Capacitary density and removable sets for Newton–Sobolev functions in metric spaces

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Abstract
In a complete metric space equipped with a doubling measure and supporting a \((1, 1)\)-Poincaré inequality, we show that every set satisfying a suitable capacitary density condition is removable for Newton–Sobolev functions.

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1 Introduction
Removable sets for Sobolev functions is a classical topic that has been widely studied over several decades, see e.g. [6, 11, 14, 17, 18, 30, 33, 34]. A closed set \(A \subset \mathbb{R}^n\) with zero Lebesgue measure is said to be removable for the Sobolev class \(W^{1,p}\) if \(W^{1,p}(\mathbb{R}^n \setminus A) = W^{1,p}(\mathbb{R}^n)\) as sets. We always consider \(1 \leq p < \infty\). Removable sets have been characterized by various conditions, which however tend to be difficult to apply in practice, e.g. as null sets for condenser capacities; see the discussion in Koskela [20, p. 292]. Indeed, the paper [20] gave a more concrete sufficient condition for removability, proving that so-called \(p\)-porous subsets of a hyperplane are removable for \(W^{1,p}\). It was also shown there that removability is equivalent with \(\mathbb{R}^n \setminus A\) supporting a \((1, p)\)-Poincaré inequality; we can express the latter by saying that \(A\) is removable for the \((1, p)\)-Poincaré inequality.

In Koskela–Shanmugalingam–Tuominen [21] this type of removability result was extended to Ahlfors regular metric measure spaces \((X, d, \mu)\) supporting a \((1, p)\)-Poincaré inequality, with \(1 < p < \infty\). We will give definitions in Sect. 2. Given \(t > 1\), one says that a set \(A \subset X\) is \(t\)-porous if for every \(x \in A\) there exist arbitrarily small radii \(r > 0\) such that...
\[ A \cap (B(x, tr) \setminus B(x, r)) = \emptyset. \]

In [21, Theorem A], compact \( t \)-porous sets for sufficiently large \( t \) were shown to be removable for the \((1, p)\)-Poincaré inequality, \( 1 < p < \infty \). In [3] and [14], suitable porosity conditions were also shown to be sufficient for removability for quasiconformal mappings. The above \( t \)-porosity condition is a very concrete, but also quite strong requirement, as very large annuli are required to have empty intersection with the set \( A \).

In the current paper, we give the following removability result for Newton–Sobolev functions \( N^{1,p} \), which are an extension of Sobolev functions to metric spaces. In this result, porosity is replaced by a capacitary density condition, which is also a very concrete, but much less restrictive assumption.

**Theorem 1.1** Suppose \((X, d, \mu)\) is a complete metric space equipped with the doubling measure \( \mu \) and supporting a \((1, 1)\)-Poincaré inequality. Suppose \( A \subset X \) is such that

\[
\liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} < c_\ast \quad \text{for } 1\text{-q.e. } x \in X
\]

and for a constant \( c_\ast > 0 \) depending only on the doubling constant and the constants in the Poincaré inequality. Then \( A \) is removable for \( N^{1,p}(X) \) for all \( 1 \leq p < \infty \), and \( X \setminus A \) supports a \((1, 1)\)-Poincaré inequality.

We will observe that if \( A \) satisfies the aforementioned \( t \)-porosity with large enough \( t \), then it also satisfies (1.2), so in particular Theorem 1.1 essentially extends the results of [21] to the case \( p = 1 \). But of course (1.2) is satisfied also by many non-porous sets. We also prove a version of Theorem 1.1 for the more general class of functions of bounded variation (BV) in Corollary 4.10. In fact, the proof of Theorem 1.1 also relies on BV theory and specifically on a new Federer-type characterization of sets of finite perimeter proved in [23].

### 2 Notation and definitions

In this section we introduce the basic notation, definitions, and assumptions that are employed in the paper.

#### 2.1 Measure theory and Newton–Sobolev functions

Throughout this paper, \((X, d, \mu)\) is a complete metric space that is equipped with a metric \( d \) and a Borel regular outer measure \( \mu \) that satisfies a doubling property, meaning that there exists a constant \( C_d \geq 1 \) such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]

for every ball \( B(x, r) := \{ y \in X : d(y, x) < r \} \), with \( x \in X \) and \( r > 0 \). Since some conditions that we consider hold only for balls with radius less than \( \text{diam } X \), for simplicity we assume that \( \text{diam } X > 0 \), that is, \( X \) consists of at least 2 points. When a property holds outside a set of \( \mu \)-measure zero, we say that it holds at almost every (a.e.) point. By a measurable set we always mean a \( \mu \)-measurable set. We say that \( X \), or \( \mu \), is Ahlfors \( Q \)-regular with \( Q > 1 \) if there exists a constant \( C_A \geq 1 \) such that for all \( x \in X \) and \( 0 < r < 2 \text{diam } X \), we have

\[
\frac{1}{C_A} r^Q \leq \mu(B(x, r)) \leq C_A r^Q.
\]
We will never assume Ahlfors $Q$-regularity in this paper, but we give the definition for reference, since it is assumed in certain results in the literature.

All functions defined on $X$ or its subsets will take values in $[-\infty, \infty]$. A function $u$ defined on a measurable set $H \subset X$ is said to be in the class $L^1_{\text{loc}}(H)$ if for every $x \in H$ there exists $r > 0$ such that $u \in L^1(B(x, r) \cap H)$. Other local spaces of functions are defined analogously.

By a curve we mean a rectifiable continuous mapping from a compact interval of the real line into $X$. The length of a curve $\gamma$ is denoted by $\ell_\gamma$. We will assume every curve $\gamma$ to be parametrized by arc-length (see e.g. [8, Theorem 3.2]), so that the curve is $\gamma : [0, \ell_\gamma] \to X$. A nonnegative Borel function $g$ on $X$ is said to be an upper gradient of a function $u$ in a set $H \subset X$ if for all nonconstant curves $\gamma$ in $H$, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds := \int_0^{\ell_\gamma} g(\gamma(s)) \, ds,$$

where $x$ and $y$ are the end points of $\gamma$. We interpret $\infty - \infty = \infty$ and $-\infty - (-\infty) = -\infty$. We also express (2.2) by saying that the pair $(u, g)$ satisfies the upper gradient inequality on the curve $\gamma$. Upper gradients were originally introduced in [12].

We always consider $1 \leq p < \infty$, with a heavy focus on the case $p = 1$. The $p$-modulus of a family of curves $\Gamma$ is defined by

$$\text{Mod}_p(\Gamma) := \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho$ on $X$ such that $\int_\gamma \rho \, ds \geq 1$ for every curve $\gamma \in \Gamma$. A property is said to hold for $p$-almost every curve if it fails only for a curve family with zero $p$-modulus. If $g$ is a nonnegative $\mu$-measurable function on a $\mu$-measurable set $H \subset X$ and the pair $(u, g)$ satisfies the upper gradient inequality on $p$-a.e. curve $\gamma$ in $H$, then we say that $g$ is a $p$-weak upper gradient of $u$ in $H$.

Denote the characteristic function of a set $A \subset X$ by $\chi_A$. If $N \subset X$ with $\mu(N) = 0$, then by using the Borel function $\rho := \infty \chi_{N'}$ for a Borel set $N' \supset N$ with $\mu(N') = 0$, we find that

$$\text{Mod}_p(\{\gamma : [0, \ell_\gamma] \to X : L^1(\gamma^{-1}(N)) > 0\}) = 0.$$

Given a $\mu$-measurable set $H \subset X$, we let

$$\|u\|_{N^1, p(H)} := \|u\|_{L^p(H)} + \inf \|g\|_{L^p(H)},$$

where the infimum is taken over all $p$-weak upper gradients $g$ of $u$ in $H$. Then we define the Newton–Sobolev space

$$N^{1, p}(H) := \{u : \|u\|_{N^1, p(H)} < \infty\}.$$

This space was first introduced in [32]. If $H$ is an open subset of $\mathbb{R}^n$, then $u \in L^p(H)$ is in the classical Sobolev space $W^{1, p}(H)$ if and only if a suitable pointwise representative of $u$ is in $N^{1, p}(H)$, and then the quantity $\|u\|_{N^{1, p}(H)}$ agrees with the classical Sobolev norm, see [4, Theorem A.2, Corollary A.4]. We understand Newton–Sobolev functions to be defined at every $x \in H$, because this is needed for upper gradients to make sense. It is known that for every $u \in N^{1, p}_{\text{loc}}(H)$ there exists a minimal $p$-weak upper gradient $g_u$ of $u$ in $H$, satisfying $g_u \leq g$ a.e. in $H$ for every $p$-weak upper gradient $g \in L^p_{\text{loc}}(H)$ of $u$ in $H$, see [4, Theorem 2.25]. Usually we use the notation $g_u$, but since there can also be a dependence on the set $H$, we sometimes use the notation $g_{u, H}$.
For truncations $u_M := \max\{-M, \min\{M, u\}\}$, we clearly have

$$g_{u_M} \leq g_u \text{ a.e.}$$  \hspace{1cm} (2.4)

For two functions $u, v \in N^{1,p}_{\text{loc}}(H) \cap L^\infty(H)$, for the minimal $p$-weak upper gradients we have the Leibniz rule

$$g_{uv} \leq |u|g_v + |v|g_u \text{ a.e. in } H;$$  \hspace{1cm} (2.5)

see [4, Theorem 2.15].

