A Novel Extension of Randomly Weighted Average

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Abstract We study a well-known problem concerning a random variable $Z$ uniformly distributed between two independent random variables. A new extension has been introduced for this problem and fairly large classes of randomly weighted average distributions are identified by their generalized Stieltjes transforms. In this article we employ the Schwartz distribution theory for finding distributions of this extension; we also study some of their properties.

Keywords Schwartz theory on distributional derivatives \cdot GTSP-N \cdot Semicircle distribution \cdot Randomly weighted average \cdot Stieltjes transform

1 Introduction

Van Assche (1987) introduced the notion of a random variable $Z$ uniformly distributed between two independent random variables $X_1$ and $X_2$, which arose in studying the distribution of products of random $2 \times 2$ matrices for stochastic search of global maxima. By letting $X_1$ and $X_2$ to have identical distributions, he derived that: (i) for $X_1$ and $X_2$ on $[-1, 1]$, $Z$ is uniform on $[-1, 1]$ if and only if $X_1$ and $X_2$ have arcsin distribution; and (ii) $Z$ possesses the same distribution as $X_1$ and $X_2$ if and only if $X_1$ and $X_2$ are degenerated or have a Cauchy distribution. Soltani and Homei (2009a) extended Van Assche’s results as follows: They put $X_1, \ldots, X_n$ to be independent, and considered for $n \geq 2$

$$S_n(R_1, \ldots, R_{n-1}) = R_1X_1 + R_2X_2 + \cdots + R_{n-1}X_{n-1} + R_nX_n,$$

(1.1)

where random proportions are $R_i = U_{(i)} - U_{(i-1)}$ (for any $i \in \{1, \ldots, n-1\}$) and $R_n = 1 - \sum_{i=1}^{n-1} R_i$, $U_{(1)}, \ldots, U_{(n-1)}$ are order statistics from a uniform distribution.
on $[0,1]$, and $U_{(0)} = 0$. Soltani and Roozegar (2012), defined fairly large classes of randomly weighted average (RWA) distributions by using new random weights. These are cuts of $[0,1]$ by $U_{(k_1)}, \ldots, U_{(k_n-1)}$, where

$$R(k_j) = U(k_j) - U(k_{j-1}), \ j = 1, \ldots, n, \hspace{10pt} \text{and} \hspace{10pt} U(k_0) = 0 \hspace{10pt} \text{and} \hspace{10pt} U(k_n) = 1,$$

and

$$S_n^* (k_1, \ldots, k_{n-1}) = \sum_{j=1}^{n} R(k_j) X_j, \hspace{10pt} k_n = n^*_0. \hspace{10pt} (1.2)$$

They employ generalized Stieltjes transform for RWA’s distributions and observe some interesting results. In this article, we follow the work of Homei (2012) for finding RWA’s distributions by using Schwartz distribution theory.

2 Some Earlier Results

In this section, we first review some results of Homei (2012) and then modify them a little bit to fit in our framework for getting Theorem 1. We provide the conditional distribution of $S_n^* (k_1, \ldots, k_n) = \sum_{j=1}^{n} R(k_j) X_j$ for given $(X_1, \ldots, X_n) = (x_1, \ldots, x_n)$ at $z$, denoted by $k(z|x_1, \ldots, x_n)$. At first we assume $x_1 > x_2 > \ldots > x_n > 0$, but later we will remove this restriction on $x_1, \ldots, x_n$. We recall that $(U_{(1)}, \ldots, U_{(n)})$ is the order statistics of a random sample $U_1, \ldots, U_n$ for the uniform $[0,1]$. The sequence of indices $\{k_1, \ldots, k_{n-1}\}$ is an ordered subsequence of $\{1, \ldots, n^* - 1\}$. Thus $\{U_{k_1}, \ldots, U_{k_{n-1}}\} \subset \{U_{(1)}, \ldots, U_{(n^* - 1)}\}$ and the increments $R(k_j)$ are defined by $R(k_j) = U(k_j) - U(k_{j-1})$, $j = 1, \ldots, n - 1$, where $U(k_0) = 0$ and $U(k_n) = 1$. Since $\sum_{j=1}^{n} R(k_j) = 1$, then $k(z|x_1, \ldots, x_n)$ can be expressed as

