THE CULLER-SHALEN SEMINORMS OF THE \((-2, 3, n)\) PRETZEL KNOT

THOMAS W. MATTMAN

Abstract. We show that the $\text{SL}_2(\mathbb{C})$-character variety of the \((-2, 3, n)\) pretzel knot consists of two (respectively three) algebraic curves when $3 \nmid n$ (respectively $3 \mid n$) and give an explicit calculation of the Culler-Shalen seminorms of these curves. Using this calculation, we describe the fundamental polygon and Newton polygon for these knots and give a list of Dehn surgeries yielding a manifold with finite or cyclic fundamental group. This constitutes a new proof of property P for these knots.

Introduction

Let $M = S^3 \setminus K$ denote the exterior of a hyperbolic knot $K$. In [CGLS], [CS1], and [CS2], Culler and Shalen construct a norm on the vector space $V = H_1(\partial M; \mathbb{R})$ which is a powerful tool in the study of Dehn surgery on $K$. In particular, they show that if surgery along slope $\alpha$ results in a manifold $M(\alpha)$ having cyclic fundamental group and $\alpha$ is not a boundary slope, then $\alpha$ has minimal norm. Boyer and Zhang [BZ1] extended this work by arguing that if $\alpha$ is not a boundary slope and $\pi_1(M(\alpha))$ is finite, then, again, $\alpha$ must have small norm. Taking advantage of this observation, we [BMZ] proved that the \((-2, 3, 7)\) pretzel knot admits only four finite surgeries and that its $\text{SL}_2(\mathbb{C})$-character variety consists of two algebraic curves. In the current article we generalize these results to hyperbolic pretzel knots of the form \((-2, 3, n)\).

In particular, we argue that the only examples of non-trivial cyclic or finite surgeries on such a knot are the five surgeries on the \((-2, 3, 7)\) and \((-2, 3, 9)\) pretzels found by Fintushel and Stern [FS] and Bleiler and Hodgson [BH]. Thus, we provide an alternate proof of Delman’s [Dl] result that non-trivial surgeries on these knots yield manifolds with infinite fundamental group when $n < 0$. Moreover, we extend this observation to the case $n \geq 11$. As for the number of curves in the $\text{SL}_2(\mathbb{C})$-character variety, we prove that there are two when $3 \nmid n$ and three otherwise.

The essential new ingredient is the use of the Seifert fibred surgeries of the \((-2, 3, n)\) pretzel knots. Although the Culler-Shalen norm of a slope $\alpha$ may be very large when $M(\alpha)$ is a Seifert fibred space, it is nonetheless possible to bound that norm in terms of the Seifert indices of $M(\alpha)$ (see [BH]). Similarly, the Seifert structure of the 2-fold branched cyclic cover of a pretzel knot also constrains the norm. Combining these constraints with knowledge of the boundary slopes (HO, Du) allows us to explicitly work out the Culler-Shalen norms of the hyperbolic

\textbf{Mathematics Subject Classification.} Primary 57M25, 57R65.

\textbf{Key words and phrases.} pretzel knot, Culler-Shalen seminorm, character variety, fundamental polygon, Newton polygon, Dehn surgery, A-polynomial.

Research supported in part by grants from NSERC (Canada), FCAR (Québec), and RIMS (Kyoto).
$(-2, 3, n)$ pretzel knots. Given these norms, we classify the finite and cyclic Dehn surgeries, enumerate the curves in the $SL_2(\mathbb{C})$-character variety, and construct the fundamental polygon and Newton polygon of the $A$-polynomial for these knots.

Section 1 provides some preliminary definitions and a brief review of the theory of Culler-Shalen seminorms. With these in hand, we state our results more explicitly and conclude that section with an outline of the paper.

1. Preliminaries, Results, and Outline

The $(-2, 3, n)$ Pretzel Knot. Let $K_n$ denote the $(−2, 3, n)$ pretzel knot (see Figure 1). If $n$ is even, $K_n$ is a link. Also, $K_1, K_3$ and $K_5$ are torus knots and therefore not hyperbolic. So we will assume that $n$ is an odd integer, $n \neq 1, 3, 5$.

Let $\pi$ denote the fundamental group of $M = S^3 \setminus K_n$ and $\hat{\pi}$ that of its 2-fold cover $\hat{M}$. The 2-fold branched cyclic cover will be denoted by $\Sigma_2$ and we will make strong use of the fact that $\pi_1(\Sigma_2) = \hat{\pi}/(\mu^2) = \hat{\pi}/(\hat{\mu})$ where $\mu \in \pi$ is the class of a meridian of $K_n$ and $\hat{\mu} \in \hat{\pi}$ the class of the loop in $\hat{M}$ which (double) covers that meridian. Similarly $\lambda$ will denote the class of a preferred longitude of $K_n$ and $\hat{\lambda} \in \hat{\pi}$ its lift. At the same time, $\pi_1(\Sigma_2)$ is a central extension of the triangle group $\Delta(2, 3, |n|)$. This is because $\Sigma_2$ is Seifert fibred with base orbifold $B = S^2(2, 3, |n|)$ and $\pi_1(\partial B) = \Delta(2, 3, |n|)$.

The manifold $M$ is small in the sense that it contains no closed essential surfaces [O]. Essential surfaces therefore meet $\partial M$ in a non-empty set of parallel curves each having the same slope. Slopes which can be obtained in this manner are called boundary slopes.

The Character Variety and Culler-Shalen Seminorms. We refer the reader to [CGLS Chapter 1] and [BZ2] for a more detailed exposition. In particular, we restrict ourselves here to the case of small hyperbolic knots.

Let $R = \text{Hom}(\pi, SL_2(\mathbb{C}))$ denote the set of $SL_2(\mathbb{C})$-representations of the fundamental group of $M$. Then $R$ is an affine algebraic set, as is $X$, the set of characters of representations in $R$. Since $M$ is small, the irreducible components of $X$ are curves [CGLS Proposition 2.4]. Moreover, for each component $R_i$ of $R$ which contains an irreducible representation, the corresponding curve $X_i$ induces a non-zero seminorm $\| \cdot \|_i$ on $V = H_1(\partial M; \mathbb{R})$ [BZ2 Propositon 5.7] via the following construction.

For $\gamma \in \pi$, define the regular function $I_\gamma : X \to \mathbb{C}$ by $I_\gamma(\chi_\rho) = \chi_\rho(\gamma) = \text{trace}(\rho(\gamma))$. By the Hurewicz isomorphism, a class $\gamma \in L = H_1(\partial M; \mathbb{Z})$ determines an element of $\pi_1(\partial M)$, and therefore an element of $\pi$ well-defined up to conjugacy.
The function $f_{\gamma} = I_{\gamma}^2 - 4$ is again regular and so can be pulled back to $\tilde{X}_i$, the smooth projective variety birationally equivalent to $X_i$. For $\gamma \in L$, $\|\gamma\|_i$ is the degree of $f_{\gamma} : \tilde{X}_i \to \mathbb{C}P^1$. The seminorm is extended to $V$ by linearity. We will call a seminorm constructed in this manner a Culler-Shalen seminorm.

If no $f_{\gamma}$ is constant on $X_i$, then $\|\cdot\|_i$ is in fact a norm (rather than just a seminorm) and we shall refer to $X_i$ as a norm curve. Since $M$ is hyperbolic, there is a representation into $\text{PSL}_2(\mathbb{C})$ carrying the hyperbolic structure. This lifts to an $\text{SL}_2(\mathbb{C})$-representation and the curve on which the character of this representation lies is a norm curve \cite{CGLS}. We will refer to it as the canonical curve. If $X_i$ is not a norm curve, then there is a boundary slope $r$ such that $f_r$ is constant on $X_i$. In this case, we will call $X_i$ an $r$-curve.

The minimal norm $s_i = \min\{\|\gamma\|_i : \gamma \in L, \|\gamma\|_i > 0\}$ is an even integer, as is $S = \sum_i s_i$, the sum being taken over the curves $X_i \subset X$. We will denote the sum of the Culler-Shalen seminorms by $\|v\|_T$ (here $v \in V$), i.e., $\|v\|_T = \sum_i \|v\|_i$. Since the sum includes the norm on the canonical curve, $\|\cdot\|_T$ is a norm (not just a seminorm).

The power of Culler-Shalen seminorms is illustrated by the following two theorems which relate them to finite and cyclic surgeries.

**Theorem 0.1** (Corollary 1.14 \cite{CGLS}). If $\alpha$ is not a boundary slope and $\pi_1(M(\alpha))$ is cyclic, then $\|\alpha\|_i = s_i$.

**Theorem 0.2** (Theorem 2.3 \cite{BZ1}). If $\alpha$ is not a boundary slope and $\pi_1(M(\alpha))$ is finite, then $\|\alpha\|_i \leq \max(2s_i, s_i + 8)$.

An important example is trivial surgery, i.e., surgery along the meridian $\mu$. Since $\mu$ is not a boundary slope of $K_n$ \cite{BZ2} and $M(\mu) = S^3$ has cyclic fundamental group, $\|\mu\|_i = s_i$ for each Culler-Shalen seminorm and $\|\mu\|_T = S$.

We will also be making use of the strong connection with boundary slopes. Using meridian-longitude coordinates, the slopes of $K_n$ are parameterized by $\mathbb{Q} \cup \{1/0\}$. The distance $\Delta(a/b, c/d)$ between two such slopes $a/b$ and $c/d$ is their minimal geometric intersection number $|ad - bc|$. In the case of a knot, such as $K_n$, for which $\mu$ is not a boundary slope, we can rewrite Lemma 6.2 of \cite{BZ2} as follows.

(Not that the underlying idea is implicit in \cite{CGLS} Chapter 1.)

**Lemma 0.3** (Lemma 6.2 \cite{BZ2}).

$$\|\gamma\|_i = 2\sum_j a_j^i \Delta(\gamma, \beta_j)$$

where the $a_j^i$ are non-negative integers and the sum is over the set of boundary slopes $\beta_j$.

On a norm curve $X_i$, at least two of the $a_j^i$ are non-zero. On an $r$-curve, $r = \beta$ is a boundary slope, and only the $a_j^i$ corresponding to $\beta$ is non-zero. Thus, the Culler-Shalen seminorm on an $r$-curve is of the form $\|\gamma\|_i = s_i \Delta(\gamma, r)$. Since $\mu = 1/0$ must have minimal norm $s_i$, it follows that $r$ is an integral boundary slope.

**Results and Outline of the paper.** Our results rest on Propositions \ref{prop1.3} and \ref{prop1.3}, both of which apply to more general classes of knots (with only minor changes to the proofs). Therefore, these may be of independent interest.
Proposition 1.1. Let $\tilde{\rho}_0$ be an irreducible $\text{PSL}_2(\mathbb{C})$-representation of $\tilde{\pi}$ which factors through $\Delta(2,3,|n|)$. Then $\tilde{\rho}_0$ has a unique extension to $\pi$.

This can be generalized to three-tangle Montesinos knots $m = K(a/p, b/q, c/r)$ by replacing $\Delta(2,3,|n|)$ with the triangle group $\Delta(p,q,r)$ (see [Mt1] for details).

Remark: Note that $\Delta(p,q,r)$ is the orbifold fundamental group of $B$, the base orbifold of the 2-fold branched cyclic cover of $m$. Essentially, Proposition 1.1 says the $\text{PSL}_2(\mathbb{C})$ character variety of $\pi_1^{\text{orb}}(B) = \Delta(p,q,r)$ includes into that of the knot $m$: $\tilde{X}(\pi_1^{\text{orb}}(B)) \subset \tilde{X}(m)$.

Although we cannot prove such an inclusion for more general Montesinos knots, this nonetheless suggests that the character variety of a Montesinos knot $m$ is largely determined by that of its associated orbifold $B$. In particular, note that the dimension of the $\text{SL}_2(\mathbb{R})$ character variety $X_\mathbb{R}$ of $\pi_1^{\text{orb}}(B)$ grows linearly with the number of tangles in $m$. Indeed, as a real variety, $X_\mathbb{R}$ includes the Teichmüller space of $B$ which is isomorphic to $\mathbb{R}^{2(t-3)}$ where $t$ is the number of tangles in $m$. Based on this evidence, we conjecture that the character varieties of $m$ show the same behaviour.

Conjecture 1.2. The dimensions of the $\text{PSL}_2(\mathbb{C})$- and $\text{SL}_2(\mathbb{C})$-character varieties of Montesinos knots $m = m(a_1/b_1, a_2/b_2, \ldots, a_i/b_i)$ grow linearly with the number of tangles $t$.

Proposition 1.3. The minimum of the total norm $\| \cdot \|_T$ is $S = 3(|n - 2| - 1)$.

More generally, for a $(-2, p, q)$ pretzel knot (see [Mt1]),

$$S = |pq| - (|p| + |q|) + |pq - 2(p + q)|.$$

The specific property of the $(-2, 3, n)$ pretzel knots used in our argument is the existence of two Seifert fibred surgeries at slopes $2n+4$ and $2n+5$, first observed by Bleiler and Hodgson [BH]. Using the work of Ben Abdelghani and Boyer [B], we can determine the norm of these slopes.

Proposition 1.4. The total norm of the $2n+4$ Seifert fibred surgery is $\|2n+4\|_T = S + 3(|n - 6| - 1)$.

Proposition 1.5. The total norm of the $2n+5$ Seifert fibred surgery is $\|2n+5\|_T = S + 4(|n - 5| - 2)$.

A final ingredient is the connection with boundary slopes (Lemma 0.3). The boundary slopes can be found using the methods of [FO, Dn] and with those in hand, the calculation of the seminorms comes down to determining the integers $a_j^i$ of Lemma 0.3.

A careful analysis of the possible values for the $a_j^i$ allows us to deduce our Main Theorem.

Theorem 1.6. The $\text{SL}_2(\mathbb{C})$ character variety of the hyperbolic $(-2,3,n)$ pretzel knot $K_n$ contains a curve of reducible characters and a norm curve $X_0$. If $n \mid 3$, there is in addition an $r$-curve $X_1$ with $r = 2n + 6$ and $s_1 = 2$. The Culler-Shalen norm $\| \cdot \|_0$ for the norm curve is as follows.

If $3 \nmid n$, then $s_0 = 3(|n - 2| - 1)$ and

$$\|\gamma\|_0 = 2[\Delta(\gamma,16) + 2\Delta(\gamma,\frac{n^2 - n - 5}{n - 3})] + \frac{n - 5}{2}\Delta(\gamma,2n + 6)$$
when \( n \geq 7 \) and
\[
\| \gamma \|_0 = 2[\Delta(\gamma, 10) + \frac{1 - n}{2} \Delta(\gamma, 2n + 6) + \Delta(\gamma, 2(n + 1)^2/n)]
\]
when \( n \leq -1 \).

If \( 3 \mid n \), then \( s_0 = 3|n - 2| - 5 \) and
\[
\| \gamma \|_0 = 2[\Delta(\gamma, 16) + 2\Delta(\gamma, \frac{n^2 - n - 5}{n-3}) + \frac{n - 7}{2} \Delta(\gamma, 2n + 6)]
\]
when \( n \geq 7 \) and
\[
\| \gamma \|_0 = 2[\Delta(\gamma, 10) - \frac{n + 1}{2} \Delta(\gamma, 2n + 6) + \Delta(\gamma, 2(n + 1)^2/n)]
\]
when \( n \leq -1 \).

Once we have this theorem, we can determine which slopes of \( K_n \) are candidate cyclic or finite surgery slopes as these are either of small norm, or else boundary slopes (Theorems 1.1 and 1.2). We conclude that the only non-trivial cyclic or finite surgeries on these \( K_n \) are the five on the \((-2, 3, 7)\) and \((-2, 3, 9)\) pretzel knots discovered by Fintushel and Stern (see \([FS]\)) and Bleiler and Hodgson \([BH]\). (Indeed, these are the only non-trivial finite or cyclic surgeries on any non-torus \((p, q, r)\) pretzel knot, see \([Mt2]\).) More precisely, we have

**Proposition 1.7.** If the \((-2, 3, n)\) pretzel knot \( K_n \) admits a non-trivial cyclic or finite surgery, then one of the following holds.
- \( K_n \) is torus, in which case \( n = 1, 3, \) or \( 5, \)
- \( n = 7, \) in which case \( 18 \) and \( 19 \) are cyclic fillings while \( 17 \) is a finite, non-cyclic filling, or
- \( n = 9, \) in which case \( 22 \) and \( 23 \) are finite, non-cyclic fillings.

