BOUNDARY CONDITIONS AND HETEROTIC CONSTRUCTION IN TOPOLOGICAL MEMBRANE THEORY

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ABSTRACT

Using the topological membrane approach to string theory, we suggest a geometric origin for the heterotic string. We show how different membrane boundary conditions lead to different string theories. We discuss the construction of closed oriented strings and superstrings, and demonstrate how the heterotic construction naturally arises from a specific choice of boundary conditions on the left and right boundaries of a cylindrical topological membrane.

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1. Introduction

The heterotic string construction [1, 2], in which left and right sectors are taken from different string theories (critical bosonic and superstrings in the original papers [1]), has no clear geometrical meaning within string theory itself. Although the left and right sectors may each separately be viewed geometrically as a two-dimensional conformal field theory on a two-dimensional random surface (the worldsheet), the heterotic construction remains somewhat hybrid. The purpose of this letter is to suggest a geometric origin for the heterotic string using the topological membrane (TM) approach to string theory [3, 4]. It has recently been shown that dualities exist between different types of string theories, including dualities between heterotic and non-heterotic strings [5]. These dualities relate the strong coupling behaviour of one string theory to another at weak coupling. Hence, a weakly coupled heterotic string is dual to some strongly coupled string theory. From this point of view, we hope that in obtaining a better understanding of the geometric nature of the heterotic string at weak coupling, we might gain further insight into the strong coupling behaviour of other string theories.

The basic idea behind the TM approach to string theory is to fill in the string worldsheet and view it as the boundary of a three-manifold. If inside the three-manifold there is a topologically massive gauge theory (TMGT) [6], it induces the chiral gauged Wess-Zumino-Novikov-Witten (WZNW) action on the boundary worldsheet. This picture is based upon the remarkable connection between Chern-Simons and conformal field theories discovered in [7] and elaborated in [8]. Based partly upon the earlier work of [9], it was then found [3, 4] that the boundary degrees of freedom are also induced in TMGT, where massive vector particles propagate in the bulk. It is interesting to note that the same picture can be used to describe edge excitations in quantum Hall systems with boundaries. A nice theory of chiral edge states based on this approach was constructed in [10]. In some sense, gapless edge states in planar condensed matter systems are the “solid-state model” for chiral string sectors.

The basic property allowing a three-dimensional string construction is the chiral nature of the induced conformal fields on the boundary. To obtain a realistic closed string theory, of course, we need both left- and right-movers. We can accomplish this by using an annular or cylindrical topological membrane with left (L) and right (R) boundaries. The bulk TMGT then induces a chiral WZNW model with left-movers on L and right-movers on R. The crucial idea is that we can independently excite the left and right sectors by choosing different boundary conditions on the left and right boundaries — and this
is possible only in TMGT, where the gauge field propagates in the bulk, and not in a pure Chern-Simons theory where there are no such propagating degrees of freedom in the bulk, only topological ones. This is why the topological membrane is so important: by specifying different membrane boundary conditions (which we shall discuss in detail later), we can construct different closed strings, including both type II theories and also the heterotic string.

We begin by showing how TMGT induces chiral bosonic edge states and how their appearance depends upon the boundary conditions for the gauge field in the bulk. In Section 3 we show how the supersymmetric topological membrane leads to the appearance of fermions on the boundary. In this way we incorporate worldsheet SUSY. In Section 4 we show how different membrane boundary conditions lead to different string constructions. For each of these constructions we calculate the spectrum in the bulk of the membrane and the resultant Casimir energy. As we shall see in Section 5, this energy distinguishes between the heterotic construction and all the others. We conclude by comparing our results with recent work in M-theory. For simplicity, in this letter we shall only consider the abelian theory. We hope to give the non-abelian generalization in a future publication.

2. Bosonic edge states from TMGT

Let us consider how TMGT induces chiral bosonic edge states and how their appearance depends upon the boundary conditions for the gauge field. There are numerous papers in which this question was discussed, here we follow the lines of [11]. The abelian theory is defined by the action

\[ S_1 = -\frac{1}{4\gamma} \int_\mathcal{M} \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \frac{k}{16\pi} \int_\mathcal{M} \epsilon^{\mu\nu\alpha} F_{\mu\nu} A_{\alpha}. \]  

(2.1)

If the three-manifold \( \mathcal{M} \) is compact, then varying the action with respect to \( A_{\nu} \) yields the equations of motion

\[ \partial_\mu F^{\mu\nu} + \frac{m}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0, \]  

(2.2)

where \( m = \gamma k / 4\pi \). The Bianchi identity \( \partial_\mu F^{\mu} = 0 \) on the dual field strength \( F^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} \) follows immediately from (2.2). Moreover, taking the curl of (2.2) shows that the gauge field is indeed massive:

\[ (\partial^2 - m^2) F^\mu = 0, \]  

(2.3)
with one propagating degree of freedom (recall that a gauge field in \(d\)-dimensional spacetime describes \(d-2\) degrees of freedom, i.e. \(3-2=1\) in our case).

