Rational $q \times q$ Carathéodory Functions and Central Non-negative Hermitian Measures

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We give an explicit representation of central measures corresponding to finite Toeplitz non-negative definite sequences of complex $q \times q$ matrices. Such measures are intimately connected to central $q \times q$ Carathéodory functions. This enables us to prove an explicit representation of the non-stochastic spectral measure of an arbitrary multivariate autoregressive stationary sequence in terms of the covariance sequence.

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1. Introduction

If $\kappa$ is a non-negative integer or if $\kappa = \infty$, then a sequence $(C_j)_{j=-\kappa}^{\kappa}$ of complex $q \times q$ matrices is called Toeplitz non-negative definite if, for each non-negative integer $n$ with $n \leq \kappa$, the block Toeplitz matrix $T_n := [C_{j-k}]_{j,k=0}^{n}$ is non-negative Hermitian. In the second half of the 1980’s, the first two authors intensively studied the structure of Toeplitz non-negative definite sequences of complex $q \times q$ matrices in connection with interpretations in the languages of stationary sequences, Carathéodory interpolation, orthogonal matrix polynomials etc. (see [8, 9] and also [5] for a systematic treatment of several aspects of the theory).

In particular, it was shown in [8, Part I] (see also [9, Section 3.4]) that the structure of the elements of a Toeplitz non-negative definite sequence of complex $q \times q$ matrices is described in terms of matrix balls which are determined by all preceding elements. Amongst these sequences there is a particular subclass which plays an important role, namely the so-called class of central Toeplitz non-negative definite sequences of complex $q \times q$ matrices. These sequences are characterized by the fact that starting with some index all further elements of the sequences coincide with the center of the matrix ball in question. Central Toeplitz non-negative definite sequences possess several interesting extremal properties (see [8, Parts I–III]) and a remarkable recurrent structure (see [5, Thm. 3.4.3]).
In view of the matrix version of a classical theorem due to Herglotz (see, e.g. [3, Thm. 2.2.1]), the set of all Toeplitz non-negative definite sequences coincides with the set of all sequences of Fourier coefficients of $q \times q$ non-negative Hermitian Borel measures on the unit circle $T := \{ z \in \mathbb{C} : |z| = 1 \}$ of $\mathbb{C}$. If $(C_j)_{j=-\infty}^{\infty}$ is a Toeplitz non-negative definite sequence of complex $q \times q$ matrices and if $\mu$ denotes the unique $q \times q$ non-negative Hermitian Borel measure on $T$ with $(C_j)_{j=-\infty}^{\infty}$ as its sequence of Fourier coefficients then we will call $\mu$ the spectral measure of $(C_j)_{j=-\infty}^{\infty}$. In the special case of a central Toeplitz positive definite sequence of complex $q \times q$ matrices, i.e., if for each non-negative integer $n$ the block Toeplitz matrix $T_n := [C_{j-k}]_{j,k=0}^{n}$ is positive Hermitian, in [8, Part III] (see also [5, Section 3.6]), we stated an explicit representation of its spectral measure. In particular, it turned out that in this special case its spectral measure is absolutely continuous with respect to the linear Lebesgue-Borel measure on the unit circle and that the corresponding Radon-Nikodym density can be expressed in terms of left or right orthogonal matrix polynomials.

The starting point of this paper was the problem to determine the spectral measure of a central Toeplitz non-negative definite sequence of complex matrices. An important step on the way to the solution of this problem was gone in the paper [11], where it was proved that the matrix-valued Carathéodory function associated with a central Toeplitz non-negative definite sequence of complex matrices is rational and, additionally, concrete representations as quotient of two matrix polynomials were derived. Thus, the original problem can be solved if we will be able to find an explicit expression for the Riesz-Herglotz measure of a rational matrix-valued Carathéodory function. This question will be answered in Thm. 2.14 As a first essential consequence of this result we determine the Riesz-Herglotz measures of central matrix-valued Carathéodory functions (see Thm. 5.10). Reformulating Thm. 5.10 in terms of Toeplitz non-negative definite sequences, we get an explicit description of the spectral measure of central Toeplitz non-negative definite sequences of complex matrices (see Thm. 5.11).

In the final Section 6, we apply Thm. 5.11 to the theory of multivariate stationary sequences. In particular, we will be able to express explicitly the non-stochastic spectral measure of a multivariate autoregressive stationary sequence by its covariance sequence (see Thm. 6.2).

2. On the Riesz-Herglotz measure of rational matrix-valued Carathéodory functions

In this section, we give an explicit representation of the Riesz-Herglotz measure of an arbitrary rational matrix-valued Carathéodory function.

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all real numbers, the set of all integers, the set of all non-negative integers, and the set of all positive integers, respectively. Throughout this paper, let $p, q \in \mathbb{N}$. If $\mathcal{X}$ is a non-empty set, then by $\mathcal{X}^{q \times p}$ we denote the set of all $q \times p$ matrices each entry of which belongs to $\mathcal{X}$. The notation $\mathcal{X}^q$ is short for $\mathcal{X}^{q \times 1}$. If
\( \mathcal{X} \) is a non-empty set and if \( x_1, x_2, \ldots, x_q \in \mathcal{X} \), then let

\[
\text{col}(x_j)_{j=1}^q := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}.
\]

For every choice of \( \alpha, \beta, \in \mathbb{R} \cup \{-\infty, +\infty\} \), let \( \mathbb{Z}_{\alpha, \beta} := \{m \in \mathbb{Z} : \alpha \leq m \leq \beta\} \). We will use \( I_q \) and \( O_{q \times p} \) for the unit matrix belonging to \( \mathbb{C}^{q \times q} \) and the null matrix belonging to \( \mathbb{C}^{q \times p} \), respectively. For each \( A \in \mathbb{C}^{q \times q} \), let \( \text{Re} A := \frac{1}{2}(A + A^*) \) and \( \text{Im} A := \frac{1}{2i}(A - A^*) \) be the real part and the imaginary part of \( A \), respectively. If \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \), then a sequence \( (C_j)_{j=-\kappa}^\kappa \) of complex \( q \times q \) matrices is called \textit{Toeplitz non-negative definite} (resp. \textit{Toeplitz positive definite}) if, for each \( n \in \mathbb{Z}_{0, \kappa} \), the block Toeplitz matrix

\[
T_n := [C_{j-k}]_{j,k=0}^n
\]

is non-negative Hermitian (resp. positive Hermitian). Obviously, if \( m \in \mathbb{N}_0 \), then \( (C_j)_{j=-m}^m \) is Toeplitz non-negative definite (resp. Toeplitz positive definite) if the block Toeplitz matrix \( T_m = [C_{j-k}]_{j,k=0}^m \) is non-negative Hermitian (resp. positive Hermitian).

Let \( \Omega \) be a non-empty set and let \( \mathfrak{A} \) be a \( \sigma \)-algebra on \( \Omega \). A mapping \( \mu \) whose domain is \( \mathfrak{A} \) and whose values belong to the set \( \mathbb{C}_{\geq}^{q \times q} \) of all non-negative Hermitian complex \( q \times q \) matrices is said to be a \textit{non-negative Hermitian} \( q \times q \) measure on \((\Omega, \mathfrak{A})\) if it is countably additive, i.e., if \( \mu(\cup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \mu(A_k) \) holds true for each sequence \( (A_k)_{k=1}^\infty \) of pairwise disjoint sets which belong to \( \mathfrak{A} \). The theory of integration with respect to non-negative Hermitian measures goes back to Kats [15] and Rosenberg [17].

In particular, we will turn our attention to the set \( \mathcal{M}_q^\sigma(\mathbb{T}) \) of all non-negative Hermitian \( q \times q \) measures on \((\mathbb{T}, \mathfrak{B}_\mathbb{T})\), where \( \mathfrak{B}_\mathbb{T} \) is the \( \sigma \)-algebra of all Borel subsets of the unit circle \( \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} \) of \( \mathbb{C} \).

Non-negative Hermitian measures belonging to \( \mathcal{M}_q^\sigma(\mathbb{T}) \) are intimately connected to the class \( \mathcal{C}_q(\mathbb{D}) \) of all \( q \times q \) Carathéodory functions in the open unit disk \( \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} \) of \( \mathbb{C} \). A \( q \times q \) matrix-valued function \( \Phi : \mathbb{D} \to \mathbb{C}^{q \times q} \) which is holomorphic in \( \mathbb{D} \) and which fulfills \( \text{Re} \Phi(z) \in \mathbb{C}_{\geq}^{q \times q} \) for all \( z \in \mathbb{D} \) is called \( q \times q \) \textit{Carathéodory function in} \( \mathbb{D} \). The matricial version of a famous theorem due to F. Riesz and G. Herglotz illustrates the mentioned interrelation:

**Theorem 2.1.** (a) Let \( \Phi \in \mathcal{C}_q(\mathbb{D}) \). Then there exists one and only one measure \( \mu \in \mathcal{M}_q^\sigma(\mathbb{T}) \) such that

\[
\Phi(z) - i \text{Im} \Phi(0) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \mu(d\zeta) \tag{2.1}
\]

for each \( z \in \mathbb{D} \). For every choice of \( z \) in \( \mathbb{D} \), furthermore,

\[
\Phi(z) - i \text{Im} \Phi(0) = C_0^{[\mu]} + 2 \sum_{j=1}^{\infty} C_j^{[\mu]} z^j
\]

where

\[
C_j^{[\mu]} := \int_{\mathbb{T}} \zeta^{-j} \mu(d\zeta), \tag{2.2}
\]
for each $j \in \mathbb{Z}$ are called the Fourier coefficients of $\mu$.

(b) Let $H$ be a Hermitian complex $q \times q$ matrix and let $\mu \in \mathcal{M}_\infty^q(\mathbb{T})$. Then the function $\Phi: \mathbb{D} \to \mathbb{C}^{q \times q}$ defined by

$$\Phi(z) := \int_\mathbb{T} \frac{\zeta + z}{\zeta - z} \mu(d\zeta) + iH$$

belongs to $\mathcal{C}_q(\mathbb{D})$ and fulfills $\operatorname{Im} \Phi(0) = H$.

A proof of Thm. 2.1 is given, e.g., in [5] Thm. 2.2.2, pp. 71/72. If $\Phi \in \mathcal{C}_q(\mathbb{D})$, then the unique measure $\mu \in \mathcal{M}_\infty^q(\mathbb{T})$ which fulfills (2.1) for each $z \in \mathbb{D}$ is said to be the Riesz-Herglotz measure of $\Phi$.

Let $\delta_u$ be the Dirac measure on $(\mathbb{T}, \mathcal{B}_\mathbb{T})$ with unit mass at $u \in \mathbb{T}$.

Example 2.2. Let $u \in \mathbb{T}$ and $W \in \mathbb{C}^{q \times q}$. Then Thm. 2.1 yields that the function $\Phi: \mathbb{D} \to \mathbb{C}^{q \times q}$ defined by $\Phi(z) := \frac{u + z}{u - z} W$ belongs to $\mathcal{C}_q(\mathbb{D})$ with Riesz-Herglotz measure $\mu := \delta_u W$. The Fourier coefficients of $\mu$ are given by $C_{\zeta}^j[\mu] = u^{-j} W$ for all $j \in \mathbb{Z}$ and the function $\Phi$ admits the representation $\Phi(z) = [1 + 2 \sum_{j=1}^{\infty} (zu)^j]W$ for all $z \in \mathbb{D}$.

Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the column space and the null space of a $p \times q$ complex matrix $A$, respectively.

Lemma 2.3. Let $\Phi \in \mathcal{C}_q(\mathbb{D})$ with Riesz-Herglotz measure $\mu$. For all $z \in \mathbb{D}$,

$$\mathcal{R}(\Phi(z) - i \operatorname{Im} \Phi(0)) = \mathcal{R}(\mu(\mathbb{T})) = \mathcal{R}(\operatorname{Re} \Phi(z))$$

and

$$\mathcal{N}(\Phi(z) - i \operatorname{Im} \Phi(0)) = \mathcal{N}(\mu(\mathbb{T})) = \mathcal{N}(\operatorname{Re} \Phi(z)).$$

Proof. Let $z \in \mathbb{D}$. Since $\operatorname{Re}(\Phi(z) - i \operatorname{Im} \Phi(0)) = \operatorname{Re} \Phi(z) \in \mathbb{C}^{q \times q}$, we obtain from [12] Lem. A.8, parts (a) and (b) then $\mathcal{R}(\operatorname{Re} \Phi(z)) \subseteq \mathcal{R}(\Phi(z) - i \operatorname{Im} \Phi(0))$ and $\mathcal{N}(\Phi(z) - i \operatorname{Im} \Phi(0)) \subseteq \mathcal{N}(\operatorname{Re} \Phi(z))$. In view of (2.1), the application of [13] Lem. B.2(b)] yields $\mathcal{R}(\Phi(z) - i \operatorname{Im} \Phi(0)) \subseteq \mathcal{R}(\mu(\mathbb{T}))$ and $\mathcal{N}(\mu(\mathbb{T})) \subseteq \mathcal{N}(\Phi(z) - i \operatorname{Im} \Phi(0))$. From (2.1), we get $\operatorname{Re} \Phi(z) = \int_\mathbb{T} (1 - |z|^2)/|\zeta - z|^2 \mu(d\zeta)$. Since $1 - |z|^2)/|\zeta - z|^2 > 0$ for all $\zeta \in \mathbb{T}$, the application of [13] Lem. B.2(b)] yields $\mathcal{R}(\operatorname{Re} \Phi(z)) = \mathcal{R}(\mu(\mathbb{T}))$ and $\mathcal{N}(\operatorname{Re} \Phi(z)) = \mathcal{N}(\mu(\mathbb{T}))$, which completes the proof.

Now we consider the Riesz-Herglotz measures for a particular subclass of $\mathcal{C}_q(\mathbb{D})$. In particular, we will see that in this case, the Riesz-Herglotz measure is absolutely continuous with respect to the linear Lebesgue measure $\lambda$ defined on $\mathcal{B}_\mathbb{T}$ and that the Radon-Nikodym density can be always chosen as a continuous function on $\mathbb{T}$.

By a region of $\mathbb{C}$ we mean an open, connected, non-empty subset of $\mathbb{C}$. For all $z \in \mathbb{C}$ and all $r \in (0, +\infty)$, let $K(z; r) := \{w \in \mathbb{C} : |w - z| < r\}$.

Lemma 2.4. Let $\mathcal{D}$ be a region of $\mathbb{C}$ such that $K(0; r) \subseteq \mathcal{D}$ for some $r \in (1, +\infty)$ and let $F: \mathcal{D} \to \mathbb{C}^{q \times q}$ be holomorphic in $\mathcal{D}$ such that the restriction $\Phi$ of $F$ onto $\mathbb{D}$ belongs to $\mathcal{C}_q(\mathbb{D})$. Then the Riesz-Herglotz measure $\mu$ of $\Phi$ admits the representation

$$\mu(B) = \frac{1}{2\pi} \int_B \operatorname{Re} F(\zeta) \Delta(d\zeta),$$

where $\Delta$ is the Radon-Nikodym density of $\mu$. 


for each $B \in \mathfrak{B}_T$.

A proof of Lem. 2.4 can be given by use of a matrix version of an integral formula due to H. A. Schwarz (see, e.g., [5, p. 71]).

In particular, Lem. 2.4 contains full information on the Riesz-Herglotz measures of that functions belonging to $C_q(D)$ which are restrictions onto $D$ of rational matrix-valued functions without poles on $T$. Our next goal is to determine the Riesz-Herglotz measure of functions belonging to $C_q(D)$ which are restrictions onto $D$ of rational matrix-valued functions having poles on $T$. First we are going to verify that in this case all poles on $T$ have order one. Our strategy of proving this is based on the following fact:

**Lemma 2.5.** Let $\Phi \in C_q(D)$ with Riesz-Herglotz measure $\mu$. For each $u \in T$, then

$$\mu(\{u\}) = \lim_{r \to 1-0} \frac{1-r}{2} \Phi(ru).$$  \hspace{1cm} (2.3)

A proof of Lem. 2.5 is given, e.g., in [6, Lem. 8.1]. As a direct consequence of Lem. 2.5 we obtain:

**Remark 2.6.** Let $D$ be a region of $C$ such that $K(0; r) \subseteq D$ for some $r \in (1, +\infty)$ and let $F: D \to \mathbb{C}^{q \times q}$ be holomorphic such that the restriction $\Phi$ of $F$ onto $D$ belongs to $C_q(D)$. Then the Riesz-Herglotz measure $\mu$ of $\Phi$ fulfills $\mu(\{u\}) = O_{q \times q}$ for all $u \in T$.

**Proposition 2.7.** Let $D$ be a region of $C$ such that $K(0; r) \subseteq D$ for some $r \in (1, +\infty)$ and let $F$ be a $q \times q$ matrix-valued function meromorphic in $D$ such that the restriction $\Phi$ of $F$ onto $D$ belongs to $C_q(D)$. Furthermore, let $u \in T$ be a pole of $F$. Then $u$ is a simple pole of $F$ with $\text{Res}(F, u) = -2u\mu(\{u\})$ and

$$\lim_{r \to 1-0} [(ru - u)F(ru)] = -2u\mu(\{u\}),$$  \hspace{1cm} (2.4)

where $\text{Res}(F, u)$ is the residue of $F$ at $u$ and $\mu$ is the Riesz-Herglotz measure of $\Phi$.

**Proof.** Because of Lem. 2.5 we have (2.3), which implies (2.4). Denote by $k$ the order of the pole $u$ of $F$. Then $k \in \mathbb{N}$ and

$$\lim_{z \to u} (z - u)^k F(z) = A \neq O_q,$$  \hspace{1cm} (2.5)

In the case $k > 1$, we infer from (2.4) that

$$\lim_{r \to 1-0} [(ru - u)^k F(ru)] = \lim_{r \to 1-0} [(ru - u)^{k-1} \left(\lim_{r \to 1-0} [(ru - u)F(ru)]\right)] = O_{q \times q},$$

which contradicts (2.5). Thus $k = 1$ and the application of (2.4) completes the proof. \hspace{1cm} $\square$

Since every complex-valued function $f$ meromorphic in a region $D$ of $C$ can be written as $f = g/h$ with holomorphic functions $g, h: D \to C$, where $h$ does not vanish identically in $D$ (see, e.g., [3, Thm. 11.46]), we obtain:
Remark 2.8. For every $p \times q$ matrix-valued function $F$ meromorphic in a region $D$ of $\mathbb{C}$, there exist a holomorphic matrix-valued function $G: D \rightarrow \mathbb{C}^{p \times q}$ and a holomorphic function $h: D \rightarrow \mathbb{C}$ which does not vanish identically in $D$, such that $F = h^{-1}G$.

If $f$ is holomorphic at a point $z_0 \in \mathbb{C}$, then, for each $m \in \mathbb{N}_0$, we write $f^{(m)}(z_0)$ for the $m$th derivative of $f$ at $z_0$.

Lemma 2.9. Let $F$ be a $p \times q$ matrix-valued function meromorphic in a region $D$ of $\mathbb{C}$. In view of Rem. 2.8, let $G: D \rightarrow \mathbb{C}^{p \times q}$ and $h: D \rightarrow \mathbb{C}$ be holomorphic such that $h$ does not vanish identically in $D$ and that $F = h^{-1}G$ holds true. Suppose that $w \in D$ is a zero of $h$ with multiplicity $m \geq 0$. Then $w$ is a pole (including a removable singularity) of $F$, the order $k$ of the pole $w$ fulfills $0 \leq k \leq m$, and $h^{(m)}(w) \neq 0$ holds true. For all $\ell \in Z_{k,m}$, furthermore,

$$
\lim_{z \to w} [(z - w)^\ell F(z)] = \frac{m!}{(m - \ell)!h^{(m)}(w)} G^{(m-\ell)}(w). \quad (2.6)
$$

Proof. Obviously $w$ is a pole (or a removable singularity) of $F$ and $k$ fulfills $0 \leq k \leq m$. Since $h$ is holomorphic, there is an $r \in (0, +\infty)$ such that $K := K(w; r)$ is a subset of $D$ and $h(z) \neq 0$ for all $z \in K \setminus \{w\}$. Then $F$ is holomorphic in $K \setminus \{w\}$. Let $\ell \in Z_{k,m}$. Then there is a holomorphic function $\Phi_{\ell}: K \rightarrow \mathbb{C}^{p \times q}$ such that $F(z) = (z - w)^{-\ell} \Phi_{\ell}(z)$ for all $z \in K \setminus \{w\}$. Consequently,

$$
\lim_{z \to w} [(z - w)^\ell F(z)] = \Phi_{\ell}(w). \quad (2.7)
$$

Since $w$ is a zero of $h$ with multiplicity $m \geq \ell$, there exists a holomorphic function $\eta_{\ell}: D \rightarrow \mathbb{C}$ such that $h(z) = (z - w)^{\ell} \eta_{\ell}(z)$ holds true for all $z \in D$. Furthermore, we have

$$
h(z) = \sum_{j=m}^{\infty} \frac{h^{(j)}(w)}{j!} (z - w)^j
$$

for all $z \in K$, where $h^{(m)}(w) \neq 0$. Thus, for all $z \in K$, we conclude

$$
\eta_{\ell}(z) = \sum_{j=m}^{\infty} \frac{h^{(j)}(w)}{j!} (z - w)^{j-\ell}.
$$

Comparing the last equation with the Taylor series representation of $\eta_{\ell}$ centered at $w$, we obtain $\eta_{\ell}^{(s)}(w) = 0$ for all $s \in Z_{0,m-\ell-1}$ and

$$
\frac{\eta_{\ell}^{(m-\ell)}(w)}{(m-\ell)!} = \frac{h^{(m)}(w)}{m!}.
$$

Using the general Leibniz rule for differentiation of products, we get then

$$
(\eta_{\ell} \Phi_{\ell})^{(m-\ell)}(w) = \sum_{s=0}^{m-\ell} \left( \begin{array}{c} m-\ell \\ s \end{array} \right) \eta_{\ell}^{(s)}(w) \left[ \Phi_{\ell}^{(m-\ell-s)}(w) \right] = \frac{(m-\ell)!h^{(m)}(w)}{m!} \Phi_{\ell}(w),
$$

6
which, in view of $h^{(m)}(w) \neq 0$, implies
\[
\Phi_\ell(w) = \frac{m!}{(m - \ell)!h^{(m)}(w)}(\eta_\ell \Phi_\ell)^{(m-\ell)}(w).
\] (2.8)

Obviously, we have
\[
\eta_\ell(z) \Phi_\ell(z) = \eta_\ell(z) [(z - w)^\ell F(z)] = h(z) F(z) = G(z)
\]
for all $z \in K \setminus \{w\}$. Since $G$ is holomorphic, by continuity, this implies $(\eta_\ell \Phi_\ell)(z) = G(z)$ for all $z \in K$ and, hence $(\eta_\ell \Phi_\ell)^{(m-\ell)}(w) = G^{(m-\ell)}(w)$. Thus, from (2.7) and (2.8) we finally obtain (2.9).

