Dg Manifolds, Formal Exponential Maps and Homotopy Lie Algebras

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Abstract: This paper is devoted to the study of the relation between ‘formal exponential maps,’ the Atiyah class, and Kapranov $L_∞[1]$ algebras associated with dg manifolds in the $C^∞$ context. We prove that, for a dg manifold, a ‘formal exponential map’ exists if and only if the Atiyah class vanishes. Inspired by Kapranov’s construction of a homotopy Lie algebra associated with the holomorphic tangent bundle of a complex manifold, we prove that the space of vector fields on a dg manifold admits an $L_∞[1]$ algebra structure, unique up to isomorphism, whose unary bracket is the Lie derivative with respect to the homological vector field, whose binary bracket is a 1-cocycle representative of the Atiyah class, and whose higher multibrackets can be computed by a recursive formula. For the dg manifold $(T^0,1\mathcal{X},\bar{\partial})$ arising from a complex manifold $\mathcal{X}$, we prove that this $L_∞[1]$ algebra structure is quasi-isomorphic to the standard $L_∞[1]$ algebra structure on the Dolbeault complex $\Omega^0\cdot(\mathcal{X})$.

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1. Introduction

This paper, which is a sequel to [38], is devoted to the study of some differential geometric aspects of dg manifolds in the \( C^\infty \) context. Dg manifolds (a.k.a. \( Q \)-manifolds [1,39,46]) have increasingly attracted attention recently due to their relevance in various fields of mathematics, in particular, mathematical physics. They first appeared in the mathematical physics literature in the study of BRST operators used to describe gauge symmetries. They play an essential role in the so called AKSZ formalism in the study of sigma model quantum field theories [1,13]. They arise naturally in a variety of situations in differential geometry, Lie theory, representation theory and homotopy algebras [21,28,54–56]. They are closely related to the emerging fields of derived differential geometry [5,11,12,24,40,41,50] and higher Lie algebroids [4,7–9,20,22,23,44,48,54,56] (see also [47, Letters 7 and 8]).

Recall that a dg manifold is a \( \mathbb{Z} \)-graded manifold \( \mathcal{M} \) endowed with a homological vector field, i.e. a degree +1 derivation \( Q \) of \( C^\infty(\mathcal{M}) \) satisfying \( [Q, Q] = 0 \). When the underlying \( \mathbb{Z} \)-graded manifold \( \mathcal{M} \) is a \( \mathbb{Z} \)-graded vector space, a dg manifold is equivalent to a finite-dimensional curved \( L_\infty \) algebra (or more precisely a curved \( L_\infty[1] \) algebra). Any complex manifold naturally gives rise to a dg manifold, as does any foliation of a smooth manifold. See Examples 2.2 and 2.3.

The exponential map plays an important role in classical differential geometry. In graded geometry, it turns out that a certain ‘formal exponential map’ is more useful. Let us describe examples, which illustrate the concept of ‘formal exponential map’ we have in mind. First of all, let us recall the relation between exponential map and Poincaré–Birkhoff–Witt isomorphism (PBW isomorphism in short) in classical Lie theory. Let \( G \) be a Lie group and let \( g \) be its Lie algebra. The space \( D_0'(g) \) of distributions on \( g \) with support \{0\} is canonically identified with the symmetric tensor algebra \( S(g) \), while the space \( D'_e(G) \) of distributions on \( G \) with support \{e\} is canonically identified with the universal enveloping algebra \( U(g) \). The classical Lie-theoretic exponential map \( \exp : g \to G \), which is a local diffeomorphism near 0, can be used to push forward the distributions on the Lie algebra to distributions on the Lie group. The induced isomorphism \( S(g) \cong D_0'(g) \to D'_e(G) \cong U(g) \) is precisely the symmetrization map realizing the well known PBW isomorphism. This construction has an analogue for smooth manifolds. However, it requires a choice of affine connection. Given a smooth manifold \( M \), let \( R \) denote its algebra of smooth real-valued functions \( C^\infty(M) \). Each affine connection \( \nabla \) on \( M \) determines an exponential map

\[
\exp^\nabla : T_M \to M \times M, \tag{1}
\]

which is a local diffeomorphism of fiber bundles

\[
\begin{array}{ccc}
T_M & \xrightarrow{\exp^\nabla} & M \times M \\
\pi & \downarrow & \downarrow \text{pr}_1 \\
M & \xrightarrow{id} & M
\end{array}
\]
from a neighborhood of the zero section of $T_M$ to a neighborhood of the diagonal $\Delta$ in $M \times M$. The space of fiberwise distributions on the vector bundle $\pi : T_M \to M$ with support the zero section can be identified, as an $R$-coalgebra, to $\Gamma(S(T_M))$. On the other hand, the space of fiberwise distributions on the fiber bundle $pr_1 : M \times M \to M$ with support the diagonal $\Delta$ can be identified, as an $R$-coalgebra, to the space $\mathcal{D}(M)$ of differential operators on $M$. Pushing distributions forward through the exponential map (1), we obtain an isomorphism of $R$-coalgebras

$$\text{pbw}^\nabla : \Gamma(S(T_M)) \to \mathcal{D}(M),$$

called PBW map [30,31]. In other words, $\text{pbw}^\nabla$ is the fiberwise $\infty$-order jet (along the zero section) of the exponential map (1) arising from the connection $\nabla$. Therefore, one can consider it as a ‘formal exponential map’ associated with the affine connection $\nabla$.

We have the following explicit formula for $\text{pbw}^\nabla$:

$$\text{pbw}^\nabla(X_0 \odot \cdots \odot X_k)(f) = \left. \frac{d}{dt_0} \right|_0 \left. \frac{d}{dt_1} \right|_0 \cdots \left. \frac{d}{dt_k} \right|_0 f(\exp^\nabla(t_0X_0 + t_1X_1 + \cdots + t_kX_k)),$$

for all $X_0, X_1, \ldots, X_k \in \Gamma(T_M)$ and $f \in C^\infty(M)$.

It turns out that the map $\text{pbw}^\nabla$ admits a nice recursive characterization which can be described in a purely algebraic way [30,31] involving the connection $\nabla$, but not the exponential map (1). Therefore, despite the geometric origin of the map $\text{pbw}^\nabla$, this ‘formal exponential map’ still makes sense algebraically in a much wider context. By way of this purely algebraic description, the ‘formal exponential map’ was extended to the context of $\mathbb{Z}$-graded manifolds over the field $\mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) by Liao–Stiénon [32]. The PBW map:

$$\text{pbw}^\nabla : \Gamma(S(T_M)) \to \mathcal{D}(\mathcal{M}) \quad (2)$$

arising from an affine connection $\nabla$ on a $\mathbb{Z}$-graded manifold $\mathcal{M}$ can be thought of as the induced formal exponential map (or the fiberwise $\infty$-order jet) of a ‘virtual exponential map’:

$$\exp^\nabla : T\mathcal{M} \to \mathcal{M} \times \mathcal{M}. \quad (3)$$

Now, let $(\mathcal{M}, Q)$ be a dg manifold. Then, both $\Gamma(S(T_M))$ and $\mathcal{D}(\mathcal{M})$ in (2) are dg coalgebras over the dg ring $(C^\infty(\mathcal{M}), Q)$—see Propositions 3.2 and 3.3. Here $(\Gamma(S(T_M)), L_Q)$ can be understood as the space of fiberwise dg distributions on the dg vector bundle $\pi : T\mathcal{M} \to \mathcal{M}$ with support the zero section—the homological vector field on $T\mathcal{M}$ is the complete lift $\hat{Q}$ of the homological vector field $Q \in \mathfrak{X}(\mathcal{M})$ [38,51]. On the other hand, $(\mathcal{D}(\mathcal{M}), L^\mathcal{D}_{\hat{Q}})$ can be identified with the space of fiberwise dg distributions on the dg fiber bundle $pr_1 : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ with support the diagonal $\Delta \subset \mathcal{M} \times \mathcal{M}$—the homological vector field on $\mathcal{M} \times \mathcal{M}$ is $(Q, Q)$. Recall that for an ordinary smooth manifold $M$, equipped with a vector field $Q$, the exponential map (1) arising from a choice of affine connection $\nabla$ on $M$ identifies the complete lift $\hat{Q} \in \mathfrak{X}(TM)$ of $Q \in \mathfrak{X}(M)$ with the vector field $(Q, Q) \in \mathfrak{X}(M \times M)$ if and only if the connection $\nabla$ is invariant under the flow of $Q$. Similarly, one may wonder whether the ‘virtual exponential map’ (3) is a morphism of dg manifolds. On the level of fiberwise $\infty$-order jets, this is equivalent to asking whether the map $\text{pbw}^\nabla : (\Gamma(S(T_M)), L_Q) \to (\mathcal{D}(\mathcal{M}), L^\mathcal{D}_{\hat{Q}})$ is an isomorphism.

\textsuperscript{1} See [57].
of dg coalgebras over \((C^\infty(M), Q)\). As in classical geometry, one expects that this would be true if the affine connection \(\nabla\) on \(M\) is invariant under the (virtual) flow of the homological vector field \(Q\); in other words, if the Atiyah class of the dg manifold \((M, Q)\) vanishes.

Our first main theorem confirms this assertion:

**Theorem A** (Theorem 3.5). Let \((M, Q)\) be a dg manifold. The Atiyah class \(\alpha_{(M, Q)}\) vanishes if and only if there exists a torsion-free affine connection \(\nabla\) on \(M\) such that

\[
\text{pbw}^\nabla : (\Gamma(S(T_M)), L_Q) \to (\mathcal{D}(M), L_Q^D)
\]

is an isomorphism of dg coalgebras over \((C^\infty(M), Q)\).

The Atiyah class of a dg manifold was first introduced by Shoikhet \([49]\) in terms of Lie algebra cohomology and 1-jets of tangent bundles, appeared also in the work of Lyakhovich–Mosman–Sharapov \([34, \text{Footnote 6}\]), and was studied systematically in 

[38]. The Atiyah class of dg manifolds plays a crucial role in the Kontsevich–Duflo type theorem for dg manifolds \([33, 51]\). Below we recall its definition in terms of affine connections \([38]\).

Let \((M, Q)\) be a dg manifold. Given an affine connection \(\nabla\) on \(M\), consider the \((1, 2)\)-tensor \(\text{At}^\nabla_{(M,Q)} \in \Gamma(M; T^\vee_M \otimes \text{End}(T_M))\) of degree +1 defined by the relation

\[
\text{At}^\nabla_{(M,Q)}(X, Y) = [Q, \nabla_X Y] - \nabla_{[Q,X]} Y - (-1)^{|X|} \nabla_X [Q, Y],
\]

for any homogeneous vector fields \(X, Y \in \mathcal{X}(M)\). Since \(L_Q(\text{At}^\nabla_{(M,Q)}) = 0\), the element \(\text{At}^\nabla_{(M,Q)}\) is a 1-cocycle called the **Atiyah cocycle** associated with the affine connection \(\nabla\). The cohomology class

\[
\alpha_{(M, Q)} := [\text{At}^\nabla_{(M,Q)}] \in H^1(\Gamma(M; T^\vee_M \otimes \text{End}(T_M))^*, L_Q)
\]

does not depend on the choice of connection \(\nabla\), and therefore is an intrinsic characteristic class called **Atiyah class** of the dg manifold \((M, Q)\) \([38]\)—see Proposition 2.5.

As shown by the pioneering work of Kapranov \([25, 42, 43]\), the Atiyah class of a holomorphic vector bundle gives rise to \(L_\infty[1]\) algebras. These \(L_\infty[1]\) algebras play an important role in derived geometry \([10, 36, 42]\) and the construction of Rozansky–Witten invariants \([25, 27, 43, 45, 53]\).

It is natural to expect that the Atiyah cocycle of a dg manifold gives rise to an \(L_\infty[1]\) algebra in a similar fashion. This is indeed true: the following theorem was announced in \([38]\), but a proof was omitted. We will give a complete proof in the present paper.

**Theorem B** (Theorem 4.4). Let \((M, Q)\) be a dg manifold. Each choice of an affine connection \(\nabla\) on \(M\) determines an \(L_\infty[1]\) algebra structure on the space of vector fields \(\mathcal{X}(M)\). While the unary bracket \(\lambda_1 : S^1(\mathcal{X}(M)) \to \mathcal{X}(M)\) is the Lie derivative \(L_Q\) along the homological vector field, the higher multibrackets \(\lambda_k : S^k(\mathcal{X}(M)) \to \mathcal{X}(M)\), with \(k \geq 2\), arise as the composition

\[
\lambda_k : S^k(\mathcal{X}(M)) \to \Gamma(S^k(T_M)) \xrightarrow{R_k} \mathcal{X}(M)
\]

induced by a family of sections \(\{R_k\}_{k \geq 2}\) of the vector bundles \(S^k(T^\vee_M) \otimes T_M\) starting with \(R_2 = -\text{At}^\nabla_{(M,Q)}\).

Furthermore, the \(L_\infty[1]\) algebra structures on \(\mathcal{X}(M)\) arising from different choices of connections are all isomorphic.
The $L_\infty[1]$ algebras arising in this way are called the Kapranov $L_\infty[1]$ algebras of the dg manifold. Our proof of Theorem B is very much inspired by Kapranov’s construction [25, Theorem 2.8.2]. The dg algebra dual (over $(C^\infty(\mathcal{M}), Q)$) to the dg coalgebra $(D(\mathcal{M}), L_\mathcal{D})$ can be considered as a kind of ‘algebra of functions’ on the ‘formal neighborhood’ of the diagonal $\Delta$ of the product dg manifold $(\mathcal{M} \times \mathcal{M}, (Q, Q))$. By construction, pbw $^\mathcal{V}$ is a formal exponential map identifying a ‘formal neighborhood’ of the zero section of $T_{\mathcal{M}}$ to a ‘formal neighborhood’ of the diagonal $\Delta$ of the product manifold $\mathcal{M} \times \mathcal{M}$. The dg coalgebra structure on $D(\mathcal{M})$ associated with the homological vector field $(Q, Q)$ on $\mathcal{M} \times \mathcal{M}$ can be pulled back through this formal exponential map so as to obtain a dg coalgebra $(\Gamma(S(T_{\mathcal{M}})), \delta^\mathcal{V})$, which in turn induces an $L_\infty[1]$ algebra on $\mathcal{X}(\mathcal{M})$.

The Kapranov $L_\infty[1]$ algebra of a dg manifold as in Theorem B is completely determined by the Atiyah 1-cocycle and the sections

$$R_k \in \Gamma(S^k(T^\mathcal{V}_{\mathcal{M}}) \otimes T_{\mathcal{M}}) \cong \Gamma(\text{Hom}(S^k(T_{\mathcal{M}}), T_{\mathcal{M}}))$$

for $k \geq 3$. It is thus natural to wonder whether the $R_k$’s can be described explicitly.

For the $L_\infty[1]$ algebra structure on the Dolbeault complex $(\Omega^{0,\cdot}(T^1_{\mathcal{X}}), \overline{\partial})$ associated with the Atiyah class of the holomorphic tangent bundle $T_\mathcal{X}$ of a Kähler manifold $\mathcal{X}$, Kapranov showed that the multibrackets can be described explicitly by a very simple formula [25]: Eq. (4) below. Consider the $\mathbb{C}$-linear extension of the Levi-Civita connection of the Kähler manifold $\mathcal{X}$; this is a $T_{\mathcal{X}}^0$-connection $\nabla$ on $T_{\mathcal{X}}^0$. Since $\mathcal{X}$ is Kähler, $\nabla$ induces a $T_{\mathcal{X}}^0$-connection on $T_{\mathcal{X}}^1$, also denoted by $\nabla$, which decomposes as the sum $\nabla = \nabla^\mathcal{V} + \nabla^1,0$ of the canonical flat $T_{\mathcal{X}}^0$-connection $\nabla^\mathcal{V}$ on $T_{\mathcal{X}}^1$ and some $T_{\mathcal{X}}^1$-connection $\nabla^1,0$ on $T_{\mathcal{X}}^1$. Since $\nabla^1,0$ is torsion-free and $d^{\nabla^1,0} \circ d^{\nabla^1,0} = 0 \in \Omega^2(\text{End} \ T_{\mathcal{X}}^1)$, the curvature of $\nabla$ is $R^\mathcal{V} = [d^{\nabla^\mathcal{V}}, d^{\nabla^1,0}]$, which equals to $R_2 \in \Omega^0,1(S^2(T^1_{\mathcal{X}})^{\mathcal{V}} \otimes T^1_{\mathcal{X}})$, the Dolbeault representative of the Atiyah 1-cocycle of the holomorphic tangent bundle $T_{\mathcal{X}}$. Kapranov [25, Theorem 2.6] proved that, for $k \geq 2$, the $k$-th multibracket $\lambda_k$ on the Dolbeault complex $(\Omega^{0,\cdot}(T^1_{\mathcal{X}}), \overline{\partial})$ is the composition of the wedge product

$$\Omega^{0,j_1}(T^1_{\mathcal{X}}) \otimes \cdots \otimes \Omega^{0,j_k}(T^1_{\mathcal{X}}) \rightarrow \Omega^{0,j_1+\cdots+j_k}(T^1_{\mathcal{X}} \otimes^k)$$

with the map

$$\Omega^{0,j_1+\cdots+j_k}(T^1_{\mathcal{X}} \otimes^k) \rightarrow \Omega^{0,j_1+\cdots+j_k+1}(T^1_{\mathcal{X}})$$

induced by

$$R_k \in \Omega^0,1(S^k(T^1_{\mathcal{X}})^{\mathcal{V}} \otimes T^1_{\mathcal{X}}) \subset \Omega^{0,1}(\text{Hom}(T^1_{\mathcal{X}} \otimes^k, T^1_{\mathcal{X}})),$$

and that, for $k \geq 3$,

$$R_k = d^{\nabla^1,0} R_{k-1} \in \Omega^0,1(S^k(T^1_{\mathcal{X}})^{\mathcal{V}} \otimes T^1_{\mathcal{X}}).$$

If $\mathcal{X}$ is a mere complex manifold rather than a Kähler manifold, the relation between the $R_k$’s is more complicated: it involves the Atiyah 1-cocycle $R_2$, the curvature of $\nabla^1,0$, and their higher covariant derivatives. Nevertheless, recursive computations are still possible as shown in [31].

In the present paper, we prove that a similar characterization of the higher multibrackets holds for the Kapranov $L_\infty[1]$ algebra of a dg manifold:
Theorem C (Theorem 4.7).

1. The sections \( R_n \in \Gamma \left( S^n \left( T_X^{\vee} \right) \otimes T_M \right) \), with \( n \geq 3 \), are completely determined, by way of a recursive formula, by the Atiyah cocycle \( A_{(M,Q)}^\nabla \), the curvature \( R^{\nabla} \), and their higher covariant derivatives—see (39).

2. In particular, if \( R^{\nabla} = 0 \), then \( R_2 = -A_{(M,Q)}^\nabla \) and \( R_n = \frac{1}{n} \overline{\nabla} R_{n-1} \), for all \( n \geq 3 \).

Finally, we investigate the Kapranov \( L_\infty[1] \) algebras arising from two classes of examples of dg manifolds: those corresponding to finite-dimensional \( L_\infty[1] \) algebras as described in Example 2.2, and those corresponding to manifolds endowed with integrable distributions, which include not only foliated manifolds but also complex manifolds as described in Example 2.3. For the dg manifold \((g[1], d_{CE})\) associated with a finite-dimensional \( L_\infty[1] \) algebra \( g[1] \), we prove that the multibrackets of the Kapranov \( L_\infty[1] \) algebra structure on \( \mathcal{X}(g[1]) \cong \text{Hom} \left( S(g[1]), g[1] \right) \) can be expressed in terms of the multibrackets of the \( L_\infty[1] \) algebra \( g[1] \)—see Proposition 5.8. We also compute the Atiyah class of the dg manifold \((g[1], d_{CE})\) in terms of Chevalley–Eilenberg cohomology of \( g[1] \) with values in the tensor product of adjoint and coadjoint modules \((g[1])^\vee \otimes g[1]\)—see Proposition 5.6. For the dg manifold \((F[1], d_F)\) arising from an integrable distribution \( F \subset T_K M \) on a smooth manifold \( M \), we show that the Kapranov \( L_\infty[1] \) algebra structure on \( \mathcal{X}(F[1]) \) is quasi-isomorphic to the \( L_\infty[1] \) algebra \( \Omega^*_F (T_K M / F) \) arising from the Lie pair \((T_K M, F)\), which was studied extensively in \([15,30,31]\). In particular, for the dg manifold \((T^{0,1}_X[1], \overline{\partial})\) associated with a complex manifold \( X \), the Kapranov \( L_\infty[1] \) algebra structure on \( \mathcal{X}(T^{0,1}_X[1]) \) is quasi-isomorphic to the \( L_\infty[1] \) algebra structure on the Dolbeault complex \((\Omega^0 \cdot (T^1_X), \overline{\partial})\) associated with the Atiyah class of the holomorphic tangent bundle \( T_X \). Moreover, each map \( \phi_k \) in the quasi-isomorphism \( \{ \phi_k \}_{k \geq 1} \) is \( \Omega^0 \cdot \)-multilinear—see Corollary 5.15.

Note that Bandiera \([2,3]\) proved that, when \( X \) is a Kähler manifold, the Kapranov \( L_\infty[1] \) algebra structure on \( \Omega^0 \cdot \cdot (T^{1,0}_X) \) is homotopy abelian over the field \( \mathbb{C} \). It would be interesting to investigate if the \( L_\infty[1] \) algebra structure of Theorem B on the space \( \mathcal{X}(\mathcal{M}) \) of vector fields over a dg manifold \((\mathcal{M}, Q)\) is homotopy abelian over the field \( \mathbb{K} \), possibly by extending the techniques developed in \([2,3]\).

