Some Consequences of Restrictions on Digitally Continuous Functions

Laurence Boxer *

Abstract

We study the consequences of some restrictions on digitally continuous functions. One of our results modifies easily to yield an analogous result for topological spaces.

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1 Introduction

If \( f : X \to Y \) is a continuous function between topological spaces, and \( \emptyset \neq A \subset X \), it is often true that knowledge of \( f|_A \) tells us little about \( f|_{X \setminus A} \). A digital image is often a model of an object in Euclidean space, and the concept of a digitally continuous function is modeled on the “preservation of nearness” notion of a Euclidean continuous function; however, when we consider a continuous function \( f : (X, \kappa) \to (Y, \lambda) \) between digital images, we often find that knowledge of \( f|_A \) tells us much about \( f|_{X \setminus A} \). In this paper, we continue the work of fixed point theory for digital images (see [24, 15, 18, 11, 12, 13, 14]) and coincidence theory for digital images (see [1]) by examining how restrictions placed on \( f|_A \) limit \( f|_{X \setminus A} \).

2 Preliminaries

Let \( \mathbb{N} \) denote the set of natural numbers; \( \mathbb{N}^* = \{0\} \cup \mathbb{N} \), the set of nonnegative integers; \( \mathbb{Z} \), the set of integers; and \( \mathbb{R} \), the set of real numbers. \#\( X \) will be used for the number of members of a set \( X \).

*Department of Computer and Information Sciences, Niagara University, New York, 14109, USA; and Department of Computer Science and Engineering, State University of New York at Buffalo
boxer@niagara.edu
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2.1 Adjacencies

Material in this section is largely quoted or paraphrased from [18].

A digital image is a pair \((X, \kappa)\) where \(X \subset \mathbb{Z}^n\) for some \(n\) and \(\kappa\) is an adjacency on \(X\). Thus, \((X, \kappa)\) is a graph for which \(X\) is the vertex set and \(\kappa\) determines the edge set. Usually, \(X\) is finite, although there are papers that consider infinite \(X\). Usually, adjacency reflects some type of “closeness” in \(\mathbb{Z}^n\) of the adjacent points. When these “usual” conditions are satisfied, one may consider a subset \(Y\) of \(\mathbb{Z}^n\) (typically, an \(n\)-dimensional cube) containing \(X\) as a model of a black-and-white “real world” image in which the black points (foreground) are represented by the members of \(X\) and the white points (background) by members of \(Y \setminus \{X\}\).

We write \(x \leftrightarrow_\kappa y\), or \(x \leftrightarrow y\) when \(\kappa\) is understood or when it is unnecessary to mention \(\kappa\), to indicate that \(x\) and \(y\) are \(\kappa\)-adjacent. Notations \(x \leftrightarroweq_\kappa y\), or \(x \leftrightarroweq y\) when \(\kappa\) is understood, indicate that \(x\) and \(y\) are \(\kappa\)-adjacent or are equal.

The most commonly used adjacencies are the \(c_u\)-adjacencies, defined as follows. Let \(X \subset \mathbb{Z}^n\) and let \(u \in \mathbb{Z}, 1 \leq u \leq n\). Then for points \(x = (x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) = y\) we have \(x \leftrightarrow_{c_u} y\) if and only if

- for at most \(u\) indices \(i\) we have \(|x_i - y_i| = 1\), and
- for all indices \(j\), \(|x_j - y_j| \neq 1\) implies \(x_j = y_j\).

The \(c_u\)-adjacencies are often denoted by the number of adjacent points a point can have in the adjacency. E.g.,

- in \(\mathbb{Z}\), \(c_1\)-adjacency is \(2\)-adjacency;
- in \(\mathbb{Z}^2\), \(c_1\)-adjacency is \(4\)-adjacency and \(c_2\)-adjacency is \(8\)-adjacency;
- in \(\mathbb{Z}^3\), \(c_1\)-adjacency is \(6\)-adjacency, \(c_2\)-adjacency is \(18\)-adjacency, and \(c_3\)-adjacency is \(26\)-adjacency.

In this paper, we mostly use the \(c_1\) and \(c_2\) adjacencies in \(\mathbb{Z}^2\).

Let \(x \in (X, \kappa)\). We use the notations

\[ N(X, x, \kappa) = \{ y \in X \mid y \leftrightarrow_\kappa x \} \]

and

\[ N^*(X, x, \kappa) = \{ y \in X \mid y \leftrightarroweq_\kappa x \} = N(X, x, \kappa) \cup \{x\}. \]

We say \(\{x_i\}_{i=0}^k \subset (X, \kappa)\) is a \(\kappa\)-path (or a path if \(\kappa\) is understood) from \(x_0\) to \(x_k\) if \(x_i \leftrightarrow_\kappa x_{i+1}\) for \(i \in \{0, \ldots, k-1\}\), and \(k\) is the length of the path.

A subset \(Y\) of a digital image \((X, \kappa)\) is \(\kappa\)-connected [24], or connected when \(\kappa\) is understood, if for every pair of points \(a, b \in Y\) there exists a \(\kappa\)-path in \(Y\) from \(a\) to \(b\).
2.2 Digitally continuous functions

Material in this section is largely quoted or paraphrased from [18].

We denote by id or id$_X$ the identity map id$(x) = x$ for all $x \in X$.

Definition 2.1. [24] Let $(X, \kappa)$ and $(Y, \lambda)$ be digital images. A function $f : X \rightarrow Y$ is $(\kappa, \lambda)$-continuous, or digitally continuous when $\kappa$ and $\lambda$ are understood, if for every $\kappa$-connected subset $X'$ of $X$, $f(X')$ is a $\lambda$-connected subset of $Y$. If $(X, \kappa) = (Y, \lambda)$, we say a function is $\kappa$-continuous to abbreviate “$(\kappa, \kappa)$-continuous.”

Theorem 2.2. [4] A function $f : X \rightarrow Y$ between digital images $(X, \kappa)$ and $(Y, \lambda)$ is $(\kappa, \lambda)$-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_{\kappa} y$ then $f(x) \leftrightarrow_{\lambda} f(y)$.

Theorem 2.3. [4] Let $f : (X, \kappa) \rightarrow (Y, \lambda)$ and $g : (Y, \lambda) \rightarrow (Z, \mu)$ be continuous functions between digital images. Then $g \circ f : (X, \kappa) \rightarrow (Z, \mu)$ is continuous.

Definition 2.4. Let $A \subset X$. A $\kappa$-continuous function $r : X \rightarrow A$ is a retraction, and $A$ is a retract of $X$, if $r(a) = a$ for all $a \in A$.

A function $f : (X, \kappa) \rightarrow (Y, \lambda)$ is an isomorphism (called a homeomorphism in [3]) if $f$ is a continuous bijection such that $f^{-1}$ is continuous.

We use the following notation. For a digital image $(X, \kappa)$,

$$C(X, \kappa) = \{ f : X \rightarrow X \mid f \text{ is } \kappa\text{-continuous} \}.$$ 

Given $f \in C(X, \kappa)$, a point $x \in X$ is a fixed point of $f$ if $f(x) = x$. We denote by Fix$(f)$ the set $\{ x \in X \mid x$ is a fixed point of $f \}$. A point $x \in X$ is an almost fixed point [24] or an approximate fixed point [15] of $f$ if $x \leftrightarrow_{\kappa} f(x)$.

We use the projection functions $p_1, p_2 : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ defined for $(x, y) \in \mathbb{Z}^2$ by $p_1(x, y) = x$, $p_2(x, y) = y$. These functions are $(c_1, c_1)$-continuous and $(c_2, c_1)$-continuous [22].

2.3 Freezing and cold sets

Material in this section is largely quoted or paraphrased from [11].

Knowledge of Fix$(f)$ for $f \in C(X, \kappa)$ can tell us much about $f|_{X\setminus \text{Fix}(f)}$. This motivates the study of freezing and cold sets.

Definition 2.5. [11] Let $(X, \kappa)$ be a digital image. We say $A \subset X$ is a freezing set for $X$ if given $g \in C(X, \kappa)$, $A \subset \text{Fix}(g)$ implies $g = \text{id}_X$. If no proper subset of a freezing set $A$ is a freezing set for $(X, \kappa)$, then $A$ is a minimal freezing set.

Definition 2.6. [11] A $\subset X$ is a cold set for the connected digital image $(X, \kappa)$ if given $g \in C(X, \kappa)$ such that $g|_A = \text{id}_A$, then for all $x \in X$, $g(x) \in N^*(X, x, \kappa)$.

Remark 2.7. [11] A freezing set is a cold set.

Definition 2.8. [12] Let $X \subset \mathbb{Z}^n$. 

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• The boundary of $X$ with respect to the $c_i$ adjacency, $i \in \{1, 2\}$, is
  \[ Bd_i(X) = \{ x \in X \mid \text{there exists } y \in \mathbb{Z}^n \setminus X \text{ such that } y \leftrightarrow_{c_i} x \}. \]
  $Bd_1(X)$ is what is called the boundary of $X$ in [23]. This paper uses both $Bd_1(X)$ and $Bd_2(X)$.

• The interior of $X$ with respect to the $c_i$ adjacency is $\text{Int}_i(X) = X \setminus Bd_i(X)$.

**Theorem 2.9.** [11] Let $X \subset \mathbb{Z}^n$ be finite. Then for $1 \leq u \leq n$, $Bd_1(X)$ is a freezing set for $(X, c_u)$.

**Theorem 2.10.** [11] Let $X = \prod_{i=1}^n [0, m_i] \subset \mathbb{Z}^n$. Let $A = \prod_{i=1}^n \{0, m_i\}$.

• Let $Y = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{Z}^n$ be such that $X \subset Y$. Let $f : X \to Y$ be $c_1$-continuous. If $A \subset \text{Fix}(f)$, then $X \subset \text{Fix}(f)$.

• $A$ is a freezing set for $(X, c_1)$; minimal for $n \in \{1, 2\}$.

**Theorem 2.11.** [11] Let $X = \prod_{i=1}^n [0, m_i] \subset \mathbb{Z}^n$, where $m_i > 1$ for all $i$. Then $Bd_1(X)$ is a minimal freezing set for $(X, c_n)$.

