NONPARAMETRIC ESTIMATION OF THE DERIVATIVE OF THE REGRESSION FUNCTION: APPLICATION TO SEA SHORES WATER QUALITY

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Abstract. This paper is devoted to the nonparametric estimation of the derivative of the regression function in a nonparametric regression model. We implement a very efficient and easy to handle statistical procedure based on the derivative of the recursive Nadaraya-Watson estimator. We establish the almost sure convergence as well as the asymptotic normality for our estimates. We also illustrate our nonparametric estimation procedure on simulated and real life data associated with sea shores water quality and valvometry.

1. Introduction

Environmental and water protection should be tackled as a top priority of our society. It is forecasted that in 2035, nearly 60% of the world’s population will live within 65 miles of the sea front (Haslett (2001)). Water quality monitoring is therefore fundamental especially on the coastline. On the one hand, marine pollution comes mostly from land based sources. On the other hand, this pollution can lead to the collapse of coastal ecosystems and cause public health issues. In this context, there is a critical need to develop a real-time reliable field assay to monitor the water quality within a decision making process. Among them, bioindicators are more and more commonly used. Endemic species are the most suitable bioindicators for the assessment of the quality of the coastal environment. For example, oysters, a well-known filter-feeding mollusc, feature a relevant sentinel organism to evaluate water quality. These animals being sedentary, they can witness the water quality evolution in a specific location.

The interest in investigating the bivalves activities by recording the valve movements has been explored for water quality surveillance. This area of interest is known as valvometry. The basic idea of valvometry is to use the bivalves ability to close its shell when exposed to a contaminant as an alarm signal (e.g. Doherty et al. 1987; Nagai et al. 2006; Sow et al. 2011). Thus, recording the shell gaping activity of oysters is an effective method to study their behavior when facing water pollution (e.g. Riisgard et al. 2006; Garcia-March et al. 2008). Nowadays, valvometric techniques produce high-frequency data, enabling online and in situ studies of the behavior of bivalve molluscs. They allow autonomous long-term recordings of valve movements without interfering their normal behavior. The goal of this paper
is to propose a nonparametric statistical procedure based on the estimation of the derivative of the regression function in order to evaluate the velocity of the valve opening/closing activity.

A wide range of literature is available on nonparametric estimation of a regression function. We refer the reader to Nadaraya 1989, Tsybakov 2009 and Devroye & Lugosi 2012 for some excellent books on density and regression function estimation. Here, we shall focus our attention on the Nadaraya-Watson estimator of the regression function (Nadaraya 1964 and Watson 1964). The almost sure convergence of this estimator was established by Noda 1976, while its asymptotic normality was proven by Schuster 1972. Later, Choi et al. 2000 proposed three data-sharpening versions of the Nadaraya-Watson estimator in order to reduce the asymptotic variance in the central limit theorem.

In this paper, we investigate an alternative approach, based on three recursive versions of the Nadaraya-Watson estimator (see Ahmad & Lin 1976; Bercu et al. 2012; Devroye & Wagner 1980; Johnston 1982; Wand & Jones 1995; Duflo 1997). These recursive versions allows us to update the estimate with new collected information during the monitoring process. Consequently, it is possible to avoid the need to recompute a new final estimate from the whole data set. To the best of our knowledge, no references are available on the derivative of the recursive Nadaraya-Watson estimator. Our first goal is to study the asymptotic behavior of the derivative of those three estimators. Our second goal is to illustrate our nonparametric estimation procedure on high-frequency valvometry data, in order to detect irregularities or abnormal behaviors of bivalves.

The paper is organized as follows. Section 2 deals with our nonparametric estimation procedure of the derivative of the regression function. We establish in Section 3 the pointwise almost sure convergence as well as the asymptotic normality of our estimators and we compare their asymptotic variances. Section 4 is devoted to simulation results to study the performance of our recursive procedure. Section 5 presents an application for the survey of aquatic system using high-frequency valvometry. All the proofs of the nonparametric theoretical results are postponed to Appendices A and B.

## 2. Nonparametric estimation of the derivative

The relationship between the distance of two electrodes \( Y_n \) and the time of the measurement \( X_n \) can be seen as a nonparametric regression model given, for all \( n \geq 1 \), by

\[
Y_n = f(X_n) + \varepsilon_n
\]

(2.1)

where \( \varepsilon_n \) are unknown random errors. In all the sequel, we assume that \( X_n \) is a sequence of independent and identically distributed random variables with positive probability density function \( g \). Our purpose is to estimate the derivative of the unknown regression function \( f \) which is directly associated with the velocity of the valve opening/closing activities of the oysters. For example, in an inhospitable
environment, oysters behavior will be altered. Consequently, detecting changes of the closing and opening speed can provide insights about the health of oysters and so can be used as bioindicators of the water quality.

We recall that the Nadaraya-Watson estimator of the link function \( f \) is defined as

\[
\hat{f}_{n}^{NW}(x) = \frac{\sum_{k=1}^{n} Y_k K \left( \frac{x - X_k}{h_n} \right)}{\sum_{k=1}^{n} K \left( \frac{x - X_k}{h_n} \right)},
\]

where the kernel \( K \) is a chosen probability density function and the bandwidth \((h_n)\) is a sequence of positive real numbers decreasing to zero. In our situation, we focus our attention on the recursive version of the Nadaraya-Watson estimator (Duflo 1997) of \( f \) given, for any \( x \in \mathbb{R} \), by

\[
\hat{f}_{n}(x) = \frac{\sum_{k=1}^{n} Y_k K \left( \frac{x - X_k}{h_k} \right)}{\sum_{k=1}^{n} \frac{1}{h_k} K \left( \frac{x - X_k}{h_k} \right)}.
\]

The denominator should, of course, be taken positive. It coincides with the recursive version of the Parzen-Rosenblatt estimator (Parzen 1962; Rosenblatt 1956) of the probability density function \( g \). For any \( x \in \mathbb{R} \), denote

\[
\hat{h}_{n}(x) = \frac{1}{n} \sum_{k=1}^{n} \frac{Y_k}{h_k} K \left( \frac{x - X_k}{h_k} \right) \quad \text{and} \quad \hat{g}_{n}(x) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{h_k} K \left( \frac{x - X_k}{h_k} \right),
\]

which can be recursively calculated as

\[
\hat{h}_{n}(x) = \frac{n-1}{n} \hat{h}_{n-1}(x) + \frac{Y_n}{nh_n} K \left( \frac{x - X_n}{h_n} \right)
\]

and

\[
\hat{g}_{n}(x) = \frac{n-1}{n} \hat{g}_{n-1}(x) + \frac{1}{nh_n} K \left( \frac{x - X_n}{h_n} \right).
\]

This modification allows dynamic updating of the estimates.

