Some properties of renormalons in gauge theories

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Abstract

We find the explicit operatorial form of renormalon-type singularities in abelian gauge theory. Local operators of dimension six take care of the first U.V. renormalon, non local operators are needed for I.R. singularities. In the effective lagrangian constructed with these operators non local imaginary parts appearing in the usual perturbative expansion at large orders are cancelled.
I. INTRODUCTION

It is well known that in field theory the perturbative series diverge [1] and are asymptotic at best so that the question of the reliability of the perturbative expansion is raised. In particular the factorial growth of the perturbative coefficients has been related to the existence of instanton solutions [2] and, in this way, a great number of results has been found (for a review see [3]). The perturbative expansion for renormalizable field theories is plagued by further divergences, the so-called renormalons, which show up as singularities on the real axis of the Borel transform of the perturbative series [4,5]. Moreover this kind of problems has recently gained a great interest [6–13] and many questions, mainly concerning the asymptotic behaviour of perturbative expansions in QCD, still wait for an answer.

The standard technique to cope with divergent series is the introduction of the Borel transform. If a function $F(\alpha)$ admits an asymptotic expansion and satisfies certain analytic properties [14,15], it can be uniquely reconstructed from the Borel transform $\tilde{F}(b)$ defined by:

$$F(\alpha) = \int_0^\infty db \ e^{-b/\alpha} \tilde{F}(b)$$

(1.1)

The presence of singularities (poles or cuts) on the positive real axis of the Borel transform forces to give a contour prescription (for example a principal value prescription) in order to define the perturbative expansion via the integral (1.1). This results in an ambiguity in the sum of the perturbative series given by the integration above and below the positive real axis in the Borel plane.

To be definite, a singularity of the form:

$$\tilde{F}(b) \sim \frac{1}{(b-b_0)^{1-s}}$$

(1.2)

gives rise to an imaginary part in the function $F(\alpha)$ of the form:

$$\text{Im} F(\alpha) \sim \alpha^s e^{-b_0/\alpha}$$

(1.3)

which in turn is related, via a dispersion relation, to the large order behaviour of the coefficients $f_n$ of the perturbative series:

$$f_n \sim _{n \to \infty} \text{const} \left( \frac{1}{b_0} \right)^n n^s \Gamma(n - s)$$

(1.4)

The renormalon-type singularities occur for

$$b_0 = \pm n/\beta_2$$

(1.5)

where $n$ is a positive integer and $\beta_2$ is the first term of the $\beta$-function and the sign plus refers to the ultraviolet renormalons while the sign minus to the infrared ones. These
divergences are usually the closest ones to the origin and give the leading asymptotics for the perturbative series.

In the following we want to study the structure and a possible classification of the singularities due to renormalons, in the framework of QED with a large number of flavours [16].

The aim of the first part of the paper is to present a prescription which could allow to take systematically into account the imaginary parts (see eq. (1.3)) coming from ultraviolet renormalons. The idea stems from a conjecture made for the first time by Parisi for the $\phi^4$ theory [17] and explicitly checked in the $1/N$ limit of this theory [17–19].

This conjecture relates the imaginary parts coming from the renormalon at $b = n/\beta^2$ to the insertion at zero momentum of local operators of dimension $d = 2n + 4$. Renormalization group arguments fix the form of the imaginary parts from which the asymptotic behaviour (1.4) is extracted (with given $b_0$ and $s$) up to an overall normalization constant.

We implement this conjecture adding gauge-invariant operators as imaginary counterterms in the lagrangian. The feasibility of such a description for the renormalon-type singularities relies on the fulfilment of two conditions:

- A unique choice must exist for the operators in order to remove the imaginary parts of all Green functions.
- In the effective theory described by the lagrangian with the counterterms, non local singularities in the Borel transformed Green functions must be absent.

The second half of the paper concerns the problem of infrared renormalons [20–22]. In the case of QCD a connection has been conjectured between infrared renormalons and the admissible multilocal gauge-invariant operators which can be seen as the dual of the local operators of the theory [21]. Then from the absence of local gauge-invariant operators of dimension two, the absence of the infrared renormalon at $b = -1/\beta^2$ has been argued. Recently the possibility of the existence of this renormalon has been raised [7,8] and this would suggest the presence of a $1/q^2$ term in the operator product expansion for euclidean correlation functions [23] and would spoil the connection with the gauge-invariant operators.