### 2.4 Digital disks and bounding curves

Material in this section is largely quoted or paraphrased from [12].

We say a finite $c_2$-connected set $S = \{x_i\}_{i=1}^n \subset \mathbb{Z}^2$ is a (digital) line segment if the members of $S$ are collinear.

We say a segment with slope of $\pm 1$ is slanted. An axis-parallel segment is horizontal or vertical.

**Remark 2.12.** [12] A digital line segment must be axis-parallel or slanted.

A closed curve is a path $S = \{s_i\}_{i=0}^m$ such that $s_0 = s_m$, and $0 < |i - j| < m$ implies $s_i \neq s_j$. If

\[ N(S, x_0, \kappa) = N(S, x_m, \kappa) = \{x_1, x_{m-1}\} \quad \text{and} \]

\[ 1 \leq i < m \text{ implies } N(S, x_i, \kappa) = \{x_{i-1}, x_{i+1}\}, \]

$S$ is a cycle. We may also refer to a cycle as a (digital) $\kappa$-simple closed curve. For a simple closed curve $S \subset \mathbb{Z}^2$ we generally assume

• $m \geq 8$ if $\kappa = c_1$,

• $m \geq 4$ if $\kappa = c_2$.

These requirements are necessary for the Jordan Curve Theorem of digital topology, below, as a $c_1$-simple closed curve in $\mathbb{Z}^2$ must have at least 8 points to have a nonempty finite complementary $c_2$-component, and a $c_2$-simple closed curve in $\mathbb{Z}^2$ must have at least 4 points to have a nonempty finite complementary $c_1$-component. Examples in [23] show why it is desirable to consider $S$ and $\mathbb{Z}^2 \setminus S$ with different adjacencies.
Theorem 2.13. [23] (Jordan Curve Theorem for digital topology) Let \( \{\kappa, \kappa'\} = \{c_1, c_2\} \). Let \( S \subset \mathbb{Z}^2 \) be a simple closed \( \kappa \)-curve such that \( S \) has at least 8 points if \( \kappa = c_1 \) and such that \( S \) has at least 4 points if \( \kappa = c_2 \). Then \( \mathbb{Z}^2 \setminus S \) has exactly 2 \( \kappa' \)-connected components.

One of the \( \kappa' \)-components of \( \mathbb{Z}^2 \setminus S \) is finite and the other is infinite. This suggests the following.

Definition 2.14. [12] Let \( S \subset \mathbb{Z}^2 \) be a \( c_2 \)-closed curve such that \( \mathbb{Z}^2 \setminus S \) has two \( c_1 \)-components, one finite and the other infinite. The union \( D \) of \( S \) and the finite \( c_1 \)-component of \( \mathbb{Z}^2 \setminus S \) is a (digital) disk. \( S \) is a bounding curve of \( D \). The finite \( c_1 \)-component of \( \mathbb{Z}^2 \setminus S \) is the interior of \( S \), denoted \( \text{Int}(S) \), and the infinite \( c_1 \)-component of \( \mathbb{Z}^2 \setminus S \) is the exterior of \( S \), denoted \( \text{Ext}(S) \).

Notes:

- If \( D \) is a digital disk determined as above by a bounding \( c_2 \)-closed curve \( S \), then \( (S, c_1) \) can be disconnected. See Figure 1.
- There may be more than one closed curve \( S \) bounding a given disk \( D \). See Figure 2. When \( S \) is understood as a bounding curve of a disk \( D \), we use the notations \( \text{Int}(S) \) and \( \text{Int}(D) \) interchangeably.
- Since we are interested in finding minimal freezing or cold sets and since it turns out we often compute these from bounding curves, we may prefer those of minimal size. A bounding curve \( S \) for a disk \( D \) is minimal if there is no bounding curve \( S' \) for \( D \) such that \#\( S' \) < \#\( S \).
- In particular, a bounding curve need not be contained in \( \text{Bd}_1(D) \). E.g., in the disk \( D \) shown in Figure 2(i), \( (2, 2) \) is a point of the bounding curve; however, all of the points \( c_1 \)-adjacent to \( (2, 2) \) are members of \( D \), so by Definition 2.14 \( (2, 2) \notin \text{Bd}_1(D) \). However, a bounding curve for \( D \) must be contained in \( \text{Bd}_2(D) \).
- In Definition 2.14, we use \( c_2 \) adjacency for \( S \) and we do not require \( S \) to be simple. Figure 2 shows why these seem appropriate.
  - The \( c_2 \) adjacency allows slanted segments in bounding curves and makes possible a bounding curve in subfigure (ii) with fewer points than the bounding curve in subfigure (i) in which adjacent pairs of the bounding curve are restricted to \( c_1 \) adjacency.
  - Neither of the bounding curves shown in Figure 2 is a \( c_2 \)-simple closed curve. E.g., non-consecutive points of each of the bounding curves, \((0, 1)\) and \((1, 0)\), are \( c_2 \)-adjacent. The bounding curve shown in Figure 2(ii) is clearly also not a \( c_1 \)-simple closed curve.
- A closed curve that is not simple may be the boundary \( \text{Bd}_2 \) of a digital image that is not a disk. This is illustrated in Figure 3.

More generally, we have the following.
Figure 1: The \( c_1 \)-disk \( D = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| < 2\} \). The bounding curve \( S = \{(x, y) \in \mathbb{Z}^2 \mid |x| + |y| = 1\} = D \setminus \{(0, 0)\} \) is not \( c_1 \)-connected.

Figure 2: Two views of \( D = [0, 3]^2_\mathbb{Z} \setminus \{(3, 3)\} \), which can be regarded as a \( c_1 \)-disk with either of the closed curves shown in dark as a bounding curve. 
(i) The dark line segments show a \( c_1 \)-simple closed curve \( S \) that is a bounding curve for \( D \). Note the point \((2, 2)\) in the bounding curve shown. By Definition 2.3, \((2, 2) \notin \text{Bd}_1(D)\); however, \((2, 2) \in \text{Bd}_2(D)\).
(ii) The dark line segments show a \( c_2 \)-closed curve \( S \) that is a minimal bounding curve for \( D \).

Figure 3: \( D = [0, 6]^6_\mathbb{Z} \times [0, 2]_\mathbb{Z} \setminus \{(3, 3)\} \) shown with a bounding curve \( S \) in dark segments. \( D \) is not a disk with either the \( c_1 \) or the \( c_2 \) adjacency, since with either of these adjacencies, \( \mathbb{Z}^2 \setminus S \) has two bounded components, \( \{(1, 1), (2, 1)\} \) and \( \{(4, 1), (5, 1)\} \).
Figure 4: \[12\] \( p \in \overline{uv} \) in a bounding curve, with \( \overline{uv} \) slanted. Note \( u \not\leftrightarrow c_1 \) \( p \not\leftrightarrow c_1 \) \( v \), \( p \leftrightarrow c_2 \) \( c \not\leftrightarrow c_1 \) \( p \), \( \{p, c\} \subset N(\mathbb{Z}^2, c_1, b) \cap N(\mathbb{Z}^2, c_1, d) \). If \( X \) is slant-thick at \( p \) then \( c \in X \). (Not meant to be understood as showing all of \( X \).)

**Definition 2.15.** \[12\] Let \( X \subset \mathbb{Z}^2 \) be a finite, \( c_i \)-connected set, \( i \in \{1, 2\} \). Suppose there are pairwise disjoint \( c_2 \)-closed curves \( S_j \subset X \), \( 1 \leq j \leq n \), such that

- \( X \subset S_1 \cup \text{Int}(S_1) \);
- for \( j > 1 \), \( D_j = S_j \cup \text{Int}(S_j) \) is a digital disk;
- no two of \( S_1 \cup \text{Ext}(S_1), D_2, \ldots, D_n \)
  are \( c_1 \)-adjacent or \( c_2 \)-adjacent; and
- we have
  \[ \mathbb{Z}^2 \setminus X = \text{Ext}(S_1) \cup \bigcup_{j=2}^{n} \text{Int}(S_j). \]

Then \( \{S_j\}_{j=1}^{n} \) is a set of bounding curves of \( X \).

Note: As above, a digital image \( X \subset \mathbb{Z}^2 \) may have more than one set of bounding curves.

### 2.5 Thickness

A notion of “thickness” in a digital image \( X \), introduced in \[12\], means, roughly speaking, \( X \) is “locally” like a disk.

Our definition of thickness depends on a notion of an “interior angle” of a disk. We have the following.

**Definition 2.16.** \[12\] Let \( s_1 \) and \( s_2 \) be sides of a digital disk \( X \subset \mathbb{Z}^2 \), i.e., maximal digital line segments in a bounding curve \( S \) of \( X \), such that \( s_1 \cap s_2 = \{p\} \subset X \). The **interior angle of \( X \) at \( p \)** is the angle formed by \( s_1 \), \( s_2 \), and \( \text{Int}(S) \).

**Definition 2.17.** \[12\] Let \( X \subset \mathbb{Z}^2 \) be a digital disk. Let \( S \) be a bounding curve of \( X \) and \( p \in S \).
Figure 5: [12] (1) \( \angle apb \) is a 90\(^\circ\) \((\pi/2 \text{ radians})\) angle of a bounding curve of \( X \) at \( p \in A_1 \), with horizontal and vertical sides. If \( X \) is 90\(^\circ\)-thick at \( p \) then \( q \in \text{Int}(X) \). (Not meant to be understood as showing all of \( X \).)

(2) \( \angle apb \) is a 90\(^\circ\) \((\pi/2 \text{ radians})\) angle between slanted segments of a bounding curve. If \( X \) is 90\(^\circ\)-thick at \( p \) then \( q \in \text{Int}(X) \) and therefore \( q' \in X \). (Not meant to be understood as showing all of \( X \)).

