Finite burden in multivalued algebraically closed fields

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Abstract

We prove that an expansion of an algebraically closed field by $n$ arbitrary valuation rings is $\text{NTP}_2$, and in fact has finite burden. It fails to be $\text{NIP}$, however, unless the valuation rings form a chain. Moreover, the incomplete theory of algebraically closed fields with $n$ valuation rings is decidable.

1 Introduction

Fix an integer $n$. Consider the theory $ACv^n F$ of 1-sorted structures $(K, +, \cdot, O_1, \ldots, O_n)$, where $(K, +, \cdot) \models ACF$ and each $O_i$ is (a unary predicate for) a valuation ring on $K$.

Our main results are as follows:

1. The (incomplete) theory $ACv^n F$ is decidable: there is an algorithm which inputs $\varphi$ and outputs whether $ACv^n F \vdash \varphi$.

2. If $M \models T$, then $M$ has finite burden, hence is strong, $\text{NTP}_2$.

3. If $M \models T$, then $M$ is $\text{NIP}$ if and only if the valuation rings are pairwise comparable.

Chapter 11 of [2] considered the more restrictive class of structures in which the $O_i$ are non-trivial and independent. The resulting theory turns out to be the model companion of the theory of fields with $n$ valuation rings. In this paper, we generalize the results of [2] by eliminating the assumptions of independence and non-triviality.

Rather than working directly with models of $ACv^n F$, it is more convenient to work with certain definitional expansions which are better behaved—for example, they are model complete. We briefly summarize the situation.

By a finite tree, we shall mean a finite poset $(P, \leq)$ containing a minimal element $\bot$, such that every interval $[\bot, p]$ is a chain. A branch of $P$ is a subposet of the form $\{x \in P : x \geq a\}$ where $a$ is a minimal element of $P \setminus \{\bot\}$. The tree $P$ can be written as a disjoint union

$$P = \{\bot\} \cup P_1 \cup \cdots \cup P_n$$

where $P_1, \ldots, P_n$ are the distinct branches of $P$. Each branch $P_i$ is itself a finite tree.
To any finite tree $P$, we shall associate a theory $T_P$. A model of $T_P$ is an algebraically closed field $(K,+,\cdot,\mathcal{O}_p : p \in P)$ with a valuation ring $\mathcal{O}_p$ for each $p \in P$, satisfying some axioms. The important properties are

1. If $P_1, \ldots, P_n$ are the branches of $P$, then a model of $T_P$ is essentially an algebraically closed field $K$ with $n$ independent non-trivial valuations $\mathcal{O}_1, \ldots, \mathcal{O}_n$, and a $T_{P_i}$-structure on the residue field of $\mathcal{O}_i$.

2. Every algebraically closed multivalued field $(K,+,\cdot,\mathcal{O}_1, \ldots, \mathcal{O}_n)$ admits a definitional expansion to a model of $T_P$, essentially by adding unary predicates for the joins $\mathcal{O}_i \cdot \mathcal{O}_j$.

The first point allows us to mimic the arguments used in the case of independent valuations. The second point relates the theories $T_P$ and $ACv^n F$.

The paper is outlined as follows. In §2 we consider the general setting of multi-valued fields with residue structure, and derive a relative model completeness result in the case of independent non-trivial valuations. Essentially, we prove the following: if $T_1, \ldots, T_n$ are model-complete theories expanding $ACF$, then model completeness holds in the theory of algebraically closed fields with $n$ independent non-trivial valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_n$ with $T_i$-structure on the residue field of $\mathcal{O}_i$. See Lemma 2.7.

In §3 we introduce the aforementioned theory $T_P$ and apply the results of §2 to prove that $T_P$ is model complete. Moreover, we show that $T_P$ is the model companion of a simpler theory $T^0_P$. (Models of $T^0_P$ are exactly the subfields of models of $T_P$.)

In §5 we prove that $T^0_P$ has the amalgamation property over algebraically closed bases. From this, we deduce several consequences, such as the usual criterion for elementary equivalence: two models $M_1, M_2$ of $T_P$ are elementarily equivalent iff the substructures $\text{Abs}(M_1)$ and $\text{Abs}(M_2)$ are isomorphic. This in turn yields decidability of $T_P$. The proof of amalgamation in $T^0_P$ relies on an amalgamation lemma in $ACVF$, which we prove in §4. The lemma says that when amalgamating valued fields, we have complete freedom in how we amalgamate the residue fields.

The rest of the paper is devoted to the classification-theoretic dividing lines $NTP_2$ and $NIP$. In §7 we define a canonical Keisler measure on the set of complete types extending any quantifier-free type. More precisely, given any model $K \models T^0_P$, we define a Keisler measure on the space of completions of $T_P \cup \text{diag}(K)$. (This is a variant of the Keisler measure defined in §11.4 of [2].) Some of the key properties of the Keisler measure rely on an analysis in §6 of extensions of nested valuation rings in certain diagrams of fields. The analysis is notationally confusing, but not deep.

In §9 we verify that models of $T_P$ have finite burden, using a minor lemma proven in §8. In §10 we turn to the matter of $NIP$, reviewing the argument from [2] §11.5.1 that algebraically closed fields with independent valuations cannot be $NIP$. We conclude in §11 by discussing different directions in which the results can probably be generalized.
1.1 Notation

We will generally use the letter $\mathcal{O}$ for valuation rings, $\mathfrak{m}$ for their maximal ideals, and lowercase roman letters (such as $k$, $\ell$) for residue fields.

If $K$ is a valued field with valuation ring $\mathcal{O}$, we let $\text{res}\mathcal{O}$ denote the residue field. We also write $\text{res}K$ for the residue field, if $\mathcal{O}$ is clear from context. We will also use $\text{res}(x)$ to denote the residue of $x$.

When multiple valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are in play, we will use subscripts to indicate which residue map we are talking about: $\text{res}_i(x)$ denotes the residue of $x$ in the $i$th residue field $k_i = \text{res} \mathcal{O}_i$.

If $\mathcal{O}_1 \subseteq \mathcal{O}$ are two valuation rings on a field $K$, we let $\mathcal{O}_1 \div \mathcal{O}$ denote the unique valuation ring on $\text{res} \mathcal{O}$ whose composition with $\mathcal{O}$ is $\mathcal{O}_1$.

Two non-trivial valuation rings $\mathcal{O}_1, \mathcal{O}_2$ are independent if they induce distinct topologies. An equivalent condition is that $\mathcal{O}_1 \cdot \mathcal{O}_2 = K$. Here, $\mathcal{O}_1 \cdot \mathcal{O}_2$ denotes the join—the smallest valuation ring on $K$ containing both $\mathcal{O}_1$ and $\mathcal{O}_2$. It happens to agree with the setwise product

$$\mathcal{O}_1 \cdot \mathcal{O}_2 = \{x \cdot y : x \in \mathcal{O}_1 \text{ and } y \in \mathcal{O}_2\}.$$

We let $\text{Val}(K)$ denote the poset of all valuation rings on $K$. If $\mathcal{O} \in \text{Val}(K)$, we let $\text{Val}(K|\mathcal{O})$ denote the subset

$$\text{Val}(K|\mathcal{O}) := \{\mathcal{O} \in \text{Val}(K) : \mathcal{O} \subseteq \mathcal{O}\}.$$

The poset $\text{Val}(K|\mathcal{O})$ is canonically isomorphic to $\text{Val}(\text{res} \mathcal{O})$ via the map

$$\text{Val}(K|\mathcal{O}) \to \text{Val}(\text{res} \mathcal{O})$$

$$\mathcal{O}' \mapsto \mathcal{O}' \div \mathcal{O}.$$

2 Multi-valued fields with residue structure

If $X$ is a set and $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are topologies on $X$, we say that $\mathcal{T}_1, \ldots, \mathcal{T}_n$ are jointly independent if the diagonal embedding

$$X \hookrightarrow (X, \mathcal{T}_1) \times \cdots \times (X, \mathcal{T}_n)$$

has dense image. In other words, if $U_i$ is a non-empty $\mathcal{T}_i$-open for each $i$, then $\bigcap_{i=1}^n U_i$ is non-empty.

**Fact 2.1.** Let $K$ be an algebraically closed field. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be pairwise independent non-trivial valuation rings on $K$. Let $V \subseteq \mathbb{A}^n_K$ be an irreducible affine variety. Then the metric topologies on $V$ are jointly independent.

This should be a classical result, but I had trouble finding the original reference; a later proof is in [2] Theorem 11.3.1.

**Lemma 2.2.** Let $(K, \mathcal{O}) \leq (K', \mathcal{O}')$ be an extension of models of ACVF. Let $k \leq k'$ be the corresponding residue field extension. Then any $K$-definable subset of $(k')^n$ is quantifier-free $k$-definable in the pure ring structure on $k'$.
Proof. This follows from the 3-sorted quantifier elimination in ACVF. We shall also see another proof later (Remark 4.9).

Corollary 2.3. Let \((K, O) \leq (K', O')\) be an extension of models of ACVF, and \(k \leq k'\) be the residue field extension. Let \(V \subseteq \mathbb{A}^{n+m}\) be a quasi-affine variety over \(K\). There is a quantifier-free formula \(R(\vec{\xi})\) in the language of rings over \(k\) such that

\[
R(K) = \{\text{res}(\vec{x}) : (\vec{x}, \vec{y}) \in V(K)\}
\]

\[
R(K') = \{\text{res}(\vec{x}) : (\vec{x}, \vec{y}) \in V(K')\}
\]

where \(\vec{x}, \vec{\xi}\) are \(n\)-tuples, \(\vec{y}\) is an \(m\)-tuple, and \(\text{res}(\vec{x})\) is understood componentwise.

Proof. This follows from Lemma 2.2; the same formula \(R\) works for both \((K, O)\) and \((K', O')\) by model completeness of ACVF.

In this section, we shall always consider \(n\)-fold multivalued fields in the \((n + 1)\)-sorted language \((K, k_1, \ldots, k_n)\) with

- Field structure on each \(K\) and \(k_i\).
- Unary predicates \(O_1, \ldots, O_n\) for the valuation rings.
- Partial maps \(\text{res}_i : K \twoheadrightarrow k_i\).

Remark 2.4. Let \((K, k_1, \ldots, k_n)\) be an \(n\)-fold multivalued field, and let \((K', k'_1, \ldots, k'_n)\) be an extension. Suppose \(K\) is e.c. in \(K'\).

1. If \(K'\) is an algebraically closed field, then \(K\) is an algebraically closed field.
2. If \(O'_i\) is non-trivial, then \(O_i\) is non-trivial.
3. If \(O'_i\) and \(O'_j\) are independent, then \(O_i\) and \(O_j\) are independent.

Proof. 1. If \(K'\) is algebraically closed and \(K\) is not, take a monic polynomial \(P(X) \in K[X]\) without a solution in \(K\). Then

\[
\exists x : P(x) = 0
\]

holds in \(K'\), but not in \(K\), contrary to existential closedness.

2. Suppose \(O'_i\) is non-trivial but \(O_i\) is trivial. Then

\[
\exists x : x \notin O_i
\]

holds in \(K'\) but not in \(K\).
3. Suppose, say, \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) fail to be independent. Then \( \mathcal{O}_0 = \mathcal{O}_1 \cdot \mathcal{O}_2 \) is a non-trivial valuation ring. Let \( m_i \) denote the maximal ideal of \( \mathcal{O}_i \). Then

\[
\begin{align*}
\mathfrak{m}_0 &\subset \mathfrak{m}_1 \subset \mathcal{O}_1 \subset \mathcal{O}_0, \\
\mathfrak{m}_0 &\subset \mathfrak{m}_2 \subset \mathcal{O}_2 \subset \mathcal{O}_0.
\end{align*}
\]

Because \( \mathcal{O}_0 \) is non-trivial, there is some non-zero \( \varepsilon \in \mathfrak{m}_0 \). Then

\[
\varepsilon \mathcal{O}_1 \subseteq \varepsilon \mathcal{O}_0 \subseteq \mathfrak{m}_0
\]

\[
\varepsilon \mathcal{O}_2 \subseteq \varepsilon \mathcal{O}_0 \subseteq \mathfrak{m}_0.
\]

So the existential statement

\[
\exists x, y : x \in \mathcal{O}_1 \land y \in \mathcal{O}_2 \land 1 + \varepsilon x = \varepsilon y
\]

is false in \( K \), as \( \mathfrak{m}_0 \) and \( 1 + \mathfrak{m}_0 \) are disjoint. On the other hand, this existential statement is true in \( K' \) by approximation (Fact 2.1) on the line \( \{(x, y) : 1 + \varepsilon x = \varepsilon y\} \).

\[\square\]

**Lemma 2.5.** Let \((K, \mathcal{O}_1, \ldots, \mathcal{O}_n)\) be a field with \( n \) valuations. Then we can embed \((K, \mathcal{O}_1, \ldots, \mathcal{O}_n)\) into a larger \( n \)-valued field \((K', \mathcal{O}'_1, \ldots, \mathcal{O}'_n)\) such that

1. \( K' \) is algebraically closed.
2. Each \( \mathcal{O}'_i \) is non-trivial.
3. The \( \mathcal{O}'_i \) are pairwise independent.

*Proof.* The class of \( n \)-fold multivalued fields is an \( \forall \exists \)-elementary class, so we may assume \((K, \mathcal{O}_1, \ldots, \mathcal{O}_n)\) is e.c. among fields with \( n \)-valuations. We claim that \( K \) has the desired properties of \( K' \). By Remark 2.4 it suffices to produce extensions of \( K \) having each of the properties separately.

1. By the Chevalley Extension Theorem (\( \square \) Theorem 3.1.1), we can extend each \( \mathcal{O}_i \) to a valuation ring \( \mathcal{O}'_i \) on \( K^{alg} \). Then \((K^{alg}, \mathcal{O}'_1, \ldots, \mathcal{O}'_n)\) is an extension in which \( K' \) is algebraically closed.

2. Suppose, say, \( \mathcal{O}_1 \) is trivial. Let \( K((T)) \) be the Laurent field extension, and let \( \mathcal{O}'_1 \) be the discrete valuation ring \( K[[T]] \). Then \( \mathcal{O}'_1 \) extends the trivial valuation \( \mathcal{O}_1 \) on \( K \). For \( i \neq 1 \), let \( \mathcal{O}'_i \) be an arbitrary extension of \( \mathcal{O}_i \) to \( K((T)) \). Then \((K((T)), \mathcal{O}'_1, \ldots, \mathcal{O}'_n)\) is an extension in which \( \mathcal{O}'_i \) is non-trivial.

