. In the $\text{SL}_2(\mathbb{C})$ and $U_q(\mathfrak{sl}_2)$ cases, explicit formulas for the 6j-symbols are computed in [CFS95]. As far as the author is aware, the only other case where explicit 6j-symbols are known is $G$-graded vector spaces for $G$ a finite group. In this case, associators are cohomology classes in $H^3(G, \mathbb{C}^\times)$.

In this note, we compute a large number of 6j symbols inside the tensor category consisting of polynomial representations of GL($\infty$). This tensor category is studied in detail by Sam and Snowden in [SS16, SS15, SS12]. More precisely, we have:

**Theorem 1.** Let $\lambda \subseteq \mu$ be partitions such that $\mu \setminus \lambda$ has two boxes not contained in a single row or column. Then

$$(j_{\lambda}^{\mu})^{-1} = \left( \begin{array}{cc} k & -k \\ k+1 & k-1 \end{array} \right).$$

Where $k$ is the axial distance in the skew partition $\mu \setminus \lambda$. These 6j symbols are uniquely defined up to scaling.

We describe computations in tensor categories using string diagrams. A good introduction to string diagrams is [Sel11]. For a complete and rigorous introduction to tensor categories, we direct the reader to [EGNO15].

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2. What are 6j symbols?

**Definition 2.** Let $X$ be a semi-simple tensor category. Index the simple objects with a set $\Lambda$. Choose a basis for each $X(\mu, \lambda \otimes \nu)$ denoted by

$$
\begin{array}{c}
\lambda \\
\mid \\
\mu
\end{array},
\begin{array}{c}
\lambda \\
\mid \\
\nu
\end{array},
\begin{array}{c}
\lambda \\
\mid \\
e_1
\end{array},
\begin{array}{c}
\lambda \\
\mid \\
e_2
\end{array}, \ldots
$$

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and let

\[ \lambda, \nu, \mu, e_1, \ldots \]

be the dual basis of \( X(\lambda \otimes \nu, \mu) \). We call these diagrams **tree string diagrams**. It is important to notice that the tree string diagrams are not canonically defined.

**Definition 3.** Pick a distinguished simple object \( X \in X \). The **fusion graph** of \( X \) has vertices \( \Lambda \) and the edges from \( \lambda \) to \( \mu \) are the distinguished basis vectors in \( X(\mu, \lambda \otimes X) \).

**Proposition 4.** Fix \( \lambda \in \Lambda \). Then \( X(\lambda, X^\otimes n) \) has dimension the number of length \( n \) paths from the tensor unit to \( \lambda \) in the fusion graph for \( X \). Moreover, an explicit basis is given by string diagrams of the form

\[
\begin{array}{c}
X \\
X \\
X \\
X \\
X \\
\lambda
\end{array}
\]

we call such string diagrams **tree basis vectors**

**Proof.** Decompose \( X^\otimes n \) using the fusion graph for \( X \). \( \square \)

**Definition 5.** Since \( X \) is semi-simple, the Artin-Wedderburn theorem implies that \( \text{End}(X^\otimes n) \) is a product of matrix algebras. Proposition 4 implies that in the tree string basis, the matrix units in \( \text{End}(X^\otimes n) \) look like

\[
\begin{array}{c}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_n \\
\lambda
\end{array}
\]

Equivalently, the irreducible representations of \( \text{End}(X^\otimes n) \) are parameterized by the simple objects in \( X \) which have a length \( n \) path from the tensor unit in the fusion graph for \( X \). The string diagrams defined in proposition 4 form a basis for the corresponding representation.
Definition 6. Fix $\lambda_1, \lambda_2, \lambda_3, \mu \in \Lambda$. Then we have two bases for $X(\mu, \lambda_1 \otimes \lambda_2 \otimes \lambda_3)$:

$$
\begin{cases}
\lambda_1 \\
\alpha \\
e_1 \\
e_2 \\
\mu
\end{cases}
\quad \leftrightarrow 
\begin{cases}
\lambda_3 \\
\beta \\
f_1 \\
f_2 \\
\mu
\end{cases}
$$

