On Conformal Radii of Non-Overlapping Simply Connected Domains

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Abstract. The paper deals with the following open problem stated by V.N. Dubinin. Let $a_0 = 0$, $|a_1| = \ldots = |a_n| = 1$, $a_k \in B_k \subset \mathbb{C}$, where $B_0, \ldots, B_n$ are disjoint domains. For all values of the parameter $\gamma \in (0, n]$ find the exact upper bound for $r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k)$, where $r(B_k, a_k)$ is the conformal radius of $B_k$ with respect to $a_k$. For $\gamma = 1$ and $n \geq 2$ the problem was solved by V.N. Dubinin. In the paper the problem is solved for $\gamma \in (0, \sqrt{n}]$ and $n \geq 2$ for simply connected domains.

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Introduction

In geometric function theory of complex variable extremal problems on non-overlapping domains are well-known classic direction (see, for example, [1–26]). A lot of such problems are reduced to determination of the maximum of product of inner radii on the system of non-overlapping domains satisfying a certain conditions. Paper of M.A. Lavrent’ev “On the theory of conformal mappings” [10] was initial impetus for such direction, in which, was first proposed and solved the problem of maximizing the product conformal radii of two non-overlapping simply connected domains. Namely, he proved the following assertion [10]: let $a_1$ and $a_2$ be some fixed points in the complex plane $\mathbb{C}$, $B_k$, $a_k \in B_k$, $k = 1, 2$ be any non-overlapping domains in $\mathbb{C}$. Then for functions $w = f_k(0)$, $k = 1, 2$, which regular in the circle $|z| < 1$ and univalently mapping it to the domain $B_k$ such that $f_k(0) = a_k$, we have inequality

$$|f_1'(0)| \cdot |f_2'(0)| \leq |a_1 - a_2|^2.$$ 

Moreover, for domains $B_k$, which have classical Green’s function equality in this inequality is attended if and only if domains $B_1, B_2$ are limited by circle $\frac{z-a_1}{z-a_2} = C$, where $C$ is an arbitrary positive constant. Lavrent’ev used this result to some aerodynamics problems.

It follows from the proof of this theorem, as a corollary, the well-known statement of Koebe-Bieberbach in theory of univalent functions. Based on these elementary estimates are obtained a number of new estimates for functions realizing a conformal mapping of a disc onto domains with certain special properties. Estimates of this type are fundamental to solving some metric problems arising when considering the correspondence of boundaries under a conformal mapping. Also, on the basis of the results concerning various extremal properties of conformal mappings, the problem of the representability of functions of a complex variable by a uniformly convergent series of polynomials is solved. Themes connected with the study of problems on non-overlapping domains was developed in papers [1–26].

Until 1974 a system of points $a_k$, $k = 1, n$, of the complex plane were fixed. In 1968 P.M. Tamrazov put forward the idea, that we can provide to the points $a_k$, $k = 1, n$, some freedom. In 1975, in accordance with this idea, G.P. Bakhtina first set and solved a number of extremal problems in classroom mutually non-overlapping domains with so-called free poles in her dissertation. The considering problem in the paper is the problem of this kind.
Let \( N \) and \( \mathbb{R} \) be the sets of natural and real numbers, respectively, let \( \mathbb{C} \) be the complex plane, and let \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) be its one-point compactification, \( \mathbb{R}^+ = (0,\infty) \).

Let \( r(B, a) = |f'(0)| \) be the conformal radius of the simply connected domain \( B \subset \overline{\mathbb{C}} \) relative to a point \( a \in B \), where \( w = f(z) \) is a univalent conformal mapping of the unit circle onto the domain \( B \subset \overline{\mathbb{C}}, \ a = f(0) \) (see, for example, [1, 10, 11, 12, 13, 14]).

