Pluricanonical Periods over Compact Riemann Surfaces of Genus at least 2

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Abstract
This article attempts to generalize Riemann’s bilinear relations on compact Riemann surface of genus \( \geq 2 \).

Contents

1 Introduction 1
2 Bol’s Observation 2
3 Decomposition of Cohomology Group 3
4 Ideas in Eichler’s explicit proof 4
5 Generalization of Riemann Relations 6
6 Period Relations among different tensor powers 10

1 Introduction
This article is an attempt to generalize Riemann’s bilinear relations on compact Riemann surface of genus \( \geq 2 \), which may lead to new structures in the theory of hyperbolic Riemann surfaces. No significant result is obtained, the article serves to bring the readers’ attention to the observation made in [Bol], and some easy consequences.
2 Bol’s Observation

Throughout this article we shall use $X$ to denote a compact Riemann surface of genus $g \geq 2$. By the Uniformization Theorem,

$$X = \mathcal{H}/\Gamma$$

where $\mathcal{H}$ is the upper half plane in $\mathbb{C}$ and $\Gamma$ is a discrete subgroup of $\text{Aut}(\mathcal{H}) = \mathbb{P}\text{SL}(2, \mathbb{R})$. Let $K$ be the canonical line bundle of $X$. Let $K^m$ and $K^{1-m}$ be the $m$-th and $(1-m)$-th tensor power of $K$ respectively. We first observe that the following map from the global sections of $K^m$ to $K^{1-m}$ is well-defined:

$$\begin{align*}
H^0(X, K^m) &\overset{d_{1-2m}}{\longrightarrow} H^0(X, K^{1-m}) \\
f(z)(dz)^m &\longmapsto f^{(1-2m)}(z)(dz)^{1-m}
\end{align*}$$

To verify our claim, we have to show that if $f(z)(dz)^m$ is invariant under the action of $\Gamma$, then $f^{(1-2m)}(z)(dz)^{1-m}$ should also be invariant under $\Gamma$. Indeed, let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any element of $\Gamma$, we have

$$f(Az)d(Az)^m = f(z)(dz)^m$$

implies

$$f(Az) = f(z)(cz + d)^{2m}$$

and so by the Cauchy integral formula,

$$f^{(1-2m)}(Az) = \frac{(1-2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r(Az)} \frac{f(\zeta)}{(\zeta - Az)^{2-2m}} d\zeta$$

$$= \frac{(1-2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r(z)} \frac{f(A\xi)}{(A\xi - Az)^{2-2m}} d(A\xi)$$

$$= \frac{(1-2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r(z)} \frac{f(\xi)(c\xi + d)^{2m}}{(A\xi - Az)^{2-2m}} d(A\xi)$$

Notice that

$$A\xi - Az = \frac{\xi - z}{(c\xi + d)(cz + d)}$$

2
therefore

\[
\frac{(1 - 2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r(z)} \frac{f(\xi)(\xi + d)^{2m}}{(\xi - z)^{2-2m}} d(\xi A) = 0
\]

It is proved in Gunning's paper that if we differentiate an \( m \)-canonical section for \( k \)-times, then we would get back another pluricanonical section if and only if \( k = 1 - 2m \).

## 3 Decomposition of Cohomology Group

**Proposition 3.1.** The following sequence of sheaf maps is exact:

\[
0 \longrightarrow \text{Ker}(d^{1-2m}) \longrightarrow \mathcal{O}(X, K^m) \xrightarrow{d^{1-2m}} \mathcal{O}(X, K^{1-m}) \longrightarrow 0
\]

where \( \mathcal{O} \) means sheaf of germs of local holomorphic sections.

