ROZANSKY-WITTEN THEORY

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Abstract. Rozansky and Witten proposed in 1996 a family of new three-dimensional topological quantum field theories, indexed by compact (or asymptotically flat) hyperkähler manifolds. As a byproduct they proved that hyperkähler manifolds also give rise to Vassiliev weight systems. These may be thought of as invariants of hyperkähler manifolds, so the theory is of interest to geometers as well as to low-dimensional topologists. This paper surveys the geometrical construction of the weight systems, how they may be integrated into the framework of Lie algebra weight systems (joint work with Simon Willerton), their applications, and an approach to a rigorous construction of the TQFTs (joint work with Justin Sawon and Simon Willerton).

1. Introduction

The 1996 paper by Rozansky and Witten [RzW] opened up a very interesting new area of geometry and topology. They wrote down a physical path integral, a sigma model involving integration over the space of all maps from a closed oriented 3-manifold $M^3$ into a hyperkähler manifold $X^{4n}$, which should give rise to a numerical invariant $Z_X(M) \in \mathbb{C}$, at least in the case when $X$ is compact or asymptotically flat. We can view this as a construction of a family of invariants $Z_X(-)$ of 3-manifolds, or of a family of invariants $Z_{(-)}(M)$ of hyperkähler manifolds. Part of the appeal of the theory is the marriage of such different worlds.

Subsequent work by Kontsevich, Kapranov, Hitchin and Sawon [Ko2, Ka, Sa, HS] has developed the theory from the geometer’s point of view. In particular, Kapranov showed that the hyperkähler condition is unnecessarily strong, and that the theory works for holomorphic symplectic manifolds. This is the point of view we take here.

There are three natural contexts for Rozansky-Witten theory, each with its own motivations:

1. Geometry. Perhaps the most concrete application of Rozansky-Witten theory is to study hyperkähler manifolds. Compact hyperkähler manifolds seem quite rare, though as yet there is no guess at a classification theorem. The Rozansky-Witten invariants amount to various complicated computations involving the curvature tensor of the manifold, and as such can be related to characteristic numbers. By combining the Rozansky-Witten invariants with the wheeling theorem of Bar-Natan, Le and Thurston [Thu], Hitchin and Sawon obtained a formula for the $L^2$-norm of the curvature in terms of a topological invariant, which is described in section 9. One might hope to extract (perhaps using the TQFT structure) further identities of this type, and obtain more constraints on the topology of compact hyperkähler manifolds.

A more extensive discussion of the geometrical point of view and its potential applications, together with a list of problems, appears in [RS]; the present paper will concentrate on the other two viewpoints.

2. Vassiliev theory. A compact holomorphic symplectic manifold $(X^{4n}, \omega)$, together with a holomorphic vector bundle $E \to X$, defines a Vassiliev weight system $w_{X,E} : \mathcal{A} \to \mathbb{C}$ on the usual algebra $\mathcal{A}$ of Jacobi diagrams with an external circle. This should be compared with the...
more familiar construction: a metric (for example, semisimple) Lie algebra \( \mathfrak{g} \), together with a representation \( \mathcal{V} \), defines a weight system \( w_{\mathfrak{g}, \mathcal{V}} : \mathcal{A} \to \mathbb{C} \).

In sections 2-4 I will describe the differential-geometric construction of the weight systems, following Kapranov. In sections 5-8 I describe how Simon Willerton and I have integrated these two rather different-looking constructions, using the derived category of coherent sheaves on \( X \).

Our original motivation in studying Rozansky-Witten theory was in fact to try to understand better the algebra \( \mathcal{A} \) and its relatives. All our intuition about such diagram algebras seems to be derived from the *invariant theory of Lie algebras*, but as Vogel [Vo] has shown, there is more to them than that. In Rozansky-Witten theory, diagrams behave more like *cohomology classes*, and this seems a valuable alternative point of view. Whether we will gain true insight remains to be seen.

3. TQFT. A compact holomorphic symplectic manifold \((X^{4n}, \omega)\) defines a 3-dimensional *topological quantum field theory* \( Z_X \), a functor from the category of 3-dimensional cobordisms to the category of finite-dimensional complex vector spaces. Again, there is a more familiar example: the *Chern-Simons theory* of Witten, Reshetikhin and Turaev [Wi, RT], in which a semisimple Lie algebra \( \mathfrak{g} \) and a root of unity \( q = e^{2\pi i/r} \) define a TQFT \( Z_{\mathfrak{g}, q} \). The work of Rozansky and Witten does not actually give a rigorous construction of such a theory, but Justin Sawon, Simon Willerton and I are currently working on providing one: this is described in sections 10 and 11.

This theory is a really non-trivial new kind of TQFT in three dimensions, and exhibits important differences from Chern-Simons theory, most notably that it is *not semi-simple*, and therefore does not satisfy the usual kinds of gluing and splitting axioms. Though many generalisations of the original axioms of TQFT have been investigated since their introduction twelve years ago, there were until now no compelling examples to make such study worthwhile.

The TQFT also appears to be related to quantum cohomology, quantum \( K \)-theory, and mirror symmetry. Perhaps the techniques of 3-dimensional TQFT will prove useful in these areas, and this theory will finally provide a link between the mysterious world of quantum and Vassiliev invariants and the more concrete geometric world of Gromov-Witten-type invariants, which seems to include almost all the other modern topological invariants.

2. Weight systems

The *Kontsevich integral* is an invariant of framed oriented knots in \( S^3 \) which takes values in (the completion of) a certain rational, graded, algebra \( \mathcal{A} \). This algebra is defined as the rational span of vertex-oriented trivalent *Jacobi diagrams* which have a preferred oriented circle, modulo the vertex-antisymmetry and IHX/STU relations. The relations are like the skein relations in knot theory: they relate diagrams which differ only locally. The pictures below shows an example Jacobi diagram and the three relations. In planar pictures such as these, vertex orientations are taken to be anticlockwise. We grade diagrams by their total number of vertices; this is *twice* the conventional grading.
A weight system is a linear map $\mathcal{A} \to \mathbb{Q}$, or perhaps to some other finite-dimensional vector space. Such maps enable us to study the structure of $\mathcal{A}$, and by composing with the Kontsevich integral, to construct more manageable scalar-valued knot invariants, the Vassiliev invariants. This is described by Bar-Natan [BN].