Further we consider the following system of points \( A_n := \{a_k \in \mathbb{C}, k = \overline{1,n} \}, n \in N, n \geq 2 \), satisfying the conditions \(|a_k| \in \mathbb{R}^+, k = \overline{1,n} \) and

\[ 0 = \arg a_1 < \arg a_2 < \cdots < \arg a_n < 2\pi. \]

Define the numbers \( \alpha_k, k = \overline{1,n} \), as follows

\[ \alpha_1 := \frac{1}{\pi} (\arg a_2 - \arg a_1), \alpha_2 := \frac{1}{\pi} (\arg a_3 - \arg a_2), \ldots, \alpha_n := \frac{1}{\pi} (2\pi - \arg a_n). \]

And let \( \alpha_0 = \max_k \alpha_k \).

Consider one open an extremal problem which was formulated in [1] in the list of unsolved problems and then repeated in monograph [14].

**Problem.** Consider the product

\[ I_n(\gamma) = r^\gamma(B_0,0) \prod_{k=1}^n r(B_k, a_k), \]

where \( B_0, B_1, \ldots, B_n \) \((n \geq 2)\) are pairwise disjoint domains in \( \overline{\mathbb{C}} \), \( a_0 = 0, |a_k| = 1, k = \overline{1,n} \) and \( 0 < \gamma \leq n \). Show that it attains its maximum at a configuration of domains \( B_k \) and points \( a_k \) possessing rotational \( n \)-symmetry.

Presently, this task is not completely solved, its solutions for certain particular cases are only known. In [1] the problem was solved for any \( n \geq 2 \) and \( \gamma = 1 \). In [15] – for any \( \gamma > 1 \) but starting with some unknown number \( n \) in advance. The next step in the study of this problem was finding possible solutions for type of restrictions \( 1 < \gamma \leq n^\alpha \) where \( 0 < \alpha < 1 \). In [16] the problem was solved for \( n \geq 8 \) and \( 1 < \gamma \leq \sqrt{n} \), in [17] – for \( n \geq 12 \) and \( 1 < \gamma \leq n^{0.45} \).

In paper [18] this problem was solved for any natural \( n \geq 5 \) and \( 0 < \gamma \leq n \) but for condition \( \alpha_0 < \frac{2}{\sqrt{\gamma}} \). So we will consider only configuration of domains \( D_k \) and points \( d_k \) for which \( \alpha_0 > \frac{2}{\sqrt{\gamma}} \).

Results and Proofs

For simply connected domains we obtained the following result.

**Theorem 1.** Let \( n \in N, n \geq 2 \) and \( \gamma \in (1, \sqrt{n}] \). Then for any different points \( a_k, k = \overline{1,n} \), which lie on the unit circle \(|w| = 1\) and any system of non-overlapping simply connected domains \( B_k, a_k \in B_k \subset \overline{\mathbb{C}}, k = \overline{0,n}, a_0 = 0 \), the following inequality holds

\[ r^\gamma(B_0,0) \prod_{k=1}^n r(B_k, a_k) \leq \left( \frac{4}{n} \right)^n \left( \frac{\gamma}{n^2} \right)^{\frac{n}{n+2}} \left( \frac{1 - \sqrt{n}}{1 + \sqrt{n}} \right)^{2\sqrt{n}}. \]  

Equality in (1) is attained, when \( a_k \) and \( B_k, k = \overline{0,n} \), are, respectively, the poles and circular domains of the quadratic differential

\[ Q(w)dw^2 = -\frac{(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}dw^2. \]

