Integrability of the Heisenberg Chains with Boundary Impurities and Their Bethe Ansatz

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Abstract

In this paper, we show the integrability of spin-1/2 XXZ Heisenberg chain with two arbitrary spin boundary Impurities. By using the fusion method, we generalize it to the spin-1 XXZ chain. Then the eigenvalues of Hamiltonians of these models are obtained by the means of Bethe ansatz method.

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1. Introduction

Recently, more and more papers have focused on the Kondo problem. It is well known that the spin dynamics of the Kondo problem is equivalent to the dynamics of the spin chain with magnetic impurities\cite{1}. Although magnetic impurities play important role in the model, they usually destroy the integrability of the system. So how to maintain the integrability of the quantum impurity system is an important problem. Fruitful achievement has been obtained based on the methods of the Bosonization and renormalization technique, conformal field theory and exact diagonalization in this field \cite{2-5}.

The quantum inverse scattering method and the Bethe ansatz technique have been powerful tools to study the integrable impurity problems in 1-dimensional physical system, such as Wang et al’s papers about the vertex model\cite{6-8} and Frahm et al and Links et al’s series papers about the $t-j$ model with impurities\cite{9-15}. The Heisenberg chain is an important model in the integrable system so many papers pay their attention to it. Andrei and Johannesson first considered the integrable Heisenberg chain with impurities under periodic boundary condition\cite{16}. Then Lee and Schlottmann generalized their results to arbitrary spin impurities\cite{17, 18}. But they have to present some unphysical terms in the Hamiltonian to maintain its integrability, though those terms may be irrelevant\cite{19}. To the open boundary condition problem, Gaudin considered the nonlinear Schrödinger model and the spin-1/2 Heisenberg chain with simple open boundaries\cite{19}, then Schulz and Alcaraz\cite{20, 21} et al. generalized it to Hubbard and other models. Wang have discussed the properties of the impurities with arbitrary spin coupled to the spin-1/2 XXX chain\cite{22}. The spin-1/2 XXZ chain coupled with spin-1/2 impurities has also been discussed in Ref.\cite{23}. In Ref.\cite{24}, the integrability of the spin-1 XXX chain with arbitrary spin impurities has been investigated.

In this paper, we study the integrability of the open Heisenberg chain coupled with arbitrary spin impurities. We discuss the spin-1/2 XXZ chain in the first part of the present paper. The spin-1 case is presented in the second part. A brief discussion about our results
is given in the last section.

2. The spin-1/2 XXZ chain

The $R$-matrix of spin-1/2 XXZ Heisenberg chain can be written as

$$
R(u) = \begin{pmatrix}
\sin(u + \eta) & 0 & 0 & 0 \\
0 & \sin u & \sin \eta & 0 \\
0 & \sin \eta & \sin u & 0 \\
0 & 0 & 0 & \sin(u + \eta)
\end{pmatrix}.
$$

(1)

This $R$-matrix is regular and satisfies the unitarity condition

$$
R(u)R(-u) = \sin(u + \eta)\sin(-u + \eta) = \rho(u).
$$

If we suppose the first and the second space of the $R$-matrix are auxiliary and quantum space respectively, and this $R$-matrix, as an operator matrix, can also be written as $L$-operator form

$$
L_n(u) = \sum_{j=1}^{4} w_j \sigma^j \otimes \sigma^j_n,
$$

(2)

where

$$
w_1 = w_2 = \frac{1}{2} \sin v,
w_4 - w_3 = \sin u,
w_4 + w_3 = \sin(u + \eta),
$$

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

The $L$-operator and $R$-matrix satisfy the following Yang-Baxter relations (YBR)

$$
R_{12}(u - v) \frac{1}{L_i} (u) \frac{2}{L_i} (v) = \frac{2}{L_i} (v) \frac{1}{L_i} (u) R_{12}(u - v),
R_{12}(u - v) \frac{1}{T} (u) \frac{2}{T} (v) = \frac{2}{T} (v) \frac{1}{T} (u) R_{12}(u - v),
$$

(3)

where $R_{12}$ acts on the auxiliary space $v_1 \otimes v_2$, and the $T$ defined by

$$
T(u) = \prod_{j=1}^{N} L_j(u)
$$

acts on quantum spaces as $v_1 \otimes v_2 \otimes \cdots \otimes v_N$. Here we have used the notation $\frac{1}{A} = A \otimes 1$, $\frac{2}{A} = 1 \otimes A$.