While on general grounds this possibility cannot be ruled out, an explicit calculation [24] shows that this pole doesn’t indeed exist within the $1/N_f$ expansion for QED. Moreover QCD is believed to have the same behaviour of QED after replacing the right $\beta_2$ coefficient.

In the spirit of the calculation of [24] we study the structure of the first infrared renormalons in QED. In particular we find the form of the multilocal counterterms compensating the imaginary parts arising from infrared renormalons and, in connection with the case of the ultraviolet renormalons, we check that a unique prescription can be given.
A short comment is in order about the use of the $1/N_f$ expansion. This kind of expansion is the natural framework for the study of renormalons in that it provides a graphic tool (see figures) for the investigation of these problems\footnote{After completion of this work, we came at vision of \cite{25} in which the authors conclude that it is not possible in general to associate the ultraviolet renormalons to a well-defined set of graphs; this is indeed possible in the large $N_f$ limit; on the other hand the calculations performed in our paper are at fixed order in $1/N_f$.}. Moreover it allows a partial resummation of the perturbative series so that one can obtain informations on the whole series and calculate quantities, like the overall normalization constant in (1.4), which cannot be found at any finite order in perturbation theory\cite{12}.

The paper is organized as follows. In Sect.II we introduce the notations and study the problem of the first ultraviolet renormalon. We find the explicit form of the counterterms and we check that all singularities can be absorbed in such a way. Moreover we verify that the non local singularities which appear in the naive perturbation theory are indeed absent when considering the lagrangian with the counterterms. In Sect.III we perform similar calculations for the infrared renormalons; the main difference is that, in this case, the operators to be added to the lagrangian are multilocal. Sec.IV is devoted to some considerations about the meaning of the lagrangian with the imaginary counterterms. Finally, the Appendix contains some technical details about the calculations with the background field formalism.

II. ULTRAVIOLET RENORMALONS

Let us begin this section with the introduction of some definitions about the $1/N_f$ expansion in QED.

Defining a rescaled coupling constant $a = \alpha N_f$, the $1/N_f$ expansion is generated, at fixed $a$, by considering a factor of $N_f^{-1/2}$ for every interaction vertex and a factor of $N_f$ for every closed fermion loop. Then the photon propagator is given at leading order by the sum of the bubble diagrams (fig.1). In the Landau gauge it reads:

$$D_{\mu\nu}(k) = -\frac{k_\mu k_\nu - k^2 g_{\mu\nu}}{k^4} D(k)$$  \hspace{1cm} (2.1)

where:

$$D(k) = \frac{1}{1 + \Pi_0(k)}$$  \hspace{1cm} (2.2)

and $\Pi_0(k)$ is the single bubble diagram. In the following we are interested in the high momentum region of integration in the Feynman integrals; so we need the expression of
\( \Pi_0(k) \) in the deep euclidean region. After renormalization in the \( \overline{\text{MS}} \) scheme, we have (all fermions are assumed to have the same mass):

\[
\Pi_0(k) \sim -\frac{a}{3\pi} \left\{ \log \frac{k^2}{\mu^2} - \frac{5}{3} \right\} + \frac{2a}{\pi} \frac{m^2}{k^2} + O \left( \frac{1}{k^4} \right)
\]  

(2.3)

From (2.2) and the definition (1.1) one can easily derive the expression for the Borel transform, with respect to \( a \), of the photon propagator; introducing an extra factor of \( a \), we have:

\[
F(k, b) \equiv (aD)(k, b) \sim e^{-5b/9\pi} \left( \frac{k^2}{\mu^2} \right)^{b/3\pi} \left[ 1 - \frac{2bm^2}{\pi k^2} + O \left( \frac{1}{k^4} \right) \right]
\]  

(2.4)

A. Imaginary part of Green functions

At next-to-leading order in \( 1/N_f \), the Green functions correspond to the graphs with one insertion of the chain of the bubbles (fig.2-4) and present renormalon-type singularities. In fact the Feynman integrals, in which the photon propagator (2.2) appears in the integrand, have an ambiguity arising from the region of integration around the Landau pole (i.e. the pole in (2.2)) which occurs, for infrared free theories (in our case \( \beta(a) = a^2/3\pi + O(1/N_f) \)) in the ultraviolet region:

\[
\Lambda^2 = \mu^2 e^{5/3} e^{1/\beta a}
\]  

(2.5)

(we are always assuming the \( \overline{\text{MS}} \) scheme). This ambiguity causes the Green functions to develop an imaginary part and is related to the leading factorial growth of the coefficients of the perturbative series in \( \alpha \). In fact it turns out that, after expanding (2.2) in powers of \( \alpha \), the coefficient of order \( n \) receives a contribution proportional to \( n! \) from the ultraviolet region of integration. These problems appear as poles on the real positive axis of the Borel transformed Green functions.

