Solution of the $SU(N)$ Vertex Model with Non-Diagonal Open Boundaries

W. Galleas and M.J. Martins

Universidade Federal de São Carlos  
Departamento de Física  
C.P. 676, 13565-905 São Carlos(SP), Brasil

Abstract

We diagonalize the double-row transfer matrix of the $SU(N)$ vertex model for certain classes of non-diagonal boundary conditions. We derive explicit expressions for the corresponding eigenvectors and eigenvalues by means of the algebraic Bethe ansatz approach.

PACS numbers: 05.50+q, 02.30.IK  
Keywords: Algebraic Bethe Ansatz, Lattice Models, Open Boundary Conditions

July 2004
The study of integrable models of statistical mechanics with arbitrary boundary conditions has gained a tremendous impulse after the work by Sklyanin [1]. This author has been able to generalize the quantum inverse scattering method [2] to include the important case of open boundaries. It turns out that an exactly solved spin chain with open boundary condition can be obtained through the following double-row transfer matrix

\[ T(\lambda) = \text{Tr}_{\mathcal{A}} \left[ K_{\mathcal{A}}^{(+)}(\lambda) T_{\mathcal{A}}(\lambda) K_{\mathcal{A}}^{(-)}(\lambda) T_{\mathcal{A}}^{-1}(-\lambda) \right] \]  

(1)

where \( T_{\mathcal{A}}(\lambda) = \mathcal{L}_{\mathcal{A}L}(\lambda) \mathcal{L}_{\mathcal{A}L-1}(\lambda) \ldots \mathcal{L}_{\mathcal{A}1}(\lambda) \) is the standard monodromy matrix that generates the corresponding closed spin chain with \( L \) sites [2].

We recall that the symbol \( \mathcal{A} \) denotes a \( N \)-dimensional auxiliary space and \( \lambda \) parameterizes the integrable manifold. The operator \( \mathcal{L}_{\mathcal{A}j}(\lambda) \) represents the bulk weights of the corresponding vertex model whose transfer matrix commutes with the spin chain Hamiltonian. The \( N \times N \) matrices \( K_{\mathcal{A}}^{(\pm)}(\lambda) \) describe the interactions at the right and left ends of the open chain. One of the simplest integrable system is the fundamental \( SU(N) \) vertex model [3] whose Boltzmann weights are given by

\[ \mathcal{L}_{\mathcal{A}j}(\lambda) = a(\lambda) \sum_{\alpha=1}^{N} \hat{e}^{(A)}_{\alpha\alpha} \otimes \hat{e}^{(j)}_{\alpha\alpha} + b(\lambda) \sum_{\alpha,\beta=1}^{N} \hat{e}^{(A)}_{\alpha\beta} \otimes \hat{e}^{(j)}_{\beta\alpha} + \sum_{\alpha,\beta=1}^{N} \hat{e}^{(A)}_{\alpha\beta} \otimes \hat{e}^{(j)}_{\beta\alpha} \]  

(2)

where \( a(\lambda) = \lambda + 1, b(\lambda) = \lambda \) and \( \hat{e}_{ij}^{(V)} \) are the usual Weyl matrices acting on the space \( V \). A quite general class of open boundary conditions for this vertex model is represented by the following \( K \)-matrices [4] [5] [6]

\[ K_{\mathcal{A}}^{(\pm)}(\lambda) = M_{\mathcal{A}}^{(\pm)} D_{\mathcal{A}}^{(\pm)}(\lambda) \left[ M_{\mathcal{A}}^{(\pm)} \right]^{-1} \quad D_{\mathcal{A}}^{(\pm)}(\lambda) = \sum_{\alpha=1}^{N} \varepsilon^{(\pm)}_{\alpha}(\lambda) \hat{e}^{(A)}_{\alpha\alpha} \]  

(3)

The elements of the \( N \times N \) matrices \( M_{\mathcal{A}}^{(\pm)} \) are arbitrary c-numbers and the functions \( \varepsilon^{(\pm)}_{\alpha}(\lambda) \) are given by

\[ \varepsilon^{(-)}_{\alpha}(\lambda) = \begin{cases} \xi_{-} + \lambda & \alpha = 1, \ldots, p \\ \xi_{-} - \lambda & \alpha = p + 1, \ldots, N \end{cases} \quad \varepsilon^{(+)}_{\alpha}(\lambda) = \begin{cases} \xi_{+} - \frac{N}{2} - \lambda & \alpha = 1, \ldots, p \\ \xi_{+} + \frac{N}{2} + \lambda & \alpha = p + 1, \ldots, N \end{cases} \]  

(4)
where \( p \) is an integer with values on the interval \( 1 \leq p \leq N \) and \( \xi \pm \) are free-parameters. Here we emphasize that each \( K \)-matrix \( (3) \) has altogether \( 2N - 1 \) arbitrary parameters characterizing the interactions at the appropriate boundary.

