**Abstract.** Extending earlier work of the authors, this is the first in a series of papers devoted to the vertex-algebraic structure of principal subspaces of standard modules for twisted affine Kac-Moody algebras. In this part, we develop the necessary theory of principal subspaces for the affine Lie algebra $A_1^{(2)}$, which we expect can be extended to higher rank algebras. As a “test case,” we consider the principal subspace of the basic $A_1^{(2)}$-module and explore its structure in depth.

1. Introduction

The theory of principal subspaces of standard modules for untwisted affine Kac-Moody algebras was initiated in influential work of Feigin and Stoyanovsky ([FS1], [FS2]) and has been further developed by several authors from different standpoints. Our approach to principal subspaces (see [CalLM1], [CalLM2], [CalLM3], [MPe] and the earlier work [CLM1], [CLM2]) is based on vertex (operator) algebra theory (cf. [B], [FLM2], [FHL], [LL]) in such a way that the principal subspaces of standard modules of a fixed level are viewed as modules for the principal subspace of the “vacuum” module, itself viewed as a vertex algebra. In these works, we have proved that all the important algebraic and combinatorial properties of principal subspaces (e.g., multi-graded dimensions), which are a priori unknown, can be extracted from standard modules by using certain natural vertex-algebraic concepts such as intertwining operators among modules (although it is fair to say that the full power of intertwining operators is yet to be explored), annihilating ideals, etc. As is customary in this area of representation theory, we often use vertex operator constructions of basic modules and then tensor products to study the (more difficult) higher level standard modules (cf. [CalLM2] and [CLM2]). We also point out that there are other types of (commutative) “principal subspaces,” which can be also studied by using our techniques (see [Pr], [Je], etc.).

In this series of papers we switch our attention to standard modules for twisted affine Lie algebras and their principal subspaces. As we shall see, in the twisted case new phenomena arise and the theory is much more subtle than in the untwisted case. This is the main reason why in this paper we focus primarily on the simplest twisted affine Lie algebra $A_2^{(2)}$, which already illustrates the difficulty of the general case. In parallel with our first paper on the untwisted $A_1^{(1)}$ case [CalLM1], we first study the level one standard $A_2^{(2)}$-module $V_L^T$ and its principal subspace $W_L^T$ and carry out the analysis in full detail. In particular, we are able to obtain a description of $W_L^T$ via generators and relations. Then, by using certain natural maps we build a short exact sequence, which gives rise to a simple recursion for the graded dimension of $W_L^T$. From this recursion we easily infer that this graded dimension is given by the number of integer partitions into distinct odd parts. We stress that the main goal of this paper is not to obtain this classical recursion but rather to set up a new twisted vertex-algebraic theory of principal subspaces that in the first test case produces a structure that naturally underlies this recursion; in [CalLM1] (and [CLM1]) it was the classical Rogers-Ramanujan recursion that we obtained from the corresponding (untwisted) vertex-algebraic structure. Interestingly, intertwining operators do not play a leading role in the present paper; instead, certain twisted operators, similar to simple current operators used in [DLM], have
turned out to be more essential. This is related to the fact that there is only one level one standard $A_2^{(2)}$-module, while for $A_1^{(1)}$ there are two, and in [CaLM1] (and [CLM1]), an intertwining operator relating them led to the basic structure. These twisted operators act naturally on the principal subspace, as certain shifting operators. In a sequel to this paper, we shall extend this work to higher-level standard $A_2^{(2)}$-modules.

Let us briefly elaborate on the contents of the paper. In Section 2, starting from an arbitrary isometry of an arbitrary positive-definite even lattice $L$, we review the relevant parts of the theory of twisted modules for lattice vertex operator algebras $V_L$. This material is mostly taken from [FLM1], [FLM2], [FLM3], [L1] and [L2]. (See also [DL1] and [BHL].) We present it in this generality as the foundation of our sequels to the present paper; but in addition, the $A_2^{(2)}$ case, the main focus of this paper, in fact exhibits the main subtleties of the general theory. Section 3 is devoted to recalling the vertex-algebraic construction of the twisted affine Lie algebra $A_2^{(2)}$ and of its basic module $V^T_L$; starting here, $L$ is the root lattice of type $A_2$. Section 4 deals with shifted modules and operators; they will become more relevant in our future publications. In Section 5 we introduce and investigate the principal subspace $W^T_L \subset V^T_L$. Section 6 contains the definitions and properties of various maps needed in Section 7 to prove a presentation of the principal subspace $W^T_L$ (see Theorem 7.1). Finally, Section 8 deals with a reformulation of one of the main results in [CaLM3], but this time, in the spirit of the present paper, without the use of intertwining operators.

2. LATTICE VERTEX OPERATOR ALGEBRAS AND TWISTED MODULES

In this section we recall, for the reader’s convenience, the vertex operator constructions associated to a general positive-definite even lattice equipped with a general isometry, which is necessarily of finite order. We review the relevant results of [L1], [FLM2], [FLM3] and [L2]. We use the notation and terminology of [FLM3] (in particular, Chapters 7 and 8) and [LL] (in particular, Sections 6.4 and 6.5). In fact, these general constructions of lattice vertex operator algebras and of twisted modules have been presented in [BHL] (for the purpose of giving an equivalence of two constructions of permutation-twisted modules), and the descriptions of the constructions and results in this section are similar to the corresponding review in [BHL] of these earlier results. In the next section we will specialize the general setting to the root lattice of $\mathfrak{sl}(3, \mathbb{C})$ and to a certain isometry of this root lattice.

We work in the following setting, under the assumptions made in Section 2 of [L1]: Let $L$ be a positive-definite even lattice equipped with a (nondegenerate symmetric) $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$, and let $\nu$ be an isometry of $L$ and $k$ a positive integer such that

$$\nu^k = 1. \quad (2.1)$$

Note that $k$ need not be the exact order of $\nu$ and, in fact, the appropriate period $k$ of $\nu$ will (necessarily) be larger than the order of $\nu$ in our specialized setting (see the next section). We also assume that if $k$ is even, then

$$\langle \nu^{k/2} \alpha, \alpha \rangle \in 2\mathbb{Z} \quad \text{for } \alpha \in L, \quad (2.2)$$

which can always be arranged by doubling $k$ if necessary. Under these assumptions we have

$$\left\langle \sum_{j=0}^{k-1} \nu^j \alpha, \alpha \right\rangle \in 2\mathbb{Z} \quad (2.3)$$

for $\alpha \in L$. This doubling procedure will be relevant in the next section, where $\nu^2 = 1$ but $k$ cannot be 2; it will be 4.
We continue to quote from [L1] and [FLM2]. Let \( \eta \) be a fixed primitive \( k \)th root of unity. The functions \( C_0 \) and \( C \) defined by

\[
C_0 : L \times L \longrightarrow \mathbb{C}^\times
\]

\[
(\alpha, \beta) \mapsto (-1)^{\langle \alpha, \beta \rangle}
\]

and

\[
C : L \times L \longrightarrow \mathbb{C}^\times
\]

\[
(\alpha, \beta) \mapsto (-1)^{\sum_{j=0}^{k-1} (\nu^j \alpha, \beta) \eta^{\sum_{j=0}^{k-1} (j \nu^j \alpha, \beta)}} = \prod_{j=0}^{k-1} (-\eta^j)^{\langle \nu^j \alpha, \beta \rangle}
\]

are bilinear into the abelian group \( \mathbb{C}^\times \) and are \( \nu \)-invariant. Clearly,

\[
C_0(\alpha, \alpha) = 1.
\]

Also,

\[
C(\alpha, \alpha) = 1,
\]

whose proof uses (2.3) and thus our assumption (2.2).

Set

\[
\eta_0 = (-1)^k \eta.
\]

Since \( C_0 \) and \( C \) are alternating bilinear maps into \( \mathbb{C}^\times \) (that is, they satisfy (2.6) and (2.7)), they determine uniquely (up to equivalence) two central extensions

\[
1 \rightarrow \langle \eta_0 \rangle \rightarrow \hat{L} \longrightarrow L \rightarrow 1
\]

and

\[
1 \rightarrow \langle \eta_0 \rangle \rightarrow \hat{L}_\nu \longrightarrow L \rightarrow 1
\]

of \( L \) by the cyclic group \( \langle \eta_0 \rangle \) with commutator maps \( C_0 \) and \( C \), respectively:

\[
aba^{-1}b^{-1} = C_0(\overline{a}, \overline{b}) \quad \text{for } a, b \in \hat{L}
\]

and

\[
aba^{-1}b^{-1} = C(\overline{a}, \overline{b}) \quad \text{for } a, b \in \hat{L}_\nu.
\]

There is a natural set-theoretic identification (which is not an isomorphism of groups unless \( k = 1 \) or \( k = 2 \)) between the groups \( \hat{L} \) and \( \hat{L}_\nu \) such that the respective group multiplications, denoted here by \( \times \) and \( \times_\nu \) to distinguish them, are related as follows:

\[
a \times b = \prod_{-k/2 < j < 0} (-\eta^j)^{\langle \nu^j a, \overline{b} \rangle} a \times_\nu b \quad \text{for } a, b \in \hat{L}.
\]

As in [L1], let

\[
e : L \longrightarrow \hat{L}
\]

\[
\alpha \mapsto e_\alpha
\]

be a normalized section of \( \hat{L} \), that is,

\[
e_0 = 1
\]

and

\[
e_\alpha = \alpha \quad \text{for all } \alpha \in L.
\]
Similarly, we have $e : L \rightarrow \hat{L}_\nu$, $\alpha \mapsto e_\alpha$ a normalized section of $\hat{L}_\nu$. The function
\begin{equation}
\epsilon_C : L \times L \rightarrow \langle \eta_0 \rangle
\end{equation}
defined by
\begin{equation}
\epsilon_\alpha \epsilon_\beta = \epsilon_C(\alpha, \beta)\epsilon_{\alpha + \beta}
\end{equation}
is a normalized 2-cocycle associated with the commutator map $C$, that is,
\begin{equation}
\epsilon_C(\alpha, \beta)\epsilon_C(\alpha + \beta, \gamma) = \epsilon_C(\beta, \gamma)\epsilon_C(\alpha, \beta + \gamma),
\end{equation}
\begin{equation}
\epsilon_C(0, 0) = 1,
\end{equation}
\begin{equation}
\epsilon_C(\alpha, \beta)\epsilon_C(\beta, \alpha) = C(\alpha, \beta).
\end{equation}
Also, the function
\begin{equation}
\epsilon_{C_0} : L \times L \rightarrow \langle \eta_0 \rangle
\end{equation}
defined by
\begin{equation}
\epsilon_{C_0}(\alpha, \beta) = \prod_{-k/2 < j < 0} (-\eta^{-j})^{(\nu^j, \alpha, \beta)}\epsilon_C(\alpha, \beta)
\end{equation}
is a normalized 2-cocycle associated with the commutator map $C_0$. Then
\begin{equation}
\epsilon_\alpha \epsilon_\beta = \epsilon_{C_0}(\alpha, \beta)\epsilon_{\alpha + \beta}
\end{equation}
(by (2.13), (2.16) and (2.21)) and
\begin{equation}
\epsilon_{C_0}(\alpha, \beta)\epsilon_{C_0}(\beta, \alpha) = C_0(\alpha, \beta)
\end{equation}
(recall formula (4.6) in [L1]).

There exists an automorphism $\hat{\nu}$ of $\hat{L}$ (fixing $\eta_0$) such that
\begin{equation}
\hat{\nu}a = \nu a
\end{equation}
for $a \in \hat{L}$, that is, $\hat{\nu}$ is a lifting of $\nu$. The map $\hat{\nu}$ is also an automorphism of $\hat{L}_\nu$ satisfying
\begin{equation}
\hat{\nu}a = \nu a
\end{equation}
for $a \in \hat{L}_\nu$.

We may and do choose $\hat{\nu}$ so that
\begin{equation}
\hat{\nu}a = a \quad \text{if} \quad \nu a = a.
\end{equation}
Then
\begin{equation}
\hat{\nu}^k = 1.
\end{equation}
See [L1] for these nontrivial facts.

Following the treatments in [L1] and [FLM2] (see also [FLM3] and [DL1]) we will construct a vertex operator algebra $V_L$ equipped with an automorphism $\hat{\nu}$, using the central extension $\hat{L}$. We will then use the central extension $\hat{L}_\nu$ to construct $\hat{\nu}$-twisted modules for $V_L$.

Embed $L$ canonically in the $C$-vector space
\begin{equation}
\mathfrak{h} = C \otimes Z L
\end{equation}
and extend the $Z$-bilinear form on $L$ to a $C$-bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$. Viewing $\mathfrak{h}$ as an abelian Lie algebra, we have the corresponding affine Lie algebra
\begin{equation}
\hat{\mathfrak{h}} = \mathfrak{h} \otimes C[t, t^{-1}] \oplus Ck
\end{equation}
with the brackets
\begin{equation}
[\alpha \otimes t^m, \beta \otimes t^n] = (\alpha, \beta)m\delta_{m+n,0}k \quad \text{for} \quad \alpha, \beta \in \mathfrak{h}, \quad m, n \in Z
\end{equation}
and
\[(2.30)\] 
\[[\mathbf{k}, \hat{\mathfrak{h}}] = 0.\]

There is a \(\mathbb{Z}\)-grading on \(\hat{\mathfrak{h}}\), called the \textit{weight grading}, given by
\[
\text{wt}(\alpha \otimes t^m) = -m \quad \text{and} \quad \text{wt}\ k = 0
\]
for \(\alpha \in \mathfrak{h}\) and \(m \in \mathbb{Z}\). Consider the following subalgebras of \(\hat{\mathfrak{h}}\):
\[
\hat{\mathfrak{h}}^+ = \mathfrak{h} \otimes t\mathbb{C}[t] \quad \text{and} \quad \hat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]
\]
and
\[
\hat{\mathfrak{h}}_Z = \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C}k.
\]

The latter is a Heisenberg algebra, in the sense that its commutator subalgebra equals its center, which is one-dimensional. Form the induced \(\hat{\mathfrak{h}}\)-module
\[(2.31)\]
\[M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}k)} \mathbb{C} \simeq S(\hat{\mathfrak{h}}^-) \quad \text{(linearly)},\]
where \(\mathfrak{h} \otimes \mathbb{C}[t]\) acts trivially on \(\mathbb{C}\) and \(k\) acts as \(1\). This is an irreducible \(\hat{\mathfrak{h}}_Z\)-module, \(\mathbb{Z}\)-graded by weights:
\[
M(1) = \bigoplus_{n \geq 0} M(1)_n,
\]
where \(M(1)_n\) denotes the homogeneous subspace of weight \(n\). Note that \(\text{wt} 1 = 0\) (by \(1\) we mean \(1 \otimes 1\)).

Now form the induced \(\hat{L}\)-module
\[(2.32)\]
\[\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\{\eta_0\}]} \mathbb{C} \simeq \mathbb{C}[L] \quad \text{(linearly)}.\]

For \(a \in \hat{L}\), write
\[\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}.\]

The space \(\mathbb{C}\{L\}\) is \(\mathbb{C}\)-graded:
\[(2.33)\]
\[\text{wt } \iota(a) = \frac{1}{2} \langle \alpha, \alpha \rangle \quad \text{for } a \in \hat{L}.\]

In particular, we have
\[(2.34)\]
\[\text{wt } \iota(1) = 0\]
and
\[(2.35)\]
\[\text{wt } \iota(e_\alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle.\]

The action of \(\hat{L}\) on \(\mathbb{C}\{L\}\) is given by
\[(2.36)\]
\[a \cdot \iota(b) = \iota(ab)\]
for \(a, b \in \hat{L}\). There is also a grading-preserving action of \(\mathfrak{h}\) on \(\mathbb{C}\{L\}\) given by
\[(2.37)\]
\[h \cdot \iota(a) = \langle h, \alpha \rangle \iota(a)\]
for \(h \in \mathfrak{h}\). Define the operator \(x^h\) as follows:
\[(2.38)\]
\[x^h \cdot \iota(a) = x^{\langle h, \alpha \rangle} \iota(a)\]
for \(h \in \mathfrak{h}\).

Set
\[(2.39)\]
\[V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\} \simeq S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L] \quad \text{(linearly)}\]
and set $1 = 1 \otimes \iota(1)$. Give $V_L$ the tensor product weight grading:

$$V_L = \prod_{n \in \mathbb{C}} (V_L)_n.$$ 

We have that $\hat{L}$, $\hat{h}_Z$, $\mathfrak{h}$, $x^h$ ($h \in \mathfrak{h}$) act naturally on $V_L$ by acting on either $M(1)$ or $\mathbb{C}\{L\}$ as indicated above. In particular, $\mathfrak{k}$ acts as $1$.

Throughout this paper, we shall use the symbol $x$ as a formal variable, and symbols such as $x_i$ (along with $x$), $y_i$, $q$ and $t$ (already used above) will be used for independent commuting formal variables. (Note that $x$, etc., will sometimes refer to other things, in context.)