**Proof.** The only thing we need to show is that \( d^{1-2m} \) is locally surjective. Fix any \( \tau_0 \in \mathcal{H} \). For any \( \varphi(\tau)(d\tau)^{1-m} \in \mathcal{O}(X, K^{1-m}) \), we define

\[
\Phi(\tau) = \frac{1}{(-2m)!} \int_{\tau_0}^{\tau} (\tau - \sigma)^{-2m} \varphi(\sigma) d\sigma
\]

Then it can be verified inductively that

\[
\Phi^{(1-2m)}(\tau) = \varphi(\tau)
\]

So the local \( m \)-canonical form \( \Phi(\tau)(d\tau)^m \) would be a preimage. \( \square \)

From the short exact sequence above, we get the following long exact sequence:

\[
\cdots \rightarrow H^0(X, K^m) \rightarrow H^0(X, K^{1-m}) \rightarrow H^1(X, \text{Ker}(d^{1-2m})) \rightarrow H^1(X, K^m) \rightarrow H^1(X, K^{1-m}) \rightarrow \cdots
\]

Note that we have the following 2 facts:
1. $H^0(X, K^m) = 0$ for $m < 0$ because in this case $K^m$ is a negative line bundle.

2. $H^0(X, K^m) \cong H^1(X, K^{1-m})$ and $H^1(X, K^m) \cong H^0(X, K^{1-m})$ by Serre Duality Theorem.

Hence

$$H^1(X, \text{Ker}(d^{1-2m})) \cong H^0(X, K^{1-m}) \oplus H^1(X, K^m)$$

$$\cong H^0(X, K^{1-m}) \oplus H^0(X, K^{1-m})$$

Notice that the above is a generalization of Hodge decomposition, i.e.,

$$H^1(X, \mathbb{C}) \cong H^0(X, K) \oplus H^1(X, \mathcal{O})$$

$$\cong H^0(X, K) \oplus H^0(X, K)$$

4 Ideas in Eichler’s explicit proof

The decomposition in the previous section is proved in [Eichler] by explicit construction. The following was introduced in his work:

For any holomorphic function $\varphi$ on $\mathcal{H}$ and $n \in \mathbb{Z}$, define

$$(\varphi[A]^{-n})(z) = \varphi(Az)(cz + d)^{-n}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

**Proposition 4.1.** Let $\varphi(z)(dz)^{1-m}$ be in $H^0(X, K^{1-m})$ and define

$$\Phi(\tau) = \int_{\tau_0}^{\tau} (\tau - \sigma)^{-2m} \varphi(\sigma) d\sigma$$

(as constructed in the proof of Proposition (3.1).) Then $\Phi^{(1-2m)}(z) = \varphi(z)$. We claim that $(\Phi[A]^{-2m} - \Phi)$ is a polynomial in $z$ of degree $\leq -2m$.

**Proof.** By the Cauchy integral formula, we have

$$\varphi(z) = \frac{(1 - 2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r(z)} \frac{\Phi(\zeta)}{(\zeta - z)^{2-2m}} d\zeta$$
and so
\[ \varphi(Az) = \frac{(1 - 2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r(z)} \frac{\Phi(\zeta)}{(\zeta - Az)^{2-2m}} d\zeta \]
\[ = \frac{(1 - 2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r'(A^{-1}z)} \frac{\Phi(A\zeta)}{(A\zeta - Az)^{2-2m}} d(A\zeta) \]
\[ = \frac{(1 - 2m)!}{2\pi \sqrt{-1}} \int_{\partial B_r'(A^{-1}z)} \frac{\Phi(A\zeta)}{(\zeta - z)^{2-2m}} (c\xi + d)^{-2m} d\xi \cdot (cz + d)^{2-2m} \]
\[ = (\Phi[A]^{-2m})(1-2m)(z) \cdot (cz + d)^{2-2m} \]

On the other hand, since \( \varphi \in H^0(X, K^{1-m}) \), we have
\[ \varphi(Az) = \varphi(z) \cdot (cz + d)^{2-2m} \]
\[ = \Phi(1-2m)(z) \cdot (cz + d)^{2-2m} \]

Hence \( (\Phi[A]^{-2m} - \Phi)(1-2m)(z) = 0 \) and so it is a polynomial of degree \( \leq -2m \).

We shall denote the polynomial \( (\Phi[A]^{-2m} - \Phi)(z) \) by \( \Omega_A(z) \), and call it the “period” of \( \varphi \) with respect to \( A \). For a fixed \( \varphi \in H^0(X, K^{1-m}) \), we have the following relation among periods with respect to different \( A, B \in \Gamma \).