The diagonalization of the transfer matrix \( (1, 2) \) for general non-diagonal \( K \)-matrices \( (3, 4) \) is a tantalizing problem due to the difficulty of finding a suitable reference state to perform a Bethe ansatz analysis. However, progress on this matter has recently been done in the literature, most of it concentrated on the eight \( [7] \) and six \( [6, 9, 10] \) vertex models. The \( Z_N \) Belavin model is to our knowledge the only multistate vertex system investigated so far with non-diagonal open boundaries \( [11] \). Though its bulk weights are known to reduce in the isotropic limit to those of the \( SU(N) \) vertex model \( (2) \), the same does not occur for the boundary \( K \)-matrices. In fact, the elliptic \( K \)-matrices associated to the \( Z_N \) Belavin model \( [12] \) have fewer free-parameters, which totals \( N + 1 \), as compared to that contained in the isotropic \( K \)-matrices \( (3, 4) \). Therefore, for \( N \geq 3 \) the \( SU(N) \) vertex model with open boundaries is indeed a genuine integrable system that deserves to be studied independently. We suspect that this situation extends to many isotropic integrable vertex models based on higher rank symmetries.

The purpose of this work is to show that the diagonalization of the transfer matrix \( (1, 2) \) of the \( SU(N) \) vertex model in the case \( M^{(+)}_A = M^{(-)}_A \) can be mapped on a similar eigenvalue problem with the diagonal boundaries \( D^{(\pm)}_A(\lambda) \). This constraint does not imply that the right and left \( K \)-matrices are the same because the parameters \( \xi \pm \) are still unrelated. This observation not only allows us to solve the eigenvalue problem for \( 2N \) independent boundary parameters but also makes it possible the relation between eigenvectors of seemingly different open boundaries. In order to see that we insert the terms \( M^{(\pm)}_A \left[ M^{(\pm)}_A \right]^{-1} \) all over the trace of the double-row transfer matrix \( (1) \), permitting us to rewrite it as

\[
T(\lambda) = \text{Tr}_A \left[ D^{(+)}_A(\lambda) \bar{T}_A(\lambda) D^{(-)}_A(\lambda) \bar{T}^{-1}_A(-\lambda) \right]
\]  

where the new monodromy \( \bar{T}_A(\lambda) = \bar{L}_{AL}(\lambda) \bar{L}_{AL-1}(\lambda) \ldots \bar{L}_{A1}(\lambda) \) whose gauge transformed \( \bar{L} \)-
operators are given by
\[ \tilde{L}_{Aj} = \left[ M_A^{(-)} \right]^{-1} L_{Aj}(\lambda) M_A^{(-)} \]  \hspace{1cm} (6)

Further progress is made by reversing the gauge transformation (6) with the help of the following quantum space transformation
\[ U_j^{-1} \tilde{L}_{Aj}(\lambda) U_j = L_{Aj}(\lambda) \]  \hspace{1cm} (7)

where \( U_j = \text{Id} \otimes \ldots \otimes M_A^{(-)} \otimes \text{Id} \ldots \otimes \text{Id} \) and \( \text{Id} \) is the \( N \times N \) identity matrix. This remarkable property \[13\] can now be used to define a new double-row operator \( \bar{T}(\lambda) \)
\[ \bar{T}(\lambda) = \prod_{j=1}^{L} U_j^{-1} T(\lambda) \prod_{j=1}^{L} U_j \]  \hspace{1cm} (8)

By using the canonical transformation \[8\] together with the assumed constraint \( M_A^{(+)} = M_A^{(-)} \) between the right and the left \( K \)-matrices we find that \( \bar{T}(\lambda) \) becomes
\[ \bar{T}(\lambda) = \text{Tr}_A \left[ D_A^{(+)}(\lambda) T_A(\lambda) D_A^{(-)}(\lambda) T_A^{-1}(\lambda) \right] \]  \hspace{1cm} (9)

which is precisely the double-row transfer matrix of the \( SU(N) \) vertex model with diagonal \( K \)-matrices \( D_A^{(\pm)}(\lambda) \).