For $\alpha \in \mathfrak{h}$, $n \in \mathbb{Z}$, we write $\alpha(n)$ for the operator on $V_L$ associated with $\alpha \otimes t^n \in \hat{h}$:

$$\alpha \otimes t^n \mapsto \alpha(n)$$

and we set

$$(2.40) \quad \alpha(x) = \sum_{n \in \mathbb{Z}} \alpha(n)x^{-n-1}.$$ 

Using the formal exponential series $\exp(\cdot)$ define

$$(2.41) \quad E^\pm(\alpha, x) = \exp \left( \sum_{n \in \pm \mathbb{Z}} \frac{\alpha(n)}{n}x^{-n} \right) \in (\text{End} V_L)[[x, x^{-1}]]$$

for any $\alpha \in \mathfrak{h}$. As in [FLM2], for $a \in \hat{L}$, set

$$(2.42) \quad Y(\iota(a), x) = \overset{\circ}{\circ} \sum_{n \neq 0} \frac{-\pi(n)}{n} \overset{\circ}{\circ} \alpha x \pi,$$

where by $\overset{\circ}{\circ} \cdot \overset{\circ}{\circ}$ we mean a normal ordering procedure in which the operators $\bar{a}(n)$ ($n < 0$) and $a \in \hat{L}$ are placed to the left of the operators $\bar{a}(n)$ ($n \geq 0$) and $x^a$ before the expression is evaluated. By (2.41) the vertex operator (2.42) becomes

$$(2.43) \quad Y(\iota(a), x) = E^-(\bar{a}, x)E^+(\bar{a}, x)ax^a.$$ 

In particular, using the section $e$ we have the operator

$$(2.44) \quad Y(\iota(e_\alpha), x) = E^-(e_\alpha, x)E^+(e_\alpha, x)e_\alpha x^\alpha$$

for $\alpha \in L$. For $\alpha \in L$ and $n \in \mathbb{Z}$ we define the operators $x_\alpha(n)$ by the expansion

$$(2.45) \quad Y(\iota(e_\alpha), x) = \sum_{n \in \mathbb{Z}} x_\alpha(n)x^{-n-\frac{(\alpha, \alpha)}{2}}.$$ 

More generally, for an element $v = \alpha_1(-n_1) \cdots \alpha_m(-n_m) \otimes \iota(a) \in V_L$ with $\alpha_1, \ldots, \alpha_m \in \mathfrak{h}$, $n_1, \ldots, n_m > 0$ and $a \in \hat{L}$, we set

$$(2.46) \quad Y(v, x) = \overset{\circ}{\circ} \left( \frac{1}{(n_1 - 1)!} \left( \frac{d}{dx} \right)^{n_1 - 1} \alpha_1(x) \right) \cdots \left( \frac{1}{(n_m - 1)!} \left( \frac{d}{dx} \right)^{n_m - 1} \alpha_m(x) \right) Y(\iota(a), x) \overset{\circ}{\circ}$$

and this gives a well-defined linear map

$$V_L \to (\text{End} V_L)[[x, x^{-1}]]$$

$$v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \quad v_n \in \text{End} V_L.$$ 

Set

$$(2.47) \quad \omega = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} h_i(-1) h_i(-1) 1,$$
where \( \{h_i\} \) is an orthonormal basis of \( \mathfrak{h} \).

By Chapter 8 of [FLM3], \( V_L = (V_L, Y, 1, \omega) \) is a simple vertex operator algebra of central charge equal to \( \dim L \) (cf. Theorem 6.5.3 in [LL]). We also have that \( V_L \) is independent, up to an isomorphism of vertex operator algebras preserving the \( \mathfrak{h} \)-module structure, of the central extension \( \mathfrak{h} \) subject to (2.11) and on the choices of \( k > 0 \) and the primitive root \( \eta \) (cf. Proposition 6.5.5 and also Remarks 6.5.4 and 6.5.6 in [LL]).

We will now extend the automorphism \( \hat{\nu} \) of \( \hat{L} \) in a natural way to an automorphism of period \( k \), also denoted by \( \hat{\nu} \), of \( V_L \): The automorphism \( \nu \) of \( L \) acts in a natural way on \( \mathfrak{h} \), on \( \mathfrak{h} \) and on \( M(1) \), preserving the gradings. We have

\[
\nu(u \cdot m) = \nu(u) \cdot \nu(m)
\]

for \( u \in \mathfrak{h} \) and \( m \in M(1) \). The automorphism \( \hat{\nu} \) of \( \hat{L} \), extended naturally to \( \mathbb{C}\{L\} \), satisfies the conditions

\[
\hat{\nu}(h \cdot \iota(a)) = \nu(h) \cdot \hat{\nu}(a),
\]

\[
\hat{\nu}(x^h \cdot \iota(a)) = x^{\nu(h)} \cdot \hat{\nu}(a)
\]

and

\[
\hat{\nu}(a \cdot \iota(b)) = \hat{\nu}(a) \cdot \hat{\nu}(b)
\]

for \( h \in \mathfrak{h} \) and \( a, b \in \hat{L} \). We take our extension \( \hat{\nu} \) on \( V_L \) to be \( \nu \circ \hat{\nu} \). It preserves the grading and we have

\[
\hat{\nu}(a \cdot v) = \hat{\nu}(a) \cdot \hat{\nu}(v)
\]

\[
\hat{\nu}(u \cdot v) = \nu(u) \cdot \hat{\nu}(v)
\]

\[
\hat{\nu}(x^h \cdot v) = x^{\nu(h)} \cdot \hat{\nu}(v)
\]

for \( a \in \hat{L}, u \in \mathfrak{h}, h \in \mathfrak{h} \) and \( v \in V_L \). It follows that \( \hat{\nu} \) is an automorphism of the vertex operator algebra \( V_L \).

In the remainder of this section we review the construction of the \( \hat{\nu} \)-twisted modules for \( V_L \) following [LL] and [FLM2] (see also [DLM]). Using the primitive \( k \)-th root of unity \( \eta \), where \( k \) is our choice of period of the isometry \( \nu \), and the vector space \( \mathfrak{h} \), set

\[
\mathfrak{h}(n) = \{h \in \mathfrak{h} | \nu h = \eta^n h\} \subset \mathfrak{h}
\]

for \( n \in \mathbb{Z} \), so that

\[
\mathfrak{h} = \bigsqcup_{p \in \mathbb{Z}/k\mathbb{Z}} \mathfrak{h}(p).
\]

Here we identify \( \mathfrak{h}(n \mod k) \) with \( \mathfrak{h}(n) \) for \( n \in \mathbb{Z} \). For \( p \in \mathbb{Z}/k\mathbb{Z} \) consider the \( p \)-th projection

\[
P_p : \mathfrak{h} \rightarrow \mathfrak{h}(p)
\]

and for \( h \in \mathfrak{h} \) and \( n \in \mathbb{Z} \) set \( h(n) = P_{n \mod k} h \).

Consider the \( \nu \)-twisted affine Lie algebra associated with \( \mathfrak{h} \) (viewed as an abelian Lie algebra) and \( \langle \cdot, \cdot \rangle \):

\[
\mathfrak{h}[\nu] = \bigsqcup_{n \in \mathbb{Z}} \mathfrak{h}(kn) \otimes t^n \oplus \mathbb{C}k
\]

with

\[
[\alpha \otimes t^m, \beta \otimes t^n] = (\alpha, \beta)m\delta_{m+n,0}k
\]
for $\alpha \in \mathfrak{h}_{(km)}$, $\beta \in \mathfrak{h}_{(kn)}$, and $m, n \in \frac{1}{k}\mathbb{Z}$, and
\begin{equation}
(2.59) \quad [k, \mathfrak{h}[\nu]] = 0.
\end{equation}
This algebra is $\frac{1}{k}\mathbb{Z}$-graded by weights:
\begin{equation}
(2.60) \quad \text{wt } (\alpha \otimes t^m) = -m, \quad \text{wt } k = 0
\end{equation}
for $m \in \frac{1}{k}\mathbb{Z}$, $\alpha \in \mathfrak{h}_{(km)}$. Notice that for $\nu$ the identity automorphism, $\mathfrak{h}[\nu]$ is the same as the untwisted affine Lie algebra $\hat{\mathfrak{g}}$ defined in (2.28). Consider the following subalgebras of $\mathfrak{h}[\nu]$
\[\hat{\mathfrak{g}}[\nu]^+ = \bigoplus_{n>0} \mathfrak{h}_{(kn)} \otimes t^n, \quad \hat{\mathfrak{g}}[\nu]^- = \bigoplus_{n<0} \mathfrak{h}_{(kn)} \otimes t^n\]
and
\[\hat{\mathfrak{g}}[\nu]_{\frac{1}{k}\mathbb{Z}} = \hat{\mathfrak{g}}[\nu]^+ \oplus \hat{\mathfrak{g}}[\nu]^- \oplus \mathbb{C}k,\]
which is a Heisenberg subalgebra of $\hat{\mathfrak{g}}[\nu]$. As in (2.31) form the induced $\hat{\mathfrak{g}}[\nu]$-module
\begin{equation}
(2.61) \quad S[\nu] = U(\hat{\mathfrak{g}}[\nu]) \otimes U(\bigoplus_{n \geq 0} \mathfrak{h}_{(kn)} \otimes t^n \otimes \mathbb{C}k) \mathbb{C},
\end{equation}
where $\bigoplus_{n \geq 0} \mathfrak{h}_{(kn)} \otimes t^n$ acts trivially on $\mathbb{C}$ and $k$ acts as 1. This is an irreducible $\hat{\mathfrak{g}}[\nu]_{\frac{1}{k}\mathbb{Z}}$-module, which is linearly isomorphic to the symmetric algebra $S(\hat{\mathfrak{g}}[\nu]^-)$. As in Section 6 of [DL1] we give the module $S[\nu]$ the natural $\mathbb{Q}$-grading by weights compatible with the action of $\hat{\mathfrak{g}}[\nu]$ and such that
\begin{equation}
(2.62) \quad \text{wt } 1 = \frac{1}{4k^2} \sum_{j=1}^{k-1} j(k-j) \dim \mathfrak{h}(j)
\end{equation}
(see also formula (2.20) in [BHL]). The reason for choosing this shifted grading will be justified later by the action of the operator $L^\nu(0)$.

Continuing to follow [LL], denote by $N$ the orthogonal complement of $\mathfrak{h}_{(0)}$ in $L$:
\begin{equation}
(2.63) \quad N = (1 - P_0)\mathfrak{h} \cap L = \{ \alpha \in L \mid \langle \alpha, \mathfrak{h}_{(0)} \rangle = 0 \}.
\end{equation}
Let
\begin{equation}
(2.64) \quad M = (1 - \nu)L \subset N.
\end{equation}
Since $\sum_{j=0}^{k-1} \nu^j \alpha \in \mathfrak{h}_{(0)}$ for any $\alpha \in \mathfrak{h}$, the commutator map (2.5) becomes
\begin{equation}
(2.65) \quad C_N(\alpha, \beta) = \eta^{\sum_{j=0}^{k-1} j(\nu^j \alpha, \beta)}
\end{equation}
for $\alpha, \beta \in N$. Let
\begin{equation}
(2.66) \quad R = \{ \alpha \in N \mid C_N(\alpha, N) = 1 \}.
\end{equation}
Note that $M \subset R$. For any subgroup $Q$ of $L$ we denote by $\hat{Q}$ the subgroup of $\hat{L}_\nu$ obtained by pulling back $Q$. By Proposition 6.1 of [LL], there exists a unique homomorphism $\tau: M \rightarrow \mathbb{C}^\times$ such that $\tau(\eta_0) = \eta_0$ and $\tau(a\hat{\nu}\hat{a}^{-1}) = \eta^{-\sum_{j=0}^{k-1} (\nu^j \pi, \pi)/2}$ for $a \in \hat{L}_\nu$.

Let us now recall the classification of the irreducible $\hat{N}$-modules:

**Proposition 2.1.** (Proposition 6.2 of [LL]) There are exactly $|R/M|$ extensions of $\tau$ to a homomorphism $\chi: \hat{R} \rightarrow \mathbb{C}^\times$. For each $\chi$, there is a unique (up to equivalence) irreducible $\hat{N}$-module on which $\hat{R}$ acts according to $\chi$, and every irreducible $\hat{N}$-module on which $M$ acts according to $\tau$ is equivalent to one of these. Every such module has dimension $|N/R|^2$.
Let $T$ be an irreducible $\hat{N}$-module. Form the induced $\hat{L}_\nu$-module

\begin{equation}
U_T = \mathbb{C}[\hat{L}_\nu] \otimes_{\mathbb{C}[\hat{N}]} T \simeq \mathbb{C}[L/N] \otimes T,
\end{equation}

where $\hat{L}_\nu$ and $h(0)$ act as follows:

\begin{align}
a \cdot b \otimes u &= ab \otimes u, \\
h \cdot b \otimes u &= \langle h, b \rangle b \otimes u
\end{align}

for $a, b \in \hat{L}_\nu$, $u \in T$, $h \in h(0)$. As operators on $U_T$,

\begin{equation}
ah a = a(\langle h, \alpha \rangle + h).
\end{equation}

For $h \in h(0)$ define the End $U_T$-valued formal Laurent series $x^h$ by:

\begin{equation}x^h \cdot b \otimes u = x^{\langle h, \alpha \rangle} b \otimes u.
\end{equation}

Then for $a \in \hat{L}_\nu$ we have

\begin{equation}x^h a = ax^{\langle h, \alpha \rangle + h} \quad \text{for} \quad h \in h(0)
\end{equation}

and

\begin{equation}\eta^h a = \eta^{\langle h, \alpha \rangle + h} a \quad \text{for} \quad h \in h(0) \quad \text{with} \quad \langle h, L \rangle \in \mathbb{Z}.
\end{equation}

Moreover, as operators on $U_T$,

\begin{equation}\hat{\nu} a = a \eta^{-\sum_{j=0}^{k} \nu_j \pi_j - \sum_{j=0}^{k-1} \nu_{j+1} \pi_j / 2}
\end{equation}

for $a \in \hat{L}_\nu$.

Since the projection map $P_0$ (see (2.56)) induces an isomorphism from $L/N$ onto $P_0 L$, we have a natural isomorphism

\begin{equation}U_T \simeq \mathbb{C}[P_0 L] \otimes T
\end{equation}

of $\hat{h}[\nu]$-modules. We also have

\begin{equation}U_T = \prod_{\alpha \in P_0 L} U_\alpha,
\end{equation}

where

\begin{equation}U_\alpha = \{ u \in U_T \mid h \cdot u = \langle h, \alpha \rangle u \quad \text{for} \quad h \in h(0) \}
\end{equation}

and

\begin{equation}a \cdot U_\alpha \subset U_{\alpha + \pi(0)} \quad \text{for} \quad a \in \hat{L}_\nu, \alpha \in P_0 L.
\end{equation}

Consider the $\mathbb{C}$-grading on $U_T$ given by

\begin{equation}\text{wt} \ u = \frac{1}{2} \langle \alpha, \alpha \rangle \quad \text{for} \quad u \in U_\alpha, \alpha \in P_0 L.
\end{equation}

Set

\begin{equation}V^T_L = S[\nu] \otimes U_T = \left(U(\hat{h}[\nu]) \otimes_{U(\mathbb{C}[\hat{h}(0) \otimes \mathbb{C}])} \mathbb{C} \right) \otimes \left(\mathbb{C}[\hat{L}_\nu] \otimes_{\mathbb{C}[\hat{N}]} T \right) \simeq S(\hat{h}[\nu]) \otimes \mathbb{C}[P_0 L] \otimes T,
\end{equation}

on which $\hat{L}_\nu$, $\hat{h}[\nu]_{\frac{1}{2} \mathbb{Z}}$, $h(0)$ and $x^h$ for $h \in h(0)$ act naturally on either $S[\nu]$ or $U_T$ as described above. The space $V^T_L$ is graded by weights using the weight gradings of $S[\nu]$ and $U_T$, as described above.
For $\alpha \in \mathfrak{h}$ and $n \in \frac{1}{k}\mathbb{Z}$, write $\alpha^\vartheta(n)$ or $\alpha_{(kn)}(n)$ for the operator on $V^T_L$ associated with $\alpha_{(kn)} \otimes t^n \in \hat{\mathfrak{h}}[\nu]$:

\[ \alpha_{(kn)} \otimes t^n \mapsto \alpha^\vartheta(n), \]

and set

\[ \alpha^\vartheta(x) = \sum_{n \in \frac{1}{k}\mathbb{Z}} \alpha^\vartheta(n)x^{-n-1} = \sum_{n \in \frac{1}{k}\mathbb{Z}} \alpha_{(kn)}(n)x^{-n-1}. \]

Consider the formal Laurent series $E^\pm(\alpha, x) \in (\text{End} \, V_L)[[x^{1/k}, x^{-1/k}]]$ (recall (2.41)). We have

\[ E^+(\alpha, x_1)E^-(\beta, x_2) = E^-(\beta, x_2)E^+(\alpha, x_1) \prod_{p \in \mathbb{Z}/k\mathbb{Z}} \left( 1 - \eta^{-1/k}x_1^{1/k} \right)^{\langle \nu p \alpha, \beta \rangle} \]

for $\alpha, \beta \in \mathfrak{h}$. Let

\[ \sigma(\alpha) = \begin{cases} \prod_{0 < j < k/2} (1 - \eta^{-j})^{\langle \nu \alpha, \alpha \rangle} 2^{2^{(a/k, 2\alpha, \alpha)}/2} & \text{if } k \in 2\mathbb{Z} \\ \prod_{0 < j < k/2} (1 - \eta^{-j})^{\langle \nu \alpha, \alpha \rangle} & \text{if } k \in 2\mathbb{Z} + 1. \end{cases} \]

Now for $a \in \hat{L}$ define the $\hat{\nu}$-twisted vertex operator $Y^\vartheta(\iota(a), x)$ acting on $V^T_L$ as follows:

\[ Y^\vartheta(\iota(a), x) = k^{-\langle \alpha, \vartheta \rangle}/2 \sigma(\vartheta) \sum_{n \neq 0} \frac{\zeta_{\alpha}^{x-n}}{\zeta_{n \vartheta}^{\nu \alpha + \vartheta(\nu \alpha) + (\vartheta(\nu \alpha), \vartheta(\nu \alpha))/2 - (\vartheta(\nu \alpha))/2}}, \]

where we view $\alpha$ in the right hand side of (2.83) as an element of $\hat{L}_\nu$ using the set-theoretic identification between $\hat{L}$ and $\hat{L}_\nu$ given in (2.13). By using (2.41) we have

\[ Y^\vartheta(\iota(a), x) = k^{-\langle \alpha, \vartheta \rangle}/2 \sigma(\vartheta)E^-(\vartheta, x)E^+(\vartheta, x)ax^{\nu \alpha + \vartheta(\nu \alpha) + (\vartheta(\nu \alpha), \vartheta(\nu \alpha))/2 - (\vartheta(\nu \alpha))/2}. \]

Define the component operators $x^\vartheta_\alpha(n)$ for $n \in (1/k)\mathbb{Z}$ and $\alpha \in L$ by the expansion

\[ Y^\vartheta(\iota(e_\alpha), x) = \sum_{n \in (1/k)\mathbb{Z}} x^\vartheta_\alpha(n)x^{-n - (\alpha, \alpha)/2}. \]

For $v = \alpha_1(-n_1) \cdots \alpha_m(-n_m) \cdot \iota(a) \in V_L$, set

\[ W(v, x) = \sum_0 \frac{1}{(n_1 - 1)!} \left( \frac{d}{dx} \right)^{n_1 - 1} \alpha_1^\vartheta(x) \cdots \frac{1}{(n_m - 1)!} \left( \frac{d}{dx} \right)^{n_m - 1} \alpha_m^\vartheta(x) Y^\vartheta(\iota(a), x) \sum_0, \]

giving a well-defined linear operator on $V^T_L$ depending linearly on $v \in V_L$ (as in (2.46)).

