Gelfand-Fuchs cohomology and M-Theory as a Topological Quantum Field Theory

Ioannis P. ZOIS

Mathematical Institute, 24-29 St. Giles’, Oxford OX1 3LB

1izzois@maths.ox.ac.uk and izzois@cc.uoa.gr
Abstract

We propose a Lagrangian density for M-Theory which is purely topological using Gelfand-Fuchs cohomology which characterises up to homotopy $\Gamma_q$-structures and hence foliations in particular. Then using S-duality, we conjecture on the existence of certain plane fields on $S^{11}$. Finally we calculate the partition function of the theory.

PACS classification: 11.10.-z; 11.15.-q; 11.30.-Ly

Keywords: Gelfand-Fuchs cohomology; M-Theory; Haefliger structures

To my brother Demetrios and to E.
0.1 Short review of current status in M-Theory

We shall start with a brief overview of "old" superstring theory, namely the picture around the end of '80's before advances in dualities and p-branes.

What we actually had was not one but five consistent superstring theories in D=10, namely types I, IIA, IIB, heterotic SO(32) and heterotic $E_8 \times E_8$ which was an embarrassment of richness. These theories were related via a pertutbative symmetry called T-duality when compactified in D=9 (this duality is a symmetry of theories with compactified spatial dimensions where in the case of Calabi-Yau 3-folds T-duality takes the form of mirror symmetry). Moreover we believed that D=10 N=1 supergravity (various versions of it) was the low energy limit (or infinite string tension limit) of these superstring theories in D=10.

There was however another piece of knowledge available which was somehow "forgotten", and that was D=11 N=1 supergravity. This theory emerged as follows: assuming that spacetime carries a metric with Minkowski signature, namely that there is only one timelike coordinate and assuming also that there are zero modes with no greater than 2 spin, then supersymmetry puts restrictions on dimensionality of spacetime, hence $D_{\text{max}} = 11$. This made phenomenologists over the years to study all possible representations of super-Poincare in various dimensions up to D=11. One of the possibilities then was this maximal D=11 N=1 supergravity. However this theory was rather unpopular because it was overshadowed by superstring theory in D=10 (with which no apparent relation was known) but also perhaps for a more important reason: it was proved to be non-renormalisable. At this point we make a prothysteron: this defect is not possible to be overcome and one reminiscent of it is the fact that the supermembrane in D=11 has continuous spectrum; there is however an attempt to incorporate this feature in the framework of matrix models.

Situation changed rapidly around mid '90's (or perhaps earlier), the most important advances being:

1.) It was proved that various versions of D=10 N=1 supergravity admit strings as solutions, a fact that now puts the two theories on a rather equal footing.

2.) New evidence for a type of non-perturbative symmetry called S-duality was found. This symmetry interchanges the strong/weak coupling regions of a theory as well as elementary/soliton solutions (and hence topology
and dynamics; it is essentially a Hodge star duality between field strengths). This S-duality emerged as a generalisation of electric/magnetic duality in Maxwell theory and as further evidence of Montonen-Olive duality in supersymmetric Yang-Mills theories. Now eventually S-duality has lead to string/5-brane duality in D=10 and to membrane/5-brane duality in D=11 (let us restrict ourselves to the old brane-scan for the moment and forget D-branes also).

3.) The observation that the five different superstring theories in D=10 was an artifact of perturbation theory; these theories should emerge from one non-perturbative theory in D=11 which was called M-Theory.

4.) The relation between supermembranes in D=11 with D=11 N=1 supergravity as well as with D=10 superstring theories; the later is done by a process which now comes under the name "dimensional reduction process". This is a general scheme which has been generalised to incorporate arbitrary dimension D and arbitrary p-branes.

5.) Applying S-duality in D=11 supermembranes we get the membrane/5-brane duality in D=11, something which we mentioned earlier.

(see references [1], [2], [3], [4], [5], [6], [7] and references therein).

0.2 A topological Lagrangian for M-Theory

Hence we have something called M-Theory consisting of membranes and 5-branes living on an 11-manifold which is non perturbative. This theory has a very intriguing feature: we can only extract information about it from its limiting theories, namely either from D=11 N=1 supergravity or from superstrings in D=10. This is so because this theory is genuinely non perturbative for a reason which lies in the heart of manifold topology:

Let us recall that in string theory, the path integral involves summation over all topologically distinct diagrams (same for point particles of course). Strings are 1-branes hence in time they swep out a 2-manifold. At the tree level then we need all topologically distinct simply connected 2-manifolds (actually there is only one, as topology tells us) and for loop corrections, topology again says that topologically distinct non simply connected 2-manifolds are classified by their genus, so we sum up over all Riemann surfaces with different genus.

