2D Navier-Stokes equation in Besov spaces of negative order

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Abstract

The Navier–Stokes equation in the bidimensional torus is considered, with initial velocity in the Besov spaces $B_{p,r}^{-s+2-\frac{2}{r}}$ and forcing term in $L^r(0,T;B_{p,q}^{-s})$ for suitable indices $s,r,p,q$. Results of local existence and uniqueness are proven in the case $-1 < -s + 2 - \frac{2}{r} < 0$ and of global existence in the case $-\frac{1}{2} < -s + 2 - \frac{2}{r} < 0$.

Key words: Navier-Stokes equations, weak solutions, existence uniqueness and regularity theory.

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1 Introduction

Analysis of classical or weak solutions to the Navier-Stokes equation in a two-dimensional bounded domain has been widely considered in the literature. A lot of work has been done on solutions with finite energy, because of their physical meaning. However, infinite energy solutions are important for other reasons. On the one hand, analysis of self-similar solutions in the whole space involves velocity fields with infinite energy because of the no rapid enough decreasing to infinity (see, e.g., [Ca, GP] and references therein). On the other hand, invariant (equilibrium) Gibbs measures are known for two-dimensional hydrodynamics in bounded domains (see, e.g.,
for the deterministic Euler equation and [DPD] [AFD] for a stochastic Navier–Stokes equation, whereas [AC] deals with both the viscous and inviscous problems. The velocity fields with finite energy are negligible with respect to these invariant measures. This makes interesting the analysis of infinite energy solutions in a bidimensional bounded spatial domain.

The aim of this paper is to investigate the Navier–Stokes evolution problem in a bounded domain $D$ of $\mathbb{R}^2$ with initial velocity and forcing term of low regularity. The force will be integrable in time to some power with value in some Besov space $B^\sigma_{pq}$ of negative order $\sigma$ (for the space variable) and the initial velocity will belong to some other Besov space of negative order. We point out the importance of the force; for instance, this allows to consider problems related to the Navier–Stokes equation with a stochastic forcing term. (See, e.g., [DPD] for the study of auxiliary (deterministic) equations of Navier–Stokes type arising from a stochastic Navier–Stokes equation. There the forcing term has space regularity of a negative order Besov space. We also refer to the bibliography of [DPD] for other papers on the stochastic Navier–Stokes equation.) Therefore the regularity of the initial velocity is important as well as the time-space regularity of the driving force. We shall provide existence and uniqueness results (global in time), as a generalization of the classical results in Hilbert spaces (recalled in the remark at the end of Section 5).

The content of the paper is as follows. In the next section, we shall introduce the Navier–Stokes equation and the Besov spaces to work with. In Section 3 we shall deal with existence and uniqueness results of solutions to our problem on a small time interval, when the initial velocity belongs to a Besov space of negative order and the forcing term is integrable in time to some power with value into a Besov space of negative order in space (see Theorem 3.1). In Section 4 global (in time) existence results will be given; for this aim, first we shall split our problem into two auxiliary problems (considering the additive splitting of the velocity field $u = x + y$). The equation for the variable $y$ will have small initial data and small forcing term; in Proposition 4.2 existence of a unique solution $y$ will be proven. The problem for the variable $x$ will be solved in Proposition 4.4 using a priori energy estimates, as in the classical case discussed e.g. in [Te79]. Theorem 4.5 will concern the main result for the given problem in the variable $u$. Finally, we shall make some remarks on the case of a smooth forcing term in Section 5. The range of variability of the parameters involved in defining the functional spaces will be specified in Appendix A and B, providing examples of non classical solutions.
2 The Navier-Stokes equation

We consider the Navier–Stokes evolution problem, i.e., the system of equations governing the motion of an homogeneous incompressible viscous fluid

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= \varphi, \\
\nabla \cdot u &= 0,
\end{align*}
\]

where the spatial domain is the torus \( T^2 = \mathbb{R}^2/(2\pi \mathbb{Z})^2 \). In other words, we consider our problem on the square \([0, 2\pi]^2\) with periodic boundary conditions. We consider a finite time interval \([0, T]\). Here the unknowns are the velocity vector field \( u = (u_1(t, \xi), u_2(t, \xi)) \) and the pressure field of the fluid \( p = p(t, \xi) \) (for \((t, \xi) \in [0, T] \times T^2\)); \( \Delta \) is the Laplacian operator \( \Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} \), \( \nabla \) is the gradient operator \( \nabla = (\partial/\partial \xi_1, \partial/\partial \xi_2) \) and \( \cdot \) is the scalar product in \( \mathbb{R}^2 \). \( \varphi \) is a given external force and \( \nu > 0 \) is the (constant) viscosity coefficient. \( \nabla \cdot u = 0 \) is the incompressibility condition.

The basic functional spaces to set the problem are the generalized Sobolev spaces \( H^s_p \) \((s \in \mathbb{R}, 1 < p < \infty)\) defined to be “regular” subspaces of the space \( D' \) of periodic divergence-free vector distributions. To define them, we proceed in the following way. Take any \( u \in D' \). Then since \( \nabla \cdot u = 0 \) on \( T^2 \), there exists a periodic scalar distribution \( \psi \) on \( T^2 \), called the stream function, such that

\[
u = \nabla ^\perp \psi \equiv (-\partial \psi/\partial \xi_2, \partial \psi/\partial \xi_1).
\]

Decomposing \( \psi \) in Fourier series with respect to the complete orthonormal system in \( \mathbb{L}^2(T^2) \) given by \( \left\{ \frac{1}{2\pi} e^{ik \cdot \xi} \right\}_{k \in \mathbb{Z}^2} \)

\[
\psi(\xi) = \sum_{k \in \mathbb{Z}^2} \psi_k \frac{e^{ik \cdot \xi}}{2\pi}
\]

by (2) we get that \( u \) has the following Fourier series representation

\[
u(\xi) = \sum_{k \in \mathbb{Z}^2} u_k e_k(\xi), \quad u_k \in \mathbb{C}, \quad \overline{u_k} = -u_{-k},
\]

where \( e_k(\xi) = \frac{k^+}{2\pi |k|} e^{ik \cdot \xi} \). Note that \( \{e_k\}_{k \in \mathbb{Z}^2} \) is a complete orthonormal system of the eigenfunctions (with corresponding eigenvalues \( |k|^2 \)) of the operator \(-\Delta\) in \( \mathbb{L}^2(T^2)^2 = \{ u \in \mathbb{L}^2(T^2)^2 : \int_{T^2} u(\xi) \, d\xi = 0, \nabla \cdot u = 0, \) with the normal component of \( u \) being periodic on \( \partial T^2 \} \), \( k^+ = (-k_2, k_1) \).
Each \( e_k \) is a periodic divergence-free \( C^\infty \)-vector function (i.e. \( e_k \in D \)). The convergence of the series (3) depends on the regularity of the vector function \( u \), and can be used to define Sobolev spaces as in the following definition. For any \( s \in \mathbb{R}, 1 < p < \infty \), we define

\[
H^s_p = \left\{ u \in D' : u = \sum_{k \in \mathbb{Z}^2_0} u_k e_k \text{ and } \sum_{k \in \mathbb{Z}^2_0} u_k |k|^s e_k \in [L_p(T^2)]^2 \right\}
\]

(4)

\( H^s_p \) is a Banach space with norm \( \|u\|_{H^s_p} = \|\sum_{k \in \mathbb{Z}^2_0} u_k |k|^s e_k\|_{L_p} \).

Thus the unknown velocity will be considered as a function of the time variable taking values in some \( H^s_p \) space

\[
u(t) = \sum_k u_k(t) e_k
\]

Let \( P \) be the following projection from the space of periodic distributions onto the space of periodic divergence-free distributions: \( Pu = \sum_k \langle u, e_{-k} \rangle e_k \). Here \( \langle \cdot, \cdot \rangle \) is the \( ((C^\infty(T^2))^2)' - [C^\infty(T^2)]^2 \) duality bracket. Applying the projection \( P \) to the first equation (1) we get rid of the pressure term (because \( \langle \nabla p, e_{-k} \rangle = -\langle p, \nabla \cdot e_{-k} \rangle = 0 \) for any \( k \)), obtaining the following formulation of our problem

\[
u'(t) + Au(t) + B(u(t)) = f(t), \quad t \in (0, T],
\]

(5)
as an equality in the distributional sense, with some initial condition \( u(0) \) assigned. We have taken the viscosity \( \nu = 1 \), without loss of generality.

