THE UNIVERSAL COVER OF 3-MANIFOLDS BUILT FROM INJECTIVE HANDLEBODIES IS $\mathbb{R}^3$.

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Abstract. This paper gives a proof that the universal cover of a closed 3-manifold built from three $\pi_1$-injective handlebodies is homeomorphic to $\mathbb{R}^3$. This construction is an extension to handlebodies of the conditions for gluing of three solid tori to produce non-Haken Seifert fibered manifolds with infinite fundamental group. This class of manifolds has been shown to contain non-Haken non-Seifert fibered manifolds.

1. Introduction

It is a long standing conjecture that all closed $P^2$-irreducible 3-manifolds with infinite fundamental group have universal cover homeomorphic to $\mathbb{R}^3$. In 1969 this was shown to be true in the case that the manifold is Haken by F. Waldhausen in [16]. Herbert Seifert showed, in his 1932 doctoral dissertation “Topology of 3-dimensional fibred spaces”, that the conjecture is true for Seifert fibered manifolds with infinite fundamental group. In 1987 J. Hass, H. Rubinstein and P. Scott, in [9], showed that the conjecture was true if the manifold is $P^2$-irreducible and it contains an essential surface other than $S^2$ or $P^2$. If Thurston’s Geometrisation conjecture is proven, which looks likely, following the announcements of Perelman, then the conjecture will be proven as a result. However it is still an interesting question to answer, using more direct means, for topological classes of manifolds that do not appear to have any obvious unifying underlying geometry. Note also that a similar question arises for higher dimensional aspherical manifolds and the methods in this paper may extend to suitable classes of examples.

The class of manifolds being considered in this paper is said to meet the ‘disk-condition’. The construction of these manifolds has been discussed by H. Rubinstein and the author in [6]. This construction is an extension to handlebodies of the gluing of three solid tori which produces non-Haken Seifert fibered manifolds with infinite fundamental group. A definition is given in the next section. The structure given by this construction tells us a lot about this class of manifolds. In [6] it is shown that the characteristic variety of these manifolds can be constructed in a rather elegant way. Also examples of atoroidal manifolds and non-Haken non-Seifert fibered manifolds that meet the disk-condition are given. In [5] the author gave an

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algorithm to show that the word problem in the fundamental group of manifolds that meet the disk-condition is solvable. The main theorem proven in this paper is:

**Theorem 1.1.** All manifolds that meet the disk-condition have a universal cover homeomorphic to $\mathbb{R}^3$.

In 1961 M. Brown in [3] showed that if an $n$ dimensional metric space contained an infinite sequence of expanding balls, such that the union of the sequence is the space then the space is homeomorphic to $\mathbb{R}^n$. So the general idea for the proof of this theorem is to produce an infinite expanding sequence of open balls that meets Brown’s conditions, in the universal cover of manifolds that meet the disk-condition. This is done by taking the dual 2-complex to the structure in the universal cover and then carefully defining an expanding sequence of simplicial subsets, such that the interior of the corresponding manifold in the universal cover is an open ball. This approach is quite pleasing as the disk-condition is used implicitly by the observation that the dual 2-complex turns out to be a CAT(0) metric space.

2. Preliminaries and definitions

Throughout this paper we will assume that, unless stated otherwise, we are working in the PL category of manifolds and maps. Even though we will not explicitly use this structure we will use ideas that are a consequence, such as regular neighbourhoods and transversality as defined by C. Rourke and B. Sanderson in [14]. The standard definitions in this field, as given by J. Hempel in [10] or W. Jaco in [11], are used.

A manifold $M$ is closed if $\partial M = \emptyset$ and irreducible if every embedded $S^2$ bounds a ball. We will assume that, unless otherwise stated, all 3-manifolds are orientable. The reason for this is that all closed non-orientable $\mathbb{P}^2$-irreducible 3-manifolds are Haken. (A manifold is $\mathbb{P}^2$-irreducible if it is irreducible and does not contain any embedded 2-sided projective planes).

If $M$ is a 3-manifold and $S$ a surface, which is not a sphere, disk or projective plane, then a map $f : S \to M$ is called essential if the induced map $f^* : \pi_1(S) \to \pi_1(M)$ is injective. This is also known as a $\pi_1$-injective map. Also $f : S \to M$ is a proper map if $f(S) \cap \partial M = f(\partial S)$. Unless otherwise stated, a homotopy/isotopy of a proper map, is assumed to be proper. That is, at each point the map remains proper. To reduce notation an isotopy/homotopy of a surface $S \subset M$ are used without defining the map. Here we are assuming that there is a map $f : S \to M$ and we are referring to an isotopy/homotopy of $f$, however defining the map is often unnecessary and would only add to excessive book keeping.

If $H$ is a handlebody and $D$ is a properly embedded disk in $H$ such that $\partial D$ is essential in $\partial H$ then $D$ is a meridian disk of $H$. If $D$ is a proper singular disk in $H$ such that $\partial D$ is essential in $\partial H$, then it is called a singular meridian disk.
Definition 2.1. For $H$ a handlebody, $\mathcal{T}$ a set of curves in $\partial H$ and $D$ a meridian disk, let $|D|$ be the number of intersection between $D$ and $\mathcal{T}$.

By the result of Freedman, Hass and Scott in [8], if we put a metric on $\partial H$ and isotop both $\mathcal{T}$ and $\partial D$ so that $\mathcal{T}$ and $\partial D$ are length minimising in $\partial H$ then the number of intersections will be minimal. Note that when there are parallel curves in $\mathcal{T}$ we need to ‘flatten’ the metric in their neighbourhood so they remain disjoint. Let $\mathcal{D}$ be a set containing a single representative from each isotopy class of meridian disks of $H$. We can also assume that the boundary of each meridian disk in $\mathcal{D}$ is a length minimising geodesic. Therefore the number of intersections between the boundaries of two disks in $\mathcal{D}$ is minimal. In [6] there is the following lemma proven.

Lemma 2.2. Any two disks of $\mathcal{D}$ can be isotoped, leaving their boundaries fixed, so that the curves of intersections are properly embedded arcs.

We will assume from this point on that all curves in $\partial H$ and meridian disks have been isotoped to have minimal intersection.

Definition 2.3. A set $\mathcal{D} \subset \mathcal{D}$ is a system of meridian disks for $H$ if all the disks in $\mathcal{D}$ are pairwise disjoint, non-parallel and they cut $H$ up into 3-balls.

If the genus of $H$ is $g$, then a minimal system of meridian disk is a system of $g$ meridian disks that cut $H$ up into a single ball.

Definition 2.4. If $S$ is a punctured sphere and $\gamma \subset S$ is a properly embedded arc that is not boundary parallel, then $\gamma$ is a wave if both ends are in the one boundary component of $S$.

Let $H$ be a handlebody, $\mathcal{T} \subset \partial H$ be a set of essential pairwise disjoint simple closed curves and $\mathcal{D}$ be a minimal system of disks for $H$. From above we are assuming that the intersection between $\mathcal{T}$ and $\mathcal{D}$ is minimal. Let $S$ be the $2g$ punctured sphere produced when $\partial H$ is cut along the boundary curves of the disks in $\mathcal{D}$. Then $\mathcal{T}_S = S \cup \mathcal{T}$ is a set of properly embedded arcs in $S$.

Definition 2.5. $\mathcal{D}$ is called waveless system of meridian disks with respect to $\mathcal{T}$ if there are no waves in $\mathcal{T}_S$.

In [6] it is shown that given a set of essential pairwise disjoint simple closed curves in the boundary of a handlebody then a waveless system of disks can always be found.

Definition 2.6. If $H$ is a handlebody and $\mathcal{T}$ is a set of essential pairwise disjoint simple closed curves in $\partial H$, then $\mathcal{T}$ meets the $n$ disk-condition in $H$ if $|D| \geq n$ for every meridian disk $D$.

See [6] for a sufficient and necessary condition for $\mathcal{T}$ to satisfy the $n$ disk-condition in $H$. It is also shown in [6] that there is an algorithm to decide if a set of curves in $\partial H$ meets the $n$ disk-condition.
Next we are going to give a description of the construction of 3-manifolds that meet the ‘disk-condition’. Let $H_1$, $H_2$ and $H_3$ be three handlebodies. Let $S_{i,j}$, for $i \neq j$ be a sub-surface of $\partial H_i$ such that:

1. $\partial S_{i,j} \neq \emptyset$.
2. The induced map of $\pi_1(S_{i,j})$ into $\pi_1(H_i)$ is injective.
3. For $j \neq k$, $S_{i,j} \cup S_{i,k} = \partial H_i$.
4. $\mathcal{T}_i = S_{i,j} \cap S_{i,k} = \partial S_{i,j}$ is a set of disjoint essential simple closed curves that meet the $n_i$ disk-condition in $H$.
5. $S_{i,j} \subset \partial H_i$ is homeomorphic to $S_{j,i} \subset \partial H_j$.

Note that $S_{i,j}$ need not be connected. Now that we have the boundary of each handlebody cut up into essential faces we want to glue them together by homeomorphisms, $\Psi_{i,j}: S_{i,j} \rightarrow S_{j,i}$, that agree along $\mathcal{T}_i$’s. The result is a closed 3-manifold $M$, for which the image of each handlebody is embedded.