$$P(\sum_{j=1}^{n-1} c_j R(k_j) \leq z - x_n); \hspace{10pt} c_j = x_j - x_n, \hspace{10pt} j = 1, \ldots, n - 1, \hspace{10pt} (2.1)$$

the distribution $\sum_{j=1}^{n-1} c_j R(k_j)$ was derived by Weisberg (1971) as

$$P(\sum_{j=1}^{n-1} c_j R(k_j) \leq z - x_n) = 1 - \sum_{j=1}^{r} \frac{h_j^{m_j - 1}(x_j; z)}{(m_j - 1)!}, \hspace{10pt} (2.2)$$

where $m_j = k_j - k_{j-1}$, $j = 1, \ldots, n$, $k_n = n^*$, $\sum_{j=1}^{n} m_j = n^*$, $h_j^{(m_j - 1)}(c_j)$ is the $(m_j - 1)$-th derivative of

$$h_j(x; z) = \frac{(x - z)^{n^* - 1}}{c_j \prod_{j \neq j}(x - x_j)^{m_j - 1}}, \hspace{10pt} (2.3)$$

at $x$ evaluated at $x_1, x_2, \ldots, x_n$, where $r$ is the largest positive integer with $z < x_r$. The distribution of $\sum_{j=1}^{n-1} c_j R(k_j)$ in (2.2) can alternatively be expressed as

$$P(\sum_{j=1}^{n-1} c_j R(k_j) \leq z - x_n) = \sum_{j=r+1}^{n} \frac{f_j^{(m_j - 1)}(x_j; z)}{(m_j - 1)!}, \hspace{10pt} (2.4)$$
where \( r^* \) is the largest positive integer such that \( x_{r^*} \geq z \), and \( f_j^{(m_j-1)}(x_j; z) \) are the \((m_j - 1)\)-th derivatives of

\[
f_j(x; z) = \frac{(x - z)^{n^* - 1}}{\prod_{i \neq j} (x - x_i)^{m_i}},
\]

at \( x = x_j \).

By using the heaviside function, the distribution in (2.4) can be expressed as

\[
k(z | x_1, ..., x_n) = \sum_{j=1}^{n} f_j^{(m_j-1)}(x_j; z) U(z - x_j)
\]

for any set of distinct values \( x_1, ..., x_n \), and any

\[
z \in [\min(x_1, ..., x_n), \max(x_1, ..., x_n)].
\]

also

\[
k(z | x_1, ..., x_n) = \begin{cases} 0 & \text{if } z < \min(x_1, ..., x_n) \\ 1 & \text{if } z \geq \min(x_1, ..., x_n). \end{cases}
\]

Indeed, \( k(z; x_1, ..., x_n) \) is a big family of distributions that include Two-Sided Power (TSP) distributions and General Two-Sided Power (GTSP) distributions, see Soltani and Homei (2009b). The conditional kernel presented in (2.6) leads us to a fairly large class of conditional kernels. Indeed, we define

\[
k(g | x_1, ..., x_n) = \sum_{j=1}^{n} \frac{(-1)^{m_j-1}}{(m_j - 1)!} \frac{d^{m_j-1}}{dx_j^{m_j-1}} \frac{g(x_j)}{\prod_{i \neq j} (x_i - x_j)^{m_i}}.
\]

For any function \( g : \mathbb{R} \to \mathbb{C} \) the kernel given in (2.8) will be reduced to (2.6) if

\[
g_z(x) = (z - x)^{n^* - 1} U(z - x).
\]

Following Van Assche (1987), we will also apply the concept of the distribution function distributional derivative \( A^{(n)} \) for a given integer \( n \), such that

\[
\int_{-\infty}^{+\infty} \varphi(x) A^{(n)}(dx) = \frac{(-1)^n}{n!} \int_{-\infty}^{+\infty} \left[ \frac{d^n}{dx^n} \varphi(x) \right] A(dx).
\]

For certain infinitely differentiable function \( \varphi \), the following lemma provides a useful integral relation between the conditional kernel (2.8) and the distribution derivative of \( F \), the distribution of the random mixture \( S_n \cdot (k_1, ..., k_{n-1}) \).