For any open set $\Omega \subset X$, the space of Newton–Sobolev functions with zero boundary values is defined by

$$N^{1,p}_0(\Omega) := \{u|_\Omega : u \in N^{1,p}(X) \text{ with } u = 0 \text{ in } X \setminus \Omega\}.$$  

This space can be understood to be a subspace of $N^{1,p}(X)$.

For any set $A \subset X$ and $0 < R < \infty$, the Hausdorff content of codimension one is defined by

$$H^R(\Omega) := \inf \left\{ \sum_j \frac{\mu(B(x_j, r_j))}{r_j} : A \subset \bigcup_j B(x_j, r_j), r_j \leq R \right\},$$  \hspace{1cm} (2.6)

where we take the infimum over finite and countable coverings. We also define this for $R = \infty$, and there we only require $r_j < \infty$. The codimension one Hausdorff measure of $A \subset X$ is then defined by

$$H(A) := \lim_{R \to 0} H^R(A).$$

We will assume throughout the paper that $X$ supports a $(1, 1)$-Poincaré inequality, meaning that there exist constants $C_P \geq 1$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1(X)$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x,r)} |u-u_{B(x,r)}| \, d\mu \leq C_P r \int_{B(x,\lambda r)} g \, d\mu,$$  \hspace{1cm} (2.7)

where

$$u_{B(x,r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$  

Given a set $A \subset X$ with $\mu(A) = 0$, we say that the space $X \setminus A = (X \setminus A, d, \mu)$ supports a $(1, p)$-Poincaré inequality if there exist constants $C'_p \geq 1$ and $\lambda' \geq 1$ such that for every $x \in X \setminus A$, $r > 0$, $u \in L^1(X)$, and every upper gradient $g$ of $u$ in $X \setminus A$, we have

$$\int_{B(x,r)} |u-u_{B(x,r)}| \, d\mu \leq C'_p r \left( \int_{B(x,\lambda' r)} g^p \, d\mu \right)^{1/p}.$$  \hspace{1cm} (2.8)

Note that by Hölder’s inequality, the $(1, 1)$-Poincaré implies the $(1, p)$-Poincaré inequality for every $1 < p < \infty$.

### 2.2 Functions of bounded variation

Next we introduce functions of bounded variation on metric spaces, following Miranda Jr. [29]. Given an open set $\Omega \subset X$ and a function $u \in L^1_{\text{loc}}(\Omega)$, we define the total variation of

$$|u|_\Omega := \inf \left\{ \int_{\Omega} |g_u| : g_u \text{ a.e. in } \Omega \right\},$$  \hspace{1cm} (2.2.1)

where $g_u$ is an upper gradient of $u$ in $\Omega$. The variation $|u|_\Omega$ is finite if $u$ is of bounded variation in $\Omega$. The space of functions of bounded variation in $\Omega$ is denoted by $BV(\Omega)$.

Given a set $A \subset X$ with $\mu(A) = 0$, we say that the space $X \setminus A = (X \setminus A, d, \mu)$ supports a $(1, p)$-variation inequality if there exist constants $C'_v \geq 1$ and $\lambda'_v \geq 1$ such that for every $x \in X \setminus A$, $r > 0$, $u \in L^1(X)$, and every upper gradient $g$ of $u$ in $X \setminus A$, we have

$$\int_{B(x,r)} |u-u_{B(x,r)}| \, d\mu \leq C'_v r \left( \int_{B(x,\lambda'_v r)} g^p \, d\mu \right)^{1/p}.$$  \hspace{1cm} (2.2.2)

Note that by Hölder’s inequality, the $(1, 1)$-variation implies the $(1, p)$-variation inequality for every $1 < p < \infty$. 
u in $\Omega$ by

$$\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_\Omega g_{ui} \, d\mu : u_i \in N_{loc}^{1,1}(\Omega), u_i \to u \text{ in } L_{loc}^1(\Omega) \right\},$$

where each $g_{ui}$ is the minimal 1-weak upper gradient of $u_i$ in $\Omega$. We say that a function $u \in L^1(\Omega)$ is of bounded variation, and denote $u \in BV(\Omega)$, if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf\{\|Du\|(W) : A \subset W, W \subset X \text{ is open}\}.$$

In [29], pointwise Lipschitz constants were used in place of weak upper gradients, but the theory can be developed similarly with either definition. In the literature, it is sometimes also required that $u_i \in \text{Lip}_{loc}(\Omega)$ instead of $u_i \in N_{loc}^{1,1}(\Omega)$, but for us this does not make a difference, since functions in $N_{loc}^{1,1}(\Omega)$ can be approximated by functions in $\text{Lip}_{loc}(\Omega)$ in the $\| \cdot \|_{N_{loc}^{1,1}(\Omega)}$-seminorm, see [4, Theorem 5.47].

If $u \in L_{loc}^1(\Omega)$ and $\|Du\|(\Omega) < \infty$, then $\|Du\|$ is a Borel regular outer measure on $\Omega$ by [29, Theorem 3.4]. A $\mu$-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where $\chi_E$ is the characteristic function of $E$. The perimeter of $E$ in $\Omega$ is also denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

Applying the Poincaré inequality (2.7) to sequences of approximating $N_{loc}^{1,1}$-functions in the definition of the total variation, we get the following BV version: for every ball $B(x, r)$ and every $u \in L_{loc}^1(X)$, we have

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_{pr} \|Du\|(B(x, \lambda r)).$$

For a $\mu$-measurable set $E \subset X$, by considering the two cases $(\chi_E)_{B(x, r)} \leq 1/2$ and $(\chi_E)_{B(x, r)} \geq 1/2$, from the above we get the relative isoperimetric inequality

$$\min\{\mu(B(x, r) \cap E), \mu(B(x, r) \setminus E)\} \leq 2C_{pr} P(E, B(x, \lambda r)). \quad (2.9)$$

We define the lower and upper densities of a set $E \subset X$ at a point $x \in X$ as follows:

$$\theta_s(E, x) := \liminf_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \quad \text{and} \quad \theta_u(E, x) := \limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))}.$$

The measure-theoretic interior of $E \subset X$ is defined by

$$I_E := \left\{ x \in X : \theta_u(X \setminus E, x) = 0 \right\}, \quad (2.10)$$

and the measure-theoretic exterior by

$$O_E := \left\{ x \in X : \theta_s(E, x) = 0 \right\}.\quad (2.11)$$

The measure-theoretic boundary is defined as

$$\partial^s E := \left\{ x \in X : \theta_u(E, x) > 0 \text{ and } \theta_u(X \setminus E, x) > 0 \right\}.$$

It is straightforward to show that these are all Borel sets; note also that the space $X$ is always partitioned into the disjoint sets $I_E, O_E,$ and $\partial^s E$. We also let

$$E_b := \{ x \in X : \theta_s(E, x) \geq b \}, \quad b > 0.$$

The strong boundary $\Sigma_b E$, for $0 < b \leq 1/2$, is defined as $\Sigma_b E := E_b \cap (X \setminus E)_b$. 

\[ \text{Springer} \]
For an open set \( \Omega \subset X \) and a \( \mu \)-measurable set \( E \subset X \) with \( P(E, \Omega) < \infty \), we know that
\[
\mathcal{H}((\partial^* E \setminus \Sigma_\gamma E) \cap \Omega) = 0
\]  
(2.12)
for a number \( \gamma = \gamma(C_d, C_P, \lambda) > 0 \), see [1, Theorem 5.3, Theorem 5.4]. We then also know that for any Borel set \( A \subset \Omega \),
\[
P(E, A) = \int_{\partial^* E \cap A} \theta_E d\mathcal{H} = \int_{\Sigma_\gamma E \cap A} \theta_E d\mathcal{H},
\]  
(2.13)
where \( \theta_E : \Omega \to [\alpha, C_d] \) with \( \alpha = \alpha(C_d, C_P, \lambda) > 0 \), see [1, Theorem 5.3] and [2, Theorem 4.6]. In particular, \( P(E, \Omega) < \infty \) implies that \( \mathcal{H}(\partial^* E \cap \Omega) < \infty \). Federer’s characterization of sets of finite perimeter states that the converse is also true. That is, if \( E \subset X \) is a \( \mu \)-measurable set such that \( \mathcal{H}(\partial^* E \cap \Omega) < \infty \), then \( P(E, \Omega) < \infty \), see [25, Theorem 1.1]. See also Federer [7, Section 4.5.11] for the original Euclidean result.

The strong boundary can also be used to characterize sets of finite perimeter, as follows.

**Theorem 2.14** ([23, Theorem 1.1]) Let \( \Omega \subset X \) be an open set and let \( E \subset X \) be a \( \mu \)-measurable set with \( \mathcal{H}(\Sigma_\beta E \cap \Omega) < \infty \), where \( 0 < \beta \leq 1/2 \) only depends on the doubling constant of the measure and the constants in the Poincaré inequality. Then \( P(E, \Omega) < \infty \).