Now consider the case when \(\mathcal{M}\) has a boundary. We assume the boundary \(\partial \mathcal{M}\) contains the timelike direction and use light-cone coordinates \(x^\pm = x^0 \pm x^1\) for the directions on the boundary and \(x^2 = x^\perp\) for the perpendicular direction. Under arbitrary variations, \(S_1\) has no extrema:

\[
\delta S_1 = \int_\mathcal{M} \left( \frac{1}{\gamma} \partial_\mu F^{\mu \nu} - \frac{k}{8\pi} \epsilon^{\mu \lambda \nu} F_{\mu \lambda} \right) \delta A_\nu + \int_{\partial \mathcal{M}} n_\perp \Pi_\perp \delta A_i
\]

(2.4)

where \(n_\perp\) is a unit-vector normal to the boundary, \(i = +, -\) and

\[
\Pi_\perp = -\frac{1}{\gamma} F_\perp + \frac{k}{8\pi} \epsilon_\perp \delta A_j.
\]

(2.5)

To obtain a sensible path integral, with a well-defined classical limit, the boundary terms must somehow vanish. This problem is equivalent to choosing the necessary boundary conditions in order to solve the bulk equations of motion.

- **C boundary conditions:**

We may choose to fix \(A_+\) (upto gauge transformations) on \(\partial \mathcal{M}\), while allowing \(A_-\) to vary. To eliminate the surface term in (2.4) we must therefore add

\[
S_2 = -\int_{\partial \mathcal{M}} \Pi_\perp A_-
\]

(2.6)

and fix \(\delta \Pi_\perp = 0\) on \(\partial \mathcal{M}\). We denote these boundary conditions by \(C = \text{Conformal}\) and will now show that they lead to the appearance of conformal degrees of freedom on \(\partial \mathcal{M}\).

Although the combined action \(S = S_1 + S_2\) can now be extremized, it is no longer gauge invariant. The point is, however, that with the addition of \(S_2\) and upon gauge fixing, some gauge degrees of freedom become dynamical on \(\partial \mathcal{M}\). This is best seen via the Faddeev-Popov gauge-fixing procedure. Explicitly, under the \(U(1)\) gauge transformation \(A_\mu = \bar{A}_\mu + \partial_\mu \theta\), we have

\[
\delta S[\theta, \bar{A}_+, \Pi_\perp] = \frac{k}{8\pi} \int_{\partial \mathcal{M}} \partial_+ \theta \partial_- \theta + \partial_- \theta \left( \bar{A}_+ - \frac{8\pi}{k} \bar{\Pi}_\perp \right).
\]

(2.7)

In order to fix the gauge, we write the partition function as

\[
\mathcal{Z} = \int \mathcal{D}A_\mu \left( \Delta_{FP} \int \mathcal{D}\theta \delta (F(\bar{A}_\mu)) \right) e^{iS[A_\mu]}
\]

(2.8)
where $F(\bar{A}_\mu) = 0$ is the gauge-fixing condition and $\Delta_{FP}$ is the Faddeev-Popov determinant. Under the change of variables $A_\mu \rightarrow \bar{A}_\mu$, the partition function factorizes as

$$Z = \int \mathcal{D}\bar{A}_\mu \Delta_{FP} \delta\left(F(\bar{A}_\mu)\right) e^{iS[\bar{A}_\mu]} \int \mathcal{D}\theta e^{iS_B[\theta, \bar{A}_+^+, \bar{A}_-^-, \bar{\Pi}^-]}.$$  \hspace{1cm} (2.9)

The first term is the standard gauge-fixed path integral for tmgt, describing a massive photon propagating in the bulk. The second term is a surface contribution describing a chiral scalar field $\theta$ coupled to $\bar{A}_+^+$ and $\bar{\Pi}^-$ which are fixed by the boundary conditions, and so the bulk and surface contributions completely decouple. Now that we have gauge-fixed we may, without loss of generality, fix $\bar{A}_+^+$ and $\bar{\Pi}^-$ to zero on $\partial M$. Thus we obtain the path integral for chiral bosonic edge states in tmgt:

$$Z = \int \mathcal{D}\bar{A}_\mu \Delta_{FP} \delta\left(F(\bar{A}_\mu)\right) e^{iS[\bar{A}_\mu]} \int \mathcal{D}\theta e^{iS_B[\theta]},$$  \hspace{1cm} (2.10)

where

$$S_B = \frac{k}{8\pi} \int_{\partial M} \partial_+ \theta \partial_- \theta.$$  \hspace{1cm} (2.11)

may be recognized as the bosonic string action in light-cone coordinates.