In order to construct the open boundary condition consistent with the integrability, we consider the reflection equation

$$
R_{12}(u - v) \frac{1}{K} (u) R_{21}(u + v) \frac{2}{K} (v) = \frac{2}{K} (v) R_{12}(u + v) \frac{1}{K} (u) R_{21}(u - v).
$$

(4)
where $K$ is the reflecting matrix, which determines the boundary terms in the Hamiltonian. One can prove that the double-row monodromy matrix defined by $U(u) = T(u)K(u)T^{-1}(-u)$ also satisfies the reflection equation

$$R_{12}(u - v) \frac{1}{U(u)R_{21}(u + v)} \frac{2}{U(v)R_{12}(u + v)} = U(v)R_{12}(u + v) \frac{1}{U(u)R_{21}(u - v)}.$$  

(5)

The dual $K$-matrix $K^+(u)$ can be defined by the automorphism

$$\phi : K(u) \rightarrow K^+(u) = K^t(-u - \eta).$$  

(6)

It satisfies the dual reflection equation

$$R_{12}(-u + v) \frac{1}{K^+(u)R_{12}(-u - v - 2\eta)} \frac{2}{K^+(v)R_{12}(-u - v - 2\eta)} = K^+(v)R_{12}(-u - v) \frac{1}{K^+(u)R_{12}(-u + v)}.$$  

(7)

Then the transfer matrix can be defined by

$$t(u) = trK^+(u)U(u).$$  

(8)

One can check that it satisfies the commutation relation

$$[t(u), t(v)] = 0.$$  

(9)

Now we couple the spin-1/2 XXZ chain with two arbitrary spin impurities located at the ends of the system. Then the $L$-operator of the boundary cites can be written as

$$L_i(u) = \begin{pmatrix} \sin(u + \frac{\eta}{2} + d_i^+ \eta) & d_i^+ \sin \eta \\ d_i^+ \sin \eta & \sin(u + \frac{\eta}{2} - d_i^+ \eta) \end{pmatrix}, (i = a, b)$$  

(10)

where $d^\pm_i, d^z_i$ are components of an arbitrary spin $m$ of $SU_q(2)$. One can easily check that the $L$-operator satisfies the first relation of (3). It also have the unitarity relation

$$L_i(u)L_i(-u) = d^2 \sin^2 \eta \sin(u + \frac{\eta}{2}) \sin(-u + \frac{\eta}{2}) = \rho_d(u)$$  

with $d^2 = \sin(m\eta)\sin(\eta + m\eta)/\sin^2 \eta$.

Define

$$T(u) = L_b(u + c_b)L_N(u) \cdots L_2(u)L_1(u)L_a(u + c_a),$$

$$\bar{T}(u) = T^{-1}(-u) \times \text{const.}$$

$$= L_a(u - c_a)L_1(u)L_2(u) \cdots L_N(u)L_b(u - c_b).$$  

(11)

where $c_i$ are free parameters. According to Cherednik and Sklyanin’s work, the reflection matrix and its dual are defined by

$$K(u) = \text{diag} \left( \frac{\sin(\xi - u)}{\sin(\xi)}, \frac{\sin(\xi + u)}{\sin(\xi + u)} \right), K^+(u) = \text{diag} \left( \frac{\sin(\xi + u + \eta)}{\sin(\xi + u + \eta)}, \frac{\sin(\xi - u - \eta)}{\sin(\xi - u - \eta)} \right).$$  

(12)

Recalling the definition of $t(u)$ (8), one can check that the above formulas satisfy the commutation relation (9). By expanding $t(u)$ in terms of $u$, we can obtain infinite number of
conserved quantities which ensures the integrability of the model. The Hamiltonian of this model can be written as

\[ H = \frac{1}{2\rho^2N(0)\rho_d(c_a)\rho_d(c_b)tr_\tau K^+(0)} \times \frac{dt(u)}{du}|_{u=0} \]

\[ = \sum_{j=1}^{N-1} \frac{H_{j,j+1}(u)}{\rho^2(u)}|_{u=0} \]

\[ + \frac{d(L_a (u + c_a) L (u - c_a))}{2\rho_d(c_a)du}|_{u=0} + \frac{1}{L (u + c_a)} \frac{dK(1)(u)}{2\rho_d(c_a)du} L (u - c_a)|_{u=0} \]

\[ + \tau K^+ (u) L (u + c_b)\tau H_{\tau,N}(u) \tau L (u - c_b)|_{u=0} \]

\[ = \frac{1}{2\rho_d(c_b)\rho^2(u)tr_\tau K^+(u)} \]

\[ + \tau K^+ (u) dL (u + c_b) \tau L (u - c_b)|_{u=0} + \text{const.} \]

\[ (13) \]

where \( H_{j,j+1}(u) = \frac{dR_{j,j+1}(u)}{du} R_{j,j+1}(u) \). Denoting by \( T_i \) the \( i \)th term of the right hand side of (13), we have

\[ T_1 = \frac{1}{\sin \eta} \sum_{j=1}^{N-1} \left( \sigma^1_j \cdot \sigma^1_{j+1} + \sigma^2_j \cdot \sigma^2_{j+1} + \cos \eta \sigma^3_j \cdot \sigma^3_{j+1} \right) \],

\[ T_2 + T_3 = \frac{1}{2} (1 + \sigma^3_a) \cdot A_a + \frac{1}{2} (1 - \sigma^3_b) \cdot B_b + \sigma^+_a \cdot C_a d^+_a C_a + \sigma^-_a \cdot C_a d^-_a C_a,

