Quasi-invariance of the stochastic flow associated to Itô’s SDE with singular time-dependent drift

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Abstract

In this paper we consider the Itô SDE

$$dX_t = dW_t + b(t, X_t) dt, \quad X_0 = x \in \mathbb{R}^d,$$

where $W_t$ is a $d$-dimensional standard Wiener process and the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ belongs to $L^p(0, T; L^q(\mathbb{R}^d))$ with $p \geq 2$, $q > 2$ and $\frac{2}{p} + \frac{2}{q} < 1$. In 2005, Krylov and Röckner [10] proved that the above equation has a unique strong solution $X_t$. Recently it was shown by Fedrizzi and Flandoli [6] that the solution $X_t$ is indeed a stochastic flow of homeomorphisms on $\mathbb{R}^d$. We prove in the present work that the Lebesgue measure is quasi-invariant under the flow $X_t$.

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Key words: Stochastic differential equation, strong solution, flow of homeomorphisms, quasi-invariance, Zvonkin-type transformation

1 Introduction

Let $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^m \otimes \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be measurable functions, and $W_t$ a standard Wiener process in $\mathbb{R}^m$. Consider the Itô SDE

$$dX_t = \sigma(t, X_t) dW_t + b(t, X_t) dt, \quad X_0 = x \in \mathbb{R}^d. \tag{1.1}$$

It is a classical result that if the coefficients $\sigma$ and $b$ are globally Lipschitz continuous with respect to the spatial variable (uniformly in $t$), then the solution $X_t$ to (1.1) constitutes a stochastic flow of homeomorphisms. In the past years, weaker conditions on the modulus of continuity of $\sigma$ and $b$, such as log-Lipschitz continuity [2, 17], have been found which still ensure the existence of a flow of homeomorphisms.

When the diffusion coefficient $\sigma$ is uniformly non-degenerate, the equation (1.1) may have pathwise uniqueness under quite weak conditions on the drift $b$. The first result in this direction is due to Veretennikov [16], which says that if $\sigma(t, \cdot)$ is bounded Lipschitz continuous and satisfies a non-degeneracy condition, then the SDE (1.1) admits a unique strong solution once $b$ is bounded measurable. In [2], Gyöngy and Martínez generalized this result to the case where $\sigma(t, \cdot)$ is locally Lipschitz continuous, and the drift coefficient $b$ is dominated by the sum of a positive constant and an integrable function. Their method relies on a convergence result of the solutions of approximating SDEs to that of the limiting SDE, which follows from the Krylov estimate.

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Zhang improved their results in [18] by replacing the locally Lipschitz continuity of \(\sigma(t, \cdot)\) with some integrability condition.

In the influential paper [10], Krylov and Röckner considered the case where \(\sigma \equiv Id\) (the identity matrix of order \(d\), hence \(W_t\) is now a \(d\)-dimensional standard Wiener process) and the drift \(b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) satisfies

\[
\int_0^T \left( \int_{\mathbb{R}^d} |b(t, x)|^p \, dx \right)^{\frac{2}{p}} \, dt < +\infty
\]

with \(p \geq 2, q > 2\) such that

\[
\frac{d}{p} + \frac{2}{q} < 1. \tag{1.3}
\]

Hence the Itô SDE becomes

\[
dX_t = dW_t + b(t, X_t) \, dt, \quad X_0 = x \in \mathbb{R}^d. \tag{1.4}
\]

They proved that the above equation has a unique strong solution by using Yamada–Watanabe’s criterion: existence of weak solution plus pathwise uniqueness implies the existence of a unique strong solution. The regularity properties of functions in the Sobolev space \(H^{2p}_{2,q}(T)\) (see the next section for its definition) play an important role in the proof of pathwise uniqueness of (1.4). Recently, Fedrizzi and Flandoli proved that the solution \(X_t\) is indeed a stochastic flow of homeomorphisms on \(\mathbb{R}^d\) (see [6, Theorem 1.2]). Moreover, when \(v_0 \in \cap_{r \geq 1} W^{1, r}(\mathbb{R}^d)\), they showed in [5] that \(v(t, x) := v_0(X_t^{-1}(x))\) is the unique weakly differentiable solution to the SPDE

\[
dv + \langle b, \nabla v \rangle \, dt + \langle \nabla v, dW_t \rangle = \frac{1}{2} \Delta v \, dt, \quad v|_{t=0} = v_0,
\]

where \(\langle \cdot, \cdot \rangle\) is the inner product in \(\mathbb{R}^d\) (cf. [13] for related studies). When the dimension \(d = 1\) and the drift \(b\) is time-independent, Aryasovay and Pilipenko [11] obtained the Sobolev regularity of the flow \(X_t\) under the assumptions that \(b\) has linear growth and locally finite variation. We mention that similar problems were studied in [7] when \(b \in L^\infty(0, T; C^\alpha_b(\mathbb{R}^d, \mathbb{R}^d))\) for some \(\alpha \in (0, 1)\) (cf. [8] for non-constant diffusion coefficients). X. Zhang proved in [20] the stochastic homeomorphism flow property for the SDE (1.1) with uniformly non-degenerate diffusion coefficient.

Our purpose in the present work is to show that the Lebesgue measure is quasi-invariant under the flow \(X_t\) generated by (1.4) with \(b\) satisfying (1.2). Here the quasi-invariance means that, almost surely, the push-forward of the Lebesgue measure by the flow is equivalent to itself. Recall that Fedrizzi and Flandoli [14] proved in this case that, almost surely, \(X_t \in C^\alpha(\mathbb{R}^d, \mathbb{R}^d)\) for any \(\alpha \in (0, 1)\); moreover, the flow \(X_t\) is weakly differentiable in the following sense (cf. [6, Theorem 1.2]): for any \(x \in \mathbb{R}^d\), the limit

\[
\lim_{h \to 0} \frac{X(x + he_i) - X(x)}{h}
\]

exists in \(L^2(\Omega \times [0, T], \mathbb{R}^d)\), where \(\{e_1, \ldots, e_d\}\) is the canonical basis of \(\mathbb{R}^d\). However, these regularity properties of the flow \(X_t\) are not sufficient to conclude that the Lebesgue measure is quasi-invariant under the action of \(X_t\). We would like to mention that the existence and uniqueness of generalized stochastic flow associated to SDEs with coefficients in Sobolev spaces are studied in [19, 3, 21, 14], showing that the reference measure is quasi-invariant under the flow when the divergence and gradient of coefficients fulfill suitable (exponential) integrability.

To state the main result of this work, we denote by \(L^d\) the Lebesgue measure on \(\mathbb{R}^d\) and \((X_t)_{\#} L^d := L^d \circ X_t^{-1}\) the push-forward of \(L^d\) by the flow \(X_t\).
Theorem 1.1. Let \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be a time-dependent vector field such that \([1,2]\) holds with \( p \geq 2 \) and \( q > 2 \) satisfying \([1,3]\). Then for all \( t \in [0, T] \), \( (X_t)_{\#} L^d \) is equivalent to \( L^d \) almost surely; in other words, the Lebesgue measure is quasi-invariant under the stochastic flow \( X_t \) of homeomorphisms generated by \([1,4]\).

