Gaussian analysis of the CMB with the smooth tests of goodness of fit

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The study of the Gaussianity of the cosmic microwave background (CMB) radiation is a key topic to understand the process of structure formation in the Universe. In this paper, we review a very useful tool to perform this type of analysis, the Rayner & Best smooth tests of goodness of fit. We describe how the method has been adapted for its application to imaging and interferometric observations of the CMB and comment on some recent and future applications of this technique to CMB data.

1 Introduction

The study of the Gaussianity of the cosmic microwave background (CMB) fluctuations has become a very useful tool in constraining theories of structure formation. The standard inflationary scenario predicts Gaussian fluctuations whereas other competitive theories would imprint non-Gaussian signatures on the CMB (see [5] for a review). Therefore, the study of the Gaussianity of the CMB can help to discard or constrain some of these theories. Moreover, secondary effects (e.g. gravitational lensing, Rees-Sciama effect, Sunyaev-Zeldovich effect...), astrophysical emissions and systematics may as well leave non-Gaussian imprints on the CMB, which should not be confused with intrinsic non-Gaussianity.

Given the importance of this type of analysis and taking into account that different methods may be sensitive to different kinds of non-Gaussianity, many tools have been developed for the study of the temperature distribution of the CMB. Among others, they include the Minkowski functionals [24], the bispectrum [18], wavelet techniques [4], geometrical estimators [27] or smooth tests of goodness of fit [3].

The interest for this type of analysis has increased even more since the release of the WMAP data [7]. A large number of different techniques have
been applied to study whether these data follow or not a homogeneous and isotropic Gaussian random field, finding in some cases unexpected results. In particular, a significant number of works have reported deviations from Gaussianity and/or isotropy, whose origin is uncertain (e.g. [35, 17, 20, 12, 13, 15, 26, 37], see also [25] for a review).

In this paper, we review the Rayner and Best smooth tests of goodness of fit for the study of the Gaussianity of the CMB. In section 2 we describe the test and how to adapt the method for its application to CMB observations. A discussion about current and future applications to different CMB datasets is given in section 3. Finally our conclusions are summarised in section 4.

2 The Rayner and Best smooth tests of goodness of fit

Given a statistical variable \( X \) and \( n \) independent realizations \( x_i, i = 1,...,n \), we want to test if \( X \) follows a given probability density function (pdf) \( f(x) \). The smooth tests of goodness of fit (gof) allows one to discriminate between a predetermined pdf \( f(x) \) (null hypothesis) and a second one that deviates smoothly from the former (alternative hypothesis).

Among the possible forms for the alternative pdf, Rayner & Best [28, 29] consider:

\[
f_k(x, \theta) = C(\theta) \exp \left[ \sum_{i=1}^{k} \theta_i h_i(x) \right] f(x),
\]

where \( \theta = (\theta_1, ..., \theta_k) \) is a set of \( k \) parameters that allows for smooth deviations of the alternative hypothesis with respect to \( f(x) \), \( C(\theta) \) is a normalisation constant that ensures that \( f_k \) is normalised to 1 and \( h_i \) form a complete set of orthonormal functions of \( f \). Note that for \( \theta = 0 \) we recover \( f(x) \), therefore, our statistical analysis consists on testing the null hypothesis \( H_0 : \{\theta = 0\} \) versus the alternative hypothesis \( H_1 : \{\theta \neq 0\} \).

To perform this analysis, the score statistic is used. This is a quantity which is closely related to the likelihood ratio (see e.g. [28]). For the Rayner & Best smooth tests of gof, the score statistic associated to the \( k \) alternative is given by

\[
S_k = \sum_{i=1}^{k} U_i^2
\]

with \( U_i = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} h_i(x_j) \)

Large values of \( S_k \) (or of \( U_i^2 \)) reject the null hypothesis.

