Field Theory for Perfect String Fluids

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Abstract

We develop a field theory description of non-dissipative string fluids and construct an explicit mapping between field theory degrees of freedom and hydrodynamic variables. The theory generalizes both a perfect fluid (described by energy density and pressure) and pressureless string dusts (described by energy density and tension) to what we call a perfect string fluid (described by energy density, pressure and tension). The Lagrangian framework also suggests a straightforward extensions of the perfect string fluid to more general fluids (with branes, charges, interactions etc.) whose equations of motion can be obtained from the ordinary variational principle.
INTRODUCTION

Many physical, cosmological and biological systems contain extended one dimensional string-like objects. These could be the galactic filaments in the large scale structure, cosmic strings after cosmological phase transition, fundamental strings near or above the Hagedorn temperature, topological strings in liquid crystals or even more complicated objects such as polymer chains. Most of these systems are considered highly non-perturbative and are usually analyzed using numerical N-body type simulations (see e.g. [1]). There are, however, a couple of analytical methods that one could potentially adopt to describe these string-like objects. One possibility is to expand each “string” into different vibration modes and to treat these modes as an infinite tower of different “particles”. Then the problem of strings reduces to the problem of infinitely many particles which can be tackled if the infinite tower is truncated. This is the effective field theory approach taken, for example, by the string theory.

Another possibility is to first coarse-grain the network of strings and then derive equations of motion for coarse-grained fluids by following the dynamics of microscopically conserved quantities (energy, momentum, tangent vector, etc.) [2]. This is the hydrodynamic approach that was recently adopted to study, for example, cosmic strings. Note that the two approaches are complementary: the effective field theory is useful when the number of relevant vibration modes is small, and the hydrodynamic description is useful when non-equilibrium effects are suppressed. For example, in the limit of local equilibrium [3] one can show that the dusts of Nambu-Goto, chiral or more generally wiggly strings can be described using the equations for the so-called pressureless string fluid [4]. In this letter we will start from a (less traditional) hydrodynamic description, but nevertheless seek a (more traditional) field theory description of string fluids.

A field theory Lagrangian for perfect (particle) fluids was known for some time. For example, if we express the fluid velocity in terms of Clebsch potentials

\[
\mu_u^\lambda = (\nabla_\lambda \psi + \alpha \nabla_\lambda \beta) \tag{1}
\]

then the pressure can be identified with Lagrangian density which is an arbitrary function of \(\mu^2\),

\[
\mathcal{L}(\mu^2) = \mathcal{L}((\nabla_\lambda \psi + \alpha \nabla_\lambda \beta)^2) (\nabla^\lambda \psi + \alpha \nabla^\lambda \beta)) = p \tag{2}
\]
and varying with respect to the potentials $\psi, \alpha, \beta$ leads to the continuity equation and the equations of motion (see e.g. [5]). More recently a field theory description of perfect fluids with charges was proposed which also allows coupling of fluids to external fields [6]. In what follows we will describe a simple procedure of constructing field theories of even more general fluids (with pressure, tension, charges, interactions etc.) by considering conserved currents at the level of Lagrangians. Then the hydrodynamic equations of motion can be obtained from the ordinary variational principle without having to go through a coarse-graining procedure.

**PERFECT STRING FLUID**

The energy momentum tensor for a perfect fluid is

$$ T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}, \quad (3) $$

where $u$ is the unit velocity of the fluid, $p$ is the pressure, and $\rho$ is the energy density in the rest frame of $u$. The energy density $\rho$ is a function of the number densities $n_a$ indexed by $a$, and we can form the corresponding chemical potentials

$$ \mu^a \equiv \frac{\partial \rho}{\partial n_a}. \quad (4) $$

These number densities can be the density of any extensive quantity such as baryon number, charge, or entropy (in which case the chemical potential is the temperature). Following the usual relations between thermodynamic potentials the pressure can be taken as a Legendre transform of $\rho$,

$$ p = -\rho + \mu^a n_a \quad (5) $$

$$ dp = n_a d\mu^a. \quad (6) $$

In addition to the conservation equation for energy-momentum we have the continuity equations for the currents $n^\mu_a \equiv n_a u^\mu$,

$$ \nabla_\mu n^\mu_a = 0. \quad (7) $$

By using these continuity equations and (6), the conservation of $T^{\mu\nu}$

$$ \nabla_\mu [(\mu^a n_a) u^\mu u^\nu - p g^{\mu\nu}] = 0, $$
can be reduced to the following equations of motion:

\[ n^\mu \nabla_\mu (\mu^a u_a) = 0. \] \hspace{1cm} (8)

