NONTRIVIAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we present a theory for the existence of multiple nontrivial solutions for a class of perturbed Hammerstein integral equations. Our approach, rather than to work directly in cones, is to utilize the theory of fixed point index on affine cones. This approach is fairly general and covers a class of nonlocal boundary value problems for functional differential equations. Some examples are given in order to illustrate our theoretical results.

1. Introduction

The problem of the existence of solutions for functional differential equations has been discussed by a large number of researchers. The motivation for these studies, apart from a purely mathematical interest, relies in the fact that these type of equations arise quite frequently when modelling physical problems, see for example the ones illustrated in the book of Hale and Lunel [7]. Existence results have been provided, for example, by Xu and Liz [27] in the first order case and by Ntouyas, Sficas and Tsamatos [20] in the second order case; see also the work of Nussbaum [21]. The existence of positive solutions, by means of the Krasnosel’ski˘ı-Guo theorem (see for example the book of Guo and Lakshmikantham [6]) has been studied by Wang [23] in the first order case, whereas the second order case has been investigated, under local boundary conditions, by Erbe and Kong [5], Karakostas, Mavridis and Tsamatos [11] and Ma [18]. The existence of positive solutions in the nonlocal case has been studied by Karakostas, Mavridis and Tsamatos [12] and, more recently, by Karaca [10].

In particular, in the paper [12], the authors study the existence of positive solutions of the functional boundary value problem (FBVP)

\[ u''(t) + F(t, u_t) = 0, \quad t \in [0, 1], \]

with initial conditions

\[ u(t) = \psi(t), \quad t \in [-r, 0], \]

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and boundary conditions (BCs)

\[ u(0) = 0, \quad u(1) = \int_{t_1}^{t_2} u(s) dA(s), \quad t_1, t_2 \in (0, 1), \tag{1.3} \]

where \( \psi \) is assumed to be nonnegative and the above integral is meant in the Riemann-STieltjes sense and is given by a positive measure.

The methodology in [12] is to re-write the FBVP (1.1)-(1.2)-(1.3) as an Hammerstein-type integral equation of the form

\[ u(t) = \int_0^1 \hat{k}(t, s) F(s, u_s) ds, \]

with a suitable kernel \( \hat{k} \) and to use the Leggett-Williams theorem [16].

Our approach is somewhat different and we study, in the spirit of [9], the existence of nontrivial solutions (that is solutions that do not coincide with \( \psi \)) of perturbed Hammerstein integral equations of the type

\[ u(t) = \psi(t) + \int_0^1 k(t, s) g(s) F(s, u_s) ds + \gamma(t) \alpha[u], \tag{1.4} \]

where \( \alpha[\cdot] \) is a linear functional given by Stieltjes integral, namely

\[ \alpha[u] = \int_0^1 u(s) dA(s). \tag{1.5} \]

We stress that the formulation (1.5) involves a signed measure and covers the case of multi-point and integral conditions, namely

\[ \alpha[u] = \sum_{j=1}^m \alpha_j u(\eta_j) \quad \text{or} \quad \alpha[u] = \int_0^1 \phi(s) u(s) ds. \]

Multi-point and integral BCs are widely studied objects in the case of ODEs. As far as we know multi-point BCs were investigated for the first time in 1908 by Picone [22]. In 1942 Whyburn [26] wrote a review on differential equations with general BCs that included also integral BCs involving STieltjes measures. We mention also the (more recent) reviews of Ma [17] and Ntouyas [19] and the papers by Karakostas and Tsamatos [13, 14] and by Webb and Infante [24].

One advantage of studying the solutions of the perturbed integral equation (1.4) is that it provides a fairly general setting that covers, as special cases, a number of FBVPs subject to nonlocal conditions.

A new feature of the present paper is that we work in affine cones. In fact, due to the presence of the delay and of the initial datum \( \psi \), in order to investigate the solutions of the integral equation (1.4) we find convenient and natural to work in translates of cones in Banach spaces, rather than to work directly in cones. In order to do this, we provide a modification, tailored for our setting, of some classical result on the fixed point index.