As a consequence of that the operators \( T(\lambda) \) and \( \bar{T}(\lambda) \) share the same eigenvalues and furthermore if \( |\bar{\psi}\rangle \) is an eigenstate of \( \bar{T}(\lambda) \) then the corresponding eigenvector \( |\psi\rangle \) of \( T(\lambda) \) is \( \prod_{j=1}^{L} U_j |\bar{\psi}\rangle \). From now on our main task consists therefore in diagonalizing the double-row transfer matrix \( \bar{T}(\lambda) \). In this case the associated open boundaries \( D_A^{(\pm)}(\lambda) \) are diagonal and such eigenvalue problem can be tackled by standard nested Bethe ansatz approach. By now this procedure has been well explained in the literature, see for instance refs. \[14, 15\], and here we shall restrict ourselves in presenting only the essential steps of the solution. We first note that diagonal boundaries permit us to use as pseudovacuum the usual ferromagnetic state
\[ |\psi_0\rangle = \prod_{j=1}^{L} |0\rangle_j , \quad |0\rangle_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_N , \]  \hspace{1cm} (10)
The next step is to write convenient commutation rules for the elements of the double transition operator \( \Upsilon_A(\lambda) = \tilde{T}_A(\lambda)D_A^{-1}(\lambda)\tilde{T}_A^{-1}(-\lambda) \) which satisfies the following quadratic relation \[1\]

\[
\mathcal{L}_{12}(\lambda - \mu)\Upsilon_1(\lambda)\mathcal{L}_{21}(\lambda + \mu)\Upsilon_2(\mu) = \Upsilon_2(\lambda)\mathcal{L}_{12}(\lambda + \mu)\Upsilon_1(\mu)\mathcal{L}_{21}(\lambda - \mu)
\]

(11)

We proceed by looking for a representation of the operator \( \Upsilon_A(\lambda) \) that is capable to distinguish potential creation and annihilation fields over the state \(|0\rangle\). Previous experience with nested Bethe ansatz diagonalization of the \( SU(N) \) vertex model \[3\] suggests us the form

\[
\Upsilon_A(\lambda) = \begin{pmatrix}
A(\lambda) & B_1(\lambda) & \cdots & B_{N-1}(\lambda) \\
C_1(\lambda) & D_{11}(\lambda) & \cdots & D_{1N-1}(\lambda) \\
\vdots & \vdots & \ddots & \vdots \\
C_{N-1}(\lambda) & D_{N-11}(\lambda) & \cdots & D_{N-1N-1}(\lambda)
\end{pmatrix}_{N \times N}
\]

(12)

From Eqs.(11,12) it follows that out of \( N^4 \) possible commutation rules there exists three families that are of great use, namely

\[
A(\lambda)B_j(\mu) = \frac{a(\mu - \lambda)}{b(\mu - \lambda)} \frac{b(\mu + \lambda)}{a(\mu + \lambda)} B_j(\mu)A(\lambda) - \frac{b(2\mu)}{a(2\mu)} \frac{1}{b(\mu - \lambda)} B_j(\lambda)A(\mu)
\]

\[
- \frac{1}{a(\lambda + \mu)} B_i(\lambda)\tilde{D}_{ij}(\mu)
\]

(13)

\[
\tilde{D}_{ij}(\lambda)B_k(\mu) = \frac{[\tilde{\rho}^{(1)}(\lambda + \mu + 1)]^{id}_{ef} [\tilde{\rho}^{(1)}(\lambda - \mu)]^{fg}_{kj} b(\lambda + \mu + 1)b(\lambda - \mu) B_d(\mu)\tilde{D}_{eg}(\lambda)}{a(2\lambda)b(\lambda - \mu)} - \frac{[\tilde{\rho}^{(1)}(2\lambda + 1)]^{id}_{ef} b(2\mu)}{a(2\lambda)b(\lambda - \mu)} B_d(\lambda)\tilde{D}_{ek}(\mu)
\]

\[
+ \frac{b(2\mu)}{a(2\mu)} \frac{[\tilde{\rho}^{(1)}(2\lambda + 1)]^{id}_{kj} b(\lambda + \mu)}{a(2\lambda)b(\lambda + \mu)} B_d(\lambda)A(\mu)
\]