**Lemma 2.10.** Let $D$ be a region of $\mathbb{C}$ such that $K(0;r) \subseteq D$ for some $r \in (1, +\infty)$ and let $F$ be a $q \times q$ matrix-valued function meromorphic in $D$ such that the restriction $\Phi$ of $F$ onto $D$ belongs to $\mathcal{C}_q(D)$. In view of Rem. 2.8 let $G : D \to \mathbb{C}^{q \times q}$ and $h : D \to \mathbb{C}$ be holomorphic such that $h$ does not vanish identically in $D$ and that $F = h^{-1} G$ holds true. Let $u \in T$ be a zero of $h$ with multiplicity $m > 0$. Then:

(a) $u$ is either a removable singularity or a simple pole of $F$.

(b) $h^{(m)}(u) \neq 0$ and
\[
\mu(\{u\}) = \frac{-m}{2uh^{(m)}(u)} G^{(m-1)}(u),
\] (2.9)

where $\mu$ is the Riesz-Herglotz measure of $\Phi$.

(c) If there is no $z \in D$ with $G(z) = O_{q \times q}$ and $h(z) = 0$, then $u$ is a pole of $F$.

(d) $u$ is a removable singularity of $F$ if and only if $G^{(m-1)}(u) = O_{q \times q}$ or equivalently $\mu(\{u\}) = O_{q \times q}$.

**Proof.** Obviously $h^{(m)}(u) \neq 0$ and $u$ is either a removable singularity or a pole of $F$, which then is simple according to Prop. 2.7, i.e., the order of the pole $u$ of $F$ is either 0 or 1. Thus, we can chose $\ell = 1$ in Lem. 2.9 and obtain
\[
\lim_{r \to 1^-}\frac{r^u - u}F(ru) = \frac{m}{h^{(m)}(u)} G^{(m-1)}(u).
\] (2.10)

Prop. 2.7 yields (2.10). Comparing (2.9) and (2.10), we get (2.9). The rest is plain.

Now we will extend the statement of Lem. 2.10 for the case of rational matrix-valued functions. For this reason we will first need some notation.

For each $A \in \mathbb{C}^{q \times q}$, let $\det A$ be the determinant of $A$ and let $A^\sharp$ be the classical adjoint of $A$ or classical adjugate (see, e.g., Horn/Johnson [14, p. 20]), so that $AA^\sharp = (\det A)I_q$ and $A^\sharp A = (\det A)I_q$. If $Q$ is a $q \times q$ matrix polynomial, then $Q^\sharp : \mathbb{C} \to \mathbb{C}^{q \times q}$ defined by $Q^\sharp(z) := (Q(z))^\sharp$ is obviously a matrix polynomial as well.
Proposition 2.11. Let $P$ and $Q$ be complex $q \times q$ matrix polynomials such that $\det Q$ does not vanish identically and the restriction $\Phi$ of $P Q^{-1}$ onto $\mathbb{D}$ belongs to $C_q(\mathbb{D})$. Let $u \in \mathbb{T}$ be a zero of $\det Q$ with multiplicity $m > 0$. Then $u$ is either a removable singularity or a simple pole of $P Q^{-1}$. Furthermore, $(\det Q^m)(u) \neq 0$ and

$$
\mu(\{u\}) = \frac{-m}{2u(\det Q)(m)(v)(P Q^*)(m-1)(u)},
$$

where $\mu$ is the Riesz-Herglotz measure of $\Phi$.

Proof. The functions $G := PQ$ and $h := \det Q$ are holomorphic in $\mathbb{C}$ such that $h$ does not vanish identically, and $F := P Q^{-1}$ is meromorphic in $\mathbb{C}$ and admits the representation $F = h^{-1}G$. Hence, the application of Lem. 2.10 completes the proof.

Proposition 2.12. Let $Q$ and $R$ be complex $q \times q$ matrix polynomials such that $\det Q$ does not vanish identically and the restriction $\Phi$ of $Q^{-1}R$ onto $\mathbb{D}$ belongs to $C_q(\mathbb{D})$. Let $u \in \mathbb{T}$ be a zero of $\det Q$ with multiplicity $m > 0$. Then $u$ is either a removable singularity or a simple pole of $Q^{-1}R$. Furthermore, $(\det Q^m)(u) \neq 0$ and

$$
\mu(\{u\}) = \frac{-m}{2u(\det Q)(m)(v)(Q^*R)(m-1)(u)},
$$

where $\mu$ is the Riesz-Herglotz measure of $\Phi$.

Proof. Apply Prop. 2.11 to $(Q^{-1}R)^T$.

As usual, if $\mathcal{M}$ is a finite subset of $\mathbb{C}^{p \times q}$, then the notation $\sum_{A \in \mathcal{M}} A$ should be understood as $O_{p \times q}$ in the case that $\mathcal{M}$ is empty. In the following, we continue to use the notations $\delta$ and $\delta_u$ to designate the linear Lebesgue measure on $(\mathbb{T}, \mathcal{B}_\mathbb{T})$ and the Dirac measure on $(\mathbb{T}, \mathcal{B}_\mathbb{T})$ with unit mass at $u \in \mathbb{T}$, respectively. Now we are able to derive the main result of this section.

Theorem 2.13. Let $r \in (1, +\infty)$, let $\mathcal{D}$ be a region of $\mathbb{C}$ such that $K(0; r) \subseteq \mathcal{D}$, and let $F$ be a $q \times q$ matrix-valued function meromorphic in $\mathcal{D}$ such that the restriction $\Phi$ of $F$ onto $\mathbb{D}$ belongs to $C_q(\mathbb{D})$. In view of Rem. 2.8, let $G: \mathcal{D} \to \mathbb{C}^{p \times q}$ and $h: \mathcal{D} \to \mathbb{C}$ be holomorphic functions such that $h$ does not vanish identically in $\mathcal{D}$ and that $F = h^{-1}G$ holds true. Then $\mathcal{N} := \{u \in \mathbb{T}: h(u) = 0\}$ is a finite subset of $\mathbb{T}$ and the following statements hold true:

(a) For all $u \in \mathcal{N}$, the inequality $h^{(m_u)}(u) \neq 0$ holds true, where $m_u$ is the multiplicity of $u$ as zero of $h$, and the matrix

$$
W_u := \frac{-m_u}{2u h^{(m_u)}(u) G^{(m_u-1)}(u)}
$$

is well defined and non-negative Hermitian, and coincides with $\mu(\{u\})$, where $\mu$ is the Riesz-Herglotz measure of $\Phi$.  

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(b) Let $\Delta : D \setminus \mathcal{N} \to \mathbb{C}^{q \times q}$ be defined by

$$
\Delta(z) := \sum_{u \in \mathcal{N}} \frac{u + z}{u - z} W_u. \tag{2.11}
$$

Then $\Theta := F - \Delta$ is a $q \times q$ matrix-valued function meromorphic in $D$ which is holomorphic in $K(0; r_0)$ for some $r_0 \in (1, r)$ and the restrictions of $\Theta$ and $\Delta$ onto $D$ both belong to $C^{q \times q}(D)$.

(c) The Riesz-Herglotz measure $\mu$ of $\Phi$ admits for all $B \in \mathcal{B}_T$ the representation

$$
\mu(B) = \frac{1}{2\pi} \int_B \text{Re}\Theta(\zeta) \Delta(d\zeta) + \sum_{u \in \mathcal{N}} W_u \delta_u(B). \tag{2.12}
$$

**Proof.** Since $h$ is a holomorphic function in $D$ which does not vanish identically in $D$ and since $T$ is a bounded subset of the interior of $D$, the set $\mathcal{N}$ is finite.

(a) This follows from Lem. 2.10.

(b) Obviously, $\Theta$ is meromorphic in $D$. According to Lem. 2.10, each $u \in \mathcal{N}$ is either a removable singularity or a simple pole of $F$ and $\mu(\{u\}) = W_u$ holds true. Prop. 2.7 yields then

$$
\lim_{z \to u} [(z - u)F(z)] = -2uW_u \tag{2.13}
$$

for each $u \in \mathcal{N}$. Obviously, $\Theta$ is holomorphic at all points $z \in T \setminus \mathcal{N}$.

Let us now assume that $u \in \mathcal{N}$. Then $h(u) = 0$ and there is a positive real number $r_u$ such that $K := K(u; r_u)$ is a subset of $D$ and $h(z) \neq 0$ for all $z \in K \setminus \{u\}$. In particular, the restriction $\theta$ of $\Theta$ onto $K \setminus \{u\}$ is holomorphic and

$$
(z - u)\theta(z) = (z - u)F(z) + (u + z)W_u - (z - u) \sum_{\zeta \in \mathcal{N} \setminus \{u\}} \frac{\zeta + z}{\zeta - z} W_\zeta \tag{2.14}
$$

is fulfilled for each $z \in K \setminus \{u\}$. Consequently, (2.13) and (2.14) provide us

$$
O_{q \times q} = -2uW_u + (u + u)W_u - (u - u) \sum_{\zeta \in \mathcal{N} \setminus \{u\}} \frac{\zeta + z}{\zeta - z} W_\zeta,
$$

$$
= \lim_{z \to u} [(z - u)F(z)] + (u + \lim_{z \to u} z)W_u - \left(\lim_{z \to u} (z - u) - u\right) \sum_{\zeta \in \mathcal{N} \setminus \{u\}} \frac{\zeta + z}{\zeta - z} W_u
$$

$$
= \lim_{z \to u} \left[ (z - u)F(z) + (u + z)W_u - (z - u) \sum_{\zeta \in \mathcal{N} \setminus \{u\}} \frac{\zeta + z}{\zeta - z} W_\zeta \right] = \lim_{z \to u} [(z - u)\theta(z)].
$$

In view of Riemann’s theorem on removable singularities, this implies that $u$ is a removable singularity for $\theta$. In particular, $\Theta$ is holomorphic at $u$. Thus, $\Theta$ is holomorphic at each $\zeta \in T$. Taking into account $D \cap \mathcal{N} = \emptyset$, we see then that $\Theta$ is holomorphic at each point $z \in D \cup T$. Since $\Theta$ is meromorphic in $D$ and $K(0; r)$ is bounded, $\Theta$ has only a finite number of poles in $K(0; r) \setminus (D \cup T)$. Thus, there is an $r_0 \in (1, r)$ such that $\Theta$
is holomorphic in $K(0; r_0)$. In particular, the restriction $\Psi$ of $\Theta$ onto $\mathbb{D}$ is holomorphic. Because of $\mathbb{D} \cap N = \emptyset$, we get

$$\Theta(z) = F(z) - \Delta(z) = \Phi(z) - \sum_{u \in N} \frac{u + z}{u - z} W_u$$

(2.15)

for each $z \in \mathbb{D}$. Because of $\mu(\{u\}) = W_u$ for each $u \in N$, we conclude that

$$\rho := \mu - \sum_{u \in N} W_u \delta_u$$

(2.16)

fulfills $\rho(\mathfrak{B}_T) \subseteq \mathbb{C}^{q \times q}$ and, hence, that $\rho$ belongs to $\mathcal{M}_q^2(\mathbb{T})$. Since $\mu$ is the Riesz-Herglotz measure of $\Phi$, we have (2.11) for each $z \in \mathbb{D}$. Thus, we obtain from (2.15) then

$$\Theta(z) = \int \frac{\zeta + z}{\zeta - z} \mu(d\zeta) + i \Im \Phi(0) - \sum_{u \in N} \left( \int \frac{\zeta + z}{\zeta - z} \delta_u(d\zeta) \right) W_u$$

$$= \int \frac{\zeta + z}{\zeta - z} \rho(d\zeta) + i \Im \Phi(0)$$

for every choice of $z$ in $\mathbb{D}$. Consequently, from Thm. 2.11 we see that $\Psi$ belongs to $\mathcal{C}_q(\mathbb{D})$ and that $\rho$ is the Riesz-Herglotz measure of $\Psi$. Since the matrix $W_u$ is non-negative Hermitian for all $u \in N$, Thm. 2.11(b) yields in view of (2.11) furthermore, that the restriction of $\Delta$ onto $\mathbb{D}$ belongs to $\mathcal{C}_q(\mathbb{D})$ as well.

