EIGENSTRUCTURE ASSIGNMENT FOR POLYNOMIAL MATRIX SYSTEMS ENSURING NORMALIZATION AND IMPULSE ELIMINATION

PEIZHAO YU*
School of Electrical and Information Engineering, Zhengzhou University of Light Industry, Zhengzhou 450002, China

GUOSHAN ZHANG
School of Electrical and Information Engineering, Tianjin University, Tianjin 300072, China

(Communicated by Ying Yang)

Abstract. In this paper, eigenstructure assignment problems for polynomial matrix systems ensuring normalization and impulse elimination are considered. By using linearization method, a polynomial matrix system is transformed into a descriptor linear system without changing the eigenstructure of original system. By analyzing the characteristic polynomial of the desired system, the normalizable condition under feedback is given, and moreover, the parametric expressions of controller gains for eigenstructure ensuring normalization are derived by singular value decomposition. Impulse elimination in polynomial matrix systems is investigated when the normalizable condition is not satisfied. The parametric expressions of controller gains for impulse elimination ensuring finite eigenstructure assignment are formulated. The solving algorithms of corresponding controller gains for eigenstructure assignment ensuring normalization and impulse elimination are also presented. Numerical examples show the effectiveness of proposed method.

1. Introduction. Polynomial matrix systems are more general description form of linear time invariant systems [41]. Many mechanical systems can be treated as polynomial matrix systems with degree 2 [20, 2]. Some high order linear systems can be described by polynomial matrix systems [33, 18]. The study of polynomial matrix systems has been greatly developed with the development of polynomial matrix theory [14, 29]. By using polynomial matrix theory, most fundamental problems for control systems can be solved, including impulsive modes detection [15], eigenstructure assignment, impulse elimination, stabilization and decoupling [19, 25, 13, 5].

2010 Mathematics Subject Classification. Primary: 15A18, 93B18; Secondary: 93C05.
Key words and phrases. Polynomial matrix system, eigenstructure assignment, normalization, impulse elimination.

The first author is supported by the National Natural Science Foundation of China grant 61903342 and the Doctor fund project of Zhengzhou University of Light Industry grant 2017BSJJ009; The second author is supported by the National Natural Science Foundation of China grant 61473202.

* Corresponding author: Peizhao Yu.
Eigenstructure assignment is to design a controller such that resulted closed-loop system has desired eigenstructure, and the dynamic performances of control system can be improved by this approach [6]. There are a few pole assignment results for standard state space systems [41] and eigenstructure assignment results for descriptor systems [6, 32, 38]. The Sylvester matrix equations are effective tool to solve eigenvalue assignment problem of second-order systems [7]. The proportional-derivative feedback is applied for robust pole assignment of second-order systems in [8]. The velocity-acceleration feedback is used to investigate robust pole assignment of second-order systems with nonsingular leading coefficient matrix in [1]. The singular value decomposition method is proposed for eigenstructure assignment of singular second-order systems in [35]. The acceleration-displacement feedback is used to study partial eigenstructure assignment for undamped vibration systems in [40, 31]. The receptances and system matrices approaches are presented for solving partial quadratic eigenvalue assignment problem in vibration systems in [4, 22]. Based on proportional-derivative feedback, eigenstructure assignment problem is investigated for high-order linear systems by solving Sylvester matrix equations in [9, 10, 11, 12]. A multi-input state feedback control approach is put forward for partial eigenstructure assignment problems in high-order systems in [37].

Linearization is a feasible method in processing high order systems for control design [3, 17]. By this method, a polynomial matrix can be converted into an equivalent matrix pencil, which preserves the finite eigenstructure of original polynomial matrix [28]. The first and second Frobenius companion forms are usually used as the linearizations, and are also termed as companion linearizations [14]. [24] shows that companion linearization is a strong linearization which preserves the finite and infinite eigenstructure of original matrix. There are also more general classes of linearizations, for instance, Fiedler linearization [27].

Different from standard state space systems, the responses of descriptor systems may contain impulse terms, which may cause saturation of control and even destroy the systems [30, 39, 26]. Therefore, impulsive behavior of descriptor systems is usually expected to be eliminated [6]. There are some methods and results for impulse elimination in descriptor systems in last two decades [6, 21, 36]. The state feedback, state proportional-derivative feedback and derivative output feedback are most commonly used controllers for impulse elimination in descriptor systems [6]. Based on a novelly restricted system, [21] gives an impulse elimination approach by using derivative output feedback. The advantage of this kind of impulse elimination methods is that the parametric forms of controllers are available. However, parametric proportional-derivative feedback can not eliminate completely impulse behavior of descriptor systems. [36] presents a structured proportional-derivative feedback which can remove completely impulsive modes of descriptor system. Some results of impulse controllability and impulse observability for second-order systems are given in [16, 23]. These impulse elimination methods are only proposed for descriptor systems.

In this work, we will study eigenstructure assignment problems for polynomial matrix systems ensuring normalization and impulse elimination. By using linearization method, the polynomial matrix system will be transformed into a linear matrix pencil. A new treatment method will be presented to transform the linearization system into a linear closed-loop system. Based on the resulted linear closed-loop system, eigenstructure assignment problems ensuring normalization and impulse elimination will be solved by singular value decomposition method. The parametric
expressions of controllers ensuring normalization and impulse elimination will be derived. The corresponding solving algorithms for controller gain matrices will be proposed.

The paper is organized as follows. Section 2 gives some preliminaries and lemmas. Section 3 presents eigenstructure assignment for polynomial matrix systems ensuring normalization. Section 4 investigates eigenstructure assignment for polynomial matrix systems ensuring impulse elimination. Section 5 gives numerical examples to illustrate the effectiveness of proposed methods and results.

2. Preliminaries. Consider a polynomial matrix system of the following form

\[ P(s)x(s) = Bu(s), \]

where \( x(s) \in \mathbb{R}^m \) is state vector, \( u(s) \in \mathbb{R}^m \) is control input, \( P(s) \in \mathbb{R}^{m \times m} \) is polynomial matrix with

\[ P(s) = A_n s^n + A_{n-1} s^{n-1} + \cdots + A_0. \]

\( A_i \in \mathbb{R}^{m \times m} \) with \( A_n \neq 0 \) and \( B \in \mathbb{R}^{m \times r} \) are constant matrices. \( \mathbb{R} \) is the real number field. Generally, the system (1) is said to be regular, if the determinant of \( P(s) \) is not identically zero. The system (1) is said to be normal, if the determinant of \( A_n \) is nonzero. A normal system is exactly regular and not vice versa.