The standard examples of weight systems, for a long time believed to be essentially the only examples, are those coming from metric Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form $b$, and let $V$ be a finite-dimensional representation of $\mathfrak{g}$. These structures are encoded by linear maps $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{Q}$, and $\mathfrak{g} \otimes V \to V$ respectively. Using the metric to identify $\mathfrak{g} \cong \mathfrak{g}^*$ enables us to transform these into equivalent but more suitable structure tensors: a totally skew version $f \in \Lambda^3 \mathfrak{g}^*$ of the bracket; the Casimir $c \in S^2 \mathfrak{g}$; and the action $a_V \in V^* \otimes V \otimes \mathfrak{g}^*$.

A Jacobi diagram of the kind spanning $\mathcal{A}$ defines a procedure for contracting these tensors together and obtaining simply a rational number. To do this, associate a copy of $f$ with each “internal” vertex, a copy of $a_V$ with each “external” vertex (those on the preferred circle), and a copy of $c$ with each edge of the graph; now contract the $\mathfrak{g} \otimes \mathfrak{g}^*$ and $V \otimes V^*$ pairs throughout. Note that the orientation conventions correspond: we associate skew 3-tensors to skew vertices, and symmetric 2-tensors to unoriented edges. The Jacobi identity and the identity expressing that $V$ is a representation of $\mathfrak{g}$ correspond to the IHX and STU relations, showing that this evaluation descends to the algebra $\mathcal{A}$ and gives a weight system $w_{\mathfrak{g}, V} : \mathcal{A} \to \mathbb{Q}$.

3. Chern-Weil theory on a complex manifold

An attractive way to introduce Rozansky-Witten invariants is as a generalisation of the usual Chern-Weil theory. Suppose $E \to X$ is a smooth complex vector bundle on a smooth manifold $X$. Picking any smooth connection on $E$ defines a curvature form $F \in \Omega^2(X; \text{End}(E))$, from which we seek to produce topological invariants of $E$, quantities independent of the choice of connection. It is well-known that the forms

$$\text{ch}_d(E) = \frac{1}{d!} \text{tr} \left\{ \left( -\frac{F}{2\pi i} \right)^d \right\}$$

are closed, and that varying the connection alters them by coboundaries, so that their de Rham cohomology classes in $H^*(X; \mathbb{C})$ are indeed invariants, namely the components of the Chern character of $E$. The theory of characteristic classes shows that these are the only functions of the curvature
one need consider; in any case, there are no other sensible ways to combine an endomorphism-valued 2-form with itself and end up with a complex-valued differential form on $X$. A schematic picture of the form representing $\text{ch}_d(E)$ (in the case $d = 8$) is shown below. Each arrow represents a copy of $F$, the blob denoting the “2-form-ness” and the arrow shows that $F$, as a section of an endomorphism bundle has one “input” and one “output”. Eight disjoint arrows would represent the wedge product $F^8 \in \Omega^{16}(X; \text{End}(E)^{\otimes 8})$; the circle illustrates the contraction of consecutive inputs and outputs which computes the trace of the product of the endomorphisms.

If we look at a bundle with a structure group other than $GL(n, \mathbb{C})$ then there may be more subtle algebraic operations we can use; in fact, the ring of invariant polynomials on the Lie algebra parameterises the different (complex-valued) characteristic classes for $G$-bundles. For example, in the case real oriented case with structure group $SO(2n)$, the ring is generated by the traces of even powers (corresponding to the Pontrjagin classes) together with the Pfaffian, corresponding to the Euler class.

The basis of the Rozansky-Witten refinement (due, in this form, to Kapranov) occurs when $E$ is a holomorphic bundle on a complex manifold $X$. In this case there is a preferred class of connections, namely those compatible with the holomorphic structure and with some smooth hermitian metric on $E$. The curvature form $F_E$ of such a connection lies in $\Omega^{1,1}(\text{End}(E))$, but can also be thought of as an element $R_E \in \Omega^{0,1}(T^* \otimes \text{End}(E))$, where $T^*$ denotes the holomorphic cotangent bundle of $X$. Let $R_T$ denote a corresponding curvature form $R_T \in \Omega^{0,1}(T^* \otimes \text{End}(T))$ for the holomorphic tangent bundle $T$. Each of these can be pictured as a trivalent vertex with two input legs, one output leg, and a vertex carrying the “1-form-ness”; larger trivalent graphs can then be used to index the ways in which they can be combined tensorially, yielding a richer range of possibilities than in the basic Chern-Weil case.

In particular, the following three pictures denote elements of $\Omega^{0,2}(T^* \otimes T^* \otimes \text{End}(E))$ obtained by forming the exterior products $R_T \wedge R_E, R_E \wedge R_T, R_E \wedge R_E$ and then contracting tensorially according to the graph. The fundamental lemma is that the sum of these three elements is a $\bar{\partial}$-coboundary. Cohomologically, it will become the Jacobi or IHX identity.

$$= \bar{\partial}(\ldots).$$

A similar lemma shows that the element $R_T$ is symmetric in its two inputs, up to a coboundary. Kapranov shows how this identity (which is really just the Bianchi identity on $T^* \otimes E^* \otimes E$, expanded using the Leibniz identity), defines the structure of an $L_\infty$-algebra on the Dolbeault complex of forms with values in $T$. We will not use this result here, preferring to work at the level of cohomology.
4. ROZANSKY-WITTEN WEIGHT SYSTEMS

Now let \((X^{4n}, \omega)\) be a holomorphic symplectic manifold, that is a complex manifold of real dimension \(4n\), with \(\omega \in \Omega^0(\Lambda^2 T^*)\) a holomorphic non-degenerate skew 2-form on \(X\). Non-degeneracy of \(\omega\) gives a holomorphic identification \(T \cong T^*\), and we can use this to convert the curvature form \(R_T\) and symplectic form \(\omega\) itself into alternative versions

\[ f_T \in \Omega^{0,1}(T^* \otimes T^* \otimes T^*), \quad \omega^{-1} \in \Omega^0(\Lambda^2 T). \]

The 1-form \(f_T\) turns out to be totally symmetric in its three tensor indices.

