Fermion Condensates and the Trivial Vacuum of Light-Cone Quantum Field Theory

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Abstract

We discuss the definition of condensates within light-cone quantum field theory. As the vacuum state in this formulation is trivial, we suggest to abstract vacuum properties from the particle spectrum. The latter can in principle be calculated by solving the eigenvalue problem of the light-cone Hamiltonian. We focus on fermionic condensates which are order parameters of chiral symmetry breaking. As a paradigm identity we use the Gell-Mann-Oakes-Renner relation between the quark condensate and the observable pion mass. We examine the analogues of this relation in the ‘t Hooft and Schwinger model, respectively. A brief discussion of the Nambu-Jona-Lasinio model is added.

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1. In two recent publications [1, 2] the vacuum structure of the 't Hooft model [3], QCD in 1+1 dimensions in the limit of large $N_C$, was analysed in terms of light-cone (LC) wave functions. These had already been obtained in 't Hooft’s original work [3] by solving the associated Bethe-Salpeter equation which is equivalent to diagonalising the LC Hamiltonian. A remarkable result of [1, 2] is the fact that the quark condensate, a quantity indicating a non-trivial vacuum structure, can efficiently be calculated in the framework of LC quantum field theory which is generally believed to have a trivial vacuum [4, 5].

The purpose of this note is to further clarify this apparent contradiction and put the definition of condensates within LC field theory in a broader perspective.

2. Assume that we have a symmetry which is explicitly broken so that the associated current is not conserved,

$$\partial_\mu j^\mu(x) = A(x).$$  

With the help of the corresponding Ward identity [6] one can derive the following formula for an arbitrary operator $B$,

$$\langle 0|\delta B/\delta \alpha|0 \rangle = -i \int d^4x \langle 0|TA(x)B(0)|0 \rangle = -\sum_n \frac{\langle 0|A(0)|n \rangle \langle n|B(0)|0 \rangle}{m_n^2}. \tag{2}$$

Here, $\delta B/\delta \alpha$ denotes the change of $B$ under the symmetry transformation, and in the last step a complete set of states $|n \rangle$, each of mass $m_n$, has been inserted. For the case of chiral symmetry, choosing

$$A = B = 2m\bar{\psi}i\gamma_5\psi,$$  \tag{3}

one finds for a single quark flavour of mass $m$

$$\langle 0|\bar{\psi}\psi|0 \rangle = -m \sum_n \frac{|\langle 0|\bar{\psi}i\gamma_5\psi|n \rangle|^2}{m_n^2}. \tag{4}$$

In Ref.s [1, 2], this expression was used as a definition for the quark condensate in the 't Hooft model. From the above derivation, however, it is clear that (4) holds quite generally. Using the PCAC relation for the axial vector current, $j_5^\mu$,

$$\partial_\mu j_5^\mu(x) = f_\pi m_\pi^2 \pi(x), \tag{5}$$

where $\pi(x)$ is an interpolating pion field and $f_\pi$ the pion decay constant, (3) involves the pion two-point function and is easily evaluated with the result

$$f_\pi^2 m_\pi^2 = -4m \langle 0|\bar{\psi}\psi|0 \rangle. \tag{6}$$
This is the famous Gell-Mann-Oakes-Renner (GOR) relation \[7\] (to lowest order in the quark mass \(m\) \[8\]), which relates the QCD parameters, \(m\), the current quark mass, and the fermion condensate to the observables \(f_{\pi}\) and \(m_{\pi}\). We have written everything in terms of bare quantities since the right-hand side does not change under renormalisation \[9\]. Thus, \(m_{\pi}\) in \(6\) is the physical pion mass.

The GOR relation can of course also be derived from \(4\) if one replaces

\[
\bar{\psi}(x)i\gamma_5\psi(x) = \frac{1}{2m}f_{\pi}m_{\pi}^2 \pi(x) ,
\]

and assumes that, for small \(m_{\pi}\), the sum is saturated by the pion.