Define constants $c_{mnr} \in \mathbb{C}$ for $m, n \in \mathbb{N}$ and $r = 0, \ldots, k - 1$ by

\[ \sum_{m, n \geq 0} c_{mn0}x^my^n = \frac{1}{2} \sum_{j=1}^{k-1} \log \left( \frac{(1 + x)^{1/k} - \eta^{-j}(1 + y)^{1/k}}{1 - \eta^{-j}} \right), \]

\[ \sum_{m, n \geq 0} c_{mnr}x^my^n = \frac{1}{2} \log \left( \frac{(1 + x)^{1/k} - \eta^{-r}(1 + y)^{1/k}}{1 - \eta^{-r}} \right) \text{ for } r \neq 0 \]

( well-defined formal power series in $x$ and $y$). Let $\{\beta_1, \ldots, \beta_{\dim \mathfrak{h}}\}$ be an orthonormal basis of $\mathfrak{h}$, and set

\[ \Delta_x = \sum_{m, n \geq 0} \sum_{r=0}^{k-1} \sum_{j=1}^{\dim \mathfrak{h}} c_{mnr}(\nu^{-r}\beta_j)(m)\beta_j(n)x^{-m-n}. \]
Note that $\Delta x$ is independent of the choice of the orthonormal basis. Then $e^{\Delta x}$ is well-defined on $V_L$ since $c_{00r} = 0$ for all $r$, and for $v \in V_L$ we have $e^{\Delta x} v \in V_L[x^{-1}]$.

Now for $v \in V_L$, we define the $\hat{\nu}$-twisted vertex operator

\[ Y^\hat{\nu}(v, x) = W(e^{\Delta x} v, x) \]

and this yields a well-defined linear map

\[ V_L \to (\text{End} V^T_L)[[x^{1/k}, x^{-1/k}]] \]

\[ \nu \mapsto Y^\hat{\nu}(v, x) = \sum_{n \in \frac{1}{k} \mathbb{Z}} v^n \hat{x}^{-n-1}, \quad v^n \in \text{End} V^T_L. \]

By [FLM2], [FLM3] and [L2] (see also [DL1]), $V^T_L = (V^T_L, Y^\hat{\nu})$ has the structure of an irreducible $\hat{\nu}$-twisted $V_L$-module. In particular, we have the twisted Jacobi identity

\[ x^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y^\hat{\nu}(u, x_1) Y^\hat{\nu}(v, x_2) - x^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) Y^\hat{\nu}(v, x_2) Y^\hat{\nu}(u, x_1) \]

\[ = x^2 \frac{1}{k} \sum_{j \in \mathbb{Z}/k \mathbb{Z}} \delta \left( \eta \frac{j(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) Y^\hat{\nu}(Y^\hat{\nu}_{j}(u, x_0)v, x_2) \]

for $u, v \in V_L$ (the main property of a twisted module), and also, the $\hat{\nu}$-twisted operator has the property

\[ Y^\hat{\nu}(\hat{\nu} v, x) = \lim_{x^{1/k} \to \eta^{-1} x^{1/k}} Y^\hat{\nu}(v, x) \]

for $v \in V_L$. Formula (2.92) immediately generalizes to

\[ Y^\hat{\nu}(\hat{\nu}^r v, x) = \lim_{x^{1/k} \to \eta^{-r} x^{1/k}} Y^\hat{\nu}(v, x) \]

for any $r \in \mathbb{Z}$. By taking $\text{Res}_{x_0}$, the twisted Jacobi identity (2.91) immediately implies the commutator formula [FLM2]:

\[ [Y^\hat{\nu}(u, x_1), Y^\hat{\nu}(v, x_2)] = x^2 \frac{1}{k} \text{Res}_{x_0} \left( \sum_{j \in \mathbb{Z}/k \mathbb{Z}} \delta \left( \eta \frac{j(x_1 - x_0)^{1/k}}{x_2^{1/k}} \right) Y^\hat{\nu}(Y^\hat{\nu}_{j}(u, x_0)v, x_2) \right). \]

Following [DL1] (see also [BHL]) we will now justify that the weight grading of $V^T_L$ given by (2.60), (2.62) and (2.77) is the grading given by the operator $L^\hat{\nu}(0)$, where the operators $L^\hat{\nu}(n)$ for $n \in \mathbb{Z}$ are defined by

\[ Y^\hat{\nu}(\omega, x) = \sum_{n \in \mathbb{Z}} L^\hat{\nu}(n)x^{-n-2} \]

(recall (2.37)). These operators have the property

\[ [L^\hat{\nu}(m), L^\hat{\nu}(n)] = (m - n)L^\hat{\nu}(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\dim \mathfrak{g} \]

for $m, n \in \mathbb{Z}$. By Proposition 6.3 of [DL1] we have

\[ [Y^\hat{\nu}(\omega, x_1), Y^\hat{\nu}(\eta(a), x_2)] = x^2 \frac{1}{k} \left( \frac{d}{dx_2} Y^\hat{\nu}(\eta(a), x_2) \right) \delta(x_1/x_2) - \frac{1}{2} \langle a, \overline{a} \rangle x^2 \frac{1}{x_2} Y^\hat{\nu}(\eta(a), x_2) \frac{\delta}{\delta x_1} \delta(x_1/x_2) \]

for $a \in \hat{\mathfrak{l}}$. Also recall from [DL1], [BHL] and [DLeM] that

\[ L^\hat{\nu}(0)1 = \frac{1}{4k^2} \sum_{j=1}^{k-1} j(k - j) \dim \mathfrak{g}(j)1 \]
$(1 \in S[\nu])$,

$$L^\nu(0)u = \left( \frac{1}{2} \langle \alpha, \alpha \rangle + \frac{1}{4k^2} \sum_{j=1}^{k-1} j(k-j) \dim h(j) \right) u$$

for $u \in U_\alpha \subset U_T \subset V_L^T \ (\alpha \in P_0L)$, and

$$[L^\nu(0), \alpha^\nu(m)] = -m\alpha^\nu(m)$$

for $m \in \mathbb{Z}^T$ and $\alpha \in h(km)$. Thus by using the grading shift (2.62) and the weight grading defined by (2.60) and (2.77) we have

$$L^\nu(0)v = \left( \text{wt } v + \frac{1}{4k^2} \sum_{j=1}^{k-1} j(k-j) \dim h(j) \right) v$$

for a homogenous element $v \in V_L^T$.

It has been established in [L1] (see also [FLM2] and [FLM3]) that if the even lattice $L$ is the root lattice of a Lie algebra of type $A$, $D$ or $E$ then $V_L^T$ has a natural structure of module for a certain twisted affine Lie algebra. In the next section we will recall in detail the special case that for $L$ the root lattice of $\mathfrak{sl}(3, \mathbb{C})$ and for a certain isometry of this root lattice the corresponding twisted module $V_L^T$ is an irreducible $A_2^{(2)}$-module.

3. Vertex operator construction of $A_2^{(2)}$

The aim of this section is to recall the twisted vertex operator construction of the affine Lie algebra $A_2^{(2)}$ as a special case of the lattice construction recalled in the previous section, following the treatment in [L1] and [FLM2]. We will specialize the previous section to the root lattice of $\mathfrak{sl}(3, \mathbb{C})$ and an involution $\nu$ induced by a Dynkin diagram automorphism of $\mathfrak{sl}(3, \mathbb{C})$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}(3, \mathbb{C})$. Denote by $\Delta \subset \mathfrak{h}^*$ the root system and by $\{\alpha_1, \alpha_2\}$ a choice of simple roots. Take $\langle a, b \rangle = \text{tr}(ab) \ (a, b \in \mathfrak{sl}(3, \mathbb{C}))$, the standard suitably-normalized nonsingular symmetric invariant bilinear form on $\mathfrak{sl}(3, \mathbb{C})$. We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via $\langle \cdot, \cdot \rangle$, so that under this identification we have $\Delta \subset \mathfrak{h}$ (and $\alpha_1, \alpha_2 \in \mathfrak{h}$).

We now specialize the previous section to the root lattice of $\mathfrak{sl}(3, \mathbb{C})$,

$$L = \mathbb{Z}\Delta = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \subset \mathfrak{h},$$

equipped with the form $\langle \cdot, \cdot \rangle$. We take $\nu$ to be the isometry of $L$ determined by

$$\nu(\alpha_1) = \alpha_2, \quad \nu(\alpha_2) = \alpha_1,$$

corresponding to the Dynkin diagram automorphism. Although $\nu^2 = 1$, we take $k = 4$ rather than 2 as our period of $\nu$, since otherwise the assertion (2.2) would not hold. Then we have (2.2) and (2.3) with $k = 4$. Fix

$$\eta = i$$

to be our primitive $4^{th}$ root of unity. With $\eta_0 = (-1)^4\eta$ as in (2.8), we in fact have

$$\eta_0 = \eta = i.$$

Extend $\nu$ linearly to an automorphism of

$$\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L,$$

our Cartan subalgebra.

We have the two central extensions of $L$ by the cyclic group generated by $i$, $\hat{L}$ and $\hat{L}_\nu$, with $C_0$ and $C$, respectively, the commutator maps (recall (2.4), (2.5) and (2.9)--(2.12)). As before, we
choose the normalized sections $e$ of $\hat{L}$ and $\hat{L}_\nu$ that send $\alpha \in L$ to $e_\alpha \in \hat{L}$ (respectively, $\hat{L}_\nu$). We also have the normalized cocycles $\epsilon_C$ and $\epsilon_{C_0}$ (see (2.15) and (2.20)). By (2.22) we get
\[
e_\alpha e_\alpha = \epsilon_{C_0}(\alpha, \alpha_2) e_{\alpha_2 + \alpha_1} \quad \text{in } \hat{L},
\]
and since
\[
e_{C_0}(\alpha, \alpha_2) / \epsilon_{C_0}(\alpha_2, \alpha_1) = C_0(\alpha, \alpha_2) = -1
\]
by (2.23) and (2.1), we have
\[(3.6) \quad e_\alpha e_\alpha = -e_\alpha e_\alpha \quad \text{in } \hat{L}.
\]
For concreteness and convenience we shall use the following particular choice of $\epsilon_{C_0}$: Take $\epsilon_{C_0} : L \times L \to \langle i \rangle$ to be the $\mathbb{Z}$-bilinear map determined by
\[
(3.7) \quad \epsilon_{C_0}(\alpha, \alpha_2) = 1, \quad \epsilon_{C_0}(\alpha_2, \alpha_1) = -1
\]
and
\[
(3.8) \quad \epsilon_{C_0}(\alpha, \alpha_1) = \epsilon_{C_0}(\alpha_2, \alpha_2) = 1.
\]
Then, in particular,
\[
(3.9) \quad \epsilon_{C_0}(\alpha, -\alpha) = 1 \quad \text{for } \alpha = \alpha_1 \text{ or } \alpha_2.
\]
We have that (2.17)–(2.19) hold for $\epsilon_{C_0}$, that is, $\epsilon_{C_0}$ is a normalized 2-cocycle associated with the commutator map $C_0$. This 2-cocycle has the properties
\[
(3.10) \quad \epsilon_{C_0}(\alpha, \beta)^2 = 1
\]
and
\[
(3.11) \quad \epsilon_{C_0}(\alpha, \beta) = \epsilon_{C_0}(\nu_\beta, \nu_\alpha)
\]
for any $\alpha, \beta \in L$. Indeed, for $\alpha = m\alpha_1 + n\alpha_2$ and $\beta = r\alpha_1 + s\alpha_2$ with $m, n, r, s \in \mathbb{Z}$, we have
\[
\epsilon_{C_0}(\alpha, \beta) = (-1)^{nr} = \epsilon_{C_0}(\nu_\beta, \nu_\alpha),
\]
which gives (3.10)–(3.11).

As in (2.21)–(2.25), we lift the isometry (3.2) of $L$ to an automorphism $\hat{\nu}$ of $\hat{L}$ and of $\hat{L}_\nu$ fixing $i$ and satisfying (2.26), so that $\hat{\nu}^4 = 1$ (recall (2.27)). Again for concreteness, we make the following particular choice of $\hat{\nu}$:
\[
(3.12) \quad \hat{\nu} e_\alpha = \epsilon_{C_0}(\alpha, \alpha) i^{(\alpha, \alpha_1 + \alpha_2)} e_\nu e_\alpha
\]
for $\alpha \in L$. Then $\hat{\nu}$ is indeed an automorphism of $\hat{L}$, since for any $\alpha, \beta \in L$ we obtain
\[
(3.13) \quad \hat{\nu}(e_\alpha e_\beta) = \epsilon_{C_0}(\alpha, \alpha) \epsilon_{C_0}(\beta, \beta) \epsilon_{C_0}(\alpha, \beta)^2 \epsilon_{C_0}(\beta, \alpha) i^{(\alpha + \beta, \alpha_1 + \alpha_2)} e_{\nu \alpha + \nu \beta}
\]
and
\[
(3.14) \quad (\hat{\nu} e_\alpha)(\hat{\nu} e_\beta) = \epsilon_{C_0}(\alpha, \alpha) \epsilon_{C_0}(\beta, \beta) \epsilon_{C_0}(\nu \alpha, \nu \beta) i^{(\alpha + \beta, \alpha_1 + \alpha_2)} e_{\nu \alpha + \nu \beta},
\]
and by (3.10) and (3.11) we observe that (3.13) and (3.14) are equal. Using (2.13), (2.21) and the fact $\hat{\nu}$ is an automorphism of $\hat{L}$ we obtain that $\hat{\nu}$ is an automorphism of $\hat{L}_\nu$. We also have that (3.12) is a lifting of (3.2) and that it satisfies (2.26). Since
\[
\hat{\nu}^2 e_\alpha = \epsilon_{C_0}(\alpha, \alpha) \epsilon_{C_0}(\nu \alpha, \nu \alpha) i^{(\alpha, \nu \alpha, \alpha_1 + \alpha_2)} e_\alpha,
\]
by (3.10), (3.11) and the fact that $i^{(\alpha + \nu \alpha, \alpha_1 + \alpha_2)} = -1$ we obtain $\hat{\nu}^2 e_\alpha = -e_\alpha$ for any $\alpha \in L$. This confirms that
\[
(3.15) \quad \hat{\nu}^4 = 1,
\]
but note that $\hat{\nu}^2 \neq 1$:
\[
\hat{\nu}^2 = -1.
\]
Formula (3.12) yields in particular
\[ \hat{\nu}e_{\alpha_1} = ie_{\alpha_1}, \quad \hat{\nu}e_{\alpha_2} = ie_{\alpha_2}, \]
(and these two formulas determine the automorphism \( \hat{\nu} \) uniquely),
\[ \hat{\nu}e_{\alpha_1 + \alpha_2} = e_{\alpha_1 + \alpha_2}, \]
and
\[ \hat{\nu}e_{-\alpha_1} = -ie_{-\alpha_1}, \quad \hat{\nu}e_{\alpha_2} = -ie_{-\alpha_2}, \quad \hat{\nu}e_{-\alpha_1 - \alpha_2} = e_{-\alpha_1 - \alpha_2}. \]

Recall from the previous section the construction of the vector space \( V_L \) (2.39), which together with the vertex operator \( Y(\cdot, x) \) (2.46), a vacuum vector and a conformal vector forms a vertex operator algebra that has a natural \( \mathbb{Z} \)-grading by weights. Following (2.48)–(2.54) we extend the automorphism \( \hat{\nu} \) of \( \hat{L} \) given by (3.12) to an automorphism of \( V_L \) denoted by \( \hat{\nu} \) as well. This acts via \( \nu \otimes \hat{\nu} \), preserves the grading and has period 4.