\( A \) is the stokes operator \( A = -\Delta \), \( B \) is the quadratic operator \( B(u) = B(u, u) \) defined by the bilinear operator \( B(u, v) = P[(u \cdot \nabla)v] \). Notice that \( B(u, v) \) is equal to \( P[\nabla \cdot (u \otimes v)] \) because of the divergence-free condition. \( f = P\varphi \).

We are interested in the evolution problem (5) for initial data and forcing term not too regular (in space variables). For this purpose, we define a scale of spaces \( B^{s\theta}_{pq} \) consisting of the Besov spaces of periodic divergence-free vector fields. They can be introduced as real interpolation spaces (see, e.g. [BL] Theorem 6.4.5):

\[
B^{s\theta}_{pq} = (H^{s_1}_{p_1}, H^{s_2}_{p_2})_{\theta,q} \quad s^* = (1 - \theta)s_1 + \theta s_2, \quad 0 < \theta < 1
\]

(6)

for \( s^* \in \mathbb{R}, 1 < p, q < \infty \).

In particular, \( B^{0\theta}_{22} = H^{s\theta}_{2} \) and \( B^{00}_{22} = [L^2_{\text{div}}(T^2)]^2 \).
Since \( \{e_k\}_{k \in \mathbb{Z}^2} \) are the eigenvectors of the Stokes operator, with corresponding eigenvalues \( \lambda_k = |k|^2 \), then we can naturally extend \( A \) to the whole space \( \mathcal{D}' \) by the following formula: 
\[
Au = \sum_k u_k |k|^2 e_k 
\]
for any \( u \in \mathcal{D}' \) and by means of this representation it is straightforward to show that the Stokes operator \( A \) is a linear operator in \( H^s \) with domain \( H^{s+2} \); moreover it is a bijective unitary operator from \( H^{s+2} \) onto \( H^s \) for any index \( s \in \mathbb{R}, 1 < p < \infty \), and also from \( B_{pq}^{s+2} \) onto \( B_{pq}^s \) (for any \( s \in \mathbb{R}, 1 < p, q < \infty \)) (see, e.g., [L] Theorem 1.1.6 for getting linear operators in interpolation spaces).

In particular the inverse operator \( A^{-1} \) is a linear bounded operator in each space \( B_{pq}^s \). Finally, the operator \( A \) generates an analytic semigroup in each \( B_{pq}^s \). (We refer, e.g., to [Te83] for classical analysis of the Navier–Stokes equation on the torus.)

The properties of the Stokes operator are the basis to analyze equation (5) as a perturbation of the linear Stokes problem by the nonlinear operator \( B \).

To this end, the key point is to estimate the operator \( B \) in Besov spaces, as done, e.g., in [Ch96].

Remark. We believe that all what follows holds if the spatial domain is any bidimensional smooth bounded domain \( \mathbb{D} \), but we postpone this analysis to a future work. However, we point out that only in the space periodic case the expression of the Gibbs measure constructed by means of the enstrophy (given in [ARI-K]) is an invariant measure for a stochastic Navier–Stokes evolution problem. This depends on the fact that this Gibbs measure is invariant for the Euler flow both on the torus (see [ARI-K]) or in any smooth bounded domain (see [AH-K]), but the Euler and Navier–Stokes boundary conditions are the same only for \( \mathbb{D} = \mathbb{T}^2 \). (For an overview on this subject see e.g. [AFa].) For this reason and because of the interest described in the introduction, we present our problem considering the spatial domain \( \mathbb{T}^2 \). We point out that our technique does not apply in the case the spatial domain is the whole space, because the inverse of the Stokes operator is not a bounded operator in \( B_{pq}^s (\mathbb{R}^2) \). Hence, the techniques are quite different from the case of the spatial domain \( \mathbb{D} = \mathbb{R}^2 \). Moreover, also the assumptions for getting existence and uniqueness results are different from the case \( \mathbb{D} = \mathbb{R}^2 \); let us consider [BG], which is as far as we are aware of the only paper including a forcing term in the analysis of very rough solutions of the Navier–Stokes

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1 For instance, there exists a complete orthonormal system \( \{e_n\}_{n \in \mathbb{N}} \) of eigenvectors of the Stokes operator in the space of square integrable divergence-free vector fields satisfying homogeneous Dirichlet boundary condition, with associated eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \), \( \lambda_n \sim n \) as \( n \to \infty \) (see, e.g., [Te79]). This allows to analyze the linear Stokes operator as presented above for the case on the torus.
problem. Biagioni and Gramchev assume \( F \in L^1_{\text{loc}}(\mathbb{R}^+; L^q(\mathbb{R}^2)) \), \( q \in [\frac{4}{3}, 2] \), or \( F \in L^1_{\text{loc}}(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \) (setting \( F = \nabla^\perp \cdot f \), for \( f \) the forcing term in (5)).

Other results for \( \mathbb{D} = \mathbb{R}^3 \) are presented in [CP].

\[ \square \]

### 3 Local existence and uniqueness

In this section we prove a result on uniqueness and local existence for equation (5) in Besov spaces. We use a technique from [B91] based on a theorem of local diffeomorphism of [VF] (§1.1).

Let \( u \) be a vector valued function defined on the time interval \([0, T]\) for \( T < \infty \) or on \([0, \infty)\) if \( T = \infty \). Keeping in mind the notation in equation (5), consider the mapping \( \Phi \ni u \mapsto (u' + Au + B(u), u(0)) \in \mathcal{E} \to \mathcal{F}, \) for some Banach spaces \( \mathcal{E} \) and \( \mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^0 \). The Fréchet derivative \( d_a \Phi \) at the point \( a = 0 \) is given by the linear operator \( d_0 \Phi : \mathcal{E} \to \mathcal{F}, u \mapsto (u' + Au, u(0)) \). Under the assumption that the quadratic operator \( B \) from \( \mathcal{E} \) to \( \mathcal{F}^1 \) is bounded\(^2\) and that the linear operator \( d_0 \Phi \) is an isomorphism, Vishik and Fursikov [VF] observe that \( \Phi \) is analytic in a neighbourhood of 0 \( \in \mathcal{E} \) and locally \( \Phi \) has an inverse operator \( \Psi \), which is analytic in the neighbourhood \( B(\sigma_\infty) \) of \( \Phi(0) \in \mathcal{F} \) of radius \( \sigma_\infty = (4\|(d_0 \Phi)^{-1}\|_2 \|B\|^{-1}). \) This provides global existence of a unique solution \( u \in \mathcal{E} \) for small forcing term and small initial data in \( \mathcal{F} \). This solution has analytical dependence on the initial value and forcing term. We will use this technique in Proposition 4.2.

Moreover, starting from this approach, [B91] shows a result of local existence for any data in \( \mathcal{F} \) by showing that the Vishik-Fursikov procedure works also for \( T < \infty \) and by finding an estimate on \( \sigma_T \). (The sub-index \( T \) reminds that we work on the time interval \([0, T]\). The important point is that also the norms in \( \mathcal{E} \) and in \( \mathcal{F} \) will depend on \( T \).) Roughly speaking, equation (5) is seen as a perturbation by a small nonlinear term \( B \) of the linear equation (which is well posed in the Hadamard’s sense, see Proposition 3.2). The “smallness” of \( B \) is obtained choosing the time interval small enough. More

\[ \text{In practice, these spaces will be constructed from } L^\alpha(0, T; B^s_{p,q}) \text{ and } B^s_{p,q} \text{ spaces. From now on, } B^s_{p,q} \text{ denotes the space for vector valued functions defined on the torus, unless otherwise specified. By the way, we remark that in the notation for the time-integrability, the sup-index has been used, whereas in the notation for the space-integrability the inf-index has been used.} \]

\[ \text{The operator } B : \mathcal{E} \to \mathcal{F}^1 \text{ is bounded if} \]

\[ \|B\| := \sup_{\|u\|_{\mathcal{E}} \leq 1} \|B(u)\|_{\mathcal{F}^1} < \infty. \]
precisely, let the time variable vary in the finite interval \([0, T]\); assume that
the quadratic term \(B : \mathcal{E} \rightarrow \mathcal{F}^1\) has a norm bounded by
\[
\|B\| \leq C_1 T^\varepsilon
\]
for some positive constants \(C_1\) and \(\varepsilon\); then the solution \(u\) to the Navier–
Stokes equation (5) exists locally in time for any forcing term in \(\mathcal{F}^1\) and
initial velocity in \(\mathcal{F}^0\). Indeed, given \((f, u_0) \in \mathcal{F}\), let \(0 < T \leq T\) be such that
\[
\|(f, u_0)\|_{\mathcal{F}} T^\varepsilon < \frac{1}{4 \| (d_0 \Phi)^{-1} \|^2 C_1}
\]
Then \((f, u_0) \in B(\sigma T)\), the \(\mathcal{F}\)-ball of radius \(\sigma T\); according to Vishik and Fur-
sikov’s result quoted above, we conclude that there exists a unique solution
defined on the time interval \([0, T]\).