- Suppose \( p \) is in a maximal slanted segment \( \sigma \) of \( S \) such that \( p \) is not an endpoint of \( \sigma \). Then \( X \) is slant-thick at \( p \) if there exists \( c \in X \) such that (see Figure 4)

\[
c \leftrightarrow_{c_2} p \nleftrightarrow_{c_1} c, \tag{1}\]

- Suppose \( p \) is the vertex of a 90\(^\circ\) \((\pi/2 \text{ radians})\) interior angle \( \theta \) of \( S \). Then \( X \) is 90\(^\circ\)-thick at \( p \) if there exists \( q \in \text{Int}(X) \) such that

- if \( \theta \) has axis-parallel sides then \( q \leftrightarrow_{c_2} p \nleftrightarrow_{c_1} q \) (see Figure 5(1));
- if \( \theta \) has slanted sides then \( q \leftrightarrow_{c_1} p \) (see Figure 5(2)).

- Suppose \( p \) is the vertex of a 135\(^\circ\) \((3\pi/4 \text{ radians})\) interior angle \( \theta \) of \( S \). Then \( X \) is 135\(^\circ\)-thick at \( p \) if there exist \( b, b' \in X \) such that \( b \) and \( b' \) are in the interior of \( \theta \) and (see Figure 6)

\[b \leftrightarrow_{c_2} p \nleftrightarrow_{c_1} b \quad \text{and} \quad b' \leftrightarrow_{c_1} p.\]

**Definition 2.18.** [12] [14] Let \( X \subset \mathbb{Z}^2 \) be a digital disk. We say \( X \) is thick if the following are satisfied. For some bounding curve \( S \) of \( X \),

- for every maximal slanted segment of \( S \), if \( p \in S \) is not an endpoint of \( S \), then \( X \) is slant-thick at \( p \), and
- for every \( p \) that is the vertex of a 90\(^\circ\) \((\pi/2 \text{ radians})\) interior angle \( \theta \) of \( S \), \( X \) is 90\(^\circ\)-thick at \( p \), and
- for every \( p \) that is the vertex of a 135\(^\circ\) \((3\pi/4 \text{ radians})\) interior angle \( \theta \) of \( S \), \( X \) is 135\(^\circ\)-thick at \( p \).
Figure 6: ∠apq is an angle of 135° degrees (3π/4 radians) of a bounding curve of X at p, with ap ∪ pq a subset of the bounding curve. If X is 135°-thick at p then b, b' ∈ X. (Not meant to be understood as showing all of X.)

2.6 Convexity

A set X in a Euclidean space \( \mathbb{R}^n \) is convex if for every pair of distinct points \( x, y \in X \), the line segment \( xy \) from \( x \) to \( y \) is contained in \( X \). The convex hull of \( Y \subset \mathbb{R}^n \), denoted \( \text{hull}(Y) \), is the smallest convex subset of \( \mathbb{R}^n \) that contains \( Y \). If \( Y \subset \mathbb{R}^2 \) is a finite set, then \( \text{hull}(Y) \) is a single point if \( Y \) is a singleton; a line segment if \( Y \) has at least 2 members and all are collinear; otherwise, \( \text{hull}(Y) \) is a polygonal disk, and the endpoints of the edges of \( \text{hull}(Y) \) are its vertices.

A digital version of convexity can be stated for subsets of the digital plane \( \mathbb{Z}^2 \) as follows. A finite set \( Y \subset \mathbb{Z}^2 \) is (digitally) convex \([12]\) if either

- \( Y \) is a single point, or
- \( Y \) is a digital line segment, or
- \( Y \) is a digital disk with a bounding curve \( S \) such that the endpoints of the maximal line segments of \( S \) are the vertices of \( \text{hull}(Y) \subset \mathbb{R}^2 \).

3 Tools for determining fixed point sets

The following assertions will be useful in determining fixed point and freezing sets.

**Proposition 3.1.** (Corollary 8.4 of \([18]\)) Let \((X, \kappa)\) be a digital image and \( f \in C(X, \kappa) \). Suppose \( x, x' \in \text{Fix}(f) \) are such that there is a unique shortest \( \kappa \)-path \( P \) in \( X \) from \( x \) to \( x' \). Then \( P \subseteq \text{Fix}(f) \).

Lemma 3.2 below,

... can be interpreted to say that in a \( c_0 \)-adjacency, a continuous function that moves a point \( p \) also moves a point that is “behind” \( p \). E.g., in \( \mathbb{Z}^2 \), if \( q \) and \( q' \) are \( c_1 \)- or \( c_2 \)-adjacent with \( q \) left, right, above, or below \( q' \), and a continuous function \( f \) moves \( q \) to the left, right, higher, or lower, respectively, then \( f \) also moves \( q' \) to the left, right, higher, or lower, respectively \([11]\).
Lemma 3.2. Let \((X, c_u) \subset \mathbb{Z}^n\) be a digital image, 1 \(\leq u \leq n\). Let \(q, q' \in X\) be such that \(q \leftrightarrow_{c_u} q'\). Let \(f \in C(X, c_u)\).

1. If \(p_i(f(q)) > p_i(q) > p_i(q')\) then \(p_i(f(q')) > p_i(q')\).
2. If \(p_i(f(q)) < p_i(q) < p_i(q')\) then \(p_i(f(q')) < p_i(q')\).

Remark 3.3. If \(X \subset \mathbb{Z}^2\) is finite, then a set of bounding curves for \(X\) is a freezing set for \((X, c_i), i \in \{1, 2\}\).

In particular, we have:

Theorem 3.4. Let \(D\) be a digital disk in \(\mathbb{Z}^2\). Let \(S\) be a bounding curve for \(D\). Then \(S\) is a freezing set for \((D, c_1)\) and for \((D, c_2)\).

The next two results form a dual pair.

Theorem 3.5. Let \(X\) be a thick convex disk with a bounding curve \(S\). Let \(A_1\) be the set of points \(x \in S\) such that \(x\) is an endpoint of a maximal axis-parallel edge of \(S\). Let \(A_2\) be the union of slanted line segments in \(S\). Then \(A = A_1 \cup A_2\) is a minimal freezing set for \((X, c_1)\).

Theorem 3.6. Let \(X\) be a thick convex disk with a minimal bounding curve \(S\). Let \(B_1\) be the set of points \(x \in S\) such that \(x\) is an endpoint of a maximal slanted edge in \(S\). Let \(B_2\) be the union of maximal axis-parallel line segments in \(S\). Let \(B = B_1 \cup B_2\). Then \(B\) is a minimal freezing set for \((X, c_2)\).

The next two results form another dual pair, generalizing the previous pair.

Theorem 3.7. Let \(V_i \subset X \subset \mathbb{Z}^2, i \in \{1, \ldots, n\}\) where each \(V_i\) is a thick convex disk. Let \(X' = \bigcup_{i=1}^{n} V_i\). Let \(C_i\) be a bounding curve of \(V_i\). Let \(A_{1,i}\) be the set of endpoints of maximal horizontal or vertical segments of \(C_i\). Let \(A_{2,i}\) be the union of maximal slanted segments of \(C_i\). Then \(A = (X \setminus X') \cup \bigcup_{i=1}^{n} (A_{1,i} \cup A_{2,i})\) is a freezing set for \((X, c_1)\).

Theorem 3.8. Let \(V_i \subset X \subset \mathbb{Z}^2, i \in \{1, \ldots, n\}\) where each \(V_i\) is a thick convex disk. Let \(X' = \bigcup_{i=1}^{n} V_i\). Let \(C_i\) be a bounding curve of \(V_i\). Let \(B_{1,i}\) be the union of maximal horizontal and maximal vertical segments of \(C_i\). Let \(B_{2,i}\) be the set of endpoints of maximal slanted segments of \(C_i\). Then \(B = (X \setminus X') \cup \bigcup_{i=1}^{n} (B_{1,i} \cup B_{2,i})\) is a freezing set for \((X, c_2)\) (the adjacency is misprinted as \(c_1\) in \([13]\)).

4 Unifying sets

4.1 Definition and general properties

Definition 4.1. Let \((X, \kappa)\) be a digital image. Let \(A \subset X\). Suppose whenever \(f, g \in C(X, \kappa)\) are such that \(f(A) = g(A) = A\) and \(f|_A = g|_A\), we have \(f = g\). Then we say \(A\) is a unifying set for \((X, \kappa)\). \(A\) is a minimal unifying set if \(A\) is a unifying set and no proper subset of \(A\) is a unifying set for \((X, \kappa)\).
Remark 4.2. Observe:

- By taking \( g \) to be the identity function \( \text{id}_X \) in Definition 4.1, we see that a unifying set is a freezing set. We have not determined whether the converse is true.

- It is trivial that \( X \) is a unifying set for \( (X, \kappa) \). We are therefore interested in finding minimal unifying sets. In light of the above, a minimal freezing set is a “good candidate” for a minimal unifying set.

In the following, we study conditions for which a freezing set must be unifying.

The desirability of the requirement that \( f(A) = g(A) = A \) in Definition 4.1 is illustrated in the following, in which this requirement is not met.

Example 4.3. Let \( X = \{0, m\} \times \{0, n\} \mathbb{Z} \) for \( m \geq 2, n > 0 \). Let \( f, g : X \to X \) be the functions

\[
f(x, y) = (0, y), \quad g(x, y) = \begin{cases} (0, y) & \text{if } x \in \{0, m\}; \\ (1, y) & \text{if } 1 \leq x \leq m - 1. \end{cases}
\]

We take \( A = \{(0,0), (0, n), (m, 0), (m, n)\} \).

Note by Theorem 2.10, \( A \) is a minimal freezing set for \( (X, c_1) \). We see easily that \( f, g \in C(X, c_1), f|_A = g|_A, f(A) = g(A) \) is a proper subset of \( A \), and \( f \neq g \).

The following shows that unifying sets are preserved by isomorphism.

Theorem 4.4. Let \( (X, \kappa) \) and \( (Y, \lambda) \) be digital images such that there exists an isomorphism \( F : (X, \kappa) \to (Y, \lambda) \). If \( A \) is a unifying set for \( (X, \kappa) \) then \( F(A) \) is a unifying set for \( (Y, \lambda) \).

Proof. Let \( f, g \in C(Y, \lambda) \) such that \( f(F(A)) = g(F(A)) = F(A) \) and \( f|_{F(A)} = g|_{F(A)} \).