**Lemma 4.2.** For any \( A, B \in \Gamma \), \( \Omega_{AB} = \Omega_A[B]^{-2m} + \Omega_B \).

**Proof.**
\[ \Omega_{AB}(\tau) = (\Phi[A]^{-2m} - \Phi)(\tau) \]
\[ = \Phi(AB\tau) \cdot (c_{AB\tau} + d_{AB})^{-2m} - \Phi(\tau) \]
\[ = \Phi(AB\tau) \cdot (c_A(B\tau) + d_A)^{-2m} \cdot (c_B\tau + d_B)^{-2m} - \Phi(\tau) \]
\[ = (\Phi[A]^{-2m})(B\tau) \cdot (c_B\tau + d_B)^{-2m} - \Phi(\tau) \]
\[ = (\Phi[A]^{-2m} - \Phi)(B\tau) \cdot (c_B\tau + d_B)^{-2m} + \Phi(B\tau) \cdot (c_B\tau + d_B)^{-2m} - \Phi(\tau) \]
\[ = \Omega_A(B\tau) \cdot (c_B\tau + d_B)^{-2m} + \Omega_B(\tau) \]
\[ = (\Omega_A[B]^{-2m})(\tau) + \Omega_B(\tau) \]

If we interpret \( \Omega \) as a map from \( \Gamma \) to the set of polynomials in \( \tau \) with degree \( \leq -2m \) (denote this set by \( M \)), then by the above lemma, \( \Omega \) satisfies the cocycle rule for the group cohomology of \( C^\bullet(\Gamma, M) \) and hence defines a class in \( H^1(\Gamma, M) \).
5 Generalization of Riemann Relations

We begin this section by first fixing some notations. Let \( \varphi \) and \( \psi \) be in \( H^0(X,K) \), and let \( \gamma_1, \gamma_{1+g}, \gamma_{1-1}, \gamma_{1+g}, \ldots, \gamma_i, \gamma_{i+g}, \gamma_{i-1}, \gamma_{i+g}, \ldots, \gamma_g, \gamma_{2g}, \gamma_{g-1}, \gamma_{2g} \) be the labelling of the sides of a fundamental domain of \( X \) in \( \mathcal{H} \) (which is a \( 4g \)-gon \( \Omega \)). We denote \( \int_{\gamma_i} \varphi \) by \( \Pi_i(\varphi) \) for all \( i \), and call it the period of \( \varphi \) with respect to \( \gamma_i \). Similar notations for \( \psi \). Suppose also that \( \gamma_i \) is identified with \( \gamma_{i-1} \) by the element \( \alpha_i \in \pi_1(X) \) and \( \gamma_{i+g} \) is identified with \( \gamma_{i+1} \) by the element \( \beta_i \in \pi_1(X) \). We first recall the following classical result.

Theorem 5.1. (Riemann bilinear relations)

1. \( \sum_{i=1}^g (\Pi_i(\varphi)\Pi_{g+i}(\psi) - \Pi_i(\psi)\Pi_{g+i}(\varphi)) = 0 \)

2. \( \sqrt{-1} \cdot \sum_{i=1}^g \left( \Pi_i(\varphi)\Pi_{g+i}(\varphi) - \Pi_i(\varphi)\Pi_{g+i}(\varphi) \right) > 0 \)

Proof. For part (1), define \( u(\tau) := \int_{\tau_0}^\tau \varphi \) where \( \tau_0 \) is any point in \( \Omega \). Then for any \( P \) on \( \gamma_i \), there corresponds an \( \alpha_i(P) = P' \) on \( \gamma_i^{-1} \). So,

\[
\int_{\gamma_i} u \psi + \int_{\gamma_i^{-1}} u \psi = \int_{\gamma_i} (u - (u + \Pi_{i+g}(\varphi))) \psi = -\Pi_{i+g}(\varphi)\Pi_i(\psi)
\]

Similarly,

\[
\int_{\gamma_{i+g}} u \psi + \int_{\gamma_{i+g}^{-1}} u \psi = \Pi_i(\varphi)\Pi_{i+g}(\psi)
\]