In the special case where \( g \) is known, a simplified version of the Nadaraya-Watson estimator of \( f \), introduced by Johnston 1982, is given by

\[
\tilde{f}_{n}(x) = \frac{\hat{h}_{n}(x)}{g(x)}.
\]

In the same vein, an alternative estimator of \( f \) when \( g \) is known, was proposed by Wand & Jones 1995. It is defined, for any \( x \in \mathbb{R} \), by

\[
\tilde{f}_{n}(x) = \frac{1}{n} \sum_{k=1}^{n} \frac{Y_k}{g(X_k)h_k} K \left( \frac{x - X_k}{h_k} \right).
\]
The derivatives of \( \tilde{f}_n(x) \), \( \tilde{f}_n(x) \), and \( \tilde{f}_n(x) \) are given, for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \), by

\[
\tilde{f}'_n(x) = \frac{\widehat{h}'_n(x)}{\widehat{g}(x)} - \frac{\widehat{h}_n(x)\widehat{g}'(x)}{\widehat{g}^2(x)},
\]

\[
\tilde{f}''_n(x) = \frac{\widehat{h}'_n(x)}{g(x)} - \frac{\widehat{h}_n(x)g'(x)}{g^2(x)},
\]

\[
\tilde{f}''_n(x) = \frac{1}{n} \sum_{k=1}^{n} \frac{Y_k}{g(X_k)h_k^2} K' \left( \frac{x - X_k}{h_k} \right).
\]

### 3. Theoretical results

In order to investigate the asymptotic behavior of these derivative estimates, it is necessary to introduce several classical assumptions.

\( (A_1) \) The kernel \( K \) is a positive symmetric bounded function, differentiable with bounded derivative, satisfying

\[
\int_{\mathbb{R}} K(x) dx = 1, \quad \int_{\mathbb{R}} K'(x) dx = 0, \quad \int_{\mathbb{R}} xK'(x) dx = -1, \quad \int_{\mathbb{R}} x^2 K''(x) dx = 0,
\]

\[
\int_{\mathbb{R}} x^4 K(x) dx < \infty, \quad \int_{\mathbb{R}} x^4 |K'(x)| dx < \infty.
\]

\( (A_2) \) The regression function \( f \) and the density function \( g \) are bounded continuous, twice differentiable with bounded derivatives.

\( (A_3) \) The noise sequence \( (\varepsilon_n) \) and the observation times \( (X_n) \) are independent.

Moreover, \( (\varepsilon_n) \) is a sequence of independent, squared integrable, identically distributed random variables such that \( \mathbb{E}[\varepsilon_n] = 0 \) and \( \mathbb{E}[\varepsilon_n^2] = \sigma^2 \).

Furthermore, the bandwidth \( (h_n) \) is a sequence of positive real numbers, decreasing to zero, such that \( nh_n \) tends to infinity. For the sake of simplicity, we shall make use of \( h_n = 1/n^\alpha \) with \( 0 < \alpha < 1 \).

Our first result on the almost sure convergence of our estimates is as follows.

**Theorem 3.1.** Assume that \( (A_1), (A_2) \) and \( (A_3) \) hold. Then, if \( 0 < \alpha < 1/3 \), we have for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \),

\[
\lim_{n \to \infty} \tilde{f}'_n(x) = f'(x) \quad \text{a.s.}
\]

\[
\lim_{n \to \infty} \tilde{f}''_n(x) = f''(x) \quad \text{a.s.}
\]

\[
\lim_{n \to \infty} \tilde{f}'''_n(x) = f'''(x) \quad \text{a.s.}
\]

**Proof.** The proof is given in Appendix A. \( \square \)
Our second result is devoted to the asymptotic normality of our estimates. Denote
\begin{equation}
(3.4) \quad \xi^2 = \int_{\mathbb{R}} (K'(x))^2 \, dx.
\end{equation}

**Theorem 3.2.** Assume that \((A_1)\), \((A_2)\) and \((A_3)\) hold and that the sequence \((\varepsilon_n)\) has a finite moment of order \(\alpha > 2\). Then, as soon as \(1/5 < \alpha < 1/3\), we have for any \(x \in \mathbb{R}\) such that \(g(x) > 0\), the pointwise asymptotic normality
\begin{equation}
(3.5) \quad \sqrt{n h_n^3} (\hat{f}_n'(x) - f'(x)) \xrightarrow{D} \mathcal{N}
\left(0, \frac{\xi^2}{(1 + 3\alpha) g(x)} \sigma^2 \right),
\end{equation}
\begin{equation}
(3.6) \quad \sqrt{n h_n^3} (\tilde{f}_n'(x) - f'(x)) \xrightarrow{D} \mathcal{N}
\left(0, \frac{\xi^2}{(1 + 3\alpha) g(x)} (f^2(x) + \sigma^2) \right),
\end{equation}
\begin{equation}
(3.7) \quad \sqrt{n h_n^3} (\tilde{f}_n'(x) - f'(x)) \xrightarrow{D} \mathcal{N}
\left(0, \frac{\xi^2}{(1 + 3\alpha) g(x)} (f^2(x) + \sigma^2) \right).
\end{equation}

**Proof.** The proof is given in Appendix B. \(\square\)

**Remark 3.1.** One can realize that the derivative of the Nadaraya-Watson estimator \(\hat{f}_n'(x)\) is more efficient than \(\tilde{f}_n'(x)\) and \(\tilde{f}_n'(x)\) as its asymptotic variance is the smallest one. The more \(f(x)\) is far away from 0, the more one should make use of \(\hat{f}_n'(x)\).

### 4. Simulated data

This section is devoted to numerical experiments in order to evaluate the performances of our derivative estimates. The data are generated from the nonparametric regression model
\begin{equation}
(4.1) \quad Y_n = f(X_n) + \varepsilon_n,
\end{equation}
where the regression function \(f\) is defined, for all \(x \in [0, 1]\), by
\begin{equation}
(4.2) \quad f(x) = \sin(2\pi x^3)^3.
\end{equation}
The random observation \((X_n)\) is a sequence of independent random variables uniformly distributed over the interval \([0, 1]\), and the source of variation \((\varepsilon_n)\) is a sequence of independent and identically random variables sharing the same \(\mathcal{N}(0, 1)\) distribution. We implement our statistical procedure with sample size \(n = 10000\).

The simulated data associated with (4.1) are given in Figure 1.

We first illustrate the pointwise almost sure convergence of the estimator \(\hat{f}_n'(x)\) to \(f'\) for the Gaussian and Epanechnikov kernels, respectively given by
\begin{equation}
K(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad \text{and} \quad K(x) = \frac{3}{4} (1 - x^2) I_{|x| \leq 1}.
\end{equation}
The first kernel is supported on the whole real line, while the second one has a compact support. The derivative \(f'\) is given, for all \(x \in [0, 1]\), by
\begin{equation}
(4.3) \quad f'(x) = 18\pi x^2 \cos(2\pi x^3) \sin(2\pi x^3)^2.
\end{equation}
Figure 1. Simulated data $(X_n, Y_n)$ with the regression function $f$ (solid line) given in (4.2) with sample size $n = 10000$.

It is well-known that in practice, the choice of the kernel is not really significant, compared to the crucial choice of the bandwidth $h_n = 1/n^\alpha$. Figure 2 shows that one should take the value of $\alpha$ close to 0.3. In order to select an automatic choice of $\alpha$, we use the cross validation method by taking the value $\alpha$ that minimizes the cross validation function

$$CV(\alpha) = \frac{1}{n} \sum_{k=1}^{n} \left( \hat{f}'_{(-k)}(X_k, \alpha) - f'(X_k) \right)^2$$

where $\hat{f}'_{(-k)}(X_k, \alpha)$ is the estimator of $f'(X_k)$ defined by (2.9) with the couple $(X_k, Y_k)$ removed. Figure 3 displays the cross validation function for the Gaussian kernel. We observe between $\alpha = 0.3$ and $\alpha = 0.4$ a plateau leading to the value $\alpha = 0.32$ since $\alpha$ has to be smaller than $1/3$. By the same method, we also obtain $\alpha = 0.32$ for the Epanechnikov kernel.
Figure 3. Example of the cross validation function for the estimates \( \hat{f}_n'(x) \) of \( f'(x) \) with the Gaussian kernel.