In the following we intend to find the behaviour of the Borel transform of the vertex function, of the electron propagator and of the photon vacuum polarization near the first ultraviolet renormalon at \( b = 3\pi \). This will enable us to find the imaginary part of these functions and to study the connection with local gauge-invariant operators of dimension six.

Let us begin with the Borel transform of the vertex function (fig.2). In the euclidean space the term of order \( 1/N_f \) is given by:

\[
\left( \frac{\Gamma_\mu}{e} \right) = -\frac{4\pi}{N_f} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\rho[p' - \hat{k} - m]\gamma_\sigma[p - \hat{k} - m]}{[(p' - k)^2 + m^2][(p - k)^2 + m^2]} \frac{k_\rho k_\sigma - k^2\delta_\rho\sigma}{k^4} F(k, b)
\]  

(2.6)

where we have factorized out the electron charge \( e \). Renormalization is taken into account by simply introducing the renormalized bubble. In fact, since we are working in the
Landau gauge, the vertex function needs no further renormalization and this results in the absence of a pole in \( b = 0 \) in \( \tilde{\Gamma}_\mu \) as written in (2.6). Moreover introducing the term of order \( 1/k^2n \) in the expansion (2.4) of \( F(k, b) \) results in a pole in \( \tilde{\Gamma}_\mu \) at \( b = 3\pi(n + 1) \).

After some manipulations with \( \gamma \) matrices and performing the integral we obtain for the first pole:

\[
\left( \frac{\tilde{\Gamma}_\mu}{e} \right)_{b \sim 3\pi} \simeq \frac{1}{N_f} \frac{1}{8} \left( \frac{1}{\mu^2} \right)^{b/3\pi} \frac{e^{-5b/9\pi}}{b - 3\pi} \left\{ \left[ (p - p')^2 - 9m^2 - 6p \cdot p' \right] \gamma_\mu 
+ 3\hat{p}' \gamma_\mu \hat{p} - 4(p'_\mu - p_\mu)(\hat{p}' - \hat{p}) \right\}
\]  

(2.7)

From the Borel representation (1.1) we deduce that \( \Gamma_\mu \) develops an imaginary part given by the residue at the pole in \( b = 3\pi \):

\[
\text{Im } \Gamma_\mu = \pi e \text{ Res} \left\{ \left( \frac{\tilde{\Gamma}_\mu}{e} \right) e^{-b/a} \right\}_{b = 3\pi}
\]  

(2.8)

Then (2.7) yields:

\[
\text{Im } \Gamma_\mu = \frac{1}{N_f} \frac{\pi e}{8\Lambda^2} \left\{ [4q^2 - 3p^2 - 3p'^2 - 9m^2] \gamma_\mu + 3\hat{p}' \gamma_\mu \hat{p} - 4q_\mu \hat{q} \right\} + O \left( \frac{1}{\Lambda^4} \right)
\]  

(2.9)

On the mass-shell this expression reduces to:

\[
\text{Im } \Gamma_\mu = \frac{1}{N_f} \frac{\pi e}{2\Lambda^2} q^2 \gamma_\mu
\]  

(2.10)

hence there is no contribution of the first ultraviolet renormalon to the large order behaviour of the perturbative series of the anomalous magnetic moment of the electron, in agreement with the asymptotic behaviour found in [3].

From (2.9) we see that the imaginary parts arising from different renormalons are classified, at this order in \( 1/N_f \), in terms of the renormalization group invariant (i.e. \( \mu \)-independent) \( \Lambda \)-parameter which plays the role of a dimensional expansion parameter. It must be noted that this parameter encloses all the dependence of the calculation from the choice of the renormalization scheme.