(14)

\[
B_i(\lambda)B_j(\mu) = \frac{[\tilde{\rho}^{(1)}(\lambda - \mu)]^{cd}_{ji} b(\lambda - \mu)}{a(\lambda - \mu)} B_c(\mu)B_d(\lambda)
\]

(15)

where \([\tilde{\rho}^{(1)}(\lambda)]^{ij}_{kl}\) are the matrix elements of the \( SU(N - 1) \) \( \mathcal{L} \)-operator \[1\][2] and the new \( \tilde{D}_{ij}(\lambda) \) operators are conveniently defined by the relation

\[
\tilde{D}_{ij}(\lambda) = D_{ij}(\lambda) - \frac{\delta_{ij}}{a(2\lambda)} A(\lambda)
\]

(16)

\[1\] Here we recall that we have used the convention \( \tilde{\rho}^{(1)}_{12}(\lambda) = \sum_{abcd}^{N-1} \tilde{\rho}^{(1)}_{12}(\lambda) \tilde{\epsilon}^{(1)}_{ac} \otimes \tilde{\epsilon}^{(2)}_{bd}.\]
Yet another important element is the action of the fields \( A(\lambda), \tilde{D}_{ij}(\lambda) \) and \( C_i(\lambda) \) on the reference state \(|0\rangle\). This can be computed by using the triangularity property of \( \mathcal{L}_{A_j}(\lambda) \) upon \(|0\rangle\), and they are given by

\[
A(\lambda) |0\rangle = \varepsilon_1^{(-)}(\lambda) [a(\lambda)]^{2L} |0\rangle
\]
\[
\tilde{D}_{ij}(\lambda) |0\rangle = [b(\lambda)]^{2L} \left[ \varepsilon_{i+1}^{(-)}(\lambda) - \frac{\varepsilon_1^{(-)}(\lambda)}{a(2\lambda)} \right] \delta_{ij} |0\rangle
\]
\[
C_i(\lambda) |0\rangle = 0
\]  \hspace{1cm} (17)

We have now gathered the basic ingredients to perform an algebraic Bethe ansatz analysis. We suppose that further eigenstates \(|\tilde{\psi}_{n_1}\rangle\) of \( \tilde{T}(\lambda) \) can be put into the following structure

\[
|\tilde{\psi}_{n_1}\rangle = B_{a_1}(\lambda_i^{(1)}) \ldots B_{a_{n_1}}(\lambda_{n_1}^{(1)}) \mathcal{F}^{a_n_1 \ldots a_1} |0\rangle
\]  \hspace{1cm} (18)

where the indices \( a_j \) run over \( N - 1 \) possible values and the rapidities \( \{\lambda_j^{(1)}\} \) will be determined by solving the eigenvalue equation

\[
\left[ \varepsilon_1^{(+)}(\lambda) + \frac{1}{a(2\lambda)} \sum_{\alpha=1}^{N-1} \varepsilon_{\alpha+1}^{(+)}(\lambda) \right] A(\lambda) |\tilde{\psi}_{n_1}\rangle + \sum_{\alpha=1}^{N-1} \varepsilon_{\alpha+1}^{(+)}(\lambda) \tilde{D}_{\alpha\alpha}(\lambda) |\tilde{\psi}_{n_1}\rangle = \Lambda(\lambda) |\tilde{\psi}_{n_1}\rangle
\]  \hspace{1cm} (19)

By carrying on the fields \( A(\lambda) \) and \( \tilde{D}_{ii}(\lambda) \) over the multiparticle state \(|18\rangle\) we generate terms that are proportional to \(|\tilde{\psi}_{n_1}\rangle\) and those that are not, denominated unwanted terms. The first ones contribute to the eigenvalue \( \Lambda(\lambda) \) and are obtained by keeping only the first terms of the commutation rules \((13,15)\) and by requiring that the coefficients \( \mathcal{F}^{a_n_1 \ldots a_1} \) are eigenstates of an auxiliary double-row operator \( \tilde{T}^{(1)}(\lambda, \{\lambda_j^{(1)}\}) \) given by

\[
\tilde{T}^{(1)}(\lambda, \{\lambda_j^{(1)}\}) = \text{Tr}_{A^{(1)}} \left[ D_{A^{(1)}}^{(1,+)}(\lambda) T_{A^{(1)}}^{(1)}(\lambda, \{\lambda_j^{(1)}\}) D_{A^{(1)}}^{(1,-)}(\lambda) \tilde{T}_{A^{(1)}}^{(1)}(\lambda, \{\lambda_j^{(1)}\}) \right]
\]  \hspace{1cm} (20)

