EXPONENTIAL STABILITY ESTIMATES FOR THE 1D NLS

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ABSTRACT. We study stability times for a family of parameter dependent nonlinear Schrödinger equations on the circle, close to the origin. Imposing a suitable Diophantine condition (first introduced by Bourgain), we prove a rather flexible Birkhoff Normal Form theorem, which implies, e.g., exponential and sub-exponential time estimates in the Sobolev and Gevrey class respectively.

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1. Introduction

We consider families of NLS equations on the circle with external parameters of the form:

\[ iu_t + u_{xx} - V \ast u + f(x, |u|^2)u = 0, \]

where \( i = \sqrt{-1} \) and \( V \ast \) is a Fourier multiplier

\[ V \ast u = \sum_{j \in \mathbb{Z}} V_j u_j e^{ijx}, \quad (V_j)_{j \in \mathbb{Z}} \in \mathcal{W}_q^\infty, \]

living in the weighted \( \ell^\infty \) space

\[ \mathcal{W}_q^\infty := \{ V = (V_j)_{j \in \mathbb{Z}} \in \ell^\infty \mid |V|_q := \sup_{j \in \mathbb{Z}} |V_j|^{q/2} < \infty \}, \quad q \geq 0, \]

where \( \langle j \rangle := \max\{|j|, 1\} \), while \( f(x, y) \) is \( 2\pi \) periodic and real analytic in \( x \) and is real analytic in \( y \) in a neighborhood of \( y = 0 \). We shall assume that \( f(x, y) \) has a zero in \( y = 0 \). By analyticity, for some \( a, R > 0 \) we have

\[ f(x, y) = \sum_{d=1}^\infty f^{(d)}(x)y^d, \quad |f|_{a, R} := \sum_{d=1}^\infty |f^{(d)}|_a^\infty R^d < \infty, \]

where, given a real analytic function \( g(x) = \sum_{j \in \mathbb{Z}} g_j e^{ijx} \), we set \( |g|_{T_a}^{[2]} := \sum_{j \in \mathbb{Z}} |g_j|^2 e^{2\alpha |j|} \).

Note that if \( f \) is independent of \( x \) \( (1.2) \) reduces to

\[ |f|_R := \sum_{d=1}^\infty |f^{(d)}| R^d < \infty. \]

Equation \( (1.1) \) is at least locally well-posed (say in a neighborhood of \( u = 0 \) in \( H^1 \), see e.g. Lemma \[7.1\]) and has an elliptic fixed point at \( u = 0 \), so that an extremely natural question is to understand stability times for small initial data. One can informally state the problem as follows: let \( u \) be an elliptic fixed point at Equation \( (1.1) \) is at least locally well-posed (say in a neighborhood of \( |\cdot| \)).

Definition 1.1. We call stability time \( T = T(\delta) \) the supremum of the times \( t \) such that for all \( |u_0|_E \leq \delta \) one has \( u(t, \cdot) \in E \) with \( |u(t, \cdot)|_E \leq 4\delta \).

Computing the stability time \( T(\delta) \) is out of reach, so the goal is to give lower (and possibly upper) bounds. A good comparison is with the case of a finite dimensional Hamiltonian system with a non-degenerate elliptic fixed point, which in the standard complex symplectic coordinates \( u_j = \frac{1}{\sqrt{2}}(q_j + ip_j) \) is described by the Hamiltonian

\[ \sum_{j=1}^n \omega_j |u_j|^2 + O(u^3), \quad \text{where } \omega_j \in \mathbb{R} \text{ are the linear frequencies.} \]

Here if the frequencies \( \omega \) are sufficiently non degenerate, say Diophantine\( ^1 \), then one can prove exponential lower bounds on \( T(\delta) \) and, if the nonlinearity satisfies some suitable hypothesis (e.g. convexity or steepness), even super-exponential ones, see for instance [MG95], [BFN15] and reference therein.

The strategy for obtaining exponential bounds is made of two main steps. The first one consists in the so-called Birkhoff normal form procedure: after \( N \geq 1 \) steps the Hamiltonian \( (1.4) \) is transformed into

\[ \sum_{j=1}^n \omega_j |u_j|^2 + Z + R, \]

\( ^1 \)Namely \( g \) is a holomorphic function on the domain \( \mathbb{T}_a := \{ x \in \mathbb{C}/2\pi \mathbb{Z} : |\text{Im} \ x| < \alpha \} \) with \( L^2 \)-trace on the boundary.

\( ^2 \)A vector \( \omega \in \mathbb{R}^n \) is called Diophantine when it is badly approximated by rationals, i.e. it satisfies, for some \( \gamma, \tau > 0, |k \cdot \omega| \geq \gamma |k|^{-\tau}, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}. \)
where $Z$ depends only on the actions $(|u_i|^2)_{i=1}^n$ while $R = O(|u|^{2k+3})$ contains terms of order at least $2n + 3$ in $|u|$.

It is well known that this procedure generically diverges in $\mathbb{N}$, so the second step consists in finding $\mathbb{N} = \mathbb{N}(\delta)$ which minimizes the size of the remainder $R$.

The problem of long-time stability for equations \eqref{1.1} has been studied by many authors. In the context of infinite chains with a finite range coupling, we mention \cite{BFG88}. Regarding applications to PDEs (and particularly the NLS) the first results were given in \cite{Bou96a} by Bourgain, who proved polynomial bounds for the stability times in the following terms: for any $M$ there exists $s = s(M)$ such that initial data which are $\delta$-small in the $H^{s+\epsilon}$ norm stay small in the $H^{s}$ norm, for times of order $\delta^{-M}$. Afterwards, Bambusi in \cite{Bam99b} proved that superanalytic initial data stay small in analytic norm, for times of order $e^{(\ln \delta)^{-1/\epsilon}}$, where $b > 1$. Bambusi and Grebert in \cite{BG06} proved polynomial bounds for a class of tame-modulus PDEs, which includes \eqref{1.1}. More precisely, they proved that for any $N \gg 1$ there exists $p(N)$ (tending to infinity as $N \to \infty$) such that for all $p \geq p(N)$ and initial datum in $H^p$ one has $T \geq C(N, p) \delta^{-N}$. For an application to the present model we refer also to \cite{ZG17}.

Similar results were also proved for the Klein Gordon equation on Zoll manifolds in \cite{BDGS}. Successively Faou and Grebert in \cite{FG13} considered the case of analytic initial data and proved subexponential bounds of the form $T \geq e^{e^{c_1 \delta^{-c_2}}}$ for classes of NLS equations in $\mathbb{T}^d$ (which include \eqref{1.1} by taking $d = 1$). Finally, Feola and Iandoli in \cite{FI} prove polynomial lower bounds for the stability times of reversible NLS equations with two derivatives in the nonlinearity.

A closely related topic is the study of orbital stability times close to periodic or quasi-periodic solutions of \eqref{1.1}. In the case $E = H^1$, Bambusi in \cite{Bam99a} proved a lower bound of the form $T \geq e^{\delta^{-c_1}}$ for perturbations of the integrable cubic NLS close to a quasi-periodic solution. Regarding higher Sobolev norms, most results are in the periodic case. See \cite{FGL13} (polynomial bounds for Sobolev initial data) and the preprint \cite{MSW18} (subexponential bounds for subanalytic initial data).

A dual point of view is to construct special orbits for which the Sobolev norms grow as fast as possible (thus $(\text{subexponential bounds for subanalytic initial data)}$).

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1.1. **The stability results.** In this paper we recover and improve the results in \cite{BG06} (Sobolev initial data) and \cite{FGL13} (analytic and subanalytic initial data) under a different Diophantine non-resonance condition on the linear frequencies, by application of a different Birkhoff normal form approach (see the comments after Theorem 1.4). More precisely, following Bourgain \cite{Bou05} we set

\begin{equation}
\Omega_q := \left\{ \omega = (\omega_j)_{j \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z}, \sup_j |\omega_j - j^2|/|j|^q < 1/2 \right\}
\end{equation}

and, for $\gamma > 0$ we define the set of ”good frequencies” as

\begin{equation}
\mathcal{D}_{\gamma, q} := \left\{ \omega \in \Omega_q : |\omega \cdot \ell| > \gamma \prod_{n \in \mathbb{Z}} \frac{1}{(1 + |\ell_n|^2(n)^{2+q})} \quad \forall \ell \in \mathbb{Z}^2 : |\ell| < \infty \right\},
\end{equation}

Note that $\mathcal{D}_{\gamma, q}$ is large with respect to a natural probability product measure on $\Omega_q$ (see \cite{Bou05} or Lemma 4.1 in the present paper).

**Remark 1.1.** From now on we shall fix $\gamma > 0, q \geq 0$ and assume that $\omega \in \mathcal{D}_{\gamma, q}$.

We note that some non-resonance condition on the frequencies is inevitable if one wants to prove long-time stability, indeed if one takes $V = 0$ and $f(x, |u|^2) = |u|^4$ then one can exhibit orbits in which the Sobolev norm is unstable in times of order $\delta^{-q}$, see \cite{GT12, HP17}.
**Sobolev initial data.** In the case of Sobolev initial data it is fundamental to have a good control on the dependence of the stability time $T$ on the the regularity $p$. This means that results are very sensitive to which (of various equivalent) Sobolev norms one considers. Recalling that the $L^2$-norm is invariant for the equation $\frac{\partial \psi}{\partial t} = \Delta \psi - |\psi|^2 \psi$, we will consider two cases:

- In the first case we deal with the usual norm $|u_0|_{L^2} + |\partial_x^p u_0|_{L^2}$, for $p > 1$. We denote this case as $S$ (Sobolev case) and, by fixing $p = p(\delta)$, we prove sub-exponential lower bound for the stability time $T(\delta)$.

- In the second case, denoted by $M$ (Modified-Sobolev case), we consider the equivalent norm $2^p |u_0|_{L^2} + |\partial_x^p u_0|_{L^2}$.

In order to simplify the exposition and obtain better bounds, in this case we consider $1.1$ with $f$ independent of $x$ (translation invariance). Again, fixing $p = p(\delta)$, we prove exponential lower bound on the stability time $T(\delta)$.

Of course, the norms in $S$ and $M$ are equivalent with constants depending on $p$. Note that when $p$ depends on $\delta$ such constants become very important.

The main qualitative difference between $S$ and $M$ is that in the latter we are requiring that the Fourier modes $0, 1, -1$ of the initial datum have very little energy. Indeed, passing to the Fourier side $u_0(x) = \sum_{j \in \mathbb{Z}} u_{0,j} e^{ijx}$, if both $|u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \leq \delta/2$ and the extra condition $|u_{0,0}|^2 + |u_{0,1}|^2 + |u_{0,-1}|^2 \leq \delta^2 2^{-2p-2}$ hold, then one has $2^p |u_0|_{L^2} \leq \delta$.

Below we formally state our first result, which depends on some constants, denoted by $\tau_S, \delta_S, K_S, \tau_M, \delta_M, T_S$ explicitly defined in Section A of the Appendix. These constants depend only on $\gamma, q, R, |f|_{a, M}$ in the case $S$ and on $\gamma, q, R, |f|_{a, M}$ in the case $M$.

**Theorem 1.1** (Sobolev stability). Consider equation (1.1) with $f$ satisfying (1.2) for $a, R > 0$.

(\textit{S}) For any $p > 1$ such that $(p - 1)/\tau_S \in \mathbb{N}$ and any initial datum $u(0) = u_0$ satisfying

\begin{equation}
|u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \leq \delta \leq \min \left\{ \delta_S (K_S)^{-3p}, \sqrt{R}/20 \right\},
\end{equation}

the solution $u(t)$ of (1.1) with initial datum $u(0) = u_0$ exists for all times

\begin{equation}
|t| \leq \frac{\tau_S}{\delta^2} (K_S)^{-5p} \left( \frac{\delta_S}{\delta} \right)^{2(p-1)/5} \quad \text{and satisfies} \quad |u(t)|_{L^2} + |\partial_x^p u(t)|_{L^2} \leq 4\delta.
\end{equation}

(\textit{M}) Assume that $f$ in (1.1) is independent of $x$. For any $p > 1$ such that $(p - 1)/\tau_M \in \mathbb{N}$ and for any initial datum $u(0) = u_0$ satisfying

\begin{equation}
2^p |u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \leq \delta \leq \min \left\{ \frac{2\sqrt{\tau_M} \delta_M}{\sqrt{p}}, \sqrt{R}/4\sqrt{10} \right\},
\end{equation}

the solution $u(t)$ of (1.1) exists for all times

\begin{equation}
|t| \leq \frac{\tau_M}{\delta^2} \left( \frac{4\tau_M \delta_M^2}{(p - 1)\delta^2} \right)^{\frac{p-1}{p}} \quad \text{and satisfies} \quad 2^p |u(t)|_{L^2} + |\partial_x^p u(t)|_{L^2} \leq 4\delta.
\end{equation}

**Remark 1.2.** Some remarks on the optimality of Theorem 1.1 are in order.

1. We stress the fact that estimates (1.8) of case $S$ is optimal in some sense. The simplest way of showing this fact is to construct a Hamiltonian which does not preserves momentum and exhibits fast drift. In fact, if we take $\delta \geq (e^{-1}p)^{-p/2}$ then orbits performing “fast drift” in a time of order $1$ may occur. Indeed consider e.g. for $2 \leq j \in \mathbb{N}$ the family of Hamiltonians:

$$H^{(j)}(u_1, u_j) := |u_1|^2 + (j^2 + V_j)|u_j|^2 + e^{-a_j} \text{Re}(|u_1|^2 u_1 \bar{u}_j).$$

Passing to action-angle variables $u_i = \sqrt{I_i} e^{i\vartheta_i}$ we get the new Hamiltonian

$$I_1 + \omega J_j + e^{-a_j} J_1^{1/2} \sqrt{J_1} \cos(\vartheta_1 - \vartheta_j) = J_1 + \omega (J_2 - J_1) + e^{-a_j} J_1^{3/2} \sqrt{J_2 - J_1} \cos \varphi_1.$$
in the new symplectic variables $J_1 = I_1$, $J_2 = I_1 + I_j$, $\varphi_1 = \vartheta_1 - \vartheta_j$, $\varphi_2 = \vartheta_j$.

Note that this Hamiltonian has $J_2$ as constant of motion while

$$\dot{J}_1 = e^{-sj} J_1^{3/2} \sqrt{J_2 - J_1 \sin \varphi_1}. $$

In this case the norm in (1.12) reads

$$\sqrt{|u|^2 + |u_j|^2 + \sqrt{|u|^2 + j^2 |u_j|^2} = \sqrt{J_2} + \sqrt{(1-j^2p)J_1 + j^2pJ_2}. $$

Taking the initial datum $u(0) = (u_0(0), u_j(0))$ with $u_1(0) = \delta/4$, $u_j(0) = j^{-p}\delta/4$, we have that its norm is smaller than $\delta$, while $J_1$ can have a drift of order $\delta^4 j^{-p} e^{-sj}$ in a time $T$ of order 1. This means that the Sobolev norm of $u(T)$ is of order $\delta^3 e^{-sj} j^p$ hence greater than $4\delta$ if $\delta^3 e^{-sj} j^p$ is large. Maximizing on $j$ we get a constraint of the form $\delta^2 e^{-p} (a^{-1}p)^p < 1$.

Of course this pathological ”fast diffusion” phenomenon comes from the non conservation of momentum and would appear (with similar constants) also in the case $\mathbb{M}$.

2. It is very important to stress that in the case $\mathbb{S}$ restricting to translation invariant Hamiltonians would not result in significantly weaker constraints on the smallness of $\delta$ w.r.t. $p$. This can be seen in the following example. Consider the family of Hamiltonians (in three degrees of freedom)

$$K^{(j)} := |u_1|^2 + j^2 |u_j|^2 + \text{Re}(\bar{u}_0^{-1} u_1^* \bar{u}_j)$$

with the constants of motion

$$L = |u_0|^2 + |u_1|^2 + |u_j|^2, \quad M = |u_1|^2 + j |u_j|^2.$$ 

Following the same approach as in the previous example one shows that $|u_j|^2$ can have a drift of order $j^{-p} \delta^{2j}$ in a time $T$ of order 1. This means that the Sobolev norm of $u(T)$ is of order $\delta^{2j} j^p$. Maximizing on $j$ we get a constraint of the form $\delta e^{p^3 - 1} < 1$. We point out that the Hamiltonian discussed above is stable in the $\mathbb{M}$ norm for all times and for $\delta$ small independent of $p$. This is the main reason for restricting in $\mathbb{M}$ to translation invariant Hamiltonians.

From Theorem 1.1 it is straightforward to maximize over $p$ and find an optimal regularity. We stress that in the case $\mathbb{S}$ our estimate on the stability time is an increasing function of $p$, so the maximum is obtained by just fixing $p$ so that $\delta = (\mathbb{C}_p \delta)^{-3p}$. On the other hand in the case $\mathbb{M}$ there is a proper maximum.

We thus have the following result. As before our statements depend on some constants, denoted by $\tilde{\delta_8}, \tilde{\delta_9}$ explicitly defined in Subsection A. These constants depend only on $\gamma, q, a, R, |f|_{L^6 R}$ in the case $\mathbb{S}$ and on $q, R, |f|_{L^6}$ in the case $\mathbb{M}$. By $[\cdot]$ we denote the integer part.

**Theorem 1.2** (Sobolev stability: optimization).

\textbf{(S)} For any $0 < \delta \leq \tilde{\delta_8}$ and any $u_0$ such that

\begin{equation}
|u_0|_{L^2} + |\partial_x^{p} u_0|_{L^2} \leq \delta, \quad p = p(\delta) := 1 + \tau_8 \left[ \frac{1}{6 \tau_8} \ln(\delta_8/\delta) \right],
\end{equation}

the solution $u(t)$ of (1.1) with initial datum $u(0) = u_0$ exists for all times

\begin{equation}
|t| \leq \frac{T_S}{\delta} e^{\frac{\ln(\delta_8/\delta)}{\ln(\tilde{\delta_9}/\delta)}} \quad \text{and satisfies} \quad |u(t)|_{L^2} + |\partial_x^{p} u(t)|_{L^2} \leq 4\delta.
\end{equation}

\textbf{(M)} Assume that $f$ in (1.1) is independent of $x$. For any $0 < \delta \leq \tilde{\delta_9}$ and

\begin{equation}
\forall p \geq p(\delta) := 1 + \tau_8 \left[ \frac{\delta^2}{\delta^2} \right], \quad \forall u_0 \quad \text{s.t.} \quad 2^p |u_0|_{L^2} + |\partial_x^{p} u_0|_{L^2} \leq \delta,
\end{equation}

the solution $u(t)$ of (1.1) with initial datum $u(0) = u_0$ exists for all times

\begin{equation}
|t| \leq \frac{T_M}{\delta^2} e^{(\tilde{\delta_8}/\delta)^2} \quad \text{and satisfies} \quad 2^p |u(t)|_{L^2} + |\partial_x^{p} u(t)|_{L^2} \leq 4\delta.
\end{equation}

\footnote{Indeed the term $e^{-sj}$ is added in order to ensure that monomials with very high momentum give an exponentially small contribution to the Hamiltonian}
Remark 1.3. Some remarks on Theorem 1.2 are in order.

Note that \([1.13]\) is the stability time computed in \([BFG88]\) for short range couplings.

1. In our study we have only considered \textit{Gauge preserving} equations, that is PDEs which preserve the \(L^2\) norm. We believe that this is just a technical question and that we could deal with more general cases. Similarly in the case \(\mathcal{M}\) we have assumed that \(f\) in \([1.1]\) is independent of \(x\), namely \textit{momentum preserving}. Not only this simplifies the proof but as explained after Theorem 1.1 allows us much better estimates. Of course we could prove the theorem (with different constants) also for \(x\)-dependent \(f\), as in the case \(S\).

2. We will prove the case \(\mathcal{M}\) only for \(p = p(\delta)\), the general case being analogous (with the same constants!) also if \(p \geq p(\delta)\).

3. One can easily restate Theorem 1.2 in terms of the Sobolev exponent \(p\), instead of \(\delta\), since the map \(\delta \to p(\delta)\) is injective.

In this paper we have considered the simplest possible example of dispersive PDE on the circle. One can easily see that the same strategy can be followed word by word in more general cases provided that the non-linearity does not contain derivatives. A much more challenging question is to consider NLS models with derivatives in the non-linearity. As we have mentioned a semilinear case was discussed by \(CMW\). A very promising approach to Birkhoff normal form for quasilinear PDEs is the one of \([Del12]\)-\([BD18]\) which was applied to fully-nonlinear reversible NLS equations in \([FI]\). It seems very plausible that one can adapt their methods (based on paralinearizations and paradifferential calculus) to our setting, however it seems that in this case one must give up the Hamiltonian structure.

Analytic and Gevrey initial data

In this case our result is similar to \([FG13]\) in the sense that we also prove \textit{subexponential bounds} on the time. We mention however that in \([FG13]\) the control of the Sobolev norm in time is in a lower regularity space w.r.t. the initial datum. Recently we have been made aware of a preprint by Cong, Mi and Wang \(CMW\) in which the authors give subexponential bounds for subanalytic initial data of a model like \([1.1]\), very similar to ours. A difference is that in their case the non linearity contains a derivative (see the comments after Theorem 1.3) while \([FG13]\) does not contain derivatives. A much more challenging question is to consider NLS models with derivatives in the non-linearity. As we have mentioned a semilinear case was discussed by \(CMW\). A very promising approach to Birkhoff normal form for quasilinear PDEs is the one of \([Del12]\)-\([BD18]\) which was applied to fully-nonlinear reversible NLS equations in \([FI]\). It seems very plausible that one can adapt their methods (based on paralinearizations and paradifferential calculus) to our setting, however it seems that in this case one must give up the Hamiltonian structure.

Let us fix \(0 < \theta < 1\), and define the function spaces

\[
H_{p,s,a} := \left\{ u(x) = \sum_{j \in \mathbb{Z}} u_j e^{i j x} \in L^2 : |u|^2_{p,s,a} := \sum_{j \in \mathbb{Z}} |u_j|^2 e^{2 a |j| + 2 s j^\theta} < \infty \right\},
\]

with the assumption \(a \geq 0, s > 0, p > 1/2\). We remark that if \(a > 0\) this is a space of analytic functions, while if \(a = 0\) the functions have Gevrey regularity. Note that for technical reasons connected to the way in which we control the small divisors, we cannot deal with the purely analytic case \(\theta = 1\), see Lemmas 3.3, 4.2. For this reason we denote this result as \(G\) (Gevrey case). The main important difference with the cases \(S, M\) is that now the regularity \(p, s, a\) is independent of \(\delta\).

As before our result, stated below, depends on some constants \(\delta_0, \delta_3, T_6\), \textit{explicitely defined} in Subsection A, and depending only on \(\gamma, q, a, R, |f|_{a,R}, p, s, a, \theta\).

Theorem 1.3 (Gevrey Stability). Fix any \(a \geq 0, s > 0\) such that \(a + s < a\) and any \(p > 1/2\). For any \(0 < \delta \leq \delta_0\) and any \(u_0\) such that

\[
|u_0|_{p,s,a} \leq \delta,
\]

the solution \(u(t)\) of \([1.1]\) with initial datum \(u(0) = u_0\) exists for all times

\[
|t| \leq \frac{T_6}{\delta^2} (\ln \frac{1}{\delta})^{1+ \theta/4} \quad \text{and satisfies} \quad |u(t)|_{p,s,a} \leq 2\delta.
\]

\(\Box\)

Indeed, thanks to the monotonicity property of our norms (see Proposition 3.2 below) the canonical transformation putting the system in Birkhoff Normal Form (see Theorem 1.4 below) in the \(p\)-case is simply the restriction to \(H^p\) of the one of the \(p(\delta)\)-case.
Remark 1.4. Some comments on Theorem 1.3 are in order.

1. We did not make an effort to maximize the exponent $1 + \theta/4$ in the stability time. In fact, by trivially modifying the proof, one could get $1 + \theta/(2^+)$. We remark that in [CMW], in which $\theta = 1/2$, the exponent is better, i.e. it is $1 + 1/(2^+)$. 

2. As we mentioned before, the main difference w.r.t. the cases S, M is that now the regularity $p, s, a$ is independent of $\delta$, with the only requirement that $p > 1/2$ and $s > 0$. If instead we took $s$ appropriately large with $\delta$ we would get an exponential bound just like in case M.