We have, by Theorem 2.3, \( f' = F^{-1} \circ f \circ F, g' = F^{-1} \circ g \circ F \in C(X, \kappa) \), and for \( a \in A \) we have \( f \circ F(a) = g \circ F(a) \), so

\[
f'(a) = F^{-1} \circ f \circ F(a) = F^{-1} \circ g \circ F(a) = g'(a).
\]

Also, given \( b = F(a) \) for \( a \in A \), by assumption we have \( f(b) = g(b) \), hence

\[
f'(a) = F^{-1}(f(b)) = F^{-1}(g(b)) = g'(a).
\]

Since \( A \) is unifying, \( f' = g' \). Therefore,

\[
f = F \circ f' \circ F^{-1} = F \circ g' \circ F^{-1} = g,
\]

so \( F(A) \) is unifying for \( (Y, \lambda) \). \( \square \)

We have the following generalization of Proposition 3.1.
Proposition 4.5. Let $f, g : X \to Y$ such that $f$ and $g$ are both $(\kappa, \lambda)$-continuous. Suppose $x_0, x_1 \in X$ and there is a $\kappa$-path $P$ of length $n$ in $X$ from $x_0$ to $x_1$. Suppose $y_0 = f(x_0) = g(x_0), y_1 = f(x_1) = g(x_1)$, and there is a unique shortest path $Q$ of length $n$ in $Y$ from $y_0$ to $y_1$. Then $f(P) = g(P) = Q$ and $f|_P = g|_P$.

Proof. Since $f(P)$ and $g(P)$ must be $\lambda$-paths from $y_0$ to $y_1$, our uniqueness and length restrictions imply $f(P) = g(P) = Q$. Continuity implies $f|_P = g|_P$. □

4.2 Cycles

Theorem 4.6. Let $n > 4$. Let $C_n = \{x_m\}_{m=0}^{n-1} \subset Z^2$, where the members of $C_n$ are indexed circularly. Let $A = \{x_i, x_j, x_k\}$ be a set of distinct members of $C_n$ such that $C_n$ is a union of unique shorter paths determined by these points. Then $A$ is a minimal freezing set for $C_n$.

Theorem 4.7. The set $A$ of Theorem 4.6 is a unifying set for $(C_n, \kappa)$, and any $f \in C(X, \kappa)$ such that $f(A) = A$ must be an isomorphism of $(X, \kappa)$.

Proof. Let $\bar{x}_i \bar{x}_j$, $\bar{x}_i \bar{x}_k$, and $\bar{x}_j \bar{x}_k$ be the unique shorter paths in $C_n$ from $x_i$ to $x_j$, from $x_i$ to $x_k$, and from $x_j$ to $x_k$, respectively. Let $B = \{\bar{x}_i \bar{x}_j, \bar{x}_i \bar{x}_k, \bar{x}_j \bar{x}_k\}$. Let $f, g \in C(C_n, \kappa)$ such that

$$f(A) = g(A) = A \text{ and } f|_A = g|_A. \quad (2)$$

Suppose $f \neq g$. Consider the following cases.

- The members of $B$ have distinct lengths. Without loss of generality,

$$\text{length}(\bar{x}_i \bar{x}_j) < \text{length}(\bar{x}_i \bar{x}_k) < \text{length}(\bar{x}_j \bar{x}_k). \quad (3)$$

Since we have that both $f(\bar{x}_i \bar{x}_j)$ and $g(\bar{x}_i \bar{x}_j)$ are paths of length at most $\text{length}(\bar{x}_i \bar{x}_j)$ from $f(x_i) = g(x_i)$ to $f(x_j) = g(x_j)$, from $\#3$ and Proposition 1.5 $f(\bar{x}_i \bar{x}_j) = g(\bar{x}_i \bar{x}_j)$ and $f|_{\bar{x}_i \bar{x}_j} = g|_{\bar{x}_i \bar{x}_j}$ is a bijection of $\bar{x}_i \bar{x}_j$. Indeed, we must have that $f$ and $g$ coincide with id$_X$ on $\bar{x}_i \bar{x}_j$, for otherwise we would have $f(x_i) = g(x_i) = x_i, f(x_j) = g(x_j) = x_i, f(x_k) = g(x_k) = x_k$, so $f(\bar{x}_i \bar{x}_k)$ is a $\kappa$-path from $x_j$ to $x_k$, contrary to $\#3$. Then by $\#2$ we have $f|_A = g|_A = \text{id}_A$, and from Proposition 1.5 it follows that $f = g = \text{id}_X$.

- Suppose two members, but not all three, of $B$ have the same length; without loss of generality, $\text{length}(\bar{x}_i \bar{x}_j) = \text{length}(\bar{x}_i \bar{x}_k)$. Then either $f|_A = g|_A = \text{id}_A$ or $f(x_i) = g(x_i) = x_i, f(x_j) = g(x_j) = x_k,$ and $f(x_k) = g(x_k) = x_j$. Then much as above, $f = g$ is an isomorphism of $(X, \kappa)$.

- Suppose all three members of $B$ have the same length. Then $f|_A = g|_A$ is a permutation of $A$. Much as above, it follows that $f = g$ is an isomorphism of $(X, \kappa)$.

In all cases, we concluded that $f = g$ is an isomorphism of $(X, \kappa)$. Thus $A$ is a unifying set for $(X, \kappa)$. □
4.3 Trees

A tree is a connected acyclic graph \((X, \kappa)\). By acyclic we mean lacking any closed curve of more than 2 points. The degree of a vertex \(x\) in \(X\) is the number of distinct vertices \(y \in X\) such that \(x \leftrightarrow y\).

**Theorem 4.8.** \(\blacksquare\) Let \((X, \kappa)\) be a digital image such that the graph \(G = (X, \kappa)\) is a finite tree with \(#X > 1\). Let \(A\) be the set of vertices of \(G\) that have degree 1. Then \(A\) is a minimal freezing set for \(G\).

**Theorem 4.9.** Let \((X, \kappa)\) be a digital image such that the graph \(G = (X, \kappa)\) is a finite tree with \(#X > 1\). Let \(A\) be the set of vertices of \(G\) that have degree 1. Then \(A\) is a minimal unifying set for \(G\). Also, if \(f \in C(X, \kappa)\) such that \(f(A) = A\), then \(f\) is an isomorphism of \((X, \kappa)\).

*Proof.* Let \(a_0 \in A\). Since \(X\) is finite, we have that \(A\) is also finite - say, \(A = \{a_i\}_{i=0}^n\). Since \(G\) is a tree, for \(0 < i \leq n\) there is a unique shortest \(\kappa\)-path \(P_i\) in \(X\) from \(a_0\) to \(a_i\). Let \(L = \{\ell_j\}_{j=1}^m\) be the set of distinct lengths of the members of \(\{P_i\}_{i=1}^n\) with

\[\ell_1 < \ell_2 < \ldots < \ell_m.\]

Let \(L_j = \{P_i \mid \text{length}(P_i) = \ell_j\}\). Since \(A\) is finite and \(f(A) = g(A) = A\),

\[f|_A = g|_A : A \rightarrow A\]

is a bijection. (4)

Let \(f, g \in C(X, \kappa)\) be such that \(f(A) = g(A) = A\) and \(f|_A = g|_A\). Every \(P_k\) of length \(\ell_1\) is the unique shortest \(\kappa\)-path in \(X\) from \(a_0\) to some \(a_k \in A \setminus \{a_0\}\). Since \(f(P_k)\) is a path from \(f(a_0) = g(a_0)\) to \(f(a_k) = g(a_k)\), our choice of \(\ell_1\) and Proposition \(\blacksquare\) imply \(f|_{P_k} = g|_{P_k}\), \(f(P_k) = g(P_k)\) has length \(\ell_1\), and from \(\blacksquare\) that \(f|_{L_1} = g|_{L_1}\) is a bijection of \(L_1\). It follows easily that \(f|_{L_1} = g|_{L_1}\) is an isomorphism. This provides the base case of an induction argument.

Suppose \(u \in \mathbb{Z}, 0 \leq u < m; f|_{P_k} = g|_{P_k}\) for every \(P_k \in \bigcup_{j=1}^m L_j\); and

\[f|_{\bigcup_{j=1}^m L_j} = g|_{\bigcup_{j=1}^m L_j}\]

is a bijection of \(\bigcup_{j=1}^u L_j\). (5)

Now consider \(P_k \in L_{u+1}\). \(f(P_k)\) and \(g(P_k)\) are \(\kappa\)-paths in \(X\) from \(f(a_0) = g(a_0)\) to \(f(a_k) = g(a_k)\) of length at most \(\ell_{u+1}\). By \(\blacksquare\), \(f(P_k)\) and \(g(P_k)\) cannot have length less than \(\ell_{u+1}\). Therefore, each of \(f(P_k)\) and \(g(P_k)\) belongs to \(L_{u+1}\). By the uniqueness condition that defines \(L_{u+1}\) it follows that \(f|_{L_{u+1}} = g|_{L_{u+1}}\). By \(\blacksquare\), \(f|_{L_{u+1}} = g|_{L_{u+1}}\) is a bijection. It follows from the above that \(f|_{\bigcup_{j=1}^{u+1} L_j} = g|_{\bigcup_{j=1}^{u+1} L_j}\) is a bijection of \(\bigcup_{j=1}^{u+1} L_j\), and, further, an isomorphism.

This completes the induction. Since \(X = \bigcup_{j=1}^m L_j\), we have \(f = g\). Since \(f\) was chosen arbitrarily, \(A\) is a unifying set. Also, \(f\) is an isomorphism.

To show the minimality of \(A\), we see easily that for any \(a \in A\) there is a \(\kappa\)-retraction \(r : X \rightarrow X \setminus \{a\}\), so \(r\) and \(\text{id}_X\) are members of \(C(X, \kappa)\) that coincide on \(A \setminus \{a\}\), \(r(A \setminus \{a\}) = \text{id}_X(A \setminus \{a\}) = (A \setminus \{a\})\), but \(r \neq \text{id}_X\). \(\blacksquare\)
4.4 Complete graphs

Theorem 4.10. Let \((X, \kappa)\) be a digital image that is a complete graph, where \(#X > 1\). Let \(A \subset X\). Then the following are equivalent.

1. \(A = X\).
2. \(A\) is a unifying set for \((X, \kappa)\).
3. \(A\) is a freezing set for \((X, \kappa)\).

