Research Article

Ergodicity of a Nonlinear Stochastic SIRS Epidemic Model with Regime-Switching Diffusions

Hongxia Liu, Juan Li, Mengnan Chi, Jinlei Liu, and Wencai Zhao

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Wencai Zhao; zhaowencai@sdust.edu.cn

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In this paper, taking both white noises and colored noises into consideration, a nonlinear stochastic SIRS epidemic model with regime switching is explored. The threshold parameter $R_s$ is found, and we investigate sufficient conditions for the existence of the ergodic stationary distribution of the positive solution. Finally, some numerical simulations are also carried out to demonstrate the analytical results.

1. Introduction

It is well known that the incidence rate plays a crucial role in studying the dynamics of infectious disease models. In general, bilinear incidence $\beta SI$ is considered in most infectious disease models [1, 2]. For example, Li and Ma [3] conducted the qualitative analyses of the SIS epidemic model with vaccination and varying total population size. Nakata and Kuniya [4] introduced the global dynamics of a class of SEIRS epidemic models in a periodic environment. In order to effectively investigate the rapid spread of the disease, it is rewarding to consider the behavioral changes and crowding effect of the infected individuals, as well as choose appropriate parameters to prevent the unbounded contact rate. Capasso and Serio [5] proposed the saturated incidence ($\beta SI/(1 + aI)$), which is more reasonable than the bilinear incidence. For the detailed introduction of the saturated incidence, see [5].

The classical SIRS epidemic model with the saturated incidence rate is in the following form [6, 7]:

\[ \begin{align*}
\frac{dS(t)}{dt} &= A - \frac{\beta S(t)I(t)}{1 + aI(t)} - \mu S(t) + \omega R(t), \\
\frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{1 + aI(t)} - (\mu + \lambda + d)I(t), \\
\frac{dR(t)}{dt} &= \lambda I(t) - (\mu + \omega)R(t),
\end{align*} \]

where $S(t)$, $I(t)$, and $R(t)$ represent the number of susceptible, infected, and removed individuals at time $t$, respectively. $A$ denotes an input of new members into the population, $\beta$ stands for the transmission rate, $\mu$ is the natural mortality, $d$ is the death rate relative to the disease, $\lambda$ is the proportion of the infective class to the recovered class, and $\omega$ is the per capita rate of loss of immunity.

In nature, it is inevitable for a population to be affected by a variety of random factors [8, 9]. Consequently, it is crucial to consider the randomness which might exist during the transmission of disease [10, 11]. In general, there are two
types of random perturbations to be considered in ecosystem modeling: one is white noise which can be described as Brownian motion [12–14], and the other is colored noise (also called telegraph noise) which can be described through a finite-state Markov chain [15–17]. In [15], Liu et al. investigated the threshold behavior of a multigroup SIRS epidemic model with standard incidence rates and Markovian switching. Lin and Jin [16] considered a stochastic SIS epidemic model with regime switching; by verifying a Foster–Lyapunov condition, the threshold condition for the ergodicity is presented. Hu et al. [17] studied a stochastic SIS epidemic model with vaccination and nonlinear incidence under regime switching.

Motivated by the above literature, we study a nonlinear stochastic SIRS epidemic model with two kinds of random interference. The model is as follows:

\[
\begin{aligned}
    dS(t) &= \left(1 - q(\xi(t))\right)A(\xi(t)) - \frac{\beta(\xi(t))S(t)I(t)}{1 + \alpha(I(t))} - \mu(\xi(t))S(t) + \omega(\xi(t))R(t) \right) dt \\
    &\quad + \sigma_1(\xi(t))S(t)dB_1(t), \\
    dI(t) &= \left(\frac{\beta(\xi(t))S(t)I(t)}{1 + \alpha(I(t))} - \mu(\xi(t))I(t) - \lambda(\xi(t))I(t) \right) dt + \sigma_2(\xi(t))I(t)dB_2(t), \\
    dR(t) &= (q(\xi(t))A(\xi(t)) + \lambda(\xi(t))I(t) - \mu(\xi(t))R(t) - \omega(\xi(t))R(t)) dt + \sigma_3(\xi(t))R(t)dB_3(t),
\end{aligned}
\]  