In this paper, we assume that system (1) is regular, but isn’t normal, that is, \( A_n \) is a nonzero singular matrix. The objectives of this paper can be described as:

1. To design a controller such that closed-loop system is normal and has desired eigenstructure.

2. To design a controller for impulse elimination and such that resulted closed-loop system has desired finite eigenstructure.

For the research objective, we consider the following feedback controller

\[ u(s) = -(F_0 + F_1 s + \cdots + F_n s^n)x(s) + \tilde{u}(s), \]

where \( F_i \in \mathbb{R}^{r \times m}, i = 0, \cdots, n \), are the gain matrices to be determined, \( \tilde{u}(s) \) is the new control input with appropriate dimension.

Applying the feedback (2) to system (1), the resulted closed-loop system is obtained as

\[ \tilde{P}(s)x(s) = B\tilde{u}(s), \]

where

\[ \tilde{P}(s) = (A_n + BF_n)s^n + \cdots + (A_1 + BF_1)s + (A_0 + BF_0). \]

Eigenstructure assignment problem is to select appropriate feedback gain matrices for controller (2) such that the resulted closed-loop system has desired eigenstructure. In this study, based on the feedback controller (2) eigenstructure assignment ensuring normalization and impulse elimination are solved. To fulfil objectives (1) and (2), we need to linearize the system (1). In the following, we first give the definition of linearization.

**Definition 2.1.** Let \( P(s) \) is an \( m \times m \) polynomial matrix with degree \( n \), linear matrix \( L(s) = sL_1 + L_2, L_i \in \mathbb{R}^{mn \times mn} \) is called linearization of \( P(s) \), if there exist two \( mn \times mn \) unimodular polynomial matrices \( U(s), V(s) \) such that

\[ U(s)L(s)V(s) = \begin{bmatrix} P(s) & 0 \\ 0 & I_{m(n-1)} \end{bmatrix}. \]
There are many different forms for linearization. For given polynomial matrix of
system (1), there exists a linearization of the form (4), which is called companion
linearization [14].

\[
L(s) = s \begin{bmatrix} I & \\
& \ddots \\
& & I \\
& & & A_n \\
A_0 & A_1 & \cdots & A_{n-1} \end{bmatrix} + \begin{bmatrix} 0 & -I & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
& & \ddots & \ddots \\
0 & 0 & \cdots & -I \end{bmatrix}.
\] (4)

By using linearization and (4), the system (1) can be equivalently rewritten as

\[
L(s)X(s) = \bar{Bu}(s),
\] (5)

where

\[X(s) = [x^T(s), s, s^2x^T(s), \ldots, s^n x^T(s)]^T, \quad \bar{B} = [0, \ldots, 0, B^T]^T.\]

Rewrite (2) as

\[
u(s) = -[F_0, F_{11}, \ldots, F_{n-1,1}] [x^T(s), s, s^2x^T(s), \ldots, s^n x^T(s)]^T
- [F_{12}, \ldots, F_{n-1,2}, F_n] [sx^T(s), \ldots, s^n x^T(s)]^T + \tilde{u}(s),
\] (6)

where \(F_i = F_{i1} + F_{i2}, i = 1, \ldots, n - 1\), then substitute (6) into (5) to obtain that

\[(sE + A)X(s) = \bar{Bu}(s),
\] (7)

where

\[
E = \begin{bmatrix} I & \\
& \ddots \\
& & I \\
BF_{12} & BF_{22} & \cdots & A_n + BF_n \end{bmatrix},
\]

\[
A = \begin{bmatrix} 0 & -I & \cdots & 0 \\
& \ddots & \ddots & \vdots \\
& & \ddots & \ddots \\
0 & 0 & \cdots & -I \\
A_0 + BF_0 & A_1 + BF_{11} & \cdots & A_{n-1} + BF_{n-1,1} \end{bmatrix}.
\]

Thus, system (1) is transformed into (7) under feedback (2).

**Remark 1.** The eigenstructure assignment problems for high-order systems were
investigated via a class of feedback controllers in [9, 10, 11]. The feedback controller
(2) is different from ones of [9, 10, 11]. The feedback controllers of [9, 10, 11] don’t
contain \(s^n x(s)\) term and can be regarded as special cases of (2). There are also huge
differences for (1) and models of [9, 10, 11, 12] in transforming into linear closed-loop
systems. In [9, 10, 11, 12], after transforming into linear closed-loop systems,
all the feedback gain matrices are contained in matrix \(A\). In (7), feedback gain
matrices are contained in matrices \(A\) and \(E\). These differences will lead to different
treatment methods in solving eigenstructure assignment problems (see Remark 2).

To solve eigenstructure assignment problems, controllability of system (1) need
be presented. In this study, the controllability of system (1) is defined as the
controllability of its linearization system. That is to say, the R-controllability, I-
controllability and S-controllability of system (1) are defined as the R-controllability,
I-controllability and S-controllability of system (5), respectively. Refer to the con-
trollability definition of descriptor system for more detail in [6].
3. Eigenstructure assignment ensuring normalization. In this section, eigenstructure assignment problem is considered to ensure that closed-loop system is normal. The system (1) is called normalizable if there exists feedback (2) such that \( \text{degdet} \hat{P}(s) = mn \), where \( \text{degdet} \) means the degree of determinant. Our objective of this section is to derive the parametric expressions of controller gains for eigenstructure assignment ensuring normalization. The following results give normalizable condition for polynomial matrix system (1).

Lemma 3.1. The system (1) is normalizable via controller (2) if and only if \( \text{rank}[A_n, B] = m \). There exists controller (2) such that the eigenvalues of closed-loop system are nonzero if and only if \( \text{rank}[A_n, B] = m \).