Applying Lem. 2.4 shows then that $\rho(B) = \frac{1}{2\pi} \int_B \Re \Theta(\zeta) \Delta(d\zeta)$ holds true for each $B \in \mathfrak{B}_T$. Thus, from (2.16), for each $B \in \mathfrak{B}_T$, we get (2.12).

A closer look at Thm. 2.13 and its proof shows that the Riesz-Herglotz measures $\rho$ and $\sum_{u \in N} W_u \delta_u$ of $\Psi$ and the restriction of $\Delta$ onto $\mathbb{D}$, respectively, are exactly the absolutely continuous and singular part in the Lebesgue decomposition of the Riesz-Herglotz measure of $\Phi$ with respect to $\lambda$. In particular, the singular part is a discrete measure which is concentrated on a finite number of points from $\mathbb{T}$ and there is no nontrivial singular continuous part. The absolutely continuous part with respect to $\lambda$ possesses a continuous Radon-Nikodym density with respect to $\lambda$.

Theorem 2.14. Let $P$ and $Q$ be $q \times q$ matrix polynomials such that $\det Q$ does not vanish identically and that the restriction $\Phi$ of $PQ^{-1}$ onto $\mathbb{D}$ belongs to $\mathcal{C}_q(\mathbb{D})$. Then $N := \{u \in \mathbb{T} : \det Q(u) = 0\}$ is a finite subset of $\mathbb{T}$ and the following statements hold true:

(a) For all $u \in N$, the inequality $(\det Q)^{(m_u)}(u) \neq 0$ holds true, where $m_u$ is the multiplicity of $u$ as zero of $\det Q$, and

$$W_u := \frac{-m_u}{2u(\det Q)^{(m_u)}(u)}(PQ^2)^{(m_u-1)}(u)$$

is a well-defined and non-negative Hermitian matrix which coincides with $\mu(\{u\})$, where $\mu$ is the Riesz-Herglotz measure of $\Phi$. 

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(b) Let $\Delta: \mathcal{D} \setminus \mathcal{N} \rightarrow \mathbb{C}^{q \times q}$ be defined by (2.11). Then $\Theta:= PQ^{-1} - \Delta$ is a rational $q \times q$ matrix-valued function which is holomorphic in $K(0;r)$ for some $r \in (1, +\infty)$ and the restrictions of $\Theta$ and $\Delta$ onto $\mathbb{D}$ both belong to $\mathcal{C}_q(\mathbb{D})$.

(c) The Riesz-Herglotz measure $\mu$ of $\Phi$ admits the representation (2.12) for all $B \in \mathbb{B}_\mathbb{T}$.

Proof. Thm. 2.14 is an immediate consequence of Thm. 2.13 if one chooses $\mathcal{D} = \mathbb{C}$, $h = \det Q$ and $G = PQ^2$. \hfill \Box

3. On the truncated matricial trigonometric moment problem

A matricial version of a theorem due to G. Herglotz shows in particular that if $\mu$ belongs to $\mathcal{M}_\geq^q(\mathbb{T})$, then it is uniquely determined by the sequence $(C_j^{\mu})_{j=-\infty}^{\infty}$ of its Fourier coefficients given by (2.2). To recall this theorem in a version which is convenient for our further considerations, let us modify the notion of Toeplitz non-negativity. Obviously, if $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and if $(C_j)_j^{\kappa}$ is a Toeplitz non-negative definite sequence, then $C_{-j} = C_j^* \kappa$ for each $j \in \mathbb{Z}_{-\infty, \kappa}$. Thus, if $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, then a sequence $(C_j)_j^{\kappa}$ is called Toeplitz non-negative definite (resp. Toeplitz positive definite) if $(C_j)_j^{\kappa}$ is Toeplitz non-negative definite (resp. Toeplitz positive definite), where $C_{-j} := C_j^*$ for each $j \in \mathbb{Z}_{0, \kappa}$.

Theorem 3.1 (G. Herglotz). Let $(C_j)_j^{\infty} = 0$ be a sequence of complex $q \times q$ matrices. Then there exists a $\mu \in \mathcal{M}_\geq^q(\mathbb{T})$ such that $C_j^{\mu} = C_j$ for each $j \in \mathbb{N}_0$ if and only if the sequence $(C_j)_j^{\infty}$ is Toeplitz non-negative definite. In this case, the measure $\mu$ is unique.

In view of the fact that $C_j^{\mu} = (C_j^{\mu})^*$ holds true for each $\mu \in \mathcal{M}_\geq^q(\mathbb{T})$ and each $j \in \mathbb{Z}$, a proof of Thm. 3.1 is given, e.g., in [5] Thm. 2.2.1, pp. 70/71.

In the context of the truncated trigonometric moment problem, only a finite sequence of Fourier coefficients is prescribed:

**TMP**: Let $n \in \mathbb{N}_0$ and let $(C_j)_j^{n} = 0$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_\geq^q(T, (C_j)_j^{n} = 0)$ of all $\mu \in \mathcal{M}_\geq^q(\mathbb{T})$ which fulfill $C_j^{\mu} = C_j$ for each $j \in \mathbb{Z}_{0, n}$.

The answer to the question of solvability of Problem TMP is as follows:

**Theorem 3.2.** Let $n \in \mathbb{N}_0$ and let $(C_j)_j^{n} = 0$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_\geq^q(T, (C_j)_j^{n} = 0)$ is non-empty if and only if the sequence $(C_j)_j^{n} = 0$ is Toeplitz non-negative definite.

Ando [11] gave a proof of Thm. 3.2 with the aid of the Naimark Dilation Theorem. An alternate proof stated in [5] Thm. 3.4.1, p. 123 is connected to Thm. 3.3 below, which gives an answer to the following matrix extension problem:

**MEP**: Let $n \in \mathbb{N}_0$ and let $(C_j)_j^{n} = 0$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{T}(C_j)_j^{n+1} = 0$ of all complex $q \times q$ matrices $C_{n+1}$ for which the sequence $(C_j)_j^{n+1}$ is Toeplitz non-negative definite.
The description of $T[(C_j)_{j=0}^n]$, we will recall here, is given by using the notion of a matrix ball: For arbitrary choice of $M \in \mathbb{C}^{p \times q}$, $A \in \mathbb{C}^{p \times p}$, and $B \in \mathbb{C}^{q \times q}$, the set $\mathcal{R}(M; A, B)$ of all $X \in \mathbb{C}^{p \times q}$ which admit a representation $X = M + AB$ with some contractive complex $p \times q$ matrix $K$ is said to be the matrix ball with center $M$, left semi-radius $A$, and right semi-radius $B$. A detailed theory of (more general) operator balls was worked out by Yu. L. Smul'jan [18] (see also [5, Section 1.5] for the matrix case). To give a parametrization of $T[(C_j)_{j=0}^n]$ with the aid of matrix balls, we introduce some further notations. For each $A \in \mathbb{C}^{p \times q}$, let $A^\dagger$ be the Moore-Penrose inverse of $A$. By definition, $A^\dagger$ is the unique matrix from $\mathbb{C}^{q \times p}$ which satisfies the four equations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$$ 

Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(C_j)_{j=0}^n$ be a sequence of complex $q \times q$ matrices. For every $j \in \mathbb{Z}_{0, \kappa}$, let $C_{j,-} := C_j$. Furthermore, for each $n \in \mathbb{Z}_{0, \kappa}$, let

$$T_n := [C_j-k]^n_{j,k=0}, \quad Y_n := \text{col}(C_j)_{j=1}^n, \quad \text{and} \quad Z_n := [C_n, C_{n-1}, \ldots, C_1]. \quad (3.1)$$

Let

$$M_1 := O_{q \times q}, \quad L_1 := C_0, \quad \text{and} \quad R_1 := C_0. \quad (3.2)$$

If $\kappa \geq 1$, then, for each $n \in \mathbb{Z}_{1, \kappa}$, let

$$M_{n+1} := Z_nT_{n-1}^\dagger Y_n, \quad L_{n+1} := C_0 - Z_nT_{n-1}^\dagger Z_n^*, \quad \text{and} \quad R_{n+1} := C_0 - Y_n^*T_{n-1}Y_n. \quad (3.3)$$

In order to formulate an answer to Problem MEP, we observe, that, if $(C_j)_{j=0}^n$ is Toeplitz non-negative definite, then, for each $n \in \mathbb{Z}_{0, \kappa}$, the matrices $L_{n+1}$ and $R_{n+1}$ are both non-negative Hermitian (see, e.g., [5 Rem. 3.4.1, p. 122]).

**Theorem 3.3.** Let $n \in \mathbb{N}_0$ and let $(C_j)_{j=0}^n$ be a sequence of complex $q \times q$ matrices. Then $T[(C_j)_{j=0}^n] \neq \emptyset$ if and only if the sequence $(C_j)_{j=0}^n$ is Toeplitz non-negative definite. In this case, $T[(C_j)_{j=0}^n] = \mathcal{R}(M_{n+1}; \sqrt{L_{n+1}}, \sqrt{R_{n+1}})$.

A proof of Thm. [3, Thm. 1], (see also [3, Theorems 3.4.1 and 3.4.2, pp. 122/123]).

Observe that the parameters $M_{n+1}$, $L_{n+1}$, and $R_{n+1}$ of the matrix ball stated in Thm. [3] admit a stochastic interpretation (see [3 Part I]).

**Lemma 3.4.** Let $n \in \mathbb{N}$ and let $\mu \in \mathcal{M}_+^q[T, (C_j)_{j=0}^n]$, where $(C_j)_{j=0}^n$ is a Toeplitz non-negative definite sequence of complex $q \times q$ matrices. If $\text{rank} T_n \leq n$, then there exists a subset $\mathcal{N}$ of $\mathbb{T}$ with at most $nq$ elements such that $\mu(\mathbb{T} \setminus \mathcal{N}) = O_{q \times q}$.

**Proof.** Let $\mu = [\mu_{j,k}]_{j,k=1}^q$ and denote by $e_1^{(q)}, e_2^{(q)}, \ldots, e_q^{(q)}$ the canonical basis of $\mathbb{C}^q$. We consider an arbitrary $\ell \in \mathbb{Z}_{1,q}$. Then $T_n^{(\ell)} := [C_j[k]]_{j,k=0}^n$ admits the representation

$$T_n^{(\ell)} = \left[\text{diag}_{n+1}(e_\ell^{(q)})\right]^* T_n \left[\text{diag}_{n+1}(e_\ell^{(q)})\right].$$
with the block diagonal matrix \( \text{diag}_n(e^{(q)i}) \in \mathbb{C}^{(n+1)q \times (n+1)} \) with diagonal blocks \( e^{(q)i} \).

