Hall Viscosity of the Composite-Fermion Fermi Seas for Fermions and Bosons

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(Dated: July 15, 2020)

The Hall viscosity has been proposed as a topological property of incompressible fractional quantum Hall states and can be evaluated as Berry curvature. This article reports on the Hall viscosities of composite-fermion Fermi seas at \( \nu = 1/m \), where \( m \) is even for fermions and odd for bosons. A well-defined value for the Hall viscosity is not obtained by viewing the \( 1/m \) composite-fermion Fermi seas as the \( n \to \infty \) limit of the Jain \( \nu = n/(nm \pm 1) \) states, whose Hall viscosities, \((\pm n + m)\hbar \rho /4 \) \((\rho \) is the two-dimensional density), approach \( \pm \infty \) in the limit \( n \to \infty \). A direct calculation shows that the Hall viscosities of the composite-fermion Fermi sea states are finite, and also relatively stable with system size variation, although they are not topologically quantized in the entire \( \tau \) space. We find that the \( \nu = 1/2 \) composite-fermion Fermi sea wave function for a square torus yields a Hall viscosity that is expected from particle-hole symmetry and is also consistent with the orbital spin of \( 1/2 \) for Dirac composite fermions. We compare our numerical results with some theoretical conjectures.

I. INTRODUCTION

The Hall viscosity has been proposed as one of the topological characteristics of FQH states [14]. In particular, it has been proposed that it is related to the “shift” on the spherical geometry [4], as confirmed by explicit evaluations [5, 6] for the Laughlin and Jain states [7, 8]. (The shift is twice of the so-called orbital spin [9].) This article is concerned with the Hall viscosity of the composite-fermion fermi sea (CFFS). CFs form a Fermi sea when they experience a zero effective field [10–13]. The best studied CFFS is at \( \nu = 1/2 \), where electrons capture two vortices to form composite fermions. We will also consider CFFs of electrons at \( \nu = 1/4 \) and \( \nu = 1/6 \), where composite fermions bind four and six vortices. Just as fermions in the lowest Landau level capture an even number of vortices to form FQH states and CFFS, bosons in the lowest Landau level can capture an odd number of vortices to form both FQH states and CFFS [14]–[18]. We will consider CFFSs of bosons at \( \nu = 1, 1/3 \) and \( 1/5 \).

A fundamental difficulty for the determination of the Hall viscosity of the CFFS is that it does not have a gap in the thermodynamic limit, and its Hall viscosity is not expected to be topologically quantized and may be sensitive to various details, such as the geometry of the torus, the shape and size of the CFFS, and the details of the CFFS wave function. Nonetheless, irrespective of the issue of its applicability to real experiments, the Hall viscosity can be evaluated for the standard CFFS wave functions, which are very accurate representations of the actual Coulomb ground states. This article reports on these results. We evaluate the Hall viscosity through calculation Berry curvature in the \( \tau \) space following Avron, Seiler, and Zograf [1] (explained below); the method is justified by the presence of a gap at individual \( \tau \) points for a finite system. To this end, we construct CFFS wave functions for these states and show that for general \( 1/m \) several wave functions can be constructed.

For \( \nu = 1/2 \) particle-hole symmetry is an additional consideration that fixes the value of the Hall viscosity as shown by Read and Rezayi [19]; we find that our calculated value is consistent with the expected value, which is not surprising given that our wave function satisfies particle-hole symmetry to a good degree. The value of Hall viscosity at \( \nu = 1/2 \) is consistent with the orbital spin \( 1/2 \) for Dirac composite fermions [20, 21], to the extent that the conjectured relation between the Hall viscosity and the orbital spin holds for incompressible states. We also show the Hall viscosities for CFFSs are not topologically quantized in the \( \tau \) space, in stark contrast to the gapped FQH states.

The paper is organized as follows. We first briefly review the Hall viscosity for gapped FQH states and the problem for CFFS in Sec. I. Then we introduce the wave functions for both fermionic CFFS and bosonic CFFS in Sec. II. Finally, we present our results and discussions of Hall viscosity for CFFS in Sec. IV.

