On Lagrangian Formulation for Half-integer HS Fields within Hamiltonian BRST Approach

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Abstract

A recent progress in a gauge-invariant Lagrangian description of massive and massless half-integer higher-spin fields in AdS and Minkowski spaces is presented. The procedure is based on a BFV-BRST operator, encoding the initial conditions realized by constraints in a Fock space and extracting the higher-spin fields from unitary irreducible representations of the AdS (Poincare) group. The construction is applicable to higher-spin tensor fields with a multi-row Young tableaux.

1. Introduction

Problems of higher-spin (HS) field theory remain an important area of scientific research in view of their close relation to superstring theory, which operates with an infinite set of bosonic and fermionic HS fields, including massless and massive fields (for a review, see [1]). This article takes a snapshot of constructing a Lagrangian formulation (LF) for free half-integer HS fields as a starting point for an interacting HS field theory in the framework of conventional Quantum Field Theory, and is based on the results presented in [2, 3].

The methods of constructing an LF for HS fields are based on the BFV–BRST approach [4], developed in a way that applies to Hamiltonian quantization of gauge theories with a given LF, and consists in a solution of the problem inverse to that of the method [4] (as in the case of string field theory [5] and in the early papers on HS fields [6]) in the sense of constructing a gauge LF with respect to a nilpotent BFV–BRST operator $Q$. This operator, in its turn, is constructed from a system $O_\alpha$ of first-class constraints defined in an auxiliary Fock space and encoding the relations that extract the fields with a definite mass and spin from the representation spaces of the AdS or Poincare group.

2. Fermionic fields in AdS spaces

It is well-known that massive half-integer spin $s = n + \frac{1}{2}$ representations of the AdS group are realized in the space of totally symmetric tensor-spinor fields $\Phi_{\mu_1...\mu_n}(x)$, with suppressed Dirac index, satisfying the equations

$$[i\gamma^\mu \nabla_\mu - r\frac{1}{2} (n + \frac{d}{2} - 2) - m] \Phi_{\mu_1...\mu_n}(x) = 0, \quad \gamma^\mu \Phi_{\mu_2...\mu_n}(x) = 0$$

(1)

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For a simultaneous description of all half-integer HS fields, one introduces a Fock space $\mathcal{H}$ generated by creation $a^+_\mu(x)$ and annihilation $a_\mu(x)$ operators, $[a_\mu, a^+_{\nu}] = -g_{\mu\nu}$, and a set of constraints for an arbitrary vector $|\Phi\rangle \in \mathcal{H}$,

$$\tag{2} t_0|\Phi\rangle = \left(-i\tilde{\gamma}^\mu D_\mu + \tilde{\gamma}\left(m + \frac{1}{2}(g_0 - 2)\right)\right)|\Phi\rangle = 0, \quad t_1|\Phi\rangle = \tilde{\gamma}^\mu a_\mu|\Phi\rangle = 0,$$

$$|\Phi\rangle = \sum_{n=0}^{\infty} \Phi_{\mu_1...\mu_n}(x) a^{+\mu_1}...a^{+\mu_n}|0\rangle, \quad \tag{3}$$

given in terms of an operator $D_\mu$ equivalent to $\nabla_\mu$ in its action in $\mathcal{H}$, as well as in terms of fermionic operators $\tilde{t}_0, t_1$ constructed from an enlarged set of Grassmann-odd gamma-matrix-like objects $\tilde{\gamma}^\mu, \tilde{\gamma}$ ($\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}$, $\{\tilde{\gamma}^\mu, \tilde{\gamma}\} = 0, \tilde{\gamma}^2 = -1$ [2, 7]), related to conventional gamma-matrices as $\gamma^\mu = \tilde{\gamma}^\mu\tilde{\gamma}$. The validity of relations (2) is equivalent to the simultaneous fulfilment of Eqs. (1) for all fermionic HS fields $\Phi_{\mu_1...\mu_n}(x)$.