3. Suppose, say, \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) fail to be independent. Let \( \mathcal{O}'_1 \) be the valuation on \( K(T) \) obtained by composing the \( T \)-adic valuation with \( \mathcal{O}_1 \). Let \( \mathcal{O}'_2 \) be the valuation on \( K(T) \) obtained by composing the \((T + 1)\)-adic valuation with \( \mathcal{O}_2 \). Then \( \mathcal{O}'_i \) extends \( \mathcal{O}_i \) for \( i = 1, 2 \), and \( \mathcal{O}'_1 \) is independent from \( \mathcal{O}'_2 \), because the \( T \)-adic and \((T + 1)\)-adic valuations on \( K(T) \) are independent. For \( i \neq 1, 2 \) choose \( \mathcal{O}'_i \) to be an arbitrary extension of \( \mathcal{O}_i \).
Remark 2.6. If $T$ is a model complete theory and $M \preceq N \models T$, then $M$ is not e.c. in $N$ unless $M \models T$.

Proof. Suppose $M$ is e.c. in $N$. We claim that $M \preceq N$ by the Tarski-Vaught test. Let $X \subseteq N$ be a non-empty $M$-definable set in the structure $N$. By model completeness, $X = \pi(Y)$ where $\pi : N^n \to N$ is a coordinate projection and $Y \subseteq N^n$ is quantifier-free definable over $M$. Non-emptiness of $X$ implies non-emptiness of $Y$. As $M$ is e.c. in $N$, the set $Y \cap M^n$ is non-empty. Therefore $\pi(Y \cap M^n) \subseteq X \cap M$ is non-empty.

Lemma 2.7. Let $T_1, \ldots, T_n$ be model-complete expansions of ACF. Let $T$ be the theory of $(n+1)$-sorted structures $(K, k_1, \ldots, k_n)$ with field structure on $K$, residue maps $\text{res}_i : K \rightsquigarrow k_i$ for each $i$, and with $(T_i)_\forall$ structure on each $k_i$. Then $(K, k_1, \ldots, k_n)$ is existentially closed if and only if

1. $K = K^{alg}$,
2. each valuation ring $\mathcal{O}_i$ is non-trivial
3. the valuation rings $\mathcal{O}_i$ are pairwise independent, and
4. each $k_i$ is a model of $T_i$.

In particular, $T$ has a model companion.

Proof. We first show the necessity of the listed conditions. Suppose $(K, k_1, \ldots, k_n)$ is existentially closed. By Lemma 2.5 we can embed $(K, k_1, \ldots, k_n)$ into a larger multivalued field $(K', k'_1, \ldots, k'_n)$, without residue structure, such that $K'$ is algebraically closed, each $\mathcal{O}_i'$ is non-trivial, and the $\mathcal{O}_i'$ are pairwise independent.

We can take $(K', \ldots)$ to be highly saturated. Then $k'_i$ is an algebraically closed field of high transcendence degree over $k$. As $k_i \models (T_i)_\forall$ we can find a $T_i$-structure on $k'_i$ extending the $(T_i)_\forall$-structure on $k_i$. This endows $(K', k'_1, \ldots, k'_n)$ with a $T$-structure, such that $k'_i \models T_i$.

Now $(K, k_1, \ldots, k_n)$ is existentially closed in the extension $(K', k'_1, \ldots, k'_n)$. Then $k_i$ is e.c. in $k'_i$, so $k_i \models T_i$ by Remark 2.6. Similarly, by Remark 2.3 the $\mathcal{O}_i$ are non-trivial and independent, and $K = K^{alg}$. Thus $(K, k_1, \ldots, k_n)$ must satisfy the listed conditions if it is existentially closed.

Next, suppose that $(K, k_1, \ldots, k_n)$ satisfies all the listed conditions. Let $(K', k'_1, \ldots, k'_n)$ be an extension; we must show that $K$ is e.c. in $K'$. Enlarging $K'$, we may assume that $K'$ is e.c., hence satisfies the listed conditions. Suppose we are given some existential formula over $K$ which is true in $K'$; we must show it is true in $K$. By adding dummy variables, we
reduce to an existential formula of the form

$$\exists \vec{x}_1, \ldots, \vec{x}_n, \vec{y}, \vec{\xi}_1, \ldots, \vec{\xi}_n : R_0(\vec{x}_1, \ldots, \vec{x}_n, \vec{y})$$

$$\land \bigwedge_{i=1}^{n} \left( \text{res}_i(\vec{x}_i) = \vec{\xi}_n \right)$$

$$\land \bigwedge_{i=1}^{n} R_i(\vec{\xi}_i),$$

where

- $\vec{x}_1, \ldots, \vec{x}_n, \vec{y}$ are tuples from the big field sort
- $\vec{\xi}_i$ is a tuple from the $i$th residue field sort,
- $\text{res}_i(-)$ acts on vectors componentwise
- $R_0$ is a quantifier-free formula over $K$ in the pure field language
- $R_i$ is a quantifier-free formula over $k_i$ in the language of $T_i$.

The relation $R_0$ can be written as a disjunction of conjunctions; we may restrict to one of the conjunctions, reducing to the case where the existential formula has the form

$$\exists \vec{x}_1, \ldots, \vec{x}_n, \vec{y}, \vec{\xi}_1, \ldots, \vec{\xi}_n : \bigwedge_{j=1}^{m} (P_j(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) = 0)$$

$$\land (Q(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \neq 0)$$

$$\land \bigwedge_{i=1}^{n} \left( \text{res}_i(\vec{x}_i) = \vec{\xi}_n \right)$$

$$\land \bigwedge_{i=1}^{n} R_i(\vec{\xi}_i),$$

where the $P_i$ and $Q$ are polynomials over $K$.

Fix some tuple $(\vec{a}_1', \ldots, \vec{a}_n', \vec{b}', \vec{\alpha}_1', \ldots, \vec{\alpha}_n')$ in the big model $K'$ witnessing the existential statement. Adding more $P_i$, we may assume that the $P_i$ cut out an irreducible affine variety $V$ over $K$, namely the locus of $(\vec{a}_1', \ldots, \vec{a}_n', \vec{b}')$ over $K$. Let $V \setminus W$ be the Zariski open subset of $V$ cut out by $Q \neq 0$.

By Corollary 2.3 we can find quantifier-free $L_{\text{rings}}$ formulas $R'_i(\vec{\xi}_i)$ such that in both $K$ and $K'$,

$$R'_i(\vec{\xi}_i) \iff \exists (\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \in V \setminus W : \text{res}(\vec{x}_i) = \vec{\xi}_i.$$

In particular, $R'_i(\vec{a}_i')$ holds. Replacing $R_i(\vec{\xi}_i)$ with $R_i(\vec{\xi}_i) \land R'_i(\vec{\xi}_i)$, we may assume that

$$R_i(\vec{\xi}_i) \implies \exists (\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \in V \setminus W : \text{res}(\vec{x}_i) = \vec{\xi}_i. \quad (1)$$
Each residue field \( k_i \) is a model of \( T_i \), hence existentially closed in \( k'_i \). Therefore we can find \( \bar{\alpha}_1, \ldots, \bar{\alpha}_n \) in \( k_1, \ldots, k_n \) such that \( R_i(\bar{\alpha}_i) \) holds. Let \( X_i \) be

\[
X_i := \{(\vec{x}_1, \ldots, \vec{x}_n, \vec{y}) \in V(K) : \text{res}_i(\vec{x}_i) = \bar{\alpha}_i\}.
\]

Each \( X_i \setminus W \) is non-empty by (1) and choice of \( \bar{\alpha}_i \). Moreover, each \( X_i \setminus W \) is an \( \mathcal{O}_1 \)-adically open subset of \( V(K) \). By independence, the intersection \( \bigcap_i (X_i \setminus W) \) is non-empty. Let \( (\vec{a}_1, \ldots, \vec{a}_n, \vec{b}) \) be a point in the intersection. Then

1. \( (\vec{a}_1, \ldots, \vec{a}_n, \vec{b}) \) lies on \( V(K) \setminus W \).
2. \( \text{res}_i(\vec{a}_i) = \bar{\alpha}_i \) because \( (\vec{a}_1, \ldots, \vec{a}_n, \vec{b}) \in X_i \).
3. \( R_i(\bar{\alpha}_i) \) holds for each \( i \), by choice of \( \bar{\alpha}_i \).

Therefore, the existential statement holds in \( K \), witnessed by \( (\vec{a}_1, \ldots, \vec{a}_n, \vec{b}, \bar{\alpha}_1, \ldots, \bar{\alpha}_n) \). So \( K \) is existentially closed. \( \square \)

3 The theories \( T_P \) and \( T^0_P \)

**Definition 3.1.** A tree is a \( \land \)-semilattice \( P \) with bottom element \( \bot \) such that for every \( x \in P \), the interval \( [\bot, x] \) is totally ordered. A homomorphism of trees is a \( \land \)-semilattice homomorphism mapping \( \bot \) to \( \bot \).

The set \( \text{Val}(K) \) of valuation rings on a field \( K \) is naturally a tree, after reversing the order. The join operation is

\[
\mathcal{O}_1 \lor \mathcal{O}_2 = \mathcal{O}_1 \cdot \mathcal{O}_2.
\]

Fix a finite tree \( P \).

**Definition 3.2.** The theory \( T_P \) is the theory of algebraically closed fields \( K \) with injective tree homomorphisms

\[
P \leftrightarrow \text{Val}(K)^{op}
\]

\[
p \mapsto \mathcal{O}_p.
\]

In other words, a model of \( T_P \) is a structure \( (K, \mathcal{O}_p : p \in P) \) where

- \( K \) is an algebraically closed field.
- \( \mathcal{O}_p \) is a valuation ring for each \( p \in P \).
- If \( p < p' \), then \( \mathcal{O}_p \supseteq \mathcal{O}_{p'} \).
- \( \mathcal{O}_\bot = K \).
• For any \( p, p' \in P \)
\[
\mathcal{O}_{p \wedge p'} = \mathcal{O}_p \cdot \mathcal{O}_{p'}.
\]

**Example 3.3.** Let \( P \) be the flat tree \( \{1, \ldots, n, \bot\} \) in which
\[
a \wedge b = \begin{cases} 
\bot & \text{if } a \neq b \\
 a & \text{if } a = b.
\end{cases}
\]
Then a model of \( T_P \) is essentially a structure \((K, \mathcal{O}_1, \ldots, \mathcal{O}_n)\) where \( K \) is an algebraically closed field and the \( \mathcal{O}_i \) are pairwise independent non-trivial valuation rings.

**Example 3.4.** Let \( K \) be an algebraically closed field and let \( \mathcal{O}_1, \ldots, \mathcal{O}_n \) be finitely many arbitrary valuation rings on \( K \). Let \( P \) be the (finite) sub-\( \wedge \)-semilattice of \( \text{Val}(K)^{op} \) generated by \( \{\mathcal{O}_1, \ldots, \mathcal{O}_n, K\} \). Then \( P \) is a tree, and \((K, \mathcal{O}_1, \ldots, \mathcal{O}_n)\) is bi-interpretable with the model \((K, \mathcal{O}_p : p \in P) \models T_P\), where \( p \mapsto \mathcal{O}_p \) is the tautological map.

**Definition 3.5.** Let \( P \) be a finite tree. Let \( T^0_P \) be the theory whose models are structures \((K, \mathcal{O}_p : p \in P)\) where \( K \) is a field and
\[
p \mapsto \mathcal{O}_p
\]
is a weakly order-preserving map \( P \to \text{Val}(K)^{op} \) sending \( \bot \) to \( K \).
In other words, a model of \( T^0_P \) is a structure \((K, \mathcal{O}_p : p \in P)\) where

- \( K \) is a field (not necessarily algebraically closed).
- \( \mathcal{O}_p \) is a valuation ring for each \( p \in P \).
- If \( p < p' \), then \( \mathcal{O}_p \supseteq \mathcal{O}_{p'} \) (but the inclusion needn’t be strict).
- \( \mathcal{O}_\bot = K \).

Note that for any \( p, p' \in P \),
\[
\mathcal{O}_{p \wedge p'} \supseteq \mathcal{O}_p \cdot \mathcal{O}_{p'}
\]
but equality needn’t hold.

**Example 3.6.** If \( P \) is the tree of Example 3.3, then a model of \( T^0_P \) is a field \( K \) with \( n \) valuation rings on it.

**Theorem 3.7.** \( T_P \) is the model companion of \( T^0_P \).

*Proof.* Let \( a_1, \ldots, a_n \) enumerate the minimal elements of \( P \setminus \{\bot\} \). Let \( P_i \) be the subposet \( \{x \in P : x \geq a_i\} \). Note that \( P_i \) is a finite tree with bottom element \( a_i \). By induction, \( T^0_{P_i} \) is the model companion of \( T_{P_i} \). Let \( T \) be the theory of \((n+1)\)-sorted structures \((K, k_1, \ldots, k_n)\) with field structure on \( K \), residue maps \( \text{res}_i : K \hookrightarrow k_i \) for each \( i \), and with \( T^0_{P_i} \)-structure on each \( k_i \).

Given a model \((K, k_1, \ldots, k_n)\) of \( T \), we get a model \((K, \mathcal{O}_p^K : p \in P)\) of \( T^0_P \) by defining
• $\mathcal{O}_\perp$ to be $K$.

• $\mathcal{O}_p$ to be the composition of $K \twoheadrightarrow k_i$ with $\mathcal{O}_p^{k_i}$, if $p \geq a_i$.

This gives an equivalence of categories from the category of models of $T$ (with morphisms the embeddings) to the category of of models of $T^0_P$ (with morphisms the embeddings). Moreover, this equivalence of categories sends elementary embeddings to elementary embeddings in both directions.

By Lemma 2.7, $T$ has a model companion $T'$ whose models are characterized by the following additional axioms:

1. $K$ is algebraically closed.

2. Each valuation ring $\mathcal{O}_i$ is non-trivial.

3. The valuation rings $\mathcal{O}_i$ are pairwise independent.

4. Each residue field $k_i$ is a model of $T_P$.

Under the equivalence of categories, models of $T'$ correspond to models of $T_P$. Therefore, $T_P$ is the model companion of $T^0_P$. \hfill \Box

### 4 Controlled amalgamation in ACVF

**Definition 4.1.** A ring homomorphism $f: R \to K$ to a field $K$ is dominant if $K$ is generated as a field by $\text{Im}(f)$.

For a fixed ring $R$, dominant morphisms out of $R$ are classified up to equivalence by prime ideals of $R$.

By the category of fields we mean the full subcategory of the category of rings. Note that homomorphisms are embeddings.

**Definition 4.2.** Let

\[
\begin{array}{ccc}
F & \rightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \rightarrow & L
\end{array}
\]

be a diagram in the category of fields.