**Proof of the theorem 1.** Note, that the cases \( n = 2 \) and \( n = 3 \) were considered in the paper [19], the case \( n = 4 \) was considered in [20] for a more general case of multiply connected domains. Therefore, we prove theorem 1 for \( n \geq 5 \).
It is important for us to know numerical value of $I_n(\gamma)$ on the system of circular domains $D_k$ and the system of poles $d_k$, $k = 0, n$ of the quadratic differential (2). Let

$$I^0_n(\gamma) = r^\gamma(D_0, d_0) \cdot \prod_{k=1}^{n} r(D_k, d_k),$$

where $\{d_k\}_{k=0}^n$ and $\{D_k\}_{k=0}^n$ are, respectively, the poles and circular domains of the quadratic differential (2). Using theorem 5.2.3 [15] we have

$$I^0_n(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{\gamma}{n^2}\right)^\frac{n}{2}}{(1 - \frac{\gamma}{n^2})^{n+\frac{n}{2}}} \left(\frac{1 - \frac{\sqrt{n}}{n}}{1 + \frac{\sqrt{n}}{n}}\right)^{2\sqrt{n}}.$$ (3)

The following assertion is known.

**Lemma 1.** [19] Let $n \in \mathbb{N}$, $n \geq 2$, $\gamma > 0$. And let $\{B_0, B_1, B_2, \ldots, B_n\}$ be the system of pairwise non-overlapping simply connected domains such that $0 \in B_0 \subset \mathbb{C}$, $a_k \in B_k \subset \mathbb{C}$, $|a_k| = 1$, $k = 1, n$ and $r^\gamma(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) > I^0_n(\gamma)$. Then the following inequality holds

$$r(B_0, 0) \leq n^{-\frac{n}{2(n-\gamma)}} I^0_n(\gamma)^{-\frac{1}{n-\gamma}}.$$ 

Note, that in [21] the problem was solved for $n \geq 4$ and $0 < \gamma \leq n$ but for condition $\alpha_0 \leq \frac{2}{\sqrt{\gamma}}$ (see also, [22, 25]). Thus taking Lemma 1 into account we consider the case when

$$r(B_0, 0) \leq n^{-\frac{n}{2(n-\gamma)}} I^0_n(\gamma)^{-\frac{1}{n-\gamma}} \quad \text{and} \quad \alpha_0 > \frac{2}{\sqrt{\gamma}}.$$ 

Proof that

$$r^\gamma(B_0, a_0) \cdot \prod_{k=1}^{n} r(B_k, a_k) \leq \left(\frac{4}{n}\right)^n \frac{\left(\frac{\gamma}{n^2}\right)^\frac{n}{2}}{(1 - \frac{\gamma}{n^2})^{n+\frac{n}{2}}} \left(\frac{1 - \frac{\sqrt{n}}{n}}{1 + \frac{\sqrt{n}}{n}}\right)^{2\sqrt{n}} < 1.$$ 

Clearly,

$$r^\gamma(B_0, a_0) \leq n^{-\frac{n^2}{2(n-\gamma)}} I^0_n(\gamma)^{-\frac{2}{n-\gamma}}.$$ 

Taking theorem 5.2.3 [15] into account we obtain

$$\prod_{k=1}^{n} r(B_k, a_k) \leq 2^n \prod_{k=1}^{n} \alpha_k \leq 2^n \alpha_0 \left(\frac{2 - \alpha_0}{n - 1}\right)^{n-1} <$$

$$< \frac{4^n}{(n - 1)^{n-1} \sqrt{\gamma}} \cdot \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{n-1}.$$ (4)

Thus, we have the following inequality

$$r^\gamma(B_0, a_0) \cdot \prod_{k=1}^{n} r(B_k, a_k) \leq \frac{4^n}{(n - 1)^{n-1} \sqrt{\gamma}} \cdot \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{n-1} \cdot I^0_n(\gamma)^{-\frac{n}{n-\gamma}} := G_n(\gamma).$$