\[ T_4 + T_5 = \frac{1}{2} (1 + \sigma^3_b) \cdot A_b + \frac{1}{2} (1 - \sigma^3_a) \cdot B_a + \sigma^+_b \cdot C_b d^+_b C_b + \sigma^-_b \cdot C_b d^-_b C_b,

\[ A_i = \frac{1}{2\rho_d(c_i)} \left( \sin(\eta + 2d^2\eta) - \frac{2\cos \xi}{\sin \xi}(d^2 \sin^2(\eta) - \sin(d^2\eta) \sin(\eta + d^2\eta)) \right),

\[ B_i = \frac{1}{2\rho_d(c_i)} \left( \sin(\eta - 2d^2\eta) - \frac{2\cos \xi}{\sin \xi}(\sin(c_i + \frac{\eta}{2} - d^2\eta) \sin(-c_i + \frac{\eta}{2} - d^2\eta) \right),

\[ C_i = -\frac{2\cos \xi}{\rho_d(c_i) \sin \xi} \sin(c_i + \frac{\eta}{2} - \xi + d^2\eta), \text{ for } i = a, b \]

(14)

where \( \xi \) should be changed to \( \xi^+ \) when \( i = b \), and this Hamiltonian is hermitician when we choose pure imaginary \( c_i \).

To construct the algebraic Bethe ansatz, we rewrite the double-row monodromy matrix \( U(u) \) in the form

\[ U(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \]

(15)

Using the reflection equation (5), we can obtain the following commutation relation

\[ B(u)B(v) = B(v)B(u); \]

\[ A(u)B(v) = \frac{\sin(\eta + v - u) \sin(u + v)}{\sin(\eta + v + u) \sin(v - u)} B(v)A(u) \]

\[ + \frac{\sin \eta \sin(u + v)}{\sin(\eta + v + u) \sin(u - v)} B(u)A(v) \]

\[ - \frac{\sin \eta}{\sin(\eta + v + u)} B(u)D(v); \]
where $\mathcal{D}(u) = \sin(2u + \eta)\mathcal{D}(u) - \sin \eta \mathcal{A}(u)$. Using the relation (8) and (15), the transfer matrix $t(u)$ can now be written as

$$
t(u) = w_1^+ \mathcal{D}(u) + w_2^+ \mathcal{A}(u) = \frac{\sin(\xi + u + \eta)}{\sin(2u + \eta)} \mathcal{D}(u) + \frac{\sin(\xi - u + 2\eta)}{\sin(2u + \eta)} \mathcal{A}(u). \tag{17}
$$

Define the pseudo-vacuum state $|0\rangle$

$$
\sigma_i^+ |0\rangle = d^+ |0\rangle = 0, \quad (i = 1, 2, \cdots, N) \tag{18}
$$

Acting the elements of $U_r(u)$ on $|0\rangle$, we have

$$
\begin{align*}
\mathcal{C}(u)|0\rangle &= 0 \\
\mathcal{A}(u)|0\rangle &= w_1|0\rangle \\
&= \sin(\xi + u) \sin^{2N}(u + \eta) \prod_{r = \pm 1} \prod_{i = a,b} \sin(u + rc_i + \eta/2 + m\eta)|0\rangle \\
\mathcal{D}(u)|0\rangle &= w_2|0\rangle \\
&= \sin(\xi - u - \eta) \sin(2u) \sin^{2N} u \prod_{r = \pm 1} \prod_{i = a,b} \sin(u + rc_i + \eta/2 - m\eta)|0\rangle \tag{19}
\end{align*}
$$

The eigenstates of $t(u)$ can be constructed from the pseudo-vacuum state

$$
|\Omega\rangle = \prod_{i = 1}^M \mathcal{B}(u_i)|0\rangle. \tag{20}
$$

Thus we obtain the eigenvalue of $t(u)$ acting on $|\Omega\rangle$

$$
t(u)|\Omega\rangle = w_1^+ \prod_{i = 1}^M \frac{\sin(\eta + v_i - u) \sin(u + v_i)}{\sin(\eta + v_i + u) \sin(-u + v_i)} |\Omega\rangle + w_2^+ \prod_{i = 1}^M \frac{\sin(u - v_i + \eta) \sin(u + v_i + 2\eta)}{\sin(u + v_i + \eta) \sin(u - v_i)} |\Omega\rangle \tag{21}
$$

with the Bethe ansatz

$$
\frac{\sin(v_j - \xi + \eta/2) \sin(v_j + \xi - \eta/2)}{\sin(v_j + \xi + \eta/2) \sin(v_j - \xi + \eta/2)} \sin^{2N}(v_j + \frac{\eta}{2}) \times \prod_{r = \pm 1} \prod_{k = a,b} \frac{\sin(v_j + rc_k + m\eta)}{\sin(v_j + rc_k - m\eta)} = \prod_{i \neq j} \frac{\sin(v_j - v_i + \eta) \sin(v_j + v_i + \eta)}{\sin(v_j - v_i - \eta) \sin(v_j + v_i - \eta)}. \tag{22}
$$

