NON-OPTIMAL LEVELS OF A REDUCIBLE MOD $\ell$ MODULAR REPRESENTATION

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ABSTRACT. Let $\ell > 3$ be a prime and $N$ be a square-free integer prime to $\ell$. For each prime $p$ dividing $N$, let $a_p$ be either $1$ or $-1$. We give a sufficient criterion for the existence of a newform $f$ of weight 2 for $\Gamma_0(N)$ such that the mod $\ell$ Galois representation attached to $f$ is reducible and $U_p f = a_p f$ for primes $p$ dividing $N$. The main techniques used are level raising methods based on an exact sequence due to Ribet.

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1. INTRODUCTION

It has been known that newforms for congruence subgroups of $\text{SL}_2(\mathbb{Z})$ give rise to compatible systems of $\ell$-adic Galois representations, and if the $\ell$-adic Galois representations attached to two newforms are isomorphic for some prime $\ell$, then the newforms are, in fact, equal. But the corresponding statement is not true for the semisimplifications of the mod $\ell$ reductions of $\ell$-adic Galois representations attached to newforms, as different newforms can be congruent modulo $\ell$. To study the different levels from which a given modular mod $\ell$ representation $\rho$ can arise is interesting and has been discussed by several mathematicians, Carayol, Diamond, Khare, Mazur, Ribet, and Taylor in the case when $\rho$ is (absolutely) irreducible. (e.g. see [DT94].)

For simplicity, fix a prime $\ell > 3$ and let $f$ be a newform of weight 2 for $\Gamma_0(N)$ with a square-free integer $N$ prime to $\ell$. Assume that $\rho_f$, the semisimplified mod $\ell$ Galois representation attached to $f$, is reducible. Then, $\rho_f \simeq 1 \oplus \chi =: \rho$, where $\chi$ is the mod $\ell$ cyclotomic character (Proposition 2.1). In the sense of Serre [Se87], the optimal level of $\rho$ is 1 because it is unramified outside $\ell$. The main purpose of this paper is to find the possible non-optimal levels of $\rho$ as in the irreducible cases due to Diamond and Taylor [DT94]. Since we consider a newform $f$ of weight 2 and square-free level $N$ with trivial character, an eigenvalue of the Hecke operator $U_p$ of $f$ is either $1$ or $-1$ for a prime $p$ dividing $N$. So, by switching prime factors of the level, we elaborate the above problem as follows.

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curves and Shimura curves. We include some proofs of them for reader’s convenience.

In this paper, we prove the following theorems.

**Theorem 1.2** (Necessary conditions). Assume a $t$-tuple $(p_1, \ldots, p_t)$ for $s$ is admissible and let $N = \prod_{i=1}^t p_i$. Then,

1. $s \geq 1$.
2. If $s = t$, $\ell \mid \phi(N) := \prod_{i=1}^t (p_i - 1)$.
3. $p_j \equiv -1 \pmod{\ell}$ for $s < j \leq t$.

**Theorem 1.3** (Sufficient conditions). Let $N = \prod_{i=1}^t p_i$. Then, a $t$-tuple $(p_1, \ldots, p_t)$ for $s$ is admissible if one of the following holds.

1. $s = t$ is odd and $\ell \mid \phi(N)$.
2. $s + 1 = t$, $s$ is odd, and $p_1 \equiv -1 \pmod{\ell}$.
3. $s = 1$, $t > 1$, and $p_j \equiv -1 \pmod{\ell}$ for $1 < j \leq t$.
4. $s = 2$, $t > 2$ is even, and $p_j \equiv -1 \pmod{\ell}$ for $2 < j \leq t$.
5. $t$ is even, $p_1 \equiv -1 \pmod{\ell}$, and a $(t - 1)$-tuple $(p_1, \ldots, p_{t-1})$ for $s$ is admissible.
6. $s < t$, $t$ is even, and a $(t - 1)$-tuple $(p_2, \ldots, p_t)$ for $(s - 1)$ is admissible.

In most cases, the above necessary conditions are also sufficient for admissibility. On the other hand, there are some cases that the necessary conditions do not guarantee admissibility, for instance, $s = t = 2$.

Using above theorem we can get some results on admissible $t$-tuples for $t \leq 4$. When $(s, t) = (2, 2), (2, 3)$ or $(4, 4)$, we use a different argument to get some sufficient conditions on admissibility.

**Theorem 1.4** (Admissible $t$-tuples for $t \leq 4$). A $t$-tuple $(p_1, \ldots, p_t)$ for $s$ is admissible if

1. $(s, t) = (1, 1)$ if and only if $p_1 \equiv 1 \pmod{\ell}$.
2. $(s, t) = (1, 2)$ if and only if $p_2 \equiv -1 \pmod{\ell}$.
3. $(s, t) = (2, 2)$ if and only if some extra conditions hold (Theorem 2.4).
4. $(s, t) = (1, 3)$ if and only if $p_2 \equiv p_3 \equiv -1 \pmod{\ell}$.
5. $(s, t) = (2, 3)$ if some conditions hold (Theorem 6.1).
6. $(s, t) = (3, 3)$ if and only if $\ell \mid (p_1 - 1)(p_2 - 1)(p_3 - 1)$.
7. $(s, t) = (1, 4)$ if and only if $p_2 \equiv p_3 \equiv p_4 \equiv -1 \pmod{\ell}$.
8. $(s, t) = (2, 4)$ if and only if $p_2 \equiv p_3 \equiv p_4 \equiv -1 \pmod{\ell}$.
9. $(s, t) = (3, 4)$ if and only if $p_4 \equiv -1 \pmod{\ell}$.
10. $(s, t) = (4, 4)$ if some conditions hold (Theorem 6.4).

In [2] we introduce Ribet’s work, which was announced in his CRM lecture [R10]. In [3] we study level raising methods which are main tools of this paper. In [4] we present a complete proof of Ribet’s work on admissible tuples using results in the previous section.

In [5] and [6] we discuss generalization of Ribet’s work and give some examples of admissible triples for $s = 2$ and of admissible quadruples for $s = 4$.

In Appendices, we provide some known results on arithmetic of Jacobian varieties of modular curves and Shimura curves. We include some proofs of them for reader’s convenience.
1.1. **Notation.** Let $B$ be a quaternion algebra over $\mathbb{Q}$ of discriminant $D$ such that $B \otimes \mathbb{Q} \mathbb{R} \simeq M_2(\mathbb{R})$. (Hence, $D$ is the product of the even number of distinct primes.) Let $\mathcal{O}$ be an Eichler order of level $N$ of $B$, and set $\Gamma_0^D(N) = \mathcal{O}^\times$; the set of (reduced) norm 1 elements in $\mathcal{O}$. Let $X_0^D(N)$ be the Shimura curve for $B$ with $\Gamma_0^D(N)$ level structure. Let $J_0^D(N)$ be the Jacobian of $X_0^D(N)$. If $D = 1$, $X_0^D(N) = X_0^1(N)$ denotes the modular curve for $\Gamma_0(N)$ and $J_0(N) = J_0^1(N)$ denotes its Jacobian variety. (Note that if $D \neq 1$, $X_0^D(N)$ may denote by $\mathbb{H}/\Gamma_0^D(N)$, where $\mathbb{H}$ is the complex upper half plane.) By Igusa [Ig59], Deligne-Rapaport [DR93], Cerednik [Ce76], Drinfeld [Dr76], Katz-Mazur [KM83], and Buzzard [Bu97], there is an integral model $X_0^D(N)$. By the theory of Raynaud [Ra70], there is the Néron model of $J_0^D(N)$ over $\mathbb{Z}$, we denote it by $J_0^D(N)/\mathbb{Z}$. We denote by $J_0^D(N)_{/\mathbb{F}_p}$ the special fiber of $J_0^D(N)/\mathbb{Z}$ over $\mathbb{F}_p$. For a Jacobian variety $J$ over $\mathbb{Q}$, we denote by $X_p(J)$ (resp. $\Phi_p(J)$) the character group (resp. the component group) of its special fiber $J_{/\mathbb{F}_p}$ of the Néron model $J_{/\mathbb{Z}}$.

There are Hecke operators $T_p$ acting on $J_0^D(N)$, we denote by $\mathbb{T}_N^D$ the $\mathbb{Z}$-algebra of the endomorphism ring of $J_0^D(N)$ generated by all $T_p$. In the case that $D = 1$ (resp. $N = 1$), we denote by $\mathbb{T}_N$ (resp. $\mathbb{T}_D^1$) the $\mathbb{Z}$-algebra of the $\mathbb{T}$-algebra of the endomorphism ring of $J_0^D(N)$ (resp. $J_0^1(N)$). If $p$ divides $DN$, we denote by $U_p$ the Hecke operator $T_p$ on $J_0^D(N)$. For a prime $p$ dividing $DN$, there is also the Atkin-Lehner involution $w_p$ on $J_0^D(N)$. For a maximal ideal $m$ of a Hecke ring $\mathbb{T}$, we denote by $\mathbb{T}_m$ the localization of $\mathbb{T}$ at $m$, i.e.,

$$\mathbb{T}_m := \lim_{\leftarrow n} \mathbb{T}/m^n.$$ 

There are two degeneracy maps $\alpha_p, \beta_p : X_0^D(Np) \to X_0^D(N)$ for a prime $p$ not dividing $DN$. Here, $\alpha_p$ (resp. $\beta_p$) is the one induced by “forgetting the level $p$ structure” (resp. “dividing by the level $p$ structure”). For any divisor $M$ of $N$, we denote by $J_0^D(M)_M$-new the $M$-new subvariety of $J_0^D(N)$. We also denote by $(\mathbb{T}_N^D)_M$-new the image of $\mathbb{T}_N^D$ in the endomorphism ring of $J_0^D(M)_M$-new. If $M = N$, we define $J_0^D(N)_M$-new := $J_0^D(N)_N$-new and $(\mathbb{T}_N^D)_M$-new := $(\mathbb{T}_N^D)_N$-new. A maximal ideal of $\mathbb{T}_N^D$ is called $M$-new if its image in $(\mathbb{T}_N^D)_M$-new is still maximal.

In this paper, we assume that $\ell > 3$ is a prime and $N$ is a square-free integer prime to $\ell$. For such an integer $N$, we define two arithmetic functions $\phi(N)$ and $\psi(N)$, where $\phi(N) := \prod_{p|N}(p - 1)$ and $\psi(N) := \prod_{p|N}(p + 1)$.

Since we focus on Eisenstein maximal ideals of residue characteristic $\ell > 3$ in this paper, we introduce the following notation for convenience.

**Notation 1.5.** We say that for two natural numbers $a$ and $b$, $a$ is equal to $b$ up to products of powers of 2 and 3 if $a = b \times 2^x3^y$ for some integers $x$ and $y$. For two finite abelian groups $A$ and $B$, we denote by $A \sim B$ if $A_\ell := A \otimes \mathbb{Z}_\ell$, the $\ell$-primary subgroup of $A$, is isomorphic to $B_\ell$ for all primes $\ell \not| 6$.

For a module $X$, $\text{End}(X)$ denotes its endomorphism ring.

We denote by $\chi$ the mod $\ell$ cyclotomic character, i.e.,

$$\chi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{F}_\ell^\times,$$

where $\zeta_\ell$ is a primitive $\ell$-th root of unity. Note that $\chi$ is unramified outside $\ell$ and $\chi(\text{Frob}_p) \equiv p \pmod{\ell}$ for a prime $p \not= \ell$, where $\text{Frob}_p$ denotes an arithmetic Frobenius element for $p$ in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

For an ideal $m$ of $\mathbb{T}$ and a variety $A$ over a field $K$ which is a $\mathbb{T}$-module, $A[m]$ denotes the kernel of $m$ on $A$, i.e.,

$$A[m] := \{ x \in A(\overline{K}) : Tx = 0 \text{ for all } T \in m \}.$$

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2. RIBET’S WORK

In this section, we discuss Ribet’s result on reducible representations arising from modular forms of weight two for \( \Gamma_0(N) \) with a square-free integer \( N \).