**Notations and conventions.** Throughout this paper, the symbol \( \mathbb{K} \) denotes a field either \( \mathbb{R} \) or \( \mathbb{C} \).

We reserve the symbol \( M \) to denote a smooth manifold (over \( \mathbb{K} \)) exclusively. The sheaf of smooth \( \mathbb{K} \)-valued functions on \( M \) is denoted \( \mathcal{O}_M = \mathcal{O}_M^\mathbb{K} \). The algebra of globally defined smooth functions on \( M \) is \( C^\infty(M) = \mathcal{O}_M(M) \).

A \((p, q)\)-shuffle is a permutation \( \sigma \) of the set \( \{1, 2, \ldots, p+q\} \) such that \( \sigma(1) < \cdots < \sigma(p) \) and \( \sigma(p+1) < \cdots < \sigma(p+q) \). The set of \((p, q)\)-shuffles will be denoted by \( \mathfrak{S}^p_q \).

We use Sweedler’s (sumless) notation for the comultiplication \( \Delta \) in a coalgebra \( C \):

\[
\Delta(c) = \sum_{(c)} c_1 \otimes c_2 = c_1 \otimes c_2, \quad \forall c \in C.
\]

All gradings in this paper are \( \mathbb{Z} \)-gradings and \( \mathcal{M} \) will always be a finite-dimensional graded manifold. Throughout the paper, ‘dg’ means ‘differential graded.’

Given a graded vector space \( V = \bigoplus_{k \in \mathbb{Z}} V^k \), the suspension of \( V \) is the graded vector space \( V[1] = \bigoplus_{k \in \mathbb{Z}} (V[1])^k \) satisfying \( (V[1])^k = V^{k+1} \). We write \(|v|\) to denote the (internal) degree of a homogeneous element \( v \in V \).
Many equations throughout the paper have the following general shape:

$$A(X_1, X_2, \ldots, X_n) = (-1)^{\sum_{i=1}^{n} \delta(i,j) \in \mathcal{X}} |X_{\sigma(i)}| |X_{\sigma(j)}| B(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}),$$

(5)

where $X_1, X_2, \ldots, X_n$ is a finite collection of $\mathbb{Z}$-graded objects; $\sigma$ is a permutation of the set of indices $\{1, 2, \ldots, n\}$; $\mathcal{X}$ is the set of couples $(i, j)$ of elements of $\{1, 2, \ldots, n\}$ such that $i < j$ and $\sigma(i) > \sigma(j)$; and $A$ and $B$ are $n$-ary operations on the $\mathbb{Z}$-graded objects $X_1, X_2, \ldots, X_n$ whose output is an object of degree $|X_1| + |X_2| + \cdots + |X_n|$. The factor $(-1)^{\sum_{i=1}^{n} \delta(i,j) \in \mathcal{X}} |X_{\sigma(i)}| |X_{\sigma(j)}|$ appearing in the right hand side of (5) is called the Koszul sign of the permutation $\sigma$ of the graded objects $X_1, X_2, \ldots, X_n$. It will customarily be abbreviated as $\varepsilon$ since its actual value—either $+1$ or $-1$—can be recovered from a careful inspection of both sides of the equation. We will also use the more explicit abbreviation $\varepsilon(X_1, X_2, \ldots, X_n)$ if the collection of $\mathbb{Z}$-graded objects being permuted is not immediately clear. As explained by Boardman in [6], this sign is mostly inconsequential and it is not necessary to devote much attention or thought to it. In fact, the right hand side of (5) can be a sum of several terms so it would be more correct to say that the general shape of the equations is

$$A(X_1, X_2, \ldots, X_n) = \sum_k (-1)^{\sum_{i=1}^{n} \delta(i,j) \in \mathcal{X}_k} |X_{\sigma_k(i)}| |X_{\sigma_k(j)}| B_k(X_{\sigma_k(1)}, X_{\sigma_k(2)}, \ldots, X_{\sigma_k(n)}).$$

2. Preliminaries

2.1. dg manifolds. Let $M$ be a smooth manifold over $\mathbb{K}$, and $\mathcal{O}_M$ be the sheaf of $\mathbb{K}$-valued smooth functions over $M$. A graded manifold $\mathcal{M}$ with support $M$ consists of a sheaf $\mathcal{A}$ of graded commutative $\mathcal{O}_M$-algebras on $M$ such that there is a $\mathbb{Z}$-graded vector space $V$ for which

$$\mathcal{A}(U) \cong \mathcal{O}_M(U) \otimes_{\mathbb{K}} \text{Hom}_{\mathbb{K}}(S(V), \mathbb{K}) \cong \mathcal{O}_M(U) \otimes_{\mathbb{K}} \hat{S}(V^\vee)$$

for sufficiently small open subsets $U \subset M$. The space of global sections of the sheaf $\mathcal{A}$ will be denoted by $C^\infty(\mathcal{M}) = \mathcal{A}(M)$. We say a graded manifold $\mathcal{M}$ is finite-dimensional if $\dim M < \infty$ and $\dim V < \infty$. Throughout this paper, all graded manifolds will always be finite-dimensional.

Remark 2.1. In the literature, the sheaf of functions $\mathcal{A}$ over a graded manifold is defined by $\mathcal{A}(U) = \mathcal{O}_M(U) \otimes_{\mathbb{K}} S(V^\vee)$ for sufficiently small open subsets $U$ of $M$. Here, however, we allow for formal power series rather than polynomials: we define the sheaf of functions $\mathcal{A}$ by $\mathcal{A}(U) = \mathcal{O}_M(U) \otimes_{\mathbb{K}} \hat{S}(V^\vee)$ for sufficiently small open subsets $U$ of $M$. Consequently, when we write ‘dg manifold’ $(\mathcal{M}, Q)$, we actually mean formal dg manifold in Kontsevich’s sense [26, Section 4.1].

By $\mathcal{I}_A$, we denote the sheaf of ideals of $\mathcal{A}$ consisting of functions vanishing at the support $M$ of $\mathcal{M}$. That is, for sufficiently small $U \subset M$,

$$\mathcal{I}_A(U) \cong \mathcal{O}_M(U) \otimes_{\mathbb{K}} \hat{S}^{\geq 1}(V^\vee).$$

Given graded manifolds $\mathcal{M} = (M, \mathcal{A})$ and $\mathcal{N} = (N, \mathcal{B})$, a morphism $\mathcal{M} \to \mathcal{N}$ of graded manifolds consists of a pair $(f, \psi)$, where $f : M \to N$ is a morphism of smooth manifolds and $\psi : f^* \mathcal{B} \to \mathcal{A}$ is a morphism of sheaves of graded commutative $\mathcal{O}_M$-algebras such that $\psi(f^* \mathcal{B}) \subset \mathcal{I}_A$. We often use the notation $\phi : \mathcal{M} \to \mathcal{N}$ to
denote such a morphism. Then \( \psi = \phi^* \). Also, we write \( \phi^* : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{M}) \) to denote the morphism on global sections. Note that the condition \( \psi(f^*\mathcal{I}_B) \subset \mathcal{I}_A \) is equivalent to \( \psi \) being continuous with respect to the \( \mathcal{I} \)-adic topology.

Vector bundles in the category of graded manifolds are called **graded vector bundles**. A **section** of a graded vector bundle \( \Phi : \mathcal{E} \to \mathcal{M} \) is a morphism of graded manifolds \( s : \mathcal{M} \to \mathcal{E} \) such that \( \Phi \circ s = \text{id}_\mathcal{M} \). We denote the \( C^\infty(\mathcal{M}) \)-module of sections of \( \mathcal{E} \) over \( \mathcal{M} \) by the usual notation \( \Gamma(\mathcal{E}) = \Gamma(M; \mathcal{E}) \).

For a graded manifold \( \mathcal{M} \) with support \( M \), its tangent bundle \( T_M \mathcal{M} \) is a graded manifold with support \( T_M \) and is a graded vector bundle over \( \mathcal{M} \). Its sections are called **vector fields** on \( \mathcal{M} \) and the space of vector fields \( \Gamma(\mathcal{M}; T_M \mathcal{M}) = \Gamma(T_M \mathcal{M}) \) can be identified with that of graded derivations of \( C^\infty(\mathcal{M}) \). We also write \( \Gamma(\mathcal{M}; T_M \mathcal{M}) = \mathfrak{x}(\mathcal{M}) \). Observe that \( \mathfrak{x}(\mathcal{M}) \) admits a Lie algebra structure, whose Lie bracket coincides with the graded commutator

\[
[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X
\]

for homogeneous elements \( X, Y \in \mathfrak{x}(\mathcal{M}) \) regarded as derivations of \( C^\infty(\mathcal{M}) \). Indeed \( T_M \mathcal{M} \) is a graded Lie algebroid [37].

A **differential graded manifold** (dg manifold in short) is a graded manifold \( \mathcal{M} \) together with a homological vector field, i.e. a vector field \( Q \in \mathfrak{x}(\mathcal{M}) \) of degree +1 satisfying \( [Q, Q] = Q \circ Q + Q \circ Q = 0 \). For a dg manifold \((\mathcal{M}, Q)\), its tangent bundle \( T_M \mathcal{M} \) is naturally a dg manifold, with the homological vector field being the complete lift\(^2\) of \( Q \), and in fact \( T_M \mathcal{M} \) is a dg Lie algebroid over \( \mathcal{M} \) [37,38].

**Example 2.2.** Let \( \mathfrak{g} \) be a finite-dimensional Lie algebra. Then \((\mathfrak{g}[1], d_{\text{CE}})\) is a dg manifold—its algebra of functions is \( C^\infty(\mathfrak{g}[1]) \cong \Lambda^* \mathfrak{g}^\vee \) and its homological vector field \( Q \) is the Chevalley–Eilenberg differential \( d_{\text{CE}} \).

This construction admits an ‘up to homotopy’ version: Given a \( \mathbb{Z} \)-graded finite-dimensional vector space \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \), the graded manifold \( \mathfrak{g}[1] \) is a dg manifold, i.e. admits a homological vector field, if and only if \( \mathfrak{g} \) admits a structure of curved \( L^\infty \) algebra.

**Example 2.3.** Let \( M \) be a smooth manifold. Then \((T_M[1], d_{\text{dR}})\) is a dg manifold—its algebra of functions is \( C^\infty(T_M[1]) \cong \Omega^\bullet(M) \) and its homological vector field \( Q \) is the de Rham differential \( d_{\text{dR}} \). Likewise, a complex manifold \( X \) gives rise to a dg manifold \((T^0_X[1], \tilde{\partial})\) whose algebra of functions \( C^\infty(T^0_X[1]) \) is \( \Omega^{0,\bullet}(X) \) and whose homological vector field \( Q \) is the Dolbeaut differential operator \( \tilde{\partial} \).

**Example 2.4.** Let \( s \) be a smooth section of a vector bundle \( E \to M \). Then \((E[-1], \iota_s)\) is a dg manifold—its algebra of functions is \( C^\infty(E[-1]) \cong \Gamma(\Lambda^{-\bullet}E^\vee) \) and its homological vector field is \( Q = \iota_s \), the interior product with \( s \). This dg manifold can be thought of as a smooth model for the (possibly singular) intersection of \( s \) with the zero section of the vector bundle \( E \), and is often called a ‘derived intersection’, or a **quasi-smooth derived manifold** [5].

Both situations in Example 2.3 are special instances of Lie algebroids, while Example 2.4 is a special case of derived manifolds [5].

\(^2\) It is also called tangent lift in the literature [33,38].
2.2. Atiyah class. Let $\mathcal{M}$ be a graded manifold and $\mathcal{E}$ be a graded vector bundle over $\mathcal{M}$. We say a $\mathbb{K}$-linear map

$$\nabla : \mathfrak{X}(\mathcal{M}) \otimes_{\mathbb{K}} \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$$

of degree 0 is a linear connection on $\mathcal{E}$ over $\mathcal{M}$ if the following axioms are satisfied:

1. $C^\infty(\mathcal{M})$-linearity w.r.t. the first argument: $\nabla_f X s = f \nabla_X s$;
2. $\nabla_X$ is a derivation: $\nabla_X(f s) = X(f) s + (f) \cdot \omega \nabla s$.

where $f \in C^\infty(\mathcal{M})$ and $X \in \mathfrak{X}(\mathcal{M})$ are homogeneous elements, and $s \in \Gamma(\mathcal{E})$.

The covariant derivative of degree 0, defined by

$$d^\nabla : \Gamma(A^p T^\nabla_{\mathcal{M}} \otimes \mathcal{E}) \rightarrow \Gamma(A^{p+1} T^\nabla_{\mathcal{M}} \otimes \mathcal{E})$$

of (internal) degree 0, is always exist [32].

The curvature of an affine connection $\nabla$ is the $\mathbb{K}$-linear map

$$\nabla_X Y = \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X, Y]$$

for all homogeneous $\omega \in \Gamma(A^p T^\nabla_{\mathcal{M}} \otimes \mathcal{E})$ and $X_1, \ldots, X_{p+1} \in \mathfrak{X}(\mathcal{M})$. The symbol $\varepsilon = \varepsilon(\omega, X_1, \ldots, X_{p+1})$ denotes the Koszul signs arising from the reordering of the homogeneous objects $\omega, X_1, \ldots, X_{p+1}$ in each term of the right hand side.

We say $\nabla$ is an affine connection on $\mathcal{M}$ if it is a linear connection on $T_{\mathcal{M}}$ over $\mathcal{M}$. Given an affine connection $\nabla$ on $\mathcal{M}$, the $(1, 2)$-tensor $T^\nabla \in \Gamma(T^\nabla_{\mathcal{M}} \otimes T^\nabla_{\mathcal{M}} \otimes T_{\mathcal{M}})$ of degree 0, defined by

$$T^\nabla(X, Y) = \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X, Y]$$

for any homogeneous vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, is called the torsion of $\nabla$. We say an affine connection $\nabla$ is torsion-free if $T^\nabla = 0$. It is well known that affine torsion-free connections always exist [32].

The curvature of an affine connection $\nabla$ is the $(1, 3)$-tensor $R^\nabla \in \Omega^2(\mathcal{M}, \text{End}(T_{\mathcal{M}}))$ of degree 0, defined by

$$R^\nabla(X, Y) Z = \nabla_X \nabla_Y Z - (-1)^{|X||Y|} \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any homogeneous vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{M})$. If the curvature $R^\nabla$ vanishes identically, the affine connection $\nabla$ is called flat.

Let $(\mathcal{M}, Q)$ be a dg manifold. We define an operator $Q$ of degree +1 on the graded $C^\infty(\mathcal{M})$-module $\Gamma(\mathcal{M}; T^\nabla_{\mathcal{M}} \otimes \text{End}(T_{\mathcal{M}}))$:

$$Q : \Gamma(\mathcal{M}; T^\nabla_{\mathcal{M}} \otimes \text{End}(T_{\mathcal{M}}))^{op} \rightarrow \Gamma(\mathcal{M}; T^\nabla_{\mathcal{M}} \otimes \text{End}(T_{\mathcal{M}}))^{op+1}$$

by the Lie derivative along the homological vector field $Q$:

$$(Q F)(X, Y) = [Q, F(X, Y)] - (-1)^k F([Q, X], Y) - (-1)^{k+|X|} F(X, [Q, Y])$$
for any section \( F \in \Gamma(\mathcal{M}; T_M^\vee \otimes \text{End}(T_M))^k \) of degree \( k \) and any homogeneous vector fields \( X, Y \in \mathfrak{X}(\mathcal{M}) \). One can easily check that \( Q^2 = 0 \). Therefore
\[
(\Gamma(\mathcal{M}; T_M^\vee \otimes \text{End}(T_M))^\bullet, \ Q)
\]
is a cochain complex.

Now given an affine connection \( \nabla \), consider the \((1, 2)\)-tensor
\[
\text{At}^\nabla_{(\mathcal{M}, Q)} \in \Gamma(\mathcal{M}; T_M^\vee \otimes \text{End}(T_M))
\]
of degree +1, defined by
\[
\text{At}^\nabla_{(\mathcal{M}, Q)}(X, Y) = [Q, \nabla_X Y] - \nabla_{[Q, X]} Y - (-1)^{|X|} \nabla_X [Q, Y]
\]
for any homogeneous vector fields \( X, Y \in \mathfrak{X}(\mathcal{M}) \).

**Proposition 2.5** [38]. *In the above setting, the following statements hold.*

(1) *If the affine connection \( \nabla \) on \( \mathcal{M} \) is torsion-free, then*
\[
\text{At}^\nabla_{(\mathcal{M}, Q)} \in \Gamma(\mathcal{M}; S^2(T_M^\vee) \otimes T_M).
\]
*In other words,*
\[
\text{At}^\nabla_{(\mathcal{M}, Q)}(X, Y) = (-1)^{|X||Y|} \text{At}^\nabla_{(\mathcal{M}, Q)}(Y, X).
\]

(2) *The element \( \text{At}^\nabla_{(\mathcal{M}, Q)} \in \Gamma(\mathcal{M}; T_M^\vee \otimes \text{End}(T_M))^1 \) is a 1-cocycle.*

(3) *The cohomology class \([\text{At}^\nabla_{(\mathcal{M}, Q)}]\) does not depend on the choice of connection.*

The element \( \text{At}^\nabla_{(\mathcal{M}, Q)} \) is called the **Atiyah cocycle** associated with the affine connection \( \nabla \). The cohomology class
\[
\alpha(\mathcal{M}, Q) := [\text{At}^\nabla_{(\mathcal{M}, Q)}] \in H^1(\Gamma(\mathcal{M}; T_M^\vee \otimes \text{End}(T_M))^\bullet, \ Q)
\]
is called the **Atiyah class** of the dg manifold \((\mathcal{M}, Q)\) [38]. See also [49] and [34, Footnote 6].

### 3. Formal Exponential Map of dg Manifolds

#### 3.1. dg coalgebras.

Let \( \mathcal{R} \) be a graded commutative ring. A **graded coalgebra** \( C \) over \( \mathcal{R} \) is a graded \( \mathcal{R} \)-module equipped with an \( \mathcal{R} \)-linear map \( \Delta : C \to C \otimes_\mathcal{R} C \) of degree 0 called comultiplication satisfying the following conditions:

(1) (Coassociativity)
\[
(\Delta \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \Delta) \circ \Delta : C \to C \otimes_\mathcal{R} C \otimes_\mathcal{R} C.
\]

(2) (Counit) There is an \( \mathcal{R} \)-linear map \( \epsilon : C \to \mathcal{R} \) of degree 0 such that
\[
(\epsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \epsilon) \circ \Delta = \text{id}_C.
\]
Let \( \text{tw} : C \otimes_{\mathcal{R}} C \to C \otimes_{\mathcal{R}} C \) be the twist map defined by
\[
\text{tw}(c_1 \otimes c_2) = (-1)^{|c_1||c_2|} c_2 \otimes c_1,
\]
for homogeneous elements \( c_1, c_2 \in C \). A graded coalgebra \( C \) is called \textbf{cocommutative} if it satisfies \( \Delta = \text{tw} \circ \Delta \).

An \( \mathcal{R} \)-linear map \( \phi : C \to C \) satisfying
\[
\Delta \circ \phi = (\text{id}_C \otimes \phi + \phi \otimes \text{id}_C) \circ \Delta
\]
is called an \textbf{\( \mathcal{R} \)-coderivation} of the graded \( \mathcal{R} \)-coalgebra \( C \). We denote the collection of all \( \mathcal{R} \)-coderivations of \( C \) by \( \text{coDer}_\mathcal{R}(C) \).

Let \( (\mathcal{R}, d_\mathcal{R}) \) be a dg commutative ring, and \( (C, d_C) \) be a dg \( (\mathcal{R}, d_\mathcal{R}) \)-module. Then the map
\[
d_{C \otimes 2} : C \otimes_{\mathcal{R}} C \to C \otimes_{\mathcal{R}} C
\]
defined by
\[
d_{C \otimes 2}(c_1 \otimes c_2) = d_C(c_1) \otimes c_2 + (-1)^{|c_1|} c_1 \otimes d_C(c_2),
\]
for any homogeneous elements \( c_1, c_2 \in C \), is a well-defined degree +1 differential called the induced differential on \( C \otimes_{\mathcal{R}} C \).

**Definition 3.1.** Let \( (\mathcal{R}, d_\mathcal{R}) \) be a dg commutative ring. A \textbf{dg coalgebra} \( (C, d_C) \) over \( (\mathcal{R}, d_\mathcal{R}) \) is a dg \( (\mathcal{R}, d_\mathcal{R}) \)-module \( (C, d_C) \), equipped with a graded coalgebra structure on \( C \) over \( \mathcal{R} \), whose comultiplication and counit map respect the differentials. That is,
\[
\Delta \circ d_C = d_{C \otimes 2} \circ \Delta, \quad \epsilon \circ d_C = d_{\mathcal{R}} \circ \epsilon
\]
where \( \Delta : C \to C \otimes_{\mathcal{R}} C \) is the comultiplication and \( \epsilon : C \to \mathcal{R} \) is the counit map.

**3.2. dg coalgebras associated with dg manifolds.** Any dg manifold \( (\mathcal{M}, Q) \) determines a pair of dg coalgebras over the dg ring \( (C^\infty(\mathcal{M}), Q) \), namely \( \mathcal{D}(\mathcal{M}) \) and \( \Gamma(S(T\mathcal{M})) \).

Below we will briefly describe these dg coalgebra structures. In the sequel, unless specified otherwise, we will always identify \( (\mathcal{R}, d_\mathcal{R}) \cong (C^\infty(\mathcal{M}), Q) \).

First, let us consider the dg coalgebra structure on the left \( \mathcal{R} \)-module \( \mathcal{D}(\mathcal{M}) \) of differential operators on \( \mathcal{M} \).