such that

\[
T_{A^{(1)}}^{(1)}(\lambda, \{\lambda_j^{(1)}\}) = \tilde{r}_{A^{(1)}}^{(1)}(\lambda + \lambda_1^{(1)} + 1) \ldots \tilde{r}_{A^{(1)}}^{(1)}(\lambda + \lambda_{a_n_1}^{(1)} + 1)\]
\[
\tilde{T}_{A^{(1)}}^{(1)}(\lambda, \{\lambda_j^{(1)}\}) = \tilde{r}_{A^{(1)}}^{(1)}(\lambda - \lambda_1^{(1)}) \ldots \tilde{r}_{A^{(1)}}^{(1)}(\lambda - \lambda_{a_n_1}^{(1)})
\]  \hspace{1cm} (21)

where \( A^{(1)} \in C^{N-1} \) and the associated \( K \)-matrices are

\[
D_{A^{(1)}}^{(1,+)}(\lambda) = \sum_{\alpha=1}^{N-1} \varepsilon_{\alpha+1}^{(+)}(\lambda) \Delta_{A^{(1)}}^{(1)} \hspace{1cm} D_{A^{(1)}}^{(1,-)}(\lambda) = \sum_{\alpha=1}^{N-1} \left[ \varepsilon_{\alpha+1}^{(-)}(\lambda) - \frac{\varepsilon_1^{(-)}(\lambda)}{a(2\lambda)} \right] \Delta_{A^{(1)}}^{(1)}
\]  \hspace{1cm} (22)
The diagonalization of the inhomogeneous operator \( T^{(1)}(\lambda, \{\lambda_j^{(l)}\}) \) is implemented by a second Bethe ansatz which is once again parameterized by a new set of rapidities \( \lambda_1^{(2)} \ldots \lambda_{n_2}^{(2)} \). By keeping on going this procedure we are able to relate the eigenvalues \( \Lambda^{(l)}(\lambda, \{\lambda_j^{(l)}\}) \) of the transfer matrix \( \tilde{T}^{(l)}(\lambda, \{\lambda_j^{(l)}\}) \) at the nearest neighbor steps \( l \) and \( l+1 \). Since the commutation relations for the elements of the corresponding double transition operator is similar to that exhibited in Eqs. \([13\underline{15}]\) it is not difficult to derive the following recursive relation

\[
\Lambda^{(l)}(\lambda, \{\lambda_j^{(l)}\}) = Q^{(l)}(\lambda) \prod_{i=1}^{n_l} a(\lambda + \lambda_i^{(l)} + l)a(\lambda - \lambda_i^{(l)}) \prod_{i=1}^{n_{l+1}} a(\lambda_i^{(l+1)} - \lambda) b(\lambda + \lambda_i^{(l+1)} + l) \\
+ \prod_{i=1}^{n_l} b(\lambda + \lambda_i^{(l)} + l)b(\lambda - \lambda_i^{(l)}) \prod_{i=1}^{n_{l+1}} \frac{1}{b(\lambda - \lambda_i^{(l+1)})} \frac{1}{b(\lambda + \lambda_i^{(l+1)} + l + 1)} \Lambda^{(l+1)}(\lambda, \{\lambda_j^{(l+1)}\})
\]

(23)

where the functions \( Q^{(l)}(\lambda) \) are given by

\[
Q^{(l)}(\lambda) = \begin{cases} \\
\frac{\lambda(\lambda + \frac{N}{2})}{(\lambda + X)(\lambda +\frac{1}{2})} (\xi_+ - \lambda) \left( \frac{N}{2} - p + \xi_+ - \lambda \right) & l = 0, \ldots, p - 1 \\
\frac{\lambda(\lambda + \frac{N}{2})}{(\lambda + \frac{1}{2})(\lambda +\frac{1}{2})} (\xi_- - \lambda - p) \left( \frac{N}{2} + \xi_+ + \lambda \right) & l = p, \ldots, N - 1
\end{cases}
\]

(24)