We can also use Theorem 1.6 to determine the fundamental polygon \( B \) which is the disc of radius \( s_0 \) in \( H_1(\partial M; R) \). As \( B \) is dual to the Newton polygon \( N \) of the \( A \)-polynomial \([CCGLS]\), we can likewise describe \( N \) explicitly. This is significant as it remains difficult to calculate \( A \)-polynomials of knots.

In summary then, there are three main inputs for our approach. We use information about the two-fold branched cyclic cover and the boundary slopes of our knots. For example, these are both known for the Montesinos knots. We also take advantage of the two Seifert fibred slopes \( 2n + 4 \) and \( 2n + 5 \). We parlay this data into information about the \( \text{SL}_2(\mathbb{C}) \)-character variety and about cyclic and finite surgeries of the knot. In principle this procedure could be carried out for any Montesinos knot which admits a Seifert fibred (or finite or cyclic) surgery. Indeed, in our thesis \([Mt]\) we make a similar analysis of the twist knots and the \((-3, 3, n)\) pretzel knots (\( |n| \leq 6 \)) (see also \([BMZ]\)). This leads us to ask

**Question 1.8.** Are there other examples of Montesinos knots admitting Seifert fibred (or cyclic or finite) surgeries?

We expect that our method could be applied to such examples.

The structure of our paper is as follows. Propositions 1.1, 1.3, 1.4, and 1.5 are proved in Sections 2, 3, 4, and 5 respectively. These propositions are then used to prove our Main Theorem 1.6 in Section 6. In Section 7 we use the theorem to prove Proposition 1.7 and to describe the fundamental polygons and Newton polygons of the hyperbolic \((-2, 3, n)\) pretzel knots.
In this section we prove Proposition 1.1.

Proposition 1.1. Let $\rho_0$ be an irreducible $\text{PSL}_2(\mathbb{C})$-representation of $\tilde{\pi}$ which factors through $\Delta(2, 3, |n|)$. Then $\rho_0$ has a unique extension to $\pi$.

Notation: In general $\rho$ will denote an $\text{SL}_2(\mathbb{C})$-representation of $\pi$ and $\rho_0$ its restriction to $\tilde{\pi}$. The corresponding $\text{PSL}_2(\mathbb{C})$-representations will be denoted $\bar{\rho}$ and $\bar{\rho}_0$ respectively.

Proof: Suppose $\bar{\rho}$ and $\bar{\rho}'$ were two extensions. Let $\alpha \in \pi \setminus \tilde{\pi}$. For any $\beta \in \tilde{\pi}$, $A = \bar{\rho}(\alpha)^{-1}\bar{\rho}'(\alpha)$ commutes with $\bar{\rho}_0(\beta)$. So $A$ commutes with each element of $\bar{\rho}_0(\tilde{\pi})$. However, as $\bar{\rho}_0$ is irreducible, this implies $A = \pm I$. Thus, $\bar{\rho}$ and $\bar{\rho}'$ agree on $\alpha$ and hence on $\pi$.

Since $\pi = \tilde{\pi} \cup \mu \tilde{\pi}$, there will be an extension provided we can find $A \in \text{PSL}_2(\mathbb{C})$ with $A^2 = \pm I$ and such that $A\bar{\rho}_0(\beta)A^{-1} = \bar{\rho}_0(\mu \beta \mu^{-1})$ for all $\beta \in \tilde{\pi}$. Our goal then is to find such an $A$ corresponding to conjugation by $\mu$.

Let $\tau$ be the involution of the 2-fold branched cyclic cover $\Sigma_2$ and choose base points $\tilde{x} \in \partial \tilde{M}$ and $\tilde{x}_0 \in \text{Fix}(\tau) \subset \Sigma_2$. Then conjugation by $\mu$ corresponds to

$$\pi_1(\tilde{M}, \tilde{x}) \xrightarrow{\mu} \pi_1(\tilde{M}, \tilde{x})$$

$$[\alpha] \mapsto [\alpha]^{\mu} = [\mu \tau(\alpha)(\mu^{-1})]$$

where $\mu \tilde{x}$ is the lift of $\mu$ beginning at $\tilde{x}$ (see Figure 2).

On the other hand, restricted to the base orbifold $B = S^2(2, 3, |n|)$, $\tau$ is reflection in the equator and interchanges the hemispheres (see [Md, Théorème 1]). As we can see in Figure 3 this interchange has the effect of taking the generators of $\pi_1^{\text{orb}}(B) = \Delta(2, 3, |n|) = < a, b, b^2, (ab)^n >$ to their inverses.

Thus, if we take $\phi_0$ as the representation of $\Delta(2, 3, |n|)$ induced by $\bar{\rho}_0$, we have the following commutative diagram:
THE CULLER-SHALEN SEMINORMS OF THE \((-2,3,n)\) PRETZEL KNOT

Figure 3. $\tau$ action on $S^2(2,3,|n|)$

$x_0 = \text{base point}$

3. Proof of Proposition 1.3

In this section we prove Proposition 1.3. The minimum of the total norm $\| \cdot \|_T$ is $S = 3(|n - 2| - 1)$.

We break the proof down as a series of lemmas. Let us begin with an overview of the strategy of the proof.

Since $\mu$ is a cyclic surgery slope which is not a boundary slope, $\| \mu \|_T = S$ (Theorem 1.1.4) whence

$$S = \| \mu \|_T - 2\| \mu \|_T - \| \mu \|_T = \| \mu^2 \|_T - \| \mu \|_T.$$

As the seminorms are given by the degree of $f_\gamma$, we can determine $S$ by investigating the characters $x$ where the order of zero differs: $Z_x(f_{\mu^2}) > Z_x(f_\mu)$. (By Theorem 1.1.3, for all $x \in X$, $Z_x(f_{\mu^2}) \geq Z_x(f_\mu)$.) That is,

$$S = \| \mu^2 \|_T - \| \mu \|_T = \sum_{i \in \tilde{X}} \sum_{x \in \tilde{x}_i} Z_x(f_{\mu^2}) - Z_x(f_\mu).$$

We argue that such a “jumping point” $x$ is the character of an irreducible representation $\rho$ (Lemma 3.1). Then $\rho$ is either a binary dihedral representation, or...
else \( \rho_0 \), the restriction to \( \tilde{\pi} \) of the corresponding \( \text{PSL}_2(\mathbb{C}) \)-representation, is non-abelian. The number of irreducible dihedral characters (We will often refer to the characters of irreducible (dihedral, etc.) representations as irreducible (dihedral, etc.) characters.) is easily related to the Alexander polynomial and we find that there are \((|n| - 1)/2 \) of these (Lemma 3.3).

The non-abelian \( \rho_0 \)'s will become representations of \( \Delta(2, 3, |n|) \). Ben Abdelghani

and Boyer \cite{BR} have calculated the number of characters of such a triangle group. In

the case of \( \Delta(2, 3, |n|) \), there are \((|n| - 1)/2 \) \( \text{PSL}_2(\mathbb{C}) \)-characters. As each is

covered twice, there are \(|n| - 1 \) \( \text{SL}_2(\mathbb{C}) \)-characters (Lemma 3.4).

In total then, we have \((|n| - 1)/2 + |n| - 1 = 3(|n| - 2)/2 \) (recall that \( n \) is odd and not 1, 3, or 5) \( \text{SL}_2(\mathbb{C}) \)-characters \( x \) where \( Z_x(f_{\mu^2}) > Z_x(f_{\mu}) \). We finish the argument by showing that the difference in degree of zero is two at each of these points (Lemma 3.5).

Let us now tackle the details of the argument. Let \( x \in \tilde{X}_i \) with \( Z_x(f_{\mu^2}) > Z_x(f_{\mu}) \)

where \( X_i \) is an algebraic component of \( X \) containing an irreducible character. Let \( \nu : X_i \to X_i \) denote normalization (\cite{Sf}, Chapter II, §3). The birational map from \( \tilde{X}_i \) to \( X_i \) is regular at all but a finite number of points of \( \tilde{X}_i \), called ideal points. As in \cite{CGLS}, \( X_i \) may be identified with the complement of the ideal points in \( \tilde{X}_i \).

**Lemma 3.1.** If \( x \in \tilde{X}_i \) with \( Z_x(f_{\mu^2}) > Z_x(f_{\mu}) \), then \( \nu(x) = \chi_\rho \) is the character of an irreducible representation \( \rho \).

**Proof:** Suppose \( x \) were an ideal point. Since \( Z_x(f_{\mu^2}) > Z_x(f_{\mu}) \), \( f_{\mu^2}(x) = 0 \) so that \( x \) is not a pole of \( f_{\mu^2} \). A little algebra shows

\[
(3.1) \quad f_{\mu^2} = (f_{\mu})^2 + 4f_{\mu}.
\]

So, \( x \) is also not a pole of \( f_{\mu} \) and therefore \( I_\mu(x) \neq \infty \) as well. It follows that either \( M \) admits a closed essential surface, or else \( \mu \) is a boundary slope (see \cite{CGLS} Proposition 1.3.9)). However, since \( K_\mu \) is a Montesinos knot with less than four tangles, neither is true (see \cite{O} Section 1 and Corollary 4)).

Thus, we can assume \( x \in X_i \) and write \( \nu(x) = \chi_\rho \) with \( \rho \in R_i \). We wish to show that \( \rho \) is an irreducible representation. The idea is to show that, if \( \rho \) is reducible, then the tangent space at \( \rho \) is too small.

We can identify the Zariski tangent space at \( \rho \) with a subspace of the space of 1-cocycles \( Z^1(\pi; sl_2(\mathbb{C})_{AdP}) \) (see \cite{Gd} Section 1.2) or \cite{W} Section 3). We can see that \( R_i \) is four dimensional since, as we have mentioned, \( X_i \) is one dimensional (the knot being small) and by \cite{CS1} Corollary 1.5.3, \( \dim(R_i) = \dim(X_i) + 3 \). Thus, \( \dim(Z^1(\pi; sl_2(\mathbb{C})_{AdP})) \geq 4 \).

Now, suppose \( \rho \) were reducible. Then, by conjugating, we can take \( \rho \) to be a representation into the upper triangular matrices. Replace each matrix \( \rho(g) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \) by \( \rho_d(g) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \) to obtain a diagonal representation \( \rho_d \) with the same character \( \nu(x) \).

Since

\[
\begin{pmatrix} 1/n & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1/n & 0 \\ 0 & n \end{pmatrix}^{-1} = \begin{pmatrix} a & b/n^2 \\ 0 & a^{-1} \end{pmatrix},
\]

we have that \( \rho_d \) is a representation into upper triangular matrices. Thus, \( \rho_d \) is irreducible.
and $R_i$ is closed under conjugation [CS1, Proposition 1.1.1], we can find representations on $R_i$ arbitrarily close to $\rho_d$. But, as $R_i$ is closed, $\rho_d \in R_i$. So without loss of generality, we can assume $\rho$ is diagonal.

**Claim 3.2.** $\rho(\mu) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}$ and $\rho(\pi) \cong \mathbb{Z}/4$.

**Proof:** (of Claim) By, Equation 3.1, $Z_{\pi}(f_{\mu}) > Z_{\pi}(f_{\mu})$ implies $Z_{\pi}(f_{\mu}) = 0$ and therefore, trace$(\rho(\mu)) \neq \pm 2$. On the other hand, trace$(\rho(\mu^2))$ is $\pm 2$, so $\rho(\mu^2) = \pm I$ (we’re assuming that $\rho$ is diagonal). Then $\rho(\mu) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}$. Since $\pi$ is normally generated by $\mu$, $\rho(\pi) \cong \mathbb{Z}/4$. □ (Claim)

Given this, we can calculate $\dim(Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho}))$ directly. Using [BN, Theorem 1.1(i)], $\dim(H^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) = b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1$. (This argument is explained in more detail and in a more general context in the proof of Lemma 4.1.) We can also determine $\dim(B^1(\pi; sl_2(\mathbb{C})_{Ad\rho}))$ as we have the surjection

$$sl_2(\mathbb{C}) \rightarrow B^1(\pi; sl_2(\mathbb{C})_{Ad\rho})$$

$$A \mapsto (u_A : \gamma \mapsto A - Ad\rho(\gamma)(A)).$$

Since $\rho(\mu) = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix}$, the kernel is the one-dimensional set $\{A \in sl_2(\mathbb{C}) \mid A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\}$ while $sl_2(\mathbb{C})$ has dimension 3. Therefore, $\dim(B^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) = 2$ and

$$\dim(Z^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) = \dim(H^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) + \dim(B^1(\pi; sl_2(\mathbb{C})_{Ad\rho})) = 1 + 2 = 3.$$

This contradiction with our earlier estimate of the dimension of the cocycles shows that $\rho$ is irreducible. □

Thus, we can assume that $x$ is the character of an irreducible representation $\rho$. Then, by [CGLS, Proposition 1.5.2], $\rho(\mu^2) = \pm I$ and $\tilde{\rho}_0$, the induced $PSL_2(\mathbb{C})$-representation of $\tilde{\pi}$, will factor through $\pi_1(\Sigma_2)$. There are now two cases depending on whether or not $\tilde{\rho}_0$ is abelian.

**Lemma 3.3.** If $\tilde{\rho}_0$ is abelian, then $\rho$ has (binary) dihedral image in $SL_2(\mathbb{C})$. There are $(|n-6| - 1)/2$ jumping points $x$ of this type.

**Proof:** If $\tilde{\rho}_0$ is abelian, it factors through the finite group $H_1(\Sigma_2; \mathbb{Z})$. Thus, $\tilde{\rho}_0(\tilde{\pi})$ is cyclic and extending to $\pi$ and lifting we see that $\rho$ has binary dihedral image in $SL_2(\mathbb{C})$.

Furthermoe, any such dihedral representation will result in a jumping point. For let $\nu(x) = x_\rho$ be the character of the binary dihedral $SL_2(\mathbb{C})$-representation $\rho$. The corresponding $PSL_2(\mathbb{C})$ representation $\tilde{\rho}$ has as image a dihedral group normally generated by $\tilde{\rho}(\mu)$. Therefore, $\tilde{\rho}(\mu)$ is of order two and consequently $\rho(\mu) \neq \pm I$ while $\rho(\mu^2) = \pm I$. This implies trace$(\rho(\mu)) = 0$ so that $Z_{\pi}(f_{\mu^2}) > Z_{\pi}(f_{\mu^2})$.

The number of dihedral characters $d$ can be related to the Alexander polynomial $\Delta_K(t)$. Indeed, $d$ is equal to $|\Delta_{K_n}(t=1)|/2$. (See [K, Theorem 10] and recall that, up to conjugation, a dihedral subgroup of $SL_2(\mathbb{C})$ may be assumed to lie in $SU(2,\mathbb{C})$.)

Finally, by [Hj, Proposition 4.1] or [Hj, Theorem 1.2],

$$\Delta_K(t) = (t-1)/(t^{n+3}-1)/(t+1) + t(t^3+1)(t^n+1)/(t+1)^2,$$
and the number of jumping points is \( d = (|n - 6| - 1)/2 \).

**Lemma 3.4.** If \( \tilde{\rho}_0 \) is non-abelian, then \( \tilde{\rho}_0 \) factors through to give an irreducible representation of \( \Delta(2, 3, |n|) \). There are \( |n| - 1 \) jumping points \( x \) of this type.