![Figure 1. Homeomorphisms between boundaries of handlebodies.](image)

**Definition 2.7.** If $M$ is a manifold constructed from three handlebodies as above, such that $\mathcal{T}_i$ meets the $n_i$ disk-condition in $H_i$ and

$$(1) \quad \sum_{i=1,2,3} \frac{1}{n_i} \leq \frac{1}{2}$$

then $M$ meets the $(n_1,n_2,n_3)$ disk-condition. If we are not talking about a specific $(n_1,n_2,n_3)$, the manifold is said to meet just the disk-condition.

Note that if the three handlebodies are solid tori then the resulting manifold is a non-Haken Seifert fibered manifold with infinite fundamental group. In [6] it is shown that the disk condition implies that the induced map on each $\pi_1(H_i)$ into $\pi_1(M)$ is injective and thus $\pi_1(M)$ is infinite. See [6] for further and possibly more enlightening discussion of this construction, in particular in relation to non-Haken Seifert fibered manifolds with infinite fundamental group and Haken manifolds. Also in [6] and [5] there are a number of motivating examples. These included a construction for non-Haken non-Seifert fibered manifolds meeting the disk-condition. These are
produced by performing Dehn Surgery along infinite families of pretzel knots and 2-bridge knots in $S^3$.

The following definition of CAT($k$) metric spaces comes from the book [2] by M. Bridson and A. Haefliger. However, these spaces were first introduced by Alexandrov in [1]. For the purposes of this paper we are only concerned with CAT(0) spaces, so the definition has been restricted to this one case.

If $X$ is a length space, it is a metric space such that any two points are joined by a unique geodesic and the distance between the points is the given by its length. Let $p_1, p_2, p_3 \in X$ are three points and $[p_i, p_j]$ for $i \neq j$ be geodesic segments joining them. Then $\Delta = [p_1, p_2] \cup [p_2, p_3] \cup [p_1, p_3]$ is a geodesic triangle. Let $\bar{p}_1, \bar{p}_2, \bar{p}_3 \in \mathbb{E}^2$ be three points such that $d(p_i, p_j) = d(\bar{p}_i, \bar{p}_j)$. Then let the comparison triangle to $\Delta$ be $\bar{\Delta}$ the geodesic triangle in $\mathbb{E}^2$ with vertices $\bar{p}_1, \bar{p}_2, \bar{p}_3$ and the comparison point to $x \in [p_i, p_j]$ be the point $\bar{x} \in [\bar{p}_i, \bar{p}_j]$ such that $d(p_i, x) = d(\bar{p}_i, \bar{x})$. $\Delta$ satisfies the CAT(0) inequality if for all points $x, y \in \bar{\Delta}$ and comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \leq d(\bar{x}, \bar{y})$.

**Definition 2.8.** If the CAT(0) inequality holds for every geodesic triangle in a metric space $X$ then it is said to be CAT(0).

A space is said to be non-positively curved if it is locally CAT(0). That is, if every point $x \in X$ there is an $r > 0$ such that the ball of radius $r$ about $x$ is CAT(0). In [2] Bridson and Haefliger show that a metric space being simply connected and non-positively curved is a sufficient condition for it to be CAT(0). All CAT(0) metric spaces are contractible, geodesic metric spaces. Other properties of CAT(0) spaces required for this paper will be introduced as required.

### 3. Structure in the universal cover

First we need to look at the universal cover of handlebodies. Let $H$ be a handlebody with a set of curves $\mathcal{T} \subset \partial H$ that meet the $n$ disk-condition. Let $\tilde{H}$ be the universal cover of $H$ and $q : \tilde{H} \to H$ be the covering projection. Then, as can be seen in figure [3] $H$ is P.L. 3-manifold with boundary, $\tilde{H}$ is a non-compact P.L. simply connected 3-manifold with boundary and $\text{int}(\tilde{H}) \cong \mathbb{R}^3$. As each component of $\mathcal{T}$ is essential in $H$, $q^{-1}(\mathcal{T})$ is a set of simple pairwise disjoint non-compact proper curves in $\partial \tilde{H}$. Let $F \subset \partial H$ be a face produced when $\partial H$ is cut along $\mathcal{T}$. By the disk-condition we know that $F$ is essential in $H$ and thus each component of $q^{-1}(F)$ is the universal cover of $F$. Let $\tilde{F} \subset \partial \tilde{H}$ be a component of $q^{-1}(F)$. Then $\tilde{F}$ is a simply connected non-compact P.L. 2-manifold, thus $\text{int}(\tilde{F}) \cong \mathbb{R}^2$.

Note that the universal covers of these objects has not been described by properly embedding them in compact manifolds. Even though this is a more concrete description it has not been done to avoid the complications brought about by ensuring that the compactification is ‘nice’, as tackled by Simon in [15]. The following lemmas are required later in the paper when
defining a process to glue the lifts of the handlebodies together to produce a sequence of expanding open 3-cells.

Let $M_1$ and $M_2$ be 3-manifolds such that $\text{int}(M_i) \cong \mathbb{R}^3$ and $\partial M_i \cong \mathbb{R}^2$ and $f : \partial M_1 \to \partial M_2$ is a homeomorphism. By R. Kirby [12] $f$ is a stable homeomorphism and by Brown and Gluck [4] all stable homeomorphisms of $\mathbb{R}^n$ are isotopic to the identity. Then as shown by R. Lickorish in [13], the two manifolds resulting from gluings by isotopic homeomorphisms, are homeomorphic. This gives us the following lemma.

**Lemma 3.1.** Let $M_i$'s and $f$ be described as above. If $M_1 \cup_f M_2 \cong \mathbb{R}^3$, then for any homeomorphism $f' : \partial M_1 \to \partial M_2$, $M_1 \cup_f M_2 \cong \mathbb{R}^3$.

This means that whether the manifold resulting from the gluing is homeomorphic to $\mathbb{R}^3$, depends only on the structure of the initial manifolds and not the homeomorphism between their boundaries that gives the gluing. This leads us to the next lemma.

**Lemma 3.2.** If the $M_i$'s are 3-manifolds as described above such that $M_1 \cong \mathbb{R}^2 \times [0, 1]$ and $\partial M_2$ has a collar neighbourhood in $M_2$, then for any homeomorphism $f : \partial M_1 \to \partial M_2$, $M_1 \cup_f M_2 \cong \mathbb{R}^3$.

**Proof.** Let $M = M_1 \cup_f M_2$ and $U \subset M_2$ is the collar neighbourhood of $\partial M_2$. In an abuse of notation we will refer to the image of the $M_i$ in $M$ as $M_i$ and from the previous lemma we can assume that $f$ is the obvious identity. It is sufficient to show that there is a homotopy of embeddings $r : M \times I \to M$ such that $r(M, 0) = M$ and $r(M, 1) = \text{int}(M_2)$. As $U$ is the collar neighbourhood of $\partial M_2$, then $U = \partial M_2 \times [0, 1]$. As $M_1 \cong \partial M_1 \times [0, 1)$ there is an obvious homotopy $r' : (M_1 \cup U) \times I \to (M_1 \cup U)$.
where \( r'(M_1 \cup U, 0) = M_1 \cup U, r'(M_1 \cup U, 0) = \text{int}(U) \) and \( r' \) restricted to \( \partial M_2 \times \{1\} \) is the identity. Then \( r \) is the identity on \( M_2 - \overline{U} \) and equivalent to \( r' \) on \( M_1 \cup U \).

Once again let \( H \) be a genus \( g \) handlebody, \( \tilde{H} \) its universal cover and \( p : \tilde{H} \to H \) is the covering projection. \( \mathcal{T} \) is a set of curves in \( \partial H \) meeting the \( n \) disk-condition and \( F \) is a face of \( \partial H \) when cut along \( \mathcal{T} \).

**Lemma 3.3.** Let \( \tilde{F} \subset \partial \tilde{H} \) be a component of \( p^{-1}(F) \), then \( \text{int}(\tilde{H}) \cup \text{int}(\tilde{F}) \cong \mathbb{R}^2 \times [0, 1) \).

**Remark 3.4.** This lemma seems reasonably obvious from the picture, however as it is an important step it is worth giving a formal proof.

**Proof.** A necessary and sufficient condition for \( N = \text{int}(\tilde{H}) \cup \text{int}(\tilde{F}) \cong \mathbb{R}^2 \times [0, 1) \) is that any compact set in \( N \) can be contained in a closed 3-cell. Therefore we can show that \( N \) has a product structure by showing that for any compact set \( C \subset N' \) there is a submanifold that contains \( C \) and which admits a product structure.

Let \( p : \tilde{H} \to H \) be the covering projection. Let \( D \) be a minimal waveless system of meridian disks (see definition 2.5). Let \( \Gamma \) be the dual graph to \( D \), as shown in figure 2. Therefore \( \Gamma \) is a wedge of \( g \) circles with a single vertex \( x \). Note that each loop of \( \Gamma \) corresponds to generator for representation of \( \pi_1(H) \) and that \( x \) is disjoint to \( D \). If \( n(\Gamma) \) is the regular neighbourhood of \( \Gamma \) in \( H \), then \( H \cong n(\Gamma) \). Let \( \tilde{\Gamma} = p^{-1}(\Gamma) \), then \( \tilde{\Gamma} \) is an infinite tree and each vertex has order \( 2g \). As \( D \) is a set of \( g \) meridian disks that cut \( H \) up into a single 3-ball, then \( \tilde{D} = p^{-1}(D) \) is a set of pairwise disjoint properly embedded disks that cut \( \tilde{H} \) up into fundamental domains, see figure 2. There is a single vertex of \( \tilde{\Gamma} \) contained in each fundamental domain.