Suppose we take a \(v\)-vertex trivalent graph with a preferred oriented circle. By associating \(f_T\) to its internal vertices, \(\omega^{-1}\) to its edges, \(R_E\) to its external vertices and wedging/contracting accordingly, we obtain an element of \(\Omega^{0,v}(X)\). This element is \(\bar{\partial}\)-closed, and varying the connections on \(E, T\) alters it by a coboundary, giving a well-defined cohomology class. To avoid a sign ambiguity here we need to orient the edges of the graph (because \(\omega^{-1}\) is skew) and order its vertices (because the vertices carry 1-forms, which anticommute). Remarkably, such an orientation of a graph is naturally equivalent to a vertex-orientation of the kind required in defining \(A\). The IHX and STU identities are satisfied because of the Bianchi identity, and the result is that we have a weight system

\[ w_{X,E} : A^v \to H^{0,v}_\bar{\partial}(X). \]

If \(X^{4n}\) is compact then we can make a weight system \(A^{2n} \to \mathbb{C}\) by associating to a graph \(\Gamma\) the integral

\[ b_T(X, E) = \int_X w_{X,E}(\Gamma) \wedge \omega^n. \]

These numbers are the original Rozansky-Witten invariants studied by Sawon \[Sa\]. Now compact holomorphic symplectic manifolds seem quite rare: Kähler ones are hyperkähler, by a theorem of Yau, and the known examples comprise two infinite families (the Hilbert schemes of points on the 4-torus or \(K3\) surface) and two exceptional examples due to O’Grady; even non-Kähler ones are relatively few.

However, the construction of the cohomology-valued \(w_{X,E}\) described above does not require \(X\) to be compact. Although in general we may know little about the target Dolbeault cohomology groups, these cases may turn out to be the most interesting for geometers, simply because there are so many examples of (non-compact) moduli spaces arising in geometry and physics which are naturally hyperkähler, and therefore holomorphic symplectic.

5. WEIGHT SYSTEMS REVISITED

What I will now explain is how to bring together the two extremely different constructions of weight systems we have seen. The basic ingredient is a reformulation of the construction of section 2 in a way that generalises to metric Lie algebras in categories other than the category of vector spaces. We will see subsequently that a holomorphic symplectic manifold gives rise to such a category and Lie algebra.

Suppose \(\mathcal{C}\) is a symmetric \(\mathbb{C}\)-linear tensor category. This is a category whose morphism sets are complex vector spaces, and which is equipped with an associative tensor operation \(\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}\), commutative in the sense that there are isomorphisms \(\tau_{A,B} : A \otimes B \to B \otimes A\), for all pairs of objects \(A, B\), satisfying \(\tau_{A,B}\tau_{B,A} = \text{id}\). The most obvious example other than the usual category of complex vector spaces is the category of super- (or \(Z_2\)-graded) vector spaces, in which the commutativity isomorphism incorporates a sign to make odd elements anticommute with one another. It is the
The fact that this commutativity isomorphism cannot be ignored is what led Vogel and Vaintrob to the picture I am about to describe: for Lie superalgebras, the standard construction explained earlier just doesn’t work.

An object \( L \) in \( C \) is a **Lie algebra** if it is equipped with a morphism \( L \otimes L \to L \) satisfying the Jacobi identity, interpreted as a linear relation between three morphisms in \( \text{Mor}(L^{\otimes 3}, L) \). Pictorially this is described by

\[
\begin{array}{ccc}
+ & + & = 0.
\end{array}
\]

where the pictures are read from bottom to top, the trivalent vertex represents the bracket, and the quadrivalent crossing represents the commutativity morphism \( \tau_{L,L} \). Such planar pictures are very useful for describing compositions in a tensor category, and are very common in the TQFT literature. See Bakalov and Kirillov [BK], for example.

Let \( 1 \) be the unit object for the tensor product in \( C \). A **non-degenerate metric** on a Lie algebra is a pair of morphisms \( L \otimes L \to 1, 1 \to L \otimes L \), pictured using a “cap” and “cup”, which satisfy a certain “S-bend” relation. A **left module** \( M \) over \( L \) is an object with a morphism \( L \otimes M \to M \), pictured as a trivalent vertex with a special oriented line marking the legs corresponding to \( M \).

A pair \( (L, M) \) determines a weight system

\[
\begin{align*}
w_{L,M} : A & \to \text{Mor}_C(1, 1).
\end{align*}
\]
To compute it, a diagram in \( A \) must first be drawn in the plane in a Morse position with respect to the vertical coordinate. Slicing it into horizontal bands, each containing a single vertex, cup or cap, defines a way to compose the structural morphisms above, starting and ending at the trivial object \( 1 \). (An attempt at Morsifying the example diagram from section 1 is shown below.) The result turns out to be independent of the planar picture used (remember that the diagrams in \( A \) are simply abstract trivalent graphs) and satisfies the correct orientation and IHX relations.

6. **Derived categories**

The use of derived categories is necessary for the algebraic unification we are proposing, and is even more important for the construction of the TQFT. This section contains a brief description of their structure, concentrating on the fact that **morphism sets in a derived category are cohomology groups**. Because of this, the derived world is the natural place to work if one wants to compose lots of cohomology classes. For details, see Thomas [Th3] and Gelfand and Manin [GM].

Let \( C \) be an abelian category, for example the category of modules over some ring \( R \). Thus, \( C \) has direct sums, kernels and cokernels, and it makes sense to consider the category of (bounded) chain
complexes $\text{Ch}(\mathcal{C})$. The most straightforward notion of equivalence for chain complexes is *chain homotopy equivalence*, and it is straightforward to pass to the corresponding homotopy category $K(\mathcal{C})$. But this is not really the correct notion: better is to use the *quasi-isomorphisms*, the chain maps which induce isomorphisms on homology, to generate the equivalence. (Whitehead’s theorem in topology, that a map between simply-connected CW-complexes inducing isomorphisms on homology is a homotopy-equivalence, is a helpful justification here.) Now homotopy-equivalences are always quasi-isomorphisms, but the converse is not true. For example, a quasi-isomorphism from a complex of free modules to a complex of torsion modules has no inverse, and this also shows that quasi-isomorphism isn’t an equivalence relation. Thus, by *forcibly* symmetrising the quasi-isomorphisms in $K(\mathcal{C})$, we obtain a quotient category $D(\mathcal{C})$ — the derived category of $\mathcal{C}$.

