Scale-invariance, dynamically induced Planck scale and inflation in the Palatini formulation

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Based on: 2006.09124, I.D.G., A. Karam and A. Racioppi
2104.04550, I.D.G., A. Karam, T.D. Pappas and V.C. Spanos

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FRW metric:

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right) \] (1)

- Inflation is a theory of exponential expansion of space in the early universe, i.e. \( a \sim e^{Ht} \).
- \( \sim 10^{-33} - 10^{-32} \) seconds after the Big Bang.
- Solves the **horizon** and **flatness** problems.
- It can also provide a mechanism for the generation of the perturbations that have resulted in the anisotropies observed in the CMB.
Minimal Inflation

Action: EH + a scalar field

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R}{2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] , \quad (M^2_{\text{Pl}} \equiv 1) \quad (2) \]

Friedmann equations:

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3} , \quad H^2 + \dot{H}^2 = \frac{\ddot{a}}{a} = -\frac{1}{6} (\rho + 3p) \quad (3) \]

Density & Pressure:

\[ \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi) , \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (4) \]

Klein-Gordon equation:

\[ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0 \quad (5) \]
The scalar ($\mathcal{P}_\zeta$) and tensor ($\mathcal{P}_T$) power spectrum is

$$\mathcal{P}_\zeta(k) = A_s \left( \frac{k}{k_*} \right)^{n_s - 1}, \quad A_s = \frac{1}{24\pi^2} \frac{V(\phi_*)}{\epsilon_V(\phi_*)}, \quad \mathcal{P}_T = 8 \left( \frac{H}{2\pi} \right)^2 \simeq \frac{2V}{3\pi^2} \quad (6)$$

Spectral tilt ($n_s$) and tensor-to-scalar ratio ($r$)

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_\zeta(k)}{d \ln k} \simeq -6\epsilon_V + 2\eta_V, \quad r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \simeq 16\epsilon_V \quad (7)$$

We have used the potential slow-roll parameters:

$$\epsilon_V = \frac{1}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \quad \eta_V = \frac{V''(\phi)}{V(\phi)} \quad (8)$$

Number of $e$-folds

$$N(\phi) = \int_{t}^{t_{end}} H dt = \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon_H}} \simeq \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon_V}} \simeq 50 - 60 \quad (9)$$
Inflationary Observables (Planck 2018 1807.06211)

\[ n_s = 0.9649 \pm 0.0042, \quad r < 0.056 \quad \text{and} \quad A_s = (2.10 \pm 0.03) \times 10^{-9} \]
The action is

$$S = \int d^4x \sqrt{-g} \left( \frac{M_{Pl}^2 R}{2} + \frac{R^2}{12M^2} \right), \quad M \sim 10^{-5}. \tag{10}$$

After a Weyl rescaling of the metric $g_{\mu\nu}$ and a field redefinition

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{M_{Pl}^2 \tilde{R}}{2} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \tag{11}$$

where

$$V(\phi) = \frac{3}{4} M_{Pl}^4 M^2 \left[ 1 - \exp \left( -\sqrt{\frac{2}{3}} \phi/M_{Pl} \right) \right]^2. \tag{12}$$

We find for $N_* = 60$

$$n_s \simeq 1 - \frac{2}{N_*} \simeq 0.966 \quad \text{and} \quad r \simeq 12 \frac{N_*}{N_*^2} \simeq 0.0033.$$
In **metric formulation**, the metric is the only dynamical degree of freedom and the connection is the Levi-Civita: \( R_{\mu\nu} = R_{\mu\nu} (g, \partial g, \partial^2 g) \).

In **Palatini formulation**, both the metric and the connection are independent dynamical degrees of freedom \( R_{\mu\nu} = R_{\mu\nu} (\Gamma, \partial \Gamma) \).

Variation with respect to \( \Gamma \) gives

\[
\Gamma^\lambda_{\alpha\beta} = \left\{ \begin{array}{c} \lambda \\ \alpha \beta \end{array} \right\} + (1 - \kappa) \left[ \delta^\lambda_\alpha \partial_\beta \omega (\phi) + \delta^\lambda_\beta \partial_\alpha \omega (\phi) - g_{\alpha\beta} \partial^\lambda \omega (\phi) \right], \quad \omega (\phi) = \ln \sqrt{A(\phi)}
\]

where \( \kappa = 1 \) in metric and \( \kappa = 0 \) in Palatini. Performing a Weyl transformation

\[
\tilde{g}_{\mu\nu} \equiv A(\phi) g_{\mu\nu} \rightarrow \sqrt{-g} = A^{-2} \sqrt{-\tilde{g}}, \quad R = A \left( 1 - \kappa \times 6 A^{1/2} \tilde{\nabla}^\mu \tilde{\nabla}_\mu A^{-1/2} \right) \tilde{R},
\]

the action becomes

\[
S_{E}^{\text{Pal or metric}} = \int d^4 x \sqrt{-\tilde{g}} \left( \frac{1}{2} \tilde{R} - \frac{1}{2} \left( \frac{1}{A} + \kappa \times \frac{3}{2} \frac{A_\phi}{A^2} \right) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{V(\phi)}{A^2} \right).
\]

(14)
We consider the Higgs-like inflationary potential

\[ V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2, \quad A(\phi) = 1 + \xi \phi^2. \]  