What we are calling a perfect string fluid has in addition to the conserved current \( n^\mu \) (we will consider only one current for the moment) a conserved bivector \( F \),

\[ \nabla_\mu F^{\mu\nu} = 0. \] \hspace{1cm} (9)

\( F \) is also constrained to be a simple bivector, i.e. it has exactly two linearly independent eigenvectors, and the fluid velocity \( u \) is in this linear subspace. Then we can define the ‘string flux’ scalar \( \varphi \) and the normalized bivector \( \Sigma \) as the magnitude and direction of \( F \),

\[ F^{\mu\nu} = \varphi \Sigma^{\mu\nu} \] \hspace{1cm} (10)

\[ \Sigma^{\mu\nu} \Sigma_{\mu\nu} = -2. \] \hspace{1cm} (11)

The orthonormal spacelike direction \( w \) is defined from \( \Sigma \) and \( u \),

\[ w^\mu \equiv \Sigma^{\mu\nu} u_\nu, \] \hspace{1cm} (12)

in terms of which we can choose to express \( \Sigma \) as,

\[ \Sigma^{\mu\nu} = w^\mu u^\nu - u^\mu w^\nu \] \hspace{1cm} (13)

\[ u^\mu u_\mu = -w^\mu w_\mu = 1 \] \hspace{1cm} (14)

\[ u^\mu w_\mu = 0 \] \hspace{1cm} (15)

The projector \( h \) onto the linear subspace of \( u \) and \( w \) can also be defined in terms of \( \Sigma \),

\[ h^{\mu\nu} \equiv u^\mu u^\nu - w^\mu w^\nu = \Sigma^{\mu\rho} \Sigma_{\rho\nu}. \] \hspace{1cm} (16)

The conservation condition on \( F \) implies through the Frobenius theorem that \( u \) and \( w \) lie along two-dimensional integrable submanifolds that can be identified as string worldsheets. And the dual tensor to \( F \), \( \tilde{F} \) is a two-form that can be integrated to give the flux of these strings across a surface. The conservation of \( F \) just implies that the net flux of strings through any closed surface is zero.

The dual to the current \( n \), \( \tilde{n} \) will also be useful as it is a three-form that can be integrated over a volume to give the conserved charge contained. These two differential forms have a
natural interpretation in terms of Lagrangian coordinates labeling fluid particles. There
is a two-dimensional space of distinct ‘worldsheet’ submanifolds that we can label with
the coordinates $X$ and $Y$. There is an implicit map that specifies which worldsheet passes
through a given spacetime point that allows us to define $X$ and $Y$ as functions on spacetime.
The two-dimensional surfaces along which both $X$ and $Y$ take constant values are just the
worldsheets. As we will discuss later there is a great deal of symmetry in how we choose
these coordinates but we do choose them so that the measure $dX \wedge dY$ is just the string
flux. In fact this will be taken as a definition,

$$\tilde{F} \equiv dX \wedge dY,$$

(17)

and thus we define the dual bivector $F$ in (9) ultimately in terms of $X$ and $Y$ fields.

To label the particles confined to a string we need a third coordinate $Z$. The one-
dimensional spaces along which all three coordinates are constant are just the particle
worldlines. Again we will fix the measure $dX \wedge dY \wedge dZ$, taking it to be the number
density,

$$\tilde{n} \equiv dX \wedge dY \wedge dZ,$$

(18)

and so the current $n^\mu$ and thus the directions $u$ and $w$ (through (12)) are also specified in
terms of these three scalar fields (the Lagrangian coordinates).