In the case of testing if our data follow a Gaussian distribution of zero mean and unit dispersion, the \( h_i \) are given by the (normalised) Hermite Chebishev
polynomials. In this case, it is possible to write the $$U_i^2$$ quantities in terms of the moments of order $$k$$, $$\mu_k$$, of the data:

\[
\begin{align*}
U_1^2 &= n\mu_1^2 \\
U_2^2 &= \frac{n}{2}(\mu_2 - 1)^2 \\
U_3^2 &= \frac{n}{6}(\mu_3 - 3\mu_1)^2 \\
U_4^2 &= \frac{n}{24}(\mu_4 - 6\mu_2 + 3)^2 \\
\end{align*}
\]

If the Gaussian hypothesis holds, the $$U_i^2$$ follow a $$\chi^2$$ distribution when $$n \to \infty$$. This allows one to determine easily the significance of any possible deviation from Gaussianity by comparing the value of the $$U_i^2$$ of the data with a $$\chi^2$$.

We must point out that the proposed technique is designed to test if the data follow a univariate Gaussian. Thus, for optimality, it should be applied to independent data. However, the CMB signal is correlated at all scales and the noise may as well present correlations. Therefore, before applying the gof test, it is necessary to transform the data to make them as independent as possible.

One possibility is to obtain the Cholesky decomposition of the correlation matrix of the data (including signal plus noise) $$C = LL^T$$ and then multiply the $$x_i$$ by the inverse of the Cholesky matrix, i.e. $$y_i = \sum_j L_{ij}^{-1} x_j$$. The constructed $$y_i$$ are uncorrelated, have zero mean and unit dispersion. Moreover if the data are Gaussian, they also follow a normal distribution and are independent. This decorrelation technique has been used for analysing the MAXIMA data with different smooth tests of gof [11, 1, 2]. Nevertheless, the preprocessing of the data has been improved in subsequent works through the use of a signal-to-noise decomposition, which is explained in the next subsection.

2.1 Signal-to-noise decomposition

The signal to noise decomposition was introduced in the CMB field by [10], whereas [3] applied this formalism jointly with the gof test. This technique allows one to construct uncorrelated eigenmodes from the data which are also associated to a certain signal-to-noise ratio.

Let us consider a set of CMB data $$d_i, i = 1, \ldots, n$$, where $$i$$ corresponds to a given position in the sky. This can be written as

\[
d_i = s_i + n_i
\]

where $$s_i$$ and $$n_i$$ are the contributions from the CMB signal and noise, respectively. The mean values of signal and noise are assumed to be zero and their correlation matrices are given by $$S_{ij} = \langle s_i s_j \rangle$$ and $$N_{ij} = \langle n_i n_j \rangle$$ where the brackets indicate average over many realizations.

The signal-to-noise eigenmodes are defined as

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3 The form of the $$h_i$$ for other usual distributions (e.g. uniform, exponential) can be found in [28].

4 The moment of order $$k$$ of the data is defined as $$\mu_k = \sum_{j=1}^{n} y_j^k / n$$
\[ \xi = R_A L_N^{-1} d \]

where \( L_N \) is the Cholesky matrix of \( N \), i.e. \( N = L_N L_N^t \), and \( R \) is the rotation matrix that diagonalizes the matrix \( A = L_N^{-1} S L_N^{-t} \). The eigenvalues of this diagonalization are denoted by \( E_i \). Let us now construct the quantities \( y_i \):

\[ y_i = \frac{\xi_i}{\sqrt{1 + E_i}} \]

It can be shown that these quantities are uncorrelated and have zero mean and unit dispersion. Moreover, if the data \( d \) are multinormal, then the \( y_i \) are distributed according to a Gaussian pdf, since all the applied transformations are linear. In this case the \( y_i \) are also independent. Therefore we are in the optimal conditions to apply the gof tests to the quantities \( y_i \).

In addition, we also have information about the signal-to-noise ratio of the \( i \) eigenmode, which is given by \( \sqrt{E_i} \). This means that eigenmodes with low values of \( E_i \) are dominated by noise and may be discarded from the analysis. Therefore, in practice, the gof test will be applied to the subset of \( y_i \) such that its signal-to-noise ratio is greater than a given threshold, i.e. \( E_i > E_{cut} \). Thus, this decomposition allows us not only to obtain uncorrelated variables but also to select the fraction of the data where the signal contribution dominates over the noise.

### 2.2 Application to interferometer observations

The previous technique has been adapted to deal with interferometric data by [3] and applied to VSA data in [30].