It is clear then that for a perturbative quantum field theory involving p-
branes we have to sum upon all topologically distinct \((p+1)\)-dim manifolds: simply connected ones for tree level and non simply connected ones for loop corrections. Thus \textit{we must know before hand the topological classification of manifolds} in the dimension of interest. That is the main problem of manifold topology in mathematics.

But now we face a deep and intractable problem: geometry tells us, essentially via a no-go theorem which is due to Whitehead from late '40's, that: "\textit{we cannot classify non simply connected manifolds with dimension greater or equal to 4}"! Hence for \(p\)-branes with \(p\) greater or equal to 3, all we can do via perturbative methods is up to tree level!

What happens for 3-manifolds then (hence for membranes)? The answer from mathematics is that we \textit{do not know} if all 3-manifolds can be classified! So even for 2-branes it is still unclear whether perturbative methods work (up to all levels of perturbation theory)!

The outlet from this situation that we propose here is not merely to look only at non perturbative aspects of these theories, as was done up to now, but to abandon perturbative methods completely from the very beginning. There is only one way known up to now which can achieve this "radical" solution to our problem: formulate the theory as a Topological Quantum Field Theory and hence get rid of all perturbations once and for all.

Let us explain how this can be achieved.

Our approach is based on one physical "principle":

\textit{A theory containing }\(p\)-\textit{branes should be formulated on an }\(m\)-\textit{dim manifold which admits }\(\Gamma_q\)-\textit{structures, where }\(q = m - p - 1\).

\textbf{N.B.}

Although we used in our physical principle \(\Gamma_q\)-structures which are more general than foliations, we shall use both these terms meaning essentially the same structure. The interested reader may refer to [11] for example to see the precise definitions which are quite complicated. The key point however is that the difference between \(\Gamma_q\)-structures (or Haefliger structures as they are most commonly known in topology) and codim-q foliations is essentially the difference between \textit{transverse} and \textit{normal}. This does not affect any of what
we have to say, since Bott-Haefliger theory of characteristic classes is formulated for the most general case, namely \( \Gamma \)-structures. We would also like to mention the relation between \( \Gamma \)-structures and \( \Omega \)-spectra which is currently an active field in topology.

(For D-branes we need a variant of the above principle, namely we need what are called plane foliations but we shall not elaborate on this point here).

One way of thinking about this principle is that it is analogous to the “past histories” approach of quantum mechanics. Clearly in quantum level one should integrate over all foliations of a given codim.

A piece of warning here: this principle does not imply that all physical process between branes are described by foliations. Although the group of foliations is huge, in fact comparable in size with the group of local diffeomorphisms [12], and foliations can be really “very nasty”, we would not like to make such a strong statement. What is definitely true though is that some physical process are indeed described by foliations, hence at least this condition must be satisfied because of them.

Note:
Before going on further, we would like to make one crucial remark: this principle puts severe restrictions on the topology that the underlying manifold may have, in case of M-Theory this is an 11-manifold. It is also very important if the manifold is open or closed. This may be of some help, as we hope, for the compactification problem of string theory or even M-Theory, namely how we go from D=10 (or D=11) to D=4 which is our intuitive dimension of spacetime. We shall address this question in the next section. The final comment is this: this principle puts absolutely no restriction to the usual quantum field theory for point particles in D=4, e.g. electroweak theory or QCD. This is so because in this case spacetime is just \( \mathbb{R}^4 \) which is non compact and we have 0-branes (point particles) and consequently 1-dim foliations for which the integrability condition is trivially satisfied (essentially this is due to a deep result of Gromov for foliations on open manifolds, which states that all open manifolds admit codim 1 foliations; in striking contrast, closed manifolds admit codim 1 foliations iff their Euler characteristic is zero, see for example in [11], [14] or references therein).
If we believe this principle, then the story goes on as follows: we are on an 11-manifold, call it M for brevity and we want to describe a theory containing 5-branes for example (and get membranes from S-duality). Then M should admit 6-dim foliations or equivalently codim 5 foliations. We know from Haefliger that the $\Gamma_q$-functor, namely the functor of codim q Haefliger structures and in particular codim q foliations, is representable. Practically this means that we can have an analogue of Chern-Weil theory which characterises foliations of M up to homotopy using cohomology classes of M. (One brief comment for foliations: one way of describing Haefliger structures more generally is to say that they generalise fibre bundles in exactly the same way that fibre bundles generalise Cartesian product. This observation is also important when mentioning gerbes later on).

