DIMENSIONS OF NON-AUTONOMOUS MEROMORPHIC FUNCTIONS
OF FINITE ORDER

JASON ATNIP

ABSTRACT. In this paper we study two classes of meromorphic functions previously studied by Mayer in [12] and by Kotus and Urbaniśki in [10]. In particular we estimate a lower bound for the Julia set and the set of escaping points for non-autonomous additive and affine perturbations of functions from these classes. For particular classes we are able to calculate these dimensions exactly. We accomplish this by constructing non-autonomous graph directed Markov systems, which sit inside of the aforementioned non-autonomous Julia sets. We also give estimates for the eventual and eventual hyperbolic dimensions of the these non-autonomous perturbations.

1. Introduction

Much work has been done recently concerning the (autonomous) dynamics of transcendental meromorphic functions. In [12], Mayer used infinite iterated function systems to find a lower bound for the Julia set of meromorphic functions of finite order as well as their hyperbolic dimension. Previously similar techniques were used by Kotus and Urbaniśki in [8] and Roy and Urbaniśki in [18] to find a lower bound for the Hausdorff dimension of the Julia set of autonomous and random systems of elliptic functions respectively by using the theories of infinite autonomous and random iterated function systems. In a similar fashion, we will use the theory of non-autonomous conformal graph directed Markov systems as developed in [2], to find a lower and upper bound for the Hausdorff dimension of the set of escaping points as well as a lower bound for the Hausdorff dimension of the radial limit set generated from a non-autonomous family of finite order meromorphic functions. To the best of the author’s knowledge our results concerning the lower bound for the dimension of the set of escaping points is new for the given classes of meromorphic functions even in the autonomous case.

In this article, we will primarily be concerned with non-autonomous dynamics stemming from perturbations of a single meromorphic function. In particular, given sequences \((c_n)_{n \in \mathbb{N}}\) and \((\lambda_n)_{n \in \mathbb{N}}\) in \(\mathbb{C}\) and a transcendental meromorphic function \(f\) of finite order \(\rho\), we will consider additive and affine perturbations of \(f\) defined by

\[ f_n(z) = f(z) + c_n \quad \text{and} \quad \hat{f}_n(z) = \lambda_n \cdot f(z) + c_n \]

for each \(n \in \mathbb{N}\). The non-autonomous additive and affine iterates are defined respectively for each \(n \in \mathbb{N}\) with

\[ F^n_+ = f_n \circ \cdots \circ f_1 \quad \text{and} \quad F^n_{\hat{A}} = \hat{f}_n \circ \cdots \circ \hat{f}_1. \]
In particular, by taking $c_n \equiv 0$ and $\lambda_n \equiv 1$ for each $n \in \mathbb{N}$, each of our results holds for ordinary, autonomous dynamical systems. Our results apply equally well to random dynamical systems if the perturbative parameters are chosen according to some probability distribution. Additionally, by only taking the multiplicative perturbations $\lambda_n \equiv 1$ and allowing the additive perturbations $c_n \neq 0$ we see that any statement concerning function $F_A$ of non-autonomous affine perturbations also applies to the function $F_+$ of non-autonomous additive perturbations. This applies, in particular, to Theorem 1.1.

We will also give results concerning the eventual dimension and eventual hyperbolic dimension of a function. The eventual dimension of a transcendental meromorphic function is given by

$$\text{ED}(f) = \lim_{R \to \infty} \text{HD}\{z \in J(f) : |f^n(z)| > R, \forall n \geq 1\}$$

and was first introduced by Rempe-Gillen and Stallard in [16]. The concept of the eventual hyperbolic dimension of a transcendental function is a generalization of the notions of the eventual dimension and the hyperbolic dimension of Shishikura (see [19]). The eventual hyperbolic dimension was first introduced by De Zotti and Rempe-Gillen in [1] and is given by

$$\text{EHD}(f) = \lim_{R \to \infty} \sup \{\text{HD}(X) : X \text{ is a hyperbolic set for } f\}.$$.

In [15], Rempe-Gillen shows that the hyperbolic dimension of a transcendental function $f$ is equal to the Hausdorff dimension of the radial Julia set $J_r(f)$. We will prove a similar result, relating the eventual hyperbolic dimension and the Hausdorff dimension of the set

$$J_r(f, R) := \{z \in J_r(f) : |f^n(z)| > R, \forall n \geq 1\}.$$

1.1. **Statement of Results.** The goal of this article is to show that for sufficiently small perturbative values, the dimensions of the escaping and radial sets of non-autonomous additive and affine functions, $F_+$ and $F_A$, have the same upper and lower bounds as the escaping and radial sets for the original unperturbed function $f$. In other words, we may use the dimension of the autonomous dynamical system to estimate the dimension of the non-autonomous system. We now present our three main results which concern the non-autonomous perturbations of two different large classes of meromorphic functions of finite order. Our first result generalizes the results of [12].

**Theorem 1.1.** Let $f_0$ be a meromorphic function of finite order $\rho$ and suppose that the following hold.

1. $f_0$ has at least one pole $b \in f_0(\mathbb{C})$ which is not in the closure of the singular values, $\text{Sing}(f_0^{-1})$. Let $q$ be the multiplicity of $b$.
2. There are constants $s > 0$, $Q > 0$ and $\alpha > -1 - 1/q$ such that

$$|f_0'(z)| \leq Q |z|^\alpha$$

for $z \in f_0^{-1}(U), |z| \to \infty$.

where $U = B(b, s)$. 
For each $0 \leq t < \frac{\rho}{\alpha + 1 + 1/q}$ there exists $\delta > 0$ and $\varepsilon > 0$ such that if $(\lambda_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ are sequences in $\mathbb{C}$ such that

$$\lambda_n, \lambda_n^{-1} \in B(1, \delta) \quad \text{and} \quad |c_n| < \varepsilon$$

for each $n \in \mathbb{N}$, then $\text{HD}(\mathcal{J}_r(F_A)) \geq t$.

If $f_0$ is of maximal divergence type then, we may revise the above statement so that we may consider each $0 \leq t \leq \frac{\rho}{\alpha + 1 + 1/q}$, and consequently we have that there exist $\varepsilon, \delta > 0$ such that

$$\text{HD}(\mathcal{J}_r(F_A)) \geq \frac{\rho}{\alpha + 1 + 1/q}.$$

Moreover, if $f_0$ has infinitely many such poles, then $\text{EHD}(f_0) \geq \frac{\rho}{\alpha + 1 + 1/q}$. If in addition, $f_0$ is of maximal divergence type then this inequality is strict.

The following two theorems generalize the results of [10], the first of which concerns the dimension of the set of escaping points, while the second is concerned with the dimension of the radial Julia set.

**Theorem 1.2.** Let $f_0 : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function of finite order $\rho > 0$ such that the following hold

1. $\infty$ is not an asymptotic value of $f_0$.
2. There exists a co-finite set, $\mathcal{P}^*$, of poles such that $\text{dist}(\text{Sing}(f^{-1}), a) > 0$ for all $a \in \mathcal{P}^*$.
3. There exist $M \in \mathbb{N}$, $\beta \geq 0$, and $R_0 > 0$ such that for each pole $a$

$$|f_0(z)| \asymp \frac{|a|^{-\beta}}{|z - a|^{m(a)}} \quad \text{and} \quad |f_0'(z)| \asymp \frac{m(a)|a|^{-\beta}}{|z - a|^{m(a)+1}}$$

for $z \in B(a, R_0)$, where $m(a) \in \mathbb{N}$ denotes the multiplicity of the pole $a$ with $1 \leq m(a) \leq M$.

Then there is $\varepsilon > 0$ such that if $(c_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{C}$ with $|c_n| < \varepsilon$ for all $n \in \mathbb{N}$ then

$$\frac{\rho M^*}{\beta + M^* + 1} \leq \begin{cases} \text{HD}(I_\infty(F_+)) \\ \text{EHD}(F_+) \\ \text{ED}(F_+) \end{cases} \leq \frac{\rho M}{\beta + M + 1},$$

where $M^*$ is the largest integer, less than or equal to $M$, such that the sum

$$\sum_{a \in m^{-1}(M^*)} (1 + |a|)^{-t}$$

is finite for $t > \rho$ and infinite for $t < \rho$.

**Theorem 1.3.** If $f_0$ has infinitely many poles and satisfies hypotheses (2)-(3) from the previous theorem then for each $0 \leq t < \frac{\rho M^*}{\beta + M^* + 1}$ there exist $\varepsilon_t, \delta_t > 0$ such that if $(c_n)_{n \in \mathbb{N}}$
and \((\lambda_n)_{n \in \mathbb{N}}\) are sequences in \(\mathbb{C}\) such that \(|c_n| < \varepsilon\) and \(\lambda_n, \lambda_n^{-1} \in B(1, \delta)\) for each \(n \in \mathbb{N}\) then

\[
\text{HD}(J_r(F_A)) \geq \text{EHD}(F_A) \geq t.
\]

If in addition, \(f_0\) is of maximal divergence type then there exists \(\varepsilon, \delta > 0\), no longer depending on \(t\), such that if \((c_n)_{n \in \mathbb{N}}\) and \((\lambda_n)_{n \in \mathbb{N}}\) are sequences in \(\mathbb{C}\) such that \(|c_n| < \varepsilon\) and \(\lambda_n, \lambda_n^{-1} \in B(1, \delta)\) for each \(n \in \mathbb{N}\), then we have

\[
\text{HD}(J_r(F_A)) \geq \text{EHD}(F_A) \geq \frac{\rho M^*}{\beta + M^* + 1}.
\]

**Remark 1.4.** Notice that Theorem 1.2 fully characterizes the dimension of the escaping set for any meromorphic function which, in addition to satisfying the hypotheses of the theorem, has co-finitely many poles, all of which have the same multiplicity, in other words, \(M^* = M\). Additionally, we would like to point out that while the non-autonomous perturbations in Theorems 1.1 and 1.3 depend upon \(t\), Theorem 1.2 has no such requirement.

1.2. Structure of the Paper. In Section 2 we recall some useful properties of meromorphic functions as well as certain notions from the study of non-autonomous dynamics and non-autonomous graph directed Markov systems. In Section 3 we will prove the first part of Theorem 1.1 and in Section 4 we will prove the first part of Theorems 1.2 and 1.3. In Section 5 we will discuss the eventual and eventual hyperbolic dimensions of several well studied classes of functions as well as complete the proofs of our three main theorems. We will also make a connection with the eventual hyperbolic dimension and the Hausdorff dimension of the radial Julia set. Finally, in Section 6 we will provide several examples of our main theorems.

2. Preliminaries

2.1. Meromorphic Functions. In the sequel we will consider meromorphic functions of finite order \(\rho = \rho(f) < \infty\). For \(a \in \hat{\mathbb{C}}\) we define the \(a\)-points to be the collection \(f^{-1}(a) = \{z_m(a) : m \in \mathbb{N}\}\). Of particular interest will be Borel series of the form

\[
\Sigma(t, a) := \sum_{m \in \mathbb{N}} |z_m(a)|^{-t}.
\]

The exponent of convergence for the series is given by

\[
\rho_c(f, a) := \inf \{t > 0 : \Sigma(t, a) < \infty\}.
\]

A theorem of Borel shows that for all but at most two points \(a \in \hat{\mathbb{C}}\), we have that

\[\rho_c(f, a) = \rho.\]  \hfill (2.1)

We say that a meromorphic function \(f\) is of maximal divergence type if

\[
\Sigma(\rho, a) = \sum_{m \in \mathbb{N}} |z_m(a)|^{-\rho} = \infty.
\]
By \( \text{Sing}(f^{-1}) \) we denote the set of singular values, that is \( z \in \text{Sing}(f^{-1}) \) if \( z \in \mathbb{C} \) and \( z \) is a critical or asymptotic value of \( f \). In the sequel we will consider functions from the Speiser class \( \mathcal{S} \) and the Eremenko-Lyubich class \( \mathcal{B} \) where

- \( f \in \mathcal{S} \) if \( \text{Sing}(f^{-1}) \) is finite,
- \( f \in \mathcal{B} \) if \( \text{Sing}(f^{-1}) \) is bounded.

For more on these two classes of functions see [5]. In the sequel we will also require the use of the following result which is commonly known as Iversen’s Theorem.

**Lemma 2.1** (Iversen’s Theorem). Let \( f \in \mathcal{B} \) be a transcendental meromorphic function such that \( \infty \) is not an asymptotic value. Then \( f \) has infinitely many poles.

We refer the reader to the survey articles [3, 9] for a thorough treatment of the dynamics of meromorphic functions.

### 2.2. Non-Autonomous Dynamics

Let \( \mathcal{F} = \{ f_\omega \}_{\omega \in \Omega} \) be a family of meromorphic functions. Given a sequence \( \omega := (\omega_n)_{n \in \mathbb{N}} \) in \( \Omega \), we define the \( n^{\text{th}} \) iterate of the function \( F_\omega : \mathbb{C} \to \hat{\mathbb{C}} \) by

\[
F^n_\omega := f_{\omega_n} \circ \ldots \circ f_{\omega_1} : \mathbb{C} \to \hat{\mathbb{C}}.
\]

If \( \omega \) is understood we will write \( F^n \) instead of \( F^n_\omega \) and \( f_n \) instead of \( f_{\omega_n} \).

