COMMUTATIVE ALGEBRAS OF ORDINARY DIFFERENTIAL OPERATORS WITH MATRIX COEFFICIENTS

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Abstract. A classification of commutative integral domains consisting of ordinary differential operators with matrix coefficients is established in terms of morphisms between algebraic curves.

1. Introduction and the main results.

The purpose of this paper is to establish a geometric classification of commutative integral domains consisting of linear ordinary differential operators with matrix coefficients. From an integral domain of matrix ordinary differential operators we construct a morphism between two irreducible algebraic curves and certain torsion-free sheaves on them which are compatible with the morphism. Conversely, from a set of geometric data consisting of an arbitrary morphism of integral curves and torsion-free sheaves on them with vanishing cohomology groups, we construct an integral domain of matrix ordinary differential operators. These two constructions are inverse to one another. In this correspondence, arbitrary integral curves (singular curves as well) appear.

It was G. Wallenberg who first recognized the rich mathematical structure in commuting ordinary differential operators. Through an explicit computation he observed that the

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coefficients of two commuting linear ordinary differential operators of orders 2 and 3 with scalar function coefficients are given by elliptic functions. The 1903 paper [21] thus initiated the long history of attempts towards the classification of commutative subalgebras of the ring of ordinary differential operators with scalar coefficients. The classification problem was finally completed in 1990 by Mulase [13] in terms of the moduli spaces of pointed algebraic curves and vector bundles defined on them. His results are based on the earlier work of Schur [18], Burchnall and Chaundy [3], Gel’fand and Dickey [5], Krichever [8], Mumford [15], Verdier [20], Sato [17] and others. Previato and Wilson [16] also made a decisive contribution to this problem.

In 1993, Adams and Bergvelt [1] discovered a beautiful three-fold relation between maximally commutative subalgebras of the loop algebra of $gl(n, \mathbb{C})$, certain morphisms between algebraic curves, and infinite-dimensional integrable systems generalizing the KP equations. Their work gave a big impact to the further development of Li and Mulase [9] in establishing a characterization of the generalized Prym varieties in terms of the multi-component KP equations. As a byproduct, a new class of commutative algebras of ordinary differential operators with matrix coefficients was constructed from morphisms between algebraic curves.

In his dissertation research, Kimura [7] realized that a more general class of commutative subalgebras of the loop algebra than those studied by [1] and [9] would lead to morphisms between singular algebraic curves. A similar observation was made by Donagi and Markman [4].

It suggests the possibility of constructing a type of Galois theory of commutative algebras of differential operators. Let $B$ be a commutative algebra of ordinary differential operators. If we choose an element $P \in B$, then $B$ is an extension of a subalgebra $\mathbb{C}[P]$. If these algebras correspond to algebraic curves, then the inclusion relation should correspond to a morphism between the curves. Do we obtain an arbitrary morphism of curves in this way? Conversely, if we start with an arbitrary morphism of curves, can we construct a commutative algebra of ordinary differential operators and its extension that correspond to the original geometric situation?

We answer these question in this paper. The complexity of nilpotent elements and zero divisors arises in the theory of differential operators with matrix coefficients. From a geometric point of view, it is natural to consider morphisms between irreducible curves as building blocks for more general cases. So we restrict ourselves to irreducible curves, and deal only with integral domains of differential operators here. The general case will be treated elsewhere.

We establish the following two theorems in this paper.

**Theorem 1.1.** Let $f : C \rightarrow C_d$ be an arbitrary morphism of degree $n$ between reduced irreducible algebraic curves $C$ and $C_d$ defined over the field $\mathbb{C}$ of complex numbers, and let $p \in C_d$ be a smooth point of $C_d$. Then for every torsion-free rank one sheaf $\mathcal{L}$ defined on $C$ such that

$$H^0(C, \mathcal{L}) = H^1(C, \mathcal{L}) = 0,$$

we have...
there is an injective algebra homomorphism

\[ j_C : H^0(C \setminus f^{-1}(p), \mathcal{O}_C) \hookrightarrow \mathfrak{gl}(n, D) \]

of the coordinate ring of \(C\) into the ring \(\mathfrak{gl}(n, D)\) of matrix ordinary differential operators, where

\[ D = (\mathbb{C}[[x]]) \left[ \frac{d}{dx} \right] \]

is the algebra of differential operators with coefficients in the ring \(\mathbb{C}[[x]]\) of formal power series in one variable.

The next theorem gives the converse.

**Theorem 1.2.** Let \(B \subset \mathfrak{gl}(n, D)\) be a commutative integral domain consisting of matrix differential operators. We assume that \(B\) is elliptic in the sense of Verdier, namely, \(B\) has a monic element of positive order, say

\[ P = I_n \cdot \left( \frac{d}{dx} \right)^r + \text{lower order terms} \in B. \]

Consider a \(\mathbb{C}\)-subalgebra

\[ B_d \subset B \cap \mathbb{C}(\langle P^{-1/r} \rangle) \]

of \(B\). We further impose that

1. The subalgebra \(B_d\) has rank one, i.e., the greatest common divisor of the orders of operators in \(B_d\) is equal to 1;
2. \(B\) is a rank \(n\) module over \(B_d\). (Since \(B\) is integral and hence torsion-free over \(B_d\), we can define its rank.)

