Hamiltonian BFV–BRST theory of closed quantum cosmological models

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Abstract

We introduce and study a new discrete basis of gravity constraints by making use of harmonic expansion for closed cosmological models. The full set of constraints is splitted into area-preserving spatial diffeomorphisms, forming closed subalgebra, and Virasoro-like generators. Operatorial Hamiltonian BFV-BRST quantization is performed in the framework of perturbative expansion in the dimensionless parameter which is a positive power of the ratio of Planckian volume to the volume of the Universe. For the $(N+1)$-dimensional generalization of stationary closed Bianchi-I cosmology the nilpotency condition for the BRST operator is examined in the first quantum approximation. It turns out, that certain relationship between dimensionality of the space and the spectrum of matter fields emerges from the requirement of quantum consistency of the model.

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1 Introduction

At the present time we observe the explosive development in quantum gravity and string theory which have many-sided interconnections [1]. From the standpoint of the constrained dynamics [2] these theories manifest a certain similarity originating from the reparametrization invariance which is a gauge symmetry of the both. Meanwhile, the theories are usually treated in an essentially different way in respect to the perturbative expansions. Various well-advanced string models, having a trend to be thought about as dual ones, are considered nowadays as expansions around different limiting values of a

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coupling constant of an unknown fundamental superstring theory \[3\]. Oppositely, gravity has a widely recognized candidate for a fundamental theory that is Einstein action or, perhaps, its some or another extension, but the perturbative treatment is a longstanding unsolved problem of the General Relativity.

The main intention of the present work is to undertake an attempt of the Hamiltonian BFV–BRST quantization of closed cosmological models by exploiting the kinship between the string and gravity gauge symmetries.

Apparently, a phase-space constrained description is very similar for string theory and gravity while they are being carried out locally with dynamical variables and constraints defined at a point of a space-like section of world-sheet or spacetime, respectively. However, there is an essential distinction in habitual approaches to quantization of strings and gravity.

Quantizing strings, one may work with a discrete set of modes, describing string excitations, and a discrete set of constraints, which form Virasoro algebra. In doing so, one observes that quantum commutators of the constraints acquire central extension, and the quantum consistency of the theory is provided for the critical values of dimensionality and intercept. One of the most efficient methods to observe this phenomenon is the BFV-BRST Hamiltonian quantization of constrained systems \[4, 5, 6, 7\]. In the framework of this method one can show how the nilpotency condition for the BRST charge defines the values of critical parameters for strings \[8, 9\], membranes \[10\] and \(W_3\) gravity \[11\]. Actually, the use of discrete set of modes and constraints is merely a convenient tool for taking into account the topology of the compact world-sheet spacelike section, when the method is applied.

Canonical quantization of gravity and cosmology is usually treated by either perturbative consideration of path integral or operatorial quantization according to Dirac scheme with constraints imposed onto physical states. Basically, such a treatment implements continuous basis of dynamical modes and constraints, which can satisfactorily describe the dynamics locally but are hardly well-defined globally on compact spacelike sections of the spacetime. Meanwhile, the consideration of cosmological perturbations in terms of globally defined harmonics was used for the investigation of different problems in classical \[12\] and quantum \[13, 14\] cosmology. However, a globally defined discrete constraint basis for quantum cosmology of closed Universes has not been hitherto discussed. Consistent canonical quantization requires both the dynamical modes and constraints to be globally defined, since this is essential for the adjustment of the ordering of the modes to the quantum algebra of the constraints. For example, in the case of the Wick ordering of bosonic string modes, the quantum corrections to the constraint commutators give rise to the central extension of the Virasoro algebra representation in the Fock space. Then the quantum constraint algebra can be made consistent provided the proper basis of constraints is chosen (\(L\) and \(\hat{L}\) are employed instead of habitual for cosmologists lapse and shift generators \(H_\perp\), \(H_\parallel\) and moreover, only one half of \(L_n\) and \(L_m\) should annihilate physical states). At the mean time, the straightforward pursuing the Dirac scheme, with \(H_\perp\) and \(H_\parallel\) being imposed on the physical subspace, leads to a contradiction.

In this paper, we suggest a new scheme for canonical quantization of the closed cosmological models. The distinguishing features of this treatment are as follows: the decomposition of both dynamical variables and constraints into harmonics which are the
eigenfunctions of the Laplace operator for the maximally symmetric space of given topology; separation of subalgebra of area-preserving diffeomorphisms from a total set of gravity constraints whereas the rest of the constraints form Virasoro-like generators; quantization of this constrained theory is performed in the framework of the Hamiltonian BFV-BRST formalism with due regard to the ordering of gravity, matter and ghost harmonics; studying the quantum nilpotency condition for the BRST operator, we apply a certain perturbative expansion of the constraints and structure functions with a small parameter \( l_p/V^{1/N} \), where \( l_p \) is a Planck length and \( V \) is a spatial volume of the Universe, \( N \) is a dimensionality of the space. Implementing this scheme, we focus the consideration mostly at the stationary Bianchi-I type Universe, although any other closed cosmological model could be treated in a similar way. The specificity of a particular topology is encoded in the constraint algebra structure functions which appear to be expressed via Clebsch–Gordan coefficients of the corresponding symmetry group representation. In the case of Bianchi-I cosmology, these coefficients are the simplest, as the group is \( U(1)^N \). Although we take this particular type of a closed cosmological model for the sake of technical simplicity, it is, however, of a certain physical interest \([15]\). On the other hand, \( N \)-torus allows to exploit a straightforward analogy with the closed bosonic string sigma–model when the nilpotency of the quantum BRST charge is examined. The most striking outcome of this quantization procedure, being applied to the Einstein gravity coupled to the matter fields, is that the number of the matter degrees of freedom appears to be correlated with the dimensionality of the space \( N \). For example, when there are \( d \) mass less scalar fields only, already the first quantum correction gives rise to the relation between \( d \) and \( N \):

\[
d = 30 + \frac{5}{2}(N + 1)(N - 2),
\]

(1.1)

which is a necessary condition for nilpotency of the quantum BRST charge. Mention the curious fact: if one puts \( N = 1 \) (thereby torus reduces to a circle and the constraints form the Virasoro algebra) then relation (1.1) results in \( d = 25 \). This result can be naturally understood from the standpoint of the string sigma–model without Weyl invariance \([16]\). As a matter of fact, bosonic string theory should not be thought about as one-dimensional limit of the Einstein gravity coupled to a set of \( d \) scalar fields, because string possesses an extra gauge symmetry, namely Weyl invariance, besides the diffeomorphisms, whereas the mentioned \( \sigma \)-model does not have Weyl invariance. As is known \([16]\) the critical dimensionality of the \( \sigma \)-model is 25, in contrast with 26 for strings. The discrepancy in these dimensionalities can be easily explained if one remembers that Weyl invariance gauges out conformal mode of 1+1-metric, while in the non-conformal \( \sigma \)-model case this mode contributes to the Virasoro generators on an equal footing with string excitations \([16]\). Thus, in some sense, the critical dimensionality for the both cases is 26, but for the \( \sigma \)-model this number consists of 25 string coordinates and 1 gravity conformal mode.

In its basic features the suggested scheme should, probably, be actual for other closed cosmological models, including those with non-stationary classical background. But the critical relations analogous to (1.1) may change their form for each particular model.

The structure of the paper is as follows: in the second section we describe a general decomposition of the gravity constraints into the discrete set of area-preserving diffeomorphisms and Virasoro-like generators and sketch the way of the representing the structure constants of the constraint algebra via Clebsch-Gordan coefficients of the corresponding
symmetry group; in the third section we suggest the split of integer vectors on $N$-torus into two classes which furnishes a means of choosing the Wick ordering for a proper part of dynamical modes and constraints; in sec. 4 we write down explicitly the decomposition of gravity constraints in new discrete basis and the corresponding structure constants for closed $N$-dimensional Bianchi-$I$ cosmological model ($N$-torus); in sec. 5 we define perturbative expansion of the constraint operators and calculate first quantum corrections to the constrained algebra; sec. 6 is devoted to the investigation of the structure of BRST operator and examination of its nilpotency condition, as a result, we come to the critical relation between the dimensionality of the space and the spectrum of matter fields; sec. 7 contains conclusions.

2 Gravity constraint algebra for a closed Universe. Harmonic expansion, area-preserving diffeomorphisms and Virasoro-like generators

Hamiltonian description of any reparametrization-invariant theory with metric, being considered as a dynamical variable, contains a set of first-class phase-space constraints $H_\perp$ and $H_i, i = 1, \ldots, N$ where $N$ is a dimensionality of space. $H_\perp$, usually called super-Hamiltonian or Wheeler-DeWitt operator (the canonical Hamiltonian vanishes on the constraint surface for this case), generates lapse transformations of spacelike hypersurface, while $H_i$, called supermomentum, generates spatial diffeomorphisms [17]. If the gravity action does not contain higher derivatives of metric, these constraints can be represented in the following form

$$H_\perp = l_P^{-N} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{l_P^{-N}} g R + H_{\perp \text{ matter}},$$

(2.1)

$$H_i = -2 g_{ij} \pi^{jk} - (g_{ik,m} + g_{im,k} - g_{km,i}) \pi^{mk} + H_{i \text{ matter}},$$

(2.2)

where $g_{ij}$ is a metric on a spacelike section of spacetime, $\pi^{ij}$ is a momentum conjugated to metric and representing extrinsic curvature of the spacelike hypersurface, $l_P$ is the Planck length. The DeWitt supermetric $G_{ijkl}$ has the following form:

$$G_{ijkl} = \frac{1}{2} \left( g_{ik} g_{jl} + g_{il} g_{jk} - \frac{2}{N - 1} g_{ij} g_{kl} \right),$$

(2.3)

while its inverse looks as

$$G^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} - 2 g^{ij} g^{kl} \right).$$

(2.4)

$H_{\perp \text{ matter}}$ and $H_{i \text{ matter}}$ are the matter field contributions to super-Hamiltonian and supermomenta, respectively. Their explicit form depends upon a specific set of the matter fields and their interactions, although the involution relations between the constraints are
not effected by the structure of matter part of these expressions, because these relations have a purely geometric origin \[13\]. These involution relations are

\[ \{H_i(x), H_j(x')\} = g(x)g^{ij}(x) \frac{\partial}{\partial x^j} \delta(x, x') \]

\[ -g(x')g^{ij}(x') \frac{\partial}{\partial x'^j} \delta(x, x'), \tag{2.5} \]

\[ \{H_i(x), H_\perp(x')\} = H_\perp(x) \frac{\partial}{\partial x^i} \delta(x, x') \]

\[ -H_\perp(x') \frac{\partial}{\partial x'^i} \delta(x, x'), \tag{2.6} \]

\[ \{H_i(x), H_j(x')\} = H_j(x) \frac{\partial}{\partial x^j} \delta(x, x') \]

\[ -H_i(x') \frac{\partial}{\partial x'^j} \delta(x, x'). \tag{2.7} \]

Here \{,\} denotes conventional Poisson brackets. The expression for super-Hamiltonian (2.1) differs from the conventional one by the factor \( g^{1/2} \); thereby the involution relations (2.5), (2.6) also have distinctions from the commonly used. Such an unusual definition of super-Hamiltonian (which, however, sometimes occurs \[19\]) is convenient for our purposes. Notice, that the involution relations (2.5), (2.6) could not be reproduced without gravity dynamical variables, because in its essence they express the gravity gauge symmetry algebra in phase space \[20\].

