IWASAWA THEORY FOR RANKIN-SELBERG PRODUCT AT AN EISENSTEIN PRIME

SOMNATH JHA, SUDHANSHU SHEKHAR, RAVITHEJA VANGALA

Abstract. Let $p$ be an odd prime, $f$ be a $p$-ordinary newform of weight $k$ and $h$ be a normalized cuspidal $p$-ordinary Hecke eigenform of weight $l < k$. In this article, we study the $p$-adic $L$-function and $p^\infty$-Selmer group of the Rankin-Selberg product of $f$ and $h$ under the assumption that $p$ is an Eisenstein prime for $h$ i.e. the residual Galois representation of $h$ at $p$ is reducible. We show that the $p$-adic $L$-function and the characteristic ideal of the $p^\infty$-Selmer group of the Rankin-Selberg product of $f, h$ generate the same ideal modulo $p$ in the Iwasawa algebra i.e. the Rankin-Selberg Iwasawa main conjecture for $f \otimes h$ holds mod $p$. As an application to our results, we explicitly describe a few examples where the above congruence holds.

INTRODUCTION

The study of Rankin-Selberg product of two modular forms is of considerable interest in number theory from arithmetic, analytic and automorphic point of view. In this article, we study Iwasawa theory for Rankin-Selberg product of two $p$-ordinary modular forms $f$ and $h$ where the weight of $f$ is strictly greater than the weight of $h$ and $p$ is an Eisenstein prime for $h$.

Iwasawa main conjecture for elliptic curves and modular forms (at an ordinary prime) has been established (in a large number of cases) by the fundamental works of Kato [Ka] and Skinner-Urban [SU]. As a next step, it is natural to study the Iwasawa theory of Rankin-Selberg $L$-function. Let $f$ and $h$ be two $p$-ordinary modular forms. In this setting, important works have been done by Lei-Loeffler-Zerbes [LLZ], Kings-Loeffler-Zerbes [KLZ] and X. Wan [Wa]. In [LLZ, Theorem 7.5.4] and [KLZ, Corollary D], one side divisibility in the Iwasawa main conjecture was shown (i.e. the characteristic ideal of the $p^\infty$-Selmer group of $f \otimes h$ divides the $p$-adic $L$-function of $f \otimes h$) provided the residual Galois representation of $f \otimes h$ at $p$ is irreducible and assuming additional hypotheses. In [Wa, Theorem 1.2], X. Wan proves the other divisibility in the main conjecture under suitable assumptions which include $f$ is a CM form and the residual Galois representation of $h$ at $p$, say $\bar{\rho}_h$ is irreducible. Thus following the works of [LLZ], [KLZ] and [Wa], the Iwasawa main conjecture is still open in many cases; for example when (i) the residual Galois representation of $f \otimes h$ at $p$ is reducible or (ii) $f$ is a non-CM eigenform.

In our setting, $p$ is an Eisenstein prime for $h$ i.e. $\bar{\rho}_h$ is reducible, so our assumptions are complimentary to those mentioned in [LLZ], [KLZ] and [Wa]. In fact, our result in this article can be interpreted in terms of congruence of modular forms. Indeed, the congruences of modular forms (cf. [H5]) is an important topic of study in the arithmetic of elliptic curves, modular forms and Iwasawa theory for congruent modular forms has been extensively studied following the works of [GV], [Va], [EPW] etc. In this article, we combine ideas from these two topics; Iwasawa theory of the Rankin-Selberg convolution and the congruence of modular forms to arrive at our result.

We now briefly recall the terminology needed to introduce our main theorem. The precise definitions are given later at appropriate places. Let $p$ be an odd prime. Let $\mathbb{Q}_{cyc}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and $\Gamma := \text{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q})$. Let $f \in S_k(\Gamma_0(N), \eta)$ be a $p$-ordinary primitive form with $p \nmid N$ and $h \in S_l(\Gamma_0(I), \psi)$ be a normalised $p$-ordinary Hecke eigenform such that $2 \leq l < k$. Write $I = I_0 \mathfrak{p}^a$ with $p \nmid I_0$. Fix a number field $K$ containing the Fourier coefficients of $f, h$ and the values of $\eta, \psi$. Let $K_\mathfrak{p}$ denote the completion of $K$ at a prime $\mathfrak{p}$ lying above $p$ induced by the embedding $i_p: \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_\mathfrak{p}$ and $\pi$ be a uniformizer of the ring of integers $\mathcal{O}$ of $K_\mathfrak{p}$. For $g \in \{f, h\}$, let $\rho_\mathfrak{p} : G_\mathbb{Q} \to \text{Aut}(V_\mathfrak{p})$ be the $p$-adic Galois representation attached to $g$ and $T_\mathfrak{p}$ be an $\mathcal{O}$-lattice inside $V_\mathfrak{p}$. Set $A_f := V_f/T_f$ and $A_f^\vee$ be the

AMS subject classification: 11F33, 11F67, 11R23.

Keywords: Iwasawa main conjecture, $p$-adic $L$-functions, Rankin-Selberg product, Selmer group.
unramified quotient of $A_f$ defined using the $p$-ordinarity of $f$. The crucial assumption in this article is:

We have an exact sequence of $G_m$ modules, $0 \to \xi_1 \to T_h/\pi \to \xi_2 \to 0$. \hfill ($\hat{\rho}_p$-red)

As $h$ is $p$-ordinary, we may assume $\xi_2$ is unramified. We choose appropriate (Teichmüller) lifts $\xi_1$ and $\xi_2$ of $\xi_1$ and $\xi_2$ respectively as described after Lemma 2.1. Let $\chi_{\text{cyc}}$ and $\omega_p$ be the $p$-adic cyclotomic character and Teichmüller character respectively. Let $A_f$ be the discrete $O$-lattice $((V_f \otimes V_0)/(T_f \otimes T_0)) \otimes \chi_{\text{cyc}}$. Using the $p$-ordinarity and following [G1], we define the $p^\infty$-Selmer groups $S_{Gr}(A_f/Q_{cyc}), S_{Gr}(A_j/Q_{cyc})$ attached to $A_f$ and $A_j$ respectively. In this setting, Hida [H2] has constructed the $p$-adic $L$-function $\mu_{p,f \times h,j}$ attached to the Rankin-Selberg convolution $f \otimes h$ (see Theorem 1.2 for the definition). Let $\mu_{p,f,\xi,j}$ be the $p$-adic $L$-function attached to $f \otimes \xi_i$ as constructed in [MTT] with an appropriate choice of periods, as explained in Theorems 3.8 and 3.10.

Let $\Sigma$ be a finite set of primes of $Q$ containing primes dividing $pN\infty$ and $Q_{\Sigma}$ be the maximal algebraic extension of $Q$ unramified outside $\Sigma$. Define $M_0$ to be the prime to $p$-part of the conductor of $\rho_h$ and $\Sigma : = \{ \ell \text{ prime} : \ell \mid I_0/M_0 \text{ and } \ell^2 \mid M_0 \}$. Put $m : = \prod_{\ell \in M} \ell$ and $\Sigma_\infty : = \{ w \text{ is a prime in } Q_{cyc} : w \mid m \}$.

Let $\mu_{p,f \times h,j}^{\Sigma_0}, \mu_{p,f,\xi,j}^{\Sigma_0}$ and $\mu_{p,f,\xi_1, j}^{\Sigma_0}$ be the $\Sigma_0$-imprimitive $p$-adic $L$-function as defined in (32) and (31) respectively. Let $S_{Gr}(A_f/Q_{cyc}), S_{Gr}(A_j/Q_{cyc})$ be $\Sigma_0$-imprimitive Selmer group attached to $A_f$ and $A_j$ respectively. Let $G_p \subset G_Q$ be the decomposition subgroup at $p$ and for a prime $w$ in $Q_{cyc}$ let $G_{Q_{cyc,w}}$ and $I_{cyc,w}$ be the decomposition and inertia subgroup at $w$. Let $T_m$ be the trivial character of modulus $m$ i.e. $T_m(n) = 1$ if $(m, n) = 1$ and $T_m(n) = 0$ if $(m, n) > 1$. We now state some hypotheses:

($\text{irr-f}$) The residual representation $\hat{\rho}_f$ is an irreducible $G_Q$-module.

($\text{p-dist}$) $f$ is $p$-distinguished i.e. the restriction of $\rho_f$ to $G_p$ satisfies $\hat{\rho}_f|_{G_p} \cong \left( \begin{array}{cc} \rho_p & * \\ 0 & \delta_p \end{array} \right)$ with $\epsilon_p \neq \delta_p$.

($\text{Sel-tors}$) $S_{Gr}(A_j/Q_{cyc})^\vee$ is a finitely generated torsion module over the Iwasawa algebra $O[[\Gamma]]$.

The following Theorems 0.1, 0.2 and Corollary 0.3 are our main results (also see Examples 7.1-7.3):

**Theorem 0.1.** [Theorem 3.19] Let $p$ be an odd prime. Let $f \in S_k(\Gamma_0(N), \eta)$ and $h \in S_l(\Gamma_0(I), \psi)$ be normalised $p$-ordinary newforms with $p \nmid N$ and $2 \leq l < k$. Let $j$ be an integer with $1 \leq j \leq k - 2$. Assume $h$ satisfies ($\hat{\rho}_h$-red) and $f$ satisfies (irr-$f$) and (p-dist). If $\ell = 3$, then we further assume that $\Gamma_0(N)/\{ \pm 1 \}$ has no non-trivial torsion elements. Then the following congruence of $\Sigma_0$-imprimitive $p$-adic $L$-functions holds in $O[[\Gamma]]$: \begin{equation} (\mu_{p,f \times h,j}^{\Sigma_0}, \mu_{p,f,\xi,j}^{\Sigma_0}, \mu_{p,f,\xi_1, j}^{\Sigma_0}) \equiv \mod \pi. \end{equation}

**Theorem 0.2.** [Theorem 5.9] Let $p$ be an odd prime. Let $f \in S_k(\Gamma_0(N), \eta)$ and $h \in S_l(\Gamma_0(I), \psi)$ be normalised $p$-ordinary newforms with $p \nmid N$ and $2 \leq l < k$. Let $j$ be an integer with $1 \leq j \leq k - 2$. Assume $h$ satisfies ($\hat{\rho}_h$-red) and $f$ satisfies (irr-$f$) and (p-dist). In addition, we make following hypotheses:

(i) $(N, M_0) = 1$ i.e. $N$ is co-prime to $M_0$.
(ii) $\psi_{I_{cyc,w}}$ has order co-prime to $p$ for every prime $w$ in $Q_{cyc}$ dividing $pI$.
(iii) The dual Selmer group $S_{Gr}(A_j/Q_{cyc})^\vee$ satisfies (Sel-tors) and moreover the assumption ($H^2_{\xi_1, \text{van}}$ holds i.e. $H^2(Q_{\Sigma}/Q_{cyc}, A_j(\xi_1 \omega_\infty^{-j})(\pi))$ vanishes.
(iv) Let $w$ be the prime in $Q_{cyc}$ dividing $p$. If $(p - 1) \mid j$ and $H^0(G_{Q_{cyc,w}}, A_j^{\xi}(\pi), (\xi_2)) \neq 0$, then we assume (ss-red$_p$) holds i.e. $\hat{\rho}_h|_{I_{cyc,w}} \cong \xi_1 \otimes \xi_2$.

Then the following congruence of the characteristic ideals of Selmer groups holds in $O[[\Gamma]]$:
\begin{equation} C_{O[[\Gamma]]}(S_{Gr}(A_j/Q_{cyc})^\vee) \equiv C_{O[[\Gamma]]}(S_{Gr}(A_j(\xi_1 \omega_\infty^{-j}))/Q_{cyc})^\vee) \equiv C_{O[[\Gamma]]}(S_{Gr}(A_j(\xi_1 \omega_\infty^{-j}))^\vee) \mod \pi. \end{equation}

**Corollary 0.3.** Let $j$ be an integer with $1 \leq j \leq k - 2$. We keep the assumptions of both Theorems 0.1, 0.2. Furthermore, assume that the Iwasawa main conjecture holds for $f \otimes \xi_1 \omega^{-j}$ and $f \otimes \xi_2 \omega^{-j}$ over $Q_{cyc}$, that is, \begin{equation} (\mu_{p,f,\xi,j}) = C_{O[[\Gamma]]}(S_{Gr}(A_j(\xi_1 \omega_\infty^{-j}))/Q_{cyc})^\vee) \end{equation}
holds for $i = 1, 2$. Then we have the following congruence of ideals in the Iwasawa algebra $O[[\Gamma]]$:
\begin{equation} (\mu_{p,f \times h,j}) \equiv C_{O[[\Gamma]]}(S_{Gr}(A_j/Q_{cyc})^\vee) \mod \pi. \end{equation}
In particular, $\mu_{p,f \times h,j}$ is a unit in $O[[\Gamma]]$ if and only if $C_{O[[\Gamma]]}(S_{Gr}(A_j/Q_{cyc})^\vee)$ is a unit in $O[[\Gamma]]$. 

}
Remark 0.4. The Iwasawa main conjecture for modular form (as stated in Conjecture 6.1), which is needed to deduce congruence (3) from the congruences (1) and (2), is known for large class of modular forms by the results of [Ka] and [SU]. More precisely, to deduce (3) we only need Iwasawa main conjecture modulo $\pi$ for $f \otimes \xi_2 \omega_p^{−1}$ i.e. the congruence $(\mu_{i,\xi_2,\omega_p^{−1}} \equiv C_{[\Gamma]}(S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee) \mod \pi$ holds for $i = 1, 2$.

Remark 0.5. We discuss the hypotheses (Sel-tors) and $(H^2_{\xi_1, \text{van}})$ appearing as condition (iii) in Theorem 0.2.

(i) Recall the definition of fine Selmer group which was defined and studied by Coates-Sujatha (cf. [CS]):

$$R(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc}) = \text{Ker}(H^1(\mathbb{Q}_S/\mathbb{Q}_{cyc}, A_f(\xi_2 \omega_p^{−1})[\pi]) \to \oplus_{\pi \in \Sigma} H^1(\mathbb{Q}_{cyc,w}, A_f(\xi_2 \omega_p^{−1})[\pi])).$$

It was observed independently by Greenberg and Sujatha that the vanishing of $\mu$-invariant, that is, $\mu(R(A_f(\mathbb{Q}_{cyc})^\vee)) = 0$ is equivalent to $H^2(\mathbb{Q}_S/\mathbb{Q}_{cyc}, A_f(\xi_2 \omega_p^{−1})[\pi]) = 0$ (cf. [JS, (40)]). Following [CS, Conjecture A], it is expected that $\mu(R(A_f(\mathbb{Q}_{cyc})^\vee)) = 0$ always holds for any cuspidal Hecke eigenform $F$ and any prime $p$ (also see [JS]). Indeed, as explained in [CS, Corollary 3.6], the vanishing of the $\mu$-invariant $\mu(R(A_f(\mathbb{Q}_{cyc})^\vee)) = 0$ is essentially a reformulation of the classical $\mu = 0$ conjecture of Iwasawa. Thus in our setting of Theorem 0.2, the assumption $(H^2_{\xi_1, \text{van}})$ is expected to hold.

(ii) Observe that in the setting of [KLZ], [LLZ] it is known that $S_{Gr}(A_f/\mathbb{Q}_{cyc})^\vee$ is a finitely generated torsion $\mathcal{O}[\Gamma]$-module using an Euler system argument.

(iii) Note that if $S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee$ has $\mu$-invariant equal to 0 i.e. $S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee$ is a finitely generated $\mathbb{Z}_p$-module for both $i = 1, 2$, then from the exact sequence (47), it follows that $S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee/\pi$ is finite. Hence by the structure theorem of $\mathcal{O}[\Gamma]$-modules, it follows that $S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee$ is a finitely generated torsion $\mathcal{O}[\Gamma]$-module. Further, as $R(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc}) \subset S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc}), \mu(S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee = 0$ implies $\mu(R(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee) = 0$ which in turn implies $H^2(\mathbb{Q}_S/\mathbb{Q}_{cyc}, A_f(\xi_2 \omega_p^{−1})[\pi])$ vanishes.

(iv) In particular, the hypotheses (Sel-tors) and $(H^2_{\xi_1, \text{van}})$ appearing in Theorem 0.2 are satisfied whenever $S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee$ is a finitely generated $\mathbb{Z}_p$-module for $i = 1, 2$ or equivalently the $\mu$-invariant of $S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee$ vanishes for $i = 1, 2$. In fact, under (irr-f), following [G2, Conjecture 1.11], it is expected that $\mu(S_{Gr}(A_f(\xi_2 \omega_p^{−1})/\mathbb{Q}_{cyc})^\vee = 0$ for $i = 1, 2$.

An application of Corollary 0.3, we obtain several explicit examples (See §7 for details).

Example 0.6. (Example 7.1) Let $p = 11, \Delta \in S_{12}(SL_2(\mathbb{Z}))$ be the Ramanujan Delta function, $h \in S_2(\Gamma_0(23))$ be the newform of label LMFDB 23.2.a and $\chi_K$ be the quadratic character of $Q(\sqrt{-23})$. Put $f = \Delta \otimes \chi_K$. Then all the assumptions in Theorems 0.1, 0.2 and Corollary 0.3 are satisfied and we deduce that the Iwasawa main conjecture holds for $f \otimes h$ modulo $\pi$, that is, the following congruences hold in $\mathcal{O}[\Gamma]$:

$$(\mu_{f \otimes h, \omega_p^{−1}}) \equiv C_{[\Gamma]}(S_{Gr}(A_f)/\mathbb{Q}_{cyc})^\vee \mod \pi \quad \text{for } 1 \leq j \leq 10.$$

Further,

$$(\mu_{f \otimes h, \omega_p^{−1}}) = C_{[\Gamma]}(S_{Gr}(A_f)/\mathbb{Q}_{cyc})^\vee = \mathcal{O}[\Gamma] \quad \text{for } 1 \leq j \leq 10 \text{ and } j \neq 4, 5.$$

Example 0.7. (Example 7.3) Let $p = 5$. Let $E$ be the elliptic curve given by $y^2 + y = x^3 + x^2 - 9x - 15$ and $f \in S_0(\Gamma_0(19), 2)$ be the newform congruent to $E$ mod $\pi$. Let $h \in S_2(\Gamma_2(11))$ be the newform of label LMFDB label 11.2.a.a. Then all the assumptions in Theorems 0.1, 0.2 and Corollary 0.3 are satisfied and we deduce that the Iwasawa main conjecture holds for $f \otimes h$ modulo $\pi$. In fact, in this case, $\mu_{f \otimes h, \omega_p^{−1}}$ is a $p$-adic unit for every $j$ and hence we deduce the full Iwasawa main conjecture, that is,

$$(\mu_{f \otimes h, \omega_p^{−1}}) = C_{[\Gamma]}(S_{Gr}(A_f)/\mathbb{Q}_{cyc})^\vee = \mathcal{O}[\Gamma] \quad \text{for } 1 \leq j \leq 4.$$

As stated in Corollary 0.3 the congruence (3) is obtained via the congruences (1) and (2). We proceed in the analytic side and algebraic side separately to establish the congruences (1) and (2) respectively.

On the analytic side, we start by lifting the characters $\xi_1$ and $\xi_2$ to the Dirichlet characters $\xi_1$ and $\xi_2$. This enables us to define a suitable $p$-ordinary and primitive Eisenstein series $g$ such that the residual Galois representation $\bar{\rho}_g \cong \xi_1 \oplus \xi_2$ and $g|_{\ell} \equiv h|_{\ell} \mod \pi$. As $\tilde{f} := f|_{\ell}$ is not necessarily primitive, there is some technical difficulty in the construction of $p$-adic $L$-function of $f|_{\ell} \times h|_{\ell}$ (See Remark 2.5).

Iwasawa Theory for Rankin-Selberg Product at an Eisenstein Prime
for details). To circumvent this, we choose an auxiliary character $\chi$ of $m$-power conductor, such that $f|\chi, g|\chi$ and $h|\chi$ remain primitive. Further, we show that the $p$-adic Rankin-Selberg $L$-function of $f|\chi \otimes h|\chi$ is equal to the $\Sigma_0$-imprimitive $p$-adic Rankin-Selberg $L$-function of $f \otimes h$ up to multiplication by a $p$-adic unit (see Proposition 3.18). Now the congruence between the Fourier coefficients of $g$ and $h$ implies the $p$-adic Rankin-Selberg $L$-functions of $f|\chi \otimes h|\chi$ and $f|\chi \otimes g|\chi$ are congruent. Thus it suffices to show that the congruence (1) holds with the $\Sigma_0$-imprimitive $p$-adic Rankin-Selberg $L$-function of $f \otimes h$ replaced by the $p$-adic Rankin-Selberg $L$-function of $f|\chi \otimes g|\chi$.

To establish this analogue of congruence (1) for the $p$-adic $L$-function of $f|\chi \otimes g|\chi$, we suitably adopt the strategy of [Va] from the setting of elliptic modular forms to our Rankin-Selberg product setting. This can be explained in two steps. First, we express the special values of the Rankin-Selberg $L$-function of $f|\chi \otimes g|\chi$ as a product of the special values of $L$-functions of $\tilde{f}_0 \otimes \xi_1$ and $\tilde{f}_0 \otimes \xi_2$ (see Lemma 2.9). From this to arrive at the required congruence of the $p$-adic $L$-functions involves a delicate choice of periods $\Omega_f$ such that (i) The $p$-adic $L$-functions $\mu_{p,f,\xi,j}$ attached to $\tilde{f} \otimes \xi_i$ are $O$-valued for $i = 1, 2$ and $0 \leq j \leq k-2$ and (ii) The $O$-valued $p$-adic $L$-function $\mu_{p,f,\xi,j}$ differs from the product $\mu_{p,f,\xi,0} \otimes \mu_{p,f,\xi,j}$ by a $p$-adic unit. In fact, to achieve (i), we choose periods using certain parabolic cohomology groups and then show that with this choice of period, the required $p$-adic $L$-functions becomes $O$-valued (see Theorems 3.6, 3.8). On the other hand, we use results from [H3, Chapter 5] and [H5] on the adjoint $L$-function of a modular form to show that the condition (ii) is satisfied. Building up on these results, we obtain the congruence of $p$-adic $L$-functions in equation (1) in Theorems 3.15, 3.19.

On the algebraic side, from (irr-$f$) and $f$ is $p$-ordinary, it follows that $S_G^{S_0}(A_f(\xi_0^\omega p^{-j})/Q_{cyc})^\vee$ is a torsion $O[I]\right.$-module for $i = 1, 2$ ([Ka]). The key ingredient in the congruence (2) is the following exact sequence, which is proved in Section 5.5:

$$0 \to S_G^{S_0}(A_f(\xi_1^\omega p^{-j})/|Q_{cyc})^\vee \to S_G^{S_0}(A_f[|Q_{cyc})^\vee \to S_G^{S_0}(A_f(\xi_2^\omega p^{-j})/|Q_{cyc})^\vee \to 0.$$ (4)

Under (irr-$f$), we show that $S_G^{S_0}(B_j/|Q_{cyc}) = S_G^{S_0}(B_j/|Q_{cyc})$ for $B_j \in \{A_f(\xi_0^\omega p^{-j}), A_f(\xi_2^\omega p^{-j})\}$. Further, for $B_j \in \{A_f(\xi_0^\omega p^{-j}), A_f(\xi_2^\omega p^{-j})\}$, we show that $S_G^{S_0}(B_j/|Q_{cyc})^\vee$ has no non-zero pseudo-null submodule using a result of [We]. The last fact enables us to show that the base change holds for the characteristic ideals i.e. $C_{O[I]}(S_G^{S_0}(B_j/|Q_{cyc})/\pi O[I]) = C_{O[I]}((S_G^{S_0}(B_j/|Q_{cyc})(|Q_{cyc})^\vee)$ with $F = O/\pi$. Putting all these together, the congruence in (2) follows (see Theorem 5.9). We now describe the key ideas involved to derive the exact sequence (4): Tensoring the exact sequence $(\rho_{p,\text{red}}$) with $A_f \otimes \omega p^{-j}$, we obtain

$$0 \to A_f(\xi_1^\omega p^{-j})/| \to A_j/| \to A_f(\xi_2^\omega p^{-j})/| \to 0.$$ A priori, the Selmer group $S_G^{S_0}(B_j/|Q_{cyc})$ depends on $B_j$. Under hypotheses (ii) and (ii) of Theorem 0.2, we show that the local conditions appearing in the Selmer group $S_G^{S_0}(B_j/|Q_{cyc})$ depend only on $B_j/|$. With this explicit description, $S_G^{S_0}(B_j/|Q_{cyc})$ is essentially determined by the corresponding residual representation. Using this and the hypotheses (iii), (iv) of Theorem 0.2, we deduce (4).

The structure of article is as follows: In §1, we recall the basics of $p$-adic modular forms and the $p$-adic Rankin-Selberg $L$-function due to Hida [H2]. In §2, we define the Eisenstein series $g$ and show that the $p$-adic $L$-functions of $f|\chi \otimes g|\chi$ and $f|\chi \otimes h|\chi$ are congruent modulo $\pi$. We also express the special values of the Rankin-Selberg $L$-function of $f|\chi \otimes g|\chi$ as a product of the special values of the $L$-functions attached to $\tilde{f} \otimes \xi_1$ and $\tilde{f} \otimes \xi_2$. Next in §3, we make an appropriate choice of periods $\Omega_f$ and use it to obtain that the ideal generated by the $p$-adic $L$-function of $f|\chi \otimes h|\chi$ in the Iwasawa algebra is congruent to the ideal generated by the product of the $p$-adic $L$-functions associated to $\tilde{f} \otimes \xi_1$ and $\tilde{f} \otimes \xi_2$ (Theorem 3.19). We also show that the ideal generated by the $p$-adic $L$-function of $f|\chi \otimes h|\chi$ is equal to the $\Sigma_0$-imprimitive $p$-adic $L$-function of $f \otimes h$ in the Iwasawa algebra. In §4, we recall the $p^{\infty}$-Selmer groups of the Rankin-Selberg convolution $f \otimes h$ and the modular form $f \otimes \xi_1$ and further give explicit descriptions of these Selmer groups in terms of residual representations. In §5, we show that the characteristic ideal of the $p^{\infty}$-Selmer group of $f \otimes h$ is congruent to the product of the characteristic ideals attached to the $p^{\infty}$-Selmer groups of $f \otimes \xi_1$ and $f \otimes \xi_2$. In §6, we prove our main theorem (Theorem 6.4) establishing the Iwasawa main conjecture modulo $\pi$ for $f \otimes h$. We discuss a few concrete examples illustrating the result of Theorem 6.4 in §7.
Acknowledgement: S. Jha acknowledges the support of SERB MTR/2019/000996 grant. A part of this work was done at ICTS-TIFR and we acknowledge the program ICTS/ecl2022/8. We thank Aribam Chandrakant for the help with the computations on $\mu$-invariants. R. Vangala acknowledges the support of IPDF of IIT Kanpur.

1. Preliminaries and Setup

In this section, we begin by recalling $p$-adic modular forms, measures, Rankin-Selberg $L$-function and $p$-adic Rankin-Selberg $L$-function. Throughout this section $J$ denotes an arbitrary positive integer.

1.1. $p$-adic modular forms

In this subsection, we briefly recall the definition of the space of $p$-adic modular forms in the sense of Serre. For more details we refer the reader to [H1], [H2]. We fix embeddings $i_p: \mathbb{Q} \to \mathbb{Q}_p$ and $i_\infty: \mathbb{Q} \to \mathbb{C}$.

Let $M_k(\Gamma_0(J), \psi)$ (resp. $M_k(\Gamma_1(J))$) is the space of modular forms with coefficients in $\mathbb{C}$, nebentypus $\psi$ and the congruence subgroup $\Gamma_0(J)$ (resp. $\Gamma_1(J)$). For a subring $R \subset \mathbb{Q}$, let $M_k(\Gamma_0(J), \psi; R)$ (resp. $M_k(\Gamma_1(J); R)$) denote the subspace of $M_k(\Gamma_0(J), \psi)$ (resp. $M_k(\Gamma_1(J))$) consisting of all modular forms with Fourier coefficients in $R$. Also, let $S_k(\Gamma_0(J), \psi)$ (resp. $S_k(\Gamma_1(J))$) be the space of cusp forms with coefficients in $\mathbb{C}$, nebentypus $\psi$ and the congruence subgroup $\Gamma_0(J)$ (resp. $\Gamma_1(J)$).

Put $S_k(\Gamma_0(J), \psi; R) := M_k(\Gamma_0(J), \psi; R) \cap S_k(\Gamma_0(J), \psi)$, $S_k(\Gamma_1(J); R) := M_k(\Gamma_1(J); R) \cap S_k(\Gamma_1(J))$.

For every modular form $f(\mathbb{Z}) = \sum_{n \geq 0} a(n) q^n$ with Fourier coefficients in $\mathbb{Q}$, we define a $p$-adic norm $|f|_p$ by $|f|_p := \sup_{n} |a(n, f)|_p$. For a finite extension $K$ of $\mathbb{Q}$, let $K_p$ be the closure of $K$ in $\mathbb{C}_p$ induced by the embedding $i_p: \mathbb{Q} \to \mathbb{Q}_p$, where $\mathbb{C}_p$ is the completion of $\mathbb{Q}_p$ and $\mathcal{O}$ be the ring of integers of $K_p$. Let $M_k(\Gamma_0(J), \psi; K_p)$ (resp. $M_k(\Gamma_1(J); K_p)$) denote the completion of $M_k(\Gamma_0(J), \psi; K)$ (resp. $M_k(\Gamma_1(J); K)$) with respect to the norm $| \cdot |_p$ in $K_p[\mathbb{Q}]$.