In the last Section we present the application of our theoretical results to some nonlocal FBVPs.
2. Fixed points on translates of a cone

In this Section we provide some useful properties of the fixed point index on a translate of a cone \( K \), in the spirit of Remark 1 of \([1]\). These properties are used in Section 3 to prove our existence and multiplicity results for the integral equation \([1, 4]\).

We require some knowledge of the classical fixed point index for compact maps, see for example \([1, 2, 6]\) for further information. Let \( X \) be a Banach Space. A cone on \( X \) is a closed, convex subset of \( X \) such that \( \lambda x \in K \) for \( x \in K \) and \( \lambda \geq 0 \) and \( K \cap (-K) = \{0\} \). If \( \Omega \) is a bounded open subset of \( K \) (in the relative topology) we denote by \( \overline{\Omega} \) and \( \partial \Omega \) the closure and the boundary of \( \Omega \) relative to \( K \). Given \( y \in X \), we can consider the translate of a cone \( K \), namely

\[ K_y := y + K = \{ y + x : x \in K \} . \]

When \( D \) is an open bounded subset of \( X \) we write \( D_{K_y} = D \cap K_y \), an open subset of \( K \).

The following Lemma is a direct consequence of classical results from degree theory, and the proof can be carried out as in the case of cones, see for example the proof of Lemma 12.1 in the review \([2]\). We give here a detailed proof for the sake of completeness.

**Lemma 2.1.** Let \( D \) be an open bounded set with \( y \in D_{K_y} \) and \( \overline{D}_{K_y} \neq K_y \). Assume that \( F : \overline{D}_{K_y} \to K_y \) is a compact map such that \( x \neq Fx \) for \( x \in \partial D_{K_y} \). Then the fixed point index \( i_{K_y}(F, D_{K_y}) \) has the following properties.

(1) If there exists \( e \in K \setminus \{0\} \) such that \( x \neq Fx + \sigma e \) for all \( x \in \partial D_{K_y} \) and all \( \sigma > 0 \), then \( i_{K_y}(F, D_{K_y}) = 0 \).

(2) If \( \mu(x - y) \neq Fx - y \) for all \( x \in \partial D_{K_y} \) and for every \( \mu \geq 1 \), then \( i_{K_y}(F, D_{K_y}) = 1 \).

(3) Let \( D' \) be open in \( X \) with \( \overline{D'} \subset D_{K_y} \). If \( i_{K_y}(F, D_{K_y}) = 1 \) and \( i_{K_y}(F, D'_{K_y}) = 0 \), then \( F \) has a fixed point in \( D_{K_y} \setminus \overline{D'}_{K_y} \). The same result holds if \( i_{K_y}(F, D_{K_y}) = 0 \) and \( i_{K_y}(F, D'_{K_y}) = 1 \).

**Proof.** (1) Let \( \alpha = \sup\{\|x\| : x \in D_{K_y}\} \), \( \beta = \sup\{\|Fx\| : x \in D_{K_y}\} \) and let \( \gamma > \frac{\alpha + \beta}{\|e\|} \).

Define \( H : [0, 1] \times \overline{D}_{K_y} \to E \) by \( H(\lambda, x) = Fx + \lambda \gamma e \). Note that \( H \) is a compact map with values in \( K_y \). By the Homotopy invariance property, we get \( i_{K_y}(F, D_{K_y}) = i_{K_y}(F + \gamma e, D_{K_y}) \).

Assume now that \( i_{K_y}(F, D_{K_y}) \neq 0 \). Then, there exists \( \bar{x} \in D_{K_y} \) such that \( \bar{x} = F(\bar{x}) + \gamma e \). Consequently, \( \|\bar{x}\| \geq \gamma \|e\| - \|Fx\| \geq \gamma \|e\| - \beta > \alpha \), which is a contradiction. Hence, \( i_{K_y}(F, D_{K_y}) = 0 \).