Throughout this paper, we will use \( \beta \) to denote the constant from this theorem; we can assume that \( \beta \leq \gamma \). Combining Theorem 2.14 and (2.13), we obtain that for every open \( \Omega \subset X \) and \( \mu \)-measurable \( E \subset X \), we have
\[
P(E, \Omega) \leq C_d \mathcal{H}(\Sigma_\beta E \cap \Omega).
\]  
(2.15)

For a function \( u \) defined on an open set \( \Omega \subset X \), we abbreviate super-level sets by
\[
\{u > t\} := \{x \in \Omega : u(x) > t\}, \quad t \in \mathbb{R}.
\]
The following coarea formula is given in [29, Proposition 4.2]: if \( \Omega \subset X \) is open and \( u \in L^1_{\text{loc}}(\Omega) \), then
\[
\|Du\|_{\Omega} = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt.
\]  
(2.16)
The integral should be understood as an upper integral; however if either side is finite, then both sides are finite and the integrand is measurable. In this case, (2.16) moreover holds with \( \Omega \) replaced by any Borel set \( A \subset \Omega \).

### 2.3 Capacities

Recall that we always consider \( 1 \leq p < \infty \). The \( p \)-capacity of a set \( A \subset X \) is defined by
\[
\text{Cap}_p(A) := \inf \|u\|^p_{N^{1,p}(X)},
\]
where the infimum is taken over all \( u \in N^{1,p}(X) \) satisfying \( u \geq 1 \) in \( A \). If a property holds outside a set of \( p \)-capacity zero, we say that it holds \( p \)-quasieverywhere, abbreviated \( p \)-q.e.

We say that a set \( U \subset X \) is \( p \)-quasiopen if for every \( \epsilon > 0 \) there exists an open set \( G \subset X \) such that \( \text{Cap}_p(G) < \epsilon \) and \( U \cup G \) is open. By [31, Remark 3.5], if \( U \subset X \) is \( p \)-quasiopen, then
\[
\text{for } p\text{-a.e. curve } \gamma : [0, \ell_\gamma] \to X \text{ we have that } \gamma^{-1}(U) \text{ is relatively open.}
\]  
(2.17)
Denoting the family of curves $\gamma : [0, \ell_{\gamma}] \to X$ intersecting a set $A \subset X$ by $\Gamma_{A}$, by [4, Proposition 1.48] we know that
\begin{equation}
\text{if } \text{Cap}_{p}(A) = 0, \quad \text{then } \text{Mod}_{p}(\Gamma_{A}) = 0. \tag{2.18}
\end{equation}

The variational $p$-capacity of a set $A \subset W$ with respect to a bounded open set $W \subset X$ is defined by
\begin{equation}
\text{cap}_{p}(A, W) := \inf_{u \in N_{1}^{1, p}(W)} \int_{X} g_{u}^{p} d\mu,
\end{equation}
where the infimum is taken over all $u \in N_{1}^{1, p}(W)$ satisfying $u \geq 1$ in $A$, and $g_{u}$ is the minimal $p$-weak upper gradient of $u$ (in $X$). This is an outer capacity, meaning that whenever $A \subset A' \subset W$, we have
\begin{equation}
\text{cap}_{p}(A, W) = \inf_{V \text{ open}} \text{cap}_{p}(V, W),
\end{equation}
see [4, Theorem 6.19(vii)]. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see [4].

If $H \subset X$ is measurable, then
\begin{equation}
\text{if } u \in N_{1}^{1, p}(H) \text{ and } v = u \text{ p-q.e. in } H, \quad \text{then } \|v - u\|_{N_{1}^{1, p}(H)} = 0, \tag{2.19}
\end{equation}
see [4, Proposition 1.61]. We also know that
\begin{equation}
\text{if } u, v \in N_{1}^{1, p}(H) \text{ and } v = u \text{ a.e. in } H, \quad \text{then } g_{v} = g_{u} \text{ a.e. in } H, \tag{2.20}
\end{equation}
by [4, Corollary 1.49, Proposition 1.59].

For an open set $\Omega \subset X$ and $u \in N_{1}^{1, \frac{1}{2}}(\Omega)$, we know that 1-q.e. point is a Lebesgue point, that is,
\begin{equation}
\lim_{r \to 0} \int_{B(x, r)} |u - u(x)| d\mu = 0 \quad \text{for 1-q.e. } x \in \Omega; \tag{2.21}
\end{equation}
see [15, Theorem 4.1] (in this paper $\mu(X) = \infty$ is assumed, but this can be circumvented by [6, Remark 4.12]).

We will often consider the so-called precise representative
\begin{equation}
u^{*}(x) := \limsup_{r \to 0} \int_{B(x, r)} u d\mu. \tag{2.22}\end{equation}
If $u \in \text{BV}(\Omega)$ and $\|Du\|$ is absolutely continuous with respect to $\mu$, then
\begin{equation}u^{*} \in N_{1}^{1, \frac{1}{2}}(\Omega) \quad \text{with} \quad \int_{\Omega} g_{u}^{*} d\mu \leq C \|Du\|(\Omega) \tag{2.23}\end{equation}
for a constant $C \geq 1$ depending only on $C_{d}, C_{p}, \lambda$; by [10, Remark 4.7] we know that this is true for some representative of $u$, and then by (2.19) and (2.21) we know that the representative can be taken to be $u^{*}$.

For every $x \in X$ and $0 < r < \frac{1}{8} \text{diam } X$, by [4, Proposition 6.16] we have
\begin{equation}C_{\text{sob}}^{-1} \frac{\mu(B(x, r))}{r^{p}} \leq \text{cap}_{p}(B(x, r), B(x, 2r)) \leq C_{d} \frac{\mu(B(x, r))}{r^{p}}, \tag{2.24}\end{equation}
where $C_{\text{sob}} \geq 1$ is a constant from a Sobolev inequality, only depending on $C_d, C_p, \lambda$. By [4, Proposition 6.16], for every $A \subset B(x, r)$ we then also have

$$\frac{\mu(A)}{\mu(B(x, r))} \leq C_{\text{sob}} r^p \frac{\text{cap}_p(A, B(x, 2r))}{\mu(B(x, r))} \leq C_{\text{sob}} C_d \frac{\text{cap}_p(A, B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} \quad \text{by (2.24).}$$

(2.25)

Next we define the fine topology in the case $p = 1$. For the analogous definition and theory in the case $1 < p < \infty$, see e.g. the monographs [4, 28].

**Definition 2.26** We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} = 0.$$  

We say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on $X$.

We denote the 1-fine closure of a set $H \subset X$, i.e. the smallest 1-finely closed set containing $H$, by $\overline{H}$. The 1-base $b_1$ is defined as the set of points where $H$ is not 1-thin.

See [24, Section 4] for discussion on this definition, and for a proof of the fact that the 1-fine topology is indeed a topology. Using (2.24), we see that a set $A \subset X$ is 1-thin at $x \in X$ if and only if

$$\lim_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0;$$

in fact this was the formulation used in [24]. From (2.25) we get that if $U \subset X$ is a 1-finely open set, then

for every $x \in U$ we have $\theta^*(X \setminus U, x) = 0$. \hfill (2.27)

By [27, Theorem 4.6], we know that

$$\text{Cap}_1(A \setminus b_1A) = 0.$$ \hfill (2.28)

By [9, Theorems 4.3 and 5.1], for an arbitrary set $A \subset X$ we have

$$\text{Cap}_1(A) = 0 \quad \text{if and only if} \quad \mathcal{H}(A) = 0.$$ \hfill (2.29)

By [27, Corollary 6.12] combined with (2.29) we know that for an arbitrary set $U \subset X$, $U$ is 1-quasiopen $\iff U = V \cup N$ where $V$ is 1-finely open and $\text{Cap}_1(N) = 0$. \hfill (2.30)

By [5, Lemma 9.3] we then know that every 1-quasiopen and every 1-finely open set is measurable. Moreover, by combining (2.30) with (2.17) and [5, Proposition 3.5], we have that if $H \subset X$ is $\mu$-measurable and $U \subset X$ is 1-finely open, and $u \in N_{\text{loc}}^{1,1}(H)$, then for the minimal 1-weak upper gradients we have

$$g_{u, H} = g_{u, H \cap U} \quad \text{a.e. in } H \cap U.$$ \hfill (2.31)

By [22, Proposition 3.3], we know that if $A \subset W$ for an open set $W \subset X$, then

$$\text{Cap}_1(A) = \text{Cap}_1(A^1) \quad \text{and} \quad \text{cap}_1(A, W) = \text{cap}_1(A^1 \cap W, W).$$ \hfill (2.32)

Our standing assumptions throughout the paper will be the following.

**Throughout this paper we assume that $(X, d, \mu)$ is a complete metric space that is equipped with the doubling measure $\mu$ and supports a $(1, 1)$-Poincaré inequality.**
3 Preliminary results

In this section we prove and record some preliminary results.

The following lemma is fairly standard, though we do not know a specific reference.