3 The spin-1 Heisenberg chain
3.1 The Zamolodchikov-Fateev 19-vertex model

The Zamolodchikov-Fateev 19-vertex $R$-matrix\[23\] associated with the spin-1 representation of $U_q(sl_2)$\[23\] can be obtained by using the fusion method\[22, 24\]. It reads

$$R(u) = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & a_5 & 0 & a_6 & 0 \\ 0 & a_3 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 & 0 & a_7 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & a_3 \\ 0 & 0 & a_6 & 0 & a_5 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 \end{pmatrix},$$

(23)

with

$$a_1 = \sin(u + 2\eta) \sin(u + \eta), \quad a_2 = \sin u \sin(u + \eta),$$

$$a_3 = \sin(2\eta) \sin(u + \eta), \quad a_4 = \sin u \sin(u - \eta),$$

$$a_5 = \sin u \sin(2\eta), \quad a_6 = \sin \eta \sin(2\eta),$$

$$a_7 = a_2 + a_6.$$

This $R$-matrix is regular and satisfies the unitarity relation

$$\sin(u + \eta) \sin(u - \eta) \sin(u + 2\eta) \sin(u - 2\eta) = \rho(u).$$

It satisfies the Yang Baxter equation

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v).$$

(24)

Inami et al\[27\] have obtained the general solution $K(u)$ of (4). In this paper, we only adopt its diagonal form as in the Ref.\[33\].

$$K(u) \equiv \text{diag}(k_1(u), k_2(u), k_3(u))$$

$$= \text{diag}\left(1, \frac{\sin\left(\frac{u}{2} + \xi - u\right)}{\sin\left(\frac{u}{2} + \xi + u\right)} \cdot \frac{\sin\left(\frac{u}{2} + \xi - u\right)}{\sin\left(\frac{u}{2} - \xi - u\right)} \right),$$

(25)

and the corresponding dual reflection matrix takes the form

$$K^+(u) \equiv \text{diag}(k_1^+(u), k_2^+(u), k_3^+(u))$$

$$= \text{diag}\left(1, \frac{\sin\left(\frac{3\eta}{2} + \xi + u\right)}{\sin\left(-\frac{\eta}{2} + \xi + u\right)} \cdot \frac{\sin\left(\frac{3\eta}{2} + \xi + u\right)}{\sin\left(-\frac{\eta}{2} - \xi + u\right)} \right).$$

(26)

3.2 Fusion of the boundary $L$-operator

In this section we discuss the fusion procedure of the boundary $L$-operator. The permutation operator and the projection operators are defined by

$$P_{12} = R_{12}(0)/\sin(\eta)$$

$$P_{12}^- = -R_{12}(-\eta)/(2 \sin \eta)$$

$$P_{12}^+ = 1 - P_{12}^-$$

(27) (28) (29)
respectively, where the $R(u)$ is from the 6-vertex model (1). They satisfy the following properties

\[ P_{12}^2 = 1, \quad (P_{12}^+)^2 = P_{12}^+, \]
\[ P_{12}^+ P_{12}^- = P_{12}^- P_{12}^+ = 0. \]  

(30)

Now we use fusion procedure to obtain the high-dimensional $L$-operator. Taking $v = u + \eta$ in the equation (3), the YBR gives

\[ R_{12}(-\eta)L_{1d}(u)L_{2d}(u + \eta) = L_{2d}(u + \eta)L_{1d}(u)R_{12}(-\eta), \]

(31)

where $d$ represent the boundary terms, and we write $L_{ij}$ as $L_{ij}$ for convenience. Multiplying above equation by $P_{12}^+$ from the left and right respectively, we get

\[ P_{12}^+ L_{2d}(u + \eta)L_{1d}(u)P_{12}^- = 0, \]
\[ P_{12}^- L_{1d}(u)L_{2d}(u + \eta)P_{12}^+ = 0. \]

(32)  

(33)  

Define

\[ L_{<12>,d}(u) = P_{12}^+ L_{1d}(u)L_{2d}(u + \eta)P_{12}^+, \]
\[ L'_{<12>,d}(u) = P_{12}^+ L_{2d}(u)L_{1d}(u - \eta)P_{12}^+. \]

(34)  

(35)

which satisfy the YBR respectively.

\[ R_{<12>,<34>}(u - v)L_{<12>,d}(u)L_{<34>,d}(v) = L_{<34>,d}(v)L_{<12>,d}(u)R_{<12>,<34>}(u - v), \]
\[ R'_{<12>,<34>}(u - v)L'_{<12>,d}(u)L'_{<34>,d}(v) = L'_{<34>,d}(v)L'_{<12>,d}(u)R'_{<12>,<34>}(u - v), \]

(36)  