Then it is known that (see [H1, Pg 170])

$$M_k(\Gamma_0(J), \psi; K_p) = M_k(\Gamma_0(J), \psi; \mathcal{O}) \otimes_{\mathcal{O}} K_p$$

where $M_k(\Gamma_0(J), \psi; \mathcal{O})$ is the completion of $M_k(\Gamma_0(J), \psi; A)$ (resp. $M_k(\Gamma_0(J), \psi; A)$) under the norm $| \cdot |_p$ in $K_p[\mathbb{Q}]$.

Next, we consider the Hecke algebras of the space of $p$-adic modular forms. Denote by $H_k(\Gamma_0(J), \psi; R)$ (resp. $H_k(\Gamma_1(J); R)$), the Hecke algebra corresponding to the space of modular forms $M_k(\Gamma_0(J), \psi; R)$ (resp. $M_k(\Gamma_1(J); R)$). Let $h_k(\Gamma_0(J), \psi; R)$ (resp. $h_k(\Gamma_1(J); R)$) denote the Hecke algebra corresponding to the space of cusp forms $S_k(\Gamma_0(J), \psi; R)$ (resp. $S_k(\Gamma_1(J); R)$).

$$H_k(\Gamma_0(J), \psi; \mathcal{O}) = \lim_{\rightarrow} H_k(\Gamma_0(J^n), \psi; \mathcal{O})$$

and

$$h_k(\Gamma_0(J), \psi; \mathcal{O}) = \lim_{\rightarrow} h_k(\Gamma_0(J^n), \psi; \mathcal{O})$$

Definition 1.1. (Idempotent) ([H2, §2]) Let $U(p)$ be the $p$th-Hecke operator in $H_k(\Gamma_0(J^n), \psi; \mathcal{O})$ and $H_k(\Gamma_1(J^n); \mathcal{O})$. For every $n$, let $e_n$ be the idempotent in $H_k(\Gamma_0(J^n), \psi; \mathcal{O})$ and $H_k(\Gamma_1(J^n); \mathcal{O})$ defined by $\lim_{m \to \infty} U(p)^m$. The idempotent operator $e$ in $H_k(\Gamma_0(J), \psi; \mathcal{O})$ and $H_k(\Gamma_1(J); \mathcal{O})$ is defined as $\lim_{n \to \infty} e_n$.

1.2. Measures and $p$-adic Rankin-Selberg Convolution

In this subsection, we briefly recall the construction of $p$-adic Rankin-Selberg $L$-function due to Hida [H2]. For a topological ring $A$, let $C(Z_p^\times; A)$ and $LC(Z_p^\times; A)$ denote the space of continuous (resp. locally constant) functions on $Z_p^\times$ with values in $A$. Let $g(z) = \sum_{n=0}^{\infty} a(n, g)q^n \in M_k(\Gamma_0(J), \psi; \mathcal{O})$. Then we consider the arithmetic measure (see [H2, Page 36, Example b]) $\mu_p$ of weight $1$ (see [H2, (5.1a)]), defined by

$$\mu_p(\phi) = \sum_{n=1}^{\infty} \phi(n)a(n,q)q^n, \quad \forall \phi \in C(Z_p^\times; \mathcal{O}),$$

where $\phi \equiv 0$ on $Z_p \setminus Z_p^\times$. Note that $\mu_p(\phi) \in S_l(\Gamma_0(J); \mathcal{O})$ for $\phi \in LC(Z_p^\times; \mathcal{O})$. Let $J_0$ be the prime to $p$-part of $J$. Let $L$ be a positive integer such that $J_0 \mid L$. Then we have a modified arithmetic measure

Iwasa theory for Rankin-Selberg Product at an Eisenstein Prime 5
defined as $\mu_\eta^F(\phi) := \mu_\eta(\phi)[J/L_0]$, where $[J/L_0] : S_1(J_0;\mathcal{O}) \to S_1(L;\mathcal{O})$ is the linear map defined by $(f|[J/L_0])(z) = f(zL/J_0)$, $\forall f \in S_1(N_0;\mathcal{O})$. Put $Z_L = Z^\times \times (Z/LZ)^\times$. For $z \in Z_L$ we denote its component in $Z^\times_L$ by $z_\eta$. We consider the action (depending on the weight and the nebentypus of $g$) of the group $Z_L$ on $C(Z^\times_L,\mathcal{O})$ by the formula $(z \ast \phi)(x) := \langle z^{-1} \cdot \phi(z) \rangle_{z_\eta}$ for $z \in Z_L$ and $\phi \in C(Z^\times_L,\mathcal{O})$. We also consider the Kronecker measure of weight one defined by

$$2\mathcal{E}(\phi) = \sum_{n=1}^{\infty} \sum_{\substack{d|n \equiv 1 \mod (\mathcal{O},L) \equiv 1}} \text{sgn}(d)\phi(d)q^n \in \mathbb{Z}[\mathcal{O}].$$

For a given integer $k > l$ and a finite order character $\eta : Z_L \to Z^\times_L$, we consider the arithmetic measure $(\mu_\eta^F \ast E)M_k : C(Z^\times_L,\mathcal{O}) \to \mathcal{S}_k(L;\mathcal{O})$ of weight $k$ and character $\eta$ defined by convolution of $\mu_\eta^F$ and $E$ as follows:

$$(\mu_\eta^F \ast E)M_k(x) := \int_{Z^\times_L} \int_{Z_L} \eta(z)z^{-1} \cdot x \phi(x) dE(z) d\mu_\eta^F(x) = \int_{Z^\times_L} \int_{Z_L} \eta(z)z^{-1} \cdot dE(z) d\mu_\eta^F(x).$$

By [H2, (9.3)] (see also [Bo, Section 2]), for a finite order character $\phi \in C(Z_L;\mathcal{O})$, we have

$$(\mu_\eta^F \ast E)M_k(x) = \int_{Z^\times_L} \int_{Z_L} \eta(z)z^{-1} \cdot dE(z) d\mu_\eta^F(x) = \int_{Z^\times_L} \int_{Z_L} \eta(z)z^{-1} \cdot \psi(z^{-1} \cdot z_\eta \phi(x) dE(z) d\mu_\eta^F(x) = \int_{Z^\times_L} \int_{Z_L} \eta(z)z^{-1} \cdot \psi(z^{-1} \cdot z_\eta \phi(x) dE(z) d\mu_\eta^F(x).$$

where $3_\eta(z) = z_\eta$ and $\phi_\eta(z) = \phi(3_\eta(z))$.

Let $f(z) = \sum_{n=1}^{\infty} a(n,f)q^n$ be a normalized Hecke eigenform of weight $k > l$, level $N_f$ and nebentypus $\eta$. We assume $f$ is $p$-ordinary i.e. $|\psi_p(a(n,f))|_p = 1$. We define the $p$-stabilization of $f$ by

$$f_0(z) = \begin{cases} f(z) & \text{if } p \mid N_f, \\ f(z) - \beta_j f(pz) & \text{otherwise}, \end{cases}$$

where $\beta_j$ is the unique root of $X^2 - a(p,f)X + \eta(p)p^{k-1} = 0$ with $|\beta_f|_p < 1$. It is well-known that the level of $f_0$, $N_{f_0} = N_f$ if $p \mid N_f$ and $N_{f_0} = p N_f$ if $p \nmid N_f$ and nebentypus of $f_0$ is $\eta$. Note that $u_f := a(p,f_0)$ is the unique $p$-adic unit root of the Hecke polynomial of $f$ at $p$. For simplicity, by a slight abuse of notation, we will denote $N_f$ by $N$. We assume that the Fourier coefficients of $f$ and the values of $\eta$ lie in $\mathcal{O}$. Hence we obtain a surjective homomorphism $\phi_f : h_k(\Gamma_0(N_f),\eta;\mathcal{O}) \to \mathcal{O}$ induced by $T(n) \to a(n,f)$.

Assume that the map $\phi_f$ induces the decomposition

$$h_k(\Gamma_0(N_f),\eta;K_p) = K_p \oplus A.$$  

It is known that the above splitting holds if $f$ is primitive. We will need the decomposition in (8) for a twist of certain specific modular form and it will be made explicit in §2. Let 1 be the idempotent attached to the first summand. We fix a constant $c(f_0) \in \mathcal{O}$ such that $c(f_0)1_{f_0} \in h_k(\Gamma_0(N_f),\eta;\mathcal{O})$. The idempotent $1_{f_0}$ induces a map

$$l_{f_0} : C(S_k(N_\eta,K_p) \to K_p$$

defined by $l_{f_0}(c) = a(1,c)_{f_0}$ where $c \in H_k(\Gamma_0(N_\eta),\eta;\mathcal{O})$ is the idempotent operator defined in Definition 1.1 and $x \in S_k(N_\eta,K_p)$. It follows from [H1, Proposition 4.1] that $C(S_k(\Gamma_0(N_\eta);\mathcal{O}),K_p)$ and the map $l_{f_0}$ is well-defined. Let $L$ be the least common multiple of $J_0$ and $N$ where $J_0$ denotes prime to $p$ part of $J$. Let $\omega_p : (Z_p/\mathbb{Z}_p)^\times \to Z^\times_p$ denote the Teichmüller character and $\chi$ be a Dirichlet character of $(Z_p/\mathbb{Z}_p)^\times$. For an integer $r \geq 1$ and a finite order character $\epsilon : \{1 + p\mathbb{Z}_p\} \to \mathcal{O}^\times$ of conductor $p^r$, let $T_{rL/L} : M_k(\Gamma_0(L),\epsilon;\mathcal{O}) \to M_k(\Gamma_0(L),\epsilon;\mathcal{O})$ denote the trace operator considered in [H2, VI,(1.7)]. For $g \in M_k(\Gamma_0(L),\phi)$, put $g^\phi(z) = \sum_{n=0}^{\infty} a(n,g)q^n \in M_k(\Gamma_0(L),\psi^{-1})$. Take $c_\chi = \eta$ and define the measure $\mu_f \times g$ as follows:

$$\mu_{f \times g}(\phi) := c(f_0) \circ l_{f_0} \circ T_{rL/L} \circ c((\mu_\eta^F \ast E)M_k(\phi^{-1})).$$

where $\phi$ denotes the composition of maps. By the choice of $c(f_0)$, it follows that $\mu_{f \times g}$ is an $\mathcal{O}$-valued measure i.e. $\mu_{f \times g}(\phi) \in \mathcal{O}$, for all $\phi \in C(Z^\times_L,\mathcal{O})$.

The Rankin-Selberg $L$-function of $f \otimes g$ is defined by

$$D_{f \otimes g}(s,f,g) := L_{f \otimes g}(2s + 2 - k - l,\psi_\eta)L(s,f,g) := L_{f \otimes g}(2s + 2 - k - l,\psi_\eta) \sum_{n=1}^{\infty} a(n,f)a(n,g)n^{-s},$$

where $L_{f \otimes g}(s) := \sum_{n=1}^{\infty} a(n,f)a(n,g)n^{-s}$.
where $L_{\text{JN}}(2s + 2 - k - l, \psi \eta)$ denotes the Dirichlet $L$-function of $\psi \eta$ with the Euler factors at the primes dividing $JN$ omitted from its Euler product (see [H1, §1]). For $f, h \in S_k(\Gamma_0(N), \chi)$, define the Petersson inner product of $f$ and $h$ by $(f, h)_N := \int_{N=1} f(z) \overline{h(z)} e^{2\pi i k z} dz$, where $\eta \equiv \text{Im}(z)$.

For integers $N$ and $J$, let $[N, J]$ denote the least common multiple of $N$ and $J$. With the notation as above we recall the following theorem:

**Theorem 1.2.** ([H2, Theorem 5.1 and Section 8], [Bo, Theorem 2.9]) Let $f \in S_k(\Gamma_0(N), \eta)$ be a normalised $p$-ordinary eigenform with $p \nmid N$ and $g \in M_l(\Gamma_0(Jp^{\alpha}))$ with $k > l$ and $(J, p) = 1$. Assume that (8) holds. For every finite order character $\phi \in C(Z_p^\times, \hat{Q}_p)$ and $0 \leq j \leq k - l - 1$, the $O$-valued measure $\mu_{f \times g}(x_p^j \phi) = \int_{Z_p^\times} x_p^j \phi \, d\mu_{f \times g} = c(f_0) \frac{\gamma_p^{3j/2} \gamma_p^{(2-k)/2} a(p, f_0)^{1-\beta}}{(2\pi)^{\frac{3}{2} + j + 2\beta + j + 2l}} \frac{D_{\text{JNp}}(l + j, f_0, \mu_{p^\alpha} \phi)(\tau_{Jp^{\beta}})}{\langle \phi \rangle^{\beta + 2j + \frac{j}{2} + j + 2l}},$

where $\beta$ is the smallest positive integer such that $\mu_{p^\alpha}(\phi^{-1}) | \tau_{Jp^{\beta}} \in M_l(\Gamma_1(Jp^{\beta}))$, $\tau_{Jp^{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $t = [N, J][N^{k/2} J(1+2j)/2 \Gamma(1+j) \Gamma(j + 1)]$.

### 1.3. Simplifying the $L$-function

In this subsection, we express the Rankin-Selberg $L$-function $D_{\text{JNp}}(l + j, f_0, \mu_{p^\alpha}) \langle \tau_{Jp^{\beta}} \rangle$ appearing in Theorem 1.2 in terms of $D_{\text{JNp}}(l + j, f_0, \mu_{p^\alpha})$.

For $g \in M_l(\Gamma_0(J^p \psi), \psi)$ with $p \nmid J$, we denote the characteristic polynomial of $g$ at $p$ by $P_p(g, T)$. More precisely, let

$$P_p(g, T) = \begin{cases} \psi(p) p^{-l} T^2 - a(p, g) T + 1 & \text{if } \alpha \neq 0, \\ 1 - a(p, g) T & \text{if } \alpha = 0. \end{cases} \quad (10)$$

Note that $P_p(g, p^{-s})$ gives the Euler factor at $p$ appearing in the $L$-function $L(s, g)$. For a newform $g \in S_l(\Gamma_0^p \psi^{p^\alpha}, \psi)$, let $W(g)$ be the root number of $g$. $g \mid \tau_{Jp^{\beta}} = W(g) g^{p^\alpha}$ (cf. [H2, Page 37] and [Mi, Theorem 4.6.15]). We first treat the case $\phi$ is the trivial character of $Z_p^\times$ and $g$ is a cuspidal form.

**Lemma 1.3.** Let $f \in S_k(\Gamma_0(N), \eta)$ be a normalised $p$-ordinary eigenform with $p \nmid N$. Let $g \in S_l(\Gamma_0(J^p \psi), \psi)$ be a $p$-ordinary newform with $p \nmid J$ and $2 \leq l < k$. Let $\beta$ be as defined in Theorem 1.2. For $0 \leq j \leq k - l - 1$, we have $\beta = \max \{\alpha, 1\} + 1$ and

$$p^{\beta(j+2)}/2 D_{\text{JNp}}(l + j, f_0, \mu_{p^\alpha} \langle \tau_{Jp^{\beta}} \rangle) = p^{\beta(j+2)}/2 u_p^{\beta-\alpha} W(g^p) P_p(g, p^{j-1} u_p^{j-1}) D_{\text{JNp}}(l + j, f_0, g).$$

**Proof.** Since $g^p \in S_l(J^{p^\alpha}, \tilde{\psi})$ is a Hecke eigenform, we have (see [H2, Line 1 - Page 46])

$$g^p \mid_{\text{tp}} = \begin{cases} g^p - a(p, g^p) g^p \mid |p| + \tilde{\psi}(p) p^{-l} g^p \mid |p^2| & \text{if } \alpha \neq 0, \\ g^p - a(p, g^p) g^p \mid |p| & \text{if } \alpha = 0. \end{cases} \quad (11)$$

As $g$ is primitive, for every $i \geq 0$, we have

$$g^p \mid |p^i \langle \tau_{Jp^{\beta}} \rangle = p^{-\beta-1} g^p \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = p^{-\beta-1} g^p \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = W(g^p) p^{\beta-\alpha} u_p^{\beta-1} g^p \mid |p^\beta-1|.$$\quad (12)

Since $a(p, g) \neq 0$ and $g$ is primitive, we have $g$ is not super-cuspidal at $p$ and $g$ is $p$-minimal (cf. remark after [H6, Lemma 10.1]).

**Case $\alpha = 0$:** By [H2, Lemma 5.2(i)], we have $\beta = 2 = \max \{\alpha, 1\} + 1$. Therefore by (11) and (12), we have

$$D_{\text{JNp}}(l + j, f_0, \mu_{p^\alpha} \langle \tau_{Jp^{\beta}} \rangle) = D_{\text{JNp}}(l + j, f_0, g^p \langle \tau_{Jp^{\beta}} \rangle - a(p, g^p) D_{\text{JNp}}(l + j, f_0, g^p \mid |p^\beta\langle \tau_{Jp^{\beta}} \rangle) + \tilde{\psi}(p) p^{-l-1} D_{\text{JNp}}(l + j, f_0, g^p \mid |p^{\beta-1}|) + \tilde{\psi}(p) p^{-l-1} D_{\text{JNp}}(l + j, f_0, g \mid |p^{\beta-2}|)).$$

Since $L(l + j, f_0, g \mid |p^i|) = p^{-i(l+j)} u_p^i L(l + j, f_0, g)$ and $\beta = 2$, we get

$$D_{\text{JNp}}(l + j, f_0, \mu_{p^\alpha} \langle \tau_{Jp^{\beta}} \rangle) = W(g^p) D_{\text{JNp}}(l + j, f_0, g) \begin{pmatrix} p^{-l-2} u_p^2 - p^{-l-1} a(p, g^p) u_p^1 + p^{-1} \tilde{\psi}(p) \\ e^{-i(l+j)} \end{pmatrix} = p^{-l-2} u_p^2 W(g^p) P_p(g^p, p^{j-1} u_p^{j-1}) D_{\text{JNp}}(l + j, f_0, g).$$\quad (13)
Case $\alpha \neq 0$: By [H2, Lemma 5.2(i)], $\beta = \alpha + 1$. Then it follows from (11) and (12) that
\[
D_{JNP}(l + j, f_0, g_{\mathfrak{p}}(\psi))(\tau_{JNP}) = W(g^p)^{\left(p^j/2 D_{JNP}(l + j, f_0, g)[p]\right)} - p^{-1/2}a(p, g^p)D_{JNP}(l + j, f_0, g).
\]
By a similar argument as in (13), we have
\[
D_{JNP}(l + j, f_0, g_{\mathfrak{p}}(\psi))(\tau_{JNP}) = p^{-1/2}W(g^p)D_{JNP}(l + j, f_0, g)\left(p^{-1/2}u_j - a(p, g^p)\right)
\]
\[
= p^{-1/2}u_j W(g^p)P_{p}(g^p, p^ju_j) D_{JNP}(l + j, f_0, g).
\]
Noting that $\beta = \alpha + 1$ finishes the proof. 

Let $v_p(\cdot)$ be the $p$-adic valuation on $\mathbb{Q}$ with $v_p(p) = 1$. For every Dirichlet character $\psi$, let $\text{cond}(\psi)$ denote the conductor of $\psi$ and $\psi_p$ denote the $p$-part of $\psi$ i.e. $\psi = \psi_p \psi'$. Let $\psi_p$ be a character of $p$-power conductor and the conductor of $\psi'$ is co-prime to $p$. For a Dirichlet character $\chi$ and a modular form $g(z) = \sum_{n \geq 0} a(n, g)q^n$, let $\{g(z)\chi(z) := \sum_{n \geq 1} \chi(n)a(n, g)q^n\}$ denote the twist of $g$ by $\chi$. We next treat the case $\phi$ is a non-trivial finite order character of $\mathbb{Z}_p^\times$ and $g$ is a cuspid form.

**Lemma 1.4.** Let $f \in S_k(\Gamma_0(N), \eta)$ be a normalised $p$-ordinary eigenform with $p \nmid N$. Let $g \in S_{i}(\Gamma_0(Jp^\infty), \psi)$ be a $p$-ordinary newform with $p \nmid J$ and $2 \leq l < k$. Let $\beta$ be as defined in Theorem 1.2. Then for every non-trivial finite order character $\phi$ of $\mathbb{Z}_p^\times$ with $\phi \neq \psi_p$ and $0 \leq j \leq k - 1$, we have $\beta = v_p(\text{cond}(\phi)) + v_p(\text{cond}(\phi'\psi))$ and
\[
D_{JNP}(l + j, f_0, g)(\tau_{JNP}) = W(g^p)D_{JNP}(l + j, f_0, g[\phi]).
\]

**Proof.** As observed in Lemma 1.3 we have $g^p$ is primitive and $\beta = v_p(\text{cond}(\phi)) + v_p(\text{cond}(\phi'\psi))$. Now the lemma follows. 

The following result is an immediate consequence of Theorem 1.2 and Lemmas 1.3, 1.4.

**Corollary 1.5.** Let $f \in S_k(\Gamma_0(N), \eta)$ be a normalised $p$-ordinary newform with $p \nmid N$. Let $g \in S_{i}(\Gamma_0(Jp^\infty), \psi)$ be a $p$-ordinary newform with $p \nmid J$ and $2 \leq l < k$. Let $\beta$ be as defined in Theorem 1.2. Set $t(f, g) = [N, J]N^{k/2}f^{(l+2)/2}\Gamma(l + j)\Gamma(j + 1)$. Then we have

(i) For $0 \leq j \leq k - 1$ and $\phi = \psi_p$, we have $\beta = \max(\alpha, 1) + 1$ and
\[
\mu_{f \times g}(x_{\mathfrak{p}}) = c(f_0)\mu_{f}(x_{\mathfrak{p}}) \sum_{l=0}^{\alpha} \frac{1}{2^{l+1}} \left(\frac{\tau}{\mathfrak{p}}\right)^{-l} \sum_{\mathfrak{p} \neq \mathfrak{p}} D_{JNP}(l + j, f_0, g[\phi]).
\]

(ii) For $0 \leq j \leq k - 1$ and $\phi \neq \psi_p$, we have $\beta = v_p(\text{cond}(\phi)) + v_p(\text{cond}(\phi'\psi))$ and
\[
\mu_{f \times g}(x_{\mathfrak{p}}) = c(f_0)\mu_{f}(x_{\mathfrak{p}}) \sum_{l=0}^{\alpha} \frac{1}{2^{l+1}} \left(\frac{\tau}{\mathfrak{p}}\right)^{-l} \sum_{\mathfrak{p} \neq \mathfrak{p}} D_{JNP}(l + j, f_0, g[\psi]).
\]

We also need the analogue of the above result in the case $g$ is an Eisenstein series. For any Dirichlet character $\psi$, let $\psi_0$ denote the primitive character associated to $\psi$ and $G(\psi) := \sum_{(a, q) = 1} \psi_0(a)e^{2\pi ia/\sigma}$. Denote the Gauss sum of $\psi$. Furthermore, for a prime $r$, let $\text{cond}_r(\psi)$ denote the $r$-part of $\text{cond}(\psi)$. Let $\chi_{cyc} : G_{Q} \rightarrow \mathbb{Z}_p^\times$ denote the $p$-adic cyclotomic character. We now define the Eisenstein series considered in this article.

**Lemma 1.6.** Let $\theta$ and $\varphi$ be primitive $p$-adic Dirichlet characters modulo $u$ and $v$ respectively with $\theta\varphi(-1) = (-1)^\beta$. Then
\[
E_{\mathfrak{u}}(\theta, \varphi)(z) = \delta_{1}(u)L(0, \varphi) + \delta(v)L(1 - l, \theta) + \delta_{2}(u, v) \sum_{n = 0}^{\infty} \frac{\theta(d)\varphi(n)/d}{2\pi i(z - \frac{3}{2})} q^n \in M_{l}(uv, \theta\varphi),
\]
where
\[
\delta_{1}(u) = \begin{cases} 2^{-1} & \text{if } l = u = 1, \\ 0 & \text{otherwise}, \end{cases} \quad \delta_{2}(u, v) = \begin{cases} 2^{-1} & \text{if } l = 2 \text{ and } u = v = 1, \\ 0 & \text{otherwise}, \end{cases} \quad \delta(v) = \begin{cases} 2^{-1} & \text{if } v = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Further, we have

(i) $(E_{\mathfrak{u}}(\theta, \varphi) | \tau_{uv})(z) = (uv^{-1})^{l/2} \varphi(-1)G(\theta)/G(\bar{\theta}) E_{\mathfrak{u}}(\varphi, \bar{\theta})(z)$, where $\tau_{uv} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(ii) $L(s, E_{\mathfrak{u}}(\theta, \varphi)) = L(s - l + 1, \theta)L(s, \varphi)$.

(iii) The Galois representation $\rho_{E_{\mathfrak{u}}(\theta, \varphi)} : G_{Q} \rightarrow \text{GL}_{2}(\mathbb{Q}_{p})$ is isomorphic to $\theta \chi_{cyc} \oplus \varphi$. Furthermore, the representation $\rho_{E_{\mathfrak{u}}(\theta, \varphi)}$ is unramified at all primes not dividing $uv$. 

Proof. The main assertion and part (i) are immediate from [H2, Lemma 5.3]. Part (ii) can be proved following the argument in [Mi, Theorem 4.7.1]. The part (iii) follows from [DS, Theorem 9.6.6]. □

Let \( \theta \) and \( \varphi \) be primitive Dirichlet characters with \( \theta \varphi(-1) = (-1)^l \). Set the “root number” of the \( E_4(\theta, \varphi) \)
\[
W(E_4(\theta, \varphi)) = (\text{cond}(\theta)/\text{cond}(\varphi))^{1/2} \varphi(-1)G(\varphi)/G(\theta).
\]
(14)
We now prove the following analogue of Lemma 1.3 in the case \( g \) is an Eisenstein series.

**Lemma 1.7.** Let \( f \in S_k(\Gamma_0(N), \eta) \) be a normalised \( p \)-ordinary eigenform with \( p \nmid N \). Let \( \theta \) and \( \varphi \) be primitive Dirichlet characters of conductors \( u, v \) respectively with \( \theta \varphi(-1) = (-1)^l \) and \( 2 \leq l < k \). Set \( uv = Jp^p \) with \( p \nmid J \). Let \( g = E_2(\theta, \varphi) \in M_1(\Gamma_0(Jp^p), \theta \varphi) \) and \( g' = E_2(\varphi, \theta) \in M_1(\Gamma_0(Jp^p), \varphi \theta) \). Assume that \( g \) is ordinary at \( p \). Let \( \beta \) be as defined in Theorem 1.2. For \( 0 \leq j \leq k - l - 1 \), we have
\[
p^{\beta(j+2)/2}D_{\text{JN}p}(l + j, f_0, \mu_p(\rho))|_{\tau_{p,j}} = p^{\beta-1/2}u_f^{-1}W(g)p^e Jp^p(Jp^p(l + j, f_0, g'))
\]
where \( \beta = \max\{s, 1\} + 1 \) is as defined in Theorem 1.2 and \( W(g') = (u/v)^{1/2} \varphi(-1)G(\varphi)/G(\theta) \).