Then there exist a degree \(n\) morphism of integral algebraic curves

\[ f : C \longrightarrow C_d, \]

a torsion-free rank one sheaf \(\mathcal{L}\) defined over \(C\), and a smooth point \(p \in C_d\) such that the construction of Theorem 1.1 gives back to \(B\). Moreover, the collection of the data \((B_d, B, P)\) satisfying the above conditions and the set of geometric data \(\langle f : C \rightarrow C_d, \mathcal{L}, p, z, \psi \rangle\) satisfying the conditions of Theorem 1.1 are in one-to-one correspondence.

**Remark 1.** The ground field should be an algebraically closed field of characteristic 0. We take it to be just the field of complex numbers.

**Remark 2.** If we start with a non-integral curve \(C\) and a morphism onto an integral curve \(C_d\) with the same conditions as in Theorem 1.1, then we obtain a non-integral commutative subalgebra of \(\mathfrak{gl}(n, D)\).

This paper is organized as follows. In Section 2, we study commutative subalgebras of a formal loop algebra and give a method of constructing morphisms between algebraic
curves from the data of commutative loop algebras. In Section 3, we establish the converse direction, i.e., from a set of geometric data consisting of a morphism between integral curves and vector bundles on them, we give a set of algebraic data which are commutative subalgebras of a loop algebra and a point of an infinite Grassmannian. The relation between matrix ordinary differential operators and the loop algebras is studied in Section 4. More detailed study has been given in [9]. In Section 5 we complete the proof of Theorem 1.1. Our theory includes the spectral curves of [2], [7], [11] as special cases. In Section 6, we show that from each spectral curve we can construct a commutative integral domain of matrix ordinary differential operators. The proof of Theorem 1.2 is completed in Section 7.

2. Formal loop algebras and algebraic curves.

Let \( \mathbb{C} \) be the field of complex numbers and \( \mathbb{C}[[z]] \) the ring of formal power series in one variable \( z \). The field of formal Laurent series, \( \mathbb{C}((z)) \), is the field of fractions of \( \mathbb{C}[[z]] \). There is a \( \mathbb{C} \)-vector space direct sum decomposition

\[
\mathbb{C}((z)) = \mathbb{C}[z^{-1}] \oplus z \cdot \mathbb{C}[[z]].
\]

The \( z \)-adic topology of \( \mathbb{C}((z)) \) is introduced by defining \( z^k \cdot \mathbb{C}[[z]], k \in \mathbb{Z} \), as the system of open neighborhoods of 0 in \( \mathbb{C}((z)) \). An element \( a \in \mathbb{C}((z)) \) is said to have order \( m \), which is denoted by \( \text{ord}(a) = m \), if \( a \in z^{-m} \cdot \mathbb{C}[[z]] \setminus z^{-m+1} \cdot \mathbb{C}[[z]] \). The negative sign indicates that we are considering the pole order of elements of \( \mathbb{C}((z)) \).

We fix a positive integer \( n \) throughout the paper. Let us consider the formal loop algebra \( gl(n, \mathbb{C}((z))) \). There is a natural filtration

\[
\text{gl}(n, \mathbb{C}((z))) = \bigcup_{k \in \mathbb{Z}} z^k \cdot \text{gl}(n, \mathbb{C}[[z]])
\]

compatible with the addition and multiplication:

\[
\begin{align*}
+ & : z^k \cdot \text{gl}(n, \mathbb{C}[[z]]) \times z^\ell \cdot \text{gl}(n, \mathbb{C}[[z]]) \rightarrow z^{k+\ell} \cdot \text{gl}(n, \mathbb{C}[[z]]) \\
\cdot & : z^k \cdot \text{gl}(n, \mathbb{C}[[z]]) \times z^\ell \cdot \text{gl}(n, \mathbb{C}[[z]]) \rightarrow z^{k+\ell} \cdot \text{gl}(n, \mathbb{C}[[z]])
\end{align*}
\]

The main object of this paper is a (commutative) integral domain \( A \subset \text{gl}(n, \mathbb{C}((z))) \). We denote

\[
A^{(k)} = A \cap \left( z^{-k} \cdot \text{gl}(n, \mathbb{C}[[z]]) \right).
\]

Thus there is a natural filtration

\[
A = \bigcup_{k \in \mathbb{Z}} A^{(k)}
\]

compatible with the addition and multiplication in \( A \). Let us choose a scalar diagonal \( \mathbb{C} \)-subalgebra \( A_d \) of \( A \):

\[
\mathbb{C} \subset A_d \subset A \cap \left( \mathbb{C}((z)) \cdot I_n \right).
\]
We identify $A_d$ as a subalgebra of $\mathbb{C}((z))$. Since $A_d \subset A$ and $A$ is integral, $A$ is a torsion-free module over $A_d$. We define
\[
\text{rank}(A_d) = \gcd\{\text{ord}(a) \mid a \in A_d\}.
\]

In order to avoid the extra complexity, we assume that the commutative integral domain $A$ and a scalar diagonal subalgebra $A_d$ satisfy the following conditions:

**Condition 2.1.**

(1) $\text{rank}(A_d) = 1$.