The $6j$ symbols are the entries in the change of basis matrix $(j_{\mu}^{\lambda_1, \lambda_2, \lambda_3})^{e_1, e_2}_{f_1, f_2}$. In other words, they are a coordinate representation of the associator. They must satisfy some algebraic relations which correspond to the pentagon axiom and the unit axiom. From the $6j$-symbols and the Grothendieck ring, you can recover the tensor category. Therefore, the $6j$-symbols are coordinates on the moduli stack of semi-simple tensor categories with a fixed Grothendieck ring.

3. Young Semi-normal form

Definition 7. Let $X$ be a semi-simple tensor category with distinguished object $X$. If $\sigma \in \text{End}(X^{\otimes 2})$ then we have

$$
\begin{align*}
\lambda & \quad \sigma \\
a & \quad b \\
\mu & \quad = \sum_{fg} m_{fg, ab}(\sigma)
\end{align*}
$$

We call the matrix $m(\sigma)$ a semi-normal form for $\sigma$.

Definition 8. We define the category $S$ which has objects the natural numbers and morphisms

$$
S(m, n) = \begin{cases}
S_m & m = n \\
0 & \text{otherwise}
\end{cases}
$$

where $S_n$ is the symmetric group with simple reflections $g_1, \ldots, g_{n-1}$. The inclusion $S_m \otimes S_n \to S_{m+n}$ defined by $g_i \otimes g_j \mapsto g_i \sigma_{m+j}$ equips $S$ with a tensor structure. We define $\mathcal{S} \subseteq [S^{\text{op}}, \text{Vec}]$ to be the idempotent completion of $S$. The monoidal structure on $S$ extends to $\mathcal{S}$ via day convolution. The category $\mathcal{S}$ can be described as the polynomial representations of $\text{GL}(\infty)$ as defined by Sam and Snowden in [SS16]. The Grothendieck ring for $\mathcal{S}$ has basis given by partitions and multiplication given by the Littlewood-Richardson rule. A special case of the
Littlewood-Richardson rule is the Pieri rule:

$$\lambda \otimes \square = \sum_{\lambda \subset \mu^{n+1}} \mu$$

This implies that the tree basis vectors

are in bijection (up to scaling) with standard skew tableaux of shape $\mu \setminus \lambda$. We shall abuse notation and identify these tree basis vectors with the corresponding standard skew tableaux. Suppose that $\lambda \subseteq \mu \vdash n + 2$ are partitions such that $\mu \setminus \lambda$ is not contained in a single row or column. Then there are exactly two partitions which satisfy $\lambda \subseteq \nu \subseteq \mu$. Call them $\nu$ and $\nu'$. The multiplicity space $S(\lambda \otimes \square \otimes \square \mu)$ is 2-dimensional with basis

The semi-normal form for $g_1$ is well known:

$$m(g_1) = \begin{pmatrix} -1/k & 1 \\ 1 - 1/k^2 & 1/k \end{pmatrix}$$

where $k$ is the axial distance in $\mu \setminus \lambda$: 
Identify the trivalent vertex $e$ with the matrix unit inside $\text{End}(X^\otimes 2)$. Consider the action of $e$ on the two bases

On the left basis, $e$ acts via the semi-normal form $m(e)$. On the right basis, $e$ is a projection. Inside $\mathcal{S}$, this implies the matrix of eigenvectors $vectors$ of $m(e) = 1/2(e + m(g_1))$ equals $(j_\mu^\lambda\square)^{-1}$. Therefore we have

$$(j_\mu^\lambda\square)^{-1} = \begin{pmatrix} k & \frac{-k}{k+1} \\ \frac{k+1}{k} & 1 \end{pmatrix}.$$ 

This proves theorem \ref{thm:main}.

**References**

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