We point out that it is indeed not difficult to show that, almost surely, the push-forward \((X_t)_{\#} L^d \) is absolutely continuous with respect to \( L^d \), based on the estimates in \([5]\) (see the proof of Proposition 3.1 in the current paper). The difficult part lies in the proof of that the Radon–Nikodym density \( K_t := \frac{d(X_t)_{\#} L^d}{dL^d} \) is everywhere positive, that is, the two measures are equivalent. To achieve this purpose, we shall make use of the Zvonkin-type transformation introduced in \([5]\) to get a new SDE which has more regular coefficients. We first prove that the Lebesgue measure is quasi-invariant under the flow generated by this SDE, then we transfer the quasi-invariance property to the original flow \( X_t \). Using this method, we do not need the existence of generalized divergence of the drift \( b \), in contrast to \([13]\) Theorem 1.1. The same idea does work to extend our result to the more general case studied by X. Zhang \([20]\) (of course, we have to extend \([5]\) Lemmas 3 and 5) to this setting. We don’t want to do such technical extensions in this short note.

The organization of this paper is as follows. In Section 2 we recall some known results which are critical for proving Theorem 1.1. In particular, we introduce the Zvonkin-type transformation (also called Itô–Tanaka trick) used by Fedrizzi and Flandoli \([5,6]\) to prove the existence of unique strong solution to \([1,4]\). Following the ideas in \([12,19]\), we first prove in Section 3 the quasi-invariance of the flow \( Y_t \) which is the strong solution to the transformed SDE \([2,7]\), then we transfer this property to the solution \( X_t \) through a \( C^1 \)-diffeomorphism.

2 Notations and preliminary results

In this section we first introduce some notations of function spaces and then collect some known results which are crucial for our paper. The main references are \([5,6]\).

For \( p \geq 1 \), \( L^p(\mathbb{R}^d) \) denotes the usual space of (possibly vector-valued) functions on \( \mathbb{R}^d \) which are Lebesgue integrable of order \( p \). Let \( f(t,x) \) be a function of time and space, we will use superscripts to characterize the time-part of the norm and subscripts for the space-part: we will have \( L^p_\beta(S,T) = L^q(S,T; L^p(\mathbb{R}^d)) \). For simplicity, \( L^q_p(T) := L^q_p(0,T) \). We also need some notations of Sobolev spaces: \( W^{\alpha,p}(\mathbb{R}^d) \) is the usual Sobolev space, and

\[
H^\alpha_\beta(T) = L^q(0,T; W^{\alpha,p}(\mathbb{R}^d)), \quad H^\beta_p(T) = W^{\beta,q}(0,T; L^p(\mathbb{R}^d)).
\]

Finally, \( H^\alpha_{\beta,p}(T) = H^\alpha_{\beta,p}(T) \cap H^\beta_p(T) \).

Now we state the following result concerning the existence and uniqueness of weak solutions to \([1,4]\) (cf. \([6]\) Theorem 2.5 and Corollary 2.6).

Theorem 2.1. Assume that \( b \in L^p_\beta(T) \) with \( p,q \) satisfying \([1,3]\). Then

(i) for fixed \( x \in \mathbb{R}^d \), there exist processes \( X_t, W_t \) defined for \( t \in [0,T] \) on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) such that \( W_t \) is a \( d \)-dimensional \( (\mathcal{F}_t)\)-Wiener process and \( X_t \) is an \( (\mathcal{F}_t)\)-adapted, continuous, \( d \)-dimensional process for which

\[
\mathbb{P} \left( \int_0^T |b(t, X_t)|^2 \, dt < \infty \right) = 1 \tag{2.1}
\]

and almost surely, for all \( t \in [0,T] \),

\[
X_t = x + W_t + \int_0^t b(s, X_s) \, ds.
\]
(ii) weak uniqueness holds for the equation (1.4) in the class of solutions satisfying (2.1); moreover, if \( f \in L_{\tilde{p}}^q(T) \) with \( \frac{d}{p} + \frac{2}{q} < 1 \), then for any \( k \in \mathbb{R} \), there exists a constant \( C_f \) depending on \( \|f\|_{L_{\tilde{p}}^q(T)} \) such that
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E} \left[ e^{k \int_0^T |f(t,X_t)|^2 \, dt} \right] \leq C_f. \tag{2.2}
\]

The next result (see [6, Theorem 3.3]) concerning the regularity of solutions to the backward parabolic system (2.3) plays an important role.

**Theorem 2.2.** Let \( \lambda > 0 \) and \( p \geq 2, q > 2 \) such that (1.3) holds. Take two vector fields \( b, f \in L^q_p(T) \). Then in \( H^{q,2}_{2,p}(T) \) there exists a unique solution of the backward parabolic system
\[
\partial_t u + \frac{1}{2} \Delta u + b \cdot \nabla u - \lambda u + f = 0, \quad u(T, x) = 0. \tag{2.3}
\]
Moreover, there exists a finite constant \( N \) depending only on \( d, p, q, T, \lambda \) and \( \|b\|_{L^q_p(T)} \) such that
\[
\|u\|_{H^{q,2}_{2,p}(T)} := \|\partial_t u\|_{L^q_p(T)} + \|u\|_{\bar{H}^{q,2}_{2,p}(T)} \leq N \|f\|_{L^q_p(T)}. \tag{2.4}
\]

We also have

**Lemma 2.3.** Let \( u_\lambda \) be the solution of (2.3). Then
\[
\sup_{t \leq T} \|\nabla u_\lambda\|_{\infty} \to 0 \quad \text{as} \quad \lambda \to \infty,
\]
where \( \|\cdot\|_{\infty} \) is the supremum norm in the space \( C(\mathbb{R}^d) \) of continuous functions.

In view of the above lemma, we fix \( \lambda > 0 \) such that
\[
\sup_{t \leq T} \|\nabla u_\lambda\|_{\infty} \leq \frac{1}{2}. \tag{2.5}
\]
Define
\[
\phi_\lambda(t, x) = x + u_\lambda(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\]
The properties of the map \( \phi_\lambda \) are collected in the next proposition (cf. [6, Lemma 3.5]).

**Proposition 2.4.** The following statements hold:

(i) uniformly in \( t \in [0, T] \), \( \phi_\lambda(t, \cdot) \) has bounded first derivatives which are Hölder continuous;

(ii) for every \( t \in [0, T] \), \( \phi_\lambda(t, \cdot) \) is a \( C^1 \)-diffeomorphism on \( \mathbb{R}^d \);

(iii) \( \phi_\lambda^{-1}(t, \cdot) := (\phi_\lambda(t, \cdot))^{-1} \) has bounded first spatial derivatives, uniformly in \( t \);

(iv) \( \phi_\lambda \) and \( \phi_\lambda^{-1} \) are jointly continuous in \( (t, x) \).