For \( n \in \mathbb{Z} \) set
\[ h(n) = \{ x \in h \mid \nu(h) = i^n h \} \subset h, \]
such that
\[ h = \coprod_{n \in \mathbb{Z}/4 \mathbb{Z}} h(n). \]

We identify \( h(n \mod 4) \) with \( h(n) \). In view of (3.2) extended linearly to \( h \) we have:
\[ h(0) = \{ s(\alpha_1 + \alpha_2) \mid s \in \mathbb{C} \}, \]
\[ h(2) = \{ s(\alpha_1 - \alpha_2) \mid s \in \mathbb{C} \} \]
and
\[ h(1) = h(3) = 0. \]

Using the \( n \)th projection (2.56) set \( h(n) = P_{n \mod 4} h \) for \( h \in h \) and \( n \in \mathbb{Z} \). For \( j = 1, 2 \), we have
\[ (\alpha_j)(0) = \frac{1}{2}(\alpha_1 + \alpha_2), \quad (\alpha_j)(2) = \frac{1}{2}(\alpha_1 - \alpha_2) \]
and
\[ (\alpha_j)(1) = (\alpha_j)(3) = 0. \]

Form the \( \nu \)-twisted affine Lie algebra associated to the abelian Lie algebra \( h \):
\[ \hat{h}[\nu] = \coprod_{n \in \mathbb{Z}} h(n) \otimes t^n \oplus \mathbb{C}k = \coprod_{n \in \frac{1}{4} \mathbb{Z}} h(4n) \otimes t^n \oplus \mathbb{C}k \]
such that \( k \) is a central element and
\[ [\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} k \]
for \( m, n \in \frac{1}{4} \mathbb{Z} \) and \( \alpha \in h(4m), \beta \in h(4n) \). By (3.21)–(3.23) we have
\[ \hat{h}[\nu] = \coprod_{n \in \mathbb{Z}} h(0) \otimes t^n \oplus \coprod_{n \in \frac{1}{4} \mathbb{Z}} h(2) \otimes t^{n} \oplus \mathbb{C}k \]
\[ = h(0) \otimes \mathbb{C}[t, t^{-1}] \oplus h(2) \otimes t^{1/2} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k \]
This algebra is \( \frac{1}{2} \mathbb{Z} \)-graded by weights:
\[ \text{wt}(\alpha \otimes t^m) = -m, \quad \text{wt} k = 0 \]
for $m \in (1/2)\mathbb{Z}$ and $\alpha \in \mathfrak{h}(4m)$. Consider the Heisenberg subalgebra of $\hat{\mathfrak{h}}[\nu]$, 
\[
\hat{\mathfrak{h}}[\nu]_{\frac{1}{4}\mathbb{Z}} = \prod_{n \in \frac{1}{4}\mathbb{Z}, n \neq 0} \mathfrak{h}(4n) \otimes t^n \oplus \mathbb{C}k
\]
and the subalgebras 
\[
\hat{\mathfrak{h}}[\nu]^\pm = \prod_{n \in \frac{1}{4}\mathbb{Z}, \pm n > 0} \mathfrak{h}(4n) \otimes t^n.
\]
The induced $\hat{\mathfrak{h}}[\nu]$-module (2.61) becomes 
\[
S[\nu] = U(\hat{\mathfrak{h}}[\nu]) \otimes_{U(\prod_{n \geq 0} \mathfrak{h}(4n) \otimes t^n \oplus \mathbb{C}k)} \mathbb{C} \simeq S(\hat{\mathfrak{h}}[\nu]^-) \]
and this is irreducible as an $\hat{\mathfrak{h}}[\nu]_{\frac{1}{4}\mathbb{Z}}$-module. The module $S[\nu]$ is $\mathbb{Q}$-graded such that 
\[
\text{wt } 1 = \frac{1}{16}
\]
by (2.62).

Recall from the previous section the spaces $N$, $M$ and $R$ (see (2.63), (2.64) and (2.66)). In our setting we have 
\[
N = M = \{s(\alpha_1 - \alpha_2) \mid s \in \mathbb{Z}\}
\]
and 
\[
C_N(\alpha, \beta) = 1 \text{ for } \alpha, \beta \in N
\]
(cf. (2.65)). Thus
\[
N = M = R,
\]
and so,
\[
\hat{N} = \hat{M} = \hat{R}.
\]
(Here we use our notation for pulling back a subgroup of $L$ introduced in Section 2.) By Proposition 6.1 in [L1] there is a unique homomorphism $\tau : \hat{M} = \hat{N} \to \mathbb{C}^\times$ such that 
\[
\tau(i) = i, \quad \tau(a\bar{a}^{-1}) = i^{-\sum_{j=0}^{3}(\nu^j\bar{a}, \bar{a})/2}.
\]
Denote by $\mathbb{C}_\tau$ the one-dimensional $\hat{N}$-module $\mathbb{C}$ with character $\tau$ and write
\[
T = \mathbb{C}_\tau.
\]
This is the unique (up to equivalence) irreducible $\hat{N}$-module given by Proposition 6.2 in [L1] (see also Proposition 2.1). The induced $\hat{L}_\nu$-module (2.67) becomes
\[
U_T = \mathbb{C}[\hat{L}_\nu] \otimes_{\mathbb{C}[\hat{N}]} T \simeq \mathbb{C}[L/N],
\]
and this is graded by weights (see (2.77)). There are the natural actions of $\hat{L}_\nu$, $\mathfrak{h}(0)$ and $x^h$ for $h \in \mathfrak{h}(0)$ on $U_T$ (see (2.68), (2.69) and (2.71)).

As in (2.78) set
\[
V_T = S[\nu] \otimes U_T \simeq S(\hat{\mathfrak{h}}[\nu]^-) \otimes \mathbb{C}[L/N],
\]
on which $\hat{L}_\nu$, $\hat{\mathfrak{h}}[\nu]_{\frac{1}{4}\mathbb{Z}}$, $\mathfrak{h}(0)$ and $x^h$ ($h \in \mathfrak{h}(0)$) act.

Using the operators $\alpha^\hat{\rho}(n)$ (or $\alpha(4n)(n)$) on $V_T$ defined in (2.79) set
\[
\alpha^\hat{\rho}(x) = \sum_{n \in \frac{1}{4}\mathbb{Z}} \alpha^\hat{\rho}(n)x^{-n-1}
\]
for \( \alpha \in \mathfrak{h} \) and \( n \in \frac{1}{4} \mathbb{Z} \). Note that by (3.24) and (3.25) we have
\[
\alpha_{j}^{\hat{\nu}}(x) = \sum_{n \in \frac{1}{4} \mathbb{Z}} \alpha_{j}^{\hat{\nu}}(n)x^{-n-1}
\]
for \( j = 1, 2 \).

The normalizing factor (2.82) becomes
\[
\sigma(\alpha) = (1 + i)^{\langle \nu, \alpha \rangle}\frac{\sigma(\alpha)}{2}
\]
for any \( \alpha \in \mathfrak{h} \). Consider the \( \hat{\nu} \)-twisted vertex operator (2.83) acting on \( V_{L}^{T} \) for \( e_{\alpha} \in \hat{L} \) and its reformulation in terms of the formal exponential series \( E^{\pm}(\cdot, x) \),
\[
Y_{\hat{\nu}}(t(e_{\alpha}), x) = 4^{-\langle \alpha, \alpha \rangle/2}\sigma(\alpha)E^{(-\alpha, x)}E^{+}(-\alpha, x)e_{\alpha}x^{\alpha(0)+\langle \alpha(0), \alpha(0) \rangle/2-\langle \alpha, \alpha \rangle/2}.
\]

As we have mentioned in the previous section, \( V_{L}^{T} \) together with a vertex operator obtained in a canonical way from (3.37) has a natural structure of a \( \hat{\nu} \)-twisted module for the vertex operator algebra \( V_{L} \). Consider the component operators \( x_{\alpha}^{\hat{\nu}}(n) \) for \( n \in (1/4) \mathbb{Z} \) and \( \alpha \in L \) such that
\[
Y_{\hat{\nu}}(t(e_{\alpha}), x) = \sum_{n \in (1/4) \mathbb{Z}} x_{\alpha}^{\hat{\nu}}(n)x^{-n-\langle \alpha, \alpha \rangle/2}.
\]

Following Section 9 of [L1] (see also [FK] and [S]), we now define a nonassociative algebra \( (\mathfrak{g}, [\cdot, \cdot]) \) over \( \mathbb{C} \) as follows:
\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}x_{\alpha},
\]
with \( \mathfrak{h} \) as in (3.5) and \( \{x_{\alpha}\}_{\alpha \in \Delta} \) a set of symbols, such that
\[
[h, x_{\alpha}] = \langle h, \alpha \rangle x_{\alpha} = -[x_{\alpha}, h], \quad [h, h] = 0,
\]
\[
[x_{\alpha}, x_{\beta}] = \begin{cases} 
\varepsilon_{C_{0}}(\alpha, -\alpha)\alpha & \text{if } \alpha + \beta = 0 \\
\varepsilon_{C_{0}}(\alpha, \beta)x_{\alpha+\beta} & \text{if } \langle \alpha, \beta \rangle = -1 \\
0 & \text{if } \langle \alpha, \beta \rangle \geq 0
\end{cases}
\]
for \( h \in \mathfrak{h} \) and \( \alpha, \beta \in \Delta \), where as in the previous section, \( \varepsilon_{C_{0}} \) is any normalized 2-cocycle associated to the commutator map (2.4). Then \( \mathfrak{g} \) is a Lie algebra and in fact is a copy of \( \mathfrak{sl}(3, \mathbb{C}) \). The form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h} \) extends naturally to a nonsingular symmetric invariant bilinear form on \( \mathfrak{g} \) by:
\[
\langle h, x_{\alpha} \rangle = \langle x_{\alpha}, h \rangle = 0,
\]
\[
\langle x_{\alpha}, x_{\beta} \rangle = \begin{cases} 
\varepsilon_{C_{0}}(\alpha, -\alpha) & \text{if } \alpha + \beta = 0 \\
0 & \text{if } \alpha + \beta \neq 0
\end{cases}
\]
Using our particular choice of the normalized 2-cocycle \( \varepsilon_{C_{0}} \) we have
\[
[x_{\alpha}, x_{\beta}] = \begin{cases} 
\alpha & \text{if } \alpha + \beta = 0 \text{ and } \alpha = \alpha_{1} \text{ or } \alpha_{2} \\
x_{\alpha+\beta} & \text{if } \alpha = \alpha_{1} \text{ and } \beta = \alpha_{2} \\
0 & \text{if } \langle \alpha, \beta \rangle \geq 0
\end{cases}
\]
and
\[
\langle x_{\alpha}, x_{\beta} \rangle = \begin{cases} 
1 & \text{if } \alpha + \beta = 0 \text{ and } \alpha = \alpha_{1} \text{ or } \alpha_{2} \\
0 & \text{if } \alpha + \beta \neq 0
\end{cases}
\]
(see (3.7)–(3.9)).

Continuing to follow [L1], we define the function
\[
\psi : \mathbb{Z}/4\mathbb{Z} \times L \rightarrow \langle i \rangle
\]
by the condition
\[
\hat{\nu}^{p}(t(e_{\alpha})) = \psi(p, \alpha)t(e_{\nu p_{\alpha}}),
\]
where we are using our particular choices of \( \hat{\nu} \) (now extended to \( \mathbb{C}\{L\} \)) and of the section \( e \) (and hence of the 2-cocycle \( \epsilon_{e_0} \)); recall that \( \hat{\nu}^4 = 1 \). Using (3.12) and (3.16)–(3.18) we have

\[
\psi(0, \alpha) = 1, \quad \psi(1, \alpha) = i, \quad \psi(2, \alpha) = -1, \quad \psi(3, \alpha) = -i \quad \text{for} \quad \alpha \in \{\alpha_1, \alpha_2\},
\]

(3.42) \quad \psi(0, -\alpha) = 1, \quad \psi(1, -\alpha) = -i, \quad \psi(2, -\alpha) = -1, \quad \psi(3, -\alpha) = i \quad \text{for} \quad \alpha \in \{\alpha_1, \alpha_2\},

and

\[
\psi(p, \alpha) = 1 \quad \text{for} \quad \alpha = \pm(\alpha_1 + \alpha_2), \quad 0 \leq p \leq 3.
\]

We extend the linear automorphism \( \nu \) of \( \mathfrak{h} \) to a linear automorphism, which we call \( \hat{\nu} \), of \( \mathfrak{g} \), as follows:

\[
\hat{\nu}_\alpha x = \psi(1, \alpha)x_{\nu\alpha}
\]

for \( \alpha \in \Delta \). Then

\[
\hat{\nu}^p x = \psi(p, \alpha)x_{\nu\alpha}
\]

for \( 0 \leq p \leq 3 \),

\[
\hat{\nu}^4 = 1 \quad \text{on} \quad \mathfrak{g},
\]

and \( \hat{\nu} \) preserves \([\cdot, \cdot]\) and \( \langle \cdot, \cdot \rangle \). We have

\[
\hat{\nu} x_{\alpha_1} = ix_{\alpha_2}, \quad \hat{\nu} x_{\alpha_2} = ix_{\alpha_1}, \quad \hat{\nu} x_{\alpha_1 + \alpha_2} = x_{\alpha_1 + \alpha_2},
\]

(3.47) \quad \hat{\nu} x_{-\alpha_1} = -ix_{-\alpha_2}, \quad \hat{\nu} x_{-\alpha_2} = -ix_{-\alpha_1}, \quad \hat{\nu} x_{-\alpha_1 - \alpha_2} = x_{-\alpha_1 - \alpha_2}

from (3.42)–(3.47).

For \( n \in \mathbb{Z} \), set

\[
\mathfrak{g}(n) = \{x \in \mathfrak{g} \mid \hat{\nu}(x) = i^n x\},
\]

the \( i^n \)-eigenspace for \( \hat{\nu} \). Form the \( \hat{\nu} \)-twisted affine Lie algebra associated to \( \mathfrak{g} \) and \( \hat{\nu} \),

\[
\mathfrak{g}[\hat{\nu}] = \coprod_{n \in \mathbb{Z}} \mathfrak{g}(n) \otimes t^n \oplus \mathbb{C} k = \coprod_{n \in \mathbb{Z}/4 \mathbb{Z}} \mathfrak{g}(4n) \otimes t^n \oplus \mathbb{C} k,
\]

where

\[
[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n, 0} k
\]

and

\[
[k, \mathfrak{g}[\hat{\nu}]] = 0
\]

for \( m, n \in \mathbb{Z}/4 \mathbb{Z} \) and \( x \in \mathfrak{g}(4n) \) and \( y \in \mathfrak{g}(4n) \) (cf. [K]). This is a copy of the twisted affine Lie algebra \( A^{(2)}_1 \) and it is \( \mathbb{Z}/4 \mathbb{Z} \)-graded. Set

\[
\mathfrak{g}[\hat{\nu}] = \mathfrak{g}[\hat{\nu}] \oplus \mathbb{C} d,
\]

where the action of \( d \) is given by \([d, x \otimes t^n] = nx \otimes t^n \) for \( x \in \mathfrak{g} \) and \( n \in (1/4) \mathbb{Z} \) and \([d, k] = 0\).

The next theorem gives the \( \mathfrak{g}[\hat{\nu}] \)-module structure of \( V^T_L \):

**Theorem 3.1.** (Theorem 9.1 of [L]; see also Theorem 3 of [FLM2]) The representation of \( \mathfrak{h}[\nu] \) on \( V^T_L \) extends uniquely to a Lie algebra representation of \( \mathfrak{g}[\hat{\nu}] \) on \( V^T_L \) such that

\[
(x_{\alpha}(4n)) \otimes t^n \mapsto x_{\alpha}(n)
\]

for all \( n \in \mathbb{Z}/4 \mathbb{Z} \) and \( \alpha \in L \). Moreover, \( V^T_L \) is an irreducible \( \mathfrak{g}[\hat{\nu}] \)-module.
Throughout the rest of this paper, for $x \in \mathfrak{g}$ and $n \in \frac{1}{4}\mathbb{Z}$ we will write $x^\nu(n)$ for the action of $x(4n) \otimes t^n \in \hat{\mathfrak{g}}[\hat{\nu}]$ on any $\hat{\mathfrak{g}}[\hat{\nu}]$-module. In particular, we have the operators $x_\alpha^\nu(n)$ for $\alpha \in L$. Also, sometimes we write $x^\nu(n)$ for the Lie algebra element $x(4n) \otimes t^n$, and it will be clear from the context whether $x^\nu(n)$ is an operator or a Lie algebra element.