The comultiplication
\[
\Delta : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}) \otimes_{\mathcal{R}} \mathcal{D}(\mathcal{M}) \quad (7)
\]
is defined by
\[
(\Delta D)(f \otimes g) = D(f \cdot g),
\]
where \( f, g \in C^\infty(\mathcal{M}) \) and \( D \in \mathcal{D}(\mathcal{M}) \).

The differential \( \mathcal{L}_Q^D : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}) \) is defined as the commutator with \( Q \), which is also the Lie derivative along the homological vector field \( Q \):
\[
\mathcal{L}_Q^D(D) = [Q, D] = Q \cdot D - (-1)^{|D|} D \cdot Q \quad (8)
\]
for any $D \in \mathcal{D}(\mathcal{M})$, where $[-,-]$ denotes the commutator on $\mathcal{D}(\mathcal{M})$.

The induced differential on $\mathcal{D}(\mathcal{M}) \otimes_R \mathcal{D}(\mathcal{M})$ is again the Lie derivative $\mathcal{L}_Q^D$, which coincides with $[Q,-]$, with $[-,-]$ being the Gerstenhaber bracket on polydifferential operators on $\mathcal{M}$.

The counit map

$$\epsilon : \mathcal{D}(\mathcal{M}) \to C^\infty(\mathcal{M})$$

(9)
is the canonical projection, which evaluates a differential operator $D$ on the constant function 1.

Note that $\mathcal{D}(\mathcal{M})$ admits a natural ascending filtration by the order of differential operators

$$C^\infty(\mathcal{M}) = \mathcal{D}^{\leq 0}(\mathcal{M}) \subset \cdots \subset \mathcal{D}^{\leq n}(\mathcal{M}) \subset \cdots$$

where $\mathcal{D}^{\leq n}(\mathcal{M})$ denotes the space of differential operators of order $\leq n$. The following proposition can be easily verified.

**Proposition 3.2.** For any dg manifold $(\mathcal{M}, Q)$, the space of differential operators $\mathcal{D}(\mathcal{M})$ on $\mathcal{M}$, equipped with the comultiplication $\Delta$, the differential $\mathcal{L}_Q^D$ and the counit $\epsilon$ as in (7), (8) and (9), is a filtered dg cocommutative coalgebra over $(C^\infty(\mathcal{M}), Q)$.

Next we describe the dg coalgebra structure on the left $\mathcal{R}$-module $\Gamma(\mathcal{S}(T\mathcal{M}))$.

The comultiplication

$$\Delta : \Gamma(\mathcal{S}(T\mathcal{M})) \to \Gamma(\mathcal{S}(T\mathcal{M})) \otimes_R \Gamma(\mathcal{S}(T\mathcal{M}))$$

is given by

$$\Delta(X_1 \circ \cdots \circ X_n) = 1 \otimes (X_1 \circ \cdots \circ X_n) + (X_1 \circ \cdots \circ X_n) \otimes 1 + \sum_{k=1}^{n-1} \sum_{\sigma \in \mathfrak{S}^{n-k}_k} \epsilon \cdot (X_{\sigma(1)} \circ \cdots \circ X_{\sigma(k)}) \otimes (X_{\sigma(k+1)} \circ \cdots \circ X_{\sigma(n)}),$$

(10)

where $X_1, \cdots, X_n \in \Gamma(\mathcal{T}\mathcal{M})$. The symbol $\mathfrak{S}^{n-k}_k$ denotes the set of all $(k, n-k)$-shuffles and the symbol $\epsilon := \epsilon(X_1, X_2, \cdots, X_n)$ denotes the Koszul signs arising from the reordering of the homogeneous objects $X_1, X_2, \cdots, X_n$ in each term of the right hand side.

The differential

$$\mathcal{L}_Q : \Gamma(\mathcal{S}(T\mathcal{M})) \to \Gamma(\mathcal{S}(T\mathcal{M}))$$

(11)
is the Lie derivative along the homological vector field $Q$. The induced differential on $\Gamma(\mathcal{S}(T\mathcal{M})) \otimes_R \Gamma(\mathcal{S}(T\mathcal{M})) \cong \Gamma(\mathcal{S}(T\mathcal{M}) \otimes \mathcal{S}(T\mathcal{M}))$ is again the Lie derivative $\mathcal{L}_Q$.

The counit map

$$\epsilon : \Gamma(\mathcal{S}(T\mathcal{M})) \to C^\infty(\mathcal{M})$$

(12)
is the canonical projection.

Note that $\Gamma(\mathcal{S}(T\mathcal{M}))$ admits a canonical ascending filtration

$$C^\infty(\mathcal{M}) = \Gamma(\mathcal{S}^{\leq 0}(T\mathcal{M})) \subset \cdots \subset \Gamma(\mathcal{S}^{\leq n}(T\mathcal{M})) \subset \cdots.$$
Proposition 3.3. For any dg manifold $(M, Q)$, the space $\Gamma(S(T_M))$, equipped with the comultiplication $\Delta$, the differential $\mathcal{L}_Q$ and the counit map $\epsilon$ as in (10), (11) and (12), is a filtered dg cocommutative coalgebra over $(C^\infty(M), Q)$.

3.3. Formal exponential map of a dg manifold. Let $M$ be a finite-dimensional graded manifold and $\nabla$ be an affine connection on $M$. A purely algebraic description of the Poincaré–Birkhoff–Witt map has been extended to the context of $\mathbb{Z}$-graded manifolds by Liao–Stiénon [32]. As pointed out in the introduction, for an ordinary smooth manifold, the PBW map is a formal exponential map. In the same way, one can think of the PBW map of a $\mathbb{Z}$-graded manifold as an induced formal exponential map of ‘the virtual exponential map’

$$\exp^\nabla : T_M \rightarrow M \times M$$

by taking fiberwise $\infty$-jets.

Recall that the Poincaré–Birkhoff–Witt map

$$\text{pbw}^\nabla : \Gamma(S(T_M)) \rightarrow \mathcal{D}(M)$$

is defined by the inductive formula [32]:

$$\text{pbw}^\nabla(f) = f, \quad \forall f \in C^\infty(M);$$

$$\text{pbw}^\nabla(X) = X, \quad \forall X \in \mathfrak{X}(M);$$

and

$$\text{pbw}^\nabla(X_1 \circ \cdots \circ X_n) = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \left( X_k \text{pbw}^\nabla(X^{[k]}) - \text{pbw}^\nabla(\nabla X_k X^{[k]}) \right),$$

where $X^{[k]} = X_1 \circ \cdots \circ \hat{X}_k \circ \cdots \circ X_n \in \Gamma(S^{n-1}(T_M))$ for homogeneous vector fields $X_1, \ldots, X_n \in \mathfrak{X}(M)$ and $\varepsilon_k = (-1)^{|X_k|(|X_1| + \cdots + |X_{k-1}|)}$ is the Koszul sign.

Theorem 3.4 [32]. The map $\text{pbw}^\nabla$ is an isomorphism of graded coalgebras from $\Gamma(S(T_M))$ to $\mathcal{D}(M)$ over $C^\infty(M)$.

Now, we assume there exists a homological vector field $Q$ on $M$ so that $(M, Q)$ is a dg manifold. Then, both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ in (14) are dg coalgebras over $(C^\infty(M), Q)$, according to Propositions 3.2 and 3.3. We think of the elements of the dg coalgebra $(\Gamma(S(T_M)), \mathcal{L}_Q)$ as fiberwise dg distributions on the dg vector bundle $\pi : T_M \rightarrow M$ with support the zero section—the homological vector field on $T_M$ is the complete lift $\hat{Q}$ of the homological vector field $Q$ on $M$ [38, 51]. Likewise, we think of the elements of the dg coalgebra $(\mathcal{D}(M), \mathcal{L}_D^Q)$ as fiberwise dg distributions on the dg fiber bundle $\text{pr}_1 : M \times M \rightarrow M$ with support the diagonal $\Delta$—the homological vector field on $M \times M$ is $(Q, Q)$. On the level of fiberwise $\infty$-jets, the fact that the virtual exponential map (13) is a map of dg manifolds is equivalent to the map $\text{pbw}^\nabla : (\Gamma(S(T_M)), \mathcal{L}_Q) \rightarrow (\mathcal{D}(M), \mathcal{L}_D^Q)$ being an isomorphism of dg coalgebras over $(C^\infty(M), Q)$. This consideration leads to the following
Theorem 3.5 Let \((\mathcal{M}, Q)\) be a dg manifold. The Atiyah class \(\alpha(\mathcal{M}, Q)\) of the dg manifold \((\mathcal{M}, Q)\) vanishes if and only if there exists a torsion-free affine connection \(\nabla\) on \(\mathcal{M}\) such that

\[
\text{pbw}^\nabla : \left(\Gamma(S(T\mathcal{M})), \mathcal{L}_Q\right) \to \left(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q^D\right)
\]
is an isomorphism of dg coalgebras over \((C^\infty(\mathcal{M}), Q)\).

Remark 3.6. A similar theorem pertaining to the Atiyah class of Lie pairs was obtained in [31, Theorem 5.10]. It would be interesting to establish a result that encompasses both [31, Theorem 5.10] and Theorem 3.5 under a unified framework.

In order to prove Theorem 3.5, we consider the linear map

\[
C^\nabla : \Gamma(S(T\mathcal{M})) \to \mathcal{D}(\mathcal{M})
\]
defined by

\[
C^\nabla := \mathcal{L}_Q^D \circ \text{pbw}^\nabla - \text{pbw}^\nabla \circ \mathcal{L}_Q. \tag{17}
\]

One can easily check that \(C^\nabla\) is a \(C^\infty(\mathcal{M})\)-linear map of degree +1. Moreover, for \(n \geq 0\),

\[
C^\nabla(\Gamma(S^{\leq n}(T\mathcal{M}))) \subseteq \mathcal{D}^{\leq n-1}(\mathcal{M}).
\]

The following proposition indicates that \(C^\nabla\) can be completely determined by a recursive formula.

Proposition 3.7. Let \((\mathcal{M}, Q)\) be a dg manifold, and let \(\nabla\) be a torsion-free affine connection on \(\mathcal{M}\). Then the map \(C^\nabla\) satisfies

\[
C^\nabla(f) = 0; \tag{18}
\]

\[
C^\nabla(X) = 0; \tag{19}
\]

\[
C^\nabla(X \odot Y) = -\text{At}^\nabla_{(\mathcal{M}, Q)}(X, Y), \tag{20}
\]

for all \(f \in C^\infty(\mathcal{M}), X, Y \in \mathfrak{X}(\mathcal{M})\), and, for \(n \geq 3\), it satisfies the recursive formula

\[
C^\nabla(X) = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \left[ (-1)^{|X_k|} X_k \cdot C^\nabla(X^{[k]}) - C^\nabla(\nabla X_k X^{[k]}) \right] - \frac{2}{n} \sum_{i < j} \varepsilon_i \varepsilon_j (-1)^{|X_i| - |X_j|} \text{pbw}^\nabla \left( \text{At}^\nabla_{(\mathcal{M}, Q)}(X_i, X_j) \odot X^{[i, j]} \right), \tag{21}
\]

where \(X = X_1 \odot \cdots \odot X_n \in \Gamma(S^n(T\mathcal{M}))\) denotes the symmetric tensor product of \(n\) homogeneous vector fields \(X_1, \ldots, X_n \in \mathfrak{X}(\mathcal{M})\); \(X^{[k]} = X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n\) for any \(1 \leq k \leq n\); \(X^{[i, j]} = X_1 \odot \cdots \odot \widehat{X_i} \odot \cdots \odot \widehat{X_j} \odot \cdots \odot X_n\) for any \(1 \leq i < j \leq n\); and \(\varepsilon_k = (-1)^{|X_k|(|X_1|+\cdots+|X_{k-1}|)}\) is the Koszul sign arising from the reordering \(X_1, X_2, \ldots, X_n \mapsto X_k, X_1, X_2, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n\).

We now prove Theorem 3.5 based on Proposition 3.7.
Proof of Theorem 3.5. Observe that, according to Proposition 2.5, \( \alpha_{(\mathcal{M}, Q)} \) vanishes if and only if there exists an affine connection \( \nabla' \) for which \( \mathcal{A}_Q(\nabla') = 0 \). It follows from \( \mathcal{L}_Q(\nabla') = \mathcal{A}_Q(\nabla') = 0 \) that \( \mathcal{L}_Q(T) = 0 \). Therefore, if the Atiyah cocycle of the affine connection \( \nabla' \) vanishes, then so does the Atiyah cocycle of the torsion-free connection \( \nabla = \nabla' - \frac{1}{2} T \)

\[
\mathcal{A}_\nabla(\nabla) = \mathcal{L}_Q(\nabla) = \mathcal{L}_Q(\nabla') = \mathcal{A}_Q(\nabla') = 0.
\]

Thus, it suffices to prove that \( \mathcal{C}_\nabla = 0 \) if and only if \( \mathcal{A}_\nabla(\nabla) = 0 \).

Assume that \( \mathcal{C}_\nabla = 0 \). By Proposition 3.7, we have

\[
\mathcal{A}_\nabla(\nabla)(X, Y) = -\mathcal{C}_\nabla(X \odot Y) = 0
\]

for all \( X, Y \in \mathfrak{X}(\mathcal{M}) \).

Conversely, assume that \( \mathcal{A}_\nabla(\nabla) = 0 \). Then we have \( \mathcal{C}_\nabla(X \odot Y) = 0 \) by Proposition 3.7. Hence \( \mathcal{C}_\nabla(Y) = 0 \) for all \( Y \in \Gamma(S^{\leq 2}(T\mathcal{M})) \). Moreover, Eq. (21) can be written as

\[
\mathcal{C}_\nabla(X) = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \left[ (-1)^{|X_k|} X_k \cdot \mathcal{C}_\nabla(X^{[k]}) - \mathcal{C}_\nabla(\nabla X_k X^{[k]}) \right], \quad \forall X \in \left( \Gamma(S^{\geq 3}(T\mathcal{M})) \right).
\]

Therefore, \( \mathcal{C}_\nabla = 0 \) by the inductive argument.

3.4. Proof of Proposition 3.7. Now we turn to the proof of Proposition 3.7. We will divide the proof into several lemmas.

Lemma 3.8. Under the same hypothesis as in Proposition 3.7, Eqs. (18), (19) and (20) hold.

Proof. Equations (18) and (19) follow immediately from Eq. (15).

To prove Eq. (20), let \( X, Y \in \mathfrak{X}(\mathcal{M}) \) be homogeneous vector fields. Since \( \nabla \) is torsion-free, we have

\[
\nabla_X Y - (-1)^{|X||Y|} \nabla_Y X = [X, Y] = XY - YX.
\]

It then follows from Eq. (16) that

\[
\text{pbw}^\nabla(X \odot Y) = XY - \nabla_X Y.
\]

From there, we obtain

\[
\mathcal{L}_Q^\nabla \circ \text{pbw}^\nabla(X \odot Y) = [Q, X]Y + (-1)^{|X|} X[Q, Y] - [Q, \nabla_X Y]
\]

and

\[
\text{pbw}^\nabla \circ \mathcal{L}_Q(X \odot Y) = \text{pbw}^\nabla \left( [Q, X] \odot Y + (-1)^{|X|} X \odot [Q, Y] \right)
\]

\[
= \left( [Q, X]Y - \nabla_{[Q, X]} Y \right) + (-1)^{|X|} (X[Q, Y] - \nabla_X [Q, Y]).
\]
As a result, we have
\[
C^\nabla (X \odot Y) = (\mathcal{L}^D_Q \circ \text{pbw}^\nabla - \text{pbw}^\nabla \circ \mathcal{L}_Q)(X \odot Y) \\
= - \left( [Q, \nabla X Y] - \nabla_{[Q, X]} Y - (-1)^{|X|} \nabla_X [Q, Y] \right) \\
= - \text{At}^\nabla_{(\mathcal{M}, Q)}(X, Y).
\]

In the sequel, we adopt the following notations. For any \( X = X_1 \odot \cdots \odot X_n \in \Gamma(S^n(T \mathcal{M})) \), we write \( X^{[k]} = X_1 \odot \cdots \odot \widehat{X}_k \odot \cdots \odot X_n \); for \( i \neq j \), we write \( X^{[i,j]} = X_1 \odot \cdots \odot \widehat{X}_i \odot \cdots \odot \widehat{X}_j \odot \cdots \odot X_n \), and for all \( 1 \leq i \leq n \), \( X^{[i,i]} = 0 \).

**Lemma 3.9.** Under the same hypothesis as in Proposition 3.7, for all \( n \geq 3 \) and \( X = X_1 \odot \cdots \odot X_n \in \Gamma(S^n(T \mathcal{M})) \), we have
\[
\mathcal{L}_Q^D \circ \text{pbw}^\nabla(X) = \frac{1}{n} \sum_{k=1}^{n} \left( \varepsilon \cdot [Q, X_k] \cdot \text{pbw}^\nabla(X^{[k]}) + \varepsilon \cdot X_k \cdot \mathcal{L}_Q^D(\text{pbw}^\nabla(X^{[k]})) \right) \\
- \frac{1}{n} \sum_{k=1}^{n} \varepsilon \cdot \mathcal{L}_Q^D(\text{pbw}^\nabla(\nabla_{X_k} X^{[k]})) \\
\text{and} \\
\text{pbw}^\nabla \circ \mathcal{L}_Q(X) \\
= \frac{1}{n} \sum_{k=1}^{n} \left( \varepsilon \cdot [Q, X_k] \cdot \text{pbw}^\nabla(X^{[k]}) + \varepsilon \cdot X_k \cdot \text{pbw}^\nabla(\mathcal{L}_Q(X^{[k]})) \right) \\
- \varepsilon \cdot \text{pbw}^\nabla(\mathcal{L}_Q(\nabla_{X_k} X^{[k]})) + \frac{1}{n} \sum_{i < j} \varepsilon \cdot \text{pbw}^\nabla \left( 2 \text{At}^\nabla_{(\mathcal{M}, Q)}(X_i, X_j) \odot X^{[i,j]} \right).
\]

In the two equations above and in the proof of the Lemma as well, the symbol \( \varepsilon = \varepsilon(Q, X_1, \cdots, X_n) \) denotes the Koszul signs arising from the reordering of the homogeneous objects \( Q, X_1, \cdots, X_n \) in each term of the right hand sides.

**Proof.** The formula for \( \mathcal{L}_Q^D \circ \text{pbw}^\nabla(X) \) is immediate from Eq. (16).

Next, we will compute \( \text{pbw}^\nabla \circ \mathcal{L}_Q(X) \). Since \( \mathcal{L}_Q(X) = \sum_{k=1}^{n} \varepsilon \cdot ([Q, X_k] \odot X^{[k]}) \), applying Eq. (16), we have
\[
\text{pbw}^\nabla \circ \mathcal{L}_Q(X) = \frac{1}{n} \left( A_1 - A_2 + B - C \right),
\]
where
\[
A_1 := \sum_{k=1}^{n} \varepsilon \cdot [Q, X_k] \cdot \text{pbw}^\nabla(X^{[k]}), \\
A_2 := \sum_{k=1}^{n} \varepsilon \cdot \text{pbw}^\nabla(\nabla_{[Q, X_k]} X^{[k]}),
\]
\[ B := \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot X_i \cdot \text{pbw}^\nabla (\{Q, X_k\} \odot X^{[i,k]}), \]

\[ C := \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\nabla X_i (\{Q, X_k\} \odot X^{[i,k]})). \]  

(22)

First, by changing the order of summation, we obtain

\[ B = \sum_{i=1}^{n} \sum_{k=1}^{n} \varepsilon \cdot X_i \cdot \text{pbw}^\nabla (\{Q, X_k\} \odot X^{[i,k]}) \]

\[ = \sum_{i=1}^{n} \varepsilon \cdot X_i \cdot \text{pbw}^\nabla (\mathcal{L}_Q (X^{[i]})) . \]  

(23)

We also can write

\[ A_2 = \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\nabla \{Q, X_k\} X_i) \odot X^{[k,i]} \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\nabla \{Q, X_k\} X_j) \odot X^{[i,k]} . \]  

(24)

Now we also have

\[ \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\{Q, X_k\} \odot \nabla X_i X^{[i,k]} ) \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\{Q, X_k\} \odot \nabla X_i X_j X^{[i,k,j]} ) \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\nabla X_i X_j \odot [Q, X_k] \odot X^{[i,k,j]} ) \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\nabla X_i X_j \odot \mathcal{L}_Q X^{[i,j]} ) \]

\[ = \sum_{i=1}^{n} \sum_{k=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\nabla X_i X_k \odot \mathcal{L}_Q X^{[i,k]} ). \]

Therefore, it follows that

\[ \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\{Q, \nabla X_i X_k\} \odot X^{[i,k]} ) + \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\{Q, X_k\} \odot \nabla X_i X^{[i,k]} ) \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\{Q, \nabla X_i X_k\} \odot X^{[i,k]} ) + \sum_{i=1}^{n} \sum_{k=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\nabla X_i X_k \odot \mathcal{L}_Q X^{[i,k]} ). \]
\[= \sum_{i=1}^{n} \sum_{k=1}^{n} \varepsilon \cdot \text{pbw}^\nabla \mathcal{L}_Q(\nabla_{X_i} X_k \odot X^{[i,k]}) \]
\[= \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla (\mathcal{L}_Q(\nabla_{X_i} X^{[i]})). \tag{25} \]

Moreover,
\[ C = \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla \left( (\nabla_{X_i} [Q, X_k]) \odot X^{[i,k]} \right) \]
\[+ \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla \left( [Q, X_k] \odot \nabla_{X_i} X^{[i,k]} \right). \tag{26} \]

Then by combining Eqs. (24), (25) and (26) and using the definition of Atiyah cocycles, we obtain
\[ A_2 + C = \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla \left( (\nabla_{X_i} [Q, X_k] - \mathcal{A}_2(\mathcal{L}_Q \circ \text{pbw}^\nabla \circ \nabla_{X_i} X_k)) \odot X^{[i,k]} \right) \]
\[+ \sum_{k=1}^{n} \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla \left( [Q, X_k] \odot \nabla_{X_i} X^{[i,k]} \right) \]
\[= \sum_{i=1}^{n} \varepsilon \cdot \text{pbw}^\nabla \left( \mathcal{L}_Q(\nabla_{X_i} X^{[i]}) \right) - \sum_{i<j} \varepsilon \cdot \text{pbw}^\nabla \left( 2 \mathcal{A}_2(\mathcal{L}_Q \circ \text{pbw}^\nabla \circ \nabla_{X_i} X_j) \odot X^{[i,j]} \right). \tag{27} \]

The conclusion thus follows from Eqs. (22), (23), and (27).

