EMBEDDED MINIMAL AND CONSTANT MEAN CURVATURE 
ANNULUS TOUCHING SPHERES

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Abstract. We show that a compact embedded constant mean curvature annulus in \( \mathbb{R}^3 \) tangent to two spheres of same radius along its boundary curves and having non-vanishing Gaussian curvature is part of a Delaunay surface. In special, if the annulus is minimal, then the annulus is part of a catenoid. Secondly we show that a compact embedded constant mean curvature annulus with negative (respectively, positive) Gaussian curvature meeting a sphere tangentially and a plane in constant contact angle \( \geq \pi/2 \) (respectively, \( \leq \pi/2 \)) is part a Delaunay surface. In special, if the annulus is minimal and the contact angle is \( \geq \pi/2 \), then it is part of a catenoid.

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Catenoid is the only nonplanar minimal surface of rotation in \( \mathbb{R}^3 \) [3]. Rotational surfaces of constant mean curvature in \( \mathbb{R}^3 \) are called the Delaunay surfaces: cylinders, spheres, unduloids and nodoids. Therefore catenoid and Delaunay surface meet every plane, which is perpendicular to the axis of rotation, in constant contact angle. Conversely, if a compact embedded minimal or constant mean curvature surface meets two parallel planes in constant contact angles, then the surface is part of a catenoid or part of a Delaunay surface. This can be proved by using the Alexandrov’s moving plane argument [12] to planes perpendicular to the parallel planes. A compact immersed minimal annulus meeting two parallel planes in constant contact angles is also part of a catenoid. This result is not true for constant mean curvature surfaces: Wente had constructed examples of immersed constant mean curvature annuli in a slab or in a ball meeting the boundary planes or the boundary sphere perpendicularly [10]. Compared to the above first case, we may ask whether a compact minimal annulus or a compact embedded constant mean curvature annulus meeting two spheres in constant contact angles is part of a catenoid or part of a plane. In [9], it is shown that if a compact embedded minimal annulus meets two concentric spheres perpendicularly then the minimal annulus is part of a plane.

In this paper, we show that a compact embedded constant mean curvature annulus \( \mathcal{A} \) in \( \mathbb{R}^3 \) meeting two spheres \( S_1 \) and \( S_2 \) of same radius \( \rho \) tangentially and having non-vanishing Gaussian curvature \( K \) is part of a Delaunay surface. More precisely,
depending on the values of \( K \) and the mean curvature \( H \) we have three cases: i) 
\( K < 0 \) and \( H > -1/\rho \), in which case \( \mathcal{A} \) is part of a unduloid if \( H < 0 \), part of a catenoid if \( H = 0 \) and part of a nodoid if \( H > 0 \), ii) \( K > 0 \) and \( -1/\rho < H < -1/2\rho \), in which case \( \mathcal{A} \) is part of a unduloid, and iii) \( K > 0 \) and \( H < -1/\rho \), in which case \( \mathcal{A} \) is part of a nodoid. In the first two cases, \( \mathcal{A} \) stays outside of the balls \( B_1 \) and \( B_2 \) bounded by \( S_1 \) and \( S_2 \). If iii) holds, then \( \mathcal{A} \subset B_1 \cap B_2 \).

We also show that a compact embedded constant mean curvature annulus \( \mathcal{B} \) in \( \mathbb{R}^3 \) with negative (respectively, positive) Gaussian curvature meeting a unit sphere tangentially and a plane in constant contact angle \( \geq \pi/2 \) (respectively, \( \leq \pi/2 \)) is part of a Delaunay surface. In special, a compact embedded minimal annulus in \( \mathbb{R}^3 \) meeting a sphere tangentially and a plane in constant contact angle \( \geq \pi/2 \) is part of a catenoid.

To prove Theorem 1 and 2, we use the \(-\rho\)-parallel surface \( \tilde{\mathcal{A}} \) of \( \mathcal{A} \) (respectively, \( \tilde{\mathcal{B}} \) of \( \mathcal{B} \)): the parallel surface of \( \mathcal{A} \) (respectively, of \( \mathcal{B} \)) with distance \( \rho \) in the direction to the centers of the spheres. We use the Alexandrov’s moving plane argument [2], [6] to prove that \( \mathcal{A} \) and \( \tilde{\mathcal{B}} \) are rotational. Since \( \mathcal{A} \) and \( \tilde{\mathcal{B}} \) are the parallel surfaces of \( \mathcal{A} \) and \( \mathcal{B} \) respectively, \( \mathcal{A} \) and \( \mathcal{B} \) are also rotational and, hence, are part of a Delaunay surface or part of a catenoid.