It is characterised by the universal property that any functor defined on $\text{Ch}(\mathcal{C})$ which takes quasi-isomorphisms to isomorphisms (the most obvious example being the homology-group functors $h_i: \text{Ch}(\mathcal{C}) \to \mathcal{C}$) factors through $D(\mathcal{C})$. The *objects* of the derived category are the same as those of $\text{Ch}(\mathcal{C})$. In particular, objects of the original category $\mathcal{C}$ may be identified with chain complexes whose only non-zero term lies in degree zero.

Under suitable conditions, a *functor* $F: \mathcal{A} \to \mathcal{B}$ between abelian categories may also be derived to a functor $D(F): D(\mathcal{A}) \to D(\mathcal{B})$. The *classical derived functors* associated to $F$ are just the composites of the homology functors $h_i$ with $D(F)$. For example, if $\mathcal{C}$ is the category of $R$-modules and $\mathcal{A}$ the category of abelian groups, then the “$R$-invariant part” functor $\Gamma = \text{Hom}_R(1, -): \mathcal{C} \to \mathcal{A}$ has a derived functor denoted $R\Gamma$, whose classical derived functors $H^i = h^iR\Gamma$ are the standard cohomology functors for $R$-modules, and are computable using resolutions. (In classical homological algebra there is an inessential distinction between applying such a functor to an object of $\mathcal{A}$, computing e.g. the cohomology of a module, and to an object of $D(\mathcal{A})$, computing e.g. the hypercohomology of a complex of modules.) A second example is that the bifunctor $\text{Hom}_R(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{A}$ has a derived functor $D(\mathcal{C})^{\text{op}} \times D(\mathcal{C}) \to D(\mathcal{A})$ denoted $R\text{Hom}$ or Ext, whose classical derived functors are the bifunctors $\text{Ext}_R^i(-, -)$.

The *morphisms* in $D(\mathcal{C})$ are cohomology groups, in fact

$$\text{Mor}_{D(\mathcal{C})}(A, B) = \text{Ext}^0_{\mathcal{C}}(A, B).$$

Observing that for objects of $\mathcal{C}$ (rather than complexes) $\text{Ext}^0_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$ shows that $\mathcal{C}$ is embedded in $D(\mathcal{C})$ as a full subcategory. More generally, we have

$$\text{Mor}_{D(\mathcal{C})}(A, B[i]) = \text{Ext}^i_{\mathcal{C}}(A, B).$$

Under this identification, the Yoneda product of cohomology classes

$$\text{Ext}^i(A, B) \otimes \text{Ext}^j(B, C) \to \text{Ext}^{i+j}(A, C).$$

becomes *composition* of morphisms, shifted so as to make sense. The construction of weight systems is, in effect, a complicated Yoneda product, which is far better expressed as a composition of morphisms.

7. ROZANSKY-WITTEN WEIGHT SYSTEMS

Let $X$ be a holomorphic symplectic manifold. We know that we can construct weight systems $w_{X,E}$ out of holomorphic vector bundles $E$ on $X$. In order to fit these into the framework of section 5, we want to view these bundles as *modules* over a Lie algebra in some category.

Ignore the symplectic structure for now and think of $X$ simply as a complex manifold. Let $\mathcal{O}_X$ be its *structure sheaf*, the sheaf of germs of holomorphic functions on $X$. It is a sheaf of *rings*, and the sheaves of germs of sections of vector bundles are examples of sheaves of $\mathcal{O}_X$-*modules*, in fact *locally-free* ones. If we consider the more general *coherent sheaves of $\mathcal{O}_X$-modules*, those which are
locally quotients of finite-rank locally-free sheaves, then we obtain an abelian category. It is then a natural step to pass to the derived category, which we denote by $D(X)$.

The objects of $D(X)$, then, are bounded chain complexes of coherent sheaves, though thinking of them as complexes of holomorphic vector bundles won’t do too much harm. The morphism sets are cohomology groups: in particular, for a pair of sheaves $E, F$ (viewed as complexes concentrated in degree zero, as described earlier) we have

$$\text{Mor}_{D(X)}(E, F[i]) = \text{Ext}^i(E, F),$$

a convenient description of the usual sheaf cohomology Ext-groups. This language subsumes the earlier differential-geometric language: in fact, when $E$ is a holomorphic vector bundle, we can identify the Dolbeault cohomology groups, via sheaf cohomology groups, as morphism sets:

$$H^{0,i}_\bar{\partial}(X; E) = H^i(E) = \text{Ext}^i(O_X, E) = \text{Mor}_{D(X)}(O_X, E[i]).$$

Notice that $O_X$ is the unit object in the tensor category $D(X)$, and that the last term in this sequence may be thought of as a kind of “space of invariants” of the object $E[i]$, by analogy with the situation in a module category.