**Proof of Proposition 3.7.** Equations (18), (19) and (20) have been proved in Lemma 3.8. It remains to prove Eq. (21). According to Lemma 3.9, we have
\[ \mathcal{L}_Q^D \circ \text{pbw}^\nabla (X) - \text{pbw}^\nabla \circ \mathcal{L}_Q(X) \]
\[= \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|} X_k \cdot (\mathcal{L}_Q^D \circ \text{pbw}^\nabla - \text{pbw}^\nabla \circ \mathcal{L}_Q)(X^{[k]}) \]
\[- \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k (\mathcal{L}_Q^D \circ \text{pbw}^\nabla - \text{pbw}^\nabla \circ \mathcal{L}_Q)(\nabla_{X_k} X^{[k]}) \]
\[- \frac{1}{n} \sum_{i<j} \varepsilon_i \varepsilon_j (-1)^{|X_i| \cdot |X_j|} \text{pbw}^\nabla \left( 2 \mathcal{A}_2(\mathcal{L}_Q \circ \text{pbw}^\nabla \circ \nabla_{X_i} X_j) \odot X^{[i,j]} \right). \]

This concludes the proof of Proposition 3.7.

\[ \square \]

### 4. Atiyah Class and Homotopy Lie Algebras

This section is devoted to the study of homotopy Lie algebras associated with the Atiyah class of dg manifolds.
4.1. Kapranov $L_\infty[1]$ algebras of dg manifolds. The Atiyah class of a holomorphic vector bundle is closely related to $L_\infty[1]$ algebras as shown by the pioneer work of Kapranov [25, 42, 43]. These $L_\infty[1]$ algebras play an important role in derived geometry [10, 36, 42] and the construction of Rozansky–Witten invariants [25, 27, 43, 45, 53].

In this section, following Kapranov [25], we show that the Atiyah class of a dg manifold is related to $L_\infty[1]$ algebras in a similar fashion. We refer to [17, Sections 4 and 5] for an interpretation in terms of derived categories.

Let $(\mathcal{M}, Q)$ be a dg manifold and let $\nabla$ be an affine connection on $\mathcal{M}$. The Lie derivative $\mathcal{L}_Q^\nabla$ along the homological vector field $Q$ is a degree +1 coderivation of the dg coalgebra $\mathcal{D}(\mathcal{M})$ over $(C^\infty(\mathcal{M}), Q)$ according to Proposition 3.2.

Transferring $\mathcal{L}_Q^\nabla$ from $\mathcal{D}(\mathcal{M})$ to $\Gamma(S(T\mathcal{M}))$ by the graded coalgebra isomorphism $\text{pbw}^\nabla$ in (14), we obtain a degree +1 coderivation $\delta^\nabla$ of $\Gamma(S(T\mathcal{M}))$:

$$\delta^\nabla := (\text{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q^\nabla \circ \text{pbw}^\nabla.$$ (28)

Therefore $(\Gamma(S(T\mathcal{M})), \delta^\nabla)$ is a dg coalgebra over the dg ring $(C^\infty(\mathcal{M}), Q)$.

Finally, dualizing $\delta^\nabla$ over $(C^\infty(\mathcal{M}), Q)$, we obtain a derivation of degree +1:

$$D^\nabla : \Gamma(\hat{\mathcal{S}}(T\mathcal{M})) \to \Gamma(\hat{\mathcal{S}}(T\mathcal{M})).$$ (29)

Here we used the identification $\Gamma(\hat{\mathcal{S}}(T\mathcal{M})) \cong \text{Hom}_{C^\infty(\mathcal{M})}(\Gamma(S(T\mathcal{M})), C^\infty(\mathcal{M}))$.

The following theorem was first announced in [38], but a proof was omitted. We will present a complete proof below.

**Theorem 4.1.** Let $(\mathcal{M}, Q)$ be a dg manifold, and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$.

(i) The operator $D^\nabla$ is a derivation of degree +1 of the graded algebra $\Gamma(\hat{\mathcal{S}}(T\mathcal{M}))$ satisfying $(D^\nabla)^2 = 0$. Thus $(\Gamma(\hat{\mathcal{S}}(T\mathcal{M})), D^\nabla)$ is a dg algebra.

(ii) There exists a sequence of degree +1 sections $R_k \in \Gamma(S^k(T\mathcal{M}) \otimes T\mathcal{M})$, $k \geq 2$, whose first term $R_2$ equals to $-\text{At}^\nabla_{(\mathcal{M}, Q)}$, such that

$$D^\nabla = \mathcal{L}_Q + \sum_{k=2}^{\infty} R_k,$$

where each $\tilde{R}_k : \Gamma(\hat{\mathcal{S}}(T\mathcal{M})) \to \Gamma(\hat{\mathcal{S}}(T\mathcal{M}))$ denotes the $R$-linear degree +1 derivation corresponding to $\tilde{R}_k$.

(iii) Different choices of torsion-free affine connections $\nabla$ induce isomorphic dg algebras $(\Gamma(\hat{\mathcal{S}}(T\mathcal{M})), D^\nabla)$.

**Remark 4.2.** The graded algebra $\Gamma(\hat{\mathcal{S}}(T\mathcal{M}))$ can be thought of as the graded algebra of functions on a graded manifold $\hat{T}\mathcal{M}$ with support $M$ and $D^\nabla$ as a homological vector field on $\hat{T}\mathcal{M}$. Note that $T\mathcal{M}$ and $\hat{T}\mathcal{M}$ are different graded manifolds: the support of $T\mathcal{M}$ is $T_M$ while the support of $\hat{T}\mathcal{M}$ is $M$.

Before we prove this theorem, we need to recall some basic notations.

Recall that given a graded commutative algebra $R$ and a graded $R$-module $V$, the symmetric tensor algebra $(S_R(V), \mu)$ over $R$ admits a canonical graded coalgebra structure $\Delta : S_R(V) \to S_R(V) \otimes_R S_R(V)$ defined by [31]
\[ \Delta(v_1 \odot \cdots \odot v_n) = 1 \otimes (v_1 \odot \cdots \odot v_n) + (v_1 \odot \cdots \odot v_n) \otimes 1 \]
\[
+ \sum_{k=1}^{n-1} \sum_{\sigma \in \mathcal{S}_k} \varepsilon(\sigma) (v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)})
\]

for homogeneous elements \( v_1, \ldots, v_n \in V \). Here the symbol \( \varepsilon = \varepsilon(v_1, v_2, \ldots, v_n) \) denotes the Koszul signs arising from the reordering of the homogeneous objects \( v_1, v_2, \ldots, v_n \) in each term of the right hand side.

The following lemma is standard.

**Lemma 4.3** [31,35]. Let \( R \) be a graded commutative algebra and \( V \) be an \( R \)-module. There is a natural isomorphism of \( R \)-modules

\[
\text{coDer}_R(S_R(V)) \cong \prod_{k=0}^{\infty} \text{Hom}_R(S^k_R(V), V).
\]

More explicitly, the correspondence between a sequence of maps \( \{q_k\}_{k \geq 0} \) with \( q_k \in \text{Hom}_R(S^k_R(V), V) \) and a coderivation \( Q \in \text{coDer}_R(S_R(V)) \) is given by

\[
Q(v_1 \odot \cdots \odot v_n) = q_0(1) \odot v_1 \odot \cdots \odot v_n + q_n(v_1 \odot \cdots \odot v_n) \odot 1
+ \sum_{k=1}^{n-1} \sum_{\sigma \in \mathcal{S}_k} \varepsilon(\sigma) q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}
\]

(30)

for homogeneous vectors \( v_1, \ldots, v_n \in V \).

For a given graded \( R \)-coalgebra \((C, \Delta)\) and a graded \( R \)-algebra \((A, \mu)\), the convolution product \( \ast \) on the graded vector space \( \text{Hom}_R(C, A) \) is defined by

\[ f \ast g = \mu \circ (f \otimes g) \circ \Delta \]

for all \( f, g \in \text{Hom}_R(C, A) \). It is clear that \( (\text{Hom}_R(C, A), \ast) \) is a graded \( R \)-algebra. In particular, since \( S_R(V) \) is both a graded coalgebra and a graded algebra, the space of \( R \)-linear maps \( \text{Hom}_R(S_R(V), S_R(V)) \) admits a convolution product:

\[
(f \ast g)(v) = \sum_{(v)} (-1)^{|g|(|v_1|)} f(v(1)) \odot g(v(2)),
\]

(31)

where \( v \in S_R(V) \) and \( \Delta(v) = \sum_{(v)} v(1) \otimes v(2) \).

Using the above notation (31), we may write Eq. (30) as

\[
Q = \sum_{k=0}^{\infty} (q_k \ast \text{id}_{S_R(V)})
\]

(32)

where the map \( q_k : S_R(V) \to S^k_R(V) \) is defined by the following commutative diagram:

\[
\begin{array}{ccc}
S_R(V) & \xrightarrow{q_k} & S^k_R(V) \\
\downarrow \text{pr}_k & & \uparrow \\
S^k_R(V) & \xrightarrow{\tilde{q}_k} & S^1_R(V).
\end{array}
\]

(33)

Here \( \text{pr}_k : S_R(V) \to S^k_R(V) \) denotes the canonical projection. We write \( \text{id}_S \) for \( \text{id}_{S_R(V)} \) below if there is no confusion. We are now ready to give a detailed proof of Theorem 4.1.
Proof of Theorem 4.1. For (i), by construction, it is clear that the operator $D^\nabla$ in (29) is indeed a degree +1 derivation. Since $Q$ is a homological vector field, from (28), it follows that $(\delta^\nabla)^2 = 0$. Therefore $(D^\nabla)^2 = 0$.

To prove (ii), consider the case when $\mathcal{R} = C^\infty(\mathcal{M})$ and $V = \Gamma(\mathcal{T}_\mathcal{M})$ in Lemma 4.3. Recall that $C^\infty$ in (17) is $\mathcal{R}$-linear, and $\text{pbw}^\nabla : \Gamma(S(T_\mathcal{M})) \to \mathcal{D}(\mathcal{M})$ is an isomorphism of graded coalgebras over $\mathcal{R}$. Since $\mathcal{L}_Q \in \text{coDer}_{\mathcal{R}}(\Gamma(S(T_\mathcal{M})))$ and $\mathcal{L}_Q^D \in \text{coDer}_{\mathcal{R}}(\Gamma(S(T_\mathcal{M})))$, it thus follows that

$$(\text{pbw}^\nabla)^{-1} \circ C^\nabla = (\text{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q^D \circ \text{pbw}^\nabla - \mathcal{L}_Q \in \text{coDer}_{\mathcal{R}}(\Gamma(S(T_\mathcal{M}))).$$

Since both $\mathcal{L}_Q^D$ and $\mathcal{L}_Q$ are of degree +1 and $\text{pbw}^\nabla$ is of degree 0, it follows from Lemma 4.3 and Eq. (32) that there exists a sequence of degree +1 sections $R_k \in \Gamma(S^k(T_\mathcal{M}) \otimes T_\mathcal{M})$, $k \geq 0$, such that

$$(\text{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q^D \circ \text{pbw}^\nabla - \mathcal{L}_Q = \sum_{k=0}^{\infty} (\tilde{R}_k \ast \text{id}).$$

(34)

Here we think of $R_k$ as an $\mathcal{R}$-linear map $R_k : \Gamma(S^k(T_\mathcal{M})) \to \Gamma(T_\mathcal{M})$, and $\tilde{R}_k : \Gamma(S(T_\mathcal{M})) \to \Gamma(S(T_\mathcal{M}))$ is then defined as in Diagram (33).

From Eqs. (18), (19) and (20), it follows that

$$R_0 = 0, \quad R_1 = 0, \quad \text{and} \quad R_2 = - \text{At}^\nabla_{(\mathcal{M}, Q)}.$$  (35)

Thus the conclusion follows immediately from taking the $\mathcal{R}$-dual of (34).

Finally, assume that $\nabla'$ is another torsion-free affine connection on $\mathcal{M}$. Let $\phi := (\text{pbw}^\nabla)^{-1} \circ \text{pbw}^\nabla$. From Propositions 3.2, 3.3 and Theorem 3.4, it follows that

$$\phi : (\Gamma(S(T_\mathcal{M})), \delta^\nabla) \xrightarrow{\sim} (\Gamma(S(T_\mathcal{M})), \delta^\nabla')$$

(36)

is an isomorphism of dg coalgebras over $(C^\infty(\mathcal{M}), Q)$. Dualizing $\phi$ over the dg algebra $(C^\infty(\mathcal{M}), Q)$, we obtain an isomorphism of dg algebras over $(C^\infty(\mathcal{M}), Q)$:

$$\phi^\nabla : (\Gamma(\tilde{S}(T_\mathcal{M})), D^\nabla) \xrightarrow{\sim} (\Gamma(\tilde{S}(T_\mathcal{M})), D^\nabla).$$

This concludes the proof of the theorem.

Indeed, following Kapranov [25], one may consider $(\Gamma(\tilde{S}(T_\mathcal{M})), D^\nabla)$ as the ‘dg algebra of functions’ on the ‘formal neighborhood’ of the diagonal $\Delta$ of the product dg manifold $(\mathcal{M} \times \mathcal{M}, (Q, Q))$: the PBW map $\text{pbw}^\nabla$ is, by construction, a formal exponential map identifying a ‘formal neighborhood’ of the zero section of $T_\mathcal{M}$ to a ‘formal neighborhood’ of the diagonal of $\mathcal{M} \times \mathcal{M}$ as $\mathbb{Z}$-graded manifolds and Eq. (28) asserts that $D^\nabla$ is the homological vector field obtained on $T_\mathcal{M}$ by pullback of the vector field $(Q, Q)$ on $\mathcal{M} \times \mathcal{M}$ through this formal exponential map. The reader is invited to compare Theorem 4.1 with [25, Theorem 2.8.2].

As an immediate consequence, we are ready to prove the main result of this section.
Theorem 4.4. Let $(\mathcal{M}, Q)$ be a dg manifold. Each choice of an affine connection $\nabla$ on $\mathcal{M}$ determines an $L_\infty[1]$ algebra structure on the space of vector fields $\mathfrak{X}(\mathcal{M})$. While the unary bracket $\lambda_1 : S^1(\mathfrak{X}(\mathcal{M})) \to \mathfrak{X}(\mathcal{M})$ is the Lie derivative $L_Q$ along the homological vector field, the higher multibrackets $\lambda_k : S^k(\mathfrak{X}(\mathcal{M})) \to \mathfrak{X}(\mathcal{M})$, with $k \geq 2$, arise as the composition

$$\lambda_k : S^k(\mathfrak{X}(\mathcal{M})) \to \Gamma(S^k(T\mathcal{M})) \xrightarrow{R_k} \mathfrak{X}(\mathcal{M})$$

induced by a family of sections $\{R_k\}_{k \geq 2}$ of the vector bundles $S^k(T_\mathcal{M}) \otimes T\mathcal{M}$ starting with $R_2 = -A_{(\nabla)}(\mathcal{M}, Q)$.

Furthermore, the $L_\infty[1]$ algebra structures on $\mathfrak{X}(\mathcal{M})$ arising from different choices of affine connections are all isomorphic.

For clarity, we point out that $S^k(\mathfrak{X}(\mathcal{M}))$ denotes the symmetric tensor product over the field $\mathbb{K}$ of $k$ copies of $\mathfrak{X}(\mathcal{M})$. While $\lambda_1$ is merely a $\mathbb{K}$-linear endomorphism of $\mathfrak{X}(\mathcal{M})$, we note that, for all $k \geq 2$, the multibracket $\lambda_k$ is $C^\infty(\mathcal{M})$-linear in each of its $k$ arguments.

**Proof.** The first part follows immediately from the fact that $(\Gamma(S(T\mathcal{M})), \delta^\nabla)$ is a dg coalgebra over $(C^\infty(\mathcal{M}), Q)$.

The uniqueness is a direct consequence of Theorem 4.1 as well. Indeed, it is easier to derive it using the dg coalgebra $(\Gamma(S(T\mathcal{M})), \delta^\nabla)$. If $\nabla'$ is another torsion-free affine connection on $\mathcal{M}$, we know that $\phi : (\Gamma(S(T\mathcal{M})), \delta^\nabla) \xrightarrow{\cong} (\Gamma(S(T\mathcal{M})), \delta^\nabla')$ as in (36) is an isomorphism of dg coalgebras over $(C^\infty(\mathcal{M}), Q)$. Thus it follows that the sequence of maps $\{\phi_k\}_{k \geq 1}$ defined by the composition

$$\phi_k : S^k(\mathfrak{X}(\mathcal{M})) \to \Gamma(S^k(T\mathcal{M})) \xrightarrow{\phi} \Gamma(S(T\mathcal{M})) \xrightarrow{pr_1} \Gamma(T\mathcal{M}) = \mathfrak{X}(\mathcal{M})$$

is an isomorphism of $L_\infty[1]$ algebras. Indeed, it is simple to see from (15) that the linear term $\phi_1$ is the identity map.

This $L_\infty[1]$ algebra structure on $\mathfrak{X}(\mathcal{M})$ is called the **Kapranov** $L_\infty[1]$ **algebra** of the dg manifold $(\mathcal{M}, Q)$.

### 4.2. Recursive formula for multibrackets.

It is clear that the Kapranov $L_\infty[1]$ algebra of a dg manifold in Theorem 4.4 is completely determined by the Atiyah 1-cocycle and

$$R_k \in \Gamma(S^k(T_\mathcal{M}) \otimes T\mathcal{M}) \cong \Gamma(\text{Hom}(S^k(T\mathcal{M}), T\mathcal{M}))$$

for $k \geq 3$.

Recall that, for the $L_\infty[1]$ algebra on the Dolbeault complex $\Omega^0,\bullet(T_X^{1,0})$ associated with the Atiyah class of the tangent bundle $T_X$ of a Kähler manifold $X$, Kapranov showed that the multibrackets can be described explicitly by a very simple formula: (4). For a general complex manifold, it was proved in [31] that they can be computed recursively as well. It is thus natural to ask if one can describe the multibrackets in Theorem 4.4 explicitly.

In what follows, we will give a characterization of these multibrackets, or equivalently all terms $R_k$, $k \geq 2$, by showing that they are completely determined, via a recursive formula, by the Atiyah cocycle $A_{(\nabla)}(\mathcal{M}, Q)$, the curvature $R^\nabla$, and their higher covariant derivatives.
We need to introduce some notations first.

By \( d^\nabla R_{n-1} \in \Gamma(S^n(T\mathcal{M}) \otimes T\mathcal{M}) \), we denote the symmetrized covariant derivative of \( R_{n-1} \). That is, for any \( X \in \Gamma(S^n(T\mathcal{M})) \),

\[
(d^\nabla R_{n-1})(X) = \sum_{k=1}^{n} \varepsilon_k (\nabla_{X_k} R_{n-1})(X^{[k]}) = \sum_{k=1}^{n} \varepsilon_k \left( (-1)^{|X_k|} \nabla_{X_k}(R_{n-1}(X^{[k]})) - R_{n-1}(\nabla_{X_k}X^{[k]}) \right),
\]

(37)

Here \( \varepsilon_k = (-1)^{|X_k|(|X_1|+\cdots+|X_{k-1}|)} \) is the Koszul sign.

Let \( B^\nabla : \Gamma(T\mathcal{M} \otimes S(T\mathcal{M})) \to \Gamma(S(T\mathcal{M})) \) be the map defined by

\[
B^\nabla(Y;X) = (\text{pbw}^\nabla)^{-1} \left( Y \cdot \text{pbw}^\nabla(X) \right) - \nabla_Y X,
\]

(38)

for all \( Y \in \mathcal{X}(\mathcal{M}) \) and \( X \in \Gamma(S^n(T\mathcal{M})) \). The following can be verified directly.

**Lemma 4.5.** The map \( B^\nabla \) is well defined and \( \mathcal{R} \)-linear. Hence it is indeed a bundle map \( B^\nabla : T\mathcal{M} \otimes S(T\mathcal{M}) \to S(T\mathcal{M}) \).

As we will see below, the map \( B^\nabla \) is completely determined by the curvature \( R^\nabla \) and its higher covariant derivatives.

Let

\[
\Gamma(\tilde{\mathcal{S}}(T\mathcal{M})) \otimes_\mathcal{R} \Gamma(S(T\mathcal{M})) \xrightarrow{(-1,-)} \mathcal{R}
\]

be the duality pairing defined by

\[
\langle \alpha_1 \odot \cdots \odot \alpha_q | X_1 \odot \cdots \odot X_p \rangle = \begin{cases} 
\sum_{\sigma \in S_p} \varepsilon(\sigma) \langle \alpha_1|X_{\sigma(1)}\rangle \cdots \langle \alpha_p|X_{\sigma(p)}\rangle & \text{if } p = q \\
0 & \text{if } p \neq q
\end{cases}
\]

for all homogeneous elements \( \alpha_1, \ldots, \alpha_q \in \Gamma(T\mathcal{M}) \) and \( X_1, \ldots, X_p \in \Gamma(T\mathcal{M}) \). The symbol \( \varepsilon(\alpha) = \varepsilon(\alpha_1, \alpha_2, \cdots, \alpha_p, X_1, X_2, \cdots, X_p) \) denotes the Koszul signs arising from the reordering of the homogeneous objects \( \alpha_1, \alpha_2, \cdots, \alpha_p, X_1, X_2, \cdots, X_p \) in each term of the right hand side.