where \( q \) is a fraction of vaccinated individuals for newborns. The incidence rate \( \alpha \) contains the crowding effect of the infected individuals and should not be disturbed by the noises in the environment. \( B_i(t) (i = 1, 2, 3) \) denotes one-dimensional standard Brownian motion, and \( \sigma_i (i = 1, 2, 3) \) is the intensity of white noise. \( \xi(t) \) is a right-continuous Markov chain taking values in \( \mathcal{H} = \{1, 2, \ldots, m\} \), and the generator matrix of \( \xi(t) \) is \( \Gamma = (\gamma_{ij})_{1 \leq i, j \leq m} \). The details of the Markov chain are presented in [18], which we omit here.

In this paper, the dynamic behaviors of stochastic differential system (2) are discussed. In Section 2, we get the conditions for the extinction and persistence in mean of the infected. In Section 3, we investigate the ergodicity of system (2) by constructing a suitable Lyapunov function. Finally, numerical simulations are given in Section 4.

2. The Extinction and Persistence of the Disease

In system (2), let \( N(t) = S(t) + I(t) + R(t) \); then, we have

\[
\begin{aligned}
    dN(t) &= [A(\xi(t)) - \mu(\xi(t))N(t)] dt + \sigma_1(\xi(t))S(t)dB_1(t) \\
    &\quad + \sigma_2(\xi(t))I(t)dB_2(t) \\
    &\quad + \sigma_3(\xi(t))R(t)dB_3(t).
\end{aligned}
\]  

From Lemma 2.1 and Lemma 2.2 of [19], we have the following.

**Lemma 1.** For any initial value \( (S(0), I(0), R(0), 0, 0) \in \mathbb{R}^2 \times \mathcal{M} \), the solution \( (S(t), I(t), R(t)) \) of system (2) has the following properties:

1. \( \lim_{t \to \infty} (S(t) + I(t) + R(t))/t = 0 \) a.s.
2. \( \lim_{t \to \infty} \left( \int_0^t S(u)dB_1(u)/t \right) = 0 \), \( \lim_{t \to \infty} \left( \int_0^t I(u)dB_2(u)/t \right) = 0 \) a.s.

**Definition 1**

1. If \( \lim_{t \to \infty} I(t) = 0 \), then the disease tends to be extinct
2. If \( \lim_{t \to \infty} \inf (1/t)E \int_0^t I(z)dz > 0 \), then the disease tends to be persistent in mean

Define a matrix

\[
M = \begin{pmatrix}
\mu(1) - \gamma_{11} & \cdots & -\gamma_{1m} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\gamma_{m1} & \cdots & \mu(m) - \gamma_{nm} & 0 & \cdots & 0 \\
-\omega(1) & \cdots & 0 & \mu(1) + \omega(1) - \gamma_{11} & \cdots & -\gamma_{1m} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -\omega(m) & -\gamma_{m1} & \cdots & \mu(m) + \omega(m) - \gamma_{nm}
\end{pmatrix}.
\]
Following Lemma 1 in [20], we obtain that $M$ is a nonsingular $M$-matrix. By Theorem 2.10 (see Mao and Yuan [21]), for vector $\beta = (\beta(1), \beta(2), \ldots, \beta(m), 0, 0, \ldots, 0)^T \in \mathbb{R}^{2m}$, the equation $M\eta = \beta$ has a unique solution:

\[
\begin{align*}
\beta(k) - d_1(k)\mu(k) + \sum_{l \leq d} \gamma_{kl}d_1(l) &= 0, \quad k = 1, 2, \ldots, m, \\
d_1(k)\omega(k) - d_2(k)(\mu(k) + \omega(k)) + \sum_{l \leq d} \gamma_{kl}d_2(l) &= 0, \quad k = 1, 2, \ldots, m.
\end{align*}
\]