Similar considerations hold for the fermion propagator (fig.3). Its Borel transform at order \( 1/N_f \) reads:

\[
\tilde{\Sigma}(p, b) = \frac{4\pi}{N_f} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\nu[(\hat{p} + \hat{k}) - m] \gamma_\mu}{(p + k)^2 + m^2} \frac{k^2 \delta_{\mu\nu} - k_\mu k_\nu}{k^4} F(k, b)
\]  

(2.11)

Isolating the pole at \( b = 3\pi \) we have:

\[
\tilde{\Sigma}(p, b) \simeq \frac{1}{N_f} \left( \frac{1}{\mu^2} \right)^{b/3\pi} \frac{e^{-5b/9\pi}}{b - 3\pi} \left\{ \frac{9}{8} m^2 \hat{p} + \frac{9}{4} m^2 + \frac{3}{8} p^2 \hat{p} + \frac{9}{2} \frac{bm^3}{\pi} \right\}
\]  

(2.12)
This yields an imaginary part given by:

\[ \text{Im } \Sigma(p) = \frac{1}{N_f \Lambda^2} \left\{ -\frac{9}{8} m^2 \tilde{p} - \frac{3}{8} \mu^2 \tilde{p} + \frac{63}{4} m^3 \right\} + O \left( \frac{1}{\Lambda^4} \right) \tag{2.13} \]

It is straightforward to verify that (2.13) and (2.13) satisfy the Ward identity:

\[ q_\mu \text{Im } \left( \frac{\Gamma_\mu}{e} \right) = \text{Im } \Sigma(p') - \text{Im } \Sigma(p) \tag{2.14} \]

Finally, the imaginary part of the photon vacuum polarization can be deduced from Beneke [24]. Exploiting gauge-invariance for the proper photon propagator we have:

\[ \Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2) \tag{2.15} \]

with:

\[ \Pi = \Pi_0 + \Pi_1 \frac{1}{N_f} + O \left( \frac{1}{N_f^2} \right) \tag{2.16} \]

\( \Pi_0 \) is the single bubble, while \( \Pi_1 \) corresponds to the diagrams of fig.4. The result found by Beneke in the massless limit is:

\[ \left( \frac{\Pi_1}{a} \right)(Q^2, b) = -\frac{1}{6\pi^2} e^{-5b/9\pi} \left( \frac{Q^2}{\mu^2} \right)^{b/3\pi} D(b/3\pi) \tag{2.17} \]

with:

\[ D(u) \sim \frac{2}{3} \frac{1}{(1 - u)^2} + \frac{11}{9} \frac{1}{1 - u} + O(1) \tag{2.18} \]

From this expression one can infer the imaginary part of the proper photon propagator:

\[ \text{Im } \Pi_{1,\mu\nu}(q^2) = -\frac{a}{3 \Lambda^2} (q_\mu q_\nu - q^2 g_{\mu\nu}) \left( \log \frac{q^2}{\mu^2} - \frac{5}{3} \right) \]

\[ + \frac{q^2}{\Lambda^2} (q_\mu q_\nu - q^2 g_{\mu\nu}) \left( \frac{11}{18} a + \pi \right) + O \left( \frac{1}{\Lambda^4} \right) \tag{2.19} \]

The non local singularity corresponding to the term proportional to \( \log q^2 \) in the above formula could in principle destroy the connection with the gauge-invariant counterterms. Its role will be discussed in the following.

**B. Connection with local operators of dimension six**

Our basic result is the explicit value of the imaginary parts of the vertex function (eq.(2.9)), of the fermion propagator (eq.(2.13)) and of the photon vacuum polarization.
(eq. (2.19)) in correspondence of the first ultraviolet renormalon. We want to stress again that all our calculations have been performed at fixed order in $1/N_f$ and all the following considerations rely upon this kind of expansion in which renormalon-type problems have a more direct interpretation.

Let us now turn to the main point of our investigation i.e. the connection with local gauge-invariant operators. On simply dimensional grounds and exploiting renormalization group arguments one can argue that the imaginary parts given by (2.9), (2.13) and (2.19) are proportional to the insertion of operators of dimension six. We want to check if it is possible to give a prescription in terms of local gauge-invariant operators added as imaginary counterterms in the lagrangian. Let us study the structure of the counterterms.