In order to construct a Lie algebra in this category we need a sheaf-theoretic version of the curvature form of a bundle. Suppose $E$ is a holomorphic vector bundle on $X$. Its Atiyah class $\alpha_E \in \text{Ext}^1(E, E \otimes T^*)$ is the extension class (obstruction to splitting) arising from the exact sequence

$$0 \to E \otimes T^* \to JE \to E \to 0,$$

where $JE$ denotes the bundle of 1-jets of sections of $E$. (See [A] for the details.) It corresponds to the cohomology classes of the curvature forms $R_E$ and $F_E$ used in section 3 under the isomorphisms

$$\text{Ext}^1(E, E \otimes T^*) \cong H^1(E^* \otimes E \otimes T^*) \cong H^{0,1}_\bar{\partial}(E^* \otimes E \otimes T^*) \cong H^{1,1}_\bar{\partial}(\text{End}(E)).$$

For our purposes it is better to make the identification

$$\text{Ext}^1(E, E \otimes T^*) \cong \text{Ext}^1(E \otimes T, E) \cong \text{Mor}_{D(X)}(E \otimes T, E[1]),$$

because this begins to look like the structural map of a right module. Applying the shift functor $[-1]$ to each side produces a morphism

$$\alpha_E : E \otimes T[-1] \to E.$$

Repeating this for the tangent bundle $T$, with a shift of $[-2]$ gives

$$\alpha_T : T[-1] \otimes T[-1] \to T[-1].$$

This is in fact a Lie bracket: it is skew rather than symmetric, because of the parity shift, and the earlier Bianchi identity becomes the Jacobi identity. It is possible to extend the definition of the Atiyah class $\alpha_E$ to arbitrary coherent sheaves and even complexes of them, so that we have the following theorem.

**Theorem.** If $X$ is a complex manifold then the object $T[-1]$ is a Lie algebra in the category $D(X)$, and all other objects in $D(X)$ are modules over it.

Note that this is true for any complex manifold. So what about the symplectic structure? By viewing $\omega$ first as as a morphism $O_X \to \Lambda^2 T^*$ and then dualising and shifting judiciously, we construct a pair of morphisms

$$\omega : T[-1] \otimes T[-1] \to O_X[-2], \quad \omega^{-1} : O_X \to T[-1] \otimes T[-1][2].$$

These are actually symmetric rather than skew, again as a result of the interaction of the shift with the notion of commutativity.
Because these morphisms have non-trivial “degrees”, namely the outstanding shifts $[\pm 2]$, we have to alter the category $D(X)$ in order to be able to interpret them correctly as defining a non-degenerate metric. Define the graded derived category $\tilde{D}(X)$ to have the same set of objects as $D(X)$, but with the space of morphisms $A \to B$ being the graded vector space $\text{Ext}^*(A,B)$, instead of just $\text{Ext}^0(A,B)$. Composition of morphisms in $\tilde{D}(X)$ is graded bilinear. After this “deformation”, we can view $\omega$ as a genuine morphism $T[-1] \otimes T[-1] \to O_X$, and hence as a metric.

This deformation seems pleasing rather than puzzling if one compares it with the deformation quantization of the category of representations of a semisimple Lie algebra arising from the Knizhnik-Zamolodchikov equation, which is explained further in section 10. In summary:

**Theorem.** If $X$ is a holomorphic symplectic manifold then $T[-1]$ is a metric Lie algebra in the graded derived category $\tilde{D}(X)$, and all other objects are modules over it.

From this theorem and the methods of section 5 we get weight systems $w_{X,E} : A \to H^*(O_X)$ for each object $E$ of $D(X)$.

### 8. Wheels and wheeling

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ its universal enveloping algebra, and $S(\mathfrak{g})$ its symmetric algebra. The Poincaré-Birkhoff-Witt theorem says that the map $\text{PBW} : S(\mathfrak{g}) \to U(\mathfrak{g})$, given by including $S(\mathfrak{g})$ into the tensor algebra $T(\mathfrak{g})$ and then projecting, is an isomorphism of (filtered) vector spaces. It is equivariant, so gives an isomorphism $S(\mathfrak{g})^0 \cong U(\mathfrak{g})^0 = Z(\mathfrak{g})$.

Now $S(\mathfrak{g}), U(\mathfrak{g})$ are algebras, and though they aren’t isomorphic as algebras (one is commutative and one non-commutative!), their invariant parts are isomorphic as algebras, though not by the restriction of the PBW map. The isomorphism is called the **Duflo isomorphism**, and is built from the function

$$j^\frac{1}{2}(x) = \det^{\frac{1}{2}} \left\{ \frac{\sinh(\text{ad} x/2)}{\text{ad} x/2} \right\}$$

on the Lie algebra $\mathfrak{g}$. One can define a map $S(\mathfrak{g}) \to S(\mathfrak{g})$ by viewing this function as lying in the completion of $S\mathfrak{g}^*$ and applying the contraction $S(\mathfrak{g})^* \otimes S(\mathfrak{g}) \to S(\mathfrak{g})$, which is a kind of convolution or “cap product”. The Duflo theorem is that the composite

$$S(\mathfrak{g})^0 \xrightarrow{j^\frac{1}{2}} S(\mathfrak{g})^0 \xrightarrow{\text{PBW}} U(\mathfrak{g})^0$$

is an isomorphism.

Bar-Natan, Le and Thurston [Thu] discovered a purely diagrammatic version of this statement. Let $\mathcal{B}$ be the space spanned by Jacobi diagrams with both trivalent and unitrivalent vertices, subject to the usual orientation convention and relations. It is a bi-graded rational commutative algebra under disjoint union: write $\mathcal{B}^{v,l}$ for the part with $v$ internal trivalent vertices, and $l$ legs.

This algebra $\mathcal{B}$ plays the role of $S(\mathfrak{g})^0$: the original algebra $\mathcal{A}$, with connect-sum of diagrams as its product, plays the role of $U(\mathfrak{g})^0$. The analogue of the PBW map is $\chi : \mathcal{B} \to \mathcal{A}$, defined by sending an $l$-legged diagram to the average of the $l!$ diagrams obtained by attaching its legs in all possible orders to an oriented circle. This is an isomorphism of rational graded vector spaces [BN], but not of algebras.

The analogue of the function $j^\frac{1}{2}$ is the special **wheels element**

$$\Omega = \exp \left\{ \sum_{i=1}^{\infty} b_{2i} w_{2i} \right\} ,$$
living in the completion of $\mathcal{B}$. Here, $w_{2i}$ is a wheel diagram (a circular hub with $2i$ legs) and the $b_{2i}$ are versions of Bernoulli numbers such that replacing $w_{2i}$ by the function $\text{tr}(\text{ad} x)^{2i} \text{ recovers } j^\frac{1}{2}(x)$.

