BISEP ARABLE EXTENSIONS ARE NOT NECESSARILY FROBENIUS

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Abstract. We give necessary and sufficient conditions on an Ore extension $A[x;\sigma,\delta]$, where $A$ is a finite dimensional algebra over a field $F$, for being a Frobenius extension over the ring of commutative polynomials $F[x]$. As a consequence, as the title of this paper highlights, we provide a negative answer to a problem stated by Caenepeel and Kadison.

1. Introduction

Frobenius extensions where introduced by Kasch [9, 10], and by Nakayama and Tsuchuku [13, 14] as a generalization of the well known notion of Frobenius algebra. Of course the underlying idea was to recover the duality theory of Frobenius algebras in a more general setting. The notion of separable extension comes from the generalization of the well known notion of separable field extension. The classical definition of separable ring extension is due to Hirata and Sugano in [7]. Both notions, Frobenius and separable, have been extended to more general framework in category theory.

As it is explained in the Introduction of [1], deep connections between separable and Frobenius extensions were found from the very beginning. For instance, Eilenberg and Nakayama show in [2] that finite dimensional semisimple algebras over a field are symmetric, hence Frobenius. The extension of this result to algebras over commutative rings has as key result in a paper of Endo and Watanabe, concretely they show that separable, finitely generated, faithful and projective algebras over a commutative ring are symmetric [3, Theorem 4.2]. Their ideas where connected to separable extensions, as defined in [7], by Sugano, who shows that separable and centrally projective extensions are Frobenius, see [17, Theorem 2]. However, as Caenepeel and Kadison say “it is implicit in the literature that there are several cautionary examples showing separable extensions are not always Frobenius extensions in the ordinary untwisted sense”. They provide one of these examples in [1, §4] under the stronger hypothesis that the extension is split, but the Frobenius property is lost because the provided extension is not finitely generated. Split extensions are naturally considered since separability and splitting can be viewed as particular cases of the notion of separable module introduced in [18], see also [8]. Biseparable extensions are therefore considered because they contains both notions of separable and split extensions under the same module theoretic approach. Biseparable extensions are finitely generated and projective, hence the example they provide is not a counter example of their main question: “Are biseparable extensions Frobenius?”

In this paper we develop some techniques based in the Ore extensions introduced in [15] to provide a counter example to the previous question. Our example also gives a negative answer the same question but considering Frobenius extensions of the second kind as introduced by Nakayama and Tsuchuku in [14].

This paper is structured as follows. In section 2 we recall precise definitions of Frobenius and biseparable extensions, and we recall again the main question we are going to answer. In section 3 Frobenius extensions are lifted under Ore extensions, while similar results are obtained in section 4 for biseparable extensions. Finally, in section 5 the full counter example is built.

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2. Preliminares

We recall the notions of Frobenius, separable and split extensions. All along the paper B and C are arbitrary unital rings, whilst we reserve the letter A for denoting an algebra over a field \( \mathbb{F} \). Following, for instance, \cite{G recruitment}, a unital ring extension \( C \subseteq B \) is said to be Frobenius if B is a finitely generated projective right C-module and there exists an isomorphism \( B \cong B^* = \text{Hom}(B, C) \) of \( C - B \)-bimodules. Here, by \( \text{Hom}(B, C) \), we denote the set of morphisms of right C-modules from B to C. The additive group \( B^* \) is endowed with the standard \( C - B \)-bimodule structure given by \((\chi b)(u) = c(\chi(bu)) \) for any \( \chi \in B^* \), \( c \in C \) and \( b, u \in B \).

The notion of a Frobenius extension is right-left symmetric as observed in \cite{G recruitment} §1, page 11], i.e. \( C \) is Frobenius if \( B \) is a finitely generated projective left C-module and there exists an isomorphism \( B \cong B^* \) of \( B - C \)-bimodules, where \( B^* = \text{Hom}(C, B) \) is a \( B - C \)-bimodule in a analogous way.

This is a generalization of the well-known notion of Frobenius algebra over a field, namely, a finite dimensional \( \mathbb{F} \)-algebra A is Frobenius if the following equivalent conditions hold:

1. there exists an isomorphism of right (or left) \( A \)-modules \( A \cong A^* \)
2. there exists an associative and non-degenerate \( \mathbb{F} \)-bilinear form \( \langle -,- \rangle : A \times A \rightarrow \mathbb{F} \).
3. there exists a linear functional \( \varepsilon : A \rightarrow \mathbb{F} \) whose kernel does not contain a non zero right (or left) ideal.

Remark 2.1. The bijection between Frobenius forms \cite{G characteristic} and Frobenius functionals \cite{G recruitment} on A is as follows. If \( \langle -,- \rangle : A \times A \rightarrow \mathbb{F} \) is a Frobenius form, hence the rule \( \varepsilon(a) = \langle 1, a \rangle \) for any \( a \in A \) defines a Frobenius functional \( \varepsilon : A \rightarrow \mathbb{F} \). Conversely, if \( \varepsilon : A \rightarrow \mathbb{F} \) is a Frobenius functional, set \( \langle a, b \rangle = \varepsilon(ab) \) for any \( a, b \in A \) in order to get a Frobenius form.

The correspondence between Frobenius functionals \cite{G characteristic} and left \( A \)-isomorphisms \cite{G recruitment} is given as follows. For any Frobenius functional \( \varepsilon \), we may define \( \alpha : A \rightarrow A^* \) as \( \alpha(a)(b) = \varepsilon(ab) \) for any \( a, b \in A \), which becomes a left \( A \)-isomorphism. Conversely, for any left \( A \)-isomorphism \( \alpha : A \rightarrow A^* \), the rule \( \varepsilon(a) = \alpha(a)(1) \) for any \( a \in A \) provides a Frobenius functional \( \varepsilon \). See \cite{G split} Theorem 3.15 for full details. In particular, for each \( \mathbb{F} \)-basis \( \{a_1, \ldots, a_r \} \) of \( A \) there exists an \( \mathbb{F} \)-basis \( \{b_1, \ldots, b_r \} \) of \( A \) such that \( \{\alpha_1(b_1), \ldots, \alpha_1(b_r) \} \) is the dual basis of \( \{a_1, \ldots, a_r \} \), i.e.

\[
\varepsilon(b_j a_i) = \alpha_1(b_j)(a_i) = \delta_{ij}.
\]

Following \cite{G split}, the extension \( C \subseteq B \) is called separable if the canonical multiplication map

\[
\mu : B \otimes_C B \rightarrow B, \quad b_1 \otimes b_2 \mapsto b_1 b_2
\]
splits as a morphism of \( B \)-bimodules, i.e. there exists \( p \in B \otimes_C B \) such that \( bp = pb \) for all \( b \in B \) and \( \mu(p) = 1 \). The splitting map is therefore determined by \( 1 \mapsto p \).

Finally, \( C \subseteq B \) is called split if the inclusion map \( C \rightarrow B \) splits as a morphism of \( B \)-bimodules, i.e. there exists a \( C \)-bimodule morphism \( \xi : B \rightarrow C \) such that \( \xi(1) = 1 \).