• An amalgamation of $K_1$ and $K_2$ over $F$ is a diagram

\[
\begin{array}{ccc}
F & \rightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \rightarrow & L
\end{array}
\]

extending the given diagram. When the maps $K_i \to L$ are clear, one says that $L$ is an amalgamation of $K_1$ and $K_2$ over $F$. 10
• Two amalgamations $L$ and $L'$ are equivalent if there is an isomorphism $L \to L'$ such that

\[
\begin{array}{c}
K_1 \\
\downarrow \ \\
L \\
\downarrow \ \\
K_2
\end{array}
\longrightarrow
\begin{array}{c}
L' \\
\downarrow \ \\
K_2
\end{array}
\]

commutes.

• An amalgamation $L$ is reduced if $L$ is the compositum $K'_1 K'_2$, where $K'_i$ is the image of $K_i \to L$. Equivalently, $L$ is reduced if the morphism

$$K_1 \otimes_F K_2 \to L$$

is dominant.

• The reduction of an amalgamation $L$ is the subfield $K'_1 K'_2$ of $L$, where $K'_i$ is the image of $K_i \to L$.

• An amalgamation type is an equivalence class of reduced amalgamations, or equivalently, a prime ideal in $K_1 \otimes_F K_2$.

• The amalgamation type of an amalgamation $L$ is the equivalence class of the reduction, or equivalently, $\ker K_1 \otimes_F K_2 \to L$.

• If $K_1 \otimes_F K_2$ is a domain, the independent amalgamation type is the amalgamation type corresponding to the zero ideal $(0) \leq K_1 \otimes_F K_2$, and an independent amalgamation is one of independent type.

**Lemma 4.3.** Let

\[
\begin{array}{c}
K_0 \\
\downarrow \ \\
K_2
\end{array}
\longrightarrow
\begin{array}{c}
K_1
\end{array}
\]

be a diagram of embeddings of valued fields $(K_i, \mathcal{O}_i)$. Then the natural ring homomorphism

$$\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \to K_1 \otimes_{K_0} K_2$$

is injective.

**Proof.** Because $\mathcal{O}_1$ is torsionless as an $\mathcal{O}_0$-module, it is flat. Therefore, the natural map

$$\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \to \mathcal{O}_1 \otimes_{\mathcal{O}_0} K_2$$

is an injection. Similarly, $K_2$ is a flat $\mathcal{O}_0$-module, so

$$\mathcal{O}_1 \otimes_{\mathcal{O}_0} K_2 \to K_1 \otimes_{\mathcal{O}_0} K_2$$

is also injective.
is injective. Finally, the map

\[ K_1 \otimes_{O_0} K_2 \to K_1 \otimes_{K_0} K_2 \]

is an isomorphism because \( O_0 \to K_0 \) is a (category-theoretic) epimorphism and tensor products are pushouts in the category of rings.

**Remark 4.4.** If \( f : A \hookrightarrow B \) is an injective homomorphism of rings, then every minimal prime of \( A \) extends to a prime of \( B \). Indeed, if \( \mathfrak{p} \) is a minimal prime of \( A \), let \( S = A \setminus \mathfrak{p} \). Injectivity of \( f \) implies that \( f(S) \) is a multiplicative subset of \( B \) not containing zero, so the localization \( f(S)^{-1}B \) is non-trivial. Any prime ideal of \( f(S)^{-1}B \) pulls back to a prime in \( A \) contained in \( \mathfrak{p} \), hence equal to \( \mathfrak{p} \) by minimality.

**Remark 4.5.** The category of valued fields and embeddings is equivalent to the category of valuation rings and injective local homomorphisms (i.e., injective ring homomorphisms \( f : O_1 \to O_2 \) such that \( f^{-1}(m_2) = m_1 \)).

**Lemma 4.6.** Let

\[
\begin{array}{ccc}
K_0 & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \longrightarrow & L
\end{array}
\]

be a diagram of embeddings of valued fields. Let \( O_i \) and \( k_i \) be the valuation ring and residue field of \( K_i \). Given any amalgamation type \( \tau \) of \( k_1 \) and \( k_2 \) over \( k_0 \), there exists an amalgamation of valued fields:

\[
\begin{array}{ccc}
K_0 & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \longrightarrow & L
\end{array}
\]

such that

1. If \( \ell \) denotes the residue field of \( L \), then the amalgamation type of \( \ell \) over \( k_1 \) and \( k_2 \) is \( \tau \).
2. If \( K_1 \otimes_{K_0} K_2 \) is a domain, then \( L \) is an independent amalgamation.
3. \( L \) is a reduced amalgamation of \( K_1 \) and \( K_2 \).

**Proof.** Requirement (3) is trivial to arrange, by replacing \( L \) with its subfield generated by \( K_1 \) and \( K_2 \). So we will forget requirement (3).

Let \( \mathfrak{n} \) be the prime ideal of \( k_1 \otimes_{k_0} k_2 \) associated to the amalgamation type \( \tau \). Let \( \mathfrak{p}_1 \) be the pullback of \( \mathfrak{n} \) under the surjective ring homomorphism

\[ O_1 \otimes_{O_0} O_2 \to k_1 \otimes_{k_0} k_2. \]
Consider the commutative diagram of sets for \( i = 1, 2 \):

\[
\begin{array}{ccc}
\text{Spec } k_i & \longrightarrow & \text{Spec } k_i \otimes_{k_0} k_2 \\
\downarrow & & \downarrow \\
\text{Spec } O_i & \longrightarrow & \text{Spec } O_1 \otimes_{O_0} O_2.
\end{array}
\]

There is only one point in the top left set, and it maps to the maximal ideal \( m_i \in \text{Spec } O_i \). Since \( p_1 \) comes from \( n \) in the top left, the restriction of \( p_1 \) to \( O_i \) must be \( m_i \).

Now let \( p_0 \) be some minimal prime in \( O_1 \otimes_{O_0} O_2 \), chosen to lie below \( p_1 \). Consider the commutative diagram

\[
\begin{array}{ccc}
\text{Spec } K_i & \longrightarrow & \text{Spec } K_1 \otimes_{K_0} K_2 \\
\downarrow & & \downarrow \\
\text{Spec } O_i & \longrightarrow & \text{Spec } O_1 \otimes_{O_0} O_2.
\end{array}
\]

By Lemma 4.3 and Remark 4.4, \( p_0 \) comes from an element of the top right corner. Again, \( \text{Spec } K_i \) has only one point and it maps to the zero ideal in \( \text{Spec } O_i \), so the restriction of \( p_0 \) to \( O_i \) must then be the zero ideal.

By the Chevalley Extension Theorem ([1] Theorem 3.1.1) there is a valuation ring \((O_3, m_3)\) and a homomorphism

\[ O_1 \otimes_{O_0} O_2 \rightarrow O_3 \]

under which \( m_3 \) and \((0)\) pull back to \( p_1 \) and \( p_0 \), respectively. This yields a diagram

\[
\begin{array}{ccc}
O_0 & \longrightarrow & O_1 \\
\uparrow & & \uparrow \\
O_2 & \longrightarrow & O_3
\end{array}
\]

Under the composition

\[ O_1 \rightarrow O_1 \otimes_{O_0} O_2 \rightarrow O_3, \]

the prime ideals \( m_3 \) and \((0)\) pull back to \( p_1 \) and \( p_0 \), and then to \( m_1 \) and \((0)\), respectively. It follows that \( O_1 \rightarrow O_3 \) is an injective local homomorphism. The same holds for \( O_2 \rightarrow O_3 \) similarly. By Remark 4.5 we get a diagram of valued fields

\[
\begin{array}{ccc}
K_0 & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \longrightarrow & L
\end{array}
\]

From the induced diagram

\[
\begin{array}{ccc}
O_1 \otimes_{O_0} O_2 & \longrightarrow & k_1 \otimes_{k_0} k_2 \\
\downarrow & & \downarrow \\
O_3 & \longrightarrow & \ell
\end{array}
\]

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we see that the kernel of $k_1 \otimes_{k_0} k_2 \to \ell$ pulls back to $p_1$ on $\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2$. So $\ker k_1 \otimes_{k_0} k_2 \to \ell$ and $n$ have the same image under the injection $\text{Spec } k_1 \otimes_{k_0} k_2 \hookrightarrow \text{Spec } \mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2$. Therefore, the kernel equals $n$, so the amalgamation type of $\ell$ is $\tau$ as desired.

Finally, suppose that $K_1 \otimes_{K_0} K_2$ is a domain. By Lemma 4.3, the map $\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \to K_1 \otimes_{K_0} K_2$ is injective, so $\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2$ is a domain. The minimal prime $p_0$ must then be the zero ideal $(0)$. Now, in the diagram

\[
\begin{array}{c}
\mathcal{O}_0 \\
\downarrow \\
\mathcal{O}_2 \\
\downarrow \\
\mathcal{O}_3
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow
\end{array}
\begin{array}{c}
\mathcal{O}_1 \\
\downarrow \\
\mathcal{O}_2 \\
\downarrow \\
\mathcal{O}_3
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow
\end{array}
\begin{array}{c}
K_0 \\
\downarrow \\
K_1 \\
\downarrow \\
K_2 \\
\downarrow
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow \\
\rightarrow \\
\downarrow
\end{array}
\begin{array}{c}
K_1 \otimes_{K_0} K_2
\end{array}
\]

every ring is a domain, and moreover every morphism is an injection:

- The maps $\mathcal{O}_i \to K_i$ are injections by definition.
- The map $\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \to K_1 \otimes_{K_0} K_2$ is an injection by Lemma 4.3.
- The maps out of $K_0, K_1, K_2$ are injections because the $K_i$ are fields.
- The maps $\mathcal{O}_i \to \mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2$ are injective because the diagram commutes, and the other path from $\mathcal{O}_i$ to $K_1 \otimes_{K_0} K_2$ is made of injections.
- The map $\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2 \to \mathcal{O}_3$ is injective because the pullback of $(0)$ under this map was $p_0 = (0)$, by choice of $\mathcal{O}_3$.

Therefore, the above diagram belongs to the category of domains and embeddings. Applying
the functor $\text{Frac}(\cdot)$ yields the diagram

$$
\begin{array}{c}
K_0 \longrightarrow K_1 \\
\downarrow \quad \downarrow \\
K_0 \quad \quad \text{Frac}(\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2) \\
\downarrow \quad \downarrow \\
K_2 \longrightarrow \text{Frac}(K_1 \otimes_{K_0} K_2) \\
\end{array}
$$

This diagram contains three amalgamations of $K_1$ and $K_2$ over $K_0$, namely $L$, $\text{Frac}(\mathcal{O}_1 \otimes_{\mathcal{O}_0} \mathcal{O}_2)$ and $\text{Frac}(K_1 \otimes_{K_0} K_2)$. Moreover, the diagram shows that they have the same amalgamation type. By definition, $\text{Frac}(K_1 \otimes_{K_0} K_2)$ has the independent amalgamation type. Thus $L$ is also an independent amalgamation.

**Remark 4.7.** Lemma 4.6 can be used to prove quantifier elimination in ACVF. The Lemma implies that the class of valued fields has the amalgamation property. By abstract nonsense, it only remains to prove that models of ACVF are 1-e.c., in other words,

$$M_1 \models \exists x : \varphi(x; \vec{b}) \implies M_2 \models \exists x : \varphi(x; \vec{b})$$

for any extension $M_1 \leq M_2$ of models, tuple $\vec{b}$ from $M_1$, and quantifier-free formula $\varphi(x; \vec{y})$ with $x$ a singleton. The 1-e.c. property can be verified in a straightforward fashion from the swiss cheese decomposition of quantifier-free definable sets.

**Remark 4.8.** Lemma 4.6 implies amalgamation for the class of two-sorted structures $(K, k)$ where $K$ is a valued field and $k$ is an extension of the residue field. Indeed, suppose we are given a diagram

$$
\begin{array}{c}
(K_0, k_0) \longrightarrow (K_1, k_1) \\
\downarrow \\
(K_2, k_2) \\
\end{array}
$$

of embeddings of such structures. First, amalgamate $k_1$ and $k_2$ over $k_0$ into a monster model $\mathcal{M}$ of $\text{ACF}$:
This induces an amalgamation of the residue fields:

\[
\begin{array}{c}
\text{res } K_0 \rightarrow \text{res } K_1 \\
\downarrow \quad \downarrow \\
\text{res } K_2 \rightarrow \mathbb{M}.
\end{array}
\]

(2)

By Lemma 4.6 one can then amalgamate the valued fields

\[
\begin{array}{c}
K_0 \rightarrow K_1 \\
\downarrow \quad \downarrow \\
K_2 \rightarrow L
\end{array}
\]

in such a way that

\[
\begin{array}{c}
\text{res } K_0 \rightarrow \text{res } K_1 \\
\downarrow \quad \downarrow \\
\text{res } K_2 \rightarrow \text{res } L
\end{array}
\]

has the same amalgamation type as (2). Because the amalgamation types agree, there is an embedding of \( \text{res } L \) into \( \mathbb{M} \) such that the diagram commutes:

\[
\begin{array}{c}
\text{res } K_0 \rightarrow \text{res } K_1 \\
\downarrow \quad \downarrow \\
\text{res } K_2 \rightarrow \text{res } L \\
\downarrow \quad \downarrow \\
k_0 \rightarrow k_1 \\
\downarrow \quad \downarrow \\
k_2 \rightarrow \mathbb{M}.
\end{array}
\]

The embedding of \( \text{res } L \) into \( \mathbb{M} \) yields a structure \((L, \mathbb{M})\) and the above diagram means that

\[
\begin{array}{c}
(K_0, k_0) \rightarrow (K_1, k_1) \\
\downarrow \quad \downarrow \\
(K_2, k_2) \rightarrow (L, \mathbb{M})
\end{array}
\]

commutes.

Remark 4.9. Amalgamation for the 2-sorted structures \((K, k)\) implies a sort of quantifier elimination for 2-sorted ACVF. Specifically, if \((K, k)\) is one of these two-sorted structures (possible with \(k\) strictly greater than \(\text{res } K\)) then any two embeddings of \((K, k)\) into a model of ACVF have the same type. This implies Lemma 2.2.
Lemma 4.10. Let $K_0 = K_0^{alg}$ and let

\[
\begin{array}{ccc}
K_0 & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \longrightarrow & K_3
\end{array}
\]

be an independent amalgamation of fields. Let $\mathcal{O}_1, \mathcal{O}_2$ be valuation rings on $K_1, K_2$ having the same restriction $\mathcal{O}_0$ to $K_0$. Then there is a valuation ring $\mathcal{O}_3$ on $K_3$ extending $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the induced amalgamation of residue fields

\[
\begin{array}{ccc}
k_0 & \longrightarrow & k_1 \\
\downarrow & & \downarrow \\
k_2 & \longrightarrow & k_3
\end{array}
\]

is independent.