(2) As a torsion-free $A_d$-module, $A$ is of rank $n$.

**Theorem 2.2.** Let $(A_d, A)$ satisfy Condition 2.1. If $A$ satisfies further that
\[
A^{(-1)} = A \cap \left(z \cdot gl(n, \mathbb{C}[z])\right) = 0,
\]
then the pair $(A_d, A)$ defines a morphism
\[
f : C \rightarrow C_d
\]
of degree $n$ between two reduced irreducible complete algebraic curves $C$ and $C_d$.

**Proof.** Let $A_d^{(m)} = A_d \cap A^{(m)}$. We note that
\[
\text{dim}_\mathbb{C} A_d^{(m)} / A_d^{(m-1)} \leq 1.
\]

Since $A_d$ is of rank 1, there are elements $a$ and $b$ in $A_d$ such that $\gcd(\text{ord}(a), \text{ord}(b)) = 1$. Let
\[
N_{A_d} = \{\text{ord}(h) \mid h \in A_d\} \subset \mathbb{N} = \{0, 1, 2, 3, \ldots\}.
\]

Then it is easy to show that $N_{A_d}$ contains all integers greater than
\[
\text{ord}(a) \cdot \text{ord}(b) - \text{ord}(a) - \text{ord}(b).
\]

This fact, together with (2.2) and that $N_{A_d}$ does not contain any negative integers, imply that

(1) $A_d$ is a finite module over $\mathbb{C}[a, b]$, and

(2) $b$ is algebraic over $\mathbb{C}[a]$.

Thus $\text{Spec}(A_d)$ is an affine algebraic curve. Following [9], [12], [13] and [15], we define a graded algebra
\[
gr(A_d) = \bigoplus_{m=0}^{\infty} A_d^{(m)}
\]
and a projective scheme

\[(2.3) \quad C_d = \text{Proj}(gr(A_d)).\]

Let

\[A_d \left[ \frac{1}{a} \right]_0 = \left\{ \frac{h}{a^m} \mid m \geq 0, h \in A_d, \ord(h) \leq m \cdot \ord(a) \right\} \subset \mathbb{C}[z].\]

Then

\[A_d \cap A_d \left[ \frac{1}{a} \right]_0 = A_d^{(0)} = \mathbb{C},\]

because \(A_d^{(-1)} = 0\). Therefore,

\[(2.4) \quad \text{Proj}(gr(A_d)) = \text{Spec}(A_d) \cup \text{Spec} \left( A_d \left[ \frac{1}{a} \right]_0 \right),\]

because a regular function on \(\text{Spec}(A_d)\) that is also regular on \(\text{Spec} \left( A_d \left[ \frac{1}{a} \right]_0 \right)\) is a constant. Thus these two affine subschemes cover the whole projective scheme \(\text{Proj}(gr(A_d))\). Since \(\text{Spec}(A_d)\) is an affine curve, \(C_d\) is a complete algebraic curve. As is shown in the literature cited above, \(C_d\) is a one-point completion of \(\text{Spec}(A_d)\). The \(z\)-adic formal completion of

\[A_d \left[ \frac{1}{a} \right]_0 \subset \mathbb{C}[z] \]

is equal to \(\mathbb{C}[z]\). The fact that

\[A_d^{(0)} = A_d \cap \mathbb{C}[z] = \mathbb{C}\]

shows that

\[(2.5) \quad C_d = \text{Spec}(A_d) \cup \{p\},\]

where \(p \in C_d\) is the unique nonsingular geometric point of \(\text{Spec}(\mathbb{C}[z])\). Since it is the point of \(C_d\) defined by an equation \(z = 0\), \(p\) is a (rational) nonsingular point of \(C_d\). Thus \(C_d\) is a one-point completion of \(\text{Spec}(A_d)\) by a nonsingular point.

Similarly, we define

\[gr(A) = \bigoplus_{m=0}^{\infty} A^{(m)}.\]

Since \(gr(A)\) is a finite-rank module over \(gr(A_d)\),

\[(2.6) \quad C = \text{Proj}(gr(A))\]

is a complete algebraic curve, and the natural inclusion

\[gr(A_d) \subset gr(A)\]
defines a surjective morphism

\[ f : C \longrightarrow C_d. \]

Since \( A \) is of rank \( n \) over \( A_d \), the degree of the morphism \( f \) is also \( n \). This completes the proof.

