FROM SUBCATEGORIES TO THE ENTIRE MODULE CATEGORIES

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Abstract. In this paper we show that how the representation theory of subcategories (of the module category over an Artin algebra) can be connected to the representation theory of all module over some algebra. The subcategories dealing with are some certain subcategories of the morphism category (including submodule categories) and the Gorenstein projective modules over (relative) stable Auslander algebras. These two kinds of subcategories, as will be seen, are closely related to each other. It is shown that to compute the almost split sequences in the subcategories it is enough to do the computation in the module category of some algebra which is known and easier to work. Then as an application the most part of Auslander-Reiten quiver of the subcategories is obtained only by the Auslander-Reiten quiver of an appropriate algebra and next adding the remaining vertices and arrows in an obvious way.

1. Introduction

The most basic problem in representation theory of Artin algebras is to classify the indecomposable finitely generated modules. Since the general problem is known to be very difficult, special attention might be paid to some well-behaved subcategories. In this paper we have plane to investigate simultaneously the subcategory of Gorenstein projective modules over the relative stable Auslander algebras and some certain subcategories of the morphism category. As will be discussed in below, there is a nice relationship between these two types considered subcategories in our paper. The notion of Gorenstein projective modules was first defined by Maurice Auslander in the mid-sixties, see [AB], for finitely generated projective $\Lambda$-modules over a Noetherian ring, called $G$-dimension zero. Later they have found important applications in commutative algebra, algebraic geometry, singularity theory and relative homological algebra. Enochs and Jenda in [EJ] generalized the notion of Gorenstein projective modules to (not necessarily finitely generated) modules over any ring $R$, under the name of Gorenstein projective modules. Studying of (finitely generated) Gorenstein projective modules has attracted attention in the setting of Artin algebras, see for instance [AR2, AR3, Ha, CSZ, RZ1, RZ2] and etc. More recently, Ringel and Zhang in [RZ17], was shown that there is a nice bijection between the indecomposable module over a finite dimensional hereditary algebra over an algebraic closed filed and the indecomposable of the stable category of Gorenstein projective modules over the tensor algebra of the algebra of dual numbers and the given hereditary algebra. Another subcategories we are interested in this paper are subcategories of the morphism category $H(\Lambda)$ of Artin algebra $\Lambda$ arising by a quasi-resolving subcategory $\mathcal{X}$ of the category $\text{mod-}\Lambda$ of finitely generated right modules over $\Lambda$. A subcategory $\mathcal{X}$ of mod-$\Lambda$ is said to be quasi-resolving if it is closed under kernels of epimorphisms in $\mathcal{X}$, finite direct sums, and containing proj-$\Lambda$, the category of

2010 Mathematics Subject Classification. 16G10, 16G60, 16G50.

Key words and phrases. relative stable Auslander algebra, triangular matrix algebra, Gorenstein projective module, almost split sequence, Auslander-Reiten quiver.
finitely generated modules. Denote by \( S_X(\Lambda) \) the subcategory of \( H(\Lambda) \) consisting of monomorphisms in \( \text{mod-}\Lambda \) with terms in \( X \) such that whose cokernels belong to \( X \). When \( X = \text{mod-}\Lambda \), the subcategory \( S_{\text{mod-}}(\Lambda) \), simply \( S(\Lambda) \), becomes the submodule category, which have been studied extensively by Ringel and Schmidmeier in [RS1, RS2], and also a generalization of their works given in [XZZ]. In addition, a surprising link, established in [KLM], between the stable submodule category with the singularity theory via weighted projective lines of type \((2, 3, p)\) is discovered.

It is worth noting that when \( X \) is equal to the subcategory \( \text{Gprj-}\Lambda \) of Gorenstein projective \( A \)-modules in \( \text{mod-}\Lambda \), then \( S_{\text{Gprj-}}(\Lambda) \) is nothing else than the subcategory of Gorenstein projective module over \( T_2(\Lambda) \), upper triangular \( 2 \times 2 \) matrix algebra over \( \Lambda \), here \( \text{mod-}T_2(\Lambda) \) is identified by \( H(\Lambda) \).

Our main purpose in this paper is to study the Auslander-Reiten theory, as a powerful tool in the modern representation theory, over the above-mentioned subcategories by approaching with module category over some Artin algebras. The Auslander-Reiten theory for the module categories over an Artin algebra is much more known than some subcategories. At least, the Auslander-Reiten translation in the module categories is computed, that is, \( D\text{Tr}r \). Such a connection can be helpful to transfer those known results to the subcategories. This aim is inspired by the work of [RSZ] as follows: In [RSZ] the authors make a connection between \( S(k[x]/(x^n)) \) and the module category \( \text{mod-}\Pi_{n-1} \) of the preprojective algebra \( \Pi_{n-1} \) of type \( A_{n-1} \) via defining two functors which originally come from the works of Auslander-Reiten and Li-Zhang. We know that \( \Pi_{n-1} \) is the stable Auslander algebra of representation-finite algebra \( k[x]/(x^n) \) (see [DR], Theorem 3, Theorem 4 and Chapter 7). Hence those functors appeared in [RSZ] are indeed functors from \( S(k[x]/(x^n)) \) to \( \text{Aus(mod-}k[x]/(x^n)) \). We recall that the stable Auslander algebra \( \text{Aus(mod-}A) \) of a representation-finite algebra \( A \) is the endomorphism algebra \( \text{End}_A(M) \) in the sable category \( \text{mod-}\Lambda \), where \( M \) is a basic representation generator for \( A \), i.e., the additive closure \( \text{add-M} \) of \( M \) in \( \text{mod-}\Lambda \) is the entire category, and each indecomposable direct summand of \( M \) has multiplicity one.

Recall also, for an additive category \( C \), \( \text{mod-}C \) denote the category of finitely presented functors over \( C \). It is known that in case that \( C \) is generated by an object \( C \), i.e., \( \text{add-}C = C \), then the evaluation functor on \( C \) induces an equivalence of categories \( \text{mod-}C \simeq \text{mod-}\text{End}_C(C) \). Assume \( X \) is a quasi-resolving subcategory of \( \text{mod-}\Lambda \). The observation given in [RSZ] motivate us to define a functor from \( S_X(\Lambda) \) to \( \text{mod-}X \), denote by \( \Psi_X \), see Construction 3.1. Here \( X \) is the stable category of \( X \), that is, the quotient of \( X \) by the ideal of morphisms factoring through a projective module. This is a relative version of the functor \( \alpha \) defined in [RSZ] in a functorial language. analogue to \( \alpha \), the functor \( \Psi_X \) is full, dense and objective, then \( \Psi_X \) induces an equivalence between the quotient category \( S_X(\Lambda)/U \) and \( \text{mod-}X \), where \( U \) is the ideal generated by the objects in \( S_X(\Lambda) \) in the form of \((X \xrightarrow{1} X)\) or \((0 \to X)\), where \( X \) runs through objects in \( X \) (see Theorem 3.2). Following [RSZ], an additive functor \( F : A \to B \) will be said to be objective provided any morphism \( f : A \to A' \) in \( A \) with \( F(f) = 0 \) factors through an object \( C \) that \( F(C) = 0 \). With some additional conditions on \( X \), see Setup 5.3, we will observe in Section 5 the almost split sequences in \( S_X(\Lambda) \) is preserved by \( \Psi_X \). We mean here all almost exact sequences in \( S(X) \) except a special class of almost split sequences are mapped into the ones in \( \text{mod-}X \) by \( \Psi_X \). Next, since the notion of the Auslander-Reiten quivers is based on the almost split sequences, we can conclude that the Auslander-Reiten quiver \( \Gamma_X \) of \( \text{mod-}X \) can be considered as a valued full subquiver of \( \Gamma_{S_X(A)} \) such that contains all vertices in \( \Gamma_{S_X(A)} \) except those of vertices corresponding to the isomorphism classes of the indecomposable objects having the form \((X \xrightarrow{1} X)\) and \((0 \to X)\). Therefore, the task to find the full of \( \Gamma_{S_X(A)} \)
is only to add the remaining vertices corresponding to the isomorphism classes of the indecomposable objects in $S_X(\Lambda)$ having the simplest structure among the other indecomposable objects in $S_X(\Lambda)$, and arrows attached them. In the case that $\mathcal{C}$ is of finite representation type, i.e., $\mathcal{C} = \text{add-}X$ for some $X \in \mathcal{C}$, we have $\text{mod-}\mathcal{C} \simeq \text{mod-}\text{Aus}(\mathcal{C})$, where $\text{Aus}(\mathcal{C}) = \text{End}_A(M)$ and $M$ is a basic representation generator of $\mathcal{C}$. The algebra $\text{Aus}(\mathcal{C})$ is called the relative stable Auslander algebra with respect to the subcategory $\mathcal{C}$. Note that the relative stable Auslander algebras $\text{Aus}(\mathcal{C})$ induced by a subcategory of finite representation type $\mathcal{C}$ are isomorphism of algebras. Thus, the notation $\text{Aus}(\mathcal{C})$ is well-defined up to isomorphism of algebras. Therefore, in this way, the representation theoretic properties of the subcategory $S_X(\Lambda)$ via the functor $\Psi_X$ can be approached by the module category over the associated relative stable Auslander algebra $\text{Aus}(\mathcal{C})$.

In parallel with the subcategories in the form $S_X(\Lambda)$, we are also interested in considering the subcategory $\text{Gprj-}\mathcal{C}$ of Gorenstein projective functors in the category of finitely presented modules over the associated relative stable Auslander algebra $\text{Aus}(\mathcal{C})$. Thus, the notation $\text{Aus}(\mathcal{C})$ can be approached by the module category over the associated relative stable Auslander algebra $\text{Aus}(\mathcal{C})$. Because $\mathcal{G}$ is of finite representation type, we are looking for, the module category over the relative stable Auslander algebra $\text{Aus}(\mathcal{C})$ via the functor $\Psi_X$.

Specializing our two approaches in the above, we reach the following interesting corollary.

**Corollary 1.1.** Let $\Lambda$ be a Gorenstein algebra of finite representation type. Let $\text{Aus}(\text{mod-}\Lambda)$ and $\text{Aus}(\text{Gprj-}\Lambda)$ be, respectively, the stable Auslander algebra and stable Cohen-Macaulay Auslander algebra of $\Lambda$. Set $g\Gamma_{\Lambda} := \text{Gprj-}\Gamma_{\text{mod-}\Lambda}$ and $\Psi_{\mathcal{G}} := \Psi_{\text{Gprj-}\Lambda}$. Then we have the following assertions.

1. There are the following functors

$$
\text{Gprj-}\Gamma_{2}(\Lambda) \xrightarrow{\Psi_{\mathcal{G}}} \text{mod-Aus}(\text{Gprj-}\Lambda) \xrightarrow{\Psi_{\mathcal{G}}} \text{Gprj-Aus}(\text{mod-}\Lambda),
$$
where the functor $\Psi_G$ is full, dense and objective, and the functor $\varphi \Gamma_\Lambda$ is fully faithful, almost dense and exact. Moreover, each of the functors $\varphi \Gamma_\Lambda, \Psi_G$ “preserves the almost split sequences”;

(2) The Auslander-Reiten quiver $\Gamma_{\text{Aus}(\text{Gprj}\Lambda)}$ of the module category over $\text{Aus}(\text{Gprj}\Lambda)$ is a full valued subquiver of the Auslander-Reiten quiver $\Gamma_{\text{Gprj}\Lambda^{-2}(\Lambda)}$ of the subcategory $\text{Gprj}\Lambda^{-2}(\Lambda)$ and also the Auslander-Reiten quiver $\Gamma_{\text{Gprj}\Lambda^{-\text{mod}\Lambda}}$ of the subcategory of Gorenstein projective modules over $\text{Aus}(\text{Gprj}\Lambda)$, i.e.,

$$\Gamma_{\text{Gprj}\Lambda^{-2}(\Lambda)} \hookrightarrow \Gamma_{\text{Aus}(\text{Gprj}\Lambda)} \hookrightarrow \Gamma_{\text{Gprj}\Lambda^{-\text{mod}\Lambda}}.$$

Since $\text{Gprj}\Lambda$ is a triangulated category, $\text{Aus}(\text{Gprj}\Lambda)$ is a self-injective algebra, and moreover with complexity at most one by Theorem 4.7. Based on the above corollary the study of the subcategory of Gorenstein projective modules over $T_2(\Lambda)$ and $\text{Aus}(\text{mod}\Lambda)$ is connected to module category over a self-injective algebra with complexity one. The class of self-injective algebras is one of the important classes in the representation theory of Artin algebras which are well-understood. Hence the knowledge concerning the class of self-injective algebras by the above observation may carry over into the subcategory of Gorenstein projective modules.

The paper is organized as follows. In section 2, some important facts and notions, including functor categories, Gorenstein projective modules and triangular matrix algebras, which will be needed in the paper are recalled. The other basic facts needed we will try to collect them in the beginning of the corresponding sections. In section 3, we will define the functor $\Psi_\mathcal{X}$ and Theorem 3.2 will be proved. In section 4, firstly, an explicit description of $n$-th syzygies of the functors in $\text{mod}\mathcal{A}$ is given in terms of their projective resolutions in $\text{mod}\mathcal{X}$, see Corollaries 4.2 and 4.4. Secondly, the description has different applications. The important one is to give a characterization of Gorenstein projective functor in $\text{mod}\mathcal{X}$ (Theorem 4.13) and using this characterization to define the functor $\Psi_{\mathcal{X}}$, which is the second important functor in the paper. Among other results, our description also will be used to provide some information (Proposition 4.6) about the complexity of the relative stable Auslander algebras $\text{Aus}(\mathcal{X})$. In section 5, we will begin with the definition of almost split sequences for the subcategories which are closed under extensions and then the functors $\Psi_\mathcal{X}$ and $\Psi_{\mathcal{Y}}$ are used as a tool to transfer the almost split sequences between the corresponding categories. In section 6, the notion of the Auslander-Reiten quivers for a Krull-Schmidt category is introduced and then as a direct consequence of the results of section 5, Theorem 6.1 is stated to give a comparison between the Auslander-Reiten quivers $\Gamma_\mathcal{Y}, \Gamma_{\text{Gprj}\mathcal{X}}$ and $\Gamma_{\text{Gprj}\Lambda}(\Lambda)$. Also, some certain components of $\text{Gprj}\Lambda$ and $\text{Gprj}\Lambda^{-2}(\Lambda)$ are studied and then it is proved that the number of such components is a derived invariant. In the last section, we will provide several examples to show that how our results help to draw the Auslander-Reiten quiver of the subcategories and to determine the representation type of the submodule categories.

2. Preliminaries

2.1. Functors category. Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ a subcategory of $\mathcal{A}$. We denote by $\text{Hom}_{\mathcal{A}}(X, Y)$ the set of morphisms from $X$ to $Y$. Denote by $\text{ind-} \mathcal{A}$ the set of isomorphisms classes of indecomposable objects in $\mathcal{A}$. An (right) $\mathcal{A}$-module is a contravariant additive functor from $\mathcal{A}$ to the category of abelian groups. We call an $\mathcal{A}$-module $F$ finitely presented if there exists an exact sequence $\text{Hom}_{\mathcal{A}}(-, X) \xrightarrow{f} \text{Hom}_{\mathcal{A}}(-, Y) \to F \to 0$. We denote by $\text{mod-} \mathcal{A}$ the category of finitely presented $\mathcal{A}$-modules. We call $\mathcal{C}$ contravariantly (resp. covariantly) finite in $\mathcal{A}$ if $\text{Hom}_{\mathcal{A}}(-, X)[c]$ (resp. $\text{Hom}_{\mathcal{A}}(-, X)[c]$) is a finitely generated $\mathcal{C}$-module for any $X$ in $\mathcal{A}$. 

We call \( \mathcal{C} \) functorially finite if it is contravariantly and covariantly finite. It is known that if \( \mathcal{C} \) is a contravariantly finite subcategory of abelian category \( \mathcal{A} \), then \( \text{mod-}\mathcal{C} \) is an abelian category.

Let \( \mathcal{A} \) be an abelian category with enough projectives and \( \mathcal{C} \) consists of all projective objects of \( \mathcal{A} \). We consider the stable category of \( \mathcal{C} \), denoted by \( \underline{\mathcal{C}} \). The objects of \( \underline{\mathcal{C}} \) are the same as the objects of \( \mathcal{C} \), which we usually denote by \( X \) when an object \( X \in \mathcal{C} \) considered as an object in the stable category, and the morphisms are given by \( \text{Hom}_\mathcal{C}(X,Y) = \text{Hom}_\mathcal{C}(X,Y)/P(X,Y) \), where \( P(X,Y) \) is the subgroup of \( \text{Hom}_\mathcal{C}(X,Y) \) consisting of those morphisms from \( X \) to \( Y \) which factor through a projective object in \( \mathcal{A} \). We also denote by \( \overline{f} \) the residue class of \( f : X \to Y \) in \( \text{Hom}_\mathcal{C}(X,Y) \). In order to simplify, we will use \((- , X) \), resp. \((- , X) \), for the representable functor \( \text{Hom}_\mathcal{C}(- , X) \), resp. \( \text{Hom}_\mathcal{C}(- , X) \), in \( \text{mod-}\mathcal{C} \), resp. \( \text{mod-}\mathcal{C} \). It is well-known that the canonical functor \( \pi : \mathcal{C} \to \underline{\mathcal{C}} \) induces a fully faithful functor functor \( \pi^* : \text{mod-}\underline{\mathcal{C}} \to \text{mod-}\mathcal{C} \). Hence due to this embedding we can identify the functors in \( \text{mod-}\underline{\mathcal{C}} \) as functors in \( \text{mod-}\mathcal{C} \).

We assume throughout this paper that \( \Lambda \) is an Artin algebra over commutative artinian ring \( k \). A subcategory \( \mathcal{X} \) of \( \text{mod-}\Lambda \), the category of finitely generated right \( \Lambda \)-modules, is always a full subcategory of \( \text{mod-}\Lambda \) closed under isomorphisms, finite direct sums and direct summands.

The subcategory \( \mathcal{X} \) is called of finite representation type, or simply representation-finite, if \( \text{mod-}\Lambda \) is of finite representation type. If \( \mathcal{X} \) is of finite representation type, then it admits a representation generator, i.e., there exists \( X \in \mathcal{X} \) such that \( \mathcal{X} = \text{add-}X \), the subcategory of \( \text{mod-}\Lambda \) consisting of all direct summands of all finite direct sums of copies of \( X \). It is known that \( \text{add-}X \) is a functorially finite subcategory of \( \text{mod-}\Lambda \).

To avoid complicated notations, for \( \text{prj-}\Lambda \subset \mathcal{X} \subset \text{mod-}\Lambda \), we show \( \text{Hom}_\mathcal{X}(X,Y) \), resp. \( \text{Hom}_\Lambda(X,Y) \), by \( \text{Hom}_\mathcal{X}(X,Y) \) and \( \text{Hom}_\Lambda(X,Y) \) respectively. Set \( \text{Aus}(\mathcal{X},X) = \text{End}_\Lambda(X) \) whenever \( \mathcal{X} \) is a subcategory with representation generator \( X \). Clearly \( \text{Aus}(\mathcal{X},X) \) is an Artin algebra. It is known that the evaluation functor \( \zeta_X : \text{mod-}\mathcal{X} \to \text{mod-}\text{Aus}(\mathcal{X},X) \) defined by \( \zeta_X(F) = F(X) \), for \( F \in \text{mod-}\mathcal{X} \), gives an equivalence of categories. It also induces an equivalence of categories \( \text{mod-}\mathcal{X} \simeq \text{mod-}\text{Aus}(\mathcal{X},X) \), only by the restriction. Recall that \( \text{Aus}(\mathcal{X},X) = \text{End}_\Lambda(X)/\mathcal{P} \), where \( \mathcal{P} = \mathcal{P}(X,Y) \). The Artin algebra \( \text{Aus}(\mathcal{X},X) \), resp. \( \text{Aus}(\mathcal{X},X), \) is called the relative, resp. stable, Auslander algebra of \( \Lambda \) with respect to the subcategory \( \mathcal{X} \) and with the representation generator \( X \). For the case \( \mathcal{X} = \text{mod-}\Lambda \), we obtain the stable Auslander algebras. In fact, if \( X' \) is another representation generator of \( \mathcal{X} \), then \( \text{Aus}(\mathcal{X},X) \), resp. \( \text{Aus}(\mathcal{X},X') \), and \( \text{Aus}(\mathcal{X},X'), \) resp. \( \text{Aus}(\mathcal{X},X') \), are Morita equivalent. But if both are basic, i.e., the multiplicity of the indecomposable direct summand to be at most one, in this case, stronger situation occurs, that is, \( \text{Aus}(\mathcal{X},X) \simeq \text{Aus}(\mathcal{X},X'), \) resp. \( \text{Aus}(\mathcal{X},X') \simeq \text{Aus}(\mathcal{X},X'), \) as isomorphism of algebras. If no ambiguity may rise, for simplicity, \( \text{Aus}(\mathcal{X}), \text{resp.} \text{Aus}(\mathcal{X}) \), usually means the relative, resp. stable, Auslander algebra \( \text{Aus}(\mathcal{X},X), \text{resp.} \text{Aus}(\mathcal{X},X) \), of \( \Lambda \) with respect to \( \mathcal{X} \) for some basic representation generator of \( \mathcal{X} \).

