The Possibilistic Horn Non-Clausal Knowledge Bases

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Abstract

Possibilistic logic is the most extended approach to handle uncertain and partially inconsistent information. Regarding normal forms, advances in possibilistic reasoning are mostly focused on clausal form. Yet, the encoding of real-world problems usually results in a non-clausal (NC) formula and NC-to-clausal translators produce severe drawbacks that heavily limit the practical performance of clausal reasoning. Thus, by computing formulas in its original NC form, we propose several contributions showing that notable advances are also possible in possibilistic non-clausal reasoning.

Firstly, we define the class of Possibilistic Horn Non-Clausal Knowledge Bases, or \( \text{H}_\Sigma \), which subsumes the classes: possibilistic Horn and propositional Horn-NC. \( \text{H}_\Sigma \) is shown to be a kind of NC analogous of the standard Horn class.

Secondly, we define Possibilistic Non-Clausal Unit-Resolution, or \( \text{UR}_\Sigma \), and prove that \( \text{UR}_\Sigma \) correctly computes the inconsistency degree of \( \text{H}_\Sigma \) members. \( \text{UR}_\Sigma \) had not been proposed before and is formulated in a clausal-like manner, which eases its understanding, formal proofs and future extension towards non-clausal resolution.

Thirdly, we prove that computing the inconsistency degree of \( \text{H}_\Sigma \) members takes polynomial time. Although there already exist tractable classes in possibilistic logic, all of them are clausal, and thus, \( \text{H}_\Sigma \) turns out to be the first characterized polynomial non-clausal class within possibilistic reasoning.

We discuss that our approach serves as a starting point to developing uncertain non-clausal reasoning on the basis of both methodologies: DPLL and resolution.

Keywords: Possibilistic Logic; Horn; Non-Clausal; Inconsistency; Tractability; Resolution; DPLL; Satisfiability Testing; Logic Programming.

1 Introduction

Possibilistic logic is the most popular approach to represent and reason with uncertain and partially inconsistent knowledge. Regarding normal forms, the encoding of real-world problems does usually not result in a clausal formula and although a possibility non-clausal formula is theoretically equivalent to some possibilistic clausal formula [26, 22], approaches needing clausal form transformations are practically infeasible or have experimentally shown to be highly inefficient as discussed below.

Two kinds of clausal form transformation are known: (1) one is based on the repetitive application of the distributive laws to the input non-clausal formula until a logically equivalent clausal formula is obtained; and (2) the other transformation, Tsetin-transformation [49], is based on recursively substituting sub-formulas in the input non-clausal formula by fresh literals until obtaining an equi-satisfiable, but not equivalent, clausal formula.
The first transformation blows up exponentially the formula size, and since real-world problems have a large number of variables and connectives, the huge dimension of the resulting clausal formulas prevents even highly-efficient state-of-the-art solvers from attaining solutions in a reasonable time.

The second kind of transformation also involves a number of drawbacks. The Tseitin-transformation usually produces an increase of formula size and number of variables, and also a loss of information about the formula’s original structure. Besides in most cases, the normal form is not unique. Deciding how to perform the transformation enormously influences the solving process and it is usually impossible to predict which strategy is going to be the best, as this depends on the concrete solver used and on the kind of problem which should be solved. Further, Tseitin-transformation keeps the satisfiability test but losses the logical equivalence, which rules out its usage in many real-world problems.

We abandon the assumption that the input formula should be transformed to clausal form and directly process it in its original structure. Since real-world problems rarely occur in clausal form, we allow an arbitrary nesting of conjunctions and disjunctions and only limit the scope of the negation connective. The non-clausal form considered here is popularly called negation normal form (NNF), and can be obtained deterministically and causing only a negligible increase of the formula size.

Developing methods for NC reasoning is an actual concern in the principal fields of classical logic, namely satisfiability solving \[\text{[51, 43]},\] logic programming \[\text{[17, 14]},\] theorem proving \[\text{[31, 53]},\] and quantified boolean formulas \[\text{[30, 13]},\] and in many other fields (see \[\text{[40]},\] and the references thereof). And within non-classical logics, NC formulas with different functionalities have been studied in a profusion of languages: signed many-valued logic \[\text{[47, 8, 58]},\] Lukasiewicz logic \[\text{[42]},\] Levesque’s three-valued logic \[\text{[15]},\] Belnap’s four-valued logic \[\text{[15]},\] M3 logic \[\text{[1]},\] fuzzy logic \[\text{[35]},\] fuzzy description logic \[\text{[34]},\] intuitionistic logic \[\text{[55]},\] modal logic \[\text{[55]},\] lattice-valued logic \[\text{[60]},\] and regular many-valued logic \[\text{[39]},\] We highlight the proposal in \[\text{[49, 50]}\] as is the only existing approach, to our knowledge, to deal with possibilistic non-clausal formulas, concretely within the answer set programming field.

On the other side, the Horn clausal formulas are pivotal elements of our proposed possibilistic reasoning approach towards combining non-clausal expressiveness with high efficiency. Horn formulas are recognized as central for deductive databases, declarative programming, and more generally, for rule-based systems. In fact, Horn formulas have received a great deal of attention since 1943 \[\text{[45, 38]}\] and, at present, there is a broad span of areas within artificial intelligence relying on them, and their scope covers a fairly large spectrum of realms spread across many logics and a variety of reasoning settings.

Regarding possibilistic Horn formulas, computing their inconsistency degree is a tractable problem \[\text{[41]},\] and even almost-lineal \[\text{[3]},\] Related to this standard Horn class but going beyond clausal form, we present a novel possibilistic class, denoted \(\text{H}_\Sigma\), that is in NC form and that we call Horn Non-Clausal (Horn-NC). We show that \(\text{H}_\Sigma\) is a sort of non-clausal analogous of the possibilistic Horn class. Besides the latter, \(\text{H}_\Sigma\) also subsumes the class of propositional Horn-NC formulas recently presented \[\text{[40]},\]

From a computational view, we prove that computing the inconsistency degree of \(\text{H}_\Sigma\) members is a tractable problem. This result signifies that polynomiality in our context is preserved when upgrading both from clausal to non-clausal form and from propositional to possibilistic logic. Polynomiality is preserved when upgrading from clausal to non-clausal form because both classes possibilistic Horn \[\text{[41]}\] and possibilistic Horn-NC are
tractable. Similarly, polynomiality is preserved when upgrading from propositional to possibilistic logic because both classes propositional Horn-NC \[40\] and possibilistic Horn-NC are tractable.

In summary, our contributions are: introducing the hybrid class of *Possibilistic Horn Non-Clausal Knowledge Bases*, or \( \mathcal{P}_\Sigma \), and then, proving that computing their inconsistency degree is a polynomial problem. Our contributions are outlined next.

**Firstly**, the syntactical Horn-NC restriction is determined by lifting the Horn clausal restriction “a formula is Horn if all its clauses have any number of negative literals and at most one positive literal”, to the non-clausal level in the following manner: “a propositional NC formula is Horn-NC if all its disjunctions have any number of negative disjuncts and at most one non-negative disjunct”. By extending such definition to possibilistic logic, we establish straightforwardly that: a possibilistic NC knowledge base is Horn-NC only if all its propositional formulas are Horn-NC. Accordingly, \( \mathcal{P}_\Sigma \) is defined as the class of *Possibilistic Horn-NC Knowledge Bases*. Note that \( \mathcal{P}_\Sigma \) naturally subsumes the standard possibilistic Horn clausal class.

The set relations that the new class \( \mathcal{P}_\Sigma \) bears to the standard possibilistic classes Horn (\( \mathcal{H}_\Sigma \)), Non-Clausal (\( \mathcal{NC}_\Sigma \)) and Clausal (\( \mathcal{C}_\Sigma \)) are depicted in Fig. 1. Specifically, we show the next relationships of \( \mathcal{P}_\Sigma \) with \( \mathcal{H}_\Sigma \) and \( \mathcal{NC}_\Sigma \): (1) \( \mathcal{P}_\Sigma \) and \( \mathcal{H}_\Sigma \) are related in that \( \mathcal{P}_\Sigma \) subsumes syntactically \( \mathcal{H}_\Sigma \) but both classes are semantically equivalent; and (2) \( \mathcal{P}_\Sigma \) and \( \mathcal{NC}_\Sigma \) are related in that \( \mathcal{P}_\Sigma \) contains all NC bases whose clausal form is Horn. Thus, in view of (1) and (2) relations, \( \mathcal{P}_\Sigma \) is a sort of NC analogous of \( \mathcal{H}_\Sigma \).

**Secondly**, we establish the inferential calculus *Possibilistic Non-Clausal Unit-Resolution*, or \( \mathcal{UR}_\Sigma \), and then prove that \( \mathcal{UR}_\Sigma \) correctly computes the inconsistency degree of the bases in the class \( \mathcal{P}_\Sigma \). NC unit-resolution for propositional logic has been recently presented \[40\] and \( \mathcal{UR}_\Sigma \) is its generalization to possibilistic logic. \( \mathcal{UR}_\Sigma \) is formulated in a clausal-like fashion, which contrasts with the functional-like fashion of the existing (full) non-clausal resolution \[46\]. We argue that our clausal-like formulation eases the understanding of \( \mathcal{UR}_\Sigma \), the building of the required formal proofs and the future generalization of \( \mathcal{UR}_\Sigma \) to determine Non-Clausal Resolution for possibilistic and for other uncertainty logics.

**Thirdly**, we prove that computing the consistency degree of \( \mathcal{P}_\Sigma \) members has polynomial complexity. There indeed exist polynomial classes in possibilistic logic but all of them are clausal \[41\], and so, the tractable non-clausal fragment was empty. We think that this is just a first tractable result in possibilistic reasoning and that the approach presented here will serve to widen the tractable possibilistic non-clausal fragment.
Below we give an specific possibilistic non-clausal base $\Sigma$, whose suffix notation will be detailed in Section 2 and wherein $P, Q, \ldots$ and $\neg P, \neg Q, \ldots$ are positive and negative literals, respectively, and $\phi_1, \phi_2$ and $\phi_3$ are non-clausal propositional formulas. We will

$$\varphi = \{ \land P (\lor \neg Q \{ \land (\lor \neg P \neg Q) \{ \land (\lor \phi_1 \{ \land \phi_2 \neg P\}) Q\} \} \phi_3 \}$$

$$\Sigma = \{ \langle \varphi : 0.8 \rangle \langle P : 0.8 \rangle \langle \neg Q : 0.6 \rangle \langle R : 0.6 \rangle \langle \phi_1 : .3 \rangle \langle \phi_3 : 1 \rangle \}$$

display that $\Sigma$ is Horn-NC when $\phi_3$ is Horn-NC and at least one of $\phi_1$ or $\phi_2$ is negative.

Recapitulating, the list of properties of $\overline{H}_\Sigma$ is given below, where the last two properties have been shown in [40] for propositional logic but are inherited by $\overline{H}_\Sigma$:

- Computing the inconsistency degree of $\overline{H}_\Sigma$ is tractable.
- $\overline{H}_\Sigma$ subsumes syntactically the possibilistic Horn class.
- $\overline{H}_\Sigma$ is equivalent semantically to the possibilistic Horn class.
- $\overline{H}_\Sigma$ contains all possibilistic NC bases whose clausal form is Horn.
- $\overline{H}_\Sigma$ is linearly recognizable [40].
- $\overline{H}_\Sigma$ is strictly succincter than the possibilistic Horn class [40].

The presented approach serves as starting point to develop approximate non-clausal reasoning based on (1) DPLL and (2) resolution: (1) $\overline{UR}_\Sigma$ paves the way to define DPLL in NC since its NC Unit-Propagation is based on NC Unit-Resolution, i.e. $\overline{UR}_\Sigma$; and (2) the existing NC resolution [46] presents some deficiencies derived from its functional-like formalization, such as not precisely defining the potential resolvents. Our clausal-like formalization of $\overline{UR}_\Sigma$ skips such deficiencies and signifies a step forward towards defining NC resolution for at least those uncertainty logics for which clausal resolution is already defined, e.g. possibilistic logic [23, 24].

This paper is organized as follows. Section 2 and 3 present background on propositional non-clausal formulas and on possibilistic logic, respectively. Section 4 defines the class $\overline{H}_\Sigma$. Section 5 introduces the calculus $\overline{UR}_\Sigma$. Section 6 provides examples illustrating how $\overline{UR}_\Sigma$ computes $\overline{H}_\Sigma$ members. Section 7 provides the formal proofs of the correctness of $\overline{UR}_\Sigma$ and of the tractability of $\overline{H}_\Sigma$. Section 8 focuses on related and future work. Last section summarizes the main contributions.

2 Propositional Non-Clausal Logic

This section presents some terminologies used in this paper and background on non-clausal (NC) propositional logic (see [9] for a complete background). We present first the needed syntactical concepts and then the semantical ones. We begin by introducing the language.

Definition 2.1. The NC language is formed by the sets: constants $\{ \bot, \top \}$, propositions $P = \{ P, Q, R, \ldots \}$, connectives $\{ \neg, \lor, \land \}$ and auxiliary symbols $\{, \}$.

Next we describe the required elements relative to clausal formulas.

\footnote{Succinctness was defined in [33].}
Definition 2.2. \( X \) (resp. \( \neg X \)) with \( X \in P \) is a positive (resp. negative) literal. \( \mathcal{L} \) is the set of literals. Constants and literals are atoms. \( \langle \lor \ell_1 \ell_2 \ldots \ell_k \rangle \), the \( \ell_i \) being literals, is a clause. A clause with at most one positive literal is Horn. \( \{ \land C_1 C_2 \ldots C_n \} \), the \( C_i \) being clauses, is a clausal formula. \( \mathcal{C} \) and \( \mathcal{H} \) are the set of clausal and Horn formulas, respectively.

Note. We firstly justify our chosen notation of non-clausal formulas before defining them. Thus, for the sake of readability of non-clausal formulas, we will employ:

1. The prefix notation as it requires only one \( \lor/\land \)-connective per formula, while infix notation requires \( k-1 \), \( k \) being the arity of the involved \( \lor/\land \)-connective.
2. Two formula delimiters (see Definition 2.3), \( \langle \lor \ldots \rangle \) for disjunctions and \( \{ \land \ldots \} \) for conjunctions, to better distinguish them inside non-clausal formulas.

So our next definition is that of non-clausal formulas\(^2\), whose differential feature is that the connective \( \neg \) can occur only in front of propositions, i.e. at atomic level.

Definition 2.3. The non-clausal formulas over a set of propositional variables \( P \) is the smallest set \( \mathcal{NC} \) such that the following conditions hold:

- \( \{ \bot, \top \} \cup \mathcal{L} \subset \mathcal{NC} \).
- If \( \forall i \in \{1, \ldots, k\}, \varphi_i \in \mathcal{NC} \) then \( \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \in \mathcal{NC} \).
- If \( \forall i \in \{1, \ldots, k\}, \varphi_i \in \mathcal{NC} \) then \( \{ \lor \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \in \mathcal{NC} \).
- \( \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \) and any \( \varphi_i \) are called conjunction and conjunct, respectively.
- \( \{ \lor \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \) and any \( \varphi_i \) are called disjunction and disjunct, respectively.
- \( \{ \bigcirc \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \) stands for both \( \{ \lor \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \) and \( \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \).

Example 2.4. \( \varphi_1 \) to \( \varphi_3 \) below are NC formulas, while \( \varphi_4 \) is not. We will show that \( \varphi_2 \) is Horn-NC while \( \varphi_1 \) is not Horn-NC, and as \( \varphi_3 \) includes \( \varphi_1 \), then \( \varphi_3 \) is not Horn-NC either. On the other side, the example in the Introduction is Horn-NC under certain conditions.

- \( \varphi_1 = \{ \land (\lor \neg P \ Q \ \bot) \ (\lor Q \ {\land} \neg R \ S \ \top) \} \)
- \( \varphi_2 = (\lor \ {\land} \neg P \ \top) \ {\land} (\lor \neg P \ R) \ {\land} Q \ (\lor P \ \neg S) \} \ {\land} \bot \ Q \}
- \( \varphi_3 = (\lor \ \varphi_1 \ {\land} Q \ (\lor \varphi_1 \ \neg Q \ \varphi_2) \} \ {\land} \varphi_2 \ \top \ \varphi_1 \}
- \( \varphi_4 = \neg(\lor \varphi_1 \ \varphi_2) \)

Definition 2.5. Sub-formulas are recursively defined as follows. The unique sub-formula of an atom is the atom itself, and the sub-formulas of a formula \( \varphi = \{ \bigcirc \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \) are \( \varphi \) itself plus the sub-formulas of the \( \varphi_i \)'s.

Example 2.6. The sub-formulas of a clausal formula are the formula itself plus its clauses, literals and constants.

\(^2\)Also called "negation normal form formulas" in the literature.
Definition 2.7. NC formulas are modeled by trees if: (i) the nodes are: each atom is a leaf and each occurrence of a $\land/\lor$-connective is an internal node; and (ii) the arcs are: each sub-formula $[\odot \varphi_1 \ldots \varphi_i \ldots \varphi_k]$ is a $k$-ary hyper-arc linking the node of $\odot$ with, for every $i$, the node of $\varphi_i$ if $\varphi_i$ is an atom and with the node of its connective otherwise.

Example 2.8. The tree of $\{\land \neg R \lor \{\land \neg P \lor \perp \lor \neg P \neg R\}\}$ is given in Fig. 2.

Remark. Directed acyclic graphs (DAGs) generalize trees and allow for important savings in space and time. Our approach also applies when NC formulas are represented and implemented by DAGs. Nevertheless, for simplicity, we will use formulas representable by trees in the illustrative examples throughout this article.