Proof. 1) \(\Rightarrow\) 2) \(\Rightarrow\) 3): These implications are noted in Remark [4.2]

3) \(\Rightarrow\) 1): Suppose otherwise. Then there exists \(x_0 \in X \setminus A\). Let \(x_1 \in X \setminus \{x_0\}\).

Let \(g : X \to X\) be defined by

\[
g(x) = \begin{cases} 
  x & \text{for } x \neq x_0; \\
  x_1 & \text{for } x = x_0.
\end{cases}
\]

Since \((X, \kappa)\) is a complete graph, \(g \in C(X, \kappa)\). Note \(g|_A = \text{id}_A\). But since \(g(x_0) \neq x_0\), we have a contradiction of the assumption that \(A\) is freezing. The contradiction gives us the desired conclusion.

4.5 Rectangles in \(\mathbb{Z}^2\) with axis-parallel sides and \(c_1\)

In this section, we study unifying sets for digital rectangles with axis-parallel edges in \(\mathbb{Z}^2\), using the \(c_1\) adjacency.

Proposition 4.11. [14] Let \(X \subset \mathbb{Z}^2\). Let \(S\) be a minimal bounding curve for \(X\). Let \(p_0\) be the vertex of an interior angle of \(S\), formed by axis-parallel edges \(E_1\) and \(E_2\) of \(S\), of measure 90° (\(\pi/2\) radians). Let \(A\) be any of a freezing set for \((X, c_1)\), a cold set for \((X, c_1)\), a freezing set for \((X, c_2)\), or a cold set for \((X, c_2)\). Let \(X\) be 90°-thick at \(p_0\). Then \(p_0 \in A\).

Proposition 4.12. Let \(m > 1\), \(n > 1\), and \(X = [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}\). Let \(A \subset X\). Then \(A\) is a freezing set for \((X, c_1)\) if and only if

\[
A' = \{(0, 0), (m, 0), (0, n), (m, n)\} \subset A.
\]

Therefore, \(A'\) is the only minimal freezing set for \((X, c_1)\).

Proof. If \(A\) is a freezing set, then by Proposition [4.11] \(A' \subset A\). Since \(A'\) is a freezing set by Theorem [3.7] it follows that \(A'\) is unique as a minimal freezing set.

If \(A' \subset A\) then, since \(A'\) is a freezing set, \(A\) is a freezing set [11].

Theorem 4.13. Let \(X = [-m, m]_{\mathbb{Z}} \times [-n, n]_{\mathbb{Z}}\). Let

\[
A = \{(-m, -n), (-m, n), (m, -n), (m, n)\}.
\]

Then \(A\) is a unifying set for \((X, c_1)\). Further, every \(f \in C(X, c_1)\) such that \(f(A) = A\) is an isomorphism.
Consider the following cases.

- $m < n$. Since $f(A) = g(A) = A$, we have that $f(B)$, $g(B)$, $f(T)$, and $g(T)$ are $c_1$-paths of length at most $2m$ between distinct members of $A$, and since the closest distinct members of $A$ are joined by paths of length $2m$, $f(B)$, $g(B)$, $f(T)$, and $g(T)$ are paths of length $2m$. Therefore, $f(B \cup T) = g(B \cup T) = B \cup T$. Continuity implies that for all $(x, y) \in B \cup T$, one of the following holds:
  - $f(x, y) = g(x, y) = (x, y)$, or
  - $f(x, y) = g(x, y) = (x, -y)$, or
  - $f(x, y) = g(x, y) = (-x, y)$, or
  - $f(x, y) = g(x, y) = (-x, -y)$.

Suppose the first case, $f(x, y) = g(x, y) = (x, y)$ for $(x, y) \in B \cup T$. Each $(x, y) \in X$ lies on the unique shortest $c_1$-path between $b = (x, -n)$ and $t = (x, n)$. Since $f(b) = g(b) = b$ and $f(t) = g(t) = t$, we must have $f(x, y) = g(x, y) = (x, y)$ by Proposition 4.5. Thus $f = g = \text{id}_X$.

- $m > n$. This case is similar to the case $m < n$, yielding the conclusion that $f = g$ is an isomorphism of $(X, c_1)$.

- $m = n$. In this case we have either $f(B \cup T) = g(B \cup T) = B \cup T$ or $f(B \cup T) = g(B \cup T) = L \cup R$. In the former case, $f|_{B \cup T}$ and $g|_{B \cup T}$ are given by one of the four possibilities listed above; in the latter case, one of the following holds. For $(x, y) \in B \cup T$,
  - $f(x, y) = g(x, y) = (y, x)$, or
  - $f(x, y) = g(x, y) = (y, -x)$, or
  - $f(x, y) = g(x, y) = (-y, x)$, or
  - $f(x, y) = g(x, y) = (-y, -x)$.

An argument like that used above shows that in each of these cases, $f = g$ is an isomorphism of $(X, c_1)$.

Thus all cases lead to the conclusion that $f = g$, hence $A$ is unifying; and that $f \in C(X, c_1)$ such that $f(A) = A$ implies $f$ is an isomorphism of $(X, c_1)$. \qed
4.6 Rectangles in \( \mathbb{Z}^2 \) with slanted sides and \( c_2 \)

In this section, we study unifying sets for digital rectangles with slanted edges in \( \mathbb{Z}^2 \), using the \( c_2 \) adjacency. Our assertions are dual to those of section 4.5 and have proofs with common elements.

**Proposition 4.14.** Let \( X \) be a digital rectangle in \( \mathbb{Z}^2 \) with slanted edges. Let \( B \subset X \). Let \( B' \) be the set of endpoints of edges of \( X \). Then \( B \) is a freezing set for \( (X, c_2) \) if and only if \( B' \subset B \). Therefore, \( B' \) is the only minimal freezing set for \( (X, c_2) \).

**Proof.** By Theorem 4.4, there is no loss of generality in assuming

\[
B' = \{(0,0), (m,m), (n,-n), (m+n,m-n)\} \text{ for some } m,n \in \mathbb{N}.
\]

If \( B \) is a freezing set, then by Proposition 4.11, \( B' \subset B \). Since \( B' \) is a freezing set by Theorem 4.3, it follows that \( B' \) is unique as a minimal freezing set. \( \square \)

**Theorem 4.15.** Let \( X \) be the digital rectangle with edges in the set

\[
B = \{(0,0), (m,m), (n,-n), (m+n,m-n)\}.
\]

Then \( B \) is a unifying set for \( (X, c_2) \). Further, every \( f \in C(X, c_2) \) such that \( f(B) = B \) is an isomorphism.

**Proof.** Let \( LR \) (lower right) be the edge of \( X \) from \( (n,-n) \) to \( (m+n,m-n) \). Let \( UL \) (upper left) be the edge of \( X \) from \( (0,0) \) to \( (m,m) \). Let \( LL \) (lower left) be the edge of \( X \) from \( (0,0) \) to \( (n,-n) \). Let \( UR \) (upper right) be the edge of \( X \) from \( (m,m) \) to \( (m+n,m-n) \). For \( m < n \), there are distinct isomorphisms \( F_1, F_2, F_3, F_4 : S \to S \), where

\[
S = LR \cup UL \cup LL \cup UR
\]

is the bounding curve of \( X \), where \( F_1 = \text{id}_S \), \( F_2 \) reverses the orientations of \( UL \) and \( LR \), \( F_3 \) interchanges \( UL \) and \( LR \) while preserving their orientations, and \( F_4 \) interchanges \( UL \) and \( LR \) and reverses their orientations.

Consider the following cases.

- \( m < n \). Since \( f(B) = g(B) = B \), we have that \( f(UL), g(UL), f(LR), \) and \( g(LR) \) are \( c_2 \)-paths of length at most \( m \) between distinct members of \( B \), and since the closest distinct members of \( B \) are joined by paths of length \( m, f(UL), g(UL), f(LR), \) and \( g(LR) \) are paths of length \( m \). Therefore, \( f(UL \cup LR) = g(UL \cup LR) = UL \cup LR \). Proposition 4.15 implies that for all \( (x,y) \in UL \cup LR, f(x,y) = g(x,y) = F_i(x,y) \) for some index \( i \).

Suppose the first case, \( f(x,y) = g(x,y) = (x,y) \) for \( (x,y) \in UL \cup LR \). Each \( (x,y) \in X \) lies on the unique shortest \( c_2 \)-path (a slanted path) between some \( d_1 \in UL \) and some \( d_2 \in LR \). Since \( f(d_j) = g(d_j) \) for \( j \in \{1,2\} \), we must have \( f(x,y) = g(x,y) = (x,y) \) by Proposition 4.4. Thus \( f = g = \text{id}_X \). Similarly, \( f = g \) is an isomorphism of \( (X, c_2) \) in the other cases.
• $m > n$. This case is similar to the case $m < n$, and we similarly conclude that $f = g$ is an isomorphism of $(X, c_2)$.

• $m = n$. Here, in addition to the isomorphisms $F_1, F_2, F_3, F_4$ discussed above, we also have isomorphisms $R_1, R_2, R_3, R_4$ of $(X, c_2)$ that rotate the edges of $X$ by $90^\circ$ ($\pi/2$ radians) either clockwise or counterclockwise, either preserving or reversing the orientations of both $UL$ and $LR$. An argument like that used above shows that in each of these cases, $f = g$ is an isomorphism of $(X, c_2)$.

Thus all cases lead to the conclusion that $f = g$, hence $B$ is unifying; and that $f \in C(X, c_2)$ such that $f(B) = B$ implies $f$ is an isomorphism of $(X, c_2)$. □

4.7 Generalized normal product

In this section, we consider unifying sets for Cartesian products of digital images using the normal product adjacency.

We have the following generalization of the normal product adjacency [2] for the Cartesian product of two graphs.

**Definition 4.16.** [25, 8] Let $u, v \in \mathbb{N}$, $1 \leq u \leq v$. Let $(X_i, \kappa_i)$ be digital images, $i \in \{1, \ldots, v\}$. Let $x_i, y_i \in X_i$, $x = (x_1, \ldots, x_v)$, $y = (y_1, \ldots, y_v)$. Then $x \leftrightarrow y$ in the generalized normal product adjacency $NP_u(\kappa_1, \ldots, \kappa_v)$ if for at least 1 and at most $u$ indices $i$, $x_i \leftrightarrow \kappa_i$, $y_i$ and for all other indices $j$, $x_j = y_j$.

