Abstract

In a totally ordered set the notion of sorting a finite sequence is defined through a suitable permutation of the sequence’s indices. In this paper we prove a simple formula that explicitly describes how the elements of a sequence are related to those of its sorted counterpart. As this formula relies only on the minimum and maximum functions we use it to define the notion of sorting for lattices. A major difference of sorting in lattices is that it does not guarantee that sequence elements are only rearranged. However, we can show that other fundamental properties that are associated with sorting are preserved.

1 Introduction

Let \((X, \leq)\) be a totally ordered set and \(x \in X^n\) a sequence \(x_1, \ldots, x_n\) of length \(n\) in \(X\). There exists for each such sequence a permutation \(\varphi\) of \([1, n] = \{1, \ldots, n\}\) such that \(x \circ \varphi \in X^n\) is a nondecreasing sequence. If \(x\) is injective, then \(\varphi\) is uniquely determined, and vice versa. However, regardless whether there is exactly one permutation, the rearrangement \(x \circ \varphi\) is uniquely determined and one thus can refer to it as the result of nondecreasing sorting \(x\) which we denote as \(x^\uparrow\). We remark that \(x^\uparrow\) is the only nondecreasing sequence of length \(n\) in which each element of \(X\) appears as often as in \(x\).

Nondecreasing sorting defines a map \(x \mapsto x^\uparrow\) from \(X^n\) to the subset of nondecreasing sequences. This map has several interesting properties. First of all, it is idempotent, that is,

\[
(x^\uparrow)^\uparrow = x^\uparrow
\]  

and thus a projection. This implies also that the map \(x \mapsto x^\uparrow\) is surjective. Secondly, for each permutation \(\psi\) of \([1, n]\) we have

\[
(x \circ \psi)^\uparrow = x^\uparrow
\]
In Section 2 we prove Identity (4) that explicitly describes how the elements of $x_1^\uparrow, \ldots, x_n^\uparrow$ are related to $x_1, \ldots, x_n$. This formula only uses the minimum and maximum functions on finite sets. Based on this observation, we define in Section 3 the notion of sorting of sequences in a lattice through simply replacing the minimum/maximum operations by the infimum/supremum operations. We also show that sorting in lattices in general not just reorders the elements of a sequence but really changes them. However, we also prove that our definition satisfies other properties that are associated with sorting.

2 A Formula for Sorting

Let $(X, \leq)$ be a totally ordered set, then each nonempty finite subset $A$ of $X$ contains a least and a greatest element [11, R. 6.5]. We also speak of the minimum and maximum of $A$ and refer to these special elements as $\bigwedge A$ and $\bigvee A$, respectively. The following inequalities hold for all $a \in A$

$$\bigwedge A \leq a \leq \bigvee A$$ (3)

For $A = \{x, y\}$ we use the notation $x \land y$ and $x \lor y$ to denote the minimum and maximum, respectively.

The main results of this paper depend on a particular family of finite sets.

**Definition 2.1.** For $k \in [1, n]$ we denote with

$$\mathbb{N}_k(n) := \{A \subset [1, n] \mid |A| = k\}$$

the set of subsets of $[1, n]$ that contain exactly $k$ elements. There are $\binom{n}{k}$ such subsets.

**Proposition 2.2.** Let $(x_1, \ldots, x_n)$ be a sequence in a totally ordered set, then the following identity holds for the elements of the sequence $(x_1^\uparrow, \ldots, x_n^\uparrow)$

$$x_k^\uparrow = \bigwedge_{I \in \mathbb{N}_k(n)} \bigvee_{i \in I} x_i$$ (4)

Before we prove Proposition 2.2 we introduce an abbreviation for the right hand side of Identity (4). For a sequence $x$ of length $n$ we define

$$x_k := \bigwedge_{I \in \mathbb{N}_k(n)} \bigvee_{i \in I} x_i \quad \text{for } 1 \leq k \leq n$$ (5)
With this notation Proposition 2.2 reads $x^\triangle = x^\wedge$.

Here are some simple observations about the elements of $x^\triangle$.