In fact it is proved that the correct cohomology to classify Haefliger structures up to homotopy (and hence foliations which constitute a particular example of Haefliger structures) is the Gelfand-Fuchs cohomology. This is a result of Bott and Haefliger, essentially generalising an earlier result due to Godbillon and Vey which was dealing only with codim 1 foliations, [15].

Now we have a happy coincidence: the Bott-Haefliger class for a codim 5 foliation (which, recall, is what we want for 5-branes on an 11-manifold) is exactly an 11-form, something that fits well with using it as a Lagrangian density!

The construction for arbitrary codim q foliations goes as follows: let $F$ be a codim q foliation on an m-manifold M and suppose its normal bundle $\nu(F)$ is orientable. Then $F$ is defined by a global decomposable q-form $\Omega$. Let $\{(U_i, X_i)\}_{i \in I}$ be a locally finite cover of distinguished coordinate charts on M with a smooth partition of unity $\{\rho_i\}$. Then set

$$\Omega = \sum_{i \in I} \rho_i dx_i^{m-q+1} \wedge ... \wedge dx_i^m$$

Since $\Omega$ is integrable,

$$d\Omega = \theta \wedge \Omega,$$  \hspace{1cm} (1)

where $\theta$ some 1-form on M. The $(2q+1)$-form
\[ \gamma = \theta \wedge (d\theta)^q, \quad (2) \]

is closed and its de Rham cohomology class is independent of all choices involved in defining it, depending only on homotopy type of \( F \). That’s the class we want.

Clearly for our case we are on an 11-manifold dealing with 5-branes, hence 6-dim foliations, hence codim 5 and thus the class \( \gamma \) is an 11-form.

This construction can be generalised to arbitrary \( \Gamma_q \)-structures as a mixed de Rham-Cech cohomology class and thus gives an element in \( H^{2q+1}(B\Gamma_q^r; \mathbb{R}) \), where \( B\Gamma_q^r \) is the classifying space for \( \Gamma_q^r \)-structures. Note that in fact the BHGV class is a cobordism invariant of codim q foliations of compact \( (2q+1) \)-dim manifolds. This construction gives one computable characteristic class for foliations. Optimally we would like a generalisation of the Chern-Weil construction for \( GL_q \). That is we would like an abstract GDA with the property that for any codim q foliation \( F \) on a manifold \( M \) there is a GDA homomorphism into the de Rham algebra on \( M \), defined in terms of \( F \) such that the induced map on cohomology factors through a universal map into \( H^*(B\Gamma_q^r; \mathbb{R}) \). This algebra is nothing more than the Gelfand-Fuchs Lie coalgebra of formal vector fields in one variable.

More concretely, let \( \Gamma \) be a transitive Lie-pseudogroup acting on \( \mathbb{R}^n \) and let \( a(\Gamma) \) denote the Lie algebra of formal \( \Gamma \) vector fields associated to \( \Gamma \). Here a vector field defined on on \( U \subset \mathbb{R}^n \) is called a \( \Gamma \) vector field if the local 1-parameter group which it engenders is \( \Gamma \) and \( a(\Gamma) \) is defined as the inverse limit

\[ a(\Gamma) = \lim_{\rightarrow} a^k(\Gamma) \]

of the k-jets at 0 of \( \Gamma \) vector fields. In the pseudogroup \( \Gamma \) let \( \Gamma_0 \) be the set of elements of \( \Gamma \) keeping 0 fixed and set \( \Gamma_0^k \) equal to the k-jets of elements in \( \Gamma_0 \).

Then the \( \Gamma_0^k \) form an inverse system of Lie groups and we can find a subgroup \( K \subset \lim_{\rightarrow} \Gamma_0^k \) whose projection on every \( \Gamma_0^k \) is a maximal compact subgroup for \( k > 0 \). This follows from the fact that the kernel of the projection \( \Gamma_0^{k+1} \rightarrow \Gamma_0^k \) is a vector space for \( k > 0 \). The subgroup \( K \) is unique up to conjugation and its Lie algebra \( k \) can be identified with a subalgebra of \( a(\Gamma) \).