We let \( \mathcal{J}(F_\omega) \) be the set of points in \( \mathbb{C} \) such that the iterates \( (F^n_\omega) \) form a normal family on some neighborhood, and let \( \mathcal{I}(F_\omega) = \mathcal{J}(F_\omega)^c \). Then \( \mathcal{J}(F_\omega) \) and \( \mathcal{I}(F_\omega) \) are the non-autonomous Fatou and Julia sets associated with the fiber \( \omega \), respectively. By

\[
I_\infty(F_\omega) = \left\{ z \in \mathcal{I}(F_\omega) : \lim_{n \to \infty} F^n_\omega(z) = \infty \right\}
\]

we denote the set of escaping points, i.e. the subset of the Julia set consisting of points which escape to infinity under iteration of \( F_\omega \). The non-autonomous radial Julia set associated with a given sequence \( \omega \), denoted by \( \mathcal{J}_r(F_\omega) \), is the set of all points \( z \in \mathcal{J}(F_\omega) \) such that \( F^n_\omega(z) \) is defined for all \( n \in \mathbb{N} \) and there is some \( \delta > 0 \) such that for infinitely many \( n \in \mathbb{N} \), the disk \( B(F^n_\omega(z), \delta) \) can be pulled back univalently along the orbit of \( z \).

**Lemma 2.2.** If \( \mathfrak{F} = (f_n)_{n \in \mathbb{N}} \) is a sequence of meromorphic functions and \( \xi \in \mathbb{C} \) is a point such that there exists a sequence \( \xi_k \to \xi \), \( \xi_k \neq \xi \), and there is a subsequence \( (n_j)_{j \in \mathbb{N}} \) such that

\[
\lim_{j \to \infty} |(F^{n_j})'(\xi_k)| = \infty
\]

where \( (F^{n_j}(\xi_k))_{j \in \mathbb{N}} \) is bounded for all \( k \geq 1 \), then \( \xi \in \mathcal{J}(F) \).

**Proof.** By way of contradiction suppose \( \xi \in \mathcal{J}(F) \). Then there is some neighborhood \( U \ni \xi \) such that the iterates \( F^n|_U : U \to \hat{\mathbb{C}} \) forms a normal family. Without loss of generality, suppose that \( F^{n_j}|_U \) converges uniformly to some function \( g : U \to \hat{\mathbb{C}} \) with \( g(\xi) \in \mathbb{C} \). Choose \( \delta > 0 \) sufficiently small such that \( g(B(\xi, \delta)) \subseteq B(g(\xi), 1) \). Then we can choose
\[ \delta' < \delta \] sufficiently small such that \( g'(B(\xi, \delta')) \subseteq B(g'(\xi), 1) \). Now we choose \( k \) large such that \( \xi_k \in B(\xi, \delta') \). Then we have

\[ 1 + |g'(\xi)| \geq |g'(\xi_k)| = \lim_{j \to \infty} |(F^n)'(\xi_k)| = \infty, \]

which is a contradiction. Thus we must have that \( \xi \in \mathcal{J}_\Lambda. \)

The main technique used throughout this article will be to build a non-autonomous iterated function system which sits comfortably within the Julia set. We now recall some properties of non-autonomous IFSs and their generalization non-autonomous graph directed Markov systems.

2.3. Non-Autonomous Graph Directed Markov Systems. We will begin by considering the directed multigraph \((V, E, i, t)\) such that there is a sequence of finite subsets \( V_n \subseteq V \) for \( n \geq 0 \) such that \( V = \bigcup_{n \geq 0} V_n \) and for each \( n \geq 1 \) we have that \( v \in V_n \) if and only if there is an edge \( e \in E \) such that \( t(e) = v \) and \( i(e) \in V_{n-1} \), where \( i(e) \) and \( t(e) \) respectively represent the initial and terminal vertices of the edge \( e \). We let \( I^n \) denote the collection of edges which connect vertices originating in \( V_n \) and terminating in \( V_{n-1} \) and for each \( n \in \mathbb{N} \) we let

\[ A^{(n)} : I^{(n)} \times I^{(n+1)} \to \{0, 1\} \]

be the incidence matrix defined by the property that if \( A^{(n)}_{ab} = 1 \) then \( t(a) = i(b) \). We then let \( I^n \) be the set of all admissible words of length \( n \), i.e.

\[ I^n = \{ \omega = \omega_1 \omega_2 \ldots \omega_n : \omega_j \in I^j \text{ and } A_{\omega_j \omega_{j+1}} = 1 \text{ for } 1 \leq j \leq n - 1 \}. \]

For each \( v \in V_n \) we consider a non-empty, compact, connected set \( X_v^{(n)} \subseteq \mathbb{R}^d \) which is regularly closed, that is \( X_v^{(n)} = \text{Int}(X_v^{(n)}) \).

**Definition 2.3.** A non-autonomous conformal graph directed Markov system (NCGDMS) \( \Phi \) is a sequence of maps, incidence matrices, and spaces together with a multigraph denoted by

\[ \Phi = \left\{ (\Phi^{(n)})_{n \in \mathbb{N}} : (A^{(n)})_{n \in \mathbb{N}} : (X^{(n)})_{n \geq 0} : (V_n)_{n \geq 0} : (I^{(n)})_{n \in \mathbb{N}} : E, i, t \right\} \]

where

\[ \Phi^{(n)} = \left\{ \varphi^{(n)}_e : X^{(n)}_{t(e)} \to X^{(n-1)}_{i(e)} \right\}_{e \in I^{(n)}}, \]

such that the following hold:

1. (Open Set Condition): \( \varphi^{(n)}_a (\text{Int}(X^{(n)}_{t(a)})) \cap \varphi^{(n)}_b (\text{Int}(X^{(n)}_{t(b)})) = \emptyset \) for all \( n \in \mathbb{N} \), and \( a \neq b \in I^{(n)} \).
2. (Conformality): For all \( v \in V_n \) there is an open and connected \( W_v^{(n)} \supseteq X_v^{(n)} \) (independent of \( j \)) such that for each \( j \in I^{(n)} \) with \( t(j) = v \), the map \( \varphi^{(n)}_j \) extends to a \( C^1 \) conformal diffeomorphism of \( W_v^{(n)} \) into \( W_{i(j)}^{(n-1)} \). Moreover, we can assume that

\[ \text{diam}(W_v^{(n)}) \leq 2 \text{ diam}(X_v^{(n)}) \]
for each \( n \in \mathbb{N} \) and each \( v \in V_n \).

(3) (Uniform Contraction): There is a constant \( \eta \in (0, 1) \) such that
\[
\left| (\varphi_{\omega_j^{j+m}})'(x) \right| \leq \eta^m
\]
for all sufficiently large \( m \in \mathbb{N} \), all \( \omega \in I^\infty \), all \( j \geq 1 \), and all \( x \in X_{l(\omega_{j+m})}^{(j+m)} \), where
\[
\omega_{j+m} = \omega_j \omega_{j+1} \ldots \omega_{j+m}.
\]

(4) (Bounded Distortion): There is \( K \geq 1 \) such that for all \( m \in \mathbb{N} \), for any \( k \leq m \), for all \( \omega \in I_{k,m} \),
\[
\left| (\varphi_{\omega_k^m}(\omega))'(x) \right| \leq K \left| (\varphi_{\omega_k^m}(\omega))'(y) \right|
\]
for all \( x, y \in X_{l(\omega)}^{(m)} \).

(5) (Geometry Condition): There exists \( N \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \) and all \( v \in V_n \) there exist \( \Gamma_1^{(n)}, \ldots, \Gamma_N^{(n)} \subseteq W_v^{(n)} \) such that each of the \( \Gamma_j^{(n)} \) are convex, and
\[
X_v^{(n)} \subseteq \bigcup_{j=1}^N \Gamma_j^{(n)}.
\]

We also suppose there exists \( \vartheta > 0 \) such that for each \( x \in X_v^{(n)} \) we have that
\[
B(x, \vartheta \cdot \text{diam}(X_v^{(n)})) \subseteq W_v^{(n)}.
\]

(6) (Uniform Cone Condition): There exist \( \alpha, \gamma > 0 \) with \( \gamma < \frac{\pi}{2} \) such that for every \( n \in \mathbb{N} \), every \( v \in V_n \), and every \( x \in X_v^{(n)} \) there is an open cone
\[
\text{Con}(x, u_x, \gamma, \alpha \cdot \text{diam}(X_v^{(n)})) \subseteq \text{Int}(X_v^{(n)})
\]
with vertex \( x \), direction vector \( u_x \), central angle of measure \( \gamma \), and altitude \( \alpha \cdot \text{diam}(X_v^{(n)}) \) comparable to diam(\( X_v^{(n)} \)).

(7) (Diameter Condition): For each \( n \in \mathbb{N} \) we have
\[
\lim_{n \to \infty} \frac{1}{n} \log d_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \sup_{k \geq 0} \log \frac{d_{k+n}}{d_k} = 0,
\]
where
\[
d_n = \min \{ \text{diam}(X_v^{(n)}) : v \in V_n \} \quad \text{and} \quad \overline{d}_n = \max \{ \text{diam}(X_v^{(n)}) : v \in V_n \}.
\]
Furthermore, we assume that
\[
\lim_{n \to \infty} \frac{1}{n} \log \#V_n = 0.
\]

A NCGDMS \( \Phi \) is called stationary if the sequence of sets \( X^{(n)} \) is constant, i.e. if \( X^{(n)} = X^{(m)} \) for all \( n, m \in \mathbb{N} \). In other words, the collection \( X \) of compact connected spaces does not depend on the time \( n \). To emphasize when a particular NCGDMS is not stationary, we will call that system non-stationary. If the collections \( V_n \) are singletons for every \( n \in \mathbb{N} \) and if the matrices \( A^{(n)} \) contain only ones, i.e. every letter at time \( n + 1 \) is allowed to
follow every letter at time \( n \), then we refer to the system \( \Phi \) as a non-autonomous conformal iterated function system (NCIFS).

A NCGDMS \( \Phi \) is called finite if the collections \( \Phi^{(n)} \) are finite for each \( n \), and infinite otherwise. \( \Phi \) is said to be uniformly finite if there is a constant \( M > 0 \) such that \( \#I^{(n)} < M \) for each \( n \in \mathbb{N} \).

**Remark 2.4.** Notice that if each of the spaces \( X^{(n)} \) are convex then the Uniform Cone Condition and the Geometry Condition hold. Furthermore, Koebe’s Distortion Theorem implies that the Bounded Distortion Property holds for dimension \( d = 2 \). If the system \( \Phi \) is in fact a stationary NCIFS, then the Uniform Cone Condition and Diameter Condition are automatically satisfied.

The limit set of a NCGDMS \( \Phi \) is defined as

\[
J_\Phi = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^n} \varphi_\omega(X).
\]

For each \( n \in \mathbb{N} \) and \( 0 \leq t \leq d \) we define the potential functions

\[
Z_n(t) = \sum_{\omega \in I^n} \| (\varphi_\omega)' \|^t \quad \text{and} \quad Z^{(n)}(t) = \sum_{i \in I^{(n)}} \| (\varphi_i^{(n)})' \|^t,
\]

where we take \( \| \cdot \| \) to denote the sup norm. Bounded distortion implies that

\[
Z_n(t) \geq K^{-nt} Z^{(1)}(t) \cdots Z^{(n)}(t).
\]

The lower pressure function can then be defined as

\[
P(t) = \liminf_{n \to \infty} \frac{1}{n} \log Z_n(t).
\]

We say that **Bowen’s formula holds** for the system \( \Phi \) if the Hausdorff dimension of the limit set coincides with the Bowen dimension of the limit set, that is if

\[
\text{HD}(J_\Phi) = B_\Phi,
\]

where the Bowen dimension \( B_\Phi \) is given

\[
B_\Phi := \sup \{ t \geq 0 : P(t) \geq 0 \} = \inf \{ t \geq 0 : P(t) \leq 0 \} = \sup \{ t \geq 0 : Z_n(t) \to \infty \}.
\]

**Definition 2.5.** We say that a NCGDMS \( \Phi \) is subexponentially bounded if

\[
\lim_{n \to \infty} \frac{1}{n} \log \#I^{(n)} = 0.
\]

We say the \( \Phi \) is **finitely primitive** if there is a constant \( p \in \mathbb{N} \) such that for each \( n \in \mathbb{N} \), there is a finite set \( \Lambda_n \subseteq I^{n+1,n+p} \) such that the following hold.

1. For all \( a \in I^{(n)} \), \( b \in I^{(n+p+1)} \) there is \( \lambda(a,b) := \lambda \in \Lambda_n \) such that \( a\lambda b \in I^{n,n+p+1} \).
2. There is some constant \( Q > 0 \) such that for each \( m \in \mathbb{N} \) and for each \( \lambda \in \Lambda_m \) we have \( Q \leq \| (\varphi^{m+1}_{\lambda(m+1,m+p)})' \| \).

**Remark 2.6.** Note that any finite system for which the incidence matrices \( A^{(n)} \) consist solely of ones, e.g. NCIFS, is automatically finitely primitive and both of the constants \( p \) and \( Q \) may be taken to be zero.
In the sequel we will make use of the following two theorems.

**Theorem 2.7** (Corollary 9.4 of [2]). Let $\Phi$ be a finite NCGDMS that is finitely primitive and subexponentially bounded. If there is some constant $M > 0$ such that for each $n \in \mathbb{N}$ we have
\[
Z_{(n)}(t) = \sum_{i \in I^{(n)}} \left\| \left( \varphi_i^{(n)} \right)' \right\|^t \leq M
\]
then Bowen’s formula holds.

We can do much better in the case of NCIFS. While we still require that the system be subexponentially bounded, we can drop any and all assumptions on the derivatives of the maps $\varphi^{(n)}$.

**Theorem 2.8.** If $\Phi$ is a finite, subexponentially bounded NCIFS, then Bowen’s formula holds.

See [17] for a proof in the stationary setting, and [2] for a proof in the non-stationary setting.

Throughout the article, for $R > 0$ we let
\[
B_R = \{ z \in \mathbb{C} : |z| > R \}.
\]

We will also use the symbols $\asymp$ and $\lesssim$ to denote comparable values, by which we mean that $A \asymp B$ if and only if there is some constant $C \geq 1$ such that $C^{-1}A \leq B \leq CA$ and $A \lesssim B$ if and only if there is $C \geq 1$ such that $A \leq CB$.