In \cite{G split} Definition 2.4, the notion of a separable module is extended to the concept of biseparable module. When particularizing to ring extensions, \cite{G split} Lemma 3.3] says that \( C \subseteq B \) is called to be biseparable if one of the following equivalent conditions holds:

1. \( B \) is biseparable as \( B - C \)-bimodule and finitely generated projective as left \( C \)-module.
2. \( B \) is biseparable as \( C - B \)-bimodule and finitely generated projective as right \( C \)-module.
3. \( B \) is biseparable as \( B - C \)-bimodule and as \( C - B \)-bimodule.
4. \( C \subseteq B \) is split, separable and finitely generated projective as left \( C \)-module and as right \( C \)-module.

Henceforth, motivated by the arguments provided in the Introduction, the following question is stated in \cite{G split}:

**Problem 2.2.** \cite{G split} Problem 3.5] Are biseparable extensions Frobenius?

The main aim of this paper is to build an example of a ring extension which is biseparable and not Frobenius, giving a negative answer to Problem 2.2. Throughout the paper we assume that \( A \) is a finite dimensional \( \mathbb{F} \)-algebra of dimension \( r \). Let also denote by \( \sigma : A \rightarrow A \) an algebra \( \mathbb{F} \)-automorphism and \( \delta : A \rightarrow A \) an \( \mathbb{F} \)-linear \( \sigma \)-derivation on \( A \), i.e. \( \sigma(ab) = \sigma(a)\delta(b) + \delta(a)b \) for all \( a, b \in A \). We denote by \( R \) the ring of (commutative) polynomials \( \mathbb{F}[x] \) and by \( S \) the Ore extension
We may then consider \( f = a \in A \) written on the left whose product is twisted by the rule \( xa = \sigma(a)x + \delta(a) \) for any \( a \in A \). This notation is fixed throughout the rest of the paper.

We give conditions on \( \sigma \) and \( \delta \) in order to get that \( R \subseteq S \) inherits the corresponding properties (separable, split, Frobenius) from \( F \subseteq A \). A precise construction of \( A, \sigma \) and \( \delta \) will lead to the counterexample.

3. Lifting Frobenius extensions

Given \( a \in A, n \geq 0 \) and \( 0 \leq i \leq n \), we denote by \( N^n_i(a) \) the coefficient of degree \( i \) when multiplying \( x^n \) on the right by \( a \) in \( S \). That is to say,

\[
x^n a = \sum_{i=0}^{n} N^n_i(a)x^i.
\]

We may then consider \( \mathbb{F} \)-linear operators \( N^n_i : A \to A \) for any \( i \) and \( n \) with \( 0 \leq i \leq n \). These can be described as the sum of all possible products with \( i \) factors of \( \sigma \) and \( n - i \) factors of \( \delta \). For instance, \( N^n_0 = \text{id}_A, N^n_1 = \delta \) and \( N^n_i = \sigma^n \) and \( N^n_{n-1} = \sigma^n - \sigma^{n-1} \delta \). We set \( N^n_i = 0 \) whenever \( i < 0 \) or \( i \geq n + 1 \). These maps were introduced in [12], where \( N^n_i \) is denoted by \( f^n \).

We need the following technical result.

**Lemma 3.1.** Let \( \{a_1, \ldots, a_r\} \) be an \( \mathbb{F} \)-basis of \( A \). The following statements hold.

1. \( \{a_1, \ldots, a_r\} \) is a right basis of \( S \) over \( R \) and \( R' \cong S \) as right \( R \)-modules.
2. \( \{a_1, \ldots, a_r\} \) is a left basis of \( S \) over \( R \) and \( R' \cong S \) as left \( R \)-modules.

**Proof.** (1) Straightforward since there is no interaction between \( z \) and the elements in \( A \).

(2) It is well known that \( S^{op} = A^{op}[x; \sigma^{-1}, -\delta \sigma^{-1}] \) [see e.g. [3] page 39, Exercise 2R]. Observe that \( \{a_1, \ldots, a_r\} \) is also a basis of \( A^{op} \) as an \( \mathbb{F} \)-vector space, hence by (1) \( \{a_1, \ldots, a_r\} \) is a basis of \( S^{op} \) as right \( R^{op} \)-module. Hence \( \{a_1, \ldots, a_r\} \) is a basis of \( S \) as left \( R \)-module. \( \square \)

**Theorem 3.2.** There exists a bijective correspondence between

1. right \( S \)-isomorphisms from \( S \) to \( S^* \).
2. Frobenius structures on the \( \mathbb{F} \)-algebra \( A \).

**Proof.** Let \( \varepsilon : A \to \mathbb{F} \) be a Frobenius functional on \( A \). We define \( \alpha_\varepsilon : S \to S^* \) as follows. For any \( f = \sum_{i=0}^{n} f_i x^i \in S \), set

\[
\alpha_\varepsilon(f)(1) = \sum_{i=0}^{n} \varepsilon(f_i)x^i,
\]

and, in general, for any \( g \in S, \alpha_\varepsilon(f)(g) = \alpha_\varepsilon(fg)(1) \). The map is well-defined, \( \alpha_\varepsilon(f) \) is \( R \)-linear for all \( f \in S \). Indeed, let \( f, g \in S \) and a monomial \( ax^n \in R \). Suppose that \( fg = \sum_{i=0}^{n} m_i x^i \) in \( S \),
Then, by (3), for all $g$

\[ g = \sum_{i=0}^{t} m_i x^i
\]

so that, by $F$-linearity, $\alpha_\varepsilon(f)$ is a right $R$-morphism for any $f \in S$. Additionally, observe that, for any $f, g, h \in S$, $\alpha_\varepsilon(fh)(g) = \alpha_\varepsilon(f)(hg)$ so that $\alpha_\varepsilon$ is a right $S$-morphism.

Let $f = \sum_{i=0}^{n} a_i x^i \in S$ with $a_n \neq 0$ such that $\alpha_\varepsilon(f) = 0$. Then, for any $b \in A$,

\[ 0 = \alpha_\varepsilon(f)(b) = \sum_{i=0}^{n} \alpha_\varepsilon(a_i x^i b)(1) = \sum_{i=0}^{n} \alpha_\varepsilon\left(a_i \sum_{j=0}^{i} N_j^i(b) x^j\right)(1) = \sum_{i=0}^{n} \sum_{j=0}^{i} \varepsilon(a_i N_j^i(b)) x^j. \]

In particular, $\varepsilon(a_n N_n^i(b)) = \varepsilon(a_n \sigma^n(b)) = 0$ for any $b \in A$. Since $\sigma$ is an automorphism, $\varepsilon(a_n b) = 0$ for each $b \in A$ and then the kernel of $\varepsilon$ contains the right ideal generated by $a_n$, a contradiction. Thus $\alpha_\varepsilon$ is injective.

Notice that, for any $c \in F$ and any $f, g \in S$,

\[ \alpha_\varepsilon(cf)(g) = \alpha_\varepsilon(fc)(g) = \alpha_\varepsilon(f)(cg) = \alpha_\varepsilon(f)(gc) = \alpha_\varepsilon(f)(g)c = c \alpha_\varepsilon(f)(g). \]

Hence $\alpha_\varepsilon$ is an $F$-$S$-monomorphism.

