Abstract

Given a lamination endowed with a Riemannian metric we introduce the notion of a multiplicative cocycle. Next, we define the Lyapunov exponents of such a cocycle with respect to a harmonic probability measure directed by the lamination. We also prove an Oseledec multiplicative ergodic theorem in this context. This theorem implies the existence of an Oseledec decomposition almost everywhere which is holonomy invariant. Moreover, in the case of differentiable cocycles we establish effective integral estimates for the Lyapunov exponents. These results find applications in the geometric and dynamical theory of laminations. They are also applicable to (not necessarily closed) laminations with singularities. Interesting holonomy properties of a generic leaf of a foliation are obtained. The main ingredients of our method are the theory of Brownian motion, the analysis of the heat diffusions on Riemannian manifolds, the ergodic theory in discrete dynamics and a geometric study of laminations.

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1 Introduction

We first recall the definition of a multiplicative cocycle in the context of discrete dynamics. Let \( T \) be a measurable transformation of a probability measure space \((X, \mathcal{B}, \mu)\). Assume that \( \mu \) is \( T \)-invariant (or equivalently, \( T \) preserves \( \mu \)), that is, \( \mu(T^{-1}B) = \mu(B) \) for all \( B \in \mathcal{B} \) (or equivalently, \( T_*\mu = \mu \)).

**Definition 1.1.** Let \( \mathbb{G} \) be either \( \mathbb{N} \) or \( \mathbb{Z} \). In case \( \mathbb{G} = \mathbb{Z} \) we assume further that \( T \) is bi-measurable\(^1\). A measurable function \( \mathcal{A} : X \times \mathbb{G} \to \text{GL}(d, \mathbb{R}) \) is called a **multiplicative cocycle over** \( T \) or simply a **cocycle** if for every \( x \in X \), \( \mathcal{A}(x, 0) = \text{id} \) and the following **multiplicative law** holds

\[
\mathcal{A}(x, m + k) = \mathcal{A}(T^k(x), m)\mathcal{A}(x, k), \quad m, k \in \mathbb{G}.
\]

Throughout the article we use the notation \( \log^+ := \max(0, \log) \). Moreover, the **angle** between two subspaces \( V, W \) of \( \mathbb{R}^d \) (resp. \( \mathbb{C}^d \)) is, by definition,

\[
\angle(V, W) := \min \{ \arccos \langle v, w \rangle : v \in V, w \in W, \|v\| = \|w\| = 1 \}.
\]

Here \( \langle \cdot, \cdot \rangle \) (resp. \( \| \cdot \| \)) denotes the standard Euclidean inner product (resp. Euclidean norm) of \( \mathbb{R}^d \) or of \( \mathbb{C}^d \). Now we are able to state the classical Oseledec Multiplicative Ergodic Theorem \([30, 24, 32]\).

**Theorem 1.2.** Let \( T \) be as above and let \( \mathcal{A} : X \times \mathbb{G} \to \text{GL}(d, \mathbb{R}) \) be a cocycle such that the real-valued functions \( x \mapsto \log^+ \|\mathcal{A}^\pm(x, 1)\| \) are \( \mu \)-integrable. Then there exists \( Y \in \mathcal{B} \) with \( TY \subset Y \) and \( \mu(Y) = 1 \) such that the following properties hold:

(i) There is a measurable function \( m : Y \to \mathbb{N} \) with \( m \circ T = m \).

(ii) For each \( x \in Y \) there are real numbers

\[
\chi_m(x) < \chi_{m(x)-1}(x) < \cdots < \chi_2(x) < \chi_1(x)
\]

with \( \chi_i(Tx) = \chi_i(x) \) when \( 1 \leq i \leq m(x) \), and the function \( x \mapsto \chi_i(x) \) is measurable on \( \{ x \in Y : m(x) \geq i \} \). These numbers are called the Lyapunov exponents associated to the cocycle \( \mathcal{A} \) at the point \( x \).

(iii) For each \( x \in Y \) there is a decreasing sequence of linear subspaces

\[
\{0\} \equiv V_{m(x)+1}(x) \subset V_{m(x)}(x) \subset \cdots \subset V_2(x) \subset V_1(x) = \mathbb{R}^d,
\]

of \( \mathbb{R}^d \) such that \( \mathcal{A}(x, 1)V_i(x) = V_i(Tx) \) and that \( x \mapsto V_i(x) \) is a measurable map from \( \{ x \in Y : m(x) \geq i \} \) into the corresponding Grassmannian of \( \mathbb{R}^d \). This

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\(^1\) An invertible map \( T \) is said to be bi-measurable if both \( T \) and \( T^{-1} \) are measurable.
sequence of subspaces is called the Lyapunov filtration associated to the cocycle $A$ at the point $x$.

(iv) For each $x \in Y$ and $v \in V_i(x) \setminus V_{i+1}(x)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\|A(x,n)v\|}{\|v\|} = \chi_i(x).$$

(v) Suppose now that $\mathbb{G} = \mathbb{Z}$ and that $T$ is bi-measurable invertible. Then, for every $x \in Y$, there exists $m(x)$ linear subspaces $H_1(x), \ldots, H_{m(x)}(x)$ of $\mathbb{R}^d$ such that

$$V_j(x) = \bigoplus_{i=j}^{m(x)} H_i(x),$$

with $A(x,1)H_i(x) = H_i(Tx)$, and $x \mapsto H_i(x)$ is a measurable map from $\{x \in Y : m(x) \geq i\}$ into the corresponding Grassmannian of $\mathbb{R}^d$. Moreover,

$$\lim_{n \to \pm \infty} \frac{1}{|n|} \log \frac{\|A(x,n)v\|}{\|v\|} = \pm \chi_i(x),$$

uniformly on $v \in H_i(x) \setminus \{0\}$, and the following limit holds

$$\lim_{n \to \infty} \frac{1}{n} \log \sin \left| \angle \left( H_S(T^n x), H_{N \setminus S}(T^n x) \right) \right| = 0,$$

where, for any subset $S$ of $N := \{1, \ldots, m(x)\}$, we define $H_S(x) := \bigoplus_{i \in S} H_i(x)$.

The definition of cocycles and Theorem 1.2 can also be formulated using the action of $\text{GL}(d, \mathbb{C})$ on $\mathbb{C}^d$ instead of the action of $\text{GL}(d, \mathbb{R})$ on $\mathbb{R}^d$. The above fundamental theorem together with Pesin’s work in [31] constitute the nonuniform hyperbolicity theory of maps. This theory becomes now one of the major parts of the general dynamical theory and one of the main tools in studying highly sophisticated behavior associated with “deterministic chaos”. Nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents. Namely, a dynamical system is nonuniformly hyperbolic if it admits an invariant measure such that the Lyapunov exponents associated to a certain representative cocycle of the system are nonzero almost everywhere.

The ergodic theory of laminations is not so developed as that of maps or flows, perhaps because of at least two reasons. The first one is that laminations which have invariant measures are rather scarce. In fact, invariant measures for maps should be replaced by harmonic measures for laminations. On the other hand, there is the additional problem with “time”. In the dynamics of laminations, the concept of linearly or time-ordered trajectories are replaced with the vague notion of multi-dimensional futures for points, as defined by the leaves through the points. The geometry of the leaves thus plays a fundamental role in the study of lamination dynamics, which is a fundamentally new aspect of the subject, in contrast to the study of diffeomorphisms, or $\mathbb{Z}$-actions. This is the second reason.
The reader is invited to consult the surveys by Fornæss-Sibony [17], by Ghys [21], and by Hurder [19] for a recent account on this subject. We mention here the approach using Brownian motion which has been first introduced by Garnett [20] in order to explore the dynamics of laminations. This method has been further pursued by many authors (see, for example, [3] [22] etc).

The purpose of this article is to establish an Oseledec multiplicative ergodic theorem for laminations. This will be a starting attempt in order to develop a nonuniform hyperbolicity theory for laminations. The natural framework of our study is a given lamination endowed with a Riemannian metric which directs a harmonic probability measure. Our purpose consists of two tasks. The first one is to formulate a good notion of (multiplicative) cocycles. Secondly, we define the Lyapunov exponents for such cocycles and prove an Oseledec multiplicative ergodic theorem in this context. The main examples of laminations we have in mind come from two sources. The first one consists of all compact smooth laminations. It is easy to endow each such a lamination with a transversally continuous Riemannian metric. The second source comprises (possibly singular) foliations by Riemann surfaces in the complex projective space \( \mathbb{P}^k \) or in algebraic manifolds. Such a foliation often admits a canonical Riemannian metric, namely, the Poincaré metric. However, this metric is transversally measurable, it is continuous only in some good cases (see [12] [17]). Our main examples of cocycles are the holonomy cocycles (or their tensor powers) of such foliations.

A recent result in our direction is obtained by Candel in [3] who defines the Lyapunov exponent of additive cocycles. To state his result in the context of multiplicative cocycles we need to introduce some terminology and definition. A precise formulation will be recalled in Section 2 below. Let \((X, \mathcal{L})\) be a lamination endowed with a Riemannian metric tensor on leaves. Let \(\Omega := \Omega(X, \mathcal{L})\) be the space consisting of all continuous paths \(\omega : [0, \infty) \to X\) with image fully contained in a single leaf. Consider the semi-group \((T^t)_{t \in \mathbb{R}^+}\) of shift-transformations \(T^t: \Omega \to \Omega\) defined for all \(t, s \in \mathbb{R}^+\) by

\[
T^t(\omega)(s) := \omega(s + t), \quad \omega \in \Omega.
\]

For \(x \in X\), let \(\Omega_x\) be the subspace consisting of all paths in \(\Omega\) starting from \(x\). We endow \(\Omega_x\) with a canonical probability measure: the Wiener measure \(W_x\). Let \(\alpha\) be a closed one-form on the leaves of \((X, \mathcal{L})\). Define a map \(A: \Omega \times \mathbb{R}^+ \to \mathbb{C}^*\) by

\[
A(\omega, t) := e^{\int_{\omega[0, t]} \alpha}.
\]

Clearly, the following multiplicative property holds \(A(\omega, s + t) = A(T^t\omega, s)A(\omega, t)\) for all \(s, t \in \mathbb{R}^+\). \(A\) is called the multiplicative cocycle associated to \(\alpha\).

\(A\) can be defined in the following manner. Let \(L\) be a leaf of \((X, \mathcal{L})\). Since \(\alpha\) is closed on \(L\), it is exact when lifted to the universal cover \(\tilde{\mathcal{L}} \to L\) of \(L\), that is, there is a complex-valued function \(f\) on \(\tilde{\mathcal{L}}\) such that \(df = \pi^*\alpha\). Then, if
\(\omega\) is a path in \(L\),
\[A(\omega, t) := e^{f(\tilde{\omega}(t)) - f(\tilde{\omega}(0))},\]
where \(\tilde{\omega}\) is any lift of \(\omega\) to \(\tilde{L}\). The value of \(A(\omega, t)\) is independent of the lift \(\tilde{\omega}\) and \(f\). It depends only on \(\alpha\) and the homotopy class of the curve \(\omega|_{[0, t]}\). Let \(\Delta\) be the Laplace operator associated to the metric tensor of \((X, \mathcal{L})\). Consider the operator \(\delta\) which sends every closed one-form \(\alpha\) to the function
\[\delta\alpha(x) := (\tilde{\Delta}f)(\tilde{x}),\]
where \(x\) is a point in the leaf \(L\), \(\tilde{x}\) is a lift of \(x\) to \(\tilde{L}\), i.e, \(\tilde{x} \in \pi^{-1}(x)\), \(f\) is related to \(\alpha\) as above, and \(\tilde{\Delta}\) is the lift to \(\tilde{L}\) of the Laplace operator \(\Delta\) on \(L\).

The following result of Candel gives the asymptotic value of multiplicative cocycles of dimension 1 (see [3, Section 8]).

**Theorem 1.3.** Let \((X, \mathcal{L})\) be a compact \(C^2\)-smooth lamination endowed with a transversally continuous Riemannian metric. Let \(\mu\) be a harmonic probability measure directed by \((X, \mathcal{L})\). Let \(\alpha\) be a closed one-form such that both \(\alpha\) and \(\delta\alpha\) are bounded. Then for \(\mu\)-almost every \(x \in X\), the asymptotic value
\[\chi(x) := \lim_{t \to \infty} \frac{1}{t} \log \|A(\omega, t)\|\]
exists for \(W_x\)-almost every \(\omega \in \Omega_x\). The real numbers \(\chi(x)\) is called the Lyapunov exponent associated to the cocycle \(A\) at \(x\). Moreover,
\[\int_X \chi(x) d\mu(x) = \Re \int_X \delta\alpha(x) d\mu(x),\]
where \(\Re\) denotes the real part of a complex number. If, moreover, \(\mu\) is ergodic, then \(\chi(x)\) is constant for \(\mu\)-almost every \(x \in X\).

An immediate application of Candel’s theorem is the case where \((X, \mathcal{L})\) is a foliation of transversal (real or complex) dimension 1 and \(A\) is its holonomy cocycle (see [2, 21]). Deroin [11] also obtains some similar results in the last case. Since \(A\) takes its values in \(\mathbb{C}^*\) which is naturally identified with \(\text{GL}(1, \mathbb{C})\), Candel’s result may be considered as Oseledec’s theorem for compact \(C^2\)-smooth laminations with a continuous Riemannian metric in the case \(d = 1\). Our purpose may be rephrased as generalizing Candel’s theorem to the context of more general laminations in arbitrary cocycle dimension \(d\). More concretely, we need to

• introduce a large class of laminations for which neither the compactness of \(X\) nor the transversal smoothness of the associated metric is required, the new class should include not only compact smooth laminations, but also (possibly singular) foliations by Riemann surfaces in algebraic manifolds;

• introduce a notion of (multiplicative) cocycles in arbitrary dimensions which is natural and which captures the essential features of Candel’s definition of cocycles in dimension 1 as well as the definition of cocycles for maps;
• construct an Oseledec decomposition and Lyapunov exponents at almost every point in the spirit of Theorem 1.2 and Theorem 1.3.

To overcome the major difficulty with time-ordered trajectories we follow partly the approach by Garnett and Candel using the Wiener measure $W_x$ on $\Omega_x$. More precisely, our idea is that the asymptotic behavior of a cocycle $A$ at a point $x$ and a vector $v \in \mathbb{R}^d$ is determined by the asymptotic behavior of $A(\omega, \cdot)v$ where $\omega$ is a typical path in $\Omega_x$ that is, it is an element of a certain subset of $\Omega_x$ of full $W_x$-measure. However, in order to make this idea work in the context of arbitrary dimension $d$ we have to develop new techniques based on the so-called leafwise Lyapunov exponents and the Lyapunov forward and backward filtrations. These techniques are partly inspired by Ruelle’s work in [32]. Another crucial ingredient is the construction of weakly harmonic measures which maximize (resp. minimize) some Lyapunov exponents functionals. The next important tool is a procedure of splitting invariant sub-bundles. We also need to establish new measure and ergodic theories on the sample-path space $\Omega$. Since the description of our method is rather involved, we postpone it until the two next sections. We are inspired by the methods of Ledrappier, Walters [26, 35] in discrete dynamics. We also improve the random ergodic theorems of Kakutani [23] and adapt it to the context of laminations.

Let us review shortly the main results of this work. A full development and explanation will be given in Section 2 and Section 3 below. A cocycle of dimension $d$ on a lamination $(X, \mathcal{L})$ is a map $A$ defined on $\Omega \times \mathbb{G}$ ($\Omega := \Omega(X, \mathcal{L})$ and $\mathbb{G} \in \{\mathbb{N}, \mathbb{R}^+\}$) with matrix-valued $GL(d, K)$ ($K \in \{\mathbb{R}, \mathbb{C}\}$) satisfying identity, homotopy, multiplicative and measurable laws. Now we are in the position to state the Main Theorem in an incomplete and informal formulation:

**Theorem 1.4.** Let $(X, \mathcal{L}, g)$ be a lamination satisfying some reasonable standing hypotheses. Let $A : \Omega \times \mathbb{G} \to GL(d, K)$ be a cocycle. Let $\mu$ be a harmonic probability measure. Assume that $A$ satisfies some integrability condition with respect to $\mu$. Then there exists a leafwise saturated Borel set $Y \subset X$ with $\mu(Y) = 1$ such that the following properties hold:

(i) There is a measurable function $m : Y \to \mathbb{N}$ which is leafwise constant.

(ii) For each $x \in Y$ there exists a decomposition of $\mathbb{K}^d$ as a direct sum of $K$-linear subspaces

$$\mathbb{K}^d = \bigoplus_{i=1}^{m(x)} H_i(x),$$

such that $A(\omega, t)H_i(x) = H_i(\omega(t))$ for all $\omega \in \Omega_x$ and $t \in \mathbb{G}$. Moreover, the map $x \mapsto H_i(x)$ is a measurable map from $\{x \in Y : m(x) \geq i\}$ into the Grassmannian of $\mathbb{K}^d$. Moreover, there are real numbers

$$\chi_{m(x)}(x) < \chi_{m(x)-1}(x) < \cdots < \chi_2(x) < \chi_1(x)$$

such that the function $x \mapsto \chi_1(x)$ is measurable and leafwise constant on $\{x \in Y : \chi_{m(x)}(x) < \cdots < \chi_2(x) < \chi_1(x)\}$. 


\(m(x) \geq i\), and
\[
\lim_{t \to \infty, t \in \mathbb{G}} \frac{1}{t} \log \left\| A(\omega, t)v \right\| = \chi_i(x),
\]
uniformly on \(v \in H_i(x) \setminus \{0\}\), for \(W_x\)-almost every \(\omega \in \Omega_x\). The numbers
\(\chi_{m(x)}(x) < \chi_{m(x)-1}(x) < \cdots < \chi_2(x) < \chi_1(x)\)
are called the Lyapunov exponents associated to the cocycle \(A\) at the point \(x\).

(iii) For \(S \subset N := \{1, \ldots, m(x)\}\) let \(H_S(x) := \oplus_{i \in S} H_i(x)\). Then
\[
\lim_{t \to \infty, t \in \mathbb{G}} \frac{1}{t} \log \sin \left| \angle(H_S(\omega(t)), H_{N \setminus S}(\omega(t))) \right| = 0
\]
for \(W_x\)-almost every \(\omega \in \Omega_x\).

(iv) If, moreover, \(\mu\) is ergodic, then the functions \(Y \ni x \mapsto m(x)\) as well as
\(Y \ni x \mapsto \dim H_i(x)\) and \(Y \ni x \mapsto \chi_i(x)\) are all constant. In this case
\(\chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1\) are called the Lyapunov exponents associated to the cocycle \(A\).

It is worthy noting that the decomposition \(\mathbb{K}^d = \oplus_{i=1}^{m(x)} H_i(x)\) in (ii) depends
only on \(x\), in particular, it does not depend on paths \(\omega \in \Omega_x\). We will see later that
Theorem 1.5 (i)–(iii) is the abridged version of Theorem 3.7 whereas Theorem 1.4 (iv) is the abridged version of Corollary 3.8 below. Moreover, Theorem 1.4 seems to be the analogue of Theorem 1.2 in the context of laminations. Further remarks as well as applications of the Main Theorem will be given after Theorem 3.7 below.

We will also see later that Theorem 3.11 below generalizes Theorem 1.3 to the context of cocycles of arbitrary dimensions. Since the framework of the former theorem requires a good deal of preparations, we prefer to state it in the full form and to discuss its applications in Section 3. Moreover, Theorem 9.20 below gives a characterization of Lyapunov spectrum in the spirit of Ledrappier’s work in [26]. All our results demonstrate that there is a strong analogue between the dynamical theory of maps and that of laminations.

The article is organized as follows. In Section 2 we develop the background for our study. In particular, we introduce a new \(\sigma\)-algebra \(\mathcal{A}\) on \(\Omega\) which is one of the main objects of our study. Section 3 is started with the notion of multiplicative cocycles. Next, we state the main results and their corollaries. This section is ended with an outline of our method. The measure and ergodic theories on the measure space \((\Omega, \mathcal{A})\) will be developed in Section 4 and Appendix below.

Our techniques as well as the proofs of the main theorems and their corollaries are presented in Sections 5–9.

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2 Background

In this section, we develop the background for our study. We first introduce the notion of Riemannian laminations, the main objects of our study. Some basic properties of these objects and related notions such as positive harmonic measures and covering laminations are also presented. Next, we recall the heat equations on leaves and review the theory of Brownian motion in the context of Riemannian laminations. This preparation allows us to develop a new measure theory for some sample-path spaces. A comprehensive and modern exposition on the theory of laminations could be found in the two volumes by Candel-Conlon [4, 5] or in Walczak’s book [34].

2.1 (Riemannian) laminations and Laplacians

Let $X$ be a separable locally compact space. Consider an atlas $\mathcal{L}$ of $X$ with charts

$$
\Phi_i : U_i \rightarrow B_i \times T_i ,
$$

where $T_i$ is a locally compact metric space, $B_i$ is a domain in $\mathbb{R}^n$ and $U_i$ is an open subset of $X$.

**Definition 2.1.** We say that $(X, \mathcal{L})$ is a lamination (of dimension $n$) or equivalently continuous lamination (of dimension $n$) if all $\Phi_i$ are homeomorphism and all the changes of coordinates $\Phi_i \circ \Phi_j^{-1}$ are of the form

$$
(x, t) \mapsto (x', t'), \quad x' = \Psi(x, t), \quad t' = \Lambda(t)
$$

where $\Psi, \Lambda$ are continuous functions. Moreover, we say that $(X, \mathcal{L})$ is $C^k$-smooth for some $k \in \mathbb{N} \cup \{\infty\}$ if $\Psi$ is $C^k$-smooth with respect to $x$ and its partial derivatives of any order $\leq k$ with respect to $x$ are jointly continuous with respect to $(x, t)$.

In Section 9 below the following weak form of laminations is needed.

**Definition 2.2.** We say that $(X, \mathcal{L})$ is a measurable lamination of dimension $n$ if all $\Phi_i : U_i \rightarrow B_i \times T_i$ are bijective and bi-(Borel) measurable and if all the changes of coordinates $\Phi_i \circ \Phi_j^{-1}$ are of the form

$$
(x, t) \mapsto (x', t'), \quad x' = \Psi(x, t), \quad t' = \Lambda(t)
$$

where $\Psi, \Lambda$ are Borel measurable functions and $\Phi_i \circ \Phi_j^{-1}$ is homeomorphic on the variable $x$ when $t$ is fixed. Moreover, we say that $(X, \mathcal{L})$ is $C^k$-smooth for some $k \in \mathbb{N} \cup \{\infty\}$ if $\Psi$ is $C^k$-smooth with respect to $x$. 

The open set \( U_i \) is called a flow box and the manifold \( \Phi_i^{-1}\{t = c\} \) in \( U_i \) with \( c \in T_i \) is a plaque. The property of the above coordinate changes insures that the plaques in different flow boxes are compatible in the intersection of the boxes. A leaf \( L \) is a minimal connected subset of \( X \) such that if \( L \) intersects a plaque, it contains the plaque. So, a leaf \( L \) is a connected real manifold of dimension \( n \) immersed in \( X \) which is a union of plaques. For a point \( x \in X \) let \( L_x \) denote the leaf passing through \( x \). We will only consider oriented laminations, i.e. the case where the \( \Phi_i \) preserve the canonical orientation on \( \mathbb{R}^n \). A transversal in a flow box is a closed set of the box which intersects every plaque in one point. In particular, \( \Phi_i^{-1}(\{x\} \times T_i) \) is a transversal in \( U_i \) for any \( x \in B_i \). In order to simplify the notation, we often identify \( T_i \) with \( \Phi_i^{-1}(\{x\} \times T_i) \) for some \( x \in B_i \) or even identify \( U_i \) with \( B_i \times T_i \) via the map \( \Phi_i \).

When \( X \) is a Riemannian manifold and the leaves of \( \mathcal{L} \) are manifolds immersed in \( X \), we say that \( (X,\mathcal{L}) \) is a foliation. Moreover, \( (X,\mathcal{L}) \) is called a transversally \( C^k \)-smooth foliation if there is an atlas \( \mathcal{L} \) of \( X \) with charts

\[
\Phi_i : U_i \rightarrow B_i \times T_i,
\]

with \( T_i \) an open set of some \( \mathbb{R}^d \) (or \( \mathbb{C}^d \)) such that \( \Psi_i \) is a diffeomorphism of class \( C^k \). For a transversally \( C^1 \)-smooth foliation \( (X,\mathcal{L}) \), a transversal section is a submanifold \( S \) of \( X \) such that for every flow box \( U \) and for every plaque \( P \) of \( U \), either \( S \cap U \) does not intersect \( P \), or \( S \cap U \) is transverse to \( P \) at their unique intersection.

We introduce the class of \( C^k \)-smooth (real or complex-valued) functions defined on a \( C^k \)-smooth lamination \( (X,\mathcal{L}) \). Let \( Z \) be a separable, locally compact metrizable space, and let \( U \) be an open subset of the product \( \mathbb{R}^n \times Z \). A function \( f : U \rightarrow \mathbb{R} \) is said to be \( C^k \)-smooth at a point \( (x_0, z_0) \) if there is a neighborhood of this point of the form \( D \times Z_0 \) such that the function \( z \mapsto f(\cdot, z) \in C^k(D) \) is continuous on \( Z \), where \( C^k(D) \) has the topology of uniform convergence on compact subsets of all derivatives of order \( \leq k \). The function \( f \) is said to be \( C^k \)-smooth in \( U \) if it is \( C^k \)-smooth at every point of \( U \). Generalizing this definition, given an open subset \( U \) of \( X \), a function \( f : U \rightarrow \mathbb{R} \) (or \( \mathbb{C} \)) is said to be \( C^k \)-smooth if for any \( C^k \) atlas with charts

\[
\Phi_i : U_i \rightarrow B_i \times T_i,
\]

the functions \( f_i := f \circ \Phi_i^{-1} : \Phi_i(U_i \cap U) \rightarrow \mathbb{R} \) (or \( \mathbb{C} \)) are \( C^k \)-smooth in the previous sense. Denote by \( C^k(U) \) the space of \( C^k \)-smooth (real or complex-valued) functions defined on \( U \). Moreover, \( C^k_0(U) \) denotes those elements of \( C^k(U) \) which are compactly supported in \( U \).

We say that a triplet \( (X,\mathcal{L},g) \) consisting of a \( C^2 \)-smooth (measurable) lamination \( (X,\mathcal{L}) \) and a tensor \( g \) on leaves of \( \mathcal{L} \) is a Riemannian lamination, if \( g \) is a Riemannian metric on \( (X,\mathcal{L}) \) which is transversally measurable. More precisely, a tensor \( g \) on leaves of \( \mathcal{L} \) is said to be a Riemannian metric on \( (X,\mathcal{L}) \) if, using
the charts $\Phi_i : \mathcal{B}_i \to \mathcal{B} \times \mathcal{T}_i$, $g$ can be expressed as a collection of tensors $(\omega_i)$ with the following properties:

- (Metric condition) $\omega_i$ is defined on $\mathcal{B} \times \mathcal{T}_i$ and has the following expression
  $$\omega_i := \sum_{p,q=1}^n g_{ij}^p(x,t) dx_p \otimes dx_q, \quad x = (x_1, \ldots, x_n) \in \mathcal{B}_i, \ t \in \mathcal{T}_i$$
  where the matrix of functions $(g_{ij}^p)$ is symmetric and positive definite, and the functions $(g^p_{jk})$ are $C^2$-smooth with respect to $(x,t)$;

- (Compatibility condition) $(\Phi_i \circ \Phi_j^{-1})_* \omega_j = \omega_i$. We often write $\omega_i := (\Phi_i)_* g$.

Moreover, $g$ is said to be transversally measurable (resp. transversally continuous) if

- (Measurable (resp. continuity) condition) the functions $(g^p_{jk})$ are Borel measurable (resp. continuous) with respect to $(x,t)$.

Roughly speaking, transversal measurability (resp. transversal continuity) means that the metric depends in a measurable (resp. continuous) way on transversals. Since $X$ is paracompact, we can use a partition of unity in order to construct a Riemannian metric tensor $g$ on any $C^2$-smooth lamination $(X, \mathcal{L})$ such that $g$ is transversally continuous.

Let $(X, \mathcal{L}, g)$ be a Riemannian measurable lamination. Then $g$ induces a metric tensor $g|_L$ on each leaf $L$ of $(X, \mathcal{L})$, and thus a corresponding Laplacian $\Delta_L$. If $u$ is a function on $X$ that is of class $C^2$ along each leaf, then $\Delta u$ is, by definition, the aggregate of the leafwise Laplacians $\Delta_L u$.

### 2.2 Covering laminations

The covering lamination $(\widetilde{X}, \widetilde{\mathcal{L}})$ of a lamination $(X, \mathcal{L})$ is, in some sense, its universal cover. We give here its construction. For every leaf $L$ of $(X, \mathcal{L})$ and every point $x \in L$, let $\pi_1(L, x)$ denotes as usual the first fundamental group of all continuous closed paths $\gamma : [0, 1] \to L$ based at $x$, i.e. $\gamma(0) = \gamma(1) = x$. Let $[\gamma] \in \pi_1(L, x)$ be the class of a closed path $\gamma$ based at $x$. Then the pair $(x, [\gamma])$ represents a point in $(\widetilde{X}, \widetilde{\mathcal{L}})$. Thus the set of points of $\widetilde{X}$ is well-defined. The leaf $\widetilde{L}$ passing through a given point $(x, [\gamma]) \in \widetilde{X}$, is by definition, the set

$$\widetilde{L} := \{(y, [\delta]) : \ y \in L_x, \ [\delta] \in \pi_1(L, y)\},$$

which is the universal cover of $L_x$. We put the following topological structure on $\widetilde{X}$ by describing a basis of open sets. Such a basis consists of all sets $\mathcal{N}(U, \alpha)$, $U$ being an open subset of $X$ and $\alpha$ being a homotopy on $U$. Here a homotopy $\alpha$ on $U$ is a continuous function $\alpha : U \times [0, 1] \to X$ such that $\alpha_x := \alpha(x, \cdot)$ is a closed path in $L_x$ based at $x$ for all $x \in U$ (that is, $\alpha_x[0, 1] \subset L_x$ and $\alpha(x, 0) = \alpha(x, 1) = x, \forall x \in U$), and

$$\mathcal{N}(U, \alpha) := \{(x, [\alpha_x]) : \ x \in U\}.$$
The projection $\pi : \tilde{X} \to X$ is defined by $\pi(x, [\gamma]) := x$. It is clear that $\pi$ is locally homeomorphic. Let $\Phi_i : U_i \to \mathbb{B}_i \times T_i$ be a chart of the atlas $\mathcal{L}$ of the lamination $(X, \mathcal{L})$. By shrinking $U_i$ if necessary, we may assume without loss of generality that there is a homotopy $\alpha_i$ on $U_i$. Consider the following chart on $\tilde{X}$:

$$\Phi_{i, \alpha} : N(U_i, \alpha) \to \mathbb{B}_i \times T_i$$

given by $\Phi_{i, \alpha}(\tilde{x}) = \Phi_i(\pi(\tilde{x})), \tilde{x} \in N(U_i, \alpha)$. Using these charts an atlas $\tilde{\mathcal{L}}$ of $\tilde{X}$ is well-defined. Since $\pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L})$ maps leaves to leaves, $(\tilde{X}, \tilde{\mathcal{L}})$ inherits the differentiable structure on leaves and the lamination structure from $(X, \mathcal{L})$. If $(X, \mathcal{L}, g)$ is a Riemannian lamination, then we equip $(\tilde{X}, \tilde{\mathcal{L}})$ with the metric tensor $\pi^* g$ so that $(\tilde{X}, \tilde{\mathcal{L}}, \pi^* g)$ is also a Riemannian lamination. We call $\pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L})$ the covering lamination projection of $(X, \mathcal{L})$.

### 2.3 Heat kernels, (weakly) harmonic measures and Standing Hypotheses

For every point $x$ in an arbitrary leaf $L$ of a Riemannian measurable lamination $(X, \mathcal{L}, g)$, consider the heat equation on $L$

$$\frac{\partial p(x, y, t)}{\partial t} = \Delta_g p(x, y, t), \quad \lim_{t \to 0} p(x, y, t) = \delta_x(y), \quad y \in L, \ t \in \mathbb{R}_+.$$ 

Here $\delta_x$ denotes the Dirac mass at $x$, and the limit is taken in the sense of distribution, that is,

$$\lim_{t \to 0+} \int_L p(x, y, t) \phi(y) d\text{Vol}_L(y) = \phi(x)$$

for every smooth function $\phi$ compactly supported in $L$, where $\text{Vol}_L$ denotes the volume form on $L$ induced by the metric tensor $g|_L$.

The smallest positive solution of the above equation, denoted by $p(x, y, t)$, is called the heat kernel. Such a solution exists when $L$ is complete and of bounded geometry (see, for example, [8, 5]). We say that $(X, \mathcal{L}, g)$ satisfies Hypothesis (H1) if the leaves of $\mathcal{L}$ are all complete and of uniformly bounded geometry with respect to $g$. The assumption of uniformly bounded geometry means that there are real numbers $r > 0$, and $a, b$ such that for every point $x \in X$, the injectivity radius of the leaf $L_x$ at $x$ is $\geq r$ and all sectional curvatures belong to the interval $[a, b]$. Assuming this hypothesis then the heat kernel $p(x, y, t)$ exists on all leaves. The heat kernel gives rise to a one parameter family $\{D_t : t \geq 0\}$ of diffusion operators defined on bounded functions on $X$:

$$D_t f(x) := \int_{L_x} p(x, y, t) f(y) d\text{Vol}_{L_x}(y), \quad x \in X.$$  \hfill (2.1)
We record here the semi-group property of this family: \( D_0 = \text{id} \) and \( D_{t+s} = D_t \circ D_s \) for \( t, s \geq 0 \).

We note the following relation between the diffusion in a complete Riemannian manifold \( L \) and in its universal cover \( \tilde{L} \). In this article we often identify the fundamental group \( \pi_1(L) \) with the group of deck-transformations of the covering projection \( \pi: \tilde{L} \to L \). Recall that \( \tilde{L} \) is endowed with the metric \( \pi^*(g|_L) \). The Laplace operator \( \Delta \) on \( L \) lifts to \( \tilde{\Delta} \) on \( \tilde{L} \), which commutes with \( \pi \). To the operator \( \tilde{\Delta} \) is associated the heat kernel \( \tilde{p}(\tilde{x}, \tilde{y}, t) \), which is related to \( p(x, y, t) \) on \( L \) by

\[
p(x, y, t) = \sum_{\gamma \in \pi_1(L)} \tilde{p}(\tilde{x}, \gamma \tilde{y}, t)
\]

(2.2)

where \( \tilde{x}, \tilde{y} \) are lifts of \( x \) and \( y \) respectively. Moreover, the heat kernel is invariant under the deck-transformations, that is,

\[
\tilde{p}(\gamma \tilde{x}, \gamma \tilde{y}, t) = \tilde{p}(\tilde{x}, \tilde{y}, t)
\]

(2.3)

for all \( \gamma \in \pi_1(L) \) and \( \tilde{x}, \tilde{y} \in \tilde{L} \) and \( t \geq 0 \). As an immediate consequence of identity (2.3), we obtain the following relation between \( D_t \) and the heat diffusions \( \tilde{D}_t \) on \( \tilde{L} \).

**Proposition 2.3.** For every bounded function \( f \) defined on \( L \) and every \( t \in \mathbb{R}^+ \),

\[
\tilde{D}_t(f \circ \pi) = (D_t f) \circ \pi \quad \text{on } \tilde{L}.
\]

The following definitions will be used throughout the article.

**Definition 2.4.** We say that \((X, \mathcal{L}, g)\) satisfies Hypothesis \((H2)\) if \((X, \mathcal{L})\) is a lamination and if \( \Delta u \) is locally bounded for every \( u \in \mathcal{C}^2_0(X) \).

We say that \((X, \mathcal{L}, g)\) satisfies the Standing Hypotheses if this triplet satisfies both Hypotheses \((H1)\) and \((H2)\).

Let \( \mu \) be a positive finite Borel measure on \( X \).

\( \mu \) is called weakly harmonic if the following property is satisfied:

(i) \( \int_X \Delta f \, d\mu = \int_X f \, d\mu \) for all bounded measurable functions \( f \) defined on \( X \) and all \( t \in \mathbb{R}^+ \).

When \((X, \mathcal{L}, g)\) satisfies the Standing Hypotheses, \( \mu \) is called harmonic if it is weakly harmonic and if it satisfies the following additional property:

(ii) \( \int_X \Delta f \, d\mu = 0 \) for all functions \( f \in \mathcal{C}^2_0(X) \).

**Remark 2.5.** It is worthy making the following comment on property (ii). Hypothesis \((H2)\) guarantees that the function \( \Delta u \) is bounded, hence \( \mu \)-integrable for every \( u \in \mathcal{C}^2_0(X) \).

When \((X, \mathcal{L}, g)\) is a compact \( \mathcal{C}^2 \)-smooth lamination endowed with a transversally continuous Riemannian metric \( g \), it is known (see, for instance, [5, 24]) that (i) is equivalent to (ii), that is, weak harmonic measures are harmonic. On the other
hand, the same equivalence holds when $(X,\mathcal{L})$ is the regular part of a compact foliation by Riemann surfaces with linearizable singularities (see [12, 28]).

It is worthy noting here that $\|D_t f\|_{L^\infty(X)} \leq \|f\|_{L^\infty(X)}$ for every bounded function $f$ and every $t \in \mathbb{R}^+$ because $\int_{L_t} p(x, y, t) d\text{Vol}(y) = 1$ (see Chavel [8]). This, combined with (i) and an interpolation argument, implies that $\|D_t f\|_{L^q(X,\mu)} \leq \|f\|_{L^q(X,\mu)}$ for every $1 \leq q \leq \infty$, $t \in \mathbb{R}^+$ and every function $f \in L^q(X,\mu)$. In other words, the norm of the operator $D_t$ on $L^q(X,\mu)$ is $\leq 1$.

We have the following decomposition (see Proposition 4.7.9 in [34]).

Proposition 2.6. Let $\mu$ be a positive harmonic measure on $X$. Let $U \simeq B \times \mathbb{T}$ be a flow box as above which is relatively compact in $X$. Then, there is a positive Radon measure $\nu$ on $\mathbb{T}$ and for $\nu$-almost every $t \in \mathbb{T}$ there is a positive harmonic function $h_t$ on $B$ such that if $K$ is compact in $B$ the integral $\int_{\mathbb{T}} \|h_t\|_{L^1(K)} d\nu(t)$ is finite and

$$\int f d\mu = \int_{\mathbb{T}} \left( \int_{B} h_t(y) f(y, t) d\text{Vol}(y) \right) d\nu(t)$$

for every continuous compactly supported function $f$ on $U$. Here $\text{Vol}(y)$ denotes the Lebesgue volume form on $B$.

Proof. The local decomposition is provided by the disintegration of the measure with respect to the fibration $\pi : U \simeq B \times \mathbb{T} \to \mathbb{T}$ which is constant on the leaves. This allows to find a measure $\nu := \pi_* (\mu|_U)$ on $\mathbb{T}$ and a measurable assignment of a probability measure $\lambda_t$ on $B \times \{t\}$ and $\delta_t$ is the Dirac mass at $t$. Since $\theta$ is harmonic we deduce from Definition 2.4 (ii) and from the regularity results for weak solutions to elliptic differential equations that $\lambda_t = h_t \text{Vol}_t$, where $h_t$ is a harmonic function on the plaque $B \times \{t\}$ and $\text{Vol}_t$ is the Riemannian volume form on this plaque. Using this representation, the Choquet integral representation theorem [9] allows us to conclude the proof.

Throughout the article unless otherwise specified we assume that $(X,\mathcal{L},g)$ satisfies the Standing Hypotheses (H1) and (H2).
2.4 Brownian motion and Wiener measures without holonomy

In this subsection we follow the expositions given in [5] and [3]. Recall first the following terminology. An algebra $\mathcal{A}$ on a set $\Omega$ is a family of subsets of $\Omega$ such that $\Omega \in \mathcal{A}$ and that $X \setminus A \in \mathcal{A}$ and $A \cap B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$. If, moreover, $\mathcal{A}$ is stable under countable intersections, i.e., $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$ for any sequence $(A_n)_{n=1}^{\infty} \subseteq \mathcal{A}$, then $\mathcal{A}$ is said to be a $\sigma$-algebra. The ($\sigma$-) algebra generated by a family $\mathcal{F}$ of subsets of $\Omega$ is, by definition, the smallest ($\sigma$-) algebra containing $\mathcal{F}$.

If $\Omega$ is a topological space, then the Borel ($\sigma$-) algebra of $\Omega$ is denoted by $B(\Omega)$.

Let $(X, \mathcal{L}, g)$ be a Riemannian lamination satisfying the Standing Hypotheses. Let $\Omega := \Omega(X, \mathcal{L})$ be the space consisting of all continuous paths $\omega : [0, \infty) \to X$ with image fully contained in a single leaf. This space is called the sample-path space associated to $(X, \mathcal{L}, g)$. Observe that $\Omega$ can be thought of as the set of all possible paths that a Brownian particle, located at $\omega(0)$ at time $t = 0$, might follow as time progresses. The heat kernel will be used to construct a family $\{W_x\}_{x \in X}$ of probability measures on $\Omega$.

The construction of the measures on $\Omega$ needs to be done first in the space of all maps from the half-line $[0, \infty)$ into $X$, which is denoted by $X([0, \infty))$. The natural topology of this space is the product topology, but its associated Borel $\sigma$-algebra is too large for most purposes. Instead, we will use the $\sigma$-algebra $\mathcal{C}$ generated by cylinder sets. Recall that a cylinder set is a set of the form

$$C = C(t_1, B_1 : 1 \leq i \leq m) := \{ \omega \in X([0, \infty)) : \omega(t_i) \in B_i, \ 1 \leq i \leq m \},$$

where $m$ is a positive integer and the $B_i$ are Borel subsets of $X$, and $0 \leq t_1 < t_2 < \cdots < t_m$ is a set of increasing times. In other words, $C$ consists of all elements of $X([0, \infty))$ which can be found within $B_i$ at time $t_i$.

The structure of the measure space $(X([0, \infty)), \mathcal{C})$ is best understood by viewing it as an inverse limit. To do so, let the collection of finite subsets of $[0, \infty)$ be partially ordered by inclusion. Associated to each finite subset $F$ of $[0, \infty)$ is the measure space $(X^F, \mathfrak{F}^F)$, where $\mathfrak{F}^F$ is the Borel $\sigma$-algebra of the product topology on $X^F$. Each inclusion of finite sets $E \subset F$ canonically defines a projection $\pi_{EF} : X_F \to X_E$ which drops the finitely many coordinates in $F \setminus E$. These projections are continuous, hence measurable, and consistent, for if $E \subset F \subset G$, then $\pi_{EF} \circ \pi_{FG} = \pi_{EG}$. The family $\{(X^F, \mathfrak{F}^F), \pi_{EF} | E \subset F \subset [0, \infty) \mbox{ finite}\}$ is an inverse system of spaces, and its inverse limit is $X([0, \infty))$ with canonical projections $\pi_F : X([0, \infty)) \to X^F$. The $\sigma$-algebra $\mathcal{C}$ generated by the cylinder sets is the smallest one making all the projections $\pi_F$ measurable.

For each $x \in X$, a probability measure $W_x$ on the measure space $(X([0, \infty)), \mathcal{C})$ will now be defined. If $F = \{0 \leq t_1 < \cdots < t_m\}$ is a finite subset of $[0, \infty)$ and
$C^F := B_1 \times \cdots \times B_m$ is a cylinder set of $(X^F, \mathcal{X}^F)$, define

$$W_x^F(C^F) := \left( D_{t_1}(\chi_{B_1} D_{t_2-t_1}(\chi_{B_2} \cdots \chi_{B_{m-1}} D_{t_{m-1}-t_{m-2}}(\chi_{B_m}) \cdots) \right)(x), \quad (2.4)$$

where $\chi_{B_i}$ is the characteristic function of $B_i$ and $D_t$ is the diffusion operator given by (2.1). It is an obvious consequence of the semi-group property of $D_t$ that if $E \subset F$ are finite subsets of $[0, \infty)$ and $C^E$ is a cylinder subset of $X^E$, then

$$W_x^E(C^E) = W_x^F(\pi_{EF}^{-1}(C^E)).$$

Let $\mathcal{G}$ be the (non $\sigma$-) algebra generated by the cylinder sets in $X^{[0,\infty)}$. The above identity implies that $W_x$ extends to a measure on $\mathcal{G}$, hence to an outer measure on the family of all subsets of $X^{[0,\infty)}$. The $\sigma$-algebra of sets that are measurable with respect to this outer measure contains the cylinder sets, hence contains the $\sigma$-algebra $\mathcal{C}$. The Carathéodory-Hahn extension theorem then guarantees that the restriction of this outer measure to $\mathcal{C}$ is the unique measure agreeing with $W_x$ on the cylinder sets. This measure $W_x$ gives the set of paths $\omega \in X^{[0,\infty)}$ with $\omega(0) = x$ total probability.

**Theorem 2.7.** The subset $\Omega$ of $X^{[0,\infty)}$ has outer measure 1 with respect to $W_x$.

**Proof.** The case where $X$ is a single leaf has been proved in Appendix C4 in [5]. Note that here is the place where we make use of the Hypothesis (H1). The general case of a lamination $(X, \mathcal{L})$ follows almost along the same lines. More precisely, Lemma C.4.2 in [5] still holds in the context of a lamination $(X, \mathcal{L})$ noting that given a countable subset $F$ of $[0, \infty)$, then

$$W_x \left( \{ \omega \in X^{(0,\infty)} : \omega(t) \not\in L_x \text{ for some } t \in F \} \right) = 0.$$

Let $\widetilde{\mathcal{A}} := \widetilde{\mathcal{A}}(\Omega) = \widetilde{\mathcal{A}}(X, \mathcal{L})$ be the $\sigma$-algebra on $\Omega$ consisting of all sets $A$ of the form $A = C \cap \Omega$, with $C \in \mathcal{C}$. Then we define

$$W_x(A) = W_x(C \cap \Omega) := W_x(C). \quad (2.5)$$

$W_x$ is well-defined on $\widetilde{\mathcal{A}}$. Indeed, the $W_x$-measure of any measurable subset of $X^{[0,\infty)} \setminus \Omega$ is equal to 0 by Theorem 2.7. If $C, C' \in \mathcal{C}$ and $C \cap \Omega = C' \cap \Omega$, then the symmetric difference $(C \setminus C') \cup (C' \setminus C)$ is contained in $X^{[0,\infty)} \setminus \Omega$, so $W_x(C) = W_x(C')$. Hence, $W_x$ produces a probability measure on $(\Omega, \widetilde{\mathcal{A}})$. We say that $A \in \widetilde{\mathcal{A}}$ is a cylinder set (in $\Omega$) if $A = C \cap \Omega$ for some cylinder set $C \in \mathcal{C}$. The measure space $(\Omega, \widetilde{\mathcal{A}})$ has been thoroughly investigated in the works of Candel and Conlon in [5] [3]. We record here a useful property of cylinder sets (in $\Omega$).
Proposition 2.8. 1) If $A$ and $B$ are two cylinder sets, then $A \cap B$ is a cylinder set and $\Omega \setminus A$ is a finite union of mutually disjoint cylinder sets. In particular, the family of all finite unions of cylinder sets forms an algebra on $\Omega$.

2) If $A$ is a countable union of cylinder sets, then it is also a countable union of mutually disjoint cylinder sets.

Proof. Part 1) follows easily from the definition of cylinder sets.

To prove Part 2) let $A = \bigcup_{n=1}^{\infty} A_n$ with $A_n$ a cylinder set. Write $A = \bigcup_{n=1}^{\infty} B_n$, where $B_n := A_n \setminus B_{n-1}$ for $n > 1$. Then $B_n \cap B_m = \emptyset$ for $n \neq m$. On the other hand, using Part 1) we can show by induction on $n$ that each $B_n$ is a finite union of mutually disjoint cylinder sets. This proves Part 2). \qed

2.5 Wiener measures with holonomy

Let $(X, \mathcal{L}, g)$ be a Riemannian lamination satisfying the Standing Hypotheses and let $\Omega := \Omega(X, \mathcal{L})$. The measure space $(\Omega, \tilde{\mathcal{A}})$ defined in the previous subsection turns out to be incapable to detect the holonomy of the leaves. Here is a simple example.

Example 2.9. Given two points $x_0$, $x_1$ in a common leaf $L$, the cylinder set

$$C = C\left(\{0, \{x_0\}\}, \{1, \{x_1\}\}\right) = \{\omega \in \Omega : \omega(0) = x_0, \omega(1) = x_1\}$$

does not distinguish the homotopy type of the path $\omega|_{[0,1]}$ in $L$.

Now we introduce a new $\sigma$-algebra $\mathcal{A}$ on $\Omega$ which contains $\tilde{\mathcal{A}}$ and which has the advantage of taking into account the holonomy of the leaves. This new object will play a vital role in this article. Let $\pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L})$ be the covering lamination projection. It is a leafwise map. Fix an arbitrary $\tilde{x} \in \tilde{X}$ and $x := \pi(\tilde{x}) \in X$. Let $\Omega_x = \Omega_x(X, \mathcal{L})$ be the space of all continuous leafwise paths starting at $x$ in $(X, \mathcal{L})$, that is,

$$\Omega_x := \{\omega \in \Omega : \omega(0) = x\}.$$

Analogously, let $\tilde{\Omega}_x = \Omega_x(\tilde{X}, \tilde{\mathcal{L}})$ be the space of all continuous leafwise paths starting at $\tilde{x}$ in $(\tilde{X}, \tilde{\mathcal{L}})$. Every path $\omega \in \Omega_x$ lifts uniquely to a path $\tilde{\omega} \in \tilde{\Omega}_x$ in the sense that $\pi \circ \tilde{\omega} = \omega$, that is, $\pi(\tilde{\omega}(t)) = \omega(t)$ for all $t \geq 0$. In what follows this bijective lifting is denoted by $\pi_x^{-1} : \Omega_x \to \tilde{\Omega}_x$. So $\pi \circ (\pi_x^{-1}(\omega)) = \omega, \omega \in \Omega_x$. By Subsection 2.4, we construct a $\sigma$-algebra $\tilde{\mathcal{A}}(\tilde{\Omega})$ on $\tilde{\Omega} := \Omega(\tilde{X}, \tilde{\mathcal{L}})$ which is the $\sigma$-algebra generated by all cylinder sets in $\tilde{\Omega}$.

Definition 2.10. Let $\mathcal{A} = \mathcal{A}(\Omega)$ be the $\sigma$-algebra generated by all sets of following family

$$\{\pi \circ \tilde{A} : \text{cylinder set } \tilde{A} \text{ in } \tilde{\Omega}\}.$$
where \( \pi \circ \tilde{A} := \{ \pi \circ \tilde{\omega} : \tilde{\omega} \in \tilde{A} \} \).

For a point \( x \in X \), we apply the previous definition to the lamination consisting of a single leaf \( L := L_x \) with its covering projection \( \pi : \tilde{L} \to L \). By setting \( \Omega := \Omega(L) \) and \( \tilde{\Omega} := \Omega(\tilde{L}) \) in this context, we can define the \( \sigma \)-algebra \( \mathcal{A}(L) := \mathcal{A}(\Omega(L)) \). Let \( \mathcal{A}_x \) be the restriction of \( \mathcal{A}(L) \) on the set \( \Omega_x \). So \( \mathcal{A}_x \) is a \( \sigma \)-algebra on \( \Omega_x \).

Observe that \( \tilde{\mathcal{A}} \subset \mathcal{A} \) and that the equality holds if all leaves of \((X, \mathcal{L})\) have trivial holonomy. Moreover, we also have that \( \mathcal{A}_x \subset \mathcal{A} \). Note that \( \mathcal{A}(\tilde{\Omega}) = \mathcal{A}(\tilde{\Omega}) \).

Now we construct a family \( \{ W_x \} \) of probability Wiener measures on \((\Omega, \mathcal{A})\).

Let \( x \) be a point in \( X \) and \( C \) an element of \( \mathcal{A} \).

Then we define the so-called Wiener measure

\[
W_x(C) := W_{\tilde{x}}(\pi^{-1}_{\tilde{x}} C),
\]

where \( \tilde{x} \) is a lift of \( x \) under the projection \( \pi : \tilde{L} \to L = L_x \), and

\[
\pi^{-1}_{\tilde{x}}(C) := \{ \pi^{-1}_{\tilde{x}} \omega : \omega \in C \cap \Omega_x \},
\]

and \( W_{\tilde{x}} \) is the probability measure on \((\Omega(\tilde{L}), \mathcal{A}(\tilde{L}))\) given by (2.5).

The following result will be proved in Appendix.

**Theorem 2.11.** For each element \( B \in \mathcal{A} \), the bounded function \( X \ni x \mapsto W_x(B) \in [0, 1] \) is Borel measurable.

Given a \( \sigma \)-finite positive Borel measure \( \mu \) on \( X \), we construct a \( \sigma \)-finite positive measure \( \tilde{\mu} \) on \((\Omega, \mathcal{A})\) as follows.

\[
\tilde{\mu}(B) := \int_X \left( \int_{\omega \in B \cap \Omega_x} dW_x \right) d\mu(x) = \int_X W_x(B) d\mu(x), \quad B \in \mathcal{A}.
\]

The measure \( \tilde{\mu} \) is called the Wiener measure with initial distribution \( \mu \). Note that when \( \mu \) is a probability measure, so is \( \tilde{\mu} \).

The following result gives basic properties of the measures \( W_x \) and \( \tilde{\mu} \).

**Proposition 2.12.** We keep the above hypotheses and notation.

(i) \( \pi^{-1}_{\tilde{x}} C \in \mathcal{A}(\tilde{\Omega}) \) and the value of \( W_x(C) \) defined in (2.6) is independent of the choice of \( \tilde{x} \). Moreover, \( W_x \) is a probability measure on \((\Omega, \mathcal{A})\).

(ii) If \( \mu \) is a \( \sigma \)-finite positive measure (resp. a probability measure), then \( \tilde{\mu} \) given in (2.7) is \( \sigma \)-finite positive measure (resp. a probability measure) on \((\Omega, \mathcal{A})\).

Since the proof of this proposition is somehow technical, we postpone it to Subsection 10.3 and 10.7 in Appendix below for the sake of clarity.

The following result relates the weak harmonicity of probability measures defined on \((X, \mathcal{L})\) to the invariance of the corresponding measures on \( \Omega(X, \mathcal{L}) \).

---

2 When a lamination \((X, \mathcal{L})\) consists of a single leaf \( L \), i.e. \((X, \mathcal{L}) = (L, L)\), we often write \( \Omega(L) \) instead of \( \Omega(L, L) \).
Theorem 2.13. If $\mu$ is a probability weakly harmonic measure on $(X, \mathcal{L})$, then $\bar{\mu}$ is $T^n$-invariant on $(\Omega, \mathcal{A})$ for all $t \in \mathbb{R}^+$.

The proof of Theorem 2.13 will be provided in Appendix.

3 Statement of the main results

First, we introduce a notion of multiplicative cocycles for Riemannian laminations. Next, we state our main results as well as their applications. Finally we outline their proofs.

3.1 Multiplicative cocycles

Observe that the orbit $L_x$ of a point $x \in X$ by a transformation $T : X \to X$ is the set $\{T^n x : n \in \mathbb{N}\}$, and hence can be ordered by the unique map $\omega : \mathbb{N} \to L_x$ given by $\omega(n) = T^n x$, $n \in \mathbb{N}$. The case of laminations is quite different: the orbit $L_x$ of a point $x \in X$ by a lamination $(X, \mathcal{L})$ is, as expected, the whole leaf passing through $x$. However, this leaf is a manifold and hence it cannot be time-ordered by $\mathbb{N}$. Therefore, it is natural to replace the unique map $\omega$ in the context of a transformation $T$ by the space $\Omega_x$. Hence, a plausible (multiplicative) cocycle on $(X, \mathcal{L})$ should be a multiplicative map $A : \Omega \times \mathbb{R}_+ \to GL(d, \mathbb{R})$ such that $A(\omega, 0) = id$ for all $\omega \in \Omega$. Obviously, this temporary definition is still not good enough. Indeed, since the space $\Omega$ is too large, there are plenty of pairs $(\omega, t)$ consisting of an $\omega \in \Omega$ and a $t \in \mathbb{R}$ such that $\omega(0) = \omega(t)(= x)$, should we request that

$$A(\omega, t) = A(\omega, 0)$$

At this stage the topology of the leaf $L_x$ comes into play as suggested to us by Candel definition of cocycles for $GL(1, \mathbb{R})$. So it is quite natural to assume the last identity when $\omega|_{[0,t]}$ is null-homotopic in $L_x$, and hence the matrix $A(\omega, t)$ should depend only on the class of homotopy of paths $\omega|_{[0,t]}$ with two fixed ends-points $\omega(0)$ and $\omega(t)$. So a reasonable definition of (multiplicative) cocycles should reflect the topology of the leaves of $(X, \mathcal{L})$. The notion of homotopy for paths in $\Omega$ can be made precise as follows.

Definition 3.1. Let $\omega_1 : [0, t_1] \to X$ and $\omega_2 : [0, t_2] \to X$ be two continuous paths with image fully contained in a single leaf $L$ of $(X, \mathcal{L})$. We say that $\omega_1$ is homotopic to $\omega_2$ if there exists a continuous map $\omega : I \to L$ with $I := \{(t, s) \in \mathbb{R}_+ \times [0, 1] : 0 \leq t \leq (1-s)t_1 + st_2\}$ such that $\omega(0, s) = \omega_1(0) = \omega_2(0)$ and $\omega((1-s)t_1 + st_2, s) = \omega_1(t_1) = \omega_2(t_2)$ for all $s \in [0, 1]$, and $\omega(\cdot, 0) = \omega_1$ and $\omega(\cdot, 1) = \omega_2$. In other words, the path $\omega_1$ may be deformed continuously on $L$ to $\omega_2$, the two ends of $\omega_1$ being kept fixed during the deformation.
Definition 3.2. Let $G$ be either $\mathbb{N}t_0$ (for some $t_0 > 0$) or $\mathbb{R}^+$. A (multiplicative) cocycle on $\Omega$ is a map

$$A : \Omega \times G \to \text{GL}(d, \mathbb{R})$$

such that

1. (identity law) $A(\omega, 0) = \text{id}$ for all $\omega \in \Omega$;
2. (homotopy law) if $\omega_1, \omega_2 \in \Omega$ and $t_1, t_2 \in G$ such that $\omega_1(0) = \omega_2(0)$ and $\omega_1(t_1) = \omega_2(t_2)$ and $\omega_1|_{[0,t_1]}$ is homotopic to $\omega_2|_{[0,t_2]}$, then

$$A(\omega_1, t_1) = A(\omega_2, t_2);$$

3. (multiplicative law) $A(\omega, s + t) = A(T^t(\omega), s)A(\omega, t)$ for all $s, t \in G$ and $\omega \in \Omega$;
4. (measurable law) $A(\cdot, t) : \Omega \ni \omega \mapsto A(\omega, t)$ is measurable for every $t \in G$.

Observe that if $A : \Omega \times G \to \text{GL}(d, \mathbb{R})$ is a cocycle, then the map $A^{-1} : \Omega \times G \to \text{GL}(d, \mathbb{R})$, defined by $A^{-1}(\omega, t) := (A(\omega, t))^\dagger$, is also a cocycle, where $A^\dagger$ (resp. $A^{-1}$) denotes as usual the transpose (resp. the inverse) of a square matrix $A$.

As a fundamental example, we define the holonomy cocycle of a $C^1$ transversally smooth foliation $(X, \mathcal{L})$ of codimension $d$ in a Riemannian manifold $(X, g)$. Let $T_x(L)$ be the tangent bundle to the leaves of the foliation, i.e., each fiber $T_x(L)$ is the tangent space $T_x(L_x)$ for each point $x \in X$. The normal bundle $N(L)$ is, by definition, the quotient of $T(X)$ by $T(L)$, that is, the fiber $N_x(L)$ is the quotient $T_x(X)/T_x(L)$ for each $x \in X$. Observe that the metric $g$ on $X$ induces a metric (still denoted by $g$) on $N(L)$. For every transversal section $S$ at a point $x \in X$, the tangent space $T_x(S)$ is canonically identified with $N_x(L)$ through the composition $T_x(S) \hookrightarrow T_x(X) \to T_x(X)/T_x(L)$.

For every $x \in X$ and $\omega \in \Omega_x$ and $t \in \mathbb{R}^+$, let $h_{\omega,t}$ be the holonomy map along the path $\omega|_{[0,t]}$ from a fixed transversal section $S_0$ at $\omega(0)$ to a fixed transversal section $S_t$ at $\omega(t)$ (see Subsection 10.3 below). Using the above identification, the derivative $Dh_{\omega,t} : T_{\omega(0)}(S_0) \to T_{\omega(t)}(S_t)$ induces a map (still denoted by) $Dh_{\omega,t} : N_{\omega(0)}(L) \to N_{\omega(t)}(L)$. The last map depends only on the path $\omega|_{[0,t]}$; in particular, it does not depend on the choice of transversal sections $S_0$ and $S_t$.

An identifier $\tau$ of $(X, \mathcal{L})$ is a suitable smooth map which associates to each point $x \in X$ a linear isometry $\tau(x) : N_x(L) \to \mathbb{R}^d$, that is, a linear morphism such that

$$\|\tau(x)v\| = \|v\|, \quad v \in N_x(L), \quad x \in X.$$
Here we have used the Euclidean norm on the left-hand side and the \( g \)-norm on the right hand side. We identify every fiber \( N_x(L) \) with \( \mathbb{R}^d \) via \( \tau \). The holonomy cocycle \( A \) of \( (X, L) \) with respect to the identifier \( \tau \) is defined by

\[
A(\omega, t) := \tau(\omega(t)) \circ (Dh_{\omega,t})(\omega(0)) \circ \tau^{-1}(\omega(0)), \quad \omega \in \Omega, \ t \in \mathbb{R}^+.
\]

**Proposition 3.3.** The holonomy cocycle \( A \) is a multiplicative cocycle.

**Proof.** Since \( h_{\omega,0} = \text{id} \), we have that \( A(\omega, 0) = \text{id} \). To prove that the homotopy law is fulfilled, let \( \omega_1, \omega_2 \in \Omega \) and \( t_1, t_2 \in \mathbb{R}^+ \) such that \( \omega_1(0) = \omega_2(0) = x \) and \( \omega_1(t_1) = \omega_2(t_2) \) and \( \omega_1|_{[0,t_1]} \) is homotopic to \( \omega_2|_{[0,t_2]} \). By Proposition 2.3.2 in [1], \( h_{\omega_1,t_1} = h_{\omega_2,t_2} \) on an open neighborhood of \( x \) in a fixed transversal section through \( x \). Hence, \( (Dh_{\omega_1,t_1})(x) = (Dh_{\omega_2,t_2})(x) \), which proves the homotopy law.

For \( s, t \in \mathbb{R}^+ \) and \( \omega \in \Omega \), we have, by the chain rule,

\[
(Dh_{\omega,s+t})(\omega(0)) = (Dh_{T^s(\omega),s})(\omega(t)) \circ (Dh_{\omega,t})(\omega(0)).
\]

Combining this and the definition of \( A \), the multiplicative law follows.

The measurable law is an immediate consequence of Proposition 3.4 below.

\[ \square \]

One can perform basic operations on the category of cocycles such as the tensor product, the direct sum and the wedge-product. In this article we are only concerned with the last operation. Let \( A_1 \) and \( A_2 \) be two cocycles defined on \( \Omega \times G \) with values in \( \text{GL}(d_1, \mathbb{R}) \) and \( \text{GL}(d_2, \mathbb{R}) \) respectively. Then their wedge-product is the map \( A_1 \wedge A_2 : \Omega \times G \to \text{GL}(\mathbb{R}^{d_1} \wedge \mathbb{R}^{d_2}) \) given by the formula

\[
(A_1 \wedge A_2)(\omega, t)(v_1 \wedge v_2) := A(\omega, t)v_1 \wedge A(\omega, t)v_2, \quad \omega \in \Omega, \ t \in G, \ v_1 \in \mathbb{R}^{d_1}, \ v_2 \in \mathbb{R}^{d_2}.
\]

The operation is defined analogously when \( A_1 \) and \( A_2 \) are with values in \( \text{GL}(d_1, \mathbb{C}) \) and \( \text{GL}(d_2, \mathbb{C}) \) respectively. We leave to the reader to prove the following result:

**Proposition 3.4.** \( A_1 \wedge A_2 \) is a (multiplicative) cocycle.

### 3.2 First Main Theorem and applications

The following notions are needed.

**Definition 3.5.** Let \( (S, \mathcal{S}, \nu) \) be a \( \sigma \)-finite positive measure space. A subset \( Z \subset S \) is said to be of **null \( \nu \)-measure** (resp. of **full \( \nu \)-measure**) if \( \nu(Z) = 0 \) (resp. \( \nu(S \setminus Z) = 0 \)).

**Definition 3.6.** Let \( (X, L, g) \) be a (Riemannian) lamination.

A subset \( Z \subset X \) is said to be **leafwise saturated** if \( a \in Z \) implies that the whole leaf \( L_a \) is contained in \( Z \).

A function \( f \) defined on a leafwise saturated set \( Y \subset X \) is called **leafwise constant** if it is constant on each restriction of \( f \) to \( L_a \) for each \( a \in Y \).

A positive finite Borel measure \( \mu \) on \( X \) is said to be **ergodic** if every leafwise saturated measurable subset of \( X \) either has full \( \mu \)-measure or null \( \mu \)-measure.
Now we are in the position to state our first (or abstract) Oseledec Multiplicative Ergodic Theorem for Riemannian laminations.

**Theorem 3.7.** Let \((X, \mathcal{L}, g)\) be a lamination satisfying the Standing Hypotheses. Let \(\mu\) be a harmonic probability measure. Let \(G\) be either \(\mathbb{N}_0\) or \(\mathbb{R}^+\), where \(t_0 > 0\) is a given number. Let \(A : \Omega \times G \to \text{GL}(d, \mathbb{R})\) be a cocycle on \(\Omega\). Assume that \(A\) satisfies the integrability condition, that is,

- if \(G = \mathbb{N}_0\) then \(\int_\Omega \log^+ \|A^\pm(t, t_0)\|d\mu_\omega(\omega) < \infty\);
- if \(G = \mathbb{R}^+\), then \(\int_\Omega \sup_{t \in [0, t_0]} \log^+ \|A^\pm(t, t)\|d\mu_\omega(\omega) < \infty\).

Then there exists a measurable function \(m : Y \to \mathbb{N}\) which is leafwise constant.

(i) There is a measurable function \(m : Y \to \mathbb{N}\) which is leafwise constant.

(ii) For each \(\omega \in X\) there exists a decomposition of \(\mathbb{R}^d\) as a direct sum of linear subspaces

\[\mathbb{R}^d = \bigoplus_{i=1}^{m(\omega)} H_i(x),\]

such that \(A(\omega, t)H_i(x) = H_i(\omega(t))\) for all \(\omega \in \Omega_x\) and \(t \in G\). Moreover, the map \(x \mapsto H_i(x)\) is a measurable map from \(\{x \in Y : m(x) \geq i\}\) into the Grassmannian of \(\mathbb{R}^d\). Moreover, there are real numbers

\[\chi_m(x) < \chi_{m-1}(x) < \cdots < \chi_2(x) < \chi_1(x)\]

such that the function \(x \mapsto \chi_i(x)\) is measurable and leafwise constant on \(\{x \in Y : m(x) \geq i\}\), and

\[\lim_{t \to \infty, t \in G} \frac{1}{t} \log \frac{\|A(\omega, t)v\|}{\|v\|} = \chi_i(x),\]  

(3.1)

uniformly on \(v \in H_i(x) \setminus \{0\}\), for \(W_x\)-almost every \(\omega \in \Omega_x\). The numbers \(\chi_m(x) < \chi_{m-1}(x) < \cdots < \chi_2(x) < \chi_1(x)\) are called the Lyapunov exponents associated to the cocycle \(A\) at the point \(x\).