Definition 2.9. An interpretation $\omega$ maps the formulas NC into the truth-value set $\{0, 1\}$ and is extended from propositional variables $P$ to formulas NC via the rules below, where $X \in P$ and $\varphi_i \in NC$, $1 \leq i \leq k$. We will denote $\Omega$ the universe of interpretations.

- $\omega(\perp) = \omega(\lor) = 0$ and $\omega(\top) = \omega(\{\land\}) = 1$.
- $\omega(X) + \omega(\neg X) = 1$.
- $\omega(\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) = \max\{\omega(\varphi_i) : 1 \leq i \leq k\}$.
- $\omega(\land \varphi_1 \ldots \varphi_i \ldots \varphi_k) = \min\{\omega(\varphi_i) : 1 \leq i \leq k\}$.

Definition 2.10. $\varphi$ and $\varphi'$ being formulas, some well-known semantical notions follow. An interpretation $\omega$ is a model of $\varphi$ if $\omega(\varphi) = 1$. If $\varphi$ has a model then it is consistent and otherwise inconsistent. $\varphi$ and $\varphi'$ are (logically) equivalent, denoted $\varphi \equiv \varphi'$, if $\forall \omega, \omega(\varphi) = \omega(\varphi')$. $\varphi'$ is logical consequence of $\varphi$, denoted $\varphi \models \varphi'$, if $\forall \omega, \omega(\varphi) \leq \omega(\varphi')$.

Next, some well-known rules allowing to simplify formulas are supplied.

Definition 2.11. Constant-free, equivalent formulas are straightforwardly obtained by recursively applying to sub-formulas the simplifying rules below:
• Replace \((\lor T \varphi)\) with \(T\).
• Replace \(\{\land \bot \varphi\}\) with \(\bot\).
• Replace \(\{\land T \varphi\}\) with \(\varphi\).
• Replace \((\lor \bot \varphi)\) with \(\varphi\).

Example 2.12. The constant-free, equivalent NC formula of \(\varphi_2\) in Example 2.4 is:

\[
\varphi = (\lor \neg P \{\land (\lor \neg P R) \{\land Q (\lor P \neg S)\}\})
\]

Remark. For simplicity and since free-constant, equivalent formulas are easily obtained, hereafter we will consider only free-constant formulas.

3 Necessity-Valued Possibilistic Logic

Let us have a brief refresher on necessity-valued possibilistic logic (the reader may consult [22, 25, 26] for more details).

3.1 Semantics

At the semantic level, possibilistic logic is defined in terms of a posibilistic distribution \(\pi\) on the universe \(\Omega\) of interpretations, i.e. an \(\Omega \to [0, 1]\) mapping which intuitively encodes for each \(\omega \in \Omega\) to what extent it is plausible that \(\omega\) is the actual world. \(\pi(\omega) = 0\) means that \(\omega\) is impossible, \(\pi(\omega) = 1\) means that nothing prevents \(\omega\) from being true, whereas \(0 < \pi(\omega) < 1\) means that \(\omega\) is only somewhat possible to be the real world. Possibility degrees are interpreted qualitatively: when \(\pi(\omega) > \pi(\omega')\), \(\omega\) is considered more plausible than \(\omega'\). A posibilistic distribution \(\pi\) is normalized if \(\exists \omega \in \Omega, \pi(\omega) = 1\), i.e. at least one interpretation is entirely plausible.

A possibility distribution \(\pi\) induces two uncertainty functions from the formulas \(\text{NC}\) to \([0, 1]\), called possibility and necessity functions and noted \(\Pi\) and \(N\), respectively, which allow us to rank formulas. \(\Pi\) is defined by Dubois et al. (1994) [22] as:

\[
\Pi(\varphi) = \max\{\pi(\omega) \mid \omega \in \Omega, \omega \models \varphi\},
\]

and evaluates the extent to which \(\varphi\) is consistent with the beliefs expressed by \(\pi\). The dual necessity measure \(N\) is defined by:

\[
N(\varphi) = 1 - \Pi(\neg \varphi) = \inf\{1 - \pi(\omega) \mid \omega \in \Omega, \omega \not\models \varphi\},
\]

and evaluates the extent to which \(\varphi\) is entailed by the available beliefs [22]. So the lower the possibility of an interpretation that makes \(\varphi\) False, the higher the necessity degree of \(\varphi\). \(N(\varphi) = 1\) means \(\varphi\) is a totally certain piece of knowledge, whereas \(N(\varphi) = 0\) expresses the complete lack of knowledge of priority about \(\varphi\). Note that always \(N(T) = 1\) for any possibility distribution, while \(\Pi(T) = 1\) (and, related, \(N(\bot) = 0\)) only holds when the possibility distribution is normalized, i.e. only normalized distributions can express consistent beliefs [22].

A major property of \(N\) is Min-Decomposability: \(\forall \varphi, \psi, N(\varphi \land \psi) = \min(N(\varphi), N(\psi))\). However, for disjunctions only \(N(\varphi \lor \psi) \geq \max(N(\varphi), N(\psi))\) holds. Further, one has \(N(\varphi) \leq N(\psi)\) if \(\varphi \models \psi\), and hence, \(N(\varphi) = N(\psi)\) if \(\varphi \equiv \psi\).
3.2 Syntactics

A possibilistic formula is a pair \( \langle \varphi : \alpha \rangle \in \mathcal{NC} \times (0, 1] \), where \( \varphi \) is a propositional NC formula, \( \alpha \in (0, 1] \) expresses the certainty that \( \varphi \) is the case, and it is interpreted as the semantic constraint \( N(p) \geq \alpha \). So formulas of the form \( \langle \varphi : 0 \rangle \) are excluded. A possibilistic base \( \Sigma \) is a collection of possibilistic formulas \( \Sigma = \{ \langle \varphi_i : \alpha_i \rangle \mid i = 1, \ldots, k \} \) and corresponds to a set of constraints on possibility distributions. The classical knowledge base associated with \( \Sigma \) is denoted as \( \Sigma^* \), i.e., \( \Sigma^* = \{ \varphi \mid \langle \varphi : \alpha \rangle \in \Sigma \} \). \( \Sigma \) is consistent if and only if \( \Sigma^* \) is consistent. It is noticeable that, due to Min-Decomposability, a possibilistic logic base can be easily put in clausal form.\(^3\)

Typically, there can be many possibility distributions that satisfy the set of constraints \( N(\varphi) \geq \alpha \) but we are usually only interested in the least specific possibility distribution, i.e., the possibility distribution that makes minimal commitments, namely, the greatest possibility distribution w.r.t. the following ordering: \( \pi \) is a least specific possibility distribution compatible with \( \Sigma \) if for any \( \pi' \), \( \pi' \neq \pi \), compatible with \( \Sigma \), one has \( \forall \omega \in \Omega, \pi(\omega) \geq \pi'(\omega) \).

Such a least specific possibility distribution always exists and is unique.\(^2\)

Thus, for a given \( \langle \varphi : \alpha \rangle \), possibilistic distributions should consider that an \( \omega \) that makes \( \varphi \ True \) is possible at the maximal level, say 1, while an \( \omega \) that makes \( \varphi \ False \) is possible at most at level 1 − \( \alpha \). Thus the semantic counterpart of a base \( \Sigma \), or the least specific distribution \( \pi_{\Sigma} \) is defined by, \( \forall \omega, \omega' \in \Omega \):

\[
\pi_{\Sigma}(\omega) = \begin{cases} 
1 & \text{if } \forall \varphi_i, \alpha_i \in \Sigma, \omega \models \varphi_i \\
\min\{1 - \alpha_i \mid \omega \not\models \varphi_i, \langle \varphi_i, \alpha_i \rangle \in \Sigma\} & \text{otherwise}
\end{cases}
\]

Proposition 3.1. Let \( \Sigma \) be a possibilistic base. For any possibility distribution \( \pi \) on \( \Omega \), \( \pi \) satisfies \( \Sigma \) if and only if \( \pi \leq \pi_{\Sigma} \).

Proposition 3.1 says that \( \pi_{\Sigma} \) is the least specific possibility distribution satisfying \( \Sigma \) and it has been shown in reference \(^{22}\).

3.3 Syntactic Deduction

This subsection introduces some few notions about deduction in possibilistic logic and starts by the well-known possibilistic inference rules to be handled in this article:

Definition 3.2. We define below three rules, where \( \ell \in \mathcal{L}; \varphi, \psi \in \mathcal{NC} \) and \( \alpha, \beta \in (0, 1] \). The first is possibilistic resolution \(^{23}\),\(^{24}\); the second rule is Min-Decomposability; and the third rule, Max-Necesity, follows from the semantic constraint meaning of \( \langle \varphi : \alpha \rangle \).

- **Resol**: \( \langle (\lor \ell \varphi) : \alpha \rangle, \langle (\lor \neg \psi \beta) : \beta \rangle \vdash \langle (\lor \varphi \psi) : \min\{\alpha, \beta\} \rangle \).
- **MinD**: \( \langle \varphi : \alpha \rangle, \langle \psi : \beta \rangle \vdash \langle (\land \varphi \psi) : \min\{\alpha, \beta\} \rangle \).
- **MaxN**: \( \langle \varphi : \alpha \rangle, \langle \varphi : \beta \rangle \vdash \langle \varphi : \max\{\alpha, \beta\} \rangle \).

Before formulating the soundness and completeness theorem in possibilistic logic, we need to introduce the next concept of \( \alpha \)-cut; we call the \( \alpha \)-cut (resp. strict \( \alpha \)-cut) of \( \Sigma \), denoted \( \Sigma_{\geq \alpha} \) (resp. \( \Sigma_{>\alpha} \)), the set of classical formulas in \( \Sigma \) having a necessity degree at least equal to \( \alpha \) (resp. strictly greater than \( \alpha \)), namely \( \Sigma_{\geq \alpha} = \{ \varphi \mid \langle \varphi : \beta \rangle \in \Sigma, \beta \geq \alpha \} \) (resp. \( \Sigma_{>\alpha} = \{ \varphi \mid \langle \varphi : \beta \rangle \in \Sigma, \beta > \alpha \} \)).

\(^3\)Nevertheless, as said previously, this translation can blow up exponentially the size of formulas and so can dramatically reduce the overall efficiency of the clausal reasoner.
Theorem 3.3. The following soundness and completeness theorem holds:

$$\Sigma \models_\pi \langle \varphi : \alpha \rangle \iff \Sigma \vdash_{\text{Res}} \langle \varphi : \alpha \rangle \iff \Sigma_{\geq \alpha} \models \varphi \iff \Sigma_{\geq \alpha} \vdash \varphi$$

where $\models_\pi$ means any $\omega$ compatible with $\Sigma$ is also compatible with $\langle \varphi : \alpha \rangle$, or formally, $\forall \omega, \pi_\Sigma(\omega) \leq \pi_{\{\langle \varphi : \alpha \rangle\}}(\omega)$. $\vdash_{\text{Res}}$ relies on the repeated use of possibilistic resolution.

The last half of the above expression reduces to the soundness and completeness theorem of propositional logic applied to each level cut of $\Sigma$, which is an ordinary propositional base.

3.4 Partial Inconsistency

The inconsistency degree of a base $\Sigma$ in terms of its $\alpha$-cut can be equivalently defined as the largest weight $\alpha$ such that the $\alpha$-cut of $\Sigma$ is inconsistent:

$$\text{Inc}(\Sigma) = \max\{\alpha \mid \Sigma_{\geq \alpha} \text{ is inconsistent}\}.$$  

Inc($\Sigma$) = 0 entails $\Sigma^*$ is consistent. In [22], the inconsistency degree of $\Sigma$ is defined by the least possibility distribution $\pi_\Sigma$, concretely $\text{Inc}(\Sigma) = 1 - \sup_{\omega \in \Omega} \pi_\Sigma(\omega)$.

To check whether $\varphi$ follows from $\Sigma$, one should add $\langle \neg \varphi : 1 \rangle$ to $\Sigma$ and then check whether $\Sigma \cup \{\langle \neg \varphi : 1 \rangle\} \models \langle \bot : \alpha \rangle$. Equivalently the maximum $\alpha$ s.t. $\Sigma \models \langle \varphi : \alpha \rangle$ is given by the inconsistency degree of $\Sigma \cup \{\langle \neg \varphi : 1 \rangle\}$, i.e. $\Sigma \models \langle \varphi : \alpha \rangle$ iff $\alpha = \text{Inc}(\Sigma \cup \{\langle \neg \varphi, 1 \rangle\})$.

Proposition 3.4. The next statements are proven in [22]:

$$\Sigma \models \langle \varphi : \alpha \rangle \iff \Sigma \cup \{\langle \neg \varphi : 1 \rangle\} \models \langle \bot : \alpha \rangle \iff \alpha = \text{Inc}(\Sigma \cup \{\langle \neg \varphi, 1 \rangle\}) \iff \Sigma_{\geq \alpha} \vdash \varphi.$$  

This result shows that any deduction problem in possibilistic logic can be viewed as computing an inconsistency degree.

3.5 Clausal and Non-Clausal Bases

According to previous definitions, to each class of propositional formulas corresponds a class of possibilistic bases. Below, we define the possibilistic classes handled here and after recall the complexity of computing the inconsistency degree of their members.

Definition 3.5. A possibilistic base $\Sigma = \{\langle \varphi_i : \alpha_i \rangle \mid i = 1, \ldots, k\}$ is called Horn, clausal or NC if all its formulas $\varphi_i, 1 \leq i \leq 1$, are Horn, clausal or NC, respectively. $H_\Sigma, C_\Sigma$ and $NC_\Sigma$ denote, respectively, the classes of possibilistic Horn, clausal and NC bases.

In this article we define in Definition 4.19 a novel possibilistic class, i.e. the possibilistic Horn-NC class. Next we just define acronyms associated to the problems of computing the inconsistency degree of the four mentioned possibilistic classes.

Definition 3.6. Horn-INC, CL-INC, Horn-NC-INC and NC-INC denote respectively the problems of computing the inconsistency degree of Horn, clausal, Horn-NC and NC bases.

Complexities. Regarding the complexities of the previous problems, we have:

- Clausal Pbs.: CL-INC is Co-NP-complete [41] and Horn-INC is polynomial [41].
- NC-INC is Co-NP-complete. This claim stems from: (i) Theorem 3.3 applies to both clausal and NC bases; and (ii) checking whether an interpretation is a model of an NC propositional formula is polynomial as for clausal formulas.
- Horn-NC-INC has polynomial complexity as proven in Section 7.
4 The Possibilistic Horn-NC Class: $\overline{H}_\Sigma$

This section defines the class $\overline{H}_\Sigma$ of Possibilistic Horn Non-Clausal (Horn-NC) bases and states its properties and relationships with other possibilistic classes. The proofs were given in [40] but are provided in an Appendix for the sake of the paper being self contained. $\overline{H}_\Sigma$ subsumes the next two classes:

- possibilistic Horn, or $H_\Sigma$; and
- propositional Horn-NC, or $\overline{H}$ (recently presented [40]).

We first define the latter, i.e. the class $\overline{H}$ of propositional Horn-NC formulas, which is the propositional component of the new possibilistic class $\overline{H}_\Sigma$ to be introduced.

4.1 Simple Definition of $\overline{H}$

Below, we define $\overline{H}$ in a simple way, and in the next subsection, will give its detailed definition by taking a closer look to this simple definition. We start by defining the negative formulas, which generalize the negative literals in the clausal framework.

**Definition 4.1.** A non-clausal formula is negative if it has uniquely negative literals. We will denote $N^-$ the set of negative formulas.

**Example 4.2.** Trivially negative literals are basic negative formulas. Another example of negative NC formula is $(\lor \{\land \neg P \neg R\} \{\land \neg S (\lor \neg P \neg Q)\}) \in N^-$. □

Next we upgrade the Horn pattern “a Horn clause has (any number of negative literals and) at most one positive literal” to the NC context in the next straightforward way:

**Definition 4.3.** An NC formula is Horn-NC if all its disjunctions have any number of negative disjuncts and at most one non-negative disjunct. We denote $H$ the class of Horn-NC formulas.

Clearly the class $\overline{H}$ subsumes the Horn class $H$. From Definition 1.3 it follows trivially that all sub-formulas of any Horn-NC are Horn-NC too. Yet, the converse does not hold: there are non-Horn-NC formulas whose all sub-formulas are Horn-NC.

**Example 4.4.** One can see that $\varphi_1$ below has only one non-negative disjunct and so $\varphi_1$ is Horn-NC, while $\varphi_2$ is not Horn-NC as it has two non-negative disjuncts.

- $\varphi_1 = (\lor \{\land \neg Q \neg S\} \{\land R P\})$.
- $\varphi_2 = (\lor \{\land \neg Q S\} \{\land R \neg P\})$. □

**Example 4.5.** We now consider both $\varphi$ in Example 2.12 (copied below) and $\varphi'$ below too, which results from $\varphi$ by just switching its literal $\neg P$ for $P$:

- $\varphi = (\lor \neg P \{\land (\lor \neg P R) \{\land Q (\lor P \neg S)\}\})$
- $\varphi' = (\lor P \{\land (\lor \neg P R) \{\land Q (\lor P \neg S)\}\})$

All disjunctions of $\varphi$, i.e. $(\lor \neg P R)$, $(\lor P \neg S)$ and $\varphi$ itself (By Definition 2.5, $\varphi$ is a sub-formula of $\varphi$), have exactly one non-negative disjunct; so $\varphi$ is Horn-NC. Yet, $\varphi' = (\lor P \phi)$, $\phi$ being non-negative, has two non-negative disjuncts; thus $\varphi'$ is not Horn-NC. □
4.2 Detailed Definition of $\overline{H}$

Before giving a fine-grained definition of $\overline{H}$, we individually and inductively specify:

- Horn-NC conjunctions, in Lemma 4.6, and
- Horn-NC disjunctions, in Lemma 4.8.

and subsequently, we *compactly* specify $\overline{H}$ by merging both specifications.