Let \( C \) be a compact set in \( N \). Then there is a simply connected manifold \( N' \subset \tilde{H} \) made up of a finite number of fundamental domains of \( \tilde{H} \) such that \( N' \) contains \( C \). Let \( T \subset \tilde{\Gamma} \) be the minimal connected tree such that \( T \) contains all the vertices contained in \( N' \). Therefore \( T \) is a compact tree and \( n(T) \subset \text{int}(H) \), its regular neighbourhood, is a closed three cell. Let \( B \) be a fundamental domain in \( N' \) such that \( B \cap \tilde{F} \neq \emptyset \). As \( \tilde{F} \) is \( \pi_1 \)-injective in \( H \) and \( D \) is waveless, \( B \cap \tilde{F} \) is a disk. Let \( \{D_i\} \) be the maximal subset of \( \tilde{D} \cap B \) such that \( D_i \cap \tilde{F} \neq \emptyset \). \( B \) can be fibered by \( (B \cap \tilde{F}) \times [0, 1] \) such that each \( D_i \) is vertical. Thus all the fundamental domains of \( N' \) that have nonempty intersection with \( \tilde{F} \) can be similarly fibered so the fiberings agree on \( \tilde{D} \) in their intersections. This fibered can then be easily extended to the fundamental domains of \( N' \) that have empty intersection with \( \tilde{F} \). We now have the product structure \( (N' \cap \tilde{F}) \times [0, 1] \) on \( N' \) and thus the product structure \( \text{int}(N' \cap \tilde{F}) \times [0, 1] \) on \( \text{int}(N') \cup \text{int}(\tilde{F} \cap N') \). Thus \( n(T) \) can be expanded in \( \text{int}(N') \cup \text{int}(\tilde{F} \cap N') \) to contain \( C \).  

Let \( M \) be a 3-manifold which can be constructed by a gluing of three handlebodies, \( H_i \)'s, that meets the disk-condition and \( \mathcal{T} = \bigcap H_i \) is the set
of triple curves. Also let $X$ be the $2$-complex $\bigcup \partial H_i$. Note that this is not the usual use of $2$-complex, as $X$ is constructed by gluing surfaces along their boundaries. These surfaces can easily be cut up into cells. Let $\tilde{M}$ be the universal cover with $p : \tilde{M} \to M$ the covering projection. As the induced map of $\pi_1(H_i)$ into $\pi_1(M)$ is injective and $H_i$ is embedded in $M$, each component of $p^{-1}(H_i)$ is the universal cover of $H_i$. As each face of $X$ is also $\pi_1$-injective, $\tilde{X} = p^{-1}(X)$ is a $2$-complex whose faces are non-compact ‘ideal’ polyhedron. Note that if a face $X$ is not an annulus then its lifts in $\tilde{X}$ will have infinite order as an ideal polygon.

4. Dual $2$-complex

Next we wish to construct the dual $2$-complex to $\tilde{X}$. There are two ways of viewing this object. One is actually embedded in $\tilde{M}$. The second is an abstract view where it exists by itself. For the majority of this proof it does not matter which view we use, so most of the time no distinction will be drawn. However, each view does have its advantage at some point in this argument.

Let $C$ be the dual $2$-complex to $\tilde{X}$. The embedded view of $C$ is that it has a vertex at the center of each lift of a handlebody. There is an edge dual to each lift of a face of $X$ in $\tilde{M}$ and a face dual to each lift of a curve in $\mathcal{T}$. As each curve in $\mathcal{T}$ intersects three lifts of handlebodies, each face of $C$ is a triangle. The abstract view of $C$ is that each vertex corresponds to a lift of a handlebody. An edge between two vertices corresponds to the gluing, along the lift of a face of $X$, between the two corresponding universal covers of handlebodies. Finally a face of $C$ corresponds to the gluing of the lift of three handlebodies along the lift of a triple curve. Clearly the resulting $2$-complex in each case is identical. As all the faces in $C$ correspond to the lifts of triple curves thus they are all triangular.

We can put a metric on $C$ by assuming all the faces are geodesics triangles, where the corner angles correspond to the disk-condition satisfied by the handlebody corresponding to the vertex. If the vertex corresponds with a lift of the handlebody $H_i$ such that $\mathcal{T}$ meets the $n_i$ disk-condition in $H_i$, then the internal angle of the face at the vertex is $2\pi/n_i$, as shown in figure 4. As each face has a vertex associated with each of the handlebodies, the sum of the internal angles of each face is always at most $\pi$. Thus all the faces are similar and are either Euclidean or Hyperbolic triangles. Note if the $M$ meets the $(6,6,6),(4,8,8)$ or $(4,6,12)$ disk-condition then all the faces are Euclidean geodesic triangles. We will always assume that the shortest edge of each face has length $1$. By Bridson and Haefliger [2], as the number of isometry classes of simplices of $C$ is finite, $C$ is a complete length space, meaning that the distance between any two points is the length of the shortest path joining them. If the genus of a handlebody is at least two then there are an infinite number of lifts of its faces in the boundary of a lift of the handlebody, thus the corresponding vertex in $C$ is going to have infinite
order. Similarly if a face is not an annulus, all the edges in $C$ corresponding to lifts of it will have infinite order. Let $C^1$ be the 1-skeleton of $C$ and $C^0$ be the set of vertices of $C$.

![Figure 3. A face of $C$.](image)

The simplest examples for $C$ is when $M$ is a Seifert fibered space. As each handlebody is a solid torus, a vertex that is at the center of a lift of the solid torus $H_i$ has order $n_i$ and as each face of $X$ is an annulus every edge has order two. Thus $C$ is a tessellation of either $\mathbb{E}^2$ or $\mathbb{H}^2$ by isomorphic geodesic triangles, depending on the disk-condition.

Only the base disk-conditions $((6, 6, 6), (4, 8, 8)$ and $(4, 6, 12))$ need be considered. The reason for this is that the disk-condition is used by the non-positively curved structure it induces and thus the minimal cases are somehow the worst. Also if $M$ satisfies the $(n_1, n_2, n_3)$ disk-condition, such that $\sum \frac{1}{n_i} < \frac{1}{2}$, then $M$ also meets at least one of the base disk-conditions. This means that we can assume the faces of $C$ are Euclidean geodesic triangles.

$C$ seems fairly unfriendly as it can have infinite degree on its vertices and edges, however it turns out not too bad as it is CAT(0). From Bridson and Haefliger [2] a length space being simply connected and non-positively curved is sufficient to show that $C$ is CAT(0).

**Lemma 4.1.** If $M$ is a manifold that meets the disk-condition, $\tilde{M}$ is its universal cover and $C$ is the dual 2-complex to $\tilde{X}$, as described above, then $C$ is simply connected.

**Proof.** The idea of this proof is to show that there is a retraction of $\tilde{M}$ onto $C$, therefore as $\tilde{M}$ is simply connected then $C$ is simply connected. Let $\tilde{T} = p^{-1}(T)$ and $\tilde{X} = p^{-1}(X)$. Through out this proof we will refer to the embedded model of $C$. We will also assume that it in general position with respect to $\tilde{X}$.

First we want to show there is a retraction of $\tilde{T}$ into $C$. Each component of $\tilde{T}$ is an open arc that intersects $C$ exactly once in the interior of a 2-simplex. Moreover each face intersect exactly one component of $\tilde{T}$. This means that for any face $F$ of $\tilde{X}$ that $F \cap C$ is a tree that cuts $\tilde{F}$ up into 4-gons with a single vertex removed, as shown in figure 4. There is a retraction
where each component $\alpha \subset \tilde{T}$ is sent to the point $\alpha \cap C$. That is $r_{\tilde{T}}(\tilde{T}) = \tilde{T} \cap C$.

Next we want to show there is a retraction of the lifts of the faces of $X$ to $C$. Let $A$ be a face of $C$ and $A'$ a face adjacent to $A$, such that $a = A \cap \tilde{T}$ and $a' = A' \cap \tilde{T}$ can be joined by a simple arc in $\tilde{M}$ otherwise disjoint from $C$. Let $\gamma$ and $\gamma'$ be the open arcs from $\tilde{T}$ such that $\gamma \cap A = a$ and $\gamma' \cap A = a'$. Let $\beta$ be the simple sub-arc of the graph $\tilde{X} \cap C$ that joins $a$ and $a'$. Thus the retraction of $\gamma$ to $a$ and $\gamma'$ to $a'$ can be extended to a retraction of $F'$ to $\beta$. Thus there is a retraction of $F$ to the tree $\tilde{C} \cap F$. We can then extend this to a retraction $r_{\tilde{X}} : \tilde{X} \to C$, where $r_{\tilde{X}}(\tilde{X}) = \tilde{X} \cap C$, which is the dual graph to $C$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{retraction.png}
\caption{Retraction of $\tilde{X}$ into $C$.}
\end{figure}

As we get $\tilde{M}$ from $\tilde{X}$ by gluing in open 3-cells to $\tilde{X} \cup C$, the retraction $r_{\tilde{X}}$ can be extended to a retraction $r : \tilde{M} \to C$. Thus $C$ and $\tilde{M}$ are homotopy equivalent and $C$ is simply connected.