**Proof:** Since \( \tilde{\rho}_0 \) is a non-abelian \( \mathrm{PSL}_2(\mathbb{C}) \)-representation of \( \pi_1(\Sigma_2) \), it factors through to give an irreducible representation of \( \Delta(2, 3, |n|) \), the fundamental group of the base orbifold of \( \Sigma_2 \). (Recall that \( \pi_1(\Sigma_2) \) is a central extension of \( \Delta(2, 3, |n|) \).)

By [BB, Proposition D], the number of \( \mathrm{PSL}_2(\mathbb{C}) \)-characters of \( \Delta(p, q, r) \) is

\[
\begin{align*}
(p - \left\lfloor \frac{p}{2} \right\rfloor - 1)(q - \left\lfloor \frac{q}{2} \right\rfloor - 1)(r - \left\lfloor \frac{r}{2} \right\rfloor - 1) + \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{r}{2} \right\rfloor + [\frac{\gcd(p, q)}{2}] + [\frac{\gcd(p, r)}{2}] + [\frac{\gcd(q, r)}{2}] + 1
\end{align*}
\]

where \( \left\lfloor x \right\rfloor \) denotes the largest integer less than or equal to \( x \). This count includes the reducible characters. As the character of a reducible representation is also the character of an abelian representation, we see that the reducible characters correspond to representations of \( H_1(\Delta(p, q, r)) = \mathbb{Z}/a \oplus \mathbb{Z}/(b/a) \) where \( a = \gcd(p, q, r) \) and \( b = \gcd(pq, pr, qr) \). So the number of reducible \( \mathrm{PSL}_2(\mathbb{C}) \)-characters of \( \Delta(p, q, r) \) is

\[
\begin{align*}
\left\lfloor \frac{p}{2} \right\rfloor + 1, & \quad \text{if } a \equiv 1 \pmod{2} \\
\left\lfloor \frac{q}{2} \right\rfloor + 2, & \quad \text{if } a \equiv 0 \pmod{2}.
\end{align*}
\]

Thus, there are \((|n| - 1)/2\) irreducible \( \mathrm{PSL}_2(\mathbb{C}) \)-characters of \( \Delta(2, 3, |n|) \). The corresponding representations each extend to an irreducible representation \( \tilde{\rho}_0 \) of \( \tilde{\pi} \). These in turn can be extended to \( \pi \) by Proposition [1.4]. Moreover, as we see from the proof of that proposition, any representation \( \tilde{\rho} \) which extends \( \tilde{\rho}_0 \) is such that \( \tilde{\rho}(\mu) \) has order two. Thus, the irreducible representations of \( \Delta(2, 3, |n|) \) all lead to jumping points where \( Z_x(f_\nu) > Z_x(f_\chi) \). As these \( \mathrm{PSL}_2(\mathbb{C}) \)-characters are covered twice in \( \mathrm{SL}_2(\mathbb{C}) \) ([BZ1, Lemma 5.5]), we have \( |n| - 1 \) \( \mathrm{SL}_2(\mathbb{C}) \) jumping points of this type.

Combining the two previous lemmas, we have \( 3(|n| - 2| - 1)/2 \) characters \( \nu(x) \) where the degree of zero jumps. We can complete the argument by showing that the jump is two and that each \( \nu(x) \) is a simple point of \( X \) (i.e., \( \nu(x) \) is on a unique irreducible component of \( X \) and smooth on that component, see [Sf, Chapter 2 §2]).

**Lemma 3.5.** At each jumping point \( x \), \( \nu(x) \) is a simple point of \( X \), and the jump is two,

\[
Z_x(f_\nu) - Z_x(f_\chi) = 2.
\]

**Proof:** We have argued that \( \nu(x) = \chi_\rho \) is the character of an irreducible representation \( \rho \). Moreover, either \( \rho \) is dihedral, or else \( \tilde{\rho}_0 \) factors through \( \Delta(2, 3, |n|) \). Dihedral representations of 2-bridge knots have been analyzed by Tanguay [Ta] and we will adopt his arguments to our situation.

For representations going through \( \Delta(2, 3, |n|) \), we plan to follow the argument of [BZ1, Section 4] (see also [BB, Theorem A]). The essential requirements in this approach are that \( \rho(\pi_1(\partial M)) \not\subset \{ \pm I \} \), and that \( \nu(x) \) is a simple point of \( X \). However, as Claim [3.6] below suggests, we will be frustrated if \( \rho \) is octahedral.

Therefore, we will break the problem into three cases: \( \tilde{\rho}_0 \) factors through the triangle group \( \Delta(2, 3, |n|) \) and \( \rho \) is not octahedral; \( \rho \) is dihedral; and \( \rho \) is octahedral.
Case 1: $\rho_0$ factors through $\Delta(2,3,|n|)$ and $\rho$ is not octahedral

To apply the method of [BZ1, Section 4], we need to show that $\rho(\pi_1(\partial M)) \not\subset \{\pm I\}$. The following claim starts us on the way.

Claim 3.6. Suppose $\rho_0$ factors through $\Delta(2,3,|n|)$. If $\rho_0(\pi_1(\partial \widetilde{M})) = \{\pm I\}$ then $\rho$ is a (binary) octahedral representation.

Proof: (of Claim) We’ve argued that $\hat{\rho}_0(I) = \rho_0(\mu^2) = \pm I$, so we are lead to investigate the image of $\check{\lambda}$, the lift of $\lambda$ to $\hat{M}$. Trotter [14] has explained how to find the image of $\check{\lambda}$ in $\Delta(p,q,r)$ in the case of a $(p,q,r)$ pretzel knot with $p,q,r$ all odd. Following the analogous procedure in our case, we find that $\check{\lambda}$ projects to $c^k a b^{k+1} b c b c \in \Delta(2,3,|n|) = \langle a, b, c, a^2, b^3, c^n, abc^{-1} \rangle$, where $k = \lfloor \frac{|n|}{2} \rfloor$.

We can take

$$\tilde{\rho}_0(c) = \pm \left( \begin{array}{cc} \omega & 0 \\ 0 & \omega^{-1} \end{array} \right), \quad \tilde{\rho}_0(b) = \pm \left( \begin{array}{cc} u & 1 \\ u(1-u) & 1-u \end{array} \right)$$

where $\omega = e^{\pi i j/|n|}$, $1 \leq j \leq \lfloor \frac{|n|}{2} \rfloor$ and $u \in \mathbb{C}$ (compare [BZ2, Example 3.2]). Since $a = cb^{-1}$ is of order two, $\text{trace}(\tilde{\rho}_0(a)) = 0$ whence $u = \omega^2/(\omega^2 - 1)$.

Now,

$$\tilde{\rho}_0(c^k a b^{k+1} b c b c) = \rho_0(c^{k+1} b^{-1} c^{k+1}) = \pm \left( \begin{array}{cc} (-1)^j(1-u)\omega & -1 \\ 1+u(u-1) & (-1)^j u/\omega \end{array} \right)$$

while

$$\tilde{\rho}_0(bcb) = \pm \left( \begin{array}{cc} \omega u^2 + (u-u^2-1)/\omega & \omega u + (1-u)/\omega \\ (u^2-u^3-u)\omega + \frac{u^3-2u^2+2u-1}{\omega} & (u-u^2-1)\omega + \frac{1-u^3}{\omega} \end{array} \right)$$

and so, after substituting $u = \omega^2/(\omega^2 - 1)$,

$$\text{trace}(\tilde{\rho}_0(\check{\lambda})) = \pm 2\left( \frac{\omega}{\omega^2-1} \right)^2 \left( \frac{\omega^6+1}{\omega^3} \right) - (-1)^j \left( \frac{\omega^4+1}{\omega^2} \right)$$

$$= \pm \frac{1}{\sin^2(\pi j/|n|)} \left( \cos(3\pi j/|n|) - (-1)^j \cos(2\pi j/|n|) \right)$$

Thus, $\text{trace}(\tilde{\rho}_0(\check{\lambda})) = \pm 2$ only if $\pm \sin^2(\pi j/|n|)$ is $\cos(5\pi j/2|n|) \cos(\pi j/2|n|)$ or $\sin(5\pi j/2|n|) \sin(\pi j/2|n|)$. The only choice consistent with our conditions on $j$ and $n$ is that $j/|n| = 1/3$.

In other words, as long as $j/|n| \neq 1/3$, we are assured that $\text{trace}(\tilde{\rho}_0(\check{\lambda})) \neq \pm 2$ and, therefore, that $\tilde{\rho}_0(\pi_1(\partial \widetilde{M})) \neq \{\pm I\}$.

On the other hand, if $j/|n| = 1/3$ (this is possible only when $3 \nmid n$), then $\rho$ is a (binary) octahedral representation. $\square$ (Claim)

By Claim 3.6, $\rho_0(\pi_1(\partial \widetilde{M})) \neq \{\pm I\}$. Then, $\rho_0(\pi_1(\partial \widetilde{M})) \not\subset \{\pm I\}$ and $\rho(\pi_1(\partial M)) \not\subset \{\pm I\}$ as well.

The other requirement of [BZ1, Section 4] is that $\nu(x)$ be simple. We first show that the corresponding character $y = \chi_{\rho_0}$ is smooth in $Y$, the character variety for $\check{\pi}$. As the Zariski tangent space at $\rho_0$ can be identified with a subspace of the cocycles, we proceed by investigating the group cohomology.

Claim 3.7. $Z^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{\Ad_{\rho_0}}) \cong Z^1(\Delta(2,3,|n|); sl_2(\mathbb{C})_{\Ad_{\rho_0}})$.

Proof: (of Claim) The Seifert structure of $\Sigma_2$ gives the exact sequence

$$0 \longrightarrow F \longrightarrow \pi_1(\Sigma_2) \xrightarrow{\phi} \Delta(2,3,|n|) \longrightarrow 1$$
where $F = \langle h \rangle \cong \mathbb{Z}$ is the group of a regular fibre $h$. The projection $\phi$ induces a homomorphism $\Phi : Z^1(\Delta(2,3,|n|); sl_2(\mathbb{C})_{Ad\rho_0}) \to Z^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\tilde{\rho}_0})$.

To construct the inverse, we show that, for each $u \in Z^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\tilde{\rho}_0})$, $u(h) = 0$. Indeed, for all $g \in \pi_1(\Sigma_2)$, $u(hg) = u(gh)$. On the other hand, a straightforward calculation shows that the cohomology of the triangle group is trivial (for example, see [Mt1, Lemma 5.1.3]).

Interpreted in this context, the proposition says “simple points of $X$ are trivial.”

Putting it together,

$$\phi : H^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\rho_0}) \cong H^1(\Delta(2,3,|n|); sl_2(\mathbb{C})_{Ad\tilde{\rho}_0}).$$

Now, since the PSL$_2(\mathbb{C})$-representation $\tilde{\rho}_0$ and the SL$_2(\mathbb{C})$-representation $\rho_0$ result in exactly the same adjoint action on $sl_2(\mathbb{C})$, we see that

$$\dim_{\mathbb{C}}(H^1(\pi_1(\Sigma_2); sl_2(\mathbb{C})_{Ad\tilde{\rho}_0})) = 0$$

as well. So we can proceed as in [BZ], Section 4] to show that $H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\tilde{\rho}_0})$ has dimension one and hence that $\tilde{\pi}$ is simple in $\mathbb{Y}$.

The following lemma allows us to relate the smoothness of $y$ to that of $\nu(x)$.

**Lemma 3.8.** Let $\rho$ be an SL$_2(\mathbb{C})$-representation of a finitely generated group $\pi$ and $\rho_0$ the restriction to a normal subgroup of finite index $\tilde{\pi}$. Then

$$\dim_{\mathbb{C}}(H^1(\pi; sl_2(\mathbb{C})_{Ad\rho_0})) \leq \dim_{\mathbb{C}}(H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\tilde{\rho}_0})).$$

**Proof:** The Lyndon - Hochschild - Serre spectral sequence gives us the exact sequence (see [R], Theorem 11.5))

$$0 \to H^1(\pi/\tilde{\pi}; (sl_2(\mathbb{C})_{Ad\rho_0})^\tilde{\pi}) \to H^1(\pi; sl_2(\mathbb{C})_{Ad\rho_0}) \to H^1(\tilde{\pi}; (sl_2(\mathbb{C})_{Ad\tilde{\rho}_0})^{\pi/\tilde{\pi}}),$$

where $A^G = \{ a \in A \mid g \cdot a = g, \forall g \in G \}$ denotes the set of fixed points of the module $A$ under the group action $G$. Now, $H^1(\pi/\tilde{\pi}; (sl_2(\mathbb{C})_{Ad\rho_0})^{\pi/\tilde{\pi}}) = 0$ since $\pi/\tilde{\pi}$ is finite and $(sl_2(\mathbb{C})_{Ad\rho_0})^{\tilde{\pi}}$ is a complex vector space. So we have

$$H^1(\pi; sl_2(\mathbb{C})_{Ad\rho_0}) \to H^1(\tilde{\pi}; sl_2(\mathbb{C})_{Ad\tilde{\rho}_0})^{\pi/\tilde{\pi}} \to H^1(\pi; sl_2(\mathbb{C})_{Ad\rho_0}).$$

In our case, the proposition shows that $\dim_{\mathbb{C}}(H^1(\pi; sl_2(\mathbb{C})_{Ad\rho_0})) \leq 1$ whence $\nu(x)$ is a smooth point of $X_i$ and a simple point of $X$.

**Remark:** We have been using the ideas of [BZ], Section 4] whereby, under appropriate conditions, $x = \chi_\rho$ is simple in $X(\pi)$ exactly when $\dim_{\mathbb{C}}H^1(\pi; sl_2(\mathbb{C})_{Ad\rho_0}) = 1$. Interpreted in this context, the proposition says “simple points of $X(\tilde{\pi})$ lift to simple points of $X(\pi)$.”
Thus, if ρ factors through Δ(2, 3, |n|) and is not octahedral, then ψ(x) is a simple point of X and ρ(π1(∂M)) ⊄ {±I}. Following the reasoning of [BZI, Section 4] (see also [BH, Theorem A]), we conclude that Zx(fu) − Zx(fv) = 2. This proves Lemma 3.5 in the case that ŋ factors through Δ(2, 3, |n|) and ρ is not octahedral.

Case 2: ρ is dihedral

Here we follow the approach outlined by Tanguay in his thesis [Ta]. Since that document remains unpublished, we give a summary of his argument. As in the previous case, we appeal to [BZI, Section 4] (or [BH, Theorem A]). Most of the work goes into proving that H1(π; sl2(ℂ)Adρ) has dimension 1 when ρ is a dihedral representation (Claim 3.9). Once this is established, [BZI, Lemma 4.5] shows that ν(x) is a simple point of X. Moreover, as the non-abelian group ρ(π) is generated by ρ(π1(∂M)), it follows that ρ(π1(∂M)) ⊄ {±I}. Lemmas 4.6 through 4.9 of [BZI] then imply Zx(fx) = 2 for each non-trivial γ ∈ L such that fx(x) = 0. In particular, we’ve already argued that ρ(μ2) = ±I (see proof of Lemma 3.3) whence fx(x) = 0, and therefore, Zx(fx) = 2. On the other hand, trace(ρ(μ)) = 0 (proof of Lemma 3.3) implies Zx(fx) = 0. Thus, ν(x) is simple and the jump at x, Zx(fx) − Zx(fv), is two.

Therefore, the following claim will suffice to prove Lemma 3.5 in the case that ρ is dihedral.

Claim 3.9. \( \dim H^1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1 \) for ρ dihedral.

Proof: (of Claim) In [BN, Theorem 1.1(i)], Boyer and Nicas explain how the dimension of the cohomology with coefficients twisted by a cyclic representation can be written in terms of certain covering spaces of the manifold. The plan is to adapt that argument to the present situation of a dihedral representation.