(37)

where $R_{<12>,<34>}(u - v)$ is the fused $R$-matrix [24], acts on $V_{<12>} \otimes V_{<34>}$. Here we give the proof for $R(u)$:

\[
\text{LHS} = R_{<12>,<34>}(u - v)P_{12}^+ L_{1d}(u)L_{2d}(u + \eta)P_{12}^+ P_{34}^+ L_{3d}(v)L_{4d}(v + \eta)P_{34}^+
\]
\[
= P_{12}^+ P_{34}^+ R_{14}(u - v - \eta)R_{13}(u - v)P_{34}^+ P_{34}^+ R_{24}(u - v)R_{23}(u - v + \eta)\times P_{34}^+ P_{12}^+ L_{1d}(u)L_{2d}(u + \eta)L_{3d}(v)L_{4d}(v + \eta)P_{12}^+ P_{34}^+
\]
\[
= R_{14}(u - v - \eta)R_{13}(u - v)R_{24}(u - v)R_{23}(u - v + \eta)L_{1d}(u)L_{2d}(u + \eta)\times L_{3d}(v)L_{4d}(v + \eta)P_{12}^+ P_{34}^+
\]
\[
= R_{14}(u - v - \eta)R_{24}(u - v)R_{13}(u - v)L_{1d}(u)L_{3d}(v)L_{2d}(u + \eta)\times R_{23}(u - v + \eta)L_{4d}(v + \eta)P_{12}^+ P_{34}^+
\]
\[
= L_{3d}(v)L_{4d}(v + \eta)L_{1d}(u)L_{2d}(u + \eta)R_{14}(u - v - \eta)R_{13}(u - v)\times R_{24}(u - v)R_{23}(u - v + \eta)P_{34}^+ P_{12}^+
\]
\[
= P_{34}^+ L_{3d}(v)L_{4d}(v + \eta)P_{34}^+ P_{12}^+ L_{1d}(u)L_{2d}(u + \eta)P_{12}^+ P_{34}^+ R_{14}(u - v - \eta)\times R_{13}(u - v)P_{34}^+ P_{34}^+ R_{24}(u - v)R_{23}(u - v + \eta)P_{12}^+ P_{34}^+
\]
\[
= L_{<34>,d}(v)L_{<12>,d}(u)R_{<12>,<34>}(u - v)\]
\[
= \text{RHS}. \]

(38)

The proof for the other formula is similar. Substituting relation (10) into (33) and taking the transformation

\[
L_{<12>,d}(u) \mapsto (1 - \sqrt{2\cos \eta} \ 1)L_{<12>,d}(u) \begin{pmatrix} 1 & 1 \\ \sqrt{2\cos \eta} & 1 \end{pmatrix},
\]

(39)
we have

\[
L_{<12>}(u) = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix},
\]

with

\[
\begin{align*}
    a_{11} &= \sin(u + d^z \eta) \sin(u + \eta + d^z \eta), \\
    a_{12} &= d^- \sqrt{2} \cos \eta \sin(u + d^z \eta) \sin \eta, \\
    a_{13} &= (d^-)^2 \sin^2 \eta, \\
    a_{21} &= \sqrt{2} \cos \eta \sin(u + d^z \eta) \sin \eta d^+, \\
    a_{22} &= \sin u \sin(u + \eta) + d^2 \sin^2 \eta - 2 \sin^2 (d^z \eta) \cos \eta, \\
    a_{23} &= \sqrt{2} \cos \eta \sin(u - d^z \eta) \sin \eta d^-, \\
    a_{31} &= (d^+)^2 \sin^2 \eta, \\
    a_{32} &= d^+ \sqrt{2} \cos \eta \sin(u - d^z \eta) \sin \eta, \\
    a_{33} &= \sin(u - d^z \eta) \sin(u + \eta - d^z \eta).
\end{align*}
\]

This \( L \)-operator also satisfies the unitarity relation with

\[
\rho_d(u) = \frac{1}{4} \{ (-d^2 - \cos(2u - \eta) + \cos \eta + d^2 \cos(2\eta)) \} \\
\times \{ -d^2 - \cos(2u + \eta) + \cos \eta + d^2 \cos(2\eta) \}.
\]

### 3.3 The Hamiltonian of this model

Define

\[
T(u) = L_b(u + c_b) L_N(u) \cdots L_2(u) L_1(u) L_a(u + c_a),
\]

\[
\bar{T}(u) = T^{-1}(-u) \times \text{const.}
\]

\[
T(u) = L_a(u - c_a) L_1(u) L_2(u) \cdots L_N(u) L_b(u - c_b),
\]

where \( c_a \) and \( c_b \) are constant. The spin-1 \( L \)-operator is obtained from the \( R \)-matrix (23) by assigning the second space to be the quantum space, and \( L_i(u) \) (\( i = a, b \)) is given by (40). The Hamiltonian of this model is as same as the spin-1/2 case (13). Here we give \( T_1 \) as

\[
T_1 = \frac{1}{\sin(2\eta)} \sum_{j=1}^{N-1} \left\{ \left( \frac{s_j \cdot \tilde{s}_{j+1}}{\cos \eta} - \frac{(s_j \cdot \tilde{s}_{j+1})^2}{\cos^2 \eta} \right) + (1 - \cos \eta) \left( s_j^z s_{j+1}^z s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+ \right) \right. \\
-\left. (1 - \cos(2\eta)) \left( s_j^z s_{j+1}^z - (s_j^z)^2 (s_{j+1}^z)^2 + (s_j^z)^2 + (s_{j+1}^z)^2 \right) \right\}
\]