We are now ready to state the Zvonkin-type transformation used in [7, 5, 6] to prove the existence of unique strong solution to Itô’s SDE with irregular drift coefficient. Replacing \( f \) by \( b \) in the parabolic equation (2.3), that is, we consider
\[
\partial_t u_\lambda + \frac{1}{2} \Delta u_\lambda + b \cdot \nabla u_\lambda = \lambda u_\lambda - b, \quad u_\lambda(T, x) = 0. \tag{2.6}
\]
In the following we shall fix a $\lambda > 0$ such that $[2.5]$ holds, and we omit the subscript $\lambda$ to simplify notations. Denote by $\phi_t(x) = x + u(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$. By Itô’s formula,

$$
\begin{align*}
    du(t, X_t) &= \frac{\partial u}{\partial t}(t, X_t) dt + \nabla u(t, X_t)(b(t, X_t) dt + dW_t) + \frac{1}{2} \Delta u(t, X_t) dt \\
    &= \lambda u(t, X_t) dt - b(t, X_t) dt + \nabla u(t, X_t) dW_t.
\end{align*}
$$

Define the new process $Y_t = \phi_t(X_t(\phi_0^{-1})) = X_t(\phi_0^{-1}) + u(t, X_t(\phi_0^{-1}))$. Then (we write $X_t$ instead of $X_t(\phi_0^{-1})$ to save notations)

$$
\begin{align*}
    dY_t &= b(t, X_t) dt + dW_t + \lambda u(t, X_t) dt - b(t, X_t) dt + \nabla u(t, X_t) dW_t \\
    &= \lambda u(t, X_t) dt + (Id + \nabla u(t, X_t)) dW_t,
\end{align*}
$$

where $Id$ is the identity matrix of order $d$. Since $\phi_t$ is a $C^1$-diffeomorphism, we can define

$$
\begin{align*}
    \tilde{\sigma}(t, y) &= Id + \nabla u(t, \phi_t^{-1}(y)), \\
    \tilde{b}(t, y) &= \lambda u(t, \phi_t^{-1}(y)), \quad (t, y) \in [0, T] \times \mathbb{R}^d.
\end{align*}
$$

Therefore $Y_t$ satisfies the new Itô’s SDE

$$
\begin{align*}
    dY_t &= \tilde{\sigma}(t, Y_t) dW_t + \tilde{b}(t, Y_t) dt, \quad Y_0 = x. \tag{2.7}
\end{align*}
$$

We shall see in the next result that the coefficients of SDE (2.7) are more regular than those of (1.4), thus it is easier to be treated.

**Proposition 2.5.** We have $\nabla \tilde{b} \in C([0, T], C_b(\mathbb{R}^d))$ and

$$
\tilde{\sigma} \in C([0, T], C_b(\mathbb{R}^d)) \cap L^q(0, T; W^{1, p}(\mathbb{R}^d)).
$$

Based on the regularity of the coefficients $\tilde{\sigma}, \tilde{b}$ and applying Krylov-type estimates, Fedrizzi and Flandoli first proved the pathwise uniqueness of solutions to the equation (2.7), and then translated this result to the solution $X_t$ of the original equation (1.4) via the transformation $\phi_t$. Thus by Yamada–Watanabe’s criterion, both equations (2.7) and (1.4) have a unique strong solution. The following theorem is the main result in [6] (see Theorem 1.2 there).

**Theorem 2.6.** Under the assumption of Theorem 2.1 the equation (1.4) has a unique strong solution $X_t$ which defines a stochastic flow of homeomorphisms and is $\alpha$-Hölder continuous for every $\alpha < 1$.

From Proposition 2.5 and Sobolev’s embedding theorem, we see that, for a.e. $t \in [0, T]$, $\tilde{\sigma}(t, \cdot)$ is Hölder continuous of order $\theta < 1$. Therefore we cannot directly apply the classical results (see for instance [11] Lemma 4.3.1) to conclude that the solution $Y_t$ to (2.7) leaves the reference measure quasi-invariant. To show our main theorem, we need to do some approximation arguments. Here are the necessary preparations (see [4] Lemma 12]).

**Lemma 2.7.** Let $b^n$ be a sequence of smooth vector fields converging to $b$ in $L^2_0(T)$, and $u^n$ the solution to (2.6) with $b$ replaced by $b^n$. Then we have

(i) $u^n(t, x)$ and $\nabla u^n(t, x)$ converge pointwise in $(t, x)$ to $u(t, x)$ and $\nabla u(t, x)$ respectively, and the convergence is uniform on compact sets;

(ii) $\lim_{n \to \infty} \|u^n - u\|_{H^2_{L^p}(T)} = 0$;

(iii) there exists a $\lambda$ for which $\sup_{n \geq 1} \sup_{t, x} |\nabla u^n(t, x)| \leq \frac{1}{2}$;
(iv) $\sup_{n \geq 1} \| \nabla^2 u^n \|_{L^p_b(T)} \leq C$.

Let $Y^n_t(x)$ be the flow associated to $\varphi^n_t$ with the coefficients
\[
\tilde{\sigma}^n(t, y) = I d + \nabla u^n(t, \phi^n_t^{-1}(y)), \quad \tilde{b}^n(t, y) = \lambda u^n(t, \phi^n_t^{-1}(y)),
\]
where $\phi^n_t(x) = x + u^n(t, x)$. For $R > 0$, $B_R$ denotes the ball in $\mathbb{R}^d$ centered at the origin with radius $R$. We have

**Proposition 2.8.** In the situation of Lemma 2.4, for every $R > 0$ and $k \geq 2$, we have
\[
\lim_{n \to \infty} \sup_{t \leq T} \sup_{x \in B_R} \mathbb{E}(|Y^n_t(x) - Y_t(x)|^k) = 0.
\]

The same convergence holds for the inverse flows $Y^n_{t-1}$ and $Y^{-1}_t$. Moreover, there exists $C_k$ independent on $n \geq 1$ such that
\[
\mathbb{E} \left[ \sup_{t \leq T} |Y^n_t(x)|^k \right] \leq C_k(1 + |x|^k).
\]

**Proof.** The first assertion was shown in the proof of [5, Lemma 3] (see p.1336), while the second one is a consequence of [5, Lemma 3] and the relation $Y^n_{t-1} = \phi^n_0(X^n_{t-1}(\phi^{-1}_n))$, where $X^n_t$ is the flow associated to (1.4) with $b$ replaced by $b_n$, and $X^{n-1}_t$ is its inverse flow. As for the last estimate, it is a slight improvement of [5, (26)] by removing $\sup_{t \leq T}$ into the expectation: this follows from the uniform growth of the coefficients $\tilde{\sigma}^n, \tilde{b}^n$ and classical moment estimates. \qed

### 3 Proof of the main result

In this section we first prove that the Lebesgue measure is quasi-invariant under the stochastic flow $Y_t$ generated by the new equation (2.7), following the ideas in [12, 19]. Then we transfer this property to the solution $X_t$ of the original SDE (1.4) by using the diffeomorphism $\phi_t : x \mapsto x + u(t, x)$, where $u$ solves the parabolic equation (2.6).

We start by recalling the setting. Let $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a time-dependent vector field verifying the assumption of Theorem 1.1 and $u(t, x)$ the solution to the parabolic system (2.6).