An important thing to note about the use of Lagrangian coordinates is that the continuity
equations (7) and (9) are satisfied by construction,

$$d\tilde{n} = 0$$

(19)

$$d\tilde{F} = 0.$$  

(20)

To get a complete set of equations of motion we only need to add the conservation of the
ergy-momentum tensor which is specified by choosing a Lagrangian as a functional of the
$X, Y, Z$ fields. In particular, the Lagrangian $\mathcal{L}(\varphi, n)$ will only depend on the scalar fields
through the combinations $\varphi^2$ and $n^2$,

$$\varphi^2 = \frac{1}{2} \tilde{F}^{\lambda\mu} \tilde{F}_{\lambda\mu},$$

(21)

$$n^2 = -\frac{1}{3!} \tilde{n}^{\lambda\mu\nu} \tilde{n}_{\lambda\mu\nu}.$$  

(22)
Varying the Lagrangian by \( g_{\mu\nu} \) we find \( T^{\mu\nu} \):

\[
T^{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \mathcal{L} g^{\mu\nu} = 2 \left[ \frac{\partial \mathcal{L}}{\partial \phi^2} \phi^2 (g^{\mu\nu} - h^{\mu\nu}) + \frac{\partial \mathcal{L}}{\partial n^2} n^2 (g^{\mu\nu} - u^{\mu} u^{\nu}) \right] - \mathcal{L} g^{\mu\nu}
\]

\[
= (\rho + p) u^{\mu} u^{\nu} - (\tau + p) w^{\mu} w^{\nu} - pg^{\mu\nu}
\]

where we define

\[
\rho \equiv -\mathcal{L}
\]

\[
p \equiv \mathcal{L} - \mathcal{L}_{,\phi^2} - \mathcal{L}_{,n^n},
\]

and the new thermodynamic potential \( \tau \) related to the string tension,

\[
\tau \equiv -\mathcal{L} + \mathcal{L}_{,n^n}.
\]

One can also think about the tension term as the leading anisotropic correction to the pressure.

If \( \mathcal{L} \) does not depend on \( \phi \) the perfect string fluid reduces to the ordinary perfect fluid. This approach to perfect fluids in terms of a variational principle and Lagrangian coordinates is well established (see [7] for a review). Usually the variational principle is expressed by varying the worldlines in the action through diffeomorphisms. But as we show later, we can also treat \( X, Y, Z \) as ordinary scalar fields which can be varied independently to produce the equations of motion. A similar field theory perspective for perfect fluids is found in [6].

Another simplification occurs if \( \mathcal{L} \) does not depend on \( n \). This case was earlier studied by Kopczynski who also made use of a Lagrangian as a function of \( \phi^2 \) [8]. However he used a diffeomorphic variational principle and did not express \( \bar{F} \) in terms of scalar fields. Because of this, the conservation of \( F \) and the restriction of \( F \) to simple bivectors were both enforced through the use of Lagrange multipliers.

**PRESSURELESS STRING FLUID**

The inspiration behind the Kopczynski fluid came from the case where the pressure vanishes, in which case the string fluid further reduces to a model studied by Stachel in which the submanifolds behave as independent Nambu-Goto strings [9]. More recently it
was shown that coarse-graining an interacting network of Nambu-Goto strings in the limit of local equilibrium leads to a pressureless string fluid where the submanifolds behave as wiggly strings [4].

To gain a better understanding of the pressureless case, first note that \( \tilde{n} = \tilde{F} \wedge dZ \) involves a factor of \( \varphi \) and so it may be helpful to define a factored number density \( \nu \),

\[
n \equiv \varphi \nu
\]

(27)

From (26), the condition for the pressure to vanish is expressed through the derivative with respect to \( \varphi \) at constant \( \nu \),

\[
\mathcal{L} = \varphi \left( \frac{\partial \mathcal{L}}{\partial \varphi} \right)_\nu \equiv -\varphi U(\nu),
\]

(28)

where \( U \) is some function of \( \nu \) alone. Similarly we can define a modified tension,

\[
T \equiv U - U, \nu = \varphi^{-1} \tau,
\]

(29)

so that the energy momentum tensor is just

\[
T^{\mu \nu} = \varphi (U u^\mu u^\nu - T w^\mu w^\nu).
\]

(30)