On the other hand,

\[
\int_{\partial \Omega} u \psi = \int_{\Omega} d(u \psi) = 0
\]

since \( u \psi \) is \( d \)-closed. Thus we get part (1) by summing over \( i \). For part (2), consider \( \int_{\partial \Omega} \overline{u} \varphi \) and apply similar trick. \( \square \)
Now we generalise this result for pluricanonical sections \( \varphi \) and \( \psi \) in \( H^0(X, K^{1-m}) \) with \( m < 0 \). We denote \( \gamma_1 \) to be starting from the point \( \tau_1 \) to the point \( \tau_2 \). \( \gamma_{1+g} \) from \( \tau_2 \) to \( \tau_3 \), \( \gamma_{1}^{-1} \) from \( \tau_3 \) to \( \tau_4 \), \( \gamma_{1+g}^{-1} \) from \( \tau_4 \) to \( \tau_5 \). By diagram chasing, we know that

\[
\begin{align*}
\tau_4 &= \alpha_1 \tau_1 \\
\tau_3 &= \beta_1^{-1} \tau_4 = \beta_1^{-1} \alpha_1 \tau_1 \\
\tau_2 &= \alpha_1^{-1} \tau_3 = \alpha_1^{-1} \beta_1^{-1} \alpha_1 \tau_1 \\
\tau_5 &= \beta_1 \tau_2 = \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_1 \tau_1 = \beta_1 \alpha_1^{-1} \beta_1^{-1} \tau_4
\end{align*}
\]

Define

\[
\Phi(\tau) := \frac{1}{(-2m)!} \int_{\tau_1}^{\tau} (\tau - \sigma)^{-2m} \varphi(\sigma) d\sigma
\]

and

\[
\Psi(\tau) := \frac{1}{(-2m)!} \int_{\tau_1}^{\tau} (\tau - \sigma)^{-2m} \psi(\sigma) d\sigma
\]

then \( \Phi^{(1-2m)}(\tau) = \varphi(\tau) \) and \( \Psi^{(1-2m)}(\tau) = \psi(\tau) \). We now investigate the relations between the "periods" of \( \varphi \) and \( \psi \).

Observe that \( \Phi(\tau)\psi(\tau)d\tau \) is a global holomorphic 1-form on \( \mathcal{H} \). Define

\[
I(\varphi, \psi) := \int_{\partial \Omega} \Phi(\tau)\psi(\tau)d\tau = \sum_{i=1}^{9} \left( \int_{\gamma_i} + \int_{\gamma_i^{-1}} + \int_{\gamma_{1+g}} + \int_{\gamma_{1+g}^{-1}} \right) \Phi(\tau)\psi(\tau)d\tau
\]

By Stoke’s theorem, \( I(\varphi, \psi) = 0 \). On the other hand,

\[
\begin{align*}
&\int_{\gamma_1} \Phi(\tau)\psi(\tau)d\tau + \int_{\gamma_1^{-1}} \Phi(\tau)\psi(\tau)d\tau \\
&= \int_{\gamma_1} \Phi(\tau)\psi(\tau)d\tau - \int_{\gamma_1} \Phi(\alpha_1 \tau)\psi(\alpha_1 \tau)d(\alpha_1 \tau) \\
&= \int_{\gamma_1} \Phi(\tau)\psi(\tau)d\tau - \int_{\gamma_1} \Phi(\alpha_1 \tau) \cdot \psi(\tau)(c_{\alpha_1} \tau + d_{\alpha_1})^{2-2m} \cdot (c_{\alpha_1} \tau + d_{\alpha_1})^{-2}d\tau \\
&= \int_{\gamma_1} (\Phi(\tau) - \Phi(\alpha_1 \tau)(c_{\alpha_1} \tau + d_{\alpha_1})^{-2m}) \psi(\tau)d\tau \\
&= - \int_{\gamma_1} \Omega_{\alpha_1}(\tau)\psi(\tau)d\tau
\end{align*}
\]
Similarly,

\[
\int_{\gamma_{1+g}} \Phi(\tau)\psi(\tau)d\tau + \int_{\gamma_{1+g}} \Phi(\tau)\psi(\tau)d\tau = -\int_{\gamma_{1+g}} \Omega_{\beta_1}(\tau)\psi(\tau)d\tau
\]