Note, to prove the theorem 1 for some $n$ and $\gamma$, it suffices to show that the inequality $G_n(\gamma) < 1$ holds. So, we show that for $n \geq 5$ the inequality $G_n(\sqrt{n}) < 1$ holds.
\[ G_n(\sqrt{n}) = \frac{4^n}{(n-1)^{n-1}n^{\frac{3}{2}}} \left(1 - \frac{1}{n^2}\right)^{n-1} \frac{1}{n^{\frac{n\sqrt{\pi}}{2(n-\sqrt{n})}}} \frac{1}{I_n^0(\sqrt{n})^{n-\sqrt{n}}} = \]
\[ = \frac{4^n}{(n-1)^{n-1}n^{\frac{3}{2}}} \left(1 - \frac{1}{n^2}\right)^{n-1} \frac{1}{n^{\frac{n\sqrt{\pi}}{2(n-\sqrt{n})}}} \left(\frac{n}{4}\right)^{n\frac{2}{\sqrt{n-1}}} \times \]
\[ \times \left(1 - \frac{1}{n^2}\right)^{n-1} \left(1 - \frac{1}{n^2}\right)^{\frac{3}{n^2+1}^{\sqrt{n-1}}} \left(1 - \frac{1}{n^2}\right)^\left(\frac{n^2+1}{\sqrt{n-1}}\right)^{\left(1 + \frac{1}{n^2}\right)\left(1 - \frac{1}{n^2}\right)^{\frac{2n^3}{\sqrt{n-1}}}}. \]

Obviously, for \( n \geq 5 \) we have
\[ \left(1 + \frac{1}{n-1}\right)^{n-1} < e, \]
\[ \left(1 - \frac{1}{n^2}\right)^{\frac{n^2+1}{\sqrt{n-1}}} < 1. \]

Expression \( \left(1 + \frac{1}{n-1}\right)^{\frac{2n^3}{\sqrt{n-1}}} \) decreases with increasing \( n \), so for \( n \geq 5 \) we have
\[ \left(1 + \frac{1}{n-1}\right)^{\frac{2n^3}{\sqrt{n-1}}} < 30. \]

Expression \( n^{\frac{2n+3\sqrt{n+3}}{4(\sqrt{n-1})}} \left(1 - \frac{1}{n^2}\right)^{n-1} \) also decreases with increasing \( n \), and we obtain that for \( n \geq 5 \) it does not exceed 0.01. Summing obtained estimates we have
\[ G_n(\sqrt{n}) < e \cdot 30 \cdot 0.01 < 1. \quad (5) \]

Thus, for \( \gamma = \sqrt{n} \) the theorem 1 is proved. Further consider the validity of the theorem for \( 1 < \gamma < \sqrt{n} \). The following equality holds
\[ (I_n^0(\gamma))' = I_n^0(\gamma) \left(\frac{1}{n} \ln \left(\frac{4\gamma}{n^2 - \gamma}\right) + \frac{1}{\sqrt{\gamma}} \ln \left(\frac{n - \sqrt{\gamma}}{n + \sqrt{\gamma}}\right)\right). \]

It is not difficult to obtain the following lemma.

**Lemma 2.** For \( n \geq 5 \) and \( 1 < \gamma < n \) the function \( G_n(\gamma) \) monotonically increases.

**Proof of the lemma 2.** Using the logarithmic derivative, we investigate the monotonicity of the function \( G_n(\gamma) \).
\[ \ln (G_n(\gamma)) = \ln \left(\frac{4^n}{(n-1)^{n-1}n^{\frac{3}{2}}} \left(1 - \frac{1}{n^2}\right)^{n-1} \frac{1}{n^{\frac{n\sqrt{\pi}}{2(n-\sqrt{n})}}} \frac{1}{I_n^0(\sqrt{n})^{n-\sqrt{n}}} \right) = \]
\[ = -\frac{n\gamma}{2(n-\gamma)} \ln n - \frac{n}{n - \gamma} \ln I_n^0(\gamma). \]
Respectively,
\[
\left( \ln\left( G_n(\gamma) \right) \right)' = -\frac{1}{2\gamma} + \frac{n - 1}{2\gamma(\sqrt{n} - 1)} - \frac{n^2 \ln n}{2(n - \gamma)^2} - \frac{n \ln I^0_n(\gamma)}{(n - \gamma)^2} - \frac{n}{n - \gamma} I^0_n(\gamma)'.
\]