Then, we define a parameter

\[
R_x = \sum_{k \leq d} \pi_k \left[ d_1(k)(1 - q(k))A(k) + d_2(k)q(k)A(k) - \left(\mu(k) + \lambda(k) + \frac{\sigma^2(k)}{2}\right) \right].
\]

**Theorem 1.** For any initial value $(S(0), I(0), R(0), r(0)) \in \mathbb{R}^4 \times \mathbb{M}$, assume that $\alpha + d_1 - d_2 > 0$; then,

1. The disease $I(t)$ of system (2) is extinct when $R_x < 0$.
2. The disease $I(t)$ of system (2) is persistent in mean when $R_x > 0$.

**Proof.** Define the function

\[
U(S(t), I(t), R(t), k) = U_1(S(t), I(t), R(t), k) + U_2(I(t)),
\]

where

\[
U_1(S(t), I(t), R(t), k) = d_1(k)(S(t) + I(t)) + d_2(k)R(t),
\]

\[
U_2(I(t)) = \ln I(t) + \alpha I(t).
\]

Then, applying Itô’s formula, we have

\[
LU_1 = d_1(k)(1 - q(k))A(k) - \mu(k)S(t) - (\mu(k) + \lambda(k))I(t) + \omega(k)R(t)
\]

\[
+ d_2(k)[q(k)A(k) + \lambda(k)I(t) - (\mu(k) + \omega(k))R(t)] + (S(t) + I(t)) \sum_{l = 1}^{m} \gamma_{kl}d_1(l) + R(t) \sum_{l = 1}^{m} \gamma_{kl}d_2(l)
\]

\[
= d_1(k)(1 - q(k))A(k) + d_2(k)q(k)A(k) + \left[-d_1(k)\mu(k) + \sum_{l = 1}^{m} \gamma_{kl}d_1(l)\right]S(t)
\]

\[\quad - \left[(\mu(k) + \lambda(k))d_1(k) - \lambda(k)d_2(k) - \sum_{l = 1}^{m} \gamma_{kl}d_1(l)\right]I(t)
\]

\[\quad + \left[d_1(k)\omega(k) - d_2(k)(\mu(k) + \omega(k)) + \sum_{l = 1}^{m} \gamma_{kl}d_2(l)\right]R(t)
\]

\[
= d_1(k)(1 - q(k))A(k) + d_2(k)q(k)A(k) - \beta(k)S(t) - [\beta(k) + \lambda(k)d_1(k) - \lambda(k)d_2(k)]I(t),
\]

\[
LU_2 = \frac{\beta(k)S(t)}{1 + \alpha I(t)} \left(\mu(k) + \lambda(k) + \frac{\sigma^2(k)}{2}\right) + \frac{\alpha \beta(k)S(t)I(t)}{1 + \alpha I(t)} - \alpha(\mu(k) + \lambda(k))I(t)
\]

\[
= \beta(k)S(t) - \alpha(\mu(k) + \lambda(k))I(t) - \left(\mu(k) + \lambda(k) + \frac{\sigma^2(k)}{2}\right).
\]
By (10) and (11), one can obtain that

\[
LU = d_1(k)(1-q(k))A(k) + d_2(k)q(k)A(k) - \left(\mu(k) + \lambda(k) + \frac{\sigma^2(k)}{2}\right) \\
- [\beta(k) + \lambda(k)d_1(k) - \lambda(k)d_2(k) + a\mu(k) + a\lambda(k)]I(t) \\
= d_1(k)(1-q(k))A(k) + d_2(k)q(k)A(k) - \left(\mu(k) + \lambda(k) + \frac{\sigma^2(k)}{2}\right) \\
- [\beta(k) + \lambda(k)(\alpha + d_1(k) - d_2(k)) + a\mu(k)]I(t). 
\]

