The Abstract Machinery of Interaction
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Abstract
We provide an original reformulation of the IAM, an abstract machine introduced by Danos and Regnier based on Girard’s Geometry of Interaction. We present our machine as acting directly on λ-terms, rather than on linear logic proof nets. The inductive nature of the λ-calculus allows direct and self-contained proofs of soundness and adequacy. Moreover, we improve on the literature showing that the denotational semantics induced by our machine is invariant for open terms and erasing steps, solving a notorious issue of the Geometry of Interaction.

1 Introduction
The advantage, and at the same time the drawback, of the λ-calculus is its distance from low-level, implementative details. It comes with just one rule, β-reduction, and with no indications about how to implement it on low-level machines. It is an advantage when reasoning about programs expressed as λ-terms. It is a drawback, instead, when one wants to implement the λ-calculus, or to do complexity analyses, because β-steps are far from being atomic operations. In particular, terms can grow exponentially with the number of β-steps, a degeneracy known as size explosion, which is why β-reduction cannot be reasonably implemented, at least if one sticks to an explicit representation of λ-terms.

Environment Machines. Implementations solve this issue by evaluating the λ-calculus up to sharing of sub-terms, where sharing is realized through a data structure called environment, collecting the sharing annotations generated by the machine during the execution, one for each encountered β-redex. For common weak evaluation strategies (i.e. that do not inspect in the scope of λ-abstractions) such as call-by-name/value/need, the number of β-steps is a reasonable time cost model [BG95, SGM02, DLM08]. Environment machines—whose most famous examples are Landin’s SECD [Lan65], Felleisen and Friedman’s CEK [FF86] or Krivine’s KAM [Kri07]—can be extended to open terms and optimized in such a way that they run within a linear overhead with respect to the number of β-steps [ASC15, AG17]. Said differently, they respect the time cost model (see [Acc18a] for an overview). For space, the situation is different. Only very recently the problem has been tackled [FKR19] and some preliminary and limited results have appeared. Then, environment machines store information for every β-step, therefore using space linear in time, which is the worst possible use of space.

Beyond Environments. In practice, frameworks based on the λ-calculus are invariably implemented using environments. Nonetheless, the lack of a fixed execution schema for the λ-calculus leaves open, in theory, the possibility of alternative implementation schemes. The theory of linear logic indeed provides a completely different style of abstract machines, rooted in Girard’s Geometry of Interaction [Gir89] (shortened to GoI in the following). These GoI machines were pioneered by Danos and Regnier and Mackie in the nineties [Mac95, DR99]. The basic idea is that the machine does not use environments, while it keeps track of information that allows retrieving previous β-redexes, by using a data structure called token, saving information about the history of the computation. The key point is that the token does not store information about every single β-redex, thus disentangling space-consumption from time-consumption. In other words, GoI machines are good candidates for space-efficient implementation schemes, as first shown by Dal Lago and Schöpp [DLS10]. The price to pay is that the machine wastes

1On sequential models space cannot exceed time, as one needs a unit of time to use a unit of space.
a lot of time to retrieve β-redexes, so that time is sacrificed for space. The same, however, happens with space-sensitive Turing machines.

**The Interaction Abstract Machine.** The original GoI machine is the Interaction Abstract Machine (IAM) by Danos and Regnier [DR99, DHR96], formulated on linear logic proof-nets as a reversible, bideterministic automaton. In [DHR96], Danos, Herbelin, and Regnier prove that the IAM is sound with respect to *linear head evaluation* (shortened to LHE), a refinement of head evaluation, arising from the linear logic decomposition of the λ-calculus. Soundness of GoI machines amounts to show that they provide a denotational semantics invariant by LHE—see Sect. 6 for an overview of the difference with environment machines. Danos and Regnier’s proof of soundness for the IAM is indirect, as it follows from a sequence of results relating the IAM to AJM games, AJM games to HO games, HO games to another abstract machine, the PAM, and finally the PAM to LHE.

The understanding of the IAM requires advanced expertise in linear logic, and it is thus out of scope for many people with an interest in the implementation of functional languages. Moreover, the soundness proof requires further expertise in game semantics, and it is not as neat as for environment machines. The aim of this paper is to recast the IAM in a setting not requiring any background in linear logic or game semantics, and to guide the reader through the subtleties of the IAM. At the same time, providing a clean and self-contained technical development, and improving on some of the properties of the IAM.

**The Interaction Abstract Lambda Machine.** The first contribution of the paper is a formulation of the IAM as a machine acting directly on λ-terms rather than on linear logic proof nets. The starting point of our Interaction Abstract Lambda Machine (IALM) is seeing a position in the code $t$ (what is usually the position of the token on the proof net representation of $t$) as a pair $(u, C)$ of a sub-term $u$ and a context $C$ such that $C⟨u⟩=t$. These positions are nothing else but a readable presentation of pointers.

**Direct Soundness and Adequacy Proofs.** We do more than simply change syntax. As Danos *et al.*, we show that the IALM provides a denotational semantics of LHE. In contrast to them, however, our proof of soundness is self-contained, and based on a variation over Sands’ *improvements* [San96], a natural notion of bisimulation. We also give a clean and direct proof of *adequacy*, that is, the fact that the machine semantics is not empty if and only if LHE terminates. A contribution of this work, we believe, precisely lies in the way in which we state and prove these results.

A further difference with Danos and Regnier’s and Mackie’s work is that we model our machine on the call-by-name translation of the λ-calculus in linear logic, while they use the call-by-value one. This way, we solve the mismatch between the strategy w.r.t. we are going to prove the machine correct and the underlying logical cut-elimination dynamics.

**Open Terms and Erasing Steps.** A relevant point is that our semantics is invariant when considering open terms and erasing steps. This is a novelty, since GoI is usually not invariant in such a setting. The improvement is better explained referring to the GoI presentation in terms of paths in a proof net. Our approach considers only paths starting at the conclusion of the net corresponding to the output of the λ-term, while the IAM considers paths between any two conclusions of the net. The intuitionistic nature of λ-terms indeed gives a privileged point of view, that turns out to be the essential ingredient for the invariance of the interpretation for open terms with respect to erasing steps.

It may be argued that the AJM game model can be seen as the GoI restricted to paths starting from the output, and that it is well known that the AJM interpretation is invariant for open terms and erasing steps. GoI and AJM games are different, however, in that games prescribe player and opponent to follow a protocol, through switching conditions, which are not there in the GoI. We find remarkable that soundness for open terms can be obtained without relying on switching conditions, but on a refined notion of initial state.

**This Paper in Perspective.** This paper is only the last chapter of a long-time endeavor by the authors directed at understanding complexity measures and implementation schemas for the λ-calculus. We

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2For the acquainted reader, they play a role akin to the initial labels in Lévy’s labeled λ-calculus, itself having deep connections with the IAM [ADLR94].
provide an original reformulation of the original machine by Danos and Regnier, together with a neat self-contained development and improved properties (invariance for open terms and erasing steps). The aim is to set the ground for a formal(izable), robust, and systematic study of GoI machines and their complexity.

**Related Work on GoI.** This is certainly not the first paper on the GoI and the λ-calculus. Indeed, the literature on the topic and its applications is huge, and goes from Girard’s original papers [Gir89], to Abramsky et al’s reformulation using the Int-construction [AHS02], Danos and Regnier’s using path algebras [DR93], Ghica’s applications to circuit synthesis [Ghi07], together with extensions by Hoshino, Muroya, and Hasuo to languages with various kinds of effects [HMHT14], and Laurent extension to the additive connectives of linear logic [Lau01]. In all these cases, the GoI interpretation, even when given on λ-terms, goes through linear logic (or symmetric monoidal categories) in an essential way.

The GoI has also been studied in relationship with implementation of functional languages, by Gonthier, Abadi and Levy as a proof methodology in the study of optimal implementations [GAL92], and by Mackie with his GoI machine for PCF [Mac95] and Gödel System T [Mac17]. Recently, the space-efficiency studied by Dal Lago and Schöpp [DLS16] has been exploited by Mazza in [Maz15] and, together with Terui, in [MT15]. Dal Lago and coauthors have also introduced variants of the IAM acting on proof nets for a number of extensions of the λ-calculus [DLPHY14, DLFVY15, DLFVY17, DLTY17]. Curien and Herbelin study abstract machines related to game semantics and the IAM in [CH98, CH07]. Muroya and Ghica have recently studied the GoI in combination with rewriting and abstract machines in [MG17].

**Related Work on Environment Machines.** Environment machines for the λ-calculus have been recently closely scrutinized as for their time efficiency. Before 2014, the topic had been mostly neglected—the only two counterexamples being Blelloch and Greiner in 1995 [BG95] and Sands, Gustavsson, and Moran in 2002 [SGM02]. Since 2014—motivated by advances by Accattoli and Dal Lago on time cost models for the λ-calculus [ADL16]—Accattoli and co-authors have explored time analyses of environment machines from different angles [ABM14, AG17, AB17, ACGSC19].

### 2 A Gentle Introduction to the Geometry of Interaction

This section is an informal introduction to Girard’s Geometry of Interaction as implemented by the IALM, the abstract machine we are introducing in this paper. Many details are left out, and shall be covered in the next sections.

**Preliminaries.** The IALM implements head reduction, the simple reduction defined as:

\[ \lambda x_1 \ldots x_k.(\lambda y.t)ur_1 \ldots r_h \rightarrow_h \lambda x_1 \ldots x_k.t\{y\leftarrow u\}r_1 \ldots r_h. \]

The meaning of “implement” is explained a bit here, and more extensively in Sect. 6. Moreover, the IALM rather implements a linear variant of \(\rightarrow_h\), but for now the difference does not matter.

An essential point is that the initial code \(t\) of the machine never changes. The IALM only moves over it, in a local way, with no rewriting of the code and without ever substituting terms for variables. The current position in the code \(t\) is represented as a pair \((C, u)\) where \(C\) is a context and \(C(u) = t\). Moreover, the state of the machine is represented by two stacks, called log and tape respectively, whose functioning shall be explained soon.

In contrast to environment machines, that are either weak (that is, never enter abstractions) or strong (they enter into all abstractions), the IALM is incrementally strong, that is, it has a finer mechanism that allows entering into some of the abstractions. The number of head abstractions that a run of the IALM can cross, called the depth of the run, is specified at the beginning by the content of the tape, coded in unary: \(n\) is represented with \(n\) occurrences of the distinguished symbol \(\bullet\). Note the difference with environment machines: once the code \(t\) is fixed, such machines have only one initial state, while the IALM has a family of initial states, one for each depth.

A tricky point is that, given \(t\), the IALM does not compute the whole head normal form \(\text{hnf}(t)\) of \(t\), but only the head variable of \(\text{hnf}(t)\). This is very much in accordance with the idea of head reduction,
in which the arguments of the head variable are never touched. More about this shall be explained in Sect. 6.

Before giving an example run, we need one last concept. Beyond the current position and the token, a IALM state has a direction, ↓ or ↑. When the direction is downwards (↓), the machine looks for the head variable of the subterm. When it is upwards (↑), the IALM looks for the argument the found head variable would be substituted for under head evaluation (explanations below).

An Example of IALM run. Suppose one wants to evaluate the term $t := ((λz.λx.x)w)(λy.y)$, whose head normal form is $λy.y$. We know that the head variable $y$ in $\text{hnf}(t)$ is under one abstraction. Then, to find it, we have to run the IALM at depth 1, that is, starting from $(t, ⟨·⟩, ǫ, •, ↓)$. We expect $y$ as the result of the computation.

Let’s then consider the first four transitions of such a computation, that perform a visit of the leftmost branch of $t$, called the spine, until a variable is found.

| Sub-term         | Context | Log | Tape | Direction |
|------------------|---------|-----|------|-----------|
| $((λz.λx.x)w)(λy.y)$ | $⟨·⟩$   | $ε$ | •    | ↓         |
| $(λz.λx.x)w$     | $⟨·⟩(λy.y)$ | $ε$ | ••   | ↓         |
| $λz.λx.x$        | $⟨·⟩w(λy.y)$ | $ε$ | •••• | ↓         |
| $λx.x$           | $((λz.⟨·⟩)w)(λy.y)$ | $ε$ | ••   | ↓         |
| $x$              | $((λz.λx.⟨·⟩)w)(λy.y)$ | $ε$ | •    | ↓         |

Note the pushing and popping of •: one of the tasks of the tape is to account for the abstractions and applications encountered along the spine: the symbol • is pushed on applications, and pulled on abstractions (when the direction is ↓), so that the crossing of a β-redex leaves the stack unchanged. We shall say that the IALM searches up to β-redexes. Note also that, contrary to environment machines, arguments of the encountered β redexes are not saved, this way saving space, and disentangling space from time.

Once in the state $(x, ((λz.λx.⟨·⟩)w)(λy.y), ǫ, •)$, the IALM switches to phase ↑, and the machine starts to check whether $x$ would be substituted during head evaluation. In the KAM, it is enough to look up the environment, while in the IALM, this needs to be reconstructed, because encountered β redexes were not recorded. This is done by the next four steps, where again the search is up to β-redexes.

| Sub-term         | Context          | Log     | Tape | Direction |
|------------------|------------------|---------|------|-----------|
| $x$              | $((λz.λx.⟨·⟩)w)(λy.y)$ | $ε$     | •    | ↓         |
| $λx.x$           | $((λz.⟨·⟩)w)(λy.y)$ | $ε$     | ••   | ↑         |
| $λz.λx.x$        | $⟨·⟩w(λy.y)$     | $ε$     | ••   | ↑         |
| $(λz.λx.x)w$     | $⟨·⟩(λy.y)$      | $ε$     | ••   | ↑         |
| $λy.y$           | $((λz.λx.x)w)⟨·⟩$ | $(x, λx.⟨·⟩, ǫ)$ | •    | ↓         |

Some further crucial aspects of the IALM show up here.

- **Phases**: the IALM starts looking for the term that may be substituted for $x$, from a natural place, namely the λ-abstraction binding $x$. One needs to keep track of which of the (possibly many) occurrences of the bound variable one is coming from. This is done by simply pushing on the tape the position of the found occurrence of $x$ (w.r.t. its binder), and by switching the machine in upward mode ↑.

- **Locality**: as well as all the other ones, also the transition from the occurrence to the binder is local if occurrences are implemented as pointers to their binders, as in the proof net representation of λ-terms, see Section [11] for a precise comparison.

- **Log**: the upward journey is guided by the context (note the blue color). In the example, a term that would be substituted is found, namely $λy.y$. Observe that the log gets touched for the first time. Its role is to keep enough information as to potentially backtrack to the position in its entry, called logged position, as it shall be better explained in the next section.
• **Succeed or iterate:** in general, if the machine finds no term to substitute on \( x \), then the logged position shall not be removed from the tape, providing the result of the run—that is, the head variable. If instead a term \( u \) to substitute is found, as in the example, then the process starts over, switching to \( \downarrow \) phase and looking for the head variable of \( u \).

