MULTI-COMPONENT KdV HIERARCHY, V-ALGEBRA
AND NON-ABELIAN TODA THEORY

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ABSTRACT

I prove the recently conjectured relation between the $2 \times 2$-matrix differential operator $L = \partial^2 - U$, and a certain non-linear and non-local Poisson bracket algebra (V-algebra), containing a Virasoro subalgebra, which appeared in the study of a non-abelian Toda field theory. In particular, I show that this V-algebra is precisely given by the second Gelfand-Dikii bracket associated with $L$. The Miura transformation is given which relates the second to the first Gelfand-Dikii bracket. The two Gelfand-Dikii brackets are also obtained from the associated (integro-) differential equation satisfied by fermion bilinears. The asymptotic expansion of the resolvent of $(L - \xi)\Psi = 0$ is studied and its coefficients $R_l$ yield an infinite sequence of hamiltonians with mutually vanishing Poisson brackets. I recall how this leads to a matrix KdV hierarchy which here are flow equations for the three component fields $T, V^+, V^-$ of $U$. For $V^\pm = 0$ they reduce to the ordinary KdV hierarchy. The corresponding matrix mKdV equations are also given, as well as the relation to the pseudo-differential operator approach. Most of the results continue to hold if $U$ is a hermitian $n \times n$-matrix. Conjectures are made about $n \times n$-matrix, $m^{\text{th}}$-order differential operators $L$ and associated $V_{(n,m)}$-algebras.
1. Introduction

During the last ten years there has been made tremendous progress in understanding the close relations between integrable hierarchies of KdV type (see e.g. ref. [1]) and field theory, in particular conformal field theory. Early examples include the correspondence between the second hamiltonian structure of the KdV equation and the Virasoro algebra [2,3], with the free field realization of the latter providing the Miura transformation for the former. (Analogous results hold for the supersymmetric extensions, see e.g. [4]). This pattern was generalized with the discovery of the relation between Toda field theories, $W_m$-algebras and the $m^{th}$ Gelfand-Dikii hierarchies ($m^{th}$ reduction of KP) [5,6,7]. The general association of a hamiltonian structure (Poisson bracket) with a differential operator is mainly due to Gelfand and Dikii [8,9] and Drinfel’d and Sokolov [10]. The aim of the present paper is to elaborate on yet another generalization of these patterns.

In the recent study of the non-abelian Toda field theory as a model for strings propagating on a black hole background [11], a new Poisson bracket algebra of the conserved currents was discovered. It includes the Virasoro algebra with classical central charge as a subalgebra. This algebra contains, besides the stress-tensor $T$, two other spin-two currents $V^+$ and $V^-$. It is a non-linear and non-local algebra and was called $V$-algebra to emphasize the similarities (non-linearity) and differences (non-locality) with the well-known $W$-algebras [12]. The algebra reads:

\[
\begin{align*}
\gamma^{-2}\{T(\sigma), T(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'}) [T(\sigma')\delta(\sigma - \sigma')] - \frac{1}{2}\delta'''(\sigma - \sigma') \\
\gamma^{-2}\{T(\sigma), V^\pm(\sigma')\} &= (\partial_\sigma - \partial_{\sigma'}) [V^\pm(\sigma')\delta(\sigma - \sigma')] \\
\gamma^{-2}\{V^\pm(\sigma), V^\pm(\sigma')\} &= \epsilon(\sigma - \sigma')V^\pm(\sigma)V^\pm(\sigma') \\
\gamma^{-2}\{V^\pm(\sigma), V^{\mp}(\sigma')\} &= -\epsilon(\sigma - \sigma')V^\pm(\sigma)V^{\mp}(\sigma') \\
&\quad + (\partial_\sigma - \partial_{\sigma'}) [T(\sigma')\delta(\sigma - \sigma')] - \frac{1}{2}\delta'''(\sigma - \sigma') .
\end{align*}
\]

Here $\gamma^2$ is a (classically arbitrary) scale factor related to the classical central charge $c$ by $c = 12\pi \gamma^{-2}$, as can be easily seen by looking at the modes $L_n = \gamma^{-2} \int_{-\pi}^{\pi} d\sigma [T(\tau, \sigma) + \frac{1}{4}] e^{in(\tau + \sigma)}$. To be definite, the coordinate $\sigma$ is supposed to take values on the unit circle $S^1$, and $\epsilon(\sigma - \sigma')$ is defined to be the unique periodic, antisymmetric and translationally invariant function obeying $\partial_\sigma\partial_{\sigma'}\epsilon(\sigma - \sigma') = 2\delta'(\sigma - \sigma')$, namely $\epsilon(\sigma - \sigma') = \theta(\sigma - \sigma') - \theta(\sigma' - \sigma) - \frac{\sigma - \sigma'}{\pi}$.
= \frac{1}{\pi i} \sum_{m \neq 0} \frac{1}{m} e^{im(\sigma - \sigma')}$. However, for many of the results presented here, $\sigma \in S^1$ is not essential, and can be replaced by $\sigma \in \mathbb{R}$.

In ref. [11] a free field realization of the $T$ and $V^\pm$ was given (see section 3 below). Using this realization it was shown that simple (vertex-operator type) exponentials $\psi_1$ and $\psi_2$ of the free fields satisfy a matrix differential equation

\[
\begin{bmatrix}
\partial^2_{\sigma} - \begin{pmatrix} T & -\sqrt{2}V^+ \\
-\sqrt{2}V^- & T \end{pmatrix}
\end{bmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = 0.
\] (1.2)

Below, I will often use the matrix notation

\[
L\Psi = 0, \quad L = \partial^2 - U, \quad U = \begin{pmatrix} T & -\sqrt{2}V^+ \\
-\sqrt{2}V^- & T \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\
\psi_2
\end{pmatrix}
\] (1.3)

where, if not indicated otherwise, $\partial \equiv \partial_{\sigma}$, and $U$ and $\Psi$ depend on $\sigma$ (as well as on other parameters $t_1, t_2, \ldots$ which will not be relevant until section 6). The dependence on $\tau$, inherited from the conformal field theory is trivial since $U$ and $\Psi$ only depend on the combination $\sigma + \tau$. Actually, as usual (see below), this is the first flow of the matrix KdV hierarchy: $\frac{\partial}{\partial t_1} U = \partial_\sigma U$, so that $\tau$ is identified with $t_1$.

The plan of this paper is fairly well described by the abstract.