(iii) For \(S \subset N := \{1, \ldots, m(x)\}\) let \(H_S(x) := \bigoplus_{i \in S} H_i(x)\). Then

\[\lim_{t \to \infty, t \in G} \frac{1}{t} \log \sin |\angle(H_S(\omega(t)), H_{N \setminus S}(\omega(t)))| = 0\]  

(3.2)

for \(W_x\)-almost every \(\omega \in \Omega_x\).

Some remarks are in order.

- The decomposition of \(\mathbb{R}^d\) as a direct sum of subspaces \(\mathbb{R}^d = \bigoplus_{i=1}^{m(x)} H_i(x)\), given in (ii) is said to be the Oseledec decomposition at a point \(x \in Y\). If we apply Theorem \[1.7\] to the shift-transformation \(T\) acting on the probability measure space \((\Omega(X, \mathcal{L}), \mathcal{A}, \mu)\), we obtain a weaker conclusion that for \(\mu\)-almost every path \(\omega\), there is an Oseledec decomposition at the point \(x = \omega(0)\). But this decomposition depends on each path \(\omega \in \Omega_x\). A remarkable point of Theorem \[3.7\] is that the following stronger statement still holds: for each point \(x\) in a leafwise saturated set \(Y \subset X\) of full \(\mu\)-measure, we have a common Oseledec
decomposition at the point \( x \) for \( W_x \)-almost every path \( \omega \in \Omega_x \). We can even show that for each \( x \in Y \), there is a set \( \mathcal{F}_x \subset \Omega_x \) of full \( W_x \)-measure such that identity (3.1) and identity (3.2) above hold for all \( \omega \in \mathcal{F}_x \).

- The identity \( A(\omega, 1)H_i(\omega(0)) = H_i(\omega(1)) \) for every continuous leafwise path \( \omega \) contained in \( Y \) is known as the holonomy invariant property of the Oseledecc decomposition at each point of \( Y \).

- The definition of the cocycle and Theorem 3.7 can be formulated using the action of \( \text{GL}(d, \mathbb{C}) \) on \( \mathbb{C}^d \) instead of the action of \( \text{GL}(d, \mathbb{R}) \) on \( \mathbb{R}^d \). Consequently, we obtain an Oseledecc decomposition of \( \mathbb{C}^d \) as a direct sum of complex subspaces \( \mathbb{C}^d = \bigoplus_{i=1}^{m(x)} H_i(x) \) at every point \( x \in Y \).

We deduce from Theorem 3.7 the following important consequence.

**Corollary 3.8.** We keep the hypotheses of Theorem 3.7 and suppose in addition that \( \mu \) is ergodic. Then there are a leafwise saturated Borel set \( Y \subset X \) of full \( \mu \)-measure, an integer \( m \geq 1 \), and \( m \) real numbers \( \chi_i \) and \( d_i \) integers for \( 1 \leq i \leq m \) such that the conclusion of Theorem 3.7 holds for \( Y \) and that \( m(\omega) = m \) and \( \chi_i(x) = \chi_i \) and \( \dim H_i(x) = d_i \) for every \( x \in Y \) and \( 1 \leq i \leq m \). Moreover,

\[
\chi_1 = \lim_{t \to \infty, t \in G} \frac{1}{t} \log \| A(\omega, t) \| \quad \text{and} \quad \chi_m = - \lim_{t \to \infty, t \in G} \frac{1}{t} \log \| A^{-1}(\omega, t) \|
\]

for \( \bar{\mu} \)-almost every \( \omega \in \Omega \).

Now we apply Theorem 3.7 in order to investigate the \( k \)-fold exterior product \( A^{\wedge k} (1 \leq k \leq d) \) of a given cocycle \( A \). Recall that \( A^{\wedge k} \) is a map defined \( \Omega \times \mathbb{G} \) with values in \( \text{GL}((\mathbb{R}^d)^k) \) (resp. \( \text{GL}((\mathbb{C}^d)^k) \)), given by the formula

\[
A^{\wedge k} := A \wedge \cdots \wedge A \quad (k \text{ times}).
\]

We keep the hypotheses and notation of Corollary 3.8. Consider \( d \) functions \( \chi : \Omega \times (\mathbb{R}^d)^k \to \mathbb{R} \) for \( 1 \leq k \leq d \), given by

\[
\chi(\omega; v_1, \ldots, v_k) := \limsup_{t \to \infty, t \in G} \frac{1}{t} \| A(\omega, t)v_1 \wedge \cdots \wedge A(\omega, t)v_k \| \quad (3.3)
\]

for \( \omega \in \Omega \) and \( v_1, \ldots, v_k \in \mathbb{R}^d \).

**Corollary 3.9.** There exist a leafwise saturated Borel set \( Y \subset X \) of full \( \mu \)-measure and \( d \) functions \( \chi : Y \times (\mathbb{R}^d)^k \to \mathbb{R} \) for \( 1 \leq k \leq d \) such that all the conclusions of Corollary 3.8 hold for \( Y \) and that the following properties also hold:

(i) For each \( x \in Y \) there exists a set \( \mathcal{F}_x \subset \Omega_x \) of full \( W_x \)-measure such that for any vectors \( v_1, \ldots, v_k \in \mathbb{R}^d \) and every \( \omega \in \mathcal{F}_x \) the right hand side in formula (3.3) is, in fact, a true limit. Moreover,

\[
\chi(x; v_1, \ldots, v_k) = \chi(\omega; v_1, \ldots, v_k), \quad \omega \in \mathcal{F}_x.
\]
The number $\chi(x; v_1, \ldots, v_k)$ is called the $k$ dimensional Lyapunov exponent of the vectors $v_1, \ldots, v_k$ at $x$.

(ii) $\chi(x; v) = \chi_i$ for $v \in (\oplus_{j=1}^{m} H_j(x)) \setminus (\oplus_{j=i+1}^{m} H_j(x))$.

(iii) if $v_1, \ldots, v_k \in \bigcup_{i=1}^{m} H_i(x)$ and $v_1 \wedge \cdots \wedge v_k \neq 0$, then

$$\chi(x; v_1, \ldots, v_k) = \sum_{i=1}^{k} \chi_i(x; v_i).$$

(iv) For each $x \in Y$ we have the following Oseledec decomposition for the cocycle $A^k$ at $x$:

$$\mathbb{R}^d = \oplus_{1 \leq i_1, \ldots, i_k \leq m} H_{i_1}(x) \wedge \cdots \wedge H_{i_k}(x).$$

In particular, the Lyapunov exponents of $A^k$ form the set

$$\{\chi'_{i_1} + \cdots + \chi'_{i_k} : 1 \leq i_1 < \cdots < i_k \leq d\},$$

where $\chi'_{d} \leq \cdots \leq \chi'_{1}$ are exactly the Lyapunov exponents $\chi_m < \cdots < \chi_1$, each $\chi_i$ being counted with multiplicity $d_i$. In particular, we have that

$$\lim_{t \to \infty, \ t \in G} \frac{1}{t} \log \|A(\omega, t)^{\wedge k}\| = \sum_{i=1}^{k} \chi'_i, \quad 1 \leq k \leq d.$$

Another important consequence of Theorem 3.7 is a characterization of Lyapunov spectrum in the spirit of Ledrappier’s work in [26]. We will establish this result in Theorem 9.20 in Section 9 below after developing necessary materials.

### 3.3 Second Main Theorem and applications

In order to state the Second Main Theorem we need to introduce some new notions. Let $(X, \mathcal{L})$ be a Riemannian lamination satisfying the Standing Hypotheses and set $\Omega := \Omega(X, \mathcal{L})$ and let $G$ be either $\mathbb{N}s$ (for some $s > 0$) or $\mathbb{R}^+$.

**Definition 3.10.** Let $\mathcal{A} : \Omega \times G \to \text{GL}(d, \mathbb{R})$ be a map that satisfies the identity, homotopy and multiplicative laws in Definition 3.2. Fix an arbitrary element $t_0 \in G \setminus \{0\}$. In any flow box $\Phi_i : U_i \to B_i \times T_i$ with $B_i$ simply connected, consider the map $\alpha_i : B_i \times B_i \times T_i \to \text{GL}(d, \mathbb{R})$ defined by

$$\alpha_i(x, y, t) := \mathcal{A}(\omega, t_0),$$

where $\omega$ is any leaf path such that $\omega(0) = \Phi_i^{-1}(x, t), \omega(1) = \Phi_i^{-1}(y, t)$ and $\omega[0, t_0]$ is contained in the simply connected plaque $\Phi_i^{-1}(. , t)$. We say that $\alpha_i$ is the local expression of $\mathcal{A}$ on the flow box $\Phi_i$. By the homotopy law in Definition 3.2, the local expression of $\mathcal{A}$ on the flow box $\Phi_i$ does not depend on the choice of $t_0 \in G \setminus \{0\}$.
Suppose now that \((X, L)\) is smooth lamination of class \(C^k\) \((k \in \mathbb{N})\). Then

a map \(A\) as above is said to be \(C^{l+\epsilon}\)-differentiable cocycle for some \(l \in \mathbb{N}\) with \(l < k\) and some \(0 \leq \epsilon \leq 1\) if, for any flow box \(\Phi_t\) of a \(C^k\)-smooth atlas for \((X, L)\), the local expression of \(A\) is \(C^{l+\epsilon}\)-differentiable. Clearly, this definition does not depend on the choice of a smooth atlas for \((X, L)\).

Given a \(C^2\)-differentiable cocycle \(A\), we define two functions \(\bar{\delta}(A)\), \(\delta(A) : X \to \mathbb{R}\) as well as four quantities \(\bar{\chi}_{\text{max}}(A), \chi_{\text{max}}(A), \bar{\chi}_{\text{min}}(A), \chi_{\text{min}}(A)\) as follows.

Consider the function \(f_{u,x} : K \to \mathbb{R}\) defined by

\[
f_{u,x}(y) := \log \frac{\|A(\omega, 1)u\|}{\|u\|}, \quad y \in K, \ u \in \mathbb{R}^d \setminus \{0\},
\]

where \(\omega \in \Omega\) is any path such that \(\omega(0) = x, \omega(1) = y\) and that \(\omega[0,1]\) is contained in \(K\). Then define

\[
\bar{\delta}(A)(x) := \sup_{u \in \mathbb{R}^d : \|u\|=1} (\Delta f_{u,x})(x) \quad \text{and} \quad \delta(A)(x) := \inf_{u \in \mathbb{R}^d : \|u\|=1} (\Delta f_{u,x})(x),
\]

where \(\Delta\) is, as usual, the Laplacian on the leaf \(L_x\) induced by the metric tensor \(g\) on \((X, L)\). We also define

\[
\bar{\chi}_{\text{max}} = \bar{\chi}_{\text{max}}(A) := \int_X \bar{\delta}(A)(x)d\mu(x), \quad \chi_{\text{max}} = \chi_{\text{max}}(A) := \int_X \delta(A)(x)d\mu(x);
\]

\[
\bar{\chi}_{\text{min}} = \bar{\chi}_{\text{min}}(A) := -\bar{\chi}_{\text{max}}(A^{-1}), \quad \chi_{\text{min}} = \chi_{\text{min}}(A) := -\chi_{\text{max}}(A^{-1}).
\]

Note that our functions \(\bar{\delta}, \bar{\delta}\) are the multi-dimensional generalizations of the operator \(\delta\) introduced by Candel [3] which has been recalled in Section II.

There is another equivalent characterization of ergodicity for the class of all harmonic probability measures on a compact \(C^2\)-smooth lamination \((X, L)\). This class forms a compact convex cone in the space of all Radon measures on \(X\).

Proposition 2.6.18 in [5] says that the ergodic measures are exactly the extremal members of this cone.

We are in the position to state our second main result.

**Theorem 3.11.** Let \((X, L)\) be a compact \(C^2\)-smooth lamination endowed with a transversally continuous Riemannian metric \(g\). Let \(\mu\) be a harmonic probability measure which is ergodic. Let \(A : \Omega \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{R})\) be a \(C^1\)-differentiable cocycle. Then there exists a leafwise saturated Borel set \(Y \subset X\) of full \(\mu\)-measure and a number \(m \in \mathbb{N}\) and \(m\) integers \(d_1, \ldots, d_m \in \mathbb{N}\) such that the following properties hold:

(i) For each \(x \in Y\) there exists a decomposition of \(\mathbb{R}^d\) as a direct sum of linear subspaces

\[
\mathbb{R}^d = \oplus_{i=1}^m H_i(x),
\]
such that \( \dim H_i(x) = d_i \) and \( A(\omega, t)H_i(x) = H_i(\omega(t)) \) for all \( \omega \in \Omega_x \) and \( t \in \mathbb{G} \). Moreover, \( x \mapsto H_i(x) \) is a measurable map from \( Y \) into the Grassmannian of \( \mathbb{R}^d \).
Moreover, there are real numbers \( \chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1 \) such that
\[
\lim_{t \to \infty, t \in \mathbb{R}^+} \frac{1}{t} \log \frac{\| A(\omega, t)v \|}{\|v\|} = \chi_i,
\]
uniformly on \( v \in H_i(x) \setminus \{0\} \), for \( W_x \)-almost every \( \omega \in \Omega_x \), where \( \| \cdot \| \) denotes any norm in \( \mathbb{R}^d \). The numbers \( \chi_m < \chi_{m-1} < \cdots < \chi_2 < \chi_1 \) are called the Lyapunov exponents of the cocycle \( A \).

(ii) For \( S \subset N := \{1, \ldots, m\} \) let \( H_S(x) := \oplus_{i \in S} H_i(x) \). Then
\[
\lim_{t \to \infty, t \in \mathbb{R}^+} \frac{1}{t} \log \sin \left| \angle (H_S(\omega(t)), H_{N \setminus S}(\omega(t))) \right| = 0
\]
for \( W_x \)-almost every \( \omega \in \Omega_x \).

(iii) If, in addition, \( A \) is \( C^{2+\epsilon} \)-differentiable for some \( \epsilon > 0 \), then
\[
\chi_{\max} \leq \chi_1 \leq \bar{\chi}_{\max} \quad \text{and} \quad \chi_{\min} \leq \chi_m \leq \bar{\chi}_{\min}.
\]

Theorem 3.11 generalizes Theorem 1.3 to the higher dimensions. On the other hand, assertion (iii) of Theorem 3.11 combined with Corollary 3.9 implies effective integral estimates for Lyapunov exponents of a \( C^{2+\epsilon} \) differentiable cocycle.

**Corollary 3.12.** Under the hypothesis and the notation of Theorem 3.11 (iii) let \( \chi_1' \leq \cdots \leq \chi_d' \) be the Lyapunov exponents \( \chi_m < \cdots < \chi_1 \), each \( \chi_i \) being counted with multiplicity \( d_i \). Then
\[
\chi_{\max}(A^\wedge k) \leq \sum_{i=1}^k \chi_i' \leq \bar{\chi}_{\max}(A^\wedge k), \quad 1 \leq k \leq d.
\]

When \( k = d \), \( A^{\wedge d} \) is a cocycle of dimension 1, and hence Corollary 3.12 gives that
\[
\sum_{i=1}^d \chi_i' = \bar{\chi}_{\max}(A^{\wedge k}) = \bar{\chi}_{\max}(A^{\wedge k}).
\]
So we obtain an effective integral formula for the sum of all Lyapunov exponents counted with multiplicity.

Now we apply Theorem 3.11 to the holonomy cocycle of a compact \( C^2 \) transversally smooth foliation \( (X, \mathcal{L}) \) of codimension \( d \) in a Riemannian manifold \( (X, g) \).
Let \( N(\mathcal{L}) \) be the normal bundle of this foliation. We say that a leaf \( L \) is holonomy invariant if there exists a measurable decomposition of \( x \ni L \mapsto N(\mathcal{L})_x \) into the direct sum of \( d \) lines \( H_1(x) \oplus \cdots \oplus H_d(x) \) such that these lines are invariant.
with respect to the differential of the holonomy map along every closed continuous path. More concretely, the last invariance means that for every \( x \in L \) and for every path \( \gamma \in \Omega \) with \( \gamma(0) = \gamma(1) = x \), it holds that \( Dh_{\gamma,1}H_i(x) = H_i(x) \), \( i = 1, \ldots, d \). Clearly, if \( L \) has trivial holonomy (i.e. \( h_{\gamma,1} = id \) for every \( x \in L \) and every path \( \gamma \) as above), then it is holonomy invariant. However, the converse statement is, in general, not true.

We get the following consequence of Theorem 3.11.

**Corollary 3.13.** Let \( \mu \) be an ergodic harmonic probability measure directed by a compact \( C^2 \) transversally smooth foliation \((X, L)\) of codimension \( d \) in a Riemannian manifold \((X, g)\). Suppose that the holonomy cocycle of \((X, L)\) admits \( d \) distinct Lyapunov exponents with respect to \( \mu \). Then for \( \mu \)-almost every \( x \in X \), the leaf \( L_x \) is holonomy invariant.

It is relevant to mention here a well-known theorem due to G. Hector, D.-B.-A. Epstein, K. Millet and D. Tischler (see Theorem 2.3.12 in [4]) which states that a generic leaf of a lamination has trivial holonomy. Recall that a subset of leaves of \((X, L)\) is said to be generic if its union contains a countable intersection of open dense leafwise saturated sets of \( X \). This theorem may be viewed as a topological counterpart of Corollary 3.13.

In forthcoming papers [28, 29] we will investigate the holonomy cocycle of a (possibly singular) foliation by hyperbolic Riemann surfaces. In particular, we will establish Theorem 3.7 and 3.11 and find geometric interpretations of Lyapunov exponents. We will also compare our characteristic exponents with other definitions in the literature. In the context of foliations by hyperbolic Riemann surfaces, the holonomy of leaves is closely related to the uniformizations of leaves and their Poincaré metric. This subject has been received a lot of attention in the recent years (see for example [2, 6, 12, 13, 14, 16, 17, 18, 27]).

### 3.4 Plan of the proof

We use the method of Brownian motion which is initiated by Garnett [20] and is developed further by Candel [3]. More precisely, we want to prove a Multiplicative Ergodic Theorem for the shift-transformations \( T_t \) \((t > 0)\) defined on the sample-path space \( \Omega(X, L) \) such that the Oseledec decomposition exists at almost every point \( x \in X \), that is, such a decomposition is common for \( W_x \)-almost every path \( \omega \in \Omega_x \). Section 4 is devoted to some aspects of the measure theory and the ergodic theory on sample-path spaces. The results of this section will be used throughout the article. Some of these results are stated in this section, but their proofs are given in Appendix below. Section 5 focuses the study of Lyapunov exponents on a single leaf. In that section we establish a Lyapunov filtration, that is, a weak form of an Oseledec decomposition, at almost every point in a single leaf. The main ingredients are an appropriate definition of leafwise Lyapunov...
exponents using the Brownian motion and the Markov property of stochastic
processes.

Following Ruelle’s proof of Oseledec’s theorem (see [32]) we need to construct
a forward filtration and a backward filtration at almost every point so that these
filtrations are compatible with the considered cocycle. Section 6 introduces the
notion of an invariant bundle. The usefulness of this notion is illustrated by a
splitting theorem which reduces the study of Lyapunov exponents of a cocycle
to that of splitting bundles which are easier to handle. Using the results of the
previous sections and appealing to an argument of Walters in [35] we prove the
existence of Lyapunov forward filtrations in Section 7. The existence of Lyapunov
backward filtrations is much harder to obtain; it will be established in Section 8
thanks to an involved calculus on heat diffusions. It will be shown in Section 9
that the intersection of these two filtrations forms the Oseledec decomposition.
To do this we develop a new technique of constructing weakly harmonic measures
and of splitting invariant sub-bundles. This approach is inspired by the similar
device of Ledrappier [26] and Walters [35] in the context of discrete dynamics.
The proofs of the main results as well as their corollaries are also presented in
this section.

4 Preparatory results

The first part of this section deals with measurability questions that arise in the
study of a Riemannian lamination \((X, \mathcal{L}, g)\) satisfying the Standing Hypotheses.
In particular, we give a sufficient and simple criterion for multiplicative cocycles.
The remaining part of the section discusses the Markov property of the Brownian
motion.

Before going further we fix several standard notion and terminology on Mea-
sure Theory which will be used throughout this article (see, for example, [15, 7]
for more details). A positive measure space \((S, \mathcal{S}, \nu)\) is said to be finite (resp.
\(\sigma\)-finite) if \(\nu(S) < \infty\) (resp. if there exists a sequence \((S_n)_{n=1}^{\infty} \subset \mathcal{S}\) such that
\(\nu(S_n) < \infty\) and \(S = \bigcup_{n=1}^{\infty} S_n\)).

Let \((S, \mathcal{S}, \nu)\) be a \(\sigma\)-finite positive measure space. A subset \(N \subset S\) is said
to be negligible if there exists \(A \in \mathcal{S}\) such that \(N \subset A\) and \(\nu(A) = 0\). The
\(\nu\)-completion of \(\mathcal{S}\) is the \(\sigma\)-algebra generated by \(\mathcal{S}\) and the negligible sets, it
is denoted by \(\mathcal{S}_\nu\). The elements of \(\mathcal{S}_\nu\) are said \(\nu\)-measurable. The measure \(\nu\)
admits a unique extension (still denoted by \(\nu\)) to \(\mathcal{S}_\nu\), and the measure space
\((S, \mathcal{S}_\nu, \nu)\) is said to be the completion of \((S, \mathcal{S}, \nu)\). The measure space \((S, \mathcal{S}, \nu)\)
is said to be complete if \(\mathcal{S}_\nu = \mathcal{S}\). When \(S\) is a topological space, \(\mathcal{B}(S)\) denotes
as usual the \(\sigma\)-algebra of Borel sets of \(S\).

Let \((T, \mathcal{I})\) and \((S, \mathcal{I})\) be two measurable spaces. A function \(\sigma : T \to S\) is
said to be measurable if \(\sigma^{-1}(A) \in \mathcal{I}\) is for every \(A \in \mathcal{I}\). Suppose in addition
that \((T, \mathcal{I}, \mu)\) is a positive \(\sigma\)-finite measure space. Then a function \(\sigma : T \to S\)
is said to be $\mu$-measurable if $\sigma^{-1}(A)$ is $\mu$-measurable for every $A \in \mathcal{F}$.

### 4.1 Measurability issue

Let $\pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L})$ be the covering lamination projection. A set $A \subset X$ is said to be a cylinder image if $A = \pi \circ \tilde{A}$ for some cylinder set $\tilde{A} \subset \tilde{\Omega} := \Omega(\tilde{X}, \tilde{\mathcal{L}})$.

Recall from Definition 2.10 that the $\sigma$-algebra $\mathcal{A}$ (resp. $\tilde{\mathcal{A}}$) on $\Omega := \Omega(X, \mathcal{L})$ is generated by all cylinder images (resp. by all cylinder sets) and that for a point $x \in X$, let $\mathcal{A}_x$ be the restriction of $\mathcal{A}$ on $\Omega_x$.

Recall from Definition 2.10 that the $\sigma$-algebra $\mathcal{A}$ (resp. $\tilde{\mathcal{A}}$) on $\Omega := \Omega(X, \mathcal{L})$ is generated by all cylinder images (resp. by all cylinder sets) and that for a point $x \in X$, let $\mathcal{A}_x$ be the restriction of $\mathcal{A}$ on $\Omega_x$.

Proposition 4.1. (i) For every $x \in X$ and for every $A \in \mathcal{A}_x$, there exists a decreasing sequence $(A_n)$, each $A_n$ being a countable union of mutually disjoint cylinder images such that $A \subset A_n$ and that $W_x(A_n \setminus A) \to 0$ as $n \to \infty$.

(ii) Suppose in addition that $X = \tilde{X}$. So cylinder images coincide with cylinder sets, and hence $\mathcal{A}_x = \mathcal{A}$. Then for every $A \in \mathcal{A}$, there exists a decreasing sequence $(A_n)$, each $A_n$ being a countable union of mutually disjoint cylinder sets such that $A \subset A_n$ and that $\bar{\mu}(A_n \setminus A) \to 0$ as $n \to \infty$.

Proposition 4.1 will play an important role in the sequel. Since the proof of this proposition is somehow technical, we postpone it to Subsection 10.3 and 10.7 in Appendix below for the sake of clarity. Note, however, that Proposition 4.1 together with Proposition 2.12 give fundamental properties of the measures $W_x$ and $\bar{\mu}$.

Proposition 4.2. Let $S$ be a topological space. For any measurable set $F$ of the measurable space $(\Omega \times \mathcal{S}, \mathcal{A} \otimes \mathcal{B}(S))$ let $\Phi(F)$ be the function

$$X \times S \ni (x, s) \mapsto W_x(\{\omega \in \Omega_x : (\omega, s) \in F\}).$$

Then $\Phi(F)$ is measurable.

Proof. We argue as in the proof of Proposition 10.11 by replacing the integral with the family of Wiener measures.

Let $\mathfrak{A}$ be the family of all sets $A = \bigcup_{i \in I} \Omega_i \times S_i$, where $\Omega_i \in \mathcal{A}$ and $S_i \in \mathcal{B}(S)$, and the index set $I$ is finite. Note that $\mathfrak{A}$ is an algebra on $\Omega \times S$ which generates the $\sigma$-algebra $\mathcal{A} \otimes \mathcal{B}(S)$. Moreover, each such set $A$ can be expressed as a disjoint finite union $A = \bigcup_{i \in I} \Omega_i \times S_i$. Using the above expression for such a set $A$, we infer that

$$\Phi(A)(x, s) = \sum_{i \in I} W_x(\Omega_i) 1_{S_i}(s), \quad (x, s) \in X \times S.$$ 

On the other hand, by Theorem 2.11 $X \ni x \mapsto W_x(\Omega_i)$ is Borel measurable. Consequently, $\Phi(A)$ is measurable for all $A \in \mathfrak{A}$.  

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Let \( \mathcal{A} \) be the family of all sets \( A \subset \Omega \times S \) such that \( \Phi(A) \) is measurable. The previous paragraph shows that \( \mathcal{A} \subset \mathcal{A} \).

Next, suppose that \( (A_n)_{n=1}^{\infty} \subset \mathcal{A} \) and that either \( A_n \searrow A \) or \( A_n \nearrow A \). By Lebesgue dominated convergence, we get that either \( \Phi(A_n) \searrow \Phi(F) \) or \( \Phi(A_n) \nearrow \Phi(A) \). So \( \Phi(A) \) is also measurable. Hence, \( A \in \mathcal{A} \). Consequently, by Proposition 10.10, \( \mathcal{A} \otimes \mathcal{B}(S) \subset \mathcal{A} \). In particular, \( \Phi(A) \) is well-defined and measurable for each \( A \in \mathcal{A} \otimes \mathcal{B}(S) \). This completes the proof.

For every \( x \in X \) let \( L := L_x \) be the leaf passing through \( x \), or more generally let \( (L, g) \) be a complete Riemannian manifold of bounded geometry. Recall from Section 2 that \( \Omega(L) \) (resp. \( \Omega_x \)) is the space of continuous paths \( \omega : [0, \infty) \to L \) (resp. the subspace of \( \Omega(L) \) consisting of all paths originated at \( x \)). Recall also that \( \Omega(L) \) (resp. \( \Omega_x \)) is endowed with the \( \sigma \)-algebra \( \mathcal{A}(L) = \mathcal{A}(\Omega(L)) \) (resp. \( \mathcal{A}_x = \mathcal{A}(\Omega_x) \)). Let \( W_x \) be the probability Wiener measure on \( \Omega_x \). For any function \( f : \Omega(L) \to \mathbb{R} \cup \{\pm\} \), let \( \operatorname{ess} \sup f \) denote the essential supremum of \( f \) with respect to \( W_x \), that is,

\[
\operatorname{ess} \sup_{\omega \in \Omega_x} f(\omega) := \inf_{E \in \mathcal{A}(\Omega_x), W_x(E) = 1} \sup_{\omega \in E} f(\omega).
\]

(4.1)

For any measurable function \( f : \Omega \to [-\infty, \infty] \), define two functions \( \overline{f} \) and \( \underline{f} \) by

\[
\overline{f}(x) := \operatorname{ess} \sup_{\omega \in \Omega_x} f(\omega) \quad \text{and} \quad \underline{f}(x) := \operatorname{ess} \inf_{\omega \in \Omega_x} f(\omega), \ x \in X.
\]

We are in the position to state the second main result of this section.

**Proposition 4.3.** Let \( f \) be a measurable function on \( \Omega \). Then \( \overline{f} \) and \( \underline{f} \) are measurable on \( X \).

**Proof.** Since \( \overline{-f} = -\overline{f} \), we only need to prove that \( \overline{f} \) is measurable. In the proof of Proposition 4.3 below we need to show that \( \{x \in X : \overline{f} \leq r\} \) is a measurable set in \( X \) for all real number \( r \). Observe that

\[
\{x \in X : \overline{f} \leq r\} = \{x \in X : W_x(A_r) = 1\},
\]

where \( A_r := \{\omega \in \Omega : f(\omega) \leq r\} \in \mathcal{A} \). On the other hand, applying Theorem 2.11 yields that the function \( X \ni x \mapsto W_x(A_r) \) is measurable. Hence, \( \{x \in X : \overline{f} \leq r\} \) is measurable as desired.

The following result shows that the measurable law of a cocycle is equivalent to the measurability of its local expressions on each flow boxes. The latter condition is very easy to check in practice.

**Proposition 4.4.** Let \( \mathbb{G} \) be either \( \mathbb{N}_{t_0} \) (for some \( t_0 > 0 \)) or \( \mathbb{R}^+ \). Let \( \mathcal{A} : \Omega(X, \mathcal{S}) \times \mathbb{G} \to \text{GL}(d, \mathbb{R}) \) be a map which satisfies the identity, homotopy and multiplicative laws in Definition 3.12. Then \( \mathcal{A} \) is a multiplicative cocycle if and only if the local expression of \( \mathcal{A} \) on every flow box is measurable (see Definition 3.10 above).
We postpone the proof of Proposition 4.4 to Subsection 10.7 in Appendix below.

4.2 Markov property of Brownian motion

We establish some facts concerning Brownian motion using the Wiener measures with holonomy. Let \((L, g)\) be a complete Riemannian manifold of bounded geometry. Let \(F\) be a measurable bounded function defined on \(\Omega(L)\) and let \(x \in L\) be a point.

The expectation of \(F\) at \(x\) is the quantity

\[
E_x[F] := \int F(\omega) dW_x(\omega).
\]

Let \(\mathcal{F}\) be a \(\sigma\)-subalgebra of \(\mathcal{A}(L) = \mathcal{A}(\Omega(L))\). The conditional expectation of the function \(F\) with respect to \(\mathcal{F}\) at \(x\) is a function \(E_x[F|\mathcal{F}]\) defined on \(\Omega(L)\) and measurable with respect to \(\mathcal{F}\) such that

\[
\int_A E_x[F|\mathcal{F}](\omega) dW_x(\omega) = \int_A F(\omega) dW_x(\omega)
\]

for all \(A \in \mathcal{F}\). Note that \(E_x[F|\mathcal{F}]\) is unique in the "\(W_x\)-almost everywhere" sense. Therefore, we may restrict ourselves to all \(A \in \mathcal{F} \cap \mathcal{A}_x\).

For \(r \geq 0\) let \(\pi_r : \Omega(L) \rightarrow L\) be the projection given by \(\pi_r(\omega) := \omega(r)\), \(\omega \in \Omega(L)\). For \(s \geq 0\) let \(\mathcal{F}_s\) be the smallest \(\sigma\)-algebra making all the projections \(\pi_r : \Omega(L) \rightarrow L\) with \(0 \leq r \leq s\) measurable. For \(t \geq 0\) let \(\mathcal{F}_{t+} := \bigcap \mathcal{F}_s\).

The Markov property says the following

**Theorem 4.5.** Let \(F\) be a measurable bounded function defined on \(\Omega(L)\). Then for every \(x \in L\) and \(t > 0\) the following equality \(E_x[F \circ T^t|\mathcal{F}_{t+}] = E_x[F] \circ \pi_t\) holds \(W_x\)-almost everywhere, i.e.,

\[
E_x[F \circ T^t|\mathcal{F}_{t+}](\omega) = E_{\omega(t)}[F]
\]

holds for \(W_x\)-almost \(\omega \in \Omega(L)\).

**Proof.** Let \(\pi : \tilde{L} \rightarrow L\) be the universal cover of \(L\). Fix arbitrary points \(x \in L\), and \(\tilde{x} \in \pi^{-1}(x)\), and a number \(t > 0\). Consider the function \(\tilde{F} : \Omega(\tilde{X}, \tilde{L}) \rightarrow \mathbb{R}\) given by

\[
\tilde{F}(\tilde{\omega}) := F(\pi \circ \tilde{\omega}), \quad \tilde{\omega} \in \Omega(\tilde{X}, \tilde{L}).
\]

For every element \(A \in \mathcal{F}_{t+} \cap \mathcal{A}_x\), let \(\tilde{A} := \pi_{\tilde{x}}^{-1}(A)\). The Markov property for Brownian motion without holonomy (see, for instance, Theorem C.3.4 in [5]), applied to \(\tilde{L}\) with the reference point \(\tilde{x}\), yields that

\[
\int_{\tilde{A}} (\tilde{F} \circ T^t) dW_{\tilde{x}}(\tilde{\omega}) = \int_{\tilde{A}} E_{\omega(t)}[\tilde{F}] dW_{\tilde{x}}(\tilde{\omega})
\]

References: [5]
Moreover, using the bijective lifting \( \pi_x^{-1} : \Omega_x \rightarrow \tilde{\Omega}_x \), and applying Lemma 10.35 (iii) below, we see easily that

\[
E_{\tilde{\omega}(t)}[\tilde{F}] = E_{\omega(t)}[F], \quad \tilde{\omega} := \pi_x^{-1}(\omega), \; \omega \in \Omega_x.
\]

By a similar argument, we also obtain that

\[
\int_A (\tilde{F} \circ T^t) dW_{\tilde{x}}(\tilde{\omega}) = \int_A (F \circ T^t) dW_x(\omega).
\]

Combining the last three identities, we infer that, for every element \( A \in F_{t+} \cap \mathcal{A}_x \),

\[
\int_A (F \circ T^t) dW_x(\omega) = \int_A E_{\omega(t)}[F] dW_x(\omega).
\]

It is worthy noting that the continuity of the sample paths in \( \Omega(L) \) plays the crucial in the proof of Theorem 1.5. As an important consequence of Markov property, the following result relates the ergodicity of harmonic probability measures defined on \( (X, \mathcal{L}) \) to that of the corresponding extended measures on \( \Omega := \Omega(X, \mathcal{L}) \).

**Theorem 4.6.** Let \( \mu \) be a probability measure harmonic on \( (X, \mathcal{L}, g) \). Then \( \mu \) is ergodic if and only if \( \bar{\mu} \) is ergodic for some (equivalently, for all) shift-transformation \( T^t \) with \( t \in \mathbb{R}^+ \setminus \{0\} \).

We postpone the proof of Theorem 4.6 to Subsection 10.11 in Appendix below. Note however that the analogue version of this theorem in the case of the \( \sigma \)-algebra \( \mathcal{A} := \mathcal{A}(\Omega) \) has been outlined in Theorem 3 in [20].

## 5 Leafwise Lyapunov exponents

We first introduce an alternative definition of Lyapunov exponents in the discrete version of the First Main Theorem and then study this notion on a fixed leaf of a lamination. This approach permits us to apply the Brownian motion theory more efficiently. Consequently, we obtain important invariant properties of Lyapunov exponents.

In this section \( (X, \mathcal{L}, g) \) is a Riemannian lamination equipped and \( A : \Omega(X, \mathcal{L}) \times G \rightarrow GL(d, \mathbb{R}) \) is a cocycle with \( G := Nt_0 \) for some \( t_0 > 0 \). Suppose without loss of generality that \( t_0 = 1 \), that is, \( G = \mathbb{N} \). Consider the function \( \chi : X \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\pm \infty\} \) defined by

\[
\chi(x, v) := \text{ess. sup}_{\omega \in \Omega_x} \chi(\omega, v), \quad (x, v) \in X \times \mathbb{R}^d, \quad (5.1)
\]
where, for each fixed \((x, v) \in X \times \mathbb{R}^d\), the operator \(\text{ess. sup}_{\omega \in \Omega_x} f(\omega)\) has been defined in (1.1) and
\[
\chi(\omega, v) := \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v\|,
\quad \omega \in \Omega_x.
\]
The following elementary lemma will be useful.

**Lemma 5.1.** Let \((L, g)\) be a complete Riemannian manifold of bounded geometry and \(f : \Omega(L) \to \mathbb{R} \cup \{\pm\}\) a measurable function. Then, for every \(x \in L\), there exists a set \(E \in \mathcal{A}(\Omega_x)\) of full \(W_x\)-measure such that
\[
\text{ess. sup}_{\omega \in \Omega_x} f(\omega) = \sup_{\omega \in E} f(\omega).
\]
In particular, for every set \(Z \subset \Omega_x\) of null \(W_x\)-measure,
\[
\sup_{\omega \in E} f(\omega) = \sup_{\omega \in E \setminus Z} f(\omega).
\]

**Proof.** For every \(n \in \mathbb{N} \setminus \{0\}\) let \(E_n \in \mathcal{A}(\Omega_x)\) of full \(W_x\)-measure such that
\[
\sup_{\omega \in E_n} f(\omega) \leq \text{ess. sup}_{\omega \in \Omega_x} f(\omega) + \frac{1}{n}.
\]
Setting \(E := \cap_{n \geq 1} E_n\), we see that \(W_x(E) = 1\) and \(\sup_{\omega \in E} f(\omega) \leq \text{ess. sup}_{\omega \in \Omega_x} f(\omega)\).

Hence, the first equality of the lemma follows. The second equality follows by combining the first one with the equality \(W_x(E \setminus Z) = 1\). \(\square\)

The fundamental properties of \(\chi\) are given below.

**Proposition 5.2.** (i) \(\chi\) is a measurable function.
(ii) \(\chi(x, 0) = -\infty\) and \(\chi(x, v) = \chi(x, \lambda v)\) for \(x \in X, v \in \mathbb{R}^d, \lambda \in \mathbb{R} \setminus \{0\}\). So we can define a function, still denoted by \(\chi\), defined on \(X \times \mathbb{P}(\mathbb{R}^d)\) by
\[
\chi(x, [v]) := \chi(x, v), \quad x \in X, \ v \in \mathbb{R}^d \setminus \{0\},
\]
where \([\cdot] : \mathbb{R}^d \setminus \{0\} \to \mathbb{P}(\mathbb{R}^d)\) which maps \(v \mapsto [v]\) is the canonical projection.
(iii) \(\chi(x, v_1 + v_2) \leq \max\{\chi(x, v_1), \chi(x, v_2)\}\), \(x \in X, v_1, v_2 \in \mathbb{R}^d\).
(iv) For all \(x \in X\) and \(t \in \mathbb{R} \cup \{\pm\}\) the set
\[
V(x, t) := \{v \in \mathbb{R}^d : \chi(x, v) \leq t\}
\]
is a linear subspace of \(\mathbb{R}^d\). Moreover, \(s \leq t\) implies \(V(x, s) \subset V(x, t)\).
(v) For every \(x \in X\), \(\chi(x, \cdot) : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}\) takes only finite \(m(x)\) different values
\[
\chi_{m(x)}(x) < \chi_{m(x)-1}(x) < \cdots < \chi_2(x) < \chi_1(x).
\]
(vi) If, for \( x \in X \), we define \( V_i(x) \) to be \( V(x, \chi_i(x)) \) for \( 1 \leq i \leq m(x) \), then

\[
\{0\} \equiv V_{m(x)+1}(x) \subset V_{m(x)}(x) \subset \cdots \subset V_2(x) \subset V_1(x) \equiv \mathbb{R}^d
\]

and

\[
v \in V_i(x) \setminus V_{i+1}(x) \iff \sup_{\omega \in E_x} \limsup_{n \to \infty} \frac{1}{n} \log \| A(\omega, n)v \| = \chi_i(x)
\]

for some set \( E_x \in \mathcal{A}(\Omega_x) \) of full \( W_x \)-measure, \( E_x \) depends only on \( x \) (but it does not depend on \( v \in \mathbb{R}^d \)).

**Proof.** Since we know by the measurable law in Definition 3.2 that \( A(\cdot, n) \) is measurable on \( \Omega(X, \mathcal{L}) \) for every \( n \in \mathbb{N} \), the function \( \Omega(X, \mathcal{L}) \times \mathbb{R}^d \ni (\omega, v) \mapsto \chi(\omega, v) \) is also measurable. Consequently, assertion (i) follows from Proposition 4.3. The proof of (ii) is clear.

Now we turn to assertion (iii). By Lemma 5.1 pick sets \( E_0, E_1, E_2 \in \mathcal{A}(\Omega_x) \) of full \( W_x \)-measure such that

\[
\chi(x, v_1) = \sup_{\omega \in E_1} \limsup_{n \to \infty} \frac{1}{n} \log \| A(\omega, n)v_1 \|,
\]

where we put \( v_0 := v_1 + v_2 \). Now setting \( E := E_0 \cap E_1 \cap E_2 \), \( E \) is an element of \( \mathcal{A}(\Omega_x) \) of full \( W_x \)-measure.

The following elementary result is needed.

**Lemma 5.3.** If \( a_n, b_n \geq 0 \) for \( n \geq 1 \) then

\[
\limsup_{n \to \infty} \frac{1}{n} \log (a_n + b_n) = \max \left\{ \limsup_{n \to \infty} \frac{1}{n} \log a_n, \limsup_{n \to \infty} \frac{1}{n} \log b_n \right\},
\]

and

\[
\liminf_{n \to \infty} \frac{1}{n} \log (a_n + b_n) \geq \max \left\{ \liminf_{n \to \infty} \frac{1}{n} \log a_n, \liminf_{n \to \infty} \frac{1}{n} \log b_n \right\}.
\]

Using the equality of the above lemma it follows that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \| A(\omega, n)(v_1 + v_2) \| \leq \max \left\{ \limsup_{n \to \infty} \frac{1}{n} \log \| A(\omega, n)v_1 \|, \limsup_{n \to \infty} \frac{1}{n} \log \| A(\omega, n)v_2 \| \right\}.
\]

Taking the supremum of both sides of the last inequality over all \( \omega \in E \) and using Lemma 5.1 we obtain \( \chi(x, v_1 + v_2) \leq \max \{ \chi(x, v_1), \chi(x, v_2) \} \), which proves (iii).

Each \( V(x, t) \) is a linear subspace of \( \mathbb{R}^d \) by (ii) and (iii). The inclusion \( V(x, t) \subset V(x, s) \) for \( s \leq t \) is also clear. Hence, (iv) follows.

Fix \( x \in X \). Since \( s < t \) implies \( V(x, s) \subset V(x, t) \) and hence \( \dim V(x, s) \leq \dim V(x, t) \), we can enumerate all the values of \( t : \chi_{m(x)}(x) < \chi_{m(x)-1}(x) < \cdots < \chi_2(x) < \chi_1(x) \), where \( t \mapsto \dim V(x, t) \) changes. Therefore, \( \chi(x, \cdot) \) can only take the values \( \chi_{m(x)}(x), \chi_{m(x)-1}(x), \ldots, \chi_2(x), \chi_1(x) \). This proves (v).
To prove (vi) it suffices to find a set $E_x \in \mathcal{A}(\Omega_x)$ with the required properties. To do this fix a point $x \in X$ and a basis $\{v_1, \ldots, v_d\}$ of $\mathbb{R}^d$ such that $\{v_1, \ldots, v_{k_j}\}$ is a basis of $V_{m(x)-j+1}(x)$ for $j = 1, \ldots, m(x)$, where $k_j := \dim V_{m(x)-j+1}(x)$. By Lemma 5.1, pick a set $E_j \in \mathcal{A}(\Omega_x)$ of full $W_x$-measure such that

$$
\chi(x, v_i) = \sup_{\omega \in E_i} \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v_i\|
$$

Now set $E_x := \bigcap_{j=1}^d E_j$. The desired conclusion follows from the above equalities for $\chi(x, v_i)$ and from (ii), (iii) and Lemma 5.1.

Now let $L$ be a fixed leaf of a lamination $(X, \mathcal{L})$ and Vol the Lebesgue measure induced by its Riemannian metric. Fix a point $x \in L$.

**Proposition 5.4.** (i) For any measurable set $A \subset \Omega(L)$,

$$W_x(A) \leq \int_{y \in L} p(x, y, 1)W_y(T(A)) \text{Vol}(y).$$

If, moreover, $T^{-1}(T(A)) = A$, then the above inequality is an equality.

(ii) Given a set $A \subset \Omega_x$ of full $W_x$-measure, then for Vol-almost every $y \in L$, $T(A)$ is of full $W_y$-measure.

**Proof.** Consider two bounded measurable functions $F, G : \Omega(L) \to \mathbb{R}$ defined by

$$F(\omega) := \begin{cases} 1, & \omega \in A; \\ 0, & \omega \notin A. \end{cases}$$

and

$$G(\omega) := \begin{cases} 1, & \omega \in T(A); \\ 0, & \omega \notin T(A). \end{cases}$$

It is clear that $F \leq G \circ T$. Moreover, if $T^{-1}(T(A)) = A$ then $F = G \circ T$. Consequently, we have that

$$W_x(A) = \mathbb{E}_x[F] = \mathbb{E}_x[\mathbb{E}_x[F|\mathcal{F}_{1+}]] \leq \mathbb{E}_x[\mathbb{E}_x[G \circ T|\mathcal{F}_{1+}]].$$

where the second equality holds by the projection rules of the expectation operation (see Theorem C.1.6 in [5]), the inequality follows from the estimate $F \leq G \circ T$. By the Markov property (see Theorem 4.5), we get that

$$\mathbb{E}_x[G \circ T|\mathcal{F}_{1+}] = \mathbb{E}_x[G] \circ \pi_1.$$

Inserting this into the previous inequalities we obtain that

$$W_x(A) \leq \mathbb{E}_x[\mathbb{E}_x[G \circ T|\mathcal{F}_{1+}]] = \mathbb{E}_x[\mathbb{E}_x[G] \circ \pi_1].$$
This, combined with
\[ E_y[G] = \int_{\omega \in T(A)} dW_y(\omega) = W_y(T(A)), \quad y \in L, \]
implies that
\[ W_x(A) \leq E_x[E_y[G] \circ \pi_1] = \int_{y \in L} p(x, y, 1) W_y(T(A)) \textup{Vol}(y), \]
which proves the first assertion.

The second assertion follows by combining the first one, and the identity
\[ \int_{y \in L} p(x, y, 1) \textup{Vol}(y) = 1, \]
and the estimate \( 0 \leq W_y(T(A)) \leq 1 \).

Here is the main result of this section.

**Proposition 5.5.** Let \( L \) be a leaf of \((X, \mathcal{L})\) and \( A \) a cocycle on \((X, \mathcal{L})\). For every \( x \in X \), let \( m(x) \), \( \chi_j(x) \), \( V_i(x) \) be given by Proposition 5.2. Then there exist a number \( m \in \mathbb{N} \) and \( m \) integers \( 1 \leq d = d_0 < \cdots < d_1 = d \) and \( m \) real numbers \( \chi_m < \chi_{m-1} < \cdots < \chi_1 \) and a subset \( Y \subset L \) with the following properties:

(i) \( \textup{Vol}(L \setminus Y) = 0 \);
(ii) for every \( x \in Y \) and every \( 1 \leq j \leq m \), we have \( m(x) = m \) and \( \chi_j(x) = \chi_j \) and \( \text{dim} \ V_j(x) = d_j \); 
(iii) for every \( x, y \in Y \), and every \( \omega \in \Omega_x \) such that \( \omega(1) = y \), we have \( A(\omega, 1) V_i(x) = V_i(y) \) with \( 1 \leq i \leq m \).

Prior to the proof it is worthy noting that that property (iii) is a primitive version of the holonomy invariance of the Oseledec decomposition.

**Proof.** First we prove that there is a constant \( \chi_1 \) such that \( \chi_1(x) = \chi_1 \) for Vol-
almost every \( x \in L \). Let \( \{e_1, \ldots, e_d\} \) be the canonical basis of \( \mathbb{R}^d \). Since we know that \( \chi : L \times \mathbb{R}^d \to \mathbb{R} \) is measurable, it follows that \( \chi_1(x) = \sup_{u \in \mathbb{R}^d} \chi(x, u) = \sup_{1 \leq j \leq d} \chi(x, e_j) \) is also measurable. Let
\[ \chi_1 := \text{ess. sup}_{x \in L} \chi_1(x) := \inf_{E \subset L : \textup{Vol}(L \setminus E) = 0} \sup_{x \in E} \chi_1(x). \]

Fix a point \( x \in L \). By Lemma 5.1, there exists a set \( A \in \mathcal{A}(\Omega_x) \) of full \( W_x \)-measure such that
\[ \chi_1(x) = \max \sup_{1 \leq j \leq d} \chi(\omega, e_j). \]

On the one hand, it follows from the definition that
\[ \chi(T\omega, A(\omega, 1)v) = \chi(\omega, v), \quad (\omega, v) \in \Omega(X, \mathcal{L}) \times \mathbb{R}^d. \quad (5.2) \]
Note that \( \{ A(\omega, 1)e_j : 1 \leq j \leq d \} \) forms a basis of \( \mathbb{R}^d \). Therefore, we infer from (ii) and (iii) of Proposition 5.2 that for \( y = \omega(1) \),

\[
\chi_1(y) = \max_{1 \leq j \leq d} \chi(y, A(\omega, 1)e_j).
\]

On the other hand, by Proposition 5.4 for Vol-almost every \( y \in L \), \( T(A) \) is of full \( W_y \)-measure. Consequently, for such \( y \),

\[
\chi_1(y) \leq \max_{1 \leq j \leq d} \sup_{\omega \in T(A)} \chi(\omega, A(\omega, 1)e_j) = \max_{1 \leq j \leq d} \sup_{\omega \in A} \chi(\omega, e_j) = \chi_1(x).
\]

Hence \( \chi_1(x) \geq \chi_1(y) \) for Vol-almost every \( y \in L \). So \( \chi_1(x) \geq \chi_1 \). On the other hand, by definition, \( \chi_1(x) \leq \chi_1 \) for Vol-almost every \( x \in L \). Therefore, there exists a Borel set \( Y_1 \subset L \) such that \( \text{Vol}(L \setminus Y_1) = 0 \) and that \( \chi_1(x) = \chi_1 \) for every \( x \in Y_1 \).

Consider

\[
A_2 := \{ (x, v) \in Y_1 \times \mathbb{R}^d : \chi(x, v) < \chi_1 \} \subset \text{Leb}(Y_1) \times \mathcal{B}(\mathbb{R}^d),
\]

where \( \mathcal{B}(\mathbb{R}^d) \) denotes, as usual, the Borel \( \sigma \)-algebra, and \( \text{Leb}(Y_1) \) denotes the completion of the Borel \( \sigma \)-algebra of \( Y_1 \) equipped with the Lebesgue measure, \( Y_1 \) being endowed with the induced topology from \( L \). Let \( \Pi_1 : Y_1 \times \mathbb{R}^d \to Y_1 \) be the natural projection, then by Theorem 10.3 below, \( \Pi_1(A_2) \subset \text{Leb}(L) \). Also \( \Pi_1(A_2) = \{ x \in Y_1 : m(x) > 0 \} \). If \( \text{Vol}(\Pi_1(A_2)) = 0 \), then the proof of the proposition is complete with \( m = 1 \) and \( Y = Y_1 \) and \( V_1 = \mathbb{R}^d \).

Suppose now that \( \text{Vol}(\Pi_1(A_2)) > 0 \). For \( y \in L \), let \( V_2(y) \) be the proper vector subspace \( A_2 \cap \Pi_1^{-1}(y) \) of \( \mathbb{R}^d \) with the convention that \( V_2(y) = \{ 0 \} \) if \( y \notin \Pi_1(A_2) \). Fix a point \( x \) in the set \( \Pi_1(A_2) \). So \( \text{dim} V_2(x) > 0 \). There are two cases to consider.

**Case 1:** \( L \) is simply connected.

Fix a basis \( u_1(x), \ldots, u_k(x) \) of \( V_2(x) \). For every \( y \in L \), let \( V'(y) := A(\omega, 1)V_2(x) \) and \( u_j(y) := A(\omega, 1)u_j(x) \), where \( 1 \leq j \leq k \) and \( \omega \) is any element of \( \Omega_x \) such that \( \omega(1) = y \). The simple connectivity of \( L \) and the homotopy law for \( A \) ensure that this definition is independent of the choice of \( \omega \). Note that

\[
\chi_2(x) = \sup \{ \chi(x, v) : (x, v) \in A_2 \} = \max_{1 \leq j \leq k} \chi(x, u_j(x)).
\]

By Lemma 5.1, there is a set \( A \in \mathcal{A}(\Omega_x) \) of full \( W_x \)-measure such that

\[
\chi_2(x) = \max_{1 \leq j \leq k} \sup_{\omega \in A} \chi(\omega, u_j(x)).
\]

By Proposition 5.4, for Vol-almost every \( y \in L \), \( T(A) \) is of full \( W_y \)-measure. Consequently, using this and (5.2) we infer that, for all such \( y \),

\[
\sup_{v \in V'(y)} \chi(y, v) \leq \max_{1 \leq j \leq k} \sup_{\omega \in T(A)} \chi(\omega, u_j(y)) = \max_{1 \leq j \leq k} \sup_{\omega \in A} \chi(\omega, u_j(x)) = \chi_2(x) < \chi_1.
\]
Hence, the above inequality implies that \( V'(y) \subset V_2(y) \) for Vol-almost every \( y \in L \). Since \( \mathcal{A} \) is with values in \( \text{GL}(d, \mathbb{R}) \), we have clearly that \( \dim V'(y) = \dim V_2(x) > 0 \). Thus, \( \dim V_2(y) \geq \dim V'(y) = \dim V_2(x) > 0 \) for all such \( y \). So all such \( y \) belong to \( \Pi_1(\Lambda_2) \). Summarizing what has been done so far, we have shown that \( \text{Vol}(L \setminus \Pi_1(\Lambda_2)) = 0 \) and that for each \( x \in \Pi_1(\Lambda_2) \), \( 0 < \dim V_2(x) \leq \dim V_2(y) < d \) for Vol-almost every \( y \in \Pi_1(\Lambda_2) \). So there is an integer \( d_2 < d \) and a set \( Y_x \subset L \) such that \( \text{Vol}(L \setminus Y_x) = 0 \) and that for every \( y \in Y_x \), \( \dim V_2(y) = d_2 \) and \( V_2(y) = V'(y) \). This, combined with the previous estimate \( \sup_{v \in V'(y)} \chi(y, v) \leq \chi_2(y) \), implies that \( \chi_2(y) \leq \chi_2(x) \) for \( y \in Y_x \). Using that Vol-almost every \( x \) is contained in \( \Pi_1(\Lambda_2) \), we may find a Borel set \( Y_2 \subset Y_1 \) and \( \chi_2 \in \mathbb{R} \cup \{\pm \infty\} \) such that \( \text{Vol}(L \setminus Y_2) = 0 \) and \( \chi_2(x) = \chi_2 \) for every \( x \in Y_2 \).

**Case 2:** \( L \) is not necessarily simply connected.

The holonomy problem arises. More concretely, given two points \( x \) and \( y \in L \) and two paths \( \omega_1, \omega_2 \in \Omega(L) \) such that \( \omega_1(0) = \omega_2(0) = x \) and \( \omega_1(t_1) = \omega_2(t_2) = y \), then \( \mathcal{A}(\omega_1, t_1) V_2(x) \) is not necessarily equal to \( \mathcal{A}(\omega_2, t_2) V_2(x) \).

Let \( \pi : \tilde{L} \to L \) be the universal cover. Fix \( x \in L \) and let \( \tilde{x} \in \tilde{L} \) be a lifting of \( x \). Recall from Lemma 10.35 (ii) below that \( \pi_{\tilde{x}}^{-1} : \Omega_{\pi} \to \tilde{\Omega}_x \) is a canonical identification of the two paths spaces which identifies the respective Wiener measures \( W_x \) and \( W_{\tilde{x}} \) on them. More precisely, for \( E \in \mathcal{A} \left( \Omega_{\pi} \right) \), we have that \( E := \pi(E) \in \mathcal{A} \Omega_x \) and \( W_{\tilde{x}}(E) = W_x(E) \).

We construct a cocycle \( \tilde{A} \) on \( \tilde{L} \) as follows:

\[
\tilde{A}(\tilde{\omega}, t) := A(\pi(\tilde{\omega}), t), \quad t \in \mathbb{R}^+, \ \tilde{\omega} \in \Omega(\tilde{L}).
\]

For \( \tilde{x} \in \tilde{L} \) we define \( V_2(\tilde{x}) \) relative to the cocycle \( \tilde{A} \) thanks to Proposition 5.2. Using the above canonical identification and the definition of \( \tilde{A} \), we see that

\[
\sup_{\tilde{\omega} \in \tilde{E}} \chi(\tilde{\omega}, v) = \sup_{\omega \in E} \chi(\omega, v)
\]

for every \( E \in \mathcal{A} \left( \Omega_{\pi} \right) \) and \( v \in \mathbb{R}^d \). By taking the infimum of the above equality over all \( E \) of full \( W_{\tilde{x}} \)-measure, we get that \( \chi(\tilde{x}, v) = \chi(x, v) \). Hence,

\[
V_2(\tilde{x}) = V_2(x) = V_2(\pi(\tilde{x})). \tag{5.3}
\]

Since the cocycle \( \tilde{A} \) is defined on the simply connected manifold \( \tilde{L} \), we may apply Case 1. Consequently, there is a set \( \tilde{Y}_2 \subset \tilde{L} \) such that \( \text{Vol}(\tilde{L} \setminus \tilde{Y}_2) = 0 \) and that the assertions (i)–(iii) hold for \( m = 2 \). Now let \( x \) and \( y \) be two points in \( Y_2 := \pi(\tilde{Y}_2) \) and let \( \omega_1, \omega_2 \in \Omega(L) \) be two paths such that \( \omega_1(0) = \omega_2(0) = x \) and \( \omega_1(t_1) = \omega_2(t_2) = y \). Since \( x \in Y_2 \), we fix a lift \( \tilde{x} \in \tilde{Y}_2 \) of \( x \). Let \( \tilde{\omega}_1 := \pi_{\tilde{x}}^{-1}(\omega_1) \), \( \tilde{\omega}_2 := \pi_{\tilde{x}}^{-1}(\omega_2) \), and \( \tilde{y}_1 := \tilde{\omega}_1(t_1) \), \( \tilde{y}_2 := \tilde{\omega}_2(t_2) \). We consider two subcases.

**Subcase 2a:** Both \( \tilde{y}_1 \) and \( \tilde{y}_2 \) belong to \( \tilde{Y}_2 \).

By assertion (iii) and (5.3) we get that

\[
\mathcal{A}(\omega_1, t_1) V_2(x) = \tilde{A}(\tilde{\omega}_1, t_1) V_2(\tilde{x}) = V_2(\tilde{y}_1) \text{ and } \mathcal{A}(\omega_2, t_2) V_2(x) = \tilde{A}(\tilde{\omega}_2, t_2) V_2(\tilde{x}) = V_2(\tilde{y}_2).
\]
Since \( \pi(\tilde{y}_1) = \pi(\tilde{y}_2) = y \), we obtain, by (5.3) again, that \( V_2(\tilde{y}_1) = V_2(\tilde{y}_2) = V_2(y) \). Hence, \( \mathcal{A}(\omega_1, t_1)V_2(x) = \mathcal{A}(\omega_2, t_2)V_2(x) \). So there is no holonomy problem in this subcase.

**Subcase 2b:** Either \( \tilde{y}_1 \) or \( \tilde{y}_2 \) is outside \( \tilde{Y}_2 \).

Assume without loss of generality that \( t_1 = t_2 = 1 \). Since \( \text{Vol}(\tilde{L} \setminus \tilde{Y}_2) = 0 \), it follows that \( \text{Vol}(L \setminus Y_2) = 0 \). Consequently, by re-parameterizing \( \omega_1|_{[0,1]} \) and \( \omega_2|_{[0,1]} \) and by replacing \( \omega_1|_{[0,1]} \) (resp. \( \omega_2|_{[0,1]} \)) by a path of the same homotopy class if necessary, we may choose \( z \in Y \) close to \( y \) such that

- \( \omega_1(1/2) = \omega_2(1/2) = z \) and \( \tilde{z}_1 := \tilde{\omega}_1(1/2) \in \tilde{Y}_2, \tilde{z}_2 := \tilde{\omega}_2(1/2) \in \tilde{Y}_2 \);
- \( \omega_1|_{[1/2,1]} \) is homotopic with \( \omega_2|_{[1/2,1]} \) in \( L_x \).

By the first \( \bullet \) we may apply Subcase 2a to \( \tilde{z}_1 \) and \( \tilde{z}_2 \) in place of \( \tilde{y}_1 \) and \( \tilde{y}_2 \). Hence, using (5.3) we obtain that

\[
\mathcal{A}(\omega_1, 1/2)V_2(x) = \mathcal{A}(\omega_2, 1/2)V_2(x) = V_2(\tilde{z}_1) = V_2(\tilde{z}_2) = V_2(z).
\]

On the other hand, the second \( \bullet \) implies that

\[
\mathcal{A}(T^{1/2}\omega_1, 1/2)V_2(z) = \mathcal{A}(T^{1/2}\omega_2, 1/2)V_2(z) = V_2(y).
\]

Combining the equalities in the last two lines and appealing to the multiplicative law of \( \mathcal{A} \), we get that \( \mathcal{A}(\omega_1, 1)V_2(x) = \mathcal{A}(\omega_2, 1)V_2(x) \). This completes Subcase 2b. Hence, the proposition is proved for \( m \leq 2 \).

Consider

\[
\Lambda_3 := \{(x, v) \in Y_1 \times \mathbb{R}^d : \chi(x, v) < \chi_2\} \subset \text{Leb}(Y_1) \times \text{Leb}(\mathbb{R}^d).
\]

Let \( V_3(y) \) be the proper vector subspace \( \Lambda_3 \cap \Pi_1^{-1}(y) \) of \( \mathbb{R}^d \) for \( y \in \Pi_1(\Lambda_3) \), and let \( V_3(y) := \{0\} \) otherwise. We argue as above and use that \( \dim V_3(y) < \dim V_2(y) < \dim V_1(y) \) when \( \text{Vol}(\Pi_1(\Lambda_3)) > 0 \). Consequently, the proposition is proved for \( m \leq 3 \). We continue this process. It will be finished after a finite \( m \) steps. This completes the proof. \( \square \)

## 6 Splitting subbundles

In this section we are given a Riemannian lamination \((X, \mathcal{L}, g)\) satisfying the Standing Hypotheses and a harmonic probability measure \( \mu \) directed by \((X, \mathcal{L})\). We also fix a number \( d \in \mathbb{N} \) and let \( G := \mathbb{N} \).

**Definition 6.1.** A *measurable bundle of rank* \( k \) is a Lebesgue measurable map \( V : Y \ni x \mapsto V_x \) of \( Y \) into the Grassmannian \( \text{Gr}_k(\mathbb{R}^d) \) of vector subspaces of dimension \( k \) for some \( k \leq d \), where \( Y \subset X \) is a subset of full \( \mu \)-measure. A measurable bundle \( U \) of rank \( l : Y \ni x \mapsto U_x \) is said to be a *measurable subbundle of* \( V \) if \( U_x \subset V_x, x \in X \). The trivial bundle on \( Y \) is defined by \( Y \ni x \mapsto \mathbb{R}^d \), and is denoted by \( Y \times \mathbb{R}^d \).
For a subset \( Y \subset X \) of full \( \mu \)-measure, let
\[
\Omega(Y) := \{ \omega \in \Omega(X, \mathcal{L}) : \pi_n \omega \in Y, \ \forall n \in \mathbb{N} \},
\]
where \( \pi_n : \Omega(X, \mathcal{L}) \to X \) is, as usual, the projection given by \( \pi_n \omega := \omega(n) \), \( \omega \in \Omega(X, \mathcal{L}) \). Given a cocycle \( A : \Omega(X, \mathcal{L}) \times \mathbb{N} \to \text{GL}(d, \mathbb{R}) \) and a subset \( Y \subset X \) of full \( \mu \)-measure, a measurable sub-bundle \( Y \ni x \mapsto V_x \) of \( Y \times \mathbb{R}^d \) is said to be \( A \)-invariant if
\[
A(\omega, n)V_{\omega(0)} = V_{\omega(n)}, \quad \omega \in \Omega(Y).
\]

Using formula (2.4) we see easily that for a subset \( \Omega(\bar{\omega})(X) \ni \omega \neq \omega' \), the map \( \omega \mapsto V_{\omega(0)} \) is an \( A \)-invariant sub-bundle of rank \( d_1 \) of \( Y \times \mathbb{R}^d \) for \( 1 \leq i \leq m \).

For a subset \( Y \subset X \) of full \( \mu \)-measure, \( \Omega(Y) \) is a subset of \( \Omega(X, \mathcal{L}) \) of full \( \bar{\mu} \)-measure.

We may rephrase Proposition 5.5 as follows.

**Corollary 6.2.** Suppose that \( \mu \) is ergodic. Let \( A \) be a cocycle on \( (X, \mathcal{L}) \). Then there exist a Borel set \( Y \subset X \) of full \( \mu \)-measure and a number \( m \in \mathbb{N} \) and \( m \) integers \( 1 \leq d_m < d_{m-1} < \cdots < d_1 = d \) and \( m \) real numbers \( \chi_m < \chi_{m-1} < \cdots < \chi_1 \) with the following properties:

(i) \( m(x) = m \) for every \( x \in Y \);

(ii) the map \( Y \ni x \mapsto V_i(x) \) is an \( A \)-invariant sub-bundle of rank \( d_i \) of \( Y \times \mathbb{R}^d \) for \( 1 \leq i \leq m \);

(iii) for every \( x \in Y \) and \( 1 \leq i \leq m \), \( \chi_i(x) = \chi_i \).

**Proof.** By Proposition 5.5, for each leaf \( L \) of \( (X, \mathcal{L}) \) we can find a subset \( Y_L \subset L \) and an integer \( m_L \) such that \( \text{Vol}(L \setminus Y_L) = 0 \) and that all properties (i)–(iii) hold for \( m_L \) maps \( L \ni x \mapsto V_i(x) \) with \( 1 \leq i \leq m_L \). Let \( Y := \bigcup Y_L \), the union being taken over all leaves of \( (X, \mathcal{L}) \). So \( Y \) is of full \( \mu \)-measure. Consider the leafwise constant function \( \tilde{m} : X \to \mathbb{N} \) given by \( \tilde{m} := m_L \) on any leaf \( L \). So \( \tilde{m}(x) = m(x) \) for \( \mu \)-almost every \( x \in X \). By the ergodicity of \( \mu \), \( \tilde{m} \) is equal to a constant \( m \) \( \mu \)-almost everywhere. By removing from \( Y \) a subset of null \( \mu \)-measure if necessary, we may assume that \( m(x) = m \) for all \( x \in Y \). This proves assertion (i).