The fact, which we need later, from [Au2, Chapter 2], is any simple functor \( S \) in \( \text{mod-}\mathcal{X} \) is isomorphic to \( (- , X)/r(- , X) \), where \( X \) is an indecomposable non-projective module in \( \mathcal{X} \) and \( r(- , X) \) is the radical functor of \( (- , X) \) in \( \text{mod-}\mathcal{X} \). The notions of the radical of a functor and also simple functors are defined in analogy with of those in the module category of rings. Further, if \( \mathcal{X} \) is functorially finite and closed under extensions, then \( S \) has the following minimal projective resolution in \( \text{mod-}\mathcal{X} \)

\[
0 \to (-, Z) \xrightarrow{(-f)} (-, Y) \xrightarrow{(-g)} (-, X) \to S \to 0,
\]

where \( 0 \to Z \xrightarrow{f} Y \xrightarrow{g} X \to 0 \) is an almost split sequence in \( \mathcal{X} \), see [AS] for details and definition of almost split sequences for subcategories.
2.2. Gorenstein projective objects. Let $\mathcal{A}$ be an abelian category with enough projectives. A complex

$P^\bullet : \ldots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \rightarrow \ldots$

of projective objects in $\mathcal{A}$ is said to be totally acyclic provided it is acyclic and the Hom complex $\text{Hom}_\Lambda(P^\bullet, Q)$ is also acyclic for any projective object $Q$ in $\mathcal{A}$. An object $M$ in $\mathcal{A}$ is said to be Gorenstein projective provided that there is a totally acyclic complex $P^\bullet$ of projective objects over $\mathcal{A}$ such that $M \cong \ker(d^0)$. We denote by $\text{Gprj-}\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of all Gorenstein projective objects in $\mathcal{A}$. Whenever, for an additive category $\mathcal{C}$, $\text{mod-}\mathcal{C}$ is abelian, we will use $\text{Grpj}-\mathcal{C}$ to show the Gorenstein projective object in $\text{mod-}\mathcal{C}$. Also, for simplicity, when $\mathcal{A} = \text{mod-}\Lambda$, the subcategory of Gorenstein projective module in $\text{mod-}\Lambda$ is shown by $\text{Gprj-}\Lambda$.

An Artin algebra $\Lambda$ is of finite Cohen-Macaulay type, or simply, CM-finite, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective $\Lambda$-modules. Clearly, $\Lambda$ is a CM-finite algebra if and only if there is a finitely generated module $E$ such that $\text{Gprj-}\Lambda = \text{add-E}$. In this case, $E$ is called to be a Gorenstein projective representation generator of $\text{Gprj-}\Lambda$. If $\text{gldim } \Lambda < \infty$, then $\text{Gprj-}\Lambda = \text{prj-}\Lambda$, so $\Lambda$ is CM-finite. If $\Lambda$ is self-injective, then $\text{Gprj-}\Lambda = \text{mod-}\Lambda$, so $\Lambda$ is CM-finite if and only if $\Lambda$ is representation-finite. If $E$ is a basic Gorenstein projective representation generator of $\text{Gprj-}\Lambda$, then the relative (stable) Auslander algebra $\text{Aus}(\text{Gprj-}\Lambda, E) = \text{End}_\Lambda(E)$, resp. $\text{Aus}(\text{Gprj-}\Lambda, E) = \text{End}_\Lambda(E)$, in the literature is usually called the (stable) Cohen-Macaulay Auslander algebra of $\Lambda$. The reason of such a naming might be since in some context use “Cohen-Macaulay” instead of “Gorenstein-projective”.

2.3. Triangular matrix algebras. Let $\mathcal{A}$ be an abelian category and let $H(\mathcal{A})$ be the morphism category over $\mathcal{A}$. Indeed, the objects in $H(\mathcal{A})$ are the morphisms in $\mathcal{A}$, and morphisms are given by the commutative diagrams. We can consider objects in $H(\mathcal{A})$ as the representations over the quiver $A_2 : v \rightarrow w$ by objects and morphisms in $\mathcal{A}$, usually denoted by $\text{rep}(A_2, \mathcal{A})$. In case that $\mathcal{A} = \text{mod-}\Lambda$ we know by a general fact the category $\text{rep}(A_2, \text{mod-}\Lambda)$, or simply $\text{rep}(A_2, \Lambda)$, is equivalent to the category of finitely generated right module over the path algebra $\Lambda A_2 \simeq T_2(\Lambda)$, where $T_2(\Lambda) = \left[ \begin{array}{cc} \Lambda & \Lambda \\ 0 & \Lambda \end{array} \right]$, upper triangular $2 \times 2$ matrix algebra over $\Lambda$. Since the categories $\text{rep}(A_2, \Lambda)$, or $H(\text{mod-}\Lambda)$, simply $H(\Lambda)$, are equivalent to $\text{mod-}T_2(\Lambda)$, then by these equivalences we can naturally define the notion of Gorenstein projective representations (morphisms) in $\text{rep}(A_2, \Lambda)$ ($H(\text{mod-}\Lambda)$), coming from the concept of Gorenstein projective module over $T_2(\Lambda)$. There is the following local characterization of Gorenstein projective representations in $\text{rep}(A_2, \Lambda)$:

**Lemma 2.1.** (EHS, Theorem 3.5.1) or [LZ, Theorem 5.1]) Let $X \xrightarrow{f} Y$ be a representation in $\text{rep}(A_2, \Lambda)$. Then $X \xrightarrow{f} Y$ is a Gorenstein projective representation if and only if (1) $X, Y$ and coker $f$ are in $\text{Gprj-}\Lambda$, and (2) $f$ is a monomorphism.

Throughout of the paper, we completely free use the identification between objects in $H(\Lambda)$ and modules in $\text{mod-}T_2(\Lambda)$.

3. An equivalence

let $\mathcal{A}$ be an abelian category with enough projectives in this section. Following [MT], a subcategory $\mathcal{C}$ of $\mathcal{A}$ is called quasi-resolving if it contains the projective objects of $\mathcal{A}$, closed under finite direct sums and closed under kernels of epimorphisms in $\mathcal{C}$. Moreover, a quasi-resolving
subcategory \( C \) of \( A \) is called \textit{resolving} if it is closed under isomorphisms, direct summands and closed under extensions. In this case, as mentioned in [MT, Proposition 2.11], \( \text{mod-}C \) is an abelian category with enough projectives, although \( \text{mod-}C \) might not to be always abelian. In this section we show that for a resolving subcategory of \( A \), the category \( \text{mod-}C \) of finitely presented functors over \( C \) is realized by the (additive) quotient category of a subcategory of the morphism category of \( A \) modulo a relation generated by some objects.

Assume \( C \) is a full subcategory of an additive category \( D \). Denote by \( C \), the ideal of morphism in \( D \) which factor through an object in \( C \). The quotient category \( D/C \) has the same objects as \( D \) but Hom-space of morphisms

\[
\text{Hom}_{D/C}(X,Y) := \text{Hom}_D(X,Y)/[C](X,Y).
\]

For a given subcategory \( C \) of an abelian category \( A \), we assign the subcategory \( S_C(A) \) of \( H(A) \) consisting of morphism \( A \xrightarrow{f} B \) satisfying:

(i) \( f \) is a monomorphism;
(ii) \( A, B \) and \( \text{Coker}(f) \) belong to \( X \).

In the case that \( A = \text{mod-}\Lambda \), for \( X \subseteq \text{mod-}\Lambda \) we show \( S_X(\text{mod-}\Lambda) = S_X(\Lambda) \).

We define a functor \( \Psi_C : S_C(A) \to \text{mod-}C \) respect to a subcategory \( C \) of \( A \) as follows.

\textbf{Construction 3.1.} Taking an object \( A \xrightarrow{f} B \) of \( S_C(A) \), then we have the following short exact sequence

\[
0 \to A \xrightarrow{f} B \to \text{Coker } f \to 0
\]

in \( A \), this in turn gives the following short exact sequence

\[
(*) \quad 0 \to (-, A) \xrightarrow{(-, f)} (-, B) \to (-, \text{Coker } f) \to F \to 0
\]

in \( \text{mod-}C \). In fact, \((*)\) corresponds to a projective resolution of \( F \) in \( \text{mod-}C \). We define \( \Psi_C(A \xrightarrow{f} B) := F \).

For morphism: Let \( \sigma = (\sigma_1, \sigma_2) \) be a morphism from \( A \xrightarrow{f} B \) to \( A' \xrightarrow{f'} B' \), it makes the following commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & A & \xrightarrow{f'} & A' & \xrightarrow{f'} & B' & \xrightarrow{f} & B & \xrightarrow{f} & \text{Coker } f & \xrightarrow{f} & \text{Coker } f' & \xrightarrow{f} & 0 \\
\downarrow{\sigma_1} & & \downarrow{\sigma_2} & & \downarrow{\sigma_3} & & & & \downarrow{\sigma_3} & & & & \downarrow{\sigma_3} & & \\
0 & \xrightarrow{f'} & A' & \xrightarrow{f'} & B' & \xrightarrow{f'} & \text{Coker } f' & \xrightarrow{f'} & 0,
\end{array}
\]

that \( \sigma_3 \) can be determined uniquely by \( \sigma_1 \) and \( \sigma_2 \). By applying the Yoneda functor, the above diagram gives the following diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{(-, f)} & (-, A) & \xrightarrow{(-, f')} & (-, A') & \xrightarrow{(-, f')} & (-, A') & \xrightarrow{(-, f)} & (-, B) & \xrightarrow{(-, f)} & (-, B) & \xrightarrow{(-, f)} & (-, \text{Coker } f) & \xrightarrow{(-, \text{Coker } f)} & (-, \text{Coker } f') & \xrightarrow{(-, \text{Coker } f')} & (-, \text{Coker } f') & \xrightarrow{(-, \text{Coker } f)} & F' & \xrightarrow{(-, \text{Coker } f')} & 0.
\end{array}
\]

in \( \text{mod-}C \). Define \( \Psi_C(\sigma) := (-, \sigma_3) \), which is obtained uniquely by \( \sigma_1 \) and \( \sigma_2 \).

The following result was first stated in the unpublished work [H1] by the author.
Theorem 3.2. Let \( A \) be an abelian category with enough projectives. Let \( \mathcal{C} \) be a quasi-resolving subcategory of \( A \). Consider the full subcategory \( \mathcal{V} \) of \( S_{\mathcal{C}}(A) \) formed by all finite direct sums of objects in the form of \( (C \xrightarrow{id} C) \) or \( (0 \to C) \), that \( C \) runs through objects in \( \mathcal{C} \). Then the functor \( \Psi_{\mathcal{C}} \), defined in the above construction, induces the following equivalence of categories

\[
S_{\mathcal{C}}(A)/\mathcal{V} \simeq \text{mod-} \mathcal{C}.
\]

Proof. The functor \( \Psi_{\mathcal{C}} \) is dense. Let \( F \in \text{mod-} \mathcal{C} \). By identifying \( F \) as an object in \( \text{mod-} \mathcal{C} \) we have a projective presentation \((-,B) \xrightarrow{\sim g} (-,D) \to F \to 0\) such that \( g \) is an epimorphism. The assumption \( \mathcal{C} \) being closed under kernels of epimorphisms implies \( \text{Ker}(g) \) belongs to \( \mathcal{C} \). Hence we get the projective resolution \( 0 \to (-,\text{Ker}(g)) \to (-,B) \to (-,D) \to F \to 0 \), in \( \text{mod-} \mathcal{C} \) and consequently \( \Psi_{\mathcal{C}}(\text{Ker}(g) \to B) = F \). Any morphism between functors in \( \text{mod-} \mathcal{C} \) is considered as a morphism in \( \text{mod-} \mathcal{C} \), so can be lifted to the corresponding projective resolutions. Then by using Yoneda’s Lemma we can obtain a morphism in \( S_{\mathcal{C}}(A) \) to prove fullness. If \( \Psi_{\mathcal{C}}(A \xrightarrow{f} B) = 0 \), then by definition,

\[
0 \to (-,A) \xrightarrow{(-,f)} (-,B) \xrightarrow{(-,g)} (-,\text{Coker } f) \to 0.
\]

Now by putting \( \text{Coker } f \), as it belongs to \( \mathcal{C} \), in the above short exact sequence we get

\[
0 \to A \xrightarrow{f} B \to \text{Coker } f \to 0
\]

is splitting. This gives us \( A \xrightarrow{f} B \) being isomorphic to the object \((A \to \text{Im } f) \oplus (\text{Ker } g) \to 0\) in \( \text{H}(A) \), and so \( f \) belongs to \( \mathcal{V} \). Note that \( \text{Ker } g \in \mathcal{C} \): as \( A \simeq \text{Im } f \) then \( \text{Im } f \) is in \( \mathcal{C} \), and on the other hand, by the following split short exact sequence

\[
0 \to \text{Ker } g \to B \simeq \text{Im } f \oplus \text{Ker } g \to \text{Im } f \to 0,
\]

and using our assumption being closed under kernels of epimorphisms proves our claim. Assume that \( \Psi_{\mathcal{C}}(\sigma) = 0 \), for \( \sigma = (\sigma_1, \sigma_2) : (A \xrightarrow{i} B) \to (A' \xrightarrow{i'} B') \) in \( S_{\mathcal{C}}(A) \). Therefore, we have

\[
0 \longrightarrow (-,A) \xrightarrow{(-,f)} (-,B) \longrightarrow (-,\text{Coker } f) \longrightarrow F \longrightarrow 0
\]

Since the first (resp. second) row of the above diagram is a projective resolution for \( F \) (resp. \( F' \)) in \( \text{mod-} \mathcal{C} \). Then by considering the following commutative diagram

\[
\cdots \longrightarrow (-,A) \xrightarrow{(-,f)} (-,B) \longrightarrow (-,\text{Coker } f) \longrightarrow \cdots
\]

\[
\cdots \longrightarrow (-,A') \xrightarrow{(-,f')} (-,B') \longrightarrow (-,\text{Coker } f') \longrightarrow \cdots
\]

as a chain map in \( \mathcal{C}^b(\text{mod-} \mathcal{C}) \), the category of bonded complexes on \( \text{mod-} \mathcal{C} \), should be null-homotopic. Hence by using a standard argument, the above chain map should be factored through a projective complex in \( \mathcal{C}^b(\text{mod-} \mathcal{C}) \) as follows:
This factorization as the above gives us a factorization of morphism $\sigma$ through the direct sum of $A' \to A'$ and $0 \to B'$ in $S_C(A)$. So $\Psi_C$ is an objective functor in the sense of [RZ].

So $\Psi_C$ is full, dense and objective. Hence by [RZ, Appendix], $\Psi_C$ induces an equivalence between $S_C(A)/V$ and mod-$\mathcal{C}$. So we are done. \hfill $\square$

In fact, the above theorem is a relative version of the equivalences given in [RZ] and [E].

**Remark 3.3.** As we were informed later by an unknown referee the above equivalence is a consequence of [B, Proposition 3.8 and Corollary 3.9 (i)] in a completely different approach. But what is important more for us here not really the equivalence itself, as will be shown in the next sections, to see that how the functor $\Psi_X$ helps to transfer the representation theory from $S_X(\Lambda)$ to mod-$\mathcal{X}$, for a quasi-resolving subcategory $\mathcal{X}$ of mod-$\Lambda$.

Let $\mathcal{X}$ be a quasi-resolving subcategory of mod-$\Lambda$ and of finite representation type. Denote by $\mathcal{C}$ the full additive subcategory of $S_X(\Lambda)$ consisting of all indecomposable objects in $S_X(\Lambda)$ not isomorphic to an object of the form either $X \to X$ or $0 \to X$ with $X$ indecomposable module in $\mathcal{X}$. Then based on the the equivalence in Theorem 3.2 and in view of Lemma 5.1, one can see there is a bijection between the modules in $\mathcal{C}$ and the indecomposable modules in mod-Aus($\mathcal{X}$). Hence, $S_X(\Lambda)$ is a subcategory of finite representation type of $H(\text{mod-}\Lambda)$ if and only if the algebra Aus($\mathcal{X}$) is representation-finite. To make easier for giving reference let us state the above observation as the following theorem:

**Theorem 3.4.** Let $\mathcal{X} \subset \text{mod-}\Lambda$ be a subcategory of finite representation and quasi-resolving. Then, the following conditions are equivalent

1. The relative stable Auslander algebra $\text{Aus}(\mathcal{X})$ is of finite representation type;
2. The subcategory $S_X(\Lambda)$ of mod-$T_2(\Lambda)$ is of finite representation type.

By using Lemma 4.12, we observe $S_{\text{Gprj}-\Lambda}(\Lambda) \simeq \text{Gprj-T}_2(\Lambda)$. Thus by Theorem 3.4, we can say for a CM-finite algebra $\Lambda$: $T_2(\Lambda)$ is CM-finite if and only if the stable Cohen-Macaulay Auslander algebra, Aus($\text{Gprj}-\Lambda$), is representation-finite. In particular, if assume $\Lambda$ is a self-injective of finite representation type, then $T_2(\Lambda)$ is CM-finite if and only if the stable Auslander algebra, Aus($\text{mod-}\Lambda$), is representation-finite.

Let us give an easy application by the above observation in the following example.

**Example 3.5.** Let $A = k\mathcal{Q}/I$ be a quadratic monomial algebra, i.e. the ideal $I$ is generated by paths of length two. By [CSZ, Theorem 5.7], $\text{Gprj-}A \simeq T_1 \times \cdots \times T_n$ such that the underlying categories of triangulated categories $T_i$ are equivalent to semisimple abelian categories mod-$k^{d_i}$, for some natural numbers $d_i$. Hence mod-$\text{Gprj-}A$ is a semisimple abelian category and consequently Aus($\text{Gprj-}A$) a semisimple Artin algebra. Thus Theorem 3.4 implies $T_2(\Lambda)$ is CM-finite.
4. Syzygies

We begin this section by giving an explicit description of the syzygies (up to projective summands) of any functor $F$ in mod-$\mathcal{X}$ via their projective resolutions in mod-$\mathcal{X}$, here $\mathcal{X}$ is a quasi-resolving subcategory of mod-$\Lambda$. Among other immediate results, this description helps us to give a classification of the non-projective Gorenstein projective functors in mod-$\mathcal{X}$ via their projective resolutions in mod-$\mathcal{X}$. To have such a classification we need some more conditions on $\mathcal{X}$. By such a classification, as our main aim of this section, we will define a fully faithful (and almost dense whenever $\mathcal{X}$ is of finite representation type) functor from mod-$\mathcal{Y}$ to Gprj-$\mathcal{X}$, where $\mathcal{Y} = \mathcal{X} \cap \text{Gprj-}\Lambda$.

Given a $\Lambda$-module $M$, denote the kernel of the projective cover $P_M \to M$ by $\Omega_{\Lambda}(M)$. $\Omega_{\Lambda}(M)$ is called the first syzygy of $M$. We let $\Omega_{\Lambda}^0(M) = M$ and then inductively for each $i \geq 1$, set $\Omega_{\Lambda}^i(M) = \Omega_{\Lambda}(\Omega_{\Lambda}^{i-1}(M))$. Similarly, whenever for an additive category $\mathcal{C}$, mod-$\mathcal{C}$ is a semi-perfect abelian, i.e., an abelian category such that any object has projective cover, one can define the $n$-the syzygy of a functor $F$ in mod-$\mathcal{C}$, and denoted by $\Omega_{\mathcal{C}}^n(F)$. Let $\mathcal{X}$ be a contravariantly finite and quasi-resolving subcategory of mod-$\Lambda$. Then $\mathcal{X}$ and $\mathcal{X}$ both are varieties of annul, in the scene of [Au1]. On the other hand, since for each $\mathcal{X}$, resp. $\mathcal{X}$, in $\mathcal{X}$, resp. $\mathcal{X}$, $\text{End}_\Lambda(\mathcal{X})$, resp. $\text{End}_\Lambda(\mathcal{X})$, is clearly semi-perfect ring, then by [Au1, Corollary 4.13], mod-$\mathcal{X}$ and mod-$\mathcal{X}$ both have projective covers.