**Lemma 3.1** Let \( v \in L^1(X) \) be a pointwise defined function and let \( D_j \subset X \), \( j \in \mathbb{N} \), be measurable sets such that \( \text{Cap}_1(D_j) \to 0 \), and suppose that \( v \in N^{1,1}(X \setminus D_j) \) with

\[
\int_{X \setminus D_j} g_{v,X \setminus D_j} \, d\mu \leq M < \infty \quad \text{for all } j \in \mathbb{N}.
\]

Then \( v \in N^{1,1}(X) \) with \( \int_X g_v \, d\mu \leq M \).

Recall that \( g_{v,X \setminus D_j} \) denotes the minimal 1-weak upper gradient of \( v \) in \( X \setminus D_j \).

**Proof** By passing to a subsequence (not relabelled), we can assume that \( \text{Cap}_1(D_j) \leq 2^{-j} \). Then by replacing the sets \( D_j \) with \( \bigcup_{k=j}^{\infty} D_k \), we can assume that \( D_{j+1} \subset D_j \). From now on, we understand each \( g_{v,X \setminus D_j} \) to be pointwise defined. From the definition of minimal 1-weak upper gradients, we get \( g_{v,X \setminus D_j} \leq g_{v,X \setminus D_{j+1}} \) a.e. in \( X \setminus D_j \). We define pointwise \( g := \limsup_{j \to \infty} g_{v,X \setminus D_j} \) in \( X \setminus \bigcap_{j=1}^{\infty} D_j \), and then \( \int_X g \, d\mu \leq M \) by (3.2); note that \( \text{Cap}_1 \left( \bigcap_{j=1}^{\infty} D_j \right) = 0 \) and thus also \( \mu \left( \bigcap_{j=1}^{\infty} D_j \right) = 0 \). Moreover, if we denote by \( \Gamma \) the family of curves intersecting \( \bigcap_{j=1}^{\infty} D_j \), we have \( \text{Mod}_1(\Gamma) = 0 \) by (2.18). Note also that \( g \geq g_{v,X \setminus D_j} \) a.e. in \( X \setminus D_j \), for all \( j \in \mathbb{N} \). The family of curves \( \gamma : [0, \ell_{\gamma}] \to X \setminus D_j \) for which \( (v, g_{v,X \setminus D_j}) \) does not satisfy the upper gradient inequality on \( \gamma \), and the family of curves \( \gamma : [0, \ell_{\gamma}] \to X \setminus D_j \) for which

\[
\mathcal{L}^1(\gamma^{-1}((g < g_{v,X \setminus D_j}))) > 0,
\]

are \( \text{Mod}_1 \)-negligible for all \( j \in \mathbb{N} \), by (2.3). Consider a curve \( \gamma : [0, \ell_{\gamma}] \to X \setminus \bigcap_{j=1}^{\infty} D_j \) outside these exceptional families. Since \( D_j \) is a decreasing sequence, \( \gamma \) is in \( X \setminus D_j \) for some \( j \in \mathbb{N} \). Hence we get

\[
|v(\gamma(0)) - v(\gamma(\ell_{\gamma}))| \leq \int_\gamma g_{v,X \setminus D_j} \, ds \leq \int_\gamma g \, ds.
\]

Thus \( g \) is a 1-weak upper gradient of \( v \) in \( X \), and we have the result. \( \square \)

The following lemma is given in [26, Lemma 3.3].

**Lemma 3.3** Let \( G \subset X \) with \( \text{Cap}_1(G) < \infty \) and let \( \varepsilon > 0 \). Then there exists an open set \( W \supset G \) with \( \text{Cap}_1(W) \leq C(\text{Cap}_1(G) + \varepsilon) \) and a function \( \eta \in N_0^{1,1}(W) \) with \( 0 \leq \eta \leq 1 \) on \( X \), \( \eta = 1 \) in \( G \), and \( \|\eta\|_{N_0^{1,1}(X)} \leq C(\text{Cap}_1(G) + \varepsilon) \), for a constant \( C \) depending only on \( C_d, C_F, \lambda \).

Next we record an isoperimetric inequality. By [4, Corollary 3.8], there exist constants \( C > 0 \) and \( \sigma > 0 \) depending only on \( C_d \) such that for all \( x \in X \) and \( 0 < r < R < \text{diam} \, X/2 \), we have

\[
\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq C \left( \frac{r}{R} \right)^\sigma.
\]

(3.4)

Thus there exists a constant \( L > 2 \) depending only on \( C_d \) such that for every \( x \in X \) and every \( 0 < r < \text{diam} \, X/(2L) \), we have

\[
\mu(B(x, 2r)) \leq \frac{1}{2} \mu(B(x, Lr)).
\]
Then for any measurable set $H \subset B(x, 2r)$, by the relative isoperimetric inequality (2.9) we have
\[
\mu(H) = \mu(H \cap B(x, Lr)) \\
\leq 2C_P Lr P(H, B(x, L\lambda r)) \\
\leq 2C_P Lr P(H, X).
\] (3.5)

Next we note that the following consequence of the Borel regularity of $\mu$ holds. This can be proved similarly as in e.g. [27, Lemma 4.3].

**Lemma 3.6** Let $A \subset X$. Then there exists a Borel set $A^* \supset A$ such that
\[
\mu(A \cap B(x, r)) = \mu(A^* \cap B(x, r))
\]
for every ball $B(x, r) \subset X$.

Denote by $\lceil b \rceil$ the smallest integer at least $b \in \mathbb{R}$. Let
\[
c_* := \frac{\alpha \beta}{8C_d^{2+\lceil \log_2\lambda \rceil}C_P C_{sob} L},
\] (3.7)
where $\alpha$ is defined after (2.13) and can be assumed to take values in $(0, 1)$, $0 < \beta \leq 1/2$ is from Theorem 2.14, and $L \geq 2$ is from before (3.5). Recall that we also assume $C_P, C_{sob} \geq 1$ (for convenience). The number $0 < c_* < 1$ is the constant used in our main Theorem 1.1.

The removable sets that we consider are expected to have zero $\mu$-measure. For this reason, the following basic fact is useful. Recall that if a property holds outside a set of 1-capacity zero, we say that it holds at 1-quasi every point, abbreviated “1-q.e.”

**Lemma 3.8** Suppose $A \subset X$ is such that
\[
\liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} < c_* \text{ for 1-q.e. } x \in X.
\]
Then $\mu(A) = 0$.

**Proof** By (2.25), we have
\[
\liminf_{r \to 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} < C_{sob} C_d c_* = \frac{\alpha \beta}{8C_d^{1+\lceil \log_2\lambda \rceil}C_P L} < 1 \text{ for 1-q.e. } x \in X.
\]
By Lemma 3.6, we find a Borel set $A^* \supset A$ such that
\[
\mu(A \cap B(x, r)) = \mu(A^* \cap B(x, r))
\]
for every ball $B(x, r) \subset X$. Thus in fact
\[
\liminf_{r \to 0} \frac{\mu(A^* \cap B(x, r))}{\mu(B(x, r))} < 1
\]
for 1-q.e. $x \in X$, and then also for a.e. $x \in X$. By the Lebesgue differentiation theorem, see e.g. [13, p. 77], it follows that necessarily $\mu(A^*) = 0$ and then also $\mu(A) = 0$. \qed

The following standard fact can be proved as in e.g. [16, Lemma 2.6]; note that when $D \subset X$ is Borel, the restriction
\[
\mathcal{H}[D(A) := \mathcal{H}(A \cap D), \quad A \subset X,
\]
is a Borel regular outer measure by [13, Lemma 3.3.13].
Lemma 3.9 Let $D \subset X$ be a Borel set with $\mathcal{H}(D) < \infty$. Then for $\mathcal{H}$-a.e. $x \in X \setminus D$, we have
\[
\lim_{r \to 0} \frac{\mathcal{H}(D \cap B(x, r))}{\mu(B(x, r))} = 0.
\]

\section{4 Proof of Theorem 1.1}

In this section we prove the removability Theorem 1.1, as well as a version for BV functions.

We define removable sets for Newton–Sobolev and BV functions as follows. As usual, we consider $1 \leq p < \infty$.

\textbf{Definition 4.1} Let $A \subset X$ such that $\mu(A) = 0$. We say that $A$ is removable for $N^{1,p}(X)$ if for every $u \in N^{1,p}(X \setminus A)$ there exists $v \in N^{1,p}(X)$ such that $v = u$ a.e. in $X$. We say that $A$ is isometrically removable for $N^{1,p}(X)$ if moreover $g_v = g_u$ a.e. in $X$ for every such $v$.

Note that more precisely, above we denote the minimal $p$-weak upper gradient $g_u = g_u, X \setminus A$. If $g_v = g_u$ a.e. in $X$ for some $v$ as above, then it is in fact true for every such $v$, by (2.20). Note also that always $g_v \geq g_v, X \setminus A = g_u, X \setminus A$ a.e. by (2.20).