Proof. Let

\[
\text{det} \hat{P}(s) = \alpha_{mn}s^{mn} + \alpha_{mn-1}s^{mn-1} + \cdots + \alpha_1s + \alpha_0.
\]

Note that

\[
\alpha_{mn} = \text{det}(A_n + BF_n), \alpha_0 = \text{det}(A_0 + BF_0), \prod_{i=1}^{mn} s_i = \frac{\alpha_0}{\alpha_{mn}},
\]

where \( s_i \) for \( i = 1 : mn \) are eigenvalues of closed-loop system. It follows that \( \text{degdet} \hat{P}(s) = mn \) if and only if \( \text{det}(A_n + BF_n) \neq 0 \). \( s_i \) for \( i = 1 : mn \) are nonzero if and only if \( \text{det}(A_0 + BF_0) \neq 0 \). It is obtained directly from the result of Lemma 1 of [36] that there exists \( F_n \) such that \( \text{det}(A_n + BF_n) \neq 0 \) if and only if \( \text{rank}[A_n, B] = m \). Similarly, \( s_i \) for \( i = 1 : mn \) are nonzero if and only if \( \text{rank}[A_0, B] = m \). \( \square \)

Assume that desired eigenstructure of closed-loop system is as follows:

\[
J = \text{diag}(J_1, J_2, \cdots, J_p), J_i = \text{diag}(J_{i1}, J_{i2}, \cdots, J_{iq_i}),
\]

\[
J_{ij} = \begin{bmatrix} s_i & 1 \\ \vdots & \vdots \\ s_i & 1 \\ \end{bmatrix}_{p_{ij} \times p_{ij}}, \quad (8)
\]

where \( J_{ij}, j = 1, 2, \cdots, q_i \), are the \( q_i \) Jordan blocks associated with the eigenvalue \( s_i \). \( J \) is the Jordan matrix associated with all the finite eigenvalues of closed-loop system. In what follows, without loss of generality, assume that \( s_i \) is nonzero. It is seen that the closed-loop system has \( mn \) finite eigenvalues, that is,

\[
\sum_{i=1}^{p} \sum_{j=1}^{q_i} p_{ij} = mn.
\]

Define that

\[
V = [V_1, V_2, \cdots, V_p], V_i = [V_{i1}, V_{i2}, \cdots, V_{iq_i}], V_{ij} = [v_{ij}^1, v_{ij}^2, \cdots, v_{ij}^{p_{ij}}]. \quad (9)
\]

are eigenvector matrices corresponding to Jordan matrices in (8). \( V_i \) is an eigenvector matrix associated with the eigenvalue \( s_i \), where \( v_{ij}^k \in \mathbb{C}^{mn} \), \( k = 1, 2, \cdots, p_{ij} \). \( \mathbb{C} \) is the complex number field. In order to derive parametric expressions of feedback gains, we give an important lemma.

Lemma 3.2 ([35]). Let \( M \in \mathbb{R}^{m \times m} \), \( N \in \mathbb{R}^{m \times d} \), \( X \in \mathbb{C}^{m+d} \), \( Y \in \mathbb{C}^m \), \( \text{rank}[M, N] = m \). Then all the parametric solutions of equation

\[
[M, N]X = Y; \quad (10)
\]
can be expressed by
\[ X = S \left[ \begin{array}{c} \Sigma^{-1}U \quad Y \\ G \end{array} \right], \]
where \( S \in \mathbb{C}^{(m+d)\times(m+d)} \), \( U \in \mathbb{C}^{m\times m} \) are unitary matrices. \( \Sigma \) is the singular value matrix of \([M,N]\). \( G \in \mathbb{C}^d \) is arbitrary column vector.

**Theorem 3.3.** Assume that system (1) is \( R \)-controllable and \( \text{rank}[A_n,B] = m \), then based on the prescribed eigenstructure (8), the parametric expressions of gains controller (2) for normalization are given by
\[ F_i = F_{i1} + F_{i2}, \quad i = 1, \ldots, n-1, \]
\[ [F_0, F_{11}, \ldots, F_{n-1,1}] = HV^{-1}, \]
\[ [F_{12}, \ldots, F_{n-1,2}, F_n] = WJ^{-1}V^{-1}, \]
where \( V \) is given by (9) and \( W, H \in \mathbb{C}^{r\times mn} \) are defined by
\[ W = [W_1, W_2, \ldots, W_p], W_i = [W_{i1}, W_{i2}, \ldots, W_{ip}], W_{ij} = [w_{ij}^1, w_{ij}^2, \ldots, w_{ij}^p], \]
\[ H = [H_1, H_2, \ldots, H_p], H_i = [H_{i1}, H_{i2}, \ldots, H_{ip}], H_{ij} = [h_{ij}^1, h_{ij}^2, \ldots, h_{ij}^p], \]
with
\[ v_{ij,1}^l = S_{i,12}G_{ij}^l - S_{i,11}U_i \sum_{k=1}^n A_k \left( \sum_{\tau=1}^k S_i^{k-\tau} v_{ij,\tau}^{l-1} \right), \]
\[ v_{ij,k}^l = s_i v_{ij,k-1}^l + v_{ij,k-1}^{l-1}, \quad k = 2, \ldots, n, \]
\[ w_{ij}^l = S_{i,22}G_{ij}^l - S_{i,21}U_i \sum_{k=1}^n A_k \left( \sum_{\tau=1}^k S_i^{k-\tau} v_{ij,\tau}^{l-1} \right), \]
\[ h_{ij}^l = S_{i,32}G_{ij}^l - S_{i,31}U_i \sum_{k=1}^n A_k \left( \sum_{\tau=1}^k S_i^{k-\tau} v_{ij,\tau}^{l-1} \right). \]

\[ [(v_{ij,1}^l)^T, \ldots, (v_{ij,n}^l)^T]^T = v_{ij}^l, v_{ij,k}^l \in \mathbb{C}^m, v_{ij,0}^l = 0, \quad i = 1, \ldots, p, \quad j = 1, \ldots, q_i, \quad l = 1, \ldots, p_{ij}. \]