1. CONSTANT MEAN CURVATURE ANNULUS MEETING SPHERES TANGENTIALLY

In the following, we may assume that the spheres have radius 1. Let \( \mathcal{A} \) be a compact embedded annulus with constant mean curvature \( H \) and meeting two unit spheres \( S_1 \) and \( S_2 \) tangentially along the boundary curves \( \gamma_1 \) and \( \gamma_2 \). We fix the unit normal \( N \) of \( \mathcal{A} \) to point away from the centers of the spheres. Let \( Y: A(1, R) \to \mathbb{R}^3 \) be a conformal parametrization of \( \mathcal{A} \) from an annulus \( A(1, R) = \{(x, y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2 + y^2} \leq R\} \). We define \( X \) by \( X = Y \circ \exp \) on the strip \( \mathcal{B} = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \log R\} \). Then \( X \) is periodic with period \( 2\pi \). Let \( z = u + iv \) and \( \lambda^2 := |X_u|^2 = |X_v|^2 \).

Let \( h_{ij} \), \( i, j = 1, 2 \), be the coefficients of the second fundamental form of \( X \) with respect to \( N \). Note that the Hopf differential \( \phi(z) dz^2 = (h_{11} - h_{22} - 2ih_{12}) dz^2 \) is holomorphic for constant mean curvature surfaces [6]. The theorem of Joachimstahl [4] says that \( \gamma_1 \) and \( \gamma_2 \) are curvature lines of \( \mathcal{A} \). Hence \( h_{12} \equiv 0 \) on \( u = 0 \) and \( u = \log R \). Since \( h_{12} \) is harmonic and periodic, we have \( h_{12} \equiv 0 \) on \( \mathcal{B} \). This implies that \( z \) is a conformal curvature coordinate and \( h_{11} - h_{22} = \text{constant} \) [7]. Let \( c = h_{11} - h_{22} \). If \( \mathcal{A} \) is minimal, then we have \( K < 0 \) and \( c = 2h_{11} > 0 \) by the choice of \( N \). When \( H = -1 \), \( \mathcal{A} \) is part of the unit sphere \( S_1 = S_2 \) by the boundary comparison principle for mean curvature operator [5]. We assume that \( H \neq -1 \) in the following. The principal curvatures of \( \mathcal{A} \) are

\[
\kappa_1 = H + \frac{c}{2\lambda^2} \quad \text{and} \quad \kappa_2 = H - \frac{c}{2\lambda^2}.
\]
We parameterize $\gamma_1$ and $\gamma_2$ by $\gamma_1(v) = X(0, v)$ and $\gamma_2(v) = X(\log R, v)$ for $v \in [0, 2\pi)$. In the following, we assume that $A$ has non-vanishing Gaussian curvature.

**Lemma 1.** Each $\gamma_i(v), i = 1, 2$, has constant speed $\sqrt{c/(2(1+H))}$ and $\kappa_2$ is $-1$ on $\gamma_1$ and $\gamma_2$. As spherical curves, $\gamma_1$ and $\gamma_2$ are convex. On $A \setminus \partial A$, we have $\lambda^2 < c/(2(1+H))$ when $K < 0$ and $\lambda^2 > c/(2(1+H))$ when $K > 0$.

**Proof.** The curvature vector of $\gamma_1(v)$ is

$$\vec{\kappa} = \frac{1}{|X_v|} \frac{d}{dv} \left( \frac{X_v}{|X_v|} \right) = \frac{1}{|X_v|^2} X_{vv} - \frac{X_v}{|X_v|^4} \left( X_v \cdot X_{vv} \right)$$

$$= \frac{1}{\lambda^2} \left( -\frac{\lambda u}{\lambda} X_u + h_{22} N \right).$$

Let the center of $S_1$ be the origin of $\mathbb{R}^3$. Since $A$ is tangential to $S_1$ along $\gamma_1$, we have $N(0, v) = X(0, v) = \gamma_1(v)$ on $\gamma_1$. Since $\gamma_1$ is on the unit sphere $S_1$, the curvature vector $\vec{\kappa}$ of $\gamma_1$ satisfies $(\vec{\kappa} \cdot \gamma_1)(v) = -1$. Hence we have $\kappa_2 = \frac{h_{22}}{\lambda^2} = -1$ on $\gamma_1$. Since $\lambda^2 = |\gamma_1|^2$ on $\gamma_1$, we have $|\gamma_1| = \sqrt{c/(2(1+H))}$ from (1). By choosing the center of $S_2$ as the origin of $\mathbb{R}^3$, we get the results for $\gamma_2$.