Once the argument \( \lambda y.y \) is found, the IALM now looks for its head variable \( y \). Please note that this is possible because of the \( \bullet \) on the tape. Otherwise, i.e. if the initial state were \((t, (\cdot), \epsilon, \epsilon)\), the IALM would be stuck in this *final* state, signaling that \( t \rightarrow_n \lambda y.u \) for a term \( u \). Indeed, each \( \bullet \) in the initial state allows for the inspection of one head lambda of \( \text{hnf}(t) \).

| Sub-term | Context | Log | Tape | Direction |
|----------|---------|-----|------|-----------|
| \( \lambda y.y \) | \(((\lambda z. \lambda x.x)w)(\cdot)\) | \( (x, \lambda x.(\cdot), \epsilon) \) | \( \bullet \) | \( \downarrow \) |
| \( y \) | \(((\lambda z. \lambda x.x)w)(\lambda y.(\cdot))\) | \( (x, \lambda x.(\cdot), \epsilon) \) | \( \epsilon \) | \( \downarrow \) |
| \( \lambda y.y \) | \(((\lambda z. \lambda x.x)w)(\cdot)\) | \( (x, \lambda x.(\cdot), \epsilon) \) | \( (y, \lambda y.(\cdot), \epsilon) \) | \( \uparrow \) |
| \( \cdots \) | \( \cdots \) | \( \cdots \) | \( \cdots \) | \( \cdots \) |
| \(((\lambda z. \lambda x.x)w)(\lambda y.y)\) | \( (\cdot) \) | \( \epsilon \) | \( (y, \lambda y.(\cdot), \epsilon) \) | \( \uparrow \) |

The head variable \( y \) is found in two steps. After that, the machine switches to \( \uparrow \) phase and runs again though the same path, thus arriving again at the root of the term \( t \). The IALM then stops and gives its output: \( y \), the head variable of \( \text{hnf}(t) \), is on the tape.

### 3 The Interaction Abstract Lambda Machine

In this section we introduce the data structures used by the IALM and its transition rules.

**Terms and Levelled Contexts.** Let \( V \) be a countable set of variables. Terms of the \( \lambda \)-calculus are defined as follows.

\[
\lambda \text{-terms} \quad t, u, r \ ::= \quad x \in V \mid \lambda x.t \mid tu.
\]

Free and bound variables are defined as usual: \( \lambda x.t \) binds \( x \) in \( t \). Terms are considered modulo \( \alpha \)-equivalence, and \( t[x\leftarrow u] \) denotes capture-avoiding (meta-level) substitution of all the free occurrences of \( x \) in \( t \). The study of the IALM requires a notion of context more informative than the usual one, introduced next.

**Leveled contexts**

\[
\begin{align*}
C_0 & \ ::= \langle \cdot \rangle \mid \lambda x.C_0 \mid C_0 t ; \\
C_{n+1} & \ ::= \lambda x.C_{n+1} \mid C_{n+1} t \mid tC_n .
\end{align*}
\]

The index \( n \) in \( C_n \) counts the number of arguments into which the hole \( \langle \cdot \rangle \) is contained in \( C_n \). Such an index has a natural interpretation in linear logic terms. According to the standard (call-by-name) translation of the \( \lambda \)-calculus into linear logic proof nets, in a context \( C_n \), the hole lies inside exactly \( n \) !-boxes. Contexts of level 0 are also called head contexts and are denoted by \( H, K, G \). The level of a context shall be omitted when not relevant to the discussion—note that any ordinary context can be written in a unique way as a leveled context, so that the omission is anyway harmless.

The *plugging* \( C_n(t) \) of a term \( t \) in \( C_n \) is defined by replacing the hole \( \langle \cdot \rangle \) with \( t \), potentially capturing free variables of \( t \). Plugging \( C_n(C_m) \) of a context for a context is defined similarly. A position \( \langle \cdot \rangle \) of (level \( n \)) in a term \( u \) is a pair \( (t, C_n) \) such that \( C_n(t) = u \).

**Logs and Logged Positions.** The IALM relies on two mutually recursive notions, namely logged positions and logs: a logged position is a position \( (t, C_n) \) together with a log\(^3\) \( L_n \), that is a list of logged positions, having length \( n \).

**Logged positions**

\[
p ::= (t, C_n, L_n)
\]

**Logs**

\[
L_0 ::= \epsilon \quad L_{n+1} ::= p \cdot L_n
\]

\(^3\)In computer science logs are traces that can only grow, while here they also shrink. The terminology suggests a tracing mechanism—trace is avoided because related to categorical formulations of the GoI.
The set of logged positions is $\mathcal{P}$, and we use $\cdot$ also to concatenate logs, writing, e.g., $L_n \cdot L$. Intuitively, logs contain some minimal information for backtracking to the associated position.

**Tape, Token, Direction, State.** The tape $T$ is a finite sequence of elements of two kinds, namely logged positions, and occurrences of the special symbol $\cdot$, needed to cross abstractions and applications. A token is a log plus a tape. A state of the machine is given by a position and a token, together with a mode of operation called direction.

**Definition 3.1 (IALM State).** A state $s$ of the IALM is a quintuple $(t, C, L, T, d)$ where:

1. $t$ is a $\lambda$-term: the code term;
2. $C$ is a context: the code context;
3. $L$ is an element of $\mathcal{P}^*$: the log;
4. $T$ is an element of $\{(\cdot) \cup \mathcal{P}\}^*$: the tape;
5. $d$ is an element in $D = \{\uparrow, \downarrow\}$: the direction.

Directions shall be represented mostly via colors: the code term in red, to represent $\downarrow$, and the code context in blue, to represent $\uparrow$. This way, the fifth component is often omitted.

**Initial States.** The IALM starts on states of the form $s_{t,k} := (t, (\cdot), \epsilon, \cdot^k)$, where $t$ is a term, $k \geq 0$ is the depth of the state, and $\epsilon$ is the empty log. Intuitively, the machine evaluates the term $t$ being allowed to inspect up to $k$ $\lambda$-abstractions of the head normal form of $t$. Note that there are many initial states for a given term $t$, one for each tape $\cdot^k$.

**Transitions.** The transitions of the IALM are in Fig. 1. Their union is noted $\rightarrow_{IALM}$ and a state $s_{t,k}$ is reachable if $s_{t,k} \rightarrow_{IALM}^* s$ for an initial state $s_{t,k}$. The basic idea is that $\downarrow$-states $(t, C, L, T)$ are queries about the head variable of (the head normal form of) $t$ and $\uparrow$-states $(t, C, L, T)$ are queries about the argument of an abstraction. Next, we explain how the transitions realize three entangled mechanisms of the machine.

1. **Search up to $\beta$-redexes:** note that $\rightarrow_{s_1}$ skips the argument and adds a $\cdot$ on the tape. The idea is that $\cdot$ keeps track that an argument has been encountered—it’s identity is however forgotten.

   Then $\rightarrow_{s_2}$ does the dual job: it skips an abstraction when the IO stacks carries a $\cdot$, that is, the

| Sub-term | Context | Log | Tape | \rightarrow |
|----------|---------|-----|------|------------|
| $ut$     | $C$     | $L$ | $T$  | $\cdot T$  | $\rightarrow_{s_1}$ |
| $u$      | $C(\cdot)$ | $L$ | $\cdot T$ | $\rightarrow_{s_2}$ |
| $\lambda x.t$ | $C$ | $L$ | $\cdot T$ | $\rightarrow_{s_3}$ |
| $t$      | $C(\lambda x.\cdot)$ | $L$ | $T$  | $\rightarrow_{s_4}$ |
| $x$      | $C(\lambda x.D_n)$ | $L_n \cdot L$ | $T$ | $\rightarrow_{var}$ |
| $\lambda x.D_n(x)$ | $C$ | $L$ | $(x, \lambda x.D_n, L_n) \cdot T$ | $\rightarrow_{br2}$ |

Figure 1: IALM transitions.
trace of a previously encountered argument. This mechanism thus realizes search up to \( \beta \)-redexes, that is, without recording them and leaving the tape unchanged. Note that \( \to_\beta \) and \( \to_\Delta \) realize the same during the \( \uparrow \) phase.

2. Finding variables and arguments: when the head variable \( x \) of the active subterm is found, transition \( \to_{\text{var}} \) switches the current direction, and the machine starts looking for potential substitutions for \( x \). The IALM then moves to the position of the binder \( \lambda x \) of \( x \), and starts exploring the context \( C \), looking for the first argument up to \( \beta \)-redexes. The relative position of \( x \) w.r.t. \( \lambda \) binder is recorded in a new logged position that is added to the tape. Since the machine moves out of a context of level \( n \), namely \( D_n \), the logged position contains the first \( n \) logged positions of the log.

Roughly, this is an encoding of the run that led from the level of \( \lambda x.D \) at hand, in case the machine would later need to backtrack. When the argument \( t \) for the abstraction binding the variable \( x \) in \( p \) is found, transition \( \to_{\text{arg}} \) starts looking for the head variable of \( t \), switching phase. Note that moving to \( t \), the level increases, and that the logged position \( p \) is moved from the tape to the log. The idea is that \( p \) is now a completed argument query, and it becomes part of the history of how the machine got to the current position, to be potentially used for backtracking.

3. Backtracking: it is started by transition \( \to_{\text{bt}} \). The idea is that the search for an argument of the \( \uparrow \)-phase has to temporarily stop, because there are no arguments left at the current level. The search of the argument then has to be done among the arguments of the variable occurrence that triggered the search, encoded in \( p \). Then the machine enters into backtracking mode, which is denoted by a \( \beta \)-phase with a logged position on the tape, to reach the position in \( p \). Backtracking is over when \( \to_{\text{bt}} \) is fired. The \( \uparrow \)-phase and the logged position on the tape mean that the IALM is backtracking. In fact, the machine is not looking for the head variable of the current subterm \( \lambda x.t \), it is rather going back to the variable position in the tape, to find its argument. This is realized by moving to the position in the tape and changing direction. Moreover, the log \( L_n \) encapsulated in the logged position is put back on the global log. An invariant shall guarantee that the logged position on the tape always contains a position relative to the active abstraction.

Example 3.2. We provide an example of a IALM run that exhibits backtracking. Let us consider the \( \lambda \)-term \( t := (\lambda x.xx)(\lambda y.y) \). We evaluate \( t \) according to weak head reduction, thus starting from the state \((t, (), \epsilon, \epsilon)\). The first steps of the computation are needed to reach the head variable, namely \( x \).

| Sub-term     | Context | Log  | Tape | Direction |
|--------------|---------|------|------|-----------|
| \((\lambda x.xx)(\lambda y.y)\) | \((\cdot)\) \((\lambda y.y)\) | \(\epsilon\) | \(\epsilon\) | \(\downarrow\) |
| \(\lambda x.xx\) | \((\cdot)\) \((\lambda y.y)\) | \(\epsilon\) | \(\cdot\) | \(\downarrow\) |
| \(xx\) | \((\lambda x.\cdot)\) \((\lambda y.y)\) | \(\epsilon\) | \(\cdot\) | \(\downarrow\) |
| \(x\) | \((\lambda x.\cdot x)\) \((\lambda y.y)\) | \(\epsilon\) | \(\cdot\) | \(\uparrow\) |

Once the head variable \( x \) has been found, the machine switches to upward mode \( \uparrow \) in order to find its argument \( \lambda y.y \).

| Sub-term     | Context | Log  | Tape | Direction |
|--------------|---------|------|------|-----------|
| \(x\) | \((\lambda x.\cdot x)(\lambda y.y)\) | \(\epsilon\) | \(\cdot\) | \(\downarrow\) |
| \(\lambda x.xx\) | \((\cdot)\) \((\lambda y.y)\) | \(\epsilon\) | \((x, \lambda x.\cdot x)\) \(\cdot\) | \(\uparrow\) |

Intuitively, the first occurrence of \( x \) has been substituted for \( \lambda y.y \), thus forming a new virtual \( \beta \)-redex \((\lambda y.y)\cdot x\). Indeed, a \( \cdot\) is on top of the tape, thus allowing the IALM to inspect \( \lambda y.y \), reaching its head variable \( y \).

| Sub-term     | Context | Log  | Tape | Direction |
|--------------|---------|------|------|-----------|
| \(\lambda y.y\) | \((\lambda x.xx)(\cdot)\) | \((x, \lambda x.\cdot x, \epsilon)\) | \(\cdot\) | \(\downarrow\) |
| \(y\) | \((\lambda x.xx)(\lambda y.\cdot)\) | \((x, \lambda x.\cdot x, \epsilon)\) | \(\epsilon\) | \(\downarrow\) |
| \(\lambda y.y\) | \((\lambda x.xx)(\cdot)\) | \((x, \lambda x.\cdot x, \epsilon)\) | \((y, \lambda y.\cdot, \epsilon)\) | \(\uparrow\) |
Once the head variable \( y \) has been found, the machine, in upward mode \( \uparrow \), starts looking for the argument of \( y \) from its binder \( \lambda y.y \). However, \( \lambda y.y \) was not the left side of an application forming a \( \beta \)-redex. Indeed, it was virtually substituted for the first occurrence of \( x \), in the log, thus creating the virtual redex \( (\lambda y.y)x \). Its argument is thus the second occurrence of \( x \). The IALM is able to retrieve it, walking again the path towards the variable \( \lambda y.y \) has been virtually substituted for, namely the first occurrence of \( x \), saved in the log. This is what we call backtracking.

| Sub-term | Context | Log       | Tape       | Direction |
|----------|---------|-----------|------------|-----------|
| \( \lambda y.y \) | \( (\lambda x.x)() \) | \( (x, \lambda x.()x, \epsilon) \) | \( (y, \lambda y.(), \epsilon) \) | \( \uparrow \) |
| \( \lambda x.xx \) | \( ()(\lambda y.y) \) | \( \epsilon \) | \( (x, \lambda x.()x, \epsilon) \cdot (y, \lambda y.(), \epsilon) \) | \( \downarrow \) |
| \( x \) | \( (\lambda x.()x)(\lambda y.y) \) | \( \epsilon \) | \( (y, \lambda y.(), \epsilon) \) | \( \uparrow \) |
| \( x \) | \( (\lambda x.x())(\lambda y.y) \) | \( (y, \lambda y.(), \epsilon) \) | \( \epsilon \) | \( \downarrow \) |

Notice that we are able to backtrack because we saved the occurrence of the substituted variable in the token, otherwise the machine would not be able to know which occurrence of \( x \) is the right one. Of course, when the first occurrence of \( x \) is reached the IALM, now again in upward mode \( \uparrow \), finds immediately its argument, that is the second occurrence of \( x \). At this point the machine looks for the argument of this last occurrence of \( x \), finding, of course, again \( \lambda y.y \).

| Sub-term | Context | Log       | Tape       | Direction |
|----------|---------|-----------|------------|-----------|
| \( x \) | \( (\lambda x.x())(\lambda y.y) \) | \( (y, \lambda y.(), \epsilon) \) | \( \epsilon \) | \( \downarrow \) |
| \( \lambda x.xx \) | \( ()(\lambda y.y) \) | \( \epsilon \) | \( (x, \lambda x.()x, \epsilon) \cdot (y, \lambda y.(), \epsilon) \) | \( \uparrow \) |
| \( \lambda y.y \) | \( (\lambda x.x)() \) | \( (x, \lambda x.()x, \epsilon) \cdot (y, \lambda y.(), \epsilon) \) | \( \epsilon \) | \( \downarrow \) |

The computation then stops, signalling that \( t \) has weak head normal form. Please notice that the position on the log has now a nested structure. Indeed it carries information about the virtual substitutions already performed.