**Proof.** Since \( g' \) is a Hecke eigenform, we have (see [H2, Page 46])
\[
g'\mid_{\text{Jp}} = \begin{cases} g' - a(p, g')g'|p| + \bar{\theta}(p)p^{l-1}g'|p|^2 & \text{if } s \neq 0, \\ g' - a(p, g')g'|p| & \text{if } s = 0. \end{cases}
\]
(15)
For every \( i \geq 0 \), by Lemma 1.6 we obtain
\[
(g')\mid_{\text{Jp}} = p^{-i/2}(\theta)^{i/2} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) p^{i/2} = p^{-i/2}p^{(\beta-\delta-1)/2}g'|p|^\delta \]
\[
= W(g)p^{(\beta-\delta-2)/2}g'|p|^\delta
\]
(16)

**Case s = 0:** We have \( p \nmid u, v \). By [Mi, Lemma 4.6.1] and (15), we have \( \beta \leq 2 \). We claim that \( \beta = 2 \). If \( \beta = 0 \), then by (15) we have \( (a(p, g')g'|p| + \bar{\theta}(p)p^{l-1}g'|p|^2) \in M_1(\Gamma_0(J), \theta \bar{\varphi}) \subset M_1(\Gamma_0(Jp^p), \theta \bar{\varphi}) \). By [Mi, Theorem 4.6.4(1)], we have \( \bar{\theta}(p)p^{l-1}g'|p| \in M_1(\Gamma_0(J), \theta \bar{\varphi}) \). Since \( \bar{\theta}(p) \neq 0 \) in this case, we get \( g'|p| \in M_1(\Gamma_0(J), \theta \bar{\varphi}) \). Since \( p \nmid J \), this leads to contradiction according to [Mi, Lemma 4.6.1(2)]. Hence \( \beta \geq 1 \). If \( \beta = 1 \), then \( \bar{\theta}(p)p^{l-1}g'|p|^2 \in M_1(\Gamma_0(Jp^p), \theta \bar{\varphi}) \). Hence by [Mi, Theorem 4.6.4(1)], we have \( g'|p| \in M_1(\Gamma_0(J), \varphi \theta) \). By a similar argument as before, this again leads to a contradiction. Therefore by (15) and (16), we have
\[
D_{\text{JN}p}(l + j, f_0, \mu_p(\rho)|_{\tau_{p,j}} = D_{\text{JN}p}(l + j, f_0, g') - a(p, g')D_{\text{JN}p}(l + j, f_0, g') + \bar{\theta}(p)p^{l-1}D_{\text{JN}p}(l + j, f_0, g')
\]
\[
= W(g')\left( p^{l/2}D_{\text{JN}p}(l + j, f_0, g') \right) - a(p, g')D_{\text{JN}p}(l + j, f_0, g') + \bar{\theta}(p)p^{l-1}D_{\text{JN}p}(l + j, f_0, g')
\]
\[
= W(g)p^{l/2}u_f^{-1}W(g)p^{\delta-1/2}u_f^{-1} = p^{-1/2}u_f^{-1}D_{\text{JN}p}(l + j, f_0, g')
\]
(17)

**Case s \neq 0:** Since \( g \) is \( p \)-ordinary, we have \( a(p, g) \neq 0 \). Thus \( p \nmid u, v \). Again by [Mi, Lemma 4.6.1], we have \( \beta \leq s + 1 \). If \( \beta = s \), then by (15) and (16) we have \( g' \mid_{\text{Jp}} \in M_1(\Gamma_0(Jp^p), \theta \bar{\varphi}) \). By [Mi, Theorem 4.6.4(2)], we get \( g = 0 \) which is a contradiction. Hence \( \beta = s + 1 \). Then it follows from (15) and (16) that
\[
D_{\text{JN}p}(l + j, f_0, \mu_p(\rho)|_{\tau_{p,j}} = W(g)p^{l/2}D_{\text{JN}p}(l + j, f_0, g') - a(p, g')D_{\text{JN}p}(l + j, f_0, g')
\]
By a similar argument as in (17), we have
\[
D_{\text{JN}p}(l + j, f_0, \mu_p(\rho)|_{\tau_{p,j}} = p^{-l/2}W(g)p^{l/2}D_{\text{JN}p}(l + j, f_0, g') - a(p, g')D_{\text{JN}p}(l + j, f_0, g')
\]
\[
= p^{-l/2}u_f^{-1}W(g)p^{\delta-1/2}u_f^{-1} = p^{-l/2}u_f^{-1}W(g)p^{\delta-1/2}u_f^{-1}D_{\text{JN}p}(l + j, f_0, g).
\]
Noting that \( \beta = s + 1 \) finishes the proof. □

We next treat the analogue of Lemma 1.4 in the case \( g \) is an Eisenstein series.
Lemma 1.8. Let $f, g$ and $g'$ be as in Lemma 1.7. Let $\phi$ be a non-trivial finite order character of $\mathbb{Z}_p^\times$ with $\phi \neq \bar{\theta}_p, \bar{\varphi}_p$. Let $\beta$ be as defined in Theorem 1.2. For every $0 \leq j \leq k - l - 1$, we have
\[
D_{JN}\sum_{l+j, f_0, g|\phi}(\bar{\phi}^l(\bar{\sigma})) = W(g\phi^l|\phi)D_{JN}(l+j, f_0, g|\phi)
\]
where $\beta = v_p(\text{cond}(\theta|\phi)) + v_p(\text{cond}(\varphi|\phi))$ and $W(g\phi^l|\phi) = (\text{cond}(\theta|\phi)/\text{cond}(\varphi|\phi))^{1/2}\varphi(1)G(\bar{\phi}^l)/G(\theta|\phi)$.

Proof. Recall that $\bar{\theta}_0$ and $\bar{\varphi}_0$ is the primitive character associated to $\bar{\theta}_0$ and $\bar{\varphi}_0$ respectively. We claim that $a(n, g'|\phi) = a(n, E_1(\bar{\theta}_0, \bar{\varphi}_0))$. If $p \nmid n$, then it is easy to check that above equality holds. If $p | n$, then $a(n, g'|\phi) = 0$. Since $\phi \neq \theta_p, \varphi_p$, it follows that $(\bar{\theta}_0, \bar{\varphi}_0) = 0$. Hence $g'|\phi) = E_1(\bar{\theta}_0, \bar{\varphi}_0)$. Noting that $G(\bar{\varphi}_0) = G(\bar{\varphi}_0) = G(\bar{\varphi}_0)$, the lemma now follows from Lemma 1.6.

We now deduce the following result which relates the p-adic Rankin-Selberg L-function with the product of L-function attached to modular form.

Corollary 1.9. Let $f \in S_k(\Gamma_0(N), \eta)$ be a normalised p-ordinary newform with $p \nmid N$. Let $\theta$ and $\varphi$ be primitive Dirichlet characters of conductors $u, v$ respectively with $\theta \varphi(-1) = -1$ and $2 \leq l < k$. Set $uv = Jp^l$ with $p \nmid J$. Let $g = E_1(\theta, \varphi) \in M_1(\Gamma_0(Jp^l), \theta \varphi)$ and $g' = E_1(\varphi, \theta) \in M_1(\Gamma_0(Jp^l), \varphi \theta)$. Assume that $g$ is ordinary at $p$. Let $\beta$ be as defined in Theorem 1.2. Set $l(f, g) = [N, J]_{N}^{k/2}, J^{l+2}/2(1+1)I$. Then we have

(i) For $0 \leq j \leq l - 1$ and $\phi = \bar{\theta}_p$, we have $\beta = \text{max}\{s, l\} + 1$ and
\[
\mu_{f \times g}(x_0 f_p) = c(f_0)l(f, g)p^{s+2(l+2)/2}u^{(1+1)/2}p^{(2-k)/2}W(g\phi^l|\phi)f_p(g, p^{2-k}/2)\frac{L(l+j, f_0, \varphi)L(l+j, f_0, \varphi)}{(2i)^{k+1+2}p^{(2-k)/2}u^{(1+1)/2}f_p|\text{cond}(\theta|\phi), f_p|\text{cond}(\varphi|\phi)}},
\]
where $W(g\phi^l|\phi) = (\text{cond}(\theta|\phi)/\text{cond}(\varphi|\phi))^{1/2}\varphi(1)G(\bar{\varphi}_0)/G(\theta|\phi)$.

(ii) For $0 \leq j \leq l - 1$ and $\phi \neq \bar{\theta}_p, \bar{\varphi}_p$, we have $\beta = v_p(\text{cond}(\theta|\phi)) + v_p(\text{cond}(\varphi|\phi))$ and
\[
\mu_{f \times g}(x_0 f_p) = c(f_0)l(f, g)p^{s+2(l+2)/2}u^{(1+1)/2}p^{(2-k)/2}W(g\phi^l|\phi)f_p(g, p^{2-k}/2)\frac{L(l+j, f_0, \varphi)L(l+j, f_0, \varphi)}{(2i)^{k+1+2}p^{(2-k)/2}u^{(1+1)/2}f_p|\text{cond}(\theta|\phi), f_p|\text{cond}(\varphi|\phi)}},
\]
where $W(g\phi^l|\phi) = (\text{cond}(\theta|\phi)/\text{cond}(\varphi|\phi))^{1/2}\varphi(-1)G(\bar{\varphi}_0)/G(\theta|\phi)$.

Proof. By Lemma 1.6(ii), we get $L(l+j, f_0, \varphi) = L(l+j, f_0, \varphi)L(l+j, \varphi)$. Thus by Rankin-Selberg method (See [S1, Lemma 1]), we obtain $D_{JN}(l+j, f_0, g|\phi) = L(l+j, f_0, \theta|\phi)L(1/j, f_0, \varphi|\phi)$. Now the corollary follows from Theorem 1.2, Lemma 1.7 and Lemma 1.8.

2. Towards the congruence of the p-adic L-functions

Fix $f(z) = \sum_{n=1}^{\infty} a(n, f)q^n \in S_k(\Gamma_0(N), \eta)$ a primitive form. We shall assume throughout this article that $p \nmid N$ and $f$ is p-ordinary. Let $f_0$ be the p-stabilization of $f$ as in (7). For an integer $J$, let $f_0$ denote the prime to $p$-part of $J$. For a normalised Hecke eigenform $g$ with nebentypus $\varphi$, let $K_0$ be the number field generated by the Fourier coefficients of $g$ and the values of $\varphi$.

We also fix $h(z) = \sum_{n=1}^{\infty} a(n, h)q^n \in M_1(\Gamma_0(2), \psi)$, a normalized newform of weight $2 \leq l < k$. Assume that $h$ is p-ordinary. Put $I = I_0p^l$ where $I_0$ is co-prime to $p$. Let $K$ be a number field containing $F$ and $K_0$. Let $K_0$ denote the completion of $K$ at a prime $p$ lying above $p$ induced by the embedding $i_p : \mathbb{Q} \rightarrow \mathbb{Q}_p$ and $O_{K_0}$ be the ring of integers of $K_0$ and $\pi$ be a uniformizer of $O_{K_0}$. Let $(\rho, V_0)$ be the $p$-adic Galois representation attached to the modular form $h$ (see Theorem 4.1). Choose a rank two $O_{K_0}^\times$-submodule $T_h$ of $V_h$ which is invariant under the action of the absolute Galois group $G_\mathbb{Q} := \text{Gal}(\mathbb{Q}/\mathbb{Q})$. Let $\rho_p : G_\mathbb{Q} \rightarrow \text{GL}_2(T_h/\pi)$ be the reduction of $\rho$ modulo $\pi$. Assume that the semi-simplification of $T_h/\pi$ is isomorphic to $\xi_1 \oplus \xi_2$, i.e.

\[
0 \rightarrow \xi_1 \rightarrow T_h/\pi \rightarrow \xi_2 \rightarrow 0.
\]

In this section, we use the residual characters $\xi_1$ and $\xi_2$ to construct an Eisenstein series $g$ congruent to $h$. We then express the special values of the Rankin-Selberg L-function $f_0 \otimes g$ as a product of the special values of L-function associated to $f_0 \otimes \xi_1$ and $f_0 \otimes \xi_2$. In the next section, using these results we shall show that the p-adic Rankin-Selberg L-function associated to $f_0 \otimes g$ is congruent to the product of p-adic L-functions of $f \otimes \xi_1$ and $f \otimes \xi_2$. We begin by constructing the Dirichlet characters $\xi_i$ whose reduction is $\xi_i$ for $i = 1, 2$.\]
Lemma 2.1. Let $\xi : (\mathbb{Z}/v\mathbb{Z})^\times \to \mathbb{F}_p^\times$ be a character. Then there exists a Dirichlet character $\xi : (\mathbb{Z}/v\mathbb{Z})^\times \to \mu_{p-1}$ such that the reduction of $\xi$ equals $\xi$, where $\mu_{p-1} := \{z \in \mathbb{C} : z^{p-1} = 1\}$. Further, if the conductor of $\xi$ equals $v_0$ with $p \nmid v_0$, then the conductor of $\xi$ is $v_0p^a$ with $a = \min\{a, 1\}$.

Proof. Without any loss of generality, assume that $\xi$ is primitive. Since $\mathbb{F}_p^\times$ doesn't contain non-trivial $p$th root of unity, we get that the image of $p$-Sylow subgroup of $(\mathbb{Z}/v\mathbb{Z})^\times$ is trivial under $\xi$. As $\xi$ is primitive, we obtain $p^2 \parallel v$ and $v = v_0p^a$ with $a \in \{0, 1\}$ and $p \nmid v_0$. Choose a finite extension $L/\mathbb{Q}_p$ such that $(\mathcal{O}_L/\pi)^\times \cong \mathbb{F}_p^\times$. Composing $\xi$ with the Teichmüller character $\omega : (\mathcal{O}_L/\pi)^\times \to \mu_{p-1}$ we obtain a lift $\xi : (\mathbb{Z}/v\mathbb{Z})^\times \to \mu_{p-1}$, and we have $\xi \equiv \xi \mod p$. If the conductor of $\xi$ is strictly less than $v$, then it would imply that the conductor of $\xi$ is strictly less than $v$. Thus the conductor of $\xi$ is $v_0p^a$. This finishes the proof. ∎

Since $h$ is $p$-ordinary, either $\xi_1$ or $\xi_2$ is unramified at $p$ (see Theorem 4.1(ii)). Without any loss of generality, assume $\xi_2$ is unramified at $p$. Since $\xi_2(\text{Frob}_p) \neq 0$, we denote the arithmetic Frobenius at $p$, we get $p$ doesn't divide the conductor of $\xi_2$. From Lemma 2.1, it follows that there exists a Dirichlet character $\xi_2$ whose reduction equals $\xi_2$ and the conductor of $\xi_2$ is $M_2$ with $p \nmid M_2$. Similarly, we can lift $\xi_1\omega r^{-1}$ to a Dirichlet character $\xi_1\omega r^{-1}$ such that the conductor of $\xi_1\omega r^{-1}$ is of the form $M_1p^s$ with $(M_1, p) = 1$ and $s \in \{0, 1\}$. Since the conductor of $(\Gamma/h/\pi)$ divides the conductor of $\Gamma$, we obtain $M_1M_2 | I_0$. Set $M := M_1M_2p^s$. Then $I_0 = M_1M_2$ and $M_0 | I_0$. Put $\Sigma_0 = \{r \mid r \mid (I_0/M_0) \text{ or } r^2 \mid (M_0)\}$ and set

$$m := \prod_{r \in \Sigma_0} r. \quad (19)$$

Note that $\xi_1\xi_2\omega r^{-1}(-1) = \tilde{\psi}(-1) = (-1)$. As $p$ is odd, $\xi_1\xi_2\omega r^{-1}(-1) = (-1)$. Defining $g(z) := E_l(\xi_1\omega r^{-1} - \xi_2(z))$. (20)

From Lemma 1.6, we have $g(z) \in M_l(\Gamma_0(M_1), \xi_2\omega r^{-1})$ and $g(z) = \sum_{n \geq 0} a(n, g)q^n$, with $a(n, g) = \sum_{d|n} \xi_2(d)\xi_2(n/d)\omega r^{-1}(d)\xi_2(d)^{n/d-1}$ for all $n \geq 1$. Further, we have $\rho_g \equiv \xi_1 \equiv \xi_2 \equiv (\Gamma/h/\pi)^{ss}$. As $\xi_2(p) \neq 0$, it follows that $a(p, g) = \xi_2(p) + \xi_2^{2p}\omega r^{-1}(p)$ is a $p$-adic unit. Hence $g$ is $p$-ordinary. Recall that for an integer $J$, $\xi_J$ denotes the trivial character of modulus $J$. We now show that $g|_{\text{rmp}}$ and $h|_{\text{rmp}}$ are congruent modulo $\pi$.

Lemma 2.2. With the notation as above, we have $h|_{\text{rmp}} \equiv g|_{\text{rmp}} \mod \pi$.

Proof. Let $r$ be a prime. Since $(\Gamma/h/\pi)^{ss}$ and $(\Gamma/l/\pi)^{ss}$ are the same, we get $a(r, h|_{\text{rmp}}) = a(r, h) \equiv a(r, g) = a(r, g|_{\text{rmp}}) \mod \pi$, $\forall \ r \mid \text{mp}M_0$. If $r = p$ or $r \mid m$, then $a(r, h|_{\text{rmp}}) = a(r, g|_{\text{rmp}})$. Next, consider $r \mid M_0$ and $r \not\mid \text{mp}$. This forces $r \mid I_0/M_0, r \mid M_0$ and $r \mid I_0$. Since $\xi_1, \xi_2$ are both ramified at $r$, we obtain $h|_{\text{rmp}} \equiv \xi_2|_{\text{rmp}} \equiv \xi_2|_{\text{rmp}} \equiv a(r, g) \equiv a(r, g|_{\text{rmp}}) \mod \pi$. If $r \mid M_0$ and $r \nmid M_1$, then $a(r, g) = \xi_1(r)\omega r^{-1}(r)^{-1} \equiv \xi_1(r) \mod p$. Again by a similar argument, we deduce $a(r, h) = \xi_2(\text{Frob}_p) \equiv \xi_1(\text{Frob}_p) \equiv a(r, h) \mod \pi$. Thus in either case, we have $h|_{\text{rmp}} = a(r, h) \equiv a(r, g) \equiv a(r, g|_{\text{rmp}}) \mod \pi$. This proves the lemma. ∎

We note the following observation made while proving in the lemma above:

Lemma 2.3. If $r \mid I_0$ and $r \mid \text{mp}$, then $r|I_0, r|\text{M}_0$ and $r|\text{cond}(\psi)$.

For every prime $r$, let $v_r(\cdot)$ be the $r$-adic valuation on $\mathbb{Q}$ with $v_r(r) = 1$. Recall that $f \in S_k(\Gamma_0(N), \eta)$ and $h \in S_k(\Gamma_0(I_0p^m), \psi)$ are primitive with $p \not\mid NI_0$. Also $g = E_l(\xi_1\omega r^{-1}, \xi_2) \in M_l(\Gamma_0(M_0p^s), \xi_2\omega r^{-1})$ with $p \not\mid M_0$. For every prime $r \mid m$, set $n_r = \max\{v_r(\psi), v_r(\xi_1), v_r(\xi_2)\} + 2$ and $n = \max\{n_r\}$. For every prime $r \mid m$, choose a primitive character $\chi_r$ of conductor $r^m$ and set $\chi = \prod_{r|m} \chi_r$ be a Dirichlet conductor $m^a$. Using that $v_r(\text{cond}(\chi)) = v_r(\text{cond}(\chi_r)) > \max\{v_r(\psi), v_r(\xi_1), v_r(\xi_2)\}$, it is easy to check that the Dirichlet character $\chi$ satisfies the following conditions:
(i) $\chi$ is primitive and conductor of $\chi$ equals $m^n$ with $n \geq 2$.
(ii) For every prime $r \mid m$, we have $v_r(\text{cond}(\eta \chi)) = v_r(\text{cond}(\chi))$ and $v_r(\text{cond}(\chi)) > v_r(N)$.
(iii) For every prime $r \mid m$, we have $v_r(\text{cond}(\psi \chi)) = v_r(\text{cond}(\chi))$ and $v_r(\text{cond}(\chi)) > v_r(I_0p^\infty)$.
(iv) For every prime $r \mid m$, we have $v_r(\text{cond}(\xi_2 \chi)) = v_r(\text{cond}(\chi))$, $v_r(\text{cond}(\xi_1 \omega_p^{-1} \chi)) = v_r(\text{cond}(\chi))$ and $v_r(\text{cond}(\chi)) > v_r(M_0p^\infty)$.

Throughout this article we denote the level of $f|\chi$, $g|\chi$, $h|\chi$ by $N_f|\chi$, $N_g|\chi$ and $N_h|\chi$ respectively. We describe the values of $N_f|\chi$, $N_g|\chi$ and $N_h|\chi$ in the following lemma.

Lemma 2.4. We have $f|\chi \in S_k(\Gamma_0(N),\eta \chi)$ and $g|\chi \in S_l(\Gamma_0(N),\psi \chi)$ are newforms with $N_{f|\chi} = [N, \text{cond}(\chi)][\text{cond}(\eta \chi)]$ and $N_{g|\chi} = [I_0p^\infty, \text{cond}(\chi)][\text{cond}(\psi \chi)]$. Also, we have $g|\chi = E(\xi_1 \omega_p^{-1} \chi, \xi_2 \chi) \in M_i(\bar{N}_{g|\chi}, \xi_1 \xi_2 \omega_p^{-1} \chi^2)$ with $N_{g|\chi} = [M_0p^\infty, \text{cond}(\chi)][\text{cond}(\psi \chi)]$. Furthermore, we have $[N_{f|\chi}, N_{h|\chi}p^{-\infty}] = [N_{f|\chi}, N_{g|\chi}p^{-\infty}]$.

Proof. The first assertion follows from [AL, Theorem 4.1]. The second assertion follows from the property (iv) listed above. To verify the last assertion we do case by case analysis and show the $r$-adic values of $[N_{f|\chi}, N_{g|\chi}]$ and $[N_{f|\chi}, N_{h|\chi}]$ are the same. As $M_0p^\infty \mid I_0p^\infty$, we need to consider the cases (a) $r \mid m$, (b) $r \nmid m$ but $r \mid I_0$ and (c) $r \nmid I_0$ and $r \mid N$. If $r$ is prime and $r \mid m$, then by property (ii)-(iv) listed above, we have $v_r([N_{f|\chi}, N_{g|\chi}]) = 2v_r(\text{cond}(\chi)) = v_r([N_{f|\chi}, N_{h|\chi}])$. If $r$ is a prime $r \nmid m$ but $r \mid I_0$, then by Lemma 2.3 we have $v_r([N_{f|\chi}, N_{g|\chi}]) = v_r([N_{f|\chi}, N_{h|\chi}]) = v_r([N_{f|\chi}, N_{h|\chi}])$. If $r$ is a prime $r \mid I_0$ but $r \mid N$, then $v_r([N_{f|\chi}, N_{g|\chi}]) = v_r([N_{f|\chi}, N_{h|\chi}])$. This finishes the proof of the last assertion.

Remark 2.5. If $F \in S_k(\Gamma_0(N), \phi; K)$ is primitive, then the Hecke algebra $h_ξ(\Gamma_0(N), \chi; K)$ splits (as described in (8)). This splitting of Hecke algebra is essential in the construction of $p$-adic Rankin-Selberg $L$-function $f_{\chi, \psi}$, by Hida (See Theorem 1.2). As $f_{l|m}$ is not necessarily primitive, a priori one needs to assume the splitting of Hecke algebra $h_ξ(\Gamma_0(N), \eta; m; K)$ to guarantee the existence of the $p$-adic $L$-function $f_{l|m,h|l|m}$. As $f|\chi$ is primitive, the map $\phi_{f|\chi}$ induces the splitting of the corresponding Hecke algebra. This ensures the existence of the $p$-adic $L$-functions $f_{\chi|\eta,h|\chi}$ and $f_{\chi|\psi,h|\chi}$ and Theorem 3.10 holds for $f|\chi$. We later show that the ideal generated by “twisted” $p$-adic $L$-function $f_{\chi|\eta,h|\chi}$ is same as the ideal generated by $\Sigma_0$-imprimitive $p$-adic $L$-function $\sum_{\chi}^\infty$ in the Iwasawa algebra.

Using the congruence in Lemma 2.2 we deduce the following congruence of $p$-adic measures:

Lemma 2.6. Let $f \in S_k(\Gamma_0(N), \eta)$ be a $p$-ordinary newform and $g, h$ be as before. Let $\chi$ be a primitive Dirichlet character be as above. Then for every finite order character $\phi \in C(\mathbb{Z}_p^\times; \bar{Q}_p)$ and $0 \leq j \leq k-1$, we have

$$\mu_{f|\chi \times h|\chi}(x_p^j \phi) \equiv \mu_{f|\chi \times h|\chi}(x_p^j \phi) \mod p.$$  

Proof. By Lemma 2.2, we have $a(n,h|\chi) \equiv a(n,g|\chi) \mod p$ for all $p \nmid n$. As $\phi(p) = 0$, for every finite order character $\phi$ of $\mathbb{Z}_p^\times$, it follows from (5) that $\mu_{g|\chi}(\phi) \equiv \mu_{h|\chi}(\phi) \mod p$. By Lemma 2.4, we have $L = \text{lcm}(N_{f|\chi}, N_{g|\chi}) = \text{lcm}(N_{f|\chi}, N_{h|\chi})$. Hence $\mu_{p|\chi}(\phi) \equiv \mu_{g|\chi}(\phi) \mod p$. Since $\psi \equiv \xi_1 \xi_2 \omega_p^{-l} \mod p$, it follows that $E(\eta \chi^2 \cdot \psi^1 \chi^{-2} \cdot (\phi^{-2}) \bar{3}^{l-2j-1}) = E(\eta \chi^2 \cdot \xi_1 \chi^{-2} \cdot (\phi^{-2}) \bar{3}^{l-2j-1}) \mod p$. Thus we deduce from (6) that $(\mu^{L_{\text{g|\chi}}}_{p|\chi} \star E)_{\eta \chi^2,k}(x_p^j \phi) \equiv (\mu^{L_{\text{h|\chi}}}_{p|\chi} \star E)_{\eta \chi^2,k}(x_p^j \phi) \mod p$. By [H2, Proposition 7.8], it follows that the linear map $c_{f_0|\chi}^{\text{f|\chi}}$ preserves integrality. Since the trace operator $T_{\text{Tr}/N}$ also preserves integral structures the lemma follows from (9).
where \( t' = (N_{1|N}/N)^{k/2}(N_{h|X}/I_{0|0})^{(l+2)/2}[N_{f|X}, N_{h|X}]/[N, I_{0|0}] \) and \( L_{\Sigma}(l + j, f, h) \) denotes the Euler factor of \( D_{1|N}p(l + j, f, h) \) at \( r \).

Proof. Note that the span of \{finite order characters of \( \mathbb{Z}_p^* \) \} \( \{\tilde{\psi}_p\} \) is dense in \( C(\mathbb{Z}_p^*, \hat{\mathbb{Q}}_p) \). Thus it is enough to show that the statement of lemma holds for all \( \phi \in \{\text{finite order characters of } \mathbb{Z}_p^* \} \setminus \{\tilde{\psi}_p\} \). By Corollary 1.5(i), we have

\[
\mu_{f|X\times h|X}(x_0^2) = c(f_0|X|)(f|X, h|X)p^{(l+2)/2}(\chi(p)u_0)^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(h, p^ju_f(x(p)))
\]

\[
\cdot \frac{D_{N_{f|X}, N_{h|X}}(l + j, f_0|X, h|X)}{(2i)^{k+1+2}p^{(l+2)/2}}(\frac{f_0|X, h|X}{f|X, h|X})^{1-\alpha}p^{(2-k)/2}W^g(\chi)P_g(h, p^ju_f(x(p))\cdot \frac{D_{N_{f|X}, N_{h|X}}(l + j, f_0|X, h|X)}{(2i)^{k+1+2}p^{(l+2)/2}}(\frac{f_0|X, h|X}{f|X, h|X})^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(h, p^ju_f(x(p)))
\]

Since \( P_{f}(h, p^ju_f(x(p))) = P_{f}(h, p^ju_f(x(p)) \text{ and twisting } f, h \text{ by } \chi \text{ doesn't change the value of } \alpha \), we have

\[
\mu_{f|X\times h|X}(x_0^2) = c(f_0|X|)(f|X, h|X)p^{(l+2)/2}(\chi(p)u_0)^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(h, p^ju_f(x(p)))
\]

\[
\cdot \frac{D_{N_{f|X}, N_{h|X}}(l + j, f_0|X, h|X)}{(2i)^{k+1+2}p^{(l+2)/2}}(\frac{f_0|X, h|X}{f|X, h|X})^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(h, p^ju_f(x(p))\cdot \frac{D_{N_{f|X}, N_{h|X}}(l + j, f_0|X, h|X)}{(2i)^{k+1+2}p^{(l+2)/2}}(\frac{f_0|X, h|X}{f|X, h|X})^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(h, p^ju_f(x(p)))
\]

Now the lemma follows from Corollary 1.5(i) in the case \( \chi = \iota_p \). The remaining case \( \phi \neq \iota_p, \tilde{\psi}_p \) can be proved similarly using Corollary 1.5(ii).

In the next section, using Eq. (21), we show that \( \mu_{f|X\times h|X} \) and the \( \Sigma_0 \)-imprimitive \( p \)-adic L-function \( \mu_{f|X\times g|X} \) generate the same ideal in the Iwasawa algebra.

To ease the we denote \( \tilde{f} = f|\iota_p \) and \( \tilde{f}_0 = f_0|\iota_p \). Since the conductor of \( \chi \) is a positive power of \( m \), we have \( \chi \tilde{\chi} = \iota_m \). We now relate the \( p \)-adic Rankin L-function \( \mu_{f|X\times g|X} \) to the product of the \( L \)-functions attached to \( \tilde{f}_0|\iota_l \) and \( \tilde{f}_1 \).

**Lemma 2.9.** Let \( f \in S_{k}(\Gamma_0(N), \eta) \) be a normalised \( p \)-ordinary newform with \( p \nmid N \). Let \( g = E_l(\xi_1\omega_p^{1-l}, \xi_2) \) and \( g' = E_l(\xi_2, \xi_1\omega_p^{1-l}) \). Set \( f|X, g|X) = [N_{f|X}, N_{g|X}[N_{f|X}^{k/2}(N_{g|X})^{l+2}/2] \Gamma(l + j) \Gamma(j + 1) \). Let \( \phi \) be a finite order character on \( \mathbb{Z}_p^* \) and \( 0 \leq j \leq k - l - 1 \).

(i) If \( \phi = \iota_p \), then we have \( \beta = 2 \) and

\[
\mu_{f|X\times g|X}(x_0^2) = c(f_0|X|)(f|X, g|X)p^{(l+2)/2}(\chi(p)u_0)^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(g, p^ju_f(x(p)))
\]

\[
\cdot \frac{L(j + 1, \tilde{f}_0, \tilde{g}_0, \xi_2|X, \xi_1\omega_p^{1-l})}{(2i)^{k+1+2}p^{(l+2)/2}}(\frac{f_0|X, g|X}{f|X, g|X})^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(g, p^ju_f(x(p)))
\]

where \( W(\chi(p)) = (\text{cond}\xi_1\omega_p^{1-l}\chi)/\text{cond}(\xi_2\chi)^{1/2}\xi_2\chi(-1)\xi_2\chi/G(\xi_2\chi)/G(\xi_1\omega_p^{1-l}\chi) \).