Considering compact manifolds, one can express phase space variables and constraints among them via a discrete set of coefficients of harmonic expansion. The functional basis of the expansion is formed by eigenfunctions of the Laplace operators defined in the maximally symmetric space of a topology under consideration. For the case of Bianchi I cosmology these functions are simply elements of Fourier expansion on a torus, for the case of Bianchi IX cosmology – hyperspherical functions \[12\]. From the viewpoint of local consideration this harmonic representation is completely equivalent to the original continuous parameterization of the phase space and constraints, but the benefit of this treatment is that the expansion, being well-defined on the manifold, reflects explicitly all the topological distinctions between different types of cosmological models. The significance of this topological specificity becomes obvious at comparing open and closed bosonic strings. Locally, both the models are described by the same canonical variables and constraints, whereas the difference transparently manifests itself in the spectrum of modes and constraints upon the Fourier expansion. Really, the dynamical modes of closed string are left- and right-moving excitations \( a^n, \bar{a}^m \) subjected to two copies of Virasoro constraints \( L_n \) and \( \bar{L}_m \), while in the case of the open string the dynamical modes are the standing waves with a single set of Virasoro constraints \( L_n \).

Now, we shall try to push the consideration of the constrained Hamiltonian formalism along the lines of harmonic representation for the closed cosmological models. The super–Hamiltonian can be written as

\[ H_\perp(x) = \sum_{(n)} H_\perp^{(n)}(x^0) Q^{(n)}(x), \tag{2.8} \]
where $Q^{(n)}(x)$ is a scalar–type harmonic of the corresponding Laplace operator, labeled by a multi-index $(n)$. The discrete set of constraints $H^{(n)}$ is given by

$$
H^{(n)} = \int dV H_{\perp}(x)Q^*(n)(x),
$$

(2.9)

where $dV$ denotes the covariant volume element of the manifold; the harmonics are normalized

$$
\int dV Q^{(n)}(x)Q^*(n)(x) = \delta^{(n)(m)},
$$

(2.10)

The structure of supermomentum expansion is more complicated:

$$
H_i(x) = \sum_{(n)} H^{(n)}_{\parallel} \partial_i Q^{(n)}(x) + \sum_{(A)} H^{(A)} S^{(A)}_i(x),
$$

(2.11)

here $S^{(A)}_i(x)$ stand for the transverse vector harmonics obeying the relations

$$
\nabla^i S^{(A)}_i(x) = 0.
$$

(2.12)

So, two types supermomentum constraint harmonics appear: the longitudinal $H_{\parallel}$ and the transversal $H^{(A)}$ labeled by another set of multi–indices $(A)$. The inverse relations for $H_{\parallel}$ and $H^{(A)}$ are

$$
H^{(n)}_{\parallel} = \int dV H_i(x) \frac{\partial^i Q^*(n)(x)}{\lambda(n)},
$$

(2.13)

$$
H^{(A)} = \int dV H_i(x) S^{*(A)}(x),
$$

(2.14)

where $\lambda^{(n)}$ is an eigenvalue of the Laplace operator corresponding to the eigenfunction $Q^{(n)}(x)$. Transversal vector harmonics are normalized

$$
\int dV S^{(A)}_i(x) S^{(B)}_i(x) = \delta^{(A)(B)}.
$$

(2.15)

Transversal constraint’s harmonics $H^{(A)}$ define corresponding continuous subset of supermomentum constraints

$$
\tilde{H}_i(x) = \sum_{(A)} H^{(A)} S^{(A)}_i(x).
$$

(2.16)

Constraints $\tilde{H}_i(x)$ are obviously divergenceless

$$
\nabla^i \tilde{H}_i(x) = 0,
$$

(2.17)

where the covariant derivative is defined with respect to the metric of the maximally symmetric space, which we shall call as a background metrics in what follows.

It is easy to check that the Poisson bracket of $\tilde{H}_i(x)$ among themselves gives $\tilde{H}_i(x)$ again

$$
\{\tilde{H}_i(x), \tilde{H}_j(x')\} = \tilde{H}_j(x) \frac{\partial}{\partial x^i} \delta(x, x') - \tilde{H}_i(x') \frac{\partial}{\partial x'^j} \delta(x, x')
$$

(2.18)

Usually, it is suggestive to consider the contractions of constraints to arbitrary "weight–functions". In the case of $\tilde{H}_i(x)$, these constraint functionals

$$
\int dV \tilde{H}_i(x) f^i(x) \equiv \tilde{H}(f),
$$

(2.19)
where $f^i(x)$ can be chosen divergenceless, form the closed algebra with respect to Poisson bracket

$$\{\tilde{H}(f_1), \tilde{H}(f_2)\} = \tilde{H}([f_1, f_2]) = [f_1, f_2] = f_1^i \frac{\partial}{\partial x^j} f_2^j - f_2^i \frac{\partial}{\partial x^j} f_1^j. \quad (2.20)$$

The Lie bracket of divergenceless vector fields $f_1, f_2$, apparently, is a divergenceless too. The commutation relations (2.20) are typical for generators of area–preserving diffeomorphisms introduced in the paper [21] and being intensively studied in relationship with p-brane theories [22]. However, the transformations generated by $\tilde{H}$ do not preserve the volume of the physical space, i.e. $\{\tilde{H}, g\} \neq 0$, where $g = \det g_{ij}$ is a genuine (not a background) metric. In principle, one can introduce divergenceless projection of supermomentum with respect to the genuine metric $g_{ij}$, and such a projection does generate area-preserving transformations. However, these projections of the constraints do not form a closed algebra with respect to Poisson bracket. This disclosure is caused by the fact that the projector depends upon the metric and so it is not invariant with respect to a metric shift. So, the Poisson bracket of two area-preserving supermomentum constraints does not give an area-preserving projection of a supermomentum constraint again. For our purposes it is essential to employ the constraint basis, with the closed subalgebra of spatial transversal diffeomorphisms, and we adopt for this reason the basis of longitudinal and transversal constraints. The latter we shall refer to as area-preserving, because they form the closed subalgebra of this type, although they employ a background metric instead of a genuine one. The longitudinal part of supermomentum will be combined with the super-Hamiltonian discrete constraints (2.9) to construct Virasoro-like generators.

In continuous basis these generators read

$$H_{\perp}(x) \pm \frac{1}{\Delta^{1/2}} \nabla^i H_i(x), \quad (2.21)$$

where the covariant derivative $\nabla^i$ and Laplace operator $\Delta$ are defined in respect to background metric. Similarly, area-preserving diffeomorphisms in continuous parameterization are

$$\left(\delta_j^i - \nabla_j \frac{1}{\Delta} \nabla^i\right) H_i. \quad (2.22)$$

Dynamical variables are also expanded into the appropriate set of harmonics. For example, a scalar field expansion is

$$\varphi(t, x) = \sum_{(n)} \varphi^{(n)}(t) Q^{(n)}(x), \quad (2.23)$$

where $Q^{(n)}$ is the same set of scalar harmonics as in the expansion of $H_{\perp}(x)$ (2.8). The expansion of metrics is more complicated:

$$g_{ij}(t, x) = \sum_{(n)} a^{(n)}(t) g_{ij}^{(0)}(x) Q^{(n)}(x) + \sum_{(n)} b^{(n)}(t) \left(\frac{\nabla_i \nabla_j}{\lambda^{(n)}} - g_{ij}^{(0)} \frac{N}{N}\right) Q^{(n)}(x)$$

$$+ \sum_{(A)} c^{(A)}(t) \left(\nabla_i S_{ij}^{(A)}(x) + \nabla_j S_{ij}^{(A)}(x)\right) + \sum_{(A)} d^{(A)}(t) G_{ij}^{(A)}(x), \quad (2.24)$$

where $g_{ij}^{(0)}(x)$ is the background metric, $G_{ij}^{(A)}(x)$ is the set of transverse traceless tensor harmonics:

$$g^{ij} g_{ij} = 0; \quad \nabla^i G_{ij} = 0. \quad (2.25)$$
Similar expansions can be written for the rest of canonical variables. For the case of $S^3$ this multipole expansion \( (2.24) \) was described in detail in \cite{12}. The expansion for $g_{ij} \ (2.24)$ is involved not only into super-Hamiltonian \( (2.1) \) and supermomenta \( (2.2) \), but also into the structure functions of constraint algebra via relation \( (2.5) \), thereby the multipole modes $a^{(n)}(t), b^{(n)}(t), c^{(A)}(t)$ and $d^{(\alpha)}(t)$ enter the involution coefficients. However, in the framework of the perturbative consideration adopted in this paper for the calculation of quantum corrections, the first approximation can be obtained neglecting the multipoles from the involution structure functions. This will be explained in more detail in Sec. 5, where quantum BRST charge will be studied for the case of Bianchi-I cosmology.

Let us discuss the multipole independent part of the structure functions. One may easily see that they are proportional to the integrals of three harmonic functions of such types:

\[
\int dV Q^{(n)}(x)Q^{(m)}(x)Q^{*(l)}(x),
\int dV Q^{(n)}(x)Q^{(m)}(x)S^{(A)}_{i}(x), \int dV Q^{(n)}(x)S^{(A)}_{i}(x)S^{(B)}_{j}(x),
\int dV S^{(A)}_{i}(x)S^{(B)}_{j}(x)S^{(C)}_{k}(x). \tag{2.26}
\]

The integrals \( (2.26) \) can be expressed via bilinear combinations of Clebsch-Gordan coefficients of the corresponding symmetry group by making use of Wigner-Eckart theorem (see e.g. \cite{23}).

If the expansion of constraints into the harmonics of dynamical modes is considered, then the coefficients at bilinear combinations of multipoles are expressed in terms of the similar integrals, which in their turn can be reduced to the Clebsch-Gordan coefficients. As a matter of fact, only quadratic combinations of harmonics in the constraints may contribute to the constant part of the first quantum correction to the constraint involution relations. This contribution can usually be very instructive, e.g.: for the case of string \cite{8, 9}, membranes \cite{10} or $W_3$-gravity \cite{11} it is the correction, which fixes the critical parameters.

From the viewpoint of obtaining the critical parameters for closed cosmological models, the method is basically the same for any group, but the simplest case of Bianchi-I cosmology allows to avoid the cumbersome expressions involving summing of bilinear combinations of Clebsch-Gordan coefficients.