The transformation $\phi_t$ and the coefficients $\tilde{\sigma}, \tilde{b}$ are defined as in Section 2. As mentioned in the last section, the diffusion coefficient $\tilde{\sigma}$ of (2.7) is only Hölder continuous, which makes it impossible to directly apply the existing results to conclude the quasi-invariance of $Y_t : \mathbb{R}^d \to \mathbb{R}^d$. Therefore we take a sequence $\{b^n\}_{n \geq 1}$ of smooth vector fields with compact supports in $[0, T] \times \mathbb{R}^d$ such that
\[
\lim_{n \to \infty} \|b^n - b\|_{L^p(T)} = 0.
\]

Denote by $X^n_t$ the flow of diffeomorphisms generated by (1.4) with $b$ replaced by $b^n$, and $X^{n-1}_t$ its inverse flow. We include the following estimate which was proved in [5, Lemma 5]: for every $k \geq 1$,
\[
\sup_{n \geq 1} \sup_{t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}(\| \nabla X^{n-1}_t(x) \|^k) < +\infty.
\]

Let $u^n$ be the solution to (2.7) with $b$ replaced by $b^n$. Then $u^n$ is smooth with bounded derivatives (cf. [7, Theorem 2] where $u^n$ has bounded derivatives up to order 2 when $b^n \in L^\infty(0, T; C^2_b(\mathbb{R}^d, \mathbb{R}^d))$; note that the parabolic equation (9) in [7] is not accompanied with the boundary condition $u(T, \cdot) = 0$). By Lemma 2.4(iii), we shall fix $\lambda$ big enough such that
\[
\sup_{t \leq T} \| \nabla u(t, \cdot) \|_{\infty} \vee \sup_{n \geq 1} \sup_{t \leq T} \| \nabla u^n(t, \cdot) \|_{\infty} \leq \frac{1}{2}.
\]
Let \(\tilde{\sigma}^n(t, y)\) and \(\tilde{b}^n(t, y)\) be defined as in (2.8). We consider the Itô SDE

\[
dY^n_t = \tilde{\sigma}^n(t, Y^n_t) \, dW_t + \tilde{b}^n(t, Y^n_t) \, dt, \quad Y^n_0 = x.
\]

Since the coefficients \(\tilde{\sigma}^n(t, y)\) and \(\tilde{b}^n(t, y)\) are smooth with bounded spatial derivatives, uniformly in \(t \in [0, T]\), we know that \(Y^n_t\) is a flow of diffeomorphisms on \(\mathbb{R}^d\). The inverse flow is denoted by \(Y^{n,-1}_t\). Moreover, \(Y^n_t = \phi^n_t(X^n_0)\) with \(\phi^n_t(x) = x + u^n(t, x)\).

In the sequel, we denote by \((Y^n_t)_\# L^d = L^d \circ Y^{n,-1}_t\) and \((Y^{n,-1}_t)_\# L^d = L^d \circ Y^n_t\) the push-forwards of the Lebesgue measure \(L^d\) by the flows \(Y^n_t\) and \(Y^{n,-1}_t\). Then it is well-known that

\[
\rho^n_t := \frac{d(Y^n_t)_\# L^d}{dL^d} = |\det \nabla Y^n_t| \quad \text{and} \quad \rho^n_{t} := \frac{d(Y^{n,-1}_t)_\# L^d}{dL^d} = |\det \nabla Y^n_t|.
\]

The following simple relation holds:

\[
\rho^n_t(x) = \left[\rho^n_t(Y^{n,-1}_t(x))\right]^{-1}. \quad (3.4)
\]

Moreover, by [11, Lemma 4.3.1] (see also [19, (2.2)]), the density function \(\rho^n_t\) has the following explicit expression:

\[
\tilde{\rho}_t^n(x) = \exp \left\{ \int_0^t \left( \text{div}(\tilde{\sigma}^n)(s, Y^n_s(x)), dW_s \right) + \int_0^t \left[ \text{div}(\tilde{b}^n) - \frac{1}{2} \langle \nabla \tilde{\sigma}^n, (\nabla \tilde{\sigma}^n)^\top \rangle \right](s, Y^n_s(x)) \, ds \right\},
\]

where \(\text{div}(\tilde{\sigma}^n) = (\text{div}(\tilde{\sigma}^n, 1), \ldots, \text{div}(\tilde{\sigma}^n, d))\) is a vector-valued function whose components are the divergences of the column vectors of \(\tilde{\sigma}^n\), and \(\langle \nabla \tilde{\sigma}^n, (\nabla \tilde{\sigma}^n)^\top \rangle = \sum_{k=1}^d \sum_{j,k=1}^d (\partial_j \tilde{\sigma}^n_{ik})(\partial_k \tilde{\sigma}^n_{ik})\). Next, noting that \(x = \tilde{\phi}_t^n(\phi^{n,-1}_t(x)) = \phi^{n,-1}_t(x) + u^n(t, \phi^{n,-1}_t(x))\), thus

\[
Id = \nabla \phi^{n,-1}_t(x) + \nabla u^n(t, \phi^{n,-1}_t(x)) \nabla \phi^{n,-1}_t(x).
\]

As a result, for any \(x \in \mathbb{R}^d\), we have by (3.3) that \(\| \cdot \|_{\text{op}}\) is the operator norm

\[
1 = \|Id\|_{\text{op}} \geq \| \nabla \phi^{n,-1}_t(x) \|_{\text{op}} - \| \nabla u^n(t, \phi^{n,-1}_t(x)) \|_{\text{op}} \| \nabla \phi^{n,-1}_t(x) \|_{\text{op}} \geq \frac{1}{2} \| \nabla \phi^{n,-1}_t(x) \|_{\text{op}},
\]

that is,

\[
\sup_{t \leq T} \| \nabla \phi^{n,-1}_t(x) \|_{\text{op}} \leq 2. \quad (3.6)
\]

Combining this estimate with (3.2) and the relation \(Y^{n,-1}_t = \phi^n_0(X^{n,-1}_0(\phi^{n,-1}_t))\), we obtain

\[
\sup_{n \geq 1} \sup_{t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left( \rho^n_t(x)^k \right) = \sup_{n \geq 1} \sup_{t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left( |\det \left( \nabla Y^n_t(x) \right)|^k \right) < \infty, \quad \text{for all } k \geq 1. \quad (3.7)
\]

Now we are ready to show that the Lebesgue measure is absolutely continuous under the action of the flow \(Y_t\) generated by (2.7).

**Proposition 3.1 (Absolute continuity under the flow \(Y_t\)).** Assume the condition of Theorem [17]. Then for any \(t \in [0, T]\), the push-forward \((Y_t)_\# L^d\) of the Lebesgue measure is absolutely continuous with respect to \(L^d\). Moreover, the Radon–Nikodym density \(\rho_t := \frac{d(Y_t)_\# L^d}{dL^d}\) satisfies

\[
\sup_{t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left( \rho_t(x)^k \right) < \infty, \quad \text{for all } k \geq 1. \quad (3.8)
\]
Proof. Based on the estimate \eqref{eq:3.7} and Proposition \ref{prop:2.8}, this result is a consequence of \cite[Lemma 3.5]{19}. We include its proof here for the reader’s convenience. Proposition \ref{prop:2.8} implies that, up to a subsequence, $Y^n_t(\omega, x)$ converges to $Y_t(\omega, x)$ for $(\mathbb{P} \otimes L^d)$-a.e. $(\omega, x)$ as $n \to \infty$. We fix any $N > 0$ and let $C_N(\mathbb{R}^d, \mathbb{R}_+)$ be the collection of nonnegative continuous functions with support in $B_N$. Then for any $\varphi \in C_N(\mathbb{R}^d, \mathbb{R}_+)$, by Fubini’s theorem and Fatou’s lemma, it holds for a.s. $\omega \in \Omega$ that

$$
\int_{\mathbb{R}^d} \varphi(Y_t(x)) \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} \varphi(Y^n_t(x)) \, dx = \liminf_{n \to \infty} \int_{\mathbb{R}^d} \varphi(y) \rho^n_t(y) \, dy =: \liminf_{n \to \infty} J^n_{\varphi}(\omega). \tag{3.9}
$$