2.1. Reducible mod \( \ell \) Galois representations arising from newforms. Let \( \ell > 3 \) be a prime and let \( N \) be a square-free integer prime to \( \ell \). Let \( f \) be a newform of weight 2 for \( \Gamma_0(N) \). Assume that \( \rho_f \), the semisimplified mod \( \ell \) Galois representation associated to \( f \), is reducible. Then, Proposition 2.1 (Ribet). \( \rho_f \) is isomorphic to \( 1 \oplus \chi \), where \( \chi \) is the mod \( \ell \) cyclotomic character.

Proof. Since \( \rho_f \) is reducible, it is the direct sum of two 1-dimensional representations. Let \( \alpha, \beta : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}^\times \) be the corresponding characters, where \( \mathbb{F} \) is some finite field of characteristic \( \ell \). As is well known, the hypothesis that \( N \) is square-free implies that the representation \( \rho_f \) is semistable outside \( \ell \) in the sense that inertia subgroups of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) for primes other than \( \ell \) act unipotently in the representation \( \rho_f \). It follows that \( \alpha \) and \( \beta \) are unramified outside \( \ell \). Accordingly, each of these two characters is some power of \( \chi \). If \( \alpha = \chi^i \) and \( \beta = \chi^j \), the two exponents \( i \) and \( j \) are determined modulo \( (\ell - 1) \) by the restrictions of \( \alpha \) and \( \beta \) to an inertia group for \( \ell \) in \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Using the results of \([\text{Ed92}]\), one sees easily that these exponents can only be 0 and 1 (up to permutation). \( \square \)

2.2. Admissible tuples. Fix a prime \( \ell > 3 \). Fix \( t \), the number of prime factors of \( N \), and \( s \in \{1, \cdots , t\} \), the number of plus signs. (\( s \) might be zero but in the theorem below, we will show that \( s \neq 0 \).

We seek to characterize \( t \)-tuples \((p_1, \cdots , p_t)\) of distinct primes for \( s \) so that there is a newform \( f \) of level \( N = \prod_{i=1}^t p_i \) with weight two and trivial character such that

1. \( \rho_f \simeq 1 \oplus \chi \), where \( \rho_f \) is the semisimplified mod \( \ell \) Galois representation associated to \( f \),
2. \( U_{p_i} f = f \) for \( 1 \leq i \leq s \), and
3. \( U_{p_j} f = -f \) for \( s < j \leq t \).

We call these \( t \)-tuples for \( s \) admissible. When we discuss admissible tuples, we always fix a prime \( \ell > 3 \) and assume that the level \( N \) is prime to \( \ell \).

2.3. Result on admissible tuples. In this subsection, we introduce the work of Ribet on admissible tuples, which was announced in his CRM lecture \([\text{R10}]\). For a proof, see \([\text{R14}]\).

Theorem 2.2 (Ribet). Let a \( t \)-tuple \((p_1, \cdots , p_t)\) for \( s \) be admissible and let \( N = \prod_{i=1}^t p_i \). Then the following hold.

1. \( s \geq 1 \).
2. If \( s = t \), \( \ell \mid \phi(N) \).
3. For \( s < j \leq t \), \( p_j \equiv -1 \) (mod \( \ell \)).

Assume that a \( t \)-tuple \((p_1, \cdots , p_t)\) for \( s \) is admissible. If \( s = t \), then \( \ell \mid \phi(N) \). And if \( s + 1 = t \), then \( p_t \equiv -1 \) (mod \( \ell \)). Ribet proved that this is also a sufficient condition when \( s \) is odd.

Theorem 2.3 (Ribet). Let \( N = \prod_{i=1}^t p_i \). Then, a \( t \)-tuple \((p_1, \cdots , p_t)\) for \( s \) is admissible if one of the following holds.
(1) If \( s = t \) and \( s \) is odd, \( \ell \mid \phi(N) \).
(2) If \( s + 1 = t \) and \( s \) is odd, \( p_t \equiv -1 \pmod{\ell} \).

By the Theorem 2.2 and 2.3, a single \((p)\) for \( s = 1 \) is admissible if and only if \( p \equiv 1 \pmod{\ell} \).

When \( t = 2 \), a pair \((p, q)\) for \( s = 1 \) is admissible if and only if \( q \equiv -1 \pmod{\ell} \). On the other hand, we only have a necessary condition for admissibility of a pair \((p, q)\) for \( s = 2 \) that \( \ell \mid \phi(pq) = (p - 1)(q - 1) \). Without loss of generality, we assume that \( p \equiv 1 \pmod{\ell} \). Then,

**Theorem 2.4 (Ribet).** A pair \((p, q)\) for \( s = 2 \) is admissible if and only if \( q \equiv 1 \pmod{\ell} \) or \( q \) is an \( \ell \)-th power modulo \( p \).

When \( t \) is even, Ribet proved a level raising theorem.

**Theorem 2.5 (Ribet).** Assume that a \((t - 1)\)-tuple \((p_1, \ldots, p_{t-1})\) for \( s \) is admissible and \( t \) is even. Then, a \( t \)-tuple \((p_1, \ldots, p_t)\) for \( s \) is admissible if and only if \( p_t \equiv -1 \pmod{\ell} \).

### 3. Level Raising Methods

In his paper [R84], Ribet studied the kernel of the map

\[
\gamma_p : J_0(N) \times J_0(N) \to J_0(Np)
\]

which is induced by the degeneracy maps. He also computed the intersection of the \( p \)-new subvariety and the \( p \)-old subvariety of \( J_0(Np) \). Diamond and Taylor generalized Ribet’s result [DT94], and they determined non-optimal levels of irreducible mod \( \ell \) modular representations by level raising methods. However, we cannot directly use their methods to find non-optimal levels of \( 1 \oplus \chi \). The reason is basically that the kernels of their level raising maps are “Eisenstein”, and we do not know \( U_p \) actions on the kernels for primes \( p \) dividing the level. Instead, we introduce new level raising methods based on an exact sequence due to Ribet.

#### 3.1. Equivalent condition.

Let \( m \) be a maximal ideal of the Hecke ring \( \mathbb{T}_N \) of residue characteristic \( \ell \). Let \( \rho_m \) be the semisimplified mod \( \ell \) Galois representation associated to \( m \). Assume \( p \) is a prime not dividing \( N \). We call level raising occurs for \( m \) (from level \( N \) to level \( Np \)) if there is a maximal ideal \( n \) of \( \mathbb{T}_{Np} \) such that

1. \( n \) is \( p \)-new, and
2. \( \rho_n \), the semisimplified mod \( \ell \) representation associated to \( n \), is isomorphic to \( \rho_m \).

Let \( \mathbb{T} := \mathbb{T}_{Np} \). Since \( \mathbb{T}_{\text{p-old}} \) is isomorphic to \( \mathbb{T}_N[U_p]/(U_p^2 - T_p U_p + p) \), a maximal ideal \( m \) of \( \mathbb{T}_N \) can be regarded as a maximal ideal of \( \mathbb{T}_{\text{p-old}} \) once we choose a root \( \gamma \) of the polynomial \( X^2 - T_p X + p \pmod{m} \) so that \( U_p - \gamma \in m \). By abusing notation, let \( m \) be a maximal ideal of \( \mathbb{T} \) whose image in \( \mathbb{T}_{\text{p-old}} \) is \( m \). If level raising occurs for \( m \), \( m \) is also \( p \)-new. In other words, the image of \( m \) in \( \mathbb{T}_{\text{p-new}} \) is also maximal.

To detect level raising, Ribet showed that all congruences between \( p \)-new and \( p \)-old forms can be detected geometrically by the intersection between the \( p \)-old and the \( p \)-new parts of the relevant Jacobian.

**Theorem 3.1 (Ribet).** Let \( J := J_0(Np) \). As before, assume that \( Np \) is prime to \( \ell \) and \( p \nmid N \). Let \( m \) be a maximal ideal of \( \mathbb{T}_{Np} \) of residue characteristic \( \ell \) which is \( p \)-old. Then level raising occurs for \( m \) if and only if

\[
J_{\text{p-old}} \bigcap J_{\text{p-new}}[m] \neq 0.
\]

**Proof.** Let \( \Omega := J_{\text{p-old}} \bigcap J_{\text{p-new}} \). If \( \Omega[m] \neq 0 \), \( J_{\text{p-new}}[m] \) is not zero, which implies that \( m \) is \( p \)-new.

Conversely, assume \( \Omega[m] = 0 \). Consider the following exact sequence

\[
0 \longrightarrow \Omega \longrightarrow J_{\text{p-old}} \times J_{\text{p-new}} \longrightarrow J \longrightarrow 0.
\]
Let $e = (1, 0) \in \text{End}(J_{p-old}) \times \text{End}(J_{p-new})$. If $e \not\in \text{End}(J)$, $J \not\cong J_{p-old} \times J_{p-new}$. Thus, $\Omega[m] = 0$, which means that $\mathfrak{m}$ is not in the support of $\Omega$, implies that $e \in \text{End}(J) \otimes_T \mathfrak{T}_m$. Moreover, $e \in T \otimes_Z \mathbb{Q}$ because $\Omega$ is finite. Thus,

$$e \in (T \otimes_Z \mathbb{Q}) \bigcap (\text{End}(J) \otimes_T \mathfrak{T}_m).$$

The intersection $(T \otimes_Z \mathbb{Q}) \bigcap (\text{End}(J) \otimes_T \mathfrak{T}_m)$ is equal to the localization of the saturation of $T$ in $\text{End}(J)$ at $\mathfrak{m}$. Since $T$ is saturated in $\text{End}(J)$ locally at $\mathfrak{m}$ by the theorem of Agashe, Ribet, and Stein [ARS12], $e \in \mathfrak{T}_m$.

If $\mathfrak{m}$ is also a maximal ideal after projection $T \to T_{p-new}$, the injection $T \hookrightarrow T_{p-old} \times T_{p-new}$ is not an isomorphism after localizing at $\mathfrak{m}$. Thus, $e = (1, 0) \in T_{p-old} \times T_{p-new}$ cannot be in $\mathfrak{T}_m$, which is a contradiction. Therefore $\mathfrak{m}$ is not $p$-new. \hfill \Box

**Remark 3.2.** The theorem of Agashe, Ribet, and Stein is as follows,

**Theorem 3.3 (Agashe, Ribet, and Stein).** Let $\ell$ be the characteristic of $\mathbb{T}_N/\mathfrak{m}$. Then, $\mathbb{T}_N$ is saturated in $\text{End}(J_0(N))$ locally at $\mathfrak{m}$ if

1. $\ell \nmid N$, or
2. $\ell \mid N$ and $\ell \equiv \pm 1 \pmod{\mathfrak{m}}$.

In our case, the level $Np$ is prime to $\ell$, so $\mathbb{T}_{Np}$ is saturated in $\text{End}(J_0(Np))$ locally at $\mathfrak{m}$.

When we consider Jacobians of Shimura curves, we don’t have the $q$-expansion principle, so the saturation property of the Hecke algebra is difficult to prove. However, we can prove the following.

**Proposition 3.4.** Let $\mathbb{T} := \mathbb{T}_q^p$, $J := J_0^p(q)$, and $\mathfrak{m} := (\ell, U_p - 1, U_q - 1, U_r + 1, T_s - s - 1 : \text{for primes } s \nmid pqr) \subset \mathbb{T}$. Assume that $p \not\equiv 1 \pmod{\ell}$ and $q \not\equiv 1 \pmod{\ell}$. Then, $\mathbb{T}$ is saturated in $\text{End}(J)$ locally at $\mathfrak{m}$.