- Since $(X, \leq)$ is a total order, we know that each element of $x^\triangle$ belongs to $x$.
- In particular, we see that $x^\triangle_1$ is the least element and $x^\triangle_n$ the greatest element of $x$, respectively.

The following lemma states that $x^\triangle$ is a nondecreasing sequence.

**Lemma 2.3.** If $x$ is a sequence of length $n$ in a totally ordered set $(X, \leq)$, then $x^\triangle$ is a nondecreasing sequence.

**Proof.** Let $1 \leq k < n$ and $I$ be an arbitrary subset of $[1, n]$ with $k + 1$ elements. If $J$ is a subset of $I$ with $k$ elements, then we have

$$x^\triangle_k = \bigwedge_{L \in \binom{I}{k}} \bigvee_{i \in L} x_i \leq \bigvee_{j \in J} x_j \quad \text{by Inequality (3)}$$

$$\leq \bigvee_{i \in I} x_i \quad \text{by } J \subset I$$

Since $I$ is an arbitrary set of $k + 1$ elements we obtain from here

$$x^\triangle_k \leq \bigwedge_{L \in \binom{I}{k+1}} \bigvee_{i \in L} x_i = x^\triangle_{k+1}$$

$\Box$

**Remark 2.4.** Note that in the proof of Lemma 2.3 we have only used the fact that the minimum of a set is a lower bound for all elements of that set.

**Proof of Proposition 2.2** We will show that for each $k$ with $k \in [1, n]$ both

$$x^\triangle_k \leq x^\wedge_k$$

and

$$x^\wedge_k \leq x^\triangle_k$$

hold.

Let $\varphi$ be a permutation of $[1, n]$ with

$$x^\wedge = x \circ \varphi$$

and let $J \subset [1, n]$ be the subset for which

$$J = \varphi([1, k])$$

3
holds.

From the fact that $J$ contains exactly $k$ elements and that $x^\uparrow$ is nondecreasing we conclude

$$x^\wedge_k = \bigwedge_{i \in \binom{\varphi^{-1}(B)}{k}} \bigvee_{i \in I} x_i \leq \bigvee_{j \in J} x_j$$

by Inequality (3)

$$= \bigvee_{j \in J} x^\uparrow(\varphi^{-1}(j))$$

by Identity (6)

$$= \bigvee_{i \in [1,k]} x^\uparrow_i$$

by Identity (7)

According to Lemma 2.3, the sequence $x^\uparrow$ is nondecreasing and we obtain

$$= x^\uparrow_k$$

which finishes the first part of the proof.

Conversely, we conclude from the definition of $x^\wedge$ and the fact that $(X, \leq)$ is a total order that there exists a subset $B$ of $[1, n]$ with exactly $k$ elements such that

$$x^\wedge_k = \bigwedge_{i \in \binom{\varphi^{-1}(B)}{k}} \bigvee_{i \in I} x_i = \bigvee_{i \in B} x_i$$

by Identity (6)

$$= \bigvee_{i \in B} x^\uparrow(\varphi^{-1}(i))$$

by Identity (6)

$$= \bigvee_{j \in \varphi^{-1}(B)} x^\uparrow_j$$

holds. Since $x^\uparrow$ is nondecreasing we have

$$\bigvee_{j \in \varphi^{-1}(B)} x^\uparrow_j = x^\uparrow_m$$

where

$$m = \bigvee(\varphi^{-1}(B))$$

is the greatest element of $\varphi^{-1}(B)$. Thus, we have

$$x^\wedge_k = x^\uparrow_m$$

(8)

However, since $\bigvee(\varphi^{-1}(B))$ is a subset of $[1, n]$ that has exactly $k$ elements we have

$$k \leq m$$

and since $x^\uparrow$ is nondecreasing

$$x^\uparrow_k \leq x^\uparrow_m$$

From this and Identity (8) we conclude

$$x^\uparrow_k \leq x^\wedge_k$$

which completes the proof.
3 Sorting in Lattices

Let \((X, \leq)\) be a partial order that is also a lattice \((X, \wedge, \vee)\) \([2]\), that is, for each \(x, y \in X\) there exists the infimum \(x \wedge y\) and the supremum \(x \vee y\). These operations are commutative and associative. Moreover, they satisfy the so-called absorption properties for all \(x, y \in X\)

\[ x \vee (x \wedge y) = x \]
\[ x \wedge (x \vee y) = x \]