3. Affine Perturbations

Let $f_0$ be a meromorphic function of finite order $\rho$ and suppose that the following hold.

1. $f_0$ has at least one pole $b \in f_0(\mathbb{C})$ which is not in the closure of the singular values, $\text{Sing}(f_0^{-1})$. Let $b$ be such a pole and let $q$ be the multiplicity of $b$.
2. There are constants $s > 0$, $Q > 0$ and $\alpha > -1 - 1/q$ such that
   \[
   |f_0'(z)| \leq Q |z|^\alpha \quad \text{for} \quad z \in f_0^{-1}(U), |z| \to \infty.
   \]
   where $U = B(b, s)$.

The proof of the main theorem of this section will rely on our ability to construct a finite NCIFS, contained the non-autonomous Julia set $\mathcal{J}(F_A)$, for which we can find a suitable lower bound for its Hausdorff dimension.

**Theorem 3.1.** Let $f_0$ be a meromorphic function of finite order $\rho$ satisfying conditions [1] and [2] above with pole $b$ and neighborhood $U_0 = B(b, s_0)$. For each $0 \leq t < \frac{\rho}{\alpha + 1 + 1/q}$ there exists $\delta_t > 0$ and $\varepsilon_t > 0$ such that if $(\lambda_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ are sequences in $\mathbb{C}$ such that
\[
\lambda_n, \lambda_n^{-1} \in B(1, \delta_t) \quad \text{and} \quad |c_n| < \varepsilon_t
\]
for each $n \in \mathbb{N}$, then $\text{HD}(\mathcal{J}_t(F_A)) \geq t$. 
If in addition $f_0$ is of maximal divergence type, then there exist $\varepsilon, \delta > 0$ such that if $(\lambda_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ are sequences in $\mathbb{C}$ such that

$$\lambda_n, \lambda_n^{-1} \in B(1, \delta) \quad \text{and} \quad |c_n| < \varepsilon$$

for each $n \in \mathbb{N}$, then $\text{HD}(J_\varphi(F_A)) \geq \frac{\rho}{\alpha + 1 + 1/q}$.

Proof. Suppose $b$ is a pole of $f_0$ with multiplicity $q$. Notice that there is some neighborhood, $W$, of $b$ such that there is some function $g_0$ with $g_0(b) \neq 0$, which is analytic on $U_0$ and $f_0(z) = \frac{g_0(z)}{(z-b)^q}$. Without loss of generality we assume that $s_0$ is sufficiently small such that the following hold.

(i) No singular values of $f_0$ belongs to $U^* := B(b, 2s_0)$, i.e. $\text{Sing}(f_0^{-1}) \cap U^* = \emptyset$.

(ii) For each $w \in U^* \setminus \{b\}$ we have

$$|f'_0(w)| \leq \frac{1}{|w-b|^{q+1}} \sim |f_0(w)|^{1+1/q}.$$  \hspace{1cm} (3.2)

Now let

$$V := f_0(U_0 \setminus \{b\}),$$

which is a nonempty neighborhood of $\infty$. Choose $R > 0$ sufficiently large such that $B_R \subseteq V$. Define the $b$-points for $f_0$ to be the set $\{z_m^{(0)} : m \in \mathbb{N}\} = f_0^{-1}(b)$. Then $z_m^{(0)} \to \infty$ as $m \to \infty$. For each $z_m^{(0)} \in f_0^{-1}(b) \cap V$ and let $\varphi_m^{(0)}$ denote the inverse branch of $f_0$ such that $\varphi_m^{(0)}(b) = z_m^{(0)}$. In Claim 3.1 of [12], Mayer shows that there exists $M_0 \in \mathbb{N}$ such that for all $m \geq M_0$,

$$\varphi_m^{(0)}(U_0) \subseteq B_R \subseteq V.$$

As $f_0$ is an unbranched covering of $V$, for each $m \geq M_0$, we define $\psi_m^{(0)}$ to be an inverse branch of $f_0$ defined on $\varphi_m^{(0)}(U_0)$. Mayer then shows that the infinite autonomous iterated function system given by

$$\Phi_0 = \{\gamma_m^{(0)} : U_0 \to U_0 : m \geq M_0\},$$

where $\gamma_m^{(0)} := \psi_m^{(0)} \circ \varphi_m^{(0)}$, is such that the limit set $J_{\Phi_0}$ of $\Phi_0$ is contained within the Julia set $J(f_0)$. Mayer is then able to estimate that

$$\text{HD}(J_{\Phi_0}) \geq \frac{\rho}{\alpha + 1 + 1/q}$$

by showing that

$$\sum_{m \geq M_0} |(\gamma_m^{(0)})'(b)| \geq \sum_{m \geq M_0} |z_m^{(0)}|^{-t(\alpha + 1 + 1/q)}.$$  \hspace{1cm} \text{(2.1)}

In view of (2.1), we see that $\frac{\rho}{\alpha + 1 + 1/q}$ is the critical exponent of the full series

$$\sum_{m \geq 0} |z_m^{(0)}|^{-t(\alpha + 1 + 1/q)}.$$
Thus, we obtain the desired result by applying Theorem 3.15 of [11], which says that the Hausdorff dimension of the limit set of an infinite autonomous IFS $S = \{\gamma_j : X \to X\}_{j \in \mathbb{N}}$ on a set $X$ is at least as large as

$$\theta := \inf \left\{ t > 0 : \sum_{j \in \mathbb{N}} |\gamma_j'(x)|^t < \infty, x \in X \right\}.$$  

(3.3)

Let

$$0 < r < \frac{s_0}{16K^2},$$  

(3.4)

where $K$ is the distortion constant, and set $U_1 = B(b, r)$. Since $z_m^{(0)} \to \infty$, we take $M_1 \geq M_0$ sufficiently large such that

$$\psi_m^{(0)}(\varphi_m^{(0)}(U_0)) \subseteq U_1 \subseteq U_0$$  

(3.5)

for all $m \geq M_1$.

Fix $t < \frac{\rho}{\alpha+1+1/q} \leq 2$. Then

$$\sum_{m \geq M_1} |z_m^{(0)}|^{-t(\alpha+1/q)} = \infty,$$  

(3.6)

and thus there is some $N_t \in \mathbb{N}$, depending on $t$, such that

$$\sum_{m = M_1}^{M_1+N_t} |z_m^{(0)}|^{-t(\alpha+1/q)} \geq 2^{1+2(4+2/q)}K^2Q^2L$$  

(3.7)

where $L \geq 1$ is the comparability constant coming from (3.2).

Now we choose $\varepsilon_t, \delta_t > 0$, which depend on $t$, such that the following hold.

- $\delta_t < \min \left\{ \frac{s_0}{s_0}, \frac{s_0}{8} \cdot \frac{1}{2} \right\}$.
- $\varepsilon_t < \min \left\{ \frac{1}{2}, \delta_t \right\}$.
- $(1 + \delta_t) [r + \delta_t(1 + |b|)] < s_0$.
- For all $M_1 \leq m \leq M_1 + N_t$ we have

$$0 < \delta_t < \frac{-1 + \sqrt{1 + 4\frac{s_0 - r}{1+r+|b|}}}{2}$$

and that one can see that (3.8) is non-vacuous by rewriting as a cubic equation and then applying Descartes’ rule of signs in view of (3.4).

For each $n \in \mathbb{N}$, we suppose that $|c_n| < \varepsilon_t$ and that both $\lambda_n, \lambda_n^{-1} \in B(1, \delta_t)$. Furthermore, for each $w \in \mathbb{C}$, define

$$f_n(w) = \lambda_n f_0(w) + c_n.$$
We now claim that for each \( n \in \mathbb{N} \) we have that \[ b \in \frac{U_1 - c_n}{\lambda_n} \subseteq U_0, \]
where \( \frac{U_1 - c_n}{\lambda_n} := \left\{ \frac{w - c_n}{\lambda_n} : w \in U_1 \right\} \). To see this we simply calculate
\[
\left| \frac{w - c_n}{\lambda_n} - b \right| \leq \left| \frac{w - c_n}{\lambda_n} - \frac{b - c_n}{\lambda_n} \right| + \left| \frac{b - c_n}{\lambda_n} - b \right|
\]
\[
= |\lambda_n^{-1}| \left| w - b \right| + |\lambda_n^{-1}| \left| b - c_n - \lambda_n b \right|
\]
\[
\leq (1 + \delta_t) [r + |c_n| + |1 - \lambda_n| |b|]
\]
\[
\leq (1 + \delta_t) [r + \delta_t (1 + |b|)] < s_0.
\]

Now for each \( n \in \mathbb{N} \) and \( m \geq M_1 \) we define the inverse branch \( \varphi_m^{(n)} \) of \( f_n \) on \( U_1 \) by
\[ \varphi_m^{(n)}(w) = \varphi_m^{(0)} \left( \frac{w - c_n}{\lambda_n} \right), \]
and hence, by definition, we have that
\[ \varphi_m^{(n)}(U_1) \subseteq \varphi_m^{(0)}(U_0) \]
for all \( m \geq M_1 \) and \( n \in \mathbb{N} \). We also note that condition \( \textbf{31} \) above ensures that
\[ |(\varphi_m^{(n)})'(w)| \geq K^{-1} |\varphi_m^{(n)}(w)|^{-\alpha}, \]
for each \( w \in U_1 \). For each \( n \geq 0 \) and \( m \geq M_1 \), set
\[ z_m^{(n)} = \varphi_m^{(n)}(b). \]
Now since \( b \in \frac{U_1 - c_n}{\lambda_n} \) we have that \( z_m^{(0)} = \varphi_m^{(0)}(b) = \varphi_m^{(n)}(\lambda_n (b + c_n)) \in \varphi_m^{(n)}(U_1) \) for all \( m \geq M_1 \) and \( n \in \mathbb{N} \). In particular, applying Koebe’s Distortion Theorem, we have that
\[ |z_m^{(n)} - z_m^{(0)}| \leq \operatorname{diam}(\varphi_m^{(n)}(U_1)) \leq 2Kr \left| (\varphi_m^{(n)})'(b) \right| \leq 2 |\lambda_n^{-1}| K^2 r \left| (\varphi_m^{(0)})'(b) \right|. \]
Furthermore, we also have that for each \( w \in U_1 \)
\[ |\varphi_m^{(n)}(w) - z_m^{(n)}| \leq 2 |\lambda_n^{-1}| K^2 r \left| (\varphi_m^{(0)})'(b) \right| \leq (1 + \delta_t) 2K^2 r \left| (\varphi_m^{(0)})'(b) \right|. \]
Thus for \( w \in U_1 \) we have that
\[
\left| \varphi_m^{(n)}(w) - z_m^{(0)} \right| \leq \left| \varphi_m^{(n)}(w) - z_m^{(n)} \right| + \left| z_m^{(n)} - z_m^{(0)} \right|
\]
\[
\leq |\lambda_n^{-1}| \left( |c_n| + \left| \varphi_m^{(n)}(w) - z_m^{(n)} \right| + \left| z_m^{(n)} - \lambda_n z_m^{(n)} \right| \right) + \left| z_m^{(n)} - z_m^{(0)} \right|
\]
\[
\leq (1 + \delta_t) \left[ \varepsilon_t + (1 + \delta_t) 2K^2 r \left| (\varphi_m^{(0)})'(b) \right| + \delta_t \left| z_m^{(n)} \right| + 2K^2 r \left| (\varphi_m^{(0)})'(b) \right| \right].
\]
Now since we can write
\[ |z_m^{(n)}| \leq |z_m^{(n)} - z_m^{(0)}| + |z_m^{(0)}| \leq (1 + \delta_t) 2K^2 r \left| (\varphi_m^{(0)})'(b) \right| + |z_m^{(0)}|, \]
thus, upon grouping like terms, we have that
\[
\left| \frac{\varphi_m^{(n)}(w) - c_{n-1}}{\lambda_{n-1}} - z_m^{(0)} \right| \leq (1 + \delta_t) \left[ \delta_t + \delta_t \left| z_m^{(0)} \right| + (\delta_t^2 + 2\delta_t + 2)2K^2r \left| (\varphi_m^{(0)})'(b) \right| \right].
\]

In light of (3.3), we see that
\[
\left| \frac{\varphi_m^{(n)}(w) - c_{n-1}}{\lambda_{n-1}} - z_m^{(0)} \right| < \frac{s_0}{4} \left| (\varphi_m^{(0)})'(b) \right|
\]
for all \(M_1 \leq m \leq M_1 + N_t\). Thus, by Koebe’s Distortion Theorem, we have that
\[
\frac{\varphi_m^{(n)}(U_1) - c_{n-1}}{\lambda_{n-1}} \subseteq B \left( z_m^{(0)}, \frac{s_0}{4} \left| (\varphi_m^{(0)})'(b) \right| \right) \subseteq \varphi_m^{(0)}(U_0)
\]
for all \(n \in \mathbb{N}\) and \(M_1 \leq m \leq M_1 + N_t\). Define \(\psi_m^{(n)} : \varphi_m^{(n+1)}(U_1) \to U_0\) by
\[
\psi_m^{(n)}(z) = \psi_m^{(0)} \left( \frac{z - c_n}{\lambda_n} \right).
\]

In view of (3.3), and by our choice of \(M_1\), have that \(\psi_m^{(n)}\) is well defined and moreover
\[
\psi_m^{(n-1)}(\varphi_m^{(n)}(U_1)) \subseteq U_1
\]
for all \(n \in \mathbb{N}\) and all \(M_1 \leq m \leq M_1 + N_t\). Then \(\psi_m^{(n)}\) is the inverse branch of \(f_n\) restricted to \(\varphi_m^{(n+1)}(U_1)\). Now let
\[
U_m^{(n)} := \psi_m^{(2n-1)} \circ \varphi_m^{(2n)}(U_1) \subseteq U_1,
\]
and define
\[
\gamma_m^{(n)} := \psi_m^{(2n-1)} \circ \varphi_m^{(2n)} : U_1 \to U_1
\]
for all \(n \in \mathbb{N}\) and \(M_1 \leq m \leq M_1 + N_t\). Letting \(I_m^{(n)} = \{ m \in \mathbb{N} : M_1 \leq m \leq M_1 + N_t \}\), for each \(n \in \mathbb{N}\) we set
\[
\Phi^{(n)} := \{ \gamma_m^{(n)} : U_1 \to U_m^{(n)} : m \in I_m^{(n)} \},
\]
and let
\[
\Phi := (\Phi^{(n)})_{n \in \mathbb{N}}.
\]
Since the images of the inverse branches are disjoint, the open set condition is satisfied, and as \(\#I_m^{(n)} = N_t + 1\) for each \(n \in \mathbb{N}\), we have that \(\Phi\) is a uniformly finite NCIFS. Thus Bowen’s formula holds, i.e.
\[
\text{HD}(J_\Phi) = B_\Phi.
\]

Now to see that \(J_\Phi \subseteq J(F_A)\) as desired, suppose \(z \in J_\Phi\). Since \(|\gamma'_m(z)| \to 0\) as \(|\omega| = n \to \infty\) then we see \(|(F_{\lambda}^{2n})'(z)| \to \infty\) as \(n \to \infty\). Thus, in view of Lemma 2.2, we see that the limit set \(J_\Phi\) is contained in the Julia set \(J(F_A)\). Furthermore, by construction, we have that \(J_\Phi \subseteq J(\varphi_m^{(n)})\).