\textbf{Definition 4.2} Let $A \subset X$ be closed such that $\mu(A) = 0$. We say that $A$ is removable for $\text{BV}(X)$ if for every $u \in \text{BV}(X \setminus A)$ we have $u \in \text{BV}(X)$. We say that $A$ is isometrically removable for $\text{BV}(X)$ if moreover $\|Du\|(X) = \|Du\|(X \setminus A)$.

Note that we understand BV functions to be defined only almost everywhere, and so it is not necessary to talk about an extension $v$ here. Moreover, note that we assume $A$ to be closed, because BV functions are defined only in open sets.

The cornerstone of our removability results is the following proposition, which essentially says that $A$ is removable for sets of finite perimeter.

\textbf{Proposition 4.3} Let $A \subset X$ be a closed set such that
\[
\liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} < c_* \quad \text{for } I\text{-q.e. } x \in A. \tag{4.4}
\]

Let $E \subset X$ be a $\mu$-measurable set with $\mathcal{H}(\partial^* E \setminus A) < \infty$. Then $\mathcal{H}(\Sigma_\beta E \cap A) = 0$.

\textbf{Proof} The inequality in (4.4) holds outside a set $N \subset A$ with $\text{Cap}_1(N) = 0$, and thus also $\mathcal{H}(N) = 0$ by (2.29). Fix $x \in \Sigma_\beta E \cap A \setminus N$ (assuming it exists). There is a sequence $r_j \searrow 0$ such that
\[
\limsup_{j \to \infty} \frac{\text{cap}_1(A \cap B(x, r_j), B(x, 2r_j))}{\text{cap}_1(B(x, r_j), B(x, 2r_j))} < c_* \quad (4.5)
\]

By the definition of the variational $1$-capacity, and recalling that it is an outer capacity, for every $r > 0$ we find a function $v_r \in N^{1,1}_r(B(x, 2r))$ such that $v_r \geq 1$ in a neighborhood of $A \cap B(x, r)$, and
\[
\int_X g_{v_r} \, d\mu \leq \text{cap}_1(A \cap B(x, r), B(x, 2r)) + \mu(B(x, r)).
\]

Here $v_r \in N^{1,1}(X) \subset \text{BV}(X)$ with $\|Dv_r\|(X) \leq \int_X g_{v_r} \, d\mu$, and then by the coarea formula (2.16) we find a set $A(x, r) := \{v_r > t\}$ for some $0 < t < 1$, for which
\[
P(A(x, r), X) \leq \|Dv_r\|(X) \leq \text{cap}_1(A \cap B(x, r), B(x, 2r)) + \mu(B(x, r)), \quad (4.6)
\]
and also \( A(x, r) \subset B(x, 2r) \), and \( A \cap B(x, r) \) is contained in the interior of \( A(x, r) \). By (2.13),

\[
\limsup_{j \to \infty} \frac{\mathcal{H}(\partial^* A(x, r_j))}{\mu(B(x, r_j))} \leq \frac{1}{\alpha} \limsup_{j \to \infty} \frac{P(A(x, r_j), X)}{\mu(B(x, r_j))} \\
\leq \frac{1}{\alpha} \limsup_{j \to \infty} \frac{\text{cap}_1(A \cap B(x, r_j), B(x, 2r_j))}{\mu(B(x, r_j))} \quad \text{by (4.6)}
\]

\[
\leq C_d \limsup_{j \to \infty} \frac{\text{cap}_1(A \cap B(x, r_j), B(x, 2r_j))}{\text{cap}_1(B(x, r_j), B(x, 2r_j))} \quad \text{by (2.24)}
\]

\[
\leq C_d \frac{\beta}{\alpha} \quad \text{by (4.5)}
\]

\[
\leq \frac{\beta}{8C_P C_d^{[\log_2 \lambda]}} \quad \text{by (3.7)}.
\]

By the doubling property of \( \mu \),

\[
\limsup_{j \to \infty} \frac{\mu(A(x, r_j))}{\mu(B(x, r_j/\lambda))} \leq C_d^{[\log_2 \lambda]} \limsup_{j \to \infty} \frac{\mu(A(x, r_j))}{\mu(B(x, r_j))} \\
\leq 2C_P LC_d^{[\log_2 \lambda]} \limsup_{j \to \infty} \frac{P(A(x, r_j), X)}{\mu(B(x, r_j))} \quad \text{by (3.5)}
\]

\[
\leq 2C_P LC_d^{1+[\log_2 \lambda]} c_* \quad \text{by (4.7) middle 3 inequalities} \leq \beta/2 \quad \text{by (3.7)}.
\]

Combining this with the fact that \( x \in \Sigma_\beta E \), we get

\[
\liminf_{j \to \infty} \frac{\mu(B(x, r_j/\lambda) \setminus (E \cup A(x, r_j)))}{\mu(B(x, r_j))} \geq \frac{1}{C_d^{[\log_2 \lambda]}} \liminf_{j \to \infty} \frac{\mu(B(x, r_j/\lambda) \setminus (E \cup A(x, r_j)))}{\mu(B(x, r_j/\lambda))} \\
\geq \frac{\beta}{2C_d^{[\log_2 \lambda]}},
\]

and using again the fact that \( x \in \Sigma_\beta E \), we have in total

\[
\liminf_{j \to \infty} \frac{\mu(B(x, r_j/\lambda) \cap (E \cup A(x, r_j)))}{\mu(B(x, r_j))} \geq \frac{\beta}{C_d^{[\log_2 \lambda]}} \quad \text{and}
\]

\[
\liminf_{j \to \infty} \frac{\mu(B(x, r_j/\lambda) \setminus (E \cup A(x, r_j)))}{\mu(B(x, r_j))} \geq \frac{\beta}{2C_d^{[\log_2 \lambda]}}.
\]

Thus by (2.9) and (2.13),

\[
\liminf_{j \to \infty} \frac{\mathcal{H}(\partial^*(E \cup A(x, r_j)) \cap B(x, r_j))}{\mu(B(x, r_j))} \geq \frac{\beta}{4C_P C_d^{[\log_2 \lambda]}}.
\]

From the definition of the measure-theoretic interior and boundary (2.10), (2.11), it is straightforward to verify that

\[
\partial^*(E \cup A(x, r_j)) \cap B(x, r_j) \subset \left( \partial^* E \setminus I(x, r_j) \right) \cup \partial^* A(x, r_j) \cap B(x, r_j)
\]

\[
\subset \left( \partial^* E \cup A \right) \cup \partial^* A(x, r_j) \cap B(x, r_j).
\]
Thus
\[
\liminf_{j \to \infty} r_j \frac{\mathcal{H}((\partial^* E \setminus A) \cap B(x, r_j))}{\mu(B(x, r_j))} \\
\geq \liminf_{j \to \infty} r_j \frac{\mathcal{H}(\partial^* (E \cup A(x, r_j)) \cap B(x, r_j))}{\mu(B(x, r_j))} - \limsup_{j \to \infty} r_j \frac{\mathcal{H}(\partial^* A(x, r_j) \cap B(x, r_j))}{\mu(B(x, r_j))}
\geq \frac{\beta}{8C_P C_d [\log_2 \lambda]} > 0 \text{ by (4.7), (4.9).}
\]

Recall that we have considered an arbitrary point \( x \in \Sigma_\beta E \cap A \setminus N \). However, since \( \partial^* E \setminus A \) is a Borel set, by Lemma 3.9 we have that
\[
\lim_{r \to 0} r \frac{\mathcal{H}((\partial^* E \setminus A) \cap B(x, r))}{\mu(B(x, r))} = 0
\]
for \( \mathcal{H} \)-a.e. \( x \in X \setminus (\partial^* E \setminus A) \), in particular for \( \mathcal{H} \)-a.e. \( x \in A \). It follows that \( \mathcal{H}(A \cap \Sigma_\beta E \setminus N) = 0 \), and thus in fact \( \mathcal{H}(A \cap \Sigma_\beta E) = 0 \). \( \square \)

Now we can prove the following removability result for BV functions.

**Corollary 4.10** Suppose \( A \subset X \) is a closed set such that
\[
\liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} < c_* \quad \text{for } 1\text{-q.e. } x \in A.
\]
Then \( A \) is isometrically removable for BV(\( X \)).

**Proof** Let \( u \in \text{BV}(X \setminus A) \). By Lemma 3.8, we have \( \mu(A) = 0 \), and so \( \|u\|_{L^1(X)} = \|u\|_{L^1(X \setminus A)} \). Using the coarea formula (2.16), and noting that some integrals below are upper integrals because measurability is not clear, we estimate
\[
\|Du\|(X \setminus A) = \int_{-\infty}^\infty P([u > t], X \setminus A) \, dt
\geq \alpha \int_{(\infty, \infty)} \mathcal{H}(\partial^* [u > t] \cap (X \setminus A)) \, dt \quad \text{by (2.13)}
\geq \alpha \int_{(\infty, \infty)} \mathcal{H}(\Sigma_\beta [u > t]) \, dt \quad \text{by Proposition 4.3}
\geq \alpha C_d^{-1} \int_{(-\infty, \infty)} P([u > t], X) \, dt \quad \text{by (2.13)}
= \alpha C_d^{-1} \|Du\|\(X\).
\]
Thus \( u \in \text{BV}(X) \). Note also that from the above estimates, we obtain that \( \mathcal{H}(\partial^* [u > t] \setminus A) < \infty \) for a.e. \( t \in \mathbb{R} \). By again applying the coarea formula (2.16), we get
\[
\|Du\|(A) = \int_{-\infty}^\infty P([u > t], A) \, dt
\leq C_d \int_{-\infty}^\infty \mathcal{H}(\Sigma_\beta [u > t] \cap A) \, dt \quad \text{by (2.13)}
= 0
\]
by Proposition 4.3. Thus in fact \( \|Du\|\(X \setminus A\) = \|Du\|\(X\) \), meaning that \( A \) is isometrically removable for BV(\( X \)). \( \square \)
Remark 4.11  Note that here we assumed $A$ to be closed, which made the proof rather straightforward. As mentioned before, the class of BV functions is only defined in open sets, unlike Newton–Sobolev functions which are natural to define in any $\mu$-measurable set. One could consider suitable definitions of the BV class also in non-open sets, but we choose not to go in this direction here.