3. One could consider initial data with an intermediate regularity between Sobolev and Gevrey and compute stability times. A good example (suggested to us by Z. Hani) could be the space $h$ (1.18) and is equivalent to $H^{\delta - 3 + \ln(\ln(1/\delta))}$. Following the proof of Theorem 1.3 almost verbatim one can get an estimate of the type $T \geq C\delta^{-3 + \ln(\ln(1/\delta))}$.

1.2. The Birkhoff Normal Form. Our results are based on a Birkhoff normal form procedure, which we now describe. Let us pass to the Fourier side via the identification

$$u(x) = \sum_{j \in \mathbb{Z}} u_j e^{ijx} \mapsto u = (u_j)_{j \in \mathbb{Z}},$$

where $u$ belongs to some complete subspace of $\ell^2$. More precisely, given a real sequence $w = (w_j)_{j \in \mathbb{Z}}$, with $w_j \geq 1$ we consider the Hilbert space $^{5}$

$$H_w := \left\{ u := (u_j)_{j \in \mathbb{Z}} \in \ell^2(\mathbb{C}) : \|u\|_w^2 := \sum_{j \in \mathbb{Z}} |w_j|^2 |u_j|^2 < \infty \right\},$$

and fix the symplectic structure to be

$$\omega := \left\{ i \sum_{j} du_j \wedge d\bar{u}_j \right\}.$$

In this framework the Hamiltonian of (1.1) is

$$H_{\text{NLS}}(u) := D_\omega + P, \quad \text{where}$$

$$D_\omega := \sum_{j \in \mathbb{Z}} \omega_j |u_j|^2, \quad P := \int_T F(x, |u(x)|^2) dx, \quad F(x, y) := \int_0^y f(x, s) ds.$$

As examples of $H_w$ we consider:

S) (Sobolev case) $w_j = |j|^p$, which is isometrically isomorphic, by Fourier transform, to $H_{p,0,0}$ defined in (1.16) and is equivalent to $H^p$ equipped with the norm $| \cdot |_{L^2} + |\partial_x^p \cdot |_{L^2}$ with equivalence constants independent of $p$ (see 2.8).

M) (Modified-Sobolev case) $w_j = |j|^p$, where $|j| := \max\{|j|, 2\}$; this space is equivalent to $H^p$ equipped with the norm $2^p | \cdot |_{L^2} + |\partial_x^{2^p} \cdot |_{L^2}$ with equivalence constants independent of $p$ (see 2.9).

G) (Gevrey case) $w_j = |j|^p e^{\alpha|j|+\beta j}^s$, which is isometrically isomorphic, by Fourier transform, to $H_{p,s,a}$ defined in (1.16).

Here and in the following, given $r > 0$, by $B_r(h_w)$ we mean the closed ball of radius $r$ centered at the origin of $h_w$. 

---

5 Endowed with the scalar product $(u, v)_{h_w} := \sum_{j \in \mathbb{Z}} w_j^2 u_j \bar{v}_j.$
Definition 1.2 (majorant analytic Hamiltonians). For \( r > 0 \), let \( \mathcal{A}_r(\mathfrak{h}_w) \) be the space of Hamiltonians
\[
H : B_r(\mathfrak{h}_w) \to \mathbb{R}
\]
such that there exists a pointwise absolutely convergent power series expansion\(^6\)
\[
H(u) = \sum_{\alpha, \beta \in \mathbb{N}^Z, |\alpha| + |\beta| < \infty} H_{\alpha, \beta} u^\alpha \bar{u}^\beta, \quad u^\alpha := \prod_{j \in \mathbb{Z}} u_j^{\alpha_j}
\]
with the following properties:

(i) Reality condition:
\[
H_{\alpha, \beta} = H_{\beta, \alpha};
\]
(ii) Mass conservation:
\[
H_{\alpha, \beta} = 0 \quad \text{if} \quad |\alpha| \neq |\beta|,
\]
namely the Hamiltonian Poisson commutes with the mass \( \sum_{j \in \mathbb{Z}} |u_j|^2 \).

Finally, given \( H \) as above, we define its majorant \( \tilde{H} : B_r(\mathfrak{h}_w) \to \mathbb{R} \) as
\[
\tilde{H}(u) = \sum_{\alpha, \beta \in \mathbb{N}^Z, |\alpha| + |\beta| < \infty} |H_{\alpha, \beta}| u^\alpha \bar{u}^\beta.
\]

We also define the subspace of normal form Hamiltonians
\[
\mathcal{K} := \left\{ Z \in \mathcal{A}_r(\mathfrak{h}_w) : Z(u) = \sum_{\alpha \in \mathbb{N}^Z} Z_{\alpha, \alpha} |u|^2 \alpha \right\}.
\]

Note that \( Z_{\alpha, \alpha} \in \mathbb{R} \) for every \( \alpha \in \mathbb{N}^Z \) by condition (1.21).

In the following we will also deal with a smaller class of Hamiltonians, namely the ones which have the momentum \( \sum_{j \in \mathbb{Z}} j |u_j|^2 \) as additional first integral.

Definition 1.3. We say that a Hamiltonian \( H \in \mathcal{A}_r(\mathfrak{h}_w) \) preserves momentum when
\[
H_{\alpha, \beta} = 0 \quad \text{if} \quad \sum_{j \in \mathbb{Z}} (\alpha_j - \beta_j) \neq 0,
\]
namely the Hamiltonian \( H \) Poisson commutes with \( \sum_{j \in \mathbb{Z}} j |u_j|^2 \).

Note that if the nonlinearity \( f \) in equation (1.1) does not depend on the variable \( x \), then the Hamiltonian \( P \) in (1.20) preserves momentum.

We now state a Birkhoff Normal Form Theorem for the Hamiltonian in (1.20). Fix any \( N \geq 1 \) and consider the space \( \mathfrak{h}_w \) where \( w \) is one of the following three cases, where \( \tau, \tau_1 \) are fixed positive constants defined in (A):

S) (Sobolev case) \( w_j = \langle j \rangle^{1 + \tau_1} \);  
M) (Modified-Sobolev case) \( w_j = |j|^{1 + \tau_1} \), where \( |j| := \max(|j|, 2) \);  
G) (Gevrey case) \( w_j = e^{a|j| + s(j)^\theta(j)^p} \) with \( p > 1/2, s > 0, 0 \leq a < a \).

As before we define in Subsection [A] below the constants \( \tau, \tau_1, C_1, C_2, C_3 \), corresponding to the cases S, M, G respectively. We remark that these constants depend on \( N \geq 1 \).

Theorem 1.4 (Birkhoff Normal Form). Fix any \( N \geq 1 \) and consider the space \( \mathfrak{h}_w \) where \( w \) is one of the three above cases: S, M, G. Consider the Hamiltonian (1.20), assuming, only in the case M, that \( f \) does not depend on

\(^6\)As usual given a vector \( k \in \mathbb{Z}^Z, |k| := \sum_{j \in \mathbb{Z}} |k_j| \).
x (momentum conservation). Then for any $0 < r \leq \mathbf{r}$ there exists two close to identity invertible symplectic change of variables

$$
\Psi, \Psi^{-1} : B_r(\mathbf{h}_\omega) \mapsto \mathbf{h}_\omega, \quad \sup_{|u|_\mathbf{h}_\omega \leq r} |\Psi \pm 1(u)| - u |_{\mathbf{h}_\omega} \leq C_1 r^3 \leq \frac{1}{8} r, 
$$

$$
\Psi \circ \Psi^{-1} u = \Psi^{-1} \circ \Psi u = u, \quad \forall u \in B_{\mathbf{h}_\omega}^r(\mathbf{h}_\omega)
$$

(1.25)

such that in the new coordinates

$$
H \circ \Psi = D_\omega + Z + R,
$$

for suitable majorant analytic Hamiltonians $Z, R \in A_r(\mathbf{h}_\omega), Z \in \mathcal{K}$, satisfying the estimate

$$
\sup_{|u|_\mathbf{h}_\omega \leq r} |X_Zu| \leq C_2 r^3, \quad \sup_{|u|_\mathbf{h}_\omega \leq r} |X_Ru| \leq C_3 r^{20+3},
$$

(1.26)

$X_Z$ (resp. $X_R$), being the hamiltonian vector field generated by the the majorant of $Z$ (resp. $R$). Moreover, in the case $\mathbf{m}, R$ preserves momentum.

The proof of our Birkhoff normal form result is based on a procedure which, while following the line of previous works such as [BG06] and [FG13], takes a slightly different point of view. Broadly speaking the core is the following: as already noticed in [FG13] small divisor estimates and hence stability are simpler to prove for translation invariant PDEs (i.e. Hamiltonian systems which preserve the momentum). Considering this fact we introduce an appropriate norm, which weights non-momentum preserving monomial exponentially. This norm is rather cumbersome and depends on many parameters (see comments in the next page) but we show that it is very well suited for performing Birkhoff normal form steps for dispersive PDEs on the circle. This rather simple idea, allows us a very good control of the small divisors by generalizing the estimates by Bourgain in [Bou05].

As a byproduct our normal forms are simpler, in the sense that they are functions only of the linear actions, and it is relatively easy to compute all the constants.

Above we stated Theorem 1.4 only in the cases $\mathbf{s}, \mathbf{m}, \mathbf{g}$, but our method is quite versatile we thus formulate a Birkhoff Norm Form result in the general contest of weighted Hilbert spaces, see Theorem 5.1. Once we have the Birkhoff theorem, Theorem 1.4 follows directly.

1.3. About the norms. After a brief presentation of the symplectic structure relevant for NLS equations we start, in Section 2, by defining the subspace of $A_r(\mathbf{h}_\omega)$ of $\eta$-majorant regular Hamiltonians denoted by $\mathcal{H}_{r, \eta}(\mathbf{h}_\omega)$ and defined by the condition that an appropriate majorant norm of the associated Hamiltonian vector field is bounded (see Definition 2.2). The parameters $r > 0, \eta \geq 0$ have the following role: $r$ controls the radius of analiticity of the Hamiltonian vector field as a function from $\mathbf{h}_\omega$ to itself, while $\eta$ ensures that the terms (monomials) which do not preserve momentum are exponentially small.

We note that for Hamiltonians which preserve momentum the dependence on the parameter $\eta$ is trivial and can be omitted. Indeed in the latter case the norm coincides with the usual majorant norm, see for instance [BBP14]. Another interesting point is that on the space of polynomials our norm controls the majorant-tame norm defined in [BG06] (see Proposition 3.3). Although this fact is not needed in our proof we find it an interesting remark (it was pointed out to us by A. Maspero), since most proofs of Sobolev stability strongly rely on majorant-tame properties of the Hamiltonians.

Our norm is well suited for measuring Hamiltonian vector fields, indeed in Subsection 2.2 we show that it is closed with respect to Poission brackets (as a scale in $r$). Moreover a Hamiltonian with small norm defines a well posed symplectic change of variables on a ball of $\mathbf{h}_\omega$. Furthermore if $\mathbf{h}_\omega$ is closed by convolution then the nonlinearity $P$ of the NLS Hamiltonian (1.20) is in $\mathcal{H}_{r, \eta}(\mathbf{h}_\omega)$ for appropriate $r, \eta$. This is discussed in Subsection 2.3 in the cases $\mathbf{s}, \mathbf{m}, \mathbf{g}$.

Anyway we think that the main point is that our norm has explicit (and for us quite surprising) monotonicity properties. In Section 3 we first give an abstract result, which ensures that $\mathcal{H}_{r, \eta}(\mathbf{h}_\omega) \subseteq \mathcal{H}_{r', \eta'}(\mathbf{h}_\omega)$ under appropriate relations among $r', r, \eta', \eta, \mathbf{w}, \mathbf{w}'$, while in Proposition 3.2 we specify to the three cases $\mathbf{s}, \mathbf{m}, \mathbf{g}$.

\footnote{The Banach case could be treated as well.}
Finally our norm is well suited for the control of the solution of the homological equation \( \{D_\omega, S\} = R \). In Section 4 we first give an abstract result, which ensures that if \( R \in \mathcal{H}_{r, \eta}(\mathfrak{h}_\omega) \) then \( S \in \mathcal{H}_{r', \eta'}(\mathfrak{h}_{\omega'}) \) (for an appropriate choice of \( r', \eta', \omega' \)) and satisfies a quantitative bound. Then, in Proposition 4.4 we specify to our three cases \( S, \mathfrak{M}, \mathfrak{G} \).

At this point we have all the ingredients needed to perform the steps of Birkhoff normal form; this is done in Section 5 in an abstract setting. Finally we specify to our three cases \( S, \mathfrak{M}, \mathfrak{G} \) and prove Theorem 1.4 in Section 6 and Theorems 1.2 and 1.3 in Section 8.

The appendices are devoted to giving full details of the most technical proofs.

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## 2. Functional setting and symplectic structure

### 2.1. Spaces of Hamiltonians.

As explained in the Introduction our weighted spaces \( \mathfrak{h}_\omega \) are contained in \( \ell^2(\mathbb{C}) \), so we endow them with the standard symplectic structure coming from the Hermitian product on \( \ell^2(\mathbb{C}) \).

We identify \( \ell^2(\mathbb{C}) \) with \( \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \) through \( u_j = (x_j + iy_j)/\sqrt{2} \) and induce on \( \ell^2(\mathbb{C}) \) the structure of a real symplectic Hilbert space\(^8\) by setting, for any \( (u^{(1)}, u^{(2)}) \in \ell^2(\mathbb{C}) \times \ell^2(\mathbb{C}) \),

\[
\langle u^{(1)}, u^{(2)} \rangle = \sum_{j} \left( x_j^{(1)} x_j^{(2)} + y_j^{(1)} y_j^{(2)} \right), \quad \omega(u^{(1)}, u^{(2)}) = \sum_{j} \left( y_j^{(1)} x_j^{(2)} - x_j^{(1)} y_j^{(2)} \right),
\]

which are the standard scalar product and symplectic form \( \Omega = \sum_j dy_j \wedge dx_j \).

For convenience and to keep track of the complex structure, one often writes the vector fields and the differential forms in complex notation, that is

\[
\Omega = i \sum_j du_j \wedge d\bar{u}_j, \quad X_H^{(j)} = i \frac{\partial}{\partial u_j} H
\]

where the one form and vector field are defined through the identification between \( \mathbb{C} \) and \( \mathbb{R}^2 \), given by

\[
du_j = \frac{1}{\sqrt{2}}(dx_j + idy_j), \quad d\bar{u}_j = \frac{1}{\sqrt{2}}(dx_j - idy_j),
\]

\[
\frac{\partial}{\partial u_j} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{u}_j} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).
\]

**Definition 2.1** (\( \eta \)-majorant analytic Hamiltonians). For \( \eta \geq 0, r > 0 \) let \( \mathcal{A}_{r, \eta}(\mathfrak{h}_\omega) \subseteq \mathcal{A}_r(\mathfrak{h}_\omega) \) be the subspace of majorant analytic Hamiltonians (recall Definition 1.2) such that

\[
H_{\eta}(u) = \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}| e^{\eta|\pi(\alpha - \beta)|} u^{\alpha} \bar{u}^{\beta}
\]

is point-wise absolutely convergent on \( B_r(\mathfrak{h}_\omega) \) and

\[
\pi(\alpha - \beta) := \sum_{j \in \mathbb{Z}} j(\alpha_j - \beta_j).
\]

Note that \( \pi(\alpha - \beta) \) is the eigenvalue of the adjoint action of the momentum Hamiltonian \( \sum_{j \in \mathbb{Z}} j|u_j|^2 \) on the monomial \( u^{\alpha} \bar{u}^{\beta} \). The exponential weight \( e^{\eta|\pi(\alpha - \beta)|} \) is added in order to ensure that the monomials which do not preserve momentum must have an exponentially small coefficient.

---

\(^8\)We recall that given a complex Hilbert space \( H \) with a Hermitian product \( \langle \cdot, \cdot \rangle \), its realification is a real symplectic Hilbert space with scalar product and symplectic form given by

\[
\langle u, v \rangle = 2\text{Re}(u, v), \quad \omega(u, v) = 2\text{Im}(u, v).
\]
The Hamiltonian functions being defined modulo a constant term, we shall assume without loss of generality that $H(0) = 0$.

We will say that a Hamiltonian $H(u) \in A_{r,\eta}(h_\eta)$ is $\eta$-regular if $X_{H_{\eta}} : B_r(h_\eta) \to h_\eta$ and is uniformly bounded, where $X_{H_{\eta}}$ is the vector field associated to the $\eta$-majorant Hamiltonian in (2.3). More precisely we give the following

**Definition 2.2** ($\eta$-regular Hamiltonians). For $\eta \geq 0, r > 0$ let $\mathcal{H}_{r,\eta}(h_\eta)$ be the subspace of $A_{r,\eta}(h_\eta)$ of those Hamiltonians $H$ such that

$$|H|_{\mathcal{H}_{r,\eta}(h_\eta)} = |H|_{r,\eta,w} := r^{-1}\sup_{|w|_{h_\eta} \leq r} |X_{H_{\eta}}|_{h_\eta} < \infty.$$

We shall show that this guarantees that the Hamiltonian flow of $H$ exists at least locally and generates a symplectic transformation on $h_\eta$, i.e. $h_\eta$ is an invariant subspace for the dynamics.

**Remark 2.1.** We note that if $H$ preserves momentum, then $|H|_{r,\eta,w} = |H|_{r,0,w}$ does not depend on $\eta$ and coincides with the majorant norm of a regular Hamiltonian as defined in [BBP13, Definition 2.6], when $I = \emptyset$. Actually this is nothing but the restriction to Hamiltonian vector fields of the $\eta$-momentum majorant norm defined in [BBP14, Definition 2.3] when $I = \emptyset$.

**Remark 2.2.** By mass conservation and since $H(0) = 0$, it is straightforward to prove that the norm $|\cdot|_{r,\eta,w}$ is increasing in the radius parameter $r$ (see also Proposition 2.1).

Note that if $|H|_{\mathcal{H}_{r,\eta}(h_\eta)} < \infty$ then $H$ admits an analytic extension $\hat{H}$, that is

$$(u_+, u_-) \in B_r(\ell^2(C)) \times B_r(\ell^2(C)) \to \hat{H}(u_+, u_-) : H(u) = \hat{H}(u, \bar{u}),$$

whose Taylor series expansion is

$$\hat{H}(u_+, u_-) = \sum_{|\alpha|, |\beta| \in \mathbb{N}^2} H_{\alpha,\beta} u_+^\alpha u_-^\beta.$$

where we denote by $\sum^*$ the sum restricted to those $\alpha, \beta : |\alpha| = |\beta| < \infty$.

One can see that

$$\frac{\partial}{\partial u_j} H(u) = \frac{\partial \hat{H}(u_+, u_-)}{\partial u_{-j}} |_{u_+ = \bar{u}_- = u}.$$

Let us now define two fundamental subspaces of $\mathcal{H}_{r,\eta}(h_\eta)$.

**Definition 2.3** (Range and Kernel). Let $\mathcal{R}$ (for Range) and $\mathcal{K}$ (for Kernel) the following subspaces of $\mathcal{H}_{r,\eta}(h_\eta)$

(2.3) $\mathcal{R} = \mathcal{R}_{r,\eta}(h_\eta) := \{ H \in \mathcal{H}_{r,\eta}(h_\eta) \mid H = \sum_{\alpha \neq \beta} H_{\alpha,\beta} u_+^\alpha u_-^\beta \}$

(2.4) $\mathcal{K} = \mathcal{K}_{r,\eta}(h_\eta) := \{ H \in \mathcal{H}_{r,\eta}(h_\eta) \mid H = \sum_{\alpha \in \mathbb{N}^2} H_{\alpha,\alpha} u_+^{2\alpha} \}$

We thus can write $\mathcal{H}_{r,\eta}(h_\eta) = \mathcal{R}_{r,\eta}(h_\eta) \oplus \mathcal{K}_{r,\eta}(h_\eta)$.

Note that $\mathcal{R}$ and $\mathcal{K}$ define continuous projection operators with

(2.5) $|\PiKH|_{r,\eta,w}, |\PiKH|_{r,\eta,w} \leq |H|_{r,\eta,w}$

**Example 1** (Notation for the Gevrey case). In the case $h_\eta = h_{p,s,a}$ we denote the space of $\eta$-regular Hamiltonians $\mathcal{H}_{r,\eta}(h_{p,s,a})$ by $\mathcal{H}_{r,p,s,a,\eta}$ with norm

(2.6) $|H|_{r,p,s,a,\eta} := r^{-1}\sup_{|u|_{p,s,a} \leq r} |X_{H_\eta}|_{p,s,a},$

namely

$$|\cdot|_{r,p,s,a,\eta} = |\cdot|_{\mathcal{H}_{r,\eta}(h_{p,s,a})}.$$
Definition 2.4 (Modified Sobolev space). Fix $w = |j|^p$ where

$$|j| := \max\{|j|, 2\}$$

and consider $h^p := h_w$ endowed with the norm

$$\|u\|_{p}^2 := \sum_{j \in \mathbb{Z}} |j|^{2p}|u_j|^2.$$  

This norm is equivalent to the norm

$$|u|_{p}^2 := \sum_{j \in \mathbb{Z}} \langle j \rangle^{2p}|u_j|^2,$$

since

$$|\cdot|_p \leq \|\cdot\|_p \leq 2^p \cdot |\cdot|_p.$$  

Remark 2.3. Note that, identifying by the Fourier transform (1.17) the function $u(x)$ with the sequence of its Fourier coefficients $u$, we have

$$|u(x)|_{p,0,0} = |u|_p.$$  

Moreover

$$|u|_p \leq |u(x)|_{L^2} + |\partial_x^p u(x)|_{L^2} \leq 2|u|_p$$

and

$$\|u\|_p \leq 2^p|u(x)|_{L^2} + |\partial_x^p u(x)|_{L^2} \leq 2\|u\|_p.$$  

Note that here we write $u(x)$ and $u$ to distinguish the function $u(x)$ from the sequence $u$ of its Fourier coefficients; however in the rest of the paper we simply write $u$ to denote the function too.

We now introduce the subspace of $\mathcal{H}_{r,0}(h^p)$ of those Hamiltonians preserving momentum.

Definition 2.5 (momentum preserving regular Hamiltonians). Given $r > 0, p \geq 0$ let $\mathcal{H}_{r,0}(h^p)$ be the space of point-wise absolutely convergent Hamiltonians on $\|u\|_p \leq r$ which preserves momentum and such that

$$\|H\|_{r,p} := r^{-1} \left( \sup_{\|u\|_p \leq r} \|X_H\|_p \right) < \infty,$$

namely

$$\|\cdot\|_{r,p} = |\cdot|_{\mathcal{H}_{r,0}(h^p)}.$$  

2.2. Poisson structure and Hamiltonian flows. The scale $\{\mathcal{H}_{r,\eta}(h_w)\}_{r>0}$ is a Banach-Poisson algebra in the following sense

Proposition 2.1. For $0 < \rho \leq r$ and $\eta > 0$ we have

$$\{F,G\}_{r,\eta,w} \leq 4 \left( 1 + \frac{r}{\rho} \right) |F|_{r+\rho,\eta,w}G|_{r+\rho,\eta,w}.$$  

Proof. It is essentially contained in [BBP13]. See in particular Lemma 2.16 of [BBP13] with $n = 0$ (no action variables here) and no $s$ and $s'$ (no actions variable here). Note that the constant in Lemma 2.16 is 8, instead of 4 in the present paper, because of the presence there of action variables which scale different from the cartesian ones (namely $(2^r)^2$ instead of $2^r$). Recall also the required properties of the space $E$ (named $h_w$ in the present paper) mentioned after Definition 2.5. □

The following Lemma is a simple corollary

Note that on the preserving momentum subspace $\mathcal{H}_{r,\eta}(h_w)$ coincides with $\mathcal{H}_{r,0}(h_w)$ for every $\eta$. 