Contraction (we will again write a cap product) with a diagram $C \in \mathcal{B}$ is the operation $\cap : \mathcal{B} \to \mathcal{B}$ given by summing over all attachments of legs of $C$ to legs of the target diagram. This extends to an action of the completion of $\mathcal{B}$. The wheeling theorem is then that

$$\mathcal{B} \xrightarrow{\Omega \cap} \mathcal{B} \xrightarrow{\chi} \mathcal{A}$$

is an isomorphism of algebras.

A metric Lie algebra $\mathfrak{g}$ defines weight systems $\mathcal{B} \to S(\mathfrak{g})^\oplus$ and $\mathcal{A} \to U(\mathfrak{g})^\oplus$, setting up a commuting diagram which intertwines the wheeling and Duflo isomorphisms (see [BGRT] for the details). (It seems peculiar here that the Duflo isomorphism itself is true for all Lie algebras, not just the metric ones.)

This whole picture has an analogue for holomorphic symplectic manifolds. We construct in [RW] a pair of objects $S, U$ in $\tilde{D}(X)$ which are the symmetric and universal enveloping algebras of the Lie algebra $\mathfrak{T}[-1]$, in the sense that they are associative algebras with appropriate universal properties. The description of $S$ is straightforward, because one can take tensor powers and form symmetrisers inside $\tilde{D}(X)$. Constructing $U$ is not so straightforward, because $\tilde{D}(X)$ is not abelian and the usual quotient construction therefore doesn’t make sense. As one might expect there is also a PBW isomorphism $S \cong U$ which does not respect their structure as associative algebras.

The “invariant parts” of these two algebras are the algebras $\text{Mor}(\mathcal{O}_X, S)$ and $\text{Mor}(\mathcal{O}_X, U)$, which are just the total cohomology groups of these objects and are finite-dimensional graded complex algebras. They appear in Kontsevich’s remarkable paper [Ko1] under the names $HT^\ast$ (cohomology of polyvector fields) and $HH^\ast$ (Hochschild cohomology). To be precise, we have

$$\text{Mor}(\mathcal{O}_X, S) = \bigoplus_n HT^n(X) = \bigoplus_n \bigoplus_{i+j=n} H^i(\Lambda^j \mathcal{T})$$

$$\text{Mor}(\mathcal{O}_X, U) = \bigoplus_n HH^n(X) = \bigoplus_n \text{Ext}^n_{\mathcal{O}}(O_{\Delta}, O_{\Delta})$$

where $O_{\Delta}$ is the structure sheaf of the diagonal in $X \times X$. The algebra structures here are the natural ones: wedge product and Yoneda product, respectively. The usual Dolbeault cohomology $H^\ast(\Lambda^\ast \mathcal{T})$ acts by contraction on $HT^\ast$, and Kontsevich proved that cap product with the root-A-hat-class, the characteristic class corresponding to the power series

$$\left\{ \frac{\sinh(x/2)}{x/2} \right\}^\frac{1}{2}$$

gives a Duflo-style isomorphism of algebras

$$HT^\ast \xrightarrow{\hat{\Delta}} HT^\ast \xrightarrow{\text{PBW}} HH^\ast.$$
9. The Hitchin-Sawon theorem

**Theorem.** Let $X^{4n}$ be a compact hyperkähler manifold. Then there is an identity relating the root-A-hat-genus of $X$, its volume and the $L^2$-norm $\|R\|$ of its Riemann curvature tensor:

$$\hat{A}^2[X] = \frac{1}{(192\pi^2)^n} \frac{\|R\|^{2n}}{\text{vol}(X)^{n-1}}.$$ 

This theorem, proved in [HS], is a striking identity between a topological and a geometric invariant of $X$. Its proof is a consequence of the wheeling theorem, and shows the potential of Rozansky-Witten theory in geometry.

The basic idea of the proof is worth explaining. We use the Rozansky-Witten weight systems in the form $w_X : B^{v,l} \to H^{l,v}_0(X)$. There are two numerical evaluations one can consider: if $\Gamma \in B^{2n,2n}$ is a diagram with $2n$ internal vertices and $2n$ legs then we can define

$$c_X(\Gamma) = \int_X w_X(\Gamma),$$

whereas if $\Gamma \in B^{2n,0}$ is a “closed” trivalent diagram with $2n$ vertices then we can define

$$b_X(\Gamma) = \int_X w_X(\Gamma) \wedge \omega^n.$$ 

The key observation is that $c_X(\Gamma) = b_X(\text{cl}(\Gamma))$ (up to normalisations and signs) when $\text{cl}(\Gamma)$ is the closure of $\Gamma$, the element obtained by summing over all ways of pairing up the legs of $\Gamma$. (Remember that we are always really working with linear combinations of diagrams.) This is a straightforward consequence of the shuffle product formula for the top power $\omega^n$ of a symplectic form evaluated on a list of $2n$ vectors.

If $\Gamma$ is chosen to be a polywheel diagram, that is, a disjoint union of wheels with $2n$ legs in total, then $w_X(\Gamma)$ turns out to be a product of Chern character terms with total degree $2n$, and so $c_X(\Gamma)$ is a linear combination of Chern numbers. In particular, for the special wheels element $\Omega$ we get the root-A-hat polynomial $w_X(\Omega) = \hat{A}^2(TX)$, and hence $c_X(\Omega) = \hat{A}^2[X]$.

On the other hand, the closure $\text{cl}(\Omega)$ may be computed using Jacobi diagram techniques worked out by Bar-Natan, Le and Thurston [Thu], and it turns out to be $\exp(\Theta/24)$, where $\Theta$ is the two-vertex theta-graph. A standard result in hyperkähler geometry expresses this graph in terms of the norm of the curvature and gives the result.

Huybrechts used this identity to prove a finiteness theorem for hyperkähler manifolds [H]. It is to be hoped that further identities of a similar nature will emerge from the TQFT structure of RW theory, and help in the programme of classification of hyperkähler manifolds.