Proof. Because $K_0$ is algebraically closed, so is $k_0$. Therefore, $k_1 \otimes_{k_0} k_2$ is a domain, so it makes sense to talk about the independent amalgamation type on the residue fields. By Lemma 4.6 there is some amalgamation of valued fields

\[
\begin{array}{ccc}
K_0 & \longrightarrow & K_1 \\
\downarrow & & \downarrow \\
K_2 & \longrightarrow & L
\end{array}
\]

such that

- $L$ is an independent amalgamation of $K_1$ and $K_2$ over $K_0$.
- $\text{res} L$ is an independent amalgamation of $k_1$ and $k_2$ over $k_0$.
- $L$ is a reduced amalgamation of $K_1$ and $K_2$ over $K_0$.

By assumption, $K_3$ also has the independent amalgamation type, so its reduction is isomorphic to $L$. Therefore, there is an embedding of $L$ into $K_3$ such that the following diagram of pure fields commutes:
Let $\mathcal{O}_3$ be any valuation ring on $K_3$ extending the valuation ring on $L$. Then the above diagram becomes a diagram of valued fields. Moreover, the induced diagram of residue fields looks like

$$
\begin{array}{c}
\xymatrix{
k_0 \ar[r] & k_1 \ar[d] & \\
k_2 \ar[r] & \text{res } L & \\
\text{res } K_3
}\end{array}
$$

Thus the amalgamation type of res $K_3$ over $k_1$ and $k_2$ is the same as res $L$, namely the independent type.

Recall that $\text{Val}(K)$ denotes the poset of valuation rings on $K$, and $\text{Val}(K|\mathcal{O})$ denotes the subposet of valuation rings below a given $\mathcal{O} \in \text{Val}(K)$. If $\mathcal{O}' \in \text{Val}(K|\mathcal{O})$, then $\mathcal{O}' \div \mathcal{O}$ denotes the valuation ring on res $\mathcal{O}$ whose composition with $\mathcal{O}$ is $\mathcal{O}'$.

**Remark 4.11.**

1. If $L/K$ is an extension of fields, there is a restriction map

$$\text{Val}(L) \to \text{Val}(K)$$

$$\mathcal{O} \mapsto \mathcal{O} \cap K$$

2. If $L/K$ is an extension of fields, if $\mathcal{O}_L$ is a valuation ring on $L$, and if $\mathcal{O}_K = \mathcal{O}_L \cap K$ is the restriction to $K$, then there is a restriction map $\text{Val}(L|\mathcal{O}_L) \to \text{Val}(K|\mathcal{O}_K)$. Moreover, the diagram commutes

$$
\begin{array}{c}
\xymatrix{
\text{Val}(L|\mathcal{O}_L) \ar[r] & \text{Val}(K|\mathcal{O}_K) \ar[d] \\
\text{Val}(L) & \text{Val}(K) \ar[l]
}\end{array}
$$

where the vertical maps are inclusions.

3. If $K$ is a valued field and $\mathcal{O}_K$ is a valuation ring on $K$ with residue field $k = \text{res } \mathcal{O}_K$, then there is a bijection

$$\text{Val}(K|\mathcal{O}_K) \to \text{Val}(k)$$

$$\mathcal{O} \mapsto \mathcal{O} \div \mathcal{O}_K.$$

4. If $L/K$ is an extension of fields, if $\mathcal{O}_L$ is a valuation ring on $L$, if $\mathcal{O}_K = \mathcal{O}_L \cap K$ is the restriction to $K$, and if $\ell \to k$ is the residue field embedding $\text{res } \mathcal{O}_L \to \text{res } \mathcal{O}_K$, then
the diagram

\[
\begin{array}{c}
\text{Val}(L|O_L) \twoheadrightarrow \text{Val}(K|O_K) \\
\text{Val}(\ell) \twoheadrightarrow \text{Val}(k)
\end{array}
\]

commutes, where the vertical maps are the bijections of \([3]\) and the horizontal maps are the restriction maps of \([2]\).

**Lemma 4.12.** Let

\[
(F, O_F) \twoheadrightarrow (K_1, O_{K_1}) \\
(K_2, O_{K_2}) \twoheadrightarrow (L, O_L)
\]

be a diagram of valued fields. Suppose \(F\) is algebraically closed and \(\text{res } O_L\) is an independent amalgamation of \(\text{res } O_{K_1}\) and \(\text{res } O_{K_2}\) over \(\text{res } O_F\). Given \(O'_F \subseteq O_F, O'_{K_1} \subseteq O_{K_1}\), and \(O'_{K_2} \subseteq O_{K_2}\) such that \(O'_{K_1}\) and \(O'_{K_2}\) restrict to \(O'_F\), there is some \(O'_L \subseteq O_L\) extending \(O'_{K_1}\) and \(O'_{K_2}\) such that \(\text{res } O'_L\) is an independent amalgamation of \(\text{res } O'_{K_1}\) and \(\text{res } O'_{K_2}\) over \(\text{res } O'_F\).

**Proof.** This follows from Remark 4.11.4 and Lemma 4.10. Specifically, Remark 4.11.4 shows the commutativity of the diagram

\[
\begin{array}{c}
\text{Val}(F|O_F) \twoheadrightarrow \text{Val}(K_1|O_{K_1}) \\
\text{Val}(K_2|O_{K_2}) \twoheadrightarrow \text{Val}(L|O_L) \\
\text{Val}(\text{res } O_F) \twoheadrightarrow \text{Val}(\text{res } O_{K_1}) \\
\text{Val}(\text{res } O_{K_2}) \twoheadrightarrow \text{Val}(\text{res } O_L)
\end{array}
\]

where the vertical maps are bijections. The problem we need to solve is on the upper plane, but the diagram allows us to move the problem to the lower plane. One concludes by applying Lemma 4.10 to the diagram

\[
\begin{array}{c}
\text{res } O_F \twoheadrightarrow \text{res } O_{K_1} \\
\text{res } O_{K_2} \twoheadrightarrow \text{res } O_L.
\end{array}
\]

\[
\square
\]
5 Amalgamation in $T_P$

**Proposition 5.1.** Let $P$ be a finite tree. Let

$$
\begin{array}{c}
K_0 \\
| \\
K_2
\end{array}
\begin{array}{c}
\to
\\
\to
\\
\to
\end{array}
\begin{array}{c}
K_1
\end{array}
$$

be a diagram of embeddings of models of $T^0_P$, with $K_0 = K^\text{alg}_0$. Then the diagram can be completed to a diagram

$$
\begin{array}{c}
K_0 \\
\to
\\
K_2
\end{array}
\begin{array}{c}
\to
\\
\to
\\
\to
\end{array}
\begin{array}{c}
K_1 \\
K_3
\end{array}
$$

of embeddings of models of $T^0_P$. Furthermore, $K_3$ can be chosen to be an independent amalgamation of $K_1$ and $K_2$.

**Proof.** Let $K_3 = \text{Frac}(K_1 \otimes_{K_0} K_2)$. Let $O^p_i$ and $k^p_i$ denote the $p$th valuation ring and residue field on $K_i$. One chooses $O^p_3$ on $K_3$ by upwards recursion on $p$, ensuring that $\text{res } O^p_3$ is an independent amalgamation of $\text{res } O^p_1$ and $\text{res } O^p_2$ over $\text{res } O^p_0$ at each step. This is possible by Lemma 4.12.

**Corollary 5.2.** In $T_P$, field-theoretic algebraic closure agrees with model-theoretic algebraic closure.

**Proof.** Suppose $M \models T_P$ and $K = K^\text{alg} \leq M$. Suppose $a \in \text{acl}(K)$. We claim $a \in K$. Take a second copy $M'$ of $M$, amalgamated with $M$ independently over $K$ inside a third model $M'' \models T_P$. By model completeness, $M' \preceq M'' \succeq M$. Let $X$ be the $K$-definable finite set of conjugates of $a$. Then $a \in X(M) = X(M'') = X(M')$, so $a \in M \cap M'$. In the ACF reduct,

$$
M' \downarrow_K M \implies a \downarrow_K a \implies a \in \text{acl}(K) \implies a \in K.
$$

**Corollary 5.3.** Let $M_1, M_2$ be two models of $T_P$, let $K_i$ be an algebraically closed subfield of $M_i$, and $f : K_1 \to K_2$ be an isomorphism of $T^0_P$-structures. Then $f$ is a partial elementary map.

**Proof.** Amalgamate $M_1$ and $M_2$ over $K$ and use model completeness of $T_P$.

**Definition 5.4.** If $P$ is a finite tree, then $T^\text{alg}_P$ is $T^0_P$ plus the axiom that $K \models ACF$.

So $T^0_P \models T^\text{alg}_P \models T_P$.

**Corollary 5.5.** $T_P$ is the model completion of $T^\text{alg}_P$. 

20
Corollary 5.6. Let $M, M'$ be two models of $T_P$. Then $M \equiv M'$ if and only if $\text{Abs}(M) \cong \text{Abs}(M')$, where $\text{Abs}(M)$ denotes the substructure of “absolute numbers,” i.e., elements algebraic over the prime field.

The only valuation ring on $\mathbb{F}_p^{alg}$ is the trivial one, because $(\mathbb{F}_p^{alg})^\times$ is torsion. Therefore,

Corollary 5.7. If $M, M' \models T_P$ and $\text{char}(M) = \text{char}(M') > 0$, then $M \equiv M'$.

Corollary 5.8. Let $K$ be a model of $T^0_P$, let $\varphi(\bar{x})$ be a sentence in the language of $T_P$, and let $\bar{a}$ be a tuple from $K$. There is a finite normal extension $L/K$ such that for $M \models T_P$ extending $K$, whether or not $\varphi(\bar{a})$ holds in $M$ is determined by the induced $T^0_P$ structure on (the copy of) $L$ in $M$.

Corollary 5.9. The (incomplete) theory $T_P$ is decidable: there is an algorithm which takes a sentence $\varphi$ and determines whether $T_P \models \varphi$.

Proof. As $T_P$ is c.e., it suffices to show that the set of sentences consistent with $T_P$ is c.e.

Let $\chi$ be a function from $P$ to $\{0, 2, 3, 5, 7, \ldots\}$ satisfying the requirement that $\chi(x) = p \implies \chi(y) = p$ for $x \leq y \in P$ and $p \neq 0$. For any such $\chi$, let $T_{P,\chi}$ be $T_P$ plus axioms asserting that $\text{char}(\text{res}\mathcal{O}_x) = \chi(x)$ for all $x \in P$. Define $T^0_{P,\chi}$ similarly. Each $T_{P,\chi}$ is consistent, so it suffices to show that the set of sentences consistent with $T_{P,\chi}$ is c.e., uniformly in $\chi$.

When $\chi(\bot) > 0$, the theory $T_{P,\chi}$ is complete by Corollary 5.7 and therefore decidable. So assume $\chi(\bot) = 0$.

Let $a_1, \ldots, a_n$ enumerate the minimal $a \in P$ such that $\chi(a) > 0$, and let $p_i = \chi(a_i)$.

Claim 5.10. Let $M_1, M_2$ be two models of $T^0_{P,\chi}$, algebraic over the prime field. Let $f : M_1 \to M_2$ be an isomorphism of the underlying fields. Then $f$ is an isomorphism of $T^0_{P,\chi}$-structures if and only if $f$ sends $\mathcal{O}_{a_i}^{M_1}$ to $\mathcal{O}_{a_i}^{M_2}$ for $i = 1, \ldots, n$.

Proof. Without loss of generality, $M_1$ and $M_2$ have the same underlying field $K$ and $f$ is the identity map $id_K : K \to K$. The “only if” direction is clear. For the “if” direction, note that the non-trivial valuation rings on $K$ are pairwise incomparable and mixed characteristic, because $K$ is algebraic over the prime field. Therefore, $\mathcal{O}_a$ must be trivial when $\chi(a) = 0$, and $\mathcal{O}_a$ must equal $\mathcal{O}_{a_i}$ when $a \geq a_i$. So the $\mathcal{O}_{a_i}$ determine the other valuation rings. □

Let $\Psi$ be the set of sentences of the form

$$\exists x : Q(x) = 0 \land R_1(x) \land \cdots \land R_n(x),$$

where $Q(X) \in \mathbb{Q}[X]$ is a monic irreducible polynomial, $R_i(x)$ is a quantifier-free predicate only involving the $a_i$th valuation ring, and $ACVF_{\mathbb{F}_p} \models \exists x : Q(x) = 0 \land R_i(x)$. The set $\Psi$ is c.e., because the set of monic irreducible polynomials is c.e.

Claim 5.11. A sentence $\varphi$ is consistent with $T_{P,\chi}$ if and only if $T_{P,\chi} \cup \{\psi\} \models \varphi$ for some $\psi \in \Psi$.  

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Proof. For the “if” direction, we only need to show that the sentences
\[ \psi := (\exists x : Q(x) = 0 \land R_1(x) \land \cdots \land R_n(x)) \]
are consistent with \( T_{P,\chi} \). Fix a copy of \( Q_{\text{alg}} \) and a root \( \alpha \) of \( Q(X) \). For each \( i \), we can find a valuation ring \( O_i \) on \( Q_{\text{alg}} \) of mixed characteristic \((0, p_i)\), such that
\[ (Q_{\text{alg}}, O_i) \models R_i(\alpha). \]
Indeed, first choose an arbitrary valuation ring \( O' \) of mixed characteristic \((0, p_i)\), use the assumption on \( R_i \) to find \( \alpha' \in Q_{\text{alg}} \) such that
\[ (Q_{\text{alg}}, O') \models Q(\alpha') = 0 \land R_i(\alpha'), \]
and then move \( \alpha' \) and \( O' \) to \( \alpha \) and \( O_i \) by an automorphism in \( \text{Gal}(\mathbb{Q}) \). Now
\[ (Q_{\text{alg}}, O_1, \ldots, O_n) \models \exists x : Q(x) = 0 \land R_1(x) \land \cdots \land R_n(x), \]
witnessed by \( \alpha \). Expand \( (Q_{\text{alg}}, O_1, \ldots, O_n) \) to a model of \( T_{P,\chi}^0 \) as in the proof of Claim 5.10 and then extend to a model \( M \models T_{P,\chi} \). Then \( M \models \psi \).