The term proportional to $(q^2 \gamma_\mu - q_\mu q)$ in (2.9) can be seen as the insertion of the gauge-invariant operator $(\partial_\mu F_{\mu\nu}) \bar{\psi} \gamma_\nu \psi$; hence we deduce the presence of the following counterterm in the lagrangian (the coefficient is uniquely determined):

$$\Delta L_1 = -i \frac{1}{N_f} \frac{\pi e}{2 \Lambda^2} (\partial_\mu F_{\mu\nu}) \bar{\psi} \gamma_\nu \psi$$

Moreover, from the term proportional to $m^2$ in (2.9) and (2.13) we see that the operators $m^2 \Lambda^{-2} \bar{\psi} \partial \psi$ and $em^2 \Lambda^{-2} \bar{\psi} \hat{A} \bar{\psi}$ combine in a gauge-invariant form and this corresponds to the following lagrangian counterterm:

$$\Delta L_2 = -i \frac{1}{N_f} \frac{9\pi}{8} \frac{m^2}{\Lambda^2} \bar{\psi} (i\partial - e \hat{A}) \psi$$

The fundamental check involves the imaginary part of the photon vacuum polarization which contains a term proportional to $\log q^2$. In fact this term cannot be cancelled by the direct insertion of a local operator but ought to arise from the insertion of a local operator in a loop diagram. One can easily convince himself that the only operator that can do this job must have the form of (2.20). We then compute the insertion of $\Delta L_1$ on the vacuum polarization as given by the diagram of fig.5. This diagram is of order $1/N_f$ since it contains a term $1/N_f$ from $\Delta L_1$, a term $1/N_f$ from the two vertices and a term $N_f$ from the fermion loop. It gives the following contribution to the imaginary part of the vacuum polarization (in the $\overline{\text{MS}}$ scheme):

$$i \frac{1}{N_f} \frac{a}{3} \frac{q^2}{\Lambda^2} (q_\mu q_\nu - q^2 g_{\mu\nu}) \left( \log \frac{q^2}{\mu^2} - \frac{5}{3} \right)$$

This contribution exactly cancels the logarithmic term in $\text{Im} \Pi_1$.

To summarize, we have explicitly checked that, at next-to-leading order in $1/N_f$, the imaginary parts due to the ultraviolet renormalon at $b = 3\pi$ can be cancelled by the contribution of local gauge-invariant operators of dimension six. Indeed it is possible to give a realization of this connection at lagrangian level.
III. INFRARED RENORMALONS

In this section we deal with the so-called infrared renormalons i.e. singularities on the real axis of the Borel transform at $b\beta_2 = -n$ which arise from the low momentum region of integration in the Feynman integrals. As already stated these singularities are on the negative real axis of the Borel variable for infrared free theories like QED or QCD in the limit of large number of flavours and give rise to a sign alternating Borel summable contribution to the perturbative series (see eq.(1.4)).

On the contrary, for asymptotically free theories like QCD the infrared renormalons destroy the Borel summability of the perturbative series, a phenomenon usually related to the presence of non-perturbative effects, the condensates, which cause the physical vacuum to have a non trivial structure. Hence the study of infrared renormalons in QCD is of primary interest especially for the comprehension of the operator product expansion which is usually introduced in order to parametrize non-perturbative effects [23].

Nevertheless it is common wisdom that the structure of the infrared renormalons in QCD can be obtained from the study of QED, after the substitution of the value of $\beta_2$ of QCD. This means that the structure of the infrared renormalons is insensitive to the presence of non-perturbative contributions in the operator product expansion which in the case of QED is a pure reorganization of the perturbative series but depends only on the nature of the gauge-invariant operators of the theory, in agreement with the conjecture made by Parisi [20].

In the following we intend to perform an explicit check of the connection with gauge-invariant operators, in analogy with the calculations of the previous section concerning ultraviolet renormalons. The basic difference in the case of infrared renormalons is that this connection can be implemented by adding imaginary multilocal gauge-invariant operators to the lagrangian. These multilocal operators must be considered as the “dual” of the local operators appearing in the operator product expansion (for a rigorous definition see [20]). Our aim is to find the form and the normalization of these operators and to check the combinatoric.