- $\tilde{C}$ boundary conditions:

Alternatively, we could choose to fix $\bar{A}_-$ on $\partial M$ while allowing $\bar{A}_+^+$ to vary. In order to extremize $S_1$ we must add

$$\tilde{S}_2 = -\int_{\partial M} \Pi_+ \bar{A}_+ = 0$$  \hspace{1cm} (2.12)

and fix $\delta \Pi_+ = 0$ on $\partial M$. We denote these boundary conditions by $\tilde{C}$. Applying the FP gauge-fixing procedure we obtain

$$\tilde{S}_B = -\frac{k}{8\pi} \int_{\partial M} \partial_+ \theta \partial_- \theta.$$  \hspace{1cm} (2.13)

Since parity reverses the sign of $k$ and interchanges $A_+^+ \leftrightarrow A_-^-$, we see that $C$ and $\tilde{C}$ boundary conditions are related by a parity transformation. Hence we need only consider $C$ boundary conditions and take $k$ to be positive.

- $N$ boundary conditions:

Now consider the case where both $\bar{A}_+^+$ and $\bar{A}_-^-$ are fixed on $\partial M$. We denote these boundary conditions by $N=\text{NON-CONFORMAL}$ because they do not induce conformal edge states. Since both $\bar{A}_+^+$ and $\bar{A}_-^-$ are fixed on $\partial M$, the surface variations in (2.4) vanish and so $S_1$
has a well-defined classical extrema. If we now gauge fix just $S_1$ using the FP procedure, the integral over the gauge group becomes

$$\int \mathcal{D}\theta e^{i\delta S_1[\theta, \bar{\Lambda}\pm]} = \int \mathcal{D}\theta e^{\frac{ik}{8\pi} \int_{\partial\mathcal{M}} \theta \epsilon^{\mu\nu}\partial_{\mu}A_{\nu}} \sim \delta (F_{\pm}) \mid_{\partial\mathcal{M}}. \quad (2.14)$$

If $\theta$ is non-compact, we get a $\delta$-functional which tells us that the boundary conditions for $\bar{\Lambda}_+$ and $\bar{\Lambda}_-$ must be compatible with the constraint $F_{\pm} = 0$ on $\partial\mathcal{M}$. (Note that if $\theta$ is compact we get the constraint mod $2\pi$). This constraint holds when the boundary $\partial\mathcal{M}$ is perfectly conducting, for which the tangential component of the field strength must vanish. We conclude that $n$ boundary conditions have no dynamical degrees of freedom on $\partial\mathcal{M}$. It is important to note that we can only impose $n$ boundary conditions in the full TMGT: in pure Chern-Simons theory $[\bar{\Lambda}_+, \bar{\Lambda}_-] = i\hbar$, and so we cannot simultaneously fix both $\bar{\Lambda}_+$ and $\bar{\Lambda}_-$ on $\partial\mathcal{M}$.

3. Fermionic edge states from SUSY TMGT

We now show how the supersymmetric topological membrane leads to chiral fermionic edge states and, hence, worldsheet SUSY. The idea is simple: under SUSY transformations, the action is invariant only up to total derivatives — in order to maintain SUSY for manifolds with boundaries, we must have chiral fermions on the boundary. In [12] and [4] it was shown how to obtain super-WZNW models from pure Chern-Simons theories. Here we consider the supersymmetric extension of abelian TMGT using the conventions given in Appendix A of [12].

SUSY TMGT is described by the bulk action $S = S_1 + S_3$ where

$$S_3 = \frac{1}{2\gamma} \int_{\mathcal{M}} \bar{\Lambda}_\gamma^\nu \partial_\nu \lambda - \frac{k}{8\pi} \int_{\mathcal{M}} \bar{\lambda} \lambda. \quad (3.1)$$

Ignoring surface terms for the moment, $S$ is invariant under the $N = 1$ SUSY transformations:

$$\begin{align*}
\delta A_\mu & = -\bar{\eta} \gamma_\mu \lambda + \bar{\eta} \partial_\mu \psi \\
\delta \lambda & = \epsilon^{\mu\nu\rho} \partial_\nu A_\rho \gamma_\mu \eta,
\end{align*} \quad (3.2)$$

where $\eta$ is a global Grassman parameter. The second term in (3.2) is equivalent to a gauge transformation parameterized by $\theta(x) = \bar{\eta} \psi(x)$ and, as we shall see, is needed to maintain SUSY when $\mathcal{M}$ has a boundary.
If $\partial \mathcal{M} \neq 0$, then the fermionic action $S_3$ has no extrema:

$$\delta S_3 = \int_{\mathcal{M}} \delta \bar{\lambda} \left( \frac{1}{\gamma} \gamma^\mu \partial_\mu \lambda - \frac{k}{4\pi} \lambda \right) + \frac{i}{2\gamma} \int_{\partial \mathcal{M}} (\lambda_+ \delta \lambda_- + \lambda_- \delta \lambda_+)$$  \hspace{1cm} (3.4)

and so we must fix either $\lambda_-$ or $\lambda_+$ on the boundary to ensure a well-defined classical limit. Since $\{\lambda_-, \lambda_+\} = i\hbar$, we cannot simultaneously fix both $\lambda_-$ and $\lambda_+$ on $\partial \mathcal{M}$. As we shall soon see, SUSY dictates whether $\lambda_-$ or $\lambda_+$ is fixed on $\partial \mathcal{M}$. Secondly, since two $N = 1$ SUSY transformations generate translations in all three directions, translation in the direction perpendicular to the boundaries cannot be avoided unless we explicitly break the $N = 1$ SUSY to an $N = 1/2$ chiral SUSY. Hence we must set either $\eta_+ = 0$ or $\eta_- = 0$.

First consider the case $\eta_+ = 0$.

i) $\eta_+ = 0$ :

The restricted SUSY transformations are:

$$\delta \bar{A}_+ = -i \eta_+ \lambda_+ = 0$$  \hspace{1cm} (3.5)

$$\delta \bar{A}_- = i \eta_- \lambda_-$$  \hspace{1cm} (3.6)

$$\delta \lambda_- = 2F_{+\perp} \left( \begin{array}{c} \eta_- \\ 0 \end{array} \right) + F_{\perp+} \left( \begin{array}{c} 0 \\ \eta_- \end{array} \right) .$$  \hspace{1cm} (3.7)

We will show that $c$ boundary conditions combined with SUSY leads to the supersymmetric WZ model on $\partial \mathcal{M}$, while for $n$ boundary conditions there are no such dynamical edge states.

- **N boundary conditions:**

Recall from Section 2 that $n$ boundary conditions are $\bar{A}_+$ and $\bar{A}_-$ fixed on $\partial \mathcal{M}$ along with the constraint (2.14) that $F_{+\perp} = 0$ on $\partial \mathcal{M}$. Since $F_{+\perp} = 0$ on $\partial \mathcal{M}$, it follows from Eq. (3.7) that $\delta \lambda_- = 0$ on $\partial \mathcal{M}$. Moreover, since $\bar{A}_-$ is fixed on $\partial \mathcal{M}$, Eq. (3.6) implies that we must fix $\lambda_- = 0$ on $\partial \mathcal{M}$. Hence the surface variations in (3.4) vanish and $S_3$ has a well-defined classical limit. Under the restricted SUSY transformations (3.5) – (3.7), the bulk action $S$ is invariant up to surface terms:

$$\delta S = \frac{i}{\gamma} \int_{\partial \mathcal{M}} 2F^{+\perp} \eta_- \lambda_+ + F^{\perp-} \eta_- \lambda_- .$$  \hspace{1cm} (3.8)

The first term vanishes because $F^{+\perp} = 0$ on $\partial \mathcal{M}$ and the second term vanishes because $\lambda_- = 0$ on $\partial \mathcal{M}$. Including $\delta A_\mu = \bar{\eta} \partial_\mu \psi$ in the SUSY transformations, we obtain an
additional surface variation:

\[ \delta S = -\frac{ik}{4\pi} \int_{\partial M} F^{+ -} \eta_- \psi_+ \]  

which again vanishes because \( F^{+ -} = 0 \) on \( \partial M \). Hence we conclude that for \( n \) boundary conditions there are no dynamical edge states, neither bosonic nor fermionic, and susy is maintained in the bulk.

- **C boundary conditions:**

In Section 2 we found that bosonic edge states are induced if we fix \( \bar{A}_+ \) and \( \bar{\Pi}^{\perp-} \) on \( \partial M \) and include the surface term \( S_2 \) in the action. Since \( \bar{A}_- \) is allowed to vary, rather than fixing \( \lambda_- = 0 \) in (3.6), we should instead fix \( \lambda_+ = 0 \) on \( \partial M \) in order to extremize \( S_3 \). Fixing \( \lambda_+ \) on \( \partial M \) in turn implies by (3.7) that \( F^{\perp+} = 0 \) on \( \partial M \). Under the restricted susy transformations (3.5) – (3.7), the bulk action \( S \) is again invariant upto the surface terms given in (3.8). These terms now vanish, however, by virtue of the C boundary conditions. The first term vanishes because \( \lambda_+ = 0 \) on \( \partial M \). The second term vanishes because \( F^{\perp+} = F^{\perp+} = 0 \) on \( \partial M \). Including \( \delta A_\mu = \bar{\eta} \partial_\mu \psi \) in the susy transformations, we obtain an additional surface variation:

\[ \delta S = -\frac{ik}{4\pi} \int_{\partial M} \partial_+ \theta \eta_- \partial_- \psi_+ \]  

which does not vanish. To maintain susy we must therefore supplement the bosonic action (2.11) with the surface term

\[ S_F = \frac{k}{8\pi} \int_{\partial M} \psi_+ \partial_- \psi_+ \quad \text{with} \quad \delta \psi_+ = i \partial_- \theta \eta_- , \]  

which may be recognized as the chiral fermionic action of superstring theory.

ii) \( \eta_- = 0 \):

In this case the restricted susy transformations are:

\[ \delta \bar{A}_+ = -i \eta_+ \lambda_+ \]  
\[ \delta \bar{A}_- = i \eta_- \lambda_- = 0 \]  
\[ \delta \lambda = -2F^{+ -} \begin{pmatrix} 0 \\ \eta_+ \end{pmatrix} - F^{\perp -} \begin{pmatrix} \eta_- \\ 0 \end{pmatrix} . \]

Fermionic edge states of opposite chirality are obtained if we impose C boundary conditions.
• Ĉ boundary conditions:

Recall that for Ĉ boundary conditions we fix $\tilde{A}_-$ and $\tilde{\Pi}^{\perp+}$ on $\partial M$ and add $\tilde{S}_2$ to the action. Since $\tilde{A}_+$ is allowed to vary in (3.12), rather than fixing $\lambda_+ = 0$ on $\partial M$, we should instead fix $\lambda_- = 0$ on $\partial M$ in order to extremize $S_3$. Fixing $\lambda_-$ on $\partial M$ in turn implies by (3.14) that $F_{\perp-} = 0$ on $\partial M$. Under the restricted susy transformations (3.12) – (3.14), the bulk action is invariant up to surface terms:

$$\delta S = -\frac{i}{\gamma} \int_{\partial M} 2F_{+-} \eta_+ \lambda_- + F^{\perp+} \eta_+ \lambda_+ .$$  \hfill (3.15)

The first term vanishes because $\lambda_- = 0$ on $\partial M$ and the second term vanishes because $F^{\perp+} = F_{\perp-} = 0$ on $\partial M$. Under $\delta A_\mu = \bar{\eta} \partial_\mu \psi$ we obtain a non-vanishing surface term:

$$\delta S = -\frac{ik}{4\pi} \int_{\partial M} \partial_- \theta \eta_+ \partial_+ \psi_- .$$  \hfill (3.16)

To maintain susy we must add the fermionic action

$$\tilde{S}_F = \frac{k}{8\pi} \int_{\partial M} \psi_- \partial_+ \psi_- \quad \text{with} \quad \delta \psi_- \big|_{\partial M} = i \partial_- \theta \eta_+ ,$$  \hfill (3.17)

whose chirality is opposite to the action (3.11) obtained using Ĉ boundary conditions. This agrees with our earlier observation that parity interchanges Ĉ and Ĉ boundary conditions.

4. String constructions

Recall that the Neveu-Schwarz–Ramond model [13], which exhibits 2D worldsheet susy, contains both left and right movers on the string worldsheet. We therefore consider a cylindrical topological membrane with left (L) and right (R) boundaries. Depending on the boundary conditions, the bulk TMGT may then induce left movers on L and right movers on R. The full left-right symmetric string worldsheet is obtained by gluing the separate left and right worldsheets traced out by L and R. In this letter we shall only consider closed oriented strings (i.e. type II and heterotic) and so L and R must have the same orientation upon gluing. To obtain both left and right movers we must impose Ĉ boundary conditions on both L and R. We denote this construction by CC. Note that the string construction based on CC boundary conditions has no (global) susy since we must simultaneously set $\eta_- = \eta_+ = 0$. 

8
Type I theories, on the other hand, are based on unoriented open and closed strings. We should obtain unoriented closed strings in the TM approach by gluing L and R with opposite orientations. We hope to give the precise details of this gluing in a future publication. We should also point out that an open string construction has been obtained from pure Chern-Simons gauge theory on a three dimensional $\mathbb{Z}_2$ orbifold. Motivated by string duality, it would be interesting to consider the full TMGT on a $\mathbb{Z}_2$ orbifold.