2. The Gelfand-Dikii brackets associated with $L = \partial^2 - U$

In this section, following refs. [9,10,6,13], I define the first and second Gelfand-Dikii brackets (hamiltonian structures) associated with the $2 \times 2$-matrix second-order differential operator $L$. Then I show that the second Gelfand-Dikii bracket is identical to the $V$-algebra (1.1).

Let $f$ and $g$ be polynomial functionals on the space of second-order differential operators $L$, i.e. polynomial functionals of $U$ (and its derivatives). One defines the pseudo-differential

\* To be precise, the equation of ref. [11] is a differential equation in $u = \tau + \sigma$. However, for the present purpose, we can consider the $\psi_1, \psi_2$ to be exponentials of the left-moving (i.e. $u$-dependent) parts of the free fields only, so that $\partial_\sigma \psi_i(u) = \partial_u \psi_i(u)$, and the equation given here is equivalent to that given in ref. [11].
operator

\[ X_f = \partial^{-1}X_1 + \partial^{-2}X_2 , \quad X_1 = \frac{\delta f}{\delta U} \]  

(2.1)

where \( \partial^{-1} \partial = \partial \partial^{-1} = 1 \) and \( \frac{\delta}{\delta U} \) is defined as

\[ \frac{\delta}{\delta U} = \left( \begin{array}{cc} \frac{1}{2} \frac{\delta}{\delta T} & -\frac{1}{\sqrt{2}} \frac{\delta}{\sqrt{2} \delta V} \\ -\frac{1}{\sqrt{2}} \frac{\delta}{\sqrt{2} \delta V} & \frac{1}{2} \frac{\delta}{\delta T} \end{array} \right) \]  

(2.2)

so that \( \frac{\delta}{\delta U} \int \text{tr} \, U^n = nU^{n-1} \), and \( X_2 \) is determined (cf. \([6,9,13]\)) by requiring \( \frac{\delta}{\delta U} \) res\([L,X_f] = 0 \).

As usual, the residue of a pseudo-differential operator, denoted res, is the coefficient of \( \partial^{-1} \).

One has

\[ X'_2 = \frac{1}{2} \left( \frac{\delta f}{\delta U} \right)^{''} + \frac{1}{2} \left[ U, \frac{\delta f}{\delta U} \right] \]  

(2.3)

Integrating this equation yields \( X_2 \). Here, one observes a new feature as compared to the scalar case: since in general \( \left[ U, \frac{\delta f}{\delta U} \right] \neq 0 \), \( X_2 \) will be given by a non-local expression involving an integral. This is the origin of the non-local terms (\( \sim \epsilon(\sigma - \sigma') \)) in the \( V \)-algebra (1.1).

The first Gelfand-Dikii bracket is defined by

\[ \{ f, g \}_{GD1} = a \int d\sigma \text{tr} \text{ res} \left( [L,X_f]_+X_g \right) \]  

(2.4)

where \([ \cdot, \cdot \] is the commutator and the subscript + indicates, as usual, to take only the differential operator part (no negative powers of \( \partial \)). The parameter \( a \) is a real, non-zero scale factor, to be fixed later on. It is easy to obtain

\[ \{ f, g \}_{GD1} = -2a \int d\sigma \text{tr} \frac{\delta f}{\delta U} \frac{\delta g}{\delta U} \]  

(2.5)

I assume that one can freely integrate by parts, which is true in particular if \( \sigma \in S^1 \). If \( \sigma \in \mathbb{R} \), one has to require that \( U \) vanishes sufficiently fast (e.g. exponentially) as \( \sigma \to \pm \infty \).

---

\( \dagger \) On the circle \( S^1 \) e.g. \( \partial^{-1} \) is well defined on functions without constant Fourier mode, i.e. \( f(\sigma) = \sum_{m \neq 0} f_m e^{-im\sigma} \). One easily sees that \( (\partial^{-1} f)(\sigma) = \int d\sigma' \frac{1}{2} \epsilon(\sigma - \sigma') f(\sigma') \) with \( \epsilon(\sigma - \sigma') = \frac{1}{\pi} \sum_{m \neq 0} \frac{1}{m} e^{im(\sigma - \sigma')} \).

\( \ddagger \) This condition is necessary since the coefficient of \( \partial \) in \( L \) vanishes.

\( \S \) using the well-known formula \( \partial^{-1} f = \sum_{n=0}^{\infty} (-)^n f^{(n)} \partial^{-1-n} \)
The second Gelfand-Dikii bracket is usually more interesting. In analogy with the standard procedure [9,10,6,13] I define it as follows.

\[
\{f, g\}_{GD2} = a \int d\sigma \, \text{tr} \, \text{res} \left( L(X_f L) + X_g - (L X_f) + L X_g \right) .
\]  

(2.6)

It is straightforward to obtain

\[
\{f, g\}_{GD2} = a \int d\sigma \, \text{tr} \left( \frac{1}{2} \delta f \frac{\partial^3 \delta g}{\partial U^3} + \frac{1}{2} \left[ U, \delta g \right] \left( \partial^{-1} \left[ U, \delta f \right] \right) \right.

\[
- \frac{\delta f}{\delta U} (U \partial + \frac{1}{2} U') \frac{\delta g}{\delta U} + \frac{\delta g}{\delta U} (U \partial + \frac{1}{2} U') \frac{\delta f}{\delta U} \right)
\]  

(2.7)

where the \( \partial^{-1} \) is meant to act only on \( [U, \frac{\delta f}{\delta U}] \).

Inserting the definitions of the 2 \times 2-matrices \( U \) and \( \frac{\delta f}{\delta U} \) one obtains

\[
\{f, g\}_{GD1} = -\frac{1}{2} \int d\sigma \left( \frac{\delta f}{\delta T} \frac{\partial}{\partial \delta T} + \frac{\delta g}{\delta V^+} \frac{\partial}{\partial \delta V^+} + \frac{\delta g}{\delta V^-} \frac{\partial}{\partial \delta V^-} \right)
\]  

(2.8)

and

\[
\{f, g\}_{GD2} = -\frac{a}{2} \int d\sigma \left[ - \frac{1}{2} \frac{\delta f}{\delta T} \frac{\partial^3 \delta g}{\partial T^3} - \frac{1}{2} \frac{\delta f}{\partial V^+} \frac{\partial^3 \delta g}{\partial V^+} - \frac{1}{2} \frac{\delta f}{\partial V^-} \frac{\partial^3 \delta g}{\partial V^-} + T \left( \frac{\delta f}{\delta T} \frac{\partial}{\partial \delta T} + \frac{\delta f}{\delta V^+} \frac{\partial}{\partial \delta V^+} + \frac{\delta f}{\delta V^-} \frac{\partial}{\partial \delta V^-} \right) \right.