Now we estimate a lower bound of the Hausdorff dimension of the limit set of \(\Phi\) analogous to \(\theta\) given in (3.3) for the autonomous setting. For more see [2].
Let $u^{(n)}_m = \lambda_{2n}b + c_{2n}$ for each $n \in \mathbb{N}$ and $M_1 \leq m \leq M_1 + N_t$. Notice that

$$\varphi_m^{(2n)}(w^{(n)}_m) = z^{(0)}_m.$$

Then

$$Z_{(n)}(t) \geq \sum_{m=M_1}^{M_1 + N_t} |(\gamma_m^{(n)})'(w^{(n)}_m)|^t$$

$$= \sum_{m=M_1}^{M_1 + N_t} \left| (\psi_m^{(2n-1)}(z^{(0)}_m))^t \right| \left( \varphi_m^{(2n)}(w^{(n)}_m) \right)^t$$

$$= \sum_{m=M_1}^{M_1 + N_t} \left| f_{2n-1}'(\psi_m^{(2n-1)}(z^{(0)}_m)) \right|^{-t} \left| f_{2n}'(z^{(0)}_m) \right|^{-t}$$

$$\geq (1 - \delta_t)^{2t} \sum_{m=M_1}^{M_1 + N_t} \left| f_0'(\psi_m^{(2n-1)}(z^{(0)}_m)) \right|^{-t} \left| f_0'(z^{(0)}_m) \right|^{-t}.$$

As $b \neq \psi_m^{(2n-1)}(z^{(0)}_m) \in U_t$ and $u^{(n)}_m \in U_t$, applying (3.1), (3.2), and (3.7) we see

$$Z_{(n)}(t) \geq Q^{-t}(1 - \delta_t)^{2t} \sum_{m=M_1}^{M_1 + N_t} \left| f_0'(\psi_m^{(2n-1)}(z^{(0)}_m)) \right|^{-t} \left| z^{(0)}_m \right|^{-t \alpha}$$

$$\geq Q^{-t}L^{-1}(1 - \delta_t)^{2t} \sum_{m=M_1}^{M_1 + N_t} \left| f_0'(\psi_m^{(2n-1)}(z^{(0)}_m)) \right|^{-t(1+1/q)} \left| z^{(0)}_m \right|^{-t \alpha}$$

$$= Q^{-t}L^{-1}(1 - \delta_t)^{2t} \sum_{m=M_1}^{M_1 + N_t} \left| f_0 \left( \psi_m^{(0)} \left( \frac{z^{(0)}_m - c_{2n-1}}{\lambda_{2n-1}} \right) \right) \right|^{-t(1+1/q)} \left| z^{(0)}_m \right|^{-t \alpha}$$

$$= Q^{-t}L^{-1}(1 - \delta_t)^{2t} \sum_{m=M_1}^{M_1 + N_t} \left| \lambda_{2n-1}^{-t(1+1/q)} \sum_{m=M_1}^{M_1 + N_t} \left| z^{(0)}_m - c_{2n-1} \right|^{-t(1+1/q)} \left| z^{(0)}_m \right|^{-t \alpha}$$

$$\geq Q^{-t}L^{-1}(1 - \delta_t)^{t(3+1/q)} \sum_{m=M_1}^{M_1 + N_t} \left| z^{(0)}_m - c_{2n-1} \right|^{-t(1+1/q)} \left| z^{(0)}_m \right|^{-t \alpha}.$$

Given that $t < 2$, $\delta_t < \frac{1}{2}$, and that $\left| z^{(0)} - c_{2n-1} \right| \leq 2 \left| z^{(0)}_m \right|$ for all $n \in \mathbb{N}$ and $M_1 \leq m \leq M_1 + N_t$, as if this were not the case we could otherwise take $M_1$ sufficiently large, we have

$$Z_{(n)}(t) \geq Q^{-t}L^{-1}2^{-t(1+1/q)}(1 - \delta_t)^{t(3+1/q)} \sum_{m=M_1}^{M_1 + N_t} \left| z^{(0)}_m \right|^{-t(1+1/q)} \left| z^{(0)}_m \right|^{-t \alpha}.$$
Thus, in light of (2.2) we see
\[ Z_n(t) \geq 2^n, \]
which in turn implies that \( P(t) > 0 \), and hence \( \text{HD}(J_\Phi) \geq t \).

Now if \( f_0 \) is also of maximal divergence type, then taking \( t = \rho = \frac{\rho}{\alpha + 1 + 1/q} \) we may return to (3.6) and notice that the sum still diverges. Hence we may continue on with the proof from that point as written, and thus reach the conclusion that
\[ \text{HD}(J_r(F_A)) \geq \frac{\rho}{\alpha + 1 + 1/q}, \]
which finishes the proof. \( \square \)

**Remark 3.2.** We would like to point out that as \( t \) approaches the critical value \( \frac{\rho}{\alpha + 1 + 1/q} \) the perturbative values \( \varepsilon_t, \delta_t \), which both depend upon \( t \), tend towards zero. Indeed, as \( t \) increases towards the critical exponent, the number \( N_t \) increases to infinity, which means that the maximum modulus of the points \( z_m^{(0)} \) increases towards infinity, which means that the perturbative values \( \varepsilon_t \) and \( \delta_t \) must simultaneously decrease to zero.

This is precisely the reason why we are unable to show that the hyperbolic dimension of the affine perturbations are at least as great as \( \frac{\rho}{\alpha + 1 + 1/q} \). However, the following corollary, which was first proven by Mayer in [12], follows from our Theorem 3.1 in the setting of autonomous systems.

**Corollary 3.3.** Let \( f \) be a meromorphic function of finite order \( \rho \) which satisfies the hypotheses of the previous theorem, including the constants \( \alpha \) and \( q \). Then
\[ \text{HD}(J_r(f)) \geq \frac{\rho}{\alpha + 1 + 1/q}. \]
If in addition, \( f \) is of maximal divergence type then the inequality becomes strict.

### 4. Different Transcendental Meromorphic

We now follow Kotus and Urbański’s paper [10]. Let \( \mathcal{P} = \mathcal{P}(f) = f^{-1}(\infty) \) denote the set of all poles of the function \( f \). Let \( m \) be the function on the set of poles \( \mathcal{P} \) which assigns to each pole \( a \), its multiplicity \( m(a) \). In this section we will consider a transcendental meromorphic function, \( f_0 : \mathbb{C} \rightarrow \hat{\mathbb{C}} \), of finite order \( \rho > 0 \) such that the following hold.

1. \( \infty \) is not an asymptotic value of \( f_0 \).
2. There exists a co-finite set \( \mathcal{P}^* \subseteq \mathcal{P} \) such that \( \text{dist}(\text{Sing}(f^{-1}), a) > 0 \) for all \( a \in \mathcal{P}^* \).
(3) There exist \( M \in \mathbb{N}, \beta \geq 0, \) and \( R_0 > 0 \) such that for each \( a \in \mathcal{P} \)
\[
|f_0(z)| \preceq \frac{|a|^{-\beta}}{|z-a|^{m(a)}} \quad \text{and} \quad |f'_0(z)| \preceq \frac{m(a)|a|^{-\beta}}{|z-a|^{m(a)+1}}
\]
for \( z \in B(a, R_0) \), where \( m(a) \in \mathbb{N} \) with \( 1 \leq m(a) \leq M \).

Note that Lemma 2.1 implies that \( f_0 \) has infinitely many poles. As \( m : \mathcal{P} \to \mathbb{N} \) takes on only finitely many values, there is \( M \in \mathbb{N} \) such that \( m(a) \leq M \) for each \( a \in \mathcal{P} \) and there is a largest integer \( M^* \leq M \) such that the sum
\[
\sum_{a \in m^{-1}(M^*)} (1 + |a|)^{-t}
\]
is finite for \( t > \rho \) and infinite for \( t < \rho \).

**Theorem 4.1.** If \( f_0 \) satisfies the above conditions (1)-(3), then there is \( \varepsilon > 0 \) such that if \( (c_n)_{n \in \mathbb{N}} \) is a sequence in \( \mathbb{C} \) with \( |c_n| < \varepsilon \) for all \( n \in \mathbb{N} \) then
\[
\HD(I_{\infty}(F_+)) \leq \frac{\rho M}{\beta + M + 1}.
\]

**Remark 4.2.** The idea behind the proof relies on fact that \( \infty \) is not an asymptotic value of \( f_0 \), nor \( f_n \) for any \( n \). This means that points which escape to infinity under iterates of \( F_c \) must remain close to poles.

**Proof.** Take \( R_0 \) sufficiently large such that
\[
\text{dist}(\text{Sing}(f_0^{-1}), \mathcal{P}^*) > 2R_0.
\]
Let \( 0 < S < R_0 \) and choose \( 0 < \varepsilon < R_0 - 2S \). Hypothesis (3) on \( f_0 \) above implies that as \( a \) ranges over \( \mathcal{P}^* \), the balls \( B(a, R_0) \) are mutually disjoint. Now for \( R \) sufficiently large, say \( R \geq R_1 \geq R_0 \), we have that \( B_R \subseteq f_0(B(a, R_0)) \) for each \( a \in \mathcal{P}^* \) since
\[
|a|^{-\beta} R_0^{-m(a)} \lesssim |a|^{-\beta} \lesssim 1.
\]
Now, let \( R_2 \geq R_1 \) sufficiently large such that
\[
B_{R_1} \subseteq f_0(B(a, R_0)) \quad \text{and} \quad \text{dist}(\text{Sing}(f_0^{-1}), a) > 2R_0
\]
for all \( a \in \mathcal{P} \cap B_{R_2} \). For each \( a \in \mathcal{P} \) and \( R > 0 \) we let \( B_a(R) \) denote the connected component of \( f_0^{-1}(B_R) \) which contains \( a \). Then for \( R, |a| > R_2 \) we have
\[
B_a(R) \subseteq B(a, R_0).
\]
Now hypothesis (3) also implies that there is a constant \( L \geq 1 \) such that for all \( a \in \mathcal{P} \) and all \( R \geq 2R_0 \) we have
\[
\text{diam}(B_a(R)) \leq LR^{-1/m(a)}|a|^{-\beta/m(a)}.
\]
Now choose \( R_3 > R_2 \) sufficiently large such that for all \( R \geq R_3 \) we have
\[
\text{diam}(B_a(R)) \leq LR^{-1/m(a)}|a|^{-\beta/m(a)} \leq LS^{-1/m(a)}|a|^{-\beta/m(a)}.
\]
If $U \subseteq B_{R_1} \setminus \{\infty\} \cap \bigcup_{a \in \mathcal{P}} B(a, 2R_0)$ is an open and simply-connected, then all holomorphic inverse branches $f_{a,0,U,j}^{-1}$ of $f_0$, which take $U$ into $B(a, R_0)$, are all well defined for $1 \leq m(a) \leq M$. Hypothesis (3) then allows us to write

$$\left| (f_{a,0,U,j}^{-1})'(z) \right| \leq |z|^{-\frac{m(a)+1}{m(a)}} |a|^{-\frac{\beta}{m(a)}}$$

for $z \in U$. Now let $K \geq 1$ be the comparability constant for the previous equation (4.4).