### 3 Wick ordering and basis of integer vectors on $N$-torus

In contrast to ordinary quantum mechanics, quantum theories with an infinite number of degrees of freedom may be inequivalent for a different choices of operator symbols. Of course, different symbols of operators (different orderings) are formally linked by unitary transformations, but an appearance of ultraviolet divergences may destroy the unitarity
of the transformation. For example, Wick symbols of the Virasoro constraints in bosonic string theory yield the nonvanishing finite central extension, whereas Weyl symbol of the same constraints does not reveal this correction to the commutation relations. So, the choice of ordering appears to be not technical, but fundamental step, defining the theory.

The common wisdom is that the modes of oscillating behaviour should be quantized by Wick ordering, while those, having non-oscillating dynamics (usually, zero modes) are ordered by Weyl rule.

To provide an opportunity of Wick quantization of the part of the constraints and Fock representation for the model we should introduce some kind of split of the set of $N$-dimensional wave vectors on $N$-torus. In the case of string the similar problem can be resolved in a very simple way because there are one-dimensional vectors (i.e. integer numbers) $n$. One merely considers positive integer numbers as the numbers corresponding to the annihilation operators $a_n$ and creation operators $a_n^+$ and negative integer numbers as numbers corresponding to the creation operators $a_n^+$ and annihilation operators $a_n$. In turn, ghost operators corresponding to Virasoro constraints $L_n$ and $\bar{L}_{-n}$, where $n$ is positive are treated as creation operators while the ghost operators corresponding to constraints $\bar{L}_{-n}$ and $L_n$ are treated as annihilation operators.

An attempt to consider Virasoro-like constraint subset for multidimensional cases should imply some splitting of integer vectors

$$\vec{n} = (n_1, n_2, \ldots, n_N)$$

into positive and negative ones. The plausible way to do it, from our point of view, is suggested in the paper [24].

Let us fix in some way the order of main axis of our torus. Then, if the first wave number $n_1$ is positive, we shall call the vector positive and if $n_1$ is negative we shall call the vector negative. If $n_1 = 0$ we should turn to the second integer number $n_2$ and to call the vector $\vec{n}$ positive or negative depending on sign of $n_2$. If both numbers $n_1$ and $n_2$ are equal to zero, the sign of vector $\vec{n}$ is determined by the sign of $n_3$ and so on. One can easily see that all the vectors $\vec{n}$ must belong to one of these two classes excepting the zero-vector $(0, 0, \ldots, 0)$. Below we shall implement a notation

$$sgn(\vec{n}) = 1, \text{ if } \vec{n} \text{ is positive}$$
$$sgn(\vec{n}) = -1, \text{ if } \vec{n} \text{ is negative.}$$

(3.2)

As a matter of fact we shall consider the integer vectors weighted by radii of $N$-torus:

$$\vec{n} = \left(\frac{n_1}{R_1}, \ldots, \frac{n_N}{R_N}\right).$$

(3.3)

In what follows we shall basically use the so called “last element” limit, i.e. we shall reduce our calculations mainly to the case then only number $n_N$ differs from zero. It will give us an opportunity to make all the calculations explicitly.


4 Constraint algebra of Bianchi-I model in harmonic representation

In this section, we write down explicitly Virasoro-like and area-preserving diffeomorphism constraints for Bianchi-I model in harmonic representation and study their commutation relations.

\[
L(n) = \frac{1}{2} \left( H_\perp(n) + \frac{\text{sgn}(\bar{n})n^iH_i(n)}{|n|} \right), \quad (4.1)
\]

\[
\bar{L}(n) = \frac{1}{2} \left( H_\perp(n) - \frac{\text{sgn}(\bar{n})n^iH_i(n)}{|n|} \right), \quad (4.2)
\]

where

\[
H_\perp(n) = \int d^N x H_\perp(x) \exp(i\bar{n}x), \quad (4.3)
\]

\[
H_i(n) = \int d^N x H_i(x) \exp(i\bar{n}x), \quad (4.4)
\]

Here, positive and negative constraint harmonics \( L(n), \bar{L}(n) \) and \( L(-n), \bar{L}(-\bar{n}) \) are complex conjugated:

\[
L^*(n) = L(-\bar{n}) \quad \text{and} \quad \bar{L}^*(\bar{n}) = \bar{L}(-\bar{n}). \quad (4.5)
\]

Let us write down the involution relations among these constraints, omitting the gravity harmonic contributions from structure coefficients:

\[
[L(n), \bar{L}(\bar{m})] = U^{L(n+\bar{m})}_{L(n)L(\bar{m})} L(n + \bar{m}) + U^{\bar{L}(n+\bar{m})}_{L(n)L(\bar{m})} \bar{L}(n + \bar{m}) + U^{\bar{H}_i(n+\bar{m})}_{L(n)L(\bar{m})} \bar{H}_i(n + \bar{m})
\]

\[
= \frac{1}{4} \left( 1 + \frac{\text{sgn}(\bar{n})\text{sgn}(\bar{m})\bar{m}\bar{n}}{|\bar{m}|^2|\bar{n}|} \right)
\times \left\{ (\text{sgn}(\bar{n})|\bar{n}| - \text{sgn}(\bar{m})|\bar{m}|) + \frac{\text{sgn}(\bar{n} + \bar{m})(\bar{n}^2 - \bar{m}^2)}{|\bar{n} + \bar{m}|} \right\} L(n + \bar{m})
\]

\[
+ \left( \text{sgn}(\bar{n})|\bar{n}| - \text{sgn}(\bar{m})|\bar{m}| \right) - \frac{\text{sgn}(\bar{n} + \bar{m})(\bar{n}^2 - \bar{m}^2)}{|\bar{n} + \bar{m}|} \bar{L}(n + \bar{m})
\]

\[
+ \left( n_i - m_i - \frac{(n_i^2 - m_i^2)(n_i + m_i)}{(n_i + m_i)^2} \right) H_i(n + \bar{m}) \}, \quad (4.6)
\]

\[
[L(n), \bar{L}(\bar{m})] = U^{L(n+\bar{m})}_{L(n)L(\bar{m})} L(n + \bar{m}) + U^{\bar{L}(n+\bar{m})}_{L(n)L(\bar{m})} \bar{L}(n + \bar{m}) + U^{\bar{H}_i(n+\bar{m})}_{L(n)L(\bar{m})} \bar{H}_i(n + \bar{m})
\]

\[
= \frac{1}{4} \left( 1 - \frac{\text{sgn}(\bar{n})\text{sgn}(\bar{m})\bar{m}\bar{n}}{|\bar{m}|^2|\bar{n}|} \right)
\times \left\{ (\text{sgn}(\bar{n})|\bar{n}| + \text{sgn}(\bar{m})|\bar{m}|) + \frac{\text{sgn}(\bar{n} + \bar{m})(\bar{n}^2 - \bar{m}^2)}{|\bar{n} + \bar{m}|} \right\} L(n + \bar{m})
\]

\[
+ \left( \text{sgn}(\bar{n})|\bar{n}| + \text{sgn}(\bar{m})|\bar{m}| \right) - \frac{\text{sgn}(\bar{n} + \bar{m})(\bar{n}^2 - \bar{m}^2)}{|\bar{n} + \bar{m}|} \bar{L}(n + \bar{m})
\]

\[
+ \left( n_i - m_i - \frac{(n_i^2 - m_i^2)(n_i + m_i)}{(n_i + m_i)^2} \right) H_i(n + \bar{m}) \}, \quad (4.7)
\]
\[ [\bar{L}(\tilde{n}), L(\bar{m})] = U_{\bar{L}(\bar{m})L(\tilde{n})}^L(\bar{m} + \tilde{n}) L(\tilde{n} + \bar{m}) + U_{\bar{L}(\bar{m})L(\tilde{n})}^{\bar{L}(\tilde{n} + \bar{m})} \bar{L}(\tilde{n} + \bar{m}) + U_{L(\tilde{n})L(\bar{m})}^{\bar{H}_i(\bar{m} + \tilde{n})} \bar{H}_i(\bar{m} + \tilde{n}) \]
\[ = \frac{1}{4} \left( 1 + \frac{\text{sgn}(\tilde{n}) \text{sgn}(\bar{m}) \tilde{n} \bar{m}}{\tilde{n} \bar{m}} \right) \times \left\{ \left( \text{sgn}(\tilde{n}) |\tilde{n}| - \text{sgn}(\tilde{n}) |\tilde{n}| + \frac{\text{sgn}(\tilde{n} + \bar{m})(\tilde{n}^2 - \bar{m}^2)}{|\tilde{n} + \bar{m}|} \right) L(\tilde{n} + \bar{m}) \right. \]
\[ + \left( \text{sgn}(\tilde{n}) |\tilde{n}| + \text{sgn}(\tilde{n}) |\tilde{n}| - \frac{\text{sgn}(\tilde{n} + \bar{m})(\tilde{n}^2 - \bar{m}^2)}{|\tilde{n} + \bar{m}|} \right) \bar{L}(\tilde{n} + \bar{m}) \]
\[ + \left. \left( n_i - m_i - \frac{\tilde{n}^2 - \bar{m}^2}{(\tilde{n} + \bar{m})^2} (n_i + m_i) \right) H_i(\tilde{n} + \bar{m}) \right\}. \quad (4.8) \]

In the right-hand side of the relations (4.6) - (4.8) appear supermomenta constraints \( H_i(\tilde{n} + \bar{m}) \) contracted with vector
\[ \tilde{n} - \bar{m} - \frac{(\tilde{n}^2 - \bar{m}^2)(\tilde{n} + \bar{m})}{(\tilde{n} + \bar{m})^2}, \]
which is orthogonal to \((\tilde{n} + \bar{m})\). Thus, we have seen, that in the involution relations for Virasoro-like constraints \( L(\tilde{n}) \) and \( \bar{L}(\tilde{n}) \) appear area-preserving diffeomorphisms. So, the constraints \( L, \bar{L} \) do not form a closed algebra.