Just as conjunctions of Horn clausal formulas are Horn too, likewise conjunctions of Horn-NC formulas are Horn-NC too, which is straightforwardly formalized next.

**Lemma 4.6.** Conjunctions of Horn-NC formulas are Horn-NC as well, formally:

$$\{\land \varphi_1 \ldots \varphi_i \ldots \varphi_k\} \in \overline{H} \text{ iff for } 1 \leq i \leq k, \varphi_i \in \overline{H}.$$  

*Proof.* It is obvious that if all sub-formulas $\varphi_i$ individually verify Definition 4.3 so does a conjunction thereof, and vice versa. 

**Example 4.7.** If $H$ is Horn, $\phi_1$ is $\phi_1$ from Example 4.4 and $\phi_2$ is $\phi$ from Example 4.5, i.e. $\phi_1$ and $\phi_2$ are Horn-NC, then for instance $\varphi_1 = \{\land H \phi_1 \phi_2\}$ is Horn-NC.

In order to give now a detailed definition of $\overline{H}$, we verify that Definition 4.3 can be equivalently reformulated in the next inductive manner: “an NC is Horn-NC if all its disjunctive sub-formulas have any number of negative disjuncts and one disjunct is Horn-NC”. This leads to the next formalization and statement.

**Lemma 4.8.** A NC disjunction $\varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k)$ with $k \geq 1$ disjuncts pertains to $\overline{H}$ iff $\varphi$ has $k - 1$ negative disjuncts and one Horn-NC disjunct, formally

$$\varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in \overline{H} \text{ iff } \exists i \text{ s.t. } \varphi_i \in \overline{H} \text{ and } \forall j \neq i, \varphi_j \in \overline{N}.$$  

*Proof.* See Appendix.

The next claims follow trivially from Lemma 4.8:

- Horn clauses are non-recursive Horn-NC disjunctions.
- NC disjunctions with all negative disjuncts are Horn-NC.
- NC disjunctions with $k \geq 2$ non-negative disjuncts are not Horn-NC.

Next, we first reexamine, bearing Lemma 4.8 in mind, the formulas from Example 4.4 included in Example 4.9 and then those from Example 4.5 included in Example 4.10.

**Example 4.9.** Below we analyze $\varphi_1$ and $\varphi_2$ from Example 4.4.

- $\varphi_1 = (\lor \{\land \neg Q \neg S\} \{\land R P\})$.
  - Clearly $\{\land \neg Q \neg S\} \in \overline{N}$.
  - By Lemma 4.6 $\{\land R P\} \in \overline{H}$.
  - By Lemma 4.8 $\varphi_1 \in \overline{H}$.
- $\varphi_2 = (\lor \{\land \neg Q S\} \{\land R \neg P\})$.
  - Obviously $\{\land \neg Q S\} \notin \overline{N}$ and $\{\land R \neg P\} \notin \overline{N}$.
  - According to Lemma 4.8 $\varphi_2 \notin \overline{H}$. 

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Example 4.10. Consider again $\varphi$ and $\varphi'$ from Example 4.5 and recall that $\varphi'$ results from $\varphi$ by just switching its literal $\neg P$ for $P$. Below we check one-by-one whether or not the sub-formulas of both $\varphi$ and $\varphi'$ are in $\overline{H}$.

- By Lemma 4.8, $(\lor \neg P R) \in \overline{H}$.
- By Lemma 4.8, $(\lor P \neg S) \in \overline{H}$.
- By Lemma 4.6, $\{\land Q (\lor P \neg S)\} \in \overline{H}$.
- By Lemma 4.6, $\phi = \{\land (\lor \neg P R) \{\land Q (\lor P \neg S)\}\} \in \overline{H}$.
- Using previous formula $\varphi$, we have $\varphi = (\lor \neg P \phi)$. Since $\neg P \in N^-$ and $\phi \in \overline{H}$, by Lemma 4.8 $\varphi \in \overline{H}$.
- The second formula in Example 4.5 is $\varphi' = (\lor P \phi)$.
  - Since $P, \phi \not\in N^-$, by Lemma 4.8 $\varphi' \notin \overline{H}$. \square

By using Lemmas 4.6 and 4.8 the class $\overline{H}$ is syntactically, compactly and inductively defined as follows.

Definition 4.11. We define the set $\widehat{H}$ over the set of propositional variables $P$ as the smallest set such that the conditions below hold, where $k \geq 1$ and $L$ is the set of literals.

1. $L \subset \widehat{H}$.
2. If $\forall i, \varphi_i \in \widehat{H}$ then $\{\land \varphi_1 \ldots \varphi_i \ldots \varphi_k\} \in \widehat{H}$.
3. If $\varphi_i \in \widehat{H}$ and $\forall j \not= i, \varphi_j \in N^-$ then $(\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in \widehat{H}$.

Theorem 4.12. We have that $\widehat{H} = \overline{H}$.

Proof. See Appendix. \square

Theorem 4.12 below states that $\widehat{H}$ and $\overline{H}$ indeed coincide, namely Definition 4.11 is the recursive and compact definition of the class $\overline{H}$ of Horn-NC formulas. Besides, inspired by Definition 4.11 in [40] a linear algorithm is designed that recognizes whether a given NC $\varphi$ is Horn-NC and, such property is inherited by the possibilistic Horn-NC formulas.

Example 4.13. Viewed from Definition 4.11 we analyze $\varphi$ and $\varphi'$ from Example 4.10:

- By (3), $(\lor \neg P R) \in \overline{H}$.
- By (3), $(\lor P \neg S) \in \overline{H}$.
- By (2), $\{\land Q (\lor P \neg S)\} \in \overline{H}$.
- By (2), $\phi = \{\land (\lor \neg P R) \{\land Q (\lor P \neg S)\}\} \in \overline{H}$.
- By (3), $\varphi = (\lor \neg P \phi) \in \overline{H}$.
- By (3), $\varphi' = (\lor P \phi) \notin \overline{H}$. \square

Example 4.14. If we assume that $\varphi_1, \varphi_2$ and $\varphi_3$ are negative and $\varphi_4$ and $\varphi_5$ are Horn-NC, then according to Definition 4.11 four examples of Horn-NC formulas follow.
• By (3), $\varphi_6 = (\lor \varphi_1 \varphi_4) \in \overline{H}$.
• By (2), $\varphi_7 = \{\land \varphi_1 \varphi_3 \varphi_6\} \in \overline{H}$.
• By (3), $\varphi_8 = (\lor \varphi_1 \varphi_2 \varphi_7) \in \overline{H}$.
• By (2), $\varphi_9 = \{\land \varphi_6 \varphi_7 \varphi_8\} \in \overline{H}$.

Next, we analyze a more complete example, concretely $\varphi$ from the Introduction.

**Example 4.15.** Let us take $\varphi$ below, wherein $\varphi_1, \varphi_2$ and $\varphi_3$ are NC formulas:

$$\varphi = \{\land P (\lor \neg Q \{\land (\lor \neg P \neg Q \ R) (\lor \varphi_1 \{\land \varphi_2 \neg P\}) \} \} \varphi_3 \}.$$  

The disjunctions of $\varphi$ and the proper $\varphi$ can be rewritten as follows:

- $\psi_1 = (\lor \neg P \neg Q \ R)$.
- $\psi_2 = (\lor \varphi_1 \{\land \varphi_2 \neg P\})$.
- $\psi_3 = (\lor \neg Q \{\land \psi_1 \psi_2 \ Q \}$).
- $\varphi = \{\land P \psi_3 \varphi_3\}$.

We analyze one-by-one such disjunctions and finally the proper $\varphi$:

- $\psi_1$: Trivially, $\psi_1$ is Horn, so $\psi_1 \in \overline{H}$.
- $\psi_2$: $\psi_2 \in \overline{H}$ if $\varphi_1, \varphi_2 \in \overline{H}$ and if at least one of $\varphi_1$ or $\varphi_2$ is negative.
- $\psi_3$: $\psi_3 \in \overline{H}$ if $\psi_2 \in \overline{H}$ (as $\psi_1 \in \overline{H}$).
- $\varphi$: $\varphi \in \overline{H}$ if $\psi_2, \psi_3 \in \overline{H}$ (as $\psi_3 \in \overline{H}$ if $\psi_2 \in \overline{H}$).

Summarizing the conditions on $\varphi$ and on $\psi_2$, we have that:

- $\varphi$ is Horn-NC if $\varphi_3, \varphi_1$ and $\varphi_2$ are Horn-NC and if at least one of $\varphi_1$ or $\varphi_2$ is negative.

If we consider that $\varphi$ implicitly verifies Definition 4.3 (all sub-formulas of a Horn-NC are Horn-NC), then we conclude that $\varphi$ is Horn-NC if at least one of $\varphi_1$ or $\varphi_2$ is negative.

### 4.3 Properties of the Class $\overline{H}$

An important feature of Horn-formulas is the following:

**Theorem 4.16.** Applying $\lor/\land$-distributivity to a Horn-NC $\varphi$ results in a Horn formula.

*Proof.* See Appendix. ■

We already saw that syntactically $\overline{H}$ subsumes $H$, but besides, $\overline{H}$ is semantically related to $H$ as Theorem 4.17 claims.

**Theorem 4.17.** $\overline{H}$ and $H$ are semantically equivalent: each formula in a class is logically equivalent to some formula in the other class.

*Proof.* By Theorem 4.16 for every $\varphi \in \overline{H}$ there exists $H \in \overline{H}$ such that $\varphi \equiv H$. The converse follows from the fact that $H \subset \overline{H}$. ■
The next theorem makes it explicit how the classes Horn-NC and NC are related.

**Theorem 4.18.** \( \mathcal{H} \) contains the next NC fragment: if applying \( \land / \lor \) distributivity to an NC formula \( \varphi \) results in a Horn formula, then \( \varphi \) is in \( \mathcal{H} \).

*Proof.* See Appendix. \( \blacksquare \)

The syntactical and semantical properties exhibited by \( \mathcal{H} \) affirmed by the last three theorems suggest that \( \mathcal{H} \) is a kind of NC analogous of the standard Horn class \( \mathcal{H} \).

### 4.4 The Definition of \( \mathcal{H}_\Sigma \)

Finally, from \( \mathcal{H} \), we straightforwardly define the class \( \mathcal{H}_\Sigma \) of possibilistic Horn-NC bases.

**Definition 4.19.** A possibilistic Horn-NC formula is a pair \( \langle \varphi : \alpha \rangle \), where \( \varphi \in \mathcal{H} \) and \( \alpha \in (0 \ 1] \). A possibilistic Horn-NC base \( \Sigma \) is a set of possibilistic Horn-NC formulas. \( \mathcal{H}_\Sigma \) denotes the class of possibilistic Horn-NC bases.

**Example 4.20.** We take the next Horn-NCs: \( \varphi \) from Example 4.5 and \( \varphi_9 \) from Example 4.14. By \( \varphi' \) we denote \( \varphi \) from Example 4.15 considering that the specified conditions warranting that \( \varphi \) is Horn-NC are met. An example of a possibilistic Horn-NC base is:

\[ \{ \langle P : .8 \rangle, \langle \varphi : .8 \rangle, \langle \varphi_9 : .5 \rangle, \langle \varphi' : .9 \rangle, \langle \neg Q, .1 \rangle \} \]

**Corollary 4.21.** \( \mathcal{H}_\Sigma \) and \( \mathcal{H}_\Sigma \) are semantically equivalent: each formula in a class is equivalent to some formula in the other class.

*Proof.* It follows from the definitions of \( \mathcal{H}_\Sigma \) and \( \mathcal{H}_\Sigma \) and Theorem 4.17. \( \blacksquare \)

**Corollary 4.22.** \( \mathcal{H}_\Sigma \) is the next NC\( _\Sigma \) fragment: if \( \langle \varphi : \alpha \rangle \in NC_\Sigma \) and applying \( \land / \lor \) distributivity to \( \varphi \) results in a Horn formula, then \( \langle \varphi : \alpha \rangle \in \mathcal{H}_\Sigma \).

*Proof.* It follows from the definitions of \( \mathcal{H}_\Sigma \) and \( NC_\Sigma \) and Theorem 4.18. \( \square \)

**Remark.** Since \( \mathcal{H} \) is the NC analogous of \( \mathcal{H} \) so is \( \mathcal{H}_\Sigma \) of \( \mathcal{H}_\Sigma \).

### 5 Possibilistic NC Unit-Resolution \( \mathcal{UR}_\Sigma \)

Possibilistic clausal resolution was defined in the 1980s [23] [24] but possibilistic non-clausal resolution has not been proposed yet. This section is a step forward towards its definition as we define Possibilistic Non-Clausal Unit-Resolution, denoted \( \mathcal{UR}_\Sigma \), which is an extension of the calculus presented in [40] for propositional logic. The main inference rule of \( \mathcal{UR}_\Sigma \) is called \( UR_\Sigma \), and while the other rules in \( UR_\Sigma \) are simple, \( UR_\Sigma \) is somewhat involved and so is presented progressively as follows:

- for quasi-clausal Horn-NC bases in Subsection 5.1 and
- for nested Horn-NC bases in Subsection 5.2

Afterwards, Subsection 5.3 describes \( \mathcal{UR}_\Sigma \), which besides \( UR_\Sigma \), comprises: (a) the propositional rule \( UR_P \), which is \( UR_\Sigma \) adapted to propositional logic, (b) the propositional simplification rules, and (c) the possibilistic rules \( MinD \) and \( MaxN \). Subsection 5.4 gives the algorithm to obtain \( \text{Inc}(\Sigma) \) which combines \( \mathcal{UR}_\Sigma \) with \( \alpha \)-cuts of the input \( \Sigma \). To end this section, Subsection 5.5 gives two further inferences rules, not needed for warranting the completeness of \( \mathcal{UR}_\Sigma \). We recall that \( \bot \) and \( (\lor) \) are equivalent (see Definition 2.9).
5.1 Quasi-Clausal NC Unit-Resolution

We start our presentation with propositional formulas and then switch to possibilistic bases. Assume propositional formulas with the quasi-clausal pattern below in which $\ell$ and $\neg \ell$ are any literal and its negated one, and the $\varphi$’s and the $\phi$’s are formulas:

$$\{ \land \varphi_1 \ldots \varphi_{l-1} \ell \varphi_{l+1} \ldots \varphi_i \land (\lor \varphi_1 \ldots \varphi_{j-1} \neg \ell \varphi_{j+1} \ldots \varphi_k) \varphi_{i+1} \ldots \varphi_n \}$$

We say that these formulas are quasi-clausal because if the $\varphi$’s and $\phi$’s were clauses and literals, respectively, then such formulas would be clausal. It is not hard to see that a quasi-clausal formula is equivalent to a formula of the kind:

$$\{ \land \varphi_1 \ldots \varphi_{l-1} \ell \varphi_{l+1} \ldots \varphi_i \land (\lor \varphi_1 \ldots \varphi_{j-1} \neg \ell \varphi_{j+1} \ldots \varphi_k) \varphi_{i+1} \ldots \varphi_n \}$$

and thus, one can derive the next simple inference rule for propositional formulas:

$$\frac{\ell \land (\lor \varphi_1 \ldots \varphi_j \neg \ell \varphi_{j+1} \ldots \varphi_k)}{(\lor \varphi_1 \ldots \varphi_j \varphi_{j+1} \ldots \varphi_k)}$$

(1)

Notice that for clausal formulas, Rule (1) coincides with clausal unit-resolution.

Now let us switch to possibilistic bases. The setting in which NC unit-resolution is applicable is when $\Sigma$ has two Horn-NC formulas such that one is a unit clause $\langle \ell : \alpha \rangle$ and the other has the pattern: $\langle \{ \land \varphi_1 \ldots \varphi_i (\lor \varphi_1 \ldots \varphi_{j-1} \neg \ell \varphi_{j+1} \ldots \varphi_k) \varphi_{i+1} \ldots \varphi_n \} : \beta \rangle$. Namely, as $\Sigma$ is an implicit conjunction of its formulas, then $\Sigma$ contains a conjunction:

$$\langle \ell : \alpha \rangle \land \langle \{ \land \varphi_1 \ldots \varphi_{i-1} (\lor \varphi_1 \ldots \varphi_{j-1} \neg \ell \varphi_{j+1} \ldots \varphi_k) \varphi_{i+1} \ldots \varphi_n \} : \beta \rangle$$

(2)

In this setting and by using Min-Decomposability, i.e. $N(\varphi \land \psi) = \min(N(\varphi), N(\psi))$ (Definition 5.2), one can easily derive the next possibilistic inference:

$$\frac{\langle \ell : \alpha \rangle \land \langle \{ \lor \varphi_1 \ldots \varphi_j \neg \ell \varphi_{j+1} \ldots \varphi_k \} : \beta \rangle}{\langle \{ \lor \varphi_1 \ldots \varphi_j \varphi_{j+1} \ldots \varphi_k \} : \min\{\alpha, \beta\} \rangle}$$

(3)

The soundness of (3) follows immediately from the property Min-Decomposability. If $D(\neg \ell)$ stands for $(\lor \varphi_1 \ldots \varphi_j \varphi_{j+1} \ldots \varphi_n)$, then the previous rule can be concisely rewritten as:

$$\frac{\langle \ell : \alpha \rangle \land \langle \{ \lor \neg \ell \ D(\neg \ell) \} : \beta \rangle}{D(\neg \ell) : \min\{\alpha, \beta\}}$$

(4)

Notice that the previous rule amounts to substituting the formula referred to by the right conjunct in the numerator with the formula in the denominator, and in practice, to just eliminate $\neg \ell$ and update the necessity weight. Let us illustrate these notions.