The space $V^T_L$ is endowed with the tensor product weight grading. As we mentioned in the previous section, this grading is given by the action of $L^\nu(0)$. By (2.96) and (2.97) we have

$$L^\nu(0)1 = \frac{1}{16}1,$$

and we write

$$\text{wt } 1 = \frac{1}{16},$$

where by 1 we mean $1 \otimes 1 \in S[\nu] \otimes U_T$ (recall (3.32) and (3.33)), and by using (2.99) we have

$$L^\nu(0)v = \left(\text{wt } v + \frac{1}{16}\right)v$$

for a homogeneous element $v \in V^T_L$.

Taking $a = e_\alpha$ in (2.95) we obtain

$$[L^\nu(0), x_\alpha^\nu(n)] = \left( -n - 1 + \frac{1}{2}(\alpha, \alpha) \right)x_\alpha^\nu(n),$$

and thus

$$\text{wt } x_\alpha^\nu(n) = -n - 1 + \frac{1}{2}(\alpha, \alpha)$$

for any $n \in (1/4)\mathbb{Z}$.

The space $V^T_L$ has also a charge grading given by the eigenvalues of the operator $(\alpha_1 + \alpha_2)(0)$, and this grading is compatible with the weight grading. Thus $x_\alpha^\nu(n)$ has charge $\langle \alpha_1 + \alpha_2, \alpha \rangle$ for any $n \in (1/4)\mathbb{Z}$, where $x_\alpha^\nu(n)$ is viewed again as either an operator or as an element of $U(\hat{\alpha}[\hat{\nu}])$.

Now as a consequence of (2.92) and (2.93) (recall that in our specialized setting, $k = 4$ and $\eta = i$), we obtain the following linear relations among the operators $x_\alpha^\nu(m)$ for $\alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ and $m \in \frac{1}{4}\mathbb{Z}$:

**Lemma 3.2.** We have

$$x_\alpha^\nu(m) = 0 \text{ if } \alpha \in \{\alpha_1, \alpha_2\} \text{ and } m \in \frac{1}{2}\mathbb{Z},$$

(3.57)

$$x_{\alpha_2}^\nu(m) = x_{\alpha_1}^\nu(m) \text{ if } m \in \frac{1}{4} + \mathbb{Z},$$

(3.58)

$$x_{\alpha_2}^\nu(m) = -x_{\alpha_1}^\nu(m) \text{ if } m \in \frac{3}{4} + \mathbb{Z}$$

and

$$x_{\alpha_1 + \alpha_2}^\nu(m) = 0 \text{ if } m \in \frac{1}{4}\mathbb{Z} \setminus \mathbb{Z}.$$ (3.59)

**Proof.** We first show that (3.56) holds for $\alpha_1$. By taking $r = 2$ and $v = \iota(e_{\alpha_1})$ in (2.93) and by using (3.41) for $p = 2$ and $\alpha = \alpha_1$ we obtain

$$Y^\nu(-\iota(e_{\alpha_1}), x) = \lim_{x^{1/4} \to -x^{1/4}} Y^\nu(\iota(e_{\alpha_1}), x).$$
This immediately yields
\[- \sum_{m \in (1/4)\mathbb{Z}} \mathcal{X}^\nu_alpha_{alpha_1}(m)x^{-m-1} = \sum_{m \in (1/4)\mathbb{Z}} (-1)^{-4m}\mathcal{X}^\nu_alpha_{alpha_1}(m)x^{-m-1},\]
and thus (3.56) for \(alpha_1\). The proof of the same formula for \(alpha_2\) instead of \(alpha_1\) is completely analogous.

Next we use (2.92) for \(v = i(\epsilon_{alpha_1})\) together with (3.41) for \(p = 1\) and \(alpha = alpha_1\) and thus we get
\[iY^\nu_i(\epsilon_{alpha_2}, x) = \lim_{x^{1/4-\nu_1-1/4}} Y^\nu_i(\epsilon_{alpha_1}, x).\]

This implies
\[\sum_{m \in (1/4)\mathbb{Z}} ix^\nu_alpha_{alpha_2}(m)x^{-m-1} = \sum_{m \in (1/4)\mathbb{Z}} i^{-4m}x^\nu_alpha_{alpha_1}(m)x^{-m-1},\]
which gives (3.57) and (3.58).

By using (2.92) for \(v = i(\epsilon_{alpha_1+alpha_2})\) together with (3.41) for \(p = 1\) and \(alpha = alpha_1 + alpha_2\) we obtain
\[\sum_{m \in (1/4)\mathbb{Z}} x^\nu_alpha_{alpha_1+alpha_2}(m)x^{-m-1} = \sum_{m \in (1/4)\mathbb{Z}} i^{-4m}x^\nu_alpha_{alpha_1+alpha_2}(m)x^{-m-1},\]
which implies formula (3.59).

Note that by the above lemma and (4.38) we have
\[(3.60) \quad Y^\nu_i(\epsilon_{alpha}, x) = \sum_{m \in 1/4 + 1/2\mathbb{Z}} x^\nu_alpha_{alpha}(m)x^{-m-1} \text{ for } alpha \in \{alpha_1, alpha_2\}\]
and
\[(3.61) \quad Y^\nu_i(\epsilon_{alpha_1+alpha_2}, x) = \sum_{m \in \mathbb{Z}} x^\nu_alpha_{alpha_1+alpha_2}(m)x^{-m-1}.\]

By using the commutator formula (2.91) for twisted vertex operators we obtain the following brackets:

**Lemma 3.3.** For \(m, n \in 1/4 + 1/2\mathbb{Z}\) we have
\[(3.62) \quad [x^\nu_alpha_{alpha_1}(m), x^\nu_alpha_{alpha_2}(n)] = \frac{1}{2} x^\nu_alpha_{alpha_1+alpha_2}(m + n),\]
\[(3.63) \quad [x^\nu_alpha_{alpha_1}(m), x^\nu_alpha_{alpha_1}(n)] = -\frac{i}{4}(i^{-4m} - (-i)^{-4m})x^\nu_alpha_{alpha_1+alpha_2}(m + n),\]
\[(3.64) \quad [x^\nu_alpha_{alpha_2}(m), x^\nu_alpha_{alpha_2}(n)] = \frac{i}{4}(i^{-4m} - (-i)^{-4m})x^\nu_alpha_{alpha_1+alpha_2}(m + n),\]
and for \(m \in \mathbb{Z}, n \in 1/4\mathbb{Z}\) and \(alpha \in \{alpha_1, alpha_2, alpha_1 + alpha_2\}\) we have
\[(3.65) \quad [x^\nu_alpha_{alpha_1+alpha_2}(m), x^\nu_alpha_{alpha_2}(n)] = 0.\]

For \(n \in \mathbb{Z}\) we shall identify \(\mathfrak{g}(n \text{ mod } 4)\) with \(\mathfrak{g}(n)\), the \(n^\nu\)-eigenspace of \(nu\) in \(\mathfrak{g}\) (recall (3.49)). By (3.2) and (3.45) we have
\[(3.66) \quad \mathfrak{g}(0) = \mathbb{C}x_{alpha_1+alpha_2} \oplus \mathbb{C}(alpha_1 + alpha_2) \oplus \mathbb{C}x_{-alpha_1-alpha_2} = \mathfrak{g}(4m),\]
\[(3.67) \quad \mathfrak{g}(1) = \mathbb{C}(x_{alpha_1 + alpha_2}) \oplus \mathbb{C}(x_{-alpha_1 - x_{-alpha_2}}) = \mathfrak{g}(4m+1),\]
\[(3.68) \quad \mathfrak{g}(2) = \mathbb{C}(alpha_1 - alpha_2) = \mathfrak{g}(4m+2),\]
\[(3.69) \quad \mathfrak{g}(3) = \mathbb{C}(x_{alpha_1 - alpha_2}) \oplus \mathbb{C}(x_{-alpha_1 + x_{-alpha_2}}) = \mathfrak{g}(4m+3)\]
for any \( m \in \mathbb{Z} \), and
\[
g = \prod_{p \in \mathbb{Z}/4\mathbb{Z}} g_{(p)}.\]

Note that the twisted affine Lie algebra \((3.50)\) corresponding to \( g \) and to the automorphism \( \hat{\nu} \) of \( g \) decomposes as
\[
\hat{g}[\hat{\nu}] = g(0) \otimes \mathbb{C}[t, t^{-1}] \oplus g(1) \otimes \mathbb{C}[t, t^{-1}]^{1/4} \oplus g(2) \otimes \mathbb{C}[t, t^{-1}]^{1/2} \oplus g(3) \otimes \mathbb{C}[t, t^{-1}]^{3/4} \oplus \mathbb{C}k.
\]

For any \( \hat{\nu} \)-stable Lie subalgebra \( u \) of \( g \), we shall write \( \hat{u}[\hat{\nu}] \) for the correspondingly defined twisted affine Lie subalgebra of \( \hat{g}[\hat{\nu}] \). In particular, we may write \( \hat{h}[\hat{\nu}] \) (recall \((3.26)\)) as \( \hat{h}[\hat{\nu}] \). Take the \( \hat{\nu} \)-stable Lie subalgebra \((3.70)\)
\[
n = \mathbb{C}x_{\alpha_1} \oplus \mathbb{C}x_{\alpha_2} \oplus \mathbb{C}x_{\alpha_1 + \alpha_2}
\]
of \( g \), and consider the twisted affinization of \( n \),
\[
\hat{n}[\hat{\nu}] = \prod_{r \in \mathbb{Z}} n_{(r)} \otimes t^r \oplus \mathbb{C}k = \prod_{r \in \mathbb{Z}/4\mathbb{Z}} n_{(4r)} \otimes t^r \oplus \mathbb{C}k \subset \hat{g}[\hat{\nu}],
\]
where for \( r \in \mathbb{Z} \), \( n_{(r)} \) is the \( t^r \)-eigenspace of \( n \) for \( \hat{\nu} \) (as in \((3.49)\)). As in \([\text{CalLM1}] \)–\([\text{CalLM3}]\), we drop the 1-dimensional space \( \mathbb{C}k \) in \((3.71)\) and use instead the subalgebras
\[
\hat{n}[\hat{\nu}] = \prod_{r \in \mathbb{Z}} n_{(r)} \otimes t^r = \prod_{r \in \mathbb{Z}/4\mathbb{Z}} n_{(4r)} \otimes t^r
\]
and
\[
\hat{n}[\hat{\nu}] = \prod_{r \geq 0} n_{(r)} \otimes t^r = \prod_{r \in \mathbb{Z}/4\mathbb{Z}, r \geq 0} n_{(4r)} \otimes t^r
\]
and
\[
\hat{n}[\hat{\nu}] = \prod_{r < 0} n_{(r)} \otimes t^r = \prod_{r \in \mathbb{Z}/4\mathbb{Z}, r < 0} n_{(4r)} \otimes t^r
\]
of \( \hat{n}[\hat{\nu}] \) (note that the form \( \langle \cdot, \cdot \rangle \) vanishes on \( n \)).

4. Shifted operators

In this section we shift the \( \hat{\nu} \)-twisted vertex operators \((2.83)\) and \((2.89)\) by using an element of \( h(0) \) that lies in the rational span of the lattice \( L \), and we recall a shifted twisted vertex operator construction of \( A_2^{(2)} \). We follow Section 10 of \([\text{L1}]\).

Fix
\[
\gamma \in h(0).
\]
Let \( a, b \in \hat{L}_\nu, u \in T \) and \( h \in h(0) \). Define a \( \gamma \)-shifted action of \( h(0) \) on \( U_T \) by
\[
h^\gamma \cdot b \otimes u = \langle h, \bar{b} + \gamma \rangle b \otimes u
\]
and the \( \text{End} U_T \)-valued formal Laurent series \( x^{h^\gamma} \) by
\[
x^{h^\gamma} \cdot b \otimes u = x^{\langle h, \bar{b} + \gamma \rangle} b \otimes u.
\]
Then we have
\[
x^{h^\gamma} a = ax^{\langle h, \bar{a} + \gamma \rangle + h} \quad \text{for} \quad h \in h(0).
\]
Recall \((2.69), (2.71)\) and \((2.73)\).
Now \[ U_T = \prod_{\alpha \in P_0 L} U_\alpha, \]
where
\[ U_\alpha = \{ u \in U_T \mid h^\gamma \cdot u = (h, \alpha + \gamma)u \text{ for } h \in \mathfrak{h}(0) \}. \]
The space \( U_T \), and hence \( V^T_L \), has a new \( \mathfrak{h}[\nu] \)-structure, with \( \mathfrak{h}[\nu] \) acting trivially and \( \mathfrak{h}(0) \) as in (4.1). Continuing to follow \( \text{[L1]} \) we define the \( \gamma \)-shifted \( \hat{\nu} \)-twisted vertex operator on \( V^T_L \) for \( a \in \hat{L} \), as follows:
\[
(4.4) \quad Y^\hat{\nu},\gamma(\iota(a), x) = k^{-\pi^2/2}\sigma(\alpha)E^\gamma(-\bar{\alpha}, x)E^\gamma(-\bar{\alpha}, x)ax^\gamma(0, \bar{\pi}(0) + \bar{\pi}(0)/2 - \bar{\pi}/2).
\]
Note that
\[
(4.5) \quad Y^\hat{\nu},\gamma(\iota(a), x) = Y^\hat{\nu}(\iota(a), x)x^{\bar{\pi}(0), \gamma}.
\]
Define the component operators \( x^\hat{\nu},\gamma(\alpha, x) \) for \( \alpha \in L \) and \( \gamma \in \mathbb{Z} \) by
\[
(4.6) \quad Y^\hat{\nu},\gamma(\iota(\epsilon_\alpha), x) = \sum_{n \in \mathbb{Z}} x^\hat{\nu},\gamma(\alpha, x)x^{-n-(\alpha, \alpha)/2}.
\]
Then for \( n \in \frac{1}{k} \mathbb{Z} \) we have
\[
(4.7) \quad x^\hat{\nu},\gamma(\alpha, n) = x^\hat{\nu}(\alpha + (\alpha(0), \gamma)),
\]
as operators on \( V^T_L \).

For \( v = \alpha_1(-n_1) \cdots \alpha_m(-n_m) \cdot \iota(a) \in V_L \) with \( \alpha_1, \ldots, \alpha_m \in \mathfrak{h}, n_1, \ldots, n_m > 0 \) and \( a \in \hat{L} \), one can define the shifted version of the \( \hat{\nu} \)-twisted vertex operators (2.89):
\[
(4.8) \quad Y^\hat{\nu},\gamma(v, x) = W^\gamma(e^{dx}v, x),
\]
where
\[
(4.9) \quad W^\gamma(v, x) = \circ \left( \frac{1}{(n_1 - 1)!} \frac{d}{dx} \right)^{n_1 - 1} \hat{\alpha}_1(x) \cdots \left( \frac{1}{(n_m - 1)!} \frac{d}{dx} \right)^{n_m - 1} \hat{\alpha}_m(x) Y^\hat{\nu},\gamma(\iota(a), x) \circ.
\]

**Remark 4.1.** The operators (4.4) correspond to the shifted operators \( Y^\gamma(a, \zeta) \) in \( \text{[L1]} \). By replacing \( \zeta \) by \( x^{-\frac{\pi^2}{2k}} \) in \( Y^\gamma(a, \zeta) \) and then by multiplying by \( x^{-\frac{\bar{\pi}x}{2k}} \) one obtains the \( \gamma \)-shifted \( \hat{\nu} \)-twisted vertex operator \( Y^\hat{\nu},\gamma(\iota(a), x) \). Here we identify \( a \) with \( \iota(a) \).

Now suppose that \( \gamma \) lies in the rational span of \( L \). Choose \( \bar{k} \in \mathbb{Z} \) such that
\[
(4.10) \quad \bar{k}(\alpha, \gamma) \in \frac{1}{k} \mathbb{Z}
\]
for \( \alpha \in L \).