For $n \geq 0$, let $\Omega_{\mathcal{X}}^n(\mathcal{X})$ denote the subcategory of mod-$\Lambda$ consisting of all modules $M$ such that $M \cong Q \oplus N$, where $Q \in \text{prj-}\Lambda$ and $N = \Omega_{\mathcal{X}}^n(\mathcal{X})$ for some $\mathcal{X}$ in $\mathcal{X}$.

**Proposition 4.1.** Let $\mathcal{X} \subseteq \text{mod-}\Lambda$ be a contravariantly finite and quasi-resolving subcategory of mod-$\Lambda$. Let $F \in \text{mod-}\mathcal{X}$ and $0 \to (\mathcal{X}) \to (\mathcal{X}) \to F \to 0$ be a projective resolution of $F$ in mod-$\mathcal{X}$. Then, there is a short exact sequence $0 \to G \to (\mathcal{X}) \to F \to 0$, in mod-$\mathcal{X}$, such that $G$ has the following projective resolution $0 \to (\mathcal{X}) \to (\mathcal{X}) \to (\mathcal{X}) \to G \to 0$ in mod-$\mathcal{X}$, where $P_C$ is the projective cover of $C$ in mod-$\Lambda$.

**Proof.** Since $F$ vanishes on projective modules, then we have the exact sequence $0 \to (\mathcal{X}) \to (\mathcal{X}) \to F \to 0$ in mod-$\mathcal{X}$. Letting $G$ be the kernel of the epimorphism $(\mathcal{X}) \to F \to 0$, then we have the short exact sequence $0 \to G \to (\mathcal{X}) \to F \to 0$ in mod-$\mathcal{X}$. Consider the following pull-back diagram in mod-$\mathcal{X}$:
By using an standard argument, we get the following commutative diagram with exact columns and rows and the middle row splitting:

\[
\begin{array}{c}
0 \\
\downarrow \\
\rightarrow (-, A) \rightarrow D_1 \rightarrow H_1 \rightarrow 0 \\
\downarrow \\
\rightarrow (-, B) \oplus (-, C) \rightarrow (-, C) \rightarrow 0 \\
\downarrow \\
\rightarrow K_1 \rightarrow M \rightarrow (-, C) \rightarrow 0 \\
\end{array}
\]

Note that in the above digram \( H_1 \) is obtained by the minimal projective resolution \( 0 \rightarrow (-, \Omega_A(C)) \rightarrow (-, P_C) \rightarrow (-, C) \rightarrow (-, C) \rightarrow 0 \) in mod-\( X \). Also, we have the following pull-back diagram in mod-\( X \):

\[
\begin{array}{c}
0 \\
\downarrow \\
\rightarrow (-, B) \rightarrow (-, B) \oplus (-, C) \rightarrow (-, C) \rightarrow 0 \\
\downarrow \\
\rightarrow G \rightarrow M \rightarrow (-, C) \rightarrow 0 \\
\end{array}
\]
Again, we have the following pull-back diagram in mod-\(\mathcal{X}\):

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & (-, \Lambda(C)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

By gluing the short exact sequences 0 → \(D_1 \rightarrow (-, B) \rightarrow G \rightarrow 0\) and 0 → \((-, \Omega(C)) \rightarrow (-, A ⊕ P_C) \rightarrow D_1 \rightarrow 0\), we get the desired projective resolution in the statement.

In particular, the minimal projective resolution \(\Omega(\mathcal{X})F\) of \(F\) in mod-\(\mathcal{X}\) is a direct summand of the projective resolution appeared in the statement of above proposition.

By iterating use of the above proposition we get the following corollary.

**Corollary 4.2.** Let \(\mathcal{X} \subseteq \text{mod-}\Lambda\) be a contravariantly finite and quasi-resolving subcategory of mod-\(\Lambda\). Let \(F \in \text{mod-}\mathcal{X}\) and 0 → \((-, A) \rightarrow (-, B) \rightarrow (-, C) \rightarrow F \rightarrow 0\) be a projective resolution of \(F\) in mod-\(\mathcal{X}\). Then, for each \(n > 0\), we have the following situations.

1. If \(n = 3k\), then \(\Omega_{\Lambda}^k(F) \simeq G\) in the stable category mod-\(\mathcal{X}\), where \(G\) is settled in the following short exact sequence

   \[
   0 \rightarrow (-, \Lambda^k(A)) \rightarrow (-, \Lambda^k(B) \oplus Q) \rightarrow (-, \Lambda^k(C) \oplus P) \rightarrow G \rightarrow 0,
   \]

   for some projective modules \(P\) and \(Q\) in prj-\(\Lambda\), in mod-\(\mathcal{X}\).

2. If \(n = 3k + 1\), then \(\Omega_{\Lambda}^k(F) \simeq G\) in the stable category mod-\(\mathcal{X}\), where \(G\) is settled in the following short exact sequence

   \[
   0 \rightarrow (-, \Lambda^{k+1}(C)) \rightarrow (-, \Lambda^k(A) \oplus Q) \rightarrow (-, \Lambda^k(B) \oplus P) \rightarrow G \rightarrow 0,
   \]

   for some projective modules \(P\) and \(Q\) in prj-\(\Lambda\), in mod-\(\mathcal{X}\).

3. If \(n = 3k + 2\), then \(\Omega_{\Lambda}^k(F) \simeq G\) in the stable category mod-\(\mathcal{X}\), where \(G\) is settled in the following short exact sequence

   \[
   0 \rightarrow (-, \Lambda^{k+1}(B)) \rightarrow (-, \Lambda^{k+1}(C) \oplus Q) \rightarrow (-, \Lambda^k(A) \oplus P) \rightarrow G \rightarrow 0,
   \]

   for some projective modules \(P\) and \(Q\) in prj-\(\Lambda\), in mod-\(\mathcal{X}\).

As a direct consequence of the above result, we obtain the following sufficient condition for \(\mathcal{X}\) to be regular in terms of \(S(\Lambda)\). Assume that \(\mathcal{C}\) is an additive category such that mod-\(\mathcal{C}\) is abelian with enough projectives. Recall that \(\mathcal{C}\) is said to be regular if every object of mod-\(\mathcal{C}\) has finite projective dimension.
Proposition 4.3. Let $\mathcal{X} \subseteq \text{mod-}\Lambda$ be a contravariantly finite and quasi-resolving subcategory of \text{mod-}\Lambda. If for each object $A \xrightarrow{f} B$ in $\mathcal{S}_\mathcal{X}(\Lambda)$, either of the modules $A, B$ and $\text{Cok}(f)$ has a finite projective dimension, then $\mathcal{X}$ is regular. In particular, if $\mathcal{X}$ is of finite representation type, then the relative stable Auslander algebra $\text{Aus}(\mathcal{X})$ is a regular algebra.

Proof. Assume that one of the modules $A, B$ and $\text{Cok}(f)$ of a given monomorphism $A \xrightarrow{f} B$ in $\mathcal{S}_\mathcal{X}(\Lambda)$ has a finite projective dimension, namely $A$. We can choose sufficiently large positive integer $3k + 1$ such that $\text{Of}_{3k+1}(A)$ is projective. Next, Corollary 4.2 implies that $\text{Of}_{\mathcal{X}}(F)$ is a direct summand of $(-, \text{Of}_{\Lambda}(B))$, and consequently $\text{Of}_{\mathcal{X}}(F)$ is a projective object. So we are done.  

The projective resolutions of the syzygy functors given in Corollary 4.2 are not necessarily minimal. In the next results, we will try to obtain the projective resolutions of the ones in Corollary 4.2 to be more close to the minimal ones.

Lemma 4.4. The projective module $P$ appeared in the first term of the projective resolution of $G$ stated in Corollary 4.2, in each case, is redundant.

Proof. To prove the statement we shall use this general fact: Assume that functor $H$ in $\text{mod-}\mathcal{X}$ has the projective resolution $0 \rightarrow (-, M) \rightarrow (-, L) \rightarrow (-, N \oplus P') \rightarrow F \rightarrow 0$ in $\text{mod-}\mathcal{X}$ with projective module $P'$. Then there is a projective resolution in the following form of $H$

$$0 \rightarrow (-, M) \rightarrow (-, L) \rightarrow (-, N) \rightarrow F \rightarrow 0.$$  

Consider the following pull-back diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & M & \rightarrow & U & \rightarrow & N & \rightarrow & 0 \\
| & | & | & | & | & | & | \\
0 & \rightarrow & M & \rightarrow & L & \rightarrow & N \oplus P' & \rightarrow & 0 \\
| & | & | & | & | & | & | \\
P' & \rightarrow & P' & \rightarrow & 0 & \rightarrow & 0
\end{array}
$$

where the monomorphism in the last column is the natural injection and the middle row is the induced short exact in $\text{mod-}\Lambda$ from the projective resolution of $H$. The above diagram induces
the following commutative diagram by applying the Yoneda functor

\[
\begin{array}{cccccc}
0 & \rightarrow & (-, M) & \rightarrow & (-, U) & \rightarrow & (-, N) & \rightarrow & H_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \phi & & \\
0 & \rightarrow & (-, M) & \rightarrow & (-, L) & \rightarrow & (-, N \oplus P') & \rightarrow & H & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
(-, P') & \rightarrow & H/H_0 & \rightarrow & 0 & & & & & & \\
\end{array}
\]

in mod-\(\mathcal{X}\). To get \(\phi\) being a monomorphism we did only a digram chasing. Since \(H\) belongs to mod-\(\mathcal{X}\) then also \(H/H_0\) is contained in mod-\(\mathcal{X}\). Hence whose minimal projective resolution in mod-\(\mathcal{X}\) is coming from a short exact sequence in mod-\(\Lambda\), namely, \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\). But since we have the epimorphism \((-, P') \rightarrow H/H_0\), by the above diagram, hence \((-, C)\) must be a direct summand of \((-, P')\), so \(C\) is projective. Consequently, \(H/H_0 = 0\) and so \(H = H_0\). Thus we are done. \(\square\)

By inductively using Proposition 4.1 and Lemma 4.4, we get the following nicer projective resolutions than the given ones in Corollary 4.2.

**Corollary 4.5.** Let \(\mathcal{X} \subseteq \text{mod-}\Lambda\) be a contravariantly finite and quasi-resolving subcategory of mod-\(\Lambda\). Let \(F \in \text{mod-}\mathcal{X}\) and \(0 \rightarrow (-, A) \rightarrow (-, B) \rightarrow (-, C) \rightarrow F \rightarrow 0\) be a projective resolution of \(F\) in mod-\(\mathcal{X}\). Then, for each \(n > 0\), we have the following situations.

1. If \(n = 3k\), then \(\Omega_n\mathcal{X}(F) \simeq G\) in the stable category mod-\(\mathcal{X}\), where \(G\) is settled in the following short exact sequence

\[
0 \rightarrow (-, \Omega_k^k(A)) \rightarrow (-, \Omega_k^k(B) \oplus Q) \rightarrow (-, \Omega_k^k(C)) \rightarrow G \rightarrow 0,
\]

where \(Q\) is the projective cover of \(\Omega_k^k(A)\).

2. If \(n = 3k + 1\), then \(\Omega_n\mathcal{X}(F) \simeq G\) in the stable category mod-\(\mathcal{X}\), where \(G\) is settled in the following short exact sequence

\[
0 \rightarrow (-, \Omega_k^{k+1}(C)) \rightarrow (-, \Omega_k^k(A) \oplus Q) \rightarrow (-, \Omega_k^k(B)) \rightarrow G \rightarrow 0,
\]

where \(Q\) is the projective cover of \(\Omega_k^{k+1}(C)\).

3. If \(n = 3k + 2\), then \(\Omega_n\mathcal{X}(F) \simeq G\) in the stable category mod-\(\mathcal{X}\), where \(G\) is settled in the following short exact sequence

\[
0 \rightarrow (-, \Omega_k^{k+1}(B)) \rightarrow (-, \Omega_k^{k+1}(C) \oplus Q) \rightarrow (-, \Omega_k^{k+1}(A)) \rightarrow G \rightarrow 0,
\]

where \(Q\) is the projective cover of \(\Omega_k^{k+1}(B)\).

Let us give another easy consequences of our investigation of the syzygies of the functors to construct algebras with with complexity of each modules at most one.
4.1. **The complexity.** We recall the definition of complexity of a module which was first introduced by Alperin and Evens in the realm of group algebras in [AE]. Let $M$ be a module in mod-$\Lambda$ with minimal projective resolution

$$\ldots \rightarrow P^n \xrightarrow{\delta} P^{n-1} \rightarrow \ldots \rightarrow P^0 \xrightarrow{\delta_0} M \rightarrow 0.$$ 

The $i$-th Betti number of $M$, $\beta^i_\Lambda(M)$, equals the number of indecomposable direct summands of $P^i$. Finally, the complexity of $M$ over $\Lambda$ is defined as

$$\text{cx}_\Lambda(M) = \inf\{t \in \mathbb{N} \mid \exists \alpha \in \mathbb{R} \quad \beta^i_\Lambda(M) \leq \alpha t^{-1} \text{ for } i \gg 0\}.$$ 

In fact, the complexity of a module measures the growth of its minimal projective resolution. Notice $\text{cx}_\Lambda(M) = 1$ means that the Betti numbers are bounded, $\text{cx}_\Lambda(M) = 0$ means that that $M$ has finite projective dimension.

**Proposition 4.6.** Let $\mathcal{X} \subseteq \text{mod-}\Lambda$ be a quasi-resolving subcategory of finite representation type. If for any $X \in \mathcal{X}$, $\text{cx}_\Lambda(X) \leq 1$, then all (finitely generated) modules over the relative stable Aus($\mathcal{X}$) has complexity at most one.

**Proof.** Let $F$ be a functor in mod-$\mathcal{X}$ with the following projective resolution

$$0 \rightarrow (-, A) \rightarrow (-, B) \rightarrow (-, C) \rightarrow F \rightarrow 0,$$

in mod-$\mathcal{X}$. Let $n > 0$. Assume first $n = 3k$ for some $k$. By Corollary 4.5, there is a $G$ in mod-$\mathcal{X}$ such that $\Omega^n_\Lambda(F) \simeq G$ in the stable category $\text{mod-Aus(}\mathcal{X}\text{)}$, and with the following projective resolution

$$0 \rightarrow (-, \Omega^k_\Lambda(A)) \rightarrow (-, \Omega^k_\Lambda(B) \oplus Q) \rightarrow (-, \Omega^k_\Lambda(C)) \rightarrow G \rightarrow 0 \ (\dagger)$$

Moreover in view of the construction of $G$, we have an exact sequence $0 \rightarrow G \rightarrow P^{n-1} \rightarrow \ldots \rightarrow P^0 \rightarrow G \rightarrow 0$ in mod-$\mathcal{X}$ with projective functors $P^n$ (in mod-$\mathcal{X}$). Hence by the minimality property, $\Omega^n_\Lambda(F)$ is indeed a direct summand of $G$. The sequence $(\dagger)$ gives the epimorphism $(-, \Omega^k_\Lambda(C)) \rightarrow \Omega^k_\Lambda(F) \rightarrow 0$, and since $\Omega^k_\Lambda(F)$ is in mod-$\mathcal{X}$, it induces the epimorphism $(-, \Omega^{2k}_\Lambda(C)) \rightarrow \Omega^{2k}_\Lambda(F) \rightarrow 0$ in mod-$\mathcal{X}$. The latter epimorphism implies $\beta^{n\text{Aus(}\mathcal{X}\text{)}}_{\Lambda}(F)$ is less than or equal to the number of all indecomposable summands of $\Omega^n_\Lambda(C)$. But this number is less than or equal to $\beta^k_\Lambda(C)$, and consequently by the assumption there is a fixed integer, say $m$, such that $\beta^k_\Lambda(C) \leq m$. Hence we have a bound for the Betti numbers of $F$ in mod-$\mathcal{X}$, which this means $\text{cx}_\Lambda(\text{Aus}(\mathcal{X})(F)) \leq 1$. The proof of the other cases of $n$ is similar, so we skip their proofs. $\square$

Further, for some special cases by the above result we can construct self-injective algebras with complexity of each module at most one.

**Theorem 4.7.** Let $\Lambda$ be a CM-finite algebra. Then every module over the stable Cohen-Macaulay Auslander algebra $\text{Aus(}\text{Gprj-}\Lambda\text{)}$, that is, a self-injective algebra, has complexity at most one.

**Proof.** The syzygy functor $\Omega_\Lambda$ preserves indecomposability over non-projective Gorenstein projective modules, see [C, Lemma 2.2]. Then for an indecomposable module $G$ in Gprj-$\Lambda$, the set $\{\Omega^n_\Lambda(M) \mid n \geq 0\}$ is an infinite set of indecomposable modules in Gprj-$\Lambda$. Since, by assumption, Gprj-$\Lambda$ has only finitely many indecomposable modules up to isomorphisms. Hence we conclude that $\Omega^r(M) \simeq \Omega^s(M)$ for some $r > s \geq 1$, and hence $\Omega^{r-s}_\Lambda(M) \simeq M$. Therefore, any module in Gprj-$\Lambda$ is periodic, and clearly has complexity at most one. So Proposition 4.6 finishes the proof. $\square$

As next application we shall give a characterization of Gorenstein projective functors in mod-$\mathcal{X}$, whence $\mathcal{X}$ satisfying sufficiently nice proprieties.
4.2. Gorenstein projective functors. Recall that an Artin algebra $\Lambda$ is said to be Gorenstein if the injective dimension $\Lambda$ as right and left module is finite. In this case, one can prove the equality $\text{id}_\Lambda \Lambda = \text{id}_\Lambda \Lambda = n$, where $n$ is the common value, and we say that $\Lambda$ is an $n$-Gorenstein algebra.

**Proposition 4.4.** Let $X \subseteq \text{mod-} \Lambda$ be a contravariantly finite and quasi-resolving subcategory of $\text{mod-} \Lambda$. Assume that there is $n \geq 0$ such that $\Omega^n(X)$ is contained in $\text{Gprj-} \Lambda$. If $F$ is an indecomposable non-projective Gorenstein projective object in $\text{mod-} X$, then there is a projective resolution of $F$ in the following form

$$0 \to (-, A) \to (-, B) \to (-, C) \to F \to 0$$

such that all the modules $A, B$ and $C$ are Gorenstein projective $\Lambda$-modules.

**Proof.** By definition, $F$ is isomorphic, in the stable category $\text{mod-} X$, to $3n$-syzygy of some functor $G$ in $\text{mod-} X$. Consider a projective resolution of $G$ in $\text{mod-} X$ as the following

$$0 \to (-, M) \to (-, N) \to (-, L) \to G \to 0.$$ 

By Corollary 4.2, the $3n$-th syzygy $\Omega_{\Lambda}^{3n}(G) \simeq G'$ in $\text{mod-} X$, such that $G'$ has the following projective resolution

$$0 \to (-, \Omega_{\Lambda}^n(M)) \to (-, \Omega_{\Lambda}^n(N) \oplus Q) \to (-, \Omega_{\Lambda}^n(L) \oplus P) \to G' \to 0$$

in $\text{mod-} X$, for some projective modules $P$ and $Q$. Hence $G' \simeq F$ in $\text{mod-} X$. So there is $X$ and $Y$ in $X$ such that $G' \oplus (-, X) \simeq F \oplus (-, Y)$ in $\text{mod-} X$. Moreover, by our assumption $\Omega_{\Lambda}^n(M), \Omega_{\Lambda}^n(N)$ and $\Omega_{\Lambda}^n(L)$ lie in $\text{Gprj-} \Lambda$. But since $\text{mod-} X$ is a Krull-Schmidt category, see [K, Corollary 4.4], and $F$ is an indecomposable non-projective object, then $F$ has to be isomorphic to a direct summand of $G'$. As we have seen in the above $G'$ has a projective resolution in $\text{mod-} X$ such that whose terms are presented by the Gorenstein projective modules, then clearly any direct summand of $G'$ so is. Thus $F$ has the desired projective resolution. \hfill \Box

By help of the above result we can construct many CM-finite algebras.

**Proposition 4.9.** Let $X \subseteq \text{mod-} \Lambda$ be a quasi-resolving subcategory of finite representation type. Assume that $\Omega^n(X) \subseteq \text{Gprj-} \Lambda$ for some $n \geq 0$. If $T_2(\Lambda)$ is CM-finite, then the relative stable Auslander algebra $\text{Aus}(X)$ is CM-finite.