**Proof.** It suffices to find a free $\mathfrak{T}_m$-module of finite rank on which $\text{End}(J)$ operates by functoriality (as in the paper [ARS12]).

Let $Y$ (resp. $L$, $X$) be the character group of $J_0^p(q)$ at $r$ (resp. $J_0(pqr)$, $J_0(pq)$ at $p$). By Ribet [R90], there is an exact sequence

$$0 \longrightarrow Y \longrightarrow L \longrightarrow X \oplus X \longrightarrow 0.$$ 

Let $a$ (resp. $b$) be the corresponding Eisenstein ideal to $\mathfrak{m}$ in $\mathbb{T}_{pqr}$ (resp. $\mathbb{T}_{pq}$). Since $p \not\equiv 1 \pmod{\ell}$ and a pair $(p, q)$ for $s = 2$ is not admissible, $b$ is not $p$-new, so $X \oplus X$ does not have support at $b$. (Note that the action of $\mathbb{T}_{pq}$ on $X$ factors through $\mathbb{T}_{pq}$.) Thus, $Y_m \simeq L_a$. Since $L/aL$ is of dimension 1 over $\mathbb{T}_{pqr}/\mathfrak{a}$ by Theorem B.2 and $L$ is of rank 1 over $\mathbb{T}_{pqr}$ in the sense of Mazur [M77], $L_a$ is free of rank 1 over $(\mathbb{T}_{pqr})_a$ by Nakayama’s lemma. So, $Y_m \simeq L_a$ is also a free module of rank 1 over $\mathfrak{T}_m$. \hfill \Box

**Remark 3.5.** Ribet provides the above proof.

Using the above proposition and the proof of Theorem 3.1 we can prove the following theorem.

**Theorem 3.6.** Let $\mathbb{T} := \mathbb{T}_q^p$, $J := J_0^p(q)$, and $\mathfrak{m} := (\ell, U_p - 1, U_q - 1, U_r + 1, T_s - s - 1 : \text{for primes } s \nmid pqr) \subset \mathbb{T}$. Assume that $p \not\equiv 1 \pmod{\ell}$ and $q \not\equiv 1 \pmod{\ell}$. Then, level raising occurs for $\mathfrak{m}$ if and only if

$$J_{p-old} \cap J_{q-new}[\mathfrak{m}] \neq 0.$$
3.2. The intersection of the $p$-old subvariety and the $p$-new subvariety. As in the previous subsection, let $p$ be a prime not dividing $N$. Let $\Omega$ be the intersection of the $p$-old subvariety and the $p$-new subvariety of $J_0(Np)$. By the degeneracy maps, we have the following maps

$$J_0(N) \times J_0(N) \xrightarrow{\gamma_p} J_0(Np) \xrightarrow{\delta_p} J_0(N) \times J_0(N).$$

The composition of the above two maps is the matrix

$$\delta_p := \begin{pmatrix} p+1 & T_p \\ T_p & p+1 \end{pmatrix}.$$

Let $\Delta$ be the kernel of the above composition $\delta_p$, i.e.

$$\Delta := J_0(N)^2[\delta_p] = \{(x, y) \in J_0(N)^2 : (p+1)x = -T_py \text{ and } T_px = -(p+1)y\}.$$

Let $\Sigma$ be the kernel of $\gamma_p$. Then $\Delta$ contains $\Sigma$ and is endowed with a canonical non-degenerate alternating $\mathbb{G}_m$-valued pairing. Let $\Sigma^\perp$ be the orthogonal to $\Sigma$ relative to this pairing. Then, $\Sigma^\perp$ contains $\Sigma$ and we have the formula

$$\Omega = \Sigma^\perp / \Sigma.$$

For more details, see [R84].

We define $\Delta^+$ and $\Delta^-$ as follows.

$$\Delta^+ := \{(x, -x) \in J_0(N)^2 : x \in J_0(N)[T_p - p - 1]\}$$

and

$$\Delta^- := \{(x, x) \in J_0(N)^2 : x \in J_0(N)[T_p + p + 1]\}.$$

They are eigensubspaces of $\Delta$ for the Atkin-Lehner operator $w_p$. ($w_p$ acts on $J_0(N)^2$ by swapping its components.) If we ignore 2-primary subgroups, $\Delta \sim \Delta^+ \oplus \Delta^-$. Furthermore we have the filtrations as follows.

$$0 \subset \Sigma^+ \subset (\Sigma^\perp)^+ \subset \Delta^+$$

and

$$0 \subset \Sigma^- \subset (\Sigma^\perp)^- \subset \Delta^-.$$

Since $\Delta / \Sigma^\perp$ is the $\mathbb{G}_m$-dual of $\Sigma$ and $\Sigma$ is an antidiagonal embedding of the Shimura subgroup of $J_0(N)$ by Ribet [R84], $\Sigma^+ \sim \Sigma$ and $\Sigma^- \sim 0$. Thus,

$$(\Sigma^\perp)^- \sim \Delta^-.$$

By the map $\gamma_p$, $(\Sigma^\perp)^+$ maps to $(\Sigma^\perp)^+/\Sigma$ and $(\Sigma^\perp)^- \sim \Delta^-$ maps to $\Delta^-$ (up to 2-primary subgroups). Since $\Sigma^\perp / \Sigma$ lies in $J_0(Np)p_{\text{new}}$, $U_p + w_p = 0$. Hence, $(\Sigma^\perp)^+ / \Sigma$ (resp. $\Delta^-$) corresponds to the $+1$ (resp. $-1$)-eigenspace of $\Omega$ for the $U_p$ operator (up to 2-primary subgroups).

3.3. Ribet’s exact sequence. Let $X_p(J)$ (resp. $\Phi_p(J)$) be the character group (resp. the component group) of $J$ at $p$. By the degeneracy maps, there is a Hecke equivariant map between component groups

$$\Phi_p(J_0^D(Np)) \times \Phi_p(J_0^D(Np)) \to \Phi_p(J_0^D(Npq)),$$

where $q$ is a prime not dividing $NDp$. Let $K$ (resp. $C$) be the kernel (resp. the cokernel) of the above map. We recall Theorem 4.3 of [R90].

**Theorem 3.7** (Ribet). There is a Hecke equivariant exact sequence

$$0 \longrightarrow K \longrightarrow X \oplus X / \mu_q(X \oplus X) \longrightarrow \Psi \longrightarrow C \longrightarrow 0,$$

where

$$X := X_p(J_0^D(Np)), \quad \Psi := \Phi_q(J_0^{Dpq}(N)), \quad \text{and} \quad \mu_q := \begin{pmatrix} q+1 & T_q \\ T_q & q+1 \end{pmatrix}.$$
If we ignore 2-,3-primary subgroups of \( K \) (resp. \( C \)), it is isomorphic to \( \Phi_p(J_0^D(Np)) \) (resp. \( \mathbb{Z}/(q + 1)\mathbb{Z} \)). For a prime \( \ell \geq 3 \), we denote by \( A_\ell \mathbb{A}_q \otimes \mathbb{Z}_\ell \). We can decompose the above exact sequence into the eigenspaces by the action of the \( U_q \) operator as follows.

+1 eigenspaces : \[
0 \to \Phi_p(J_0^D(Np))_\ell \to (X/(T_q - q + 1)X)_\ell \to \Psi_\ell^+ \to 0.
\]

−1 eigenspaces : \[
0 \to (X/(T_q + q - 1)X)_\ell \to \Psi_\ell^- \to (\mathbb{Z}/(q + 1)\mathbb{Z})_\ell \to 0,
\]

where \( \Psi_\ell^+ \) (resp. \( \Psi_\ell^- \)) denotes +1 (resp. −1) eigenspace for \( U_q \) operator on \( \Psi \). For more details, see [R90] or Appendix A.

4. PROOF OF RIBET’S WORK

Even though Ribet’s work has been explained in many lectures (e.g. [R10]), a complete proof has not been published yet. In this section, we present it based on his idea.

By Mazur’s approach, using ideals in the Hecke algebra \( \mathbb{T}_N \), proving admissibility of a \( t \)-tuple \((p_1, \ldots, p_t)\) for \( s \) is equivalent to showing that a maximal ideal \( m \) is new, where \( m = (\ell, U_{p_i} - 1, U_{p_j} + 1, T_r - r - 1 : 1 \leq i \leq s, s < j \leq t, \) for all primes \( r \nmid N := \prod_{i=1}^t p_i \)).

To prove that \( m \) is new, we seek a \( \mathbb{T}^{new}_N \)-module \( A \) such that \( A[m] \neq 0 \). (Since \( \mathbb{T}^{new}_N \) has 1, \( A[\mathbb{T}^{new}_N] = 0 \).)

**Proof of Theorem 2.2**

(1) When \( s = 0 \) and \( t = 1 \), Mazur proved that a single \((N)\) is not admissible [M77]. Ribet generalized his result to the case \( s = 0 \) and \( t = 2 \) [R08]. Ribet’s method also works for all \( t > 2 \). We present a proof of non-existence of admissible triples for \( s = 0 \) and \( t > 3 \).

Assume a triple \((p, q, r)\) for \( s = 0 \) is admissible and \( f \) is a newform of level \( pqr \) such that

(a) \( \rho_f \simeq 1 \oplus \chi \) and
(b) \( U_kf = -f \) for \( k = p, q, \) and \( r \).

This implies that \( p \equiv q \equiv r \equiv -1 \) (mod \( \ell \)) by (3) below.

Let \( e \) be the normalized Eisenstein series of weight 2 and level 1,

\[
e(\tau) := -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma(n)x^n,
\]

where \( \sigma(n) = \sum_{d|n, d > 0} d \) and \( x = e^{2\pi i \tau} \). The filtration of \( e \) (mod \( \ell \)) is \( \ell + 1 \) (cf. [M77], [Se72], [Sw73]); in other words, \( e \) (mod \( \ell \)) cannot be expressed as a sum of mod \( \ell \) modular forms of weight 2 and level prime to \( \ell \). Raising the level of \( e \), we can get Eisenstein series of weight 2 and level \( pqr \).

**Definition 4.1.** For any modular form \( g \) of level \( N \) and a prime \( p \) which does not divide \( N \),

\[
[p]^+(g)(z) := g(z) - pg(pz) \quad \text{and} \quad [p]^-(g)(z) := g(z) - g(pz).
\]

Recall Proposition 2.8 of [Y14].

**Proposition 4.2.** Let \( g \) be an Eisenstein series of weight 2 and level \( N \) which is an eigenform for all Hecke operators, then \([p]^+(g)\) (resp. \([p]^-(g)\)) is an Eisenstein series of weight 2 and level \( Np \) such that the eigenvalue of \( U_p \) is 1 (resp. \( p \)).
Proof of Theorem 2.3.