\section*{Definition 4.2}
For $x \in C$ let the \textbf{star} of $x$, denoted $st(x)$, be the union of all simplices containing $x$. Let the \textbf{open star}, denoted $st^o(x)$, be the union of $x$ and of the interiors of all the simplices adjacent to $x$. Then the \textbf{link} of $x$, denoted $lk(x)$, is the union of all simplices in $st(x)$ that do not contain $x$.

\section*{Definition 4.3}
For simplices $A, B \subset C$, if $A \subset B$ then $B$ is said to be adjacent to $A$.

\section*{Definition 4.4}
If $D$ is a simplicial complex and $e \subset D$ is an edge, then $e$ is said to be a \textbf{free edge} if it is adjacent to exactly one face.

Note that $C$ does not have any free edges, and thus for $x \subset C$ some vertex, $lk(x)$ is the union of all the free edges of $st(x)$. 

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\bibitem{2} R. Dijkgraaf, "The geometry of the quantum world and the

\end{thebibliography}
Lemma 4.5. $C$ is non-positively curved.

As each face is a non-positively curved geodesic triangle, if $C$ is going to have positive curvature then it is going to be at a vertex. From Bridson and Haefliger [2] showing that every non-trivial path in the link of a vertex has length at least $2\pi$ is a sufficient condition for $C$ to be non-positively curved.

Proof. For $v \in C^0$, we can think of $lk(v)$ as a graph in $S^2$, where each edge of $lk(v)$ represents a face in $C$ and a vertex in $lk(v)$ represents an edge in $C$. If $v$ corresponds to a lift of a handlebody that meets the $n$ disk-condition then we can give each edge in $lk(v)$ a length of $\frac{2\pi}{n}$. This means that the length of an edge in $lk(v)$ is the same as the angle between the other two edges in $C$ of the triangle represented by the edge. If $\alpha$ is a loop in $lk(v)$, then let $|\alpha|$ be the length of $\alpha$. Let $\tilde{v}$ be the lift of a handlebody in $\tilde{M}$ corresponding to $v$. Using the embedded view of $C$, a loop $\alpha'$ in $\Gamma = C \cap \partial \tilde{v} \cong lk(v)$, that is essential in $\Gamma$, and thus $\partial H$, will project down to a curve that is the boundary of a singular meridian disk $H$. Thus by the disk condition $\alpha'$ will intersect $C^1$ at least $n$ times. From the obvious retraction of $st(v)$ onto $C \cap \tilde{v}$, for any non-trivial loop $\alpha \subset lk(v)$, $|\alpha| \geq 2\pi$. From Bridson and Haefliger [2] we know this is a sufficient condition for $C$ to be non-positively curved. $\square$

A direct corollary of the previous two lemmas is:

Corollary 4.6. If $C$ is the 2-complex described above, then it is CAT(0).

5. Expanding sequence in $C$

As $C$ is CAT(0) any two points in $C$ can be joined by a unique geodesic. Also as $C$ has no free edges, that is all edges are adjacent to at least two faces, any geodesic arc can be extended. Thus we can define a metric ball in $C$ around some vertex $v \subset C^0$.

Definition 5.1. Let a metric ball of radius $r$ centered at $v$ be $B_{r,v} = \{x \in C : d(x,v) \leq r\}$ and a level sphere be $S_{r,v} = \{x \in C : d(x,v) = r\}$. A level sphere is said to be regular if it is transverse to the simplicial structure of $C$, otherwise it is said to be critical. Also for $S_{r,v}$ critical, let $V_{r,v} = S_{r,v} \cap C^0$.

As $C$ is CAT(0), $B_{r,v}$ is convex and contractible. Also note that if a level sphere is regular then it is disjoint from $C^0$. A level sphere is a connected graph, that is separating in $C$ and whose edges are circular arcs. Therefore as the edges of $C$ are geodesic, for $r > 0$, $B_{r,v}$ is never simplicial. If there was a sequence of simplicial metric balls, then they would correspond to a sequence of expanding submanifolds in $\tilde{M}$ and thus all that would remain to prove theorem 1.1 is that the interior of these manifolds is homeomorphic to $\mathbb{R}^3$. Unfortunately this is not the case and further work is required to generate an expanding sequence of simplicial subsets of $C$.

As any geodesic arc can be extended, we can produce an infinite sequence of expanding metric balls in $C$. We do this by showing that the radii of critical spheres around a vertex are discrete. This means that if we take
the infinite sequence of expanding critical spheres about a vertex, that the union of the sequence is $\mathcal{C}$.

**Lemma 5.2.** Critical level spheres do not accumulate at a finite radius $r$.

**Remark 5.3.** A version of this lemma for CAT(0) polyhedral complexes is proved by Bridson and Haefliger in [2], pg 122.

**Proof.** The idea of this proof is to show that the distances between any two vertices in $\mathcal{C}$ form a discrete set in $\mathbb{R}$. That is any range in $\mathbb{R}$ contains only finitely many possible distances. Let $\triangle$ be the Euclidean geodesic triangle isometric to each face of $\mathcal{C}$ and $\mathbf{T}$ be the tessellation of $\mathbb{E}^2$ by $\triangle$. Therefore any disk of finite radius contains a finite number of vertices. This means that if $D$ is the set of all possible distances between vertices in $\mathcal{T}$, then there are only finitely many elements of $D$ smaller than any finite $r > 0$. In other words $D$ is discrete in $\mathbb{R}$. Also note that as the shortest edge length of $\triangle$ is always 1, then for any $d \in D$ $d \geq 1$.

Let $\gamma$ be a geodesic in $\mathcal{C}$, between two vertices and $l(\gamma)$ be its length. Note that $l(\gamma)$ is finite. If $\gamma$ is disjoint to $\mathcal{C}$ other than at its ends, then it is isometric to a geodesic between vertices of $\mathcal{T}$. This means that the possible lengths of such a geodesic are discrete. If $\gamma$ intersect vertices of $\mathcal{C}$ other than at its ends, it must do so at most $l(\gamma)$ times, as the minimum distance between vertices is 1. Each sub-geodesic of $\gamma$ which is disjoint from $\mathcal{C}$ other than at its ends is isometric to a geodesic between vertices of $\mathcal{T}$. Therefore if $l(\gamma) \leq r$, $l(\gamma) = \sum_{i=1}^{m} d_i$ where $d_i \in D$ and $m \leq r$. However, as $l(\gamma) \leq r$ each $d_i \leq r$. There are only a finite set of such $d_i$’s. Thus, as there is only a finite number of finite sums from a finite set of numbers, the possible lengths of $\gamma$ are finite if its length is at most $r$. This means that the possible lengths of geodesics between vertices in $\mathcal{C}$ are a discrete set. □

**Definition 5.4.** Two vertices $v, v' \in \mathcal{C}$ are said to be neighbours if they are both adjacent to the same edge.

**Lemma 5.5.** For a vertex $v \subset \mathcal{C}$, $st(v)$ is convex.

**Proof.** Clearly $st(v)$ is connected as any two points in it can be joined by a path through $v$. Note that $lk(v)$ is the union of all the free edges of $st(v)$. As the shortest edge length in $\mathcal{C}$ is always 1, $\mathbf{S}_{\frac{1}{2}, v}$ is contained in $st(v)$. Moreover, $lk(v)$ is homeomorphic to $\mathbf{S}_{\frac{1}{2}, v}$. Therefore we know that $lk(v)$ is connected and separating. Let $l$ be the length of the longest edges adjacent to $v$. This means that if $M$ meets the $(6, 6, 6)$ disk-condition, then $l = 1$, if it meets the $(4, 6, 12)$ disk-condition, then $l = \sqrt{3}$ or 2 and if it meets the $(4, 8, 8)$ disk-condition then $l = 1$ or $\sqrt{2}$. Let $A = st(v) \cap \mathbf{S}_{l, v} \subset lk(v)$, that is, all the vertices in $st(v)$ distance $l$ from $v$. If all the edges adjacent to $v$ are the same length then $A$ contains all the vertices of $lk(v)$. If the edges adjacent to $v$ are two different lengths then, as each handlebody in $\mathcal{H}$ is embedded, if a vertex in $lk(v)$ is in $A$ then all its neighbouring vertices in $lk(v)$ are not and vice versa. Thus $A$ is separating in $B_{l, v} - st(v)$. 

If \( st(v) \) is not convex it must contain two points, \( x \) and \( y \), such that \( \alpha \), the image of the geodesic joining them, is not contained entirely in \( st(v) \). As \( lk(v) \) is separating, \( \alpha \) must intersect it at least twice and an even number of times. We will also assume that \( \alpha \) is transverse to \( C \). As \( B_{l,v} \) is convex and \( A \) is separating in \( B_{l,v} - st(v) \) there is \( \alpha' \), a connected sub-arc of \( \alpha \), such that \( \alpha' \cap st(v) = \{ x', y' \} \subset lk(v) \) where \( x' \) and \( y' \) are points contained in a single component of \( lk(v) - A \). This means they are either contained in the same edge of \( lk(v) \) or in two adjacent edges of \( lk(v) \).