It is convenient to describe area-preserving diffeomorphisms as
\[ \bar{H}_v(\tilde{n}) = v_i H_i(\tilde{n}), \quad \tilde{n} v \tilde{n} = 0. \quad (4.9) \]

introducing an arbitrary vector \( \tilde{v} \), being orthogonal to the integer vector \( \tilde{n} \). The other involution relations are as follows:
\[ [\bar{H}_v(\tilde{n}), L(\bar{m})] = U_{\bar{H}(\tilde{v},\tilde{n})L(\bar{m})}^{\bar{H}(\tilde{v},\tilde{n})} L(\tilde{n} + \bar{m}) + U_{\bar{H}(\tilde{v},\tilde{n})L(\bar{m})}^{\bar{H}(\tilde{v},\tilde{n})} \bar{L}(\tilde{n} + \bar{m}) + U_{\bar{H}(\tilde{v},\tilde{n})L(\bar{m})}^{\bar{H}(\tilde{v},\tilde{n})} \bar{H}_v(\tilde{n} + \bar{m}) \]
\[ = -\frac{1}{2} (\tilde{v} \tilde{m}) \left( 1 + \frac{\text{sgn}(\tilde{n}) \text{sgn}(\tilde{n} + \bar{m}) |\tilde{n}|}{|\tilde{n} + \bar{m}|} \right) L(\tilde{n} + \bar{m}) \]
\[ -\frac{1}{2} (\tilde{v} \tilde{m}) \left( 1 - \frac{\text{sgn}(\tilde{n}) \text{sgn}(\tilde{n} + \bar{m}) |\tilde{n}|}{|\tilde{n} + \bar{m}|} \right) \bar{L}(\tilde{n} + \bar{m}) \]
\[ + \frac{1}{2} \frac{\text{sgn}(\tilde{n})}{|\tilde{n}|} \left( (\tilde{n} \tilde{m}) v_i - (\tilde{v} \tilde{m}) m_i + \frac{\tilde{m}^2 (\tilde{v} \tilde{m}) (n_i + m_i)}{(\tilde{n} + \bar{m})^2} \right) H_i(\tilde{n} + \bar{m}), \quad (4.10) \]
\[ [H_v(\tilde{n}), \bar{L}(\tilde{m})] = U_{\bar{H}(\tilde{v},\tilde{n})L(\bar{m})}^H(\tilde{v},\tilde{n}) L(\tilde{n} + \bar{m}) + U_{\bar{H}(\tilde{v},\tilde{n})L(\bar{m})}^H(\tilde{v},\tilde{n}) \bar{L}(\tilde{n} + \bar{m}) + U_{\bar{H}(\tilde{v},\tilde{n})L(\bar{m})}^H(\tilde{v},\tilde{n}) \bar{H}_v(\tilde{n} + \bar{m}) \]
\[ = -\frac{1}{2} (\tilde{v} \tilde{m}) \left( 1 - \frac{\text{sgn}(\tilde{n}) \text{sgn}(\tilde{n} + \bar{m}) |\tilde{n}|}{|\tilde{n} + \bar{m}|} \right) L(\tilde{n} + \bar{m}) \]
\[ -\frac{1}{2} (\tilde{v} \tilde{m}) \left( 1 - \frac{\text{sgn}(\tilde{n}) \text{sgn}(\tilde{n} + \bar{m}) |\tilde{n}|}{|\tilde{n} + \bar{m}|} \right) \bar{L}(\tilde{n} + \bar{m}) \]
\[ + \frac{1}{2} \frac{\text{sgn}(\tilde{n})}{|\tilde{n}|} \left( (\tilde{n} \tilde{m}) v_i - (\tilde{v} \tilde{m}) m_i - \frac{\tilde{m}^2 (\tilde{v} \tilde{m}) (n_i + m_i)}{(\tilde{n} + \bar{m})^2} \right) H_i(\tilde{n} + \bar{m}), \quad (4.11) \]
\[ [H_\n\vec{n}, H_\vec{m}(\vec{m})] = U_{\vec{n}, \vec{m}}(\vec{n}, \vec{m}) H_{\vec{n} + \vec{m}} = (\vec{m} \vec{w}) H_\n(\vec{n} + \vec{m}) - (\vec{n} \vec{v}) H_\vec{m}(\vec{n} + \vec{m}). \]  

(4.12)

Concluding this section we should notice that the formulae (4.1) and (4.2) contain an ambiguity concerning zero modes \( L(0) \) and \( \bar{L}(0) \) which have to be described additionally. It is convenient define them as

\[ L(0) = \frac{1}{2} (H_\perp(0) + H_N(0)) \]  

(4.13)

and

\[ \bar{L}(0) = \frac{1}{2} (H_\perp(0) - H_N(0)) \]  

(4.14)

to provide a proper correspondence with a genuine Virasoro algebra in the limiting case \( N = 1 \).

5 Quantum corrections to the algebra of Virasoro-like constraints

In Sec. 3 we have introduced the split of integer vectors on \( N \)-torus into positive and negative ones. Now we shall apply this decomposition to define creation and annihilation operators for gravity and matter fields, and to split the Virasoro-like constraints into positive and negative ones. The latter decomposition will further define the natural Wick ordering for the ghost variables upon the construction of quantum BRST charge. Then we calculate the \( c \)-number part of the quantum correction to the commutator \([L(\vec{n}), L(-\vec{n})]\), which is of crucial importance from the standpoint of the nilpotency condition on BRST operator.

To begin with, we consider the massless scalar field contributions to constraints \( H_\perp(\vec{n}) \) and \( H_i(\vec{n}) \) which, in turn, define Virasoro-like constraints. Scalar contributions into super-Hamiltonian and supermomentum are as follows:

\[ H_{\perp \text{ scalar}} = \frac{1}{2} p^2 + \frac{1}{2} g g^{ij} \varphi_i \varphi_j, \]  

(5.1)

\[ H_i \text{ scalar} = p \varphi_i. \]  

(5.2)

Now, let us expand \( \varphi \) and its conjugate momentum \( p \) via creation and annihilation operators:

\[ \varphi = \frac{\varphi_0}{\sqrt{V}} + \sum_{\vec{k} > 0} \frac{1}{\sqrt{2 \omega_{\vec{k}} V}} (a_{\vec{k}} \exp(-i\vec{k} \vec{x}) + \bar{a}_{\vec{k}} \exp(i\vec{k} \vec{x})) + a_{\vec{k}}^\dagger \exp(i\vec{k} \vec{x}) + \bar{a}_{\vec{k}}^\dagger \exp(-i\vec{k} \vec{x})), \]  

(5.3)

\[ p = \frac{p_0}{\sqrt{V}} + i \sum_{\vec{k} > 0} \frac{\omega_{\vec{k}}}{2V} (-a_{\vec{k}} \exp(-i\vec{k} \vec{x}) - \bar{a}_{\vec{k}} \exp(i\vec{k} \vec{x})) + a_{\vec{k}}^\dagger \exp(i\vec{k} \vec{x}) + \bar{a}_{\vec{k}}^\dagger \exp(-i\vec{k} \vec{x})). \]  

(5.4)
Here $V$ is the volume of $N$-torus:

$$V = (2\pi)^N R_1 R_2 \cdots R_N, \quad (5.5)$$

where, $R_i$ are radii of the torus, $\omega_k = \sqrt{\vec{k}^2}$.

Substituting Eqs. (5.3)–(5.4) into Eqs. (5.1)–(5.2) and keeping only quadratic contributions one can write down the following expressions for $H_{\perp \text{scalar}}(\vec{n})$ and $H_{i \text{ scalar}}(\vec{n})$:

$$H_{\perp \text{scalar}} \vec{n} = ip_0 (\vec{a}_\vec{n}^+ - a_{\vec{n}}) \sqrt{\frac{\omega_{\vec{k}}}{2}}$$

$$- \frac{1}{2} \sum_{0<\vec{k}<\vec{n}} (a_{\vec{k}} a_{\vec{n}-\vec{k}} + a_{\vec{k}}^+ a_{\vec{n}-\vec{k}}^+) \left( \frac{\sqrt{\omega_{\vec{k}} \omega_{\vec{n}-\vec{k}}}}{2} + \frac{\vec{k}(\vec{n} - \vec{k})}{2\sqrt{\omega_{\vec{k}} \omega_{\vec{n}-\vec{k}}}} \right)$$

$$+ \frac{1}{2} \sum_{\vec{k}>0} (a_{\vec{n}+\vec{k}} \vec{a}_{\vec{k}} + a_{\vec{n}+\vec{k}}^+ \vec{a}_{\vec{k}}^+) \left( \frac{(\vec{n} + \vec{k})\vec{k}}{\omega_{\vec{n}+\vec{k}}^2} - \frac{\sqrt{\omega_{\vec{n}+\vec{k}} \omega_{\vec{k}}}}{\sqrt{\omega_{\vec{n}+\vec{k}}^2 \omega_{\vec{k}}}} \right)$$

$$+ \frac{1}{2} \sum_{\vec{k}>0} (a_{\vec{n}+\vec{k}} a_{\vec{n}+\vec{k}}^+ + a_{\vec{n}+\vec{k}}^+ a_{\vec{n}+\vec{k}}) \left( \frac{\sqrt{\omega_{\vec{n}+\vec{k}}^2 \omega_{\vec{k}}}}{\sqrt{\omega_{\vec{n}+\vec{k}}^2 \omega_{\vec{k}}}} + \frac{(\vec{n} + \vec{k})\vec{k}}{\sqrt{\omega_{\vec{n}+\vec{k}}^2 \omega_{\vec{k}}}} \right), \quad (5.6)$$

$$H_{i \text{ scalar}} \vec{n} = -ip_0 \frac{n_i}{2\omega_{\vec{n}}} (a_{\vec{n}} + \vec{a}_{\vec{n}}^+)$$

$$+ \sum_{0<\vec{k}<\vec{n}} (\vec{a}_{\vec{k}}^+ \vec{a}_{\vec{n}-\vec{k}} - a_{\vec{n}} a_{\vec{n}-\vec{k}}^+) \frac{(n - k)_i}{2} \sqrt{\frac{\omega_{\vec{k}}}{\omega_{\vec{n}-\vec{k}}}}$$

$$+ \sum_{\vec{k}>0} (a_{\vec{n}+\vec{k}} \vec{a}_{\vec{k}} + a_{\vec{n}+\vec{k}}^+ \vec{a}_{\vec{k}}^+) \frac{k_i}{2} \sqrt{\frac{\omega_{\vec{n}+\vec{k}}}{\omega_{\vec{k}}}}$$

$$- \sum_{\vec{k}>0} (\vec{a}_{\vec{k}} a_{\vec{n}+\vec{k}} + a_{\vec{n}+\vec{k}}^+ \vec{a}_{\vec{k}}^+) \frac{(n + k)_i}{2} \sqrt{\frac{\omega_{\vec{k}}}{\omega_{\vec{n}+\vec{k}}}}$$

$$+ \sum_{\vec{k}>0} (a_{\vec{k}} a_{\vec{n}+\vec{k}}^+ + a_{\vec{k}}^+ \vec{a}_{\vec{n}+\vec{k}}) \frac{(n + k)_i}{2} \sqrt{\frac{\omega_{\vec{k}}}{\omega_{\vec{n}+\vec{k}}}}$$

$$- \sum_{\vec{k}>0} (\vec{a}_{\vec{k}}^+ a_{\vec{n}+\vec{k}}^+ + a_{\vec{k}}^+ \vec{a}_{\vec{n}+\vec{k}}) \frac{k_i}{2} \sqrt{\frac{\omega_{\vec{n}+\vec{k}}}{\omega_{\vec{k}}}}. \quad (5.7)$$

The dynamical metric does not contribute into the expansions (5.6) and (5.7) in quadratic approximation because the background value of scalar field is zero. As a matter of fact, such a choice of the background is the only consistent for the stationary Bianchi-I universe at a classical level.