In analogy with the preceding section we deal with QED at next-to-leading order in $1/N_f$. The only difference is that now we consider the massless case; then the proper photon propagator at leading order has the following form in the $\overline{\text{MS}}$ scheme:

$$\Pi_0(k) = -\frac{a}{3\pi} \left( \log \frac{k^2}{\mu^2} - \frac{5}{3} \right)$$

while the Borel transform of $D(k)$ is:

$$F(k, b) = e^{-\frac{5b}{9\pi}} \left( \frac{k^2}{\mu^2} \right)^{b/3\pi}$$

The expression for the Borel transformed functions is the same as before. From (2.6) and
we see that \( \tilde{\Gamma}_\mu \) has infrared singularities for \( b = -3\pi n \); expanding around the first two poles we have:

\[
\left( \frac{\tilde{\Gamma}_\mu}{e} \right) \approx \frac{1}{N_f} \left( \frac{1}{\mu^2} \right)^{b/3\pi} e^{-5b/9\pi} \left\{ \frac{9}{8} \frac{1}{b+3\pi} \tilde{p}^\mu \tilde{p}^\nu - \frac{1}{4} \frac{1}{b+6\pi} \left[ \frac{2q^2}{p^2 p'^2} \hat{\gamma}_\mu \hat{p}' + 3\gamma_\mu - 5 \frac{p'_\mu p'_\nu}{p'^2} - 5 \frac{p_\mu \hat{p}'}{p'^2} - \frac{p'_\mu \hat{p}}{p'^2} - \frac{p'_\mu \hat{p}}{p'^2} \right] \right\}
\]

(3.3)

This yields an imaginary part given by:

\[
\text{Im } \Gamma_\mu = \frac{1}{N_f} \frac{9\pi e}{8} \frac{\Lambda^2}{\hat{p}^\mu} \frac{\gamma_\mu}{\hat{p}'} - \frac{1}{N_f} \frac{\pi e}{4} \frac{\Lambda^4}{p^2 p'^2} \left( \frac{2q^2}{p^2 p'^2} \hat{\gamma}_\mu \hat{p}' + 3\gamma_\mu - 5 \frac{p'_\mu \hat{p}'}{p'^2} - 5 \frac{p_\mu \hat{p}'}{p'^2} - \frac{p'_\mu \hat{p}}{p'^2} - \frac{p'_\mu \hat{p}}{p'^2} \right) + O(\Lambda^6)
\]

(3.4)

While for the ultraviolet renormalons the expansion parameter was \( \Lambda^{-2} \), now it is \( \Lambda^2 \); in fact we are simulating ultraviolet free theories which present a Landau pole at small momenta (in (2.5) we must take \( \beta_2 < 0 \)).

The imaginary part of the fermion propagator can be obtained in a similar way. We have:

\[
\text{Im } \Sigma(p) = \frac{1}{N_f} \frac{9\pi e}{8} \Lambda^2 \frac{1}{\hat{p}^\mu} \gamma_\mu \frac{1}{\hat{p}'} - \frac{1}{N_f} \frac{\pi e}{4} \frac{\Lambda^4}{p^2 p'^2} \left( \frac{2q^2}{p^2 p'^2} \hat{\gamma}_\mu \hat{p}' + 3\gamma_\mu - 5 \frac{p'_\mu \hat{p}'}{p'^2} - 5 \frac{p_\mu \hat{p}'}{p'^2} - \frac{p'_\mu \hat{p}}{p'^2} - \frac{p'_\mu \hat{p}}{p'^2} \right) + O(\Lambda^6)
\]

(3.5)

Even (3.4) and (3.5) satisfy the Ward identity (2.14).

From [24] we deduce the following imaginary part for the photon vacuum polarization:

\[
\text{Im } \Pi_{1\mu\nu} = \frac{3a}{4} \frac{\Lambda^4}{q^4} (q_\mu q_\nu - q^2 g_{\mu\nu}) + O(\Lambda^6)
\]

(3.6)

The absence of the term of order \( \Lambda^2 \) reflects the absence of the first renormalon pole at \( b = 3\pi \) and this is a first evidence for the connection with gauge-invariant operators [7,8,24].

We want to study how the imaginary parts (3.4), (3.5) and (3.6) can be compensated by the contributions of multilocal operators.