An application of generalized Itô’s formula yields

\[
U(S(t), I(t), R(t), \xi(t)) \\
= U(S(0), I(0), R(0), \xi(0)) + \int_0^t LU(S(z), I(z), R(z), \xi(z))dz + \int_0^t d_1(\xi(z))\sigma_1(\xi(z))S(z)dB_1(z) \\
+ \int_0^t [1 + (\alpha + d_1(\xi(z)))I(z)]\sigma_2(\xi(z))dB_2(z) + \int_0^t d_2(\xi(z))\sigma_3(\xi(z))R(z)dB_3(z) \\
+ \int_0^t \int_R U(S(z), I(z), R(z), i_0 + \ell(\xi(z), l)) - U(S(z), I(z), R(z), \xi(z))\mu(dz, dl), 
\]

where \(\mu(dz, dl) = \gamma(dz, dl) - \mu(dl)dz\) is a martingale measure. By using Lemma 1.9 of [21], we can get

\[
E[U(S(t), I(t), R(t), \xi(t))] = E[U(S(0), I(0), R(0), \xi(0))] + E\left[\int_0^t LU(S(z), I(z), R(z), \xi(z))dz\right]. 
\]

Then,

\[
E\left[\frac{U(S(t), I(t), R(t), \xi(t))}{t}\right] \\
= E\left[\frac{U(S(0), I(0), R(0), \xi(0))}{t}\right] \\
+ E\left[\frac{1}{t} \int_0^t [d_1(\xi(z))(1-q(\xi(z)))A(\xi(z)) + d_2(\xi(z))q(\xi(z))A(\xi(z)) - \left(\mu(\xi(z)) + \lambda(\xi(z)) + \frac{\sigma^2(\xi(z))}{2}\right)]dz\right] \\
- E\left[\frac{1}{t} \int_0^t [\beta(\xi(z)) + \lambda(\xi(z))(\alpha + d_1(\xi(z)) - d_2(\xi(z))) + a\mu(\xi(z))]I(z)dz\right].
\]
Hence, 
\[
\frac{R_2}{\beta + \lambda (a + \tilde{\alpha}_1 - \tilde{\alpha}_2) + a\mu} \leq \lim_{t \to \infty} \inf \frac{1}{t} E \int_0^t I(z)dz \\
\leq \lim_{t \to \infty} \sup \frac{1}{t} E \int_0^t I(z)dz \\
\leq \frac{R_2}{\beta + \lambda (a + \tilde{\alpha}_1 - \tilde{\alpha}_2) + a\mu}.
\]  
\tag{16}

In conclusion, based on the condition \(a + \tilde{\alpha}_1 - \tilde{\alpha}_2 > 0\), we have the following:

1. If \(R_2 < 0\), then \(\lim_{t \to \infty} I(t) = 0\), a.s.
2. If \(R_2 > 0\), then \(\lim_{t \to \infty} \inf \frac{1}{t} E \int_0^t I(z)dz \geq (R_2/\beta + \lambda (a + \tilde{\alpha}_1 - \tilde{\alpha}_2) + a\mu) > 0\), and the disease \(I(t)\) of system (2) is persistent in mean \(D\).

Remark 1. According to Theorem 1, if the intensity of white noise is large enough that the condition \(R_2 < 0\) holds, then the disease dies out with probability 1. Conversely, if \(R_2 > 0\), the disease of system (2) is persistent in mean. This means that the presence of environmental noise is conducive to disease control.

3. Ergodic Stationary Distribution of System (2)

The study of the ergodicity and stationary distribution has been widely concerned by many scholars [22, 23]. In this section, in order to investigate the ergodic property of system (2), we establish a suitable Lyapunov function with Markov conversion.

Theorem 2. For any initial value \((S(0), I(0), R(0), \xi(0)) \in R^2 \times M\), if \(R_2 > 0\), stochastic process \((S(t), I(t), R(t), \xi(t))\) of system (2) is ergodic and has a unique stationary distribution in \(R^2 \times M\).

