Spatiotemporal bifurcations in plasma drift-waves

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Abstract

Experimental data from an experiment on drift-waves in plasma is presented. The experiment provides a space-time diagnostic and has a control parameter that permits the study of the transition from a stable plasma to a turbulent plasma. The biorthogonal decomposition is used to analyse the data. We introduce the notion of complex modulation for two-dimensional systems. We decompose the real physical system into complex modulated monochromatic travelling waves and give a simple model describing the speed doubling observed in the data as the control parameter increases.

Key words: turbulence, drift-waves, bifurcation, PACS: 52.35.R, 52.35.K, 47.20.K.

1 introduction

The stability of spatially extended dynamical systems and the transition towards spatio-temporal chaos is still a wide open problem. Considerable progress has been made recently by models for pattern formation [1] and advanced three-wave interaction schemes [2,3]. The plasma turbulence phenomenon [4] is considered to be one possible application to the concept of spatio-temporal chaos. In particular, drift wave turbulence is of special interest for the understanding of anomalous transport in magnetically confined plasmas [5]. For the investigation of the transition from a stable plasma to drift wave turbulence, the study of nonlinear model descriptions is unambiguous[6].
In the present paper, we present a simple dynamical model for spatio-temporal bifurcations towards chaos. It is obtained from the numerical analysis of new experimental observations of the space-time structure of current-driven drift waves in cylindrical geometry.

In Section 2, drift waves are briefly introduced. The experimental arrangement as well as the diagnostic tools for the measurement of the spatio-temporal structure of regular and turbulent drift waves are described. For the present investigation, three paradigmatic data sets are considered for three different values of the control parameter: one single propagating mode, nonlinear interaction of drift modes, and fully developed turbulence. These different dynamical states result after successive bifurcations caused by an increase of an appropriately chosen control parameter. The powerful tool of biorthogonal decomposition [7,8] is used to analyse the experimental data. The results of this analysis provide the basis for the model description. In Section 3, the complex modulations of two-dimensional spatio-temporal systems are defined and the importance of modulated monochromatic travelling waves (modulated MTW) is underlined. The modulated MTWs are identified in Section 4 and characterized in Section 5. A speed doubling phenomenon is observed. A low-dimensional model is provided in Section 6.

2 Experimental investigations

Drift waves are universal instabilities of magnetically confined plasmas. In the present study we consider a cylindrical plasma column with a constant axial homogenous magnetic field $\vec{B} = B\hat{z}$ and a radial electron density profile $n_e(r)$. A fluid description [9] of the plasma shows that the electrons drift in the azimuthal direction $\vec{\theta}$ with diamagnetic drift velocity

$$\vec{v}_{\text{dia}} = -\frac{k_B T_e}{eB} L^{-1} \vec{\theta}$$

where $L^{-1} = \frac{1}{n_e(r)} \frac{dn_e(r)}{dr}$ describes the inverse gradient length of the density profile. From the thermodynamic point of view, the diamagnetic drift provides a source of free energy for the drift instability [10]. The drift instability propagates as a plasma wave in the azimuthal direction with diamagnetic velocity. The wave number is predominantly in the $\vec{\theta}$-direction, but has a small $\vec{z}$-component to allow the electrons to flow freely along the magnetic field lines. The frequency of the instability is roughly given by $\omega^* = (m/r_0)v_{\text{dia}}$, where $r_0$ is the position of the maximum density gradient and $m$ is the azimuthal mode number. The linear analysis of a cylindrical, weakly ionized plasma has revealed that the growth rate of drift instabilities is strongly enhanced by
electron neutral collisions as well as by an additional electron drift along the $\vec{z}$-axis [11]. Generally, only the onset regime of growing drift instability allows a linear description. In the nonlinear regime, a large variety of new dynamical phenomena become important (for a detailed review see Refs. [5,6]). Of particular interest is the transition from a stable plasma to turbulence, because fluctuation-induced transport plays a decisive role in the plasma edge physics of fusion devices [12,13].

The experimental investigations are carried out in the central section of a large magnetized triple plasma device. A schematic drawing is shown in Fig. 1. A homogenous quiet plasma is produced in two independent multidipole-confined discharge chambers [14]. For the present studies, argon is used as the filling gas and the degree of ionization is below 0.1%. The magnetized central section is separated from the plasma chambers by two electrically isolated mesh grids (transparency 50%). The plasma produced in the discharge chambers enters the central section by (i) diffusion and (ii) drift. The diffusion process (i) is based on the axial density gradient between the plasma chamber and the central section. The effect of magnetic mapping by the fringing magnetic field lines at the end of the central section is minimized by a particular magnetic compensation technique [15]. An additional electron drift (ii) is superimposed by biasing the grid positively with respect to the plasma potential in the source chamber. This electron drift is known to destabilize drift waves (see above) and is consequently an appropriate control parameter for the dynamics of the system. Experimentally, the axial electron drift is varied by the bias of the injection grid $U_{G1}$ (cf. Fig. 1). The second grid is considered as a plasma loss surface and remains at anode potential, i.e.: $U_{G2} = 0$. In order to be close to the threshold value for the onset of the drift instability, only one plasma chamber is operated and a gradient-driven electron drift is present. The most important discharge parameters and the plasma parameters for the present measurements are summarized in Table 1. The radial profiles of the electron-density, plasma potential, and electron-temperature are plotted in Fig. 2. A more detailed description of the experiment can be found in Ref. [16]. The spatiotemporal structure of regular and turbulent drift waves is investigated by an azimuthally arranged multi-channel Langmuir probe array [16]. Each single Langmuir probe provides the temporal fluctuations of the plasma density. For the present experiment, a circular arrangement of $N = 64$ probes at constant radial and axial position is used. The fixed radial position of the
Table 1
List of discharge parameters and plasma parameters for which the experimental investigation is performed.