Since \( \int_{\partial\Omega} \Phi(\tau)\psi(\tau)d\tau = 0 \), if we write explicitly, for \( 1 \leq i \leq g \),

\[
\Omega_{\alpha_i}(\tau) = \sum_{\mu=0}^{-2m} c_{\mu}(\alpha_i) \cdot \tau^\mu \quad ; \quad \Omega_{\beta_i}(\tau) = \sum_{\mu=0}^{-2m} c_{\mu}(\beta_i) \cdot \tau^\mu
\]

then by the above computation for \( i = 1 \), we know that

\[
0 = I(\varphi, \psi) = \sum_{i=1}^{g} \sum_{\mu=0}^{-2m} \left( c_{\mu}(\alpha_i) \cdot \int_{\gamma_i} \tau^\mu \psi(\tau)d\tau + c_{\mu}(\beta_i) \cdot \int_{\gamma_{i+g}} \tau^\mu \psi(\tau)d\tau \right)
\]

To deal with the integrals in the above expression, we have the following

**Proposition 5.2.**

\[
\int_{\gamma_1} \sigma^\mu \psi(\sigma)d\sigma = (-1)^\mu \mu!(-2m - \mu)!(-x_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1))
\]

\[
\int_{\gamma_{1+g}} \sigma^\mu \psi(\sigma)d\sigma = (-1)^\mu \mu!(-2m - \mu)! (x_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1) - x_{-2m-\mu}(\alpha_1^{-1} \beta_1))
\]

where the coefficients \( x_{-2m-\mu}(A) \) come from the “periods” of \( \psi \) with respect to corresponding \( A \in \Gamma \), namely,

\[
\Omega'_A(\tau) := (\Psi[A]^{-2m} - \Psi)(\tau) = \sum_{\mu=0}^{-2m} c_{\mu}'(A) \cdot \tau^\mu
\]

8
Proof. We first give an integral expression for the period:

\[ \Omega'_A(\tau) := \sum_{\mu=0}^{2m} c'_\mu(A) \tau^\mu := (\Psi[A]^{-2m} - \Psi)(\tau) \]

\[ = \frac{1}{(-2m)!} \int_{\tau_1}^{\tau} (A\tau - \sigma)^{-2m} \psi(\sigma) d\sigma \cdot (c\tau + d)^{-2m} - \int_{\tau_1}^{\tau} (\tau - \sigma)^{-2m} \psi(\sigma) d\sigma \]

\[ = \frac{1}{(-2m)!} \int_{A^{-1}\tau_1}^{\tau} (A\tau - A\sigma')^{-2m} \cdot \psi(A\sigma') \cdot d(A\sigma') \cdot (c\tau + d)^{-2m} - \int_{\tau_1}^{\tau} (\tau - \sigma)^{-2m} \psi(\sigma) d\sigma \]

\[ = \frac{1}{(-2m)!} \int_{A^{-1}\tau_1}^{\tau} (\tau - \sigma')^{-2m}(c\sigma' + d)^2(c\tau + d)^2 \cdot \psi(\sigma')(c\sigma' + d)^2 (c\sigma' + d)^{-2} \cdot d\sigma' \]

\[ \cdot (c\tau + d)^{-2m} - \frac{1}{(-2m)!} \int_{\tau_1}^{\tau} (\tau - \sigma)^{-2m} \psi(\sigma) d\sigma \]

If we put \( A = \alpha_1^{-1} \beta_1 \alpha_1 \), then we would get

\[ \Omega'_{\alpha_1^{-1} \beta_1 \alpha_1}(\tau) = \sum_{\mu=0}^{2m} c'_\mu(\alpha_1^{-1} \beta_1 \alpha_1) \tau^\mu = \frac{1}{(-2m)!} \int_{\gamma_1}^{\tau} (\tau - \sigma)^{-2m} \psi(\sigma) d\sigma \]

comparing coefficients of \( \tau \) gives

\[ \int_{\gamma_1} \sigma^\mu \psi(\sigma) d\sigma = (-1)^\mu!(-2m - \mu)!c'_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1) \]

Similarly if we put \( A = \alpha_1^{-1} \beta_1 \alpha_1 \) and \( A = \alpha_1^{-1} \beta_1 \) respectively, we get

\[ \int_{\tau_2} \sigma^\mu \psi(\sigma) d\sigma = (-1)^\mu!(-2m - \mu)!c'_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1) \]

and

\[ \int_{\tau_3} \sigma^\mu \psi(\sigma) d\sigma = (-1)^\mu!(-2m - \mu)!c'_{-2m-\mu}(\alpha_1^{-1} \beta_1) \]