Using the same argument for \( m \) maps \( Y \ni x \mapsto V_i(x) \) with \( 1 \leq i \leq m \), the corollary follows.

The purpose of this section is to split an \( A \)-invariant bundle into a direct sum of \( A \)-invariant components. This splitting will enable us to apply the ergodic Birkhoff theorem in the next sections.

**Lemma 6.3.** Let \( h : \Omega(X, \mathcal{L}) \to [0, \infty) \) be a measurable function such that \((h - h \circ T)^+ \in L^1(\bar{\mu})\). Then \( \frac{1}{n} h(T^n \omega) \to 0 \) for \( \bar{\mu} \)-almost every \( \omega \in \Omega(X, \mathcal{L}) \).

**Proof.** Observe that \( h(T^n \omega) = h(\omega) - \sum_{i=0}^{n-1} (h - h \circ T)(T^i \omega) \). Since \((h - h \circ T)^+ \in L^1(\bar{\mu})\), the classical Birkhoff ergodic theorem gives that \( \lim_{n \to \infty} \frac{1}{n} h(T^n \omega) \) exists for \( \bar{\mu} \)-almost every \( \omega \in \Omega(X, \mathcal{L}) \), but could take the value \( \infty \). We need to prove
that this limit is equal to 0 almost everywhere. To do this let \( A_k := \{ \omega \in \Omega(X, \mathcal{X}) : |h(\omega)| \leq k \} \) for \( k \in \mathbb{N} \). Then \( \bigcup_{k=1}^{\infty} A_k = \Omega(X, \mathcal{X}) \). If \( \bar{\mu}(A_k) > 0 \) then by the recurrence theorem, for \( \bar{\mu} \)-almost every \( \omega \in A_k \), there exist \( n_1(\omega) < n_2(\omega) < \cdots \) with \( T^{n_i}(\omega) \in A_k \), \( i \geq 1 \). Hence, \( |h(T^{n_i}(\omega))| \leq k \) and so \( \liminf_{n \to \infty} \frac{1}{n} |h(T^n \omega)| = 0 \). This holds for \( \bar{\mu} \)-almost every \( \omega \in \bigcup_{k=1}^{\infty} A_k \). \( \square \)

In what follows \( Y \) denotes the set of full \( \mu \)-measure given by Corollary \ref{corollary:measure}. For \( x \in Y \) let \( \Omega_x(Y) \) denotes the space of all paths in \( \Omega(Y) \) originated at \( x \), that is, \( \Omega_x(Y) := \Omega(Y) \cap \Omega_x \). For a matrix \( A \in \text{GL}(d, \mathbb{R}) \) and a vector subspace \( U \subset \mathbb{R}^d \), let \( \|A|_U\| \) be the Euclidean norm of the linear homomorphism \( A|_U : U \to \mathbb{R}^d \).

**Lemma 6.4.** Let \( A \) be a cocycle such that \( \int_{\Omega(X, \mathcal{X})} \log^+ \|A(\omega, 1)|d\bar{\mu}(\omega) < \infty \). Suppose that \( \mu \) is ergodic and that \( Y \ni x \mapsto U(x) \) is a measurable \( A \)-invariant sub-bundle of \( Y \times \mathbb{R}^d \).

(i) Then \( \lim_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)|_U(\pi_{\omega})\| \) exists and is constant for \( \bar{\mu} \)-almost every \( \omega \in \Omega(Y) \), but the limit could be \(-\infty\);

(ii) Suppose that the value of the above limit is less than or equal to \( \alpha \in \mathbb{R} \). For \( \epsilon > 0 \) define

\[
a_\epsilon(\omega) := \sup_{n \in \mathbb{N}} \left( \|A(\omega, n)|_U(\pi_{\omega})\| \cdot e^{-n(\alpha + \epsilon)} \right).
\]

Then \( \lim_{n \to \infty} \frac{1}{n} \log a_\epsilon(T^n \omega) = 0 \) for \( \bar{\mu} \)-almost every \( \omega \in \Omega(Y) \).

**Proof.** For \( n \in \mathbb{N} \) let \( f_n : \Omega(Y) \to \mathbb{R} \) defined by

\[
f_n(\omega) := \log \|A(\omega, n)|_U(\pi_{\omega})\|, \quad \omega \in \Omega(Y).
\]

By the hypothesis, \( \int_{\Omega(Y)} f_1^+(\omega) d\bar{\mu}(\omega) < \infty \). Since \( A \) is a cocycle and the sub-bundle \( x \mapsto U(x) \) is \( A \)-invariant, we see that

\[
f_{n+m}(\omega) \leq f_n(\omega) + f_m(T^m \omega), \quad \omega \in \Omega(Y).
\]

Applying the subadditive ergodic theorem \cite{25} to the sequence \( (f_n) \), assertion (i) follows.

We turn to assertion (ii). By the choice of \( \alpha \) we have that \( 0 \leq a_\epsilon(\omega) < \infty \). Also

\[
\frac{a_\epsilon(\omega)}{a_\epsilon(T\omega)} \leq \max \left( \|A(\omega, 1)|_U(\pi_{\omega})\| \cdot e^{-(\alpha + \epsilon)}, 1 \right)
\]

so that

\[
\log a_\epsilon(\omega) - \log a_\epsilon(T\omega) \leq \max \left( \log^+ \|A(\omega, 1)|_U(\pi_{\omega})\| - (\alpha + \epsilon), 0 \right).
\]

Recall from the hypothesis that \( \int_{\Omega(X, \mathcal{X})} \log^+ \|A(\omega, 1)|d\bar{\mu}(\omega) < \infty \). Hence, \( \omega \mapsto \left( \log a_\epsilon(\omega) - \log a_\epsilon(T\omega) \right)^+ \) is \( \bar{\mu} \)-integrable and we can apply Lemma \ref{lemma:almost_equal} \( \square \)
For two vector subspaces $A, B$ of $\mathbb{R}^d$, let $\text{Hom}(A, B)$ denote the vector space of all linear homomorphisms from $A$ to $B$. Now we are in the position to state the main result of this section.

**Theorem 6.5.** Let $\mu$ be an ergodic harmonic probability measure, and $\mathcal{A} : \Omega(X, \mathcal{F}) \times N \rightarrow \text{GL}(d, \mathbb{R})$ a cocycle, and $Y \subset X$ a set of full $\mu$-measure. Assume that $\int_{\Omega(X, \mathcal{F})} \log^+ \|\mathcal{A}(\omega, 1)\|d\mu(\omega) < \infty$. Assume also that $Y \ni x \mapsto U(x)$ and $Y \ni x \mapsto V(x)$ are two measurable $\mathcal{A}$-invariant sub-bundles of $Y \times \mathbb{R}^d$ with $V(x) \subset U(x)$, $x \in Y$. Define a new measurable sub-bundle $Y \ni x \mapsto W(x)$ of $Y \times \mathbb{R}^d$ by splitting $U(x) = V(x) \oplus W(x)$ so that $W(x)$ is orthogonal to $V(x)$ with respect to the Euclidean inner product of $\mathbb{R}^d$. Let $\alpha, \beta$ be two real numbers with $\alpha < \beta$ such that

- $\chi(x, v) \leq \alpha$ for every $x \in Y$, $v \in V(x) \setminus \{0\}$;
- $\chi(\omega, w) \geq \beta$ for every $x \in Y$, every $w \in W(x) \setminus \{0\}$ and for every $\omega \in \mathcal{G}_{x,w}$.

Here $\mathcal{G}_{x,w}$ is a subset of $\Omega(Y)$ depending on $x$ and $w$ with $W_x(\mathcal{G}_{x,w}) > 0$, and the functions $\chi(x, v)$ and $\chi(\omega, w)$ have been defined in (5.7).

Let $\mathcal{A}(\omega, 1)|_{U(\pi_0)} : U(\pi_0) \rightarrow U(\pi_1)$ induce the linear maps $C(\omega) : W(\pi_0) \rightarrow W(\pi_1)$ and $B(\omega) : W(\pi_0) \rightarrow V(\pi_1)$ by

$$A(\omega, 1)w = B(\omega)w \oplus C(\omega)w, \quad \omega \in \Omega(Y), \ w \in W(x).$$

(i) Then the map $C$ defined on $\Omega(Y) \times \mathbb{N}$ by the formula

$$C(\omega, n) := C(T^n\omega) \in \text{Hom}(W(\pi_1\omega), W(\pi_n\omega)), \quad \omega \in \Omega(Y), \ n \in \mathbb{N},$$

satisfies $C(\omega, m + k) = C(T^k\omega, m)C(\omega, k)$, $m, k \in \mathbb{N}$. Moreover, $C(\omega, n)$ is invertible.

There exists a subset $Y'$ of $Y$ of full $\mu$-measure with the following properties:

(ii) for each $x \in Y'$ and for each $w \in W(x) \setminus \{0\}$, there exists a set $\mathcal{F}_{x,w} \subset \mathcal{G}_{x,w}$ such that $W_x(\mathcal{F}_{x,w}) = W_x(\mathcal{G}_{x,w})$ and that for each $v \in V(x)$ and each $\omega \in \mathcal{F}_{x,w}$, we have

$$\chi(\omega, v \oplus w) = \chi(\omega, w) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|C(\omega, n)w\|;$$

(iii) if for some $x \in Y'$ and some $w \in W(x) \setminus \{0\}$ and some $v \in V(x)$ and some $\omega \in \mathcal{F}_{x,w}$ the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|C(\omega, n)w\|$ exists, then $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|A(\omega, n)(v \oplus w)\|$ exists and is equal to the previous limit.

**Proof.** We use $v$ to denote a general element of some $V(x)$ and $w$ a general element of some $W(x)$. Using the multiplicative property of the cocycle $\mathcal{A}$, we obtain the following formula, for $\omega \in \Omega_x(Y)$,

$$A(\omega, n)(v \oplus w) = (A(\omega, n)v + D(\omega, n)w) \oplus C(\omega, n)w,$$

where $D(\omega, n) : W(\pi_0\omega) \rightarrow V(\pi_n\omega)$ is given by

$$D(\omega, n) := \sum_{i=0}^{n-1} A(T^{i+1}\omega, n - i - 1) \circ B(T^i\omega) \circ C(\omega, i).$$
To prove assertion (i) pick an arbitrary \( w \in W(x) \). Using (6.1) and the assumption that both maps \( x \mapsto U(x) \), \( x \mapsto V(x) \) are \( A \)-invariant sub-bundles of \( Y \times \mathbb{R}^d \), we see that \( C(\omega, n)w \) is the image of \( A(\omega, n)(w) \) by the projection of \( U_{\pi_n} = V_{\pi_n} \oplus W_{\pi_n} \) onto the second summand. Hence,

\[
A(\omega, m + k)(V_{\pi_\omega} \oplus w) = V_{\pi_{m+k}} \oplus C(\omega, m + k)w.
\]

Moreover, using the \( A \)-invariant assumption again we have that

\[
A(\omega, m + k)(V_{\pi_\omega} \oplus w) = A(T^k \omega, m)A(\omega, k)(V_{\pi_0} \oplus w) = A(T^k \omega, m)(V_{\pi_\omega} \oplus C(\omega, k)w)
= V_{\pi_{m+k}} \oplus C(T^k \omega, m)C(\omega, k)w.
\]

This, combined with the previous equality, implies assertion (i).

Now we prove assertions (ii) and (iii). Lemma 5.3, applied to identity (6.1), yields that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)(v + w)\| = \max \left( \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v + D(\omega, n)w\|, \limsup_{n \to \infty} \frac{1}{n} \log \|C(\omega, n)w\| \right). \tag{6.2}
\]

Letting \( v = 0 = w \neq 0 \) in (6.2), we deduce that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)w\| = \max \left( \limsup_{n \to \infty} \frac{1}{n} \log \|D(\omega, n)w\|, \limsup_{n \to \infty} \frac{1}{n} \log \|C(\omega, n)w\| \right). \tag{6.3}
\]

For \( \epsilon > 0 \) let \( a_\epsilon(\omega) := \sup_{n \in \mathbb{N}} \|A(\omega, n)|_{V_{\pi_\omega}}\| e^{-n(a + \epsilon)} \). By the first assumption \( \bullet \), we may apply Lemma 6.4 to \( a_\epsilon(\omega) \), and Lemma 6.3 to \( h(\omega) := \|A(\omega, 1)\| \), \( m \in \mathbb{N} \) be a sequence decreasing strictly to 0. By Lemma 6.3 and Lemma 6.3 we may find, for each \( m \geq 1 \), a subset \( \Omega_m \) of \( \Omega(Y) \) of full \( \mu \)-measure such that \( \frac{1}{n} a_m(T^n \omega) \to 0 \) and \( \frac{1}{n} \log \|A(T^n \omega, 1)\| \to 0 \) for all \( \omega \in \Omega_m \). For every \( x \in Y \) set \( \mathcal{F}_x := \Omega_x \cap \bigcap_{m=1}^\infty \Omega_m \subset \Omega_x(Y) \). Since \( \cap_{m=1}^\infty \Omega_m \) is full \( \mu \)-measure, there exists a subset \( Y' \subset Y \) of full \( \mu \)-measure such that for every \( x \in Y' \), \( \mathcal{F}_x \) is of full \( W_x \)-measure. By the first assumption \( \bullet \) combined with Proposition 5.2 (ii)-(iii), for every \( x \in Y' \), there exists a set \( \mathcal{F}_{x, w} \subset \mathcal{F}_x \) of full \( W_x \)-measure such that, for every \( \omega \in \mathcal{F}_x \),

\[
\chi(\omega, v) \leq \alpha < \beta, \quad v \in V(x). \tag{6.4}
\]

By the second assumption \( \bullet \), for every \( x \in Y' \) and for every \( w \in W(x) \setminus \{0\} \), there exists a set \( \mathcal{F}_{x, w} := \mathcal{G}_{x, w} \cap \mathcal{F}_x \subset \Omega_x(Y) \) such that, for every \( \omega \in \mathcal{F}_{x, w} \),

\[
\alpha < \beta \leq \chi(\omega, w). \tag{6.5}
\]
Since $W_x(\mathcal{F}_x) = 1$, we see that $W_x(\mathcal{F}_{x,w}) = W_x(\mathcal{G}_{x,w})$. We will prove that for every $x \in Y'$ and for every $w \in W(x) \setminus \{0\}$, and for every $\omega \in \mathcal{F}_x$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \|C(\omega, n)w\| = \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)w\|.$$  \hspace{1cm} (6.6)

Let $\tau$ be the left side limit. By (6.3) and (6.5), $\tau$ is smaller than the right hand side. By (6.4), $\alpha$ is strictly smaller than the right hand side. So $\max(\tau, \alpha)$ is smaller than the right hand side of (6.6). Hence, by (6.3) again, $\limsup_{n \to \infty} \frac{1}{n} \log \|D(\omega, n)w\| \geq \max(\tau, \alpha)$. We will prove that

$$\limsup_{n \to \infty} \frac{1}{n} \log \|D(\omega, n)w\| \leq \max(\tau, \alpha).$$  \hspace{1cm} (6.7)

Taking (6.7) for granted, the above reasoning shows that the inequality (6.7) is, in fact, an equality. Hence, it will follow from (6.3) that $\limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)w\| = \max(\tau, \alpha)$. Recall again from (6.4) that $\limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)w\| > \alpha$. Hence,

$$\tau > \alpha$$  \hspace{1cm} (6.8)

and (6.6) follows. So the proof of (6.6) is reduced to the proof of (6.7).

In order to show (6.7) fix an arbitrary $m \geq 1$. Then there exists $N$ depending on $\omega, w$ and $m$ such that $n \geq N$ implies that $\|\mathcal{C}(\omega, n)w\| < e^{n(\tau + \epsilon_m)}$. If we write $\mathcal{L}(\omega, n) : V(\pi_0, \omega) \to V(\pi_n, \omega)$ instead of $A(\omega, n)|_{V(\pi_0, \omega)}$, then

$$\|D(\omega, n)w\| \leq \sum_{i=0}^{n-1} \|\mathcal{L}(T^{i+1}\omega, n - i - 1)\| \cdot \|B(T^i\omega)\| \cdot \|C(\omega, i)w\|$$

$$\leq n \max_{0 \leq i \leq n-1} \|\mathcal{L}(T^{i+1}\omega, n - i - 1)\| \cdot \|B(T^i\omega)\| \cdot \|C(\omega, i)w\|$$

$$= n\|\mathcal{L}(T^{i_n+1}\omega, n - i_n - 1)\| \cdot \|B(T^{i_n}\omega)\| \cdot \|C(\omega, i_n)w\|$$

for some $0 \leq i_n \leq n - 1$, which depends also on $\omega$ and $w$. Note that $(i_n)$ is an increasing sequence.

**Case 1:** $(i_n)$ is unbounded.

So $i_n \geq N$ for $n$ large enough. Consequently, we have that

$$\frac{1}{n} \log^+ \|B(T^{i_n}\omega)\| \leq \frac{i_n}{n} \frac{1}{i_n} \log^+ \|A(T^{i_n}\omega, 1)\| \leq \frac{1}{i_n} \log^+ \|A(T^{i_n}\omega, 1)\| \to 0$$

by the membership $\omega \in \mathcal{F}_x$ and by Lemma 6.3, and

$$\frac{1}{n} \log a_{\epsilon_m}(T^{i_n+1}\omega) = \frac{i_n + 1}{n} \frac{1}{i_n + 1} \log a_{\epsilon_m}(T^{i_n+1}\omega) \to 0$$

by Lemma 6.4. These inequalities, combined with the above estimate for $\|D(\omega, n)w\|$, imply that

$$\frac{1}{n} \log \|D(\omega, n)w\| \leq \frac{1}{n} \log n + \frac{1}{n} \log a_{\epsilon_m}(T^{i_n+1}\omega) + \frac{n - 1 - i_n}{n} (\alpha + \epsilon_m)$$

$$+ \frac{1}{n} \log^+ \|B(T^{i_n}\omega, n)\| + \frac{i_n}{n} (\tau + \epsilon_m).$$

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So \( \limsup_{n \to \infty} \frac{1}{n} \log \|D(\omega, n)w\| \leq \max(\tau, \alpha) + \epsilon_m \). By letting \( m \to \infty \), we get (6.7).

**Case 2:** \((i_n)\) is bounded, say \( i_n \leq M \) for all \( n \).

We see easily that
\[
\frac{1}{n} \log \|D(\omega, n)w\| \leq \frac{1}{n} \log n + \max_{0 \leq i \leq M} \frac{1}{n} \log \|L(T^{i+1}\omega, n - i - 1)\|
+ \max_{0 \leq i \leq M} \frac{1}{n} \log \|B(T^i\omega)\| + \max_{0 \leq i \leq M} \frac{1}{n} \log \|C(\omega, i)\|.
\]

Since on the right hand side, the \( \limsup \) of the second term is smaller than \( \alpha \) by (6.4), whereas other terms tend to 0 as \( n \to \infty \), it follows that
\[
\frac{1}{n} \log \|D(\omega, n)w\| \leq \alpha,
\]
proving (6.7). Hence, the proof of (6.6) is complete.

Lemma 5.3, applied to the first term in the right hand side of (6.2), yields that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v + D(\omega, n)w\| \leq \max \left( \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v\|, \limsup_{n \to \infty} \frac{1}{n} \log \|D(\omega, n)w\| \right).
\]

Observe in the last line that the first term in the right hand side is smaller than \( \alpha \) by (6.4), whereas the second term \( \leq \max(\tau, \alpha) \) by (6.7). This, combined with (6.8), implies that the left hand side of the last line is \( \leq \tau \). This, coupled with (6.2) and (6.6), gives that for every \( x \in Y' \) and \( \omega \in \mathcal{F}_x \),
\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)(v \oplus w)\| = \limsup_{n \to \infty} \frac{1}{n} \log \|C(\omega, n)w\|
= \tau = \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)w\|, \quad \forall v \in V(\pi_0\omega), \ w \in W(\pi_0\omega).
\]

This proves assertion (ii).

Now suppose that \( \limsup_{n \to \infty} \frac{1}{n} \log \|C(\omega, n)w\| \) exists for some \( x \in Y' \), some \( \omega \in \mathcal{F}_x \) and some \( w \in W(\pi_0\omega) \backslash \{0\} \). By the inequality in Lemma 5.3 and (6.1), we have, for every \( v \in V(x) \), that
\[
\liminf_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)(v \oplus w)\|
\geq \max(\liminf_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v\|, \liminf_{n \to \infty} \frac{1}{n} \log \|D(\omega, n)w\|, \liminf_{n \to \infty} \frac{1}{n} \log \|C(\omega, n)w\|)
\geq \liminf_{n \to \infty} \frac{1}{n} \log \|C(\omega, n)w\|.
\]

This, combined with (6.9), implies assertion (iii).
7 Lyapunov forward filtrations

The first part of this section makes the reader familiar with some new terminology and auxiliary results which are constantly present in this work. The second part is devoted to two Oseledec type theorems. The first one is a direct consequence of Oseledec Multiplicative Ergodic Theorem 1.2. The second theorem is the main result of this section. The proof of the latter theorem occupies the last parts of the section where new techniques such as totally invariant sample-path sets and stratifications are introduced. We will see in this proof that the holonomy of the leaves comes into action. Before proceeding further we need the following terminology. Let \( T \) be a measurable transformation defined on a measurable space \( \Omega \). A measurable set \( F \subset \Omega \) is said to be \( T \)-invariant (resp. \( T \)-totally invariant) if \( T^{-1}F = F \) (resp. \( TF = T^{-1}F = F \)). When \( T \) is surjective, a set \( F \) is \( T \)-invariant if and only if it is \( T \)-totally invariant.

7.1 Oseledec type theorems

Consider a lamination \((X, \mathcal{L})\) satisfying the Standing Hypotheses endowed with a harmonic probability measure \( \mu \) which is ergodic. Consider also a (multiplicative) cocycle \( A : \Omega \times \mathbb{N} \to \text{GL}(d, \mathbb{R}) \), where \( \Omega := \Omega(X, \mathcal{L}) \). Assume that \( \int_{\Omega} \log^+ \|A^{i+1}(\omega, 1)\|d\mu(\omega) < \infty \). By Theorem 4.6, \( \bar{\mu} \) is ergodic with respect to \( T \) acting on the measure space \((\Omega, A, \bar{\mu})\). Consequently, we deduce from Theorem 1.2 (i)-(iv) the following result.

Theorem 7.1. There exists a subset \( \Phi \) of \( \Omega \) of full \( \bar{\mu} \)-measure and a number \( l \in \mathbb{N} \) and \( l \) integers \( 1 \leq r_1 < r_{l-1} < \cdots < r_1 = d \) and \( l \) real numbers \( \lambda_1 < \lambda_{l-1} < \cdots < \lambda_1 \) such that the following properties hold:

(i) For each \( \omega \in \Phi \) there are linear subspaces

\[
\{0\} \equiv V_{i+1}(\omega) \subset V_i(\omega) \subset \cdots \subset V_2(\omega) \subset V_1(\omega) = \mathbb{R}^d,
\]

of \( \mathbb{R}^d \) such that \( A(\omega, 1)V_i(\omega) = V_i(T\omega) \) and that \( \dim V_i(\omega) = r_i \) for all \( \omega \in \Phi \). Moreover, \( \omega \mapsto V_i(\omega) \) is a measurable map from \( \Phi \) into the Grassmannian of \( \mathbb{R}^d \).

(ii) For each \( \omega \in \Phi \) and \( v \in V_i(\omega) \setminus V_{i+1}(\omega) \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \frac{\|A(\omega, n)v\|}{\|v\|} = \lambda_i,
\]

for every \( \omega \in \Phi \).

(iii) \( \lambda_1 = \lim_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)\| \) for every \( \omega \in \Phi \).

On the other hand, by Corollary 6.2 there is a Borel set \( Y \subset X \) of full \( \mu \)-measure and there are integers \( m \geq 1 \), \( 1 \leq d_m < \cdots < d_1 = d \) and real numbers \( \chi_m < \cdots < \chi_1 \) such that \( m(x) = m \), \( \dim V_i(x) = d_i \) and \( \chi_i(x) = \chi_i \) for every \( x \in Y \). Moreover, \( Y \ni x \mapsto \dim V_i(x) \) is an \( A \)-invariant sub-bundle of \( Y \times \mathbb{R}^d \).
The purpose of this section is to unify the above two results. More concretely, we prove the following

**Theorem 7.2.** Under the above hypotheses and notation we have that \( m \leq l \) and \( \{\chi_1, \ldots, \chi_m\} \subset \{\lambda_1, \ldots, \lambda_l\} \) and \( \chi_1 = \lambda_1 \). Moreover, there exists a Borel set \( Y_0 \subset Y \) of full \( \mu \)-measure such that for every \( x \in Y \) and for every \( u \in V_i(x) \setminus V_{i+1}(x) \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)u\| = \chi_i
\]

for \( W_x \)-almost every \( \omega \in \Omega_x \). In particular,

\[
\chi(\omega, u) = \chi(x, u) = \chi_i
\]

for \( W_x \)-almost every \( \omega \in \Omega_x \). Here the functions \( \chi(\omega, u) \) and \( \chi(x, u) \) are defined in \((5.1)\).

This result may be considered as the first half of Theorem 3.7.

**Proof of Theorem 7.2.** The proof is divided into several steps.

**Step 1:** Construction of a Borel set \( Y_0 \subset Y \) of full \( \mu \)-measure. Proof that \( \{\chi_1, \ldots, \chi_m\} \subset \{\lambda_1, \ldots, \lambda_l\} \) and \( \chi_1 = \lambda_1 \).

By Corollary 6.2 and Theorem 7.1, there is a Borel set \( Y_0 \subset Y \) of full \( \mu \)-measure such that for every \( x \in Y_0 \), the set \( \Phi \cap \Omega_x \) is of full \( W_x \)-measure. By the definition and by Lemma 5.1, we obtain, for \( x \in Y_0 \) and \( u \in \mathbb{R}^d \setminus \{0\} \), a set \( E \subset \Omega_x \cap \Phi \) of full \( W_x \)-measure such that

\[
\chi(x, u) = \operatorname{ess. \ sup}_{\omega \in \Omega_x} \chi(\omega, u) = \sup_{\omega \in E} \chi(\omega, u) = \{\lambda_1, \ldots, \lambda_l\}. 
\]

This implies that \( \{\chi_1, \ldots, \chi_m\} \subset \{\lambda_1, \ldots, \lambda_l\} \). In particular, we get that \( m \leq l \) and \( \chi_1 \leq \lambda_1 \). Therefore, in order to prove that \( \chi_1 = \lambda_1 \), it suffices to show that \( \chi_1 \geq \lambda_1 \). By Proposition 5.2, for every \( x \in Y_0 \) there exists a set \( E_x \in \mathcal{A}(\Omega_x) \) of full \( W_x \)-measure such that

\[
v \in V_i(x) \setminus V_{i+1}(x) \iff \sup_{\omega \in E_x} \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v\| = \chi_i.
\]

Therefore, by the definition we get that for every \( v \in \mathbb{R}^d \setminus \{0\} \), for every \( x \in Y_0 \) and every \( \omega \in E_x \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v\| \leq \max\{\chi_m, \ldots, \chi_1\} = \chi_1.
\]

Hence, \( \lambda_1 \leq \chi_1 \), as desired.

**Step 2:** By shrinking \( Y_0 \) a little, for every \( x \in Y_0 \) and every \( u \in V_m(x) \setminus \{0\} \), the equalities \( \chi(\omega, u) = \chi(x, u) = \chi_m \) hold for \( W_x \)-almost every \( \omega \in \Omega_x \).

The proof of Step 2 will be given in Subsection 7.3 below.

**Step 3:** End of the proof.
Consider the orthogonal decomposition $V_{m-1}(x) = V_m(x) \oplus W(x)$ with respect to the Euclidean inner product in $\mathbb{R}^d$. We will apply Theorem 6.5 to the cocycle $\mathcal{A}$ in the following setting:

$$V(x) := V_m(x) \quad \text{and} \quad U(x) := V_{m-1}(x), \quad x \in Y.$$ 

In order to ensure the two • conditions in Theorem 6.5 we choose $\alpha := \chi_m$ and $\beta > \alpha$ so that

$$\beta < \min \{ \lambda \in \{ \lambda_1, \ldots, \lambda_l \} : \lambda > \alpha \}.$$ 

This choice, coupled with Theorem 7.1 and Corollary 6.2, guarantees that the hypotheses of Theorem 6.5 are fulfilled. Consequently, we deduce from assertion (ii) of the latter theorem that

$$\text{ess. sup}_{\omega \in \Omega_x(Y)} \limsup_{n \to \infty} \frac{1}{n} \log \| C(\omega, n)w \| = c_m$$

for $\mu$-almost every $x \in X$ and for all $w \in W(x) \setminus \{0\}$.

By Theorem 10.5 there is a bimeasurable bijection between the $\mathcal{A}$-invariant sub-bundle $Y \ni x \mapsto W(x)$ of rank $d_{m-1} - d_m$ and $Y \times \mathbb{R}^{d_{m-1} - d_m}$ covering the identity and which is linear on fibers. Using this bijection, it follows from Theorem 6.5 (i) that $C$ is multiplicative cocycle induced by $\mathcal{A}$. Since $Y \ni x \mapsto W(x)$ is a measurable $\mathcal{A}$-invariant sub-bundle and $\| C(\omega, 1) \| \leq \| \mathcal{A}(\omega, 1) \|$ and $\| C(\omega, 1)^{-1} \| \leq \| \mathcal{A}(\omega, 1)^{-1} \|$, we infer from the $\bar{\mu}$-integrability of $\omega \mapsto \mathcal{A}(\omega, 1)$ that $\omega \mapsto C(\omega, 1)$ is also $\bar{\mu}$-integrable. Consequently, this, together with (7.1) allows us to apply to the cocycle $C$ the same arguments used in Step 2. Hence, we can show that, for $\mu$-almost every $x \in X$ and for every $w \in W(x) \setminus \{0\}$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\| C(\omega, n)w \|}{\| w \|} = c_m$$

for $W_x$-almost every $\omega \in \Omega_x$. Then, by assertion (iii) of Theorem 6.5 we get that, for $\mu$-almost every $x \in X$ and for every $v \in V_{m-1}(x) \setminus V_m(x)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\| \mathcal{A}(\omega, n)v \|}{\| v \|} = c_m$$

for $W_x$-almost every $\omega \in \Omega_x(Y)$. So, for $\mu$-almost every $x \in X$ and for every $v \in V_{m-1}(x) \setminus V_m(x)$,

$$\chi(\omega, v) = \chi(x, v) = c_m$$

for $W_x$-almost every $\omega \in \Omega_x(Y)$.

The next case where $v \in V_{m-2}(x) \setminus V_{m-1}(x)$ can be proved in the same way. Repeating the above process a finite number of times, the proof of the theorem is thereby completed. □
7.2 Fibered laminations and totally invariant sets

In this subsection we introduce a special weak form of laminations: the fibered laminations. Next we study totally invariant measurable sample-path sets of these new objects. This, together with the study of stratifications given in the next subsection, will be the main ingredients in the proof of Theorem 7.2. This subsection is inspired by the proof of Theorem 4.6 given in Appendix below. Now let \((X, \mathcal{L}, g)\) be a Riemannian lamination endowed with a harmonic probability measure \(\mu\). Let \(\pi : \left(\widetilde{X}, \widetilde{\mathcal{L}}\right) \to (X, \mathcal{L})\) be the covering lamination projection.

**Definition 7.3.** A weakly fibered lamination \(\Sigma\) over \((X, \mathcal{L}, g, \mu)\) is the data of a Hausdorff topological space \(\Sigma\) and a measurable projection \(\iota : \Sigma \to \widetilde{X}\) such that for every \(y \in \Sigma\), there exists a set \(\Sigma_y \subset \Sigma\) satisfying the following properties (i)–(iii):

(i) \(y \in \Sigma_y\), and if \(y_1, y_2 \in \Sigma_y\) then \(\Sigma_{y_1} = \Sigma_{y_2}\);

(ii) the restriction \(\iota|_{\Sigma_y} : \Sigma_y \to L_{\iota(y)}\) is homeomorphic, where \(L_x\) is as usual the leaf of the lamination \((\widetilde{X}, \widetilde{\mathcal{L}})\) passing through \(\tilde{x}\);

(iii) the cardinal of every fiber \(\iota^{-1}(\tilde{x})\) \((\tilde{x} \in \widetilde{X})\) is at most countable.

Each set \(\Sigma_y\) \((y \in \Sigma)\) is called a leaf of \(\Sigma\). Since \((\widetilde{X}, \widetilde{\mathcal{L}}, \pi^*g)\) is a Riemannian lamination, we equip each leaf \(\Sigma_y\) with the differentiable structure by pulling back via \(\iota|_{\Sigma_y}\) the differentiable structure on \(L_{\iota(y)}\). Similarly, we endow each leaf \(\Sigma_y\) with the metric tensor \((\iota|_{\Sigma_y})^*(\pi^*g|_{L_{\iota(y)}})\).

Let \(\iota : \Sigma \to \widetilde{X}\) be a weakly fibered lamination over \((X, \mathcal{L}, g, \mu)\). Observe that every leaf \(\Sigma_y\) is simply connected as it is diffeomorphic to the leaf \(L_{\iota(y)}\) which is simply connected by the construction of \((\widetilde{X}, \widetilde{\mathcal{L}})\). Consequently, we can carry out the constructions given in Subsection 2.4 and (equivalently) in Subsection 2.5 above. More specifically, we construct the sample-path space \(\Omega(\Sigma)\) consisting of all continuous paths \(\omega : [0, \infty) \to \Sigma\) with image fully contained in a single leaf. For each \(y \in \Sigma\), let \(\Omega_y\) be the subspace consisting of all continuous paths \(\omega : [0, \infty) \to \Sigma_y\) with \(\omega(0) = y\). Using cylinder sets we construct the \(\sigma\)-algebra \(\mathcal{A}(\Sigma) := \mathcal{A}(\Omega(\Sigma))\), and the \(\sigma\)-algebra \(\mathcal{A}_y := \mathcal{A}(\Sigma_y)\), and the Wiener probability measures \(W_y\) on \((\Omega_y, \mathcal{A}_y)\) for each \(y \in \Sigma\).

A weakly fibered lamination \(\Sigma\) over \((X, \mathcal{L}, g, \mu)\) is said to be a fibered lamination if it satisfies the following additional property:

(iv) for every set \(A \in \mathcal{A}(\Sigma)\), the image \(\tau \circ A \subset \Omega(X, \mathcal{L})\) is \(\bar{\mu}\)-measurable, where, for a set \(A \subset \Omega(\Sigma)\), \(\tau \circ A := \{\tau \circ \omega : \omega \in A\}\).

In what follows let \(\iota : \Sigma \to \widetilde{X}\) be a fibered lamination over \((X, \mathcal{L})\). Let \(\tau := \pi \circ \iota : \Sigma \to X\). So \(\tau\) maps leaves to leaves and the cardinal of every fiber \(\tau^{-1}(x)\) \((x \in X)\) is at most countable because of Definition 7.3 (iii) and of the fact that the cardinal of every fiber \(\pi^{-1}(x)\) \((x \in X)\) is at most countable. Consider the \(\sigma\)-finite measures \(\nu := \tau^*\mu\) on \((\Sigma, \mathcal{B}(\Sigma))\) and \(\bar{\nu} := \tau^*\bar{\mu}\) on \((\Omega(\Sigma), \mathcal{A}(\Sigma))\).

For a measurable function \(f : \Sigma \to \mathbb{R}^+\), consider the maximal function on fibers
$M[f]: X \to \mathbb{R}^+$ given by

$$M[f](x) := \sup_{y \in \tau^{-1}(x)} f(y), \quad x \in X,$$

with the convention that $M[f](x) := 0$ if $\tau^{-1}(x) = \emptyset$. For a measurable function $F: \Omega(\Sigma) \to \mathbb{R}^+$ on $(\Omega(\Sigma), \mathcal{A}(\Sigma))$, consider the $\ast$-norm:

$$\|F\|_\ast := \int_X M[f] d\mu, \quad (7.2)$$

where the function $f: \Sigma \to \mathbb{R}^+$ is defined by

$$f(y) := \int_{\Omega_y} F(\omega) dW_y(\omega), \quad y \in \Sigma.$$

**Remark 7.4.** Clearly, $\|F\|_\ast = 0$ if and only if for $\mu$-almost every $x \in X$, and for every $y \in \tau^{-1}(x)$, it holds that $F(\omega) = 0$ for $W_y$-almost every $\omega$.

**Lemma 7.5.** For every $A \in \mathcal{A}(\Sigma)$, it holds that $\bar{\mu}(\tau \circ A) \leq \bar{\nu}(A)$.

*Proof.* Let $A' := \tau \circ A$. By Lemma 10.30, $A'$ belongs to the $\bar{\mu}$-completion of $\mathcal{A}$. Note that for every $x \in X$ and every $\omega' \in A' \cap \Omega_x$, there exists $y \in \tau^{-1}(x)$ and $\omega \in A$ such that $\omega' = \tau \circ \omega$. Hence, for $\mu$-almost every $x \in X$,

$$W_x(A') \leq \sum_{y \in \tau^{-1}(x)} W_y(A), \quad x \in X.$$

Integrating both sides of the above inequality and using that $\nu = \tau^* \mu$, the lemma follows. \qed

**Lemma 7.6.** For a measurable functions $F, G: \Omega(\Sigma) \to \mathbb{R}^+$ on $(\Omega(\Sigma), \mathcal{A}(\Sigma))$, it holds that

$$\|F + G\|_\ast \leq \|F\|_\ast + \|G\|_\ast \quad \text{and} \quad \|F\|_\ast \leq \int_{\Omega(\Sigma)} F d\bar{\nu}.$$

*Proof.* Consider the functions $f, g: \Sigma \to \mathbb{R}^+$ defined by

$$f(y) := \int_{\Omega_y} F(\omega) dW_y(\omega) \quad \text{and} \quad g(y) := \int_{\Omega_y} G(\omega) dW_y(\omega), \quad y \in X.$$

The first assertion is an immediate consequence of the estimate

$$\sup_{y \in \tau^{-1}(x)} (f(y) + g(y)) \leq \sup_{y \in \tau^{-1}(x)} f(y) + \sup_{y \in \tau^{-1}(x)} g(y), \quad x \in X.$$

The second one follows from

$$\sup_{y \in \tau^{-1}(x)} f(y) \leq \sum_{y \in \tau^{-1}(x)} f(y), \quad x \in X.$$
Now we are in the position to state the main result of this subsection.

**Theorem 7.7.** 1) For any sets $A, B \in \mathcal{A}(\Sigma)$, we have

$$
\liminf_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} (F \circ T^k)G \right\|_* \leq \|F\|_*\|G\|_*,$$

where $F := 1_A$ and $G := 1_B$.

2) If $A \in \mathcal{A}(\Sigma)$ is $T$-totally invariant, where $T$ is the shift-transformation on $\Omega(\Sigma)$, then $\|1_A\|_*$ is equal to either 0 or 1.

We are inspired by Kakutani’s method in the proof of [23, Theorem 3].

**Proof.** Assuming first Part 1) we will prove Part 2). Indeed, applying Part 1) to functions $F = G = 1_A$ yields that

$$
\liminf_{n \to \infty} \frac{1}{n} \left\| \sum_{k=0}^{n-1} (1_A \circ T^k)1_A \right\|_* \leq \|1_A\|_*^2.
$$

Since $A = T^{-1}(A)$, the left-hand side is equal to $\|1_A\|_*$. Hence, we obtain that $\|1_A\|_* = \|1_A\|_*^2$. So $\|1_A\|_*$ is equal to either 0 or 1, as desired.

To prove Part 1) we first assume that $A, B \in \mathcal{C}$, where $\mathcal{C}$ denotes the (non $\sigma$-) algebra generated by all cylinder sets in $\Omega(\Sigma)$. We know from Proposition 2.8 that each element of $\mathcal{C}$ may be represented as the finite union of mutually disjoint cylinder sets. Therefore, we may write

$$
A := \bigcup_{p \in P} A^p := \bigcup_{p \in P} C(\{t_i, A_i^p\} : m) \quad \text{and} \quad B := \bigcup_{q \in Q} B^q := \bigcup_{q \in Q} C(\{s_j, B_j^q\} : l),
$$

where the cylinder sets on the right hand sides are mutually disjoint and the index set $P$ and $Q$ are finite. Consequently, for $k \geq k_0 := s_l$, we have that

$$(F \circ T^k) \cdot G = \sum_{p \in P, q \in Q} 1_{C_p^q},$$

where $C_p^q$ is the cylinder set $C(\{s_1, B_1\}, \ldots, \{s_l, B_l\}, \{t_1 + k, A_1\}, \ldots, \{t_m + k, A_m\} : l + m)$. By (2.4), we get, for every $y \in \Sigma$, that

$$
W_y(C_{p,q}) = \left( D_{s_1} (\chi_{B_1^p} \cdots \chi_{B_{s_1-1}^p} D_{s_1-s_1-1} (\chi_{B_1^p} D_{t_1+k-s_l} (\chi_{A_1^p} D_{t_2-t_l} (\chi_{A_2^p} \cdots \chi_{A_{m-s_1}^p} D_{t_m-t_m-s_1} (\chi_{A_m^p} (\cdots)))))(y).
$$

Consider the function $H : \Sigma \to [0,1]$ given by

$$
H(y) := \sum_{p \in P} D_{t_1} (\chi_{A_1^p} \cdots \chi_{A_{m-s_1}^p} D_{t_m-t_m-s_1} (\chi_{A_m^p} (\cdots)))(y) = W_y(\bigcup_{p \in P} A^p).
$$

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Consider also the linear integral operator $\mathcal{D} : L^\infty(\Sigma) \to L^\infty(\Sigma)$ given by

$$\mathcal{D}(f) := \sum_{q \in Q} D_s \left( \chi_{B_1^q} \cdots \chi_{B_{s_1}^q} \cdots \chi_{B_{s_2}^q} \cdots \chi_{B_{s_1}^q} \cdots (\chi_{B_1^q} f) \cdots \right), \ f \in L^\infty(\Sigma).$$

Summarizing what has been done so far, we have shown that for every $k \geq k_0$ and every $y \in \Sigma$,

$$\int_{\Omega_y} F(T^k \omega) G(\omega) dW_y = \mathcal{D}(D_{k-s_t} H)(y), \text{ where } H(y) = W_y(A).$$

Observe that $H \leq K \circ \tau$, where $K : X \to \mathbb{R}^+$ is given by $K := M[W_*(A)]$, the function $W_*(A) : \tilde{X} \to [0,1]$ being given by $\tilde{X} \ni \tilde{x} \mapsto W_{\tilde{x}}(A)$. This, combined with the previous equality, implies that for all $n > k_0$,

$$\int_{\Omega_y} \frac{1}{n} \sum_{k=k_0}^{n-1} (F \circ T^k) \cdot G dW_y \leq \mathcal{D} \left( \frac{1}{n} \sum_{k=k_0}^{n-1} D_{k-s_t} K \circ \pi \right)(y), \ y \in \Sigma. \quad (7.3)$$

Since $0 \leq K \leq 1$, it follows that $\sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} D_k K \leq 1$. On the other hand, by Akcoglu’s ergodic theorem $\frac{1}{n} \sum_{k=0}^{n-1} D_k K$ converges $\mu$-almost everywhere to $\int_X K d\mu = \|1_A\|_\ast$. Putting these altogether and using the explicit formula of $\mathcal{D}$, we deduce from Lebesgue’s dominated convergence that, for $\mu$-almost every $x \in X$ and for every $y \in \tau^{-1}(x) \subset \Sigma$,

$$\lim_{n \to \infty} \mathcal{D} \left( \frac{1}{n} \sum_{k=k_0}^{n-1} D_{k-s_t} K \circ \pi \right)(y) = \lim_{n \to \infty} \mathcal{D} \left( \frac{1}{n} \sum_{k=0}^{n-1} D_{k-s_t} K \circ \pi \right)(y) = \mathcal{D}(\|1_A\|_\ast \cdot 1)(y),$$

where $1$ is the function identically equal to $1$ on $\Sigma$. The right hand side is equal to

$$\|1_A\|_\ast \mathcal{D}(1)(y) = \|1_A\|_\ast W_y(B).$$

This, coupled with (7.3), implies that, for $\mu$-almost every $x \in X$ and for every $y \in \tau^{-1}(x)$,

$$\limsup_{n \to \infty} \int_{\Omega_y} \frac{1}{n} \sum_{k=0}^{n-1} (F \circ T^k) \cdot G dW_y \leq \|1_A\|_\ast W_y(B). \quad (7.4)$$

By Fatou’s lemma the left hand side is greater than $\int_{\Omega_y} \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (F \circ T^k) \cdot G dW_y$. Consequently, Part 1) follows.

It remains to treat the general case where $A, B \in \mathcal{A}(\Sigma)$. Recall that all leaves of $\Sigma$ are simply connected. Therefore, by Proposition 4.1 (ii), for every $A, B \in \mathcal{A}(\Sigma)$ there exist two sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ such that each $A_n$ (as well as each $B_n$) is a countable union of elements in $\mathcal{E}$ and that $A \subset A_n$,
On the one hand, it follows from (7.5) that \( \bar{\nu}(A \setminus A') < \frac{\epsilon}{4} \), \( \bar{\nu}(A \setminus A'') < \frac{\epsilon}{4} \), \( \bar{\nu}(B_n \setminus B) < \frac{\epsilon}{4} \), \( \bar{\nu}(B_n \setminus B') < \frac{\epsilon}{4} \).

Hence,

\[
\bar{\nu}(A \setminus A') \leq \bar{\nu}(A \setminus A'') < \frac{\epsilon}{4} \quad \text{and} \quad \bar{\nu}(B \setminus B') \leq \bar{\nu}(B_n \setminus B') < \frac{\epsilon}{4}.
\]

Using this and applying Proposition 4.1 (ii) to both sets \( A \setminus A' \) and \( B \setminus B' \), we obtain two sets \( A'' \) and \( B'' \), each of them being a countable union of cylinder sets, such that

\[
A \setminus A' \subset A'', \quad \bar{\nu}(A'') < \epsilon/2 \quad \text{and} \quad B \setminus B' \subset B'', \quad \bar{\nu}(B'') < \epsilon/2.
\]

Let \( A'' := \tau \circ (A'') \) and \( B' := \tau \circ (B'') \). By Lemma 7.5

\[
\bar{\mu}(A'') \leq \bar{\nu}(A''') < \frac{\epsilon}{2} \quad \text{and} \quad \bar{\mu}(B'') \leq \bar{\nu}(B''') < \frac{\epsilon}{2}.
\]

On the one hand, it follows from (7.5) that

\[
\bar{\nu}(A' \setminus A) < \frac{\epsilon}{4} \quad \text{and} \quad \bar{\nu}(B' \setminus B) < \frac{\epsilon}{4}.
\]

This, combined with Lemma 7.6 implies that

\[
\|1_{A'}\|_* \leq \|1_A\|_* + \|1_{A' \setminus A}\|_* < \|1_A\|_* + \epsilon/4 \quad \text{and} \quad \|1_{B'}\|_* \leq \|1_B\|_* + \epsilon/4. \tag{7.7}
\]

On the other hand, since \( 1_A - 1_{A'} \leq 1_{A''} \leq 1_{\tau^{-1}(A'')} \) and \( 1_B - 1_{B'} \leq 1_{B''} \leq 1_{\tau^{-1}(B'')} \), we deduce that, for every \( x \in X \) and \( y \in \tau^{-1}(x) \) and \( n \geq 1 \),

\[
\int_{\Omega_y} \frac{1}{n} \sum_{k=0}^{n-1} (F \circ T^k) \cdot G dW_y \leq \int_{\Omega_y} \frac{1}{n} \sum_{k=0}^{n-1} (1_{A'} \circ T^k) \cdot 1_{B'} dW_y \\
+ \int_{\Omega_y} \left( \frac{1}{n} \sum_{k=0}^{n-1} (1_{\tau^{-1}(A'')} \circ T^k) \right) 1_{B} dW_y + \int_{\Omega_y} \left( \frac{1}{n} \sum_{k=0}^{n-1} (F \circ T^k) \right) 1_{B \setminus B'} dW_y.
\]

Since \( A', B' \in \mathcal{C} \), it follows from the previous case (see (7.4)) that the \( \limsup_{n \to \infty} \) of first term on the right hand side is \( \leq \|1_{A'}\|_* W_y(B') \). Since \( F \leq 1 \), the third term is bounded from above by

\[
\int_{\Omega_y} 1_{B \setminus B'} dW_y = W_y(B \setminus B') \leq W_y(B'') \leq W_x(B'').
\]
The second term is dominated by
\[
\int_{\Omega_y} \frac{1}{n} \sum_{k=0}^{n-1} (1_{\tau^{-1}A'} \circ T^k) dW_y = \int_{\Omega_x} \frac{1}{n} \sum_{k=0}^{n-1} (1_{A'} \circ T^k) dW_x,
\]
where we recall that \( x = \tau(y) \). Consequently,
\[
\int_{\Omega_y} \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (F \circ T^k) \cdot G dW_y \leq \limsup_{n \to \infty} \int_{\Omega_y} \frac{1}{n} \sum_{k=0}^{n-1} (F \circ T^k) \cdot G dW_y \leq \|1_{A'}\|_{\ast} W_y(B') + W_x(B'') + \limsup_{n \geq 1} \int_{\Omega_x} \frac{1}{n} \sum_{k=0}^{n-1} (1_{A'} \circ T^k) dW_x.
\]

On the other hand, by Theorem 4.6 and Fatou’s lemma, the \( \limsup \) on the last line \( \leq \int_{\Omega} 1_{A'} d\bar{\mu} = \bar{\mu}(A) \) for \( \mu \)-almost every \( x \in X \). Summarizing what has been done sofar we have shown that
\[
\left\| \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (F \circ T^k) G \right\|_{\ast} \leq \|1_{A'}\|_{\ast} \|1_{B'}\|_{\ast} + \int_X W_x(B'') d\mu(x) + \bar{\mu}(A '').
\]

Using (7.6) the last line is dominated by \( \|1_{A'}\|_{\ast} \|1_{B'}\|_{\ast} + \epsilon \), which is, in turn, bounded by \( \|1_{A}\|_{\ast} \|1_{B}\|_{\ast} + 4\epsilon \) in virtue of (7.7). Since \( \epsilon > 0 \) is arbitrarily chosen, Part 1) in the general case where \( A, B \in \mathcal{A}(\Sigma) \) follows. \( \square \)

7.3 Cylinder laminations and end of the proof

Let \((X, \mathcal{L}, g)\) be a lamination satisfying the Standing Hypotheses and set \( \Omega := \Omega(X, \mathcal{L}) \) as usual. The purpose of this subsection is to complete Step 2 in the proof of Theorem 7.2.

Definition 7.8. For every \( 1 \leq k \leq d \), the cylinder lamination of dimension \( k \) of a cocycle \( \mathcal{A} : \Omega(X, \mathcal{L}) \times \mathbb{G} \to \text{GL}(d, \mathbb{R}) \), denoted by \((X_{k, \mathcal{A}}, \mathcal{L}_{k, \mathcal{A}})\), is defined as follows. The ambient topological space of the cylinder lamination is \( X \times \text{Gr}_k(\mathbb{R}^d) \) which is independent of \( \mathcal{A} \). Its leaves are defined as follows. For a point \( (x, U) \in X \times \text{Gr}_k(\mathbb{R}^d) \) and for every simply connected plaque \( K \) of \((X, \mathcal{L})\) passing through \( x \), we define the plaque \( \mathcal{K} \) of \((X \times \text{Gr}_k(\mathbb{R}^d), \mathcal{L}_{k, \mathcal{A}})\) passing through \( (x, U) \) by
\[
\mathcal{K} = \mathcal{K}(K, x, U) := \{(y, \mathcal{A}(\omega, 1)U) : y \in K, \omega \in \Omega_x, \omega(1) = y, \omega[0, 1] \subset K\},
\]
where we also denote by \( \mathcal{A}(\omega, 1) \) its induced action on \( \text{Gr}_k(\mathbb{R}^d) \), that is,
\[
\mathcal{A}(\omega, 1)U := \{\mathcal{A}(\omega, 1)u : u \in U\}.
\]

Note that the projection on the first factor \( \text{pr}_1 : X \times \text{Gr}_k(\mathbb{R}^d) \to X \) maps \( \mathcal{K} \) onto \( K \) homeomorphically. We endow the plaque \( \mathcal{K} \) with the metric \((\text{pr}_1|_{\mathcal{K}})^*(g|_{\mathcal{K}})\).
By this way, the leaves of \((X \times \text{Gr}_k(\mathbb{R}^d), \mathcal{L}_{k,A})\) is equipped with the metric \(\text{pr}^*g\), and hence \((X \times \text{Gr}_k(\mathbb{R}^d), \mathcal{L}_{k,A}, \text{pr}^*g)\) is a Riemannian measurable lamination. The Laplacian and the one parameter family \(\{D_t : t \geq 0\}\) of the diffusion operators are defined using the leafwise metric \(\text{pr}^*g\).

Since the local expression of \(\mathcal{A}\) on flow boxes is, in general, only measurable, the cylinder lamination \((X \times \text{Gr}_k(\mathbb{R}^d), \mathcal{L}_{k,A})\) is a measurable lamination in the sense of Definition 2.2. Let \(\Omega_{k,A} := \Omega(X_{k,A}, \mathcal{L}_{k,A})\).

Clearly, when \(k = d\) we have that \((X, \mathcal{L}) \equiv (X_{d,A}, \mathcal{L}_{d,A})\).

In what follows let \(\mathcal{A} : \Omega \times N \rightarrow \text{GL}(d, \mathbb{R})\) be a (multiplicative) cocycle. Now we discuss the notion of saturations. The \((\text{leafwise})\) saturation of a set \(Z \subset X\) in the lamination \((X, \mathcal{L})\) is the leafwise saturated set

\[
\text{Satur}(Z) := \bigcup_{x \in Z} L_x.
\]

For a set \(\Sigma \subset X \times \text{Gr}_k(\mathbb{R}^d)\), the saturation of \(\Sigma\) with respect to the cocycle \(\mathcal{A}\) is the saturation of \(\Sigma\) in the lamination \((X_{k,A}, \mathcal{L}_{k,A})\).

We have the following natural identification.

**Lemma 7.9.** The transformation \(\Omega_{k,A} \rightarrow \Omega \times \text{Gr}_k(\mathbb{R}^d)\) which maps \(\eta\) to \((\omega, U(0))\), where \(\eta(t) = (\omega(t), U(t)), t \in [0, \infty)\), is bijective.

**Proof.** The identification, follows from the fact that \(\eta\) is uniquely determined in term of \(\omega\) and \(U(0)\). Indeed, we have that

\[
\eta(t) = (\omega(t), U(t)) = (\omega(t), \mathcal{A}(\omega, t)(U(0))).
\]

\[\square\]

Let \(T\) be as usual the shift-transformation on \(\Omega_{k,A}\). A set \(\widehat{F} \subset \Omega_{k,A}\) is said to be \(T\)-totally invariant if \(T \widehat{F} = T^{-1} \widehat{F} = F\). Using Lemma 7.9 we may define \(T\) and \(T^{-1}\) on \(\Omega \times \text{Gr}_k(\mathbb{R}^d)\) as follows:

\[
T(\omega, u) := (T\omega, \mathcal{A}(\omega, 1)u), \quad (\omega, u) \in \Omega \times \text{Gr}_k(\mathbb{R}^d);
\]

\[
T\widehat{F} := \{T(\omega, u) : (\omega, u) \in \widehat{F}\} \quad \text{and} \quad T^{-1} \widehat{F} := \{(\omega, u) : T(\omega, u) \in \widehat{F}\}.
\]

Here \(\widehat{F}\) is a subset of \(\Omega \times \text{Gr}_k(\mathbb{R}^d)\). Given a set \(\widehat{F} \subset \Omega \times \text{Gr}_k(\mathbb{R}^d)\), let \(F\) be the projection of \(\widehat{F}\) onto the first factor, that is,

\[
F := \{\omega \in \Omega : \exists u \in \mathbb{P}(\mathbb{R}^d) : (\omega, u) \in \widehat{F}\}.
\]

We see easily that if \(\widehat{F}\) is \(T\)-totally invariant, so is \(F\).

By Theorem 10.5 below there is a bimeasurable bijection between the \(\mathcal{A}\)-invariant sub-bundle \(Y \ni x \mapsto V_m(x)\) of rank \(d_m\) and \(Y \times \mathbb{R}^{d_m}\) covering the identity and which is linear on fibers. Therefore, we may assume without loss...
of generality that $V_m(x) = \mathbb{R}^{dm}$ everywhere in Step 2. By Corollary 6.2 (iii), we have, for every $x \in Y$ and for every $v \in V_m(x)$, that
\[
\text{ess. sup}_{\omega \in \Omega_x(Y)} \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega,n)v\| = \chi_m.
\] (7.8)

Consider the following set \( \hat{F} \subset \Omega \times \mathbb{P}(\mathbb{R}^{dm}) \):
\[
\hat{F} := \{ (\omega, u) \in \Omega \times \mathbb{P}(\mathbb{R}^{dm}) : \chi(\omega, u) < \chi_m \}.
\] (7.9)

Pick an arbitrary \((\omega, u) \in \hat{F}\). So \(\chi(\omega, u) < \chi_m\). Let \(\eta \in \Omega\) be an arbitrary path such that \(T\eta = \omega\) and choose \(v \in \mathbb{P}(\mathbb{R}^{dm})\) such that \(u = A(\eta,1)v\). We get that \(\chi(\eta, v) = \chi(\omega, u) < \chi_m\). Hence, \((\eta, v) \in \hat{F}\). So we have just shown that \(T^{-1}\hat{F} \subset \hat{F}\). Similarly, we also obtain that \(T\hat{F} \subset \hat{F}\). That is, \(\hat{F}\) is \(T\)-totally invariant. So the image \(F := \pi_1(\hat{F})\) of \(\hat{F}\) onto the first factor \(\Omega\) is \(T\)-totally invariant.

For each integer \(1 \leq k \leq d_m\) let
\[
\mathcal{N}_k := \{ x \in X : \exists U \in \text{Gr}_k(\mathbb{R}^{dm}) \text{ such that } W_x(\mathcal{F}_{x,U}) > 0 \},
\]
where, for each point \(x \in X\) and each vector subspace of dimension \(k\) in \(\mathbb{R}^{dm}\),
\[
\mathcal{F}_{x,U} := \{ \omega \in \Omega_x : \forall u \in U : (\omega,u) \in \hat{F} \},
\]
where \(\hat{F}\) is given by (7.9). Note that
\[
\mathcal{N}_{d_m} \subset \mathcal{N}_{d_m-1} \subset \cdots \subset \mathcal{N}_1 \subset X.
\]

**Lemma 7.10.** Let \(k\) be an integer with \(1 \leq k \leq d_m\).
1) Both map \(M_k : X \times \text{Gr}_k(\mathbb{R}^{dm}) \to [0,1]\) given by
\[
M_k(x,U) := W_x(\{ \omega \in \Omega : \forall u \in U : (\omega,u) \in \hat{F} \}),
\]
and map \(N_k : X \to [0,1]\) given by
\[
N_k(x) := \sup_{U \in \mathcal{G}_k(x)} W_x(\mathcal{F}_{x,U}), \quad x \in X,
\]
are measurable.
2) \(\mathcal{N}_k\) is measurable and \(\text{Vol}_{L_x}(L_x \cap \mathcal{N}_k) > 0\) for every \(x \in \mathcal{N}_k\).
3) If \(\mu(\mathcal{N}_k) = 0\) if and only if \(\mu(\text{Satur}(\mathcal{N}_k)) = 0\).
Lemma 7.11. If \( \chi_m \neq \lambda_l \), then \( \mu(\mathcal{N}_{\lambda_m}) = 0 \).

Proof. By Theorem 7.11, we may find \( k \) measurable functions \( b_1, \ldots, b_k : \text{Gr}_k(\mathbb{R}^{dm}) \to \text{Gr}_1(\mathbb{R}^{dm}) \) such that for each \( U \in \text{Gr}_k(\mathbb{R}^{dm}) \), the \( k \) lines \( b_1(U), \ldots, b_k(U) \) span \( U \).

For every \( 1 \leq i \leq k \) consider the following subset of \( \Omega \times \text{Gr}_k(\mathbb{R}^{dm}) \):

\[
F_i := \{ (\omega, U) \in \Omega \times \text{Gr}_k(\mathbb{R}^{dm}) : \chi(\omega, b_i(U)) < \chi_m \}.
\]

Since \( \chi \) and \( b_i \) are measurable functions, each \( F_i \) is a measurable subset of \( \Omega \times \text{Gr}_k(\mathbb{R}^{dm}) \). Let \( F_0 := \cap_{i=1}^k F_i \). So \( F_0 \) is also measurable. Observe that

\[
M_k(x, U) = W_x(\{ \omega \in \Omega : (\omega, U) \in F_0 \}), \quad (x, U) \in X \times \text{Gr}_k(\mathbb{R}^{dm}).
\]

On the other hand, the function \( X \times \text{Gr}_k(\mathbb{R}^{dm}) \ni (x, U) \mapsto W_x(\{ \omega \in \Omega : (\omega, U) \in F_0 \}) \) is measurable by Proposition 4.2. Consequently, \( M_k \) is measurable.

To prove that \( N_k \) is measurable observe that

\[
W_x(\mathcal{F}_{x,U}) = W_x(\{ \omega \in \Omega : (\omega, U) \in F_0 \}).
\]

We deduce that

\[
N_k(x) = \sup_{U \in \text{Gr}_k(\mathbb{R}^{dm})} W_x(\{ \omega \in \Omega : (\omega, U) \in F_0 \}).
\]

Recall that the function \( X \times \text{Gr}_k(\mathbb{R}^{dm}) \ni (x, U) \mapsto W_x(\{ \omega \in \Omega : (\omega, U) \in F_0 \}) \) is measurable. Consequently, applying Lemma 10.2 in Appendix to the last equality yields that \( N_k \) is measurable. This completes Part 1).

Since \( \mathcal{M}_k = \{ x \in X : N_k(x) > 0 \} \), the measurability of \( \mathcal{M}_k \) follows from Part 1). To prove the other assertion of Part 2) fix \( x_0 \in \mathcal{M}_k \) and \( U_0 \in \text{Gr}_l(\mathbb{R}^{dm}) \) such that \( \mathcal{F}_{x_0,U_0} > 0 \). In other words,

\[
W_{x_0}(F_0 \cap \Omega((L_{k,A})(x_0,U_0))) > 0,
\]

where \( (L_{k,A})(x_0,U_0) \) denotes the leaf of \( (X_{k,A}, L_{k,A}) \) passing through the point \( (x_0, U_0) \) and \( \Omega((L_{k,A})(x_0,U_0)) \) denotes the space of all continuous paths \( \omega \) defined on \([0,\infty)\) with image fully contained in this leaf.

Since \( F \) is \( T \)-totally invariant, it is easy to see that so is \( F_0 \). Consequently, applying Proposition 5.3 (i) yields that for \( L := L_{x_0}, \)

\[
\text{Vol}(\{ x \in L : W_x(F_0 \cap \Omega((L_{k,A})(x_0,U_0))) > 0 \}) > 0,
\]

proving the last assertion of Part 2).

Using Part 2), Part 3) follows from Part 3) of Proposition 10.29.

\[\square\]
For each integer $1 \leq k \leq d_m$ and each point $x \in X$, let

$$
\mathcal{U}_k(x) := \{U \in \text{Gr}_k(\mathbb{R}^{d_m}) : W_x(\mathcal{F}_{x,U}) > 0\}.
$$

Now we arrive at the following stratifications.

**Lemma 7.12.** Let $x \in X \setminus \mathcal{N}_{k+1}$. Then for every $U, V \in \mathcal{U}_k(x)$ with $U \neq V$, it holds that $W_x(\mathcal{F}_{x,U} \cap \mathcal{F}_{x,V}) = 0$. In particular, $0 < \sum_{U \in \mathcal{U}_k(x)} W_x(\mathcal{F}_{x,U}) \leq 1$ and hence the cardinal of $\mathcal{U}_k(x)$ is at most countable.

**Proof.** Suppose that there exist $U, V \in \mathcal{U}_k(x)$ such that $U \neq V$, and that $W_x(\mathcal{F}_{x,U} \cap \mathcal{F}_{x,V}) > 0$. Let $W$ be the vector space spanned by both $U$ and $V$. Since $U \neq V$, $W$ is of dimension $\geq k + 1$. Let $w$ be an arbitrary element in $W$. So we may find $u \in U$ and $v \in V$ such that $w = u + v$. Arguing as in the proof of Proposition 5.2 (iii), we get that

$$
\chi(\omega, u + v) \leq \max\{\chi(\omega, u), \chi(\omega, v)\}, \quad \omega \in \Omega.
$$

Consequently, for every $\omega \in \mathcal{F}_{x,U} \cap \mathcal{F}_{x,V}$, we infer that

$$
\chi(\omega, w) = \chi(\omega, u + v) \leq \max\{\chi(\omega, u), \chi(\omega, v)\} < \chi_m.
$$

Hence, $\mathcal{F}_{x,U} \cap \mathcal{F}_{x,V} \subset \mathcal{F}_{x,W}$. This, combined with the assumption that $W_x(\mathcal{F}_{x,U} \cap \mathcal{F}_{x,V}) > 0$, implies that $W_x(\mathcal{F}_{x,W}) > 0$, that is, $x \in \mathcal{N}_{k+1}$, which contradicts the hypothesis. Hence, the first assertion of the lemma follows.

The second assertion follows from the first one since $W_x$ is a probability measure. To prove that the cardinal of $\mathcal{U}_k(x)$ is at most countable, consider, for each $N \geq 1$, the following subset of $\mathcal{U}_k(x)$:

$$
\mathcal{U}_k^N(x) := \{U \in \mathcal{U}_k(x) : W_x(\mathcal{F}_{x,U}) > 1/N\}.
$$

By the first assertion, the cardinal of $\mathcal{U}_k^N(x)$ is at most $N$. Since $\mathcal{U}_k(x) = \bigcup_{N=1}^{\infty} \mathcal{U}_k^N(x)$, the last assertion of the lemma follows. \(\Box\)

**End of the proof of Step 2 of Theorem 7.2.** If $\chi_m = \lambda_l$, then by Theorem 7.1 for $\mu$-almost every $x \in X$, we have that $\chi(\omega, v) = \lambda_l = \chi_m$ for all $v \in V_m(x)$ and that $V_l(\omega) = V_m(x)$ for $W_x$-almost every $\omega$. Hence, Step 2 is finished. Therefore, in the sequel we assume that $\chi_m \neq \lambda_l$. Consequently, by Lemma 7.11 we get that $

\mu(\mathcal{N}_{d_m}) = 0.$

In the remaining part of the proof we let $k$ descend from $d_m - 1$ to 1. So we begin with $k = d_m - 1$ and recall that $\mu(\mathcal{N}_{d_m}) = 0$. The remaining proof is divided into two sub-steps.