\[
+ \frac{\delta f}{\delta T} \frac{\partial}{\partial \delta T} + \frac{\delta f}{\delta V^+} \frac{\partial}{\partial \delta V^+} + \frac{\delta f}{\delta V^-} \frac{\partial}{\partial \delta V^-} \left( f \leftrightarrow g \right) \right)

\[
+ V^+ \left( \frac{\delta f}{\delta T} \frac{\partial}{\partial \delta T} + \frac{\delta f}{\delta V^+} \frac{\partial}{\partial \delta V^+} + \frac{\delta f}{\delta V^-} \frac{\partial}{\partial \delta V^-} \right) \left( f \leftrightarrow g \right)
\]  

(2.9)

Then, taking \( f, g \) to be \( T, V^+ \) or \( V^- \) one has the

**Proposition 2:** The second Gelfand-Dikii bracket (2.9) coincides with the \( V \)-algebra (1.1) provided one chooses

\[
a = -2\gamma^{-2} .
\]  

(2.10)

Henceforth I will adopt this choice and the only Poisson bracket used is the second Gelfand-Dikii bracket, unless otherwise stated.
3. Free field realization and the Miura transformation

In ref. [11] the $V$-algebra (1.1) was obtained from a field theoretic realization. After a series of field redefinitions, which need not concern us here, the following free field realization of the generators was obtained

$$T = \frac{1}{2} \sum_{i=1}^{3} (\varphi_i')^2 - \frac{1}{\sqrt{2}} \varphi_3''$$

$$V^\pm = \frac{1}{2} (\sqrt{2} \varphi_3' - \partial) \left[ e^{\mp i\sqrt{2} \varphi_2} (\varphi_1' \mp i \varphi_2') \right]$$

(3.1)

where, for the present purpose, $\partial = \partial_\sigma$, $\varphi_i' = \partial \varphi_i$ and $\varphi_i = \varphi_i(\tau + \sigma)$ is a chiral half of a free field. Then it was shown in ref. [11], and can easily be verified again, that the transformation (3.1) maps the following free-field Poisson brackets to the brackets (1.1):

$$\{ \varphi_i(\sigma), \varphi_j(\sigma') \} = -\frac{\gamma^2}{2} \delta_{ij} \epsilon(\sigma - \sigma') .$$

(3.2)

Note that this implies

$$\{ \varphi_i'(\sigma), \varphi_j'(\sigma') \} = \gamma^2 \delta_{ij} \delta'(\sigma - \sigma')$$

(3.3)

or, if $\sigma, \sigma' \in S^1$, $i\{ \varphi_n^+, \varphi_m^- \} = n \delta^{ij} \delta_{n+m,0}$ for the Fourier moded defined by $\varphi_j'(\sigma) = \gamma(2\pi)^{-1/2} \sum_n \varphi_{n}^j e^{-in\sigma}$. These are harmonic oscillator Poisson brackets. Of course, (3.3) has the structure of the first Gelfand-Dikii bracket (2.8). Thus (3.1) maps the first Gelfand-Dikii bracket to the second one, or, in other words:

**Proposition 3:** The transformation (3.1) provides a Miura transformation for the two hamiltonian structures associated with $L = \partial^2 - U$.

The Miura transformation (3.1) can be nicely written in matrix form. Define

$$P \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -p_3 & \sqrt{2}p_+ \\ \sqrt{2}p_- & -p_3 \end{pmatrix}$$

$$p_3 = \varphi_3', \quad p_\pm = \frac{1}{\sqrt{2}} e^{\mp i\sqrt{2} \varphi_2} (\varphi_1' \mp i \varphi_2') .$$

(3.4)

* Of course, one could replace $\partial_\sigma$ by $\partial_u = \frac{1}{2} (\partial_\sigma + \partial_\tau)$ everywhere and consider $\varphi_i$ to be the full free field, i.e. a sum of a right- and a left-moving part. This is the approach of ref. [11].
Then

\[ U = P^2 + P' \] (3.5)

and the Poisson brackets of the \( p \) are

\[
\begin{align*}
\gamma^{-2}\{p_3(\sigma), p_3(\sigma')\} &= \delta'(\sigma - \sigma') \\
\gamma^{-2}\{p_3(\sigma), p_\pm(\sigma')\} &= 0 \\
\gamma^{-2}\{p_\pm(\sigma), p_\pm(\sigma')\} &= \epsilon(\sigma - \sigma')p_\pm(\sigma)p_\pm(\sigma') \\
\gamma^{-2}\{p_\pm(\sigma), p_\mp(\sigma')\} &= -\epsilon(\sigma - \sigma')p_\pm(\sigma)p_\mp(\sigma') + \delta'(\sigma - \sigma') .
\end{align*}
\] (3.6)

The matrix Miura transformation \(^\dagger\) (3.5) allows us to rewrite the differential operator \( L \) as

\[ L = \partial^2 - U = (\partial + P)(\partial - P) . \] (3.7)

Note that (3.1) implies that \( V^+ \) and \( V^- \) are complex conjugate if the \( \varphi_j \) are considered real. All equations are compatible with this assumption. In all of the following, I will suppose that \((V^+)^* = V^-\). In particular, this implies that \( U \) is a hermitian matrix, so that \( L \) is a hermitian operator. Let me also note that one solution of the equation \( L\Psi = 0 \) is easily obtained. If \( T \) and \( V^\pm \) are given by (3.1), a solution is [11]

\[
\begin{align*}
\psi_1 &= \exp\left[\frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2 - \varphi_3)\right] \\
\psi_2 &= \exp\left[\frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2 - \varphi_3)\right] = \psi_1^* .
\end{align*}
\] (3.8)

\(^\dagger\) Strictly speaking, as a mapping between the two Gelfand-Dikii brackets, the Miura transformation is the transformation (3.1) from the \( \varphi_j' \) to the \( T, V^\pm \). Since, due to the \( \epsilon(\sigma - \sigma')\)-terms, the Poisson brackets (3.6) of the \( p \) are not exactly the first Gelfand-Dikii bracket, equation (3.5) is not really a Miura transformation. However, by abuse of language, it is convenient to refer to it nevertheless as the matrix Miura transformation.
4. The integro-differential equation for (fermion) bilinears

It is often useful to introduce a spectral parameter $\lambda$ and consider the eigenvalue problem

$$-L\Psi = \lambda\Psi$$  \hspace{1cm} (4.1)