Example 5.1. Let $\Sigma$ be a base including $\varphi_1$ and $\varphi_2$ below, where $\phi$ is a formula:

- $\varphi_1 = \langle P : .8 \rangle$
- $\varphi_2 = \langle \{ \land \phi (\lor \neg R \neg P S) (\lor S \{ \land Q \neg P \} \ R) \} : .6 \rangle$.

Taking $P$ in $\varphi_1$ and the left-most $\neg P$ in $\varphi_2$, we have $D(\neg P) = (\lor \neg R S)$, and by applying

$$\Sigma \leftarrow \Sigma \cup \langle \{ \land \phi (\lor \neg R S) (\lor S \{ \land Q \neg P \} \ R) \} : .6 \rangle$$

Rule (4) to $\varphi_2$, the above formula is deduced and added to the base $\Sigma$. □
We now extend our analysis from formulas with pattern \( (\lor -\ell \ D(-\ell) ) : \beta \) to formulas with pattern \( (\lor C(-\ell) \ D(-\ell) ) : \beta \) wherein \( C(-\ell) \) is the maximal sub-formula that becomes false when \(-\ell\) is false, namely, \( C(-\ell) \) is: (i) the maximal sub-formula, and (ii) equivalent to a conjunction of the kind \(-\ell \land \psi\). In other words, \( C(-\ell) \) is the maximal sub-formula "conjunctively linked" to \(-\ell\).

For instance If the input base \( \Sigma \) contains \( \ell : \alpha \) and another formula of the kind:

\[
\langle (\lor \varphi_1 \ {\land} \phi_1 \ {\land} -\ell \ (\lor \phi_2 -P)) \phi_3 \rangle \varphi_2 : \beta
\]

then \( C(-\ell) = \{ \land \phi_1 \ {\land} -\ell \ (\lor \phi_2 -P) \} \phi_3 \} \) because:

- (ii) \( C(-\ell) \) is equivalent to \(-\ell \land \psi = -\ell \land \{ \land \phi_1 \ (\lor \phi_2 -P) \} \phi_3 \}; and
- (i) no sub-formula \( C'(-\ell) \) bigger than \( C(-\ell) \) verifies \( C'(-\ell) \equiv -\ell \land \psi' \).

Clearly, if \(-\ell\) becomes false so does \( C(-\ell) \) = \( \{ \land \phi_1 \ {\land} -\ell \ (\lor \phi_2 -P) \} \phi_3 \} \).

**Remark.** \( C(-\ell) \) contains \(-\ell\) but \( D(-\ell) \) excludes it.

**Example 5.2.** The formula given below is an extension of \( \varphi_2 \) from Example 5.1 in which, by clarity, its previous sub-formula \( (\lor S \ {\land} -Q -P) \) is denoted \( \phi_1 \) and the previous literal \(-P\) is now extended to the formula \( \{ -P \ (\lor S -R) \} \) including \(-P\):

\[
\varphi = \langle (\land \phi (\lor -R \ {\land} -P \ (\lor S -R) ) \ S) \phi_1 \ R \rangle : .6
\]

Taking the left-most \(-P\) \( \phi_1 \) has also another literal \(-P\), \( \varphi \) has a sub-formula with pattern \( (\lor C(-P) \ D(-P) ) \), in which \( C(-P) = \{ -P \ (\lor S -R) \} \) and \( D(-P) = (\lor -R S) \).

Regarding the inference rule, we have that when \( \Sigma \) has both a unitary clause \( \ell : \alpha \) and another formula \( (\varphi : \beta) \) such that \( \varphi \) has the pattern \( (\lor C(-\ell) \ D(-\ell) ) \), then the possibilistic NC unit-resolution rule is easily obtained by extending Rule (4) as follows:

\[
\frac{\langle \ell : \alpha \rangle \land \langle (\lor C(-\ell) \ D(-\ell) ) : \beta \rangle}{\langle D(-\ell) : \min(\alpha, \beta) \rangle} \tag{5}
\]

The soundness of (5) follows from \( \ell \land C(-\ell) \equiv \bot \) and its proof is given in Section 4. Fig. 3 depicts Rule (5) where the left and right trees represent, respectively, the numerator and denominator of (5).

![Fig. 3. Depicting Rule (5).](image)

**Example 5.3.** Rule (5) with \( \varphi_1 = \langle P : .3 \rangle \) and with \( \varphi \) from Example 5.2 derives:

\[
\Sigma \leftarrow \Sigma \cup \{ (\land \phi (\lor -R S) \phi_1 \ R) : .3 \}
\]
5.2 Nested NC Unit-Resolution

Coming back to the almost-clausal formulas expressed in (2) and extending its literal $\neg \ell$ to $\mathcal{C}(\neg \ell)$, we now rewrite them compactly as indicated below, where $\Pi$ and $\Pi'$ denote a concatenation of formulas, namely $\Pi = \varphi_1 \ldots \varphi_{i-1}$ and $\Pi' = \varphi_{i+1} \ldots \varphi_n$:

$$\langle \ell : \alpha \rangle \land \{ \langle \land \Pi (\lor \mathcal{C}(\neg \ell) \land \mathcal{D}(\neg \ell)) \Pi' : \beta \rangle \}$$

We now analyze the nested Horn-NC bases $\Sigma$ to which NC unit-resolution can be indeed applied. That is, $\Sigma$ must have a unit-clause $\langle \ell : \alpha \rangle$ and a possibilistic nested Horn-NC formula, denoted $\langle \Pi : \beta \rangle$, with a syntactical pattern of the nest kind:

$$\langle [\land \Pi_1 \ldots \land \Pi_k (\lor \mathcal{C}(\neg \ell) \land \mathcal{D}(\neg \ell)) \Pi'_k \ldots \Pi'_1 : \beta \rangle$$

where all the $\Pi_j$'s and $\Pi'_j$'s are concatenations of formulas, e.g. for the nesting level $j, 1 \leq j \leq k$, we have $\Pi_j = \varphi_{j_1} \ldots \varphi_{j_{j-1}}$ and $\Pi'_j = \varphi_{j_{j+1}} \ldots \varphi_{j_{k_j}}$. Since the presence of formulas in the base $\Sigma$ means that they are conjunctively linked, then one has:

$$\langle \ell : \alpha \rangle \land \{ [\land \Pi_1 \ldots [\land \Pi_k (\lor \mathcal{C}(\neg \ell) \land \mathcal{D}(\neg \ell)) \Pi'_k \ldots \Pi'_1 : \beta \} \} \}$$

By following the same principle that led us to Rule (5) and taking into account that $N(\varphi_1 \land \varphi_2) = \min\{N(\varphi_1), N(\varphi_1)\}$, one obtains the nested NC unit-resolution rule:

$$\langle \ell : \alpha \rangle \land \{ [\land \Pi_1 \ldots [\land \Pi_k (\lor \mathcal{C}(\neg \ell) \land \mathcal{D}(\neg \ell)) \Pi'_k \ldots \Pi'_1 : \beta \} \} \}_{\min\{\alpha, \beta\}}$$

Recapitulating, Rule (7) indicates that if the Horn-NC $\Sigma$ has two formulas such that one is a unit clause $\langle \ell : \alpha \rangle$ and the other $\langle \Pi : \beta \rangle$ has the pattern of the right conjunct in the numerator, then $\Pi$ can be replaced with the formula in the denominator. In practice, applying (7) amounts to just removing $\mathcal{C}(\neg \ell)$ from $\Pi$ and updating the necessity weight.

We now denote $\Pi$ the right conjunct in the numerator of (7) and by $\Pi > (\lor \mathcal{C}(\neg \ell) \land \mathcal{D}(\neg \ell))$ denote that $(\lor \mathcal{C}(\neg \ell) \land \mathcal{D}(\neg \ell))$ is a sub-formula of $\Pi$. Rule (7) above can be compacted, giving rise to a more concise formulation of $UR_{\Sigma}$:

$$\langle \ell : \alpha \rangle \land \{ \Pi > (\lor \mathcal{C}(\neg \ell) \land \mathcal{D}(\neg \ell)) : \beta \} \}_{UR_{\Sigma}}$$

The soundness of the rule $UR_{\Sigma}$ follows from $\ell \land \mathcal{C}(\neg \ell) \equiv \bot$ and is proven in Section 7. Two simple examples illustrating how $UR_{\Sigma}$ works are Examples [6.1] and [6.2]. Two more complete examples are Examples [6.3] and [6.6] but they employ other inferences and mechanisms relative to $UR_{\Sigma}$ given in the remaining of this section.

Remark. It is not difficult to check that, for clausal formulas, $UR_{\Sigma}$ coincides with the standard possibilistic clausal unit-resolution [22, 26]. This clausal-like formulation of NC unit-resolution contrasts with the functional-like one of classical NC resolution handled until now in the literature and presented in [46] (see also [5]). We believe that our version, as previously said, is more suitable to understand, implement and formally analyze.

---

4 The notation $[\land \varphi_1 \ldots \varphi_k]$ was introduced in Definition [23] bottom.
5.3 The Calculus $\mathcal{UR}_\Sigma$

Besides $\mathcal{UR}_\Sigma$, the calculus $\mathcal{UR}_\Sigma$ also includes the rules: (a) propositional NC unit-resolution, or $\mathcal{UR}_P$, which is $\mathcal{UR}_\Sigma$ but applied inside propositional formulas, (b) the rules to simplify propositional formulas, and (c) the possibilistic rules Min-Decomposability (MinD) and Max-Necessity (MaxN) mentioned in Definition 3.2

5.3.1 Propositional NC Unit-Resolution

A major difference between computing the inconsistency degree of clausal and non-clausal bases is that the unity members of the former, i.e. clauses, are always consistent, while a non-clausal formula can itself be inconsistent. That is, the input $\Sigma$ can contain a formula $\langle \Pi : \alpha \rangle$ where $\Pi$ is inconsistent, and if so, $\langle \Pi : \alpha \rangle$ is equivalent to $\langle \bot : \alpha \rangle$, which brings to:

**Proposition 5.4.** If $\langle \Pi : \alpha \rangle \in \Sigma$ and $\Pi$ is inconsistent then $\text{Inc}(\Sigma) \geq \alpha$.

*Proof.* By definition $\text{Inc}(\Sigma) = \max\{\beta | \Sigma \geq \beta \text{ is inconsistent}\}$. Trivially if $\Pi$ is inconsistent then $\bot \in \Sigma \geq \alpha$, and thus, $\Sigma \geq \alpha$ is inconsistent. So $\text{Inc}(\Sigma) \geq \alpha$. ■

Hence, first of all, the propositional formula $\Pi$ of each $\langle \Pi : \alpha \rangle \in \Sigma$ must be checked for consistency. If $\Pi$ is inconsistent, then, by definition, $\text{Inc}(\Sigma)$ is the maximum of $\alpha$ and the inconsistency degree of the strict $\alpha$-cut of $\Sigma$. Thus, one can remove from $\Sigma$ all formulas $\langle \Pi : \beta \rangle$ such that $\beta \leq \alpha$ and search whether $\text{Inc}(\Sigma_{>\alpha}) > 0$.

The inference rule $\mathcal{UR}_P$ testing the consistency of a propositional $\Pi$, where $\langle \Pi : \alpha \rangle \in \Sigma$, is easily derived from $\mathcal{UR}_\Sigma$ by considering that the conjunction of a unit clause $\ell$ and of a formula $\Pi$ containing $C(\neg \ell)$ happens inside $\Pi$. Thus $\mathcal{UR}_P$ is as follows:

\[
\begin{align*}
\langle \ell \land \Pi \succ (\lor \ C(\neg \ell) \ D(\neg \ell)) : \alpha \rangle & \quad \mathcal{UR}_P \\
\langle \Pi \succ \neg \ell : \alpha \rangle
\end{align*}
\]

(8)

A complete example through which we show how $\mathcal{UR}_P$ proceeds testing the consistency of propositional NC formulas is Example 6.3 in the next section, and Example 6.4 illustrates the effects of applying Proposition 5.4.

5.3.2 Simplification Formulas Rules

Each application of the previous $\mathcal{UR}_\Sigma$ and $\mathcal{UR}_P$ demands the subsequent application of trivial logical simplifications of propositional formulas. For instance, $\langle \lor \varphi (\lor P (\lor \neg R \phi)) \rangle$ and $\langle \lor P \{\land (\lor \varphi)\} \rangle$ can be obviously substituted by $\langle \lor \varphi P \neg R \phi \rangle$ and $\langle \lor P \rangle$, respectively. Next we formalize such kind of simplification rules. Being $\Pi$ the propositional formula of a possibilistic formula $\langle \Pi : \alpha \rangle$ in a base $\Sigma$, the first two rules below simplify formulas by (upwards) propagating ($\lor$) from sub-formulas to formulas:

\[
\begin{align*}
\langle \Pi \succ (\lor \phi_1 \ldots \phi_{i-1} (\lor) \phi_{i+1} \ldots \phi_k) : \alpha \rangle & \quad \lor \\
\langle \Pi \succ (\lor \phi_1 \ldots \phi_{i-1} \phi_{i+1} \ldots \phi_k) : \alpha \rangle \\
\langle \Pi \succ \{\land (\lor) \varphi_1 \ldots \varphi_{i-1} (\lor) \varphi_{i+1} \ldots \varphi_k\} : \alpha \rangle & \quad \land \\
\langle \Pi \succ (\lor) : \alpha \rangle
\end{align*}
\]

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The next two rules remove redundant connectives. The first one removes a connective $\circ$ if it is applied to a single formula, i.e. $[\circ \phi_1]$, and the second one removes a connective if it is inside another equal connective, i.e. applies to sub-formulas with the pattern $[\circ \phi_1 \phi_2 \ldots \phi_n \varphi_{i+1} \ldots \varphi_k]$, $\circ_1 = \circ_2$. So the formal rules are:

$$\langle \Pi \succ [\circ_1 \varphi_1 \ldots \varphi_{i-1} [\circ_2 \phi_1 \varphi_{i+1} \ldots \varphi_k] : \alpha] \circ \phi \rangle$$

$$\langle \Pi \succ [\circ_1 \varphi_1 \ldots \varphi_{i-1} \phi_1 \varphi_{i+1} \ldots \varphi_k] : \alpha \rangle$$

5.3.3 Possibilistic Rules

We recall the possibilistic rules in Definition 3.2. We pay attention to the case in which a conjunction $\langle \{\land \varphi_1 \ldots \varphi_k\} : \alpha \rangle$ is deduced. It is clear that, in this case, we can deduce that the necessity weight of each individual conjunct $\varphi$ is $\alpha$:

$$\langle \{\land \varphi_1 \ldots \varphi_k\} : \alpha \rangle \vdash \langle \varphi_1 : \alpha \rangle, \ldots, \langle \varphi_k : \alpha \rangle \quad \text{MinD}$$

The last needed rule to be included in $\mathcal{UR}_\Sigma$ is $\text{MaxN}$:

$$\langle \varphi : \alpha \rangle, \langle \varphi : \beta \rangle \vdash \langle \varphi : \max\{\alpha, \beta\} \rangle \quad \text{MaxN}$$

5.3.4 The Calculus $\mathcal{UR}_\Sigma$

The calculus $\mathcal{UR}_\Sigma$ is composed of all the above inference rules:

**Definition 5.5.** We define $\mathcal{UR}_\Sigma$ as the calculus formed by $\mathcal{UR}_\Sigma$, $\mathcal{UR}_P$, the rules $\text{MinD}$ and $\text{MaxN}$, and the simplification rules, namely

$$\mathcal{UR}_\Sigma = \{ \mathcal{UR}_\Sigma, \mathcal{UR}_P, \text{MinD}, \text{MaxN}, \bot \lor, \bot \land, \circ, \circ \circ \}.$$  

Examples 6.1 and 6.2 are simple examples of how $\mathcal{UR}_\Sigma$ proceeds. Example 6.5 and its continuation Example 6.6 are quite complete examples. Example 6.4 illustrates how $\mathcal{UR}_\Sigma$ searches for just one empty clause $\langle \bot : \alpha \rangle$ and Example 6.6 determines $\text{Inc}(\Sigma)$.

**Remark.** Having established possibilistic NC unit-resolution, the procedure NC unit-propagation for possibilistic NC formulas can be designed, and on top of it, the possibilistic NC DPLL scheme can be defined.