(ii) If \( \phi \neq \iota_p, \xi_1\omega_p^{1-l} \), then we have \( \beta = \nu_p(\text{cond}\xi_1\omega_p^{1-l}\phi)/\nu_p(\text{cond}\xi_2\phi) \) and

\[
\mu_{f|X\times g|X}(x_0^2) = c(f_0|X|)(f|X, g|X)p^{(l+2)/2}(\chi(p)u_0)^{1-\beta}p^{(2-k)/2}W(\chi(p))P_g(g, p^ju_f(x(p)))
\]

\[
\cdot \frac{L(j + 1, \tilde{f}_0, \tilde{g}_0, \xi_2|X, \xi_1\omega_p^{1-l} \phi)}{(2i)^{k+1+2}p^{(l+2)/2}}(\frac{f_0|X, g|X}{f|X, g|X})^{1-\alpha}p^{(2-k)/2}W(\chi(p))P_g(g, p^ju_f(x(p)))
\]

where \( W(\chi(p)) = (\text{cond}\xi_1\omega_p^{1-l}\chi)/\text{cond}(\xi_2\chi)^{1/2}\xi_2\chi\phi(-1)\xi_2\chi\phi/G(\xi_2\chi)/G(\xi_1\omega_p^{1-l}\chi)\).

Proof. Note that \( u_f(x) = \chi(p)u_f \) and \( \chi \) has conductor prime to \( p \). Since we are twisting \( f, g \) and \( \xi_1\omega_p^{1-l}\phi, \xi_2\phi \) by \( \chi \), the \( p \)-part of the level and conductor remain the same. Further, from (10), we have \( P_g(g^\phi|X, p^ju_f(x(p)) = P_g(g^\phi, p^ju_f(x(p))) \). Now the lemma follows immediately from Corollary 1.9.

3. Periods and the congruences of \( p \)-adic \( L \)-functions

In this section, we show that the \( p \)-adic Rankin-Selberg \( L \)-function \( \mu_{f|X\times g|X} \) is congruent to the product of the \( p \)-adic \( L \)-functions of \( \tilde{f}_0 \otimes \xi_1 \) and \( \tilde{f}_0 \otimes \xi_2 \). In order to do this, we make an appropriate choice of periods in §3.1. We also show that the ideal generated by the \( p \)-adic \( L \)-function \( \mu_{f|X\times h|X} \) is same as the ideal generated by the \( \Sigma_0 \) imprimitive \( p \)-adic \( L \)-function of \( \mu_{f|X\times h} \) in the Iwasawa algebra.

3.1. Relation between the periods

We begin by recalling an algebraic result due to Shimura.
Theorem 3.1. ([S1, Theorem 1]) Let $F \in S_k(\Gamma_1(N))$ be a normalised Hecke eigenform. There exist complex periods $\Omega_F^+$ and $\Omega_F^-$ such that for every Dirichlet character $\theta$, we have

$$\frac{L(j, F, \theta)}{(2\pi i)^j G(\theta)\Omega^\theta_{\text{even}}(1-j, \theta^{-1}(-1))} \in \overline{Q}, \quad \text{for} \quad 1 \leq j \leq k-1.$$ \hspace{1cm} (22)

Further, we have $\Omega_F^\pm_{\text{reg}}(p) \in \overline{Q} \Omega_F^\pm$ are well-defined up to an element of $\overline{Q}^\times$.

We next recall the $p$-adic $L$-function associated to an eigenform. See [MTT, §14] and [SU, §3.4.4].

Theorem 3.2. Let $F = \sum a(n, F)q^n \in S_k(\Gamma_0(N), \varphi)$ be a $p$-ordinary eigenform and $\Omega_F^+, \Omega_F^-$ be complex periods satisfying (22). Then there exist a bounded measure $\mu_F$ such that for a finite order character $\phi$ of $(\mathbb{Z}/L\mathbb{Z}) \times \mathbb{Z}_p^\times$, with $p \nmid L$ and $0 \leq j \leq k-2$, we have

$$\mu_F(x_j^0 \phi) = \int_{(\mathbb{Z}/L\mathbb{Z}) \times \mathbb{Z}_p^\times} x_j^0 \phi \, d\mu_F = \frac{\epsilon_p(u_F, x_j^0 \phi) \text{cond}(\phi)^{j+1} L(j+1, F, \phi)}{(2\pi i)^{j+1} G(\phi) \Omega^\phi_{\text{even}}(1-j, \phi^{-1}(-1))}. \hspace{1cm} (23)$$

Here $\epsilon_p(u_F, x_j^0 \phi) = \left(1 - \frac{\partial \omega(p)}{\partial p}(p,j)\right) \left(1 - \frac{\partial \omega(p,j)}{\partial p}\right)$ is the $p$-adic multiplier, $\phi_0$ is the primitive character associated to $\phi$ and $u_F$ is the unique $p$-adic unit root of $X^2 - a(p, F)X + \varphi(p)p^{\ell-1}$.

Observe that for $l \leq j \leq k-1$, we have $(-1)^l \xi_2 \phi(0) (-1)^l (\xi F_{\omega^j_p}(-1)) = (-1)^{l+1} \xi_2 \phi(-1) = (1)^{l+1} \xi_2 \phi(0) = (1)^l = 1$. Hence $\text{sgn}(-1)^l \xi_2 \phi(0) = -\text{sgn}(1)^l \xi_2 \phi(0)$. Recall that $f_0$ is the $p$-stabilization of the $p$-ordinary newform $f$ and $\tilde{f}_0 := f_0|_{\text{triv}}$. Thus, for $l \leq j \leq k-1$, we have

$$\frac{L(j, f_0, \xi_2 \phi)}{(2\pi i)^{j+1} G(\xi_2 \phi)\Omega^\phi_{\text{odd}}(j)} \cdot \frac{L(j + 1, f_0, \xi_2 \phi)}{(2\pi i)^{j+1} G(\xi_2 \phi)\Omega^\phi_{\text{odd}}(j)} \in \overline{Q}.$$

In view of Lemma 2.9, to compare the $p$-adic Rankin $L$-function $\mu_{f_0 \otimes h_0 \otimes \xi}$ with the product of $p$-adic $L$-functions corresponding to $f_0 \otimes \xi_1$ and $f_0 \otimes \xi_2$, it suffices to compare $c(f_0|_{\text{triv}})/(f_0|_{\text{triv}})^{p\text{triv}}(\Gamma_s|_{\text{triv}}) = c(f_0|_{\text{triv}})/(f_0|_{\text{triv}})^{p\text{triv}}(\Gamma_s|_{\text{triv}})$. For a Dirichlet character $\varphi$ modulo $N$, we denote by $L(n, \varphi; A)$ the $\Gamma_0(N)$-module $\text{Hom}(\mathbb{Z}_p^\times, \mathbb{Q}_p)/\mathbb{Q}$, which we recall is the stabilizer of $s$ in $\Gamma$. Define the parabolic cohomology

$$H^p_p(\Gamma_s, M) = \ker \left\{ \text{res} : H^p(\Gamma_s, \widetilde{M}) \to H^p(\partial \Gamma_s, \widetilde{M}) \right\},$$

where $\text{res}$ are the corresponding restriction maps. Under the isomorphism $H^1(\Gamma, M) \cong H^1(\Gamma, \overline{M})$, we have $H^p_p(\Gamma_s, \widetilde{M}) \cong H^p_p(\Gamma_s, \overline{M})$. There is a well-defined action of Hecke algebra on the cohomology groups $H^p_p(\Gamma_s, \widetilde{M})$ and $H^p_p(\Gamma_s, \overline{M})$ (see [S2, §8.3] or [H4, §6.3]).
Now this action coincides with the action defined on [H4, Page 177]. To ease the notation, we denote \( \widetilde{L(n, A)} \) by \( L(n, A) \) (resp. \( \mathbb{C} \)). Let \( S_k(\Gamma) \) be the space of anti-holomorphic cusp forms on \( \Gamma \), where \( \bar{\mathbb{C}} \) is the complex conjugate of \( \mathbb{C} \).

Theorem 3.3. ([Eichler-Shimura] [H4, §6.2, Theorem 1]) With the notation as above. The maps

\[
\delta : S_k(\Gamma) \to H^1_p(\Gamma, L(k-2, \mathbb{C})) \quad \text{and} \quad \delta : S_k(\Gamma) \to H^1_p(\Gamma, L(k-2, \mathbb{C}))
\]

are Hecke equivariant isomorphisms.

Lemma 3.4. Let \( F \in S_k(\Gamma(0), \eta) \) be a normalised eigenform. Let \( \varphi \) be a primitive Dirichlet character of conductor \( D \). Then we have

\[\int_{\mathbb{R}^2} \bar{\varphi}(F(z)) \omega(F(z)) \, dz = \int_{\mathbb{R}^2} \varphi(F(z)) \omega(F(z)) \, dz, \quad \forall \gamma \in \Gamma(0)(\mathbb{C}).\]

Theorem 3.5. ([H4, ¶3.2]) Let the notation be as above. For a field \( K \) of characteristic zero, we have \( H^p_p(\Gamma_0(N), L(n, \varphi; K)) \) is a free module of rank one over \( h_k(\Gamma_0(N), \varphi; K) \) (resp. \( h_k(\Gamma_0(N); K) \)).

Recall that for a number field \( K \) and a prime \( p \) in \( K \) dividing \( p \), \( \mathcal{O} = \mathcal{O}_K \) is the valuation ring of \( K \).
Suppose $1/n! \in A$. Define a pairing $\left[\cdot, \cdot\right]$ on $L(n, \varphi; A) \times L(n, \varphi; A) \to A$ by [H4, §6.2, (2a)]

$$\left[\sum_{j=0}^{n} a_j X^j Y^{n-j}, \sum_{j=0}^{n} b_j X^j Y^{n-j}\right] = \sum_{j=0}^{n} (-1)^j \binom{n}{j} a_j b_{n-j}.$$  

Note that $[(z + X + Y)^n, (z + X + Y)^n] = \sum_{j=0}^{n} (-1)^j \binom{n}{j} z^{n-j} = (z - \overline{z})^n$. The above pairing is perfect as $n!$ is invertible in $A$. Write $X_N := X_{\Gamma_0(N)}$ to ease the notation. By Poincaré duality, the above pairing $\left[\cdot, \cdot\right]$ induces a pairing ([H4, §6.2, (3a)!]):

$$H^1_\ast(X_N, \mathbb{L}(n, \varphi; \mathbb{C})) \times H^1_\ast(X_N, \mathbb{L}(n, \varphi; \mathbb{C})) \to H^2_\ast(X_N, \mathbb{C}) \cong \mathbb{C},$$

where the first map is the wedge product and the last map is integrating the 2-form on $X_N$. We continue to denote this pairing by $\left[\cdot, \cdot\right]$. Further, for $x \in H^1_\ast(\Gamma_0(N), (k - 2, \varphi; \mathbb{C}))$, define $(x|\tau_N)(\gamma) := \tau_N \cdot \gamma \tau_N^{-1}$, for all $\gamma \in \Gamma_0(N)$. This in turn defines an action of $\tau$ on $H^1_\ast(X_N, \mathbb{L}(n, \varphi; \mathbb{C})) \cong H^1(\Gamma_0(N), (n, \varphi; \mathbb{C}))$. Consider the pairing $\langle x, y \rangle := [x, y|\tau_N] = [x|\tau_N, y]$, where $\tau_N = \begin{psmallmatrix} 0 & -1 \\ 1 & 0 \end{psmallmatrix}$. We need the following version of [H3, Theorem 5.16]:

**Theorem 3.6.** Let $F(z) = \sum \alpha(n, F)q^n \in S_k(\Gamma_0(N), \varphi; \mathbb{K})$ be a normalised Hecke eigenform and $a(n, F) \in \mathcal{O}$. Let $\phi : H^1_\ast(\Gamma_0(N), (k - 2, \varphi; \mathbb{C})) \to \mathbb{K}$ be the homomorphism induced by $T(n) \to a(n, F)$. Fix a generator $\xi$ of $H^1_\ast(\Gamma_0(N), (k - 2, \varphi; \mathbb{C}))$ and $\beta \in \mathbb{K}, \alpha \in \mathbb{C}$.

(i) $\delta(F) = \pm \phi \cdot \beta \cdot \epsilon(F) = \alpha \beta \frac{\zeta}{\xi}.$

(ii) Further, we have $2^k \zeta + 2^{-1} (F^\varphi|\tau_N, F)_N = \alpha \beta \Omega_F \langle \xi^+, \xi^- \rangle.$

**Proof.** The first assertion follows from Lemma 3.5 and the fact that $H^1_\ast(\Gamma_0(N), (k - 2, \varphi; \mathbb{C})) = H^1_\ast(\Gamma_0(N), (k - 2, \varphi; \mathbb{C}))^\pm$. Further, we have $2^k \zeta + 2^{-1} (F^\varphi|\tau_N, F)_N = \alpha \beta \Omega_F \langle \xi^+, \xi^- \rangle.$ The assertion (ii) was proved in [H3, Theorem 5.16] when $F$ is primitive. The proof extends to this case, we omit the details.

**Remark 3.7.** The periods $\Omega_F^\pm$ chosen in Theorem 3.6 are some times refereed as canonical periods (cf. [H4, Page 187]).

We next show that the periods chosen in Theorem 3.6 have the property that the measure $\mu_F(\cdot)$ in Theorem 3.2 is $O$-valued. To do this, it suffices to show that the modular symbol attached to $F$, when multiplied by $1/\Omega_F^\pm$, is integral (see [MTT], [Ki]).

Let $\Delta = \text{Div}(P^1(\mathbb{Q}))$ denote the group of divisors generated on $P^1(\mathbb{Q})$. Let $\Delta_0$ denote the subgroup of $\Delta$ consisting of divisors of degree zero. Recall that $A$ is a ring with $1/2 \in A$ and $M$ is an $A[G]$-module. For $\gamma \in GL_2(\mathbb{Q}) \cap M_2(\mathbb{Z})$ and $[r] \in P^1(\mathbb{Q})$, define

$$\langle \gamma \cdot \Psi([r]) \rangle = \gamma \Psi(\{r\}) \cdot \gamma^{-1}, \quad \forall \Psi \in \text{Hom}_2(\Delta, \mathbb{M}) \text{ or } \text{Hom}_2(\Delta_0, \mathbb{M}).$$

Let $\text{Symb}_B(M) := \text{Hom}_B(\Delta_0, \mathbb{M})$ be the group of modular symbols and $\text{BSymb}_B(M) := \text{Hom}_B(\Delta, M)$ be the group of boundary modular symbols. There is a natural restriction map res : $\text{BSymb}_B(M) \to \text{Symb}_B(M)$. There is a well-defined action of Hecke algebra on $\text{Symb}_B(M)$ (see [MTT, §4]). By [GS, Theorem 4.3] (see also [Va, (9)]), we have the following exact sequence of Hecke modules

$$0 \to H^0(\Gamma, (L, n, A)) \to \text{BSymb}_B(L(n, A)) \xrightarrow{\text{res}} \text{Symb}_B(L(n, A)) \xrightarrow{\text{res}} H^1(\Gamma, L(n, A)) \to 0.$$  

(25)

Following [GS], [Ki] and [Va] we now show in Theorem 3.8 that the measure in Theorem 3.2 is $O$-valued with the choice of periods $\Omega^\pm_F$ as specified in Theorem 3.6.

**Theorem 3.8.** Let $F \in S_k(\Gamma_0(N), \varphi; \mathbb{O})$ be a normalised $p$-ordinary eigenform with the residual Galois representation at $p$ is irreducible. Let $\Omega_F^\pm$ be the periods as defined in Theorem 3.6. Then with these choices of periods $\Omega_F^\pm$, the measure $\mu_F$ satisfies the interpolation property (23) i.e. for every finite order character $\phi$ of $\mathbb{Z}/L\mathbb{Z}^\ast$, we have

$$\mu_F(x_0^j \phi) = \int_{(\mathbb{Z}/L\mathbb{Z})^\ast \times \mathbb{Z}_p^\ast} x_0^j \phi \frac{1}{u_F} \left(1 - \frac{\phi(p)\varphi(p)p^{k-2}}{u_F} \right) \left(1 - \frac{\phi(p^j)}{u_F} \right) \frac{\text{cond}(\phi)^{j+1} \text{val}(\phi) L(j + 1, F, \phi)}{(-2\pi i)^{j+1} G(\phi) \Omega_F^\pm \langle(-1)^{j+1} \cdot (-1)^{j+1} \rangle}.$$  

Furthermore, $\mu_F$ is an $O$-valued measure.

**Proof.** By [Va, 1.6], there exist $\Delta^\pm \in \text{Symb}_B(\Gamma)(L(k - 2, \mathbb{O}))$ with $\Theta(\Delta^\pm(F)) = \delta(F) \pm \Omega^\pm_F$. Let $\mu$ be the measure as defined in [GS, (4.16),(4.17)] (See [Ki, §4.2]), attached to these modular symbols $\Delta^\pm_F$. By
\[ Va, (11) \] and \[ GS, \text{ Theorem 4.18} \] (See also \[ Ki, \text{ Theorem 4.8} \]), it follows that \( \mu \) satisfies the interpolation formula given in (23). Hence the measure \( \mu_F \) obtained in Theorem 3.2 equals \( \mu \). That the measure \( \mu_F \) is an \( \mathcal{O} \)-valued measure follows from \[ Ki, \text{ Lemma 4.3} \].

**Definition 3.9. (Module of congruences)** Let \( R \) be a finite, flat and reduced algebra over \( A \) and \( B = \text{Frac}(A) \). Moreover, assume that we are given a map \( \lambda : R \to A \) such that it induces an \( A \)-algebra decomposition \( R \otimes_A B \cong B \otimes X \). Let \( 1_X \) be the idempotent corresponding to the first summand. Put \( a = \ker(R \to X) = 1_X R \cap R, \ S = \text{Im}(R \to X) \) and \( b = \ker(\lambda) \). The module of congruences \( C_0(\lambda) \) is defined by

\[
C_0(\lambda) = (R/a) \otimes_{R \otimes A} \mathcal{O}_K \cong \frac{R}{a} \oplus b \cong \lambda(R)/\lambda(a) \cong 1_X R/a \cong S/b.
\]

If \( R \) is Gorenstein i.e. \( \text{Hom}_A(R, A) \cong R \) as \( R \)-modules, then \( C_0(\lambda) = A/c(\lambda) \) for some \( c(\lambda) \in A \) (cf. \[ H5, \text{ Lemma 6.9} \]). In particular, we have \( c(\lambda)1_X \in R \). Note that \( c(\lambda) \) is well defined up to a unit in \( A \).

For \( F \in S_k(\Gamma_0(N), \varphi) \) and \( \Omega_F^{\pm} \) as in Theorem 3.8, we now determine \( \langle F|\tau_N, F \rangle_N/\Omega_F^{\pm}\Omega_F^\sigma \) explicitly up to a \( p \)-adic unit.

Recall that if \( F \in S_k(\Gamma_0(N), \varphi; K) \) is either a normalised \( p \)-ordinary newform or \( p \)-stabilization of a normalised \( p \)-ordinary newform, then the splitting \( \lambda_k(\Gamma_0(N), \varphi; K) \cong K_p \oplus X \) holds (See (8)) and we have the following result due to Hida (cf. \[ H3, \text{ Theorem 5.20} \]).

**Theorem 3.10.** Let \( p \) be odd and \( k \geq 3 \). Let \( F \in S_k(\Gamma_0(N), \varphi; K) \)  be either a normalised \( p \)-ordinary newform or \( p \)-stabilization of a normalised \( p \)-ordinary newform. We assume \( p > 3 \), if \( \Gamma_0(N)/\{ \pm 1 \} \) has non-trivial torsion elements. Let \( \tilde{p} \) be the residual representation of \( F \) and \( \Omega_F^{\pm} \) as in Theorem 3.8. If \( F \) satisfies \( \text{(irr.-f)} \) and \( \text{(p-dist)} \), we have

\[
c(F) = \frac{\Omega_F^{\pm}\Omega_F^\sigma}{2^{k-1}N^{k/2-1}(F|\tau_N, F)_N} \text{ is a } p \text{-adic unit.}
\]

We next compare the periods \( \Omega_F^{\pm}\Omega_F^\sigma \) and \( \langle (f_0|\chi)^{\sigma}|\tau_{N|f_0|}\chi, f_0|\chi \rangle_{N|f_0|} \).

**Lemma 3.11.** Let \( f \in S_k(\Gamma_0(N), \eta) \). Assume \( f \) satisfies \( \text{(p-dist)} \) and \( \text{(irr.-f)} \). Then the following quantity

\[
\omega(f_0|\chi)^{\sigma}|\tau_{N|f_0|}\chi, f_0|\chi \rangle_{N|f_0|} \text{ is a } p \text{-adic unit.}
\]

**Proof.** Applying Theorem 3.10 for \( f_0|\chi \), we have \( \langle (f_0|\chi)^{\sigma}|\tau_{N|f_0|}\chi, f_0|\chi \rangle_{N|f_0|} = p^{(2-k)/2}c(f_0|\chi)\Omega_f^+\Omega_f^- \) up to a \( p \)-adic unit. So we need to show that \( \Omega_f^+\Omega_f^- \) differs from \( \Omega_f^+\Omega_f^- \) by a \( p \)-adic unit. To ease the notation, denote the conductor of \( \chi \) by \( C_\chi \). Let \( \text{tw}_{N|f_0|}\chi : H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \to H^1_p(\Gamma_0(N|C_\chi|f_0), L(k-2, \eta \chi^2; W)) \) be defined as in (24). By Theorem 3.6, we have \( \delta(f_0|\chi)^{\pm}/\Omega_f^\pm \) is a basis of \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \). By Lemma 3.4, we have \( \text{tw}_{N|f_0|}\chi(\delta(f_0|\chi)^{\pm}/\Omega_f^\pm) = G(\chi)\delta(f_0|\chi)^{\pm}/\Omega_f^\pm \). Thus the image of \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \) under \( \text{tw}_{N|f_0|}\chi \) lies inside \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \).

Next consider the map \( \text{tw}_{N|f_0|}\chi : H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \to H^1_p(\Gamma_0(N|C_\chi|f_0), L(k-2, \eta \chi^2; W)) \). From Lemma 3.4, it follows that \( \text{tw}_{N|f_0|}\chi(\delta(f_0|\chi)^{\chi(-1)}/\Omega_f^\chi) = \text{tw}_{N|f_0|}\chi(\delta(f_0|\chi)^{\chi(-1)}/\Omega_f^\chi) = G(\chi)\delta(f_0|\chi)^{\chi(-1)/\Omega_f^\chi} \). Hence it is a \( p \)-adic unit. Finally, \( \text{tw}_{N|f_0|}\chi \) is surjective. As the \( \mathcal{W} \)-modules, \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \) and \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \) are rank 2, we get \( \text{tw}_{N|f_0|}\chi \) is an isomorphism. Since \( \delta(f_0|\chi)^{\chi(-1)/\Omega_f^\chi} \) is a basis of \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \), we get \( \text{tw}_{N|f_0|}\chi \) is surjective. As the \( \mathcal{W} \)-modules, \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \) and \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \) are rank 2, we get \( \text{tw}_{N|f_0|}\chi \) is an isomorphism. Since \( \delta(f_0|\chi)^{\chi(-1)/\Omega_f^\chi} \) is a basis of \( H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; W)) \cap H^1_p(\Gamma_0(N|f_0|), L(k-2, \eta \chi^2; K)) \), we get \( \text{tw}_{N|f_0|}\chi(\delta(f_0|\chi)^{\chi(-1)/\Omega_f^\chi} = G(\chi)\delta(f_0|\chi)^{\chi(-1)/\Omega_f^\chi} = G(\chi)\delta(f_0|\chi)^{\chi(-1)/\Omega_f^\chi} \). Since \( \delta(f_0|\chi)^{\chi(-1)/\Omega_f^\chi} \) is also
Lemma 3.12. Let \( f \in S_k(\Gamma_0(N), \eta) \) be a \( p \)-ordinary newform and \( \tilde{f}_0 = f_0|_{I_m} \). Assume \( f \) satisfies (\( p \)-dist) and (irr-f). Then the following quantity

\[
\frac{c(f_0|\chi)}{c(f_0)} \cdot \frac{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}
\]

is a \( p \)-adic unit. This proves the lemma.

Proof. Applying Lemma 3.10, the stated ratio equals \( \Omega_{\tilde{f}_0}^+ \Omega_{\tilde{f}_0}^- / \Omega_{f_0|\chi}^+ \Omega_{f_0|\chi}^- \) up to a \( p \)-adic unit. Consider the following maps

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(NP), L(k - 2, \eta; W)) \to H^1_p(\Gamma_0(NC^2p), L(k - 2, \eta^2; W))
\]

and

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)) \to H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)).
\]

As in Lemma 3.11, we can show

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(NP), L(k - 2, \eta; W)) \to H^1_p(\Gamma_0(NP), L(k - 2, \eta; W)) \to H^1_p(\Gamma_0(NP), L(k - 2, \eta^2; W)) \to H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)) \to H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)).
\]

are injective and \( tw_{NP, \chi} \) is an isomorphism. Further by Lemma 3.4, we have \( tw_{NP, \chi}(\delta(f_0)^\pm) = G(\chi)(\delta(f_0)|\chi)^{\pm \chi(\chi)} \) and \( tw_{NP, \chi}(\delta(f_0)|\chi)^{\pm \chi(\chi)} \). The following quantity

\[
\frac{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}
\]

is a \( p \)-adic unit. This proves the lemma.

Proof. Applying Lemma 3.10, the stated ratio equals \( \Omega_{\tilde{f}_0}^+ \Omega_{\tilde{f}_0}^- / \Omega_{f_0|\chi}^+ \Omega_{f_0|\chi}^- \) up to a \( p \)-adic unit. Consider the following maps

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(NP), L(k - 2, \eta; W)) \to H^1_p(\Gamma_0(NC^2p), L(k - 2, \eta^2; W))
\]

and

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)) \to H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)).
\]

are injective and \( tw_{NP, \chi} \) is an isomorphism. Further by Lemma 3.4, we have \( tw_{NP, \chi}(\delta(f_0)^\pm) = G(\chi)(\delta(f_0)|\chi)^{\pm \chi(\chi)} \) and \( tw_{NP, \chi}(\delta(f_0)|\chi)^{\pm \chi(\chi)} \). The following quantity

\[
\frac{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}
\]

is a \( p \)-adic unit. This proves the lemma.

Proof. Applying Lemma 3.10, the stated ratio equals \( \Omega_{\tilde{f}_0}^+ \Omega_{\tilde{f}_0}^- / \Omega_{f_0|\chi}^+ \Omega_{f_0|\chi}^- \) up to a \( p \)-adic unit. Consider the following maps

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(NP), L(k - 2, \eta; W)) \to H^1_p(\Gamma_0(NC^2p), L(k - 2, \eta^2; W))
\]

and

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)) \to H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)).
\]

are injective and \( tw_{NP, \chi} \) is an isomorphism. Further by Lemma 3.4, we have \( tw_{NP, \chi}(\delta(f_0)^\pm) = G(\chi)(\delta(f_0)|\chi)^{\pm \chi(\chi)} \) and \( tw_{NP, \chi}(\delta(f_0)|\chi)^{\pm \chi(\chi)} \). The following quantity

\[
\frac{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}
\]

is a \( p \)-adic unit. This proves the lemma.