For a positive integer \( n \), let \( e_n(\tau) := e(n\tau) \). Let \( P := e + e_p \equiv [p]^{+}(e) \pmod{\ell} \) (resp. \( Q := e + e_q, R := e + e_r \)) be the mod \( \ell \) Eisenstein series of weight 2 and level \( p \) (resp. \( q, r \)). Raising the level, \( e_p + e_{pq} \) (resp. \( e_{pq} + e_{pqr} \)) is a mod \( \ell \) modular form of weight 2 and level \( pq \) (resp. \( pqr \)). Therefore,

\[
e + e_{pqr} = P - (e_p + e_{pq}) + (e_{pq} + e_{pqr})
\]

is also a mod \( \ell \) modular form of weight 2 and level \( pqr \). Let \( E := \sum_{n|pqr} e_n \) and \( F := \sum_{n|pqr} (-1)^{\omega(n)} e_n = [p]^{-} \circ [q]^{-} \circ [r]^{-}(e) \), where \( \omega(n) \) is the number of distinct prime factors of \( n \). Since \( p \equiv q \equiv r \equiv -1 \pmod{\ell} \), \( E \) is congruent modulo \( \ell \) to

\[
[p]^{-} \circ [q]^{-} \circ [r]^{-}(e) = \sum_{n|pqr} (-1)^{\omega(n)} n e_n.
\]

Thus \( E \) and \( F \) are mod \( \ell \) eigenforms for all Hecke operators, and \( E \) is a mod \( \ell \) modular form of weight 2 and level \( pqr \). Moreover,

\[
U_p(E) \equiv U_q(E) \equiv U_r(E) \equiv 1 \pmod{\ell} \quad \text{and} \quad U_p(F) \equiv U_q(F) \equiv U_r(F) \equiv -1 \pmod{\ell}.
\]

Since \( F \) and \( f \) have the same Fourier expansion at \( i\infty \) modulo \( \ell \), by the \( q \)-expansion principle, they are equal modulo \( \ell \). In other words, \( F \) is a mod \( \ell \) modular form of weight 2 and level \( pqr \). However, this is a contradiction because

\[
-8e = E - F - 2P - 2Q - 2R - 2(e + e_{pqr})
\]

is also a mod \( \ell \) modular form of weight 2 and level \( pqr \).

(2) Let \( N = \prod_{i=1}^{\ell} p_i \) and let \( G = [p_1]^{+} \circ \cdots \circ [p_\ell]^{+}(e) \) be an Eisenstein series of level \( N \). Then the constant term of the Fourier expansion of \( G \) at \( i\infty \) is \((-1)^{t+1} \frac{\phi(N)}{24} \). (Proposition 2.13 of [Y14].)

Assume that a \( t \)-tuple \((p_1, \cdots, p_t)\) for \( s = t \) is admissible. Then there is a newform \( f \) of level \( N \) such that \( \rho_f \simeq 1 \oplus \chi \) and \( U_{p_i} f = f \) for \( 1 \leq i \leq t \). Since \( G \) and \( f \) have the same eigensystem modulo a maximal ideal above \( \ell \), the Fourier expansion of \( G - f \) at \( i\infty \) is congruent to \((-1)^{t+1} \frac{\phi(N)}{24} \) modulo a maximal ideal. Hence, \((-1)^{t+1} \frac{\phi(N)}{24} \) is 0 modulo \( \ell \) because we assume that \( \ell > 3 \) and there is no modular form of weight 2 and level \( N \) whose Fourier expansion at \( i\infty \) is constant. (cf. Lemma 5.10 of chapter II of [M77].)

(3) Assume that a \( t \)-tuple \((p_1, \cdots, p_t)\) for \( s \) is admissible. Then, there is a newform \( f = \sum a_n q^n \) of level \( N \) such that \( p_1 f \simeq 1 \oplus \chi, U_{p_i} f = f \) for \( 1 \leq i \leq s \), and \( U_{p_{t-j}} f = -f \) for \( s < j \leq t \). The semisimplification of the local representation \( \rho_f |_{\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \) for a prime divisor \( p \) of \( N \) is \( \epsilon \oplus \epsilon \chi \), where \( \epsilon \) is an unramified quadratic character such that \( \epsilon(Frob_{p}) = a_p \). Thus, for \( s < j \leq t \), \( \epsilon(Frob_{p_j}) = a_{p_j} = -1 \), and

\[
\rho_f(Frob_{p_j}) = 1 + p_j \equiv -1 - p_j \pmod{\ell},
\]

which implies that \( p_j \equiv -1 \pmod{\ell} \).

\[\square\]

Proof of Theorem 2.3

(1) Let \( m := (\ell, U_{p_i} - 1, T_r - r - 1 : 1 \leq i \leq t, \text{ for primes } r \nmid N) \) be an Eisenstein maximal ideal of \( \mathbb{T}_N \). It is enough to show that \( m \) is new.

Assume that \( \ell \nmid \phi(N) \). Let \( p = p_1 \) and \( D = N/p \). Let \( \Phi_p := \Phi_p(J^0_D(p)) \) be the component group of \( J^0_D(p)/\mathbb{F}_p \). Since \( \ell \nmid \phi(N) \), \( \Phi_p[m] \neq 0 \) by Proposition [A3] and [A3]. By Theorem 3.10 of [R90] and the monodromy exact sequence \([A, 1]\) in the Appendix \( A \), the action of the Hecke ring on \( \Phi_p \) factors through \((\mathbb{T}_p^D)^{p\text{-new}}\), so \( m \) is \( p \)-new in \( \mathbb{T}_p^D \).

By the Jacquet-Langlands correspondence, \( m \) is new.
Proof of Theorem 2.4. Let \( \eta \) be an Eisenstein maximal ideal of \( \mathbb{T}_N \). Assume that \( q \equiv -1 \pmod{\ell} \). Let \( p := p_1 \) and \( D := N/pq \) (if \( s = 1 \), set \( D = 1 \)). Since the number of distinct prime factors of \( D \) is even, there are Shimura curves \( X_0^D(p) \), \( X_0^D(pq) \), and \( X_0^{Dpq}(1) \). By the Ribet’s exact sequence in \[3.3\] we have

\[
\Phi_p(J_0^D(p)) \times \Phi_p(J_0^D(p)) \xrightarrow{\gamma_q} \Phi_p(J_0^D(pq)) \quad C \quad 0
\]

and

\[
\Phi_q(J_0^{Dpq}(1)) \xrightarrow{C} 0.
\]

By Corollary \[\text{A.6}\] \( C \) is annihilated by \( n \). Therefore, \( n \) is a proper maximal ideal of \( \mathbb{T}_Dpq \). By the Jacquet-Langlands correspondence, \( n \) is a new maximal ideal of \( \mathbb{T}_{Dpq} \). In other words, the given \( t \)-tuple for \( s \) is admissible.

\[\square\]

Proof of Theorem 2.4. Let \( I := (U_p - 1, T_r - r - 1 : \text{for primes } r \neq p) \) be the Eisenstein ideal of \( \mathbb{T} := \mathbb{T}_p \) and \( m := (\ell, I) \). By Mazur, \( I_m := I \otimes \mathbb{T}_m \) is a principal ideal and it is generated by \( \eta_q := T_q - q - 1 \) for a good prime \( q \) (Proposition 16.6 of Chap II of \[M77\]). Moreover, \( q \) is a good prime if and only if \( q \equiv 1 \pmod{\ell} \) and \( q \) is not an \( \ell \)-th power modulo \( p \) (cf. Theorem 11 of loc. cit.). In other words, the above condition on \( q \) is equivalent to the condition that \( \eta_q \) is not a generator of \( I_m \).

Assume that \( \eta_q \) is not a generator of \( I_m \). By the Ribet’s exact sequence in \[3.3\] we have

\[
0 \quad \Phi_\ell \quad (X/\eta_qX)_\ell \quad (\Psi^\ell)_\ell \quad 0,
\]

where \( \Phi := \Phi_p(J_0(p)) \), \( X := X_p(J_0(p)) \), and \( \Psi := \Phi_q(J_0^{pq}(1)) \). By Mazur, \( \Phi \) is a free module of rank 1 over \( \mathbb{T}/I \) and \( X_m \) is free of rank 1 over \( \mathbb{T}_m \) (§II.11 of loc. cit.). Since \( (X/I \times) \otimes \mathbb{T}_m = X_m/I_mX_m \cong (\mathbb{T}/I)_m \), if \( \eta_q \) is not a generator of \( I_m \), then \( \#(X/\eta_qX)_m > \#(X/I \times)_m = \#(\mathbb{T}/I)_m = \#(\Phi)_m \). In other words, after localizing the above exact sequence at \( m \), \( \Psi^+_n \neq 0 \), where \( n = (\ell, U_p - 1, U_q - 1, T_r - r - 1 : \text{for primes } r \neq pq) \) is an ideal of \( \mathbb{T}_{pq} \) corresponding to \( m \). Thus, \( n \) is maximal. By the Jacquet-Langlands correspondence, \( n \) is new.

Conversely, \( \eta_q \) is a local generator of \( I_m \). Let \( \Omega \) be the intersection of the \( q \)-old subvariety and the \( \eta_q \)-new subvariety of \( J_0(p) \). Let \( \Delta := J_0(p)\delta_{pq}^{2}\delta_{pq}^{r} \) and let \( \Sigma \) be the kernel of \( \gamma_q \) as in \[3.2\] We have a filtration of \( \Delta^+ \)

\[
0 \subseteq \Sigma \subseteq (\Sigma^+)^+ \subseteq \Delta^+
\]

and \( \Delta^+ \) is isomorphic to \( J_0(p)[\eta_q] \). Since \( \eta_q \) is a generator of \( I_m \), \( (\Delta^+)_m \) is isomorphic to \( J_0(p)[I]_m \). By Mazur (loc. cit.), \( J_0(p)[I] \) is free of rank 2 and \( \Sigma \) is free of rank 1 over \( \mathbb{T}/I \). Thus, the m-primary subgroup of \( (\Sigma^+)^+/\Sigma = 0 \) because \( \Delta^+/(\Sigma^+)^+ = \mathbb{G}_m \)-dual of \( \Sigma \). In other words, the m-primary subgroup of \( \Omega \) is 0, i.e., \( m \) is not in the support of \( \Omega \). By Theorem 3.1 \( m \) is not \( q \)-new. In other words, a pair \( (p, q) \) for \( s = 2 \) is not admissible.

Proof of Theorem 2.3. Assume that a \((t - 1)\)-tuple \((p_1, \cdots, p_{t-1})\) for \( s \) is admissible and \( t \) is even.

Let \( p = p_1, D = \prod_{j=2}^{t-1} p_j \), and \( q = p_1 \). (If \( t = 2 \), let \( D = 1 \).) Since the number of prime factors of \( D \) is even, there are Shimura curves \( X_0^D(p) \), \( X_0^D(pq) \), and \( X_0^{Dpq}(1) \). If a \((t-1)\)-tuple \((p_1, \cdots, p_{t-1}, q)\) for \( s \) is admissible, then \( q \equiv -1 \pmod{\ell} \) by Theorem 2.2.

Conversely, assume \( q \equiv -1 \pmod{\ell} \). Since a \((t - 1)\)-tuple \((p_1, \cdots, p_{t-1})\) for \( s \) is admissible, there is a new Eisenstein maximal ideal \( m := (\ell, U_p - 1, U_p + 1, T_r - r - 1 : 1 \leq i \leq s, s < j \leq t - 1, \text{for primes } r \not\mid pD) \) in \( \mathbb{T}_p \). Let \( X := X_p(J_0^D(p)) \) be the character group of \( J_0^D(p)/\mathbb{F}_p \). Then, by the Ribet’s exact sequence in \[3.3\] we have

\[
0 \quad (X/(T_q + q + 1)X)_\ell \quad (\Psi^-)_\ell \quad 0.
\]
Theorem 5.2. Because $q \equiv -1 \pmod{\ell}$, $\ell \in \mathbb{m}$, and $T_q - q - 1 \in \mathbb{m}$, $T_q + q + 1 \in \mathbb{m}$. By the Jacquet-Langlands correspondence and the fact that $(\mathbb{T}_p^D)^{p\text{-new}}$ acts faithfully on $X$, $X/mX \neq 0$. Therefore $(X/(T_q + q + 1)X)_\ell$ has support at $m$, so $\Psi^{-n}[n] \neq 0$, where $n := (\ell, U_{p_1} - 1, U_{p_2} + 1, T_r - r - 1 : 1 \leq i \leq s, s < j \leq t$, for primes $r \nmid Dpq)$ is maximal. In other words, $n$ is maximal. By the Jacquet-Langlands correspondence, $n$ is new.