with

\[
\frac{1}{\cos \eta} s_j \cdot \tilde{s}_{j+1} = \frac{1}{2} s_j^- s_{j+1}^+ + \frac{1}{2} s_j^+ s_{j+1}^- + \cos \eta \frac{\sin(s_j^z \eta) \sin(s_{j+1}^z \eta)}{\sin^2 \eta},
\]

and

\[
\begin{align*}
s^+ &= \sqrt{2} \cos \eta \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
s^- &= \sqrt{2} \cos \eta \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
s^z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\end{align*}
\]
\[ T_2 + T_3 = \frac{1}{2} (s_z + s_z^2) A_a + \sin \eta s_a s^+ D_a d^+ + (s^+)^2 E_a (d^-)^2 \]
\[ + \sin \eta s_z s d^+ D_a + \left( \frac{1}{2 \cos \eta} (s^+ s^+ + s^- s^+ \right) - s_z^2) B_a - \sin \eta s_z F_a d^- \]
\[ + (s^-)^2 (d^+)^2 E_a - \sin \eta s_z s^- d^+ F_a + \frac{1}{2} (s_z^2 - s_z) C_a, \]

\[ T_4 + T_5 = \frac{1}{2} (s_z + s_z^2) A_b + \sin \eta s_z s^+ D_b d^- + (s^+)^2 E_b (d^-)^2 \]
\[ + \sin \eta s_z s d^+ D_b + \left( \frac{1}{2 \cos \eta} (s^+ s^+ + s^- s^+ \right) - s_z^2) B_b - \sin \eta s_z F_b d^- \]
\[ + (s^-)^2 (d^+)^2 E_b - \sin \eta s_z s^- d^+ F_b + \frac{1}{2} (s_z^2 - s_z) C_b, \]

with

\[ A_i = \frac{1}{2 \rho \rho_1 (c_i)} \left\{ \cos(2c_i) \cos \eta \sin(\eta + 2d^z \eta) - \frac{1}{2} \sin(2\eta + 4d^z \eta) \right. \]
\[ + [d^2 \sin^2 \eta - \sin(d^z \eta) \sin(d^z \eta)] \cdot [2 \cos \eta \sin(2\eta + 2d^z \eta) \]
\[ \left. - 4 \cos(\frac{\eta}{2} + \xi) \cos \eta \sin(c_i + \eta + d^z \eta) \sin(-c_i + \eta + d^z \eta) \sin(\frac{\eta}{2} + \xi) \right. \]
\[ + \left. \frac{2 \sin(2\xi)(d^2 \sin^2 \eta - \sin(\eta + d^z \eta)) \sin(2\eta + d^z \eta))}{\sin(\eta + d^z \eta) \sin(\eta + d^z \eta) \sin(\frac{\eta}{2} + \xi)} \right\}, \]

\[ B_i = \frac{1}{2 \rho \rho_1 (c_i)} \left\{ 2 \sin \eta \sin(2c_i) [d^2 \sin^2 \eta - \cos \eta \sin(d^z \eta)] + \sin(2\eta)(\sin^2(2d^z \eta) - \sin^2 c_i) \right. \]
\[ - \frac{2 \cos(\frac{\eta}{2} + \xi)}{\sin(\eta + \xi)} \times (d^2 \sin^2 \eta + \sin c_i \sin(c_i + \eta) - 2 \cos \eta \sin^2(d^z \eta) \]
\[ \times (d^2 \sin^2 \eta + \sin c_i \sin(c_i - \eta) - 2 \cos \eta \sin^2(d^z \eta) \]
\[ - \frac{4 \sin(2\xi) \cos \eta \sin(c_i - d^z \eta) \sin(-c_i - d^z \eta))(d^2 \sin^2 \eta - \sin(d^z \eta)) \sin(\eta + d^z \eta)}{\sin\frac{\eta}{2} - \xi - \sin\frac{\eta}{2} + \xi} \right\}, \]

\[ C_i = \frac{1}{2 \rho \rho_1 (c_i)} \left\{ \cos(2c_i) \cos \eta \sin(\eta - 2d^z \eta) - \frac{1}{2} \sin(2\eta - 4d^z \eta) \right. \]
\[ + [d^2 \sin^2 \eta + \sin(d^z \eta) \sin(\eta - d^z \eta)] \cdot [2 \cos \eta \sin(2\eta - 2d^z \eta) \]
\[ + \left. \frac{4 \cos(\frac{\eta}{2} + \xi) \cos \eta \sin(c_i + \eta - d^z \eta) \sin(c_i - \eta + d^z \eta) \sin(\frac{\eta}{2} + \xi)}{\sin(\eta - d^z \eta) \sin(\eta - d^z \eta) \sin(\frac{\eta}{2} + \xi)} \right\}, \]