**Proof.** Since the number of indecomposable projective modules, up to isomorphisms, over an Artin algebra is finite, so it is enough to concentrate on indecomposable non-projective Gorenstein projective modules. Assume that $F$ is an indecomposable non-projective Gorenstein projective functor in $\text{mod-} X$. Let $0 \to (-, A) \xrightarrow{(-, s_F)} (-, B) \to (-, C) \to F \to 0$ be a minimal projective resolution of $F$ in $\text{mod-} X$. By Proposition 4.8, $A, B$ and $C$ must be Gorenstein projective $\Lambda$-modules. Then $s_F$ belongs to $\mathcal{S}_{\text{Gprj-} \Lambda}(\Lambda)$, or the same $\text{Gprj-} T_2(\Lambda)$, by Lemma 2.1. But $s_F$ is indecomposable, as an object in $\text{H}(\Lambda)$, by lemma 5.1, and further uniquely, up to isomorphism, determined by $F$ because of the minimality property. Now this bijection (sending $F$ into $s_F$) and in view of our assumption complete the proof. \hfill \Box

In the sequel, we shall investigate the converse of Proposition 4.8. We need the following lemma for our investigation.
Lemma 4.10. Let $\mathcal{X} \subseteq \text{mod}-\Lambda$ be a contravariantly finite subcategory and quasi-resolving subcategory. Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence in $\text{mod-}\mathcal{X}$. If we have the following projective resolution of $F_1$ and $F_2$ in $\text{mod-}\mathcal{X}$,

$$0 \to (-, A) \to (-, B) \to (-, C) \to F_1 \to 0,$$

$$0 \to (-, A') \to (-, B') \to (-, C') \to F_2 \to 0.$$

Then $F_3$ has a projective resolution in the following form

$$0 \to (-, D) \to (-, C \oplus B') \to (-, C') \to F_3 \to 0,$$

where $D \oplus A \simeq A' \oplus B$.

Proof. Consider the following pull-back diagram:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
K_2 & K_2 & \\
\downarrow & \downarrow & \downarrow \\
0 & W_1 & (-, C') \\
\downarrow & \downarrow & \downarrow \\
0 & F_1 & F_2 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
\]

where the second column is obtained by the projective resolution of $F_2$. Then by using a standard argument, we get the following diagram with exact columns and rows and the middle row splitting:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & W_2 & \ldots \to K_1 & \ldots \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & W_2 & \ldots \to \oplus (-, C) & \ldots \to (-, C) & \ldots \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & K_2 & W_1 & F_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
where the first and third column, respectively, are obtained by the projective resolution of $F_2$ and $F_1$. Consider the following pull-back diagram:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
(-, A) & (-, A) \\
\downarrow & \downarrow \\
0 & (-, A') & (-, A' + B) & (-, B) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & (-, A') & W_2 & K_1 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

Now by gluing suitable short exact sequences of the above diagrams, we get the following exact sequence

\[
0 \to (-, A) \to (-, A' \oplus B) \to (-, B' \oplus C) \to (-, C') \to F_3 \to 0.
\]

On the other hand, we know that projective dimension of $F_3$ is at most two in mod-$\mathcal{X}$, hence the kernel $K$ of $(-, B' \oplus C) \to (-, C)$ must be projective in mod-$\mathcal{X}$. Since $\mathcal{X}$ is closed under direct summands then there is $D$ in $\mathcal{X}$ such that $K \cong (-, D)$. So we have the desired sequence in the statement and the latter assertion follows, in view of the Yoneda lemma, from the split short exact sequence $0 \to (-, A) \to (-, A' \oplus B) \to (-, D) \to 0$ in mod-$\mathcal{X}$. □

Let $C$ be an additive category such that mod-$C$ is abelian. For any $F$ and $G$ in mod-$C$, we mean by $\text{Ext}^1_C(F, G)$, the set of isomorphism classes of extensions of $F$ by $G$.

**Lemma 4.11.** Let $\mathcal{X} \subseteq \text{mod-}\Lambda$ be a contravariantly finite and quasi-resolving subcategory. If $F$ in mod-$\mathcal{X}$ has a projective resolution in mod-$\mathcal{X}$ as the following

\[
0 \to (-, A) \to (-, B) \to (-, C) \to F \to 0
\]

with Gorenstein projective modules $A, B, C$ in mod-$\Lambda$, then $\text{Ext}^1_{\mathcal{X}}(F, (-, X)) = 0$ for any $X \in \mathcal{X}$

**Proof.** Assume $F$ is a functor in mod-$\mathcal{X}$ having a projective resolution as in the statement. Since mod-$\mathcal{X}$ is a subcategory of mod-$\Lambda$ closed under extensions, then this allows us for computing extension group of two objects in mod-$\mathcal{X}$ do only in mod-$\mathcal{X}$. Hence $\text{Ext}^1_{\mathcal{X}}(F, (-, X)) \cong \text{Ext}^1_{\mathcal{X}}(F, (-, X))$. The latter group is the homology of the middle term in the following sequence

\[
\text{Hom}_{\mathcal{A}}((-, C), (-, X)) \to \text{Hom}_{\mathcal{A}}((-, B), (-, X)) \to \text{Hom}_{\mathcal{A}}((-, A), (-, X)),
\]

where $\mathcal{A} = \text{mod-}\mathcal{X}$, obtained by applying $\text{Hom}_{\mathcal{A}}((-, (-, X)))$ on the projective resolution of $F$ in mod-$\mathcal{X}$. Now by using the Yoneda lemma the above sequence is isomorphic to the following sequence of abelian groups

\[
\text{Hom}_{\mathcal{A}}(C, X) \to \text{Hom}_{\mathcal{A}}(B, X) \to \text{Hom}_{\mathcal{A}}(A, X).
\]

So $\text{Ext}^1_{\mathcal{X}}(F, (-, X))$ is isomorphic to the homology of the middle term of the above sequence. But this sequence is exact, see [MT, Lemma 2.2], and consequently $\text{Ext}^1_{\mathcal{X}}(F, (-, X)) = 0$. □
Let \( M \) be a module in \( \text{mod-} \Lambda \), and \( n \) a positive integer. Let \( g : M \to Q_M \) be a minimal left (prj-\( \Lambda \))-approximation. Then the cokernel of \( g \) is called the \textit{first projective cosyzygy} of \( M \) and denoted by \( \Omega^{-1}_p(M) \). The \( n \)-th cosyzygy \( \Omega^{-n}_p(M) \) is defined inductively as \( \Omega^{-1}_p(\Omega^{-n-1}_p(M)) \).

We say that \( X \) is \textit{closed under projective cosyzygies}, if for any \( X \in X \), \( \Omega^{-1}_p(X) \) lies in \( X \).

**Proposition 4.12.** Let \( \mathcal{X} \subseteq \text{mod-} \Lambda \) be a contravariantly finite subcategory, including \( \text{prj-} \Lambda \), closed under kernels of epimorphisms, projective cosyzygies. If \( F \) in \( \text{mod-} \mathcal{X} \) has a projective resolution in \( \text{mod-} \mathcal{X} \) as the following

\[
0 \to (\cdot, A) \to (\cdot, B) \to (\cdot, C) \to F \to 0
\]

with Gorenstein projective modules \( A, B, C \) in \( \text{mod-} \Lambda \), then \( F \) is a Gorenstein projective functor in \( \text{mod-} \mathcal{X} \).

**Proof.** Assume \( F \in \text{mod-} \mathcal{X} \) has a projective resolution as in the statement. Since \( B \) is Gorenstein projective then by definition there is a short exact sequence as \( 0 \to B \xrightarrow{f} P \to \Omega^{-1}_p(B) \to 0 \) with \( f \) a minimal left prj-\( \Lambda \)-approximation. Consider the following push-out diagram:

\[
\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A & \to & P & \to & U & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^{-1}_p(B) & \to & \Omega^{-1}_p(B) & \to & 0
\end{array}
\]

The module \( U \) appeared in the last column is a Gorenstein projective module since \( C \) and \( \Omega^{-1}_p(B) \) both are Gorenstein projective, and also is in \( \mathcal{X} \) because by our assumption being closed under projective cosyzygies. By applying the Yoneda functor on the above push-out diagram we get the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & (\cdot, A) & \to & (\cdot, B) & \to & (\cdot, C) & \to & F & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & (\cdot, A) & \to & (\cdot, P) & \to & (\cdot, U) & \to & (\cdot, U) & \to & 0
\end{array}
\]

in \( \text{mod-} \mathcal{X} \). Note that by the Snake lemma we can see \( \phi : F \to (\cdot, U) \) is a monomorphism. Putting \( K_1 := \text{Ker}((\cdot, C) \to F) \) and \( K_2 := \text{Ker}((\cdot, U) \to (\cdot, U)) \), from the diagram \((\dagger)\) and
using the Snake lemma we get the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & (-, A) & \to & (-, B) & \to & K_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (-, A) & \to & (-, P) & \to & K_2 & \to & 0.
\end{array}
\]

The diagram (†) gives us also the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & K_1 & \to & (-, C) & \to & F' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_2 & \to & (-, U) & \to & (-, U) & \to & 0.
\end{array}
\]

in mod-\(\mathcal{X}\). Again by using the Snake lemma we reach the following short exact sequence in mod-\(\mathcal{X}\),

\[
0 \to \text{Cok } \psi \to G \to F' \to 0, \quad (\dagger\dagger)
\]

where \(G = \text{Cok}((-C) \to (-U))\) and set \(F' = \text{Cok } \phi\). Applying Lemma 4.10 for the short exact sequence \(0 \to K_1 \to K_2 \to \text{Cok } \psi \to 0\), the functor \(\text{Cok } \psi\) has the following projective resolution

\[
0 \to (-, A) \to (-, A \oplus B) \to (-, P) \to \text{Cok } \psi \to 0
\]

in mod-\(\mathcal{X}\). Then again by using Lemma 4.10 for the short exact sequence \(\dagger\dagger\) we have the following exact sequence

\[
0 \to (-, D) \to (-, P \oplus C) \to (-, U) \to F' \to 0,
\]

where \(D\) is a direct summand of \(A \oplus B\), in mod-\(\mathcal{X}\). Hence \(D\) is Gorenstein projective. So far we have obtained a short exact sequence \(\eta : 0 \to F \to (\text{-}\mathcal{U}) \to F' \to 0\) such that \(F'\) the same as \(F\) has a projective resolution in mod-\(\mathcal{X}\) such that whose all terms in the induced short exact sequence in mod-\(\Lambda\) are Gorenstein projective modules. Moreover, the short exact sequence remains exact by applying \(\text{Hom}_\Lambda\mathcal{A}((-,-\mathcal{X}))\) for any \(X \in \mathcal{X}\), see Lemma 4.11, where \(\mathcal{X} = \text{mod-\mathcal{X}}\). Inductively, we can construct an exact sequence of representable functor in mod-\(\mathcal{X}\), except possibly \(F\), as the following

\[
0 \to F \to (\text{-}\mathcal{U}_0) \to (\text{-}\mathcal{U}_1) \to \cdots
\]

in mod-\(\mathcal{X}\) such that it remains exact by applying \(\text{Hom}_\Lambda\mathcal{A}((-,-\mathcal{X}))\) for any \(X \in \mathcal{X}\). Next by gluing the above exact sequence and a projective resolution of \(F\) in mod-\(\mathcal{X}\), we get the required totally acyclic complex having \(F\) as its syzygy. So we are done. \(\square\)

By combination of the above proposition and Proposition 4.8, we have the following characterization of the indecomposable non-projective Gorenstein projective objects in mod-\(\mathcal{X}\) via their minimal projective resolutions in mod-\(\mathcal{X}\).
Theorem 4.13. Let $X \subseteq \text{mod-}\Lambda$ be a contravariantly finite and quasi-resolving subcategory being closed under projective cosyzygies. Assume that there is $n \geq 0$ such that $\Omega^n(X)$ is contained in $\text{Gprj-}\Lambda$. Let $F$ be an indecomposable non-projective functor in $\text{mod-}X$ with the following minimal projective resolution

$$0 \to (-,A) \to (-,B) \to (-,C) \to F \to 0.$$ 

Then $F$ is a Gorenstein projective object in $\text{mod-}X$ if and only if $A,B$ and $C$ are Gorenstein projective modules in $\text{mod-}\Lambda$.

Note that if $F \in \text{mod-}X$ is an indecomposable projective functor, so a Gorenstein projective functor, it is not true all the modules in the induced short exact sequence by its minimal projective resolution in $\text{mod-}X$ belong to $\text{Gprj-}\Lambda$. In fact, consider indecomposable module $X \in X \setminus \text{Gprj-}\Lambda$, then we have the following minimal projective resolution of $(-,X)$,

$$0 \to (-,\Omega\Lambda(X)) \to (-,P_X) \to (-,X) \to (-,X) \to 0$$

in $\text{mod-}X$. But $X$ does not lie in $\text{Gprj-}\Lambda$.

The Gorenstein projective dimension of an object $M$ in an abelian category $\mathcal{A}$, denoted by $\text{Gpd } M$, is defined as the infimum of the integers $n \geq 0$ such that there exits an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0,$$

in $\mathcal{A}$ with $G_i$ Gorenstein projective objects.

Corollary 4.14. Let $X \subseteq \text{mod-}\Lambda$ be the same as in Theorem 4.13. Then for any $F$ in $\text{mod-}X$,

$$\text{Gpd } F \leq 3n.$$ 

Proof. It is a direct consequence of Theorem 4.13 and Corollary 4.2. \hfill \Box

The above result recover [MT, Theorem 3.11] in our setting. We also remark that for a quasi-resolving subcategory $X$ the condition “ $\Omega^n(X)$ is contained in $\text{Gprj-}\Lambda$ and closed under cosyzygies” is called $(G_n)$ in [MT].

4.3. The extension functor. We begin by the following general construction which is essential to define another functor in similar to Construction 3.1 to make a connection between a subcategory and the module category over an Artin algebra.

Construction 4.15. Let $C'$ be a full subcategory of an additive category $\mathcal{C}$. Let $F$ be a finitely presented functor over $C'$. So there is an exact sequence $(-,C_2) \to (-,C_1) \to F \to 0$ with $C_i, i = 1,2$ in $C'$. We can naturally construct the functor $\tilde{F}$ in $\text{mod-}\mathcal{C}$, by defining $\tilde{F}(C) := \text{Cok}(\text{Hom}_{\mathcal{C}}(C,C_2) \to \text{Hom}_{\mathcal{C}}(C,C_1))$ for any $C$ in $\mathcal{C}$. Hence, by definition, we have the projective presentation $(-,C_2) \to (-,C_1) \to \tilde{F} \to 0$ in $\text{mod-}\mathcal{C}$. In the latter case, the functors $(-,C_i)$ are considered as representable functors in $\text{mod-}\mathcal{C}$. For simplicity, both cases are shown by the same notations. The definition of $\tilde{F}$ is independent of choosing a projective presentation of $F$ in $\text{mod-}\mathcal{C}$. Let $F \xrightarrow{\sigma} G$ be a morphism in $\text{mod-}\mathcal{C}$. The morphism $\sigma$ can be lifted as the following to their existing projective presentations

$$
\begin{array}{ccccccccc}
(-,C_2) & \to & (-,C_1) & \to & F & \to & 0 \\
\downarrow (-,f_2) & & \downarrow (-,f_1) & & \downarrow \sigma \\
(-,C'_2) & \to & (-,C'_1) & \to & G & \to & 0.
\end{array}
$$
in mod-\(C\). For each \(C \in \mathcal{C}\), \(\hat{\sigma}_C : \hat{F}(C) \to \hat{G}(C)\) is defined as the following

\[
\begin{array}{cccc}
\text{Hom}_\mathcal{C}(C, C_2) & \xrightarrow{f'_1} & \text{Hom}_\mathcal{C}(C, C_1) & \xrightarrow{f'_0} \text{Hom}_\mathcal{C}(C, C_0) \\
\downarrow \hat{f}_1 & & \downarrow \hat{f}_0 & & \downarrow \hat{f}_c \\
\text{Hom}_\mathcal{C}(C, C'_2) & \xrightarrow{\hat{f}'_1} & \text{Hom}_\mathcal{C}(C, C'_1) & \xrightarrow{\hat{f}'_0} \text{Hom}_\mathcal{C}(C, C'_0) \\
\end{array}
\]

Again, the definition of \(\hat{\sigma}\) is independent of choosing projective presentations for \(F\) and \(G\). So, in this way, we obtain a morphism \(\hat{\sigma}\) in mod-\(\mathcal{C}\). Now, we can define the functor \(c \cdot T_\mathcal{C} : \text{mod-}C' \to \text{mod-}\mathcal{C}\), by sending functor \(F \in \text{mod-}C'\) into \(\hat{F}\) in mod-\(C'\), and morphism \(\sigma : F \to G\) in mod-\(\mathcal{C}\) into morphism \(\hat{\sigma} : \hat{F} \to \hat{G}\) in mod-\(\mathcal{C}\). One can see easily the functor \(c \cdot T_\mathcal{C}\) is a fully faithful functor, and moreover the restriction of \(c \cdot T_\mathcal{C}\) of \(\hat{F}\) to \(C'\) is \(F\). We call this functor the extension functor from \(C'\) into \(\mathcal{C}\).

The extension functor defined in above construction might be known in the literature. We have not found a suitable reference for it.

Motivated by the conditions required in Theorem 4.13 the following setup is introduced.

**Setup 4.16.** Let \(\mathcal{X} \subseteq \text{mod-}\Lambda\) be a contravariantly finite and quasi-resolving subcategory being closed under projective cosyzygies. Also assume that there is \(n \geq 0\) such that \(\Omega^n(\mathcal{X})\) is contained in Gprj-\(\Lambda\). Denote \(\mathcal{Y} = \mathcal{X} \cap \text{Gprj-}\Lambda\). It is clearly again quasi-resolving subcategory of mod-\(\Lambda\). Note that by [MT, Theorem 1.4], the equality \(\Omega^n(\mathcal{X}) = \mathcal{X} \cap \text{Gprj-}\Lambda\) holds.

Assume \(\mathcal{X}\) and \(\mathcal{Y}\) are the same as in Setup 4.16. Specializing Construction 4.15 for \(\mathcal{Y} \subseteq \mathcal{X}\), we reach the extension functor \(\underline{\mathcal{Y}}\mathcal{X} : \text{mod-}\mathcal{Y} \to \text{mod-}\mathcal{X}\). By Theorem 4.13, see also Lemma 4.17 for more details, the essential image of \(\underline{\mathcal{Y}}\mathcal{X}\) is contained in Gprj-\(\mathcal{X}\). Hence, we indeed have the functor functor \(\underline{\mathcal{Y}}\mathcal{X} : \text{mod-}\mathcal{Y} \to \text{Gprj-}\mathcal{X}\). It is shown again by \(\underline{\mathcal{Y}}\mathcal{X}\) the induced functor from mod-\(\mathcal{Y}\) into Gprj-\(\mathcal{X}\). This is the second promised functor in the line of our purpose in the this paper. Let us remark that by identifying mod-\(\mathcal{Y}\) and mod-\(\mathcal{X}\), respectively, with the subcategory of mod-\(\mathcal{Y}\) and mod-\(\mathcal{X}\) consisting of those functors vanishing on projective modules, then we can consider \(\underline{\mathcal{Y}}\mathcal{X}\) as a restriction of the functor \(\underline{\mathcal{Y}}\mathcal{X} : \text{mod-}\mathcal{Y} \to \text{mod-}\mathcal{X}\).

**Lemma 4.17.** Let \(\mathcal{X}\) and \(\mathcal{Y}\) be as in Setup 4.16. Then the extension functor \(\underline{\mathcal{Y}}\mathcal{X}\), defined in the above, is an exact functor. Moreover, its essential image is contained in Gprj-\(\mathcal{X}\).