---

Note that on the preserving momentum subspace $\mathcal{H}_{r,\eta}(h_w)$ coincides with $\mathcal{H}_{r,0}(h_w)$ for every $\eta$.
Lemma 2.1 (Hamiltonian flow). Let $0 < \rho < r$, and $S \in \mathcal{H}_{r+\rho, \eta}(\mathfrak{h}_w)$ with

$$|S|_{r+\rho, \eta, w} \leq \delta := \frac{\rho}{8e(r+\rho)}.$$  

Then the time $1$-Hamiltonian flow $\Phi^1_S : B_r(\mathfrak{h}_w) \to B_{r+\rho}(\mathfrak{h}_w)$ is well defined, analytic, symplectic with

$$\sup_{u \in B_r(\mathfrak{h}_w)} |\Phi^1_S(u) - u|_{\mathfrak{h}_w} \leq (r+\rho)|S|_{r+\rho, \eta, w} \leq \frac{\rho}{8e}.$$  

For any $H \in \mathcal{H}_{r+\rho, \eta}(\mathfrak{h}_w)$ we have that $H \circ \Phi^1_S = e^{\{S, \cdot\}H} \in \mathcal{H}_{r, \eta}(\mathfrak{h}_w)$ and

$$e^{\{S, \cdot\}H}_{r, \eta, w} \leq 2|H|_{r+\rho, \eta, w},$$  

$$|\left( e^{\{S, \cdot\} - \text{id} \}H \right)_{r, \eta, w} | \leq \delta^{-1}|S|_{r+\rho, \eta, w}|H|_{r+\rho, \eta, w},$$  

$$\left| \left( e^{\{S, \cdot\} - \{S, \cdot\} \}H \right)_{r, \eta, w} \right| \leq \frac{1}{2}\delta^{-2}|S|^2_{r+\rho, \eta, w}|H|_{r+\rho, \eta, w}.$$  

More generally for any $h \in \mathbb{N}$ and any sequence $(c_k)_{k \in \mathbb{N}}$ with $|c_k| \leq 1/k!$, we have

$$\sum_{k \geq h} c_k \text{ad}_S^k (H)_{r, \eta, w} \leq 2|H|_{r+\rho, \eta, w}(|S|_{r+\rho, \eta, w}/2\delta)^{h},$$  

where $\text{ad}_S(\cdot) := \{S, \cdot\}.$

Proof. For brevity we set, for every $r' > 0$

$$| \cdot |_{r'} := | \cdot |_{r', \eta, w}.$$  

We use Lemma B.3 with $E \to \mathfrak{h}_w$, $X \to X$, $\delta_0 \to (r+\rho)|S|_{r+\rho}$, $r \to r+\rho$, $r_1 \to r$, $T \to 8e$. Then the fact that the time $1$-Hamiltonian flow $\Phi^1_S : B_r(\mathfrak{h}_w) \to B_{r+\rho}(\mathfrak{h}_w)$ is well defined, analytic, symplectic follows, since for any $\eta \geq 0$

$$\sup_{u \in B_{r+\rho}(\mathfrak{h}_w)} |X_S|_{\mathfrak{h}_w} \leq (r+\rho)|S|_{r+\rho} < \frac{\rho}{8e}.$$  

Regarding the estimate (2.13), again by Lemma B.3 (choosing $t = 1$), we get

$$\sup_{u \in B_r(\mathfrak{h}_w)} |\Phi^1_S(u) - u|_{\mathfrak{h}_w} \leq (r+\rho)|S|_{r+\rho} < \frac{\rho}{8e}.$$  

Estimates (2.14), (2.15), (2.16) directly follow by (2.17) with $h = 0, 1, 2$, respectively and $c_k = 1/k!$, recalling that by Lie series

$$H \circ \Phi^1_S = e^{\text{ad}_S} H = \sum_{k=0}^{\infty} \frac{\text{ad}_S^k H}{k!} = \sum_{k=0}^{\infty} \frac{H^{(k)}}{k!},$$  

where $H^{(i)} := \text{ad}_S^i (H) = \text{ad}_S(H^{(i-1)})$, $H^{(0)} := H$.

Let us prove (2.17). Fix $k \in \mathbb{N}$, $k > 0$ and set

$$r_i := r + \rho \left( 1 - \frac{i}{k} \right), \quad i = 0, \ldots, k.$$  

Note that, by the monotonicity of the norm (recall Remark 2.2)

$$|S|_{r_i} \leq |S|_{r+\rho}, \quad \forall i = 0, \ldots, k.$$  

Noting that

$$1 + \frac{kr_i}{\rho} \leq k \left( 1 + \frac{r}{\rho} \right), \quad \forall i = 0, \ldots, k.$$  

Therefore

$$|S|_{r+\rho} \leq \frac{\rho}{8e}.$$  

This completes the proof of the lemma.
by using $k$ times (2.11) we have
\[ |H^{(k)}|_r = |\{S, H^{(k-1)}\}|_r \leq 4(1 + \frac{kr}{\rho})|H^{(k-1)}|_{r_{k-1}}|S|_{r_{k-1}} \]
\[ \leq 2(2.18) \leq |H|_{r+\rho}|S|_{r+\rho}4k \prod_{i=1}^{k} \left(1 + \frac{kr_i}{\rho}\right) \leq |H|_{r+\rho} \left(4k \left(1 + \frac{r}{\rho}\right)|S|_{r+\rho}\right)^k . \]

Then, using $kk \leq c^k k!$, we get
\[ \left| \sum_{k \geq h} c_k H^{(k)} \right|_r \leq \sum_{k \geq h} |c_k||H^{(k)}|_r \leq |H|_{r+\rho} \sum_{k \geq h} \left(4k \left(1 + \frac{r}{\rho}\right)|S|_{r+\rho}\right)^k \]
\[ = |H|_{r+\rho} \sum_{k \geq h} (|S|_{r+\rho}/2\delta)^k \leq 2|H|_{r+\rho}(|S|_{r+\rho}/2\delta)^h . \]

Finally, if $S$ and $H$ satisfy mass conservation so does each $\text{ad}_S^k H$, $k \geq 1$, hence $H \circ \Phi^k_S$ too. \hfill $\Box$

2.3. Nemitskii operators. Now we show that the nonlinearities in (1.1) are bounded in the norm $|\cdot|_{r,\eta,\nu}$ in the cases $S, M, G$. For any $0 \leq p \leq p', 0 \leq s \leq s', 0 \leq a \leq a'$ we have
\[ (2.20) \]
\[ h_{p',s',a'} \subseteq h_{p,s,a} \]
and
\[ |v|_{p,s,a} \leq |v|_{p',s',a'}, \quad \forall v \in h_{p,s,a}. \]

For $p > 1$ let $\star : h_{p,s,a} \times h_{p,s,a} \to h_{p,s,a}$ be the convolution operation defined as
\[ (f, g) \mapsto f \star g := \left( \sum_{j_1, j_2 \in \mathbb{Z}, j_1 + j_2 = j} f_{j_1} g_{j_2} \right)_{j \in \mathbb{Z}} . \]

The map $\star : (f, g) \mapsto f \star g$ is continuous in the following sense:

**Lemma 2.2.** For $p > 1/2$ we have
\[ (2.21) \]
\[ |f \star g|_{p,s,a} \leq C_{\text{alg}}(p) |f|_{p,s,a} |g|_{p,s,a} . \]

The proof is given in Appendix B.1

**Lemma 2.3.** For $p > 1/2$ and $f, g \in h^p$
\[ (2.22) \]
\[ \|f \star g\|_p \leq C_{\text{alg}, p}(p) \|f\|_p \|g\|_p . \]

The proof is given in Appendix B.2

**Lemma 2.4** (Nemitskii operators). Let $p > 1/2$. (i) Fix $s \geq 0$, $a_0 \geq 0$. Consider a sequence $F^{(d)} = \left( F_{j}^{(d)} \right)_{j \in \mathbb{Z}} \in h_{p,s,a_0}$, $d \geq 1$, such that
\[ (2.23) \]
\[ \sum_{d=1}^{\infty} \sum_{d=1}^{\infty} |F_{j}^{(d)}|_{p,s,a_0} R^d < \infty \]
for some $R > 0$.

For $u = (u_j)_{j \in \mathbb{Z}}$ let $\tilde{u} = (\tilde{u}_j)_{j \in \mathbb{Z}}$ and consider the Hamiltonian
\[ H(u) = \sum_{d=1}^{\infty} \left( F_{j}^{(d)} \star \cdots \star \tilde{u} \star \cdots \star \tilde{u} \right)_{d \text{ times}} . \]
For all \((\eta,a,r)\) such that \(\eta + a \leq a_0\) and \((C_{\text{alg}}(p)r)^2 \leq R\), we have that \(H \in H_{r,p,s,a,\eta}\) and
\[
|H|_{r,p,s,a,\eta} \leq r^{-1}\sum_{d=1}^{\infty} |d| F^{(d)}|_{p,s,a_0} (C_{\text{alg}}(p)r)^{2d-1} < \infty.
\]

(ii) Analogously if \(F^{(d)}\) are constants satisfying
\[
(2.24) \quad \sum_{d=1}^{\infty} d|F^{(d)}| R^d < \infty
\]
and \((C_{\text{alg,}\eta}(p)r)^2 \leq R\), then \(H \in H_{r,p}\) with
\[
(2.25) \quad \|H\|_{r,p} \leq 2^p R^{-1}\sum_{d=1}^{\infty} d|F^{(d)}|(C_{\text{alg,}\eta}(p)r)^{2d-1} < \infty.
\]

Proof. In Appendix \[\text{[3.3]}\]. \(\square\)

Corollary 2.1. Consider the correction term \(P = \int_T F(x,|u|^2)dx\) in the NLS Hamiltonian \[\text{(1.20)}\], where the argument \(f\) in \(F\) satisfies \[\text{(1.2)}\]. Let \(p > 1/2\).

(i) For any \(a,s,\eta \geq 0\) such that \(a + \eta < a\) and any \(r > 0\) such that \(\sup (C_{\text{alg}}(p)r)^2 \leq R\), we have
\[
(2.26) \quad \|P\|_{r,p,s,a,\eta} \leq C_{\text{geo}}(p,s,a-a-\eta) \frac{(C_{\text{alg}}(p)r)^2}{R} \|f\|_{a,R} < \infty.
\]
where \(f\) and \(|f|_{a,R}\) are defined in \[\text{(1.2)}\].

(ii) If \(F\) is independent of \[\text{(1.2)}\] and \(\|f\|_{a,R}\), we have
\[
(2.27) \quad \|P\|_{r,p} \leq 2^p \frac{(C_{\text{alg,}\eta}(p)r)^2}{R} \|f\|_R < \infty.
\]

Proof. By definition (recall \[\text{(1.2)}\] and \[\text{(1.20)}\])
\[
(2.28) \quad F(x,y) = \int_0^y f(x,s)ds = \sum_{d=2}^{\infty} \int_0^y f^{(d-1)}(x) \frac{y^d}{d!} = \sum_{d=2}^{\infty} F^{(d)}(x)y^d
\]
therefore we have
\[
P = \int_T F(x,|u|^2)dx = \sum_{d=2}^{\infty} \left( F^{(d)} \ast u \ast \cdots \ast u \ast \bar{u} \ast \cdots \ast \bar{u} \right)_{d \text{ times}}_{d \text{ times}}.
\]
To each analytic function \(F^{(d)}(x)\) we associate its Fourier coefficients; we have \(\left( F^{(d)}_j \right)_{j \in \mathbb{Z}} \in h_{p,s,a_0}\) for \(a_0 := a + \eta < a\) and \(s,p \geq 0\). Indeed
\[
|F^{(d)}|_{p,s,a_0}^2 := \sum_j e^{2u_0(j+2s(j))^2} \left( j \right)^{2p} |F^{(d)}_j|^2 \leq \sum_j e^{2u_0(j+2s(j))^2} \left( j \right)^{2p} \left( j \right)^{2d-1} \frac{|F^{(d-1)}_j|^2}{d^2}
\]
\[
\leq \frac{c^2(p,s,a-a_0)}{d^2} \sum_j e^{2u_0(j+2s(j))^2} \left( j \right)^{2d-1} |F^{(d-1)}_j|^2 \leq \frac{c^2(p-a_0,s,p)}{d^2} \|f^{(d-1)}\|^2_{a_0}
\]
with
\[
c(p,s,t) := e^s + \sup_{x \geq 1} xe^{-tx+sx^2}
\]
Now condition \[\text{(1.2)}\] ensures that \[\text{(2.23)}\] holds and our claim follows, by Lemma \[\text{2.4}\] setting \(a_0 = a + \eta\).

(ii) Follows from \[\text{(2.25)}\]. \(\square\)

\[^{10}\] \(R\) defined in \[\text{(1.2)}\].

\[^{11}\] i.e. \(P\) preserves momentum and we are assuming \[\text{(1.3)}\].
3. Monotonicity properties.

Given two positive sequences \( w = (w_j)_{j \in \mathbb{Z}}, w' = (w'_j)_{j \in \mathbb{Z}} \) we write that \( w \leq w' \) if the inequality holds point wise, namely
\[
w \leq w' : \iff w_j \leq w'_j, \quad \forall j \in \mathbb{Z}.
\]
In this way if \( r' \leq r \) and \( w \leq w' \) then \( B_{r'}(h_w) \subseteq B_r(h_w) \). Consequently if \( r' \leq r, \eta' \leq \eta \) and \( w \leq w' \) then \( \mathcal{A}_{r',\eta'}(h_w) \subseteq \mathcal{A}_{r,\eta}(h_w) \).

We thus wish to study conditions on \((3.3)\) for any \( \pi \) by setting \( \mathcal{A}_{r,\eta}(h_w) \subseteq \mathcal{A}_{r',\eta'}(h_{w'} \).

We thus wish to study conditions on \((r, \eta, w), (r', \eta', w')\) (with \( r' \leq r \)) which ensure that \( \mathcal{H}_{r,\eta}(h_w) \subseteq \mathcal{H}_{r',\eta'}(h_{w'}) \).

Note that this is not obvious at all, since we are asking that an Hamiltonian vector field of \( X_H \in \mathcal{H}_{r,\eta}(h_w) \), when restricted to the smaller domain \( B_{r'}(h_{w'}) \) belongs to the smaller space \( h_{w'} \).

Let us start by rewriting the norm \( | \cdot |_{r, \eta, w} \) in a more adimensional way.

**Definition 3.1.** For any \( H \in \mathcal{H}_{r,\eta}(h_w) \) we define a map
\[
B_1(\ell^2) \to \ell^2, \quad y = (y_j)_{j \in \mathbb{Z}} \mapsto \left( Y_H^{(j)}(y; r, \eta, w) \right)_{j \in \mathbb{Z}}
\]
by setting
\[
Y_H^{(j)}(y; r, \eta, w) := \sum_{\alpha, \beta} |H_{\alpha, \beta}| \frac{(\alpha_j + \beta_j)}{2} c_{r,\eta,w}^{(j)}(\alpha, \beta) y^{\alpha + \beta - e_j}
\]
where \( e_j \) is the \( j \)-th basis vector in \( \mathbb{N}^2 \), while the coefficient
\[
c_{r,\eta,w}^{(j)}(\alpha, \beta) := r^{(\alpha + |\beta| - 2 |\eta|)\eta(\alpha - \beta)} \frac{w_j^2}{w^{\alpha + \beta}}
\]
is defined for any \( \alpha, \beta \in \mathbb{N}^2 \). For brevity, we set
\[
\sum_* := \sum_{\alpha, \beta : |\alpha| = |\beta|}.
\]
The momentum \( \pi(\cdot) \) was defined in (2.2).

**Lemma 3.1.** Let \( r, r' > 0, \eta, \eta' \geq 0, w, w' \in \mathbb{R}^Z_+ \). The following properties hold.

(i) The norm of \( H \) can be expressed as
\[
|H|_{r, \eta, w} = \sup_{|y|_{\ell^2} \leq 1} |Y_H(y; r, \eta, w)|_{\ell^2}
\]

(ii) Given \( H^{(1)} \in \mathcal{H}_{r,\eta,w} \) and \( H^{(2)} \in \mathcal{H}_{r,\eta,w} \), such that for all \( \alpha, \beta \in \mathbb{N}^2 \) and \( j \in \mathbb{Z} \) with \( \alpha_j + \beta_j \neq 0 \) one has
\[
|H^{(1)}_{\alpha, \beta}|^{(j)} c_{r,\eta,w}^{(j)}(\alpha, \beta) \leq c |H^{(2)}_{\alpha, \beta}|^{(j)} c_{r,\eta,w}^{(j)}(\alpha, \beta),
\]
for some \( c > 0 \), then
\[
|H^{(1)}|_{r^*, \eta', w'} \leq c |H^{(2)}|_{r, \eta, w}.
\]

**Proof.** See appendix [B.4].

As a corollary we get the following

**Proposition 3.1.** Let \( r, r' > 0, \eta, \eta' \geq 0, w, w' \in \mathbb{R}^Z_+ \). If
\[
C := \sup_{j \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}^2} \frac{c_{r,\eta,w}^{(j)}(\alpha, \beta)}{c_{r',\eta',w'}^{(j)}(\alpha, \beta)} < \infty,
\]
then
\[
|H|_{r^*, \eta', w'} \leq C |H|_{r, \eta, w}.
\]
In particular $|\cdot|_{r,\eta,w}$ is increasing in $r$ and $\eta$, namely if $r^* \leq r$ and $\eta^* \leq \eta$ then

$$|H|_{r^*,\eta',w} \leq C|H|_{r,\eta,w}.$$  

Moreover, if $r^* \leq r$, $\eta \leq \eta'$ and $H \in K_{r,\eta}(\mathcal{H}_w)$ then

$$|H|_{r^*,\eta',w} \leq |H|_{r,\eta,w}.$$  

Furthermore, if $H$ preserves momentum then

$$|H|_{r^*,\eta',w} \leq C_0|H|_{r,\eta,w},$$

where

$$C_0 := \sup_{\alpha,\beta \in \mathbb{N}^2, \alpha_i + \beta_j \neq 0, \sum_i i(\alpha_i - \beta_j) = 0} \frac{c_{r^*,\eta',w}(\alpha,\beta)}{c_{r,\eta,w}(\alpha,\beta)} < \infty.$$  

Proof. Inequality (3.5) directly follows from Lemma 3.1 (ii), while (3.6) follows directly by (3.2) since in the kernel $\alpha_j + \beta_j \neq 0$ implies $\alpha_j + \beta_j \geq 2$. The momentum preserving case follows analogously. □

Definition 3.2 (minimal scaling degree). We say that $H$ has minimal scaling degree $d = d(H)$ (at zero) if

$$H_{\alpha,\beta} = 0, \quad \forall \alpha,\beta : \quad |\alpha| = |\beta| \leq d,$$

$$H_{\alpha,\beta} \neq 0, \quad \forall \alpha,\beta : \quad |\alpha| = |\beta| = d + 1.$$  

We say that $d(0) = +\infty$.

Lemma 3.2. If $H \in \mathcal{H}_{r,\eta}(\mathcal{H}_w)$ with $d(H) \geq d$, then for all $r^* \leq r$ one has

$$|H|_{r^*,\eta,w} \leq \left( \frac{r^*}{r} \right)^{2d} |H|_{r,\eta,w}.$$  

Proof. Recalling (3.2), we have

$$\frac{c_{r^*,\eta,w}(\alpha,\beta)}{c_{r,\eta,w}(\alpha,\beta)} = \left( \frac{r^*}{r} \right)^{|\alpha|+|\beta|-2}.$$  

Since $|\alpha| + |\beta| - 2 \geq 2d$, the inequality follows by Proposition 3.1. □

3.1. Monotonicity of $|\cdot|_{r,\eta,p,a,\eta}$ and $|\cdot|_{r,p}$. In this section we prove properties of monotonicity for the norms used in the Sobolev, Modified Sobolev and Gevrey cases introduced in Section 1. To prove such properties we strongly rely on some notation and results introduced by Bourgain in [Bou5] and extended later on by Cong-Li-Shi-Yuan in CLSY (Definition 3.3 and Lemma 3.3 below).

Definition 3.3. Given a vector $v = (v_i)_{i \in \mathbb{Z}}$, $v_i \in \mathbb{N}$, $|v| < \infty$ we denote by $\hat{v} = \hat{n}(v)$ the vector $(\hat{n}_i)_{i \in I}$ (where $I \subset \mathbb{N}$ is finite) which is the decreasing rearrangement of

$$\{ N \ni h > 1 \text{ repeated } v_h + v_{-h} \text{ times} \} \cup \{ 1 \text{ repeated } v_1 + v_{-1} + v_0 \text{ times} \}$$

Remark 3.1. A good way of envisioning this list is as follows. Given $v = (v_i)_{i \in \mathbb{Z}}$ consider the monomial $m(v) := \prod_i x_i^{v_i}$. We can write uniquely

$$m(v) = \prod_i x_i^{v_i} = x_{j_1}x_{j_2} \cdots x_{j_{|v|}}$$

then $\hat{n}(v)$ is the decreasing rearrangement of the list $(\{j_1, \ldots, j_{|v|}\})$.

As an example, consider the case $v \neq 0$. Then, by construction there exists a unique $J \geq 0$ such that $v_J = 0$ for all $|j| > J$ and $v_J + v_{-J} \neq 0$ hence

$$v = (\ldots, 0, v_{-J}, \ldots, v_0, \ldots, v_J, 0 \ldots).$$
If $J = 0$ then
\[
\hat{n} = (1, \ldots, 1)_{v_0 \text{ times}}
\]
otherwise we have
\[
\hat{n} = \left( \underbrace{J, \ldots, J}_{v_J + \ldots + v_{J} \text{ times}}, \underbrace{J-1, \ldots, J-1}_{v_{J-1} + \ldots + v_{J-1} \text{ times}}, \underbrace{1, \ldots, 1}_{v_1 + \ldots + v_1 \text{ times}} \right)
\]

Given $\alpha, \beta \in \mathbb{N}$ with $1 \leq |\alpha| = |\beta| < \infty$, from now on we define
\[
\hat{n} = \hat{n}(\alpha + \beta).
\]
We set the even number
\[
N := |\alpha| + |\beta|,
\]
which is the cardinality of $\hat{n}$.

We observe that, given
\[
\pi = \sum_{i \in \mathbb{Z}} i(\alpha_i - \beta_i) = \sum_{h>0} h(\alpha_h - \beta_h - \alpha_{-h} + \beta_{-h}),
\]
there exists a choice of $\sigma_i = \pm 1, 0$ such that
(3.9)
\[
\pi = \sum_{l} \sigma_l \hat{n}_l.
\]
with $\sigma_l \neq 0$ if $\hat{n}_l \neq 1$. Hence,
(3.10)
\[
\hat{n}_1 \leq |\pi| + \sum_{l \geq 2} \hat{n}_l.
\]
Indeed, if $\sigma_1 = \pm 1$, the inequality follows directly from (3.9); if $\sigma_1 = 0$, then $\hat{n}_1 = 1$ and consequently $\hat{n}_l = 1 \forall l$. Since the mass is conserved, the list $\hat{n}$ has at least two elements, and the inequality is achieved.

**Lemma 3.3** (Constance generalizzato). Given $\alpha, \beta$ such that $\sum_{i} i(\alpha_i - \beta_i) = \pi \in \mathbb{Z}$, we have that setting $\hat{n} = \hat{n}(\alpha + \beta)$
\[
\sum_{i} i^{\theta} (\alpha_i + \beta_i) = \sum_{l \geq 1} \hat{n}_l^{\theta} \geq 2\hat{n}_1^{\theta} + (2 - 2^{\theta})\sum_{l \geq 3} \hat{n}_l^{\theta} - \theta|\pi|.
\]

**Proof.** In appendix [C.1](#)

The following proposition gathers the monotonicity properties of the norm $|\cdot|_{r,p,s,a,\eta}$ with respect to the parameters $p, s, a$.

**Proposition 3.2.** The following inequalities hold:

1. Variations w.r.t. the parameter $p$. For any $0 < \rho < r$, $0 < \sigma < \eta$ and $p_1 > 0$ we have
\[
|H|_{r-\rho, p+p_1, s, a, \eta-\sigma} \leq C_{\text{mon}}(r/\rho, \sigma, p_1)|H|_{r,p,s,a,\eta}.
\]
2. Variation w.r.t. the parameter $s$. For any $0 < \sigma < \eta$ we have
\[
|H|_{r,p, s+\sigma, a, \eta-\sigma} \leq |H|_{r,p,s,a,\eta}.
\]
3. Variation w.r.t. the parameter $a$. For any $0 < \sigma < \eta$
\[
|H|_{e^{-\sigma} r, p, s+a, \eta-\sigma} \leq e^{2\sigma}|H|_{r,p,s,a,\eta}.
\]
Proof. In all that follows we shall use systematically the fact that our Hamiltonians preserve the mass and are zero at the origin. These facts imply that $|\alpha| = |\beta| \geq 1$.

