1. Introduction

The Kaczmarz method is an alternating projection method for solving linear systems. Due to its simplicity, the Kaczmarz method is often used in various applications. For example, the Kaczmarz method is called Algebraic Reconstruction Technique (ART) in computed tomography [1, §1]. Here, we focus on convergence theory of an efficient randomized version of the Kaczmarz method. For given \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \), we consider the linear system \( Ax = b \). In this paper, we assume \( m > n \) and \( A \) has full column rank. For each \( k = 0, 1, \ldots \), let \( x^k \in \mathbb{R}^n \) denote the computed vector in the \( k \)-th iteration. In addition, let \( A_i \) for \( i = 1, \ldots, m \) denote the \( i \)-th row vectors of \( A \), and \( p_i := \|A_i\|_2^2 / \|A\|_F^2 \) for \( i = 1, \ldots, m \). The randomized Kaczmarz method is described as follows.

Recently, Strohmer and Vershynin [1] proved that, if the system \( Ax = b \) has a solution, i.e., the system is consistent, the randomized Kaczmarz method exponentially converges to the solution vector \( x^* \) in the mean square. More precisely, letting

\[
\alpha_{RK} := 1 - \frac{\lambda_{\text{min}}(A^TA)}{\|A\|_F^2},
\]

where \( \lambda_{\text{min}} \) denotes the smallest eigenvalue, we have

\[
\mathbb{E}[\|x^k - x^*\|_2^2] \leq \alpha_{RK} \|x^0 - x^*\|_2^2 + \frac{\|\eta\|_2^2}{\lambda_{\text{min}}(A^TA)},
\]

where \( \mathbb{E}[\cdot] \) means the expectation of the inside.

If a noise vector \( \eta \) is on the system: \( Ax^* + \eta = b \), then for the given \( A \) and \( b \), the system \( Ax = b \) has no solution in general, i.e., the system is inconsistent. In those cases, Needell provided an error analysis of the approximate solutions computed by the randomized Kaczmarz method for the first time [2]. Zouzias and Freris refined this analysis, which implies

\[
\mathbb{E}[\|x^k - x^*\|_2^2] \leq \alpha_{RK} \|x^0 - x^*\|_2^2 + \frac{\|\eta\|_2^2}{\lambda_{\text{min}}(A^TA)},
\]

for any combination of \( x^* \) and \( \eta \) satisfying \( Ax^* + \eta = b \) [3, §3.3].

Our aim is to extend the above error estimate (3) to a unifying framework for a variety of randomized algorithms in a similar manner to Gower and Richtárik [4] for consistent linear systems in the next section.

2. Gower-Richtárik method for consistent linear systems

Gower and Richtárik presented a framework for various randomized iterative methods for consistent linear systems in [4]. The framework has two parameters: a positive definite matrix \( B \in \mathbb{R}^{m \times m} \) defining inner product, and a random matrix \( S \in \mathbb{R}^{m \times q} \) corresponding to the random sampling. Their framework has the six equivalent expressions under the assumption that the system is consistent [4, §2.1]. Among them, we focus on the following two expressions. The first one is

\[
x^{k+1} = x^k - B^{-1}A^TY(Ax^k - b),
\]
where
\[ Y := S(S^T AB^{-1} A^T S)^\dagger S^T. \] (5)

Here let \( \dagger \) denote the Moore–Penrose pseudoinverse. The second one is
\[ z^k := x^k - x^*, \quad z^{k+1} = (I - B^{-1} Z)z^k, \] (6)
where \( x^* \) is the solution vector and
\[ Z := A^T S(S^T AB^{-1} A^T S)^\dagger S^T A = A^T Y A. \] (7)
The second one yields
\[ \mathbf{E}[\|z^k\|_B^2] \leq \alpha^k \|z_0\|^2_B, \] (8)
where
\[ \alpha := 1 - \lambda_{\min}(B^{-1} \mathbf{E}[Z]), \]
\[ = 1 - \lambda_{\min}(B^{-\frac{1}{2}} \mathbf{E}[Z] B^{-\frac{1}{2}}). \] (9)

The exponential convergence to the solution \( x^* \), namely \( 0 \leq \alpha < 1 \), is proved under the assumption \( \mathbf{E}[Z] \) is a positive definite matrix; see [4, §4.4] for the details.