Finally, it remains to prove that $\alpha_\varepsilon$ is surjective. Let $\{a_1, \ldots, a_r\}$ be an $F$-basis of $A$. By Lemma 3.1, $\{a_1, \ldots, a_r\}$ is an $R$-basis of $S$ as right $R$-module, hence, if we denote $\gamma_1, \ldots, \gamma_r \in S^*$ the dual elements defined by $\gamma_i(a_j) = \delta_{ij}$ for all $1 \leq i, j \leq r$, then $\{\gamma_1, \ldots, \gamma_r\}$ is a left $R$-basis of $S^*$. Let us show that $x^n \gamma_1 \in \text{Im } \alpha$ for all $n \geq 0$ and $1 \leq i \leq r$, which yields the result.

For any $n \geq 0$, since $\{\sigma^n(a_1), \ldots, \sigma^n(a_r)\}$ is an $F$-basis of $A$, by (1), there exist $b_1^{(n)}, \ldots, b_r^{(n)} \in A$ such that

\[ \varepsilon\left(b_i^{(n)} \sigma^n(a_j)\right) = \delta_{ij} \]

for all $1 \leq i, j \leq r$. For each $1 \leq i \leq r$, set

\[ g^{(i)} = \sum_{k=0}^{n} g_k^{(i)} x^k \in S, \]

where $g_n^{(i)} = b_i^{(n)}$ and, for each $0 \leq m \leq n - 1$,

\[ g_m^{(i)} = - \sum_{\ell=1}^{r} \left(b^{(m)}_{\ell} \left( \sum_{k=m+1}^{n} \varepsilon \left(g_k^{(i)} N_k^m(a_{\ell})\right) \right) \right). \]

Then, by (3), for all $1 \leq i, j \leq r$,

\[ \varepsilon\left(g_n^{(i)} \sigma^n(a_j)\right) = \varepsilon\left(b_i^{(n)} \sigma^n(a_j)\right) = \delta_{ij}. \]
and

\[ \varepsilon \left( g_m^{(i)} N_m^m(a_j) \right) = \varepsilon \left( g_m^{(i)} \sigma^m(a_j) \right) \\
= \varepsilon \left( - \sum_{\ell = 1}^{r} \left( b_{\ell}^{(m)} \left( \sum_{k = m+1}^{n} \varepsilon \left( g_k^{(i)} N_m^m(a_{\ell}) \right) \right) \right) \sigma^m(a_j) \right) \\
= - \sum_{\ell = 1}^{r} \sum_{k = m+1}^{n} \varepsilon \left( g_k^{(i)} N_m^m(a_{\ell}) \right) \varepsilon \left( b_{\ell}^{(m)} \sigma^m(a_j) \right) \\
= - \sum_{k = m+1}^{n} \varepsilon \left( g_k^{(i)} N_m^m(a_j) \right) . \]

Hence

\[ \sum_{k = m}^{n} \varepsilon \left( g_k^{(i)} N_m^m(a_j) \right) = 0 \quad (6) \]

for \( 1 \leq i, j \leq r, 0 \leq m \leq n - 1 \). Now,

\[ \alpha_{\varepsilon}(g^{(i)})(a_j) = \alpha_{\varepsilon} \left( \sum_{k = 0}^{n} g_k^{(i)} x^k a_j \right) \quad (1) \]

\[ = \alpha_{\varepsilon} \left( \sum_{k = 0}^{n} g_k^{(i)} \sum_{m = 0}^{k} N_m^m(a_j) x^m \right) \quad (1) \]

\[ = \sum_{m = 0}^{n} \sum_{k = m}^{n} \varepsilon \left( g_k^{(i)} N_m^m(a_j) \right) x^m \]

\[ = \sum_{m = 0}^{n} \sum_{k = m}^{n} \varepsilon \left( g_k^{(i)} N_m^m(a_j) \right) x^m \]

\[ = \delta_{ij} x^n, \text{ by (5) and (6),} \]

\[ = x^n \gamma_i(a_j). \]

So \( x^n \gamma_i \in \text{Im} \alpha \), as required.

Conversely, let \( \alpha : S \to S^* \) be a right \( S \)-isomorphism. Note that \( \alpha \) is also left \( F \)-linear, since \( F \) is in the center of \( S \). We set then \( \varepsilon_{\alpha} : A \to F \) as \( \varepsilon_{\alpha}(a) = \alpha(a)(1) \). We need first to show that \( \alpha(a)(1) \in F \) for any \( a \in A \). Actually, we may prove that \( \alpha(a)(b) \in F \) for any \( a, b \in A \). Let \( \{a_1, \ldots, a_r\} \) be an \( F \)-basis of \( A \). By Lemma 3.1, \( \{a_1, \ldots, a_r\} \) is also a basis of \( S \) as right \( R \)-module, hence the dual basis \( \{\gamma_1, \ldots, \gamma_r\} \subseteq S^* \), defined by \( \gamma_i(a_j) = \delta_{ij} \) for any \( 1 \leq i, j \leq r \), is a left \( R \)-basis of \( S^* \). Since \( \alpha \) is surjective, for each \( 1 \leq i \leq r \), there exists certain \( g_i \in S \) such that \( \alpha(g_i) = \gamma_i \).
Let us suppose that \( g_i = \sum_{k=0}^{n_i} b_k^i x^k \) for any \( 1 \leq i \leq r \). Therefore,

\[
\delta_{ij} = \gamma_i(a_j) = \alpha(g_i)(a_j) = \alpha \left( \sum_{k=0}^{n_i} b_k^i x^k \right)(a_j) = \sum_{k=0}^{n_i} \alpha(b_k^i x^k)(a_j) = \sum_{k=0}^{n_i} \alpha(b_k^i)(x^k a_j) = \sum_{k=0}^{n_i} \alpha(b_k^i)(N_k^m(a_j)x^m) = \sum_{k=0}^{n_i} \alpha(b_k^i)(N_k^m(a_j)x^m) = \sum_{k=0}^{n_i} \alpha(b_k^i)(N_k^m(a_j)x^m),
\]

where \( * \) comes from that \( \alpha \) is a right \( S \)-morphism, and \( \dagger \) is due to \( \alpha(b_k^i) \) is a right \( R \)-morphism for any \( k \) and \( i \). Now, if \( n_i \geq 1 \), then

\[
\alpha(b_{n_i}^i)(\sigma^{m_i}(a_j)) = 0
\]

for any \( j \in \{1, \ldots, r\} \). By Lemma [3.1] \( \{\sigma^{m_i}(a_1), \ldots, \sigma^{m_i}(a_r)\} \) is a right \( R \)-basis of \( S \), so \( \alpha(b_{n_i}^i) = 0 \) and then \( b_{n_i}^i = 0 \). Hence, \( n_i \) must be zero, so \( g_i \in A \) for each \( 1 \leq i \leq r \). Furthermore, \( \{g_1, \ldots, g_r\} \) is an \( F \)-basis of \( A \). It is enough to prove their \( F \)-linear independence. Indeed, if \( \sum_{i=1}^{r} c_i g_i = 0 \) for some \( c_i \in F \), then, for any \( j \),