5.4 Finding $\text{Inc}(\Sigma)$

The calculus $\mathcal{UR}_\Sigma$ determines just one sub-set of contradictory formulas along with its inconsistency degree. Yet, a given $\Sigma$ can typically contain many contradictory subsets, each of them induces the deduction of one empty clause $\langle \bot : \alpha \rangle$. By definition of $\text{Inc}(\Sigma) = \max\{\alpha : \Sigma_\alpha \text{is inconsistent}\}$ and by Proposition 3.4, we have that:

$$\text{Inc}(\Sigma) = \max\{\alpha : \Sigma_\alpha \text{is inconsistent}\} = \max\{\alpha \mid \Sigma \vdash \langle \bot : \alpha \rangle\}. \quad (9)$$

Our simple strategy to find $\text{Inc}(\Sigma)$ is as follows. Firstly, using $\mathcal{UR}_\Sigma$, we determine one inconsistent subset $\Sigma_1 \subseteq \Sigma$ and its $\text{Inc}(\Sigma_1) = \alpha$, which according to (9), amounts to
deducing $\langle \bot : \alpha \rangle$. In the future, we are only interested in knowing whether \( \text{Inc}(\Sigma) > \alpha \) and so, for that, we require only the strict $\alpha$-cut of $\Sigma$, i.e. $\Sigma_{>\alpha}$. Again using $\text{UR}_{\Sigma}$, we attempt to deduce $\langle \bot : \beta \rangle$ and, if it is obtained, continue with $\Sigma_{>\beta}$. These operations are recursively performed until getting a consistent base, i.e. the empty clause is no longer deduced. Then the $\alpha$-cut of the last inconsistent base is the sought $\text{Inc}(\Sigma)$. This process is algorithmically described below. \textbf{Find} should be called with $\text{Inc} = 0$.

\textbf{Find($\Sigma$, Inc)}

1. Apply $\text{UR}_{\Sigma}$ to $\Sigma$ and if $\langle \bot : \alpha \rangle$ is derived then go to (2) else Return $\text{Inc}$.

2. We search whether there exists $\beta > \alpha$ such that $\Sigma_{>\beta}$ is inconsistent. Thus, we update $\Sigma \leftarrow \{ \langle \varphi : \beta \rangle | \langle \varphi : \beta \rangle \in \Sigma, \beta > \alpha \}$ and $\text{Inc} \leftarrow \alpha$; and call \textbf{Find($\Sigma$, Inc)}.

The value $\text{Inc}$ returned by \textbf{Find} is $\text{Inc}(\Sigma)$, which is proven in Section 7. If $\text{Inc} = 0$ then the input $\Sigma$ is consistent. In Example 6.6 we illustrate the algorithmic strategy of \textbf{Find}.

\section*{5.5 Further Inferences Rules}

This last subsection presents two further inferences rules no required to ensure completeness but, since they allow shorter proofs, their appropriate management can yield significant speed-ups. These two rules are: Propositional NC Local Unit-Resolution and Possibilistic NC Hyper Unit-Resolution.

\subsection*{5.5.1 Propositional NC Local-Unit-Resolution}

$\text{UR}_P$ could also apply to propositional sub-formulas and can be used in the general framework of non-Horn-NC bases. The $\text{UR}_P$ local application means that applying $\text{UR}_P$ to sub-formulas $\varphi$ of any formula $\Pi$, where $\langle \Pi : \alpha \rangle \in \Sigma$, such that $\varphi$ has the $\text{UR}_P$ numerator pattern, should be authorized. Namely, applying $\text{UR}_P$ to sub-formulas with pattern $\varphi = \ell \land \Pi \land (\lor C(-\ell) \land D(-\ell))$ should be permitted and so, $\varphi$ could be substituted with $\ell \land \Pi \land D(-\ell)$. Hence, the formal specification of the Propositional NC Local-Unit-Resolution rule, $LUR$, for any non-Horn-NC $\varphi$ is:

\[
\begin{array}{c}
\langle \Pi \succ (\ell \land \varphi \succ (\lor C(-\ell) \land D(-\ell)) : \alpha) \\
\langle \Pi \succ (\ell \land \varphi \succ D(-\ell)) : \alpha \rangle
\end{array} \frac{LUR}
\]

This inference rule should be read: if $\langle \Pi : \alpha \rangle \in \Sigma$ and $\Pi$ has a conjunctive \textbf{sub-formula} with a literal $\ell$ conjunctively linked to a sub-formula $\varphi$ having pattern $(\lor C(-\ell) \land D(-\ell))$, then its component $C(-\ell)$ can be eliminated.

An example illustrating the functioning of the previous rule is Example 6.7.

\textbf{Remark.} The introduction of this new rule $LUR$ applicable to certain sub-formulas habilitates new sequences of inferences, and so, shorter proofs are now available.

\textbf{Proposition 5.6.} Let $\langle \Pi, \alpha \rangle \in \Sigma$. If applying $LUR$ to $\Pi$ results in $\Pi'$, then $\Pi$ and $\Pi'$ are logically equivalent.

\textbf{Proof.} The soundness of $LUR$ follows from that of $\text{UR}_P$ proved in Lemma 7.3. □

\textbf{Remark.} The obtaining of the simplification rules for their local application to sub-formulas is similarly obtained.
5.5.2 Possibilistic NC Hyper-Unit-Resolution

The given definition of possibilistic NC unit-resolution, or $UR_\Sigma$, can be extended in order to obtain Possibilistic NC Hyper-Unit-Resolution ($HUR$). Then assume that the possibilistic base has a unit-clause $\langle \ell : \alpha \rangle$ and two sub-formulas $(\lor C(\neg \ell^1) \ D(\neg \ell^1))$ and $(\lor C(\neg \ell^2) \ D(\neg \ell^2))$, where $\neg \ell^i$ denotes a specific occurrence of $\neg \ell$. The simultaneous application of NC unit-resolution with two sub-formulas is formally expressed as follows:

$$
\frac{\langle \ell : \alpha \rangle \land \Pi^1 \succ (\lor C(\neg \ell^1) \ D(\neg \ell^1)) : \beta^1 \land \Pi^2 \succ (\lor C(\neg \ell^2) \ D(\neg \ell^2)) : \beta^2}{\Pi^1 \succ D(\neg \ell^1) : \min\{\alpha, \beta^1\} \land \Pi^2 \succ D(\neg \ell^2) : \min\{\alpha, \beta^2\}}
$$

If the sub-formula $\langle \Pi^i \succ (\lor C(\neg \ell^i) \ D(\neg \ell^i)) : \beta^i \rangle$ is denoted $\langle \Pi, CD(\neg \ell), \beta \rangle^i$, then $HUR$ for $k$ sub-formulas is formally expressed below, where, for $i, 1 \leq i \leq k$, $\beta^i = \min\{\alpha, \beta^i\}$.

$$
\frac{\langle \ell : \alpha \rangle \land \langle \Pi, CD(\neg \ell), \beta \rangle^1 \land \ldots \land \langle \Pi, CD(\neg \ell), \beta \rangle^i \land \ldots \land \langle \Pi, CD(\neg \ell), \beta \rangle^k}{\langle \Pi, D(\ell), \beta \rangle^1 \land \ldots \land \langle \Pi, D(\ell), \beta \rangle^i \land \ldots \land \langle \Pi, D(\ell), \beta \rangle^k} \text{ HUR}
$$

Since $\neg \ell^i, 1 \leq i \leq k$, are literal occurrences that are pairwise different, so are the sub-formulas $CD^i$ (and so $D^i$) in the numerator and denominator of $HUR$. However, the formulas $\Pi^i$ are not necessarily different; see Example 6.8 for this concrete question and in general for checking the working of the rule $HUR$.

Remark. An NC hyper unit-resolution rule, more general than $HUR$, can be devised to include simultaneously several unit-clauses so that for each unit-clause $\langle \ell : \alpha \rangle$ several sub-formulas $\langle \Pi^i \succ (\lor C(\neg \ell^i) \ D(\neg \ell^i)) : \beta^i \rangle$ can be considered. In other words, one can consider simultaneously $k \geq 2$ unit clauses and so simultaneously apply $k \geq 2$ $HUR$ rules.

6 Illustrative Examples

This section gives examples illustrating the notions presented in the previous sections. Concretely, we provide the next examples:

- Example 6.1 a simple inconsistent possibilistic Horn-NC base.
- Example 6.2 a simple consistent possibilistic Horn-NC base.
- Example 6.3 a complete propositional Horn-NC formula.
- Example 6.4 an input base with an inconsistent propositional formula.
- Example 6.5 a complete possibilistic Horn-NC base.
- Example 6.6 an example showing the strategy of Find.
- Example 6.7 an example of NC Local Unit-Resolution.
- Example 6.8 an example of NC Hyper Unit-Resolution.

Among such examples, we highlight Examples 6.5 and 6.6 which contain a rather complete Horn-NC base, whose inconsistency degree is obtained in two phases. The first one is provided in Example 6.5 and the second in Example 6.6. All inference rules of $UR_\Sigma$ are needed as well as their combination with the strict $\alpha$-cuts of the input $\Sigma$.

Example 6.1. Let us assume that $\Sigma$ is the next possibilistic Horn-NC base:

$$
\Sigma = \{ \langle P : .8 \rangle, \langle \Pi_1 \succ (\lor \{ \land P \land \neg Q \} : .6), \langle (\lor \neg P \land \neg Q) : .7 \rangle \}
$$
• \( UR_\Sigma \) with \( \langle P : .8 \rangle \) and \( \Pi_1 \) gives rise to the next matchings:
  - \( \Pi = \Pi_1 \)
  - \( \langle \lor C(\neg P) \ D(\neg P) \rangle = \Pi_1 \)
  - \( C(\neg P) = \{ \land \neg P \ \neg Q \} \)
  - \( D(\neg P) = Q \)

• Hence, \( UR_\Sigma \) adds: \( \Sigma \leftarrow \Sigma \cup \langle \lor \neg Q \{ \land \neg P \} \rangle : .6 \)

• Applying simplifications to the last formula: \( \Sigma \leftarrow \Sigma \cup \langle Q : .6 \rangle \)

• \( UR_\Sigma \) with \( \langle Q : .6 \rangle \) and with the last formula in the initial \( \Sigma \) gives:
  - \( \Pi = \Pi_1 \)
  - \( \langle \lor C(\neg Q) \ D(\neg Q) \rangle = \Pi \)
  - \( C(\neg Q) = \neg Q \)
  - \( D(\neg Q) = \neg P \)

• Hence, \( UR_\Sigma \) adds: \( \Sigma \leftarrow \Sigma \cup \langle \lor \neg P : .6 \rangle \)

• Resolving \( \langle P : .8 \rangle \) in the input \( \Sigma \) with the last formula: \( \Sigma \leftarrow \Sigma \cup \langle \bot : .6 \rangle \)

• Therefore \( UR_\Sigma \) obtains \( \text{Inc}(\Sigma) = .6 \)

\[\text{Example 6.2.}\]

Let us assume that \( \Sigma \) is the next possibilistic Horn-NC base:
\( \Sigma = \{ \langle Q : .8 \rangle, \langle \Pi_1 = (\lor \neg Q \ \{ \land R \ (\lor \neg Q \ \{ \land S \ \neg P \}) \} : .6 \rangle, \langle (\lor \neg P \ \neg Q) : .7 \rangle \} \)

• \( UR_\Sigma \) with \( \langle Q : .8 \rangle \) and with the rightest \( \neg Q \) in \( \Pi_1 \) gives the next matchings:
  - \( \Pi = \Pi_1 \)
  - \( \langle \lor C(\neg Q) \ D(\neg Q) \rangle = (\lor \neg Q \ \{ \land \neg P \} \rangle \)
  - \( C(\neg Q) = \neg Q \)
  - \( D(\neg Q) = \{ \land R \ \neg P \} \)

• Hence, \( UR_\Sigma \) adds: \( \Sigma \leftarrow \Sigma \cup \langle (\lor \neg Q \ \{ \land R \ \{ \land S \ \neg P \}) \} : .6 \rangle \)

• After simplifications: \( \Sigma \leftarrow \Sigma \cup \langle (\lor \neg Q \ \{ \land R \ \{ \land S \ \neg P \}) \} : .6 \rangle \)

• Using again \( \langle Q : .8 \rangle \) and the last formula:
  - \( \Pi = (\lor \neg Q \ \{ \land R \ \{ \land S \ \neg P \}) \}
  - \( \langle \lor C(\neg Q) \ D(\neg Q) \rangle = \Pi \)
  - \( C(\neg Q) = \neg Q \)
  - \( D(\neg Q) = \{ \land R \ \{ \land S \ \neg P \}) \}

• Hence, \( UR_\Sigma \) adds: \( \Sigma \leftarrow \Sigma \cup \langle (\lor \{ \land R \ \{ \land S \ \neg P \}) \} : .6 \rangle \)

• After simplifications: \( \Sigma \leftarrow \Sigma \cup \langle (\lor \{ \land R \ \{ \land S \ \neg P \}) \} : .6 \rangle \)

• Applying the rule \( \text{MinD} \): \( \Sigma \leftarrow \Sigma \cup \{ \langle R : .6 \rangle, \langle S : .6 \rangle, \langle \neg P : .6 \rangle \} \).
• Using the first and last formulas in the initial \( \Sigma \): \( \Sigma \leftarrow \Sigma \cup \{ (\lor \neg P) : .7 \} \)

• Applying \( \text{MaxN} \) with \( (\neg P : .6) \) and \( (\lor \neg P) : .7 \) the former is eliminated.

• No more resolvents apply.

• The propositional component of \( \Sigma \) is consistent, so \( \text{Inc}(\Sigma) = 0. \)

Next, we give a rather elaborated propositional formula and show how the propositional NC unit-resolution, or \( UR_P \), together with the simplification rules, detect its inconsistency.

**Example 6.3.** Let us assume that the input \( \Sigma \) has a possibilistic Horn-NC \( \langle \phi : \alpha \rangle \) given below, where \( \phi_1 \) and \( \phi_2 \) are assumed to be Horn-NC formulas.

\[ \langle \phi = \{ \land (\lor R \phi_1) (\lor \neg P \{ \land (\lor \neg P \neg R) (\lor \phi_2 \{ \land \neg Q \neg P \}) R \} ) P \} : \alpha \rangle \]

The tree associated with \( \phi \) is depicted in Fig. 4, left.

![Fig. 4. Formulas \( \phi \) (left) and \( \phi' \) (right)](image)

Thus, before computing the inconsistency degree of \( \Sigma \), one needs to check whether its propositional formulas are inconsistent. We show below how \( UR_P \) checks the inconsistency of \( \phi \). \( UR_P \) with \( P \) and the right-most \( \neg P \) yields the next matchings in the \( UR_P \) numerator:

- \( \Pi = (\lor \neg P \{ \land (\lor \neg P \neg R) (\lor \phi_2 \{ \land \neg Q \neg P \}) R \} ) \)

- \((\lor C(\neg P) D(\neg P)) = (\lor \phi_2 \{ \land \neg Q \neg P \}). \)

- \( C(\neg P) = \{ \land \neg Q \neg P \} \)

- \( D(\neg P) = \phi_2 \)

Applying \( UR_P \) to \( \phi \) yields:

\[ \phi' = \langle \{ \land (\lor R \phi_1) (\lor \neg P \{ \land (\lor \neg P \neg R) (\lor \phi_2 \{ \land \neg Q \neg P \}) R \} ) \} : \alpha \} \]

The resulting tree is the right one in Fig. 4. Assume that we proceed now with a second NC unit-resolution step by picking the same \( P \) and the left-most \( \neg P \) (colored blue in Fig. 4, right). Then, the right conjunct of the numerator of \( UR_P \) is as follows:
• $\Pi = (\lor \neg P (\lor \{ \land (\lor \neg P \neg R) (\lor \phi_2) R \}) )$

• $(\lor C(\neg P) D(\neg P)) = \Pi$

• $C(\neg P) = \neg P$

• $D(\neg P) = (\lor \{ \land (\lor \neg P \neg R) (\lor \phi_2) R \})$

By applying $UR_P$ to $\varphi'$, the obtained formula is depicted in Fig. 5, left:

Fig. 5. Example 6.3 continued.

After three simplification steps, one gets the formula associated with the right tree in Fig. 5. Finally, two applications of $UR_P$ to the two pairs $R$ and $\neg R$, and $P$ and $\neg P$, lead the calculus to derive $\langle (\lor) : \alpha \rangle$.

In the next example, we illustrate the effects of finding that one of the propositional formulas of the input base is inconsistent.

Example 6.4. Let $\varphi$ be the formula from Example 6.3 and $\Sigma_1$ be $\Sigma$ from Example 6.1 and let us analyze the base $\Sigma = \Sigma_1 \cup \{ \langle \varphi : .6 \rangle \}$. Then, firstly the propositional rules of $UR_\Sigma$ are applied to each propositional Horn-NC in $\Sigma$, and in particular, to $\langle \varphi : .6 \rangle$, which, according to Example 6.3 yields $\langle (\lor) : .6 \rangle$. Then before calling $Find$, $\Sigma_1$ is reduced to $\Sigma_1 = \{ \langle P : .8 \rangle, \langle (\lor \neg P \neg Q) : .7 \rangle \}$ and then $Find$ is called with such $\Sigma_1$. Since $\Sigma_1$ is consistent, one can conclude that $Inc(\Sigma) = .6$.

We next give a complete formula and illustrate how $UR_\Sigma$ determines just one inconsistent subset $\Sigma_1$ of a Horn-NC base $\Sigma$ and its degree $Inc(\Sigma_1)$. By now, we are not concerned with finding the maximum inconsistency degree, but just in finding one inconsistent subset. Later, in Example 6.6, we will illustrate the process performed by $Find$ to obtain the inconsistency degree $Inc(\Sigma)$.

Example 6.5. Let us assume that $\Sigma$ is the next possibilistic Horn-NC base:

$\Sigma = \{ \langle P : .8 \rangle, \langle \Pi_1 : .6 \rangle, \langle \Pi_2 : .5 \rangle, \langle \{ \land \neg P \neg Q \} : .7 \rangle \}$

wherein the propositional formulas $\Pi_1$ and $\Pi_2$, both individually consistent, are as follows:

• $\Pi_1 = (\lor \{ \land \neg P \neg Q \} \{ \land Q P \})$

• $\Pi_2 = (\lor \neg Q \{ \land R (\lor \neg Q \{ \land S \neg P \}) \})$
The input base $\Sigma$ is inconsistent and below, we step-by-step provide the inferences carried out by the calculus $UR_\Sigma$ to derive one empty formula $\langle \bot : \alpha \rangle$.