**Item (1)** First we assume that $\rho \leq r/2$. By Lemma 3.1 item (ii) only need to show that, for any $0 < \rho \leq r/2$, $0 < \sigma < \eta$ and $p_1 > 0$ there exists a constant $C_{\text{mon}}$ such that

$$c_{\rho, -\rho, \rho + p_1, s, a, \eta - \sigma}^{(j)}(\alpha, \beta) \leq C_{\text{mon}} c_{\rho, -\rho, \rho + p_1, s, a, \eta - \sigma}^{(j)}(\alpha, \beta)$$

for all $j$, $\alpha, \beta$ with $|\alpha| = |\beta| \geq 1$ and $\alpha_j + \beta_j \neq 0$. In order to prove our claim we need to control

$$\sup_{j, \alpha, \beta \neq 0} \left( \prod_{i} (\alpha_i + \beta_i) \right) p_1 e^{-\sigma|\pi|} \left( \frac{r - \rho}{r} \right)^{|\alpha| + |\beta| - 2}.$$ 

We use the notations of Definition 3.3 with $\hat{n}(\alpha + \beta) \equiv \hat{n}$. Since $\alpha_j + \beta_j \neq 0$ we have that $\langle j \rangle \leq \hat{n}_1$. Note that

$$\prod_{i} (\alpha_i + \beta_i) = \prod_{l \geq 1} \hat{n}_l.$$ 

Hence

$$\langle j \rangle^2 \prod_{i} (\alpha_i + \beta_i) \leq \prod_{l \geq 2} \hat{n}_l$$

Let us call $N = |\alpha| + |\beta| \geq 2$. By (3.10) we have that

$$\langle j \rangle^2 \prod_{i} (\alpha_i + \beta_i) \leq \prod_{l \geq 2} \hat{n}_l \leq \log \left( \frac{r - \rho}{r} \right) + \log \left( \frac{r}{\rho} \right) \left( N - 1 \right) \hat{n}_2 + |\pi| \leq \prod_{l \geq 2} \hat{n}_l \leq \prod_{l \geq 2} \hat{n}_l.$$ 

We have shown that

$$\sup_{\alpha_j + \beta_j \neq 0} \langle j \rangle^2 \prod_{i} (\alpha_i + \beta_i) \leq N + |\pi|.$$ 

Since $(N + |\pi|)^{p_1} \leq 2^{p_1} (N^{p_1} + |\pi|^{p_1})$, denoting $L := \log (r - \rho)$ we repeatedly use Lemma B.1 in order to control

$$\sup_{N \geq 2, \pi \in \mathbb{Z}} (N + |\pi|)^{p_1} e^{-\sigma|\pi|} \left( \frac{r - \rho}{r} \right)^{N - 2}$$

$$\leq 2^{p_1} \left( \sup_{N \geq 2, \pi \in \mathbb{Z}} N^{p_1} e^{-\sigma|\pi|} L(N - 2) + \sup_{N \geq 2, \pi \in \mathbb{Z}} |\pi|^{p_1} e^{-\sigma|\pi|} L(N - 2) \right)$$

$$\leq 2^{p_1} \max \left\{ \left\{ \frac{p_1}{L} \right\}^{p_1}, 1 \right\} + \left\{ \frac{p_1}{\sigma} \right\}^{p_1} \leq 2^{p_1 + 1} \max \left\{ \left\{ \frac{p_1}{L} \right\}^{p_1}, \left\{ \frac{p_1}{\sigma} \right\}^{p_1}, 1 \right\}$$

$$\leq 2^{p_1 + 1} p_1 \max \left\{ \left\{ \frac{2r}{p_1} \right\}^{p_1}, \left\{ \frac{1}{\sigma} \right\}^{p_1}, 1 \right\} = C_{\text{mon}},$$

using that

$$L \geq \log (1 + \rho/r) \geq 2 \log (3/2) \rho/r \geq \rho/2r,$$

which holds since we are in the case $\rho \leq r/2$. This completes the proof in the case $\rho \leq r/2$.

Consider now the case $r/2 < \rho < r$. Using the monotonicity of the norm w.r.t. $r$ and the already proved case with $\rho = r/2$, we have

$$|H|_{r - \rho, \rho + p_1, s, a, \eta - \sigma} \leq |H|_{r/2, \rho + p_1, s, a, \eta - \sigma} \leq 2^{p_1 + 1} \max \left\{ \left\{ \frac{4p_1}{L} \right\}^{p_1}, \left\{ \frac{p_1}{\sigma} \right\}^{p_1}, 1 \right\} |H|_{r, \eta, \nu},$$

proving (1) also in the case $r/2 < \rho < r$.

**Item (2)** We need to show that

$$c_{r, p, s + \sigma, a, \eta - \sigma}^{(j)}(\alpha, \beta) \leq c_{r, p, s + \sigma, a, \eta}^{(j)}(\alpha, \beta)$$
namely that
\begin{equation}
\exp(-\sigma(\sum_i |i|^\theta (\alpha_i + \beta_i) - 2|j|^\theta + |\pi(\alpha - \beta)|)) \leq 1
\end{equation}
This follows by \ref{3.3} since
\begin{equation}
\sum_i |i|^\theta (\alpha_i + \beta_i) - 2|j|^\theta + |\pi(\alpha - \beta)| \geq (1 - \theta) \left( \sum_{l \geq 3} \hat{n}^\theta_l + |\pi| \right) \geq 0
\end{equation}
for all \(\alpha, \beta\) in \(\sum\) such that \(\alpha_j + \beta_j \neq 0\).

**Item (3)** We proceed as in item (1) - (2), our claim follows if we can show that
\begin{equation}
\sum_i (\alpha_i + \beta_i) |i| - 2|j| + |\pi| \geq \sum_{l \geq 2} \hat{n}_l - \hat{n}_1 + |\pi| - |\alpha_0 + \beta_0| \geq -(|\alpha| + |\beta|),
\end{equation}
This is proved in formula \(\text{[3.10]}\).

Incidentally we note that norm \(|.|_{r, p, s, a, \eta}\) possesses the tameness property.

**Proposition 3.3.**
\[
\sup_{|u|_{r, p, s, a} \leq r} \frac{|X_H|_{p, s, a}}{|u|_{p, s, a}} \leq C_{\text{tame}}(\rho, \eta, p)|H|_{r + p, p, s, a, \eta}
\]
**Proof.** In Appendix \ref{B.5} \(\Box\)

**Proposition 3.4.** The norm \(\|.|_{r, p}\) is monotone decreasing in \(p\), namely \(\|.|_{r, p + p_1} \leq \|.|_{r, p}\) for any \(p_1 > 0\).

**Proof.** For the norm \(\|.|_{r, p}\) the quantity in \(\text{[3.2]}\) becomes
\begin{equation}
c^{(j)}_{r, p}(\alpha, \beta) := p^{|\alpha| + |\beta| - 2} \left( \frac{|j|^2}{\prod_{i \in \mathbb{Z}} |i|^{\alpha_i + \beta_i}} \right)^p.
\end{equation}
By Lemma \ref{3.1} item (ii) we only need to show that
\begin{equation}
c^{(j)}_{r, p + p_1}(\alpha, \beta) \leq c^{(j)}_{r, p}(\alpha, \beta)
\end{equation}
for all \(j, \alpha, \beta\) with \(|\alpha| = |\beta| \geq 1\) and \(\alpha_j + \beta_j \geq 1\) (recall the momentum conservation), namely we have to prove that
\begin{equation}
\sup_{\alpha_j + \beta_j \geq 1} \frac{|j|^2}{\prod_{i \in \mathbb{Z}} |i|^{\alpha_i + \beta_i}} \leq 1.
\end{equation}
We first show that the inequality holds in the case \(j = 0, \pm 1\). Indeed we have
\[
\prod_{i} |i|^{\alpha_i + \beta_i} \geq \prod_{i} 2^{\alpha_i + \beta_i} = 2^{\sum_i \alpha_i + \beta_i} \geq 4
\]
since \(\sum_i \alpha_i + \beta_i \geq 2\) (by the fact that \(|\alpha| = |\beta| \geq 1\)).
Consider now the case \(|j| = |j| \geq 2\). Since \(\alpha_j + \beta_j \geq 1\), inequality \(\text{[3.24]}\) follows by
\begin{equation}
\sup_{j, \alpha, \beta} \frac{|j|}{\prod_{i \neq j} |i|^{\alpha_i + \beta_i}} \leq 1.
\end{equation}
By momentum conservation we have
\begin{equation}
|j| \leq \sum_{i \neq j} |i| (\alpha_i + \beta_i) \leq \sum_{i \neq j} |i| (\alpha_i + \beta_i)
\end{equation}
and \(\text{[3.25]}\) follows if we show that
\begin{equation}
\sup_{j, \alpha, \beta} \frac{\sum_{i \neq j} |i| (\alpha_i + \beta_i)}{\prod_{i \neq j} |i|^{\alpha_i + \beta_i}} \leq 1,
\end{equation}
where we can restrict the sum and the product to the indexes $i$ such that $\alpha_i + \beta_i \geq 1$. This last estimates follows by the fact that given $x_k \geq 1$

$$\sum_{2 \leq k \leq n} \frac{kx_k}{\prod_{2 \leq k \leq n} kx_k} \leq 1,$$

as it can be easily proved by induction over $n$ (noting that $n^x \geq nx$ for $n \geq 2$, and any $x \geq 1$).

4. Small divisors and homological equation

Let us consider the set of frequencies

$$\Omega_q := \{ \omega = (\omega_j)_{j \in \mathbb{Z}} \in \mathbb{R}^\mathbb{Z}, \sup_j |\omega_j - j^2(j)^q < 1/2 \};$$

this set is isomorphic to $[-1/2, 1/2]^\mathbb{Z}$ via the identification

$$\xi \mapsto \omega(\xi), \quad \omega_j(\xi) = j^2 + \frac{\xi_j}{(j)^q}.$$ 

We endow $\Omega_q$ with the probability measure $\mu$ induced by the product measure on $[-1/2, 1/2]^\mathbb{Z}$. The dependence w.r.t. the parameters $s,p$, thanks to Lemma 3.3 and formulae (3.20) and (3.17), works like an ultraviolet cut-off in the following sense. If a Hamiltonian has $\hat{n}_3 > N$ for all $H_{\alpha,\beta} \neq 0$ then its norm is of order $\leq e^{-sN} \theta_N - p$. This kind of restriction on Hamiltonians is often used in small divisor problems; since the denominators can be bounded from below in terms of $\hat{n}_3$, see for example [BG06], [FG13], then the solution of the homological equation is controlled. Actually we shall not use this kind of cut-off but instead, following Bourgain, we rely on a refined version of Lemma 3.3 (see Lemma 4.2 below), in order to deal with the small denominators.

We now define the set of Diophantine frequencies, the following definition is a slight generalization of the one given by Bourgain in [Bou05].

**Definition 4.1.** Given $\gamma > 0$ and $q \geq 0$, we denote by $D_{\gamma,q}^{\mu_1,\mu_2}$ the set of $\mu_1, \mu_2, \gamma$-Diophantine frequencies

$$D_{\gamma,q}^{\mu_1,\mu_2} := \left\{ \omega \in \Omega_q : |\omega \cdot \ell| > \gamma \prod_{n \in \mathbb{Z}} \frac{1}{(1 + |\ell_n|\mu_1(n)^{\mu_2 + q})}, \forall \ell \in \mathbb{Z}^\mathbb{Z} : 0 < |\ell| < \infty \right\}.$$ 

For all $\mu_1, \mu_2 > 1$, Diophantine frequencies are typical in $\Omega_q$ and we have the following measure estimate.

**Lemma 4.1.** For $\mu_1, \mu_2 > 1$ the exists a positive constant $C_{\text{meas}}(\mu_1, \mu_2)$ such that

$$\mu(\Omega_q \setminus D_{\gamma,q}^{\mu_1,\mu_2}) \leq C_{\text{meas}}(\mu_1, \mu_2) \gamma.$$ 

**Proof.** In Appendix C.3

Here and in the following we shall always assume that

$$\omega \in D_{\gamma,q}^{2,2}.$$ 

We will take

$$0 < \gamma \leq 1$$

and, coherently with [17], we shall write for brevity

$$D_{\gamma,q} = D_{\gamma,q}^{2,2}.$$ 

The following Lemma is the key point in the control of the small divisors appearing in the solution of the Homological equation.

---

12Denoting by $\mu$ the measure in $\Omega_q$ and by $\nu$ the product measure on $[-1/2, 1/2]^\mathbb{Z}$, then $\mu(A) = \nu(\omega^{-1}(A))$ for all sets $A \subset \Omega_q$ such that $\omega^{-1}(A)$ is $\nu$-measurable.
Lemma 4.2. Consider $\alpha, \beta \in \mathbb{N}^2$ with $1 \leq |\alpha| = |\beta| < \infty$. If
\begin{equation}
\sum_i (\alpha_i - \beta_i)^2 \leq 10 \sum_i |\alpha_i - \beta_i|,
\end{equation}
then for all $j$ such that $\alpha_j + \beta_j \neq 0$ one has
\begin{equation}
\sum_i |\alpha_i - \beta_i||i\rangle^\theta/2 \leq C_\theta \left( \sum_i |\alpha_i + \beta_i||i\rangle - 2|j\rangle + |\pi| \right),
\end{equation}
where $N = |\alpha| + |\beta|$ and $\pi = \sum_i i(\alpha_i - \beta_i)$ (recall (4.2)).

Proof. In appendix C.2 \hfill \square

Note that
\begin{equation}
\sum_i (\alpha_i - \beta_i)^2 \geq 10 \sum_i |\alpha_i - \beta_i| \quad \implies \quad |\omega \cdot (\alpha - \beta)| \geq 1.
\end{equation}

Indeed denoting $\omega_j = j^2 + \xi_j(j)^{-q}$ with $|\xi_j| \leq \frac{1}{2}$,
\begin{equation}
|\omega \cdot (\alpha - \beta)| \geq 10 \sum_j |\alpha_j - \beta_j| - \frac{1}{2} \sum_j |\alpha_j - \beta_j| \geq 1.
\end{equation}

In the remaining part of this section, on appropriate source and target spaces, we will study the invertibility of the "Lie derivative" operator
\begin{equation}
L_\omega : H \mapsto L_\omega H := \sum \langle \omega \cdot (\alpha - \beta) \rangle H_{\alpha, \beta} u^{\alpha} \bar{u}^\beta,
\end{equation}
which is nothing but the action of the Poisson bracket $\left\{ \sum_j \omega_j |u_j|^2, \cdot \right\}$ on $H$. Let us start with the following lemma.

For any $r, \eta, \omega$ and $\alpha, \beta \in \mathbb{N}^2$ let
\begin{equation}
\forall j \in \mathbb{Z} \quad c^{(j)}_{r, \eta, \omega}(\alpha, \beta) := r^{|\alpha| + |\beta| - 2} e^{\eta|\pi(\alpha - \beta)|} \frac{|u_j^2|}{\omega^{|\alpha + \beta|}}
\end{equation}
be the coefficient defined in (3.2) of Definition 3.1.

Lemma 4.3 (Homological equation). Fix $\omega \in \mathbb{D}_{\gamma,q}$. Consider two ordered weights $0 < r^* \leq r$, $0 \leq \eta' \leq \eta$, $\omega' \geq \omega$, such that
\begin{equation}
K := \gamma \sup_{j \in \mathbb{Z}, \alpha \neq \beta \in \mathbb{N}^2 \atop \alpha_j + \beta_j \neq 0} \frac{c^{(j)}_{r^*, \eta', \omega'}(\alpha, \beta)}{c^{(j)}_{r, \eta, \omega}(\alpha, \beta)|\omega \cdot (\alpha - \beta)|} < \infty,
\end{equation}
then for any $R \in \mathcal{R}_{r, \eta}(\omega)$ the homological equation
\begin{equation}
L_\omega S = R
\end{equation}
has a unique solution $S = L_\omega^{-1} R$ in $\mathcal{R}_{r^*, \eta', \omega'}(\omega')$, which satisfies
\begin{equation}
|L_\omega^{-1} R|_{r^*, \eta', \omega'} \leq \gamma^{-1} K |R|_{r, \eta, \omega}.
\end{equation}
Similarly, if $R$ preserves momentum, assuming only
\begin{equation}
K_0 := \gamma \sup_{j \in \mathbb{Z}, \alpha \neq \beta \in \mathbb{N}^2 \atop \alpha_j + \beta_j \neq 0} \frac{c^{(j)}_{r^*, \eta', \omega'}(\alpha, \beta)}{c^{(j)}_{r, \eta, \omega}(\alpha, \beta)|\omega \cdot (\alpha - \beta)|} < \infty,
\end{equation}
then for any $R \in \mathcal{R}_{r, \eta}(\omega)$ the homological equation
\begin{equation}
L_\omega S = R
\end{equation}
has a unique solution $S = L_\omega^{-1} R$ in $\mathcal{R}_{r^*, \eta', \omega'}(\omega')$, which satisfies
\begin{equation}
|L_\omega^{-1} R|_{r^*, \eta', \omega'} \leq \gamma^{-1} K_0 |R|_{r, \eta, \omega}.
we have that \( S \) also preserves momentum and

\[ |L^{-1}_\omega R|_{\tau', \eta', \omega'} \leq \gamma^{-1} K_0 |R|_{\tau, \eta, \omega}. \]

**Proof.** Given any Hamiltonian \( R \in \mathcal{R} \), the formal solution of \( L_S = R \) is given by

\[ L^{-1}_\omega R = \sum_{|\alpha| = |\beta|, \alpha \neq \beta} \frac{1}{i (\omega \cdot (\alpha - \beta))} R_{\alpha, \beta} u^\alpha \bar{u}^\beta, \]

where \( u \in B_{r'}(h_u) \). By Lemma 3.1 (ii) (applied to \( H^{(1)} = L^{-1}_\omega R \) and \( H^{(2)} = R \)) and (4.10), we get (4.11). The momentum preserving case is analogous.

### 4.1. The homological equation.

**Lemma 4.4.** Let \( \omega \in \mathcal{D}_{\gamma, q} \) and let \( 0 < \sigma < \eta, 0 < \rho < r/2 \). The following holds.

\( i \) For any \( R \in \mathcal{R}_{\tau, \eta}(h_u) \), with \( u_j = |j|^{p} e^{i j \theta} |x|^s j^\theta \), the Homological equation \( L_S = R \) has a unique solution

\[ S = L^{-1}_\omega R \in \mathcal{R}_{\tau, \eta}(h_u), \]

with \( u_j' = |j|^{p} e^{i j \theta} |x|^s j^\theta \), which satisfies

\[ |L^{-1}_\omega R|_{\tau, \eta, \omega} \leq \gamma^{-1} e^{C_1 \sigma^{-\frac{2}{3}}} |R|_{\eta, \omega}. \]

\( ii \) For any \( R \in \mathcal{R}_{\tau, \eta}(h_u) \), with \( u_j = |j|^{p} e^{i j \theta} |x|^s j^\theta \), the Homological equation \( L_S = R \) has a unique solution

\[ S = L^{-1}_\omega R \in \mathcal{R}_{\tau, \eta, \sigma}(h_u), \]

with \( u_j' = |j|^{p} e^{i j \theta} |x|^s j^\theta \), which satisfies

\[ |L^{-1}_\omega R|_{\tau, \eta, \sigma, \bar{\omega}} \leq \gamma^{-1} C_2 (r, \omega, \sigma, \tau) |R|_{\eta, \omega}. \]

\( iii \) For any preserving momentum \( R \in \mathcal{R}_{\tau, \eta, \sigma}(h_u) \), with \( u_j = |j|^{p} e^{i j \theta} |x|^s j^\theta \), the Homological equation \( L_S = R \) has a unique preserving momentum solution

\[ S = L^{-1}_\omega R \in \mathcal{R}_{\tau, 0}(h_u), \]

with \( u_j' = |j|^{p} e^{i j \theta} |x|^s j^\theta \), which satisfies

\[ \|L^{-1}_\omega R\|_{\tau, \rho, \nu} \leq \gamma^{-1} 6^{10} (4^{6} e^{C_1})^2 \|R\|_{\tau, \rho, \nu}. \]

**Proof.** In the following, we will compute for each item the corresponding \( K, K_0 \) defined in (4.10) and (4.12), and show their finiteness in order to apply Lemma 4.3 and give the explicit upper bounds entailed in (4.15)-(4.17).

**Item \( i \)** In this case

\[ K = \gamma \sup_{\beta_i \neq 0} \frac{e^{-|\sigma (\sum_i \langle i \rangle^\theta \alpha_i + \beta_i - 2 \langle j \rangle^\theta |\pi|)}}{|\omega \cdot (\alpha - \beta)|}. \]

There are two cases.

If (4.5) does not hold, then by (4.8) \( |\omega \cdot (\alpha - \beta)| \geq 1 \) and by (3.11) and (4.4) we get

\[ \frac{e^{-|\sigma (\sum_i \langle i \rangle^\theta \alpha_i + \beta_i - 2 \langle j \rangle^\theta |\pi|)}}{|\omega \cdot (\alpha - \beta)|} \leq 1 \]

and the bound is trivially achieved.

Otherwise, let us consider the case in which (4.5) holds. By applying Lemma 4.2 since \( \omega \in \mathcal{D}_{\gamma, q} \) we get:

\[ e^{-|\sigma (\sum_i \langle i \rangle^\theta \alpha_i + \beta_i - 2 \langle j \rangle^\theta |\pi|)} \]

\[ \leq e^{-C_\sigma} \sum_i |\alpha_i - \beta_i |^\frac{q}{2} \prod_i \left( 1 + (\alpha_i - \beta_i)^2 (i)^{2+q} \right) \]

\[ \leq \exp \sum_i \left[ -\frac{\sigma}{C_\sigma} |\alpha_i - \beta_i |^\frac{q}{2} + \ln \left( 1 + (\alpha_i - \beta_i)^2 (i)^{2+q} \right) \right] \]

\[ = \exp \sum_i f_i(|\alpha_i - \beta_i |) \]

(4.18)

where, for \( 0 < \sigma < 1, i \in \mathbb{Z} \) and \( x \geq 0 \), we defined

\[ f_i(x) := -\frac{\sigma}{C_\sigma} x (i)^{\frac{q}{2}} + \ln (1 + x^2 (i)^{2+q}) . \]
In order to bound (4.18), we need the following lemma, whose proof is postponed to Appendix C.3.

**Lemma 4.5.** Setting

\[
i_\ell := \left( \frac{8C_*(q + 3)}{\sigma \theta} \ln \frac{4C_*(q + 3)}{\sigma \theta} \right)^{\frac{\hat{s}}{q}}
\]

we get

\[
\sum_i f_i(\ell_i) \leq 7(q + 3)i_\ell \ln i_\ell - \frac{\sigma}{2C_*(\hat{n}_1(\ell))^{\frac{\hat{s}}{q}}}
\]

for every \( \ell \in \mathbb{Z}^2 \) with \(|\ell| < \infty\).

The inequality (4.15) follows from plugging (4.19) into (4.18) and evaluating the constant.

**Item 3)** In this case \( K \) in (4.10) is

\[
K = \gamma \sup_{j, \alpha, \beta \neq 0} \left(1 - \frac{\rho}{r}\right)^{N - 2} \left(\frac{\langle j \rangle^2}{\prod_i (\hat{n}_i)^{\alpha_i + \beta_i}}\right)^{\tau} \frac{1}{|\omega \cdot (\alpha - \beta)|}
\]

where \( N = |\alpha| + |\beta| \).

Here we two consider two cases.

If (4.5) is not satisfied then (4.8) holds and the right hand side of (4.20) is bounded by the quantity in (3.15) and it is estimated analogously.