(15)

Canonical field redefinition gives

\[ \phi(\chi) \approx \frac{1}{\sqrt{\xi}} \exp \left( \sqrt{\frac{1}{6}} \chi \right) \quad \text{(Metric)}, \quad \phi(\chi) = \frac{1}{\sqrt{\xi}} \sinh(\sqrt{\xi} \chi) \quad \text{(Palatini)} \]  

(16)

The Einstein-frame potential in terms of \( \chi \) can be expressed as

\[ U(\chi) \approx \frac{\lambda}{4\xi^2} \left( 1 + \exp \left( -\sqrt{\frac{2}{3}} \chi \right) \right)^{-2}, \quad \text{(Metric)}, \]  

(17)

\[ U(\chi) = \frac{\lambda}{4\xi^2} \tanh^4 \left( \sqrt{\xi} \chi \right), \quad \text{(Palatini)} \]  

(18)

\[ n_s \approx 1 - \frac{2}{N_*} + \frac{3}{2N_*^2}, \quad r \approx \frac{12}{N_*^2}, \quad A_s \approx \frac{\lambda N_*^2}{72\pi^2 \xi^2} \quad \text{(Metric)}, \]  

(19)

\[ n_s \approx 1 - \frac{2}{N_*} - \frac{3}{8\xi N_*^2}, \quad r \approx \frac{2}{\xi N_*^2}, \quad A_s \approx \frac{\lambda N_*^2}{12\pi^2 \xi} \quad \text{(Palatini)}. \]  

(20)
Palatini inflation with an $R^2$ term (1810.05536 Enckell et al)

1810.10418 Antoniadis et al, 1901.01794 Tenkanen, 1911.11513 I.D.G & A.B. Lahanas
2012.06831 Dimopoulos et al

Action

$$ S_J = \int d^4 x \sqrt{-g} \left[ \frac{A(\phi)}{2} R + \frac{\alpha}{2} R^2 - \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \quad (19) $$

Introducing an auxiliary field we eliminate the $R^2$ term and after a Weyl rescaling we obtain

$$ S_E = \int d^4 x \sqrt{-g} \left( \frac{R}{2} + K(\phi) X + L(\phi) X^2 - U(\phi) \right), \quad (20) $$

with $X = -1/2 \partial_\mu \phi \partial^\mu \phi$ and $K(\phi) = \frac{A(\phi)}{A^2(\phi) + 8\alpha V(\phi)}$, $L(\phi) = \frac{2\alpha}{A^2(\phi) + 8\alpha V(\phi)}$.

$$ U(\phi) = \frac{V(\phi)}{A^2(\phi) + 8\alpha V(\phi)}. $$

- Equation of motion

$$ (K + 3L\dot{\phi}^2)\ddot{\phi} + 3H(K + L\dot{\phi}^2)\dot{\phi} + U'(\phi) + \frac{1}{4}(2K' + 3L'\dot{\phi}^2)\dot{\phi}^2 = 0, \quad (21) $$

- Speed of sound

$$ c_s^2 = \frac{\partial p/\partial X}{\partial \rho/\partial X} = \frac{1 + L \dot{\phi}^2/K}{1 + 3L \dot{\phi}^2/K}, \quad (22) $$
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{\xi \phi^2 R}{2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] , \quad V(\phi) = \frac{1}{4} \lambda(\phi) \phi^4 + \Lambda^4 \] (23)

At the minimum
\[ V(v) = \frac{1}{4} \lambda(v) v^4 + \Lambda^4 = 0 , \quad v = \frac{M_P}{\sqrt{\xi}} \] (24)

Minimization: \( \beta(v) + 4\lambda(v) = 0 \). This implies

a) \( \beta(v) > 0 , \lambda(v) < 0 \)

b) \( \beta(v) = \lambda(v) = 0 \)

Taylor expansion around the VEV
\[ \lambda(\phi) = \lambda(v) + \beta(v) \ln \frac{\phi}{v} + \frac{1}{2!} \beta'(v) \ln^2 \frac{\phi}{v} + \frac{1}{3!} \beta''(v) \ln^3 \frac{\phi}{v} + \cdots , \] (25)

For the two cases we get
\[ \lambda^a(\phi) \simeq \lambda(v) + \beta(v) \ln \frac{\phi}{v} , \quad V(\phi) = \Lambda^4 \left\{ 1 + \left[ 4 \ln \left( \frac{\phi}{v} \right) - 1 \right] \frac{\phi^4}{v^4} \right\} \] (26)
\[ \lambda^b(\phi) \simeq \frac{\beta'(v)}{2} \ln^2 \frac{\phi}{v} , \quad V(\phi) = \frac{1}{8} \beta' \phi^4 \ln^2 \left( \frac{\phi}{v} \right) \] (27)
1st and 2nd order Coleman-Weinberg potentials

1st order:

\[
\text{JF } V(\phi) = \Lambda^4 \left\{ 1 + \left[ 4 \ln \left( \frac{\phi}{v} \right) - 1 \right] \frac{\phi^4}{v^4} \right\}, \quad \text{EF } \bar{U}(\bar{\zeta}) = \Lambda^4 \left( 4 \frac{\bar{\zeta}}{v} + e^{-4 \frac{\bar{\zeta}}{v}} - 1 \right)
\]