For sake of consistency with our original eigenvalue problem we set \( \lambda_j^{(0)} \equiv 0 \) for \( j = 1, \ldots, n_0 = L \). By the same token the unwanted terms generated in the eigenvalue problem of the double-row operator \( \tilde{T}^{(l)}(\lambda, \{\lambda_j^{(l)}\}) \) are cancelled out provided that the variables \( \{\lambda_j^{(l+1)}\} \) satisfy the nested Bethe ansatz equations

\[
\prod_{i=1}^{n_{l-1}} \frac{a(\lambda^{(l)}_k + \lambda^{(l-1)}_i + l - 1)}{b(\lambda^{(l)}_k + \lambda^{(l-1)}_i + l - 1)} a(\lambda^{(l)}_k - \lambda^{(l-1)}_i) Q^{(l-1)}(\lambda^{(l)}_k) b(2\lambda^{(l)}_k + l - 1) = \\
\prod_{j \neq k}^{n_l} \frac{a(\lambda^{(l)}_j + \lambda^{(l)}_j + l)}{b(\lambda^{(l)}_j + \lambda^{(l)}_j + l - 1)} a(\lambda^{(l)}_j - \lambda^{(l)}_j) \prod_{j=1}^{n_{l+1}} a(\lambda^{(l+1)}_j - \lambda^{(l)}_j) b(\lambda^{(l+1)}_j + \lambda^{(l)}_j + l) \\
\prod_{j \neq k} b(\lambda^{(l)}_j + \lambda^{(l)}_j + l - 1) a(\lambda^{(l)}_j - \lambda^{(l)}_j) a(\lambda^{(l+1)}_j - \lambda^{(l)}_j) b(\lambda^{(l+1)}_j + \lambda^{(l)}_j + l)
\]

(25)

Explicit results are now obtained by iterating Eqs. \([23\underline{25}]\) beginning at \( l = 0 \) until we reach the step \( l = N - 2 \). At such final step one has to diagonalize an inhomogeneous six vertex model with open boundaries by adapting previous results obtained by Sklyanin \([11]\). Putting together all that and by making the convenient displacements \( \lambda_j^{(l)} \rightarrow \lambda_j^{(l)} - \frac{l}{2} \) we find that the
final result for the eigenvalues $\Lambda(\lambda)$, up to a normalization factor of value $(1 - \lambda^2)^L$, are given by the expression

$$
\Lambda(\lambda) = Q^{(0)}(\lambda) [a(\lambda)]^{2L} \prod_{i=1}^{n_1} \frac{(\lambda - \lambda_i^{(1)}) - \frac{1}{2}}{(\lambda - \lambda_i^{(1)}) + \frac{1}{2}} \frac{(\lambda + \lambda_i^{(1)}) - \frac{1}{2}}{(\lambda + \lambda_i^{(1)}) + \frac{1}{2}}
$$

$$
+ \ [b(\lambda)]^{2L} \sum_{i=1}^{N-2} Q^{(l)}(\lambda) \prod_{i=1}^{n_l} \frac{(\lambda - \lambda_i^{(l)}) + \frac{l+1}{2}}{(\lambda - \lambda_i^{(l)}) + \frac{1}{2}} \frac{(\lambda + \lambda_i^{(l)}) + \frac{l+1}{2}}{(\lambda + \lambda_i^{(l)}) + \frac{1}{2}} \prod_{i=1}^{n_{i+1}} \frac{(\lambda - \lambda_i^{(l+1)}) + \frac{l-1}{2}}{(\lambda - \lambda_i^{(l+1)}) + \frac{l}{2}} \frac{(\lambda + \lambda_i^{(l+1)}) + \frac{l-1}{2}}{(\lambda + \lambda_i^{(l+1)}) + \frac{l}{2}}
$$

$$
+ \ [b(\lambda)]^{2L} Q^{(N-1)}(\lambda) \prod_{i=1}^{n_{N-1}} \frac{(\lambda - \lambda_i^{(N-1)}) + \frac{N+1}{2}}{(\lambda - \lambda_i^{(N-1)}) + \frac{N}{2}} \frac{(\lambda + \lambda_i^{(N-1)}) + \frac{N+1}{2}}{(\lambda + \lambda_i^{(N-1)}) + \frac{N}{2}}
$$

while the Bethe ansatz roots $\{\lambda_i^{(1)}, \ldots, \lambda_i^{(N-1)}\}$ satisfy the following system of non-linear equations