- We apply $UR_\Sigma$ with $\langle P : .8 \rangle$ and $(\Pi_1 : .6)$ and the next matchings:
  - $\Pi = \Pi_1$
  - $(\lor C(\neg P) D(\neg P)) = \Pi_1$
  - $C(\neg P) = \{ \land \neg P \neg Q \}$
  - $D(\neg P) = \{ \land Q P \}$

- Hence, $UR_\Sigma$ adds: $\Sigma \leftarrow \Sigma \cup \langle \lor \{ \land Q P \} \rangle : .6$

- Simplifying the last formula: $\Sigma \leftarrow \Sigma \cup \langle \land Q P \rangle : .6$

- Applying $MinD$ to the last formula: $\Sigma \leftarrow \Sigma \cup \langle Q : .6 \rangle \cup \langle P : .6 \rangle$

- Since $\langle P : .8 \rangle, \langle P : .6 \rangle \in \Sigma$, by $MaxN$: $\Sigma \leftarrow \Sigma / \langle P : .6 \rangle$

- Applying $UR_\Sigma$ with $\langle Q : .6 \rangle$ and the rightest $\neg Q$ of $\langle \Pi_2 : .5 \rangle$:
  - $\Pi = \Pi_2$
  - $(\lor C(\neg Q) D(\neg Q)) = (\lor \neg Q \{ \land S \neg P \})$
  - $C(\neg Q) = \neg Q$
  - $D(\neg Q) = \{ \land S \neg P \}$

- Thus $UR_\Sigma$ adds: $\Sigma \leftarrow \Sigma \cup \langle \lor \{ \land R (\lor \{ \land S \neg P \}) \} \rangle : .5$

- We denote $\langle \Pi_3 : .5 \rangle$ the last added formula.

- Applying $UR_\Sigma$ with again $\langle Q : .6 \rangle$ and the last formula $\langle \Pi_3 : .5 \rangle$:
  - $\Pi = \Pi_3$
  - $(\lor C(\neg Q) D(\neg Q)) = \Pi_3$
  - $C(\neg Q) = \neg Q$
  - $D(\neg Q) = \{ \land R (\lor \{ \land S \neg P \}) \}$

- Hence $UR_\Sigma$ adds: $\Sigma \leftarrow \Sigma \cup \langle \lor \{ \land R (\lor \{ \land S \neg P \}) \} \rangle : .5$

- Simplifying the last formula: $\Sigma \leftarrow \Sigma \cup \langle \{ \land R S \neg P \} \rangle : .5$

- Using the rule $InvMinD$:
  $\Sigma \leftarrow \Sigma \cup \langle \{ R : .5 \}, \langle S : .5 \}, \langle \neg P : .5 \} \rangle$

- From $\langle \neg P : .5 \rangle$ and the initial $\langle P : .8 \rangle$: $\Sigma \leftarrow \Sigma \cup \langle \lor : .5 \rangle$.

- So the (first) inconsistency degree found is $\mathbf{.5}$.

Next example continues the previous one towards, this time, computing the proper $Inc(\Sigma)$. This example illustrates the strategy of $Find$ to do so.
Example 6.6. Let us continue with Example 6.5. Since $(\forall : .5)$ was found, for checking whether $\text{Inc}(\Sigma) > .5$, all possibilistic formulas whose necessity weight is not bigger than .5 are useless, that is, one can obtain the strict .5-cut of $\Sigma$. Thus, the new base is $\Sigma_{> .5} = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1$ and $\Sigma_2$ are the strict .5-cut of the initial formulas and of the deduced formulas, respectively, and which are given below:

$$\Sigma_1 = \{ \langle P : .8 \rangle, \langle \Pi_1 : .6 \rangle, \langle \{ \land \lnot P \land Q \} : .7 \} \}$$

$$\Sigma_2 = \{ \langle \{ \lor \{ \land \Pi P \} \} : .6 \rangle, \langle \{ \land \Pi P \} : .6 \rangle, \langle Q : .6 \} \}$$

One can check that the only non-subsumed formula in $\Sigma_2$ is $\langle Q : .6 \rangle$. So, $\Sigma_2$ is reduced to $\Sigma_2 = \{ \{ \langle Q : .6 \} \}$. Now, Find newly launches the process to compute the inconsistency of $\Sigma = \Sigma_1 \cup \{ \{ Q : .6 \} \}$ with $\text{Inc} = .5$ and follows the next steps:

- Using $\langle Q : .6 \rangle$ and right-most formula in $\Sigma_1$ yields: $\langle (\lor) : .6 \rangle$.
- The new $\Sigma$ is $\Sigma = \{ \langle P : .8 \rangle, \langle \{ \land \lnot P \land Q \} : .7 \} \}$ and the new $\text{Inc}$ is .6.
- $\text{UR}^{\Sigma}$ is relaunched and finds $\langle (\lor) : .7 \rangle$.
- The new $\Sigma$ is $\{ \langle P : .8 \rangle \}$ and the new $\text{Inc}$ is .7.
- $\text{UR}^{\Sigma}$ finds $\Sigma$ is consistent.
- Hence Find returns $\text{Inc} = .7$.

Next example illustrates the application of NC Local Unit-resolution, or LUR.

Example 6.7. Consider again the formula $\varphi$ from Example 6.3:

$$\langle \varphi = \{ \land (\lor R \phi_1) (\lor \lnot P \{ \land (\lor \lnot P \lnot R) (\lor \phi_2 \{ \land \lnot Q \lnot P \}) R \} P : \alpha \} \}.$$

One can check that its sub-formula

$$\phi = (\lor \lnot P \{ \land (\lor \lnot P \lnot R) (\lor \phi_2 \{ \land \lnot Q \lnot P \}) R \}$$

has the pattern of the LUR numerator regarding $\lnot R$ and $R$. Thus LUR can be applied and so $\phi$ be replaced, after simplifications, with $\langle \{ \lor \lnot P \{ \land \lnot Q \lnot P \} R \} \rangle$ in the formula $\varphi$. In this specific example, only one literal is removed, but in a general case, big sub-formulas may be eliminated.

The next example is devoted to the rule of NC hyper unit-resolution, or HUR.

Example 6.8. Let us reconsider also the formula in previous Example 6.3 (recall that the $i$ superscript in $\lnot P_i$ denotes an specific literal occurrence of $\lnot P$):

$$\langle \{ \land (\lor R \phi_1) (\lor \lnot P^1 \{ \land (\lor \lnot P^2 \lnot R) (\lor \phi_2 \{ \land \lnot Q \lnot P^3 \}) R \} P : \alpha \} \}.$$

One can apply NC Hyper Unit-Resolution with $P$ and the three literals $\lnot P_i$. The formula $\Pi$ in the numerator of HUR is the same for the three literals, so it is noted $\Pi^{1,2,3}$, but the formulas $\langle \lor (\lnot P^i \lor (\lnot Q \lnot P_i) \rangle$ are different and are given below:

$$\Pi^{1,2,3} = (\lor \lnot P \{ \land (\lor \lnot P \lnot R) (\lor \phi_2 \{ \land \lnot Q \lnot P \}) R \}$$
– \( \bigvee C(\neg P^1) \bigwedge D(\neg P^1) = \Pi^{1,2,3} \)
– \( \bigvee C(\neg P^2) \bigwedge D(\neg P^2) = (\bigvee \neg P^2 \neg R) \)
– \( \bigvee C(\neg P^3) \bigwedge D(\neg P^3) = (\bigvee \phi_2 \{\neg Q \neg P^3\}) \)

By applying NC Hyper Unit-Resolution, one gets:

\[ \langle \{\bigwedge (\bigvee R \phi_1) \{\bigvee (\bigwedge R \phi_2 R)\} P\} : \alpha \rangle \]

After simplifying:

\[ \langle \{\bigwedge (\bigvee R \phi_1) \neg R \phi_2 R P\} : \alpha \rangle \]

Clearly, a simple NC unit-resolution deduces \( \langle (\bigvee : \alpha) \rangle \). Altogether, in this particular example, the rule HUR accelerates considerably the proof of inconsistency.

7 Correctness of \( \mathcal{UR}_\Sigma \) and Polynomiality of \( \overline{H}_\Sigma \)

This section provides the proofs of both, the correctness of \( \mathcal{UR}_\Sigma \) to determine the inconsistency degree of \( \overline{H}_\Sigma \) and of the tractability of \( \overline{H}_\Sigma \) (i.e. of Horn-NC-INC, Definition 3.6). Some proofs, as those of soundness of the simplification rules, are simple and intuitive and so are omitted. The following formal proofs are provided:

– soundness of quasi-clausal NC unit-resolution (Rule (5));
– soundness of nested NC unit-resolution \( \mathcal{UR}_\Sigma \);
– correctness of the propositional rules of \( \mathcal{UR}_\Sigma \);
– correctness of the complete \( \mathcal{UR}_\Sigma \);
– correctness of the algorithm Find; and
– polynomial complexity of \( \overline{H}_\Sigma \).

**Proposition 7.1.** Rule (5) is sound:

\[ \langle \ell : \alpha \rangle \land \langle (\bigvee C(\neg \ell) \bigwedge D(\neg \ell)) : \beta \rangle \models \langle D(\neg \ell) : \min\{\alpha, \beta\} \rangle. \]

**Proof.** Denoting \( F = \langle \ell : \alpha \rangle \land \langle (\bigvee C(\neg \ell) \bigwedge D(\neg \ell)) : \beta \rangle \), we have:

- By MinD, \( F \models \langle (\bigvee \ell \land C(\neg \ell) \ell \land D(\neg \ell)) : \min\{\alpha, \beta\} \rangle \).
- Since \( \ell \land C(\neg \ell) \models \bot \) then, \( F \models \langle \ell \land D(\neg \ell) : \min\{\alpha, \beta\} \rangle \).
- By MinD, \( F \models \langle D(\neg \ell) : \min\{\alpha, \beta\} \rangle \).

\[ \blacksquare \]

**Proposition 7.2.** The rule \( \mathcal{UR}_\Sigma \) is sound:

\[ \langle \ell : \alpha \rangle \land \langle \Pi \triangleright (\bigvee C(\neg \ell) \bigwedge D(\neg \ell)) : \beta \rangle \models \langle D(\neg \ell) : \min\{\alpha, \beta\} \rangle. \]

**Proof.** By denoting \( F = \langle \ell : \alpha \rangle \land \langle \Pi \triangleright (\bigvee C(\neg \ell) \bigwedge D(\neg \ell)) : \beta \rangle \), we have:

- By MinD, \( F \models \langle \ell \land \Pi \triangleright (\bigvee \ell \land C(\neg \ell) \ell \land D(\neg \ell)) : \min\{\alpha, \beta\} \rangle \).
- Then, \( F \models \langle \ell \land \Pi \triangleright (\bigvee \ell \land C(\neg \ell) \ell \land D(\neg \ell)) : \min\{\alpha, \beta\} \rangle \).
- Since \( \ell \land C(\neg \ell) \models \bot \) then, \( F \models \langle \ell \land \Pi \triangleright \ell \land D(\neg \ell) : \min\{\alpha, \beta\} \rangle \).

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Then, $F = \langle \ell \land \Pi \triangleright D(\neg \ell) : \min \{\alpha, \beta\} \rangle$.

By MinD, $F = \langle \Pi \triangleright D(\neg \ell) : \min \{\alpha, \beta\} \rangle$. ■

The next lemma states the correctness of the propositional rules of $UR_Σ$.

**Lemma 7.3.** Let $UR_P$ be the subset $\{ UR_P, \bot \lor, \bot \land, \odot \phi, \odot \odot \} \subset UR_Σ$. A propositional Horn-NC formula $\varphi$ is inconsistent iff $UR_P$ with input $\langle \varphi : \alpha \rangle$ derives some $\langle \bot : \alpha \rangle$.

**Proof.** We analyze below both directions of the lemma.

- $\Rightarrow$ Let us assume that $\varphi$ is inconsistent. Then $\varphi$ must have a sub-formula verifying the $UR_P$ numerator; otherwise, all complementary pairs of literals $\ell$ and $\neg \ell$ are included in disjunctions. In this case, since all disjunctions of $\varphi$, by definition of Horn-NC formula, have at least one negative literal, $\varphi$ would be satisfied by assigning to all propositions the value 0, which contradicts the initial hypothesis. Therefore, $UR_P$ is applied to $\varphi$ with two complementary literals $\ell$ and $\neg \ell$ and the resulting formula is simplified. The new formula is equivalent to $\varphi$ and has at least one literal less than $\varphi$. Hence, by induction on the number of literals of $\varphi$, we obtain that $UR_P$ ends only when $\langle (\lor) : \alpha \rangle$ is derived.

- $\Leftarrow$ Let us assume that $UR_P$ has been iteratively applied until a formula $\varphi'$ different from $\langle (\lor) : \alpha \rangle$ is obtained. Clearly, if the $UR_P$ numerator is not applicable then there is not a conjunction of a literal $\ell$ with a disjunction including $\neg \ell$. Then we have, firstly, since $UR_P$ is sound, that $\varphi$ and $\varphi'$ are equivalent. Secondly, if $\varphi'$ has complementary literals, then they are integrated in disjunctions. Thus $\varphi'$ is satisfied by assigning the value 0 to all its unassigned propositions, since, by definition of Horn-NC formula, all disjunctions have at least one negative disjunct. Therefore, since $\varphi'$ is consistent so is $\varphi$. ■

The next lemma claims the correctness of $UR_Σ$.

**Lemma 7.4.** Let $\Sigma$ be a possibilistic Horn-NC base. $UR_Σ$ derives some empty clause $\langle (\lor) : \alpha \rangle$ iff $\Sigma$ is inconsistent, and if $UR_Σ$ derives $\langle (\lor) : \alpha \rangle$ then $Inc(\Sigma) \geq \alpha$.

**Proof.** The propositional component of possibilistic NC unit-resolution verifies Lemma 7.3 and hence, $UR_Σ$ derives an empty formula $\langle (\lor) : \alpha \rangle$ iff the conjunction of the propositional formulas in the base $\Sigma$ is inconsistent, namely if $\Sigma^*$ is inconsistent. If $UR_Σ$ derives $\langle (\lor) : \alpha \rangle$, then by Lemma 7.3 $UR_Σ$ detects a subset $\Sigma_1 \subseteq \Sigma$ which is indeed inconsistent. Then by Proposition 7.2 the degree $\alpha$ found by $UR_Σ$ corresponds to $Inc(\Sigma_1)$. Since obviously $Inc(\Sigma_1) \leq Inc(\Sigma)$, the lemma holds. ■

The next lemma states the correctness of the algorithm Find.

**Lemma 7.5.** If $Find(\Sigma, 0)$ returns $\alpha$ then $Inc(\Sigma) = \alpha$.

**Proof.** We denote $\Sigma'$ and $Inc$ the variables of $Find$ in a given recursion. Let us prove that the next hypothesis holds in every call to $Find$:

$$\Sigma > Inc = \Sigma' \text{ and } Inc(\Sigma) = \max\{Inc(\Sigma'), Inc\}$$

- We check that the initial call $Find(\Sigma, 0)$ verifies the hypothesis:
- We have $\Sigma' = \Sigma$ and $\text{Inc} = 0$.
- So indeed we have $\Sigma > 0 = \Sigma'$ and $\text{Inc}(\Sigma) = \max\{\text{Inc}(\Sigma'), 0\} = \text{Inc}(\Sigma)$
- We prove that if the hypothesis holds for $k \geq 1$ then it holds for $k + 1$.
- First of all, $\text{UR}_\Sigma$ is applied to $\Sigma'$.
- By Lemma 7.4 if $\text{UR}_\Sigma$ derives $\langle \bot : \alpha \rangle$ then $\text{Inc}(\Sigma') \geq \alpha$, else $\Sigma'$ is consistent.
- Case $\Sigma'$ is consistent: $\text{Inc}(\Sigma') = 0$.
- By induction hypothesis $\text{Inc}(\Sigma) = \max\{\text{Inc}(\Sigma'), \text{Inc}\} = \text{Inc}$
- So $\text{Find}$ correctly returns $\text{Inc}(\Sigma)$ and ends.
- Case $\Sigma'$ is inconsistent: $\text{Inc}(\Sigma') = \alpha > 0$.
- By induction hypothesis $\Sigma' = \Sigma > \text{Inc}$, and so $\alpha > \text{Inc}$.
- The next $\Sigma'$, noted $\Sigma''$, and $\text{Inc}$, noted $\text{Inc}'$, are $\Sigma'' = \Sigma > \alpha$ and $\text{Inc}' = \alpha$.
- We check in (i) and (ii) that the hypothesis holds:
- (i) By induction hypothesis: $\Sigma' = \Sigma > \text{Inc}$
  - Since $\alpha > \text{Inc}$ then trivially $\Sigma' > \alpha = \Sigma > \text{Inc}$
  - Hence $\Sigma'' = \Sigma' > \alpha = \Sigma > \alpha = \Sigma > \text{Inc}'$.
- (ii) By Lemma 7.4, $\text{Inc}(\Sigma') \geq \alpha$
  - Since $\Sigma' = \Sigma > \text{Inc}$ and $\alpha > \text{Inc}$ then $\text{Inc}(\Sigma) \geq \alpha$
  - Hence $\text{Inc}(\Sigma) = \max\{\text{Inc}(\Sigma > \alpha), \alpha\} = \max\{\text{Inc}(\Sigma''), \text{Inc}'\}$
- Altogether, the hypothesis holds until $\text{Find}$ finds $\Sigma'$ consistent and then correctly returns $\text{Inc}(\Sigma)$. Hence Lemma 7.5 holds.