The following is an immediate consequence of the Fedosov construction for graded manifolds [32, Theorem 5.6 and Proposition 5.2], which is briefly recalled in Appendix A.

**Lemma 4.6.** (i) The bundle map \( B^\nabla : T\mathcal{M} \otimes S(T\mathcal{M}) \to S(T\mathcal{M}) \) in Lemma 4.5 is completely determined by the curvature \( R^\nabla \) and its higher covariant derivatives. More precisely, given any \( Y \in \mathcal{X}(\mathcal{M}) \), provided that \( B^\nabla(Y;Y) \) is known for all \( Y \in \Gamma(S^{n-1}(T\mathcal{M})) \), one can compute \( B^\nabla(Y;X) \) for any \( X \in \Gamma(S^n(T\mathcal{M})) \).

(ii) Moreover, if \( R^\nabla = 0 \), then \( B^\nabla(Y;X) = Y \odot X \), for all \( Y \in \mathcal{X}(\mathcal{M}) \) and \( X \in \Gamma(S(T\mathcal{M})) \).
Proof. (i). Let \( \nabla^\sharp_Y X = (\text{pbw} \nabla)^{-1} (Y \cdot \text{pbw} \nabla (X)) \).

Then by Eq. (38),
\[
B^\nabla (Y; X) = \nabla^\sharp_Y X - \nabla_Y X.
\]

For the rest of the proof, we follow the notation from Appendix A and, in particular, Theorem A.4. For all \( \sigma \in \Gamma (\widehat{S}(T_M)) \), we have
\[
\left( \sigma \middle| \nabla^\sharp_Y X - \nabla_Y X \right) = (-1)^{|\sigma| \cdot |Y|} \left( \nabla_Y \sigma - \nabla^\sharp_Y \sigma \middle| X \right)
\]
\[
= (-1)^{|\sigma| \cdot |Y|} \left( i_Y (d^\nabla - d^\nabla^\sharp) (\sigma) \right) \left| X \right)
\]
\[
= (-1)^{|\sigma| \cdot |Y|} \left( i_Y (\delta - \widetilde{A}^\nabla) (\sigma) \right) \left| X \right)
\]
\[
= \left( \sigma \middle| Y \circ X \right) - (-1)^{|\sigma| \cdot |Y|} \left( i_Y \widetilde{A}^\nabla (\sigma) \right) \left| X \right)
\]
\[
= \left( \sigma \middle| Y \circ X \right) - \left( \sigma \middle| (i_Y \widetilde{A}^\nabla)^T X \right).
\]

Thus it follows that
\[
B^\nabla (Y; X) = Y \circ X - (i_Y \widetilde{A}^\nabla)^T X.
\]

The conclusion thus follows from Corollary A.7.

(ii) Moreover, if \( R^\nabla = 0 \), then \( A^\nabla = 0 \) by Eq. (65), and hence we obtain
\[
B^\nabla (Y; X) = Y \circ X.
\]

The proof is complete. \( \square \)

**Theorem 4.7.** (i) The sections \( R_n \in \Gamma (S^n (T_M) \otimes T_M) \), with \( n \geq 3 \), are completely determined by the Atiyah 1-cocycle \( \text{At}^\nabla (M, Q) \), the curvature \( R^\nabla \), and their higher covariant derivatives, through the recursive formula
\[
R_n = \frac{2}{n} (\bar{R}_2 \ast \text{id}) + \frac{1}{n} \sum_{k=2}^{n-1} \left[ (d^\nabla R_k \ast \text{id}) + (1 - k) (\bar{R}_k \ast \text{id}) - B^\nabla \circ (\bar{R}_k \otimes \text{id}) \circ \Delta \right].
\]

(ii) In particular, if \( R^\nabla = 0 \), then \( R_2 = -\text{At}^\nabla (M, Q) \) and \( R_n = \frac{1}{n} d^\nabla R_{n-1} \) for all \( n \geq 3 \).

In terms of Sweedler’s notation \( \Delta X = X_{(1)} \otimes X_{(2)} \), one can rewrite Eq. (39) as follows:
\[
R_n (X) = \frac{1}{n} \sum_{k=2}^{n-1} \left[ (d^\nabla R_k (X_{(1)}) \otimes X_{(2)}) + (1 - k) (R_k (X_{(1)}) \otimes X_{(2)}) - B^\nabla (R_k (X_{(1)}); X_{(2)}) \right]
\]
\[
+ \frac{2}{n} (R_2 (X_{(1)}) \otimes X_{(2)})
\]

Now we proceed to prove Theorem 4.7. For any \( X \in \Gamma (S^n (T_M)) \), we can write
\[ C^\nabla (X) = \text{pbw}^\nabla \circ \left( (\text{pbw}^\nabla)^{-1} \circ L_Q^D \circ \text{pbw}^\nabla - L_Q \right)(X) \]

\[ = \text{pbw}^\nabla \left( \sum_{k=0}^{n} (\bar{R}_k \star \text{id})(X) \right) \quad \text{by Eq. (34)} \]

\[ = \sum_{k=2}^{n} \text{pbw}^\nabla \circ (\bar{R}_k \star \text{id})(X) \quad \text{by Eqs. (35).} \quad (40) \]

In order to simplify the notation, we introduce a sequence of maps, for \( k \geq 2 \), \( B_k^\nabla : \Gamma \left( S(TM) \right) \rightarrow \Gamma \left( S(TM) \right) \), defined by

\[ B_k^\nabla (X) = B^\nabla \circ (\bar{R}_k \otimes \text{id}) \circ \Delta (X), \quad \forall X \in \Gamma \left( S^n (TM) \right). \]

Explicitly, in terms of Sweedler’s notation \( \Delta X = X_1 \otimes X_2 \), we write

\[
B_k^\nabla (X) = B^\nabla (R_k(X_1); X_2)) \\
= (\text{pbw}^\nabla)^{-1} \left( R_k(X_1) \cdot \text{pbw}^\nabla (X_2)) - \nabla R_k(X_1)X_2 \right). \quad (41)
\]

From Lemma 4.5, it follows that \( B_k^\nabla \), with \( k \geq 2 \), is \( \mathcal{R} \)-linear. That is, \( B_k^\nabla \), with \( k \geq 2 \), is indeed a bundle map \( S(TM) \rightarrow S(TM) \).

**Proof of Theorem 4.7.** (i) First, we will prove the recursive formula (39).

Pick any element \( X = X_1 \otimes \cdots \otimes X_n \in \Gamma \left( S^n (TM) \right) \). Again, for the sake of simplicity, we use Sweedler’s notation \( \Delta X = X_1 \otimes X_2 \) and the Koszul sign \( \varepsilon_k = (-1)^{|X_k||X_1|+\cdots+|X_{k-1}|} \). For each \( l \), by Eqs. (16) and (31), we have

\[
(n - l + 1) \text{pbw}^\nabla \circ (\bar{R}_l \star \text{id})(X) \\
= (n - l + 1) \text{pbw}^\nabla (R_l(X_1) \otimes X_2) \\
= R_l(X_1) \cdot \text{pbw}^\nabla (X_2) - \text{pbw}^\nabla \left( \nabla R_l(X_1)X_2 \right) \\
+ \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|} \left( X_k \cdot \text{pbw}^\nabla \left( R_l(X_1^{(1)} \otimes X_2^{(2)}) - \text{pbw}^\nabla \left( \nabla X_2 \left( R_l(X_1^{(1)} \otimes X_2^{(2)}) \right) \right) \right) \right) \\
= R_l(X_1) \cdot \text{pbw}^\nabla (X_2) - \text{pbw}^\nabla \left( \nabla R_l(X_1)X_2 \right) \\
+ \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|} \left[ X_k \cdot \text{pbw}^\nabla \circ (\bar{R}_l \star \text{id})(X_1^{(k)}) - \text{pbw}^\nabla \left( \nabla X_2 \left( (\bar{R}_l \star \text{id})(X_1^{(k)}) \right) \right) \right].
\]

Combining it with Eq. (41), we conclude that

\[
(n - l + 1) \text{pbw}^\nabla \circ (\bar{R}_l \star \text{id})(X) - \text{pbw}^\nabla \circ B_l^\nabla (X) \\
= \sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|} \left[ X_k \cdot \text{pbw}^\nabla \circ (\bar{R}_l \star \text{id})(X_1^{(k)}) - \text{pbw}^\nabla \left( \nabla X_2 \left( (\bar{R}_l \star \text{id})(X_1^{(k)}) \right) \right) \right].
\]

Therefore,
\[(n - l + 1)(\tilde{R}_l \star \text{id})(X) - B^\nabla_l(X)\]

\[
\sum_{k=1}^{n} \varepsilon_k (-1)^{|X_k|} \left[ (\text{pbw} \nabla)^{-1} \left( X_k \cdot \text{pbw} \nabla \circ (\tilde{R}_l \star \text{id})(X^{[k]}) \right) - \nabla X_k \left( (\tilde{R}_l \star \text{id})(X^{[k]}) \right) \right].
\]

(42)

Also, for each \(l\), by Eq. (37), we have

\[
(\text{d}^\nabla \tilde{R}_l \star \text{id})(X) = \sum_{k=1}^{n} \varepsilon_k \left[ (\text{d}^\nabla \tilde{R}_l)(X_k; X^{[k]}_1 \odot X^{[k]}_2) \right]
\]

\[
= \sum_{k=1}^{n} \varepsilon_k \left[ (-1)^{|X_k|} \left( \nabla X_k \left( \tilde{R}_l(X^{[k]}_1) \odot X^{[k]}_2 \right) \right) \right] - \left( \tilde{R}_l \left( \nabla X_k X^{[k]}_2 \right) \right] \right]
\]

\[
- \sum_{k=1}^{n} \varepsilon_k \left[ (\tilde{R}_l \left( \nabla X_k X^{[k]}_1 \right) \odot X^{[k]}_2) + (-1)^{|X_k|} \left( \tilde{R}_l (X^{[k]}_1) \odot (\nabla X_k X^{[k]}_2) \right) \right]
\]

\[
= \sum_{k=1}^{n} \varepsilon_k \left[ (-1)^{|X_k|} \nabla X_k \left( (\tilde{R}_l \star \text{id})(X^{[k]}) \right) - (\tilde{R}_l \star \text{id})(\nabla X_k X^{[k]}_2) \right].
\]

According to (35), we have \(R_2 = -\text{At}^\nabla_{\{\mathcal{M}, Q\}}\). Hence

\[
\text{pbw} \nabla \circ (\tilde{R}_2 \star \text{id})(X) = - \sum_{i < j} \varepsilon_i \varepsilon_j (-1)^{|X_i| - |X_j|} \left( \text{pbw} \nabla \left( \text{At}^\nabla_{\{\mathcal{M}, Q\}}(X_i, X_j) \odot X^{[i,j]} \right) \right).
\]

(43)

By Eqs. (21) and (43), we have

\[
C^\nabla(X) - \frac{2}{n} \left( \text{pbw} \nabla \circ (\tilde{R}_2 \star \text{id})(X) \right)
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \left[ (-1)^{|X_k|} X_k \cdot C^\nabla(X^{[k]}) - C^\nabla(\nabla X_k X^{[k]}) \right]
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=2}^{n-1} \varepsilon_k \left[ (-1)^{|X_k|} X_k \cdot \text{pbw} \nabla \circ (\tilde{R}_l \star \text{id})(X^{[k]}) - \text{pbw} \nabla \circ (\tilde{R}_l \star \text{id})(\nabla X_k X^{[k]}) \right]
\]

\[
= \frac{1}{n} \sum_{k=1}^{n} \sum_{l=2}^{n-1} \varepsilon_k \left[ (-1)^{|X_k|} X_k \cdot \text{pbw} \nabla \circ (\tilde{R}_l \star \text{id})(X^{[k]}) - \text{pbw} \nabla \left( \left( \tilde{R}_l \star \text{id} \right)(X^{[k]}) \right) \right]
\]

\[
+ \frac{1}{n} \sum_{k=1}^{n} \sum_{l=2}^{n-1} \varepsilon_k \left[ (-1)^{|X_k|} \left( \text{pbw} \nabla \left( \left( \tilde{R}_l \star \text{id} \right)(X^{[k]}) \right) \right) - \text{pbw} \circ (\tilde{R}_l \star \text{id} \right) \left( \nabla X_k X^{[k]} \right) \right],
\]

where the second equality is obtained by applying Eq. (40) to \(C^\nabla(X^{[k]})\) and \(C^\nabla(\nabla X_k X^{[k]})\). It thus follows that

\[
(\text{pbw} \nabla)^{-1} \circ C^\nabla(X) = \frac{2}{n} \left( \tilde{R}_2 \star \text{id} \right)(X) = \alpha + \beta,
\]

(44)
where

\[ \alpha = \frac{1}{n} \sum_{k=1}^{n-1} \sum_{l=2}^{n-1} \epsilon_k (-1)^{|X_k|} \left[ (\text{pbw}^\nabla)^{-1} \left( X_k \cdot \text{pbw}^\nabla \circ (\tilde{R}_l \ast \text{id})(X^{(k)}) \right) - \nabla X_k \left( (\tilde{R}_l \ast \text{id})(X^{(k)}) \right) \right], \]

and

\[ \beta = \frac{1}{n} \sum_{k=1}^{n-1} \sum_{l=2}^{n-1} \epsilon_k \left[ (-1)^{|X_k|} \left( \nabla X_k \left( (\tilde{R}_l \ast \text{id})(X^{(k)}) \right) \right) - (\tilde{R}_l \ast \text{id})(\nabla X_k X^{(k)}) \right] \]

\[ = \sum_{l=2}^{n-1} \frac{1}{n} \left( d^\nabla R_l \ast \text{id} \right)(X). \quad (45) \]

Now, according to (42),

\[ \alpha - \sum_{l=2}^{n-1} (\tilde{R}_l \ast \text{id})(X) = \sum_{l=2}^{n-1} \frac{1}{n} \left( (n - l + 1)(\tilde{R}_l \ast \text{id})(X) - B_l^\nabla(X) \right) - \sum_{l=2}^{n-1} (\tilde{R}_l \ast \text{id})(X) \]

\[ = \frac{1}{n} \sum_{l=2}^{n-1} \left[ (1 - l) \left( (\tilde{R}_l \ast \text{id})(X) \right) - B_l^\nabla(X) \right]. \quad (46) \]

Equation (40) can be rewritten as

\[ R_n(X) = (\text{pbw}^\nabla)^{-1} \circ C^\nabla(X) - \sum_{k=2}^{n-1} (\tilde{R}_k \ast \text{id})(X). \]

Equations (44), (45) and (46) then yield Eq. (39).

From (35), we know that \( R_2 = -A_{t_{(\mathcal{M}, Q)}} \). According to Lemma 4.6, the bundle map \( B^\nabla \) is completely determined by the curvature \( R^\nabla \) and its higher covariant derivatives. It thus follows from the recursive formula (39) that, for any \( n \geq 3 \), \( R_n \) is determined by the \( R_k \) with \( k \leq n - 1 \), the curvature and their covariant derivatives. Thus, by inductive argument, \( R_n \) is completely determined by the Atiyah 1-cocycle, the curvature and their higher covariant derivatives.

(ii) Assume that \( R^\nabla = 0 \). By Lemma 4.6, the bundle map \( B^\nabla : T_{\mathcal{M}} \otimes S(T_{\mathcal{M}}) \rightarrow S(T_{\mathcal{M}}) \) admits the simple description \( B^\nabla(Y; X) = Y \odot X \). Thus the formula \( R_n(X) = \frac{1}{n} d^\nabla R_{n-1}(X) \) can be obtained by an induction argument, again using the recursive formula (39).

This concludes the proof of the theorem. \( \square \)

5. Examples

This section is devoted to the study of the Kapranov \( L_\infty[1] \) algebras arising from some standard classes of dg manifolds including those corresponding to \( L_\infty[1] \) algebras, foliations and complex manifolds as described in Examples 2.2 and 2.3.
5.1. dg manifolds associated with $L_\infty[1]$ algebras. Let $\mathfrak{g}$ be a finite-dimensional $L_\infty$ algebra with $d = \dim \mathfrak{g}$. Then $\mathfrak{g}[1]$ is an $L_\infty[1]$ algebra: the (canonical) symmetric coalgebra $(S(\mathfrak{g}[1]), \Delta)$ is equipped with a coderivation $\tilde{Q} \in \text{coDer}(S(\mathfrak{g}[1]))$ of degree +1 satisfying $\tilde{Q} \circ \tilde{Q} = 0$ and $\tilde{Q}(1) = 0$. Indeed, $\tilde{Q}$ is equivalent to a sequence of linear maps $q_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1], k \geq 1$, of degree +1 satisfying the generalized Jacobi identities. The map $q_k$ is called the $k$-th multibracket.

Given an $L_\infty[1]$ algebra $\mathfrak{g}[1]$, we say a vector space $\mathcal{M}$ is a $\mathfrak{g}[1]$-module if there exists a sequence of maps $\rho_k : S^k(\mathfrak{g}[1]) \otimes \mathcal{M} \to \mathcal{M}, k \geq 0$, of degree +1, satisfying the standard compatibility condition [29]. If we write

$$\rho = \sum_{k \geq 0} \rho_k : S(\mathfrak{g}[1]) \otimes \mathcal{M} \to \mathcal{M},$$

the compatibility condition is expressed explicitly as

$$\rho \circ (\text{id}_{S(\mathfrak{g}[1])} \otimes \rho) \circ (\Delta \otimes \text{id}_{\mathcal{M}}) + \tilde{Q} \otimes \text{id}_{\mathcal{M}} = 0.$$

As an obvious example, we have the trivial module $\mathcal{M} = \mathbb{K}$ together with the trivial action $\rho_k = 0$ for all $k \geq 0$. Another example is the adjoint module $\mathcal{M} = \mathfrak{g}[1]$ with the adjoint action $\rho_k : S^k(\mathfrak{g}[1]) \otimes \mathfrak{g}[1] \to \mathfrak{g}[1]$ defined by

$$\rho_k(X \otimes X) = q_{k+1}(X \otimes X),$$

where $X \in S^k(\mathfrak{g}[1]), X \in \mathfrak{g}[1]$ and $q_{k+1} : S^{k+1}(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ is the multibracket of the $L_\infty[1]$ algebra $\mathfrak{g}[1]$. That is, $\{\rho_k\}_{k \geq 0}$ is defined by the following commutative diagram

\[
\begin{array}{ccc}
S^k(\mathfrak{g}[1]) \otimes \mathfrak{g}[1] & \xrightarrow{\rho_k} & \mathfrak{g}[1] \\
\text{sym} & & \\
& S^{k+1}(\mathfrak{g}[1]) & \\
\end{array}
\]

where sym : $S^* (\mathfrak{g}[1]) \otimes \mathfrak{g}[1] \to S^{*+1}(\mathfrak{g}[1])$ is the canonical symmetrization map. Dualizing, we obtain a $\mathfrak{g}[1]$-module structure on $(\mathfrak{g}[1])^\vee$, whose action map is called coadjoint action.

Throughout this section, we denote the degree of a homogeneous element $x \in \mathfrak{g}[1]$ by $|x|$. In particular, if $\mathfrak{g}$ is a Lie algebra concentrated in degree 0, then for any $x \in \mathfrak{g}[1]$, its degree is $|x| = -1$.

The Chevalley–Eilenberg cochain complex associated with a $\mathfrak{g}[1]$-module $\mathcal{M}$ is

$$C(\mathfrak{g}[1]; \mathcal{M}) = \left( \text{Hom} \left( S(\mathfrak{g}[1]), \mathcal{M} \right), d_{CE}^{\mathcal{M}} \right),$$

where $d_{CE}^{\mathcal{M}}$ is defined by

$$d_{CE}^{\mathcal{M}}(F) = \rho \circ (\text{id} \otimes F) \circ \Delta - (-1)^{|F|} F \circ \tilde{Q},$$

for any homogeneous element $F \in \text{Hom} \left( S(\mathfrak{g}[1]), \mathcal{M} \right)$.