**Proof.** To prove the latter claim in the statement, let \(F \in \text{mod-}\mathcal{Y}\). Then there is an exact sequence in mod-\(\mathcal{Y}\)

\[
0 \to (-, A) \to (-, B) \to (-, C) \to F \to 0
\]

with modules \(A, B\) and \(C\) in \(\mathcal{Y}\). The image \(\hat{F}\) under the functor \(\underline{\mathcal{Y}}\mathcal{X}\) has the following projective resolution

\[
0 \to (-, A) \to (-, B) \to (-, C) \to \hat{F} \to 0
\]

in mod-\(\mathcal{X}\). Let us emphasis here the representable functors in the first exact sequence and the second exact sequence in the above considered ,respectively, as functors in mod-\(\mathcal{Y}\) and mod-\(\mathcal{X}\). Now by the characterization given in Theorem 4.13, we infer that \(\hat{F}\) lies in Gprj-\(\mathcal{X}\) as required. To prove the first part of the statement, we should show that image of an short exact sequence in mod-\(\mathcal{Y}\) is mapped to a short exact sequence in mod-\(\mathcal{X}\). Take a short exact sequence

\[
\eta : 0 \to F_1 \to F_2 \to F_3 \to 0 \text{ in mod-}\mathcal{Y}.
\]

Assume \(0 \to (-, X) \to (-, Y) \to (-, Z) \to F_1 \to 0\) and \(0 \to (-, X') \to (-, Y') \to (-, Z') \to F_3 \to 0\), respectively are projective resolutions of \(F_1\) and
$F_3$. By a standard argument we have the following commutative diagram in mod-$\mathcal{Y}$ with exact rows and the first three columns (from the left side hand) splitting:

![Diagram](https://example.com/diagram.png)

further, the sequence $\eta$ is settled in the rightmost column. Because of being splitting the first three columns, we can consider this part of the above diagram as a commutative diagram in mod-$\mathcal{X}$. Then by getting cokernel of the induced commutative diagram in mod-$\mathcal{X}$, we obtain a short exact sequence in mod-$\mathcal{X}$. In fact, the obtained short exact sequence in mod-$\mathcal{X}$ is the image of $\eta$ under the functor $\mathbf{Y}_{\mathcal{X}}$. So we are done. \hfill $\Box$

The functor $\mathbf{Y}_{\mathcal{X}}$ is not dense in general. In fact, if there is an indecomposable $M \in \mathcal{X} \setminus \mathcal{Y}$, then $(-, M)$ is in $\text{Gprj-}_\mathcal{X}$ but not in the essential image of $\mathbf{Y}_{\mathcal{X}}$. If it did, then $(-, M) \simeq \tilde{F}$. Let $(-, N) \rightarrow (-, K) \rightarrow F \rightarrow 0$ be a projective presentation of $F$ in mod-$\mathcal{Y}$. Hence we have the projective presentation $(-, N) \rightarrow (-, K) \rightarrow \tilde{F} \rightarrow 0$ in mod-$\mathcal{X}$. By the isomorphism, we also have the projective presentation $(-, N) \rightarrow (-, K) \rightarrow (-, M) \rightarrow 0$ of $(-, M)$ in mod-$\mathcal{X}$. On the other hand, we know $(-, P_M) \rightarrow (-, M) \rightarrow (-, M) \rightarrow 0$ is a minimal projective presentation. Considering these two projective presentations of $(-, M)$ in mod-$\mathcal{X}$ follows that $M$ is isomorphic to a direct summand of $K$. But $K$ is in $\mathcal{Y}$, this means $M \notin \mathcal{Y}$, that is a contraction.

**Definition 4.18.** A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is called almost dense if all but finitely many indecomposable objects, up to isomorphism, are in the essential image of $F$.

We summarize the above observation in the following theorem:

**Theorem 4.19.** Let $\mathcal{X}$ and $\mathcal{Y}$ be as in Setup 4.16. The extension functor $\mathbf{Y}_{\mathcal{X}}$ is a fully faithful exact functor. The essential image of the extension functor $\mathbf{Y}_{\mathcal{X}}$ lies in $\text{Gprj-}_\mathcal{X}$. Moreover, the essential image contains all indecomposable functors in $\text{Gprj-}_\mathcal{X}$ except the indecomposable projective functors $(-, X)$, where $X$ is an indecomposable module in $\mathcal{X} \setminus \mathcal{Y}$. In particular, if $\mathcal{X}$ is of finite representation type, then $\mathbf{Y}_{\mathcal{X}}$ is almost dense.

Specializing to the case when $\mathcal{X}$ is the subcategory of Gorenstein projective modules over an Artin algebra, we get the following result. For simplicity, we show the extension functor $\text{Gprj-}_A Y_{\text{mod-}A}$ by $\mathbf{Y}_A$.

**Theorem 4.20.** Let $\Lambda$ be a Gorenstein algebra. Then

1. The extension functor $\mathbf{Y}_A$ is mapped into $\text{Gprj-}_A$. Its essential image of $\mathbf{Y}_A$ contains all indecomposable functors but indecomposable functors $(-, X)$ such that $X$ is not
isomorphic to an indecomposable Gorenstein projective module. Moreover, if $\Lambda$ is of finite representation type, then the functor $\mathcal{g}T_\Lambda : \text{mod-}\text{Gprj-}\Lambda \to \text{Gprj-}\text{mod-}\Lambda$ is almost dense.

(2) The composition functor $\mathcal{g}T_\Lambda \circ \Psi_{\text{Gprj-}\Lambda} : \text{Gprj-}\text{T}_2(\Lambda)/Y \to \text{Gprj-}\text{mod-}\Lambda$, where $\Psi_{\text{Gprj-}\Lambda}$ introduced in Construction 3.1, is fully faithful and whose essential image contains all indecomposable functor except indecomposable projective functors as in (1). Moreover, if $\Lambda$ is of finite representation type, then the functor $\mathcal{g}T_\Lambda \circ \Psi_{\text{Gprj-}\Lambda}$ is almost dense.

Proof. The statement is a direct consequence of the above theorem and using this fact that over Gorenstein algebra $\Lambda$, there is $n \geq 0$ such that $\Omega^n(\text{mod-}\Lambda) = \text{Gprj-}\Lambda$. $\square$

We end this section by the following interesting result.

Corollary 4.21. Let $\Lambda$ be a Gorenstein algebra of finite representation type. Then the following conditions are equivalent.

1. $T_2(\Lambda)$ is CM-finite;
2. The stable Cohen-Macaulay Auslander algebra $\text{Aus}(\text{Gprj-}\Lambda)$ is representation-finite;
3. The stable Auslander algebra $\text{Aus}(\text{mod-}\Lambda)$ is CM-finite and Gorenstein.

Proof. It follows from Theorem 4.20, Theorem 3.4 and Corollary 4.14 along with the characterization given in Lemma 2.1. $\square$

5. Almost split sequences

In this section we will show how the functors $\Psi_X$ and $\mathcal{g}T_X$, which are introduced in the previous sections can be used to transfer the almost split sequences. First, we need to define the almost split sequences for the subcategories as follows. Let $\mathcal{C}$ be an full extension-closed subcategory of an abelian category $\mathcal{A}$. Let $\delta: 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in $\mathcal{A}$. We call $\delta$ a short exact sequence in $\mathcal{C}$ if all whose terms belong to $\mathcal{C}$. A morphism $f : X \to Y$ in $\mathcal{C}$ is called a proper monomorphism if it is a monomorphism in $\mathcal{A}$ and its cokernel in $\mathcal{A}$ belongs to $\mathcal{C}$. Dually, one can define a proper epimorphism in $\mathcal{C}$. An object $X$ in $\mathcal{C}$ is called Ext-injective if every proper monomorphism $f : X \to Y$ in $\mathcal{C}$ is a section; and Ext-projective if every proper epimorphism $g : Z \to X$ in $\mathcal{C}$ is a retraction. The class of all short exact sequences in $\mathcal{C}$ gives naturally an exact structure on $\mathcal{C}$ in the sense of Quillen. The subcategory $\mathcal{C}$ has enough projectives, if for every object $C$ in $\mathcal{C}$ there is a proper epimorphism $P \to C$ with Ext-projective object $P$. Dually, one can define the notion of having enough injectives.

Next, we recall from [AR1] some terminology and facts for the Auslander-Reiten theory. One says that $f$ in $\mathcal{C}$ is right almost split if $f : X \to Y$ is not a retraction and every non-retraction morphism $g : M \to Y$ in $\mathcal{C}$ factors through $f$; and minimal right almost split if $f$ is right minimal and right almost split. In a dual manner, one defines $f$ to be (minimal) left almost split. A short exact sequence

$$\delta: 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{C}$ is called almost split if $f$ is minimal left almost split and $g$ is minimal right almost split. Since $\delta$ is unique up to isomorphism for $X$ and $Z$, we may write $X = \tau_C Z$ and $Z = \tau_C^{-1} X$. We shall say that $\mathcal{C}$ has right almost split sequences if every indecomposable object is either Ext-projective or the ending term of an almost split sequence, dually, $\mathcal{C}$ has left almost split sequences if every indecomposable object is either Ext-injective or the starting term of an almost split sequence. We call $\mathcal{C}$ has almost split sequences if it has both left and right almost split sequences.
5.1. Exchange between the almost split sequence in $S_X(\Lambda)$ and $\text{mod-}\mathcal{X}$. Let $\mathcal{X}$ be a contravariantly finite and quasi-resolving subcategory of mod-$\Lambda$. As we have observed before mod-$\mathcal{X}$ and mod-$\mathcal{X}$ are semi-perfect, and hence by [K, Corollary 4.4] a Krull-Schmidt category. Moreover, since mod-$\mathcal{X}$ is Hom-finite, i.e., for any $F,G$ in mod-$\mathcal{X}$, $\text{Hom}_\mathcal{X}(F,G)$, where $\mathcal{A} = \text{mod-}\mathcal{X}$, is a $k$-module of finite length, then mod-$\mathcal{X}$ satisfies the bi-chain condition, in the sense of [K], hence by [K, Proposition 5.4], an functor in mod-$\mathcal{X}$ is indecomposable if and only if $\text{End}_\mathcal{X}(F)$ is local. Analogously, we have the same characterization for indecomposability of functors in $\text{mod-}\mathcal{X}$.

Consider $F \in \text{mod-}\mathcal{X}$ and since we have projective covers in mod-$\mathcal{X}$, so we can construct a minimal projective resolution for $F$ in mod-$\mathcal{X}$. Fix a minimal projective resolution in mod-$\mathcal{X}$ for $F$ as the following

$$\eta_F : 0 \to (-, A_F) \to (-, B_F) \to (-, C_F) \to F \to 0.$$ 

In this way, we have associated to any $F$ in mod-$\mathcal{X}$ with the object $(A_F \xrightarrow{s_F} B_F)$ in $S_X(\Lambda)$.

We begin with the following easily established preliminary result.

**Lemma 5.1.** $F$ is an indecomposable functor in mod-$\mathcal{X}$ if and only if $s_F$ so is in $S_X(\Lambda)$.

**Proof.** Assume $F$ is indecomposable. By definition of functor $\Psi_X$, see Construction 3.1, $\Psi_X(s_F) = F$. If $s_F$ would not be indecomposable, then it must have an indecomposable direct summand in the form of $X \xrightarrow{1} X$ or $0 \to X$. But this means that $\eta_F$ is not a minimal projective resolution, a contradiction. Proving the converse, if $F$ was not indecomposable, then there were a decomposition $F = F_1 \oplus F_2$, $F_1, F_2 \neq 0$. By the uniqueness of the minimal projective resolutions $\eta_F \simeq \eta_{F_1} \oplus \eta_{F_2}$, so we get $s_F \simeq s_{F_1} \oplus s_{F_2}$, a contradiction. We are done.

Note that when $\mathcal{X}$ is closed under extension, this follows $S_X(\Lambda)$ so is in $X(\Lambda)$. Hence $S_X(\Lambda)$ naturally gets an exact structure, so we can talk about the existence of almost split sequences in it.

The following lemma gives the structure of indecomposable Ext-projective (injective) objects in $S_X(\Lambda)$ which will be helpful later. We need some perpetrations to prove it. For a given subcategory $\mathcal{X}$ of mod-$\Lambda$ denote by $\mathcal{F}_\mathcal{X}(\Lambda)$ the subcategory of $H(\Lambda)$ consisting of all morphism $A \xrightarrow{f} B$ satisfying:

(i) $f$ is an epimorphism;

(ii) $A, B$ and Ker $f$ belong to $\mathcal{X}$.

The restrictions of the kernel and cokernel functors (see [RS2, Section 1]) clearly induces a pair of inverse equivalences

$$\text{Ker} : \mathcal{F}_\mathcal{X}(\Lambda) \to S_X(\Lambda) \quad \text{and} \quad \text{Cok} : S_X(\Lambda) \to \mathcal{F}_\mathcal{X}(\Lambda).$$

**Lemma 5.2.** Let $\mathcal{X}$ be a contravariantly finite, quasi-resolving and closed under extensions subcategory of mod-$\Lambda$.

1. A monomorphism $A \xrightarrow{f} B$ is an indecomposable Ext-projective in the category $S_X(\Lambda)$ if and only if it is isomorphic to either $P \xrightarrow{1} P$ or $0 \to P$ with indecomposable projective $P$ in mod-$\Lambda$.

2. Assume $\mathcal{X}$ has enough injectives. A monomorphism $A \xrightarrow{f} B$ is an indecomposable Ext-injective in the category $S_X(\Lambda)$ if and only if it is isomorphic to either $I \xrightarrow{1} I$ or $0 \to I$ with indecomposable Ext-injective module $I$ in the exact category $\mathcal{X}$. 
Proof. (1) Assume \( A \xrightarrow{f} B \) is an indecomposable Ext-projective in the category \( \mathcal{S}_X(\Lambda) \). Let \( \pi : P \to A \) and \( \pi' : Q \to \text{Cok} \ f \) be the projective covers of \( A \) and \( \text{Cok} \ f \), respectively. There is a morphism \( e : Q \to B \) such that \( de = \pi' \), where \( d : B \to \text{Cok} \ f \) is the canonical epimorphism. We have a proper epimorphism in the following form

\[
\begin{array}{ccc}
P & \xrightarrow{P} & A \\
\downarrow & & \downarrow \\
Q \oplus P & \xrightarrow{[f \circ e]} & B \\
\end{array}
\]

in \( \mathcal{S}_X(\Lambda) \), where \( l = [0 \ 1]^t \). The above proper epimorphism in \( \mathcal{S}_X(\Lambda) \) follows \( f \) must be in one of the forms imposed in the statement. It is also easy to see that the indecomposable objects stated in the statements are Ext-projective in \( \mathcal{S}_X(\Lambda) \).

(2) In the similar way of (1), one can show that: An object \( A \xrightarrow{f} B \) in \( \mathcal{F}_X(\Lambda) \) is an indecomposable Ext-injective in the category \( \mathcal{F}_X(\Lambda) \) if and only if it is isomorphic to either \( I_1 \xrightarrow{\psi_1 \psi_2} I \) or \( I \xrightarrow{\psi_1} 0 \) with indecomposable Ext-injective module \( I \) in the category \( X \). Applying the equivalence functor \( \text{Ker} \), defined in the above, for the characterization of Ext-injective objects in \( \mathcal{F}_X(\Lambda) \) yields the result. \( \square \)

Setup 5.3. Let \( \mathcal{X} \subseteq \mod-\Lambda \) be contravariantly finite, quasi-resolving and closed under extensions. Further, assume \( \mathcal{X} \) has enough injectives and \( \mathcal{S}_X(\Lambda) \) has almost split sequences.

In the following construction we shall explain how the almost split sequences in \( \mod-\mathcal{X} \) can be computed by doing in \( \mathcal{S}_X(\Lambda) \).

Construction 5.4. Let \( \mathcal{X} \) be as in Setup 5.3. Let \( H \) be an indecomposable non-projective functor in \( \mod-\mathcal{X} \). Then by lemma 5.1, \( s_H \) so is indecomposable in \( \mathcal{S}_X(\Lambda) \). Also, it is not a projective object in the exact category \( \mathcal{S}_X(\Lambda) \), see Lemma 5.2. Hence by our assumption there is an almost split sequence in \( \mathcal{S}_X(\Lambda) \) ending at \( s_H \), namely,

\[
\begin{array}{ccc}
\epsilon : & 0 & \xrightarrow{X_1} \\
& \downarrow \phi_1 & \downarrow \psi_1 \\
& X_2 & \xrightarrow{A_H} \\
& \downarrow \phi_2 & \downarrow \psi_2 \\
& Z_1 & \xrightarrow{B_H} \\
& \downarrow h & \downarrow s_H \\
& Z_2 & 0 \\
& \downarrow & \downarrow \\
& Cok d & \xrightarrow{Cok M} \\
& \downarrow & \downarrow \\
& Cok s_H & 0 \\
\end{array}
\]

By expanding above diagram in \( \mod-\Lambda \) we get the following commutative diagram

\[
\begin{array}{cccccc}
& & & & & (\ast) & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

Since the morphisms \([1 \ 1] : (A_H \xrightarrow{1} A_H) \to (A_H \xrightarrow{\psi_1} B_H)\) and \([0 \ 1] : (0 \to B_H) \to (A_H \xrightarrow{\psi_2} B_H)\) are non-retraction hence factor through \([\psi_1 \psi_2] \). Thus, both \( \psi_1 \) and \( \psi_2 \) are split. Consequently, \( Z_2 \cong X_2 \oplus B_H \) and \( Z_1 \cong X_1 \oplus A_H \). Now we show that \( \mu_2 \) is also split epimorphism. To this do,
consider the morphism \([\sigma_1 \sigma_2] : (\Omega \Lambda (\text{Cok } s_H) \to P) \to (A_H \overset{s_H}{\to} B_H)\) obtaining with the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega \Lambda (\text{Cok } s_H) & \longrightarrow & P & \longrightarrow & \text{Cok } s_H & \longrightarrow & 0 \\
\sigma_1 & & & & \sigma_2 & & & & \\
0 & \longrightarrow & A_H & \overset{s_H}{\longrightarrow} & B_H & \longrightarrow & \text{Cok } s_H & \longrightarrow & 0.
\end{array}
\]

here, as usual, \(P\) is the projective cover of \(\text{Cok } s_H\). But \([\sigma_1 \sigma_2]\) is not retraction, otherwise it leads to \(H \simeq (-, D)\), where \(D\) is a direct summand of \(\text{Cok } s_H\), a contradiction. Hence \([\sigma_1 \sigma_2]\) factors through \([\psi_1 \psi_2]\), and consequently \(\text{Id}_{\text{Cok } s_H}\) factors through \(\mu_2\). But this means that \(\mu_2\) is split epimorphism, so the result. Applying the Yoneda functor over the diagram \((*)\) and using this observation, as proved in the above, that is, the rows are split, then we have the following commutative diagram (applying the isomorphisms due to the split epimorphisms \((\psi_1, \psi_2, \mu_2)\) and also with abuse of the notation we denote again by \(h\) the corresponding morphism).

\[
\begin{array}{cccccc}
0 & \longrightarrow & (-, X_1) & \overset{(-, d)}{\longrightarrow} & (-, X_2) & \longrightarrow & (-, \text{Cok } d) & \longrightarrow & 0 \\
& & & & \downarrow f & & & & \\
0 & \longrightarrow & (-, X_1 \oplus A_H) & \overset{(-, h)}{\longrightarrow} & (-, X_2 \oplus B_H) & \longrightarrow & (-, \text{Cok } d \oplus C_H) & \longrightarrow & 0 \\
& & & & \downarrow g & & & & \\
0 & \longrightarrow & (-, A_H) & \overset{(-, s_H)}{\longrightarrow} & (-, B_H) & \longrightarrow & (-, C_H) & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

Note that a renaming of some notations in the diagram will be given in the last of this construction. So we have obtained in the right most of the above diagram a short exact sequence, denote by \(S_H\), in \(\text{mod-}\mathcal{X}\). It by our construction is uniquely determined with the given indecomposable non-projective object \(H\). The first term \(F\) in the short exact sequence must be non-zero. Otherwise, because of indecomposability of \(d\) as an object in \(\mathcal{S}_X(\Lambda)\), since it is the last term of an almost split sequence, then it would be isomorphic to either \(0 \to X\) or \(X \overset{1}{\to} X\) for some indecomposable \(X\) in \(\mathcal{X}\). Assume the case \(d \simeq (-, X)\) happened. We also had \(G \simeq H\). Hence the middle row of \((**\ast)\) gives a projective resolution of \(G\), and, on the other hand, the last row gives the minimal one, so because of being minimal yields the deleted projective resolution \(P_G\) of \(G\) provided in the middle row of \((**\ast)\) is a direct sum of the deleted minimal projective \(P_H\) of \(H\) with some contractible complexes. Namely, \(P_G\) in the category of complexes is isomorphic to such a decomposition

\[
P_H \oplus X_1 \oplus X_2 \oplus X_3
\]

(as objects in the category of complexes), where for each \(i = 1, 2, 3\)

\[
X_i : \cdots 0 \to 0 \to (-, A_i) \overset{(-, \text{Id}_{A_i})}{\longrightarrow} (-, A_i) \to 0 \to \cdots
\]
Proposition 5.6. Let $s$ be a module such that $s$ is an almost split sequence in $S_X$. Assume to the contrary that $s$ is not split. Then $G$ is a direct summand of the minimal projective resolution of $F$ and $G$ in the first and the last row of the diagram $(**)$, i.e.,

$$0 \to (-, A_F) \xrightarrow{(-, s_{F+H})} (-, B_F \oplus B_H) \xrightarrow{(-, r_{F+H})} (-, C_F \oplus C_H) \to G \to 0.$$  

On the other hand, the sequence in the middle row of $(**)$ is projective. Hence, due to the property of being minimal projective resolution, the deleted minimal projective resolution of $G$, given in the above, is a direct summand of the one given in the diagram $(**)$ in the Construction 5.4. By translating this fact in $S_X$, we get this decomposition in $S_X$. Therefore, based on the above facts we can assume $s$, the almost split sequence in $S_X$ ending at $F$, has the following form

$$0 \to (-, A_F) \xrightarrow{(-, s_{F+H})} (-, B_F \oplus B_H) \xrightarrow{(-, r_{F+H})} (-, C_F \oplus C_H) \to G \to 0.$$  

where $s_{F+H}$ is uniquely determined by $F$ or $H$.