This notation is intentionally similar to that used by Carter in describing “barotropic” strings [10]. The difference is that Carter is considering a single string rather than a fluid, and so all derivatives are projected into the worldsheet directions. For instance, the conservation \( \nabla_\mu F^{\mu \nu} = 0 \) corresponds to Carter’s condition for \( \Sigma \) to form a submanifold,

\[
h_\mu^\lambda \nabla_\lambda \Sigma^{\mu \nu} = 0.
\]

(31)

In general it can be proved (using both (9) and (31)) that if there is a tensor \( A^{\mu \ldots} \) where the index \( \mu \) lies in the worldsheet, then the following statements are equivalent:

\[
\nabla_\mu (\varphi A^{\mu \ldots}) = 0
\]

\[
h_\mu^\lambda \nabla_\lambda A^{\mu \ldots} = 0.
\]

(32)

Since both the conservation for \( T^{\mu \nu} \) and \( n^\mu \) are of this form, we see that \( \varphi \) decouples from equations of motion for the submanifolds themselves in the pressureless string fluid. The motion of each individual submanifold may be solved for as a barotropic string described by
the equation of state $U(\nu)$. Once $\Sigma$ has been solved for, the string flux $\varphi$ is determined by the initial values on any two-dimensional spacelike surface intersecting the submanifolds.

The connection between this variational approach and that of barotropic strings can be used to construct Lagrangians describing submanifolds of arbitrary equation of state. In particular, the coarse-grained string fluid in [4] has submanifolds obeying the wiggly string equation of state $UT = 1$. By (29), this is described by $U(\nu) = \sqrt{1 + \nu^2}$. So the Lagrangian for that model is given by

$$L = -\varphi \sqrt{1 + \nu^2} = -\sqrt{\varphi^2 + n^2},$$

where again $\varphi^2$ and $n^2$ are written in terms of the scalar fields $X, Y, Z$.

**VARIATIONAL PRINCIPLE**

Until now the Lagrangian $L$ has only been used to find the energy-momentum tensor. The conservation of $T_{\mu\nu}$ and the identities $d\tilde{n} = d\tilde{F} = 0$ are all that is needed for the equations of motion, but it is not clear that this is equivalent to requiring that the action be invariant under variations of $X, Y, Z$. To see this, first consider the $Z$ coordinate along the string. By varying the action by $\delta Z$ and recalling the definition (12) of $w$, we find the equation

$$\nabla_\nu (\varphi \Sigma_{\nu\mu} w_\mu) \equiv \nabla_\nu \Pi_\nu^Z = 0,$$

where we have defined the conserved current $\Pi_\nu^Z$ which is also the Noether current associated with translations in $Z$. Due to the decoupling of $\varphi$ through (32), this is the same as the ‘dual’ spacelike current appearing in Carter’s work on single strings (e.g. [11]), although we see it holds even for string fluids with pressure.

The equation corresponding to a variation $\delta X$ can be written as

$$X_{,\kappa} Y_{,\mu} \nabla_\lambda \left( \frac{\partial L}{\partial \varphi} \Sigma_{\lambda\mu} \right) - X_{,\kappa} Y_{[\mu} Z_{,\nu]} \nabla_\lambda \left( \frac{\partial L}{\partial n_{\nu\lambda}} \tilde{u}^{\lambda\mu} \right) = 0.$$  

(35)

Putting the $\delta Y$ and $\delta Z$ equations in the same form and combining leads ultimately to

$$-\frac{3}{2} F^{\lambda\mu} \nabla_{[\kappa} \left( \frac{\partial L}{\partial \varphi} \Sigma_{\lambda\mu]} \right) + 2n^\lambda \nabla_{[\kappa} \left( \frac{\partial L}{\partial n_{\nu\lambda]} u_{\nu]} \right) = 0.$$  

(36)

For an ordinary particle fluid the first term vanishes and we just recover the usual equations of motion [5]. The first term has a form analogous to the perfect fluid equations of motion.
and appears also in Kopczynski’s work. This equation can be shown to be equivalent to conservation of $T^{\mu\nu}$ by reversing the steps leading to (8). Although here we are considering a single current and a single string flux, adding additional dependences in the Lagrangian simply adds additional terms of the similar form.