Therefore

\[ \int_{\gamma_1} \sigma^\mu \psi(\sigma) d\sigma = (-1)^\mu!(-2m - \mu)! (c'_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1) - c'_{-2m-\mu}(\alpha_1^{-1} \beta_1)) \]

\[ \square \]
Although we get an expression for the integrals in the case \( i = 1 \), we cannot simply replace 1 by \( i \) to get similar formulas for other \( i \), the reason is that in our definition of \( \Phi \) and \( \Psi \), the coordinate of the universal cover \( \mathcal{H} \) is involved (unlike the case for Riemann bilinear relations), so the expression of \( \Phi \) and \( \Psi \) depends on our choice of the starting point of the integrals, namely, \( \tau_1 \). If one computes the above integrals for different \( i \)'s, the group elements involved would be more complicated. Indeed, by similar technique, we clearly have, for \( 2 \leq i \leq g \), \( 0 \leq \mu \leq -2m \),

\[
\int_{\tau_{i}} \sigma^\mu \psi(\sigma) d\sigma = (-1)^\mu!(-2m - \mu)!(c'_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \alpha_{i-1}^{-1} \beta_{i-1} \alpha_{i-1}^{-1} \beta_{i-1}^{-1} - c'_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \alpha_{i-1}^{-1} \beta_{i-1} \alpha_{i-1}^{-1} \beta_{i-1}^{-1} \beta_i \alpha_i))
\]

and

\[
\int_{\tau_{i}+g} \sigma^\mu \psi(\sigma) d\sigma = (-1)^\mu!(-2m - \mu)!(c'_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \alpha_{i-1}^{-1} \beta_{i-1} \alpha_{i-1}^{-1} \beta_{i-1}^{-1} \alpha_i^-1 \beta_i \alpha_i - c'_{-2m-\mu}(\alpha_1^{-1} \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \alpha_{i-1}^{-1} \beta_{i-1} \alpha_{i-1}^{-1} \beta_{i-1}^{-1} \beta_i))
\]

Having these expressions, we can put them back into the summation formula of \( I(\varphi, \psi) \) to get a relation between coefficients of the periods. We shall not bother ourselves to explicitly write down the conclusion since the expressions involved are complicated.

### 6 Period Relations among different tensor powers

In this section, we are trying to apply the same trick to get some period relations between two different pluricanonical sections \( \varphi \) and \( \psi \), but this time, they come from different tensor powers of \( K \). We let \( \varphi \in H^0(X, K^{1-m}) \) and \( \psi \in H^0(X, K^{1-n}) \) with \( n < m \leq 0 \). We show that although we are using again the same technique (i.e. Stoke’s theorem), we are not getting anything.

Define

\[
\Phi(\tau) := \frac{1}{(-2m)!} \int_{\tau_1}^{\tau} (\tau - \sigma)^{-2m} \varphi(\sigma) d\sigma
\]

and

\[
\Psi(\tau) := \frac{1}{(-2n)!} \int_{\tau_1}^{\tau} (\tau - \sigma)^{-2n} \psi(\sigma) d\sigma
\]
Consider

\[ \int_{\gamma_i} \Phi(\tau) \psi(\tau) d\tau + \int_{\gamma_i^{-1}} \Phi(\tau) \psi(\tau) d\tau \]

\[ = \int_{\gamma_i} \Phi(\tau) \psi(\tau) d\tau - \int_{\gamma_i} \Phi(\alpha_i \tau) \psi(\alpha_i \tau) d(\alpha_i \tau) \]

\[ = \int_{\gamma_i} (\Phi(\tau) - \Phi(\alpha_i \tau)(\gamma_i \tau + d_{\alpha_i})^{-2n}) \psi(\tau) d\tau \]

\[ = \int_{\gamma_i} \left[ (\Phi(\tau) - \Phi(\alpha_i \tau)(\gamma_i \tau + d_{\alpha_i})^{-2m}) (\gamma_i \tau + d_{\alpha_i})^{2m-2n} + \Phi(\tau) \left( 1 - (\gamma_i \tau + d_{\alpha_i})^{2m-2n} \right) \right] \psi(\tau) d\tau \]