Conversely, suppose that \( \phi \) holds in some model \( M \models T_{P,\chi} \). By Corollary 5.8 there is a subfield \( L \leq M \) such that \([L : \mathbb{Q}] < \infty \) and \( T_{P,\chi} \cup \text{diag}(L) \models \phi \). Let \( \alpha \) be a generator of \( L = \mathbb{Q}(\alpha) \). Then
\[ T_{P,\chi} \cup \text{qftp}(\alpha/\emptyset) \models \psi. \]
Let \( \text{qftp}_i(\alpha/\emptyset) \) be the quantifier-free type in the reduct \((M, O_{a_i})\). Then
\[ T_{P,\chi} \cup \bigcup_{i=1}^{n} \text{qftp}_i(\alpha/\emptyset) \models \text{qftp}(\alpha/\emptyset) \]
essentially by Claim 5.10. By compactness and the lemma on constants, there are quantifier-free formulas \( R_i(x) \in \text{qftp}_i(\alpha/\emptyset) \) such that
\[ T_{P,\chi} \cup \{ \exists x : R_1(x) \land \cdots \land R_n(x) \} \models \varphi. \]
Let \( Q(X) \) be the minimal polynomial of \( \alpha \) over \( \mathbb{Q} \) and let \( \psi \) be the sentence
\[ \exists x : Q(x) = 0 \land R_1(x) \land \cdots \land R_n(x). \]
Then \( T_{P,\chi} \cup \{ \psi \} \models \varphi \) a fortiori. Moreover, for any \( i \)
\[ M \models \exists x : Q(x) = 0 \land R_i(x), \]
witnessed by \( \alpha \). So the formula \( \exists x : Q(x) = 0 \land R_i(x) \) is consistent with \( \text{ACVF}_{0,p_i} \). But \( \text{ACVF}_{0,p_i} \) is complete, so \( \text{ACVF}_{0,p_i} \models \exists x : Q(x) = 0 \land R_i(x) \). Therefore \( \psi \in \Psi. \qed
Given the claim, it follows that the set of sentences consistent with $T_{P, \chi}$ is c.e., uniformly in $\chi$. Taking the union over all $\chi$, the set of sentences consistent with $T_P$ is c.e. The set of consequences of $T_P$ is trivially c.e., and so the theory is decidable.

Using Example 3.4, we deduce

**Corollary 5.12.** Let $ACv^n F$ be the theory of algebraically closed fields with $n$ valuation rings (as unary predicates). Then the incomplete theory $ACv^n F$ is decidable.

**Proof.** The only thing to check here is that we can bound the size of $P$ from the number $n$ of given valuations $O_1, \ldots, O_n$. On account of the tree structure, every valuation in $P$ is of the form $O_i \cdot O_j$ (or $K$), so there are certainly no more than $n^2 + 1$ elements in $P$. 

6 Normal and relatively closed extensions

**Definition 6.1.** Fix a diagram

\[ F \rightarrow K_1 \]

\[ K_2 \]

in the category of fields. A reduced amalgamation

\[ F \rightarrow K_1 \]

\[ K_2 \rightarrow L \]

is cozy if the maps $K_i \rightarrow L$ are isomorphisms. An amalgamation type is cozy if a representative reduced amalgamation is cozy.

**Remark 6.2.** The following are equivalent:

- Every amalgamation type of $K_1$ and $K_2$ over $F$ is cozy.
- $K_1$ and $K_2$ are (algebraic) normal extensions of $F$, isomorphic to each other over $F$.

**Lemma 6.3.** Let $L/K$ be a normal (algebraic) extension, and $O$ be a valuation ring on $K$.

1. $Aut(L/K)$ acts transitively on the set of extensions of $O$ to $L$.
2. If $O'$ is any extension of $O$ to $L$, then the residue field extension is a normal extension.
3. The residue field extension does not depend on $O'$ in the following sense: if $O'$ and $O''$ are two extensions of $O$ to $L$, then $res O'$ and $res O''$ are isomorphic over $res O$.
Proof. Let \( O_1 \) and \( O_2 \) be two (not necessarily distinct) extensions of \( O \) to \( L \). By Remark 6.2 it suffices to show that \( O_1 \) and \( O_2 \) are in the same orbit of \( \text{Aut}(L/K) \) and that every amalgamation type of \( \text{res} O_1 \) and \( \text{res} O_2 \) over \( \text{res} O \) is cozy. Given any amalgamation type \( \tau \), by Lemma 4.6 there is an amalgamation of valued fields

\[
(K, O) \rightarrow (L, O_1) \rightarrow (L, O_2) \rightarrow (L', O')
\]

such that

- \( L' \) is a reduced amalgamation of \( L \) and \( L \) over \( K \)
- \( \text{res} O' \) is an amalgamation of \( \text{res} O_1 \) and \( \text{res} O_2 \) over \( \text{res} O \), of type \( \tau \).

Then \( L' \) is a cozy amalgamation of \( L \) and \( L \) over \( K \), by normality of \( L/K \), Remark 6.2 and the fact that \( L' \) is a reduced amalgamation. If \( \sigma \) is the induced isomorphism \( L \rightarrowtail L' ightarrowtail L \), then \( \sigma \in \text{Aut}(L/K) \) and \( \sigma(O_1) = O_2 \), proving transitivity. Moreover, the fact that \( L' \) is a cozy amalgamation of \( L \) and \( L \) over \( K \) implies the same thing for the residue fields: \( \text{res} O' \) is a (reduced) cozy amalgamation of \( \text{res} O_1 \) and \( \text{res} O_2 \) over \( \text{res} O \). Therefore \( \tau \) is cozy, completing the proof.

Definition 6.4. If \( L/K \) is a finite normal extension and \( O \) is a valuation ring on \( K \), we let \( n_{O, L/K} \) denote the (finite) number of extensions of \( O \) to \( L \).

Note that \( n_{O, L/K} \) depends only on the isomorphism type of \( L \) over \( K \): if \( L'/K \) is an isomorphic extension, then \( n_{O, L'/K} = n_{O, L/K} \).

Recall that if \( L/K \) is a finite (algebraic) extension of valued fields, then the residue field extension is also finite, of degree no greater than \( [L : K] \), because one can lift a basis of \( \text{res} L \) over \( \text{res} K \) to a \( K \)-linearly independent set in \( L \).

Lemma 6.5. Let

\[
\begin{array}{ccl}
L_1 & \rightarrow & L_2 \\
K_1 & \leftarrow & K_2
\end{array}
\]

be a diagram of fields, in which \( L_2/L_1 \) and \( K_2/K_1 \) are finite normal extensions. Suppose \( K_1 \) is relatively algebraically closed in \( L_1 \). Let \( O_{L_1}, O_{K_1}, \) and \( O_{K_2} \) be valuation rings on \( L_1, K_1, \) and \( K_2 \), respectively. Suppose \( O_{K_1} \) is the restriction of \( O_{L_1} \) and \( O_{K_2} \). Then, the set of valuation rings \( O_{L_2} \) on \( L_2 \) extending both \( O_{L_1} \) and \( O_{K_2} \) is non-empty, and has size exactly

\[
\frac{n_{O_{L_2}, L_2/L_1}}{n_{O_{K_1}, K_2/K_1}}
\]
Proof. We claim that the restriction map $\rho : \text{Aut}(L_2/L_1) \to \text{Aut}(K_2/K_1)$ is surjective. Assume otherwise, and embed $L_2$ into a monster model $M$ of ACF. By elimination of imaginaries and the model-theoretic Galois correspondence, non-surjectivity implies there is $x \in \text{dcl}(K_2) \cap \text{dcl}(L_1) \setminus \text{dcl}(K_1)$. Definable closure in ACF corresponds to perfect closure. Thus, after replacing $x$ with $x^{p^k}$, we may assume $x \in K_2 \cap L_1$. Then $K_2 \cap L_1 \setminus K_1$ is non-empty, contradicting relative algebraic closure of $K_1$ in $L_1$.

Thus $\rho : \text{Aut}(L_2/L_1) \to \text{Aut}(K_2/K_1)$ is surjective. Let $V_L$ be the set of valuation rings on $L_2$ extending $O_{L_1}$ and $V_K$ be the set of valuation rings on $K_2$ extending $O_{K_1}$. Both these sets are finite. By Lemma 6.3, $\text{Aut}(L_2/L_1)$ acts transitively on $V_L$ and $\text{Aut}(K_2/K_1)$ acts transitively on $V_K$. The restriction map $V_L \to V_K$ is compatible with the action, in the sense that if $O \in V_L$ and $\sigma \in \text{Aut}(L_2/L_1)$, then

$$(\sigma \cdot O)) \cap K_2 = (\sigma|K_2) \cdot (O \cap K_2).$$

In particular, if we view $V_K$ as an $\text{Aut}(L_2/L_1)$-set via the homomorphism $\rho : \text{Aut}(L_2/L_1) \to \text{Aut}(K_2/K_1)$, then the restriction $V_L \to V_K$ is a homomorphism of $\text{Aut}(L_2/L_1)$-sets. Then $V_K$ is a transitive $\text{Aut}(L_2/L_1)$ by surjectivity of $\rho$. Because both $V_L$ and $V_K$ are transitive $\text{Aut}(L_2/L_1)$-sets, every fiber of the map $V_L \to V_K$ has the same cardinality. This cardinality must be

$$\frac{|V_L|}{|V_K|} = \frac{n_{O_{L_1},L_2/L_1}}{n_{O_{K_1},K_2/K_1}}$$

\[\square\]

Lemma 6.6. Let $L/K$ be a finite normal extension. Let $O_K \supset O'_K$ be two valuation rings on $K$. There is an integer $n_{O_K,O'_K,L/K}$ such that for any $O_L$ on $L$ extending $O_K$, the set $S$ of $O'_L \in \text{Val}(L|O_L)$ extending $O'_K$ has size exactly $n_{O_K,O'_K,L/K}$.

Furthermore, for any $O_L$, we have

$$n_{O_K,O'_K,L/K} = n_{O_K,O'_K,L/K} \cdot n_{O_L,O_K}.$$  

Proof. Let $k$ be the residue field of $O_K$. Given $O_L$, let $\ell$ be $\text{res} O_L$. By Lemma 6.3, the isomorphism type of $\ell$ over $k$ does not depend on $O_L$. Let $n_{O_K,O'_K,L/K}$ be $n_{O_K,O'_K,L/K}$; this depends only on the isomorphism type of $\ell$ over $k$, hence is independent of $O_L$.

By Remark 4.11.4 there is a diagram

$$\text{Val}(L|O_L) \xrightarrow{\sim} \text{Val}(\ell) \xrightarrow{\sim} \text{Val}(K|O_K)$$

with horizontal maps the isomorphisms

$$O \mapsto O \div O_L$$

$$O \mapsto O \div O_K$$

\[1\]Here, $\text{Aut}(-/-)$ denotes automorphisms of pure fields, not valued fields.
respectively.

The set \( S \) is the fiber of the left vertical map over \( \mathcal{O}_K' \). Via the horizontal isomorphisms, this is in bijection with the set of valuations on \( \ell \) extending \( \mathcal{O}_K' \div \mathcal{O}_K \). By definition, this set has size \( n_{\mathcal{O}_K' \div \mathcal{O}_K, \ell/k} \), the value we chose for \( n_{\mathcal{O}_K, \mathcal{O}_K', L/K} \).

\[ \square \]

Lemma 6.7. Let

\[
\begin{array}{ccc}
L_1 & 
\rightarrow & L_2 \\
\uparrow & & \uparrow \\
K_1 & \rightarrow & K_2
\end{array}
\]

be a diagram of fields, in which \( L_2/L_1 \) and \( K_2/K_1 \) are finite normal extensions. Let \( \mathcal{O}_{L_2} \) be a valuation ring on \( L_2 \) and let \( \mathcal{O}_{L_1}, \mathcal{O}_{K_2}, \mathcal{O}_{K_1} \) be the restrictions to \( L_1, K_2, \) and \( K_1 \), respectively. Suppose that \( \text{res} \mathcal{O}_{K_1} \) is relatively algebraically closed in \( \text{res} \mathcal{O}_{L_1} \). Let \( \mathcal{O}_{L_1}', \mathcal{O}_{K_1}', \) and \( \mathcal{O}_{K_2}' \) be valuation rings on \( L_1, K_1, \) and \( K_2, \) respectively, such that

\begin{itemize}
  \item \( \mathcal{O}_{L_1}' \subseteq \mathcal{O}_{L_1}, \mathcal{O}_{K_1}' \subseteq \mathcal{O}_{K_1}, \) and \( \mathcal{O}_{K_2}' \subseteq \mathcal{O}_{K_2}. \)
  \item \( \mathcal{O}_{K_1}' \) is the restriction of both \( \mathcal{O}_{L_1}' \) and \( \mathcal{O}_{K_2}' \) to \( K_1. \)
\end{itemize}

Let \( S \) be the set of valuation rings \( \mathcal{O}_{L_2}' \) on \( L_2 \) such that

\begin{itemize}
  \item \( \mathcal{O}_{L_2}' \subseteq \mathcal{O}_{L_2}. \)
  \item \( \mathcal{O}_{L_2}' \) extends both \( \mathcal{O}_{L_1}' \) and \( \mathcal{O}_{K_2}'. \)
\end{itemize}

Then \( S \) is non-empty and has cardinality exactly

\[
\frac{n_{\mathcal{O}_{L_1}, \mathcal{O}_{L_1}', L_2/L_1}}{n_{\mathcal{O}_{K_1}, \mathcal{O}_{K_1}', K_2/K_1}}
\]

where the \( n \) are as in Lemma 6.6.