Consequently,

\[
\text{rank} \, T_n^{(n)} \leq \text{rank} \, T_n \leq n.
\]

Hence, there exists a vector \( v^{(n)} \in \mathbb{C}^{n+1} \setminus \{O_{(n+1) \times 1}\} \) and \( T_n^{(n)} v^{(n)} = O_{(n+1) \times 1} \). With \( v^{(n)} = \text{col}(v_j^{(n)})_{j=0}^n \), then

\[
0 = (v^{(n)})^* T_n^{(n)} v^{(n)} = \int_{\mathcal{T}} \left| \sum_{j=0}^n v_j^{(n)} \zeta_j^j \right|^2 \mu_{\ell\ell}(d\zeta)
\]

follows. Since \( \ell \in \mathbb{Z}_{1,q} \) was arbitrarily chosen, we obtain \( \text{tr} \, \mu(\mathbb{T} \setminus \mathcal{N}) = 0 \), where \( \mathcal{N} \) consists of all modulus 1 roots of the polynomial \( \prod_{\ell=1}^q \sum_{j=0}^n v_j^{(n)} \zeta_j^j \), which is of degree at most \( nq \). Thus, by observing that \( \mu \) is absolutely continuous with respect to \( \text{tr} \, \mu \), the proof is complete.

\[\square\]

4. Central non-negative Hermitian measures

In this section, we study so-called central non-negative Hermitian measures.

Let \( \kappa \in \mathbb{N} \cup \{+\infty\} \) and let \( (C_j)^n_{j=0} \) be a sequence of complex \( q \times q \) matrices. If \( k \in \mathbb{Z}_{1,\kappa} \) is such that \( C_j = M_j \) for all \( j \in \mathbb{Z}_{k,\kappa} \), where \( M_j \) is given by (3.2) and (3.3), then \( (C_j)^n_{j=0} \) is called \textit{central of order} \( k \). If in the case \( \kappa \geq 2 \) the sequence \( (C_j)^n_{j=0} \) is additionally not central of order \( k - 1 \), then \( (C_j)^n_{j=0} \) is called \textit{central of minimal order} \( k \). If there exists a number \( \ell \in \mathbb{Z}_{1,\kappa} \) such that \( (C_j)^n_{j=0} \) is central of order \( \ell \), then \( (C_j)^n_{j=0} \) is simply called \textit{central}.

Let \( n \in \mathbb{N}_0 \) and let \( (C_j)^n_{j=0} \) be a sequence of complex \( q \times q \) matrices. Let the sequence \( (C_j)^n_{j=n+1} \) be recursively defined by \( C_j := M_j \), where \( M_j \) is given by (3.3). Then \( (C_j)^n_{j=0} \) is called \textit{central sequence corresponding to} \( (C_j)^n_{j=0} \).

\textbf{Remark 4.1.} Let \( n \in \mathbb{N}_0 \) and let \( (C_j)^n_{j=0} \) be a Toeplitz non-negative definite sequence of complex \( q \times q \) matrices. According to Thm. 3.3 then the central sequence corresponding to \( (C_j)^n_{j=0} \) is Toeplitz non-negative definite as well.

Observe that the elements of central Toeplitz non-negative definite sequences fulfill special recursion formulas (see [8] Part V, Thm. 32, p. 303 or [5] Thm. 3.4.3, p. 124). Furthermore, if \( n \in \mathbb{N}_0 \) and if \( (C_j)^n_{j=0} \) is a Toeplitz positive definite sequence of complex \( q \times q \) matrices, then the central sequence corresponding to \( (C_j)^n_{j=0} \) is Toeplitz positive definite (see [5] Thm. 3.4.1(b)).

A non-negative Hermitian measure \( \mu \) belonging to \( \mathcal{M}_\mathbb{T}^q(\mathbb{T}) \) is said to be \textit{central} if \( (C_j)^n_{j=0} \) is central. If \( k \in \mathbb{N} \) is such that \( (C_j)^n_{j=0} \) is central of \( (\text{minimal}) \) order \( k \), then \( \mu \) is called \textit{central of (minimal) order} \( k \).

\textbf{Remark 4.2.} Let \( n \in \mathbb{N}_0 \), let \( (C_j)^n_{j=0} \) be a Toeplitz non-negative definite sequence of complex \( q \times q \) matrices and let \( (C_j)^n_{j=0} \) be the central sequence corresponding to \( (C_j)^n_{j=0} \). According to Thm. 3.1 there is a unique non-negative Hermitian measure \( \mu \) belonging to \( \mathcal{M}_\mathbb{T}^q(\mathbb{T}) \) such that its Fourier coefficients fulfill \( C_j^{[n]} = C_j \) for each \( j \in \mathbb{N}_0 \).
non-negative Hermitian $q \times q$ measure $\mu$ is called the central measure corresponding to $(C_j)_{j=0}^{\infty}$.

**Proposition 4.3.** Let $n \in \mathbb{N}$ and let $(C_j)_{j=0}^{n}$ be a Toeplitz non-negative definite sequence of complex $q \times q$ matrices. Suppose $\text{rank} \, T_n = \text{rank} \, T_{n-1}$. Then there exists a finite subset $\mathcal{N}$ of $\mathbb{T}$ such that the central measure $\mu_c$ corresponding to $(C_j)_{j=0}^{n}$ fulfills $\mu(\mathbb{T} \setminus \mathcal{N}) = O_{q \times q}$.

**Proof.** We have $\mu_c \in \mathcal{M}^q_{\geq}[\mathbb{T}, (C_j)_{j=0}^{\infty}]$ where $(C_j)_{j=0}^{\infty}$ is the central Toeplitz non-negative definite sequence corresponding to $(C_j)_{j=0}^{n}$. According to [12] Prop. 2.26, we get $L_{\ell+1} = O$ for all $\ell \in \mathbb{Z}_{n, +\infty}$. In view of [12] Lem. 2.25, then $\text{rank} \, T_\ell = \text{rank} \, T_{n-1}$ follows for all $\ell \in \mathbb{Z}_{n, +\infty}$. In particular, $\text{rank} \, T_{nq} = \text{rank} \, T_{n-1} \leq nq$. Since $\mu_c$ belongs to $\mathcal{M}^q_{\geq}[\mathbb{T}, (C_j)_{j=0}^{nq}]$, the application of Lem. 3.4 completes the proof.

If $n \in \mathbb{N}$ and if $(C_j)_{j=0}^{n}$ is a Toeplitz positive definite sequence of complex $q \times q$ matrices, then the central measure corresponding to $(C_j)_{j=0}^{n}$ is the unique measure in $\mathcal{M}^q_{\geq}[\mathbb{T}, (C_j)_{j=0}^{\infty}]$ with maximal entropy (see [8] Part II, Thm. 10).

**Remark 4.4.** Let $(C_j)_{j=0}^{\infty}$ be a Toeplitz non-negative definite sequence which is a central of order 0. Then it is readily checked that $C_k = O_{q \times q}$ for each $k \in \mathbb{N}$ and that the central measure $\mu$ corresponding to $(C_j)_{j=0}^{0}$ admits the representation $\mu = \frac{1}{2\pi} C_0 \Delta$, where $\Delta$ is the linear Lebesgue measure defined on $\mathcal{B}_\mathbb{T}$.

Now we describe the central measure corresponding to a finite Toeplitz positive definite sequence of complex $q \times q$ matrices.

**Theorem 4.5.** Let $n \in \mathbb{N}_0$ and let $(C_j)_{j=0}^{n}$ be a Toeplitz positive definite sequence of complex $q \times q$ matrices. Let $T_{n}^{-1} = [t_{jk}^{[n]}]_{j,k=0}^{n}$ be the $q \times q$ block representation of $T_n^{-1}$, and let the matrix polynomials $A_n: \mathbb{C} \to \mathbb{C}^{q \times q}$ and $B_n: \mathbb{C} \to \mathbb{C}^{q \times q}$ be given by

$$A_n(z) := \sum_{j=0}^{n} t_{j0}^{[n]} z^j \quad \text{and} \quad B_n(z) := \sum_{j=0}^{n} t_{n,n-j}^{[n]} z^j. \quad (4.1)$$

Then $\det A_n(z) \neq 0$ and $\det B_n(z) \neq 0$ hold true for each $z \in \mathbb{D} \cup \mathbb{T}$ and the central measure $\mu$ for $(C_j)_{j=0}^{n}$ admits the representations

$$\mu(B) = \frac{1}{2\pi} \int_B [A_n(\zeta)]^{-*} A_n(0) [A_n(\zeta)]^{-1} \Delta(d\zeta) \quad (4.2)$$

and

$$\mu(B) = \frac{1}{2\pi} \int_B [B_n(\zeta)]^{-1} B_n(0) [B_n(\zeta)]^{-*} \Delta(d\zeta) \quad (4.3)$$

for each $B \in \mathcal{B}_\mathbb{T}$, where $\Delta$ is the linear Lebesgue measure defined on $\mathcal{B}_\mathbb{T}$.

The fact that $\det A_n(z) \neq 0$ or $\det B_n(z) \neq 0$ for $z \in \mathbb{D} \cup \mathbb{T}$ can be proved in various ways (see e.g. Ellis/Gohberg [7], Section 4.4) or Delsarte/Genin/Kamp [4], Thm. 6], and [9] Prop. 3.6.3, p. 336], where the connection to the truncated matricial trigonometric moment problem is used.
The representations (4.4) and (4.3) are proved in [8] Part III, Thm. 16, Rem. 18, pp. 332/333.

The measure given via (4.2) was studied in a different framework by Del-sarte/Genin/Kamp [4]. These authors considered a non-negative Hermitian measure \( \mu \in \mathcal{M}_q^\sigma(\mathbb{T}) \) with Toeplitz positive definite sequence \((C_j^{[\mu]})_{j=0}^\infty\) of Fourier coefficients. Then it was shown in [4, Thm. 9] that, for each \( n \in \mathbb{N}_0 \), the measure constructed via (4.2) from the Toeplitz positive definite sequence \((C_j^{[\mu]})_{j=0}^n\) is a solution of the truncated trigonometric moment problem associated with the sequence \((C_j^{[\mu]})_{j=0}^n\). The main topic of [4] is to study left and right orthonormal systems of \( q \times q \) matrix polynomials associated with the measure \( \mu \). It is shown in [4] that these polynomials are intimately connected with the polynomials \( A_n \) and \( B_n \) which were defined in Thm. 4.5.

**Proposition 4.6.** Let \( P \) be a complex \( q \times q \) matrix polynomial of degree \( n \) such that \( P(0) \) is positive Hermitian and \( \det P(z) \neq 0 \) for all \( z \in \mathbb{D} \cup \mathbb{T} \). Let \( g: \mathbb{T} \to \mathbb{C}^{q \times q} \) be defined by \( g(\zeta) := [P(\zeta)]^{-1}P(0)[P(\zeta)]^{-1} \). Then \( \mu: \mathcal{B}_T \to \mathbb{C}^{q \times q} \) defined by \( \mu(B) := \frac{1}{2\pi} \int_B g(\zeta) d\zeta \) belongs to \( \mathcal{M}_q^\sigma(\mathbb{T}) \) and is central of order \( n + 1 \).