**Sub-step 1:** If $\mu(\mathcal{N}_{k+1}) = 0$ and $k \geq 1$, then $\mu(\mathcal{N}_{k}) = 0$.

Suppose in order to reach a contradiction that $\mu(\mathcal{N}_{k}) > 0$. Let $\pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L})$ be the covering lamination projection of $(X, \mathcal{L})$, and set $\tilde{\Omega} := \Omega(\tilde{X}, \tilde{\mathcal{L}})$. Let

$$
\Sigma_k := \{ (\tilde{x}, U) \in \pi^{-1}(\mathcal{N}_k \setminus \text{Satur}(\mathcal{N}_{k+1})) \times \text{Gr}_k(\mathbb{R}^{d_m}) : U \in \mathcal{U}_k(\pi(\tilde{x})) \}. \quad (7.10)
$$
We construct a cocycle $\tilde{A}$ on $(\bar{X}, \mathcal{L})$ as follows:

$$\tilde{A}(\bar{\omega}, t) := A(\pi \circ \bar{\omega}, t), \quad t \in \mathbb{R}^+, \ \bar{\omega} \in \bar{\Omega}.$$ 

Consider the cylinder lamination of dimension $k$ $(\bar{X}_{k, \bar{A}}, \mathcal{L}_{k, \bar{A}})$ of the cocycle $\bar{A}$. Note that $\Sigma_k \subset \bar{X}_{k, \bar{A}} = \bar{X} \times \text{Gr}_k(\mathbb{R}^{dn})$. Let $\bar{\Sigma}_k$ be the saturation of $\Sigma_k$ in this lamination. The projection of $\bar{\Sigma}_k \subset \bar{X} \times \text{Gr}_k(\mathbb{R}^{dn})$ onto the first factor is denoted by $\text{pr}_1(\bar{\Sigma}_k)$. Since $\mu(\mathcal{N}_{k+1}) = 0$ and $\mu(\mathcal{N}_k) > 0$, if follows from Part 3) of Lemma 7.10 that $\mu(\mathcal{N}_k \setminus \text{Satur}(\mathcal{N}_{k+1})) > 0$. On the other hand, $\pi(\text{pr}_1(\bar{\Sigma}_k)) \subset X$ is equal to the saturation of $\mathcal{N}_k \setminus \text{Satur}(\mathcal{N}_{k+1})$ in the lamination $(X, \mathcal{L})$. Putting all these together and noting that $\mu$ is ergodic, we infer that the leafwise saturated set $\pi(\text{pr}_1(\bar{\Sigma}_k)) \subset X$ is of full $\mu$-measure.

For any point $(\bar{x}, U) \in \bar{\Sigma}_k$, let $(\bar{\Sigma}_k)(\bar{x}, U)$ be the saturation of $(\bar{x}, U)$ in $(\bar{X}_{k, \bar{A}}, \mathcal{L}_{k, \bar{A}})$, and we call it the leaf of $\bar{\Sigma}_k$ passing through $(\bar{x}, U)$. Since this leaf is also a leaf in $(\bar{X}_{k, \bar{A}}, \mathcal{L}_{k, \bar{A}})$, it is endowed with the natural metric $\text{pr}_1^* \bar{g}$, where $\bar{g} := \pi^* g$. Let $\Omega(\bar{\Sigma}_k)$ be the space of all continuous paths $\omega : [0, \infty) \to \bar{\Sigma}_k$ with image fully contained in a single leaf. Using the canonical identification given by Lemma 7.9 we identify $\Omega(\bar{\Sigma}_k)$ with a subspace of $\bar{\Omega} \times \text{Gr}_k(\mathbb{R}^d)$.

Let

$$\tilde{\hat{A}}_k := \left\{ (\omega, U) \in \bar{\Omega} \times \text{Gr}_k(\mathbb{R}^{dn}) : (\omega, U) \subset \tilde{F} \right\} .$$

where $\tilde{F}$ is given by (7.9). We can easily show that $\tilde{\hat{A}}_k$ is $T$-totally invariant. Moreover, it is also measurable by Proposition 4.2. Let

$$\tilde{\tilde{A}}_k := \left\{ (\bar{\omega}, U) \in \bar{\Omega} \times \text{Gr}_k(\mathbb{R}^{dn}) : (\pi \circ \bar{\omega}, U) \in \tilde{\hat{A}}_k \text{ and } (\bar{\omega}(0), U) \in \bar{\Sigma}_k \right\} .$$

(7.11)

**Proposition 7.13.** Suppose as in Sub-step 1 that $\mu(\mathcal{N}_{k+1}) = 0$ and $\mu(\mathcal{N}_k) > 0$.

1) $\tilde{\tilde{A}}_k$ is a $T$-totally invariant subset of $\Omega(\bar{\Sigma}_k)$.

2) $\iota : \bar{\Sigma}_k \to \bar{X}$ is a weakly fibered lamination over $(X, \mathcal{L}, g, \mu)$, where $\iota$ is the canonical projection onto the first factor.

**Proof.** To prove Part 1) observe that $\tilde{\tilde{A}}_k = \Omega(\bar{\Sigma}_k) \cap A_k$, where

$$A_k := \left\{ (\bar{\omega}, U) \in \bar{\Omega} \times \text{Gr}_k(\mathbb{R}^{dn}) : (\pi \circ \bar{\omega}, U) \in \tilde{\hat{A}}_k \right\} .$$

Since $\tilde{\tilde{A}}_k$ is $T$-totally invariant, so are $A_k$ and $\tilde{\hat{A}}_k$.

Now we turn to Part 2). Recall from the construction of $\bar{\Sigma}_k$, that it is the saturation of $\Sigma_k$ in the lamination $(\bar{X}_{k, \bar{A}}, \mathcal{L}_{k, \bar{A}})$. Consequently, we only need to check Definition 7.3 (iii), that is, the cardinal of every fiber $\iota^{-1}(\bar{x})$ ($\bar{x} \in \bar{X}$) is at most countable. To do this fix an arbitrary point $\bar{x}_0 \in \bar{X}$ and let $\iota^{-1}(\bar{x}_0) = \{(\bar{x}_0, U_i) : i \in I\}$. We need to show that the index set $I$ is at most countable.
Suppose without loss of generality that $\tilde{x}_0 \not\in \pi^{-1}(\text{Satur}(\mathcal{M}_{k+1}))$ since otherwise $\iota^{-1}(\tilde{x}_0) = \emptyset$ by the construction of $\Sigma_k$ and $\Sigma$. For each $i \in I$ there exists $(\tilde{x}_i, V_i) \in \Sigma_k$ such that $(\tilde{x}_i, V_i)$ on the same leaf as $(\tilde{x}_0, U_i)$ in $\Sigma_k$. The membership $(\tilde{x}_i, V_i) \in \Sigma_k$ implies, by the definition of $\Sigma_k$, that
\[ W_{\pi(\tilde{x}_i)}(\mathcal{P}_{\pi(\tilde{x}_i)}, V_i) > 0, \quad i \in I. \]

In other words,
\[ W_{\pi(\tilde{x}_i)}(A_k \cap \Omega((\Sigma_k)(\tilde{x}_0, U_i))) > 0, \quad i \in I, \]
where $(\Sigma_k)(\tilde{x}, U)$ denotes the leaf of $\Sigma_k$ passing through the point $(\tilde{x}, U)$, and $\Omega((\Sigma_k)(\tilde{x}, U))$ denotes the space of all continuous paths $\omega$ defined on $[0, \infty)$ with image fully contained in this leaf. By Part 1) the set $\tilde{A}_k \cap \Omega((\Sigma_k)(\tilde{x}_0, U_i))$ is $T$-totally invariant. Consequently, applying Proposition 5.4 (i) yields that for each $i \in I$,
\[ \text{Vol}\left(\left\{ \tilde{x} \in \tilde{L} : W_{\tilde{x}}(A_k \cap \Omega((\Sigma_k)(\tilde{x}_0, U_i))) > 0 \right\}\right) > 0. \]

Here $\tilde{L}$ is the leaf $\tilde{L}_{\tilde{x}_0}$ passing through $\tilde{x}_0$ in $(\tilde{X}, \tilde{g})$ and $\text{Vol}$ is the Lebesgue measure induced by the metric $\tilde{g} := \pi^* g$ on the leaf $\tilde{L}$.

Next, we cover $\tilde{L}$ by a countable family of open sets $(O_n)_{n=1}^{\infty}$ such that $0 < \text{Vol}(O_n) < 1$ for each $n$. The previous estimates show that for each $i \in I$, there is an integer $n \geq 1$ such that
\[ 0 < \int_{O_n} W_{\tilde{x}}(A_k \cap \Omega((\Sigma_k)(\tilde{x}_0, U_i))) d\text{Vol}(\tilde{x}) \leq \text{Vol}(O_n) < 1. \quad (7.12) \]

Note that $\tilde{L} \cap \pi^{-1}(\mathcal{M}_{k+1}) = \emptyset$ because $\tilde{x}_0 \not\in \pi^{-1}(\text{Satur}(\mathcal{M}_{k+1}))$. Consequently, we deduce from Lemma 7.12 that, for each $n \geq 1$ and for each $\tilde{x} \in O_n$,
\[ \sum_{i \in I} W_{\tilde{x}}(A_k \cap \Omega((\Sigma_k)(\tilde{x}_0, U_i))) \leq 1. \]

Here we make the convention that for a collection $(a_i)_{i \in I} \subset \mathbb{R}^+$,
\[ \sum_{i \in I} a_i := \sup_{J \subset I, J \text{ at most countable}} \sum_{j \in J} a_j. \]

Integrating the above inequality over $O_n$, we get that
\[ \sum_{i \in I} \int_{O_n} W_{\tilde{x}}(A_k \cap \Omega((\Sigma_k)(\tilde{x}_0, U_i))) d\text{Vol}(\tilde{x}) \leq \text{Vol}(O_n) < 1. \]

So for each $n \geq 1$, there is at most a countable number of $i \in I$ such that
\[ 0 < \int_{O_n} W_{\tilde{x}}(A_k \cap \Omega((\Sigma_k)(\tilde{x}_0, U_i))) d\text{Vol}(\tilde{x}). \]

Using this and varying $n \in \mathbb{N}$, and combining them with (7.12), the countability of $I$ follows. \[ \square \]
In what follows let \( \tilde{\iota} : \Sigma_k \to \tilde{X} \) be the above weakly fibered lamination over \((X, \mathcal{L})\). Let \( \tau := \pi \circ \tilde{\iota} : \Sigma_k \to X \). So \( \tau \) maps leaves to leaves and the cardinal of every fiber \( \tau^{-1}(x) \) \((x \in X)\) is at most countable. Consider the \( \sigma \)-finite measures \( \nu := \tau^* \mu \) on \((\Sigma_k, \mathcal{B}(\Sigma_k))\) and \( \bar{\nu} := \tau^* \bar{\mu} \) on \((\Omega(\Sigma_k), \mathcal{F}(\Sigma_k))\).

The following stronger version of Proposition 7.13 will be proved in Subsection 10.8 in Appendix below.

**Proposition 7.14.** Suppose as in Sub-step 1 that \( \mu(\mathcal{N}_{k+1}) = 0 \) and \( \mu(\mathcal{N}_k) > 0 \).
1) \( \iota : \Sigma_k \to \tilde{X} \) is a fibered lamination over \((X, \mathcal{L}, g, \mu)\).
2) \( \tilde{A}_k \) is \( \bar{\nu} \)-measurable.

Taking for granted Proposition 7.14 we resume the proof Step 2 of Theorem 7.2. By Proposition 7.14 and Part 1) of Proposition 7.13, we may apply Theorem 7.7 to \( \tilde{A}_k \). This yields that \( \|1_{\tilde{A}_k}\|_* \) is either 0 or 1. Recall that for every \( x \in \mathcal{N}_k \), there is \( U \in \text{Gr}_k(\mathbb{R}^d) \) such that \( (\omega, U) \subset \tilde{F} \) for all \( \omega \in \mathcal{F}_{x,U} \) and that \( W_x(\mathcal{F}_{x,U}) > 0 \). On the other hand, since \( \mu(\mathcal{N}_{k+1}) = 0 \) it follows from Part 3) of Lemma 7.10 that \( \mu(\text{Satur}(\mathcal{N}_{k+1})) = 0 \). This, combined with the assumption that \( \mu(\mathcal{N}_k) > 0 \), and formula (2.7), implies that \( \|1_{\tilde{A}_k}\|_* > 0 \). Hence, \( \|1_{\tilde{A}_k}\|_* = 1 \).

Consequently, we deduce from formula (7.2) that for \( \mu \)-almost every \( x \in X \),
\[
\sup_{y \in \tau^{-1}(x)} W_y(\tilde{A}_k) = 1.
\]

Using that \( \tau = \pi \circ \iota \), where \( \iota : \Sigma_k \to \tilde{X} \) is the canonical projection onto the first factor, we rewrite the above identity as follows
\[
\sup_{\tilde{x} \in \pi^{-1}(x), U \in \text{Gr}_k(\mathbb{R}^d)} W_{\tilde{x},U}(\tilde{A}_k) = 1. \tag{7.13}
\]

Note from (7.11) that \( (\tilde{\omega}', U) \in \tilde{A}_k \) if and only if \( (\tilde{\omega}'', U) \in \tilde{A}_k \) for every \( \omega \in \Omega \) and \( U \in \text{Gr}_k(\mathbb{R}^d) \) and \( \tilde{\omega}', \tilde{\omega}'' \in \pi^{-1}(\omega) \). Therefore, for every \( x \in X \) and every \( x', x'' \in \pi^{-1}(x) \), and every \( U \in \text{Gr}_k(\mathbb{R}^d) \),
\[
W_{x'} \left( \left\{ \tilde{\omega} \in \tilde{\Omega}_{x'} : (\tilde{\omega}, U) \in \tilde{A}_k \right\} \right) = W_{x''} \left( \left\{ \tilde{\omega} \in \tilde{\Omega}_{x''} : (\tilde{\omega}, U) \in \tilde{A}_k \right\} \right)
\]

because both members are equal to \( W_x(\mathcal{F}_{x,U}) \) by Lemma 10.35. This, combined with (7.13), implies that
\[
\sup_{U \in \mathcal{U}_k(x)} W_x(\mathcal{F}_{x,U}) = \sup_{U \in \text{Gr}_k(\mathbb{R}^d)} W_x(\mathcal{F}_{x,U}) = 1.
\]

Consequently, there exists a sequence \( (U_N) \subset \mathcal{U}_k(x) \) such that \( \lim_{N \to \infty} W_x(\mathcal{F}_{x,U_N}) = 1 \). On the other hand, we infer from \( \mu(\text{Satur}(\mathcal{N}_{k+1})) = 0 \) and from Lemma 61.
7.12 that \( W_x(\mathcal{F}_{x,U} \cap \mathcal{F}_{x,U'}) = 0 \) when \( U \neq U' \) for \( \mu \)-almost every \( x \in X \). Consequently, there exists an element \( U \in \mathcal{U}_k(x) \) such that \( W_x(\mathcal{F}_x,U) = 1 \). So for \( \mu \)-almost every \( x \in X \), there exists a \( k \)-dimensional subspace \( U_x \) such that \( \chi(\omega,u) < \chi_m \) for all \( u \in U(x) \) and for \( W_x \)-almost every \( \omega \). Recall from Theorem 7.1 that \( \chi(\omega,u) \in \{ \lambda_1, \ldots, \lambda_l \} \) for \( W_x \)-almost every \( \omega \). Consequently, we deduce from the definition of \( \chi(x,u) \), that \( \chi(x,u) < \chi_m \) for \( \mu \)-almost every \( x \in Y_0 \) and for every \( u \in U(x) \). But this contradicts the fact that \( \chi(x,u) = \chi_m \) for \( \mu \)-almost every \( x \in X \) and for every \( u \in \mathcal{P}(\mathbb{R}^{d_m}) \). So \( \mu(N_k) = 0 \). This completes Sub-step 1.

**Sub-step 2: End of the proof.**

We repeat Sub-step 1 by descending \( k \) from \( d_m - 1 \) to 1. Finally, we obtain that \( \mu(N_1) = 0 \). So for \( \mu \)-almost every \( x \in X \) and for all \( u \in \mathcal{P}(\mathbb{R}^{d_m}) \), \( \chi(\omega,u) = \chi_m \) for \( W_x \)-almost every \( \omega \). This completes Sub-step 2. The proof of Step 2 of Theorem 7.2 is thereby completed. □

## 8 Lyapunov backward filtrations

This section is devoted to the construction of the Lyapunov backward filtrations associated to a cocycle \( \mathcal{A} \) defined on a lamination \((X, \mathcal{L})\). The word “backward” means that we go to the past, i.e. the time \( n \) tends to \(-\infty\). We will see that some important properties of the forward filtrations also hold for the backward filtrations. However, the corresponding proof in the backward context is much harder since there are many difficulty and difference in comparison with the forward situation.

### 8.1 Extended sample-path spaces

In this subsection \((X, \mathcal{L}, g)\) is a measurable lamination satisfying Hypothesis \((H_1)\). Let \( \mu \) be a weakly harmonic probability measure on \((X, \mathcal{L})\). Recall first that the shift-transformation \( T \) is an endomorphism of the probability space \((\Omega, \mathcal{A}, \hat{\mu})\), where we denote, as usual, \( \Omega := \Omega(X, \mathcal{L}) \), \( \mathcal{A} = \mathcal{A}(\Omega) \) and \( \hat{\mu} \) is the Wiener measure with initial distribution \( \mu \) given by (2.7). Consider the natural extension \((\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mu})\) of this space which is constructed as follows (see [10]). In the sequel this extension is called the extended sample-path space of \((\Omega, \mathcal{A}, \hat{\mu})\).

Each element of \( \hat{\Omega} \) is a continuous path \( \hat{\omega} : \mathbb{R} \to X \) with image fully contained in a single leaf of \((X, \mathcal{L})\). Consider the group \((T^t)_{t \in \mathbb{Z}}\) of shift-transformations \( T^t : \hat{\Omega} \to \hat{\Omega} \) defined for all \( t, s \in \mathbb{Z} \) by

\[
T^t(\hat{\omega})(s) := \hat{\omega}(s + t), \quad \hat{\omega} \in \hat{\Omega}.
\]

Observe that all \( T^t \) are invertible and \((T^t)^{-1} = T^{-t}, t \in \mathbb{R} \). Consider also the canonical restriction \( \hat{\pi} : \hat{\Omega} \to \Omega \) which, to each path \( \hat{\omega} \), associates its restriction
on \([0, \infty)\), that is,
\[
\hat{\pi}(\hat{\omega}) := \hat{\omega}|_{[0, \infty)}, \quad \hat{\omega} \in \hat{\Omega}.
\] (8.1)

For every \(i \in \mathbb{N}\) the \(\sigma\)-algebra \(\hat{\mathcal{A}}_{-i}\) consists of all sets of the form
\[
A = A_{i,C} := \{\hat{\omega} \in \hat{\Omega} : \hat{\pi}(T^i\hat{\omega}) \in C\},
\]
where \(C \in \mathcal{A}\). In other words, \(\hat{\mathcal{A}}_{-i} = (\hat{\pi} \circ T^i)^{-1} \mathcal{A}\). So \((\hat{\mathcal{A}}_{-i})_{i=1}^{\infty}\) is an increasing sequence of \(\sigma\)-algebras on \(X\). Let \(\hat{\mathcal{A}} = \hat{\mathcal{A}}(\Omega)\) denote the \(\sigma\)-algebra generated by the union \(\bigcup_{i=1}^{\infty} \hat{\mathcal{A}}_{-i}\). On each \(\hat{\mathcal{A}}_{-i}\) there is a natural probability measure \(\hat{\mu}\) defined by
\[
\hat{\mu}(A_{i,C}) := \bar{\mu}(C),
\]
for every \(C \in \mathcal{A}\), where \(A_{i,C}\) is defined above. This relation gives a compatible family of finite-dimensional probability distribution which, according to Kolmogorov’s theorem (see [15]), may be extended to a probability measure \(\hat{\mu}\) on the \(\sigma\)-algebra \(\hat{\mathcal{A}}\). We record here a useful property of \(\hat{\mu}\)-measurable sets.

**Lemma 8.1.** Let \(A\) be a subset of \(\hat{\Omega}\). Then \(\hat{\mu}(A) = 0\) if and only if for every \(\epsilon > 0\) there exists an increasing sequence \((A_i)_{i=1}^{\infty} \subset \mathcal{A}\) such that \(A_i \in \hat{\mathcal{A}}_{-i}\) and that \(A \subset \bigcup_{i=1}^{\infty} A_i\) and that \(\hat{\mu}(A_i) < \epsilon\).

**Proof.** Consider the (non \(\sigma\)-) algebra \(\mathcal{I} := \bigcup_{i=1}^{\infty} \hat{\mathcal{A}}_{-i}\). It is clear that \(\hat{\mu}\) is finitely additive on \(\mathcal{I}\). By Theorem 12.1.2 in [15] \(\hat{\mu}\) is countably additive on \(\mathcal{I}\). So we are in the position to apply Part 1) of Proposition 10.3 Consequently, \(\hat{\mu}(A) = 0\) if and only if
\[
\inf \left\{ \sum_{i=1}^{\infty} \hat{\mu}(B_i) : B_i \in \mathcal{I}, \ A \subset \bigcup_{i=1}^{\infty} B_i \right\} = 0.
\]
Letting \(A_i := B_1 \cup \cdots \cup B_i\) and by reorganizing the elements \(A_i\) if necessary, we obtain the desired conclusion. \(\square\)

Since \(\mu\) is harmonic, it follows from Theorem 2.13 that \(\hat{\mu}\) is \(T\)-invariant. Consequently, \(\hat{\mu}\) is also \(T\)-invariant. Finally, since the probability harmonic measure \(\mu\) is ergodic on \((X, \mathcal{L})\) by our assumption, it follows from Theorem 4.6 that \(\hat{\mu}\) is ergodic for \(T\) acting on \((\Omega, \mathcal{A})\). Consequently, we deduce from [10, p. 241] that \(\hat{\mu}\) is also ergodic for \(T\) acting on \((\hat{\Omega}, \hat{\mathcal{A}})\).

Let \((L, g)\) be a complete Riemannian manifold of bounded geometry. Let \(\hat{\Omega}(L)\) be the space of continuous paths \(\omega : \mathbb{R} \to L\). For every \(i \in \mathbb{N}\), the \(\sigma\)-algebra \(\hat{\mathcal{A}}_{-i}(L)\) consists of all sets of the form
\[
A = A_{i,C} := \{\hat{\omega} \in \hat{\Omega}(L) : \hat{\pi}(T^i\hat{\omega}) \in C\},
\]
where $C \in \mathcal{A}(L)$ (see Definition 2.10 for the $\sigma$-algebra $\mathcal{A}(L)$). In other words, $\hat{\mathcal{A}}_n(L) = (\pi \circ T^n)^{-1}(\mathcal{A}(L))$. So $(\hat{\mathcal{A}}_n(L))_{n=1}^\infty$ is an increasing sequence of $\sigma$-algebras on $L$. Let $\hat{\mathcal{A}}(L)$ denote the $\sigma$-algebra on $\hat{\Omega}(L)$ generated by the union $\bigcup_{n=1}^\infty \hat{\mathcal{A}}_n(L)$.

**Definition 8.2.** A sequence $(A_n)_{n=1}^\infty \subset \hat{\mathcal{A}}(L)$ is said to be a nested covering of a set $A \in \hat{\mathcal{A}}(L)$ if $A_n \in \hat{\mathcal{A}}_n(L)$ and $A_n \subset A_{n+1}$ for every $n$, and $A \subset \bigcup_{n=1}^\infty A_n$.

For a set $A \in \hat{\mathcal{A}}_n(L)$ and a point $x \in L$, let $W_x(T^nA)$ denotes the (Wiener) $W_x$-measure of the set $(\pi \circ T^n)A \in \mathcal{A}(L)$. Recall from formula (2.1) that $\{D_t : t \in \mathbb{R}^+\}$ is the semi-group of diffusion operators on $L$. In what follows let Vol be the Lebesgue measure induced by the metric $g$ on $L$.

**Lemma 8.3.** Let $\alpha := (A_n)_{n=1}^\infty$ be a nested covering of a set $A \in \hat{\mathcal{A}}(L)$. Then the sequence of functions $\Theta_n : L \to [0,1]$ given by

$$\Theta_n(x) := D_n(W_x(T^nA_n)), \quad x \in L,$$

satisfies $0 \leq \Theta_n \leq \Theta_{n+1} \leq 1$. So the function

$$\Theta(\alpha)(x) := \lim_{n \to \infty} \Theta_n(x), \quad x \in L,$$

is well-defined.

**Proof.** By Proposition 5.4 applied to the set $T^nA_n \in \mathcal{A}(L)$, we get

$$W_x(T^nA_n) \leq \int_L p(x,y,1)W_y(T^{n+1}A_n)dVol(y) = (D_1W_\bullet(T^{n+1}A_n))(x).$$

Since $A_n \subset A_{n+1}$ and $T^{n+1}A_n$, $T^{n+1}A_{n+1} \in \mathcal{A}(L)$, it follows that

$$(D_1W_\bullet(T^{n+1}A_n))(x) \leq (D_1W_\bullet(T^{n+1}A_{n+1}))(x).$$

This, combined with the previous estimate, implies that

$$W_x(T^nA_n) \leq D_1W_\bullet(T^{n+1}A_{n+1}))(x).$$

Acting $D_n$ on both sides of the last estimate, we get $\Theta_n \leq \Theta_{n+1}$. The remaining estimates $0 \leq \Theta_n \leq 1$ are evident. 

Before going further we discuss briefly the notion of null (resp. positive or full) $W_x$-measure. Let $A \subset \Omega_x$ be a measurable set. Recall that $A$ is of null $W_x$-measure if $W_x(A) = 0$. Clearly $W_x(A) > 0$ if and only if $A$ is of null $W_x$-measure, in this case we also say that $A$ is of positive $W_x$-measure. Moreover, $W_x(A) = 1$ if and only if $\Omega_x \setminus A$ is of null $W_x$-measure. So the notion of null $W_x$-measure is sufficient in order to formulate two other notions (namely, positive
and full \( W_x \)-measure). Furthermore, while studying leafwise Lyapunov exponents using forward orbits, we make full use of the following property which is an immediate consequence of formula (2.7):

If \( A \subset \Omega \) is a subset of null \( \bar{\mu} \)-measure, then for \( \mu \)-almost every \( x \in X \), the set \( A_x := A \cap \Omega_x \) is of null \( W_x \)-measure.

This discussion also shows that if we want to study leafwise Lyapunov exponents using backward orbits, we need to find analogue notions for null, positive or full measure and an analogue property as the above one in the setting of extended paths \( \Omega \). The above discussion also suggests that we should start by finding a good notion for null measure in \( \Omega \). However, the following difficulty arises in the setting of \( \Omega(L) \) for a leaf \( L \). Indeed, we do not have at our hand evident or canonical measures for \( \Omega(L) \) as the Wiener ones for \( \Omega_x \). The following definition get around this obstacle by providing us relevant notions which models on the Wiener measure.

**Definition 8.4.** Let \( A \subset \hat{\mathcal{A}}(L) \).

- \( A \) is said to be of null measure in \( L \) if there exists a sequence of nested coverings \( (\alpha_n)_{n=1}^\infty \) of \( A \) such that \( \lim_{n \to \infty} \Theta(\alpha_n)(x) = 0 \) for Vol\((L)\)-almost every \( x \in L \).
- \( A \) is said to be of positive measure in \( L \) if it is not of null measure in \( L \).
- \( A \) is said to be of full measure in \( L \) if \( \Omega(L) \setminus A \) is of null measure in \( L \).
- We say that a property \( \mathcal{H} \) holds for almost every \( \omega \in \hat{\Omega}(L) \) if there is a set \( A \subset \hat{\Omega}(L) \) of full measure in \( L \) such that \( \mathcal{H} \) holds for every \( \omega \in A \).
- For a given \( 0 < \epsilon \leq 1 \), a nested covering \( \alpha := (A_n)_{n=1}^\infty \) of \( A \) is said to be of size smaller than \( \epsilon \) with respect to a reference point \( x_0 \in L \) if \( (D_t \Theta(\alpha))(x_0) < \epsilon \).

**Remark 8.5.** It follows immediately from the above definition that the intersection of a set of full measure and a set of positive measure in the same leaf is always nonempty.

**Proposition 8.6.** 1) Given \( A \subset \hat{\mathcal{A}}(X) \) a set of null \( \bar{\mu} \)-measure, then for \( \mu \)-almost every \( x \in X \), \( A \cap \hat{\mathcal{A}}(L_x) \) is of null measure in \( L_x \). Equivalently, given \( A \subset \hat{\mathcal{A}}(X) \) a set of full \( \bar{\mu} \)-measure, then for \( \mu \)-almost every \( x \in X \), \( A \cap \hat{\mathcal{A}}(L_x) \) is of full measure in \( L_x \).

2) If \( A \subset \hat{\mathcal{A}}(L) \) is a set of null measure in a leaf \( L \), then for every point \( x \in L \), and for every \( \epsilon > 0 \) there is a nested covering of \( A \) of size smaller than \( \epsilon \) with respect to \( x \). Conversely, if a set \( A \subset \hat{\mathcal{A}}(L) \) admits a point \( x_0 \in L \) (not necessarily belonging to \( A \)) such that for every \( \epsilon > 0 \) there is a nested covering of \( A \) of size smaller than \( \epsilon \) with respect to \( x_0 \), then \( A \) is of null measure in \( L \).

3) Let \( (A_n)_{n=1}^\infty \subset \hat{\mathcal{A}}(L) \) be a sequence of null measure in a leaf \( L \). Then \( \bigcup_{n=1}^\infty A_n \) is also of null measure in \( L \).

**Proof.** First we prove assertion 1). Fix a sequence \( (\epsilon_n) \searrow 0 \) as \( n \nearrow \infty \). By Lemma 8.1 and Definition 8.2 there exists, for every \( n \), a nested covering \( \alpha_n := (A_n)_{i=1}^\infty \)
of $A$ such that $\hat{\mu}(\bigcup_{i=1}^{\infty} A^n_i) < \epsilon_n$. Since $\mu$ is harmonic, it follows from Definition 2.4 that it is $D_l$-invariant for all $i \in \mathbb{R}^+$. So we get that

$$\int_X D_l(W_\bullet(T^i A^n_i))(x)d\mu(x) = \int_X W_\bullet(T^i A^n_i)(x)d\mu(x) = \mu(T^i A^n_i) = \hat{\mu}(T^i A^n_i)$$

because $\hat{\mu}$ is $T$-invariant and $\mu$ is $D_l$-invariant. Since we know from Lemma 8.3 that $D_l(W_\bullet(T^i A^n_i))$ converge pointwise to $\Theta(\alpha_n)$, it follows from the Lebesgue’s dominated convergence that $\int_X \Theta(\alpha_n)(x)d\mu(x) \leq \epsilon_n$. So the sequence $(\Theta(\alpha_n))_{n=1}^{\infty}$ converges in $L^1(X,\mu)$ to 0. By extracting a subsequence if necessary, we may assume without loss of generality that the sequence $(\Theta(\alpha_n))_{n=1}^{\infty}$ converges pointwise to 0 $\mu$-almost everywhere. Hence, for $\mu$-almost every $x \in X$, we have $\lim_{n \to \infty} \Theta(\alpha_n)(y) = 0$ for Vol$_{L_x}$-almost every $y \in L_x$. Clearly, for such a point $x \in X$, the sequence of nested coverings $(\alpha_{n,x})_{n=1}^{\infty}$ of $A \cap \hat{\Omega}(L_x)$ defined by

$$\alpha_{n,x} := (A^n_{i,x})_{i=1}^{\infty}, \quad A^n_{i,x} := A^n_i \cap \hat{\Omega}(L_x),$$

satisfies Definition 8.4. This finishes assertion 1).

Now we turn to the first part of assertion 2). By Definition 8.4 there exists a sequence of nested coverings $(\alpha_{n})_{n=1}^{\infty}$ of $A$ such that $\lim_{n \to \infty} \Theta(\alpha_n)(x) = 0$ for Vol$(L)$-almost every $x \in L$. Since we know from Lemma 8.3 that $0 \leq \Theta(\alpha_n)(x) \leq 1$, applying the Lebesgue’s dominated convergence yields that

$$(D_1 \Theta(\alpha_n))(x_0) = \int_L p(x_0, x, 1) \Theta(\alpha_n)(x)d\text{Vol}(x) \to 0 \quad \text{as } n \to \infty.$$ 

The first part follows. To prove the second part of assertion 2), we find, for every $n \geq 1$, a nested covering $\alpha_n$ of $A$ of size smaller than $\frac{1}{n}$ with respect to $x_0$. So $\lim_{n \to \infty} \int_L p(x_0, y, 1) \Theta(\alpha_n)(y)d\text{Vol}(y) = 0$. By passing to a subsequence if necessary, we see that $\Theta(\alpha_n)$ converges pointwise to 0 Vol$(L)$-almost everywhere in $L$.

To prove assertion 3) set $A := \bigcup_{n=1}^{\infty} A_n$, where $(A_n)_{n=1}^{\infty} \subset \hat{\omega}(L)$ is a sequence of null measure in the leaf $L$. Fix a point $x_0 \in L$ and a real number $\epsilon > 0$. Fix also a sequence $(\epsilon_n)_{n=1}^{\infty} \subset \mathbb{R}^+$ such that $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$. By the first part of assertion 2), for every $n \geq 1$, there exists a nested covering $\alpha_n := (A_i^{n})_{i=1}^{\infty}$ of $A_n$ of size smaller than $\epsilon_n$ with respect to $x_0$. Consider the nested covering $\alpha := (B_i)_{i=1}^{\infty}$ of $A$ given by

$$B_i := \bigcup_{n=1}^{\infty} A_i^{n}.$$ 

To complete the proof of assertion 3) it suffices, thanks to the second part of
assertion 2), to check that \((D_1\Theta(\alpha))(x_0) < \epsilon\). To this end we have

\[
(D_1\Theta(\alpha))(x_0) = \int_L p(x_0, y, 1)\Theta(\alpha)(y) d\text{Vol}(y)
\]

\[
= \lim_{i \to \infty} \int_L p(x_0, y, 1)D_i(W_*(T^i B_i))(y) d\text{Vol}(y)
\]

\[
\leq \lim_{i \to \infty} \int_L p(x_0, y, 1)\sum_{n=1}^{\infty} D_i(W_*(T^i A_n^i))(y) d\text{Vol}(y).
\]

The last line may be written as

\[
\sum_{n=1}^{\infty} \lim_{i \to \infty} \int_L p(x_0, y, 1)D_i(W_*(T^i A_n^i))(y) d\text{Vol}(y)
\]

\[
= \sum_{n=1}^{\infty} \int_L p(x_0, y, 1)\Theta(\alpha_n)(y) d\text{Vol}(y) < \sum_{n=1}^{\infty} \epsilon_n < \epsilon,
\]

which finishes the proof.

The following result will be very useful later on.

**Lemma 8.7.** Let \((L, g)\) be a complete Riemannian manifold of bounded geometry and \(\pi: \tilde{L} \to L\) its universal cover.

1. Let \(\tilde{\mathcal{F}} \subset \hat{\Omega}(\tilde{L})\) be a set of positive measure in \(\tilde{L}\). Then \(\pi \circ \tilde{\mathcal{F}} \subset \hat{\Omega}(L)\) is a set of positive measure in \(L\), where \(\pi \circ \tilde{\mathcal{F}} := \{\pi \circ \tilde{\omega} : \tilde{\omega} \in \tilde{\mathcal{F}}\}\).

2. A set \(\mathcal{F} \subset \hat{\Omega}(L)\) is of full measure in \(L\) if and only if the set \(\pi^{-1} \mathcal{F} \subset \hat{\Omega}(\tilde{L})\) is of full measure in \(\tilde{L}\), where \(\pi^{-1} \mathcal{F} := \{\tilde{\omega} \in \hat{\Omega}(\tilde{L}) : \pi \circ \tilde{\omega} \in \mathcal{F}\}\).

**Proof.** To prove Part 1) suppose in order to reach a contradiction that \(\mathcal{F} := \pi \circ \tilde{\mathcal{F}} \subset \hat{\Omega}(L)\) is a set of null measure in \(L\). By Definition 8.4 there exist a set \(F \subset L\) and a sequence of nested coverings \((\alpha_n)_{n=1}^{\infty}\) of \(\mathcal{F}\) such that \(\text{Vol}(L \setminus F) = 0\) and \(\lim_{n \to \infty} \Theta(\alpha_n)(x) = 0\) for every \(x \in F\).

Write \(\alpha_n = (A_n^i)_{i=1}^{\infty}\). Consider the sequence of nested coverings \((\tilde{\alpha}_n)_{n=1}^{\infty}\) of \(\pi^{-1}(\mathcal{F})\) defined by

\[
\tilde{\alpha}_n := (\tilde{A}_n^i)_{i=1}^{\infty}, \quad \tilde{A}_n^i := \pi^{-1}(A_n^i).
\]

Clearly, \(\tilde{\mathcal{F}} \subset \pi^{-1}\mathcal{F}\). To complete the proof of Part 1) it suffices to show that \(\lim_{n \to \infty} \Theta(\tilde{\alpha}_n)(\tilde{x}) = 0\) for every \(\tilde{x} \in \tilde{F} := \pi^{-1}(F)\). This will follow immediately from the equality

\[
\Theta(\tilde{\alpha}_n)(\tilde{x}) = \Theta(\alpha_n)(x), \quad x \in L, \tilde{x} \in \pi^{-1}(x), \quad (8.2)
\]

and the above mentioned property of \((\alpha_n)_{n=1}^{\infty}\).
To prove (8.2) we start with the following immediate consequence of Lemma 10.35 (i) below

\[ W_x(B) = W_{\hat{x}}(\pi^{-1}B), \quad x \in L, \ \hat{x} \in \pi^{-1}(x), \ B \in \mathcal{A}(L). \]

Using this and the equality \( \pi^{-1}(T^iA_n^i) = T^i(\pi^{-1}A_n^i) \), we get that

\[ W_y(T^iA_n^i) = W_{\hat{y}}(T^iA_n^i), \quad y \in L, \ \hat{y} \in \pi^{-1}(y). \]

This, combined with (2.2), implies that

\[ D_i(W_*(T^iA_n^i))(x) = D_i(W_*(T^i\hat{A}_n^i))(\hat{x}), \quad x \in L, \ \hat{x} \in \pi^{-1}(x). \]

Hence, (8.2) follows.

Next, we turn to the “only if” part of assertion 2). Observe that we only need to show that the set \( \hat{\Omega}(\hat{L}) \setminus \pi^{-1}(\mathcal{F}) \) is of null measure in \( \hat{L} \). Suppose the contrary in order to get a contradiction. Then by Part 1), the set \( \hat{\Omega}(\hat{L}) \setminus \mathcal{F} \) is of positive measure in \( L \), which is impossible since \( \mathcal{F} \) is of full measure in \( L \).

Finally, we establish the “if” part of assertion 2). Since \( \hat{\Omega}(\hat{L}) \setminus \pi^{-1}(\mathcal{F}) \) is of null measure in \( \hat{L} \), we deduce from Definition [8.4] that there exist a set \( \hat{F} \subset \hat{L} \) and a sequence of nested coverings \( (\hat{\alpha}_n)_{n=1}^\infty \) of \( \hat{\Omega}(\hat{L}) \setminus \pi^{-1}\mathcal{F} \) such that \( \text{Vol}(\hat{L} \setminus \hat{F}) = 0 \) and \( \lim_{n \to \infty} \Theta(\hat{\alpha}_n)(\hat{x}) = 0 \) for every \( \hat{x} \in \hat{F} \). Write \( \hat{\alpha}_n := (\hat{A}_n^i)_{i=1}^\infty. \) For a deck-transformation \( \gamma \in \pi_1(L) \) and a set \( \hat{A} \subset \hat{\Omega}(\hat{L}), \ \gamma \circ \hat{A} := \{ \gamma \circ \hat{\omega} : \ \hat{\omega} \in \hat{A} \}. \)

Replacing each \( \hat{A}_n^i \) with its subset \( \gamma \circ \hat{A}_n^i \), we may assume without loss of generality that \( \gamma \circ \hat{A}_n^i = \hat{A}_n^i \) for all \( \gamma \in \pi_1(L) \). So there exists a set \( \tilde{A}_n^i \in \mathcal{A}_{-n}(L) \) such that \( \hat{A}_n^i = \pi^{-1}(\tilde{A}_n^i) \). Consider the following sequence of nested coverings \( \alpha_n = (A_n^i)_{i=1}^\infty \) of \( \Omega(L) \setminus \mathcal{F} \). Let \( \tilde{F} := \cap_{\gamma \in \pi_1(L)} \gamma (\hat{F}), \ \tilde{F}' := \cap_{\gamma \in \pi_1(L)} \gamma (\hat{F}) \subset L. \) So we obtain that \( \pi^{-1}(\tilde{F}) \subset \tilde{F} \) and \( \text{Vol}(L \setminus \tilde{F}) = 0. \) Using (8.2) and the equality

\[ \lim_{n \to \infty} \Theta(\hat{\alpha}_n)(\hat{x}) = 0, \quad \hat{x} \in \hat{F}, \]

we see that \( \lim_{n \to \infty} \Theta(\alpha_n)(x) = 0 \) for every \( x \in F. \) This completes the proof. \( \square \)

### 8.2 Leafwise Lyapunov backward exponents and Oseledec type theorem

Let \( \mathcal{A} : \Omega(X, \mathcal{L}) \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{R}) \) be a cocycle. We extends it to a map (still denoted by) \( \hat{\mathcal{A}} : \hat{\Omega}(X, \mathcal{L}) \times \mathbb{R} \to \text{GL}(d, \mathbb{R}) \) by the following formula

\[
\mathcal{A}(\omega, t) := \begin{cases} 
\mathcal{A}(\hat{\pi} \omega, t), & t \geq 0; \\
\mathcal{A}(\hat{\pi}(T^i \omega), |t|)^{-1}, & t < 0,
\end{cases} \quad (8.3)
\]

where \( \hat{\pi} \) is given by (8.1). It can be checked that the multiplicative law

\[
\mathcal{A}(\omega, s + t) = \mathcal{A}(T^i \omega, s) \mathcal{A}(\omega, t)
\]
still holds for all $s, t \in \mathbb{R}$ and $\omega \in \hat{\Omega}(X, \mathcal{L})$. By the same way we extend a cocycle $A : \Omega(X, \mathcal{L}) \times \mathbb{N} \to \text{GL}(d, \mathbb{R})$ to a cocycle (still denoted by) $A : \hat{\Omega}(X, \mathcal{L}) \times \mathbb{Z} \to \text{GL}(d, \mathbb{R})$.

Let $(L, g)$ be a complete Riemannian manifold of bounded geometry. For any function $f : \hat{\Omega}(L) \to \mathbb{R} \cup \{\pm\}$, let $\text{ess. sup} f$ denote the essential supremum of $f$ given by the following formula

$$\text{ess. sup} f := \inf_{E} \sup_{\omega \in E} f(\omega),$$

the infimum being taken over all elements $E \in \hat{\mathcal{A}}(L)$ that are of full measure in $L$ (see Definition 8.4).

The following result is the counterpart of Lemma 5.1 in the backward setting. 

**Lemma 8.8.** Let $(L, g)$ be a complete Riemannian manifold of bounded geometry and $f : \hat{\Omega}(L) \to \mathbb{R} \cup \{\pm\}$ a measurable function. Then there exists a set $E \in \hat{\mathcal{A}}(L)$ of full measure in $L$ such that

$$\text{ess. sup} f = \sup_{\hat{\omega} \in E} f(\hat{\omega}).$$

In particular, for every subset $Z \subset \hat{\Omega}(L)$ of null measure in $L$,

$$\sup_{\omega \in E} f(\omega) = \sup_{\omega \in E \setminus Z} f(\omega).$$

**Proof.** We proceed as in the proof of Lemma 5.1 using (8.4) and assertion 3) of Proposition 8.6.

Now we will introduce the notion of Lyapunov backward exponents of the cocycle $A$ with respect to a leaf $L$. For $(x, v) \in L \times \mathbb{R}^d$, and $\omega \in \hat{\Omega}(L)$ with $\omega(0) = x$ introduce the quantity

$$\chi_{x,v}^{-}(\omega) := \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, -n)v\|.$$ 

In what follows we want to extend this definition to the case where $\omega(0)$ is not equal to $x$. Having at hands this extension, we will be able to define a function $\chi^- : L \times \mathbb{R}^d \to \mathbb{R} \cup \{-\infty, \infty\}$, which plays the analogue role in the backward setting as the function $\chi : L \times \mathbb{R}^d \to \mathbb{R} \cup \{-\infty, \infty\}$ given by (5.1) do in the forward setting. Consider two cases. 

**Case 1:** $L$ is simply connected. 

For two pairs $(x, v), (y, u) \in L \times \mathbb{R}^d$, we write $(x, v) \overset{A}{\sim} (y, u)$ if there is a path $\omega \in \Omega(L)$ with $\omega(0) = x$, $\omega(1) = y$ and $A(\omega, 1)v = u$. This is an equivalent relation. Consider the function $\chi_{x,v}^{-} : \hat{\Omega}(L) \to \mathbb{R} \cup \{-\infty, \infty\}$ defined by

$$\chi_{x,v}^{-}(\omega) := \limsup_{n \to \infty} \frac{1}{n} \log \|A(\omega, -n)u_{x,v,\omega}\|, \quad \omega \in \hat{\Omega}(L), \quad (8.5)$$
where \( u_{x,v,\omega} \in \mathbb{R}^d \) is uniquely determined by the condition that \((\omega(0), u_{x,v,\omega}) \sim (x,v)\). The uniqueness is an immediate consequence of the simple connectivity of \( L \) and the homotopy law for \( A \).

Using (8.3) and the functions \( \chi_{x,v} \), consider the function \( \chi^- : L \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm \infty\} \) given by

\[
\chi^-(x,v) := \text{ess. sup} \chi^-_{x,v}, \quad (x,v) \in L \times \mathbb{R}^d.
\]

**Case 2:** \( L \) is arbitrary.

Let \( \pi : \tilde{L} \to L \) be the universal cover. We construct a cocycle \( \tilde{A} \) on \( \tilde{L} \) as follows:

\[
\tilde{A}(\tilde{\omega},t) := A(\pi(\tilde{\omega}),t), \quad t \in \mathbb{R}, \; \tilde{\omega} \in \hat{\Omega}(\tilde{L}).
\]  

(8.6)

Since \( \tilde{L} \) is simply connected, we may apply Case 1. More exactly, we can define \( \chi^- : \tilde{L} \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm \infty\} \) by the formula

\[
\chi^-((\tilde{x},v)) := \text{ess. sup} \chi^-_{\tilde{x},v}, \quad ((\tilde{x},v)) \in \tilde{L} \times \mathbb{R}^d.
\]

Here ess. sup is defined by (8.4) and the function \( \chi^-_{\tilde{x},v} : \hat{\Omega}(\tilde{L}) \to \mathbb{R} \cup \{\pm \infty\} \) is given by

\[
\chi^-_{\tilde{x},v}(\tilde{\omega}) := \limsup_{n \to \infty} \frac{1}{n} \log \|\tilde{A}(\tilde{\omega},-n)u_{\tilde{x},v,\tilde{\omega}}\|, \quad \tilde{\omega} \in \hat{\Omega}(\tilde{L}),
\]

where \( u_{\tilde{x},v,\tilde{\omega}} \in \mathbb{R}^d \) is uniquely determined by the condition that \((\tilde{\omega}(0), u_{\tilde{x},v,\tilde{\omega}}) \sim (\tilde{x},v)\), namely, \( u_{\tilde{x},v,\tilde{\omega}} = \tilde{A}(\tilde{\eta},1)v \) for some (and hence every) path \( \tilde{\eta} \in \Omega(\tilde{L}) \) with \( \tilde{\eta}(0) = \tilde{x} \) and \( \tilde{\eta}(1) = \tilde{\omega}(0) \).

For \( x \in L \) and \( u,v \in \mathbb{P}(\mathbb{R}^d) \), we write \( u \sim^A v \) if there is a path \( \omega \in \Omega(\tilde{L}) \) with \( \omega(0), \omega(1) \in \pi^{-1}(x) \) and \( A(\omega,1)u = v \). This is an equivalent relation.

For \( x \in L \) and \( u,v \in \mathbb{P}(\mathbb{R}^d) \), let class\(_x,A(u)\) denote the (at most countable) set of all \( v \in \mathbb{P}(\mathbb{R}^d) \) such that \( u \sim^A v \).

**Definition 8.9.** • A set \( \mathcal{F} \subset \hat{\Omega}(\tilde{L}) \) is said to be invariant under deck-transformations if \( \gamma \circ \mathcal{F} = \mathcal{F} \) for all \( \gamma \in \pi_1(L) \). Clearly, this property is equivalent to the condition \( \mathcal{F} = \pi^{-1}(\mathcal{F}) \) for some set \( \mathcal{F} \subset \hat{\Omega}(L) \).

• A set \( \mathcal{F} \subset \hat{\Omega}(X,L) \) is said to be invariant under deck-transformations if \( \mathcal{F} \cap \hat{\Omega}(\tilde{L}) \) is invariant under deck-transformations for each leaf \( L \) of \((X,L)\). Clearly, this property is equivalent to the condition \( \mathcal{F} = \pi^{-1}(\mathcal{F}) \) for some \( \mathcal{F} \subset \hat{\Omega} \), where \( \pi : (\tilde{X},\tilde{L}) \to (X,L) \) is the covering lamination projection.

• The above two definitions can be adapted in a natural way to the case where \( \mathcal{F} \subset \Omega(\tilde{L}) \) and to the case where \( \mathcal{F} \subset \Omega(X,L) \).
Lemma 8.10. (i) Let $\gamma \in \pi_1(L)$ be a deck-transformation and let $\bar{x}_1, \bar{x}_2 \in \bar{L}$ and $v_1, v_2 \in \mathbb{R}^d$ be such that $\gamma(\bar{x}_1) = \bar{x}_2$ and that $\bar{A}(\gamma, 1)v_1 = v_2$, where $\gamma \in \Omega(\bar{L})$ is a path such that $\bar{\gamma}(0) = \bar{x}_1$ and $\bar{\gamma}(1) = \bar{x}_2$. Then, for every $\bar{x} \in \Omega(\bar{L})$,

$$
\chi_{\bar{x}_1, v_1}(\bar{x}) = \chi_{\bar{x}_2, v_2}(\bar{x}) \quad \text{and} \quad \chi_{\bar{x}, v_1} = \chi_{\bar{x}_2, v_1}(\gamma \bar{x}).
$$

(ii) Suppose now that $\bar{a}, \bar{b} \in \pi^{-1}(x)$ for some $x \in L$ and $u, v \in \mathbb{P}(\mathbb{R}^d)$ such that $u \sim_A v$. Then $\chi^-(\bar{a}, u) = \chi^-(\bar{b}, v)$.

(iii) For $x \in X$ and $u \in \mathbb{R}^d \setminus \{0\}$, there exists a set $\mathcal{F} = \mathcal{F}_{x, u} \subset \Omega(\bar{L})$ which is of full measure in $\bar{L}_x$ and which is invariant under deck-transformations such that for every $\bar{a}, \bar{b} \in \pi^{-1}(x)$, and every $v \in \text{class}_{x, A}(u)$, it holds that

$$
\chi^-(\bar{a}, u) := \sup_{\gamma \in \mathcal{F}} \chi_{\bar{b}, v}(\bar{x}).
$$

Proof. Note that for all $\bar{x} \in \Omega(\bar{L})$, we have that

$$(\bar{x}(0), u_{\bar{x}_1, v_1}) \sim (\bar{x}_2, v_2) \quad \text{and} \quad (\bar{x}(0), u_{\bar{x}_2, v_2}) \sim (\bar{x}_1, v_2).$$

This, combined with $\bar{x}_1, v_1 \sim (\bar{x}_2, v_2)$, implies that $u_{\bar{x}_1, v_1} = u_{\bar{x}_2, v_2}$. Hence, $\chi_{\bar{x}_1, v_1}(\bar{x}) = \chi_{\bar{x}_2, v_2}(\bar{x})$, as asserted. Next, suppose without loss of generality that $\bar{x}(0) = \bar{x}_1$. So $\chi_{\bar{x}_1, v_1}(\bar{x}) = \limsup_{n \to \infty} \frac{1}{n} \log \|\bar{A}(\gamma, -n)v_1\|$, which is also equal to $\limsup_{n \to \infty} \frac{1}{n} \log \|\bar{A}(\gamma, -n)v_1\|$, by the definition of the cocycle $\bar{A}$ in (8.6) and by the identity $\pi \circ \gamma = \pi$ on $\bar{L}$. Hence, $\chi_{\bar{x}_1, v_1}(\bar{x}) = \chi_{\bar{x}_2, v_1}(\gamma \bar{x})$. This proves the last identity of assertion (i).

We turn to the proof of assertion (ii). Since $u \sim_A v$, there exists $\bar{c} \in \pi^{-1}(x)$ such that $v := \bar{A}(\gamma, 1)u$, where $\gamma \in \Omega(\bar{L})$ is a path such that $\gamma(0) = \bar{a}$ and $\gamma(1) = \bar{c}$. We deduce from the first identity of assertion (i) and the definition of $\chi^-(x, v)$ in Case 1 that $\chi^-(\bar{a}, u) = \chi^-(\bar{c}, v)$.

Let $w := \bar{A}(\beta, 1)u$, where $\beta \in \Omega(\bar{L})$ is a path such that $\beta(0) = \bar{a}$ and $\beta(1) = \bar{b}$. Let $\bar{a}$ be a path such that $\bar{a}|_{[0, 1]}$ is the concatenation $\bar{\gamma}|_{[0, 1]} \circ \bar{\beta}|_{[0, 1]}$. Thus, $\bar{a}|_{[0, 1]}$ connects $\bar{c}$ to $\bar{b}$. By Lemma 8.8, let $\bar{E} \subset \Omega(\bar{L})$ be a set of full measure in $\bar{L}$ such that

$$
\chi^-(\bar{c}, v) = \sup_{\bar{x} \in \bar{E}} \chi_{\bar{c}, v}(\bar{x}).
$$

Let $\alpha$ be the deck-transformation sending $\bar{c}$ to $\bar{b}$. Since $\bar{E}$ is of full measure in $\bar{L}$, we can check using Definition 8.4 that $\alpha \circ \bar{E}$ is also of full measure in $\bar{L}$. This, combined with the second identity of assertion (i), implies that $\chi^-(\bar{c}, v) \geq \chi^-(\bar{b}, v)$. Arguing as above for $\alpha^{-1}$, we get that $\chi^-(\bar{c}, v) = \chi^-(\bar{b}, v)$. This, combined with the identity $\chi^-(\bar{a}, u) = \chi^-(\bar{c}, v)$, completes assertion (ii).

Now we prove assertion (iii). Observe by assertion (ii) that we may assume without loss of generality that $\bar{a} = \bar{b}$. By Lemma 8.8, for each $v \in \text{class}_{x, A}(u)$,
there is a set $\mathcal{F}_v \subset \tilde{\Omega}(\tilde{L})$ of full measure in $\tilde{L}_x$ with the following property:

$$
\chi^-(\tilde{a}, u) := \sup_{\mathcal{F}_v} \chi^-_{\tilde{a}, v}(\tilde{\omega}).
$$

Consider the set

$$
\mathcal{F} := \bigcap_{v \in \text{class}_{x,A}(u)} \left( \bigcap_{\gamma \in \pi_1(L)} \gamma \circ \mathcal{F}_v \right).
$$

Clearly, by Part 3) of Proposition 8.6 $\mathcal{F}$ is of full measure in $\tilde{L}_x$ and is invariant under deck-transformations. Using Lemma 8.8 again and the above property of $\mathcal{F}_v$, we see that $\mathcal{F}$ satisfies the conclusion of assertion (iii).}

In view of Lemma 8.10 (ii) we are able to define the function $\chi^- : L \times \mathbb{R}^d \to \mathbb{R} \cup \{\pm \infty\}$ as follows. By convention $\chi^-(x, 0) := -\infty$. For $u \in \mathbb{R}^d \setminus \{0\}$, set

$$
\chi^-(x, u) := \chi^-(\tilde{x}, v), \quad (x, v) \in L \times \mathbb{R}^d,
$$

(8.7)

where $\tilde{x}$ is an arbitrary element in $\pi^{-1}(x)$ and $v$ is an arbitrary element in $\text{class}_{x,A}(u)$.

We record here the properties of $\chi^-$. Some of them are analogous to those of $\chi$ stated in Proposition 5.2.

**Proposition 8.11.**

(i) $\chi^-$ is a measurable function.

(ii) $\chi^-(x, u) = \chi^-(y, v)$ if there exists a path $\omega \in \Omega(L)$ such that $\omega(0) = x$, $\omega(1) = y$ and $A(\omega, 1)u = v$.

(iii) $\chi^-(x, 0) = -\infty$ and $\chi^-(x, v) = \chi^-(x, \lambda v)$ for $x \in X$, $v \in \mathbb{R}^d$, $\lambda \in \mathbb{R} \setminus \{0\}$.

So we can define a function, still denoted by $\chi^-$, defined on $X \times \mathbb{P}(\mathbb{R}^d)$ by

$$
\chi^-(x, [v]) := \chi^-(x, v), \quad x \in X, \ v \in \mathbb{R}^d \setminus \{0\}.
$$

(iv) $\chi^-(x, v_1 + v_2) \leq \max\{\chi^-(x, v_1), \chi^-(x, v_2)\}$, $x \in X$, $v_1, v_2 \in \mathbb{R}^d$.

(v) For all $x \in X$ and $t \in \mathbb{R} \cup \{\pm\}$ the set

$$
V^-(x, t) := \{v \in \mathbb{R}^d : \chi^-(x, v) \leq t\}
$$

is a linear subspace of $\mathbb{R}^d$. Moreover, $s \leq t$ implies $V(x, s) \subset V(x, t)$.

(vi) For every $x \in X$, $\chi^-(x, \cdot) : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ takes only finite $m^-(x)$ different values

$$
\chi^-_{m^-(x)}(x) > \chi^-_{m^-(x) - 1}(x) > \cdots > \chi^-_2(x) > \chi^-_1(x).
$$

(vii) If, for $x \in X$, we define $V^i_-(x)$ to be $V^-(x, \chi^-_i(x))$ for $1 \leq i \leq m^-(x)$, then

$$
\{0\} \equiv V^0_-(x) \subset V^1_-(x) \subset \cdots \subset V^{m^-(x) - 1}_-(x) \subset V^{-m^-(x)}_-(x) \equiv \mathbb{R}^d
$$

and

$$
v \in V^i_-(x) \setminus V^{i-1}_-(x) \Leftrightarrow \sup \limsup_{n \to \infty} \frac{1}{n} \log \|\tilde{A}(\omega, -n)u_{x, \tilde{v}, \tilde{\omega}}\| = \chi^-_i(x)
$$

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for some (and hence every) $\tilde{x} \in \pi^{-1}(x)$ and some (and hence every) $\tilde{v} \in \text{class}_{x,A}(v)$. Here $E_x \in \hat{A}(L_x)$ is a set of full measure in $\hat{L}_x$ which is also invariant under deck-transformations, $E_x$ depends only on the point $x$ (but it does not depend on $v \in \mathbb{R}^d$).

**Proof.** Arguing as in the proof of assertion (i) in Proposition 5.2 and using the construction of $\hat{A}(L)$, the measurability of $\chi^-$ follows. Assertion (ii) is an immediate consequence of Lemma 8.10.

Applying Part 3) of Proposition 8.6 and Lemma 8.8, we proceed as in the proof of assertions (ii)–(v) of Proposition 5.2. Consequently, assertions (iii)–(vi) follow. Arguing as in the proof of assertion (vi) of Proposition 5.2 and applying Lemma 8.10 (iii), assertion (vii) follows.

As a consequence we obtain an analogue of Proposition 5.5 in the backward setting.

**Proposition 8.12.** Let $L$ be a leaf of $(X, \mathcal{L})$ and $A$ a cocycle on $(X, \mathcal{L})$. Then there exist a number $m^- \in \mathbb{N}$ and $m^-$ integers $1 \leq d^-_1 < \cdots < d^-_{m^- - 1} < d^-_{m^-} = d$ and $m^-$ real numbers $\chi^-_1 < \cdots < \chi^-_{m^- - 1} < \chi^-_{m^-}$ with the following properties:

(i) for every $x \in L$ and every $1 \leq j \leq m^-$, we have $m^-(x) = m^-$ and $\chi^-_j(x) = \chi^-_j$ and $\dim V^-_j'(x) = d^-_j$;

(ii) for every $x, y \in L$, and every $\omega \in \Omega_x$ such that $\omega(1) = y$, we have $A(\omega, 1)V^-_i'(x) = V^-_i'(y)$ with $1 \leq i \leq m^-$.

**Proof.** It follows from the definition of $\chi^-$ and assertion (ii) of Proposition 8.11.

**Remark 8.13.** The reader should notice the difference between the conclusion of Proposition 5.5 and that of Proposition 8.12. Indeed, in the former the desired properties only hold $\text{Vol}_L$-almost everywhere, whereas in the latter these properties hold everywhere in $L$. This difference is a consequence of the rather special definition of the function $\chi^-$. Given a leaf $L$ we often write $m^-(L)$ instead of $m^-$ given by Proposition 8.12 in order to emphasize the dependence of $m^-$ on the leaf $L$. The same rule also applies as well to other quantities obtained by this proposition.
Theorem 8.14. We keep the notation introduced by Theorem 7.1. There exists a subset $\Psi$ of $\tilde{\Omega}$ of full $\mu$-measure such that the following properties hold:

(i) For each $\omega \in \Psi$ there are $l$ linear subspaces $H_1(\omega), \ldots, H_l(\omega)$ of $\mathbb{R}^d$ such that

$$V_i(\omega) = \oplus_{i=0}^{l} H_i(\omega),$$

with $A(\omega, 1)H_i(\omega) = H_i(T\omega)$, and $\omega \mapsto H_i(\omega)$ is a measurable map from $\Psi$ into the Grassmannian of $\mathbb{R}^d$.

(ii) For each $\omega \in \Psi$ and $v \in H_i(\omega) \setminus \{0\}$, and for every leafwise saturated Borel set $Y$ of positive measure in $X$, there is a set $F \subset x$ such that for every $\omega$, $\lambda_i = -\lim_{m \to \infty} \frac{1}{n} \log \|A(\omega, -n)\|$ for every $\omega \in \Psi$.

Proof. By Corollary 5.6.2 from Proposition 5.5. Consequently, we infer from Proposition 8.12 the following:

Corollary 8.15. There exist a leafwise saturated Borel set $Y \subset X$ of full $\mu$-measure and a number $m^- \in \mathbb{N}$ and $m^-$ integers $1 \leq d_1^- < d_2^- < \cdots < d_m^- = d$ and $m^-$ real numbers $\chi_1^- < \chi_2^- < \cdots < \chi_m^-$ with the following properties:

(i) $m^-(L_x) = m^-$ for every $x \in Y$;

(ii) the map $Y \ni x \mapsto V_i^-(x)$ is an $A$-invariant sub-bundle of rank $d_i^-$ of $Y \times \mathbb{R}^d$ for $1 \leq i \leq m$;

(iii) for every $x \in Y$ and $1 \leq i \leq m$, $\chi_i^-(L_x) = \chi_i^-$.

Now we are in the position to state the analogue version of Theorem 7.2 in the backward setting.

Theorem 8.16. Under the above hypotheses and notation we have that $m^- \leq l$ and $\{\chi_1^- , \ldots , \chi_m^-\} \subset \{-\lambda_1 , \ldots , -\lambda_l\}$ and $\chi_m^- = -\lambda_l$. Moreover, there exists a leafwise saturated Borel set $Y_0 \subset Y$ of full $\mu$-measure such that for every $x \in Y_0$ and for every $v \in V_i^-(x) \setminus V_{i-1}^-(x)$ with $1 \leq i \leq m^-$, and for every $\bar{x} \in \pi^{-1}(x)$, there is a set $\mathcal{F} = \mathcal{F}_{\bar{x},v} \subset \tilde{\Omega}(\bar{L}_x)$ of positive measure in $\tilde{L}_x$ such that $\chi_{\bar{x},v}(\bar{\omega}) = \chi_i^-$ for every $\bar{\omega} \in \mathcal{F}$.

Proof. By Corollary 8.15 and Theorem 8.14 and Part 1) of Proposition 8.6, there is a leafwise saturated Borel set $Y_0 \subset Y$ of full $\mu$-measure such that for every $x \in Y_0$, the set $\Psi \cap \tilde{\Omega}(\bar{L}_x)$ is of full measure in $\bar{L}_x$. By Lemma 8.10 (iii), we obtain, for each $x \in Y_0$ and $v \in \mathbb{R}^d \setminus \{0\}$, a set $\tilde{E} \subset \tilde{\Omega}(\bar{L}_x)$ which is of full
measure in $\tilde{L}_x$ and which is also invariant under deck-transformations such that for an arbitrary \( \tilde{x} \in \pi^{-1}(x) \),

$$\chi^-(x, v) = \text{ess. sup}_{\tilde{\omega}} \chi^{-}_{\tilde{x},v} (\tilde{\omega}) = \sup_{\tilde{\omega} \in \tilde{E}} \chi^{-}_{\tilde{x},v} (\tilde{\omega}). \quad (8.8)$$

Let \( E := \pi \circ \tilde{E} \subset \hat{\Omega}(L) \). So \( \pi^{-1}(E) = \tilde{E} \). By Lemma 8.7, \( E \) is of full measure in the leaf \( L_x \). Recall from above that \( \Psi \cap \tilde{\Omega}(L) \) is of full measure in \( L_x \). So by Part 1) of Proposition 8.6 and by Part 2) of Lemma 8.7 again, the set \( E \cap \Psi \) is of full measure in the leaf \( L_x \), and the set \( \pi^{-1}(E \cap \Psi) \) is of full measure in the leaf \( \tilde{L}_x \). Hence, by Theorem 8.14 (ii), we have, for \( \tilde{\omega} \in \pi^{-1}(E \cap \Psi) \), that

$$\chi^{-}_{\tilde{x},v} (\tilde{\omega}) = \lim_{n \to \infty} \frac{1}{n} \log \| A(\omega, n) u \| \in \{-\lambda_1, \ldots, -\lambda_l\}. \quad (8.9)$$

Here \( \omega := \pi \circ \tilde{\omega} \) and \( u := u_{\tilde{x},v,\tilde{\omega}} \). This, combined with (8.8) and Corollary 8.15, implies that \( \{\chi_1^{-}, \ldots, \chi_n^{-}\} \subset \{-\lambda_1, \ldots, -\lambda_l\} \). In particular, we get that \( m^- \leq l \) and \( \chi_m^- \leq -\lambda_l \). Therefore, in order to prove that \( \chi_m^- = -\lambda_l \), it suffices to show that \( \chi_m^- \geq -\lambda_l \). By Proposition 8.11 (vii), for every \( x \in Y \), there exists set \( \tilde{E}_x \in \mathcal{A}(\tilde{L}_x) \) which is full measure in \( \tilde{L}_x \) and which is also invariant under deck-transformations such that

$$v \in V_i^{-}(x) \setminus V_i^{-1}(x) \iff \sup_{\tilde{\omega} \in \tilde{E}_x} \lim_{n \to \infty} \frac{1}{n} \log \| \tilde{A}(\tilde{\omega}, -n) u_{\tilde{x},v,\tilde{\omega}} \| = \chi_i^-.$$

Let \( E_x := \pi \circ \tilde{E}_x \). By Lemma 8.7, \( E_x \) is of full measure in \( L_x \). Therefore, by the definition we get, for every \( v \in \mathbb{R}^d \setminus \{0\} \), for every \( x \in Y \) and every \( \omega \in E_x \), that

$$\limsup_{n \to \infty} \frac{1}{n} \log \| A(\omega, -n) u_{\tilde{x},v,\tilde{\omega}} \| = \limsup_{n \to \infty} \frac{1}{n} \log \| \tilde{A}(\tilde{\omega}, -n) u_{\tilde{x},v,\tilde{\omega}} \| \leq \max\{\chi_1^-, \ldots, \chi_n^-\} = \chi_m^-,$$

where \( \tilde{\omega} \) is any path in \( \hat{\Omega}(\tilde{L}) \) such that \( \pi \circ \tilde{\omega} = \omega \). Hence, \( \chi_m^- \geq -\lambda_l \), as desired.