(28)

For \( v \ll 1 \) (i.e. \( \xi \gg 1 \)) and \( \bar{\zeta} > 0 \), the potential becomes

\[
\bar{U}(\bar{\zeta}) \approx a_\zeta \bar{\zeta}, \quad \text{with} \quad a_\zeta = 4 \frac{\Lambda^4}{v}.
\]

(29)

For \( v \gg 1 \) (i.e. \( \xi \ll 1 \)), the potential reduces to

\[
\bar{U}(\bar{\zeta}) \approx \frac{m^2}{2} \bar{\zeta}^2, \quad \text{with} \quad m = m_1 = 4 \frac{\Lambda^2}{v}.
\]

(30)

2nd order:

\[
\text{JF } V(\phi) = \frac{1}{8} \beta' \phi^4 \ln^2 \left( \frac{\phi}{v} \right), \quad \text{EF } \bar{U}(\bar{\zeta}) = \frac{m^2}{2} \bar{\zeta}^2, \quad \text{with} \quad m^2 = m_2^2 = \frac{\beta' v^2}{4}.
\]

(31)
Adding an $\frac{\alpha}{2} R^2$

Red $\rightarrow$ 1st order
Blue $\rightarrow$ 2nd order

$$U = \frac{\bar{U}}{1 + 8\alpha\bar{U}}$$
with $\bar{U} = V/A^2$. 
Inflationary predictions

Values of $\alpha = 0, 10^7, 10^8, 10^9$ and $10^{10}$
1403.4226 A. Salvio & A. Strumia, 1512.05890 A. Farzinnia & S. Kouwn, 2104.04550 I.D.G., A. Karam, T.D. Pappas & V.C. Spanos

The model: $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_X$

Extra particles: 3 RH neutrinos, 1 gauge boson, 1 scalar field

- $\mathcal{L}_{\text{Yukawa}}^{\text{BSM}} = - y_{ij}^{D} \bar{\ell}_L^i H N_R^j - \frac{1}{2} y_{M}^{i} \Phi \overline{N}_{R}^{iC} N_R^i + h.c$
- $\mathcal{L}_{\text{scalar}}^{\text{BSM}} = - \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} \lambda_\phi \phi^4 + \frac{1}{4} \lambda_{h \phi} h^2 \phi^2$
- $\mathcal{L}_{\text{SM}}$ with no Higgs mass term
- $\mathcal{L}_{\text{gravity}} = \frac{1}{2} \left( \xi_\phi \phi^2 + \xi_h h^2 \right) g^{\mu \nu} R_{\mu \nu} (\Gamma) + \frac{\alpha}{2} R^2 + \frac{\beta}{2} R_{\mu \nu} R^{\mu \nu}$

$\rightarrow$ Dynamical generation $M_P^2 = \xi_\phi v_\phi^2 + \xi_h v_h^2$
Inflationary action

\[ S_J = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left[ \left( \xi \phi^2 + \xi_h h^2 \right) g^{\mu\nu} R_{\mu\nu} + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right] \right. \]
\[ \left. - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\mu h \partial_\nu h - V^{(0)}(\phi, h) \right\} , \quad (32) \]

with

\[ V^{(0)}(\phi, h) = \frac{1}{4} \left( \lambda_\phi \phi^4 - \lambda_h \phi^2 h^2 \phi^2 + \lambda_h h^4 \right) . \]

After a Weyl rescaling

\[ g^{\mu\nu} \longrightarrow \Omega^2 g^{\mu\nu}, \Omega^2 = \xi \phi^2 + \xi_h h^2, \quad R \longrightarrow \Omega^2 R \]

we obtain

\[ S_{IF} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left[ g^{\mu\nu} R_{\mu\nu} + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right] \right. \]
\[ \left. - \frac{1}{2\Omega^2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2\Omega^2} g^{\mu\nu} \partial_\mu h \partial_\nu h - \frac{V^{(0)}(\phi, h)}{\Omega^4} \right\} . \quad (33) \]
**Gildener-Weinberg approach**

Gildener & Weinberg, 1976

The intermediate frame potential is

\[ U^{(0)}(\phi, h) \equiv \frac{V^{(0)}(\phi, h)}{\Omega^4} = \frac{(\lambda_\phi \phi^4 - \lambda_{h\phi} h^2 \phi^2 + \lambda_h h^4)}{4 (\xi_\phi \phi^2 + \xi_h h^2)^2}. \] (34)

**Flat direction (FD):** \( \partial_\phi U^{(0)}(\phi, h) = \partial_h U^{(0)}(\phi, h) = 0 \) gives the extremization condition, \( v_h = \sqrt{\frac{\lambda_{h\phi} \xi_\phi + 2 \lambda_\phi \xi_h}{\lambda_{h\phi} \xi_h + 2 \lambda_h \xi_\phi}} v_\phi \)

Along the FD \( U^{(0)}_{\text{min}} \equiv U^{(0)}(v_\phi, v_h) = \frac{(4 \lambda_h \lambda_\phi - \lambda_{h\phi}^2) M_P^4}{16 [\lambda_\phi \xi_h^2 + \xi_\phi (\lambda_{h\phi} \xi_h + \lambda_h \xi_\phi)]}. \) (35)