Let us consider an interferometer observing a small region of the sky at frequency \( \nu \), for which the flat-sky approximation is valid. In this case the complex visibility, which is the response of the interferometer at the considered frequency, is given by

\[ V(\mathbf{u}, \nu) = \int P(\hat{x}, \nu) B(\hat{x}, \nu) \exp(i2\pi \mathbf{u} \cdot \hat{x}) d\hat{x} \]

where \( \hat{x} \) corresponds to the angular position of the observed point on the sky and \( \mathbf{u} \) is the baseline vector in units of the wavelength of the observed radiation. \( P(\hat{x}, \nu) \) is the primary beam of the antennas (normalized to unity at its peak) and \( B(\hat{x}, \nu) \) corresponds to the brightness distribution on the sky.

Of course, for a realistic instrument, the effect of instrumental noise should be also taken into account. Therefore, the \( i \)th baseline \( \mathbf{u}_i \) of the interferometer will measure

\[ d(\mathbf{u}_i, \nu) = V(\mathbf{u}_i, \nu) + n(\mathbf{u}_i, \nu) \]

where \( n(\mathbf{u}_i, \nu) \) corresponds to the instrumental noise of the \( \mathbf{u}_i \) visibility.

Let be \( N \) the total number of complex visibilities observed by the interferometer. Since the measured quantities are complex, the number of elements
that constitute the data are \( N_d = 2N \), corresponding to the real and imaginary parts of each observed visibility.

Testing the Gaussianity of the measured visibilities is equivalent to testing the joint Gaussianity of their real and imaginary parts. Therefore the signal-to-noise decomposition can be applied directly to these quantities (so we will have a total of \( N_d \) eigenmodes). The correlation matrix \( S \) of the real and imaginary parts of \( V(u_i, \nu) \) (i.e. the correlation matrix of the signal) can be computed following the work of [21] whereas the noise correlation matrix is determined by the characteristics of the instrument. Once the signal-to-noise eigenmodes have been obtained, the gof technique can be applied to test the Gaussianity of these quantities (or of a subset of them with the highest signal-to-noise ratio).

As in the previous case, if the data are distributed as a multinormal, the constructed eigenmodes are independent and follow a Gaussian distribution of zero mean and unit dispersion.

A complementary analysis can also be performed on the phases of the decorrelated visibilities. If the data are Gaussian, the phases should follow a uniform distribution. This can be tested using the Rayner & Best smooth tests of gof by considering the appropriate \( h_i \) in equation (2) (see [28, 3] for details). However, [3] found that, for their considered examples, the phase analysis was less sensitive to deviations from Gaussianity than the test based on the real and imaginary parts of the visibilities.

2.3 Some comments about the method

One of the advantages of the Gaussianity analysis based on the gof test and the signal-to-noise formalism is that it is well suited for the study of many different kinds of CMB observations. In particular, it can be adapted to deal with most of the problematics found in real data. For instance, it is not affected by the presence of holes in the data or by the use of irregular masks and it can easily deal with anisotropic and/or correlated noise. Also, as already explained, it can be applied to imaging or interferometric data. Another interesting feature of the method is that it allows one to choose that fraction of the data with a signal-to-noise ratio above a certain threshold. In addition, as will be discussed in the next section, it is a very sensitive technique, being able to detect different type of deviations in the data (such as intrinsic non-Gaussianity, systematic effects or anisotropy of the local power spectrum).

The main shortcoming of the technique is the large amount of CPU required to calculate the signal-to-noise eigenmodes, since it involves the diagonalization of large matrices (of size \( n \times n \), where \( n \) is the number of data to be analysed). However, the method uses only a fraction of the eigenmodes (those whose signal-to-noise ratio is higher than a given threshold) and therefore it is not necessary to obtain all the eigenmodes and eigenvalues of the problem. To take advantage of this fact, [30] proposes the use of the Arnoldi algorithm which significantly speeds the calculation of the required \( y_i \). This method is based on the construction of a matrix \( H \) of dimension \( m \times m \) (with \( m < n \))
such that it is possible to construct a good approximation to certain eigenvectors and eigenvalues of $A$ from those of $H$. In particular, the eigenvectors that are well approximated correspond to those with higher eigenvalues. From these quantities it is also possible to construct those eigenmodes with higher signal-to-noise ratio, i.e., those that are kept for the analysis (see [30, 31] for details). This means that we have significantly reduced the computational cost of the analysis, since we are working with a matrix of size $m \times m$ instead of $n \times n$.