Finally, the existence of a set \( \mathcal{F} = \mathcal{F}_{\tilde{x},v} \subset \hat{\Omega}(\tilde{L}_x) \) with the desired property stated in the theorem is an immediate consequence of combining (8.8) and (8.9) and the equality \( \chi_m^- = -\lambda_l \) and Corollary 8.15. \( \square \)

Finally, we conclude the section with the following backward version of Theorem 6.5.

**Theorem 8.17.** Let \( Y \subset X \) be a set of full \( \mu \)-measure. Assume that \( Y \ni x \mapsto U(x) \) and \( Y \ni x \mapsto V(x) \) are two measurable \( \mathcal{A} \)-invariant sub-bundles of \( Y \times \mathbb{R}^d \) with \( V(x) \subset U(x) \), \( x \in Y \). Define a new measurable sub-bundle \( Y \ni x \mapsto W(x) \) of \( Y \times \mathbb{R}^d \) by splitting \( U(x) = V(x) \oplus W(x) \) so that \( W(x) \) is orthogonal to \( V(x) \) with
respect to the Euclidean inner product of $\mathbb{R}^d$. Using (8.7), we define the cocycle $\tilde{A}$ on $\tilde{\Omega} \times \mathbb{Z}$ in terms of $A$. Let $\alpha, \beta$ be two real numbers with $\alpha < \beta$ such that

- $\chi^-(x, v) \leq \alpha$ for every $x \in Y$, $v \in V(x) \setminus \{0\}$;
- $\chi^-(\tilde{x}, w(\tilde{\omega})) \geq \beta$ for every $x \in Y$, every $\tilde{x} \in \pi^{-1}(x)$, every $w \in W(x)$, and for every $\tilde{\omega} \in \tilde{\mathcal{F}}_{\tilde{x}, w}$. Here $\tilde{\mathcal{F}}_{\tilde{x}, w} \subset \tilde{\Omega}(\tilde{L}_x)$ (depending on $\tilde{x}$ and $w$) is of positive measure in $\tilde{L}_x$, the function $\chi^-(x, v)$ (resp. $\chi^-(\tilde{x}, w(\tilde{\omega}))$) is defined in (8.7) (resp. in (8.2)).

Let $A(\omega, -1)|_{U(\pi_0\omega)} : U(\pi_0\omega) \to U(\pi_{-1}\omega)$ induce the linear maps $C(\omega) : W(\pi_0\omega) \to W(\pi_{-1}\omega)$ and $B(\omega) : W(\pi_0\omega) \to V(\pi_{-1}(\omega))$ by

$$A(\omega, -1)w = B(\omega)w \oplus C(\omega)w, \quad \omega \in \tilde{\Omega}, \ w \in W(x).$$

(i) Then the map $C$ defined on $\tilde{\Omega} \times (-\mathbb{N})$ by the formula

$$C(\omega, -n) := C(T^{-n}\omega) \in \text{Hom}(W(\pi_0\omega), W(\pi_{-n}\omega)), \quad \omega \in \tilde{\Omega}, \ n \in \mathbb{N},$$

satisfies $C(\omega, -(m + k)) = C(T^{-k}\omega, m)C(\omega, -k)$, $m, k \in \mathbb{N}$. Moreover, $C(\omega, -n)$ is invertible.

Using (8.6), we define the cocycle $\tilde{\mathcal{C}}$ on $\tilde{\Omega} \times \mathbb{Z}$ in terms of $\mathcal{C}$. Then there exists a subset $Y'$ of $Y$ of full $\mu$-measure with the following properties:

(ii) For each $x \in Y'$ and for each $\tilde{x} \in \pi^{-1}(x)$ and for each $w \in W(x) \setminus \{0\}$, there exists a set $\tilde{\mathcal{F}}_{\tilde{x}, w} \subset \tilde{\mathcal{F}}_{\tilde{x}, w}$ such that $\tilde{\mathcal{F}}_{\tilde{x}, w}$ is of positive measure in $\tilde{L}_x$ and that for each $v \in V(x)$ and each $\tilde{\omega} \in \tilde{\mathcal{F}}_{\tilde{x}, w}$, we have

$$\chi^-(\tilde{x}, v \oplus w(\tilde{\omega})) = \chi^-(\tilde{x}, w(\tilde{\omega})) = \limsup_{n \to \infty} \frac{1}{n} \log \|\tilde{\mathcal{C}}(\tilde{\omega}, -n)u_{\tilde{x}, w(\tilde{\omega})}\|;$$

(iii) if for some $x \in Y'$ and some $w \in W(x) \setminus \{0\}$ and some $v \in V(x)$ and some $\tilde{\omega} \in \tilde{\mathcal{F}}_{\tilde{x}, w}$ the limit $\lim_{n \to \infty} \frac{1}{n} \log \|\tilde{\mathcal{C}}(\tilde{\omega}, -n)u_{\tilde{x}, v \oplus w(\tilde{\omega})}\|$ exists, then

$$\lim_{n \to \infty} \frac{1}{n} \log \|\tilde{A}(\tilde{\omega}, -n)u_{\tilde{x}, v \oplus w(\tilde{\omega})}\|$$

exists and is equal to the previous limit.

Proof. Using the multiplicative property of the cocycle $A$, we obtain the following formula, which is the backward version of (6.1) in Theorem 6.3 above: for $\omega \in \Omega$ and $n \in \mathbb{N}$,

$$A(\omega, -n)(v \oplus w) = (A(\omega, -n)v + D(\omega, -n)w) \oplus C(\omega, -n)w,$$  \hfill (8.10)

where $D(\omega, -n) : W(\pi_0\omega) \to V(\pi_{-n}\omega)$ is given by

$$D(\omega, -n) := \sum_{i=0}^{n-1} A(T^{-(i+1)}\omega, -(n - i - 1)) \circ B(T^{-i}\omega) \circ C(\omega, -i).$$
Next, observe that Lemma \ref{lem:6.4} and Lemma \ref{lem:6.3} remain valid if we replace $\Omega(X, \mathcal{L})$, $T$ and $\bar{\mu}$ with $\hat{\Omega}(X, \mathcal{L})$, $T^{-1}$ and $\check{\mu}$ respectively. For $\epsilon > 0$ let
\[
a_\epsilon(\omega) := \sup_{n \in \mathbb{N}} \left( \|A(\omega, -n)\|_{V(\pi_0 \omega)} \cdot e^{-n(\alpha + \epsilon)} \right).
\]
By the first assumption $\bullet$, we may apply Lemma \ref{lem:6.4} to $a_\epsilon(\omega)$, and Lemma \ref{lem:6.3} to $h(\omega) := \|A(\omega, -1)\|$, $\omega \in \hat{\Omega}(X, \mathcal{L})$. Let $(\epsilon_m)_{m=1}^\infty$ be a sequence decreasing strictly to 0. By Lemma \ref{lem:6.4} and Lemma \ref{lem:6.3} we may find, for each $m \geq 1$, a subset $\hat{\Omega}_m$ of $\hat{\Omega}(Y)$ of full $\check{\mu}$-measure such that $\frac{1}{n} a_{\epsilon_m}(T^{-n} \omega) \to 0$ and $\frac{1}{n} \log \|A(T^{-n} \omega, 1)\| \to 0$ for all $\omega \in \hat{\Omega}_m$. For every $x \in Y$ set $\hat{\mathcal{F}}_x := \hat{\Omega}(L_x) \cap \bigcap_{m=1}^{\infty} \hat{\Omega}_m \subset \hat{\Omega}(L_x)$. Since $\bigcap_{m=1}^{\infty} \hat{\Omega}_m$ is of full $\check{\mu}$-measure, it follows from Part 1) of Proposition \ref{prop:8.6} that there exists a subset $Y' \subset Y$ of full $\mu$-measure such that for every $x \in Y'$, $\hat{\mathcal{F}}_x$ is of full measure in $L_x$. By the first assumption $\bullet$ combined with Proposition \ref{prop:8.11} (iii)-(iv), for every $x \in Y'$, there exists a set $\hat{\mathcal{F}}_x \subset \hat{\mathcal{F}}_x$ of full measure in $L_x$ such that, for every $\hat{x} \in \pi^{-1}(x)$ and for every $\bar{\omega} \in \pi^{-1}(\hat{x})$,
\[
\chi_{\hat{x},v}(\bar{\omega}) \leq \alpha < \beta, \quad v \in V(x).
\]
(8.11)
By the second assumption $\bullet$, for every $x \in Y'$ and for every $\hat{x} \in \pi^{-1}(x)$ and for every $w \in W(x) \setminus \{0\}$, there exists a set $\hat{\mathcal{F}}_{\hat{x},w} := \hat{\mathcal{F}}_{\hat{x},w} \cap \pi^{-1} \hat{\mathcal{F}}_x \subset \hat{\Omega}(L_x)$ such that, for every $\omega \in \hat{\mathcal{F}}_{\hat{x},w}$,
\[
\alpha < \beta \leq \chi_{\hat{x},w}(\bar{\omega}).
\]
(8.12)
Since $\hat{\mathcal{F}}_x$ is of full measure in $L_x$, we infer from Part 2) of Lemma \ref{lem:8.7} that $\pi^{-1} \hat{\mathcal{F}}_x$ is of full measure in $\tilde{L}_x$. Since $\hat{\mathcal{F}}_{\hat{x},w}$ is the intersection of a set of positive measure and a set of full measure in $\tilde{L}$, we deduce from Remark \ref{rem:8.5} that $\hat{\mathcal{F}}_{\hat{x},w}$ is of positive measure in $\tilde{L}_x$.

Using (8.10)-(8.11)-(8.12) and making the necessary changes (for example, $n$ is replaced with $-n$), we argue as in the proof of Theorem \ref{thm:6.8} from (6.6) to the end of that proof.

\section{Proof of the main results}

In this section we prove the First Main Theorem (Theorem \ref{thm:3.7}) and the Second Main Theorem (Theorem \ref{thm:3.11}) as well as their corollaries. In addition we also give a Ledrappier type characterization of Lyapunov spectrum. The section is organized as follows. In Subsection \ref{subsec:9.1} we introduce some terminology, notation and auxiliary results which will be of constant use later on. Subsection \ref{subsec:9.2} introduces two important techniques. The first one is constructing weakly harmonic measures which maximize (resp. minimize) certain Lyapunov exponent functionals. Using the first technique we develop the second one which aims at splitting invariant sub-bundles (see Theorem \ref{thm:9.12} and Theorem \ref{thm:9.18} below). Having at
hand all needed tools and combining them with the results established in Section 7 and Section 8 above, Subsection 9.3 and Subsection 9.4 are devoted to the proof of the main results of this work.

In what follows, for a linear (real or complex) vector space $V$, we denote by $\mathbb{P}V$ its projectivisation.

### 9.1 Canonical cocycles and specializations

Consider a lamination $(X, \mathcal{L}, g)$ satisfying the Standing Hypotheses endowed with a harmonic probability measure $\mu$ which is ergodic. Set $\Omega := \Omega(X, \mathcal{L})$ and $\hat{\Omega} := \hat{\Omega}(X, \mathcal{L})$. So $\hat{\mu}$ is invariant and ergodic with respect to $T$ acting on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$. We also consider a (multiplicative) cocycle $\mathcal{A} : \Omega \times \mathbb{N} \rightarrow \text{GL}(d, \mathbb{R})$.

In this subsection we consider the cylinder lamination of dimension 1 $(X_A, \mathcal{L}_A) := (X_{1,A}, \mathcal{L}_{1,A})$. We identify $\text{Gr}_d(\mathbb{R}^d)$ with $\mathbb{P}(\mathbb{R}^d)$ and write $P := \mathbb{P}(\mathbb{R}^d)$. So $X_\mathcal{A} \equiv X \times P$ and we will write $(X \times P, \mathcal{L}_A)$ instead of $(X_A, \mathcal{L}_A)$. Let $\Omega := \Omega(X, \mathcal{L})$, $\hat{\Omega} := \hat{\Omega}(X, \mathcal{L})$, $\hat{\Omega}_A := \hat{\Omega}(X_A, \mathcal{L}_A)$ and $\hat{\Omega}_A := \hat{\Omega}(X_A, \mathcal{L}_A)$. Let $\pi : (\hat{X}, \mathcal{L}) \rightarrow (X, \mathcal{L})$ be the covering lamination projection. We have the following natural identifications. Let $\hat{\mathcal{O}} := \Omega((\hat{X}, \mathcal{L}))$.

**Lemma 9.1.** 1) The transformation $\Omega_A \rightarrow \Omega(X, \mathcal{L}) \times P$ which maps $\eta$ to $(\omega, u(0))$, where $\eta(t) = (\omega(t), u(t)), t \in [0, \infty)$, is bijective.

2) The transformation $\hat{\Omega}_A = \hat{\Omega}(X, \mathcal{L}) \times P$ which maps $\hat{\eta}$ to $(\hat{\omega}, u(0))$, where $\hat{\eta}(t) = (\hat{\omega}(t), u(t)), t \in [0, \infty)$, is bijective.

**Proof.** The first part is Lemma 7.9 in the special case when $k = 1$.

Part 2) can be proved in exactly the same way as Part 1). □

Using Lemma 9.1 we construct the canonical cocycle of dimension 1 on $(X \times P, \mathcal{L}_A)$ as follows. For $(\omega, u) \in \hat{\Omega} \times P$ and $t \in \mathbb{R}^+$, let

$$C_A((\omega, u), t) := \|A(\omega, t)u\|, \quad u \in P,$$

where the right hand side is given by

$$\|A(\omega, t)u\| := \frac{\|A(\omega, t)\tilde{u}\|}{\|\tilde{u}\|}, \quad \tilde{u} \in \mathbb{R}^d \setminus \{0\}, \quad u = [\tilde{u}],$$

with $[\cdot] : \mathbb{R}^d \setminus \{0\} \rightarrow P$ the canonical projection. Since $\mathcal{A}$ is a cocycle, the above definition implies that $\log C_A$ is an additive cocycle, that is,

$$\log C_A((\omega, u), n+m) = \log C_A(T^n(\omega, u), m) + \log C_A((\omega, u), n), \quad (\omega, u) \in \hat{\Omega}_A, \quad n \in \mathbb{Z}.$$

Given a point $x \in X$, let $\pi : \tilde{L} \rightarrow L = L_x$ be the universal cover of the leaf $L_x$ and let $\tilde{x} \in \tilde{L}$ be such that $\pi(\tilde{x}) = x$. We construct a cocycle $\tilde{A}$ on $\tilde{L}$ as follows:

$$\tilde{A}(\tilde{\omega}, t) := A(\pi(\tilde{\omega}), t), \quad t \in \mathbb{R}, \quad \tilde{\omega} \in \Omega(\tilde{L}).$$
Given an element $u \in \mathbb{P}(\mathbb{R}^d)$, the \textit{specialization} of $A$ at $(\bar{L}, \bar{x}; u)$ is the function $f = f_{u,\bar{x}} : \bar{L} \to \mathbb{R}$ defined by

$$f_{u,\bar{x}}(\bar{y}) := \log \|\bar{A}(\bar{\omega}, 1)u\|, \quad \bar{y} \in \bar{L},$$

where $\bar{\omega} \in \bar{\Omega}_{\bar{x}}$ is any path such that $\bar{\omega}(1) = \bar{y}$. This definition is well-defined because of the homotopy law for $A$ and of the simple connectivity of $\bar{L}$.

Let $\bar{z} \in \bar{L}$. Let $v \in P$ such that $(\bar{x}, u) \bar{A} (\bar{z}, v)$, i.e., $v = \bar{A}(\bar{\omega}, 1)u$, where $\bar{\omega} \in \bar{\Omega}_{\bar{x}}$ is any path such that $\bar{\omega}(1) = \bar{z}$. Let $\bar{\eta} \in \bar{\Omega}_{\bar{x}}$ be such that $\bar{\eta}(1) = \bar{y}$. We concatenate $\bar{\omega}|_{[0,1]}$ and $\bar{\eta}$ in order to obtain a path

$$\bar{\xi}(t) := \begin{cases} 
\bar{\omega}(2t), & 0 \leq t \leq 1/2; \\
\bar{\eta}(2t - 1), & t \geq 1/2.
\end{cases}$$

Note that $\bar{\xi} \in \bar{\Omega}_{\bar{x}}$ and $\bar{\xi}(1) = \bar{y}$. Therefore, using the multiplicative law and homotopy law for $A$, we see that

$$f_{v,\bar{z}}(\bar{y}) = \log \|\bar{A}(\bar{\eta}, 1)v\| = \log \|\bar{A}(\bar{\bar{\xi}}, 1)u\| - \log \|\bar{A}(\bar{\bar{\omega}}, 1)u\|$$

$$= f_{u,\bar{x}}(\bar{y}) - f_{u,\bar{x}}(\bar{z}).$$

Using the homotopy law for $\bar{A}$ and using the simple connectivity of $\bar{L}$, we see that

$$f_{u,\bar{x}}(t\bar{\omega}(\bar{t})) = \log \|\bar{A}(\bar{\bar{\omega}}, t)u\|, \quad \bar{\omega} \in \bar{\Omega}_{\bar{x}}(\bar{L}), \; t \in \mathbb{R}^+.$$  

This, combined with the identity $E_x[f \circ \pi_t(\omega)] = D_t f(\bar{x})$, implies that

$$E_x[\log \|A(\cdot, t)u\|] = E_x[\log \|\bar{A}(\cdot, t)u\|] = (D_t f_{u,\bar{x}})(\bar{x}) = (D_t f_{v,\bar{z}})(\bar{x}) - f_{v,\bar{z}}(\bar{x}),$$

where the last equality follows from (9.4).

Now we compare the specializations with the function $f_{v,y_0}$ constructed in (3.4). Recall that we fix an arbitrary point $y_0 \in L$ and an arbitrary point $\bar{y}_0 \in \pi^{-1}(y_0)$. Let $v := A(\omega, 1)u$, where $\omega \in \Omega_{\bar{x}}$ is any path such that $(\pi^{-1}_x\omega)(1) = \bar{y}_0$. Let $y$ be an arbitrary point in a simply connected, connected open neighborhood of $y_0$. On this neighborhood a branch of $\pi^{-1}$ such that $\pi^{-1}(y_0) = \bar{y}_0$ is well-defined. Set $\bar{y} := \pi^{-1}(y)$. We see that the function $f_{v,y_0}$ constructed in (3.4) satisfies

$$f_{v,y_0}(y) = f_{v,\bar{y}_0}(\bar{y}).$$

### 9.2 Weakly harmonic measures and splitting invariant sub-bundles

We recall from [35] some results about dual spaces. Let $(X, \mathcal{B}(X), \mu)$ be a probability Borel space. Let $E$ be a separable Banach space with dual space $E^*$. 

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Let \( L_\mu^1(E) \) be the space of all \( \mu \)-measurable functions \( f : X \to E \) such that
\[
\|f\| := \int_X \|f(x)\|d\mu(x) < \infty.
\]
This is a Banach space with the norm \( f \mapsto \|f\| \), where two functions \( f \) and \( g \) are identified if \( f = g \) for \( \mu \)-almost everywhere. Let \( L_\mu^\infty(E^*, E) \) be the space of all maps \( f : X \to E^* \) for which the function \( X \ni x \mapsto f_x(v) \) is bounded and measurable for each \( v \in E \), where two such functions \( f, g \) are identified if \( X \ni x \mapsto f_x(v) \) and \( X \ni x \mapsto g_x(v) \) are equal \( \mu \)-almost everywhere for every \( v \in E \). This is a Banach space with the norm
\[
\|f\|_\infty := \text{ess. sup}_{x \in X} \|f_x\| = \inf_{\gamma \in \mathcal{M}(X)} \sup_{x \in Y} \|f_x\|,
\]
which is finite by the principle of uniform boundedness. Consider the map \( \Lambda : L_\mu^\infty(E^*, E) \to (L_\mu^1(E))^* \), given by
\[
(\Lambda \gamma)(f) := \int_X \gamma_x(f_x)d\mu(x),
\]
where \( \gamma : X \to E^* \) which maps \( x \mapsto \gamma_x \) is in \( L_\mu^\infty(E^*, E) \), and \( f : X \to E \) which maps \( x \mapsto f_x \) is in \( L_\mu^1(E) \). By \([1]\) \( \Lambda \) is an isomorphism of Banach spaces. In what follows, for a locally compact metric space \( \Sigma \), we denote by \( \mathcal{M}(\Sigma) \) the space of all Radon measures on \( \Sigma \) with mass \( \leq 1 \).

We will be interested in the case where \( E := \mathcal{C}(P, \mathbb{R}) \) for a compact metric space \( P \). So \( \mathcal{M}(P) \) is the closed unit ball of \( E^* \). The set \( L_\mu^\infty(\mathcal{M}(P)) \) of all measurable maps \( \alpha : X \to \mathcal{M}(P) \) is contained in the unit ball of \( L_\mu^\infty(E^*, E) \), and is closed with respect to the weak-star topology \( L_\mu^\infty(E^*, E) \). Hence, \( L_\mu^\infty(\mathcal{M}(P)) \) is compact with respect to this topology. The set \( L_\mu^\infty(\mathcal{M}(P)) \) can be identified with
\[
\mathcal{M}_\mu(X \times P) := \{ \lambda \in \mathcal{M}(X \times P) : \lambda \text{ projects to } m \text{ on } X \}.
\]
via the map \( L_\mu^\infty(\mathcal{M}(P)) \ni \nu \mapsto \lambda \in \mathcal{M}(X \times P) \), where for \( X \ni x \mapsto f_x \) in \( L_\mu^1(\mathcal{C}(P, \mathbb{R})) \), we have
\[
\int_{X \times P} f_x(u)d\lambda(x,u) = \int_X \left( \int_P f_x(u)d\nu_x(u) \right)d\mu(x). \tag{9.7}
\]

In the remaining part of the section let \((X, \mathcal{L}, g)\) be a Riemannian satisfying the Standing Hypotheses, and let \( P := \mathbb{P}(\mathbb{R}^d) \). For each cocycle \( \mathcal{A} : \Omega(X, \mathcal{L}) \times \mathbb{G} \to \text{GL}(d, \mathbb{R}) \), we consider its cylinder lamination of dimension 1, denoted by \((X_{\mathcal{A}}, \mathcal{L}_{\mathcal{A}})\), which is given by
\[
(X_{\mathcal{A}}, \mathcal{L}_{\mathcal{A}}) := (X \times \text{Gr}_1(\mathbb{R}^d), \mathcal{L}_{1, \mathcal{A}}),
\]
where the lamination on the right hand side is given by Definition 7.8. Using the identification \( \text{Gr}_1(\mathbb{R}^d) = \mathbb{P}(\mathbb{R}^d) = P \), we may write \( X_{\mathcal{A}} = X \times P \).
Definition 9.2. Let $\mathcal{A}$ be a cocycle. A positive finite Borel measure $\nu$ on $X \times P$ is said to be $\mathcal{A}$-weakly harmonic if and only if $\int_{X\times P} D_1 f d\nu = \int_{X\times P} f d\nu$ for all bounded measurable functions $f$ defined on $X \times P$.

Let $\mu$ be a harmonic probability measure on $X$. Denote by $\text{Har}_\mu(X \times P)$ the convex closed cone of all elements $\nu$ in $L_\mu(M(P))$ which, under the identification (9.7), are $\mathcal{A}$-weakly harmonic positive finite measures on $X \times P$.

When the cocycle $\mathcal{A}$ is clear from the context, we often write “weakly harmonic” (resp. $\text{Har}_\mu(X \times P)$) instead of “$\mathcal{A}$-weakly harmonic” (resp. $\text{Har}_\mu(X \times P)$).

An element $\nu \in \text{Har}_\mu(X \times P)$ is said to be extremal if it is an extremal point of this convex closed cone, that is, if $\nu = t\nu_1 + (1 - t)\nu_2$ for some $0 < t < 1$ and $\nu_1, \nu_2 \in \text{Har}_\mu(X \times P)$, then $\nu_1$ and $\nu_2$ are constants times of $\nu$. Clearly, the set of extremal points of $\text{Har}_\mu(X \times P)$ which are also probability measures is always nonempty.

Remark 9.3. Observe that the cylinder lamination $(X, L_{\mathcal{A}})$ is a measurable lamination. So we can speak of weakly harmonic measures in the sense of Definition 2.4 on $(X, L_{\mathcal{A}})$. In fact, this notion is stronger than the notion of $\mathcal{A}$-weakly harmonic measures since the former requires the invariance of the heat diffusions for all $t \geq 0$, whereas the latter demands this invariance for only $t = 1$.

Proposition 9.4. Let $\nu \in \text{Har}_\mu(X \times P)$ be an extremal element. Then $\nu$ is ergodic. In particular, there exists an element of $\text{Har}_\mu(X \times P)$ which is an ergodic probability measure.

Proof. Suppose in order to reach a contradiction that $\nu$ is not ergodic and $\nu(X \times P) = 1$. Then there is a leafwise saturated Borel set $Y \subset X \times P$ with $0 < \nu(Y) < 1$. Consider two probability measures

$$
\nu_1 := \frac{1}{\nu(Y)} \nu|_Y \quad \text{and} \quad \nu_2 := \frac{1}{1 - \nu(Y)} \nu|(X \times P) \setminus Y.
$$

Clearly, $\nu = \nu(Y)\nu_1 + (1 - \nu(Y))\nu_2$. Moreover, using Definition 9.2 and the assumption that $Y$ is leafwise saturated Borel set, we can show that both $\nu_1$ and $\nu_2$ belong to $\text{Har}_\mu(X \times P)$. Hence, $\nu$ is not extremal, which is the desired contradiction. \qed

From now on we fix a harmonic probability measure $\mu$ which is ergodic and assume the integrability condition $\int_{\Omega(X, \mathcal{A})} \log^+ \|\mathcal{A}(\omega, 1)\| d\mu(\omega) < \infty$. Consider the functions $\varphi$ and $\varphi_n : X \times P \to \mathbb{R}$ given by

$$
\varphi(x, u) := \int_{\Omega_x} \log \|\mathcal{A}(\omega, 1)u\| dW_x(\omega),
$$

$$
\varphi_n := \frac{1}{n} \sum_{i=0}^{n-1} D_i \varphi. \quad (9.8)
$$
Lemma 9.5. For every $t \geq 0$, the operators $D_t : L_\mu(\mathcal{M}(P)) \to L_\mu(\mathcal{M}(P))$ and $D_t : L^1_\mu(\mathcal{C}(P, \mathbb{R})) \to L^1_\mu(\mathcal{C}(P, \mathbb{R}))$ are contractions.

Proof. Let $E := \mathcal{C}(P, \mathbb{R})$. Let $\nu \in L_\mu(\mathcal{M}(P))$ and $x \in X$ and $t \geq 0$. For any positive measure $\lambda$ on $P$ let $||\lambda||$ denotes its mass. Then, for $\mu$-almost every $x \in X$, we have that

$$||D_t \nu||_x = \int_{L_x} p(x, y, t) d\nu(y) \leq \int_{L_x} p(x, y, t) ||\nu||_\infty d\nu(y) = ||\nu||_\infty$$

Hence, $||D_t \nu||_\infty \leq ||\nu||_\infty$.

Now we turn to the second assertion. For $\psi \in L^1_\mu(E)$ and $x \in X$, we have that

$$||D_t \psi(x)||_E = \int_{L_x} p(x, y, t) ||\psi(y)||_E d\nu(y).$$

Integrating both sides with respect to $\mu$ over $X$, we get that

$$||D_t \psi||_{L^1_\mu(E)} = \int_X D_t(||\psi(x)||_E) d\mu(x) = \int_X ||\psi(x)||_E d\mu(x) = ||\psi||_{L^1_\mu(E)} < \infty,$$

where $||\psi(x)||_E$ is the function $X \ni x \mapsto ||\psi(x)||_E$, and where the first equality holds since $\mu$ is harmonic. Hence, to complete the second assertion, it suffices to show that given each $t \geq 0$ and $\psi \in L^1_\mu(E)$, we have $(D_t \psi)(x) \in E$ for $\mu$-almost every $x \in X$. To do this observe from the last argument that for $\mu$-almost every $x \in X$, we have that $(D_t(||\psi(x)||_E))(x) < \infty$. Fix such a point $x$ and let $\pi : \tilde{L} \to L = L_x$ be the universal cover, and fix a point $\tilde{x}$ that projects to $x$. We are reduced to the following problem:

Let $\psi : \tilde{L} \times P \to \mathbb{R}$ be a measurable function such that

- $\psi(\tilde{y}, \cdot)$ is continuous on $P$ for every $\tilde{y} \in \tilde{L}$;
- $\int_{\tilde{L}} \max_P |\psi(\tilde{y}, \cdot)| < \infty$.

Then the function $P \ni u \mapsto \int_{\tilde{L}} p(\tilde{x}, \tilde{y}, t)\psi(\tilde{y}, u)$ is continuous.

Since the conclusion of the problem follows easily from the Lebesgue’s dominated convergence theorem, the proof is complete.

We obtain the following ergodic property of the canonical cocycle $C_A$ of a cocycle $A$.

Theorem 9.6. Let $\nu$ be an element of $\text{Har}_\mu(X \times P)$ and let $\alpha_0 := \int_{X \times P} \varphi d\nu$. Then there exists a leafwise constant measurable function $\alpha : X_A = X \times P \to \mathbb{R}$ with the following properties:

(i) $\lim_{n \to \infty} \frac{1}{n} \log C_A((\omega, u), n) = \alpha(\omega)$ for $\nu$-almost every $(x, u) \in X \times P$ and for $W_x$-almost every $\omega \in \Omega_x$ (or equivalently, $\lim_{n \to \infty} \frac{1}{n} \log C_A((\omega, u), n) = \alpha(\omega(0))$ for $\nu$-almost every $(\omega, u) \in \Omega_A$);
(ii) \( \lim_{n \to \infty} \frac{1}{n} \log C_A((\hat{\omega}, u), -n) = -\alpha(\hat{\omega}(0)) \) for \( \nu \)-almost every \((\hat{\omega}, u) \in \hat{\Omega}_A\);

(iii) \[ \lim_{n \to \infty} \int_{\hat{\Omega}_A} \frac{1}{n} \log C_A((\hat{\omega}, u), n) d\hat{\nu}(\hat{\omega}, u) = \lim_{n \to \infty} \int_{\hat{\Omega}_A} \frac{1}{n} \log C_A((\omega, u), n) d\hat{\nu}(\omega, u) = \alpha_0 \]

and \( \lim_{n \to \infty} \int_{\hat{\Omega}_A} \frac{1}{n} \log C_A((\hat{\omega}, u), -n) d\hat{\nu}(\hat{\omega}, u) = -\alpha_0. \)

**Proof.** First we consider the case when \( \Omega_A \to \mathbb{R} \) given by \( f(\omega, u) := \log C_A((\omega, u), 1), (\omega, u) \in \Omega_A \) (resp. the function \( f : \hat{\Omega}_A :\to \mathbb{R} \) given by the function \( f(\hat{\omega}, u) := \log C_A((\hat{\omega}, u), 1), (\hat{\omega}, u) \in \hat{\Omega}_A \)).

Observe that for \( n \in \mathbb{N} \) and \( \omega \in \Omega_A \) (resp. for \( n \in \mathbb{Z} \) and \( \hat{\omega} \in \hat{\Omega}_A \)),

\[
\log C_A(\omega, n) = \sum_{i=0}^{n-1} f(T^i(\omega)) \quad \text{(resp. } \log C_A(\hat{\omega}, n) = \sum_{i=0}^{n-1} f(T^i(\hat{\omega})) \text{)}.
\]

Consequently, by Corollary [10.37] we get a real number \( \alpha \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \log C_A((\omega, u), n) = \alpha \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log C_A((\hat{\omega}, u), n) = \alpha
\]

for \( \nu \)-almost every \((\omega, u) \in \Omega_A \) and for \( \nu \)-almost every \((\hat{\omega}, u) \in \hat{\Omega}_A \). Moreover, by Birkhoff’s ergodic theorem,

\[
\lim_{n \to \infty} \int_{\Omega_A} \frac{1}{n} \log C_A((\omega, u), n) d\nu(\omega, u) = \lim_{n \to \infty} \int_{\hat{\Omega}_A} \frac{1}{n} \log C_A((\hat{\omega}, u), n) d\hat{\nu}(\hat{\omega}, u) = \alpha.
\]

Consider the function \( g : \hat{\Omega}_A :\to \mathbb{R} \) given by the function \( g(\hat{\omega}, u) := \log C_A((\hat{\omega}, u), -1), (\hat{\omega}, u) \in \hat{\Omega}_A \). Observe that

\[
\log C_A(\cdot, -n) = \sum_{i=0}^{n-1} g(T^{-i}(\cdot)), \quad n \in \mathbb{N}.
\]

Consequently, by Corollary [10.37] we get a real number \( \beta \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \log C_A((\hat{\omega}, u), -n) = \beta
\]

for \( \nu \)-almost every \((\hat{\omega}, u) \in \hat{\Omega}_A \) and that

\[
\lim_{n \to \infty} \int_{\hat{\Omega}_A} \frac{1}{n} \log C_A((\hat{\omega}, u), -n) d\hat{\nu}(\hat{\omega}, u) = -\beta.
\]

On the other hand, since \( \nu \) is \( T \)-invariant, we also get that

\[
-\beta = \lim_{n \to \infty} \int_{\hat{\Omega}_A} \frac{1}{n} \log C_A(T^n(\hat{\omega}, u), -n) d\hat{\nu}(\hat{\omega}, u) = -\lim_{n \to \infty} \int_{\hat{\Omega}_A} \frac{1}{n} \log C_A((\hat{\omega}, u), n) d\hat{\nu}(\hat{\omega}, u) = -\alpha,
\]

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where the second equality follows from the identity $C_A(T^n(\omega, u), -n)C_A((\omega, u), n) = C_A((\omega, u), 0) = 1$. This implies that $\alpha = \beta$.

Now we consider the general case where $\nu$ is not necessarily ergodic. It suffices to apply Choquet decomposition theorem in order to decompose $\nu$ into extremal measures. By Proposition 9.4, these measures are ergodic. Therefore, applying the previous case to each component measure of this decomposition and combining the obtained results, the theorem follows. \hfill $\square$

**Lemma 9.7.** $\varphi \in L^1_\mu(\mathcal{E}(P, \mathbb{R}))$, where $\varphi$ is given by (9.8).

**Proof.** Recall from (9.8) that

$$\varphi(x, u) = \int_{\Omega_x} \log \|A(\omega, 1)u\|dW_x(\omega), \quad (x, u) \in X \times P.$$  

By the integrability condition, for $\mu$-almost every $x \in X$, $\int_{\Omega_x} \log \|A(\omega, 1)\|dW_x(\omega) < \infty$. Putting this together with the continuity of each map $P \in u \mapsto \log \|A(\omega, 1)u\|$, we may apply the Lebesgue’s dominated convergence theorem. Consequently, $\varphi(x, \cdot)$ is continuous on $P$ for such a point $x$, and

$$\|\varphi\|_{L^1_\mu(\mathcal{E}(P, \mathbb{R}))} \leq \int_{x \in X} \left( \int_{\Omega_x} \log \|A(\omega, 1)\|dW_x(\omega) \right) d\mu(x) < \infty,$$

where the last inequality holds by the integrability condition. This completes the proof. \hfill $\square$

**Lemma 9.8.** For every $n \geq 1$,

$$\varphi_n(x, u) = \frac{1}{n} \int_{\Omega_x} \log \|A(\omega, n)u\|dW_x(\omega), \quad (x, u) \in X \times P,$$

where $\varphi_n$ is given by (9.8).

**Proof.** Fix an arbitrary point $x_0 \in X$ and an arbitrary element $u \in P$. Let $\pi : \tilde{L} \to L = L_{x_0}$ be the universal cover, and fix a point $\tilde{x}_0 \in \tilde{L}$ such that $\pi(\tilde{x}_0) = x_0$. Let $f = f_{u, \tilde{x}_0}$ be the specialization of $A$ at $(\tilde{L}, \tilde{x}_0; u)$ given by (9.3). For every $n \geq 1$ let

$$\psi_n(\tilde{x}) = \frac{1}{n} \int_{\Omega_{\tilde{x}}} \log \|\tilde{A}(\tilde{\omega}, n)u_{\tilde{x}}\|dW_{\tilde{x}}(\tilde{\omega}), \quad \tilde{x} \in \tilde{L},$$

where $u_{\tilde{x}} \in P$ is determined by $(\tilde{x}, u_{\tilde{x}}) \xrightarrow{\mathcal{A}} (\tilde{x}_0, u)$. We only need to show that $\psi_n(\tilde{x}_0) = \frac{1}{n} \sum_{i=0}^{n-1}(D_i\psi_1)(\tilde{x}_0)$. To do this recall from (9.5) that

$$\psi_n(\tilde{x}) = E_{\tilde{x}} \left[ \log \|\tilde{A}(\cdot, n)u_{\tilde{x}}\| \right] = D_n f(\tilde{x}) - f(\tilde{x}).$$

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We deduce from the above identity that

\[ \psi_n(\bar{x}) = (D_n f)(\bar{x}) - f(\bar{x}) = \frac{1}{n} \sum_{i=0}^{n-1} D_i (D f - f)(x_0) = \frac{1}{n} \sum_{i=0}^{n-1} (D_i \psi_1)(\bar{x}), \]

as desired. \qed

**Lemma 9.9.** Let \((\nu_n)_{n=1}^{\infty} \subset L_\mu(\mathcal{M}(P)).\)

1) Then there is a subsequence \((\nu_{n_j})_{j=1}^{\infty}\) such that \(\frac{1}{n_j} \sum_{i=0}^{n_j-1} D_i \nu_{n_j}\) converges weakly to a measure \(\nu \in \text{Har}_\mu(X \times P).\) In particular, there is always a probability measure which belongs to \(\text{Har}_\mu(X \times P).\)

2) Suppose in addition that for each \(n,\) for \(\mu\)-almost every \(x \in X,\) \((\nu_n)_x\) is a Dirac mass at some point \(u_n(x) \in P.\) Then the above sequence \((n_j)_{j=1}^{\infty}\) satisfies

\[ \lim_{j \to \infty} \int_X \varphi_{n_j}(x, u_{n_j}(x)) d\mu(x) = \int_{X \times P} \varphi d\nu, \]

where \(\varphi_n\) and \(\varphi\) are given by (9.8).

**Proof.** Let \(\nu^n := \frac{1}{n} \sum_{i=0}^{n-1} D_i \nu_n, \) \(n \geq 1.\) By Lemma 9.5, \(\nu^n \in L_\mu(\mathcal{M}(P)).\) The sequence \((\nu^n)_{n=1}^{\infty}\) has a convergent sequence in the weak star topology. Therefore, there is \(n_j \nearrow \infty\) and \(\nu \in L_\mu(\mathcal{M}(P))\) such that

\[ \int \psi d\nu^{n_j} \to \int \psi d\nu, \quad \forall \psi \in L_\mu^1(\mathcal{C}(P, \mathbb{R})). \]

To prove that \(\nu\) is \(\mathcal{A}\)-weakly harmonic we need to show that \(\int D\psi d\nu = \int \psi d\nu.\) Since \(\psi \in L_\mu^1(\mathcal{C}(P, \mathbb{R}))\) it follows from Lemma 9.5 that \(D_1 \psi \in L_\mu^1(\mathcal{C}(P, \mathbb{R})).\)

Therefore, applying the last limit to both \(\psi\) and \(D\psi,\) we need to show that

\[ \int \psi d\nu^{n_j} - \int \psi d\nu \to 0, \quad \forall \psi \in L_\mu^1(\mathcal{C}(P, \mathbb{R})). \]

Observe that

\[ \int \psi d\nu^{n_j} - \int \psi d\nu^{n_j} = \frac{1}{n_j} \sum_{i=0}^{n_j} \int \psi d(D_i \nu_{n_j}) - \frac{1}{n_j} \sum_{i=0}^{n_j-1} \int \psi d(D_i \nu_{n_j}) \]

\[ = \frac{1}{n_j} \int \psi d(D_{n_j} \nu_{n_j}) - \frac{1}{n_j} \int \psi d\nu_{n_j}. \]

By Lemma 9.5 both terms in the last line tends to 0 as \(n_j \nearrow \infty.\) Hence, \(\nu\) is \(\mathcal{A}\)-weakly harmonic, proving Part 1.

Since we know by Lemma 9.7 that \(\varphi \in L_\mu^1(\mathcal{C}(P, \mathbb{R}))\) it follows from Part 1) that \(\int \varphi d\nu^{n_j} \to \int \varphi d\nu.\) Using the explicit formula for \(\nu^{n_j}\) and \(\nu_{n_j}\) and \(\varphi_{n_j},\) the leaf-hand side is equal to

\[ \int \varphi \left( \frac{1}{n_j} \sum_{i=0}^{n_j-1} D_i \nu_{n_j} \right) = \int \frac{1}{n_j} \sum_{i=0}^{n_j-1} D_i \varphi d\nu_{n_j} = \int \varphi_{n_j} d\nu^{n_j}. \]

This completes the proof. \qed
Lemma 9.10. Let \( \nu \in \text{Har}_h(X \times P) \). Let \( Q \) be Borel subset of \( X \times P \) such that \( \nu(Q) > 0 \). Let \( \alpha \) and \( \beta \) be two real numbers such that for every \((x, u) \in Q\), we have

(i) \( \chi(\omega, u) = \alpha \) for \( W_x \)-almost every \( \omega \in \Omega_x \);
(ii) \( \chi_{\tilde{x}, u}(\tilde{\omega}) = -\beta \) for every \( \tilde{x} \in \pi^{-1}(x) \) and for every \( \tilde{\omega} \in \tilde{F}_{\tilde{x}, u} \), where \( \tilde{F}_{\tilde{x}, u} \subset \tilde{\Omega}(\tilde{L}_x) \) (depending on \( \tilde{x} \) and \( u \)) is of positive measure in \( \tilde{L}_x \).

Here the function \( \chi(\omega, u) \) (resp. \( \chi_{\tilde{x}, u}(\tilde{\omega}) \)) has been defined in (8.5) (resp. in (8.3)). Then \( \alpha = \beta \).

Proof. First, by Proposition 9.4 we may assume without loss of generality that \( \nu \) is ergodic. Let \( \gamma := \int_{X \times P} \varphi d\nu \). By Theorem 9.6 we have that \( \lim_{n \to \infty} \frac{1}{n} \log \mathcal{C}_A((\omega, u), n) = \gamma \) for \( \nu \)-almost every \((x, u) \) and \( W_x \)-almost every \( \omega \). This, combined with assumption (i), implies that \( \gamma = \alpha \).

By Theorem 9.6 again, \( \lim_{n \to \infty} \frac{1}{n} \log \mathcal{C}_A((\tilde{\omega}, u), -n) = -\gamma \) for \( \tilde{\nu} \)-almost every \( (\tilde{\omega}, u) \in \tilde{\Omega}_A \). This, coupled with Part 1) of Proposition 8.6 applied to the cylinder lamination \((X \times P, \mathcal{L}_A)\), implies that for \( \nu \)-almost every \((x, u) \in X \times P\), the set

\[
F_{x,u} := \left\{ (\tilde{\omega}, v) \in \tilde{\Omega}(L_x) \times P : (x, u) \text{ and } (\tilde{\omega}(0), v) \text{ are on the same leaf of } (X_A, \mathcal{L}_A) \right\}
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathcal{C}_A((\tilde{\omega}, v), -n) = -\gamma
\]

is of full measure in \( L_x \). So by Part 2) of Lemma 8.7, \( \tilde{F}_{x,u} := \pi^{-1}F_{x,u} \subset \tilde{\Omega}(\tilde{L}_x) \) is of full measure in \( \tilde{L}_x \). Note that for an arbitrary \( \tilde{\omega} \in \tilde{\Omega}(\tilde{L}_x) \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathcal{C}_A((\tilde{\omega}, v), -n) = \chi_{\tilde{y}, v}(\tilde{\omega}),
\]

where \( \tilde{y} \) is an arbitrary point in \( \pi^{-1}(y) \) with \( y := \tilde{\omega}(0) \) and \( \tilde{\omega} := \pi_{\tilde{y}}^{-1}\tilde{\omega} \). Consequently, for \( \nu \)-almost every \((x, u) \in X \times P\), the set

\[
\left\{ \tilde{\omega} \in \tilde{\Omega}(\tilde{L}_x) : \exists \tilde{x} \in \pi^{-1}(x), \exists v \in P : (\tilde{x}, u) \text{ and } (\tilde{\omega}(0), v) \right\}
\]

are on the same leaf of \( (\tilde{X}_A, \tilde{\mathcal{L}}_A) \) and \( \chi_{\tilde{\omega}(0), v}(\tilde{\omega}) = -\gamma \)

is of full measure in \( \tilde{L}_x \). On the other hand, by assumption (ii), for each \((x, u) \in Q\) and for each \( \tilde{x} \in \pi^{-1}(x) \), the set

\[
\left\{ \tilde{\omega} \in \tilde{\Omega}(\tilde{L}_x) : \chi_{\tilde{x}, u}(\tilde{\omega}) = -\beta \right\}
\]

are on the same leaf of \( (\tilde{X}_A, \tilde{\mathcal{L}}_A) \) and \( \chi_{\tilde{\omega}(0), v}(\tilde{\omega}) = -\beta \)
is of positive measure in $\tilde{L}_x$. Recall from Remark \REF{8.5} that the intersection of a set of full measure and a set of positive measure in the leaf $\tilde{L}_x$ is nonempty. Applying this to the last two subsets of $\tilde{\Omega}(\tilde{L}_x)$ and using the assumption that $\nu(Q) > 0$ yields that for every path $\tilde{\omega}$ in their intersection,

$$-\gamma = \chi_{\tilde{\omega}(0),\nu}(\tilde{\omega}) = -\beta.$$ 

Hence, $\gamma = \beta$, which implies that $\alpha = \beta = \gamma$. \hfill \Box

**Corollary 9.11.** Let $Y \subset X$ be a Borel set of full $\mu$-measure. Let $\alpha$ and $\beta$ be two real numbers such that for every $x \in Y$ and $u \in P$, we have

(i) $\chi(\omega, u) = \alpha$ for $W_x$-almost every $\omega \in \Omega_x$;

(ii) $\chi_{\tilde{x},u}(\tilde{\omega}) = -\beta$ for every $\tilde{x} \in \pi^{-1}(x)$ and for every $\tilde{\omega} \in \hat{\mathcal{F}}_{\tilde{x},u}$, where $\hat{\mathcal{F}}_{\tilde{x},u} \subset \hat{\Omega}(\tilde{L}_x)$ (depending on $\tilde{x}$ and $u$) is of positive measure in $\tilde{L}_x$.

Then $\alpha = \beta$.

**Proof.** By Part 1) of Lemma \REF{9.9} let $\nu$ be a probability measure on $X \times P$ which belongs to $\text{Har}_\mu(X \times P)$. Set $Q := Y \times P$. Since $Y \subset X$ is a Borel set of full $\mu$-measure, it follows that $\nu(Q) = 1$. Consequently, applying Lemma \REF{9.10} yields that $\alpha = \beta$. \hfill \Box

Now we arrive at the first result on splitting invariant bundles.

**Theorem 9.12.** Let $Y \subset X$ be a Borel set of full $\mu$-measure. Assume also that $Y \ni x \mapsto U(x)$ and $Y \ni x \mapsto V(x)$ are two measurable $\mathcal{A}$-invariant subbundles of $Y \times \mathbb{R}^d$ with $V(x) \subset U(x)$ for all $x \in Y$. Let $\alpha, \beta, \gamma$ be three real numbers with $\alpha < \beta$ such that

1) $\chi^-(x, u) = \gamma$ for every $x \in Y$, every $u \in U(x) \setminus \{0\}$;

2) $\chi(\omega, v) = \alpha$ for every $x \in Y$, every $v \in V(x) \setminus \{0\}$ and for $W_x$-almost every $\omega \in \Omega_x$;

3) $\chi(\omega, u) = \beta$ for every $x \in Y$, every $u \in U(x) \setminus V(x)$, and for $W_x$-almost every $\omega \in \Omega_x$.

Here the function $\chi(\omega, u)$ (resp. $\chi^-(x, u)$) has been defined in \REF{5.7} (resp. in \REF{8.7}). Then $\beta = -\gamma$ and $V(x) = \{0\}$ for $\mu$-almost every $x \in Y$.

**Proof.** We are in the position to apply Theorem \REF{6.5}. Define a new measurable sub-bundle $Y \ni x \mapsto W(x)$ of $Y \times \mathbb{R}^d$ by splitting $U(x) = V(x) \oplus W(x)$ so that $W(x)$ is orthogonal to $V(x)$ with respect to the Euclidean inner product of $\mathbb{R}^d$. By \REF{6.1}, the linear map $\mathcal{A}(\omega, n)$ induce two other linear maps $\mathcal{C}(\omega, n) : W(\pi_0\omega) \to W(\pi_n\omega)$ and $\mathcal{D}(\omega, n) : W(\pi_0\omega) \to V(\pi_n\omega)$ satisfying

$$\mathcal{A}(\omega, n)(v + w) = (\mathcal{A}(\omega, n)v + \mathcal{D}(\omega, n)w) \oplus \mathcal{C}(\omega, n)w \quad (9.9)$$

for $x \in Y$, $v \in V(x)$, $w \in W(x)$, $\omega \in \Omega_x(Y)$. By Part (i) of Theorem \REF{6.5} $\mathcal{C}$ defined on $\tilde{\Omega}(Y) \times \mathbb{N}$ satisfies the multiplicative law

$$\mathcal{C}(\omega, m + k) = \mathcal{C}(T^k\omega, m)\mathcal{C}(\omega, k), \quad m, k \in \mathbb{N}.$$ 

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Moreover, \( C(\omega, n) \) is invertible. Using assumption 2) and 3) and the inequality \( \alpha < \beta \), we may apply Part (ii) of Theorem 6.5. Consequently, there exists a subset \( Y' \) of \( Y \) of full \( \mu \)-measure with the following properties: for every \( x \in Y' \) and \( w \in P W(x) \), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log \| C(\omega, n) w \| = \beta,
\]

(9.10)

for every \( \omega \in \mathcal{F}_{x,w} \), where \( \mathcal{F}_{x,w} \subset \Omega_x(Y) \) is a set of full \( W_x \)-measure.

By Theorem 10.5, there is a bimeasurable bijection between the bundle \( Y \ni x \mapsto W(x) \) and \( Y \times \mathbb{R}^d \) with \( \dim U(x) - \dim V(x) = d' \) covering the identity and which is linear on fibers. Using this and applying Lemma 9.9, we may find an ergodic weakly harmonic probability measure \( \lambda \) living on the leafwise saturated set (with respect to \( C \)) \( \{ (x, PW(x)) : x \in X \} \). Using \( \lambda \) and (9.10) and applying Theorem 9.6 yields that

\[
\lim_{n \to \infty} \frac{1}{n} \int_X \left( \int_{\Omega_x} \left( \int_{u \in PW(x)} \log \| C(\omega, n) u \| d\lambda_x(u) \right) dW_x(\omega) \right) d\mu(x) = \beta.
\]

(9.11)

On the other hand, let

\[
M_n(x) := \sup_{u \in P} \varphi_n(x,u),
\]

where \( P \) stands, as usual, for \( \mathbb{P}(\mathbb{R}^d) \). Set

\[
\Delta_n := \{ (x,u) \in X \times P : \varphi_n(x,u) = M_n(x) \in \mathcal{B}(X) \times \mathcal{B}(P) \}.
\]

We have \( \text{pr} : \Delta_n \to X \), where \( \text{pr} : X \times P \to X \) is the natural projection. Since for each \( x \in X \), \( \{ u \in P : (x,u) \in \Delta_n \} \) is closed, we can choose by Theorem 10.1 a measurable map \( u_n : X \to \mathbb{R}^d \) such that \( (x,u_n(x)) \in \Delta_n \) for all \( x \in X \). We may apply Lemma 9.9 to the sequence \( (u_n)_{n=1}^\infty \). Consequently, we obtain an ergodic measure \( \nu \) on \( X \times P \) and a real number \( \beta' \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \int_{\Omega \times P} \log \| C(\omega,n) u \| d\nu(\omega,u) = \beta'.
\]

and that \( \lim_{n \to \infty} \frac{1}{n} \int \log \| C(\omega,n) u \| \, d\nu(\omega,u) = \beta' \) for \( \nu \)-almost every \( (w,u) \in \Omega \times P \). Recall from (9.9) that \( \| A(\omega,n) u \| \geq \| C(\omega,n) w \| \) for all \( w \in W(x) \). Hence, \( \log \frac{\| A(\omega,n) u \|}{\| u \|} \geq \log \frac{\| C(\omega,n) u \|}{\| u \|} \) for \( u \in W(x) \). Consequently, we deduce from Lemma 9.8 that

\[
\int_X \varphi_n(x,u_n(x)) d\mu(x) = \frac{1}{n} \int_X \left( \int_{\Omega_x} \log \| A(\omega,n) u_n(x) \| dW_x(\omega) \right) d\mu(x).
\]
By the choice of $u_n$ and the fact that each $\lambda_x$ is a probability measure on $\mathbb{P}(W(x))$ for $\mu$-almost every $x \in X$, the right hand side is greater than
\[
\frac{1}{n} \int_X \left( \int_{v \in \mathbb{P}(W(x))} \left( \int_{\Omega_x} \log \|A(\omega, n)v\|dW_x(\omega) \right) d\lambda_x(v) \right) d\mu(x) \\
\geq \frac{1}{n} \int_X \left( \int_{v \in \mathbb{P}(W(x))} \left( \int_{\Omega_x} \log \|C(\omega, n)v\|dW_x(\omega) \right) d\lambda_x(v) \right) d\mu(x) \\
= \frac{1}{n} \int_X \left( \int_{v \in \mathbb{P}(W(x))} \log \|C(\omega, n)v\|dW_x(\omega) \right) d\lambda_x(v) d\mu(x),
\]
By (9.11) the limit when $n \to \infty$ of the last expression is equal to $\beta$. This, combined with Lemma 9.9 implies that $\beta' = \int_{X \times P} \phi d\nu \geq \beta$. Note that by 2) and 3) and the assumption $\alpha \leq \beta$ and applying Theorem 9.6 to $\nu$, we have $\beta' \leq \beta$. So $\beta' = \beta$. So $\lim_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)v\| = \beta$ for $\nu$-almost every $(\omega, v) \in \Omega \times P$.

Let $Q := \{(x, v) : x \in X, v \in \mathbb{P}(\mathbb{R}^d \setminus V(x))\}$. This is a leafwise saturated Borel set. Since $\nu$ is ergodic, $\nu(Q)$ is either 0 or 1.

Case $\nu(Q) = 0$: then $\nu((X \times P) \setminus Q) = 1$. Hence, for $\mu$-almost every $x$, there exists $u \in V(x)$, such that $\lim_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)u\| = \beta$ for $W_x$-almost every $\omega$. This, combined with assumption 2), implies that $V(x) = \{0\}$ for $\mu$-almost every $x \in X$. Consequently, we deduce from Lemma 9.10 and 1) and 3) that $\beta = -\gamma$, as desired.

Case $\nu(Q) = 1$: then for $\mu$-almost every $x$, there exists $u \in \mathbb{R}^d \setminus V(x)$, such that $\lim_{n \to \infty} \frac{1}{n} \log \|A(\omega, n)u\| = \beta$ for $W_x$-almost every $\omega$. We are in the position to apply Lemma 9.10. Consequently, we get that $\beta = -\gamma$.

It remains to show that $V(x) = \{0\}$ for $\mu$-almost every $x \in X$. Suppose the contrary. So dim $V(x) = d' \geq 1$ for $\mu$-almost every $x \in X$. Restricting $A(\omega, \cdot)$ on $V(x)$ for every $w \in \Omega_x$, and applying Lemma 9.10, we get that $\alpha = -\gamma$. Hence, $\alpha = \beta = -\gamma$, which contradicts the hypothesis that $\alpha \leq \beta$.

Proposition 9.13. Let $Y \subset X$ be a Borel set of full $\mu$-measure and $Y \ni x \mapsto V(x)$ a measurable $A$-invariant sub-bundle of $Y \times \mathbb{R}^d$ such that dim $V(x) < d$ for all $x \in Y$. Let $\alpha, \beta, \gamma$ be three real numbers with $\alpha < \beta$ such that
1) $\chi(\omega, u) = \gamma$ for every $x \in Y$, every $u \in \mathbb{R}^d \setminus \{0\}$, and for $W_x$-almost every $\omega \in \Omega$;
2) $\chi^{-}(x, v) = \alpha$ for every $x \in Y$, every $v \in V(x) \setminus \{0\}$;
3) $\chi^{-}(x, u) = \beta$ for every $x \in Y$, every $u \in \mathbb{R}^d \setminus V(x)$.

Here the function $\chi(x, u)$ (resp. $\chi^{-}(x, u)$) has been defined in (5.7) (resp. in (8.7)). Then $\beta = -\gamma$ and $V(x) = \{0\}$ for $\mu$-almost every $x \in Y$.

Observe that if $V(x) = \{0\}$ for $\mu$-almost every $x \in Y$, then we deduce from Corollary 9.11 and 1) and 3) that $\beta = -\gamma$, as desired. Therefore, we only consider the case where dim $V(x) \geq 1$ for every $x \in Y$ in the sequel.

Prior to the proof of this result we need to introduce some preparation. Define a sub-bundle $Y \ni x \mapsto W(x)$ of $Y \times \mathbb{R}^d$ and the multiplicative $C(\omega, n)$ :
$W(\pi_0 \omega) \to W(\pi_n \omega)$ by (9.9), where $U(x) := \mathbb{R}^d$, $x \in Y$. Next, using formula (8.3) we extend $A$ and $C$ to $\Omega(Y) \times \mathbb{Z}$ such that its extension still satisfies the multiplicative law. By assumption 3), for every $x \in Y$, every $\tilde{x} \in \pi^{-1}(x)$, every $w \in W(x)$, there is a set $\bar{G}_{x,w} \subset \Omega(\bar{L}_x)$ of positive measure in $\bar{L}_x$ such that $\chi_{x,w}(\tilde{\omega}) = \beta$ for every $\tilde{\omega} \in \bar{G}_{x,w}$. This, combined with assumption 2) and the inequality $\alpha < \beta$, allows us to apply Part (ii) of Theorem 8.17. Consequently, there exists a subset $Y'$ of $Y$ of full $\mu$-measure with the following properties: for every $x \in Y'$ and every $\tilde{x} \in \pi^{-1}(x)$ and every $w \in W(x) \setminus \{0\}$, we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \|C(\tilde{\omega}, -n)w\| = \beta,$$  \hspace{1cm} (9.12)

for every $\tilde{\omega} \in \bar{G}_{x,w}$, where $\bar{G}_{x,w} \subset \Omega(\bar{L}_x)$ is of positive measure in $\bar{L}_x$.

On the other hand, by Theorem 10.5, there is a bimeasurable bijection between the bundle $Y \ni x \mapsto W(x)$ and $Y \times \mathbb{R}^{d-d'}$ with $\dim V(x) = d'$ covering the identity and which is linear on fibers. Using this and applying Lemma 9.9, we may find an ergodic weakly harmonic probability measure $\lambda$ living on the leafwise saturated set (with respect to $C$) $\{x, \mathbb{P}(W(x)) : x \in X\}$.

For $(x, u) \in X \times P$ such that $u \not\in \mathbb{P}(V(x))$, let $\text{pr}_x u := \text{pr}_x \tilde{u} \in \mathbb{P}(W(x))$, where $\tilde{u} \in \mathbb{R}^{d} \setminus \{0\}$ such that $\|\tilde{u}\| = u$ and $\text{pr}_x \tilde{u}$ is the component of $\tilde{u}$ in $W(x)$ from the direct sum decomposition $\tilde{u} \in V(x) \oplus W(x)$. Recall that for $\omega \in \Omega$ and $t \in \mathbb{R}$, let

$$m(\omega, t) := \min_{u \in V(\omega(0)) \setminus \{0\}} \log \frac{\|A(\omega, t)v\|}{\|v\|}.$$  \hspace{1cm} (9.13)

Consider the functions $\psi_n : X \times P \to \mathbb{R}$ ($n \geq 1$) given by

$$\psi_n(x, u) := \begin{cases} \int_{\Omega_x} \min \{1/n \cdot \log \|C(\omega, n)\text{pr}_x u\|, 1/n \cdot m(\omega, n)\} dW_x(\omega), & u \not\in \mathbb{P}(V(x)) \text{;} \\ 1/n \cdot m(\omega, n) & u \in \mathbb{P}(V(x)). \end{cases}$$  \hspace{1cm} (9.14)

The functions $\psi_n$ are, in general, only upper semi-continuous with respect to the variable $u \in P$. For each $n \geq 1$ we also consider the following continuous regularizations $(\psi_{n,N})_{N=1}^{\infty}$ of $\psi_n$, which are defined by

$$\psi_{n,N}(x, u) := \begin{cases} \int_{\Omega_x} \min \{1/n \cdot \log \|C(\omega, n)\text{pr}_x u\|, 1/n \cdot m(\omega, n) - 1/N\} dW_x(\omega), & u \not\in \mathbb{P}(V(x)) \text{;} \\ 1/n \cdot m(\omega, n) - 1/N & u \in \mathbb{P}(V(x)). \end{cases}$$  \hspace{1cm} (9.15)

The properties of the functions $\psi_n$ and $\psi_{n,N}$ are collected in the following result.

**Lemma 9.14.** For each $n \geq 1$ $\psi_{n,N} \nearrow \psi_n$ as $N \nearrow \infty$. Moreover, $\psi_{n,N} \in L^1(\mathcal{E}(P, \mathbb{R}))$.

**Proof.** The limit $\psi_{n,N} \nearrow \psi_n$ as $N \nearrow \infty$ follows from the definition of $\psi_{n,N}$ and $\psi_n$ given in (9.14)-(9.15). Next, we show that $\psi_{n,N}(x, \cdot) \in \mathcal{E}(P, \mathbb{R})$ for each
Proof. Fix an arbitrary \( x \in \tilde{\Omega} \) where \( \varphi \) varies in a small neighborhood of \( u \) where \( x \). Using this and arguing as in the proof of Lemma 9.17 we can show that \( \psi_{n,N} \in L_{\mu}^1(\mathcal{C}(P,\mathbb{R})) \).

\[ \psi_{n,N} \geq \frac{1}{n} \sum_{i=0}^{n} D_{i} \psi_{1,N} \]

Lemma 9.15. For every \( n, N \geq 1 \)

Proof. Fix an arbitrary \( N \geq 1 \) and an arbitrary element \( u \in P \). Set \( L := L_{x_0} \). Let \( \pi : \tilde{L} \to L \) be a universal cover, and fix a point \( \tilde{x}_0 \in \tilde{L} \) such that \( \pi(\tilde{x}_0) = x_0 \). For every \( n \geq 1 \) and \( \tilde{x} \in \tilde{L} \)

\[ \psi_n(\tilde{x}) := \begin{cases} \int_{\Omega_{\tilde{x}}} \min \{1/n \cdot \log ||C(\tilde{\omega},n)pr_{\tilde{x}}(u_{\tilde{x}})||, 1/n \cdot m(\omega,n) - 1/N \} dW_{\tilde{x}}(\omega), & u \notin \tilde{P}(\tilde{x}_0), \\ 1/n \cdot m(\omega,n) - 1/N, & u \in \tilde{P}(\tilde{x}_0), \end{cases} \]

where \( u_{\tilde{x}} \in P \) is determined by \( (\tilde{x},u_{\tilde{x}}) \sim (\tilde{x}_0,u) \). We only need to show that

\[ \psi_n(\tilde{x}_0) \geq \frac{1}{n} \sum_{i=0}^{n-1} (D_{i} \psi_{1})(\tilde{x}_0), \]

where \( D_{i} \) are the diffusion operators on \( \tilde{L} \). Consider also the functions \( F_n : \tilde{\Omega} \to \mathbb{R} \) defined by: for \( \omega \in \tilde{\Omega}_{\tilde{x}} (\tilde{x} \in \tilde{L}) \)

\[ F_n(\omega) := \begin{cases} \min \{1/n \cdot \log ||C(\omega,n)pr_{\tilde{x}}(u_{\tilde{x}})||, 1/n \cdot m(\omega,n) - 1/N \}, & u \notin \tilde{P}(x_0), \\ 1/n \cdot m(\omega,n) - 1/N, & u \in \tilde{P}(x_0). \end{cases} \]

Set \( G := F_1 \). The following elementary result is needed.

Lemma 9.16. For \( n \geq 1 \) let \( a_0, \ldots, a_{n-1}, m_0, \ldots, m_{n-1} \) be \( 2n \) real numbers. Then

\[ \sum_{i=0}^{n-1} \min \{a_i, m_i\} \leq \min \{\sum_{i=0}^{n-1} a_i, \sum_{i=0}^{n-1} m_i\}. \]

By the multiplicative law of cocycles and by Lemma 9.16 applied to

\[ a_i := \log ||C(T^{i}\omega,1)pr_{\tilde{x}}(u_{\tilde{x}})|| \quad \text{and} \quad m_i := m(T^{i}\omega) - 1/N \]

for each \( \omega \in \Omega(\tilde{L}) \) and \( 0 \leq i \leq n - 1 \), we have that

\[ F_n(\omega) \geq \frac{1}{n} \sum_{i=0}^{n-1} G \circ T^{i}(\omega). \]
Recall from Subsection 4.2 that, for $s \geq 0$, $\mathcal{F}_s$ is the smallest $\sigma$-algebra making all the projections $\pi_r : \Omega(L) \to \tilde{L}$ with $0 \leq r \leq s$ measurable, and $\mathcal{F}_{t^+} := \cap_{s>t} \mathcal{F}_s$ for $t \geq 0$. Consequently, we obtain, for every $\tilde{x} \in \tilde{L}$, that

$$E_{\tilde{x}}[F_n] \geq \frac{1}{n} \sum_{i=0}^{n-1} E_{\tilde{x}}[G \circ T^i] = \frac{1}{n} \sum_{i=0}^{n-1} E_{\tilde{x}}[G \circ T^i | \mathcal{F}_{i^+}],$$

where, the equality holds by the projection rules of the expectation operation (see Theorem C.1.6 in [5]). By the Markov property (see Theorem 4.5 above), we get that

$$E_{\tilde{x}}[G \circ T^i | \mathcal{F}_{i^+}] = E_{\ast}[G] \circ \pi_i.$$ 

Inserting this into the previous inequality we obtain that

$$E_{\tilde{x}}[F_n] \geq \frac{1}{n} \sum_{i=0}^{n-1} E_{\tilde{x}}[E_{\ast}[G] \circ \pi_i].$$

This, combined with the obvious identities

$$E_{\tilde{y}}[G] = \psi_1(\tilde{y}) \quad \text{and} \quad E_{\tilde{x}}[F_n] = \psi_n(\tilde{x}) \quad \tilde{x}, \tilde{y} \in L,$$

implies that

$$\psi_n(\tilde{x}) \geq \frac{1}{n} \sum_{i=0}^{n-1} E_{\tilde{x}}[\psi_1 \circ \pi_i].$$

On the other hand, we know from (9.5) that $E_{\tilde{x}}[\psi_1 \circ \pi_i] = (\tilde{D}_1 \psi_1)(\tilde{x})$. Inserting this into the last estimate and setting $\tilde{x} := \tilde{x}_0$, the lemma follows.