Proof. Let \(x(t) = \ln S(t), y(t) = \ln I(t), \) and \(z(t) = \ln R(t)\). System (2) can be transformed into

\[
\begin{align*}
    dx(t) &= \left(1 - q(\xi(t))A(\xi(t)) \frac{\beta(\xi(t))e^{\gamma(t)}}{1 + ae^{\gamma(t)}} - \mu(\xi(t)) + \frac{\omega(\xi(t))e^{\gamma(t)}}{e^{\alpha(t)}} - \frac{1}{2}\sigma_1^2(\xi(t))\right)dt + \sigma_1(\xi(t)) dB_1(t), \\
    dy(t) &= \left[\beta(\xi(t))e^{\gamma(t)} \frac{\lambda(\xi(t))e^{\gamma(t)}}{e^{\alpha(t)}} - \mu(\xi(t)) - \lambda(\xi(t)) - \frac{1}{2}\sigma_2^2(\xi(t))\right]dt + \sigma_2(\xi(t)) dB_2(t), \\
    dz(t) &= \left[q(\xi(t))A(\xi(t)) \frac{\lambda(\xi(t))e^{\gamma(t)}}{e^{\alpha(t)}} + \mu(\xi(t)) - \omega(\xi(t)) - \frac{1}{2}\sigma_3^2(\xi(t))\right]dt + \sigma_3(\xi(t)) dB_3(t).
\end{align*}
\]

The ergodic property of system (2) is the same as that of system (17). We need to prove that system (17) satisfies the three conditions of Lemma 2.2 in [18]. Obviously, conditions (1) and (2) are satisfied. Hence, we only need to prove condition (3).

Define

\[
H(x, y, z, k) = \frac{1}{\theta + 1}(e^x + e^y + e^z)^{\theta + 1} - a[d_1(k)(e^x + e^y) + d_2(k)e^z + y + ae^y + \tau(k)] - x - z, 
\]

where \(\theta \in (0, 1), a^2 = \max_{x \in [3, \infty]} \sigma^2, \) and \(a > 0\) satisfy \(\tilde{\mu} - (\theta/2)a^2 > 0, -aR_2 + 2\tilde{\mu} + \tilde{\omega} + (1/2)\sigma_1^2 + (1/2)\sigma_2^2 \leq -2.\) \(\tau_k\) will be determined later. There exists a unique point \((x_0, y_0, z_0, k)\) which is the minimum value of \(H(x, y, z, k).\) Set

\[
V(x, y, z, k) = H(x, y, z, k) - \min H(x_0, y_0, z_0, k).
\]

Denote

\[
V_1 = \frac{1}{\theta + 1}(e^x + e^y + e^z)^{\theta + 1},
\]

\[
V_2 = -[d_1(k)(e^x + e^y) + d_2(k)e^z + y + ae^y + \tau(k)],
\]

\[
V_3 = -x,
\]

\[
V_4 = -z.
\]

An application of generalized Itô’s formula yields
Let \( k \) be the solution of the following Poisson system:

\[
LV_1 = (e^x + e^y + e^z)^3 (A(k) - \mu(k)e^x - \mu(k)e^y - \mu(k)e^z)
\]

\[
+ \frac{\theta}{2} (e^x + e^y + e^z)^{\theta - 1} \left( \sigma^2_1(k)e^{2x} + \sigma^2_2(k)e^{2y} + \sigma^2_3(k)e^{2z} \right)
\]

\[
\leq 3^\theta A(k) \left( e^{bx} + e^{by} + e^{bz} - (\mu(k)e^{x}) - (\mu(k)e^{y}) - (\mu(k)e^{z}) \right)
\]

\[
+ \frac{\theta}{2} \sigma^2_1(k)e^{(\theta + 1)x} + \frac{\theta}{2} \sigma^2_2(k)e^{(\theta + 1)y} + \frac{\theta}{2} \sigma^2_3(k)e^{(\theta + 1)z}
\]

\[
= \left( \mu(k) - \frac{\theta}{2} \sigma^2_1(k) \right) e^{(\theta + 1)x} - \left( \mu(k) - \frac{\theta}{2} \sigma^2_2(k) \right) e^{(\theta + 1)y}
\]

\[
- \left( \mu(k) - \frac{\theta}{2} \sigma^2_3(k) \right) e^{(\theta + 1)z} + 3^\theta A(k) (e^{bx} + e^{by} + e^{bz})
\]  