In any case, we want to stress the fact that in \(4\) and \(6\) above a vacuum quantity, the condensate, is expressed in terms of the particle spectrum. So, once the spectrum is known, after, say, diagonalising the LC Hamiltonian by one of the various methods on the market \[10\], we can translate back properties of the spectrum into properties of the vacuum. In view of that, we suggest to deemphasize the role of the vacuum, which is natural to the extent that most of its properties are not directly observable. This is particularly true within the LC framework, where the vacuum state seems to decouple completely from the particle states. Similar ideas have been put forward long ago, in the context of chiral symmetry in the (LC) parton model, by Susskind et al. \[11\]: “In this framework the spontaneous symmetry breakdown must be attributed to the properties of the hadron’s wavefunction and not to the vacuum” \[12\]. A related point of view has also been taken more recently in \[13\].

3. Before we pursue the program just outlined we would like to remark that the ‘master equation’ \(2\) cannot be derived by strictly sticking to the LC framework. To this end note that the first term in \(2\) can be written with the help of the charge

\[
Q(x^0) = \int d^3x j^0(x^0, x) ,
\]

the generator of the symmetry, as

\[
\langle 0|\delta B/\delta a|0\rangle = -i\langle 0|\left[ B, Q \right]|0\rangle ,
\]

where the commutator is evaluated at equal time \(x^0 = 0\). This expression cannot be directly translated into the LC language by replacing the ordinary charge \(Q(x^0)\) by the LC or “light-like” charge,

\[
Q(x^+) = \int dx^-d^2x_\perp j^+(x^+, x^- , x_\perp) ,
\]
and evaluating the commutator at equal light-cone time \( x^+ \). The reason for this is a peculiar property of LC charges: they annihilate the vacuum, irrespective of whether they generate a symmetry or not [4, 14], which is in accordance with the triviality of the LC vacuum. Thus, the right-hand side of (3), evaluated on a null-plane, is always zero. Hence, that part of the operator \( \delta B/\delta \alpha \) having a non-vanishing vacuum expectation value (VEV) cannot be obtained by an infinitesimal transformation generated by the light-like charge \( Q \). For example, in the LC sigma model, the relation

\[
\left[ \pi, Q_5 \right] = -i\sigma
\]  

(11)
does only hold for those modes of the field operators, \( \pi \) and \( \sigma \), having non-vanishing LC three-momenta, \((p^+, p_\perp) \neq 0\) [15]. These non-zero modes do not have a VEV. Thus, the VEV of (11) vanishes on both sides, as it should.

The moral is that we have to assume the validity of the identity (3) for any possible choice of quantisation hypersurface, in particular for one tangent to the LC, i.e. a null plane. This means in particular, that we can use a complete set of eigenstates of the light-cone Hamiltonian in the last term of (3), as was done in [1, 2].

4. The ‘t Hooft model is the simplest theory where use can be made of LC wave functions. This is due to the fact that, in the limit of large \( N_C \), the coupling to higher Fock states is suppressed by negative powers of \( N_C \). Therefore, the LC eigenvalue problem closes in the quark-anti-quark sector and the sum in (3) is saturated by one-meson states. For the matrix elements in (4) one finds [1, 2, 16, 17]

\[
F_n \equiv \langle 0 | \bar{\psi} i\gamma_5 \psi | n \rangle = \sqrt{N_C} \frac{m^2}{\pi} \int_0^1 dx \frac{\phi_n(x)}{x(1-x)} .
\]  

(12)
The \( \phi_n(x) \) are the LC wave functions of the mesonic bound states and are obtained as solutions of ‘t Hooft’s equation,

\[
m^2_n \phi_n(x) = \frac{m^2}{x(1-x)} \phi_n(x) + \mu_0^2 \int dy \frac{\phi_n(x) - \phi_n(y)}{(x-y)^2} .
\]  