For our purposes we need the cohomology of basic elements rel \( K \) in \( a(\Gamma) \), namely \( H(a(\Gamma); K) \) which is defined as follows: Let \( A\{a^k(\Gamma)\} \) denote the
algebra of multilinear alternating forms on \( a^k(\Gamma) \) and let \( A\{a(\Gamma)\} \) be the direct limit of the \( A\{a^k(\Gamma)\} \). The bracket in \( a(\Gamma) \) induces a differential on \( A\{a(\Gamma)\} \) and we write \( H\{a(\Gamma)\} \) for the resulting cohomology group. The relative group \( H^*(a(\Gamma); K) \) is now defined as the cohomology of the subcomplex of \( A\{a(\Gamma)\} \) consisting of elements which are invariant under the natural action of \( K \) and annihilated by all inner products with elements of \( k \). Then the result is:

Let \( F \) be a \( \Gamma \)-foliation on \( M \). There is an algebra homomorphism

\[
\phi : H\{a(\Gamma); K\} \rightarrow H(M; \mathbb{R})
\]

which is a natural transformation on the category \( C(\Gamma) \).

The construction of \( \phi \) is as follows:

Let \( P^k(\Gamma) \) be the differential bundle of \( k \)-jets at the origin of elements of \( \Gamma \). It is a principal \( \Gamma^k_0 \)-bundle. On the other hand \( \Gamma \) acts transitively on the left on \( P^k(\Gamma) \). Denote by \( A(P^\infty(\Gamma)) \) the direct limit of the algebras \( A(P^k(\Gamma)) \) of differential forms on \( P^k(\Gamma) \). The invariant forms wrt the action of \( \Gamma \) constitute a differential subalgebra denoted \( A_{\Gamma} \). One can then prove that it is actually isomorphic to \( A(a(\Gamma)) \).

Now let \( F \) be a foliation on \( M \) and let \( P^k(F) \) be the differentiable bundle over \( M \) whose fibre at every point say \( x \in M \) is the space of \( k \)-jets at this point of local projections that vanish on \( x \). This is a \( \Gamma^k_0 \)-principal bundle. Its restriction is isomorphic to the inverse image of the bundle \( P^k(\Gamma) \), hence the differential algebra of \( \Gamma \)-invariant forms on \( P^k(\Gamma) \) is mapped in the algebra \( A(P^k(F)) \) of differential forms on \( P^k(F) \). If we denote by \( A(P^\infty(F)) \) the direct limit of \( A(P^k(F)) \) we get an injective homomorphism \( \phi \) of \( A(a(\Gamma)) \) in \( A(P^\infty(F)) \) commuting with the differential.

This homomorphism is compatible with the action of \( K \), hence induces a homomorphism on the subalgebra of \( K \)-basic elements. But the algebra \( A(P^k(F); K) \) of \( K \)-basic elements in \( A(P^k(F)) \) is isomorphic to the algebra of differential forms on \( P^k(F)/K \) which is a bundle over \( M \) with contractible fibre \( \Gamma^k_0/K \). Hence \( H(A(P^k(F); K)) \) is isomorphic via the de Rham theorem to \( H(M; \mathbb{R}) \). The homomorphism \( \phi \) is therefore obtained as the composition

\[
H(a(\Gamma); K) \rightarrow H(A(P^\infty(F); K)) = H(M; \mathbb{R})
\]

But we think that is enough with abstract nonsense formalism. Let us make our discussion more down to earth:
Consider the GDA (over $\mathbb{R}$)

$$WO_q = \wedge(u_1, u_3, ..., u_{2(q/2)-1}) \otimes P_q(c_1, ..., c_q)$$

with $du_i = c_i$ for odd $i$ and $dc_i = 0$ for all $i$ and

$$W_q = \wedge(u_1, u_2, ..., u_q) \otimes P_q(c_1, ..., c_q)$$

with $du_i = c_i$ and $dc_i = 0$ for $i=1, ..., q$ where $deg u_i = 2i - 1$, $deg c_i = 2i$ and $\wedge$ denotes exterior algebra, $P_q$ denotes the polynomial algebra in the $c_i$'s mod elements of total degree greater than $2q$. The cohomology of $W_q$ is the Gelfand Fuchs cohomology of the Lie algebra of formal vector fields in $q$ variables. We note that the ring structure at the cohomology level is trivial, that is all cup products are zero. Then the main result is that there are homomorphisms

$$\phi : H^*(WO_q) \to H^*(B\Gamma_q^r; \mathbb{R})$$

$$\tilde{\phi} : H^*(W_q) \to H^*(\tilde{B}\Gamma_q^r; \mathbb{R})$$

for $r \geq 2$ with the following property ($B\Gamma_q^r$ denotes the classifying space for framed foliations): If $F$ is a codim $q$ $C^r$ foliation of a manifold $M$, there is a GDA homomorphism

$$\phi_F : WO_q \to \wedge^*(M)$$

into the de Rham algebra on $M$, defined in terms of the differential geometry of $F$ and unique up to chain homotopy, such that on cohomology we have $\phi_F = f^* \circ \phi$, where $f : M \to B\Gamma_q^r$ classifies $F$. If the normal bundle of $F$ is trivial, there is a homomorphism

$$\tilde{\phi}_F : W_q \to \wedge^*(M)$$

with analogous properties.