| parameter                        | symbol | value               |
|----------------------------------|--------|---------------------|
| magnetic field                   | $B$    | 70mT                |
| neutral gas pressure (argon)     | $p$    | $5.6 \cdot 10^{-2}$Pa |
| discharge voltage                | $U_d$  | 60V                 |
| discharge current                | $I_d$  | 13A                 |
| plasma radius (FWHM)             | $R$    | 0.1m                |
| plasma length                    | $L$    | 1.8m                |
| $e$ drift velocity               | $v_{d,e}$ | $\leq 0.4v_{th,e}$ |
| electron density (center)        | $n_e$  | $5 \cdot 10^{10}$cm$^{-3}$ |
| electron temperature (center)    | $T_e$  | 1.5eV               |
| ion temperature (center)         | $T_i$  | 0.03eV              |

probes, imposed by technical constraints, doesn’t allow to address here the interesting problem of the radial profile of the perturbation. The probe array provides the temporal evolution of the spatial structure of drift waves which propagate mainly in the azimuthal direction. The temporal resolution is given by the maximum sample rate of the acquisition system ($\Delta t = 1\mu s$), and the spatial resolution is given by the azimuthal angle between each two probes ($\Delta x = 2\pi/64$). The data is stored as an $N \times M$-matrix $u_{i,j} = n_e(i\Delta x, j\Delta t)$ where $N = 64$ (space) and $M = 2048$ (time). Note that investigations of spatiotemporal phenomena can be done with one probe by a method based on conditional averaging [17].

In order to study the bifurcation behaviour of drift waves, the data sets of three representative dynamical states are considered: $\mathcal{U}_1$ a single monochromatic drift mode, $\mathcal{U}_2$ drift modes with nonlinear interaction, $\mathcal{U}_3$ turbulent drift waves. The different dynamical states $\mathcal{U}_k$ are recorded successively by increasing the accessible control parameter $U_{G1}$. In Fig. 3, the three data sets $\mathcal{U}_k$ are shown as greyscale plots. Data set $\mathcal{U}_1$ [Fig. 3(a)] is a single propagating mode, data set $\mathcal{U}_2$ [Fig. 3(b)] shows interacting drift modes, and data set $\mathcal{U}_3$ [Fig. 3(c)] is the state of strong drift wave turbulence. These three data sets are the basis for the following analysis and model description. For this purpose the plasma
is considered as a spatiotemporal dynamical system, whose state is described by a function \( u_\epsilon(x, t) \) where \( \epsilon \) represents the complete set of experimental parameters. For the present investigations only one parameter, the grid bias \( U_{G1} \), is varied. The grid bias has been chosen as control parameter because it allows to consider that all other parameters remain in good approximation constant.

Each dynamical state given by the data set \( \mathcal{U}_k \) corresponds to a control parameter value \( \epsilon_k \).

The analysis tool used here is the biorthogonal decomposition (BOD). This tool provides a way to study in the space and time properties simultaneously. We present here just the most important features of the BOD. (For more details see Ref. [7])

Suppose that our system is described by a function \( u(x, t) \) defined on a spatial range \( X \) and a temporal interval \( T \). In the experimental situation, \( X \) is the domain of the azimuthal angle \( x \) in cylindrical coordinates, i.e. \( X = [0, 2\pi] \). The biorthogonal decomposition provides the smallest linear subspace \( \chi(X) \) containing the phase space trajectory \( \xi_t \) (described as time \( t \) runs) defined by :

\[
\forall x \in X, \ \xi_t(x) = u(x, t).
\]  

(2)

The set of all the vectors \( \xi_t \) is the trajectory and the evolution of \( \xi_t \) is the dynamics of the system.

The biorthogonal decomposition also provides the smallest linear subspace \( \chi(T) \) containing the spatial structure \( \xi_x \) (described as the spatial position \( x \) varies) defined by :

\[
\forall t \in T, \ \xi_x(t) = u(x, t).
\]  

(3)

In the present paper, the \( L^2 \) scalar product is used to define \( H(X) \) and \( H(T) \), the Hilbert spaces of the functions of \( x \) defined on \( X \), and the functions of \( t \) defined on \( T \) respectively. The BOD is the spectral analysis of the operator \( U \), which acts from \( H(X) \) into \( H(T) \) :

\[
(U\phi)(t) = \int u(x, t)\phi(x)dx,
\]  

(4)

where \( U \) defines a one–to–one relation between the vectors of \( \chi(X) \) and \( \chi(T) \), the orthogonal complements of the kernels of \( U \) and its adjoint \( U^* \).
If \( U \) is compact, as it is here the BOD decomposes \( u(x, t) \) into temporal and spatial orthogonal modes and \( u(x, t) \) can be written as follows:

\[
    u(x, t) = \sum \alpha_n \psi_n(t) \phi_n(x),
\]

with \( \alpha_1 \geq \alpha_2 \geq \ldots \geq 0 \), and the orthogonality relations \((\phi_n, \phi_m) = \delta_{n,m}\) and \((\psi_n, \psi_m) = \delta_{n,m}\). The \( \phi_n \) are called topos, and the \( \psi_n \) chronos.