It is easy to see from Eq. (5.6) that in the case when $\vec{n} = 0$, Hamiltonian takes the diagonal form

$$H_{\perp \text{scalar}} \vec{a} = \frac{p_0^2}{2} + \frac{1}{2} \sum_{\vec{k}>0} \omega_{\vec{k}} (a_{\vec{k}}^+ a_{\vec{k}} + a_{\vec{k}}^+ a_{\vec{k}}^+). \quad (5.8)$$

The choice of creation and annihilation operators, given by Eqs. (5.3) and (5.4), proves itself in the suggestive form of the super-Hamiltonian.
Using formulae (5.9)–(5.17) one can get the expression for the contribution of the scalar fields into the central extension (we mean here that zero modes for scalar field and its conjugate momentum, as well as zero modes for other fields, are ordered according to Weyl rule, while other modes are Wick-ordered)

\[ [L(\bar{n}), L(-\bar{n})]_{c.e.\, scalars} = \frac{1}{2} \sum_{k=1}^{n_i} \left( \frac{\omega_k(n - k)}{\bar{n} - \bar{k}} \right) + \frac{1}{8} \sum_{k>0} \left( \frac{(\bar{n} + \bar{k})\bar{k}}{\sqrt{\bar{n} + \bar{k}}} \right) - \sqrt{\omega_{\bar{n} + \bar{k}} \omega_{\bar{k}}} \left( k_i \sqrt{\omega_{\bar{n} + \bar{k}}} - (n + k)_i \sqrt{\omega_{\bar{k}}} \right) \right] . \tag{5.9} \]

In the “last-element” limit (see sec. 3) the formula (5.9) is reduced to

\[ [L(\bar{n}), L(-\bar{n})]_{c.e.\, scalars} = \frac{1}{2} \sum_{k=1}^{n-1} k(n - k) = \frac{1}{12} n(n^2 - 1) , \tag{5.10} \]

and one can see that the contribution of scalar fields has a string-like structure. In the case when the model includes \( d \) massless scalar fields each of them gives the same contribution to the commutator \([L(\bar{n}), L(-\bar{n})]\), and thereby the right-hand side of (5.11) is multiplied by \( d \).

Consider now Wick representation for the metric and the quadratic part of gravity constraints. It is convenient to rewrite quadratic contribution of gravity variables into the constraints (2.1) and (2.2) in the following form:

\[ H_i = g_{ab} \pi^{c d}_{e} F_{i c d}^{a b} \, e + g_{ab} \pi^{c d}_{e} F_{i c d}^{a b} + \frac{1}{l_P} C_{i c d f}^{a b c d f} g_{ab} g_{c d f} + \frac{1}{l_P^2} D_{i c d f}^{a b c d f} g_{ab} g_{c d f} . \tag{5.11} \]

Here, in the Bianchi-I flat background, we have:

\[ E_{i c d}^{a b} = -2 \delta_{i (a} \delta_{d) e} \delta_{c b} , \tag{5.13} \]

\[ F_{i c d}^{a b} = \delta_{i c}^{a (c} \delta_{b d)} e - 2 \delta_{i (c} \delta_{b d)} e , \tag{5.14} \]

where

\[ \delta_{i c d}^{a b} \equiv \delta_{i (c}^{a} \delta_{d)}^{b} = \frac{1}{2} \left( \delta_{c}^{a} \delta_{d}^{b} + \delta_{c}^{b} \delta_{d}^{a} \right) . \tag{5.15} \]

\[ C_{i c d f}^{a b c d f} = \delta_{i c d}^{a} \delta_{f}^{b} - \delta_{i c}^{d} \delta_{f}^{e} \delta_{d}^{b} - \delta_{i d}^{c} \delta_{f}^{e} \delta_{c}^{b} + 2 \delta_{i (c} \delta_{d) f} \delta_{c d f} + 2 \delta_{i (c} \delta_{d) (e} \delta_{f) d} \delta_{e f} . \tag{5.16} \]

\[ D_{i c d f}^{a b c d f} = \frac{1}{4} \delta_{i c d}^{a} \delta_{f}^{b} - \frac{1}{2} \delta_{i c}^{d} \delta_{f}^{e} \delta_{d}^{b} - \frac{1}{2} \delta_{i d}^{c} \delta_{f}^{e} \delta_{c}^{b} + \frac{3}{4} \delta_{i (c} \delta_{d) (e} \delta_{f) d} \delta_{e f} + \frac{1}{2} \delta_{i (c} \delta_{d) (e} \delta_{f) d} \delta_{e f} . \tag{5.17} \]
In Eqs. (5.13)–(5.17) we have symmetrization in couples of indices $a$ and $b$, $c$ and $d$ and in Eq. (5.16) we have also the symmetrization in indices $e$ and $f$.

Now let us expand the metric $g_{ab}$ and the conjugate momentum $\pi^{ab}$ in terms of operators of creation and annihilation corresponding to different harmonics:

$$ g_{ab}(\vec{x}) = \sqrt{\frac{2l_p^{N-1}}{V}} g_{ab}(0) + \sum_{\vec{k}} \sqrt{\frac{2l_p^{N-1}}{V}} \sqrt{\frac{1}{2\omega_{\vec{k}}}} (a_{ab}(\vec{k}) \exp(-i\vec{k}\vec{x}) + a_{ab}^+(\vec{k}) \exp(i\vec{k}\vec{x})) + a_{ab}(\vec{k}) \exp(i\vec{k}\vec{x}) \right),$$

$$ \pi^{cd}(\vec{x}) = \sqrt{\frac{2l_p^{N-1}}{V}} \pi^{cd}(0) + \sum_{\vec{k}} iG^{ab,cd} \sqrt{\frac{1}{2l_p^{N-1}V}} \sqrt{\frac{\omega_{\vec{k}}}{2}} (-a_{ab}(\vec{k}) \exp(-i\vec{k}\vec{x}) + a_{ab}^+(\vec{k}) \exp(i\vec{k}\vec{x})), \right)$$

It is the choice of creation and annihilation operators which provides the diagonal form for the part of Hamiltonian $H_\perp(\vec{0})$ describing physical degrees of freedom for gravitons, i.e. transverse-traceless modes.

Now one can write down the $n$-th component of super-Hamiltonian (4.3) and super-momentum (4.4) in terms of expansions (5.18) and (5.19):

$$ H_\perp(\vec{0}) = \frac{i}{2} \sqrt{2\omega_{\vec{n}} \pi^{ab}(0)} (-a_{ab}(\vec{n}) + a_{ab}^+(\vec{n})) $$

$$ \frac{2n_{e.n}}{\sqrt{2\omega_{\vec{n}}}} C^{ab,cd,ef} g_{ab}(0)(a_{cd}(\vec{n}) + a_{cd}^+(\vec{n})) $$

$$ - \sum_{0<\vec{k}<\vec{n}} (a_{ab}(\vec{k}) a_{cd}(\vec{n} - \vec{k}) + a_{ab}^+(\vec{k}) a_{cd}^+(\vec{n} - \vec{k})) $$

$$ \times \left( \frac{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}}{4} G^{ab,cd} - \frac{(n-k)\epsilon(n-k)}{2\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}} C^{ab,cd,ef} + \frac{k_{e,f}}{2\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}} C^{cd,ab,ef} \right) $$

$$ + \frac{k_{e,f}}{2\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}} D^{ab,cd,ef} + \frac{(n-k)\epsilon(k_{e,f})}{2\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}} D^{cd,ab,ef} $$

$$ + \sum_{0<\vec{k}<\vec{n}} (a_{ab}(\vec{k}) a_{cd}^+ (\vec{n} - \vec{k}) + a_{ab}^+(\vec{k}) a_{cd}(\vec{n} - \vec{k})) $$

$$ \times \left( \frac{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}}{4} G^{ab,cd} - \frac{(n-k)\epsilon(n-k)}{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}} C^{ab,cd,ef} - \frac{k_{e,f}}{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} - \vec{k}}}} D^{ab,cd,ef} \right) $$

$$ + \sum_{\vec{k}>0} (a_{ab}(\vec{n} + \vec{k}) a_{cd}^+(\vec{k}) + a_{ab}^+(\vec{n} + \vec{k}) a_{cd}(\vec{k})) $$

$$ \times \left( \frac{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} + \vec{k}}}}{4} G^{ab,cd} - \frac{k_{e,f}}{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} + \vec{k}}}} C^{ab,cd,ef} + \frac{(n+k)\epsilon(k_{e,f})}{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} + \vec{k}}}} D^{ab,cd,ef} \right) $$

$$ + \sum_{\vec{k}>0} (a_{ab}(\vec{n} + \vec{k}) a_{cd}^+(\vec{k}) + a_{ab}^+(\vec{n} + \vec{k}) a_{cd}(\vec{k})) $$

$$ \times \left( -\frac{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} + \vec{k}}}}{4} G^{ab,cd} - \frac{k_{e,f}}{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} + \vec{k}}}} C^{ab,cd,ef} + \frac{(n+k)\epsilon(k_{e,f})}{\sqrt{\omega_{\vec{k}} \omega_{\vec{n} + \vec{k}}}} D^{ab,cd,ef} \right) $$
Indeed, due to definition of creation and annihilation operators (5.18), (5.19) in which momentum $\pi^j$ is proportional to $1/l_P^{(N-1)/2}$ while $g_{ij}$ is proportional to $l_P^{(N-1)/2}$ the dependence on $l_P$ disappears from super-Hamiltonian in quadratic approximation. $k$-th order
of expansion in the metric (super-Hamiltonian contains only quadratic terms in momenta) is proportional to $\kappa^{k-2}$. Thus, the terms of $k$-th order in creation and annihilation operators are proportional to the same power of the expansion parameter $\kappa$. So, this expansion in $a, a^+, \bar{a}, \bar{a}^+$ can be thought about as plausible for the stationary Universe whose size is greater than Planck length.

