Fan-type degree condition restricted to triples of induced subgraphs ensuring Hamiltonicity

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Abstract
In 1984, Fan gave a sufficient condition involving maximum degree of every pair of vertices at distance two for a graph to be Hamiltonian. Motivated by Fan’s result, we say that an induced subgraph $H$ of a graph $G$ is $f$-heavy if for every pair of vertices $u, v \in V(H)$, $d_H(u, v) = 2$ implies that $\max\{d(u), d(v)\} \geq n/2$. For a given graph $R$, $G$ is called $R$-$f$-heavy if every induced subgraph of $G$ isomorphic to $R$ is $f$-heavy. For a family $\mathcal{R}$ of graphs, $G$ is $\mathcal{R}$-$f$-heavy if $G$ is $R$-$f$-heavy for every $R \in \mathcal{R}$. In this note we show that every 2-connected graph $G$ has a Hamilton cycle if $G$ is $\{K_1, 3, P_7, D\}$-$f$-heavy or $\{K_1, 3, P_7, H\}$-$f$-heavy, where $D$ is the deer and $H$ is the hourglass. Our result is a common generalization of previous theorems of Broersma et al. and Fan on Hamiltonicity of 2-connected graphs.

Keywords: Combinatorial problem; Hamilton cycle; Fan-type degree condition; Induced subgraph; Claw

AMS Subject Classification (2000): 05C38 05C45

1 Introduction
We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

Let $G$ be a graph and $H$ be a subgraph of $H$. For two vertices $x, y \in V(H)$, a shortest $(x, y)$-path in $H$ means that a path connecting $x$ and $y$ with all vertices in $H$. The distance between $x$ and $y$ in $H$, denoted by $d_H(x, y)$, is the length of a shortest $(x, y)$-path in $H$. If $H = G$, we use $d(x, y)$ instead of $d_G(x, y)$.

A graph is called Hamiltonian if it contains a Hamilton cycle, i.e., a cycle passing through all its vertices. Checking whether a given graph is Hamiltonian or not is a notorious $NP$-complete decision problem. Thus graphists drew their attention to find sufficient

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conditions for the existence of Hamilton cycles in graphs. The following sufficient condition for the existence of Hamilton cycles in 2-connected graphs is well known.

**Theorem 1** (Fan [10]). Let $G$ be a 2-connected graph on $n$ vertices. If $\max\{d(x), d(y)\} \geq n/2$ for every pair of vertices $x$ and $y$ with $d(x, y) = 2$, then $G$ is Hamiltonian.

There is another kind of sufficient conditions for Hamiltonicity of graphs, called forbidden subgraph condition. Before we state some of these results, we first introduce some terminology and notation.

Let $G$ be a graph and $H$ be a subgraph of $G$. If $H$ contains every edge $xy \in E(G)$ with $x, y \in V(H)$, then $H$ is called an induced subgraph of $G$. For a given graph $R$, $G$ is $R$-free if $G$ contains no induced subgraph isomorphic to $R$. For a family $\mathcal{R}$ of graphs, $G$ is $\mathcal{R}$-free if $G$ is $R$-free for every $R \in \mathcal{R}$. The graph $K_{1,3}$ is called a claw. Its only vertex with degree 3 is called the center, and other vertices are the end vertices of the claw. Throughout this note, instead of $K_{1,3}$-free, we use the more common term claw-free.

The following are two results on forbidden subgraph conditions for Hamiltonicity of graphs.

**Theorem 2** (Broersma and Veldman [6]). Let $G$ be a 2-connected graph. If $G$ is claw-free and $\{P_7, D\}$-free, then $G$ is Hamiltonian. (see Fig. 1)

**Theorem 3** (Faudree, Ryjáček and Schiermeyer [11]). Let $G$ be a 2-connected graph. If $G$ is claw-free and $\{P_7, H\}$-free, then $G$ is Hamiltonian. (see Fig. 1)

![Fig. 1. Graphs $D$ and $H$.](image)

Let $G$ be a graph on $n$ vertices. A vertex $v$ of $G$ is called heavy if $d(v) \geq n/2$. Following [5], an induced claw of $G$ is called 2-heavy if at least two of its end vertices are heavy. The graph $G$ is 2-heavy if all induced claw of $G$ are 2-heavy. Thus 2-heavy graphs can be seen as graphs by restricting Fan’s condition to every induced claw.