In a lattice, the infimum and supremum exist for every finite subset \(A\) \([2, p. 4]\) and are denoted by \(\wedge A\) and \(\vee A\), respectively. Note, however, that for a finite subset \(A\) in a general lattice neither the infimum nor the supremum necessarily belong to \(A\). If \((X, \leq)\) is a total order, then \(\wedge\) and \(\vee\) are the minimum and maximum functions. This means, our notation is consistent with that of Section\([2]\).

An essential observation is that for a sequence \(x\) of length \(n\) in a lattice the value

\[ \bigvee_{i \in \mathbb{N}} \bigwedge_{i \in I} x_i \]

is well-defined for \(k \in [1, n]\). This motivates the following definition.

**Definition 3.1.** If \(x\) is a sequence of length \(n\) in a lattice \((X, \wedge, \vee)\), then we refer to \(x^\triangledown\) as defined by Identity (5) as \(x\) nondecreasingly sorted with respect to the lattice \((X, \wedge, \vee)\).

The following lemma states that for \(x^\triangledown\) is indeed a nondecreasing sequence with respect to the partial order \((X, \leq)\) of the lattice.

**Lemma 3.2.** If \(x\) is a finite sequence in a lattice \((X, \wedge, \vee)\) with associated partial order \((X, \leq)\), then Identity (5) defines a nondecreasing sequence \(x^\triangledown\).

**Proof.** In order to prove this lemma we can proceed exactly as in the proof of Lemma \([2, 3]\) where \((X, \leq)\) is a total order. As noted in Remark \([2, 4]\) we have used only the fact that \(\wedge A\) is a lower bound of \(A\) which by definition also holds for lattices. □
A simple consequence of Identity (5) and Lemma 3.2 is that sorting in lattices respects lower and upper bounds of the original sequence.

**Lemma 3.3.** Let $x = (x_1, \ldots, x_n)$ be a finite sequence in a lattice $(X, \land, \lor)$ with associated partial order $(X, \leq)$. If for $1 \leq i \leq n$ holds

$$a \leq x_i \leq b,$$

then

$$a \leq x_i^\triangle \leq b$$

holds for $1 \leq i \leq n$ as well.

**Proof.** From Identity (5) (see also Identity (9)) follows that $x_n^\triangle$ is the supremum of the elements $x_1, \ldots, x_n$. Thus, we have $x_n^\triangle \leq b$. Lemma 3.2 ensures that $x_n^\triangle$ is the largest element of $X^\triangle$. Thus we have $x_i^\triangle \leq b$ for $1 \leq i \leq n$. The case for the lower bound $a$ is treated analogously. 

**3.1 Examples**

When applying Identity (5) it is sometimes convenient to use a slightly more explicit way to write the elements of $X^\triangle$.

$$x_i^\triangle = x_1 \land \ldots \land x_n$$

$$x_2^\triangle = \bigwedge_{1 \leq i < j \leq n} x_i \lor x_j$$

$$\vdots$$

$$x_k^\triangle = \bigwedge_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \lor \ldots \lor x_{i_k} \quad (9)$$

$$\vdots$$

$$x_n^\triangle = x_1 \lor \ldots \lor x_n.$$

**Example 3.4.** Consider the finite set $X = \{x, y, z\}$. Figure 1 shows the lattice of all subsets of $X$.

Let $x$ be the sequence

$$a = (\{x\}, \{y\}, \{z\})$$
then

\[ a^\triangle = (\emptyset, \emptyset, X) \]

Thus, \( a^\triangle \) is a nondecreasing sequence that consists of elements that are completely different from those of \( a \).