Note that \( A^{(0)} \) is a \( \mathbb{C} \)-subalgebra of \( A \). It is an integral domain, and since \( A^{(-1)} = 0 \), it is a subalgebra of \( gl(n, \mathbb{C}) \). Therefore, every element of \( A^{(0)} \) is algebraic over \( \mathbb{C} \), and hence

\[ A^{(0)} = \mathbb{C}. \]

Let us define

\[ A \left[ \frac{1}{a} \right]_0 = \left\{ \frac{h}{a^m} \mid m \geq 0, \ h \in A^{(m, \text{ord}(a))} \right\} \subset gl(n, \mathbb{C}[[z]]). \]

Then

\[ A \cap A \left[ \frac{1}{a} \right]_0 = A^{(0)} = \mathbb{C}. \]

Thus

\[ C = \text{Proj}(gr(A)) = \text{Spec}(A) \cup \text{Spec} \left( A \left[ \frac{1}{a} \right]_0 \right) \]

because of the same reason as in (2.4). However, this time \( C \) is neither a one-point completion of \( \text{Spec}(A) \) nor a completion by smooth points. The \( z \)-adic formal completion of \( A \left[ \frac{1}{a} \right]_0 \) is

\[ A \left[ \frac{1}{a} \right]^\wedge_0 = A \left[ \frac{1}{a} \right]_0 \widehat{\otimes}_{A_d \left[ \frac{1}{a} \right]_0} \mathbb{C}[[z]], \]

which is a module of rank \( n \) over \( \mathbb{C}[[z]] \) because of the second item of Condition 2.1. The formal scheme

\[ \text{Spec} \left( A \left[ \frac{1}{a} \right]^\wedge_0 \right) = \widehat{C}_{f^{-1}(p)} \]

is the formal completion of \( C \) along the divisor \( f^{-1}(p) \). In general, it is not smooth. If we consider even a larger extension

\[ A \left[ \frac{1}{a} \right]^\wedge_0 \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)) \],

then it is an \( n \)-dimensional vector space over the field \( \mathbb{C}((z)) \). We can choose a \( \mathbb{C}((z)) \)-linear basis for the above vector space from \( A \left[ \frac{1}{a} \right]_0 \), say \( X_1, \cdots, X_n \). Then

\[ \mathbb{C}[[z]][X_1, \cdots, X_n] \subset A \left[ \frac{1}{a} \right]^\wedge_0. \]
If there is an element \( X \in A \left[ \frac{1}{a} \right]_0 \) such that
\[
\mathbb{C}[[z]][X] = A \left[ \frac{1}{a} \right]_0 ,
\]
then
\[
\tilde{C}_{f^{-1}(p)} = \text{Spec}(\mathbb{C}[[z]][X]) = \text{Spec} \left( \frac{\mathbb{C}[[z]][t]}{(ch_X(t))} \right),
\]
where \( ch_X(t) \in \mathbb{C}[[z]][t] \) is the characteristic polynomial of \( X \). This is the case of a spectral curve, where the covering of \( \text{Spec}(\mathbb{C}[[z]]) \) at infinity is defined by the characteristic polynomial \( ch_X(t) \). It can well be singular, but the singularity of \( \tilde{C}_{f^{-1}(p)} \) is far more general than those appearing in the case of spectral curves.

3. The geometric data and the Grassmannian.

The geometric data we deal with in this section are the following: Let \( C \) be a reduced irreducible algebraic curve, and
\[
f : C \longrightarrow C_d
\]
an algebraic morphism of degree \( n \) onto another reduced irreducible algebraic curve \( C_d \). Here we assume that everything is defined over the field \( \mathbb{C} \). We choose, once and for all, a smooth point \( p \in C_d \) and an isomorphism
\[
\tilde{C}_{d,p} \sim \text{Spec}(\mathbb{C}[[z]])
\]
of the formal completion \( \tilde{C}_{d,p} \) of \( C_d \) at the divisor \( p \) and the formal scheme \( \text{Spec}(\mathbb{C}[[z]]) \). The isomorphism (3.2) amounts to give a formal coordinate \( z \) of \( C_d \) such that the equation \( z = 0 \) defines the point \( p \).

Let \( L \) be a torsion-free sheaf on \( C \) of rank 1, and let \( F = f_*L \) be its push-forward, which is a torsion-free sheaf of rank \( n \) on \( C_d \). We choose a formal trivialization
\[
\psi : F|_{\tilde{C}_{d,p}} \sim \mathcal{O}_{\tilde{C}_{d,p}}(-p)^{\oplus n}
\]
of the vector bundle \( F|_{\tilde{C}_{d,p}} \) on the formal completion. We also use the same notation \( \psi \) for
\[
\psi : H^0(\tilde{C}_{d,p}, F|_{\tilde{C}_{d,p}}) \sim H^0(\tilde{C}_{d,p}, \mathcal{O}_{\tilde{C}_{d,p}}(-p)^{\oplus n}) = (\mathbb{C}[[z]] \cdot z)^{\oplus n}.
\]
Because of the definition, \( F = f_*\mathcal{L} \) is an \( \mathcal{O}_C \)-module of rank 1. Thus we have an \( \mathcal{O}_{C_d} \)-algebra homomorphism

\[
(3.4) \quad f_*\mathcal{O}_C \rightarrow \mathcal{E}nd_{\mathcal{O}_{C_d}}(F).
\]