**Orthogonal rotation:** \[
\begin{pmatrix}
\phi \\
h
\end{pmatrix} = \begin{pmatrix}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{pmatrix} \begin{pmatrix}
s \\
\sigma
\end{pmatrix}.
\]

**Mixing angle:** \( \omega \equiv \arctan \left( \frac{v_h}{v_\phi} \right) \).
The one-loop corrections along the flat direction for the canonical field \( s_c \) at the scale \( \Lambda \) may be written as

\[
U^{(1)}(s_c) = A \, s_c^4 + B \, s_c^4 \ln \frac{s_c^2}{\Lambda^2},
\]

where in our model

\[
A = \frac{1}{64\pi^2v_s^4} \left\{ M_h^4 \left( \ln \frac{M_h^2}{v_s^2} - \frac{3}{2} \right) + 6M_W^4 \left( \ln \frac{M_W^2}{v_s^2} - \frac{5}{6} \right) + 3M_Z^4 \left( \ln \frac{M_Z^2}{v_s^2} - \frac{5}{6} \right) \\
+ 3M_X^4 \left( \ln \frac{M_X^2}{v_s^2} - \frac{5}{6} \right) - 6M_{NR}^4 \left( \ln \frac{M_{NR}^2}{v_s^2} - 1 \right) - 12M_t^4 \left( \ln \frac{M_t^2}{v_s^2} - 1 \right) \right\},
\]

\[
B = \frac{M^4}{64\pi^2v_s^4}, \quad M^4 = M_h^4 + 3M_X^4 + 6M_W^4 + 3M_Z^4 - 6M_{NR}^4 - 12M_t^4,
\]

Minimizing (36), we can determine the scale \( \Lambda \) as

\[
\Lambda = v_s \exp \left[ \frac{A}{2B} + \frac{1}{4} \right].
\]

Then, we can express the one-loop correction as

\[
U^{(1)}(s_c) = \frac{M^4}{64\pi^2v_s^4} s_c^4 \left[ \ln \frac{s_c^2}{v_s^2} - \frac{1}{2} \right].
\]
One-loop effective potential

We now require that the full one-loop effective potential is zero at $v_s$. Then

$$U_{\text{eff}}(v_s) = U_{\text{min}}^{(0)} + U^{(1)}(v_s) = 0,$$

(38)

which finally yields

$$U_{\text{eff}}(s_c) = \frac{M^4}{128\pi^2} \left[ \frac{s_c^4}{v_s^4} \left( 2 \ln \frac{s_c^2}{v_s^2} - 1 \right) + 1 \right].$$

Don’t forget that

$$v_s^2 = \frac{M_P^2}{\xi_s},$$

with $\xi_s \equiv \xi_\phi \cos^2 \omega + \xi_h \sin^2 \omega$.

Finally, the effective action along the FD written explicitly in terms of the inflaton field reads

$$S_{IF} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left[ g^{\mu\nu} R_{\mu\nu} + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right] - \frac{1}{2} g^{\mu\nu} \partial_\mu s_c \partial_\nu s_c - U_{\text{eff}}(s_c) \right\}.$$  

(39)
The IF action can be cast in the form

\[ S_{IF} = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} C(g_{\mu \nu}, R_{\mu \nu}) + \mathcal{L}_m(g_{\mu \nu}, s_c, \partial_\mu s_c) \right], \quad (40) \]

where we have defined

- \( C(g_{\mu \nu}, R_{\mu \nu}) = g^{\mu \nu} R_{\mu \nu} + \alpha R^2 + \beta R_{\mu \nu} R^{\mu \nu} \)
- \( \mathcal{L}_m(g_{\mu \nu}, s_c, \partial_\mu s_c) = -\frac{1}{2} g^{\mu \nu} \partial_\mu s_c \partial_\nu s_c - U_{\text{eff}}(s_c) \)
Einstein frame representation

See Jaakko Annala’s 2020 Master thesis for more details on this calculation.

Now, upon introducing the auxiliary field $\Sigma_{\mu\nu}$ the action becomes

$$
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} C(g_{\mu\nu}, \Sigma_{\mu\nu}, s_c) + \frac{1}{2} \frac{\partial C}{\partial \Sigma_{\mu\nu}} (R_{\mu\nu} - \Sigma_{\mu\nu}) + \mathcal{L}_m(g_{\mu\nu}, s_c) \right].
$$

(41)

We introduce the new variable $\sqrt{-qq^{\mu\nu}} = \sqrt{-g} \frac{\partial C}{\partial \Sigma_{\mu\nu}}$ thus the action can be written as

$$
S = \int d^4x \left\{ \frac{\sqrt{-q}}{2} q^{\mu\nu} R_{\mu\nu} - \frac{\sqrt{-g}}{2} \left[ \frac{\partial C}{\partial \Sigma_{\mu\nu}} \Sigma_{\mu\nu}(q_{\mu\nu}, g_{\mu\nu}, s_c) - C(q_{\mu\nu}, g_{\mu\nu}, s_c) - 2\mathcal{L}_m(g_{\mu\nu}, s_c) \right] \right\}
$$

(42)