\( U_i \) and \( S_i \) are unitary matrices and
\[ S_i = \begin{bmatrix} S_{i,11} & S_{i,12} \\ S_{i,21} & S_{i,22} \\ S_{i,31} & S_{i,32} \end{bmatrix}. \]
\( G_{ij}^l \in \mathbb{C}^{2r} \) are parameter vectors with \((G_{ij}^k)^* = G_{ij}^k\) when \( s_i^* = s_i \) for all \( i, h. \)

**Proof.** It is seen from the closed-loop system (7) and the relationship of eigenvectors chain associated with eigenvalues that
\[ E(s_i v_{ij}^l + v_{ij}^{l-1}) + Av_{ij}^l = 0, \]
that is,
\[ \begin{bmatrix} I & & & \\ & \ddots & & \\ BF_{12} & BF_{22} & \cdots & A_n + BF_n \\ 0 & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I \end{bmatrix} \begin{bmatrix} s_i v_{ij,1}^l + v_{ij,1}^{l-1} \\ s_i v_{ij,2}^l + v_{ij,2}^{l-1} \\ \vdots \\ s_i v_{ij,n}^l + v_{ij,n}^{l-1} \end{bmatrix} = 0, \]
\[ \begin{bmatrix} v_{ij,1}^l \\ v_{ij,2}^l \\ \vdots \\ v_{ij,n}^l \end{bmatrix} = 0, \]
where \( v^l_{ij} = [(v^l_{ij,1})^T, (v^l_{ij,2})^T, \cdots, (v^l_{ij,n})^T]^T \). Then, it follows immediately that
\[
v^l_{ij,k} = s_i v^l_{ij,k-1} + v^{l-1}_{ij,k-1}, k = 2, \cdots, n, \tag{14}
\]
\[
B \sum_{k=1}^{n-1} F_{k,2}(s_i v^l_{ij,k} + v^{l-1}_{ij,k}) + (A_n + B F_n)(s_i v^l_{ij,n} + v^{l-1}_{ij,n})
+ (A_0 + B F_0)v^l_{ij,1} + \sum_{k=1}^{n-1} (A_{k-1} + B F_{k,1})v^l_{ij,k+1} = 0. \tag{15}
\]

It is seen from (14) that
\[
v^l_{ij,k} = s_i^{-1}v^l_{ij,1} + \sum_{r=1}^{k-1} s_i^{-1-r}v^{l-1}_{ij,r}, k = 2, \cdots, n. \tag{16}
\]

Substituting (16) into (15), one can deduce
\[
\sum_{k=0}^{n} A_k s_i^k v^l_{ij,1} + B[F_{1,2}, \cdots, F_n](s_i v^l_{ij} + v^{l-1}_{ij}) + B[F_0, \cdots, F_{n-1,1}]v^l_{ij} = -\sum_{k=1}^{n} A_k \left( \sum_{r=1}^{k} s_i^{-r}v^{l-1}_{ij,r} \right). \tag{17}
\]

Let
\[
[F_{1,2}, \cdots, F_{n-1,2}, F_n](s_i v^l_{ij} + v^{l-1}_{ij}) = w^l_{ij}, [F_0, F_{11}, \cdots, F_{n-1,1}]v^l_{ij} = h^l_{ij}. \tag{18}
\]

Then it follows immediately from (18) that
\[
[F_{1,2}, \cdots, F_{n-1,2}, F_n] V J = W, [F_0, F_{11}, \cdots, F_{n-1,1}] V = H. \tag{19}
\]

where \( W, H \in \mathbb{C}^{r \times mn} \) with
\[
W = [W_1, W_2, \cdots, W_p], W_i = [W_{i1}, W_{i2}, \cdots, W_{iq}], W_{ij} = [w^1_{ij}, w^2_{ij}, \cdots, w^{p_{ij}}_{ij}],
\]
\[
H = [H_1, H_2, \cdots, H_p], H_i = [H_{i1}, H_{i2}, \cdots, H_{iq}], H_{ij} = [h^1_{ij}, h^2_{ij}, \cdots, h^{p_{ij}}_{ij}].
\]

That is
\[
[F_{1,2}, \cdots, F_{n-1,2}, F_n] = W J^{-1} V^{-1}, [F_0, F_{11}, \cdots, F_{n-1,1}] = HV^{-1}.
\]

So, (17) can be equivalently rewritten as
\[
[P(s_i), B, B] \left[(v^l_{ij,1})^T, (w^l_{ij})^T, (h^l_{ij})^T\right]^T = -\sum_{k=1}^{n} A_k \left( \sum_{r=1}^{k} s_i^{-r}v^{l-1}_{ij,r} \right). \tag{20}
\]

The system (1) is R-controllable. Thus, \( \text{rank}[P(s_i), B, B] = \text{rank}[P(s_i), B] = m. \)

By Lemma 3.2
\[
[(v^l_{ij,1})^T, (w^l_{ij})^T, (h^l_{ij})^T]^T = S_l \left[ -\Sigma^{-1} U_l \sum_{k=1}^{n} A_k \left( \sum_{r=1}^{k} s_i^{-r}v^{l-1}_{ij,r} \right) \right], \tag{21}
\]

where \( S_l \in \mathbb{C}^{(m+2r) \times (m+2r)}, U_l \in \mathbb{C}^{m \times m} \) are unitary matrices. \( \Sigma_i \) is the singular value matrix of \( [P(s_i), B, B] \). There exists a parameter vector \( G \) for each \( v^l_{ij} \) by Lemma 3.2, denote the \( G \) with \( G^l_{ij} \). Then \( G^l_{ij} \in \mathbb{C}^{2r} \) and
\[
v^l_{ij,1} = S_{l,12} G^l_{ij} - S_{l,11} \Sigma^{-1} U_l \sum_{k=1}^{n} A_k \left( \sum_{r=1}^{k} s_i^{-r}v^{l-1}_{ij,r} \right),
\]
\[
w^l_{ij} = S_{l,22} G^l_{ij} - S_{l,21} \Sigma^{-1} U_l \sum_{k=1}^{n} A_k \left( \sum_{r=1}^{k} s_i^{-r}v^{l-1}_{ij,r} \right),
\]
\[
h^l_{ij} = S_{l,32} G^l_{ij} - S_{l,31} \Sigma^{-1} U_l \sum_{k=1}^{n} A_k \left( \sum_{r=1}^{k} s_i^{-r}v^{l-1}_{ij,r} \right),
\]
where
\[
\begin{bmatrix}
S_{i,11} & S_{i,12} \\
S_{i,21} & S_{i,22} \\
S_{i,31} & S_{i,32}
\end{bmatrix} \equiv S_i.
\]

It is worth to note that conjugated eigenvalues correspond to conjugated eigenvectors. Therefore, if \( s^*_h = s_i \), then \( G^k_{hj} \) and \( G^k_{ij} \) should be selected for conjugated vectors. Besides, \( G^k_{ij} \) may also be selected to ensure that \( V \) is not singular.