The Gaussian curvature $K$ satisfies

$$\Delta \log \lambda = -K \lambda^2,$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$. We can rewrite this equation as

$$\lambda \Delta \lambda = |\nabla \lambda|^2 - K \lambda^4.$$

Since $\lambda_u(0, v) = 0$ and $\lambda_u(\log R, v) = 0$ and $K \neq 0$, $\lambda$ does not have interior maximum when $K < 0$, and does not have interior minimum when $K > 0$. Since $\lambda^2 = c/(2(1+H))$ on $\gamma_1$ and $\gamma_2$, it follows that $\lambda^2 < c/(2(1+H))$ on $A \setminus \partial A$ when $K < 0$ and $\lambda^2 > c/(2(1+H))$ when $K > 0$. Moreover we have $\lambda_u \leq 0$ on $u = 0$ and $\lambda_u \geq 0$ on $u = \log R$ when $K < 0$ and $\lambda_u \geq 0$ on $u = 0$ and $\lambda_u \leq 0$ on $u = \log R$ when $K > 0$. Since $\frac{X}{|X|} \in TS_i$ is perpendicular to $\gamma_i$, the geodesic curvature of $\gamma_i$ as a spherical curve is $\vec{\kappa} \cdot \frac{X}{|X|} = -\frac{\lambda}{\lambda^2}$. Hence $\gamma_1$ and $\gamma_2$ are convex as spherical curves. $\square$

**Remark 1.** If $\lambda^2 \equiv c/(2(1+H))$ on $A$, then $K \equiv 0$ and $A$ is part of a cylinder.

2. $-1$-parallel surface

The $-1$-parallel surface $\bar{A}$ of $A$ is defined by

$$\bar{X} = X - N.$$

The image of $\gamma_1$ (respectively, of $\gamma_2$) in $\bar{A}$ is a point corresponding to the center of $S_1$ (respectively, of $S_2$). We denote the centers of $S_1$ and $S_2$ by $O$ and $O_2$ for
simplicity. We fix the unit normal $N$ of $\hat{A}$ to be $N$. Since $z = u + iv$ is a curvature coordinate of $X$, we have

\[
\hat{X}_u = \left(1 + \frac{h_{11}}{\lambda^2}\right) X_u \quad \text{and} \quad \hat{X}_v = \left(1 + \frac{h_{22}}{\lambda^2}\right) X_v.
\]

Since $\kappa_2 = -1$ on $\gamma_i$ (Lemma 1), $\hat{X}$ is singular for $u = 0$ and $u = \log R$. By Lemma 1, we have $\lambda^2 \neq c/2(1 + H)$ on $A \setminus \partial A$, which implies that $1 + \kappa_2 \neq 0$ on $A \setminus \partial A$. When $K < 0$, we have $\kappa_1 > 0$ on $A \setminus \partial A$. Hence $\hat{X}$ is regular for $0 < u < \log R$ and we have $H > -1$.

Now suppose that $K > 0$. Since $\kappa_2 = -1$ on $\gamma_i$ (Lemma 1), we have $\kappa_1 < 0$ and $H < -1/2$. We consider two cases separately: $H < -1$ and $-1 < H < -1/2$. If $H < -1$, then $c < 0$ from $\lambda^2 = c/2(1 + H) > 0$ on $\gamma_i$. Hence we have $\kappa_1 < -1$, which implies that $\hat{X}$ is regular for $0 < u < \log R$. If $-1 < H < -1/2$, then we must have $c > 0$. This implies that $1 + \kappa_1 \neq 0$. Otherwise we have $0 < 2\lambda^2(1 + H) = -c$, which contradicts $c > 0$. Hence $\hat{X}$ is regular for $0 < u < \log R$.

**Remark 2.** When $K < 0$ or $K > 0$ and $-1 < H < -1/2$, $A$ stays outside of the balls $B_1$ and $B_2$ bounded by $S_1$ and $S_2$. If $K > 0$ and $H < -1$, then $A \subset B_1 \cap B_2$.