(with $\lambda \in \mathbb{R}$, since $U$ is a hermitian matrix). It is suggestive to think of this as a Schrödinger equation for a fermion, and thus natural to consider also fermion bilinears. Define

$$\chi_1 = (\psi_1^*\psi_1 + \psi_2^*\psi_2) = \Psi^+\Psi$$
$$\chi_2 = -\sqrt{2}\psi_1^*\psi_2 = -\sqrt{2}\Psi^+\sigma_+\Psi$$
$$\chi_3 = -\sqrt{2}\psi_2^*\psi_1 = -\sqrt{2}\Psi^+\sigma_-\Psi.$$  \hspace{1cm} (4.2)

The reason to choose the Pauli matrices $\sigma_+$, $\sigma_-$ and the unit matrix (rather than $\sigma_3$) lies in the form of the matrix $U = T - \sqrt{2}\sigma_-V^+ - \sqrt{2}\sigma_-V^-$. Let $\chi$ be the vector with components $\chi_1, \chi_2, \chi_3$. One then has the following

**Lemma 4:** Assuming that $(V^+)^* = V^-$, $\chi$ satisfies the following integro-differential equation

$$\Delta \chi = \lambda N \chi, \quad N = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \partial$$  \hspace{1cm} (4.3)

where $\Delta$ is the operator\(^\dagger\)

$$\Delta = -\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \partial^3 + \partial \begin{pmatrix} T & V^+ & V^- \\ V^+ & 0 & T \\ V^- & T & 0 \end{pmatrix} + \begin{pmatrix} T & V^+ & V^- \\ V^+ & 0 & T \\ V^- & T & 0 \end{pmatrix} \partial$$  \hspace{1cm} (4.4)

\(^\dagger\) As usual, $\partial$ and $\partial^{-1}$ are meant to act on everything on their right in $\Delta \chi$. For example, $(V^-\partial^{-1}V^-\chi_2)(\sigma) = V^-(\sigma) \int d\sigma' \frac{1}{2} \epsilon(\sigma - \sigma')V^-(\sigma')\chi_2(\sigma')$. 

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The matrix elements of this operator are simply the second Gelfand-Dikii brackets:

\[
\langle \sigma | \Delta | \sigma' \rangle = \gamma^{-2} \begin{pmatrix}
\{ T(\sigma), T(\sigma') \} & \{ T(\sigma), V^+(\sigma') \} & \{ T(\sigma), V^-(\sigma') \} \\
\{ V^+(\sigma), T(\sigma') \} & \{ V^+(\sigma), V^+(\sigma') \} & \{ V^+(\sigma), V^-(\sigma') \} \\
\{ V^-(\sigma), T(\sigma') \} & \{ V^-(\sigma), V^+(\sigma') \} & \{ V^-(\sigma), V^-(\sigma') \}
\end{pmatrix}
\]

while the matrix elements \( \langle \sigma | N | \sigma' \rangle \) obviously are the first Gelfand-Dikii brackets.

**Proof:** Equation (4.5) follows obviously from (4.4) using (1.1). To show that (4.3) is satisfied is equally straightforward. For example, one of the equations is

\[
[\partial^3 - 2\partial(T - \lambda) - 2(T - \lambda)\partial] \psi_1^* \psi_2 + \sqrt{2}(\partial V^- + V^- \partial)(\psi_1^* \psi_1 + \psi_2^* \psi_2) = -4V^- \partial^{-1}(V^+ \psi_1^* \psi_2 - V^- \psi_2^* \psi_1)
\]

Using (4.1) in component form, i.e. \([\partial^2 - (T - \lambda)] \psi_1 + \sqrt{2}V^+ \psi_2 = 0, [\partial^2 - (T - \lambda)] \psi_2 + \sqrt{2}V^- \psi_1 = 0\), and the complex conjugate equations (with \((V^+)^* = V^-\)) one easily sees that the l.h.s. of (4.6) equals \(\sqrt{2}V^-(\psi_1^* \partial \psi_1 - \partial \psi_1^* \psi_1 - \psi_2^* \partial \psi_2 + \partial \psi_2^* \psi_2)\). This equals the r.h.s. of (4.6) since one has \(\sqrt{2}\partial(\psi_1^* \partial \psi_1 - \partial \psi_1^* \psi_1 - \psi_2^* \partial \psi_2 + \partial \psi_2^* \psi_2) = 4V^- \psi_2^* \psi_1 - 4V^+ \psi_1^* \psi_2\). The two other equations are proven in the same way.

Let me emphasize again that the integro-differential equation for the fermion bilinears has reproduced the first \((N)\) and second \((\Delta)\) Gelfand-Dikii brackets. This is not too surprising in view of the important role played by fermion bilinears in the usual KP hierarchy [14].

## 5. Asymptotic expansion of the resolvent

This section is inspired by the classical work of Gelfand and Dikii [8] where the asymptotic expansion of the resolvent is given for scalar \(U\). Most results of [8] have a straightforward generalisation to the present matrix case, although some of the proofs have to be modified. The matrix case has been studied by Olmedilla, Martinez Alonso and Guil [15] who, in particular, were the first to prove Proposition 5.2 below. Since our proof is very compact, and in order

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\(\S\) I use “scalar” as opposed to “matrix”. Of course, under a coordinate transformation, it is most convenient to view \(U\) as transforming as a quadratic differential with an additional Schwarzian derivative term.
to keep this paper self-contained, it will be presented here. For a hermitian $n \times n$-matrix $U$
define the resolvent $R$ as

$$R(x, y; \xi) = \langle x | (-\partial_x^2 + U + \xi)^{-1} | y \rangle$$  \hspace{1cm} (5.1)$$

which is a solution of

$$(-\partial_x^2 + U(x) + \xi) R(x, y; \xi) = \delta(x - y) . \hspace{1cm} (5.2)$$