**Lemma 1** The \((n^* - 1)\)-th distributional derive of the distribution \( F \) and the conditional kernel (2.8) are subject to

\[
\int_{\mathbb{R}} g(x) dF^{(n^* - 1)}(x) = \int_{\mathbb{R}^n} k(g | x_1, ..., x_n) \prod_{i=1}^{n} F_{X_i}(dx_i),
\]

for any infinitely differential function \( g \) for which the integrals are finite.
Proof First we verify (2.10) with \( g_z(x) = (z - x)^{n^*-1} U(z - x) \), where \( z \) is a fixed real number. Indeed,

\[
U(z - x) = \frac{(-1)^{n^*-1}}{(n^* - 1)!} \frac{d^{n^*-1}}{dx^{n^*-1}} \left[ (z - x)^{n^*-1} U(z - x) \right]
\]

\[
= \frac{(-1)^{n^*-1}}{(n^* - 1)!} \frac{d^{n^*-1}}{dx^{n^*-1}} g_z(x),
\]

thus

\[
P(S_n^*(k_1, ..., k_{n-1}) \leq z) = \int_{\mathbb{R}} U(z - x) dF(x)
\]

\[
= \int_{\mathbb{R}^n} k(z|x_1, ..., x_n) \prod_{i=1}^{n} F_{X_i}(dx_i),
\]

therefore,

\[
\left( \frac{-1}{n^* - 1} \right)^{n^*-1} \int_{\mathbb{R}} \left[ \frac{d^{n^*-1}}{dx^{n^*-1}} g_z(x) \right] dF(x) = \int_{\mathbb{R}^n} k(g_z|x_1, ..., x_n) \prod_{i=1}^{n} F_{X_i}(dx_i).
\]  \hfill (2.11)

Now (2.11) together with (2.9) will lead us to

\[
\int_{\mathbb{R}} g_z(x) dF^{(n^*-1)}(x) = \int_{\mathbb{R}^n} k(g_z|x_1, ..., x_n) \prod_{i=1}^{n} F_{X_i}(dx_i).
\]  \hfill (2.12) \square

Since the conditional kernel \( k(g \mid x_1, ..., x_n) \) is linear in \( g \), by using an argument similar to the one given by Van Assche (1987) we can enlarge the class of functions for which (2.10) holds for the class of infinitely differentiable functions where the corresponding integrals are finite. The Stieltjes transform has appeared to be an appropriate tool for investigating how the distributions of the random mixture \( S_n^*(k_1, ..., k_{n-1}) \) is related to the distributions of \( X_1, ..., X_n \). The Stieltjes transform of a distribution \( H \) is defined by

\[
S(H, z) = \int_{\mathbb{R}} \frac{1}{z - x} H(dx), \ z \in \mathbb{C} \cap (\text{supp} H)^c,
\]  \hfill (2.13)

where \( \text{supp} H \) stands for the support of \( H \). The following is the main theorem of this section that expresses the Stieltjes transform of the random mixture with respect to those of \( X_1, ..., X_n \).

**Theorem 1** Assume \( X_1, ..., X_n \) are independent and continuous. Then for any complex number \( z \in \mathbb{C} \cap \bigcap_{i=1}^{n} (\text{supp} F_{X_i})^c \) the following identity holds:

\[
\left( \frac{-1}{n^* - 1} \right)^{n^*-1} \frac{d^{n^*-1}}{dz^{n^*-1}} S(F, z) = \prod_{i=1}^{n} \left( \frac{-1}{m_i - 1} \right)^{m_i-1} \frac{d^{m_i-1}}{dz^{m_i-1}} S(F_{X_i}, z).
\]
**Proof** Let \( h_z(x) = \frac{1}{z - x} \). From (2.12) we have

\[
S(F^{(n^* - 1)}, z) = \int_{\mathbb{R}^n} k(h_z | x_1, \ldots, x_n) \prod_{i=1}^n F_{X_i}(dx_i).
\]