In what follows, (4) is called the Gower-Richtárik method for inconsistent systems in the next section because a variety of randomized algorithms arise as special cases of the iteration (4). In this section, let us see that the Gower-Richtárik method reproduces various randomized iterative methods with various \( B \) and \( S \). In particular, the randomized Kaczmarz method is associated with \( B = I \) \( \in \mathbb{R}^{n \times n} \) and \( S = e_i \), where \( e_i \) \( (i = 1, \ldots, m) \) are the ith columns of the identity matrix \( I \) \( \in \mathbb{R}^{m \times m} \) and every \( i \) is selected with probability \( p_i := \|A_i\|_2^2/\|A\|^2 \). Thus, we see
\[ x^{k+1} = x^k - B^{-1} T S(S^T AB^{-1} A^T S)^\dagger S^T (A x^k - b) \]
which is equivalent to line 3 in Algorithm 1. Moreover, (8) reproduces (2) with these parameters as
\[ \alpha = 1 - \lambda_{\min}(B^{-1} \mathbf{E}[Z]) \]
\[ = 1 - \lambda_{\min} \left( \sum_{i=1}^m p_i A_i^T e_i (e_i^T A A^T e_i) e_i A_i^T \right) \]
\[ = 1 - \lambda_{\min} \left( \sum_{i=1}^m \frac{\|A_i\|_2^2 A_i^T A_i}{\|A_i\|_2^2} \right) \]
\[ = 1 - \lambda_{\min}(A^T A) \frac{\|A\|^2}{\|A\|^2} = \alpha_{\text{RRK}}. \]

As another example, the Gower-Richtárik method reproduces randomized coordinate descent in least square cases in the situation for \( B = A^T A \) and \( S = A e_i \), where \( e_i \) \( (i = 1, \ldots, n) \) are the ith columns of the identity matrix \( I \) \( \in \mathbb{R}^{n \times n} \) and every \( i \) is selected with probability \( p_i := \|A_i\|_2^2/\|A\|^2 \). The iteration is
\[ x^{k+1} = x^k - B^{-1} A^T S(S^T AB^{-1} A^T S)^\dagger S^T (A x^{k} - b) \]
\[ = x^k - \frac{A_i^T (A x^{k} - b)}{\|A_i\|_2^2} e_i, \] (10)

where \( A_i \) for \( i = 1, \ldots, n \) mean the ith columns of the matrix \( A \). This algorithm reads as follows. Regarding this algorithm, noting (8) with these parameters, we can obtain
\[ \mathbf{E}[\|A x^k - b\|_B^2] \leq \alpha_{\text{CD}} \|A x^0 - b\|^2_B, \] (11)
where \( \alpha_{\text{CD}} := \alpha_{\text{RRK}} \) in (1); see [4, §3.7] for the details.

3. Error analysis of the Gower-Richtárik method for inconsistent systems

The six update expressions of the framework of Gower and Richtárik [4, §2.1] are not equivalent when the system is inconsistent. However, among them, (4) still reproduces various randomized iterative methods even if the system is inconsistent. Thus the remaining task is to establish their error estimates. Hence, we heed the following two facts: (i) such an estimate has been already established for a special case, the randomized Kaczmarz, as seen in (3); on the other hand, (ii) if we limit ourselves to consistent systems, the way to elevate the simple estimate (2) for the randomized Kaczmarz to more general case (8) has been revealed as shown above. With these observations, let us commence by focusing on the derivation of (3) for the randomized Kaczmarz method. First of all, noting line 3 in Algorithm 1, we have
\[ x^{k+1} = x^k + (I - B^{-1} Z) (x^k - x^*) + \frac{\eta_i}{\|A_i\|_2^2} A_i^T, \] (12)
where \( A x^* + \eta = b \).