Let ρ(π) = D_{4m}, the binary dihedral group of order 4m. Then \( Ad\rho(\pi) \subset Aut(SL_2(\mathbb{C})) \) is isomorphic to \( D_{2m} \), the dihedral group of order 2m. In analogy with [BN, Theorem 1.1(i)], Tanguay [Ta] shows that the Betti number \( b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) \) can be related to the Betti numbers of several covers of \( M \):

\[
b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = b_1(\bar{\pi}; \mathbb{C}) - b_1(\pi; \mathbb{C}) + \frac{1}{\phi(m)} \sum_{d|m} \mu(\frac{m}{d}) b_1(\pi_d; \mathbb{C}),
\]

where the \( \pi_d \) are the kernels of the maps

\[
\pi \xrightarrow{Ad\rho} D_{2m} \to D_{2d},
\]

and \( \phi \) and \( \mu \) are the Euler and Möbius Functions respectively. Now, \( b_1(\pi; \mathbb{C}) = 1 \) [EL, Exercise 2.E.6] and \( b_1(\bar{\pi}; \mathbb{C}) = 1 \) [EL, Section 8.D]. So we can show that \( b_1(\pi; sl_2(\mathbb{C})_{Ad\rho}) = 1 \) by arguing that \( b_1(\pi_d; \mathbb{C}) = d \).

Let \( \tilde{M}_d \) be the covering of \( M \) corresponding to \( \pi_d \). Then \( \tilde{M}_d \) also covers \( \tilde{M} \) and this covering may be extended to an orbifold covering \( \Sigma_d \to \Sigma_2 \). We will argue that \( b_1(\Sigma_d) = 0 \). Then, since \( \Sigma_d \) is obtained from \( \tilde{M}_d \) by filling along \( d \) tori, \( 0 = b_1(\Sigma_d) \geq b_1(\bar{M}_d) - d \) whence \( b_1(\bar{M}_d) \leq d \). On the other hand, as \( \bar{M}_d \) has \( d \) toral boundary components, Lefschetz duality allows us to argue that \( b_1(\pi_d) = \dim(H_1(\bar{M}_d; \mathbb{C})) \geq d \). Therefore, \( b_1(\pi_d) = d \), as required.
It remains to show that \( b_1(\Sigma_d) = 0 \), and this is where we must introduce some
new ideas beyond those used by Tanguay. We have the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & E & \longrightarrow & \pi_1(\Sigma_d) & \longrightarrow & \pi_1^{\text{orb}}(B_d) & \longrightarrow & 1 \\
\downarrow & & \downarrow P' & & \downarrow P & & \downarrow P'' & & \\
0 & \longrightarrow & F & \longrightarrow & \pi_1(\Sigma_d) & \longrightarrow & \Delta(2, 3, |n|) & \longrightarrow & 1
\end{array}
\]

where the horizontal rows are the exact sequences arising from the Seifert structure
of \( \Sigma_d \) and \( \Sigma_2 \). \( E \cong F \cong \mathbb{Z} \) represent regular fibres, and \( B_d \) is the base orbifold
of \( \Sigma_d \). Now, \( \text{Im}(P) \) is normal since \( P \) is a regular cover. This implies \( \text{Im}(P'') \)
is normal. Since \( E \) and \( F \) are abelian, \( \text{Im}(P') \) is also normal. Thus, the cokernels will
be groups and we can use the Snake Lemma to obtain the exact sequence

\[
\text{ker}(P'') \xrightarrow{\delta} \text{coker}(P') \xrightarrow{\alpha} \text{coker}(P) \xrightarrow{\beta} \text{coker}(P'') \longrightarrow 1.
\]

Here, \( \text{ker}(P'') = 0 \) since \( B_d \rightarrow \Delta(2, 3, |n|) \) is an orbifold covering space. Thus, \( \alpha \)
is injective. Since \( P \) comes from the dihedral covering \( \tilde{M}_d \rightarrow \tilde{M} \rightarrow M \), we see that
\( \text{coker}(P) \cong \mathbb{Z}/d \). By the injectivity of \( \alpha \), \( \text{coker}(P') \cong \mathbb{Z}/a \) where \( a \mid d \). On the other
hand, as the degree \( d \) of the Seifert cover \( \Sigma_d \rightarrow \Sigma_2 \) is the product of the degree \( a \)
in the fibres and the degree of the orbifold cover \( c \), we see that \( \text{card}(\text{coker}(P'')) = c = d/a \). However, since \( \text{Im}(\alpha) = \text{ker}(\beta) \), \( \text{Im}(\beta) \) also has cardinality \( c \):

\[
\text{coker}(P'') = \text{Im}(\beta) \cong \text{coker}(P)/\text{ker}(\beta) \cong (\mathbb{Z}/d)/(\mathbb{Z}/a) \cong \mathbb{Z}/c.
\]

The projection \( \Delta(2, 3, |n|) \rightarrow \text{coker}(P'') \cong \mathbb{Z}/c \) is an abelian representation, and as such factors through
\( H_1(\Delta(2, 3, |n|)) = \mathbb{Z}/b \), where \( b = \gcd(3n, 6, 2n) = \gcd(3, n) = 1 \) or 3. So the covering \( B_d \rightarrow S^2(2, 3, |n|) \) is either trivial, or else of
degree 3 whence \( B_d \) is either \( S^2(2, 3, |n|) \) or else \( S^2(2, 2, 2, 3, |n|/3) \). Given \( B_d \), we have an explicit formula (see [BuZ, Equation 12.31]) for \( \pi_1(\Sigma_d) \) involving the orders
of the cone points. We can then show that \( H_1(\Sigma_d; \mathbb{Z}) \) is torsion by examining its
order ideal (see [R], Section 8.B]). Therefore, \( b_1(\Sigma_d) = 0 \) as required.

This completes the proof of the claim. \( \square \) (Claim)

**Case 3:** \( \rho \) is octahedral

The approach here is much like that used for the dihedral representations. In this
case,

\[
b_1(\pi; sl_2(\mathbb{C}))_{A_{d6}} = b_1(\tilde{\pi}; \mathbb{C}) - b_1(\pi; \mathbb{C})
\]

where \( \tilde{\pi} = \rho^{-1}(D_6) \) with \( D_6 \) a dihedral subgroup of index four in the octahedral
group \( S_4 \). Of course \( b_1(\pi; \mathbb{C}) = 1 \) as before, so we will want to argue that \( b_1(\tilde{\pi}; \mathbb{C}) = 2 \).

Let \( \tilde{M} \) be the covering of \( M \) corresponding to \( D_6 \). Then \( \tilde{M} \) is an irregular
covering of degree 4 which also covers \( M \). As before, we extend the covering \( \tilde{M} \rightarrow \tilde{M} \) to a degree two map between Seifert spaces: \( \tilde{\Sigma} \rightarrow \Sigma_2 \). This leads to a diagram
quite similar to that for the dihedral representation:

\[
\begin{array}{cccccc}
0 & \longrightarrow & E & \longrightarrow & \pi_1(\tilde{\Sigma}) & \longrightarrow & \pi_1^{\text{orb}}(\tilde{B}) & \longrightarrow & 1 \\
\downarrow & & \downarrow P' & & \downarrow P & & \downarrow P'' & & \\
0 & \longrightarrow & F & \longrightarrow & \pi_1(\Sigma_2) & \longrightarrow & \Delta(2, 3, |n|) & \longrightarrow & 1
\end{array}
\]
In this case, \( \hat{B}, \) the base orbifold of \( \hat{\Sigma}, \) is either a \( 1 - 1 \) or a \( 2 - 1 \) cover of \( S^2(2,3,|n|). \) In particular it is a regular cover and \( \text{coker}(P') \) is either trivial or cyclic of order 2. Since \( \text{coker}(P') \) is abelian, it is also a factor of \( H_1(\Delta(2,3,|n|)) \cong \mathbb{Z}/b \) where \( b = \gcd(3, n) = 1 \) or 3. Thus, \( \hat{B} = S^2(2,3,|n|) \) as well and, as in the dihedral case, we find \( b_1(\hat{\Sigma}) = 0. \) Since \( \hat{M} \) has two boundary components, we may now argue that \( b_1(\hat{\Sigma}; C) = 2. \) Thus, \( b_1(\pi; s\ell_2(\mathbb{C})_{Ad\rho}) = 1 \) and \( \nu(x) \) is again a smooth point of \( X_i \) (and a simple point of \( X \)) yielding a jump of two.

This completes the proof of Lemma 3.3 and with it, the proof of Proposition 1.3.

4. PROOF OF PROPOSITION 1.4

In this section, we prove

**Proposition 1.4.** The total norm of the \( 2n+4 \) Seifert fibred surgery is \( \|2n+4\|_T = S + 3(|n - 6| - 1). \)

**Proof:** Bleiler and Hodgson [BH Proposition 17] have shown that \( 2n + 4 \) surgery on the \( (-2,3,n) \) pretzel knot \( K_n \) results in a manifold which is Seifert fibred over \( S^2(2,4,|n-6|). \) (Actually, there is a small error in the statement of their proposition which refers to “\( 4n + 14 \) surgery on the \( (-2,3,2n+7) \) pretzel knot.” It should read “\( 4n+18 \) surgery on the \( (-2,3,2n+7) \) pretzel knot.”) We can find the total Culler-Shalen norm of this slope in much the same way as we calculated \( S \) above.

Recall that

\[
\|2n + 4\|_i = \sum_{x \in X_i} Z_x(f_{2n+4})
\]

where \( Z_x(f) \) denotes the order of zero of \( f \) at \( x. \) Since the meridian \( \mu \) of \( K_n \) is not a boundary slope, \( Z_x(f_\mu) \leq Z_x(f_{2n+4}) \) for each \( x \) ([CGLS, Proposition 1.1.3]). This suggests that we approach the calculation of the total norm \( \|2n + 4\|_T \) by comparison with \( \|\mu\|_T = S : 

\[
(4.4) \quad \|2n + 4\|_T = S + \sum_i \sum_{x \in X_i} (Z_x(f_{2n+4}) - Z_x(f_\mu)).
\]

We first show that the “jumping points” \( x \), where \( Z_x(f_\mu) < Z_x(f_{2n+4}) \), are characters of irreducible representations (Lemma 4.1). We next show that there are \( 3(|n - 6| - 1)/2 \) such characters (Lemma 4.3) and finally that the jump at each such character is two (BH, Theorem A).

**Lemma 4.1.** If \( x \in \tilde{X}_i \) with \( Z_x(f_{2n+4}) > Z_x(f_\mu) \), then \( \nu(x) = \chi_\rho \) is the character of an irreducible representation \( \rho. \)

**Proof:** Since \( M \) is small and \( 2n + 4 \) is not a strict boundary class, we may apply [CGLS, Proposition 1.6.1] to see that \( Z_x(f_{2n+4}) = Z_x(f_\mu) \) at ideal points. So we can assume \( x \in X'_i. \) Then \( \nu(x) = \chi_\rho \) is the character of a representation \( \rho \in R_i. \) We wish to show that \( \rho \) is an irreducible representation. The idea is to show that, if \( \rho \) is reducible, then the tangent space (which is a subspace of the space of 1-cocycles of \( \pi \) by [CGLS, Section 1.2] or [W, Section 3]) is too small.

As in the proof of Lemma 3.1, we can argue that the dimension of \( Z^1(\pi; s\ell_2(\mathbb{C})_{Ad\rho}) \) is at least 4.

On the other hand, if \( \rho \) were reducible, then, since \( R_i \) is closed and invariant under conjugation, we can assume that \( \rho \) is diagonal. Now, \( Z_x(f_{2n+4}) > Z_x(f_\mu) \)
implies \( \rho(2n+4) = \pm I \) (\cite[Proposition 1.5.4]{GLS}). So, if we take \( \bar{\rho} \) as the PSL\(_2(\mathbb{C})\)-representation corresponding to \( \rho \), then \( \bar{\rho} \) factors through \( H_1(M(2n+4); \mathbb{Z}) \cong \mathbb{Z}/(2n+4) \). Since \( \rho(\pi) \) is normally generated by \( \rho(\mu) \) and \( \rho \) is diagonal, we see that

\[
\rho(\mu) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}
\]

with \( \eta^{2(2n+4)} = 1 \).

**Claim 4.2.** \( b_1(\pi; sl_2(\mathbb{C})_{\text{Ad}\rho}) = 1 \).

**Proof:** (of Claim) By \cite[Theorem 1.1(i)]{BN},

\[
b_1(\pi; sl_2(\mathbb{C})_{\text{Ad}\rho}) = b_1(\pi; \mathbb{C}) + 2b_1(\pi; \mathbb{C}_\beta)
\]

where \( \beta = \eta^2 \) is a \((2n+4)\)th root of unity. Of course \cite[Exercise 2.E.6]{BN}, \( b_1(\pi; \mathbb{C}) = 1 \). Now, \( \mathbb{C}_\beta \in \mathbb{C} \) with the \( \mathbb{Z} \)-action induced by \( t \cdot c = c\beta \) where \( t \) is a generator of \( \mathbb{Z} \). Since \( \pi \) surjects onto \( H_1(M; \mathbb{Z}) \cong \mathbb{Z} \), this gives a \( \pi \)-action on \( \mathbb{C} \).

We can also think of \( H_1(M; \mathbb{Z}) \) as acting on \( \tilde{M} \), the infinite cyclic cover of \( M \), and define a \( \mathbb{C}[t, t^{-1}] \)-module structure on \( H_1(\tilde{M}; \mathbb{C}) \) (see \cite[Section 7.A]{R}). In this context \( H^1(\pi(\tilde{M}); \mathbb{C}_\beta) = \ker(H^1(\tilde{M}; \mathbb{C}) \otimes \mathbb{C} \beta \rightarrow H^1(\tilde{M}; \mathbb{C})) \) where \( t - \beta \) represents multiplication by \( t - \beta \). Since the Alexander polynomial \( \Delta_{K_\beta}(t) \) is the generator of \( H^1(\tilde{M}; \mathbb{C}) \) as a \( \mathbb{C}[t, t^{-1}] \)-module, we can argue that \( \ker(t - \beta) = 0 \) if \( \Delta_{K_\beta}(\beta) \neq 0 \).

It is not difficult to show that \( \Delta_{K_\beta}(t) \) admits no roots which are \( 2n+4 \)th roots of unity (see Lemma \cite{J}). Thus \( H^1(\pi; \mathbb{C}_\beta) = 0 \) and \( b_1(\pi; \text{sl}_2(\mathbb{C})_{\text{Ad}\rho}) = 1 \). \( \square \)(Claim)

We can now argue as in the proof of Lemma \cite{J} that \( \dim(B^1(\pi; \text{sl}_2(\mathbb{C})_{\text{Ad}\rho})) = 2 \) and \( \dim(\text{Z}(\pi; \text{sl}_2(\mathbb{C})_{\text{Ad}\rho})) = 3 \). This contradicts our earlier estimate for the dimension of the cocycles and we conclude that \( \rho \) is an irreducible representation. \( \square \)

**Lemma 4.3.** There are \( 3|n-6| - 1 \) jumping points \( x \) where \( Z_x(f_{2n+4}) > Z_x(f_{\mu}) \). Moreover, at each jumping point, \( \nu(x) \) is a simple point of \( X \).

**Proof:** The plan is to argue that such a jumping point \( \nu(x) = \chi_{\rho} \) gives rise to a PSL\(_2(\mathbb{C})\)-representation \( \bar{\rho} \) which factors through \( \Delta(2, 4, |n-6|) \). Conversely, every such \( \bar{\rho} \) factoring through \( \Delta(2, 4, |n-6|) \) leads to a jumping point \( \nu(x) \). Since the \( \nu(x) \) are simple (and therefore each lies on only one curve in \( X \)), we can count the number of jumping points simply by counting the number of PSL\(_2(\mathbb{C})\)-representations of \( \Delta(2, 4, |n-6|) \). Here are the details:

By \cite[Proposition 1.5.4]{GLS}, \( Z_x(f_{2n+4}) > Z_x(f_{\mu}) \) implies \( \rho(2n+4) = \pm I \). Therefore, the corresponding PSL\(_2(\mathbb{C})\) representation \( \rho \) factors through \( \pi_1(M(2n+4)) \). As this is an irreducible PSL\(_2(\mathbb{C})\)-representation (and therefore either abelian or else with image \( \mathbb{Z}/2 \times \mathbb{Z}/2 \)) it must kill the center of \( \pi_1(M(2n+4)) \) and factor through to give an irreducible PSL\(_2(\mathbb{C})\)-representation \( \bar{\rho}' \) of \( \Delta(2, 4, |n-6|) \), the base orbifold of \( M(2n+4) \).