$$
\left[ \frac{(\lambda_k^{(1)} + \frac{1}{2})}{(\lambda_k^{(1)} - \frac{1}{2})} \right]^{2L} \Theta^{(1)}(\lambda_k^{(1)}) = \prod_{j \neq k}^{n_1} \frac{(\lambda_k^{(1)} - \lambda_j^{(1)}) + 1}{(\lambda_k^{(1)} - \lambda_j^{(1)}) - 1} \frac{(\lambda_k^{(1)} + \lambda_j^{(1)}) + 1}{(\lambda_k^{(1)} + \lambda_j^{(1)}) - 1} \times \prod_{j=1}^{n_2} \frac{(\lambda_j^{(2)} - \lambda_k^{(2)}) + \frac{1}{2}}{(\lambda_j^{(2)} - \lambda_k^{(2)}) - \frac{1}{2}} \frac{(\lambda_j^{(2)} + \lambda_k^{(2)}) - \frac{1}{2}}{(\lambda_j^{(2)} + \lambda_k^{(2)}) + \frac{1}{2}}
$$

$$
\prod_{i=1}^{n_{l-1}} \frac{(\lambda_k^{(l)} - \lambda_i^{(l-1)}) + \frac{1}{2}}{(\lambda_k^{(l)} - \lambda_i^{(l-1)}) - \frac{1}{2}} \frac{(\lambda_k^{(l)} + \lambda_i^{(l-1)}) + \frac{1}{2}}{(\lambda_k^{(l)} + \lambda_i^{(l-1)}) - \frac{1}{2}} \Theta^{(l)}(\lambda_k^{(l)}) = \prod_{j \neq k}^{n_1} \frac{(\lambda_k^{(l)} - \lambda_j^{(l)}) + 1}{(\lambda_k^{(l)} - \lambda_j^{(l)}) - 1} \frac{(\lambda_k^{(l)} + \lambda_j^{(l)}) + 1}{(\lambda_k^{(l)} + \lambda_j^{(l)}) - 1} \times \prod_{j=1}^{n_{l+1}} \frac{(\lambda_j^{(l+1)} - \lambda_k^{(l+1)}) + \frac{1}{2}}{(\lambda_j^{(l+1)} - \lambda_k^{(l+1)}) - \frac{1}{2}} \frac{(\lambda_j^{(l+1)} + \lambda_k^{(l+1)}) - \frac{1}{2}}{(\lambda_j^{(l+1)} + \lambda_k^{(l+1)}) + \frac{1}{2}}
$$

$$
\prod_{i=1}^{n_{N-2}} \frac{(\lambda_k^{(N-1)} - \lambda_i^{(N-2)}) + \frac{1}{2}}{(\lambda_k^{(N-1)} - \lambda_i^{(N-2)}) - \frac{1}{2}} \frac{(\lambda_k^{(N-1)} + \lambda_i^{(N-2)}) + \frac{1}{2}}{(\lambda_k^{(N-1)} + \lambda_i^{(N-2)}) - \frac{1}{2}} \Theta^{(N-1)}(\lambda_k^{(N-1)}) = \prod_{j \neq k}^{n_1} \frac{(\lambda_k^{(N-1)} - \lambda_j^{(N-1)}) + 1}{(\lambda_k^{(N-1)} - \lambda_j^{(N-1)}) - 1} \frac{(\lambda_k^{(N-1)} + \lambda_j^{(N-1)}) + 1}{(\lambda_k^{(N-1)} + \lambda_j^{(N-1)}) - 1}
$$

where the function $\Theta^{(l)}(\lambda)$ is given by

$$
\Theta^{(l)}(\lambda) = \begin{cases} 
(\lambda - \frac{1}{2} + \xi_p) (\lambda + \frac{1}{2} - \xi_p) 
\quad \text{if } l = p \\
1 
\quad \text{otherwise}
\end{cases}
$$

We would like to close this letter with the following remarks. The same strategy described above works when one of the boundaries is purely free and the other stays arbitrary with
$2N - 1$ independent parameters. This is for instance the case of $K^{(+)}_{A}(\lambda) = \text{Id}$ and $K^{(-)}_{A}(\lambda)$ an arbitrary $K$-matrix (311). It turns out that the corresponding Bethe ansatz results for this choice of boundaries are obtained from Eqs. (26-30) by taking their $\xi^{+} \rightarrow \infty$ limit. We note that similar result has recently been reported in ref. [16] however on the basis of the analytical Bethe ansatz method and solely for case of diagonal boundaries.