3 Modulated travelling waves

In this section we discuss the conditions for a two–dimensional system to be considered a modulated travelling wave. The introduction of modulated travelling waves is of interest as we shall see in the drift wave turbulence, where the decomposition of a high–dimensional system into a set of two–dimensional sub–systems allows us to focus on the dominating properties of the dynamical behaviour. The way in which a monochromatic travelling wave may be deformed in our case of study is now analysed in the following theorem for which we need first to introduce some simple definitions concerning a complex formalism for the corresponding modulations.

Let us now specify what is meant by two–dimensional projections of a N–dimensional system be described by its BOD

\[
    u(x, t) = \sum_{k=1}^{N} a_k \psi_k(t) \phi_k(x). \tag{6}
\]

defined on the space interval \( X = [0, 2\pi] \) and on the time interval \( T = [t_0, t_1] \).

**Definition 1** The **projection** of a system \( u(x, t) \) onto two vectors of index \( m \) and \( n \) is the two dimensional system \( u_{m,n}(x, t) \) described by:

\[
    u_{m,n}(x, t) = a_m \psi_m(t) \phi_m(x) + a_n \psi_n(t) \phi_n(x), \tag{7}
\]

Thus the projection of the dynamics \( \xi_t \) (associated with the spatio–temporal system \( u(x, t) \)) onto the eigenvectors of index \( m \) and \( n \) is simply the projection of \( \xi_t \) onto the two topos \( \phi_m \) and \( \phi_n \), i.e.

\[
    \forall x \in X, \quad \xi_{m,n}^t(x) = u_{m,n}(x, t). \tag{8}
\]

The projection of the spatial structure (also given by the spatio–temporal
Fig. 4. The operator $U$ maps the subspace of $\chi(X)$ spanned by $\phi_1$ and $\phi_2$ into the subspace of $\chi(T)$ spanned by $\psi_1$ and $\psi_2$.

system $u(x,t)$ onto the eigenvectors of index $m$ and $n$ is simply the projection of $\xi_x$ onto the two chronos $\psi_m$ and $\psi_n$.

$$\forall t \in T, \xi_x^{m,n}(t) = u_{m,n}(x,t)$$

The schematic drawing Fig 4 illustrates this projection process of the dynamics and the spatial structure.

**Definition 2** A **modulatrix** $M$ is a pair of complex valued continuous functions $M(x)$ and $N(t)$. $M$ is called the spatial modulatrix, and $N$ the temporal modulatrix.

Note that each complex–valued continuous function is the parametric representation of an arc. Using the vocabulary of complex analysis (see for instance [18]), the arc $\Gamma_X$ is defined as

$$\Gamma_X : M = M(x), 0 \leq x \leq 2\pi.$$  \hspace{1cm} (10)

**Definition 3** A **modulatrix** $M = (M,N)$ is a **continuous phase modulatrix** if $M$ and $N$ can be written as

$$M(x) = A(x)e^{iF(x)}$$ \hspace{1cm} (11)

and

$$N(t) = B(t)e^{iG(t)},$$ \hspace{1cm} (12)

where $A$, $B$, $F$, and $G$ are continuous functions. It will be called a **regular continuous phase modulatrix** if moreover the increase of the argument of $F$, $D_F = F(2\pi) - F(0)$, and the increase of the argument of $G$, $D_G = G(t_1) - G(t_0)$, are both equal to zero.

It is known that a continuous complex–valued function can be written as a product of a continuous modulus function and a continuous argument function if the function is never equal to 0 (see [18]). The modulus function is unique and two different argument functions differ by a constant integral multiple of $2\pi$. The real numbers $D_F = F(2\pi) - F(0)$ and $D_G = G(t_1) - G(t_0)$ are independent of the choice of the argument functions $F$ and $G$ [18].

**Definition 4** The **spatial complexification** of two–dimensional system $u(x,t)$
whose BOD is

\[ u(x,t) = \alpha_1 \psi_1(t) \phi_1(x) + \alpha_2 \psi_2(t) \phi_2(x) \]  

(13)

is

\[ Z(x) = \alpha_1 \phi_1(x) + i\alpha_2 \phi_2(x). \]  

(14)

The corresponding **temporal complexification** is

\[ Y(t) = \alpha_1 \psi_1(t) + i\alpha_2 \psi_2(t). \]  

(15)

Note that the spatial complexification is a representation of the spatial structure in the complex plane, and the temporal complexification is a representation of the dynamics in the complex plane.