**End of the proof of Proposition 9.13.** Next, applying Lemma 9.10 and using (9.12), we deduce that

$$\lim_{n \to \infty} \frac{1}{n} \int_X \left( \int_{\Omega X} \left( \int_{u \in PW(x)} \log \|C((\omega, u), n)\| d\lambda_x(u) \right) dW_x(\omega) \right) d\mu(x)$$

$$= \lim_{n \to \infty} \int \frac{1}{n} \log \|C(\omega, u)\| d\tilde{\lambda}(\omega, u) = - \lim_{n \to \infty} \int \frac{1}{n} \log \|C(\omega, -n)u\| d\tilde{\lambda}(\omega, u) = -\beta.$$ 

(9.16)

Now let

$$m_n(x) := \inf_{u \in PW(x)} \psi_n(x, u).$$

Set

$$\Delta_n := \{(x, u) \in X \times P : \psi_n(x, u) = m_n(x) \in \mathcal{B}(X) \times \mathcal{B}(P)\}.$$

We have $pr : \Delta_n \to X$, where $pr : X \times P \to X$ is the natural projection. Since for each $x \in X$, $\{u \in P : (x, u) \in \Delta_n\}$ is closed, we can choose by Theorem 10.1
a measurable map \( u_n : X \to \mathbb{R}^d \) such that \((x, u_n(x)) \in \Delta_n\) for all \(x \in X\). We may apply Lemma 9.9 to the sequence \((u_n)_{n=1}^\infty\). Consequently, we obtain a probability measure \( \nu \in \text{Har}_\mu(X \times P) \) and a sequence \((n_j) \nearrow \infty\) as \(j \nearrow \infty\) such that

\[
\lim_{j \to \infty} \int_X \left( \frac{1}{n_j} \sum_{i=0}^{n_j-1} D_i \psi \right) (x, u_{n_j}(x)) d\mu(x) = \int_{X \times P} \psi d\nu \tag{9.17}
\]

for every \( \psi \in L^1_\mu(\mathcal{C}(P, \mathbb{R})) \). The following result is needed.

**Lemma 9.17.** \( \lim_{n \to \infty} \int_X m_n(x) d\mu(x) \leq -\beta \).

**Proof.** Observe from (9.14) that for \( v \in \mathbb{P}W(x) \),

\[
\psi_n(x, v) \leq \int_{\Omega_x} \log \|C(\omega, n)v\| dW_x(\omega).
\]

Using this we get that

\[
\int_X m_n(x) d\mu(x) = \int_X \psi_n(x, u_n(x)) d\mu(x)
\]

\[
\leq \frac{1}{n} \int_X \left( \int_{\mathbb{P}W(x)} \left( \int_{\Omega_x} \log \|C(\omega, n)v\| dW_x(\omega) \right) d\lambda_x(v) \right) d\mu(x),
\]

where the inequality follows from the construction of \( u_n \) and the fact that each \( \lambda_x \) is a probability measure on \( \mathbb{P}W(x) \) for \( \mu \)-almost every \( x \in X \). By (9.16), the limit of the last integral as \( n \to \infty \) is equal to \(-\beta\). Hence, the lemma follows.

Resuming the proof of Proposition 9.13, we deduce from Lemma 9.15 that for an arbitrary \( N \geq 1 \),

\[
\int_X \psi_{n,N}(x, u_n(x)) d\mu(x) \geq \int_X \frac{1}{n} \sum_{i=0}^{n} D_i \psi_{1,N}(x, u_n(x)) d\mu(x).
\]

By (9.17) and Lemma 9.14, the right hand side tends to \( \int_{X \times P} \psi_{1,N} d\nu \) as \( n \to \infty \). On the other hand, by Lemma 9.14, \( \psi_{n,N} \leq \psi_n \) and \( \psi_{1,N} \nearrow \psi_1 \) as \( N \to \infty \). Putting these estimates together and letting \( N \to \infty \), yields that

\[
\lim_{n \to \infty} \int_X \psi_n(x, u_n(x)) d\mu(x) \geq \int_{X \times P} \psi_1 d\nu.
\]

In other words,

\[
\lim_{n \to \infty} \int_X m_n(x) d\mu(x) \geq \int_{X \times P} \psi_1 d\nu. \tag{9.18}
\]

Let \( Q := \{(x, u) \in X \times P : u \notin \mathbb{P}V(x)\} \). Note that \( Q \) is leafwise saturated. By Lemma 9.10 applied to \( \{(x, u) : u \in V(x)\} \), and assumption 1) and 2), we get that

\[
\gamma = -\alpha. \tag{9.19}
\]
There are two cases to consider.  

**Case** $\nu(Q) > 0$.  

Since $Q$ is leafwise saturated, the probability measure $1/\nu(Q) \cdot \nu|_Q$ is weakly harmonic supported on $Q$. By Lemma 9.10 and assumption 1) and 3), we get that $\beta = -\gamma$. This, combined with (9.19), implies that $\alpha = \beta$ which contradicts the assumption that $\alpha < \beta$. Hence, this case cannot happen.  

**Case** $\nu(Q) = 0$.

Lemma 9.17, combined with (9.18), implies that $\int_{x \times P} \psi_1 d\nu \leq -\beta$. Since $\nu$ is supported on $(X \times P) \setminus Q = \{(x, u) : u \in V(x)\}$, it follows from the last estimate and the formula of $\psi_1$ in (9.14) that

$$
\int_{x \in X} \left( \int_{\omega \in \Omega_x} m(\omega, 1) dW_x(\omega) \right) \leq -\beta.
$$

By assumption 2), the left hand side is equal to $\gamma$. This, combined with (9.19), implies that $-\alpha \leq -\beta$. But this contradicts the assumption $\alpha < \beta$. Hence, the second case cannot happen.  

Now we arrive at the second main result of this subsection. This theorem, together with Theorem 9.12, constitute the indispensable toolkit in order to obtain splitting invariant sub-bundles in the next subsections.  

**Theorem 9.18.** Let $Y \subset X$ be a Borel set of full $\mu$-measure and $1 \leq k \leq d$ an integer. Assume that $Y \ni x \mapsto V^{-i}(x)$ for $1 \leq i \leq k$ and $Y \ni x \mapsto U(x) = V^0(x)$ are $(k + 1)$ measurable $A$-invariant sub-bundles of $Y \times \mathbb{R}^d$ such that

$$
\{0\} = V^{-k}(x) \subset \cdots \subset V^{-1}(x) \subsetneq U(x), \quad x \in Y.
$$

Let $\alpha_1, \ldots, \alpha_k$ and $\gamma$ be $(k + 1)$ real numbers with $\alpha < \beta$ such that

1) $\chi(\omega, u) = \gamma$ for every $x \in Y$, every $u \in U(x) \setminus \{0\}$, and for $W_x$-almost every $\omega \in \Omega$;  
2) $\chi^-(x, v) = \alpha_i$ for every $1 \leq i \leq k$, every $x \in Y$, every $v \in V^{-(i-1)}(x) \setminus V^{-i}(x)$.  

Then $\alpha_1 = -\gamma$ and $V^{-k}(x) = \cdots = V^{-1}(x) = \{0\}$ for all $x \in Y$.  

**Proof.** Suppose without loss of generality that the sequence $(V^{-i}(x))_{i=0}^k$ is strictly decreasing in $i$. There are two cases to consider.  

**Case:** $k = 1$. By Theorem 10.5, there is a bimeasurable bijection between the bundle $Y \ni x \mapsto U(x)$ and $Y \times \mathbb{R}^d$ with dimension $d$ covering the identity and which is linear on fibers. Using this bijection, we are able to apply Corollary 9.11. Consequently, $\alpha_1 = -\gamma$, as asserted.  

**Case:** $k > 1$. So $\{0\} \subsetneq V^{-(k-1)}(x) \subsetneq V^{-(k-2)} \subset U(x)$ for each $x \in Y$. By Theorem 10.5, there is a bimeasurable bijection $\Lambda$ from the bundle $Y \ni x \mapsto V^{-(k-2)}(x)$ onto $Y \times \mathbb{R}^{d''}$ with dimension $V^{-(k-2)}(x) = d''$ covering the identity and which is linear on fibers. Using this bijection, we are in the position to apply Proposition 9.13 to the following situation: $d$ is replaced with $d'$, $V(x) := \Lambda(x, V^{-(k-1)}(x))$. Consequently, we obtain that $V(x) = 0$, hence $V^{-(k-1)}(x) = 0$ for all $x \in Y$, which is a contradiction. So this case cannot happen.  

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9.3 First Main Theorem and Ledrappier type characterization of Lyapunov spectrum

Assume without loss of generality that $\mu$ is ergodic. We are in the position to apply the results obtained in Section 7 and Section 8 and Subsection 9.2. The proof of Theorem 3.7 is divided into two cases.

**Case I: $\mathbb{G} = \mathbb{N}_t$ for some $t_0 > 0$.**

Without loss of generality we may assume that $t_0 = 1$, that is, $\mathbb{G} = \mathbb{N}$. In what follows we will make full use of the results as well as the notation given in:

- Theorem 8.14 and Theorem 8.16 in the backward setting;
- Theorem 7.1 and Theorem 7.2 in the forward setting;
- Theorem 9.12 and Theorem 9.18 for splitting invariant sub-bundles.

For example, $\Phi \subset \Omega(X, \mathcal{L})$ is the set of full $\bar{\mu}$-measure introduced by Theorem 7.1. This case is divided into 4 steps.

**Step 1:** Proof that $\chi_m = \lambda_l$. Moreover, we have that $V_m(x) = V_l(\omega)$ for $\mu$-almost every $x \in X$ and for $W_\omega$-almost every path $\omega \in \Psi$.

By Theorem 8.16, $\lambda_l = -\chi_m^-$. So it is sufficient to show that $\chi_m = -\chi_m^-$. Recall from Theorem 7.2 that for $\mu$-almost every $x \in X$,

$$V_m(x) := \left\{ v \in \mathbb{R}^d : \lim_{n \to \infty} \frac{1}{n} \log \| A(\omega, n)v \| = \chi_m^- W_\omega \text{-almost every } \omega \in \Omega(L_x) \right\}.$$  

(9.20)

Moreover, by Theorem 7.2, $\chi_m \in \{\lambda_1, \ldots, \lambda_l\}$. On the other hand, recall also from Theorem 8.16 that for $\mu$-almost every $x \in X$, and for every $\bar{x} \in \pi^{-1}(x)$, and for every $v \in \mathbb{R}^d \setminus V_{m^{-1}}(x)$, there exists a set $\tilde{F}_{\bar{x}, v} \subset \tilde{\Omega}(\tilde{L}_x)$ of positive measure in $\tilde{L}_x$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \log \| \tilde{A}(\bar{\omega}, -n)u_{\bar{x}, v, \bar{\omega}} \| = \chi_m^-$$  

(9.21)

for every $\bar{\omega} \in \tilde{F}_{\bar{x}, v}$.

Therefore, we deduce from Theorem 7.1 and Theorem 8.14 and Part 1) of Proposition 8.6 that there exists $1 \leq s \leq l$ such that $V_m(\omega(0)) = V_s(\omega) = \bigoplus_{j=s}^{l} H_j(\omega)$ for $\mu$-almost every $x \in X$ and for almost every $\omega \in \tilde{\Omega}(L_x)$ (see Definition 8.4 for the notion of almost everywhere in $\tilde{\Omega}(L_x)$). Similarly, by Theorem 8.16, $\chi_{m^{-1}_{-1}} \in \{-\lambda_1, \ldots, -\lambda_l\}$ and $\chi_{m^-} = -\lambda_l$. Consequently, we deduce from Theorem 8.14 and Part 1) of Proposition 8.6 that there exists $1 \leq t < l$ such that $V_{m^{-1}_{-1}}(\omega(0)) := \bigoplus_{j=1}^{t} H_j(\omega)$ for $\mu$-almost every $x \in X$ and for almost every $\omega \in \tilde{\Omega}(L_x)$. This, combined with the previous decomposition of $V_m(\omega(0))$, implies that for $\mu$-almost every $x \in X$, and for almost every $\omega \in \tilde{\Omega}(L_x)$,

$$\{0\} \neq H_t(\omega) \subset V_m(\omega(0)) \setminus V_{m^{-1}_{-1}}(\omega(0)).$$

Note by Corollary 6.2 and Corollary 8.15 that $Y \ni x \mapsto V_m(x)$ and $Y \ni x \mapsto V_{m^{-1}_{-1}}(x)$ with $0 \leq i \leq m^-$ are measurable $\mathcal{A}$-invariant bundles.
Next, we are in the position to apply Theorem 9.18 to the following context: the measurable $\mathcal{A}$-invariant bundle $Y \ni x \mapsto U(x)$ is given by $U(x) := V_m(x)$, and its $m^-$ $\mathcal{A}$-invariant sub-bundles $Y \ni x \mapsto V^{-i}(x)$ are given by $V^{-i}(x) := V_m(x) \cap V_{m-i}^{-}(x)$. Restricting $\mathcal{A}(\omega, \cdot)$ on $V_m(x)$ for $\omega \in \Omega_x$, and applying Proposition 9.13 yields that $\chi_m = -\chi_{m^-}$, as desired. So $\chi_m = \lambda$. Therefore, we deduce from Theorem 7.1 and 9.20 that $V_m(x) = V_l(\omega)$ for $\mu$-almost every $x \in X$ and for $W_2$-almost every path $\omega$.

**Step 2:** Proof that $m = l$. Moreover, for every $1 \leq i \leq m$, we have that $\chi_i = \lambda_i$ and that for $\mu$-almost every $x \in X$, it holds that $V_i(x) = V_i(\omega)$ for $W_x$-almost every path $\omega \in \Phi$.

We will use a duality argument. Applying Theorem 7.1 to the cocycle $\mathcal{A}^*-1$, we obtain $l^*$ Lyapunov exponents $\lambda^*_1 < \cdots < \lambda^*_l$ and the Lyapunov forward filtration

$$\{0\} \equiv V_{l^*+1}(\omega) \subset V_{l^*}(\omega) \subset \cdots \subset V_{2^*}(\omega) \subset V_{1^*}(\omega) = \mathbb{R}^d,$$

for $\bar{\mu}$-almost every $\omega \in \Omega(X, \mathcal{L})$. Applying Step 1 to the cocycle $\mathcal{A}^*-1$, we obtain, for $\mu$-almost every $x \in X$, a space $V^*(x) \subset \mathbb{R}^d$, such that $V^*_i(\omega) = V^*(x)$ for $W_x$-almost every $\omega$.

By 32 we know that $l^* = l$ and $\{-\lambda_1, \ldots, -\lambda_2\}$ is the orthogonal complement of $V_{l+2}(\omega)$. Applying Step 1 to the cocycle $\mathcal{A}^*-1$, we obtain, for $\mu$-almost every $x \in X$, a space $V^*(x) \subset \mathbb{R}^d$, such that $V^*_i(\omega) = V^*(x)$ for $W_x$-almost every $\omega$. So $V^*(x)$ is the orthogonal complement of $V_{2}(\omega)$ in $\mathbb{R}^d$ for $\bar{\mu}$-almost every $\omega \in \Omega(X, \mathcal{L})$. In particular, $V^*_i(\omega)$ is the orthogonal complement of $V_{2}(\omega)$ in $\mathbb{R}^d$ for $\bar{\mu}$-almost every $\omega \in \Omega(X, \mathcal{L})$.

Recall from the previous paragraph that $V^*_i(\omega) = V^*(x)$ for $W_x$-almost every $\omega$. So $V^*(x)$ is the orthogonal complement of $V_{2}(\omega)$ in $\mathbb{R}^d$ for $W_x$-almost every $\omega$. Let $V^*_2(x)$ be the orthogonal complement of $V^*(x)$ in $\mathbb{R}^d$. We deduce that $V_{2}(\omega) = V^*_2(x)$ for $\mu$-almost every $x \in X$ and for $W_x$-almost every $\omega$. This, combined with the definition of $V_{2}(x)$, implies that $V_{2}(\omega) = V^*_2(x)$ for $\mu$-almost every $x \in X$. So there exists a Borel set $Y \subset X$ of full $\mu$-measure such that $Y \ni x \mapsto V_{2}(x)$ is an $\mathcal{A}$-invariant measurable bundle of rank $r_2$. By Part 3) of Theorem 10.5 there is a bimeasurable bijection between the $\mathcal{A}$-invariant sub-bundle $Y \ni x \mapsto V_{2}(x)$ and $Y \times \mathbb{R}^{r_2}$ covering the identity and which is linear on fibers. Using this bijection, the restriction of $\mathcal{A}$ on $V_{2}(x)$, $x \in Y$, becomes a cocycle $\mathcal{A}'$ on $\mathbb{R}^{r_2}$. Note that the Lyapunov exponents of $\mathcal{A}'$ are $\lambda \leq \cdots \leq \lambda_2$.

We repeat the previous argument to $\mathcal{A}'$ and using the above bijection. Consequently, we may find a Borel set $Y \subset X$ of full $\mu$-measure such that $V^*_3(x) = V^*_3(x)$ for every $x \in Y$ and for $W_x$-almost every $\omega$.

By still repeating this argument $(l-3)$-times, we may find a Borel set $Y \subset X$ of full $\mu$-measure $1 \leq i \leq l$ such that $V_i(\omega) = V_i(x)$ for every $1 \leq i \leq l$ and every $x \in Y$ and for $W_x$-almost every $\omega$. In particular, $m = l$.

**Step 3:** Proof that $m = m^- = l$. Moreover, for every $1 \leq i \leq m$ we have that $\chi_i = \lambda_i = -\chi_{i^-}$ and that for $\mu$-almost every $x \in X$, there exists a space $H_i(x) \subset \mathbb{R}^d$ such that $H_i(\omega) = H_i(x)$ for $W_x$-almost every path $\omega \in \Phi$.

Recall from Step 2 that $m = l$ and $\chi_i = \lambda_i$ for $1 \leq i \leq m$. First we will prove that $\chi_{m^-} = \chi_{m-1}$. Combining Step 2 and Theorem 8.14 and Part 1) of
Proposition 8.6, we get that
\[ V_{m-1}(\omega(0)) = H_m(\omega) \oplus H_{m-1}(\omega) \quad (9.22) \]
for \( \mu \)-almost every \( x \in X \) and for almost every \( \omega \in \hat{\Omega}(L_x) \).

Recall from Theorem 8.16 that \( \chi_{m^-} = -\lambda_m \) and \( \chi_{m^- - 1}, \chi_{m^- - 2} \in \{-\lambda_1, \ldots, -\lambda_{m-1}\} \).

Consequently, we deduce from Theorem 8.14 and Part 1) of Proposition 8.6 that there exists \( 1 \leq t \leq m - 1 \) such that
\[ V_{m^- - 1}(\omega(0)) = \bigoplus_{j=1}^{t} H_j(\omega) \quad (9.23) \]
for \( \mu \)-almost every \( x \in X \) and for almost every \( \omega \in \hat{\Omega}(L_x) \). In particular, we get \( \chi_{m^- - 1} = -\lambda_t \).

In order to prove that \( \chi_{m^- - 1} = -\lambda_{m-1} \), it suffices to show that the possibility \( t < m - 1 \) cannot happen since \( t \geq m - 1 \) implies that \( t = m - 1 \) and hence \( \chi_{m^- - 1} = -\lambda_t = -\lambda_{m-1} \).

Suppose in order to reach a contradiction that \( t < m - 1 \). Using the decompositions (9.22)–(9.23) and noting that \( V_{m^-}(x) = \mathbb{R}^d \), we have that, for \( \mu \)-almost every \( x \in X \),
\[ \{0\} \neq H_{m-1}(x) \subset (V_{m-1}(x) \cap V_{m^-}(x)) \quad \text{and} \quad V_{m-1}(x) \cap V_{m^- - 1}(x) = \{0\}. \]

Therefore, we are in the position to apply Theorem 9.12 to the following context: the \( \mathcal{A} \)-invariant bundle \( x \mapsto U(x) \) is given by \( U(x) := V_{m-1}(x) \cap V_{m^-}(x) \), and its \( \mathcal{A} \)-invariant subbundle \( x \mapsto V(x) \) is given by \( V(x) := V_m(x) \). Restricting \( \mathcal{A}(\omega, \cdot) \) on \( V_{m-1}(x) \cap V_{m^-}(x) \) for \( \omega \in \Omega_x \), and applying Theorem 9.12 yields that \( V(x) = 0 \), hence \( V_{m}(x) = 0 \) for all \( x \in Y \), which is impossible. Thus we have shown that \( \chi_{m^- - 1} = -\lambda_{m-1} \). Consequently, we deduce from Theorem 8.14 and Part 1) of Proposition 8.6 that \( V_{m^- - 1}(\omega(0)) := \bigoplus_{j=1}^{m-1} H_j(\omega) \) for \( \mu \)-almost every \( x \in X \) and for almost every \( \omega \in \hat{\Omega}(L_x) \).

So there exists a Borel set \( Y \subset X \) of full \( \mu \)-measure such that \( Y \ni x \mapsto V_{m^- - 1}(x) \) is an \( \mathcal{A} \)-invariant measurable bundle of rank \( d - d_m \). By Part 3) of Theorem 10.5, there is a bimeasurable bijection between the \( \mathcal{A} \)-invariant subbundle \( Y \ni x \mapsto V(x) \) and \( Y \times \mathbb{R}^{d-d_m} \) covering the identity and which is linear on fibers. Using this bijection, the restriction of \( \mathcal{A} \) on \( V_{m^- - 1}(x), x \in Y \), becomes a cocycle \( \mathcal{A}' \) on \( \mathbb{R}^{d-d_2} \). Note that the Lyapunov exponents of \( \mathcal{A}' \) are \( \lambda_{l-1} < \cdots < \lambda_1 \).

We repeat the previous argument to \( \mathcal{A}' \) and using the above bijection. More specifically, consider \( U(x) := V_{m-2}(x) \cap V_{m^- - 1}(x) \) and \( V(x) := V_{m-1}(x) \) for each \( x \in Y \). Consequently, we may find a Borel set \( Y \subset X \) of full \( \mu \)-measure such that \( V_{m^- - 2}(\omega(0)) := \bigoplus_{j=1}^{m-2} H_j(\omega) \) for every \( x \in Y \) and for \( W_x \)-almost every \( \omega \in \Phi \).

By still repeating this argument \((m-3)\)-times, we may find a Borel set \( Y \subset X \) of full \( \mu \)-measure \( 1 \leq i \leq l' \), such that \( V_i(\omega(0)) := \bigoplus_{j=1}^{l'-1} H_j(\omega) \) for every \( 1 \leq i \leq m \) and every \( x \in Y \) and for \( W_x \)-almost every \( \omega \in \Phi \). In particular, \( m^- = m \).
Setting $H_i(x) := V_i(x) \cap V_i^{-}(x)$, we deduce that $H_i(\omega) = H_i(x)$ and for $W_x$-almost every path $\omega \in \Phi$.

**Step 4: End of the proof.**

First, observe that by Step 3, $H_i(x) \subset V_i(x) \setminus V_{i+1}(x)$ for $\mu$-almost every $x \in X$. Recall also from Step 3 that for such a point $x$, $V_i(\omega) = V_i(x)$ for $W_x$-almost every $\omega \in \Phi$. Consequently, by Theorem 7.2 $\lim_{n \to \infty} \frac{1}{n} \log \frac{\|A(\omega,n)\|}{\|e\|} = \chi_i$, for $W_x$-almost every $\omega \in \Phi$ and for every $v \in H_i(x)$.

Next, we will prove the following weaker version of assertion (iii):

**There exists a set $Y \subset X$ of full $\mu$-measure such that for every subset $S \subset N := \{1, \ldots, m\}$,**

$$
\lim_{n \to \infty} \frac{1}{n} \log \sin \left| \angle \left( H_S(\omega(n)), H_{N \setminus S}(\omega(n)) \right) \right| = 0
$$

for every $x \in Y$ and $W_x$-almost every path $\omega \in \Phi$.

Although the argument is standard, we still reproduce it here for the sake of completeness. To this end consider the function

$$
\phi(\omega) := \log \sin \left| \angle \left( H_S(\omega(0)), H_{N \setminus S}(\omega(0)) \right) \right|, \quad \omega \in \Omega.
$$

Observe that

$$
|\phi(T^i \omega) - \phi(\omega)| \leq \log \max \{\|A(\omega, 1)\|, \|A^{-1}(\omega, 1)\|\}.
$$

So $\phi \circ T - \phi$ is $\mu$-integrable. Hence, our desired conclusion follows from Lemma 6.3.

Summarizing what has been done in Step 4, we have shown that there exists a (not necessarily saturated) Borel set $Y \subset X$ of full $\mu$-measure such that all assertions (i)–(iii) of Theorem 3.7 hold. Moreover, for each $x \in Y$, there exists a set $\mathcal{F}_x \subset \Omega_x$ of full $W_x$-measure such that identity (3.1) and identity (3.2) hold for all $\omega \in \mathcal{F}_x$. It remains to show that by shrinking the set $Y$ a little we can find such a set $Y$ which is also leafwise saturated.

The following result is needed.

**Lemma 9.19.** Let $\Xi \subset \Omega$ be a $T$-totally invariant subset of full $\bar{\mu}$-measure. Then there exists a leafwise saturated Borel subset $Y \subset X$ of full $\mu$-measure such that for every $y \in Y$, $\Xi$ is of full $W_y$-measure.

**Proof.** We say that a set $Z \subset X$ is almost leafwise saturated if $a \in Z$ implies that the whole leaf $L_a$ except a null Lebesgue measure set is contained in $Z$, where the Lebesgue measure on $L_a$ is induced by the Riemannian metric $g$ on $L_a$. Since $\Xi \subset \Omega$ is $T$-totally invariant, $T(\Omega \setminus \Xi) = \Omega \setminus \Xi$. On the other hand, we deduce from $\bar{\mu}(\Xi) = 1$ that $\bar{\mu}(\Omega \setminus \Xi) = 0$. Hence, $\bar{\mu}(T(\Omega \setminus \Xi)) = 0$. So there exists an almost leafwise saturated subset $Z \subset X$ of full $\mu$-measure such that for every $x \in Z$, $W_x(T(\Omega \setminus \Xi)) = 0$. Let $Y$ be the leafwise saturation of $Z$. Clearly,
\[ \mu(Y) = 1. \] By shrinking \( Y \) a little if necessary we may assume that \( Y \) is a Borel set. Let \( y \) be an arbitrary point in \( Y \). Since for Vol-almost every \( x \in L_y \) we have \( x \in Z \) it follows that \( W_x(T(\Omega \setminus \Xi)) = 0 \) for such a point \( x \). Consequently, by Proposition 5.4 (i), we get that

\[ W_y(\Omega \setminus \Xi) \leq \int_{x \in L_y} p(x, y, 1) W_x(T(\Omega \setminus \Xi))d\text{Vol}(x) = 0. \]

Hence, \( \Xi \) is of full \( W_y \)-measure for all \( y \in Y \).

Now we resume the proof of the First Main Theorem. By shrinking the set \( Y \subset X \) a little we may assume without loss of generality that \( Y \) is almost leafwise saturated of full \( \mu \)-measure. Let \( Y' \) be the leafwise saturation of \( Y \). Using the action of \( A \) on \( \mathbb{R}^d \), we can extend \( m \) functions \( Y \ni x \mapsto H_i(x) \) to \( m \) functions \( Y' \ni x \mapsto H_i(x) \) as follows: given any point \( x' \in Y' \), we find a point \( x \in L_{x'} \cap Y \) and set \( H_i(x') = A(\omega, 1)H_i(x) \) for any path \( \omega \in \Omega_x \) with \( \omega(1) = x' \). This extension is well-defined (i.e. no monodromy problem occurs) because \( Y \ni x \mapsto H_i(x) \) is \( A \)-invariant. Consider the set

\[ \Xi := \left\{ \omega \in \Omega(X, \mathcal{L}) : \lim_{n \to \infty} \frac{1}{n} \log \frac{\|A(\omega, n)u\|}{\|u\|} = \chi_i, \, \forall u \in H_i(x) \setminus \{0\}, \, \forall 1 \leq i \leq m, \right\} \]

\[ \& \lim_{n \to \infty} \frac{1}{n} \log \sin |\angle(H_S(\omega(n)), H_{N \setminus S}(\omega(n)))| = 0, \quad \forall S \subset N := \{1, \ldots, m\}. \]

By Step 3 as well as the assertion established in the preceding paragraph, \( \Xi \) is of full \( \bar{\mu} \)-measure. On the other hand, using that \( x \mapsto H_i(x) \) is \( A \)-invariant, it is straightforward to see that \( \Xi \) is \( T \)-totally invariant. Therefore, applying Lemma 9.19 yields a leafwise saturated Borel set \( Y'' \subset Y' \) of full \( \mu \)-measure such that \( \Xi \) is of full \( W_y \)-measure for every \( y \in Y'' \). This completes the proof of Theorem 3.7 in the case \( \mathbb{G} = \mathbb{N} \). In the sequel we write \( Y \) instead of \( Y'' \) for simplicity.

**Case II:** \( \mathbb{G} = \mathbb{R}^+ \).

We only need to establish assertion (ii) and (iii) of Theorem 3.7. Without loss of generality we may assume that \( t_0 = 1 \). By the hypothesis the function \( F : \Omega(X, \mathcal{L}) \to \mathbb{R}^+ \) given by

\[ F(\omega) := \sup_{t \in [0,1]} \left| \log \|A^{\pm 1}(\omega, t)\| \right|, \quad \omega \in \Omega(X, \mathcal{L}), \]

is \( \bar{\mu} \)-integrable. Therefore, by Birkhoff’s ergodic theorem \( \frac{1}{n} F \circ T^n \) converge to 0 \( \bar{\mu} \)-almost everywhere when the integers \( n \) tend to \( \infty \). On the other hand, for \( n \leq t < n + 1 \) and for \( u \in \mathbb{R}^d \setminus \{0\} \), we have that

\[ \left| \log \frac{\|A(\omega, t)u\|}{\|u\|} - \log \frac{\|A(\omega, n)u\|}{\|u\|} \right| \leq \log \max\{\|A(T^n \omega, t-n)\|, \|A^{-1}(T^n \omega, t-n)\|\}. \]
The right hand side is bounded by \((F \circ T^n)(\omega)\). This, coupled with the convergence of \(\frac{1}{n} \log \|A(\omega, t)u\|\) to \(\chi_i\) when \(u \in H_i(x) \setminus \{0\}\) and the integers \(n\) tend to \(\infty\) for \(W_x}\)-almost every \(\omega \in \Omega_x\), and with the convergence of \(\frac{1}{n} F \circ T^n(\omega)\) to 0 for \(\bar{\mu}\)-almost everywhere \(\omega\), implies that
\[
\lim_{t \to \infty, \, t \in \mathbb{R}^+} \frac{1}{t} \log \frac{\|A(\omega, t)u\|}{\|u\|} = \chi_i, \quad u \in H_i(x) \setminus \{0\},
\]
for \(\bar{\mu}\)-almost everywhere \(\omega\). Consequently, it is sufficient to apply Lemma 9.19 in order to conclude assertion (ii).

We turn to the proof of assertion (iii). Fix a subset \(S \subset N := \{1, \ldots, m\}\) and consider the function
\[
\phi(\omega) := \log \sin \left| \angle \left( \left[ H_S(\omega(t)), H_{N \setminus S}(\omega(t)) \right] \right) \right|, \quad \omega \in \Omega.
\]
Observe that, for \(n \leq t < n + 1\),
\[
\left| \phi(T^t \omega) - \phi(T^n \omega) \right| \leq \log \max \{\|A(T^n \omega, t - n)\|, \|A^{-1}(T^n \omega, t - n)\|\}.
\]
The right hand side is bounded by \((F \circ T^n)(\omega)\). This, coupled with the limit
\[
\lim_{n \to \infty} \frac{1}{n} \log \sin \left| \angle \left( \left[ H_S(\omega(t)), H_{N \setminus S}(\omega(t)) \right] \right) \right| = 0
\]
and with the convergence of \(\frac{1}{n} F \circ T^n(\omega)\) to 0 for \(\bar{\mu}\)-almost everywhere \(\omega\), implies that
\[
\lim_{t \to \infty, \, t \in \mathbb{R}^+} \frac{1}{t} \log \sin \left| \angle \left( \left[ H_S(\omega(t)), H_{N \setminus S}(\omega(t)) \right] \right) \right| = 0
\]
for \(\bar{\mu}\)-almost everywhere \(\omega\). Using this and applying Lemma 9.19 again, assertion (iii) follows. \(\square\)

**Proof of Corollary 3.8**

Let \(Y\) be the set given by Theorem 3.7. Part (i) and (ii) of this theorem implies that the functions \(m\) and \(\chi_i\) are leafwise constant. Using the assumption that \(\mu\) is ergodic and removing from \(Y\) a null \(\mu\)-measure set if necessary, the conclusion (except the two identities) of the corollary follows.

If \(G = \mathbb{N}\), the two identities of the corollary hold by Ruelle’s work [32]. If \(G = \mathbb{R}^+\), we argue as in Case II of the proof of Theorem 3.7. \(\square\)

**Proof of Corollary 3.9**

Using the remark following Theorem 3.7 the case \(k = 1\) of the corollary is exactly Corollary 3.8. Now we consider the case \(k > 1\). If in assertion (iii) of the corollary we replace \(\chi(x; v_1, \ldots, v_k)\) by \(\chi(\omega; v_1, \ldots, v_k)\), \(\omega \in \Omega_x\), then assertions (ii), (iii) and (iv) of the corollary follow from Ruelle’s work [32]. So it suffices to prove assertion (i). To this end we apply assertion (ii) of Theorem 3.7 to the cocycle \(A^k\). Consequently, we may find a leafwise saturated Borel set \(Y \subset X\) of...
full $\mu$-measure such that for every $x \in Y$, there exists a set $\mathcal{F} \subset \Omega_x$ of full $W_x$-measure such that if $v_1, \ldots, v_k \in \mathbb{R}^d$ are fixed, then $\chi(\omega; v_1, \ldots, v_k)$ is constant for all $\omega \in \mathcal{F}_x$. We denote by $\chi(x; v_1, \ldots, v_k)$ this common value. Assertion (i) follows. □

We arrive at the spectrum description in terms of weakly harmonic measures. We are inspired by Ledrappier [26, Proposition 5.1, pp. 328-329] who studies the case of maps.

**Theorem 9.20.** We keep the hypotheses, notation and conclusions of Corollary 10.38. So $Y$ is a leafwise saturated Borel set of full $\mu$-measure given by this corollary.

1) For each probability measure which is also an ergodic element $\nu \in \text{Har}_\mu(X \times \mathbb{P}(\mathbb{R}^d))$, there is a unique integer $1 \leq i \leq m$ such that
   
   (i) $\int_{X \times \mathbb{P}(\mathbb{R}^d)} \varphi \, d\bar{\nu} = \chi_i$;
   
   (ii) $\nu$ is supported by the total space of the $A$-invariant subbundle $Y \ni x \mapsto \mathbb{P}(H_i(x))$, i.e.,
   
   $\nu \{ (x, u) : x \in X \& u \in \mathbb{P}(H_i(x)) \} = 1$.

2) Conversely, for each $1 \leq i \leq m$, there exists such a measure $\nu$.

   In particular, the spectrum (i.e. the set of all Lyapunov exponents) of $A$ is the set of values of $\int_{X \times \mathbb{P}(\mathbb{R}^d)} \varphi \, d\bar{\nu}$ as $\nu$ runs over all probability measures which are also ergodic elements in $\text{Har}_\mu(X \times \mathbb{P}(\mathbb{R}^d))$.

**Proof.** Recall from Step 3 and Step 4 in Case I in the proof of Theorem 3.7 that

$$H_i(\omega) = H_i(x) \quad (9.24)$$

for every $x \in Y$ and for $W_x$-almost every $\omega \in \Omega$. Next, applying Corollary 10.37 to $\nu$ yields that $\tilde{\nu}$ is $T$-ergodic on $\Omega_{1,A}$ and $\tilde{\nu}$ is $T$-ergodic on $\widehat{\Omega}_{1,A}$, where $\tilde{\nu}$ is the Wiener measure with initial distribution $\nu$ given by (2.7), and $\tilde{\nu}$ is the natural extension of $\nu$ on $\widehat{\Omega}_{1,A}$. Recall from Lemma 9.1 that $\Omega_{1,A} \equiv \Omega \times \mathbb{P}(\mathbb{R}^d)$ and $\widehat{\Omega}_{1,A} \equiv \widehat{\Omega} \times \mathbb{P}(\mathbb{R}^d)$. Using all these and applying [26] Proposition 5.1, pp. 328-329 to the ergodic map $T$ acting on $(\Omega_{1,A}, \tilde{\nu})$ (resp. $(\widehat{\Omega}_{1,A}, \tilde{\nu})$) yields a unique integer $i$ with $1 \leq i \leq m$ such that $\int_{\Omega \times \mathbb{P}(\mathbb{R}^d)} \varphi \, d\tilde{\nu} = \chi_i$ and that

$$\tilde{\nu} \{ (\omega, u) : \omega \in \Omega \& u \in \mathbb{P}(H_i(\omega(0))) \} = 1.$$

Combining the former equality with (2.7), assertion (i) follows. The latter equality, coupled with identity (9.24) and (2.7), implies assertion (ii), thus proving Part 1).

Now we turn to Part 2). By Theorem 10.5, there is a bimeasurable bijection between the bundle $Y \ni x \mapsto H_i(x)$ and $Y \times \mathbb{R}^d$ covering the identity and which is linear on fibers. Using this and applying Part 1) of Lemma 9.9 we may find an
ergodic weakly harmonic probability measure $\nu$ living on the leafwise saturated subset $\{(x, \mathbb{P}H_i(x)) : x \in Y\}$ of the lamination $(X_{1,A}, \mathcal{L}_{1,A})$, that is,

$$\nu(\{(x, \mathbb{P}H_i(x)) : x \in Y\}) = 1).$$

Arguing as in the proof of Part 1), Part 2) follows.

9.4 Second Main Theorem and its corollaries

Let $(X, \mathcal{L})$ be a lamination satisfying the Standing Hypotheses. In this subsection we will combine Theorem 3.7 and Candel’s results [3] in order to establish Theorem 3.11. Let $\text{dist}$ be the distance function induced by the Riemannian metric on every leaf. Following Candel [3] we introduce the following terminology

**Definition 9.21.** A cocycle $A : \Omega(X, \mathcal{L}) \times \mathbb{R}^+ \to \text{GL}(d, \mathbb{R})$ is said to be *moderate* if there exist constants $C, R > 0$ such that

$$\log \|A^{\pm 1}(\omega, t)\| \leq C \text{dist}(\omega(t), \omega(0)) + R, \quad \omega \in \Omega, \ t \in \mathbb{R}^+. $$

Here is a simple sufficient condition for a moderate cocycle.

**Lemma 9.22.** If $A$ is $C^1$-differentiable cocycle on a compact $C^1$ smooth lamination, then $A$ is moderate.

**Proof.** Choose a finite covering of $X$ by flow boxes $\Phi_i : U_i \to \mathbb{B}_i \times T_i$ with $\mathbb{B}_i$ simply connected. In any flow box $\Phi_i$, let $\alpha_i : \mathbb{B}_i \times \mathbb{B}_i \times T_i \to \text{GL}(d, \mathbb{R})$ be the local expression of $A$ (see Definition 3.10 with the choice $t_0 := 1$). So

$$\alpha_i(x, y, s) = A(\omega, 1), \quad (x, y, s) \in \mathbb{B}_i \times \mathbb{B}_i \times T_i,$$

where $\omega$ is any leaf path such that $\omega(0) = \Phi_i^{-1}(x, s), \omega(1) = \Phi_i^{-1}(y, s)$ and $\omega[0, 1]$ is contained in the simply connected plaque $\Phi_i^{-1}(\cdot, s)$. We deduce from this formula that $\alpha_i(x, x, s) = \text{id}$. Consequently, $\|\alpha_i(x, y, s) - \text{id}\| \leq C \|x - y\|$ for a finite constant $C$ independent of the flow box $\Phi_i$. This implies the desired conclusion.

We will prove the following

**Proposition 9.23.** Let $A$ be a moderate cocycle. Then the function $F : \Omega(X, \mathcal{L}) \to \mathbb{R}^+$ defined by

$$F(\omega) := \sup_{t \in [0,1]} \log^+ \|A^{\pm 1}(\omega, t)\|, \quad \omega \in \Omega(X, \mathcal{L}),$$

is \(\bar{\mu}\)-integrable.
Proof. Since $\mathcal{A}$ is moderate, we get that
\[
\log^+ \|\mathcal{A}^\pm(\omega, t)\| \leq C \text{dist}(\omega(0), \omega(t)) + R, \quad \omega \in \Omega(X, \mathcal{L}), \ t \in \mathbb{R}^+.
\]
Therefore,
\[
\int_{\Omega(X, \mathcal{L})} \sup_{t \in [0, 1]} \log^+ \|\mathcal{A}^\pm(\omega, t)\| d\bar{\mu}(\omega) \leq R + C \int_{\Omega} \sup_{t \in [0, 1]} \text{dist}(\omega(0), \omega(t)) d\bar{\mu}(\omega).
\]
By formula (2.7) we may rewrite the integral on the right hand side as
\[
\int_X \left( \int_{\Omega} \sup_{t \in [0, 1]} \text{dist}(\omega(0), \omega(t)) dW_x(\omega) \right) d\mu(x).
\]
We will prove that the inner integral is bounded from above by a constant independent of $x$. This will imply that the function $\omega \mapsto R + C \sup_{t \in [0, 1]} \text{dist}(\omega(0), \omega(t))$ is $\bar{\mu}$-integrable, and hence so is the function $F$. To this end we focus on a single $L$ passing through a given fixed point $x$. Observe that
\[
\int_{\Omega_x} \sup_{t \in [0, 1]} \text{dist}(\omega(0), \omega(1)) dW_x(\omega) = \int_0^\infty W_x\{\omega \in \Omega_x : \sup_{t \in [0, 1]} \text{dist}(\omega(0), \omega(t)) > s\} ds.
\]
The following estimate is needed.

**Lemma 9.24.** There is a finite constant $c > 0$ such that for all $s \geq 1$,
\[
W_x \left\{ \omega \in \Omega(X, \mathcal{L}) : \sup_{t \in [0, 1]} \text{dist}(\omega(0), \omega(t)) > s \right\} < ce^{-s^2}.
\]

**Proof.** It follows by combining Lemma 8.16 and Corollary 8.8 in [3]. \qed

Resuming the proof of Proposition 9.23, Lemma 9.24 applied to the right hand side of the last inequality, shows that the integral
\[
\int_{\Omega_x} \sup_{t \in [0, 1]} \text{dist}(\omega(0), \omega(t)) dW_x(\omega)
\]
is bounded from above by a constant independent of $x$. This completes the proof. \qed

Now we are able to prove Theorem 3.11. The proof is divided into two steps.

**Step I:** Proof of assertions (i) and (ii).

By Lemma 9.22, $\mathcal{A}$ is moderate. By Proposition 9.23
\[
\int_{\Omega(X, \mathcal{L})} \sup_{t \in [0, 1]} \|\mathcal{A}^{\pm}(\omega, t)\| d\bar{\mu}(\omega) < \infty.
\]
Consequently, we are able to apply Theorem 3.7. Hence, assertions (i) and (ii) of Theorem 3.11 follow.

**Step II:** Proof of assertions (iii).

First we will prove that $\chi_{\text{max}}(\mathcal{A}) \leq \chi_1 \leq \bar{\chi}_{\text{max}}(\mathcal{A})$. In fact, we only show that $\chi_1 \leq \bar{\chi}_{\text{max}}(\mathcal{A})$ since the inequality $\chi_1 \geq \chi_{\text{max}}(\mathcal{A})$ can be proved in the same way. The proof is divided into several sub-steps.

**Sub-step II.1:** Proof that for every $x \in Y$ and $u \in \mathbb{P}(\mathbb{R}^d)$ and $t > 0$,

$$
\int_{\Omega_x} \log \|A(\omega, t)u\|dW_\omega(x) \leq \int_0^t (D_0\delta(A))(x)ds.
$$

(9.25)

To prove (9.25) we fix an arbitrary point $x \in X$ and an arbitrary $u \in \mathbb{P}(\mathbb{R}^d)$. Let $\pi : \tilde{L} \to L$ be the universal cover of the leaf $L := L_x$ and fix $\tilde{x} \in \tilde{L}$ that projects to $x$. Recall that the bijective lifting $\pi^{-1}_x : \Omega_x \to \hat{\Omega}_x$ identifies the two path-spaces canonically. Following (9.3) consider the specialization $f : \tilde{L} \to \mathbb{R}$ of $A$ at $(\tilde{L}, \tilde{x}; u)$ defined by

$$
f(\tilde{y}) := \log \|A(\omega, 1)u\|,
\hfill (9.26)
$$

where $\omega \in \Omega_x$ is any path such that $(\pi^{-1}_x(\omega))(1) = \tilde{y}$.

Fix an arbitrary point $y_0 \in L$ and an arbitrary point $\tilde{y}_0 \in \pi^{-1}(y_0)$. Let $v := A(\omega, 1)u \in \mathbb{P}(\mathbb{R}^d)$, where $\omega \in \Omega_x$ is any path such that $(\pi^{-1}_x(\omega))(1) = \tilde{y}_0$. Let $y$ be an arbitrary point in a simply connected, connected open neighborhood of $y_0$. On this neighborhood a branch of $\pi^{-1}$ such that $\pi^{-1}(y_0) = \tilde{y}_0$ is well-defined. Set $\tilde{y} := \pi^{-1}(y)$. By (9.4), we have that

$$
f_v(\tilde{y}_0) = f(\tilde{y}) - f(\tilde{y}_0),
\hfill (9.26)
$$

where $f$ is defined by (9.26). This, combined with formula (3.5) and (9.6), implies that

$$
\bar{\delta}(\mathcal{A})(y_0) = \Delta_y f_v(\tilde{y}_0) = (\bar{\Delta}f)(\tilde{y}_0).
\hfill (9.27)
$$

In summary, we have proved the following crucial estimate:

$$
\bar{\delta}(\mathcal{A})(y) \geq (\Delta f)(\tilde{y}), \quad y \in L, \; \tilde{y} \in \pi^{-1}(y).
\hfill (9.27)
$$

Consider the cocycle $\tilde{A}$ on $\tilde{L}$ defined by

$$
\tilde{A}(\tilde{\omega}, t) := A(\pi(\tilde{\omega}), t), \quad t \in \mathbb{R}^+, \; \tilde{\omega} \in \Omega(\tilde{L}).
\hfill (9.27)
$$

Using the homotopy law for $\tilde{A}$ and using the simple connectivity of $\tilde{L}$, we see that

$$
f(\pi(t(\tilde{\omega}))) = \log \|\tilde{A}(\tilde{\omega}, t)u\|,
\hfill (9.27)
$$

where $\tilde{\omega} \in \Omega_{\tilde{x}}(\tilde{L})$, $t \in \mathbb{R}^+$. Consequently, we infer from (9.3) with $(\tilde{z}, v) = (\tilde{x}, u)$ that

$$
E_x[\log \|A(\cdot, t)u\|] = E_x[\log \|\tilde{A}(\cdot, t)u\|] = (D_0f)(\tilde{x}) = (D_tf)(\tilde{x}) - f(\tilde{x}),
\hfill (9.28)
$$

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where the last equality holds because of $f(\tilde{x}) = 0$.

Recall from Definition 8.3 in [3] that a function $h$ defined on a complete Riemannian manifold $M$ with distance function $\text{dist}$ is said to be moderate if

$$\log |h(y) - h(z)| \leq C \text{dist}(y, z) + R, \quad y, z \in M,$$

for some constants $C, R > 0$. We need the following result.

**Lemma 9.25.** If $f, |df|, \Delta f$ are moderate functions on $\tilde{L}$, then

$$(D_{t}f)(\tilde{x}) - f(\tilde{x}) = \int_{0}^{t} D_{s}\Delta f(\tilde{x}) ds.$$  

**Proof.** It follows from the proof of Proposition 8.11 and Theorem 8.13 in [3]. Note that here is the place where we make use of the hypotheses that $(X, \mathcal{L})$ is a compact $C^{2}$-differentiable lamination and the leafwise metric $g$ is transversally continuous as Proposition 8.11 in [3] requires. 

**Lemma 9.26.** Assume that the cocycle $A$ is $C^{2+\epsilon}$-differentiable for some $\epsilon > 0$. Then $f, |df|, \Delta f$ are moderate functions on $\tilde{L}$, where $f$ is the function defined by (9.3), $x \in X$ being fixed.

**Proof.** We only prove that $\Delta f$ is moderate since the other assertions can be proved similarly. For every $y_{0} \in L_{x}$, let $v \in \mathbb{P}(\mathbb{R}^{d})$ be defined just after (9.26). So the function $f_{v,y_{0}}$ constructed in (3.4) is well-defined on any simply connected neighborhood of $y_{0}$ in $L_{x}$. We will show that there is a constant $C > 0$ independent of $x$ and $y_{0}$ such that

$$|\Delta_{y}f_{v,y_{0}}(y) - \Delta_{y}f_{v,y_{0}}(y_{0})| \leq C \text{dist}(y, y_{0})$$  

(9.29)

for every $y$ in any simply connected plaque passing through $y_{0}$. By (9.6) and by the compactness of the lamination $(X, \mathcal{L})$ as well as Hypothesis (H1), (9.29) will easily imply that $\Delta f$ is moderate.

To prove (9.29) let $\Phi : U \to \mathcal{B} \times \mathbb{T}$ be a flow box containing $y_{0}$. Let $\alpha : \mathcal{B} \times \mathcal{B} \times \mathbb{T} \to \text{GL}(d, \mathbb{R})$ be the local expression of $A$ on a (see Definition 3.10).

So there is $s \in \mathbb{T}$ such that

$$\alpha(y_{0}, y, s) = A(\eta, 1), \quad (y_{0}, y, s) \in \mathcal{B} \times \mathcal{B} \times \mathbb{T},$$

where $\eta$ is any leaf path such that $\eta(0) = \Phi_{1}^{-1}(y_{0}, s), \eta(1) = \Phi_{1}^{-1}(y, s)$ and $\eta[0, 1]$ is contained in the simply connected plaque $\Phi_{1}^{-1}(\cdot, s)$. Since $A$ is $C^{2+\epsilon}$-differentiable, we deduce from the last equality that $\alpha(y_{0}, \cdot, s) \in C^{2+\epsilon}$. This, combined with the equality

$$f_{v,y_{0}}(y) = \log \|A(\eta, 1)v\| = \log \|\alpha(y_{0}, y, s)v\| = \log \|\alpha(y_{0}, y, s)v\| - \log \|\alpha(y_{0}, y_{0}, s)v\|$$

implies (9.29).
Coming back the proof of assertion (iii), recall from the hypotheses that $A$ is $C^{2+\epsilon}$-differentiable. Therefore, by Lemma 9.26 the function $f$ defined by (9.3) satisfies the hypotheses of Lemma 9.25. Consequently, for every $u \in \mathbb{P}(\mathbb{R}^d)$,

$$\int_{\Omega_x} \log \|A(\omega,t)u\|dW_x(\omega) = \int_0^t D_s f(\tilde{x})ds \leq \int_0^t (D_s \delta(A))(x)ds, \quad x \in X, \ t > 0,$$

where the equality holds by combining Lemma 9.25 and (9.28), and the inequality holds by an application of inequality (9.27). This proves (9.25).

**Sub-step II.2:** End of the proof of the inequality $\chi_1 \leq \bar{\chi}_{\max}(A)$.

By Theorem 9.20 there exists a probability measure $\nu$ which is an ergodic element of $\text{Har}_\mu(X \times P)$ such that $\int_{X \times P} \varphi d\nu = \chi_1$. Consequently, using this together with Theorem 9.6 and formula (9.1) for the canonical cocycle $C$, we infer that

$$\int_{\Omega \times \mathbb{P}(\mathbb{R}^d)} \log \|A(\omega,1)u\|d\tilde{\nu}(\omega,u) = \chi_1.$$ 

Using formula (9.7) we rewrite the left hand side as

$$\int_X \left( \int_{u \in \mathbb{P}(\mathbb{R}^d)} \int_{\Omega_x} \log \|A(\omega,1)u\|dW_x(\omega) d\nu'_x(u) \right) d\mu(x).$$

Next, applying inequality (9.25) to the inner integral and recalling that each $\nu_x$ is a probability measure on $\mathbb{P}(\mathbb{R}^d)$, we deduce from the last two equalities that

$$\chi_1 \leq \int_X \int_0^1 (D_s \delta(A))(x)dsd\mu(x).$$

On the other hand, since $\mu$ is harmonic, we get that

$$\int_{X_0} (D_s \delta(A))(x)dsd\mu(x) = \int_{X_0} \delta(A)(x)d\mu(x), \quad s > 0.$$ 

Combining this and the last inequality and formula (3.6) together, it follows that $\chi_1 \leq \bar{\chi}_{\max}(A)$.

Now we turn to the proof of $\chi_{\min} \leq \chi_m \leq \bar{\chi}_{\min}$. Recall from (3.2) that the Lyapunov exponents of the cocycle $A^{<i}$ are $-\chi_1 < \cdots < -\chi_m$. Hence, what has been done before shows that $\chi_{\max}(A^{<1}) \leq -\chi_m \leq \bar{\chi}_{\max}(A^{<1})$. This, coupled with (3.6), completes the proof.

**Proof of Corollary 3.12.**

By Corollary 3.8 and Corollary 3.9 the maximal Lyapunov exponent $\chi_1(A^{\wedge k})$ of the cocycle $A^{\wedge k}$ is equal to the sum $\sum_{i=1}^k \chi_i$. On the other hand, by Theorem 3.11 $\chi_{\max}(A^{\wedge k}) \leq \chi_1(A^{\wedge k}) \leq \bar{\chi}_{\max}(A^{\wedge k})$. This, combined with the last equality, finishes the proof.

**Proof of Corollary 3.13.**
We apply Theorem 3.11 to the holonomy cocycle of the foliation \((X, \mathcal{L})\). Since we know by hypothesis that this cocycle admits \(d\) distinct Lyapunov exponents with respect to \(\mu\), it follows that the integer \(m\) given by Theorem 3.11 coincides with \(d\). Hence, in the Oseledec decomposition in assertion (i) of this theorem we have that \(\dim H_i(x) = 1, 1 \leq i \leq d\) for every \(x\) in a leafwise saturated Borel set \(Y \subset X\) of full \(\mu\)-measure. Clearly, for such a point \(x\) the leaf \(L_x\) is holonomy invariant. \(\square\)

10 Appendix: Measure and Ergodic theories on sample-path spaces

Given a lamination \((X, \mathcal{L}, g)\) we first develop measure and ergodic theories on the sample-path space \(\Omega(X, \mathcal{L})\) endowed with the \(\sigma\)-algebra \(\mathcal{A}\) (introduced in Subsection 2.5) and the extended sample-path space \(\hat{\Omega}(X, \mathcal{L})\) endowed with the \(\sigma\)-algebra \(\hat{\mathcal{A}}\) (introduced in Subsection 8.1). Next, we prove Theorem 2.11, Proposition 2.12, Theorem 2.13, Proposition 4.1, Theorem 4.6 and Proposition 7.14. These results have been stated and used in the previous sections. The section is divided into several subsections. Note that the study of the \(\sigma\)-algebra \(\tilde{\mathcal{A}}\) (defined in Subsection 2.4) in the same context is thoroughly investigated in the works \([4, 5, 3]\). The main difference between the measure theory with \(\tilde{\mathcal{A}}\) and that with \(\mathcal{A}\) is that the holonomy phenomenon plays a vital role in the latter context but not in the former one. In what follows \(\sqcup\) denotes the disjoint union.

10.1 Multifunctions and measurable selections

We start the first part of this section with a review on the theory of measurable multifunctions as presented in the lecture notes by Castaing and Valadier in \([7]\) and as developed in Walters’ article \([35]\). Consider a measurable space \((T, \mathcal{T})\) and a separable locally complete metric space \(S\). A multifunction \(\Gamma\) from \(T\) to \(S\) associates to each \(t \in T\) a nonempty subset \(\Gamma(t) \subset S\). The graph of such a multifunction \(\Gamma\) is \(G(\Gamma) := \{(t, s) \in T \times S : s \in \Gamma(t)\}\). We say that a multifunction \(\Gamma\) is measurable if its graph \(G(\Gamma)\) belongs to the product of two \(\sigma\)-algebras \(\mathcal{T} \otimes \mathcal{B}(S)\). Here \(\mathcal{B}(S)\) denotes, as usual, the Borel \(\sigma\)-algebra of \(S\).

An important problem in the theory of multifunctions is to prove the existence of a measurable selection of \(\Gamma\): a selection is a map \(\sigma : T \rightarrow S\) such that \(\sigma(t) \in \Gamma(t)\) for all \(t \in T\).

The following result from Theorem III.6 in \([7]\) plays an important role in this work.

**Theorem 10.1.** Let \((T, \mathcal{T})\) be a measurable space and \(S\) a separable complete metric space, and \(\Gamma\) be a multifunction map \(T\) to non-empty closed subsets of \(S\).
If for each open set \( U \) in \( S \), \( \Gamma^{-}(U) := \{ t \in T : \Gamma(t) \cap U \neq \emptyset \} \) belongs to \( \mathcal{T} \), then \( \Gamma \) admits a measurable selection.

When \( \Gamma \) is non necessarily closed-valued, the following von Neumann-Aumann-Sainte Beuve measurable selection theorem is very useful.

**Theorem 10.2.** Let \( (T, \mathcal{T}) \) be a measurable space and \( S \) a separable locally complete metric space. Let \( \Gamma \) be a measurable multifunction from \( T \) to \( S \). Then \( \Gamma \) admits a selection \( \sigma : T \to S \) such that \( \sigma \) is \( \nu \)-measurable for every positive finite measure \( \nu \) on \( (T, \mathcal{T}) \).

**Proof.** The hypothesis on \( S \) implies that there exists a metric \( d \) on \( S \) which induces the same topology such that \( (S, d) \) is complete. Hence, \( S \) is a Polish space, in particular, it is Suslin. Hence, the theorem is a consequence of Theorem III.22 in [7]. \( \square \)

We also have the following measurable projection theorem.

**Theorem 10.3.** (see [7, p. 75]) Let \( (X, \mathcal{S}, \mu) \) be a complete probability measure and let \( Y \) be a complete separable metric space endowed with the Borel \( \sigma \)-algebra \( \mathcal{B}(Y) \). Consider the natural projection of \( X \times Y \) onto \( X \). Then for every \( Z \in \mathcal{S} \otimes \mathcal{B}(Y) \), the projection of \( Z \) onto \( X \) is in \( \mathcal{S} \).

The following result is useful.

**Lemma 10.4.** Let \( (T, \mathcal{T}, \mu) \) be a complete probability measure space and \( S \) a complete separable metric space endowed with the Borel \( \sigma \)-algebra \( \mathcal{B}(S) \). Let \( \phi : T \times S \to \mathbb{R} \) be a measurable function. Then the function \( T \ni t \mapsto M(t) := \sup_{s \in S} \phi(t, s) \) is measurable.

**Proof.** We only need to check that \( \{ t \in T : M(t) > \alpha \} \) is measurable for all \( \alpha \in \mathbb{R} \). Note that the last set is equal to the image under the projection onto \( T \) of the set \( \{(t, s) \in T \times S : \phi(t, s) > \alpha \} \). This image is measurable by Theorem 10.3. \( \square \)

We also need the following weaker version of Theorem 7 in [35].

**Theorem 10.5.** Let \( (X, \mathcal{S}, \mu) \) be a complete probability measure space and let \( x \mapsto V_x \) of \( X \) into \( \text{Gr}_k(\mathbb{R}^d) \) be a map, where \( 0 \leq k \leq d \) are given integers. Then the following are equivalent:

1) \( x \mapsto V_x \) is a measurable map, where \( \text{Gr}_k(\mathbb{R}^d) \) is endowed with the Borel \( \sigma \)-algebra \( \mathcal{B}(\text{Gr}_k(\mathbb{R}^d)) \).

2) There are measurable maps \( v_1, \ldots, v_k : X \to \mathbb{R}^d \) such that for all \( x \in X \), \( \{v_1(x), \ldots, v_k(x)\} \) is a basis for \( V_x \).

3) There is a bi-measurable bijection \( \Lambda \) from \( \{(x, v) : v \in V_x\} \) onto \( X \times \mathbb{R}^d \) which is linear on fibers and covers the identity map of \( X \), that is, for each \( x \in X \), \( \Lambda(x, \cdot) \) is an invertible linear transformation from \( V_x \) onto \( \{x\} \times \mathbb{R}^d \).
10.2 $\sigma$-algebras: approximations and measurability

In this subsection we recall some results of the measure theory and prove some new ones. A *simple function* on a measurable space $(S, \mathcal{S})$ is any finite sum

$$f := \sum a_i 1_{A_i}, \quad \text{where } a_i \in \mathbb{R}, \ A_i \in \mathcal{S}.$$  

The following result is elementary.

**Proposition 10.6.** Let $f$ be a measurable bounded function on $(S, \mathcal{S})$. Then there exists two sequences of simple functions $(g_n)_{n=1}^{\infty}$ and $(h_n)_{n=1}^{\infty}$ such that $g_n \searrow f$ and $h_n \nearrow f$.

**Definition 10.7.** Let $(S, \mathcal{S}, \nu)$ be a positive measure space and $B$ a family of elements of $\mathcal{S}$. We say that a subset $A \subset S$ is *approximable* by $B$ if there exists a sequence $(A_n)_{n=1}^{\infty}$ of subsets of $S$ such that

- each $A_n$ is a countable union of elements of $B$ and that $A \subset A_n$ and $\nu(A_n \setminus A) \to 0$ as $n \to \infty$.
- the sequence $(A_n)_{n=1}^{\infty}$ is decreasing, i.e., $A_{n+1} \subset A_n$ for all $n$.

We say that a family $D$ of elements of $\mathcal{S}$ is approximable by $B$ if each element of $D$ is approximable by $B$.

**Proposition 10.8.** 1) For any set $Z$ and algebra $\mathcal{B}$ of subsets of $Z$, any countably additive functions $\nu$ from $\mathcal{B}$ into $[0, \infty]$ extends to a measure (still denoted by $\nu$) on the $\sigma$-algebra $\mathcal{S}$ generated by $\mathcal{B}$. More explicitly, $\nu$ is defined as follows:

$$\nu(A) := \inf \left\{ \sum_{i=1}^{\infty} \nu(B_i) : B_i \in \mathcal{B}, \ A \subset \bigcup_{i=1}^{\infty} B_i \right\}, \quad A \in \mathcal{S}.$$  

2) If, moreover, $\nu$ is $\sigma$-finite, then $\mathcal{S}$ is approximable by $\mathcal{B}$.

**Proof.** Part 1) follows from Theorem 3.1.4 in [15].

Part 2) in the case when $\nu$ is finite follows immediately from the formula in Part 1).

When $\nu$ is $\sigma$-finite, we fix a sequence $(Z_m)_{m=1}^{\infty} \subset \mathcal{S}$ such that $Z = \bigcup_{m=1}^{\infty} Z_m$ and $\nu(Z_m) < \infty$ for each $m$. Next, we apply the previous case to each set $Z_m$ in order to obtain a decreasing sequence $(A_{mn})_{n=1}^{\infty}$ of subsets of $Z_m$ such that each $A_{nn}$ is a countable union of elements of $\mathcal{B}$ and that $A \cap Z_m \subset A_{mn}$ and $\nu(A_{mn} \setminus (A \cap Z_m)) < 2^{-(n+m)}$. Now letting $A_n := \bigcup_{m=1}^{\infty} A_{mn}$ for each $n \geq 1$, we see easily that the sequence $(A_n)_{n=1}^{\infty}$ satisfies the desired conclusion.

The following criterion is very useful.

**Proposition 10.9.** Let $(S, \mathcal{S}, \nu)$ be a $\sigma$-finite positive measure space and $\mathcal{S}^0$ and $\mathcal{B}$ two families of elements of $\mathcal{S}$. Assume in addition that the intersection of two sets of $\mathcal{B}$ may be represented as a countable union of sets in $\mathcal{B}$ and that...
\(\mathcal{D}^0\) is approximable by \(\mathcal{B}\). Starting from \(\mathcal{D}^0\), we define inductively the sequence of families \((\mathcal{D}^N)_{N=1}^\infty\) and a new family \(\mathcal{D}\) of elements of \(\mathcal{I}\) as follows:

\[
\mathcal{D}^{2k+1} := \{ A \in \mathcal{I} : A = \bigcup_{n=1}^\infty A_n, \quad A_n \in \mathcal{D}^{2k} \}, \quad k \in \mathbb{N}; \\
\mathcal{D}^{2k} := \{ A \in \mathcal{I} : A = \bigcap_{n=1}^\infty A_n, \quad A_n \in \mathcal{D}^{2k-1} \text{ and } A_{n+1} \subset A_n \}, \quad k \in \mathbb{N}; \\
\mathcal{D} := \bigcup_{k=1}^\infty \mathcal{D}^k.
\]

Then the \(\sigma\)-algebra generated by all elements of \(\mathcal{D}\) is also approximable by \(\mathcal{B}\).

**Proof.** We prove by induction on the index \(N\) of \(\mathcal{D}^N\) that \(\mathcal{D}^N\) is approximable by \(\mathcal{B}\). By the assumption, \(\mathcal{D}^N\) is approximable by \(\mathcal{B}\) for \(N = 0\). Suppose that \(\mathcal{D}^{N-1}\) is approximable by \(\mathcal{B}\) for some \(N \geq 1\). Let \(A \in \mathcal{D}^N\). We need to show that \(A\) is approximable by \(\mathcal{B}\). There are two cases.

**Case** \(N = 2k + 1\):

Since \(A\) can be represented as \(A = \bigcup_{n=1}^\infty A_n\) with \(A_n \in \mathcal{D}^{N-1}\), we obtain, by the hypothesis of induction, for each \(n\), a decreasing sequence \((A_{mn})_{m=1}^\infty\) such that \(A_n \subset A_{mn}\) and that each \(A_{mn}\) is a countable union of elements of \(\mathcal{B}\) and that \(\nu(A_{mn} \setminus A_n) < 2^{-(n+m)}\). Now letting \(B_m := \bigcup_{n=1}^\infty A_{mn}\), we see easily that \((B_m)_{m=1}^\infty\) is decreasing, \(A \subset B_m\) and \(\nu(B_m \setminus A) < 2^{-m}\) as desired.

**Case** \(N = 2k\):

Since \(A\) can be represented as \(A = \bigcap_{n=1}^\infty A_n\) with \(A_n \in \mathcal{D}^{N-1}\) and \(A_{n+1} \subset A_n\), we obtain, by the hypothesis of induction, for each \(n\), a sequence \((B_n)_{n=1}^\infty\) such that each \(B_n\) is a countable union of elements of \(\mathcal{B}\) and that \(A_n \subset B_n\) and \(\nu(B_n \setminus A_n) < 2^{-n}\). Replacing each \(B_n\) with \(B_n \cap B_{n-1} \cap \cdots \cap B_1\) and using the assumption that the intersection of two sets of \(\mathcal{B}\) may be represented as a countable union of sets in \(\mathcal{B}\), we may assume in addition that \(B_{n+1} \subset B_n\). Since \(\nu(A_n \setminus A) \to 0\), it follows that \(\nu(B_n \setminus A) \to 0\). This completes the proof in the last case.