\[
0 = \alpha \left( \sum_{i=1}^{r} c_i g_i \right)(a_j) = \sum_{i=1}^{r} c_i \alpha(g_i)(a_j) = \sum_{i=1}^{r} c_i \gamma_i(a_j) = c_j,
\]

because of \( \alpha \) is left \( F \)-linear. Let then \( a, b \in A \) with \( a = \sum_{j=1}^{r} a_j c_j \) and \( b = \sum_{i=1}^{r} d_i g_i \), where \( c_i, d_j \in F \) for any \( 1 \leq i \leq r \),

\[
\alpha(b)(a) = \alpha \left( \sum_{j=1}^{r} d_j g_j \right) \left( \sum_{j=1}^{r} a_j c_j \right) = \sum_{i=1}^{r} d_i \alpha(g_i) \left( \sum_{j=1}^{r} a_j c_j \right) = \sum_{i=1}^{r} d_i \sum_{j=1}^{r} \alpha(g_i)(a_j)c_j = \sum_{i=1}^{r} d_i \sum_{j=1}^{r} \gamma_i(a_j)c_j = \sum_{i=1}^{r} d_i c_i \in F.
\]

Then \( \varepsilon_\alpha \) is well-defined. It is clear that \( \varepsilon_\alpha \) is \( F \)-linear. Let \( b \in A \) such that the right ideal generated by \( b \) is in \( \text{Ker} \quad \varepsilon_\alpha \). Hence \( \alpha(b)(a) = \alpha(ba)(1) = \varepsilon_\alpha(ba) = 0 \) for any \( a \in A \). In particular \( \alpha(b)(a_j) = 0 \) for all \( 1 \leq j \leq r \) and, by Lemma 3.1, it follows that \( \alpha(b) = 0 \). Therefore \( b = 0 \) since \( \alpha \) is an isomorphism. Thus \( \varepsilon_\alpha \) is a Frobenius functional.
It remains to prove that both constructions are inverse one to each other. Indeed, let $\varepsilon$ be a Frobenius form on $A$. Following the above notation, for any $a \in A$,

$$
\varepsilon_a(a) = \alpha_x(a)(1) = \varepsilon(a).
$$

On the other hand, let $\alpha$ be a $S$-right isomorphism from $S$ to $S^*$. For any $f, g \in S$, where $fg = \sum_{i=0}^{n} m_i x^i$,

$$
\alpha_{\varepsilon_a}(f)(g) = \alpha_{\varepsilon_a}(fg)(1)
$$

$$
= \sum_{i=0}^{n} \varepsilon_a(m_i)x^i
$$

$$
= \sum_{i=0}^{n} \alpha(m_i)(1)x^i
$$

$$
= \sum_{i=0}^{n} \alpha(m_i)(x^i)
$$

$$
= \sum_{i=0}^{n} \alpha(m_i x^i)(1)
$$

$$
= \alpha \left( \sum_{i=0}^{n} m_i x^i \right)(1)
$$

$$
= \alpha(fg)(1)
$$

$$
= \alpha(f)(g),
$$

so that $\alpha_{\varepsilon_a} = \alpha$.

Condition (1) in Theorem 3.2 is quite close to the notion of Frobenius extension, removing the need of being left $R$-module morphism. We have not found in the literature that this condition has been introduced and studied. For this reason let us now introduce semi Frobenius extensions.

**Definition 3.3.** A unital ring extension $C \subseteq B$ is said to be right (resp. left) semi Frobenius if $B$ is a finitely generated projective right (resp. left) $C$-module and there exists an isomorphism $B \cong B^*_{\text{op}}$ of right $B$-modules (resp. an isomorphism $B \cong B_{\text{op}}^*$ of left $B$-modules).

Our aim now is to prove that $A$ is a Frobenius algebra over $F$ if and only if the extension $R \subseteq S$ is left or right semi Frobenius.

**Theorem 3.4.** Let $A$ be an $F$-algebra. The following are equivalent:

1. $A$ is a Frobenius $F$-algebra,
2. the extension $R \subseteq S$ is right semi Frobenius,
3. the extension $R \subseteq S$ is left semi Frobenius.

**Proof.** The equivalence between (1) and (2) is Theorem 3.2.

In order to check the equivalence (1) if and only if (3), observe that $A$ is Frobenius if and only if $A^{\text{op}}$ is Frobenius. By Theorem 3.2, $A^{\text{op}}$ is a Frobenius $F$-algebra if and only if $F[x] \subseteq A^{\text{op}}[x;\sigma^{-1},-\delta\sigma^{-1}]$ is right semi Frobenius. Since $F[x] = R = R^{\text{op}}$ and $S^{\text{op}} = A^{\text{op}}[x;\sigma^{-1},-\delta\sigma^{-1}]$ (see e.g. [6], page 39, Exercise 2R), it follows that $A^{\text{op}}$ is a Frobenius $F$-algebra if and only if $R \subseteq S$ is left semi Frobenius.

**Remark 3.5.** Although, by Theorems 3.2 and 3.4, $R \subseteq S$ is left semi Frobenius if and only if it is right semi Frobenius, it is an open question to know if, in general, the notion of semi Frobenius extension is left-right symmetric, as it does for Frobenius extensions, see [13], §1, page 11.

We now refine the latter results in the realm of Frobenius extensions.

**Theorem 3.6.** There exists a bijective correspondence between the sets of

1. $R-S$-isomorphisms from $S$ to $S^*$.
2. Frobenius functionals $\varepsilon : A \rightarrow F$ satisfying $\varepsilon \sigma = \varepsilon$ and $\varepsilon \delta = 0$.
3. Frobenius forms $\langle -,- \rangle : A \times A \rightarrow F$ satisfying the conditions $\langle a,b \rangle = \langle \sigma(a),\sigma(b) \rangle$ and $\langle \delta(a),\delta(b) \rangle + \langle \delta(a),b \rangle = 0$ for all $a,b \in A$. 

Proof. In order to prove the bijection between (1) and (2), by Theorem 3.2, it is enough to show that left $R$-linearity on the right $S$-isomorphism $\alpha : S \to S^\ast$ implies the conditions described in (2) on the Frobenius functional $\varepsilon$, and vice versa. Let us suppose that $\alpha : S \to S^\ast$ is left $R$-linear. Following the notation of the proof of Theorem 3.2 for each $a \in A$,

$$
\varepsilon_\alpha(a)x = x\varepsilon_\alpha(a) \\
= x\alpha(a)(1) \\
= \alpha(xa)(1) \\
= \alpha(\sigma(a)x + \delta(a))(1) \\
= \alpha(\sigma(a)(x)(1) + \alpha(\varepsilon(a))(1) \\
= \alpha(\sigma(a))x + \alpha(\varepsilon(a))(1) \\
= \varepsilon_\alpha(\sigma(a))x + \varepsilon_\alpha(\delta(a)),
$$

so that $\varepsilon_\alpha(\sigma(a)) = \varepsilon_\alpha(a)$ and $\varepsilon_\alpha(\delta(a)) = 0$.