If the complexifications are never zero, the argument function can be introduced by the following definitions:

**Definition 5** A spatial complexification \( Z(x) \) is a **phase continuous complexification** if it can be written as:

\[ Z(x) = C(x)e^{iQ(x)}. \]  

(16)

**Definition 6** A temporal complexification \( Y(t) \) is a **phase continuous complexification** if it can be written as:

\[ Y(t) = D(t)e^{iR(t)}. \]  

(17)

Of particular interest is the case in which \( X \) is the circle.

**Definition 7** Let \( u(x,t) \) be a two–dimensional system defined on a circle \( X \). Its spatial complexification \( Z(x) \) is assumed to be always non–zero and continuous on \( X \). Then \( Z(x) \) can be written as \( Z(x) = C(x)e^{iQ(x)} \). As \( Z(0) = Z(2\pi) \) we have \( Q(2\pi) = Q(0) + n2\pi \) where \( n \) is an integer. The number \( n \) is called the **spatial winding number** of the system.

Let us now introduce the notion of modulation of a system:

**Definition 8** Let \( u(x,t) \) be a two–dimensional system which BOD is given by 13. Let \( M = (M(x), N(t)) \) be a (regular) modulatrix. A spatiotemporal (regular) modulation of \( u(x,t) \) is the system \( u'(x,t) \) defined by

\[ u'(x,t) = \alpha_1' \psi_1'(t) \phi_1'(x) + \alpha_2' \psi_2'(t) \phi_2'(x) \]  

(18)
with
\[ \alpha'_1 \phi'_1(x) + i \alpha'_2 \phi'_2(x) = M(x)(\alpha_1 \phi_1(x) + i \alpha_2 \phi_2(x)) \] (19)

and
\[ \alpha'_1 \psi'_1(t) + i \alpha'_2 \psi'_2(t) = N(t)(\alpha_1 \psi_1(t) + i \alpha_2 \psi_2(t)). \] (20)

Note that in general \( \psi'_1, \phi'_1, \psi'_2, \) and \( \phi'_2 \) are not eigenvectors of \( u' \).

Let us investigate the effect of a modulation on the dimension of the system.

**Theorem 9** Let \( u(x,t) \) be a system described by two phase continuous complexifications \( Z(x) = C(x)e^{iQ(x)} \) and \( Y(t) = D(t)e^{iR(t)} \). Let \( M = (M,N) \) be a continuous phase modulatrix with \( M(x) = A(x)e^{iF(x)} \) and \( N(t) = B(t)e^{iG(t)} \). The dimension of the modulated system is reduced to one if
\[ \forall x \in X, \; Q(x) + F(x) = z_1 \] (21)

or
\[ \forall t \in T, \; R(t) + G(t) = z_2 \] (22)

where \( z_1 \) and \( z_2 \) are two complex numbers. Otherwise the dimension of the modulated system remains equal to two.

**Proof.** Let us first apply the spatial modulation. Let \( Z'(x) \) be the spatial complexification of the spatially modulated system \( u'(x,t) \) :
\[ Z'(x) = M(x)Z(x) \] (23)

The dimension of the spatial structure is reduced only if it is embedded in a segment, i.e., if \( Z' \) can be written as \( Z'(x) = C'(x)e^{iq_1} \) where \( q_1 \) is real. In this case,
\[ u'(x,t) = C'(x) \cos q_1 \psi_1(t) + C'(x) \sin q_1 \psi_2(t), \] (24)

it is clear that the new system is one-dimensional, more precisely
\[ u'(x,t) = \alpha'_1 \psi'_1(t) \phi'_1(x) \] (25)
with $\alpha'_1 = ||C'||$, $\phi_1 = C'/\alpha'_1$, and $\psi'_1 = \cos q_1 \psi_1 + \sin q_1 \psi_2$.

However,

$$C'(x)e^{iq_1} = A(x)e^{iF(x)}C(x)e^{iQ(x)} \tag{26}$$

This equation implies $\forall x \in X, Q(x) + F(x) = e^{iq_1}$, where $z_1 = e^{iq_1}$. Conversely, if we assume that $M(x) = A(x)e^{i(q_1 - F(x))}$, then the modulated system becomes one–dimensional. We can show in the same way that the reduction of the dimension of the dynamics after a temporal modulation is equivalent to (22). $\square$

**Theorem 10** A regular spatial modulation does not change the spatial winding number.

**PROOF.** Let $Z(x) = C(x)e^{iQ(x)}$ be a complexification with a winding number $n$. We then have $Q(2\pi) = Q(0) + n2\pi$. Let $M(x) = A(x)e^{iF(x)}$ be a regular spatial modulation. Therefore, because $X$ is here the circle, $F(2\pi) = F(0)$. The spatial complexification of the modulated system is by definition

$$Z'(x) = C'(x)e^{iQ'(x)} \tag{27}$$

with $C'(x) = C(x)A(x)$ and $Q'(x) = Q(x) + F(x)$. Then using the values of $F(x)$ and $Q(x)$ at $x = 0$ and $x = 2\pi$ we get $Q'(2\pi) = Q'(0) + n2\pi$. $\square$

Note that this theorem implies that in order to change the winding number by modulation, a non–regular modulation is required.