Although we have concentrated our attention on $N \geq 3$ $SU(N)$ models, similar idea is also applicable with success to the $SU(2)$ XXX spin chain. In this special case it is more convenient to start by inserting $M_{A}^{(+)} \left[ M_{A}^{(+)} \right]^{-1}$ all over the double-row transfer matrix (1,2) and after reversing the transformed $\tilde{L}$-operators we impose that $\left[ M_{A}^{(+)} \right]^{-1} K^{(-)}_{A}(\lambda) M_{A}^{(+)}$ is a triangular matrix. This allows us to carry out the algebraic Bethe ansatz for a single constraint between the six possible boundary parameters, reproducing what has been found earlier for the XXZ chain [2]. However, this method has the clear advantage of relating the eigenfunctions of the XXX chain having all possible boundary terms with five free parameters to that with only one off-diagonal and suitable diagonal boundary terms. We hope that this relationship could be of utility in physical applications such as in the study of the scaling behaviour of symmetric exclusion processes [17].

The possibility of undoing gauge transformed $L$-operators appears to be a general property of isotropic vertex models [13]. Therefore, we expect that the method devised in this work will be useful in the solution of the eigenspectrum of a variety of isotropic systems with certain non-diagonal open boundaries. Interesting examples would be the case of soliton non-preserving boundaries [16, 18] for the conjugated representation of the $SU(N)$ as well as general boundaries for vertex models invariant by the $O(N)$ and $Sp(2N)$ Lie algebras.

\footnote{This construction clearly leads us to fewer constrained boundary parameters than the condition $M_{A}^{(+)} = M_{A}^{(-)}$ but when $N \geq 3$ a Bethe ansatz analysis has eluded us so far.}
Acknowledgements

W. Galleas thanks FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo) for financial support. The work of M.J. Martins has been supported by the Brazilian Research Council-CNPq and FAPESP.

References

[1] E.K. Sklyanin, *J.Phys.A:Math.Gen.* 21 (1988) 2375

[2] L.A. Takhtajan and L.D. Faddeev, *Russian Math. Surveys*, 34 (1979) 11; V.E. Korepin, G. Izergin and N.M. Bogoliubov, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993

[3] P.P. Kulish and N.Y. Reshetikhin, *J.Phys.A: Math.Gen.* 16 (1983) L591; O. Babelon, H.J. de Vega and C.M. Viallet, *Nucl.Phys.B*, 200 (1982) 266

[4] H.J. de Vega and A. Gonzales-Ruiz, *J.Phys.A:Math.Gen.* 27 (1994) 6129; J. Abad and M. Rios, *Phys.Lett.B* 352 (1995) 92

[5] P.P. Kulish, [hep-th/9507070](http://arxiv.org/abs/hep-th/9507070), M. Mintchev, E. Ragoucy and P. Sorba, *J.Phys.A:Math.Gen.* 34 (2001) 8345

[6] L. Mezincescu and R. Nepomechie, *Int.J.Mod.Phys.A* 6 (1991) 5231 and addendum.

[7] H. Fan, B.Y. Hou, K.J. Shi and Z.X. Yang, *Nucl.Phys.B* 478 (1996) 723

[8] R.I. Nepomechie, *J.Stat.Phys.* 111 (2003) 1363; *J.Phys.A:Math.Gen.* 37 (2004) 433

[9] J. Cao, H.Q. Lin, K.J. Shi and Y. Wang, *Nucl.Phys.B* 663 (2003) 487

[10] J. de Gier and P. Pyatov, *JSTAT* 03 (2004) P002

[11] W.L. Yang and R. Sasaki, *Nucl.Phys.B* 679 (2004) 495
[12] H. Fan, B.Y. Hou, G.L. Li and K.J. Shi, Phys.Lett.A 250 (1998) 79

[13] G.A.P. Ribeiro and M.J. Martins, nlin.SI/0406021

[14] A. Foester and M. Karowski, Nucl.Phys.B 396 (1993) 611; H.J. de Vega and A. Gonzalez-Ruiz, Nucl.Phys.B 424 (1994) 468

[15] R.H. Yue, H. Fan and B.Y. Hou, Nucl.Phys.B 462 (1996) 167; G.L. Li, R.H. Yue and B.Y. Hou, Nucl.Phys.B 586 (2000) 711

[16] D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, math-ph/0406021

[17] R.B. Stinchcombe and G.M. Schütz, Phys.Rev.Lett. 75 (1995) 140; F.C. Alcaraz and V. Rittenberg, Phys.Lett.B 314 (1993) 377.

[18] A. Doikou, J.Phys.A:Math.Gen. 33 (2000) 4755