Conversely, if $\varepsilon : A \to F$ verifies $\varepsilon\sigma = \varepsilon$ and $\varepsilon\delta = 0$, for any $f, g \in S$, where $fg = \sum_{i=0}^{n} m_i x^i$,

$$
\alpha_\varepsilon(fg)(g) = \alpha_\varepsilon(fxg)(1) \\
= \alpha_\varepsilon \left( \sum_{i=0}^{n} \varepsilon(\sigma(m_i)x^{i+1}) \right)(1) + \alpha_\varepsilon \left( \sum_{i=0}^{n} \varepsilon(\delta(m_i)x^i) \right)(1) \\
= \sum_{i=0}^{n} \varepsilon(\sigma(m_i)x^{i+1}) + \sum_{i=0}^{n} \varepsilon(\delta(m_i)x^i) \\
= \sum_{i=0}^{n} \varepsilon(m_i)x^{i+1} \\
= x \sum_{i=0}^{n} \varepsilon(m_i)x^i \\
= x\alpha_\varepsilon(fg)(1) \\
= x\alpha_\varepsilon(f)(g),
$$

and thus $\alpha_\varepsilon$ is left $R$-linear. The bijection between (2) and (3) follows from the bijection between Frobenius forms and Frobenius functionals explained in Remark 2.1. $\square$

**Corollary 3.7.** $R \subseteq S$ is Frobenius if and only if there exists a Frobenius functional $\varepsilon : A \to F$ verifying $\varepsilon\sigma = \varepsilon$ and $\varepsilon\delta = 0$.

We finish the section showing a family of examples of left and right semi Frobenius, but not Frobenius, extensions.

**Example 3.8.** Let $p$ be a prime number and $F_p$ the finite field of $p$ elements. Consider some $n > 1$, and the field extension $F_p \subseteq F_{p^n}$. Then $F_{p^n}$ is a Frobenius $F_p$-algebra. Let $\tau : F_{p^n} \to F_{p^n}$ be the Frobenius automorphism, i.e. $\tau(x) = x^p$ for any $x \in F_{p^n}$. Then, there exists $\alpha \in F_{p^n}$ such that $\{\alpha, \tau(\alpha), \ldots, \tau^{n-1}(\alpha)\}$ is an $F_p$-basis of $F_{p^n}$. We set then the $\tau$-derivation $\delta : F_{p^n} \to F_{p^n}$ given by

$$
\delta(b) = (\tau(b) - b) \frac{\alpha}{\tau(\alpha) - \alpha}
$$

for any $b \in F_{p^n}$. By Corollary 3.7, $F_p[x] \subseteq F_{p^n}[x; \sigma, \delta]$ is left and right semi-Frobenius. Nevertheless, it is not Frobenius. Indeed, by Theorem 3.6, $F_p[x] \subseteq F_{p^n}[x; \tau, \delta]$ is Frobenius if and only if there exists a Frobenius functional $\varepsilon : F_{p^n} \to F_p$ such that $\varepsilon\tau = \varepsilon$ and $\varepsilon\delta = 0$. But, in such a case, since $\delta(\alpha) = \alpha$,

$$
0 = \varepsilon(\delta(\alpha)) = \varepsilon(\alpha) = \varepsilon(\tau(\alpha)) = \cdots = \varepsilon(\tau^{n-1}(\alpha)).
$$

So that $\varepsilon = 0$. 
4. Lifting Biseparable Extensions

In this section we aim to provide conditions for ensuring that the extension $R \subseteq S$ is biseparable. Since, by [1, Lemma 3.3] and Lemma 3.1, this is so if and only if $R \subseteq S$ is separable and split, we deal with both notions independently. Let us first analyze the property of being split.

**Proposition 4.1.** Let $C \subseteq B$ be a ring extension, $\sigma : B \rightarrow B$ an automorphism of $B$ and $\delta : B \rightarrow B$ a $\sigma$-derivation on $B$ such that $\sigma(C) \subseteq C$ and $\delta(C) \subseteq C$. Suppose that $C \subseteq B$ is split and $\xi : B \rightarrow C$ is a $C$-bimodule morphism with $\xi\sigma = \sigma\xi$, $\xi\delta = \delta\xi$ and $\xi(1) = 1$, then $C[x; \sigma, \delta] \subseteq B[x; \sigma, \delta]$ is split.

**Proof.** We define $\hat{\xi} : B[x; \sigma, \delta] \rightarrow C[x; \sigma, \delta]$ as, for any $f = \sum_{i=0}^{n} b_{i}x^{i} \in B[x; \sigma, \delta],$

$$\hat{\xi}(f) = \sum_{i=0}^{n} \xi(b_{i})x^{i}.$$

We prove that $\xi$ is a $C[x; \sigma, \delta]$-bimodule morphism. Let then $a \in C$ and $f = \sum_{i=0}^{n} b_{i}x^{i} \in B[x; \sigma, \delta],$

$$\hat{\xi}(xf) = \hat{\xi} \left( \sum_{i=0}^{n} b_{i}x^{i+1} + \delta(b_{i})x^{i} \right) = \sum_{i=0}^{n} \xi(b_{i})x^{i+1} + \sum_{i=0}^{n} \xi(\delta(b_{i}))x^{i} = \sum_{i=0}^{n} \sigma(b_{i})x^{i+1} + \sum_{i=0}^{n} \delta(\xi(b_{i}))x^{i} = x \sum_{i=0}^{n} \xi(b_{i})x^{i} = x \hat{\xi}(f),$$

and

$$\hat{\xi}(af) = \hat{\xi} \left( \sum_{i=0}^{n} ab_{i}x^{i} \right) = \sum_{i=0}^{n} \xi(ab_{i})x^{i} = a \sum_{i=0}^{n} \xi(b_{i})x^{i} = a \hat{\xi}(f),$$

so $\hat{\xi}$ is left $C[x; \sigma, \delta]$-linear. Analogously,

$$\hat{\xi}(fx) = \hat{\xi} \left( \sum_{i=0}^{n} b_{i}x^{i+1} \right) = \sum_{i=0}^{n} \xi(b_{i})x^{i+1} = \left( \sum_{i=0}^{n} \xi(b_{i})x^{i} \right) x = \hat{\xi}(f)x,$$

and

$$\hat{\xi}(fa) = \hat{\xi} \left( \sum_{i=0}^{n} b_{i} \left( \sum_{k=0}^{i} N_{k}^{i}(a)x^{k} \right) \right) = \sum_{i=0}^{n} \sum_{k=0}^{i} \xi(b_{i}N_{k}^{i}(a))x^{k} = \sum_{i=0}^{n} \sum_{k=0}^{i} \xi(b_{i})N_{k}^{i}(a)x^{k} = \sum_{i=0}^{n} \xi(b_{i}) \left( \sum_{k=0}^{i} N_{k}^{i}(a)x^{k} \right) = \sum_{i=0}^{n} \xi(b_{i})x^{i} a = \hat{\xi}(f)a,$$
so \( \hat{\xi} \) is right \( C[x; \sigma, \delta] \)-linear. Clearly, \( \hat{\xi}(1) = 1 \), and thus \( B[x; \sigma, \delta] \subseteq C[x; \sigma, \delta] \) is split. \( \Box \)

**Corollary 4.2.** If there exists an \( \mathbb{F} \)-linear map \( \xi : A \to \mathbb{F} \) such that \( \xi(1) = 1 \), \( \xi \sigma = \xi \) and \( \xi \delta = 0 \), then \( R \subseteq S \) is split.