**Theorem 11** Let $u(x,t)$ be a two–dimensional system defined on the circle $X$ whose spatial complexification $Z(x)$ and its temporal complexification $Y(t)$ never vanish.

Then there exists one unique regular modulatrix and a pair consisting of an integer $k$ and a real $\omega$ such that $u(x,t)$ is the modulation of a monochromatic travelling wave $u_0(x,t) = \cos(kx)\cos(\omega t) + \sin(kx)\sin(\omega t)$.

**PROOF.** Let us write the spatial complexification $Z(x)$ as

$$Z(x) = C(x)e^{iQ(x)} \tag{28}$$
and the temporal complexification as

\[ Y(t) = D(t)e^{iR(t)} \]  

(29)

We set

\[ k = \frac{1}{2\pi}D_Q \]  

(30)

and

\[ \omega = \frac{1}{t_1 - t_0}D_R , \]  

(31)

where \( D_Q = Q(2\pi) - Q(0) \) and \( D_R = R(t_1) - R(t_0) \) are the increases of the arguments of \( Z \) and \( Y \) respectively. The numbers \( k \) and \( \omega \) are unique, because \( D_Q \) and \( D_R \) are unique. Because \( X \) is a circle, the winding number \( k \) is an integer.

Let us define :

\[ M(x) = C(x)e^{i(Q(x)-kx)} \]  

(32)

and

\[ N(t) = D(t)e^{i[R(t)-\omega t]} \]  

(33)

The modulatrix \( \mathcal{M} = (M, N) \) is regular and \( u(x, t) \) is the modulation of a system \( u_0(x, t) \) by \( \mathcal{M} \). The spatial complexification and the temporal complexification of \( u_0(x, t) \) are respectively \( Z_0(x) = e^{ikx} \) and \( Y_0(t) = e^{i\omega t} \). The system \( u_0(x, t) \) is thus of the form

\[ u_0(x, t) = \cos(kx)\cos(\omega t) + \sin(kx)\sin(\omega t). \]  

(34)

The functions \( Q(x) \) and \( R(t) \) are defined modulo \( 2\pi \), so \( C(x) \) and \( D(t) \) are unique. Thus the functions \( M \) and \( N \) are unique. \( \square \)

This theorem provides a way to consider a two–dimensional experimental system as a spatio–temporal modulation of a monochromatic travelling wave. In particular, a continuous deformation of a wave is described by a modulatrix \( \mathcal{M}_\epsilon = (M_\epsilon, N_\epsilon) \) that depends on the control parameter \( \epsilon \). In the case where \( X \) is the circle, we can define the winding number, and the jump in the winding
Fig. 5. Plot of the weights for the data in y-log scale (left) and plot of the ratios $a_{n+1}/a_n$ with respect to $n$ (right). (a) data $U_1$ (b) data $U_2$ (c) data $U_3$. They exhibit the degeneracy of the weights which may be associated to a spatio-temporal symmetry.

number that is observed for a certain value of the control parameter will correspond to a bifurcation. Note the analogy of the complexifications with the loops in the order parameter space in the study of the topological defects [19].

Our complex modulation is a generalisation of the real modulation introduced in [20]. The modulations introduced in [20] are not appropriate for the description of the deformations of a function $\phi_1(x)$ (resp. $\psi_1(t)$) that has a zero which shifts as $\epsilon$ varies. Instead, we can describe the deformation of such a function provided that we find a complementary function $\phi_2(x)$ (resp. $\psi_2(t)$) that never vanishes for the value of $x$ (resp. $t$) where $\phi_1(x)$ (resp. $\psi_1(t)$) has a zero. Note that in the absence of resonances, as in the case of real modulatrix $M = (M, N)$, the new eigenvectors are the modulation of the non perturbed system, as in Ref. [20].

4 Identifying the modulated monochromatic travelling waves

The aim of this section is to simplify the description of the dynamical behaviour of the present system by the identification of the two-dimensional structures existing in the system for different values of the control parameter $\epsilon$. In order to identify two-dimensional subsystems whose spatial and temporal complexification never vanishes, the idea that this system is a deformed monochromatic travelling wave (MTW) is used. More precisely, a MTW has two basic properties: (i) the degeneracy of the weights and (ii) the Fourier transform of the chronos and topos are delta functions.

We show in Fig. 5 the plot of the weight on a logarithmic scale. In the analysis of the plot of weights, we have to look for pairs of eigenvalues that are degenerated. The degeneracy of pairs is obvious only for the first two eigenvalues of each data set. However, if the ratio $a_n/a_{n+1}$ is plotted versus $n$, the degeneracy of all weights can be quantified (see Fig. 5).

We are thus looking for values of $n$ such that $a_n/a_{n+1}$ is close to unity. This plot provides a way to couple eigenvalues: two eigenvalues $a_n$ and $a_{n+1}$ are considered to be coupled if the ratio $a_n/a_{n+1}$ forms a local maximum in the plot ratios. Following the previous rule, we obtain the pairs listed in Table. 2 that we call structures.