First of all it is evident that the terms of order \( \Lambda^2 \) in (3.4) and in (3.5) are both proportional to the insertion on the functions calculated at leading order (without the chain of bubbles) of the multilocal operator:

\[
\Delta S_1 = -i \frac{1}{N_f} \frac{9\pi}{16} \Lambda^2 \int d^4 x \bar{\psi} \gamma_\mu \psi(x) \int d^4 y \bar{\psi} \gamma_\mu \psi(y)
\]

(3.7)

It is pleasant that a unique choice of the normalization of the operator takes care of both the contributions.
Let us now turn to eq. (3.6). When considering the operator product expansion for the vacuum polarization, the first non trivial operator to be considered is $F_{\mu\nu} F_{\mu\nu}$. It has dimension four and thus gives a contribution proportional to $q^{-4}$ that should compensate the imaginary part of the vacuum polarization arising from the infrared renormalon at $b_\beta = -2$ [22]. We want to extract this contribution adding to the action the multilocal operator dual to $F^2$:

$$\Delta S_2 = i \frac{C}{12} \Lambda^4 \int d^4 x F_{\mu\nu}(x) \int d^4 y F_{\mu\nu}(y)$$

(3.8)

The constant $C$ is fixed by the condition that the insertion of $\Delta S_2$ on $\Pi_0$ cancel the imaginary part of $\Pi_1$.

In general the insertion of $\Delta S_2$ on some correlation function can be computed with the Schwinger operator method in the so-called fixed-point gauge; the details of the calculation will be given in the appendix. The result for the insertion of $\Delta S_2$ on $\Pi_{0\mu\nu}$ is:

$$-i \frac{C}{N_f} \frac{2a^2}{3} \Lambda^4 (q_{\mu} q_{\nu} - q^2 g_{\mu\nu})$$

(3.9)

Comparing this expression with eq. (3.6), we find for the constant $C$ the value:

$$C = \frac{9}{8a}$$

(3.10)

As a fundamental check for the consistence of the theory we try to extract the same constant $C$ from the imaginary part of the other functions. In fact, while there is no contribution of $\Delta S_2$ on the fermion propagator (this is consistent with the form of Im $\Sigma$), one can compute the insertion of $\Delta S_2$ on $\Gamma_{\mu}$ at $q = 0$. Referring again to the appendix for the details of the calculation we find the following contribution:

$$i \frac{eC}{N_f} \frac{2\pi a^2}{3} \frac{1}{p^4} \left( \gamma_{\mu} - \frac{4p_{\mu} \hat{p}}{p^2} \right)$$

(3.11)

This must be confronted with the term proportional to $\Lambda^4$ at $q = 0$ in Im $\Gamma_{\mu}$:

$$- \frac{1}{N_f} \frac{3\pi e}{4} \frac{1}{p^4} \left( \gamma_{\mu} - \frac{4p_{\mu} \hat{p}}{p^2} \right)$$

(3.12)

Hence the same coefficient $C$ found in (3.10) eliminates this imaginary term.

For the sake of completeness we have extracted the coefficient $C$ directly from the imaginary part of $\langle F_{\mu\nu} F_{\mu\nu} \rangle$; from the diagram of fig.6 we find for the Borel transform:

$$\langle a\tilde{F}^2 \rangle \simeq \frac{9}{8\pi} \frac{1}{b + 6\pi} e^{-5b/9\pi} \left( \frac{1}{\mu^2} \right)^{b/3\pi}$$

(3.13)

which implies:
\[
\text{Im} \langle F_{\mu\nu} F_{\mu\nu} \rangle = -\frac{9}{8a} \Lambda^4 + O(\Lambda^6) \tag{3.14}
\]

Moreover the insertion of \( \Delta S_2 \) on \( \langle F_{\mu\nu} F_{\mu\nu} \rangle \) is simply given by \( C \) so that even this imaginary part is cancelled with the choice \( \text{[3.10]} \).

**IV. CONCLUSIONS**

We have shown in a non trivial context that renormalon-type singularities can be absorbed in an effective lagrangian of the form:

\[
\mathcal{L}_{\text{eff}} = \mathcal{L} + i\Delta \mathcal{L} \tag{4.1}
\]

\( \Delta \mathcal{L} \) is uniquely determined by the request of having real correlation functions in the euclidean region. Local counterterms correspond to ultraviolet renormalons while multilocal counterterms to infrared ones.

The imaginary part in \( \mathcal{L}_{\text{eff}} \) restores unitarity which is violated in the original theory for the presence of renormalons.

The description in terms of an effective lagrangian makes sense only if the resulting theory doesn’t present non local singularities. We have checked that this is the case for QED in the framework of the \( 1/N_f \) expansion.