**Proof.** Observe that any finite dimensional \( \mathbb{F} \)-algebra \( A \) is split, since there is an \( \mathbb{F} \)-basis of \( A \) containing the element 1. Hence, it follows from Proposition 4.1 since \( \sigma|_{\mathbb{F}} = \text{id}_{\mathbb{F}} \) and \( \delta|_{\mathbb{F}} = 0 \). \( \Box \)

The transfer of separability in Ore extensions is studied in [5]. For brevity, we use the following notation. If \( C \subseteq B \) is a ring extension, we denote by \( \sigma^\otimes \) and \( \delta^\otimes \) the maps

\[
\sigma^\otimes : B \otimes_C B \to B \otimes_C B \quad \text{and} \quad \delta^\otimes : B \otimes_C B \to B \otimes_C B,
\]

\[
b_1 \otimes b_2 \mapsto \sigma(b_1) \otimes \sigma(b_2)
\]

\[
\delta^\otimes : B \otimes_C B \to B \otimes_C B \quad \text{and} \quad \delta^\otimes : B \otimes_C B \to B \otimes_C B,
\]

\[
b_1 \otimes b_2 \mapsto \sigma(b_1) \otimes \delta(b_2) + \delta(b_1) \otimes b_2
\]

for any \( b_1, b_2 \in B \). By [5, Lemma 27], \( \sigma^\otimes \) and \( \delta^\otimes \) are well defined.

**Proposition 4.3** ([5, Theorem 29]). Let \( C \subseteq B \) be a ring extension, \( \sigma : B \to B \) an automorphism of \( B \) and \( \delta : B \to B \) a \( \sigma \)-derivation on \( B \) such that \( \sigma(C) \subseteq C \) and \( \delta(C) \subseteq C \). If \( C \subseteq B \) is separable and there exists a separability element \( p \) verifying \( \sigma^\otimes(p) = p \) and \( \delta^\otimes(p) = 0 \), then \( C[x; \sigma, \delta] \subseteq B[x; \sigma, \delta] \) is separable.

In [4, Theorem 8] a converse result of Proposition 4.3 is provided when \( \delta = 0 \). Here we generalize part of this result when \( \delta \) is an inner \( \sigma \)-derivation. So, for the rest of this section, \( \sigma : A \to A \) is an \( \mathbb{F} \)-linear automorphism and \( \delta_{\sigma,b} : A \to A \) is a \( \sigma \)-derivation defined by

\[
\delta_{\sigma,b}(a) = ba - \sigma(a)b
\]

for some \( b \in A \). Hence \( R = \mathbb{F}[x] \) and \( S = A[x; \sigma, \delta_{\sigma,b}] \). Recall that we have fixed an \( \mathbb{F} \)-basis \( \{a_1, \ldots, a_r\} \) of \( A \).

**Lemma 4.4.** The set \( \{a_i \otimes_R a_j x^k \mid 1 \leq i, j \leq r, k \geq 0\} \) is an \( \mathbb{F} \)-basis of \( S \otimes_R S \). Consequently, the map

\[
\varphi : S \otimes_R S \to \bigoplus_{k \geq 0} (A \otimes_{\mathbb{F}} A)x^k, \quad (a_i \otimes_R a_j x^k) \mapsto (a_i \otimes_{\mathbb{F}} a_j)x^k
\]

is an \( \mathbb{F} \)-isomorphism that provides an \( \mathbb{N} \)-grading on \( S \otimes_R S \) as an \( \mathbb{F} \)-vector space.

**Proof.** It can be derived from Lemma 3.1 and [16, Corollary 8.5] that \( S \otimes_R S \) is a free right \( R \)-module with basis \( \{a_i \otimes_R a_j \mid 1 \leq i, j \leq r\} \), hence \( \{a_i \otimes_R a_j x^k \mid 1 \leq i, j \leq r, k \geq 0\} \) is an \( \mathbb{F} \)-basis. Consequently \( \varphi \) is an isomorphism because \( \{(a_i \otimes_{\mathbb{F}} a_j)x^k \mid 1 \leq i, j \leq r, k \geq 0\} \) is a basis of \( \bigoplus_{k \geq 0} (A \otimes_{\mathbb{F}} A)x^k \).

**Proposition 4.5.** If \( R \subseteq S \) is separable and \( \delta \) is inner, then \( \mathbb{F} \subseteq A \) is separable.

**Proof.** Let \( p \in S \otimes_R S \) be a separability element. We do not loose generality if we assume \( p = \sum_{i=1}^{r} \sum_{j=0}^{m} a_i \otimes_R g_{ij} x^j \). Let \( a \in A \). Since \( ap = pa \) we have

\[
\sum_{i=1}^{r} \sum_{j=0}^{m} a_i a_i \otimes_R g_{ij} x^j = \sum_{i=1}^{r} \sum_{j=0}^{m} \sum_{k=0}^{m} a_i \otimes_R g_{ij} N^i_k(a)x^k
\]

\[
= \sum_{i=1}^{r} \sum_{k=0}^{m} \sum_{j=0}^{m} a_i \otimes_R g_{ij} N^i_k(a)x^k.
\]

By Lemma 4.4 and by applying \( \varphi \), we get that, for all \( 0 \leq \ell \leq m \),

\[
\sum_{i=1}^{r} a_i \otimes_{\mathbb{F}} g_{i\ell} = \sum_{i=1}^{r} \sum_{j=0}^{m} a_i \otimes_{\mathbb{F}} g_{ij} N^i_k(a) \in A \otimes_{\mathbb{F}} A.
\]
Multiplying on the right by $b^\ell$ and adding all the obtained identities we have
\[
\sum_{\ell=0}^{m} \sum_{i=1}^{r} a_i \otimes F g_{i\ell} b^\ell = \sum_{\ell=0}^{m} \sum_{i=1}^{r} \sum_{j=\ell}^{m} a_i \otimes F g_{ij} N_j^\ell(a) b^\ell \\
= \sum_{i=1}^{r} \sum_{j=0}^{m} \sum_{\ell=0}^{m} a_i \otimes F g_{ij} N_j^\ell(a) b^\ell.
\]
Since $ba = \sigma(a)b + \delta_{\sigma,b}(a)$, it follows that
\[
\sum_{\ell=0}^{j} N_j^\ell(a) b^\ell = b^\ell a,
\]
hence
\[
\sum_{i=1}^{r} \sum_{j=0}^{m} a_i \otimes F g_{ij} b^\ell = \sum_{i=1}^{r} \sum_{j=0}^{m} a_i \otimes F g_{ij} b^\ell a.
\]
So $\hat{p} = \sum_{i=1}^{r} \sum_{j=0}^{m} a_i \otimes F g_{ij} b^\ell$ satisfies $a\hat{p} = \hat{p}a$ for all $a \in A$. Now, since
\[
1 = \mu(p) = \sum_{i=1}^{r} \sum_{j=0}^{m} a_i g_{ij} x^i \in A[x; \sigma, \delta_{\sigma,b}],
\]
it follows that $1 = \sum_{i=1}^{r} a_i g_{ij} x^i$ and $0 = \sum_{i=1}^{r} a_i g_{ij}$ for all $1 \leq j \leq m$. Therefore $\mu(\hat{p}) = 1$ and $\hat{p}$ is a separability element for $F \subseteq A$.