Now we prove the following theorem, which involves (an improved version of) the case $p = 1$ of our main Theorem 1.1.

Theorem 4.12  Suppose $A \subset X$ is such that
\[
\liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} < c_\ast \quad \text{for 1-q.e. } x \in X.
\]
Then $A$ isometrically removable for $N^{1,1}(X)$.

Proof  Note that $\mu(A) = 0$ by Lemma 3.8. First assume that $X \setminus A$ is 1-finely open.

Let $u \in N^{1,1}(X \setminus A)$. First suppose that $-M \leq u \leq M$ for some $M > 0$. Since $X \setminus A$ is 1-quasiopen by (2.30), we find an open set $G \subset X$ such that $(X \setminus A) \cup G$ is open and thus $A \setminus G$ is closed, and $\text{Cap}_1(G)$ can be chosen arbitrarily small. Let $\varepsilon > 0$. Using Lemma 3.3, in fact we also find an open set $U \supset G$ such that $\text{Cap}_1(U) < \varepsilon$, and a function $\eta \in N_0^{1,1}(U)$ such that $0 \leq \eta < 1$ on $X$, $\eta = 1$ in $G$, and $\|\eta\|_{N^{1,1}(X)} < \varepsilon$. Define
\[
v := (1 - \eta)u.
\]
We first observe that
\[
v \in N^{1,1}(X \setminus A) \quad \text{with} \quad g_v, X \setminus A \leq g_u, X \setminus A + Mg_{\eta, X} \tag{4.13}
\]
by the Leibniz rule (2.5); we understand these minimal 1-weak upper gradients to be pointwise defined. Consider a curve $\gamma : [0, \ell_\gamma] \to (X \setminus A) \cup G$; recall that the latter set is open. Excluding a $\text{Mod}_1$-negligible family, we can assume that the pair $(v, g_v, X \setminus A)$ satisfies the upper gradient inequality on all subcurves of $\gamma$ that are in $X \setminus A$; see [4, Lemma 1.34(c)]. By (2.17) and (2.30), excluding another $\text{Mod}_1$-negligible family, we can assume that $\gamma^{-1}(X \setminus A)$ and $\gamma^{-1}(G)$ are relatively open sets.

Now $[0, \ell_\gamma]$ is a compact set that is covered by the two relatively open sets $\gamma^{-1}(X \setminus A)$ and $\gamma^{-1}(G)$. By the Lebesgue number lemma, there exists a number $\delta > 0$ such that every subinterval of $[0, \ell_\gamma]$ with length at most $\delta$ is contained either in $\gamma^{-1}(X \setminus A)$ or in $\gamma^{-1}(G)$. Choose $m \in \mathbb{N}$ such that $\ell_\gamma/m \leq \delta$ and consider the subintervals $I_j := [j\ell_\gamma/m, (j + 1)\ell_\gamma/m], j = 0, \ldots, m - 1$. If $I_j \subset \gamma^{-1}(X \setminus A)$, then by our assumptions on $\gamma$,
\[
|v(j\ell_\gamma/m) - v((j + 1)\ell_\gamma/m)| \leq \int_{j\ell_\gamma/m}^{(j+1)\ell_\gamma/m} g_v, X \setminus A(\gamma(s)) \, ds.
\]
Otherwise $I_j \subset \gamma^{-1}(G)$. Recall that $v = 0$ in $G$. Then
\[
|v(j\ell_\gamma/m) - v((j + 1)\ell_\gamma/m)| = 0.
\]
Adding up the inequalities for $j = 0, \ldots, m - 1$, by (4.13) we conclude that the upper gradient inequality holds for the pair $(v, g_u, X \setminus A + Mg_{\eta, X})$ on the curve $\gamma$, that is,
\[
|v(\gamma(0)) - v(\gamma(\ell_\gamma))| \leq \int_\gamma (g_u, X \setminus A + Mg_{\eta, X}) \, ds.
\]
We conclude that $v \in N^{1,1}((X \setminus A) \cup G)$ with $g_v, (X \setminus A) \cup G \leq g_u, X \setminus A + M g_{\eta, X}$, and so (we drop the sets from the subscripts)

$$\int_{(X \setminus A) \cup G} g_v \, d\mu \leq \int_{X \setminus A} g_u \, d\mu + M \int_X g_{\eta} \, d\mu \leq \int_{X \setminus A} g_u \, d\mu + M \varepsilon. \quad (4.14)$$

Using the coarea formula (2.16), we estimate

$$\int_{(X \setminus A) \cup G} g_v \, d\mu \geq \|Dv\|((X \setminus A) \cup G)$$

$$= \int_{\infty}^{\infty} P([v > t], (X \setminus A) \cup G) \, dt$$

$$\geq \alpha \int_{(-\infty, \infty)} \mathcal{H}(d^*\{v > t\} \cap [(X \setminus A) \cup G]) \, dt \quad \text{by (2.13)}$$

$$\geq \alpha \int_{(-\infty, \infty)} \mathcal{H}(\Sigma_\beta\{v > t\}) \, dt \quad \text{by Proposition 4.3}$$

$$\geq \alpha C_d^{-1} \int_{(-\infty, \infty)} P([v > t], X) \, dt \quad \text{by (2.15)}$$

$$= \alpha C_d^{-1} \|Dv\|(X). \quad (4.15)$$

We obtain $v \in BV(X)$. Since $v \in N^{1,1}((X \setminus A) \cup G)$, obviously $\|Dv\|$ is absolutely continuous with respect to $\mu$ in the open set $(X \setminus A) \cup G$. But we also have by the coarea formula (2.16) and (2.13) that

$$\|Dv\|(A \setminus G) \leq C_d \int_{-\infty}^{\infty} \mathcal{H}(\Sigma_\beta\{v > t\} \cap (A \setminus G)) \, dt = 0$$

by Proposition 4.3, and so $\|Dv\|$ is absolutely continuous with respect to $\mu$ in the entire $X$. By (2.23), we get $v^* \in N^{1,1}(X)$ with

$$\|Dv\|(X) \geq \frac{1}{C} \int_X g_{v^*} \, d\mu \geq \frac{1}{C} \int_{X \setminus U^1} g_{v^*, X \setminus U^1} \, d\mu = \frac{1}{C} \int_{X \setminus U^1} g_{u^*, X \setminus U^1} \, d\mu,$$

where the last equality follows from the fact that $v^* = u^*$ everywhere in $X \setminus U^1$ by (2.27). Combining this with (4.15) and (4.14), we get

$$\int_{X \setminus A} g_u \, d\mu + M \varepsilon \geq \frac{\alpha C_d^{-1}}{C} \int_{X \setminus U^1} g_{u^*, X \setminus U^1} \, d\mu.$$

Note that by (2.21), $v$ has Lebesgue points 1-q.e. in the open set $(X \setminus A) \cup G$. Thus clearly

$$u^* = v^* = v = u \quad 1\text{-q.e. in } (X \setminus A) \setminus U^1.$$

Note that $\text{Cap}_1(U^1) < \varepsilon$ by (2.32). Taking the limit as $\varepsilon \to 0$, by Lemma 3.1 we get $u^* \in N^{1,1}(X)$ with

$$\int_{X \setminus A} g_u \, d\mu \geq \frac{\alpha C_d^{-1}}{C} \int_X g_{u^*} \, d\mu, \quad (4.16)$$

and

$$u^* = u \quad 1\text{-q.e. in } X \setminus A. \quad (4.17)$$
Next consider a general (possibly unbounded) \( u \in N^{1,1}(X \setminus A) \). We have for the truncations \( u_M := \max\{-M, \min\{M, u\}\} \) that
\[
\int_{X \setminus A} g_u \, d\mu \geq \int_{X \setminus A} g_{u_M} \, d\mu \geq \frac{\alpha C_d^{-1}}{C} \int_X g(u_M)^* \, d\mu \quad \text{by (4.16)}.
\]
Note that each \((u_M)^*\) has Lebesgue points 1-q.e. in \( X \) by (2.21). Consider a point \( x \in X \) that is a Lebesgue point for all \((u_M)^*\). Then \((u_M)^*(x) = (u_{M+1})^*(x)\) for all \( M \in \mathbb{N} \). Moreover, \( g(u_M)^* \) is an a.e. increasing sequence by (2.4) and (2.19). Letting \( M \to \infty \), by [4, Proposition 2.4] we get for \( v := \lim_{M \to \infty} (u_M)^* \) (note that the limit exists 1-q.e.) that \( v \in N^{1,1}(X) \) and
\[
\int_{X \setminus A} g_u \, d\mu \geq \frac{\alpha C_d^{-1}}{C} \int_X g_v \, d\mu.
\]
Moreover, by (4.17) we get \((u_M)^* = u_M\) 1-q.e. in \( X \setminus A \), and thus
\[
v = u \quad \text{1-q.e. in } X \setminus A,
\]
and in particular \( v = u \) a.e. in \( X \). Since \( X \setminus A \) is 1-finely open, by (2.31) and (2.20) we get
\[
g_v = g_v, X \setminus A = g_u, X \setminus A \quad \text{a.e. in } X.
\]
Thus \( A \) is isometrically removable for \( N^{1,1}(X) \).