$R(x, y; \xi)$ is well-defined for any $\xi$ such that $-\xi$ is not an eigenvalue $\lambda_\alpha$ of $-\partial_x^2 + U$. If the spectrum is discrete, as I will suppose, $R(x, y; \xi)$ is a meromorphic function in the complex $\xi$-plane. Let $R_0$ be the resolvent for $U = 0$, e.g. for $x, y \in \mathbb{R}$ one has $R_0(x, y; \xi) = \frac{1}{2\sqrt{\xi}} e^{-\sqrt{\xi}|x-y|}$. Then $R$ has the (formal) expansion $R = R_0 \sum_{k=0}^{\infty} (-)^k (UR_0)^k$. The $k = 1$ term e.g. is $-\int dz R_0(x, z; \xi) U(z) R_0(z, y; \xi)$, and developing $U(z) = \sum \frac{(z-x)^l}{l!} U^{(l)}(x)$ one finds, for $x = y$, the sum $-\sum_{n=1}^{\infty} \frac{U^{(2n-2)}(x)}{4n \xi^{n+1/2}}$. One can proceed in the same way with the terms $k > 1$. For other boundary conditions, e.g. $x, y \in S^1$, similar considerations apply. As a result one obtains the

**Lemma 5.1:** The restriction of the resolvent to the diagonal, $R(x; \xi) \equiv R(x, x; \xi)$ has an asymptotic expansion for $\xi \to \infty$ of the form

$$R(x; \xi) = \sum_{n=0}^{\infty} \frac{R_n[u]}{\xi^{n+1/2}} . \hspace{1cm} (5.3)$$

This equation is to be understood as an equality of the asymptotic expansions in powers of $\xi^{-1/2}$, disregarding any terms that vanish exponentially fast as $\xi \to \infty$. (These exponentially vanishing terms do depend on the specific boundary conditions, while the coefficients of the asymptotic expansion do not [8].) The coefficients $R_n[u] \equiv R_n(x)$ are differential polynomials in $U$ (i.e. polynomials in $U$ and its derivatives). It also follows from the above construction that terms like e.g. $U^2 U''$ appear in a symmetrized form $\frac{1}{2} (U^2 U'' + U'' U^2)$. In particular, if $U$ is a hermitian $2 \times 2$-matrix with $U_{11} = U_{22}$, the same is true for $R(x; \xi)$ and $R_n(x)$.

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*For the case $x, y \in [-\pi, \pi]$ with periodic boundary conditions one has $R_0(x, y; \xi) = \frac{1}{2\sqrt{\xi}} \frac{\cosh \sqrt{\xi}(\pi - |x-y|)}{\sinh \sqrt{\xi} \pi}$ and one finds for the $k = 1$ term $-\sum_{n=1}^{\infty} \frac{U^{(2n-2)}(x)}{4n \xi^{n+1/2}} + O(e^{-\sqrt{\xi}\pi})$, which coincides with the result for $x, y \in \mathbb{R}$ up to terms that vanish exponentially fast as $\xi \to \infty$.
It is well-known from standard quantum mechanics that if \( \{ \Psi_\alpha(x) \} \) is a complete set of orthonormal eigenfunctions* of the eigenvalue problem

\[
(-\partial^2_x + U(x))\Psi_\alpha(x) = \lambda_\alpha \Psi_\alpha(x) ,
\]

i.e. satisfying \( \sum_\alpha \Psi_\alpha(x)\Psi^*_\alpha(y) = \delta(x-y) \), then the resolvent has the spectral decomposition

\[
R(x,y;\xi) = \sum_\alpha \frac{\Psi_\alpha(x)\Psi^*_\alpha(y)}{\xi + \lambda_\alpha} .
\]

Note that it follows from the spectral decomposition and the normalization of the \( \Psi_\alpha \) that

\[
\int \text{tr} \ R(x;\xi)dx = \sum_\alpha \frac{1}{\xi + \lambda_\alpha} .
\]

**Proposition 5.2:** For a \( n \times n \) hermitian matrix \( U \) let the resolvent \( R(x,y;\xi) \) be a solution of (5.2). Then \( R \equiv R(x;\xi) = R(x,x;\xi) \) satisfies

\[
R''' - 2(UR' + R'R) - (U'R + RU') + [U,\partial^{-1}[U,R]] = 4\xi R'
\]

(5.7)

(where \( R' \equiv \partial_x R(x;\xi) \) etc.) and the coefficients \( R_n \) of the asymptotic expansion (5.3) satisfy

\[
4R'_{n+1} = R'''_n - 2(UR'_n + R'_n U) - (U'R_n + R_n U') + [U,\partial^{-1}[U,R_n]] .
\]

(5.8)

**Proof:** It is straightforward to see, using (5.2), that \( R(x,y;\xi) \) satisfies the following differential equation in \( x \) and \( y \):

\[
(\partial_x + \partial_y)^3 R(x,y;\xi) - 2U(x)(\partial_x + \partial_y)R(x,y;\xi) - 2((\partial_x + \partial_y)R(x,y;\xi))U(y)
- U'(x)R(x,y;\xi) - R(x,y;\xi)U'(y) + U(x)(\partial_x - \partial_y)R(x,y;\xi)
- ((\partial_x - \partial_y)R(x,y;\xi))U(y) = 4\xi(\partial_x + \partial_y)R(x,y;\xi) .
\]

(5.9)

Note that all \( \delta'(x-y) \)-terms cancel. Next, since \( (\partial_x + \partial_y)(\partial_x - \partial_y)R(x,y;\xi) = U(x)R(x,y;\xi) - R(x,y;\xi)U(y) \) one has \( (\partial_x - \partial_y)R(x,y;\xi) = (\partial_x + \partial_y)^{-1}(U(x)R(x,y;\xi) - R(x,y;\xi)U(y)) \). Inserting this into eq. (5.9), taking \( x = y \) and observing that \((\partial_x + \partial_y)^k R(x,y;\xi))|_{x=y} = \partial^{k}_{x} R(x;\xi) \) one obtains the desired equation (5.7). Equation (5.8) then is an immediate consequence of the expansion (5.3).

* Recall that I suppose a discrete spectrum \( \{ \lambda_\alpha \} \). This avoids, among other things, unnormalizable eigenfunctions. However, since most of the results involve only formal algebra in \( U \) and its derivatives, this is not really essential (cf. ref. [8]).
Although not obvious from the recurrence relation (5.8), one is guaranteed by Lemma 5.1 that this can be integrated to yield all $R_n$ as differential polynomials\footnote{Indeed, the last term, related to the non-local terms in the $V$-algebra, and caused by the non-commutativity of the matrices, makes it even less obvious than in the scalar case.} in $U$. The first few are (for $U$ a hermitian $n \times n$-matrix)

\begin{align*}
R_0 & = \frac{1}{2} \\
R_1 & = - \frac{1}{4} U \\
R_2 & = \frac{1}{16} (3U^2 - U'') \\
R_3 & = - \frac{1}{64} (10U^3 - 5UU'' - 5U''U - 5U'^2 + U^{(4)}) \\
R_4 & = \frac{1}{256} (35U^4 - 21U^2U'' - 21UU''U - 28UU''U - 28U'^2U - 28UU'^2 - 14U''UU'' \\
& \quad + 7UU^{(4)} + 7U^{(4)}U + 14U''U'' + 14U''''U + 21U'^2 + U^{(6)}) .
\end{align*}