Notice that, considering only the weight distribution is not sometimes suffi-
Table 2
Pairs obtained after the analysis of the weights

| data set | 1–2 | 3–4 | – | 6–7 | 9–10 | 12–13 | 14–15 | 16–17 |
|----------|-----|-----|---|-----|------|-------|-------|-------|
| U₁       |     |     |   |     |      |       |       |       |
| U₂       |     |     |   |     |      |       |       |       |
| U₃       |     |     |   |     |      |       |       |       |

Fig. 6. Modulus squared Fourier transforms of the chronos. The three columns correspond to the three data sets. The x-axis is the frequency. The frequency unit is $d\omega = 2048 \times 10^{-6} s^{-1}$. The y axis is labelled by the eigenvector index.

Fig. 7. Modulus squared Fourier transforms of the topos. The three columns correspond to the three data sets. The x-axis is the spatial frequency. The frequency unit is the spatial loop. The y axis is labelled by the eigenvector index.

cient to decide the pairing. Such is, for instance, the case in data $U₃$ for the $aₙ$, $n = 3, 4, 5$. A careful look to the corresponding chronos and topos, specially by considering their Fourier transform, will give, in this case an unambiguous pairing.

The next step in the identification of modulated MTW is to consider pairs of functions that have the same spatial (temporal) frequencies. An appropriate way to detect such modulated sine and cosine functions is to apply a Fourier transform to the eigenfunctions. The plot of the square modulus of the Fourier transforms of the chronos and topos are represented in Fig. 6 and Fig. 7.

A single–peaked spectrum is found for the first two chronos and topos of each data set. Pairs of eigenvectors are found by inspection: each two Fourier spectra with almost or exactly the same structure indicates the presence of a pair. Using the conditions on the Fourier transform, Table 2 can be improved. Table 3 give the new list of pairs. The pairs are labelled $w_k$, where $k$ is the index of the wave associated with the pair of eigenvectors.

Table 3
Pairs obtained after the analysis of the Fourier spectrum of the eigenvectors. The modulated MTW associated with those pairs are labeled $w_k$, where $k$ is the index of the wave.

| data set | $w₁$ | $w₂$ | $w₃$ | $w₄$ | $w₅$ | $w₆$ | $w₇$ | $w₈$ |
|----------|------|------|------|------|------|------|------|------|
| U₁       | 1–2  | 3–4  | 5–6  | 7–8  | 9–10 | 12–13 | 14–15 | 16–17 |
| U₂       | 1–2  | 3–4  | 5–6  | 7–8  | 9–10 | 12–13 | 14–15 | 16–17 |
| U₃       | 1–2  | 3–4  | 5–6  | 7–8  | 9–10 | 12–13 | 14–15 | 16–17 |

Note that the Eigenvector 11 for the sets $U₁$ and $U₂$ and Eigenvector 7 for the set $U₃$ do not belong to a pair.
Fig. 8. Spatial structures associated with the first eight modulated travelling waves. a) Data set $\mathcal{U}_1$, b) Data set $\mathcal{U}_2$, c) Data set $\mathcal{U}_3$. Each square subfigure is labelled in the corner by the index of the eigenvectors spanning the structure.

Note also that the degeneracy of the weights $a_6$ and $a_7$ doesn’t correspond to a modulated travelling wave. This is because the energy of the eigenfunction 7 which is a global oscillation of the plasma without any propagation is close to the energy of the third travelling wave.

5 Organization of the different modulated MTWs

In the previous section, two-dimensional subsystems which are modulated travelling waves were identified. In this section, these waves are studied with respect to both the eigenvector indexes and the control parameter.

First the spatial and temporal frequencies $k$ and $\omega$ are determined. Actually, Theorem 11 provides an explicit way to compute $k$ and $\omega$. However, this direct approach is cumbersome in practice. Therefore the winding number $k$ is obtained in a different manner: The first eight spatial structures associated with the first eight modulated travelling waves are shown in Fig. 8. The winding number becomes easier to inspect in the plot of the spatial structures (as in the case of the Fourier transform) as the control parameter becomes larger. This shows that the spatial modulation becomes more regular as $\epsilon$ increases.

Fig. 8 shows that the winding number for the data set $\mathcal{U}_1$ is well defined only for the first wave $w_1$ (The labelling is defined in the Table 3). For the data set $\mathcal{U}_2$, the spatial winding number is defined for the three first waves, and for the data set $\mathcal{U}_3$ it is well defined for all eight waves studied. The winding numbers found are shown in the Table 4. The temporal frequencies $\omega$ associated with these waves are determined from the Fourier spectra (Fig. 6). A broad Fourier spectra corresponds to the presence of zeros in the temporal complexifications. The $\omega$ is well-defined from the Fourier spectra only for the first wave of the data set $\mathcal{U}_1$ and $\mathcal{U}_2$, and for the first three waves for the data set $\mathcal{U}_3$.