These are quite general results. The crucial point is the choice of the
spaces \(\mathcal{E}\) and \(\mathcal{F}\). In [B91] regular Hilbert spaces are considered: \(\mathcal{E} = \{u \in L^2(0, T; H_2^\delta) : u' \in L^2(0, T; H_0^\delta)\}\) and \(\mathcal{F} = L^2(0, T; H_2^\delta) \times H_2^\delta\). Here
we consider \(\mathcal{E} = \{u \in L^r(0, T; B_r^{-s+2}) : u' \in L^r(0, T; B_r^{-s})\}\) and \(\mathcal{F} = L^r(0, T; B_r^{-s}) \times B_r^{-s+2-\frac{2}{p}}\)
with \(1 < p, q, r < \infty\) and \(s \in \mathbb{R}\). The norms are
defined as
\[
\|(f, u_0)\|_{\mathcal{F}} = (\int_0^T \|f(t)\|_{B_r^{-s+2}} dt)^{1/r} + \|u_0\|_{B_r^{-s+2}}
\]
\[
\|u\|_{\mathcal{E}} = (\int_0^T \|u(t)\|_{B_r^{-s+2}} dt + \int_0^T \|u'(t)\|_{B_r^{-s+2}} dt)^{1/r}
\]
We are especially interested in the case \(-s + 2 - \frac{2}{r} < 0\).

In this framework, the main result in this section is as follows.

**Theorem 3.1** Let
\[
\mathcal{E} = \{u \in L^r(0, T; B_r^{-s+2}) : u' \in L^r(0, T; B_r^{-s})\}
\]
\[
\mathcal{F}^1 = L^r(0, T; B_r^{-s}), \quad \mathcal{F}^0 = B_r^{-s+2-\frac{2}{p}}, \quad \mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^0
\]
with \(1 < p, q, r < \infty\), \(s \in \mathbb{R}\).

Then
\[i)\] The map \(d_0 \Phi : \mathcal{E} \ni u \mapsto (u' + Au, u(0)) \in \mathcal{F}\) is well defined and
continuous. Moreover, the operator \(d_0 \Phi\) is an isomorphism of Banach
spaces \(\mathcal{E}\) and \(\mathcal{F}\).
ii) If the parameters \( r, p, q, s \) satisfy the following conditions

\[
\begin{align*}
  r &\leq q \quad (8) \\
  s - 1 &< \frac{2}{p} \quad (9) \\
  1 - s &< \frac{2}{p} \quad (10) \\
  s + \frac{2}{p} + \frac{2}{r} &< 3 \quad (11) \\
  2 &< s + \frac{2}{p} + \frac{2}{r} \quad (12) \\
  3 &< s + \frac{2}{p} + \frac{4}{r} \quad (13)
\end{align*}
\]

then the map \( \mathcal{E} \ni u \mapsto B(u) \in \mathcal{F}^1 \) is well defined and bounded. Therefore \( \Phi : \mathcal{E} \to \mathcal{F} \) is analytic. Moreover, there exist positive constants \( C_1 \) and \( \varepsilon \) such that

\[
\|B\| \leq C_1 T^\varepsilon.
\]

iii) Under the same assumptions as in ii), for any forcing term \( f \in L^r(0, T; B_p^{-s}) \) and initial data \( u_0 \in B_p^{-s+2-\frac{2}{r}} \), there exists a time interval \( [0, T] \subseteq [0, T] \) and a unique solution \( u \) to equation (5) with initial velocity \( u_0 \), defined on the time interval \( [0, T] \) with \( T > 0 \) satisfying (7), and

\[
\begin{align*}
  u &\in L^r(0, T; B_p^{-s+2}) \cap C([0, T]; B_p^{-s+2-\frac{2}{r}}) \\
  u' &\in L^r(0, T; B_p^{-s})
\end{align*}
\]

The solution \( u \) depends analytically on the data \( u_0 \) and \( f \).

The properties i) of the linear operator \( d_0\Phi \) follow from a general result:

**Proposition 3.2** Let \( T \in (0, \infty) \), \( 1 < p, q, r < \infty \) and \( s \in \mathbb{R} \).

For any \( f \in L^r(0, T; B_p^{-s}) \) and \( u_0 \in B_p^{-s+2-\frac{2}{r}} \), there exists a unique \( u \in W^{1, r}(0, T; B_p^{-s+2}) \) such that

\[
\begin{align*}
  u'(t) + Au(t) &= f(t), \quad t \in (0, T] \\
  u(0) &= u_0
\end{align*}
\]

Moreover, the functions \( u', u \) depend continuously on the data \( f \) and \( u_0 \), that is there exists a positive constant \( c \) such that

\[
\left( \int_0^T (\|u(t)\|_{B_p^{-s+2}}^r + \|u'(t)\|_{B_p^{-s}}^r) dt \right)^{1/r} \leq c \left( \int_0^T \|f(t)\|_{B_p^{-s}}^r dt \right)^{1/r} + \|u_0\|_{B_p^{-s+2-\frac{2}{r}}}
\]

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Finally, the space \( W^{1,r}(0,T) \) is continuously embedded in the space \( C_b([0,T];B_{p,r}^{-s+2-\frac{2}{r}}) \), that is there exists a positive constant \( c \) such that

\[
\|u\|_{C_b([0,T];B_{p,r}^{-s+2-\frac{2}{r}})} \leq c \|u\|_{W^{1,r}(0,T)}
\]

and therefore the initial condition makes sense.

All the constants\(^4\) depend only on \( r,p,q,s \).

For the proof, see, e.g., [B95] Proposition 4.1 (based on [DV]). The assumptions on the linear operator \( A \) and on the space \( B_{p,q}^{-s} \) are fulfilled; namely, the properties of the Stokes operator recalled in Section 2 and the fact that \( B_{p,q}^{-s} \) (\( 1 < p,q < \infty, s \in \mathbb{R} \)) is a UMD Banach space (i.e. has the Unconditional Martingale Difference property). The last part of Proposition 3.2 is obtained by interpolation, bearing in mind the interpolation result for Besov spaces \( B_{p,r}^{-s+2-\frac{2}{r}} = (B_{p,q}^{-s},B_{p,q}^{-s+2})_{\frac{1}{r},r} \) for \( 1 < r < \infty, 1 \leq p,q \leq \infty, s \in \mathbb{R} \).

This corresponds to the first part i) stated in Theorem 3.1. Assuming i) and ii), then part iii) is proven as described at the beginning of this section. Hence, the proof of Theorem 3.1 is complete as soon as we prove ii). This is given by the following result for the nonlinear equation.