Now, as in the proof of Lemma \cite{J},

\[
H^1(\pi_1(M(2n+4)); \text{sl}_2(\mathbb{C})_{\text{Ad}\rho}) \cong H^1(\Delta(2, 4, |n-6|); \text{sl}_2(\mathbb{C})_{\text{Ad}\bar{\rho}'})
\]

is trivial. Thus, arguing as in \cite[Section 4]{BZ}, we can deduce that \( \nu(x) \) is a smooth point of \( X_1 \) (and in fact a simple point of \( X \)) so that \( \nu^{-1}(\nu(x)) = x \). Therefore, the jumping points \( \nu(x) = \chi_{\rho} \) where \( Z_x(f_{2n+4}) > Z_x(f_{\mu}) \) are simple points of \( X \) and correspond to irreducible PSL\(_2(\mathbb{C})\) characters \( \rho \) which factor through \( \Delta(2, 4, |n-6|) \).

Conversely, any such representation induces a jumping point. This is immediate if the representation is diagonalizable on \( \pi_1(\partial M) \) since then \( \rho(\mu) \) is of finite order,
but not ±1. On the other hand, ρ(2n + 4) = ±1. Thus Zx(f_{2n+4}) > 0 = Zx(f_μ).
If ρ(π_1(∂M)) is parabolic, we can appeal to [BH, Theorem A].

So, to find the number of jumping points, we must count the irreducible $PSL_2(C)$-characters of $\Delta(2, 4, |n - 6|)$. By Equations 3.4 and 3.3, there are $|n-6|-1$ such. Since $d = (\Delta,K_0(-1)-1)/2 = (|n-6|-1)/2$ (see Proof of Lemma 3.3), half of these are dihedral characters. Now, dihedral characters are covered once in $SL_2(C)$ while other characters are covered twice ([BZ], Lemma 5.5). Thus we have $3(|n-6|-1)/2$ $SL_2(C)$-characters coming from irreducible $PSL_2(C)$-representations which factor through $\Delta(2, 4, |n - 6|)$. As we have argued, this is the number of jumping points where $Z_x(f_{2n+4}) > Z_x(f_μ)$. □

To complete the proof of Proposition 1.4, we use [BH, Theorem A] which says that the jump is two at each jumping point. So in Equation 4.4 we have $3(|n-6|-1)/2$ jumping points, each providing a jump of two. We conclude that $\|2n+4\|_T = S + 3(|n-6|-1)$ and Proposition 1.4 is proved. □

5. Proof of Proposition 1.5

In this section, we prove

**Proposition 1.5.** The total norm of the $2n+5$ Seifert fibred surgery is $\|2n+5\|_T = S + 4(|n-5|-2)$.

**Proof:** Bleiler and Hodgson [BH, Proposition 16] have shown that $2n+5$ surgery on the $(-2,3,n)$ pretzel knot $K_n$ results in a manifold which is Seifert fibred over $S^2(3,5,|n-5|/2)$ The argument here is essentially identical to that in the previous section. In particular,

\[
\|2n+5\|_T = S + \sum_i \sum_{x \in X_i} (Z_x(f_{2n+5}) - Z_x(f_μ)).
\]

The main difference is that now $x$ may be the character of a reducible representation. The argument of Lemma 4.3 depended on the Alexander polynomial having no zeroes at a $(2n+4)$th root of unity. However, it may have a zero at a $(2n+5)$th root of unity. To clarify this point, we begin by investigating the roots of the Alexander polynomial.

**Lemma 5.1.** Let $\Delta_{K_n}(t)$ be the Alexander polynomial of the $(-2,3,n)$ pretzel knot $K_n$. (In particular, $n$ is odd.) Suppose $\Delta_{K_n}(\zeta) = 0$ where $\zeta$ is a primitive $m$th root of unity. Then one of the following is true.

- 3 | $n$ and $m = 6$.
- 10 | $(n-1)$ and $m = 10$.
- 12 | $(n-3)$ and $m = 12$.
- 15 | $(n-5)$ and $m = 15$.

**Proof:** The Alexander polynomial never admits zeroes which are prime power roots of unity. Indeed, by [BuZ, Theorem 8.21], $H_1(Σ_m; Z)$ is finite iff no root of the Alexander polynomial is an $m$th root of unity. Here $Σ_m$ denotes the $m$-fold branched cyclic cover of the knot (see [R], Section 10.C]). Using the Milnor [M] sequence we can show that $b_1(Σ_m) = 0$ whenever $m$ is a prime power.

It is straightforward to argue that there are no zeroes at $m$th roots of unity when $m \geq 18$. This leaves $m = 6, 10, 12, 14,$ and $15$. Since the value of $\Delta_{K_n}(\zeta)$ depends only on the value of $n \mod m$, one need only make the calculation for each of the
$m$ equivalence classes to verify that there are no roots when $m = 14$, and roots in the other cases ($m = 6, 10, 12, 15$) only as given in the statement of the lemma.

In particular, if $\Delta_K(t)$ admits a $2n + 5$ root of unity, then $m \mid 2n + 5$ for one of $m = 6, 10, 12$ or 15. However, if $6 \mid 2n + 5$, then $3 \nmid n$. Similarly, $m = 10$ and $m = 12$ are not feasible. However, if $m = 15$, we must have $15 \mid 2n + 5$ and $15 \mid (n - 5)$ which implies $n \equiv 5 \pmod{30}$. Indeed, one can verify that when $n \equiv 5 \pmod{30}$, the Alexander polynomial admits primitive 15th roots of unity as zeroes and that these are simple (i.e., not repeated) zeroes.

So, we are left to examine two cases.

Case 1: $n \not\equiv 5 \pmod{30}$

In this case, the argument is identical to that of Proposition 4.4, so we will omit most of the details. The jumping points correspond to irreducible $\text{PSL}_2(\mathbb{C})$-characters of $\Delta(3, 5, |n - 5|/2)$, the base orbifold of $M(2n + 5)$. By Equations 3.2 and 3.3, there are $|n - 5| - 2$ such. Since $H_1(M(2n + 5); \mathbb{Z}) \cong \mathbb{Z}/(2n + 5)$ has odd order, none of these are dihedral characters. Therefore, they are covered twice in $\text{SL}_2(\mathbb{C})$ (\cite[Lemma 5.5]{BZ1}). Since each contributes a jump of two (\cite[Theorem A]{BB}), we have proved the proposition in this case.

Case 2: $n \equiv 5 \pmod{30}$

As in the proof of Lemma 4.1, we can apply \cite[Proposition 1.6.1]{CGLS} to deduce that at a jumping point $x$ (i.e., where $\text{Z}_s(f(2n + 5)) = \text{Z}_s(f_n)$), $\nu(x) = \chi_\rho$ is the character of a representation $\rho$. The difficulty is that now $\rho$ may be reducible. We will consider irreducible and reducible jumping points separately.

**Lemma 5.2.** There are $2(|n - 5| - 6)$ jumping points where $\nu(x)$ is the character of an irreducible representation. These are simple points of $X$ and each contributes a jump of two.

**Proof:** If $\nu(x) = \chi_\rho$ is the character of an irreducible representation, we can follow the arguments of the proof of Lemma 4.1 to see that $x$ is a simple point of $X$, and that the corresponding $\text{PSL}_2(\mathbb{C})$-representation $\tilde{\rho}$ factors through $\Delta(3, 5, |n - 5|/2)$, the group of the base orbifold of $M(2n + 5)$. Moreover, every such $\text{PSL}_2(\mathbb{C})$-representation leads to a jumping point.

By Equations 3.2 and 3.3, there are $|n - 5| - 6$ $\text{PSL}_2(\mathbb{C})$-characters of $\Delta(3, 5, |n - 5|/2)$. As $M(2n+5)$ has odd degree homology, none of these are dihedral characters, and they are therefore covered twice in $\text{SL}_2(\mathbb{C})$. Finally, \cite[Theorem A]{BB} shows that the jump is two at each of these characters.

**Lemma 5.3.** There are 8 jumping points where $\nu(x)$ is the character of a reducible representation. These are simple points of $X$ and each contributes a jump of two.

**Proof:** If we compare the current case, $n \equiv 5 \pmod{30}$, with the previous case, $n \not\equiv 5 \pmod{30}$, we see that there are four fewer irreducible $\text{PSL}_2(\mathbb{C})$-characters of $\Delta(3, 5, |n - 5|/2)$ ($|n - 5| - 6$ versus $|n - 5| - 2$). On the other hand, we now have 4 reducible characters corresponding to representations with image $\mathbb{Z}/15$. The strategy is to show that these also yield jumping points.

Since $n \equiv 5 \pmod{30}$, then $|n - 5|/2 = 15k$ and $H_1(\Delta(2, 3, |n|)) \cong \mathbb{Z}/15$. As we have mentioned, $\Delta_K(t)$ admits primitive 15th roots of unity as zeroes and they are simple zeroes of $\Delta_K(t)$. 


Let $\xi = e^{2\pi ji/15}$ be a primitive 15th root of unity and let $\rho$ be the reducible $\text{SL}_2(\mathbb{C})$ representation of $\pi$ induced by

$$\rho(\mu) = \begin{pmatrix} e^{\pi ji/15} & 0 \\ 0 & e^{-\pi ji/15} \end{pmatrix}.$$

Then $\rho(\mu)^{15} = \pm I$ and $\rho$ is a cover of one of the reducible $\text{PSL}_2(\mathbb{C})$ representations of $\Delta(2,3,|n|)$ projecting onto $\mathbb{Z}/15$. In other words, we can think of $\rho$ as a representation of $\pi$ which factors through $M(2n + 5)$. Corresponding to the eight primitive 15th roots of unity, we have eight $\text{SL}_2(\mathbb{C})$ characters. We will show that the jump at each of these characters is two.

Frohman and Klassen [FK, Theorem 1.1] show that such a representation $\rho$ is the endpoint of an arc of irreducible representations. So $\rho \in R_s$, a component of the $\text{SL}_2(\mathbb{C})$-representation variety containing an irreducible representation. The corresponding character $x = \chi_\rho$ lies on the curve $X_t = t(R_s)$.

Since $\xi = e^{2\pi ji/15}$ is a primitive 15th root of unity, $\chi_\rho(\mu) = 2 \cos(\pi ji/15) = \pm 2$. So $Z_x(f_{\mu}) = 0$, $x$ is a non-trivial character, and, moreover, $x(\pi_1(\partial M)) \neq \{ \pm 2 \}$. (A character is trivial if $\chi(\pi) \subset \{ \pm 2 \}$.) See [FK, Section 3.2].) On the other hand, since $\rho$ factors through $M(2n + 5)$, $\rho(2n + 5) = I$ and $Z_x(f_{2n+5}) > 0$. So $Z_x(f_{2n+5}) > Z_x(f_\mu)$ and there is a jump at $x$.

Now, Proposition 1.5.2 of [CGLS] shows that there is a non-abelian representation $\rho' \in R_t$ with character $x$ and $\rho'(2n + 5) = \pm I$. Since $x(\pi_1(\partial M)) \neq \{ \pm 2 \}$, we see that $\rho'(\pi_1(\partial M)) \nsubseteq \{ \pm I \}$. Finally, as in the proof of Lemma 4.1, we can argue that $H^1(\pi_1(M(2n+5)); \text{SL}_2(\mathbb{C})_{Ad_{\rho'}}) = 0$. These are then simple points of $X$ ([FK, Section 4]) and each provides a jump, $Z_x(f_{2n+5}) - Z_x(f_\mu)$, of two ([BR, Theorem A]).

Combining Lemmas 5.2 and 5.3 we see that we have also proved Proposition 1.3 in the case $n \equiv 5 \pmod{30}$. This completes the proof of the proposition.

### 6. Proof of Main Theorem

In this section we prove

**Theorem 1.6.** The $\text{SL}_2(\mathbb{C})$ character variety of the hyperbolic $(-2,3,n)$ pretzel knot $K_n$ contains a curve of reducible characters and a norm curve $X_0$. If $n \not|\ 3$, there is in addition an $r$-curve $X_1$ with $r = 2n + 6$ and $s_1 = 2$. The Culler-Shalen norm $\| \cdot \|_0$ for the norm curve is as follows.

If $3 \nmid n$, then $s_0 = 3(\lfloor n/2 \rfloor - 1)$ and

$$\| \gamma \|_0 = 2[\Delta(\gamma, 16) + 2\Delta(\gamma, \frac{n^2 - n - 5}{n-3}) + \frac{n-5}{2}\Delta(\gamma, 2n + 6)]$$

when $n \geq 7$ and

$$\| \gamma \|_0 = 2[\Delta(\gamma, 10) + \frac{1-n}{2}\Delta(\gamma, 2n + 6) + \Delta(\gamma, 2(n+1)^2/n)]$$

when $n \leq -1$.

If $3 \mid n$, then $s_0 = 3(n/2 - 5)$ and

$$\| \gamma \|_0 = 2[\Delta(\gamma, 16) + 2\Delta(\gamma, \frac{n^2 - n - 5}{n-3}) + \frac{n-7}{2}\Delta(\gamma, 2n + 6)]$$
when \( n \geq 7 \) and
\[
\|\gamma\|_0 = 2[\Delta(\gamma, 10) - \frac{n+1}{2}\Delta(\gamma, 2n+6) + \Delta(\gamma, 2(n+1)^2/n)]
\]
when \( n \leq -1 \).

**Proof:** By Lemma 14.3, the Culler-Shalen seminorms can be written in terms of the boundary slopes \( \beta_j \). Using the methods of [IC, DS] there are four boundary slopes:\n\( \beta_1 = 0, \beta_2 = 2n+6, \beta_3 = 16 \) (respectively 10), and \( \beta_4 = \frac{n^2 - n - 5}{2} \) (respectively \( 2(n+1)^2/n \)) when \( n \geq 7 \) (respectively \( n \leq -1 \)). Thus,
\[
\|\gamma\|_i = 2[a\Delta(\gamma, \beta_1) + a^2\Delta(\gamma, \beta_2) + a^3\Delta(\gamma, \beta_3) + a^4\Delta(\gamma, \beta_4)],
\]
and finding the seminorms comes down to determining the non-negative integers \( a_j \) (\( j = 1, 2, 3, 4 \)). We will frequently omit the \( i \) super- and subscripts in the following.

Propositions 1.3, 1.4, and 1.5 imply the following inequalities for each Culler-Shalen seminorm \( \| \cdot \| \).

\[
\|\mu\| = s \leq 3|\|n-2\| - 1|
\]
(6.6)
\[
s \leq \|2n+4\| \leq s + 3|\|n-6\| - 1\| \text{ and } 
\]
\[
s \leq \|2n+5\| \leq s + 4|\|n-5\| - 2\|.
\]

**Lemma 6.1.** If \( n \geq 7 \) or \( n \leq -11 \) there is exactly one norm curve \( X_0 \) in the character variety \( X \).

If \( n \geq 7 \), the Culler-Shalen norm on \( X_0 \) is given by the coefficients \( a_1 = 0 \), \( a_3 = 1 \), \( a_4 = 2 \), and \( 0 \leq a_2 \leq (n - 5)/2 \).

If \( n \leq -11 \), the set of coefficients is one of the following three types.

1. \( a_1 = 0, a_3 = a_4 = 1 \) and \( 0 \leq a_2 \leq (1 - n)/2 \).
2. \( a_1 = a_4 = 1, a_3 = 0 \) and \( 0 \leq a_2 \leq (1 - n)/2 \).
3. \( a_1 = 1, a_3 = a_4 = 0 \) and \( 0 < a_2 \leq (n + 25)/2 \).

Moreover, Type 3 can occur only if \( n \geq -23 \).