Broersma et al. [5] extended Theorems [2] and [3] to a larger class of 2-heavy graphs.

**Theorem 4** (Broersma, Ryjáček and Schiermeyer [5]). Let $G$ be a 2-connected graph. If $G$ is 2-heavy, and moreover, $\{P_7, D\}$-free or $\{P_7, H\}$-free, then $G$ is Hamiltonian.
Let $G$ be a graph and $H$ be an induced subgraph. We say that $H$ is $f$-heavy if for every pair of vertices $u, v \in V(H)$, $d_H(u, v) = 2$ implies that $\max\{d(u), d(v)\} \geq n/2$. For a given graph $R$, $G$ is called $R$-$f$-heavy if every induced subgraph of $G$ isomorphic to $R$ is $f$-heavy. For a family $\mathcal{R}$ of graphs, $G$ is called $\mathcal{R}$-$f$-heavy if $\mathcal{R}$-$f$-heavy for every $R \in \mathcal{R}$. Note that every $R$-free graph is also $R$-$f$-heavy, and that a graph is 2-heavy is equivalent to that it is claw-$f$-heavy.

By relaxing forbidden subgraph conditions to conditions in which the subgraphs are allowed, but where Fan-type degree condition is imposed on these subgraphs if they appear, we extend Theorem 4 as follows.

**Theorem 5.** Let $G$ be a 2-connected graph. If $G$ is $\{K_{1,3}, P_7, D\}$-$f$-heavy or $\{K_{1,3}, P_7, H\}$-$f$-heavy, then $G$ is Hamiltonian.

**Remark 1.** It is easily seen that every graph satisfying the condition of Theorem 1 or 4 also satisfies the one of Theorem 5. Furthermore, The graph provided in the following is a Hamiltonian graph satisfying the condition of Theorem 5, but not the one of Theorem 1 or 4.

Let $n \geq 16$ be an even integer and $K_{n/2} + K_{n/2-7}$ denotes the union of two complete graphs $K_{n/2}$ and $K_{n/2-7}$. We construct the graph $G$ with $V(G) = V(K_{n/2} + K_{n/2-7}) \cup \{x, y, z, u, v, w, t\}$ and $E(G) = E(K_{n/2} + K_{n/2-7}) \cup \{xy, xz, yz, yw, wu, zt, tv\} \cup \{xx', yx', zx': x' \in V(K_{n/2})\} \cup \{uy', vy': y' \in V(K_{n/2-7})\}$.

This fact shows that Theorem 5 indeed strengthens Theorems 1 and 4.

In the next section, we will give the proof of Theorem 5. Some concluding remarks will be given in Section 3.

## 2 Proof of Theorem 5

Before giving the proof of Theorem 5, we introduce some additional terminology, and will list two useful lemmas.

Let $G$ be a graph and $C$ be a cycle of $G$. We denote by $\overrightarrow{C}$ the cycle with a given orientation, and by $\overleftarrow{C}$ the same subgraph with the reverse orientation. For two vertices $x, y \in V(C)$, $\overrightarrow{C}[x, y]$ is denoted by the consecutive vertices from $x$ to $y$ in $C$ by the direction specified by $\overrightarrow{C}$, and $\overleftarrow{C}[y, x]$ is the same vertices with the reverse order. For a vertex $x \in V(C)$, $x^+$ denotes the successor of $x$ on $\overrightarrow{C}$, and $x^-$ denotes its predecessor. Similarly, for a path $P$ and $x, y \in V(P)$, $P[x, y]$ denotes the subpath of $P$ from $x$ to $y$.