Figure 4 illustrates the good approximation of \( \hat{f}_n'(x) \) to \( f'(x) \) for the two kernels. Hereafter, we recall from Theorem 3.2 that the asymptotic variance of our estimates depends on the integral \( \xi^2 \) defined in (3.4). Consequently, we select in our simulations the kernel with the smaller \( \xi^2 \) value. It is easy to compute the values of \( \xi^2 \) for the Gaussian and Epanechnikov kernels. They are respectively given by \( \xi^2 = 1/4\sqrt{\pi} \approx 0.1410 \) and \( \xi^2 = 3/2 \). One can also observe that the Gaussian kernel has the smallest \( \xi^2 \) value comparing to all commonly used kernels. Therefore, we shall use in all the sequel the Gaussian kernel.

After selecting \( \alpha \) by cross validation, Figure 5 shows that the three estimators \( \hat{f}_n'(x) \), \( \tilde{f}_n'(x) \), and \( \hat{f}_n''(x) \) approaches perfectly well the true derivative \( f'(x) \).

In order to illustrate the pointwise asymptotic normality of our estimates, we implement a simulation study based on \( N = 2000 \) realizations. We numerically check the asymptotic normality at points \( x = 0.4 \) and \( x = 0.9 \) for our three estimators.
One can see in Figure 6 that the distributions of our three estimators are normally distributed and centered around 0. We observe the effect of $f^2(x)$ in the asymptotic variance of $\hat{f}_n(x)$ and $\tilde{f}_n(x)$. Indeed, for $x = 0.4$ we have $f^2(x) = 0.0036$, while for $x = 0.9$ we have $f^2(x) = 0.9489$ which explains the differences between the asymptotic variances.

One can observe in Figure 7 the different behavior of the asymptotic variance $\hat{f}_n(x)$ in comparison with the two others estimators. Once again, a high variability coincides with a large value of $f^2(x)$.

Finally, our numerical experiments illustrate the good performances and also the robustness of our statistical procedure for heavy-tailed error distributions Durrieu & Briollais 2009 (data not shown). We also observed that the mean squared error of $\hat{f}_n$ is much more smaller than the mean squared error of the non-recursive version of the Nadaraya-Watson estimator. In term of asymptotic variance, it is clear that $\hat{f}_n$ performs better than $\tilde{f}_n$ and $\tilde{f}_n^\prime$. Consequently, we choose to make use of $\hat{f}_n$ to estimate the derivative $f'$ for our real life data experiments.

5. High-frequency valvometry data

The motivation of this paper is to monitor sea shores water quality. For that purpose, we study bivalves activities by recording the valve movements. We use a high frequency, noninvasive valvometry electronic system developed by the UMR CNRS 5805 EPOC laboratory in Arcachon (France). The electronic principle of valvometry is described by Tran et al. 2003, Chambon et al. 2007 and on the website http://molluscan-eye.epoc.u-bordeaux1.fr. This electronic system works autonomously without human intervention for a long period of time (at least one full year). Each animal is equipped with two light coils (sensors), of approximately 53mg each (unembedded), fixed on the edge of each valve. One of the coils emits a high-frequency, sinusoidal signal which is received by the other coil. The strength of the electric field produced between the two coils being proportional to the inverse of
distance between the point of measurement and the center of the transmitting coil, the distance between coils can be measured and the accuracy of the measurements is a few $\mu$m.

For each sixteen animals, one measurement is received every 0.1s (10 Hz). So, the activity of oyster is measured every 1.6s and each day, we obtain 864,000 triplets of data: the time of the measurement, the distance between the two valves and the
Figure 7. Boxplots of the three estimates $\hat{f}_n'(x)$, $\tilde{f}_n'(x)$ and $\check{f}_n'(x)$ of $f'(x)$ versus $x$. The solid line represents the underlying derivative function $f'(x)$. 
animal number. A first electronic card in a waterproof case next to the animals manages the electrodes and a second electronic card handles the data acquisition. The valvometry system uses a GSM/GPRS modem and Linux operating system for the data storage, the internet access, and the data transmission. After each 24h period or any other programmed period of time, the data are transmitted to a workstation server and then inserted in a SQL database which is accessible with the software R (R Development Core Team 2015) or a text terminal.

Several valvometric systems have been installed around the world: southern lagoon of New Caledonia, Spain, Ny Alesund Svalbard at 1300 km from the north pole, the north east of Murmansk in Russia on the Barents sea and at several sites in France with various species but we concentrate here on the Locmariaquer site situated in south Brittany based on sixteen oysters placed in a single bag. Locmariaquer (GPS coordinates 47°34 N, 2°56 W) is an important oyster farming area located near the narrow tidal pass which connects the gulf of Morbihan to the ocean, on the right side of the Auray river’s mouth. Thus, oysters are close to the seasonal high traffic of the navigation channel and are potentially exposed to pollution as chemical residues of intensive agricultural practices.

As argued in Ahmed et al. 2015, Durrieu et al. 2015 and Durrieu et al. 2016, pollution can affect the activity of oysters and in particular the shells opening and closing velocities and so the movement speeds can be considered as an indicator of the animal stress activity since its movements are associated to aquatic system perturbations. In Ahmed et al. 2015, the authors propose an interesting deterministic alternative method for the estimation of movement velocity based on differentiator estimators.

An example of valves activity and opening/closing velocity recordings June 2, 2011 is depicted in Figure 8. Figure 9 displays for the same day the plot of the estimate \( \hat{f}'_n \) of \( f' \) of the valve closing and opening velocity for one oyster at the Locmariaquer site. The bandwidth parameter was selected by the cross validation method described in the previous section.

To visualize the opening and closing velocity estimations of the 16 oysters from the 63th to the 243th days of 2011, we represent in Figure 10 for each oyster and each day the estimator \( \hat{f}'_n(x) \) of the closing and opening velocities \( f'(x) \) at time \( x \) over a period of time of 24 hours using a customized color table: the yellow color is associated to the class of the smallest velocities, the green color to the class of intermediate velocities and the red color to the class of the largest velocities. This graphical representation reveals distinct clusters of behaviors. We observe in the Figure 10 white lines from the 207th to the 210th days, corresponding to a power outage on the site due to a storm. Before the 100th days, the animals have a normal regular activity. The most red zone between the 100th days and the 125th days can be explained by a sudden change in temperature in the environment associated to the modifications of the specific activity of two enzymatic biomarkers (Glutathione-S-transferase and Acetylcholinesterase) meaning a possible pollution as described in Durrieu et al. 2016. We observed an intense activity of closing at the bottom of the
Figure 8. A typical example of valvometric data for one oyster the June 2, 2011. In the left hand side, relationship between the opening amplitude (in millimeters) and the time of the experiment (over 24 hours period). In the right hand side, the closing and opening velocity (millimeters per second) according to time (over the same period).

Figure 9. The dashed line displays for June 2, 2011, the estimated $f'(x)$ using estimator $f_n(x)$ versus the time $x$ and the solid lines represent the observed speeds of valve openings and closings. The closing and opening velocity are measured in millimeters per second.
Figure (days $\geq 210$) associated to a spawning activity.