For two poles $a_1, a_2 \in B_{2R_4}$ we denote by $f_{a_1, a_2,j}^{-1} : B(a_2, 2R_0) \to \mathbb{C}$, $j = 1 \leq j \leq m(a_1)$, all inverse branches of $f_0$. It then follows that

$$f_{a_1, a_2,j}^{-1}(B(a_2, R_0)) \subseteq B_{a_1}(2R_3 - R_0) \subseteq B_{a_1}(R_3) \subseteq B(a_1, S) \subseteq B(a_1, R_0).$$

Now let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $|c_n| < \epsilon$ for all $n \in \mathbb{N}$ and define $f_n(z) = f_0(z) + c_n$ for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Furthermore, let the function $F_+ : \mathbb{C} \to \hat{\mathbb{C}}$ be defined by

$$F_+^n(z) = f_n \circ \cdots \circ f_1(z)$$

for all $n \in \mathbb{N}$. By our choice of $S$ and $\epsilon > 0$ we have that $z - c_n \in B(a, R_0)$ for all $z \in B(a, 2S)$ and $n \in \mathbb{N}$. Thus, for poles $a_1, a_2 \in B_{2R_3}$ the inverse branches $f_{n,a_1,a_2,j}^{-1} : B(a_2, 2S) \to \mathbb{C}$, $1 \leq j \leq m(a_1)$, are well defined and given by

$$f_{n,a_1,a_2,j}^{-1}(z) = f_{a_1, a_2,j}^{-1}(z - c_n)$$

for $z \in B(a_2, 2S)$. Moreover, in view of (4.5), we have that

$$f_{n,a_1,a_2,j}^{-1}(B(a_2, S)) \subseteq B(a_1, S)$$

for each $n \in \mathbb{N}$ and $1 \leq j \leq m(a_1)$. Now set

$$I_R(F) := \{ z \in \mathbb{C} : |F_+^n(z)| > R \text{ for all } n \geq 1 \}.$$

Since $\sum_{a \in \mathcal{P}} |a|^{-u}$ converges if and only if $u > \rho$, then given $t > \frac{\rho M}{\beta+M+1}$, there is $R_4 \geq R_3$ sufficiently large such that

$$MK^t \sum_{a \in \mathcal{P} \setminus B_{R_4}} |a|^{-t} \frac{\frac{\beta+M+1}{M}}{M} \leq 1.$$

Let $R_5 > 4R_4$ and define $I := \mathcal{P} \cap B_{R_4}$. Now in view of (4.5) and (4.6), it follows that for every $n \in \mathbb{N}$ and $R > 2R_5$ the family of sets

$$W_n = \left\{ f_{1,a_0,a_1,j_0}^{-1} \circ \cdots \circ f_{n,a_{n-1},a_n,j_{n-1}}^{-1}(B_{a_n}(R/2)) : a_i \in I, 1 \leq j_i \leq m(a_i), i = 0, \ldots, n \right\}$$

is well defined and covers $I_R(F_+)$. To see this we note that since $\infty$ is not an asymptotic value for $f_n$, each of the connected components of the inverse images of $B_R$ under $f_n$ contain neighborhoods of poles.

In light of (4.3) and (4.4), we can write the following estimate

$$\Sigma_n = \sum_{a_0 \in I} \sum_{j_0=1}^{m(a_0)} \cdots \sum_{a_{n-1} \in I} \sum_{j_{n-1}=1}^{m(a_{n-1})} \sum_{a_n \in I} \text{diam}_t(f_{1,a_0,a_1,j_0}^{-1} \circ \cdots \circ f_{n,a_{n-1},a_n,j_{n-1}}^{-1}(B_{a_n}(R/2)))$$
Remark 4.3. Thus we must have \( \text{HD}(I_R(F_+)) \leq L^t(2/S)^{t/M} \), Now setting \( I_{R,e}(F_+) := \left\{ z \in \mathbb{C} : \liminf_{n \to \infty} |F^n(z)| > R \right\} = \bigcup_{n \geq 1} F^{-n}(I_R(F_+)) \), we see that \( \text{HD}(I_{\infty}(F_+)) \leq \text{HD}(I_{R,e}(F_+)) = \text{HD}(I_R(F_+)) \leq t \). Letting \( t \to \frac{\rho M}{\beta + M + 1} \), provides the desired result. 

Thus \( (4.7) \) gives us that \( \Sigma \leq L^t(2/S)^{t/M} \). For each \( n \in \mathbb{N} \), since the diameters of the sets of the covers \( W_n \) converge to 0 uniformly as \( n \to \infty \), we can estimate the Hausdorff measure to be \( \text{HD}(I_R(F_+)) \leq L^t(2/S)^{t/M} \). Thus we must have \( \text{HD}(I_R(F_+)) \leq t \). Now setting \( I_{R,e}(F_+) := \left\{ z \in \mathbb{C} : \liminf_{n \to \infty} |F^n(z)| > R \right\} = \bigcup_{n \geq 1} F^{-n}(I_R(F_+)) \), we see that \( \text{HD}(I_{\infty}(F_+)) \leq \text{HD}(I_{R,e}(F_+)) = \text{HD}(I_R(F_+)) \leq t \). Letting \( t \to \frac{\rho M}{\beta + M + 1} \), provides the desired result. 

Remark 4.3. It should be noted that in the previous theorem, and in the theorem to follow, our choice of the perturbative value \( \varepsilon \) no longer depends upon \( t \), it as it does in Theorem 3.1.

Together with Theorem 4.1, the following theorem completes the proof of the first part of Theorem 3.2.

Theorem 4.4. If \( f_0 \) satisfy the same hypotheses (1)-(3) as in the previous theorem, then \( \text{HD}(I_{\infty}(F_+)) \geq \frac{\rho M^*}{\beta + M^* + 1} \).
Proof. In order to prove Theorem 4.4 we follow the insights of Remark 4.2 in order to construct a NCGDMS which is contained in $I_{\infty}(F_+)$. Let $R_0, \ldots, R_4, S, \varepsilon$ be as in the previous proof. Then for $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{C}$ with $|c_n| < \varepsilon$ and two poles $a_1, a_2 \in B_{2R_4}$ we have

\begin{equation}
 f_{n,a_1,a_2,1}^{-1}(\overline{B}(a_2, S)) \subseteq \overline{B}(a_1, S).
\end{equation}

Enumerate the set

\[ \mathcal{P} := \mathcal{P} \cap B_{2R_4} \cap m^{-1}(M^*) \]

in such a way that $|a_n| \leq |a_{n+1}|$ for each $n \in \mathbb{N}$. Since $\sum_{a \in \mathcal{P}} |a|^{-u}$ converges if and only if $u > \rho$, then for a fixed $t < \frac{\rho M^*}{\beta + M^* + 1}$, let the number $\xi_{n,t} = \xi_n$, depending on $t$, be the least integer such that

\begin{equation}
 \sum_{j=n}^{\xi_n} |a_j|^{-t \frac{\beta + M^* + 1}{M}} \geq 2K^4,
\end{equation}

where $K$ is defined as before, to be the constant of comparability coming from (4.4). As $|a_n| \to \infty$ as $n \to \infty$ we see that $\xi_n \to \infty$ as $n \to \infty$ as well. Define $\beta_n := \xi_{n+1} - \xi_n$. By definition we have

\begin{equation}
 \gamma_k := \sum_{i=1}^{k} \beta_i = \xi_{k+1} - \beta_1.
\end{equation}

Without loss of generality, we may assume that $\beta_1 > 1$, otherwise we may increase $R_4$ to be sufficiently large. Now we seek to define a NCGDMS whose limit set sits inside of the set of escaping points. To that end, we begin by defining the directed multigraph on which our system operates. Let $V = \mathbb{N}$ be the set of vertices. We define the vertices at each time $n \geq 0$ inductively as follows. Let $V_0 = \{1, \ldots, \beta_1\}$. For each $1 \leq j \leq \gamma_1$ take $V_j = \{1, \ldots, \beta_1 + j\}$. If $j = \beta_1 = \gamma_1$, then we have in particular, $V_{\gamma_1} = \{1, \ldots, \xi_2\}$. Now for $\gamma_1 + j \leq \gamma_2$ let $V_{\gamma_1 + j} = \{2, \ldots, \xi_2 + j\}$. In general for $\gamma_k < n \leq \gamma_{k+1}$ with $n = \gamma_k + j$ for $1 \leq j \leq \beta_{k+1}$ we set

\[ V_n = \{k + 1, \ldots, \xi_{k+1} + j\}. \]

Notice that, by construction, we have that

\begin{equation}
 \sum_{a \in V_n} |a|^{-t \frac{\beta + M^* + 1}{M}} \leq 2K^4 + |a_1|
\end{equation}

for each $n \in \mathbb{N}$. Now let $I^{(n)} = V_{n-1} \times V_n$ for each $n \in \mathbb{N}$ and define the functions

\[ i((j,k)) = j \quad \text{and} \quad t((j,k)) = k \]

for $(j,k) \in I^{(n)}$ to be the initial and terminal vertices, respectively, of the edge $(j,k)$. We take the incidence matrices $A^{(n)}$ to be composed of all ones. In other words, this means that we allow all possible infinite words $\omega = \omega_1 \omega_2 \ldots$ where $\omega_n \in I^{(n)}$ for each $n \in \mathbb{N}$. Finally, letting

\[ X^{(n)} = \{ \overline{B}(a_j, S) : j \in V_n \} \]
for each $n \geq 0$ and taking

$$
\Phi^{(n)} := \left\{ \varphi_{(j,k)}^{(n)} : \overline{B}(a_k, S) \to \overline{B}(a_j, S) | (j,k) \in I^{(n)} \right\}
$$

for each $n \in \mathbb{N}$, where $\varphi_{(j,k)}^{(n)} = f_{n,j,k,1}^{-1}$, defines a finite, non-stationary non-autonomous conformal graph directed Markov system

$$
\Phi = \left( (\Phi^{(n)})_{n \in \mathbb{N}} : (A^{(n)})_{n \in \mathbb{N}} : (X^{(n)})_{n \geq 0} : (V_n)_{n \geq 0} : (I^{(n)})_{n \in \mathbb{N}} : i, t \right).
$$

Indeed, since each of the sets $\overline{B}(a_n, S)$ is convex, we have that the Uniform Cone, and Geometry Conditions are immediately satisfied. Furthermore the diameters are constant and the collections $V_n$ grow subexponentially, so we also have that the Diameter Condition is immediately satisfied as well.

By construction, we have that

$$
J_\Phi := \bigcap_{n=1}^{\infty} \bigcup_{\omega \in I^{(n)}} \varphi_\omega(X) \subseteq \mathcal{I}_\infty(F_+)
$$

since every pole is eventually discarded from the construction in favor of a pole of higher modulus. Furthermore, we have that $\Phi$ is finitely primitive and subexponentially bounded.

As we wish to estimate the Hausdorff dimension of the limit set by means of the pressure function, we seek to apply Theorem 2.7 in order to show that Bowen’s formula holds. In order to do that, we first calculate $Z_{(n)}(t)$. In view of (1.11), for $n \in \mathbb{N}$, we have that

$$
Z_{(n)}(t) = \sum_{(a,b) \in I^{(n)}} \left\| (\varphi_{(a,b)}^{(n)})^t \right\|^t = \sum_{(a,b) \in V_{n-1} \times V_n} \left\| (f_{n,a,b,1}^{-1})^t \right\|^t
= K^t \sum_{(a,b) \in V_{n-1} \times V_n} |b|^{-\frac{M^{n+1}}{M^t}} |a|^{-\frac{\beta}{M^t}}
= K^t \sum_{a \in V_{n-1}} |a|^{-\frac{\beta}{M^t}} \sum_{b \in V_n} |b|^{-\frac{M^{n+1}}{M^t}}
\leq K^2 (2K^4 + |a_1|)^2.
$$

Now that we have that Bowen’s formula holds, we aim to find a lower bound for $B_\Phi$. Again we turn to estimating $Z_{(n)}(t)$, but this time we seek a lower bound. Thus, for $n \in \mathbb{N}$ we have

$$
Z_{(n)}(t) = \sum_{(a,b) \in I^{(n)}} \left\| (\varphi_{(a,b)}^{(n)})^t \right\|^t = \sum_{(a,b) \in V_{n-1} \times V_n} \left\| (f_{n,a,b,1}^{-1})^t \right\|^t
\geq K^{-t} \sum_{(a,b) \in V_{n-1} \times V_n} |b|^{-t \frac{M^{n+1}}{M^t}} |a|^{-t \frac{\beta}{M^t}}
= K^{-t} \sum_{a \in V_{n-1}} |a|^{-t \frac{\beta}{M^t}} \sum_{b \in V_n} |b|^{-t \frac{M^{n+1}}{M^t}}.
$$

Now, concerning $n, n-1$, there are three cases to consider.
(i) \( n, n - 1 \neq \gamma_k \) for any \( k \in \mathbb{N} \).
(ii) There is \( k \in \mathbb{N} \) such that \( n = \gamma_k, n - 1 < \gamma_k \).
(iii) There is \( k \in \mathbb{N} \) such that \( n - 1 = \gamma_k, n > \gamma_k \).

In each case we see that the sets \( V_n \) and \( V_{n-1} \) are the same but for at most two points. Specifically we have that

- (i) \( V_{n-1} = \{k+1, \ldots, \xi_{k+1} + j\} \) and \( V_n = \{k+1, \ldots, \xi_{k+1} + j + 1\} \),
- (ii) \( V_{n-1} = \{k, \ldots, \xi_{k+1} - 1\} \) and \( V_n = \{k, \ldots, \xi_{k+1}\} \),
- (iii) \( V_{n-1} = \{k, \ldots, \xi_{k+1}\} \) and \( V_n = \{k+1, \ldots, \xi_{k+1} + 1\} \),

where in case (i) we have that \( n - 1 = \gamma_k + j \) for some \( 1 \leq j < \beta_{k+1} \). Letting \( V^*_n = V_{n-1} \cap V_n \),

the following calculation holds in each of the three cases. Continuing from (4.12) we see

\[
Z_n(t) \geq K - t \sum_{a \in V_{n-1}} |a|^{-t \frac{\beta}{M}} \sum_{b \in V_n} |b|^{-t \frac{\lambda^*}{M+1}}
\]

\[
\geq K - t \sum_{a \in V^*_n} |a|^{-t \frac{\beta}{M^*}} \sum_{a \in V^*_n} |a|^{-t \frac{\lambda^*}{M^*+1}}
\]

\[
\geq K - t \sum_{a \in V^*_n} |a|^{-t \frac{\beta + M^* + 1}{M^*+1}} \geq 2^{4-t}.
\]

Applying (2.2) we see that

\[
Z_n(t) \geq K - nt Z(1)(t) \cdots Z(n)(t) \geq 2^n K^{4n-2nt} \geq 2^n.
\]

Thus for \( t < \frac{\beta M^*}{\beta + M^*+1} \) we have that \( P(t) > 0 \), which implies that

\[
t \leq \text{HD}(J_\Phi) \leq \text{HD}(I_{\infty}(F_+)).
\]

Letting \( t \to \frac{\beta M^*}{\beta + M^*+1} \) finishes the proof. \( \square \)

**Remark 4.5.** We should point out that although we have chosen to present Theorem 4.4 within the generality of non-autonomous dynamics, the previous result, to the best of the author’s knowledge, was not previously known even in the autonomous case.