To obtain a Hermitian BFV–BRST charge, it is necessary to deduce a set of first-class quantities $O_I : \{O_a\} \subset \{O_I\}$ closed under the operations of a) Hermitian conjugation with respect to an odd scalar product $\langle \tilde{\Psi}|\Phi\rangle$ [2] and b) supercommutator multiplication $[\cdot, \cdot]$. As a result of the first step in obtaining the set of $\{O_a\}$, the original massive half-integer HS symmetry superalgebra, $\{o_I\} = \{\tilde{t}_0, t_1, t_1^+, l_1, l_1^+, g_0, \tilde{t}_0\}$, $i = 1, 2$,

$$\tag{4} (t_1^+, g_0) = (\tilde{\gamma}^\mu a^+_\mu, -a^+_\mu a^+ + \frac{D}{2}), \quad (l_1, l_1^+) = -i(a^\mu, a^{+\mu})D_\mu, \quad (l_2, l_2^+) = \frac{1}{2}(a^\mu a_\mu, a^{+\mu}a^+),$$

$$\tag{5} \tilde{t}_0 = g^{\mu\nu}(D_\nu D_\mu - \Gamma^a_{\mu\nu}D_a) - r\left(g_0 + t_1^+t_1 + \frac{d(d-3)}{4}\right) + \left(m + r\tilde{\gamma}(g_0 - 2)\right)^2,$$

contains a central charge $\tilde{m} = (m - 2\sqrt{r})$, a subset of 6 second-class constraints $\{o_a\} = \{t_1, t_1^+, l_1, l_1^+\}$, a quantity $g_0$, composing, together with $\tilde{m}$, an invertible supermatrix $\|\{o_a, o_0\}\|$, and satisfies some quadratic algebraic relations.

To convert the subsystem $o_a$ into the first-class system $O_a$, we apply an additive conversion procedure for nonlinear superalgebras developed in [8], which consists, first of all, in constructing additional (with respect to $o_I$) parts $o'_I$ acting in a new Fock space $\mathcal{H}'$ generated by fermionic $f, f^+$ and bosonic $b, b^+$, $i = 1, 2$, creation and annihilation operators, so that the converted constraints $O_I = o_I + o'_I$ satisfy a new algebra: $\{O_I, O_J\} \sim O_K$.

The condition of additivity, $\{o_I, o'_I\} = 0$, cannot be fulfilled for $o_I$ due to the presence of the $\tilde{\gamma}$-matrix in its definition. Therefore, we pass to another basis of constraints, $o_I \rightarrow \bar{o}_I = u'_I o_I$, by means of a nondegenerate transformation, such that only $\tilde{t}_0, \tilde{t}_0$ are changed, $t_0 = -i\tilde{\gamma}^\mu D_\mu, l_0 = -\tilde{t}_0$, and such that after the additive conversion $\bar{O}_I = \bar{o}_I + o'_I$ we can make an inverse transformation ($u^{-1}$ being enlarged in $\mathcal{H} \otimes \mathcal{H}'$ to $U^{-1}$) of the converted constraints $O_I$ to $\bar{O}_I = (U^{-1})^T \bar{O}_I$. Next, the construction of the additional parts $o'_I$ is based on the condition $[\bar{O}_I, \bar{O}_J] \sim \bar{O}_K$ that implies an unambiguous form of the superalgebra of $\{o'_I\}$:

$$\tag{6} [o'_I, o'_J]_s = f^k_{ij}o'_k - (-1)^{\varepsilon(o_m)\varepsilon(o_n)}f^{km}o'_m o'_n,$$

(with the Grassmann parity $\varepsilon(o_m) = 0, 1$ respectively for bosonic and fermionic $o_m$), provided that the form of one of $\{o'_I\}$ is given by $[\bar{o}_I, \bar{o}_J]_s = f^k_{ij}\bar{o}_k + f^{km}_{ij}\bar{o}_k\bar{o}_m$ [2]. Then, the enlarged central charge $\bar{M} = \tilde{m} + m'$ vanishes, whereas the explicit expressions for $o'_I$

1Where $r$ is the radius of a $d$-dimensional AdS space $r = \frac{R}{d(d-3)}$, with $R$ being the scalar curvature, $g_{\mu\nu}$ having the mostly minus signature, and Dirac’s matrices satisfying the relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. 

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in terms of the operators $f, f^+, b_i, b_i^+$ and new constants $m_0, h$ [they are to be determined later from the condition of reproducing the correct form of Eqs.(2)] are presented in [2] and can be found following the method described in [7] for totally-symmetric half-integer HS fields in flat spaces, as well as in [9] for massive integer HS fields in AdS spaces extended to the case of the Verma module construction for nonlinear superalgebras.