**Proposition 3.3** Suppose the real numbers \( s,p,q,r \in (1,\infty) \) satisfy the conditions (8-13) of Theorem 3.1. Then there exist constants \( \varepsilon > 0 \) and \( C_1 > 0 \) such that for all \( T > 0 \)

\[
\left( \int_0^T \|B(u(t))\|_{B_{p,q}^{-s}}^{1/r} dt \right)^{1/r} \leq C_1 T^{\varepsilon} \|u\|_{W^{1,r}(0,T)}^2, \quad u \in W^{1,r}(0,T).
\]

**Proof.** First, let us show that there exists a pair \( (a,b) \) of real numbers such that

\[
2 - \frac{2}{r} - s < a,b \quad (15)
\]

\[
a,b < \frac{2}{p} \quad (16)
\]

\[
a + b = \frac{2}{p} + 1 - s \quad (17)
\]

\[
\frac{r}{2} [(a + b) + 2(s - 2 + \frac{2}{r})] < 1 \quad (18)
\]

Before discussing these inequalities, let us remark that (9) and (17) imply that

\[
a + b > 0 \quad (19)
\]

\(^4\)We make the convention to denote different constants by the same symbol \( c \), unless we want to mark them for further reference.
We begin from (12), written as
\[ 2 - \frac{2}{r} - s < \frac{2}{p} \]
which grants that there are solutions to (15)-(16).
Secondly, (18) is equivalent with
\[ a + b < 4 - 2s - \frac{2}{r} \]  
\[ (18') \]
Since by (11) (written as \( \frac{2}{p} + 1 - s < 4 - 2s - \frac{2}{r} \)) any solution to (17) satisfies (18') and hence (18), we only need to show that the system (15)-(16)-(17) has at least one solution. We look for a solution such that \( a = b \). (Anyway, it is not difficult to see that \( a = b \) is not the only possible solution.) For this it is enough that
\[ 2 - \frac{2}{r} - s < \frac{1}{p} + \frac{1}{2} - \frac{s}{2} < \frac{2}{p} \]
The second of this inequalities reads
\[ 1 - s < \frac{2}{p} \]
which is (10).
The first one reads
\[ 2 - \frac{2}{r} < \frac{1}{p} + \frac{1}{2} + \frac{s}{2} \]
which is equivalent to (13). Thus, system (15 18) has at least a solution.
Define
\[ \alpha = \frac{r}{2} [a + s - 2 + \frac{2}{r}] \]
\[ \beta = \frac{r}{2} [b + s - 2 + \frac{2}{r}] \]
By (15) we have \( \alpha, \beta > 0 \) and by (18) we have \( \alpha + \beta < 1 \). In particular \( \alpha, \beta < 1 \).
We are now ready to finish the proof of Proposition 3.3. First we estimate the bilinear operator by means of Bony’s paraproducts techniques, as given in [Ch96], Corollary 1.3.1. Because \( a \) and \( b \) satisfy (16)-(17)-(19), we have
\[ \|B(u(t))\|_{B_{p,q}^{-s}} = \|\nabla \cdot [u(t) \otimes u(t)]\|_{B_{p,q}^{-s}} \]
\[ \leq \|u(t) \otimes u(t)\|_{B_{p,q}^{-s+1}} \]
\[ \leq c \|u(t)\|_{B_{p,q}^0} \|u(t)\|_{B_{p,q}^0} \]  
\[ (20) \]
Secondly, we use well known results on Besov spaces as interpolation spaces (see, e.g., [BL]) to get
\[ \|u\|_{B_{p,q}^0} \leq c \|u\|_{B_{p,q}^0}^{1-\alpha} \|u\|_{B_{p,q}^{-s+2}}^\alpha \|u\|_{B_{p,q}^{-s+2}} \]
∥u∥_{B^s_{p,q}} ≤ c ∥u∥_{B_{p,q}^{-s+2-\frac{2}{r}}}^{1-\beta} ∥u∥_{B_{p,q}^{-s+2}}^\beta

with the interpolation parameters α, β defined above. Here and in the following, c denotes different constants.

We use these inequalities to continue the estimate of the quadratic operator from the last line of (20):

\[ ∥B(u(t))∥_{B_{p,q}^{-s}} ≤ c ∥u(t)∥_{B_{p,q}^{s+2-\frac{2}{r}}}^{2-α-β} ∥u∥_{B_{p,q}^{s+2}}^{α+β} \]

\[ ≤ c ∥u(t)∥_{B_{p,q}^{s+2-\frac{2}{r}}}^{2-α-β} ∥u∥_{B_{p,q}^{s+2}}^{α+β}, \quad \text{for } r ≤ q \]  

(21)

Since α + β < 1, then

\[ (\int_0^T ∥B(u(t))∥_{B_{p,q}^{-s}}^r dt)^{1/r} \leq c ∥u∥_{C([0,T];B_{p,q}^{s+2-\frac{2}{r}})}^{2-α-β} \left(\int_0^T ∥u(t)∥_{B_{p,q}^{s+2}}^{(α+β)r} dt\right)^{1/r} \]

\[ ≤ c ∥u∥_{C([0,T];B_{p,q}^{s+2-\frac{2}{r}})}^{2-α-β} T^{(1-α-β)/r} \left(\int_0^T ∥u(t)∥_{B_{p,q}^{s+2}}^{r} dt\right)^{(α+β)/r} \]

\[ ≤ c ∥u∥_{W^{1,r}(0,T)}^{2} T^{(1-α-β)/r} \]

where we have used Hölder’s inequality for the time integral in the second line and the embedding (14) of Proposition 3.2 in the third line.

We point out that conditions (9) and (11) are against each other; namely, rewritten down for the regularity value of the initial data, they are

\[ -s + 2 - \frac{2}{r} > -\frac{2}{r} - \left(\frac{2}{p} - 1\right) \quad \text{and} \quad -s + 2 - \frac{2}{r} > \frac{2}{p} - 1 \]

It follows that

\[ -s + 2 - \frac{2}{r} > \max \left\{ -\frac{2}{r} - \left(\frac{2}{p} - 1\right), \left(\frac{2}{p} - 1\right) \right\} \]

By the computations (40) in Appendix A, we get that \(-s + 2 - \frac{2}{r} > -1\) at least. This imposes a restriction on the admissible initial velocity to solve equation (5) locally in time.

4 Global existence and uniqueness

We want to show that the local solution constructed in the previous section exists on the whole time interval [0, T]. To prove this, we split our problem into two subproblems, considering two auxiliary variables x and y such that
\[ u = x + y. \]

Following [GP], let decompose the data as

\[ u_0 = x_0 + y_0 \quad \text{and} \quad f = g + h. \]

The problem for the variable \( y \) will have small forcing term \( h \) and small initial data \( y_0 \). Time-global existence and uniqueness will be proved by means of Vishik and Fursikov’s technique.

The problem for the variable \( x \) will have more regular data: initial data \( x_0 \in \mathcal{D} \) and force \( g \in L^2(0, T; H^{-1}_2) \). Time-global existence and uniqueness will be proved by means of an a priori estimate of the energy.

By the very definition of Besov spaces \([9]\), the space of periodic divergence-free smooth functions \( \mathcal{D} \) is dense in any \( B^{s}_{p,q} \). Therefore, we have the following Lemma for the splitting of the data.

**Lemma 4.1** Let \( f \in L^r(0, T; B^{s}_{p,q}) \) with \( 1 \leq r, p, q < \infty \) and \( \sigma \in \mathbb{R} \). Then for any \( \varepsilon > 0 \) there exist functions \( g^\varepsilon \in C^\infty([0, T]; \mathcal{D}) \) and \( h^\varepsilon \in L^r(0, T; B^{s}_{p,q}) \) such that \( \|h^\varepsilon\|_{L^r(0,T;B^s_{r,q})} < \varepsilon \) and \( f = g^\varepsilon + h^\varepsilon \).

The case of constant (in time) functions gives the splitting for the initial data.

We proceed now in this way. The two subproblems read

\[
\begin{align*}
\left\{ \begin{array}{l}
x'(t) + Ax(t) + B(x(t), x(t)) \\
\quad + B(x(t), y(t)) + B(y(t), x(t)) = g(t), \\
x(0) = x_0
\end{array} \right. & \quad t \in (0, T) \tag{22} \\
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
y'(t) + Ay(t) + B(y(t), y(t)) = h(t), \\
y(0) = y_0
\end{array} \right. & \quad t \in (0, T) \tag{23} \\
\end{align*}
\]

For the latter one, we can choose \( h \) and \( y_0 \) sufficiently small (written as \( \ll 1 \) below), in order to have the following result.

**Proposition 4.2** Let the assumptions \((8)-(13)\) of Theorem 3.1 be satisfied.

Then, given any \( y_0 \in B^{-s+2-\frac{2}{r}}_{p,r} \), \( h \in L^r(0, T; B^{-s}_{p,q}) \) with \( \|y_0\|_{B^{-s+2-\frac{2}{r}}_{p,r}} \ll 1 \), \( \|h\|_{L^r(0,T;B^{-s}_{p,q})} \ll 1 \), there exists a unique solution \( y \) to equation \((23)\) on the time interval \([0, T]\), such that

\[
\begin{align*}
y & \in L^r(0,T;B_{p,q}^{-s}) \cap C([0,T];B_{p,r}^{-s+2-\frac{2}{r}}), \quad \text{and} \quad y' \in L^r(0,T;B_{p,q}^{-s}). \tag{24}
\end{align*}
\]
Proof. Global existence for small initial data and small forcing term is obtained as described at the beginning of section 3 by means of Vishik and Fursikov’s technique. Therefore, parts i) and ii) of Theorem 3.1 entails this Proposition.