In the previous two theorems, we were only able to prove a lower bound for the set of escaping points for additive perturbations of \( f_0 \). As we shall see in the following theorem, our choice of \( \delta > 0 \), for multiplicative perturbations, depends on the modulus of the pole at time \( n \). In the following theorem we will need only finitely many poles to obtain our lower bound. However, to estimate the set of escaping points, we require infinitely many poles whose moduli are going to infinity, which would require us to choose \( \delta = 0 \) and leading to only trivial multiplicative perturbations, i.e. \( \lambda_n \equiv 1 \). We encountered a similar situation for our choice of \( \delta \) in Theorem 3.1.

The following theorem differs from the previous two in two main ways. First, we no longer require that \( \infty \) is not an asymptotic value for \( f_0 \), but rather instead, we will only need that \( f_0 \) has infinitely many poles. Second, much like in the proof of Theorem 3.1, our choice of our perturbative values \( \varepsilon, \delta \) will again depend upon the value of \( t \).
Theorem 4.6. Let $f_0 : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function with infinitely many poles that satisfies hypotheses (2)–(3). Then for each $0 \leq t < \frac{\rho M^*}{\beta + M^* + 1}$ there exist $\varepsilon_t, \delta_t > 0$ such that if $(c_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ are sequences in $\mathbb{C}$ such that $|c_n| < \varepsilon_t$ and $\lambda_n, \lambda_n^{-1} \in B(1, \delta_t)$ for each $n \in \mathbb{N}$ then

$$\text{HD}(\mathcal{J}_t(F_A)) \geq t.$$ 

If in addition, $f_0$ is of maximal divergence type then there exists $\varepsilon, \delta > 0$, no longer depending on $t$, such that

$$\text{HD}(\mathcal{J}_t(F_A)) \geq \frac{\rho M^*}{\beta + M^* + 1}.$$

Proof. With the exception of the choice of $\varepsilon$, the proof runs the same as the proof of Theorem 4.1 up to (4.5), i.e. let $R_0, \ldots, R_4, S$ be the same such that we have

$$f_{0,a_1,a_2}^{-1}(B(a_2, R_0)) \subseteq B_{a_1}(2R_3 - R_0) \subseteq B_{a_1}(R_3) \subseteq B(a_1, S) \subseteq B(a_1, R_0).$$

Again let

$$\mathcal{P} := \mathcal{P} \cap B_{2R_4} \cap m^{-1}(M^*) = \{a_0, a_1, \ldots\}$$

be enumerated, such that $|a_n| \leq |a_{n+1}|$ for all $a_n \in \mathcal{P}$ and all $n \geq 0$. Now since $\sum_{a \in \mathcal{P}} |a|^{-u}$ converges if and only if $u > \rho$, then for $t < \frac{\rho M^*}{\beta + M^* + 1}$, there is some $N_t \in \mathbb{N}$, depending on $t$, such that

$$\sum_{n=1}^{N_t} |a_n|^{-\frac{2 + M^* + 1}{M^*}} \geq 2K^6 |a_0|^{-\frac{2 + M^* + 1}{M^*}}.$$

Let $I = \{a_1, \ldots, a_{N_t}\}$. Choose $\varepsilon_t, \delta_t > 0$ such that the following hold

- $\varepsilon_t < \delta_t$.
- $\delta_t < \frac{R_0 - 2S}{2S}$.
- $(1 + \delta_t)(\delta_t(1 + |a|) + S) < \frac{R_0}{2}$ for all $a \in I$.

Let $(c_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ be sequences in $\mathbb{C}$ such that $|c_n| < \varepsilon_t$ and $\lambda_n, \lambda_n^{-1} \in B(1, \delta_t)$ for each $n \in \mathbb{N}$ and define $f_n : \mathbb{C} \to \hat{\mathbb{C}}$ to be the affine perturbation of $f_0$ at time $n$ given by

$$f_n(z) = \lambda_n f_0(z) + c_n.$$

By our choice of $\varepsilon_t, \delta_t$ we have that for each $a \in I$ and each $z \in B(a, S)$

$$\frac{z - c_n}{\lambda_n} \in B(a, R_0).$$

Indeed,

$$\left| \frac{z - c_n}{\lambda_n} - a \right| \leq |\lambda_n^{-1}| (|c_n| + |z - a| + |1 - \lambda_n| |a|)$$

$$\leq (1 + \delta_t) (\varepsilon_t + S + \delta_t |a|)$$

$$\leq (1 + \delta_t) (\delta_t(1 + |a|) + S) < \frac{R_0}{2}.$$
The requirement that \( \delta_t < \frac{R_0 - 2S}{2S} \) ensures that such a \( \delta_t \) exists. As it implies that \((1 + \delta_t)S < R_0/2\), we see that solving (4.16) reduces to choosing
\[
0 < \delta_t < \frac{-1 + \sqrt{1 + 2R_0(1 + |a_N|)^{-1}}}{2}.
\]
Now for each \( a \in I \) we fix inverse branches of \( f_n \)
\[f_n^{-1}_{a,a,a_0,1} : \overline{B}(a, S) \to \mathbb{C} \quad \text{and} \quad f_n^{-1}_{a,a_0,1} : \overline{B}(a_0, S) \to \mathbb{C}.
\]
Together (4.13) and (4.15) gives us that
\[f_n^{-1}_{a,a_0,1}(\overline{B}(a, S)) \subseteq \overline{B}(a_0, S) \quad \text{and} \quad f_n^{-1}_{a,a_0,1}(\overline{B}(a_0, S)) \subseteq \overline{B}(a, S).
\]
For each \( n \in \mathbb{N} \) and \( a \in I \) we let the function \( \varphi_a^{(n)} \) be defined by
\[
\varphi_a^{(n)} := f_{2n-1,a,a_0,1}^{-1} \circ f_{2n,a,a_0,1}^{-1} : \overline{B}(a_0, S) \to \overline{B}(a, S).
\]
Then each of the functions \( \varphi_a^{(n)} \) is a contraction, and as there are only finitely many of them, they are in fact uniformly contracting. Then the collection
\[
\Phi = \{ \varphi_a^{(n)} \}_{n \in \mathbb{N}} = \{ \{ \varphi_a^{(n)} : a \in I \} \}_{n \in \mathbb{N}}
\]
forms a stationary NCIFS in the style of [17], for which Bowen’s formula holds. The limit set \( J_\Phi \) of the NCIFS \( \Phi \) is given by
\[
J_\Phi = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in F_n} \varphi_\omega(\overline{B}(a_0, S)).
\]
As \(|(\varphi_\omega)'(\zeta)| \to 0\) for \(|\omega| = n \to \infty\), we have that \(|F^{2n}(\zeta)| \to \infty\) as \( n \to \infty\). Thus Lemma 2.2 implies that \( J_\Phi \subseteq \mathcal{J}(F_A) \). Now for each \( n \in \mathbb{N} \) we estimate
\[
Z_{(n)}(t) = \sum_{a \in I} \| (\varphi_a^{(n)})' \|^t = \sum_{a \in I} \| (f_{2n-1,a,a_0,1}^{-1} \circ f_{2n,a,a_0,1}^{-1})(\varphi_a^{(n)})' \|^t
\geq K^{-t} \sum_{a \in I} \| (f_{2n-1,a,a_0,1})' \|^t \| (f_{2n,a,a_0,1})' \|^t
\geq K^{-2t} \sum_{a \in I} |a|^t \rho_{M^*}^{1+1/\alpha} |a|^{-t \frac{1+M^*+1}{M^*}}
\geq 2K^{-2t} |a|^t \rho_{M^*}^{1+1/\alpha}.
\]
Thus for each \( n \in \mathbb{N} \) we have
\[
Z_n(t) \geq K^{-nt} Z_{(1)}(t) \cdots Z_{(n)}(t) \geq 2^n K^{6n-3nt} |a_0|^{n(2-t) \frac{1+M^*+1}{M^*}} \geq 2^n.
\]
Thus \( P(t) > 0 \) and hence, \( \text{HD}(J_\Phi) \geq t \), which finishes the proof of the first statement.

Now if in addition \( f_0 \) is of maximal divergence type, then for \( t = \frac{\rho_{M^*}}{\beta + M^* - 1} \) we have that the sum
\[
\sum_{a_n \in \mathcal{P}} |a_n|^{t \frac{1+M^*+1}{M^*}} = \infty,
\]
and as such, we are able to find \( N_t > \infty \) as in (4.14). Continuing the proof from there in the same manner as before, we see that there is \( \varepsilon, \delta > 0 \) such that
\[
\frac{\rho M^*}{\beta + M^* + 1} \leq \text{HD}(J_\Phi) \leq \text{HD}(J_r(F_A)),
\]
completing the proof. \( \square \)

**Remark 4.7.** Note that, as in the proof of Theorem 3.1, our choices of \( \varepsilon, \delta \) must go to zero as \( t \) approaches the critical exponent unless we know that the function \( f_0 \) is of maximal divergence type. This is precisely because in the case where \( f_0 \) is of maximal divergence type we are assured a finite number \( N_t \) such that the sums in question, (3.6) and (4.14), are sufficiently large. If \( f_0 \) is not of maximal divergence type, then we must choose \( N_t \) equal to \( \infty \) which necessarily means that the values \( \varepsilon, \delta \) must be equal to zero as they are tied to the value of \( N_t \) in an inverse manner.

5. **Eventual Dimensions**

In this section we collect together several results, some of which are new and some which are already known, concerning the eventual dimension and the eventual hyperbolic dimension of several classes of transcendental functions. In particular, we provide results for the two main classes which have already been discussed.

The *eventual dimension* of a function \( f \), given by
\[
\text{ED}(f) = \lim_{R \to \infty} \text{HD}(\{z \in \mathcal{J}(f) : |f^n(z)| > R, \forall n \geq 1\}),
\]
was first introduced by Rempe-Gillen and Stallard in [16], though it had been used implicitly before by several authors. The following proposition was proven by Rempe-Gillen and Stallard first in the case of transcendental entire functions, however their same proof holds more generally for transcendental meromorphic functions.

**Proposition 5.1.** Let \( f \) be a transcendental meromorphic function. Then
\[
(5.1) \quad \text{HD}(I_\infty(f)) \leq \text{ED}(f) \leq \text{HD}(J(f)).
\]

In [4], Bergweiler and Kotus show that for a transcendental meromorphic function \( f \in \mathcal{B} \) of finite order \( \rho \) such that \( \infty \) is not an asymptotic value and there is some \( M \in \mathbb{N} \) such that the multiplicity of co-finitely many poles is at most \( M \) then
\[
\text{HD}(I_\infty(f)) \leq \text{ED}(f) \leq \frac{2M\rho}{2 + M\rho}.
\]
In fact, they provide a function \( f \) such that
\[
\text{HD}(I_\infty(f)) = \frac{2M\rho}{2 + M\rho} \quad \text{and} \quad \text{HD}\left(\left\{z \in \mathbb{C} : \liminf_{n \to \infty} |f^n(z)| \geq R\right\}\right) > \frac{2M\rho}{2 + M\rho}
\]
for all \( R > 0 \). In particular, we see that there is a transcendental meromorphic function \( f \) such that
\[
\text{HD}(I_\infty(f)) < \text{ED}(f).
\]
This of course shows that the first inequality of (5.1) may in fact be strict and the two quantities need not be equal.

The notion of the \textit{eventual hyperbolic dimension} of a function $f$ was introduced by De Zotti and Rempe-Gillen in [1] and is given by

$$EHD_1(f) = \sup \{\text{HD}(X) : X \subseteq B_R \text{ is hyperbolic for } f\},$$

where the set $X \subseteq \mathbb{C}$ is \textit{hyperbolic} for $f$ if $X$ is compact and forward invariant such that for some $n \in \mathbb{N}$ and some $\lambda > 1$ we have

$$|(f^n)'|_X > \lambda.$$

In [15] Rempe-Gillen shows that the hyperbolic dimension of a function $f$ is the same as the Hausdorff dimension of its radial Julia set, i.e.

$$\text{HypDim}(f) = \text{HD}(J_r(f)).$$

We now seek to show that the same relationship between the dimension of hyperbolic sets and the dimension of the radial Julia set is also true for the eventual hyperbolic dimension.

Rempe-Gillen actually proves his result in more generality for Ahlfors islands maps, which as the name suggests, exhibit the islands property. However, as we are primarily interested in transcendental meromorphic functions we will only require the classical Ahlfors five island theorem.

**Theorem 5.2** (Ahlfors Five Island Theorem). Let $f$ be a transcendental meromorphic function and let $D_1, \ldots, D_5$ be simply connected domains in $\mathbb{C}$ with disjoint closures. Then there exists $1 \leq j \leq 5$ and, for any $R > 0$, a simply connected domain $U \subseteq B_R$ such that $f|_U$ is a conformal isomorphism onto $D_j$.

Moreover, if $f$ has only finitely many poles then we need only consider three simply connected domains $D_i$ rather than five.

We shall also require the use of the following lemma and definition.

**Lemma 5.3** (Observation 2.12 in [15]). Suppose $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a meromorphic map and $B$ is a backwards invariant set, i.e. $f^{-1}(B) \subseteq B$. If $U$ is an open set which intersects $J(f)$, then

$$\text{HD}(B) = \text{HD}(B \cap U).$$

**Definition 5.4.** Given a meromorphic function $f$ and a point $z \in J_r(f)$, then the disk $B(w, \delta)$, for some $w \in \mathbb{C}$ and $\delta > 0$, is called a \textit{disk of univalence} for $z$ if it has the following properties.