\[ = - \int_{\gamma_i} \Omega_{\alpha_i}(\tau)(\gamma_i \tau + d_{\alpha_i})^{2m-2n} \psi(\tau) d\tau + \int_{\gamma_i} \Phi(\tau) \psi(\tau) d\tau \]

\[ + \int_{\gamma_i} \Phi(\alpha_i^{-1} \tau)(\gamma_i \tau + d_{\alpha_i}) \psi(\tau) d\tau \]

\[ = - \int_{\gamma_i} \Omega_{\alpha_i}(\tau)(\gamma_i \tau + d_{\alpha_i})^{2m-2n} \psi(\tau) d\tau + \int_{\gamma_i} \Phi(\tau) \psi(\tau) d\tau \]

\[ + \int_{\gamma_i} \Phi(\alpha_i^{-1} \tau)(\gamma_i \tau + d_{\alpha_i}) \psi(\tau) d\tau \]

\[ = - \int_{\gamma_i} \Omega_{\alpha_i}(\tau)(\gamma_i \tau + d_{\alpha_i})^{2m-2n} \psi(\tau) d\tau + \int_{\gamma_i} \Phi(\tau) \psi(\tau) d\tau \]

\[ + \int_{\gamma_i} \Phi(\alpha_i^{-1} \tau)(\gamma_i \tau + d_{\alpha_i}) \psi(\tau) d\tau \]
Cancelling the repeated terms on both sides gives

\[ \int_{\gamma_i} \Omega_{\alpha_i}(\tau)(c_{\alpha_i} \tau + d_{\alpha_i})^{2m-2n} \psi(\tau) d\tau = \int_{\gamma_{i-1}} \Omega_{\alpha_{i-1}}(\tau) \psi(\tau) d\tau \]

The above formula can be deduced directly from Lemma (4.2).

Again, if we write explicitly

\[ \Omega_{\alpha_i}(\tau)(c_{\alpha_i} \tau + d_{\alpha_i})^{2m-2n} = \sum_{\mu=0}^{-2n} d_{\mu}(\alpha_i) \tau^\mu ; \quad \Omega_{\alpha_{i-1}}(\tau) = \sum_{\nu=0}^{-2m} c_{\nu}(\alpha_{i-1}) \tau^\nu \]

then we have

\[ \sum_{\mu=0}^{-2n} d_{\mu}(\alpha_i) \int_{\gamma_i} \sigma^\mu \psi(\sigma) d\sigma = \sum_{\nu=0}^{-2m} c_{\nu}(\alpha_{i-1}) \int_{\gamma_{i-1}} \sigma^\nu \psi(\sigma) d\sigma \]

To deal with the integrals, by the discussion at the end of the previous section we know that for \(2 \leq i \leq g\), \(0 \leq \mu \leq -2n\),

\[ \int_{\gamma_i} \sigma^\mu \psi(\sigma) d\sigma = (-1)^\mu \mu!(-2n - \mu)! \left( c'_{-2n-\mu}(\alpha_i^{-1} \beta_i \alpha_i \beta_i^{-1} \ldots \alpha_{i-1}^{-1} \beta_i \alpha_{i-1} \beta_i^{-1}) - c'_{-2n-\mu}(\alpha_i^{-1} \beta_i \alpha_i \beta_i^{-1} \ldots \alpha_{i-1}^{-1} \beta_i \alpha_{i-1} \beta_i^{-1}) \right) \]

and similarly

\[ \int_{\gamma_{i-1}} \sigma^\nu \psi(\sigma) d\sigma = (-1)^\mu \mu!(-2n - \mu)! \left( c'_{-2n-\mu}(\alpha_i^{-1} \beta_i \alpha_i \beta_i^{-1} \ldots \alpha_{i-1}^{-1} \beta_i \alpha_{i-1} \beta_i^{-1} \right) - c'_{-2n-\mu}(\alpha_i^{-1} \beta_i \alpha_i \beta_i^{-1} \ldots \alpha_{i-1}^{-1} \beta_i \alpha_{i-1} \beta_i^{-1}) \right) \]

We can apply this trick to get \(2g\) relations corresponding to \(\gamma_1, \ldots, \gamma_g, \gamma_{1+g}, \ldots, \gamma_{g+g}\).
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