It is easily seen that
\[
-\frac{1}{2\gamma} + \frac{n - 1}{2\gamma(\sqrt{n} - 1)} = \frac{n - \sqrt{n}}{2\gamma(\sqrt{n} - 1)} > 0.
\]

From Lemma 1 we obtain
\[
-\frac{n^2 \ln n}{2(n - \gamma)^2} - \frac{n \ln I^0_n(\gamma)}{(n - \gamma)^2} - \frac{n}{n - \gamma} \left( \ln I^0_n(\gamma) \right)' = -\frac{n}{2(n - \gamma)^2} \times \left( n \ln n + 2 \ln I^0_n(\gamma) + \frac{2(n - \gamma)}{n} \ln \left( \frac{4\gamma}{n^2 - \gamma} \right) + \frac{2(n - \gamma)}{\sqrt{n}} \ln \left( \frac{n - \sqrt{n}}{n + \sqrt{n}} \right). \right.
\]

Note that
\[
n \ln n + 2 \ln I^0_n(\gamma) < 0
\]
and
\[
\frac{2(n - \gamma)}{n} \ln \left( \frac{4\gamma}{n^2 - \gamma} \right) + \frac{2(n - \gamma)}{\sqrt{n}} \ln \left( \frac{n - \sqrt{n}}{n + \sqrt{n}} \right) < 0
\]
for \( n \) and \( \gamma \) which satisfy the conditions of the lemma. Thus,
\[
-\frac{n^2 \ln n}{2(n - \gamma)^2} - \frac{n \ln I^0_n(\gamma)}{(n - \gamma)^2} - \frac{n}{n - \gamma} \left( \ln I^0_n(\gamma) \right)' > 0.
\]

Taking (6) and (7) into consideration we have the following inequality
\[
\left( \ln\left( G_n(\gamma) \right) \right)' > 0.
\]

Lemma 2 is proved.

Further using Lemma 2 and inequality (5), we obtain the statement of the theorem 1 for \( \alpha_0 > \frac{2}{\sqrt{\gamma}}. \)

Finally, using results of the papers [18, 21, 22] we also obtain that the theorem 1 is valid for \( \alpha_0 \leq \frac{2}{\sqrt{\gamma}}. \)

Thus, theorem 1 is proved.

Note that for some number \( n \) we can get stronger results. In the following theorem we take \( \gamma_n \) as the root of the equation \( G_n(\gamma) = 1 \) with an accuracy to 0.1.

**Theorem 2.** Let \( n = 5, 10, \gamma_5 = 3.2, \gamma_6 = 3.8, \gamma_7 = 4.2, \gamma_8 = 4.7, \gamma_9 = 5.1, \gamma_{10} = 5.4. \) Then for any \( \gamma \in (1, \gamma_n] \) inequality (1) holds, where domains \( B_k \) and points \( a_k \) are the same as in Theorem 1.

**Proof of the theorem 2.** In order to prove the theorem we use the following inequality
\[
G_n(\gamma) := \frac{4^n}{n^{\gamma} (\gamma - 1)^{\gamma}} \left( 1 - \frac{1}{\sqrt{\gamma}} \right)^{n-1} \leq 1,
\]
where the value \( I^0_n(\gamma) \) is determined by the formula (3). Substituting in (8) specific values of \( n = 5, 10 \) we obtain

\[
G_5(3.2) \approx 0.9612 < 1, \quad G_6(3.8) \approx 0.9962 < 1, \\
G_7(4.2) \approx 0.8879 < 1, \quad G_8(4.7) \approx 0.9624 < 1, \\
G_9(5.1) \approx 0.9731 < 1, \quad G_{10}(5.4) \approx 0.9001 < 1.
\]

Further, using Lemma 2 we have statement of theorem 2. Thus theorem 2 is proved.
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