**Proof.** Obviously, \( \mu \) belongs to \( \mathcal{M}_q^\sigma(\mathbb{T}) \). Let \((C_j)_{j=-\infty}^\infty \) be the Fourier coefficients of \( \mu \). According to [10] Lem. 2, then \( T_n^\mathbb{T} \) is positive Hermitian, i.e. the sequence \((C_j)_{j=0}^n \) is Toeplitz positive definite, and \( P \) coincides with the matrix polynomial \( A_n \) given in (4.1). In view of Thm. 4.5, thus \( \mu \) is the central measure corresponding to \((C_j)_{j=0}^n \). In particular, \((C_j)_{j=0}^\infty \) is the central sequence corresponding to \((C_j)_{j=0}^n \) and therefore \((C_j)_{j=0}^\infty \) is central of order \( n + 1 \). Hence, \( \mu \) is central of order \( n + 1 \). \( \square \)

Using [10] Lem. 3 instead of [10] Lem. 2], one can analogously prove the following dual result:

**Proposition 4.7.** Let \( Q \) be a complex \( q \times q \) matrix polynomial of degree \( n \) such that \( Q(0) \) is positive Hermitian and \( \det Q(z) \neq 0 \) for all \( z \in \mathbb{D} \cup \mathbb{T} \). Let \( h: \mathbb{T} \to \mathbb{C}^{q \times q} \) be defined by \( h(\zeta) := [Q(\zeta)]^{-1}Q(0)[Q(\zeta)]^{-1} \). Then \( \mu: \mathcal{B}_T \to \mathbb{C}^{q \times q} \) defined by \( \mu(B) := \frac{1}{2\pi} \int_B h(\zeta) d\zeta \) belongs to \( \mathcal{M}_q^\sigma(\mathbb{T}) \) and is central of order \( n + 1 \).

**Proposition 4.8.** Let \( n \in \mathbb{N}_0 \) and let \( \mu \in \mathcal{M}_q^\sigma(\mathbb{T}) \) be central of order \( n + 1 \) with Fourier coefficients \((C_j)_{j=-\infty}^\infty \) such that the sequence \((C_j)_{j=0}^n \) is Toeplitz positive definite. Then the matrix polynomials \( A_n \) and \( B_n \) given by (4.1) fulfill \( \det A_n(z) \neq 0 \) and \( \det B_n(z) \neq 0 \) for all \( z \in \mathbb{D} \cup \mathbb{T} \), and \( \mu \) admits the representations (4.4) and (4.3) for all \( B \in \mathcal{B}_T \).

**Proof.** Since \( \mu \) is central of order \( n + 1 \), the sequence \((C_j)_{j=0}^\infty \) is central of order \( n + 1 \). In particular, \((C_j)_{j=0}^n \) is the central sequence corresponding to \((C_j)_{j=0}^n \). Hence, \( \mu \) is the central measure corresponding to \((C_j)_{j=0}^n \). The application of Thm. 4.5 completes the proof. \( \square \)

In the general situation of an arbitrarily given Toeplitz non-negative definite sequence \((C_j)_{j=0}^n \) of complex \( q \times q \) matrices, the central measure corresponding to \((C_j)_{j=0}^n \) can also be represented in a closed form. To do this, we will use the results on matrix-valued Carathéodory functions defined on the open unit disk \( \mathbb{D} \) which were obtained in Section 2.
5. Central matrix-valued Carathéodory functions

In this section, we recall an explicit representation of the Riesz-Herglotz measure of an arbitrary central matrix-valued Carathéodory function.

**Remark 5.1.** Let \((C_j)_{j=0}^{\infty}\) be a Toeplitz non-negative definite sequence of complex \(q \times q\) matrices and let \((\Gamma_j)_{j=0}^{\infty}\) be given by

\[
\Gamma_0 := C_0 \quad \text{and} \quad \Gamma_j := 2C_j \tag{5.1}
\]

for each \(j \in \mathbb{N}\). Furthermore, let \(\mu \in \mathcal{M}_{q \geq [T, (C_j)_{j=0}^{\infty}]}\). In view of \(\Gamma_0^* = \Gamma_0\), Theorems 3.1 and 2.1 show that \(\mu\) belongs to \(\mathcal{M}_{q \geq [T, (C_j)_{j=0}^{\infty}]}\) if and only if \(\mu\) is the Riesz-Herglotz measure of the \(q \times q\) Carathéodory function \(\Phi : \mathbb{D} \to \mathbb{C}^{q \times q}\) defined by

\[
\Phi(z) := \int_T \frac{\zeta + z}{\zeta - z} \mu(d\zeta). \tag{5.2}
\]

The well-studied matricial version of the classical Carathéodory interpolation problem consists of the following:

**CIP:** Let \(\kappa \in \mathbb{N}_0 \cup \{+\infty\}\) and let \((\Gamma_j)_{j=0}^{\kappa}\) be a sequence of complex \(q \times q\) matrices. Describe the set \(\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^{\kappa}]\) of all \(\Phi \in \mathcal{C}_q(\mathbb{D})\) such that \(\frac{1}{j!}\Phi^{(j)}(0) = \Gamma_j\) holds true for each \(j \in \mathbb{Z}_{0,\kappa}\).

In order to formulate a criterion for the solvability of Problem CIP, we recall the notion of a Carathéodory sequence. If \(\kappa \in \mathbb{N}_0 \cup \{+\infty\}\), then a sequence \((\Gamma_j)_{j=0}^{\kappa}\) is called a \(q \times q\) Carathéodory sequence if, for each \(n \in \mathbb{Z}_{0,\kappa}\), the matrix \(\text{Re} S_n\) is non-negative Hermitian, where \(S_n\) is given by

\[
S_n := \begin{bmatrix}
\Gamma_0 & 0 & \ldots & 0 & 0 \\
\Gamma_1 & \Gamma_0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Gamma_{n-1} & \Gamma_{n-2} & \ldots & \Gamma_0 & 0 \\
\Gamma_n & \Gamma_{n-1} & \ldots & \Gamma_1 & \Gamma_0 \\
\end{bmatrix}. \tag{5.3}
\]

**Theorem 5.2.** Let \(\kappa \in \mathbb{N}_0 \cup \{+\infty\}\) and let \((\Gamma_j)_{j=0}^{\kappa}\) be a sequence of complex \(q \times q\) matrices. Then \(\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^{\kappa}] \neq \emptyset\) if and only if \((\Gamma_j)_{j=0}^{\kappa}\) is a \(q \times q\) Carathéodory sequence.

In the case \(\kappa = \infty\), Thm. 5.2 is a consequence of Theorems 2.1 and 2.1. In the case \(\kappa \in \mathbb{N}_0\), a proof of Thm. 5.2 can be found, e.g., in [8, Part I, Section 4].

**Corollary 5.3.** Let \((\Gamma_j)_{j=0}^{\infty}\) be a sequence of complex \(q \times q\) matrices. Then \(\Phi : \mathbb{D} \to \mathbb{C}^{q \times q}\) defined by

\[
\Phi(z) := \sum_{j=0}^{\infty} z^j \Gamma_j \tag{5.4}
\]

belongs to \(\mathcal{C}_q(\mathbb{D})\) if and only if \((\Gamma_j)_{j=0}^{\infty}\) is a \(q \times q\) Carathéodory sequence.
Proof. Apply Thm. [5.2]

Remark 5.4. If $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and a sequence $(\Gamma_j)^n_{j=0}$ of complex $q \times q$ matrices are given, then it is readily checked that $(\Gamma_j)^n_{j=0}$ is a $q \times q$ Carathéodory sequence if and only if the sequence $(C_j)^n_{j=0}$ defined by

$$C_0 := \text{Re} \Gamma_0 \quad \text{and} \quad C_j := \frac{1}{2} \Gamma_j \quad (\text{5.5})$$

for each $j \in \mathbb{Z}_{1,\kappa}$ is Toeplitz non-negative definite.

Let $\kappa \in \mathbb{N} \cup \{+\infty\}$, let $(\Gamma_j)^n_{j=0}$ be a sequence of complex $q \times q$ matrices, and let the sequence $(C_j)^n_{j=0}$ be given by (5.5) for all $j \in \mathbb{Z}_{1,\kappa}$. If $k \in \mathbb{Z}_{1,\kappa}$ is such that $(C_j)^n_{j=0}$ is central of (minimal) order $k$, then $(\Gamma_j)^c_{j=0}$ is called C-central of (minimal) order $k$. If there exists a number $\ell \in \mathbb{Z}_{1,\kappa}$ such that $(\Gamma_j)^n_{j=0}$ is C-central of order $\ell$, then $(\Gamma_j)^n_{j=0}$ is simply called C-central.

Let $n \in \mathbb{N}_0$, let $(\Gamma_j)^n_{j=0}$ be a sequence of complex $q \times q$ matrices, and let the sequence $(C_j)^n_{j=0}$ be given by (5.5) for all $j \in \mathbb{Z}_{0,n}$. Let the sequence $(\Gamma_j)^\infty_{j=n+1}$ be given by $\Gamma_j := 2C_j$, where $(\Gamma_j)^n_{j=0}$ is the central sequence corresponding to $(C_j)^n_{j=0}$. Then $(\Gamma_j)^\infty_{j=0}$ is called the C-central sequence corresponding to $(\Gamma_j)^n_{j=0}$.

Remark 5.5. Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)^n_{j=0}$ be a $q \times q$ Carathéodory sequence. According to Remarks [5.4] and [4.1] then the C-central sequence corresponding to $(\Gamma_j)^n_{j=0}$ is a $q \times q$ Carathéodory sequence.

Let $\Phi \in \mathcal{C}_q(\mathbb{D})$ with Taylor series representation (5.1). If $k \in \mathbb{N}$ is such that $(\Gamma_j)^\infty_{j=0}$ is C-central of (minimal) order $k$, then $\Phi$ is called central of (minimal) order $k$. If there exists a number $\ell \in \mathbb{N}$ such that $\Phi$ is central of order $\ell$, then $\Phi$ is simply called central.

Remark 5.6. Let $n \in \mathbb{N}_0$, let $(\Gamma_j)^n_{j=0}$ be a $q \times q$ Carathéodory sequence, and let $(\Gamma_j)^\infty_{j=0}$ be the C-central sequence corresponding to $(C_j)^n_{j=0}$. According to Rem. [5.7] and Cor. [5.3] then $\Phi : \mathbb{D} \rightarrow \mathbb{C}^{q \times q}$ given by (5.4) belongs to $\mathcal{C}_q(\mathbb{D})$. This function $\Phi$ is called the central Carathéodory function corresponding to $(\Gamma_j)^n_{j=0}$.

Remark 5.7. Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)^n_{j=0}$ be a Toeplitz non-negative definite sequence of complex $q \times q$ matrices. Further, let $\mu \in \mathcal{M}_>^q(\mathbb{T})$. From Rem. [5.1] one can see then that $\mu$ is the central measure corresponding to $(\Gamma_j)^n_{j=0}$ if and only if $\Phi : \mathbb{D} \rightarrow \mathbb{C}^{q \times q}$ defined by (5.2) is the central Carathéodory function corresponding to the sequence $(\Gamma_j)^n_{j=0}$ given by (5.1) for each $j \in \mathbb{Z}_{0,n}$.

Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)^n_{j=0}$ be a sequence of complex $q \times q$ matrices such that $\mathcal{C}_q[\mathbb{D},(\Gamma_j)^n_{j=0}] \neq \emptyset$. Then Theorems [5.2] and [5.3] indicate that

$$\frac{1}{(n+1)!}\Phi^{(n+1)}(0) : \Phi \in \mathcal{C}_q[\mathbb{D},(\Gamma_j)^n_{j=0}] = \mathbb{R} \left(2M_{n+1}, \sqrt{2L_{n+1}}, \sqrt{2R_{n+1}}\right),$$

where $(C_j)^n_{j=0}$ is given by (5.5) for all $j \in \mathbb{Z}_{0,n}$ (see also [8, Part I, Thm. 1]).

Remark 5.8. In the case $n = 0$, i.e., if only one complex $q \times q$ matrix $\Gamma_0$ with $\text{Re} \Gamma_0 \in \mathbb{C}^{q \times q}$ is given, the central Carathéodory function corresponding to $(\Gamma_j)^{0}_{j=0}$ is the constant function (defined on $\mathbb{D}$) with value $\Gamma_0$ (see [11, Rem. 1.1]).
The first and second authors showed in [11] that in the general case the central Carathéodory function corresponding to a $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$ is a rational matrix-valued function and constructed explicit right and left quotient representations with the aid of concrete $q \times q$ matrix polynomials. To recall these formulas, we introduce several matrix polynomials which we use if $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and a sequence $(C_j)_{j=0}^n$ of complex $q \times q$ matrices are given.

For all $m \in \mathbb{N}_0$ let the matrix polynomial $e_m$ be defined by

$$e_m(z) := [z^0 I_q, z^1 I_q, z^2 I_q, \ldots, z^m I_q].$$

Let $\Gamma_0 := \text{Re} C_0$. For each $j \in \mathbb{Z}_{1,\kappa}$, we set $\Gamma_j := 2C_j$ and $C_{-j} := C_j^*$. For each $n \in \mathbb{Z}_{0,\kappa}$, let the matrices $T_n$, $Y_n$ and $S_n$ be defined by (3.1) and (5.3). Furthermore, for each $n \in \mathbb{Z}_{0,\kappa}$, let the matrix polynomials $a_n$ and $b_n$ be given by

$$a_n(z) := \Gamma_0 + z e_{n-1}(z) S_{n-1}^\dagger T_{n-1} Y_n \quad \text{and} \quad b_n(z) := I_q - z e_{n-1}(z) T_{n-1}^\dagger Y_n.$$  

(5.6)

Now we see that central $q \times q$ Carathéodory functions admit the following explicit quotient representations expressed by the given data:

**Theorem 5.9** ([11, Thm. 1.2]). Let $n \in \mathbb{N}$, let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence, and let $\Phi$ be the central Carathéodory function corresponding to $(\Gamma_j)_{j=0}^n$. Then the matrix polynomials $a_n$ and $b_n$ given by (5.6) fulfill $\det b_n(z) \neq 0$ and $\Phi(z) = a_n(z)[b_n(z)]^{-1}$ for all $z \in \mathbb{D}$.

Observe that further quotient representations of $\Phi$ are given in [11] Theorems 1.7 and 2.3 and Prop. 4.7.

Obviously, the set

$$\mathcal{N}_n := \{v \in \mathbb{T}: \det b_n(v) = 0\}$$

is finite. For each $v \in \mathcal{N}_n$, let $m_v$ be the multiplicity of $v$ as a zero of $\det b_n$. Then $(\det b_n)^{(m_v)}(v) \neq 0$ for each $v \in \mathcal{N}_n$, so that, for each $v \in \mathcal{N}_n$, the matrix

$$X_{n,v} := \frac{-m_v}{2v(\det b_n)^{(m_v)}(v)}(a_n b_n^*)(v)$$

and the matrix-valued functions $\Delta_n: \mathbb{C} \setminus \mathcal{N}_n \to \mathbb{C}^{q \times q}$ given by

$$\Delta_n(z) := \sum_{v \in \mathcal{N}_n} \frac{v + z}{v - z} X_{n,v}$$

and

$$\Lambda_n := a_n b_n^{-1} - \Delta_n$$

are well defined.

Thm. [5.9] shows that the central Carathéodory function $\Phi$ corresponding to a $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$ is a rational matrix-valued function. Thus, combining Theorems [5.9] and [2.14] yields an explicit expression for the Riesz-Herglotz measure of $\Phi$. 

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Theorem 5.10. Let \( n \in \mathbb{N} \) and let \((\Gamma_j)_{j=0}^n\) be a \( q \times q \) Carathéodory sequence. Then the Riesz-Herglotz measure \( \mu \) of the central Carathéodory function corresponding to \((\Gamma_j)_{j=0}^n\) admits the representation

\[
\mu(B) = \frac{1}{2\pi} \int_B \text{Re} \Lambda_n(\zeta) \lambda(d\zeta) + \sum_{v \in \mathcal{N}_n} X_{n,v} \delta_v(B)
\]

(5.9)

for all \( B \in \mathfrak{B}_T \), where \( \Lambda_n \) is given via (5.8) and where \( \lambda \) is the linear Lebesgue measure defined on \( \mathfrak{B}_T \).

Proof. Use Theorems 5.9 and 2.14. \( \square \)

Now we reformulate Thm. 5.10 in the language of central measures.

Theorem 5.11. Let \( n \in \mathbb{N} \) and let \((C_j)_{j=0}^n\) be a Toeplitz non-negative definite sequence of complex \( q \times q \) matrices. Then the central measure \( \mu \) for \((C_j)_{j=0}^n\) admits the representation (5.9) for all \( B \in \mathfrak{B}_T \).

Proof. In view of \( \text{Re} C_0 = C_0 \), the assertion follows immediately from Remarks 5.14 and 5.1 and Thm. 5.10. \( \square \)

The following examples show in particular that central measures need neither be continuous with respect to the Lebesgue measure nor be discrete measures.

Example 5.12 (cf. Rem. 4.4). The sequence \((C_j)_{j=0}^\infty\) given by \( C_0 := 1 \) and \( C_j := 0 \) for all \( j \in \mathbb{N} \) is obviously Toeplitz non-negative definite. Since \( M_1 = 0 = C_1 \) and \( M_{k+1} = Z_k T_{k-1}^T Y_k = O_{1 \times k} \cdot T_{k-1}^T \cdot O_{k \times 1} = 0 = C_{k+1} \) for all \( k \in \mathbb{N} \), it is the central sequence corresponding to \((C_j)_0^0\) and it is central of order 0. It is readily seen that \( \frac{1}{\pi} \lambda \) is the central measure corresponding to \((C_j)_0^0\) and that \( \Phi : \mathbb{D} \to \mathbb{C} \) defined by \( \Phi(z) = 1 \) is the central Carathéodory function corresponding to \((\Gamma_j)_0^0\), where \( \Gamma_0 := 1 \).

Example 5.13. The sequence \((C_j)_{j=0}^\infty\) given by \( C_j := 1 \) is obviously Toeplitz non-negative definite. Since \( C_1 \neq 0 = M_1 \) and \( M_{k+1} = Z_k T_{k-1}^T Y_k = 1_k \cdot (k-2) 1_k \cdot 1_k = 1 \) for all \( k \in \mathbb{N} \), where \( 1_k := \text{col}(1)_k \), it is the central sequence corresponding to \((C_j)_0^1\) and it is central of order 1. It is readily seen that \( \delta_1 \) is the central measure corresponding to \((C_j)_0^1\) and that \( \Phi : \mathbb{D} \to \mathbb{C} \) defined by \( \Phi(z) = (1+z)/(1-z) \) is the central Carathéodory function corresponding to \((\Gamma_j)_0^1\), where \( \Gamma_0 := 1 \) and \( \Gamma_1 := 2 \).

Remark 5.14. Let \( \kappa \in \mathbb{N}_0 \cup \{+\infty\} \) and let \((C_j)_0^\kappa\) and \((D_j)_0^\kappa\) be Toeplitz non-negative definite sequences of complex \( q \times q \) matrices and complex \( p \times p \) matrices, respectively. Then the sequence \( \text{diag}(C_j, D_j)_{j=0}^\kappa \) is Toeplitz non-negative definite.

Remark 5.15. Let \( \kappa \in \mathbb{N} \cup \{+\infty\} \) and \( k, \ell \in \mathbb{Z}_{1,\kappa} \). Let \((C_j)_0^\kappa\) be a sequence of complex \( q \times q \) matrices central of order \( k \) and let \((D_j)_0^\kappa\) be a sequence of complex \( p \times p \) matrices central of order \( \ell \). Then the sequence \( \text{diag}(C_j, D_j)_{j=0}^\kappa \) is central of order \( \max\{k, \ell\} \).

Example 5.16. In view of Examples 5.12 and 5.13 one can easily see from Remarks 5.14 and 5.15 that the sequence \((C_j)_{j=0}^\infty\) given by \( C_0 := I_2 \) and \( C_j := [0 \ 1] \) for all \( j \in \mathbb{N} \) is Toeplitz non-negative definite, central of order 1 and, thus, it coincides with the central
sequence corresponding to \((C_j)_{j=0}^\infty\). It is readily seen that \(\begin{bmatrix} \frac{\pi}{2} & 0 \\ 0 & \delta_1 \end{bmatrix}\) is the central measure corresponding to \((C_j)_{j=0}^1\) and that \(\Phi: \mathbb{D} \to \mathbb{C}^{2 \times 2}\) defined by \(\Phi(z) = \begin{bmatrix} 0 & 0 \\ 1/(1+z) & 1/(1-z) \end{bmatrix}\) is the central Carathéodory function corresponding to \((\Gamma_j)_{j=0}\), where \(\Gamma_0 := I_2\) and \(\Gamma_1 := \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\).

**Remark 5.17.** Let \(\kappa \in \mathbb{N}_0 \cup \{+\infty\}\), let \((C_j)_{j=0}^\infty\) be a Toeplitz non-negative definite sequence of complex \(q \times q\) matrices and let \(U\) be a unitary \(q \times q\) matrix. Then, formula (A.1) below shows that the sequence \((U^* C_j U)_{j=0}^\infty\) is Toeplitz non-negative definite.

**Example 5.18.** Let the sequence \((C_j)_{j=0}^\infty\) be given by \(C_0 := I_2\) and \(C_j := \frac{1}{4}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\) for all \(j \in \mathbb{N}\). With the unitary matrix \(U := \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\) we have \(C_0 = U^* I_2 U\) and \(C_j = U^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\) for all \(j \in \mathbb{N}\). In view of Example 5.16, one can then easily see from Rem. 5.17 and Lem. A.2 that the sequence \((C_j)_{j=0}^\infty\) is Toeplitz non-negative definite, central of order 1, and thus it coincides with the central sequence corresponding to \((C_j)_{j=0}^1\). Furthermore, \(\frac{1}{4}\begin{bmatrix} \frac{\pi}{2} \lambda + s_1 & -\sqrt{3}\lambda - r_1 \\ -\sqrt{3}\lambda - r_1 & \frac{\pi}{2} \lambda - s_1 \end{bmatrix}\) is the central measure corresponding to \((C_j)_{j=0}^1\) and \(\Phi: \mathbb{D} \to \mathbb{C}^{2 \times 2}\) defined by \(\Phi(z) = \frac{1}{4}\begin{bmatrix} 3 + \frac{\pi}{2} \lambda & \sqrt{3} (1 - \frac{\pi}{2} \lambda) \\ -\sqrt{3} (1 - \frac{\pi}{2} \lambda) & 1 + 3 \frac{\pi}{2} \lambda \end{bmatrix}\) is the central Carathéodory function corresponding to \((\Gamma_j)_{j=0}^1\), where \(\Gamma_0 := I_2\) and \(\Gamma_1 := \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}\).