(21)

\[
LV_2 = -d_1(k) (1 - q(k)) A(k) - d_2(k) q(k) A(k) + \mu(k) + \lambda(k) + \frac{\sigma^2_1(k)}{2} - \sum_{l=1}^{m} y_{kl} \tau(l)
\]

\[
+ e^x \left[ d_1(k) \mu(k) - \beta(k) - \sum_{l=\text{d.c.}} y_{kl} d_1(l) \right] + e^y \left[ (d_1(k) + \alpha) (\mu(k) + \lambda(k)) - d_2(k) \lambda(k) - \sum_{l=\text{d.c.}} y_{kl} d_2(l) \right]
\]

\[
+ e^z \left[ -d_1(k) \omega(k) + d_2(k) (\mu(k) + \omega(k)) - \sum_{l=\text{d.c.}} y_{kl} d_2(l) \right]
\]

\[
= -G_{ok} - \sum_{l=1}^{m} y_{kl} \tau(l) + e^x \left[ \beta(k) + (d_1(k) - d_2(k)) \lambda(k) + \alpha (\mu(k) + \lambda(k)) \right],
\]  

(22)

where we used equation (4) and

\[
G_{ok} = d_1(k) (1 - q(k)) A(k) + d_2(k) q(k) A(k) - \mu(k) - \lambda(k) - \frac{\sigma^2_1(k)}{2}
\]

(23)

Let \( \tau = (\tau_1, \ldots, \tau_m)^T \) be the solution of the following Poisson system:

\[
LV_2 = -\sum_{k=1}^{m} \pi_k G_{ok} + e^x \left[ \beta(k) + (d_1(k) - d_2(k)) \lambda(k) + \alpha (\mu(k) + \lambda(k)) \right],
\]  

(25)

Substituting this equality into (22), we obtain

\[
\Gamma \tau = \left( \sum_{k=1}^{m} \pi_k G_{ok} \right) \overrightarrow{1} - \begin{pmatrix} G_{o1} \\ \vdots \\ G_{om} \end{pmatrix},
\]  

(24)

where \( G = (G_{o1}, G_{o2}, \ldots, G_{om})^T \).
Figure 1: Simulations of the solution \((S(t), I(t), R(t))\) to stochastic model \((2)\) with \((S(0), I(0), R(0)) = (1, 0.7, 0.8)\). (a) \(k = 1, \sigma^2(1) = 0.05\). (b) \(k = 2, \sigma^2(2) = 0.1\).

Figure 2: Simulations of the solution \((S(t), I(t), R(t))\) to stochastic model \((2)\) with \((S(0), I(0), R(0)) = (1, 0.7, 0.8)\). (a) \(k = 1, \sigma^2(1) = 1.2\). (b) \(k = 2, \sigma^2(2) = 1.2\).
Combining (21) and (25)–(27), we have

\[
LV(x, y, z, k) = LV_1 + aLV_2 + LV_3 + LV_4
\]

\[
\leq -\left(\bar{\mu} - \frac{\theta_2}{2}\sigma^2\right)e^{(\theta_1)x} + 3\bar{\beta}Ae^{\theta x} - \left(1 - \bar{q}\right)\bar{A}\frac{e^x}{\varepsilon} - \left(\bar{\mu} - \frac{\theta_2}{2}\sigma^2\right)e^{(\bar{\theta}_1)y}
\]

\[
+ 3\beta_Ae^{\theta y} + ae^y \left[\bar{\beta} + \bar{\lambda}\left(\bar{d}_1 - \bar{d}_2\right) + \alpha(\bar{\mu} + \bar{\lambda})\right] + e^y
\]

\[
- \left(\bar{\mu} - \frac{\theta_2}{2}\sigma^2\right)e^{(\bar{\theta}_1)z} - \left(\bar{q}\bar{A}\right)e^z
\]

\[
- aR + 2\bar{\mu} + \bar{\omega} + \frac{1}{2}\sigma^2 + \frac{1}{2}\beta^2.
\]