By \eqref{eq:3.7}, there exists a subsequence still denoted by $n$ and a $\rho^{(0)}_t \in L^\infty(\mathbb{R}^d; L^k(\Omega))$ satisfying \eqref{eq:3.8} such that

$$\rho^n_t \text{ weakly * converges to } \rho^{(0)}_t \text{ in } L^\infty(\mathbb{R}^d; L^k(\Omega)).$$

Since $\rho^n_t$ also converges weakly to $\rho^{(0)}_t$ in $L^2(\Omega \times B_N)$, by Banach–Saks theorem, there is another subsequence still denoted by $n$ such that its Cesàro mean $\hat{\rho}^n_t := \frac{1}{n} \sum_{k=1}^{n} \rho^n_t$ converges strongly to $\rho^{(0)}_t$ in $L^2(\Omega \times B_N)$. Therefore, up to a subsequence, $\hat{\rho}^n_t(\omega)$ converges to $\rho^{(0)}_t(\omega)$ in $L^2(B_N)$ for a.s. $\omega$. Hence

$$\int_{\mathbb{R}^d} \varphi(Y_t(x)) \, dx \leq \liminf_{n \to \infty} J^n_{\varphi}(\omega) \leq \lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(y) \hat{\rho}^n_t(y) \, dy \quad \text{as } n \to \infty.
$$

Combining this limit together with \eqref{eq:3.9} gives us

$$\int_{\mathbb{R}^d} \varphi(Y_t(x)) \, dx \leq \lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(y) \hat{\rho}^n_t(y) \, dy = \int_{\mathbb{R}^d} \varphi(y) \rho^{(0)}_t(y) \, dy.
$$

The separability of $C_N(\mathbb{R}^d, \mathbb{R}_+)$ implies that there is a common full set $\hat{\Omega}$ such that the above inequality holds for all $\omega \in \hat{\Omega}$ and $\varphi \in C_N(\mathbb{R}^d, \mathbb{R}_+)$. Since $N > 0$ is arbitrary, we conclude that $(Y_t)^{\#} L^d$ is absolutely continuous with respect to $L^d$ and the density function $\rho_t \leq \hat{\rho}^{(0)}_t$. Hence the estimate \eqref{eq:3.8} holds.

To show that the push-forward $(Y_t)^{\#} L^d$ is in fact equivalent to $L^d$, we intend to give in the following an explicit expression for the Radon–Nikodym density $\rho_t$. To this end, we shall prove that the density functions $\hat{\rho}^n_t$ defined in \eqref{eq:3.5} are convergent to

$$\hat{\rho}_t(x) = \exp \left\{ \int_0^t \langle \text{div}(\bar{\sigma})(s, Y_s(x)), dW_s \rangle + \int_0^t \left[ \text{div}(\bar{b}) - \frac{1}{2}(\nabla \bar{\sigma}, (\nabla \bar{\sigma})^*) \right](s, Y_s(x)) \, ds \right\} \tag{3.10}
$$

in some sense. We need the technical result below.

**Lemma 3.2.** Let $f \in L^2_b(T)$ with $\tilde{p}, \tilde{q}$ satisfying \eqref{eq:1.3}. Then for any $k \geq 1$, there exists a constant $C_{k,f}$ depending on $k$ and $\|f\|_{L^2_b(T)}$ such that

$$\sup_{n \geq 1} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left[ e^{k \int_0^T |f(t,Y^n_t(x))|^2 \, dt} \right] \leq C_{k,f}.
$$

**Proof.** This result is a consequence of Theorem \ref{thm:2.1}. Indeed, since $X^n_t$ is the solution to \eqref{eq:1.4} with $b$ replaced by $b^n$, then \eqref{eq:2.2} implies

$$\sup_{n \geq 1} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left[ e^{k \int_0^T |f(t,X^n_t(x))|^2 \, dt} \right] \leq C_{k,f}, \tag{3.11}
$$
Proof. We denote by $Y_i^n$ is related to $X_i^n$ by the diffeomorphism $\phi_i^n: Y_i^n = \phi_i^n(X_i^n(\phi_0^{n-1})).$ Thus

$$\mathbb{E} \left[ e^{k \int_0^T |f(t, Y_i^n(x))|^2 dt} \right] = \mathbb{E} \left[ \exp \left( k \int_0^T |f(t, \phi_i^n(X_i^n(\phi_0^{n-1}(x))))|^2 dt \right) \right].$$

Consider the new function $g^n(t, x) = f(t, \phi_i^n(x))$. By the change of variable,

$$\int_0^T \left( \int_{\mathbb{R}^d} |g^n(t, x)|^{\frac{p}{p-1}} dx \right)^{\frac{p}{p}} dt = \int_0^T \left( \int_{\mathbb{R}^d} |f(t, y)|^{\frac{p}{p-1}} \det (\nabla \phi_i^{n-1}(y)) dy \right)^{\frac{p}{p}} dt. \quad (3.12)$$

Inequality (3.6) implies

$$\sup_{n \geq 1} \sup_{y \in \mathbb{R}^d} \left| \det (\nabla \phi_i^{n-1}(y)) \right| < \infty,$$

which, together with (3.12), leads to

$$\sup_{n \geq 1} \|g^n\|_{L^p(\mathcal{F})} < \infty.$$  

Combining this estimate with (3.11) completes the proof. \hfill \Box

We now prove the following result which is analogous to [12, Lemma 3.5].

**Lemma 3.3** (Uniform estimate of Radon–Nikodym densities). For any $k \in \mathbb{R}$, it holds

$$\sup_{n \geq 1} \sup_{t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left[ (\hat{\rho}_i^n(x))^k \right] < +\infty.$$

**Proof.** We denote by $\xi^n = \text{div} (\tilde{b}^n) - \frac{1}{2} \langle \nabla \tilde{\sigma}^n, (\nabla \tilde{\sigma}^n)^* \rangle$ to simplify notations. Then

$$(\hat{\rho}_i^n(x))^k = \exp \left\{ k \int_0^t \langle \text{div}(\tilde{\sigma}^n)(s, Y_s^n(x)), dW_s \rangle + k \int_0^t \xi^n(s, Y_s^n(x)) \, ds \right\}$$

$$= \exp \left\{ k \int_0^t \langle \text{div}(\tilde{\sigma}^n)(s, Y_s^n(x)), dW_s \rangle - k^2 \int_0^t |\text{div}(\tilde{\sigma}^n)(s, Y_s^n(x))|^2 \, ds \right\}$$

$$\times \exp \left\{ \int_0^t (2|\text{div}(\tilde{\sigma}^n)|^2 + k\xi^n)(s, Y_s^n(x)) \, ds \right\}.$$ 

Thus by Cauchy’s inequality,

$$\mathbb{E} \left[ (\hat{\rho}_i^n(x))^k \right] \leq \left( \mathbb{E} \exp \left\{ 2k \int_0^t \langle \text{div}(\tilde{\sigma}^n)(s, Y_s^n(x)), dW_s \rangle - 2k^2 \int_0^t |\text{div}(\tilde{\sigma}^n)(s, Y_s^n(x))|^2 \, ds \right\} \right)^{\frac{1}{2}}$$

$$\times \left( \mathbb{E} \exp \left\{ 2 \int_0^t (2|\text{div}(\tilde{\sigma}^n)|^2 + k\xi^n)(s, Y_s^n(x)) \, ds \right\} \right)^{\frac{1}{2}}. \quad (3.13)$$