Since \( \mathcal{L} \) is torsion-free over \( \mathcal{O}_C \), the homomorphism (3.4) is injective. The formal trivialization \( \psi \) gives the matrix representation of

\[
\text{End}(F) = H^0(C_d, \mathcal{E}nd_{\mathcal{O}_{C_d}}(F)) \subset H^0(\widehat{C}_{d,p}, \mathcal{E}nd_{\mathcal{O}_{C_d}}(F)|_{\widehat{C}_{d,p}}) = \text{End}(F|_{\widehat{C}_{d,p}})
\]

around the smooth point \( p \):

\[
\psi^* \otimes \psi : \text{End}(F|_{\widehat{C}_{d,p}}) \sim \rightarrow gl(n, \mathbb{C}[[z]]).
\]

In particular, the coordinate ring of the curve \( C \) admits an embedding \( h : H^0(C \setminus f^{-1}(p), \mathcal{O}_C) \cong H^0(C_d \setminus \{p\}, f_*\mathcal{O}_C) \hookrightarrow H^0(C_d \setminus \{p\}, \mathcal{E}nd_{\mathcal{O}_{C_d}}(F)) \hookrightarrow gl(n, \mathbb{C}((z))) \)

into a formal loop algebra. We denote by \( A \) its image:

\[
(3.5) \quad A = h(H^0(C \setminus f^{-1}(p), \mathcal{O}_C)) \subset gl(n, \mathbb{C}((z))).
\]

Let

\[
(3.6) \quad W = \psi(H^0(C_d \setminus \{p\}, F)) \subset \mathbb{C}((z))^\oplus n,
\]

and

\[
(3.7) \quad \gamma_W : W \rightarrow \frac{\mathbb{C}((z))^\oplus n}{(\mathbb{C}[[z]] \cdot z)^\oplus n}
\]

be the natural projection. Since \( \psi \) gives an isomorphism

\[
H^0(\widehat{C}_{d,p}, F|_{\widehat{C}_{d,p}}) \cong (\mathbb{C}[[z]] \cdot z)^\oplus n,
\]

we have a canonical isomorphism

\[
H^0(C_d, F) \cong \text{Ker}(\gamma_W).
\]

Similarly, the covering cohomology computation

\[
H^1(C_d, F) \cong \frac{H^0(\widehat{C}_{d,p} \setminus \{p\}, F|_{\widehat{C}_{d,p}})}{H^0(C_d \setminus \{p\}, F) + H^0(\widehat{C}_{d,p}, F|_{\widehat{C}_{d,p}})}
\]
gives

\[(3.9) \quad H^1(C_d, \mathcal{F}) \cong \text{Coker}(\gamma_W).\]

For more detail of this computation, see [13]. Motivated by (3.8) and (3.9), we define the infinite-dimensional Grassmannian by

\[Gr_n = \{ W \subset \mathbb{C}((z))^{\oplus n} \mid \gamma_W \text{ of (3.7) is Fredholm} \}.\]

We note that the Fredholm condition automatically implies that \( W \) is a closed subset of \( \mathbb{C}((z))^{\oplus n} \) with respect to the product topology of the Krull topology in \( \mathbb{C}((z)) \). The big-cell \( Gr^+_n \) is defined to be the subset of \( Gr_n \) consisting of points \( W \) such that \( \gamma_W \) is an isomorphism.

Let us denote by \( \mathcal{G} \) the collection of the geometric data

\[(3.10) \quad \langle f : C \to C_d, \mathcal{L}, p, z, \psi \rangle\]

described above. We identify

\[(3.11) \quad \langle f' : C' \to C'_d, \mathcal{L}', p', z', \psi' \rangle\]

with (3.10) if there is an isomorphism

\[j : C_d \overset{\sim}{\to} C'_d\]

such that (3.10) is the pull back of (3.11) via \( j \). We have shown that a collection of geometric data (3.10) gives rise to a collection

\[(3.12) \quad \langle i : A_d \hookrightarrow A, W \rangle\]

of algebraic data, where \( A \) and \( W \) are as in (3.5) and (3.6), and

\[(3.13) \quad A_d = h(f^*(H^0(C_d \setminus \{p\}, \mathcal{O}_{C_d}))) \subset A \subset \text{gl}(n, \mathbb{C}((z))).\]

Of course by definition \( A_d \subset \mathbb{C}((z)) \), and the inclusion \( A_d \subset A \) is the scalar diagonal embedding. The set of all algebraic data (3.12) is denoted by \( \mathcal{A} \). We call the map

\[(3.13) \quad \mu : \mathcal{G} \ni \langle f : C \to C_d, \mathcal{L}, p, z, \psi \rangle \mapsto \langle i : A_d \hookrightarrow A, W \rangle \in \mathcal{A}\]

the cohomology map, because of the association (3.5) and (3.6).