For \( a \in L \) we define the operator \( Y^\hat{\nu},\gamma,\bar{k}(\iota(a), x) \) as follows:
\[
(4.11) \quad Y^\hat{\nu},\gamma,\bar{k}(\iota(a), x) = Y^\hat{\nu},\gamma(\iota(a), x^{\bar{k}}).
\]
For \( \alpha \in L \) and \( n \in \frac{1}{kk} \mathbb{Z} \) define the operators \( x^\hat{\nu},\gamma,\bar{k}(\alpha, n) \) by
\[
(4.12) \quad Y^\hat{\nu},\gamma,\bar{k}(\iota(\epsilon_\alpha), x) = \sum_{n \in \frac{1}{kk} \mathbb{Z}} x^\hat{\nu},\gamma,\bar{k}(\alpha, n)x^{-n-(\alpha, \alpha)/21}.
\]
For the rest of this section we specialize $L$ to the root lattice of $\mathfrak{sl}(3, \mathbb{C})$ and $\nu$ as in (3.2). From the previous section, $k = 4$ and $\eta = i$. Let $\eta_\gamma \in \mathbb{C}$ be a primitive $4k$th root of unity such that

$$\eta_\gamma^k = i. \quad (4.13)$$

Define

$$\psi_\gamma : \mathbb{Z}/4k\mathbb{Z} \times L \to \langle \eta_\gamma \rangle \quad (4.14)$$

by

$$\psi_\gamma(p, \alpha) = \eta_\gamma^{-4kp(\alpha, \gamma)} \psi(p, \alpha) \quad (4.15)$$

(recall (3.40)). Denote by $\hat{\nu}_\gamma$ the shifted version of (3.45):

$$\hat{\nu}_\gamma h = \nu h \quad \text{for} \quad h \in \mathfrak{h} \quad (4.16)$$

and

$$\hat{\nu}_\gamma x_\alpha = \eta_\gamma^{-4k(\alpha, \gamma)} \hat{\nu} x_\alpha \quad \text{for} \quad \alpha \in L, \quad (4.17)$$

where $\hat{\nu}$ is as in (3.45). Then $\hat{\nu}_\gamma$ is a Lie algebra automorphism of $\mathfrak{g}$ such that

$$\hat{\nu}_\gamma x_\alpha = \psi_\gamma(1, \alpha)x_\nu x_\alpha \quad \text{for} \quad \alpha \in \Delta \quad (4.18)$$

and

$$\hat{\nu}_\gamma^{4k} = 1 \quad \text{on} \quad \mathfrak{g}. \quad (4.19)$$

For $n \in \mathbb{Z}$ set

$$\mathfrak{h}(n) = \{ h \in \mathfrak{h} \mid \hat{\nu}_\gamma h = \eta_\gamma^n h \}. \quad (4.20)$$

Consider the $\hat{\nu}_\gamma$-twisted affine Lie algebra associated with $\mathfrak{h}$ and $\hat{\nu}_\gamma$:

$$\hat{\mathfrak{h}}[\hat{\nu}_\gamma] = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h}(n) \otimes t^{\frac{n}{4k}} \mathbb{C}k = \bigoplus_{n \in \frac{1}{4k}\mathbb{Z}} \mathfrak{h}(4kn) \otimes t^n \mathbb{C}k \quad (4.21)$$

with the usual brackets. Since the Lie algebras $\hat{\mathfrak{h}}[\hat{\nu}_\gamma]$ and $\mathfrak{h}[\nu]$ are isomorphic, $V_L^T$ is an $\hat{\mathfrak{h}}[\hat{\nu}_\gamma]$-module. Now consider $\hat{\mathfrak{g}}[\hat{\nu}_\gamma]$, the $\hat{\nu}_\gamma$-twisted affine Lie algebra associated to $\mathfrak{g}$ and $\hat{\nu}_\gamma$:

$$\hat{\mathfrak{g}}[\hat{\nu}_\gamma] = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(n) \otimes t^{\frac{n}{4k}} \mathbb{C}k = \bigoplus_{n \in \frac{1}{4k}\mathbb{Z}} \mathfrak{g}(4kn) \otimes t^n \mathbb{C}k, \quad (4.22)$$

where

$$\mathfrak{g}(n) = \{ x \in \mathfrak{g} \mid \hat{\nu}_\gamma x = \eta_\gamma^n x \} \quad (4.23)$$

for $n \in \mathbb{Z}$, with the corresponding brackets. This is a copy of $A_2^{(2)}$. Recall (4.12), the component operators of the vertex operator $Y^{\hat{\nu}_\gamma, \gamma, k}(e_\alpha, x)$. The analogue of Theorem 3.1 holds:

**Theorem 4.2.** (Theorem 10.1 of [L1]) The representation of $\hat{\mathfrak{h}}[\nu_\gamma]$ on $V_L^T$ extends uniquely to a Lie algebra representation of $\hat{\mathfrak{g}}[\hat{\nu}_\gamma]$ on $V_L^T$ such that

$$(x_\alpha(4kn) \otimes t^n) \mapsto \hat{x}_\alpha^{\hat{\nu}_\gamma, k}(n)$$

for all $n \in \frac{1}{4k}\mathbb{Z}$ and $\alpha \in L$, and this representation is irreducible.
Now assume that \( \tilde{k} = 1 \) and \( \gamma \) is an element of \( \mathfrak{h}(0) \) that lies in the rational span of \( L \) and such that it satisfies (4.10). From now on we will denote by \( V^T_L \) the \( \hat{\mathfrak{g}}[\hat{\nu}] \)-module \( V^T_L \). It was proved in [LiII] (see also Proposition 2.14 in [DLM]) that \( (V^T_L, Y^\gamma) \) is a \( \sigma_{\tilde{y}} \)-twisted module for \( V^T_L \), where
\[
\sigma_{\gamma} = e^{-2\pi i \gamma},
\]
an automorphism of \( V_L \), and the vertex operator \( Y^\gamma \) is as in (4.4) and (4.8). Recall the component operators \( x^\gamma_\alpha(n) \) for \( \alpha \in L, n \in \frac{1}{k} \mathbb{Z}, \) and (4.4).

Similarly, \( (V^\gamma_L, Y^\gamma) \) is a \( \sigma_{\gamma} \)-twisted module for \( V_L \), where \( V^\gamma_L \) is the vector space \( V_L \), and for \( a \in \hat{L} \),
\[
Y^\gamma(a(x), x) = e^{-(\bar{\alpha}, x)} ax^\alpha \gamma,
\]
and, more generally, for \( v = \alpha_1(-n_1) \cdots \alpha_m(-n_m) \in V_L \) with \( \alpha_1, \ldots, \alpha_m \in \mathfrak{h}, n_1, \ldots, n_m > 0 \) and \( a \in L \), we set
\[
Y^\gamma(a, x) = x^\alpha_1(n_1) \cdots x^\alpha_m(n_m) x^{-\langle(\alpha, \alpha)\rangle \gamma/2}.
\]
Define the component operators \( x^\gamma_\alpha(n) \) for \( \alpha \in L \) and \( n \in (1/k) \mathbb{Z} \):
\[
Y^\gamma(a, x) = \sum_{n \in (1/k) \mathbb{Z}} x^\gamma_\alpha(n)x^{-\langle(\alpha, \alpha)\rangle \gamma/2}.
\]
Note that
\[
Y^\gamma(a, x) = Y(a, x)x^{\langle(\alpha, \alpha)\rangle \gamma}.
\]
By Proposition 2.15 in [DLM] we have that the \( \sigma_{\gamma} \)-twisted \( V_L \) module \( (V^\gamma_L, Y^\gamma) \) is naturally isomorphic to the \( \sigma_{\gamma} \)-twisted \( V_L \) module \( V_{L+\gamma} \).

5. Principal subspaces of standard \( A_2^{(2)} \)-modules

Recall the twisted affine Lie algebras \( \hat{\mathfrak{g}}[\hat{\nu}], \hat{\mathfrak{g}}[\tilde{\nu}] \) and their subalgebras \( \bar{\mathfrak{g}}[\hat{\nu}], \bar{\mathfrak{g}}[\tilde{\nu}] \) and \( \bar{\mathfrak{g}}[\tilde{\nu}]_- \) (see (3.50)–(3.51) and (3.7)–(3.74)). The notion of principal subspace of a highest weight module for an affine Lie algebra was introduced in [CalLM2] as a straightforward generalization of the principal subspace of a standard module for an untwisted affine Lie algebra of type \( A \) as in [FS1], [FS2].

**Definition 5.1.** For any standard \( \hat{\mathfrak{g}}[\hat{\nu}] \)-module \( V \) we define its principal subspace \( W \) to be
\[
W = U(\bar{\mathfrak{n}}[\hat{\nu}]) \cdot v,
\]
where \( v \) is a highest weight vector of \( V \).

**Remark 5.2.** This definition can be used to define principal subspaces for an arbitrary twisted affine Lie algebra \( \hat{\mathfrak{g}}[\hat{\nu}] \), whenever the automorphism playing the role of \( \hat{\nu} \) preserves the subalgebra generalizing \( \mathfrak{n} \), and not only \( A_2^{(2)} \). If necessary, we can also relax the condition that \( v \) be a highest weight vector and instead allow \( v \) to be a more general vector (see [CalLM3]).

In particular, we have the principal subspace, denoted by \( W^T_L \), of \( V^T_L \),
\[
W^T_L = U(\bar{\mathfrak{n}}[\hat{\nu}]) \cdot v_\Lambda,
\]
where
\[
\Lambda \in (\mathfrak{h}(0) \oplus \mathbb{C}k \oplus \mathbb{C}d)^* \quad \text{is the fundamental weight of } \hat{\mathfrak{g}}[\hat{\nu}] \quad \text{defined by } \langle \Lambda, k \rangle = 1, \langle \Lambda, h(0) \rangle = 0 \quad \text{and } \langle \Lambda, d \rangle = 0, \quad \text{and } v_\Lambda \text{ is a highest weight vector of } V^T_L.
\]
Then
\[
W^T_L = U(\bar{\mathfrak{n}}[\hat{\nu}]) \cdot v_\Lambda.
\]
We shall take our highest weight vector $v_\Lambda$ of $V_L^T$ to be
\begin{equation}
(5.3) \quad v_\Lambda = 1 \in V_L^T,
\end{equation}
where by 1 we mean $1 \otimes 1 \in S[\nu] \otimes U_T$ (recall (3.32) and (3.33)).

Consider the surjective map
\begin{equation}
(5.4) \quad F_\Lambda : U(\hat{\mathfrak{g}}[\hat{\nu}]) \rightarrow V_L^T
\end{equation}
and denote by $f_\Lambda$ the restriction of $F_\Lambda$ to $U(\hat{\mathfrak{g}}[\hat{\nu}])$:
\begin{equation}
(5.5) \quad f_\Lambda : U(\hat{\mathfrak{g}}[\hat{\nu}]) \rightarrow W_L^T
\end{equation}
\[a \mapsto a \cdot v_\Lambda.\]

One of the main goals of this paper is to give a precise description of the kernel $\text{Ker} f_\Lambda$, and thus a presentation of the principal subspace $W_L^T$.

As in [CalLM1–CalLM3] we will use principal subspaces of generalized Verma modules. Set
\begin{equation}
(5.6) \quad N_L^T = U(\hat{\mathfrak{g}}[\hat{\nu}]) \otimes_{U(\hat{\mathfrak{g}}[\hat{\nu}]) \geq 0} \mathbb{C} v_N^N,
\end{equation}
where
\[
\hat{\mathfrak{g}}([\hat{\nu}])_{\geq 0} = \left( \prod_{n \geq 0} \mathfrak{g}(n) \otimes t^{n/4} \right) \oplus \mathbb{C} k.
\]
Define the principal subspace of the generalized Verma module $N_L^T$,
\begin{equation}
(5.7) \quad W_L^{T,N} = U(\hat{\mathfrak{g}}[\hat{\nu}]) \cdot v_N^N \subset N_L^T.
\end{equation}

We have the following surjective maps:
\begin{equation}
F_\Lambda^N : U(\hat{\mathfrak{g}}[\hat{\nu}]) \rightarrow N_L^T
\end{equation}
\[a \mapsto a \cdot v_\Lambda^N,\]
\begin{equation}
(5.5) \quad f_\Lambda^N : U(\hat{\mathfrak{g}}[\hat{\nu}]) \rightarrow W_L^{T,N}
\end{equation}
\[a \mapsto a \cdot v_\Lambda^N\]
and
\begin{equation}
\Pi_\Lambda : N_L^T \rightarrow V_L^T
\end{equation}
\[a \cdot v_\Lambda^N \mapsto a \cdot v_\Lambda,\]
\begin{equation}
\pi_\Lambda : W_L^{T,N} \rightarrow W_L^T
\end{equation}
\[a \cdot v_\Lambda^N \mapsto a \cdot v_\Lambda,\]

**Theorem 5.3.** On the standard $\hat{\mathfrak{g}}[\hat{\nu}]$-module $V_L^T$ we have:
\begin{equation}
(5.8) \quad \lim_{x_1^{1/4} \rightarrow x_1^{1/4}} (x_1^{1/2} + x_2^{1/2}) Y^{\hat{\nu}}(\iota(e_{\alpha_j}), x_1) Y^{\hat{\nu}}(\iota(e_{\alpha_j}), x_2) = 0 \quad \text{for } j = 1, 2,
\end{equation}
\begin{equation}
(5.9) \quad \lim_{x_1^{1/4} \rightarrow x_1^{1/4}} (x_1^{1/2} - x_2^{1/2}) Y^{\hat{\nu}}(\iota(e_{\alpha_1}), x_1) Y^{\hat{\nu}}(\iota(e_{\alpha_2}), x_2) = 0,
\end{equation}
\begin{equation}
(5.10) \quad Y^{\hat{\nu}}(\iota(e_{\alpha_1 + \alpha_2}), x)^2 = 0
\end{equation}
and
\begin{equation}
(5.11) \quad Y^{\hat{\nu}}(\iota(e_{\alpha_j}), x) Y^{\hat{\nu}}(\iota(e_{\alpha_1 + \alpha_2}), x) = 0 \quad \text{for } j = 1, 2.
\end{equation}
Proof. This follows by the use of \((2.81)\). \(\square\)

By analogy with the corresponding constructions in \([\text{CalLM1}, \text{CalLM3}]\), we introduce the following formal infinite sums indexed by \(t \in \frac{1}{4}\mathbb{Z}\):

\[
R_{j;t} = \sum_{n_1, n_2 \in (1/4) + (1/2)\mathbb{Z}} \left( x^{\hat{\nu}}_{\alpha_j} \left( n_1 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_j} (n_2) + x^{\hat{\nu}}_{\alpha_j} (n_1)x^{\hat{\nu}}_{\alpha_j} \left( n_2 + \frac{1}{2} \right) \right) \text{ for } j = 1, 2,
\]

\[
R_{1;2;t} = \sum_{n_1, n_2 \in (1/4) + (1/2)\mathbb{Z}} \left( x^{\hat{\nu}}_{\alpha_1} \left( n_1 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_2} (n_2) - x^{\hat{\nu}}_{\alpha_2} (n_1)x^{\hat{\nu}}_{\alpha_1} \left( n_2 + \frac{1}{2} \right) \right),
\]

\[
R_{2;1;t} = \sum_{n_1, n_2 \in (1/4) + (1/2)\mathbb{Z}} \left( x^{\hat{\nu}}_{\alpha_2} \left( n_1 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_1} (n_2) - x^{\hat{\nu}}_{\alpha_1} (n_1)x^{\hat{\nu}}_{\alpha_2} \left( n_2 + \frac{1}{2} \right) \right),
\]

and

\[
R_{1,2;j;t} = \sum_{m, n \in (1/4) + (1/2)\mathbb{Z}} x^{\hat{\nu}}_{\alpha_1 + \alpha_2} (m)x^{\hat{\nu}}_{\alpha_j} (n) \text{ for } j = 1, 2.
\]

**Corollary 5.4.** For any \(t \in (1/4)\mathbb{Z}\), \(R_{j;t}, R_{1;2;t}, R_{2;1;t}, R_{1,2;1;t}\) and \(R_{1,2;2;t}\) applied to any vector of \(V^T_L\) give finite sums that equal zero.

**Remark 5.5.** Note that by Lemma \([3.2]\) we have \(R_{2;t} = \pm R_{1;t}, R_{1;2;t} = \pm R_{1;1;t}, R_{2;1;t} = \pm R_{1;1;t}\) and \(R_{1,2;2;t} = \pm R_{1,2;1;t}\).