Combining this result with the fact that $B\Gamma_q^0$ is contractible, we deduce that a foliation is essentially determined by the structure of its normal bundle; the Chern classes of the normal bundle are contained in the image of the map $\phi$ above but we have additional non trivial classes in the case of foliations.
(which are rather difficult to find though), one of which is this BHGV class which we constructed explicitly and it is the class we use as a Lagrangian density which is purely topological since its degree fits nicely for describing 5-branes.

There is an alternative approach due to Simons [19] which avoids passing to the normal bundle using circle coefficients. What he actually does is to associate to a principal bundle with connection a family of characteristic homomorphisms from the integral cycles on a manifold to $S^1$ and then defining an extension denoted $K^2_k$ of $H^{2k}(BGL_q; \mathbb{Z})$. This approach is related to gerbes. A gerbe over a manifold is a construction which locally looks like the Cartesian product of the manifold with a line bundle. Clearly it is a special case of foliations (remember our previous comment on foliations). However this approach actually suggests that they might be equivalent, if the approach of Bott-Haefliger is equivalent to that of Simons, something which is not known.

Now the conjecture is that the partition function of this Lagrangian is related to the invariant introduced in [9].

In order to establish relation with physics, we must make some identifications. The 1-form $\theta$ appearing in the Lagrangian has no direct physical meaning. In physics it is assumed that a 5-brane gives rise to a 6-form gauge field denoted $A_6$ whose field strength is simply

$$dA_6 = F_7$$  \hspace{1cm} (3)

The only way we can explain geometrically this is that this 6-form is the Poincare dual of the 6-chain that the 5-brane sweeps out as it moves in time.

We know that since we have S-duality between membranes and 5-branes, in an obvious notation one has

$$F_7 = * F_4$$  \hspace{1cm} (4)

which is the S-duality relation, where

$$F_4 = dA_3$$  \hspace{1cm} (5)

Observe now that the starting point for 5-brane theory is $A_6$ where the starting point to construct the BHGV class was the 5-form $\Omega$. How are they related?
There are three obvious possibilities:
I. \(d\Omega = A_6\) That would imply that \(A_6\) is pure gauge.
II. \(dF_4 = \Omega\) This is trivial because it implies \(d\Omega = 0\), hence \(d\Omega = \theta \wedge \Omega = 0\).
III. The only remaining possibility is
\[
* A_6 = \Omega
\] (6)

We call this “reality condition”. So now in principle we can substitute equations (6) and (1) into (2) and get an expression for the Lagrangian which involves the gauge field \(A_6\).

The Euler-Lagrange equations which are actually analogous to D=11 N=1 supergravity Euler-Lagrange equations (see equation 8 below) read:
\[
d * d\theta + \frac{1}{5} (d\theta)^5 = 0
\] (7)

The on-shell relation with D=11 N=1 supergravity is established as follows: recall that the bosonic sector of this supergravity theory is
\[
\int F_4 \wedge F_4 \wedge A_3
\]
where \(F_4 = dA_3\) with Euler-Lagrange equations
\[
d * F_4 + \frac{1}{2} F_4 \wedge F_4 = 0
\] (8)

Constraining \(A_3\) via (8), by (5), (4), (3), (6) and (1) we get a constraint for \(\theta\) which can be added to the class \(\gamma\) as a Lagrange multiplier.

Let us note that a Lagrangian density with only the BHGV class would give as Euler-Lagrange equations the condition that \(\theta\) is closed.

In principle, one must end up with an equivalent theory starting with membranes (that’s due to S-duality), provided of course a suitable class was found. Clearly the BHGV class for a membrane would be a 17-form.

The final comment is this: it is a rather intriguing feature of this approach that although we start with 5-branes which give rise to 6-forms as gauge potentials, we end up with a characteristic class for foliations as a Lagrangian density which involves 1-forms. If we see this 1-form \(\theta\) appearing
in the Lagrangian as a gauge potential, that would imply the existence of point particles which are really the fundamental elements. This idea is in accordance to what was described in [27] for instance, since matrix models use point particles’ degrees of freedom.