If (4.5) holds instead, by applying formula (3.17), Lemma 4.2 and the fact that \( \omega \in \mathbb{D}_{r, q} \) we get:

\[
\left(\frac{\langle j \rangle^2}{\prod_i (\hat{n}_i)^{\alpha_i + \beta_i}}\right)^{\tau} \frac{1}{|\omega \cdot (\alpha - \beta)|} \leq \left(\frac{\langle j \rangle^2}{\prod_i (\hat{n}_i)^{\alpha_i + \beta_i}}\right)^{\tau} \prod_i \left(1 + |\alpha_i - \beta_i|^2\right)^{2+q}
\]

\[
\leq \left(\frac{N + |\pi|}{\prod_{i=3}^{\infty} \hat{n}_i}\right)^{\tau} \left(e^{27(1 + |\pi|)^3N_6 \prod_{i=3}^{\infty} \hat{n}_i}\right)^2 + 1
\]

\[
\leq e^{27(2+q)(N + |\pi|)^{\tau}} \leq e^{27(2+q)(N + |\pi|)^{3\tau}}
\]

By using Lemma B.1 (just like explained in detail in formula (3.18) with \( p_1 = 3\tau \)), \( K \) in (4.20) is bounded by

\[
e^{27(2+q)(N + |\pi|)^{3\tau}} \left(1 - \frac{\rho}{r}\right)^{N - 2} e^{-\sigma|\pi|}
\]

\[
\leq e^{27(2+q)2^{3\tau+1}(3\tau)^{3\tau}} \max \left\{ \left(\frac{2r}{\rho}\right)^{3\tau}, \left(\frac{1}{\sigma}\right)^{3\tau}, 1 \right\}
\]

**Item 6)** Note that in this case the constant in (4.10) amounts to

\[
K_0 = \gamma \sup_{j, \alpha, \beta \neq 0} \left(\frac{\langle j \rangle^2}{\prod_i (\hat{n}_i)^{\alpha_i + \beta_i}}\right)^{\tau} \frac{\gamma}{|\omega \cdot (\alpha - \beta)|}
\]

since (recall (3.3)) we have

\[
c^{(j)}_{r, 0, u}(\alpha, \beta) = r^{N-2} \left(\frac{\langle j \rangle^2}{\prod_i (\hat{n}_i)^{\alpha_i + \beta_i}}\right)^{p+\tau_1},
\]

\[
c^{(j)}_{r, 0, u}(\alpha, \beta) = r^{N-2} \left(\frac{\langle j \rangle^2}{\prod_i (\hat{n}_i)^{\alpha_i + \beta_i}}\right)^{p}.
\]
We have two cases. If (4.8) holds $K_0 \leq \gamma$ by (3.24).

Otherwise (4.5) holds and, therefore, (4.7) (note that here $\pi = 0$) applies, giving

$$K_0 \leq \sup \left( \frac{|j|^2}{\prod_i |i|^{\alpha_i + \beta_i}} \right)^{t_1} \prod_i (1 + |\alpha_i - \beta_i|^{2/2+q})$$

$$\leq \sup \left( \frac{|j|^2}{\prod_i |i|^{\alpha_i + \beta_i}} \right)^{t_1} e^{27(2+q)} N^{6(2+q)} \prod_{l=3}^N \tau_l^{(2+q)}$$

since $\omega \in D_{\gamma,q}$. We claim that

$$N \leq 4 \prod_{l=3}^N [\tau_l]^{\frac{1}{2+q}}. \quad (4.21)$$

Indeed if $N = 2$, the inequality is trivial. Since $N$ is even we have to consider only the case $N \geq 4$, which follows by Lemma B.1. Recalling (3.16) we have

$$\prod_i |i|^{\alpha_i + \beta_i} = \prod_{l=1}^N [\tau_l]. \quad (4.22)$$

Then

$$\sup_{j, \alpha, \beta, \alpha_i + \beta_i, l} \left( \frac{|j|^2}{\prod_i |i|^{\alpha_i + \beta_i}} \right) \leq \frac{[\tau_l]^{2/2+q}}{\prod_{l=2}^N [\tau_l]^{\frac{1}{2+q}}} \leq \frac{1}{\prod_{l=3}^N [\tau_l]}$$

where the last inequality holds by momentum conservation. Then

$$K_0 \leq 2^{\tau_1-1} \left( \frac{1}{\prod_{l=2}^N [\tau_l]} \right)^{t_1} \left( \frac{(\sum_{l=3}^{t_1} [\tau_l]^{\frac{1}{2+q}})^{\frac{1}{2+q}}} {\prod_{l=2}^N [\tau_l]^{\frac{1}{2+q}}} \right)$$

$$\leq 2^{\tau_1-1} (4^{6e^{27}})^{2+q} \left( 1 + \frac{(\sum_{l=3}^{t_1} [\tau_l]^{\frac{1}{2+q}})^{\frac{1}{2+q}}} {\prod_{l=2}^N [\tau_l]^{\frac{1}{2+q}}} \right)$$

by Lemma B.2 with $a = 1/2$. The estimate on $K_0$, hence inequality (4.17) follows. \hfill \square

5. Abstract Birkhoff Normal Form

In this section we prove an abstract Birkhoff normal form theorem; its setting is flexible and easy to adapt in the three cases of our interest $S, M, G$. The normal form will be proved iteratively by means of the following Lemma, which constitutes the main step of the procedure.

**Lemma 5.1.** Fix $\omega \in D_{\gamma,q}$. Let $r > r' > 0, \eta \geq \eta' \geq 0, w \leq w'$. Consider

$$H = D_\omega + Z + R, \quad Z \in K_{r,\eta}(h_\omega), \quad R \in R_{r,\eta}(h_\omega), \quad d(Z) \geq 1, \quad d(R) \geq d \geq 1.$$

Assume that (3.4) and (4.10) hold and that

$$|R|_{r,\eta,w} \leq \frac{\gamma \delta}{K}, \quad \text{with} \quad \delta := \frac{r - r'}{16er'} \quad (5.1)$$

Then there exists a change of variables

$$\Phi : B_{r'}(h_{w'}) \to B_r(h_{w}), \quad (5.2)$$

such that

$$H \circ \Phi = D_\omega + Z' + R', \quad Z' \in K_{r',\eta'}(h_{w'}), \quad R' \in R_{r',\eta'}(h_{w'}), \quad d(Z') \geq 1, \quad d(R') \geq d + 1.$$

\(^{13}\) Using that $(a + b)^{r_1} \leq 2^{r_1-1}(a^{r_1} + b^{r_1})$ for $a, b \geq 0, r_1 \geq 1$.

\(^{14}\) $K$ is the constant in (4.10).
Moreover, \[ |Z'|_{r',\eta',w'} \leq |Z|_{r,\eta,w} + (\gamma \delta)^{-1} K |R|_{r,\eta,w} (C |R|_{r,\eta,w} + |Z|_{r,\eta,w}) , \]
(5.3) \[ |R'|_{r',\eta',w'} \leq (\gamma \delta)^{-1} K |R|_{r,\eta,w} (C |R|_{r,\eta,w} + |Z|_{r,\eta,w}) . \]

Finally, for \( \omega^2 \geq \omega' \), assume the further conditions
\[ \sup_{\gamma \in \mathbb{Z}, \alpha \neq \beta \in \mathbb{N}^2 \atop \alpha_j + \beta_j \neq 0} \frac{c_j(\alpha, \beta)}{c_r(\omega, \alpha, \beta)} \leq K^2 < \infty , \quad r^* := \frac{r' + r}{2} \]
and
\[ |R|_{r,\eta,w} \leq \gamma \delta K^2 / K^2 \]
(5.5)

Then
\[ \Phi |_{B_r(h_{\omega'})} : B_{r'}(h_{\omega'}) \to B_r(h_{\omega}) , \]
(5.6)

Moreover if \( R \) preserves momentum, assuming only that
\[ K^2_0 := \sup_{\gamma \in \mathbb{Z}, \alpha \neq \beta \in \mathbb{N}^2 \atop \alpha_j + \beta_j \neq 0} \frac{c_j(\alpha, \beta)}{c_r(\omega, \alpha, \beta)} \leq K^2 < \infty \]
and that (5.1), (5.5) hold with \( K_0, K^2_0 \) instead of \( K, K^2 \) we have that \( R' \) preserves momentum and (5.6) holds with \( K^2_0 \) instead of \( K^2 \).

Proof. By Lemma \[4.3\] let \( S = L^{-1}_w R \) in \( \mathcal{R}_{r',\eta'}(h_{\omega'}) \) be the unique solution of the homological equation \( L_w S = R \) on \( B_{r'}(h_{\omega'}) \). Note that \( d(S) \geq d \). We have
\[ |S|_{r',\eta',w'} \leq \gamma^{-1} K |R|_{r,\eta,w} . \]
(5.8)

We now apply Lemma \[2.1\] with \( (r, \eta, \omega) \mapsto (r', \eta', \omega') \) and \( \rho := r^* - r' \). Note that (5.1) and (5.8) imply (2.12). We define \( \Phi := \Phi_2 \) and compute
\[ H' := H \circ \Phi = D_w + Z + (e^{S_{r'}} - \text{id}) - \{ S_{r'} \} D_w + (e^{S_{r'}} - \text{id})(Z + R) = D_w + Z - \sum_{j=2}^{\infty} \frac{(\text{ad} S_{r'})^{j-1}}{j!} R + (e^{S_{r'}} - \text{id})(Z + R) . \]

We now set
\[ Z' = \Pi_R H' - D_w , \quad R' = \Pi_R H' . \]

Since the scaling degree is additive w.r.t. Poisson brackets, we have that \( d(Z') \geq 1 \) and \( d(R') \geq d + 1 \). By (2.17)
\[ |Z'|_{r',\eta',w'} \leq |Z|_{r',\eta',w'} + (\gamma \delta)^{-1} K |R|_{r,\eta,w} (|R|_{r,\eta,w} + |Z|_{r,\eta,w}) , \]
\[ |R'|_{r',\eta',w'} \leq (\gamma \delta)^{-1} K |R|_{r,\eta,w} (|R|_{r,\eta,w} + |Z|_{r,\eta,w}) . \]

Since \[4.10\] holds we can apply Proposition \[3.1\] by (3.5) and (3.6) we get
\[ |R|_{r',\eta',w'} \leq C |R|_{r,\eta,w} , \quad |Z|_{r',\eta',w'} \leq |Z|_{r,\eta,w} . \]

Finally assume (5.5) and (5.4). By Lemma \[4.3\] let \( S^2 = L^{-1}_w R \) in \( \mathcal{R}_{r',\eta'}(h_{\omega'}) \) be the solution of the homological
equation \( L_\omega S^2 = R \) on \( B_{r^*}(h_\omega) \subseteq B_{r^*}(h_\omega') \). Since \( S \) and \( S^2 \) solve the same linear equation on \( B_{r^*}(h_\omega') \), we have that \( S^2 = S \mid B_{r^*}(h_\omega) \).

By (4.11) we get

\[ (5.9) \quad |S|_{r^*,\eta',\omega'} \leq \gamma^{-1} K^4 |R|_{r,\eta,\omega}. \]

We now apply Lemma 2.1 with \((r,\eta,\omega) \sim (r',\eta',\omega')\) and \(\rho := r^* - r'\). Note that (5.5) and (5.9) imply (2.12). Then (5.6) follows by (2.13) and (5.9).

The momentum preserving case is analogous. \(\square\)

We are now ready to state and prove the abstract Birkhoff normal form theorem, from which Theorem 1.4 will follow, as a particular case in the Gevrey, Sobolev Modified Sobolev settings.

Fix any natural \(N > 1\), \(\eta \geq 0\) and sequence of weights \(w_0 \leq w_1 \leq \cdots \leq w_N\). For any given \(r > 0\) we set

\[ (5.10) \quad r_n = (2 - \frac{n}{N})r, \quad \eta_n = (1 - \frac{n}{N})\eta, \quad 0 \leq n \leq N, \quad r^*_n = \frac{r_{n+1} + r_n}{2}, \quad 0 \leq n < N. \]

For brevity we set

\[ (5.11) \quad h_n := h_\omega^n, \quad \mathcal{H}_n := \mathcal{H}_{r_n,\eta_n}(h_n), \quad 0 \leq n \leq N, \quad \mathcal{H}_{n,*} := \mathcal{H}_{r_n,\eta_n+1}(h_{n+1}), \quad 0 \leq n < N, \]

and, correspondingly, \(\mathcal{R}_n, \mathcal{K}_n, \mathcal{R}_{n,*}, \mathcal{K}_{n,*}\) and

\[ (5.12) \quad |\cdot|_n := |\cdot|_{r_n,\eta_n,w_n}, \quad |\cdot|_{n,*} := |\cdot|_{r_n,\eta_n,w_{n+1}}. \]

**Assumption 1.** Assume that

\[ (5.13) \quad \tilde{C} := \max \left\{ 1, \sup_{0 \leq n < \delta} \sup_{\substack{j,\alpha,\beta \geq 0 \\ \alpha_j + \beta \neq 0}} \frac{c^{(j)}(r_n,\eta_n+1,w_{n+1})(\alpha,\beta)}{c^{(j)}(r_n,\eta_n,w_n)(\alpha,\beta)} \right\} < \infty, \]

\[ (5.14) \quad \tilde{K} := \max \left\{ 1, \sup_{0 \leq n < \delta} \sup_{\substack{j,\alpha,\beta \geq 0 \\ \alpha_j + \beta \neq 0}} \frac{c^{(j)}(r_n,\eta_n+1,w_{n+1})(\alpha,\beta)}{c^{(j)}(r_n,\eta_n,w_n)(\alpha,\beta)|\omega \cdot (\alpha - \beta)|} \right\} < \infty, \]

\[ (5.15) \quad \tilde{K}^\sharp := \max \left\{ 1, \sup_{0 \leq n < \delta} \sup_{\substack{j,\alpha,\beta \geq 0 \\ \alpha_j + \beta \neq 0}} \frac{c^{(j)}(r_n,\eta_n,w_n)(\alpha,\beta)}{c^{(j)}(r_n,\eta_n,w_n)(\alpha,\beta)|\omega \cdot (\alpha - \beta)|} \right\} < \infty. \]

In the case of momentum preserving hamiltonians we define \(\tilde{C}_0, \tilde{K}_0, \tilde{K}_0^\sharp\) as in (5.13)-(5.15) with the further condition \(\sum_i i(\alpha_i - \beta_i) = 0\); and we only assume that such constants are bounded.

*Remark 5.1.* Recalling (3.2) we note that the constants \(\tilde{C}, \tilde{K}, \tilde{K}^\sharp\) (as well as \(\tilde{C}_0, \tilde{K}_0, \tilde{K}_0^\sharp\)) do not depend on \(r\). They only depend on \(w_{n,j}/w_{0,j}\).

**Lemma 5.2.** By Assumption (5.13) we have the monotonicity properties

\[ (5.16) \quad \mathcal{H}_0 \subseteq \mathcal{H}_{0,*} \subseteq \cdots \subseteq \mathcal{H}_n \subseteq \mathcal{H}_{n,*} \subseteq \mathcal{H}_{n+1} \subseteq \cdots \subseteq \mathcal{H}_\delta, \]

with estimates

\[ H \in \mathcal{H}_n \quad \Rightarrow \quad |H|_{n,*} \leq \tilde{C}|H|_n, \quad 0 \leq n \leq i \leq N - 1 \]

\[ H \in \mathcal{K}_n \quad \Rightarrow \quad |H|_{n,*} \leq |H|_n, \quad 0 \leq n \leq i \leq N - 1. \]
Proof. We apply Proposition 3.1 with

\[ r, \eta, \varpi \leadsto r_n, \eta_n, \varpi_n, \quad r^*, \eta^*, \varpi^* \leadsto r_n^*, \eta_n^*, \varpi_n^* \]

by noting that the bound (3.4) follows from (5.13). The bounds in (5.17) follow from (3.5) and (3.6). The chain of inclusions (5.16) follows. \[ \square \]

**Theorem 5.1 (Abstract Birkhoff Normal Form).** Consider a Hamiltonian of the form

\[ H = D_\omega + G, \quad D_\omega = \sum_j \omega_j |u_j|^2 \]

with \( \omega \in D_{r,q}, \ G \in \mathcal{H}_{r,\eta}(\mathbb{h}_{s_0}), \ r > 0, \ \eta \geq 0 \) and such that \( d(G) \geq 1 \). Set

\[ r_* := \sqrt{\frac{\gamma}{2 |G|_{r,\eta,s_0}}}. \]

For any \( N \in \mathbb{N}_+ \), under Assumption \( \check{\gamma} \) set

\[ \bar{r} = \bar{r}(\check{\gamma}) := \min \left\{ \frac{r_*}{\sqrt{N \max\{\check{C}_K, \check{K}\}^2}}, \frac{\bar{r}}{2} \right\}. \]

Then for all \( 0 < r \leq \bar{r} \) there exists a symplectic change of variables

\[ \Psi : \ B_r(\mathbb{h}_{s_0}) \mapsto B_{2r}(\mathbb{h}_{s_0}), \quad \sup_{u \in B_r(\mathbb{h}_{s_0})} |\Psi(u) - u|_{s_0} \leq \check{C}_1 r^3, \quad \check{C}_1 := \frac{\check{K}^2}{2^7 e \bar{r}^2}, \]

such that in the new coordinates

\( H \circ \Psi = D_\omega + Z + R, \quad Z \in K_{r,0,s_0}, \ R \in \mathcal{R}_{r,0,s_0}, \ d(Z) \geq 1, \ d(R) \geq N, \)

\[ |Z|_{r,0,s_0} \leq \check{C}_2 r^2, \quad \check{C}_2 := \frac{\gamma}{2^8 e \bar{r}^2}, \]

\[ |R|_{r,0,s_0} \leq \check{C}_3 r^{2(\check{\gamma}+1)}, \quad \check{C}_3 := \frac{\gamma}{2^9 e \bar{r}^2} \left( \frac{\check{K} \bar{r}^{\check{\gamma}+1}}{4 \bar{r}^2} \right)^{\check{\gamma}}. \]

In the case that \( G \) preserves momentum, the same result holds with \( \check{C}_0, \check{K}_0, \check{K}_0^\sharp \) instead of \( \check{C}, \check{K}, \check{K}^\sharp \); moreover also \( R \) preserves momentum.

Proof. We will prove the thesis inductively. Let us start by noticing that

\[ \bar{r} = \min \left\{ \frac{\bar{r}}{8 \sqrt{|G|_{r,\eta,s_0}}} \sqrt{\frac{\gamma \check{\delta}}{\max\{\check{C}_K, \check{K}\}^2}}, \frac{\bar{r}}{2} \right\}, \quad \check{\delta} := \frac{1}{32 e \bar{r}^2} \]

and, for all \( 0 < r \leq \bar{r} \), let us set

\[ \varepsilon := \gamma^{-1} \left( \frac{2 r}{\bar{r}} \right)^2 |G|_{r,\eta,s_0} = \frac{1}{2^9 e} \left( \frac{r}{r_*} \right)^2. \]

From definition (5.19) we thus deduce that

\[ 8 \varepsilon \max\{\check{C}_K, \check{K}^\sharp\} \check{\delta}^{-1} \leq 1. \]

Recalling the notations introduced in (5.10)-(5.12), by Lemma (3.2) we have

\[ \gamma^{-1} |G|_0 \leq \varepsilon, \]

hence, setting \( Z^{(0)} := \Pi_X G \) and \( R^{(0)} := \Pi_R G \), from (2.5) it follows that

\[ \gamma^{-1} |Z^{(0)}|_0, \ \gamma^{-1} |R^{(0)}|_0 \leq \varepsilon. \]
Finally, using the same strategy as above, we also get

\[ \gamma^{-1}|Z^{(n)}|_n \leq \varepsilon \sum_{h=0}^{n} 2^{-h} \gamma^{-1}|R^{(n)}|_n \leq \varepsilon^{n+1} \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\right)^n \leq 2^{-n}\varepsilon. \]

Fix any \( k < n \). Let us assume that we have constructed \( H^{(0)}, \ldots, H^{(k)} \) satisfying (5.23) for all \( 0 \leq n \leq k \). We want to apply Lemma 5.1 with

\[ H, r, \eta, w \sim H^{(k)}, r_k, \eta_k, \omega_k \text{ and } \tilde{r}, \tilde{\eta}, \tilde{w}, d \sim r_k+1, \eta_k+1, \omega_k+1, k+1. \]

By construction the bounds (5.4), (4.10) and (5.4) hold since \( C \leq \hat{C}, K \leq \hat{K}, K^\sharp \leq \hat{K}^\sharp \), where \( \hat{C}, \hat{K}, \hat{K}^\sharp \) were defined in (5.13), (5.14), (5.15). We just have to verify that (5.1) holds, namely

\[ |R^{(k)}|_k \leq \frac{\gamma r_k - r_{k+1}}{16\varepsilon r_k}. \]

In fact, by applying the inductive hypothesis (5.23) and the smallness condition (5.22), we get

\[ |R^{(k)}|_k \leq \gamma \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\right)^{k+1} \leq \frac{\varepsilon}{2k} \leq \frac{\gamma}{16\varepsilon K} \leq \frac{\gamma r_k - r_{k+1}}{16\varepsilon r_k}. \]

The verification of (5.5) is completely analogous.

So, by applying Lemma 5.1, we construct a change of variable \( \Phi_k \) as in (5.2) with

\[ \Phi_k : B_{r_{k+1}}(\omega_{k+1}) \rightarrow B_{r_k}(\omega_{k+1}). \]

Let us now set

\[ H^{(k+1)} = D_{\omega} + Z^{(k+1)} + R^{(k+1)} := H_k \circ \Phi_k \]

with \( Z^{(k+1)} \in K_{k+1}, R^{(k+1)} \in R_{k+1} \) and \( d(Z^{(k+1)}) \geq 1, d(R^{(k+1)}) \geq k+2 \). It remains to prove the bounds in the second line of (5.23) (with \( n = k+1 \)). By (5.3) we have

\[ |Z^{(k+1)}|_{k+1} \leq |Z^{(k)}|_k + (\gamma \hat{\delta})^{-1} \hat{K} |R^{(k)}|_k (\hat{C}|R^{(k)}|_k + |Z^{(k)}|_k), \]

\[ |R^{(k+1)}|_{k+1} \leq (\gamma \hat{\delta})^{-1} \hat{K} |R^{(k)}|_k (\hat{C}|R^{(k)}|_k + |Z^{(k)}|_k). \]

By substituting the inductive hypothesis (5.23), we have the following chain of inequalities

\[ \gamma^{-1}|R^{(k+1)}|_{k+1} \leq \hat{\delta}^{-2} \hat{K} \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\varepsilon\right)^{k+1} \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\varepsilon\right)^{k+1} \leq \hat{\delta}^{-1} \hat{K} \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\varepsilon\right)^{k+1} \left(\hat{C} + 2\right) \leq \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\varepsilon\right)^{k+1} \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\varepsilon\right)^{k+1}, \]

which proves the bound on \( R^{(n)} \) in (5.23) for any \( n \). En passant, we note that

\[ \gamma \varepsilon \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\varepsilon\right) = \frac{\gamma}{2^9 e r^2} \left(\frac{\hat{C}\hat{K} N}{4r^2}\right)^n \varepsilon^{2(n+1)}. \]

Finally, using the same strategy as above, we also get

\[ \gamma^{-1}|Z^{(k+1)}|_{k+1} \leq \varepsilon \left(\sum_{h=0}^{k} 2^{-h} + \left(4\hat{C}\hat{K}\hat{\delta}^{-1}\varepsilon\right)^{k+1} \right) \leq \varepsilon \sum_{h=0}^{k+1} 2^{-h}, \]

which completes the proof of the inductive hypothesis (5.23), and remark that

\[ \varepsilon \sum_{h=0}^{n} 2^{-h} = \frac{\varepsilon^2}{2^9 e r^2} (1 - 2^{-n+1}). \]
By (5.6) we have

\[ \Phi_k : B_{rk+1}(h_0) \to B_{rk}(h_0), \]

(5.27)

\[ \sup_{u \in B_{rk+1}(h_0)} |\Phi_k(u) - u|_{h_0} \leq r_k \gamma^{-1} \hat{R}^2 |R(k)|_k. \]

In conclusion we define

\[ \Psi := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{N-1} : B_r(h_0) \to B_{2r}(h_0). \]

Since we have

\[ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{N-1} = (\Phi_0 - \text{id}) \circ \Phi_1 \circ \cdots \circ \Phi_{N-1} + (\Phi_1 - \text{id}) \circ \Phi_2 \circ \cdots \circ \Phi_{N-1} + \cdots, \]

By (5.27) we get

\[ \sup_{u \in B_r(h_0)} |\Psi(u) - u|_{h_0} \leq 2N \gamma^{-1} \hat{R}^2 \sum_{k=0}^{N-1} 2^{-k} - \hat{R}^2 \varepsilon, \]

proving (5.20). We finally set \( Z = Z_N, R = R_N \) and the estimates (5.21) follow by (5.25)-(5.26).

6. PROOF OF THEOREM 1.4

Theorem 1.4 is a particular case of the general Theorem 5.1 in the usual three cases S, M, G. We have to check Assumption 1.