From Eqs. (5.20), (2.2), (4.1) it follows that the contribution of graviton modes into the central extension of the commutator $[L(\bar{n}), L(-\bar{n})]$ looks as follows:

$$[L(\bar{n}), L(-\bar{n})]_{c.e. \text{ gravitons}} = \frac{n_i}{|\bar{n}|} \times \left\{ \begin{array}{l}
\frac{1}{4} G_{ab,a'b'} G_{cd,c'd'} E_{i}^{a'b',c'd',e'} n_i \omega_{\bar{n}} \\
+ \frac{n_i n_f n_{e'}}{2 \omega_{\bar{n}}} C_{ab,c'd'} G_{cd,a'f} F_{i}^{a'b',c'd',e'} \\
+ \sum_{0 < k < \bar{n}} \left\{ \frac{1}{4} G_{ab,a'b'} G_{cd,c'd'} E_{i}^{a'b',c'd',e'} \omega_{\bar{n} - k} \right. \\
+ \frac{1}{4} G_{ab,a'b'} G_{cd,c'd'} F_{i}^{a'b',c'd',e'} \omega_{\bar{n} - k} \\
+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} C_{ab,c'd'} E_{i}^{a'b',c'd',e'} k_e k_f k_{e'} \omega_{\bar{n} - k} \\
+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} C_{ab,c'd'} F_{i}^{a'b',c'd',e'} (n - k) \omega_{\bar{n} - k} \\
+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} D_{ab,c'd'} E_{i}^{a'b',c'd',e'} k_e (n - k) \omega_{\bar{n} - k} \\
+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} D_{ab,c'd'} F_{i}^{a'b',c'd',e'} (n - k) \omega_{\bar{n} - k} \\
+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} D_{ab,c'd'} D_{ab,c'd'} F_{i}^{a'b',c'd',e'} (n - k) \omega_{\bar{n} - k} \\
+ \left. \sum_{k > 0} \left\{ \frac{1}{4} G_{ab,a'b'} G_{cd,c'd'} G_{ab,c'd'} E_{i}^{a'b',c'd',e'} ((n + k) \omega_{\bar{n} - k} - k_e \omega_{\bar{n} - k}) \\
+ \frac{1}{4} G_{ab,a'b'} G_{cd,c'd'} F_{i}^{a'b',c'd',e'} (\omega_{\bar{n} + k} (n + k) - \omega_{\bar{n} + k} k^{'e'}) \\
+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} C_{ab,c'd'} E_{i}^{a'b',c'd',e'} \left( \frac{(n + k) e (n + k) f (n + k) e'}{\omega_{\bar{n} + k}} - \frac{k_e k_f k_{e'}}{\omega_{\bar{n} + k}} \right) \\
+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} C_{ab,c'd'} F_{i}^{a'b',c'd',e'} \left( \frac{k_e k_f (n + k) e'}{\omega_{\bar{n} + k}} - \frac{(n + k) \omega_{\bar{n} + k}}{\omega_{\bar{n} + k}} \right) \right. \right\} 
\end{array} \right\}$$

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\begin{align*}
&+ \frac{1}{2} G_{ab,c'd'} G_{cd,a'b'} C_{ab,cd,ef} F_{i}^{ab',c'd',e'} \left( \frac{(n + k)e(n + k)f(n + k)e'}{\omega_{n+k}^{2}} - \frac{k_e k_f k_i}{\omega_k^{2}} \right) \\
&+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} D_{i}^{ab',c'd',e'} \left( \frac{(n + k)e(n + k)f(n + k)e'}{\omega_{n+k}^{2}} - \frac{k_e n}{\omega_k^{2}} \right) \\
&+ \frac{1}{2} G_{ab,c'd'} G_{cd,a'b'} D_{i}^{ab',c'd',e'} \left( \frac{k_e(n + k)f(n + k)e'}{\omega_{n+k}^{2}} - \frac{(n + k)e(n + k)f(n + k)e'}{\omega_k^{2}} \right) \\
&+ \frac{1}{2} G_{ab,a'b'} G_{cd,c'd'} D_{i}^{ab',c'd',e'} \left( \frac{k_e(n + k)f(n + k)e'}{\omega_{n+k}^{2}} - \frac{(n + k)e(n + k)f(n + k)e'}{\omega_k^{2}} \right) \\
&+ \frac{1}{2} G_{ab,c'd'} G_{cd,a'b'} D_{i}^{ab',c'd',e'} \left( \frac{(n + k)e(n + k)f(n + k)e'}{\omega_k^{2}} - \frac{k_e n}{\omega_{n+k}^{2}} \right) \\
&\times \left( \frac{(n + k)e(n + k)f(n + k)e'}{\omega_k^{2}} - \frac{k_e(n + k)f(n + k)e'}{\omega_{n+k}^{2}} \right) \right) \right). \quad (5.22)
\end{align*}

It is easy to get the contractions of structure tensors $G, C, D, E$ and $F$ defined by formulae (2.3),(2.4), (5.13)–(5.17). These contractions look as follows:

\begin{align*}
G_{ab,a'b'} G_{cd,a'b'} G_{cd,ef} F_{i}^{ab',c'd',e'} &= \frac{(N - 2)(N + 1)}{2} \delta_i^{e'}, \quad (5.23) \\
G_{ab,a'b'} G_{cd,a'b'} C_{ab,cd,ef} E_{i}^{ab',c'd',e'} &= G_{ab,c'd'} G_{cd,ef} C_{ab,cd,ef} F_{i}^{ab',c'd',e'} \\
&= (N - 2)(N + 1)(\delta_i^{e'} \delta^{ef} - \delta_i^{e} \delta^{ef}), \quad (5.24) \\
G_{ab,a'b'} G_{cd,a'b'} C_{ab,cd,ef} F_{i}^{ab',c'd',e'} &= G_{ab,c'd'} G_{cd,ef} C_{ab,cd,ef} F_{i}^{ab',c'd',e'} \\
&= -\frac{(N - 2)(N + 1)}{2(N - 1)\delta_i^{e'} \delta^{ef}} \left( (N - 3)\delta_i^{e'} \delta^{ef} + 2\delta_i^{e} \delta^{ef} \right), \quad (5.25) \\
G_{ab,a'b'} G_{cd,a'b'} D_{i}^{ab',c'd',e'} &= \frac{3N^2 - 3N - 8}{4(N - 1)} \delta_i^{e'} \delta^{ef} \\
&- \frac{N^2 - N - 4}{4(N - 1)} \delta_i^{e} \delta^{ef} \frac{N^2 - 3}{2(N - 1)} \delta_i^{e} \delta^{ef}, \quad (5.26) \\
G_{ab,c'd'} G_{cd,a'b'} D_{i}^{ab',c'd',e'} &= \frac{3N^2 - 3N - 8}{4(N - 1)} \delta_i^{e'} \delta^{ef} \\
&- \frac{N^2 - N - 4}{4(N - 1)} \delta_i^{e} \delta^{ef} \frac{N^2 - 3}{2(N - 1)} \delta_i^{e} \delta^{ef}, \quad (5.27) \\
G_{ab,a'b'} G_{cd,a'b'} D_{i}^{ab',c'd',e'} &= \frac{-3N^3 + 6N^2 + N - 20}{8(N - 1)} \delta_i^{e'} \delta^{ef} \\
&- \frac{N^2 - 9}{8(N - 1)} \delta_i^{e} \delta^{ef} + \frac{N^2 + 2N + 5}{8(N - 1)} \delta_i^{e} \delta^{ef}, \quad (5.28)
\end{align*}
\[ G_{ab,c'd'} G_{cd,a'b'} D^{ab,cd,f} F_{i}^{a'b',c'd',e'} = \frac{-3N^3 + 6N^2 + N - 20}{8(N - 1)} \delta_i^{e'} \delta_{ef} \]
\[ - \frac{N^2 - 9}{8(N - 1)} \delta_i^e \delta_{ee'} + \frac{N^2 + 2N + 5}{8(N - 1)} \delta_i^e \delta_{fe'}. \]

(5.29)

In the “last-element limit” (see sec.3), the expression (5.22) is reduced to the much more simple one

\[ [L(\bar{n}), L(-\bar{n})]_{c.e. \text{ gravitons}} = \]
\[ (GGGE)_{l.e.} \frac{1}{4} \left( n^2 + \sum_{k=1}^{n-1} k^2 - \sum_{k=1}^{\infty} k^2 + \sum_{k=n+1}^{\infty} k^2 \right) \]
\[ + (GGGF)_{l.e.} \left( \frac{1}{4} \sum_{k=1}^{n-1} k(n - k) \right) \]
\[ + (GGCE)_{l.e.} \left( \frac{1}{2} \sum_{k=1}^{n-1} \left( \frac{k^3}{n - k} + k(n - k) \right) \right) \]
\[ + 1 \sum_{k=1}^{\infty} \left( - \frac{k^3}{n + k} + \frac{(n + k)^3}{k} \right) \]
\[ + (GGCF)_{l.e.} \left( \frac{1}{2} n^2 + \sum_{k=1}^{n-1} k^2 \right) \]
\[ + (GGDE)_{l.e.} \left( \sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{\infty} k^2 - \sum_{k=n+1}^{\infty} k^2 \right) \]
\[ + (GGDF)_{l.e.} \left( \sum_{k=1}^{n-1} k(n - k) \right). \]

(5.30)

Here the notations \((GGGE)_{l.e.}\) etc. denote the corresponding contractions taken in the “last-element” limit i.e. in the case when \(e = f = e' = N\). It is easy to see that the coefficient at \((GGGE)_{l.e.}\) in Eq. (5.30) vanishes. The structure at \((GGCE)_{l.e.}\) is very inconvenient for calculation, but fortunately

\[(GGGE)_{l.e.} = 0.\]

Other sums can be calculated due to simple formulae

\[ \sum_{k=1}^{n-1} k(n - k) = \frac{n(n^2 - 1)}{6}, \]

(5.31)

(5.32)

(5.33)
Using Eqs. (5.31)–(5.33) one reduces Eq. (5.30) to

\[ [L(\vec{n}), L(-\vec{n})]_{c.e. \, gravitons} = \]

\[ n^3 \left( \frac{1}{24} (GGGF)_{l.e.l.} + \frac{1}{3} (GGCF)_{l.e.l.} + \frac{2}{3} (GGDE)_{l.e.l.} + \frac{1}{6} (GGDF)_{l.e.l.} \right) \]

\[ + n \left( -\frac{1}{24} (GGGF)_{l.e.l.} + \frac{1}{6} (GGCF)_{l.e.l.} + \frac{1}{3} (GGDE)_{l.e.l.} - \frac{1}{6} (GGDF)_{l.e.l.} \right) \]  (5.34)

We shall be interested mainly in the coefficient at \( n^3 \) (because as it is well-known the coefficients at \( n \) can be adjusted to the necessary values by inclusion of intercept into \( L_0 \)), however we shall write down the full expression for the graviton contribution into the quantum correction to \([L(\vec{n}), L(-\vec{n})]\). Now we shall calculate (5.34) using the following formulae for the contractions in the “last-element” limit:

\[(GGGF)_{l.e.l.} = \frac{(N-2)(N+1)}{2}, \]  (5.35)

\[(GGCF)_{l.e.l.} = -\frac{(N-2)(N+1)}{2}, \]  (5.36)

\[(GGDE)_{l.e.l.} = -\frac{1}{2}, \]  (5.37)

\[(GGDF)_{l.e.l.} = -\frac{3(N-2)(N+1)}{8}. \]  (5.38)

Substituting Eqs. (5.35)–(5.38) into Eq. (5.34) we have

\[ [L(\vec{n}), L(-\vec{n})]_{c.e. \, gravitons} = -\frac{1}{12} \left( 4 + \frac{5}{2} (N-2)(N+1) \right) n^3 \]

\[- \frac{1}{12} \left( 2 + \frac{1}{2} (N-2)(N+1) \right) n. \]  (5.39)

Notice, that the appearance of the expression \((N-2)(N+1)/2\) in the right-hand side of Eq. (5.39) does not look surprising because this expression is nothing but the number of gravitons. What is less trivial that the coefficient at the number of gravitons differs from the one at the number of matter fields and even has another sign. Moreover, there is a constant contribution in the right-hand side of Eq. (5.39), which is independent from the dimensionality of space.