Finally we remove the assumption that \( X \setminus A \) is 1-finely open. Let \( A \subset X \) be as in the statement of the theorem, and let \( u \in N^{1,1}(X \setminus A) \). The set \( X \setminus \overline{A} \) is 1-finely open, and by (2.32), we have
\[
\liminf_{r \to 0} \frac{\text{cap}_1(\overline{A} \setminus B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} < c_*
\]
for 1-q.e. \( x \in X \). We of course have \( u \in N^{1,1}(X \setminus \overline{A}) \), and note that \( \mu(\overline{A}) = 0 \) by Lemma 3.8. By the first part of the proof, we find \( v \in N^{1,1}(X) \) with \( v = u \) a.e. in \( X \) and \( g_v = g_v, X \setminus \overline{A} \) a.e. in \( X \). Formula (2.20) and the fact that \( A \subset \overline{A} \) imply that necessarily \( g_v \geq g_v, X \setminus A = g_u, X \setminus A \geq g_u, X \setminus \overline{A} \) a.e. in \( X \), and so we also get \( g_v = g_u, X \setminus A \) a.e. in \( X \). Thus \( A \) is isometrically removable for \( N^{1,1}(X) \). \( \square \)

As mentioned in the introduction, there is a well known equivalence between a set being removable for Newton–Sobolev functions, and being removable for the Poincaré inequality. We will use a version of this equivalence given in the following two theorems from the recent paper [6]. The first of these theorems is given by the implication \((c) \Rightarrow (f)\) in [6, Theorem 5.4].

**Theorem 4.18** Let \( A \subset X \) be such that \( \mu(A) = 0 \). If the set \( A \) is isometrically removable for \( N^{1,1}(X) \), then the space \( X \setminus A \) supports a \((1, 1)\)-Poincaré inequality.

Recall the definition from (2.8). The second theorem is given by the implication \((f) \Rightarrow (a)\) in [6, Theorem 5.4].

**Theorem 4.19** Let \( A \subset X \) be such that \( \mu(A) = 0 \), and let \( 1 \leq p < \infty \). Suppose that the space \( X \setminus A \) supports a \((1, p)\)-Poincaré inequality. Then the set \( A \) is removable for \( N^{1,p}(X) \).
Proof of Theorem 1.1 By Theorem 4.12, the set $A$ is isometrically removable for $N^{1,1}(X)$. Then by Theorem 4.18, $X \setminus A$ supports a $(1, 1)$-Poincaré inequality. Then by Hölder’s inequality, $X \setminus A$ also supports a $(1, p)$-Poincaré inequality.

Finally by Theorem 4.19, the set $A$ is also removable for $N^{1,p}(X)$ for all $1 < p < \infty$. □

5 Discussion

In this final section, we make remarks and in particular compare our results with the existing literature.

First we comment on the role of the new Federer-type characterization of sets of finite perimeter given in Theorem 2.14. We observe that if we were relying on the ordinary Federer’s characterization (explained right before Theorem 2.14), then in (4.8) we would need to be considering a point $x \in \partial^* E$ instead of $x \in \Sigma_\beta E$, and then there would be no guarantee even that

$$\limsup_{j \to \infty} \frac{\mu(B(x, r_j/\lambda) \setminus (E \cup A(x, r_j)))}{\mu(B(x, r_j))} > 0,$$

and so the rest of the proof would not work. In order to make it work, instead of the capacitary density condition (1.2) we would need to consider the stronger condition

$$\lim_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} = 0 \quad \text{for 1-q.e. } x \in X.$$ (5.1)

However, this would mean that $\text{Cap}_1(b_1 A) = 0$ (recall Definition 2.26), and then it would follow that $\text{Cap}_1(A) = 0$, by (2.28). By (2.18), 1-a.e. curve then has empty intersection with $A$, and so $A$ is clearly isometrically removable for $N^{1,1}(X)$. Thus this situation is actually very straightforward, owing to the fact that (5.1) is a strong condition that forces $A$ to be very small. But replacing “lim” by “lim inf” and 0 by a small constant $c_*$, as we do in (1.2), makes the situation much more intricate.

Now we compare our results with the literature. According to Theorem A of Koskela–Shanmugalingam–Tuominen [21], in a metric space $(X, d, \mu)$ with $\mu$ doubling and supporting a $(1, p)$-Poincaré inequality with $1 < p < \infty$, and with $X$ satisfying a suitable connectivity condition, if $A \subset X$ is a compact set satisfying the $t$-porosity condition

$$\text{for every } x \in A, \quad A \cap (B(x, tr) \setminus B(x, r)) = \emptyset \quad \text{for arbitrarily small } r > 0$$ (5.2)

with a sufficiently large $t > 1$, then $X \setminus A$ also supports a $(1, p)$-Poincaré inequality. In Theorem B of [21] (and in the discussion after the theorem) it is then noted that the aforementioned connectivity condition holds in particular if $X$ is complete and Ahlfors $Q$-regular (recall (2.1)), with $Q > 1$, and $1 < p \leq Q$. Korte [19, Theorem 3.3] shows that the connectivity condition can in fact be obtained only assuming the upper mass bound

$$\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq C_0 \left( \frac{r}{R} \right)^Q$$ (5.3)

for all $x \in X$ and $0 < r \leq R < \text{diam } X$, and constants $Q > 1$, $C_0 > 0$. (Recall that we considered this type of condition already in (3.4).) Now we get the following extension of the results of [21] to the case $p = 1$. Recall that for us, completeness, doubling, and $(1, 1)$-Poincaré are standing assumptions on the space.
Corollary 5.4 Assume that the upper mass bound \((5.3)\) holds for all \(x \in X\) and \(0 < r \leq R < \text{diam} X\), and with \(Q > 1\). Suppose that a closed set \(A \subseteq X\) is \(t\)-porous with sufficiently large \(t > 1\), depending only on \(C_d, C_P, \lambda, C_0\), and \(Q\). Then \(X \setminus A\) supports a \((1, 1)\)-Poincaré inequality, and \(A\) is removable for \(N^{1-p}(X)\), with \(1 \leq p < \infty\).

Proof Let \(x \in A\). Using the \(t\)-porosity, we find a sequence \(r_j \searrow 0\) such that \(A \cap (B(x, tr_j) \setminus B(x, r_j)) = \emptyset\) for every \(j \in \mathbb{N}\). Thus

\[
\liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} \leq \liminf_{j \to \infty} \frac{\text{cap}_1(B(x, r_j), B(x, 2tr_j))}{\text{cap}_1(B(x, tr_j), B(x, 2tr_j))}
\]

\[
\leq C_{\text{cap}} C_d \liminf_{j \to \infty} \frac{\mu(B(x, r_j))/r_j}{\mu(B(x, tr_j))/(tr_j)} \quad \text{by (2.24)}
\]

\[
\leq C_{\text{cap}} C_d C_0 t \liminf_{j \to \infty} \left( \frac{r_j}{tr_j} \right)^Q
\]

\[
= C_{\text{cap}} C_d C_0 t^{1-Q}.
\]

In fact this holds for every \(x \in X\), since \(A\) is closed and so for \(x \in X \setminus A\) the limit is trivially zero. Now, \(C_{\text{cap}} C_d C_0 t^{1-Q} < c_*\) for large enough \(t > 1\), and so the capacitary density condition of Theorem 1.1 holds, and this implies both claims. \(\square\)

The proofs of Theorems A and B in [21] are based on a construction where the \((1, p)\)-Poincaré inequality is applied to chains of balls situated in the annuli that do not intersect \(A\). The advantage of our techniques is of course that we only require the capacitary density condition, instead of the strong assumption of very large annuli having empty intersection with \(A\). Thus Corollary 5.4 gives an improved removability result also in the case \(1 < p < \infty\), though only in spaces that support a \((1, 1)\)-Poincaré inequality.

The porosity condition is a very geometric condition. In closing, we examine the possibility of expressing the capacitary density condition also in a geometric form, using the Hausdorff content \((2.6)\). The comparability between the \(1\)-capacity and the Hausdorff content is generally a well known fact, and so we only sketch the proof.