**Proof:** First, suppose \( n \geq 7 \). Then Equations 6.6 become

(6.7) \[
2[a_1 + a_2 + a_3 + \frac{n-3}{2}a_4] = s \leq 3(n-3);
\]

(6.8) \[
s \leq 2[(2n+4)a_1 + 2a_2 + (2n-12)a_3 + a_4] \leq s + 3(n-7) \text{ and }
\]

(6.9) \[
s \leq 2[(2n+5)a_1 + a_2 + (2n-11)a_3 + \frac{n-5}{2}a_4] \leq s + 4(n-7).
\]

It will be useful to subtract \( s \) from each of the last two equations:

(6.10) \[
0 \leq (2n+3)a_1 + a_2 + (2n-13)a_3 - \frac{n-5}{2}a_4 \leq 3(n-7)/2,
\]

(6.11) \[
0 \leq (2n+3)a_1 + (2n-12)a_3 - a_4 \leq 2(n-7)
\]

Since \( a_i \geq 0 \), Equation 6.7 implies \( a_4 \leq 3 \). Moreover, in order to have a norm (rather than a seminorm), we would need at least two of the \( a_i \) to be greater than 0. This condition further restricts \( a_4 \leq 2 \).

Given \( a_4 \leq 2 \), Equation 6.11 implies \( (2n+3)a_1 \leq 2(n-6) \) so that \( a_1 = 0 \). Then, the same equation implies \( a_3 \leq 1 \). We will argue that, in fact, \( a_4 = 2 \) and \( a_3 = 1 \).

Suppose instead that \( a_4 \leq 1 \). Since \( a_1 = 0 \), Equation 6.11 becomes \( (2n-12)a_3 \leq 2n-13 \) so that \( a_3 = 0 \). This is a contradiction since if \( a_1 \) and \( a_3 \) are both zero,
then Equation 6.11 in fact says \( a_4 = 0 \) as well, and only \( a_2 \) is non-zero. This would mean that \( \| \cdot \| \) is not a norm.

Therefore, \( a_4 = 2 \). Since \( a_1 = 0 \), Equation 6.11 implies that \( a_3 > 0 \). Thus \( a_3 = 1 \). Finally, given these values, Equation 6.10 can be rearranged to see that \( 0 \leq a_2 \leq (n - 5)/2 \). This implies \( s = 2n - 4 + 2a_2 \), \( \|2n + 4\| = s + 2(n - 8) + 2a_2 \) and \( \|2n + 5\| = s + 4(n - 7) \).

Suppose there were two norm curves, \( X_1 \) and \( X_2 \). Then each would have norm as described in the previous paragraph. In particular \( s_1, s_2 \geq 2n - 4 \). But then \( S \geq s_1 + s_2 > 3(|n - 2| - 1) = 3(n - 3) \) which contradicts Proposition 1.3. Therefore, there can be at most one norm curve. On the other hand, since the \((-2,3,n)\) pretzel knot is hyperbolic, we know that there is a norm curve in its character variety, namely the canonical curve. Therefore, there is exactly one norm curve when \( n \geq 7 \). Moreover, the coefficients for that curve are \( a_1 = 0 \), \( a_3 = 1 \), \( a_4 = 2 \), and \( 0 \leq a_2 \leq (n - 5)/2 \).

For \( n \leq -3 \) there are several possible solutions to Equations 6.6. (Although Lemma 6.1 refers to \( n \leq -11 \), we include solutions for \( n \leq -3 \) for future reference. On the other hand, solutions for \( n = -1 \) are not necessary since the \((-2,3,-1)\) pretzel knot is a twist knot and its Culler-Shalen seminorms were worked out in [BMZ].)

To be precise, there are four possible solutions which lead to a norm (rather than just a seminorm).

1. \( a_1 = 0 \), \( a_3 = a_4 = 1 \) and \( 0 \leq a_2 \leq (1 - n)/2 \). Then \( s = 2(1 - n) + 2a_2 \), \( \|2n + 4\| = 4(4 - n) + 4a_2 = s + 2(7 - n) + 2a_2 \) and \( \|2n + 5\| = 2(7 - 3n) + 2a_2 = s + 4(3 - n) \).
2. \( a_1 = a_4 = 1 \), \( a_3 = 0 \) and \( 0 \leq a_2 \leq (1 - n)/2 \). Then \( s = 2(1 - n) + 2a_2 \), \( \|2n + 4\| = -4(n + 1) + 4a_2 = s - 2(n + 3) + 2a_2 \) and \( \|2n + 5\| = -6(n + 1) + 2a_2 = s - 4(n + 2) \).
3. If \( n \geq -23 \), \( a_1 = 1 \), \( a_3 = a_4 = 0 \) and \( 0 < a_2 \leq (n + 25)/2 \). Then \( s = 2 + 2a_2 \), \( \|2n + 4\| = -2(2n + 4) + 4a_2 = s - 2(2n + 5) + 2a_2 \) and \( \|2n + 5\| = -2(2n + 5) + 2a_2 = s - 2(2n + 6) \).
4. If \( n = -3 \), \( a_1 = a_4 = 0 \) and \( a_2 = a_3 = 1 \). Then \( s = 4 \), \( \|2n + 4\| = 28 \) and \( \|2n + 5\| = 24 \).

Since \( s \geq 2(1 - n) \) for curves of Type 1 and 2, and since \( 2 \times 2(1 - n) > S \), there is at most one norm curve of Type 1 or 2 in the character variety. To complete the argument for \( n \leq -11 \), we must show that there cannot be a norm curve of Type 3 in the presence of any other norm curve.

For example, suppose there were norm curves \( X_1 \) and \( X_2 \) of Types 1 and 3 respectively. Then, \( \|2n + 5\|_1 = s_1 + 4(3 - n) \) which implies all the jumping points for \( 2n + 5 \) surgery lie on the curve \( X_1 \) (see proof of Proposition 1.3). We’ve shown that the jumping points are simple (see Lemmas 5.2 and 5.3), so they cannot lie on any other curve. Thus, on \( X_2 \), \( 2n + 5 \) surgery must have minimal norm: \( \|2n + 5\|_2 = s_2 \). This is a contradiction.

Similar arguments show that combinations of Type 3 with Type 2 or Type 3 with Type 3 are also not feasible. Therefore, there can be at most one norm curve when \( n \leq -11 \). On the other hand, since the \((-2,3,n)\) pretzel knot is hyperbolic,
there is at least one norm curve, namely the canonical curve. So there is exactly one norm curve when \( n \leq -11 \) and its norm is of Type 1, 2, or 3.

Lemma 6.1 largely completes the proof of the proposition in case \( n \geq 7 \) or \( n \leq -11 \). What remains is to analyze the \( r \)-curves. Recall that \( r \) must be an integral boundary slope and that the Culler-Shalen seminorm on an \( r \)-curve is \( ||\gamma||_i = s_i \Delta(\gamma, r) \).

**Lemma 6.2.** Let \( n \geq 7 \) or \( n \leq -11 \). If \( 3 \mid n \), there is a unique \( r \)-curve with \( r = 2n + 6 \). Moreover, the minimal norm for this curve is \( s = 2 \). If \( 3 \nmid n \), there is no \( r \)-curve.

**Proof:** We will argue that \( K_n \) admits an \( r \)-curve only if \( r = 2n + 6 \). For this, we’ll use the graph manifold structure of \( M(2n + 6) \).

**Claim 6.3.** \( M(2n + 6) = M_1 \cup M_2 \), is the union of two Seifert fibred manifolds \( M_1 \) and \( M_2 \) along a torus. \( M_1 \) is Seifert fibred over \( \Delta^2(2, 2) \) and \( M_2 \) is Seifert fibred over \( \Delta^2(3, |n - 3|/2) \). Moreover, regular fibres intersect once on the boundary torus.

**Proof:** We can construct \( M_1 \) by thickening the spanning surface of Figure 4. The surface is a punctured Klein bottle so \( M_1 \) is a twisted I-bundle over the Klein bottle. We denote the complementary manifold \( S^3 \setminus N(M_1) \) by \( M_2 \). We can get a better handle on the Seifert structure of the \( M_i \) (\( i = 1, 2 \)) using the ideas of Dean [Dn].

A key observation [Dn, Lemma 2.2.1] is that an irreducible Haken manifold, like \( M_i \), with fundamental group \( G_{m,n} = \langle x, y \mid x^m y^n \rangle \) is Seifert fibred, the base orbifold being a disc with cone points of order \( m \) and \( n \). Since \( M_1 \) is obtained from the obvious genus 2 handlebody \( H \) in Figure 3 by adding a 2-handle along the knot, we can compute it’s fundamental group. Indeed, with respect to the generators \( a, b \) of \( \pi_1(H) \), the knot represents the relator \( b^{-1}ab^{-1}a^{-1} \). Making the change of basis, \( b^{-1}a \to c, a \to d^{-1} \), the relator becomes \( c^2d^2 \). Thus \( M_1 \) is Seifert fibred over \( \Delta^2(2, 2) \) with \( (b^{-1}a)^2 \) or \( a^2 \) representing a regular fibre and fundamental group \( \pi_1(M_1) = \langle c, d \mid c^2d^2 \rangle \).

A similar argument allows us to identify \( M_2 \) using the generators \( x \) and \( y \) of the complementary handlebody \( H’ \) (see Figure 3). In this context, the knot represents the word \( yxy^{(n-1)/2}xy^{(n-1)/2}x \). After the change of basis \( y^{(n-1)/2}x \to w, y^{-1} \to z \), the word becomes \( z^{(n-3)/2}w^3 \). Thus \( M_2 \) is Seifert fibred over \( \Delta^2(3, |n - 3|/2) \) and has fundamental group \( \pi_1(M_2) = \langle w, z \mid z^{(n-3)/2}w^3 \rangle \). Moreover, a regular fibre corresponds to \( (xy^{(n-1)/2})^3 \).

We can now argue that the fibres intersect once on the common boundary of \( M_1 \) and \( M_2 \). Indeed, Figure 3 shows how a fibre of \( M_1 \) representing \( a^2 \) and a fibre of \( M_2 \) representing \( (xy^{(n-1)/2})^3 \) have intersection number one. Note also that the \( M_1 \) fibre...
\[ ρ \] becomes \( y^{-2}x^{-1} = y^{(n-5)/2}(xy^{(n-1)/2})^{-1} = z^{(5-n)/2}w^{-1} \) in \( π_1(M_2) \) whereas the \( M_2 \) fibre \( (xy^{(n-1)/2})^3 \) goes to \( b^{-1}ab^{-1}ab^{-1}a = (b^{-1}a)^3a^{-1} = e^d \). \( \Box \) (Claim)

**Claim 6.4.** If \( 3 \mid n \), the \( \text{PSL}_2(\mathbb{C}) \)-character variety \( \hat{X}(M(2n+6)) \) contains exactly one curve. If \( 3 \nmid n \), then \( \dim(\hat{X}(M(2n+6))) = 0 \).

**Proof:** (of Claim) We will argue that irreducible \( \text{PSL}_2(\mathbb{C}) \)-characters of \( M(2n+6) \) are either isolated or else factor through \( \mathbb{Z}/2 \ast \mathbb{Z}/3 \). The result then follows from [BZ2] Example 3.2.

An irreducible representation \( \bar{ρ} : M(2n+6) \to \text{PSL}_2(\mathbb{C}) \) will be non-abelian or else have image \( \mathbb{Z}/2 \ast \mathbb{Z}/2 \). On the other hand, if it’s abelian, it also factors through \( H_1(M(2n+6); \mathbb{Z}) \cong \mathbb{Z}/(2n+6) \) and there is no cyclic group which contains \( \mathbb{Z}/2 \ast \mathbb{Z}/2 \). Therefore, if \( \bar{ρ} \) is irreducible, it’s also non-abelian. Let \( \bar{ρ}_i : π_1(M_i) \to \text{PSL}_2(\mathbb{C}) \) \((i = 1, 2)\) be the induced representations and let \( h_i \in π_1(M_i) \) be the class of a regular fibre. A little algebra shows that if \( \bar{ρ}_i \) is non-abelian, then \( \bar{ρ}_i(h_i) = ±I \) (see [Mt1, Claim 5.2.3]).

**Subclaim 6.5.** If \( \bar{ρ}(π_1(T)) \not\subset \{±I\} \), then \( χ_{\bar{ρ}} \) is isolated in \( \hat{X}(M(2n+6)) \).

**Proof:** (of Subclaim) Let us assume \( \bar{ρ}(π_1(T)) \not\subset \{±I\} \). We wish to show that \( χ_{\bar{ρ}} \) is then isolated in \( \hat{X}(M(2n+6)) \). Since regular fibres intersect once on \( T \), their images generate \( \bar{ρ}(π_1(T)) \). Therefore, in order to satisfy \( \bar{ρ}(π_1(T)) \not\subset \{±I\} \), at least one \( \bar{ρ}_i \) is abelian with \( \bar{ρ}_i(h_i) \neq ±I \).

For example, suppose \( \bar{ρ}_2 \) is abelian and \( \bar{ρ}_1 \) is not. As above, \( \bar{ρ}_1(h_1) = ±I \). Since the glueing torus \( T \) contains regular fibres, we can assume \( h_1 \in π_1(T) \). As the \( \bar{ρ}_i \)'s agree on the intersection \( π_1(T) \), \( \bar{ρ}_2(h_1) = ±I \) as well. However, we’ve seen earlier (in the proof of Claim 6.3) that a regular fibre \( h_1 \) represents the word \( z^{(5-n)/2}w^{-1} \) in \( π_1(M_2) \). Since this word is killed, \( \bar{ρ}_2 \) factors through \( \pi_1(M_2)/⟨h_1⟩ = ⟨w, z \mid w^3z^{(n-3)/2}, z^{(5-n)/2}w^{-1}⟩ = ⟨z \mid z^{6-n}⟩ \) which is cyclic of order \( |n - 6| \).

This means that \( \bar{ρ}_2(h_2) \) is of finite odd order (remember that \( n \) is odd, so that \( n - 6 \neq 0 \)). We can conjugate so that \( \bar{ρ}_2(h_2) = ± \begin{pmatrix} η & 0 \\ 0 & 1/η \end{pmatrix} \) with \( η \neq ±1, ±i \).

Now, since \( \bar{ρ}_1(h_1) = ±I \), \( \bar{ρ}_1 \) factors through the orbifold group \( π_1^{orb}(B_1) = ⟨c, d \mid \)

![Figure 5. The handlebody H.](image)
Figure 6. The regular fibres $a^2$ and $(xy^{(n-1)/2})^3$ intersect once inside the circle.

c^2, d^2$ where $B_1$ is the base orbifold of $M_1$. Again, $\bar{\rho}_1(h_2) = \bar{\rho}_2(h_2)$ is of finite order dividing $|n - 6|$ and represents the word $c^3d$. Thus $\bar{\rho}_1$ factors through $\langle c, d | c^2, d^2, (cd)^{|n-6|} \rangle$ which is dihedral of order $2|n - 6|$. Also, $\bar{\rho}_1(h_2) = \bar{\rho}_2(h_2) = \pm \begin{pmatrix} \eta & 0 \\ 0 & 1/\eta \end{pmatrix}$ is in the image of the cyclic subgroup which is therefore diagonal.
We have now given a rather specific description of $\tilde{\rho}$. Restricted to $\pi_1(M_2)$, it is cyclic of order dividing $|n - 6|$ and diagonal. Restricted to $\pi_1(M_1)$ it factors through $D_{2|n-6}$ with the cyclic subgroup having image in the diagonal matrices. Moreover, $\pm \left( \begin{array}{cc} \eta & 0 \\ 0 & 1/\eta \end{array} \right)$ with $\eta \neq \pm 1, \pm i$ is common to the images of $\pi_1(M_1)$ and $\pi_1(M_2)$. There are only a finite number of characters consistent with such a representation. Thus, characters of this form are isolated in the sense that they cannot form a curve.

Similar arguments apply when $\tilde{\rho}_1$ is abelian and $\tilde{\rho}_2$ is not or when $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are both abelian. That is, in each of these cases, we find only a finite number of isolated characters which, therefore, do not form a curve. 