## 4 Properties of the IALM

Here we first discuss a few invariants of the data structures of the machine, and then we analyze final states and the semantic interpretation defined by the IALM.

**The Code Invariant.** An inspection of the rules shows that the machine travels on a \( \lambda \)-term without altering it:

**Proposition 4.1** (Code Invariant). If \( (t, C, L, T, d) \xrightarrow{IALM} (u, D, L', T', d') \), then \( C(t) = D(u) \).

**The Balance Invariant.** Given a state \( (t, C, L, T, d) \), the log and the tape, i.e. the token, verify two easy invariants. The log \( L \), together with the position \( (t, C) \), forms a logged position, i.e. the length of \( L \) is exactly the level of the code context \( C \). Note that, then, the length of \( L \) is exactly the number of (linear logic) boxes in which the code term is contained.

About the tape, note that every time the machine switches from a \( \downarrow \)-state to an \( \uparrow \)-state (or vice versa), a logged position is pushed, or popped, from the tape \( T \). Thus, for reachable states, the number of logged positions in \( T \) gives the direction of the state. These invariants are formalized by the balance invariant below. Given a direction \( d \) we use \( d^n \) for the direction obtained by switching \( d \) exactly \( n \) times (i.e., \( \downarrow^0 = \downarrow, \uparrow^0 = \uparrow, \downarrow^{n+1} = \uparrow^n \) and \( \uparrow^{n+1} = \downarrow^n \)).

**Lemma 4.2** (Balance Invariant). Let \( s = (t, C_n, L, T, d) \) be a reachable state and \( |T|_p \) the number of logged positions in \( T \). Then \( (t, C_n, L) \) is a logged position and \( d = \downarrow^{|T|_p} \).

**Proof.** By induction on the execution \( s_0 \xrightarrow{IALM} s \) from the initial state \( s_0 \). If \( k = 0 \), \( s = i = (t, (), \epsilon, k) \). Clearly \( () \) is a level 0 context, and \( |L| = 0 \). Moreover, \( |T|_p = 0 \) and \( \downarrow^0 = \downarrow \). Now, let us consider a IAM run of length \( k > 0 \) and let \( \{s_h\}_{0 \leq h \leq k} \) be the sequence of states of this run. By induction hypothesis \( s_{k-1} = (t, C_n, T, L, d) \) is a logged position i.e \( |L| = n \) and \( \downarrow^{|T|_p} = d \). We can show, by cases, that the Lemma holds for \( s_k \).
• $d = \downarrow$.

  $t = ur$. Then $s_h = (u, C(\langle \rangle r), L, \bullet \cdot T)$. $C(\langle \rangle r)$ is a context of level $n = |L|$ and both $|T|_e$ and $d$ are unchanged.

  $t = \lambda x. u$ and $T = \bullet \cdot T'$. Then $s_h = (u, C(\lambda x. \langle \rangle), L, T')$. $C(\lambda x. \langle \rangle)$ is a context of level $n = |L|$ and both $|T|_e$ and $d$ are unchanged.

  $t = \lambda x. D_m(x)$ and $T = (x, \lambda x. D_m, L') \cdot T'$. Then $s_h = (x, C(\lambda x. D_m), L' \cdot L, T')$. $C(\lambda x. D_m)$ is a context of level $n + m = |L| + |L'|$ and since $\downarrow |T|_e = \downarrow$, then $\downarrow |T|_e + 1 = \uparrow$.

  $t = x, C = C_m(\lambda x. D_l)$ and $L = L_l \cdot L'$. Then $s_h = (x, D_l(x), C_m, L', (x, x, D_l, L_l) \cdot T)$. Since $m + l = |L|$, then $|L'| = m$ and since $\downarrow |T|_e = \downarrow$, then $\downarrow |T|_e + 1 = \uparrow$.

  $t = x, C = C_m(\langle D_l[x \leftarrow u] \rangle)$ and $L = L_l \cdot L'. \overline{\downarrow}$. Then $s_h = (u, C_m(\langle D_l[x \leftarrow u] \rangle), (x, D_l[x \leftarrow u], L_l) \cdot L', T)$. Since $m + l = |L|$, then $|L'| = m + 1$ which the level of $C_m(\langle D_l[x \leftarrow \rangle \rangle)$. Both $|T|_e$ and $d$ are unchanged.

  $t = u[x \leftarrow r]$. Then $s_h = (u, C(\langle \rangle[x \leftarrow r]), L, T)$. $C(\langle \rangle[x \leftarrow r])$ is context of level $n = |L|$. Both $|T|_e$ and $d$ are unchanged.

• $d = \uparrow$. The proof is equivalent to the one above.

Note that the tape $T$ of a balanced $\uparrow$-state always contains at least one logged position, which is why it can be seen as the answer to a query about the head variable.

The Exhaustible State Invariant. The study of the IALM requires to prove that some bad configurations never arise. On states such as $(\lambda x. D(x), C, L, p \cdot T)$, transition $\to_{IALM}$ requires the logged position $p$ to have shape $(x, \lambda x. D, L')$, that is, to contain a position isolating an occurrence of $x$ in $\lambda x. D(x)$, otherwise the machine is stuck. The exhaustive state invariant guarantees that the machine never gets stuck for this reason. The invariant being technical and involved, it is developed in Section 5. Nonetheless, it is one of the main technical contributions of this work.\footnote{Notice that proofs are already carried out in the more general framework of the linear substitution calculus, to be introduced in Section 6.} We only mention its main consequence.

Proposition 4.3 (Logged Positions Never Block the IALM). Let $s$ be a reachable state. If $s = (\lambda x. D(x), C, L, p \cdot T)$ then $p = (x, \lambda x. D, L')$.

Reversibility. The proof of Proposition 4.3 relies on a key property of the IALM, that is, bi-determinism, or reversibility: for each state $s$ there is at most one state $s'$ such that $s' \to_{IALM} s$. The property follows by simply inspecting the rules. Moreover, a run can be reverted by simply switching the direction.

Proposition 4.4 (Reversibility). If $(t, C, L, T, d) \to_{IALM} (u, D, L', T', f)$ then $(u, D, L', T', f') \to_{IALM} (t, C, L, T, d')$.

Final States. If the IALM starts on the initial state $s_{t,k} = (t, \langle \rangle, e, \bullet^k)$ the execution may either never stop or end in one of three possible final states. To explain them, let $\lambda x_0 \ldots \lambda x_j (y_{u_1} \ldots u_j)$ be the head normal form $\text{hnf}(t)$ of $t$. The three kinds of final states of the machine are:

• Failure $(\lambda x.t, C, L, e)$: this is the machine way of saying that $i > k$, that is, $\text{hnf}(t)$ has more head abstraction than those that the depth $k$ of the initial state $s_{t,k}$ allows to explore.

• Open success $(y, C, L, \bullet^y)$: the machine found the head variable, and it is the free variable $y$. Note that if $y$ is instead bound by a $\lambda$-abstraction, then the machine is not stuck.

\footnote{Exhaustible states play a role similar to Asperti and Laneve's legal paths [AL95] (for equivalent notions see [ADL94]), to which they are probably equivalent—the exact relationship is left to future work.}
\* Bound success \((t, \langle \cdot \rangle, L, \bullet^m \cdot p \cdot \bullet^n)\): the head variable has been found and it is \(y = x_m\). When the machine \(l\)-travels on the head variable \(y\), and it is abstracted, the logged position \(p\) containing \(x_m\) is put on the tape and the direction switches—the answer has been found. The sequence \(\bullet^m\) on top of tape in the final state comes from the \(\uparrow\) backtracking along the spine of \(\text{hnf}(t)\) for the equivalent of \(m\) abstractions, each one adding one \(\bullet\). At this point the IALM stops. Thus the abstraction binding \(y\) is \(\lambda x_m\).

Noticeably, the exhaustible state invariant and Proposition 4.3 together guarantee that the final states of the IALM can only have one of these three shapes.

The Semantics. The characterization of final states induces a semantic interpretation of terms, that we are going to show to be sound and adequate with respect to (linear) head evaluation.

**Definition 4.5** (IALM Semantics). We define the IALM semantics of \(\lambda\)-terms by way of a family of functions \([t]_k : \Lambda \to (\mathbb{N} \times \mathbb{N}) \cup \{\uparrow, \downarrow\} \cup V\), where \(k \in \mathbb{N}\), defined as follows.

\[
[t]_k = \begin{cases} 
(h, j) & \text{if } (t, \langle \cdot \rangle, \epsilon, \bullet^k) \text{ is terminating in the state } (t, \langle \cdot \rangle, \epsilon, \bullet^h \cdot p \cdot \bullet^j), \\
\uparrow & \text{if } (t, \langle \cdot \rangle, \epsilon, \bullet^k) \text{ is terminating in the state } (x, C, L, \bullet^k), \\
\downarrow & \text{if } (t, \langle \cdot \rangle, \epsilon, \bullet^k) \text{ is terminating in the state } (\lambda x. u, C, L, \epsilon), \\
\bot & \text{otherwise.}
\end{cases}
\]

**Pumping.** An important property that is used in the proofs is a sort of pumping lemma for the IALM tape \(T\). Intuitively, the IALM consumes the next entry of the initial input only when the question asked by the previous one(s) has been fully answered.

**Lemma 4.6** (Pumping). If \((t, C, L, \epsilon, d) \to^n_{IALM} (u, D, L', T, f)\), then \((t, C, L, T', d) \to^n_{IALM} (u, D, L', T', f)\).

**Proof.** We proceed by induction on \(n\). Thus we have that if \((t, C, L, \epsilon, d) \to^{n-1}_{IALM} (u, D, L', T, f)\), then \((t, C, L, T', d) \to^n_{IALM} (u, D, L', T', T', f)\). The proof now proceeds analyzing all possible transitions from \((u, D, L', T, f)\) and \((u, D, L', T', f)\). The key point is that every transition of the IALM consumes at most 1 element of the tape. This is why the pushed stack \(T'\) gets never touched. \(\square\)

**Monotonicity of Runs.** Last, increasing the input from \(\bullet^k\) to \(\bullet^{k+1}\) increases the length of the IALM run, except if the run of \(\bullet^k\) was successful.

We write \(|t|_k\) for the length of the IALM run of initial state \(s_{t,k} := (t, \langle \cdot \rangle, \epsilon, \bullet^k)\), that is for the length of the maximum sequence of transitions \(s_{t,k}\), if the IALM terminates, and \(|t|_k = \infty\) if the machine diverges. The next lemma compares run lengths, for which we consider that \(i < \infty\) for every \(i \in \mathbb{N}\) and \(\infty < \infty\). We also write \(s^n_{t,k}\) for the state such that \(s_{t,k} \to^n_{IALM} s^n_{t,k}\), if it exists.

**Lemma 4.7** (Monotonicity of runs). The length of runs cannot decrease if the input increases, that is, \(|t|_k \leq |t|_{k+1}\). Moreover, if \(|t|_k = n \in \mathbb{N}\) and the final state \(s^n_{t,k}\) is bound (resp. open) successful then \(|t|_k = |t|_h\) for every \(h > k\) and the final state \(s^n_{t,k}\) is bound (resp. open) successful.

**Proof.** Let \(s_{t,k} \to^n_{IALM} (u, C, L, T, d) = s^n_{t,k}\). By the pumping lemma (Lemma 4.6), if \(s_{t,k+1} \to^n_{IALM} (u, C, L, T, d) = s^n_{t,k+1}\). If \(|t|_k = \infty\) then \(s_{t,k} \to^n_{IALM} s^n_{t,k}\) for every \(n \in \mathbb{N}\) and so \(s_{t,k+1} \to^n_{IALM} s^n_{t,k+1}\), that is, \(|t|_{k+1} = \infty = |t|_k|\).

If \(|t|_k = n \in \mathbb{N}\) then \(s^n_{t,k}\) is final. Two cases. If \(s^n_{t,k}\) is an approximating final state \((\lambda x.t, C, L, \epsilon)\) then \(s^n_{t,k+1} = (\lambda x.t, C, L, \bullet)\) which can make a transition, that is, \(|t|_k < |t|_{k+1}\. If instead \(s^n_{t,k}\) is a bound successful final state \((t, \langle \cdot \rangle, L, \bullet^m \cdot p \cdot \bullet^n)\) then \(s^n_{t,k+1} = \langle t,\cdot \rangle, L, \bullet^m \cdot p \cdot \bullet^{n+1}\) which is also a successful final state, and \(|t|_{k+1} = |t|_k\). Similarly for an open successful final state. A straightforward induction then shows that the same holds for every other \(h > k\). \(\square\)
5 The Exhaustible State Invariant

This section defines and proves a key invariant of reachable IALM states. The intuition is that whenever a logged position \( p \) occurs in such a state, it is there for a reason, because no logged position occur in initial states, and transitions only add logged positions to which the machine is supposed to come back to. In particular, one can somehow revert the process which is responsible for having placed \( p \) in the state, and exhaust \( p \).

Why It Is Needed. The exhaustive state invariant is meant to show that some undesirable configurations never arise. On states such as \((\lambda x.D(x), C, L, p \cdot T)\) the IALM requires the logged position \( p \) to have the shape \((x, ax.D, L')\), that is, to be associated to a position isolating an occurrence of \( x \) in \( \lambda x.D(x) \), otherwise the machine is stuck. Similarly, on states such as \((t, C(D(x)[x \leftarrow \langle \cdot \rangle]), p \cdot L, T)\) the position of \( p \) is expected to isolate an occurrence of \( x \) in \( D(x) \), or the machine is stuck. Luckily, the machine never gets stuck for these reasons, and exhaustive states are the technical tool to prove it. Therefore, the invariant is needed to characterize the final states of the IALM.

One could redefine the transitions of the IALM asking—for these states—to jump to whatever variable position is in the logged position \( p \). In particular, one can somehow revert \((t, C)\) that they are stable by head translations of the position \((t, C)\).

Let \( m \) be the context \( C \) and \( k \) numbers \( p \) and \( \kappa \) logged positions \( p \) and \( k \) terms \( x \) and \( \lambda \). This section defines and proves a key invariant of reachable IALM states. The intuition is that whenever a logged position \( p \) occurs in such a state, it is there for a reason, because no logged position occur in initial states, and transitions only add logged positions to which the machine is supposed to come back to. In particular, one can somehow revert the process which is responsible for having placed \( p \) in the state, and exhaust \( p \).

Why It Is Needed. The exhaustive state invariant is meant to show that some undesirable configurations never arise. On states such as \((\lambda x.D(x), C, L, p \cdot T)\) the IALM requires the logged position \( p \) to have the shape \((x, ax.D, L')\), that is, to be associated to a position isolating an occurrence of \( x \) in \( \lambda x.D(x) \), otherwise the machine is stuck. Similarly, on states such as \((t, C(D(x)[x \leftarrow \langle \cdot \rangle]), p \cdot L, T)\) the position of \( p \) is expected to isolate an occurrence of \( x \) in \( D(x) \), or the machine is stuck. Luckily, the machine never gets stuck for these reasons, and exhaustive states are the technical tool to prove it. Therefore, the invariant is needed to characterize the final states of the IALM.