10. Link invariants

Let us turn now to the topological invariants arising from Rozansky-Witten theory. Recall that a semisimple Lie algebra $\mathfrak{g}$, together with a representation $V$, defines a weight system $w_{\mathfrak{g},V} : \mathcal{A} \to \mathbb{C}$, and that composing this with the Kontsevich integral should give us an invariant of framed knots in $S^3$. Because the Kontsevich integral actually lies in the completion of $\mathcal{A}$, we must first introduce a variable $h$ of degree two and complete the weight system into a graded map

$$w_{\mathfrak{g},V} : \hat{\mathcal{A}} \to \mathbb{C}[[h]],$$

from which we obtain a $\mathbb{C}[[h]]$-valued invariant $Z_{\mathfrak{g},V}$ of framed knots in $S^3$. 

There is an alternative formulation of this invariant using the framework of ribbon categories, invented by Turaev \cite{Turaev} to handle the invariants such as the Jones polynomial which arise from quantum groups. A **ribbon category** is a braided tensor category with a compatible notion of duality. It gives rise to representations of the category of framed "coloured" tangles: one assigns an object of the category (a "colour") to each string of a tangle, and then composes the elementary structural morphisms in the category (braidings and dualities) according to a Morse-theoretic slicing of the tangle into crossings, cups and caps. In particular, one gets invariants of framed coloured links, with values in the endomorphisms of the unit object of the category.

The **Knizhnik-Zamolodchikov equation** builds an interesting ribbon category from a semisimple Lie algebra. We start with the usual symmetric tensor category of modules over \( g \), tensor it with \( \mathbb{C}[[\hbar]] \), and then use the monodromy of the KZ equation to introduce a new, braided tensor (and in fact ribbon) structure. The resulting category has non-trivial associator morphisms (it is sometimes called a quasitensor category), but turns out to be equivalent to the strictly associative category of representations of the quantum group \( U_\hbar(g) \); see Bakalov and Kirillov \cite{BK}, for example. Applying Turaev’s machinery to this category gives back the \( \mathbb{C}[[\hbar]] \)-valued invariant \( Z_{g,V} \) of framed knots.

It turns out that there is an analogous picture for a holomorphic symplectic manifold \( X \). As we have seen, a vector bundle or sheaf \( E \) on \( X \) gives a weight system \( w_{X,E} : A \to H^\ast(O_X) \), and therefore a \( H^\ast(O_X) \)-valued Vassiliev invariant \( Z_{X,E} \) of framed knots.

**Theorem.** The graded derived category \( \tilde{D}(X) \) of a holomorphic symplectic manifold \( X \) may be given a ribbon structure so that the associated invariant of a knot coloured by \( E \in D(X) \) is \( Z_{X,E} \).

The starting point of this construction is the usual derived category \( D(X) \), which is a symmetric tensor category. The analogue of tensoring with \( \mathbb{C}[[\hbar]] \) is its replacement by the graded version \( \tilde{D}(X) \), in which the shift \([2]\) plays the role of \( \hbar \). The \([2]\) is attached to the symplectic form \( \omega \), just as in the Lie algebra case, \( \hbar \) may be thought of as attached to the metric. We then use the Kontsevich integral, which underlies the KZ equation, to define the ribbon structure \( \tilde{D}(X) \).

By construction, the tensor structure on \( \tilde{D}(X) \) is not strictly associative. We don’t know whether it is possible to make a gauge transformation, as Drinfeld does in the case of quantum groups \cite{Drinfeld}, to a form which is strictly associative and has a local, but more complicated, braiding. Drinfeld’s transformation is defined purely algebraically, and it is not clear how to derive it from geometry in the way we would like. Equally remarkably, the \( \mathbb{C}[[\hbar]] \)-valued invariants coming from Lie algebras turn out to be the expansions, on setting \( q = e^{\hbar} \), of Laurent polynomials in \( q \). Is there some analogous hidden structure to the Rozansky-Witten link invariants?

11. **TQFT**

The basis of Rozansky and Witten’s work is the path integral which defines the partition function for a closed oriented 3-manifold \( M \):

\[
Z_X(M) = \int_{\text{Map}(M,X)} e^{iS} d\phi.
\]

The integral is over all smooth maps \( \phi : M \to X \), and the action \( S \) is an expression involving a Riemannian metric on \( M \) and the hyperkähler metric on \( X \). It turns out to be independent of the metric on \( M \), meaning that the associated quantum field theory, too, is **topological**. The link invariants described above may also be formulated using the path integral, by insertion of suitable “Wilson loop” observables.

We would like to have a rigorous construction of this TQFT, with which we can calculate and explore. That is, we would like to build a TQFT which shares the predicted properties of the
genuine physical theory; a “Reshetikhin-Turaev invariant” for our “Witten invariant”. This is underway in joint work with Justin Sawon and Simon Willerton.

To do it we have to work combinatorially. We will build the TQFT by specifying its values on elementary pieces of surfaces and 3-manifolds, calculate for larger manifolds by means of decompositions into such pieces, and prove that our specified elementary data satisfies the coherence relations which ensure that different decompositions of a 3-manifold compute the same invariant. This is not, in all honesty, a very satisfactory method, but there’s little reasonable alternative.

A standard TQFT, such as Witten’s Chern-Simons theory [Wi], is a functor from the category of (2+1)-dimensional cobordisms to the category of vector spaces. It is typically a tensor functor, taking disjoint unions to tensor product; and a unitary one, taking surfaces to hermitian vector spaces. There is a way to enhance such a TQFT into an extended or (1+1+1)-dimensional theory, in which we also assign a category to each closed 1-manifold. In the usual theories this category is semisimple, with finitely many simple objects, and is promptly replaced by a set of colours corresponding to these objects. In Chern-Simons theory, the category for a single circle is the (truncated) representation category of a quantum group at a root of unity.

The Rozansky-Witten TQFT is somewhat different, and requires a different formalism, which has been studied by Freed, Segal, Tillmann and Khovanov [Ti, Kh], amongst others. In this formalism we view a (1+1+1)-dimensional TQFT as a functor from the 2-category of 3-cobordisms with corners, to the 2-category of linear categories. Thus, a 1-manifold is sent to a category; a “vertical” 2-dimensional cobordism between 1-manifolds defines a functor between these categories; and a “horizontal” 3-dimensional cobordism between two surfaces which have common lower and upper 1-manifolds defines a natural transformation of functors. There’s no good reason here for imposing a tensor product axiom, unitarity, or semisimplicity of the categories associated to 1-manifolds.