(13)
The variables \( x, y \) represent the LC momentum fraction carried by one of the quarks in the meson, the integral on the right-hand side is performed with a principal value prescription, and \( \mu_0^2 = g^2 N_C/2\pi \) is the basic mass scale of the theory. Integrating (13) over \( x \) one finds the important relation [14]

\[
m^2_n \int_0^1 dx \phi_n(x) = m^2 \int_0^1 dx \frac{\phi_n(x)}{x(1-x)}. 
\]  

(14)

\footnote{our LC conventions (with \( a \) an arbitrary four-vector) are: \( a^\pm = (a^0 \pm a^3)/\sqrt{2}, (a^1, a^2) = a_\perp \).}
The left-hand side is basically the wave function at the origin \((x^- = 0)\), which is expressed in terms of an integral dominated by the infrared tails, \(x \to 0 \) or \(1\).

In the chiral limit, \(m \to 0\), the sum in (4) is saturated by the meson of lowest mass, which we will call the ‘pion’. The condensate is thus given by

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{m F_\pi^2}{m_\pi^2},
\]

and one has to calculate \(F_\pi\) from (12) and \(m_\pi\). This has been done in [17] using ‘t Hooft’s ansatz for the wave function,

\[
\phi_\pi(x) \simeq x^\beta (1 - x)^\beta,
\]

where \(\beta\) is determined from ‘t Hooft’s equation (13), yielding

\[
\beta \frac{\pi}{\sqrt{3}} = \frac{m}{\mu_0}.
\]

We have explicitly displayed the factor \(\pi/\sqrt{3}\), which will play a peculiar role later. Inserting (16) into (14), one obtains for the ‘pion’ mass

\[
m_\pi^2 = \frac{2m^2}{\beta} = 2 \frac{\pi}{\sqrt{3}} m \mu_0,
\]

and for \(F_\pi\),

\[
F_\pi = \sqrt{\frac{N_C}{\pi}} \frac{\pi}{\sqrt{3}} \mu_0.
\]

Inserting (18) and (19) into (15), one easily finds the condensate [1, 2, 13, 17]

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{N_C}{\sqrt{12}} \mu_0 = -\frac{N_C}{2\pi} \frac{\pi}{\sqrt{3}} \mu_0.
\]

As was to be expected, all three dimensionful quantities, \(m_\pi\), \(F_\pi\) and \(\langle 0 | \bar{\psi} \psi | 0 \rangle\), are expressed in terms of the basic scale \(\mu_0\). It is, however, interesting to write these in terms of the parameter \(\beta\) characterising the LC wave functions. One obtains

\[
m_\pi^2 = \frac{2m^2}{\beta},
\]

\[
F_\pi = \sqrt{\frac{N_C}{\pi}} \frac{m}{\beta},
\]

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{N_C}{2\pi} \frac{m}{\beta}.
\]

Thus, all ‘low energy parameters’ are expressible in terms of \(\beta\) and the quark mass \(m\) which are equivalent to \(\mu_0\), as is evident from [17].
One last way of expressing the low energy parameters is in terms of dimensionless fractions, \(i.e.\) by measuring all masses in units of the scale \(\mu_0\),

\[
\frac{m_\pi^2}{\mu_0^2} = 2 \frac{m}{\mu_0} \frac{\pi}{\sqrt{3}} , \quad (24)
\]

\[
\frac{F_\pi}{\mu_0} = \sqrt{N_C/\pi} \frac{\pi}{\sqrt{3}} , \quad (25)
\]

\[
\frac{\langle 0 | \bar{\psi} \psi | 0 \rangle}{\mu_0} = -\frac{N_C}{2\pi} \frac{\pi}{\sqrt{3}} . \quad (26)
\]

The above way of writing everything once more exhibits the mysterious factor \(\pi/\sqrt{3}\). We will come back to this issue later.

Finally, to make contact with the GOR relation (1), we insert (23) into (21) which yields the GOR relation in the \('t\) Hooft model,

\[
\frac{N_C}{\pi} m_\pi^2 = -4m \langle 0 | \bar{\psi} \psi | 0 \rangle . \quad (27)
\]

This leads to yet another definition of the condensate, namely in terms of the 'pion' mass,

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{N_C}{4\pi} \frac{\partial}{\partial m} m_\pi^2(m) \bigg|_{m=0} . \quad (28)
\]

In contrast to the GOR relation (27) which holds to first order in the quark mass \(m\), (28) is an \textit{exact} statement.