Proof. Applying Lemma 3.10, the stated ratio equals \( \Omega_{\tilde{f}_0}^+ \Omega_{\tilde{f}_0}^- / \Omega_{f_0|\chi}^+ \Omega_{f_0|\chi}^- \) up to a \( p \)-adic unit. Consider the following maps

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(NP), L(k - 2, \eta; W)) \to H^1_p(\Gamma_0(NC^2p), L(k - 2, \eta^2; W))
\]

and

\[
tw_{NP, \chi} : H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)) \to H^1_p(\Gamma_0(Nf_1 \chi p), L(k - 2, \eta^2; W)).
\]

are injective and \( tw_{NP, \chi} \) is an isomorphism. Further by Lemma 3.4, we have \( tw_{NP, \chi}(\delta(f_0)^\pm) = G(\chi)(\delta(f_0)|\chi)^{\pm \chi(\chi)} \) and \( tw_{NP, \chi}(\delta(f_0)|\chi)^{\pm \chi(\chi)} \). The following quantity

\[
\frac{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}{\langle f_0|\tau_{NP}, \tilde{f}_0 \rangle_{NP}}
\]

is a \( p \)-adic unit. This proves the lemma.
Theorem 3.15. Let \( f \in S_k(\Gamma_0(N), \eta) \) be a \( p \)-ordinary and newform with \( p \nmid N \). Let \( g = E_i(\xi_2 \omega_p^{j-1}, \xi_1) \) and \( t(f, g, \chi) \) be as in Lemma 2.9. Set \( \tilde{f}_0 = f_0|_{m} \). Suppose that \( f \) satisfies the assumptions (irr-f) and (p-dist). Then for every finite order character \( \phi \) of \( \mathbb{Z}_p^* \) and \( 0 \leq \beta \leq k - l - 1 \), we have

\[
\mu_{f|\chi,\xi(g)(x_p^\beta)} \equiv (\ast) \mu_{f_0}(x_p^\beta \xi_1 \phi) \mu_{f_0}(x_p^\beta \xi_2 \phi) \mod \pi.
\]

Here \( (\ast) \) is the \( p \)-adic unit given by

\[
(\ast) = \frac{t(f|\chi,\xi(g)(x_p^\beta))}{\prod_{j=1}^k \Gamma(j+\beta) \Gamma(j+\beta+1)} \cdot \frac{p^{(2-k)/2} \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi}.
\]

Proof. By the \( p \)-adic Weierstrass preparation theorem, it is sufficient to show that the congruence holds for all finite order characters of \( \mathbb{Z}_p^* \) with \( \phi \neq (\xi_1 \omega_p^{j-1})_0 \). Note that \( f_p(g, p^\beta u_p^{j-1}) = (1 - \xi_2 \omega_p^{j-1}(p)p^\beta u_p^{j-1}) \cdot \chi_p f_0(x_p^\beta \xi_2 \phi) \), where \( \chi_p f_0(x_p^\beta \xi_2 \phi) \) is defined in Theorem 3.2 and \( u_f = a(p, f_0) = a(p, f_0) \). Thus by Lemma 2.9(ii), we get

\[
\mu_{f|\chi,\xi(g)(x_p^\beta)} = \left( \frac{t(f|\chi,\xi(g)(x_p^\beta))}{\prod_{j=1}^k \Gamma(j+\beta) \Gamma(j+\beta+1)} \cdot \frac{p^{(2-k)/2} \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \right) \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi}.
\]

If \( \phi \neq \xi_2 \phi \), then \( \xi_1 \omega_p^{j-1}(p) \phi = 0 \neq \xi_2 \phi \). Thus \( \chi_p f_0(x_p^\beta \xi_2 \phi) = 1 \). By Lemma 2.9(ii), then it follows from Lemma 2.9(ii) that

\[
\mu_{f|\chi,\xi(g)(x_p^\beta)} = \left( \frac{t(f|\chi,\xi(g)(x_p^\beta))}{\prod_{j=1}^k \Gamma(j+\beta) \Gamma(j+\beta+1)} \cdot \frac{p^{(2-k)/2} \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \right) \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi}.
\]

Applying Theorem 3.8 and choosing \( L = M_p \) to be the \( p \)-part of \( \text{cond}(\xi_1 \omega_p^{j-1}) \cdot \text{cond}(\xi_2 \phi) \), we obtain

\[
\mu_{f_0}(x_p^{j-1} \xi_1 \omega_p^{j-1} \phi) \mu_{f_0}(x_p^{j-1} \xi_2 \phi) \equiv \left( \frac{t(f|\chi,\xi(g)(x_p^\beta))}{\prod_{j=1}^k \Gamma(j+\beta) \Gamma(j+\beta+1)} \cdot \frac{p^{(2-k)/2} \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \right) \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \mod \pi.
\]

Substituting (28) in (26) and (27), we get

\[
\mu_{f|\chi,\xi(g)(x_p^\beta)} = \left( \frac{t(f|\chi,\xi(g)(x_p^\beta))}{\prod_{j=1}^k \Gamma(j+\beta) \Gamma(j+\beta+1)} \cdot \frac{p^{(2-k)/2} \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \right) \cdot \frac{\phi^\beta \xi_2 \phi}{\xi_2 \phi} \cdot \frac{\xi_2 \phi}{\xi_2 \phi} \mod \pi.
\]

Since \( x_p^{j-1} \xi_1 \omega_p^{j-1} \phi \equiv 1 \mod p \) and \( \mu_{f_0}(\phi) \) is \( \mathcal{O} \)-valued, we obtain \( \mu_{f_0}(x_p^{j-1} \xi_1 \omega_p^{j-1} \phi) \equiv \mu_{f_0}(x_p^{j-1} \xi_1 \phi) \mod \pi \). It remains to show
is a $p$-adic unit. The second term is a $p$-adic unit by Lemma 3.11. For the first quantity, using the formula (14), we get

$$\frac{W(g^p|\phi)}{W(g^p)} = \chi(-1)^{\frac{\cond(\xi\omega_p^{-1}\tilde{\chi})}{\cond(\xi_2\tilde{\phi})}} \frac{\cond(\xi_2\tilde{\phi})^{1/2} G(\xi_1\omega_p^{-1}\tilde{\chi})}{\cond(\xi_2\tilde{\phi})^{1/2} G(\xi_1\omega_p^{-1}\tilde{\chi})} \frac{\chi_2(\tilde{\phi})}{\chi_2(\phi)}.$$

Since $\chi$ has conductor prime to $p$, we have $\cond(\xi_1\omega_p^{-1}\tilde{\chi})/\cond(\xi_2\tilde{\phi})$ and $\cond(\xi_2\tilde{\phi})/\cond(\xi_2\tilde{\phi})$ are co-prime to $p$ for all $\phi$. By Lemma 3.14, it follows $G(\xi_1\omega_p^{-1}\tilde{\chi})/G(\xi_2\tilde{\phi})$ and $G(\xi_1\omega_p^{-1}\tilde{\chi})/G(\xi_1\omega_p^{-1}\tilde{\chi})$ are $p$-adic units. Observe that $t(f|\chi, g|\chi)/\Gamma(l+j)\Gamma(j) = [N_f|\chi, N_g|\chi]N_f^{k/2}(N_g|\chi)^{(l+2j)/2}$ is a $p$-adic unit. Now the theorem follows. \qed

Remark 3.16. Note that for every Dirichlet character $\varphi$, we have $L(j, \tilde{f}_0, \varphi) = \left(1 - \frac{\mu(\varphi)p^{k-1-j}}{\mu_j}\right) L(j, \tilde{f}_1, \varphi)$. Taking $\Omega^\pm := \Omega_{\tilde{f}_0}^\pm$ in Theorem 3.2, we get a bounded measure $\mu_j$. It follows that $\mu_j = \mu_{\tilde{f}_0}$, with $\mu_{\tilde{f}_0}$ as defined in Theorem 3.8. So we may replace the $O$-valued measure $\mu_{\tilde{f}_0}$ in Theorem 3.15 by $O$-valued measure $\mu_j$.

We next show that the root number of a primitive form $h$ twisted by $\chi$ is equal to the root number of $h$ up to a $p$-adic unit.

Lemma 3.17. Let $h \in S_l(I, \psi)$ be a $p$-ordinary newform. Then for every finite order character $\phi$ of $\mathbb{Z}_p^\times$ we have, $W(h^p|\phi)/W(h^p)$ is a $p$-adic unit.

Proof. As in [H2, §5] we write $W(h^p|\phi) = W_p(h^p|\phi)W'(h^p|\phi)$, where $W_p(h^p|\phi)$ is the local $\varepsilon$-factor of $h^p|\phi$ at $p$. Similarly, let $W(h^p|\chi) = W_p(h^p|\chi)/W'(h^p|\chi)$. From [H2, (5.4 a)-(5.4 c)], it follows that $W(h^p|\phi), W'(h^p|\phi)$ are $p$-adic units. Hence it is enough to show that $W_p(h^p|\chi)/W(h^p|\phi)$ is a $p$-adic unit.

Let $I = I_0p^a$ with $p \nmid I_0$. Note that $h$ is $p$-minimal and the local automorphic representation of $h$ at $p$, say $\pi_p(h)$, is not supercuspidal. Also we have $\pi_p(h)$ is principal series (respectively special) implies that $\pi_p(h|\chi)$ is principal series (respectively special). Thus by [H2, (5.5 b)], we have

$$W_p(h|\chi) = \begin{cases} \chi(p)^{\alpha}W_p(h) & \text{if } \pi_p(h) \text{ is principal}, \\
\chi(p)W_p(h) & \text{if } \pi_p(h) \text{ is special}. \end{cases}$$

If $\phi$ is non-trivial character of $\mathbb{Z}_p^\times$, then it follows by [H2, (5.5 c)] that

$$W_p(h^p|\chi) = \begin{cases} \chi(p)^{\nu_p(\cond(\phi))}W_p(h|\phi) & \text{if } \pi_p(h) \text{ is principal}, \\
\chi(p)^{\nu_p(\cond(\phi))}W_p(h|\phi) & \text{if } \pi_p(h) \text{ is special}. \end{cases}$$

Thus from above we have $W_p(h^p|\chi)$ differs from $W_p(h^p|\phi)$ by a $p$-adic unit. \qed

Let $\Sigma_0$ be the set of all rational primes dividing $m$ and $\mu_{f \times h}$ be the $O$-valued measure as in Theorem 1.2. For every finite order character $\phi$ of $\mathbb{Z}_p^\times$ and $0 \leq j \leq k - l - 1$, set

$$\mu_{f \times h}^{\Sigma_0}(x_p^j|\phi) := \mu_{f \times h}(x_p^j|\phi) \times \prod_{r \in \Sigma_0} L_r(l + j, \tilde{f}_0, g|\phi). \quad (30)$$

We next show that the $p$-adic $L$-function $\mu_{f|\chi \times h|\chi}$ generates the same ideal as the imprimitive $p$-adic $L$-function $\mu_{f \times h}^{\Sigma_0}$.

Lemma 3.18. Let the notation be as in Lemma 2.8. Then for every finite order character $\phi$ of of $\mathbb{Z}_p^\times$ and $0 \leq j \leq k - l - 1$, the values $\mu_{f|\chi \times h|\chi}(x_p^j|\phi)$ and $\mu_{f \times h}^{\Sigma_0}(x_p^j|\phi)$ differ by a $p$-adic unit.

Proof. By Lemma 3.10 and Lemma 3.12, we have

$$\frac{c(f_0|\chi)}{c(f_0)} \langle f_0 |--(f_0) \rangle \tau_{N_p, f_0}N_p$$

is a $p$-adic unit. By Lemma 3.17, we have $W(h^p|\phi)/W(h^p)$ is $p$-adic unit. Now the lemma follows from Lemma 2.8. \qed
Let \( \mu_{f_0}(\cdot) \) be the \( p \)-adic \( L \)-function as defined in Theorem 3.8. For \( 0 \leq j \leq k - 2 \) and Dirichlet character \( \varphi \), consider the measure \( \mu_{p,f,\varphi,j}^\Sigma \) on \( 1 + p\mathbb{Z}_p \) satisfying the interpolation property

\[
\mu_{p,f,\varphi,j}^\Sigma (\phi) = \mu_{f_0}(x_p^j \varphi \phi) = \frac{e_p(u_f, \varphi x_p^j)}{u_f \Gamma(\varphi \phi)} \frac{\text{cond}(\varphi \phi)\ell+1 \ell!}{L(j+1, f, \varphi \phi)} \frac{(2\pi i)^{\ell+1} \Omega_{f_0}^{\text{irr}}(-1)^{\ell}(\varphi \phi(-1))}{(2\pi i)^{\ell+1} \Omega_{f_0}^{\text{irr}}(-1)^{\ell}(\varphi \phi(-1))}
\]

(31)

for every finite order character \( \phi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times \) of \( p \)-power order.

For \( l - 1 \leq j \leq k - 2 \), consider the measure \( \mu_{p,f,h,j}^\Sigma \) on \( 1 + p\mathbb{Z}_p \) satisfying the interpolation property:

\[
\mu_{p,f,h,j}^\Sigma (\phi) = \mu_{f_0}^\Sigma (x_p^j \varphi \phi) = \mu_{f,h,j}^\Sigma (x_p^j \varphi \phi) = \prod_{r \in \Sigma} L_r(j+1, f_0, g | \phi)
\]

(32)

for every finite order character \( \phi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times \) of \( p \)-power order. Recall that \( \mathbb{Q}_{\text{cyc}} \) is the cyclotomic \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \) and \( \Gamma = \text{Gal}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q}) \cong \mathbb{Z}_p \). Then it is known that \( \mu_{p,f,\varphi,j}^\Sigma \) (resp. \( \mu_{p,f,h,j}^\Sigma \)) corresponds to a power series in the Iwasawa algebra \( \mathcal{O}[\Gamma] \) which we continue to denote by \( \mu_{p,f,\varphi,j}^\Sigma \) (resp. \( \mu_{p,f,h,j}^\Sigma \)).

**Theorem 3.19.** Let \( f \in S_k(\Gamma_0(N), \eta) \) be a \( p \)-ordinary newform with \( p \nmid N \). Let \( h \in S_l(\Gamma_0(I), \psi) \) be a \( p \)-ordinary eigenform such that \( 2 \leq l < k \) and \( (T_h/\pi)^{\mathbb{A}} \cong \xi_1 \otimes \xi_2 \). Suppose that \( f \) satisfies the assumptions \((\text{irr-f})\) and \((\text{p-dist})\). Then for \( l - 1 \leq j \leq k - 1 \), we have the following congruence of ideals in the Iwasawa algebra \( \mathcal{O}[\Gamma] \)

\[
(\mu_{p,f,h,j}^\Sigma) \equiv (\mu_{p,f,\xi_1,j}^\Sigma)(\mu_{p,f,\xi_2,j}^\Sigma) \mod \pi.
\]

**Proof.** This follows from Lemma 2.6, Theorem 3.15 and Lemma 3.18. \( \square \)

4. Selmer group of Modular form and Rankin-Selberg product

In this section, we discuss the \( p^\infty \)-Greenberg Selmer group attached to a modular form and the Rankin-Selberg product. We prove a control theorem for the \( p^\infty \)-Selmer group attached to the Rankin-Selberg product and also discuss explicit description of various \( p^\infty \)-Selmer groups that appear.

4.1. Background on Selmer groups

We recall the following notation from \( \S 2 \): For a normalised eigenform \( F \) with nebentypus \( \varphi \), \( K_F \) denotes the corresponding number field. Let \( L \) be a number field containing \( K_F \), \( p \) is a prime ideal in \( L \) dividing \( p \) (compatible with \( \varphi_p \)), \( L_p \) be the completion of \( L \) at \( p \) and \( \pi_L \) denote a uniformizer of the ring of integers \( \mathcal{O}_L \) of \( L_p \). Also, recall \( \chi_{\text{cyc}} : \mathbb{Q}_p \to \mathbb{Z}_p^\times \) is the \( p \)-adic cyclotomic character. For a discrete \( \mathbb{Z}_p[\Gamma] \)-module \( \mathcal{M} \), let \( \mathcal{M}^\wedge := \text{Hom}_{\text{cont}}(\mathcal{M}, \mathbb{Q}_p) \) be the Pontryagin dual of \( \mathcal{M} \).

**Theorem 4.1.** (Eichler, Shimura, Deligne, Mazur-Wiles, Wiles) Let \( F(z) = \sum_{n = 1}^{\infty} a(n, F)q^n \in S_k(\Gamma_0(A), \varphi) \) be a \( p \)-ordinary newform with \( k \geq 2 \). Then there exists a Galois representation \( \rho_F : \mathbb{G}_L \to \text{GL}_2(L_p) \) such that

(i) For all primes \( r \nmid A_p \), \( \rho_F \) is unramified and for the (arithmetic) Frobenius \( \text{Frob}_r \), at \( r \), we have

\[
\text{trace}(\rho_F(\text{Frob}_r)) = a(r, F), \quad \det(\rho_F(\text{Frob}_r)) = \varphi(\chi_{\text{cyc}}(\text{Frob}_r))^{k-1} = \varphi(r)^{k-1}.
\]

It follows (by the Chebotarev density theorem) that \( \det(\rho_F) = \varphi(\chi_{\text{cyc}})^{k-1} \).

(ii) As \( F \) is \( p \)-ordinary, let \( \text{up} \) be the unique \( p \)-adic unit root of \( X^2 = a(p, F)X + \varphi(p)^{k-1} \). Let \( \lambda_F \) be the unramified character with \( \text{Frob}_p \mid \text{up} \). Then

\[
\rho_F|_{G_p} \sim \left( \begin{array}{cc} \lambda_F^{-1} & \varphi(\chi_{\text{cyc}}) \\ 0 & \lambda_F \end{array} \right).
\]

Let \( V_F \cong L_p^{\oplus 2} \) denote the representation space of \( p \rho_F \). By compactness of \( \mathbb{G}_L \), there exists a \( G_\mathbb{Q} \) invariant \( \mathcal{O}_{L_p} \)-lattice \( T_F \) of \( V_F \). Let \( \rho_F : G_\mathbb{Q} \to \text{GL}_2(\mathcal{O}_{L_p}/\pi_L) \) be the residual representation of \( p \rho_F \).

Recall \( f \in S_k(\Gamma_0(N), \eta) \) and \( h \in S_l(\Gamma_0(J_0p^\infty), \psi) \) are \( p \)-ordinary newforms with \( p \nmid N \). Let \( K \) be a number field containing \( K_f \) and \( K_h \). To ease the notation, we denote \( \mathcal{O}_{K_0} \) by \( \mathcal{O}_K \) by \( \pi_K \) by \( \pi \) and \( \mathcal{O}/\pi \) by \( \mathbb{F} \). Let \( \Sigma \) be a finite set of primes of \( \mathbb{Q} \) such that \( \Sigma \nmid \{\ell \mid \ell \nmid pf\} \) and \( \mathbb{Q}_\Sigma \) be the maximal algebraic extension of \( \mathbb{Q} \) unramified outside \( \Sigma \). Set \( G_\Sigma(\mathbb{Q}) := \text{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}) \). Let us take \( \mathfrak{g} \in \{f, h\} \) and set \( A_\mathfrak{g} := V_\mathfrak{g}/T_\mathfrak{g} \). For a character \( \varphi \), let \( V_\mathfrak{g}(\varphi) := V_\mathfrak{g} \otimes_{\mathbb{Q}_p} \varphi \) and \( T_\mathfrak{g}(\varphi) := T_\mathfrak{g} \otimes_{\mathbb{Q}_p} \varphi \). Further, put
where both $T^+_g$ (resp. $V^+_g$) and $T^+_g$ (resp. $V^+_g$) are free $\mathcal{O}$-modules (resp. $K_{p}$-vector spaces) of rank 1 and the action of $G_0$ on $T^+_g$ and $V^+_g$ is unramified at $p$. Following [G1], for a modular form $g$ of weight $k \geq 2$, define a filtration on $V_{g}(j)$ with $0 \leq j \leq k - 2$, by

$$
F^s(V_{g}(j)) = \begin{cases} 
V_{g}(j) & \text{if } s \leq -j, \\
V_{g}^+(j) & \text{if } -j + 1 \leq s \leq k - 1 - j, \\
0 & \text{if } s \geq k - j.
\end{cases}
$$

We have a corresponding filtration on $T_{g}(j)$. Also, define $A^+_g(j) = (V_{g}^+(j)/T_{g}^+(j))$ and $A^-_g(j) = A^+_g(j)/A^+_g(j)$.

Set $V := V_f \otimes K_v V_h$ and $T := T_f \otimes \mathcal{O} T_h$. Then we have an induced filtration on $V$ and $T$ respectively:

$$
0 \subset V^+_f \otimes \mathcal{K}_{p} V^+_h \subset V^+_f \otimes \mathcal{K}_{p} V_h \subset V^+_f \otimes \mathcal{K}_{p} V_h + V_f \otimes \mathcal{K}_{p} V^+_h \subset V_f \otimes \mathcal{K}_{p} V_h,
$$

$$
0 \subset T^+_f \otimes \mathcal{T}_{h} \subset T^+_f \otimes \mathcal{T}_{h} + T_f \otimes \mathcal{O} T_h \subset T_f \otimes \mathcal{O} T_h.
$$

For every $l - 1 \leq j \leq k - 2$, we define $V_{j} := V \otimes_{\mathcal{K}_{p}} \chi_{\text{cyc}}(j) T_{j} := T \otimes_{\mathcal{O}} \chi_{\text{cyc}}(j)$ and $A_{j} := \frac{V_{j}}{T_{j}}$. Note that $A_j = \frac{T_f(\chi_{\text{cyc}}(j)) \otimes \mathcal{O} A_h \otimes T_h(\chi_{\text{cyc}}(j)) \otimes \mathcal{O} A_f}$. The action of $I_p$ on the successive quotients in our filtration (34) is given by $I_{p}^{\xi} \cdot \chi_{\text{cyc}}^{j} \cdot I_{p}^{L} \cdot \chi_{\text{cyc}}^{j}$. In particular, for $l - 1 \leq j \leq k - 2$, we have the following filtration of $V_{j} = V_f \otimes V_{h}(j)$

$$
F^s(V_{j} \otimes V_{h}(j)) = \begin{cases} 
V_{j} \otimes V_{h}(j) & \text{if } s \leq -j, \\
V_{j}^+ \otimes V_{h}(j) + V_{j} \otimes V_{h}^+(j) & \text{if } -j + 1 \leq s \leq l - 1 - j, \\
V_{j}^+ \otimes V_{h}(j) & \text{if } l - j \leq s \leq k - 1 - j, \\
V_{j}^+ \otimes V_{h}^+(j) & \text{if } k - j \leq s \leq k + l - 2 - j, \\
0 & \text{if } s \geq k + l - 1 - j.
\end{cases}
$$

Similarly, define a filtration on $T_{f} \otimes T_{h}(j)$. Following [G1], we have $F^+(V_{j}) = V_{j}^+ \otimes V_{h}(j)$ and $F^+(T_{j}) = T_{j}^+ \otimes T_{h}(j)$. Finally, we put

$$
F^+(A_{j}) := F^+(V_{j})/F^+(T_{j}) \quad \text{and} \quad A^{-}_{j} = A_{j}/F^+(A_{j}).
$$

Note that $A^{-}_{j} = T_{h} \otimes A^{-}_{f}(j)$, where $A^{-}_{j}(j) = A_{f}(j)/A^{-}_{j}(j)$.

Let $L$ be a number field and $\Sigma_{L}$ be the set of all primes in $L$ lying above $\Sigma$. Put $G_{0}(L) = \text{Gal}(L_{\Sigma_{L}}/L)$. For every prime $v \in \Sigma_{L}$ and $B_{j} \in \{A_{j}(\xi_{v}\omega_{p}^{-1}), A_{j}(\xi_{v}\omega_{p}^{-2})\}$, let choose a submodule $H_{1}^{l}(L_{v}, B_{j}) \subset H^{1}(L_{v}, B_{j}) := H^{1}(G_{L_{v}}, B_{j})$. For this choice, we define $\hat{v}$-Selmer group $S_{1}(B_{j}/L)$ as

$$
S_{1}(B_{j}/L) := \ker \left( H^{1}(G_{\Sigma_{L}}, B_{j}) \rightarrow \prod_{v \in \Sigma_{L}} H^{1}(L_{v}, B_{j}) \right).
$$

For any $r \in \mathbb{N}$, there is a natural inclusion

$$
0 \rightarrow B_{j}[\pi^{r}] \overset{i_{v}}{\rightarrow} B_{j}
$$

Next we define $\pi_{r}$ $\hat{v}$-Selmer group $S_{1}(B_{j}[\pi^{r}]/L)$ as

$$
S_{1}(B_{j}[\pi^{r}]/L) := \ker \left( H^{1}(G_{\Sigma_{L}}, B_{j}[\pi^{r}]) \rightarrow \prod_{v \in \Sigma_{L}} H^{1}(L_{v}, B_{j}[\pi^{r}]) \right),
$$

where $H^{1}(L_{v}, B_{j}[\pi^{r}]) := i_{v}^{-1}(H^{1}(L_{v}, B_{j}))$ for every $v \in \Sigma_{L}$. Here $i_{v}^{*} : H^{1}(L_{v}, B_{j}[\pi^{r}]) \rightarrow H^{1}(L_{v}, B_{j})$ is induced from $i_{v}$ in (36). For a prime $v$ in $L$, let $G_{L_{v}}$ and $I_{v}$ denote the decomposition group and inertia group at $v$ respectively. For a $G_{L}$-module $M$ we sometimes denote the $i^{th}$-cohomology group $H^{i}(G_{L}, M)$ by $H^{i}(L, M)$. Similarly, for fields $L' \subset L \subset \bar{Q}$, let $H^{i}(L/L', M) := H^{i}(\text{Gal}(L/L'), M)$. 
Definition 4.2. Let $B_j \in \{A_f(\xi_1\omega_p^{-j}), A_j, A_f(\xi_2\omega_p^{-j})\}$, $L$ be a number field and $v \in \Sigma_L$. For $\dagger \in \{\text{Gr, str}\}$, define
\[
H^1_{\dagger}(L, B_j) := \begin{cases}
\text{Ker}(H^1(G_L, B_j) \to H^1(I_v, B_j)) & \text{if } v \nmid p \text{ and } \dagger \in \{\text{Gr, str}\}, \\
\text{Ker}(H^1(G_L, B_j) \to H^1(I_v, B_j)) & \text{if } v | p \text{ and } \dagger = \text{Gr}, \\
\text{Ker}(H^1(G_L, B_j) \to H^1(G_v, B_j)) & \text{if } v | p \text{ and } \dagger = \text{str}.
\end{cases}
\]

Recall that $Q_{\text{cyc}}$ is the cyclotomic $\mathbb{Z}_p$-extension of $Q$ and $\Gamma := \text{Gal}(Q_{\text{cyc}}/Q) \cong \mathbb{Z}_p$. Set $Q_n := Q_{\text{cyc}}^{\mathbb{Z}_p^n}$ for every $n \geq 1$. Define 
\[
S_t(B_j/Q_{\text{cyc}}) := \lim_{\pi} S_t(B_j/Q_n) \quad \text{and} \quad S_t(B_j[\pi^n]/Q_{\text{cyc}}) := \lim_{\pi} S_t(B_j[\pi^n]/Q_n).
\]

Let $Q_{\text{cyc}, w}$ denote the completion at $w$ for a prime $w$ in $Q_{\text{cyc}}$ and set $\Sigma^\infty$ to be the primes in $Q_{\text{cyc}}$ lying above $\Sigma$. It can be checked that for $r \geq 1$, we have 
\[
S_t(B_j[\pi^n]/Q_{\text{cyc}}) = \text{Ker}\left(H^1(G_{Q_{\text{cyc}}}, B_j[\pi^n]) \to \prod_{w \in \Sigma^\infty} H^1(Q_{\text{cyc}, w}, B_j[\pi^n])\right),
\]
where $H^1(Q_{\text{cyc}, w}, B_j[\pi^n])$ is as defined in Definition 4.2.

Theorem 4.3. The kernel and the cokernel of the map $S_t(A_j/Q_n) \to S_t(A_j/Q_{\text{cyc}})^{\mathbb{Z}_p^n}$ are finite and uniformly bounded independent of $n$, for $\dagger \in \{\text{str, Gr}\}$.

**Proof.** Consider the commutative diagram
\[
\begin{array}{ccc}
0 & \to & S_t(A_j/Q_n) \\
\downarrow & & \downarrow \\
0 & \to & S_t(A_j/Q_{\text{cyc}})^{\mathbb{Z}_p^n} \\
\end{array}
\]
\[
\begin{array}{ccc}
H^1(G_{Q_n}, A_j) & \to & \prod_{v \in \Sigma^\infty} H^1(Q_{\text{cyc}, w}, A_j[\mathbb{Z}_p^n]) \\
\downarrow & & \downarrow \\
H^1(G_{Q_{\text{cyc}}}, A_j[\mathbb{Z}_p^n]) & \to & \prod_{v \in \Sigma^\infty} H^1(Q_{\text{cyc}, w}, A_j[\mathbb{Z}_p^n]).
\end{array}
\]

To show $\ker(r_{\Gamma,n})$ is finite and uniformly bounded for every $n$, by [Oc, Theorem 3.5(1)], it is enough to show that $H^0(Q_n, V_f) = 0$, for all $n$. In fact, it suffices to show that $H^0(Q_{n,v}, V_f) = 0$, for $v | p$. This will follow if we can establish for $v | p$

\[
H^0(Q_{n,v}, V_f^+ \otimes V_h(-j)) = 0 = H^0(Q_{n,v}, V_f^- \otimes V_h(-j)).
\]

Further, to establish the second equality in (37), it suffices to show, for $v | p$

\[
H^0(Q_{n,v}, V_f^+ \otimes V_h^+-(-j)) = 0 = H^0(Q_{n,v}, V_f^- \otimes V_h^+(j)).
\]

Let $K' = Q_{n,v}(\mu_{p^n}) = Q_{n,v}(\mu_{p^n})$. Then $G_{K'}$ acts on $V_f^+ \otimes V_h^+(j)$ and $V_f^- \otimes V_h^+(j)$ by $\lambda_f \lambda_h^j \theta$ and $\lambda_f \lambda_h^j \theta$ respectively where $\theta$ is a finite order character. Note that as $f, h$ are ordinary at $p$, by Ramakrishnan–Petersson conjecture, $|\lambda_f(\text{Frob}_v)| = p_{v}^{\frac{1}{2g}}$ and $|\lambda_h(\text{Frob}_v)| = p_{v}^{-\frac{1}{2g}}$, where $p_v$ is the cardinality of the residue field at $v$. Thus looking at the action of Frobenius at the prime $v$ dividing $p$, we deduce

\[
H^0(G_{K'}, V_f^+ \otimes V_h^+(j)) = H^0(G_{K'}, V_f^- \otimes V_h^+(j)) = 0 \quad \text{and} \quad (38) \text{ follows.}
\]

We can deduce the first vanishing result in (37) similarly. Putting these together, we get $\ker(r_{\Gamma,n})$ is finite and uniformly bounded.