II. HALL VISCOSITY AS BERRY CURVATURE

The Hall viscosity is a bulk property of quantum Hall fluid. It is the geometrical response to the strain rate applied to the fluid. In theoretical calculation, the strain rate can be simulated by putting the fluid on a torus and adiabatically deforming the shape of the torus while preserving its area. A torus is equivalent to a parallelogram on a complex plane with periodic boundary conditions in both directions, \( L_1 \) along the real axis and \( L_2 = L_1 \tau \), where the modular parameter \( \tau \) is a complex number [22]. The total area is given by \( V = L_1^2 \tau_2 = 2\pi N_\phi \ell^2 \), through which there are \( N_\phi \) flux quanta passing. (A flux quantum is defined as \( \phi_0 = \hbar /e \), and the magnetic length as \( \ell = \sqrt{\hbar c / e B} \).) It was shown by Avron, Seiler, and Zograf [1] that the Hall viscosity can be computed as Berry curvature through adiabatic deformation of the geometry of the torus:

\[
\eta^A = -\frac{\hbar \tau_2^2}{V} \mathcal{F}_{\tau_1, \tau_2},
\]
Given by
\[ \mathcal{F}_{r_1,r_2} = -2\mathrm{Im}\left( \frac{\partial \Psi}{\partial r_1} \frac{\partial \Psi}{\partial r_2} \right). \]  
(2)

Here \( \Psi \) is the many-particle ground state on the torus. Based on Eq. 1, Read proposed that \( \eta^A \) is given by
\[ \eta^A = S^2 \frac{N}{4 \sqrt{V}}. \]  
(3)

where the "shift" \( S \) is a topological quantum number, given by \( S = \frac{N}{2} - N_0 \); i.e. \( S \) is the offset of flux quanta needed to form a ground state with \( N \) particles on a sphere [4, 19].

The shift is a manifestation of the orbital spin [9] which is given by \( S/2 \). The relation Eq. 3 has been proved for various incompressible FQH states with methods of plasma analogy, Chern-Simons theory, matrix models etc. [3, 4, 19, 23, 25]. Most recently, Pu, Frenling, and Jain [6] found this relation can be proved for \( \nu = \pm \frac{1}{m+n} \) by using the Jain wave functions [26] and certain natural and justified assumptions.

In this article, we are concerned with the Hall viscosity of gapless states. A large group of gapless states in FQH systems are compressible Fermi-liquid like states, which are described as composite-fermion Fermi seas (CFFSs). As the derivation of Eq. 1 is based on adiabatic transformations, one may question if it can be used to characterize the Hall viscosity for gapless states. Here, it is important to note that the Hall viscosity is defined as a Berry curvature, which requires a finite gap only in the vicinity of a certain value of \( \tau \) [27]. (In contrast, the Berry phase would require integration over the entire \( \tau \) space.) One may argue that there is a finite gap for the individual value of \( \tau \) for the compressible Fermi-liquid like states of finite size, even though there may be level crossings as a function of \( \tau \) and for certain values of \( \tau \) the chosen wave function may no longer represent the ground state accurately. Therefore, the Berry curvature in Eq. 1 is still well-defined for a given model state, provided it corresponds to the ground state for the value of \( \tau \) under consideration.

One may think that the Hall viscosity of the CFFS at \( \nu = 1/m \) may be trivially obtained by viewing it as the \( n \to \infty \) limit of the Jain states \( \nu = n/(mn \pm 1) \). For the Jain states, the Hall viscosity is given by Eq. 3 with \( S = \pm n + m \). This would imply that the Hall viscosity of \( \nu = 1/m \) is \( +\infty \) when coming along the sequence \( \nu = n/(mn + 1) \) and \( -\infty \) along the sequence \( \nu = n/(mn - 1) \). This obviously leads to a contradiction. We find below a finite Hall viscosity by a calculation directly at \( \nu = 1/m \). Nonetheless, the question of how to reconcile the Hall viscosity of the \( \nu = 1/m \) CFFS with the Hall viscosity of nearby Jain states remains an interesting open issue.