The next three propositions prove that finding $\text{Inc}(\Sigma)$ of any Horn-NC $\Sigma$ is polynomial.

**Proposition 7.6.** If $\Sigma$ is a possibilistic Horn-NC base, then $\text{UR}_\Sigma$ with input $\Sigma$ performs at most $n^2$ inferences, $n$ being the number of symbols (size) of $\Sigma$.

**Proof.** On the one hand, each rule of $\text{UR}_\Sigma$ adds a formula $\langle \varphi : \alpha \rangle$, where $\varphi$ is a sub-formula of a propositional formula $\Pi$ of a possibilistic formula $\langle \Pi : \beta \rangle$ in the current base. Hence, the current base always contains only sub-formulas from $\Sigma$. On the other hand, the weight $\alpha$ of added formulas $\langle \varphi : \alpha \rangle$ is the minimum of two weights in the current base. Hence, the weights of formulas in the current base always come from $\Sigma$. Thus, the maximum number of deduced formulas is $m \times k$, where $m$ is the number of sub-formulas in $\Sigma$ and $k$ is the number of different weights in $\Sigma$. Hence, the maximum number of inferences performed by $\text{UR}_\Sigma$ is $m \times k$. If $n$ is the size of $\Sigma$, then $m, k \leq n$ and so the proposition holds. ■

**Proposition 7.7.** If $\Sigma$ is a possibilistic Horn-NC base, then $\text{Find}(\Sigma, 0)$ performs at most $n^2$ recursive calls, $n$ being the number of symbols of $\Sigma$.

**Proof.** $\text{Find}$ stops when it detects that the current base is consistent. If it is inconsistent then $\langle (\lor : \alpha) \rangle$ is deduced, and for future calls, $\text{Find}$ cancels the set $\{\langle \phi : \beta \rangle \mid \langle \phi : \beta \rangle \in \Sigma, \beta \leq \alpha\}$. This set trivially contains at least one formula as $\langle (\lor : \alpha) \rangle$ has been derived. On the other hand and as discussed in the previous proof, the maximum number of inserted formulas in the base is at most $m \times k$, where $m$ is the number of sub-formulas in $\Sigma$ and $k$
is the number of different weights in $\Sigma$. Thus, the number of performed recursive calls is at most $m \times k$. Since $m, k \leq n$, then Proposition 7.7 holds.

The next proposition states that the overall complexity to determine $\text{Inc}(\Sigma)$, $\Sigma \in \overline{\mathcal{H}}_\Sigma$, is polynomial.

**Proposition 7.8.** If $\Sigma$ is a possibilistic Horn-NC base, then computing $\text{Inc}(\Sigma)$ takes polynomial time, i.e. the class $\overline{\mathcal{H}}_\Sigma$ is polynomial.

**Proof.** On the one hand, by Propositions 7.6 and 7.7, the number of performed inferences in each recursive call and the number of recursive calls are both polynomial. Hence the total number of inferences is polynomially bounded. On the other hand, it is not hard to find a data structure so that each inference in $\mathcal{UR}_\Sigma$ can be polynomially performed w.r.t the size of $\Sigma$. Therefore, the complexity to determine $\text{Inc}(\Sigma)$ is polynomial.

**Remark.** Determining a tight polynomial degree of the worst-case complexity of computing a Horn-NC base, is planned for future work (see Section 8). It should be mentioned that the polynomial degree of computing their counterpart Horn clausal bases has not been specified either.

## 8 Related and Future Work

In this section we briefly discuss related work in a number of possibilistic logical contexts and succinctly propose objectives towards which our future work can be oriented.

□ Discovering polynomial NC classes.

In propositional logic, the valuable contribution to clausal efficiency of the conjunction of Horn formulas and Horn-SAT algorithms is reflected by the fact that the highly efficient DPLL solvers embed a Horn-SAT-like algorithm, so-called Unit Propagation. Hence, searching for polynomial (clausal) super-classes of the Horn class in propositional logic has been a key issue for several decades towards improving clausal reasoning and has led to a great number of such classes being currently known: hidden-Horn, generalized Horn, Q-Horn, extended-Horn, etc. (see [32, 40] for short reviews). So it is arguable that, just as the tractable clausal fragment has helped to grow overall clausal efficiency, likewise widening the tractable non-clausal fragment would grow overall non-clausal efficiency. Indeed, inspired by such polynomial non-clausal classes, efficient non-clausal algorithms can be devised. Nevertheless, we stress that the current tractable non-clausal fragment is almost empty, which signifies a manifest disadvantage with respect to its clausal counterpart. We will extrapolate the previous argumentation to possibilistic logic and determine further NC subclasses whose inconsistency-degree computing have polynomial complexity. In our next work, we will search for polynomial classes obtained by lifting the possibilistic renameable (or hidden) Horn formulas to the NC level.

□ Designing low-degree polynomial algorithms.

As we have seen in Section 7 the number of inferences required by $\mathcal{UR}_\Sigma$ and the number of recursive calls to $\textbf{Find}$ are both bounded by $O(n^2)$, where $n$ is the symbol number of the input base. Also, all inferences in $\mathcal{UR}_\Sigma$ can be reasonably performed in $O(n)$. Altogether, the complexity of determining $\text{Inc}(\Sigma)$ of Horn-NC bases is in $O(n^5)$. 30
This complexity, though polynomial, is of course no satisfactory for real-world applications where the size of formulas can be relatively huge. However no much care has been taken in the proofs of Section 7 because the goal was proving tractability, and so, first of all, a fine-grained analysis of complexity is pending. This analysis will permit likely to precise a tighter polynomial-degree complexity. Nevertheless, the work presented here should be resumed in order to notably decrease the polynomial-degree. Also, advances in several directions are obliged such as finding suitable data-structure and proposing optimized algorithms. Finally, we mention that this optimization effort has not been done so far in the clausal framework either. In fact, no explicit upper bounds of complexity algorithms that compute possibilistic clausal bases are available.

- □ Combining necessity and possibility measures.

Our approach considers only necessity-valued formulas but it can be extended to bases where formulas can have associated a possibility or a necessity measure. Such logical context has already been studied by previous authors [37, 41, 20]. If a formula is possibility-valued \(\langle \varphi : \Pi(\varphi) \geq \alpha \rangle\) it can be converted into the follow: the hypothesis holding necessity-valued formula \(\langle \neg \varphi : N(\neg \varphi) \leq 1 - \alpha \rangle\). Hence, bases having both possibility and necessity formulas can be converted into equivalent necessity formulas, where the threshold \(\alpha\) is additionally accompanied by the indication of whether the formula must have a necessity level either greater or smaller than \(\alpha\). On the other side, resolution for possibility-necessity formulas is well-known [22, 25, 26], but its extension to the NC level towards defining a possibility/necessity NC unit-resolution calculus seems not to be trivial and is an open question.

- □ Computing general NC bases:

Since \(\mathcal{H}_\Sigma\) is an NC sub-class, computing arbitrary NC bases is a natural continuation of the presented approach. As said in the Introduction, real-world problems are generally expressed in NC form, and so clausal reasoning requires the usage of a previous NC-to-clausal transformation. However, such transformation is highly inadvisable as it increases the formula size and number of variables, and losses the logical equivalence and the formula’s original structure. Besides the clausal form is not unique and how to guide the nondeterministic process towards a “good” clausal formula is not known. So, real-world efficiency is reached if formulas are computed in its original non-clausal form. Here we have favored non-clausal reasoning with contributions which particularly facilitate deduction based on resolution and DPLL. Brief discussions for future work related to how compute possibilistic general NC formulas by means of resolution and DPLL can be found below in the items “□ Defining NC Resolution” and “□ Defining NC DPLL”.

- □ Developing NC logic programming.

Possibilistic logic programming has received notable attention, and in fact, after the pioneer work in [21], a number of approaches e.g. [10] [3] [4] [2] are available in the state-of-the-art. However, all of them focus exclusively on the clausal form and so are Horn-like programs where the body of rules is a conjunction of propositions and the head is a single proposition. Our future work in this research direction will be oriented to show that \(\mathcal{P}_\Sigma\) and \(\mathcal{UR}_\Sigma\) allow to handle non-clausal logic programs, and more concretely:

1. To conceive a language for Possibilistic Non-Clausal Logic Programming denoted \(\mathcal{LP}_\Sigma\). So, instead of Horn-like rules, in \(\mathcal{LP}_\Sigma\), one may handle Horn-NC-like rules wherein bodies and heads of rules are NC formulas with slight syntactical restrictions (issued from \(\mathcal{P}_\Sigma\)).
In fact, one can check that a rule with syntax \( \langle \Pi \rightarrow \text{Hnc} : \alpha \rangle \), where \( \Pi \) is an NC formula with only positive literals and \( \text{Hnc} \) is a Horn-NC formula, is indeed a possibilistic Horn-NC formula. Therefore a set of such kind of rules, i.e. a program, is a Horn-NC base.

2) To answer queries with an efficiency qualitatively comparable to the clausal one, i.e. polynomial, which is possible thanks to the next two features: (i) an \( \mathcal{LP}_\Sigma \) program belongs to \( \overline{\mathcal{H}_\Sigma} \) and \( \mathcal{H}_\Sigma \) is a polynomial class; and (ii) the bases in \( \mathcal{H}_\Sigma \) have only one minimal model, since, as aforementioned, they are equivalent to a Horn clausal formula.

\( \square \) Developing NC answer set programming.

After the works in possibilistic logic programming, a succession of works on possibilistic answer set programming has been carried out, started by \[48\] and continued with e.g. \[51, 52, 18, 19, 6, 7\]. Although, like in possibilistic logic programming, they also focus on the clausal form, there exists an exception as indicated in the Introduction. Indeed, NC possibilistic logic has been formerly dealt with by the authors in \[49, 50\]. However, in this work no effectiveness issues are addressed. Instead, the authors extend, to possibilistic logic, proper concepts of non-clausal answer set programming within classical logic as originally defined in \[44\]; so their aim is distinct from ours. Our future work in this research direction will be oriented to show the scope of the expressiveness of the class \( \mathcal{H}_\Sigma \) for possibilistic NC answer set programming and to analyze the efficiency allowed by \( \mathcal{H}_\Sigma \). Although answering queries in possibilistic answer set programming is an intractable problem \[48\], we do not rule out the possibility of finding tractable sub-classes.

\( \square \) Partially ordered possibilistic logic.

In this work we have assumed that possibility and necessity measures, which rank interpretations and formulas, were in the real unit-interval \([0, 1]\), i.e. the available information is supposed to be totally ordered. Thus for two different necessity values \( \alpha, \beta \), we have always that either \( \alpha > \beta \) or \( \alpha < \beta \). However, considering a more real-world context with only partial orders has already been studied \[11, 12, 16\]. For instance, partial orders avoid comparing unrelated pieces of information, which happens when we merge multiple sources information and the merged pieces do not have a shared reference for uncertainty. So, considering a partial pre-order on interpretations by means of a partial pre-order on formulas is a more real-world scenario to which the presented method is planned to be generalized.

\( \square \) Casting richer logics into possibilistic logic.

In our logical framework, propositional logic is casted into possibilistic logic. Casting richer logics into possibilistic logic has already been carried out by several authors, for instance, Alsinet et al. \[3\] cast Gödel logic many-valued logic in a possibilistic framework and several researchers \[37, 20, 57\] do similarly with description logics. See reference \[25\] to know other embeddings. Since the Horn-NC class has already been defined in \[39\] for regular many-valued logic, our next aim in this research line will be to embed regular-many-valued into possibilistic logic. Specifically, we will attempt to define the class of regular-many-valued possibilistic Horn-NC bases as well as to generalize other notions presented in this article such as defining regular-many-valued possibilistic NC Unit-resolution. An open question is whether polynomiality is preserved when two uncertainty logics are combined within the non-clausal level.

\( \square \) Defining NC resolution.

The formalization of the existing NC resolution \[46\] (see also \[5\]) that dates back to the
1980s is somewhat confusing. Its functional-like definition has important weaknesses such as not precisely identifying the available resolvents or requiring complex formal proofs of its logical properties. Another symptom of its barriers is that, contrary to our approach, it has not led so far, to define either NC unit-resolution or NC hyper-resolution. Our definition of NC unit-resolution is clausal-like because, as stated in Section 5 in presence of clausal formulas, $UR_Σ$ coincides with clausal resolution [23, 24]. We believe that our kind of definition is fairly well oriented to define (full) NC Resolution and to generalize it to some uncertainty logics, and similarly, to be analyzed and to formally prove its logical properties.

□ Defining NC DPLL.

DPLL for possibilistic clausal formulas have already been studied [22, 41] but DPLL for possibilistic non-clausal formulas has received no attention. We argue below that the present article supposes an important step forward to specify the scheme NC DPLL for possibilistic logic. The principle of DPLL relies on: (1) the procedure Unit-Propagation; (2) a suitable heuristic to choice the literal $\ell$ on which performs branching; and (3) the determination of the formulas $\Sigma \land \ell$ and $\Sigma \land \neg \ell$, namely the formulas on which DPLL should split the search. One can do the next observations: (1) NC Unit-Propagation is based on NC unit-resolution, and (2) formulas $\Sigma \land \ell$ and $\Sigma \land \neg \ell$ are obtained with NC unit-resolution. Hence, the unique aspect not studied here is that of the (2) heuristic to choice the branching literal. Thus, for future work, we will propose the DPLL schema for possibilistic non-clausal reasoning after studying heuristics and taking a closer look to other aspects presented in this article.

□ Generalized Possibilistic Logic

A restriction in standard possibilistic logic is that only conjunctions of weighted formulas are allowed. Dubois et al. [28, 29, 27] have conceived the Generalized Possibilistic Logic (GPL) in which the disjunction and the negation of standard possibilistic formulas are handled. In this logic, for instance the formula $(\lor \langle \varphi : \alpha \rangle \langle \psi : \beta \rangle)$ belongs to the GPL’s language, and since possibility-valued formulas are also smoothly embedded in GPL, formulas like the next one belong to GPL:

\[
\{ \land \langle (\lor P Q) : N \rangle \langle \neg Q : N .75 \rangle \neg \langle (\lor \neg P R) : N \rangle \langle \lor (\neg R : \Pi .75) \langle \varphi : \Pi .25 \rangle \rangle \}
\]

One can see that in GPL, connectives can be internal or external (GPL is a two-tired logic) with their corresponding and different semantics, and also, GPL formulas can be expressed in NC form. In fact, observe that the previous formula is non-clausal. In [29] the authors prove that the satisfiability problem associated to GPL formulas is $NP$-complete. It is clear that the standard possibilistic Horn formulas are encapsulated in GPL and so are the Horn-NC formulas defined here. We think that in GPL, sub-classes of external Horn formulas can also be defined, as well as sub-classes that are both internally and externally Horn. Thus GPL turns out to be an interesting logic in the sense that it embeds a variety of classes of Horn-like formulas which potentially could be lifted to the NC level. So for future work, we will study the different classes of Horn-NC GPL formulas that are definable and then attempt to prove their complexity. It will be challenging to determine when polynomiality is preserved in the NC level. Of course, once we have mastered the solution of GPL Horn-NC-like formulas, the solving of unrestricted non-clausal GPL formulas can be envisaged by proposing inference and solving mechanisms.
Finding Models or Inconsistency Subsets.

Our method focuses on exclusively determining $\text{Inc}(\Sigma)$. However, one important issue for increasing theoretical and practical interest is the obtaining of models or contradictory subsets of the knowledge. In some frameworks as for instance, when the knowledge base is not definitive and is in an experimentation phase, the only data of $\text{Inc}(\Sigma)$ may be of not much help. For example, if one expects the knowledge base to be consistent and the consistency checker finds it is inconsistent, knowing the knowledge subset causing contradiction, called "witness" in the literature, can be necessary. Thus for future work, we will envisage deductive calculi oriented to providing witnesses as a return data. In this context, a more complicated problem is the determination of whether a knowledge base has exactly one model or one inconsistent subset, since determining if a problem has a unique solution is computationally more expensive than testing if has at least one solution (see [56], Chapter 17).

9 Conclusions

As the encoding of practical problems is usually expressed in non-clausal form, restricting deductive systems to handle clausal formulas obliges them to use non-clausal-to-clausal transformations, which are very expensive in terms of: increase of formula size and number of variables, and loss of logical equivalence and original formula’s structure. Further, the clausal form is not unique and however no insights are available to guide towards a "good" clausal form the non-clausal-to-clausal transformation. These drawbacks deprive clausal reasoning systems to efficiently perform in real-world applications.

To overcome such limitations and avoid the costs induced by the normal form transformation, we process formulas in non-clausal form, concretely in negation normal form (NNF). This form allows an arbitrary nesting of conjunctions and disjunctions and only limits the scope of the negation connective. NNF can be obtained deterministically and solely causes a negligible, easily assumable increase of the formula size.

Thus, along the lines of previous works in propositional and regular many-valued logics [40, 39], we have extrapolated the previous argumentation to possibilistic logic, the most extended approach to deal with knowledge impregnated of uncertainty and presenting partial inconsistencies. Thus our first contribution has been lifting the possibilistic Horn class to the non-clausal level obtaining a new possibilistic class, which has been called Horn Non-Clausal, denoted $\Pi_{\Sigma}$ and shown that it is a sort of non-clausal analogous of the standard Horn class. Indeed, we have proven that $\Pi_{\Sigma}$ subsumes syntactically the Horn class and that both classes are semantically equivalent. We have also proven that all possibilistic NC bases whose clausal form is Horn belong to $\Pi_{\Sigma}$.