5. The massive Schwinger model (QED in 1+1 dimensions \[18, 19, 20\]) has first been analysed along the lines of \('t\) Hooft by Bergknoff \[21\], who calculated the bound state spectrum using LC methods. These results have been refined and extended recently by several authors \[22, 23\]. In the massive Schwinger model, one cannot straightforwardly make use of the relations (2) and (4) in order to define the condensate. This is due to the fact that the axial current is anomalous \[25, 26\],

\[
\partial_\mu j_5^\mu = -\frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} + 2m \bar{\psi} i\gamma_5 \psi , \quad (29)
\]

so that it is \textit{not} conserved in the chiral limit \(m \to 0\). The condensate can, however, be obtained exactly by bosonisation of the massless Schwinger model \[27, 28\],

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{1}{2\pi} e^\gamma \mu_0 , \quad (30)
\]

\[2\]There have also been many attempts to obtain the exact (operator) solution for the \textit{massless} Schwinger model within LC quantisation. Due to the singular nature of massless fields in 1+1 dimensions, in particular on the LC, the necessary efforts are considerably larger than within ordinary quantisation on \(x^0 = 0\) \[24\].
where $\gamma = 0.577...$ denotes Euler’s constant. On the other hand, the ‘pion’ mass has recently been calculated up to second order in the fermion mass $m$ [29], from which one can derive an expression for the condensate analogous to (28). We only need the first order result [27],

$$
m_\pi^2(m) = \mu_0^2 - 4\pi m \langle 0 | \bar{\psi} \psi | 0 \rangle + O(m^2) = \mu_0^2 + 2e^\gamma m\mu_0 + O(m^2),
$$

(31)

where $\mu_0 = e/\sqrt{\pi}$ is the basic mass scale of the Schwinger model [3]. From (31) it is evident that the Schwinger model analogue of (28) holds, namely

$$
\langle 0 | \bar{\psi} \psi | 0 \rangle = -\frac{1}{4\pi} \frac{\partial}{\partial m} m_\pi^2(m) \bigg|_{m=0}.
$$

(32)

All one has to do to obtain (32) from (28), is to replace $N_C$ by one. This suggests that expression (31) for the ‘pion’ mass squared can be derived in the same way as for the ‘t Hooft model. In some sense this is not too surprising, since the bound state equations of both the ‘t Hooft and the Schwinger model (in the two-particle or valence sector) can be written in a unified way [31],

$$
m_n^2 \phi_n(x) = \frac{m^2}{x(1-x)}\phi_n(x) + \alpha \mu_0^2 \int_0^1 dx \phi_n(x) + \mu_0^2 \int dy \frac{\phi_n(x) - \phi_n(y)}{(x-y)^2}.
$$

(33)

In the ‘t Hooft model, $\alpha = 0$, and in the Schwinger model $\alpha = 1$. The scale parameters $\mu_0$ are given by $\mu_0^2 = g^2 N_C / 2\pi$ and $\mu_0^2 = e^2 / \pi$, respectively. By performing exactly the same steps as for the ‘t Hooft model one finds the ‘pion’ mass squared [21],

$$
m_\pi^2 = \alpha \mu_0^2 + 2\frac{\pi}{\sqrt{3}} m\mu_0.
$$

(34)