\[ D_i = \frac{\sin(c_i - \frac{\eta}{2} - d^z \eta + \xi)}{2 \sin(\frac{\eta}{2} - \xi) \sin(\frac{\eta}{2} + \xi)} \]
\[ \times \left[ (2d^2 \sin^2 \eta - \cos \eta) \sin(\frac{\eta}{2} - \xi) - \cos(2c_i) \sin(\frac{\eta}{2} + \xi) + \sin \eta \cos(\frac{3\eta}{2} + 2d^z \eta - \xi) \right], \]

\[ E_i = \frac{\sin^3 \eta \sin(c_i - \frac{3\eta}{2} - d^z \eta + \xi) \sin(c_i - \frac{\eta}{2} - d^z \eta + \xi)}{2 \cos \eta \sin(\frac{\eta}{2} - \xi) \sin(\frac{\eta}{2} + \xi)}, \]

\[ F_i = \frac{\sin(c_i - \frac{\eta}{2} - d^z \eta + \xi)}{2 \sin(\frac{\eta}{2} - \xi) \sin(\frac{\eta}{2} + \xi)}. \]
\[ \times \left[ -2d^2 \sin^2 \eta - \cos \eta \sin \left( \frac{\eta}{2} + \xi \right) + \cos(2c_i) \sin \left( \frac{\eta}{2} - \xi \right) - \sin \eta \cos \left( \frac{\eta}{2} - 2d^2 \eta + \xi \right) \right], \]

where \( i = a, b \), and \( \xi \) should be changed to \( \xi^+ \) when \( i = b \). As in the spin-1/2 case, one can check that the Hamiltonian is hermitian when we choose pure imaginary \( c_1 \).

### 3.4 The Bethe ansatz for this model

To construct the algebraic Bethe ansatz, we define the pseudo-vacuum state \(|0\rangle\) as

\[
s_i^+ |0\rangle = d^+ |0\rangle = 0, \quad (i = 1, 2, \cdots, N) \\
d^2 |0\rangle = m |0\rangle. \tag{43}
\]

And as before, we write \( T(u) \) as

\[
T(u) = \begin{pmatrix} \mathcal{A}_1(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{A}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{A}_3(u) \end{pmatrix}. \tag{44}
\]

In order to simplify our calculation, we introduce the following transformations

\[
\tilde{A}_2(u) = A_2(u) - \frac{\sin(2\eta)}{\sin(2u+2\eta)} A_1(u) \tag{45}
\]

\[
\tilde{A}_3(u) = A_3(u) - \frac{\sin(2\eta)}{\sin(2u)} A_2(u) - \frac{\sin \eta \sin(2\eta)}{\sin(2u+\eta) \sin(2u+2\eta)} A_1(u) \tag{46}
\]

It is easy to show

\[
\mathcal{C}_1(u)|0\rangle = 0, \quad \mathcal{B}_i(u)|0\rangle \neq 0, \quad (i = 1, 2, 3), \\
A_1(u)|0\rangle = w_1|0\rangle, \quad \tilde{A}_2(u)|0\rangle = w_2|0\rangle, \quad \tilde{A}_3(u)|0\rangle = w_3|0\rangle,
\]

with

\[
w_1 = \sin^{2N} (u+\eta) \sin^{2N} (u+2\eta) \\
\times \prod_{i=a,b} \prod_{r=\pm 1} \sin(u+rc_i+mn\eta) \sin(u+rc_i+\eta+mn\eta),
\]

\[
w_2 = \frac{\sin(2u) \sin(\xi-u-\frac{3\eta}{2})}{\sin(2u+2\eta) \sin(\xi+u+\frac{\eta}{2})} \sin^{2N} u \sin^{2N} (u+\eta) \\
\times \prod_{i=a,b} \prod_{r=\pm 1} [\sin(u+rc_i) \sin(u+rc_i+\eta) + d^2 \sin^2 \eta - 2 \sin^2 (mn\eta) \cos \eta],
\]

\[
w_3 = \frac{\sin(-\xi+u+\frac{\eta}{2}) \sin(-\xi+u+\frac{3\eta}{2}) \sin(\eta-2u)}{\sin(\xi+u+\frac{\eta}{2}) \sin(-\xi+u+\frac{\eta}{2}) \sin(\eta+2u)} \sin^{2N} u \sin^{2N} (u-\eta) \\
\times \prod_{i=a,b} \prod_{r=\pm 1} \sin(u+rc_i-mn\eta) \sin(u+rc_i+\eta-mn\eta).
\]