Indeed, our theory has none of these. Its basic feature is that a 1-manifold consisting of \(k\) circles gets sent to the graded derived category \(\hat{D}(X^k)\): clearly \(\hat{D}(X)\) is not semisimple and it is not true that \(\hat{D}(X \times X) \cong \hat{D}(X) \times \hat{D}(X)\).

To construct the functor associated to a surface, we break it using Morse theory into elementary pieces, and write down the basic functors associated to the different types of handles. For example, a 2-handle gives the pushforward functor \(\hat{D}(X) \to \hat{D}(pt)\); a 0-handle, the functor \(\hat{D}(pt) \to \hat{D}(X)\) which sends \(C\) to \(O_X\); and an index 1-handle which joins two circles, the functor \(\hat{D}(X \times X) \to \hat{D}(X)\) obtained by taking derived tensor product with the structure sheaf of the diagonal, then pushing forward once.

To prove independence of the decomposition we have to use the ribbon structure on \(\hat{D}(X)\), and some extra properties resembling modularity which arise from diagrammatic identities and the Kontsevich integral. These are enough to check that the Moore-Seiberg equations are satisfied. To define the invariants for elementary 3-manifolds is more complicated, and uses Walker’s framework [Wa].

It is possible to recover a traditional (2+1)-dimensional TQFT functor which sends surfaces to vector spaces instead of functors from our axioms. The empty 1-manifold is sent to \(\hat{D}(pt)\), which is essentially the category of graded vector spaces. A closed surface defines a functor \(\hat{D}(pt) \to \hat{D}(pt)\), which when applied to the generating object \(C\) in \(\hat{D}(pt)\) outputs the desired graded vector space.

Using the basic data specified above, we can compute that the graded vector space associated to a closed surface \(\Sigma_g\) is isomorphic (non-canonically) to the cohomology \(H^*(\Lambda^*T)^{\otimes g}\), as postulated by Rozansky and Witten. An action of the mapping class group on this space emerges from the Moore-Seiberg equations. The superdimensions of these spaces are given for the sphere and torus
by the Todd genus and Euler characteristic of $X$, respectively, and they vanish for $g \geq 2$. The space associated to $S^2$, which is naturally an associative algebra in any TQFT, is in fact the ring $H^*(O_X)$, which for an irreducible hyperkähler $X$ is the truncated polynomial ring $\mathbb{C}[\bar{\omega}]/(\bar{\omega}^{n+1})$. It is not semisimple, which shows that the TQFT cannot be unitary.

This TQFT has a close relationship with the Le-Murakami-Ohtsuki invariant $\text{LMO}$, which is a kind of extension of the Kontsevich integral to 3-manifolds, with values in the algebra of trivalent Jacobi diagrams. In principle, the TQFT invariant of a closed 3-manifold should equal the LMO invariant, evaluated using a weight system coming from $X$; the TQFT itself should be the evaluation of the universal TQFT constructed by Murakami and Ohtsuki $\text{MO}$, which underlies the LMO invariant. However, their TQFT is somewhat badly behaved axiomatically, and there are subtle differences in normalisation which make it unsatisfactory to try to use this approach as a definition. The $(1+1+1)$-dimensional framework seems to work so well when we are building the specific theory associated to $X$ that it is better to use it as a foundation, and to make connections with the universal theory in retrospect.

12. Future directions

In [RS] we give an extensive discussion of the potential applications of RW invariants to the geometry of hyperkähler (and certain other types of) manifolds, and conclude with a problem list. Further discussion of the interaction with Vassiliev theory, the theory of quantum invariants and TQFT will appear in [RSW]. But it is worth mentioning a few problems here, just to give the flavour of possible future research.

What is the meaning of the $\hat{A}^+_1$ genus?

This genus appears in Kontsevich’s theorem and the Hitchin-Sawon theorem, as we have seen. But it also appears in physics, in the context of cohomological $D$-brane charge. It seems natural that these two occurrences are related, but how and why? (With impeccable timing, physicists are also now adopting derived categories of coherent sheaves as basic structures in string theory!) A straightforward question is whether the root-A-hat genus, like the honest A-hat genus, has interesting integrality properties, and can be interpreted for manifolds with some appropriate geometric structure as the index of a natural elliptic operator.

Are the Rozansky-Witten weight systems new, and can their associated Vassiliev invariants detect orientation of knots?

Vogel showed that the primary examples of Vassiliev weight systems, those coming from complex semisimple Lie algebras and superalgebras, do not span the whole space of weight systems. But we do not yet know whether the Rozansky-Witten weight systems lie outside their span or not.

It’s easy to prove that the Vassiliev invariants coming from Lie algebra weight systems are unable to distinguish knots from their reverses. It is in fact thought likely that no Vassiliev invariants can separate knots from their reverses, and there is an alternative purely diagrammatic statement of this conjecture. But the proof that Lie algebra weight systems fail doesn’t work for Rozansky-Witten invariants, so there is still potential here.

Is there a geometric quantization approach to the vector spaces $Z(\Sigma_g)$?

These vector spaces, as we have seen, can be constructed from a combinatorial approach to the TQFT. This is a reasonable but crude approach: much better would be to give a direct geometric (and completely functorial) construction. In Witten’s Chern-Simons theory with gauge group $G$, there is an approach via geometric quantization of the moduli space of flat $G$-connections on a surface. Is there an analogue in Rozansky-Witten theory? A straightforward guess is that it might
be possible to define a virtual structure sheaf of the moduli space of holomorphic maps from a closed Riemann surface \( \Sigma_g \) to \( X \), and take its cohomology as the graded vector space. We would then require a “projectively flat connection”, a system of coherent isomorphisms between these sheaves, as the complex structure on \( \Sigma_g \) varies. Going a level deeper, how does one construct the category \( \tilde{D}(X) \) functorially from a given circle and manifold \( X \)? In the Chern-Simons case, one uses the representation category of the group of loops from the circle to \( G \). What is the analogue? We would require surfaces with boundary to generate functors between such categories by means of a Fourier-Mukai transform operation.

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