(2) Define \( H : [0, 1] \times \overline{D}_{K_y} \to E \) by \( H(\lambda, x) = (1 - \lambda)u + \lambda F(x) \). Observe that \( H \) is a compact map with values in \( K_y \). Thus, by the Homotopy invariance and Normalization properties, we have \( i_{K_y}(F, D_{K_y}) = i_{K_y}(u, D_{K_y}) = 1 \).

(3) This is a consequence of the Additivity and Solution properties. \( \square \)

3. Nontrivial solutions for a class of perturbed integral equations

Given a compact interval \( I \subset \mathbb{R} \), by \( C(I, \mathbb{R}) \) we mean the Banach space of the continuous functions defined on \( I \) with the usual supremum norm. Since we work with functions defined
on different intervals (usually $I = [-r, 0]$ or $I = [-r, 1]$ with $r > 0$), for sake of clarity the norm of $u \in C(I, \mathbb{R})$ will be denoted by $\|u\|_I$.

Given $r > 0$ and a continuous function $u : J \to \mathbb{R}$, defined on a real interval $J$, and given $t \in \mathbb{R}$ such that $[t-r, t] \subseteq J$, we adopt the standard notation $u_t : [-r, 0] \to \mathbb{R}$ for the function defined by $u_t(\theta) = u(t + \theta)$.

Let us consider the following integral equation in the space $C([-r, 1], \mathbb{R})$:
\[
u(t) = \psi(t) + \int_0^1 k(t, s)g(s)F(s, u_s) \, ds + \gamma(t)\alpha[u] =: \mathcal{F}u(t), \quad (3.1)
\]
where
\[
\alpha[u] = \int_0^1 u(s) \, dA(s).
\]

We require the following assumptions on the maps $F$, $k$, $\psi$, $\gamma$, $\alpha$ and $g$ that occur in (3.1):

1. (C1) The function $\psi : [-r, 1) \to \mathbb{R}$ is continuous and such that $\psi(t) = 0$ for all $t \in [0, 1]$.
2. (C2) The kernel $k : [-r, 1] \times [0, 1] \to \mathbb{R}$ is measurable, verifies $k(t, s) = 0$ for all $t \in [-r, 0]$ and almost every (a. e.) $s \in [0, 1]$, and for every $\bar{r} \in [0, 1]$ we have
\[
\lim_{t \to \bar{r}} |k(t, s) - k(\bar{r}, s)| = 0 \quad \text{for a.e. } s \in [0, 1].
\]
3. (C3) There exist a subinterval $[a, b] \subseteq (0, 1]$, a measurable function $\Phi$ with $\Phi \geq 0$ a.e., and a constant $c_1 = c_1(a, b) \in (0, 1]$ such that
\[
|k(t, s)| \leq \Phi(s) \quad \text{for all } t \in [0, 1] \text{ and a.e. } s \in [0, 1],
\]
\[
k(t, s) \geq c_1 \Phi(s) \quad \text{for all } t \in [a, b] \text{ and a.e. } s \in [0, 1].
\]
4. (C4) The function $g : [0, 1] \to \mathbb{R}$ is measurable, $g(t) \geq 0$ a.e. $t \in [0, 1]$, and satisfies that
\[
g \Phi \in L^1[0, 1] \text{ and } \int_0^1 \Phi(s)g(s) \, ds > 0.
\]
5. (C5) $F : [0, 1] \times C([-r, 0], \mathbb{R}) \to [0, \infty)$ is an operator that satisfies some Carathéodory-type conditions (see also [7]); namely, for each $\phi$, $t \mapsto F(t, \phi)$ is measurable and for a.e. $t$, $\phi \mapsto F(t, \phi)$ is continuous. Furthermore, for each $R > 0$, there exists $\varphi_R \in L^\infty[0, 1]$ such that
\[
F(t, \phi) \leq \varphi_R(t) \quad \text{for all } \phi \in C([-r, 0], \mathbb{R}) \text{ with } \|\phi\|_{[-r, 0]} \leq R, \text{ and a.e. } t \in [0, 1].
\]
6. (C6) $A$ is of bounded variation (var$(A) < +\infty$), and $K_A(s) := \int_0^1 k(t, s) \, dA(t) \geq 0$ for a.e. $s \in [0, 1]$.
7. (C7) The function $\gamma : [-r, 1] \to \mathbb{R}$ is continuous, $\gamma \neq 0$ and such that $\gamma(t) = 0$ for all $t \in [-r, 0]$; moreover, $0 \leq \alpha[\gamma] < 1$ and there exists $c_2 \in (0, 1]$ such that $\gamma(t) \geq c_2 \|\gamma\|_{[0, 1]}$ for all $t \in [a, b]$.