**Remark 4.17.** For $u = v = 2$, the generalized normal product adjacency coincides with the normal product adjacency. Sabidussi [25] uses strong for what we call the generalized normal product adjacency; we prefer the latter name, as “strong” also appears in the literature for what we call the normal product adjacency.

The following generalizes a result in [16, 7].

**Theorem 4.18.** [8] Let $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$, $1 \leq i \leq v$. Then the product map

$$f = \Pi_{i=1}^v f_i : (\Pi_{i=1}^v X_i, NP_v(\kappa_1, \ldots, \kappa_v)) \to (\Pi_{i=1}^v Y_i, NP_v(\lambda_1, \ldots, \lambda_v))$$

given by $f(x_1, \ldots, x_v) = (f_1(x_1), \ldots, f_v(x_v))$ is continuous if and only if each $f_i$ is continuous.

**Theorem 4.19.** Let $\emptyset \neq A_i \subset X_i$, where $(X_i, \kappa_i)$ is a digital image, $1 \leq i \leq v \in \mathbb{N}$. Let $A = \Pi_{i=1}^v A_i$, $X = \Pi_{i=1}^v X_i$. If $A$ is a unifying set for $(X, NP_v(\kappa_1, \ldots, \kappa_v))$ then for each $i$, $A_i$ is a unifying set for $(X_i, \kappa_i)$.

**Proof.** Suppose $A$ is a unifying set for $(X, NP_v(\kappa_1, \ldots, \kappa_v))$. For all $i$, let $f_i, g_i \in C(X_i, \kappa_i)$ be such that $f_i(A_i) = g_i(A_i) = A_i$ and $f_i|_{A_i} = g_i|_{A_i}$. Then by Theorem 4.18, $f = f_1 \times \cdots \times f_v$ and $g = g_1 \times \cdots \times g_v$ are members of $C(X, NP_v(\kappa_1, \ldots, \kappa_v))$. Further, given $a = (a_1, \ldots, a_v) \in A$, there exist $a_i' \in A_i$ such that $f_i(a_i') = g_i(a_i') = a_i$, and therefore we have $f(A) = g(A) = A$ and $f|_A = g|_A$. Since $A$ is unifying, we have $f = g$, and therefore $f_i = g_i$ for all $i$. Thus $A_i$ is unifying. □
5 Shy maps that are retractions

Shy maps in digital topology were introduced in [5] and studied further in [6, 17, 7, 8, 9]. A version of shy maps for topological spaces was introduced in [10].

Definition 5.1. [5] Let \( f : (X, \kappa) \to (Y, \lambda) \) be a continuous function of digital images. We say \( f \) is shy if

- for each \( y \in f(X) \), \( f^{-1}(y) \) is connected, and
- for every \( y_0, y_1 \in f(X) \) such that \( y_0 \) and \( y_1 \) are adjacent, \( f^{-1}\{y_0, y_1\} \) is connected.

Theorem 5.2. [6] Let \( f : (X, \kappa) \to (Y, \lambda) \) be a continuous function between digital images. Then \( f \) is shy if and only if \( f^{-1} : Y \to X \) is a connectivity preserving multivalued function (i.e., given \( \lambda \)-connected \( A \subset f(Y) \), \( f^{-1}(A) \) is \( \kappa \)-connected).

We say a point \( p \) of a connected graph \( G = (X, \kappa) \) is an articulation point of \( G \) if \( (X \setminus \{p\}, \kappa) \) is not connected.

Theorem 5.3. Let \( (X, \kappa) \) be a digital image. Let \( \emptyset \neq R \subset X \). Let \( \emptyset \neq A \subset R \) such that for each \( \kappa \)-component \( K \) of \( X \setminus R \) there exists \( p \in A \) such that \( p \) is an articulation point for \( (K \cup R, \kappa) \). Then there is a unique function \( r : X \to R \) that is a shy \( \kappa \)-retraction.

Proof. For \( x \in X \setminus R \), let \( p_x \in A \) be the articulation point for the union of \( R \) and the \( \kappa \)-component \( K_x \) of \( X \setminus R \) containing \( x \). Let \( r : X \to X \) be the function

\[
\begin{align*}
r(x) &= \begin{cases} 
x & \text{if } x \in R; 
p_x & \text{if } x \in X \setminus R.
\end{cases}
\end{align*}
\]

Clearly, \( r(X) = R \) and \( r|_R = \text{id}_R \). It is easily seen that \( r^{-1}(p_x) \setminus \{p_x\} = K_x \) is a \( \kappa \)-component of \( X \setminus R \), and \( r^{-1}(y) = \{y\} \) for \( y \in R \setminus T \). It follows that \( r \in C(X, \kappa) \), that \( r \) is a retraction of \( X \) to \( R \), and \( r \) is shy.

Suppose \( f \in C(X, \kappa) \) is a shy retraction of \( X \) to \( R \). If there exists \( x_0 \in X \setminus R \) such that \( x_1 = f(x_0) \neq p_{x_0} \), then \( p_{x_0} \) separates the points \( x_0, x_1 \in f^{-1}(x_1) \), contrary to the assumption that \( f \) is shy. The uniqueness of \( r \) as a shy retraction follows.

Corollary 5.4. Let \( (X, \kappa) \) be a digital image that is a tree. Let \( (R, \kappa) \) be a nonempty subtree of \( (X, \kappa) \). Then there is a unique function \( r : X \to R \) that is a shy \( \kappa \)-retraction.

Proof. It is trivial that if \( R = X \), we can take \( r = \text{id}_X \). Otherwise, we can take

\[
A = \{x \in R \mid x \text{ is a leaf of } R \text{ and not a leaf of } X\}.
\]

The assertion follows from Theorem 5.3. □

For topological spaces, we have the following.
Definition 5.5. [10] Let $X$ and $Y$ be topological spaces and let $f : X \to Y$. Then $f$ is shy if $f$ is continuous and for every path-connected $Y' \subset f(X)$, $f^{-1}(Y')$ is a path-connected subset of $X$. □

By using an argument similar to the proof of Theorem 5.3 we get the following.

Theorem 5.6. Let $X$ be a topological space. Let $\emptyset \neq A \subset R \subset X$ such that each $p \in A$ separates $R$ and a component of $X \setminus R$. Then there is a unique continuous function $r : X \to R$ that is a shy retraction.

6 Approximate fixed points

Suppose $A \subset X$ and $A$ is a $\kappa$-freezing set for $X$. By definition, if $f \in C(X, \kappa)$ and $A \subset \text{Fix}(f)$, then $f = \text{id}_X$, i.e., $X = \text{Fix}(f)$. If we weaken the hypothesis so that instead of assuming $A \subset \text{Fix}(f)$ we assume every point of $A$ is an approximate fixed point of $f$, might we reach the weaker conclusion that every point of $X$ is an approximate fixed point of $f$? The answer is not generally affirmative; we give a counterexample below. We also examine basic examples for which an affirmative answer is shown.

6.1 Wedge of cycles

In this section, we show that a wedge of cycles $X$ can support a freezing set $A$ and a continuous self-map $f$ such that every point of $A$ is an approximate fixed point of $f$, but not every point of $X$ is an approximate fixed point of $f$.

Theorem 6.1. [11] Let $C_m$ and $C_n$ be cycles, with $m > 4$, $n > 4$, where $C_m = \{x_i\}_{i=0}^{m-1}$, $C_n = \{x'_i\}_{i=0}^{n-1}$, with the members of $C_m$ and $C_n$ indexed circularly. Let $x_0 = x'_0$ be the wedge point of $X = C_m \lor C_n$. Let $x_i, x_j \in C_m$ and $x'_k, x'_p \in C_n$ be such that $C_m$ is the union of unique shorter paths determined by $x_i, x_j, x_0$ and $C_n$ is the union of unique shorter paths determined by $x'_k, x'_p, x'_0$. Then $A = \{x_1, x_j, x'_k, x'_p\}$ is a freezing set for $X$.

Example 6.2. Let $X = C_6 \lor C_5$, where $C_6 = \{x_i\}_{i=0}^{5}$ and $C_5 = \{x'_i\}_{i=0}^{n-1}$ are $c_2$-simple closed curves in $\mathbb{Z}^2$, with the members of $C_6$ and $C_5$ indexed circularly. By Theorems 4.6 and 6.1 if $k$ and $p$ are chosen so that $\{x'_0, x'_k, x'_p\}$ is a freezing set for $C_n$, then we can take $A = \{x_2, x_4, x'_k, x'_p\}$ to be a freezing set for $(X, c_2)$. Now take $f : X \to X$ to be the function

$$f(x) = \begin{cases} 
  x_0 & \text{if } x = x_3; \\
  x_1 & \text{if } x = x_2; \\
  x_5 & \text{if } x = x_4; \\
  x & \text{otherwise.}
\end{cases}$$

See Figure 7. One sees easily that $f \in C(X, c_2)$, that every member of $A$ is a $c_2$-approximate fixed point of $f$, but $x_3$ is not a $c_2$-approximate fixed point of $f$. 

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This is a contradiction, since \( x \in \{q_0, q_1, q_2\} \) for which \( x \geq n \) for some \( n \in \mathbb{N} \), as in the Example. The cycle with points \( p = (x, y) \) for \( x \leq 0 \) represents \( C_6 \), for which \( \{x_0, x_2, x_4\} \) is a \( c_2 \)-freezing set; the cycle with points \( p = (x, y) \) for \( x \geq 0 \) represents \( C_m \) (here, \( m = 8 \), and \( \{x'_0, x'_2, x'_4\} \) is a \( c_2 \)-freezing set for \( C_8 \), so \( A = \{x, x_4, x'_4, x'_0\} \) is a \( c_2 \)-freezing set for \( C_6 \cup C_8 \)). Arrows connect \( p \) and \( f(p) \) for points \( p \notin \text{Fix}(f) \). Each point of \( A \) is a \( c_2 \)-approximate fixed point of \( f \).