**Lemma 5.3:** Let $U$ be a general hermitian $n \times n$-matrix. Then

\[
\frac{\delta}{\delta U(x)} \int \frac{1}{2} \operatorname{tr} R_n(y) dy = - \frac{2n}{4} - \frac{1}{4} R_{n-1} .
\] (5.10)

**Proof:** From (5.6) one has $\frac{\delta}{\delta U(x)} \int \operatorname{tr} R(y; \xi) dy = \frac{\delta}{\delta U(x)} \sum_\alpha \frac{1}{\xi + \lambda_\alpha} = - \sum_\alpha \frac{\delta \lambda_\alpha}{\delta U(x)} \frac{1}{(\xi + \lambda_\alpha)^2}$. Standard first order perturbation theory gives $\delta \lambda_\alpha = \int \Psi_\alpha^+ \delta U \Psi_\alpha$, hence

\[
\frac{\delta \lambda_\alpha}{\delta U} = \Psi_\alpha^+ \Psi_\alpha ,
\] (5.12)

where $(\delta/\delta U)_{ij} = \delta/\delta U_{ji}$. Now, from (5.5) it then follows that

\[
\frac{\delta}{\delta U(x)} \int \operatorname{tr} R(y; \xi) dy = \frac{\partial}{\partial \xi} R(x; \xi) .
\] (5.13)

Inserting the expansion (5.3) gives the desired result. Note that when $U$ is not a general hermitian matrix, some care has to be taken when defining $(\delta/\delta U)$. For example, for the $2 \times 2$-matrix $U$ of eq. (1.3), $(\delta/\delta U)$ is defined by (2.2), and it is easy to see that eq. (5.13) continues to hold.
Now let me return to the special case where $U$ and hence also $R(x; \xi)$ both are hermitian $2 \times 2$-matrices with equal $(1, 1)$ and $(2, 2)$ elements. It is then convenient to define a correspondence with 3-component vectors as

$$M = \begin{pmatrix} a & -\sqrt{2}c \\ -\sqrt{2}b & a \end{pmatrix} \iff \tilde{M} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$ (5.14)

This definition, together with the spectral decomposition (5.5), as well as the property $R(x; \xi)_{11} = R(x; \xi)_{22}$, and the definition (4.2) of $\chi$ ($\chi_{\alpha}^\alpha = \Psi_{\alpha}^+ \Psi_{\alpha}$ etc) imply

$$\tilde{R}(x; \xi) = \frac{1}{2} \sum \frac{\chi_{\alpha}^\alpha}{\xi + \lambda_{\alpha}}.$$ (5.15)

Then we have the following

**Corollary 5.4:** It is a simple matter of writing out the components explicitly to see that, if $U$ is the familiar $2 \times 2$ matrix, eqs. (5.7) and (5.8) are equivalent to the following integro-differential equation

$$\Delta \tilde{R}(x; \xi) + \xi N \tilde{R}(x; \xi) = 0$$ (5.16)

where $\Delta$ and $N$ are defined in eqs. (4.3) and (4.4), and for the coefficients $R_n(x)$

$$\Delta \tilde{R}_n + N \tilde{R}_{n+1} = 0.$$ (5.17)

6. The infinite sequence of hamiltonians

In this section, I will construct an infinite sequence of hamiltonians $H_n$ with $\{H_n, H_m\} = 0$, see also [15]. (For a discrete, i.e. difference version of the matrix differential operator $L$ an infinite sequence of conserved quantities was constructed in [16].) The definitions (2.5) and (2.7) of the first and second Gelfand-Dikii brackets are sufficiently general to allow for arbitrary hermitian $n \times n$-matrices $U$. It is also obvious how to generalise the Proposition 6.3 below to this case. For simplicity of presentation, however, here I will restrict myself again to the case where $U$ is the hermitian $2 \times 2$-matrix of eq. (1.3).
That such a sequence of hamiltonians with \( \{H_n, H_m\} = 0 \) exists follows from the next

**Lemma 6.1**: The second Gelfand-Dikii bracket of any two spectral parameters \( \lambda \) (eigenvalues of \( -\partial^2 + U \)) vanishes:

\[
\{\lambda_\alpha, \lambda_\beta\} = 0 . \tag{6.1}
\]

**Proof**: If \( \lambda_\alpha = \lambda_\beta \) the bracket vanishes trivially. So let \( \lambda_\alpha \neq \lambda_\beta \). One has by (5.12) and the definition (4.5) of \( \Delta \)

\[
\gamma^{-2} \{\lambda_\alpha, \lambda_\beta\} = \int dx \, dy \, (\chi_\alpha(x))^T \langle x|\Delta|y\rangle \chi_\beta(y) = \int dx \, (\chi_\alpha(x))^T (\Delta \chi_\beta)(x) . \tag{6.2}
\]

By lemma 4, eq. (4.3), this equals \( \lambda_\beta \int dx \, (\chi_\alpha(x))^T (N \chi_\beta)(x) \). It follows from the antisymmetry of the Poisson brackets (or directly from (4.4)) that \( \Delta \) is an antisymmetric operator, as is obviously also \( N \). Thus \( \gamma^{-2} \{\lambda_\alpha, \lambda_\beta\} \) also equals \( \lambda_\alpha \int dx \, (\chi_\alpha(x))^T (N \chi_\beta)(x) \). For \( \lambda_\alpha \neq \lambda_\beta \) this implies \( \int dx \, (\chi_\alpha(x))^T (N \chi_\beta)(x) = 0 \) and hence \( \{\lambda_\alpha, \lambda_\beta\} = 0 \).

**Corollary 6.2**: The first Gelfand-Dikii bracket of any two spectral parameters \( \lambda \) vanishes, too.

**Proof**: This bracket is given by \( \int dx \, (\chi_\alpha(x))^T (N \chi_\beta)(x) \) which was just shown to vanish.