Consequently, take \(\rho\) sufficiently large, and let \(U = (-\rho, \rho)\times(-\rho, \rho)\times(-\rho, \rho);\) then, we have \(LV(x, y, z, k) \leq -1\) for any \((x, y, z, k) \in U \times \mathcal{M}\). According to Lemma 2.2 in [18], we know that \((x(t), y(t), z(t), \xi(t))\) is ergodic and positive recurrent.

This proof is completed. \(\square\)

4. Conclusions and Numerical Simulations

This paper investigated a nonlinear epidemic disease model with two kinds of noise disturbances. The threshold of extinction and persistence in mean is obtained.

Consequently, take \(\rho\) sufficiently large, and let \(U = (-\rho, \rho)\times(-\rho, \rho)\times(-\rho, \rho);\) then, we have \(LV(x, y, z, k) \leq -1\) for any \((x, y, z, k) \in U \times \mathcal{M}\). According to Lemma 2.2 in [18], we know that \((x(t), y(t), z(t), \xi(t))\) is ergodic and positive recurrent.

This proof is completed. \(\square\)

Example 1. In system (2), let \(\xi(t)\) be a right-continuous Markov chain taking values in \(\mathcal{M} = \{1, 2\}\) with the generator
\[
\Gamma = \begin{pmatrix}
-3 & 3 \\
1 & -1 \\
\end{pmatrix}
\] (29)

Then, the unique stationary distribution of \( \xi(t) \) is \( \pi = (\pi_1, \pi_2) = (1/4, 3/4) \). Let \( a = 0.2 \), and other coefficients in system (2) are selected as follows.

For \( k = i (i = 1, 2) \), \( (A(1), A(2)) = (1.3, 1.1) \), \( (q(1), q(2)) = (0.2, 0.3) \), \( (\beta(1), \beta(2)) = (0.6, 0.7) \), \( (\mu(1), \mu(2)) = (0.45, 0.5) \), \( (\lambda(1), \lambda(2)) = (0.2, 0.3) \), \( (\omega(1), \omega(2)) = (0.3, 0.2) \), \( (\sigma_1(1), \sigma_1(2)) = (0.2, 0.2) \), and \( (\sigma_3(1), \sigma_3(2)) = (0.05, 0.2) \).

Firstly, let \( (\sigma_2(1), \sigma_2(2)) = (0.05, 0.1) \). This gives \( R_s = 0.52874 > 0 \). According to Theorem 1, the infectious disease is permanent (Figure 1).

Then, let \( (\sigma_2(1), \sigma_2(2)) = (1.2, 1.2) \). This gives \( R_s = -0.1902 < 0 \). Based on Theorem 1, the disease \( I(t) \) goes extinct (Figure 2). It is easy to see that the disease tends to become extinct when the intensity of white noise increases.

Lastly, let \( (\sigma_2(1), \sigma_2(2)) = (0.05, 0.1), (\mu(1), \mu(2)) = (0.2, 0.15) \). This gives \( R_s = 3.8277 > 0 \). From Theorem 2, the stochastic process \( (S(t), I(t), R(t), \xi(t)) \) of system (2) is ergodic and has a unique stationary distribution (Figure 3). Figure 3(a) shows the Markov chain, and Figure 3(b) shows a stationary distribution of the stochastic SIRS model. From Figures 3(c)–3(e), we obtain the distributions of \( S(t), I(t) \), and \( R(t) \) in stochastic system (2), respectively. This means stochastic SIRS model (2) has an ergodic stationary distribution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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