5. An answer to a problem of Caenepeel and Kadison

In this section, with the aid of the results of Sections 3 and 4 we provide a negative answer to Problem 2.2.

Example 5.1 (Answer to Problem 2.2). Let $F_8$ be the field with eight elements described as $F_8 = F(a)$, where $a^3 + a^2 + 1 = 0$. Let $\tau$ be the Frobenius automorphism on $F_8$, that is, $\tau(c) = c^2$ for any $c \in F_8$. Observe that $\{a, a^2, a^4\}$ is an auto dual basis of the extension $F_2 \subseteq F_8$. Set $A = M_2(F_8)$, the ring of $2 \times 2$ matrices over $F_8$, and consider the $F_2$-automorphism $\sigma : A \to A$ defined as the component-by-component extension of $\tau$ to $A$. That is, $\sigma$, is given by
\[
\sigma \begin{pmatrix} x_0 & x_2 \\ x_1 & x_3 \end{pmatrix} = \begin{pmatrix} \tau(x_0) & \tau(x_1) \\ \tau(x_2) & \tau(x_3) \end{pmatrix} = \begin{pmatrix} x_0^2 & x_1^2 \\ x_2^2 & x_3^2 \end{pmatrix} \tag{7}
\]
for any $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \in A$.

We can also set the inner $\sigma$-derivation $\delta_{\sigma,M} : A \to A$ given by $\delta_{\sigma,M}(x) = Mx - \sigma(x)M$ for any $x \in A$, where
\[
M = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.
\]
Our aim is to prove that the ring extension $F_2[x] \subseteq A[x; \sigma, \delta_{\sigma,M}]$ is split and separable, and hence biseparable, but not Frobenius. For simplicity, we denote
\[
e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Hence, an $F_2$-basis of $A$ is given by $B = \{a^i e_j, \text{ such that } 0 \leq i \leq 2 \text{ and } 0 \leq j \leq 3\}$.

Let $\epsilon : A \to F_2$ be an $F_2$-linear map. If we force $\epsilon\sigma = \epsilon$, hence
\[
\epsilon(a^{2i+1} e_j) = \epsilon(a^{2i} e_j) = \epsilon(a^{2i} e_j)
\]
for any $0 \leq i \leq 2$ and $0 \leq j \leq 3$, so that $\epsilon$ is determined by four values $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in F_2$ such that $\epsilon(a^{i} e_j) = \gamma_j$ for any $0 \leq i \leq 2$ and $0 \leq j \leq 3$.

Let us then consider $\xi : A \to F_2$ the $F_2$-linear map determined by $\gamma_0 = 1, \gamma_1 = 0, \gamma_2 = 0$ and $\gamma_3 = 0$. Firstly,
\[
\xi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \xi \begin{pmatrix} a + a^2 + a^4 \\ 0 \\ a + a^2 + a^4 \end{pmatrix} = \xi(ae_0) + \xi(a^2 e_0) + \xi(a^4 e_0) + \xi(a e_3) + \xi(a^2 e_3) + \xi(a^4 e_3) = 1.
\]
On the other hand, for any $x_0, x_1, x_2, x_3 \in \mathbb{F}_8$,
\[
\delta \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \left( \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} + \begin{pmatrix} x_0^2 & x_1^2 \\ x_2^2 & x_3^2 \end{pmatrix} \right) \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}
\]
\[
= \begin{pmatrix} 0 & ax_2 \\ ax_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & ax_2^2 \\ 0 & ax_3^2 \end{pmatrix}
\]
\[
= \begin{pmatrix} 0 & ax_1^2 \\ 0 & ax_2^2 \end{pmatrix} (x_3 + x_2^3).
\]

(8)

Therefore, $\xi \delta = 0$. By Corollary 4.2, the extension $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta, M]$ is split.

Let us prove that the map $\xi$ is the only non trivial $\mathbb{F}_2$-linear map verifying the equalities $\xi \sigma = \xi$ and $\xi \delta = 0$. Let us suppose that $\varepsilon : A \rightarrow \mathbb{F}_2$ is a non zero $\mathbb{F}_2$-linear map that verifies the equation $\varepsilon \sigma = \varepsilon$. As reasoned above, it is determined by some values $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{F}_2$. Nevertheless,

- If $\gamma_1 = 1$, then $\varepsilon \delta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = 1$,
- If $\gamma_2 = 1$, then $\varepsilon \delta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = 1$,
- If $\gamma_3 = 1$, then $\varepsilon \delta \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & a + a^2 + a^4 \end{pmatrix} = 1$,

so that $\varepsilon \delta = 0$ if and only if $\gamma_0 = 1$ and $\gamma_1 = \gamma_2 = \gamma_3 = 0$. Notice that the kernel of $\xi$ contains the left ideal

$J = \left\{ \begin{pmatrix} 0 & c_2 \\ 0 & c_3 \end{pmatrix} \mid c_2, c_3 \in \mathbb{F}_8 \right\}$,

so that there is no Frobenius functional $\varepsilon : A \rightarrow \mathbb{F}_2$ verifying $\varepsilon \sigma = \varepsilon$ and $\varepsilon \delta = 0$. By Corollary 3.7, the extension $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta, M]$ is not Frobenius.