6.2 Disks in \((\mathbb{Z}^2, c_1)\)

Lemma 6.3. Let \( q_0, q_1 \in X \subset \mathbb{Z}^2 \). Suppose there is a horizontal or vertical \( c_1 \)-path \( P \) in \( X \) from \( q_0 \) to \( q_1 \). Let \( f : P \to X \) be \( c_1 \)-continuous, such that \( q_0 \) and \( q_1 \) are \( c_1 \)-approximate fixed points of \( f \). Then every member of \( P \) is a \( c_1 \)-approximate fixed point of \( f \).

Proof. Without loss of generality, \( P \) is horizontal, \( q_0 = (0, 0) \), and \( q_1 = (n, 0) \) for some \( n \in \mathbb{N} \). Suppose there exists \( q = (x, 0) \in P \) such that \( q \) is not a \( c_1 \)-approximate fixed point of \( f \). Then \( |x - p_1(f(q))| > 1 \) or \( |p_2(f(q))| > 1 \) or \( |x - p_1(f(q))| = 1 \) and \( |p_2(f(q))| = 1 \).

If \( |x - p_1(f(q))| > 1 \) then either \( p_1(f(q)) > x + 1 \) or \( p_1(f(q)) < x - 1 \).

- Suppose \( p_1(f(q)) > x + 1 \). Then by Lemma \ref{lem:c1-approximate-fixed-point}, we would have \( p_1(f(q_0)) > 1 \), contrary to the assumption that \( q_0 = (0, 0) \) is an approximate fixed point.

- If \( p_1(f(q)) < x - 1 \), then by Lemma \ref{lem:c1-approximate-fixed-point}, we would have \( p_1(f(q_1)) < n - 1 \), contrary to the assumption that \( q_1 = (n, 0) \) is an approximate fixed point.

Suppose \( |p_2(f(q))| > 1 \). Without loss of generality, \( p_2(f(q)) > 1 \), as the case \( p_2(f(q)) < 1 \) can be handled similarly. Since \( c_1 \)-adjacent points differ in only one coordinate and the \( q_i \) as approximate fixed points implies \( |p_2(f(q_i))| \leq 1 \), \( i \in \{0, 1\} \), there are at least 4 indices \( j \) for which \( p_2(f(x_j)) \neq p_2(f(x_{j+1})) \) and therefore at most \( n - 4 \) indices \( j \) for which \( p_1(f(x_j)) \neq p_1(f(x_{j+1})) \).

This is a contradiction, since \( x_0 \) and \( x_1 \) being approximate fixed points implies \( p_1(f(x_0)) \leq 1 \) and \( p_1(f(x_1)) \geq n - 1 \), so at least \( n - 2 \) indices \( j \) would satisfy \( p_1(f(x_j)) \neq p_1(f(x_{j+1})). \)

Suppose \( |x - p_1(f(q))| = 1 \) and \( |p_2(f(q))| = 1 \). Without loss of generality, \( p_1(f(q)) = x + 1 \) and \( p_2(f(q)) = y + 1 \). By the \( c_1 \)-continuity of \( f \) and Lemma \ref{lem:continuity}.

![Image of Figure 7: The map f of Example 6.2. Points are labeled by their indices as in the Example. The cycle with points p = (x, y) for x ≤ 0 represents C_6, for which {x_0, x_2, x_4} is a c_2-freezing set; the cycle with points p = (x, y) for x ≥ 0 represents C_m (here, m = 8, and {x'_0, x'_2, x'_4} is a c_2-freezing set for C_8, so A = {x, x_4, x'_4, x'_0} is a c_2-freezing set for C_6 ∪ C_8). Arrows connect p and f(p) for points p ∉ Fix(f). Each point of A is a c_2-free approximate fixed point of f.](image-url)
it follows that \( p_1(f(q_0)) \geq 1 \) and \( p_2(f(q_0)) \geq 1 \), contrary to the assumption that \( q_0 \) is a \( c_1 \)-approximate fixed point of \( f \).

Thus every case yields a contradiction brought about by assuming there is a point of \( P \) that is not an approximate fixed point of \( f \). The assertion follows. \( \square \)

**Theorem 6.4.** Let \( V_i \subset X \subset \mathbb{Z}^2 \), \( i \in \{1, \ldots, n\} \) where each \( V_i \) is a thick convex disk. Let \( X' = \bigcup_{i=1}^n V_i \). Let \( C \) be a bounding curve of \( V_i \). Let \( A_{1,i} \) be the set of endpoints of maximal axis-parallel segments of \( C \). Let \( A_{2,i} \) be the union of maximal slanted segments of \( C \).

1. \( A = (X \setminus X') \cup \bigcup_{i=1}^n (A_{1,i} \cup A_{2,i}) \) is a freezing set for \( (X, c_1) \).
2. Suppose \( f \in C(X, c_1) \) such that every point of \( A \) is a \( c_1 \)-approximate fixed point of \( f \). Then every point of \( X \) is a \( c_1 \)-approximate fixed point of \( f \).

**Proof.** Assertion 1) is Theorem 6.4. To prove assertion 2), we argue as follows.

Let \( S \) be a maximal digital segment of a bounding curve \( C_i \) for \( V_i \). If \( S \) is horizontal or vertical, then by Lemma 6.3 every point of \( S \) is a \( c_1 \)-approximate fixed point of \( f \). If \( S \) is slanted, then \( S \subset A \), so every point of \( S \) is a \( c_1 \)-approximate fixed point of \( f \). Thus each point of \( C_i \), is a \( c_1 \)-approximate fixed point of \( f \).

For \( x \in X \setminus A \), there is a horizontal segment \( P \) containing \( x \) such that the endpoints of \( P \) belong to \( \bigcup_{i=1}^n C_i \), and therefore are approximate fixed points of \( f \). By Lemma 6.3 every point of \( P \) is a \( c_1 \)-approximate fixed point of \( f \). Thus, every point of \( X \) is a \( c_1 \)-approximate fixed point of \( f \). \( \square \)

**Remark 6.5.** Theorems 6.4 and 6.4 simplify when \( X' = X \), in which case \( A = \bigcup_{i=1}^n (A_{1,i} \cup A_{2,i}) \). They might be applied in this case when \( i \neq j \) implies \( V_i \cap V_j \) is empty, a single point, or a common edge of \( V_i \) and \( V_j \).

### 6.3 Disks in \((\mathbb{Z}^2, c_2)\)

We show in this section that disks in \((\mathbb{Z}^2, c_2)\) yield results similar to those shown in section 6.2 for the \( c_1 \) adjacency.

**Lemma 6.6.** Let \( q_0, q_1 \in X \subset \mathbb{Z}^2 \). Suppose there is a slanted \( c_2 \)-path \( P \) in \( X \) from \( q_0 \) to \( q_1 \). Let \( f : P \to X \) be \( c_2 \)-continuous, such that \( q_0 \) and \( q_1 \) are \( c_2 \)-approximate fixed points of \( f \). Then every member of \( P \) is a \( c_2 \)-approximate fixed point of \( f \).

**Proof.** Without loss of generality, the slope of \( P \) is 1. Without loss of generality, \( q_0 = (0, 0) \) and \( q_1 = (n, n) \) for \( n = \text{length}(P) \). Suppose there exists \( p \in P \) that is not a \( c_2 \)-approximate fixed point of \( f \). Then \( |p_1(f(p)) - p_1(p)| > 1 \) or \( |p_2(f(p)) - p_2(p)| > 1 \).

- If \( |p_1(f(p)) - p_1(p)| > 1 \) then either \( p_1(f(p)) - p_1(p) > 1 \) or \( p_1(p) - p_1(f(p)) > 1 \).

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- If \( p_1(f(p)) - p_1(p) > 1 \) then by Lemma 3.2, \( 1 < p_1(f(q_0)) - p_1(q_0) = p_1(f(q_0)) \), contrary to the assumption that \( q_0 \) is an approximate fixed point.
- If \( p_1(p) - p_1(f(p)) > 1 \), then by Lemma 3.2, \( 1 < p_1(q_1) - p_1(f(q_1)) = n - p_1(f(q_1)), \) or \( p_1(f(q_1)) < n - 1 \), contrary to the assumption that \( q_1 \) is an approximate fixed point.

- If \( |p_2(f(p)) - p_2(p)| > 1 \) then, similarly, we obtain contradictions.

Since all cases yield contradictions, the hypothesis of a \( p \in P \) that is not a \( c_2 \)-approximate fixed point of \( f \) must be false. This completes the proof.

The following is a dual to Theorem 6.4.

**Theorem 6.7.** Let \( V_i \subset X \subset \mathbb{Z}^2, \) for \( i \in \{1, \ldots, n\} \) where each \( V_i \) is a thick convex disk. Let \( C_i \) be a bounding curve of \( V_i \). Let \( B_{1,i} \) be the union of maximal horizontal and maximal vertical segments of \( C_i \). Let \( B_{2,i} \) be the set of endpoints of maximal slanted segments of \( C_i \).

1. \( B = (X \setminus X') \cup \bigcup_{i=1}^{n} (B_{1,i} \cup B_{2,i}) \) is a freezing set for \( (X, c_2) \).

2. Suppose \( f \in C(X, c_2) \) such that every point of \( B \) is a \( c_2 \)-approximate fixed point of \( f \). Then every point of \( X \) is a \( c_2 \)-approximate fixed point of \( f \).

**Proof.** Assertion 1) is Theorem 3.5. To prove assertion 2), we argue as follows.

By Lemma 6.6, every slanted segment of \( C_i \) is made up entirely of \( c_2 \)-approximate fixed points of \( f \). From Theorem 3.6, it follows that \( C_i \) is made up entirely of \( c_2 \)-approximate fixed points of \( f \). Therefore, every point of \( B \) is a \( c_2 \)-approximate fixed point of \( f \).

Lemma 6.6 lets us conclude that if \( x \in X \) such that \( x \) lies on a slanted segment \( P \) that connects two points of \( B \), then \( x \) is a \( c_2 \)-approximate fixed point of \( f \).