By Novikov’s criterion and Lemma 3.2 to show that the first exponential is a martingale, it suffices to prove that $\text{div}(\tilde{\sigma}^n) \in L^2_\mathbb{F}(\mathcal{F})$. This is a consequence of Lemma 2.7(iv) and the definition of $\tilde{\sigma}^n$. Next by (3.3) and (3.6),

$$|\xi^n| \leq |\text{div}(\tilde{b}^n)| + \frac{1}{2} |(\nabla \tilde{\sigma}^n, (\nabla \tilde{\sigma}^n)^*)| \leq \lambda \|\nabla u^n(t, \cdot)\|_{\infty} \|\nabla \phi_i^{n-1}\|_{\infty} + \frac{1}{2} |\nabla \tilde{\sigma}^n|^2$$

$$\leq C_\lambda + \frac{1}{2} |\nabla \tilde{\sigma}^n|^2.$$
Hence the second expectation in (3.13) is dominated by
\[ \sup_{n \geq 1} \sup_{x \in \mathbb{R}^d} \mathbb{E} \exp \left\{ 2 \int_0^T (k^2 |\text{div}(\tilde{\sigma}^n)|^2 + |k \xi^n|) (s, Y_s^{n}(x)) \, ds \right\} \]
\[ \leq C_{\lambda, |k|} T \sup_{n \geq 1} \sup_{x \in \mathbb{R}^d} \mathbb{E} \exp \left\{ (2k^2 + |k|) \int_0^T |\nabla \tilde{\sigma}^n (s, Y_s^{n}(x))|^2 \, ds \right\} < \infty, \]
where the last inequality follows from Lemmas 3.2 and 2.7(4).

Now we show that the three integrals in the bracket on the right hand side of (3.10) converge to the corresponding ones in (3.11). First we have

**Proposition 3.4.** For any \( R > 0 \), it holds that
\[
\lim_{n \to \infty} \mathbb{E} \int_{B_R} \sup_{0 \leq t \leq T} \left| \int_0^t \text{div}(\tilde{b}^n)(s, Y_s^n(x)) \, ds - \int_0^t \text{div}(\tilde{b})(s, Y_s(x)) \, ds \right| \, dx = 0,
\]
\[
\lim_{n \to \infty} \mathbb{E} \int_{B_R} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \nabla \tilde{\sigma}^n, (\nabla \tilde{\sigma}^n)^\ast \rangle (s, Y_s^n(x)) \, ds - \int_0^t \langle \nabla \tilde{\sigma}, (\nabla \tilde{\sigma})^\ast \rangle (s, Y_s(x)) \, ds \right| \, dx = 0,
\]
\[
\lim_{n \to \infty} \mathbb{E} \int_{B_R} \sup_{0 \leq t \leq T} \left| \int_0^t \langle \text{div}(\tilde{\sigma}^n)(s, Y_s^n(x)), dW_s \rangle - \int_0^t \langle \text{div}(\tilde{\sigma})(s, Y_s(x)), dW_s \rangle \right| \, dx = 0.
\]

**Proof.** The proofs of the three limits have some points in common, but there are minor differences that should be taken care of. We prove them separately.

(i) We denote by \( I_n \) the term on the left hand side of the first limit. Then
\[
I_n \leq \mathbb{E} \int_0^T \int_{B_R} \left| \text{div}(\tilde{b}^n)(s, Y_s^n(x)) - \text{div}(\tilde{b})(s, Y_s^n(x)) \right| \, dx \, ds
\]
\[ + \mathbb{E} \int_0^T \int_{B_R} \left| \text{div}(\tilde{b}^n)(s, Y_s^n(x)) - \text{div}(\tilde{b})(s, Y_s^n(x)) \right| \, dx \, ds,
\]
which are written as \( I_1^n \) and \( I_2^n \) respectively. Recall that \( p \geq 2, q > 2 \) satisfies (1.3). First by Hölder’s inequality (\( p’ \) is the conjugate number of \( p \)),
\[
I_1^n \leq \int_0^T \left( \mathbb{E} \int_{B_R} 1^{p’} \, dx \right)^{\frac{1}{p’}} \left( \mathbb{E} \int_{B_R} \left| (\text{div}(\tilde{b}^n) - \text{div}(\tilde{b}))(s, Y_s^n(x)) \right|^p \, dx \right)^{\frac{1}{p}} \, ds
\]
\[ \leq C_{p,R \, T} \int_0^T \left( \mathbb{E} \int_{\mathbb{R}^d} \left| (\text{div}(\tilde{b}^n) - \text{div}(\tilde{b}))(s, y) \right|^p \rho_s^n(y) \, dy \right)^{\frac{1}{p}} \, ds.
\]
Thus by (3.7), we have
\[
I_1^n \leq C_{p,R} \int_0^T \left( \mathbb{E} \int_{\mathbb{R}^d} \left| (\text{div}(\tilde{b}^n) - \text{div}(\tilde{b}))(s, y) \right|^p \rho_s^n(y) \, dy \right)^{\frac{1}{p}} \, ds
\]
\[ \leq C_{p,R,T} \sup_{0 \leq t \leq T} \left( \mathbb{E} \int_{\mathbb{R}^d} \left| (\text{div}(\tilde{b}^n) - \text{div}(\tilde{b}))(s, y) \right|^p \rho_s^n(y) \, dy \right)^{\frac{1}{p}} \, ds
\]
\[ = C_{p,R,T} \| \text{div}(\tilde{b}^n) - \text{div}(\tilde{b}) \|_{L^p_{\mu}(T)}.
\]
By the definition of \( \tilde{b}^n, \tilde{b} \) and Lemma 2.7(ii), we conclude that
\[
\lim_{n \to \infty} I_1^n = 0.
\]
Now we deal the second term $I^n_2$. For any $\varepsilon > 0$, we can find $f \in C_c([0, T] \times \mathbb{R}^d)$ such that

$$\|\text{div}(\bar{b}) - f\|_{L^p(T)} < \varepsilon.$$ Then

$$I^n_2 \leq \mathbb{E} \int_0^T \int_{BR} |(\text{div}(\bar{b}) - f)(s, Y^n_s(x))| \, dx \, ds + \mathbb{E} \int_0^T \int_{BR} |(f - \text{div}(\bar{b}))(s, Y_s(x))| \, dx \, ds$$

$$+ \mathbb{E} \int_0^T \int_{BR} |f(s, Y^n_s(x)) - f(s, Y_s(x))| \, dx \, ds := I^n_{2,1} + I^n_{2,2} + I^n_{2,3}.$$ Analogous to the treatment of $I^n_1$, we have by Hölder’s inequality and the estimate \((3.7)\) that

$$I^n_{2,1} \leq C_{p,R} \int_0^T \left(\mathbb{E} \int_{\mathbb{R}^d} |(\text{div}(\bar{b}) - f)(s, y)|^p \rho^n_s(y) \, dy\right)^{\frac{1}{p}} \, ds$$

$$\leq C'_{p,r,T} \|\text{div}(\bar{b}) - f\|_{L^p(T)} < C'_{p,r,T}\varepsilon.$$ In the same way, by \([3, N]\), we obtain $I^n_{2,2} < C'_{p,r,T}\varepsilon$. Next the dominated convergence theorem and Proposition \(2.3\) yield $\lim_{n \to \infty} I^n_{2,3} = 0$. Summarizing these discussions, we get $\lim_{n \to \infty} I^n_2 = 0$. Combining this result with the limit \((3.14)\), we obtain the first result.