**Example 3.5.** Let us consider now the lattice \((\mathbb{N}, \text{gcd}, \text{lcm})\) where \( \text{gcd}(x, y) \) and \( \text{lcm}(x, y) \) denote the *greatest common divisor* and *least common multiple* of \( x \) and \( y \), respectively. The associated partial order of this lattice is defined by divisibility of natural numbers. Table 1 shows some examples of our definition of sorting for different sequences in \((\mathbb{N}, \text{gcd}, \text{lcm})\). Again we see that sorting in a lattice may change the elements in a sequence.

| \( x \)          | \( x^\triangle \)       |
|------------------|--------------------------|
| (1)              | (1)                      |
| (1, 2)           | (1, 2)                   |
| (1, 2, 3)        | (1, 1, 6)                |
| (1, 2, 3, 4)     | (1, 1, 2, 12)            |
| (1, 2, 3, 4, 5)  | (1, 1, 1, 2, 60)         |
| (1, 2, 3, 4, 5, 6)| (1, 1, 1, 2, 6, 60)     |
| (1, 2, 3, 4, 5, 6, 7)| (1, 1, 1, 1, 2, 6, 420)|
| (1, 2, 3, 4, 5, 6, 7, 8)| (1, 1, 1, 1, 2, 2, 12, 840)|

Table 1: Some examples of sorting in \((\mathbb{N}, \text{gcd}, \text{lcm})\)
### 3.2 Elementary Properties

In this section we prove that some well-known properties of sorting in a totally ordered set also hold for our definition of sorting in lattices.

The following lemma restates the idempotence of sorting in a totally ordered set, expressed by Identity (1), for the case of lattices.

**Lemma 3.6.** If $x$ is a finite sequence in a lattice $(X, \land, \lor)$, then

$$(x^\land)^\land = x^\land$$

**Proof.** We know from Lemma 3.2 that $x^\land$ is a nondecreasing sequence in the partial order $(X, \leq)$. Thus, the relation $\leq$ is a total order on the set

$$\{x^\land_1, \ldots, x^\land_n\} \subset X$$

In other words we can sort $x^\land$ in the classical sense. From this follows

$$x^\land = (x^\land)^\uparrow = (x^\land)^\land$$

by Identity (4) \hfill \square

We now restate the invariance of sorting under permutations—see Identity (2).

**Lemma 3.7.** If $x$ is a finite sequence in a lattice $(X, \land, \lor)$ and $\psi$ a permutation of $[1, n]$, then

$$(x \circ \psi)^\land = x^\land$$

holds.

**Proof.** We have for $k \in [1, n]$

$$(x \circ \psi)^\land_k = \bigwedge_{A \in \mathcal{P}(\{k\})} \bigvee_{i \in A} x(\psi(i))$$

$$= \bigwedge_{A \in \mathcal{P}(\{k\})} \bigvee_{j \in \psi(A)} x(j)$$

$$= \bigwedge_{B \in \mathcal{P}(\{k\})} \bigvee_{j \in B} x(j)$$

Because $\psi$ is a permutation of $[1, n]$ we find that $\psi\left(\mathbb{N}(\{k\})\right) = \mathbb{N}(\{k\})$ and conclude

$$= \bigwedge_{B \in \mathcal{P}(\{k\})} \bigvee_{j \in B} x(j)$$

$$= x^\land$$

\hfill \square
4 Conclusion

Proposition 2.2 states through Identity (4) an explicit relationship between the elements of a finite sequence in a totally ordered set to its sorted counterpart.

The author does not suggest that Identity (4) is an efficient algorithm for sorting. Since there are $2^n$ subsets of $[1, n]$, a straightforward implementation leads to an algorithm of exponential complexity. Note that the proven identity bears some similarity to the Binomial Theorem of elementary algebra. The main benefit of that proposition is not to efficiently compute $(a + b)^n$ but to serve as a means for useful transformations in proofs and algorithms.

Using Identity (4) we are able to define the notion of sorting finite sequences in lattices. Compared to sorting in a totally ordered set, sorting in lattices is a more invasive procedure because, in general, it changes sequence elements. However, the definition maintains other elementary properties that are associated with sorting.

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References

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