**Lemma 5.5.** Keep in mind all notations used in Construction 5.4. The short exact sequence $S_H$ is not split.

**Proof.** The proof is rather similar to the proof given for showing $F \neq 0$ in the construction. Assume to the contrary that $S_H$ is split. Then $G \simeq F \oplus H$. According to this fact the minimal projective resolution of $G$ in $S_X$ is a direct sum of the minimal projective resolution of $F$ and $G$ in the first and the last row of the diagram $(**)$, i.e.,

$$0 \to (-, A_F \oplus A_H) \xrightarrow{(-, s_{F+H})} (-, B_F \oplus B_H) \xrightarrow{(-, r_{F+H})} (-, C_F \oplus C_H) \to G \to 0.$$  

On the other hand, the sequence in the middle row of $(**)$ is a projective resolution of $G$ (not necessarily to be minimal). Hence, due to the property of being minimal projective resolution, the deleted minimal projective resolution of $G$, given in the above, is a direct summand of the one given in the diagram $(**)$ in the Construction 5.4. By translating this fact in $S_X$, we get this decomposition in $S_X$. Therefore, based on the above facts we can assume $s$, the almost split sequence in $S_X$ ending at $F$, has the following form

$$0 \to (-, A_F) \xrightarrow{(-, s_{F+H})} (-, B_F \oplus B_H) \xrightarrow{(-, r_{F+H})} (-, C_F \oplus C_H) \to G \to 0.$$  

where $s_{F+H}$ is uniquely determined by $F$ or $H$.

**Proposition 5.6.** Keep in mind all notations used in Construction 5.4. The short exact sequence $S_H$ is an almost split sequence.

**Proof.** By Lemma 5.1, $F$ and $H$ are indecomposable and also by the preceding lemma $S_H$ is non-split. Invoking [AR1, Theorem 2.14], although it originally is stated for abelian categories, but however the proof still works for the exact categories, to prove $\delta$ to be an almost split sequence it is enough to show that $f$ or $g$ respectively are left or right almost split. We will do it for $g$.

Let $h : H \to H'$ be a non-retraction. Considering it as a morphism in $S_X$, it can be lifted to their minimal projective resolutions and then returning to $S_X$ via the Yoneda lemma to
reach the following morphism

\[
\begin{array}{cccc}
A_{H'} & \xrightarrow{h_1} & A_{H} & \\
B_{H'} & \xrightarrow{h_2} & B_{H} & \downarrow s_{H}
\end{array}
\]

in \( S_X(\Lambda) \). The morphism \([h_1 \ h_2]\) is not retraction. Otherwise, it follows \( h \) so is, a contradiction. Hence since \( \epsilon \) is an almost split sequence, \([h_1 \ h_2]\) factors thorough via

\[
\begin{array}{cccc}
A_{H'} & \xrightarrow{h_1} & A_F \oplus A_H & \\
B_{H'} & \xrightarrow{h_2} & B_F \oplus B_H & \downarrow s_{F+H}
\end{array}
\]

Now by applying the functor \( \Psi_X \) on such a factorization, we see that the morphism \( h \) factors through \( g \) via \( \Psi_X([h_1' \ h_2']) \), as required. So we are done. \( \square \)

As we have observed in the above a recipe for constructing the almost split sequence in \( \text{mod-}\mathcal{A} \) for a given non-projective indecomposable functor \( H \) of \( \text{mod-}\mathcal{A} \) is given via computing the almost split sequence in \( S_X(\Lambda) \). Dually, we can do a similar process for a given non-injective indecomposable functor of \( \text{mod-}\mathcal{A} \).

**Proposition 5.7.** Let \( \mathcal{X} \) be the same as in Set up 5.3. Then \( \text{mod-}\mathcal{X} \) has almost split sequences.

**Proof.** It is a direct consequence of the observations provided in the above. \( \square \)

Let us in continue give some examples of subcategories satisfying the conditions of Set up 5.3. According to a theorem of Auslander and Smalø ([AS, Theorem 2.4]), one way to show that an extension-closed subcategory (of finitely generated modules over some Artin algebra) to have almost split sequences is to prove that it is functorially finite. When \( \mathcal{X} = \text{mod-}\Lambda \), in [RS2] was proved the subcategory \( S_{\text{mod-}\Lambda}(\Lambda) \), or \( S(\Lambda) \), is functorially finite in \( H(\Lambda) \). So by the above-mentioned fact \( S(\Lambda) \) has almost split sequences in \( H(\Lambda) \). In fact in [RS2], (left or right) minimal \( S(\Lambda) \)-approximations for any object in \( H(\Lambda) \) are computed explicitly. These computations provide a tool for computing the almost split sequences in \( S(\Lambda) \) as soon as we know the usual almost split sequences in \( H(\Lambda) \).

In the next result we will use the fact \( S(\Lambda) \) being functorially finite in \( H(\Lambda) \) to prove \( S_{\text{Gprj-}\Lambda}(\Lambda) \), or the same \( \text{Gprj-T}_2(\Lambda) \), so is, provided \( \text{Gprj-}\Lambda \) is contravariantly finite in \( \text{mod-}\Lambda \). Note that by a well-known result of Krause and Solberg [KS, Corollary 0.3 ] which says: a resolving contravariantly finite subcategory in \( \text{mod-}\Lambda \) is functorially finite; hence since \( \text{Gprj-}\Lambda \) is resolving then in this case it is further functorialy finite.

**Proposition 5.8.** Let \( \Lambda \) be an Artin algebra such that the subcategory of Gorenstein projective modules \( \text{Gprj-}\Lambda \) is contravariantly finite. Then \( S_{\text{Gprj-}\Lambda}(\Lambda) \) is functorially finite in \( H(\Lambda) \). In particular, \( S_{\text{Gprj-}\Lambda}(\Lambda) \) has almost split sequences.

**Proof.** Since \( S_{\text{Gprj-}\Lambda}(\Lambda) \) is indeed the subcategory of Gorenstein projective modules in \( H(\Lambda) \) so by [KS, Corollary 0.3 ] it is enough to show that \( S_{\text{Gprj-}\Lambda}(\Lambda) \) is contravariantly finite in \( H(\Lambda) \). We know \( S(\Lambda) \) is contravariantly finite in \( H(\Lambda) \), hence by making use of this fact to prove the claim it is enough to show that any object in \( S(\Lambda) \) has a right \( S_{\text{Gprj-}\Lambda}(\Lambda) \)-approximation. Take an arbitrary object \( M_1 \xrightarrow{f} M_2 \) in \( S(\Lambda) \). Hence \( f \) is a monomorphism. Let \( G_1 \xrightarrow{g} \text{Cok} \) \( f \to 0 \) be a minimal right \( \text{Gprj-}\Lambda \)-approximation of \( \text{Cok} \) \( f \) in \( \text{mod-}\Lambda \), which it exits by our assumption.
Further, set $K_1 := \text{Ker } g$, since $g$ is minimal then by Wakamutsu’s Lemma, $K_1 \in \text{Gprj-}\Lambda^\perp$, i.e., $\text{Ext}^1(G, K_1) = 0$ for any $G$ in $\text{Gprj-}\Lambda$. Consider the following pull-back diagram

Let $G_2 \xrightarrow{t} U \to 0$ be a minimal right $\text{Gprj-}\Lambda$-approximation. Again by Wakamutsu’s Lemma, $K_2 := \text{ker } t$ belongs to $\text{Gprj-}\Lambda^\perp$. Now consider the following pull-back diagram

Since both $K_1, K_2$ are in $\text{Gprj-}\Lambda^\perp$, so the first row in above digram follows $K_3$ so does. Finally, applying the Snake lemma for the following commutative diagram

yields the following short exact sequence

$0 \to K_2 \to G_3 \to M_1 \to 0$,

where $G_3 := \text{Ker } lt$, that is a Gorenstein projective module since it is a kernel of an epimorphism in $\text{Gprj-}\Lambda$. Putting together the maps obtained in the above, the following commutative diagram
can be made

\[
\begin{array}{ccccccc}
0 & 0 & 0 & (\star) \\
\downarrow & \downarrow & \downarrow & \\
0 & K_2 & G_3 & p & M_1 & \rightarrow & 0 \\
0 & K_3 & G_2 & dt & M_2 & \rightarrow & 0 \\
0 & K_1 & G_1 & q & \text{Cok} f & \rightarrow & 0 \\
0 & 0 & 0 \\
\end{array}
\]

From the above diagram, we can obtain the following short exact sequence in \(H(\Lambda)\)

\[
\begin{array}{ccccccc}
0 & K_2 & \rightarrow & G_3 & \downarrow p & M_1 & \rightarrow & 0 \\
0 & K_3 & \rightarrow & G_2 & \downarrow dt & M_2 & \rightarrow & 0 \\
0 & K_1 & \rightarrow & G_1 & \downarrow \text{it} & \text{Cok} f & \rightarrow & 0 \\
0 & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

such that the middle term lies in \(\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda)\) and \(K_2, K_3\) are in \(\text{Gprj-}\Lambda^\perp\). Consider an arbitrary object \((G \rightarrowtail G')\) in \(\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda)\). In view of the following short exact sequence

\[
\begin{array}{ccccccc}
0 & G & \downarrow i & G' & \downarrow \pi' & \text{Cok} s & \rightarrow & 0 \\
0 & \text{Im} s & \rightarrow & G & \rightarrow & 0 & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

where \(ij = s\) is an epi-mono factorization of \(s\) and \(\pi'\) the canonical epimorphism, implies \(\text{Ext}^1_{H(\Lambda)}((G \rightarrowtail G'), (K_2 \rightarrow K_3)) = 0\) for any \(s\) in \(\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda)\). Indeed, by the known adjoint pairs between \(H(\Lambda)\) and \(\text{mod-}\Lambda\) along with the vanishing of \(\text{Ext}\) for \(K_2, K_3\), we have \(\text{Ext}^1_{H(\Lambda)}((G \rightarrowtail \text{Im} s), (K_2 \rightarrow K_3)) = 0\) and \(\text{Ext}^1_{H(\Lambda)}((0 \rightarrow \text{Cok} s), (K_2 \rightarrow K_3)) = 0\), and consequently, the desired vanishing of \(\text{Ext}\) in \(H(\Lambda)\). This follows the epimorphism included in the sequence \((\dagger)\) is a right \(\mathcal{S}_{\text{Gprj-}\Lambda}(\Lambda)\)-approximation of \(f\) in \(H(\Lambda)\). Now the proof is complete.

As an immediate consequence of the above result we have the following:

**Corollary 5.9.** Assume \(\Lambda\) holds one of the following conditions

1. \(\Lambda\) is a CM-finite algebra;
2. \(\Lambda\) is a Gorenstein algebra.

Then \(\mathcal{S}_{\text{Gprj}(\Lambda)}\) host almost split sequences.

As we have seen in the above a nice connection between the almost split sequences in \(\text{mod-}\mathcal{X}\) and in \(\mathcal{S}_{\mathcal{X}}(\Lambda)\), where \(\mathcal{X}\) satisfying the conditions of Set up 5.3, are given. Now let \(\mathcal{X}'\) and \(\mathcal{Y}'\) be the same as Set up 4.16. In the rest of this section, we will give a similar connection between the almost split sequences in \(\text{Gprj-}\mathcal{X}'\), by our convention, the subcategory of Gorenstein projective functors in \(\text{mod-}\mathcal{X}'\), and in \(\text{mod-}\mathcal{Y}'\).
5.2. Exchange between the almost split sequence in \( \text{mod-}\mathcal{X} \cap \text{Gprj-}\Lambda \) and \( \text{Gprj-}\mathcal{X} \). Recall from the previous section that there is the extension functor \( \underline{\mathcal{Y}}_{\mathcal{X}} : \text{mod-}\mathcal{Y} \to \text{Gprj-}\mathcal{X} \) which is exact. For simplicity, set \( \tilde{F} := \underline{\mathcal{Y}}_{\mathcal{X}}(F) \) for any \( F \) in \( \text{mod-}\mathcal{Y} \) and also \( \tilde{f} := \underline{\mathcal{Y}}_{\mathcal{X}}(f) \) for any morphism \( f \) in \( \text{mod-}\mathcal{Y} \). In the next results, we shall show the extension functor preserves the almost split sequences.

**Proposition 5.10.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be the same as in Set up 4.16. Assume \( \eta : 0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0 \) is a short exact sequence in \( \text{mod-}\mathcal{Y} \). Then \( \eta \) is an almost split sequence in \( \text{mod-}\mathcal{Y} \) if and only if its image under \( \underline{\mathcal{Y}}_{\mathcal{X}} \)

\[ \tilde{\eta} : 0 \to \tilde{F} \xrightarrow{\tilde{f}} \tilde{G} \xrightarrow{\tilde{g}} \tilde{H} \to 0 \]

so is in \( \text{Gprj-}\mathcal{X} \).

**Proof.** We only prove the “if” part. By Proposition 4.17, \( \tilde{\eta} \) is a short exact sequence. Also, since \( \underline{\mathcal{Y}}_{\mathcal{X}} \) is fully faithful, \( \tilde{\eta} \) is non-split with indecomposable ending terms. So analogue to the proof of Proposition 5.6, it suffices to show that one of the \( \tilde{f} \) and \( \tilde{g} \) are respectively left and right almost split. We will do it for \( \tilde{g} \). Let \( h : D \to \tilde{H} \) be a non-retraction. We may assume \( D \) is indecomposable. If \( D = V \) for some \( V \) in \( \text{mod-}\mathcal{Y} \), then \( h = d \) for some \( d : V \to \tilde{H} \). The morphism \( d \) can not be a retraction because of being fully faithful of \( \underline{\mathcal{Y}}_{\mathcal{X}} \). Hence it factors through \( g \). Then by applying the functor \( \underline{\mathcal{Y}}_{\mathcal{X}} \) on the factorization we obtain the desired factorization of \( h \) thorough \( \tilde{g} \). If \( D \) does not lie in the essential image of \( \underline{\mathcal{Y}}_{\mathcal{X}} \), then \( D \simeq (-, \underline{\mathcal{X}}) \) for some \( \mathcal{X} \) not being in \( \mathcal{Y} \). In this case, the result clearly follows since \( D \) is projective.

By the above result we have some information for the middle terms of the almost split sequences in \( \text{Gprj-}\mathcal{X} \).

**Corollary 5.11.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be the same as in Set up 4.16 and further \( \text{mod-}\mathcal{Y} \) has almost split sequences. The middle term of any almost split sequence in \( \text{Gprj-}\mathcal{X} \) does not contain an indecomposable projective functor isomorphic to \((-, \underline{\mathcal{X}})\) for some \( \mathcal{X} \) not being in \( \mathcal{Y} \).

**Proof.** Let \( 0 \to A \to B \to D \to 0 \) be an almost split sequence in \( \text{Gprj-}\mathcal{X} \). Since \( D \) is not projective then there is an indecomposable non-projective \( H \) in \( \text{mod-}\mathcal{Y} \) such that \( D = \tilde{H} \). By our assumption there is the almost split sequence \( \eta : 0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0 \) ending at \( H \). By Proposition 5.10, the image of \( \eta \) under \( \underline{\mathcal{Y}}_{\mathcal{X}} \), \( \tilde{\eta} : 0 \to \tilde{F} \xrightarrow{\tilde{f}} \tilde{G} \xrightarrow{\tilde{g}} \tilde{H} \to 0 \) is an almost split sequence in \( \text{Gprj-}\mathcal{X} \). Because of the uniqueness of almost split sequence, up to isomorphism, ending at \( D \), we conclude \( B \simeq \tilde{G} \). So we are done since only the indecomposable projective functors in the form of \((-, \underline{\mathcal{X}})\) for some \( \mathcal{Y} \in \mathcal{Y} \) lie in the image of \( \underline{\mathcal{Y}}_{\mathcal{X}} \).

By putting together Proposition 5.6 and Proposition 5.10 we get the next result.

**Proposition 5.12.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be the same as in Set up 4.16 and further \( \mathcal{S}_\mathcal{Y}(\Lambda) \) has almost split sequences. Assume \( G \) is a non-projective indecomposable functor in \( \text{Gprj-}\mathcal{X} \). Then there is an almost split sequence in \( \text{Gprj-}\mathcal{X} \) ending at \( G \). Moreover, there is a non-projective indecomposable functor \( F \) in \( \text{mod-}\mathcal{X} \) such that \( G = \tilde{F} \) and

\[ \tau_{\text{Gprj-}\mathcal{X}}(G) \simeq \underline{\mathcal{Y}}_{\mathcal{X}}(\tau_{\mathcal{Y}}(F)) \simeq \underline{\mathcal{Y}}_{\mathcal{X}} \circ \Psi_{\mathcal{Y}}(\tau_{\mathcal{S}_\mathcal{Y}(\Lambda)}(s_F)). \]
6. Auslander-Reiten quivers

This section in fact is established as a direct consequence of the results of the previous section for making a connection between the Auslander-Reiten quivers of the different categories. The notion of the Auslander-Reiten quiver of a category is a convenient combinatorial tool to encode the almost split sequences. Let first introduce the Auslander-Reiten quiver of a Krull-Schmidt category. We recall that an additive category $C$ is called Krull-Schmidt category if every object decomposes into a finite direct sum of objects having local endomorphism rings. Let $C$ be a Krull-Schmidt category. We define for $C$ an associated valued quiver $\Gamma_C$, called the Auslander-Reiten quiver of $C = (\Gamma_C^0, \Gamma_C^1)$, as follows: The vertices $\Gamma_C^0$ are in one to one correspondence with isomorphism classes of the objects in $C$, are usually denoted by $[M]$, simply $M$, for indecomposable object $M$ in $C$. There is an arrow $[M] \to [N]$ in $\Gamma_C^0$ with valuation $(a, b)$ if there are a minimal right almost split morphism $M^a \oplus X \to N$ such that $X$ has no direct summand isomorphic to $M$ in $C$, and a minimal left almost split morphism $M \to N^b \oplus Y$ in $C$. If the valuation of an arrow is trivial $(1, 1)$, we only write an arrow. Assume that the Ext-projective objects in $C$ and Ext-injective objects in $C$ coincides. We denote by $\Gamma_C^1$ the valued quiver, called the stable Auslander-Reiten quiver of $C$, obtained by removing all vertices corresponding to indecomposable Ext-projective and Ext-injective objects in $C$ and the arrows attached to them. We refer to section 5 for the definitions of minimal (left) right almost split morphisms in $C$ and Ext-projective (injective) objects for the Krull-Schmidt category $C$. Although, the definitions are given there for when $C$ gets an exact structure of an ambient abelian category. But they can be still stated in this context. If $C$ gets a triangulated structure, then the Auslander-Reiten triangles come to play instead of the almost split sequences. Accordingly, the notion of the Auslander-Reiten quivers can be defined, see [Ha] for more details.

In order to simplify our notations for when $C$ is the category $\text{mod}\,D$ of finitely presented functors over category $D$, and $\text{mod}\,D$ is Krull-Schmidt, we use $\Gamma_D$ instead of $\Gamma_{\text{mod}\,D}$. Since an essential source for minimal right (left) almost split morphisms in a (exact) category $C$ are the almost split sequence, then based on the definition of the Auslander-Reiten quivers, an immediate consequence of Proposition 5.6 and Proposition 5.10, in view of Theorem 3.2 and Theorem 4.19, is our main result in this section:

\textbf{Theorem 6.1.} \hspace{1em} (a) Let $\mathcal{X}$ be the same as in Set up 5.3. Then

\begin{enumerate}
\item The valued quiver $\Gamma_\mathcal{X}$ is a full valued subquiver of $\Gamma_{\text{S}_\mathcal{X}(\Lambda)}$.
\item If $\mathcal{X}$ is of finite representation type, then $\Gamma_\mathcal{X}$ differ only finitely many vertices with $\Gamma_{\text{S}_\mathcal{X}(\Lambda)}$. In particular,

$$|\Gamma_{\text{S}_\mathcal{X}(\Lambda)}^0| = 2\text{ind} \cdot \mathcal{X} + |\Gamma_\mathcal{X}^0|.$$ 

\end{enumerate}

(b) Let $\mathcal{X}$ and $\mathcal{Y}$ be the same as in Set up 4.16 and further $\text{S}_\mathcal{Y}(\Lambda)$ has almost split sequences. Then

\begin{enumerate}
\item The valued quiver $\Gamma_\mathcal{Y}$ is a full valued subquiver of $\Gamma_{\text{Gprj}\cdot\mathcal{X}}$ and $\Gamma_{\text{S}_\mathcal{Y}(\Lambda)}$, i.e.,

$$\Gamma_{\text{S}_\mathcal{Y}(\Lambda)} \hookrightarrow \Gamma_\mathcal{Y} \hookrightarrow \Gamma_{\text{Gprj}\cdot\mathcal{X}}.$$ 

\item If $\mathcal{X}$ is of finite representation type, then $\Gamma_\mathcal{Y}$ differ only finitely many vertices with $\Gamma_{\text{S}_\mathcal{Y}(\Lambda)}$ and $\Gamma_{\text{Gprj}\cdot\mathcal{X}}$. In particular, $|\Gamma_{\text{S}_\mathcal{Y}(\Lambda)}^0| = 2\text{ind} \cdot \mathcal{Y} + |\Gamma_\mathcal{Y}^0|$ and $|\Gamma_{\text{Gprj}\cdot\mathcal{X}}^0| = |\Gamma_\mathcal{Y}^0| + |\text{ind}(\mathcal{X} \setminus \mathcal{Y})|$

\end{enumerate}

We specialize for when $\mathcal{X} = \text{mod}\,\Lambda$ and $\Lambda$ is a Gorenstein algebra. In this case $\mathcal{Y}$ is clearly equal to $\text{Gprj}\cdot\Lambda$. 