Proof. Let \( \ell_i \) and \( k_i \) denote the residue fields of \( \mathcal{O}_{L_i} \) and \( \mathcal{O}_{K_i}, \) respectively. By Remark 6.1.1.14 there is a commutative diagram

\[
\begin{array}{ccccccc}
\text{Val}(L_1|\mathcal{O}_{L_1}) & \rightarrow & \text{Val}(L_2|\mathcal{O}_{L_2}) \\
\downarrow & & \downarrow \\
\text{Val}(K_1|\mathcal{O}_{K_1}) & \rightarrow & \text{Val}(K_2|\mathcal{O}_{K_2}) \\
\text{Val}(\ell_1) & \rightarrow & \text{Val}(\ell_2) \\
\downarrow & & \downarrow \\
\text{Val}(k_1) & \rightarrow & \text{Val}(k_2)
\end{array}
\]
with vertical maps bijections. Under the bijection $\text{Val}(L_2|\mathcal{O}_{L_2}) \rightarrow \text{Val}(\ell_2)$, the set $\mathcal{S}$ corresponds to the set of $\mathcal{O}$ on $\ell_2$ restricting to $\mathcal{O}^\prime_{K_2} \div \mathcal{O}_{K_2}$ and $\mathcal{O}^\prime_{L_1} \div \mathcal{O}_{L_1}$. Now, by the commutative diagram, the fact that $\mathcal{O}^\prime_{K_2}$ and $\mathcal{O}^\prime_{L_1}$ both restrict to $\mathcal{O}^\prime_{K_1}$ implies that $\mathcal{O}^\prime_{K_2} \div \mathcal{O}_{K_2}$ and $\mathcal{O}^\prime_{L_1} \div \mathcal{O}_{L_1}$ restrict to $\mathcal{O}^\prime_{K_1} \div \mathcal{O}_{K_1}$. By assumption, $k_1$ is relatively algebraically closed in $\ell_1$, so by Lemma 6.5

$$|\mathcal{S}| = \frac{n_{\mathcal{O}_{K_2}^\prime \div \mathcal{O}_{L_2}, \ell_2/\ell_1}}{n_{\mathcal{O}_{K_1}^\prime \div \mathcal{O}_{K_1}, k_1/k_2}}.$$ 

By Lemma 6.6

$$\frac{n_{\mathcal{O}_{L_2}^\prime \div \mathcal{O}_{L_1}, \ell_2/\ell_1}}{n_{\mathcal{O}_{K_1}^\prime \div \mathcal{O}_{K_1}, k_2/k_1}} = \frac{n_{\mathcal{O}_{L_1}^\prime \div \mathcal{O}_{L_1}, \ell_2/\ell_1}}{n_{\mathcal{O}_{K_1}^\prime \div \mathcal{O}_{K_1}, k_2/k_1}}.
$$

**Definition 6.8.** Let $P$ be a finite poset.

1. Write $x \triangleright y$ if $x > y$ and there is no $z$ such that $x > z > y$.

2. A choice system on $P$ is a collection of sets $\mathcal{S}_x$ for $x \in P$ and relations $\mathcal{R}_{x,y} \subseteq \mathcal{S}_x \times \mathcal{S}_y$ for $x \triangleright y$.

3. Given a choice system on $P$ and a downwards closed subset $P' \subseteq P$, a partial choice on $P'$ is a function $f$ on $P'$ such that

$$\forall x \in P' : f(x) \in \mathcal{S}_x$$

$$\forall x \in P' \forall y < x : f(x) \mathcal{R}_{x,y} f(y).$$

We write $\Gamma(P')$ for the collection of partial choices on $P'$.

4. A choice system on $P$ is smooth at $x$ if there is a finite positive cardinal $n$ such that for any downward closed set $P' \subseteq P$ containing $x$ as a maximal element, every fiber of the restriction map

$$\Gamma(P') \rightarrow \Gamma(P' \setminus \{x\})$$

has size $n$.

**Remark 6.9.** Fix a choice system on a finite poset $P$, and let $P'$ be a downward closed subset of $P$. If the choice system is smooth at every $x \in P \setminus P'$, then every fiber of the restriction map

$$\Gamma(P) \rightarrow \Gamma(P')$$

has size $n$, for some finite positive $n$.

**Theorem 6.10.** Fix a finite tree $P$. Let $L/K$ be an extension of models of $T^0_p$. Suppose that for every $p \in P$, the $p$th residue field extension $\text{res} \mathcal{O}_p^L / \text{res} \mathcal{O}_p^K$ is relatively algebraically closed. Suppose we are given a diagram of pure fields

$$L \rightarrow L'$$

$$\text{K} \rightarrow \text{K'}$$

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where $L'/L$ and $K'/K$ are finite normal extensions. Let $S_L$ and $S_K$ be the set of extensions of the $T^0_P$-structures to $L'$ and $K'$, respectively. Then

1. The sets $S_L$ and $S_K$ are finite.
2. The restriction map $S_L \rightarrow S_K$ is surjective.
3. Every fiber of this restriction map has the same size.

Proof. Let $Q$ be the poset product of $P$ and the two-element total order $\{0, 1\}$. Note that all the relations $x \triangleright y$ in $Q$ are of the following forms:

- $(x, 1) \triangleright (x, 0)$.
- $(x, i) \triangleright (y, i)$ where $i \in \{0, 1\}$ and $y$ is the “parent” of $x$ in the tree $P$, i.e., $x \triangleright y$.

We build a choice system on $Q$ as follows:

- $S_{(x,0)}$ is the set of extensions (trivial if $x = \bot$) of $O^K_x$ to $K'$.
- $S_{(x,1)}$ is the set of extensions (trivial if $x = \bot$) of $O^L_x$ to $L'$.
- If $O^L_x \in S_{(x,1)}$ and $O^K_x \in S_{(x,0)}$, then $O^L_x \mathcal{R} O^K_x$ holds iff $O^L_x$ extends $O^K_x$.
- If $y$ is the parent of $x$ in $P$, if $O^K_y \in S_{(y,0)}$, and $O^K_x \in S_{(x,0)}$, then $O^K_y \mathcal{R} O^K_x$ holds iff $O^K_x \subseteq O^K_y$.
- If $y$ is the parent of $x$ in $P$, if $O^L_y \in S_{(y,0)}$, and $O^L_x \in S_{(x,0)}$, then $O^L_y \mathcal{R} O^L_x$ holds iff $O^L_y \subseteq O^L_x$.

If $Q' = P \times \{0\}$, then a partial choice function on $Q'$ is an extension of the $T^0_P$-structure from $K$ to $K'$, and a partial choice function on $Q$ is an extension of the $T^0_P$-structure from $L$ to $L'$. So it suffices to show that the choice system is smooth at every point $(x, i)$.

The case where $x = \bot$ is easy, so assume $x > \bot$. Let $y$ be the “parent” of $x$. If $i = 0$, smoothness at $(x, 0)$ follows by Lemma 6.6. Indeed, the number of valid choices for $O^K_x$ consistent with $O^K_y$ and $O^K_x$ is exactly $n_{O^K_y, O^K_x, K'/K}$, which does not depend on the choices.

Likewise, the case $i = 1$ follows by Lemma 6.7 the number of valid choices for $O^L_x$ consistent with $O^L_y$, $O^K_x$, and $O^L_x$ is exactly

\[
\frac{n_{O^L_y, O^L_x, L'/L}}{n_{O^K_y, O^K_x, K'/K}}
\]

Again, this does not depend on the choices so far. \qed
7 Probable truth

Theorem 7.1. There is a unique way to assign a probability $\mathbb{P}(\varphi(\bar{a})|K)$ to every model $K \models T^0_P$, tuple $\bar{a}$ from $K$, and formula $\varphi(\bar{a})$ in the language of $T_P$, satisfying the following properties:

1. $\mathbb{P}(\varphi(\bar{a})|K)$ is a rational number in $[0, 1]$.
2. $\mathbb{P}(\neg \varphi(\bar{a})|K) = 1 - \mathbb{P}(\varphi(\bar{a})|K)$.
3. $\mathbb{P}(\varphi(\bar{a})|K) + \mathbb{P}(\psi(\bar{b})|K) = \mathbb{P}(\varphi(\bar{a}) \lor \psi(\bar{b})|K) + \mathbb{P}(\varphi(\bar{a}) \land \psi(\bar{b})|K)$.
4. $\mathbb{P}(\varphi(\bar{a})|K) > 0$ if and only if $M \models \varphi(\bar{a})$ for at least one $T_P$-model $M \geq K$.
5. $\mathbb{P}(\varphi(\bar{a})|K) = 1$ if and only if $M \models \varphi(\bar{a})$ for every $T_P$-model $M \geq K$.
6. If $L/K$ is a finite normal extension of pure fields, and $L_1, \ldots, L_n$ enumerate the $T^0_P$-structures on $L$ extending the given structure on $K$, then $\mathbb{P}(\varphi(\bar{a})|K)$ is the average of $\mathbb{P}(\varphi(\bar{a})|L_i)$.
7. If $f : K_1 \rightarrow K_2$ is an isomorphism of $T^0_P$-models, and $f(\bar{a}_1) = \bar{a}_2$, then $\mathbb{P}(\varphi(\bar{a}_1)|K_1) = \mathbb{P}(\varphi(\bar{a}_2)|K_2)$.

Conditions (1-5) say, among other things, that $\mathbb{P}(-|K)$ defines a Keisler measure on the type space of embeddings of $K$ into models of $T_P$.

Proof. Let $\mathbb{P}'(-)$ be the partial function defined as follows:

1. $\mathbb{P}'(\varphi(\bar{a})|K) = 1$ if $M \models \varphi(\bar{a})$ for every $T_P$-model $M \geq K$.
2. $\mathbb{P}'(\varphi(\bar{a})|K) = 0$ if $M \models \neg \varphi(\bar{a})$ for every $T_P$-model $M \geq K$.
3. $\mathbb{P}'(\varphi(\bar{a})|K)$ is undefined otherwise.

Conditions (1) and (5) imply that $\mathbb{P}(-)$ must equal $\mathbb{P}'(-)$ when the latter is defined.

Given $K$ and $\varphi(\bar{a})$, by Corollary 5.8 there is a finite normal extension $L/K$ such that for any $T_P$-model $M \geq K$, the truth of $M \models \varphi(\bar{a})$ is determined by the $T^0_P$-structure induced on $L$. Let $L_1, \ldots, L_n$ be an enumeration of the distinct extensions of the $T^0_P$-structure from $K$ to $L$. Thus $\mathbb{P}'(\varphi(\bar{a})|L_i)$ is defined for $i = 1, \ldots, n$. Then uniqueness of $\mathbb{P}(-)$ is clear: we must set

$$\mathbb{P}(\varphi(\bar{a})|K) := \frac{\sum_{i=1}^{n} \mathbb{P}'(\varphi(\bar{a})|L_i)}{n}.$$

It remains to show that this is well-defined and satisfies the required properties.
Let $L'$ be another finite normal extension of $K$ which determines the truth of $\varphi(\bar{a})$. We claim that $L$ and $L'$ yield the same value of $P(\varphi(\bar{a})|K)$. By relating $L$ to $LL'$ and $L'$ to $LL'$, we reduce to the case where $L' \geq L$. Applying Theorem 6.10 to the diagram

\[
\begin{array}{c}
K \\
\downarrow \\
L' \\
\downarrow \\
K \\
\end{array}
\]

there is an integer $m$ such that every $L_i$ has exactly $m$ extensions $L'_{i,1}, \ldots, L'_{i,m}$ to a $T_P^0$-structure on $L'$. Then $\{L'_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq m}$ is an enumeration of the distinct $nm$-many $T_P$-structures on $L'$ extending the given structure on $K$. Moreover, every $T_P$-model extending $L'_{i,j}$ is a $T_P$-model extending $L_i$, so $P'(\varphi(\bar{a})|L'_{i,j}) = P'(\varphi(\bar{a})|L_i)$. Thus

\[
\sum_{i=1}^n \sum_{j=1}^m P'(\varphi(\bar{a})|L'_{i,j}) = \sum_{i=1}^n P'(\varphi(\bar{a})|L_i)
\]

So the definition of $P(\varphi(\bar{a})|K)$ using $L'$ agrees with that using $L$, and $P(\varphi(\bar{a})|K)$ is well-defined.

Condition (1) is clear, because we defined $P(\varphi(\bar{a})|K)$ as an average of finitely many 0’s and 1’s. For Conditions (2)-(5), choose $L$ large enough that $P'(\varphi(\bar{a})|L_i)$ and $P'(\psi(\bar{b})|L_i)$ are well-defined for all $i$. Then

\[
P'(\neg \varphi(\bar{a})|L_i) = 1 - P'(\varphi(\bar{a})|L_i)
\]

\[
P'(\varphi(\bar{a}) \lor \psi(\bar{b})|L_i) = \max(P'(\varphi(\bar{a})|L_i), P'(\psi(\bar{b})|L_i))
\]

\[
P'(\varphi(\bar{a}) \land \psi(\bar{b})|L_i) = \min(P'(\varphi(\bar{a})|L_i), P'(\psi(\bar{b})|L_i))
\]

for all $i$—in particular the left hand sides are well-defined. The desired equations then follow by averaging

\[
P'(\neg \varphi(\bar{a})|L_i) = 1 - P'(\varphi(\bar{a})|L_i)
\]

\[
P'(\varphi(\bar{a})|L_i) + P'(\psi(\bar{b})|L_i) = P'(\varphi(\bar{a}) \lor \psi(\bar{b})|L_i) + P'(\varphi(\bar{a}) \land \psi(\bar{b})|L_i)
\]

over $i = 1, \ldots, n$.

For (4), note that $P(\varphi(\bar{a})|K) > 0$ if and only if $P'(\varphi(\bar{a})|L_i) = 1$ for at least one $i$. If this holds, then extending $L_i$ to a $T_P$-model $M$, we obtain a $T_P$-model $M$ extending $K$ in which $\varphi(\bar{a})$ holds. Conversely, if $P'(\varphi(\bar{a})|L_i) = 0$ for all $i$, and $M$ is any $T_P$-model extending $K$, then $M$ extends some $L_i$, and so $M \models \neg \varphi(\bar{a})$.

Thus (4) holds. Condition (5) follows from (1) and (2).

Next consider the situation of (6). We can find a normal extension $L'$ of $K$ such that $L' \geq L$ and $P'(\varphi(\bar{a})|L')$ is defined for any extension of the $T_P^0$-structure to $L'$. As before, by an application of Theorem 6.10 we know that there is an integer $m$ such that every $L_i$ has exactly $m$ extensions $L'_{i,1}, \ldots, L'_{i,m}$ to a $T_P^0$-structure on $L'$. Then

\[
\frac{\sum_{i=1}^n P'(\varphi(\bar{a})|L_i)}{n} = \frac{\sum_{i=1}^n \sum_{j=1}^m P'(\varphi(\bar{a})|L'_{i,j})}{nm} = \frac{\sum_{i=1}^n \sum_{j=1}^m P'(\varphi(\bar{a})|L'_{i,j})}{nm} = P(\varphi(\bar{a})|K).
\]
Thus (6) holds. Finally, (7) is clear from the definition.