Both \( x' \) and \( y' \) cannot be contained in the same edge, as each edge is geodesic this would contradict the property of CAT(0) spaces that any two points are connected by a unique geodesic. Thus \( x' \) and \( y' \) must be contained in adjacent edges of \( lk(v) \), \( E_{x'} \) and \( E_{y'} \) respectively. Therefore \( z = E_{x'} \cap E_{y'} \subset lk(v) \) is a vertex. Let \( E_z \) be the edge between \( z \) and \( v \). As both \( E_{x'} \) and \( E_{y'} \) are free edges of \( st(v) \) they are contained in the boundary of a single face of \( st(v) \), \( F_{x'} \) and \( F_{y'} \) respectively, where \( F_{x'} \cap F_{y'} = E_z \). Thus \( F_{x'} \cup F_{y'} \) is convex and contains a geodesic between \( x' \) and \( y' \). This once again contradicts the CAT(0) structure of \( C \). Thus \( st(v) \) is convex. \( \square \)

**Definition 5.6.** For any set \( C \subset C \) and a point \( v \notin C \), let \( d_C(v) = \inf \{ d(y,v) : y \in C \} \).

As \( C \) is a simplicial 2-complex and its faces are Euclidean triangles there is a simplified definition for the angles between geodesics. Let \( x, y, z \subset C \) be points in \( C \) and \([x,y]\) be the geodesic arc joining the points \( x \) and \( y \). Let the **angle** between \([x,y]\) and \([x,z]\) be denoted by \( \angle_x(y,z) \). If \( x \) is not a vertex of \( C \) the \( \angle_x(y,z) \) is simply the Euclidean angle between \([x,y]\) and \([x,z]\). If \( x \) is a vertex of \( C \) then \( \angle_x(y,z) \) is the length of the shortest path in \( lk(x) \) between \([x,y]\cap lk(x)\) and \([x,z]\cap lk(x)\), as defined in the proof of lemma 4.5.

From Bridson and Haefliger [2] we get the following lemma,

**Lemma 5.7.** (Bridson and Haefliger) Let \( C \subset C \) be convex complete and \( v \notin C \) any point, then:

1. \( \{ y \in C : d(y,v) = d_C(v) \} \) is a unique point \( x \).
2. If \( y \in C \) and if \( y \neq x \) then \( \angle_x(y,v) \geq \pi/2 \), where \( x \) is as in 1.

As faces of \( C \) are convex, this lemma tells us two things. First that the subset of a face of \( C \) closest to any vertex is a single point in its boundary. Secondly that if the closest point to a vertex is contained in the interior of an edge, then the geodesic between the point and the vertex intersect is normal to the edge. This may seem rather obvious, however it can help us classify the types of intersections between a metric ball and faces.

**Lemma 5.8.** Let \( F \subset C \) be a face and \( v \) a vertex not in \( F \). Let \( E_i \), for \( i = 1, 2, 3 \), be the three edges of \( F \) and \( y_i = \{ z \in E_i : d(z,v) = d_{E_i}(v) \} \). Then at least one \( y_i \) is a vertex of \( F \).

**Proof.** The idea to this proof is to show that by assuming the lemma is false, we can find geodesics that intersect twice, giving us a contradiction.
Assume that the lemma is not true and that $y_i \in \text{int}(E_i)$ for $i = 1, 2, 3$. Note by lemma 5.7 we know that each $y_i$ is a single point and by assumption they are all different. Let $y$ be the point in $F$ closest to $v$. As $v \not\subset \text{int}(F)$, then $y$ is one of the $y_i$’s. Let it be $y_1$, thus $d(y_1, v) < d(y_i, v)$ for $i = 2$ or 3. Let $\gamma_i$ be the geodesic between $y_i$ and $v$. By lemma 5.7 we know that $\gamma_i$ is perpendicular to $E_i$. Let $F'$ be the other face adjacent to $E_1$ such that $\gamma_1 \cap F' \neq \emptyset$. Let $E_1'$ be the edge of $F'$ isometric to $E_1$, thus $E_1 = E_1'$. Two vertices of $F'$ are $x_1$ and $x_2$, let the third be $x_3'$. As $y_i \subset \text{int}(E_i)$ and $y_1$ is the point of $F$ closest to $v$, then $B_{d(y_1, v), v} \cap F = y_1$. Assume that $\gamma_i \cap E_1 = \emptyset$, for $i = 2$ or 3. Then, as any two points in $\mathcal{C}$ are connected by a unique geodesic arc, $B_{d(y_1, v), v} \cap F$ must have two components, one of which is $y_i$ and the other has $y_1$ in its interior. This contradicts convexity, thus both $\gamma_2$ and $\gamma_3$ intersect $\text{int}(E_1)$ and $\text{int}(F')$.

![Figure 5. Intersecting geodesics.](image-url)

Extend $\gamma_1$ through $F$ until it intersects at least one of $E_2$ or $E_3$. Assume that it intersects $E_2$. As $\gamma_1$ is perpendicular to $E_1$ it also intersects $E_2'$. However if $\angle_{x_2'(x_1, x_3)} \geq \pi/4$ then $\angle_{x_2'(x_3', x_3)} \geq \pi/2$, meaning that $\gamma_2$ intersects $E_3'$. Thus $\gamma_1 \cap \gamma_2 \neq \emptyset$, which contradicts the unique geodesic property of CAT(0) spaces. Therefore $\angle_{x_2'(x_1, x_3)} = \pi/6$. This means that the length of $E_1$ is either 2 or $\sqrt{3}$. If $E_1$ has length $\sqrt{3}$, then $\angle_{x_2(x_2, x_3)} = \pi/2$ and $\gamma_3 \cap F$ is parallel to $E_1$ and thus would intersect both $\gamma_1$ and $\gamma_2$. Therefore the length of $E_1$ must be 2 and $\angle_{x_2(x_2, x_3)} = \pi/3$. Thus $\angle_{x_2'(x_3', x_3)} = 2\pi/3$. This means that $\gamma_3 \cap E_3' = \emptyset$ and thus $\gamma_3 \cap E_2' \neq \emptyset$. This means that each $\gamma_i$ intersects $\text{int}(E_i')$. Let $F''$ be the other face adjacent to $E_2'$ such that $\gamma_1 \cap \text{int}(F'') \neq \emptyset$. Thus $\gamma_i \cap \text{int}(F'') \neq \emptyset$ for each $\gamma_i$. Once again let $E_1''$ be the edge of $F''$ isometric to $E_1$ and let the vertex $E_1'' \cap E_1'' = x_1''$. 

Note that $E''_3 \cap E'_3$ is a geodesic and thus $\angle_{x_1}(x''_1,x_3)=2\pi/3$. Therefore $E''_1 \cap \gamma_3 \neq \emptyset$. Also $\angle_{x_2}(x''_1,x_3)=\pi/2$, thus $(E'_3 \cup E''_3) \cap \gamma_2 \neq \emptyset$. Thus $\gamma_2$ and $\gamma_3$ intersect in int($F \cup F' \cup F''$). This takes care of all the cases, thus proving the lemma.

A direct corollary of lemma 5.8 is:

**Corollary 5.9.** If $F$ is a face of $\mathcal{C}$ and $v$ is a vertex, then for any $r>0$, $S_{r,v} \cap F$ has at most two components.

**Proof.** Suppose the corollary is not true and that for some face $F \subset \mathcal{C}$, $F \cap S_{r,v}$ has three components. By convexity of $F$ and $B_{r,v}$, $F \cap B_{r,v}$ has to be connected and convex, thus by convexity we know that $F \cap B_{r,v}$ looks like figure 6 with $F \cap S_{r,v}$ consisting of three properly embedded arcs, one running between each pair of edges of $F$. This contradicts lemma 5.8, for it implies that the closest point to $v$ for each edge is in their interior.

![Figure 6](image.png)

**Figure 6.** $F \cap B_{r,v}$ if $F \cap S_{r,v}$ has three components.

**Lemma 5.10.** If $F \cap S_{r,v}$ has two components then they cannot connect the same pair of edges.

**Proof.** Once again let $x_i$, for $i=1,2,3$, be the vertices of $F$ and $E_i$ be the edge between $x_i$ and $x_{i+1}$, where $E_3$ is the edge between $x_3$ and $x_1$. Also let $\{y_i\} = \{z \in E_i : d(z,v) = d_{E_i}(v)\}$. Let $\gamma_i$ be the geodesic that passes through $y_i$ and $v$. Note that the geodesic $\gamma_i$ is perpendicular to $E_i$. If $F \cap S_{r,v}$ connects the same pair of edges then at least two of the $y_i$’s are in $\text{int}(E_i)$. Let $y_1 \subset \text{int}(E_1)$ and $y_2 \subset \text{int}(E_2)$. Using the same argument, as was used in the proof of lemma 5.8, $\gamma_2 \cap E_1 \neq \emptyset$. Also $d(y_1,v) < d(y_2,v) \leq d(y_3,v)$. This means that $\alpha = S_{d(y_2,v),v} \cap F'$ is a circular arc with both ends in $E_1$ and $\alpha \cap E_2 = y_2$.