One could redefine the transitions of the IALM asking—for these states—to jump to whatever variable position is in the logged position \( p \). Then the IALM would not get stuck, and the invariant would not be needed for characterizing final states, but we would then need it for soundness—there is no easy way out.

First Reading? Then we suggest to skip this section, as the invariant is involved. It is nonetheless a key technical ingredient and one of the contributions of the paper. The key result used in the rest of the paper is Corollary 5.8.

Preliminaries. Exhaustible states rest on some test states for their logged positions. More specifically, each logged position \( p \) in a state \( s \) has an associated test state \( s_p \), supposed to test the exhaustibility of \( p \) in \( s \). Actually, there shall be two classes of test states, one accounting for the logged positions in the tape of \( s \), called IO states, and one for the those in the log of \( s \), called outer states.

Outer States. Let \((u, C_{n+1})\) be a position. Then, for every decomposition of \( n \) into two natural numbers \( m, k \) with \( m + k = n \), we can find contexts \( C_m \) and \( C_k \), and a term \( r \) satisfying the following conditions.

- Case \( t = C_m \langle r C_k \{u\} \rangle \). Then, the \( m + 1 \)-outer context of the position \((u, C_{n+1})\) is the context \( O_{m+1} := C_m \langle r \{\cdot \} \rangle \) of level \( m+1 \) and the \( m + 1 \)-outer position is \((C_k \{u\}, O_{m+1})\).

- Case \( t = C_m \langle r [x \leftarrow C_k \{u\}] \rangle \). Then, the \( m + 1 \)-outer context of the position \((u, C_{n+1})\) is the context \( O_{m+1} := C_m \langle r \{x \leftarrow \langle \cdot \rangle\} \rangle \) of level \( m+1 \) and the \( m + 1 \)-outer position is \((C_k \{u\}, O_{m+1})\).

It is easy to realize that any position having level \( n \) has unique \( m \)-outer context and \( m \)-outer position, for every \( 1 \leq m \leq n + 1 \), and that, moreover, outer positions are hereditary, in the following sense: the \( i \)-outer position of the \( m \)-outer position of \((u, C_{n+1})\) is exactly the \( i \)-outer position of \((u, C_{n+1})\).

Definition 5.1 (Outer State). Let \( s = (t, C_n, p_1 \cdots p_m \cdot T, d) \) be a state with \( 1 \leq m \leq n \), and \((u, O_m)\) be the \( m \)-outer position of \((t, C_n)\). The \( m \)-outer state of \( s \) is the state \( \text{out}_m(s) := (u, O_m, p_1 \cdots p_m \cdot T, d) \).

Note \( m \)-outer states do not depend on the direction of the state, nor on the underlying tape, and that they are stable by head translations of the position \((t, C_n)\), in the sense that if \( t = H(r) \) then \((t, C_n, L, T, d)\) and its head translation \((r, C_n \langle H \rangle, L, T', d)\) induce the same outer states (because the two positions have the same outer positions and their last components are themselves the same).

Lemma 5.2. Let \( s = (t, C_n, L_n, T, d) \) be a state. Then:
1. direction: the dual \((t, C_n, L_n, T, d')\) of \( s \) induces the same outer states;
2. tape: the state \((t, C_n, L_n, T', d)\) obtained from \( s \) replacing \( T \) with an arbitrary tape \( T' \) induces the same outer states;
3. head translation: if \( t = H(r) \) then the head translation \((r, C_n \langle H \rangle, L_n, T', d)\) of \( s \) induces the same outer states.
4. inclusion: if \( C_n = C_m \langle C_i \rangle \) and \( L_n = L_i \cdot L_m \) then the outer states of \((C_i \langle t \rangle, C_m, L_m, T', d)\) are outer states of \( s \).
Proof. The first three points are immediate consequences of the definition of outer state. We prove the fourth point. Let \( s' = (C_j(t), C_i, L_i, T, d) \). By induction on \( j \). If \( j = 0 \) then \( i = n \) and \( s = s' \), therefore the statement is simply says that the outer state of \( s \) is \( \text{out}(s) \), that is obviously true. Let \( j > 0 \). By i.h., the outer state \( s'' \) of \((C_{j-1}(t), C_{i+1}, p \cdot L_i, T, d)\) is \( \text{out}(s) \). Let us spell out \( s'' \). If \( C_{i+1} = C_i(\langle uC_i \rangle) \) then \( s'' = (u, C_i(\langle \cdot \rangle C_{j-1}(t)), L_i, p, \uparrow) \). Note that \( C_j = C_i(\langle u(\cdot) \rangle) \). Since \( s'' = \text{out}_i(s) \), we have \( \text{out}_{i-1}(s) = \text{out}(s'') \). Now, since outer states are stable by head translation (Point 3), we have that \( \text{out}_i(s) \) is also the outer state of the translation of \( s'' \) with respect to \( C_{j-1}(t) \), that is, of the state \((uC_{j-1}(t), C_i, L_i, p, \uparrow) = (C_j(t), C_i, L_i, p, \uparrow) \). \( \square \)

**IO States.** The second class of states we are interested in are those in which we put in evidence the logged positions in the tape.

**Definition 5.3 (IO States).** Let \( s = (t, C_n, L_n, T, d) \) be a state. Then the IO-states of \( s \) are those states which can be written as \((t, C_n, L_n, T' \cdot p, [T' \cdot \rho])\) where \( T = T_0 \cdot p \cdot T'' \) is any decomposition of the tape \( T \) of \( s \).

The notions of IO and Outer states implicitly put in evidence one logged position, which is the one of which we shall test the nature in our invariant. This is captured in the following definition.

**Definition 5.4 (IO and Outer Logged Positions).** Let \( s = (t, C_n, L_n, T, d) \) be a state. Then the IO position of \( s \) is the logged position \( p \) such that \( T = T' \cdot p \), if any. The outer logged position of \( s \), instead, is the leftmost logged position in \( L_n \).

Finally, to any logged position one can associate a set of states, dubbed its logged states, which are those coherent with the variable in the state.

**Definition 5.5 (Logged State).** Given a logged position \( p = (t, D, L') \), those states in the form \((t, C_n(D), L' \cdot L_n, \epsilon)\) are said to be generated by \( p \).

The Exhaustibility Invariant. After having introduced all the necessary preliminary ingredients, we can now state the invariant.

**Definition 5.6 (Exhaustible States).** \( \mathcal{E} \) is the smallest set of those states \( s \) such that the following conditions hold.

1. **IO Decomposition:** for any IO state \( s' \) of \( s \), it holds that \( s' \rightarrow_{IALM} s'' \in \mathcal{E} \), where \( s'' \) is generated by the IO position of \( s' \).
2. **Outer Decomposition:** for any outer state \( s' \) of \( s \), it holds that \( s' \rightarrow_{IALM} s'' \in \mathcal{E} \), where \( s'' \) is generated by the outer logged position of \( s' \).

States in \( \mathcal{E} \) are called exhaustible.

Informally, exhaustible states are those for which every outer and IO state can be exhausted, i.e., rewritten into their logged states (themselves exhaustible). The set \( \mathcal{E} \) being the smallest set of such states implies that checking that a state is exhaustible can be finitely certified, i.e. there must be a finitary proof of it.

**Lemma 5.7 (Exhaustible invariant).** Let \( s \) be a IALM reachable state. Then \( s \) is exhaustible.

The proof of Lemma 5.7 is long, but logically quite simple, being structured around a simple induction on the length of the run from the initial state to \( s \).

Proof. Let \( s = (t, (\cdot), \epsilon, \cdot, \cdot^k) \). By induction on \( k \). For \( k = 0 \) there is nothing to prove because the IO stack has no logged positions (so it does not decompose) and \( s \) has no outer state. Then suppose \( s \rightarrow_{IALM}^{k-1} s'' \rightarrow_{IALM} s' \). By i.h. \( s'' = (u, C, L, T, d) \) is exhaustible, and with this hypothesis we need to conclude that \( s' \) is exhaustible, too. There are many cases to take into account, depending on the transition used to move from \( s'' \) to \( s' \). First, suppose that \( d = \downarrow \). Cases of \( s'' \rightarrow_{IALM} s' \):
1. **Application, i.e.** \( u \equiv \text{rw} \)

\[
(rw, C, L, T) \rightarrow_{\text{1}} (\text{r}, C(\lambda x. w), L, \bullet \cdot T).
\]

We have to show that the obtained state \( s' \) is exhaustible. For outer decomposition, it follows from Lemma 5.2.3 and the i.h.: \( s'' \) is a head translation of \( s' \), and the lemma states that they have the same outer states, which are exhaustible because \( s'' \) is exhaustible by i.h.

For IO decomposition, consider a decomposition \( T = T' \cdot p \cdot T'' \). Two cases, depending on the parity of \( |T'|_p \):

(a) \( |T'|_p \) is odd. Then the exponential length of the tape \( \bullet \cdot T' \cdot p \) is even (multiplicative constants are ignored) and so the direction of the corresponding IO-state \( s'_p \) is \( \downarrow \). Note that \( s'_p \) reduces to a IO-state \( s''_p \) for \( s'' \):

\[
s'_p = (\text{r}, C(\lambda x. w), L, \bullet \cdot T' \cdot p) \rightarrow_{\text{13}} (\text{rw}, C, L, T' \cdot p) = s''_p
\]

By i.h., \( s'' \) is exhaustible, and so \( s''_p \) evolves to an exhaustible state generated by \( p \), call it \( q_p \).

We conclude by observing that \( s'_p \) evolves to \( q_p \).

(b) \( |T'|_p \) is even. Then \( \bullet \cdot T' \cdot p \) is odd, and the direction of the corresponding IO-state \( s'_p \) is \( \uparrow \). Note that the corresponding IO-state \( s''_p \) of \( s'' \) reduces to \( s'_p \):

\[
s''_p = (\text{rw}, C, L, T'' \cdot p) \rightarrow_{\text{13}} (\text{r}, C(\lambda x. w), L, \bullet \cdot T' \cdot p) = s'_p
\]

By i.h., \( s'' \) is exhaustible, then \( s''_p \) evolves to an exhaustible state generated by \( p \), call it \( q_p \).

The IAM is deterministic, so \( s'_p \) itself reduces to \( q_p \), which proves IO decomposition.

2. **Variable bound by an abstraction, i.e.** \( u = x \)

\[
s'' = (x, C(\lambda x. D_n), L_n \cdot L, T) \rightarrow_{\text{var}} (\lambda x. D_n \langle x \rangle, C, L, (x, \lambda x. D_n, L_n) \cdot T) = s'
\]

The proof that \( s' \) is exhaustible is divided in two parts:

(a) **Outer decomposition.** By Lemma 5.2.11 all outer states of \( s' \) are also outer-states of \( s'' \). Since the latter is exhaustible by i.h., then outer decomposition holds for \( s' \), too.

(b) **IO decomposition.** We need to consider various cases, corresponding to the various decompositions of the tape \( p' \cdot T \) where \( p' = (x, \lambda x. D_n, L_n) \):

i. **The logged position to test is** \( p = p' \), i.e. the first one. We are then considering a prefix of odd length of \( p' \cdot T \), so the direction of the corresponding IO-state \( s'_p \) is \( \uparrow \). Observe, however, that by definition

\[
s'_p = (\lambda x. D_n \langle x \rangle, C, L, (x, \lambda x. D_n, L_n)) \rightarrow_{\text{bt2}} (x, C(\lambda x. D_n), L_n \cdot L, \epsilon) = s''_{\uparrow}
\]

where \( s''_{\uparrow} \) is trivially generated by \( p \). Moreover, by i.h., \( s'' \) is exhaustible, a property which is easily transferred to \( s''_{\uparrow} \): the outer states are the same by Lemma 5.2.11 while \( s''_{\uparrow} \) satisfies IO decomposition trivially, because the tape is empty.

ii. **The prefix** \( T' \cdot p \) **of the tape has even length** and the direction of the corresponding IO state \( s'_p \) is \( \downarrow \). Let \( T' = (x, \lambda x. D_n, L_n) \cdot T'' \). Note that the corresponding IO-state \( s''_p \) of \( s'' \) reduces to \( s'_p \).

\[
s''_p = (x, C(\lambda x. D_n), L_n \cdot L, T'' \cdot p) \rightarrow_{\text{var}} (\lambda x. D_n \langle x \rangle, C, L, (x, \lambda x. D_n, L_n) \cdot T'' \cdot p) = s'_p
\]

By i.h., \( s'' \) is exhaustible, then \( s''_p \) evolves to an exhaustible state generated by \( p \), call it \( q_p \). The IAM is deterministic, so \( s'_p \) itself reduces to \( q_p \), which proves IO decomposition.
i. The prefix $T' \cdot p$ of the tape has odd strictly positive length and the direction of the corresponding outer state $s'_p$ is $\uparrow$. Let $T'' = (x, \lambda x.D_n, L_n) \cdot T''$. Note that $s''_p$ reduces to the corresponding outer state $s''_p$ of $s''$:

$$s''_p = (\lambda x.D_n(x), C, L, (x, \lambda x.D_n, L_n) \cdot T'' \cdot p) \rightarrow_{\text{bt2}} (x, C(\lambda x.D_n), L_n \cdot L, T'' \cdot p) = s''_p$$

We can then proceed as usual using the i.h.

3. Abstraction, i.e. $u = \lambda x.r$ and $\lambda x.D$

$$s'' = (\lambda x.r, C, L, (x, \lambda x.D_n, L') \cdot T) \rightarrow_{\text{bt2}} (x, C(\lambda x.D_n), L' \cdot L, T) = s'$$

(a) Outer Decomposition. Let $p = (x, C(\lambda x.D_n), L')$ and note that

$$s''_p = (\lambda x.r, C, L, (x, \lambda x.D_n, L')) \rightarrow_{\text{bt2}} (x, C(\lambda x.D_n), L' \cdot L, p) = s'_p$$

By i.h. $s''$ is exhaustible and so by IO decomposition $s'_p$ is exhaustible. By Lemma 5.2.2, $s'_p$ and $s'$ have the same outer states, so outer decomposition for $s'$ holds because it does for $s'_p$.