Theorem 6.2. Let \( \Lambda \) be a Gorenstein algebra. The Auslander-Reiten quiver \( \Gamma_{Gprj-\Lambda} \) of \( \text{mod-Gprj-}\Lambda \) is embedded into \( \Gamma_{Gprj-T_2(\Lambda)} \) and \( \Gamma_{\text{Gprj-mod-}\Lambda} \) as a full valued subquiver. Moreover, The stable Auslander-Reiten quiver \( \Gamma^s_{\text{Gprj-}\Lambda} \) is the same as \( \Gamma^s_{\text{Gprj-mod-}\Lambda} \).

The above theorem when is more interesting that \( \Lambda \) is of finite representation type. Then for computing the Auslander-Reiten quiver of the subcategory of Gorenstein projective modules over the associated stable Auslander algebra \( \text{Aus(}\text{mod-}\Lambda) \) and the corresponding triangular matrix algebra \( T_2(\Lambda) \), except some finitely many vertices, is enough to compute the Auslander-Reiten quiver of the corresponding stable Cohen-Macaulay Auslander algebra \( \text{Aus(Gprj-}\Lambda) \). In this case, \( \text{Aus(Gprj-}\Lambda) \) is a self-injective and moreover with complexity at most one by Theorem 4.7. There are many results about the shape of the Auslander-Reiten quiver of the self-injective algebras of finite, tame and wild representation type. For the case of finite representation, we refer to the celebrated Riedtmann’s work. Then by our results we can transfer them for the Auslander-Reiten quiver of the subcategories of Gorenstein projective modules. Therefore, in this way, we can make a connection between studying of the Gorens tein projective modules and the modules over a self-injective algebras which are well-understood. Meanwhile, we can get the Auslander-Reiten quiver of the the stable Cohen-Macaulay Auslander algebra in hand, we need only add finitely many vertices to reach \( \Gamma_{\text{Gprj-}\Lambda} \) and \( \Gamma_{\text{Gprj-mod-}\Lambda} \).

For the triangular matrix case, in the following we will provide the structure of the almost split sequences in \( S_X(\Lambda) \) ending at or starting from the indecomposable objects in the form \((0 \to C)\) or \((0 \to X)\). It is helpful for getting completely \( \Gamma_{S_X(\Lambda)} \), as an especial case \( \Gamma_{Gprj-T_2(\Lambda)} \), while we build the \( \Gamma_X \).

Lemma 6.3. Let \( X \) be the same as in Set up 5.3. Let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be an almost split sequence in \( X \). Then

1. The almost split sequence in \( S_X(\Lambda) \) ending at \((0 \to C)\) has the form

\[
\begin{array}{c c c c}
0 & \xrightarrow{0} & A & \xrightarrow{1} \\
& & \xrightarrow{i} & \xrightarrow{f} B \\
& & \xrightarrow{g} & \xrightarrow{0} C \\
& & \xrightarrow{0} & \to 0
\end{array}
\]

2. Let \( e : A \to I \) be a left minimal morphism with Ext-injective \( I \). Then the almost split sequence ending at \((C \xrightarrow{1} C)\) has the form

\[
\begin{array}{c c c c}
0 & \xrightarrow{0} & A & \xrightarrow{f} B \\
& & \xrightarrow{h} & \xrightarrow{0} C \\
& & \xrightarrow{0} & \to 0
\end{array}
\]

where \( h \) is the map \([e']^t : I \to C\) with \( e' : B \to I \) is an extension of \( e \).

3. Let \( b : P_C \to C \) be the projective cover of \( C \). Then the almost split sequence starting at \((0 \to A)\) has the form

\[
\begin{array}{c c c c}
0 & \xrightarrow{0} & A & \xrightarrow{1} \\
& & \xrightarrow{b} \Omega_A(C) \\
& & \xrightarrow{1} \Omega_A(C) \\
& & \xrightarrow{0} \Omega_A(C)
\end{array}
\]

where \( h \) is the kernel of morphism \([f b'] : A \oplus P_C \to B\), here \( b' \) is a lifting of \( b \) to \( g \).

Proof. The assertions (1) and (2) are proved in [RS2, Proposition 7.1] for when \( X = \text{mod-}\Lambda \). The argument given in there still works for our setting. To prove the assertion (3) we will use the dual result of [RS2, Proposition 7.1], see also [RS2, Proposition 7.4], which state the almost
split sequence in $\mathcal{F}_\Lambda^\Lambda$ starting at $(A \xrightarrow{1} A)$ has the form

$$
0 \xrightarrow{f} A \xrightarrow{f_1} A \oplus P_C \xrightarrow{h} P_C \xrightarrow{1} B \xrightarrow{g} B \xrightarrow{d} C \xrightarrow{1} 0
$$

where $d = [f, b']$. The short exact sequence stated in (3) is only obtained by applying the functor $\text{Ker}$ on the above one. Since an equivalence preserves the almost split sequences so one can conclude the desired result. □

From now on, let $\mathcal{X}$ be the subcategory $Gprj-\Lambda$ of Gorenstein projective modules in mod-$\Lambda$ such that it has almost split sequences. The situation of the simple functors and indecomposable projective functors in mod-$Gprj-\Lambda$ is clear, in fact, they are respectively $(-, X)$ and $S_X = (-, X)/r(-, X)$, for some indecomposable module $X$ in $Gprj-\Lambda$. In addition, they also have the following minimal projective resolution in mod-$Gprj-\Lambda$.

$$
0 \to (-, Z) \to (-, Y) \to (-, X) \to S_X \to 0
$$

such that the induced short exact sequence $0 \to Z \to Y \to X \to 0$ is an almost split sequence in $Gprj-\Lambda$, and,

$$
0 \to (-, \Omega_\Lambda(X)) \to (-, P_X) \to (-, X) \to (-, X) \to 0.
$$

On the other hand, since $Gprj-\Lambda$ is a triangulated category, then it follows the classes of projective functors and injective functors mod-$Gprj-\Lambda$ coincide. For $X \in Gprj-\Lambda$ the functor $\text{Ext}^1_\Lambda(-, X)$ in mod-$Gprj-\Lambda$ is defined as that for each $Y \in Gprj-\Lambda$, $\text{Ext}_\Lambda(-, X)(Y) := \text{Ext}^1_\Lambda(Y, X)$. There is the following minimal projective resolution in mod-$Gprj-\Lambda$

$$
0 \to (-, X) \xrightarrow{f} (-, Q_X) \to (-, \Omega_P^{-1}(X)) \to \text{Ext}^1_\Lambda(-, X) \to 0
$$

where $f$ is the minimal left prj-$\Lambda$-approximation of $X$ in mod-$\Lambda$. By the above sequence we see the isomorphism of functors $\text{Ext}^1_\Lambda(-, X) \simeq (-, \Omega_P^{-1}(X))$ in mod-$Gprj-\Lambda$.

Some components in $\Gamma_{Gprj-\Lambda}$ might consists of only a single isolated vertex. Indeed, such components are in bijection with the simple and projective-injective functors in mod-$Gprj-\Lambda$. In the sequel, we will discuss how such components in $\Gamma_{Gprj-\Lambda}$ are related to the components in $\Gamma_{Gprj-T_\Lambda^\Lambda}$.

**Proposition 6.4.** Assume that $Gprj-\Lambda$ has almost split sequences, and for an indecomposable non-projective module $X$ in $Gprj-\Lambda$, the indecomposable projective functor $(-, X)$ is simple in mod-$Gprj-\Lambda$. Then

(1) For each $i > 0$, $\Omega_\Lambda^i(X)$ has the same property as $X$, i.e., the indecomposable projective functor $(-, \Omega_\Lambda^i(X))$ is simple in mod-$Gprj-\Lambda$. Also, for each $i < 0$, we have the similar property, i.e., $(-, \Omega_P^{-i}(X))$ is simple in mod-$Gprj-\Lambda$;

(2) The short exact sequences $0 \to \Omega_\Lambda(X) \to P_X \to X \to 0$ and $0 \to X \to Q_X \to \Omega_P^{-1}(X) \to 0$ are almost split sequences in $Gprj-\Lambda$.

**Proof.** First, since $(-, X)$ is simple then the induced short exact sequence $\delta : 0 \to \Omega_\Lambda(X) \to P_X \to X \to 0$ by its minimal projective resolution in mod-$Gprj-\Lambda$ is almost split in $Gprj-\Lambda$. Let $\eta : 0 \to X \to N \to \tau_{Gprj-\Lambda}(X) \to 0$ be an almost split sequence in $Gprj-\Lambda$. We claim that $N$ is projective. Otherwise, there is an indecomposable direct summand $M$ of $N$ which is non-projective. Assume $0 \to \tau_{Gprj-\Lambda}(M) \to H \to M \to 0$ is the almost split sequence in $Gprj-\Lambda$ ending at $M$. By $\eta$ we see that $X$ is isomorphic to a direct summand of $H$. The almost split
sequence δ implies that domains of all irreducible morphisms ending at X are projective. This means \( C_{\text{Gprj}}(M) \) is a projective module, a contradiction. Hence \( \eta \) must be isomorphic to the short exact sequence \( 0 \to X \to Q_X \to \Omega_P^{-1}(X) \to 0 \). In the same argument we can show that the almost split sequence in \( \text{Gprj-A} \) ending at \( \Omega_A(X) \) is \( 0 \to \Omega_A^2(X) \to P_{1\text{A}}(X) \to \Omega_A(X) \to 0 \). Hence \( \Omega_A(X) \) has the same property as stated in the statement (1). Then using inductively these two observations shows the statements for the case \( i > 0 \) and for the case \( i < 0 \), respectively. \( \square \)

Let \( X \in \text{Gprj-A} \) be the same as in Proposition 6.4, i.e, \( (-, \underline{X}) \) is simple. Denote by \( C_X \) the component of \( \Gamma_{\text{Gprj-A}} \) containing the vertex corresponding to the isomorphism class of indecomposable \( X \). By Proposition 6.4, for each \( i \in \mathbb{Z} \), \( C_X = C_{\text{Gprj}(X)} \), and moreover, \( C_X \) is completely determined by the left minimal projective resolution and right minimal projective resolution of \( X \). Let \( n_{\text{G-A}} \) denote the number of all components in the form of \( C_X \) in the Auslander-Reiten quiver \( \Gamma_{\text{Gprj-A}} \). Similarly, denote by \( \tilde{C}_X \), the component of \( \Gamma_{\text{Gprj-T}_2(\Lambda)} \) containing the vertices corresponding to the isomorphism classes of indecomposable objects \( (0 \to X) \) and \( (X \xrightarrow{\delta} X) \). Let also \( \tilde{n}_{\text{G-A}} \) denote the number of such components in \( \Gamma_{\text{Gprj-T}_2(\Lambda)} \).

**Proposition 6.5.** Assume \( \text{Gprj-A} \) and \( \text{Gprj-T}_2(\Lambda) \) have almost split sequences. Then \( n_{\text{G-A}} = \tilde{n}_{\text{G-A}} \).

**Proof.** Let \( X \) be a module in \( \text{Gprj-A} \) such that \( (-, \underline{X}) \) is a simple functor. Consider respectively the following (left) minimal projective resolution and right minimal projective resolution of \( X \)

\[
\cdots \to P_X^1 \xrightarrow{d_X^1} P_X^0 \to X \to 0 \quad \text{and} \quad 0 \to X \xrightarrow{s_X^1} Q_X^0 \xrightarrow{s_X^0} Q_X^1 \to \cdots.
\]

Set for any \( l \geq 0 \), the canonical inclusion \( i_X^l : \Omega_P^{l+1}(X) \to P_X^l \), obtained by the left minimal projective resolution of \( X \), and for any \( l \geq 0 \), the canonical inclusion \( j_X^l : \Omega_P^{l-1}(X) \to Q_X^l \), obtained by the right minimal projective resolution of \( X \). Hence in view of Lemma 6.3, the almost split sequences ending at and starting from the indecomposable object in the form \( 0 \to X \) or \( X \xrightarrow{\delta} X \) have the following form

\[
0 \to (\Omega_A(X) \xrightarrow{i} \Omega_A(X)) \to (\Omega_A(X) \xrightarrow{i_X} P_X^0) \to (0 \to X) \to 0
\]

\[
0 \to (\Omega_A(X) \xrightarrow{i_X} P_X^0) \to (P_X^0 \xrightarrow{1} P_X^0) \oplus (0 \to X) \to (X \xrightarrow{s} X) \to 0
\]

\[
0 \to (0 \to X) \to (X \xrightarrow{i} X) \oplus (0 \to Q_X^0) \to (X \xrightarrow{j} Q_X^0) \to 0
\]

\[
0 \to (X \xrightarrow{j} X) \to (X \xrightarrow{j_X} Q_X^0) \to (0 \to \Omega_P^{-1}(X)) \to 0.
\]

By using of the above facts frequently we see that the vertices in \( \tilde{C}_X \) are corresponded to the isomorphism classes of indecomposable objects in the form \( (\Omega^i(X) \xrightarrow{i} \Omega^i(X)), (0 \to \Omega_A(X)), (0 \to H_X^i), (H_X^i \xrightarrow{1} H_X^i) \), where \( H_X^i \) is an indecomposable projective module isomorphic to a direct summand of \( P_X^i \), and \( (\Omega^i(X) \xrightarrow{i_X} P_X^{i-1}) \), for any \( i > 0 \), ans also similarly, the other indecomposable objects in \( \text{Gprj-T}_2(\Lambda) \) inducing in a same way by the right minimal projective resolution of \( X \). As we have seen the components \( C_X \) and \( \tilde{C}_X \) both completely determined by (left) minimal projective and right minimal projective resolution of \( X \). This means by sending \( C_X \) into \( \tilde{C}_X \) we obtain a bijection. Hence \( n_{\text{G-A}} = \tilde{n}_{\text{G-A}} \), as desired. \( \square \)
The result given in [Ka, Corollary 3.1] motivates us to consider the behavior of $n_{G-\Lambda}$ under derived equivalences.

We say that two algebras $\Lambda$ and $\Lambda'$ are derived equivalent, if there is a triangulated equivalence $\mathbb{D}^b(\text{mod-}\Lambda) \simeq \mathbb{D}^b(\text{mod-}\Lambda')$.

**Proposition 6.6.** Let $\Lambda$ and $\Lambda'$ are derived equivalent. If one of the algebras $\Lambda$ and $\Lambda'$ is a Gorenstein algebra with $\text{gldim}(\text{mod-Gprj-}\Lambda) = 0$, then $n_{G-\Lambda} = n_{G-\Lambda'}$.

**Proof.** It is known that the class of Gorenstein algebras are closed under derived equivalent. So we can assume that $\Lambda$ and $\Lambda'$ both are Gorenstein. Now due to [AHV, Theorem 4.1.2], $\text{Gprj-}\Lambda \simeq \text{Gprj-}\Lambda'$. Hence, we can also assume $\text{gldim}(\text{mod-Gprj-}\Lambda') = 0$. Since $\text{gldim}(\text{mod-Gprj-}\Lambda) = 0$, then any functor in $\text{mod-Gprj-}\Lambda$ is simple. This follows the Auslander-Reiten quiver $\Gamma_{\text{Gprj-T}_2(\Lambda)}$ is a disjoint union of components in the form $\tilde{C}_X$ for some indecomposable non-projective Gorenstein projective $X$ and possibly components consisting of a single isolated vertex. The number of components $\Gamma_{\text{Gprj-T}_2(\Lambda)}$ not being a single isolated vertex is exactly $n_{G-\Lambda}$ by Proposition 6.5. Similarly, this is also the case for $\Gamma_{\text{Gprj-T}_2(\Lambda')}$. By the structure of an almost split sequences in $\text{Gprj-T}_2(\Lambda)$, see proof of Proposition 6.5, we can see that the image of a given almost split sequence in $\text{Gprj-T}_2(\Lambda)$ induces an Auslander-Reiten triangle in the triangulated category $\text{Gprj-}\Lambda$. This observation yields the component $\tilde{C}_X$ inducing the component $\tilde{C}_X$ in the Auslander-Reiten quiver $\Gamma_{\text{Gprj-T}_2(\Lambda)}$ of the triangulated category $\text{Gprj-}\Lambda$. Therefore, $\Gamma_{\text{Gprj-T}_2(\Lambda)}$ is a disjoint union of $n_{G-\Lambda}$ components $\tilde{C}_X$. Clearly the components with a single isolated vertex does not not have any image in $\Gamma_{\text{Gprj-T}_2(\Lambda)}$. [As, Theorem 8.5] implies that $T_2(\Lambda)$ and $T_2(\Lambda')$ also are derived equivalent. Again by [AHV, Theorem 4.1.2], we obtain a triangulated equivalence $\text{Gprj-T}_2(\Lambda) \simeq \text{Gprj-T}_2(\Lambda')$. Note that the property of Gorensteiness of an algebra is transferred to its associated triangular matrix algebra, see [AHK, Corollary 4.3]. As a triangulated equivalence makes an quiver isomorphism between the Auslander-Reiten quivers of triangulated categories in question. Hence by the triangulate equivalence we have $\Gamma_{\text{Gprj-T}_2(\Lambda)} \simeq \Gamma_{\text{Gprj-T}_2(\Lambda')}$, which this implies $n_{G-\Lambda} = n_{G-\Lambda'}$, as desired. □

An important class of algebras with property $\text{gldim}(\text{mod-Gprj-}\Lambda) = 0$ is quadratic monomial algebras including Gentel algebras. This sort of algebras was studied in [H2] and called $\Omega_g$-algebras.

7. Examples

In a systematic way, view of the results of previous sections, we will show with some examples in this section how the Auslander-Reiten quiver of the submodule category (or the Gorenstein projective modules over a triangular matrix algebra) can be drawn by the Auslander-Reiten quiver of some relative stable Auslander algebra. In addition, a classification of some kind classes of (hereditary or self-injective Nakayama) algebras with the submodule categories of finite representation type is reduced to some accessible cases.

Throughout of this section $k$ denotes an algebraic closed field with characteristic different from 2.

In the next example we show how by having the Auslander-Reiten quiver of the stable Auslander algebra of $\Lambda$ one can build the Auslander-Reiten quiver of the submodule category $S(\Lambda)$. The point here is to add the remaining vertices corresponding to isomorphism classes of the indecomposable objects in $S(\Lambda)$ being in the form of $(X \xrightarrow{f} X)$, $(0 \rightarrow X)$ for some indecomposable $X$ in $\text{mod-}\Lambda$. To this aim, Lemma 6.3 will be helpful. For an indecomposable object $(A \xrightarrow{f} B)$
in \( S(\Lambda) \), we use the notation \("ABf"\) to show the corresponding vertex of isomorphism class of \((A \xrightarrow{f} B)\) in \( \Gamma_{S(\Lambda)} \). If no ambiguity may arise, we omit \("f"\). Hence by our convention, \( AA_1 \) and \( 0A_0 \), or simply \( AA \) and \( 0A \), respectively always show the vertices in \( \Gamma_{S(\Lambda)} \) associated with isomorphism classes of the objects \((A \xrightarrow{1} A)\) and \((0 \rightarrow A)\).