Thus the infinite set of \( \lambda_\alpha \) is in involution. However, their dependence on \( U \) is not very useful for practical purposes. On the other hand, the operator-trace of the resolvent operator, \( \int dx \, \text{tr} \, R(x; \xi) = \int dx \, \text{tr} \, \langle x|(−\partial^2 + U + \xi)^{-1}|x\rangle \), provides a very convenient functional of the eigenvalues \( \lambda_\alpha \) only (cf. eq. (5.6)). It follows from the previous lemma that

\[
\{ \int dx \, \text{tr} \, R(x; \xi), \int dy \, \text{tr} \, R(y; \xi') \} = 0 \tag{6.3}
\]

and if one defines

\[
H_n = \frac{(-4)^n}{2(2n - 1)} \int dx \, \text{tr} \, R_n(x) \tag{6.4}
\]

one has, upon inserting the expansion (5.3) into (6.3), the
Proposition 6.3: The $H_n$ are in involution:

$$\{H_n, H_m\} = 0. \quad (6.5)$$

Since this proposition is important, I will prove it once again in a different way, along the lines of [15]. By Lemma 5.3 of the previous section, eq. (5.11), one has

$$\frac{\delta H_{n+1}}{\delta U(x)} = (-4)^n R_n(x). \quad (6.6)$$

From Corollary 5.4 one then gets

Lemma 6.4: The $H_n$ satisfy the recursion relation

$$4\Delta \frac{\delta H_n}{\delta U} = N \frac{\delta H_{n+1}}{\delta U}. \quad (6.7)$$

or in components

$$2\{T, H_n\} = \frac{\partial \delta H_{n+1}}{\partial T}, \quad 2\{V^-, H_n\} = \frac{\partial \delta H_{n+1}}{\partial V^-}, \quad 2\{V^+, H_n\} = \frac{\partial \delta H_{n+1}}{\partial V^+}. \quad (6.8)$$

Thus, since $\Delta$ defines the second Gelfand-Dikii bracket, and since $\Delta$ and $N$ are antisymmetric operators, one has

$$\gamma^{-2}\{H_n, H_m\} = \int \frac{\delta H_n}{\delta U} \Delta \frac{\delta H_m}{\delta U} = \frac{1}{4} \int \frac{\delta H_n}{\delta U} N \frac{\delta H_{m+1}}{\delta U}$$

$$= -\frac{1}{4} \int \left( N \frac{\delta H_n}{\delta U} \right) \frac{\delta H_{m+1}}{\delta U} = - \int \left( \Delta \frac{\delta H_{n+1}}{\delta U} \right) \frac{\delta H_{m+1}}{\delta U} \quad (6.9)$$

Iterating this $m$ times one arrives at $\gamma^{-2}\{H_0, H_{n+m}\}$ which vanishes since $H_0$ is constant (independent of $U$). Hence $\{H_n, H_m\} = 0$ is proven again. The equalities of eq. (6.9) also show that $\{H_n, H_m\} \equiv \{H_n, H_m\}_{GD2} = \frac{1}{4}\{H_n, H_{m+1}\}_{GD1}$ i.e. the $H_n$ are in involution with respect to both Gelfand-Dikii brackets. More generally, the above Lemma 6.4 shows that for any functional $f[U]$ one has

$$4\{f, H_n\}_{GD2} = \{f, H_{n+1}\}_{GD1}. \quad (6.10)$$

The model is bihamiltonian.
The first few $H_n$ (most easily obtained from (6.4) and (5.10)) are

\[ H_1 = \frac{1}{2} \int \text{tr } U = \int T \]

\[ H_2 = \frac{1}{2} \int \text{tr } U^2 = \int (T^2 + 2V^+V^-) \]

\[ H_3 = \frac{1}{2} \int \text{tr } (2U^3 + U'^2) = \int (2T^3 + 12TV^+V^- + T'^2 + 2V^+V'-') \]

\[ H_4 = \frac{1}{2} \int \text{tr } (5U^4 + 10UU'^2 + U''^2) = \int (5T^4 + 20V^+V^- + 60T^2V^+V^- + 10TT'^2 + 20TV^+V'-' + 20T'V^+V'-' + 10T'^2 + 2V''V'^-) \]

(6.11)

**Remark 6.5:** The proof of $\{H_n, H_m\} = 0$ is based on the recursion relation (6.7) inherited from that of the coefficients $R_n$ of the resolvent. The crucial point is that $\Delta$ and $N$ give the second and first Gelfand-Dikii bracket. Comparing the recursion (5.8) for the $R_n$ for general (hermitian) $U$ with the first and second Gelfand-Dikii brackets (2.5) and (2.7) for general (hermitian) $U$ it is obvious that the proof of $\{H_n, H_m\} = 0$ can be straightforwardly generalised to $U$ a hermitian $n \times n$-matrix.

**Remark 6.6:** In the scalar case ($U$ a $1 \times 1$-matrix) one can show [3] that, if $p_\lambda$ is a solution of the Riccati equation $U - \lambda = p_\lambda^2 + p'_\lambda$, then the quantity $k(\lambda) = \exp(\int p_\lambda(x)dx)$ is an eigenvalue of the monodromy matrix of the wave function, and thus only depends on $\lambda$. Hence by the above Lemma 6.1, $\{\log k(\lambda), \log k(\lambda')\} = 0$. From the Riccati equation one has $p_\lambda = \sqrt{-\lambda} + \sum_{m=1}^{\infty} \frac{\omega^{(m)}}{(-4\lambda)^{m/2}}$ with $\omega^{(m+1)} = -(\omega^{(m)})' - \sum_{r=1}^{m-1} \omega^{(r)} \omega^{(m-r)}, \omega^{(1)} = u$. Hence $\{\int \omega^{(m)}, \int \omega^{(n)}\} = 0$. It turns out that $\int \omega^{(2m)} = 0$ while $(-)^{n+1} \int \omega^{(2n-1)}$ gives the hamiltonian $H_n$. I did not succeed to give the same type of proof in the matrix case. However, it is easy to verify for $n = 1, 2, 3$ and 4 the following

**Conjecture 6.7:** Let $U$ be a hermitian $n \times n$-matrix. Define $\Omega^{(1)} = U$ and $\Omega^{(m+1)} = -(\Omega^{(m)})' - \sum_{r=1}^{m-1} \Omega^{(r)} \Omega^{(m-r)}$. Let $\tilde{H}_n = (-)^{n+1} \frac{1}{2} \int \text{tr } \Omega^{(2n-1)}$. Then $\tilde{H}_n = H_n$.