6. The non-stochastic spectral measure of an autoregressive stationary sequence

Let \(\mathcal{H}\) be a complex Hilbert space with inner product \(\langle ., . \rangle\). For every choice of \(g = \text{col}(g^{(j)})_{j=1}^q\) and \(h = \text{col}(h^{(j)})_{j=1}^q\) in \(\mathcal{H}^q\), the Gramian \((g,h)\) of the ordered pair \([g,h]\) is defined by \((g,h) = \langle (g^{(j)},h^{(k)})_{j,k=1}^q \rangle\). A sequence \((g_m)_{m=-\infty}^\infty\) of vectors belonging to \(\mathcal{H}^q\) is said to be stationary (in \(\mathcal{H}^q\)), if, for every choice of \(m\) and \(n\) in \(\mathbb{Z}\), the Gramian \((g_m,g_n)\) only depends on the difference \(m-n\): \((g_m,g_n) = (g_{m-n},g_0)\). It is well known that the covariance sequence \((C_m)_{m=-\infty}^\infty\) of an arbitrary stationary sequence \((g_m)_{m=-\infty}^\infty\), given by \(C_m := (g_m,g_0)\) for each \(m \in \mathbb{Z}\), is Toeplitz non-negative definite, i.e., that, for each \(m \in \mathbb{N}_0\), the block Toeplitz matrix \(T_m := [C_{j-k}]_{j,k=0}^m\) is non-negative Hermitian. According to a matricial version of a famous theorem due to G. Herglotz (see Thm. 3.1 above), there exists one and only one non-negative Hermitian \(q \times q\) measure \(\mu\) defined on the set \(\mathfrak{B}_T\) of all Borel subsets of the unit circle \(T := \{ \zeta \in \mathbb{C}: |\zeta| = 1 \}\) of the complex plane \(\mathbb{C}\) such that, for each \(j \in \mathbb{Z}\), the \(j\)-th Fourier coefficient of \(\mu\) coincides with the matrix \(C_j\). Then \(\mu\) is called the non-stochastic spectral measure of \((g_j)_{j=-\infty}^\infty\).

A stationary sequence \((g_j)_{j=-\infty}^\infty\) is said to be autoregressive if there is a positive integer \(n\) such that the orthogonal projection \(\hat{g}_n\) of \(g_0\) onto the matrix linear subspace generated by \((g_{-j})_{j=1}^n\) coincides with the orthogonal projection \(\hat{g}\) of \(g_0\) onto the closed matrix linear subspace generated by \((g_{-j})_{j=1}^\infty\): \(\hat{g}_n = \hat{g}\). If \(\hat{g} \neq 0\), then the smallest positive integer \(n\) with \(\hat{g}_n = \hat{g}\) is called the order of the autoregressive stationary sequence \((g_j)_{j=-\infty}^\infty\). If \(\hat{g} = 0\), then \((g_j)_{j=-\infty}^\infty\) is said to be autoregressive of order 0.

Now we are going to give an explicit representation of the non-stochastic spectral measure of an arbitrary autoregressive stationary sequence in \(\mathcal{H}^q\), where we study the
general case without any regularity conditions. This representation is expressed in terms of the covariance sequence of the stationary sequence.

As already mentioned above, the covariance sequence \((C_j)_{j=-\infty}^{\infty}\) of an arbitrary stationary sequence \((g_j)_{j=-\infty}^{\infty}\) in \(H^q\) is Toeplitz non-negative definite. Observe that, conversely, if the complex Hilbert space \(\mathcal{H}\) is infinite-dimensional and if an arbitrary Toeplitz non-negative definite sequence \((C_j)_{j=-\infty}^{\infty}\) of complex \(q \times q\) matrices is given, then a matricial version of a famous result due to A. N. Kolmogorov [16] shows that there exists a stationary sequence \((g_j)_{j=-\infty}^{\infty}\) in \(H^q\) with covariance sequence \((C_j)_{j=-\infty}^{\infty}\) (see also [2] Thm. 7).

The interrelation between autoregressive stationary sequences and central measures is expressed by the following theorem:

**Theorem 6.1.** ([4] Part II, Thm. 9). Let \(n \in \mathbb{N}_0\) and let \((g_j)_{j=-\infty}^{\infty}\) be a stationary sequence (in \(H^q\)) with covariance sequence \((C_j)_{j=-\infty}^{\infty}\) and non-stochastic spectral measure \(\mu\). Then the following statements are equivalent:

(i) \((g_j)_{j=-\infty}^{\infty}\) is autoregressive of order \(n\).

(ii) \((C_j)_{j=0}^{\infty}\) is central of order \(n\).

(iii) \(\mu\) is central of order \(n\).

Now we are able to formulate the announced representation.

**Theorem 6.2.** Let \((g_j)_{j=-\infty}^{\infty}\) be a stationary sequence in \(H^q\) with covariance sequence \((C_j)_{j=-\infty}^{\infty}\) and let \(n \in \mathbb{N}\). Suppose that \((g_j)_{j=-\infty}^{\infty}\) is autoregressive of order \(n\). Then \(\Lambda_n\) given by (5,8) is holomorphic at each point \(u \in \mathbb{T}\) and the non-stochastic spectral measure \(\mu\) of \((g_j)_{j=-\infty}^{\infty}\) admits the representation (5,9) for all \(B \in \mathfrak{B}_{\mathbb{T}}\), where \(\Lambda\) is the linear Lebesgue measure defined on \(\mathfrak{B}_{\mathbb{T}}\), the matrix \(X_{n,v}\) is given by (5,7), and \(\delta_v\) is the Dirac measure defined on \(\mathfrak{B}_{\mathbb{T}}\) with unit mass at \(v\).

Proof. According to Thm. 6.1 the sequence \((C_j)_{j=0}^{\infty}\) is central of order \(n\) and \(\mu\) is central of order \(n\). From the definition of the non-stochastic spectral measure of \((g_j)_{j=-\infty}^{\infty}\) we know then that \(\mu\) is the central measure corresponding to \((C_j)_{j=0}^{\infty}\). Consequently, the application of Thm. 5.11 completes the proof. \(\square\)

**Remark 6.3.** Let \((g_j)_{j=-\infty}^{\infty}\) be a stationary sequence in \(H^q\) which is autoregressive of order 0. Then the non-stochastic spectral measure \(\mu\) of \((g_j)_{j=-\infty}^{\infty}\) is given by \(\mu = \frac{1}{2\pi}(g_0, g_0)\Lambda\) (see Thm. 6.1 and Rem. 4.4).

**A. Some facts from matrix theory**

**Remark A.1.** Let \(A \in \mathbb{C}^{p \times q}\). Further, let \(V \in \mathbb{C}^{m \times p}\) and \(U \in \mathbb{C}^{q \times n}\) satisfy the equations \(V^*V = I_p\) and \(UU^* = I_q\), respectively. Then \((VAV)^\dagger = U^*A^\dagger V^*\).

**Lemma A.2.** Let \(\kappa \in \mathbb{N}_0 \cup \{+\infty\}\) and let \((C_j)_{j=0}^{\kappa}\) be a sequence from \(\mathbb{C}^{q \times q}\). Let \(U \in \mathbb{C}^{q \times q}\) be unitary and let \(C_{j,U} := U^*C_jU\) for \(j \in \mathbb{Z}_0,\kappa\). For \(j \in \mathbb{Z}_0,\kappa\) let \(C_{-,j} := C_j^*\) and \(C_{-,j,U} := C_{j,U}^*\).
(a) Let \( n \in \mathbb{Z}_{0,k} \). Let \( T_n := [C_{j-k}]_{j,k=0}^n \) and \( T_{n,U} := [C_{j-k,U}]_{j,k=0}^n \). Then
\[
T_{n,U} = \left[ \text{diag}_n(U) \right]^* T_n \left[ \text{diag}_n(U) \right] \tag{A.1}
\]
and
\[
T_{n,U}^\dagger = \left[ \text{diag}_n(U) \right]^* T_n^\dagger \left[ \text{diag}_n(U) \right]. \tag{A.2}
\]

(b) Let \( n \in \mathbb{Z}_{0,k} \). Let \( Y_n \) and \( Z_n \) be given by (3.1). Furthermore let \( Y_{n,U} \) and \( Z_{n,U} \) be defined by \( Y_{n,U} := \text{col}(C_{j,U})_{j=1}^n \) and \( Z_{n,U} := [C_{n,U}, \ldots, C_{1,U}] \). Let \( M_1, L_1, \) and \( R_1 \) be given by (3.2), let \( M_{1,U} := O_{q \times q}, L_{1,U} := C_{0,U}, \) and let \( R_{1,U} := C_{0,U} \). If \( \kappa \geq 1 \), then, for each \( n \in \mathbb{Z}_{1,k} \), let \( M_{n+1}, L_{n+1}, \) and \( R_{n+1} \) be given via (3.3), let
\[
M_{n+1,U} := Z_{n,U} T_{n-1,U}^\dagger Y_{n,U}, \quad L_{n+1,U} := C_{0,U} - Z_{n,U} T_{n-1,U}^\dagger Z_{n,U} \]
and
\[
R_{n+1,U} := C_{0,U} - Y_{n,U} T_{n-1,U}^\dagger Y_{n,U}.
\]
For each \( n \in \mathbb{Z}_{0,k} \) then
\[
M_{n+1,U} = U^* M_{n+1,U}, \quad L_{n+1,U} = U^* L_{n+1,U}, \quad \text{and} \quad R_{n+1,U} = U^* R_{n+1,U}.
\]

(c) If \( k \in \mathbb{Z}_{2,k} \) and if \((C_j)^n_{j=0}\) be central of order \( k \), then \((C_{j,U})^n_{j=1}\) is central of order \( k \).

(d) If \( k \in \mathbb{Z}_{2,k} \) and if \((C_j)^n_{j=0}\) be central of minimal order \( k \), then \((C_{j,U})^n_{j=1}\) is central of minimal order \( k \).

Proof. Equation (A.1) is obvious. Since \( U \) is unitary, the matrix \( \text{diag}_n(U) \) is unitary as well. Thus, in view of Rem. A.1, formula (A.2) is an immediate consequence of (A.1). Part (a) is proved. Obviously, \( M_{1,U} = O_{q \times q} = U^* M_1 U \). Now, let \( n \in \mathbb{Z}_{1,k} \). Then, using (a) and \( \text{diag}_n(U) \text{[diag}_n(U)]^* = I_{nq}, \) we get
\[
M_{n+1,U} = Z_{n,U} T_{n-1,U}^\dagger Y_{n,U}
= [U^* C_{n,U}, \ldots, U^* C_{1,U}] [\text{diag}_n(U)]^* T_{n-1,U}^\dagger [\text{diag}_n(U)] [\text{col}(U^* C_j U)^n_{j=1}]
= U^* [C_{n,U}, \ldots, C_{1,U}] T_{n-1,U}^\dagger [\text{col}(C_j)^n_{j=1}] U = U^* Z_{n,U} T_{n-1,U} Y_{n,U} = U^* M_{n+1,U}.
\]
Analogously, the remaining assertions of (b) can be shown. The assertions stated in (c) and (d) are an immediate consequence of (b).

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