**Proposition 7.2.** Let $L/K$ be an extension of models of $T_p^0$ with the following property: for every $p \in P$, the residue field extension $\text{res} \mathcal{O}_p^L/\text{res} \mathcal{O}_p^K$ is relatively algebraically closed. Then for any formula $\varphi(\bar{a})$ with parameters $\bar{a}$ from $K$, we have

$$\mathbb{P}(\varphi(\bar{a})|L) = \mathbb{P}(\varphi(\bar{a})|K).$$

**Proof.** As in the proof of Theorem 7.1, let $\mathbb{P}'(\varphi|K)$ be 0, 1, or undefined, depending on whether $\varphi$ holds in none, all, or some of the models of $T_P$ extending $K$. Using Corollary 5.8, choose a finite normal extension $K'$ of $K$ such that $\mathbb{P}'(\varphi(\bar{a})|K')$ is defined for every $T_p^0$-structure on $K'$ extending the given structure on $K$. Let $L' = LK'$. Let $K_1', \ldots, K_n'$ enumerate the $T_p^0$-structures on $K'$ extending $K$. By Theorem 6.10, there is an integer $m$ such that for every $K_i'$, there are exactly $m$-many $T_p^0$-structures $L_{i,1}', \ldots, L_{i,m}'$ on $L'$ extending $K_i'$ and $L$. Note that $\{L_{i,j}'\}_{1 \leq i \leq n, 1 \leq j \leq m}$ is an exhaustive listing of the distinct $T_p^0$-structures on $L'$ extending $L$.

For any $i, j$, note that $\mathbb{P}'(\varphi(\bar{a})|L_{i,j}')$ is defined and equals $\mathbb{P}'(\varphi(\bar{a})|K_i')$. Indeed, if $M$ is a model of $T_P$ extending $L_{i,j}'$, then $M$ is a model of $T_P$ extending $K_i'$, so whether $M \models \varphi(\bar{a})$ must agree with $\mathbb{P}'(\varphi(\bar{a})|K_i')$. So

$$\mathbb{P}'(\varphi(\bar{a})|L_{i,j}') = \mathbb{P}'(\varphi(\bar{a})|K_i').$$

Averaging over all $i$ and $j$ immediately implies

$$\mathbb{P}(\varphi(\bar{a})|L) = \mathbb{P}(\varphi(\bar{a})|K).$$

\[\square\]

**8 Indiscernible sequences and relative closure**

**Lemma 8.1.** Let $M$ be a valued field that is a monster model of either ACVF or ACF with the trivial valuation. Let $a$ be a tuple and let

$$\ldots, b_{-1}^- b_0^- b_1^-, \ldots, \ldots, b_{-1}^- b_0^+ b_1^+, \ldots, b_{-1}^+ b_0^+ b_1^+, \ldots$$

be an $a$-indiscernible sequence of tuples, of length $3 \times \mathbb{Z}$. For $S \subseteq \mathbb{Z}$ let $b_S$ denote $\{b_i : i \in S\}$ and similarly for $b_S^+$ and $b_S^-$. Let $K_i$ and $L$ be the algebraically closed subfields of $M$ generated by $b_S^+$, $b_S^-$, and $b_S^+$, respectively. Abusing notation slightly, let $K_i(a)$ and $L(a)$ denote the perfect subfields of $M$ generated by $aK_i$ and $aL$, respectively. Then the residue field of $K_i(a)$ is relatively algebraically closed in the residue field of $L(a)$.
Proof. Without loss of generality, \( i = 0 \). Let \( k \) be the residue field of \( K_0(a) \) and \( \ell \) be the residue field of \( L(a) \). Take \( \alpha \in \ell \cap k^{alg} \setminus k \). Let \( S \) be the set of roots of the minimal polynomial of \( \alpha \) over \( k \). This is a \( K_0(a) \)-definable finite set, so it is \( F(a) \)-definable where \( F = acl(b_S^{-}b_S^{+}b_S^\ell) \), for some finite \( S \subseteq \mathbb{Z} \). Because \( \alpha \in \ell \), we can write \( \alpha \) as

\[
\alpha = \text{res} \frac{P(a,c)}{Q(a,c)}
\]

where \( P, Q \) are polynomials with integral coefficients and \( c \) is a tuple from \( L = acl(b_S^{-}b_S^{+}b_S^\ell) \). Increasing \( S \), we may assume \( c \in acl(b_S^{-}b_S^{+}b_S^\ell) \) and \( 0 \in S \). Let

\[
i_1 < \cdots < i_n < 0 < j_1 < \cdots < j_m
\]

be the elements of \( S \) in order. Note that the two sequences

\[
b_{j_m+1}, b_{j_m+2}, \ldots, b_{-3}, b_{-2}, b_{-1} \\
b_{1}, b_{2}, b_{3}, \ldots, b_{i_{n-2}}, b_{i_{n-1}}
\]

are mutually indiscernible over \( ab_S^{-}b_S^{+}b_S^\ell \), hence over \( F(a)^{alg} \). Choose \( i'_1 < \cdots < i'_n \) greater than \( j_{m} \) and \( j'_1 < \cdots < j'_m \) less than \( i_1 \). Then

\[
b_{i_1} \cdots b_{i_n} b_{j_1} \cdots b_{j_m} = T(a)^{alg} b_{i'_{1}} \cdots b_{i'_{n}} b_{j'_{1}} \cdots b_{j'_{m}}
\]

by the mutual indiscernibility. Let \( \sigma \in \text{Aut}(M/F(a)^{alg}) \) be an automorphism moving the left hand side to the right hand side. Then

\[
\sigma(b_S^{-}b_S^{+}b_S^\ell) = b_S^{-} b_{i'_{1}}^{-} \cdots b_{i'_{n}}^{-} b_{j'_{1}}^{+} \cdots b_{j'_{m}}^{+}
\]

and so

\[
\sigma(c) \in acl(b_S^{-} b_{i'_{1}}^{-} \cdots b_{i'_{n}}^{-} b_{j'_{1}}^{+} \cdots b_{j'_{m}}^{+}) \subseteq acl(b_S^{-} b_S^{+})
\]

so \( \sigma(c) \) is a tuple from \( K_0 \). Thus \( \sigma(\alpha) \) is a residue from \( K_0(a) \). Now \( \sigma \) fixes \( S \) setwise, so \( S \) intersects \( k \), a contradiction. \( \square \)

9 Finite burden

**Theorem 9.1.** Let \( N \) be the number of “leaves” in \( P \), i.e., maximal elements. Then \( T_P \) has burden no more than \( 2N \).

**Proof.** Otherwise, take a mutually indiscernible inp-pattern with \( 3 \times \mathbb{Z} \) columns and \( 2N + 1 \) rows. Let the \( i \)th row be

\[
\ldots, \varphi_i(x; b_{i_{-1}}^{-}), \varphi_i(x; b_{i_{0}}^{-}), \varphi_i(x; b_{i_{1}}^{-}), \ldots, \\
\ldots, \varphi_i(x; b_{i_{-1}}^{-}), \varphi_i(x; b_{i_{0}}^{+}), \varphi_i(x; b_{i_{1}}^{+}), \ldots,
\]

\[
\ldots, \varphi_i(x; b_{i_{-1}}^{+}), \varphi_i(x; b_{i_{0}}^{+}), \varphi_i(x; b_{i_{1}}^{+}), \ldots
\]

Let \( B^{\pm}_i \) denote the set \( \{ \ldots, b_{i_{-1}}^{\pm}, b_{i_{0}}^{\pm}, b_{i_{1}}^{\pm}, \ldots \} \).

---

\(^2\)In general, indiscernibility over \( A \) is the same thing as indiscernibility over \( acl(A) \).

\(^3\)The argument could probably be improved to get \( N \) rather than \( 2N \).
Claim 9.2. There is a mutually indiscernible array \( c_{i,j} \) of infinite tuples, such that \( c_{i,j} \) is an enumeration of \( \text{acl}(b_{i,j}B_i^+B_i^-) \).

Proof. Let \( Q = \bigcup_i (B_i^+ \cup B_i^-) \). Note that the \( b_{i,j} \) form a mutually indiscernible array over \( Q \). Take \( \hat{c}_{i,0} \) to be an enumeration of \( \text{acl}(b_{i,0}B_i^+B_i^-) \) and choose \( \hat{c}_{i,j} \) so that \( \hat{c}_{i,0}b_{i,0} \) has the same type as \( \hat{c}_{i,j}b_{i,j} \) over \( Q \). Let \( \{e_{i,j}d_{i,j}\} \) be a mutually indiscernible array over \( Q \) extracted from \( \{\hat{c}_{i,j}b_{i,j}\} \). As \( b_{i,j} \) was already mutually indiscernible over \( Q \), the array \( \{d_{i,j}\} \) has the same type as \( \{b_{i,j}\} \) over \( Q \). Choose \( \sigma \in \text{Aut}(M/Q) \) such that \( \sigma(d_{i,j}) = b_{i,j} \), and set \( \tilde{c}_{i,j} = \sigma(e_{i,j}) \).

Then the \( \{\tilde{c}_{i,j}\} \) are mutually \( Q \)-indiscernible because \( \{e_{i,j}\} \) are.

Because \( \text{tp}(\hat{c}_{i,j}b_{i,j}/Q) = \text{tp}(\hat{c}_{i,0}b_{i,0}/Q) \) for all \( j \), the same holds for the extracted array: \( \text{tp}(e_{i,j}d_{i,j}/Q) = \text{tp}(\hat{c}_{i,0}b_{i,0}/Q) \) for all \( i,j \). Therefore

\[
\tilde{c}_{i,j}b_{i,j} \equiv_Q e_{i,j}d_{i,j} \equiv_Q \hat{c}_{i,0}b_{i,0}.
\]

By choice of \( \hat{c}_{i,0} \), it follows that \( \tilde{c}_{i,j} \) is an enumeration of \( \text{acl}(b_{i,j}B_i^+B_i^-) \). \(\square\)

Fix some element \( a \) such that \( \varphi_i(a; b_{i,0}) \) holds for all \( i \).

For each \( p \in P \), consider the reduct of \( M \) to \( (K, O_q : q \leq p) \). This reduct is a model of the theory of algebraically closed fields with \( (\left\lfloor \perp, p \right\rfloor - 1) \)-many comparable valuations. This theory is an expansion of \( \text{ACVF} \) by externally definable sets (in the value group), so it has dp-rank 1.

Recall that \( N \) is the number of minimal elements in \( P \). By Lemma 4.1 in [3], we can drop no more than \( 2N \) rows and arrange that

- Each row

  \( \ldots, b_{i,0}^-; b_{i,0}; b_{i,0}^+; \ldots \)

  is \( a \)-indiscernible in every reduct \( (M, O_p) \).

- Each row

  \( \ldots, c_{i,-1}; c_{i,0}; c_{i,1}; \ldots \)

  is \( a \)-indiscernible in every reduct \( (M, O_p) \).

Since we started with \( 2N + 1 \) rows, at least one row remains. Focus on this one row, and drop the subscript \( i \)'s. We now have the following configuration:

1. The sequence

  \( \ldots, b_{-1}^-, b_0^-, b_1^-, \ldots, \)

  \( \ldots, b_{-1}, b_0, b_1, \ldots, \)

  \( \ldots, b_{1}^+, b_0^+, b_1^+, \ldots \)

  is \( a \)-indiscernible in every reduct \( (M, O_p) \).

2. The sequence

  \( \ldots, c_{-1}, c_0, c_1, \ldots \)

  is \( a \)-indiscernible in every reduct \( (M, O_p) \).
3. Each $c_i$ is an enumeration of $\text{acl}(b_Z^+b_Z^-)$.

4. The set of formulas
   $\ldots, \varphi(x; b_{-1}), \varphi(x; b_0), \varphi(x; b_1), \ldots$

   is $k$-inconsistent.

5. $\varphi(a; b_0)$ holds.

As in Lemma 8.1 let $K_i$ and $L$ be the algebraically closed subfields of $M$ generated by $b_Z^+b_Z^-$, and $b_Z^+b_Z^-$; and let $K_i(a)$ and $L(a)$ denote the perfect closures when $a$ is thrown in. Note that $c_i$ is an enumeration of $K_i$. For any $p \in P$, the $p$th residue field of $K_i(a)$ is relatively algebraically closed in $L(a)$ by Lemma 8.1 Then by Proposition 7.2

$$\mathbb{P}(\varphi(a; b_i)|K_i(a)) = \mathbb{P}(\varphi(a; b_i)|L(a)).$$

Now for any $i, j$, there is an isomorphism of multi-valued fields from $K_i(a)$ to $K_j(a)$ sending $a$ to itself and $c_i$ to $c_j$. This holds because $c_i \equiv_a c_j$ in each reduct $(M, O_p)$. It follows that

$$\mathbb{P}(\varphi(a; b_i)|L(a)) = \mathbb{P}(\varphi(a; b_0)|K_0(a)) > 0$$

for all $i$, where the inequality holds because $M \models \varphi(a; b_0)$.

Now take $N$ so large that $N \cdot \mathbb{P}(\varphi(a; b_0)|K_0(a)) > k$. Then

$$\mathbb{P}(\varphi(a; b_1)|L(a)) + \mathbb{P}(\varphi(a; b_2)|L(a)) + \cdots + \mathbb{P}(\varphi(a; b_N)|L(a)) > k$$

so by a simple probabilistic argument it follows that there is some small $M \models T_P$ extending $L(a)$ such that

$$|\{i \in \{1, \ldots, N\} : M \models \varphi(a; b_i)\}| \geq k + 1.$$

Since $L$ is algebraically closed, we can find some embedding of $M$ into $M$ over $L$. Let $a'$ be the image of this embedding. By model completeness,

$$M \models \varphi(a; b_i) \iff M \models \varphi(a'; b_i).$$

So $\varphi'(a'; b_i)$ holds for at least $k + 1$ values of $i$, contradicting $k$-inconsistency. \hfill \Box

By Example 3.4

**Corollary 9.3.** If $(K, O_1, \ldots, O_n)$ is an algebraically closed field expanded with $n$ valuation rings, the resulting structure has finite burden.
10  NIP, or lack thereof

Lemma 10.1. Let \((K, \mathcal{O}_1, \mathcal{O}_2)\) be an algebraically closed field with two independent non-trivial valuation rings. Then \((K, \mathcal{O}_1, \mathcal{O}_2)\) is not NIP, i.e., \((K, \mathcal{O}_1, \mathcal{O}_2)\) has the independence property.

Proof. Let \(p\) be a prime distinct from the characteristics of \(K\), \(\text{res} \mathcal{O}_1\), and \(\text{res} \mathcal{O}_2\). Let \(\omega \in K\) be a primitive \(p\)th root of unity. Abusing notation, we also let \(\omega\) denote its residues in \(\text{res} \mathcal{O}_1\) and \(\text{res} \mathcal{O}_2\). Let \(m_i\) denote the maximal ideal of \(\mathcal{O}_i\). For \(k \in \mathbb{Z}/p\mathbb{Z}\), let \(U_k\) and \(V_k\) denote \(\omega^k + m_1\) and \(\omega^k + m_2\). Note that the \(U_k\) are pairwise disjoint and their union is the set of \(x\) such that \(x^p \in U_0\). Similarly, the \(V_k\) are pairwise disjoint and their union is the set of \(x\) such that \(x^p \in V_0\).