We then proceed to construct a BFV-BRST operator $\tilde{Q}'$ for the system of operators $\tilde{O}_I$ in the case of the Weyl ordering for quadratic combinations of $\tilde{O}_I$ in the right-hand sides of $[\tilde{O}_I, \tilde{O}_J]$ and for the $(\mathcal{CP})$-ordering for the ghost coordinates $\mathcal{C}^I$: bosonic $q_0, q_1, q_1^+$ and fermionic $\eta_0, \eta_1, \eta_1^+$, and their conjugated momenta $P_I$: $p_0, p_1, p_0^+, p_1^+$, with the standard ghost number distribution $gh(C^I) = -gh(P_I) = 1$, providing $gh(\tilde{Q}') = 1$. In contrast to bosonic HS fields, in the given basis the operators $\{\tilde{O}_I\}$ form an open algebra with respect to $[\ ,\ ] ([\tilde{O}_I, \tilde{O}_J] = F_{IJ}^{K}(\tilde{O}, \nu')\tilde{O}_K)$, so that the nilpotent operator $\tilde{Q}'$ corresponds to a formal second-rank topological gauge theory,

$$\tilde{Q}' = O_I\mathcal{C}^I + \frac{1}{2}\mathcal{C}^{I_1}\mathcal{C}^{I_2}F_{I_1 I_2 I_3}P_J(-1)^\varepsilon(O_{I_2 + \varepsilon(O_J)}) + \frac{1}{6}\mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}F_{I_1 I_2 I_3}F_{J_1 J_2 J_3}P_{J_1}P_{J_2}P_{J_3}$$

(7)

with completely definite functions $F_{I_1 I_2 I_3}^{J_1 J_2 J_3}(\tilde{O}, \nu')$ resolving the Jacobi identity for $\tilde{Q}' [2,8]$.

A covariant extraction of the operator $G_0$ from the system $\{\tilde{O}_I\}$, in order to pass to the converted first-class constraints $\{\tilde{O}_0\}$ only, is based on the condition of independence of $\mathcal{H}_{tot} = \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}_{gh}$ of $\eta_G$ and on the elimination from $\tilde{Q}'$ of the terms proportional to $\eta_G, P_G$ [2]:

$$\tilde{Q}' = \tilde{Q} + \eta_G(\sigma + h) + BP_G, \quad \sigma + h = g_0 + g'_0 + (iq_1^+ p_1 + \sum_{k=1}^2 k\eta^{k+} P_k + h.c.);$$

(8)

the same applies to the physical vector $|\chi\rangle \in \mathcal{H}_{tot}, |\chi\rangle = |\Phi\rangle + |\Phi_A\rangle$, $|\Phi_A\rangle_{\{b_i = b_i^+ = f^+ = c = p = 0\}} = 0$, with the use of the BFV-BRST equation $\tilde{Q}'|\chi\rangle = 0$ determining the physical states:

$$\tilde{Q}|\chi\rangle = 0$$

(9)

Note that the second equation must take place in the entire $\mathcal{H}_{tot}$, thus determining the spectrum of spin values for $|\chi\rangle$, whereas the first equation is valid only in the subspace of $\mathcal{H}_{tot}$ with the zero ghost number.

The presence of an extensive gauge ambiguity in the definition of an LF permits a covariant separation in $Q$ of all the operators with second-order derivatives with respect to $x^\mu$, thus expanding $Q$ in the powers of the zero-mode pairs $q_0, p_0, \eta_0, P_0$ as follows [2]:

$$Q = q_0 \tilde{T}_0 + \eta_0 \tilde{L}_0 + i(q_1^+ q_1 - \eta_1 q_1^+) p_0 + (q_0^2 - \eta_1^+ \eta_1) P_0 + \Delta Q.$$  

(10)

As a result, due to the representation $|\chi\rangle = \sum_{k=0}^\infty q_0^k (|\chi^0_k\rangle + \eta_0 |\chi^1_k\rangle)$, the first equation in (9) takes the form

$$\Delta Q|\chi^0_0\rangle + \frac{1}{2}\{\tilde{T}_0, \eta_1^+ \eta_1\}|\chi^1_0\rangle = 0, \quad \tilde{T}_0|\chi^0_0\rangle + \Delta Q|\chi^1_0\rangle = 0,$$  

(11)

which can be deduced from the Lagrangian action

$$S = \langle \chi^0_0 | K \tilde{T}_0 | \chi^0_0 \rangle + \frac{1}{2} \langle \chi^0_0 | K \{ \tilde{T}_0, \eta_1^+ \eta_1 \} | \chi^1_0 \rangle + \langle \chi^0_0 | K \Delta Q | \chi^1_0 \rangle + \langle \chi^1_0 | K \Delta Q | \chi^0_0 \rangle.$$  

(12)

\footnote{In [2,8] it is shown that the transition $\tilde{O}_I \rightarrow O_I$ is realized in $\mathcal{H}_{tot}$ by means of a unitary transformation constructed with respect to $||U^{-1}||$, so that $\tilde{Q} = Q$ and $Q^2|\chi\rangle = 0$ if $(\sigma + h)|\chi\rangle = 0.$}
In (12), we have used an odd scalar product in $H_{tot}$ and a nondegenerate operator $K = \hat{K} \otimes K'$, which provides the Hermitian character of the operators with respect to $\langle | \rangle$, as well as the reality of $S$ (for details, see [2]). The corresponding LF of a HS field with a specific value of spin $s = n + \frac{1}{2}$ is a reducible gauge theory of $L = n - 1$-th stage of reducibility.