Concluding this section we write the contributions of some other matter fields into the central extension. The contribution of massless vector fields is quite analogous of that of scalar field. The number of local degrees of freedom of one vector field is \((N-1)\). Thus, if the model includes \(d_V\) vector fields their contribution is

\[ [L(\vec{n}), L(-\vec{n})]_{c.e. \, vectors} = \frac{d_V(N-1)}{12} n(n^2 - 1). \]  (5.40)

The Weyl or Majorana massless spinor field has \(2((N+1)/2-1)\) local degrees of freedom \(^1\) and the contribution of such fields into \([L(\vec{n}), L(-\vec{n})]_{c.e.}\) looks as

\[ [L(\vec{n}), L(-\vec{n})]_{c.e. \, spinors} = \frac{d_F}{12} 2^{((N+1)/2-1)} n(n^2 - 1). \]  (5.41)

\(^1\)It is interesting to recall that in the space–times of the exceptional dimensionality \(2+8k\), with integer \(k\), one can satisfy simultaneously Weyl and Majorana conditions and the number of degrees of freedom of spinor is \(2^{((N+1)/2-1)}\) for this case. For example, in the simplest \(1+1\)-dimensional case (i.e. for \(N = 1\)) the contribution of the spinor should be one–half of scalar contribution (see e.g. [25]).
Let us collect the contributions of all the matter fields and gravity into the central extension \([L(\vec{n}), L(-\vec{n})]\):

\[
[L(\vec{n}), L(-\vec{n})]_{c.e.} = \frac{d}{12} n^2(n^2 - 1) + \frac{d_V(N - 1)}{12} n(n^2 - 1) + \frac{d_{F2}[(N+1)/2-1]}{12} n(n^2 - 1) \\
- \frac{1}{12} \left( 4 + \frac{5}{2}(N - 2)(N + 1) \right) n^3 - \frac{1}{12} \left( 2 + \frac{1}{2}(N - 2)(N + 1) \right) n \quad (5.42)
\]

One can show by direct calculation that the commutators between area-preserving diffeomorphisms and between them and Virasoro-like generators do not contain quantum corrections. The commutators between \(L\) and \(\bar{L}\) Virasoro-like generators do not include first quantum corrections as well. It is easy to see that the quantum correction to the commutator \([\bar{L}(-\vec{n}), \bar{L}(\vec{n})]\) coincides with one for \([L(\vec{n}), L(-\vec{n})]\) given by the right-hand side of Eq. (5.42).

Thus we have seen that the algebra of constraints is disclosed already by constant contributions of the first quantum corrections. It is worth mentioning that the fact of a disclosure does not mean by itself the quantum inconsistency of the theory. A proper test for consistency is to examine the nilpotency condition for the quantum BRST charge related to the disclosed constraint algebra. This examination is performed in the next section.

6 BRST operator and critical relation between parameters of the theory

In this section we study the operatorial canonical BFV quantization [5, 6, 7] of the constrained dynamics developed in the previous section for the toroidal Bianchi-I cosmology. The general prescription of the BFV–method implies making a definite choice of the operator symbol both for the original phase space variables and ghosts. The main insight at the choice of the ghost ordering is from the due regard to the structure of the related constraints. The Virasoro–like subset of the constraints has the polarization, splitting them into the positive– and negative–frequency ones, being conjugated to each other. Thereby, it is natural to subject the related ghost variables to the Wick ordering rule. Meanwhile the ghosts attached to the area–preserving diffeomorphisms are quantized by the Weyl rule. There are several reasons for such a choice of ordering for these ghosts. On the one hand, this ordering, as it will be shown later, is compatible with the mentioned in previous section fact, that there are no quantum corrections to the commutators of the area–preserving diffeomorphisms among themselves and with the Virasoro–like constraints. On the other hand, this choice follows to some extent to a certain analogy with \(p\)–brane theory [22] where the area preserving constraints describe some residual symmetry after fixing of the light–cone gauge. Now we are in a position to write down the quantum BRST charge for the model. It looks like

\[
\Omega = \sum_{\vec{n} > 0} (\xi^{+}(\vec{n}) L(\vec{n}) + \xi(\vec{n}) L(-\vec{n}) + \bar{\xi}(\vec{n}) \bar{L}(\vec{n}) + \bar{\xi}^{+}(\vec{n}) \bar{L}(-\vec{n}))
\]
\[+c^{(0)} L(0) + \tilde{c}^{(0)} \tilde{L}(0) + \sum_{\bar{n}, \bar{v}} \bar{c}^{(\bar{n}, \bar{v})} \tilde{H}_{\bar{v}}(\bar{n})\]

\[+ \frac{1}{2} \sum_{\bar{n} > 0, \bar{m} > 0} c^{(\bar{m})} c^{(\bar{n})} P_{(\bar{n} + \bar{m})} U_{L(\bar{n}) L(\bar{m})} + \frac{1}{2} \sum_{\bar{n} > 0, \bar{m} > 0} c^{(\bar{m})} c^{(\bar{n})} P_{(\bar{n} + \bar{m})} U_{L(\bar{n}) L(\bar{m})}\]

\[+ \frac{i}{2} \sum_{\bar{n} > 0, \bar{m} > 0, \bar{v}} c^{(\bar{m})} c^{(\bar{n})} \tilde{P}_{(\bar{n} + \bar{m}, \bar{v})} U_{\tilde{H}(\bar{n} + \bar{m}, \bar{v}) L(\bar{m})} + \frac{i}{2} \sum_{\bar{n} > 0, \bar{m} > 0, \bar{v}} c^{(\bar{m})} c^{(\bar{n})} \tilde{P}_{(\bar{n} + \bar{m}, \bar{v})} U_{\tilde{H}(\bar{n} + \bar{m}, \bar{v}) L(\bar{m})}\]

\[+ \sum_{\bar{n} > 0, \bar{m} > 0} c^{(\bar{m})} P_{(\bar{n} - \bar{m})} c^{(\bar{n})} U L(\bar{n} - \bar{m}) L(\bar{m}) L(\bar{m}) L(\bar{m}) + \sum_{\bar{n} > 0, \bar{m} > 0} c^{(\bar{m})} P_{(\bar{n} - \bar{m})} c^{(\bar{n})} U L(\bar{n} - \bar{m}) L(\bar{m}) L(\bar{m})\]

\[+ \sum_{\bar{n} > 0, \bar{m} > 0, \bar{v}} c^{(\bar{m})} c^{(\bar{n})} \tilde{P}_{(\bar{n} - \bar{m}, \bar{v})} U_{\tilde{H}(\bar{n} - \bar{m}, \bar{v}) L(\bar{m})} + \sum_{\bar{n} > 0, \bar{m} > 0, \bar{v}} c^{(\bar{m})} c^{(\bar{n})} \tilde{P}_{(\bar{n} - \bar{m}, \bar{v})} U_{\tilde{H}(\bar{n} - \bar{m}, \bar{v}) L(\bar{m})}\]

\[+ \sum_{\bar{n} > 0, \bar{m} > 0, \bar{v}} c^{(\bar{m})} P_{(\bar{n} - \bar{m})} c^{(\bar{n})} U L(\bar{n} - \bar{m}) L(\bar{m}) L(\bar{m}) + \sum_{\bar{n} > 0, \bar{m} > 0, \bar{v}} c^{(\bar{m})} P_{(\bar{n} - \bar{m})} c^{(\bar{n})} U L(\bar{n} - \bar{m}) L(\bar{m}) L(\bar{m})\]
In Eq. (6.1) we should make summation over the set of \((N - 1)\) unit vectors \(\vec{v}\) enumerating area-preserving diffeomorphisms constraints \(\tilde{H}_v^{(\vec{n})}\) which are orthogonal to vector \(\vec{n}\).

The ghosts \(c^{+(\vec{n})}, c^{-(\vec{n})}\), \(P^{+(\vec{n})}\) and \(\tilde{P}^{+(\vec{n})}\) are creation operators, \(c^{-(\vec{n})}, c^{+(\vec{n})}\), \(P^{-(\vec{n})}\) and \(\tilde{P}^{-(\vec{n})}\) are annihilation operators. All these operators are Wick-ordered, while \(c^{(\vec{n},\vec{v})}, \tilde{P}^{(\vec{n},\vec{v})}, c^{(0)}, P^{(0)}, \tilde{c}^{(0)}\) and \(\tilde{P}^{(0)}\) are ordered according to Weyl rule. The latter are subjected the following non-vanishing anticommutation relations:

\[
\begin{align*}
[c^{+(\vec{n})}, P^{-(\vec{n})}] &= \delta^{(\vec{n})}, \\
[c^{-(\vec{n})}, P^{+(\vec{n})}] &= \delta^{(\vec{n})}, \\
[c^{+(\vec{n})}, P^{+(\vec{n})}] &= \delta^{(\vec{n})}, \\
[c^{-(\vec{n})}, P^{-(\vec{n})}] &= \delta^{(\vec{n})}, \\
[\tilde{c}^{(\vec{n},\vec{v})}, \tilde{P}^{(\vec{n},\vec{v})}] &= i\delta^{(\vec{n})}\delta_{(\vec{v},\vec{w})}, \\
[c^{(0)}, P^{(0)}] &= i, \\
[\tilde{c}^{(0)}, \tilde{P}^{(0)}] &= i.
\end{align*}
\]