EH sector

matter sector
Varying the previous action with respect to $g_{\mu\nu}$ will give us $g_{\mu\nu}$ as a function of $q_{\mu\nu}$, $s_c$ and $\partial_\mu s_c$. This way we obtain that

\[
\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = - \frac{1}{4(\beta + 4\alpha)} \frac{\sqrt{-q}}{\sqrt{-g}} q^{\sigma\lambda} g_{\sigma\mu} g_{\lambda\nu} 
+ \frac{1}{4\beta} g \left( q^{\sigma\lambda} q^{\rho\delta} g_{\lambda\delta} g_{\rho\nu} g_{\sigma\mu} - \frac{\alpha}{\beta + 4\alpha} q^{\delta\rho} g_{\delta\rho} q^{\sigma\lambda} g_{\sigma\mu} g_{\lambda\nu} \right) 
+ \frac{1}{2} g_{\mu\nu} \left[ \frac{1}{\beta + 4\alpha} \left( \frac{1}{2} + \frac{\alpha}{8\beta} g q^{\lambda\sigma} g_{\lambda\sigma} q^{\rho\delta} g_{\rho\delta} \right) - \frac{q}{g} \frac{1}{8\beta} g^{\lambda\sigma} q^{\delta\rho} g_{\lambda\delta} g_{\sigma\rho} \right] 
+ \frac{1}{2} g_{\mu\nu} \left( \frac{1}{2} g^{\lambda\sigma} \partial_\lambda s_c \partial_\sigma s_c + U_{\text{eff}}(s_c) \right) - \frac{1}{2} \partial_\mu s_c \partial_\nu s_c = 0.
\]

which will help us to solve the metric $g_{\mu\nu}$ in terms of the metric $q_{\mu\nu}$ and the inflaton field by applying a disformal transformation gr-qc/9211017 J.D. Bekenstein

\[
g_{\mu\nu} = A q_{\mu\nu} + B \partial_\mu s_c \partial_\nu s_c \quad (43)
\]
Final Einstein frame action

\[ S_E = \int d^4x \sqrt{-q} \left[ \frac{1}{2} q^{\mu\nu} R_{\mu\nu} + K(s_c) X_q - \bar{U}(s_c) + O(X_q^2) \right], \quad (44) \]

with \( K(s_c) = \frac{1}{1 + \tilde{\alpha} U_{\text{eff}}(s_c)} \), \( \bar{U}(s_c) = \frac{U_{\text{eff}}(s_c)}{1 + \tilde{\alpha} U_{\text{eff}}(s_c)} \) and \( \tilde{\alpha} = 2\beta + 8\alpha \).
Inflationary potential

\[
\frac{\bar{U}(s_c)}{\bar{U}(0)}
\]

\[\alpha = 0, \quad \alpha = 10^7, \quad \alpha = 10^8, \quad \alpha = 10^{8.267}, \quad \alpha = 10^9, \quad \alpha = 10^{10}, \quad \alpha = 10^{11}, \quad \alpha = 10^{12}\]
Inflationary observables

- For $\xi_s \ll 1$ and $\tilde{\alpha} = 0$ for both small field inflation (SFI) and large field inflation (LFI)

\[ n_s \simeq 1 - \frac{2}{N_*}, \quad r_0 \simeq \frac{8}{N_*}, \]  \hspace{1cm} \text{(quadratic)} \quad (45)

where $r_0$ denotes the tensor-to-scalar ratio for $\tilde{\alpha} = 0$.

- For $\xi_s \gg 1$ and $\tilde{\alpha} = 0$ we find for both SFI and LFI,

\[ n_s \simeq 1 - \frac{3}{N_*}, \]  \hspace{1cm} \text{quartic} \quad (46)

while

\[ r_0 \simeq \frac{16}{N_*} \]  \hspace{1cm} \text{(for LFI), \hspace{1cm} \text{quartic}} \quad r_0 \simeq 0 \quad \text{(for SFI), \hspace{1cm} (47)}

When $\tilde{\alpha} \neq 0$, the predictions for $n_s$ remain the same but $r$ gets modified as \text{(1810.05536 Enckell et al)}

\[ r = \frac{r_0}{1 + \tilde{\alpha}U_{\text{eff}}^*} = \frac{r_0}{1 + \frac{3}{2} \pi^2 \tilde{\alpha} A_s r_0}. \]  \hspace{1cm} \text{(48)}
Inflationary observables

In the Table $\xi_s \rightarrow 0$

| $\tilde{\alpha}$ | 0    | $10^7$ | $10^8$ | $1.85 \times 10^8$ | $10^9$ | $10^{10}$ | $10^{11}$ | $10^{12}$ |
|------------------|------|--------|--------|-------------------|--------|-----------|-----------|-----------|
| $r$              | 0.13090 | 0.12526 | 0.09022 | 0.07134 | 0.02368 | 0.00282 | 0.00029 | 0.00003  |
| $n_s$            | 0.96727 | 0.96726 | 0.96717 | 0.96711 | 0.96681 | 0.96621 | 0.96563 | 0.96517  |
| $N^\star$        | 60.6  | 60.6   | 60.4   | 60.3   | 59.8   | 58.8    | 58.0     | 57.3      |

For viable inflation $\xi_s \lesssim 4 \times 10^{-3} \Rightarrow v_s \gtrsim 15 M_P \Rightarrow v_s \simeq v_\phi \text{ as } v_h \sim \mathcal{O}(10^{-16}) M_P$. 
1 scalar field $+ R^2$ in **metric** leads to two-field inflation