Figure 10 shows also that the closing and opening velocities are the smallest (yellow zone) and highly correlated with the tidal amplitude. We have performed many other analyses of these data using extreme value theory and other nonparametric statistical methods, all of which point the same conclusion Coudret et al. 2015, Durrieu et al. 2015 and Durrieu et al. 2016. Altogether, we anticipate that this approach could have a significant contribution providing in situ instant diagnosis of the bivalves behavior and thus appears to be an effective, early warning tool in ecological risk assessment.

Figure 10. Representation of the opening and closing velocities estimation using $\hat{f}_n(x)$ from the 63th to the 243th days of 2011, considering the 16 oysters in Locmariaquer. The x-axis represents the time in a 24 hour time period and the y-axis represents the number of days since January 1, 2011.
APPENDIX A. PROOFS OF THE ALMOST SURE CONVERGENCE RESULTS.

The proofs of the almost sure convergence results rely on the following lemma. We also refer the reader to Silverman 1986 for the estimation of the derivative of the Parzen-Rosenblatt estimator.

**Lemma A.1.** Assume that \((A_1), (A_2)\) and \((A_3)\) hold. Then, the estimators \(\hat{g}_n\) and \(\hat{h}_n\), given by (2.4), satisfy for any \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} \hat{g}_n(x) = g(x) \quad \text{a.s.}
\]

\[
\lim_{n \to \infty} \hat{h}_n(x) = f(x)g(x) \quad \text{a.s.}
\]

Moreover, as soon as \(0 < \alpha < 1/3\), we also have for any \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} \hat{g}_n'(x) = g'(x) \quad \text{a.s.}
\]

\[
\lim_{n \to \infty} \hat{h}_n'(x) = (f(x)g(x))' \quad \text{a.s.}
\]

**Proof.** We shall only prove the almost sure convergence (A.4) inasmuch as (A.1) and (A.2) are well-known and the proof of (A.3) is more easy to handle and follow the same lines as the proof of (A.4). We deduce from (2.1) and (2.4) that for any \(x \in \mathbb{R}\),

\[
\hat{h}_n(x) = \frac{1}{n} \sum_{k=1}^{n} \frac{f(X_k)}{h_k} K \left( \frac{x - X_k}{h_k} \right) + \frac{1}{n} \sum_{k=1}^{n} \frac{\varepsilon_k}{h_k} K \left( \frac{x - X_k}{h_k} \right).
\]

Hence, by derivation, we have the decomposition

\[
nh_n'(x) = A_n(x) + B_n(x)
\]

where

\[
A_n(x) = \sum_{k=1}^{n} a_k(x) = \sum_{k=1}^{n} f(X_k) v_k(X_k, x),
\]

\[
B_n(x) = \sum_{k=1}^{n} b_k(x) = \sum_{k=1}^{n} \varepsilon_k v_k(X_k, x)
\]

with

\[
v_n(X_n, x) = \frac{1}{h_n^2} K' \left( \frac{x - X_n}{h_n} \right).
\]

On the one hand, we have for any \(x \in \mathbb{R}\),

\[
\mathbb{E}[a_n(x)] = \int_{\mathbb{R}} f(x_n) v_n(x_n, x) g(x_n) dx_n
\]

\[
= \frac{1}{h_n} \int_{\mathbb{R}} f(x - h_n y) g(x - h_n y) K'(y) dy.
\]
The regression function \( f \) as well as the density function \( g \) are bounded continuous and twice differentiable with bounded derivatives. Consequently, it follows from Taylor’s formula that it exist \( \theta_f, \theta_g \) in the interval \([0, 1]\) such that, for any \( x \in \mathbb{R} \),

\[
f(x - h_n y) = f(x) - h_n y f'(x) + \frac{h_n^2 y^2}{2} f''(x - h_n y \theta_f),
\]

and

\[
g(x - h_n y) = g(x) - h_n y g'(x) + \frac{h_n^2 y^2}{2} g''(x - h_n y \theta_g).
\]

By a careful analysis of each term in the product \( f(x - h_n y)g(x - h_n y) \), we deduce from (A.7) together with assumption (A_1) that

\[
\mathbb{E}[a_n(x)] = -(f(x)g(x))' \int_{\mathbb{R}} yK(y)dy + h_n f'(x)g'(x) \int_{\mathbb{R}} y^2 K'(y)dy + R_n(x)
\]

(A.8)

\[
= (f(x)g(x))' + R_n(x)
\]

where the remainder \( R_n(x) \) satisfies

\[
\sup_{x \in \mathbb{R}} |R_n(x)| = O(h_n).
\]

Consequently, (A.8) immediately leads to

(A.9)

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[A_n(x)] = (f(x)g(x))',
\]

which is the limit we are looking for. By the same token,

\[
\mathbb{E}[a_n^2(x)] = \int_{\mathbb{R}} f^2(x_n)v_n^2(x_n, x)g(x_n)dx_n
\]

\[
= \frac{1}{h_n^3} \int_{\mathbb{R}} f^2(x - h_n y)g(x - h_n y)(K'(y))^2 dy
\]

(A.10)

\[
= \frac{1}{h_n^3} \xi^2 f^2(x)g(x) + \zeta_n(x),
\]

where \( \xi^2 \) is defined in (3.4) and the remainder \( \zeta_n(x) \) is such that

\[
\sup_{x \in \mathbb{R}} |\zeta_n(x)| = O\left(\frac{1}{h_n^2}\right).
\]

Therefore, we deduce from (A.8) and (A.10) that

(A.11)

\[
\lim_{n \to \infty} \frac{1}{n^{1+3\alpha}} \text{Var}(A_n(x)) = \frac{\xi^2 f^2(x)g(x)}{1 + 3\alpha}.
\]

On the other hand, denote by \( \mathcal{F}_n \) the \( \sigma \)-algebra of the events occurring up to time \( n \), \( \mathcal{F}_n = \sigma(X_1, \varepsilon_1, \ldots, X_n, \varepsilon_n) \). Since \( (X_n) \) and \( (\varepsilon_n) \) are two independent sequences of independent and identically distributed random variables, we have for any \( x \in \mathbb{R} \),

\[
\mathbb{E}[b_n(x)|\mathcal{F}_{n-1}] = \mathbb{E}[\varepsilon_nv_n(X_n, x)|\mathcal{F}_{n-1}] = \mathbb{E}[\varepsilon_nv_n(X_n, x)] = 0.
\]

Moreover,

\[
\mathbb{E}[b_n^2(x)|\mathcal{F}_{n-1}] = \mathbb{E}[\varepsilon_n^2v_n^2(X_n, x)|\mathcal{F}_{n-1}] = \mathbb{E}[\varepsilon_n^2v_n^2(X_n, x)] = \sigma^2 \mathbb{E}[v_n^2(X_n, x)].
\]
Furthermore, we have

\[ E[v_n^2(X_n, x)] = \int_{\mathbb{R}} v_n^2(x_n, x) g(x_n) dx_n = \frac{1}{h_n^3} \int_{\mathbb{R}} g(x - h_n y) (K'(y))^2 dy \]

\[ = \frac{1}{h_n^3} \int_{\mathbb{R}} \left( g(x) - h_n y g'(x) + \frac{h_n^2 y^2}{2} g''(x) (h_n y)^2 \right) (K'(y))^2 dy \]

\[ = \frac{1}{h_n^3} \xi^2 g(x) + \Delta_n(x) \]