Ref. [19] showed that the Hall viscosity for a particle-hole symmetric state would be \( \frac{\mu N}{4V} \), no matter whether the model state is incompressible or not. This result is applicable to the CFFS at \( \nu = 1/2 \). Although it was not explicit in the construction of CFFS wave function, it was shown that the CFFS does preserve a high degree of particle-hole symmetry [28, 29], which is an exact symmetry at \( \nu = 1/2 \) in the limit of zero Landau level (LL) mixing for any two-body interaction. Our calculation below is consistent with this result. It is known that at \( \nu = 1/2 \) the coulomb ground state is well described by such a CFFS [10, 32, 34]. For CFFSs at other fillings there is no rigorous theoretical derivation of Hall viscosities to our knowledge. It is not clear if the Hall viscosities for these states are universal and how they are related to the orbital spin or the shift [35]. Based on the idea of attaching fluxes to Dirac composite fermion, Goldman and Fradkin [36] have made a conjecture for the values of Hall viscosities of general CFFSs, which are compared to our numerical results in Sec. IV.

In the remaining part of this paper, we first introduce the generic wave functions for CFFS. We then numerically calculate the Berry curvature for CFFS at filling 1/2, which has a high degree (although not exact) particle-hole symmetry, and also CFFS at other fillings including 1, 1/3, 1/4, 1/5 and 1/6.

### III. COMPOSITE FERMION FERMI SEA

We consider a parallelogram with edges \( L_1 \) and \( L_2 = L_1 \tau \). Because the physical coordinate \( z_i = x_i + iy_i \) of a particle changes with the deformation of the torus, it is convenient to define the reduced coordinate \( (\theta_{1,i}, \theta_{2,i}) \) as \( z_i = x_i + iy_i = L_1 \theta_{1,i} + L_2 \theta_{2,i} \). The reduced coordinates \( (\theta_{1,i}, \theta_{2,i}) \) remain unchanged under \( \tau \) deformations, and are therefore more convenient to use in the calculation of the Hall viscosity. Following the convention in Ref. [5, 6], we adopt the \( \tau \) gauge: \( (A_x, A_y) = B \left( y, -\frac{x}{\mu} \right) \). We impose the following periodic boundary conditions:
\[
t(L_i)\psi(z, \bar{z}) = e^{i\phi_i}\psi(z, \bar{z}) \quad i = 1, 2. \]  
(4)

The magnetic translation operator \( t \) in \( \tau \) gauge is given by
\[
t(\alpha L_1 + \beta L_2) = e^{\alpha\partial_1 + \beta\partial_2 + i2\pi N_0 \theta_1}, \]  
(5)

where \( \partial_j \equiv \partial/\partial \tau^j \).

We go over the construction of the CFFS wave function in some detail here. In particular, we show that, in general, there is more than one way to project the CFFS wave function to LLL, except for \( \nu = 1 \) and \( \nu = 1/2 \). This allows us to test the sensitivity of the Hall viscosity in the form of the CFFS wave function.

We begin with the Laughlin wave function, which will enter into the construction of the CFFS wave function. In the \( \tau \) gauge the unnormalized Laughlin wave function [37, 38] for \( N \) particles (which will be used in the construction of the CFFS wave function) at filling \( \nu = 1/m \) can be written as:
\[ \Psi_{m,k}^{\text{Lau}}[z_i, \bar{z}_i] = e^{i\pi \nu N_\phi \sum_i \theta_i^2} \left[ \vartheta \left( \frac{\partial_1 + \frac{N-1}{2}}{2} \right) \left( \frac{mZ}{L_1} \right) \prod_{i<j} \left[ \vartheta \left( \frac{1}{2} \right) \left( \frac{z_i - z_j}{L_1} \right) \right] \right]^m \] (6)

where the last factor is the torus analog of the familiar Jastrow factor \( \prod_{i<j}(z_i - z_j)^m \), and the factors preceding it ensures the correct boundary conditions and center of mass momentum.