We stress that, in particular, the assumption (C5) is crucial to prove the compactness of the operator $\mathcal{F}$ (see Theorem 3.2 below).

In the Banach space $C([-r, 1], \mathbb{R})$ we define the cone
\[
K_0 = \{ u \in C([-r, 1], \mathbb{R}) : u(t) = 0 \text{ for all } t \in [-r, 0], \min_{t \in [a, b]} u(t) \geq c\|u\|_{[-r, 1]}, \alpha[u] \geq 0 \},
\]
where \( c = \min\{c_1, c_2\} \). Note that \( K_0 \neq \{0\} \) since \( \gamma \in K_0 \) and, furthermore, that the functions in \( K_0 \) are non-negative in the subset \([a, b]\) and are allowed to change sign in \([0, 1]\).

\( K_0 \) can be seen a modification of the cone of functions introduced by Infante and Webb in [8].

The idea of incorporating the functional \( \alpha \) within the definition of the cone (this allows the use of signed measures) can be found in [25] for the case of positive functions and in [4] for the case of functions that are allowed to change sign.

We consider the following translate of the cone \( K_0 \),

\[
K_\psi = \psi + K_0 = \{ \psi + u : u \in K_0 \}.
\]

**Definition 3.1.** We define the following subsets of \( C([-r, 1], \mathbb{R}) \):

\[
K_{0, \rho} := \{ u \in K_0 : \|u\|_{[0,1]} < \rho \}, \quad V_{0, \rho} := \{ u \in K_0 : \min_{t \in [a,b]} u(t) < \rho \}
\]

and the corresponding translates

\[
K_{\psi, \rho} := \psi + K_{0, \rho}, \quad V_{\psi, \rho} := \psi + V_{0, \rho}.
\]

Observe that \( \partial K_{\psi, \rho} = \psi + \partial K_{0, \rho} \) and \( \partial V_{\psi, \rho} = \psi + \partial V_{0, \rho} \). Let us stress that a key feature of these sets is that they can be nested

\[
K_{\psi, \rho} \subset V_{\psi, \rho} \subset K_{\psi, \rho/c}.
\]

Furthermore, note that \( u \in K_\psi \) means that \( u = \psi + v \) with \( v \in K_0 \) and, therefore, we have

\[
\|u\|_{[-r,1]} = \max\{\|\psi\|_{[-r,0]}, \|v\|_{[0,1]}\}.
\]

**Theorem 3.2.** Assume that the hypotheses \((C_1)-(C_7)\) hold for some \( R > 0 \). Then \( F \) maps \( \overline{K}_{\psi, R} \) into \( K_\psi \) and is compact. When these hypotheses hold for every \( R > 0 \), \( F \) is compact and maps \( K_\psi \) into \( K_\psi \).