Next, using (4) in the Gower-Richtárik method, we modify (6) to reflect \( \eta \) in a similar manner to (12). From (4), (5), and (7), letting \( z^k := x^k - x^* \), we obtain
\[ z^k = (I - B^{-1} Z) (x^k - x^*) + B^{-1} A^T Y (b - A x^*) \]
\[ = (I - B^{-1} Z) z^{k-1} + B^{-1} A^T Y \eta, \] (13)
regarded as an extension of (6) implying the exponential convergence in consistent systems. Hence, on the basis of (13), we extend (3) in inconsistent systems. To this end, we estimate \( \|z^k\|^2_B \) with \( z^{k-1} \) and an error term as follows.

Lemma 1 Suppose that the Gower-Richtárik method expressed by (4) is applied to linear systems. For any combination of \( x^* \) and \( \eta \) satisfying \( A x^* + \eta = b \), letting \( z^k := x^k - x^* \), we have
\[ \|z^k\|^2_B = (z^{k-1})^T (B - Z) z^{k-1} + \eta^T \eta, \] (14)
where \( Y \) and \( Z \) are defined as (5) and (7), respectively.

Proof Using (13), we have
\[ (z^k)^T B z^k = (z^{k-1})^T B (I - B^{-1} Z) z^{k-1} \]
The conditional expectation of (14) is

\[ E[(z^{k-1})^T (I - Z B^{-1}) A^T Y] \]

where the last equality is due to a feature of the projection matrix \( I - B^{-1} Z \), which can be easily confirmed by direct calculations as

\[ (I - B^{-1} Z)^2 = I - B^{-1} Z. \]

In fact, for the second term of (15), we see

\[ (I - Z B^{-1}) A^T Y = A^T Y - Z B^{-1} A^T Y \]

where the second equality is due to (5) and (7). Moreover, the third term of (15) is

\[ \eta^T Y A B^{-1} A^T Y \eta = \eta^T \eta \]

from easy calculations using (5). Thus, we obtain (14).

(QED)

By taking the expectation of (14), we can prove that the next inequality holds whenever the Gower-Richtárik method expressed by (4) is applied to linear systems.

**Theorem 2** Suppose that the Gower-Richtárik method expressed by (4) is applied to linear systems, and \( E[Z] \) is positive definite, where \( Z \) is defined as (7). For each \( k = 0, 1, \ldots \), let \( x^k \) denote the computed vector in the \( k \)th iteration. For any combination of \( x^* \) and \( \eta \) satisfying \( A x^* + \eta = b \), let \( z^k := x^k - x^* \). Then, we obtain

\[ 0 \leq \alpha < 1, \]

\[ E [ \| z^k \|_b^2 | z^{k-1} ] \leq \alpha \| z^{k-1} \|_b^2 + \eta^T E[Y] \eta, \]

where \( \alpha \) and \( Y \) are defined as (9) and (5), respectively.

**Proof** The conditional expectation of (14) is

\[ E[(z^{k-1})^T (I - Z B^{-1}) A^T Y] \]

where the second term in the right-hand side is equal to

\[ (z^{k-1})^T B^{-1} E[Z] B^{-1} z^{k-1}. \]

Note that, since \( B^{-\frac{1}{2}} Z B^{-\frac{1}{2}} \) is an orthogonal projection, the eigenvalues of the expectation of the matrix are within 0 and 1. Hence, we see

\[ (z^{k-1})^T E[Z] z^{k-1} \geq \min(B^{-\frac{1}{2}} E[Z] B^{-\frac{1}{2}}) \| z^{k-1} \|^2_b, \]

\[ (z^{k-1})^T E[Z] z^{k-1} \leq \| z^{k-1} \|^2_b. \]

Using the above two inequalities and (17), we have

\[ 0 \leq \alpha < 1, \]

\[ E [ \| z^k \|_b^2 | z^{k-1} ] \leq \alpha \| z^{k-1} \|_b^2 + \eta^T E[Y] \eta, \]

where \( \alpha \) is defined as (9). Taking the expectation of (18) on \( z^{k-1} \) gives

\[ E [ \| z^k \|_b^2 ] \leq \alpha E [ \| z^{k-1} \|_b^2 ] + \eta^T E[Y] \eta. \]