3. Fermionic fields with an arbitrary Young tableaux

Let us examine the construction of an LF for spin-tensor fields characterized by a Young tableaux with 2 rows ($n_1 \geq n_2$)

$$\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) \equiv \Phi_{\mu_1...\mu_{n_1},\nu_1...\nu_{n_2}}(x) \leftrightarrow \begin{array}{cccccccc}
\mu_1 & \mu_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \mu_{n_2} \\
\nu_1 & \nu_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \nu_{n_2}
\end{array}. \quad (13)$$

The field $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ is symmetric with respect to the permutations of each type of its indices $(\mu)_{n_1},(\nu)_{n_2}$ and must obey the equations

$$\left(\gamma^\mu \partial_\mu \Phi_{(\mu)_{n_1},(\nu)_{n_2}}, \gamma^\nu \Phi_{(\mu)_{n_1},(\nu)_{n_2}}, \gamma^\nu \Phi_{(\mu)_{n_1},(\nu)_{n_2}}, \gamma^\nu \Phi_{(\mu)_{n_1},(\nu)_{n_2}}\right) = 0. \quad (14)$$

After the introduction of 2 pairs of creation and annihilation operators, we have

$$[a^i_\mu, a^{i+}_\nu] = -\eta_{\mu\nu} \delta^{ij}, \quad \delta^{ij} = diag(1,1), \quad \eta_{\mu\nu} = diag(1,-1,...,-1). \quad (15)$$

The general (Dirac-like spinor) state in this Fock space $\mathcal{H}^2 = \mathcal{H}_1 \otimes \mathcal{H}_2$ has the form

$$\ket{\Phi} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) a^{+\mu_1}_1 ... a^{+\nu_1}_1 a^{+\mu_2}_2 ... a^{+\nu_2}_2 \ket{0}. \quad (16)$$

The deduction of Eqs. (14) proceeds by means of the operators $t_0, t^i, t_{12}$ as follows:

$$t_0 \ket{\Phi} = -i \tilde{\gamma}^\mu \partial_\mu \ket{\Phi} = 0, \quad t^i \ket{\Phi} = \tilde{\gamma}^\mu a^i_\mu \ket{\Phi} = 0, \quad t_{12} \ket{\Phi} = a^{+\mu}_1 a^{\nu}_2 \ket{\Phi} = 0. \quad (17)$$

Now, we can generalize this construction to spin-tensors corresponding to a $k$-row ($k \leq [(d-1)/2]$) Young tableaux. To this end, one should introduce a Fock space $\mathcal{H}^k = \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_k$ with $k$ pairs of $a^{i+}_\mu, a^i_\mu$ and introduce operators (17), this time, however, with $i, j = 1, ..., k$ for $t_{ij} = a^{+\mu}_i a^\nu_j, i < j$. Then, an LF can be found partially according to the above-developed principles [3]. The program of an LF construction on the basis of this method for both massive and and massless fermionic HS fields with a two-row Young tableau was realized in [3].

4. Summary

In this article, we have briefly considered the construction of an LF for free massive and massless HS fields on a basis of the BFV–BRST approach. In addition, note, that the value of reducibility stage for a gauge LF increases with the number of rows of the Young tableaux. Second, there exists a possibility to eliminate the set of algebraic second-class constraints from the system of all constraints so as to reduce the amount of calculations and the form of the final LF, however, with an appearance of some off-shell algebraic conditions (such as tracelessness or $\gamma$-tracelessness). Third, it is necessary to realize a SUSY generalization of LF to HS fields, which will permit one to construct, on the basis of the research [10] of interacting bosonic HS fields, an interacting theory with fermionic HS fields.

3As opposed to the AdS space, the corresponding HS symmetry superalgebra of all the operators, including those of (17), is a Lie superalgebra.
Acknowledgements  The author thanks the organizers of the SQS’07 Workshop (JINR, Dubna, Russia) for support and hospitality.

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