0.3 Plane fields

We now pass on to the second question raised in this work, namely the restrictions on the topology of the underlying manifold of a theory containing p-branes via our physical principle.

It is clear from the definition that the existence of a foliation of certain dim, say d (or equivalently codim q = n-d) on an n-manifold (closed) depends:

a.) On the existence of a dim d subbundle of the tangent bundle

b.) On this d-dim subbundle being integrable.

The second question has been answered almost completely by Bott and in a more general framework by Thurston. Bott’s result dictates that for a codim q subbundle of the tangent bundle to be integrable, the ring of Pontrjagin classes of the subbundle with degree > 2q must be zero. There is a secondary obstruction due to Shulman involving certain Massey triple products but we shall not elaborate on this. However Bott’s result suggests nothing for question a.) above. Let us also mention that this result of Bott can be deduced by another theorem due to Thurston which states that the classifying space $B\Gamma_q^\infty$ of smooth codim q framed foliations is (q+1)-connected.

On the contrary, Thurston’s result reduces the existence of codim q > 1 foliations (at least up to homotopy) to the existence of q-plane fields. This is a deep question in differential topology, related to the problem of classification of closed manifolds according to their rank.

Now the problem of existence of q-plane fields has been answered only for some cases for spheres $S^n$ for various values of n, q [16]. In particular we know everything for spheres of dimension 10 and less. We should however mention a theorem due to Winkelnkemper [17] which is quite general in nature and talks about simply connected compact manifolds of dim n greater than 5. If n is not 0 mod 4 then it admits a so-called Alexander decomposition which under special assumptions can give a particular kind of a codim 1 foliation with $S^1$ as space of leaves and a surjection from the manifold to $S1$. 
If \( n \) is 0 mod 4 then the manifold admits an Alexander decomposition iff its signature is zero.

Let us return to string theory now: String theory works in \( D=10 \) and in this case we have the old brane-scan suggesting the string/5-brane duality. The new brane-scan contains all \( p \)-branes for \( p \leq 6 \) and some D-7 and 8-branes are thought to exist. However topology says that for a sphere in dim 10 we can have only dim 0 and dim 10 plane fields (in fact this is true for all even dim spheres), hence by Thurston only dim 0 and dim 10 foliations and then our physical principle suggests that \( S^{10} \) is ruled out as a possible underlying topological space for string theory.

What about M-Theory in \( D=11 \) then?

For the case of \( S^{11} \) then it is known that \( S^{11} \) admits a 3-plane field, hence by our physical principle a theory containing membranes can be formulated on \( S^{11} \). For \( S^{11} \) nothing is known for the existence of \( q \)-plane fields for \( q \) greater than 3. But now we apply S-duality between membranes/5-branes and conjecture that:

\[ S^{11} \text{ should admit 5-plane fields.} \]

Let us close this section with two remarks:

1. There is extensive work in foliations with numerous results which actually insert many extra parameters into their study, for example metric aspects, existence of foliations with compact leaves (all or at least one or exactly one), with leaves diffeomorphic to \( \mathbb{R}^n \) for some \( n \) etc. We do not have a clear picture for the moment concerning imposing these in physics. Let us only mention one particularly strong result due to Wall generalising a result of Reeb [5]: if a closed \( n \)-manifold admits a codim 1 foliation whose leaves are homeomorphic to \( \mathbb{R}^{n-1} \), then by Thurston we know that its Euler characteristic must vanish, but in fact we have more: it has to be the \( n \)-torus!

The interesting point however is that although all these extended objects theories in physics are expressed as \( \sigma \) models [4], hence they involve metrics on the manifold (target space) and on the worldvolumes i.e. on the leaves, in our approach the metric is only used in the reality condition (6) which makes connection with physical fields (that is some metric on the target...
space) where at the same time we do not use any metric on the source space (worldvolumes-leaves of the foliation).

2. In [9] another Lagrangian density was proposed. It is different from the one described here but they are related in an analogous way to the relation between the Polyakov and Nambu-Goto (in fact Dirac [21]) actions for the free bosonic string: extended objects basically immitate string theory and we have two formalisms: the \( \sigma \) model one which is the Lagrangian exhibited in [9] using Polyakov’s picture of \( \sigma \) models as flat principal bundles with structure group the isometries of the metric on the target space [20]; yet we also have the embedded surface picture which is the Dirac (Nambu-Goto) action and whose analogue is described in this work.

0.4 The partition function

Let us try to see what we can say about the quantum theory with the BHGV class as Lagrangian density. Hopefully we can actually say rather a lot.