5. Admissible tuples for $s = 1$ or even $t$

In this section we present new results on admissible tuples for $s = 1$ or even $t$.

Theorem 5.1. Assume $t > s$. A $t$-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible if and only if $p_i \equiv -1 \pmod{\ell}$ for $2 \leq i \leq t$.

Proof. Assume a $t$-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible. Then, by Theorem 2.2, $p_i \equiv -1 \pmod{\ell}$ for $2 \leq i \leq t$.

Conversely, assume $p_i \equiv -1 \pmod{\ell}$ for $2 \leq i \leq t$.

(1) Case 1: Assume that $t$ is odd and a $(t-1)$-tuple $(p_1, \cdots, p_{t-1})$ for $s = 1$ is admissible. Let $p := p_1$, $q := p_2$, $C = \prod_{i=3}^{t-1} p_i$, and $D = \prod_{i=3}^{t-1} p_i$. (If $t = 3$, set $C = 1$.) Let $m := (\ell, U_p - 1, U_{p_1} + 1, T_r - r - 1 : 2 \leq i \leq t - 1$, for primes $r \nmid pD)$ be a new Eisenstein maximal ideal of $\mathbb{T}_{pD}$. Let $n := (\ell, U_p - 1, U_{p_1} + 1, U_{p_2} + 1, T_r - r - 1 : 2 \leq i \leq t - 1$, for primes $r \nmid pD)$ be an Eisenstein maximal ideal of $\mathbb{T}_{pD}$. It suffices to show that $n$ is new.

Since $t$ is odd, there are Shimura curves $X_0^C(p_{p_2})$ and $X_0^Dq(p)$. By the Ribet’s exact sequence in [3, 3] we have

\[
0 \longrightarrow (X/(T_q + q + 1)X)_\ell \longrightarrow \Psi^{-n}[n] \longrightarrow (\mathbb{Z}/(q + 1)\mathbb{Z})_\ell \longrightarrow 0,
\]

where $X := X_{p_2}^C(J_0^C(p_{p_2}))$ and $\Psi := \Phi_q(J_0^D(p))$. Since $(\mathbb{T}_{pD}^{p\text{-new}})^{\text{new}}$ acts faithfully on $X$ and $m$ is new, $X/mX \neq 0$. Moreover, $T_q + q + 1 \in m$ because $q \equiv -1 \pmod{\ell}$, $\ell \in m$, and $T_q - q - 1 \in m$. Thus, $m$ is in the support of $(X/(T_q + q + 1)X)_\ell$, so $n$ is also in the support of $\Psi^{-n}[n]$. Therefore $m$ is a proper maximal ideal of $\mathbb{T}_{pD}$. If $n$ is $p$-old, then there is a new maximal ideal $(\ell, U_p, T_r - r - 1 : 2 \leq i \leq t$, for primes $r \nmid D)$ of $\mathbb{T}_{Dq}$ by the Jacquet-Langlands correspondence. In other words, a $(t-1)$-tuple $(p_2, \cdots, p_t)$ for $s = 0$ is admissible, which contradicts Theorem 2.2. Thus, $n$ is $p$-new, so by the Jacquet-Langlands correspondence $n$ is a new maximal of $\mathbb{T}_{pDq}$.

(2) Case 2: Assume that $t$ is even and a $(t-1)$-tuple $(p_1, \cdots, p_{t-1})$ for $s = 1$ is admissible.

Then, since $p_1 \equiv -1 \pmod{\ell}$, by Theorem 2.3, a $t$-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible.

When $t = 2$, by Theorem 2.3, a pair $(p_1, p_2)$ for $s = 1$ is admissible. Thus, by induction on $t$, a $t$-tuple $(p_1, \cdots, p_t)$ for $s = 1$ is admissible for all $t \geq 2$.

Using the same method as above, we can prove the following level raising theorem, which is almost complement of the case in Theorem 2.5 when $t$ is even. (This excludes the case $s = t$ only.)

Theorem 5.2. Assume $t$ is even and $t > s$. And assume that a $(t-1)$-tuple $(p_2, \cdots, p_t)$ for $(s-1)$ is admissible. Then, a $t$-tuple $(p_1, \cdots, p_t)$ is admissible for $s$.

In contrast to Theorem 2.3, there is no condition on $p_1$ for raising the level.

Proof. Let $m := (\ell, U_{p_1} - 1, U_{p_2} + 1, T_r - r - 1 : 2 \leq i \leq s, s < j \leq t$, for primes $r \nmid pD)$ be an Eisenstein maximal ideal of $\mathbb{T}_{pD}$, where $D := \prod_{k=2}^{t-1} p_k$ and $p := p_t$. By our assumption, $m$
is new. Let \( n := (\ell, U_{p_1} - 1, U_{p_2} + 1, T_s - r - 1 : 1 \leq i \leq s, s < j \leq t, \) for primes \( r \nmid Dpq \) be an Eisenstein maximal ideal of \( \mathbb{T}_{Dpq} \). It suffices to show that \( n \) is new.

Since \( t \) is even, there are Shimura curves \( X_0^D(p) \) and \( X_0^{Dpq}(1) \). By the Ribet’s exact sequence in [3.3] we have

\[
0 \longrightarrow \Phi \longrightarrow (X/(T_q - q - 1)X)_\ell \longrightarrow \Psi^+ \longrightarrow 0,
\]

where \( \Phi := \Phi_p(J_0^D(p)), X := X_p(J_0^D(p)) \), and \( \Psi := \Phi_q(J_0^{Dpq}(1)) \). Since \( m \) is new and \( (T_p^D)^{p\text{-new}} \) acts faithfully on \( X, X/mX \neq 0 \). Because \( T_q - q - 1 \in m, m \) lies in the support of \( (X/(T_q - q - 1)X)_\ell \). Since \( m \) is not in the support of \( \Phi_\ell \) by Proposition A.2 \( n \) is in the support of \( \Psi^+_\ell \). Thus, \( n \) is a maximal ideal of \( \mathbb{T}_{Dpq} \). By the Jacquet-Langlands correspondence, \( n \) is new.

**Corollary 5.3.** Assume \( t > 2 \) is even. Then, a \( t \)-tuple \((p_1, \cdots, p_t)\) for \( s = 2 \) is admissible if and only if \( p_i \equiv -1 \) (mod \( \ell \)) for \( 3 \leq i \leq t \).

**Proof.** If a \( t \)-tuple is admissible, then by Theorem 2.2 \( p_i \equiv -1 \) (mod \( \ell \)) for \( 3 \leq i \leq t \).

Conversely, assume that \( p_i \equiv -1 \) (mod \( \ell \)) for \( 3 \leq i \leq t \). By Theorem 5.1 a \((t-1)\)-tuple \((p_2, \cdots, p_t)\) for \( s = 1 \) is admissible. Thus, a \( t \)-tuple \((p_1, \cdots, p_t)\) for \( s = 2 \) is admissible. \( \square \)

### 6. Admissible Triples and Quadruples

In this section, we classify admissible triples and quadruples.

#### 6.1. Admissible Triples

By Theorem 2.2 and 2.3 a triple \((p, q, r)\) for \( s = 3 \) is admissible if and only if \( p | \phi(pqr) \). And by Theorem 5.1 a triple \((p, q, r)\) for \( s = 1 \) is admissible if and only if \( q \equiv r \equiv -1 \) (mod \( \ell \)).

However if \( s = 2 \), we cannot directly use above theorems to get admissible triples. By Theorem 2.2 if \((p, q, r)\) for \( s = 2 \) is admissible, then \( r \equiv -1 \) (mod \( \ell \)). Assume that \( r \equiv -1 \) (mod \( \ell \)). Let \( I := (U_p - 1, U_q + 1, T_s - s - 1 : \) for primes \( s \nmid pr \) be an Eisenstein ideal of \( \mathbb{T} := \mathbb{T}_{pr} \) and \( m := (\ell, I) \). Since \( r \equiv -1 \) (mod \( \ell \)), \( m \) is new maximal at level \( pr \). We want to understand admissibility of a triple \((p, q, r)\) for \( s = 2 \) by a level raising method.

**Theorem 6.1.** Assume \( p \neq 1 \) (mod \( \ell \)) and if \( q \equiv 1 \) (mod \( \ell \)), assume further that \( p \) is not an \( \ell \)-th power modulo \( q \). Let \( \eta_q := T_q - q - 1 \). Then, a triple \((p, q, r)\) for \( s = 2 \) is admissible if \( \eta_q \) is not a generator of \( I_m \).

Assume further that \( q \equiv 1 \) (mod \( \ell \)) and \( r \equiv -1 \) (mod \( \ell^2 \)). Then, a triple \((p, q, r)\) for \( s = 2 \) is not admissible if \( \eta_q \) is a generator of \( I_m \).

**Proof.** By the Ribet’s exact sequence in [3.3] we have

\[
0 \longrightarrow \Phi \longrightarrow (X/\eta_q X)_\ell \longrightarrow \Psi^+ \longrightarrow 0,
\]

where \( \Phi := \Phi_p(J_0^D(pr)), X := X_p(J_0^D(pr)), \) and \( \Psi := \Phi_q(J_0^{Dpq}(r)) \). By the Appendix A, \#\( \Phi_m = \ell^n \), where \( \ell^n \) is the power of \( \ell \) exactly dividing \( r + 1 \). By the result in [Y14], \( X_m \) is free of rank 1 over \( \mathbb{T}_m \) and \( (\mathbb{T}/I)_m \simeq \mathbb{Z}/\ell^n \mathbb{Z} \). Assume \( \eta_q \) is not a generator of \( I_m \). Then,

\[
\#(X/\eta_q X)_m > \#(X/I X)_m = \#(\mathbb{T}/I)_m = \ell^n.
\]

Thus, \( \Psi^+_m \neq 0 \), where \( n := (\ell, U_p - 1, U_q - 1, U_r + 1, T_s - s - 1 : \) for primes \( w \nmid pqr \) \( \subset \mathbb{T}_{pr}^+ \). In other words, \( n \) is maximal. If it is \( r \)-old, then a pair \((p, q)\) for \( s = 2 \) is admissible, which contradicts our assumption. Therefore, \( n \) is \( r \)-new, and by the Jacquet-Langlands correspondence, \( n \) is new.

Assume further that \( q \equiv 1 \) (mod \( \ell \)), \( r \equiv -1 \) (mod \( \ell^2 \)), and \( \eta_q \) is a local generator of \( I \), i.e., \( I_m = (\eta_q) = m_m \).
Let $K$ be the kernel of the map $\Gamma_0^p(1) \times J_0^p(1) \to J_0^p(\ell q)$ by degeneracy maps. Then, as in [3.2] we have

$$0 \subseteq K_m^+ \subseteq (K^+)_{m} \subseteq \Gamma_0^p(1)[\eta_q]_m.$$  

Since by Proposition C.4, $K^+$ contains the Skorobogatov subgroup of $\Gamma_0^p(1)$ at $r$, which is of order $r + 1$ up to products of powers of 2 and 3, $\# K_m^+ \geq \ell$. Since $(\eta_q) = I_m = m_m$, $\Gamma_0^p(1)[\eta_q]_m = \Gamma_0^p(1)[m]$ and it is of dimension 2 over $\mathbb{T}_\ell/m \simeq \mathbb{F}_\ell$ by Theorem 3.3. Because $\Gamma_0^p(1)[\eta_q]/(K^+) = \mathbb{G}_m$-dual of $K^+$, $(K^+/K)_m = 0$. Since $K^+/K$ is isomorphic to the intersection of the $q$-old subvariety and the $q$-new subvariety of $\Gamma_0^p(q)$ and $m$ is not in the support of $K^+/K$, by Theorem 3.6, level raising does not occur. Thus, $n$ is not $q$-new.