Let $G$ be a graph on $n$ vertices. Recall that a vertex of a graph $G$ is heavy if its degree is at least $n/2$. Otherwise, it is light. A cycle $C$ of $G$ is called a heavy cycle if it contains
Lemma 1 (Bollobás and Brightwell [4], Shi [15]). Let $G$ be a 2-connected graph. Then $G$ contains a heavy cycle.

Next we introduce a new concept proposed in [12] recently. In fact, it is a refinement of the closure theory of Bondy-Chvátal. For the sake of convenience, we rewrite it here. We use $\tilde{E}(G)$ to denote the set \{xy : xy \in E(G) or d(x) + d(y) \geq n, x, y \in V(G)\}. Let $k \geq 3$ be an integer. A sequence of vertices $C = v_1v_2 \ldots v_kv_1$ is called an Ore-cycle or briefly, $o$-cycle of $G$, if for every $i \in \{1, \cdots, k\}$, there holds $v_iv_{i+1} \in \tilde{E}(G)$, where the indices are taken modulo $k$.

Lemma 2 (Li, Ryjáček, Wang and Zhang [12]). Let $G$ be a graph and $C$ be an $o$-cycle of $G$. Then there exists a cycle $C'$ of $G$ such that $V(C) \subseteq V(C')$.

Proof of Theorem 5

By Lemma 1 $G$ contains a heavy cycle. Let $C$ be a longest heavy cycle of $G$, fixed an orientation. Suppose that $G$ is not Hamiltonian. Since $G$ is 2-connected, there is a path of length at least 2, internally-disjoint with $C$, that connects two vertices of $C$. Let $P = w_0w_1 \ldots w_rw_{r+1}$ be such a path with $r$ as small as possible, where $w_0 = u \in V(C)$ and $w_{r+1} = v \in V(C)$.

Claim 1. Let $x \in V(P) \setminus \{u, v\}$ and $y \in \{u^-, u^+, v^-, v^+\}$. Then $xy \not\in \tilde{E}(G)$.

Proof. Without loss of generality, assume that $y = u^-$. Suppose that $xy \in \tilde{E}(G)$. Then $C' = yxP[x, u]C[u, y]$ is an $o$-cycle containing all the vertices of $C$ and longer than $C$. By Lemma 2 there is a longer cycle containing all the vertices in $C$, that is, a longer heavy cycle in $G$, a contradiction. The other assertions can be proved similarly.

Claim 2. $u^-u^+ \in \tilde{E}(G)$, $v^-v^+ \in \tilde{E}(G)$.

Proof. Suppose that $u^-u^+ \not\in E(G)$. By Claim 1 $\{u, u^-, u^+, w_1\}$ induces a claw. By the choice of $C$, $w_1$ is light. Since $G$ is 2-heavy, we have $d(u^-) + d(u^+) \geq n$. This implies that $u^-u^+ \in \tilde{E}(G)$. Similarly, we can prove the other assertion.

Claim 3. $uv^+ \not\in \tilde{E}(G)$, $vu^+ \not\in \tilde{E}(G)$, $u^-v^- \not\in \tilde{E}(G)$, $u^+v^+ \not\in \tilde{E}(G)$.

Proof. Suppose that $uv^- \in \tilde{E}(G)$. By Claim 2 $u^+u^- \in \tilde{E}(G)$. Then $C' = uv^-C[v^-, u^+]u^+\sim\tilde{C}[u^-, v]P[v, u]$ is an $o$-cycle containing all vertices in $C$ and longer than $C$, a contradiction. Suppose $u^-v^- \in \tilde{E}(G)$. Then $C' = u^-v^-C[v^-, u]P[u, v]C[v, u^-]$ is an $o$-cycle containing all vertices in $C$ and longer than $C$, a contradiction. The other assertions can be proved similarly.
Let \( y_1 \) be the first vertex on \( \overrightarrow{C}[u,v] \) such that \( uy_1 \notin E(G) \), \( y_2 \) be the first vertex on \( \overrightarrow{C}[v,u] \) such that \( vy_2 \notin E(G) \). By Claim 3, \( uw^- \notin E(G) \) and \( vu^- \notin E(G) \). Thus, \( y_1 \) and \( y_2 \) are well-defined.