Here \( Z = \sum_i z_i \) and \( \vartheta \left( \frac{a}{b} \right) (z|\tau) \) is the Jacobi theta functions with rational characteristics, given by [39]

\[ \vartheta \left( \frac{a}{b} \right) (z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi(n+a)^2 \tau} e^{i2\pi(n+a)(z+b)}. \] (7)

The use of these \( \vartheta \) functions is advantageous because one does not need to specify the positions of the zeros. Eq. 6 has center-of-mass momentum \( k = 0, 1, 2 \cdots m - 1 \):

\[ \prod_{i=1}^N t_i(L_1/N_\phi) \Psi_{m,k}^{\text{Lau}}[z_i, \bar{z}_i] = e^{2\pi i \left( \frac{a_1 + \frac{N-1}{2}}{2} \right)} \Psi_{m,k}^{\text{Lau}}[z_i, \bar{z}_i] \] (8)

Before projection into the LLL, the CFFS wave function at filling 1/m can be written as [34]:

\[ \Psi_{m,k}^{\text{unproj}}[z_i, \bar{z}_i] = \det \left[ e^{i\mathbf{k}_n \cdot \mathbf{z}_i} \right] \Psi_{m,k}^{\text{Lau}}[z_i, \bar{z}_i] \] (9)

Here \( \det \left[ e^{i\mathbf{k}_n \cdot \mathbf{z}_i} \right] \) stands for the Slater determinant of plane waves. To satisfy the periodic boundary conditions

\[ \Psi_{m,k}^{\text{proj}}[z_i, \bar{z}_i] = e^{i\pi \nu N_\phi \sum_i \theta_i^2} \vartheta \left( \frac{\partial_1 + \frac{N-1}{2}}{2} \right) \left( \frac{mZ + i\ell^2 K}{L_1} \right) \det \left[ \hat{g}_{nl} \right] \vartheta \left( \frac{1}{2} \right) \left( \frac{z_i - z_j}{L_1} \right) \right]^m \] (14)

\[ \hat{g}_{nl} = e^{-\frac{K_i^2}{2}} (k_n + 2k_n) e^{i(k_n + k_n)z_i} e^{ik_n e^2 \partial_{z_i}} \] (15)

Here \( K \) is the sum over all occupied wave vectors \( K = \sum n k_n \). A determinant of operators \( \det \left[ \hat{g}_{nl} \right] \) is certainly not easy to compute. To avoid this, one apply the JK projection, and the wave function finally reads:

\[ \Psi_{m,k}^{\text{proj}}[\alpha] = e^{i\pi \nu N_\phi \sum_i \theta_i^2} \vartheta \left( \frac{\partial_1 + \frac{N-1}{2}}{2} \right) \left( \frac{mZ + i\ell^2 K}{L_1} \right) \det \left[ \hat{g}_{n\ell}^{(\alpha)} \right] \] (16)

\[ \hat{g}_{n\ell}^{(\alpha)} = e^{-\frac{K_i^2}{2}} (k_n + 2k_n) e^{i(k_n + k_n)z_i} \prod_{p=1}^{m/2} \prod_{j,j \neq l} \vartheta \left( \frac{1}{2} \right) \left( \frac{z_i + i\alpha_p k_n e^2 - z_j}{L_1} \right) \right]^m \] (17)