**Proposition 10.10.** Let \(\mathcal{B}\) be an algebra of subsets of a given set \(Z\). Let \(\mathcal{A}\) be a family of subsets of \(Z\) such that

- \(\mathcal{B} \subset \mathcal{A}\);
- If \((A_n)_{n=1}^\infty \subset \mathcal{A}\) such that \(A_n \subset A_{n+1}\) for all \(n\), then \(\bigcup_{n=1}^\infty A_n \in \mathcal{A}\);
- If \((A_n)_{n=1}^\infty \subset \mathcal{A}\) such that \(A_{n+1} \subset A_n\) for all \(n\), then \(\bigcap_{n=1}^\infty A_n \in \mathcal{A}\).

Then \(\mathcal{A}\) contains the \(\sigma\)-algebra generated by \(\mathcal{B}\).

**Proof.** For a collection \(T\) of subsets of \(Z\), let

- \(T_\sigma\) be all countable unions of elements of \(T\),
- \(T_\delta\) be all countable intersections of elements of \(T\),
- \(T_\kappa := (T_\delta)_\sigma\).

Now define by transfinite induction a sequence \(\mathcal{B}^m\), where \(m\) is an ordinal number, in the following manner:

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• For the base case of the definition, let $\mathcal{B}^0 := \mathcal{B}$.
• If $i$ is not a limit ordinal, then $i$ has an immediately preceding ordinal $i - 1$. Let $\mathcal{B}^i := [\mathcal{B}^{i-1}]_{i\alpha}$.
• If $i$ is a limit ordinal, set $\mathcal{B}^i = \bigcup_{j<i} \mathcal{B}^j$.

Then we can show that $\mathcal{B} := \mathcal{B}^{\omega_1}$ is the $\sigma$-algebra generated by $\mathcal{B}$, where $\omega_1$ is the first uncountable ordinal number. We can prove by transfinite induction on the ordinal number $i$ that each element $A$ of $\mathcal{B}_i$ can be written as $A = \bigcup_{n=1}^{\infty} A_n$ with $A_n \uparrow A$ as $n \uparrow \infty$ and each $A_n$ is of the form $A_n = \cap_{m=1}^{\infty} A_{nm}$ with $A_{nm} \owns A_n$ as $m \uparrow \infty$, and $A_{nm} \in \mathcal{B}^{i-1}$. This completes the proof. \(\square\)

**Proposition 10.11.** Let $(S, \mathcal{I}, \mu)$ be a positive finite measure space and $(T, \mathcal{J})$ a measurable space. Let $f : S \times T \to \mathbb{R}$ be a measurable bounded function, where $S \times T$ is endowed with the $\sigma$-algebra $\mathcal{I} \otimes \mathcal{J}$. Consider the function $F = \Phi(f) : T \to \mathbb{R}$ defined by

$$F(t) := \int_S f(s,t) d\mu(s), \quad t \in T.$$ 

Then $F$ is measurable.

**Proof.** Since $f$ is measurable and bounded, Proposition [10.6] yields a sequence of simple functions $(f_n)_{n=1}^{\infty}$ such that $f_n \uparrow f$. By Lebesgue dominated convergence, we get that $\Phi(f_n) \uparrow \Phi(f) = F$. Consequently, if all $\Phi(f_n)$ are measurable, so is $F$. Therefore, we may assume without loss of generality that $f$ is a simple function, that is,

$$f := \sum a_i 1_{A_i}, \quad \text{where } a_i \in \mathbb{R}, \ A_i \in \mathcal{I} \otimes \mathcal{J}.$$ 

This implies that

$$F = \Phi(f) = \sum a_i \Phi(1_{A_i}).$$ 

Hence, we are reduced to the case where $f := 1_A$ with $A \in \mathcal{I} \otimes \mathcal{J}$.

To prove the last fact let $\mathfrak{A}$ be the family of all sets $A = \bigcup_{i \in I} S_i \times T_i$, where $S_i \in \mathcal{I}$ and $T_i \in \mathcal{J}$, and the index set $I$ is finite. Note that $\mathfrak{A}$ is an algebra on $S \times T$ which generates the $\sigma$-algebra $\mathcal{I} \otimes \mathcal{J}$. Moreover, each such set $A$ can be expressed as a disjoint finite union $A = \bigcup_{i \in I} S_i \times T_i$. Using the above expression for such a set $A$ and the equality $f = 1_A$, we infer that

$$F(t) = \Phi(f) = \sum_{i \in I} \mu(S_i) 1_{T_i}(t), \quad t \in T.$$ 

Hence, $F$ is measurable for all $A \in \mathfrak{A}$.

Let $\mathcal{A}$ be the family of all sets $A \subset S \times T$ such that $S \owns s \mapsto 1_A(s,t)$ is measurable for all $t \in T$ and that $\Phi(1_A)$ is measurable. The previous paragraph shows that $\mathfrak{A} \subset \mathcal{A}$.

Next, suppose that $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$ and that either $A_n \owns A$ or $A_n \uparrow A$. Let $f := 1_A$ and $F := \Phi(f)$. By Lebesgue dominated convergence, we get that either $\Phi(f_n) \owns F$ or $\Phi(f_n) \uparrow F$. So $F$ is also measurable. Hence, $A \in \mathcal{A}$. Consequently,
by Proposition 10.10. \( S \otimes T \subset A \). In particular, \( F \) is well-defined and measurable for \( f := 1_A \) with \( A \in S \otimes T \). This completes the proof.

### 10.3 \( \sigma \)-algebra \( \mathcal{A} \) on a leaf

The main purpose of this subsection is to provide the necessary material in order to prove Proposition 2.12 (i) and Proposition 4.1 (i). More precisely, this subsection is devoted to the measure theory on sample-path spaces associated to a single leaf. For this purpose we need to introduce some notation and terminology as well as some preparatory results.

Fix a point \( x \in X \) and let \( L := L_x \) and \( \pi : \tilde{L} \to L \) the universal covering projection. Recall from Subsection 4.1 that a set \( A \subset \Omega(L) \) is said to be a cylinder image if \( A = \pi \circ \tilde{A} \) for some cylinder set \( \tilde{A} \subset \Omega(\tilde{L}) \).

Let \( D^1(L) \) denote following family of subsets of \( \Omega(L) \):

\[
D^1(L) := \{ A \in \Omega(L) : A = \cup_{n=1}^{\infty} A_n, \ A_n \text{ is a cylinder image} \}.
\]

Starting from \( D^1(L) \), we define inductively the sequence of families \( (D^N(L))_{N=1}^{\infty} \) and a new family \( D(L) \) of subsets of \( \Omega(L) \) as follows:

\[
D^k(L) := \{ A \in \Omega(L) : A = \cap_{n=1}^{\infty} A_n, \ A_n \in D^{k-1}(L) \text{ and } A_{n+1} \subset A_n \}, \ k \in \mathbb{N};
\]

\[
D^{k+1}(L) := \{ A \in \Omega(L) : A = \cup_{n=1}^{\infty} A_n, \ A_n \in D^k(L) \}, \ k \in \mathbb{N};
\]

\[
D(L) := \bigcup_{k=1}^{\infty} D^k(L).
\]

Note that \( (D^N(L))_{N=1}^{\infty} \) is increasing, that is, \( D^N(L) \subset D^{N+1}(L) \). The following result will be the main ingredient in the proof of assertion (i) of both Proposition 2.12 and Proposition 4.1.

**Proposition 10.12.** \( D(L) \) is an algebra.

Prior to the proof of Proposition 10.12 we need to introduce some notion. An connected open set \( U \subset L \) is said to be trivializing if we write \( \pi^{-1}(U) \) as the disjoint union of its connected components \( \tilde{U}_i \) then every restriction \( \pi|_{\tilde{U}_i} : \tilde{U}_i \to U_i \) is homeomorphic. Clearly, every simply connected domain \( U \subset L \) is trivializing. A Borel set \( \tilde{A} \subset \tilde{L} \) is said to be good if there is an open neighborhood \( \tilde{U} \) of \( \tilde{A} \) in \( \tilde{L} \) such that \( \pi \) maps \( \tilde{U} \) homeomorphically onto a trivializing open set in \( L \). A cylinder set \( \tilde{C} := C(\{t_i, \tilde{B}_i\}) \) in \( \Omega(\tilde{L}) \) is said to be good if all (Borel) sets \( \tilde{B}_i \) are good. A set \( C \subset \Omega(L) \) is said to be a good cylinder image if there is a good cylinder set \( \tilde{C} \) such that \( C = \pi \circ \tilde{C} \). Let us point out the following remarkable lifting property of good cylinder sets.

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Lemma 10.13. If $\tilde{C} := C(\{t_i, \tilde{B}_i\})$ is a good cylinder set in $\Omega(\tilde{L})$, then
\[
\pi^{-1}(\pi \circ \tilde{C}) = \bigsqcup_{\gamma \in \pi_1(L)} C(\{t_i, \gamma(\tilde{B}_i)\}).
\]

Proof. If $U$ is a trivializing set, then we have that
\[
\pi^{-1}(U) = \bigsqcup_{\gamma \in \pi_1(L)} \gamma(U)
\]
and that each restriction $\pi|_{\gamma(U)} : \gamma(U) \to U$ is homeomorphic. Using this property the lemma follows. \qed

The properties of various notion of goodness are collected in the following

Lemma 10.14. (i) If $A$ and $B$ are good cylinder images, then $A \cap B$ is a countable union of good cylinder images.

(ii) If $A$ and $B$ are good cylinder images, then $A \setminus B$ is a countable union of mutually disjoint good cylinder images.

(iii) Every cylinder image is a countable union of good cylinder images.

(iv) The intersection of two cylinder images is a countable union of cylinder images.

(v) If $A$ is a good cylinder image, then $\Omega(L) \setminus A$ is a countable union of cylinder images.

Taking for granted the lemma, we arrive at the

End of the proof of Proposition [10.12] Consider the family $\mathcal{D}^0(L)$ of all good cylinder images. First note that $\mathcal{D}^k(L) \subset \mathcal{D}^{k+1}(L)$ for all $k \in \mathbb{N}$. Using this increasing property, we deduce that to prove the proposition it is sufficient to show that:

Claim. If $A, B \in \mathcal{D}^N(L)$ for some $N \in \mathbb{N}$, then $A \cup B \in \mathcal{D}^N(L)$ and $\Omega(L) \setminus A \in \mathcal{D}^{N+1}(L)$.

For $N = 0$, Claim follows from Lemma [10.14] (v). Suppose Claim true for $N - 1$ we need to show it true for $N$. To do this fix two sets $A, B \in \mathcal{D}^N(L)$. Consider two cases.

Case 1: $N$ is even:

Let $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ be two decreasing sequences of elements in $\mathcal{D}^{N-1}(L)$ such that $A = \cap_{n=1}^\infty A_n$ and $B = \cap_{n=1}^\infty B_n$. Note from [10.1] that each $A_n$ (resp. $B_n$) is a countable union of elements in $\mathcal{D}^{N-2}(L)$. Clearly, $(A_n \cup B_n)_{n=1}^\infty$ is a decreasing sequences of elements in $\mathcal{D}^{N-1}(L)$ such that $A \cup B = \cap_{n=1}^\infty (A_n \cup B_n)$. Hence, $A \cup B \in \mathcal{D}^N(L) \subset \mathcal{D}^{N+1}(L)$.

To prove that $\Omega(L) \setminus A \in \mathcal{D}^{N+1}(L)$ we write $\Omega(L) \setminus A = \bigcup_{n=1}^\infty (\Omega(L) \setminus A_n)$. By the hypothesis of induction, each $\Omega(L) \setminus A_n$ is an element in $\mathcal{D}^N(L)$. We deduce from the last equality and from [10.1] and from the fact that $N$ is even that $\Omega(L) \setminus A \in \mathcal{D}^{N+1}(L)$, as desired.
Case 2: \( N \) is odd:

It follows from \( \text{(10.1)} \) and the oddness of \( N \) that a finite union of elements in \( \mathcal{D}^{N-1} \) is also an element in \( \mathcal{D}^{N-1} \). To complete the proof we need to show that \( \Omega(L) \setminus A \in \mathcal{D}^{N+1}(L) \) for each \( A = \bigcup_{n=1}^{\infty} A_n \) with \( A_n \in \mathcal{D}^{N-1} \). Write

\[
\Omega(L) \setminus A = \bigcap_{n=1}^{\infty} \left( \Omega(L) \setminus \bigcup_{i=1}^{n} A_i \right)
\]

By the hypothesis of induction, each \( \Omega(L) \setminus \bigcup_{i=1}^{n} A_i \) is an element in \( \mathcal{D}^N(L) \). Since \( \Omega(L) \setminus \bigcup_{i=1}^{n} A_i \) is decreasing on \( n \), it follows that \( \Omega(L) \setminus A \in \mathcal{D}^{N+1}(L) \), as desired. \( \square \)

It remains to us to establish Lemma \( \text{10.14} \).

**Proof of assertion (i) of Lemma \( \text{10.14} \).** Let \( A := \pi \circ \tilde{A}, B = \pi \circ \tilde{B} \), where \( \tilde{A} := C(\{t_i, \tilde{A}_i\} : p) \) and \( \tilde{B} := C(\{s_j, \tilde{B}_j\} : q) \), and \( \tilde{A}_i, \tilde{B}_j \) are good subsets of \( \tilde{L} \), and \( 0 \leq t_1 < t_2 < \cdots < t_p \) and \( 0 \leq s_1 < s_2 < \cdots < s_q \) are sets of increasing times. The proof is divided into two cases.

**Case 1:** The two sets of times are equal, i.e., \( p = q \) and \( t_i = s_i \) for \( 1 \leq i \leq p \).

If there is some \( i \) such that \( \pi(\tilde{A}_i) \cap \pi(\tilde{B}_i) = \emptyset \), then \( A \cap B = \emptyset \) because \( \omega \in A \cap B \) implies \( \omega(t_i) \in \pi(\tilde{A}_i) \cap \pi(\tilde{B}_i) \). If this case happens, there is nothing to prove. Therefore, we may assume that \( D_i := \pi(\tilde{A}_i) \cap \pi(\tilde{B}_i) \neq \emptyset \) for every \( 1 \leq i \leq p \). Moreover, using that \( \tilde{A}_i \) and \( \tilde{B}_i \) are good and replacing \( \tilde{A}_i \) (resp. \( \tilde{B}_i \)) by \( \tilde{A}_i \cap \pi^{-1}(D_i) \) (resp. \( \tilde{B}_i \cap \pi^{-1}(D_i) \)), we may assume without loss of generality that \( \pi(\tilde{A}_i) = \pi(\tilde{B}_i) = D_i \) for \( 1 \leq i \leq p \). Using the goodness assumption of \( \tilde{A}_i \) (resp. \( \tilde{B}_i \)) we may find, for each \( 1 \leq i \leq m \), open sets \( \tilde{U}_i, \tilde{V}_i \subseteq \tilde{L} \) and a trivializing open set \( W_i \subseteq \tilde{L} \) such that \( \tilde{A}_i \subseteq \tilde{U}_i, \tilde{B}_i \subseteq \tilde{V}_i \), and \( \pi(U_i) = \pi(V_i) = W_i \). Fix a point \( c \in D_1 \) and let \( \tilde{a} \) (resp. \( \tilde{b} \)) be the unique point in \( \pi^{-1}(c) \) lying on \( \tilde{A}_1 \) (resp. \( \tilde{B}_1 \)). Let \( \gamma \in \pi_1(L) \) be the unique deck-transformation sending \( \tilde{b} \) to \( \tilde{a} \). By shrinking \( \tilde{U}_1 \) and \( \tilde{V}_1 \) if necessary, we may assume without loss of generality that \( \gamma(\tilde{V}_1) = \tilde{U}_1 \). Setting \( \tilde{C}_i := \gamma(\tilde{B}_i), 1 \leq i \leq p \) and \( \tilde{C} := C(\{t_i, \tilde{C}_i\} : p) \),\( \) we obtain, using Lemma \( \text{10.13} \) that \( \pi \circ \tilde{C} = \pi \circ \tilde{B} = B \). Now pick an arbitrary path \( \omega \in \pi \circ \tilde{A} \cap \pi \circ \tilde{C} \). Let \( y := \pi(\omega(t_1)) \in D_1 \) and \( \tilde{y}_a = \pi^{-1}(y) \cap \tilde{A}_1 \) and \( \tilde{y}_b = \pi^{-1}(y) \cap \tilde{B}_1 \). Clearly, by Lemma \( \text{10.13} \) again, \( \pi_{\tilde{y}_a} \omega \in \tilde{A} \) and \( \pi_{\tilde{y}_b} \omega \in \tilde{B} \). This implies that \( \pi_{\tilde{y}_a}^{-1} \omega = \gamma(\pi_{\tilde{y}_b}^{-1} \omega) \in \tilde{A} \cap \tilde{C} \).

Thus we have shown that \( \pi \circ \tilde{A} \cap \pi \circ \tilde{C} \subset \pi \circ (\tilde{A} \cap \tilde{C}) \). Since the inverse inclusion is trivial, we obtain that \( \pi \circ \tilde{A} \cap \pi \circ \tilde{C} = \pi \circ (\tilde{A} \cap \tilde{C}) \). Hence,

\[
A \cap B = \pi \circ \tilde{A} \cap \pi \circ \tilde{C} = \pi \circ (\tilde{A} \cap \tilde{C}),
\]

which finishes the proof because \( \tilde{A} \cap \tilde{C} \) is a good cylinder set.

**Case 2:** The general case.
Suppose that assertion (i) is proved when the cardinal of the symmetric difference of \( \{t_1, t_2, \ldots, t_p\} \) and \( \{s_1, s_2, \ldots, s_q\} \) is \( r \). We will prove by induction that assertion (i) also holds when the cardinal of the above symmetric difference \( \leq r + 1 \). To do this consider the case where this cardinal is equal to \( r + 1 \). Pick \( t_i \) which does not belong to \( \{s_1, \ldots, s_q\} \). Let \( U_i \) be a trivializing open neighborhood of \( \pi(A_i) \). Write \( \pi^{-1}(U_i) \) as the union of its connected components \( U_{ij} \), where \( j \in J \) and the index set \( J \) is equal to either \( \mathbb{N} \) or the set \( \{0, \ldots, N\} \) for some \( N \in \mathbb{N} \). Let \( \tilde{A}_{ij} := \pi_{ij}^{-1}(\pi(A_i)) \subset U_{ij} \), where \( \pi_{ij}^{-1} \) is the inverse of the homeomorphism \( \pi|_{U_{ij}} : U_{ij} \to U_i \). For \( j \in J \) consider the following good cylinder set \( \tilde{B}^j := C(\{s_1, \tilde{B}_1\}, \ldots, \{s_q, \tilde{B}_q\}, \{t_i, \tilde{A}_{ij}\} : q + 1) \). Since the cardinal of the symmetric difference of \( \{t_1, t_2, \ldots, t_p\} \) and \( \{s_1, s_2, \ldots, s_q\} \) is \( \leq r \), it follows from the hypothesis of induction that \( \pi \circ \tilde{A} \cap \pi \circ \tilde{B}^j \) is a countable union of good cylinder images. This, combined with the equality

\[
A \cap B = \pi \circ \tilde{A} \cap \pi \circ \tilde{B} = \bigcup_{j \in J} \pi \circ \tilde{A} \cap \pi \circ \tilde{B}^j,
\]
implies Case 2, where the last equality follows from Lemma [10.13].

**Proof of assertion (ii) of Lemma [10.14].** We use the notation introduced in the proof of assertion (i). We also consider two cases.

**Case 1:** The two sets of times are equal, i.e., \( p = q \) and \( t_i = s_i \) for \( 1 \leq i \leq p \).

If \( A \cap B = \emptyset \) then one get that \( A \setminus B = A \). So assertion (ii) is trivially true. If \( A \cap B \neq \emptyset \) we proceed as in Case 1 of the proof of assertion (i). Consequently, we can show that

\[
A \setminus B = \pi \circ \tilde{A} \setminus \pi \circ \tilde{C} = \pi \circ (\tilde{A} \setminus \tilde{C}).
\]

Since we may write \( \pi \circ (\tilde{A} \setminus \tilde{C}) \) as the union of \( p \) mutually disjoint good cylinder images

\[
\bigcup_{i=1}^{p} \pi \circ C(\{t_1, \tilde{A}_1\}, \ldots, \{t_{i-1}, \tilde{A}_{i-1}\}, \{t_i, \tilde{A}_i \setminus \tilde{C}_i\}, \{t_{i+1}, \tilde{A}_{i+1}\}, \ldots, \{t_p, \tilde{A}_p\} : p),
\]
the desired conclusion follows.

**Case 2:** The general case.

Suppose that assertion (ii) is proved when the cardinal of the symmetric difference of \( \{t_1, t_2, \ldots, t_p\} \) and \( \{s_1, s_2, \ldots, s_q\} \) is \( \leq r \). We will prove by induction that assertion (i) also holds when the cardinal of the above symmetric difference \( \leq r + 1 \). As the arguments are quite similar to those given in Case 2 of the proof of assertion (i), a detailed proof is left to the interested reader.

**Proof of assertion (iii) of Lemma [10.14].** Let \( A = \pi \circ \tilde{A} \), where \( \tilde{A} := C(\{t_i, \tilde{A}_i\} : m) \) is a cylinder set in \( \Omega(\tilde{L}) \). Since for every \( \tilde{x} \in \tilde{L} \) there is an open neighborhood \( \tilde{U} \) of \( \tilde{x} \) such that \( \pi(\tilde{U}) \) is trivializing, we may write each \( \tilde{A}_i \) as a
countable union of good sets $\tilde{A}_{ij}$. Consequently, using that
\[ \tilde{A} = \bigcup_{j_1, \ldots, j_m \in \mathbb{N}} C(\{t_i, \tilde{A}_{ij}\} : m), \]
the assertion follows.

**Proof of assertion (iv) of Lemma 10.14.** It follows from combining assertion (i) and assertion (iii).

**Proof of assertion (v) of Lemma 10.14.** Let $A = \pi \circ \tilde{A}$, where $\tilde{A} := C(\{t_i, \tilde{A}_i\} : m)$ is a good cylinder set in $\Omega(L)$. If $A = \emptyset$ then we write $\Omega(L)$ as the union of the cylinder $C(\{0, \tilde{L}\})$, and hence the desired conclusion follows from assertion (iii). Therefore, we may suppose without loss of generality that all $\tilde{A}_i$ are nonempty. Let $U_i$ be a trivializing open neighborhood of $\pi(\tilde{A}_i)$ for $1 \leq i \leq m$. Write $\pi^{-1}(U_i)$ as the union of its connected components $U_{ij}$, where $j \in J$ and the index set $J$ is equal to either $\mathbb{N}$ or the set $\{0, \ldots, N\}$ with some $N \in \mathbb{N}$. Let $\tilde{A}_{ij} := \pi_{ij}^{-1}(\pi(\tilde{A}_i)) \subset U_{ij}$, where $\pi_{ij}^{-1}$ is the inverse of the homeomorphism $\pi|_{U_{ij}} : U_{ij} \to U_i$. We assume that $\tilde{A}_i = \tilde{A}_{i0}$ for $1 \leq i \leq m$. Observe that $\pi \circ C(\{t_i, \pi^{-1}(\pi(\tilde{A}_i))\} : m)$ is the disjoint union of (countable) cylinder images
\[ \pi \circ C(\{t_1, \tilde{A}_{10}\}, \{t_2, \tilde{A}_{2j_2}\}, \ldots, \{t_m, \tilde{A}_{mj_m}\} : m), \quad (j_2, \ldots, j_m) \in J^{m-1}. \]
On the other hand, assertion (iii) and the following trivial equality
\[ \Omega(L) \setminus \pi \circ C(\{t_i, \pi^{-1}(\pi(\tilde{A}_i))\} : m) = \bigcup_{i=1}^{m} \pi \circ C(\{t_i, \tilde{B}_i\} : 1), \]
where $\tilde{B}_i := \pi^{-1}(L \setminus \pi(\tilde{A}_i))$, implies that the set on the left hand side is a countable union of good cylinder images. This, combined with the previous disjoint union, implies that
\[ \Omega(L) \setminus A = \bigcup_{i=1}^{m} \pi \circ C(\{t_i, \tilde{B}_i\} : 1) \]
\[ \square \bigcup_{(j_2, \ldots, j_m) \in J^{m-1} \setminus (0, \ldots, 0)} \pi \circ C(\{t_1, \tilde{A}_{10}\}, \{t_2, \tilde{A}_{2j_2}\}, \ldots, \{t_m, \tilde{A}_{mj_m}\} : m), \]
proving assertion (iv). \hfill \square

Now we arrive at the

**End of the proof of Proposition 2.12 (i).** Fix a point $x \in X$ and let $L := L_x$, and $\mathcal{D}_x := \{ A \in \mathcal{D}(L) : A \subset \Omega_x \}$. Since we know from Lemma 10.14 that $\mathcal{D}(L)$ is an algebra, so is $\mathcal{D}_x$. For every $\tilde{x} \in \pi^{-1}(x)$, we define a probability measure $W_{\tilde{x}}^\tilde{\pi}$ on $(\Omega_x, \mathcal{A}_x)$ as follows:
\[ W_{\tilde{x}}^\tilde{\pi}(A) := W_{\tilde{x}}(\pi^{-1}A), \quad A \in \mathcal{A}_x, \]
where $W_{	ilde{x}}$ is the probability measure on $(\Omega(\tilde{L}), \mathcal{A}(\tilde{L}))$ given by (2.5). Next, we will prove that $W_{\tilde{x}}(A)$ given by formula (2.6) is well-defined for every $A \in \mathcal{A}_x$. This is equivalent to showing the following

**Claim.** $W_{\tilde{x}_1}(A) = W_{\tilde{x}_2}(A)$ for all $A \in \mathcal{A}_x$ and all points $\tilde{x}_1, \tilde{x}_2 \in \pi^{-1}(x)$, in other words, all $W_{\tilde{x}}$ with $\tilde{x} \in \pi^{-1}(x)$ coincide on $\mathcal{A}_x(L)$.

To do this our idea is to prove this coincidence on good cylinder images, then on $\mathcal{A}_1(L)$, $\mathcal{A}_2(L)$, \ldots, and finally on $\mathcal{A}(L)$.

First, consider the case where $A = \pi \circ \tilde{A}$ and $\tilde{A}$ is a good cylinder set, that is, $\tilde{A} := C(\{t_i, \tilde{A}_i\} : m)$, where $t_1 = 0$ and $\tilde{A}_1 = \{\tilde{x}_1\}$. In this case we may find, for every $1 \leq i \leq m$, an open set $\tilde{U}_i \subset \tilde{L}$ and a trivializing open set $U_i \subset L$ such that $\tilde{A}_i \subset \tilde{U}_i$ and that $\pi|_{\tilde{U}_i}$ is homeomorphic onto its image $\pi(U_i) = U_i$. Let $\gamma \in \pi_1(L)$ be the unique deck-transformation sending $\tilde{x}_1$ to $\tilde{x}_2$. Since $\tilde{A} = \pi_{\tilde{x}_1}^{-1}A$, it follows that $\pi_{\tilde{x}_2}^{-1}A = C(\{t_i, \gamma(\tilde{A}_i)\})$. So it suffices to check that

$$W_{\tilde{x}_1}(C(\{t_i, \tilde{A}_i\})) = W_{\gamma(\tilde{x}_1)}(C(\{t_i, \gamma(\tilde{A}_i)\})).$$

But this equality is an immediate consequence of the invariance under the deck-transformations of the heat kernel (see (2.3)). So all $W_{\tilde{x}}$ with $\tilde{x} \in \pi^{-1}(x)$ coincide on good cylinder images.

Next, we show that all $W_{\tilde{x}}$ with $\tilde{x} \in \pi^{-1}(x)$ coincide on the union of two good cylinder images. Indeed, let $A = B \cup C$, where $B$ and $C$ are good cylinder images. Writing $A = (B \setminus C) \cup C$ and applying Lemma [10.14] (ii) to $B \setminus C$, we may find a countable family of disjoint good cylinder images $(B_i)_{i=1}^\infty$ such that $B \setminus C = \bigcup_{i=1}^\infty B_i$. Now using the $\sigma$-additivity of a measure, we infer that

$$W_{\tilde{x}}(A) = W_{\tilde{x}}(C) + \sum_{i=1}^\infty W_{\tilde{x}}(B_i).$$

So $W_{\tilde{x}}(A)$ does not depend on $\tilde{x} \in \pi^{-1}(x)$ as desired.

In the next stage we will show that all $W_{\tilde{x}}$ coincide on each finite union of good cylinder images. Let $A = \bigcup_{i=1}^n A_i$, each $A_i$ being a good cylinder image. Writing

$$A = \left( \bigcup_{i=1}^{n-1} A_i \right) \setminus A_n,$$

and using Lemma [10.14] (ii), we show by induction on $n$ that $A$ can be expressed as a countable union of disjoint good cylinder images. Now using the $\sigma$-additivity of a measure, we infer that $W_{\tilde{x}}(A)$ does not depend on $\tilde{x} \in \pi^{-1}(x)$, as desired.

Each element $A \in \mathcal{A}_1(L)$ may be written as the union of an increasing sequence $(A_n)_{n=1}^\infty$ of subsets of $\Omega(L)$, each set $A_n$ being a finite union of good cylinder images. Using the $\sigma$-additivity of a measure again, we deduce that $W_{\tilde{x}}(A) = \lim_{n \to \infty} W_{\tilde{x}}(A_n)$. So all $W_{\tilde{x}}$ with $\tilde{x} \in \pi^{-1}(x)$ coincide on all elements of $\mathcal{A}_1(L)$.

Next, since each element of $A \in \mathcal{A}_2(L)$ may be expressed as the intersection of a decreasing sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}_1(L)$, it follows that $W_{\tilde{x}}(A) = \lim_{n \to \infty} W_{\tilde{x}}(A_n)$.
\[ \lim_{n \to \infty} W_x^\tilde{\omega}(A_n). \] So all \( W_x^\tilde{\omega} \) with \( \tilde{x} \in \pi^{-1}(x) \) coincide on \( \mathcal{D}^2(L) \). Repeating the above argument and using Proposition 10.12 we can show that all \( W_x^\tilde{\omega} \) with \( \tilde{x} \in \pi^{-1}(x) \) coincide on the algebra \( \mathcal{D}(L) = \bigcup_{k=1}^{\infty} \mathcal{D}^k(L) \), proving our claim.

We denote by \( W_x \) the restriction of \( W_x^\tilde{\omega} \) on \( \mathcal{D}(L) \) which is independent of \( \tilde{x} \in \pi^{-1}(x) \). So \( W_x \) is a countably additive function from \( \mathcal{D}(L) \) to \([0,1]\). Since the \( \sigma \)-algebra \( \mathcal{A}_x \) is generated by the algebra \( \mathcal{D}(L) \cap \mathcal{A}_x \), we deduce from Part 1) of Proposition 10.8 that \( W_x \) extends to a probability measures (still denoted by) \( W_x \) on \((\Omega_x, \mathcal{A}_x)\). This completes the proof. \( \square \)

**End of the proof of Proposition 4.1 (i).** By the construction (10.1) and assertion (iv) of Lemma 10.14 and Proposition 10.12 we are in the position to apply Proposition 2.12 (i) and Proposition 10.9. Hence, assertion (i) follows. \( \square \)

### 10.4 Holonomy maps

We will define the notion of the holonomy map along a path and the notion of flow tubes. A similar (but slightly different) formulation of the holonomy map could be found in the textbook [4, Chapter 2]. We need the following terminology and notation. A **multivalued map** \( f : Y \to Z \) is given by its graph \( \Gamma(f) \subset Y \times Z \). For each \( y \in Y \) we denote by \( f(y) \) the set \( \{ z \in Z : (y, z) \in \Gamma(f) \} \). The **domain of definition** \( \text{Dom} (f) \) of \( f \) is the set \( \{ y \in Y : f(y) \neq \emptyset \} \subset Y \) and the **range** \( \text{Range} (f) \) of \( f \) is the subset \( f(Y) := \cup_{y \in \text{Dom} (f)} f(y) \subset Z \). If, moreover, \( f \) is univalued and one-to-one, then \( \text{Dom} (f^{-1}) = \text{Range} (f) \) and \( \text{Range} (f^{-1}) = \text{Dom} (f) \).

For another multivalued map \( g : Z \to W \), we define \( \text{Dom}(g \circ f) \) as the set of all points \( y \in Y \) such that the composition \( (g \circ f)(y) \) is nonempty, i.e., the set of all \( y \in \text{Dom} (f) \) such that \( f(y) \cap \text{Dom} (g) \neq \emptyset \). The germ of a local homeomorphism \( f \) at a point \( x \in \text{Dom} (f) \) is the equivalent class of all local homeomorphisms defined on a neighborhood of \( x \) and agreeing with \( f \) on a neighborhood of \( x \).

Let \((X, \mathcal{L})\) be a lamination satisfying the Standing Hypotheses and set \( \Omega := \Omega(X, \mathcal{L}) \). Consider an atlas \( \mathcal{L} \) of \( X \) with (at most) countable and locally finite charts

\[ \Phi_\alpha : U_\alpha \to \mathbb{B}_\alpha \times T_\alpha, \]

where \( T_\alpha \) is a locally compact metric space, \( \mathbb{B}_\alpha \) is a domain in \( \mathbb{R}^k \) and \( \Phi_\alpha \) is a homeomorphism defined on an open subset \( U_\alpha \) of \( X \). A set \( S \subset X \) is said to be a **continuous transversal** \( S \) if there is a flow box \( U \) with chart \( \Phi : U \to \mathbb{B} \times T \), and an open subset \( V \) of \( T \), and a continuous map \( V \ni t \mapsto s(t) \in \mathbb{B} \) such that

\[ S = \{ \Phi^{-1}(s(t), t) : t \in V \}. \]

Note that \( S \) is a continuous image of an open subset of \( T \) and that \( S \) intersects every plaque \( \Phi^{-1}(\cdot, t) \) with \( t \in V \) of \( U \) in exactly one point and that \( S \) does not intersect other plaques of \( U \). Hence, if we fix a point \( x \in S \), then for every sufficiently small open neighborhood \( U \) of \( x \), \( S \cap U \) is still a continuous transversal.
at $x$. Consequently, if $x$ is also contained in another flow box $U'$, then by shrinking $S$ if necessary (that is, by replacing $S$ with $S \cap U$ as above, $S$ is also a continuous transversal at $x \in U'$. So the germ of a continuous transversal at a point is independent of flow boxes.

Let $\omega \in \Omega$ and set $t_0 := 0$. Let $t_1 > 0$ such that $\omega[t_0, t_1]$ is contained in a single flow box $U$. Let $S_0$ (resp. $S_1$) be a continuous transversal at $x_0 := \omega(t_0)$ (resp. at $x_1 := \omega(t_1)$). We may choose an open neighborhood $V$ of $t_0$ in $\mathbb{T}$ such that by shrinking $S_0$ and $S_1$ if necessary,

$$S_0 = \{ \Phi^{-1}(s_0(t), t) : t \in V \} \quad \text{and} \quad S_1 = \{ \Phi^{-1}(s_1(t), t) : t \in V \}.$$  

We define the holonomy map $h_{\omega,t_1} : S_0 \to S_1$ as follows:

$$h_{\omega,t_1}(x) := \Phi^{-1}(s_1(t), t), \quad x \in S_0,$$

where $t = t_x \in V$ is uniquely determined by $\Phi(x) = (s_0(t), t)$. In summary we have shown that Dom $(h_{\omega,t_1})$ (resp. Range $(h_{\omega,t_1})$) is an open neighborhood of $x_0$ in $S_0$ (resp. of $x_1$ in $S_1$). In other words, the germ at $x_0$ of $h_{\omega,t_1}$ is a well-defined homeomorphism.

Now we define the holonomy map $h_{\omega,t} : S \to S'$ in the general case. Here $S$ (resp. $S'$) is a continuous transversal at $x_0 := \omega(0)$ (resp. at $x_t := \omega(t)$). Fix a finite subdivision $0 = t_0 < t_1 < \ldots < t_k = t$ of $[0, t]$ such that $\omega[t_i, t_{i+1}]$ is contained in a flow box $U_i$. Choose a continuous transversal $S_i$ at $t_i$ such that $S_0 = S$ and $S_k = S'$. The construction given in the previous paragraph shows that $h_{T^i \omega,t_i+1-t_i}$ is a well-defined homeomorphism from an open neighborhood of $x_i$ in $S_i$ onto an open neighborhood of $x_{i+1}$ in $S_{i+1}$. The holonomy map along $\omega$ at time $t$ is, by definition, the composition

$$h_{\omega,t} := h_{T^k \omega,t_k-t_{k-1}} \circ \cdots \circ h_{T^i \omega,t_{i+1}-t_i} \circ \cdots \circ h_{\omega,t_1}.$$ 

This a a well-defined homeomorphism from an open neighborhood of $\omega(0)$ in $S$ onto an open neighborhood of $\omega(t)$ in $S'$. The germ of $h_{\omega,t}$ at $x_0$ depends only on the homotopy type of $\omega[0, t]$.

Now we introduce the notion of flow tubes which generalizes the notion of flow boxes. Flow tubes are more flexible than flow boxes. Roughly speaking, a flow tube can be as thin and as long as we like, whereas flow boxes are rigid. In some sense, a flow tube is a chain of small flow boxes. However, contrary to the flow boxes, a plaque of a given flow tube may meet several plaques of another adjacent flow tube.

**Definition 10.15.** An open set $U \subset X$ is said to be a flow tube of $(X, \mathcal{L})$ if there is a continuous transversal $\mathbb{T}$ such that for each $x \in \mathbb{T}$, there is a relatively compact open subset $U_x$ of the leaf $L_x$ with the following properties:

- $x \in U_x$ and $U_x \cap U_{x'} = \emptyset$ for $x, x' \in \mathbb{T}$ with $x \neq x'$;
- $\bigcup_{x \in \mathbb{T}} U_x = U$.  

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\[ T \] is said to be a transversal of the flow tube \( U \). For each \( x \in T \), the set \( U_x \) is said to be the plaque at \( x \) of \( U \).

The sample-path space of a flow tube \( U \) up to time \( N \geq 0 \) is, by definition, the subspace of \( \Omega := \Omega(X, \mathcal{L}) \) consisting of all \( \omega \in \Omega \) such that \( \omega[0, N] \) is fully contained in a single plaque \( U_x \) for some \( x \in T \). This space is denoted by \( \Omega(N, U) \).

Let \( \pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L}) \) be the covering lamination projection. An open set \( \tilde{U} \subset \tilde{X} \) is said to be a good flow tube in \( (\tilde{X}, \tilde{\mathcal{L}}) \) if its image \( U := \pi(\tilde{U}) \subset X \) is also a flow tube in \( (X, \mathcal{L}) \) and if the restriction of the projection \( \pi|_{\tilde{U}} : \tilde{U} \to U \) is a homeomorphism which maps each plaque of \( \tilde{U} \) onto a plaque of \( U \). For a transversal \( \mathcal{T} \) of \( U \), the set \( \mathcal{T} := \pi(\mathcal{T}) \) is a transversal of \( U \).

A pair of conjugate flow tubes \( (\tilde{U}', \tilde{U}'') \) is the data of two good flow tubes \( \tilde{U}', \tilde{U}'' \) in \( (\tilde{X}, \tilde{\mathcal{L}}) \) such that they have a common image, i.e., \( \pi(\tilde{U}') = \pi(\tilde{U}'') \).

### 10.5 Metrizability and separability of sample-path spaces

Let \( (X, \mathcal{L}, g) \) be a lamination satisfying the Standing Hypotheses. Let \( \pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L}) \) be the covering lamination projection. Let \( \Omega := \Omega(X, \mathcal{L}) \) and \( \Omega := \Omega(\tilde{X}, \tilde{\mathcal{L}}) \). The main purpose of this subsection is to prove the following result.

**Theorem 10.16.** 1) There exists a countable family of pairs of conjugate flow tubes \( \{(\tilde{U}_i', \tilde{U}_i'')\}_{i \in \mathbb{N}} \) such that for every \( N > 0 \),

\[
\left\{ (\tilde{\omega}', \tilde{\omega}'') \in \tilde{\Omega} \times \tilde{\Omega} : \pi \circ \tilde{\omega}'(t) = \pi \circ \tilde{\omega}''(t), \forall t \in [0, N] \right\}
\]

\[
\subseteq \bigcup_{i \in \mathbb{N}} \Omega(N, \tilde{U}_i') \times \Omega(N, \tilde{U}_i'').
\]

2) For each \( i \in \mathbb{N} \) let \( U_i := \pi(\tilde{U}_i') \). Then for every \( N > 0 \)

\[
\Omega = \bigcup_{i \in \mathbb{N}} \Omega(N, U_i).
\]

**Proof.** Since Part 2) is an immediate consequence of Part 1), we only need to prove the latter part. We fix an atlas \( \mathcal{L} \) of \( X \) with at most countable charts \( \Phi_\alpha : U_\alpha \to \mathcal{B}_\alpha \times T_\alpha \). Suppose without loss of generality that \( \mathcal{B}_\alpha = \mathcal{B} = [0, 1]^k \) for all \( \alpha \), where \( k \) is the real dimension of the leaves. Fix a transversal (still denoted by) \( T_\alpha := \Phi^{-1}_\alpha(\{0\} \times T_\alpha) \) for each flow box \( U_\alpha \). Since each \( T_\alpha \) is a separable metric space, we may find a countable basis \( \mathcal{T}_\alpha \) of nonempty open subsets of \( T_\alpha \). Let \( \mathcal{T} := \cup_\alpha \mathcal{T}_\alpha \). So \( \mathcal{T} \) is also countable.

Before proceeding further, look at the cube \( \mathcal{B} = [0, 1]^k \). For each \( m \in \mathbb{N} \) consider the subdivision of \( \mathcal{B} \) into \( 2^{mk} \) smaller cubes

\[
\left[ \frac{d_1}{2^m}, \frac{d_1 + 1}{2^m} \right] \times \cdots \times \left[ \frac{d_k}{2^m}, \frac{d_k + 1}{2^m} \right],
\]

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where the integers $d_1, \ldots, d_k$ range over $0, \ldots, 2^m - 1$. Each such a cube is said to be a cube of order $m$.

For each $m \in \mathbb{N}$ let $\mathcal{F}_m$ be the following family of open subsets of $X$

$$\mathcal{U}_m := \{ \Phi^{-1}_a(B \times S) : B \text{ a cube of order } m \text{ and } S \in \mathcal{F}_a \}.$$

Each element of $\mathcal{U}_m$ is said to be a small flow box of order $m$. Let $\mathcal{F} := \bigcup_{m \in \mathbb{N}} \mathcal{F}_m$. Clearly, $\mathcal{F}$ is countable. The proof is divided into several steps.

**Step 1:** Construction of the holonomy map for each small flow box.

Consider the small flow box $U := \Phi^{-1}_a(B \times S) \subset X$ of order $m$, where $B$ is a cube of order $m$ and $S \in \mathcal{F}_a$ is an open set. Let $V$ (resp. $W$) be two open subsets of $U$. Since $U$ is a flow tube with plaques $U_s := \Phi^{-1}_a(B \times \{s\})$ for $s \in S$, consider the following multivalued map $h_{U,V,W} : U \to U$ whose the graph is

$$\Gamma(h_{U,V,W}) := \{(x, y) : x \in V, w \in W \text{ and } x \text{ and } y \text{ are on the same plaque of } U \}.$$

Consider, for each $N > 0$, the sample-path space associated to $(U; V, W)$:

$$\Omega_{V,W}(N, U) := \{ \omega \in \Omega(N, U) : \omega(0) \in V \text{ and } \omega(N) \in W \}.$$

**Step 2:** Construction of the holonomy map for a chain of small flow boxes.

A chain $\mathcal{U}$ of small flow boxes is a collection of small flow boxes $(U_i)_{i=0}^p$ such that $U_i \cap U_{i+1} \neq \emptyset$ for $0 \leq i \leq p - 1$. For each $0 \leq i \leq p - 1$, let $V_i := U_i \cap U_{i-1}$ and $W_i := U_i \cap U_{i+1}$, where $U_{-1} = U_0$. So $V_i = W_i$ for $0 \leq i \leq p - 1$. The holonomy map of the chain $\mathcal{U}$ is, by definition, the multivalued map $h_{\mathcal{U}} : U_0 \to U_p$ given by

$$h_{\mathcal{U}} := h_{V_{p-1}, W_{p-1}} \circ \cdots \circ h_{V_0, W_0},$$

where each multivalued function in the right hand side is given by Step 1. Note that $\text{Dom}(h_{\mathcal{U}})$ and $\text{Range}(h_{\mathcal{U}})$ are (possibly empty) open subsets of $X$. For each $N > 0$ the sample-path space associated to the chain $\mathcal{U}$ is given by

$$\Omega(N, \mathcal{U}) := \{ \omega \in \Omega : \exists (t_i)_{i=0}^p \subset [0, N] \text{ such that } 0 = t_0 < \cdots < t_p = N$$

$$\text{and } T^i \omega \in \Omega_{V_i, W_i}(t_{i+1} - t_i, U_i) \text{ for } 0 \leq i \leq p - 1 \}.$$

If we fix a transversal $S_0$ at a point $x_0 \in \text{Dom}(h_{\mathcal{U}})$ and a transversal $S_p$ at a point $x_p \in h_{\mathcal{U}}(x_0) \subset \text{Range}(h_{\mathcal{U}})$, then $h_{\mathcal{U}}$ induces a well-defined univalued map (still denoted by $h_{\mathcal{U}}$) from an open neighborhood of $x_0$ in $S_0$ onto an open neighborhood of $x_p$ in $S_p$. The latter map is even homeomorphic. Moreover, the germ of $h_{\mathcal{U}}$ at $x_0$ coincides with the germ of $h_{\omega, N}$ at $x_0$, where $N > 0$ and $\omega \in \Omega(N, \mathcal{U})$ is a path such that $\omega(0) = x_0$ and $\omega(N) = x_p$.

**Step 3:** Construction of a flow tube associated to a good chain.

Let $N > 0$ be a fixed time. Let $\mathcal{U}$ be a chain of small flow boxes such that $\mathbb{T} := \text{Dom}(h_{\mathcal{U}}) \neq \emptyset$. For each $x \in \mathbb{T}$ the open subset of $L_x$ given by

$$\mathcal{U}_x := \{ \omega(t) : \omega \in \Omega(N, \mathcal{U}), \omega(0) = x, \ t \in [0, N] \}.$$
is said to be the plaque at $x$ associated to $U$. Observe that $x \in U_x$ and $U := \cup_{x \in \mathbb{T}} U_x$ is an open subset of $X$. We say that the chain $U$ is good if $U$ is a flow tube (see Definition 10.15) with the transversal $T$ and the plaque $U_x$ for each $x \in \mathbb{T}$ as above. In this case $U = U_U$ is said to be the flow tube associated to the good chain $U$. Clearly, a chain $U$ is good if and only if $T \neq \emptyset$ and $U_x \cap U_x' = \emptyset$ for all $x, x' \in \mathbb{T}$ with $x \neq x'$. For each $m \in \mathbb{N}$ let $\mathcal{S}_m$ be the set of all good chains each small flow box of which is of order $\geq m$. So $\mathcal{S}_m$ is countable and $(\mathcal{S}_m)_{m=0}^\infty$ is decreasing. Let $\mathcal{S} := \mathcal{S}_0$. The following result is needed.

**Lemma 10.17.** For each $N > 0$ and $\omega \in \Omega$, and for each open neighborhood $\mathbb{V}$ of $\omega[0, N]$ there exist a a good chain $U \in \mathcal{S}$ such that all small flow boxes of $U$ are contained in $\mathbb{V}$ and that $\omega \in \Omega(N, U_U)$. Moreover,

**Proof.** Using the compactness of $\omega[0, N]$ and shrinking $\mathbb{V}$ if necessary, we may suppose without loss of generality that $\mathbb{V}$ is a sufficiently small tubular neighborhood of $\omega[0, N]$ which is also a flow tube. So $\omega \in \Omega(N, \mathbb{V})$. By decreasing the size of small flow boxes if necessary (that is, by increasing $m$), we may cover $\omega[0, N]$ by a good chain $U \in \mathcal{S}_m$ such that its small flow boxes are all contained in $\mathbb{V}$ and that $\omega \in \Omega(N, U_U)$, as asserted. \hfill \square

An immediate consequence of Lemma 10.17 is that

$$\Omega = \bigcup_{U \in \mathcal{S}} \Omega(N, U_U).$$

**Step 4:** End of the proof.

Now let $(\tilde{\omega}', \tilde{\omega}'') \in \tilde{\Omega} \times \tilde{\Omega}$ such that $\pi \circ \tilde{\omega}'(t) = \pi \circ \tilde{\omega}''(t)$ for all $t \in [0, N]$. Let $\omega := \pi \circ \tilde{\omega}' \in \Omega$. Using the compactness of $\omega[0, N]$ we may choose a flow tube $\mathbb{V}$ which is a sufficiently small tubular neighborhood of $\omega[0, N]$ such that there are homotopies $\alpha'$ and $\alpha''$ on $\mathbb{V}$, that is, there are continuous functions $\alpha', \alpha'' : \mathbb{V} \times [0, N] \to X$ (see Subsection 2.2) such that

$$\tilde{\omega}'(t) = (\omega(t), [\alpha'_{\omega(t)}]) \quad \text{and} \quad \tilde{\omega}''(t) = (\omega(t), [\alpha''_{\omega(t)}]), \quad t \in [0, N].$$

Note that the cardinal of such a pair $(\alpha', \alpha'')$ is at most countable. By Lemma 10.17 we may cover $\omega[0, N]$ by a good chain $U \in \mathcal{S}$ such that its small flow boxes are all contained in $\mathbb{V}$ and that $\omega \in \Omega(N, U_U)$. Since $\mathcal{S}$ is countable we may write $U_U := U_j$ for $j \in \mathbb{N}$. Moreover, for each $j \in \mathbb{N}$, let $I_j$ be the set indexing all pairs $(\alpha'_i, \alpha''_i)$ as above. Clearly, $I_j$ is at most countable. For $i \in I_j$ let

$$\tilde{U}'_i := \{(x, [\alpha'_{ix}]) : x \in U_j\} \quad \text{and} \quad \tilde{U}''_i := \{(x, [\alpha''_{ix}]) : x \in U_j\}.$$

So $(\tilde{U}'_i, \tilde{U}''_i)$ is a pair of conjugate flow tubes. By lifting $\omega[0, N]$ to $\tilde{\omega}'[0, N]$ and $\tilde{\omega}''[0, N]$ and using that $\omega \in \Omega(N, U_U)$, we get that

$$(\tilde{\omega}', \tilde{\omega}'') \in \Omega(N, \tilde{U}'_i) \times \Omega(N, \tilde{U}''_i).$$
By taking the above membership over all \( j \in \mathbb{N} \) and all \( i \in I_j \), the theorem follows.

Although the following result is not used in this work, it is of independent interest.

**Theorem 10.18.**

1) There is a metric \( \text{dist} \) on \( \Omega := \Omega(X, \mathcal{L}) \) such that the metric space \((\Omega, \text{dist})\) is separable and that its Borel \( \sigma \)-algebra coincides with \( \mathcal{A} := \mathcal{A}(\Omega) \).

2) There is a metric \( \text{dist} \) on \( \hat{\Omega} := \hat{\Omega}(X, \mathcal{L}) \) such that the metric space \((\hat{\Omega}, \text{dist})\) is separable and that its Borel \( \sigma \)-algebra coincides with \( \hat{\mathcal{A}} := \hat{\mathcal{A}}(\Omega) \).

We only give the proof of the separability in both assertions 1) and 2). We leave to the interested reader to verify the coincidence between \( \sigma \)-algebras stated in these assertions.

**Proof.** First we prove assertion 1). Since the the induced topology on each flow box is metrizable and \( X \) is paracompact, the topological space \( X \) is metrizable. Replacing \( \rho \) by \( \frac{\rho}{1+\rho} \) if necessary, we may assume without loss of generality that there exists a metric \( \rho \) on \( X \) which induces the topology of \( X \) and which satisfies \( \rho \leq 1 \).

Consider the following metrics on \( \Omega \):

\[
\text{dist}(\omega, \omega') := \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{t \in [0,n]} \rho(\omega(t), \omega'(t)), \quad \omega, \omega' \in \Omega;
\]

\[
\text{dist}_N(\omega, \omega') := \sup_{0 \leq t \leq N} \rho(\omega(t), \omega'(t)), \quad N > 0 \text{ and } \omega, \omega' \in \Omega.
\]

For every \( \omega \in \Omega \) and \( r > 0 \) and \( N \in \mathbb{N} \setminus \{0\} \), consider the balls

\( \mathbb{B}(\omega, r) := \{ \omega' \in \Omega : \text{dist}(\omega, \omega') < r \} \) and \( \mathbb{B}_N(\omega, r) := \{ \omega' \in \Omega : \text{dist}_N(\omega, \omega') < r \} \).

Using \( \rho \leq 1 \) it is not hard to check that

\( \mathbb{B}_N(\omega, r) \subset \mathbb{B}(\omega, r) \subset \mathbb{B}(\omega, 2^N r) \).

Using this we infer that in order to show that the metric space \((\Omega, \text{dist})\) is separable, it suffices to prove that for every \( N \in \mathbb{N} \setminus \{0\} \), the space \( \Omega_N \) equipped with the metric \( \text{dist}_N \) is separable, where \( \Omega_N \) consists of all leafwise continuous paths \( \omega : [0, N] \to (X, \mathcal{L}) \). We may assume without loss of generality that \( N = 1 \).

First we consider the case where the lamination \((X, \mathcal{L})\) is reduced to a single leaf \( L \). The separability of the metric space \( \mathcal{C}([0, 1], L) \) of all continuous map \( \omega : [0, 1] \to L \) is well-known.

Next we consider the general case. Fix a countable dense sequence \((x_n)_{n=1}^{\infty} \subset X_0 \). For each \( n \geq 1 \) we fix a countable number of balls \( B(\omega_{nm}, r_{nm}) \) such that \( \omega_{nm} \subset L_{x_n} \) and that this family of balls, when restricted to the leaf \( L_{x_n} \), constitutes a basis of open subsets of \( \mathcal{C}([0, 1], L_{x_n}) \). We leave it to the interested reader
to check that the countable family \( \{ B(\omega_{nm}, r_{nm}) : n, m \geq 1 \} \) forms a basis of open subsets of \( \Omega \). Hence, the metric space \((\Omega, \text{dist})\) is separable.

To prove assertion 2) consider the following metric on \( \hat{\Omega} \):

\[
\text{dist}(\omega, \omega') := \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{t \in [-n, n]} \rho(\omega(t), \omega'(t)), \quad \omega, \omega' \in \hat{\Omega}.
\]

The rest of the proof is analogous to that of assertion 1). \(\square\)

### 10.6 The leafwise diagonal is Borel measurable

The main result of this subsection is the following

**Proposition 10.19.** The leafwise diagonal defined by

\[
\mathcal{G} := \{ (x, y) \in X^2 : x \text{ and } y \text{ are on the same leaf} \}
\]

is Borel measurable.

**Proof.** We will use the terminology and notation introduced in Subsection 10.5.

Fix an atlas \( \mathcal{L} \) of \( X \) with at most countable charts \( U_\alpha \). Fix a transversal \( T_\alpha \) for each flow box \( U_\alpha \). Let \( T := \bigcup_\alpha T_\alpha \). We also fix a countable basis \( \mathcal{T}_\alpha \) of nonempty open subsets of \( T_\alpha \). Let \( \mathcal{T} := \bigcup_\alpha \mathcal{T}_\alpha \). So \( \mathcal{T} \) is countable. Let \( \mathcal{S} \) be the (countable) set of all good chains.

Observe that two given points \( t_1 \) and \( t_2 \) are on the same leaf if and only if there exists a chain \( U \) of flow boxes and an open set \( S \in \mathcal{S} \) such that \( S \subset \text{Dom} (h_U) \) and that \( t_1 \in S \) and \( t_2 = h_U(t_1) \). Let \( \{(U_n, S_n)_{n \in I}\} \) be the (possibly empty) sequence of all possible pairs consisting of a good chain \( U_n \) and an element \( S_n \in \mathcal{S} \) such that \( S_n \subset \text{Dom} (h_{U_n}) \). Note that \( I \) is at most countable. Consider the following subset of \( T^2 \):

\[
G = \bigcup_{n \in I} \{(t, h_{U_n}(t) : t \in S_n \}.
\]

Each set \( \{(t, h_{U_n}(t) : t \in S_n \} \) is a Borel subset of \( X^2 \) because \( h_{U_n} \) is a continuous map. Hence, \( G \) is also a Borel set. We need the following result whose the proof is left to the interested reader.

**Lemma 10.20.** Let \( S \) be a Borel subset of a transversal \( T_\alpha \) of \((X, \mathcal{L})\). Then \( \text{Satur}(S) \) is also a Borel set.

Let \( (X^2, \mathcal{L}^2) \) be the product of the lamination \((X, \mathcal{L})\) with itself. More precisely, \((X^2, \mathcal{L}^2) \) is also a lamination whose the leaf \( L_{(x,x')} \) passing through a point \( (x, x') \) is \( L_x \times L_{x'} \). Observe that \( \mathcal{G} = \text{Satur}(G) \), where \( \text{Satur} \) is taken in \((X^2, \mathcal{L}^2)\). Recall that \( G \) is a Borel subset of \( T^2 \). Applying the above lemma to \( G \) yields that \( \mathcal{G} \) is a Borel set. \(\square\)
The main purpose of this subsection is to complete the proof of Proposition 2.12, Proposition 4.1 (ii) and Theorem 2.11. Let (\(X, \mathcal{L}\)) be a lamination satisfying the Standing Hypotheses and \(\pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L})\) the covering lamination projection. Set \(\Omega := \Omega(X, \mathcal{L})\) and \(\tilde{\Omega} := \Omega(\tilde{X}, \tilde{\mathcal{L}})\). Consider the \(\sigma\)-algebras \(\mathcal{A} := \mathcal{A}(\Omega)\) and \(\tilde{\mathcal{A}} := \tilde{\mathcal{A}}(\Omega)\) on \(\Omega\), and the \(\sigma\)-algebra \(\mathcal{A}(\tilde{\Omega})\) on \(\tilde{\Omega}\). Recall from Subsection 4.1 that a set \(A \subset \Omega\) is said to be a cylinder image if \(A = \pi \circ \tilde{A}\) for some cylinder set \(\tilde{A} \subset \tilde{\Omega}\). Note that \(\mathcal{A}(\tilde{\Omega}) = \tilde{\mathcal{A}}(\tilde{\Omega})\).

The following result is the main technical tool in this subsection.

**Proposition 10.21.** \(\tilde{\mathcal{A}} \subset \mathcal{A}\) and \(\pi^{-1}(\mathcal{A}) \subset \mathcal{A}(\tilde{\Omega})\) and \(\pi \circ \mathcal{A}(\tilde{\Omega}) = \mathcal{A}\).

Taking for granted Proposition 10.21 and Theorem 2.11, we arrive at the End of the proof of Proposition 2.12. Let \(C \in \mathcal{A}\), \(x \in X\) and \(\tilde{x} \in \pi^{-1}(x) \subset \tilde{X}\). By Proposition 10.21 \(\pi^{-1}C \in \mathcal{A}(\tilde{\Omega})\). So \(\pi^{-1}C = \pi^{-1}C \cap \tilde{\Omega}_{\tilde{x}} \in \mathcal{A}(\tilde{\Omega})\), as asserted.

Next, let \(\tilde{x}_1, \tilde{x}_2 \in \pi^{-1}(x)\). We have that

\[
W_{\tilde{x}_1}(\pi^{-1}C) = W_{\tilde{x}_1}(\pi^{-1}C) \quad \text{and} \quad W_{\tilde{x}_2}(\pi^{-1}C) = W_{\tilde{x}_2}(\pi^{-1}C).
\]

On the other hand, by Lemma 10.35 (i), \(W_{\tilde{x}_1}(\pi^{-1}C) = W_{\tilde{x}_2}(\pi^{-1}C)\). Putting all these together, we get that \(\tilde{W}_{\pi^{-1}C} = \tilde{W}_{\pi^{-1}C}\). So the value of \(W_x(C)\) given in (2.6) is well-defined. Using (2.6) and the fact that \(W_x\) is a probability measure on \((\Omega, \mathcal{A}(\tilde{\Omega}))\), we infer easily that \(W_x\) is a probability measure on \((\Omega, \mathcal{A})\), proving assertion (i).

For each \(B \in \mathcal{A}\), by Theorem 2.11 the bounded function \(x \mapsto W_x(B)\) is Borel measurable. Consequently, \(\tilde{\mu}(B)\) given in (2.7) is well-defined. To conclude assertion (ii) we need to show that \(\tilde{\mu}\) is countably additive function from \(\mathcal{A}\) to \(\mathbb{R}^+\). Let \(A = \bigcup_{n=1}^{\infty} A_n\), where \(A_n \in \mathcal{A}\) with \(A_n \cap A_m = \emptyset\) for \(n \neq m\). By Proposition 10.21 \(\pi^{-1}A \in \mathcal{A}(\tilde{\Omega})\). On the other hand, by Lemma 10.35 for every \(x \in X\) and for every \(\tilde{x} \in \pi^{-1}(x)\).

\[
W_x(A) = W_{\tilde{x}}(\pi^{-1}A) = \sum_{n=1}^{\infty} W_{\tilde{x}}(\pi^{-1}A_n) = \sum_{n=1}^{\infty} W_x(A_n).
\]

So integrating both sides on \((X, \mu)\), and using formula (2.7) we get \(\tilde{\mu}(A) = \sum_{n=1}^{\infty} \tilde{\mu}(A_n)\) as desired. \(\square\)

**End of the proof of Proposition 4.1 (ii).** It remains to show that \(\mathcal{A}\) is approximable by cylinder images when \(X = \tilde{X}\). In this case \(\mathcal{A} = \tilde{\mathcal{A}} := \tilde{\mathcal{A}}(\tilde{\Omega})\), in particular, cylinder images coincide with cylinder sets. Let \(\mathcal{D}\) be the family of all sets \(A \subset \Omega\) which are finite unions of cylinder sets. By Proposition 2.8, \(\mathcal{D}\) is the algebra on \(\Omega\) generated by cylinder sets. So \(\mathcal{D} \subset \mathcal{A}\). By Proposition 2.12 (ii), \(\tilde{\mu}\)
is countable additive on $\emptyset$. Consequently, applying Proposition 10.8 yields that $\mathcal{A}$ is approximable by cylinder sets. \hfill $\Box$

For the proof of Theorem 2.11 we need the following result.

Proposition 10.22. The function $\Phi : X \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$
\Phi(x, y, t) := \begin{cases} 
p(x, y, t), & x \text{ and } y \text{ are on the same leaf}; \\
0, & \text{otherwise};
\end{cases}
$$

is Borel measurable.

**Proof.** Using identity (2.2) and the fact that $\pi : \tilde{X} \to X$ is locally homeomorphic, we may assume without loss of generality that $X = \tilde{X}$, that is, all leaves of $(X, \mathcal{L})$ are simply connected. Fix a transversal $T$ and straighten all leaves passing through $T$. Using Proposition 10.19, it suffices to show that the heat kernel of the leaf $L_{\tau}$ ($\tau \in T$) given by $p_{\tau}(x, y, t) := p(x, y, t)$ with $x, y \in L_{\tau}$ and $t \in \mathbb{R}^+$, depends Borel measurably on $\tau \in T$. Let $n$ be the dimension of the leaves. By identifying with $\mathbb{R}^n$ the tangent spaces $T_{\tau}L_{\tau}$ of the leaf $L_{\tau}$ at the point $\tau \in T$, we obtain by Hopf-Rinow theorem a family of (surjective and locally diffeomorphic) exponential maps $\exp_{\tau} : \mathbb{R}^n \to L_{\tau}$, which depends measurably on the parameter $\tau$. For $N \in \mathbb{N}$ let $B_N$ be the open ball centered at 0 with radius $N$ and let $B_{N, \tau} := \exp_{\tau}(B_N)$ for $\tau \in T$. Since $B_{N, \tau}$ is a bounded regular domains in $L_{\tau}$, there exists a heat kernel $p_{N, \tau}$ for $B_{N, \tau}$. Moreover, by the construction of the heat kernel described in Appendix B.6 in [5], the family $p_{N, \tau}(\exp_{\tau}(u), \exp_{\tau}(v), s)$ depends Borel measurably on $(\tau; u, v, t) \in T \times B_N \times B_N \times \mathbb{R}^+$. Here we make a full use of the assumptions that the geometry of $L_{\tau}$ is uniformly bounded and that the leafwise complete metric $g$, when restricted to each flow box, depends Borel measurably on plaques. On the other hand, since $B_{N, \tau} \nearrow L_{\tau}$ as $N \nearrow \infty$, it is well-known (see [8] or Appendix B.6 in [5]) that

$$
p_{\tau}(\exp_{\tau}(u), \exp_{\tau}(v), t) = \sup_{N \in \mathbb{N}} p_{N, \tau}(\exp_{\tau}(u), \exp_{\tau}(v), t), (\tau; u, v, t) \in T \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+,
$$

where $p_{\tau}$ is the heat kernel of the leaf $L_{\tau}$. This formula implies the desired measurability of the function $\Phi$. \hfill $\Box$

**End of the proof of Theorem 2.11.** By Proposition 10.21, $\pi^{-1}B \in \mathcal{A}(\tilde{\Omega})$. On the other hand, by Lemma 10.35 (i), $W_{\tilde{x}}(B) = W_{\tilde{x}}(\pi^{-1}B)$ for every $\tilde{x} \in \pi^{-1}(x)$. Moreover, $\pi$ is a covering projection which is locally homeomorphic. Consequently, we are reduced to showing that for every $\tilde{B} \in \mathcal{A}(\tilde{\Omega})$, $\tilde{X} \ni \tilde{x} \mapsto W_{\tilde{x}}(\tilde{B}) \in [0, 1]$ is Borel measurable. So we may assume without loss of generality that $X = \tilde{X}$, that is, all leaves of $(X, \mathcal{L})$ are simply connected. The proof is divided into three steps.

**Step 1:** For a cylinder set $B := C(\{t_i, A_i\})$, the function $X \ni x \mapsto W_x(B)$ is measurable.

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Let $T$ be a transversal and $A_1,\ldots,A_m$ Borel subsets of $\text{Satur}(T)$. Let $f : X \rightarrow \mathbb{R}$ be a Borel measurable function. We rewrite (2.1) as follows:

$$D_t f(x) := \int_{L_x} \Phi(x,y,t)f(y)\text{dVol}_{L_x}(y), \quad x \in X,$$

where $\Phi$ is the measurable function given in Proposition 10.22 above. Next, fix a point $x_0 \in T$ and consider the measure $\mu := \text{Vol}_{L_{x_0}}$ on $L_{x_0}$. Using the straightening of leaves passing through $T$ we can write, for each $x \in T$,

$$\Phi(x,y,t)\text{dVol}_{L_x}(y) = \Phi(x,y,t)\Psi(x,y,t)\text{dVol}_{L_{x_0}}(y)$$

for some positive measurable function $\Psi$. Consequently, applying Proposition 10.11 yields that the function $\mathbb{R}^+ \times X \ni (t,x) \mapsto (D_t f)(x)$ is Borel measurable for each Borel measurable function $f : X \rightarrow \mathbb{R}$. This, combined with formula (2.4), implies that the function

$$X \ni x \mapsto W_x(B)$$

is Borel measurable, as desired.