(b) IO decomposition. As usual, we have to consider various cases, corresponding to the possible decompositions $T = T'' \cdot p \cdot T''$ of the tape.

i. $|T'_p| \text{ is odd}$, so that the prefix $T' \cdot p$ of the tape has even length and the direction of the IO-state $s'_p$ corresponding to $p$ is $\uparrow$. Note that the IO-state $s''_p$ of $s''$ reduces to the corresponding IO-state $s''_p$ of $s'$:

$$s''_p = (\lambda x.D_n(x), C, L, (x, \lambda x.D_n, L') \cdot T' \cdot p) \rightarrow_{\text{bt2}} (x, C(\lambda x.D_n), L' \cdot L, T' \cdot p) = s'_p$$

We can then proceed as usual, exploiting the determinism of the IALM and the i.h.

ii. $|T'_p| \neq 0 \text{ is even}$, so that the prefix $T' \cdot p$ of the tape has odd length and the direction of the IO-state $s'_p$ corresponding to $p$ is $\downarrow$. Note that $s''_p$ reduces to the corresponding IO-state $s''_p$ of $s''$:

$$s'_p = (x, C(\lambda x.D_n), L' \cdot L, T' \cdot p) \rightarrow_{\text{var}} (\lambda x.D_n(x), C, L, (x, \lambda x.D_n, L') \cdot T' \cdot p) = s''_p$$

Again, we can then proceed as usual using the i.h.

4. Explicit Substitution, i.e. $u = r[x\leftarrow w]$ and $\lambda x.D$

$$s'' = (r[x\leftarrow w], C, L, T) \rightarrow_{\text{es}} (r, C(\lambda x.D[x\leftarrow w]), L, T) = s'$$

For outer decomposition, it follows from Lemma 5.2.3 and the i.h.: $s''$ is a head translation of $s'$, and the lemma states that they have the same outer states, which are exhaustible because $s''$ is exhaustible by i.h.

For IO decomposition it goes exactly as the application case. We spell it out anyway. Consider a decomposition $T = T'' \cdot p \cdot T''$. Two cases, depending on the parity of $|T'_p|$: 

(a) $|T'_p|$ is odd. Then the exponential length of the tape $T' \cdot p$ is even and so the direction of the corresponding IO-state $s'_p$ is $\uparrow$. Note that $s''_p$ reduces to a IO-state $s''_p$ for $s''$:

$$s''_p = (r, C(\lambda x.D[x\leftarrow w]), L, T' \cdot p) \rightarrow_{\text{es2}} (r[x\leftarrow w], C, L, T'' \cdot p) = s''_p$$

Again, we then proceed as usual using the i.h.

(b) $|T'_p|$ is even. Then $|T' \cdot p|_p$ is odd, and the direction of the corresponding IO-state $s'_p$ of $s'$ is $\downarrow$. Note that the corresponding IO-state $s''_p$ of $s''$ reduces to $s''_p$:

$$s''_p = (r[x\leftarrow w], C, L, T' \cdot p) \rightarrow_{\text{es}} (r, C(\lambda x.D[x\leftarrow w]), L, T' \cdot p) = s'_p$$

Again, we then proceed as usual, exploiting the determinism of the IALM and the i.h.
5. Variable bound by an explicit substitution, i.e. $u = x$ and
\[ s'' = (x, C(D_n[x\leftarrow r]), L_n \cdot L, T) \rightarrow_{\text{var}2} (r, C(D_n[x\leftarrow \cdot]), (x, D_n[x\leftarrow r], L_n) \cdot L, T) = s' \]

(a) Outer decomposition: let $p := (x, D_n[x\leftarrow r], L_n)$ and $m = \lceil p \cdot L \rceil$. The $m$-outer state of $s'$ is
\[ \text{out}_m(s) = (r, C(D_n[x\leftarrow \cdot]), (x, D_n[x\leftarrow r], L_n) \cdot L, \epsilon) \]
which makes a transition
\[ \rightarrow_{\text{var}3} (x, C(D_n[x\leftarrow r]), L_n \cdot L, \epsilon) = (s'')' \]
that is a state generated by $p$, as required by outer decomposition. We have to prove that $(s'')'$ is exhaustible. IO decomposition is trivial, because the tape is empty. Outer decomposition follows from the i.h. decomposition clause by the i.h.

(b) IO decomposition: it goes exactly as in the previous ordinary cases. We spell it out anyway.

Consider a decomposition $T = T' \cdot p \cdot T''$. Two cases, depending on the parity of $|T''|_p$:

i. $|T''|_p$ is odd. Then the exponential length of the tape $T' \cdot p$ is even and so the direction of the corresponding IO-state $s'_p$ is ↑. Note that $s'_p$ reduces to a IO-state $s''_p$ for $s''$:
\[ s'_p = (r, C(D_n[x\leftarrow \cdot]), (x, D_n[x\leftarrow r], L_n) \cdot L, T' \cdot p) \rightarrow_{\text{var}3} (x, C(D_n[x\leftarrow r]), L_n \cdot L, T' \cdot p) = s''_p \]
Again, we then proceed as usual using the i.h.

ii. $|T''|_p$ is even. Then $|T' \cdot p|_p$ is odd, and the direction of the corresponding IO-state $s'_p$ is ↓. Note that the corresponding IO-state $s''_p$ of $s''$ reduces to $s'_p$:
\[ s''_p = (x, C(D_n[x\leftarrow r]), L_n \cdot L, T' \cdot p) \rightarrow_{\text{var}2} (r, C(D_n[x\leftarrow \cdot]), (x, D_n[x\leftarrow r], L_n) \cdot L, T' \cdot p) = s'_p \]
Again, we then proceed as usual, exploiting the determinism of the IALM and the i.h.

Now, suppose that $d = \uparrow$. Cases of $s'' \rightarrow_{IALM} s'$:

1. Coming from the left of an application, i.e. $C = D(\langle \cdot \rangle r)$ and
\[ s'' = (u, D(\langle \cdot \rangle r), L, p \cdot T) \rightarrow_{\text{arg}} (r, D(u\langle \cdot \rangle), p \cdot L, T) = s'. \]

The proof that $s'$ is exhaustible is divided in two parts:

(a) Outer decomposition. The outer states of $s'$ are those of $s''$ plus $(r, D(u\langle \cdot \rangle), p \cdot L, \epsilon)$. The former are line because of the i.h., while about the latter, observe that $(r, D(u\langle \cdot \rangle), p \cdot L, \epsilon)$ evolves to $(u, D(\langle \cdot \rangle r), L, p)$ which is an IO-state of $s''$. The thesis easily follows.

(b) IO decomposition. Let $T'$ be a prefix of $T$ such that $T'' = T'' \cdot p'$. Two cases:

i. $|T''|_p$ is odd, and the direction is ↓. Note that the IO-state $s''_p$ of $s''$ corresponding to $p$ reduces to an IO-state $s'_p$ of $s'$:
\[ s''_p = (r, D(u\langle \cdot \rangle), L, p \cdot T') \rightarrow_{\text{arg}} (r, D(u\langle \cdot \rangle), p \cdot L, T') = s'_p \]
We can then proceed as usual, using the i.h. and determinism of the IAM.

ii. $|T''|_p$ is even, and the direction is ↑. Note that $s'_p$ reduces to the corresponding IO-state $s''_p$ of $s'$:
\[ s'_p = (r, D(u\langle \cdot \rangle), p \cdot L, T') \rightarrow_{\text{bt}1} (r, D(u\langle \cdot \rangle), L, p \cdot T') = s''_p \]
Again, we can proceed as usual, using the i.h.
2. **Coming from the right of an application**, i.e. $C = D(r(\cdot))$ and

$$s'' = (u, D(r(\cdot)), p \cdot L, T) \rightarrow_{bt1} (r, D(\langle \cdot \rangle u), L, p \cdot T) = s'.$$

The proof that $s'$ is exhaustive is divided in two parts:

(a) **Outer decomposition**: the outer states of $s'$ are among the outer states of $s''$, so outer decomposition follows from *i.h.*

(b) **IO decomposition**. Let $T'$ be a prefix of $T$. Two cases:

i. $T' = T$ is empty. So that the tape contains only $p$, its length is odd, and the direction is $\downarrow$. The state to be proven exhaustible is

$$s'_p = (r, D(\langle \cdot \rangle u), L, p)$$

Now, note that the outer state $\text{out}_{|p|L}(s'')$ of $s''$ reduces in one step to $s'_p$:

$$\text{out}_{|p|L}(s'') = (u, D(r(\cdot)), p \cdot L, \epsilon) \rightarrow_{bt1} (r, D(\langle \cdot \rangle u), L, p)$$

By outer decomposition for $s''$, there is a state $q_p$ generated by $p$ such that $\text{out}_{|p|L}(s'') \xrightarrow{iALM} q_p$. By determinism of the IAM, $s'_p \rightarrow_{iALM} q_p$.

ii. $T' \neq T$ is non-empty. Then $T' = T'' \cdot p'$ Two cases:

A. $|T'' \cdot p'|_p$ is even, so that the tape $p \cdot T'' \cdot p'$ has odd length and the direction is $\downarrow$. Note that the IO-state $s''_{p'}$, corresponding to $p'$ of $s''$ reduces to the IO-state $s''_{p'}$ corresponding to $p'$ of $s'$:

$$s''_{p'} = (r, D(\langle \cdot \rangle u), p \cdot L, T'', p') \rightarrow_{bt1} (r, D(\langle \cdot \rangle u), L, p \cdot T'' \cdot p') = s''_{p'}$$

In this case, as usual, we can conclude by determinism of the IALM.

B. $|T'|_p$ is odd, so that the tape $p \cdot T'' \cdot p'$ has even length and the direction is $\uparrow$. Note that $s''_{p'}$ reduces to the corresponding IO-state $s''_{p'}$ of $s''$:

$$s''_{p'} = (r, D(\langle \cdot \rangle u), L, p \cdot T'' \cdot p') \rightarrow_{arg} (r, D(\langle \cdot \rangle u), p \cdot L, T'' \cdot p') = s''_{p'}$$

Again, the usual scheme allows us to conclude that IO decomposition holds.

3. **Explicit Substitution**

$$s'' = (u, C(\langle \cdot \rangle[x \leftarrow r]), L, T) \rightarrow_{es2} (u[x \leftarrow r], C, L, T) = s'$$

(a) **Outer decomposition**: by Lemma 5.2.2 (head translation), the outer states of $s'$ are outer states of $s''$, which satisfy outer decomposition by the *i.h.*

(b) **IO decomposition**: it goes exactly as for the other ordinary cases (*i.h.*, plus determinism in one of the two sub-cases).

4. **Coming from inside an explicit substitution**:

$$s'' = (u, C(D(x)[x \leftarrow \cdot]), L', L, T) \rightarrow_{var3} (x, C(D[x \leftarrow u]), L' \cdot L, T) = s'$$

(a) **Outer decomposition**: by *i.h.*, $s''$ is exhaustible, and its $|L|$ + 1-out state evolves to

$$\text{out}_{|L|+1}(s'') = (u, C(D(x)[x \leftarrow \cdot]), (x, D(x \leftarrow u), L', L, \epsilon) \rightarrow_{var3} (x, C(D[x \leftarrow u]), L' \cdot L, \epsilon) = s'_p$$

which is exhaustible. By Lemma 5.2.4, $s'_p$ and $s'$ have the same outer states, that then verify the outer decomposition clause.

(b) **IO decomposition**: since the tape is unaffected by the transition, this case goes exactly as the other ordinary ones.
Exhaustible and Final States. We are now ready to prove that the IALM never gets stuck for a mismatch of logged positions.

**Corollary 5.8 (Logged Positions Never Block the IALM).** Let $s$ be a reachable state.
1. If $s = (\lambda x. D(x), C, L, p \cdot T)$ then $p = (x, \lambda x. D, L')$.
2. If $s = (u, C[D(x)[x \leftarrow \langle y \rangle]], p \cdot L, T)$ then $p = (x, D[x \leftarrow u], L')$.

**Proof.**
1. By Lemma 5.7, $s$ is exhaustible. By IO decomposition, $(\lambda x. D(x), C, L, p) \rightarrow^*_{IALM} (y, D, L'', \epsilon)$ for some $y$, $D$, and $L''$, thus $s$ is not stuck, i.e. it can make a transition. This is possible only if $p = (x, \lambda x. D, L')$.
2. By Lemma 5.7 $s$ is exhaustible. Then, by outer decomposition, its outer state $s' = (u, C[D(x)[x \leftarrow \langle y \rangle]], p \cdot L, \epsilon)$ evolves into $(z, F, L'', \epsilon)$ for some $z$, $F$, and $L''$, thus $s'$ is not stuck, i.e. it can make a transition. This is possible only if $p = (x, D[x \leftarrow u], L')$.

6 Correctness of the IALM

In a nutshell, proving the IALM correct amounts to showing that the interpretation $[t]$ of Def. 4.5 is a sound and adequate denotational semantics for $\lambda$-terms with respect to head reduction. Soundness is the invariance of $[t]$ by head reduction. Adequacy is the fact that $[t]$ reflects the observable behavior of $t$, that is, termination in the case of weak evaluation. In the rest of this section, we compare this notion with the fundamentally different notion of correctness of environment machines.

**Soundness of Environment Machines.** An environment abstract machine $M$ executes a term $t$ according to a strategy $\rightarrow$ if from the initial state $s_1$ of code $t$ it computes a representation of the normal form $\text{hnf}(t)$. In particular, the machine somehow maintains the representation of how the strategy $\rightarrow$ modifies the term $t$ they both evaluate. Soundness is a weak bisimulation between the transitions $s \rightarrow_M s'$ of the machine and the steps $t \rightarrow u$ of the strategy. In particular, a run $\rho_t$ of the machine on $t$ passes through some states representing $u$, and the final states $s_f$ of the machine decode to $\rightarrow$-normal forms.

**Soundness of the IALM.** The IALM, and more generally GoI machines, do implement strategies, but in a different way. The IALM has many initial states, therefore many runs, for a given code $t$, one for each possible depth $k \in \mathbb{N}$. Moreover, the machine does not trace how the strategy modifies the term. If $t \rightarrow_k u$, a run of code $t$ never passes through a representation of $u$, soundness denotes something else. The idea is that, on a fixed input, the run of code $t$ is bisimilar to the run of code $u$. Notably, the latter is shorter—rewriting the code is a way of improving the associated IALM. Notice the difference with environment machines: there the bisimulation is between steps on terms and transitions on states. For the IALM, it is between transitions on states (of code $t$) and transitions on states (of code $u$).

**On Not Computing Results.** Another difference is that the IALM does not compute a code representation of the result $\text{hnf}(t)$. It recovers the micro information $[t]_k$ about it, by exploring only the immutable code $t$. This is in accordance with other models: space-sensitive Turing machines do not compute the whole output but only single bits of it. To compute the spine of $\text{hnf}(t)$, one needs to compute $[\ell]_k$ for various values of $k$, one for each abstraction of the spine, starting each time with a different input $T = \bullet^k$, and then once more for the head variable (adding a $\bullet$), if the machine ever terminates. On a head normal form $t$, the runs of the IALM become an immediate interactive reading of the spine of $t$. Inputs represent questions about the head of the normal form, the answer is encoded in the tape at the end of the run, when the run succeeds.

**Adequacy.** Soundness does not imply correctness. A trivial semantics where every object is mapped on the same element, for example, is sound but not informative. Adequacy guarantees that the interpretation $[t]$ reflects some observable aspects of $t$ and vice versa. For a head strategy in an untyped
calculus, one usually observes termination, and, if it holds, the identity of the head variable. And this is exactly what \([t]\) reflects, or is it adequate for.