Here we have the non-trivial JK projection coefficient \( \alpha = (\alpha_1, \alpha_2, \cdots \alpha_m/2) \), which was first found for \( \nu = 1/2 \) CFFS in Ref. [22] and for Jain states in Ref. [20]. For the CFFS wave function to satisfy periodic boundary conditions, they
must satisfy \( \sum p \alpha_p = m \). For \( \nu = 1/2 \), there is only one term \( \alpha_1 = 2 \), and the wave function Eq. 16 is unique. For other fermionic CFFS with \( m \geq 2 \), there is in general more than one choice of \( \alpha_p \). For instance, we can choose \((\alpha_1, \alpha_2) = (2, 2), (0, 4), (1, 3), (5, -1) \ldots \) for \( \nu = 1/4 \). A wave function similar to Eq. 16 had been used in Ref. 16 [17], which corresponds to the cases \( \alpha_{1,2-1} = m/l \) and \( \alpha_{1,2+1,3} = -m/2 = 0 \) with \( l \) being an integer. The larger class of CFFS wave functions derived above has not been reported before, and it would be interesting to ask in what sense these CFFS wave functions differ.

The JK projection can similarly be applied to bosonic CFFS with \( m \geq 3 \). Because \( m \) is now an odd number, we factor out a single power of the Jastrow factor. Eq. 16 now becomes:

\[
\Psi_{\text{proj}(b)}^{[\nu]}[\{z_i, \bar{z}_i\}] = e^{i\pi N \rho \sum_i \theta^2_i, \bar{\theta}^2_i} \left[ \prod_{j \neq i} \left( \frac{z_j - z_i}{L} \right) \phi \right] \left( \frac{m + i\ell^2 k_1}{L} \right) \det \left[ g_{nl} \right] \prod_{i<j} \phi \left( \frac{z_i - z_j}{L} \right) \right)
\]

and the condition for \( \alpha_p \) becomes: \( \sum p = (m-1)/2 \). A special case which is not covered by Eq. 18 is the bosonic CFFS at filling \( \nu = 1 \). In that case we only have one order of Jastrow factor. One might think that a wave function for the bosonic CFFS for \( \nu = 1 \) can be obtained by dividing the fermionic \( \nu = 1/2 \) CFFS by a single Jastrow factor. It turns out that that wave function is not valid. However, a closely related wave function does the job:

\[
\Psi_{\text{proj}(1)}^{[\nu]}[\{z_i, \bar{z}_i\}] = e^{i\pi N \rho \sum_i \theta^2_i, \bar{\theta}^2_i} \left[ \prod_{j \neq i} \left( \frac{z_j - z_i}{L} \right) \phi \right] \left( \frac{Z + i\ell^2}{L} \right) \det \left[ g_{nl} \right] \prod_{i<j} \phi \left( \frac{z_i - z_j}{L} \right) \right)
\]

Notice that, the argument of the theta function in the preceding equation is slightly different from that of the CFFS for \( \nu = 1/2 \).

**IV. RESULTS AND DISCUSSIONS**

We evaluated the Hall viscosities for the CFFS wave functions at \( \nu = 1, 1/2, 1/3, 1/4, 1/5 \) and \( 1/6 \), using the Berry curvature expression given in Eq. 1. The results are summarized in Table I along with the CFFS shapes we used. The CFFS shapes correspond to the global ground states for the given system sizes. We adopt the standard variational Monte Carlo method for unprojected wave functions, and the lattice Monte Carlo method [30] for LLL projected wave functions. The last digit in the brackets in Table I represents the statistical error. We present \( \eta^A \) in units of \( (\hbar \rho)^3/4 \), where \( \rho = N/V \) is the particle density. In these units, \( \eta^A \) is given by the shift \( \mathcal{S} \) for gapped FQH states according to Read’s relation Eq. 3. All results in Table I are evaluated for a square torus. As shown below, the results are quite stable as the system size increases, so long as the CFFS shapes stay circular.

As we mentioned, there are two key issues here: 1) What is the value of Hall viscosities for CFFSs? 2) Are they still topologically quantized? The fermionic CFFS at filling \( \nu = 1/2p \) can be regarded as the \( m \to \infty \) limit of Jain states at filling \( \nu = \frac{m}{2p} \). Ref. [3] shows that the Hall viscosities for Jain states are \( \eta^A = (\pm m + 2p)\frac{\hbar \rho}{4} \). Taking the limit, the Hall viscosity for the CFFS diverges to \( +\infty \) or \( -\infty \) depending on whether we approach it from below or above. Meanwhile, Table I shows that the Hall viscosities are finite and change very little with system size, although in general they are not quantized at integer values in units of \( \frac{\hbar \rho}{4} \).