(ii) We denote by $J^n$ the quantity in the second limit. Similarly we have

$$J^n \leq \mathbb{E} \int_0^T \int_{BR} |(\nabla \tilde{\sigma}^n, (\nabla \tilde{\sigma}^n)^*) - (\nabla \tilde{\sigma}, (\nabla \tilde{\sigma})^*)|(s, Y^n_s(x))| \, dx \, ds$$

$$+ \mathbb{E} \int_0^T \int_{BR} |(\nabla \tilde{\sigma}, (\nabla \tilde{\sigma})^*)(s, Y^n_s(x)) - (\nabla \tilde{\sigma}, (\nabla \tilde{\sigma})^*)(s, Y_s(x))| \, dx \, ds$$

which will be denoted by $J^n_1$ and $J^n_2$. In the following we shall assume $p > 2$ (which is the case when the dimension $d \geq 2$); in fact, the case $p = 2$ is simpler. Again by triangular inequality,

$$J^n_1 \leq \mathbb{E} \int_0^T \int_{BR} |(\nabla \tilde{\sigma}^n - \nabla \tilde{\sigma}, (\nabla \tilde{\sigma}^n)^*)|(s, Y^n_s(x))| \, dx \, ds$$

$$+ \mathbb{E} \int_0^T \int_{BR} |(\nabla \tilde{\sigma}, (\nabla \tilde{\sigma})^*) - (\nabla \tilde{\sigma}, (\nabla \tilde{\sigma})^*)|(s, Y^n_s(x))| \, dx \, ds =: J^n_{1,1} + J^n_{1,2}.$$ Since $p$ and $q$ are strictly greater than $2$, their conjugate numbers $p', q' < 2$. By Hölder’s inequality and \((3.7)\), we have for all $s \in [0, T]$,

$$\mathbb{E} \int_{BR} |\nabla \tilde{\sigma}^n(s, Y^n_s(x))|^p \, dx \leq C_{p,R} \left(\mathbb{E} \int_{BR} |\nabla \tilde{\sigma}^n(s, Y^n_s(x))|^p \, dx\right)^{\frac{1}{p}}$$

$$\leq C'_{p,R} \left(\int_{\mathbb{R}^d} |\nabla \tilde{\sigma}^n(s, y)|^p \, dy\right)^{\frac{1}{p}}.$$ Therefore

$$\int_0^T \left(\mathbb{E} \int_{BR} |\nabla \tilde{\sigma}^n(s, Y^n_s(x))|^p \, dx\right)^{\frac{1}{p}} \, ds \leq C'_{p,R} \int_0^T \left(\int_{\mathbb{R}^d} |\nabla \tilde{\sigma}^n(s, y)|^p \, dy\right)^{\frac{1}{p}} \, ds$$

$$\leq C_{p,q,R,T} \|\nabla \tilde{\sigma}^n\|_{L^p(T)}.$$ \((3.15)\)
which, by the definition of $\tilde{\sigma}^n$ and Lemma 2.7(iv), is uniformly bounded from above. In view of these discussions, an application of Hölder’s inequality leads to

$$J^n_{1,1} \leq \int_0^T \left( \mathbb{E} \int_{B_R} \left| (\nabla \tilde{\sigma}^n - \nabla \tilde{\sigma})(s, Y^n_s(x)) \right|^p \, dx \right)^{\frac{1}{p}} \left( \mathbb{E} \int_{B_R} \left| \nabla \tilde{\sigma}^n(s, Y^n_s(x)) \right|^p \, dx \right)^{\frac{1}{p}} \, ds$$

$$\leq C \int_0^T \left( \int_{\mathbb{R}^d} \left| (\nabla \tilde{\sigma}^n - \nabla \tilde{\sigma})(s, y) \right|^p \, dy \right)^{\frac{1}{p}} \left( \mathbb{E} \int_{B_R} \left| \nabla \tilde{\sigma}^n(s, y) \right|^p \, dy \right)^{\frac{1}{p}} \, ds$$

which, by (3.15) and Lemma 2.7(iv), is less than

$$C\|\nabla \tilde{\sigma}^n - \nabla \tilde{\sigma}\|_{L^p_p(T)} \|\nabla \tilde{\sigma}^n\|_{L^p_p(T)} \leq \tilde{C}\|\nabla \tilde{\sigma}^n - \nabla \tilde{\sigma}\|_{L^p_p(T)}.$$ 

Consequently, we deduce from Lemma 2.7(ii) that $\lim_{n \to \infty} J^n_{1,1} = 0$. Analogous arguments give us $\lim_{n \to \infty} J^n_{1,2} = 0$. Therefore

$$\lim_{n \to \infty} J^n_1 = 0. \tag{3.16}$$

Regarding the estimate of $J^n_2$, we first find a tensor-valued function $f \in C_c([0, T] \times \mathbb{R}^d; (\mathbb{R}^d)^{\otimes 3})$ such that $\|\nabla \tilde{\sigma} - f\|_{L^p_p(T)} < \varepsilon$, and then estimate it as below:

$$J^n_2 \leq \mathbb{E} \int_0^T \int_{B_R} \left| (\nabla \tilde{\sigma}, (\nabla \tilde{\sigma})^*) - (f, f^*) \right| (s, Y^n_s(x)) \, dx \, ds + \mathbb{E} \int_0^T \int_{B_R} \left| (f, f^*) \right| (s, Y^n_s(x)) \, dx \, ds + \mathbb{E} \int_0^T \int_{B_R} \left| (f, f^*) \right| (s, Y^n_s(x)) - (f, f^*) (s, Y^n_s(x)) \, dx \, ds.$$

For the last term, the dominated convergence theorem yields that its limit is 0 as $n \to \infty$. The first two terms can be dealt with as for $J^n_{1,1}$, and we conclude that they are bounded by $C\varepsilon$ for some $C > 0$ by the choice of the function $f$. As a result, $\lim_{n \to \infty} J^n_2 = 0$. Combining this with (3.16), we see that the second limit is also zero.