1 scalar field $+ R^2$ (or/and $R_{\mu\nu}R^{\mu\nu}$) in **Palatini** leads to one-field inflation

The effective potential is asymptotically flat

The value of $r$ becomes smaller

The values of $A_s$ and $n_s$ are "unaffected"

The Planck scale can be dynamically generated through the VEVs of the scalar fields
Thank you!
Slow-roll approximation:

\[ V(\phi) \gg \dot{\phi}^2, \quad |\ddot{\phi}| \ll |3H\dot{\phi}|, |V'|. \] (49)

The potential slow-roll parameters:

\[ \epsilon_V = \frac{1}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \quad \eta_V = \frac{V''(\phi)}{V(\phi)}. \] (50)

→ During inflation \( \epsilon_V \ll 1 \) and \( |\eta_V| \ll 1 \).

Hubble slow-roll parameter (HSRP):

\[ \epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{3\dot{\phi}^2}{\dot{\phi}^2 + 2V}, \quad \frac{\ddot{a}}{a} = H^2(1 - \epsilon_1) \] (51)

→ Inflation ends when \( \epsilon_1 = 1 \).
We know that $N_k = \ln \frac{a_{end}}{a(t)}$. This can be written as

$$N_* = 66.89 - \ln c^*_s - \ln \left( \frac{k^*}{a_0 H_0} \right) + \frac{1}{4} \left( \ln \frac{9 H^4_*}{\rho_{end}} \right) - \frac{1}{12} \ln g^*_s(\text{reh})$$

\begin{equation}
+ \frac{1 - 3w}{3(1 + w)} \left( \ln \frac{T_{reh}}{M_{Pl}} - \frac{1}{4} \ln \frac{\rho_{end}}{M^4_{Pl}} - \frac{1}{4} \ln \frac{30}{\pi^2} + \frac{\ln g^*_s(\text{reh})}{4} \right). \tag{52}
\end{equation}

- $T_{reh}$ reheating temperature
- $H_0$ Hubble constant today
- $a_0$ scale factor constant today
- $g$, $g_s$ energy and entropy dofs
- $w$ equation of state parameter
The potential is given by
\[ V(\phi) = \lambda_n \phi^n. \] (53)

The first two PSRPs are easily computed to be
\[ \epsilon_V = \frac{n^2}{2} \frac{1}{\phi^2}, \quad \eta_V = n(n-1) \frac{1}{\phi^2}. \] (54)

In the slow-roll approximation, inflation ends when \( \epsilon_1 \simeq \epsilon_V = 1 \), so \( \phi_{\text{end}} = n/\sqrt{2} \). Then
\[ n_s = 1 - \frac{n + 2}{2N_*}, \quad r = \frac{4n}{N_*}. \] (55)

Let us now consider \( N_* = 60 \)

- \( V = \frac{1}{2} m^2 \phi^2 \Rightarrow n_s \simeq 0.966 \) and \( r \simeq 0.13 \).
- \( V = \frac{1}{4} \lambda \phi^4 \Rightarrow n_s \simeq 0.950 \) and \( r \simeq 0.26 \).
The action is

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2 R}{2} + \frac{\xi h^2 R}{2} - \frac{1}{2} g^{\mu\nu} \partial_\mu h \partial_\nu h - \frac{\lambda}{4} (h^2 - v^2)^2 \right], \quad (56) \]

After a Weyl rescaling \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \) with \( \Omega^2 = 1 + \xi h^2 / M_{\text{Pl}}^2 \) and a field redefinition we obtain

\[ S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{M_{\text{Pl}}^2 \tilde{R}}{2} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U(\chi) \right], \quad (57) \]

\[
\begin{align*}
\lambda M^4 / \xi^2 / 4 &\quad \lambda v^4 / 4 \\
\lambda M^4 / \xi^2 / 16 &\quad 0
\end{align*}
\]

- \( n_s \simeq 1 - \frac{2}{N_*} \simeq 0.966 \)
- \( r \simeq \frac{12}{N_*^2} \simeq 0.0033 \)
Negligible kinetic terms

Quadratic model
Parameters: $\xi = 0.06$, $a = 10^6$ and $a = 10^{12}$.
The 1st order CW potential in is linear in the logarithmic term, therefore it corresponds to a 1-loop effective potential. As such the validity of the approximation is ensured by the requirement

$$\beta\lambda(\phi) \approx \frac{\lambda(\phi)^2}{\pi^2} \ll \beta.$$ 

On the other hand, the 2nd order CW potential in is quadratic in the logarithmic term, therefore it corresponds to a 2-loop effective potential. As such the validity of the approximation is ensured by the requirement

$$\beta_2\lambda(\phi) \approx \frac{\lambda(\phi)^4}{\pi^4} \ll \beta'.$$
Mass matrix: \( M_{ij}^2 \equiv \left. \frac{\partial^2 U^{(0)}}{\partial \Phi_i \partial \Phi_j} \right|_{\Phi^i=v_\Phi^i, \Phi^j=v_\Phi^j} \) where \( (\Phi^1, \Phi^2) = (\phi, h) \)

Mixing angle: \( \omega \equiv \arctan \left( \frac{v_h}{v_\phi} \right) = \arctan \left( \sqrt{\frac{\lambda_{\phi h} \xi_\phi + 2 \lambda_\phi \xi_h}{\lambda_{\phi h} \xi_h + 2 \lambda_h \xi_\phi}} \right) \)