Finally, let us prove that the extension is separable. Consider the element $p \in A \otimes A$ given by
\[
p = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}
\]
\[
+ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.
\]

This is a separability element of the extension $\mathbb{F}_2 \subseteq A$, since it is the composition of the separability element $a \otimes a + a^2 \otimes a^2 + a^4 \otimes a^4$ of the extension $\mathbb{F}_2 \subseteq \mathbb{F}_8$, and the separability element $c_0 \otimes c_0 + c_2 \otimes c_3$ of the extension $\mathbb{F}_8 \subseteq A$, see [5] Examples 4 and 5 and [7] Proposition 2.5. Although it is straightforward to check that $\sigma^\otimes(p) = p$ and $\delta^\otimes(p) = 0$, due to its importance in this paper, we detail explicitly all the computations. Since the Frobenius automorphism induces a permutation on $\{a, a^2, a^4\}$, it follows that
\[
\sigma^\otimes(p) = \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}
\]
\[
+ \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}
\]
\[
= p.
\]

Let us now compute $\delta^\otimes(p)$. Recall $\delta^\otimes = \sigma \otimes \delta + \delta \otimes \text{id}$. By (8) and (7), $\delta \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for each $c \in \mathbb{F}_8$, so
\[
\delta^\otimes \left( \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a^{2i+1} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^{2i} & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\delta^\otimes \left( \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \right)
\]

for $0 \leq i \leq 2$. Hence

$$\begin{align*}
\delta^\otimes(p) &= \delta^\otimes \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) + \delta^\otimes \left( \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix} \right) \\
&\quad + \delta^\otimes \left( \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a^4 & 0 \\ 0 & 0 \end{pmatrix} \right) + \delta^\otimes \left( \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \right) \\
&= \delta^\otimes \left( \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \right) + \delta^\otimes \left( \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \right) + \delta^\otimes \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \\
&= \delta^\otimes \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right) = 0.
\end{align*}$$

Moreover, by (8) and (7) again,

$$\begin{align*}
\delta^\otimes \left( \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \right) &= \begin{pmatrix} 0 & 0 \\ a^2+1 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a^2+1 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix},
\end{align*}$$

so we can follow the computations in (9) to get

$$\begin{align*}
\delta^\otimes(p) &= \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix},
\end{align*}$$

where we have used that $a^2 = 1$. The identities $a^3 = a + a^4$ and $a^5 = a^2 + a^4$ in $\mathbb{F}_8$ allow us to expand (10) in order obtain

$$\begin{align*}
\delta^\otimes(p) &= \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a + a^4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ a^2 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & a^2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.
\end{align*}$$

By Proposition 13, $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta_\sigma_M]$ is separable. Hence $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta_\sigma_M]$ is a biselectable extension which is not Frobenius.

At this point one could ask what happens if we replace the family of Frobenius extensions in Problem 2.2 by a more general family. For instance, we can consider the family of Frobenius extensions of second kind introduced in [14]. Let $C \subseteq B$ be a ring extension and let $\kappa : C \to C$ be an automorphism. There is a structure of left $C$-module on $C$ given by $a \cdot_b = \kappa(a)b$ for each $a, b \in C$. Hence, $C \subseteq B$ is said to be a $\kappa$-Frobenius extension, or a Frobenius extension
of second kind, if $B$ is a finitely generated projective right $C$-module, and there exists a $C \to B$-isomorphism from $B$ to $B^* = \text{Hom}(B_C, C_C)$. The $C \to B$-bimodule structure on $B^*$ is then given by $(ab)(c) = a \cdot f(bc) = \kappa(a)(bc)$ for any $f \in B^*$, $a \in C$ and $b,c \in B$. It is clear that a Frobenius extension of second kind is left and right semi Frobenius. A natural question that arises is then if a right and left semi Frobenius extension is a Frobenius extension of second kind. In order to answer this question, we may prove similar results to those showed in the previous sections.

**Proposition 5.2.** Let $\kappa : R \to R$ be an automorphism with $\kappa(x) = nx + n$ for some $m,n \in \mathbb{F}$ with $m \neq 0$. There exists a bijection between the sets of

1. $\mathcal{S}$-isomorphisms $\alpha : S \to S^*$.
2. Frobenius functionals $\varepsilon : A \to \mathbb{F}$ verifying $\varepsilon \sigma = m \varepsilon$ and $\varepsilon \delta = n \varepsilon$.

**Proof.** By Theorem 3.4, there exists a left $\mathcal{S}$-isomorphism $\beta : S \to S^*$ if and only if there exists a Frobenius functional $\varepsilon : A \to \mathbb{F}$. Now, analogously to the proof of Theorem 3.7,

$$\kappa(x) = m \varepsilon(a)x + n \varepsilon(a).$$

and

$$\beta(x)(a) = \beta(1)(xa) = \beta(1)(\sigma(a)x + \delta(a)) = \varepsilon(\sigma(a))x + \varepsilon(\delta(a))$$

for any $a \in A$. Hence, $\beta$ is left $R$-linear if and only if $\varepsilon \sigma = m \varepsilon$ and $\varepsilon \delta = n \varepsilon$. □

**Corollary 5.3.** $R \subseteq S$ is a Frobenius extension of second kind if and only if there exists a Frobenius functional $\varepsilon : A \to \mathbb{F}$ and $m,n \in \mathbb{F}$ with $m \neq 0$ such that $\varepsilon \sigma = m \varepsilon$ and $\varepsilon \delta = n \varepsilon$.

We can biseparable extensions Frobenius extensions of second kind? The answer is again negative.

**Example 5.4.** (Biseparable extensions are not necessary Frobenius of second kind). By the latter result, Example 5.1 also provides an example of a biseparable extension which is not Frobenius of second kind. Indeed, let $\kappa : \mathbb{F}_2[x] \to \mathbb{F}_2[x]$ be an automorphism. Hence $\kappa(x) = x + n$ for some $n \in \mathbb{F}_2$. The case $n = 0$ is already analyzed in Example 5.1. Therefore, set $\kappa(x) = x + 1$.

By Proposition 5.2, $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta_{\sigma,M}]$ is Frobenius of second kind if and only if there exists a Frobenius functional $\varepsilon : A \to \mathbb{F}_2$ verifying $\varepsilon \sigma = \varepsilon$ and $\varepsilon \delta = \varepsilon$. As reasoned in Example 5.1, $\varepsilon$ is determined by four values $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{F}_2$ such that $\varepsilon(a^i e_j) = \gamma_j$ for any $i = 0, 1, 2$ and $j = 0, 1, 2, 3$. Now,

- If $\gamma_0 = 1$, then $0 = \varepsilon \delta \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \neq \varepsilon \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = 1$,
- If $\gamma_1 = 1$, then $0 = \varepsilon \delta \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \varepsilon \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = 1$,
- If $\gamma_2 = 1$, then $0 = \varepsilon \delta \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \neq \varepsilon \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} = 1$,
- If $\gamma_3 = 1$, then $0 = \varepsilon \delta \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & a^2 \end{pmatrix} = 1$,

so that $\varepsilon \delta = \varepsilon$ if and only if $\varepsilon = 0$. By Corollary 5.3, $\mathbb{F}_2[x] \subseteq A[x; \sigma, \delta_{\sigma,M}]$ is not Frobenius of second kind. Additionally, we may state that the class of Frobenius extensions of second kind is strictly contained in the class of left and right semi Frobenius.

We can formulate the next problem.

**Problem 5.5.** Are biseparable extensions left and right semi Frobenius?

The techniques we have developed in this paper are not suitable to handle this problem. In fact, assume $R \subseteq S$ is biseparable with $\delta = \delta_{\sigma, b}$ inner. Then $\mathbb{F} \subseteq A$ is separable by Proposition 4.5. By Proposition 5.2 or 4. Theorem 4.2, $\mathbb{F} \subseteq A$ is a Frobenius extension, hence $R \subseteq S$ is right and left semi Frobenius by Theorem 5.4.

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