From the reflection equation (5), we can find the following commutation relation

\[
\mathcal{A}_1(u) \mathcal{B}_1(v) |0\rangle = \frac{\sin(u-v-2\eta) \sin(u+v)}{\sin(u+v+2\eta) \sin(u-v)} \mathcal{B}_1(v) \mathcal{A}_1(u) |0\rangle \tag{47}
\]

\[
+ \frac{\sin(2v) \sin(2\eta)}{\sin(u-v) \sin(2v+2\eta)} \mathcal{B}_1(u) \mathcal{A}_1(v) |0\rangle \\
- \frac{\sin(2\eta)}{\sin(u+v+2\eta)} \mathcal{B}_1(u) \tilde{A}_2(v) |0\rangle.
\]
\[
\begin{align*}
\tilde{t}(u)|v_1, v_2, \ldots, v_n\rangle &= (k_1^+ A_1(u) + k_2^+ A_2(u) + k_3^+ A_3(u))|v_1, v_2, \ldots, v_n\rangle \\
&= (w_1^+ A_1(u) + w_2^+ \tilde{A}_2(u) + w_3^+ \tilde{A}_3(u))|v_1, v_2, \ldots, v_n\rangle \\
&= w_1^+ w_1 \prod_{i=1}^{n} \frac{\sin(u - v_i - 2\eta)}{\sin(u + v_i)} \frac{\sin(u + v_i + \eta)}{\sin(u - v_i + \eta)} |v_1, v_2, \ldots, v_n\rangle \\
&\quad + w_2^+ w_2 \prod_{i=1}^{n} \frac{\sin(u - v_i - 2\eta)}{\sin(u + v_i + \eta)} \frac{\sin(u + v_i + \eta)}{\sin(u - v_i)} |v_1, v_2, \ldots, v_n\rangle \\
&\quad + w_3^+ w_3 \prod_{i=1}^{n} \frac{\sin(u - v_i + \eta)}{\sin(u - v_i - \eta)} \frac{\sin(u + v_i + \eta)}{\sin(u + v_i)} |v_1, v_2, \ldots, v_n\rangle 
\end{align*}
\]
where

\[ w_1^+(u) = \frac{\sin(2u + 3\eta) \sin(u - \xi^+ - \frac{\eta}{2})}{\sin(2u + \eta) \sin(u - \xi^+ + \frac{3\eta}{2})}, \]

\[ w_2^+(u) = \frac{\sin(\xi^+ + u + \frac{3\eta}{2}) \sin(u - \xi^+ - \frac{\eta}{2}) \sin(2u + 2\eta)}{\sin(-\xi^+ + u + \frac{3\eta}{2}) \sin(-u + \xi^+ - \frac{\eta}{2}) \sin(2u)}, \]

\[ w_3^+(u) = \frac{\sin(\xi^+ + u + \frac{3\eta}{2}) \sin(-u - \xi^+ - \frac{\eta}{2}) \sin(2u + 2\eta)}{\sin(-\xi^+ + u + \frac{3\eta}{2}) \sin(-u + \xi^+ - \frac{\eta}{2})}. \]

and the free parameters \( v_i, i=1,2,\cdots,n \) obey the Bethe ansatz equation

\[
\frac{\sin(\xi^+ - v_j + \frac{\eta}{2}) \sin(\xi + v_j - \frac{\eta}{2}) \sin^{2N}(v_j + \eta)}{\sin(\xi^+ + v_j + \frac{\eta}{2}) \sin(\xi - v_j - \frac{\eta}{2}) \sin^{2N}(v_j - \eta)} \prod_{k=a,b} \prod_{r=\pm 1} \frac{\sin(v_j + rc_k + m\eta)}{\sin(v_j + rc_k - m\eta)} = \prod_{i\neq j} \frac{\sin(v_j - v_i + \eta) \sin(v_j + v_i + \eta)}{\sin(v_j - v_i - \eta) \sin(v_j + v_i - \eta)}, \quad j = 1, 2, \cdots, n. \tag{52}
\]

This Bethe ansatz equation can also be derived by the means of the fusion method, we have checked that they agree with each other \([32, 33]\). With this method of Bethe ansatz, we cannot get complete eigenstates. However, it is a powerful tool to obtain the eigenvalues and Bethe ansatz equation for models which cannot be obtained with fusion method \([29]\).

4. Discussion

In this paper, we studied the integrability of the spin-1/2 and spin-1 XXZ open Heisenberg chains with boundary impurities. These models are relevant to the Kondo problem in a Luttinger liquid. By using the algebraic Bethe ansatz and its extension, we have obtained the eigenvalues of the Hamiltonians and the Bethe ansatz equations. When we let \( \vec{d} = 0 \) in this paper, one can easily check that the \( L_i(u) \) \((i = a, b)\) in formulas (10) and (40) will be identity so the present Hamiltonians and Bethe ansatz equations can be reduced to the usual ones respectively. This procedure can be generalized to the general Heisenberg chain. It is worthy to point out that the Bethe ansatz equations and eigenvalues of transfer matrices for the spin-1/2 and spin-1 XXX chains coupled with arbitrary spin impurities can be obtained by rescaling the spectral parameters \( v_j \) by \( v_j \times \eta \) and taking the limit \( \eta \to 0 \) in Bethe ansatz equations and the eigenvalues of the transfer matrices. With similar methods of Ref.\([31, 35, 36]\), the results of the present paper can also be used to calculate the boundary susceptibility, the contribution of the impurities to the specific heat and Kondo temperature, which can describe the effect of impurities to the system. We will study them in another paper \([37]\).

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