Next we show that $\text{coker}(r_{\text{str},n})$ is finite and uniformly bounded independent of $n$. Let $FR_{v,n}$ be a fixed lift of Frobenius in $G_{K_{n,v}}$. Again by [Oc, Theorem 3.5(2)], it is enough to show for all places $v | p$ in $K$, the action of $FR_{v,n}$ is non-trivial on the successive quotients of the filtration $V_f^+ \otimes \chi_{\text{cyc}}^{j\pm j}$. In our case, the action of $FR_{v,n}$ on $F_{v,\text{str}}^+ \otimes \chi_{\text{cyc}}^{j\pm j}$ is given by $\lambda_f^{j\pm 1} \lambda_h^{j\pm 1} \theta$ where $\theta$ is a finite order character. As before, we can show that the action of $FR_{v,n}$ is non-trivial. Hence we deduce $\text{coker}(r_{\text{str},n})$ is finite and uniformly bounded independent of $n$.

Next we will show $\text{coker}(r_{\text{Gr},n})$ is finite and uniformly bounded independent of $n$. The local condition defining the strict Selmer differs with the Greenberg Selmer only at primes dividing $p$. Let $\phi_{\text{Gr,str}}$ be the natural map $S_{\text{str}}(A_j/Q_{\text{cyc}}) \to S_{\text{Gr}}(A_j/Q_{\text{cyc}})$. From the definitions of Greenberg and strict Selmer group, for every $n$, there is a natural map from $\text{coker}(r_{\text{str},n}) \to \text{coker}(r_{\text{Gr},n})$ such that the order of the cokernel is bounded by the order of $\text{coker}(\phi_{\text{Gr,str}})$. Thus it suffices to show $\text{coker}(\phi_{\text{Gr,str}})$ is finite. Now the order of the cokernel is bounded by the order of $\displaystyle \bigoplus_{w | p} H^1(Q_{\text{cyc}, w}/Q_{\text{cyc}, w}, A_j^{\dagger})$ where $w$ is a prime in $Q_{\text{cyc}}$ dividing $p$ and $I_w$ is the inertia subgroup of $Q_p/Q_{\text{cyc}, w}$ at $w$. As $\text{Gal}(Q_{\text{cyc}, w}/Q_{\text{cyc}, w})$
is topologically cyclic, it suffices to show $H^0(Q_{\Sigma}^{\text{cycl}}/Q_{\Sigma}^{\text{cycl}}, A_j^{\text{cycl}}) = A_j^{\text{cycl}}$ is finite for each $w \mid p$. Thus it further reduces to show $H^0(Q_{\Sigma}^{\text{cycl}}, V_j) = 0$. This follows from the proofs of (37), (38) written above.

For any subset $\Sigma' \subseteq \Sigma$, such that $p \notin \Sigma'$ and $B_j \in \{A_j(\xi_1^{w_p^{-1}}), A_j, A_j(\xi_2^{w_p^{-1}})\}$ we define
\[S_{\text{Gr}}(B_j[\pi]/L) := \text{Ker} \left( H^1(Q_{\Sigma}/L, B_j[\pi]) \rightarrow \prod_{\nu \in \Sigma \setminus \Sigma'} H^1(L, B_j[\pi]) \right),\]
where $\Sigma'_L$ is the set of all primes in $L$ lying above $\Sigma'$.

Let $B_j \in \{A_j(\xi_1^{w_p^{-1}}), A_j, A_j(\xi_2^{w_p^{-1}})\}$. Note that if $H^0(G_L, B_j[\pi]) = 0$, then the natural map $H^1(Q_{\Sigma}/L, B_j[\pi]) \rightarrow H^1(Q_{\Sigma}, B_j[\pi])$ is an isomorphism. The following lemma is immediate from the definition of $H^1_{\text{Gr}}(L, B_j[\pi])$.

**Lemma 4.4.** Let $B_j \in \{A_j(\xi_1^{w_p^{-1}}), A_j, A_j(\xi_2^{w_p^{-1}})\}$. Assume $H^0(G_L, B_j) = 0$. Then the natural map $S_{\text{Gr}}(B_j[\pi]/L) \rightarrow S_{\text{Gr}}(B_j[\pi])$ is an isomorphism.

### 4.2. Explicit description of Selmer Groups

Recall that $f \in S_2(\Gamma_0(N), \eta)$, $h \in S_2(\Gamma_0(I), \psi)$ and let $m$ be as in (19). From now on, we choose and fix $\Sigma := \{\ell : \ell \mid N\}$. Also recall $\Sigma_\infty := \{\ell : \ell \mid m\}$. Let $\Sigma^{\infty}$ (resp. $\Sigma_\infty^{\infty}$) be the set of primes in $Q_{\Sigma}^{\text{cycl}}$ lying above $\Sigma$ (resp. $\Sigma_\infty$). In this subsection, we give a more explicit description of $H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j[\pi])$ and $H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j(\xi_1^{w_p^{-1}})[\pi])$ for $w \in \Sigma_\infty \setminus \Sigma_\infty^{\infty}$.

**Proposition 4.5.** Let $w \in \Sigma^{\infty}$ and $w \mid p$. Assume the order of $\psi|_{I_{\text{cycl}}}^{\text{cycl}}$ is co-prime to $p$. Then we have
\[H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j[\pi]) = \text{Ker} \left( H^1(G_{Q_{\Sigma}^{\text{cycl}}}, A_j[\pi]) \rightarrow H^1(I_{\text{cycl}}, A_j[\pi]) \right).
\]

**Proof.** First note that by definition of $H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j[\pi])$, we have the following exact sequence
\[0 \rightarrow H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j[\pi]) \rightarrow H^1(G_{Q_{\Sigma}^{\text{cycl}}}, A_j[\pi]) \rightarrow H^1(I_{\text{cycl}}, A_j[\pi]).\]

Consider the commutative diagram:
\[
\begin{array}{cccc}
H^1(G_{Q_{\Sigma}^{\text{cycl}}}, A_j[\pi]) & \xrightarrow{\kappa} & H^1(I_{\text{cycl}}, A_j[\pi]) \\
\downarrow{\cong} & & \downarrow{\kappa} \\
H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j[\pi]) & \xrightarrow{\kappa} & H^1(I_{\text{cycl}}, A_j[\pi]).
\end{array}
\]

From the diagram (39), we see that $H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j[\pi]) = \text{Ker}(\kappa) = \nu^{-1}(\text{Ker}(\kappa))$. Thus it suffices to show $\text{Ker}(\kappa) = 0$. As $T_{\mathfrak{f}}$ is unramified at $v$, we have $H^1(I_{\text{cycl}}, A_j[\pi]) \cong H^1(I_{\text{cycl}}, A_j[\pi] \otimes T_{\mathfrak{f}})$ and $H^1(I_{\text{cycl}}, A_j[\pi]) \cong H^1(I_{\text{cycl}}, A_j[\pi] \otimes T_{\mathfrak{f}})$. Let $\epsilon'$ be the natural map $H^1(I_{\text{cycl}}, A_j[\pi]) \rightarrow H^1(I_{\text{cycl}}, A_j[\pi])$, further, it suffices to show $\text{Ker}(\epsilon') = 0$.

As $I_{\text{cycl}}$ has $p$-th cohomological dimension 1, $H^2(I_{\text{cycl}}, A_j[\pi]) = 0 = H^2(I_{\text{cycl}}, A_j[\pi])$ and we have
\[
H^0(I_{\text{cycl}}, A_j[\pi]) \rightarrow H^1(I_{\text{cycl}}, A_j[\pi]) \rightarrow H^1(I_{\text{cycl}}, A_j[\pi]) \rightarrow H^1(I_{\text{cycl}}, A_j[\pi]) \rightarrow 0.
\]

Note as $A_j$ is unramified, $I_{\text{cycl}}$ acts on the $p$-primary group $A_j[\pi]$ by the character $\omega_p^{i}$ which has order prime to $p$. Thus either $(A_j[\pi])^{I_{\text{cycl}}} = 0$ or we have $(A_j[\pi])^{I_{\text{cycl}}} = 0$. In either case, it follows that $(A_j[\pi])^{I_{\text{cycl}}} = 0$ and hence $\epsilon_2$ is an isomorphism. As $\psi|_{\omega_p^{i}}^{-1}I_{\text{cycl}}$ has order prime to $p$, a similar argument shows that $\epsilon_1$ is also an isomorphism. By the Five lemma, it follows that $\text{Ker}(\epsilon') = 0$.

**Remark 4.6.** Assume $p > k$. Then note that $A_j[\pi] = (A_j \otimes T_h)[\pi] \cong (A_j[\pi] \otimes T_h)(j) \cong (A_j[\pi] \otimes \mathcal{O}/I_p \otimes \mathcal{O}/I_p)(j)$ and $\mathcal{O}/I_p \otimes \mathcal{O}/I_p \otimes \mathcal{O}/I_p \cong (A_j[\pi] \otimes \mathcal{O}/I_p \otimes \mathcal{O}/I_p)(j)$. The last equality is true as $p > k$ by [JMS, (3.5), (3.6)]. This shows that if $p > k$ then $A_j[\pi]$ is determined by $A_j[\pi]$ and $A_h[\pi]$.

**Proposition 4.7.** Let $w \in \Sigma^{\infty}$, $w \mid Np$ and $w \mid I$. Assume $\psi|_{I_{\text{cycl}}}^{\text{cycl}}$ has order co-prime to $p$. Then we have
\[H^1_{\text{Gr}}(Q_{\Sigma}^{\text{cycl}}, A_j[\pi]) = \text{Ker} \left( H^1(G_{Q_{\Sigma}^{\text{cycl}}}, A_j[\pi]) \rightarrow H^1(I_{\text{cycl}}, A_j[\pi]) \right).\]
Proof. From Lemma 2.3 and [H3, Theorem 3.26(iii)(a)], we have \( \rho_h|_{I_{\text{cyc},w}} \sim (q_0, q_1) \). By the definition of \( H^1_{Gr}(G_{Q_{\text{cyc},w}}, A_j[\pi]) \), we have the following exact sequence

\[
0 \to H^1_{Gr}(Q_{\text{cyc},w}, A_j[\pi]) \to H^1(I_{\text{cyc},w}, A_j[\pi]) \to H^1(I_{\text{cyc},w}, A_j).
\]

The rest of the argument is similar to the proof of Proposition 4.5. \( \square \)

**Proposition 4.8.** Recall that \( M_0 \) is the prime to \( p \)-part of \( \text{cond}(\xi_1 \omega_r^{-1}) \text{cond}(\xi_2) \). Let \( w \in \Sigma^\infty \), \( w \nmid \mathfrak{m} \), \( w \nmid \mathfrak{D} \). Assume \((N, M_0) = 1\). Then we have

\[
H^1_{Gr}(Q_{\text{cyc},w}, A_j[\pi]) = \text{Ker}\left( H^1(I_{\text{cyc},w}, A_f(j))[\pi] \otimes \frac{T_h}{\pi} \to \text{Gr}_{Q_{\text{cyc},w}} \right).
\]

Proof. Let \( \Delta_v = G_{Q_{\text{cyc},w}} / I_{\text{cyc},w} = \text{Gal}(Q_m/Q_{\text{cyc},w}) \). Using inflation-restriction sequence, we obtain the image of \( H^1(G_{Q_{\text{cyc},w}}, A_j) \to H^1(I_{\text{cyc},w}, A_j) \) lies in \( H^1(I_{\text{cyc},w}, A_j) \). From the definition of \( H^1_{Gr}(Q_{\text{cyc},w}, A_j[\pi]) \), we have the following exact sequence

\[
0 \to H^1_{Gr}(G_{Q_{\text{cyc},w}}, A_j[\pi]) \to H^1(G_{Q_{\text{cyc},w}}, A_j[\pi]) \xrightarrow{\varphi} H^1(I_{\text{cyc},w}, A_j[\pi]) \to \text{Gr}_{Q_{\text{cyc},w}}.
\]

Since the image of \( \varphi \) is \( \pi \)-torsion, we get

\[
0 \to H^1_{Gr}(G_{Q_{\text{cyc},w}}, A_j[\pi]) \to H^1(G_{Q_{\text{cyc},w}}, A_j[\pi]) \xrightarrow{\varphi} H^1(I_{\text{cyc},w}, A_j[\pi]) \to \text{Gr}_{Q_{\text{cyc},w}}[\pi] = (H^1(I_{\text{cyc},w}, A_j[\pi]))^\Delta_v.
\]

As \( w \nmid \mathfrak{m} \), we have \( w \nmid I \). Since \( A_j = A_f \otimes T_h(j) \) and \( T_h \) is unramified at \( v \), we obtain

\[
H^1(I_{\text{cyc},w}, A_j)[\pi] = (H^1(I_{\text{cyc},w}, A_f(j)) \otimes T_h)[\pi] = H^1(I_{\text{cyc},w}, A_f(j))[\pi] \otimes T_h = H^1(I_{\text{cyc},w}, A_f(j))[\pi] \otimes \frac{T_h}{\pi}.
\]

Substituting this in (40), we get the required description of \( H^1_{Gr}(Q_{\text{cyc},w}, A_j[\pi]) \). \( \square \)

Recall that \( \xi_1 \) and \( \xi_2 \) are Dirichlet characters whose reductions are \( \xi_1 \) and \( \xi_2 \) respectively. We now prove an analogues of Propositions 4.5-4.8 for \( f \otimes \xi_1 \) and \( f \otimes \xi_2 \).

**Proposition 4.9.** Let \( w \in \Sigma^\infty \setminus \Sigma_0^\infty \). Then

(i) If \( w \mid \mathfrak{p} \) and \( \mathfrak{m} \mid \mathfrak{p} \), then for \( i = 1, 2 \),

\[
H^1_{Gr}(Q_{\text{cyc},w}, A_f(\xi_1^{-1})^i|\pi]) = \text{Ker}\left( H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_1^{-1}))|\pi]) \to H^1(I_{\text{cyc},w}, A_f(\xi_1^{-1}))|\pi]) \right).
\]

(ii) If \( w \nmid \mathfrak{m} \) and \( w \nmid \mathfrak{D} \), then for \( i = 1, 2 \),

\[
H^1_{Gr}(Q_{\text{cyc},w}, A_f(\xi_2^{-1})|\pi]) = \text{Ker}\left( H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_2^{-1}))|\pi]) \to H^1(I_{\text{cyc},w}, A_f(\xi_2^{-1}))|\pi]) \right).
\]

(iii) Assume \((N, M_0) = 1\). If \( w \mid \mathfrak{m} \) and \( w \nmid \mathfrak{D} \), then for \( i = 1, 2 \),

\[
H^1_{Gr}(Q_{\text{cyc},w}, A_f(\xi_2^{-1})|\pi]) = \text{Ker}\left( H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_2^{-1}))|\pi]) \to H^1(I_{\text{cyc},w}, A_f(\xi_2^{-1}))|\pi]) \right) \otimes \frac{T_h}{\pi}.
\]

Proof. For \( i = 1, 2 \) and \( w \in \Sigma^\infty \setminus \Sigma_0^\infty \), we have the following commutative diagram

\[
\begin{array}{ccc}
H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_1^{-1}))|\pi]) & \xrightarrow{\varphi} & H^1(I_{\text{cyc},w}, A_f(\xi_1^{-1}))|\pi]) \\
\downarrow & & \downarrow \\
H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_2^{-1}))|\pi]) & \xrightarrow{\varphi} & H^1(I_{\text{cyc},w}, A_f(\xi_2^{-1}))|\pi])
\end{array}
\]

where \( A_f(w) = A_f \) if \( w \mid \mathfrak{p} \) and \( A_f(w) = A_f \) if \( w \nmid \mathfrak{p} \). By definition, we have \( H^1_{Gr}(G_{Q_{\text{cyc},w}}, A_f[\pi]) = \ker(\kappa \circ \iota_1) = \ker(\kappa \circ \nu) \). It suffices to show \( \ker(\kappa \circ \nu) = A_f(w) (\xi_2^{-1}|\pi]_{\text{cyc},w} / \pi A_f(w) (\xi_2^{-1}|\pi]_{\text{cyc},w} \) is zero.

(i) Since \( \xi_2 \) is unramified at \( w \) and \( \omega' \) is ramified at \( w \), we get \( \xi_2 \omega'^{-1} \mathfrak{D} = 1 \leftrightarrow (p - 1) \mid j \). Thus \( A_f(w) (\xi_2^{-1}|\pi]_{\text{cyc},w} = A_f(w) (\xi_2^{-1}|\pi]_{\text{cyc},w} \) if \( (p - 1) \mid j \) and \( A_f(w) (\xi_2^{-1}|\pi]_{\text{cyc},w} = 0 \) if \( (p - 1) \nmid j \). So in either case, we have \( A_f(w) (\xi_2^{-1}|\pi]_{\text{cyc},w} \) is \( \pi \)-divisible and hence \( \ker(\kappa \circ \nu) = 0 \). Thus the assertion (i) follows from the above commutative diagram. Since \( \xi_1 \) has order prime to \( p \), by construction (see the proof of Lemma 2.1) we have \( \xi_1 \) has order prime to \( p \). Thus \( \xi_1 \omega'^{-1} |_{\text{cyc},w} = 1 \) if and only if \( \xi_1 \omega'^{-1} |_{\text{cyc},w} = 1 \mod \pi \). A similar argument as above shows that \( A_f(w) (\xi_1^{-1} |\pi]_{\text{cyc},w} \) is \( \pi \)-divisible and \( \ker(\kappa \circ \nu) = 0 \).

(ii) Let \( \mathfrak{r} \) be prime in \( \mathcal{Z} \) lying below \( w \). Then \( \mathfrak{r} \nmid \mathfrak{p} \). Since \( \mathfrak{r} \nmid \mathfrak{m} \) and \( \mathfrak{r} \mid \mathfrak{D} \), we must have \( \mathfrak{r} \mid \mathfrak{m} \) and \( \mathfrak{D} \mid \mathfrak{D} \). Since \( \text{cond}(\omega^{-1}|\xi_1) = \text{cond}(\omega^{-1}|\xi_2) \), for \( i = 1, 2 \), we get \( \text{cond}(\rho_f \otimes \xi_1^{-1}) = \text{cond}(\rho_f \otimes \xi_2^{-1}) \). Thus \( A_f(\xi_1^{-1}|\pi]_{\text{cyc},w} \) is \( \pi \)-divisible by [EPW, Lemma 4.1.2]. So \( \ker(\kappa \circ \nu) \) in the above commutative diagram is zero. Now part (ii) follows.

(iii) Note that \( \xi_1 \) is unramified at \( w \). The proof is similar to Proposition 4.8. \( \square \)
5. Congruence of the characteristic ideals

In this section, we obtain the congruence between the characteristic ideal associated to the Selmer group of \( f \otimes h \) and the product of characteristic ideals associated to the Selmer group of \( f \otimes \xi_1, f \otimes \xi_2 \).

From the exact sequence (18), it follows that we have the following exact sequence of \( G_\mathbb{Q} \)-modules

\[
0 \longrightarrow \frac{\mathbb{Q}}{\pi}(\xi_1) \longrightarrow A_h[\pi] \longrightarrow \frac{\mathbb{Q}}{\pi}(\xi_2) \longrightarrow 0.
\] (41)

Tensoring with \(- \otimes \mathcal{O} T_f(-j)\) and noting that \( T_f(-j) \) is a free \( \mathcal{O} \)-module, we get

\[
0 \longrightarrow A_f(\xi_1\omega_p^{-j})[\pi] \longrightarrow A_f[\pi] \longrightarrow A_f(\xi_2\omega_p^{-j})[\pi] \longrightarrow 0.
\] (42)

In this section, we consider the following assumption on \( f \otimes \xi_1 \)

\[
H^2(Q_\Sigma/Q_{\text{cyc}}, A_f(\xi_1\omega_p^{-j})[\pi]) = 0.
\]

\((H^2_{\xi_1\text{-van}})\)

Lemma 5.1. Assume \( f \) satisfies (irr-f), \( h \) satisfies (\( \rho_h \)-red) and \( f \otimes \xi_1 \) satisfies \((H^2_{\xi_1\text{-van}})\). Then the following sequence is exact

\[
0 \rightarrow H^1(Q_\Sigma/Q_{\text{cyc}}, A_f(\xi_1\omega_p^{-j})[\pi]) \rightarrow H^1(Q_\Sigma/Q_{\text{cyc}}, A_f[\pi]) \rightarrow H^1(Q_\Sigma/Q_{\text{cyc}}, A_f(\xi_2\omega_p^{-j})[\pi]) \rightarrow 0.
\]

Proof. The exact sequence (42) induces the following exact sequence of cohomology groups

\[
H^1(Q_\Sigma/Q_{\text{cyc}}, A_f(\xi_1\omega_p^{-j})[\pi]) \rightarrow H^1(Q_\Sigma/Q_{\text{cyc}}, A_f[\pi]) \rightarrow H^1(Q_\Sigma/Q_{\text{cyc}}, A_f(\xi_2\omega_p^{-j})[\pi]).
\]

By the assumption \( H^2(Q_\Sigma/Q_{\text{cyc}}, A_f(\xi_1\omega_p^{-j})[\pi]) = 0 \), we obtain the right most map is surjective. Note that by our assumption \( A_f[\pi] \) is an irreducible \( G_\mathbb{Q} \)-module, it follows that \( A_f(\xi_1\omega_p^{-j})[\pi] \) is also an irreducible \( G_\mathbb{Q} \)-module, hence \( H^0(Q, A_f(\xi_1\omega_p^{-j})[\pi]) = 0 \). Thus by Nakayama lemma, \( H^0(Q_{\text{cyc}}, A_f(\xi_1\omega_p^{-j})[\pi]) \) also vanishes. Hence the left most map in the above exact sequence is injective.

We now state a technical assumption, which we will need later.

Let \( w \) be the prime in \( Q_{\text{cyc}} \) dividing \( p \). Then \( \rho_h|I_{\text{cyc},w} \) is semi-simple i.e. \( \rho_h|I_{\text{cyc},w} \cong \xi_1 \oplus \xi_2 \). \((\text{ss-red}_p)\)

Recall that \( \Sigma = \{ \ell : \ell \mid p \mathcal{N} I_\infty \}, \Sigma_0 = \{ \ell : \ell \mid m \} \) and \( \Sigma^\infty, \Sigma_0^\infty \) are the corresponding primes in \( Q_{\text{cyc}} \).

Proposition 5.2. Let \((N, M_0) = 1 \) and \( j \) be an integer. Assume \( f \) satisfies (irr-f), \( h \) satisfies (\( \rho_h \)-red) and \( \psi|I_{\text{cyc},w} \) has order prime to \( p \) for \( w \mid p \mathcal{A}_h \). If \( (p - 1) \mid j \) and \( H^0(G_{Q_{\text{cyc},w}}, A_f(\xi_2[\pi])) \neq 0 \) at the prime \( w \mid p \), then we further assume that \((\text{ss-red}_p)\) holds. Then for every \( w \in \Sigma^\infty \setminus \Sigma_0^\infty \), we have the following exact sequence

\[
0 \rightarrow H^1(Q_{\text{cyc},w}, A_f(\xi_1\omega_p^{-j})[\pi]) \rightarrow H^1(Q_{\text{cyc},w}, A_f[\pi]) \rightarrow H^1(Q_{\text{cyc},w}, A_f(\xi_2\omega_p^{-j})[\pi]) \rightarrow 0.
\] (43)

Proof. As the \( p \)-th cohomological dimension of \( G_{Q_{\text{cyc},w}} \) is 1, from (42) we get the exact sequence

\[
H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_1\omega_p^{-j})[\pi]) \rightarrow H^1(G_{Q_{\text{cyc},w}}, A_f[\pi]) \rightarrow H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_2\omega_p^{-j})[\pi]) \rightarrow 0.
\] (44)

We denote \( G_{Q_{\text{cyc},w}}/I_{Q_{\text{cyc},w}} \) by \( \Delta_w \).

Case \( w \mid p \): As the \( p \)-th cohomological dimension of \( G_{Q_{\text{cyc},w}} \) is 1, the surjectivity \( H^1(G_{Q_{\text{cyc},w}}, A_f[\pi]) \rightarrow H^1(G_{Q_{\text{cyc},w}}, A_f[\pi]) \) follows from the definition \( A_f[\pi] \) in (35). By the inflation-restriction sequence, we have the \( H^1(G_{Q_{\text{cyc},w}}, A_f[\pi]) \rightarrow H^1(I_{Q_{\text{cyc},w}}, A_f[\pi]) \Delta_w \) is surjective. Hence the image of the following composition

\[
H^1(G_{Q_{\text{cyc},w}}, A_f[\pi]) \rightarrow H^1(G_{Q_{\text{cyc},w}}, A_f[\pi]) \rightarrow H^1(I_{Q_{\text{cyc},w}}, A_f[\pi]) \Delta_w
\]

is equal to \( H^1(I_{Q_{\text{cyc},w}}, A_f[\pi]) \Delta_w \). Similarly image \( (H^1(G_{Q_{\text{cyc},w}}, A_f(\xi_1\omega_p^{-j})[\pi]) \rightarrow H^1(I_{Q_{\text{cyc},w}}, A_f(\xi_1\omega_p^{-j})[\pi]) = (H^1(I_{Q_{\text{cyc},w}}, A_f(\xi_1\omega_p^{-j})[\pi]) \Delta_w \) for \( i = 1, 2 \). Thus it follows from (41) and Propositions 4.5, 4.9 that the
following diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
H^1_G(G_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi] & \rightarrow & H^1_G(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] & \rightarrow & H^1_G(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] & \rightarrow & 0 \\
0 & \rightarrow & 0 \\
H^1(G_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi] & \rightarrow & H^1(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] & \rightarrow & H^1(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] & \rightarrow & 0 \\
H^1(I_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi]^{\Delta_w} & \rightarrow & H^1(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi]^{\Delta_w} & \rightarrow & H^1(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi]^{\Delta_w} & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

is commutative. So to prove the proposition, it suffices to show \(\ker(\nu) = 0\).

If \((p - 1) \nmid j\), then \(H^0(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] = 0\). Tensoring (41) with \(A_f^\infty\), and then taking the induced long exact sequence in cohomology, we obtain

\[
0 \rightarrow H^1(I_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi] \rightarrow H^1(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] \rightarrow H^1(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] \rightarrow 0.
\]

Taking \(\Delta_w\) invariants, we obtain \(\ker(\nu) = 0\).

Next consider the case \((p - 1) \nmid j\) and \(H^0(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] \neq 0\). Then by the assumption \((\text{ss-red}_p)\), we have \(A_f^\infty \cong \left( (T_\ell/\pi^2) \right) \cong \xi_1 \oplus \xi_2\). Thus we have \(H^1(I_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi] = H^1(I_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi] \oplus H^1(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi]\). Again taking \(\Delta_w\) invariants, we get \(\ker(\nu) = 0\).

Finally consider the case \((p - 1) \nmid j\) and \(H^0(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] = 0\). Then \(H^0(\Delta_w, A_f(\xi_2 \omega_p^{-j}))[\pi] \cong \pi^2\). Thus \(H^1(\Delta_w, A_f(\xi_2 \omega_p^{-j}))[\pi] = 0\). The inflation-restriction sequence and (41), in this case we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & H^1(\Delta_w, A_f(\xi_1 \omega_p^{-j}))[\pi]^{I_{\text{cycl}, w}} \\
0 & \rightarrow & H^1(\Delta_w, A_f(\xi_2 \omega_p^{-j}))[\pi]^{I_{\text{cycl}, w}} \\
H^1(G_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi] & \rightarrow & H^1(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] & \rightarrow & H^1(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] & \rightarrow & 0
\end{array}
\]

where all the vertical maps are injective. Now the proposition follows from the snake lemma and the inflation-restriction sequence.

Case \(w \nmid I, w \nmid N\): Let \(w \nmid \ell \in \mathbb{Z}\). Since \(\ell \nmid m\) and \(\ell \nmid I_0\), it forces that \(\ell \nmid I_0/M_0\) and \(\ell \mid M = \text{cond}(\xi_1) \cdot \text{cond}(\xi_2)\). By Lemma 2.3 and [H3, Theorem 3.26], we have \(\tilde{\rho}_H|_{G_{\text{cycl}, w}} \cong \xi_1 \oplus \xi_2\). Tensoring with \(A_f(-j)\) induces following exact sequence of \(\Delta_w\)-modules

\[
0 \rightarrow H^1(I_{\text{cycl}, w}, A_f(\xi_1 \omega_p^{-j}))[\pi] \rightarrow H^1(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] \rightarrow H^1(I_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] \rightarrow 0.
\]

Note that we have an analogue of commutative diagram (45) even in this case. The assertion follows from Proposition 4.7 and Proposition 4.9(ii) by a similar argument as in the case \(w \mid p\).