Observe that when $\mathcal{M}$ is the trivial module $\mathbb{K}$, the associated Chevalley–Eilenberg cochain complex

$$C(\mathfrak{g}[1]; \mathbb{K}) = \left( \text{Hom} \left( S(\mathfrak{g}[1]), \mathbb{K} \right), d_{CE}^{\mathbb{K}} = d_{CE} \right)$$
is a dg algebra, equipped with the multiplication
\[ f \odot g = \mu_\otimes (f \otimes g) \circ \Delta : S(g[1]) \to \mathbb{K} \] (48)
for any \( f, g \in \text{Hom}(S(g[1]), \mathbb{K}) \). In other words, the dg algebra \((C^\infty(g[1]), Q)\) coincides with the Chevalley–Eilenberg cochain complex \((C(g[1]; \mathbb{K}), d_{CE})\) of the trivial \(g[1]\)-module \(\mathbb{K}\). That is, \((C(g[1]; \mathbb{K}), d_{CE})\) is the dg algebra dual to the dg coalgebra \((S(g[1]), \widehat{\Delta})\). Moreover, for any \(g[1]\)-module \(\mathfrak{M}\), the Chevalley–Eilenberg cochain complex \((C(g[1]; \mathfrak{M}), d_{\mathfrak{M}}^{\text{triv}})\) is a dg module over the dg algebra \((C^\infty(g[1]), Q)\), where the action, under the identification \(\mu_0 : \mathbb{K} \otimes \mathfrak{M} \cong \mathfrak{M}\), is given by
\[ f \cdot F = \mu_0 \circ (f \otimes F) \circ \Delta : S(g[1]) \to \mathfrak{M} \] (49)
for any \( f \in \text{Hom}(S(g[1]), \mathbb{K}) \) and \( F \in \text{Hom}(S(g[1]), \mathfrak{M}) \). In particular, this means that it satisfies the compatibility condition
\[ d_{\mathfrak{M}}^{\text{triv}}(f \cdot F) = d_{CE}(f) \cdot F + (-1)^{|f|} f \cdot d_{\mathfrak{M}}^{\text{triv}}(F). \] (50)
Therefore, the Chevalley–Eilenberg differential \(d_{\mathfrak{M}}^{\text{triv}}\) is completely determined by its restriction to the subspace \(\mathfrak{M} = \text{Hom}(S^0(g[1]), \mathfrak{M})\) of \(C(g[1]; \mathfrak{M})\), which is essentially induced by the action (47). More precisely, for any \(m \in \mathfrak{M}\),
\[ d_{\mathfrak{M}}^{\text{triv}}(m) = \sum_{k \geq 0} \rho_k (-, m) \in \text{Hom}(S(g[1]), \mathfrak{M}). \]

In particular, if \(\mathfrak{M} = g[1]\) is the adjoint module of the finite-dimensional \(L_\infty[1]\) algebra \(g[1]\) described above, the Chevalley–Eilenberg differential \(d_{CE}^{g[1]}\) (seen as an operator on \(S(g[1])^* \otimes g[1]\)) is determined by the relation
\[ d_{CE}^{g[1]}(m) = \sum_{k=1}^\infty \frac{1}{(k-1)!} x^{i_k-1} \odot \cdots \odot x^{i_1} \otimes q_k(e_{i_1} \odot \cdots \odot e_{i_k-1} \odot m), \quad \forall m \in g[1]. \] (51)
where \(\{e_1, \ldots, e_d\}\) is a basis for \(g[1]\) and \(\{x^1, \ldots, x^d\}\) is the dual basis for \((g[1])^*\). In Eq. (51) and in the remainder of the present section, we use the Einstein notation tacitly to avoid inserting summations over the indices \(i_1, \ldots, i_{k-1}\) in many equations.

**Remark 5.1.** In terms of Sweedler’s notation, we may write (48) as
\[ (f \odot g)(X) = \sum_{(X)} (-1)^{|g||X_{(1)}|} f(X_{(1)})g(X_{(2)}) \]
and (49) as
\[ (f \cdot F)(X) = \sum_{(X)} (-1)^{|F||X_{(1)}|} f(X_{(1)})F(X_{(2)}), \]
where \(f, g \in \text{Hom}(S(g[1]), \mathbb{K}), F \in \text{Hom}(S(g[1]), \mathfrak{M}), X \in S(g[1])\) are homogeneous elements and \(\Delta X = \sum_{(X)} X_{(1)} \otimes X_{(2)}\).
We now proceed to describe the Kapranov $L_{\infty}[1]$ algebra of the dg manifold $(g[1], d_{CE})$. Recall that $Q = d_{CE}$ is defined by
\[ Q(f) = d_{CE}(f) = -(-1)^{|f|} f \circ \tilde{Q} \] (52)
for any homogeneous element $f \in \text{Hom}(S(g[1]), \mathbb{K}) \cong C^\infty(g[1])$.

Let \( \{ e_1, \ldots, e_d \} \) be a basis of $g[1]$ and let \( \{ x^1, \ldots, x^d \} \) be the induced linear coordinate functions on $g[1]$ satisfying
\[ x^i(e_j) = \left\langle x^i \right| e_j \right\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \]

We also use the notation
\[ \frac{\partial}{\partial x^j} x^i := (-1)^{|x^i|} \left| x^j \right| \left\langle x^i \right| e_j \right\rangle. \] (53)

**Lemma 5.2.** Under the above notation, write the multibrackets as
\[ q_k(e_{i_1}, \ldots, e_{i_k}) = \sum_j c^j_{i_1 \ldots i_k} e_j, \quad \forall k \geq 1. \]

Then the homological vector field $Q \in \mathfrak{X}(g[1])$ can be written as
\[ Q = -\sum_j \left[ \sum^\infty_{k=1} \frac{1}{k!} c^j_{i_1 \ldots i_k} x^{i_k} \otimes \cdots \otimes x^{i_1} \frac{\partial}{\partial x^j} \right]. \]

Here, we are making tacit use of the Einstein summation convention for the indices $i_1, \ldots, i_k$.

**Proof.** As a vector field, $Q$ can be written as $Q = \sum_j Q^j \frac{\partial}{\partial x^j}$ for some $Q^j \in C^\infty(g[1])$. Then, as a derivation of $C^\infty(g[1])$, $Q$ satisfies $Q(x^j) = (-1)^{|x^j|} Q^j$ according to (53). On the other hand, according to (52), we have
\[ \left\langle Q(x^j) e_{i_1} \otimes \cdots \otimes e_{i_k} \right\rangle = -(-1)^{|x^j|} \left\langle x^j \right| \tilde{Q}(e_{i_1} \otimes \cdots \otimes e_{i_k}) \right\rangle = -(-1)^{|x^j|} c^j_{i_1 \ldots i_k} \]
for any $k \geq 1$.

Therefore, we may conclude that
\[ Q^j = -\sum^\infty_{k=1} \frac{1}{k!} c^j_{i_1 \ldots i_k} x^{i_k} \otimes \cdots \otimes x^{i_1}. \]
This completes the proof. \[ \square \]

Note that we have a canonical trivialization of the tangent bundle
\[ T_{g[1]} \cong g[1] \times g[1]. \] (54)

Hence, we have the following identification
\[ C^\infty(g[1]) \otimes g[1] \leftrightarrow \mathfrak{X}(g[1]) \rightarrow \text{Hom}(S(g[1]), g[1]) \]
\[ f \otimes e_i \leftrightarrow f \frac{\partial}{\partial x^i} \mapsto \left( X \mapsto (-1)^{|e_i|} \left| X \right| (f|X) \cdot e_i \right), \] (55)
where $f \in \text{Hom}(S(g[1]), \mathbb{K}) \cong C^\infty(g[1])$ is homogeneous and $X \in S(g[1])$. 
Lemma 5.3. **Under the identification** \( (\ref{eq:identification}) \), the Lie derivative \( \mathcal{L}_Q = [Q, -] \in \text{End}(\mathcal{X}(\mathfrak{g}[1])) \) **corresponds to the Chevalley–Eilenberg differential** \( d_{\text{CE}}^{[1]} \) (for the adjoint action of \( \mathfrak{g}[1] \) on itself) seen as an operator on \( \text{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}[1]) \).

**Proof.** Recall that the Chevalley–Eilenberg differential \( d_{\text{CE}}^{[1]} \) on \( \mathfrak{g}[1] \) satisfies \( (\ref{eq:lie_algebra_hom}) \). On the other hand, we have

\[
\mathcal{L}_Q(f \cdot F) = [Q, f \cdot F] = Q(f) \cdot F + (-1)^{|f|} f \cdot [Q, F] = Q(f) \cdot F + (-1)^{|f|} f \cdot \mathcal{L}_Q(F),
\]

for any homogeneous element \( f \in C^\infty(\mathfrak{g}[1]) \cong \text{Hom}(S(\mathfrak{g}[1]), \mathbb{K}) \) and \( F \in \mathcal{X}(\mathfrak{g}[1]) \cong \text{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}[1]) \). Since \( Q(f) = d_{\text{CE}}^{[1]}(f) \) according to Eq. \( (\ref{eq:lie_algebra_hom}) \), it suffices to prove the claim for each \( \frac{\partial}{\partial x^i}, i = 1, \ldots, d \).

We keep the notation \( Q = \sum_j Q^j \frac{\partial}{\partial x^j} \). Now, by Lemma 5.2, we have

\[
\mathcal{L}_Q \left( \frac{\partial}{\partial x^j} \right) = -(-1)^{|\frac{\partial}{\partial x^j}|} \sum_j \frac{\partial}{\partial x^j}(Q^j) \frac{\partial}{\partial x^j} = (-1) \left| \frac{\partial}{\partial x^j} \right| \sum_j \frac{\partial}{\partial x^j}(Q^j) \frac{\partial}{\partial x^j} = (-1) \left| \frac{\partial}{\partial x^j} \right| \sum_{i} \sum_{k=1}^{\infty} \frac{1}{k!} c^i_{1 \cdots k} \frac{\partial}{\partial x^i} \sum_{j} \frac{\partial}{\partial x^j} \frac{1}{(k-1)!} c^j_{1 \cdots k-1} \frac{\partial}{\partial x^j} x^k \cdots x^1 \frac{\partial}{\partial x^j}.
\]

The conclusion thus follows immediately by comparing the equation above with \( (\ref{eq:lie_algebra_hom}) \). 

The trivialization of the tangent bundle \( (\ref{eq:tangent_bundle}) \) induces an isomorphism

\[
T_{\mathfrak{g}[1]}^\vee \otimes \text{End}(T_{\mathfrak{g}[1]}) \cong \mathfrak{g}[1] \times (\mathfrak{g}[1])^\vee \otimes (\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1]
\]

of vector bundles. Lemma 5.3, comparing with \( (\ref{eq:vector_bundles}) \), indicates that we have an isomorphism of cochain complexes:

\[
(\Gamma(\mathfrak{g}[1]; T_{\mathfrak{g}[1]}^\vee \otimes \text{End}(T_{\mathfrak{g}[1]}))^\bullet, Q) \cong (\text{Hom}^\bullet(S(\mathfrak{g}[1]), \mathfrak{M}), d_{\text{CE}}^{[1]}),
\]

where \( \mathfrak{M} = (\mathfrak{g}[1])^\vee \otimes (\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1] \) is the tensor product of adjoint and coadjoint modules.

Thus we have the following

**Corollary 5.4.** Let \( (\mathfrak{M}, Q) = (\mathfrak{g}[1], d_{\text{CE}}) \) be the dg manifold corresponding to a finite-dimensional \( L^\infty[1] \) algebra \( \mathfrak{g}[1] \). There is a canonical isomorphism, for any \( k \in \mathbb{Z} \),

\[
H^k(\Gamma(T_{\mathfrak{g}[1]}^\vee \otimes \text{End}(T_{\mathfrak{g}[1]}))^\bullet, Q) \cong H^k_{\text{CE}}(\mathfrak{g}[1], (\mathfrak{g}[1])^\vee \otimes (\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1])
\]

where the right hand side stands for the Chevalley–Eilenberg cohomology of the \( L^\infty[1] \) algebra \( \mathfrak{g}[1] \) with values in \( (\mathfrak{g}[1])^\vee \otimes (\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1] \).
Remark 5.5. It is sometimes useful to use the Chevalley–Eilenberg cohomology of $L_\infty$ algebras rather than $L_\infty[1]$ algebras. Then Corollary 5.4 can be rephrased as follows.

For any finite-dimensional $L_\infty$ algebra $\mathfrak{g}$, there is a canonical isomorphism, for any $k \in \mathbb{Z}$,

$$H^k\left(\Gamma(T^\vee_{\mathfrak{g}[1]} \otimes \text{End}(T_{\mathfrak{g}[1]}))^\bullet, \mathcal{Q}\right) \cong H^k_{\text{CE}}(\mathfrak{g}, \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g}).$$

where the right hand side stands for the Chevalley–Eilenberg cohomology of the $L_\infty$ algebra $\mathfrak{g}$ with values in $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g}$. Note that there is a degree shift here.

We still keep the notation $d_{\text{CE}} = \mathcal{Q} = \sum_i Q^i \frac{\partial}{\partial x^i}$. Let $\nabla : \mathfrak{g}(1) \otimes \mathfrak{g}(1) \to \mathfrak{g}(1)$ be the trivial (torsion-free) connection: $\nabla \frac{\partial}{\partial x^i} = 0$. The corresponding Atiyah 1-cocycle $\text{At}_{(\mathfrak{g}[1], d_{\text{CE}})}^\nabla \in \Gamma(\text{Hom}(S^2(T_{\mathfrak{g}[1]}), T_{\mathfrak{g}[1]}))$ is completely determined by the relations

$$\text{At}_{(\mathfrak{g}[1], d_{\text{CE}})}^\nabla \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = -(-1)^{|x^i||x^j|} \nabla \frac{\partial}{\partial x^i} \left( L_{\mathcal{Q}} \left( \frac{\partial}{\partial x^j} \right) \right)$$

$$= \sum_l (-1)^{|x^i||x^j|} \frac{\partial^2}{\partial x^i \partial x^j} (Q^l) \frac{\partial}{\partial x^l}$$

$$= \sum_l (-1)^{|x^i||x^j|} \frac{\partial^2}{\partial x^i \partial x^j} \left( -\sum_{k=1}^l c_{i_1 \cdots i_k} x^{i_k} \otimes \cdots \otimes x^{i_l} \right) \frac{\partial}{\partial x^l}$$

$$= -\sum_l \sum_{k=2}^\infty \frac{1}{(l-2)!} c_{i_1 \cdots i_{k-2} i j} x^{i_1} \otimes \cdots \otimes x^{i_{k-2}} \otimes x^{i_j} \frac{\partial}{\partial x^l}, \quad (56)$$

for all $i, j \in \{1, \ldots, d\}$.

Let $\text{At}_{(\mathfrak{g}[1], d_{\text{CE}})}^\nabla$ be the map defined by the following commutative diagram

$$\begin{aligned}
C^\infty(\mathfrak{g}[1]) \otimes S^2(\mathfrak{g}[1]) &\xrightarrow{\sim} \Gamma(S^2(T_{\mathfrak{g}[1]})) \\
&\xrightarrow{\text{At}_{(\mathfrak{g}[1], d_{\text{CE}})}^\nabla} \Gamma(\text{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}[1])) \\
&\xrightarrow{\sim} \text{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}[1]).
\end{aligned}$$

Equation (57) implies that

$$\text{At}_{(\mathfrak{g}[1], d_{\text{CE}})}^\nabla (e_i, e_j) : \ e_{l_1} \otimes \cdots \otimes e_{l_k} \mapsto -q_{k+2}(e_i \otimes e_j \otimes e_{l_1} \otimes \cdots \otimes e_{l_k}).$$

Therefore, under the identification above, we have

$$\text{At}_{(\mathfrak{g}[1], d_{\text{CE}})}^\nabla (x, y) : \ X \mapsto -q_{n+2}(x \otimes y \otimes X),$$

for any $x, y \in \mathfrak{g}[1]$ and $X \in S^n(\mathfrak{g}[1])$. Thus, by abuse of notation, we may write

$$\text{At}_{(\mathfrak{g}[1], d_{\text{CE}})}^\nabla = -\sum_{k \geq 2} q_k.$$
Proposition 5.6. Let $g[1]$ be an $L_{\infty}[1]$ algebra with multibrackets $q_k : S^k(g[1]) \to g[1]$, $k \geq 1$. Then the Atiyah class $\alpha_{(g[1], d_{CE})}$ of the dg manifold $(g[1], d_{CE})$ is

$$\alpha_{(g[1], d_{CE})} = [-\sum_{k \geq 2} q_k] \in H^1_{CE}(g[1], (g[1])^\vee \otimes (g[1])^\vee \otimes g[1]) \cong H^1(\Gamma(T_{g[1]}^\vee \otimes \text{End}(T_{g[1]}))^\bullet, \mathbb{Q}).$$

Remark 5.7. We can rephrase Proposition 5.6 in terms of multibrackets of $L_{\infty}$ algebra $g$ instead of $L_{\infty}[1]$ algebra $g[1]$. For a finite-dimensional $L_{\infty}$ algebra $g$ equipped with multibrackets $l_k : \Lambda^k g \to g$ of degree $2 - k$ for $k \geq 1$, the Atiyah class $\alpha_{(g[1], d_{CE})}$ of the dg manifold $(g[1], d_{CE})$ is

$$\alpha_{(g[1], d_{CE})} = [\sum_{k \geq 2} l_k] \in H^0_{CE}(g, g^\vee \otimes g^\vee \otimes g) \cong H^1(\Gamma(T_{g[1]}^\vee \otimes \text{End}(T_{g[1]}))^\bullet, \mathbb{Q}),$$

where $H^0_{CE}(g, g^\vee \otimes g^\vee \otimes g)$ denotes the 0-th Chevalley–Eilenberg cohomology of the $L_{\infty}$ algebra $g$ with values in the tensor product of adjoint and coadjoint modules $g^\vee \otimes g^\vee \otimes g$.

Since the trivial connection $\nabla$ is flat, by the second part of Theorem 4.7, we know that

$$R_n = \frac{1}{n} d_n^{-\nabla} R_{n-1} \in \Gamma(\text{Hom}(S^n(T_{g[1]}), T_{g[1]}))$$

for $n \geq 3$. As the connection $\nabla$ is trivial, Eq. (37) implies that

$$d_n^{-\nabla} R_{n-1} \left( \frac{\partial}{\partial x^{i_1}} \odot \cdots \odot \frac{\partial}{\partial x^{i_n}} \right) = \sum_{k=1}^{n} \varepsilon_k (-1)^{|x^k|} \left| \nabla_{\frac{\partial}{\partial x^{i_k}}} \left( R_{n-1} \left( \frac{\partial}{\partial x^{i_1}} \odot \cdots \odot \frac{\partial}{\partial x^{i_{k-1}}} \odot \frac{\partial}{\partial x^{i_n}} \right) \right) \right|$$

Here, $\varepsilon_k = (-1)^{|x^k|} \left| (|x^1| \cdots |x^{k-1}|) \right|$ is the Koszul sign. Starting from

$$R_2 \left( \frac{\partial}{\partial x^{i_1}} \odot \frac{\partial}{\partial x^{i_2}} \right) = -(-1)^{|x^{i_1}| + |x^{i_2}|} \sum_j \frac{\partial^2 Q^j}{\partial x^{i_1} \partial x^{i_2}} \frac{\partial}{\partial x^j},$$

as in (56), we inductively obtain that

$$R_n \left( \frac{\partial}{\partial x^{i_1}} \odot \cdots \odot \frac{\partial}{\partial x^{i_n}} \right) = -(-1)^{|x^{i_1}| + \cdots + |x^{i_n}|} \sum_j \frac{\partial^n Q^j}{\partial x^{i_1} \cdots \partial x^{i_n}} \frac{\partial}{\partial x^j}.$$

According to Corollary 4.4, we obtain the following

**Proposition 5.8.** Let $g[1]$ be a finite-dimensional $L_{\infty}[1]$ algebra with multibrackets $q_k : S^k(g[1]) \to g[1]$, $k \geq 1$. Let $(\mathcal{M}, Q) = (g[1], d_{CE})$ be its corresponding dg manifold. Choose the trivial connection. Then the multibrackets $(\lambda_n)_{n \geq 1}$ of the Kapranov $L_{\infty}[1]$ algebra structure on $\text{Hom}(S(g[1]), g[1]) \cong \hat{S}(g[1])^\vee \otimes g[1]$, being identified with $\mathcal{X}(g[1])$ as in Eq. (55), are as follows.

1. The unary bracket $\lambda_1$ coincides with the Chevalley–Eilenberg differential with values in the $L_{\infty}[1]$-adjoint module $g[1]$:

$$\lambda_1 = d_{CE}^g : \hat{S}(g[1])^\vee \otimes g[1] \to \hat{S}(g[1])^\vee \otimes g[1]$$
(2) For any $n \geq 2$, $\lambda_n$ is $\hat{S}(\mathfrak{g}[1])^\vee$-linear in each of its $n$ arguments, and therefore can be considered as a $\mathbb{K}$-linear map

$$\lambda_n : S^n(\mathfrak{g}[1]) \to \hat{S}(\mathfrak{g}[1])^\vee \otimes \mathfrak{g}[1],$$

which is completely determined by

$$\lambda_n(X) = \sum_{k=n}^{\infty} q_k(X \otimes -), \quad n \geq 2,$$

where $X \in S^n(\mathfrak{g}[1])$ and $q_k(X \otimes -)$ denotes the map

$$S^{k-n}(\mathfrak{g}[1]) \ni Y \mapsto q_k(X \otimes Y) \in \mathfrak{g}[1].$$

Example 5.9. If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then the Kapranov $L_\infty$ algebra (i.e. $(-1)$-shifted Kapranov $L_\infty[1]$ algebra) of the dg manifold $(\mathfrak{g}[1], d_{\mathfrak{CE}})$ is the dgla $\Lambda \mathfrak{g}^\vee \otimes \mathfrak{g}$, where the differential is the Chevalley–Eilenberg differential $d_{\mathfrak{CE}}$ of the $\mathfrak{g}$-module $\mathfrak{g}$ (for the adjoint action), and the Lie bracket is $[\xi \otimes x, \eta \otimes y] = \xi \wedge \eta \otimes [x, y]$ for homogeneous $\xi, \eta \in \Lambda \mathfrak{g}^\vee$ and $x, y \in \mathfrak{g}$.

5.2. dg manifolds associated with complex manifolds and integrable distributions. Every complex manifold $X$ determines a dg manifold $(T_X^{0,1}, \tilde{a})$—see Example 2.3. This section is devoted to the description of the corresponding Kapranov $L_\infty[1]$ algebra. Recall that for a Kähler manifold $X$, Kapranov obtained an explicit description of an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega_X^{0,1} \otimes \mathfrak{g}$, where the unary bracket is the Dolbeault operator $\tilde{a}$ and the binary bracket is the Dolbeault cocycle of the Atiyah class of $T_X$ [25, Theorem 2.6]. Kapranov proved the existence of an $L_\infty[1]$ algebra structure associated with the Atiyah class of the holomorphic tangent bundle of any complex manifold using formal geometry and PROP [25, Theorem 4.3]. See Theorem 5.11 below for the Dolbeault representatives. Since $T_X^{0,1} \subset T_X^{0,0}$ is a complex integrable distribution, we will consider general integrable distributions over $\mathbb{K}$. Indeed such $L_\infty[1]$ algebra structures can be obtained in a more general perspective in terms of Lie pairs [31]. We recall the construction briefly below.