The crucial idea which permits a heterotic construction is that we can independently excite the left and right sectors by choosing different boundary conditions on the left and right boundaries. Recall that the right-moving sector of the heterotic string is the 10-dimensional superstring, while the left-moving sector is the 26-dimensional bosonic string which has been compactified to ten dimensions. When the left and right sectors are put together, they produce a self-consistent, ghost-free, anomaly-free, one-loop finite theory. The seemingly hybrid nature of this construction becomes more natural in the TM approach. Suppose the topological membrane has a semi-simple gauge group $G_L \times G_R$ where $G_L$ represents ordinary TMGT and $G_R$ represents SUSY TMGT. The heterotic construction is obtained if we then impose $cn$ boundary conditions on $G_L$ and $nc$ boundary conditions on $G_R$. It is appealing that the different left and right sectors of the heterotic string have a common geometric origin in the context of the topological membrane. Each of the above string constructions are illustrated in Fig. 1.

5. Bulk spectrum and Casimir energy

We now calculate the spectrum of the topological membrane in the bulk for each of the above string constructions. We consider the cylindrical membrane $I \times S^1$, where the interval $I$ has length $L$ and the two circular boundaries have radii $R$. For simplicity, we shall only consider the case where $R \gg L$ which, in the limit $R \to \infty$, is equivalent to two infinitely long parallel wires held a distance $L$ apart. The coordinate system is chosen such that the two wires are parallel to the $x_1$ spatial axis and lie, respectively, on $x_2 = 0$ and $x_2 = L$. Eq. (2.3) shows that the gauge field in the bulk has a plane-wave spectrum. Likewise, (3.4) gives the fermionic equations of motion $(\partial - m)\lambda = 0$ which, multiplied by $(\partial + m)$, shows that $\lambda_+$ and $\lambda_-$ each satisfy the wave equation $(\partial^2 - m^2)\lambda_\pm = 0$.

- **NN (no-string) boundary conditions:**
We impose the gauge condition $A_+ = 0$ so that on both boundaries the value of $A_+$ is fixed. Plane-wave solutions to (2.3) can then be constructed from the gauge potential ($A_+ = 0$):

$$A_i(x) = a_i(x_2)e^{i(\omega t - k_1 x_1)} \quad i = -, 2 \quad (5.1)$$

where the explicit form of $a_i(x_2)$ is determined by the boundary conditions. NN boundary conditions are that $A_- = 0$ at both $x_2 = 0$ and $x_2 = L$, which are satisfied by

$$a_-(x_2) = \sin k_2 x_2 \quad \text{with} \quad k_2 = n\pi/L. \quad (5.2)$$

In order to have consistent commutation relations, we must also impose $\Pi^2- \neq 0$ on $\partial \mathcal{M}$. Substituting Eq. (5.2) into (2.5) gives

$$\Pi^2- = -\frac{1}{\gamma}F^2- + \frac{k}{8\pi}e^{2-}A_+$$

$$= \frac{i}{\gamma}(\omega - k_1) a_2(x_2)e^{i(\omega t - k_1 x_1)}, \quad (5.3)$$

which is indeed non-vanishing at $x_2 = 0$ and $L$ if $a_2(x_2) = \cos k_2 x_2$. Thus the gauge potential satisfying NN boundary conditions is

$$A_i(x) = \begin{pmatrix} \sin k_2 x_2 \\ \cos k_2 x_2 \end{pmatrix} e^{i(\omega t - k_1 x_1)} \quad \text{with} \quad k_2 = n\pi/L. \quad (5.4)$$

The corresponding dual field strength is

$$F^\mu = \epsilon^{\mu \alpha \beta} \partial_\alpha A_\beta = \begin{pmatrix} (k_2 - ik_1) \cos k_2 x_2 \\ (k_2 - i\omega) \cos k_2 x_2 \\ (ik_1 - i\omega) \sin k_2 x_2 \end{pmatrix} e^{i(\omega t - k_1 x_1)} \quad (5.5)$$

with spectrum

$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^2 + k_1^2 + m^2 \quad (5.6)$$

Note that NN boundary conditions correspond to the perfectly conducting case (see discussion after (2.14)) since $F^2 = 0$ on $\partial \mathcal{M}$. For the bulk fermions, NN boundary conditions are $\lambda_- = 0$ (but $\lambda_+ \neq 0$) at both $x_2 = 0$ and $x_2 = L$. Hence

$$\begin{pmatrix} \lambda_- \\ \lambda_+ \end{pmatrix} = \begin{pmatrix} \sin k_2 x_2 \\ \cos k_2 x_2 \end{pmatrix} e^{i(\omega t - k_1 x_1)} \quad \text{with} \quad k_2 = n\pi/L, \quad (5.7)$$

and so the fermionic spectrum indeed matches the bosonic spectrum (5.6).