**Lemma 2.** The mean curvature $\hat{H}$ and the Gaussian curvature $\hat{K}$ of $\hat{A}$ satisfies

\[
(1 + H)\hat{K} = (1 + 2H)\hat{H} - H. \quad \text{On} \quad A \setminus \{O, O_2\}, \quad \text{we have}
\]

i) if $K < 0$ and $H > -1$, then $\hat{k}_1 > 0$, $\hat{k}_2 > 1$ and $\hat{H} > 1$,

ii) if $K > 0$ and $-1 < H < -1/2$, then $0 < c/2\lambda^2(1 + H) < \min\{1, -H/(1 + H)\}$, $\hat{k}_1 < 0$, $\hat{k}_2 < H/(1 + H)$ and $\hat{H} < H/(1 + H)$, and

iii) if $K > 0$ and $H < -1$, then $0 < c/2\lambda^2(1 + H) < 1$, $\hat{k}_1 > (1 + 2H)/(1 + H)$, $\hat{k}_2 > H/(1 + H)$ and $\hat{H} > H/(1 + H)$.

**Proof.** Since

\[
\hat{h}_{12} = N \cdot \hat{X}_{uv} = \left(1 + \frac{h_{11}}{\lambda^2}\right) (N \cdot X_{uv}) = 0,
\]

$(u, v)$ is a curvature coordinate (not conformal) for $\hat{A}$ except for $O$ and $O_2$. We have

\[
\hat{h}_{11} = N \cdot \hat{X}_{uu} = \left(1 + \frac{h_{11}}{\lambda^2}\right) h_{11},
\]

\[
\hat{h}_{22} = N \cdot \hat{X}_{vv} = \left(1 + \frac{h_{22}}{\lambda^2}\right) h_{22}.
\]

The principal curvatures of $\hat{A}$ are

\[
\hat{k}_1 = \frac{\kappa_1}{1 + \kappa_1} = \frac{H/(1 + H) + (c/2\lambda^2(1 + H))}{1 + (c/2\lambda^2(1 + H))},
\]

\[
\hat{k}_2 = \frac{\kappa_2}{1 + \kappa_2} = \frac{H/(1 + H) - (c/2\lambda^2(1 + H))}{1 - (c/2\lambda^2(1 + H))}.
\]
From $\kappa_1 + \kappa_2 = 2H$, we have

$$H = \frac{\hat{H} - \hat{K}}{1 - 2\hat{H} - \hat{K}} \quad \text{or} \quad (1 + H)\hat{K} = (1 + 2H)\hat{H} - H.$$  

It is straightforward to see that

$$\hat{H} = \frac{H/(1 + H) - (c/2\lambda^2(1 + H))^2}{1 - (c/2\lambda^2(1 + H))^2}.$$  

Note that $\kappa_2 < 0$ on $\mathcal{A}$. First suppose that $K < 0$. Then we have $\kappa_1 > 0$, which implies that $\tilde{\kappa}_1 = \kappa_1/(1 + \kappa_1) > 0$. Since $c/2\lambda^2(1 + H) > 1$ by Lemma 1, we have $\tilde{\kappa}_2 > 1$ and $\hat{H} > 1$.

When $K > 0$, we have $\kappa_1 = H + c/2\lambda^2 < 0$. If $-1 < H < -1/2$, then we have $c > 0$ because $\lambda^2 = c/2(1 + H) > 0$ on $\gamma_i$. It follows that $c/2\lambda^2(1 + H) < -H/(1 + H)$. By Lemma 1, we also have $c/2\lambda^2(1 + H) < 1$. Therefore we have $0 < c/2\lambda^2(1 + H) < \min\{1, -H/(1 + H)\}$. It is straightforward to see that $\tilde{\kappa}_1 < 0$ and $\tilde{\kappa}_2 < H/(1 + H) < 0$ and $\hat{H} < H/(1 + H) < 0$.

When $K > 0$ and $H < -1$, we have $c < 0$ and $0 < c/2\lambda^2(1 + H) < 1$. It is straightforward to see that $\tilde{\kappa}_1 > (1 + 2H)/(1 + H)$, $\tilde{\kappa}_2 > H/(1 + H)$ and $\hat{H} > H/(1 + H)$. □

This lemma says that $\tilde{A}$ is a linear Weingarten surface with two singular points $O$ and $O_2$ and is positively curved outside $O$ and $O_2$.