This leaves us to consider points \( p = (x_0, y_0) \in X \) such that \( p \) does not lie either on an axis-parallel segment of \( B \) or on a slanted segment \( P \) that connects two points of \( B \). Such a point must be in the interior of \( X \) and therefore is \( c_2 \)-adjacent to its 4 \( c_1 \)-neighbors \( q_1 = (x_0 - 1, y_0), q_2 = (x_0 + 1, y_0), q_3 = (x_0, y_0 - 1), \) and \( q_4 = (x_0, y_0 + 1) \), each of which lies on a slanted segment joining members of \( S \) (see Figure 8). Therefore, by Lemma 6.6, \( q_1, q_2, q_3, \) and \( q_4 \) are approximate fixed points of \( f \).

Suppose \( p \) is not a \( c_2 \)-approximate fixed point of \( f \). Then either \( |p_1(f(p)) - x_0| > 1 \) or \( |p_2(f(p)) - y_0| > 1 \).

- Suppose \( |p_1(f(p)) - x_0| > 1 \). Then either \( p_1(f(p)) - x_0 > 1 \) or \( x_0 - p_1(f(p)) > 1 \).
  - Suppose \( p_1(f(p)) - x_0 > 1 \). Then by the continuity of \( f \) and Lemma 3.2, \( p_1(q_1) - p_1(f(q_1)) > 1 \), contrary to \( q_1 \) being an approximate fixed point of \( f \).
Figure 8: The point (3,2) in the digital image shown above does not lie on a slanted segment that joins 2 points of the boundary curve shown darkly.

- Suppose $x_0 - p_1(f(p)) > 1$. Then by the continuity of $f$ and Lemma 6.2, $p_1(q_2) - p_1(f(q_2)) > 1$, contrary to $q_2$ being an approximate fixed point of $f$.

- Similarly, we obtain a contradiction if $|p_2(f(p)) - y_0| > 1$.

Since all cases yield a contradiction when we assume $p$ is not a $c_2$-approximate fixed point of $f$, this hypothesis must be incorrect. The assertion follows.

**Remark 6.8.** Theorems 3.8 and 6.7 simplify when $X' = X$, in which case $B = \bigcup_{i=1}^{n}(B_{1,i} \cup B_{2,i})$. They might be applied in this case when $i \neq j$ implies $V_i \cap V_j$ is empty, a single point, or a common edge of $V_i$ and $V_j$.

### 6.4 Trees

**Theorem 6.9.** Let $(X, \kappa)$ be a digital image such that the graph $G = (X, \kappa)$ is a finite tree with $\#X > 1$. Let $E$ be the set of vertices of $G$ that have degree 1. Then $E$ is a minimal freezing set for $G$.

**Lemma 6.10.** Let $(X, \kappa)$ be a digital image such that the graph $G = (X, \kappa)$ is a finite tree. Let $f \in C(X, \kappa)$. Let $a, b \in X$ be such that $a$ and $b$ are $\kappa$-approximate fixed points of $f$. Let $P$ be the unique shortest path in $G$ from $a$ to $b$. Then $f(P) \subset P$ and every point of $P$ is a $\kappa$-approximate fixed point of $f$.

**Proof.** Let $P = \{x_i\}_{i=0}^{n}$ such that $x_0 = a$, $x_n = b$, and $x_i \leftrightarrow x_j$ if and only if $|i - j| = 1$.

- Suppose $f(a) = a$. Let us show that

$$f(x_{n-1}) \in \{x_{n-1}, b\} \subset P.$$ (6)

We know that $f(b) \in N^*(X, b, \kappa)$. If (6) is false, then $f(P) = P \cup \{f(b)\}$ is the unique shortest path in $G$ from $a$ to $f(b)$. But $P \cup \{f(b)\}$ has length $n + 1$, and $\#P = n + 1$ implies $\#f(P) \leq n$, so we have a contradiction brought about by negating (6). Thus (6) is established.
Since $G$ is acyclic, we must have $f(P) \subset P$. Now suppose for some $k$ that $x_k$ is not an approximate fixed point of $f$. Then $f(x_k) = x_m$ for some $m$ such that $|k - m| > 1$. Without loss of generality, $m - k > 1$. Then by continuity and since $G$ is acyclic, $f(x_k)$ must “pull” $f(a) = f(x_0)$ so that $f(a) = x_t$ for some $t > 1$, contrary to $a \in \text{Fix}(f)$. The contradiction establishes that each point of $P$ must be an approximate fixed point of $f$.

- Suppose $f(a) \not\in P$. Recall we are assuming $f(b) \in N^*(X, b, \kappa)$, so $f(b) \in \{x_{n-1}, b\}$ or $f(b) \not\in P$. We claim $f(b) = x_{n-1}$. For otherwise, $f(P) = \{f(a) \not= x_0, a = x_0, x_1, \ldots, x_n = b, f(b)\}$ where $f(b)$ may be equal to $b$, so $\#f(P) \in \{n + 2, n + 3\}$ while $\#P = n + 1$, a contradiction. Therefore, $f(b) = x_{n-1}$. By the acyclicity of $G$, it follows that $f(P) = \{f(a)\} \cup \{x_i\}_{i=0}^m$, where $m \in \{n - 1, n\}$. As in the case $f(a) = a$, it follows that every point of $P$ is an approximate fixed point of $f$.

- Suppose $f(a) \in P \setminus \{a\}$. Since $a$ is an approximate fixed point of $f$, it follows that $f(a) = x_1$. It follows as in the case $f(a) \not\in P$ that every point of $P$ is an approximate fixed point of $f$.

This establishes the assertion. \hfill \Box

**Theorem 6.11.** Let $(X, \kappa)$ be a digital image such that the graph $G = (X, \kappa)$ is a finite tree with $\#X > 1$. Let $E$ be the set of vertices of $G$ that have degree 1. Then given a freezing set $A$ for $G$, we have $E \subset A$.

**Proof.** Since $\#X > 1$, we can choose $x_0 \in E \setminus \{a\}$ to be a root for $G$. Then the function $f : X \to X$ given by

$$f(x) = \begin{cases} x & \text{for } x \not= a; \\ \text{parent}(a) & \text{for } x = a, \end{cases}$$

is easily seen to be a member of $C(X, \kappa)$ such that $f|_{E \setminus \{a\}} = \text{id}_{E \setminus \{a\}}$ and $f(a) \not= a$. Thus $E \setminus \{a\}$ cannot be a freezing set for $G$. The assertion follows. \hfill \Box

**Theorem 6.12.** Let $(X, \kappa)$ be a digital image such that the graph $G = (X, \kappa)$ is a finite tree with $\#X > 1$. Let $A$ be a freezing set for $G$. Suppose $f \in C(X, \kappa)$ is such that for each $a \in A$, $a$ is an approximate fixed point of $f$. Then for all $x \in X$, $x$ is an approximate fixed point of $f$.

**Proof.** Let $E$ be the set of vertices of $G$ that have degree 1. By Theorem 6.11 $E \subset A$, and by Theorem 6.10, $E$ is a freezing set. Therefore, there is no loss of generality in assuming $A = E$.

Let $f \in C(X, \kappa)$ such that for each $e \in E$, $e$ is an approximate fixed point of $f$. We can choose $x_0 \in E$ as a root of $X$. Since $x \in X$ implies $x$ is on the unique shortest path in $G$ from $x_0$ to some $e \in E$, it follows from Lemma 6.10 that $x$ is an approximate fixed point of $f$. \hfill \Box
6.5 Cycles

Theorem 6.13. Let \((C_n, \kappa)\) be a digital cycle of \(n\) distinct points, \(n \in \mathbb{N}, n \geq 3\), with \(C_n = \{x_i\}_{i=0}^{n-1}\), such that \(x_i \leftrightarrow \kappa x_j\) if and only if \(j = (i \pm 1) \mod n\). Let \(A = \{x_u, x_v, x_w\}\) be a set of distinct members of \(C_n\) such that \(C_n\) is a union of unique shorter paths determined by these points. Let \(f \in C(C_n, \kappa)\) be such that every member of \(A\) is an approximate fixed point of \(f\). Then every member of \(C_n\) is an approximate fixed point of \(f\), and \(f\) is an isomorphism.

Proof. Note by Theorem 4.6, \(A\) is a minimal freezing set for \((C_n, \kappa)\).

First, we show that \(f\) must be a surjection. Without loss of generality, \(0 \leq u < v < w < n\). Suppose \(B\) is the unique shorter path in \(C_n\) from \(x_u\) to \(x_v\). Since we must have \(\#f(B) \leq \#B\) and \(x_u\) and \(x_v\) are approximate fixed points, we must have \(f(x_u) \in \{x_{u-1}, x_u, x_{u+1}\}\) and \(f(x_v) \in \{x_{v-1}, x_v, x_{v+1}\}\).

Suppose \(f(x_u) = x_u\). We must have \(\#f(B) \leq \#B = v - u + 1 \leq n/2\), so \(f(x_v) \in \{x_{v-1}, x_v\}\). If \(f(x_v) = x_{v-1}\), then we must have \(f(x_w) = x_{w-1}\), hence (proceeding with increasing indices, \(\mod n\)) \(f(x_{u-1}) = x_{u-2}\), so \(f\) would be discontinuous at the adjacent pair \(x_{u-1}\) and \(x_u\). Thus we would have \(f(x_v) = x_v\) and \(f(x_w) = x_w\). Thus \(f|_A = \text{id}_A\). Since \(A\) is freezing, it follows that \(f = \text{id}_X\).

If \(f(x_u) = x_{u-1}\) or \(f(x_u) = x_{u+1}\), we can apply a rotation \(r(x_i) = x_{(i-1)} \mod n\) (respectively, \(r(x_i) = x_{(i+1)} \mod n\), which is an isomorphism). Then by the above, \(r \circ f = \text{id}_X\) is an isomorphism, so

\[
f = r^{-1} \circ r \circ f = r^{-1} \circ \text{id}_X = r^{-1}
\]

is an isomorphism, with each member of \(A\) an approximate fixed point.

Thus, in all cases, each member of \(A\) is an approximate fixed point of \(f\), which must be an isomorphism. \(\square\)

7 Further remarks

When a member of \(C(X, \kappa)\) has restricted behavior on a subset \(A\) of \(X\), the restriction may have a powerful effect on the behavior of \(f|_{X\setminus A}\). We have examined instances of this phenomenon with respect to freezing and cold sets, retractions, and shy maps, on a variety of basic digital images.

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