6.1.1. Examples. When $N$ is prime, $\eta_q$ is a local generator of an Eisenstein ideal $I = (T_r - r - 1 : \text{for primes } r \neq N) \subset \mathbb{T}_N$ at $m := (\ell, I)$ if and only if $q \equiv 1$ (mod $\ell$) and $q$ is not an $\ell$-th power modulo $N$. However, when $N$ is composite, we don’t know what congruence implies local generation.

Consider the easiest case. As in Theorem 6.1, we assume that $r \equiv -1$ (mod $\ell$) and $p \neq 1$ (mod $\ell$). Assume further that $r \equiv -1$ (mod $\ell^2$). In this case, $I_m = m_m$. Let $f(z) = \sum a_nz^n$ be a newform of weight 2 for $\Gamma_0(pr)$ whose mod $\ell$ Galois representation is reducible such that $a_p = 1$, $a_r = -1$, where $x = e^{2\pi i r}$. If $a_q \equiv q + 1$ (mod $\ell^2$), $\eta_q := T_q - q - 1 \in m_m^2$, so $\eta_q$ is not a local generator of $I$ at $m$.

Moreover, in our examples below, all newforms are defined over $\mathbb{Q}$, i.e., $\mathbb{T}_{pr} = \mathbb{Z}$ and $m = \ell\mathbb{Z}$. Thus, $\eta_q$ is not a local generator if and only if $\eta_q \equiv 0$ (mod $\ell^2$). In the examples below, we follow the notation in Stein's table [50].

1. Admissibility of $(2, q, 19)$ for $s = 2$ when $\ell = 5$.

A newform $f$ of level $pr = 38$ (as above) is $E[38, 2]$. Let $a_n$ be the eigenvalue of $T_n$ for $E[38, 2]$. Then $a_q \equiv 1 + q$ (mod 25) when $q = 23, 41, 97, 101, 109, 113, 149, 151, 193, 199, 239, 241, 251, 257, 277, 347, 359, 431, 479$ for primes $q < 500$. Since only $(2, 151), (2, 241), (2, 251), (2, 431)$ for $s = 2$ are admissible, a triple $(2, q, 19)$ for $s = 2$ is admissible if

$$q = 23, 41, 97, 101, 109, 113, 149, 151, 193, 199, 239, 257, 277, 347, 359, \text{ or } 479$$

for $q < 500$.

Remark 6.2. A newform of level $2 \times 23 \times 19$ with $a_2 = a_{23} = 1$ and $a_{19} = -1$ is $E[874, 8]$.

2. Admissibility of $(3, q, 19)$ for $s = 2$ when $\ell = 5$.

A newform $f$ of level $pr = 57$ (as above) is $E[57, 3]$. Let $b_n$ be the eigenvalue of $T_n$ for $E[57, 3]$. Then $b_q \equiv 1 + q$ (mod 25) when $q = 41, 97, 101, 167, 197, 251, 257, 269, 313, 349, 409, 419, 431$, and 491 for primes $q < 500$. Since only $(3, 41), (3, 431)$, and $(3, 491)$ for $s = 2$ are admissible, a triple $(3, q, 19)$ for $s = 2$ is admissible if

$$q = 97, 101, 167, 197, 251, 257, 269, 313, 349, 409, \text{ or } 419$$

for $q < 500$.

3. Admissibility of $(2, q, 29)$ for $s = 2$ when $\ell = 5$.

A newform $f$ of level $pr = 58$ (as above) is $E[58, 1]$. Let $c_n$ be the eigenvalue of $T_n$ for $E[58, 1]$. Then $c_q \equiv 1 + q$ (mod 25) when $q = 89, 97, 137, 151, 181, 191, 223, 241, 251, 347, 367, 401, 431, 433$, and 491 for primes $q < 500$. Since only $(2, 151), (2, 241), (2, 251), \text{ and } (2, 431)$ for $s = 2$ are admissible, a triple $(2, q, 29)$ for $s = 2$ is admissible if

$$q = 89, 97, 137, 151, 181, 191, 223, 347, 367, 401, 433, \text{ or } 491$$

for $q < 500$.  


(4) Admissibility of \((2, q, 13)\) for \(s = 2\) when \(\ell = 7\).

A newform \(f\) of level \(pr = 26\) (as above) is \(E[26, 2]\). Let \(d_q\) be the eigenvalue of \(T_q\) for \(E[26, 2]\). Then \(d_q \equiv 1 + q \pmod{49}\) when \(q = 43, 101, 223, 229, 233, 269, 307, 311,\) and 349 for primes \(q < 500\). Since a pair \((2, q)\) for \(s = 2\) is not admissible for \(q < 500\), a triple \((2, q, 13)\) for \(s = 2\) is admissible if

\[ q = 43, 101, 223, 229, 233, 269, 307, 311,\) or 349

for \(q < 500\).

Remark 6.3. In the last case, a pair \((2, q)\) for \(s = 2\) is admissible when \(q = 631\) and \(q = 673\). As before,

\[ d_{631} \equiv 1 + 631 \pmod{49} \text{ and } d_{691} \equiv 1 + 691 \pmod{49}. \]

In other words, computations tells that if a pair \((p, q)\) for \(s = 2\) is admissible then \(\eta_q := T_q - q - 1\) is not a local generator of \(I\) at \(m\).

6.2. Admissible quadruples. By Theorem 5.1 a quadruple \((p, q, r, w)\) for \(s = 1\) is admissible if and only if \(q \equiv r \equiv w \equiv -1 \pmod{\ell}\). And by Corollary 5.3 a quadruple \((p, q, r, w)\) for \(s = 2\) is admissible if and only if \(r \equiv w \equiv -1 \pmod{\ell}\). Moreover, by Theorem 2.2 and 2.3 a quadruple \((p, q, r, w)\) for \(s = 3\) is admissible if and only if \(w \equiv -1 \pmod{\ell}\).

Even though we don’t know what are the necessary and sufficient conditions for admissibility of quadruples for \(s = 4\), we can mimic the strategy for the case \((s, t) = (2, 3)\).

By Theorem 2.2, a quadruple \((p, q, r, w)\) for \(s = 4\) is admissible only if \(\ell \mid \phi(pqrw)\). So, without loss of generality, assume that \(p \equiv 1 \pmod{\ell}\). Assume further that \(\ell^2 \nmid \phi(pqr)\). Let \(I := (U_p - 1, U_q - 1, U_r - 1, T_k - k - 1 : \text{ for primes } k \nmid pqr)\) be an Eisenstein ideal of \(T := T_{pqr}\) and \(m := (\ell, I)\). By Theorem 2.3 \(m\) is new.

Theorem 6.4. A quadruple \((p, q, r, w)\) for \(s = 4\) is admissible if \(\eta_w := T_w - w - 1\) is not a local generator of \(I\) at \(m\).

Before we prove the above theorem, we need some lemmas about the character group and the component group of \(J_0^\eta(p)\) over \(\mathbb{F}_p\).

Lemma 6.5. Let \(\Phi := \Phi_p(J_0^{\eta}(p))\). Then, the order of \(\Phi\) is \(\phi(pqr)\) up to products of powers of 2 and 3 and \(\Phi \otimes T_m \simeq \mathbb{Z}/\ell\mathbb{Z}\).

Proof. This follows from Proposition A.2 and A.3.

Lemma 6.6. Let \(X := X_p(J_0^{\eta}(p))\). Then, \(#(X/IX) \otimes T_m \geq #(T/I) \otimes T_m = \ell\).

Proof. Since \(X\) is a \((T_{pqr})^{p_{\text{new}}\text{-mod}}\text{-module of rank 1 in the sense of Mazur (§II.6 in [M77])}, \)# \((X/IX) \otimes (T_{pqr})^{p_{\text{new}}} \geq #(T_{pqr}/I) \otimes (T_{pqr})^{p_{\text{new}}}\). By the Janquet-Langlands correspondence, \((T_{pqr})^{p_{\text{new}}} \simeq T_{\text{new}}\) and \((T_{pqr})^{p_{\text{new}}} \simeq T_{\text{new}}\). Since \(I_m = m_n\) from the assumption \(\ell^2 \nmid \phi(pqr)\), \((T_{pqr}/I) \otimes (T_{pqr})^{p_{\text{new}}} \simeq (T_{\text{new}}/m) \otimes T_{\text{new}} \simeq \mathbb{F}_\ell\). Thus, the result follows.

Proof of Theorem 6.4. It suffices to show that an ideal \(n := (\ell, U_p - 1, U_q - 1, U_r - 1, U_w - 1, T_k - k - 1 : \text{ for primes } k \nmid pqrw)\) of \(T_{pqr}\), is new. By the Ribet’s exact sequence in §3.3 we have

\[
\begin{array}{c}
0 \rightarrow \Phi_{\ell} \rightarrow (X/\eta_w X)_{\ell} \rightarrow \Psi_{\ell}^+ \rightarrow 0,
\end{array}
\]

where \(\Psi := \Phi_w(J_0^{pqrw}(1))\). Assume that \(\eta_w\) is not a local generator of \(I\) at \(m\). Since \(\eta_w\) is not a local generator of \(I\) at \(m\), We have

\[ #(X/\eta_w X) \otimes T_m > #(X/IX) \otimes T_m \geq \ell = \Phi \otimes T_m. \]

Thus, \(n\) is in the support of \(\Psi\), which means that \(n\) is a maximal ideal of \(T_{pqr}\). By the Janquet-Langlands correspondence, \(n\) is new.
6.2.1. Examples. Consider the easiest case. As in Theorem 6.4, we assume that \( p \equiv 1 \pmod{\ell} \) and \( \ell \not\mid (q - 1)(r - 1) \). Assume further that \( p \not\equiv 1 \pmod{\ell^2} \). In this case, \( J_m = m_m \).

Let \( f(\tau) = \sum a_n \tau^n \) be a newform of weight 2 for \( \Gamma_0(pqr) \) whose mod \( \ell \) Galois representation is reducible such that \( a_p = a_q = a_r = 1 \), where \( x = e^{2\pi ir} \). If \( a_w \equiv w + 1 \pmod{m^2} \), \( \eta_w := T_w - w - 1 \in \mathbb{m}^2 \), so \( \eta_w \) is not a local generator of \( I \) at \( m \).

Moreover, in our examples below, all newforms are defined over \( \mathbb{Q} \), i.e., \( \mathcal{T}_{pqr} = \mathbb{Z} \) and \( m = \ell \mathbb{Z} \). Thus, \( \eta_w \) is not a local generator if and only if \( \eta_w \equiv 0 \pmod{\ell^2} \). In the examples below, we follow the notation in Stein’s table \( \text{[St]} \).

1. Admissibility of \((11, 2, 3, w)\) for \( s = 4 \) when \( \ell = 5 \).
   A newform \( f \) of level \( pqr = 66 \) (as above) is \( E[66, 2] \). Let \( a_n \) be the eigenvalue of \( T_n \) for \( E[66, 2] \). Then \( a_w \equiv 1 + w \pmod{25} \) when \( w = 47, 53, 97, 101, 103, 127, 115, 71, 271, 307, 317, \) and 431 for primes \( w < 500 \). Thus, a quadruple \((11, 2, 3, w)\) for \( s = 4 \) is admissible if
   \[
   w = 47, 53, 97, 47, 53, 97, 101, 103, 127, 151, 271, 307, 317, \text{ or } 431
   \]
   for \( w < 500 \).

2. Admissibility of \((31, 2, 3, w)\) for \( s = 4 \) when \( \ell = 5 \).
   A newform \( f \) of level \( pqr = 186 \) (as above) is \( E[186, 3] \). Let \( b_n \) be the eigenvalue of \( T_n \) for \( E[186, 3] \). Then \( b_w \equiv 1 + w \pmod{25} \) when \( w = 19, 43, 59, 67, 71, 101, 109, 113, 131, 157, 181, 191, 227, 281, 283, 307, 331, 349, 359, 421, 431, \) and 443 for primes \( w < 500 \). Thus, a quadruple \((31, 2, 3, w)\) for \( s = 4 \) is admissible if
   \[
   w = 19, 43, 59, 67, 71, 101, 109, 113, 131, 157, 181, 191, 227, 281, 283, 307, 331, 349, 359, 421, 431, \text{ or } 443
   \]
   for \( w < 500 \).