**Lemma 3.** $\tilde{A}$ is embedded.

Proof. Let $\nu(v) = \frac{X_v}{|X_v|}$. Note that $\nu$ is a closed curve in the unit sphere $S_1$. We claim that $\nu$ is convex as a spherical curve. Otherwise, there is a great circle $\eta$ intersecting the image of $\nu$ at no less than 3 points $\nu(v_1), \ldots, \nu(v_n)$. ($\nu$ may map an interval $(v_0, v_1) \subset [0, 2\pi)$ into a single point. We choose $v_i$’s in such a way that $\nu$ maps no two $v_i$’s to the same point.) Each $\nu(v_i)$ determines a great circle $S^1_{v_i} \subset S_1$ contained in the plane perpendicular to $\nu(v_i)$. At each $\gamma_1(v_i)$, $\gamma_1$ is tangent to $S^1_{v_i}$. Since $\eta$ and $S^1_{v_i}$ are perpendicular, $\gamma_1$ cannot be convex when $n \geq 3$. Hence $\nu$ intersect every geodesic of $S_1$ at no more than two points. This shows that $\nu$ is convex as a spherical curve. Similarly, $\frac{X_v}{|X_v|}(\log R, v)$ is also convex as a spherical curve.

Since $\tilde{A}$ is a parallel surface of $\mathcal{A}$, the tangent cone $Tan(O, \tilde{A})$ of $\tilde{A}$ at $O$ is the cone formed by rays from $O$ through $\nu$. Since $\nu$ is a convex spherical curve, $Tan(O, \tilde{A})$ is convex. This shows that a small neighborhood of $O$ in $\tilde{A}$ is embedded and nonnegatively curved as a metric space [11]. Similarly, there is a neighborhood of $O_2$ in $\tilde{A}$ which is embedded and nonnegatively curved as a metric space.

Hadamard showed that a closed surface in $\mathbb{R}^3$ with strictly positive Gaussian curvature is the boundary of a convex body [6]. In particular, $S$ is embedded.
Alexandrov generalized Hadamard’s theorem to nonnegatively curved metric spaces [1]. Since $\tilde{A}$ is a nonnegatively curved closed metric space, $\tilde{A}$ is embedded.

Remark 2. We have $\nu_{\tilde{\nu}} = \frac{\lambda_u}{\lambda} X_u$. At points where $\lambda_u \neq 0$, the curvature vector of $\nu$ is

$$\vec{\kappa}_{\nu} = \frac{1}{\lambda_u} \left( -\frac{\lambda_v}{\lambda} X_u + h_{22} N \right).$$

The geodesic curvature of $\nu$ as a spherical curve $\vec{\kappa}_{\nu} \cdot N = \frac{h_{22}}{\lambda_u}$.

3. Main results

We use the Alexandrov’s moving plane argument [2], [6] to prove the theorems.

Theorem 1. A compact embedded constant mean curvature annulus $A$ with non-vanishing Gaussian curvature and meeting two spheres $S_1$ and $S_2$ of same radius tangentially is part of a Delaunay surface. In special, if $A$ is minimal, then $A$ is part of a catenoid.

Proof. We suppose that the radius of $S_1$ and $S_2$ is 1. By Lemma 2 and Lemma 3, $\tilde{A}$ is a compact embedded surface with two singular points $O$ and $O_2$ and satisfying $(1 + H) \tilde{K} = (1 + 2H) \tilde{H} - H$ at regular points. A small neighborhood of a regular point of $\tilde{A}$ can be represented as the graph of a function $f(x, y)$ satisfying

$$2(1 + H)(f_{xx} f_{yy} - f_{xy}^2) + 2H(1 + f_x^2 + f_y^2)^2 = (1 + 2H) \left((1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}\right) (1 + f_x^2 + f_y^2)^2.$$  \hspace{1cm} (5)

This equation can be rewritten as

$$\det \left(2(1 + H) D^2 f + A(Df)\right) = W^4,$$  \hspace{1cm} (6)

where $A(Df) = -(1 + 2H) \begin{pmatrix} (1 + f_y^2) W & f_x f_y W \\ f_x f_y W & (1 + f_y^2) W \end{pmatrix}$ and $W = \sqrt{1 + f_x^2 + f_y^2}$. The equation (6) is elliptic with respect to $f$ if $2(1 + H) D^2 f + A(Df)$ is positive definite. Since $\det \left(2(1 + H) D^2 f + A(Df)\right) = W^4 > 0$, (6) is elliptic if

$$\Tr \left(2(1 + H) D^2 f + A(Df)\right) = 2(1 + H) \Delta f - (1 + 2H)(2 + f_x^2 + f_y^2) W$$  \hspace{1cm} (7)

is strictly positive.

First we consider the case $K < 0$. Since $\tilde{H} > 1$ by Lemma 2, we have

$$\Delta f + f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} > 2W^{3/2},$$  \hspace{1cm} (8)

for $f$ representing $\tilde{A}$. We may assume that $f$ is defined on $B(0, \epsilon) \subset T_p \tilde{A}$ so that $\nabla f(0) = 0$ and $D^2 f$ is diagonal. For sufficiently small $\epsilon = \epsilon(p)$, (8) implies that (7) is strictly positive. Hence (6) is elliptic with respect to $f$ representing $\tilde{A}$.