**Remark 5.6.** We can write \((5.12)\) as follows:

\[
R_{j;t} = \sum_{n_1, n_2 \leq -1/4} \left( x^{\hat{\nu}}_{\alpha_j} \left( n_1 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_j} (n_2) + x^{\hat{\nu}}_{\alpha_j} (n_1)x^{\hat{\nu}}_{\alpha_j} \left( n_2 + \frac{1}{2} \right) \right) + a,
\]

where \(a \in U(\bar{n}[\hat{\nu}])\bar{n}[\hat{\nu}]_+\). Indeed, by \((3.63)\) in Lemma \([3.3]\) we have

\[
x^{\hat{\nu}}_{\alpha_1} \left( n_1 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_1} (n_2) = x^{\hat{\nu}}_{\alpha_1} (n_2)x^{\hat{\nu}}_{\alpha_1} \left( n_1 + \frac{1}{2} \right) + \frac{i}{4}(i^{-4n_1} - (-i)^{-4n_1})x^{\hat{\nu}}_{\alpha_1 + \alpha_2} \left( n_1 + n_2 + \frac{1}{2} \right)
\]

and

\[
x^{\hat{\nu}}_{\alpha_1} (n_1)x^{\hat{\nu}}_{\alpha_1} \left( n_2 + \frac{1}{2} \right) = x^{\hat{\nu}}_{\alpha_1} \left( n_2 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_1} (n_1) - \frac{i}{4}(i^{-4n_1} - (-i)^{-4n_1})x^{\hat{\nu}}_{\alpha_1 + \alpha_2} \left( n_1 + n_2 + \frac{1}{2} \right)
\]

for \(n_1, n_2 \in \frac{1}{4}\mathbb{Z}\), and thus

\[
x^{\hat{\nu}}_{\alpha_1} \left( n_1 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_1} (n_2) + x^{\hat{\nu}}_{\alpha_1} (n_1)x^{\hat{\nu}}_{\alpha_1} \left( n_2 + \frac{1}{2} \right) = x^{\hat{\nu}}_{\alpha_1} (n_2)x^{\hat{\nu}}_{\alpha_1} \left( n_1 + \frac{1}{2} \right) + x^{\hat{\nu}}_{\alpha_1} \left( n_2 + \frac{1}{2} \right) x^{\hat{\nu}}_{\alpha_1} (n_1),
\]

which belongs to \(U(\bar{n}[\hat{\nu}])\bar{n}[\hat{\nu}]_+\) for \(n_1 \geq 1/4\) or \(n_2 \geq 1/4\). This proves \((5.17)\) for \(j = 1\). Similarly one can show \((5.17)\) with \(j = 2\).
We obtain analogous results for \((5.13)\) and \((5.14)\).
We now truncate each of the formal sums above, as follows:
\begin{align}
R^0_{j; t} &= \sum_{n_1, n_2 \in (1/4) + (1/2)\mathbb{Z}} \left( x^\nu_{\alpha_j} \left( n_1 + \frac{1}{2} \right) x^\nu_{\alpha_j} (n_2) + x^\nu_{\alpha_j} (n_1) x^\nu_{\alpha_j} \left( n_2 + \frac{1}{2} \right) \right) \quad \text{for } j = 1, 2,
\end{align}
\begin{align}
R^0_{1; 2; t} &= \sum_{n_1, n_2 \in (1/4) + (1/2)\mathbb{Z}} \left( x^\nu_{\alpha_1} \left( n_1 + \frac{1}{2} \right) x^\nu_{\alpha_2} (n_2) - x^\nu_{\alpha_1} (n_1) x^\nu_{\alpha_2} \left( n_2 + \frac{1}{2} \right) \right),
\end{align}
\begin{align}
R^0_{2; 1; t} &= \sum_{n_1, n_2 \in (1/4) + (1/2)\mathbb{Z}} \left( x^\nu_{\alpha_2} \left( n_1 + \frac{1}{2} \right) x^\nu_{\alpha_1} (n_2) - x^\nu_{\alpha_2} (n_1) x^\nu_{\alpha_1} \left( n_2 + \frac{1}{2} \right) \right),
\end{align}
\begin{align}
R^0_{1; 2; t} &= \sum_{m_1, m_2 \in \mathbb{Z}} x^\nu_{\alpha_1 + \alpha_2} (m_1) x^\nu_{\alpha_1 + \alpha_2} (m_2)
\end{align}
and
\begin{align}
R^0_{1; 2; j; t} &= \sum_{m \in \mathbb{Z}, n \in (1/4) + (1/2)\mathbb{Z}} x^\nu_{\alpha_1 + \alpha_2} (m) x^\nu_{\alpha_j} (n) \quad \text{for } j = 1, 2.
\end{align}

We will often view \((5.18), (5.19), (5.20), (5.21)\) and \((5.22)\) as elements of \(U(\hat{\nu}[\hat{\nu}])\) rather than as endmorphisms of a \(\hat{g}[\hat{\nu}]\)-module. We have
\begin{align}
R^0_{2; t} &= \pm R^0_{1; t}, \\
R^0_{2; 1; t} &= -R^0_{1; 2; t} = \pm R^0_{1; t}, \\
R^0_{1; 2; 1; t} &= \pm R^0_{1; 2; 1; t}.
\end{align}
Set
\begin{align}
J = \sum_{t \geq 1/2} U(\hat{\nu}[\hat{\nu}]) R^0_{1; t} + \sum_{t \geq 2} U(\hat{\nu}[\hat{\nu}]) R^0_{1; 2; t} + \sum_{t \geq 5/4} U(\hat{\nu}[\hat{\nu}]) R^0_{1; 2; 1; t},
\end{align}
the left ideal of \(U(\hat{\nu}[\hat{\nu}])\) generated by the elements \((5.18), (5.21)\) and \((5.22)\) for \(j = 1\). Denote by \(I_\Lambda\) the left ideal
\begin{align}
I_\Lambda = J + U(\hat{\nu}[\hat{\nu}]) \hat{\nu}[\hat{\nu}]
\end{align}
of \(U(\hat{\nu}[\hat{\nu}])\). Recall the surjective map \((5.5)\). We will prove that the kernel of \(f_\Lambda\) is \(I_\Lambda\).

**Remark 5.7.** We have
\[ L^\circ(0) \ker f_\Lambda \subset \ker f_\Lambda \]
and
\[ \text{wt } R^0_{j; t} = \text{wt } R^0_{1; 2; t} = \text{wt } R^0_{1; 2; j; t} = t \]
for \(j = 1, 2\) and \(t \in (1/4)\mathbb{Z}\) (recall \((5.5)\)). Note that \(R^0_{j; t}\) has charge 2, \(R^0_{1; 2; t}\) has charge 4 and \(R^0_{1; 2; j; t}\) has charge 3 for \(j = 1, 2, t \in (1/4)\mathbb{Z}\). The space \(\ker f_\Lambda\) is also graded by charge. Hence \(\ker f_\Lambda\) and \(I_\Lambda\) are graded by both weight and charge and these two gradings are compatible.
6. Certain morphisms

Let
\[ \gamma = \frac{1}{2}(\alpha_1 + \alpha_2) = \alpha_{1(0)} \in \mathfrak{h}_{(0)}. \]

Denote by \( \theta(\cdot) \) the character of the root lattice \( L \),

\[ \theta : L \to \mathbb{C}^\times, \]

such that
\[ \theta(\alpha_1) = -i, \quad \theta(\alpha_2) = i. \]

Define the following map \( \tau_{\gamma, \theta} \) on \( \bar{n}[\nu] \):

\[ \tau_{\gamma, \theta} : \bar{n}[\nu] \to \bar{n}[\nu] \tag{6.2} \]

\[ x_\alpha^\nu(m) \mapsto \theta(\alpha)x_\alpha^\nu(m + \langle \alpha, \gamma \rangle). \tag{6.3} \]

This is a Lie algebra automorphism (see Lemma 3.3), and it extends to an automorphism of \( U(\bar{n}[\nu]) \), which we also denote by \( \tau_{\gamma, \theta} \):

\[ \tau_{\gamma, \theta} : U(\bar{n}[\nu]) \to U(\bar{n}[\nu]). \tag{6.4} \]

We have

\[ \tau_{\gamma, \theta}(x_{\alpha_1+\alpha_2}^\nu(m_1) \cdots x_{\alpha_1+\alpha_2}^\nu(m_r)x_{\alpha_1}^\nu(n_1) \cdots x_{\alpha_1}^\nu(n_s)) \]

\[ = \theta(\alpha_1)^s x_{\alpha_1+\alpha_2}^\nu(m_1 + 1) \cdots x_{\alpha_1+\alpha_2}^\nu(m_r + 1)x_{\alpha_1}^\nu(n_1 + 1/2) \cdots x_{\alpha_1}^\nu(n_s + 1/2) \]

\[ = (-i)^s x_{\alpha_1+\alpha_2}^\nu(m_1 + 1) \cdots x_{\alpha_1+\alpha_2}^\nu(m_r + 1)x_{\alpha_1}^\nu(n_1 + 1/2) \cdots x_{\alpha_1}^\nu(n_s + 1/2) \]

for \( m_1, \ldots, m_r \in \mathbb{Z} \) and \( n_1, \ldots, n_s \in \frac{1}{k}\mathbb{Z} \). Note that

\[ \tau_{\gamma, \theta}^{-1} = \tau_{-\gamma, \theta}. \tag{6.5} \]

**Lemma 6.1.** We have

\[ \tau_{\gamma, \theta} \left( I_\Lambda + U(\bar{n}[\nu])x_{\alpha_1}^\nu \left( -\frac{1}{4} \right) \right) = I_\Lambda. \tag{6.6} \]

**Proof.** We shall first show that

\[ \tau_{\gamma, \theta} \left( I_\Lambda + U(\bar{n}[\nu])x_{\alpha_1}^\nu \left( -\frac{1}{4} \right) \right) \subset I_\Lambda. \tag{6.7} \]

Note that

\[ \tau_{\gamma, \theta} \left( U(\bar{n}[\nu])\bar{n}[\nu]^+_1 + U(\bar{n}[\nu])x_{\alpha_1}^\nu \left( -\frac{1}{4} \right) \right) \subset U(\bar{n}[\nu])\bar{n}[\nu]^+_1. \tag{6.8} \]

We also have

\[ \tau_{\gamma, \theta}(R_{1,1}^0) = \theta(\alpha_1)^2 R_{1,1}^0, \quad \tau_{\gamma, \theta}(R_{1,1}^1) = \theta(\alpha_1)^2 R_{1,1}^1 \]

where \( a \in U(\bar{n}[\nu])\bar{n}[\nu]^+_1 \), \( t \geq 3/2 \),

\[ \tau_{\gamma, \theta}(R_{1,2}^0) = \theta(\alpha_1)^2 R_{1,2}^0, \quad \tau_{\gamma, \theta}(R_{1,2}^1) = \theta(\alpha_1)^2 R_{1,2}^1 \]

and

\[ \tau_{\gamma, \theta}(R_{1,2}^2) = \theta(\alpha_1)^2 R_{1,2}^2 + a, \quad a \in U(\bar{n}[\nu])\bar{n}[\nu]^+_1, \quad t \geq 4 \]

and thus

\[ \tau_{\gamma, \theta}(R_{1,2}^0) \in U(\bar{n}[\nu])\bar{n}[\nu]^+_1 \text{ for } 5/4 \leq t \leq 9/4, \]

\[ \tau_{\gamma, \theta}(R_{1,2}^0) = \theta(\alpha_1)^2 R_{1,2}^0 + a, \quad a \in U(\bar{n}[\nu])\bar{n}[\nu]^+_1, \quad t \geq 11/4, \]

and thus

\[ \tau_{\gamma, \theta}(J) \subset I_\Lambda \tag{6.9} \]
Now (6.9) and (6.10) imply (6.8).

By using Lemma 3.3 we get
\[ x^\rho_{\alpha_1 + \alpha_2}(-1) = aR^0_{1;1} + bx^\rho_{\alpha_1} \left( \frac{5}{4} \right) x^\rho_{\alpha_1} \left( \frac{1}{4} \right) + cx^\rho_{\alpha_1} \left( \frac{3}{4} \right) x^\rho_{\alpha_1} \left( -\frac{1}{4} \right) \in I_\Lambda + U(\bar{n}[\hat{\nu}])x^\rho_{\alpha_1} \left( -\frac{1}{4} \right). \]

Then one can see that
\[ U(\bar{n}[\hat{\nu}])\bar{n}[\hat{\nu}]_+ \subset \tau_{\gamma,\theta} \left( I_\Lambda + U(\bar{n}[\hat{\nu}])x^\rho_{\alpha_1} \left( -\frac{1}{4} \right) \right). \]

By the above computations we also have
\[ J \subset \tau_{\gamma,\theta} \left( I_\Lambda + U(\bar{n}[\hat{\nu}])x^\rho_{\alpha_1} \left( -\frac{1}{4} \right) \right), \]

and thus we obtain
\[ I_\Lambda \subset \tau_{\gamma,\theta} \left( I_\Lambda + U(\bar{n}[\hat{\nu}])x^\rho_{\alpha_1} \left( -\frac{1}{4} \right) \right). \]

□

Define the linear map
\[ \psi_{\gamma,\theta} : U(\bar{n}[\hat{\nu}]) \to U(\bar{n}[\hat{\nu}]), \]
\[ a \mapsto \tau_{\gamma,\theta}^{-1}(a)x^\rho_{\alpha_1}(-1/4). \]

**Lemma 6.2.** We have
\[ \psi_{\gamma,\theta}\tau_{\gamma,\theta}(I_\Lambda) \subset I_\Lambda. \]

**Proof.** For any \( a \in U(\bar{n}[\hat{\nu}]) \) we have \( \psi_{\gamma,\theta}\tau_{\gamma,\theta}(a) = ax^\rho_{\alpha_1}(-1/4) \). First notice that
\[ \psi_{\gamma,\theta}\tau_{\gamma,\theta}(U(\bar{n}[\hat{\nu}])\bar{n}_+) \subset U(\bar{n}[\hat{\nu}])\bar{n}_+ \subset I_\Lambda. \]

Using Lemma 3.3 one can check that
\[ \psi_{\gamma,\theta}\tau_{\gamma,\theta}(R^0_{1;1,t}) = x^\rho_{\alpha_1}(-1/4)R^0_{1;1,t} + bR^0_{1,2:1,t+1/4} + c, \]
where \( b \) is a nonzero constant and \( c \in U(\bar{n}[\hat{\nu}])\bar{n}_+ \),
\[ \psi_{\gamma,\theta}\tau_{\gamma,\theta}(R^0_{1,2:1,t}) = x^\rho_{\alpha_1}(-1/4)R^0_{1,2:1,t}, \]
and
\[ \psi_{\gamma,\theta}\tau_{\gamma,\theta}(R^0_{1,2:1,t}) = x^\rho_{\alpha_1}(-1/4)R^0_{1,2:1,t} + dR^0_{1,2:1,t+1/4}, \]
where \( d \) is a nonzero constant. Thus
\[ \psi_{\gamma,\theta}\tau_{\gamma,\theta}(J) \subset I_\Lambda. \]

Now (6.15) and (6.16) prove (6.14). □

Consider the linear map
\[ e_{\alpha_1} : V_L^T \to V_L^T, \]
and its restriction to the principal subspace of \( V_L^T \),
\[ e_{\alpha_1} : W_L^T \to W_L^T. \]

Since
\[ e_{\alpha_1}x^\rho_{\alpha}(m) = C(\alpha, -\alpha_1)x^\rho_{\alpha}(m - (\alpha(0), \alpha_1))e_{\alpha_1} \]
and
\[ e_{\alpha_1} \cdot 1 = 4/\sigma(\alpha_1)x^\rho_{\alpha_1}(-1/4) \cdot 1, \]
we have

\[(6.19) \quad e_{\alpha_1} (a \cdot 1) = A^{(\cdot), (\cdot)}_{C_{(\cdot)}} \psi_{\gamma, \rho} (a) \cdot 1,\]

where \(a \in U(\mathfrak{n}[\hat{\nu}])\) and \(A^{(\cdot), (\cdot)}_{C_{(\cdot)}}\) is a nonzero constant depending on the commutator map \((2.5)\), the map \((2.82)\) and the character map \((6.1)\).

We now introduce a twisted version of the \(\Delta\)-map of \([15, 2]\). See more about the (untwisted) \(\Delta\)-map in the last section of this paper. Let \(\lambda_1\) and \(\lambda_2\) be the fundamental weights of \(\mathfrak{sl}(3)\). Then

\[\lambda_1 = \frac{2}{3} \alpha_1 + \frac{1}{3} \alpha_2\]

and

\[\lambda_1(0) = \frac{1}{2} (\alpha_1 + \alpha_2), \quad \lambda_1(1) = \frac{1}{6} (\alpha_1 - \alpha_2).\]

Set

\[(6.20) \quad \Delta^T (\lambda_1, -x) = i^{2 \lambda_1(0)} x^{\lambda_1(0)} E^+ (-\lambda_1, x),\]

where \(i^{2 \lambda_1(0)} = i^{\alpha_1 + \alpha_2}\) and \(x^{\lambda_1(0)} = x^{(1/2)(\alpha_1 + \alpha_2)}\) are operators on the space \(U_T\) and thus on the space \(V^T_L\) (see \((2.71)\) and \((2.72)\)), and

\[E^+ (-\lambda_1, x) = \exp \left( \sum_{n \in \frac{1}{2} \mathbb{Z}_+} \frac{-\lambda_1(n)}{n} x^{-n} \right) \in (\text{End} V^T_L)[[x^{-1/4}]]\]

(see \((2.41)\)), so that

\[\Delta^T (\lambda_1, -x) \in (\text{End} V^T_L)[[x^{1/4}, x^{-1/4}]].\]

Denote by \(\Delta^T_c (\lambda_1, -x)\) the constant term of \(\Delta^T (\lambda_1, -x)\). For \(m_1, \ldots, m_r, n_1, \ldots, n_s \in \frac{1}{2} \mathbb{Z},\) we have

\[\Delta^T_c (\lambda_1, -x) (x^{\alpha_1 + \alpha_2} (m_1) \cdots x^{\alpha_1 + \alpha_2} (m_r) x^{\alpha_1} (n_1) \cdots x^{\alpha_1} (n_s) \cdot 1)\]

\[= \Delta^T_c (\lambda_1, -x) (x^{\alpha_1 + \alpha_2} (m_1) \cdots x^{\alpha_1 + \alpha_2} (m_r) x^{\alpha_1} (n_1) \cdots x^{\alpha_1} (n_s) \cdot 1)\]

\[= (-i)^s x^{\alpha_1 + \alpha_2} (m_1 + 1) \cdots x^{\alpha_1 + \alpha_2} (m_r + 1) x^{\alpha_1} (n_1 + 1/2) \cdots x^{\alpha_1} (n_s + 1/2) \cdot 1,\]

where we have used that

\[E^+ (-\lambda_1, x) E^- (-\alpha_1, x) = (1 - x^{1/2} / x^{1/2}) E^- (-\alpha_1, x) E^+ (-\lambda_1, x)\]

and

\[E^+ (-\lambda_1, x) E^- (-\alpha_1 - \alpha_2, x) = (1 - x / x) E^- (-\alpha_1 - \alpha_2, x) E^+ (-\lambda_1, x).\]

Thus we have the linear map

\[(6.21) \quad \Delta^T_c (\lambda_1, -x) : W^T_L \rightarrow W^T_L\]

\[a \cdot 1 \rightarrow \tau_{\gamma, \rho} (a) \cdot 1.\]
7. Main results

**Theorem 7.1.** We have

\[ \text{Ker } f_{\Lambda} = I_{\Lambda}, \]

or equivalently

\[ \text{Ker } \pi_{\Lambda} = I_{\Lambda} \cdot v_{\Lambda}^N. \]

**Proof:** The proof is analogous to the proof of the presentation of the principal subspaces in [CalLM3]. Also recall Remark 4.1 of [CalLM3], where we have compared the subtleties of that proof with those of the corresponding proof in [CalLM1].