Appendix A. The Component Group of \( J^0_p(Np) \) over \( \mathbb{F}_p \)

In their paper \text{[DR93]}, Deligne and Rapoport studied integral models of modular curves. Buzzard extended their result to the case of Shimura curves \text{[Bu97]}. In this appendix, we explain the special fiber of \( J^0_p(Np) \) over \( \mathbb{F}_p \) for a prime \( p \nmid DN \) and the Hecke actions on its component group. Assume that \( N \) is a square-free integer prime to \( D \).

A.1. The special fiber \( J^0_p(Np) \) over \( \mathbb{F}_p \).

Proposition A.1 (Deligne-Rapoport model). \( X^0_p(Np) / \mathbb{F}_p \) consists of two copies of \( X^0_p(N) / \mathbb{F}_p \). They meet transversally at supersingular points.

Let \( S \) be the set of supersingular points of \( X^0_p(Np) / \mathbb{F}_p \). Then \( S \) is isomorphic to the set of isomorphism classes of right ideals of an Eichler order of level \( N \) of the definite quaternion algebra over \( \mathbb{Q} \) of discriminant \( Dp \) (cf. \text{[R90]}). By the theory of Raynaud \text{[Ra70]}, we have the special fiber \( J^0_p(Np) / \mathbb{F}_p \) of the Néron model of \( J^0_p(Np) / \mathbb{Z} \) at \( p \). It satisfies the following exact sequence
\[
0 \longrightarrow J^0 \longrightarrow J^0_p(Np) / \mathbb{F}_p \longrightarrow \Phi_p(J^0_p(Np)) \longrightarrow 0,
\]
where \( J^0 \) is the identity component and \( \Phi_p(J^0_p(Np)) \) is the component group. Moreover, \( J^0 \) is an extension of \( J^0_p(N) \times J^0_p(N) / \mathbb{F}_p \) by \( T \), the torus of \( J^0_p(Np) / \mathbb{F}_p \). The Cartier dual of \( T \), \( \text{Hom}(T, \mathbb{G}_m) \), is called the character group \( X := X_p(J^0_p(Np)) \). It is isomorphic to the group of degree 0 elements in the free abelian group \( \mathbb{Z}^S \) generated by the elements of \( S \). (Note that, the degree of an element in \( \mathbb{Z}^S \) is the sum of its coefficients.) There is a natural pairing of \( \mathbb{Z}^S \) such that
\[
\langle s, t \rangle := \frac{|\text{Aut}(s)|}{2} \delta_{st},
\]
for any \( s, t \in S \).
where $\delta_s$ is the Kronecker $\delta$-function. This pairing induces an injection $X \hookrightarrow \text{Hom}(X, \mathbb{Z})$ and the cokernel of it is isomorphic to $\Phi_p(J_0^D(Np))$ by Grothendieck [Gr72]. We call the following exact sequence the monodromy exact sequence

(A.1) \[ 0 \xrightarrow{} X \xrightarrow{i} \text{Hom}(X, \mathbb{Z}) \xrightarrow{} \Phi_p(J_0^D(Np)) \xrightarrow{} 0. \]

For more details, see [R90].

A.2. Hecke actions on $\Phi_p(J_0^D(Np))$. By the Proposition 3.8 of [R90], the Frobenius automorphism $\text{Frob}_p$ on $X$ is equal to the operator $T_p$ on it. $\text{Frob}_p$ sends $s \in S$ to some other $s' \in S$, or might fix $s$. For elements $s, t \in S$ the above map $i$ sends $s - t$ to $\phi_s - \phi_t$, where

$$
\phi_s(x) := \langle s, x \rangle \quad \text{for any } x \in S.
$$

Thus in the group $\Phi_p(J_0^D(Np))$, $\phi_s = \phi_t$ for any $s, t \in S$. Since for all $s \in S$, the elements $\frac{2}{\#\text{Aut}(s)}\phi_s$ generate $\text{Hom}(X, \mathbb{Z})$ and $\#\text{Aut}(s)$ is a divisor of 12, $\Phi_p(J_0^D(Np))$ is isomorphic to the cyclic subgroup generated by the image of $\phi_s$ for some $s \in S$. (cf. Proposition 3.2 of [R90].)

**Proposition A.2.** For a prime divisor $r$ of $Dp$ (resp. $N$), $U_r - 1$ (resp. $U_r - r$) annihilates $\Phi$. Moreover, for a prime $r$ not dividing $DNp$, $T_r - r - 1$ annihilates $\Phi$.

**Proof.** On $\Phi$, $\phi_s = \phi_t$. Thus, $U_p(\phi_s) = \phi_t = \phi_s$, where $t = \text{Frob}_p(s)$. Since $S$ is isomorphic to the set of isomorphism classes of right ideals on an Eichler order of level $N$ in the definite quaternion algebra over $\mathbb{Q}$ of discriminant $Dp$, the set of supersingular points of $X_0^{Dp/q}(Nq)_{/\mathbb{F}_q}$ is again $S$ for a prime $q | D$. In other words, the character group of $J_0^{Dp/q}(Nq)_{/\mathbb{F}_q}$ does not depend on the choice of a prime divisor $q$ of $Dp$. (Hence the same is true for the component group by the monodromy exact sequence.) Using the same description as above, we have $U_r(\phi_s) = \phi_s$ for a prime $r$ not dividing $DNp$.

Since the degree of the map $U_r$ is $r$ for a prime divisor $r$ of $N$, $U_r(\phi_s) = \sum a_i \phi_{s_i}$ and $\sum a_i = r$. Therefore $U_r(\phi_s) = r \phi_s$ because in $\Phi$, $\phi_s = \phi_{s_i}$ for all $i$. Similarly, $T_r(\phi_s) = (r + 1)\phi_s$ for a prime $r$ not dividing $DNp$. \hfill \square

A.3. The order of $\Phi$.

**Proposition A.3.** The order of $\Phi$ is equal to $\phi(Dp)\psi(N)$ up to products of powers of 2 and 3.

**Proof.** Let $n$ be the order of $\Phi$. Therefore, for any degree 0 divisor $t = \sum a_i s_i$, $n\phi_s(t) = 0$. We decompose $n$ as a sum $\sum n_i$ for non-negative integers $n_i$. Then,

$$
n\phi_s(t) = \left( \sum n_j \phi_s \right) \left( \sum a_i s_i \right) = \sum n_j \left( \phi_{s_j} \left( \sum a_i s_i \right) \right) = \sum n_j a_j \frac{\#\text{Aut}(s_j)}{2} = 0.
$$

Therefore, for any $i \neq j$, we have $n_i \frac{\#\text{Aut}(s_i)}{2} = n_j \frac{\#\text{Aut}(s_j)}{2} = c$ by taking $t = s_i - s_j$. Since $n > 0$, each $n_i$ is positive and it is equal to $\frac{2c}{\#\text{Aut}(s_i)}$, where $2c$ is the smallest positive integer which makes $\frac{2c}{\#\text{Aut}(s_i)}$ an integer for all $i$. Since $\#\text{Aut}(s_i)$ divides 12, 2$c$ is a divisor of 12. Thus,

$$
n = \sum_{s \in S} \frac{12}{\#\text{Aut}(s)}
$$

up to products of powers of 2 and 3.

Recall Eichler’s mass formula. (cf. Corollary 5.2.3 of [Vi80].)
Proposition A.4 (mass formula). Let $S$ be the set of isomorphism classes of right ideals of an Eichler order of level $N$ in a definite quaternion algebra of discriminant $Dp$ over a number field $K$. Then,
\[
\sum_{s \in S} \frac{\# R^s}{\# \text{Aut}(s)} = 2^{1-d} \times |\zeta_K(-1)| \times h_K \times \phi(Dp) \psi(N),
\]
where $R$ is the ring of integers of $K$, $\zeta_K$ is the Dedekind zeta function of $K$, $d$ is the degree of $K$ over $\mathbb{Q}$, and $h_K$ is the class number of $K$.

In our case $K = \mathbb{Q}$, so $|\zeta_K(-1)| = \frac{1}{12}$, $d_K = h_K = 1$, and $\# R^s = 2$. Thus, the result follows.

A.4. Degeneracy maps between component groups. Let $q$ be a prime not dividing $DNp$. Let $\Phi$ (resp. $\Phi'$) be the cyclic subgroup of the component group of $J^D_0(Np)$ (resp. $J^D_0(Npq)$) at $p$ generated by the image of $\phi_s$ for some $s$ as above.

Let $\gamma_q : J^D_0(Np) \times J^D_0(Np) \to J^D_0(Npq)$ be the map defined by $\gamma_q(a, b) = \alpha_q(a) + \beta_q(b)$, where $\alpha_q, \beta_q$ are the two degeneracy maps $X^D_0(Npq) \to X^D_0(Np)$. Then, $\gamma_q$ induces a map $\gamma : \Phi \times \Phi \to \Phi'$.

Proposition A.5. Let $K$ (resp. $C$) be the kernel (resp. cokernel) of $\gamma$.
\[
0 \to K \to \Phi \times \Phi \to \Phi' \to C \to 0.
\]
Then, $K \sim \Phi$ and $C \simeq \mathbb{Z}/(q + 1)\mathbb{Z}$.

Proof. Let $S$ (resp. $S'$) be the set of supersingular points of $X^D_0(Np)/\mathbb{F}_p$ (resp. $X^D_0(Npq)/\mathbb{F}_p$). Let $\Phi = \langle \phi_s \rangle$ (resp. $\Phi' = \langle \phi_t \rangle$) for some $s \in S$ (resp. $t \in S'$). Since the degree of $\alpha_q^*(s)$ is $q + 1$, $\alpha_q^*(s) = \sum a_i t_i$ for some $t_i \in S'$, where $\sum a_i = q + 1$. Because $\phi_{t_i} = \phi_t$ in $\Phi'$, $\alpha_q^*(\phi_s) = (q + 1)\phi_t$. By the same argument as above, $\beta_q^*(\phi_s) = (q + 1)\phi_t$. Thus, the image of $\gamma$ is generated by $(q + 1)\phi_t$ and $C \simeq \mathbb{Z}/(q + 1)\mathbb{Z}$. Since $\alpha_q^*(\phi_s) = \beta_q^*(\phi_s)$, $(a, -a) \in K$ for any $a \in \Phi$. By comparing orders, we have that the orders of $K$ and $\Phi$ are equal up to products of powers of 2 and 3.

Corollary A.6. For primes $r \mid Dp$ (resp. $r \mid N$, $r \nmid DNpq$), $U_r - 1$ (resp. $U_r - r$, $T_r - r - 1$) annihilates $C$. Moreover, $U_q + 1$ annihilates $C$.

Proof. This follows from [A.2].

Remark A.7. If $\ell > 3$ is prime, then the $\ell$-primary part of $K$ (resp. $C$) is equal to the one of the kernel $K''$ (resp. the cokernel $C''$) of
\[
0 \to K'' \to \Phi_p(J^D_0(Np)) \times \Phi_p(J^D_0(Npq)) \to \Phi_p(J^D_0(Npq)) \to C'' \to 0,
\]
because $\Phi_p(J^D_0(Np))$ and $\Phi$ are equal up to $2$-, $3$- primary subgroups.