When $-1 < H < -1/2$, (7) is automatically satisfied.
Now we consider the case $K > 0$ and $H < -1$. Since $\tilde{H} > H/(1 + H)$ by Lemma 2, we have

$$\Delta f + f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} > \frac{2H}{1 + H} W^{3/2}. \quad (9)$$

Assuming that $f$ is defined on $B(0, \epsilon) \subset T_p \tilde{A}$ with $\nabla f(0) = \vec{0}$ and $D^2 f$ is diagonal, (9) implies that $\Delta f - \frac{1 + 2H}{2(1 + H)(2 + f_x^2 + f_y^2)} W$ is strictly positive for sufficiently small $\epsilon$. Then $\det \left( -2(1 + H)D^2 f - A(Df) \right) = W^4$ is elliptic for $f$ representing $\tilde{A}$. The ellipticity of (6) for $f$ representing $\tilde{A}$ enables us to use the maximum principle and the boundary point lemma [5].

Since $\tilde{A}$ is convex and embedded, we can use Alexandrov’s moving plane argument [2], [6] to show that $\tilde{A}$ is rotational as follows. Let $\Pi_\theta$ be the plane containing the line segment $OO_2 \subset \mathbb{R}^3$ and making angle $\theta$ with a fixed vector $\vec{E}$ which is perpendicular to $OO_2$. Fix a positive constant $L$ such that each plane $\Pi_{\theta}^L$, which is parallel to $\Pi_\theta$ with distance $L$ from $\Pi_\theta$, does not meet $\tilde{A}$ for all $\theta$. Let $\Pi_{\theta}^l$ be the plane between $\Pi_{\theta}^L$ and $\Pi_\theta$ with distance $l$ from $\Pi_\theta$. When $\Pi_{\theta}^l$ intersects $\tilde{A}$, we reflect the $\Pi_{\theta}^L$ side part of $\tilde{A}$ about $\Pi_{\theta}^l$. Let us denote this reflected surface $\tilde{A}_{\theta, l}^{ref}$. As we decrease $l$ from $L$, there might be the first $l_\theta \geq 0$ for which $\tilde{A}_{\theta, l}^{ref}$ is tangent to $\tilde{A}$ at an interior point or at a boundary point of $\partial \tilde{A}_{\theta, l}^{ref}$. We call this point as the first touch point. If there is no nonnegative $l$ with the first touch point, we repeat the process for $\Pi_{\theta}^{L+\pi}$ to find $l_{\theta+\pi}$ which must be positive. At the first touch point, we apply the comparison principles for [5] to see that the part of $\tilde{A}$ in the $\Pi_\theta$ side and $\tilde{A}_{\theta, l}^{ref}$ are identical and, hence, $l_\theta = 0$. This implies that $\Pi_\theta$ is a symmetry plane for $\tilde{A}$. Since $\theta$ can be chosen arbitrarily, $\tilde{A}$ should be rotational and, hence, $\tilde{A}$ is also rotational. Since the Delaunay surfaces and the catenoid are the only nonplanar rotational minimal and constant mean curvature surfaces, $\tilde{A}$ is part of a Delaunay surface or part of a catenoid. □

We used the embeddedness of $A$ in proving that $\tilde{A}$ is embedded. Whether there is a non-embedded minimal or constant mean curvature annulus meeting two unit spheres tangentially is an interesting question. Moreover we raise the following questions.

1. Is a compact immersed minimal annulus or a compact embedded minimal or constant mean curvature surface meeting a sphere perpendicularly or in constant contact angles part of a catenoid or part of a Delaunay surface? Nitsche showed that an immersed disk type minimal or constant mean curvature surface meeting a sphere in constant contact angle is either a flat disk or a spherical cap [8].

2. Is a compact immersed minimal annulus or a compact embedded minimal or constant mean curvature surface meeting two spheres in constant contact angles part of a catenoid or a plane or part of a Delaunay surface?
3. Is a compact immersed minimal or constant mean curvature annulus or a compact embedded minimal or constant mean curvature surface meeting a sphere and a plane in constant contact angles part of a catenoid or part of a Delaunay surface? We give an affirmative answer to this problem in a special case in the following.

**Theorem 2.** A compact embedded constant mean curvature annulus $B$ with negative (respectively, positive) Gaussian curvature meeting a sphere tangentially and a plane in constant contact angle $\geq \pi/2$ (respectively, $\leq \pi/2$) is part of a Delaunay surface. In special, if $B$ is minimal and the constant contact angle is $\geq \pi/2$ then $B$ is part of a catenoid.