**Claim 4.** Let \( w \in \{ w_1, \ldots , w_r \} \), \( x \in C[u^+, y_1] \) and \( y \in C[v^+, y_2] \). Then we have

1. \( wx \notin \overrightarrow{E}(G), wy \notin \overrightarrow{E}(G) \); 
2. \( uy \notin \overrightarrow{E}(G), vx \notin \overrightarrow{E}(G) \); 
3. \( xy \notin \overrightarrow{E}(G) \).

**Proof.** (1) Suppose that \( wx \in \overrightarrow{E}(G) \). By Claim 1, \( x \neq u^+ \) and this implies that \( ux^- \in E(G) \). Then \( C' = P[u, w] \overrightarrow{C}[x, u^-] u^- u^+ \overrightarrow{C}[u^+, x^-] x^- u \) is an \( o \)-cycle longer than \( C \) and contains all vertices in \( C \), a contradiction. The other assertion can be proved similarly.

(2) Suppose that \( uy \in \overrightarrow{E}(G) \). By Claim 3, \( y \neq v^+ \) and this implies that \( vy^- \in E(G) \). Then \( C' = uy \overrightarrow{C}[y, u^-] u^- u^+ \overrightarrow{C}[u^+, v^-] v^- v^+ \overrightarrow{C}[v^+, y^-] y^- v P[v, u] \) is an \( o \)-cycle longer than \( C \) and contains all vertices in \( C \), a contradiction. The other assertion can be proved.

(3) Suppose that \( xy \in \overrightarrow{E}(G) \). By Claim 2, \( u^- u^+ \in \overrightarrow{E}(G) \) and \( v^- v^+ \in \overrightarrow{E}(G) \). Now \( C' = P[u, v] vy \overrightarrow{C}[y, v^+] v^+ v^- \overrightarrow{C}[v^-, x] xy \overrightarrow{C}[y, u^-] u^- u^+ \overrightarrow{C}[u^+, x^-] x^- u \) (if \( x \neq u^+ \) and \( y \neq v^+ \)) or \( C' = P[u, v] vy \overrightarrow{C}[y, v^+] v^+ v^- \overrightarrow{C}[v^-, u^+] u^+ uy \overrightarrow{C}[y, u] \) (if \( x = u^+ \) and \( y \neq v^+ \)) or \( C' = P[u, v] \overrightarrow{C}[v, x] xv \overrightarrow{C}[v, u^-] u^- u^+ \overrightarrow{C}[u^+, x^-] x^- u \) (if \( x \neq u^+ \) and \( y = v^+ \)) is an \( o \)-cycle longer than \( C \) and contains all vertices in \( C \), a contradiction.

**Claim 5.** \( u^- u^+ \in E(G) \) or \( v^- v^+ \in E(G) \).

**Proof.** Suppose that \( u^- u^+ \notin E(G) \) and \( v^- v^+ \notin E(G) \). By Claim 2, we have \( d(u^-) + d(u^+) \geq n \) and \( d(v^-) + d(v^+) \geq n \). Thus, we obtain \( d(u^-) + d(v^-) \geq n \) or \( d(u^+) + d(v^+) \geq n \), contradicting Claim 3.

By Claim 3, without loss of generality, we assume that \( u^- u_1 \in E(G) \).

**Claim 6.** \( uv \in E(G) \).

**Proof.** Suppose that \( uv \notin E(G) \). By Claim 1, \( \{ y_1, y_1^-, u, w_1, \ldots , w_r, v, y_2, y_2^- \} \) induces a \( P_{6+r} \), where \( r \geq 1 \). Since \( G \) is \( P_7 \)-heavy, \( G \) is also \( P_{6+r} \)-heavy. By the choice of \( C \), \( w_1 \) and \( w_r \) are light. It follows that \( y_1^- \) and \( y_2^- \) are heavy, and this implies \( y_1^- y_2^- \in \overrightarrow{E}(G) \), contradicting Claim 1(3).