Let \(W\) be the definable set \(\{x^p : x \in U_0 \cap V_0\}\). We claim that the relation
\[
\varphi(x; y) \iff x + y \in W
\]
has the independence property. Let \(\epsilon_1, \ldots, \epsilon_n\) be \(n\) distinct elements in \(m_1 \cap m_2\). Consider the affine variety \(C\) in \(n + 1\) variables \((x_1, \ldots, x_n, y)\) cut out by the equations
\[
x_i^p = y + \epsilon_i
\]

Claim 10.2. \(C\) is irreducible.

Proof. It suffices to show that the ring
\[
K[X_1, \ldots, X_n, Y]/(X_1^p - Y - \epsilon_1, X_2^p - Y - \epsilon_2, \ldots, X_n^p - Y - \epsilon_n)
\]
is an integral domain. This follows from the more general property: if \(R\) is a unique factorization domain, if \(F = \text{Frac}(R)\), if \(p \in \mathbb{N}\) is a prime distinct from \(\text{char}(F)\), if \(R\) contains a primitive \(p\)th root of unity, and if \(q_1, \ldots, q_n\) are elements of \(R\) generating distinct prime ideals, then \(S := R[X_1, \ldots, X_n]/(X_1^p - q_1, \ldots, X_n^p - q_n)\) is an integral domain. First note that \(S\) is a free \(R\)-module with basis the monomials \(X_1^{s_1} \cdots X_n^{s_n}\) with \(0 \leq s_i < p\). Therefore \(S\) injects into
\[
S' := S \otimes_R F = F[X_1, \ldots, X_n]/(X_1^p - q_1, \ldots, X_n^p - q_n).
\]
Let \(L\) be the Galois extension of \(F\) obtained by adding \(p\)th roots to \(q_1, \ldots, q_n\); this is Galois because \(R\) has the primitive \(p\)th roots of unity. Then \(S'\) and \(L\) are finite \(F\)-algebras, and there is a surjection \(S' \to L\). It suffices to show that \(\dim_F S' = [L : F]\). There is an injection \(\text{Gal}(L/F) \to (\mathbb{Z}/p\mathbb{Z})^n\) determined by the faithful action of \(\text{Gal}(L/F)\) on the \(p\)th roots of the \(q_i\). If this injection fails to be onto, we can find a nonzero vector \((s_1, \ldots, s_n) \in (\mathbb{Z}/p\mathbb{Z})^n\) complementary to the image. Then \(\text{Gal}(L/F)\) fixes \(t = \prod_{i=1}^n q_i^{s_i/p}\), so \(t \in F\). But then
\[
t^p = \prod_{i=1}^n q_i^{s_i}
\]
is a \(p\)th power in \(F\), contradicting unique factorization in \(R\), as the \(s_i\) are not all congruent to 0 modulo \(p\).
Unwinding, it follows that the image of \( \text{Gal}(L/F) \to (\mathbb{Z}/p\mathbb{Z})^n \) is all of \((\mathbb{Z}/p\mathbb{Z})^n\), so \( \text{Gal}(L/F) \) has size at least \( p^n \), so \([L:F] \geq p^n = \dim_F S'\), so \( S' \to L \) is an isomorphism, so \( S' \) is a field, so \( S \) is an integral domain. \( \square \)

For any function \( \eta : [n] \to [p] \), let \( U_\eta \) be the set of \((\vec{x},y) \in C\) such that \( x_i \in U_{\eta(i)} \) for every \( i \).

**Claim 10.3.** For any \( \eta \), the set \( U_\eta \) is non-empty.

**Proof.** Take arbitrary \( y \in 1 + m_1 \cap m_2 \). It suffices to prove that for any \( i, k \), there is an \( x_i \in U_k \) such that \( x_i^p = y + \epsilon_i \). Because \( K \) is algebraically closed, \( y \) has at least one \( p \)th root \( z \). One checks that \( z \in U_k^\prime \) for some \( k' \). Multiplying by \( \omega^{k-k'} \) yields a \( p \)th root in \( U_k^\prime \). \( \square \)

Similarly, define \( V_\eta \) to be the set of \((\vec{x},y) \in C\) such that \( x_i \in V_{\eta(i)} \) for every \( i \). Then \( V_\eta \) is likewise non-empty. Note that \( U_\eta \) and \( V_\eta \) are open subsets of \( C \) with respect to the topologies induced by \( O_1 \) and \( O_2 \), respectively. By Fact 2.1 and Claim 10.2 above, it follows that \( U_\eta \cap V_{\eta'} \neq \emptyset \) for any \( \eta, \eta' \). Now given \( S \subseteq \{1, \ldots, n\} \), choose \( \eta, \eta' \) such that \( S = \{i : \eta(i) = \eta'(i)\} \) and choose \((\vec{x},y) \in U_\eta \cap V_{\eta'}\).

**Claim 10.4.** For any \( i \),

\[
y + \epsilon_i \in W \iff \eta(i) = \eta'(i) \iff i \in S.
\]

**Proof.** First suppose \( \eta(i) = \eta'(i) = k \). Then \( x_i \in U_k \cap V_k \), so \( \omega^{-k}x_i \in U_0 \cap V_0 \). Thus \( y + \epsilon_i \) is the \( p \)th power of an element of \( U_0 \cap V_0 \), namely \( \omega^{-k}x_i \). So \( y + \epsilon_i \in W \) by definition of \( W \).

Conversely, suppose \( y + \epsilon_i \in W \). Then there is some \( z \in U_0 \cap V_0 \) such that \( z^p = y + \epsilon_i = (x_i)^p \). So \( x_i = \omega^k z \) for some \( k \in \mathbb{Z}/p\mathbb{Z} \). Then \( x_i \in U_k \cap V_k \), so \( \eta(i) = k = \eta'(i) \). \( \square \)

This last claim immediately implies that the relation

\[
\varphi(x; y) \iff x + y \in W
\]

has the independence property. \( \square \)

**Theorem 10.5.** A model \( K \models T_P \) is NIP if and only if \( P \) is totally ordered.

**Proof.** First suppose \( P \) is not totally ordered. Take two incomparable elements \( p_1, p_2 \) and let \( p_0 = p_1 \land p_2 \). Then \( O_{p_0}^K \) is the join of \( O_{p_1}^K \) and \( O_{p_2}^K \). It follows that

\[
O_{p_1}^K \div O_{p_0}^K, \quad O_{p_2}^K \div O_{p_0}^K
\]

are two independent non-trivial valuations on \( \text{res} O_{p_0}^K \). These valuation rings and the field \( O_{p_0}^K \) are interpretable, so \( K \) interprets an algebraically closed field with two independent valuations, and therefore fails NIP by the Lemma.
Conversely, suppose $P$ is totally ordered. Let $\top$ be the greatest element of $P$. Then every $\mathcal{O}_p$ is a coarsening of $\mathcal{O}_\top$. Let $\Gamma$ be the value group of $\mathcal{O}_\top$. The two-sorted structure $(K, \mathcal{O}_\top, \Gamma)$ is bi-interpretable with the C-minimal theory ACVF, hence NIP. Every convex subgroup of $\Gamma$ is externally definable. Therefore, in the Shelah expansion of $(K, \mathcal{O}_0, \Gamma)$, every convex subgroup of $\Gamma$ is definable, and every coarsening of $\mathcal{O}_\top$ is definable. Consequently, the original structure $(K, \mathcal{O}_p : p \in P)$ is interpretable in the (NIP) Shelah expansion of $(K, \mathcal{O}_\top, \Gamma)$.

\[ \square \]

**Corollary 10.6.** A structure $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$ with $K = K^{alg}$ is NIP iff the $\mathcal{O}_i$ are pairwise comparable.

## 11 Open questions

From here, there are several evident directions for potential generalization.

### 11.1 Improving the bound on burden

If $P$ is a tree with $n$ leaves, we have shown that $T_P$ has burden at most $2n$. This is probably suboptimal; the correct value should be $n$.

### 11.2 Multi-valued fields with residue structure

If the $T_i$ in Lemma 2.7 have finite burden, must the model companion then have finite burden? If so, this would give a more direct proof that $T_P$ has finite burden.

### 11.3 Forking and dividing

Can we characterize forking in the theory $T_P$? Does forking equal dividing? In the case where $P = \{\bot, 1, \ldots, n\}$, i.e., the case of $n$ independent non-trivial valuations, forking was characterized in [2] §11.6. Specifically, $A \downarrow B C$ holds in the structure $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$ if and only if $A \downarrow B C$ holds in each ACVF reduct $(K, \mathcal{O}_i)$. Moreover, forking equals dividing. It would be natural to generalize these results to the non-independent setting.

### 11.4 Real closed and $p$-adically closed fields

Chapter 11 of [2] also considered the setting of $(K, \mathcal{O}_1, \ldots, \mathcal{O}_n)$, where $K$ is real closed or $p$-adically closed and the $\mathcal{O}_i$ are independent non-trivial valuation rings, independent from the canonical topology on $K$. Under these assumptions, the structure has finite burden. It seems that one should be able to drop these independence assumptions. For example, the theory of real closed fields $(K, +, \cdot, \mathcal{O}_1, \ldots, \mathcal{O}_n)$ with $n$ valuation rings ought to have finite burden and be decidable.

The appropriate analogue of $T_P$ should be the following. Let $P$ be a non-trivial finite tree and $\rho$ be a distinguished leaf (maximal element). Define $T_{(P,\rho)}^R$ recursively as follows.
Let $P_1, \ldots, P_n$ be the branches of $P$; without loss of generality $P_1$ is the branch containing $\rho$.

- If $P_1 = \{\rho\}$, then a model of $T_{(P,\rho)}^R$ should consist of
  1. A real closed field $K$.
  2. Non-trivial valuation rings $O_2, \ldots, O_n$ on $K$, independent from each other and from the order topology on $K$. (This implies that each $\text{res}O_i$ is algebraically closed.)
  3. A $T_{P_1}$ structure on each $\text{res}O_i$.

- If $P_1$ is non-trivial, then a model of $T_{(P,\rho)}^R$ should consist of
  1. A real closed field $K$.
  2. Non-trivial independent valuation rings $O_1, \ldots, O_n$, where $(K,O_1) \models \text{RCVF}$, i.e., $O_1$ is a convex subgroup. (This ensures that $\text{res}O_1$ is real closed and $\text{res}O_i$ is algebraically closed for $i > 1$.)
  3. A $T_{(P_1,\rho)}^R$-structure on $\text{res}O_1$.
  4. A $T_{P_1}$-structure on $\text{res}O_i$ for $i > 1$.

Something similar should work for $p$-adically closed fields.

### 11.5 Bounded PRC and PpC fields

Let $T_{n,m}$ be the theory of existentially closed fields with $n$ valuations and $m$ orderings. In Chapter 11 of [2], the theory $T_{n,m}$ was shown to have finite burden. The case $T_{n,1}$ is the aforementioned real closed field with $n$ independent valuations.

The case $T_{0,m}$ of $m$ orderings and no valuations is a special case of a theorem of Montenegro. Recall that a field $K$ is bounded if it has finitely many Galois extensions of degree $d$, for every $d$. In her dissertation [3], Montenegro proved that bounded pseudo real closed (PRC) fields have finite burden. The models of $T_{0,m}$ turn out to be a subset of the bounded PRC fields.

A model of $T_{n,m}$ is probably equivalent to a model of $T_{0,m}$ with $n$ independent valuation rings, independent from the order topologies. More generally, there is work in progress by Montenegro and Rideau-Kikuchi which should show that if $K$ is a bounded pseudo-real closed field, and $O_1, \ldots, O_n$ are $n$ independent valuations on $K$, independent from all the orderings, then $(K,O_1, \ldots, O_n)$ has finite burden.

It would be natural to ask whether the independence assumption can be dropped: given a bounded PRC field $K$ and finitely many arbitrary valuation rings $O_1, \ldots, O_n$, does the resulting structure $(K,O_1, \ldots, O_n)$ have finite burden?

More generally, one can replace “PRC” with “pseudo $p$-adically closed”, or a mixture of the two, and the above discussion goes through (including the citations).
11.6 Dp-minimal fields

After [sic] \cite{3} and \cite{2} §11 were completed, dp-minimal fields were completely classified (\cite{2} §9). In the preceding discussions, can we replace RCF and pCF with other dp-minimal theories of fields? For example,

**Conjecture 11.1.** If $K$ is a dp-minimal pure field and $O_1, \ldots, O_n$ are valuation rings on $K$, then $(K, O_1, \ldots, O_n)$ has finite burden.

Recall that, up to elementary equivalence, dp-minimal fields come in three types:

1. Hahn series $F((T^\Gamma))$ where $F$ is a local field of characteristic 0 ($\mathbb{R}, \mathbb{Q}_p$, or a finite extension), and where $\Gamma$ is a dp-minimal ordered abelian group. Examples:
   \[
   \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{C}((T)), \mathbb{Q}_p(\sqrt{-1}), \mathbb{Q}_p((T))
   \]

2. Hahn series $F_{\text{alg}}(\mathbb{F}_p((T^\Gamma)))$, where $\Gamma$ is a $p$-divisible dp-minimal ordered abelian group.

3. The mixed characteristic analogue of (2).

Conjecture 11.1 seems likely when $K$ is of type (1); for types (2-3) positive characteristic may cause additional problems.

There may be some analogue of PRC and PpC for other dp-minimal complete theories of fields. One could then generalize Conjecture 11.1 to the bounded pseudo dp-minimal setting.

Also, in the dp-minimal case, Conjecture 11.1 seems plausible even when $K$ is a dp-minimal expansion of a field, because of the known compatibility between definable sets and the canonical topology.

11.7 Fields of finite dp-rank or finite burden

Since we are stepping outside inp-minimality, we may as well conjecture

**Conjecture 11.2.** If $K$ is a field of finite dp-rank, and $O_1, \ldots, O_n$ are valuation rings on $K$, then $(K, O_1, \ldots, O_n)$ has finite burden.

This is probably intractable until dp-finite fields are classified.

Assuming one can complete the analogy

real closed : pseudo real closed :: dp-finite : ?

then one would also hope for an analogue of Conjecture 11.2 in the “bounded pseudo dp-finite” setting, though “bounded” needs to be changed (dp-finite fields themselves need not be bounded!).

While we are here, we may as well make a very general conjecture:

**Conjecture 11.3.** If $K$ is a field of finite burden, possibly with extra structure, and $O$ is a valuation ring on $K$, then $(K, O)$ has finite burden.

There is no real approach to proving this, short of classifying the fields of finite burden.
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