The angle is measured between the outward conormal of $B$ and the outward conormal of the bounded domain in $\Pi$ bounded by the boundary curve. Since the proof of this theorem is similar to that of Theorem 1, we omit some details which was previously proved.

Proof. Let us denote the sphere by $S_2$ and the plane by $\Pi$. We may assume that the radius of $S_2$ is 1. Let $\alpha$ be the constant contact angle between $B$ and $\Pi$. If $\alpha = \pi/2$, then we can reflect $B$ about $\Pi$ to get a constant mean curvature annulus meeting two unit spheres tangentially. Hence $B$ is part of a catenoid or a Delaunay surface by Theorem 1.

In the following, we assume that $\alpha \neq \pi/2$. As in the case for $S$ in §1, there is a conformal parametrization $X$ of $B$ from a strip $\{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq \log R\}$ for which $z = u + iv$ is a curvature coordinate. We fix the normal $N$ of $B$ to point away from the center of $S_2$. Let $c_1(v) = X(0, v)$ be on $\Pi$ and $c_2(v) = X(\log R, v)$ be on $S_2$ with $\partial X_3/\partial u > 0$ along $c_1$. As in Lemma 1, $c_2$ has constant speed $\sqrt{c/2(1 + H)}$ and $\kappa_2 = -1$ along $c_2$. Since $K \neq 0$ on $B$ and $z = u + iv$ is a curvature coordinate, we have $\kappa_2 < 0$ on $c_1$. The curvature of $c_1$ is $|\vec{\kappa}| = -\kappa_2/\sin \alpha > 0$, which shows that $c_1$ is locally convex. Since $c_1$ is a Jordan curve, it is convex.

First, we assume that $K < 0$ and $\alpha > \pi/2$. Since $\frac{X}{|X|} \cdot \frac{X_3}{|X_3|} = \cos \alpha < 0$ on $c_1$, it follows from (2) that $\lambda_u > 0$ on $c_1$. Since $\lambda_v(\log R, v) = 0$ (cf. Lemma 1), it follows from (3) that $\lambda_u \geq 0$ on $c_2$. Otherwise, $\lambda$ will have an interior maximum, which contradicts (4). Hence we have $\lambda^2 < c/2(1 + H)$ on $B \setminus c_2$. Note that $\kappa_1 > 0$ and $\kappa_2 < 0$ in $B$. From $\lambda_u \leq 0$ on $c_2$, we see that $c_2$ is convex as a spherical curve (cf. Lemma 1). Arguing as in the proof of Lemma 3, we see that $\frac{X}{|X|}(\log R, v)$ is also convex as a spherical curve.

When $K > 0$ and $\alpha < \pi/2$, we have $\frac{X}{|X|} \cdot \frac{X_3}{|X_3|} = \cos \alpha > 0$ on $c_1$. Hence $\lambda_u < 0$ on $c_1$. Since $\lambda_u(\log R, v) = 0$, it follows from (3) that $\lambda$ does not have interior minimum. Then we have $\lambda_u \leq 0$ on $c_2$ and $\lambda^2 > c/2(1 + H)$ on $B \setminus c_2$. Note that $\kappa_1 < 0$ and $\kappa_2 < 0$ in $B$. From $\lambda_u \leq 0$ on $c_2$, it follows that $c_2$ is convex as a
spherical curve. Moreover $\frac{X_u}{|X_u|}(\log R, v)$ is convex as a spherical curve (cf. Lemma 3).

Let $\tilde{B}$ be the $-1$-parallel surface of $B$. As in §2, we can show that $\tilde{B}$ is regular except for $O_2$: the image of $c_2$, and $H > -1$ when $K < 0$ and $H < -1/2$ when $K > 0$. As in Lemma 2, we see that mean curvature $\tilde{K} > \alpha$. A convex Jordan curve.

Let $\tilde{\Pi}$ be the plane parallel to $\Pi$ and containing $\tilde{K}$ of $\tilde{B}$ satisfies $(1 + H)\tilde{K} = (1 + 2H)\tilde{H} - H$ and $i)$ if $K < 0$ and $H > -1$, then $\tilde{\kappa}_1 > 0$, $\tilde{\kappa}_2 > 1$ and $\tilde{H} > 1$, $ii)$ if $K > 0$ and $-1 < H < -1/2$, then $0 < c/2\lambda^2(1 + H) < \min\{1, -H/(1 + H)\}$, $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_2 < H/(1 + H)$ and $\tilde{H} < H/(1 + H)$, and $iii)$ if $K > 0$ and $H < -1$, then $0 < c/2\lambda^2(1 + H) < 1$, $\tilde{\kappa}_1 > (1 + 2H)/2(1 + H)$, $\tilde{\kappa}_2 > H/(1 + H)$ and $\tilde{H} > H/(1 + H)$.