A natural question is whether the $H_n$ are the only hamiltonians and whether this set is complete. Preliminary considerations of some simple other functionals of $U$, i.e. of $T, V^+$ and $V^-$ seem to indicate that this is indeed the case.
7. Hierarchies of matrix KdV* and matrix mKdV flows

As usual one defines an infinite hierarchy of flows by

\[ \frac{\partial U}{\partial t_r} = \gamma^{-2}\{U, H_r\} \right. \]  \hspace{1cm} (7.1)

Since \( \{H_r, H_s\} = 0 \) it follows from the Jacobi identity that all flows commute. Also, as usual, the flow in \( t_1 \) is trivial: \( \partial U/\partial t_1 = \partial_{\sigma} U \), implying that \( U \) is a function of \( \sigma + t_1 \) (and \( t_2, t_3, \ldots \)) only. Thus \( t_1 \) can be identified with the “world-sheet time” \( \tau \). The flow in \( t_2 \) gives the matrix generalisation of the KdV equation:

\[ \frac{\partial U}{\partial t_2} = (3U^2 - U''\}' \right. \]  \hspace{1cm} (7.2)

or in components

\[ \frac{\partial T}{\partial t_2} = (3T^2 - T'' + 6V^+ V^-)\}' \right. \quad \frac{\partial V^\pm}{\partial t_2} = (6TV^\pm - V^{\pm\prime\prime})\}' \] \hspace{1cm} (7.3)

Just as the Virasoro algebra is a subalgebra of the \( V \)-algebra (1.1), the KdV equation is simply obtained by setting \( V^\pm = 0 \). Note that since all \( H_n \) are symmetric in \( V^+ \) and \( V^- \) any non-local term (\( \sim \epsilon (\sigma - \sigma') \)) that might appear cancels in all flow equations, and the latter are always partial differential equations.‡

Using the Miura transformation of section 3, eq. (3.5), one also has the hierarchy of matrix mKdV flows \( \frac{\partial P}{\partial t_r} = \gamma^{-2}\{P, H_r\} \), in particular

\[ \frac{\partial P}{\partial t_2} = \left( \frac{1}{2} P^3 - P'\}' \right. \] \hspace{1cm} (7.4)

or in components

\[ \frac{\partial p_3}{\partial t_2} = (p_3^3 + 6p_3 p_+ p_- - p_3''\}' \right. \quad \frac{\partial p^\pm}{\partial t_2} = (3p_3^2 p^\pm + 2p_3^2 p_\mp - p_\pm''\)' \] \hspace{1cm} (7.5)

---

* Matrix KdV equations were already considered a long time ago by Calogero and Degasperis [17].
† As before, \( \{\cdot, \cdot\} \) is meant to be the second Gelfand-Dikii bracket. But since \( 4\frac{\partial U}{\partial t_r} = 4\gamma^{-2}\{U, H_r\}_{GD2} = \gamma^{-2}\{U, H_{r+1}\}_{GD1} \), one can also use the first Gelfand-Dikii bracket and the next higher hamiltonian instead.
‡ This also follows from the equivalence with the first Gelfand-Dikii bracket which is local, see previous footnote.
8. Relation with the pseudo-differential operator approach

As an alternative method to study the hierarchy of matrix KdV flows one can use the pseudo-differential operator method [18]. In the scalar case this is very standard technology (see e.g. [19]), while here some attention has to be paid to the non-commutativity of $U, U'$ etc. I will only sketch how to obtain the matrix KdV equation (flow in $t_2$). Starting with $L = \partial^2 - U$ one defines the square root

$$L^{\frac{1}{2}} = \partial - \frac{1}{2}U\partial^{-1} + \frac{1}{4}U'\partial^{-2} - \frac{1}{8}(U^2 + U'')\partial^{-3} + \ldots \quad (8.1)$$

so that

$$\left(L^{\frac{3}{2}}\right)_+ = \partial^3 - \frac{3}{2}U\partial - \frac{3}{4}U' \quad (8.2)$$

and

$$[\left(L^{\frac{3}{2}}\right)_+, L] = \frac{1}{4}(3UU' + 3U'U - U''') = \frac{1}{4}(3U^2 - U'''') \quad (8.3)$$

so that the matrix KdV equation reads as expected

$$\frac{1}{4}\frac{\partial L}{\partial t_2} = [\left(L^{\frac{3}{2}}\right)_+, L]. \quad (8.4)$$

9. Conclusions and generalisations

In this paper, I have shown that the recently discovered non-linear and non-local $V$-algebra (1.1) is obtained as the second Gelfand-Dikii hamiltonian structure based on the second-order $2 \times 2$-matrix differential operator $L = \partial^2 - U$. I observed that the non-locality of the $V$-algebra is a direct consequence of the non-commutativity of matrices. The Miura transformation is given by the free field representation and relates the first and second Gelfand-Dikii brackets. It is shown that $-L\Psi = \lambda\Psi$ implies an integro-differential equation for bilinears in $\Psi$. This latter equation naturally reproduces the two Gelfand-Dikii brackets. Then the asymptotic expansion of the (matrix) resolvent is studied, and the integro-differential equation for the bilinears translates into a recursion relation for the coefficients $R_n$. Defining an infinite series of hamiltonians as integrals of matrix-traces of the $R_n$, the recursion relation of the latter
becomes a recursion for the Hamiltonians. This immediately implies that the infinite sequence of Hamiltonians is in involution. The corresponding flows all commute and give the matrix generalisations of the KdV (and via the Miura transformation also of the mKdV) hierarchy. Finally, I outlined the connection with the matrix pseudo-differential operator approach of ref. [18].

There are obvious generalisations of the previous results. The generalisation to the case where \( U \) is a hermitian \( n \times n \)-matrix was already mentioned at several occasions. More generally, one can study the \( n \times n \)-matrix, \( m \)th-order differential operator \( L = \partial^m - U_{(2)} \partial^{m-2} - U_{(3)} \partial^{m-3} - \ldots - U_{(m)} \), maybe with certain constraints on the form of the matrices \( U_{(l)} \). (The matrix \( U \) in this paper was constrained by \( \text{tr} \sigma_3 U = 0 \).) The corresponding Gelfand-Dikii algebra could be called \( V_{(n,m)} \)-algebra, so that the above \( V \)-algebra would be \( V_{(2,2)} \), while the usual \( W_N \)-algebras are \( V_{(1,N)} \). There has been recently very interesting work by Kac and van de Leur [18] on the \( n \)-component KP-hierarchy and the corresponding \( m \)th reductions, mainly (but not exclusively) in the pseudo-differential operator formalism. The present approach, which is more field theoretic and insists on the \( V \)-algebraic aspects and the Hamiltonian structures, should be viewed as complementary.

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