Case \(w \nmid N\) and \(w \nmid I_0\): We claim that \((T_\ell/\pi^2) \cong \xi_1 \oplus \xi_2\) as \(G_{\text{cycl}, w}\)-modules. Since \((N, M_0) = 1\) and \(w \nmid N\), we get \(w \nmid M_0\). Again using \(v \nmid m\), we obtain \(w \nmid I_0\). Thus \(T_\ell/\pi^2\) is unramified at \(w\) and the action of \(G_{\text{cycl}, w}\) factors through \(\Delta_w\). As the order of \(\xi_1, \xi_2\) are co-prime to \(p\), the action of \(G_{\text{cycl}, w}\) on \(T_\ell/\pi^2\) further factors through \(G_{\text{cycl}, w}/B\) such that \(B\) is a finite index normal subgroup of \(G_{\text{cycl}, w}\) containing \(I_{\text{cycl}, w}\). Since \(G_{\text{cycl}, w}/B\) has order co-prime to \(p\), we get \(T_\ell/\pi^2\) is a semi-simple \(G_{\text{cycl}, w}\)-module. Hence \(T_\ell/\pi^2 \cong \xi_1 \oplus \xi_2\) as \(G_{\text{cycl}, w}\)-module. Now the proposition in this case follows from Propositions 4.8, 4.9 using similar arguments as in previous two cases.

\[\square\]

**Remark 5.3.** We will apply Proposition 5.2 in Theorem 5.9 and Theorem 6.4 for \(l - 1 \leq j \leq k - 2\). In that case, the assumption \((\text{ss-red}_p)\) in Proposition 5.2 is required whenever \(\omega_p^j = 1\) and \(H^0(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] \neq 0\). In particular, either \(p > k - 1\) or \(H^0(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] = 0\) holds, then the assumption \((\text{ss-red}_p)\) is not required.

**Proposition 5.4.** Let \((N, M_0) = 1\) and \(\psi|_{I_{\text{cycl}, w}}\) has order co-prime to \(p\) for all primes \(w \nmid pI_0\) in \(\mathbb{Q}_{cycl}\). Assume that \(f\) satisfies \((\text{irr}_f)\), \(h\) satisfies \((\text{ss-red}_p)\) and \(f \otimes \xi_1\) satisfies \((H_{\xi_1}^\infty\text{-van})\). If \((p - 1) \nmid j\) and \(H^0(G_{\text{cycl}, w}, A_f(\xi_2 \omega_p^{-j}))[\pi] \neq 0\) at the prime \(w \mid p\), then we further assume that \((\text{ss-red}_p)\) holds for \(w\). Then
we have the following exact sequence
\[ 0 \to S^{\Sigma}_{Gr}(A_f(\xi \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc}) \to S^{\Sigma}_{Gr}(A_j[\pi]/\mathbb{Q}_{cyc}) \to S^{\Sigma}_{Gr}(A_f(\xi 2 \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc}) \to 0. \]  
(46)

Proof. By Lemma 5.1 and Proposition 5.2, we have the following commutative diagram
\[
\begin{array}{cccccc}
0 & \to & H^1(G_{\Sigma}/\mathbb{Q}_{cyc}, A_f(\xi \omega_p^{-j})[\pi]) & \to & H^1(G_{\Sigma}/\mathbb{Q}_{cyc}, A_j[\pi]) & \to & H^1(G_{\Sigma}/\mathbb{Q}_{cyc}, A_f(\xi 2 \omega_p^{-j})[\pi]) & \to 0 \\
0 & \to & \prod_{w \in \Sigma \setminus \Sigma^0} H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) & \to & \prod_{w \in \Sigma \setminus \Sigma^0} H^1(G_{\Sigma,w}/A_j[\pi]) & \to & \prod_{w \in \Sigma \setminus \Sigma^0} H^1(G_{\Sigma,w}/A_f(\xi 2 \omega_p^{-j})[\pi]) & \to 0 \\
\end{array}
\]

Applying the snake lemma to the above commutative diagram, we get the following exact sequence
\[ 0 \to S^{\Sigma}_{Gr}(A_f(\xi \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc}) \to S^{\Sigma}_{Gr}(A_j[\pi]/\mathbb{Q}_{cyc}) \to S^{\Sigma}_{Gr}(A_f(\xi 2 \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc}) \to \text{coker}(\epsilon_1). \]

To prove the lemma, we need to show coker(\epsilon_1) is zero. In this setting, the following global to local map
\[ H^1(G_{\Sigma}/\mathbb{Q}_{cyc}, A_f(\xi \omega_p^{-j})[\pi]) \to \bigoplus_{w | p} H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) \bigoplus_{w \notin \Sigma^0 \setminus \Sigma^\infty} H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) \]
is surjective. Indeed, the surjectivity of such global to local maps has been extensively studied by Greenberg (cf. [G3, §5.3]). In our setting above, it can also be conveniently found in [LS, Theorem 5.2]. Using the inflation-restriction sequence and ∆_w is topologically cyclic, we obtain the following surjective map
\[ H^1(G_{\Sigma}/\mathbb{Q}_{cyc}, A_f(\xi \omega_p^{-j})[\pi]) \to \bigoplus_{w | p} H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) \Delta^w \bigoplus_{w \notin \Sigma^0 \setminus \Sigma^\infty} H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) \]

Since
\[ H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) \to H^1(I_{\Sigma,w}, A_f(\xi \omega_p^{-j})[\pi]) \Delta^w \quad \text{for } w \mid p \text{ (see Diagram (45))}, \]

and
\[ H^1(G_{\Sigma}/\mathbb{Q}_{cyc}, A_f(\xi \omega_p^{-j})[\pi]) \to \bigoplus_{w | p} H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) \bigoplus_{w \notin \Sigma^0 \setminus \Sigma^\infty} H^1(G_{\Sigma,w}/A_f(\xi \omega_p^{-j})[\pi]) \]
is also surjective. This shows ε_1 is surjective. Hence the right exactness follows in (46). □

Lemma 5.5. We keep the setting and hypotheses as in Proposition 5.4. Then we have the following exact sequence
\[ 0 \to S^{\Sigma}_{Gr}(A_f(\xi \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc})^\vee \to S^{\Sigma}_{Gr}(A_j[\pi]/\mathbb{Q}_{cyc})^\vee \to S^{\Sigma}_{Gr}(A_f(\xi 2 \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc})^\vee \to 0. \]  
(47)

Proof. We claim that H^0(G_{\Sigma,cyc}, A_f(\xi \omega_p^{-j})[\pi]) = 0, for i = 1, 2 and H^0(G_{\Sigma,cyc}, A_j) = 0. From the assumption A_f(\pi)[\pi] (and hence A_f(\xi \omega_p^{-j})[\pi]) is an irreducible G_\Sigma-module, we have H^0(G_{\Sigma}, A_f(\xi \omega_p^{-j})[\pi]) = 0, for i = 1, 2. From Nakayama’s Lemma, it follows that H^0(G_{\Sigma,cyc}, A_f(\xi \omega_p^{-j})[\pi]) = 0, for i = 1, 2. From the exact sequence (42), it follows that H^0(G_{\Sigma,cyc}, A_f(\pi)[\pi]) = 0. For B_j \in \{A_f(\xi \omega_p^{-j}), A_j, A_f(\xi \omega_p^{-j})\}, applying Lemma 4.4 we get S^{\Sigma}_{Gr}(B_j[\pi]/\mathbb{Q}_{cyc}) = S^{\Sigma}_{Gr}(B_j[\Sigma]/\mathbb{Q}_{cyc}). Then it follows from (46) that
\[ 0 \to S^{\Sigma}_{Gr}(A_f(\xi \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc}) \to S^{\Sigma}_{Gr}(A_j[\pi]/\mathbb{Q}_{cyc}) \to S^{\Sigma}_{Gr}(A_f(\xi 2 \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc}) \to 0 \]  
(48)

is exact. Taking the Pontryagin dual, we obtain the lemma. □

As f is ordinary at p, it follows from a deep result of Kato [Ka, Theorem 17.4] that the dual Selmer groups S^{\Sigma}_{Gr}(A_f(\xi \omega_p^{-j})[\pi]/\mathbb{Q}_{cyc})^\vee are torsion O[\Gamma]-modules for i = 1, 2, (co-tors).

By the assumption (Sel-tors.), we have S^{\Sigma}_{Gr}(A_f[\pi]/\mathbb{Q}_{cyc})^\vee is also a torsion O[\Gamma]-module. Thus S^{\Sigma}_{Gr}(B_j[\pi]/\mathbb{Q}_{cyc}) is co-torsion O[\Gamma]-module for B_j \in \{A_f(\xi \omega_p^{-j}), A_j, A_f(\xi 2 \omega_p^{-j})\}. We next prove that S^{\Sigma}_{Gr}(B_j[\pi]/\mathbb{Q}_{cyc})^\vee has no non-zero pseudo-null O[\Gamma]-submodules.

Proposition 5.6. Let B_j \in \{A_f(\xi \omega_p^{-j}), A_j, A_f(\xi 2 \omega_p^{-j})\}. Assume that f satisfies (irr-f) and (Sel-tors) holds for A_j. Then S^{\Sigma}_{Gr}(B_j[\pi]/\mathbb{Q}_{cyc})^\vee has no non-zero pseudo-null O[\Gamma]-submodule.

Proof. We apply [We, Proposition 1.8] to deduce S^{\Sigma}_{Gr}(B_j[\pi]/\mathbb{Q}_{cyc})^\vee has no non-zero pseudo-null O[\Gamma]-submodule. Following [We], define Sel_{\Sigma}(\mathbb{Q}_{cyc}, B_j) := \text{Ker}(H^1(G_{\Sigma}(\mathbb{Q}_{cyc}), B_j) \to \bigoplus_{w \in \Sigma} \text{Loc}_w), where Loc_w = H^1(I_{\Sigma,cyc,w}, B_j) \text{ if } w \nmid p \text{ and Loc}_w = (\text{image}(H^1(I_{\Sigma,cyc,w}, B_j) \to H^1(I_{\Sigma,cyc,w}, B_j^\vee))) \text{ if } w \nmid p. For w \nmid p, G_{\Sigma,cyc,w}/I_{\Sigma,cyc,w} \text{ has finite order prime to } p, \text{ thus } H^1(G_{\Sigma,cyc,w}/I_{\Sigma,cyc,w}, B_j^\vee) = 0. \text{ Thus by the}
inflation-restriction sequence we get $H^i(Q_{\text{cyc}, B_j}) \to H^i(I_{\text{cyc}, w}, B_j)$ is injective. With this, it is easy to see that $\text{Sel}_\tau(Q_{\text{cyc}, B_j}) = S_{\text{Gr}}(B_j/Q_{\text{cyc}})$.

We now claim that $\text{Hom}(B_j(-1)[\pi], F)^{G_\sigma} = 0$. Since $A_f[\pi]$ is an irreducible $G_\sigma$-module, we have $\text{Hom}(A_f(\xi_1, m^{-1}-1)[\pi], F)^{G_\sigma} = 0$. Thus from the exact sequence (42) it follows that $\text{Hom}(A_f(-1)[\pi], F)^{G_\sigma} = 0$. Since $(\text{Hom}(B_j, K_{\pi}(1)) \otimes K_{\pi})[\pi] \cong \text{Hom}(B_j(-1)[\pi], F)$, we get $(\text{Hom}(B_j, K_{\pi}(1)) \otimes K_{\pi})^{G_\sigma}[\pi] = 0$. Note that $(\text{Hom}(B_j, K_{\pi}(1)) \otimes K_{\pi})^{G_\sigma} = 0$ implies that $H^0(G_{Q_{\text{cyc}}}, \text{Hom}(B_j, K_{\pi}(1)) \otimes K_{\pi}) = 0$ by Nakayama’s lemma. Now the proposition follows from [We, Proposition 1.8].

Since $S_{\text{Gr}}(B_j/Q_{\text{cyc}})$ is a co-torsion $O[[\Gamma]]$-module, we have $S_{\text{Gr}}^0(B_j/Q_{\text{cyc}})^{\vee}$ is a finitely generated torsion $O[[\Gamma]]$-module for $B_j \in \{A_f(\xi_1, m^{-1}-1), A_f(\xi_2, m^{-1}-1)\}$. We next study relation between $S_{\text{Gr}}(B_j/Q_{\text{cyc}})$ and $S_{\text{Gr}}^0(B_j/Q_{\text{cyc}})$.

**Lemma 5.7.** Let the hypotheses be as in Lemma 5.5 and also assume (Sel-tors) holds for $A_f$. Then for $B_j \in \{A_f(\xi_1, m^{-1}-1), A_f(\xi_2, m^{-1}-1)\}$, we have the following exact sequence

$$0 \longrightarrow S_{Gr}(B_j/Q_{\text{cyc}}) \longrightarrow S_{Gr}^0(B_j/Q_{\text{cyc}}) \longrightarrow \prod_{w \in \Sigma_{\text{cyc}}} H^1(I_{\text{cyc}, w}, B_j) \longrightarrow 0.$$  

(49)

**Proof.** For every $w \in \Sigma_{\text{cyc}}$, $G_{Q_{\text{cyc}}, I_{\text{cyc}, w}}$ is a pro-$\ell$ group with $\ell \neq p$. Thus $H^1(G_{Q_{\text{cyc}}, I_{\text{cyc}, w}}, B_j^{I_{\text{cyc}, w}}) = 0$ and $H^2_{\text{Gr}}(G_{Q_{\text{cyc}, I_{\text{cyc}, w}}}, B_j^{I_{\text{cyc}, w}}) = 0$ for $w \in \Sigma_{\text{cyc}}$. We first treat the case $B_j = A_f(\xi_1, m^{-1}-1), A_f(\xi_2, m^{-1}-1)$. Let $\mu_{\text{cyc}} := \{ \xi \in \mathbb{C} : \xi^p = 1 \text{ for some } r \in \mathbb{Z} \}$ and $A_f(\xi_1, m^{-1}-1)^* := \text{Hom}(T_f(\xi_1, m^{-1}-1), S_{\text{cyc}}^p)$. From (irr-f) and dimo(T_f) = 2, we have $\text{Hom}(T_f(\xi_1, m^{-1}-1), S_{\text{cyc}}^p)$ is an irreducible $G_\sigma$-module. By topological Nakayama lemma, $H^0(G_{Q_{\text{cyc}}}, \text{Hom}(T_f(\xi_1, m^{-1}-1), S_{\text{cyc}}^p)) = 0$. As $\text{Hom}(T_f(\xi_1, m^{-1}-1), S_{\text{cyc}}^p) \cong \text{Hom}(T_f(\xi_1, m^{-1}-1), S_{\text{cyc}}^p)$, it follows that $H^0(G_{Q_{\text{cyc}}}, A_f(\xi_1, m^{-1}-1)^*[\pi]) = 0$. Thus we obtain $H^0(G_{Q_{\text{cyc}}}, A_f(\xi_1, m^{-1}-1)^*[\pi])/\pi = 0$. Since $(A_f(\xi_1, m^{-1}-1)^*[\pi]$ is a finitely generated $Z_{\text{cyc}}$-module, by Nakayama lemma, we get $H^0(G_{Q_{\text{cyc}}}, A_f(\xi_1, m^{-1}-1)^*[\pi])/\pi = 0$. Now the result in this case follows from [GV, Corollary 2.3] (See also [SU, Proposition 3.14]).

We next treat the case $B_j = A_f$. Tensoring (18) with $T_f(j)$, we obtain that

$$0 \rightarrow T_f(\xi_1^-)^{\vee}/\pi \rightarrow T_f(\xi_1^-)^{\vee}/\pi \rightarrow T_f(\xi_1^-)^{\vee}/\pi.$$  

Applying $\text{Hom}_{Z_{\text{cyc}}}(\cdot, \mu_{\text{cyc}})$ and taking $G_{Q_{\text{cyc}}}$-invariants, we obtain

$$0 \rightarrow \text{Hom}_{Z_{\text{cyc}}}(T_f(\xi_1^-)^{\vee}/\pi, \mu_{\text{cyc}}^{\vee})^{G_{Q_{\text{cyc}}}} \rightarrow \text{Hom}_{Z_{\text{cyc}}}(T_f(\xi_1^-)^{\vee}/\pi, \mu_{\text{cyc}}^{\vee})^{G_{Q_{\text{cyc}}}} \rightarrow \text{Hom}_{Z_{\text{cyc}}}(T_f(\xi_1^-)^{\vee}/\pi, \mu_{\text{cyc}}^{\vee})^{G_{Q_{\text{cyc}}}}.$$  

Since $\text{Hom}_{Z_{\text{cyc}}}(T_f(\xi_1^-)^{\vee}/\pi, \mu_{\text{cyc}}^{\vee})^{G_{Q_{\text{cyc}}}} = H^0(G_{Q_{\text{cyc}}}, \text{Hom}_{Z_{\text{cyc}}}(T_f(\xi_1^-)^{\vee}/\pi, \mu_{\text{cyc}}^{\vee})) = 0$ for $i = 1, 2$, it follows that $H^0(G_{Q_{\text{cyc}}}, \text{Hom}_{Z_{\text{cyc}}}(T_f(\xi_1^-)^{\vee}/\pi, \mu_{\text{cyc}}^{\vee})) = 0$. By a similar argument as above, we get $H^0(G_{Q_{\text{cyc}}}, \text{Hom}(T_f, \mu_{\text{cyc}}^{\vee}))$ vanishes. Now the lemma, under (Sel-tors) can be deduced adopting the arguments in [GV, Corollary 2.3].

We now show that the $\Sigma_\text{cyc}$-imprimitive Selmer groups $S_{\text{Gr}}^\Sigma(B_j/Q_{\text{cyc}})^{\vee}$ have no non-zero pseudo-null $O[[\Gamma]]$ submodules.

**Corollary 5.8.** Let the hypotheses be as in Lemma 5.5 and also assume (Sel-tors) holds for $A_f$. Then the modules $S_{\text{Gr}}^\Sigma(A_f(\xi_1, m^{-1}-1)/Q_{\text{cyc}})^{\vee}$ for $i = 1, 2$ and $S_{\text{Gr}}^\Sigma(A_f(\xi_2, m^{-1}-1)/Q_{\text{cyc}})^{\vee}$ have no non-zero pseudo-null $O[[\Gamma]]$ submodules.

**Proof.** Let $B_j \in \{A_f(\xi_1, m^{-1}-1), A_f(\xi_2, m^{-1}-1)\}$. We note that, $\prod_{w \in \Sigma_{\text{cyc}}} H^1(I_{\text{cyc}, w}, B_j)^{\vee}$ has no non-zero pseudo-null submodule as $B_j$ is divisible and $G_{Q_{\text{cyc}}, I_{\text{cyc}, w}}$ has $p$-th cohomological dimension 1. Now the lemma follows from Proposition 5.6 and (49).

For a finitely generated torsion $O[[\Gamma]]$ (resp. $\mathbb{F}[[\Gamma]]$) module $M$, we denote the characteristic ideal of $M$ over $O[[\Gamma]]$ (resp. $\mathbb{F}[[\Gamma]]$) by $C_{O[[\Gamma]]}(M)$ (resp. $C_{\mathbb{F}[[\Gamma]]}(M)$). Note that by the assumption (Sel-tors) we have $S_{\text{Gr}}(A_f/Q_{\text{cyc}})^{\vee}$ is a torsion $O[[\Gamma]]$-module. Also by (co-tors), we have $S_{\text{Gr}}^\Sigma(A_f(\xi_1, m^{-1}-1)/Q_{\text{cyc}})^{\vee}$ are finitely generated torsion $O[[\Gamma]]$-modules for $i = 1, 2$. Thus we can consider their characteristic ideal. From Lemma 5.8 and [SU, Corollary 3.8(i), Corollary 3.21(iii)] we have

$$C_{O[[\Gamma]]}(S_{\text{Gr}}^\Sigma(B_j/Q_{\text{cyc}})^{\vee}) (mod \pi) = C_{\mathbb{F}[[\Gamma]]}(S_{\text{Gr}}^\Sigma(B_j/Q_{\text{cyc}})^{\vee}/\pi)$$

(50)

for $B_j \in \{A_f(\xi_1, m^{-1}-1), A_f(\xi_2, m^{-1}-1)\}$. Using this in (47), we deduce the following theorem:

**Theorem 5.9.** Let $(N, M_0) = 1$ and $\psi|I_{\text{cyc}}$ has order prime to $p$ for $w | pI_0$, where $\psi$ is the nebentypus of $h$. Assume $f$ satisfies (irr-f), $h$ satisfies (p-h, red), $f \otimes \xi_1$ satisfies ($H_2^\Sigma$-van) and (Sel-tors) holds for
If \((p - 1) \mid j \) and \(H^0(G_{\text{cyclic}}, A_f(T_r)[\xi_2]) \neq 0\) at the prime \(w \mid p\), then we further assume that (ss-red) holds. Then for \(l - 1 \leq j \leq k - 2\), we have
\[
C_{\mathcal{O}[[r]]}(S^v_{\text{Gr}}(A_f/\mathbb{Q}_{\text{cyclic}}) \phi) = C_{\mathcal{O}[[r]]}(S^v_{\text{Gr}}(A_f(\xi_2 \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}})) \mod \pi. \tag{51}
\]

**Remark 5.10.** As \(f\) is \(p\)-ordinary and \(\hat{\rho}_f\) is irreducible, it follows that \(C_{\mathcal{O}[[r]]}(S^v_{\text{Gr}}(A_f(\xi_2 \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}))\) is independent of the choice of the lattice \(T_f\) (see [SU, Page 34]). Thus by Theorem 5.9, it follows that \(C_{\mathcal{O}[[r]]}(S^v_{\text{Gr}}(A_f/\mathbb{Q}_{\text{cyclic}}) \phi)\) is also independent of the choice of the lattice \(T = T_f \otimes T_g\).

6. **Iwasawa Main Conjecture modulo \(\pi\)**

Recall that \(\mu_{p,f,x,j}\) and \(\mu_{p,f \times h,j}\) are the \(p\)-adic \(L\)-functions attached to \(f \otimes \varphi\) and \(f \otimes h\), (see (31), (32)). We now recall the Iwasawa Main Conjecture for modular forms (see Greenberg [G1]):

**Conjecture 6.1.** (Greenberg Iwasawa Main Conjecture for modular forms) Let \(F \in S_k(\Gamma_1(N))\) be a \(p\)-ordinary eigenform and \(\varphi\) be a Dirichlet character. For every critical value \(0 \leq j \leq k - 1\), we have
\[
(\mu_{p,F,\varphi,j}) = C_{\mathcal{O}[[r]]}(S_{\text{Gr}}(A_F(\varphi \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi).
\]

**Remark 6.2.** Conjecture 6.1 is known for a large class of modular forms combining the results of Kato [Ka] and Skinner-Urban [SU].

We next show that the Iwasawa main conjecture i.e. (IMC) for \(f \otimes \xi_i\) implies the \(\Sigma_0\)-imprimitive Iwasawa main conjecture for \(f \otimes \xi_i\).

**Lemma 6.3.** Let \(f \in S_k(\Gamma_0(N), \eta)\) be a \(p\)-ordinary eigenform with \(p \nmid N\) and \(f\) satisfies (irr-f). For every critical value \(0 \leq j \leq k - 1\) and \(i = 1, 2\), we have
\[
(\mu_{p,f,\xi_i,j}) = C_{\mathcal{O}[[r]]}(S_{\text{Gr}}(A_f(\xi_i \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi) \Rightarrow (\mu_{p,f,\xi_i,j}^\Sigma) = C_{\mathcal{O}[[r]]}(S^\Sigma_{\text{Gr}}(A_f(\xi_i \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi).
\]

**Proof.** By [GV, Proposition 2.4] (see also [SU, Lemma 3.13]), for every prime \(\ell \neq p\), the characteristic polynomial of \(\prod_{\ell \mid k} H^1(\mathbb{Q}_{\text{cyclic}}, A_f(\xi_i \omega_p^{-j}))\) equals \(P_{\ell,i}(\ell^{-j-1} \gamma)\), where \(P_{\ell,i}(X) = \det(I - X \text{Frob}_\ell | V_f(\xi_i)^{\ell^i})\) and \(\gamma\) is a topological generator of \(\Gamma = \text{Gal}(\mathbb{Q}_{\text{cyclic}}/\mathbb{Q})\).

Note that from the exact sequence (49), \(S_{\text{Gr}}(A_f(\xi_i \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi\) is torsion over \(\mathcal{O}[[\Gamma]]\) implies the same for \(S^\Sigma_{\text{Gr}}(A_f(\xi_i \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi\). Also it follows that
\[
C_{\mathcal{O}[[r]]}(S^\Sigma_{\text{Gr}}(A_f(\xi_i \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi) = \prod_{\ell \in \Sigma_0} P_{\ell,i}(\ell^{-j-1} \gamma) C_{\mathcal{O}[[r]]}(S_{\text{Gr}}(A_f(\xi_i \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi).
\]

Observe that \(P_{\ell,i}(X) = \det(I - X \text{Frob}_\ell | V_f(\xi_i)^{\ell^i})\) gives the Euler factor of the \(L\)-function of \(f \otimes \xi_i\) at \(\ell\). By (31), we have \(\mu_{p,f,\xi_i,j}^\Sigma = \mu_{p,f,\xi_i,j} \prod_{\ell \in \Sigma_0} P_{\ell,i}(\ell^{-j-1} \gamma)\). Now multiplying both sides of \((\mu_{p,f,\xi_i,j}) = C_{\mathcal{O}[[r]]}(S_{\text{Gr}}(A_f(\xi_i \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi) \phi\) by \(\prod_{\ell \in \Sigma_0} P_{\ell,i}(\ell^{-j-1} \gamma)\), the lemma follows. \(\Box\)

We now state our main result on the Iwasawa Main Conjecture for Rankin-Selberg \(L\)-function mod \(\pi\). By the assumption (Sel-tors), we have \(S_{\text{Gr}}(A_f/\mathbb{Q}_{\text{cyclic}}) \phi\) is a torsion \(\mathcal{O}[[\Gamma]]\)-module (also see Remark 0.5)

**Theorem 6.4.** We assume that (IMC) holds for \(f \otimes \xi_1\) and \(f \otimes \xi_2\) with \(l - 1 \leq j \leq k - 2\). Let \(f \in S_k(\Gamma_0(N), \eta), h \in S_l(\Gamma_0(I), \psi)\) be \(p\)-ordinary newforms. We assume all the hypotheses of Theorem 3.19 as well as Theorem 5.9 hold. Then for every \(l - 1 \leq j \leq k - 2\), we have
\[
(\mu_{p,f \times h,j}) \equiv C_{\mathcal{O}[[r]]}(S_{\text{Gr}}(A_f/\mathbb{Q}_{\text{cyclic}}) \phi) \mod \pi. \tag{52}
\]

**Proof.** By Theorem 3.19, we have \((\mu_{p,f,\xi_1,j}) \equiv C_{\mathcal{O}[[r]]}(S^\Sigma_{\text{Gr}}(A_f(\xi_2 \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi) \mod \pi, \) for all \(l - 1 \leq j \leq k - 2\). By (IMC) and Lemma 6.3, we have \((\mu_{p,f,\xi_1,j}^\Sigma) = C_{\mathcal{O}[[r]]}(S^\Sigma_{\text{Gr}}(A_f(\xi_2 \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi) \mod \pi\). Thus we have \((\mu_{p,f,\xi_1,j}^\Sigma) = C_{\mathcal{O}[[r]]}(S^\Sigma_{\text{Gr}}(A_f(\xi_2 \omega_p^{-j})/\mathbb{Q}_{\text{cyclic}}) \phi) \mod \pi\). Now it follows from Theorem 5.9 that
\[
(\mu_{p,f,\xi_1,j}^\Sigma) \equiv C_{\mathcal{O}[[r]]}(S^\Sigma_{\text{Gr}}(A_f/\mathbb{Q}_{\text{cyclic}}) \phi) \mod \pi. \tag{53}
\]
By a similar argument as in Lemma 6.3, using [GV, Proposition 2.4] and (49), we obtain

$$C_{Ω[I]}(S_{G Hir}(A_j/Q_{cyc})^\psi) = C_{Ω[I]}(S_{G Hir}(A_j/Q_{cyc})^\psi) \prod_{i \in \Sigma_0} P_i(\ell^{-j-1} \gamma), \quad \forall \ 1 \leq j \leq k - 2.$$ 

where $P_\ell(X) = \det(I - XFG_{[\ell]} \mathcal{Y}^H)$ and $\gamma$ is a topological generator of $\Gamma = \text{Gal}(Q_{cyc}/Q)$. Note that the Euler factor for the Rankin-Selberg $L$-function of $f \otimes h$ at $\ell$ is given by $P_\ell(X)$. Also from (30), we have $\mu_{p,f,x,h,\jmath} = \mu_{p,f,x,h,\jmath} \prod_{i \in \Sigma_0} P_i(\ell^{-j-1} \gamma)$. Following the proof of [GV, Proposition 2.4], we deduce the polynomial $P_\ell(X)$ has $\mu$-invariant zero, that is, $P_\ell(X) \neq 0 \mod \pi$. Cancelling $P_\ell(\ell^{-j-1} \gamma)$ on either side of the congruence (53), the desired congruence modulo $\pi$ follows.

In a special case when either $\mu_{p,f,x,h,\jmath}$ or $C_{Ω[I]}(S_{G Hir}(A_j/Q_{cyc})^\psi)$ is a unit, the congruence in (52) leads to the following:

**Corollary 6.5.** Let the setting be as in Theorem 6.4. Then $\mu_{p,f,x,h,\jmath}$ is a unit in the Iwasawa algebra $\mathcal{O}[[\Gamma]]$ if and only if $C_{Ω[I]}(S_{G Hir}(A_j/Q_{cyc})^\psi)$ is a unit in the Iwasawa algebra $\mathcal{O}[[\Gamma]]$.