(A.12)

where \( \xi^2 \) is defined in (3.4) and the remainder \( \Delta_n(x) \) is such that

\[ \sup_{x \in \mathbb{R}} |\Delta_n(x)| = O\left( \frac{1}{h_n^2} \right). \]

Consequently, denoting

\[ W_n(x) = \sum_{k=1}^n v_k^2(X_k, x), \]

it follows from (A.12) that

(A.13) \[ \lim_{n \to \infty} \frac{1}{n^{1+3\alpha}} E[W_n(x)] = \frac{\xi^2 g(x)}{1 + 3\alpha}. \]

We are now in the position to prove the almost sure convergence (A.4). The decomposition (A.5) can be rewritten as

(A.14) \[ n\hat{h}'_n(x) = M_n^A(x) + \mathbb{E}[A_n(x)] + B_n(x), \]

where \( M_n^A(x) = A_n(x) - \mathbb{E}[A_n(x)] \). One can observe that \( M_n^A(x) \) and \( B_n(x) \) are both square integrable martingale difference sequences with predictable quadratic variations respectively given by \( \langle M^A(x) \rangle_n = \text{Var}(A_n(x)) \) and \( \langle B(x) \rangle_n = \sigma^2 \mathbb{E}[W_n(x)] \).

Consequently, (A.11) together with (A.13) immediately lead to

(A.15) \[ \lim_{n \to \infty} \frac{\langle M^A(x) \rangle_n}{n^{1+3\alpha}} = \frac{\xi^2 f^2(x) g(x)}{1 + 3\alpha} \quad \text{and} \quad \lim_{n \to \infty} \frac{\langle B(x) \rangle_n}{n^{1+3\alpha}} = \frac{\sigma^2 \xi^2 g(x)}{1 + 3\alpha}. \]

Hence, we obtain from the strong law of large numbers for martingales given e.g. by Theorem 1.3.15 of Duflo (1997) that, for any \( \gamma > 0 \), \( (M_n^A(x))^2 = o(n^{1+3\alpha}(\log n)^{1+\gamma}) \) a.s. and \( (B_n(x))^2 = o(n^{1+3\alpha}(\log n)^{1+\gamma}) \) a.s. Therefore, as \( 0 < \alpha < 1/3 \), it ensures that, for any \( x \in \mathbb{R} \)

(A.16) \[ \lim_{n \to \infty} \frac{1}{n} M_n^A(x) = 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} B_n(x) = 0 \quad \text{a.s.} \]

Finally, we deduce from decomposition (A.14) together with (A.9) and (A.16) that for any \( x \in \mathbb{R} \),

\[ \lim_{n \to \infty} \hat{h}'_n(x) = (f(x)g(x))' \quad \text{a.s.} \]

Thus Lemma A.1 is proven. \( \square \)
Proof of Theorem 3.1. We shall now proceed to the proof of the Theorem 3.1. It clearly follows from relation (2.9) and Lemma A.1 that for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \),
\[
\lim_{n \to +\infty} \frac{\hat{f}'_n(x)}{\hat{g}'_n(x)} = \lim_{n \to +\infty} \frac{(\hat{h}'_n(x) - \hat{h}'_n(x)\hat{g}'_n(x))}{\hat{g}'_n(x)} = \frac{(f(x)g(x)g'(x)}{g(x)} - \frac{f(x)g(x)g'(x)}{g^2(x)} \quad \text{a.s.}
\]
\[
= \frac{f'(x)g(x) + f(x)g'(x) - f(x)g'(x)}{g(x)} = f'(x) \quad \text{a.s.}
\]
By the same token, relation (2.10) and Lemma A.1 immediately lead to
\[
\lim_{n \to +\infty} \frac{\hat{f}'_n(x)}{\hat{g}'_n(x)} = \lim_{n \to +\infty} \frac{(\hat{h}'_n(x) - \hat{h}'_n(x)\hat{g}'_n(x))}{\hat{g}'_n(x)} = f'(x) \quad \text{a.s.}
\]
It only remains to prove (3.3). We obtain from relation (2.11) that
\[
(A.17) \quad n\hat{f}'_n(x) = C_n(x) + D_n(x)
\]
where
\[
C_n(x) = \sum_{k=1}^{n} c_k(x) = \sum_{k=1}^{n} \frac{f(X_k)}{g(X_k)} v_k(X_k, x),
\]
\[
D_n(x) = \sum_{k=1}^{n} d_k(x) = \sum_{k=1}^{n} \frac{\varepsilon_k}{g(X_k)} v_k(X_k, x).
\]
As in the proof of Lemma A.1, we find that for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \),
\[
(A.18) \quad \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[C_n(x)] = f'(x) \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n^{1+3\alpha}} \text{Var}(C_n(x)) = \frac{\xi^2 f^2(x)}{(1 + 3\alpha)g(x)}.
\]
Hereafter, we split \( n\hat{f}'_n(x) \) into three terms
\[
(A.19) \quad n\hat{f}'_n(x) = M_n^C(x) + \mathbb{E}[C_n(x)] + D_n(x)
\]
where \( M_n^C(x) = C_n(x) - \mathbb{E}[C_n(x)] \). One can observe that \( (M_n^C(x)) \) and \( (D_n(x)) \) are both square integrable martingale difference sequences with predictable quadratic variations satisfying, for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \),
\[
(A.20) \quad \lim_{n \to \infty} \frac{\langle M_n^C(x) \rangle_n}{n^{1+3\alpha}} = \frac{\xi^2 f^2(x)}{(1 + 3\alpha)g(x)} \quad \text{and} \quad \lim_{n \to \infty} \frac{\langle D(x) \rangle_n}{n^{1+3\alpha}} = \frac{\xi^2 \sigma^2}{(1 + 3\alpha)g(x)}.
\]
Therefore, we deduce from the strong law of large numbers for martingales that, as soon as \( 0 < \alpha < 1/3 \),
\[
(A.21) \quad \lim_{n \to \infty} \frac{1}{n} M_n^C(x) = 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} D_n(x) = 0 \quad \text{a.s.}
\]
Finally, it follows from (A.19) together with (A.18) and (A.21) that for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \),
\[
\lim_{n \to \infty} \frac{\hat{f}'_n(x)}{\hat{g}'_n(x)} = f'(x) \quad \text{a.s.}
\]
which achieves the proof of Theorem 3.1. \( \square \)
Appendix B. Proofs of the asymptotic normality results.

In order to prove Theorem 3.2, we shall make use of the central limit theorem for martingales given e.g. by Theorem 2.1.9 of Duflo (1997). First of all, we focus our attention on convergence (3.7) since it is the easiest convergence to prove.