By repeating (19), we can obtain

\[ E [ \| z^k \|_b^2 ] \leq \alpha^k E [ \| z^0 \|_b^2 ] + (1 + \alpha + \cdots + \alpha^{k-1}) \eta^T E[Y] \eta \]

\[ \leq \alpha^k E [ \| z^0 \|_b^2 ] + \eta^T E[Y] \eta. \]

This completes the proof.

(QED)

We have proved that the inequality (16) holds whenever the Gower-Richtárik method is applied to linear systems, where \( x^* \) and \( \eta \) satisfy \( A x^* + \eta = b \). Recall that \( A \) has full column rank in this paper. Thus, (7) implies that \( E[Z] \) is always positive definite in any algorithm using some random sampling, as \( E[Y] \) must be positive definite. The inequality (16) is a natural extension of (3) for the randomized Kaczmarz method as follows.

**Corollary 3** The inequality (16) reproduces the inequality (3) when the Gower-Richtárik method reproduces the randomized Kaczmarz method.

**Proof** Suppose that \( B = I \) and \( S = e_i \) in the Gower-Richtárik method, where every \( i \) is selected with probability \( p_i := \| A e_i \|_F^2 / \| A \|_F^2 \) as the randomized Kaczmarz method. Then, we have

\[ E[Y] = \sum_{i=1}^m p_i e_i (A^T A e_i) e_i^T \]

\[ = \sum_{i=1}^m \| A e_i \|_F^2 e_i e_i^T = I / \| A \|_F^2, \]

\[ \alpha = 1 - \lambda_{\min}(B^{-1} E[Z]), \]

\[ = 1 - \lambda_{\min}(A^T E[Y] A) = \alpha_{NK}, \]

\[ \eta^T E[Y] \eta / (1 - \alpha) = \| \eta \|_b^2 / \lambda_{\min}(A^T E[Y] A) = \| \eta \|_b^2 / \lambda_{\min}(A^T A). \]

Therefore, (16) reproduces (3).

(QED)

Next, we consider the randomized coordinate descent method, described in Algorithm 2.

**Corollary 4** Suppose that \( B = A^T A \) and \( S = A e_i \), where every \( i \) is selected with probability \( p_i := \| A e_i \|_F^2 / \| A \|_F^2 \) in (4), corresponding to Algorithm 2. Then, the inequality (16) produces

\[ E [ \| A z^k \|_b^2 ] \leq \alpha_{CD} \| A z^0 \|_b^2 + \| A^T \eta \|_b^2 / \lambda_{\min}(A^T A), \]

where \( z_k := x^k - x^* \) and \( \alpha_{CD} := 1 - \lambda_{\min}(A^T A) / \| A \|_F^2. \)

**Proof** From easy calculations, we have

\[ E[Y] = \sum_{i=1}^m p_i A e_i (e_i A^T A (A^T A)^{-1} A^T A e_i) e_i^T A^T. \]
\begin{align*}
  &\sum_{i=1}^{m} \frac{\|A_{i}\|_{F}^{2}}{\|A_{i}\|_{F}^{2}} \frac{A_{i}e_{i}^{T}A^{T}}{\|A_{i}\|_{F}^{2}} = \frac{AA^{T}}{\|A\|_{F}^{2}},
  \\
  &\alpha = 1 - \lambda_{\min}(A^{T}A)^{-1}A^{T}E[Y]A
  \\
  &\quad = 1 - \frac{\lambda_{\min}(A^{T}A)}{||\eta||_{2}^{2}},
  \\
  &\frac{\eta^{T}E[Y]\eta}{1 - \alpha} = \frac{\|A^{T}\eta\|_{2}^{2}}{\lambda_{\min}(A^{T}A)}.
\end{align*}

Therefore, we obtain (20).