On the other hand

\[
k(h_z | x_1, \ldots, x_n) = \sum_{j=1}^n (-1)^{m_j-1} \frac{d^{m_j-1}}{dz^{m_j-1}} \frac{1}{(z-x_j)^{m_j}} \prod_{i \neq j} (x_i - x_j)^{m_i},
\]

which is equal to

\[
- \prod_{j=1}^n \frac{(-1)^{m_j}}{(z-x_j)^{m_j}}.
\]

Therefore,

\[
S(F^{(n^* - 1)}, z) = - \prod_{j=1}^n (-1)^{m_j} \int_{\mathbb{R}} \frac{1}{(z-x_j)^{m_j}} F_{X_j}(dx_j).
\]

But since

\[
\frac{d^{m_j-1}}{dz^{m_j-1}} \frac{1}{z-x} = (-1)^{m_j-1}(m_j-1)! \frac{1}{(z-x)^{m_j}},
\]

we obtain that

\[
S(F^{(n^* - 1)}, z) = (-1)^{n^*-1} \prod_{j=1}^n \frac{(-1)^{m_j-1}}{(m_j-1)!} \frac{d^{m_j-1}}{dz^{m_j-1}} S(F_{X_j}, z).
\]

Also we note that

\[
S(F^{(n^* - 1)}, z) = \int \frac{1}{z-x} F^{(n^* - 1)}(dx)
\]

\[
= \frac{(-1)^{n^*-1}}{(n^*-1)!} \int (n^*-1)! \frac{1}{(z-x)^{n^*}} F(dx)
\]

\[
= \frac{1}{(n^*-1)!} \frac{d^{n^*-1}}{dz^{n^*-1}} \int \frac{1}{z-x} F(dx)
\]

\[
= \frac{1}{(n^*-1)!} \frac{d^{n^*-1}}{dz^{n^*-1}} S(F, z),
\]

giving the result. The proof of the theorem is now complete. \( \square \)

### 3 Some properties

In this section some examples for applications of Theorem 1 are presented, and then some properties are introduced which follow from the properties of \( X \)’s.
3.1 Characterization

Let us now apply Homei’s results in (1.2).

**Theorem 2** Let $X_1$, $X_2$ and $X_3$ be i.i.d random variables on $[-1, 1]$. Then

(i) For $m_1 = m_2 = m_3 = 1$ we have
   
   (a) $S_3$ has semicircle distribution on $[-1, 1]$ if and only if $X_1$, $X_2$ and $X_3$ have arcsin distribution.
   
   (b) $X_1$, $X_2$ and $X_3$ have semicircle distribution on $[-1, 1]$ if and only if $S_3$ has power semicircle distribution on $[-1, 1]$, i.e.,
   
   $$f(z) = \frac{16}{5\pi} (1 - z^2)^{\frac{5}{2}}, \quad -1 \leq z \leq 1.$$
   
   (c) $X_3$ has arcsin distribution if and only if $S_3$ has power semicircle distribution on $[-1, 1]$, i.e.,
   
   $$f(z) = \frac{8}{3\pi} (1 - z^2)^{\frac{3}{2}}, \quad -1 \leq z \leq 1.$$

(ii) For $m_1 = 3, m_2 = m_3 = 1$ we have
   
   (d) if $X_2, X_3$ have arcsin distribution, then $X_3$ has semicircle distribution on $[-1, 1]$ if and only if $S_3$ has power semicircle distribution on $[-1, 1]$, i.e.,
   
   $$f(z) = \frac{8}{3\pi} (1 - z^2)^{\frac{3}{2}}, \quad -1 \leq z \leq 1.$$
   
   (iii) For $m_1 = 1, m_2 = 1, m_3 = 2$ we have
   
   (e) $X_1, X_2$ and $X_3$ have arcsin distribution on $[-1, 1]$ if and only if $S_3$ has a semicircle distribution on $[-1, 1]$.