Proof of convergence (3.7). It follows from (A.19) that
\[
\sqrt{nh_n^3}(\tilde{f}_n(x) - f'(x)) = \frac{\sqrt{nh_n^3}}{n} \left( M_n(x) + \mathbb{E}[C_n(x)] + D_n(x) - nf'(x) \right),
\]
which implies the martingale decomposition
\[
\sqrt{nh_n^3}(\tilde{f}_n(x) - f'(x)) = \frac{1}{\sqrt{n^{1+3\alpha}}}(\langle e, M_n(x) \rangle + \tilde{R}_n(x))
\]
where
\[
e = \left( \frac{1}{1} \right), \quad M_n(x) = \left( \frac{M_n^C(x)}{D_n(x)} \right),
\]
and the remainder
\[
\tilde{R}_n(x) = \mathbb{E}[C_n(x)] - nf'(x) = \sum_{k=1}^{n} (\mathbb{E}[c_k(x)] - f'(x)).
\]
It follows from Taylor’s formula that it exists \(\theta_f \in ]0, 1[\) such that, for any \(x \in \mathbb{R}\),
\[
\mathbb{E}[c_n(x)] = \int_{\mathbb{R}} f(x)v_n(x_n, x)dx_n = \frac{1}{h_n} \int_{\mathbb{R}} f(x - h_ny)K'(y)dy
\]
\[
= f'(x) + \frac{h_n}{2} \int_{\mathbb{R}} f''(x - h_ny\theta_f)y^2K'(y)dy,
\]
where \(v_n\) is defined in (A.6). Since \(f''\) is bounded, we have
\[
\sup_{x \in \mathbb{R}} |\mathbb{E}[c_n(x)] - f'(x)| \leq M_f \tau^2 h_n
\]
where
\[
M_f = \sup_{x \in \mathbb{R}} |f''(x)| \quad \text{and} \quad \tau^2 = \frac{1}{2} \int_{\mathbb{R}} y^2 |K'(y)|dy.
\]
Hence, we deduce from (B.2) and (B.3) that
\[
\sup_{x \in \mathbb{R}} |\tilde{R}_n(x)| \leq \tau^2 M_f \sum_{k=1}^{n} h_k.
\]
However, it is easily seen that
\[
\sum_{k=1}^{n} h_k \leq \frac{1}{1 - \alpha} n^{1-\alpha}.
\]
Therefore, as soon as \(\alpha > 1/5\), we obtain that
\[
\sup_{x \in \mathbb{R}} |\tilde{R}_n(x)| = o(\sqrt{n^{1+3\alpha}}).
\]
Hereafter, the predictable quadratic variation (Duflo (1997)) of the two-dimensional real martingale \((\mathcal{M}_n(x))\) is given, for all \(n \geq 1\), by the diagonal matrix

\[
\langle \mathcal{M}(x) \rangle_n = \begin{pmatrix} \langle MC(x) \rangle_n & 0 \\ 0 & \langle D(x) \rangle_n \end{pmatrix}.
\]

Then, it follows from (A.20) that for any \(x \in \mathbb{R}\) such that \(g(x) > 0\),

\[
\lim_{n \to \infty} \frac{1}{n^{1+3\alpha}} \langle \mathcal{M}(x) \rangle_n = \frac{\xi^2}{(1 + 3\alpha)g(x)} \begin{pmatrix} f^2(x) & 0 \\ 0 & \sigma^2 \end{pmatrix}.
\]

Furthermore, it is not hard to see that the martingale \((\mathcal{M}_n(x))\) satisfies the Lindeberg condition. As a matter of fact, we have assumed that the sequence \((\varepsilon_n)\) has a finite moment of order \(p > 2\). Let \(a > 0\) be such that \(p = 2(1 + a)\). If we denote \(\Delta \mathcal{M}_n(x) = \mathcal{M}_n(x) - \mathcal{M}_{n-1}(x)\), we have for all \(n \geq 1\),

\[
E[\|\Delta \mathcal{M}_n(x)\|^p|\mathcal{F}_{n-1}] = E\left[\left(\left(\Delta M_n^C(x)\right)^2 + (\Delta D_n(x))^2\right)^{1+a}|\mathcal{F}_{n-1}\right]
\leq 2^a E\left[|\Delta M_n^C(x)|^p + |\Delta D_n(x)|^p\right]|\mathcal{F}_{n-1}].
\]

On the one hand,

\[
E[|\Delta M_n^C(x)|^p|\mathcal{F}_{n-1}] = E[|c_n(x) - E[c_n(x)]|^p|\mathcal{F}_{n-1}]
\leq 2^{p-1}(E[|c_n(x)|^p] + E[|c_n(x)|^p]).
\]

However, as \(f'\) is bounded, it follows from (B.3) that

\[
\sup_{x \in \mathbb{R}}|E[c_n(x)]| \leq m_f + M_f \tau^2
\]

where \(m_f = \sup_{x \in \mathbb{R}}|f'(x)|\). Consequently, it exists a positive constant \(C_p\) such that

\[
\sup_{x \in \mathbb{R}}|E[c_n(x)]|^p \leq C_p.
\]

Moreover,

\[
E[|c_n(x)|^p] = \int_{\mathbb{R}} \frac{f(x_n)^p}{g(x_n)^{p-1}}|v_n(x_n, x)|^pdx_n
= \frac{1}{h_n^{2p-1}} \int_{\mathbb{R}} \frac{f(x - h_n y)^p}{g(x - h_n y)^{p-1}}|K'(y)|^pdy.
\]

Hence, for any \(x \in \mathbb{R}\) such that \(g(x) > 0\), it exist a positive constant \(c_p\) such that

\[
E[|c_n(x)|^p] \leq \frac{c_p}{h_n^{2p-1}}.
\]

Therefore, we deduce from (B.7) together with (B.8) and (B.9) that for any \(x \in \mathbb{R}\) such that \(g(x) > 0\),

\[
E[|\Delta M_n^C(x)|^p|\mathcal{F}_{n-1}] \leq 2^{p-1}\left(\frac{c_p}{h_n^{2p-1}} + C_p\right).
\]

On the other hand, we have

\[
E[|\Delta D_n(x)|^p|\mathcal{F}_{n-1}] = E[|d_n(x)|^p|\mathcal{F}_{n-1}] = E[|\varepsilon_n|^p|w_n(x_n, x)]^p|\mathcal{F}_{n-1}]
\]
where
\[ w_n(X_n, x) = \frac{v_n(X_n, x)}{g(X_n)} = \frac{1}{h_n^2 g(X_n)} K' \left( \frac{x - X_n}{h_n} \right). \]

We infer from assumption \((A_3)\) that
\[(B.11) \quad \mathbb{E}[|\Delta D_n(x)|^p | \mathcal{F}_{n-1}] = \mathbb{E}[|\varepsilon_n|^p] \mathbb{E}[|w_n(X_n, x)|^p].\]

However, the sequence \((\varepsilon_n)\) has a finite moment of order \(p > 2\) means that it exists a positive constant \(E_p\) such that \(E_p = \mathbb{E}[|\varepsilon_n|^p]\). Moreover, as in the proof of \((B.9)\), we obtain that for any \(x \in \mathbb{R}\) such that \(g(x) > 0\), it exist a positive constant \(w_p\) such that
\[(B.12) \quad \mathbb{E}[|w_n(X_n, x)|^p] \leq \frac{w_p}{h_n^{2p-1}}.\]

Hence, it follows from \((B.11)\) and \((B.12)\) that for any \(x \in \mathbb{R}\) such that \(g(x) > 0\),
\[(B.13) \quad \mathbb{E}[|\Delta D_n(x)|^p | \mathcal{F}_{n-1}] \leq \frac{E_p w_p}{h_n^{2p-1}}.\]