- CC (oriented closed superstring) boundary conditions:
susy edge states are induced on both boundaries if we impose cc boundary conditions, namely $A_- \neq 0$ at both $x_2 = 0$ and $x_2 = L$. This occurs if

$$a_-(x_2) = \cos k_2 x_2 \quad \text{with} \quad k_2 = n\pi/L.$$  \hfill (5.8)

Simultaneously, we must fix $\Pi^{2-} = 0$ on both boundaries. Hence we must have $a_2(x_2) = \sin k_2 x_2$ in (5.3). The gauge potential satisfying cc boundary conditions is

$$A_i(x) = \begin{pmatrix} \cos k_2 x_2 \\ \sin k_2 x_2 \end{pmatrix} e^{i(\omega t - k_1 x_1)} \quad \text{with} \quad k_2 = n\pi/L.$$  \hfill (5.9)

The corresponding dual field strength is

$$F^\mu = - \begin{pmatrix} (ik_1 + k_2) \sin k_2 x_2 \\ (i\omega + k_2) \sin k_2 x_2 \\ (i\omega - ik_1) \cos k_2 x_2 \end{pmatrix} e^{i(\omega t - k_1 x_1)}$$  \hfill (5.10)

with spectrum identical to Eq. (5.6). cc boundary conditions on the fermions are $\lambda_- = 0$ (but $\lambda_+ \neq 0$) at $x_2 = 0$ and $x_2 = L$. Hence

$$\begin{pmatrix} \lambda_- \\ \lambda_+ \end{pmatrix} = \begin{pmatrix} \cos k_2 x_2 \\ \sin k_2 x_2 \end{pmatrix} e^{i(\omega t - k_1 x_1)} \quad \text{with} \quad k_2 = n\pi/L,$$  \hfill (5.11)

which verifies that the cc spectrum is indeed supersymmetric.

- **NC (heterotic string) boundary conditions:**

The gauge potential satisfying nc boundary conditions is given by (5.4) with $k_2 = (n + 1/2)\pi/L$. As before, there will be no bosonic edge states induced on the boundary $x_2 = 0$ since $A_-(x_2 = 0) = 0$. However, because now the spectrum is half integer, $A_-(x_2 = L) \neq 0$ and so edge states will be induced on the boundary $x_2 = L$. Moreover, we have $\Pi_2-(x_2 = L) = 0$, which ensures the action (2.1) has a well-defined classical limit. The bosonic spectrum for the nc case is

$$\omega_n^2 = \left( (n + \frac{1}{2}) \frac{\pi}{L} \right)^2 + k_1^2 + m^2 \quad (\text{NC & CN}).$$  \hfill (5.12)

The nc boundary conditions for fermions are mixed. We must set $\lambda_- = 0$ (but $\lambda_+ \neq 0$) at $x_2 = 0$ and $\lambda_+ = 0$ (but $\lambda_- \neq 0$) at $x_2 = L$. Hence

$$\begin{pmatrix} \lambda_- \\ \lambda_+ \end{pmatrix} = \begin{pmatrix} \sin k_2 x_2 \\ \cos k_2 x_2 \end{pmatrix} e^{i(\omega t - k_1 x_1)} \quad \text{with} \quad k_2 = \left( n + \frac{1}{2} \right) \frac{\pi}{L},$$  \hfill (5.13)
which matches the bosonic spectrum (5.12).

We now calculate the Casimir energy for each of the above string constructions. Since the \( \text{NN} \) and \( \text{CC} \) spectra are supersymmetric, we immediately know that the no-string (\( \text{NN} \)) and normal closed string (\( \text{CC} \)) constructions have zero Casimir energy. Likewise, the supersymmetric right sector of the heterotic (\( \text{NC} \)) construction has zero Casimir energy. However, the bosonic (\( \text{CN} \)) left sector is not supersymmetric and so may have a non-zero Casimir energy. Defined as the infinite sum over zero modes, this Casimir energy is

\[
E_0 = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{dk_2}{2\pi} \left\{ \left( n + \frac{1}{2} \frac{\pi}{L} \right)^2 + k_2^2 + m^2 \right\}^{1/2} \quad (5.14)
\]

Replacing the power \( 1/2 \) in the integrand with \( -s/2 \), where \( s \) is a complex variable, the integral can be evaluated as the analytic continuation of the Euler beta function, yielding

\[
E_0 = \lim_{s \to -1} \frac{\pi^{1/2} \Gamma(s-1/2)}{4L^{2-s} \Gamma(s/2)} \sum_{n=-\infty}^{\infty} \left\{ (n + 1/2)^2 + q \right\}^{1-s/2} \quad (5.15)
\]

where \( q = (mL/\pi)^2 \). The analytic continuation of this series, known as an Epstein-Hurwitz zeta function, is \[13\]:

\[
\sum_{n=-\infty}^{\infty} \left\{ (n + \delta)^2 + q \right\}^{-t} = \pi^{1/2} \frac{\Gamma(t - 1/2)}{\Gamma(t)} q^{(1/2-t)} + \frac{4\pi^t}{\Gamma(t)} q^{1/4-t/2} \times \sum_{n=1}^{\infty} n^{t-1/2} \cos(2\pi n\delta) K_{t-1/2}(2\pi n \sqrt{q}) \quad (5.16)
\]

where \( K_{\nu} \) is the modified Bessel function of the second kind. Substituting \( t = -(1-s)/2 \) and then taking the limit \( s \to -1 \), we obtain the regularized Casimir energy density

\[
E_0 = -\frac{1}{6\pi} \frac{1}{L^3} \left\{ (mL)^3 + 3\pi^{-1/2}(mL)^{3/2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}} K_{-3/2}(2nL) \right\} \quad (5.17)
\]

The first term in (5.17) is independent of the separation \( L \) between the wires and can be dropped. Rewriting the second term using the identity \[14\]

\[
K_{-3/2}(z) = K_{3/2}(z) = \sqrt{\frac{z}{2}} e^{-z} z^{-3/2} (1 + z) \quad (5.18)
\]

gives a positive Casimir energy

\[
E_0 = -\frac{1}{8\pi} \frac{1}{L^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} e^{-2nmL} (1 + 2nmL) \quad . \quad (5.19)
\]
The resulting Casimir force (per unit length) is

$$F(L) = (-\partial/\partial L) \{ L \times E_0(L) \}$$

$$= -\frac{1}{2\pi} \frac{1}{L^3} \sum_{n=1}^{\infty} (-1)^n e^{-2\pi nmL} \left\{ \frac{1}{2n^3} + \frac{mL}{n^2} + \frac{(mL)^2}{n} \right\}.$$ \hspace{1cm} (5.20)

We see that the heterotic construction is dynamically distinguished by the fact that it has a repulsive Casimir force and so, in flat space, the membrane expands and will continue to do so until we can no longer assume $R >> L$. Whether or not the effects of the cylindrical geometry will stabilize this expansion depends on the geometrodynamics of the topological membrane [11, 17]. We note that modular invariance on the worldsheet should also be connected with the gravitational sector of the topological membrane and the cancellation of the global gravitational anomaly in the bulk. We hope to address these issues in future work.

6. Conclusions

In this letter we have shown how different boundary conditions in the topological membrane theory lead to different string constructions. For certain boundary conditions, it was shown how some bosonic gauge degrees of freedom become dynamical on the membrane boundary, which was given an interpretation as the string worldsheet. Worldsheet fermions were incorporated in this picture by using a supersymmetric topological membrane. The appeal of this approach is that the different left and right sectors of the heterotic string have a clear geometrical interpretation as a specific choice of membrane boundary conditions — and this choice is only possible in the full TMG, and not in a pure Chern-Simons theory. By calculating the spectrum in the bulk, we then showed that the supersymmetric oriented closed string (CC) and no-string (NN) constructions have zero Casimir energy. The heterotic (NC) construction, on the other hand, has a positive Casimir energy. Although the physical implications of this result are not yet clear to us, it is apparent, at least from the bulk perspective, that the TM approach distinguishes the heterotic construction. In turn, this could have interesting consequences for string duality and the proposal that all ten-dimensional string theories have a common eleven-dimensional origin, i.e. $M$-theory.

Hořava and Witten [18] have argued that the strong coupling limit of the ten-dimensional $E_8 \times E_8$ heterotic string is $M$-theory compactified on $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$. Their construction
shares some amusing similarities with the TM approach, although we emphasize that the two approaches are fundamentally different. In the Hořava-Witten approach, space-time ideas such as eleven-dimensional supergravity play a prominent role in determining the structure of the theory. For instance, the $E_8 \times E_8$ heterotic “string” is seen to be a cylindrical two-brane with one boundary attached to each boundary of space-time. Cancellation of gauge and gravitational anomalies then requires there to be one $E_8$ gauge group on each boundary of spacetime. In contrast, the emphasis of the TM approach is on the worldsheet properties of the open topological membrane. Nevertheless, it would be interesting to see if the extra dimension of TM theory could somehow be embedded in eleven-dimensional $M$-theory.

It has also very recently been shown that Chern-Simons couplings arise in the effective world-volume action for a type IIA superstring Dirichlet two-brane [19], which is supposed to descend from the membrane of $M$-theory. Moreover, the five-brane of $M$-theory has been interpreted as a $D$-brane of an open supermembrane in eleven dimensions [20]. Motivated by these results, it would be interesting to see if the topological membrane discussed in this letter has a role in $M$-theory.

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Figure 1: String constructions. i) no-string  ii) oriented closed string  iii) heterotic string.