The phase speed of a modulated wave is defined as the ratio of the temporal frequency $\omega$ to the spatial frequency $k$ of the associated unmodulated monochromatic wave. Considering the first waves given in Tab. 4 it is noted that a speed doubling occurs when the control parameter $\epsilon$ increases from the value that belongs to the data set $\mathcal{U}_2$ to that of the data set $\mathcal{U}_3$. Indeed, the phase speed associated with the waves $w_1$, $w_2$, and $w_3$ in the data set $\mathcal{U}_3$ is very close to twice the speed of the first travelling wave of the data $\mathcal{U}_2$. In the next section, a simple model for this spatio-temporal bifurcation is discussed.
Table 4
Spatial winding numbers $k$ and temporal frequencies $\omega$ associated with the first eight waves. The wave number $i$ is labelled wave $w_i$.

|      | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ |
|------|-------|-------|-------|-------|-------|-------|-------|-------|
| data set $U_1$ | $k = 2$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $\omega = 20$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| data set $U_2$ | $k = 2$ | $k = 3$ | $k = 3$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $\omega = 22$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| data set $U_3$ | $k = 2$ | $k = 1$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ |
| $\omega = 44$ | $\omega = 25$ | $\omega = 63$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |

Fig. 9. Graylevel plots of the two-dimensional structures. Each column is associated with a data set. For each column, the letters a) . . . h) indicate the wave $w_1, \ldots, w_8$ that is plotted. For each figure, the $x$–axis is time and the $y$–axis space.

Another way to study the effect of the spatial modulation is to plot a graylevel plot of the two dimensional restriction $u_{m,n}(x,t)$ associated with the modulated wave. The graylevel plots for the first eight waves of each data set is presented in Fig. 9.

In those plots a phase spatial modulation implies a distortion of the wave front, this distortion depending only on $x$, and an amplitude spatial modulation corresponding to global crashes in the amplitude of the waves, those crashes having the same intensity for fixed positions. In the graylevel plot, the phase modulation is easier to observe than in the probe structure, where a phase modulation implies only a non-uniform spacing of the vectors $\xi_x$.

The graylevel plots show that in the data sets $U_1$ and $U_2$, the first wave has a phase defect that locally makes the wave front more vertical. The other waves are strongly phase modulated with strong tearing of the wave fronts. However, the wave fronts of the waves in the data set $U_2$ are more regular than in the data set $U_1$. In the data set $U_3$, all the strong phase defects have disappeared or been reduced.

In the gray-level plots the strong amplitude modulations can be observed as well. The global crashes of the amplitude of the wave correspond to an amplitude modulation close to zero. This amplitude modulation is itself chaotic.

The data set $U_3$ is thus a state where chaos is essentially present in time. The spatial structures are, on the contrary, fairly regular. The temporal chaos is then due to a chaotic modulation acting on a non–chaotic spatio–temporal structure, i.e. a travelling wave. Note that, furthermore, in this case we are
faced to an homogeneous turbulence [21] as it is seen in Fig. 6. Also note that the chaotic modulation of monochromatic waves has a close relationship with the three–wave interaction model of the drift wave instability [6]. However, it one may ask if the restriction to three waves is pertinent here. Indeed, our study reveals a set of at least eight waves relevant to the data set $U_3$.

It is easier to study the spatial amplitude modulation in the probe structure plots [Fig. 8]. The modulation of the first wave increases with $\epsilon$. In contrast, the next waves become more regular as $\epsilon$ increases. The spatial structure of the data set $U_3$ coincides with a large scale of eigenvectors ($m = 3, \ldots, 19$).

To study these large scale phenomena, the plot of the weights is considered [Fig. 5]. Neglecting the pairwise degeneracy, the weights $a_n$ decrease exponentially with $n$ in well defined regions. In particular, in the distribution of the weights for the data set $U_3$, the boundaries of such a region are given by the indices $n_1 = 7$ and $n_2 = 20$. These boundaries correspond to a pronounced broadening of the Fourier spectrum in the low–frequency regime [cf. Fig. 6]. In this region, the spatial structures have a well defined winding number as shown in Fig. 8. Note that the spectrum of the chronos is bounded by a frequency near 60 in the data set $U_1$ and $U_2$. In the data set $U_3$ this bound has been shifted to 250. Even in the eigenvectors with a broad spectrum, i.e. those with an index greater than 23, the frequencies higher than 250 are nonexistent. However, the spatial frequencies are not bounded and increase with the index of the eigenvector.

**Remark 12** Even if the radial dependency of the electric field is not directly accessible to the present diagnostic, we may wonder that such effects is present giving rise to a collective rotation of the whole plasma column and therefore to a shift in the (Fourier) dispersion relation for the waves. This effect seems present in the frequencies reported in Table 4, for the data $U_3$. However, since we restrict ourselves to each two dimensional structures, this fact would have no effect in our analysis. It simply changes the position of $\chi(T)$ in the space of the functions of time $H(X)$ by a global rotation. We thank one of the referees to have pointed us this question.

6 Model for the speed doubling

In the previous section, it was noted that a speed doubling occurs when the control parameter increases. The data set $U_2$ corresponded to a state before this bifurcation and the data set $U_3$ corresponded to a state after the bifurcation. In this section we give a simple model for this speed doubling. Our model is built with two two–dimensional structures.
Fig. 10. A spatial modulation can imply a topological change in the spatial structure, which corresponds to a change of the speed of the modulated travelling wave. The spatial structures, the trajectories, and a sketch of the crest of the wave are plotted for four values of the control parameter.