**Proof** (a) For the “only if” part we assume that $S_3$ has semicircle distribution on $[-1, 1]$; then $\frac{1}{2}S''(F_Z, z) = (z^2 - 1)^{-\frac{3}{2}}$. For the “if” part assuming that $X_1, X_2$ and $X_3$ have arcsin distribution it follows from Theorem 1 that $S(F_X, z) = (z^2 - 1)^{-\frac{1}{2}}$.

(b) By Theorem 1, we have
   
   $$\frac{1}{2} S''(F_Z, z) = 8(z - \sqrt{z^2 - 1})^3 \quad \text{(for the “if” part)},$$
   
   $$S(F_X, z) = (2(z - \sqrt{z^2 - 1}))^{\frac{1}{2}} \quad \text{(for the “only if” part)}.$$

(c) By Theorem 1, we have
   
   $$\frac{1}{2} S''(F_Z, z) = 4(z - \sqrt{z^2 - 1})^2 S(F_X, z) \quad \text{(for the “if” part)},$$
   
   $$S(F_X, z) = \frac{1}{\sqrt{z^2 - 1}} \quad \text{(for the “only if” part)}.$$

(d) By Theorem 1, we have for the “if” part:
   
   $$\frac{2}{(z^2 - 1)^{\frac{3}{2}}} = S''(F_X, z) \quad \text{and for the “only if” part:}$$
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\[ S(F_X, z) = \left( \frac{z^2}{(z^2 - 1)^2} - \frac{1}{\sqrt{z^2 - 1}} \right) \left( \frac{1}{\sqrt{z^2 - 1}} \right) \left( \frac{1}{\sqrt{z^2 - 1}} \right). \]

(e) By Theorem 1, we have (for the “if” part)

\[ \frac{1}{6} S'''(F, z) = S(F_{X_1}, z)S(F_{X_2}, z)S'(F_{X_3}, z) = \frac{-z}{(z^2 - 1)^2}, \quad \text{and} \]

\[ S(F_{X_i}, z) = \left( \frac{1}{\sqrt{z^2 - 1}} \right)^2 \left( \frac{-z}{(z^2 - 1)^2} \right) \quad i = 1, 2, 3 \quad \text{(for the “only if” part)}. \]

\[ \square \]

The work of Homei (2012) has been summarized in Theorem 3 and Remark 1.

**Theorem 3** If \( X_1 \) and \( X_2 \) are independent random variables with a common distribution \( F_{X_i} \), then the characterizations of \( S_n^* (k_1) \) for \( (m_1 = 1, m_2 = 1) \) and \( (m_1 = 1, m_2 = 2) \) are identical.

**Proof** We note that \( X_1 \) and \( X_2 \) have a common distribution function \( F_{X_i} \). By using Theorem 1 for \( (m_1 = 1, m_2 = 2) \), we have

\[ -\frac{1}{2} S''(F_{S_n^*(k_1)}, z) = S(F_X, z)S'(F_X, z), \]

and so

\[ -S''(F_{S_n^*(k_1)}, z) = \frac{d}{dz} S^2(F_X, z), \]

and

\[ -S'(F_{S_n^*(k_1)}, z) = S^2(F_X, z). \quad (3.1) \]

We note that the Stieltjes transform tends to zero when \( z \) is sufficiently large. In that case the constant in the differential equation will be zero. The equation (3.1) is exactly the equation obtained by Van Assche (1987) when \( X_1 \) and \( X_2 \) have a common distribution; so his results hold in our framework as well. \[ \square \]

**Remark 1** Let \( S_{n^*} \) be the randomly weighted average given in (1.2). Assume random variables \( X_1, X_2 \) are independent and continuous, \( X_i \sim F_{X_i}, i = 1, 2 \). Then

\[ B(n_1, n_2)S^{(n_1+n_2-1)}(F_Z, z) = -S^{(n_1-1)}(F_{X_1}, z)S^{(n_2-1)}(F_{X_2}, z) \]

holds for any \( z \in \mathbb{C} \cap \bigcap_{i=1}^2 (\text{supp} F_{X_i})^c \). \[ \triangle \]
3.2 Limit properties

Let \(X_1, X_2, \cdots\) be a sequence of independent identically random variables and \(\{R_i\}\) be a sequence as (1.1). It is natural to ask whether \(S_n(R_1, \ldots, R_{n-1})\) converges in probability or not. The present subsection is devoted to those sorts of questions.