Consider the upper mass bound

\[
\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq C_0 \frac{r}{R} \quad \text{for all} \quad x \in X, \quad 0 < r \leq 4R < \text{diam} X,
\]

and for some constant \(C_0 > 0\). This is a fairly mild requirement, since roughly speaking it says that the space has dimension at least 1 at every point \(x\).

Proposition 5.6 Suppose the upper mass bound \((5.5)\) holds, and let \(A \subseteq X\). Then for every \(x \in X\), we have

\[
\frac{1}{C} \liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} \leq \liminf_{r \to 0} \frac{\mathcal{H}_\infty(A \cap B(x, r))}{\mu(B(x, r))/r}
\]

\[
\leq C \liminf_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))}
\]

for a constant \(C \geq 1\) depending only on \(C_d, C_P, \lambda, C_0\).

Proof By (2.24), the denominators are comparable for small \(r\). To prove the first inequality, let \(\varepsilon > 0\) and consider a covering \(\{B_j = B(x_j, r_j)\}_j\) of \(A \cap B(x, r)\) for which \(\sum_j \frac{\mu(B_j)}{r_j} < \)
\( \mathcal{H}_\infty(A \cap B(x, r)) + \varepsilon \). We can assume that each ball \( B_j \) intersects \( A \cap B(x, r) \). If \( r_j \geq r/4 \) for some \( j \), then we have

\[
\frac{\mu(B_j)}{r_j} \geq \frac{1}{C_d^4} \frac{\mu(B(x, r_j))}{r_j} \geq \frac{1}{C_0 C_d^4} \frac{\mu(B(x, r))}{r} \quad \text{by (5.5),}
\]

and so

\[
\frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\text{cap}_1(B(x, r), B(x, 2r))} \leq 1 \leq C_0 C_d^4 \frac{\mathcal{H}_\infty(A \cap B(x, r)) + \varepsilon}{\mu(B(x, r))/r}.
\]

Thus we can assume \( r_j \leq r/4 \) for all \( j \). Then

\[
\text{cap}_1(A \cap B(x, r), B(x, 2r)) \leq \sum_j \text{cap}_1(B_j, B(x, 2r)) \leq \sum_j \text{cap}_1(B_j, 2B_j) \quad \text{since } 2B_j \subset B(x, 2r) \leq C_d \sum_j \frac{\mu(B_j)}{r_j} \quad \text{by (2.24)} \leq C_d (\mathcal{H}_\infty(A \cap B(x, r)) + \varepsilon).
\]

Letting \( \varepsilon \to 0 \), we get the first inequality.

To prove the second inequality, let \( \varepsilon > 0 \). Consider a set \( A(x, r) \subset B(x, 2r) \) with \( A \cap B(x, r) \subset A(x, r) \subset B(x, 2r) \) and \( P(A(x, r), X) \leq \text{cap}_1(A \cap B(x, r), B(x, 2r)) + \varepsilon \), just as on page 13. If \( r > 0 \) is small, then applying [15, Theorem 3.1], we find balls \( \{B_j = B(x_j, r_j)\}_j \) covering the interior of \( A(x, r) \), with

\[
\mathcal{H}_\infty(A \cap B(x, r)) \leq \sum_j \frac{\mu(B_j)}{r_j} \leq C P(A(x, r), X) \leq C (\text{cap}_1(A \cap B(x, r), B(x, 2r)) + \varepsilon)
\]

for a constant \( C \) depending only on \( C_d, C_P, \lambda \). Letting \( \varepsilon \to 0 \), we get the second inequality.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Ambrosio, L.: Fine properties of sets of finite perimeter in doubling metric measure spaces, Calculus of variations, nonsmooth analysis and related topics. Set-Valued Anal. 10(2–3), 111–128 (2002)
2. Ambrosio, L., Miranda, M. Jr., Pallara, D.: Special functions of bounded variation in doubling metric measure spaces, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta, (2004)
3. Balogh, Z., Koskela, P.: Quasiconformality, quasisymmetry, and removability in Loewner spaces, With an appendix by Jussi Väisälä. Duke Math. J. 101(3), 554–577 (2000)
4. Björn, A., Björn, J.: Nonlinear Potential Theory on Metric Spaces, EMS Tracts in Mathematics, vol. 17, p. xii+403. European Mathematical Society (EMS), Zürich (2011)
5. Björn, A., Björn, J.: Obstacle and Dirichlet problems on arbitrary nonopen sets in metric spaces, and fine topology. Rev. Mat. Iberoam. 31(1), 161–214 (2015)
6. Björn, A., Björn, J., Lahti, P.: Removable sets for Newtonian Sobolev spaces and a characterization of \( p \)-path almost open sets. Rev. Mat. Iberoam. (2023). https://doi.org/10.4171/RMI/1419
7. Federer, H.: Geometric Measure Theory, Die Grundlehren der Mathematischen Wissenschaften, Band, vol. 153, p. xiv+676. Springer-Verlag Inc., New York (1969)

8. Hajłasz, P.: Sobolev spaces on metric-measure spaces, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 173–218, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, (2003)

9. Hakkarainen, H., Kinnunen, J.: The BV-capacity in metric spaces. Manuscr. Math. 132(1–2), 51–73 (2010)

10. Hakkarainen, H., Kinnunen, J., Lahti, P., Lehtelä, P.: Relaxation and integral representation for functionals of linear growth on metric measures spaces. Anal. Geom. Metr. Spaces 4, 288–313 (2016)

11. Hedberg, L.I.: Removable singularities and condenser capacities. Ark. Mat. 12, 181–201 (1974)

12. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. 181(1), 1–61 (1998)

13. Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J.: Sobolev Spaces on Metric Measure Spaces. An Approach Based on Upper Gradients, New Mathematical Monographs, vol. 27, p. xii+434. Cambridge University Press, Cambridge (2015)

14. Kaufman, R., Wu, J.M.: On removable sets for quasiconformal mappings. Ark. Mat. 34(1), 141–158 (1996)

15. Kinnunen, J., Korte, R., Shanmugalingam, N., Tuominen, H.: Lebesgue points and capacities via the boxing inequality in metric spaces. Indiana Univ. Math. J. 57(1), 401–430 (2008)

16. Kinnunen, J., Korte, R., Shanmugalingam, N., Tuominen, H.: Pointwise properties of functions of bounded variation in metric spaces. Rev. Mat. Complut. 27(1), 41–67 (2014)

17. Jones, P., Smirnov, S.: Removability theorems for Sobolev functions and quasiconformal maps. Ark. Mat. 38(2), 263–279 (2000)

18. Kolsrud, T.: Condenser capacities and removable sets in $W^{1,p}$. Ann. Acad. Sci. Fenn. Ser. A I Math. 8(2), 343–348 (1983)

19. Korte, R.: Geometric implications of the Poincaré inequality. Results Math. 50(1–2), 93–107 (2007)

20. Koskela, P.: Removable sets for Sobolev spaces. Ark. Mat. 37(2), 291–304 (1999)

21. Koskela, P., Shanmugalingam, N., Tuominen, H.: Removable sets for the Poincaré inequality on metric spaces. Indiana Univ. Math. J. 49(1), 333–352 (2000)

22. Lahti, P.: A Federer-style characterization of sets of finite perimeter on metric spaces. Calc. Var. Partial Differ. Equ. 56(5), 22 (2017)

23. Lahti, P.: A new Federer-type characterization of sets of finite perimeter in metric spaces. Arch. Ration. Mech. Anal. 236(2), 801–838 (2020)

24. Lahti, P.: A notion of fine continuity for BV functions on metric spaces. Potential Anal. 46(2), 279–294 (2017)

25. Lahti, P.: Federer’s characterization of sets of finite perimeter in metric spaces. Anal. PDE 13(5), 1501–1519 (2020)

26. Lahti, P.: Quasiopen sets, bounded variation and lower semicontinuity in metric spaces. Potential Anal. 52(2), 321–337 (2020)

27. Lahti, P.: The Choquet and Kellogg properties for the fine topology when $p = 1$ in metric spaces. J. Math. Pures Appl. 9(126), 195–213 (2019)

28. Malý, J., Ziemer, W.: Fine Regularity of Solutions of Elliptic Partial Differential Equations, Mathematical Surveys and Monographs, vol. 51, p. xiv+291. American Mathematical Society, Providence RI (1970)

29. Miranda, M., Jr.: Functions of bounded variation on “good” metric spaces. J. Math. Pures Appl. (9) 82(8), 975–1004 (2003)

30. Ntalampikos, D.: A removability theorem for Sobolev functions and detour sets. Math. Z. 296(1–2), 41–72 (2020)

31. Shanmugalingam, N.: Harmonic functions on metric spaces. Illinois J. Math. 45(3), 1021–1050 (2001)

32. Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoam. 16(2), 243–279 (2000)

33. Väisälä, J.: Removable sets for quasiconformal mappings. J. Math. Mech. 19, 49–51 (1969/1970)

34. Wu, J.-M.: Removability of sets for quasiconformal mappings and Sobolev spaces. Complex Var. Theory Appl. 37(1–4), 491–506 (1998)

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