Concerning equation (22), we would like to show existence and uniqueness on the whole time interval \([0, T]\). We already know that there exist \(\bar{T} \in (0, T]\) and a function \(x = u - y \in L^r(0, \bar{T}; B^{-s+2-\frac{2}{q}}_{p,q})\), \(x' \in L^r(0, \bar{T}; B^{-s}_p)\). To show that \(x\) is indeed a solution to (22) is enough to show that all the terms in (22) make sense. This is done, for the nonlinear terms, analogously as in (20)-(21). For instance

\[
\|B(x, y)\|_{B^{-s}_{p,q}} \leq c \|x\|_{B^{-s+2-\frac{2}{q}}_{p,q}} \|y\|_{B^{-s+2-\frac{2}{q}}_{p,q}} \leq \|x\|_{B^{-s+2-\frac{2}{q}}_{p,q}} \|y\|_{B^{-s+2-\frac{2}{q}}_{p,q}}.
\]

Condition (11) says that \(2p + 2r + s < 3\) (i.e. \(2p + 2r - 1 < -s + 2\) and \(2p - 1 < -s + 2 - \frac{2}{r}\)). Hence \(B^{-s+2}_{p,q} \subset B^{-s+\frac{2}{q}-\frac{1}{q}}_{p,q}\) and \(B^{-s+2-\frac{2}{q}}_{p,q} \subset B^{-\frac{2}{q}-\frac{1}{q}}_{p,q}\) and therefore

\[
\exists \bar{T} \in (0, T]\text{ and } x \text{ such that}
\]

\[
x \in L^r(0, \bar{T}; B^{\frac{2}{q}+\frac{2}{q}-\frac{1}{q}}_{p,q}) \cap C([0, \bar{T}); B^{-\frac{2}{q}-\frac{1}{q}}_{p,q}),
\]

\[
x' \in L^r(0, \bar{T}; B^{\frac{2}{q}+\frac{2}{q}-\frac{1}{q}}_{p,q}).
\]

Now we look for a priori estimates. By [GP] (Lemma 1.1), we have the following energy estimate

**Lemma 4.3** Let \(2 \leq p < \infty, 2 < q < \infty, \frac{2}{p} + \frac{2}{q} - 1 > 0\). Then, for any \(\epsilon > 0\) there exists a constant \(c_\epsilon > 0\) such that

\[
\left| \int_0^T \langle B(x(t)), y(t) \rangle dt \right| \leq \epsilon \int_0^T \|x(t)\|^2_{H^2} dt + c_\epsilon \int_0^T \|x(t)\|^2_{H^2} \|y(t)\|^q_{B^{-\frac{2}{q}+\frac{2}{q}-\frac{1}{q}}_{p,q}} dt.
\]

A precisation on the proof is required. In fact, Gallagher and Planchon in [GP] work in the whole space. Anyway, the technique used by them for the spatial domain \(\mathbb{D}\) is \(\mathbb{R}^2\), works also in our case \(\mathbb{D} = \mathbb{T}^2\). Indeed, let \(u = \sum_k u_k e_k\) with \(e_k = e_k(\xi)\) defined for \(\xi \in \mathbb{R}^2\); the Littlewood–Paley decomposition (see, e.g., [Ch98] and references therein) gives

\[
\Delta_m u = \sum_{2^n < |k| \leq 2^{n+1}} u_k e_k
\]
where $\Delta_m$ means the convolution with a function $\psi_m$ whose Fourier transform $\hat{\psi}_m$ has support in $\{\xi \in \mathbb{R}^2 : 2^m < |\xi| \leq 2^{m+1}\}$. Hence, the proof of $[GP]$ based on Bony’s paraproduct and Bernstein’s inequality$^\text{5}$ is valid also if we deal with the Besov spaces

$$B^s_{p,q}(\mathbb{T}^2) = \{ u = \sum_{k \neq 0} u_k e_k \in \mathcal{D}' : \sum_{m \in \mathbb{N}} \left( 2^{ms} \| \Delta_m u \|_{L_p(\mathbb{T}^2)} \right)^q < \infty \}$$

and the $L_p(\mathbb{T}^2)$-norms appear at the place of the $L_p(\mathbb{R}^2)$-norms of $[GP]$. Moreover, the norms $\| \nabla u \|_{L_2(\mathbb{T}^2)}$ and $\| u \|_{H^s_2(\mathbb{T}^2)}$ are equivalent. Indeed, $\mathbb{T}^2$ is a bounded domain and $\langle u, 1 \rangle = 0$ for $u \in \mathcal{D}'$.

We now look for a priori estimates for the unknown $x$. Let us multiply both sides of the first equation (22) by $x$ and integrate in space and in time. Two terms vanish, namely $\langle B(y(t), x(t)), x(t) \rangle = 0$ and $\langle B(x(t), x(t)), x(t) \rangle = 0$, see e.g. [Te79]. Moreover $\langle B(x(t), y(t)), x(t) \rangle = - \langle B(x(t), x(t)), y(t) \rangle$ and $\langle Ax(t), x(t) \rangle = \| x(t) \|_{H^1_2}^2$. We then have

$$\frac{1}{2} \| x(T) \|_{H^2_2}^2 + \int_0^T \| x(t) \|_{H^1_2}^2 \, dt = \frac{1}{2} \| x_0 \|_{H^2_2}^2 + \int_0^T \langle B(x(t)), y(t) \rangle \, dt + \int_0^T \langle g(t), x(t) \rangle \, dt$$

$$\leq \frac{1}{2} \| x_0 \|_{H^2_2}^2 + c \int_0^T \| x(t) \|_{H^2_2}^2 \| y(t) \|_{B^\frac{2}{p} + \frac{2}{q} - 1}^q \, dt$$

$$+ \frac{1}{2} \int_0^T \| x(t) \|_{H^1_2}^2 \, dt + c \int_0^T \| g(t) \|_{H^2_2}^2 \, dt \quad (26)$$

In this way, the required bounds are obtained by means of Gronwall’s lemma, as soon as we can find $\tilde{p} \geq 2, \tilde{q} > 2$ with $\frac{2}{p} + \frac{2}{q} - 1 > 0$, such that

$$\int_0^T \| y(t) \|_{B^\frac{2}{p} + \frac{2}{q} - 1}^q \, dt < \infty,$$

where $y$ is the solution to problem (23). Proposition 4.2 provides $y \in L^r(0, T; B^{-s+2}_{p,q})$. Thus we need to show that

$$L^r(0, T; B^{-s+2}_{p,q}) \subseteq L^\tilde{q}(0, T; B^{\frac{2}{p} + \frac{2}{q} - 1}_{\tilde{p}, \tilde{q}}) \quad (27)$$

for some $\tilde{p} \geq 2, \tilde{q} > 2$ with $\frac{2}{p} + \frac{2}{q} - 1 > 0$.

$^5$See, e.g., [Ch98] for the definition and properties of Bony’s paraproduct and [N] for Bernstein’s inequality.
Further conditions on the parameters $r, p, s$ are required in order that (27) holds.
First, there is the embedding $L^r(0, T) \subseteq L^\tilde{q}(0, T)$ if

$$r \geq \tilde{q}$$

(28)
since the time interval is finite.

On the other hand, the space embedding $B^{-s+2}_{p,q} \subseteq B^{-\frac{2}{p} + \frac{2}{q} - 1}_{p,q}$ holds if

$$-s + 2 - \frac{2}{p} \geq \frac{2}{q} - 1$$

(29)

$$1 \leq q \leq \tilde{q} \leq \infty, \quad 1 \leq p \leq \tilde{p} \leq \infty$$

(see [BL] Theorem 6.5.1).