The starting point will be the crucial observation that the original Godbillon-Vey class for codim 1 foliations is simply

\[ AdA \]

, which is actually the "Abelian part" of the 3-dim Chern-Simmons form. This "topological term" was for the first time used in physics to describe some peculiar properties of the spin and statistics of skyrmions, see [23]. At that moment of course it was not realised that this was actually a characteristic class for foliations. Skyrmions are soliton solutions of 3-dim non-linear \( \sigma \) model with target the group \( O(3) \) and source, in most cases \( \mathbb{R}^3 \) compactified to \( S^3 \). At that paper it was described as the Hopf invariant, emerging from the first known Hopf fibration \( S^2 \to S^3 \). This class is also a characteristic class for flat foliations of bundles. The picture is consistent because from Polyakov we know that non-linear \( \sigma \) models can be seen as flat bundles (for more details on this see [2]).

The Hopf invariant is related to linking number between curves in \( \mathbb{R}^3 \). The non-abelian 3-dim generalisation was related to knots and the Jones’ polynomial in [3]. One of the main differences in these two cases is the
"Dirac quantisation condition" of the parameter needed in the non-abelian case.

This observation make as to believe that the answer to the problem of quantising the BHGV class in 11-dim is actually reduced to quantisation of Abelian Chern-Simons theory. The answer was given by Schwarz [25]: the result is the "Ray-Singer analytic torsion" of the Laplacian on our 11-manifold. As far as we know it is not yet proved the long standing conjecture that this actually coincides with the combinatorial Reidemeister torsion of algebraic topology.

Let us make some remarks on this point: Schwarz’s result is actually more general than what we need in this particular case. He uses the notion of elliptic resolvent of a linear functional. In the special case where this elliptic resolvent is actually an "ellitpic complex", he proves that the partition function of the functional is actually the analytic torsion of the correspond- ing elliptic complex. In the even more special case where the functional is actually of the form

$$\omega^r (d\omega)^{n-r-1}$$

where \(\omega\) is some real valued form (see [25] for more details), which is actually the case for the BHGV class, then the corresponding elliptic complex is the de Rham complex, hence the result.

Some properties of the Ray-Singer torsion (denoted \(T\) in the sequel) might be helpfull:

1. \(T = 0\) for even dim manifolds.
2. The torsion of a Cartesian product equals torsion of first factor raised to the power equal to the Euler characteristic of second factor (simply connected case).
3. For complex manifolds one has to use the Dolbeault complex. In this case for a Riemannian surface (complex dim 1) with genus 1 \(T\) can be expressed in terms of elliptic zeta functions, for higher genus one uses Selberg’s zeta function.
4. For the complex torus of complex dim greater than 1, one has that \(T = 1\).
5. For Hopf surfaces (complex dim 2) one has that the torsion equals the torsion of the torus which is its fibre over the sphere (for these results and
The above discussion actually dictates that our foliation picture to describe branes should rather correspond to a free, i.e. non-interacting theory. The natural generalisation then for interacting 5-branes would be to consider the full 11-dim Chern-Simons theory with some non-abelian gauge group. This theory has been proved to be completely soluble in dim 3 using geometric quantisation methods. We know that for large $k$, using stationary phase approximation (following terminology of [3]), the theory becomes abelian. The problem in dim 11 is considerably harder though. The main point is that in dim 3 one can reduce the problem essentially to Riemann surfaces by performing ”surgeries” on the original 3-manifold. We do not know for the moment if these surgeries can be performed in dim 11. Moreover, in contrast to dim 2 case, geometric quantisation for 10-manifolds is still at its infancy as a theory.

Another point to consider is the following: if one wants to get a non-abelian generalisation as described above, one might loose S-duality. This is an abelian symmetry, namely the isomorphism given by the Hodge star operator for real valued forms. Although we have observed in [14] that the Hodge isomorphism holds for flat bundle valued forms (even with a non-abelian structure group), it brakes down in the general case. Let us recall however that this is the on-shell case for Chern-Simons theory (i.e. the Euler-Lagrange equations simply read that the connection is flat).

Since the partition function of the BHGV class gives the Ray-Singer analytic torsion, this is a topological invariant. Hence if we could find an appropriate 11-form which could characterise foliations with codim 8, namely to start with membranes insted of 5-branes, the result would be the same. Hence S-duality holds. In other words it does not really matter how we foliate our manifold, clearly a manifold can be foliated in many different ways and in many different codimensions, since the partition function of the characteristic class which describes the foliation is a topological invariant. This guarantees S-duality.