By simple geometry if $M$ meets either the $(6,6,6)$ or $(4,8,8)$ disk-conditions, then the radius of $\alpha$ has to be less than 1. This would imply that $\gamma_2$ intersects a vertex $x \subset B_{d(y_2,v),v}$ such that $d(y_2,x) < 1$. The only such vertex is $x_1$, however that would contradict $d(y_2,v) \leq d(y_3,v)$.

If $M$ meets the $(4,6,12)$ disk-condition and either $E_1$ or $E_2$ have length 1, then the radius of $\alpha$ has to be less than 1. This would once again mean that $x$ is $x_1$. Let $F'$ be the other face adjacent to $E_1$ such that $\gamma_1 \cap \text{int}(F') \neq \emptyset$ and let $x'$ be the vertex of $F'$ disjoint to $F$. Then $\gamma_2 \cap F' \neq \emptyset$. If $E_1$ has
length $\sqrt{3}$, then $E_2$ must have length 2. In this case then the radius of $\alpha$ must be less than $2/\sqrt{3}$, once again this would mean that $x$ is $x_1$. Therefore $E_1$ must have length 2 and $E_2$ has length $\sqrt{3}$. In this case $\alpha$ must have radius less than 2. Therefore $x$ is either $x_1$ or $x'$. Once again it cannot be $x_1$. If $x = x'$ then $d(x, x_1) = 1$, once again contradicting that $d(y_2, v) \leq d(y_3, v)$.

Thus having looked at all the cases the lemma has been proved. □

Lemma 5.11. Let $r > 0$ where $S_{r,v}$ is critical, then for $0 < \epsilon < r - \sqrt{r^2 - \frac{1}{4}}$ the level sphere $S_{r-t,v}$, where $t \in (0, \epsilon]$ transversely intersects each edge between any vertex in $V_{r,v}$ (see definition 5.7) and any of its neighbouring vertices contained in $B_{r,v}$.

Proof. The idea of this proof is to show that for any vertex $x \in V_{r,v}$ and an edge $E \subset B_{r,v}$ adjacent to $x$, that there is at least one point $z \in E$ such that $d(z, v) \leq \sqrt{r^2 - \frac{1}{4}}$. Let $y \in B_r$ be the vertex at the other end of $E$. Let $\gamma_x$ and $\gamma_y$ be the geodesic between $v$ and $x$ or $y$ respectively. Then as $E$ is a geodesic, $\Delta = E \cup \gamma_x \cup \gamma_y$ is a geodesic triangle.

Let $x'$, $y'$ and $v'$ be points in $E^2$ such that $d(x', v') = d(x, v) = r$, $d(y', v') = d(y, v) \leq r$ and $d(x', y') = d(x, y)$. For points $a$ and $b$ in $E^2$, let $ab$ be the image of the geodesic arc between $a$ and $b$. Therefor $\Delta' = x'v'\cup y'v'\cup x'y'$ is a geodesic triangle in $E^2$. We refer to $\Delta'$ as the comparison triangle to $\Delta$, see figure 7. As $C$ is CAT(0) we know that for any point $z \subset E$ and $z' \subset x'y'$, such that $d(z, x) = d(z', x')$, then $d(z, v) \leq d(z', v')$. Let $z$ be the point half way along $E$, that is $d(x, z) = \frac{1}{2}d(x, y) \geq \frac{1}{2}$, as the shortest edge length in $C$ is always 1 . Let $z' \subset xy$ be the corresponding point in $\Delta'$, that is $d(z', x') = d(z, x)$. $d(z', v')$ is maximal when $d(y', v') = r$ and $d(x', y') = 1$, therefore $d(z, v) \leq d(z', v') \leq \sqrt{r^2 - \frac{1}{4}}$. □

![Figure 7. Comparison triangle.](image-url)
So far we can produce a sequence of expanding balls in $C$. However for $r > 0$ these balls are not simplicial, thus there is no corresponding manifold in $\tilde{M}$, for there is no obvious way to have part of a gluing along a face or around a triple curve. Thus we need to produce a sequence of expanding simplicial sets.

**Definition 5.12.** Let the simplicial ball of radius $r$ centered at $v$, denoted $B_{r,v}^s$ be the maximal simplicial subset of $B_{r,v}$, the metric ball of radius $r$.

If $M$ meets the $(4, 8, 8)$ disk-condition and $v \subset C_0$ is a vertex such that it has adjacent edges of length 1 and $\sqrt{2}$. Then, for $1 \leq r < \sqrt{2}$, $B_{r,v}^s$ is the set of all edges adjacent to $v$ with length 1. This is not a convex set in $C$. Therefore the name simplicial ball is a little misleading, as these objects are not necessarily convex.

**Lemma 5.13.** There is a retraction of $B_{r,v}$ onto $B_{r,v}^s$.

*Proof.* The idea of this proof is to show that any simplex not entirely contained in $B_{r,v}$ can be retracted into its boundary. Let $A \subset C$ be a simplex such that $A \cap B_{r,v} \neq \emptyset$ and $A \cap B_{r,v} \neq A$. Then as both $A$ and $B_{r,v}$ are convex and $A$ is compact, then $A \cap B_{r,v}$ must be convex, compact and thus connected. Therefore if $A$ is an edge there are two possible cases. The first is that $A \cap B_{r,v}$ is a compact connected set in int($A$), the second is that a vertex of $A$ is contained in $A \cap B_{r,v}$.

If $A$ is a face then the arcs $A \cap S_{r,v}$ are circular arcs. By lemma 5.9 we know that $A \cap S_{r,v}$ has at most two components. If $A \cap S_{r,v}$ is a single arc running between different edges of $A$, then type 1 and type 2 in figure 8 are the two possibilities for $A \cap B_{r,v}$. If $A \cap S_{r,v}$ is a single arc with both ends in the one edge $E$, then by convexity we know that $A \cap B_{r,v}$ must be the disk between the arc $A \cap S_{r,v}$ and $E$, as shown in type 3 of figure 8. If $A \cap S_{r,v}$ contains two arcs, by convexity neither can be homotopic to an edge of $A$ and by lemma 5.10 they cannot connect the same edges. Therefore $A \cap S_{r,v}$ must look like type 4 of figure 8.

Let $E \subset C$ be an edge such that $E \cap B_{r,v} \subset \text{int}(E)$. Then if $F \subset C$ is a face adjacent to $E$, $F \cap S_{r,v}$ either has one arc parallel to $E$ or has two arcs. Therefore $F \cap B_{r,v}$ is either of type 3 or type 4. However $E$ cannot have more than one adjacent face whose intersection with $B_{r,v}$ is of type 4, for if it did then any point in $E \cap B_{r,v}$ would have more then one geodesic path to $v$. However all the adjacent faces of $E$ cannot have intersections of type 3 with $B_{r,v}$ for the circular arcs of $S_{r,v}$ are centered at a vertex in $C$. This means that $E$ must have one adjacent face that has an intersection of type 4 with $B_{r,v}$ and all its other adjacent faces must have intersection of type 3 with $B_{r,v}$.

Therefore for any face $F$ adjacent to $E$ that has an intersection of type 3 with $B_{r,v}$, there is a retraction of $B_{r,v} \cap F$ into $E$. We then do this for all such faces adjacent to $E$. Now we can retract the resulting set, $B$, to remove any intersection with $E$. This means that the one face adjacent to...
E that has an intersection of type 4 with $B_{r,v}$ now has an intersection of type 2 with $B$. We can then repeat this process to remove all intersections of type 3 and type 4 from $B$.

This means that for any face $F \subset C$ not entirely contained in $B$, that the intersection between them is either of type 1 or type 2. Clearly in either case $B$ can be retracted so that it only intersects $F$ in its boundary. Now the only simplices that are not entirely in $B$ are edges with one vertex contained in $B$. Therefore we can retract $B$ so that its intersection with such edges is just the vertex. Therefore $B = B_{r,v}^s$ is now simplicial.

Note that if a vertex is contained in $B_{r,v}$ then it is in $B_{r,v}^s$ and if two neighbouring vertices are contained in $B_{r,v}$, then the edge joining them is contained in $B_{r,v}^s$. The next corollary is a direct result of lemmas 5.2, 5.11 and 5.13.

**Corollary 5.14.** For any $r > 0$ such that $S_{r,v}$ is a critical level sphere, there is an $0 < \epsilon < r - \sqrt{r^2 - \frac{1}{4}}$, such that every vertex $x \subset C$, where $d(x,v) < r$, is contained in $B_{r-\epsilon,v}^s$.

Along with the geodesic extension property, this corollary tells us that there is an infinite sequence of engulfing simplicial balls in $C$.

6. **Expanding sequence in $\tilde{M}$**

All that remains to prove theorem 1.1 is to show that the interior of the sub-manifold in $\tilde{M}$ that corresponds to a simplicial ball in $C$ is homeomorphic.
to \(\mathbb{R}^3\). We do this by defining a process to get from \(B^s_{r,-\epsilon,v}\) to \(B_{r,v}\), such that at each step the interior of the corresponding sub-manifold in \(\tilde{M}\) is an open ball.