In this expression, the first term on the right-hand side is due to the anomaly. The parameter $\alpha$ thus ‘measures’ the strength of the anomaly, which in the ‘t Hooft model is absent ($\alpha = 0$). Comparing (31) and (34) one notes, however, a difference: in the second expression, the factor $e^\gamma$ is replaced by the ubiquitous $\pi/\sqrt{3}$. Numerically, one has

$$
\pi/\sqrt{3} = 1.814, \quad e^\gamma = 1.781,
$$

(35)

so that the difference is about 2% [21]. There are two possible sources for this discrepancy. First, there might be contributions from higher Fock sectors. These, however, vanish in the large-$N_c$ limit (‘t Hooft model) as well as in the chiral limit (Schwinger model). In [3] and the following we set the $\theta$-angle [13, 20] equal to zero. Including a non-vanishing vacuum angle turns out to be a rather non-trivial task. A first attempt has appeared recently [30].
both these limits, the associated ‘pion’ is exactly two-particle. The second approximation is the use of ‘t Hooft’s ansatz (19) for the wavefunctions which yields a good description for the endpoint behaviour, \( x \to 0 \) or 1, but not for intermediate values of \( x \). While the error should be small as the condensate is dominated by the endpoint behaviour, we nonetheless believe that neglecting the non-asymptotic regions in \( x \) is the main reason for the 2% discrepancy. In principle, this can be checked by using numerical methods like those of Harada et al. [23]. These authors went much beyond ‘t Hooft’s ansatz (using an elaborate wave function basis) and included up to six-particle states. In practice, however, it turns out to be rather difficult to precisely determine the small-\( m \) behaviour of \( m^2_\pi \) as the numerical data do not converge very well in this region [32].

Using (32) to obtain the condensate from (34), one finds

\[
\langle 0| \bar{\psi}\psi |0 \rangle = -\frac{1}{2\pi} \frac{\pi}{\sqrt{3}} \mu_0 ,
\]

which is equivalent to (20) after the appropriate replacements. Interestingly, the condensate is completely independent of the anomaly. Switching off the latter by (artificially) putting \( \alpha \) equal to zero also for the Schwinger model, would not change the value of \( \langle 0| \bar{\psi}\psi |0 \rangle \).

Finally, we would like to remark, that, if the solution of the Schwinger model bound state equation yields a condensate which is off by 2%, there is no reason to believe that exactly the same procedure yields a correct value for the condensate in the ‘t Hooft model. Thus we conjecture, that the factor \( \pi/\sqrt{3} \) should be replaced by \( e^\gamma \) everywhere. In particular, the ‘t Hooft model condensate (20) would then become

\[
\langle 0| \bar{\psi}\psi |0 \rangle = -\frac{N_C}{2\pi} e^\gamma \mu_0 .
\]

6. One remaining question is whether the above discrepancy can be regarded as physical. To answer this we concentrate on the Schwinger model (\( \alpha=1 \)) in what follows and write for the condensate (36) more generally

\[
\langle 0| \bar{\psi}\psi |0 \rangle = -c\mu_0 ,
\]

and for the ‘pion’ mass (34) in units of \( \mu_0 \),

\[
\frac{m_\pi}{\mu_0} = 1 + 2\pi c \frac{m}{\mu_0} + O(m/\mu_0)^2 .
\]

The question then can be reformulated: is there a unique, physical value for \( c \)? In a recent publication [33], for example, the authors claim that the constant \( c \) is dependent on the renormalization scheme used. They do not, however, give this dependence explicitly.
A convenient way to analyse this issue is the following. We use the bosonised (sine-Gordon) Hamiltonian (density) [34]

\[
\mathcal{H}(m, \mu_0) = N_{\mu_0} \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} \mu_0^2 \phi^2 - cm\mu_0 \cos(\sqrt{4\pi}\phi) \right]
\]

\[
\equiv N_{\mu_0} \left[ \mathcal{H}_0 + \frac{1}{2} \mu_0^2 \phi^2 - cm\mu_0 \cos(\sqrt{4\pi}\phi) \right],
\]

(40)

where \(N_{\mu_0}\) denotes normal-ordering with respect to the scale \(\mu_0\). The energy density in the vacuum of mass \(\mu_0\) is then

\[
\mathcal{E}(m, \mu_0) = \langle 0, \mu_0 | \mathcal{H}(m, \mu_0) | 0, \mu_0 \rangle = -cm\mu_0 .
\]

(41)