1. For infinitely many $n \in \mathbb{N}$, $f^n(z) \in B(w, \delta)$ and $B(w, 2\delta)$ pulls back univalently to $z$ under $f^n$.
2. If $U$ is the component of $f^{-n}(B(w, \delta))$ containing $z$ then

$$\psi_n := \sup_{\zeta \in U} |(f^n)'(\zeta)| < \infty.$$
Koebe’s Distortion Theorem then gives that
\[ B \left( z, \frac{1}{4} \vartheta_n^{-1} \right) \subseteq U \subseteq B(z, K \vartheta_n^{-1}) \].
Furthermore, this implies that
\[ \lim_{n \to \infty} \vartheta_n = \infty. \]

**Remark 5.5.** Suppose \( D \) is a disk of univalence for \( z \) and \( z \not\in D \). Then there is a sequence \((n_k)_{k \in \mathbb{N}}\) in \( \mathbb{N} \) such that \( f^{n_k}(z) \in D \) for each \( k \in \mathbb{N} \). So in particular, \( D \) is a disk of univalence for \( f^{n_k}(z) \) for each \( k \geq 1 \). Therefore, there are elements of \( D \) for which \( D \) is a disk of univalence.

Rempe-Gillen proved in [15] that such disks exist. We are now ready to prove our result concerning the eventual hyperbolic dimension. The proof follows pretty closely with the one given by Rempe-Gillen but with a few minor changes. We include it for the sake of completeness.

**Theorem 5.6.** Given a meromorphic function \( f : \mathbb{C} \to \hat{\mathbb{C}} \), the quantities \( \text{EHD}_1(f) \) and
\[ \text{EHD}_2(f) := \lim_{R \to \infty} \text{HD} \left( \{ z \in \mathcal{J}_r(f) : |f^n(z)| > R, \forall n \geq 1 \} \right) \]
exist and are equal. We call their common value the eventual hyperbolic dimension of \( f \), and denote it by
\[ \text{EHD}(f) = \text{EHD}_1(f) = \text{EHD}_2(f). \]

**Proof.** First note that the limit in the definition of \( \text{EHD}_2(f) \) is well defined as the sets
\[ \mathcal{J}_r(f, R) := \{ z \in \mathcal{J}_r(f) : |f^n(z)| > R, \forall n \geq 1 \} \]
are descending as \( R \to \infty \).

Now given that every hyperbolic set \( X \) for \( f \) is contained in \( \mathcal{J}_r(f) \), we automatically have that
\[ \text{EHD}_1(f) \leq \text{EHD}_2(f). \]

Now for the opposite inequality begin by fixing \( R > 0 \). Let \( \varepsilon > 0 \) and set \( d := \text{EHD}_2(f) \). Then \( d = \lim_{R \to \infty} d_R \), where \( d_R := \text{HD}(\mathcal{J}_r(f, R)) \). Setting
\[ \text{HypDim}(f, R) := \sup \{ \text{HD}(X) : X \subseteq B_R \text{ is hyperbolic for } f \}, \]
we will show that \( \text{HypDim}(f, R) \geq \hat{d}_R := d_R - \varepsilon \).

Pick a countable basis for the topology of \( \hat{\mathbb{C}} \) which consists of rounded simply connected disks. Let \( \{ D_i \}_{i \in \mathbb{N}} \) be the collection of these disks which are contained in \( B_R \) and let \( \mathcal{J}_r(D_i) \) be the set of points in \( \mathcal{J}_r(f, R) \) for which \( D_i \) is a disk of univalence. Then clearly if \( z \in \mathcal{J}_r(D_i) \) and \( w \in \mathbb{C} \) with \( f(w) = z \) then either \( w \in \mathcal{J}_r(D_i) \) or \( w \) is a critical point. This
implies that $J_r(D_i)$ is the union of a backward invariant set, $\tilde{J}_r(D_i)$, and a countable set, i.e.

$$J_r(D_i) = \tilde{J}_r(D_i) \cup \bigcup_{n=0}^{\infty} f^n(\text{Crit}(f))$$

Lemma 5.3 then gives that for any non-empty open set $U$ which intersects $\tilde{J}_r(D_i)$ we must have that

$$\text{HD}(J_r(D_i)) = \text{HD}(\tilde{J}_r(D_i)) = \text{HD}(\tilde{J}_r(D_i) \cap U).$$

Now given that $J_r(f, R)$ is a countable union of the $J_r(D_i)$, we choose $j \in \mathbb{N}$ such that $\text{HD}(J_r(D_j)) > \hat{d}_R$ and set $D := D_j$. Now take $A = \tilde{J}_r(D) \cap D$, which is not only non-empty, given Remark 5.3 above, but we also have that $\text{HD}(A) > \hat{d}_R$. The definition of Hausdorff dimension and measure gives that there exists a $\delta > 0$ and a cover, $\{V_j\}_{j=1}^{\infty}$, of $A$ of sets of diameter less than $\delta$ (in fact we may take these sets $V_j$ to be balls of radius $\delta/2$ if desired) such that

$$\sum_{j=1}^{\infty} \text{diam}(V_j)^{\hat{d}_R} > (10K)^{\hat{d}_R}.$$

Now for each $a \in A$ take $n_a \in \mathbb{N}$ sufficiently large such that $U_a$, the component of $f^{-n_a}(D)$ containing $a$, is contained in $D$ with $U_a \subseteq D$ and set

$$\vartheta_a := \sup_{z \in U_a} |(f^{n_a})'(z)| > \frac{10K}{\delta}.$$

Then $A$ is covered by open balls of the form $B(a, 5K\vartheta_a^{-1})$, so applying the “5r”-covering theorem (see for example [6], Theorem 1.2) we can find a subsequence $a_j$ such that the balls $B(a_j, 5K\vartheta_a^{-1})$ are disjoint and we have that

$$A \subseteq \bigcup_{j} B(a_j, 5K\vartheta_a^{-1}).$$

But each of these balls has diameter $10K\vartheta_a^{-1}$, which is less than $\delta$. Thus we have

$$\sum_{j=1}^{\infty} \left(\frac{10K}{\vartheta_a^{-1}}\right)^{\hat{d}_R} > (10K)^{\hat{d}_R},$$

which implies that

$$\sum_{j=1}^{\infty} (\vartheta_a^{-1})^{\hat{d}_R} > 1.$$ 

Thus taking $N \in \mathbb{N}$ sufficiently large, we have

$$(5.2) \quad \sum_{j=1}^{N} (\vartheta_a^{-1})^{\hat{d}_R} > 1.$$
Now for each \(1 \leq j \leq N\), let \(\varphi_j\) be the inverse branch of \(f_n^a_j\) which maps \(D\) onto \(U_{a_j}\). Letting \(\Phi_R\) be the finite autonomous iterated function system generated by these maps \(\varphi_j\), we see that the limit set \(J_{\Phi_R}\) is contained in \(A\), and (5.2) gives that there is some \(\eta > 0\) such that for each \(n \in \mathbb{N}\) we have

\[
Z(n)(\hat{d}_R) \geq \sum_{j=1}^{N} \left( \inf_{x \in \mathbb{D}} |\varphi_j'(x)| \right)^{\hat{d}_R} > 1 + \eta > 1.
\]

Thus we see that \(P(\hat{d}_R) > 0\) and hence \(\text{HD}(J_{\Phi_R}) \geq \hat{d}_R\). \(J_{\Phi_R}\) together with finitely many forward-images forms our desired hyperbolic set showing that

\[
\text{HypDim}(f, R) \geq d_R - \varepsilon.
\]

Letting \(R \to \infty\), we have

\[
\text{EHD}_1(f) \geq \text{EHD}_2(f) - \varepsilon,
\]

which, as this holds for each \(\varepsilon > 0\), completes the proof. \(\square\)

**Remark 5.7.** Notice that the notion of eventual dimension immediately generalizes to include all non-autonomous functions and even though the idea of a hyperbolic set is not clear for non-autonomous dynamics, in light of the previous theorem we take the eventual hyperbolic dimension of a general non-autonomous function to be the Hausdorff dimension of its radial Julia set.

Clearly by definition, specifically the definition of \(\text{EHD}_2(f)\), we have that

(5.3) \quad \text{EHD}(f) \leq \text{ED}(f) \quad \text{and} \quad \text{EHD}(f) \leq \text{HD}(J_r(f)).

Together with (5.3), the following theorem completes the proof of Theorem 1.2. It’s proof follows from the proofs of Theorems 4.1 and 4.4 by letting \(R_0 \to \infty\) as each proof relies on the construction of a NCGDMS contained sufficiently well within \(B_{R_0}\).

**Theorem 5.8.** Suppose \(f_0\) satisfies the hypotheses of Theorem 4.4. Then there is \(\varepsilon > 0\) such that if \((c_n)_{n \in \mathbb{N}}\) is a sequence in \(\mathbb{C}\) with \(|c_n| < \varepsilon\) for all \(n \in \mathbb{N}\) then

\[
\frac{\rho M^*}{\beta + M^* + 1} \leq \text{EHD}(F_{+}) \leq \text{ED}(F_{+}) \leq \frac{\rho M}{\beta + M + 1}.
\]

The same alteration made to the proof of Theorem 4.6 i.e. letting \(R_0 \to \infty\), gives the following theorem which together with (5.3) and Theorem 4.6 finally completes the proof of Theorem 1.3.

**Theorem 5.9.** Let \(f_0 : \mathbb{C} \to \hat{\mathbb{C}}\) be a transcendental meromorphic function with infinitely many poles that satisfies hypotheses (2)-(3). Then for each \(0 \leq t < \frac{\rho M^*}{\beta + M^* + 1}\) there exist \(\varepsilon_t, \delta_t > 0\) such that if \((c_n)_{n \in \mathbb{N}}\) and \((\lambda_n)_{n \in \mathbb{N}}\) are sequences in \(\mathbb{C}\) such that \(|c_n| < \varepsilon_t\) and \(\lambda_n, \lambda_n^{-1} \in B(1, \delta_t)\) for each \(n \in \mathbb{N}\) then

\[
\text{EHD}(F_A) \geq t.
\]
If in addition, \( f_0 \) is of maximal divergence type then there exists \( \varepsilon, \delta > 0 \), no longer depending on \( t \), such that if \((c_n)_{n \in \mathbb{N}} \) and \((\lambda_n)_{n \in \mathbb{N}} \) are sequences in \( \mathbb{C} \) such that \( |c_n| < \varepsilon \) and \( \lambda_n, \lambda_n^{-1} \in B(1, \delta) \) for each \( n \in \mathbb{N} \) then

\[
\text{EHD}(F_A) \geq \frac{\rho M^*}{\beta + M^* + 1}.
\]

The following corollary is essentially a summary of our results in the best case scenario and follows directly from Theorems 1.2 and 1.3, Remark 1.4, and (5.3).

**Corollary 5.10.** Suppose \( f_0 \) satisfies the hypotheses of Theorem 4.1. Additionally suppose that we have that \( M = M^* \). Then there is \( \varepsilon > 0 \) such that if \((c_n)_{n \in \mathbb{N}} \) is a sequence in \( \mathbb{C} \) with \( |c_n| < \varepsilon \) for all \( n \in \mathbb{N} \) then

\[
\text{HD}(I_{\infty}(F_+)) = \text{EHD}(F_+) = \text{ED}(F_+) = \frac{\rho M}{\beta + M + 1} \leq \text{HD}(J_r(F_+)).
\]

If in addition \( f_0 \) is of maximal divergence type then there are \( \varepsilon, \delta > 0 \) such that if \((c_n)_{n \in \mathbb{N}}, (\lambda_n)_{n \in \mathbb{N}} \) are sequences in \( \mathbb{C} \) with \( |c_n| < \varepsilon \) and \( \lambda_n, \lambda_n^{-1} \in B(1, \delta) \) for all \( n \in \mathbb{N} \) then

\[
\text{HD}(I_{\infty}(F_+)) = \text{EHD}(F_+) = \text{ED}(F_+) = \frac{\rho M}{\beta + M + 1} \leq \text{EHD}(F_A) \leq \text{HD}(J_r(F_A)).
\]

The following theorem was mentioned briefly in [4] as a consequence of Mayer’s technique from [12], though no formal proof was given. We now give a short proof of the following theorem, which along with Theorem 3.1, completes the proof of Theorem 1.1.

**Theorem 5.11.** Let \( f \) be a meromorphic function of finite order \( \rho \) and suppose that the following hold.

1. \( f \) has infinitely many poles \( b_i \in f(\mathbb{C}) \) which is not in the closure of the singular values, \( \text{Sing}(f^{-1}) \). Let \( q_i \) be the multiplicity of \( b_i \) and suppose that \( q_i \leq q < \infty \) for each \( i \geq 1 \).

2. There are uniform constants \( s > 0, Q > 0 \) and \( \alpha > -1 - 1/q \) such that for each \( i \in \mathbb{N} \)

\[
|f'(z)| \leq Q |z|^\alpha \quad \text{for} \quad z \in f^{-1}(U_i), |z| \to \infty.
\]

where \( U_i = B(b_i, s) \).

Then \( \text{EHD}(f) \geq \frac{\rho}{\alpha + 1 + 1/q} \). If in addition \( f \) is of maximal divergence type, then this inequality is strict.