**Definition 6.1.** For a simplicial set \(C\) and a vertex \(x \subset C\), let \(\text{adj}_C(x) = \text{st}(x) \cap \text{lk}(x)\), that is the set of simplices in \(C\) adjacent to \(x\), and \(\text{adj}_C^0(x) = \text{st}^0(x) \cap \text{lk}(x)\).

Note we can get \(\text{adj}_C^0(x)\) from \(\text{adj}_C(x)\) by removing its intersection with \(\text{lk}(x)\). Also by lemma 5.5 we know that for some vertex \(x \in V_{r,v}\) and \(\epsilon < r - \sqrt{r^2 - \frac{1}{4}}\), the level sphere \(S_{r,-\epsilon,v}\) intersects every edge of \(\text{adj}_C^0(x)\) transversely.

**Lemma 6.2.** For \(x \in V_{r,v}\), \(\text{adj}_{B_{r,v}}^0(x)\) is convex.

**Remark 6.3.** The idea behind this proof is the same as that used for the proof of lemma 5.5.

**Proof.** Let \(\Gamma = \text{lk}(x) \cap \text{adj}_{B_{r,v}}(x)\) and \(\Pi\) be the set of free edges of \(\text{adj}_{B_{r,v}}(x)\) not in \(\Gamma\), that is the free edges adjacent to \(x\). Let \(D = B_{r,v} \cap \text{st}(x)\). By convexity of \(B_{r,v}\) and \(\text{st}(x)\), \(D\) is convex. Also note that \(\text{adj}_{B_{r,v}}(x) \subset D\). Let \(C = D - \text{adj}_{B_{r,v}}(x)\) and \(C_i\)'s be the components of \(C\). As all the vertices in \(D\) are in \(\text{adj}_{B_{r,v}}(x)\), then \(\text{adj}_{B_{r,v}}(x) \cap \mathbb{C}^0 = \emptyset\). Also \(C \cap \mathbb{C}^1 \subset \text{lk}(x)\) and each edge in the \(\text{lk}(x)\) is free in \(\text{st}(x)\). Therefore each \(C_i\) is contained in a single face of \(\text{st}(x)\). For any \(C_i\), \(\overline{C_i} \cap \text{adj}_{B_{r,v}}(x)\) is some edge \(E_i \subset \Pi\). Therefore if \(\text{adj}_{B_{r,v}}(x)\) is not convex then there must be two points \(\{x,y\} \subset \text{adj}_{B_{r,v}}(x)\) such that the image, \(\alpha\), of the geodesic joining them is not entirely contained in \(\text{adj}_{B_{r,v}}(x)\). However \(D\) is convex so \(\alpha \subset D\). This means that for at least one \(C_i\), that \(C_i \cap \alpha \neq \emptyset\). Moreover, this means that \(\alpha\) must intersect \(E_i\) at least twice, which contradicts the unique geodesic property of CAT(0) spaces. Thus \(\text{adj}_{B_{r,v}}(x)\) is convex. \(\square\)

**Lemma 6.4.** Let \(F\) be a face of \(C\) then all three vertices of \(F\) cannot be in \(V_{r,v}\).

**Proof.** By lemma 5.11 we know that if \(F\)’s three vertices are in \(V_{r,v}\) then there is an \(\epsilon > 0\) such that \(F \cap S_{r,-\epsilon,v}\) has three components. However by lemma 5.9 we know this is not possible. \(\square\)

**Lemma 6.5.** If \(x\) and \(x'\) are in \(V_{r,v}\) and neighbouring vertices, then the edge between them is free in \(\text{adj}_{B_{r,v}}(x)\).

**Proof.** Let \(E\) be the edge between \(x\) and \(x'\). Assume that the lemma is not true and that \(E\) is adjacent to more than one face in \(\text{adj}_{B_{r,v}}(x)\). By lemma 6.4 we know that the third vertex in any such face is contained in \(B_{r,-\epsilon,v}\). However as the metric ball is retractible in \(C\) we know that for all but one of these faces in \(\text{adj}_{B_{r,v}}(x)\) adjacent to \(E\) that their closest point to \(X\) is in \(\text{int}(E)\). Let \(F\) be such a face, that is \(y = \{z \in F : d_F(v) = d(z,v)\} \subset \text{int}(E)\).
By lemma 5.7 we know that \( y \) is a single point. Let \( x'' \) be the third vertex of \( F \). Then \( d(x'', v) < r \). Let \( E_1 \) be the edge between \( x \) and \( x'' \) and let \( E_2 \) be the edge between \( x' \) and \( x'' \). Also let \( y_i = \{ z \in E_i : d_{E_i}(v) = d(z, v) \} \). As \( F \subset B_{r, v} \) and \( d(x, v) = d(x', v) = r \) we know that \( y_1 \neq x \) and \( y_2 \neq x' \). The closest point of \( F \) to \( v \) is \( \text{int}(E) \) and for \( d_F(v) < t < r \) \( S_{t,v} \cap F \) is a set of circular arcs. Then it is not possible for \( y_1 = y_2 = x'' \) and at least one \( y_i \) must be in \( \text{int}(E) \). Let it be \( E_1 \). If \( y_2 = x'' \) then \( S_{t,v} \cap F \), for \( d_{E_1}(v) < t < d(x'', v) \), has two components connecting the same pair of edges. However by lemma 5.10 we know that this is not possible. Therefore \( y_2 \subset \text{int}(E_2) \).

Let \( d_{E_1}(v) \geq d_{E_2}(v) \), then \( S_{t,v} \cap F \), for \( d_{E_1}(v) < t < d(x'', v) \) has three components. However by lemma 5.9 we know this is not possible. \( \square \)

To produce the process for expanding from one simplicial ball to the next we need to define a simplicial join.

**Definition 6.6.** Let \( v \) be a vertex of \( C \) and \( \Gamma \) be a graph in \( \text{lk}(v) \), then their simplicial cone, \( a \ast \Gamma \), is the simplicial subset of \( st(v) \) produced by coning \( \Gamma \) to \( v \), as shown in figure 9.

Note that in combinatorial topology this process is more generally known as a simplicial join. If \( \Gamma = \text{lk}(x) \), then \( st(x) = v \ast \Gamma \).

![Figure 9. Simplicial cone.](image)

**Definition 6.7.** Let \( x \in C^0 \) and \( T \subset \text{lk}(x) \) is a connected tree, then using the metric on \( \text{lk}(x) \), as defined in the proof of lemma 4.5 the diameter of \( T \) is \( \text{diam}(T) = \max \{ d(a, b) : a, b \in T \} \).

For some simplicial set \( C \subseteq C \) let \( \tilde{C} \) be the corresponding sub-manifold of \( \tilde{M} \). The following lemma gives the ‘move’ used to expand a simplicial set in \( C \) so that the interior of the corresponding set in \( \tilde{M} \) remains homeomorphic to \( \mathbb{R}^3 \).

**Lemma 6.8.** Let \( x \in C^0 \) and \( T \subset \text{lk}(x) \) be a connected tree such that \( \text{diam}(T) \leq \pi \), then the interior of the manifold in \( \tilde{M} \) corresponding to \( x \ast T \) is homeomorphic to \( \mathbb{R}^3 \).

**Proof.** Let \( F = \text{int}(\tilde{T} \cap \tilde{x}) \subset \tilde{M} \). By lemmas 3.1 and 3.2 it is sufficient to show that \( \text{int}(\tilde{T}) \cong \mathbb{R}^3 \) and \( F \cong \mathbb{R}^2 \). Let \( E \) be an edge of \( T \) and \( a, b \subset E \) the two vertices of \( E \). Then \( \tilde{E} \) is two lifts of handlebodies glued along a face \( G = \tilde{a} \cap \tilde{b} \subset \tilde{X} \). As \( a \) and \( b \) are joined by single edge in \( E \), \( \text{int}(G) \cong \mathbb{R}^2 \). By
lemmas \( \text{3.1} \) and \( \text{3.2} \). Let \( T', T'' \subseteq T \) be connected trees such that \( T'' \) is produced from \( T' \) by adding a single edge. Let \( a \) be the vertex contained in \( T'' \) but not in \( T' \). As \( a \) is joined to \( T' \) by a single edge, \( \text{int}(\tilde{a} \cap \tilde{T}') \cong \mathbb{R}^2 \). Therefore if \( \tilde{T}' \cong \mathbb{R}^3 \), then \( \tilde{T}'' \cong \mathbb{R}^3 \). Thus by using induction \( \tilde{T} \cong \mathbb{R}^3 \).

Taking the embedded view of \( \mathcal{C} \), \( \partial \tilde{x} \cap (x \ast T) \) is a graph dual to \( F \) and homeomorphic to \( T \). As \( \text{diam}(T) \leq \pi \) by the disk-condition \( F \) is a missing boundary disk in \( \partial \tilde{x} \) and thus \( \text{int}(F) \cong \mathbb{R}^2 \). \( \square \)

The following corollary is a generalisation of the previous lemma.