This expression can be used to give yet another definition of the condensate [35]. Regarding the fermion mass \(m\) as a parameter, the Feynman-Hellmann theorem leads to

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle_{\mu_0} = \frac{\partial}{\partial m} \mathcal{E}(m, \mu_0) = -c\mu_0 ,
\]

(42)

which is the same as (38). What happens when the scale is changed, say, from \(\mu_0\) to \(\mu\)? The answer can be obtained from Coleman’s work on the sine-Gordon model [34], where one finds

\[
\mathcal{H}(m, \mu_0) = N_{\mu} \left[ \mathcal{H}_0 + \frac{1}{2} \mu_0^2 \phi^2 - cm\mu_0 \cos(\sqrt{4\pi}\phi) \right]
\]

\[-\frac{1}{8\pi} \left( \mu_0^2 + \mu_0^2 \ln \frac{\mu^2}{\mu_0^2} - \mu^2 \right) .
\]

(43)

The vacuum energy density becomes

\[
\mathcal{E}(m, \mu) = \langle 0, \mu | \mathcal{H}(m, \mu_0) | 0, \mu \rangle = -\frac{1}{8\pi} \left( \mu_0^2 + \mu_0^2 \ln \frac{\mu^2}{\mu_0^2} - \mu^2 \right) - cm\mu .
\]

(44)

yielding the condensate at scale \(\mu\),

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle_{\mu} = \frac{\partial}{\partial m} \mathcal{E}(m, \mu) = -c\mu .
\]

(45)

The scale dependence of the condensate is thus very simple and can be expressed in terms of the renormalisation group (RG) equation,

\[
\frac{\partial}{\partial \mu} \langle 0 | \bar{\psi} \psi | 0 \rangle_{\mu} = -c .
\]

(46)

The RG invariant quantity, which does not get renormalised, thus being the analogue of \(m\langle 0 | \bar{\psi} \psi | 0 \rangle\) in (8), is therefore
\[
\langle 0 | \bar{\psi} \psi | 0 \rangle_\mu = -c . \quad (47)
\]
The conclusion is that the constant \( c \) defines the physical, RG invariant value of the condensate and by itself does not depend on the renormalisation scale \( \mu \). As the condensates (30) and (36) are evaluated at the same scale \( \mu_0 \), we regard the discrepancy (35) as being relevant and calling for an explanation.

7. It is obvious that in higher dimensions the renormalisation program becomes much more involved, in particular for the bound state equations yielding the LC wave functions [36]. An exception from this rule might be effective theories which need not be renormalisable and are thus meaningful only below some physical cutoff \( \Lambda \). A prominent example is the Nambu-Jona-Lasinio (NJL) model [37] describing the dynamical breakdown of chiral symmetry. It has a chirally symmetric four-fermion interaction, but (beyond a critical coupling, \( g > g_c \)) the vacuum breaks chiral symmetry resulting in a non-vanishing fermion condensate. In mean-field approximation this condensate determines the mass gap

\[
m - m_0 = -2g \langle 0 | \bar{\psi} \psi | 0 \rangle_m , \quad (48)
\]
between the current quarks with mass \( m_0 \) and the constituent quarks with mass \( m \) (dynamical mass generation). The constituent mass is obtained self-consistently from the gap equation (48), and this is how all the non-perturbative physics enters. The condensate itself is calculated perturbatively, i.e. in a Dirac vacuum for free fermions of mass \( m \). Again, the Feynman-Hellmann theorem is very helpful. Integrating over all the one-particle energies of the Dirac sea, one finds

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle_m = \frac{\partial}{\partial m} \mathcal{E}(m) = \frac{\partial}{\partial m} \int_{-\infty}^{0} \frac{dk^+}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{m^2 + k_\perp^2}{2k^+}
\]

\[
= -\frac{m}{8\pi^3} \int_{0}^{\Lambda} \frac{dk^+}{k^+} \int d^2 k_\perp , \quad (49)
\]