Fig. 11. Model for the speed doubling. The gray level plot of the modulated MTW is shown for four values of $\epsilon$. The corresponding spatial structures are shown in the right column.

We first describe one of the two structures. The figure Fig. 10 shows how a spatial modulation can change the winding number and the speed of the eigenstructure.

The phase speed (cf. Section 5) is defined as a ratio. In order to define the states to model uniquely, the speed alone is not sufficient. The model structure describes an evolution from a state described by the function $u_{-1}(x,t) = \cos(2x)\cos(\omega t) - \sin(2x)\sin(\omega t)$ to a state described by the function $u_1(x,t) = \cos(x)\cos(\omega t) + \sin(x)\sin(\omega t)$.

Let us consider a temporal complexification which is independant on $\epsilon$ defined by

$$Y_0(t) = e^{i\omega t}. \quad (35)$$

and the corresponding spatial complexification by

$$Z_\epsilon(x) = (1 - \epsilon)e^{i2x} + (1 + \epsilon)e^{ix}. \quad (36)$$

These spatial and temporal complexifications correspond for $\epsilon = -1$ to the complexifications of $u_{-1}(x,t)$ and for $\epsilon = 1$ to the complexification of $u_1(x,t)$. Fig. 11 shows the plots for four values of the control parameter $\epsilon$, the spatial complexification $Z_\epsilon(x)$ in the right column, and the gray level representation of the two dimensionnal system $u_\epsilon(x,t)$.

The spatial complexification is never equal to zero except for the value $\epsilon = 0$. For this value, the spatial complexification vanishes for $x = \pi$. For negative $\epsilon$ the winding number is 2, and for positive $\epsilon$ it is 1. However, for $\epsilon = 0$, the winding number is not defined. A bifurcation in the winding number thus occurs.

A regular spatial modulation is just defined (i) before the bifurcation by dividing the spatial complexification $Z_\epsilon(x)$ by $Z_{-1}(x)$ (ii) after the bifurcation by dividing the spatial complexification $Z_\epsilon(x)$ by $Z_1(x)$. The spatial modulation
before the bifurcation $M_b^b(x)$ is defined by

$$M_b^b(x) = (1 - \epsilon) + (1 + \epsilon)e^{-ix}.$$  (37)

$M_b^b(x)$ is a regular modulation because $(1 - \epsilon) > (1 + \epsilon)$ when $\epsilon < 0$. The spatial modulation after the bifurcation $M_a^a(x)$ is defined by

$$M_a^a(x) = (1 + \epsilon) + (1 - \epsilon)e^{ix}.$$  (38)

$M_a^a(x)$ is a regular modulation after the bifurcation because $(1 + \epsilon) > (1 - \epsilon)$ when $\epsilon < 0$. The spatial defect which occurs at $x = \pi$ for $\epsilon = 0$ permits the wave front to change its shape as shown in Fig. 11. The two (unnormalized) topos are the real part and the imaginary part of the spatial complexification

$$\phi_1(x) = (1 - \epsilon) \cos(2x) + (1 + \epsilon) \cos(x),$$  (39)

$$\phi_2(x) = (1 - \epsilon) \sin(2x) + (1 + \epsilon) \sin(x).$$  (40)

The chronos are left unchanged

$$\psi_1(t) = \cos(\omega t),$$  (41)

$$\psi_2(x) = \sin(\omega t).$$  (42)

Both the spatial behaviour (see Fig 8, Fig 9)—i.e. the bifurcation of a winding number from the value 2 to the value 1—and the temporal behaviour (see Fig 6, Fig 9)—i.e. a frequency $\omega$ roughly independent of the control parameter $\epsilon$—shows that this structure corresponds to the modulated MTW $w_1$ for the data sets $U_1$ and $U_2$ and $w_2$ for the data sets $U_3$. (The notations $w_k$ were introduced in the Table 3).

In the same way we model the structure corresponding to the second pair in $U_1$ and $U_2$ and the first in $U_3$. Notice that in this case the bifurcation is due to the simultaneous deformation of the temporal modulation and of the corresponding spatial modulation.

Finally, the model is built by gluing the two structures weighted by two corresponding eigenvalues (the relative energies of the two structures). According to the general theory of bifurcations described by the BOD [22], these energies cross at the bifurcation and this is the reason why the energies of the two structures are interchanged when passing from $U_2$ to $U_3$. 

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7 Summary and Conclusions

The BOD of the experimental data of the drift–waves experiment showed the importance of the two–dimensional sub–systems. It has been shown that the two–dimensional structures are complex–modulated monochromatic travelling waves (modulated MTW). A spatial regularity counterpart of a temporal chaos has been discovered in the most turbulent data set. The most important feature of the evolution of the system with the control parameter $\epsilon$, i.e. the speed doubling, has been modeled as well. The model consists of a pair of simple two–dimensional structures that undergo an exchange of energy as the value of the control parameter varies.

A study of new experimental data will be done in the future in order to improve the model and better characterize the bifurcation. The regular behaviour also needs to be better understood. Furthermore, the connection between the modulated MTW and the waves–interactions models for the drift–waves will be investigate in the future.

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