In order to compute the inconsistency degree of $\Pi_{\Sigma}$ members, we have established the calculus Possibilistic Non-Clausal Unit-Resolution, denoted $\mathcal{UR}_{\Sigma}$. We formally proved that $\mathcal{UR}_{\Sigma}$ correctly computes the inconsistency degree of any $\Pi_{\Sigma}$ base. $\Pi_{\Sigma}$ was nonexistent in the literature and extends the propositional logic calculus given in [40] to possibilistic logic.

After having specified $\Pi_{\Sigma}$ and $\mathcal{UR}_{\Sigma}$, we have studied the computational problem of computing the inconsistency degree of $\Pi_{\Sigma}$ via $\mathcal{UR}_{\Sigma}$ and determined that it is polynomial, and hence, $\Pi_{\Sigma}$ is the first found class to be possibilistic, non-clausal and polynomial.

Our formulation of $\mathcal{UR}_{\Sigma}$ is unambiguously clausal-like since, when applied to clausal
formulas, $UR_\Sigma$ indeed coincides with clausal unit-resolution. This aspect is relevant in the sense that it lays the foundations towards redefining NC resolution in a clausal-like manner which could avoid the barriers caused by the existing functional-like definition (see related work). We believe that this clausal-like definition of NC resolution will allow to generalize it to some other uncertainty logics.

Finally, in this work we also attempted to show that effective NC reasoning for possibilistic and for some other uncertainty logics is an open research field and, in view of our outcomes, we consider the presented research line is rather promising. A symptom of such consideration is the possibility of our method to be extended to different possibilistic logic contexts giving rise to a number of future research directions that were briefly discussed:

Computing possibilistic arbitrary NC bases; discovering additional tractable NC subclasses; conceiving low-degree-polynomial algorithms; extending generalist possibilistic logic to NC; combining necessity and possibility measures; considering partially ordered possibility measures; developing possibilistic NC logic programming; developing possibilistic NC answer set programming; casting richer logics in possibilistic logic; defining possibilistic NC resolution; defining possibilistic NC DPLL; and finding models.

Some of the above listed future objectives can also be searched in the context of other non-classical logics such as Lukasiewicz logics, Gödel logic, product logic, etc.

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10 Proofs of Section 4

Lemma 4.8 A NC disjunction \( \varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \) with \( k \geq 1 \) disjuncts pertains to \( \mathcal{H} \) iff \( \varphi \) has \( k - 1 \) negative disjuncts and one Horn-NC disjunct, formally

\[
\varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in \mathcal{H} \quad \text{iff} \quad \exists i \text{ s.t. } \varphi_i \in \mathcal{H} \quad \text{and} \quad \forall j \neq i, \varphi_j \in \mathcal{N}^-.\]
\begin{proof} \textbf{If:} As the formulas \( \forall j, j \neq i, \varphi_j \) have no positive literals, the non-negative disjunctions of \( \varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \) are those of \( \varphi_i \) plus \( \varphi \) and \( \varphi \) themselves. Given that by hypothesis \( \varphi_i \in \overline{\mathcal{H}} \) and that \( \forall j, j \neq i, \varphi_j \) has no positive literals then all of them pertain to \( \overline{\mathcal{H}} \). \textbf{Iff:} It can be easily proved by contradiction that if any of the two conditions of the lemma are unsatisfied, i.e. (i) \( \exists i, \varphi_i \notin \overline{\mathcal{H}} \) or (ii) \( \exists j, j \neq i, \varphi_i, \varphi_j \notin \mathcal{N}^- \), then \( \varphi \notin \overline{\mathcal{H}} \). \end{proof}

\textbf{Theorem 4.12.} We have that \( \mathcal{H} = \overline{\mathcal{H}} \).

\hspace{1em} \textbf{Proof.} We prove first \( \mathcal{H} \subseteq \overline{\mathcal{H}} \) and then \( \mathcal{H} \supseteq \overline{\mathcal{H}} \).

- \( \mathcal{H} \subseteq \overline{\mathcal{H}} \) is easily proven by structural induction as outlined below:

  1. \( \mathcal{L} \subseteq \overline{\mathcal{H}} \) trivially holds.

  2. The non-recursive \( \mathcal{H} \) conjunctions are literal conjunctions, which trivially verify Definition 4.3 and so are in \( \overline{\mathcal{H}} \). Assume that \( \mathcal{H} \subseteq \overline{\mathcal{H}} \) holds until a given inductive step and that \( \varphi_i \in \overline{\mathcal{H}} \), \( \varphi_i \in \overline{\mathcal{H}} \), \( 1 \leq i \leq k \). In the next recursion, any \( \varphi = \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \) may be added to \( \mathcal{H} \). On the other hand, by induction hypothesis, we have \( \varphi_i \in \overline{\mathcal{H}} \), \( 1 \leq i \leq k \), and so by Lemma 4.6, \( \varphi \in \overline{\mathcal{H}} \). Therefore \( \mathcal{H} \subseteq \overline{\mathcal{H}} \) holds.

- \( \mathcal{H} \supseteq \overline{\mathcal{H}} \). Given that the structures to define \( \mathcal{N} \mathcal{C} \) and \( \mathcal{H} \) in Definition 2.3 and Definition 4.11 respectively, are equal, the potential inclusion of each NC formula \( \varphi \) in \( \mathcal{H} \) is systematically considered. Further, the statement: if \( \varphi \in \overline{\mathcal{H}} \) then \( \varphi \in \mathcal{H} \), is proven by structural induction on the depth of formulas, by applying a reasoning similar to that of the previous \( \mathcal{H} \subseteq \overline{\mathcal{H}} \) case and by also using Lemmas 4.6 and 4.8. \end{proof}

\textbf{Proof of Theorems 4.16 and 4.18.} Before proving both theorems, the preliminary Theorem 10.2 is required.

\textbf{Definition 10.1.} For every \( \varphi \in \mathcal{N} \mathcal{C} \), we define \( cl(\varphi) \) as the unique clausal formula that results from applying \( \lor/\land \) distributivity to \( \varphi \) until a clausal formula, viz. \( cl(\varphi) \), is obtained.

\textbf{Theorem 10.2.} Let \( \varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in \overline{\mathcal{H}} \). \( cl(\varphi) \in \mathcal{H} \) iff \( \varphi \) has \( k - 1 \) negative disjuncts and one disjunct s.t. \( cl(\varphi_i) \in \mathcal{H} \), formally:

\[ cl((\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k)) \in \mathcal{H} \iff (1) \exists i, \text{ s.t. } cl(\varphi_i) \in \mathcal{H} \text{ and } (2) \forall j, j \neq i, \varphi_j \in \mathcal{N}^- \).

\hspace{1em} \textbf{Proof.} \textbf{If-then.} By refutation: let \( cl((\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k)) \in \mathcal{H} \) and prove that if (1) or (2) are violated, then \( cl(\varphi) \notin \mathcal{H} \).

- (1) \( \exists i, \text{ s.t. } cl(\varphi_i) \in \mathcal{H} \)
  - If we take the case \( k = 1 \), then \( \varphi = \varphi_1 \).
  - But \( cl(\varphi_1) \notin \mathcal{H} \) implies \( cl(\varphi) \notin \mathcal{H} \).

- (2) \( \exists j, j \neq i, \varphi_j \notin \mathcal{N}^- \)
  - Suppose that, besides \( \varphi_i, \varphi_j \notin \mathcal{N}^- \) with \( j \neq i \).
  - We take \( k = 2, \varphi_1 = P \) and \( \varphi_2 = Q \).
  - So, \( \varphi = (\lor \varphi_1 \varphi_2) = (\lor P Q) \), and hence \( cl(\varphi) \notin \mathcal{H} \).

\end{enumerate}
Only-If. Without loss of generality, we take \((\bigvee \varphi_1 \ldots \varphi_i \ldots \varphi_{k-1}) = \varphi^- \in \mathcal{N}^-\) and \(\varphi_k \in \overline{\mathcal{H}}\), and prove:

\[
\text{cl}(\varphi) = \text{cl}((\bigvee \varphi_1 \ldots \varphi_i \ldots \varphi_{k-1} \varphi_k)) = \text{cl}((\bigvee \varphi^- \varphi_k)) \in \mathcal{H}.
\]

- To obtain \(\text{cl}(\varphi)\), one must obtain first \(\text{cl}(\varphi^-)\) and \(\text{cl}(\varphi_k)\), and so
  \[
  (i) \quad \text{cl}(\varphi) = \text{cl}((\bigvee \varphi^- \varphi_k)) = \text{cl}((\bigvee \text{cl}(\varphi^-) \text{cl}(\varphi_k))).
  \]

- By definition of \(\varphi^- \in \mathcal{N}^-\),
  \[
  (ii) \quad \text{cl}(\varphi^-) = \{\wedge D_1^- \ldots D_{m-1}^- D_m^-\}; \text{ the } D_i^-'s \text{ are negative clauses.}
  \]

- Since \(\varphi_k \in \overline{\mathcal{H}}\),
  \[
  (iii) \quad \text{cl}(\varphi_k) = H = \{\wedge h_1 \ldots h_{n-1} h_n\}; \text{ the } h_i's \text{ are Horn clauses.}
  \]

- By (i), (ii) and (iii),
  \[
  \text{cl}(\varphi) = \text{cl}((\bigvee \{\wedge D_1^- \ldots D_{m-1}^- D_m^-\} \{\wedge h_1 \ldots h_{n-1} h_n\})).
  \]

- Applying \(\lor/\land\) distributivity to \(\text{cl}(\varphi)\) and noting \(C_i = (\bigvee D_i^- h_i)\),
  \[
  \text{cl}(\varphi) = \text{cl}((\bigwedge \{\bigwedge C_1 \ldots C_i \ldots C_n\} (\bigvee \{\bigwedge D_2^- \ldots D_{m-1}^- D_m^-\} H)) ).
  \]

- Since the \(C_i = (\bigvee D_i^- h_i)\)'s are Horn clauses,
  \[
  \{\wedge C_1 \ldots C_i \ldots C_n\} = H_1 \in \mathcal{H}.
  \]

- For \(j < m\) we have,
  \[
  \text{cl}(\varphi) = \text{cl}(\{\bigwedge H_1 \ldots H_{j-1}H_j (\bigvee \{\bigwedge D_{j+1}^- \ldots D_{m-1}^- D_m^-\} H)\} ).
  \]

- For \(j = m\), \(\text{cl}(\varphi) = \{\bigwedge H_1 \ldots H_{m-1} H_m H\} = H' \in \mathcal{H}.
  \]

- Hence \(\text{cl}(\varphi) \in \mathcal{H}\). \(\blacksquare\)

**Theorem 4.16.** \(\forall \varphi \in \overline{\mathcal{H}} \text{ we have cl}(\varphi) \in \mathcal{H}\).

**Proof.** We consider Definition 4.11 of \(\overline{\mathcal{H}}\). The proof is done by structural induction on the depth \(r(\varphi)\) of any \(\varphi \in \overline{\mathcal{H}}\) and defined below, where \(\ell\) is a literal:

\[
r(\varphi) = \left\{ \begin{array}{ll}
0 & \varphi = [\bigcirc \ell_1 \ldots \ell_{k-1} \ell_k] \text{ or } \varphi = \ell.
1 + \max\{r(\varphi_1), \ldots, r(\varphi_{k-1}), r(\varphi_k)\} & \varphi = [\bigcirc \varphi_1 \ldots \varphi_{k-1} \varphi_k].
\end{array} \right.
\]

- **Base Case:** \(r(\varphi) = 0\).
  - Clearly, \(r(\varphi) = 0\) entails \(\varphi = [\bigcirc \ell_1 \ldots \ell_{k-1} \ell_k] \in \mathcal{H}\) and \(\varphi = \ell \in \mathcal{H}\).
  - So \(\text{cl}(\varphi) = \varphi \in \mathcal{H}\).

- **Induction hypothesis:** \(\forall \varphi, r(\varphi) \leq n, \ \varphi \in \overline{\mathcal{H}} \text{ entails } \text{cl}(\varphi) \in \mathcal{H}\).

- **Induction proof:** \(r(\varphi) = n + 1\).
  By Definition 4.11 lines (2) ad (3) below arise:
(2) $\varphi = \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \}$, where $k \geq 1$.

- By definition of $r(\varphi)$,
  
  $r(\varphi) = n + 1$ entails $1 \leq i \leq k$, $r(\varphi_i) \leq n$.

- By induction hypothesis,
  
  $\varphi_i \in \overline{\mathcal{H}}$ and $r(\varphi_i) \leq n$ entail $cl(\varphi_i) \in \mathcal{H}$.

- It is obvious that,
  
  $cl(\varphi) = \{ \land \ cl(\varphi_1) \ldots cl(\varphi_i) \ldots cl(\varphi_k) \}$.

- Therefore,
  
  $cl(\varphi) = \{ \land H_1 \ldots H_i \ldots H_k \} = H \in \mathcal{H}$.

(3) $\varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_{k-1} \varphi_k) \in \overline{\mathcal{H}}$, where:

- $k \geq 1$, $0 \leq i \leq k - 1$, $\varphi_i \in \mathcal{N}^-$ and $\varphi_k \in \overline{\mathcal{H}}$.

- By definition of $r(\varphi)$,
  
  $r(\varphi) = n + 1$ entails $r(\varphi_k) \leq n$.

- By induction hypothesis,
  
  $d(\varphi_k) \leq n$ and $\varphi_k \in \overline{\mathcal{H}}$ entail $cl(\varphi_k) \in \mathcal{H}$.

- By Theorem 10.2 only-if,
  
  $0 \leq i \leq k - 1$, $\varphi_i \in \mathcal{N}^-$ and $cl(\varphi_k) \in \mathcal{H}$ entail:
  
  $cl( (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_{k-1} \varphi_k) ) \in \mathcal{H}$.

$\blacksquare$

**Theorem 4.18.** $\forall \varphi \in \mathcal{NC}$: if $cl(\varphi) \in \mathcal{H}$ then $\varphi \in \overline{\mathcal{H}}$.

**Proof.** It is done by structural induction on the depth $d(\varphi)$ of $\varphi$ defined as

$$d(\varphi) = \begin{cases} 0 & \varphi \in \mathcal{C}. \\ 1 + \max \{ d(\varphi_1), \ldots, d(\varphi_i), \ldots, d(\varphi_k) \} & \varphi = [\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k]. \end{cases}$$

- **Base case:** $d(\varphi) = 0$ and $cl(\varphi) \in \mathcal{H}$.
  
  - $d(\varphi) = 0$ entails $\varphi \in \mathcal{C}$.
  
  - If $\varphi \notin \mathcal{H}$, then $cl(\varphi) \notin \mathcal{H}$, contradicting the assumption.
  
  - Hence $\varphi \in \mathcal{H}$ and so by Definition 4.11 $\varphi \in \overline{\mathcal{H}}$.

- **Inductive hypothesis:** $\forall \varphi \in \mathcal{NC}$, $d(\varphi) \leq n$, $cl(\varphi) \in \mathcal{H}$ entail $\varphi \in \overline{\mathcal{H}}$.

- **Induction proof:** $d(\varphi) = n + 1$.

  By Definition 2.3 of $\mathcal{NC}$, cases (i) and (ii) below arise.

  (i) $cl(\varphi) = cl( \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \} ) \in \mathcal{H}$ and $k \geq 1$.  

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Since $\varphi$ is a conjunction, $1 \leq i \leq k$, $cl(\varphi_i) \in \mathcal{H}$.

- By definition of $d(\varphi)$,
  \[ d(\varphi) = n + 1 \quad \text{entails} \quad 1 \leq i \leq k, \quad d(\varphi_i) \leq n. \]

- By induction hypothesis,
  \[ 1 \leq i \leq k, \quad d(\varphi_i) \leq n, \quad cl(\varphi_i) \in \mathcal{H} \quad \text{entail} \quad \varphi_i \in \overline{\mathcal{H}}. \]

- By Definition 4.11 line (2),
  \[ 1 \leq i \leq k, \quad \varphi_i \in \overline{\mathcal{H}} \quad \text{entails} \quad \varphi \in \overline{\mathcal{H}}. \]

(ii) $cl(\varphi) = cl((\lor \varphi_1 \ldots \varphi_{i-1} \varphi_{i} \ldots \varphi_{k-1} \varphi_k)) \in \mathcal{H}$ and $k \geq 1$.

- By Theorem 10.2 if-then,
  \[ 0 \leq i \leq k - 1, \quad \varphi_i \in \mathcal{N}^\ominus \quad \text{and} \quad cl(\varphi_k) \in \mathcal{H}. \]

- By definition of $d(\varphi)$,
  \[ d(\varphi) = n + 1 \quad \text{entails} \quad d(\varphi_k) \leq n. \]

- By induction hypothesis,
  \[ d(\varphi_k) \leq n \quad \text{and} \quad cl(\varphi_k) \in \mathcal{H} \quad \text{entail} \quad \varphi_k \in \overline{\mathcal{H}}. \]

- By Definition 4.11 line (3),
  \[ 0 \leq i \leq k - 1, \quad \varphi_i \in \mathcal{N}^\ominus \quad \text{and} \quad \varphi_k \in \overline{\mathcal{H}} \quad \text{entail}: \]
  \[ (\lor \varphi_1 \ldots \varphi_{i-1} \varphi_{i} \ldots \varphi_{k-1} \varphi_k) = \varphi \in \overline{\mathcal{H}}. \]

\[\blacksquare\]