(QED)

Note that the inequality (20) indicates that Algorithm 2 computes the least square solution of \( \min \|b - Ax\|_{2}^{2} \) because \( A^{T}\eta = A^{T}(b - Ax^{*}) = 0 \) whenever \( x^{*} \) is the least square solution.

4. Numerical experiments

In this section, we present numerical results to illustrate the theoretical error estimate in (16). Our computing environment is as follows:

- OS: Windows 7 Home Premium
- CPU: Intel(R) Core i7-3630QM CPU 2.40GHz
- Memory: 6GB
- Software: MATLAB R2014b

To generate a noisy linear system \( Ax^{*} + \eta = b \), we used an \( m \times n \) random dense matrix \( A \) generated with the MATLAB command \texttt{randn(m,n)}, which returns an \( m \times n \) matrix of normally distributed random numbers. The vector \( x^{*} \) is generated with \texttt{randn(n,1)} as the approximate solution vector. In addition, the noise vector \( \eta \) with \( ||\eta||_{2} = 0.1 \) is generated using \texttt{randn(m,1)}, i.e., \( \texttt{eta=randn(m,1)} \) and \( \texttt{eta=0.1*eta/norm(eta)} \). Moreover, let \( b := Ax^{*} + \eta \). We set \( m = 1000 \) and \( n = 500 \).

To verify Theorem 2, we tested Algorithms 1 and 2 as the examples of the Gower-Richtárik method. For each algorithm, we ran 10 trials with \( 1.2 \times 10^{5} \) iterations for a fixed starting vector \( x^{0} = 0 \in \mathbb{R}^{n} \). For each \( k \), the average result \( \|x^{k} - x^{*}\|_{2}^{2} \) of 10 trials is interpreted as an approximation of \( \mathbb{E}[\|x^{k} - x^{*}\|_{2}^{2}] = \mathbb{E}[\|z^{k}\|_{2}^{2}] \) in (16).

Firstly, we present the result of the randomized Kaczmarz (RK) method, described in Algorithm 1. In Fig. 1, the blue curve shows the right-hand side of (16), equivalent to that of (3). The red curve shows \( \mathbb{E}[\|x^{k} - x^{*}\|_{2}^{2}] \) in Algorithm 1. We see that the right-hand side of (16) is an upper bound of \( \mathbb{E}[\|x^{k} - x^{*}\|_{2}^{2}] \).

Secondly, let us see the case of the randomized coordinate descent (CD) method, described in Algorithm 2, which is compared with CGLS in Fig. 2 as in [1, §4.2] since both algorithms converge to the least square solution. However, iteration numbers of CGLS are multiplied by \( n \) in Fig. 2 to equalize the numbers of operations. The result shows that the error estimate in (16), namely (20), is also numerically supported. Similarly to [2, §3], we observed that the errors of CD are fairly reduced for \( k \leq n \) as in Table 1, while CD is slower than CGLS.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\text{method} & \text{CD} & \text{CGLS} & \text{RK} & \text{CGLS} \\
\hline
\text{Algorithm 1} & 0.12 & 0.36 & 0.12 & 0.12 \\
\text{Algorithm 2} & 0.12 & 0.36 & 0.12 & 0.12 \\
\hline
\end{tabular}
\caption{Convergence of \( \mathbb{E}[\|x^{k} - x^{*}\|_{2}^{2}] / \|A(x^{0} - x^{*})\|_{2}^{2} \).}
\end{table}

References

[1] T. Strohmer and R. Vershynin, A randomized Kaczmarz algorithm with exponential convergence, J. Fourier Anal. Appl., 15 (2009), 262–278.
[2] D. Needell, Randomized Kaczmarz solver for noisy linear systems, BIT, 50 (2010), 395–403.
[3] A. Zouzias and N. M. Freris, Randomized extended Kaczmarz for solving least squares, SIAM J. Matrix Anal. Appl., 34 (2013), 773–793.
[4] R. M. Gower and P. Richtárik, Randomized iterative methods for linear systems, SIAM J. Matrix Anal. Appl., 36 (2015), 1660–1690.