We recall assumption (11) of Theorem 3.1:

$$s + \frac{2}{p} + \frac{2}{r} < 3$$

The choice $\tilde{q} = r$, in order to satisfy (28), makes that the conditions (11) and (29) become identical (to be precise, there is the difference $< or \leq$, which makes (11) a slightly stronger than (29)). Thus we have

$$\|y\|_{B^{\frac{2}{p} + \frac{2}{q} - 1}_{p,q}} \leq c \|y\|_{B^{-s+2}_{p,q}}$$

(30)

Summing up, choosing $\tilde{q} = r$ and $\tilde{p} = p$ in (26) and assuming (8-13) with the additional conditions $r > 2$, $\frac{2}{p} + \frac{2}{r} - 1 > 0$, the proper estimates follow. More precisely, from (26) we first have that

$$\|x(T)\|_{H^0_2}^2 \leq \|x_0\|_{H^0_2}^2 + c \int_0^T \|x(t)\|_{H^0_2}^2 \|y(t)\|_{B^{-s+2}_{p,q}}^{r} dt + c \int_0^T \|g(t)\|_{H^{-1}_2}^2 dt$$

Gronwall’s lemma gives

$$\sup_{0 \leq t \leq T} \|x(t)\|_{H^0_2}^2 \leq C(\|x_0\|_{H^0_2}, \|y\|_{L^r(0,T;B^{-s+2}_{p,q})}, \|g\|_{L^2(0,T;H^{-1}_2)}) < \infty$$

The last result in conjunction with (26) gives

$$\int_0^T \|x(t)\|_{H^0_2}^2 dt \leq C(\|x_0\|_{H^0_2}, \|y\|_{L^r(0,T;B^{-s+2}_{p,q})}, \|g\|_{L^2(0,T;H^{-1}_2)}) < \infty$$

Finally

$$\|x\|_{L^\infty(0,T;H^0_2)} + \|x\|_{L^2(0,T;H^0_2)} \leq C(\|x_0\|_{H^0_2}, \|y\|_{L^r(0,T;B^{-s+2}_{p,q})}, \|g\|_{L^2(0,T;H^{-1}_2)})$$

(31)
Use now the embedding theorem in Besov spaces

$$H_2^\sigma \equiv B_{22}^\sigma \subseteq \left(\begin{array}{c}
(\rho \geq 2) B_{p2}^{\sigma-1+\frac{2}{p}} \\
(\rho \geq 2) B_{p\rho}^{\sigma-1+\frac{2}{p}}
\end{array}\right)$$

so to get from (31) that

$$\|x\|_{L^\infty(0,T;B_{p\rho}^{2})} + \|x\|_{L^2(0,T;B_{p\rho}^{2})} \leq C\left(\|x_0\|_{H_2^0}, \|y\|_{L^r(0,T;B_{p\rho}^{\sigma+2})}, \|g\|_{L^2(0,T;H_2^{-1})}\right)$$

By (complex) interpolation (that is $[L^2, L^\infty]_{1-\frac{2}{r}} = L^r$ and $[B_{p\rho}^{2}, B_{p\rho}^{2-1}]_{1-\frac{2}{r}} = B_{p\rho}^{\frac{2}{r}+\frac{2}{r}-1}$ for $2 < r < \infty$), we obtain that

$$\|x\|_{L^\infty(0,T;B_{p\rho}^{2})} + \|x\|_{L^r(0,T;B_{p\rho}^{2})} \leq C\left(\|x_0\|_{H_2^0}, \|y\|_{L^r(0,T;B_{p\rho}^{\sigma+2})}, \|g\|_{L^2(0,T;H_2^{-1})}\right)$$

Comparing (32) with (25), we get that the solution $x$ exists on the whole time interval $[0, T]$. We have therefore proven the following result.

**Proposition 4.4** Let the assumptions (8-13) of Theorem 3.1 be satisfied and moreover assume $r > 2$ and $\frac{2}{p} + \frac{2}{r} - 1 > 0$. Then given any $x_0 \in D$ and $g \in L^2(0, T; H_2^{-1})$, there exists a unique solution $x$ to equation (22) on the time interval $[0, T]$ such that

$$x \in L^r(0, T; B_{p\rho}^{\frac{2}{r}+\frac{2}{r}-1}) \cap C([0, T]; B_{p\rho}^{\frac{2}{r}-1})$$

$$x' \in L^r(0, T; B_{p\rho}^{\frac{2}{r}+\frac{2}{r}-3})$$

**Remark.** Notice that more regularity on $g$ does not improve the regularity of $x$, because of the presence of $y$ (which has a role similar to an external force in equation (22)). For this reason, we assume $g \in L^2(0, T; H_2^{-1})$ instead of the other possible choice $g \in C^\infty([0, T]; D)$. On the other hand, the initial data $x_0$ is chosen very smooth in order to consider without problems the continuity in time in the next results.

We combine Proposition 4.2 and Proposition 4.4 and, bearing in mind the embedding used to show (25) (that is to show that $x$ is less regular than $y$), we get that

$$u = x + y \in L^r(0, T; B_{p\rho}^{\frac{2}{r}+\frac{2}{r}-1}) \cap C([0, T]; B_{p\rho}^{\frac{2}{r}-1})$$

$$u' = x' + y' \in L^r(0, T; B_{p\rho}^{\frac{2}{r}+\frac{2}{r}-3})$$

(33)
This implies that there exists a function $u$, given by $u = x + y$, with the regularity specified in (33). This is indeed a solution to equation (5). In fact, bearing in mind equations (22) and (23), we notice that the function $u = x + y$ solves the equation (5), thanks to the fact that the nonlinearity $B(u(t))$ is well defined; and this is so, because Chemin’s result to estimate the quadratic term guarantees that this exists if $u$ belongs to some Besov space of positive index and from (33) we have that $u(t) \in B^{\frac{2}{p} + \frac{2}{r} - 1}_{p,q}$ with $\frac{2}{p} + \frac{2}{r} - 1 > 0$ (for a.e. $t$). Hence, $u$ is the sought global solution.

We sum up all the results proven so far and state our main theorem.

**Theorem 4.5** For any forcing term $f \in L^r(0,T;B^{-s}_{p,q})$ and initial velocity $u_0 \in B_{p,r}^{-s+2-\frac{2}{r}}$ with

\[
\begin{align*}
    s &\in \mathbb{R}, & 1 < p, q < \infty \\
    2 &< r \leq q \\
    \frac{2}{p} + \frac{2}{r} - 1 &> 0 \\
    -\frac{2}{p} &< s - 1 < \frac{2}{p} \\
    2 &< s + \frac{2}{p} + \frac{2}{r} < 3 \\
    3 &< s + \frac{2}{p} + \frac{4}{r} 
\end{align*}
\]

there exists a unique solution $u$ to equation (5) on the time interval $[0,T]$ such that

\[
\begin{align*}
    u &\in L^r(0,T;B_{p,q}^{\frac{2}{p}+\frac{2}{r}-1}) \cap C([0,T];B_{p,r}^{\frac{2}{p}-1}) \\
    u' &\in L^r(0,T;B_{p,q}^{\frac{2}{p}+\frac{2}{r}-3}) 
\end{align*}
\]

Moreover, there exists a (strictly) positive $T \leq T$ such that the above solution belongs to $L^2(0,T;B_{p,q}^{-s+2}) \cap C([0,T];B_{p,r}^{-s+2-\frac{2}{r}})$. Hence strong continuity for $t \to 0$ holds.

**Remark.** In Appendix B, examples fulfilling all these assumptions will be given. The assumption $r > 2$ imposes that $-s + 2 - \frac{2}{r} > -\frac{1}{2}$, as shown by (41) in Appendix A.

**Proof.** What remains to be proven is the uniqueness result. Let us denote by $\mathcal{U}$ the set of functions $u$ satisfying the conditions (33). Consider two solutions $u, \tilde{u} \in \mathcal{U}$ and denote by $\delta$ the difference $u - \tilde{u}$. Then $\delta \in \mathcal{U}$ and it satisfies the equation

\[
\begin{align*}
    \delta'(t) + A\delta(t) + B(u(t),\delta(t)) + B(\delta(t),\tilde{u}(t)) &= 0, & t \in (0,T) \\
    \delta(0) &= 0
\end{align*}
\]
This is a linear equation in $\delta$. We analyze this equation as a linear Stokes problem with a (linear) perturbation term. If the perturbation $B(u, \cdot) + B(\cdot, \tilde{u})$ is good enough (mainly, small for small time $T$, so that this gives a small perturbation of the well-posed linear parabolic equation), then there exists a unique solution. This will hold on a small time interval; but since $\delta \equiv 0$ is a solution, then we get that the unique solution is the zero one on a small time interval. Starting again from the zero value, we can proceed in the same way to cover the whole time interval.