The solving processes of controller (2) for eigenstructure assignment ensuring normalization are presented by the following algorithm.

**Algorithm 3.4.** Input: A R-controllable polynomial matrix system with the form (1) satisfying \( \text{rank}[A_n, B] = m \).

**Step 1:** Select the parameter vectors \( G^l_{ij} \), \( i = 1, \cdots, p \); \( j = 1, \cdots, q \); \( l = 1, \cdots, p \). If \( s^*_h = s_i \) for some \( i, h \), then take \( (G^k_{hj})^* = G^k_{ij} \).

**Step 2:** For each prescribed eigenvalue \( s_i \), the matrix \( [P(s_i), B, B] \) is decomposed by SVD method, then \( \Sigma_i, U_i, S_i \) and partitioned matrices \( S_{i,11}, S_{i,12}, S_{i,21}, S_{i,22}, S_{i,31}, S_{i,32} \) are obtained.

**Step 3:** Compute the vectors \( v_{l,k}^l, w_{l,j}^l, h_{l,j}^l, i = 1, \cdots, p; j = 1, \cdots, q; l = 1, \cdots, p \) by (13). Then the matrices \( J, V, W \) and \( H \) are constructed by (8), (9), (11) and (12), respectively.

**Step 4:** Compute the gain matrix using
\[
[F_0, F_{11}, \cdots, F_{n-1,1}] = HV^{-1}, [F_{12}, \cdots, F_{n-1,2}, F_n] = WJ^{-1}V^{-1}.
\]

Then \( F_1 = F_{11} + F_{12}, i = 1, \cdots, n - 1 \). Thus, the feedback controller (2) is obtained.

**Remark 2.** The eigenstructure assignment problems for high-order systems were investigated in [9, 10, 11, 12] by solving a generalized Sylvester matrix equation, while the eigenstructure assignment problems of this study are solved by singular value decomposition method. Besides, eigenstructure assignment for polynomial matrix systems ensuring impulse elimination can also be solved well by singular value decomposition method, and this problem will be considered in following section.

4. **Eigenstructure assignment ensuring impulse elimination.** The impulsive behavior is an important characteristic in descriptor systems. The response of singular polynomial matrix system may contain impulse terms. According to descriptor systems theory, we say that polynomial matrix system has impulsive modes if its state solution contains an impulse term. A polynomial matrix system is called impulse-free, or equivalently, having no impulsive modes if there are no impulse terms in the state solution.

In this section, we consider the eigenstructure assignment problem for (1) ensuring impulse elimination via feedback (2). The objective is to derive the parametric expressions of feedback gains and give the corresponding solving algorithms. In order to present the main methods and results, we first give some useful results about impulse-free and impulse elimination.
Lemma 4.1. [15] The system (1) is impulse-free if and only if
\[
\begin{bmatrix}
A_n & \cdots & A_1 & A_0 \\
\vdots & \ddots & \vdots & \vdots \\
A_n & A_{n-1} & & \\
A_n & A_{n-1} & & A_n
\end{bmatrix}
- \begin{bmatrix}
A_n & \cdots & A_2 & A_1 \\
\vdots & \ddots & \vdots & \vdots \\
A_n & A_{n-1} & & \\
A_n & A_{n-1} & & A_n
\end{bmatrix}
= n.
\]

The Lemma 4.1 is a rank criterion for a polynomial matrix system to be impulse-free. It is difficult to verify the impulse-freeness condition by this rank criterion when designing controller for eigenstructure assignment. To design controller for eigenstructure assignment ensuring impulse elimination, we need linearize system (1). Therefore, the following lemma is needed.

Lemma 4.2. If the system (5) is impulse-free, then the system (1) is impulse-free.

Lemma 4.2 can be obtained from [34]. By Lemma 4.2, we will derive the parametric expressions of controller gains for finite eigenstructure assignment ensuring impulse elimination. For the sake of discussion, in what follows, we assume that \( n_1 \) is the number of finite eigenvalues for closed-loop systems, and \( n_2 = mn - n_1 \), then \( n_2 \) is the number of infinite eigenvalues. In standard descriptor linear systems, we know that controllability is a necessary and sufficient condition for arbitrary eigenstructure assignment by using the proportional state feedback. The impulse elimination problem of this study is solved based on descriptor systems theory, thus controllability can ensure arbitrary finite eigenstructure assignment in polynomial matrix systems. Therefore, the following results can be obtained.

Theorem 4.3. Assume that the system (1) is S-controllable. Then based on prescribed finite eigenstructure (8), the parametric expressions of gains controller (2) for impulse elimination and arbitrary finite eigenstructure assignment can be expressed as
\[
F_i = F_{i1} + F_{i2}, i = 1, \cdots, n - 1,
\]
\[
[F_0 F_{11} \cdots F_{n-1,1}] = [H_f, H_\infty][V_f, V_\infty]^{-1},
\]
\[
[F_{12} \cdots F_{1,2} F_n] = [W_f J_f^{-1}, W_\infty][V_f, V_\infty]^{-1}.
\]
where \( V_f \in \mathbb{C}^{mn \times n_1} \), \( W_f, H_f \in \mathbb{C}^{r \times n_1} \) with the forms of (9), (11) and (12), respectively. \( V_\infty \equiv [0, \cdots, 0, V_\infty^T]^T \) with \( V_\infty \in \mathbb{C}^{m \times n_2} \), and \( [V_\infty^T, W_\infty^T]^T \in \ker[A_n, B] \) with \( W_\infty \in \mathbb{C}^{r \times n_2} \), \( H_\infty \in \mathbb{C}^{r \times n_2} \) is any given matrix.