Among these cases, the CFFS at \( \nu = 1/2 \) has an additional symmetry, namely an exact particle-hole symmetry for any two-body interaction confined in the LLL. The CFFS wave function satisfies this symmetry to a high degree but not exactly. Refs. [19, 48] show that the particle-hole symmetric ground state has \( \eta^A = \frac{\hbar \rho}{4} \), independent of whether the ground state is compressible or incompressible. Table I shows that the Hall viscosity for the LLL projected CFFS wave function at filling 1/2 is quantized at \( \frac{\hbar \rho}{4} \) for a square torus with the same accuracy as found for the incompressible Jain states [6]. This shows that the small particle-hole asymmetric part in the CFFS wave function does not change the Hall viscosity appreciably for \( \tau = i \). Levin and Son [21] showed that if the particle-hole symmetry is not spontaneously broken, there is an exact relationship between the Hall conductivity and the susceptibility which can be derived
through Dirac CF theory. This relation predicts the orbital spin of \( \nu = 1/2 \) CFFS to be \( s = 1/2 \) by making use of Galilean invariance \([39, 50]\). The orbital spin is half of the shift. Levin and Son also showed that from HLR theory one can derive a similar relation, which predicts \( s = 1 \) for \( \nu = 1/2 \) CFFS. Our result is consistent with orbital spin predicted by the Dirac CF theory provided that the Hall viscosity and orbital spin for the CFFS are also related through Read's conjecture.

How about the Hall viscosities of CFFS at other fillings which do not have particle-hole symmetry? Ref. \([27]\) argued that the Hall viscosity of \( \nu = 1/m \) CFFS is \( \eta^A = \frac{m \hbar}{2} \), i.e. the same value as Laughlin states for all fillings (even without particle-hole symmetry) in thermodynamic limit if it is evaluated through Eq. \([1]\) at \( \tau = i \).

Our numerical result is close to that prediction only for the bosonic CFFS wave function at \( \nu = 1 \) in LLL. We found that the Hall viscosities are not quantized at integer values of \( \nu \) through Eq. \([3]\) in general. In fact, they are more or less around (not accurately at) the values \( \eta^A = \frac{m \hbar}{2} \) for \( \nu = 1/m \), which corresponds to \( \nu = m/2 \) (except the \( \nu = 1 \) case just mentioned). Ref. \([40]\) gives a conjecture for the Hall viscosities of CFFS which says \( \eta^A = \frac{\hbar}{2} (m - 1) \) for \( \nu = 1/m \). For \( \nu = 1/2 \) it corresponds to the orbital spin of Dirac composite fermion and also agrees with our numerical results.

The second question is whether the Hall viscosity calculated through Eq. \([3]\) is still topologically quantized for CFFS, just as the gapped FQH states. We found the answer is no. This can be seen from two aspects. First, Table \([1]\) shows that the Hall viscosities are different for different CFFS wave functions at the same filling, i.e., it depends on whether the wave functions are projected into the LLL or not, and if so, how they are projected. However, different CFFS wave functions at a given filling actually describe the same topological phase, so they should yield the same value if the Hall viscosity is topologically quantized. Indeed, for the Jain states we found that the projected and unprojected CF wave functions have the same Hall viscosities if the system is big enough \([6]\). Secondly, Refs. \([3, 19]\) have shown that the topologically quantized Hall viscosities would yield the same value independent of at which \( \tau \) point the Berry curvature is evaluated, i.e. it is independent of the shape of the torus. We test whether this is still true for CFFS, and the results are summarized in Table \([1]\). For a gapped FQH state, two ground states at different \( \tau \) values are adiabatically connected, in the sense that we can get one from the other by just changing \( \tau \) in the wave function. This is apparently not true for the CFFS. When the shape of the torus is deformed, the shape of the CFFS also changes. A circular CFFS at \( \tau = 1/2 \) would become an elliptical CFFS at \( \tau' \). While the elliptical CFFS might still have the most compact shape in that given momentum sector, it is no longer the global ground state for all momentum sectors. As shown in Ref. \([31]\), the ground states in high energy momentum sectors are not very accurately described by the CFFS wave functions. Therefore, we changed the CFFS configuration in \( k \) space to make the CFFS as circular as possible for \( \tau = e^{i\pi/3}, e^{i\pi/4} \) considered here. As Table \([1]\) shows, the Hall viscosity for \( \tau = e^{i\pi/3} \) is very close to the Hall viscosity for \( \tau = i \) shown in Table \([1]\) which is remarkable because they are two different ground states that are not adiabatically connected.