Lemma 6.1. Consider the constants introduced in Assumption 1. We have the usual three cases:

S) When

\[ w_{n,j} = w_{0,j} \beta \eta^\tau, \quad \tau := \tau_0 (2 + q) \]

we have

\[ \hat{C} \leq C_{\text{con}}(4N, \eta/N, \tau), \quad \hat{K} \leq C_2(4N, \eta/N, \tau), \]

\[ \hat{R}^2 \leq C_2(4N, \eta/N, N\tau). \]

M) When

\[ w_{n,j} = w_{0,j} \beta \eta^{\tau_1}, \quad \tau := \tau_0 (2 + q) \]

and assuming the momentum conservation, we have

\[ \hat{C}_0 = 1, \quad \hat{R}_0, \hat{R}_0^2 \leq 6^{\tau_1} (4\beta e^{27})^{2+q}. \]

G) When

\[ w_{n,j} = w_{0,j} e^{\eta^\tau} \beta \eta^\gamma \]

we have

\[ \hat{C} = 1, \quad \hat{K}, \hat{K}^2 \leq e^{C_1(\eta^\tau)^\beta}. \]

Proof: S) The computation of \( \hat{C} \) follows from (3.14); the ones of \( \hat{K}, \hat{R}^2 \) again from Lemma 4.4.

M) The computation of \( \hat{C}_0 \) follows from (3.23); the ones of \( \hat{K}_0, \hat{R}_0^2 \) again from Lemma 4.4.

G) The computation of \( \hat{C} \) follows from (3.12); the ones of \( \hat{K}, \hat{R}^2 \) from Lemma 4.4.

We use Theorem 5.1 with

(6.1)

\[ G \sim P, \quad w \sim w. \]

We distinguish the usual three cases.

S) Set

(6.2)

\[ w_{n,j} := (j)^{1+\tau_n}, \quad \eta := \eta/a, \quad \tau := \sqrt{\hat{R}}. \]

\[ C_{\text{alg}}(1). \]

\[ ^{16} \text{Recall Remark 5.1.} \]
Then Assumption 1 is satisfied by Lemma 6.1. We have that
\begin{equation}
|P|_{r, \eta, w_0} = |P|_{r, 1, 0, 0, a/2} \leq C_{\text{12a}}(1, 0, a/2)|f|_{a, R}
\end{equation}

Noting that
\begin{align}
C_{\text{non}}(4N, \eta/N, \tau) &= 2^{2r+1}r!N\max\{4, (1/2\eta)\}\tau = 2(\sqrt{\tau} \max\{4, (1/2\eta)\})N \tau, \\
C_2(4N, \eta/N, \tau) &= e^{27(2+q)}(12\tau \max\{4, (1/2\eta)\})N^{3\tau} \\
C_2(4N, k_1, \tau) &= e^{27(2+q)}(12\tau \max\{4, (2\eta)^{-1}\})N^{3\tau},
\end{align}

we have that for \( N \geq 3 \)
\begin{equation}
C_{\text{non}}(4N, \eta/N, \tau)C_2(4N, \eta/N, \tau) \leq \sqrt{C_2(4N, \eta/N, M\tau)}.
\end{equation}

By (5.18)
\begin{equation}
r_* \geq \frac{\sqrt{\gamma R}}{C_{\text{alg}}(1)\sqrt{2^{11}eC_{\text{12a}}(1, 0, a/2)|f|_{a, R}}} \geq d_8.
\end{equation}

Then, recalling (5.19) and (6.5),
\begin{equation}
\hat{r} \geq r(S).
\end{equation}
Moreover
\begin{equation}
\hat{C}_1 \leq C_1(S), \quad \hat{C}_2 \leq C_2(S)
\end{equation}
and, recalling (6.21),
\begin{equation}
\hat{C}_3 \leq C_3(S).
\end{equation}
Finally
\begin{equation}
C_1(S)(r(S))^2 \leq \frac{1}{8},
\end{equation}
proving the last inequality in (1.25).

Set
\begin{equation}
w_{n,j} := [j]^{1+r}n, \quad \eta := 0, \quad \bar{r} := \frac{\sqrt{R}}{C_{\text{alg}}(1)} = \sqrt{R/10}.
\end{equation}

Then Assumption 1 is satisfied by Lemma 6.1. We have that
\begin{equation}
|P|_{\bar{r}, \eta, w_0} = |P|_{\bar{r}, 1, 0, 0, a/2} \leq 2|f|_{R}
\end{equation}
By (5.18)
\begin{equation}
r_* \geq \frac{\sqrt{\gamma R}}{2^6\sqrt{10c}|f|_{R}}
\end{equation}
Then, recalling (5.19)
\begin{equation}
\hat{r} \geq r(M).
\end{equation}
Moreover
\begin{equation}
\hat{C}_1 \leq C_1(M), \quad \hat{C}_2 \leq C_2(M), \quad \hat{C}_3 \leq C_3(M), \quad C_1(M)(r(M))^2 \leq \frac{1}{8}.
\end{equation}

Set
\begin{equation}
w_{n,j} := e^{a|j| + (s + (\eta - 1)\eta)|j|^\eta}n, \quad \eta := \eta_8 := \min\left\{\frac{a - a_2}{2}, s\right\}, \quad \bar{r} := \frac{\sqrt{R}}{C_{\text{alg}}(p)}.
\end{equation}
Then Assumption 1 is satisfied by Lemma 6.1. We have that
\begin{equation}
|P|_{\bar{r}, \eta, w_0} = |P|_{\bar{r}, p, s - \eta, a, \eta} \leq C_{\text{12a}}(p, s - \eta, a - a - \eta)|f|_{a, R}
\end{equation}
By (5.18) \[ r_* \geq \delta_8. \]

Then, recalling (5.19) and Lemma 6.1
\[ \hat{r} \geq \mathbf{r}(\mathcal{G}), \quad \hat{c}_1 \leq \mathbf{c}_1(\mathcal{G}), \quad \hat{c}_2 \leq \mathbf{c}_2(\mathcal{G}), \quad \hat{c}_3 \leq \mathbf{c}_3(\mathcal{G}), \quad \mathbf{c}_1(\mathcal{G}(\mathbf{r}(\mathcal{G}))^2 \leq \frac{1}{8}. \]

The proof of Theorem 1.4 is now completed.

7. PROOF OF THEOREM 1.1

Theorem 1.1 follows from Theorem 1.4.

We need the following auxiliary result, whose proof is postponed to the Appendix.

Lemma 7.1. On the Hilbert space \( \mathfrak{h}_u \) consider the dynamical system
\[ \dot{v} = X_N + X_R, \quad v(0) = v_0, \quad |v_0|_w \leq \frac{3}{4} r, \]
where \( N \in \mathcal{A}_v(\mathfrak{h}_u) \) and \( R \in \mathcal{H}_{r,\eta}(\mathfrak{h}_u) \) for some \( r > 0, \eta \geq 0 \). Assume that
\[ \Re(X_N, v)_{\mathfrak{h}_u} = 0. \]

Then
\[ |v(t)|_w < \frac{r}{8}, \quad \forall |t| \leq \frac{1}{8} |R|_{r,\eta,w}. \]

Let us now prove Theorem 1.1 starting with the case S).

S) Set \( r : = 2\delta. \)

Recalling (6.2) and (6.1), by Corollary 2.1 solutions of the PDE (1.1) in the Sobolev space \( \mathfrak{h}_{p,0,0} \), correspond, by Fourier identification (1.17), to orbits of the Hamiltonian System (1.20) in the space (recall Definition 2.4)
\[ \mathfrak{h}_u = \mathfrak{h}^p \quad \text{with} \quad w_j = \langle j \rangle^p \quad (\text{and} \quad |\cdot|_w = |\cdot|_p). \]

An initial datum \( u_0 \) satisfying \( |u_0|_{L^2} + |\partial_x^p u_0|_{L^2} \leq \delta \) corresponds to \( u_t \in \mathfrak{h}^p \) with \( |u_0|_p \leq \delta \) by (2.8). We want to apply the Birkhoff Normal Form Theorem 1.4 with
\[ N = (p - 1)/\tau_3. \]

Recalling the definition of \( \mathbf{r}(S,N) \) and noting that \( C_{1/2}(1) = 2\sqrt{1 + \pi^2/3} \), we have to verify that, for any \( N \geq 1 \)
\[ \delta_S(k_p)^{-3p} \leq \frac{d_s}{2 \sqrt{\mathcal{N} c_2(4N, a/2N, N\tau)}}. \]

Indeed
\[ \frac{d_s}{2 \sqrt{\mathcal{N} c_2(4N, a/2N, N\tau)}} \leq \frac{\delta_S}{2 \sqrt{2e^{-2(2+q)/2} / \sqrt{N} (\sqrt{12\tau N} \max \{2, a^{-1/2}\})^{3\tau}}} \]
\[ = \frac{d_s}{2 \sqrt{2e^{-2(2+q)/2} / \sqrt{N} \max \{2, a^{-1/2}\}}} (p - 1)^{-3(p-1)-1/2} \]

and (7.2) follows since \( p^{-3p} < (p - 1)^{-3(p-1)-1/2} \) for \( p > 1 \).

Then by Theorem 1.4 and setting \( v_0 := \Psi^{-1}(u_0) \), by (1.25) we get
\[ |v_0|_p = |\Psi^{-1}(u_0) - u_0|_p + |u_0|_p \leq \frac{1}{8} r + \frac{1}{2} r = \frac{5}{8} r. \]

\[ \text{We still denote it by } u_0. \]
By (1.26) and (6.4)
\[
|R|_{r,0,ω} \leq C_3(S,N)(2δ)^{2(δ+1)}
\]
\[
= δ^2 2^{10} C_{\text{fam}}(1,0, a/2)|f|_{a,R} \left( \frac{NC_{\text{fam}}(4N, a/2N, τ)C_2(4N, a/2N, τ)δ^2}{ds^4} \right)^{N/4}
\]
\[
= δ^2 2^{10} C_{\text{fam}}(1,0, a/2)|f|_{a,R} \left( \frac{4e^{27(2+q)3^{3τ}}(4 \max \{4, (1/a)\})^{4τ}δ^2}{τds^4} \right)^{N/4} \left( \frac{τ}{N} \right)^{4τ+1}(τN)
\]
We claim that (remember that N = (p − 1)/τ)
\[
|R|_{r,0,ω} < \frac{1}{8} \frac{δ^2}{T_S(K_0p)^{5p}} \left( \frac{δ}{δ_2} \right)^{2Np}
\]
This holds true since
\[
2^{10} C_{\text{fam}}(1,0, a/2)|f|_{a,R} \left( \frac{4e^{27(2+q)3^{3τ}}(4 \max \{4, (1/a)\})^{4τ}δ^2}{τds^4} \right)^{N/4} \left( \frac{τ}{N} \right)^{4τ+1}(τN)
\]
noting that (p − 1)^{4τ+1}(p−1) < p^{5p} for p > 1 (recall that τ ≥ 15). We apply Lemma 7.1 with N → Dω + Z, η ≈ 0; then by (7.1)
\[
|v(t)|_p \leq |v(t)|_p - |v_0|_p + |v_0|_p \leq \frac{r}{8} + \frac{5}{8}r = \frac{3}{4}r, \quad ∀ |t| \leq (C_0p)^{-\frac{4τ+1}{2τ}}p \left( 1 - \frac{1}{δ} \right)^{2τ}\frac{2(τ−1)}{τ}.
\]
Since Ψ is symplectic we have that u(t) = Ψ(v(t)); then by (1.25)
\[
|u(t)|_p \leq |Ψ(v(t)) - v(t)|_p + |v(t)|_p \leq \frac{r}{8} + \frac{3}{4}r < 2δ, \quad ∀ |t| \leq \frac{T_S}{δ^2}(K_0p)^{-5p} \left( \frac{δ_2}{δ} \right)^{2(τ−1)}\frac{2τ}{τ}.
\]
Finally by (2.8) we get
\[
|u(t)|_{L^2} + |∂_t^p u(t)|_{L^2} \leq 2|u(t)|_p \leq 4δ, \quad ∀ |t| \leq \frac{T_S}{δ^2}(K_0p)^{-5p} \left( \frac{δ_2}{δ} \right)^{2(τ−1)}\frac{2τ}{τ},
\]
proving (1.9).

M) It is similar to the previous case but now
\[
\mathfrak{h}_s = h^0 \quad \text{with} \quad w_j = |j|^p \quad (\text{and} \quad |·|_ω = ||·||_p).
\]
An initial datum u_0 satisfying 2^p|u_0|_{L^2} + |∂_t^p u_0|_{L^2} ≤ δ corresponds to u_0 ∈ h^0 with ||u_0||_p ≤ δ by (2.9). Now we can apply the Birkhoff Normal Form Theorem 1.1 with N = (p − 1)/τ since, for any N ≥ 1
(7.3)
\[
\min \left\{ \frac{2\sqrt{τ_δ}δ^2}{\sqrt{p}}, \frac{\sqrt{R}}{4\sqrt{10}} \right\} \leq \frac{1}{2} \frac{T_S}{δ^2}(M,N) = \min \left\{ \frac{2δ_2}{\sqrt{p}}, \frac{\sqrt{R}}{4\sqrt{10}} \right\}.
\]
Proceeding as in the case S and noting that now
\[
8|R|_{r,0,ω} ≤ 8C_3(S,N)(2δ)^{2(δ+1)} = 5 \cdot 2^9 \left( \frac{Nδ^2}{4δ^2_k} \right)^N δ^2 = \frac{1}{T_S} \left( \frac{(p−1)δ^2}{4τδ^2_k} \right)^{\frac{p−1}{8}} δ^2,
\]
we get
\[
||u(t)||_p ≤ 2δ, \quad ∀ |t| \leq \frac{T_S}{δ^2} \left( \frac{4τδ^2_k}{(p−1)δ^2} \right)^{\frac{p−1}{8}}.
\]
Finally by (2.9) we get
\[
2^p|u(t)|_{L^2} + |∂_t^p u(t)|_{L^2} ≤ 2||u(t)||_p \leq 4δ, \quad ∀ |t| \leq \frac{T_S}{δ^2} \left( \frac{4τδ^2_k}{(p−1)δ^2} \right)^{\frac{p−1}{8}}.
\]
proving (1.11).

8. Proof of Theorems 1.2 and 1.3

We start considering Theorem 1.2 case S.

S) We start by noticing that for $3p \ln(k_S p) \leq \ln(\delta_S / \delta)$ the function $\frac{T_S}{\delta^2} (k_S p)^{5p} \left( \frac{\delta_S}{\delta} \right)^{\frac{2(p-1)}{\rho}}$ is increasing in $p$.

Let us check that $p(\delta)$ defined in (1.12) satisfies (1.8), namely, passing to the logarithms and setting $y := \ln(\delta_S / \delta)$, we have to check that $3p \ln(k_S p) \leq y$. Indeed we have

$$3p \ln(k_S p) \leq 3 \left(1 + \frac{1}{6 \ln(y)}\right) \left(\ln(k_S) + \ln(1 + \frac{1}{6 \ln(y)})\right) \leq y$$

provided that\(^{18}\)

$$y \geq \max\{k_S, 40\}.$$

Now we have to show that

$$\frac{T_S}{\delta^2} e^{\frac{3}{4} \ln(y)} (k_S p)^{5p} \left( \frac{\delta_S}{\delta} \right)^{\frac{2(p-1)}{\rho}}$$

which amounts to

$$e^{\frac{3}{4} \ln(y)} (k_S p)^{5p} e^{\frac{2(p-1)}{\rho} y} \leq 1$$

or equivalently

$$\frac{y^2}{4\tau \ln y} + 5p \ln(k_S p) - \frac{2(p-1)}{\tau} y \leq 0.$$

Substituting $1 + \frac{y^2}{6 \ln y} - \frac{y^2}{3 \ln y}$ we get

$$\frac{y^2}{4\tau \ln y} + 5p \ln(k_S p) - \frac{2(p-1)}{\tau} y \leq \frac{y^2}{4\tau \ln y} + \frac{5}{3} \ln(y) - \frac{2}{3} \ln(y) - 2y(1 + \frac{y}{6\tau \ln y} - 1)$$

$$\leq -\frac{y^2}{12\tau \ln y} + \frac{5}{3} \ln(y) - \frac{11}{3} y < 0$$

if

$$y \geq 50\tau \ln(k_S / 3), \quad \frac{y}{\ln(y)} \geq 88\tau.$$

Note that the last inequality holds if $y \geq 88\tau^2$ (recall that $\tau \geq 15$). Recollecting the condition that $y$ has to satisfy is

$$y \geq \max\{k_S, 50\tau \ln(k_S / 3), 88\tau^2\},$$

namely $\delta \leq \delta_S$.

M) Since we are assuming $\delta \leq \delta_S$ we have that $p$ defined in (1.14) satisfies $p > 1$ and that (1.10) holds. Then Theorem 1.1 applies and (1.15) follows by\(^{19}\) (1.11).

We finally prove Theorem 1.3.

We proceed as in the proof of Theorem 1.2. Let us choose

$$(8.1) \quad \mathbb{N}(r) := \left[ 2\ln \left( \frac{2\delta_S}{r} \right)^{\theta/4} \right] = \left[ 2\ln \delta_S \frac{\theta}{2} \right].$$

Recalling (6.9) and (6.1), by Corollary 2.1 solutions of the PDE (1.7) in the space $h_{p,s,a}$, correspond, by Fourier identification (1.17), to orbits of the Hamiltonian System (1.20) in the space

$$h_v \quad \text{with} \quad w_j = e^{a[|j| + s(j)]^\theta / j}.$$

\(^{18}\) Note that the function

$$y \mapsto y - 3 \left(1 + \frac{1}{6 \ln(y)}\right) \left( \ln y + \ln(1 + \frac{1}{6 \ln(y)})\right)$$

is positive for $y \geq 40$.

\(^{19}\) Noting that $(4x/|x|)^{\alpha} \geq e^x$ for $x = \delta_S^2 / \delta^2 \geq 1$. 

We claim that an initial datum \( u_0 \) satisfying \( |u_0|_{p,s,a} \leq \delta \) corresponds to \( u_0 \in B_\delta \) with \( |u_0|_\nu \leq \delta \).

We claim that \( r \leq 2c_0 \delta \) implies

\[
(8.2) \quad \frac{rN e^{C_1 \left( \frac{r}{\pi r} \right)^2}}{2 \delta^2} \leq 1.
\]

Indeed we have \( N(r) \geq N_0 \) and by (8.1) \( r \leq 2\delta_0 e^{-\frac{1}{2}(\ln(r)/2)^{4/\theta}} \) and (8.2) follows if we show that the function

\[
N \to e^{-\frac{1}{2}(\ln(r)/2)^{4/\theta}} N e^{C_1 \left( \frac{r}{\pi r} \right)^2}
\]

is \( \leq 1 \) for \( N \geq N_0 \). This is true since the function is decreasing for \( N \geq N_0 \) and is \( \leq 1 \) for \( N = N_0 \). This proves the claim (8.2).

Then by (1.26) and (8.2)

\[
|R|_{r,0,0,0} \leq C_3(q) r^{2q+1} \leq \frac{r^2}{27 e^{2 \delta_0^2}} \left( \frac{r}{2 \delta_0} \right)^{2(\ln \frac{r}{\theta})} \leq \frac{\gamma \delta^2}{27 e^{2 \delta_0^2}} \left( \frac{\delta}{\delta_0} \right)^{2(\ln \frac{r}{\theta})},
\]

since \( N(r) \geq (\ln \frac{2\delta_0}{r})^{4/\theta} = (\ln \frac{\delta}{\theta})^{4/\theta} \). Proceeding as in the proof of Theorem 1.2 we get

\[
|u(t)|_{p,s,a} \leq 2\delta, \quad \forall \; |t| \leq \frac{T_0}{\delta^2} e^{(\ln \frac{\delta}{\theta})^{1+\theta/4}}.
\]

The proof of Theorem 1.3 is completed.

APPENDIX A. CONSTANTS.

In this subsection are listed all the constants appearing along the paper. We first introduce some auxiliary constants. Given \( t, \sigma, \zeta > 0, p > 1/2, 0 < \theta < 1, s, q \geq 0 \), we set\(^{21}\): \( C_{\text{non}}(t, \sigma, p) := 2^{p+1} p^p \max \left\{ (2t)^p, \sigma^{-p}, 1 \right\}, \)

\[
C_{\text{alg}}(p) := 2^p \left( \sum_{i \in \mathbb{Z}} \langle i \rangle^{-2p} \right)^{1/2},
\]

\[
C_{\text{alg},p}(p) := \sqrt{4 + 2 \frac{2p+1}{2p-1}},
\]

\[
C_{\text{linx}}(p,s,t) := C_{\text{alg}}(p) \left( e^s + \sup_{x \geq 1} x^s e^{-tx+sx^s} \right),
\]

\[
C_* := 13/(1-\theta),
\]

\[
C_1 := 28 \theta^{-1} (q+3) \left( 8(q+3)C_*\theta^{-1} \right)^{\frac{3}{2}} \left( \ln (8(q+3)C_*\theta^{-1}) \right)^{\frac{5}{2}} + 1,
\]

\[
C_2(t, \sigma, \zeta) := e^{27(2+q)C_{\text{non}}(t, \sigma, 3\zeta)},
\]

\[
\tau := \tau_0(2+q), \quad \tau_0 := 15/2, \quad \tau_1 := 2 \left( \tau_0 + \frac{3}{2\ln 2} \right)(2+q)
\]

\(^{20}\)We still denote it by \( u_0 \).

\(^{21}\)Regarding \( C_{\text{linx}} \), note that

\[
\sup_{x \geq 1} x^p e^{-tx+sx^s} \leq \exp \left( (1-\theta) \left( \frac{e^s}{p} \right)^{1/p} \right) \max \left\{ \frac{p}{e(1-\theta)t}, e^{-\frac{(1-\theta)q}{p}} \right\}^p.
\]
Here are the constants appearing in Theorem 1.1

\[ \tau_S := \tau, \]
\[ \delta_S := \frac{\sqrt{\gamma R} \left( \sqrt{3} \max \{2, a^{-1/2}\} \right)^3}{2^7 e^{27(2+\eta)/2} \sqrt{C_{\text{hex}}(1, 0, a/2)} |f|_{a,R}}, \]
\[ k_S := \frac{\sqrt{12}}{7} \max \{2, a^{-1/2}\}, \]
\[ T_S := \frac{e^{R \tau} 4\delta_3^6}{3 \cdot 2^{16} C_{\text{hex}}(1, 0, a/2) |f|_{a,R}}, \]
\[ K_S := \left( \frac{1}{2} 2^{3\tau^{6\tau} + 6} \right)^{1/(5\tau)}, \]
\[ \tau_M := \tau_1, \]
\[ \delta_M := \frac{\sqrt{\gamma R}}{\sqrt{5 \cdot 2^{11} e^{6\tau^7} (4^9 e^{27})^{2+\eta} |f|_R}}, \]
\[ T_M := \frac{R}{5 \cdot 2^9 |f|_R}. \]

Here are the constants appearing in Theorem 1.2

\[ \delta_S := \min \left\{ \delta_3 \exp \left( - \max \{k_3, 50 \tau \ln(k_3/3), 88 \tau^2\} \right), \frac{\sqrt{R}}{10} \right\}, \]
\[ \delta_M := \min \left\{ \delta_M, \frac{\sqrt{R}}{4 \sqrt{10}} \right\}. \]

Here are the constants appearing in Theorem 1.3

\[ \delta_0 := \min \left\{ c_0 \delta_0, \frac{\sqrt{R}}{4 C_{\text{alg}}(p)} \right\}, \]
\[ \delta_0 := \frac{\sqrt{\gamma R}}{C_{\text{alg}}(p) \sqrt{2^{11} e^{C_{\text{hex}}(p, s - \eta_0, a - a - \eta_0) |f|_{a,R}}} \}, \]
\[ \eta_0 := \min \left\{ \frac{a - a}{2}, s \right\}, \]
\[ c_0 := \exp \left( - \max \left\{ \frac{16 (4 C_1)^\theta}{\eta_0^2}, 2^{2q+1} \right\} \right)^{4/\theta}, \]
\[ T_0 := \frac{2^4 e \delta_0^6}{\gamma}. \]

Finally, with respect to the three cases \( S, M, G \), we define the constants \( r, C_1, C_2, C_3 \) of Theorem 1.4. To stress their dependence on the three cases \( S, M, G \), we denote them with \( r(S), r(M), r(G), C_1(S), C_1(M), C_1(G) \) etc. or also \( r(S, N) \) etc. when we want to emphasise the dependence on \( N \):

\[ r(S) := r(S, N) := \min \left\{ \frac{d_3}{\sqrt{N C_2(4 N, a/2, N \tau)}}, \frac{\sqrt{R}}{5} \right\}, \]
\[ C_1(S) := \frac{C_2(4 N, a/2 N \tau)}{2^7 e d_3^2}, \]
\[ C_2(S) := \frac{\gamma}{2^8 e d_3^2}, \]
\[ C_3(S) := C_3(S, N) := \frac{2^{8} C_{\text{hex}}(1, 0, a/2) |f|_{a,R}}{e R} \left( \frac{N C_{\text{mon}}(4 N, a/2 N \tau) C_2(4 N, a/2 N \tau)}{4 d_3^2} \right)^N \]
\[ d_3 := \frac{\sqrt{\gamma R}}{\sqrt{2^{11} C_{\text{hex}}(1, 0, a/2) |f|_{a,R}}}. \]
Proof of Lemma 2.2.