From the theoretical point of view it would be interesting to study the global structure of \( \mathcal{L}_{\text{eff}} \) and to understand how the introduction of imaginary counterterms may influence the renormalization of the theory.
APPENDIX A: BACKGROUND FIELD TECHNIQUE

We want to shortly review the background field formalism which enables to compute the correlation functions in an external field \[26,27\].

The Schwinger method enables to enforce gauge-invariance at all stages of the calculation; it is based on the introduction of a basis of eigenvectors of the coordinate operator:

\[
X_\mu |x\rangle = x_\mu |x\rangle \tag{A1}
\]

\[
\langle x | P_\mu | y \rangle = -i \frac{\partial}{\partial x_\mu} \delta(x - y) + e A_\mu(x) \delta(x - y) \tag{A1}
\]

In addition we perform our calculation in the so-called fixed-point gauge \[28\]:

\[
x_\mu A_\mu(x) = 0 \tag{A2}
\]

In this gauge the vector potential is written in terms of the field strenght and of its covariant derivatives calculated in \(x = 0\); in momentum space we have:

\[
A_\mu(k) = \int d^4 z e^{ikz} A_\mu(z) \approx -i \frac{(2\pi)^4}{2} F_{\rho\mu}(0) \frac{\partial}{\partial k_\rho} \delta^{(4)}(k) + \ldots \tag{A3}
\]

From this formulas it is easy to derive the expression for the fermion propagator in an external field:

\[
S(q) = \int dx \langle x | \frac{1}{P + q} | 0 \rangle = \frac{1}{q} - \frac{1}{q^4} e^{\tilde{F}_{\rho\sigma} q_\rho \gamma_\sigma \gamma_5} + \ldots \tag{A4}
\]

where \(\tilde{F}_{\mu\nu}\) is the dual of \(F_{\mu\nu}\).

We want to find the contributions of \(\langle F^2 \rangle\) to the proper photon propagator and to the vertex function; for this purpose we must use the randomness of the field i.e.:

\[
\langle F_{\mu\nu}(0)F_{\rho\sigma}(0) \rangle = \frac{1}{12} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \langle F_{\alpha\beta}F_{\alpha\beta} \rangle \tag{A5}
\]

The insertion of \(\langle F^2 \rangle\) on \(\Pi(q^2)\) has been computed many times in the literature (see for example \[23\]); the expression for the photon propagator in an external field is given by (using \(A4\) and \(A5\)):

\[
\Pi_{\mu\nu}(q) = ie^2 \int d^4 x e^{iqx} \text{Tr}\{\gamma_\mu S(x, 0)\gamma_\nu S(0, x)\} = e^2 (q_\mu q_\nu - q^2 g_{\mu\nu}) \left\{ -\frac{1}{12\pi^2} \log q^2 - \frac{e^2}{24\pi^2 q^4} \langle F^2 \rangle \right\} \tag{A6}
\]

The contribution of \(\langle F^2 \rangle\) to the vertex function at zero momentum transfer comes from the insertion of a single \(A_\mu\) on every propagator (see fig.7):
\[ V_\mu^{(F^2)}(p) = -2e^3 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{1}{\hat{p} + k_1 + k_2} \hat{A}(k_2) \frac{1}{\hat{p} + k_1} \gamma_\mu \frac{1}{\hat{p} + k_1} \hat{A}(k_1) \frac{1}{\hat{p}} \]  

(A7)

From (A3) and (A5) we have, considering the one-particle irreducible function:

\[ \Gamma^{(F^2)}_\mu(p) = \frac{1}{6} e^3 \frac{(F^2)}{p^4} \left[ \gamma_\mu - 4 \frac{p_\mu \hat{p}}{p^2} \right] \]  

(A8)

which corresponds to eq.(3.11)
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FIGURE CAPTIONS

**Fig 1.** The photon propagator at leading order.

**Fig 2.** The vertex function at order $1/N_f$.

**Fig 3.** The fermion propagator at order $1/N_f$.

**Fig 4.** The photon vacuum polarization at order $1/N_f$.

**Fig 5.** Insertion of $\Delta \mathcal{L}_1$ (the crossed circle) on the vacuum polarization.

**Fig 6.** Graph contributing to the vacuum expectation value of $F_{\mu\nu} F^{\mu\nu}$.

**Fig 7.** Contribution of $F^2$ to the vertex function at zero momentum transfer.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406006v1
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406006v1
FIG. 4

FIG. 5

FIG. 6

FIG. 7