Making use of Wick theorem, one may establish that due to the terms with two contractions of ghost operators, the requirement of vanishing of the squared BRST operator \(\Omega^2 = 0\) should give rise to the following nonvanishing corrections to the quantum involution relations of the constraint operators:

\[
\begin{align*}
[L^{(\vec{n})}, L^{(-\vec{n})}]_{c.e. \text{ ghosts}} &= \frac{1}{2} U^{L^{(\vec{n})}}_{L^{(0)}} U^{L^{(0)}}_{L^{(-\vec{n})}} L^{(-\vec{n})} + \frac{1}{2} U^{L^{(0)}}_{L^{(\vec{n})}} L^{(-\vec{n})} U^{L^{(-\vec{n})}}_{L^{(0)}} \\
+ &\sum_{0 < k < \vec{n}} U^{L^{(\vec{n}-\vec{k})}}_{L^{(\vec{n})}} L^{(-\vec{k})} U^{L^{(-\vec{k})}}_{L^{(\vec{n}-\vec{k})}} L^{(-\vec{n})} \\
+ &\sum_{k > 0} U^{L^{(\vec{n}+\vec{k})}}_{L^{(\vec{n})}} L^{(\vec{k})} L^{(-\vec{n}+\vec{k})} L^{(-\vec{n})} U^{L^{(-\vec{n}-\vec{k})}}_{L^{(\vec{n}-\vec{k})}} L^{(-\vec{k})} L^{(-\vec{n})}; \\
\end{align*}
\]

\[
\begin{align*}
[\tilde{L}^{(-\vec{n})}, \tilde{L}^{(\vec{n})}]_{c.e. \text{ ghosts}} &= \frac{1}{2} U^{L^{(-\vec{n})}}_{L^{(-\vec{n})}} U^{L^{(0)}}_{L^{(-\vec{n})}} \tilde{L}^{(-\vec{n})} + \frac{1}{2} U^{L^{(0)}}_{L^{(-\vec{n})}} \tilde{L}^{(-\vec{n})} U^{L^{(\vec{n})}}_{L^{(0)}} L^{(-\vec{n})} \\
+ &\sum_{0 < k < -\vec{n}} U^{L^{(-\vec{n}-\vec{k})}}_{L^{(-\vec{n})}} \tilde{L}^{(-\vec{k})} U^{L^{(-\vec{k})}}_{L^{(-\vec{n}-\vec{k})}} \tilde{L}^{(-\vec{n})} \\
+ &\sum_{k < 0} U^{L^{(-\vec{n}+\vec{k})}}_{L^{(-\vec{n})}} L^{(\vec{k})} L^{(-\vec{n}+\vec{k})} L^{(-\vec{n})} U^{L^{(-\vec{n}-\vec{k})}}_{L^{(-\vec{n}-\vec{k})}} L^{(-\vec{k})} L^{(-\vec{n})}.
\end{align*}
\]
Straightforward but tedious consideration shows that the nonvanishing quantum corrections arise only in commutators \([L(\vec{n}), L(-\vec{n})]\) and \([L(-\vec{n}), L(\vec{n})]\). Thus, the rest of the commutators do not get c-number quantum corrections. Moreover, all the contributions to Eqs. (6.3), (6.4) of structure constants involving area-preserving diffeomorphisms are mutually canceled.

Left–hand sides of the relations (6.3) and (6.4) represent the quantum corrections to the corresponding commutators calculated in the preceding section, while the right–hand sides consist of the quantum ghost contributions required by the nilpotency condition. Left–hand sides are determined by a specific realization of the constraint algebra and, in particular, depend upon the specific spectrum of the fields involved into the original Lagrangian, whereas the right-hand sides do not depend on the specific realization of the constraints, being completely fixed only by the structure constants of the algebra. So, these equalities, following the nilpotency of the quantum BRST charge, may constitute in principle the nontrivial restrictions on the theory parameters.

It was shown in the preceding section non-trivial corrections to quantum involution relations (see Eq. (5.42) depending on matter content of the theory appear only in commutators \([L(\vec{n}), L(-\vec{n})]\) (and, symmetrically, in \([L(-\vec{n}), L(\vec{n})]\)). Here, we have an analogous structure of quantum corrections to involution relations being originated from quantum commutators between the terms in \(\Omega\) which are of third order in ghost variables. In this approximation, it is sufficient to keep only the constant parts of the structure coefficients because the multipole harmonics may contribute into the central extension only if the terms with three or more contractions are accounted for. These terms represent the quantum corrections of a higher order.

Now we should calculate the right–hand side of Eq. (6.3) explicitly (calculation of (6.4) does not give us an additional information).

Substituting into Eq. (6.3) explicit expressions for structure constant we have:

\[
[L(\vec{n}), L(-\vec{n})]_{c.e. \text{ ghosts}} = 2\vec{n}^2 + \sum_{0 < k < \vec{n}} \frac{1}{4|\vec{n}| |\vec{k}|^2 (\vec{n} - \vec{k})^2} \\
\times \left\{ 2|\vec{n} - \vec{k}| \vec{n}^4 \vec{k}^2 - |\vec{n} - \vec{k}| \vec{n}^2 \vec{k}^4 - |\vec{n} - \vec{k}| \vec{n}^2 \vec{k}^2 (\vec{n} \vec{k}) \\
+ 2|\vec{n} - \vec{k}| \vec{n}^2 (\vec{n} \vec{k})^2 + 2|\vec{n} - \vec{k}| \vec{k}^4 (\vec{n} \vec{k}) \\
-3|\vec{n} - \vec{k}| \vec{k}^2 (\vec{n} \vec{k})^2 - |\vec{n} - \vec{k}| (\vec{n} \vec{k})^3 + 4\vec{n}^4 |\vec{k}| (\vec{n} \vec{k}) \\
+\vec{n}^2 |\vec{k}|^5 - \vec{n}^2 |\vec{k}|^3 (\vec{n} \vec{k}) - 6\vec{n}^2 |\vec{k}| (\vec{n} \vec{k})^2 \\
-2|\vec{k}|^5 (\vec{n} \vec{k}) + 5|\vec{k}|^3 (\vec{n} \vec{k})^2 - |\vec{k}| (\vec{n} \vec{k})^3 \right\} \\
+ \sum_{k > 0} \frac{1}{2|\vec{n}| \vec{k}^2 (\vec{n} + \vec{k})^2} \\
\times \left\{ 2|\vec{n} + \vec{k}| \vec{n}^4 \vec{k}^2 - |\vec{n} + \vec{k}| \vec{n}^2 \vec{k}^4 + |\vec{n} + \vec{k}| \vec{n}^2 \vec{k}^2 (\vec{n} \vec{k}) \\
+ 2|\vec{n} + \vec{k}| \vec{n}^2 (\vec{n} \vec{k})^2 - 2|\vec{n} + \vec{k}| \vec{k}^4 (\vec{n} \vec{k}) \\
-3|\vec{n} + \vec{k}| \vec{k}^2 (\vec{n} \vec{k})^2 + |\vec{n} + \vec{k}| (\vec{n} \vec{k})^3 - 4\vec{n}^4 |\vec{k}| (\vec{n} \vec{k}) \\
+\vec{n}^2 |\vec{k}|^5 + \vec{n}^2 |\vec{k}|^3 (\vec{n} \vec{k}) - 6\vec{n}^2 |\vec{k}| (\vec{n} \vec{k})^2 \right\}
\]

\(^2\)The general fact of appearance the corrections which emerge from the quantum contribution of ghost commutators has been first studied in the paper [7].
\[ +2|\vec{k}|^5(\vec{n}\vec{k}) + 5|\vec{k}|^3(\vec{n}\vec{k})^2 + |\vec{k}|(\vec{n}\vec{k})^3 \].  

Going in Eq. (6.5) to the “last-element limit” we come to the following expression

\[ [L(\vec{n}), L(-\vec{n})]_{c.e.\ ghosts} = 2n^2 + \sum_{k=1}^{n-1} (2n-k)(n+k) = \frac{13}{6}n^3 - \frac{1}{6}n. \]  

It is easy to see that the result (6.6) coincides with the well-known expression for the Virasoro algebra.

Now, equating the coefficient at \( n^3 \) with an analogous coefficient in Eq. (5.42) we come to the following critical relation between the dimensionality of space and the matter content of the theory:

\[ d + d_V(N-1) + d_F 2^{(N-1)/2-1}] = 30 + \frac{5}{2}(N-2)(N+1). \]  

Thus, we may claim that this short relation eventually represents a necessary condition for a nilpotency of the BRST charge (6.1) in the first quantum approximation. As is seen it imposes some correlations between number of scalars \( d \), vectors \( d_V \), spinors \( d_F \) and dimensionality of the space.

7 Conclusions

In the Conclusions let us summarize briefly the key points of suggested construction and make some comments about the results.

We have suggested a treatment of closed cosmological models by the harmonic expansion of gravity and matter canonical variables and the constraints. As basis harmonics we have chosen eigenfunctions of Laplacian on the maximally symmetric manifold of the topology under consideration. In doing so the structure constants of the involution relations of constraints algebra could be expressed via Clebsch–Gordan coefficients of the corresponding symmetry group. This provides the general background for the multipole expansion of constrained Hamiltonian dynamics of closed cosmological models, and their subsequent quantization. In this paper, however, we restricted the further analysis to the case of \( N \)-dimensional generalization of the stationary closed Bianchi - I cosmological model (\( N \)-torus). Except the technical simplicity, this choice of the particular model was motivated by the fact that it has the stationary classical solutions.

Besides the multipole expansion, the construction implies splitting of the constraint set into the area-preserving diffeomorphism generators, forming a closed subalgebra, and the Virasoro-like generators.

Performing the quantization we have constructed the BRST operator, defining the Wick ordering for the gravity, matter and Virasoro-like ghost excitations and the Weyl one for zero-modes and for the ghosts related to area-preserving diffeomorphisms. Our further objective was an examination of the nilpotency condition for the BRST charge. Actually, in the nilpotency equation \( \Omega^2 = 0 \), we have managed to calculate the constant part of the first quantum correction to the coefficient quadratic in ghosts. In doing so
we applied the expansion of constraints and structure functions in powers of the dimensionless parameter \( \kappa = \sqrt{\frac{4(\alpha - 1)}{N}} \). Finally, we derived from the nilpotency condition the relation between the dimensionality of the space \( N \) and the spectrum of matter fields (6.7). Although, the restrictions imposed by Eq. (6.7) do not predict the unique spectrum and spatial dimensionality, one can observe some curious consequences, for example: stationary Bianchi-I closed cosmology is inconsistent without matter at any positive integer \( N \). Notice, also, that in \( N = 3 \) case, the admissible matter spectra do not contain sufficient number of degrees of freedom to be compatible with the Standard Model or its generalizations. It is not too disappointing, because the stationary closed Bianchi-I model itself could hardly be expected to describe the observable cosmology. On the other hand, the very existence of such a relation allows to hope that analogous conditions, being deduced for more realistic models (with non-stationarity, other topology, massive particles and, perhaps, with an extended symmetry), can display a better correspondence with the realistic particle physics.

One more (a little bit speculative) way of fine-tuning the number of matter degrees of freedom to the dimensionality of space could be found in a generalized Kaluza-Klein ideology. In this way, Eq. (5.7) can be thought of as valid for the fundamental multidimensional theory. That provides much more opportunities for the matter spectrum in low-dimensional effective theory by means of a proper compactification.

While an opportunity of interplay between matter content of the Universe and its geometrical characteristics is of certain interest, we would like to think that the designed scheme of Hamiltonian treatment and quantization of closed cosmological models may open some other promising prospects for studying various aspects of quantum cosmology.

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