## 7 Micro-Step Refinement

The proof of correctness of the IALM is unfortunately very hard to be directly carried out with respect to head reduction: this is specified using meta-level substitutions, here noted \(t[x\leftarrow u]\), which is a macro operation, potentially making many copies of \(u\) and modifying \(t\) in many places. It is very hard—if possible at all—to define explicitly a bisimulation of IALM runs (as required for soundness) that relates states whose code is modified by meta-level substitution.

We are then led to switch to \textit{linear head reduction}, a refinement of head reduction in which substitution is performed in \textit{micro-steps}, replacing only the head variable occurrence, and keeping the substitution suspended for all the other occurrences. This is also the approach followed by Danos and Regnier [DR99]. We depart from their approach, however, in the way we formally define linear head reduction. We adopt a formulation where the suspension of the substitution if formalized via a sharing constructor, which is more compact but conflates different concepts and makes the technical development less clean.

An important point is that head reduction and its linear variant are observationally equivalent, that is, one terminates on \(t\) if and only if the other terminates on \(t\), and they do produce the same head variable.

\textit{The Adopted Presentation.} Linear head reduction (shortened to LHR) was introduced by Mascari & Pedicini and Danos & Regnier [MP94, DR04] as a strategy on proof nets. It is to proof nets for the \(\lambda\)-calculus what head evaluation is to the \(\lambda\)-calculus [AK10].

The presentation adopted here, noted \(\sim\), was introduced by Accattoli [Acc12, Acc18b], formulated as a strategy in a \(\lambda\)-calculus with explicit sharing, the \textit{linear substitution calculus} \(\mathbb{LSC}\). The LSC presentation of \(\sim\) is isomorphic to the one on proof nets [Acc18b], while the one used by Danos and Regnier—although closely related to proof nets—is not, it is isomorphic only up to Regnier’s \(\sigma\)-equivalence [Reg94].

LSC Terms and Levelled contexts. Let \(V\) be a countable set of variables. Terms of the \textit{linear substitution calculus} (LSC) are defined by the following grammar.

\[
\begin{align*}
\text{LSC TERMS} & \quad \begin{array}{c}
\alpha \in \mathcal{V} & | & \lambda x.t & | & tu & | & t[x\leftarrow u].
\end{array} \\
\end{align*}
\]

The construct \(t[x\leftarrow u]\) is called an \textit{explicit substitution} or ES, not to be confused with the meta-level substitution \(t[x\leftarrow u]\). As it is standard, \(t[x\leftarrow u]\) binds \(x\) in \(t\), but not in \(u\)—terms are still considered up to \(\alpha\)-conversion. Levelled contexts are also naturally extended to the LSC.

\[
\begin{align*}
\text{LEVELLED CONTEXTS} & \quad \begin{array}{c}
C_0 & ::= & \langle \cdot \rangle & | & \lambda x.C_0 & | & C_0t & | & C_0[x\leftarrow t]; \\
C_{n+1} & ::= & \lambda x.C_{n+1} & | & C_{n+1}t & | & C_{n+1}[x\leftarrow t] & | & tC_n & | & t[x\leftarrow C_n].
\end{array} \\
\end{align*}
\]

Contexts and Plugging. The LSC makes a crucial use of contexts to define its operational semantics. First of all, we need \textit{substitution contexts}, that simply packs together ES:

\[
\begin{align*}
\text{SUBSTITUTION CONTEXTS} & \quad \begin{array}{c}
S & ::= & \langle \cdot \rangle & | & S[x\leftarrow t].
\end{array} \\
\end{align*}
\]

When plugging is used for substitution contexts, we write it in a post-fixed manner, that is \(\langle t \rangle S\), to stress that the ES actually appears on the right of \(t\).

Linear Head Reduction. The LSC comes with a notion of reduction that resembles the decomposed, micro-step process of cut-elimination in linear logic proof-nets. Essentially, the meta-level substitution

\footnote{The LSC is a subtle reformulation of Milner’s calculus with explicit substitutions [Mil07, KC08], inspired by Accattoli and Kesner structural \(\lambda\)-calculus [AK10].}
that turn a garbage collection complete the evaluation:

\[ H(t) \rightarrow_{\text{lhnf}} H(t)[x \leftarrow t] \]

Two micro substitutions on the head, followed by two steps of LSC, by simply considering (logged) positions with respect to the extended syntax, and adding the 4 → \( \beta \) up to that

ES, that is turning them into meta-level substitutions. A linear head normal form has the unfolding \( lhnf \) of \( S \) may be surrounded by a substitution context \( \lambda x \)

same shape \( \lambda x \) surrounds \( S \) unfolding the term in (1) one obtains the head normal form \( \top \)

that we do not have any restriction on closed terms, and thus the number of linear head evaluation in the literature, in particular its relationship with head evaluation is well known. On a given term \( \top \) it to stress that the IALM is invariant also with respect to → \( \beta \) definition of

In plugging \( t \) in \( H \), rule \( \rightarrow_{\text{ls}} \) may perform on-the-fly renaming of bound variables in \( H \), to avoid capture of free variables of \( t \). Often, the literature does not include rule \( \rightarrow_{\text{gc}} \), responsible for erasing steps, in the definition of \( \rightarrow \). The reason is that \( \rightarrow_{\text{gc}} \) is strongly normalizing and it can be postponed. We include it to stress that the IALM is invariant also with respect to \( \rightarrow_{\text{gc}} \) on open terms.

Note that our definition of \( \rightarrow \) allows more than one \( \rightarrow \) redex at a time in a term. It is not a problem, as \( \rightarrow \) has the diamond property—this is standard.

Relationship with Head Evaluation, and Normal Forms. Linear head evaluation is studied at length in the literature, in particular its relationship with head evaluation is well known. On a given term \( t \), linear head evaluation \( \rightarrow \) terminates on the linear head normal form \( \text{lhnf}(t) \) if and only if head evaluation \( \rightarrow_{h} \) terminates on the head normal form \( \text{hnf}(t) \). Moreover, \( \text{hnf}(t) \) is obtained from \( \text{lhnf}(t) \) by simply unfolding ES, that is, turning them into meta-level substitutions. A linear head normal form has the same shape \( \lambda x_{1} . . . . \lambda x_{k} . (y_{t_{1}} . . . . . t_{h}) \) of a head normal form but for the fact that each spine sub-term may be surrounded by a substitution context \( S \), that is, they have the cumbersome shape (where \( S_{i} \) surrounds \( \lambda x_{1} . . . . \lambda x_{k} . (y_{t_{1}} . . . . . t_{h}) \) and \( S'_{j} \) surrounds \( y_{t_{1}} . . . . . t_{j} \)):

\[
(\lambda x_{1} . . . . \lambda x_{k} . (((S_{i_{1}} . . . . . S_{i_{h}}))S_{b} . . . . . S_{2} ))S_{1}
\]

\( \text{(1)} \)

where none of the ES in \( S_{i} \) and \( S'_{j} \) binds \( y \) (otherwise there would be a \( \rightarrow_{\text{ls}} \) redex). Unfolding the ES of a linear head normal form produces a head normal form having the same spine structure, that is, with the same abstractions, the same head variable and the same number of arguments—concretely, unfolding the term in (1) one obtains the head normal form \( \lambda x_{1} . . . . \lambda x_{k} . (y_{u_{1}} . . . . . u_{h}) \) for some \( u_{1} . . . . . u_{h} \). Therefore, in the paper we shall refer to a \( \rightarrow \)-normal term up to substitution \( \lambda x_{1} . . . . \lambda x_{k} . (y_{u_{1}} . . . . . u_{h}) \) meaning that we harmlessly ignore the substitution contexts around the spine sub-terms. Please note that we do not have any restriction on closed terms, and thus the number of \( \lambda \)-abstractions in the spine of \( \text{hnf}(t) \) and \( \text{hnf}(t) \) could also be 0.

Example 7.1. We provide here an example of LHR sequence. Consider the following 3 steps:

\[
(\lambda x . xx)(\lambda y . y) \rightarrow_{\text{db}} (xx)[x \leftarrow \lambda y . y] \rightarrow_{\text{ls}} ((\lambda y . y)[x \leftarrow \lambda y . y] \rightarrow_{\text{db}} y[y \leftarrow x][x \leftarrow \lambda y . y]
\]

that turn a \( \beta \)/multiplicative redex into a ES, substitute on the head variable occurrence, and continue with another multiplicative step. Two micro substitution steps on the head, followed by two steps of garbage collection complete the evaluation:

\[
y[y \leftarrow x][x \leftarrow \lambda y . y] \rightarrow_{\text{ls}} x[x \leftarrow \lambda y . y] \rightarrow_{\text{ls}} (\lambda y . y)[y \leftarrow x][x \leftarrow \lambda y . y] \rightarrow_{\text{gc}} \lambda z . z
\]

Additional IALM transitions. The IALM presented in the previous sections is easily adapted to the LSC, by simply considering (logged) positions with respect to the extended syntax, and adding the 4 transitions for ES in Fig. 2. Transitions \( \rightarrow_{\text{es}} \) and \( \rightarrow_{\text{es2}} \) simply skips ES during search—now search is up to \( \beta \)-redexes and ES. Transition \( \rightarrow_{\text{var2}} \) shortcuts the search of the term \( u \) to substitute for \( x \), given that \( u \) is already available in \( [x \leftarrow u] \). Therefore, the machine stays in the ↓ phase and moves to evaluate
Proposition 8.2. Let $\mathcal{R}$ be an improvement on two DTS $\mathcal{S}$ and $\mathcal{Q}$, and $s\mathcal{R}q$.

1. Termination equivalence: $s \in \mathcal{S}_1$ if and only if $q \in \mathcal{Q}_1$.
2. Improvement: $|s| \geq |q|$.

8 Im-Proving Soundness and Adequacy

Bisimulations and Improvements. We now introduce the notion of improvement—a refinement of the classical notion of bisimulation inspired by Sands [San96]—we are going to use to prove soundness of the IALM. They are weak-bisimulations between two transition systems preserving termination and guaranteeing that whenever $s$ and $q$ are related and terminating then $q$ terminates in no more steps than $s$—the no-more-steps part implies that the definition is asymmetric in the way it treats the two transition systems.

Preliminaries for Bisimulations. A deterministic transition system (DTS) is a pair $\mathcal{S} = (S, \mathcal{T})$, where $S$ is a set of states and $\mathcal{T} : S \rightarrow S$ a partial function. If $\mathcal{T}(s) = s'$, then we write $s \rightarrow s'$, and if $s$ rewrites in $n$ steps then we write $s \rightarrow^n s'$. We note with $\mathcal{F}_S$ the set of final states, i.e. the subset of $S$ containing all $s \in S$ such that $\mathcal{T}(s)$ is undefined. A state $s$ is terminating if there exists $n \geq 0$ and $s' \in \mathcal{F}_S$ such that $s \rightarrow^n s'$. We call $\mathcal{S}_1$ the set of terminating states of $\mathcal{S}$ and by $\mathcal{S}_1$ we denote $S \setminus \mathcal{S}_1$. The evaluation length map $|\cdot| : S \rightarrow \mathbb{N} \cup \{\infty\}$ is defined as $|s| := n$ if $s \rightarrow^n s'$ and $s' \in \mathcal{F}_S$, and $|s| := \infty$ if $s \in \mathcal{S}_1$.

Definition 8.1 (Improvement). Given two DTS $\mathcal{S}$ and $\mathcal{Q}$, a relation $\mathcal{R} \subseteq S \times Q$ is an improving bisimulation, or simply an improvement, if $(s, q) \in \mathcal{R}$ implies the followings, schematized in Fig. 8.

- Final state left: if $s \in \mathcal{F}_S$, then $q \in \mathcal{F}_Q$.
- Final state right: if $q \in \mathcal{F}_Q$, then $s \rightarrow^n s'$, for some $s' \in \mathcal{F}_S$ and $n \geq 0$.
- Transition left: if $s \rightarrow s'$, then there exists $s'', q', n, m$ such that $s' \rightarrow^m s''$, $q \rightarrow^n q'$, $s'' \mathcal{R} q'$ and $n \leq m + 1$.
- Transition right: if $q \rightarrow q'$, then there exists $s', q'', n, m$ such that $s \rightarrow^m s'$, $q' \rightarrow^n q''$, $s' \mathcal{R} q''$ and $m \geq n + 1$.

What improves along an improvement is the number of transitions required to reach a final state, if any.

Proposition 8.2. Let $\mathcal{R}$ be an improvement on two DTS $\mathcal{S}$ and $\mathcal{Q}$, and $s\mathcal{R}q$.

1. Termination equivalence: $s \in \mathcal{S}_1$ if and only if $q \in \mathcal{Q}_1$.
2. Improvement: $|s| \geq |q|$.
\begin{proof}

1. $\Rightarrow$. Let us suppose $s \in S_1$ and let $n$ be the number of steps that $s$ needs to terminate. We proceed by induction on $n$. If $n = 0$, $s \in F_S$ and since $s R q$, $q \in F_Q$ and thus $q \in Q_1$. If $n = h > 0$, then $s \rightarrow s'$, and thus there exists $s'', q', k, j$ such that $q \rightarrow^k q'$, $s' \rightarrow^j s''$, $s'' R q''$ and $k \leq j + 1$. Since $s''$ terminates in less than $h - 1$ steps, by induction hypothesis $q' \in Q_1$ and thus also $q \in Q_1$.

$\Leftarrow$. Let us suppose $q \in Q_1$ and let $n$ be the number of steps that $q$ needs to terminate. We proceed by induction on $n$. If $n = 0$, $q \in F_Q$ and since $s R q$, $s \in S_1$. If $n = h > 0$, then $s \rightarrow q'$, and thus there exists $s', q'', k, j$ such that $s \rightarrow^k s'$, $q' \rightarrow^j q''$, $s'' R q''$ and $k \geq j + 1$. Since $q''$ terminates in less than $h$ steps, by induction hypothesis $s' \in S_1$ and thus also $s \in S_1$.

2. If $s \in S_1$ and $q \in Q_2$, then $|s| = |q| = \infty$. Let us consider the other case, i.e. when $s \in S_1$ and $q \in Q_1$. We proceed by induction on $|s|$. If $|s| = 0$, then $q \in F_Q$ and thus also $|q| = 0$. If $|s| = n > 0$, then $s \rightarrow s'$ and there exists $s'', q', m, l$ such that $q \rightarrow^m q'$, $s' \rightarrow^l s''$, $s'' R q''$ and $m \leq l + 1$. By i.h., $|s''| \geq |q'|$. Thus, since $m \leq l + 1$, then $|s| = |s''| + l + 1 \geq |q'| + m = |q|$.

$\square$

The idea is to exploit improvements to prove the soundness of the IALM, i.e. its invariance by (linear) head reduction. In particular, we are able to devise an improvement $\triangleright$ in such a way that if $t \rightarrow u$, then initial states of code $t$ and $u$ are related by $\triangleright$, i.e. $s_{t,k} = \langle t, (\cdot), (\cdot), (\cdot) \rangle \triangleright (u, (\cdot), (\cdot), (\cdot)) = s_{u,k}$. Moreover, the improvement $\triangleright$, besides termination, preserves also the structure of the tape. This way, if $s_{t,k} \triangleright s_{u,k}$, then $[t]_k = [u]_k$ for each $k \geq 0$.