(iii) Finally, we denote by $K^n$ the quantity in the last limit. By Burkholder’s inequality,

$$K^n \leq C \int_{B_R} \mathbb{E} \left[ \left( \int_0^T \left| \text{div}(\tilde{\sigma}^n)(s, Y^n_s(x)) - \text{div}(\tilde{\sigma})(s, Y^n_s(x)) \right|^2 \, ds \right)^{\frac{1}{2}} \right] \, dx \leq C_R \left( \int_0^T \mathbb{E} \int_{B_R} \left| \text{div}(\tilde{\sigma}^n)(s, Y^n_s(x)) - \text{div}(\tilde{\sigma})(s, Y^n_s(x)) \right|^2 \, dx \, ds \right)^{\frac{1}{2}},$$

where the second inequality follows from Cauchy’s inequality. It suffices to estimate the term in the big bracket which will be denoted by $\tilde{K}^n$. We have

$$\tilde{K}^n \leq 2 \int_0^T \mathbb{E} \int_{B_R} \left| (\text{div}(\tilde{\sigma}^n) - \text{div}(\tilde{\sigma}))(s, Y^n_s(x)) \right|^2 \, dx \, ds + 2 \int_0^T \mathbb{E} \int_{B_R} \left| \text{div}(\tilde{\sigma})(s, Y^n_s(x)) - \text{div}(\tilde{\sigma})(s, Y^n_s(x)) \right|^2 \, dx \, ds =: \tilde{K}^n_1 + \tilde{K}^n_2.$$

Again we assume $p > 2$ in condition (1.3). Then by Hölder’s inequality and (3.7),

$$\tilde{K}^n_1 \leq 2 \int_0^T \left( \mathbb{E} \int_{B_R} 1 \, dx \right)^{1 - \frac{2}{p}} \left( \mathbb{E} \int_{B_R} \left| (\text{div}(\tilde{\sigma}^n) - \text{div}(\tilde{\sigma}))(s, Y^n_s(x)) \right|^p \, dx \right)^{\frac{1}{p}} \, ds \leq C_{p,R} \left( \mathbb{E} \int_{\mathbb{R}^d} \left| \text{div}(\tilde{\sigma}^n) - \text{div}(\tilde{\sigma})(s, y) \right|^p \, dy \right)^{\frac{1}{p}} \, ds \leq C_{p,R,q,T} \|\text{div}(\tilde{\sigma}^n) - \text{div}(\tilde{\sigma})\|_{L^p_p(T)}^2.$$
which, due to Lemma 2.7(ii), goes to 0 as $n$ increases to infinity. The treatment of $K^n_2$ is analogous to that of $I^n_2$, hence we omit it. The proofs are finally completed. 

We are at the position of proving

**Proposition 3.5** (Quasi-invariance under the flow $Y_t$). For any $t \in [0, T]$, the push-forward $(Y_t)_# L^d$ of the Lebesgue measure $L^d$ by the flow $Y_t$ is equivalent to $L^d$; moreover,

$$
\rho_t(x) := \frac{d(Y_t)_# L^d}{dL^d}(x) = \left[ \bar{\rho}_t(Y_t^{-1}(x)) \right]^{-1},
$$

where the Radon–Nikodym density $\bar{\rho}_t(x)$ is defined in (3.10).

**Proof.** By Proposition 3.4 there is a subsequence still denoted by $n$ such that for $(\mathbb{P} \times L^d)$-almost all $(\omega, x)$,

$$
\lim_{n \to \infty} \bar{\rho}_t^n(\omega, x) = \bar{\rho}_t(\omega, x) \text{ uniformly in } t \in [0, T].
$$

Taking into account the uniform bound proved in Proposition 3.3 we have for any $k \geq 1$ and $R > 0$ that

$$
\lim_{n \to \infty} \mathbb{E} \int_{B_R} |\bar{\rho}_t^n(x) - \bar{\rho}_t(x)|^k \, dx = 0. \quad (3.17)
$$

Now for any $\varphi, \psi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$, we have $\mathbb{P}$-a.s.,

$$
\int_{\mathbb{R}^d} \varphi(Y_t^{-n}(x)) \psi(x) \, dx = \int_{\mathbb{R}^d} \varphi(y) \psi(Y_t^n(y)) \bar{\rho}_t^n(y) \, dy \quad \text{for all } t \leq T.
$$

By (3.17) and Proposition 2.8 up to a subsequence, the two sides of the above equality are convergent in $L^1(\mathbb{P})$ for any fixed $t \in [0, T]$. Thus we get for $\mathbb{P}$-a.s. $\omega$,

$$
\int_{\mathbb{R}^d} \varphi(Y_t^{-1}(x)) \psi(x) \, dx = \int_{\mathbb{R}^d} \varphi(y) \psi(Y_t(y)) \bar{\rho}_t(y) \, dy. \quad (3.18)
$$

Since $C_c(\mathbb{R}^d, \mathbb{R}_+)$ is separable, we can find a full set $\Omega_t \subset \Omega$ such that for all $\omega \in \Omega_t$, the above identity holds for any $\varphi, \psi \in C_c(\mathbb{R}^d, \mathbb{R}_+)$. Noting that $\mathbb{P}$-a.s., $\bar{\rho}_t(y)$ is positive for all $(t, y) \in [0, T] \times \mathbb{R}^d$, we finish the proof by applying [19] Lemma 3.4(iii).

Finally we can prove the main result of this paper.

**Proof of Theorem 1.1.** Fix any $\varphi \in C_c(\mathbb{R}^d)$. Since $X_t = \phi_t^{-1}(Y_t(\phi_0))$, we have $\mathbb{P}$-a.s.,

$$
\int_{\mathbb{R}^d} \varphi(X_t(x)) \, dx = \int_{\mathbb{R}^d} \varphi(\phi_t^{-1}(Y_t(\phi_0)(x))) \, dx = \int_{\mathbb{R}^d} \varphi(\phi_t^{-1}(Y_t(y))) \cdot |\det(\nabla \phi_0^{-1}(y))| \, dy.
$$

Applying Propositions 3.5 and 3.4 leads to

$$
\int_{\mathbb{R}^d} \varphi(X_t(x)) \, dx = \int_{\mathbb{R}^d} \varphi(\phi_t^{-1}(x)) \rho_t(x) \cdot |\det(\nabla \phi_0^{-1}(Y_t^{-1}(x)))| \, dx
$$

$$
= \int_{\mathbb{R}^d} \varphi(y) \rho_t(\phi_t(y)) \cdot |\det(\nabla \phi_t(y))| \cdot |\det(\nabla \phi_0^{-1}(Y_t^{-1}(\phi_t(y))))| \, dy
$$

$$
= \int_{\mathbb{R}^d} \varphi(y) \left[ \bar{\rho}_t(Y_t^{-1}(\phi_t(y))) \right]^{-1} \cdot |\det(\nabla \phi_t(y))| \cdot |\det(\nabla \phi_0^{-1}(Y_t^{-1}(\phi_t(y))))| \, dy.
$$

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Therefore, for $\mathbb{P}$-a.s. $\omega$, $(X_t)_{#} \mathcal{L}^d$ is absolutely continuous with respect to $\mathcal{L}^d$ with the Radon–Nikodym density

$$K_t(x) := \frac{d(Y_t)_{#} \mathcal{L}^d}{d\mathcal{L}^d}(x) = \left[\hat{\rho}_t(Y_t^{-1}(\phi_t(x)))\right]^{-1} \left|\det(\nabla\phi_t(x))\right| \cdot \left|\det \left[\nabla \phi_t^{-1}(Y_t^{-1}(\phi_t(x)))\right]\right|.$$

Noticing that $\hat{\rho}_t$ is positive everywhere and $\nabla\phi_t(x)$ is non-degenerate for all $x \in \mathbb{R}^d$, we see that the Radon–Nikodym density $K_t(x)$ is positive for all $x \in \mathbb{R}^d$. Consequently, $(X_t)_{#} \mathcal{L}^d$ is equivalent to $\mathcal{L}^d$; in other words, the Lebesgue measure is quasi-invariant under the action of the flow $X_t$ generated by (1.4). \hfill \square

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