**Proof.** Since \( |b_i| \to \infty \) as \( i \to \infty \), Theorem 3.1 allows us to construct an autonomous iterated function system \( \Phi_i \) contained in \( J_r(f) \cap B_{R_i} \), where \( R_i = |b_i| - 2s \), such that

\[
\text{HD}(J_{\Phi_i}) \geq t
\]

for each \( t < \frac{\rho}{\alpha + 1 + 1/q} \). Letting \( i \to \infty \), and subsequently \( R_i \to \infty \), finishes the proof of the first part.
Now if $f$ is of maximal divergence type then the IFS $\Phi_i$ is hereditarily regular and it thus follows from Theorem 3.20 of [11] that we may sharpen our estimate so that

$$\text{HD}(J_{\Phi_i}) > t$$

for each $t \leq \frac{\rho}{\alpha+1+1/q}$. Again, letting $i \to \infty$ finishes the proof. \hfill $\square$

**Remark 5.12.** The proof gives more. In fact we see that for each $0 \leq t < \frac{\rho}{\alpha+1+1/q}$ and each $i \in \mathbb{N}$ there exists $\varepsilon_{i,t}, \delta_{i,t} > 0$ such that if $(\lambda_n)_{n \in \mathbb{N}}$ and $(c_n)_{n \in \mathbb{N}}$ are sequences in $\mathbb{C}$ with $\lambda_n, \lambda_n^{-1} \in B(1, \delta_{i,t})$ and $|c_n| < \varepsilon_{i,t}$ for each $n \in \mathbb{N}$ then

$$\text{HD}(\mathcal{J}_r(F_{A,i,t}) \cap B_{R_i}) \geq t,$$

where $R_i = |b_i| - 2s$.

It is worth noting that $\varepsilon_{i,t}$ and $\delta_{i,t}$ depend not only on $t$, but also on $|b_i|$, and in particular we have that $\lim_{i \to \infty} \delta_{i,t} = 0$ and, provided $f_0$ is not of maximal divergence type, $\lim_{t \to \frac{\rho}{\alpha+1+1/q}} \delta_{i,t} = 0$, as well as the respective statements for $\varepsilon_{i,t}$. So, we are unable to find non-autonomous perturbations that work uniformly for each pole $b_i, i \in \mathbb{N}$ and each $0 \leq t < \frac{\rho}{\alpha+1+1/q}$.

6. **Examples**

As the calculation of the perturbative values $\varepsilon, \delta$ can be quite complicated, for each of the following examples we will instead show that the necessary hypotheses are satisfied in order to apply our theorems.

**Example 6.1 (Periodic Functions).** The polynomial growth condition of (3.1) is satisfied for every periodic function $f$ with $\alpha = 0$. Therefore, we may apply Theorem 1.1 for any periodic function such that there exists a pole $b \not\in \text{Sing}(f - 1)$. With additional information, such as the existence of infinitely many poles, we may apply Theorems 1.2 and 1.3. The following example produces a class of such periodic functions.

**Example 6.2 (Rational Exponentials).** Let $f(z) = R(e^z)$, where $R$ is a rational function such that $R(0) \neq \infty$ and $R(\infty) \neq \infty$. Then $f$ is a simply periodic function with finitely many poles in each strip of periodicity. Furthermore, $\text{Sing}(f^{-1}) = \{R(0), R(\infty)\}$ and it is easy to check that we can apply Theorems 1.2 and 1.3 with $\rho = 1$ and $\beta = 0$.

In particular

$$f_0(z) = \mu(\tan(z))^m, \quad m \in \mathbb{N} \text{ and } \mu \in \mathbb{C}^*$$

is such a function. Moreover, since each of the poles are of multiplicity $m$, we can find $\varepsilon, \delta > 0$ such that

$$\text{HD}(I_{\infty}(F_+)) = \text{EHD}(F_+) = \text{ED}(F_+) = \frac{m}{m+1} \leq \text{HD}(\mathcal{J}_r(F_+)),$$
and
\[ \text{HD}(J_r(F_A)) \geq \text{EHD}(F_A) \geq \frac{m}{m + 1}. \]

This second inequality is precisely the inequality obtained for the autonomous case in \([7, 12]\). For autonomous dynamics, we can improve the inequality concerning the hyperbolic dimension. It follows from \([20]\) that \( \text{HD}(J_r(f_0)) > 1 \).

**Example 6.3 (Elliptic Functions).** Elliptic functions have been a subject of much study lately. Previously, the non-autonomous case of elliptic functions has been covered in \([2]\) while the autonomous and random cases have been discussed in \([12, 8, 10, 18]\), but is also covered more generally by our theorems here.

If \( f_0 \) is an elliptic function then by definition we have that there exists \( w_1, w_2 \in \mathbb{C} \) with \( \Im\left(\frac{w_1}{w_2}\right) > 0 \) such that \( f(z) = f(\zeta) \) if and only if \( \zeta = z + nw_1 + mw_2 \) for some \( n, m \in \mathbb{Z} \). Then we have that \( \rho = 2, \beta = 0 \), and so applying Theorem 1.2 we have that there exist \( \varepsilon, \delta > 0 \) such that
\[ \text{HD}(I_\infty(F_+)) = \text{EHD}(F_+) = \text{ED}(F_+) = \frac{2q}{q + 1} \leq \text{HD}(J_r(F_+)), \]
where \( q \) is the maximum multiplicity of each of the poles of \( f_0 \). As \( f_0 \) is of maximal divergence type, Theorem 1.3 gives that there exist \( \varepsilon, \delta > 0 \) such that
\[ \text{HD}(J_r(F_A)) \geq \text{EHD}(F_A) \geq \frac{2q}{q + 1}. \]

**Example 6.4 (Exponential Elliptics).** In \([14]\) Mayer and Urbański show that the Julia set of a function of the form
\[ f(z) = \mu e^{g(z)} \text{ for } \mu \in \mathbb{C}^*, \]
where \( g \) is an elliptic function, has Hausdorff dimension equal to 2. In \([12]\), Mayer shows that such functions also have hyperbolic dimension equal to 2. Unfortunately functions of this form do not satisfy the hypotheses of Theorem 1.1. However, functions of the form
\[ f_0^{(d)}(z) = \mu \left(1 + \frac{g(z)}{d}\right)^d \text{ for } \mu \in \mathbb{C}^*, d \in \mathbb{N}, \]
where \( g \) is an elliptic function, do satisfy the hypotheses of Theorem 1.1 with \( \rho = 2, \alpha = 0 \), and the maximum multiplicity of poles equal to \( dq \), where \( q \) is the maximum multiplicity of the poles of \( g \). Then we have that for any \( t < \frac{2d}{dq+1} \) there exist \( \varepsilon_t, \delta_t > 0 \) such that
\[ \text{HD}(J_r(F_A^{(d)})) \geq t, \]
where \( F_A^{(d)} \) is the function of non-autonomous affine perturbations of \( f_0^{(d)} \). As \( \varepsilon_t, \delta_t \) are independent of \( d \), letting \( d \to \infty \) we see that for any \( t < 2 \) there exist \( \varepsilon_t, \delta_t > 0 \) such that
\[ \text{HD}(J_r(F_A)) \geq t, \]
where \( F_A^{(d)} \) is the function of non-autonomous affine perturbations of the exponential elliptic function \( f \).
Example 6.5 (Polynomial Schwarzian Derivative). Recall that the Schwarzian derivative of a function \( f \) is given by
\[
S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.
\]
Exponential and tangent functions are examples of classes which have constant Schwarzian derivative. Examples for which \( S(f) \) is a polynomial are
\[
f(z) = \int_0^z e^{Q(w)} dw,
\]
where \( Q(w) \) is a polynomial, and
\[
f(z) = \frac{a A_i(z) + b B_i(z)}{c A_i(z) + d B_i(z)} \text{ with } ad - bc \neq 0,
\]
where \( A_i, B_i \) are the Airy functions of the first and second kind respectively. If \( S(f_0) = P \), a polynomial of degree \( d \), then one can show that it satisfies the hypotheses of Theorem 1.1 with \( \rho = d/2 + 1 \) and \( \alpha = d/2 \) and is even of maximal divergence type (see Section 2.4 of [13] for details). Applying Theorem 1.1 we have that there are \( \varepsilon, \delta > 0 \) such that
\[
\text{HD}(J_r(F_A)) \geq \text{EHD}(F_A) \geq \frac{d + 2}{d + 4} \geq \frac{1}{2}.
\]

Example 6.6. Let
\[
f_0(z) = \frac{1}{z \sin(z)}.
\]
Then \( f_0 \) is a meromorphic function of order \( \rho = 1 \) with infinitely many poles,
\[
\mathcal{P} = \{ n\pi : n \in \mathbb{Z} \},
\]
all of which are simple except for 0. We also have that \( \infty \) is not an asymptotic value and the set of singular values consists of the lone asymptotic value \( z = 0 \) and infinitely many critical values of the form \( v_n = \pm \frac{2}{(2n+1)\pi} \) for \( n \in \mathbb{Z} \), which implies that \( f_0 \in \mathcal{B} \). One can then show that \( \beta = 1 \) and that each pole has multiplicity equal to 1. As \( f_0 \) is of maximal divergence type, we may apply Theorems 1.2 and 1.3 to obtain that there exist \( \varepsilon, \delta > 0 \) such that
\[
\text{HD}(J_r(F_+)) = \text{EHD}(F_+) = \text{ED}(F_+) = \frac{1}{3} \leq \text{HD}(J_r(F_+))
\]
and
\[
\text{HD}(J_r(F_A)) \geq \text{EHD}(F_A) \geq \frac{1}{3}.
\]

Example 6.7. Let
\[
f_0(z) = \frac{1}{z \cos(\sqrt{z})}.
\]
Then $f_0$ is a meromorphic function with infinitely many poles

$$\mathcal{P} = \{0\} \cup \left\{ \frac{(2n + 1)\pi}{2} : n \in \mathbb{N} \right\},$$

all of which are simple and have multiplicity identically equal to 1. One can easily check that $\rho = 1/2$ and $\beta = 1/2$ and that $f_0$ is of maximal divergence type. Now since we have that the singular values of $f_0$ contain a single asymptotic value, $z = 0$ and infinitely many critical values $v_n$ such that

$$|v_n| = \frac{1}{(n\pi + (n\pi)^{-1}) \cos(\sqrt{n\pi + (n\pi)^{-1}})}.$$

As the right hand side tends towards 0 as $|n| \to \infty$ we have that $f_0 \in \mathcal{B}$, and consequently we are able to apply Theorems 1.2 and 1.3. Thus there exists $\varepsilon, \delta > 0$ such that

$$\text{HD}(I_{\infty}(F_+)) = \text{EHD}(F_+) = \text{ED}(F_+) = \frac{1}{5} \leq \text{HD}(J_r(F_+))$$

and

$$\text{HD}(J_r(F_A)) \geq \text{EHD}(F_A) \geq \frac{1}{5}.$$

REFERENCES

[1] Alexandre Dezotti and Lasse Rempe-Gillen. The eventual hyperbolic dimension of entire functions, 2014.

[2] Jason Atnip. Non-autonomous conformal graph directed Markov systems. arXiv:1706.09978 [math], June 2017.

[3] Walter Bergweiler. Iteration of meromorphic functions. Am. Math. Soc. Bull. New Ser., 29(2):151–188, 1993.

[4] Walter Bergweiler and Janina Kotus. On the Hausdorff dimension of the escaping set of certain meromorphic functions. Transactions Am. Math. Soc., 364(10):5369–5394, 2012.

[5] A. É. Ermenko and M. Yu. Lyubich. Dynamical properties of some classes of entire functions. Univ. de Grenoble. Annales de l’Institut Fourier, 42(4):989–1020, 1992.

[6] Juha Heinonen. Lectures on analysis on metric spaces. Springer, New York, 2001.

[7] Janina Kotus. On the Hausdorff dimension of Julia sets of meromorphic functions. II. Bull. de la Société; mathématique de France, 123(1):33–46, 1995.

[8] Janina Kotus and Mariusz Urbański. Hausdorff dimension and Hausdorff measures of Julia sets of elliptic functions. Bull. Lond. Math. Soc., 35(02):269–275, March 2003.

[9] Janina Kotus and Mariusz Urbański. Fractal measures and ergodic theory of transcendental meromorphic functions. In Transcendental dynamics and complex analysis, volume 348 of London Math. Soc. Lecture Note Ser., pages 251–316. Cambridge Univ. Press, Cambridge, 2008. DOI: 10.1017/CBO9780511735233.013.

[10] Janina Kotus and Mariusz Urbański. Hausdorff dimension of radial and escaping points for transcendental meromorphic functions. Ill. J. Math., 52(3):1035, 2008.

[11] R. Daniel Mauldin and Mariusz Urbański. Dimensions and measures in infinite iterated function systems. Proc. Lond. Math. Soc., 3(1):105–154, 1996.

[12] Volker Mayer. The size of the Julia set of meromorphic functions. Math. Nachrichten, 282(8):1189–1194, August 2009.
[13] Volker Mayer and Mariusz Urbański. *Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order*, volume 203 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2010.

[14] Volker Mayer, Mariusz Urbański, and others. Exponential elliptics give dimension two. *Ill. J. Math.*, 49(1):291–294, 2005.

[15] Lasse Rempe. Hyperbolic dimension and radial Julia sets of transcendental functions. *Proc. Am. Math. Soc.*, 137(4):1411–1420, 2009.

[16] Lasse Rempe and Gwyneth M. Stallard. Hausdorff dimensions of escaping sets of transcendental entire functions. *Proc. Am. Math. Soc.*, 138(05):1657–1665, December 2009. arXiv: 0904.3072.

[17] Lasse Rempe-Gillen and Mariusz Urbański. Non-autonomous conformal iterated function systems and Moran-set constructions. *Transactions Am. Math. Soc.*, 368(3):1979–2017, 2016.

[18] Mario Roy and Mariusz Urbański. Random graph directed Markov systems. *Discret. Contin. Dyn. Syst*, 30(1):261–298, 2011.

[19] Mitsuhiro Shishikura. The boundary of the Mandelbrot set has Hausdorff dimension two. *Astérisque*, (222):7, 389–405, 1994.

[20] Bartłomiej Skorulski. The existence of conformal measures for some transcendental meromorphic functions. *Complex dynamics*, 396:169–201, 2006. DOI: 10.1090/conm/396/07402.

---

Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA

E-mail address: jason.atnip@unt.edu

Web: http://atnipmath.com