**Theorem 8.3** (Soundness, via improvements). If $t \rightarrow u$, then $[t]_k = [u]_k$ for each $k \geq 0$.

To prove adequacy, one direction is easy, namely if $\rightarrow$ terminates then the IALM succeeds—by soundness, it is enough to prove that the IALM succeeds on $\rightarrow$ normal forms, which is immediate. The other direction is subtler. It is proved by contra-position, that is, we show that if $\rightarrow$ diverges on $t$ then no run of the IALM on $t$ ends in a successful state—careful: the run does not necessarily diverge, we shall explain this after the adequacy theorem.

The proof is obtained via a quantitative analysis of the improvements used to prove soundness, showing that the length of runs strictly decreases along $\rightarrow$. This way, intuitively, if $\rightarrow$ diverges on $t$, then also the length of the run starting from $t$ must be infinite (modulo the need of $\bullet$ on the tape).

**Theorem 8.4** (Adequacy). Let $t$ be a LSC term. Then $t$ has $\rightarrow$-normal form iff there exists $k > 0$ such that either $[t]_k = \langle m, l \rangle$ for some $m, l \geq 0$ or $[t]_k = x$ for some $x \in V$.

**Terminating without Success.** It is possible that $\rightarrow$ diverges on $t$ and all the runs of the IALM terminate on $t$ without ever succeeding. The idea is that the IALM performs a fine analysis of the $\rightarrow$ evaluation of $t$, approximating $\rightarrow$ while incrementally building the Lévy-Longo tree of $t$. On a looping term such as $\Omega$ the IALM does diverge. On a non-terminating term such as $\lambda x. \lambda y. x x$ the IALM does not diverge, it gives $[\lambda x. \lambda y. x x]_k = \perp$ for each $k \geq 0$. Note in fact that $\lambda x. \lambda y. x x$, i.e. $\lambda x. \lambda y. x x$ has an infinite number of abstractions in its limit normal form and thus an infinite number of $\bullet$ on the tape would be needed to inspect them all. One the contrary, $\Omega$ has no head lambda in its limit normal form and thus $[\Omega]_k = \perp$ for each $k \geq 0$.
9 Improving Soundness, Concretely

In this section we are going to show how the improvement \(\bowtie\) mentioned in the previous section is concretely defined. This way, we will be able to prove soundness.

The improvement \(\bowtie\) has to relate a state whose code contains a \(\sim\)-redex with the state whose code contains the corresponding reduct. Since \(\sim\) is the union of the three rewriting rules \(\sim_{dB}\), \(\sim_{ls}\) and \(\sim_{gc}\), we are going to define \(\bowtie\) as \(\bowtie_{dB}\cup\bowtie_{ls}\cup\bowtie_{gc}\).

9.1 Creation of Explicit Substitutions

Here we define a relation \(\bowtie_{dB}\) on IALM states induced by rule \(\sim_{dB}\), and prove it to be an improvement.

Steps, Positions, and Context Rewriting. Lifting a step \(t \sim_{dB} u\) to a relation between a IALM state \(s\) of code \(t\) and a state \(q\) of code \(u\) requires changing all positions relative to \(t\) in \(s\) to positions relative to \(u\) in \(q\). A first point to note is that we also have to change all the positions in the token, so that \(\bowtie_{dB}\) has to relate positions, logged positions, tape, log, and states.

A second more technical aspect is that one needs to extend linear head evaluation to contexts. Let us explain why. Consider a step \(t \sim_{dB} u\) where—for simplicity—the redex is at top level and the associated state \((\langle \lambda x.r \rangle Sw, \langle \cdot \rangle, \epsilon, \epsilon)\) has an empty token. This should be \(\bowtie_{dB}\)-related to a state \((\langle r[x\leftarrow w] \rangle S, \langle \cdot \rangle, \epsilon, \epsilon)\).

Let’s have a look at how the two states evolve:

\[
\begin{align*}
(\langle \lambda x.r \rangle Sw, \langle \cdot \rangle, \epsilon, \epsilon) &\quad \bowtie_{dB} \quad (\langle r[x\leftarrow w] \rangle S, \langle \cdot \rangle, \epsilon, \epsilon) \\
(\langle \lambda x.r \rangle S, \langle \cdot \rangle w, \epsilon, \bullet) &\quad \downarrow S \\
(\lambda x.r, Sw, \epsilon, \bullet) &\quad \downarrow (r, (\langle \cdot \rangle) Sw, \epsilon, \epsilon) \\
(r, (\langle \cdot \rangle) Sw, \epsilon, \epsilon) &\quad \downarrow (r, (\langle \cdot \rangle[x\leftarrow w]) S, \epsilon, \epsilon)
\end{align*}
\]

To close the diagram, we need \(\bowtie_{dB}\) to relate the two bottom states. Note that their relation can be seen as a \(\sim_{dB}\) step involving the contexts of the two positions. Therefore we extend the definition of \(\sim_{dB}\) to contexts adding the following top level clause (then included in \(\sim_{dB}\) via a closure by head contexts):

\[(\langle \lambda x.C \rangle St \Rightarrow_{dB} \langle C[x\leftarrow t] \rangle S)\]

The new clause, in turn, requires a further extension of \(\sim_{dB}\). Consider:

\[
\begin{align*}
(x, (\lambda x.D_n)u, L_n, \epsilon) &\quad \bowtie_{dB} \quad (x, D_n[ x\leftarrow u], L_n, \epsilon) \\
(\lambda x.D_n(x), \langle \cdot \rangle u, (x, (\lambda x.D_n)u, L_n)) &\quad \downarrow u, (\lambda x.D_n(x)) \langle \cdot \rangle \langle x, (\lambda x.D_n)u, L_n, \epsilon) \\
(u, (\lambda x.D_n(x)) \langle \cdot \rangle) &\quad \downarrow (u, D_n(x) \langle x\leftarrow \cdot \rangle), (x, D_n[ x\leftarrow u], L_n, \epsilon)
\end{align*}
\]

We are then led to add the following clause to \(\sim_{dB}\) (closed by head contexts):

\[(\lambda x.t) SC \Rightarrow_{dB} \langle t[x\leftarrow C] \rangle S\]

Note that in the two local bisimulation diagrams the right side is shorter. This is typical of when the machine travels through the redex. Outside of the redex, however, the two sides have the same length, as the next example shows—example that also motivates a further extension of \(\sim_{dB}\) to contexts. Consider the case where \(t \sim_{dB} u\) and the diagram is:
We then need to extend \(-\circ_{db}\) so that \(t(\cdot) \rightarrow_{db} u(\cdot)\). A similar situation happens also when entering an ES with transition \(\rightarrow_{var2}\). To close these diagrams, we add two further cases of reduction on contexts. Note that this time they have to be expressed via steps on terms (then included in \(-\circ_{db}\) via a closure by head contexts), as their direct definition would require contexts with two holes.

Note that \(\circ_{db}\) contains all pairs \((\langle t, \cdot \rangle, \epsilon, p, \cdot \rangle, \langle u, \cdot \rangle, \epsilon, p, \cdot \rangle)\), i.e. all the initial states containing a \(db\)-redex and its corresponding reduct.

The proof of the next theorem is a tedious easy check of diagrams.

**Theorem 9.2.** \(\circ_{db}\) is an improvement between IALM states.

### 9.2 Linear Substitution

Along the lines of the previous sub-section, here we define the candidate improvement \(\circ_{ls}\) on states induced by rule \(-\circ_{ls}\), and prove it an improvement.

**Steps, Positions, and Context Rewriting.** As for \(-\circ_{db}\), we are led to extend the rewriting relation to contexts. There are however some new subtleties. Given \(t \rightarrow_{db} u\) and a position \((r, C)\) for \(t\), for \(\circ_{db}\) the redex in \(t\) falls always entirely either in \(t\) or \(C\). If \(t \rightarrow_{ls} u\), instead, the redex can be split between the two. Consider the following diagram (where to simplify we assume the step to be at top level and the token to be empty).

\[
\begin{array}{c}
(t, \langle \cdot \rangle, r, \epsilon, p) \rightarrow_{db} (u, \langle \cdot \rangle, r, \epsilon, p) \\
(r, t(\cdot), r, \epsilon) \rightarrow_{db} (r, u(\cdot), p, \epsilon)
\end{array}
\]

To close it, we have to \(\circ_{db}\)-relate the two bottom states, where the pattern of the redex/reduct is split between the two parts of the position. This shall motivate clause rdx2 in the definition of \(\circ_{ls}\).

---

7\(\Gamma\) is a meta-variable that stands either for a log \(L\) or for a tape \(T\).
The new rule comes with consequences. Consider the following diagram, where the two starting states are related by the new clause for $\triangleright_{1s}$:

\[
\begin{array}{c}
(x, H[x\leftarrow t], \epsilon, \epsilon) \\
\downarrow \\
(t, H[x\leftarrow t], \epsilon, \epsilon)
\end{array}
\]

To close the diagram, as usual, we have to $\triangleright_{1s}$-relate them. There are, however, two delicate points. First, we cannot see the context $H(x)[x\leftarrow \epsilon]$ as making a $\text{co}_{1s}$ step towards $H[x\leftarrow t]$, because $t$ does not occur in $H(x)[x\leftarrow \epsilon]$. For that, we have to introduce a variant of $\text{co}_{1s}$ on contexts that is parametric in $t$ (and more general than the one to deal with the showed simplified diagram):

\[H(x)[x\leftarrow C] \rightarrow_{1s,t} H(C)[x\leftarrow C(t)].\]

The second delicate point of diagram (2) is that the extension of $\triangleright_{1s}$ has to also $\triangleright_{1s}$-relate signature stacks of different length, namely $\epsilon$ and $(x, H[x\leftarrow t], \epsilon)$. This happens because positions of the two states do isolate the same term, but at different depths, as one is in the ES. Then the definition of $\triangleright_{1s}$ has two new clauses, one for signatures and one for states, to handle such a case. Luckily, the mismatch in length of signature stacks is at most 1.

Last, as for $\triangleright_{db}$, we need the following two clauses of reduction on contexts (then included in $\text{co}_{1s}$ via a closure by head contexts).

\[
\begin{array}{c}
t \text{co}_{1s} u \\
t \text{co}_{1s} u \end{array}
\]

Definition 9.3. Binary relation $\triangleright_{1s}$ is defined by the following rules.

\[
\begin{array}{c}
\frac{t \text{co}_{1s} u \quad (t, H) \triangleright_{1s} (u, H)}{r_{dx_{1s}}} \\
\frac{C \text{co}_{1s} D \quad (t, C) \triangleright_{1s} (t, D)}{c_{tx_{1s}}} \\
\frac{K = K'(G[x \leftarrow t]) \quad H(x)[x \leftarrow C] \rightarrow_{1s,t} H(C)[x\leftarrow C(t)]}{r_{dx2}} \\
\frac{(x, C, L) \triangleright_{1s} (x, D, L')}{t_{ok1_{1s}}} \\
\frac{(x, C, L) \triangleright_{1s} (x, D, L')}{t_{ok2_{1s}}} \\
\frac{L \triangleright_{1s} L'}{t_{ok3_{1s}}} \\
\frac{(x, C) \triangleright_{1s} (x, D)}{t_{sig1_{1s}}} \\
\frac{(x, C, L) \triangleright_{1s} (x, D, L')}{t_{sig2_{1s}}} \\
\frac{(t, C, L, T, d) \triangleright_{1s} (u, D, L', T', d')}{t_{state1_{1s}}} \\
\frac{(t, C, L, T, d) \triangleright_{1s} (u, D, L', T', d')}{t_{state2_{1s}}} \\
\end{array}
\]

Theorem 9.4. $\triangleright_{1s}$ is an improvement between IALM states.

9.3 Garbage Collection

Steps, Positions, and Context Rewriting. The candidate improvement $\triangleright_{gc}$ induced by $\text{co}_{gc}$ requires an extension of $\text{co}_{gc}$ with a rule on contexts which is similar to the parametric one for $\text{co}_{1s}$. Let $t[x\leftarrow u] \text{co}_{gc} t$ and consider:

\[
\begin{array}{c}
(t[x\leftarrow u], (\cdot), \epsilon, \epsilon) \\
\downarrow \\
(t, (\cdot)[x\leftarrow u], \epsilon, \epsilon)
\end{array}
\]

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To close the diagram, we extend the definition of $\rightarrow_{\text{gc}}$ to context with the following parametric rule (closed by head contexts):

$$C[x←u] \rightarrow_{\text{gc},t} C \quad \text{if } x \notin \text{fv}(t).$$

We also need, as for $\triangleright_{\text{gb}}$ and $\triangleright_{\text{is}}$, the rules (closed by head contexts):

$$t \rightarrow_{\text{gc}} u \quad \frac{t \rightarrow_{\text{gc}} u}{tC \rightarrow_{\text{gc}} uC} \quad \frac{t[x←C] \rightarrow_{\text{gc}} u[x←C]}{t \rightarrow_{\text{gc}} u}$$

**Definition 9.5.** Binary relation $\triangleright_{\text{gc}}$ is defined by the following rules.

$$\begin{array}{c}
\frac{t \rightarrow_{\text{gc}} u}{(t,H) \triangleright_{\text{gc}} (u,H)} \\
\frac{C \rightarrow_{\text{gc}} D}{(t,C) \triangleright_{\text{gc}} (t,D)} \\
\frac{C \rightarrow_{\text{gc}} D}{(t,C) \triangleright_{\text{gc}} (t,D)} \\
\frac{T \rightarrow_{\text{gc}} T'}{(x,C) \rightarrow_{\text{gc}} (x,D)} \\
\frac{T \rightarrow_{\text{gc}} T'}{(x,C) \rightarrow_{\text{gc}} (x,D)} \\
\frac{L \rightarrow_{\text{gc}} L'}{(t,C,L) \rightarrow_{\text{gc}} (t,D,L')} \\
\frac{d = d'}{(t,C,L,T,d) \rightarrow_{\text{gc}} (u,D,L,T,d')}
\end{array}$$

**Theorem 9.6.** $\triangleright_{\text{gc}}$ is an improvement between IALM states.

### 9.4 The Soundness Theorem

Let’s go back to soundness. We have to show the improvements of the previous sections do imply soundness. Consider $\triangleright = \triangleright_{\text{gb}} \cup \triangleright_{\text{is}} \cup \triangleright_{\text{gc}}$, that is an improvement because its components are. Consequently, if $t \rightarrow u$, then the IALM run on $u$ improves the one on $t$, that is,

$$s_{t,k} = (t, \langle \cdot \rangle, \epsilon, \bullet^k) \triangleright (u, \langle \cdot \rangle, \epsilon, \bullet^k) = s_{u,k}.$$

Improvements transfer more than termination/divergence along linear head evaluation $\rightarrow$. They also give bisimilar, structurally equivalent tapes, proving the invariance of the semantics, that is, soundness.