From (3.8) and (3.9) we see that the image \( \langle i : A_d \hookrightarrow A, W \rangle \) of \( \mu \) has a point \( W \) of the big-cell Grassmannian \( Gr^+_n \) if and only if

\[(3.14) \quad H^0(C, \mathcal{L}) = H^1(C, \mathcal{L}) = 0,\]
because
\[ H^i(C, \mathcal{L}) \cong H^i(C_d, \mathcal{F}) . \]
Let us denote by \( \mathcal{G}^+ \) the set of geometric data satisfying (3.14), and by \( \mathcal{A}^+ \) the set of algebraic data such that \( W \in Gr_n^+ \). Then we have a map
\[ \mu : \mathcal{G}^+ \rightarrow \mathcal{A}^+ . \]
If \( C \) is non-singular, then a general line bundle \( \mathcal{L} \) on \( C \) of degree \( g(C) - 1 \) satisfies (3.14), where \( g(C) \) denotes the genus of \( C \). If \( C \) is singular, then let
\[ r : \tilde{C} \rightarrow C \]
be the normalization of \( C \). Choose a line bundle \( \tilde{\mathcal{L}} \) on \( \tilde{C} \) with vanishing cohomology groups, and define
\[ \mathcal{L} = r_*(\tilde{\mathcal{L}}) . \]
Then \( \mathcal{L} \) is a torsion-free sheaf of rank 1 on \( C \) satisfying (3.14).

4. The Grassmannian and the pseudodifferential operators.

In order to embed the coordinate ring \( A \) of an algebraic curve \( C \) into the ring of matrix ordinary differential operators, we need the theory of matrix pseudodifferential operators. Let us denote by
\[ E = (\mathbb{C}[[x]])((\partial^{-1})) \]
the set of all pseudodifferential operators with coefficients in \( \mathbb{C}[[x]] \), where \( \partial = d/dx \). This is an associative algebra and has a natural filtration
\[ E^{(m)} = (\mathbb{C}[[x]])[[\partial^{-1}]] \cdot \partial^m \]
by the order of operators. We can identify \( \mathbb{C}((z)) \) with the set of pseudodifferential operators with constant coefficients, where \( z = \partial^{-1} \):
\[ \mathbb{C}((z)) = \mathbb{C}((\partial^{-1})) \subset E . \]
There is also a canonical projection
\[ (4.1) \quad \rho : E \rightarrow E/Ex \cong \mathbb{C}((\partial^{-1})) = \mathbb{C}((z)) , \]
where \( Ex \) is the left-maximal ideal of \( E \) generated by \( x \). In an explicit form, this projection is given by
\[ (4.2) \quad \rho : E \ni P = \sum_{m \in \mathbb{Z}} \partial^m \cdot a_m(x) \mapsto \sum_{m \in \mathbb{Z}} a_m(0)z^{-m} \in \mathbb{C}((z)) . \]
It is obvious from (4.1) that $\mathbb{C}((z))$ is a left $E$-module. The action is given by
\[
P \cdot v = P \cdot \rho(Q) = \rho(PQ),
\]
where $v \in \mathbb{C}((z)) = E/E x$ and $Q \in E$ is a representative of the equivalence class such that $\rho(Q) = v$. We also use the notations
\[
\begin{align*}
D &= (\mathbb{C}[[x]])[\partial] \\
E^{(-1)} &= (\mathbb{C}[[x]])[[\partial^{-1}]] \cdot \partial^{-1},
\end{align*}
\]
which are the set of linear ordinary differential operators and the set of pseudodifferential operators of negative order with scalar coefficients, respectively. Note that there is a natural right $(\mathbb{C}[[x]])$-module direct sum decomposition
\[
E = D \oplus E^{(-1)}.
\]
According to this decomposition, we write $P = P^+ \oplus P^-$, where $P \in E$, $P^+ \in D$, and $P^- \in E^{(-1)}$.

Now consider the matrix algebra $gl(n, E)$, which is the algebra of pseudodifferential operators with coefficients in matrix valued functions. This algebra acts on the vector space $V = \mathbb{C}((z))^{\oplus n} \cong (E/E x)^{\oplus n}$ from the left. It therefore induces an infinitesimal action of $gl(n, E)$ on the Grassmannian $Gr_n$. The decomposition (4.3) induces
\[
V = \mathbb{C}[z^{-1}]^{\oplus n} \oplus (\mathbb{C}[[z]] \cdot z)^{\oplus n}
\]
after identifying $z = \partial^{-1}$, and the base point $\mathbb{C}[z^{-1}]^{\oplus n}$ of the Grassmannian $Gr_n$ is the residue class of $D^{\oplus n}$ via the projection $E^{\oplus n} \to E^{\oplus n} / (E x)^{\oplus n}$. Therefore, the $gl(n, D)$-action on $V$ preserves $\mathbb{C}[z^{-1}]^{\oplus n}$. The two theorems we need are the following:

**Theorem 4.1.** A pseudodifferential operator $P \in gl(n, E)$ with matrix coefficients is a differential operator, i.e. $P \in gl(n, D)$, if and only if
\[
P \cdot \mathbb{C}[z^{-1}]^{\oplus n} \subset \mathbb{C}[z^{-1}]^{\oplus n}.
\]

**Theorem 4.2.** Let $S \in gl(n, E)$ be a monic zero-th order pseudodifferential operator of the form
\[
S = I_n + \sum_{m=1}^{\infty} s_m(x) \partial^{-m},
\]
where $s_m(x) \in gl(n, \mathbb{C}[[x]])$. Then the map
\[
\sigma : \Sigma \ni S \mapsto W = S^{-1} \cdot \mathbb{C}[z^{-1}]^{\oplus n} \in Gr_n^+
\]
gives a bijective correspondence between the set $\Sigma$ of pseudodifferential operators of the form (4.4) and the big-cell $Gr_n^+$ of the Grassmannian.