Given that varying the scalar fields leads to the correct equations of motion, we can consider the fields to be degrees of freedom in a Hamiltonian sense — indeed this can be worked out explicitly for lower dimensional string fluids. Noether’s theorem also holds for the symmetries under relabeling, and the conservation of these currents can be taken to be equivalent to the equations of motion. In the interests of space, we will not dwell on these points here. But for the reader that wishes to rederive the currents note that an arbitrary function $G(X, Y)$ can be added to $Z$, and the relabeling symmetries of $X$ and $Y$ are equivalent to the symplectic transformations for a single degree of freedom. These can be generated by an arbitrary function $H(X, Y)$ as in Hamilton’s equations.

**GENERALIZED FLUIDS**

The variational approach to a perfect string fluid lends itself easily to many generalizations. By changing the number of Lagrangian coordinate fields and choosing the form of the wedge products in the action we can have multidimensional objects of arbitrary dimensions interacting. A simple example is if we have the Lagrangian depend on a single scalar field $X$ in the combination $-g^{\mu\nu}X_{,\mu}X_{,\nu}$. In three spatial dimensions this can be considered to be a coordinate labeling submanifolds identified as domain walls. Indeed in the pressureless case where

$$L = -\sqrt{-g^{\mu\nu}X_{,\mu}X_{,\nu}},$$

(37)

it can be shown (with a method analogous to that of Stachel for a string fluid) that the submanifolds obey the standard equation of motion for domain walls (see e.g. [12]). By reintroducing dependence in the Lagrangian on $\tilde{F}$ and $\tilde{n}$ (involving the same field $X$) we can then study a fluid of particles or strings confined to domain walls. This can be extended to higher dimensional objects if we increase the spacetime dimension.

In ordinary perfect fluids the equation of state typically is also dependent on a conserved entropy density $n_s$. If the Lagrangian simply depends on an additional field $Z_s$ in the
combination $\tilde{n}_s = dX \wedge dY \wedge dZ_s$ the entropy can flow in an independent direction from the velocity $u$ of the particles. This approach can be used to couple the fluid to any independent current confined to the worldsheets.

If instead we wish for a current like entropy to flow in the same direction as the particles there are two valid options. The entropy per particle $s$ is conserved along the wordlines, so it is a function $s(X, Y, Z)$. Then we can form the entropy density current as

$$\tilde{n}_s = s(X, Y, Z)dX \wedge dY \wedge dZ,$$

which points in the direction $u$ and is automatically conserved. A Lagrangian depending on $n^2_s$ then can be varied by $X, Y, Z$ (but not $s$ itself) leading to the correct equations of motion. This approach is similar to that taken in the diffeomorphic approach to ordinary fluids [7], and we do not introduce an extra degree of freedom in the sense of a field having a conjugate momentum. However the function $s$ appears directly in the Lagrangian, which breaks the relabeling symmetry of $X, Y, Z$.

An alternate approach to implementing currents flowing with the velocity was suggested in the context of coupling a perfect fluid to an electromagnetic field [6]. If the Lagrangian depends on an extra scalar field $Z_s$ in the combination $m_s \equiv \tilde{n}_s \wedge dZ_s$, the Noether current associated with shifts in $Z_s$ points in the direction of the fluid velocity. The quantity $m_s$ itself is related to the chemical potential associated with the Noether current, and the quantity $\mathcal{L}_{m_s}$ is constant along the worldlines and equal to the function $s(X, Y, Z)$ above.

We can generalize this approach in the string fluid by also introducing two additional scalar fields $X_s, Y_s$ in the Lagrangian in the combination $m'_s \equiv \tilde{F} \wedge dX_s \wedge dY_s$. The two Noether currents associated with the new fields are equivalent to the conservation of a single antisymmetric tensor $F_s$ that has the same direction $\Sigma$ as $F$. The quantity $\mathcal{L}_{m'_s}$ is constant on the worldsheet, and so we can represent it by a function $s(X, Y)$. So similarly to the previous case, this new interaction is equivalent to a Lagrangian with dependence on an additional string flux $\tilde{F}_s = s(X, Y)\tilde{F}$. These ideas can be extended further to introduce additional fluxes of domain walls and higher dimensional objects as well.

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