Given the Subclaim, the only way to construct a curve in $\tilde{X}(M(2n + 6))$ is by making use of representations $\tilde{\rho}$ which kill the glueing torus $T$ and therefore factor through 

$$
\pi_1(M(2n + 6))/\pi_1(T) = (\pi_1(M_1) *_{\pi_1(T)} \pi_1(M_2))/\pi_1(T)
= \pi_1(M_1)/\pi_1(T) * \pi_1(M_2)/\pi_1(T)
= \mathbb{Z}/2 * \mathbb{Z}/g.
$$

where $g = \gcd(3, |n-3|/2)$. If $3 \nmid n$, then $g = 1$ and this is an abelian representation, contradicting an earlier assumption. So there is no curve in $\tilde{X}(M(2n + 6)$ when $3 \nmid n$. If $g = 3$ (i.e., if $3 \mid n$), we see that we are looking at representations of $\mathbb{Z}/2 * \mathbb{Z}/3$. Since $\tilde{X}(\mathbb{Z}/2 * \mathbb{Z}/3)$ contains exactly one curve (see [BZ2, Example 3.2]), we conclude that this is also the case for $\tilde{X}(M(2n + 6))$.

So, if $3 \mid n$, $\dim_{\mathbb{C}}(\tilde{X}(2n + 6)) = 0$ and there can be no $r$-curve with $r = 2n + 6$ (compare [BZ2, Example 5.10]).

On the other hand, if $3 \mid n$, there is a unique curve in $\tilde{X}(M(2n + 6))$. Moreover, since the representation $\tilde{\rho}_{1/2}$ (see [BZ2, Example 3.2] or Equation 6.12 below) is dihedral, this curve contains the character of a dihedral representation. It follows that the curve is covered by a unique curve in the $\text{SL}_2(\mathbb{C})$-character variety $X(M(2n + 6))$ (see [BZ1, Lemma 5.5]). Thus there is exactly one $r$-curve, call it $X_1$, with $r = 2n + 6$ when $3 \mid n$.

**Claim 6.6.** The minimal norm $s_1$ of $X_1$ is two.

**Proof:** (Claim) Recall that $s_1 = \|\mu\|_1$ is the degree of $f_\mu$ (Theorem 1.1). So we need to understand the image of $\mu$ under the composition $\pi \to \pi_1(M(2n + 6)) \to \mathbb{Z}/2 * \mathbb{Z}/3 = \langle c, d \mid c^2, d^3 \rangle$. We can construct $\mu$ in terms of a curve $\gamma$ in the genus two surface which connects points on opposite sides of the knot; see Figure 7. The idea is that we can break up the meridian as the sum of a loop in $M_1$ and a loop in $M_2$. We will then show that those project to $c$ and $d$ respectively. The first loop is $\gamma$ plus a small arc joining the two endpoints of $\gamma$ in the interior of $H$, the obvious genus two handlebody. The second loop is $\gamma$ plus a small arc joining the two endpoints of $\gamma$ and passing through the complementary genus two handlebody $H'$.

In $\pi_1(M_1)$, $\gamma$ represents $ab^{-1}$ which is conjugate to $b^{-1}a$ and therefore projects onto the generator of $\mathbb{Z}/2 = \pi_1(M_1)/\pi_1(T)$. In $\pi_1(M_2)$, $\gamma$ represents $xy^{(n-1)/2}$ which projects to the generator of $\mathbb{Z}/3$. Thus $\mu$ is mapped to $cd$ in $\mathbb{Z}/2 * \mathbb{Z}/3$.

Let $\tilde{X}_1$ be the unique curve in $\tilde{X}(\mathbb{Z}/2 * \mathbb{Z}/3)$. In [BZ2, Example 3.2], the authors construct a double covering $\mathbb{C} \to \tilde{X}_1$ given by mapping $z \in \mathbb{C}$ to the character of
\[ \bar{\rho}_z : \begin{pmatrix} c \\ d \end{pmatrix} \mapsto \begin{pmatrix} z^2 - 1 \\ z - 1 \end{pmatrix}, \quad \bar{\rho}_z : \begin{pmatrix} c \\ d \end{pmatrix} \mapsto \begin{pmatrix} z^2 - 1 \\ z - 1 \end{pmatrix}. \] (6.12)

Since \( \text{trace}(\bar{\rho}_z(c)) = i(2z - 1) \), we see that \( f_{\text{cd}}(\chi_{\bar{\rho}_z}) = -(2z - 1)^2 - 4 \) has degree 2. As this is a double covering of \( \bar{X}_1 \) by \( C \), the corresponding character on \( \bar{X}_1 \) has degree 1. Finally, lifting to the curve \( X_1 \) in \( X(M) \) which double covers \( \bar{X}_1 \subset \bar{X}(M(2n + 6)) \subset \bar{X}(M(2n + 6)) \), we deduce \( s_1 = \deg f_{\mu} = 2 \). (Claim)

Thus far, we’ve constructed an \( r \)-curve \( X_1 \) for \( r = 2n + 6 \) when \( 3 \mid n \) and shown that there is none when \( 3 \nmid n \). It remains to argue that there are no \( r \)-curves for the other integral boundary slopes.

For \( n \geq 7 \), the integral boundary slopes are 0, 16, and \( 2n + 6 \). We will show that 0 and 16 do not admit \( r \)-curves. Recall (Lemma 6.1) that \( \|2n+5\| = s_0 + 4(|n-5|-2) \), i.e., all the jumping points for the \( 2n + 5 \) surgery are on the norm curve \( X_0 \). So we would have \( \|2n+5\| = s_i \) on any \( r \)-curve \( X_1 \). However, if \( r = 0 \) for example, \( \|2n+5\| = s_i \Delta(2n+5,0) = (2n+5)s_i \). So there can be no \( r \)-curve for \( r = 0 \). Similarly, there can be no \( r = 16 \) curve. Analogous arguments show that when \( n \leq -11 \), there is no \( r \)-curve with \( r = 0 \) or 10. Thus \( r = 2n + 6 \) is the only candidate for an \( r \)-curve amongst the integral boundary slopes when \( n \leq -11 \) as well.

This completes the proof of Lemma 6.2. □

Thus when \( n \geq 7 \) or \( n \leq -11 \), there is exactly one norm curve. There will be one \( r \)-curve when \( 3 \mid n \) and otherwise there are no additional curves containing irreducible characters. Since the set of reducible characters forms a complex line, we see that \( X(K_n) \), the character variety of the knot \( K_n \), consists of two (three) curves when \( 3 \nmid n \) \( (3 \mid n) \) and \( n \geq 7 \) or \( n \leq -11 \). To complete the proof of the Main Theorem for these \( n \), we need only verify that the Culler-Shalen seminorms are as stated.

If \( n \geq 7 \) we know the norm on the norm curve \( X_0 \) up to the coefficient \( a_0 \). If \( 3 \nmid n \), then there is no \( r \)-curve and the norm curve \( X_0 \) is the only one contributing to the total norm \( \| \cdot \| \). In particular \( s_0 = S = 3(|n-2|-1) = 3(n-3) \). Therefore,
\( a_2 = (n - 5)/2 \) and the Culler-Shalen norm on \( X_0 \) is as stated in the theorem. If \( 3 \mid n \), there is also an \( r \)-curve \( X_1 \) with \( s_1 = 2 \). In this case \( S = s_0 + s_1 \Rightarrow s_0 = S - s_1 = 3(n - 3) - 2 \). This implies \( a_2 = (n - 7)/2 \).

If \( n \leq -11 \), we have several candidates (Types 1, 2, and 3 of Lemma 6.1) for the norm \( \| \cdot \|_0 \) on the norm curve \( X_0 \). If \( 3 \nmid n \), there are no other curves and \( \| \cdot \|_0 \) is the total norm. In particular, it must satisfy Propositions 1.3, 1.4 and 1.5. The only possibility is \( \| \cdot \|_0 \) of Type 1 with \( a_2 = (1 - n)/2 \). If \( 3 \mid n \), the total norm is a combination of \( \| \gamma \|_1 = 2\Delta(2n + 6, \gamma) \) on \( X_1 \) and the norm \( \| \cdot \|_0 \) on the norm curve \( X_0 \). Again, the only choice satisfying Propositions 1.3, 1.4 and 1.5 is \( \| \cdot \|_0 \) of Type 1 with \( a_2 = -(n + 1)/2 \).

This completes the proof of the Main Theorem for the cases \( n \geq 7 \) and \( n \leq -11 \). There remain the cases \( n = -9, -7, \ldots, -1 \).

\( n = -9 \) Here \( 3 \mid n \), so, following the reasoning of Lemma 6.2, there is an \( r \)-curve \( X_1 \) with \( r = 2n + 6 = -12 \) and \( s_1 = 2 \). Moreover, there is no \( r \)-curve with \( r = 0 \) or 10.

As for the norm curves, if there’s a norm curve of Type 1 or 2, it’s the only norm curve. For example, a norm curve \( X_0 \) of Type 1 has \( \|2n + 5\|_0 = s_0 + 4(3 - n) \) so that all the jumping points for this surgery are on \( X_0 \). This means any other norm curve would have minimal norm for this surgery slope: \( \|2n + 5\|_i = s_i \). However, this is not true for any norm curve, be it of Type 1, 2, or 3.

On the other hand, we need a new type of argument to show that two (or more) curves of Type 3 is also not a feasible arrangement. Suppose then that there were two Type 3 norm curves \( X_0 \) and \( X_2 \). Since \( S = 30 \) and \( s_1 = 2 \), we see that

\[
28 \leq s_0 + s_2 = 4 + 2(a_2^0 + a_2^2),
\]

where we have given the \( a_2 \)'s superscripts showing which curve they come from. This implies \( a_2^0 + a_2^2 \geq 12 \). But then

\[
\|2n + 4\|_0 + \|2n + 4\|_2 = s_0 + s_2 - 4(2n + 5) + 2(a_2^0 + a_2^2) \geq s_0 + s_2 + 52 + 24
\]

which contradicts the equation \( \|2n + 4\|_T = S + 3(|n - 6| - 1) = S + 42 \). Thus, we see that there is exactly one norm curve and one \( r \)-curve when \( n = -9 \). Since these must combine to give the total norm, the norm curve is of Type 1 with \( a_2 = 4 = -(n + 1)/2 \).

\( n = -7 \) In this case there is no \( r \)-curve for \( r = 2n + 6 = -8 \). By examining the norm of the \( 2n + 5 = -9 \) slope, we see that if there is a norm curve of Type 1, it is the only curve in \( X \) containing an irreducible character (in particular, there are no \( r \)-curves).

Similarly, if there is a norm curve of Type 2, then there is no \( r \)-curve with \( r = 10 \).

However, we cannot immediately eliminate the possibility of an \( r \)-curve for \( r = 0 \). Indeed, suppose that there were a Type 2 norm curve \( X_0 \), together with an \( r \)-curve \( X_1 \) with \( r = 0 \). For the norm curve \( X_0 \),

\[
s_0 = 2(1-n)+2a_2 = 16+2a_2, \quad \|2n+4\|_0 = \| -10 \|_0 = s_0 - 2(n+3) + 2a_2 = s_0 + 8 + 2a_2
\]

and \( \|2n + 5\|_0 = \| -9 \|_0 = s_0 - 4(n + 2) = s_0 + 20 \).
On the other hand, for the $r$-curve $X_1$, $\|2n+4\|_1 = \| -10\|_1 = s_1 + 3(5-n) - 8 = s_1 + 28$ and also $\|2n+4\|_1 = \| -10\|_1 = s_1 \Delta(-10, 0) = 10s_1$. This implies

$$10s_1 \leq s_1 + 28$$

$$\Rightarrow 9s_1 \leq 28$$

$$\Rightarrow s_1 \leq 28/9$$

Since $s_1$ is an even integer, we see that $s_1 = 2$. Similarly, an examination of the $-9$ slope also leads us to the conclusion that $s_1 = 2$. So we cannot eliminate the possibility of an $r$-curve with $r = 0$ directly as we did earlier. We need a new type of argument to handle this situation. We need to analyze the possible combinations of curves.

For example, suppose $X$ contained a Type 2 norm curve $X_0$ and one $r$-curve $X_1$ for $r = 0$ and no other norm or $r$-curves. Then $\| -9\|_0 = s_0 + 20$ and

$$\| -9\|_1 = s_1 \Delta(-9, 0)$$

$$= 9s_1$$

$$= s_1 + 8s_1$$

$$= s_1 + 16.$$ 

So

$$\| -9\|_T = \| -9\|_0 + \| -9\|_1$$

$$= s_0 + s_1 + 36$$

$$= S + 36$$

$$< S + 40$$

$$= S + 4(|n - 5| - 2).$$

Thus, if we assume that these are the only two curves, we see that we cannot account for all the jumping points associated with the $-9$ slope. Therefore, this is not a possible configuration for $X$. By analyzing the possible combinations of norm curves and $r$-curves in this way, we see that the only possibility consistent with Propositions 1.3, 1.4, and 1.5 is that there is exactly one norm curve of Type 1 with $a_2 = 4 = (1-n)/2$ and no $r$-curves.

$n = -5$ A similar analysis shows that $X$ contains no $r$-curves and exactly one norm curve of Type 1 with $a_2 = 3 = (1-n)/2$.

$n = -3$ Since $3 \mid n$, we know (see Lemma 6.2) that there is an $r$-curve $X_1$ for $r = 2n + 6 = 0$ with

$$s_1 = 2, \|2n+4\|_1 = \| -2\|_1 = s_1 + 2 \text{ and } \|2n+5\|_1 = \| -1\|_1 = s_1.$$ 

If we follow the same strategy as in the previous cases we find that there are two possible configurations:

I. In addition to the $r$-curve there is one Type 1 norm curve $X_0$ with

$$s_0 = 10, \| -2\|_0 = s_0 + 22 \text{ and } \| -1\|_0 = s_0 + 24.$$ 

II. Here there is a Type 2 norm curve $X_0$:

$$s_0 = 8, \| -2\|_0 = s_0 \text{ and } \| -1\|_0 = s_0 + 4.$$
as well as an additional $r$-curve $X_2$ with $r = 10$:

$$s_2 = 2, \| -2 \| = s_2 + 22 \text{ and } \| -1 \| = s_2 + 20.$$

Both configurations are consistent with Propositions $1.3$, $1.4$, and $1.5$. $S = 3(|n - 2| - 1) = 12, \| -2 \| = S + 3(|n - 6| - 1) = S + 24$, and $\| -1 \| = S + 4(|n - 5| - 2) = S + 24$.

In order to show that the second configuration does not arise, note that $X_2$, the PSL$_2(\mathbb{C})$ analogue of $X_2$, would include into $\bar{X}(M(10))$ (see Example 5.10). Now, $M(−1)$ is Seifert fibred over $S^2(3, 4, 5)$ and the jumping points for $\| −1 \|$ come from the six irreducible PSL$_2(\mathbb{C})$-characters of $\Delta(3, 4, 5)$. If the second configuration is valid, five of these characters are on $X_2$ and therefore come from representations lying in $\bar{X}(M(10))$. We will argue that at least two of them do not.

Indeed two of the characters correspond to representations which factor through $\Delta(2, 3, 5)$ which has order 60. On the other hand, if such a representation $\bar{\rho}$ is also in $\bar{X}(M(10))$, then it annihilates both the 10 and the $-1$ slopes. In other words, the kernel of $\bar{\rho}$ contains an index eleven subgroup of $\pi_1(\partial M)$. Therefore, $\bar{\rho}(\pi_1(\partial M))$ is either $\mathbb{Z}/11$ or else trivial. On the other hand, $\bar{\rho}(\pi_1(\partial M))$ also factors through $\Delta(2, 3, 5)$. Thus $\bar{\rho}(\pi_1(\partial M))$ is trivial and since $\pi_1(M)$ is normally generated by the peripheral group, $\bar{\rho}(\pi_1(M)) = \{±I\}$ as well. This contradicts the fact that $\bar{\rho}$ is an irreducible representation. Therefore, the irreducible representations which factor through $\Delta(2, 3, 5)$ are not in $\bar{X}(M(10))$. This shows that the second configuration is not possible.