**Lemma 2** Let \(\{X_k\}\) be a sequence of independent, identically distributed random variables with \(E(|X_k|) < \infty\) and \(E(X_k) = \mu\). Let \(a_{nk}\) satisfy these conditions:

(i) \(\lim_{n \to \infty} a_{nk} = 0\) for every \(k\),

(ii) \(\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 1\), and

(iii) \(\sum_{k=1}^{\infty} a_{nk} \leq M\) for all \(n\).

If \(\max |a_{nk}| \to 0\) as \(n \to \infty\), then \(\sum_k a_{nk} X_k \to \mu\) in probability.

**Proof** See Pruitt (1966).

**Lemma 3** Assume \(\{R_i\}\) is given by (1.1). Then \(\max R_i \to 0\), w.p.1, as \(n \to \infty\).

**Proof** It was shown that the maximum spacings tend to zero when the sample is taken from uniform distribution on \([0, 1]\), see Devroye (1981). Then obviously \(R_i \to 0\), w.p.1, as \(n \to \infty\).

We are now in a position to state the converges in probability on \(S_n(R_1, \ldots, R_{n-1})\).

**Theorem 4** Let \(\{X_n\}\) be a sequence of independent, identically distribution with \(E(|X_k|) < \infty\), \(E(X_k) = \mu\) and \(P(\max R_i \to 0) = 1\). Then:

\[S_n(R_1, \ldots, R_{n-1}) \to \mu\text{ in probability.}\]

**Proof** The proof of theorem is an easy consequence of the above lemmas.

3.3 Equality

One may guess that \(S_n(R_1, \ldots, R_{n-1})\) have minimum variance (as is the case for \(\mathcal{F}\)) but following theorem shows that this is not true.

**Theorem 5** \(S_n(R_1, \ldots, R_{n-1})\) does not have minimum variance in class \(\mathcal{C}\).

\[\mathcal{C} = \{S_n(W_1, \ldots, W_{n-1}) : W_1 = V(i) - V(i-1), V_n = 1, V_0 = 0\},\]

where \(V_i\) have a power distribution with parameter \(\theta\).

**Proof** We have \(V_\theta(S_n(W_1, \ldots, W_{n-1})) = (\sum_{i=1}^{n} EW_i^2)\sigma^2\), and variance for RWA in class \(\mathcal{C}\) is

\[V_\theta(S_n(W_1, \ldots, W_{n-1})) = \sigma^2 \left( \frac{2n\theta}{\theta + 2} - \frac{2n\theta}{n\theta + 1} + 1 \right) - \sum_{i=1}^{n-1} \sum_{k=0}^{n-i-1} \frac{2n!\theta^2(-1)^{n-i-1-k}}{(i\theta + 1)(i - 1)!k!(n - i - 1 - k)(n\theta - k\theta + 2)}\).


Differentiating $V_\theta(S_n(W_1, ..., W_n))$ with respect to $\theta$ and evaluating at $\theta = 1$, requires a tedious work, which finally results in

$$\frac{\partial V_\theta(S_n(W_1, ..., W_{n-1}))}{\partial \theta} |_{\theta=1} = \frac{2n - 2(n + 2) \sum_{i=2}^{n+1} \frac{1}{i}}{(n + 1)(n + 2)^2}.$$ 

When $n \geq 2$ the last expression is negative, so in $\theta = 1$, $V_\theta(S_n(W_1, ..., W_{n-1}))$ is decreasing. On the other hand, this function on $\theta$ is continuous, so there is some $\theta$ in which the value of function is less than the value at $\theta = 1$. □

As can be seen from the below figure, the function becomes increasing when the interval $(1, 2)$ is omitted from the domain. Thus in that case the minimum value of the function on the near domain is achieved at $\theta = 1$.

**Fig. 1** $V_\theta((S_n(W_1, ..., W_n)))$ \( \theta \geq 1, \ n = 10, \ n = 20, \ n = 40. \)
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