Consequently, we deduce from \((B.6)\) together with \((B.10)\) and \((B.13)\) that for any \(x \in \mathbb{R}\) such that \(g(x) > 0\), one can find a positive constant \(M_p\) such that, for all \(n \geq 1\),
\[(B.14) \quad \mathbb{E}[\|\Delta \mathcal{M}_n(x)\|^p | \mathcal{F}_{n-1}] \leq \frac{M_p}{h_n^{2p-1}} \quad \text{a.s.}\]

We recall that \(p = 2(1 + \alpha)\). For any \(\varepsilon > 0\), if \(\mathcal{A}_k(x, \varepsilon, n) = \{\|\Delta \mathcal{M}_k(x)\| \geq \varepsilon \sqrt{n^{1 + 3\alpha}}\}\), we have from \((B.14)\),
\[\frac{1}{n^{1 + 3\alpha}} \sum_{k=1}^{n} \mathbb{E}[\|\Delta \mathcal{M}_k(x)\|^2 I_{\mathcal{A}_k(x, \varepsilon, n)} | \mathcal{F}_{k-1}] \leq \frac{1}{\varepsilon^{p-2} n^b} \sum_{k=1}^{n} \mathbb{E}[\|\Delta \mathcal{M}_k(x)\|^p | \mathcal{F}_{k-1}] \]
\[\leq \frac{M_p}{\varepsilon^{p-2} n^b} \sum_{k=1}^{n} \frac{1}{h_k^{2p-1}} \quad \text{a.s.}\]
\[\leq \frac{M_p n^c}{\varepsilon^{p-2}} \quad \text{a.s.}\]

where \(b = (a + 1)(1 + 3\alpha)\) and \(c = a(\alpha - 1)\). Since \(c < 0\), the Lindeberg condition is clearly satisfied. Finally, we can conclude from the central limit theorem for martingales (Duflo (1997)) that for any \(x \in \mathbb{R}\) such that \(g(x) > 0\),
\[(B.15) \quad \frac{1}{\sqrt{n^{1 + 3\alpha}}} \mathcal{M}_n(x) \xrightarrow{D} \mathcal{N}(0, \Gamma(x)),\]

where
\[\Gamma(x) = \frac{\xi^2}{(1 + 3\alpha)g(x)} \begin{pmatrix} f^2(x) & 0 \\ 0 & \sigma^2 \end{pmatrix}.\]

Hence, \((B.1)\) together with \((B.4)\) and \((B.15)\) immediately leads to
\[\sqrt{n h_n^2 (\tilde{f}_n(x) - f'(x))} \xrightarrow{D} \mathcal{N}(0, \frac{\xi^2}{1 + 3\alpha} \frac{f^2(x) + \sigma^2}{g(x)}).\]
Proof of convergence (3.6). It follows from (2.4) that

(B.16) \[ n\hat{h}_n(x) = P_n(x) + Q_n(x) = M_n^P(x) + E[P_n(x)] + Q_n(x) \]

where \( M_n^P(x) = P_n(x) - E[P_n(x)] \),

\[ P_n(x) = \sum_{k=1}^{n} p_k(x) = \sum_{k=1}^{n} f(X_k)u_k(X_k, x), \]

\[ Q_n(x) = \sum_{k=1}^{n} q_k(x) = \sum_{k=1}^{n} \varepsilon_k u_k(X_k, x) \]

with

\[ u_n(X_n, x) = \frac{1}{h_n} K \left( \frac{x - X_n}{h_n} \right). \]

Hence, for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \), we obtain from (2.10), (A.14) and (B.16)

\[ n(\hat{f}_n(x) - f'(x)) = \frac{1}{g(x)} \left( M_n^A(x) + E[A_n(x)] + B_n(x) \right) \]

\[ - \frac{g'(x)}{g^2(x)} \left( M_n^P(x) + E[P_n(x)] + Q_n(x) \right) - nf'(x) \]

which leads to the martingale decomposition

(B.17) \[ \sqrt{n} h_n^3 (\hat{f}_n(x) - f'(x)) = \frac{1}{\sqrt{n^{1+3\alpha}}} \left( \langle \hat{c}(x), M_n(x) \rangle + \tilde{R}_n(x) \right) \]

where\n
\[ \hat{c}(x) = \frac{1}{g^2(x)} \begin{pmatrix} g(x) \\ g(x) \\ -g'(x) \\ -g'(x) \end{pmatrix}, \quad M_n(x) = \begin{pmatrix} M_n^A(x) \\ B_n(x) \\ M_n^P(x) \\ Q_n(x) \end{pmatrix}, \]

and the remainder

\[ \tilde{R}_n(x) = \frac{1}{g(x)} E[A_n(x)] - \frac{g'(x)}{g^2(x)} E[P_n(x)] - nf'(x). \]

We saw in (A.8) that \( E[a_n(x)] = (f(x)g(x))' + R_n(x) \) where \( \sup_{x \in \mathbb{R}} |R_n(x)| = O(h_n) \). By the same token, \( E[P_n(x)] = f(x)g(x) + \zeta_n(x) \) where \( \sup_{x \in \mathbb{R}} |\zeta_n(x)| = O(h_n^2) \). Therefore, as soon as \( \alpha > 1/5 \), we obtain that

(B.18) \[ \sup_{x \in \mathbb{R}} |\tilde{R}_n(x)| = O \left( \sqrt{\sum_{k=1}^{n} h_k} \right) = o(\sqrt{n^{1+3\alpha}}). \]

Furthermore, as in the proof of (A.15) and (B.5), the predictable quadratic variation of the four-dimensional real martingale \((M_n(x))\) satisfies, for any \( x \in \mathbb{R} \),

(B.19) \[ \lim_{n \to \infty} \frac{1}{n^{1+3\alpha}} \langle \mathcal{M}(x) \rangle_n = \Gamma(x) \]
where $\Gamma(x)$ is the four-dimensional covariance matrix given by

$$\Gamma(x) = \frac{\xi^2 g(x)}{(1 + 3\alpha)} \begin{pmatrix} f^2(x) & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Moreover, via the same lines as in the proof of (B.14), we can also show that $(\mathcal{M}_n(x))$ satisfies the Lindeberg condition. Finally, we find from the central limit theorem for martingales (Duflo (1997)) that for any $x \in \mathbb{R}$,

$$\frac{1}{\sqrt{n^{1+3\alpha}}} \mathcal{M}_n(x) \xrightarrow{D} \mathcal{N}(0, \Gamma(x)),$$

which implies, from (B.17) and (B.18), that for any $x \in \mathbb{R}$ such that $g(x) > 0$,

$$\sqrt{n h^3_n (\hat{f}_n'(x) - f'(x))} \xrightarrow{D} \mathcal{N}\left(0, \frac{\xi^2}{1 + 3\alpha} \frac{f^2(x) + \sigma^2}{g(x)}\right).$$  

**Proof of convergence (3.5).** First of all, for any $x \in \mathbb{R}$, denote $h(x) = f(x)g(x)$. It follows from (2.3) and (2.4) that for any $x \in \mathbb{R}$ such that $g(x) > 0$,

$$\hat{f}_n(x) - f(x) = \frac{\hat{h}_n(x)}{\hat{g}_n(x)} - \frac{h(x)}{g(x)} = \frac{1}{g(x)\hat{g}_n(x)} \left(\hat{h}_n(x)g(x) - h(x)\hat{g}_n(x)\right)$$

$$= \frac{1}{g(x)\hat{g}_n(x)} \left(g(x)\left(\hat{h}_n(x) - h(x)\right) - h(x)\left(\hat{g}_n(x) - g(x)\right)\right)$$