Proof. The closed-loop system (7) is impulse-free. It follows that the infinite eigenstructure satisfies
\[
EV_\infty = -AV_\infty J_\infty = 0, \quad \text{rank} V_\infty = n_2.
\]
Thus,
\[
\begin{bmatrix}
I \\
\vdots \\
BF_{12} & BF_{22} & \cdots & A_n + BF_n
\end{bmatrix}
\begin{bmatrix}
V_{\infty 1} \\
V_{\infty 2} \\
\vdots \\
V_{\infty n}
\end{bmatrix}
= 0,
\]
that is, \( \text{rank} V_{\infty} = n_2 \), and
\[
V_{\infty i} = 0, i = 1, \cdots, n - 1, [A_n, B] [V_{\infty}^T, (F_n V_{\infty})^T]^T = 0.
\]
Obviously, \( \dimker[A_n, B] = m + r - \text{rank}[A_n, B] \geq n_2 \), where \( \dimker \) means the dimensions of the kernel space. There exists \( V_{\infty n} \) with
\[
[V_{\infty n}^T, (F_n V_{\infty n})^T]^T \in \ker[A_n, B].
\]
Let \( F_n V_{\infty n} = W_{\infty} \), then
\[
[F_{12}, \cdots, F_{n-1,2}, F_n][0, \cdots, 0, V_{\infty n}^T]^T = W_{\infty}.
\] (26)
Let \( V_{\infty} \equiv [0, \cdots, 0, V_{\infty n}^T]^T \), then
\[
[F_{12}, \cdots, F_{n-1,2}, F_n]V_{\infty} = W_{\infty}.
\] (27)

For finite eigenstructure assignment, we can obtain similar results with Theorem 3.3,
\[
[F_0, F_{11}, \cdots, F_{n-1,1}] V_f = H_f,
\]
(28a)
\[
[F_{12}, \cdots, F_{n-1,2}, F_n] V_f = W_f J_f^{-1}.
\]
(28b)

For any given matrix \( H_{\infty} \), let
\[
[F_0, F_{11}, \cdots, F_{n-1,1}] V_{\infty} = H_{\infty},
\] (29)
by (28a) and (29), we obtain (22b). Combining (27) and (28b) to produce (22c).

\[\square\]

\[\text{FIGURE 1. The responses of } x \text{ and } \dot{x} \text{ for normalization}\]

The solving processes of gains controller (2) for finite eigenstructure assignment ensuring impulse elimination are presented by the following algorithm.
EIGENSTRUCTURE ASSIGNMENT FOR POLYNOMIAL MATRIX SYSTEMS 261

0 2 4 6 8 10
−30
−20
−10
0
10
20
time
x“(t)

x”1(t) x”2(t)

0 2 4 6 8 10
−200
−100
0
100
200
time
x(3)(t)

x(3)1(t) x(3)2(t)

Figure 2. The responses of ¨ x and x(3) for normalization

Algorithm 4.4. Input: An S-controllable polynomial matrix system with the form (1).

Step 1: Select the parameter vectors G_l^i, j = 1, · · · , p; j = 1, · · · , q_i; l = 1, · · · , p_ij. If s^*_i = s_i for some i, h, then take (G^k_h) = G^k_l_j.

Step 2: For each prescribed eigenvalue s_i, the matrix [P(s_i), B, B] is decomposed by SVD method, then Σ_i, U_i, S_i and partitioned matrices S_i, S_i,1, S_i,2, S_i,21, S_i,22, S_i,31, S_i,32 are obtained.

Step 3: Compute the vectors v_l^i,j,k, w_l^i,j, h_l^i,j, i = 1, · · · , p; j = 1, · · · , q_i; l = 1, · · · , p_ij by (13). Then the matrices J_f, V_f, W_f and H_f are constructed by (8), (9), (11) and (12), respectively.

Step 4: Solve the kernel space of [A_n, B] and take [V_T∞, W_T∞]^T ∈ ker[A_n, B] with V∞ ∈ C^{m×n_2}, W∞ ∈ C^{r×n_2}. Then V∞, W∞ are obtained. Let V∞ ≡ [0, · · · , 0, V^T_T∞]^T. Take any matrix H∞ with H∞ ∈ C^{r×n_2}.

Step 5: Compute the gains matrices using

[F_0, F_{11}, · · · , F_{n−1,1},] = [H_f, H∞][V_f, V∞]^{-1}, [F_{12}, · · · , F_{n−1,2}, F_n] = [W_f J_f^{-1}, W∞][V_f, V∞]^{-1}.

Then F_i = F_{i1} + F_{i2}, i = 1, · · · , n − 1. Thus, the feedback controller (2) for impulse elimination is obtained.

5. Numerical examples. In this section, we will present some examples to illustrate the results.
Example 1. Consider a time-invariant system
\[ A_4x^{(4)}(t) + A_3x^{(3)}(t) + A_2\dot{x}(t) + A_1x(t) + A_0x(t) = Bu(t). \]
Its eigenstructure is identical with the system (1) with
\[
A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix},
A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.
\]
Obviously, \( \text{rank} A_4 = 1 \) implies that given system is singular. The characteristic polynomial of open-loop system is \( 4s^4 + 2s^2 + 2s + 2 \). To normalize the system, we need assign 8 eigenvalues of closed-loop system. The eigenvalues of closed-loop system are chosen as \( s_{1,2} = -2 \pm i; s_{3,4} = -5; s_{5,6} = -4 \pm 4i; s_{7,8} = -3 \).