However, the Hall viscosity for \( \tau = e^{i\pi/4} \) clearly deviates from the values of a square torus. Hence, we conclude that the Hall viscosity of the CFFS is not topologically quantized in the \( \tau \) space.

Ref. \([6]\) presents a way to analytically derive the Hall viscosity for the CF wave functions. That approach is based on the assumption that the overall normalization factor does not contribute to Hall viscosity. For the CFFS wave function in Eq. \([9]\) the normalization of the Slater determinant is independent of \( \tau \), because \( e^{i \mathbf{k} \cdot \mathbf{r}} = e^{-\frac{\hbar}{2} (n_1 s_1 + n_2 s_2)} \) if \( \mathbf{k} = n_1 \mathbf{b}_1 + n_2 \mathbf{b}_2 \) and \( \tau = \theta_1 L + \theta_2 L \tau \). Therefore, the aforementioned assumption for CFFS means:

\[
\lim_{N \to \infty} \frac{1}{N} \left( \frac{\partial}{\partial \tau_2} \right) \ln Z = 0. \tag{22}
\]

\[
Z = \left( \int \prod_i d^2 r_i \left| \Psi_{\text{proj}} \right|^2 \right)^{-\frac{1}{2}} \left( \int \prod_i d^2 r_i \left| \Psi_{\text{unproj}} \right|^2 \right)^{-\frac{1}{2}} \tag{23}
\]

If the assumption in Eq. \([22]\) is also true for the CFFS, then the Hall viscosity of the CFFS would be \( \eta^A = \frac{\hbar}{2} \) at \( \nu = 1/m \) following the approach in Ref. \([6]\). The results in Table \([1]\) clearly show that this is not the case. It is interesting to note that while the overall normalization factor does not contribute to the Hall viscosity for gapped Jain states, it makes a nontrivial contribution to the Hall viscosity of the CFFSs.

V. CONCLUSION

In this paper, we calculated the Hall viscosity of CFFS wave functions at various fillings by evaluating the Berry curvature in the \( \tau \) space. We found that the Hall viscosities at \( \tau = i \) are finite and fluctuate very little with system size. Especially, the CFFS wave function at filling \( 1/2 \) in LLL has Hall viscosity very close to \( \eta^A = \frac{\hbar}{2} \) for a particle-symmetric state \([19]\). This value is also consistent with the \( 1/2 \) orbital spin of Dirac composite fermions \([21]\). We also compared our numerical results with more general theoretical conjectures \([27, 36]\) for other fillings but have not found a perfect consistency. We also note that the Hall viscosities depend on whether the CFFS wave functions are project or not, and the way of projection. By evaluating the Hall viscosity at different positions in \( \tau \) space, we show the Hall viscosity defined through Berry curvature of CFFS is not topologically quantized.