Let $F \subseteq T_{\mathbb{K}}M$ be an integrable distribution. Then $(F[1], d_F)$ is a dg manifold, whose algebra of smooth functions $C^\infty(F[1], \mathbb{K})$ is identified with $\Omega_F := \Gamma(\Lambda F^\vee)$ and whose homological vector field is the leafwise de Rham differential, i.e. the Chevalley–Eilenberg differential $d_F : \Omega_F^* \to \Omega_F^{*+1}$ of the Lie algebroid $F$. It is well known that the normal bundle $B := T_{\mathbb{K}}M / F$ is naturally an $F$-module: the $F$-action on $B$ is the Bott connection [14] defined by

$$\nabla^\text{Bott}_a b = q([a, \tilde{b}]),$$

for all $a \in \Gamma(F)$, $b \in \Gamma(B)$ and $\tilde{b} \in \Gamma(T_{\mathbb{K}}M)$ such that $q(\tilde{b}) = b$. Here $q : T_{\mathbb{K}}M \to B$ denotes the canonical projection. Let $\mathcal{D}(M)$ be the space of $\mathbb{K}$-linear differential operators on $M$, and $R = C^\infty(M; \mathbb{K})$ be the space of $\mathbb{K}$-valued smooth functions on $M$. Then $\mathcal{D}(M)$ is an $R$-coalgebra equipped with the standard coproduct

$$\Delta : \mathcal{D}(M) \to \mathcal{D}(M) \otimes_R \mathcal{D}(M).$$

(58)
Let $\mathcal{D}(M)\Gamma(F) \subseteq \mathcal{D}(M)$ be the left ideal of $\mathcal{D}(M)$ generated by $\Gamma(F)$. Since

$$\Delta(\mathcal{D}(M)\Gamma(F)) \subseteq \mathcal{D}(M) \otimes_R \mathcal{D}(M)\Gamma(F) + \mathcal{D}(M)\Gamma(F) \otimes_R \mathcal{D}(M),$$

the coproduct (58) descends to a well-defined coproduct over $R$

$$\Delta : \mathcal{D}(B) \rightarrow \mathcal{D}(B) \otimes_R \mathcal{D}(B), \quad (59)$$

on the quotient space $\mathcal{D}(B) := \frac{\mathcal{D}(M)\Gamma(F)}{\mathcal{D}(M)\Gamma(M,F)}$. Hence $\mathcal{D}(B)$ is an $R$-algebra as well, called the $R$-algebra of differential operators transverse to $F$ [52].

It is well known that $\mathcal{D}(B)$ is an $F$-module [30,31]: the $F$-action is given by

$$a \cdot u = \overline{a \circ u}, \quad (60)$$

for all $a \in \Gamma(F)$ and $u \in \mathcal{D}(M)$—the symbol $\overline{\cdot}$ denotes the image of $\cdot$ under the quotient map $\mathcal{D}(M) \rightarrow \mathcal{D}(B)$. Here $\circ$ denotes the composition of differential operators. Moreover, $F$ acts on $\mathcal{D}(B)$ by coderivations. The associated Chevalley–Eilenberg differential

$$d^{\Omega_F}_F : \Omega^*_F(\mathcal{D}(B)) \rightarrow \Omega^{*+1}_F(\mathcal{D}(B))$$

is a coderivation of the $\Omega_F$-linear coproduct

$$\Delta : \Omega_F(\mathcal{D}(B)) \rightarrow \Omega_F(\mathcal{D}(B)) \otimes_{\Omega_F} \Omega_F(\mathcal{D}(B))$$

extending the coproduct (59) on $\mathcal{D}(B)$. Thus $(\Omega_F(\mathcal{D}(B)), d^{\Omega_F}_F, \Delta)$ is a dg coalgebra over $(\Omega_F, d_F)$.

Let $j : B \rightarrow T_KM$ be a splitting of the short exact sequence of vector bundles (over $M$)

$$0 \rightarrow F \overset{i}{\rightarrow} T_KM \overset{q}{\rightarrow} B \rightarrow 0. \quad (61)$$

Choose a torsion-free linear connection $\nabla^B$ on the normal bundle $B$, i.e. a linear $T_KM$-connection on $B$ satisfying the condition:

$$\nabla^B_X(q(Y)) - \nabla^B_Y(q(X)) - q([X,Y]) = 0,$$

for all $X, Y \in \Gamma(T_KM)$. It is known [31, Lemma 5.2] that a torsion-free linear connection $\nabla^B$ is necessarily an extension of the Bott representation of $F$ on $B$, that is, $\nabla^B_{i(a)} \overline{X} = \nabla^B_{a} \overline{X}$ for all $a \in \Gamma(F)$ and $\overline{X} \in \Gamma(B)$.

According to [30,31], the pair $(j, \nabla^B)$ determines an isomorphism of $R$-coalgebras

$$\overline{pbw} : \Gamma(S(B)) \rightarrow \mathcal{D}(B),$$

called PBW isomorphism for the Lie pair $(T_KM, F)$, which is defined recursively by the relations

$$\overline{pbw}(f) = f, \quad \forall f \in R,$$
$$\overline{pbw}(b) = j(b), \quad \forall b \in \Gamma(B),$$

and

$$\overline{pbw}(b_1 \odot \cdots \odot b_n) = \frac{1}{n} \sum_{k=1}^n \left\{ j(b_k) \cdot \overline{pbw}(b^{[k]}) - \overline{pbw} (\nabla^B_{j(b_k)}(b^{[k]})) \right\},$$

where

$$b^{[k]} = \frac{1}{k} \sum_{j=0}^{k-1} \binom{k}{j} b_1^{j} \odot \cdots \odot b_n^{k-j}.$$
where we keep the notation from (60) and $b^{(k)} = b_1 \circ \cdots \circ b_{k-1} \circ b_{k+1} \circ \cdots \circ b_n$. Extending this isomorphism of $R$-coalgebras $\Omega_F$-linearly, we obtain an isomorphism of $\Omega_F$-coalgebras

$$\text{pbw}: \Omega_F(S(B)) \to \Omega_F(D(B)).$$  \hspace{1cm} (62)

Transferring the coderivation $d_{F}^S$ of $\Omega_F(D(B))$ to $\Omega_F(S(B))$ via the isomorphism (62), we obtain a degree +1 coderivation $\tilde{\delta}$ of $\Omega_F(S(B))$:

$$\tilde{\delta} := (\text{pbw})^{-1} \circ d_{F}^S \circ \text{pbw}: \Omega_F(S(B)) \to \Omega_F^{*+1}(S(B)).$$

Thus $(\Omega_F(S(B)), \tilde{\delta}, \Delta)$ is a dg coalgebra over $(\Omega_F, d_F)$.

By dualizing $\tilde{\delta}$ over the dg algebra $(\Omega_F, d_F)$, we obtain a degree +1 derivation

$$D: \Omega_F^*(\widehat{S}(B^\vee)) \to \Omega_F^{*+1}(\widehat{S}(B^\vee)).$$

According to [31, Theorem 5.7], it can be expressed as

$$D = d_F^{\mathcal{V}Bott} + \sum_{k=2}^{\infty} \tilde{\mathcal{R}}_k,$$

where

1. $d_F^{\mathcal{V}Bott}$ is the Chevalley–Eilenberg differential corresponding to the Bott connection of $F$ on $\widehat{S}(B^\vee)$;
2. for any $k \geq 2$, $\tilde{\mathcal{R}}_k : \Omega_F^*(\widehat{S}(B^\vee)) \to \Omega_F^{*+1}(\widehat{S}(B^\vee))$ is the $\Omega_F^*$-linear degree +1 derivation acting by contraction induced from a section $\mathcal{R}_k \in \Omega_F^1(\widehat{S}(B^\vee) \otimes B)$;
3. $\mathcal{R}_2 \in \Omega_F^1(S(B^\vee) \otimes B)$ is the Atiyah 1-cocycle $\mathcal{A}_2^{\mathcal{V}Bott}_{T\kappa M/F}$ associated with the connection $\nabla^B$ defined by

$$\mathcal{R}_2(a, \tilde{X}) = \nabla^B_a \nabla^B_{\tilde{X}} - \nabla^B_{\tilde{X}} \nabla^B_a - \nabla^B_{[a, X]},$$

for all $a \in \Gamma(F)$ and $X \in \Gamma(T_{\kappa M})$, where $\tilde{X} \in \Gamma(B)$ denotes the image of $X$ under the quotient map $T_{\kappa M} \to T_{\kappa M}/F$.

A priori, $\mathcal{R}_2$ belongs to $\Omega_F^1(B^\vee \otimes \text{End}(B))$, but the torsion-free assumption guarantees that it is indeed an element in $\Omega_F^1(S(B^\vee) \otimes B)$. Its cohomology class

$$\alpha_{T\kappa M/F} \in \mathbb{H}^1_{\mathcal{CE}}(F, B^\vee \otimes \text{End}(B))$$

is independent of the choice of $\nabla^B$ and is called the Atiyah class of the Lie pair $(T_{\kappa M}, F)$ [14]. Note that $\Omega_F^*(\widehat{S}(B^\vee))$ is the algebra of functions on $F[1] \oplus B$. Thus $(F[1] \oplus B, D)$ is a dg manifold with support $M$, called a Kapranov dg manifold associated with the Lie pair $(T_{\kappa M}, F)$ [31]. One can prove that the various Kapranov dg manifold structures on $F[1] \oplus B$ resulting from all possible choices of splitting and connection are all isomorphic.

**Theorem 5.10** [31, Theorem 5.7]. Let $F \subseteq T\kappa M$ be an integrable distribution. The choice of a splitting $j : B \to T\kappa M$ of the short exact sequence (61) and a torsion-free linear connection $\nabla^B$ of the vector bundle $B$ determines an $L_\infty[1]$ algebra structure on the graded vector space $\Omega_F^*(B)$ defined by a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of multibrackets such that each $\lambda_k$, with $k \geq 2$, is $\Omega_F$-multilinear, and
• the unary bracket $\lambda_1$ is the Chevalley–Eilenberg differential $d^\nabla_{Bott}$ associated with the Bott connection $\nabla_{Bott}$ of $F$ on $B$;
• the binary bracket $\lambda_2$ is the map

$$\lambda_2 : \Omega^j_F(B) \otimes \Omega^j_F(B) \to \Omega^{j+j+1}_F(B)$$

induced by the Atiyah 1-cocycle $R_2 \in \Omega^1_F(S^2(B^\vee) \otimes B)$;
• for every $k \geq 3$, the $k$-th multibracket $\lambda_k$ is the composition of the wedge product

$$\Omega^j_F(B) \otimes \cdots \otimes \Omega^k_F(B) \to \Omega^{j+\cdots+j+k}(B^\otimes)$

with the map

$$\Omega^{j+\cdots+j+k}_F(B^\otimes) \to \Omega^{j+\cdots+j+k+1}_F(B)$$

induced by an element $R_k \in \Omega^1_F(S^k(B^\vee) \otimes B) \subset \Omega^2_F((B^\vee)^k \otimes B)$.

Moreover, the $L_\infty[1]$ algebra structure on $\Omega^\bullet_F(B)$ is unique up to isomorphisms in the sense that the various $L_\infty[1]$ algebra structures resulting from all possible choices of splitting and connection are all isomorphic.

Any such $L_\infty[1]$ algebra structure on $\Omega^\bullet_F(B)$ is called a Kapranov $L_\infty[1]$ algebra of the integrable distribution $F$.

As a special case, consider a complex manifold $X$. The subbundle $F = T^{0,1}_X \subset T_C X$ is an integrable distribution, and the normal bundle $B := T_C X / T^{0,1}_X$ is naturally identified with $T^{1,0}_X$. Moreover, the Chevalley–Eilenberg differential associated with the Bott $F$-connection on $T^{1,0}_X$ becomes the Dolbeault operator

$$\bar{\partial} : \Omega^{0,\bullet}_X(T^{1,0}_X) \to \Omega^{0,\bullet+1}(T^{1,0}_X).$$

The following is an immediate consequence of Theorem 5.10, which extends Kapranov’s construction for Kähler manifolds [25, Theorem 2.6] to all complex manifolds.

**Theorem 5.11** [31, Theorem 5.24]. For a given complex manifold $X$, any torsion-free $T^{1,0}_X$-connection $\nabla^{1,0}$ on $T^{1,0}_X$ determines an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T^{1,0}_X)$ such that

• the unary bracket $\lambda_1$ is the Dolbeault operator

$$\bar{\partial} : \Omega^{0,j}(T^{1,0}_X) \to \Omega^{0,j+1}(T^{1,0}_X);$$

• the binary bracket $\lambda_2$ is the map

$$\lambda_2 : \Omega^{0,j_1}(T^{1,0}_X) \otimes \Omega^{0,j_2}(T^{1,0}_X) \to \Omega^{0,j_1+j_2+1}(T^{1,0}_X)$$

induced by the Dolbeault representative of the Atiyah 1-cocycle $R_2 \in \Omega^{1}(S^2(T^{1,0}_X) \otimes T^{1,0}_X)$;
• for every $k \geq 3$, the $k$-th multibracket $\lambda_k$ is the composition of the wedge product
\[
\Omega^{0,j_1}(T^{1,0}_X) \otimes \cdots \otimes \Omega^{0,j_k}(T^{1,0}_X) \to \Omega^{0,j_1+\cdots+j_k}(T^{1,0}_X)^\otimes k
\]
with the map
\[
\Omega^{0,j_1+\cdots+j_k}(T^{1,0}_X)^\otimes k \to \Omega^{0,j_1+\cdots+j_k+1}(T^{1,0}_X)
\]
induced by an element $R_k$ of the subspace $\Omega^{0,1}(S^k((T^{1,0}_X)^\vee) \otimes T^{1,0}_X)$ of $\Omega^{0,1}(S^k((T^{1,0}_X)^\vee) \otimes T^{1,0}_X)$ completely determined by the Atiyah 1-cocycle $R_2$, the curvature of $\nabla^{1,0}$, and their higher covariant derivatives.

Moreover, the $L_\infty[1]$ algebra structure on $\Omega^{0,\bullet}(T^{1,0}_X)$ is unique up to isomorphisms.

Now we are ready to consider the Kapranov $L_\infty[1]$ algebra of the dg manifold $(F[1], d_F)$. Let
\[
\tilde{\Phi} : D(F[1]) \to \Omega_F(D(B))
\]
be the map defined by $\tilde{\Phi}(D) = \pi_*(D)$, where $\pi_* : D(F[1]) \to \Omega_F \otimes_R D(M)$ is the pushforward map
\[
\pi_*(D)(f) = D(\pi^* f), \quad \forall D \in D(F[1]), \forall f \in R
\]
arising from the projection $\pi : F[1] \to M$ and $\pi_*(D) \in \Omega_F(D(B))$ denotes the class of $\pi_*(D)$ in $\Omega_F \otimes_R D(M) \cong \Omega_F(D(B))$.

**Theorem 5.12** [16,52]. There exists a contraction of dg $\Omega_F$-modules
\[
\tilde{\Phi} : \Omega_F(D(M)) \to \Omega_F(D(B)),
\]
where the projection $\tilde{\Phi}$ is a morphism of $\Omega_F$-coalgebras.

Choose a torsion-free affine connection $\nabla$ on $F[1]$. We write
\[
\text{pbw} : \Gamma(S(T_{F[1]})) \to D(F[1])
\]
for the corresponding Poincaré–Birkhoff–Witt map as in (14).

Conjugation by the PBW maps $\text{pbw}$ and $\overline{\text{pbw}}$, respectively, on the left hand side and on the right hand side of $\text{(63)}$ yields

**Corollary 5.13.** There exists a contraction of dg $\Omega_F$-modules
\[
\tilde{\Phi} : \Gamma(S(T_{F[1]})) \to D(F[1]), \quad \text{pbw}^{-1} \circ \tilde{\Phi} \circ \text{pbw}^{-1} \circ \text{pbw}_{\text{pbw}} = \text{d}_F^\text{pbw},
\]
where the projection $\Phi := \text{pbw}^{-1} \circ \tilde{\Phi} \circ \text{pbw}$ is a morphism of $\Omega_F$-coalgebras.
The projection $\Phi$ determines a sequence of maps $\{\phi_k\}_{k \geq 1}$ making the diagrams

$$S^k_{\mathbb{K}}(\mathfrak{X}(F[1])) \xrightarrow{\phi_k} \Omega_F(B)$$

$$\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Gamma(S(T_F[1])) \xrightarrow{\Phi} \Omega_F(S(B))$$

(commutative. Note that $\phi_1 : \mathfrak{X}(F[1]) \to \Omega_F(B)$ is the composition

$$\mathfrak{X}(F[1]) \xrightarrow{\pi_*} \Omega_F(T_{\mathbb{K}}M) \xrightarrow{q} \Omega_F(B).$$

**Theorem 5.14.** Let $F \subseteq T_{\mathbb{K}}M$ be an integrable distribution. Then the sequence of $\Omega_F$-multilinear maps $\{\phi_k\}_{k \geq 1}$ defined by the commutative diagrams (64) constitutes a quasi-isomorphism from the Kapranov $L_\infty[1]$ algebra $\mathfrak{X}(F[1])$ arising from the dg manifold $(F[1], d_F)$ to the Kapranov $L_\infty[1]$ algebra $\Omega^*_F(B)$ arising (as in Theorem 5.10) from the integrable distribution $F$.

As an immediate consequence, we have

**Corollary 5.15.** Consider the dg manifold $(T_X^{0,1}[1], \bar{\partial})$ arising, as in Example 2.3, from a complex manifold $X$. The Kapranov $L_\infty[1]$ algebra $\mathfrak{X}(T_X^{0,1}[1])$ is quasi-isomorphic to the $L_\infty[1]$ algebra $\mathfrak{X}(T_X^{0,1}[1])$—see Theorem 5.11. The quasi-isomorphism $\{\phi_k\}_{k \geq 1}$, in which each map $\phi_k$ is $\Omega_X^{0,\bullet}$-multilinear, is given by (64) (with $F = T_X^{0,1}$ and $B = T_X^{1,0}$), and in particular $\phi_1 : \mathfrak{X}(T_X^{0,1}[1]) \to \Omega^{0,\bullet}(T_X^{1,0})$ is the composition

$$\mathfrak{X}(T_X^{0,1}[1]) \xrightarrow{\pi_*} \Omega^{0,\bullet}(T_{\mathbb{K}}^C) \xrightarrow{\text{pr}} \Omega^{0,\bullet}(T_X^{1,0}).$$

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**Appendix A. Fedosov Construction for Graded Manifolds**

This section is to give a brief description of the Fedosov construction for graded manifolds. We refer readers to [18,19,32] for more details.

Throughout this section, $\mathcal{M}$ is a finite-dimensional graded manifold and $\nabla$ is a torsion-free affine connection on $\mathcal{M}$. By abuse of notation, the induced linear connection on $\tilde{S}(T\mathcal{M})$ is denoted by the same symbol. The associated covariant derivative is written $d\nabla : \Omega^*(\tilde{S}(T\mathcal{M})) \to \Omega^{*+1}(\tilde{S}(T\mathcal{M}))$.

Consider the map $\nabla^\sharp : \mathfrak{X}(\mathcal{M}) \times \Gamma(S(T_\mathcal{M})) \to \Gamma(S(T_\mathcal{M}))$ defined by

$$\nabla^\sharp_Y X = (\text{pbw} \nabla)^{-1}(Y \cdot \text{pbw} \nabla(X))$$

for all $Y \in \mathfrak{X}(\mathcal{M})$ and $X \in \Gamma(S(T_\mathcal{M}))$.

**Lemma A.1.** The above map $\nabla^\sharp$ defines a flat connection on $S(T\mathcal{M})$. 


Abusing notation, we write the same symbol $\nabla^\frac{1}{2}$ to denote the induced flat connection on $\widehat{S}(T_{\mathcal{M}}^\vee)$. The associated covariant derivative $d^\nabla^\frac{1}{2} : \Omega^*(\widehat{S}(T_{\mathcal{M}}^\vee)) \to \Omega^{*+1}(\widehat{S}(T_{\mathcal{M}}^\vee))$ satisfies $(d^\nabla^\frac{1}{2})^2 = 0$.

We use the identification

$$\Omega^p(\widehat{S}(T_{\mathcal{M}}^\vee)) \cong \Gamma(\Lambda^p(T_{\mathcal{M}}^\vee) \otimes \widehat{S}(T_{\mathcal{M}}^\vee)) \cong \Gamma(\text{Hom}(\Lambda^p(T_{\mathcal{M}}^\vee) \otimes S(T_{\mathcal{M}}^\vee), \mathbb{K})).$$

The total degree of $\omega \in \Omega^p(\widehat{S}(T_{\mathcal{M}}^\vee))$ is $p + |\omega|$, where $p$ is the cohomological degree and $|\omega|$ is the internal degree of $\omega$.