7. Examples

In this section, we illustrate Theorem 6.4 and Corollary 6.5.

**Example 7.1.** We take $p = 11$. Let $\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$ be the Ramanujan Delta function. We have $\Delta \in S_{22}(S_{22}(Z))$ and $r(11)$ is 11-adic unit. Consider the quadratic character $\chi_K = (\frac{\cdot}{\mathbb{Q}})$ associated to the field $K = \mathbb{Q}(\sqrt{-23})$. Let $f := \Delta \otimes \chi_K$. Then $f \in S_{22}(\Gamma_0(23^2))$ is a primitive $p$-ordinary form. Moreover, the residual representation $\rho_{\Delta}$ is an irreducible $G_{\mathbb{Q}}$-module.

Having chosen $f$, we next choose the candidate for the normalised cusp form $h$.

Consider the Eisenstein series $g(z) = E_2(z) - 23E_2(3z)$, where $E_2(z) = \frac{1}{2} + \sum_{n=1}^{\infty} \sigma(n) q^n$. Then by a result of Mazur [Ma, Proposition 5.12(ii)], there exists a normalised Hecke eigenform $\tilde{h} \in S_2(\Gamma_0(23))$ with LMFDB label 23.2.a such that $a(n,h) \equiv a(n,g) \mod p$, where $p = (11, -\sqrt{23}) \subset \mathbb{Z}[[1 + \sqrt{23}]/2]$. By [DS, Theorem 9.6.6], we have $\tilde{\rho}_h \simeq \tilde{\rho}_g \simeq \tilde{\omega}_p \oplus 1$. Thus $\xi_1 = \tilde{\omega}_p$ and $\xi_2 = 1$. By our construction, $\xi_1 = \omega_p$ and $\xi_2 = 1$. Thus by (20), we have $g(z) = E(1,1)(z) = E_2(z) - pE_2(pz)$, where $L_p$ is the trivial character modulo $p = 11$. As $\text{cond}(\rho_h) = 23$ and $\text{cond}(\rho_h) = 11$, we obtain $m = 23$ by (19). Thus $\Sigma = \{11, 23, \infty\}$ and $\Sigma_0 = \{23\}$.

As $\rho_{\Delta}$ is irreducible and $p$-distinguished it follows that $\tilde{\rho}_{\rho_h} (= \tilde{\rho}_{\Delta} \otimes \chi_K)$ is also irreducible and $p$-distinguished. Hence the hypotheses of Theorem 3.19 are satisfied.

Next we check the hypotheses of Theorem 5.9. We have $N = N_f = 23^2$, $M_0 = 1$, $(N,M_0) = 1$ and the nebentypus of $h$ i.e. $\psi$ is trivial. Also $A_f[\pi] = A_{\Delta}[\pi] \otimes \chi_K$ is an irreducible $G_{\mathbb{Q}}$-module. Note that Frobp acts on $A_f(\xi_2)$ by the scalar $a(p,f)\xi_2(p) = \chi_K(p)(p)a(p,\Delta) = -\tau(p) = -\tau(11) \neq 1 \mod p$. Thus $H^0(cyc,v,\xi_2) = 0$ so we need not check (ss-red,p).

It remains to check that (Sel-tors) and $(\Sigma^2 - \text{van})$ hold to conclude all the conditions in Theorem 5.9 are met. For this, by Remark 0.5, it suffices to show that $\mu(S_{G Hir}(A_f(\xi_2)/Q_{cyc})^\psi) = 0$ for $i = 1, 2$. Moreover, as $\xi_1 = \omega_p$ and $\xi_2 = 1$, it further reduces to show $\mu(S_{G Hir}(A_f(\xi_2)/Q_{cyc})^\psi) = 0$ for $1 \leq j \leq 10$.

Consider the elliptic curve $E := X_0(11)$ of conductor 11 over $\mathbb{Q}$ (LMFDB label 11.2.a.a) defined by

$$y^2 + y = x^3 - x^2 - 10x - 20.$$ 

Let $f_E$ be the modular form corresponding to $E$ under modularity theorem. For $p = 11$, there is a $\Lambda$-adic form whose specialisation at $k = 2$ is $f_E$ and at $k = 12$ is $\Delta$ [GS, Page 409]. Hence the residual Galois representation of $E$ and $\Delta$ at $p = 11$ are isomorphic. Since $f$ and $f_E \otimes \chi_K$ lie in the same branch of Hida family, it is enough to know that $\mu(S_{G Hir}(f_E \otimes \chi_K \omega_p^{-j}/Q_{cyc})^\psi) = 0$ (cf. [EPW, Theorem 4.3.3]).

By a deep result of Kato [Ka, Theorem 17.4] and [EPW, Theorem 4.3.3] it follows

$$\mu(S_{G Hir}(f_E \otimes \chi_K \omega_p^{-j}/Q_{cyc})^\psi) = \mu(S_{G Hir}(A_f(\xi_2)/Q_{cyc})^\psi) \leq \mu^{an}(f \otimes \omega_p^{-j}) = \mu^{an}(f_E \otimes \chi_K \omega_p^{-j}).$$

Thus it suffices to show that the analytic $\mu$-invariant vanishes i.e. the $p$-adic $L$-function $\mu_{p,f_E,\chi_K^{-j},j}(T)$ is not divisible in the Iwasawa algebra $\mathcal{O}[[T]]$ for all $0 \leq j \leq 9$. We now compute $\mu_{p,f_E,\chi_K^{-j},j}(0)$ for $0 \leq j \leq 9$. 

Let the periods $\Omega_{f_E \otimes \chi_K}^\pm$ be as in Theorem 3.6. From [MTT] (cf. [Ch, Section 5.1]), we get
\[
\mu_{p, f_E \otimes \chi_K, j}(0) = \begin{cases} \frac{1}{\Omega_{f_E \otimes \chi_K}} \sum_{n=1}^{p-1} \omega_p^j(b) x^{\alpha_n}(\omega_p^n(b/p)) & \text{if } \omega_p^j \neq 1, \\ (1 - \alpha)^{-1} f_E \otimes \chi_K & \text{otherwise,} \end{cases}
\]
where $x^\pm$ are the ± modular symbols associated to $f_E \otimes \chi_K$, as defined by Manin and α is a root of $p^3$-Hecke polynomial of $f_E \otimes \chi_K$ with $|\alpha| = 1$. Using SAGE, we compute $x^\pm$. The values are

| $x^+$ | 1/11 | 2/11 | 3/11 | 4/11 | 5/11 | 6/11 | 7/11 | 8/11 | 9/11 | 10/11 |
|-------|------|------|------|------|------|------|------|------|------|------|
| $x^+$ | 2    | 0    | 5    | 0    | 10   | 0    | 5    | 0    | 5    | 0    |
| $x^-$ | 0    | 0    | -5   | 0    | 5    | 0    | -5   | 0    | 5    | 0    |

Let $\zeta_{10}$ be a primitive root of unity and $\omega_p(2) = \zeta_{10}$ and $\omega_p(-1) = -1$. Then it follows $\omega_p(3) = \omega_p(2)^3 \omega_p(-1) = -\zeta_{10}^2$, $\omega_p(4) = \zeta_{10}^4$ and $\omega_p(5) = \omega_p(2)^5 = \zeta_{10} = \zeta_{10}^5 - \zeta_{10}^4 + \zeta_{10} - 1$. Thus from (55),
\[
\mu_{p, f_E \otimes \chi_K, j}(0) \sim \begin{cases} \left(2 + 5\omega_p^j(3) + 5\omega_p^j(4), \right) & \text{if } j \text{ is even and } 10 \not| j, \\ \left(5(-\omega_p^j(3) + \omega_p^j(4)), \right) & \text{if } j \text{ is odd,} \end{cases}
\]
where $\sim$ denotes up to multiplication by a $p$-adic unit. Taking $j = 0, 1, 2, \ldots, 9$ we obtain
\[
\begin{align*}
\mu_{p, f_E \otimes \chi_K, j}(0) & \sim 5\zeta_{10}^3 + 5\zeta_{10}^5 & \mu_{p, f_E \otimes \chi_K, 2}(0) & \sim 5\zeta_{10}^3 - 5\zeta_{10}^5 - 3 \\
\mu_{p, f_E \otimes \chi_K, 0}(0) & \sim -5\zeta_{10}^3 + 5\zeta_{10}^5 - 10\zeta_{10} + 5 & \mu_{p, f_E \otimes \chi_K, 4}(0) & \sim -5\zeta_{10}^3 + 5\zeta_{10}^5 + 2 \\
\mu_{p, f_E \otimes \chi_K, 3}(0) & \sim 0 & \mu_{p, f_E \otimes \chi_K, 6}(0) & \sim -5\zeta_{10}^3 + 5\zeta_{10}^5 + 2 \\
\mu_{p, f_E \otimes \chi_K, 9}(0) & \sim 5\zeta_{10}^3 - 5\zeta_{10}^5 + 10\zeta_{10} - 5 & \mu_{p, f_E \otimes \chi_K, 8}(0) & \sim 5\zeta_{10}^3 - 5\zeta_{10}^5 - 3 \\
\mu_{p, f_E \otimes \chi_K, 0}(0) & \sim -5\zeta_{10}^3 - 5\zeta_{10}^5 & \mu_{p, f_E \otimes \chi_K, 9}(0) & \sim 1.
\end{align*}
\]
For $j = 5$, we have $E \otimes \chi_K \omega_p^j$ is an elliptic curve with LMFB label 64009.d2. Further $\mu_{p, f_E \otimes \chi_K, 5}(T) = Tu(T)$, for some $u(T) \in \mathcal{O}[T]$ using SAGE. Thus $\mu^\ell_{an}(f_E \otimes \chi K \omega_p^j) = 0$ for $1 \leq j \leq 10$. Thus we have completed checking all the hypotheses of Theorem 5.9.

Finally, we need to check the Iwasawa main conjecture for $f \otimes \omega_p^j$ for $i = 1, 2$ to verify Theorem 6.4. We have $\prod_{j=5}^9 \mu_{p, f_E \otimes \chi_K, j}(0) \sim 3003125$, which is a 11-adic unit. This shows that $\prod_{j=0}^9 \mu_{p, f_E \otimes \chi_K, j}(T) = Tu(T)$ for some unit $u(T) \in \mathcal{O}[T]$. By (54), we get $\mu(S_G(f_{A_E \otimes \chi K}(\omega_p^j)/\mathcal{Q}_{cyc})) = \mu_{an}(f_E \otimes \chi K \omega_p^j) = 0$. Now we compute the $\lambda^\ell_{an}(f_E \otimes \chi K \omega_p^j)$ and $\lambda^{alg}(f_E \otimes \chi K \omega_p^j)$ namely the analytic and algebraic $\lambda$-invariants of $f_E \otimes \chi K \omega_p^j$. Once again, it follows from the above computation that $\lambda^\ell_{an}(f_E \otimes \chi K \omega_p^j) = 0$ for $j \neq 5$ and $\lambda_{an}(f_E \otimes \chi K \omega_p^j) = 1$ for $j = 5$. By a similar argument as in (54), we get $\lambda^{alg}(f_E \otimes \chi K \omega_p^j) = \lambda^\ell_{an}(f_E \otimes \chi K \omega_p^j) = 0$ for $1 \leq j \leq 10$ and $j \neq 5$. From the above calculations, we have $\lambda^\ell_{an}(f_E \otimes \chi K \omega_p^j) = 1$. Since the $j$-invariant of the elliptic curve attached to elliptic curve associated to $f_E \otimes \chi K \omega_p^j$ is not an integer, it follows from [Ne, Theorem 1](applied for $k, k_0, k'$ there equal to Q) that $\lambda^{alg}(f_E \otimes \chi K \omega_p^j) \equiv 1 \mod 2$. By the result of Kato $\lambda^{alg}(f_E \otimes \chi K \omega_p^j) \leq \lambda^\ell_{an}(f_E \otimes \chi K \omega_p^j) = 1$. Hence $\lambda^{alg}(f_E \otimes \chi K \omega_p^j) = \lambda^\ell_{an}(f_E \otimes \chi K \omega_p^j) = 1$ and the Greenberg Iwasawa main conjecture holds for $f_E \otimes \chi K \omega_p^j$. Thus the Greenberg Iwasawa main conjecture holds for $f_E \otimes \chi K$ (cf. [EPW, Theorem 5.1.7]). By [EPW, Corollary 5.1.4], it follows that Iwasawa main conjecture also holds for $f \otimes \omega_p^j$.

Hence for $f = \Delta \otimes \chi_K$ and $h \in S_2(\Gamma_0(23))$ of LMFB label 23.2.a, the hypotheses of Theorem 6.4 are met. Therefore by same theorem, we conclude that the Iwasawa main conjecture holds $f \otimes h$ modulo $\pi$, that is
\[
\mu_{p, f \otimes h, j}(T) = (C_{\mathcal{O}[T]}(S_{G_f}(A_j)/\mathcal{Q}_{cyc})) \mod \pi \text{ for } 1 \leq j \leq 10.
\]
Further $\mu^\ell_{an}(f \otimes \omega_p^j) = 0$ and $\lambda^\ell_{an}(f \otimes \omega_p^j) = 0$ for $j \neq 5$, it follows that $\mu_{p, f, j, i}(T)$ is unit whenever $j \neq 5$. Applying Theorem 3.19, we obtain $\mu_{p, f \otimes h, j}$ is a unit whenever $j \neq 4, 5$. Thus, it follows from (56) that $C_{\mathcal{O}[T]}(S_{G_f}(A_j)/\mathcal{Q}_{cyc})$ is a unit in $\mathcal{O}[T]$ for $j \neq 4, 5$ and
\[
\mu_{p, f \otimes h, j} = (C_{\mathcal{O}[T]}(S_{G_f}(A_j)/\mathcal{Q}_{cyc})) = \mathcal{O}[T] \text{ for } 1 \leq j \leq 10 \text{ and } j \neq 4, 5.
\]
Further, if $j = 5$ we have $\mu^\ell_{an}(f \otimes \omega_p^j) = 0$ and $\lambda^\ell_{an}(f \otimes \omega_p^j) = 1$. Thus $(\mu_{p, f \otimes h, j}) \equiv C_{\mathcal{O}[T]}(S_{G_f}(A_j)/\mathcal{Q}_{cyc}) \equiv (T) \mod \pi$, for some unit $u(T) \in \mathcal{O}[T]$ for $j = 4, 5$.

**Example 7.2.** Let $p = 5$. Let $f(z) = \sum a(n, f)q^n \in S_6(\Gamma_0(52))$ be the cusp form given by LMFB label 52.6.a.a. We have $f(z) = q - 9q^2 - 3q^3 + 53q^4 - 218q^9 - 702q^{11} + \cdots$. Thus $f$ is $p$-ordinary and $p \nmid N_f$. 
Consider the Eisenstein series $g'(z) = E_2(z) - 11E_2(11z)$, where $E_2(z) = \frac{1}{12} + \sum_{n=1}^{\infty} \sigma(n)q^n$. Then by a result of Mazur [Ma, Proposition 5.12 (ii)], there exists a normalised Hecke eigenform $h \in S_2(\Gamma_0(11))$ (LMFDB label 11.2.a.a) such that $a(n, h) \equiv a(n, g') \mod 5$.

We verify Theorem 6.4 for $f \otimes h$.

By [DS, Theorem 9.6.6], we have $\rho_p \simeq \bar{\rho}_p \simeq 1$. Thus $\xi_1 = \bar{\omega}_p$ and $\xi_1 = 2$. By our construction, $\xi_1 = \omega_p$ and $\xi_2 = 1$. Thus $g(z) = E(1p, 1)(z) = E_2(z) - \bar{p}E_2(\bar{p}z)$, where $1p$ is the trivial character modulo $p = 5$. As $\text{cond}(\rho_p) = 11$ and $\text{cond}(\bar{\rho}_p) = 5$, we obtain $m = 11$ by (19). Thus $K = \mathbb{Q}$, $\Sigma = \{5, 11, \infty\}$ and $\Sigma_0 = \{11\}$.

Let $F$ be the $\Lambda$-adic newform passing through $f$. Specializing $F$ at $(2, \omega_p)$ gives a weight 2 cusp form, say, $f_2$. Then it known that $f_2 \in S_2(\Gamma_0(260), \varepsilon_{260})$ is congruent to $f$ modulo 5 and $f_2$ is either a newform or 5-stabilisation of a newform in $S_2(\Gamma_0(52), \varepsilon_{52})$. From the LMFDB data, the space of newforms of level 260, weight 2 and trivial nebentypus $S_2^{\text{new}}(\Gamma_0(260), \varepsilon_{260})$ has dimension 4. Also, every newform in $S_2^{\text{new}}(\Gamma_0(260), \varepsilon_{260})$ has 5th Fourier coefficient equal to 1. However $a(5, f) = -3 \neq 1 \mod 5$. Thus $f_2$ must be the 5-stabilization of a newform in $S_2(\Gamma_0(52), \varepsilon_{52})$. From LMFDB, $S_2^{\text{new}}(\Gamma_0(52), \varepsilon_{52})$ has dimension 1 and is spanned by cusp form given by LMFDB label 52.2.a.a. Further this cusp form corresponds to the elliptic curve $E$ given by $y^2 = x^3 + 10$ and we denote it by $f_E$. Note that the residual representation $\bar{\rho}_E$ is an irreducible $G_{\mathbb{Q}}$-module and so is $\bar{\rho}_p$. Thus $f$ satisfies $(irr-f)$. It follows from Theorem 4.1 (iii) that $f$ is $p$-distinguished. Hence the hypotheses of Theorem 3.19 hold for $f$.

Next we check the hypotheses of Theorem 5.9. We have $N = N_f = 52$, $M_0 = 1$, so $(N, M_0) = 1$ and the nebentypus of $h$ i.e. $\psi$ is trivial. Note that Frobp acts on $A_f(\xi_2)$ by the scalar $a(p, f)\xi_2(p) = a(p, f) = -3$. As $a(p, f)\xi_2 \neq 1 \mod 5$, it follows that $H^0(G_{Q, \psi}, A_f(\xi_2)) = 0$ and we need not check (ss-redp).

As explained in Example 7.1, it suffices to check that $\mu^m(f_E \otimes \xi_2^{-1}) = 0$ for $i = 1, 2$ and $1 \leq j \leq 4$ to conclude all the conditions in Theorem 5.9 are met. Since $\xi_1 = \omega_p$ and $\xi_2 = 1$, it is enough to show $\mu^m(f_E \otimes \omega_2^{-1}) = 0$ for $1 \leq j \leq 4$. To do this we proceed as in Example 7.1. Using computations on SAGE, we get

| $x^\pm$ | 1/5 | 2/5 | 3/5 | 4/5 |
|--------|-----|-----|-----|-----|
| $x^+$  | 1   | 1   | 1   | 1   |
| $x^-$  | 1   | 1   | -1  | -1  |

where $x^\pm$ are the $\pm$ modular symbols associated to $f_E$, as defined by Manin.

Let $\zeta_4$ be a primitive root of unity and $\omega_p(2) = \zeta_4$ and $\omega_p(-1) = -1$. Then it follows $\omega_p(3) = -\zeta_4$.

Using the analogue of (55) in this case, we have

\[
\mu_{p,f_E,1,j}(0) \sim \begin{cases} 
2 + \omega_2(2) + \omega_2(3) 
\text{if } j = 2, \\
2 + \omega_2(2) - \omega_2(3) 
\text{if } j = 3, 
\end{cases}
\]

and $\mu_{p,f_E,1,1}(0) = 2(1 + \zeta_4)$.

For $j = 2$, $E \otimes \omega_2^2$ is the elliptic curve given by LMFDB label 1300.d1 with additive reduction at $p = 5$. From the LMFDB, we observe that $E \otimes \omega_2^2$ has analytic rank 1 i.e. $L(s, E \otimes \omega_2^2)$ has a simple zero at $s = 1$. Using SAGE, we get $\mu^m(f_E \otimes \omega_2^2) = 0$ and $\lambda^m(f_E \otimes \omega_2^2) = 1$. Thus $\mu^m(f_E \otimes \omega_2^2) = 0$ for $1 \leq j \leq 4$ and all the hypotheses of Theorem 5.9 are satisfied.

Again by [Ka, Theorem 17.4], we get $\mu^m(f_E \otimes \omega_2^2) = \mu^m(f_E \otimes \omega_2^0) = 0$ and $\lambda^m(f_E \otimes \omega_2^2) = \lambda^m(f_E \otimes \omega_2^0) = 0$ for $1 \leq j \leq 4$ and $j \neq 2$. As explained in Example 7.1, it follows from [Ka, Theorem 17.4] and $p$-parity conjecture (See [Ne, Theorem 1]) that $\lambda^m(f_E \otimes \omega_2^0) = \lambda^m(f_E \otimes \omega_2^2) = 1$. Hence Iwasawa main conjecture holds for $f_E \otimes \omega_2^0$ for $1 \leq j \leq 4$. By [EPW, Corollary 5.1.4] it follows that Iwasawa main conjecture also holds for $f \otimes \omega_2^0$.

Hence the hypotheses of Theorem 6.4 are met and the Iwasawa main conjecture holds $f \otimes h$ modulo 5. Further $\mu^m(f \otimes \omega_2^0) = 0$ and $\lambda^m(f \otimes \omega_2^0) = 0$ for $j \neq 2$, it follows that $\mu_{p,f,E,1,j}(T)$ is unit whenever $j \neq 2$. Applying Theorem 3.19, we obtain $\mu_{p,f,E,1,j}(T)$ is a unit whenever $j \neq 1, 2$. Thus it follows from (56) that $C_{O[[T]]}(S_{Gr}(A_j/Q_{cyc})^\vee)$ is a unit in $O[[T]]$ for $j \neq 1, 2$ and

$$\mu_{p,f\times h,j} = (C_{O[[T]]}(S_{Gr}(A_j/Q_{cyc})^\vee)) = O[[T]] \quad \text{for } j = 3, 4.$$ (58)

Also $(\mu_{p,f\times h,j}) \equiv C_{O[[T]]}(S_{Gr}(A_j/Q_{cyc})^\vee) \equiv (T) \mod \pi$ for $j = 1, 2$. 


Example 7.3. Let \( p = 5 \). Let \( E \) be the elliptic curve given by \( y^2 + y = x^3 + x^2 - 9x - 15 \). By modularity theorem, \( E \) corresponds to a weight 2 cuspidal form \( f_E = q - 2q^3 - 2q^4 + 3q^5 - q^7 + q^9 + 3q^{11} + \cdots \) (LMFDB label 19.2.a.a). Then \( f_E \) is 5-ordinary and \( \rho_{f_E} \) is irreducible. Let \( F \) be the \( \Lambda \)-adic cuspidal form passing through \( f_E \). Specializing \( F \) at \((6, \zeta_6)\) gives a weight 6 cuspidal form of the family, say \( f_6 \). Then it is known that \( f_6 \in S_6(\Gamma_0(5), t_{85}) \) is congruent to \( f_E \) modulo 5. From the LMFDB data, \( S^\text{new}_6(\Gamma_0(5), t_{85}) \) doesn't contain any 5-ordinary cuspidal form. Hence \( f_6 \) is 5-stabilisation of a newform \( f \) in \( S_6(\Gamma_0(19), \chi_{19}) \). This newform \( f \) belongs to newform orbit given by LMFDB label 19.6.a.d.

Let \( f \) be as above and take \( h \in S_2(\Gamma_0(11)) \) (LMFDB label 11.2.a.a) as described in Example 7.2. We verify Theorem 6.4 for \( f \otimes h \).

Since \( f \equiv f_E \) it follows that \( f \) satisfies (irr-\( f \)) and (p-dist). Again in this case, the hypotheses of Theorem 3.19 hold for \( f \).

Next we check the hypotheses of Theorem 5.9. In this case \( N = N_f = 19, M_0 = 1, (N, M_0) = 1 \) and the nebentypus of \( h \) i.e. \( \psi \) is trivial. Again we compute \( a(p, f_E) \not\equiv 1 \mod 5 \) and as explained in the previous examples, it follows that \( H^0(I_{\text{cyc}, v}, \mathcal{A}_f^\infty(\zeta_2)) = 0 \), so we need not check (ss-red\(_p\)).

Also, as in Example 7.1 and 7.2, it suffices to check that \( \mu^{\text{alg}}(f_E \otimes \xi(\omega_p)^{-1}) = 0 \) for \( i = 1, 2 \) to conclude all the conditions in Theorem 5.9 are met. Since \( \xi = \omega_p \) and \( \xi_2 = 1 \), it is enough to show \( \mu(S^\text{new}_6(\mathcal{A}_f(\omega_p^{-1})/\mathbb{Q}_2)) = 0 \) for \( 1 \leq j \leq 4 \). Again take \( \Omega_{f_E}^p \) be as in Theorem 3.6. Proceeding as in the previous two examples, we compute via SAGE the values of the modular symbols \( x^\pm \):

\[
\begin{array}{cccc}
\downarrow & 1/5 & 2/5 & 3/5 & 4/5 \\
x^- -1/2 & 1 & 0 & -1/2 \\
x^+ & 1/2 & 0 & 0 & -1/2
\end{array}
\]

Let \( \zeta_4 \) be a primitive root of unity such that \( \omega_p(2) = \zeta_4 \). Then it follows \( \omega_p(3) = -\zeta_4 \). Thus

\[\mu_{p, f_E, 1, j}(0) \sim \begin{cases} (-1 + \omega_p^j(2) + \omega_p^j(3)), & \text{if } j \text{ is even and } 4 \nmid j, \\ 1, & \text{if } j \text{ is odd}, \end{cases}\]

here \( \sim \) denotes equality up to multiplication by a \( p \)-adic unit. Taking \( j = 0, 1, 2, 3 \) we obtain

\[
\begin{align*}
\mu_{p, f_E, 1, j}(0) &\sim 1 & \mu_{p, f_E, 1, 2}(0) &\sim -3 \\
\mu_{p, f_E, 1, 3}(0) &\sim 1 & \mu_{p, f_E, 1, 0}(0) &\sim 1/3.
\end{align*}
\]

Hence \( \mu^{\text{alg}}(f_E \otimes \omega_p^j) = 0 \) and \( \lambda^{\text{alg}}(f_E \otimes \omega_p^j) = 0 \) for \( 1 \leq j \leq 4 \). Thus we have verified all the hypotheses of Theorem 5.9.

Once again by [Ka, Theorem 17.4], it follows that \( \mu^{\text{alg}}(f_E \otimes \omega_p^j) = \mu^{\text{alg}}(f_E \otimes \omega_p^j) = 0 \) and \( \lambda^{\text{alg}}(f_E \otimes \omega_p^j) = \lambda^{\text{alg}}(f_E \otimes \omega_p^j) = 0 \) for \( 1 \leq j \leq 4 \). It follows that the Iwasawa main conjecture holds for \( f \otimes \omega_p^j \) and \( \mu_{p, f, 1, j} \) is a \( p \)-adic unit for all \( 1 \leq j \leq 4 \).

Hence the hypotheses of Theorem 6.4 are met and the Iwasawa main conjecture holds \( f \otimes h \) modulo \( \pi \). Further, applying Theorem 3.19, we obtain \( \mu_{p, f \otimes h, j} \) is a unit for all \( j \). Thus, it follows from (56) that \( C_{[\Omega]}(\mathcal{S}_G(\mathcal{A}_j)/\mathbb{Q}_{\text{cyc}}^\infty) \) is a unit in \( \mathcal{O}[[\Gamma]] \) for all \( j \) and

\[
(\mu_{p, f \otimes h, j}) = (C_{[\Omega]}(\mathcal{S}_G(\mathcal{A}_j)/\mathbb{Q}_{\text{cyc}}^\infty)) = \mathcal{O}[[\Gamma]] \quad \text{for } 1 \leq j \leq 4.
\]
Introduction to the arithmetical theory of automorphic functions

[S2] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Publ. Math. Soc. Japan, No. 11.

[SU] C. Skinner and E. Urban, The Iwasawa main conjectures for GL_2, Invent. Math. 195(1) (2014), 1–277.

[SZ] C. Skinner and W. Zhang, Indivisibility of Heegner points in the multiplicative case, preprint arXiv:1407.1099.

[VA] V. Vatsal, Canonical periods and congruence formulae, Duke J. Math. 98 (1999), 397–419.

[We] T. Weston, Iwasawa invariants of Galois deformations, Manuscripta Math. 118(2) (2005), 161–180.

[Wa] X. Wan, Iwasawa main conjecture for Rankin-Selberg p-adic L-functions, Algebra and Number Theory 14(2) (2020), 383–483.

SOMNATH JHA, DEPARTMENT OF MATHEMATICS AND STATISTICS, IIT KANPUR, KANPUR-208016, INDIA
Email address: jhasom@iitk.ac.in

SUDHANSHU SHEKHAR, DEPARTMENT OF MATHEMATICS AND STATISTICS, IIT KANPUR, KANPUR-208016, INDIA
Email address: sudhansh@iitk.ac.in

RAVITHRA VANGALA, DEPARTMENT OF MATHEMATICS, IISc BANGALORE, BANGALORE-560012, INDIA
Email address: ravithreav@iisc.ac.in