As it stands, the integral is of course divergent and requires regularisation. In the most straightforward manner one chooses \( m^2 / \Lambda \leq k^+ \leq \Lambda \) and \( |k_\perp| \leq \Lambda \), so that the condensate becomes

\[
\langle 0 | \bar{\psi} \psi | 0 \rangle_m = -\frac{m}{8\pi^2} \int_{m^2 / \Lambda}^{\Lambda} \frac{dk^+}{k^+} \int d(k_\perp^2) = -\frac{m}{8\pi^2} \Lambda^2 \ln \frac{\Lambda^2}{m^2} . \quad (50)
\]

Plugging this result into the gap equation (18) (setting for simplicity \( m_0 = 0 \) in what follows) one finds for the dynamical mass squared,
\[
m^2(g) = \Lambda^2 \exp \left( -\frac{4\pi^2}{g\Lambda^2} \right).
\]

The critical coupling is determined by the vanishing of this mass, \(m(g_c) = 0\), and from (51) we find the surprising result

\[
g_c = 0.
\]  

This result, however, is wrong since one knows from the conventional treatment of the model that the critical coupling is finite of the order \(\pi^2/\Lambda^2\), both for covariant and non-covariant cutoff [37]. In addition, it is quite generally clear that in the free theory \((g = 0)\) chiral symmetry is not broken \((for m_0 = 0)\) and, therefore, this should not happen for arbitrarily small coupling, either. The remedy is once more to use an information from the ordinary calculation of the condensate. We translate the non-covariant, but rotationally invariant, three-vector cutoff, \(|k| \leq \Lambda\), into LC coordinates [38], which leads to

\[
0 \leq k^2_\perp \leq 2\Lambda k^+ - m^2 - (k^+)^2, \quad \frac{m^2}{2\Lambda} \leq k^+ \leq 2\Lambda.
\]

Note that the transverse cutoff becomes a polynomial in \(k^+\). The \(k_\perp\)-integration thus has to be performed first. For the condensate this yields an analytic structure different from (50),

\[
\langle 0|\bar{\psi}\psi|0 \rangle_m = -\frac{m}{8\pi^2} \left( 2\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right),
\]

where we have neglected subleading terms in the cutoff \(\Lambda\). From (54), one infers the correct cutoff dependence of the critical coupling,

\[
g_c = \frac{2\pi^2}{\Lambda^2}.
\]

The moral of this calculation is that even in a non-renormalisable theory like the NJL model, the LC regularisation prescription is a subtle issue. In order to get a physically sensible result the transverse cutoff has to be \(k^+\)-dependent. Clearly, this dependence cannot be arbitrary but should be constrained from dimensional and symmetry considerations. For renormalisable theories, such arguments have been given by Perry and Wilson [36]. In the example above, it was (ordinary) rotational invariance that solved the problem.

The condensate (54) was already obtained in [38], where, however, a slightly more complicated cut-off was used. In that work, the condensate was defined covariantly in terms of the fermion propagator at the origin, \(S_F(x = 0)\),

\[
\langle 0|\bar{\psi}\psi|0 \rangle_m = -i\text{Tr}S_F(0) = -i\frac{m}{4\pi^4} \int \frac{d^4k}{k^2 - m^2 + i\epsilon}.
\]
Performing the integration over $k^-$ with the appropriate cutoff leads to (54).

Within the NJL model, an illustrating analogy to magnetic systems can be made. Chiral symmetry corresponds to rotational symmetry, the vacuum energy density to the Gibbs free energy, and the mass $m$ to an external magnetic field. The order parameter measuring the rotational symmetry breaking is the magnetisation. It is obtained by differentiating the free energy with respect to the external field. This is the analogue of expression (49) as derived from the Feynman-Hellmann theorem.

It would be very interesting to relate the NJL condensate to LC wave functions in the same spirit as for the two dimensional models above. To this end one has to solve the (pseudoscalar) bound state equation not only for the pion mass (as was done in [38] in the chiral limit) but also for the associated eigenfunctions of the LC Hamiltonian. Work in this direction is underway.

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