To normalize by feedback (2), the parameter vectors are chosen as
\[
G_1 = G_2 = [-1, 2, 2, 2]^T, G_3 = G_4 = [3, -2, 0, 1]^T, \\
G_5 = G_6 = [-5, 4, 3, -1]^T, G_7 = G_8 = [3, 4, -1, 0]^T.
\]
Then by Theorem 3.3, the feedback gains are
\[
F_0 = \begin{bmatrix} 34.3388 & 11.0916 \\ 54.9206 & 2.6466 \end{bmatrix},
F_1 = \begin{bmatrix} -1.4105 & 6.7316 \\ 53.1747 & -0.7203 \end{bmatrix},
F_2 = \begin{bmatrix} -22.5429 & 0.9368 \\ 21.3340 & -0.0542 \end{bmatrix},
F_3 = \begin{bmatrix} -7.1685 & 0.0503 \\ 4.8974 & -0.0023 \end{bmatrix},
F_4 = \begin{bmatrix} -2.2621 & 0.0081 \\ 0.4994 & 0.0026 \end{bmatrix}.
\]
The errors of closed-loop eigenvalues are, respectively,
\[
\{10^{-14} \times (2.4203 - 0.2323i), 10^{-14} \times (-8.4821 + 0.3021i), 10^{-6} \times 1.6492, \\
10^{-12} \times (-1.0925 - 2.182i), 10^{-13} \times (-2.8244 + 2.247i), 10^{-7} \times (-9.1797)\}.
\]
The zero input response curves of closed-loop system are presented in Figure 1 and Figure 2, where the initial values are given by \( x_0 = [-18, 12]^T, \dot{x}_0 = [5, -1]^T, \ddot{x}_0 = [5, -2]^T, x_0^{(3)} = [1, -3]^T \). The Figure 1 and Figure 2 show that the state and their every order derivative of closed-loop system are asymptotically stable by replacing the eigenstructure of open-loop system.

Example 2. Consider a singular polynomial matrix system with coefficient matrices
\[
A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & 2 \\ 0 & 5 \end{bmatrix},
A_1 = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 2 & -3 \end{bmatrix}.
\]
The system has impulsive modes by Lemma 4.1. Since \( \text{rank}[A_4, B] = 2 \), the system can not be normalized via feedback. According to Theorem 4.3, the impulsive modes of system can be eliminated. We select closed-loop finite eigenvalues as follows
\[
s_{1,2} = -2 \pm i, s_{3,4} = -5 \pm 3i, s_{5,6} = -4 \pm 4i, s_{7,8} = -1 \pm 2i, s_{9,10} = -3 \pm i, s_{11} = -6.
\]
The errors of closed-loop finite eigenvalues are, respectively, 

\[ EV_{\infty} = 0 \text{ with the error grade } 10^{-14}. \]

Then by Theorem 4.3, the gains are

\[
F_0 = \begin{bmatrix} 623.35 & -141.16 & 68.98 \\ 332.18 & -67.53 & 19.99 \end{bmatrix}, \quad F_{11} = \begin{bmatrix} 1035 & -285.8 & 18.3 \\ 452 & -0.1151 & 3.5 \end{bmatrix}, \\
F_{12} = \begin{bmatrix} 180.75 & -55.39 & 22.17 \\ 150.60 & -26.16 & 11.5 \end{bmatrix}, \quad F_{21} = \begin{bmatrix} 774.91 & -115.59 & 2.51 \\ 262.63 & -40.85 & 0.27 \end{bmatrix}, \\
F_{22} = \begin{bmatrix} 320.99 & -103.42 & 5.61 \\ 220.10 & -53.54 & 2.71 \end{bmatrix}, \quad F_{31} = \begin{bmatrix} 13.3846 & 0.2365 & 0.1473 \\ -6.2737 & 0.5559 & -0.0042 \end{bmatrix}, \\
F_{32} = \begin{bmatrix} 243.39 & -40.51 & 0.74 \\ 142.77 & -21.51 & 0.32 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1.1915 & -0.0117 & 0.0421 \\ 1.5811 & -0.0041 & 0.0146 \end{bmatrix}.
\]

The responses of \( x \) for impulse elimination are shown in Figure 3 and Figure 4. It is known from the Figure 3 and Figure 4 that there are no impulse responses in closed-loop system.
The state and their every order derivative of closed-loop system are asymptotically stable by replacing the eigenstructure of open-loop system.

6. Conclusions. In this paper, we have considered eigenstructure assignment for polynomial matrix systems ensuring normalization and/or impulse elimination. The parametric gain matrices of normalization controller for eigenstructure assignment were formulated by singular value decomposition method. We designed controller for impulse elimination in singular polynomial matrix system when the system can not be normalized. The parametric expressions of controller for finite eigenstructure assignment ensuring impulse elimination were derived for singular polynomial matrix system. This study proposed a new treatment method for feedback controller to transform linearization system into a closed-loop system. Based on the transformation form, eigenstructure assignment problems for polynomial matrix systems ensuring normalization and impulse elimination are solved well by singular value decomposition method. A second-order mechanical system can be treated as polynomial matrix system with degree 2. The results of this paper can be useful for mechanical systems and high-order linear systems models in applications.

Acknowledgments. The authors would like to thank the referees for their constructive comments and suggestions.

REFERENCES

[1] T. H. S. Abdelaziz, Robust pole placement for second-order linear systems using velocity-plus-acceleration feedback, IET Control Theory and Applications, 7 (2013), 1843–1856.
[2] T. H. S. Abdelaziz, Parametric approach for eigenstructure assignment in descriptor second-order systems via velocity-plus-acceleration feedback, *Journal Dynamic Systems, Measurement, and Control*, 136 (2014), 044505.

[3] E. N. Antoniou and S. Vologiannidis, A new family of companion forms of polynomial matrices, *Electronic Journal of Linear Algebra*, 11 (2004), 78–87.

[4] Z. J. Bai and Q. Y. Wan, Partial quadratic eigenvalue assignment in vibrating structures using receptances and system matrices, *Mechanical Systems and Signal Processing*, 88 (2017), 290–301.

[5] M. Chu and N. D. Buono, Total decoupling of general quadratic pencils, Part II: Structure preserving isospectral flows, *Journal of Sound and Vibration*, 309 (2008), 112–128.

[6] G. R. Duan, *Analysis and Design of Descriptor Linear Systems*, New York: Springer; 2010.

[7] G. R. Duan, Parametric eigenstructure assignment in second-order descriptor linear systems, *IEEE Transactions on Automatic Control*, 49 (2004), 1789–1794.

[8] G. R. Duan and Y. L. Wu, Robust pole assignment in matrix descriptor second-order linear systems, *Transactions of the Institute of Measurement and Control*, 27 (2005), 279–295.