**Example 7.1.** Let \( \Lambda \) be the path algebra of the quiver \( A_3 : 1 \rightarrow 2 \rightarrow 3 \). The Auslander-Reiten quiver \( \Gamma_{\Lambda} \) of \( \Lambda \) is

\[
\begin{array}{c}
\alpha \\
\beta \\
\end{array}
\begin{array}{c}
P_1 = I_3 \\
P_2 \rightarrow I_2 \\
P_3 \leftarrow S_2 \leftarrow I_1
\end{array}
\]

As usual, the \( P_i, I_i \) and \( S_i \), respectively, show the indecomposable projective, injective and simple module corresponding to vertex \( i \). Then the stable Auslander algebra \( A = \text{Aus}(\text{mod-}\Lambda) \) is given by the following quiver

\[
\begin{array}{c}
I_1 \xrightarrow{\alpha} I_2 \xrightarrow{\beta} S_2
\end{array}
\]

bound by \( \alpha \beta = 0 \). The Auslander-Reiten quiver of \( A \) is of the form

\[
\begin{array}{c}
011 \\
010 \\
001 \leftrightarrow 010 \leftrightarrow 110 \leftrightarrow 100
\end{array}
\]

Hence the indecomposable functors

\((-, S_2)/\text{rad}(-, S_2), (-, I_2), (-, I_2)/\text{rad}(-, S_2), (-, I_1)\) and \((-, I_1)/\text{rad}(-, I_1)\)

in \( \text{mod-}\text{mod-}\Lambda \) are mapped under the evaluation functor to the indecomposable modules in \( \text{mod-}\Lambda \) with dimension vectors \( 001, 011, 010, 110 \) and \( 100 \), respectively. For transforming our results to \( S(\Lambda) \) we need to compute the minimal projective resolution of the indecomposable functor in \( \text{mod-}\text{mod-}\Lambda \). The case for an indecomposable projective functor clear it is induced by getting minimal projective resolution in \( \text{mod-\Lambda} \), and for a simple functor it is obtained by computing almost split sequence in \( \text{mod-\Lambda} \). Hence by help of the \( \Gamma_{\Lambda} \) we have the following minimal projective resolution in \( \text{mod-}\text{mod-}\Lambda \):

\[
\begin{align*}
0 & \rightarrow (-, P_2) \rightarrow (-, P_1 \oplus S_2) \rightarrow (-, I_2) \rightarrow (-, I_2)/\text{rad}(-, I_2) \rightarrow 0 \\
0 & \rightarrow (-, P_3) \rightarrow (-, P_2) \rightarrow (-, S_2) \rightarrow (-, S_2)/\text{rad}(-, S_2) \rightarrow 0 \\
0 & \rightarrow (-, S_2) \rightarrow (-, I_2) \rightarrow (-, I_1) \rightarrow (-, I_1)/\text{rad}(-, I_1) \rightarrow 0 \\
0 & \rightarrow (-, S_3) \rightarrow (-, P_1) \rightarrow (-, I_2) \rightarrow (-, I_2) \rightarrow 0 \\
0 & \rightarrow (-, P_2) \rightarrow (-, P_1) \rightarrow (-, I_1) \rightarrow (-, I_1) \rightarrow 0
\end{align*}
\]

We can consider \( \Gamma_{\Lambda} \) as subquiver of \( \Gamma_{S(\Lambda)} \) by Theorem 6.1. Based on the embedding any vertex in \( \Gamma_{\Lambda} \), which is identified by an indecomposable functor in \( \text{mod-\text{mod-}\Lambda} \), is mapped into a vertex of \( \Gamma_{S(\Lambda)} \) corresponding to the isomorphism class of the monomorphism in \( S(\Lambda) \) induced by the minimal projective resolution of the indecomposable functor in \( \text{mod-\Lambda} \) (as listed in the above). Now by using this fact and in view of Lemma 6.3, the Auslander-Reiten quiver of \( S(\Lambda) \) as follows.
Note that the vertices in the above diagram (or else where) which are presented by the same symbol have to be identified.

**Example 7.2.** Let $Q$ be a quiver of type $D_n, E_6, E_7$ or $E_8$, then the stable Auslander-Reiten quiver of $\Lambda$ contains a subquiver as the following

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\]

In fact, it is induced by an almost split sequence with no projective direct summand in the middle term. Then the bound quiver of the stable Auslander algebra of $\Lambda$ contain the above subquiver with the induced mesh relation. Since the algebra defined by the bound subquiver is of infinite representation type, hence in view of Theorem 3.4, $S(kQ)$ is of infinite representation type in these types of $Q$.

**Example 7.3.** Let $A_n$ be a quiver with $n$ vertices and with the following linear orientation

\[
Q : 1 \to 2 \to \cdots \to n
\]

We know that the Auslander-Reiten quiver of $Q$ has the following shape

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ
\end{array}
\]

with $n$ vertices in the leftmost side. We know the bounded quiver of $\text{Aus}(\text{mod-}kA_n)$ is the opposite quiver of $\Gamma^*_A$, which is obtained of $\Gamma_{A_n}$ by deleting vertices in the leftmost side, with mesh relations. This fact follows $\text{Aus}(\text{mod-}kA_n) = \text{Aus}(\text{mod-}kA_{n-1})$, where $A_{n-1}$ is obtained by $A_n$ with deleting the sink vertex $n$. The characterization given in [IPTZ], for those algebras of finite representation type with the Auslander algebra to be of finite representation type again implies $n - 1 \leq 4$. Hence $S(kA_n)$ is of finite representation type if and only if $n \leq 5$.

Due to Examples 7.2 and 7.3, checking the representation type of the submodule category over the representation-finite hereditary algebras is reduced to investigate only hereditary algebras of type $A$ and to see that whether the reflection functors preserve the representation type of the submodule categories.
Example 7.4. Let $\Lambda$ be given by the quiver

$$
\begin{array}{c}
1 \\
\alpha \\
\beta \\
2 \\
\end{array}
$$

with the relations $\alpha \beta \alpha = 0$ and $\beta \alpha \beta = 0$. The stable Auslander algebra $\Gamma$ of $\Lambda$ is given by the quiver

$$
\begin{array}{c}
1 \\
\alpha \\
\beta \\
2 \\
\epsilon \\
3 \\
\end{array}
$$

with the relations $\alpha \beta = 0$, $\beta \gamma = 0$, $\gamma \epsilon = 0$ and $\epsilon \alpha = 0$. Hence $\Gamma$ is a Nakayama algebra and so representation-finite. Therefore, by Theorem 3.4, $S(\Lambda)$ is of finite representation type.

Example 7.5. Let $\Lambda = T_2(k[x]/(x^2))$, $k[x]$ is the polynomial ring in one variable $x$ with coefficients in $k$. Then $\Lambda$ is a 1-Gorenstein algebra of finite representation type. In fact, it is a simple example of algebras considered in [GLS]. The algebra $\Lambda$ is given by the quiver

$$
\begin{array}{c}
1 \\
\alpha_1 \\
\beta_2 \\
2 \\
\end{array}
$$

with relations $\alpha_1^2 = \alpha_2^2 = 0$ and $\alpha_2 \beta = \beta \alpha_1$. Due to the computation given in [GLS, Example 13.5], the Auslander-Reiten quiver $\Gamma_\Lambda$ of $\Lambda$ look as the following

where the vertices are displayed by the composition series. So the $\text{Aus} (\text{mod-} \Lambda)$ is given by the following quiver with the mesh relations (indicated by dashed lines) and the two vertices in the left most column have to be identified with the two vertices in the rightmost column with respect to the same sign.

The indecomposable modules with composition series $1^2$, $1^2$ and $1$ are Gorenstein projective modules; take the indecomposable Gorenstein projective module $G_1$, $G_2$ and $G_3$ respectively with those composition series. Let $P_1$ and $P_2$ denote respectively the indecomposable projective
modules corresponding to the vertex 1 and 2 in the bound quiver of Λ. The Auslander-Reiten quiver $\Gamma_{\text{Gprj-}\Lambda}$ of the subcategory $\text{Gprj-}\Lambda$ is given as the following

```
  P_1
  \downarrow
  G_3 \rightarrow G_2
  \downarrow
  G_2 \rightarrow G_1 \rightarrow G_3
  \downarrow
  P_2
```

In view of $\text{Gprj-}\Lambda$, the stable Cohen-Macaulay Auslander algebra $\text{Aus(Gprj-}\Lambda)$ of Λ is given by the quiver

```
  G_2
  \downarrow
  a \rightarrow b
  \downarrow
  c
  G_3 \leftarrow G_1
```

with the relations $ab = bc = ca = 0$. The Auslander-Reiten quiver of $\text{Aus(Gprj-}\Lambda)$ is of the form

```
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
10 & 01 & 10 & 0
\end{array}
```

where the vertices are displayed by the dimension vectors. Hence the dimension vectors $\begin{pmatrix} 0 \\ 10 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 01 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ present respectively the simple functors $S_{G_3} = (-, G_3)/\text{rad}(-, G_3)$, $S_{G_1} = (-, G_1)/\text{rad}(-, G_1)$ and $S_{G_2} = (-, G_2)/\text{rad}(-, G_2)$; also the dimension vectors $\begin{pmatrix} 1 \\ 01 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ present respectively the indecomposable projective functor $(-, G_1)$, $(-, G_2)$ and $(-, G_2)$. Hence the Auslander-Reiten quiver $\Gamma_{\text{Gprj-Aus(mod-}\Lambda)}$ of the subcategory of Gorenstein projective modules over $\text{Aus(mod-}\Lambda)$, by Theorem 6.2, contains the above quiver. Take the indecomposable modules $M, N, T, U$ in $\text{mod-}\Lambda$ such that their composition series respectively are presented as $2, 1^2, 1^2, 2, 2$ and $2, 2$. In fact, they are those of indecomposable module which are not included in the image of the extension functor $gT_A$. To get the full of $\Gamma_{\text{Gprj-Aus(mod-}\Lambda)}$, we need only to see how the remaining projective indecomposable module in $\text{Gprj-Aus(mod-}\Lambda)$ being $(-, M)$, $(-, N)$, $(-, T)$ and $(-, T)$ can be added into the $\Gamma_{\text{Aus(Gprj-}\Lambda)}$. It can be easily seen that the minimal right(left) almost morphisms $X \rightarrow Y$ in $\text{mod-}\Lambda$, as presented in $\Gamma_{\Lambda}$, where $X$ or $Y \in \{M, N, T, U\}$, induce the minimal right(left) almost morphisms $(-, X) \rightarrow (-, Y)$ in $\text{Gprj-mod-}\Lambda$. These minimal right(left) almost morphisms $(-, X) \rightarrow (-, Y)$ give us the remaining part of $\Gamma_{\text{Gprj-Aus(mod-}\Lambda)}$. Now, we deduce $\Gamma_{\text{Gprj-Aus(mod-}\Lambda)}$ looks as follows:

```
\begin{array}{cccc}
(-, G_3) & (-, T) & (-, M) & (-, U) \\
(-, N) & (-, G_1) & (-, G_2) & (-, G_3) \\
S_{G_3} & S_{G_1} & S_{G_2} & S_{G_3}
\end{array}
```
Let us continue our computation in order to find the Auslander-Reiten quiver $\Gamma_{\text{Gprj-T}_2(\Lambda)}$ of the subcategory of Gorenstein projective modules over $T_2(\Lambda)$. Analogue to Example 7.1 we use the same notations to show the indecomposable objects in $\text{Gprj-T}_2(\Lambda)$. Also we need to find the minimal projective resolutions of indecomposable functors in $\text{mod-Gprj-}\Lambda$ in $\text{mod-Gprj-}\Lambda$. To do this, the almost split sequences, as indicated in $\Gamma_{\text{Gprj-}\Lambda}$ in the above, in $\text{Gprj-}\Lambda$ help us to compute the minimal projective resolution of the simple functors. We will do as follows.

$$0 \to (-, G_1) \to (-, G_2) \to (-, G_3) \to S_{G_1} \to 0$$
$$0 \to (-, G_3) \to (-, P_1 \oplus G_1) \to (-, G_2) \to S_{G_2} \to 0$$
$$0 \to (-, G_2) \to (-, G_3 \oplus P_2) \to (-, G_1) \to S_{G_1} \to 0$$
$$0 \to (-, G_3) \to (-, P_1) \to (-, G_2) \to S_{G_2} \to 0$$
$$0 \to (-, G_2) \to (-, P_1 \oplus P_2) \to (-, G_2) \to S_{G_2} \to 0$$
$$0 \to (-, G_1) \to (-, P_2) \to (-, G_1) \to S_{G_1} \to 0$$

By Theorem 6.2, we just need to add the vertices corresponding to the remaining indecomposable objects to the quiver $\Gamma_{\text{Aus(Gprj-}\Lambda)}$. This in view of lemma 6.3 is done as follows.

Recall that a finite-dimensional $k$-algebra $\Lambda$ is a Nakayama algebra if any indecomposable is uniserial, i.e. it has a unique composition series ([ARS], p.197). In this case $\Lambda$ is representation-finite. If $k$ is algebraically closed then any connected self-injective Nakayama algebra is Morita equivalent to $\Lambda(n, t), n \geq 1, n \geq 2$ ([GR], p.243), which is defined as follows.

Let $C_n$ be the cyclic quiver with $n$ vertices and $J$ the ideal generated by all arrows, and then set $\Lambda(n, t) := kC_n/J^t$ with $t \geq 2$. For the Auslander-Reiten quiver of $\Lambda(n, t)$, see [GR, Section 2] and [ARS, Page 197]. In particular, the stable Auslander-Reiten quiver of $\Lambda(n, t)$ is the quotient $\mathbb{Z}A_{t-1}/<\tau^n>$ of the repetitive quiver $\mathbb{Z}A_{t-1}$ modulo the group generated by $\tau^n$.

**Example 7.6.** Let $\Lambda(n, t)$ be a self-injective algebra as defined above. For $t = 2$, $\Lambda(n, t)$ is a quadratic algebra, hence $T_2(\Lambda(n, t))$ is of finite representation type, see Example 3.5. We assume $t > 2$. In [RS1] was shown $\mathcal{S}(\Lambda(1, t))$ is of finite representation type, or equivalently by Theorem 3.4, the $\text{Aus(mod-}\Lambda(1, t))$ is of finite representation type, if and only $t \leq 5$. If $n > 2$, then the Auslander-Reiten quiver $\Gamma_{\text{mod-}\Lambda(n, t)}$ has the following subquiver without any relation.

Trivially, this subquiver is of infinite representation type, and so the stable Auslander algebra of $\Lambda(n, t)$ is of infinite representation type, and equivalently the submodule category $\mathcal{S}(\Lambda(n, t))$ is of infinite representation type. Hence $\mathcal{S}(\Lambda(n, t))$ is of finite representation type if $n \leq 2$. 

![Diagram](image-url)
The situation for $n = 1$ is clear by Ringel and Schmidmeier’s work, so it remains to check for $n = 2$. Assume $t > 5$. Since in an obvious way there is a fully faithful functor from $\text{mod-}\text{Aus} \text{(mod-Λ(1, t))}$ to $\text{mod-}\text{Aus} \text{(mod-Λ(2, t))}$. But as we have already mentioned $\mathcal{S}(\Lambda(1, t))$ is of infinite representation type, so is $\mathcal{S}(\Lambda(2, t))$. Hence $t$ must be less than 6. Finally, to give a complete classification for which self-injective Nakayama algebra having the associated submodule category to be of finite representation type, in conjunction with Example 7.4, it is enough to check only for the cases $\Lambda(2, 4)$ and $\Lambda(2, 5)$.

8. ACKNOWLEDGMENT

The author would like to thank Hideto Asashiba for some useful comments to improve the English of the manuscript.

REFERENCES

[AE] J. Alperin, L. Evens, Representations, resolutions, and Quillen’s dimension theorem, J. Pure Appl. Algebra 22 (1981), 1-9.
[As] H. Asashiba, Gluing derived equivalences together, Adv. Math. 235 (2013), 134-160.
[AHK] J. Asadollahi, R. Hafezi and M.H. Keshavarz, Injective representations of bound quiver algebras, J. Algebra Appl. 17 (2018), no. 1, 1850001, 13 pp.
[AHV] J. Asadollahi, R. Hafezi and R. Vahed, Derived equivalences of functor categories, J. Pure Appl. Algebra 223 (2019), 1073-1096.
[Au1] M. Auslander, Representation theory of artin algebras I, Communication in Algebra, 1 (1974), pp. 177-268.
[Au2] M. Auslander, Functors and morphisms determined by objects, Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976), pp. 1244. Lecture Notes in Pure Appl. Math., Vol. 37, Dekker, New York, 1978.
[AB] M. Auslander and M. Bridger, Stable module theory, Mem. Amer. Math. Soc. 94 (1969).
[AR1] M. Auslander and I. Reiten, Representation theory of artin algebras IV, Comm. Algebra 5 (1977) 443-518.
[AR2] M. Auslander, I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86 (1991) 111-152.
[AR3] M. Auslander, I. Reiten, Cohen-Macaulay and Gorenstein algebras, Progr. Math. 95 (1991) 221-245.
[ARS] M. Auslander, I. Reiten, and Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. 36, Cambridge Univ. Press, 1995.
[AS] M. Auslander and S. O. Smalø, Almost split sequences in subcategories, J. Algebra 69(2) (1981), 426-454.
[C] X.W. Chen, Algebras with radical square zero are either self-injective or CM-free, Proceedings of American Mathematical Society, 140(1) (2012), 93-98.
[B] A. Beligiannis, On the Freyd Categories of an Additive Category, Homology, Homotopy and Applications Vol. 2, No.11 (2000), pp. 147-185.
[CSZ] X.W. Chen, D. Shen and G. Zhou, The Gorenstein-projective modules over a monomial algebra, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 148A (2018), 1115-1134.
[DR] V. Dlab, C. M. Ringel, The module theoretical approach to quasi-hereditary algebras, In: Representations of Algebras and Related Topics (ed. H. Tachikawa and S. Brenner). London Math. Soc. Lecture Note Series 168. Cambridge University Press (1992), 200-224.
[E] O. Eriksson, From submodule categories to the stable Auslander algebra, J. Algebra 486 (2017), 98-118.
[EJ] E. Enochs and O. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (1995), no. 4, 611-633.
[EHS] H. Eshraghi, R. Hafezi and Sh. Salari, Total acyclicity for complexes of representations of quivers, Comm. Algebra 41 (2013), no. 12, 4425-4441.
[GLS] C. Geiss, B. Leclerc and J. Schröer, Quivers with relations for symmetrizable Cartan matrices I: Foundations, Invent. Math. 209 (2017), no. 1, 61-158.
[GR] P. Gabriel, and C. Riedtmann, Group representations without groups, Comment. Math. Helv. 54(1979):240-287.
R. HAFEZI, Auslander-Reiten duality for subcategories, available on arXiv:1705.06684.

R. Hafezi, On Cohen-Macaulay Auslander algebras, available on arXiv:1802.05156.

D. Happel, On Gorenstein algebras, Progr. Math. 95 (1991) 389-404.

M. Kalck, Singularity categories of gentle algebras, Bull. Lond. Math. Soc. 47 (2015), no. 1, 65-74.

D. Kussin, H. Lenzing and H. Meltzer, Nilpotent operators and weighted projective lines, J. Reine Angew. Math. 685 (2013), 33-71

H. Krause, Krull-Schmidt categories and projective covers, Expo. Math. 33 (2015), no. 4, 535-549.

H. Krause and Ø. Solberg, Applications of cotorsion pairs, J. London Math. Soc. (2) 68 (2003), 631-650.

K. Igusa, M. Platzeck, G. Todorov, D. Zacharia, Auslander algebras of finite representation type. Comm. Algebra 15 (1987), no. 1-2, 377-424

X-H. Luo and P. Zhang, Monic representations and Gorenstein-projective modules, Pacific J. Math. 264 (2013), no. 1, 163-194.

H. Matsu and R. Takahashi, Singularity categories and singular equivalences for resolving subcategories, Math. Z. 285 (2017), no. 1-2, 251-286.

D. Quillen, Higher algebraic $K$-theory: I, in Algebraic $K$-theory I: Higher $K$-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), vol. 341 of Lecture Notes in Math., Springer, Berlin, 1973, 85–147.

C.M. Ringel and M. Schmidmeier, Invariant subspaces of nilpotent linear operators I, J. Reine Angew. Math. 614 (2008), 1-52.

C.M. Ringel and M. Schmidmeier, The Auslander-Reiten translation in submodule categories, Trans. Amer. Math. Soc. 360 (2008), no. 2, 691-716.

C. M. Ringel and P. Zhang, From submodule categories to preprojective algebras, Math. Z. 278 (2014), no. 1-2, 55-73.

C. M. Ringel and P. Zhang, Representations of quivers over the algebras of dual numbers, J. Algebra. 475 (2017), 327-360.

C. M. Ringel and P. Zhang, Gorenstein-projective and semi-Gorenstein-projective modules, available on arXiv:1808.01809.

C. M. Ringel and P. Zhang, Gorenstein-projective and semi-Gorenstein-projective modules. II available on arXiv:1905.04048.

B-L. Xiong, P. Zhang and Y-H. Zhang, Auslander-Reiten translations in monomorphism categories, Forum Math. 26 (2014), no. 3, 863-912.

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