Proofs of these theorems are given in [9].
5. Constructing commuting differential operators from the geometric data.

Once the theory of pseudodifferential operators is established, the passage from the geometry of curves to the ring of differential operators is straightforward. Let
\[ \langle f : C \to C_d, \mathcal{L}, p, z, \psi \rangle \in \mathcal{G}^+ \]
be a collection of geometric data with (3.14), and let
\[ \langle i : A_d \hookrightarrow A, W \rangle = \mu(\langle f : C \to C_d, \mathcal{L}, p, z, \psi \rangle) \]
be the corresponding algebraic data. Then we have
\[ W \in Gr_n^+ , \]
thus it corresponds to an invertible matrix pseudodifferential operator \( S \in \Sigma \) as in Theorem 4.2:
\[ (5.1) \quad W = S^{-1} \cdot \mathbb{C}[z^{-1}]^\oplus n . \]
Since \( W \) is an \( A \)-module, we have
\[ (5.2) \quad A \cdot W \subset W . \]
The identification \( z = \frac{1}{\partial} \) makes the coordinate ring \( A \) a subalgebra of pseudodifferential operators with constant coefficients:
\[ A \subset \text{gl}(n, \mathbb{C}(z)) = \text{gl}(n, \mathbb{C}(\frac{1}{\partial})) \subset \text{gl}(n, E) . \]
Let
\[ B = S \cdot A \cdot S^{-1} \subset \text{gl}(n, E) \]
be the conjugate algebra of \( A \) in \( \text{gl}(n, E) \) by the operator \( S \). Then from (5.1) and (5.2), we have
\[ B \cdot \mathbb{C}[z^{-1}]^\oplus n = S \cdot A \cdot S^{-1} \cdot \mathbb{C}[z^{-1}]^\oplus n \]
\[ = S \cdot A \cdot W \]
\[ \subset S \cdot W \]
\[ = \mathbb{C}[z^{-1}]^\oplus n . \]
Therefore, from Theorem 4.1, the ring \( B \) consists of differential operators:
\[ B \subset \text{gl}(n, D) . \]
This completes the proof of Theorem 1.1. We remark that the curve \( C \) does not have to be irreducible for this construction.
6. Spectral curves as an example.

The theory of spectral curves by Hitchin [6] and Beauville, Narasimhan and Ramanan [2] provides a method of giving a collection of geometric data that satisfies all the conditions of Theorem 1.1. We start with a smooth algebraic curve $C_d$ and a point $p$ of $C_d$. Choose a vector bundle $\mathcal{F}$ on $C_d$ of rank $n$ and degree $n \cdot (g(C_d) - 1)$ such that

$$H^0(C_d, \mathcal{F}) = H^1(C_d, \mathcal{F}) = 0.$$ 

We also need a formal trivialization $\psi$ of $\mathcal{F}$ as in (3.3).

Following Markman [11], let us choose a line bundle $\mathcal{N}$ on $C_d$ of degree at least $2g(C_d) - 2$. The original choice is the canonical line bundle, but any sufficiently positive $\mathcal{N}$ will do for our purpose. Finally, we have to fix a Higgs field

$$\phi \in H^0(C_d, \mathcal{N} \otimes \mathcal{E}nd(\mathcal{F})).$$

A Higgs field can be identified with a sheaf homomorphism $\phi : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{N}$, which induces a homomorphism

$$\wedge^i \phi : \bigwedge^i \mathcal{F} \rightarrow \bigwedge^i \mathcal{F} \otimes \mathcal{N}^i.$$ 

We call $\text{trace}(\wedge^i \phi) \in H^0(C_d, L^i)$ the $i$-th characteristic coefficient of $\phi$. We use the notation

$$(6.1) \quad ch(\phi) = (\text{trace}(\wedge^0 \phi), \text{trace}(\wedge^1 \phi), \cdots, \text{trace}(\wedge^n \phi)) \in \bigoplus_{i=1}^n H^0(C_d, \mathcal{N}^i)$$

to denote the characteristic coefficients of $\phi$.

Consider the sheaf of symmetric algebras $\text{Sym}(\mathcal{N}^{-1})$ on the curve $C_d$ generated by $\mathcal{N}^{-1}$. Then $\text{Spec}(\text{Sym}(\mathcal{N}^{-1}))$ is the total space $|\mathcal{N}|$ of the line bundle $\mathcal{N}$. The $i$-th characteristic coefficient $s_i \in H^i(C_d, \mathcal{N}^i)$ gives a homomorphism $s_i : \mathcal{N}^m \rightarrow \mathcal{N}^{m+i}$. Let $\mathcal{I}_s$ be the sheaf of ideals of $\text{Sym}(\mathcal{N}^{-1})$ generated by

$$(6.2) \quad \left(1 - s_1 + s_2 - s_3 + \cdots + (-1)^n s_n\right) \otimes \mathcal{N}^{-n}.$$ 