[9] G. R. Duan, Two parametric approaches for eigenstructure assignment in high-order linear systems, *J. Control Theory Appl.*, 1 (2003), 59–64.

[10] G. R. Duan, Parametric approaches for eigenstructure assignment in high-order descriptor linear systems, *IEEE Conference on Decision and Control*, 49 (2004), 1789–1795.

[11] G. R. Duan and H. H. Yu, Parametric approaches for eigenstructure assignment in high-order descriptor linear systems, *Proceedings of the 45th IEEE Conference on Decision and Control*, (2006), 1399–1404.

[12] G. R. Duan and H. H. Yu, Complete eigenstructure assignment in high-order descriptor linear systems via proportional plus derivative state feedback, *Proceedings of the 6th World Congress on Intelligent Control and Automation*, (2006), 500–505.

[13] R. Galindo, Input/output decoupling of square linear systems by dynamic two-parameter stabilizing control, *Asian Journal of Control*, 18 (2016), 2310–2316.

[14] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, New York: Academic Press; 1982.

[15] D. Henrion and J. C. Zúñiga, Detecting infinite zeros in polynomial matrices, *IEEE Transactions on Circuits and systems II-Express Briefs*, 52 (2005), 744–745.

[16] M. Hou, Controllability and elimination of impulsive modes in descriptor systems, *IEEE Transactions on Automatic Control*, 49 (2004), 1723–1727.

[17] Y. Ilyashenko and O. Romaskevich, Sternberg linearization Theorem for skew products, *Journal of Dynamical and Control Systems*, 22 (2016), 595–614.

[18] S. Johansson, B. Kågström and P. V. Dooren, Stratification of full rank polynomial matrices, *Linear Algebra and its Applications*, 439 (2013), 1062–1090.

[19] D. T. Kawano, M. Morzfeld and F. Ma, The decoupling of second-order linear systems with a singular mass matrix, *Journal of Sound and Vibration*, 332 (2013), 6829–6846.

[20] Y. Kim, H. S. Kim and J. L. Junkins, Eigenstructure assignment algorithm for mechanical second-order systems, *AIAA Journal Guidance, Control and Dynamics*, 22 (1999), 729–731.

[21] J. Li, Y. F. Teng and Q. L. Zhang, et al, Eliminating impulse for descriptor system by derivative output feedback, *Journal of Applied Mathematics*, 2014 (2014), Art. ID 265601, 15 pp.

[22] H. Liu, B. X. He and X. P. Chen, Minimum norm partial quadratic eigenvalue assignment for vibrating structures using receptance method, *Mechanical Systems and Signal Processing*, 123 (2019), 131–142.

[23] P. Losse and V. Mehrmann, Controllability and observability of second order descriptor systems, *SIAM Journal on Control and Optimization*, 47 (2008), 1351–1379.

[24] D. S. Mackey, N. Mackey and C. Mehl, et al, Vector spaces of linearizations for matrix polynomials, *SIAM Journal on Matrix Analysis and Applications*, 28 (2006), 971–1004.

[25] M. Morzfeld and F. Ma, The decoupling of damped linear systems in configuration and state spaces, *Journal of Sound and Vibration*, 330 (2011), 155–161.

[26] I. M. Stamova, Parametric stability of impulsive functional differential equations, *Journal of Dynamical and Control Systems*, 14 (2008), 235–250.

[27] F. D. Terán, F. M. Dopico and D. S. Mackey, Fiedler companion linearizations and the recovery of minimal indices, *SIAM Journal on Matrix Analysis and Applications*, 31 (2010), 2181–2204.
[28] F. D. Terán, F. M. Dopico and D. S. Mackey, Linearizations of singular matrix polynomials and the recovery of minimal indices, Electronic Journal of Linear Algebra, 18 (2009), 371–402.

[29] A. I. Vardulakis, Linear Multivariable Control: Algebraic Analysis and Synthesis Methods, Chichester: Wiley, 1991.

[30] H. Wang, S. Duan and C. Li, et al, Stability criterion of linear delayed impulsive differential systems with impulse time windows, International Journal of Control, Automation and Systems, 24 (2016), 174–180.

[31] S. Xu and J. Qian, Orthogonal basis selection method for robust partial eigenvalue assignment problem in second-order control systems, Journal of Sound and Vibration, 317 (2008), 1–19.

[32] C. L. Yang, J. Z. Liu and Y. Liu, Solutions of the generalized Sylvester matrix equation and the application in eigenstructure assignment, Asian Journal of Control, 14 (2012), 1669–1675.

[33] H. H. Yu and G. R. Duan, ESA in high-order linear systems via output feedback, Asian Journal of Control, 11 (2009), 336–343.

[34] P. Z. Yu and G. S. Zhang, Infinite zero structure of polynomial matrix and impulsive modes of polynomial matrix systems, Proceedings of the 34th Chinese Control Conference (CCC), (2015), 278–282.

[35] P. Z. Yu and G. S. Zhang, Eigenstructure assignment and impulse elimination for singular second-order system via feedback control, IET Control Theory and Applications, 10 (2016), 869–876.

[36] G. S. Zhang and W. Q. Liu, Impulsive mode elimination for descriptor systems by a structured P-D feedback, IEEE Transactions on Automatic Control, 56 (2011), 2968–2973.

[37] L. Zhang, F. Yu and X. T. Wang, An algorithm of partial eigenstructure assignment for high-order systems, Mathematical Methods in the Applied Sciences, 41 (2018), 6070–6079.

[38] B. Zhang, Eigenstructure assignment for linear descriptor systems via output feedback, Asian Journal of Control, 21 (2019), 759–769.

[39] Z. Zhang and N. Wong, Canonical projector techniques for analyzing descriptor systems, International Journal of Control, Automation and Systems, 12 (2014), 71–83.

[40] J. Zhang, H. Ouyang and J. Yang, Partial eigenstructure assignment for undamped vibration systems using acceleration and displacement feedback, Journal of Sound and Vibration, 333 (2014), 1–12.

[41] D. Z. Zheng, Linear System Theory, 2nd edition, Tsinghua University Press, Beijing, 2002.

Received May 2019; revised July 2019.
E-mail address: yupzhao@126.com
E-mail address: zhanggs@tju.edu.cn