**Corollary 6.9.** Let \( C \subset \mathcal{C} \) be a connected simplicial set such that \( \text{int}(\tilde{C}) \cong \mathbb{R}^3 \) and \( x \in C^0 \) such that \( T = \text{lk}(x) \cap C \) is a connected tree and \( \text{diam}(T) < \pi \), then the interior of the manifold corresponding to \( \mathcal{C} \cap (x \ast T) \) is homeomorphic to \( \mathbb{R}^3 \).

This comes from the observation that \( \tilde{x} \cap \tilde{C} = \tilde{x} \cap \tilde{T} \), where \( \tilde{x} \), \( \tilde{C} \) and \( \tilde{T} \) are the manifolds in \( \tilde{M} \) corresponding to \( x \), \( \mathcal{C} \) and \( T \) respectively.

Now the next step is to show that the intersection between the link of a vertex in \( V_{r,v} \) and the simplicial ball \( B_{r,v}^s \) is in fact a tree. Note that \( \text{adj}_{B_{r,v}^s}(x) = x \ast T \).

**Lemma 6.10.** If \( x \in V_{r,v} \), then \( \text{adj}_{B_{r,v}^s}(x) \cap \text{lk}(x) = T \) is a connected tree such that \( \text{diam}(T) \leq \pi \).

**Proof.** By lemma \( \text{5.2} \) we can find an \( 0 < \epsilon < r - \sqrt{r^2 - \frac{1}{4}} \) such that the level sphere \( S_{r-\epsilon,v} \) intersects every edge in \( \text{adj}_{B_{r,v}^s}(x) \) adjacent to \( x \), at least once transversely and \( B_{r-\epsilon,v}^s \) contains all vertices that are distance less than \( r \) from \( v \). Let \( \Gamma = S_{r-\epsilon} \cap \text{adj}_{B_{r,v}^s}(x) \). Then \( \Gamma = \{ \Gamma_i \} \) is a set of graphs in \( \text{adj}_{B_{r,v}^s}(x) \).

First we want to show \( \Gamma \) has a single component that separates \( x \) from all the other vertices of \( \text{adj}_{B_{r,v}^s}(x) \). As \( S_{r-\epsilon,v} \) is separating, \( \Gamma \) is separating in \( \text{adj}_{B_{r,v}^s}(x) \). However, as \( \text{adj}_{B_{r,v}^s}(x) \) is convex and thus simply connected, each component of \( \Gamma \) must be separating. Therefore as \( S_{r-\epsilon,v} \) intersects every edge of \( \text{adj}_{B_{r,v}^s}(x) \) adjacent to \( x \), that \( \Gamma \) separates \( x \) from all other vertices in \( \text{adj}_{B_{r,v}^s}(x) \).

Let \( \Gamma' \) be a subset of components of \( \Gamma \) with a minimal number of components that separate \( x \) from all other vertices in \( \text{adj}_{B_{r,v}^s}(x) \). We want to show that \( \Gamma' \) is connected, that is contains only one component. If \( \Gamma' \) is not connected then, as \( x \) is adjacent to every face in \( \text{adj}_{B_{r,v}^s}(x) \), there must be a face \( F \) of \( \text{adj}_{B_{r,v}^s}(x) \), such that \( \Gamma' \cap F \) is two embedded arcs. By lemma \( \text{5.9} \) \( \Gamma' \cap F \) cannot contain more than two. For two neighbouring vertices, \( a \) and \( b \), let \( E_{ab} \) be the edge joining them. By Lemma \( \text{6.4} \) we know that for any face of \( \text{adj}_{B_{r,v}^s}(x) \) that at most two of its vertices are in \( V_{r,v} \). First let \( F \) be a face such that two of its vertices, \( a_1 \) and \( a_2 \), are in \( B_{r-\epsilon,v} \). Then the edge \( E_{a_1a_2} \) must be contained in \( B_{r-\epsilon,v} \) and thus \( \Gamma \) does not intersect it.
As $a_i$'s are contained in $B_{r-\epsilon,v}$ and $x$ is not, $\Gamma$ intersects $E_{xa_i}$ exactly once. Therefore $F \cap \Gamma$ contains only one arc.

The second case is when only one vertex, $a$, of $F$ is contained in $B_{r-\epsilon,v}$. Therefore the third vertex $x'$ must be in $V_{r,v}$. Therefore the edges $E_{xa}$ and $E_{x'a}$ both intersect $\Gamma$ once. As both $x$ and $x'$ are not contained in $B_{r-\epsilon,v}$, by definition of $\epsilon$ we know that $E_{xx'}$ intersects $B_{r-\epsilon,v}$ in its interior. Thus $E_{xx'}$ intersects $\Gamma$ twice. This means that $\Gamma \cap F$ is two properly embedded arcs in $F$, one between $E_{xx'}$ and $E_{xa}$ and the other between $E_{xx'}$ and $E_{x'a}$.

By lemma 6.10, each component of $T$ is a tree. As both $adj_B(x)$ and $B_{r-\epsilon,v}$ are convex, $adj_B(x) \cap B_{r-\epsilon,v}$ is convex and does not contain $x$. Therefore if $\Gamma'$ contained a loop then $adj_B(x) \cap B_{r-\epsilon,v}$ must contain the disk bounded by that loop. However by the cone structure of $st(x)$ such a disk must contain $x$. Thus $\Gamma'$ is a tree.

When we cut $adj_B(x)$ along $\Gamma'$, because of the cone structure of $st(x)$, the component that doesn’t contain $x$ has the product structure $\Gamma' \times I$. Therefore $T \cong \Gamma'$. As $T \subset B_{r,v} \cap lk(x)$ and by convexity of $B_{r,v}$, $diam(T) \leq \pi$.

**Lemma 6.11.** If $x \in V_{r,v}$, then $T' = adj_B(x) \cap B_{r-\epsilon,v} \subset lk(x)$ is a connected tree and $diam(T') \leq \pi$.

**Proof.** Let $T = adj_B(x) \cap lk(x)$. Clearly $T' \subset T$, therefore $diam(T') \leq \pi$ and by lemma 6.10 each component of $T'$ is a tree, so all we need to prove is that $T'$ is connected. We can get $T'$ from $T$ by removing all the vertices $V_{r,v} \cap T$ and their adjacent edges. However we know from lemma 6.5 that the edge between any vertex $V_{r,v} \cap T$ and $x$ is free in $adj_B(x)$, therefore any vertex in $V_{r,v} \cap T$ has degree one in $T$. This means that all the vertices in $V_{r,v} \cap T$ are not separating in $T$ and thus $T'$ is connected.

**Lemma 6.12.** If the interior of the manifold in $\tilde{M}$ corresponding to $B^x_{r-\epsilon,v}$ is an open ball, then the interior of the manifold corresponding to $B^v_{r,v}$ is an open ball.

**Proof.** To prove the above lemma we will define a process for getting from $B^x_{r-\epsilon,v}$ to $B^v_{r,v}$, then show that at each step the simplicial set corresponds to a missing boundary ball in $\tilde{M}$ and then that we do indeed get $B^v_{r,v}$.

Let $V_{r,v} = \{x_1, x_2, \ldots \}$ and $C_0 = B_{r-\epsilon,v}$. Then let $\Gamma_i = lk(x_i) \cap C_{i-1}$ and $C_i = C_{i-1} \cup (\Gamma_i \ast x_i)$. To show that each $C_i$ corresponds to a missing boundary ball, by lemma 6.9 we just need to show that each $\Gamma_i$ is a connected tree.
We know that \((B_{r, v} - \epsilon, v \cap lk(x_i)) \subseteq \Gamma_i \subseteq (B_{r,v}^* \cap lk(x_i))\) and by lemma [6,10] and lemma [6,11] both \(B_{r, v} - \epsilon, v \cap lk(x_i)\) and \(B_{r,v}^* \cap lk(x_i)\) are connected trees. Also by lemma [6,5] any vertices of \(V_{r,v}\) in \(B_{r,v}^* \cap lk(x_i)\) are not separating. Therefore each \(\Gamma_i\) is a connected tree.

As \(\Gamma_i \subseteq (B_{r,v}^* \cap lk(x_i))\), then \(x_i \ast \Gamma_i \subseteq adj_{B_{r,v}^*}(x)\). Therefore we know that \(C_i \subseteq B_{r,v}^*\). If \(C_i \subset B_{r,v}^*\) then \(B_{r,v} - C_i\) must contain at least one vertex \(x_i\) and \(x\) must be in \(V_{r,v}\). Therefore once all the vertices of \(V_{r,v}\) have been added to \(B_{r, -\epsilon, v}\) by this process, the resulting simplicial set is \(B_{r,v}^*\).

Proof. (of theorem 1.1) To prove lemma 1.1 we use the previous lemma to produce an infinite sequence of expanding open balls and by Brown [3], this implies that \(\tilde{M}\) is homeomorphic to \(\mathbb{R}^3\). By lemma 5.14 for any vertex \(v \in C\) there is a sequence \(R = \{r_i\}\) such that \(B_{r_i,v} \subset B_{r_{i+1},v}^*\) and by the previous lemma we know that the interior of the manifolds corresponding to the sequence \(\{B_{r_i,v}^*\}\) of simplicial balls are open balls in \(M\). Thus we have an infinite sequence of engulfing open balls and the universal cover is homeomorphic to \(\mathbb{R}^3\). □

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