Orthogonal rotation:
\[
\begin{pmatrix}
\phi \\
h
\end{pmatrix} = \begin{pmatrix}
\cos \omega & -\sin \omega \\
\sin \omega & \cos \omega
\end{pmatrix} \begin{pmatrix}
s \\
\sigma
\end{pmatrix}
\]

Mass eigenvalues: \( m_s^2 = 0 \),
\[
m_\sigma^2 = \frac{M_P^4 (\lambda_{h \phi} \xi_h + 2 \lambda_h \xi_\phi) (2 \lambda_\phi \xi_h + \lambda_{h \phi} \xi_\phi)^2 [(\lambda_{h \phi} + 2 \lambda_\phi) \xi_h + (2 \lambda_h + \lambda_{h \phi}) \xi_\phi]}{8 v_h^2 [\lambda_\phi \xi_h^2 + \xi_\phi (\lambda_{h \phi} \xi_h + \lambda_h \xi_\phi)]^3}
\]
Along the FD $\sigma = 0$, so

$$s^2 = \phi^2 + h^2, \quad s = \frac{\phi}{\cos \omega} = \frac{h}{\sin \omega}. \quad (58)$$

and

$$\frac{1}{\Omega^2} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g^{\mu\nu} \partial_\mu h \partial_\nu h \right] = \frac{1}{\Omega^2} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu s \partial_\nu s \right],$$

where, the nonminimal coupling functional expressed in terms of $s$ has the following form:

$$\frac{1}{\Omega^2} = \frac{1}{\xi_\phi \phi^2 + \xi_h h^2} = \frac{1}{\xi_s s^2}. \quad (59)$$

In the last equation, we have defined an effective nonminimal coupling constant for the scalon as

$$\xi_s \equiv \xi_\phi \cos^2 \omega + \xi_h \sin^2 \omega. \quad (60)$$

Finally, we perform the following field redefinition in order to render the kinetic term of $s$ canonical:

$$s_c - v_c = \int_{v_s}^{s} \frac{1}{\sqrt{\xi_s}} \frac{ds'}{s'} = \frac{1}{\sqrt{\xi_s}} \ln \frac{s}{v_s}. \quad (61)$$
Minimizing (36), we can determine the scale $\Lambda$ as

$$\Lambda = v_s \exp \left[ \frac{A}{2B} + \frac{1}{4} \right].$$

(62)

Then, we can express the one-loop correction as

$$U^{(1)}(s_c) = \frac{M^4}{64\pi^2 v_s^4} s_c^4 \left[ \ln \left( \frac{s_c^2}{v_s^2} \right) - \frac{1}{2} \right].$$

(63)

From the one-loop corrections we can obtain the radiatively-generated mass for the $s$ scalar

$$m_s^2 = \frac{M^4}{8\pi^2 v_s^2}.$$

(64)
In contrast to the metric case there is now a plethora of invariants that can be constructed out of the Ricci and Riemann tensors:

\[ R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \quad \hat{R}^\mu_{\nu} = g^{\lambda\sigma} R^\mu_{\sigma\nu\lambda} \quad \text{and} \quad R'_{\mu\nu} = R^\lambda_{\lambda\mu\nu}. \]

The most general Lagrangian second order in the Riemann tensor contains 16 possible contractions and can be written as

\[
S = \int d^4x \sqrt{-g} \left[ \alpha R^2 + \beta_1 R_{\mu\nu} R^{\mu\nu} + \beta_2 R_{\mu\nu} R^{\nu\mu} + \beta_3 R_{\mu\nu} \hat{R}^{\mu\nu} + \beta_4 R_{\mu\nu} \hat{R}'^{\mu\nu} + \beta_5 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \beta_6 \hat{R}_{\mu\nu} \hat{R}'^{\mu\nu} + \beta_7 \hat{R}_{\mu\nu} R'^{\mu\nu} + \beta_8 R'_{\mu\nu} R'^{\mu\nu} + \beta_9 R_{\mu\nu} R'^{\mu\nu} + \gamma_1 R_{\mu\nu\sigma\lambda} R^{\mu\nu\sigma\lambda} + \gamma_2 R_{\mu\nu\sigma\lambda} R^{\mu\sigma\nu\lambda} + \gamma_3 R_{\mu\nu\sigma\lambda} R^{\nu\mu\sigma\lambda} + \gamma_4 R_{\mu\nu\sigma\lambda} R^{\nu\sigma\mu\lambda} + \gamma_5 R_{\mu\nu\sigma\lambda} R^{\sigma\nu\mu\lambda} + \gamma_6 R_{\mu\nu\sigma\lambda} R^{\sigma\lambda\mu\nu} \right].
\]
\( g_{\mu\nu} = A q_{\mu\nu} + B \partial_\mu s_c \partial_\nu s_c \) \hspace{1cm} (65)

- **Inverse:** \( g^{\mu\nu} = \bar{A} q^{\mu\nu} + \bar{B} q^{\mu\lambda} q^{\nu\sigma} \partial_\lambda s_c \partial_\sigma s_c \) with 
  \( \bar{A} = \frac{1}{A}, \quad \bar{B} = -\frac{B}{A^2 - 2AB X_q} \).