Therefore, we have the first configuration. There is an $r$-curve and a norm curve of Type 1 with $a_2 = 1 = -(n + 1)/2$.

$n = −1$ This knot was treated using different methods in [BMZ] (where it is identified as the twist knot $K_2$). We saw that there is one norm curve in the character variety and no $r$-curves. Moreover, the norm curve corresponds to Type 1 with $a_2 = 1 = (1 - n)/2$.

This completes the proof of the Main Theorem.

7. Applications

With the Main Theorem in hand, we look at applications. Since surgeries which result in a manifold having cyclic or finite fundamental group are of small norm, we can use our knowledge of the Culler-Shalen seminorms to understand which surgeries might be finite or cyclic. A nice way to visualize this connection is via the fundamental polygon.

Given a Culler-Shalen seminorm $\| \cdot \|$ arising from a norm curve in the character variety of a knot $K$, we call $B$, the disc of radius $s$ in $V = H_1(\partial M; \mathbb{R})$, a fundamental polygon for $K$. A fundamental polygon is a compact, convex, and finite-sided polygon with vertices which are rational multiples of boundary slopes in $L = H_1(\partial M; \mathbb{Z})$. It is symmetric ($-B = B$) and centred at $(0, 0)$. As we have shown in the Main Theorem, $K_n$ has only one norm curve $X_0$ which is therefore the canonical curve. We will refer to the associated polygon as the fundamental polygon $B$ of $K_n$.

By Theorems $1.3$ and $1.5$ a cyclic or finite surgery slope either has norm bounded by $\max(2s_i, s_i + 8)$ or else is a boundary slope. On the other hand, for a small knot like $K_n$, a cyclic or finite surgery cannot occur on a boundary slope.
Lemma 7.1. If $M$ is small and $\alpha$ is a boundary slope, then $M(\alpha)$ is not cyclic or finite.

Proof: By [CGLS, Theorem 2.0.3], $M(\alpha)$ is not finite, and it is cyclic only if $M(\alpha) \cong S^1 \times S^2$. However, Gabai [Ga] has shown that, amongst knots in $S^3$, only slope 0 surgery on the trivial knot can produce $S^1 \times S^2$.

Therefore, as long as $s_0 > 8$, all cyclic and finite surgeries of $K_n$ will lie within $2B$, the norm disk of radius $2s_0$.

Proposition 1.7. If the $(-2,3,n)$ pretzel knot $K_n$ admits a non-trivial cyclic or finite surgery, then one of the following holds.

- $K_n$ is torus, in which case $n = 1, 3, \text{ or } 5$,
- $n = 7$, in which case 18 and 19 are cyclic fillings while 17 is a finite, non-cyclic filling, or
- $n = 9$, in which case 22 and 23 are finite, non-cyclic fillings.

Remark: A torus knot admits an infinite number of cyclic fillings. A cyclic filling of a non-trivial knot in $S^3$ is necessarily finite [Ga]. The finite and cyclic surgeries of $K_7$ and $K_9$ were discovered by Fintushel and Stern (see [FS]) and Bleiler and Hodgson [BH]. The content here is that these are the only non-trivial finite or cyclic surgeries on the non-torus members of this family of knots.

Proof: Let $n \geq 7$. The case $n = 7$ is the subject of [BZ1, Example 10.1] where it is shown that there are exactly three non-trivial finite surgeries. For $n = 9$, $\max(2s_0, s_0 + 8) = 2s_0$ (Theorem 1.6), so any finite or cyclic surgery slopes must lie in $2B$. However, as we see in Figure 8, the only slopes inside $2B$ are 21, 22, 23 and $\mu = 1/0$. If 21 were a finite surgery, then it would be one with norm $2s_0$ and therefore dihedral ([BZ1, Theorem 2.3]). However, it cannot be dihedral since 21
is odd. Therefore, the \((-2, 3, 9)\) pretzel knot admits exactly two non-trivial finite surgeries: 22 and 23.

As \(n\) increases, the fundamental polygon for the norm curve maintains the same basic shape but becomes smaller (the exact coordinates of the polygon are determined by the Culler-Shalen norm \(\| \cdot \|_0\) of the Main Theorem). For \(11 \leq n \leq 19\), the only slopes inside \(2B\) are \(2n + 4, 2n + 5\) and \(\mu\) and once \(n \geq 21\), only \(2n + 4\) and \(\mu\) remain. However, since \(M(2n + 4)\) and \(M(2n + 5)\) are Seifert fibred over a hyperbolic orbifold when \(n \geq 11\), these are not finite surgeries. Thus the \((-2, 3, n)\) pretzel knots admit no non-trivial cyclic or finite surgeries when \(n \geq 11\).

Figure 9 gives the fundamental polygon of the \((-2, 3, -7)\) pretzel knot and illustrates the situation for \(n \leq -1\). For these knots, \(2B\) lies below the line \(y = 1\). Thus the only surgery slope within \(2B\) is \(\mu\). Since \(s_0 > 8\) for \(n \leq -3\), there can be no non-trivial finite or cyclic surgeries when \(n \leq -3\). For \(n = -1, s_0 = 6\) and any finite or cyclic surgery would have to lie in the polygon \(\frac{15}{4}B\) (i.e., \(\frac{15}{4} = (s_0 + 8)/s_0\)). However, as this polygon also lies below \(y = 1\), we again conclude that there are no non-trivial finite or cyclic surgeries when \(n = -1\).

This completes the proof of the proposition. \(\square\)

Note that, as the five finite fillings of the \((-2, 3, 7)\) and \((-2, 3, 9)\) pretzel knots are not simply-connected, this constitutes a proof of Property P for the \((-2, 3, n)\) pretzel knots. However, these knots are strongly invertible, so Property P was already known [BS].

As a final application, we derive the Newton polygon of the \(A\)-polynomial [CCGLS] of the \((-2, 3, n)\) knots. Recall that the Newton polygon of a polynomial \(A = \sum_{(i,j)} b_{i,j} t^i m^j \in \mathbb{Z}[t, m]\) is the convex hull in \(\mathbb{R}^2\) of \(\{(i,j) | b_{i,j} \neq 0\}\). Boyer and Zhang have shown that the Newton polygon \(N\) and the fundamental polygon \(B\) are dual in the following sense.
Theorem 7.2 (Theorem 1.4 of [BZ3]). The line through any pair of antipodal vertices of $B$ is parallel to a side of $N$. Conversely, the line through any pair of antipodal vertices of $N$ is parallel to a side of $B$.

Thus given the fundamental polygon, one can deduce $N$, at least up to scaling and translation. The conventions we use are that $N$ meets the $l$ and $m$ axes but lies in the first quadrant. The scale is provided by Shanahan’s width function.

Definition 7.3 (Definition 1.2 of [Sn]). The $p/q$ width $w(p/q)$ of $N$ is one less than the number of lines of slope $p/q$ which intersect $N$ and contain a point of the integer lattice.

We require that Shanahan’s width be half the Culler-Shalen norm: $\|p/q\|_0 = 2w(p/q)$. With these conventions, the Newton polygon is completely determined by the norm $\| \cdot \|_0$ of our Main Theorem.

The vertices of the Newton polygon are

$\begin{align*}
c &= (2(n^2 - n + 3), n - 2) \\
d &= (3n^2 - 4n - 9, 3(n - 3)/2) \\
b &= (16, 1) \\
e &= (3n^2 - 4n - 25, (3n - 11)/2) \\
a &= (0, 0) \\
f &= (n^2 - 2n - 15, (n - 5)/2)
\end{align*}$

when $n \geq 7$ and $3 \nmid n$ (see Figure 10 which may be compared with [Sn, Figure 6]);

$\begin{align*}
(0, 0), (16, 1), (n^2 - 2n - 15, (n - 5)/2), (2(n^2 - n + 3), n - 2), \\
(3n^2 - 4n - 25, (3n - 11)/2), (3n^2 - 4n - 9, 3(n - 3)/2)
\end{align*}$

when $n \geq 7$ and $3 \nmid n$ (see Figure 10 which may be compared with [Sn, Figure 6]);

$\begin{align*}
(0, 0), (16, 1), (12(n - 7), (n - 7)/2), (3(n^2 - 6n + 23), n - 2), \\
(3n^2 - 6n - 31, (3n - 13)/2), (3(n^2 - 2n - 5), (3n - 11)/2)
\end{align*}$

when $n = 3k, k \geq 3$;

$\begin{align*}
(0, (1 - 3n)/2), (10, 3(1 - n)/2), (n^2 + 2n - 3, -n), \\
(2(n^2 + 2n + 6), (3 - n)/2), (3n^2 + 6n - 1, 0), (3(n^2 + 2n + 3), 1)
\end{align*}$

when $n \leq -5$ and $3 \nmid n$ (see Figure 11);

$\begin{align*}
(0, (3n + 1)/2), (10, (1 - 3n)/2), (n^2 + 4n + 3, -n), \\
(2(n^2 + 2n + 6), (1 - n)/2), (3n^2 + 8n + 5, 0), (3n^2 + 8n + 15, 1)
\end{align*}$

when $n = 3k, k \leq -1$; and

$\begin{align*}
(0, 0), (0, 1), (4, 2), (10, 1), (14, 2), (14, 3) \text{ when } n = -1.
\end{align*}$
THE CULLER-SHALEN SEMINORMS OF THE $(-2, 3, n)$ PRETZEL KNOT

Figure 11.

$$b = (10, 3(1 - n)/2) \quad c = (2(n^2 + 2n + 6), (3 - n)/2)$$

$$d = (3(n^2 + 2n + 3), 1)$$

$$a = (0, (1 - 3n)/2)$$

$$f = (n^2 + 2n - 3, -n) \quad e = (3n^2 + 6n - 1, 0)$$

ACKNOWLEDGEMENTS

This work forms part of my Ph.D. thesis and I would like to thank my supervisor Steven Boyer for his substantial contributions and indispensable advice. Much of this was written during a visit to Nihon University in Tokyo and I am grateful to Kimihiko Motegi and the Department of Mathematics for their hospitality during my stay. I am indebted to Kurt Foster and Dave Rusin who provided suggestions about cyclotomic roots of the Alexander polynomial.

I am especially grateful to the referee for a close reading of an earlier version of this paper and for many concrete suggestions which have significantly improved the exposition.

REFERENCES

[BH] S. Bleiler and C. Hodgson, ‘Spherical space forms and Dehn filling,’ Topology 35 (1996) 809-833.

[BS] S. Bleiler and M. Scharlemann, ‘A projective plane in $\mathbb{R}^4$ with three critical points is standard. Strongly invertible knots have property $P$.’ Topology 27 (1988) 519-540.

[BB] L. Ben Abdelghani and S. Boyer, ‘A calculation of the Culler-Shalen seminorms associated to small Seifert Dehn fillings.’ Proc. London Math. Soc. 83 (2001) 235-256.

[BMZ] S. Boyer, T. Mattman and X. Zhang, ‘The fundamental polygons of twist knots and the $(-2,3,7)$ pretzel knot’, Knots ’96 Proceedings, World Scientific (1997) 159-172.

[BN] S. Boyer and A. Nicas, ‘Varieties of group representations and Casson’s invariant for rational homology 3-spheres,’ Trans. Amer. Math. Soc. 322 (1990) 507-522.

[BZ1] S. Boyer and X. Zhang, ‘Finite Dehn surgery on knots,’ J. Amer. Math. Soc. 9 (1996) 1005-1050.

[BZ2] On Culler-Shalen Seminorms and Dehn filling,’ Ann. of Math. (2) 148 (1998) 737-801. [math.GT/9811182]

[BZ3] A proof of the finite filling conjecture,’ (submitted).

[BuZ] G. Burde and H. Zieschang, Knots, de Gruyter (1985).

[CCGLS] D. Cooper, M. Culler, H. Gillet, D.D. Long and P.B. Shalen, ‘Plane curves associated to character varieties of 3-manifolds’, Invent. Math. 118 (1994), 47-84.
M. Culler, C.M. Gordon, J. Luecke and P.B. Shalen, ‘Dehn surgery on knots’, *Ann. of Math.* **125** (1987) 237-300.

M. Culler and P. Shalen, ‘Varieties of group representations and splittings of 3-manifolds’, *Ann. of Math.* **117** (1983) 109-146.

‘Bounded, separating surfaces in knot manifolds’, *Invent. Math.* **75** (1984) 537-545.

J.C. Dean, *Hyperbolic knots with small Seifert-fibered Dehn surgeries*, PhD Thesis, The University of Texas at Austin, Austin (1996).

C. Delman, ‘Constructing Essential Laminations and Taut Foliations Which Survive All Dehn Surgeries,’ (preprint).

N. Dunfield, ‘A table of boundary slopes of Montesinos knots,’ *Topology* **40** (2001) 309-315. math.GT/9901120

R. Fintushel and R. Stern, ‘Constructing lens spaces by surgery on knots,’ *Math. Z* **175** (1980) 33-51.

C.D. Frohman and E.P. Klassen, ‘Deforming representations of knot groups in SU(2),’ *Comment. Math. Helvetici.* **66** (1991) 340-361.

D. Gabai, ‘Foliations and the topology of 3-manifolds. III,’ *J. Diff. Geo.* **26** (1987) 479-536.

W. Goldman, ‘The symplectic nature of fundamental groups of surfaces,’ *Advances in Math.* **54** (1984) 200-225.

E. Hironaka, ‘The Lehmer polynomial and pretzel links,’ (to appear in *Bulletin of Can. Math. Soc.*).

A.E. Hatcher and U. Oertel, ‘Boundary slopes for Montesinos knots,’ *Topology* **28** (1989) 453-480.

E.P. Klassen, ‘Representations of knot groups in SU(2),’ *Trans. A.M.S.* **326**(2) (1991) 795-828.

N. Maruyama, ‘On Dehn surgery along a certain family of knots,’ *J. of Tsuda College* **19** (1987) 261-280.

T. Mattman, ‘The Culler-Shalen seminorms of pretzel knots,’ PhD Thesis, McGill University, Montreal (2000) available at [http://www.csuchico.edu/math/mattman](http://www.csuchico.edu/math/mattman)

T. Mattman, ‘Cyclic and finite surgeries on pretzel knots,’ math.GT/0102050

J.W. Milnor, ‘Infinite cyclic coverings,’ *Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967)* Prindle, Weber & Schmidt (1968) 115-133.

J.M. Montesinos, ‘Revêtements ramifiés de noeuds, espaces fibrés de Seifert et scindements de Heegaard,’ *Orsay Lecture Notes* (1976).

U. Oertel, ‘Closed incompressible surfaces in complements of star links,’ *Pac. J. Math.* **111** (1984) 209-230.

J. Porti, ‘Torsion de Reidemeister pour les variétés hyperboliques,’ *Mem. Amer. math. Soc.* **128** (1997).

R. Riley ‘Parabolic representations of knot groups I,’ *Proc. London Math. Soc.* (3) **24** (1972) 217-242.

R. Riley, ‘Knots and Links’ 2nd Edition, Publish or Perish (1990).

J. Rotman, *An Introduction to Homological Algebra*, Academic Press (1979).

I. Shafarevich, *Basic Algebraic Geometry*, Die Grundlehren der mathematischen Wissenschaften, Band 213, Springer-Verlag, New York 1974.

P. Shanahan, ‘Cyclic Dehn surgery and the A-polynomial,’ *Topology Appl.* **108** (2000) 7–36.

D. Tanguay, *Chirurgies Finies et Noeuds Rationnels*, PhD Thesis, UQAM, Montreal, Canada (1995).

H.F. Trotter, ‘Non-invertible knots exist,’ *Topology* **2** (1964) 275-280.

A. Weil, ‘Remarks on the cohomology of groups,’ *Ann. of Math.* **80** (1964) 149-157.

Department of Mathematics & Statistics, California State University, Chico, Chico, CA95929-0525

E-mail address: TMattman@CSUCHico.edu