We want to analyze the perturbation $B(u, \delta) + B(\delta, \tilde{u})$. We define the operators

$$\Gamma_u \delta := -B(u, \delta) \quad \tilde{\Gamma}_u \delta := -B(\delta, u)$$

and the space

$$S := \{ \delta \in L^r(0, T; B_p^{\frac{2}{p} + \frac{2}{q} - 1}) : \delta' \in L^r(0, T; B_p^{\frac{2}{p} + \frac{2}{q} - 3}) \}$$

equipped with the norm $\|\delta\|_S = \left( \int_0^T \|\delta(t)\|_{B_p^{\frac{2}{p} + \frac{2}{q} - 1}}^r \, dt + \int_0^T \|\delta'(t)\|_{B_p^{\frac{2}{p} + \frac{2}{q} - 3}}^r \, dt \right)^{1/r}$.

It is enough to consider the case with $u$, since the same works for $\tilde{u}$, because of the symmetry of Chemin’s estimates in the two arguments. We are going to show that

$$\Gamma_u : S \to L^r(0, T; B_p^{\frac{2}{p} + \frac{2}{q} - 3})$$

and

$$\|\Gamma_u \delta\|_{L^r(0, T; B_p^{\frac{2}{p} + \frac{2}{q} - 3})} \leq C_u(T) \|\delta\|_S$$

(35)

with $C_u(T) \to 0$ as $T \to 0$.

This in nothing but an application of Chemin’s estimates. In fact

$$\|B(u, \delta)\|_{B_p^{\frac{2}{p} + \frac{2}{q} - 3}} \leq \|u \otimes \delta\|_{B_p^{\frac{2}{p} + \frac{2}{q} - 2}}$$

$$\leq c \|u\|_{B_p^{\frac{2}{p} + \frac{2}{q} - 1}} \|\delta\|_{B_p^{\frac{2}{p} - 1}} \quad \text{for} \quad \frac{2}{p} + \frac{2}{q} - 1 > 0$$

$$\leq c \|u\|_{B_p^{\frac{2}{p} + \frac{2}{q} - 1}} \|\delta\|_{B_p^{\frac{2}{r} - 1}} \quad \text{for} \quad r \leq q$$

Therefore, integrating in time, we get

$$\|B(u, \delta)\|_{L^r(0, T; B_p^{\frac{2}{p} + \frac{2}{q} - 3})} \leq C_2 \|u\|_{L^r(0, T; B_p^{\frac{2}{p} + \frac{2}{q} - 1})} \|\delta\|_{C([0, T]; B_p^{\frac{2}{r} - 1})} \leq C_2 \|u\|_{L^r(0, T; B_p^{\frac{2}{p} + \frac{2}{q} - 1})} \|\delta\|_S.$$  (36)
In the last step, we have used the interpolation result (as (14)) which allows to dominate the norm in $C([0,T];B^{\frac{2}{p}+\frac{3}{r}}_{p,q})$ by the norm in $S$

$$\|u\|_{C([0,T];B^{\frac{2}{p}+\frac{3}{r}}_{p,q})} \leq C \|u\|_S. \quad (37)$$

Finally, (35) holds with

$$C_u(T) = C_2 \|u\|_{L^r(0,T;B^{\frac{2}{p}+\frac{3}{r}}_{p,q})}.$$ We remark that $C_u(T) \to 0$ as $T \to 0$.

We shall show that the problem

$$\begin{cases}
\delta'(t) + A\delta(t) = -\Gamma_u \delta(t) - \tilde{\Gamma}_u \delta(t), & t \in (0,T], \\
\delta(0) = 0
\end{cases} \quad (38)$$

has a unique solution $\delta \in S$ on a small time interval, using a contraction theorem. Let us denote by $\Upsilon$ the mapping giving the solution to the Stokes problem

$$\Upsilon : g \mapsto v \begin{cases}
v'(t) + Av(t) = g \\
v(0) = 0
\end{cases}$$

We know from Proposition 3.2 that $\Upsilon$ is an isomorphism from $L^r(0,T;B^{\frac{2}{p}+\frac{3}{r}}_{p,q})$ onto $S$ and

$$\|\Upsilon g\|_S \leq C_3 \|g\|_{L^r(0,T;B^{\frac{2}{p}+\frac{3}{r}}_{p,q})} \quad (39)$$

for some positive constant $C_3$. Therefore, by (35) and (39) we get that

$$\|\Upsilon \circ (\Gamma_u \delta + \tilde{\Gamma}_u \delta)\|_S \leq C_3 \left(C_u(T) + C_{\tilde{u}}(T)\right) \|\delta\|_S$$

This shows that the mapping $\Upsilon \circ (\Gamma_u + \tilde{\Gamma}_u) : S \to S$ is a contraction as soon as we work on the time interval $[0,T] \subseteq [0,\overline{T}]$ with $\overline{T} > 0$ chosen in such a way that

$$C_3 \left(C_u(T) + C_{\tilde{u}}(T)\right) < 1$$

Hence there exists a unique solution $\delta \in S$ to equation (34) on the time interval $[0,T]$. This must coincide with the zero function: $\delta(t) = 0$ for all $t \in [0,T]$.

Since the constants providing the contraction mapping depend only on the norms of $u$ and $\tilde{u}$ (because problem (34) is linear), we start again from $\delta(T) = 0$ and we get the same result on the time interval $[\overline{T},2\overline{T}]$ and so on to conclude the proof in a finite number of analogous steps. 

\[\blacksquare\]
5 Remarks on the case where the forcing term $f$ is smooth

In the previous section, the constraint $r > 2$ has appeared. This implies that the initial data $u_0$ is assumed to belong to the Besov space $B^{-s+2-\frac{2}{r}}_{p,r}$ with $-s + 2 - \frac{2}{r} > -\frac{1}{2}$ (see the comment at the end of section 3 and (41) in Appendix A). This restriction comes from the use of the energy estimate of Lemma 4.3. We want now to show that for $1 < r \leq 2$, when the forcing term $f$ is smooth enough (say, $f \in L^2(0,T;H_0^2)$), a classical energy estimate can be used instead of Lemma 4.3. Therefore, assuming conditions (8-13) and some more regularity on the forcing term, we prove global existence also when $1 < r \leq 2$. Notice that, when the condition $r > 2$ is removed, the initial velocity regularity index $-s + 2 - \frac{2}{r}$ can be very close to $-1$ (see also some examples of admissible values in Appendix B). This agrees with similar results obtained when there is no forcing term (see, e.g., [GP] for the problem in $\mathbb{R}^2$).

We go back to the statement of Proposition 4.2 assuming that $h = 0$ (having taken $g = f$). Since $y \in L^r(0,T;B^{-s+2}_{p,q})$, then there exists a $t_1 \in (0,T]$ as close to 0 as we want, such that

$$y(t_1) \in B^{-s+2}_{p,q}$$

When the condition $1 < r \leq 2$ is added to (8-13), then

$$-s + 2 > 0$$

because at the end of section 3 we have shown that $-s + 2 - \frac{2}{r} > -1$. Moreover, if the index regularity of the initial velocity is negative, i.e. $-s + 2 - \frac{2}{r} < 0$, then condition (11) imposes $p > 2$. At the end of Appendix B, it will be shown that this implies $B^{-s+2-\frac{2}{r}}_{p,r} \not\subseteq B^0_{2,2}$. Therefore we are not dealing with the classical problem of initial velocity with finite energy. Summing up, if $1 < r \leq 2$, $-s + 2 - \frac{2}{r} < 0$ and (8-13) hold, then

$$y(t_1) \in B^{-s+2}_{p,2} \cap B^{-s+2-\frac{2}{r}}_{p,r}$$

Since $-s + 2 - \frac{2}{r} < 0 < -s + 2$, there exists $\theta \in (0,1)$ such that

$$B^0_{p,2} = (B^{-s+2-\frac{2}{r}}_{p,r},B^{-s+2}_{p,2})_{\theta,2}$$

Therefore

$$y(t_1) \in B^0_{p,2} \subset B^0_{2,2} \equiv H^0_2$$

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because the spatial domain is bounded. (In fact, in the same way we can show that \( y \) is infinitely smooth in space and time on the time interval \((0, T]\), because there is no forcing term for \( y \). But we do not need this result.) Therefore on the time interval \([t_1, T]\), the classical Hilbert-space theory can be applied to get existence and uniqueness results (see, e.g., \[Te79\]):

\[
y \in L^2(t_1, T; H^1_2) \cap C([t_1, T]; H^0_2)
\]

By interpolation between \( L^2(t_1, T; H^1_2) \) and \( L^\infty(t_1, T; H^0_2) \), we get

\[
y \in L^4(t_1, T; H^{1/2}_2)
\]