Our work leaves some questions that need further elucidation: Is there a general relation between Hall viscosity
 TABLE I. The Hall viscosity for CFFS wave functions. The data is shown as $4\eta^i/\hbar\rho$, in which $\eta^i$ defined in Eq. 1 is calculated through Monte Carlo at $\tau = i$, and $\rho$ is the particle density $\rho = N/V$. For gapped FQH states, the quantity shown would be equivalent to shift $S$. We specify the CFFS shapes (which correspond to the global ground states) and system sizes Eq. 1 in the first two row. We specify the wave function at each giving factor by showing $\alpha$, which is defined in Sec. III. The unprojected wave function is denoted as "un". The last digit in bracket represents the statistical error.

| Number of particles | \( \nu \) | \( \alpha \) | \( 4\eta^i/\hbar\rho \) |
|---------------------|---------|----------|------------------|
| 12                  | 1/2     | un       | 1.013(1)         |
|                     |         |          | 1.012(4)         |
|                     |         |          | 1.014(1)         |
|                     |         |          | 1.011(4)         |
| 21                  | 1/4     | (2,2)    | 1.991(3)         |
|                     |         | (4,0)    | 1.801(2)         |
|                     |         |          | 1.80(1)          |
|                     |         |          | 1.822(5)         |
|                     |         |          | 1.783(6)         |
| 37                  | 1/6     | (2,2)    | 3.355(7)         |
|                     |         | (3,3,0)  | 3.147(6)         |
|                     |         |          | 3.203(8)         |
|                     |         |          | 3.28(3)          |
|                     |         |          | 3.11(4)          |
| 69                  | 1       | (1)      | 1.041(3)         |
|                     |         |          | 1.115(4)         |
|                     |         |          | 1.152(3)         |
|                     |         |          | 1.170(5)         |
| 12                  | 1/3     | (3)      | 1.336(1)         |
|                     |         |          | 1.336(7)         |
|                     |         |          | 1.346(3)         |
|                     |         |          | 1.327(7)         |
| 21                  | 1/5     | (3,2)    | 2.533(5)         |
|                     |         | (5,0)    | 2.340(4)         |
|                     |         |          | 2.35(1)          |
|                     |         |          | 2.379(5)         |
|                     |         |          | 2.28(1)          |


 of CFFS at $\tau = i$ and the shift (or orbital spin), as our result for $\nu = 1/2$ indicates? If the answer is yes, how can one interpret the values found in Table I and why do the different LLL projections yield different values? Is some flux attachment (i.e., $\alpha$ in Table I) more physical than others? We hope these problems will be addressed in the future.

ACKNOWLEDGMENTS

We are grateful to J. K. Jain for our discussion on the results and his advice on the manuscript. We also would like to thank Michael Fremling for his comments on the manuscript and D. T. Son for our discussion that inspired this work. This work was supported in part by the U. S. Department of Energy, Office of Basic Energy Sciences, under Grant No. [de-sc0005042. The numerical part of this research was conducted with Advanced CyberInfrastructure computational resources provided by The Institute for CyberScience at The Pennsylvania State University.

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The Hall viscosity for CFFS wave functions for different shapes of torus shown in the same convention as Table I. The first two CFFS shapes have hexagonal torus shapes, while the third one has circular as possible for each torus so that they correspond to the global ground states. Note that the $\tau = 37$ column has different wave vector configuration compared to the $N = 37$ column in Table I. The last digit in bracket represents the statistical error.

| Number of particles $\nu$ | $\alpha$ | $\frac{4\eta^4}{\hbar n}$ | $\nu$ | $\alpha$ | $\frac{4\eta^4}{\hbar n}$ |
|--------------------------|---------|------------------------|------|---------|------------------------|
| 1/2                      | (2)     | 1.01(2)                | 1/2  | (2)     | 1.01(2)                |
| 1/4                      | (2,2)   | 2.04(1)                | 1/4  | (2,2)   | 2.07(1)                |
| 1/6                      | (6,0,0) | 2.89(2)                | 1/6  | (6,0,0) | 2.97(2)                |
| 1/3                      | (3)     | 1.33(4)                | 1/3  | (3)     | 1.33(4)                |
| 1/5                      | (3,2)   | 2.57(1)                | 1/5  | (3,2)   | 2.62(2)                |

TABLE II. The Hall viscosity for CFFS wave functions for different shapes of torus shown in the same convention as Table I.
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