- **Determinant:** \( g = q A^3 (A - 2BX_q) \)

- **Kinetic \( q \):** \( X_q \equiv -\frac{1}{2} q^{\mu\nu} \partial_\mu s_c \partial_\nu s_c \)

- **Kinetic \( g \):** \( X_g = \bar{A} X_q - 2\bar{B} X_q^2 \)

Substituting all of these in \( \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = 0 \) and requiring that the coefficients of \( q_{\mu\nu}, \partial_\mu s_c \partial_\nu s_c \) must vanish identically we obtain
\[
\frac{1}{16\beta(4\alpha + \beta)R_5} \left( 4(4\alpha + \beta)A^2 - 4\beta A\sqrt{R_5} - 4\alpha AR_2 - (4\alpha + \beta)R_3 \\
+ 4\beta R_5 + \alpha R_2^2 \right) + \frac{U_{\text{eff}}(s_c)}{2} - \frac{X_g}{2} = 0,
\]

(66)

\[
\frac{1}{16\beta(4\alpha + \beta)R_5} \left( 4(4\alpha + \beta)R_4 - 4\beta R_1\sqrt{R_5} - 4\alpha R_2 R_1 - (4\alpha + \beta)BR_3 \\
+ 4\beta BR_5 + \alpha BR_2^2 \right) + \frac{BU_{\text{eff}}(s_c)}{2} - \frac{BX_g}{2} - \frac{1}{2} = 0,
\]

(67)

with \( R_1 = B(2A - 2BX_q) \), \ldots.

In slow-roll

\[
A = a_0 + a_1X_q + \mathcal{O}(X_q^2), \quad B = b_0 + b_1X_q + \mathcal{O}(X_q^2).
\]

\[
a_0 = \frac{1}{1 + \tilde{\alpha}U_{\text{eff}}}, \quad b_0 = \frac{(\tilde{\beta} - \tilde{\alpha})}{(1 + \tilde{\alpha}U_{\text{eff}})(1 + \tilde{\beta}U_{\text{eff}})},
\]

\[
a_1 = \frac{\tilde{\beta}}{2(1 + \tilde{\beta}U_{\text{eff}})}, \quad b_1 = \frac{(\tilde{\beta} - \tilde{\alpha})(3\tilde{\beta} - 2\tilde{\alpha} + (2\tilde{\beta} - \tilde{\alpha})(\tilde{\alpha} + \tilde{\beta})U_{\text{eff}} + \tilde{\alpha}\tilde{\beta}^2U_{\text{eff}}^2)}{(1 + \tilde{\alpha}U_{\text{eff}})(1 + \tilde{\beta}U_{\text{eff}})^3},
\]

where we have defined \( \tilde{\alpha} = 2\beta + 8\alpha \), \( \tilde{\beta} = 4\beta + 8\alpha \).
The functions $R_i$ which has been displayed in Eqs. (66)-(67) are listed below

\[
\begin{align*}
R_1 &= B \left( 2A - 2BX_q \right), \\
R_2 &= 4A - 2BX_q, \\
R_3 &= 4A^2 - 4ABX_q + 4B^2X_q^2, \\
R_4 &= A(R_1 + AB) - 2BR_1X_q, \\
R_5 &= A^3 \left( A - 2BX_q \right).
\end{align*}
\]
Inflation in the Palatini formalism

\( \alpha = 0 \)
\( \tilde{\alpha} = 10^7 \)
\( \tilde{\alpha} = 10^8 \)
Small field inflation

| $\tilde{\alpha}$ | $\xi_s^{(\text{min})}$ | $\mathcal{M}$ | $r$   | $n_s$ | $N_\star$ |
|------------------|------------------------|--------------|-------|-------|-----------|
| 0                | 0.0006267              | 0.0502432    | 0.0729636 | 0.968159 | 60.3      |
| $10^7$           | 0.0005830              | 0.0510926    | 0.0730490 | 0.968233 | 60.3      |
| $10^8$           | 0.0002017              | 0.0651665    | 0.0732724 | 0.968439 | 60.3      |

| $\tilde{\alpha}$ | $\xi_s^{(\text{max})}$ | $\mathcal{M}$ | $r$   | $n_s$ | $N_\star$ |
|------------------|------------------------|--------------|-------|-------|-----------|
| 0                | 0.0041417              | 0.0297085    | 0.0161109 | 0.957741 | 59.6      |
| $10^7$           | 0.0041389              | 0.0297168    | 0.0160355 | 0.957747 | 59.6      |
| $10^8$           | 0.0041367              | 0.0297308    | 0.0152745 | 0.957739 | 59.6      |

**Table:** For $\tilde{\alpha} \lesssim 10^{8.267} \simeq 1.85 \times 10^8$, only small field inflation yields viable values for $r$ and $n_s$. Here, we give the minimum and maximum values of $\xi_s$ for which we obtain viable predictions for various $\tilde{\alpha}$. We also give the values of $\mathcal{M}$, $r$, $n_s$ and $N_\star$ for these marginal values of $\xi_s$. 
Table: For various $\tilde{\alpha} \gtrsim 10^{8.267} \approx 1.85 \times 10^8$, and for both small and large field inflation, we give the corresponding maximum values of $\xi_s$ that yield predictions that comply with the observational bounds.