We first note that (see, e.g. Lemma 17 of Biasco-Di Gregorio, ARMA 2010) for $p > 1/2$ and every sequence $\{x_i\}_{i \in \mathbb{Z}}$, $x_i \geq 0$,

$$\left( \sum_{i \in \mathbb{Z}} x_i \right)^2 \leq c \sum_{i \in \mathbb{Z}} \frac{(i)^p (j-i)^p}{\langle j \rangle^p} x_i,$$

with $c := 4^p \sum_{i \in \mathbb{Z}} |i|^{-2p} = (C_{\text{alg}}(p))^2$. Then

$$|f \ast g|_{p,s,a}^2 \leq \sum_j e^{2s(j)^p} e^{2a|j|^p} \left( \sum_i |f_i||g_{j-i}| \right)^2 \leq c \sum_j e^{2s(j)^p} e^{2a|j|^p} \sum_i (i)^{2p} (j-i)^{2p} |f_i|^2 |g_{j-i}|^2 \leq c \sum_i e^{2s(i)^p} e^{2a|i|^p} (j-i)^{2p} e^{2s(j-i)^p} e^{2a|j-i|^p} |g_{j-i}|^2 = c |f|_{p,s,a}^2 |g|_{p,s,a}^2.$$

\[\square\]
B.2. Proof of Lemma 2.3. Set

\[ \phi(i, j) := \frac{|j|}{|i| |j - i|}, \quad \forall i, j \in \mathbb{Z}. \]

Note that

(B.1) \[ \phi(i, j) = \phi(j, i) = \phi(-i, -j). \]

We claim that

(B.2) \[ \phi(i, j) \leq 1. \]

Indeed by (B.1) we can consider only the case \( j \geq 0 \). Since \( \phi(-|i|, j) \leq \phi(|i|, j) \) we can consider only the case \( i \geq 0 \). Again by (B.1) we can assume \( j \geq i \). In particular we can take \( j > i > 0 \), (B.2) being trivial in the cases \( j = i, i = 0 \). We have

\[ \phi(i + 1, i) = \frac{i + 1}{2(i)} \leq \frac{3}{4}, \quad \phi(j, 1) = \frac{j}{2(j - 1)} \leq 1. \]

Then it remains also to discuss the case \( j - 2 \geq i \geq 2 \); we have

\[ \phi(i, j) = \frac{j}{i(j - i)} = \frac{1}{i} + \frac{1}{j - i} \leq 1, \]

proving (B.2).

For \( q \geq 0 \) set

(B.3) \[ c_q := \sup_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} (\phi(i, j))^q = \sup_{j \geq 0} \sum_{i \in \mathbb{Z}} (\phi(i, j))^q. \]

We claim that

(B.4) \[ c_q \leq 4 + 2 \frac{q + 1}{q - 1} < \infty, \quad \forall q > 1. \]

Indeed, since \( |j| / |j + 1| \leq 1 \) and \( |j| / |j - 1| \leq 3/2 \) for \( j \geq 0 \), we have\(^{22}\)

\[
\begin{align*}
c_q &= \sup_{j \geq 0} \left( \frac{|j|^q}{2^{q-1} |j + 1|^q} + \frac{1}{2^{q-1} |j - 1|^q} + \sum_{i \leq -2, 2 \leq i \leq j - 2, i \geq j + 2} (\phi(i, j))^q \right) \\
&\leq 2^{3-q} + \sup_{j \geq 0} \left( \sum_{i \geq 2} \frac{|j|^q}{i^q(j + i)^q} + \sum_{2 \leq i \leq j - 2} \left( \frac{1}{i} + \frac{1}{(j - i)} \right)^q + \sum_{i \geq j + 2} \frac{|j|^q}{i^q(j - i)^q} \right) \\
&\leq 2^{3-q} + \sum_{i \geq 2} \frac{1}{i^q} + 2^{q-1} \sum_{2 \leq i \leq j - 2} \left( \frac{1}{i^q} + \frac{1}{(j - i)^q} \right) + \sum_{i \geq j + 2} \frac{1}{(i - j)^q} \\
&\leq 4 + 2 \frac{q + 1}{q - 1},
\end{align*}
\]

using that \( (x + y)^q \leq 2^{q-1}(x^q + y^q) \) for \( x, y \geq 0 \) and that\(^{23}\)

\[ \sum_{i \geq 2} i^{-q} \leq \frac{q + 1}{2^{q(q - 1)}}. \]

Note that for every \( q, q_0 \geq 0 \) we have

(B.5) \[ c_{q_0 + q} \leq c_{q_0} \]

since

\[ c_{q_0 + q} := \sup_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} (\phi(i, j))^{q_0}(\phi(i, j))^q \leq \sup_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} (\phi(i, j))^{q_0} = c_{q_0}. \]

\(^{22}\) Note that the term \( \left( \frac{1}{i} + \frac{1}{(j - i)} \right)^q \) for \( j = 4 \) and \( i = 2 \) is 1 for every \( q \).

\(^{23}\) \[ \sum_{i \geq 2} i^{-q} \leq 2^{-q} + \int_{2}^\infty x^{-q}dx. \]
We now note that for $p > 1/2$, $j \in \mathbb{Z}$ and every sequence $\{x_i\}_{i \in \mathbb{Z}}$, $x_i \geq 0$, we have by Cauchy-Schwarz inequality
\[
\left( \sum_{i \in \mathbb{Z}} x_i \right)^2 = \left( \sum_{i \in \mathbb{Z}} (\phi(i, j))^{p} (\phi(i, j))^{-p} x_i \right)^2 \leq c_{2p} \sum_{i \in \mathbb{Z}} ((\phi(i, j))^{-p} x_i)^2,
\]
with $c_{2p}$ defined in (B.3). Using the above inequality we get
\[
\|f \ast g\|_p^2 \leq \sum_j |j|^{2p} \left( \sum_i |f_i||g_{j-i}| \right)^2 \leq c_{2p} \sum_j \sum_i |j|^{2p} |j-i|^{2p} |f_i| \sum_j |j-i|^{2p} |g_{j-i}|^2 \leq c_{2p} \|f\|_p^2 \|g\|_p^2.
\]
The proof ends recalling (B.4).

B.3. Proof of Lemma 2.4

(i) By definition the $\eta$-majorant Hamiltonian is
\[
H_\eta = \sum_d \sum_{j_0,j_1,\ldots,j_{2d}} e^{\eta|\pi_{j_1,\ldots,j_{2d}}|} |F_{d,j_0}^{(d)}| |\bar{u}_{j_1} u_{j_2} \bar{u}_{j_3} \cdots \bar{u}_{j_{2d}}|
\]
where
\[
\pi_{j_1,\ldots,j_{2d}} = \sum_{i=1}^{2d} (-1)^i j_i = -ja,
\]
hence
\[
H_\eta = \sum_d \left( F_{\eta}^{(d)} \ast u \ast \cdots \ast u \ast \bar{u} \ast \cdots \ast \bar{u} \right)_{d \text{ times}}, \quad F_{\eta}^{(d)} := \left( e^{\eta|j||F_{j}^{(d)}|} \right)_{j \in \mathbb{Z}}.
\]
consequently
\[
X_{H_\eta}^{(j)} = \sum_d \left( F_{\eta}^{(d)} \ast u \ast \cdots \ast u \ast \bar{u} \ast \cdots \ast \bar{u} \right)_{d \text{ times}}_{(d-1) \text{ times}}_{j}.
\]
Moreover
\[
|X_{H_\eta}|_{p,s,a} \leq \sum_d d(C_{alg}(p))^{2d-1} |F_{\eta}^{(d)}|_{p,s,a}(|u|_{p,s,a})^{2d-1}.
\]
Since
\[
|F_{\eta}^{(d)}|_{p,s,a} = |F^{(d)}|_{s,a+0,p} \leq |F^{(d)}|_{p,s,a_0}
\]
we get
\[
|X_{H_\eta}|_{p,s,a} \leq \sum_d d(C_{alg}(p))^{2d-1} |F^{(d)}|_{p,s,a_0}(|u|_{p,s,a})^{2d-1}.
\]
Therefore
\[
|H|_{r,p,\eta}^{(p,s,a_0,0)} = r^{-1} \sup_{|u|_{p,s,a} < r} \left| X_{H_\eta} \right|_{p,s,a} \leq r^{-1} \sum_d d|F^{(d)}|_{p,s,a_0} (C_{alg}(p)r)^{2d-1} < \infty.
\]
(ii) The proof is analogous to point (i).
Lemma B.3. Let $0 < r_1 < r$. Let $E$ be a Banach space endowed with the norm $| · |_E$. Let $X : B_r \to E$ a vector field satisfying

$$\sup_{B_r} |X|_E \leq \delta_0 .$$

Then the flow $\Phi(u, t)$ of the vector field is well defined for every $|t| \leq T := \frac{r - r_1}{\delta_0}$ and $u \in B_{r_1}$ with estimate

$$|\Phi(u, t) - u|_E \leq \delta_0 |t|, \quad \forall |t| \leq T .$$

Proof. Fix $u \in B_{r_1}$. Let us first prove that $\Phi(u, t)$ exists $\forall |t| \leq T$. Otherwise there exists a time $0 < t_0 < T$ such that $|\Phi(u, t_0)|_E < r$ for every $0 \leq t < t_0$ but $|\Phi(u, t_0)|_E = r$. Then, by the fundamental theorem of calculus (B.6)

$$\Phi(u, t_0) - u = \int_0^{t_0} X(\Phi(u, \tau)) d\tau .$$

Therefore

$$r - r_1 \leq |\Phi(u, t_0)|_E - |u|_E \leq |\Phi(u, t_0) - u|_E \leq \int_0^{t_0} |X(\Phi(u, \tau))|_E d\tau \leq \delta_0 t_0$$

$$< \delta_0 T = r - r_1 ,$$

which is a contradiction. Finally, for every $|t| \leq T$,

$$|\Phi(u, t) - u|_E \leq \left| \int_0^t \ |X(\Phi(u, \tau))|_E d\tau \right| \leq \delta_0 |t| .$$

\[ \square \]

B.4. Proof of Lemma 3.1 We first prove (i). It is easily seen that:

$$X_{H_q}^{(j)} (u) = i \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}| \beta_j \epsilon^{\eta |\pi(\alpha - \beta)|} u^\alpha \bar{u}^{\beta - e_j} .$$

Now

$$|X_{H_q}^{(j)} (u)|_q \leq |X_{H_q}^{(j)} (u)|_\varphi, \quad u = (|u_j|)_{j \in \mathbb{Z}}$$

hence, in evaluating the supremum of $|X_{H_q}^{(j)} |_q$ over $|u|_q \leq r$ we can restrict to the case in which $u = (u_j)_{j \in \mathbb{Z}}$ has all real positive components. Hence

$$|H| |_{r, \eta, \varphi} = r^{-1} \sup_{|u|_\varphi \leq r} \left| \left( \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}| \beta_j \epsilon^{\eta |\pi(\alpha - \beta)|} u^\alpha \bar{u}^{\beta - e_j} \right)_{j \in \mathbb{Z}} \right|_\varphi .$$

Then

(B.7)

$$|H| |_{r, \eta, \varphi} = \frac{1}{2r} \sup_{|u|_\varphi \leq r} \left| \left( W_{q}^{(j)} (u) \right)_{j \in \mathbb{Z}} \right|_\varphi ,$$

where

$$W_{q}^{(j)} (u) = \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}| \left( \alpha_j + \beta_j \right) \epsilon^{\eta |\pi(\alpha - \beta)|} u^\alpha \bar{u}^{\beta - e_j} ,$$

since, by the reality condition 1.21 we have

$$\sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}| \beta_j \epsilon^{\eta |\pi(\alpha - \beta)|} u^\alpha \bar{u}^{\beta - e_j} = \sum_{\alpha, \beta \in \mathbb{N}^2} |H_{\alpha, \beta}| \alpha_j \epsilon^{\eta |\pi(\alpha - \beta)|} u^\alpha \bar{u}^{\beta - e_j} = \frac{1}{2} W_{q}^{(j)} (u) .$$

By the linear map

$$L_{r, \varphi} : \ell^2 \to \mathfrak{h}_\varphi , \quad y_j \mapsto \frac{r}{y_j} y_j = u_j ,$$

\[ ^{24} \text{Namely the solution of the equation } \partial_t \Phi(u, t) = X(\Phi(u, t)) \text{ with initial datum } \Phi(u, 0) = u. \]

\[ ^{25} \text{We assume } t_0 \text{ positive, the negative case is analogous.} \]
the ball of radius 1 in $\ell^2$ is isomorphic to the the ball of radius $r$ in $\mathfrak{h}$, namely $L_{r, \eta}(B_1(\ell^2)) = B_r(\mathfrak{h})$. We have

$$Y^{(j)}_H(y; r, \eta, u) = \frac{1}{2} W^{(j)}_\eta(L_{r, \eta}y).$$

Then (i) follows.

In order to prove item (ii) we rely on the fact that, since we are using the $\eta$-majorant norm, the supremum over $y$ in the norm is achieved on the real positive cone. Moreover, given $u, v \in \ell^2$, if

$$|u_j| \leq |v_j|, \quad \forall j \in \mathbb{Z}$$

then $|u|_{\ell^2} \leq |v|_{\ell^2}$.

**B.5. Proof of Proposition 3.3.** We start by Taylor expanding $H$ in homogeneous components. The majorant analiticity implies that for a homogeneous component of degree $d$ one has

$$|H^{(d)}|_{r, p, \eta} \leq |H|_{r, p, \eta}$$

Now let us consider the polynomial map (homogeneous of degree $d - 1$) $X^{(d)}_{H(a)} : \mathfrak{h}_{p, s, a} \to \mathfrak{h}_{p, s, a}$; as is habitual we identify the polynomial map with the corresponding symmetric multilinear operator $M^{(d-1)} : \mathfrak{h}_{p, a}^{d-1} \to \mathfrak{h}_{p, s, a}$.

Since we are in a Hilbert space, one has that

$$|M|_{p, s, a}^{\text{op}} := \sup_{u_1, \ldots, u_{d-1} \in \mathfrak{h}_{p, s, a}} \|M^{(d-1)}(u_1, \ldots, u_{d-1})\|_{p, s, a} = \sup_{|u|_{p, s, a} \leq 1} |M^{(d-1)}(u, \ldots, u)|_{p, s, a}$$

$$= \sup_{|u|_{p, s, a} \leq 1} |X^{(d)}_{H(a)}|_{p, s, a} \leq r^{-d+2} |H|_{r, p, s, a, \eta}$$

for all $\eta \geq 0$. Now let us compute the tame norm on a homogeneous component, i.e.

$$\sup_{|u|_{p_0, s, a} \leq r - \rho} \|M^{(d-1)}(u^{d-1})\|_{p, s, a} = \sup_{|u|_{p_0, s, a} \leq r - \rho} \frac{|M^{(d-1)}(u^{d-1})|_{p_0, s, a}}{|u|_{p_0, s, a}}$$

where

$$N^{(d-1)}_p(u^{d-1}) = (j)^{d-1} \sum_{j_1, \ldots, j_{d-1}} |M^{(d-1)}_{j_1, \ldots, j_{d-1}}| u_{j_1} \cdots u_{j_{d-1}}$$

now setting $\pi = \sum_j j \cdot 1 - j$ we have

$$N^{(d-1)}_p(u_1, \ldots, u_{d-1}) \leq (d - 1) (j)^{d-1} \sum_{j_1, \ldots, j_{d-1}} |M^{(d-1)}_{j_1, \ldots, j_{d-1}}| u_{j_1} \cdots u_{j_{d-1}}$$

$$\leq (d - 1) \sum_{j_1, \ldots, j_{d-1}} \left( \sum_{i} j_i + |\pi| \right)^{d-1} |M^{(d-1)}_{j_1, \ldots, j_{d-1}}| u_{j_1} \cdots u_{j_{d-1}}$$

$$\leq (d - 1) 2^{d-1} C(\eta, p) \sum_{j_1, \ldots, j_{d-1}} e^{\eta|\pi|} |M^{(d-1)}_{j_1, \ldots, j_{d-1}}| u_{j_1} \cdots u_{j_{d-1}}$$

$$+ (d - 1) 2^{d-1} (d - 1)^{d-1} \sum_{j_1, \ldots, j_{d-1}} |M^{(d-1)}_{j_1, \ldots, j_{d-1}}| (j_1)^{d-1} u_{j_1} \cdots u_{j_{d-1}}$$

which means that for any $|u|_{p_0, s, a} \leq r - \rho$ one has

$$|N^{(d-1)}_p(u^{d-1})|_{p_0, s, a} \leq (d - 1) 2^{d-1} C(\eta, p) |H^{(d)}|_{r-\rho, p_0, s, a, \eta} |u|_{p_0, s, a} + 2^{d-1} (d - 1)^{d-1} |M^{(d-1)}_{p_0, s, a}(r - \rho)^{d-2} |u|_{p_0, s, a}$$

$$\leq (d - 1) 2^{d-1} C(\eta, p) + (d - 1)^{d-1} \left( 1 - \frac{\rho}{r} \right)^{d-2} |H|_{r, p, s, a, \eta} |u|_{p, s, a}$$
We conclude that
\[
\sup_{|u|_{p_0,s,a} \leq r} \frac{|X_H|_{p,s,a}}{|u|_{p,s,a}} \leq 2^{p-p_0} |H|_{r,p_0,s,a,\eta} \sum_{d \geq 2} (d-1) (C(\eta, p) + (d-1)^{p-p_0}) (1 - \frac{D}{r})^{d-2}
\]
and the thesis follows since the right hand side is convergent. \(\square\)

B.6. **Proof of Lemma 7.1.** Let us look at the time evolution of \(|v(t)|_u^2\). By construction and Cauchy-Schwarz inequality
\[
2|v(t)|_u \frac{d}{dt} |v(t)|_u = 2 \left| \frac{d}{dt} |v(t)|_u^2 \right| = 2 \left| \mathrm{Re}(v, v_t) \right| = 2 \left| \mathrm{Re}(v, X_R) \right|
\]
as long as \(|v(t)|_u \leq r\); namely
\[
(B.8) \quad \left| \frac{d}{dt} |v(t)|_u \right| \leq r |R|_{r, \eta, u}
\]
as long as \(|v(t)|_u \leq r\).

Assume by contradiction that there exists a time \(T_0 < T\) such that
\[
(B.9) \quad |v(T_0)|_u - |v_0|_u < r, \quad \forall 0 \leq t < T_0,
\]
but
\[
|v(T_0)|_u - |v_0|_u = r.
\]
Then
\[
|v(t)|_u \leq |v_0|_u + \frac{r}{8} < r, \quad \forall 0 \leq t < T_0.
\]
By (B.8) we get
\[
|v(T_0)|_u - |v_0|_u \leq r |R|_{r, \eta, u} T_0 < r,
\]
which contradicts (B.9), proving (7.1).

**Appendix C. Small divisor estimates**

C.1. **Proof of Lemma 3.3.** The fact that this (3.11) holds true when \(\pi = 0\) is proven in [Bou96b] and [CLSY]. The bound (3.11) is equivalent to proving
\[
(C.1) \quad \sum_{l \geq 3} \tilde{n}_l^\theta - 2\tilde{n}_1^\theta + \theta|\pi| - (2 - 2^\theta) \sum_{l \geq 3} \tilde{n}_l^\theta \geq 0.
\]
i.e.
\[
(C.2) \quad \sum_{l \geq 3} \tilde{n}_l^\theta - \tilde{n}_1^\theta + \theta|\pi| - (2 - 2^\theta) \sum_{l \geq 3} \tilde{n}_l^\theta \geq 0.
\]
Inequality (C.2) then follows from
\[
(C.3) \quad f(|\pi|) := \sum_{l \geq 3} \tilde{n}_l^\theta - \left( |\pi| + \sum_{l \geq 2} \tilde{n}_l \right)^\theta + \theta|\pi| - (2 - 2^\theta) \sum_{l \geq 3} \tilde{n}_l^\theta \geq 0,
\]
26The case \(T_0 < 0\) is analogous.
which we are now going to prove. We shall show that the function $f(x)$ is increasing in $x \geq 0$; then the result follows by showing $f(0) \geq 0$, which is what was proven by Yuan and Bourgain.

We now verify that $f'(x) \geq 0$. By direct computation we see that

$$f'(x) = -\theta \left( x + \sum_{l \geq 2} \hat{n}_l \right)^{\theta-1} + \theta,$$

so it suffices to prove that

$$(C.4) \quad 1 \leq \left( x + \sum_{l \geq 2} \hat{n}_l \right)^{1-\theta},$$

which is indeed true, since $\sum_{l \geq 2} \hat{n}_l \geq \hat{n}_2 \geq 1$ holds, by mass conservation.

\[\square\]

C.2. Proof of Lemma 4.2. In this subsection we will take

$$(C.5) \quad \alpha, \beta \in \mathbb{N}^Z \text{ with } 1 \leq |\alpha| = |\beta| < \infty.$$

Given $u \in \mathbb{Z}^Z$, with $|u| < \infty$, consider the set

$$\{ j \neq 0, \text{ repeated } |u_j| \text{ times} \},$$

where $D < \infty$ is its cardinality. Define the vector $m = m(u)$ as the reordering of the elements of the set above such that $|m_1| \geq |m_2| \geq \cdots \geq |m_D| \geq 1$. Given $\alpha \neq \beta \in \mathbb{N}^Z$, with $|\alpha| = |\beta| < \infty$ we consider $m = m(\alpha - \beta)$ and $\hat{n} = \hat{n}(\alpha + \beta)$. If we denote by $D$ the cardinality of $m$ and $N$ the one of $\hat{n}$ we have

$$(C.6) \quad D + \alpha_0 + \beta_0 \leq N$$

and

$$(C.7) \quad (|m_1|, \ldots, |m_D|, 1, \ldots, 1) \leq (\hat{n}_1, \ldots, \hat{n}_N).$$

Set

$$\sigma_l = \text{sign}(\alpha_{m_l} - \beta_{m_l}).$$

For every function $g$ defined on $\mathbb{Z}$ we have that

$$(C.8) \quad \sum_{i \in \mathbb{Z}} g(i) |\alpha_i - \beta_i| = g(0) |\alpha_0 - \beta_0| + \sum_{l \geq 1} g(m_l),$$

Lemma C.1. Assume that $g$ defined on $\mathbb{Z}$ is non negative, even and not decreasing on $\mathbb{N}$. Then, if $\alpha \neq \beta$,

$$(C.9) \quad \sum_{i \in \mathbb{Z}} g(i) |\alpha_i - \beta_i| \leq 2g(m_1) + \sum_{l \geq 3} g(\hat{n}_l).$$

Proof. By (C.8)

$$\sum_{i \in \mathbb{Z}} g(i) |\alpha_i - \beta_i| = g(0) |\alpha_0 - \beta_0| + \sum_{l \geq 1} g(m_l)$$

$$\leq g(1)(\alpha_0 + \beta_0) + 2g(m_1) + \sum_{l \geq 3} g(m_l)$$

and (C.9) follows by (C.6) and (C.7).
We denote as before the momentum by $\pi$ so by (C.8)
\begin{equation}
\pi = \sum_{i \in \mathbb{Z}} (\alpha_i - \beta_i)i = \sum l \sigma_l m_l
\end{equation}
and
\begin{equation}
\sum_i (\alpha_i - \beta_i)^2 = \sum l \sigma_l m_l^2.
\end{equation}
Analogously
\begin{equation}
\sum_i \mid \alpha_i - \beta_i \mid = D + \mid \alpha_0 - \beta_0 \mid \geq N.
\end{equation}
Finally note that
\begin{equation}
\sigma_l \sigma_{l'} = -1 \implies m_l \neq m_{l'}.
\end{equation}
Note that
\begin{equation}
\alpha \neq \beta \implies N \geq 3 \text{ or } \pi \neq 0,
\end{equation}
indeed, by mass conservation, $|\alpha| = |\beta| = 1$ therefore if $N = 2$ we get $\alpha - \beta = e_{j_1} - e_{j_2}$ so if $\pi = 0$ we have $\alpha = \beta$. Note also that
\begin{equation}
\alpha \neq \beta \implies D \geq 1,
\end{equation}
indeed, if $D = 0$ then $\alpha_i - \beta_i = 0$ for every $|l| \geq 1$ and, by mass conservation $\alpha_0 = \beta_0$, contradicting $\alpha \neq \beta$.