Define two operators

$$\delta : \Omega^p(\widehat{S}(T_{\mathcal{M}}^\vee)) \to \Omega^{p+1}(\widehat{S}(T_{\mathcal{M}}^\vee))$$

and

$$\mathfrak{h} : \Omega^p(\widehat{S}(T_{\mathcal{M}}^\vee)) \to \Omega^{p-1}(\widehat{S}(T_{\mathcal{M}}^\vee))$$

by

$$(\delta \omega) (X_1 \wedge \cdots \wedge X_{p+1}; Y_1 \odot \cdots \odot Y_{q-1})$$

$$= \sum_{i=1}^{p+1} (-1)^{i+1} \varepsilon \cdot \omega(X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_{p+1}; X_i \odot Y_1 \odot \cdots \odot Y_{q-1})$$

and

$$(\mathfrak{h} \omega) (X_1 \wedge \cdots \wedge X_{p-1}; Y_1 \odot \cdots \odot Y_{q+1})$$

$$= \frac{1}{p+q} \sum_{j=1}^{q+1} \varepsilon \cdot \omega(Y_j \wedge X_1 \wedge \cdots \wedge X_{p-1}; Y_1 \odot \cdots \odot \hat{Y}_j \odot \cdots \odot Y_{q+1}),$$

for all $\omega \in \Omega^p(\widehat{S}(T_{\mathcal{M}}^\vee))$ and all homogeneous $X_1, \ldots, X_{p+1}, Y_1, \ldots, Y_{q+1} \in \mathfrak{x}(\mathcal{M})$. The symbol $\varepsilon$ denotes the Koszul signs: either $\varepsilon(X_1, \cdots, X_{p+1}, Y_1, \cdots, Y_{q-1})$ or $\varepsilon(X_1, \cdots, X_{p-1}, Y_1, \cdots, Y_{q+1})$, as appropriate.

Both $\delta$ and $\mathfrak{h}$ are $C^\infty(\mathcal{M})$-linear, and $\delta$ is the Koszul operator. Observe that $\delta$ has total degree $+1$ and $\mathfrak{h}$ has total degree $-1$. However neither $\delta$ nor $\mathfrak{h}$ change the internal degree: $|\delta \omega| = |\omega|$ and $|\mathfrak{h} \omega| = |\omega|$.

Remark A.2. In [18,19,32], the operator $\mathfrak{h}$ is written as $\delta^{-1}$. We avoid this notation because $\mathfrak{h}$ is not an inverse map of $\delta$, and it is rather a homotopy operator.

Lemma A.3. The operator $\delta$ satisfies $\delta^2 = 0$. That is,

$$0 \to \Omega^0(\widehat{S}(T_{\mathcal{M}}^\vee)) \to \Omega^1(\widehat{S}(T_{\mathcal{M}}^\vee)) \to \Omega^2(\widehat{S}(T_{\mathcal{M}}^\vee)) \to \cdots$$

is a cochain complex. Moreover, the operators $\delta$ and $\mathfrak{h}$ satisfy

$$\delta \circ \mathfrak{h} + \mathfrak{h} \circ \delta = \text{id} - \pi_0,$$

where $\pi_0 : \Omega^*(\widehat{S}(T_{\mathcal{M}}^\vee)) \to C^\infty(\mathcal{M})$ is the natural projection.

We have the following theorem
Theorem A.4 [32, Theorem 5.6]. Let $\mathcal{M}$ be a finite-dimensional graded manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. Then the covariant derivative $d^{\nabla}$ decomposes as

$$d^{\nabla} = d^\nabla - \delta + \tilde{A}^\nabla,$$

where the operator $\tilde{A}^\nabla : \Omega^\bullet(\hat{S}(T^\vee_M)) \to \Omega^{\bullet+1}(\hat{S}(T^\vee_M))$ is the derivation of (total) degree +1 determined by a certain element $A^\nabla$ of $\Omega^1(\mathcal{M}, \hat{S}^{\geq 2}(T^\vee_M) \otimes T_M)$ satisfying $h \circ A^\nabla = 0$.

Remark A.5. The operator $\tilde{A}^\nabla$ increases the cohomological degree by +1 and preserves the internal degree. That is, although the total degree of $\tilde{A}^\nabla$ is +1, its internal degree is $|\tilde{A}^\nabla| = 0$.

Write

$$A^\nabla = \sum_{n \geq 2} A^\nabla_n, \quad A^\nabla_n \in \Omega^1(\mathcal{M}, S^n(T^\vee_M) \otimes T_M).$$

Let $R^\nabla \in \Omega^2(\mathcal{M}; \text{End}(T_M))$ denote the curvature of $\nabla$.

Proposition A.6. We have the following recursive formula for $A^\nabla_n$:

$$A^\nabla_2 = h \circ R^\nabla,$$

$$A^\nabla_{n+1} = h \circ \left( d^\nabla A^\nabla_n + \sum_{p+q=n} \frac{1}{2} [A^\nabla_p, A^\nabla_q] \right), \quad \forall n \geq 2.$$

Proof. According to Theorem A.4 and Lemma A.1, we have $d^{\nabla} = d^\nabla - \delta + A^\nabla$ and $(d^{\nabla})^2 = 0$.

By Lemma A.3, we know $\delta^2 = 0$ and $\delta \circ h + h \circ \delta = \text{id} - \pi_0$. Also, $(d^\nabla)^2 = R^\nabla$. Since $\nabla$ is torsion-free, we have

$$[\delta, d^\nabla] = \delta \circ d^\nabla + d^\nabla \circ \delta = 0.$$

As a result, $(d^{\nabla})^2 = 0$ implies that

$$\delta \circ A^\nabla + A^\nabla \circ \delta = R^\nabla + d^\nabla A^\nabla + \frac{1}{2} [A^\nabla, A^\nabla].$$

By applying the operator $h$, we get

$$A^\nabla = h \circ \delta \circ A^\nabla = h \circ \left( R^\nabla + d^\nabla A^\nabla + \frac{1}{2} [A^\nabla, A^\nabla] \right),$$

because $h \circ A^\nabla = 0$ and $\pi_0 \circ A^\nabla = 0$.

Since $h(\Omega^2(\hat{S}^q(T^\vee_M))) \subset \Omega^1(\hat{S}^{q+1}(T^\vee_M))$, applying the canonical projections

$$\Omega^1(\mathcal{M}, \hat{S}(T^\vee_M) \otimes T_M) \to \Omega^1(\mathcal{M}, S^n(T^\vee_M) \otimes T_M)$$

Theorem A.4 [32, Theorem 5.6]. Let $\mathcal{M}$ be a finite-dimensional graded manifold and let $\nabla$ be a torsion-free affine connection on $\mathcal{M}$. Then the covariant derivative $d^{\nabla}$ decomposes as

$$d^{\nabla} = d^\nabla - \delta + \tilde{A}^\nabla,$$

where the operator $\tilde{A}^\nabla : \Omega^\bullet(\hat{S}(T^\vee_M)) \to \Omega^{\bullet+1}(\hat{S}(T^\vee_M))$ is the derivation of (total) degree +1 determined by a certain element $A^\nabla$ of $\Omega^1(\mathcal{M}, \hat{S}^{\geq 2}(T^\vee_M) \otimes T_M)$ satisfying $h \circ A^\nabla = 0$.

Remark A.5. The operator $\tilde{A}^\nabla$ increases the cohomological degree by +1 and preserves the internal degree. That is, although the total degree of $\tilde{A}^\nabla$ is +1, its internal degree is $|\tilde{A}^\nabla| = 0$.

Write

$$A^\nabla = \sum_{n \geq 2} A^\nabla_n, \quad A^\nabla_n \in \Omega^1(\mathcal{M}, S^n(T^\vee_M) \otimes T_M).$$

Let $R^\nabla \in \Omega^2(\mathcal{M}; \text{End}(T_M))$ denote the curvature of $\nabla$.

Proposition A.6. We have the following recursive formula for $A^\nabla_n$:

$$A^\nabla_2 = h \circ R^\nabla,$$

$$A^\nabla_{n+1} = h \circ \left( d^\nabla A^\nabla_n + \sum_{p+q=n} \frac{1}{2} [A^\nabla_p, A^\nabla_q] \right), \quad \forall n \geq 2.$$

Proof. According to Theorem A.4 and Lemma A.1, we have $d^{\nabla} = d^\nabla - \delta + A^\nabla$ and $(d^{\nabla})^2 = 0$.

By Lemma A.3, we know $\delta^2 = 0$ and $\delta \circ h + h \circ \delta = \text{id} - \pi_0$. Also, $(d^\nabla)^2 = R^\nabla$. Since $\nabla$ is torsion-free, we have

$$[\delta, d^\nabla] = \delta \circ d^\nabla + d^\nabla \circ \delta = 0.$$

As a result, $(d^{\nabla})^2 = 0$ implies that

$$\delta \circ A^\nabla + A^\nabla \circ \delta = R^\nabla + d^\nabla A^\nabla + \frac{1}{2} [A^\nabla, A^\nabla].$$

By applying the operator $h$, we get

$$A^\nabla = h \circ \delta \circ A^\nabla = h \circ \left( R^\nabla + d^\nabla A^\nabla + \frac{1}{2} [A^\nabla, A^\nabla] \right),$$

because $h \circ A^\nabla = 0$ and $\pi_0 \circ A^\nabla = 0$.

Since $h(\Omega^2(\hat{S}^q(T^\vee_M))) \subset \Omega^1(\hat{S}^{q+1}(T^\vee_M))$, applying the canonical projections

$$\Omega^1(\mathcal{M}, \hat{S}(T^\vee_M) \otimes T_M) \to \Omega^1(\mathcal{M}, S^n(T^\vee_M) \otimes T_M)$$
(for each \( n \geq 2 \)) to the equality
\[
A^\nabla = \mathfrak{h} \circ \left( R^\nabla + d^\nabla A^\nabla + \frac{1}{2} [A^\nabla, A^\nabla] \right) \in \Omega^1(\mathcal{M}, \hat{\mathcal{S}}(T^\nabla_M) \otimes T_M)
\]
yields the relations
\[
\begin{align*}
A^\nabla_2 &= \mathfrak{h} \circ R^\nabla, \\
A^\nabla_{n+1} &= \mathfrak{h} \circ \left( d^\nabla A^\nabla_n + \sum_{p+q=n} \frac{1}{2} [A^\nabla_p, A^\nabla_q] \right), \quad \forall n \geq 2.
\end{align*}
\tag{65}
\]
This completes the proof. \( \square \)

**Corollary A.7.** Under the same hypothesis as in Theorem A.4, the elements \( A^\nabla_n \) with \( n \geq 2 \) are completely determined by the curvature \( R^\nabla \) and its higher covariant derivatives. In fact, the element \( A^\nabla_n \) is determined by the elements \( A^\nabla_k \) with \( k \leq n - 1 \) by way of the recursive formula (65).

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**References**

1. Alexandrov, M., Schwarz, A., Zaboronsky, O., Kontsevich, M.: The geometry of the master equation and topological quantum field theory. Int. J. Mod. Phys. A **12**(7), 1405–1429 (1997). https://doi.org/10.1142/S0217751X97001031
2. Bandiera, R.: Formality of Kapranov’s brackets in Kähler geometry via pre-Lie deformation theory. Int. Math. Res. Not. IMRN **21**, 6626–6655 (2016). https://doi.org/10.1093/imrn/rmv362
3. Bandiera, R.: Homotopy abelian \( L_\infty \) algebras and splitting property. Rend. Mat. Appl. (7) **37**(1–2), 105–122 (2016)
4. Bandiera, R., Chen, Z., Stiénon, M., Xu, P.: Shifted derived Poisson manifolds associated with Lie pairs. Commun. Math. Phys. **375**(3), 1717–1760 (2020). https://doi.org/10.1007/s00220-019-03457-w
5. Behrend, K., Liao, H.-Y., Xu, P.: Derived Differentiable Manifolds (2020). arXiv:2006.01376 [math.DG]
6. Boardman, J.M.: The principle of signs. Enseign. Math. (2) **12**, 191–194 (1966)
7. Bonavolontà, G., Poncin, N.: On the category of Lie \( n \)-algebroids. J. Geom. Phys. **73**, 70–90 (2013). https://doi.org/10.1016/j.geomphys.2013.05.004
8. Bruce, A.J.: From \( L_\infty \)-algebroids to higher Schouten/Poisson structures. Rep. Math. Phys. **67**(2), 157–177 (2011). https://doi.org/10.1016/S0034-4877(11)00010-3
9. Bruce, A.J., Grabowski, J., Vitagliano, L.: Representations up to homotopy from weighted Lie algebroids. J. Lie Theory **28**(3), 715–737 (2018)
10. Căldăraru, A., Willerton, S.: The Mukai pairing. I. A categorical approach. N. Y. J. Math. **16**, 61–98 (2010)
11. Carchedi, D., Roytenberg, D.: Homological Algebra for Superalgebras of Differentiable Functions (2012). arXiv:1212.3745 [math.AG]
12. Carchedi, D., Roytenberg, D.: On theories of superalgebras of differentiable functions. Theory Appl. Categ. **28**(30), 1022–1098 (2013)
13. Cattaneo, A.S., Schätz, F.: Introduction to supergeometry. Rev. Math. Phys. **23**(6), 669–690 (2011). https://doi.org/10.1142/S0129055X11004400
14. Chen, Z., Stiénon, M., Xu, P.: From Atiyah classes to homotopy Leibniz algebras. Commun. Math. Phys. **341**(1), 309–349 (2016). https://doi.org/10.1007/s00220-015-2494-6
15. Chen, Z., Xiang, M., Xu, P.: Atiyah and Todd classes arising from integrable distributions. J. Geom. Phys. **136**, 52–67 (2019). https://doi.org/10.1016/j.geomphys.2018.10.011
16. Chen, Z., Xiang, M., Xu, P.: Hochschild cohomology of dg manifolds associated to integrable distributions (2021). arXiv:2103.08096 [math.DG]
17. Cheng, J., Chen, Z., Ni, D.: Hopf algebras arising from dg manifolds. J. Algebra **584**, 19–68 (2021). https://doi.org/10.1016/j.jalgebra.2021.05.004
18. Dolgushev, V.A.: Covariant and equivariant formality theorems. Adv. Math. **191**(1), 147–177 (2005). https://doi.org/10.1016/j.aim.2004.02.001
19. Emmrich, C., Weinstein, A.: The differential geometry of Fedosov’s quantization. In: Lie Theory and Geometry, Vol. 123, pp. 217–239. Progr. Math. Birkhäuser Boston, Boston, MA (1994) https://doi.org/10.1007/978-1-4612-0261-5_7
20. Grabowska, K., Grabowski, J.: n-Tuple principal bundles. Int. J. Geom. Methods Mod. Phys. 15(12), 1850211, 18 (2018). https://doi.org/10.1142/S0219887818502110
21. Grabowski, J.: Modular classes of skew algebroid relations. Transform. Groups 17(4), 989–1010 (2012). https://doi.org/10.1007/s00031-012-9197-2
22. Jotz Lean, M.: Lie 2-algebroids and matched pairs of 2-representations: a geometric approach. Pac. J. Math. 301(1), 143–188 (2019). https://doi.org/10.2140/pjm.2019.301.143
23. Jotz Lean, M., Mehta, R.A., Papantonis, T: Modules and representations up to homotopy of Lie n-algebroids (2020). arXiv:2001.01101 [math.DG]
24. Joyce, D.: An introduction to d-manifolds and derived differential geometry. In: Moduli Spaces. Vol. 411, pp. 230–281. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge (2014)
25. Kapranov, M.: Rozansky–Witten invariants via Atiyah classes. Compos. Math. 115(1), 71–113 (1999). https://doi.org/10.1023/A:1000664527238
26. Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. 66(3), 157–216 (2003). https://doi.org/10.1023/B:MA TH.0000027508.00421.bf
27. Kontsevich, M.: Rozansky–Witten invariants via formal geometry. Compos. Math. 115(1), 115–127 (1999). https://doi.org/10.1023/A:1000619911308
28. Kotov, A., Strobl, T.: Characteristic classes associated to Q-bundles. Int. J. Geom. Methods Mod. Phys. 12(1), 1550006, 26 (2015). https://doi.org/10.1142/S0219887815500061
29. Lada, T., Markl, M.: Strongly homotopy Lie algebras. Commun. Algebra 23(6), 2147–2161 (1995). https://doi.org/10.1080/00927879508825335
30. Laurent-Gengoux, C., Stiénon, M., Xu, P.: Exponential map and L∞ algebra associated to a Lie pair. C. R. Math. Acad. Sci. Paris 350(17–18), 817–821 (2012). https://doi.org/10.1016/j.crma.2012.08.009
31. Laurent-Gengoux, C., Stiénon, M., Xu, P.: Poincaré–Birkhoff–Witt isomorphisms and Kapranov dg-manifolds. Adv. Math. 387, Paper No. 107792, 62 pp. (2021). https://doi.org/10.1016/j.aim.2021.107792
32. Liao, H.-Y., Stiénon, M.: Strongly homotopy Lie algebras. Commun. Algebra 23(6), 700–730 (2019). https://doi.org/10.1080/00927879508825335
33. Liao, H.-Y., Stiénon, M., Xu, P.: Formality theorem for differential graded manifolds. C. R. Math. Acad. Sci. Paris 356(1), 27–43 (2018). https://doi.org/10.1016/j.crma.2017.11.017
34. Lyakhovich, S.L., Mosman, E.A., Sharapov, A.A.: Characteristic classes of Q-manifolds: classification and applications. J. Geom. Phys. 60(5), 729–759 (2010). https://doi.org/10.1016/j.geomphys.2010.01.008
35. Manetti, M.: On some formality criteria for DG-Lie algebras. J. Algebra 438, 90–118 (2015). https://doi.org/10.1016/j.jalgebra.2015.04.029
36. Markarian, N.: The Atiyah class, Hochschild cohomology and the Riemann–Roch theorem. J. Lond. Math. Soc. (2) 79(1), 129–143 (2009). https://doi.org/10.1112/jlms/jdn064
37. Mehta, R.A.: Q-Algebroids and their cohomology. J. Symplectic Geom. 7(3), 263–293 (2009)
38. Mehta, R.A., Stiénon, M., Xu, P.: The Atiyah class of a dg-vector bundle. C. R. Math. Acad. Sci. Paris 356(1), 27–43 (2018). https://doi.org/10.1016/j.crma.2017.11.017
39. Lyakhovich, S.L., Mosman, E.A., Sharapov, A.A.: Characteristic classes of Q-manifolds: classification and applications. J. Geom. Phys. 60(5), 729–759 (2010). https://doi.org/10.1016/j.geomphys.2010.01.008
40. Pridham, J.P.: A differential graded model for derived analytic geometry. Adv. Math. 360, 106922, 29 (2020). https://doi.org/10.1016/j.aim.2019.106922
41. Pridham, J.P.: An outline of shifted Poisson structures and deformation quantisation in derived differential geometry (2018). arXiv:1804.07622 [math.DG]
42. Ramadoss, A.C.: The big Chern classes and the Chern character. Int. J. Math. 19(6), 699–746 (2008). https://doi.org/10.1142/S0129167X0804856
43. Roberts, J., Willerton, S.: On the Rozansky–Witten weight systems. Algebraic Geom. Topol. 10(3), 1455–1519 (2010). https://doi.org/10.2140/agt.2010.10.1455
44. Roytenberg, D.: On the structure of graded symplectic supermanifolds and Courant algebroids. In: Quantization, Poisson Brackets and Beyond (Manchester, 2001), Vol. 315, pp. 169–185. Contemp. Math. Amer. Math. Soc., Providence, RI (2002). https://doi.org/10.1090/conm/315/05479
45. Sawon, J.: Rozansky–Witten invariants of hyperkähler manifolds. Thesis (Ph.D.)–University of Cambridge (2000). arXiv:math/0404360 [math.DG]
46. Schwarz, A.: Geometry of Batalin–Vilkovisky quantization. Commun. Math. Phys. 155(2), 249–260 (1993)
47. Ševera, P.: Letters to Alan Weinstein about Courant algebroids (2017). arXiv:1707.00265 [math.DG]
48. Ševera, P.: Some title containing the words “homotopy” and “symplectic”, e.g. this one. In: Travaux Mathématiques. Fasc. XVI. Vol. 16, pp. 121–137. Trav. Math. Univ. Luxemb., Luxembourg (2005)
49. Shoikhet, B.: On the Duflo formula for $L_\infty$-algebras and $Q$-manifolds (1998). arXiv:math/9812009 [math.QA]

50. Spivak, D.I.: Derived smooth manifolds. Duke Math. J. 153(1), 55–128 (2010). https://doi.org/10.1215/00127094-2010-021

51. Stiénon, M., Xu, P.: Atiyah classes and Kontsevich–Duflo type theorem for DG manifolds. In: Homotopy Algebra, Deformation Theory and Quantization, Vol. 123, pp. 63–110. Banach Center Publ. Polish Acad. Sci. Inst. Math., Warsaw (2021). https://doi.org/10.4064/bc123-3

52. Vitagliano, L.: On the strong homotopy associative algebra of a foliation. Commun. Contemp. Math. 17(2), 1450026, 34 (2015). https://doi.org/10.1142/S0219199714500266

53. Voglaire, Y., Xu, P.: Rozansky–Witten-type invariants from symplectic Lie pairs. Commun. Math. Phys. 336(1), 217–241 (2015). https://doi.org/10.1007/s00220-014-2221-8

54. Voronov, T.T.: $Q$-Manifolds and higher analogs of Lie algebroids. In: XXIX Workshop on Geometric Methods in Physics, Vol. 1307, pp. 191–202. AIP Conf. Proc. Amer. Inst. Phys., Melville, NY (2010)

55. Voronov, T.T.: $Q$-Manifolds and Mackenzie theory. Commun. Math. Phys. 315(2), 279–310 (2012). https://doi.org/10.1007/s00220-012-1568-y

56. Voronov, T.T.: Graded geometry, $Q$-manifolds, and microformal geometry. Fortschr. Phys. 67(8–9), 1910023 (2019). https://doi.org/10.1002/prop.201910023

57. Yano, K., Ishihara, S.: Tangent and cotangent bundles: differential geometry. In: Pure and Applied Mathematics, No. 16, pp. ix+423. Marcel Dekker, Inc., New York (1973)

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