It is easy to see that \( I_{\Lambda} \cdot v_{\Lambda}^N \subset \text{Ker } \pi_{\Lambda}. \) Now assume that \( \text{Ker } \pi_{\Lambda} \) is not included in \( I_{\Lambda} \cdot v_{\Lambda}^N. \) Then there exists an element \( a \in U(\bar{\mathfrak{n}}[\hat{\nu}]), \) which we assume to be homogeneous with respect to the weight and charge gradings, such that

\[ a \cdot v_{\Lambda}^N \in \text{Ker } \pi_{\Lambda} \]

Then \( a \) is nonzero and nonconstant. We choose \( a \) to be an element of the smallest possible weight satisfying (7.25). Note that \( a \) has positive weight.

We first claim that

\[ a \in I_{\Lambda} + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_1}^\circ (-1/4). \]

Assume that (7.26) is not true. Then

\[ \tau_{\gamma, \theta}(a) \cdot v_{\Lambda}^N \notin I_{\Lambda} \cdot v_{\Lambda}^N. \]

Indeed, if \( \tau_{\gamma, \theta}(a) \cdot v_{\Lambda}^N \in I_{\Lambda} \cdot v_{\Lambda}^N, \) then \( \tau_{\gamma, \theta}(a) \in I_{\Lambda}, \) and by Lemma 6.1 we get \( a \in I_{\Lambda} + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_1}^\circ (-1/4), \) a contradiction. Since \( a \cdot v_{\Lambda}^N \in \text{Ker } \pi_{\Lambda}, \) then \( a \cdot 1 = 0 \) and by applying the map (6.21) we obtain

\[ \tau_{\gamma, \theta}(a) \cdot 1 = 0, \]

which implies that

\[ \tau_{\gamma, \theta}(c) \cdot 1 = 0. \]

We now claim that

\[ c \in I_{\Lambda} + U(\bar{\mathfrak{n}}[\hat{\nu}])x_{\alpha_1}^\circ (-1/4). \]

Assume that (7.31) does not hold. Then by Lemma 6.1 we have

\[ \tau_{\gamma, \theta}(c) \cdot v_{\Lambda}^N \notin I_{\Lambda} \cdot v_{\Lambda}^N. \]

On the other hand,

\[ \tau_{\gamma, \theta}(c) \cdot v_{\Lambda}^N \in \text{Ker } \pi_{\Lambda}. \]

Indeed, since \( a \in \text{Ker } f_{\Lambda} \) and \( b \in I_{\Lambda} \subset \text{Ker } f_{\Lambda} \) we get

\[ 0 = (a - b) \cdot 1 = cx_{\alpha_1}^\circ (-1/4) \cdot 1 = \frac{1}{A(\cdot, \cdot)} e_{\alpha_1}(\tau_{\gamma, \theta}(c) \cdot 1), \]

which implies that

\[ \tau_{\gamma, \theta}(c) \cdot 1 = 0. \]
Since \( \text{wt } \tau_{\gamma, \theta}(c) < \text{wt } a \), (7.32) and (7.33) contradict our choice of the element \( a \) satisfying (7.25), and therefore (7.31) holds. Then

\[
\text{(7.34)} \quad cx^b_{\alpha_1}(-1/4) \in I_A x^b_{\alpha_1}(-1/4) + I_A.
\]

Notice that \( \psi_{\gamma, \theta}(\tau_I) = I_A x^b_{\alpha_1}(-1/4) \) and thus by Lemma 6.2, we obtain that \( I_A x^b_{\alpha_1}(-1/4) \subseteq I_A \). Therefore \( cx^b_{\alpha_1}(-1/4) \in I_A \), which gives \( a \in I_A \). This shows that our initial assumption is false, and therefore we have (7.24). □

Recall the linear maps (6.18) and (6.21).

**Theorem 7.2.** We have the following short exact sequence of maps:

\[
\text{(7.35)} \quad 0 \longrightarrow W_L^T \xrightarrow{e_{\alpha_1}} W_L^T \xrightarrow{\Delta_T^c(\lambda_1, -x)} W_L^T \longrightarrow 0.
\]

**Proof:** It is obvious that \( e_{\alpha_1} \) is injective, \( \Delta_T^c(\lambda_1, -x) \) is surjective and

\[
\text{Im } e_{\alpha_1} \subseteq \text{Ker } \Delta_T^c(\lambda_1, -x).
\]

Let \( v = a \cdot 1 \in \text{Ker } \Delta_T^c(\lambda_1, -x) \) for \( a \in U(\overline{\hat{n}}[\hat{\nu}]) \). Then

\[
0 = \Delta_T^c(\lambda_1, -x)(v) = \tau_{\gamma, \theta}(a) \cdot 1.
\]

Therefore \( \tau_{\gamma, \theta}(a) \in I_A \) and by Lemma 6.1, we have

\[
a \in I_A + U(\overline{\hat{n}}[\hat{\nu}])x^b_{\alpha_1}(-1/4).
\]

Thus

\[
\text{(7.36)} \quad v = a \cdot 1 \in \text{Ker } \Delta_T^c(\lambda_1, -x) \quad \text{if and only if} \quad a \in I_A + U(\overline{\hat{n}}[\hat{\nu}])x^b_{\alpha_1}(-1/4).
\]

Let \( v = a \cdot 1 \in \text{Im } e_{\alpha_1} \) for \( a \in U(\overline{\hat{n}}[\hat{\nu}]) \). Then \( v = bx^b_{\alpha_1}(-1/4) \cdot 1 \), where \( b \in U(\overline{\hat{n}}[\hat{\nu}]) \). This implies that

\[
a \in I_A + U(\overline{\hat{n}}[\hat{\nu}])x^b_{\alpha_1}(-1/4),
\]

and we get

\[
\text{(7.37)} \quad v = a \cdot 1 \in \text{Im } e_{\alpha_1} \quad \text{if and only if} \quad a \in I_A + U(\overline{\hat{n}}[\hat{\nu}])x^b_{\alpha_1}(-1/4).
\]

Now (7.36) and (7.37) give the inclusion \( \text{Ker } \Delta_T^c(\lambda_1, -x) \subseteq \text{Im } e_{\alpha_1} \). □

As we recall from Section 3, the vector space \( V_L^T \) has compatible gradings by weight, given by the action of the Virasoro algebra operator \( L^c(0) \), and by charge, given by the eigenvalues of the operator \( \alpha_1 + \alpha_2 = (\alpha_1 + \alpha_2)(0) \). Restrict these gradings to \( W_L^T \). In order to make the degrees integers, we shall now use the weight grading given by \( 4L^c(0) \) and the charge grading given by \( \alpha_1 + \alpha_2 \). We consider the graded dimension of the principal subspace \( W_L^T \):

\[
\chi(x; q) = \text{tr}_{W_L^T} x^{\alpha_1 + \alpha_2} q^{4L^c(0)} \in q^{1/4} \mathbb{C}[[x, q]]
\]

(recall (3.52)), where \( x \) and \( q \) are commuting formal variables. As in [CalL3], in order to avoid the factor \( q^{1/4} \), we use the following slightly modified graded dimension:

\[
\chi'(x; q) = q^{-1/4} \chi(x; q) \in \mathbb{C}[[x, q]].
\]

Theorem 7.2 now implies:

**Corollary 7.3.** We have

\[
\text{(7.38)} \quad \chi'(x; q) = \chi'(xq^2; q) + xq\chi'(xq^2; q).
\]
Proof: We denote by $W_{L,k,l}^T$ the homogeneous subspace of $W_L^T$ which consists of elements of charge $k$ and weight $l$. The exact sequence from Theorem [7,2] gives

\begin{equation}
0 \longrightarrow W_{L,k-1,l-2k+1}^T \xrightarrow{e_{\alpha_1}} W_{L,k,l}^T \xrightarrow{\Delta^T_{\lambda_1,-x}} W_{L,k,l-2k}^T \longrightarrow 0,
\end{equation}

and this proves (7.38). □

Thus, as a consequence of the vertex-algebraic theory of principal subspaces in the case of twisted affine Lie algebras that we have initiated in this paper, we have obtained a recursion which characterizes the graded dimension of the principal subspace $W_L^T$. The solution of this recursion is given by (cf. [A]):

**Corollary 7.4.** We have

$$
\chi'(x; q) = \prod_{n \geq 1} (1 + xq^{2n-1}).
$$

Equivalently, $\chi'(x; q)$ is the two-variable generating function of the number of partitions of $n$ into $m$ distinct odd parts, which we denote by $p(O, m, n)$, that is,

$$
\chi'(x; q) = \sum_{m \geq 0} \sum_{n \geq 0} p(O, m, n)x^m q^n.
$$

In particular, $\chi'(1; q)$ is the generating function of the number of partitions of $n$ into distinct odd parts, which we denote by $p(O, n)$, that is,

$$
\chi'(1; q) = \sum_{n \geq 0} p(O, n)q^n.
$$

8. ANOTHER PROOF OF THE EXACTNESS OF SHORT SEQUENCES FOR PRINCIPAL SUBSPACES IN THE CASE OF UNTWISTED AFFINE LIE ALGEBRAS OF TYPES $A, D, E$

In this section, we reformulate the result proved in [CalLM3] giving canonical exact sequences for principal(-like) subspaces of the level one standard modules for the untwisted affine Lie algebras of types $A, D$ and $E$. In our reformulation, the role of the intertwining operators is played by the $\Delta$-map of [L2] (not to be confused with the $\Delta_x$ map used earlier).

As in [CalLM3], let $g$ be a finite-dimensional complex simple Lie algebra of type $A, D$ or $E$, of rank $l$. Let $\mathfrak{h}$ be a Cartan subalgebra of $g$ and let $\{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{h}^*$ be a set of simple roots. Denote by $\lambda_1, \ldots, \lambda_l$ the corresponding fundamental weights of $g$. It is convenient to set $\lambda_0 = 0$. Consider the lattice vertex operator algebra $V_L$ constructed from the root lattice $L$ of $g$ (see (2.39) and (2.46)), and consider the $V_L$-module $V_{L+\lambda_i} (= M(1) \otimes \mathbb{C}[L]e^{\lambda_i})$, $i = 0, \ldots, l$. Denote by $Y_{L+\lambda_i}$ the vertex operator that gives the $V_L$-module structure for $V_{L+\lambda_i}$. Note that $V_{L+\lambda_i}$ was denoted by $V_{L+\lambda_i}$ in [CalLM3]. Denote by $W_L$ the principal subspace of $V_L$ and by $W_{L+\lambda_i}$ the principal-like subspaces of $V_{L+\lambda_i}$ for $i = 1, \ldots, l$, introduced in [CalLM3]. We refer to [CalLM3] for details.

As in [L2], consider the linear map

\begin{equation}
\Delta(\lambda_i, x) = x^{\lambda_i}E^+(-\lambda_i, -x) \in (\mathrm{End} V_L)[[x, x^{-1}]],
\end{equation}

where $E^+(\cdot, x)$ is as in (2.31), and let

$$
\tilde{Y}_{VL}(v, x) = Y_{VL}(\Delta(\lambda_i, x)v, x) = \sum_{m \in \mathbb{Z}} \tilde{v}_m x^{-m-1}.
$$

It was proved in [L2] that $(V_L, \tilde{Y}_{VL})$ is naturally isomorphic to $(V_{L+\lambda_i}, Y_{VL+\lambda_i})$ as a $V_L$-module, as we discuss below.
In [CalLM3], we considered an (essentially unique) intertwining operator
\[ \mathcal{Y}(:, x) \in \mathcal{I} \left( \frac{V_{L+\lambda_i}}{V_{L+\lambda_i}, V_L} \right), \]
given by
\[ \mathcal{Y}(u, x)v = e^{xL(-1)}Y_{L+\lambda_i} (v, -x)u, \]
and its constant term \( \mathcal{Y}_c(:, x) \). By projection we also have the map between principal-like subspaces
\[ \mathcal{Y}_c(e^{\lambda_i}, x) : W_L \rightarrow W_{L+\lambda_i}, \]
which commutes with the generators of \( \tilde{n} \), where \( n \) is the Lie subalgebra of \( g \) spanned by the root vectors associated with the positive roots, and \( \tilde{n} \) is as before the affinization of \( n \) without central extension.

Recall from [CalLM3] the linear isomorphism of vector spaces
\[ e_{\lambda_i} : V_L \rightarrow V_{L+\lambda_i}. \]
It is slightly more convenient to consider (see [Pr])
\[ [\lambda_i] = e_{\lambda_i} \circ c(:, \lambda_i) : V_L \rightarrow V_{L+\lambda_i}, \]
where \( c(:, \cdot) \) is the (multiplicative) commutator map as in formula (2.17) of [CalLM3], extended naturally to a linear map on \( V_L \) as in formula (12.2) of [DL2]. This modified map now satisfies
\[ [\lambda_i]x_{\alpha}(m) = x_{\alpha}(m - \langle \lambda_i, \alpha \rangle)[\lambda_i] \]
for each root \( \alpha \). Moreover, this map is an isomorphism between the \( V_L \)-modules \( (V_L, \tilde{Y}_L) \) and \( (V_{L+\lambda_i}, \tilde{Y}_{L+\lambda_i}) \):
\[ [\lambda_i]^{-1}Y_{L+\lambda_i}(v, x)[\lambda_i] = \tilde{Y}_L(v, x) \]
(see [Li2]). Denote by \( \Delta_c(\lambda_i, -x) \) the constant term of \( \Delta(\lambda_i, -x) \). It defines a linear map
\[ \Delta_c(\lambda_i, -x) : W_L \rightarrow W_L. \]
Indeed, let us compute the action of \( \Delta_c(\lambda_i, -x) \) on a “monomial” in \( W_L \). For \( i = 1, \ldots, k \), we have
\[ \Delta_c(\lambda_i, -x)x_{\alpha_i}(-m_1 - 1) \cdots x_{\alpha_k}(-m_k - 1) \cdot 1 \]
\[ = \Delta_c(\lambda_i, -x)\text{Coeff}_{x_1^{m_1} \cdots x_k^{m_k}}(e^{\alpha_1}, x_1) \cdots Y(e^{\alpha_k}, x_k) \cdot 1 \]
\[ = CT_x\text{Coeff}_{x_1^{m_1} \cdots x_k^{m_k}}(-x)(1 - x_i/x)Y(e^{\alpha_1}, x_1) \cdots Y(e^{\alpha_k}, x_k) \cdot 1 \]
\[ = x_{\alpha_i}(-m_1 - 1) \cdots x_{\alpha_i}(-m_i - 1) \cdots x_{\alpha_k}(-m_k - 1) \cdot 1, \]
where \( m_1, \ldots, m_k \geq 0 \) and \( CT_x(\cdot) \) stands for the constant term. Notice that the shift occurs only for the root \( \alpha_i \), a consequence of
\[ E^+(-\lambda_i, x)E^-(-\alpha_j, x_j) = \left( 1 - \frac{x_j}{x} \right) \delta_{ij} E^+(-\alpha_j, x_j)E^-(-\lambda_i, x), \]
where \( \delta_{ij} \) is the Kronecker symbol.

We now reformulate Theorem 5.2 in [CalLM3]:

**Proposition 8.1.** The short exact sequence
\[ (8.43) \quad 0 \rightarrow W_L \xrightarrow{e_{\alpha_i}} W_L \xrightarrow{[\lambda_i]^{-1}\mathcal{Y}_c(e^{\lambda_i}, x)} W_L \rightarrow 0 \]
can be equivalently replaced by
\[ (8.44) \quad 0 \rightarrow W_L \xrightarrow{e_{\alpha_i}} W_L \xrightarrow{\Delta_c(\lambda_i, -x)} W_L \rightarrow 0. \]
Proof. For \( v \in W_L \), we have

\[
[\lambda_i]^{-1} \mathcal{Y}_c(e^{\lambda_i}, x)v = [\lambda_i]^{-1} \text{CT}_x \mathcal{Y}(e^{\lambda_i}, x)v = [\lambda_i]^{-1} \text{CT}_x \{ e^{L(-1)x}Y_{L+\lambda_i}(v, -x)e_{\lambda_i} \} \\
= \text{CT}_x e^{L(-1)x} \tilde{Y}_L(v, -x)1 = \text{CT}_x e^{L(-1)x}Y_L(\Delta(\lambda_i, -x)v, -x)1 \\
= \text{CT}_x e^{L(-1)x} e^{-L(-1)x} \Delta(\lambda_i, -x)v = \Delta_c(\lambda_i, -x)v.
\]

\[ \square \]

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