**Theorem 9.7** (Soundness, via improvements).

If $t \rightarrow u$, then $\llbracket t \rrbracket_k = \llbracket u \rrbracket_k$ for each $k \geq 0$.

**Proof.** Since $t \rightarrow u$, then $s = (t, \langle \cdot \rangle, \epsilon, \bullet^k) \triangleright (u, \langle \cdot \rangle, \epsilon, \bullet^k) = q$ by the results about improvements (Theorem 9.2, Theorem 9.3 or Theorem 9.6). First of all, since improvements transfer termination/divergence (Proposition 9.1), we have $\llbracket u \rrbracket_k = \perp$ if and only if $\llbracket u \rrbracket_k = \perp$. Now, assume that $\llbracket t \rrbracket_k \neq \perp$. Let us call $s' = (r, C, L, T, d)$ the terminal state of $s$. Since $\triangleright$ is an improvement, there exists a state $q' = (r', C', L', T', d')$ such that $q \rightarrow_{\text{IALM}} q'$ and $s \triangleright q'$. We consider different cases according to the different possible shapes of $s'$.

- $s' = (\lambda x.w, C, L, \epsilon)$, i.e. $r = \lambda x.w$, $T = \epsilon$ and $d = \perp$. Then, since $s' \triangleright q'$, $T' = \epsilon$. Moreover, either $t \rightarrow u$, and thus $r' = \lambda x.w'$ or $C \rightarrow D$ and thus $r = r'$. Then, $\llbracket t \rrbracket_k = \llbracket u \rrbracket_k = \perp$.
- $s' = (t, \langle \cdot \rangle, \epsilon, \bullet^m \cdot p \cdot l)$. Then, since $s' \triangleright q'$, $C' = \langle \cdot \rangle$, because the hole cannot $\rightarrow$-reduce. Moreover, the structure of the tape is preserved by $\triangleright$ and thus $\llbracket t \rrbracket_k = \llbracket u \rrbracket_k = \langle m, l \rangle$.
- $s' = (x, C, L, \epsilon)$. Since a variable cannot $\rightarrow$-reduce, also $r' = x$. Then $\llbracket t \rrbracket_k = \llbracket u \rrbracket_k = x$.

\[ \Box \]

### 10 Improving Adequacy, Concretely

We established soundness with respect to linear head evaluation $\rightarrow$, that is, the semantics $\llbracket \cdot \rrbracket$ is an invariant of $\rightarrow$. Here we are going to prove that $\llbracket \cdot \rrbracket$ is also adequate for $\rightarrow$, i.e. that $\llbracket t \rrbracket$ reflects some observable aspects of $t$ and vice versa. We recall the statement of the Adequacy Theorem.
Theorem 10.1 (Adequacy). Let $t$ be a LSC term. Then $t$ has $\rightarrow$-normal form if and only if there exists $k > 0$ such that either $[t]_k = \langle m, l \rangle$ for some $m, l \geq 0$ or $[t]_k = x$ for some $x \in V$.

Direction IALM to $\rightarrow$. The only if direction of the statement is easy to prove. Since $[t]_k$ is invariant by $\rightarrow$ (soundness) and $\rightarrow$ terminates on $t$ we can as well assume that $t$ is normal. The rest is given by the following proposition.

Proposition 10.2 (Reading the head variable on $\rightarrow$-normal forms). Let $t = \lambda x_0 \ldots \lambda x_n . y u_1 \ldots u_l$ be a head linear normal form up to substitution. If $y = x_m$ where $0 \leq m \leq n$, then $[t]_{n+1} = \langle m, l \rangle$, otherwise, if $y$ is free, then $[t]_{n+1} = y$.

Proof. We proceed computing $[t]_{n+1}$ explicitly, according to the definition.

$$(t, \langle \cdot \rangle, \epsilon, n + 1) \rightarrow^{n+1}_{IALM} (y u_1 \ldots u_l, \lambda x_0 \ldots \lambda x_n . \langle \cdot \rangle, \epsilon, \epsilon)$$

$$(\lambda x_m \ldots \lambda x_n . \langle \cdot \rangle y u_1 \ldots u_l, \epsilon) \rightarrow^{\alpha}_{IALM} (y, \lambda x_0 \ldots \lambda x_n . \langle \cdot \rangle u_1 \ldots u_l, \epsilon, \epsilon)$$

If $y$ is free, the IALM stops and $[t]_{n+1} = y$. Otherwise, if $y$ is bound by a $\lambda$-abstraction, i.e. $y = x_m$ for $0 \leq m \leq n$, the computation continues.

$$(t, \langle \cdot \rangle, \epsilon, n + 1) \rightarrow^{n+1}_{IALM} (x_m, \lambda x_0 \ldots \lambda x_n . \langle \cdot \rangle y u_1 \ldots u_l, \epsilon)$$

$$\rightarrow^{\alpha}_{IALM} (\lambda x_m \ldots \lambda x_n . x_m u_1 \ldots u_l, \lambda x_0 \ldots \lambda x_{m-1} . \langle \cdot \rangle, \epsilon, \epsilon)$$

$$\rightarrow^{m}_{IALM} (u, \langle \cdot \rangle, \epsilon, m, p, \epsilon)$$

where $p = (x_m, \lambda x_0 \ldots \lambda x_n . \langle \cdot \rangle u_1 \ldots u_l, \epsilon)$. Therefore $[t]_k = \langle m, l \rangle$.\qed

Direction $\rightarrow$ to IALM. The proof of the if direction of the adequacy theorem is by contra-position, that is, we prove that if the $\rightarrow$ diverges on $t$ then no run of the IALM on $t$ ends in a successful state.

The proof is obtained via a quantitative analysis of the improvements used to prove soundness, showing that the length of runs strictly decreases along $\rightarrow$. Note that improvements guarantee that the length of runs does not increase. To prove that it actually decreases one needs an additional global analysis of runs—improvements only deal with local bisimulation diagrams.

On proof nets, this decreasing property correspond to the standard fact that IAM paths passing through a cut have shorter residuals after that cut.

We recall that we write $[t]_k$ for the length of the IALM run $(t, \langle \cdot \rangle, \epsilon, n)$, with the convention that $[t]_k = \infty$ if the machine diverges.

Lemma 10.3 (The length of terminating runs strictly decreases along $\rightarrow$). Let $t \rightarrow u$ and $[t]_k \neq \infty$. There exists $k \geq 0$ such that $[t]_k > [u]_h$ for each $h \geq k$.

Proof. We treat the case of $t \rightarrow_{db} u$, the others are obtained via similar diagrams. If $t$ has a $\rightarrow_{db}$-redex then it has the shape $t = H(\langle \lambda x . r \rangle S w)$ and $u$ is in the form $u = H(\langle r[x-w] \rangle S)$. By induction on the structure of $H$ one can prove that there exist $k, n \geq 0$ such that $(t, \langle \cdot \rangle, \epsilon, n) \rightarrow^{\alpha}_{IALM} (\lambda x . r) S w, H, \epsilon, \epsilon)$ and $(u, \langle \cdot \rangle, \epsilon, m) \rightarrow^{\alpha}_{IALM} (\langle r[x-w] \rangle S, H, \epsilon, \epsilon)$.

Given such $n$ and $k$ by the pumping lemma (Lemma 4.6) also the following holds: for any $j \geq 0$, $(t, \langle \cdot \rangle, \epsilon, n + k) \rightarrow^{\alpha}_{IALM} (\langle r[x-w] \rangle S, H, \epsilon, \epsilon)$ and $(u, \langle \cdot \rangle, \epsilon, m + k) \rightarrow^{\alpha}_{IALM} (\langle r[x-w] \rangle S, H, \epsilon, \epsilon)$. Moreover, by definition of the improvement $\triangleright_{db}$ we have the following diagram.
Here we sketch how the IALM relates to the IAM, i.e. the original presentation based on linear logic proof nets, due to Danos and Regnier [DR99]. For lack of space, we avoid defining proof nets and related concepts, and focus only on the key points.

11 Comparison with the Original Proof Net Presentation

From \( s_1 \uparrow \downarrow \), the hypothesis \(|t|_h \neq \infty\), and the properties of improvements (Lemma 2), we obtain \(|s_1| \geq |s_2|\). Then, by setting \( h := k + j \), we have

\[ |t|_h = n + 1 + |S| + 1 + |s_1| > n + |S| + 1 + |s_2| = |u|_h. \]

□

Using the lemma, we prove the if direction of adequacy.

Proposition 10.4 (\( \rightarrow \)-divergence implies that the IALM never succeeds). Let \( t \) be a \( \rightarrow \)-divergent LSC term. There is no \( k \geq 0 \) such that \(|t|_k\) is successful.

Proof. By contradiction, suppose that there exists \( k \) such that \(|t|_k\) is successful. Then by soundness \(|t|_k = n \in \mathbb{N}\) and it ends on a successful state. By monotonicity of runs (Lemma 1.7), \(|t|_k = |t|_h = n\) for every \( h > k \). Since \( t \rightarrow \)-divergent, then there exists an infinite reduction sequence \( \rho : t = t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots t_k \rightarrow \cdots \). Since the length of terminating runs strictly decreases along \( \rightarrow \) for sufficiently long inputs (Lemma 10.3), for each \( i \in \mathbb{N} \) if \( t_i \rightarrow t_{i+1} \) then there exists \( k_i \) such that \(|t_i|_{k_i} > |t_{i+1}|_{k_i}\). Now, consider \( h = \max\{k_0, \ldots, k_n, k_{n+1}\} \). We have that \(|t_j|_h > |t_{j+1}|_h\) for every \( j \in \{0, 1, \ldots, n + 1\} \). Then \(|t_0|_h \geq |t_{n+1}|_h + n + 1\). Since the length of runs is non-negative, we obtain that \(|t_0|_h \geq n + 1\), which is absurd because \( h \geq k_0 \) and so \(|t_0|_h = n\).

□
Figure 5: Transition rules of Danos and Regnier’s IAM related to exponential signatures.

Essentially, the IALM corresponds to the IAM on proof nets representing λ-terms according to the call-by-name translation $\tilde{t}^\lambda$ in Fig. [DR99], and considering only paths from the principal port of the obtained net. There is a bisimulation between the IALM and such a restricted IAM, which is not strong because of some technicalities inherent to proof nets. In particular, Rule $\rightarrow_{\text{arg}}$ short-circuits the path between a variable $x$ and its abstraction $\lambda x. C_n(x)$. In proof nets, this path traverses a dereliction, exactly $n$ auxiliary doors, possibly a contraction tree, and ends on the $\text{Y}$ representing the abstraction. That being said, IALM rules correspond exactly to the actions attached to proof nets edges presented in [DR99].

To present this equivalence, we relate the original token of [DR99] with ours. In Danos and Regnier’s presentation the token is given by two stacks, called boxes stack $B$ and balancing stack $S$. They correspond exactly to our log $L$ and tape $T$, respectively. They are formed by sequences of multiplicative constants $p$ (corresponding to our •) or $q$, and by exponential signatures $\sigma$. They are defined by the following grammar:

\[
\begin{align*}
\text{Balancing stacks} & \quad S \ ::= \ \epsilon \mid p \cdot S \mid q \cdot S \mid \sigma \cdot S \\
\text{Boxes stacks} & \quad B \ ::= \ \epsilon \mid \sigma \cdot B \\
\text{Exponential signatures} & \quad \sigma, \sigma' \ ::= \ \square \mid \langle \sigma, \sigma' \rangle \mid \langle l, \sigma \rangle \mid \langle r, \sigma \rangle
\end{align*}
\]

Intuitively, exponential signatures are binary trees with $\square$, $l$ or $r$ as leaves, where $l$ and $r$ denote the left/right premise of a contraction. Fig. 5 shows the IAM transitions concerning exponential signatures that are the relevant difference with respect to the IALM.

Let us explain how the following key transition is simulated by the IAM:

\[
\begin{align*}
(x, C(\lambda x. D_n), L_n \cdot L, T) & \rightarrow_{\text{arg}} (\lambda x. D_n(x), C, L, (x, \lambda x. D_n, L_n) \cdot T).
\end{align*}
\]

The IAM does the same, just in more steps and with another syntax. Consider a token $(B_n \cdot B, S)$ approaching a variable $x$ that is $n$ boxes deeper than its binder $\lambda x. D_n(x)$. Variables are translated as dereliction links and thus we have: $(B_n \cdot B, S) \rightarrow (B_n \cdot B, \square \cdot S)$.

Then, the token travels until the binder of $x$ is found (a $\text{Y}$ in the proof net translation of the term), i.e. it traverses exactly $n$ boxes always exiting from the auxiliary doors. Moreover, for every such box a contraction could be encountered. Let us suppose for the moment that $x$ is used linearly, so that no contractions are encountered. Then the token rewrites in the following way, traversing $n$ auxiliary doors.

\[
\begin{align*}
(\sigma_1 \cdot B_{n-1} \cdot B, \square \cdot S) & \rightarrow (\sigma_2 \cdot B_{n-2} \cdot B, (\sigma_1, \square) \cdot S) \rightarrow (\sigma_3 \cdot B_{n-3} \cdot B, (\sigma_2, (\sigma_1, \square)) \cdot S) \\
& \rightarrow \cdots \rightarrow (B, (\sigma_n, \cdots (\sigma_1, \square) \cdots) \cdot S).
\end{align*}
\]

Note the perfect matching between the two formulations: in both cases the first $n$ logged positions/signatures in the log/boxes stack are removed from it and, once wrapped in a single logged position/signature, then put on the tape/balancing stack. In presence of contractions the exponential signature $\langle \sigma_n, \cdots (\sigma_1, \square) \cdots \rangle$ is interleaved by $l$ and $r$ leaves. These symbols represent nothing more than a binary code used to traverse the contraction tree of $x$. In the IALM, we use a more human readable way of representing the same information: we explicitly save the variable occurrence through its position inside its binder.

\footnote{The translation uses a recursive type $\sigma = ? \alpha^+ \text{N}\alpha$ in order to be able to represent untyped terms of the λ-calculus—this is standard. Every net has a unique conclusion labeled with $\alpha$, which is the output, and all the other conclusions have type $? \alpha^+$ and are labelled with a free variable of the term. In the abstraction case $\lambda x.t$, if $x \notin \text{fv}(t)$ then a weakening is added to represent that variable.}

\footnote{Actually, there is a minor difference: the use of the call-by-name translation allows to get rid of $q$, dual of $p$, needed in [DR99].}

\footnote{We used symbols $l$ and $r$ respectively instead of $p'$ and $q'$ for the sake of clarity.}
12 Conclusion

We (re)formulated the interaction abstract machine directly on \(\lambda\)-terms, making it conceptually closer to traditional abstract machines, and ultimately allowing direct and genuinely inductive proofs of correctness. Moreover, the resulting denotational semantics turns out being sound and adequate also for open terms and erasing steps.

The relevance of this work does not only lie in the obtained results, but also in the new formulation we give of the Geometry of Interaction and in the style of the proof techniques employed. This opens up at least two possible routes, the first one being the formalization of meta-theoretical results about the GoI, the second one instead lying on a fine analysis of the complexity of the implementation of the \(\lambda\)-calculus, in particular regarding the space-time trade-off.

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