Then the quotient $\text{Sym}(\mathcal{N}^{-1})/\mathcal{I}_s$ is a sheaf of algebras of relative dimension zero. We define the spectral curve by

$$(6.3) \quad C_s = \text{Spec}(\text{Sym}(\mathcal{N}^{-1})/\mathcal{I}_s).$$

In a more geometric language, the spectral curve $C_s$ is a subvariety of $|\mathcal{N}|$, where $|\mathcal{N}|$ is considered to be an open algebraic surface. The projection of $|\mathcal{N}|$ onto $C_d$ defines a natural map

$$f_s : C_s \rightarrow C_d$$

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of degree $n$. Its fiber at $q \in C_d$ is the set of solutions of the polynomial equation
\[
\left(1 - s_1 + s_2 - s_3 + \cdots + (-1)^n s_n\right) \otimes y^n = y^n - \bar{s}_1 y^{n-1} + \bar{s}_2 y^{n-2} - \cdots + (-1)^n \bar{s}_n = 0
\]
evaluated at $q$, where $y \in \mathcal{N}^{-1}$ is a linear coordinate of the fiber of $\mathcal{N}$, and
\[
\bar{s}_i = s_i \otimes y^i \in \mathcal{N}^i \otimes \mathcal{N}^{-i} = \mathcal{O}_C
\]
is a regular function defined locally near $q \in C_d$. This polynomial equation is also the characteristic equation of the homomorphism $\phi : \mathcal{N}^{-1} \rightarrow \mathcal{E}nd(F)$ at $q$ when $ch(\phi) = s$. Note that the homomorphism
\[
\phi : \mathcal{N}^{-1} \rightarrow \mathcal{E}nd(F)
\]
induces an algebra homomorphism
\[
Sym(\mathcal{N}^{-1}) \rightarrow \mathcal{E}nd(F).
\]
This algebra homomorphism factors through $\mathcal{O}_{C_s} = Sym(\mathcal{N}^{-1})/\mathcal{I}_s$ if $ch(\phi) = s$, which gives a $Sym(\mathcal{N}^{-1})/\mathcal{I}_s$-module structure in $\mathcal{F}$. Since the rank of $Sym(\mathcal{N}^{-1})/\mathcal{I}_s$ over $\mathcal{O}_{C_d}$ is $n$, $\mathcal{F}$ defines a line bundle over $C_s$, which we denote by $\mathcal{L}_s$. It is clear that $(f_s)_*(\mathcal{L}_s) = \mathcal{F}$. Thus we have constructed a desired geometric data
\[
\langle f_s : C_s \rightarrow C_d, \mathcal{L}_s, p, z, \psi \rangle \in G^+.
\]

### 7. From differential operators to geometry.

Let $B \subset gl(n, D)$ be an integral domain that satisfies the condition of Theorem 1.2. It is easy to show that

**Lemma 7.1.** There is an element $S \in \Sigma$ such that
\[
P = S \cdot I_n \cdot \partial^r \cdot S^{-1},
\]
where $r$ is the order of $P$ as a monic elliptic ordinary differential operator.

The operator $S$ relates the ring of differential operators to the formal loop algebras:
\[
A = S^{-1} \cdot B \cdot S \subset gl(n, \mathbb{C}((z)))
\]
where $z = \partial^{-1}$. This is because every element of $A$ commutes with $I_n \cdot \partial^r$, and hence it has constant coefficients. Similarly, we define
\[
A_d = S^{-1} \cdot B_d \cdot S \subset gl(n, \mathbb{C}((z)))
\]
Since $B_d \subset \mathbb{C}((P^{-1/r}))$, we have

$$A_d \subset I_n \cdot \mathbb{C}((z)),$$

i.e., it is a scalar diagonal subalgebra of $A$. By definition, the pair $(A, A_d)$ satisfies Condition 2.1. Thus by Theorem 2.2, we have a degree $n$ morphism

$$f : C \rightarrow C_d,$$

where $C_d$ is a one-point completion of $\text{Spec}(A_d)$ with a smooth point $p$.

We still have to construct a torsion-free rank one sheaf $\mathcal{L}$ on $C$. Note that $D^\otimes n$ is a left $gl(n, D)$ right $\mathbb{C}[[x]]$ bimodule. We consider $D^\otimes n$ as a bimodule over $B \otimes \mathbb{C}[[x]]$, accordingly. Similarly, $I_n \cdot D$ is a bimodule over $B_d \otimes \mathbb{C}[[x]]$. It is known [15] that this module structure gives rise to a rank one sheaf on $C_d \times \text{Spec}(\mathbb{C}[[x]])$. Since $D^\otimes n$ is rank $n$ over $D$ and $B$ is rank $n$ over $B_d$, we conclude that $B \otimes \mathbb{C}[[x]]$-bimodule $D^\otimes n$ defines a torsion-free rank one sheaf on $C \times \text{Spec}(\mathbb{C}[[x]])$. Restricting it to $C = C \times \{0\}$, we obtain a torsion-free rank one sheaf $\mathcal{L}$ defined on $C$. The vanishing of the cohomology groups (3.14) can be proved in a similar method developed in [9] and [13]. This completes the proof of Theorem 1.2.

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