3 Applications to CMB data

The gof tests were firstly introduced in the CMB field by [11], which carried out a Gaussianity analysis of the MAXIMA data [19]. The results showed that the data were compatible with Gaussianity (see also [1, 2]).

A more recent application of the Rayner & Best gof test has been carried out by [30], that present a Gaussianity analysis of the Very Small Array (VSA) data [34, 23, 16]. The VSA is an interferometer sited at the Teide Observatory (Tenerife) designed to observe the sky on scales going from $2^\circ$ to $10'$ and operates at frequencies between 26 and 36 GHz (see [36] for a detailed description).

In the analysis, most of the fields observed by the VSA were found to be compatible with Gaussianity. However, deviations from Gaussianity were detected in the $U_2^2$ statistic in three cases. After a thorough analysis of the possible origins of these detections, the authors concluded that one of the deviations was associated to a residual systematic effect of a few visibility points, which, when corrected, have a negligible effect on the angular power spectrum. A second detection seemed to have its origin in a deviation of the local power spectrum of the considered field with respect to the power spectrum estimated from the complete dataset. This deviation was found at angular scales around the third angular peak ($\ell = 700 - 900$). If the affected visibilities were removed, a cosmological analysis based only on this modified power spectrum and the COBE data showed no differences except for the physical baryon density, which decreased by 10 per cent and got closer to the value obtained from Big Bang Nucleosynthesis. Finally, the third deviation from Gaussianity was found in observations of the Corona Borealis supercluster region [22]. In this case, the non-Gaussianity was identified as intrinsic to the data, probably due, at least in part, to the presence of Sunyaev-Zeldovich emission in the region. This result has been later confirmed with the measurements of the MITO telescope in this region [6]. A combined maximum likelihood analysis of the MITO and the VSA data provided a weak detection of a faint signal compatible with a SZ effect, characterized by a Comptonization parameter of $y = (7.8^{+5.3}_{-4.4}) \times 10^{-6}$, at 68% CL.

An application of the gof technique to the Archeops data is currently ongoing [14]. Archeops is a balloon-borne experiment, which is dedicated to
measure the CMB temperature anisotropies from large to small angular scales [8, 9]. It has also been designed as a test bed for the forthcoming Planck high frequency instrument. The preliminary results show the good performance of the method, that is able to deal with the presence of anisotropic and correlated noise in the data.

The application of the gof technique to the WMAP data [7] is of great interest and is currently in progress. Due to the large amount of data observed by this experiment, a whole sky analysis at full resolution is unfeasible, due to the large computational resources required for the signal-to-noise decomposition. However, two types of complementary tests are possible: an analysis of the full-sky at low-resolution and a study of small regions of the sky at high resolution. Given the sensitivity of the gof tests to detect deviations from a homogeneous and isotropic Gaussian random field, this analysis could shed new light on some of the anomalies reported for the WMAP data.

4 Conclusions

We have reviewed the Rayner & Best smooth tests of goodness of fit and its applications to CMB data. One of the most interesting features of this method is that it can deal with most of the problematics found in real data such as the use of irregular masks or the presence of anisotropic and/or correlated noise. In addition, it has been adapted to deal either with imaging or interferometric observations. The main shortcoming of the technique is the large computational cost required to perform the signal-to-noise decomposition of the data. However, this problem can be significantly alleviated by the use of approximate methods such as the Arnoldi algorithm.

The recent and current applications of the gof tests to different datasets are showing its good performance. Most notably, the method has been able to detect deviations from a homogeneous and isotropic Gaussian field in the VSA data, which were associated to very different origins: residual systematics, a deviation of the local power spectrum with respect to the global one and non-Gaussianity intrinsic to the data. It is important to mention that Gaussianity analyses had already been performed in the VSA dataset using other methods [32, 33] but neither the residual systematics nor this small deviation of the power spectrum were detected. Therefore we believe that this method constitutes a very useful tool for the statistical analysis of CMB data.

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