**Lemma C.2.** Given $\alpha \neq \beta \in \mathbb{N}^2$, with $1 \leq |\alpha| = |\beta| < \infty$ and satisfying (4.5), we have
\begin{equation}
|m_1| \leq 20|\pi| + 31 \sum_{l \geq 3} \tilde{n}_l^2.
\end{equation}

**Proof.** In the case $D = 1$ by (C.10) $|\pi| = |m_1|$ and (C.16) follows. Let us now consider the case $D = 2$, i.e.
\[
\alpha - \beta = \sigma_1 e_{m_1} + \sigma_2 e_{m_2} + (\alpha_0 - \beta_0) e_0.
\]
Let us start with the case $\sigma_1 \sigma_2 = 1$. By mass conservation $|\sigma_1 + \sigma_2| = |\beta_0 - \alpha_0| = 2$. By (C.12) $N \geq 4$. Then conditions (4.5) and (C.12) imply that
\[
m_1^2 + m_2^2 \leq 20|\alpha_0 - \beta_0| = 40.
\]
Then
\[
|m_1| \leq \sqrt{40} \leq \sqrt{40} \sum_{l \geq 3} \tilde{n}_l^2
\]
since $N \geq 4$ and $\tilde{n}_l \geq 1$. When $\sigma_1 \sigma_2 = -1$ we have $m_1 \neq m_2$, $|\pi| = |m_1 - m_2| \geq 1$ and by mass conservation $\alpha_0 - \beta_0 = 0$. Then
\[
(|m_1| + |m_2|)(|m_1| - |m_2|) = m_1^2 - m_2^2 \leq 20.
\]
If $|m_1| > |m_2|$ then
\begin{equation}
|m_1| \leq 20 \leq 20|\pi|.
\end{equation}
Otherwise $m_1 = -m_2$ and, therefore, $|\pi| = 2|m_1|$, completing the proof in the case $D = 2$.

Let us now consider the case $D \geq 3$. By (4.5), (C.11) and (C.12)
\[
m_1^2 + \sigma_1 \sigma_2 m_2^2 \leq 10N + \sum_{l = 3}^D m_l^2 \leq 10N + \sum_{l = 3}^N \tilde{n}_l^2
\]
\[
= 20 + \sum_{l = 3}^N (10 + \tilde{n}_l^2) \leq 20 + 11 \sum_{l = 3}^N \tilde{n}_l^2 \leq 31 \sum_{l = 3}^N \tilde{n}_l^2.
\]
\[\text{Note that by (C.14) the r.h.s. of (C.16) is at least 20.}\]
If \( \sigma_1 \sigma_2 = 1 \) then
\[
|m_1|, |m_2| \leq \sqrt{31 \sum_{t \geq 3} \hat{n}_t^2}.
\]

If \( \sigma_1 \sigma_2 = -1 \)
\[
(|m_1| + |m_2|)(|m_1| - |m_2|) = m_1^2 - m_2^2 \leq 31 \sum_{t \geq 3} \hat{n}_t^2.
\]

Now, if \( |m_1| \neq |m_2| \) then
\[
|m_1| + |m_2| \leq 31 \sum_{t \geq 3} \hat{n}_t^2.
\]

Conversely, if \( |m_1| = |m_2| \), by \((C.13)\), \( m_1 \neq m_2 \), hence \( m_1 = -m_2 \). By substituting this relation into \((C.10)\), we have
\[
2|m_1| \leq |\pi| + \sum_{t \geq 3} |m_t| \leq |\pi| + \sum_{t \geq 3} \hat{n}_t^2,
\]
concluding the proof. \( \square \)

**Conclusion of the proof of Lemma 4.2.** As above, given \( \alpha, \beta \in \mathbb{N}^2 \), with \( 1 \leq |\alpha| = |\beta| < \infty \) we consider \( m = m(\alpha - \beta) \) and \( \hat{n} = \hat{n}(\alpha + \beta) \). Note that \( N := |\alpha + \beta| \geq 2 \).

We have\(^{28}\)
\[
\sum |\alpha_i - \beta_i| \langle i \rangle^\theta/2 \leq 2|m_1|^{\theta/2} + \sum_{t \geq 3} \hat{n}_t^{\theta/2}
\]
\[
\leq 2 \left( 20|\pi| + 31 \sum_{t \geq 3} \hat{n}_t^{\theta} \right)^{\theta/2} + \sum_{t \geq 3} \hat{n}_t^{\theta/2}
\]
\[
\leq 2(20|\pi|)^{\theta/2} + 2(31)^{\theta/2} \sum_{t \geq 3} \hat{n}_t^{\theta} + \sum_{t \geq 3} \hat{n}_t^{\theta/2}
\]
\[
(C.18)
\]

using that \( 1 - \theta \leq 2 - 2^\theta \) for \( 0 \leq \theta \leq 1 \). Then by Lemma \((3.3)\) and \((C.18)\) we get
\[
\sum |\alpha_i - \beta_i| \langle i \rangle^\theta/2 \leq \frac{13}{1 - \theta} \left( (1 - \theta)|\pi| + \sum_i \langle i \rangle^\theta (\alpha_i + \beta_i) + |\pi| - 2\hat{n}_1^{\theta} \right)
\]
\[
\leq \frac{13}{1 - \theta} \left[ \sum_i \langle i \rangle^\theta (\alpha_i + \beta_i) + |\pi| - 2(\langle 1 \rangle^\theta) \right],
\]
proving \((4.6)\). \( \square \)

Let us now prove \((4.7)\) passing to the logarithm. We have
\[
\sum_i \ln(1 + |\alpha_i - \beta_i| \langle i \rangle) = \sum_{|i| \leq 1} \ln(1 + |\alpha_i - \beta_i|) + \sum_{|i| \geq 2} \ln(1 + |\alpha_i - \beta_i| |i|)
\]
\[
(C.19)
\]
\[
\leq 3 \ln(1 + N) + \sum_{|i| \geq 2} \ln(1 + |\alpha_i - \beta_i| |i|)
\]
\[
\leq 3 \ln 2 + 3 \ln N + \frac{3}{2} \sum_{|i| \geq 2} |\alpha_i - \beta_i| \ln |i|,
\]
\(\square\)

\(^{28}\)Using that for \( x, y \geq 0 \) and \( 0 \leq c \leq 1 \) we get \( (x + y)^c \leq x^c + y^c \).
using that $1 + cx \leq \frac{3}{2} x^c$ for $c \geq 1$, $x \geq 2$. If $\alpha_i - \beta_i = 0$ for every $|i| \geq 2$ then (4.7) follows. Assume now that $\alpha_i - \beta_i \neq 0$ for some $|i| \geq 2$. By (C.14) we have

(C.20) \[ N \geq 3 \quad \text{or} \quad \pi \neq 0. \]

We claim that, when $N \geq 3$,

(C.21) \[ \ln \left( \sum_{l=3}^{N} \hat{n}_l^2 \right) \leq \ln N + \sum_{l=3}^{N} \ln \hat{n}_l^2. \]

Let $\mathcal{S} := \{ 3 \leq l \leq N, \text{s.t. } \hat{n}_l \geq 2 \}$. If $\mathcal{S} = \emptyset$ we have the equality in (C.21). Otherwise $\sum_{l \in \mathcal{S}} \hat{n}_l^2 \geq 4$ and\footnote{Use that $\ln(x+y) \leq \ln x + \ln y$ if $x, y \geq 2$.} \[ \ln \left( \sum_{l=3}^{N} \hat{n}_l^2 \right) \leq \ln \left( N + \sum_{l \in \mathcal{S}} \hat{n}_l^2 \right) \leq \ln N + \sum_{l \in \mathcal{S}} \ln \hat{n}_l^2, \]

proving (C.21).

We claim that

(C.22) \[ \ln \left( 20|\pi| + 31 \sum_{l=3}^{N} \hat{n}_l^2 \right) \leq \ln(1 + |\pi|) + \ln N + \sum_{l=3}^{N} \ln \hat{n}_l^2 + 20 + \ln 31. \]

Indeed consider first the case $\pi = 0$, then $N \geq 3$ by (C.20) and (C.22) follows by (C.21). Consider now the case $|\pi| \geq 1$. If $N < 3$ (C.22) follows (there is no sum). If $N \geq 3$ we have\footnote{Recall footnote 29.} \[ \ln \left( 20|\pi| + 31 \sum_{l=3}^{N} \hat{n}_l^2 \right) \leq \ln(1 + |\pi|) + \ln \left( 31 \sum_{l=3}^{N} \hat{n}_l^2 \right) \]

\[ \leq \ln(|\pi|) + \ln \left( \sum_{l=3}^{N} \hat{n}_l^2 \right) + 20 + \ln 31. \]

Recalling (C.21) this complete the proof of (C.22).

Let us continue the proof of (4.7). Set $g(i) := 0$ if $|i| \leq 1$ and $g(i) := \ln |i|$ if $|i| \geq 2$ and apply (C.9) to (C.19): we get

\[
\begin{align*}
\sum_{|i| \geq 2} |\alpha_i - \beta_i| \ln |i| & \leq 2 \ln |m_1| + \sum_{l \geq 3} \ln \hat{n}_l \\
& \leq 2 \ln \left( 20|\pi| + 31 \sum_{l \geq 3} \hat{n}_l^2 \right) + \sum_{l \geq 3} \ln \hat{n}_l \\
& \leq 2 \ln(1 + |\pi|) + 2 \ln N + 5 \sum_{l=3}^{N} \ln \hat{n}_l + 16.
\end{align*}
\]

Inserting in (C.19) we obtain

\[ \sum_i \ln(1 + |\alpha_i - \beta_i| |i|) \leq 3 \ln(1 + |\pi|) + 6 \ln N + \frac{15}{2} \sum_{l=3}^{N} \ln \hat{n}_l + 27. \]

concluding the proof of (4.7). \[ \square \]
C.3. Proof of Lemma 4.5 First of all we note that
\[ \sum_i f_i(|\ell_i|) = \sum_{i \text{ s.t. } \ell_i \neq 0} f_i(|\ell_i|) \]
since \( f_i(0) = 0 \). We have that\(^{31}\)
\[ f_i(x) \leq -\frac{\sigma}{C_*} \langle i \rangle \frac{q}{2} x + 2 \ln(x) + (2 + q) \ln(i) + 1, \quad \forall x \geq 1. \]
We have that
\[ \max_{x \geq 1} \left( -\frac{\sigma}{C_*} \langle i \rangle \frac{q}{2} x + 2 \ln(x) \right) = \begin{cases} -\frac{\sigma}{C_*} \langle i \rangle \frac{q}{2} & \text{if } \langle i \rangle \geq i_0, \\ -2 + 2 \ln \frac{2C_*}{\sigma} - \theta \ln\langle i \rangle & \text{if } \langle i \rangle < i_0, \end{cases} \]
where
\[ i_0 := \left( \frac{2C_*}{\sigma} \right)^{\frac{q}{2}}, \]
since the maximum is achieved for \( x = 1 \) if \( \langle i \rangle \geq i_0 \) and \( x = \frac{2C_*}{\sigma \langle i \rangle^{q/2}} \) if \( \langle i \rangle < i_0 \). Note that \( i_0 \geq e \). Then we get
\[ \sum_i f_i(|\ell_i|) = \sum_{i \text{ s.t. } \ell_i \neq 0} f_i(|\ell_i|) \leq \sum_{\langle i \rangle \geq i_0 \text{ s.t. } \ell_i \neq 0} \left( (2 + q) \ln\langle i \rangle + 1 - \frac{\sigma}{C_*} \langle i \rangle \frac{q}{2} \right) + \sum_{\langle i \rangle < i_0} \left( 2 \ln \frac{2C_*}{\sigma} + (2 + q - \theta) \ln\langle i \rangle \right). \]
We immediately have that
\[ \sum_{\langle i \rangle \geq i_0} \left( 2 \ln \frac{2C_*}{\sigma} + (2 + q - \theta) \ln\langle i \rangle \right) \leq 3i_0 \left( 2 \ln \frac{2C_*}{\sigma} + (2 + q) \ln i_0 \right) \]
\[ = 3 \left( 2 + \frac{2}{\theta} (2 + q) \right) \left( \frac{2C_*}{\sigma} \right)^{\frac{q}{2}} \ln \frac{2C_*}{\sigma}. \]
Moreover, in the case \( \langle i \rangle \geq i_0 \geq e, \)
\[ (2 + q) \ln\langle i \rangle + 1 - \frac{\sigma}{C_*} \langle i \rangle \frac{q}{2} \leq (2 + q + 1) \ln\langle i \rangle - \frac{\sigma}{C_*} \langle i \rangle \frac{q}{2} = \frac{2}{\theta} (2 + q + 1) \left( \ln\langle i \rangle \frac{q}{2} - 2C \langle i \rangle \frac{q}{2} \right) \]
where
\[ C := \frac{\sigma \theta}{4C_* (2 + q + 1)} < 1. \]
Therefore
\[ S_* := \sum_{\langle i \rangle \geq i_0 \text{ s.t. } \ell_i \neq 0} \left( (2 + q) \ln\langle i \rangle + 1 - \frac{\sigma}{C_*} \langle i \rangle \frac{q}{2} \right) \]
satisfies
\[ S_* \leq \sum_{\langle i \rangle \geq i_0 \text{ s.t. } \ell_i \neq 0} \frac{2}{\theta} (2 + q + 1) \left( \ln\langle i \rangle \frac{q}{2} - 2C \langle i \rangle \frac{q}{2} \right). \]
We have that\(^{32}\)
\[ \ln\langle i \rangle \frac{q}{2} - 2C \langle i \rangle \frac{q}{2} \leq -C \langle i \rangle \frac{q}{2}, \quad \text{when } \langle i \rangle \geq i_* := \left( \frac{2}{C} \ln \frac{1}{C} \right)^{\frac{q}{2}}. \]
Note that
\[ i_e \geq \max\{i_0, i_*\}. \]

\(^{31}\) Using that \( \ln(1 + y) \leq 1 + \ln y \) for every \( y \geq 1 \).

\(^{32}\) Using that, for every fixed \( 0 < C \leq 1 \), we have \( Cx \geq \ln x \) for every \( x \geq \frac{2}{C} \ln \frac{1}{C} \).
Therefore
\[
S_* \leq \frac{2}{\theta} (2 + q + 1) \left( \sum_{(i) < i_1} \ln(i)^2 - \sum_{(i) \geq i_2 \text{ s.t. } i \neq 0} \left( \mathcal{C}(i)^2 \right) \right) \\
\leq (2 + q + 1) \left( 3i_2 \ln i_2 - \frac{2\mathcal{C}}{\theta} M_*^2 \right).
\]
where
\[M_* := \max \{ |i| \geq i_1, \text{ s.t. } \ell_i \neq 0 \}\]
and \(M_* := 0\) if \(|\ell| = 0\) for every \(|i| \geq i_2\). In conclusion we get
\[
\sum_i f_i(|\ell_i|) \leq 3 \left( 2 + \frac{2}{\theta} (2 + q) \right) \left( \frac{2C_s}{\sigma} \right)^2 \ln \frac{2C_s}{\sigma} + (2 + q + 1) \left( 3i_2 \ln i_2 - \frac{2\mathcal{C}}{\theta} M_*^2 \right)
\]
\[
\leq 6(q + 3)i_2 \ln i_2 - \frac{\sigma}{2C_s} M_*^2
\]
\[
\leq 7(q + 3)i_2 \ln i_2 - \frac{\sigma}{2C_s} (\hat{n}_1(\ell))^2,
\]
noting that \(\hat{n}_1(\ell) = M_*\) if \(M_* \neq 0\), otherwise \(\hat{n}_1(\ell) < i_2\), and, therefore,
\[
\frac{\sigma}{2C_s} (\hat{n}_1(\ell))^2 < \frac{\sigma}{2C_s} i_2^2 \leq (q + 3)i_2 \ln i_2
\]
\[
\square
\]

C.4. Measure Estimates.

**Proof of Lemma 4.4.** For \(\ell \in \mathbb{Z}^q\) with \(0 < |\ell| < \infty\) we define
\[
\mathcal{R}_\ell := \left\{ \omega \in \Omega_q : |\omega \cdot \ell| \leq \frac{\gamma}{1 + |\ell_0|^{\mu_1}} \prod_{n \neq 0} \frac{1}{(1 + |\ell_n|^{\mu_1})^{n^{\mu_2+q}}} \right\}
\]
- if \(\ell\) is such that \(\ell_n = 0 \ \forall n \neq 0\) then
  \[
  \mu(\mathcal{R}_\ell) = \frac{\gamma}{1 + |\ell_0|^{\mu_1}}
  \]
- Otherwise: let \(s = s(\ell) > 0\) be the smallest positive index \(i\) such that \(|\ell_i| + |\ell_{-i}| \neq 0\) and \(S = S(\ell)\) be the biggest. Then we have\(^{33}\)
  \[
  \mu(\mathcal{R}_\ell) \leq \frac{\gamma s^q}{(1 + |\ell_0|^{\mu_1})} \prod_{n \neq 0} \frac{1}{(1 + |\ell_n|^{\mu_1})^{n^{\mu_2+q}}}
  \]
Let us write
\[
\frac{1}{1 + |\ell_0|^{\mu_1}} \prod_{n \neq 0} \frac{1}{(1 + |\ell_n|^{\mu_1})^{n^{\mu_2+q}}} = \frac{1}{1 + |\ell_0|^{\mu_1}} \prod_{n > 0} \frac{1}{(1 + |\ell_n|^{\mu_1})^{n^{\mu_2+q}}} \frac{1}{(1 + |\ell_{-n}|^{\mu_1})^{n^{\mu_2+q}}} \frac{1}{(1 + |\ell_{-n}|^{\mu_1})^{n^{\mu_2+q}}}
\]
\[
= \frac{1}{1 + |\ell_0|^{\mu_1}} \prod_{n \leq S(\ell)} \frac{1}{(1 + |\ell_n|^{\mu_1})^{n^{\mu_2+q}}} \frac{1}{1 + |\ell_{-n}|^{\mu_1})^{n^{\mu_2+q}}}
\]
\(^{33}\)Assume, e.g. that \(\ell_0 \neq 0\), then \(|\ell_0, \omega \cdot \ell| \geq s^{-q}\).
Now

(C.23) \[ \mu(\Omega_q \setminus \mathcal{B}_{r,q}) \leq \sum_{s} \mu(\mathcal{R}_s) = \sum_{s>0} \frac{\gamma}{1 + |\ell_0|^\mu_1} \]

(C.24) \[ + \sum_{s>0} \sum_{\ell_s(t)=s, S(t)=s} \frac{1}{1 + |\ell_0|^\mu_1} \frac{1}{1 + |\ell_s|^\mu_1|s|^\mu_2+q} \frac{1}{1 + |\ell_{-s}|\mu_1|s|^\mu_2+q} \]

(C.25) \[ + \sum_{0<s<s} \sum_{\ell_s(t)=s, S(t)=s} \frac{\gamma s^q}{1 + |\ell_0|^\mu_1} \prod_{s \leq n \leq S} \frac{1}{1 + |\ell_n|^\mu_1|n|^\mu_2+q} \frac{1}{1 + |\ell_{-n}|\mu_1|n|^\mu_2+q}. \]

Let us estimate (C.24)

\[ \sum_{s>0} \sum_{\ell_s(t) \in \mathbb{Z}} \frac{1}{1 + |\ell_0|^\mu_1} \sum_{\ell_s \ell_{-s} \in \mathbb{Z}} \frac{\gamma s^q}{1 + |\ell_s|^\mu_1|s|^\mu_2+q} \frac{1}{1 + |\ell_{-s}|\mu_1|s|^\mu_2+q} \leq c(\mu_1) \gamma \sum_{s>0} \frac{s^q}{1 + |\ell_s|^\mu_1|s|^\mu_2+q} \frac{1}{1 + |\ell_{-s}|\mu_1|s|^\mu_2+q} \]

Now since

\[ \sum_{h=1}^{\infty} \frac{1}{1 + |h|^\mu_1|n|^\mu_2+q} \leq \sum_{h=1}^{\infty} \frac{1}{h^\mu_1|n|^\mu_2+q} \leq \frac{c(\mu_1)}{|n|^\mu_2+q} \]

we have

\[ \sum_{h \in \mathbb{Z}} \frac{1}{1 + |h|^\mu_1|n|^\mu_2+q} \leq 1 + \frac{2c(\mu_1)}{|n|^\mu_2+q} \]

Then we have

\[ \sum_{\ell_s \ell_{-s} \in \mathbb{Z}} \frac{1}{1 + |\ell_s|^\mu_1|s|^\mu_2+q} \frac{1}{1 + |\ell_{-s}|\mu_1|s|^\mu_2+q} \leq \frac{c_1(\mu_1)}{|s|^\mu_2+q} \]

and consequently (C.24) is bounded by

\[ c_2(\mu_1) \gamma \sum_{s>0} |s|^b \leq c_3(\mu_1, \mu_2) \gamma. \]

Regarding the third line in (C.23), we note that for all \( n \) we have

\[ \sum_{\ell_s \ell_{-s} \in \mathbb{Z}} \frac{1}{1 + |\ell_s|^\mu_1|n|^\mu_2+q} \frac{1}{1 + |\ell_{-s}|\mu_1|n|^\mu_2+q} \leq \left(1 + 2 \frac{c(\mu_1)}{|n|^\mu_2+q}\right)^2. \]
Hence
\[
\sum_{\ell: s(\ell) = s, \quad S(\ell) = S} 1 \left( 1 + \frac{1}{|\ell_0|^{\mu_1}} \right) \prod_{s \leq n \leq S} \frac{1}{\left( 1 + \frac{1}{|\ell_n|^{\mu_1} |n|^{\mu_2 + q}} \right) \left( 1 + \frac{1}{|\ell_{-n}|^{\mu_1} |n|^{\mu_2 + q}} \right)}
\]
\[
= \sum_{\ell_0 \in \mathbb{Z}} 1 \left( 1 + \frac{1}{|\ell_0|^{\mu_1}} \right) \times \sum_{\ell_s, \ell_{-s} \in \mathbb{Z}} \frac{1}{\left( 1 + \frac{1}{|\ell_s|^{\mu_1} |S|^{\mu_2 + q}} \right) \left( 1 + \frac{1}{|\ell_{-s}|^{\mu_1} |S|^{\mu_2 + q}} \right)}
\]
\[
\times \prod_{s \leq \ell \leq S} \sum_{\ell_s, \ell_{-s} \in \mathbb{Z}} \frac{1}{\left( 1 + \frac{1}{|\ell_n|^{\mu_1} |n|^{\mu_2 + q}} \right) \left( 1 + \frac{1}{|\ell_{-n}|^{\mu_1} |n|^{\mu_2 + q}} \right)}
\]
\[
\leq \frac{C_4(\mu_1)}{S^{\mu_2 + q} S^{\mu_2 + q}} \prod_{s \leq n \leq S} \left( 1 + 2 \frac{c(\mu_1)}{|n|^{\mu_2 + q}} \right)^2
\]
\[
\leq \frac{C_5(\mu_1, \mu_2)}{S^{\mu_2 + q} S^{\mu_2 + q}} \exp \left( \sum_{n \geq 1} \ln \left( 1 + 2 \frac{c(\mu_1)}{|n|^{\mu_2 + q}} \right)^2 \right)
\]

Then, multiplying by \( \gamma s^q \) and taking the \( \sum_{0 < s < S} \), we have that also (C.25) is bounded by some constant \( C_{\text{neu}}(\mu_1, \mu_2) \gamma \).

\[\square\]

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