**Claim 7.** \( r = 1 \).

**Proof.** Suppose \( r \geq 2 \). Since \( r \geq 2 \) and by the choice of the path \( P \), we have \( w_1 v \notin E(G) \) and \( w_r u \notin E(G) \). By Claim 3, we obtain \( u^+ v \notin E(G) \) and \( u^- v \notin E(G) \). By Claim 1, \( w_1 u^- \notin E(G) \) and \( w_r v^+ \notin E(G) \). Thus each of \( \{ u, w_1, u^-, v \} \) and \( \{ v, w_r, v^-, u \} \) induces
a claw. Since each of \( \{w_1, w_r\} \) is light and \( G \) is 2-heavy, \( u^- \) and \( v^- \) are heavy. Hence \( u^- v^- \in \tilde{E}(G) \), which contradicts Claim 3.

Note that \( G \) is \( D \)-\( f \)-heavy or \( H \)-\( f \)-heavy. If \( G \) is \( D \)-\( f \)-heavy, then by Claim 4 \( \{y_1, y^-_1, u, w_1, v, y_2^-\} \) induces a \( D \). Since \( w_1 \) is light, \( y_1^- \) and \( y_2^- \) are heavy. It follows that \( y_1^- y_2^- \in \tilde{E}(G) \), which contradicts Claim 4(3). Now, we assume that \( G \) is \( H \)-\( f \)-heavy. By Claims 1 and 3 \( \{u^- v, u^+, w_1, v\} \) induces an \( H \). It follows that \( u^- \) is heavy. If \( v^- v^+ \in E(G) \), then by Claims 1 and 3 \( \{v^-, v, v^+, w_1, u\} \) induces an \( H \). Similarly, we have \( v^- \) is heavy. If \( v^- v^+ \notin E(G) \), then \( \{v^-, v, v^+, w_1\} \) induces a claw. Since \( w_1 \) is light and \( G \) is 2-heavy, we have \( v^- \) is heavy. In these two cases, we obtain \( u^- v^- \in \tilde{E}(G) \), which contradicts Claim 3.

The proof is complete.

3 Concluding remarks

In this note, we give a new sufficient condition for Hamiltonicity of graphs by restricting Fan’s condition to triples of induced subgraphs of graphs.

In fact, the idea that one can guarantee Hamiltonicity of graph by restricting Fan’s condition to pairs of induced subgraphs dated from Bedrossian, Chen and Schelp [2]. Later, Chen, Wei and Zhang [8, 9], and Li, Wei and Gao [13] got related results with this similar idea. Note that Bedrossian [1] characterized all pairs of forbidden subgraphs \( \{R, S\} \) for Hamiltonicity of 2-connected graphs. Thus we can pose this problem: which two connected graphs \( R \) and \( S \) other than \( P_3 \) imply that every 2-connected \( \{R, S\} \)-\( f \)-heavy graph is Hamiltonian? Recently, this problem has been completely solved in [14].

Brousek [7] gave a complete characterization of triples of connected graphs \( \{K_{1,3}, R, S\} \) such that a graph \( G \) being 2-connected and \( \{K_{1,3}, R, S\} \)-\( f \)-heavy is Hamiltonian. Thus we can pose the following problem naturally.

**Problem 1.** To characterize all possible triples of connected graphs \( \{K_{1,3}, R, S\} \) such that every 2-connected graph \( G \) being \( \{K_{1,3}, R, S\} \)-\( f \)-heavy is Hamiltonian.

Acknowledgement

This work is supported by NSFC (No. 11271300) and the Doctorate Foundation of Northwestern Polytechnical University (cx201326). The author would like to express gratitude to the editors and reviewers, whose invaluable suggestions have improved the presentation of this work.
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