APPENDIX B. Multiplicity one theorems

B.1. The dimension of $J_{p^r}(1)[m]$. Let $J := J^p_0(1)$ be the Jacobian of the Shimura curve $X^p_0(1)$ and $T := T^{pr}$ be the Hecke ring in $\text{End}(J)$. Assume that $p \not\equiv 1 \pmod{\ell}$ and $r \equiv -1 \pmod{\ell}$. Then, the corresponding ideal to $m := (\ell, U_p - 1, U_r + 1, T_w - w - 1 : \text{for primes } w \nmid pr) \subset T$ in $T^{pr}$ is neither $p$-old nor $r$-old. In this case we can prove multiplicity one theorem for $J[m]$.

Theorem B.1 (Ribet). $J[m]$ is of dimension 2.

For a proof, we need the following proposition.

Proposition B.2. $T_m$ is Gorenstein.
Proof. Let $Y$ be the character group of $J_{\mathbb{F}_p}$. Then by Ribet \cite{R90}, there is an exact sequence;

$$0 \rightarrow Y \rightarrow L \rightarrow X \oplus X \rightarrow 0,$$

where $L := X_r(J_0(p))$ (resp. $X := X_r(J_0(r))$) is the character group of $J_0(p)/\mathbb{F}_r$ (resp. $J_0(r)$). Since $m$ is not old, $(T_{pr})_{\mathbb{F}_r} \cong T_m$ and $X_b = 0$, where $a$ (resp. $b$) is the corresponding Eisenstein ideal to $m$ in $T_{pr}$ (resp. $T_r$). Thus, we have $Y_m \cong L_a$. Since for $a$, multiplicity one theorem holds \cite{Y14}, it implies that $L_a$ is free of rank 1 over $(T_{pr})_a$, i.e., $Y_m$ is free of rank 1 over $T_m$. By Grothendieck \cite{Gr72}, there is a monodromy exact sequence,

$$0 \rightarrow Y \rightarrow \text{Hom}(Y, \mathbb{Z}) \rightarrow \Phi \rightarrow 0,$$

where $\Phi := \Phi_p(J)$ is the component group of $J_{\mathbb{F}_p}$. After tensoring with $\mathbb{Z}_\ell$ over $\mathbb{Z}$,

$$0 \rightarrow Y \otimes \mathbb{Z}_\ell \rightarrow \text{Hom}(Y \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell) \rightarrow \Phi_\ell \rightarrow 0.$$

Using an idempotent $e_m \in \mathbb{T}_\ell := \mathbb{T} \otimes \mathbb{Z}_\ell$, we get

$$0 \rightarrow Y_m \rightarrow \text{Hom}(Y_m, \mathbb{Z}_\ell) \rightarrow \Phi_m \rightarrow 0.$$

By the Ribet’s exact sequence in \cite{R76} we have

$$0 \rightarrow K \rightarrow (X \oplus X)/(\mu_p(X \oplus X)) \rightarrow \Phi \rightarrow C \rightarrow 0.$$

Since first, second, and fourth terms vanish after localizing at $m$ (resp. $\alpha$), $\Phi_m = 0$, which implies that $Y_m \cong \text{Hom}(Y_m, \mathbb{Z}_\ell)$ is self-dual. Therefore $T_m$ is Gorenstein. \hfill \Box

Now we prove the theorem above.

Proof of Theorem \cite{B2} Let $J_m := \cup_n J_m[\mathbb{F}_n]$ be the $m$-divisible group of $J$ and let $T_mJ$ be the Tate module of $J$ at $m$, which is $\text{Hom}(J_m, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Then $T_mJ$ is free of rank 2 if and only if $J_m$ is of dimension 2 over $\mathbb{T}/m$. Since $J$ has purely toric reduction at $p$, there is an exact sequence for any $n \geq 1$ (c.f. \cite{R76})

$$0 \rightarrow \text{Hom}(Y/\ell^nY, \mu_{\ell^n}) \rightarrow J[\ell^n] \rightarrow Y/\ell^nY \rightarrow 0.$$

By taking projective limit, we have

$$0 \rightarrow \text{Hom}(Y \otimes \mathbb{Z}_\ell, \mathbb{Z}_\ell(1)) \rightarrow T_mJ \rightarrow Y \otimes \mathbb{Z}_\ell \rightarrow 0,$$

where $\mathbb{Z}_\ell(1)$ is the Tate twist. By applying idempotent $e_m$, we get

$$0 \rightarrow \text{Hom}(Y_m, \mathbb{Z}_\ell(1)) \rightarrow T_mJ \rightarrow Y_m \rightarrow 0.$$

Since $Y_m$ is free of rank 1 over $T_m$, $T_mJ$ is free of rank 2 over $T_m$. \hfill \Box

Remark B.3. By Mazur (appendix of \cite{T97}), $T_m$ is Gorenstein if and only if $J_m$ is of dimension 2.

B.2. The dimension of $J_0(pqr)[m]$. Let $J := J_0(pqr)$ and $T := T_{pqr}$. Let $L := X_p(J)$ be the character group of $J$ at $p$ and $m := (\ell, U_p - 1, U_q - 1, U_r + 1, T_s - s - s : \text{for primes } s \nmid pqr) \subset T$.

Assume that $p \not\equiv 1 \pmod{\ell}$, $q \not\equiv 1 \pmod{\ell}$, and $r \equiv -1 \pmod{\ell}$. Then,

Theorem B.4. $L/mL$ is of dimension 1 over $T/m$ and $J[m]$ is of dimension 2.
Proof. By Theorem 4.2(2) of [Y14], \( \dim J[m] = 2 \). Let \( T \) be the torus of \( J \) at \( p \). Note that \( J[m] \) is a non-trivial extension of \( \mu_2 \) by \( \mathbb{Z}/(\ell) \mathbb{Z} \) that ramified only at \( r \). Since \( \text{Frob}_p \) acts by \( pU_p \) on \( T, \mathbb{Z}/(\ell) \mathbb{Z} \) cannot be in \( T[m] \). Since the dimension of \( J[m] \) is 2, \( T[m] \) is at most of dimension 1. On the other hand, \( T \) acts faithfully on \( T \) and \( m = p \)-new because a pair \((p, r)\) for \( s = 1 \) is admissible. Accordingly, the dimension of \( T[m] \) is at least 1. Therefore \( L/mL \), which is the dual space of \( T[m] \), is of dimension 1.

\section*{Appendix C. The Skorobogatov subgroup of \( J^D_0(N) \) at \( p \)}

In his paper [Sk05], Skorobogatov introduced “Shimura coverings” of Shimura curves. Let \( B \) be a quaternion algebra over \( \mathbb{Q} \) of discriminant \( pD \) such that \( B \otimes \mathbb{R} \cong M_2(\mathbb{R}) \). Let \( \mathcal{O} \) be an Eichler order of \( B \) of level \( N \), and set \( \Gamma^D_0(N) := \mathcal{O}^\times,1 \), the set of reduced norm 1 elements in \( \mathcal{O} \). Let \( I_p \) be the unique two-sided ideal of \( \mathcal{O} \) of reduced norm \( p \). Then, \( 1 + I_p \subset \Gamma^D_0(N) \) and it defines a covering \( X \to X^D_0(N) \). By Jordan [Sk05], there is an unramified subcovering \( X \to X_p \to X^D_0(N) \) whose Galois group is \( \mathbb{Z}/((p + 1)/\epsilon(p)) \), where \( \epsilon(p) \) is 1, 2, 3, or 6. (About \( \epsilon(p) \), see page 781 of [Sk05]). Since unramified abelian coverings of \( X^D_0(N) \) correspond to subgroups of \( J^D_0(N) \), we define the Skorobogatov subgroup of \( J^D_0(N) \) from \( X_p \).

\begin{definition}
The \textbf{Skorobogatov subgroup} \( \Sigma_p \) of \( J^D_0(N) \) at \( p \) is the subgroup of \( J^D_0(N) \) which corresponds to the above unramified covering \( X_p \) of \( X^D_0(N) \).

These subgroups have similar properties to Shimura subgroups. For example,

\begin{proposition}
On \( \Sigma_p \), \( U_p \) (resp. \( U_q, U_r, T_s \)) acts by \(-1\) (resp. 1, \( r \), \( s + 1 \)) for primes \( q \mid D \), \( r \mid N \), and \( s \nmid pD \).
\end{proposition}

Proof. The proof is similar to the action of Hecke operators on Shimura subgroups. By using moduli theoretic description of \( X^D_0(N) \), the complex points of \( X \) classifies \((A, P)\) where \( A \) is a false elliptic curve with level \( N \) structure and \( P \) is a generator of \( A/I_p \). Since the level structures at primes \( r \) dividing \( D \) are compatible with the level structure at \( p \), which gives rise to our covering \( X \), the Atkin-Lehner involution \( w_r \) acts trivially on the covering group. This gives the action of \( U_q \) when \( q \) divides \( D \) because \( U_q = w_q \). Since for primes \( r \) dividing \( N \), \( U_r + w_r = \beta^*_r(\alpha_r)_s \) and \( \beta^*_r = w_r\alpha^*_r, U_r = w_r\alpha^*_r \), \( U_r = w_r(r + 1) - w_r = w_r(r + 1) - w_r = r \) on \( \Sigma_p \), where \( \alpha_r, \beta_r \) are two degeneracy maps from \( X^D_0(N) \) to \( X^D_0(N/r) \). For primes \( s \nmid pD \), \( T_s = (\beta_s)_s, \alpha^*_s = (\beta_s)_s w_s \alpha^*_s = (\beta_s)_s \beta^*_s = s + 1 \) since the image of Skorobogatov subgroups by degeneracy maps lies in Skorobogatov subgroups and \( w_s \) acts trivially.

Consider \( U_p \) on \( \Sigma_p \). The map \( U_p \) sends \((A, P) \) to \((A/A[I_p], Q)\), where \( (P, Q) = \zeta \) for some fixed primitive \( p \)-th root of unity \( \zeta \) and the pairing \((,),(-)\) on \( A[I_p] \times A[I_p] \). (About the above pairing, see [Bu97].) For \( \sigma \) in the covering group of \( X \to X^D_0(N) \), \( (A, P) \) to \((A, \sigma P)\). Thus \( U_p \sigma U_p^{-1} = \sigma^{-1} \), which implies \( U_p \sigma \) acts by \(-1\) on \( \Sigma_p \).

\begin{remark}
It might be easier than above if you consider the actions of \( w_p \) on the group of \( 2 \times 2 \) matrices as in Calegari and Venkatesh. See page 29 of [CV12].
\end{remark}

\begin{proposition}
Let \( K \) be the kernel of the map
\[
J^D_0(1) \times J^D_0(1) \to J^D_0(q)
\]
by the degeneracy maps \( \alpha^*_q \) and \( \beta^*_q \). Then \( K \) contains an antidiagonal embedding of \( \Sigma_r \).
\end{proposition}

Proof. Let \( \Sigma_r \) (resp. \( \Sigma \)) be the Skorobogatov subgroup of \( J^D_0(1) \) (resp. \( J^D_0(q) \)) at \( r \). Since \( w_q \) acts trivially on \( \Sigma \) and the image of \( \Sigma_r \) by \( \alpha^*_q \) lies in \( \Sigma, \alpha^*_q(a) + \beta^*_q(-a) = \alpha^*_q(a) + w_q(\alpha^*_q(-a)) = \alpha^*_q(a) - \alpha^*_q(a) = 0 \). Thus, \( K \) contains \( \{(a, -a) \in J^D_0(1) \times J^D_0(1) : a \in \Sigma_r \} \).

\begin{remark}
Since \( K \) contains an antidiagonal embedding of \( \Sigma_r, K[m] \neq 0 \), where \( m := (\ell, U_p - 1, U_r + 1, T_s - s - 1 : \text{for primes } s \nmid pr) \text{ if } r \equiv -1 \pmod{\ell} \).
\end{remark}