**Step 2:** Let $\mathcal{D}$ be the family of all finite unions of cylinder sets. Then for every $B \in \mathcal{D}$, the function $X \ni x \mapsto W_x(B)$ is Borel measurable.

By Proposition 2.8 we may write $B$ as a finite union of mutually disjoint cylinder sets $B = \bigsqcup_{i=1}^{k} B_i$. Since $W_x(B) = \sum_{i=1}^{k} W_x(B_i)$ and each function $X \ni x \mapsto W_x(B_i)$ is Borel measurable, the desired conclusion follows.

**Step 3:** End of the proof.

let $\mathcal{C}$ be the family of all $B \in \mathcal{A}$ such that the function $X \ni x \mapsto W_x(B)$ is Borel measurable. By Step 2, we get that $\mathcal{D} \subset \mathcal{C}$. Next, observe that if $(A_n)_{n=1}^{\infty} \subset \mathcal{C}$ such that $A_n \nearrow A$ (resp. $A_n \searrow A$) as $n \nearrow \infty$, then $A \in \mathcal{C}$ because $W_x(A_n) \nearrow W_x(A)$ (resp. $W_x(A_n) \searrow W_x(A)$). Consequently, applying Proposition 10.10 yields that $\mathcal{A} \subset \mathcal{C}$. Hence, $\mathcal{A} = \mathcal{C}$. This completes the proof.

Before proceeding to the proof of Proposition 10.21 we make some comments on our method in this subsection. The approach adopted in Subsection 10.3 which is heavily based on Lemma 10.13 and which results in Proposition 10.12 does not work in the present context of general laminations. Indeed, the holonomy phenomenon in the case of a lamination is much more complicated than that in the case of a single leaf. More concretely, instead of trivializing open sets in the context of a single leaf as in Proposition 10.12, we have to work with flow boxes in the context of laminations. However, there exists, in general, no flow box whose every plaque is simultaneously trivializing.

Our new approach consists in replacing cylinder images with directed cylinder images. Roughly speaking, directed cylinder images take into account the holonomy whereas the non-directed ones do not so.

In proving Proposition 10.21 our new approach consists of two steps. First, we use the separability and holonomy result developed in Subsection 10.3 and 10.5 above. Second, we replace cylinder images with their preimages in $\tilde{\Omega}$ and express these preimages in term of directed cylinder sets.
Definition 10.23. Let $\widetilde{\mathcal{U}}$ be a good flow tube and $N > 0$ a given time.

A directed cylinder set $\tilde{A}$ with respect to $(N, \tilde{\mathcal{U}})$ is the intersection of a cylinder set $C(\{t_i, \tilde{A}_i\} : m)$ in $\tilde{\Omega}$ and the sample-path space $\Omega(N, \tilde{\mathcal{U}})$ satisfying the time requirement $N \geq t_m = \max t_i$.

A set $A \subset \Omega$ is said to be a directed cylinder image (with respect to $(N, \tilde{\mathcal{U}})$) if $A = \pi \circ \tilde{A}$ for some directed cylinder set $\tilde{A} \subset \tilde{\Omega}$ (with respect to $(N, \tilde{\mathcal{U}})$).

We fix a countable family of pairs of conjugate flow tubes $\{\tilde{\mathcal{U}}_i, \tilde{\mathcal{U}}''_i\}_{i \in \mathbb{N}}$ satisfying the conclusion of Theorem 10.16. More concretely, let $\mathcal{U}_i$ be the common image of $\tilde{\mathcal{U}}_i$ and $\tilde{\mathcal{U}}''_i$, and let $\pi'_i := \pi|_{\tilde{\mathcal{U}}'_i} : \tilde{\mathcal{U}}'_i \to \mathcal{U}_i$ and $\pi''_i := \pi|_{\tilde{\mathcal{U}}''_i} : \tilde{\mathcal{U}}''_i \to \mathcal{U}_i$ be two homeomorphisms which maps plaques onto plaques. Then for every $N > 0$,

$$\left\{ (\tilde{\omega}', \tilde{\omega}'') \in \tilde{\Omega} \times \tilde{\Omega} : (\pi \circ \tilde{\omega}')(t) = (\pi \circ \tilde{\omega}'')(t), \forall t \in [0, N]\right\} \subset \bigcup_{i \in \mathbb{N}} \Omega(N, \tilde{\mathcal{U}}'_i) \times \Omega(N, \tilde{\mathcal{U}}''_i). \tag{10.2}$$

Now we establish some properties of directed cylinder images.

Lemma 10.24. Let $\tilde{\mathcal{U}}, \tilde{\mathcal{V}}$ be two good flow tubes and $N > 0$ a given time.

1) Then $\Omega(N, \tilde{\mathcal{U}})$ is a countable union of increasing sets, each being a countable intersection of decreasing cylinder sets.

2) $\Omega(N, \pi(\tilde{\mathcal{U}}))$ is a countable union of increasing sets, each being a countable intersection of decreasing cylinder images.

3) Every directed cylinder set (resp. image) is a countable union of increasing sets, each being a countable intersection of decreasing cylinder sets (resp. images).

4) Every cylinder image may be represented as a countable union of directed cylinder images (with respect to flow tubes $\tilde{\mathcal{U}}'_i$).

5) $\mathcal{A}$ coincides with the $\sigma$-algebra on $\Omega$ generated by directed cylinder images (with respect to flow tubes $\tilde{\mathcal{U}}'_i$).

Proof. Fix an increasing sequence $(\tilde{F}_j)_{j=0}^\infty$ of compact subsets of $\tilde{\mathcal{U}}$ such that, for every $\omega \in \Omega(N, \tilde{\mathcal{U}})$, there exists $\tilde{x} \in \tilde{T}$ and $j \in \mathbb{N}$ such that $\omega[0, N] \subset \tilde{\mathcal{U}}_{\tilde{x}} \cap \tilde{F}_j$, where $\tilde{T}$ is a transversal of $\tilde{\mathcal{U}}$. Note that $\tilde{F}_j \nearrow \tilde{\mathcal{U}}$ as $j \nearrow \infty$.

Proof of Part 1). Using the continuity of each path in a sample-path space and using the density of the rational numbers in $[0, N]$, we obtain that

$$\Omega(N, \tilde{\mathcal{U}}) = \bigcup_{j \in \mathbb{N}} \left( \bigcap_{i \in \mathbb{N}} C(\{s, \tilde{F}_j\} : s \in S_i) \right), \tag{10.3}$$

where $\{S_i : i \in \mathbb{N}\}$ is the family of all finite sets $S$ of rational numbers in $[0, N]$. Replacing $S_i$ with $S_0 \cup \cdots \cup S_i$, the last intersection does not change. Therefore, we may assume without loss of generality that $S_0 \subset S_1 \subset S_2 \cdots$, and hence $\Omega(N, \tilde{\mathcal{U}})$ is equal to a countable union of increasing sets, each being a countable
intersection of decreasing cylinder sets. This proves Part 1).

**Proof of Part 2.** In what follows set \( \mathbb{U} := \pi(\tilde{\mathbb{U}}). \) The proof of Part 2) will be complete if one can show that

\[
\Omega(N, \mathbb{U}) = \bigcup_{j \in \mathbb{N}} \left( \bigcap_{i \in \mathbb{N}} \pi \circ C(\{s, \tilde{F}_j\} : s \in S_i) \right). \tag{10.4}
\]

Let \( \omega \in \Omega(N, \mathbb{U}). \) Since \( \pi|_{\tilde{\mathbb{U}}} : \tilde{\mathbb{U}} \to \mathbb{U} \) is a homeomorphism which maps plaques onto plaques, we see that \( (\pi|_{\tilde{\mathbb{U}}})^{-1}(\omega) \in \Omega(N, \tilde{\mathbb{U}}), \) and hence \( (\pi|_{\tilde{\mathbb{U}}})^{-1}(\omega) \in C(\{s, \tilde{F}_j\} : s \in S_i) \) for every \( i \in \mathbb{N}. \) This, combined with the property of \( (\tilde{F}_j)_{j=0}^{\infty}, \) implies that \( \omega \in \bigcup_{i \in \mathbb{N}} \bigcap_{i \in \mathbb{N}} \pi \circ C(\{s, \tilde{F}_j\} : s \in S_i). \)

Conversely, we pick an arbitrary path \( \omega \in \bigcap_{i \in \mathbb{N}} \pi \circ C(\{s, \tilde{F}_j\} : s \in S_i) \) for some \( j \in \mathbb{N}, \) and show that \( \omega \in \Omega(N, \mathbb{U}). \) The choice of \( \omega \) implies that \( \omega(t) \in \mathbb{U} \) for all \( t \in \mathbb{Q} \cap [0, N]. \) Since \( \omega \) is a leafwise continuous map and \( \mathbb{U} \) is a flow tube and the intersection of \( \pi(\tilde{F}_j) \) with each plaque of \( \mathbb{U} \) is compact, we infer that \( \omega(0, N] \) is contained in a plaque of \( \mathbb{U}. \) Hence, \( \omega \in \Omega(N, \mathbb{U}), \) as desired.

**Proof of Part 3.** Let \( \bar{A} = C(\{t_i, \bar{A}_i\} : p) \) be a cylinder set in \( \tilde{\Omega}, \) and let \( A := \pi \circ \bar{A} \) its images. Fix a time \( N \geq t_p. \) Arguing as in the proof of (10.3), we see that

\[
\bar{A} \cap \Omega(N, \tilde{\mathbb{U}}) = \bigcup_{j \in \mathbb{N}} \left( \bar{A} \cap \bigcap_{i \in \mathbb{N}} C(\{s, \tilde{F}_j\} : s \in S_i) \right).
\]

So every directed cylinder set is a countable union of increasing sets, each being a countable intersection of decreasing cylinder sets. Next, arguing as in the proof of (10.4), we see that

\[
\pi \circ (\bar{A} \cap \Omega(N, \tilde{\mathbb{U}})) = A \cap \Omega(N, \mathbb{U}) = \bigcup_{j \in \mathbb{N}} \left( \bigcap_{i \in \mathbb{N}} \pi \circ (\bar{A} \cap C(\{s, \tilde{F}_j\} : s \in S_i)) \right).
\]

So every directed cylinder image is a countable union of increasing sets, each being a countable intersection of decreasing cylinder images.

**Proof of Part 4.** We first deduce from (10.2) that

\[
\tilde{\Omega} = \bigcup_{i \in \mathbb{N}} \Omega(N, \tilde{\mathbb{U}}_i).
\]

This implies that

\[
\bar{A} = \bigcup_{i \in \mathbb{N}} C(\{t_i, \bar{A}_i\} : p) \cap \Omega(N, \tilde{\mathbb{U}}_i).
\]

Acting \( \pi \) on both sides, Part 4) follows.

**Proof of Part 5.** Recall that \( \mathcal{A} \) is the \( \sigma \)-algebra generated by all cylinder images. This, coupled with Part 4) implies that \( \mathcal{A} \) is contained in the \( \sigma \)-algebra on \( \Omega \) generated by directed cylinder images (with respect to flow tubes \( \tilde{\mathbb{U}}_i \)). By Part 3) the inverse inclusion is also true. Hence, Part 5) follows. \( \square \)
The following result is very useful.

**Lemma 10.25.** For each cylinder image \( A \), its preimage \( \pi^{-1}(A) \) may be represented as \( \bigcup_{i \in \mathbb{N}} A_i \), where \( A_i \) is a directed cylinder set with respect to \( \bar{U}_i' \).

**Proof.** Let \( A = \pi \circ \tilde{A} \), where \( \tilde{A} = C(\{t_j, \tilde{A}_j\} : p) \), and fix a time \( N \geq t_p = \max t_j \). For each \( 1 \leq j \leq p \) and each \( i \in \mathbb{N} \) consider the Borel subset \( \tilde{B}_j^i \) of \( \tilde{U}_i' \) given by

\[
\tilde{B}_j^i := (\pi'_i)^{-1}(\pi''_i(\tilde{A}_j \cap \bar{U}_i')).
\]

Since \( \pi'_i := \pi|_{\bar{U}_i} : \bar{U}_i' \to U_i \) and \( \pi''_i := \pi|_{\tilde{U}_i'} : \tilde{U}_i' \to U_i \) are two homeomorphisms which maps plaques onto plaques one can show that

\[
\pi \circ \left( C(\{t_j, \tilde{B}_j^i\} : p) \cap \Omega(N, \bar{U}_i') \right) = \pi \circ \left( C(\{t_j, \tilde{A}_j\} : p) \cap \Omega(N, U_i') \right).
\]

Since the right hand side is contained in \( A \), it follows that

\[
\bigcup_{i \in \mathbb{N}} C(\{t_j, \tilde{B}_j^i\} : p) \cap \Omega(N, \bar{U}_i') \subset \pi^{-1}(A).
\]

Consequently, the proof will be complete if one can show that the above inclusion is, in fact, an equality. To do this pick an arbitrary path \( \tilde{\omega}' \in \pi^{-1}(A) \). So there exists \( \tilde{\omega}'' \in \tilde{A} \) such that \( (\pi \circ \tilde{\omega}'')(t) = (\pi \circ \tilde{\omega}'')(t) \) for all \( t \in [0, N] \). So by (10.2), there exists \( i \in \mathbb{N} \) such that \( (\tilde{\omega}', \tilde{\omega}'') \in \Omega(N, \bar{U}_i') \times \Omega(N, \tilde{U}_i') \). Hence, \( \tilde{\omega}'' \in C(\{t_j, \tilde{A}_j\} : p) \cap \Omega(N, \bar{U}_i') \) and \( \tilde{\omega}' \in C(\{t_j, \tilde{B}_j^i\} : p) \cap \Omega(N, U_i') \), as desired.

\( \square \)

Now we arrive at the

End of the proof of Proposition 10.21.

**Proof of** \( \mathcal{A} \subset \mathcal{A} \) : Let \( A := C(\{A_i, t_i\} : p) \) be a cylinder set in \( \Omega \). For each \( 1 \leq i \leq p \) let \( A_i := \pi^{-1}(A_i) \). Consider the cylinder set \( \tilde{A} := C(\{\tilde{A}_i, t_i\} : p) \) in \( \tilde{\Omega} \). We can check that \( A = \pi \circ \tilde{A} \). So every cylinder set is also a cylinder image. Hence, \( \mathcal{A} \subset \mathcal{A} \), as desired.

**Proof of** \( \pi^{-1}(\mathcal{A}) \subset \pi^{-1}(\tilde{\Omega}) \) : Recall that \( \mathcal{A} \) is the \( \sigma \)-algebra generated by all cylinder images. Consequently, the family \( \{\pi^{-1}(A) : A \in \mathcal{A}\} \) is the \( \sigma \)-algebra on \( \tilde{\Omega} \) generated by all sets of the form \( \pi^{-1}(A) \) with \( A \) a cylinder image. On the other hand, combining Lemma 10.25 and Part 3 of Lemma 10.24 it follows that \( \pi^{-1}(A) \in \pi^{-1}(\tilde{\Omega}) \) for each cylinder image \( A \). Hence, \( \pi^{-1}(\mathcal{A}) \subset \pi^{-1}(\tilde{\Omega}) \), as asserted.

**Proof of** \( \mathcal{A} = \pi \circ \mathcal{A}(\tilde{\Omega}) \) : Since \( \pi^{-1}(\mathcal{A}) \subset \pi^{-1}(\tilde{\Omega}) \) it follows that \( \mathcal{A} \subset \pi \circ \mathcal{A}(\tilde{\Omega}) \). Therefore, it remains to show the inverse inclusion \( \pi \circ \mathcal{A}(\tilde{\Omega}) \subset \mathcal{A} \). To this end consider the family

\[
\mathcal{C} := \left\{ \tilde{A} \in \mathcal{A}(\tilde{\Omega}) : \pi^{-1}(\pi \circ \tilde{A}) = \tilde{A} \text{ and } \pi \circ \tilde{A} \in \mathcal{A} \right\}.
\]
It is worthy noting that by the third • in Definition 8.9 for a set \( \tilde{A} \in \mathcal{A}(\tilde{\Omega}) \), the equality \( \pi^{-1}(\pi \circ \tilde{A}) = \tilde{A} \) holds if and only if \( \tilde{A} \) is invariant under deck-transformations.

Let \( \tilde{A} \) be a cylinder set in \( \tilde{\Omega} \). Then \( \pi \circ \tilde{A} \in \mathcal{A} \), and hence \( \pi^{-1}(\pi \circ \tilde{A}) \in \mathcal{A}(\tilde{\Omega}) \). Moreover, since \( \pi \circ (\pi^{-1}(\pi \circ \tilde{A})) = \pi^{-1}(\pi \circ \tilde{A}) \), we infer that \( \pi^{-1}(\pi \circ \tilde{A}) \in \mathcal{G} \). Let \( \mathcal{G} \) be the algebra of all finite unions of cylinder sets in \( \tilde{\Omega} \). We deduce easily from the previous discussion that \( \pi^{-1}(\pi \circ \tilde{A}) \in \mathcal{G} \) for each \( \tilde{A} \in \mathcal{G} \).

Next, if \( (\tilde{A}_n)_{n=1}^\infty \subset \mathcal{G} \) such that \( \tilde{A}_n \subset \tilde{A}_{n+1} \) for all \( n \), then \( \pi \circ (\cup_{n=1}^\infty \tilde{A}_n) = \cup_{n=1}^\infty \pi \circ \tilde{A}_n \in \mathcal{A} \). Hence, \( \cup_{n=1}^\infty \tilde{A}_n \in \mathcal{G} \).

Analogously, if \( (\tilde{A}_n)_{n=1}^\infty \subset \mathcal{G} \) such that \( \tilde{A}_{n+1} \subset \tilde{A}_n \) for all \( n \), then \( \pi \circ (\cap_{n=1}^\infty \tilde{A}_n) = \cap_{n=1}^\infty \pi \circ \tilde{A}_n \in \mathcal{A} \) since each \( \tilde{A}_n \) is invariant under deck-transformations. Consequently, we can show that \( \cap_{n=1}^\infty \tilde{A}_n \in \mathcal{G} \). Therefore, applying Proposition 10.10 yields that \( \pi \circ \mathcal{G} \subset \mathcal{A} \), where \( \mathcal{G} \) is the \( \sigma \)-algebra generated by all the sets of the form \( \pi^{-1}(\pi \circ \tilde{A}) \), with \( \tilde{A} \) a cylinder set. Using the transfinite induction and the identity \( \pi \circ (\pi^{-1}(\pi \circ \tilde{A})) = \pi \circ \tilde{A} \), it is not difficult to show that \( \pi \circ \mathcal{G} = \pi \circ \mathcal{A}(\tilde{\Omega}) \). Hence, \( \pi \circ \mathcal{G} \subset \mathcal{A}(\tilde{\Omega}) \), as desired.

**End of the proof of Proposition 4.4.** The proof is divided into three steps. In the first two steps we show that the measurability of local expressions implies the measurable law. The last step is devoted to the proof of the inverse implication. By Definition 3.10 assume without loss of generality that \( t_0 = 1 \).

To start with the first implication it is sufficient to show that the map \( \mathcal{A}(\cdot, t) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \) given by \( (\omega, u) \mapsto \mathcal{A}(\omega, t)u \) is measurable for every fixed \( t \in \mathbb{G} \). Without loss of generality we may assume that \( t = 1 \). Working with the covering lamination \( (\tilde{X}, \tilde{\mathcal{L}}) \) and transferring the results back to \( (X, \mathcal{L}) \) via the projection \( \pi \), we may also assume that \( X = \tilde{X} \), that is, all leaves are simply connected. This implies that \( \mathcal{A} = \tilde{\mathcal{A}} \). Moreover, we will make full use of the consequence that \( \mathcal{A}(\omega, t) \) depends only on \( \omega(0) \) and \( \omega(t) \) for each \( t > 0 \). Let \( O \subset \mathbb{R}^d \) be a Borel set.

**Step 1:** Given a flow box \( \Phi : U \to \mathbb{B} \times T \), the set

\[
S_{U, O} := \{ (\omega, u) \in \Omega \times \mathbb{R}^d : \omega(0) \text{ and } \omega(1) \text{ live in a common plaque of } U \text{ and } \mathcal{A}(\omega, 1)u \in O \}
\]

is measurable. To do this let \( \alpha \) be the local expression of \( \mathcal{A} \) on this flow box. By hypothesis, \( \alpha \) is measurable. Consequently, \( \{ (x, y, t, u) \in \mathbb{B} \times \mathbb{B} \times T \times \mathbb{R}^d : \alpha(x, y, t)u \in O \} \) is a measurable set. Hence, the set

\[
P := \{ (\Phi^{-1}(x, t), \Phi^{-1}(y, t), u) : (x, y, t, u) \in \mathbb{B} \times \mathbb{B} \times T \times \mathbb{R}^d : \alpha(x, y, t)u \in O \}
\]

is also measurable in \( X \times X \times \mathbb{R}^d \).

By the construction of the \( \sigma \)-algebra \( \mathcal{A} \), we see that, for any set \( Q \) belonging to the product of Borel \( \sigma \)-algebras \( \mathcal{B}(X) \times \mathcal{B}(X) \times \mathcal{B}(\mathbb{R}^d) \), the generalized cylinder

\[
C(0, 1; Q) := \{ (\omega, u) \in \Omega \times \mathbb{R}^d : (\omega(0), \omega(1), u) \in Q \}
\]
belongs to the product of \( \sigma \)-algebras \( \mathcal{A} \times \mathcal{B}(\mathbb{R}^d) \). Thus, \( C(0, 1; Q) \) is measurable. This, combined with the equality \( S_{U,O} = C(0, 1; P) \) implies that \( S_{U,O} \) is also measurable as desired.

**Step 2:** Measurability of local expressions implies measurable law.

Next, we consider a given flow tube \( U \) and define \( S_{U,O} \) as in the case of a flow box. Observe that each flow tube may be covered by a finite number of flow boxes and that \( A(\omega, 1) \) depends only on \( \omega(0) \) and \( \omega(1) \). Consequently, the argument used in Step 1 still works in the context of flow tubes using the local expression of \( A \) on different finite flow boxes that covers \( U \) and making the obviously necessary changes. So we can also prove that \( S_{U,O} \) is measurable for each flow tube \( U \).

On the other hand, as an immediate consequence of Theorem \[10.16\] there exists a countable family of flow tubes \((U_i)_{i \in \mathbb{N}}\) such that \( \Omega = \bigcup_{i \in \mathbb{N}} \Omega(1, U_i) \). Therefore,

\[
\{(\omega, u) \in \Omega \times \mathbb{R}^d : A(\omega, 1)u \in O\} = \bigcup_{i \in \mathbb{N}} S_{U_i,O}
\]

is measurable. Since this is true for each Borel set \( O \), Step 2 is complete.

**Step 3:** Measurable law implies measurability of local expressions. Let \( O \subset \mathbb{R}^d \) be a Borel set and \( U \) a flow box. Since \( A(\omega, 1) \) depends only on \( \omega(0) \) and \( \omega(1) \), we only need to show that the set

\[
\{(x, y, u) \in U^2 \times \mathbb{R}^d : x \text{ and } y \text{ live in a common plaque of } U \text{ and there is } \omega \in \Omega \text{ with } \omega(0) = x \text{ and } \omega(1) = y \text{ and } A(\omega, 1)u \in O\}
\]

is measurable. This is reduced to showing that the set \( S_{U,O} \) is measurable. Write the last set as

\[
S_{U,O} = \Omega(1, U) \cap \{A(\omega, 1)u \in O\}.
\]

The first set on the right hand side is measurable by Part 2) of Lemma \[10.24\] whereas the second one is measurable by the measurable law applied to the cocycle \( A \). This completes the proof. \( \square \)

### 10.8 Fibered laminations

The purpose of this subsection is to complete the proof of Proposition \[7.14\] which has been stated and used in Subsection \[7.3\] above. The subsection is divided into two parts. The first one discuss the relation between the saturations of Borel sets, and a harmonic probability measure. Here we realize the important difference of harmonic measures over weakly harmonic ones. The second part is devoted to the proof of Proposition \[7.14\].

To start with the first part, let \((X, \mathcal{L}, g)\) be a lamination satisfying the Standing Hypotheses. Let \( U \) be a flow tube with transversal \( T \). For a set \( Y \subset U \),

- **the projection of** \( Y \) **onto** \( T \), denoted by \( T_Y \), **is given by**

\[
T_Y := \{t \in T : U_t \cap Y \neq \emptyset\};
\]
• the **plaque-saturation** of \( Y \) in \( U \), denoted by \( \text{Satur}_U(Y) \), is given by

\[
\text{Satur}_U(Y) := \bigcup_{t \in \mathbb{T}_i: U_t \cap Y \neq \emptyset} U_t;
\]

• \( Y \) is said to be **plaque-saturated** if \( Y = \text{Satur}_U(Y) \).

Fix an (at most) countable cover of \( X \) by flow tubes \( U_i \) with transversal \( \mathbb{T}_i \) indexed by the (at most countable) set \( I \). We can use the family of flow tubes given by Part 2) of Theorem 10.16. For each \( i \in I \) set \( I_i := \{ j \in I : U_j \cap U_i \neq \emptyset \} \).

For a set \( Y \subset X \), we define an increasing sequence \( (Y_{ip})_{p=0}^{\infty} \) of plaque-saturated sets in \( U_i \) as follows: For each \( i \in I \) set

\[
Y_{ip} := \begin{cases} 
\text{Satur}_{U_i}\left( U_i \cap \bigcup_{j \in I_i} Y_{j,p-1} \right), & p \geq 1; \\
\text{Satur}_{U_i}(U_i \cap Y), & p = 0.
\end{cases} \tag{10.5}
\]

The following result allows us to compute the saturation of such a set \( Y \) using the above approximating sequence.

**Proposition 10.26.** 1) Under the above hypotheses and notation, then for each \( i \in I \), \( Y_{ip} \nearrow \text{Satur}(Y) \cap U_i \) as \( p \nearrow \infty \).

2) If, moreover, \( (\mathbb{T}_i)_{Y \cap U_i} \) is a Borel set for each \( i \in I \), then \( \text{Satur}(Y) \) is also a Borel set.

**Proof.** We leave the proof Part 1) to the interested reader since it is not difficult.

Now we turn to Part 2). By the hypothesis, \( Y_{i0} \) is a Borel set for each \( i \in I \). This, combined with (10.5), implies that each \( Y_{ip} \) is a Borel set. This, coupled with Part 1) implies that \( \text{Satur}(Y) \cap U_i \) is a Borel set, as asserted. \( \square \)

In what follows \( \mu \) is a harmonic measure on \((X, \mathcal{L}, g)\). The following elementary result is needed (see Exercise 2.4.16 in [5]).

**Lemma 10.27.** Let \( \Phi : U \to \mathbb{B} \times \mathbb{T} \) and \( \Phi' : U' \to \mathbb{B}' \times \mathbb{T}' \) be two small flows boxes with transversal \( T \) and \( T' \) respectively. Let \( \lambda \) (resp. \( \lambda' \)) be the measure defined on \( T \) (resp. on \( T' \)) which is given by the local decomposition of a harmonic measure \( \mu \) on \( U \) (resp. \( U' \)) thanks to Proposition 2.6. Assume that the change of coordinates \( \Phi \circ \Phi' \) is of the form

\[
(x, t) \mapsto (x', t'), \quad x' = \Psi(x, t), \quad t' = \Lambda(t)
\]

Then the measure \( \Lambda^* \lambda' \) is absolutely continuous with respect to \( \lambda \) on \( \text{Dom}(\Lambda) \).

Let \( \mathbb{T} \) be a transversal of a flow tube \( U \) of \((X, \mathcal{L})\). Fix a partition of unity subordinate to a finite covering of \( U \) by small flow boxes. We applying Lemma 10.27 while traveling different small flow boxes. Consequently, we obtain the following version of Proposition 2.6 in the context of flow tubes.
Proposition 10.28. Under the above hypotheses and notation, there exists a finite positive Borel measure $\lambda$ on $\mathbb{T}$ and for $\lambda$-almost every $t \in \mathbb{T}$ there is a harmonic function $h_t > 0$ defined on the plaque $U_t$ such that for every Borel set $Y \subset U$,

$$\mu(Y) = \int_Y d\mu = \int_\mathbb{T} \left( \int_{U_t} 1_Y(y) h_t(y) d\text{Vol}_t(y) \right) d\lambda(t),$$

where Vol$_t(y)$ denotes the volume form induced by the metric tensor $g$ on the plaque $U_t$.

In the sequel let $\lambda_i$ be the measure on the transversal $T_i$ when we apply Proposition 10.28 to $\mu|_{U_i}$. The relation between $\mu$ and the measure $\lambda$ associated to $\mu$ on a transversal in a flow tube is described in the following result.

Proposition 10.29. Let $Y \subset X$ be a Borel set.

1) Assume that $Y$ is contained in a flow tube $U$ with transversal $T$. Then $T_Y$ is $\lambda$-measurable, where $\lambda$ is the measure on $\mathbb{T}$ given by Proposition 10.28.

2) If $\lambda_i((T_i)_Y \cap U_i) = 0$ for every $i \in I$ then $\mu(Y) = 0$. Conversely, if $\mu(Y) = 0$ and $\text{Vol}_a(L_a \cap Y) > 0$ for every $a \in Y$, then $\lambda((T_Y \cap U) = 0$ for every flow tube $U$ with transversal $T$. Here $\text{Vol}_a$ denotes the volume form on $L_a$ induced by the metric tensor $g|_{L_a}$ and $\lambda$ is given by Proposition 10.28.

3) If $\mu(Y) = 0$ and $\text{Vol}_a(L_a \cap Y) > 0$ for every $a \in Y$, then $\mu(\text{Satur}(Y)) = 0$.

4) There exist a leafwise saturated Borel set $Z$ and a leafwise saturated set $E$ with $\mu(E) = 0$ such that $\text{Satur}(Y) = Z \cup E$ and that $T_Z$ is a Borel set for any transversal $T$ of each flow box $U$.

Proof. To prove Part 1) we assume first that $U$ is a flow box, that is, we can write $U \simeq B \times T$, where $B$ is a domain in $\mathbb{R}^n$. Applying Theorem 10.3 to $Y$ yields that $T_Y$ is $\lambda$-measurable. When $U$ is a general flow tube, we use a finite covering of $U$ by small flow boxes by applying Lemma 10.27. Part 1) follows.

To prove the first assertion of Part 2) suppose in order to get a contradiction that $\lambda_i((T_i)_Y \cap U_i) > 0$ for some $i \in I$. Next, we apply Proposition 10.26 to the saturation of $(T_i)_Y \cap U_i$. This, combined with Lemma 10.27 and using a partition of unity, we may find a flow tube $U$ with transversal $T$ and a Borel set $Z \subset U \cap Y$ and a Borel set $S \subset T \cap \text{Satur}((T_i)_Y \cap U_i)$ such that

- $\lambda(S) > 0$, where $\lambda$ is given by Proposition 10.28.
- $\text{Vol}_t(U_t \cap Z) > 0$ for each $t \in S$.

Inserting these into the equality in Proposition 10.28 we get that $\mu(Z) > 0$. Hence, $\mu(Y) > 0$, which is a contradiction. The second assertion of Part 2) is thus complete.

Now we turn to Part 3). Since $\text{Satur}(Y) = \bigcup_{i \in I} \text{Satur}((T_i)_Y \cap U_i))$, we only need to show that $\mu(\text{Satur}((T_i)_Y \cap U_i)) = 0$ for each $i \in I$. Fix such an $i_0 \in I$. By the second assertion of Part 2), we get that $\lambda_{i_0}((T_{i_0})_Y \cap U_{i_0}) = 0$. Consequently, arguing as in the first assertion of Part 2), we can show that $\mu(Z) = 0$, as desired.
Finally, we prove Part 4). Since $\text{Satur}(Y) = \bigcup_{i \in I} \text{Satur}(Y \cap U_i)$, we may assume without loss of generality that $Y$ is contained in a flow tube $U$ with transversal $T$. Let By Part 1), $T_Y$ is $\lambda$-measurable. So we can write $T_Y = S \sqcup E$, where $S \subset T$ is a Borel set and $F \subset T$ with $\lambda(F) = 0$. Here $\lambda$ is the measure on $T$ given by Proposition [10.28] Let $Z := \text{Satur}(S)$ and $E := \text{Satur}(F)$. Since $S$ is a Borel set we know by Proposition [10.26] that so is $Z$. Moreover, by the first assertion of Part 2), $\mu(E) = 0$. This finishes Part 4).

In the second part of the subsection let $\iota : \Sigma \to \tilde{X}$ be a weakly fibered lamination over $(X, \mathcal{L}, g, \mu)$ and let $\tau := \pi \circ \iota : \Sigma \to X$. Consider the $\sigma$-finite measure $\nu := \tau^*\bar{\mu}$ on $(\Omega(\Sigma), \mathcal{A}(\Sigma))$.

**Lemma 10.30.** If, for every cylinder set $A \subset \Omega(\Sigma)$, the image $\tau \circ A \subset \Omega(X, \mathcal{L})$ is $\bar{\mu}$-measurable, then Definition 7.5 (iv) is satisfied, i.e., $\Sigma$ is a fibered lamination.

**Proof.** First consider the case where $A$ is a countable union of cylinder sets $A_i$ in $\Omega(\Sigma)$. Since $\tau \circ A = \bigcup_{i=1}^{\infty} \tau \circ A_i$, and each image $\tau \circ A_i$ is $\bar{\mu}$-measurable, so is $\tau \circ A$.

Now we turn to the general case where $A \in \mathcal{A}(\Sigma)$. Note that the leaves of $\Sigma$ are all simply connected. Consequently, applying Proposition 4.1 (ii) yields a decreasing sequence $(A_n)$, each $A_n$ being a countable union of mutually disjoint cylinder sets such that $A \subset A_n$ and that $\nu(A_n \setminus A) \to 0$ as $n \to \infty$. By the previous case, $(\tau \circ A_n)_{n=1}^{\infty}$ is a decreasing sequence of $\bar{\mu}$-measurable sets containing $\tau \circ A$. Moreover, we have that by

$$\bar{\mu}(\tau \circ A_n \setminus \tau \circ A) \leq \bar{\mu}(\tau \circ (A_n \setminus A)) \leq \nu(A_n \setminus A) \to 0 \quad \text{as } n \to \infty,$$

where the second inequality holds by Lemma 7.5 Thus, $\tau \circ A$ is $\bar{\mu}$-measurable. □

Next, we generalize Definition 10.15 to the context of weakly fibered laminations.

**Definition 10.31.** A set $U_\Sigma \subset \Sigma$ is said to be a flow tube if its image in $\tilde{X}$ $\tilde{U} := \iota(U_\Sigma) \subset \tilde{X}$ is a good flow tube in $(\tilde{X}, \tilde{\mathcal{L}})$ and if $U_\Sigma = \iota^{-1}(\tilde{U})$.

So the image $U := \pi(\tilde{U})$ is also a flow tube in $(X, \mathcal{L})$ and the restriction of the projection $\pi|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism which map plaques onto plaques.

Let $T$ be a transversal of the flow tube $U \subset X$. For each $x \in T$, let $\tilde{x} := (\pi|_{\tilde{U}})^{-1}(x) \in \tilde{U}$. For such a point $\tilde{x}$ and for each $y \in \tau^{-1}(\tilde{x}) \subset U_\Sigma$, the set $U_{\Sigma,y} := \Sigma_y \cap \tau^{-1}(U_x)$ is said to be the plaque passing through $y$ of $U_\Sigma$, where $U_x$ is the plaque of $U$ passing through $x$ given by Definition 10.15.

The sample-path space of a flow tube $U_\Sigma$ up to time $N \geq 0$ is, by definition, the subspace of $\Omega(\Sigma)$ consisting of all $\omega \in \Omega(\Sigma)$ such that $\omega[0, N]$ is fully contained in a single plaque $U_{\Sigma,y}$ for some $y \in \tau^{-1}(x)$ and $x \in T$. This space is denoted by $\Omega(N, U_\Sigma)$. 135
Next, we generalize the notion of directed cylinder sets and images in Definition 10.23 to the context of weakly fibered laminations.

**Definition 10.32.** Let $U_{\Sigma}$ be a flow tube of a $\Sigma$ and $N > 0$ a given time.

A directed cylinder set $A$ with respect to $(N, \tilde{U}_{\Sigma})$ is the intersection of a cylinder set $C(\{t_j, A_j\} : m)$ in $\Omega(\Sigma)$ and the sample-path space $\Omega(N, U_{\Sigma})$ satisfying the time requirement $N \geq t_m = \max t_j$.

A set $B \subset \Omega := \Omega(X, \mathcal{L})$ is said to be a directed cylinder image (with respect to $(N, \tilde{U}_{\Sigma})$) if $B = \tau \circ A$ for some directed cylinder set $A \subset \Omega(\Sigma)$ (with respect to $(N, U_{\Sigma})$).

The following result is needed.

**Lemma 10.33.** 1) For every set $E \subset X$ with $\mu(E) = 0$ and $t \geq 0$, $\bar{\mu}(C(\{t, E\})) = 0$.
2) Every cylinder set $A := C(\{t_j, A_j\} : m)$ with $A_i$ $\mu$-measurable for each $j$ is $\bar{\mu}$-measurable.

**Proof.** Since $\mu$ is harmonic, it follows from Theorem 2.13 that $\bar{\mu}(C(\{t, E\})) = \widetilde{\mu}(C(\{0, E\}))$. On the other hand, by formula (2.7), $\bar{\mu}(C(\{0, E\})) = \mu(E) = 0$. This completes Part 1.

Writing each $A_j$ as a disjoint union $A_j = B_j \cup E_j$ with $B_j$ a Borel set and $\mu(E_j) = 0$, we get a Borel cylinder set $B := C(\{t_j, B_j\} : m) \subset A$. Moreover, $A \setminus B \subset \cup_{j=1}^{m} C(\{t_j, E_j\} : 1)$. Applying Part 1) to the right hand side yields that $\bar{\mu}(A \setminus B) = 0$. Hence, $A$ is $\mu$-measurable. $\square$

Now we arrive at

**End of the proof of Proposition 7.14.** For the sake of simplicity in what follow we write $d$ (resp. $\Sigma$) instead of $d_m$ (resp. $\Sigma_k$) which has appeared in the proof of Step 2 of Theorem 7.2 in Subsection 7.3. Let $\Sigma := \text{Satur}(\Sigma)$ in $(\tilde{X}_{k, \tilde{A}}, \tilde{\mathcal{L}}_{k, \tilde{A}})$. Recall that $\tilde{X}_{k, \tilde{A}} = \tilde{X} \times \text{Gr}_k(\mathbb{R}^d)$. By Lemma 10.30 we need to show that for every cylinder set $A \subset \Omega(\Sigma)$, the cylinder image $\tau \circ A \subset \Omega(X, \mathcal{L})$ is $\bar{\mu}$-measurable. Fix such a cylinder set $A := C(\{t_j, A_j \cap \Sigma\} : q)$, where each $A_i$ is a Borel set of $\tilde{X}_{k, \tilde{A}} = \tilde{X} \times \text{Gr}_k(\mathbb{R}^d)$. We have the following

**Lemma 10.34.** Let $U_{\Sigma}$ be a flow tube of $\Sigma$, let $T$ be a transversal of the flow tube $U := \tau(\Sigma)$ in $(X, \mathcal{L})$, and let $V_{\Sigma} \subset U_{\Sigma}$ be a plaque-saturated Borel set and $M \in \mathbb{N}$ such that
(i) $T \cup V \subset U$ is a Borel set, where $V := \tau(V_{\Sigma})$;
(ii) the cardinal of $\tau^{-1}(x) \cap V_{\Sigma}$ does not exceed $M$ for each $x \in V$.

Then $B := \tau \circ A_V$ is $\bar{\mu}$-measurable, where

$$A_V := \Omega(N, U_{\Sigma}) \cap C(\{t_j, A_j \cap V_{\Sigma}\} : q) \quad \text{and} \quad N \geq t_q.$$  \hspace{1cm} (10.6)
Taking Lemma 10.34 for granted, we resume the proof of Proposition 7.14. Combining Lemma 7.10 and the formula for $\Sigma$ given in (7.10), we get that $\Sigma = \bigcup_{l=1}^{\infty} \Sigma^l$, where $(\Sigma^l)_{l=1}^{\infty}$ is an increasing sequence of Borel subsets of $X$.

$$\Sigma^l = \{(\tilde{x}, U) \in (\tilde{X} \setminus \pi^{-1}(M_{k+1})) \times \text{Gr}_k(\mathbb{R}^d) : M_k(\pi(\tilde{x}), U) > 1/l\}.$$  

By Lemma 7.12, we get that

$$\# \{ U : (\tilde{x}, U) \in \Sigma^l \} < l, \quad \tilde{x} \in \tilde{X} \setminus \pi^{-1}(M_{k+1}). \quad (10.7)$$

Now let $\{ (\tilde{U}^l_i, \tilde{U}'_i) \}_{i \in \mathbb{N}}$ be a countable family of pairs of conjugate flow tubes in $\tilde{X}$ given by Theorem 10.16. Using the countable family of flow tubes $(\tilde{U}^l_i)_{i \in \mathbb{N}}$, we may apply Proposition 10.26 to each $\Sigma^l$ the construction of a sequence of plaque-saturated sets $\Sigma^l_{ip}$. Consequently, we have that

$$\text{Satur}(\Sigma^l) \not\supset \text{Satur}(\Sigma) = \Sigma \quad \text{and} \quad \Sigma^l_{ip} \not\supset \tilde{U}'_i \cap \text{Satur}(\Sigma^l)$$

as $l$ (resp. $p$) tends to $\infty$. Therefore, we infer that

$$\Omega(\Sigma) = \bigcup_{i \in \mathbb{N}} \Omega(N, \cup_i \Sigma) \cap C(\{ t_j, \Sigma^l_{ip} \} : q),$$

where $\cup_i = \iota^{-1}(\tilde{U}'_i)$. So

$$A = \bigcup_{(i,l,p) \in \mathbb{N}^3} \Omega(N, \cup_i \Sigma) \cap C(\{ t_j, A_j \cap \Sigma^l_{ip} \} : q).$$

Hence, to prove that $\tau \circ A$ is $\bar{\mu}$-measurable it is sufficient to show that each set

$$\tau \circ \left( \Omega(N, \cup_i \Sigma) \cap C(\{ t_j, A_j \cap \Sigma^l_{ip} \} : q) \right)$$

is $\bar{\mu}$-measurable. On the other hand, combining (10.7) with the inductive construction of $\Sigma^l_{ip}$, we see that for $l, i, p \geq 1$,

$$\# \{ U \in \text{Gr}_k(\mathbb{R}^d) : (\tilde{x}, U) \in \Sigma^l_{ip} \} < \infty, \quad \tilde{x} \in \tilde{X} \setminus \pi^{-1}(M_{k+1}).$$

Consequently, applying Lemma 10.34 to $V_\Sigma := \Sigma^l_{ip}$ yields that each set of form (10.8) is $\bar{\mu}$-measurable. This finishes the proof Proposition 7.14 modulo Lemma 10.34.

To complete Proposition 7.14 it remains to establish the

**Proof of Lemma 10.34.** Let $\lambda$ be the measure defined on $T$ described by Proposition 2.6. Since $\pi^{-1}(T_V) \times \text{Gr}_k(\mathbb{R}^d) \cap V_\Sigma$ is a Borel subset of $\tilde{X} \times \text{Gr}_k(\mathbb{R}^d)$ it follows that the map $\# : T_V \to \mathbb{N}$, which maps each $x \in T_V$ to the cardinal of the intersection $\tau^{-1}(x) \cap V_\Sigma$, is $\lambda$-measurable. Consequently, there are $M$ mutually disjoint Borel subsets $T_1, \ldots, T_M$ of $T$ and a set $E \subset T$ such that $E = \bigcup_{j=1}^{M} T_j = T_V$.  

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and \( \#(\tau^{-1}(x) \cap V_{\Sigma}) = j \) for each point \( x \in T_j \). Let \( S_j := \text{Satur}_U(T_j) \) and \( \mathcal{E} := \text{Satur}_U(E) \). By Proposition 10.29, \( \mu(\mathcal{E}) = 0 \). Let \( V_j := \tau^{-1}(S_j) \cap V_{\Sigma} \) and \( \mathcal{E} := \tau^{-1}(\mathcal{E}) \cap V_{\Sigma} \). These sets are all plaque-saturated in the flow tube \( U_{\Sigma} \). Clearly, \( V_{\Sigma} = \mathcal{E} \cup_{j=1}^M V_j \). Using this and formula (10.6) and applying Lemma 10.33, we are reduced to the case where \( V = V_M \) and \( V_1 = \cdots = V_{M-1} = \mathcal{E} = \emptyset \). In other words, we only need to treat the following situation: for every \( x \in T_V \), it holds that \( \#(\tau^{-1}(x) \cap V_{\Sigma}) = M \).

Using the last identity and applying Theorem 10.2 several times to the multifunction \( \Gamma := \{(x, y) : T_V \times V_{\Sigma} : \tau(y) = x\} \) from \( T_V \to V_{\Sigma} \), we may find \( N \) \( \lambda \)-measurable selections \( s_1, \ldots, s_M : T_V \to V_{\Sigma} \) such that

\[
\{s_1(x), \ldots, s_M(x)\} = \tau^{-1}(x) \cap V_{\Sigma}, \quad \text{for } \lambda - \text{almost every } x \in T_V.
\]

Since \( V_{\Sigma} \) is plaque-saturated in \( U_{\Sigma} \), we extend all functions \( s_1, \ldots, s_M : V \to V_{\Sigma} \) while preserving the above equality as follows:

1. \( \{s_1(x), \ldots, s_M(x)\} \subset \tau^{-1}(x) \cap V_{\Sigma} \) for each \( x \in V \);
2. if \( x \) and \( x' \) are in the same plaque in \( U \), then so are \( \tau_j(x) \) and \( \tau_j(x') \) for each \( j \).

We deduce from these extensions that \( \{s_1(x), \ldots, s_M(x)\} = \tau^{-1}(x) \cap V_{\Sigma} \) for each \( x \in V \). For each \( 1 \leq j \leq M \) let \( V_{\Sigma}^j := \{(x, s_j(x)) : x \in V\} \). So we have that \( V_{\Sigma} = \cup_{j=1}^M V_{\Sigma}^j \) and that each \( V_{\Sigma}^j \) is plaque-saturated in \( U_{\Sigma} \). This, combined with (10.6), gives that \( A_V = \cup_{j=1}^M A_V^j \), where

\[
A_V^j := \Omega(N, U_{\Sigma}) \cap C(\{t_i, A_i \cap V_{\Sigma}^j \} : p).
\]

Therefore, we infer that \( B = \tau \circ A_V = \cup_{j=1}^M \tau \circ A_V^j = \cup_{j=1}^M B^j \), where

\[
B^j := \Omega(N, U) \cap C(\{t_i, A_{ij} \} : p), \quad 1 \leq j \leq M,
\]

and each \( A_{ij} := \{x \in V : s_j(x) \in A_i\} \) is a Borel subset of \( X \). By Part 3 of Lemma 10.24, each \( B^j \) belongs to \( \mathcal{A} \). Hence, \( B = \tau \circ A_V \) belongs also to \( \mathcal{A} \) as desired.

### 10.9 Relation between a lamination and its covering lamination

Recall that \( \pi : (\tilde{X}, \tilde{\mathcal{L}}) \to (X, \mathcal{L}) \) is the canonical projection. Following Definition 2.10, let \( \Omega := \Omega(X, \mathcal{L}) \), \( \tilde{\Omega} := \Omega(\tilde{X}, \tilde{\mathcal{L}}) \) be two sample-path spaces and let \( \mathcal{A} = \mathcal{A}(\Omega) \), \( \mathcal{A}(\tilde{\Omega}) \) be two \( \sigma \)-algebras on them respectively. Note that \( \mathcal{A}(\Omega) = \mathcal{A}(\tilde{\Omega}) \). For every \( \omega \in \Omega \) let \( \pi^{-1}(\omega) \) be the set

\[
\{\tilde{\omega} \in \tilde{\Omega} : \pi \circ \tilde{\omega} = \omega\}.
\]

A function \( \tilde{f} : \tilde{X} \to \mathbb{R} \) (resp. \( \tilde{F} : \tilde{\Omega} \to \mathbb{R} \)) can be written as \( \tilde{f} = f \circ \pi \) for some function \( f : X \to \mathbb{R} \) (resp. can be written as \( \tilde{F} = F \circ \pi \) for some function
This subsection is devoted to the proof of Theorem 2.13. The following result is needed.

**Lemma 10.35.** (i) Let \( A \in \mathcal{A} \), let \( x \) be a point in \( X \), and let \( \tilde{x}_1, \tilde{x}_2 \) be two points of \( \pi^{-1}(x) \). Then
\[
W_{\tilde{x}_1}(\pi^{-1}A) = W_{\tilde{x}_2}(\pi^{-1}A) = W_x(A).
\]

(ii) Let \( \tilde{x} \in \tilde{X}, \tilde{A} \in \mathcal{A}_{\tilde{X}} \) and \( x := \pi(\tilde{x}) \). Then \( W_{\tilde{x}}(\tilde{A}) = W_x(\pi \circ \tilde{A}) \).

(iii) For a bounded measurable function \( F : \Omega (X, \mathcal{L}) \rightarrow \mathbb{R} \) let \( \tilde{F} : \Omega (\tilde{X}, \tilde{\mathcal{L}}) \rightarrow \mathbb{R} \) be the function defined by \( \tilde{F} = F \circ \pi \). To each \( x \in X \) we associate an arbitrary (but fixed) element \( \tilde{x} \in \pi^{-1}(x) \). Then for every Borel probability measure \( \mu \) on \( X \),
\[
\int_\Omega F(\omega) d\mu(\omega) = \int_X \left( \int_{\tilde{\Omega}} \tilde{F}(\tilde{\omega}) dW_{\tilde{x}}(\tilde{\omega}) \right) d\mu(x).
\]

**Proof.** Since \( A \in \mathcal{A} \), we get \( A \cap \Omega (L_x) \in \mathcal{A} (L_x) \). Consequently, combining assertion (i) of Proposition 4.11 and formula (2.6) yields that
\[
W_{\tilde{x}_1}(\pi^{-1}A) = W_{\tilde{x}_2}(\pi^{-1}(A \cap \Omega (L_x))) = W_x(A \cap \Omega (L_x)) = W_x(A).
\]
This proves assertion (i).

To prove assertion (ii) observe that
\[
W_x(\pi \circ \tilde{A}) = W_{\tilde{x}}(\pi^{-1}(\pi \circ \tilde{A})) = W_{\tilde{x}}(\pi^{-1}_{\tilde{x}}(\pi \circ \tilde{A})�)
\]
where the first equality holds by assertion (i). Since \( \tilde{A} \in \mathcal{A}_{\tilde{X}} \), we have that \( \pi^{-1}_{\tilde{x}}(\pi \circ \tilde{A}) = \tilde{A} \). Hence, \( W_x(\pi \circ \tilde{A}) = W_{\tilde{x}}(\tilde{A}) \), as asserted.

To prove assertion (iii), we first consider the case where \( F := 1_A \) for some \( A \in \mathcal{A} \). In this case, the assertion holds by combining assertion (i) above and formula (2.7). The general case deduces from the above case using Proposition 2.12 (ii). \( \square \)

### 10.10 Invariance of Wiener measures with harmonic initial distribution

This subsection is devoted to the proof of Theorem 2.13. The following result is needed.

**Lemma 10.36.** Let \( \mu \) be a Borel probability measure on \((\Omega, \mathcal{A})\). Then, for every \( t \in \mathbb{R}^+ \) and for every bounded measurable function \( F : \Omega \rightarrow \mathbb{R} \),
\[
\int_{\omega \in \Omega} F(T_t \omega) d\tilde{\mu}(\omega) = \int_{x \in X} \left( \int_{\Omega} F(\omega) dW_x(\omega) \right) d(D_t \mu)(x),
\]

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where $D_t \nu$ is the probability measure (unique in the sense of $\nu$-almost everywhere) that satisfies
\[
\int_X D_t f(x) d\mu(x) = \int_X f(x) d(D_t \mu)(x)
\]
for every bounded measurable function $f : X \to \mathbb{R}$.

Taking Lemma 10.36 for granted, we arrive at the End of the proof of Theorem 2.13. By Lemma 10.36, it is sufficient to show that $D_t \mu = \mu$. But this identity follows from the assumption that $\mu$ is weakly harmonic. \hfill \square

Proof of Lemma 10.36. Fix a time $t \geq 0$. We will show that for each element $A \in \mathcal{A}$,
\[
\int_{\omega \in \Omega} 1_A(T^t \omega) d\bar{\nu}(\omega) \leq \int_{x \in X} \left( \int_{\Omega} 1_A(\omega) dW_x(\omega) \right) d(D_t \mu)(x), \quad (10.9)
\]
where $1_A$ is, as usual, the characteristic function of $A$. We only need to prove the lemma for every function $F$ which is the characteristic function of an element $A \in \mathcal{A}$. Taking (10.9) for granted, and applying (10.9) to both $A$ and $\Omega \setminus A$ for each $A \in \mathcal{A}$, and summing up both sides of the obtained two inequalities, and noting that $1_A(\omega) + 1_{\Omega \setminus A}(\omega) = 1$ for all $\omega \in \Omega$, we deduce that (10.9) is, in fact, an equality. Using this and approximating each bounded measurable function $F : \Omega \to \mathbb{R}$ by simple functions, i.e., by functions which are a finite linear combination of characteristic functions, the lemma follows.

So it remains to establish (10.9). To this end fix an arbitrary $\epsilon > 0$ and set $A' := \pi^{-1}(A)$. By Part 2) of Lemma 10.25, $A' \in \tilde{\mathcal{A}}$. By Part 1) of Proposition 2.8, let $\mathfrak{C}$ be the algebra on $\tilde{\Omega}$ consisting of all sets which are a finite union of cylinder sets. Consider the $\sigma$-finite measure $\nu := \pi^*(D_t \mu)$ on $\tilde{X}$. Let $\tilde{\nu}$ be the Wiener measure with initial distribution $\nu$ defined on $\mathcal{A}(\tilde{\Omega})$ by formula (2.7). $\tilde{\nu}$ is countably additive on $\mathfrak{C}$. Consequently, applying Proposition 10.8 to the measure space $(\tilde{\Omega}, \mathfrak{A}(\tilde{\Omega}), \tilde{\nu})$ and the algebra $\mathfrak{C}$ yields a set $\tilde{A}$ which is a countable union of cylinder sets such that
\[
A' \subset \tilde{A} \quad \text{and} \quad \tilde{\nu}(\tilde{A} \setminus A') < \epsilon. \quad (10.10)
\]
By Part 2) of Proposition 2.8 we may write
\[
\tilde{A} = \bigcup_{p \in \mathbb{N}} \tilde{A}^p := \bigcup_{p \in \mathbb{N}} C(\{t_i^p, \tilde{A}_i^p \setminus \tilde{A}_i^p \} : m_p),
\]
where the cylinder sets $\tilde{A}^p = C(\{t_i^p, \tilde{A}_i^p \} : m_p)$ are mutually disjoint. Consequently, we obtain a disjoint countable decomposition
\[
\tilde{B} := (T^t)^{-1} \tilde{A} = \bigcup_{p \in \mathbb{N}} \tilde{B}^p := \bigcup_{p \in \mathbb{N}} C(\{t_i^p + t, \tilde{A}_i^p \} : m_p).
\]

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By (2.4), we get, for every $\tilde{x} \in \tilde{X}$, that

$$W_{\tilde{x}}(\tilde{B}^p) = \left(\tilde{D}_{t_1}^{p_1}(\chi_{\tilde{A}_1^p} \tilde{D}_{t_2}^{p_2}(\chi_{\tilde{A}_2^p} \cdots \chi_{\tilde{A}_{mp-1}^p} \tilde{D}_{t_{mp-1}}^{p_{mp-1}}(\chi_{\tilde{A}_{mp}^p}) \cdots))\right)(\tilde{x}).$$

Consider the function $\tilde{H} : \tilde{X} \to [0,1]$ given by

$$\tilde{H} := \sum_{p \in \mathbb{N}} \tilde{D}_{t_1}^{p_1}(\chi_{\tilde{A}_1^p} \tilde{D}_{t_2}^{p_2}(\chi_{\tilde{A}_2^p} \cdots \chi_{\tilde{A}_{mp-1}^p} \tilde{D}_{t_{mp-1}}^{p_{mp-1}}(\chi_{\tilde{A}_{mp}^p}) \cdots)).$$

Observe that, for every $\tilde{x} \in \tilde{X}$,

$$\int_{\tilde{X}} 1_{\tilde{A}}(T^t\tilde{\omega})dW_{\tilde{x}}(\tilde{\omega}) = W_{\tilde{x}}(\tilde{B}) = \sum_{p \in \mathbb{N}} W_{\tilde{x}}(\tilde{B}^p) = (\tilde{D}_t\tilde{H})(\tilde{x}).$$

This, combined with $\pi^{-1}(A) = A' \subset \tilde{A}$, implies that, for every $x \in X$ and every $\tilde{x} \in \pi^{-1}(x)$,

$$\int_{\Omega_x} 1_{A}(T^t\omega)dW_x(\omega) = \int_{\tilde{\Omega}_x} 1_{A'}(T^t\tilde{\omega})dW_{\tilde{x}}(\tilde{\omega}) \leq \int_{\tilde{\Omega}_x} 1_{\tilde{A}}(T^t\tilde{\omega})dW_{\tilde{x}}(\tilde{\omega}) = (\tilde{D}_t\tilde{H})(\tilde{x}).$$

Fix a measurable map $\iota : X \to \tilde{X}$ such that $\iota(x) \in \pi^{-1}(x)$. Using the last estimate and applying Lemma 10.35 (ii), we have that

$$\int_{\Omega} 1_A(T^t\omega)d\tilde{\mu}(\omega) = \int_{x \in X} \left( \int_{\Omega_x} 1_A F(T^t\omega)dW_x(\omega) \right)d\mu(x) \leq \int_X (\tilde{D}_t\tilde{H})(\iota(x))d\mu(x).$$

Therefore, the proof of (10.9) will be complete if one can show that

$$\int_X (\tilde{D}_t\tilde{H})(\iota(x))d\mu(x) \leq \epsilon + \int_{x \in X} \left( \int_{\Omega} 1_A(\omega)dW_x(\omega) \right)d(D_t\mu)(x).$$

To this end observe that, for every $\tilde{x} \in \tilde{X}$,

$$\tilde{H}(\tilde{x}) = W_{\tilde{x}}(\tilde{A}) = W_{\tilde{x}}(A') + W_{\tilde{x}}(\tilde{A} \setminus A')$$

Acting $\tilde{D}_t$ on the last equality we obtain that, for every $x \in X$,

$$\tilde{D}_t\tilde{H}(\iota(x)) = (\tilde{D}_t\tilde{W}_x(A'))(\iota(x)) + (\tilde{D}_t\tilde{W}_x(\tilde{A} \setminus A'))(\iota(x)).$$

By Lemma 10.35 $W_{\tilde{x}}(A') = W_x(A)$ for every $\tilde{x} \in \pi^{-1}(x)$. So $W_x(A')$ is constant on fibers and $W_x(A') = W_x(A) \circ \pi$. Hence, by an application of Proposition 2.3 we get that $\tilde{D}_t\tilde{W}_x(A') = (D_tW_x(A)) \circ \pi$. Putting all these together, we get that

$$\int_X (\tilde{D}_t\tilde{H})(\iota(x))d\mu(x) = \int_X (D_tW_x(A))(x)d\mu(x) + \int_X (\tilde{D}_t\tilde{W}_x(\tilde{A} \setminus A'))(\iota(x))d\mu(x).$$
Note that \( \int_X (D_t W_\ast(\tilde{A} \setminus A'))(\nu(x))d\mu(x) = \int_X W_\ast(A)d(D_t \mu) \). Consequently, (10.11) is reduced to showing that

\[ \int_X \tilde{D}_t W_\ast(\tilde{A} \setminus A')(\nu(x))d\mu(x) < \epsilon. \] (10.12)

To do this we use \( \nu = D_t \mu \) and apply Lemma 10.35 (ii) in order to obtain

\[ \nu(\tilde{A} \setminus A') = \int_X \left( \sum_{\tilde{x} \in \pi^{-1}(x)} W_{\tilde{x}}(\tilde{A} \setminus A') \right)d(D_t \mu)(x) \]

where \( K : X \to \mathbb{R}^+ \) given by

\[ K(x) := \sum_{\tilde{x} \in \pi^{-1}(x)} W_{\tilde{x}}(\tilde{A} \setminus A'), \quad x \in X. \]

Therefore, this, coupled with (10.10), gives that

\[ \int_X (D_t K)(x)d\mu(x) \leq \nu(\tilde{A} \setminus A') < \epsilon. \]

Using formula (2.22) we get that for every \( x \in X \) and \( \tilde{x} \in \pi^{-1}(x) \),

\[ (D_t K)(x) = \int_{y \in L_x} \left( \sum_{\tilde{y} \in \pi^{-1}(y)} \tilde{p}(\tilde{x}, \tilde{y}, t) \right) \left( \sum_{\tilde{y}' \in \pi^{-1}(y)} W_{\tilde{y}'}(\tilde{A} \setminus A') \right)d\text{Vol}_{L_x}(y). \]

The right hand side in the last line is greater than

\[ \int_{y \in L_x} \left( \sum_{\tilde{y} \in \pi^{-1}(y)} \tilde{p}(\tilde{x}, \tilde{y}, t) \right) W_{\tilde{y}}(\tilde{A} \setminus A')d\text{Vol}_{L_x}(y) = (\tilde{D}_t W_\ast(\tilde{A} \setminus A'))(\tilde{x}). \]

Choosing \( \tilde{x} := \nu(x) \) and integrating the last inequality over \( X \), we get that

\[ \int_X (\tilde{D}_t W_\ast(\tilde{A} \setminus A'))(\nu(x))d\mu(x) \leq \int_X (D_t K)(x)d\mu(x) \leq \nu(\tilde{A} \setminus A') < \epsilon. \]

This proves (10.12), and thereby completes the lemma. \( \square \)

### 10.11 Ergodicity of Wiener measures with ergodic harmonic initial distribution

This subsection is devoted to the proof of Theorem 4.6. We may assume without loss of generality that \( t = 1 \). The proof is divided into several steps.

**Step 1:** Ergodicity \( T \) on \( (\Omega, \mathcal{A}, \tilde{\mu}) \) implies the ergodicity of \( \mu \).
Let $A$ be a leafwise saturated Borel subset of $X$. Let $\Omega(A)$ be the set consisting of all $\omega \in \Omega(X, \mathcal{L})$ which are contained in $A$. Clearly, $\Omega(A)$ is $T$-invariant. By the ergodicity of $T$ acting on $(\Omega, \mathcal{A}, \bar{\mu})$, $\mu(\Omega(A))$ is either 1 or 0. Hence, we deduce from formula (2.7) that $\mu(A)$ is either 1 or 0.

In the rest of the proof we will suppose that $\mu$ is ergodic.

**Step 2:** Ergodicity of $\mu$ implies the ergodicity of $T$ on $(\Omega, \mathcal{A}, \bar{\mu})$.

We need to prove that $\bar{\mu}(A)$ is equal to either 0 or 1 for every $T$-totally invariant set $A \in \mathcal{A}$. Fix such a set $A$ and let $\tilde{A} := \pi^{-1}(A) \in \mathcal{A}(\tilde{\Omega})$. Consider the trivial fibered lamination $\Sigma$ over $(X, \mathcal{L}, g, \mu)$, that is, $\Sigma := \tilde{X}$ and $i : \Sigma \to \tilde{X}$ is the identity. Clearly, $\tilde{A} \in \mathcal{A}(\Sigma)$ is also $T$-totally invariant. By Theorem 7.7 $\lVert \mathbf{1}_{\tilde{A}} \rVert_\ast$ is equal to either 0 or 1. On the other hand, $\mathbf{1}_{\tilde{A}} = \mathbf{1}_A \circ \pi$ because $A := \pi^{-1}(A)$. Consequently,

$$\lVert \mathbf{1}_{\tilde{A}} \rVert_\ast = \int_{x \in \tilde{X}} W_x(A) d\mu(x) = \bar{\mu}(A).$$

Hence, $\bar{\mu}(A)$ is equal to either 0 or 1 as desired. $\square$

**Corollary 10.37.** Let $\mathcal{A}$ be a cocycle on a lamination $(X, \mathcal{L})$ and $\mu$ a harmonic probability measure on $X$ which is ergodic. Let $\nu$ be an element in $\text{Har}_\mu(X_{1,\mathcal{A}})$ which is also ergodic. Let $\hat{\nu}$ be the Wiener measure with initial distribution $\nu$ given by (2.7). Let $\hat{\nu}$ be the natural extension of $\nu$ on $\hat{\Omega}_{1,\mathcal{A}}$.

1) Then $\hat{\nu}$ is $T$-ergodic on $\Omega_{1,\mathcal{A}}$ and $\hat{\nu}$ is $T$-ergodic on $\hat{\Omega}_{1,\mathcal{A}}$.

2) Let $f : \Omega_{1,\mathcal{A}} \to \mathbb{R}$ be a $\bar{\mu}$-integrable function. Then there is a constant $\alpha$ such that for $\nu$-almost every $(x, u) \in X_{1,\mathcal{A}}$, and for $W_x$-almost every $\omega$, $\frac{1}{n} f(T^n(\omega, u)) \to \alpha$. Moreover, $\int_{\Omega_{1,\mathcal{A}}} f d\nu = \alpha$.

3) Let $\hat{\nu}$ be the natural extension of $\tilde{\nu}$ on $\hat{\Omega}_{1,\mathcal{A}}$. Let $f : \hat{\Omega}_{1,\mathcal{A}} \to \mathbb{R}$ be a $\tilde{\mu}$-integrable function. Then there is a constant $\alpha$ such that for $\tilde{\nu}$-almost every $(\tilde{\omega}, u) \in \hat{\Omega}_{1,\mathcal{A}}$, $\frac{1}{n} f(T^{-n}(\tilde{\omega}, u)) \to \alpha$. Moreover, $\int_{\hat{\Omega}_{1,\mathcal{A}}} f d\tilde{\nu} = \alpha$.

**Proof.** Part 2) and 3) follows from combining Part 1) and the Birkhoff ergodic theorem. So it suffices to prove Part 1). By [10] p. 241, if $\hat{\nu}$ is $T$-ergodic on $\Omega_{1,\mathcal{A}}$, then $\hat{\nu}$ is $T$-ergodic on $\hat{\Omega}_{1,\mathcal{A}}$. So it remains to show that $\hat{\nu}$ is $T$-ergodic on $\Omega_{1,\mathcal{A}}$.

Arguing as in Step 2 of the proof of Theorem 4.6 and replacing the lamination $(X, \mathcal{L}, g)$ (resp. the measure $\mu$) with $(X_{1,\mathcal{A}}, \mathcal{L}_{1,\mathcal{A}}, \text{pr}_1^* g)$ (resp. with $\nu \in \text{Har}_\mu(X_{1,\mathcal{A}})$) the desired conclusion follows. It is worthy noting from Definition 7.8 above that although $(X_{1,\mathcal{A}}, \mathcal{L}_{1,\mathcal{A}}, \text{pr}_1^* g)$ is only a measurable lamination, it possesses almost the same atlas as the continuous lamination $(X, \mathcal{L})$. Therefore, many properties of the latter lamination still hold in the context of the former one. $\square$
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