We now assume \( f \in L^2(0, T; B^0_{2q}) \). Since

\[
B^0_{2q} \subset B^{-s}_{p^\ast q} \quad \text{if } p > 2, -1 > -s - \frac{2}{p}
\]

\[
L^2(0, T) \subseteq L^r(0, T) \quad \text{if } 1 < r \leq 2
\]

Then in our setting we have that

\[
f \in L^2(0, T; B^0_{2q}) \subset L^r(0, T; B^{-s}_{p^\ast q})
\]

Therefore local existence results are obtained by means of Proposition 3.3. We only need an a priori estimate on the time interval \([t_1, T]\) for the unknown \( x \). This is easily obtained from the following classical estimate on the trilinear term

\[
\langle B(x), y \rangle \leq \|x\|_{L_4} \|
abla x\|_{L_2} \|y\|_{L_4} \quad \text{by Hölder inequality}
\]

\[
\leq c \|x\|_{H^{1/2}_2} \|x\|_{H^1_2} \|y\|_{H^{1/2}_2} \quad \text{by Sobolev embedding}
\]

\[
\leq c \|x\|_{H_2}^{1/2} \|x\|_{H^1_2}^{3/2} \|y\|_{H^{1/2}_2} \quad \text{by interpolation}
\]

\[
\leq \varepsilon \|x\|_{H_2}^2 + c\varepsilon \|x\|_{H^1_2}^2 \|y\|_{H^{1/2}_2}^2 \quad \text{by Young inequality}
\]

Hence in the time interval \([t_1, T]\), the unknown \( x \) does not explode in the required norms and therefore there exists a unique \( x \in L^2(t_1, T; H^1_2) \cap C([t_1, T]; H^0_2) \), \( x' \in L^2(t_1, T; H^{-1}_2) \). We remind that \( t_1 \) can be chosen close to 0 as much as we want. Since existence on any small time interval \([0, t_1]\) was already proven in section 3, this result implies the global existence. Finally, \( u \in L^2(t_1, T; H^1_2) \cap C([t_1, T]; H^0_2) \) for any \( 0 < t_1 < T \). And this solution \( u \) is unique.

Remark. If \( u_0 \in H^0_2 \) and \( f \in L^2(0, T; H^{-1}_2) \), then a classical result grants that there exists a unique solution \( u \in C([0, T]; H^0_2) \cap L^2(0, T; H^1_2) \), \( u' \in L^2(0, T; H^{-1}_2) \) (see, e.g., \[Te79\]).
Appendix A  Lower estimates on $-s + 2 - \frac{2}{r}$

Let $1 < p, q, r < \infty$ be given. Then

$$\max \left\{ -\frac{2}{r} \frac{2}{p} + 1, \frac{2}{p} - 1 \right\} = \begin{cases} \frac{2}{p} - 1 & 1 < p \leq 2, \forall r \\ \frac{2}{p} - 1 & p > 2, 1 < r \leq \frac{p}{p-2} \\ -\frac{2}{r} - \frac{2}{p} + 1 & p > 2, r > \frac{p}{p-2} \end{cases}$$

Hence, it easily follows that

$$\inf_{1 < p < \infty} \max_{1 < r < \infty} \left\{ -\frac{2}{r} \frac{2}{p} + 1, \frac{2}{p} - 1 \right\} = -1 \quad (40)$$

With some more (but elementary) work, we obtain

$$\inf_{1 < p < \infty} \max_{2 < r < \infty} \left\{ -\frac{2}{r} \frac{2}{p} + 1, \frac{2}{p} - 1 \right\} = -\frac{1}{2} \quad (41)$$

Appendix B  Admissible values for the parameters $s, p, r$ and numerical examples

We want to show that system (9-13) has a non void set of solutions. These are the conditions appearing in Theorem 3.1 providing local existence. Two more conditions are required for the global existence of Theorem 4.5 unless the forcing term is smooth enough (see Section 5). We start analyzing the less restrictive conditions (9-13).

Set $x = \frac{2}{p}$ and $y = \frac{2}{r}$. Then the system of conditions is

$$\begin{align*}
2 - s &< x + y < 3 - s \\
3 - s &< x + 2y \\
s - 1 &< x \\
1 - s &< x \\
0 &< x, y < 2
\end{align*} \quad (42)$$

Because of the range of values specified in the last line, necessary conditions for the existence of a solution to (42) are

$$\begin{align*}
2 - s &< 4 \quad \text{and} \quad 0 < 3 - s \\
3 - s &< 6 \\
s - 1 &< 2 \\
1 - s &< 2
\end{align*}$$
Hence, the admissible values for the parameter $s$ are $-1 < s < 3$. Moreover $x < 3 - s$ (from the first and the last line in (42)) and $x > s - 1$ (that is the third line in (42)) imposes the further restriction: $s < 2$.

Summing up, the admissible values for the parameter $s$ are

$$-1 < s < 2$$

We distinguish two cases.

- $-1 < s < 1$

Since $s - 1 < 1 - s$, the third line in (42) can be neglected. Representing the remaining conditions (42) on the $(x, y)$-plane, it is easy to see that there exist solutions. If we are interested in the solutions satisfying also the condition $-1 < -s + 2 - y < 0$ (for the regularity of the initial velocity), then $s$ must be positive. We give examples of parameters satisfying the above conditions.

Examples:

- for Pro. 3.3
- $s = \frac{9}{10}$, $r = \frac{40}{49}$, $p = 12$, $- s + 2 - \frac{2}{r} = -\frac{4}{5}$

- for Th. 4.5
- $s = -\frac{9}{10}$, $r = \frac{100}{49}$, $p = \frac{40}{39}$, $- s + 2 - \frac{2}{r} = \frac{48}{25}$

We do not choose $q$ since this is the less significant parameter to characterize a Besov space. Notice that $1 < r \leq 2$ in the first case, providing global existence for “regular” forcing term. The second case concerns positive index regularity $-s + 2 - \frac{2}{r}$ for the initial velocity.

- $1 \leq s < 2$

Since $s - 1 \geq 1 - s$, the forth line in (42) can be neglected. Again the graphic representation shows that there are solutions. The condition $-1 < -s + 2 - y < 0$ (for the regularity of the initial velocity) requires $x < 1$ (i.e. $p > 2$). Notice that in this case both conditions $y < 1$ (i.e. $r > 2$) and $-s + 2 - y < 0$ (i.e. $-s + 2 - \frac{2}{r} < 0$) can be fulfilled, providing parameters satisfying the assumptions of Theorem 4.5 with the initial velocity in a Besov space of negative order (not allowed in the previous case).

Examples:

- for Pro. 3.3
- $s = \frac{11}{10}$, $r = \frac{8}{7}$, $p = \frac{40}{3}$, $- s + 2 - \frac{2}{r} = -\frac{17}{20}$

- for Th. 4.5
- $s = \frac{140}{100}$, $r = \frac{200}{89}$, $p = 4$, $- s + 2 - \frac{2}{r} = -\frac{48}{100}$
- $s = \frac{11}{10}$, $r = \frac{40}{19}$, $p = 3$, $- s + 2 - \frac{2}{r} = -\frac{1}{20}$
- $s = \frac{4}{3}$, $r = 3$, $p = \frac{5}{2}$, $- s + 2 - \frac{2}{r} = 0$
- $s = \frac{19}{10}$, $r = 21$, $p = 2$, $- s + 2 - \frac{2}{r} = \frac{1}{210}$
This is the only case providing global solutions with initial velocity \( u_0 \) in Besov space of negative order \(-s + 2 - \frac{2}{r} < 0\) and force \( f \in L^r(0,T;B^{-s}_{p,q})\).

**Remark.** We remark that \(-s + 2 - \frac{2}{r} < 0\) and \( s + \frac{2}{p} + \frac{2}{r} < 3\) imposes \( \frac{2}{p} < 1\), that is \( p > 2\). This implies that there the embedding \( B^{r+s+2-\frac{2}{r}}_{r \infty} \subset B^{0}_{22} \) never holds. Hence we really deal with a generalization of the classical result for \( u_0 \in B^{0}_{22} \). An analogous statement holds for the forcing term: \( L^r(0,T;B^{-s}_{p,q}) \not\subset L^2(0,T;B^{-1}_{22}) \), because \( B^{-s}_{p,q} \not\subset B^{-1}_{22} \) for \( p > 2 \) and \( s > 1 \). □

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