(B.20)

$$= \frac{1}{\hat{g}_n(x)} \left(\hat{h}_n(x) - h(x)\right) - \frac{h(x)}{g(x)\hat{g}_n(x)} \left(\hat{g}_n(x) - g(x)\right).$$

By the same token, we obtain from (2.9) together with tedious but straightforward calculation that for any $x \in \mathbb{R}$ such that $g(x) > 0$,

$$\hat{f}_n'(x) - f'(x) = \left(\frac{\hat{h}_n(x)}{\hat{g}_n(x)} - \frac{\hat{h}_n(x)}{\hat{g}_n^2(x)}\right) - \left(\frac{h'(x)}{g(x)} - \frac{h(x)}{g^2(x)}\right)$$

$$= \frac{1}{\hat{g}_n(x)} \left(\hat{h}_n'(x) - h'(x)\right) - \frac{f'(x)}{\hat{g}_n(x)} \left(\hat{g}_n(x) - g(x)\right)$$

(B.21)

$$- \frac{\hat{h}_n(x)}{\hat{g}_n^2(x)} \left(\hat{g}_n'(x) - g'(x)\right) - \frac{g'(x)}{\hat{g}_n(x)} \left(\hat{f}_n(x) - f(x)\right).$$

Hence, we obtain from (B.20) and (B.21) that for any $x \in \mathbb{R}$ such that $g(x) > 0$,

$$\hat{f}_n'(x) - f'(x) = \frac{1}{\hat{g}_n(x)} \left(\hat{h}_n'(x) - h'(x)\right) + \frac{\hat{\ell}_n(x)}{\hat{g}_n^2(x)} \left(\hat{g}_n(x) - g(x)\right)$$

(B.22)

$$- \frac{\hat{h}_n(x)}{\hat{g}_n^2(x)} \left(\hat{g}_n'(x) - g'(x)\right) - \frac{g'(x)}{\hat{g}_n(x)} \left(\hat{h}_n(x) - h(x)\right).$$
where \( \hat{\ell}_n(x) = f(x)g'(x) - f'(x)\hat{g}_n(x) \). Therefore, we deduce from identity (B.22) the martingale decomposition

\[
(B.23) \quad \sqrt{n}h_n^2(\hat{f}'_n(x) - f'(x)) + \hat{R}_n(x) = \frac{1}{\sqrt{n}^{1+\alpha}}(\langle \hat{e}_n(x), \mathcal{M}_n(x) \rangle + \hat{R}_n(x))
\]

with

\[
\hat{e}_n(x) = \frac{1}{\hat{g}_n^2(x)} \begin{pmatrix} \hat{g}_n(x) \\ \hat{g}_n(x) \\ -\hat{h}_n(x) \\ \hat{\ell}_n(x) \\ -g'(x) \end{pmatrix}, \quad \mathcal{M}_n(x) = \begin{pmatrix} M^A_n(x) \\ B_n(x) \\ M^U_n(x) \\ M^P_n(x) \\ Q_n(x) \end{pmatrix},
\]

where the martingale difference sequences \((M^A_n(x)), (B_n(x))\) and \((M^P_n(x)), (Q_n(x))\) were previously defined in (A.14) and (B.16), while the martingale difference sequences \((M^U_n(x))\) and \((M^V_n(x))\) are given by \(M^U_n(x) = U_n(x) - \mathbb{E}[U_n(x)]\) and \(M^V_n(x) = V_n(x) - \mathbb{E}[V_n(x)]\) with

\[
U_n(x) = \sum_{k=1}^{n} u_k(X_k, x) = \sum_{k=1}^{n} \frac{1}{h_n} K \left( \frac{x - X_n}{h_n} \right), \quad V_n(x) = \sum_{k=1}^{n} v_k(X_k, x) = \sum_{k=1}^{n} \frac{1}{\hat{g}_n^2} K' \left( \frac{x - X_n}{h_n} \right).
\]

It is not hard to see that the remainder \(\hat{R}_n(x)\), which can be explicitly calculated, plays a negligible role since, as soon as \(\alpha > 1/5\),

\[
(B.24) \quad \sup_{x \in \mathbb{R}} |\hat{R}_n(x)| = o(\sqrt{n}^{1+3\alpha}).
\]

It remains to establish the asymptotic behavior of the six-dimensional real martingale \((\mathcal{M}_n(x))\). As in the proof of (B.5) and (B.19), we can show that for any \(x \in \mathbb{R}\),

\[
(B.25) \quad \lim_{n \to \infty} \frac{1}{n^{1+3\alpha}} \langle \mathcal{M}_n(x) \rangle_n = \Gamma(x)
\]

where \(\Gamma(x)\) is the six-dimensional covariance matrix given by

\[
\Gamma(x) = \frac{\xi^2 g(x)}{1 + 3\alpha} \begin{pmatrix} f^2(x) & f(x) & 0 & 0 & 0 \\ 0 & f^2(x) & 0 & 0 & 0 \\ f(x) & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Moreover, via the same lines as in the proof of (B.14), \((\mathcal{M}_n(x))\) satisfies the Lindeberg condition. Hence, we obtain from the central limit theorem for martingales (Duflo (1997)) that for any \(x \in \mathbb{R}\),

\[
(B.26) \quad \frac{1}{\sqrt{n}^{1+\alpha}} \mathcal{M}_n(x) \xrightarrow{D} \mathcal{N}(0, \Gamma(x)).
\]
Furthermore, it follows from Lemma A.1 that for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \), \( \hat{e}_n(x) \) converges a.s. to \( e(x) \) where

\[
e(x) = \frac{1}{g^2(x)} \begin{pmatrix} g(x) \\ g(x) \\ -h(x) \\ -g'(x) \\ -g'(x) \end{pmatrix}
\]

with \( \ell(x) = f(x)g'(x) - f'(x)g(x) \). Finally, we deduce from (B.23), (B.24), (B.26) and (B.27) together with Slutsky’s lemma that for any \( x \in \mathbb{R} \) such that \( g(x) > 0 \),

\[
\sqrt{n h_n^3} (\hat{f}_n'(x) - f'(x)) \xrightarrow{D} \mathcal{N}(0, \sigma^2(x))
\]

where \( \sigma^2(x) = \langle e(x), \Gamma(x)e(x) \rangle \). However, as \( h(x) = f(x)g(x) \), it is not hard to see that

\[
\sigma^2(x) = \frac{\xi^2}{(1 + 3\alpha)g^3(x)} \begin{pmatrix} 1 & f^2(x) & 0 & f(x) \\ g(x) & 0 & \sigma^2 & 0 \\ -f(x)g(x) & f(x) & 0 & 1 \\ -f(x)g(x) & -f(x) & 0 & 1 \end{pmatrix} \begin{pmatrix} g(x) \\ g(x) \\ -h(x) \\ -g'(x) \end{pmatrix}
\]

\[
= \frac{\xi^2}{(1 + 3\alpha)g(x)} \begin{pmatrix} 1 & f^2(x) & 0 & f(x) \\ g(x) & 0 & \sigma^2 & 0 \\ -f(x) & f(x) & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -f(x) \end{pmatrix}
\]

\[
= \frac{\xi^2 \sigma^2}{(1 + 3\alpha)g(x)},
\]

which completes the proof of Theorem 3.2.

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