The final comment would be that these ”topological terms” in the Lagrangians related to characteristic classes for foliations (very closely related to non-linear $\sigma$ models) are also met in some other interesting cases, namely
$\theta$ vacua and QCD, massive 3-dim Yang-Mills and 3-dim gravity as well as fermions coupled to gauge fields again in dim 3 and the non-conservation of parity (see [24], [23], [22]). All these cases refer to odd dim real manifolds. Maybe of some relevance also for even (real) dim manifolds whose dim is an odd multiple of 2, using complex structure and the Dolbeault complex.
Bibliography

[1] M. J. Duff et all: "String Solitons", Phys. Rep. 259 (1995) 213

[2] M. J. Duff: "Supermembranes", hep-th 9611203

[3] P. K. Townsend: "Four Lectures on M-Theory", hep-th 9612121
P. K. Townsend: "The D=11 supermembrane revisited", Phys. Lett. B350 (1995)

[4] B. de Wit and J. Louis: "Supersymmetry and Dualities in various dimensions", hep-th 9801132

[5] P. C. West: "Supergravity, brane dynamics and string dualities", hep-th 9811101

[6] E. Witten: "String theory dynamics in various dimensions", Nucl. Phys. B443 (1995)
N. Seiberg and E. Witten: Nucl. Phys. B426 (1994)
"Quantum Field Theory and the Jones polynomial", Commun. Math. Phys. 121 (1989), 351-399

[7] C. Montonen and D. Olive: "Magnetic monopoles as gauge particles", Phys. Lett. B72 (1977), 117
[8] R. Bott: ”Lectures on characteristic classes and foliations”, Springer LNM 279, 1972
R. Bott and A. Haefliger: ”Characteristic classes of Γ-foliations”, Bull. Am. Math. Soc. 78.6, (1972)

[9] I. P. Zois: A new invariant for σ models, hep-th 9904001 (to appear in Commun. Math. Phys., communicated by A. Connes)
I.P. Zois: “On search for the M-Theory Lagrangian”, Phys. Lett. B402 (1997)

[10] S.T. Tsou and I. P. Zois: ”Geometric interpretation of two-index potentials as twisted de Rham cohomology” (to appear in Rept. Math. Phys.)

[11] H. B. Lawson: ”Foliations”, Bull. Am. Math. Soc. 80.3, 1974

[12] W. Thurston: “Foliations and groups of diffeomorphisms”, Bull. Am. Math. Soc. 80.2 (1974)

[13] W. Thurston: ”Theory of foliations of codim greater than 1”, Comment. Math. Helvetici 49 (1974), 214-231

[14] M. L. Gromov: “Stable mappings of foliations into manifolds”, Izv. Akad. Nauk. USSR Ser. Mat. 33 (1969)

[15] C. Godbillon and J. Vey: “Un invariant des feuilletages de codim 1”, C R Acad. Sci. Paris Ser AB 273 (1971)

[16] N. Steenrod: “The topology of fibre bundles”, Princeton 1951
[17] H. E. Winkelnkemper: “Manifolds as open books”, Bull. Am. Math. Soc. 79 (1973)

[18] G. Reeb: “Feuillages, resultats anciens et nouveaux”, Montreal 1982

[19] J. Simons: “Characteristic forms and transgression”, preprint SUNY Stony Brook

[20] A.M. Polyakov: “Gauge particles as rings of glue”, Nucl. Phys. B164 (1979)

[21] P.A.M. Dirac: Proc. Roy. Soc. London A166 (1969)

[22] S. Deser et al: ”3-dim massive gauge theories”, Phys. Rev. Lett. 48, No 15 (1982)

[23] F. Wilczek and A. Zee: ”Linking numbers, spin and statistics of solitons”, Phys. Rev. Lett. 51, No 25 (1983)

[24] A.N. Redlich: ”Gauge non-invariance and parity non-conservation of 3-dim fermions”, Phys. Rev. Lett. 52, No 1 (1984)

[25] A.S. Schwarz: ”The partition function of degenerate quadratic functional and the Ray-Singer invariants”, Lett. Math. Phys. 2 (1978)

[26] D.B. Ray and I.M. Singer: ”R-torsion and the Laplacian on Riemannian manifolds”, Advances in Math. 7 (1971)
”Analytic torsion for complex manifolds”, Ann. of Math. 98 (1973)
[27] T. Banks et al: "M-Theory as a Matrix Model: A Conjecture", Phys. Rev. D55 (1997), 5112