The convexity of $\frac{X_u}{|X_u|}(\log R, v)$ as a spherical curve implies that there is a neighborhood of $O_2$ in $\tilde{B}$ which is embedded and nonnegatively curved as a metric space. Let $\tilde{\Pi}$ be the plane parallel to $\Pi$ and containing $\tilde{c}_1$. The curvature of $\tilde{c}_1$ is $|\tilde{\kappa}_2|/\sin \alpha$, which does not vanish. Hence $\tilde{c}_1$ is locally convex. Using the orthogonal projection onto $\tilde{\Pi}$, $\tilde{c}_1$ may be considered as a sin $\alpha$-parallel curve of $c_1$ in $\Pi$. Hence $\tilde{c}_1$ is also a convex Jordan curve.

Suppose that $K < 0$ and $\alpha > \pi/2$. Since $\kappa_1 > 0$, $\tilde{X}_u$ is a positive multiple of $X_u$ by (4). The positivity of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ implies that $\tilde{B}$ meets $\tilde{\Pi}$ in constant angle $\pi - \alpha$. Suppose that $K > 0$ and $\alpha < \pi/2$. If $-1 < H < -1/2$, then we have $c > 0$ and $\kappa_1 > -1$. Hence $\tilde{X}_u$ is a positive multiple of $X_u$ by (4). The negativity of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ implies that $\tilde{B}$ meets $\tilde{\Pi}$ in constant angle $\alpha$. When $K > 0$ and $H < -1$, we have $c < 0$ and $\kappa_1 < -1$. Hence $\tilde{X}_u$ is negative multiple of $X_u$ by (4). In this case, $\tilde{B}$ lies below $\tilde{\Pi}$ and $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are both positive. It is straightforward to see that $\tilde{B}$ meets $\tilde{\Pi}$ in constant angle $\alpha$.

Let $\tilde{B}$ be the singular surface obtained from $\tilde{B}$ by attaching the disk in $\tilde{\Pi}$ bounded by $\tilde{c}_1$ to $\tilde{B}$. Since $\tilde{B}$ meets $\tilde{\Pi}$ in acute angle, $\tilde{B}$ is a nonnegatively curved metric space. By Alexandrov’s generalization of Hadamard’s theorem (4), $\tilde{B}$ is the boundary of a convex body. Therefore $\tilde{B}$ is embedded. Note again that $\tilde{H}$, $\tilde{K}$, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ satisfy the statements of Lemma 2. Hence (5) is elliptic for functions representing $\tilde{B}$ locally. We can apply Alexandrov’s moving plane argument to $\tilde{B}$ using planes perpendicular to $\tilde{\Pi}$ as in the proof of Theorem 1 to see that $\tilde{B}$ is rotational. Hence $B$ is rotational and, as a result, is part of a Delaunay surface or part of a catenoid.

\[ \Box \]

References

[1] A. D. Alexandrov, Intrinsic Geometry of Convex Surfaces (in Russian). German translation: Die innere Geometrie der konvexen Flächen, Akad. Verl., Berlin, 1955.

[2] A. D. Alexandrov, Uniqueness theorems for surfaces in the large V, Amer. Math. Soc. Transl. 21 (1962), 412-416.

[3] O. Bonnet, Mémoire sur l’emploi d’un nouveau système de variables dans l’étude des surfaces courbes, J. Mathém. p. appl. (2) (1860), 153-266.
[4] M. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, New Jersey, (1976).
[5] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Classics in mathematics, Springer-Verlag, (2001).
[6] H. Hopf, *Differential Geometry in the large*, Springer, Berlin, (1989).
[7] J. McCuan, *Symmetry via spherical reflection and spanning drops in a wedge*, Pacific J. Math. 180 (1997), no. 2, 291–323.
[8] J.C.C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rat. Mech. Anal. 89 (1985), 1-19.
[9] S. Park and J. Pyo, *Minimal annuli meeting spheres perpendicularly*, in preparation.
[10] H. Wente, *Tubular capillary surfaces in a convex body*. Advances in geometric analysis and continuum mechanics (Stanford, CA, 1993), 288-298, International Press, Cambridge, MA, (1995).