**Proof.** Let \( R > 0 \) be given and let \( u \in \overline{K}_{\psi, R} \). Let us show that \( Fu - \psi \in K_0 \). First of all observe that our assumptions imply that \( Fu \) is continuous on \([-r, 1]\) and that \( Fu(t) - \psi(t) = 0 \) for \( t \in [-r, 0] \). Now, for every \( t \in [0, 1] \) we have

\[
|Fu(t) - \psi(t)| \leq \int_0^1 |k(t, s)|g(s)f(s, u_s)\, ds + |\gamma(t)|\alpha[u]
\]

\[
\leq \int_0^1 \Phi(s)g(s)f(s, u_s)\, ds + \|\gamma\|_{[0,1]}\alpha[u],
\]

moreover, since \([a, b] \subseteq (0, 1]\),

\[
\min_{t \in [a, b]} (Fu(t) - \psi(t)) \geq c_1 \int_0^1 \Phi(s)g(s)f(s, u_s)\, ds + c_2 \gamma(t)\alpha[u] \geq c||Fu - \psi||_{[-r,1]}.
\]

Furthermore, by \((C6)\) and \((C7)\), we have

\[
\alpha[Fu] = \alpha[\gamma]\alpha[u] + \int_0^1 \kappa(s)g(s)f(s, u_s)\, ds \geq 0.
\]

Therefore we have that \( Fu \in K_\psi \) for every \( u \in \overline{K}_{\psi, R} \).
To prove the compactness of \( \mathcal{F} \), let \( \{u^n\} \) be a sequence in \( C([-r,1],\mathbb{R}) \) with \( \|u^n\|_{[-r,1]} < R \). Observe that \( \|u^n_t\|_{[-r,0]} < \tilde{R} \) for all \( t \in [0,1] \), where \( \tilde{R} := R + \|\psi\|_{[-r,0]} \). Consequently, for \( t \in [-r,1] \) we have

\[
|\mathcal{F}u^n(t)| \leq |\psi(t)| + \int_0^1 |k(t,s)|g(s)F(s,u^n_s)\,ds + |\gamma(t)|\alpha[u]
\]

\[
\leq \|\psi\|_{[-r,0]} + \int_0^1 \Phi(s)g(s)\varphi_R(s)\,ds + \|\gamma\|_{[0,1]}\alpha[u].
\]

Hence, the sequence \( \{\mathcal{F}u^n\} \) is bounded. Now, let \( \varepsilon > 0 \) be given; by \((C_2)\) and the continuity of \( \psi \) and \( \gamma \), there exists \( \delta > 0 \) such that:

\[ |\psi(t_1) - \psi(t_2)| < \varepsilon \]

provided that \( t_1, t_2 \in [-r,0] \) with \( |t_1 - t_2| < \delta \);

\[ |\gamma(t_1) - \gamma(t_2)| < \varepsilon \]

provided that \( t_1, t_2 \in [0,1] \) with \( |t_1 - t_2| < \delta \);

\[ |k(t_1, s) - k(t_2, s)| < \varepsilon \]

for a.e. \( s \in [0,1] \), provided that \( t_1, t_2 \in [0,1] \) with \( |t_1 - t_2| < \delta \).

Therefore we have:

\[ |\mathcal{F}u^n(t_1) - \mathcal{F}u^n(t_2)| \leq |\psi(t_1) - \psi(t_2)| \leq \varepsilon, \]

if \( t_1, t_2 \in [-r,0] \) with \( |t_1 - t_2| < \delta \);

\[ |\mathcal{F}u^n(t_1) - \mathcal{F}u^n(t_2)| \leq \int_0^1 |k(t_1, s) - k(t_2, s)|g(s)F(s,u^n_s)\,ds + |\gamma(t_1) - \gamma(t_2)|\alpha[u^n] \]

\[ \leq \varepsilon \int_0^1 g(s)\varphi_R(s)\,ds + \varepsilon R \text{var}(A), \]

if \( t_1, t_2 \in [0,1] \) with \( |t_1 - t_2| < \delta \);

\[ |\mathcal{F}u^n(t_1) - \mathcal{F}u^n(t_2)| \leq |\mathcal{F}u^n(t_1) - \mathcal{F}u^n(0)| + |\mathcal{F}u^n(0) - \mathcal{F}u^n(t_2)| \]

\[ \leq \varepsilon \left( 1 + \int_0^1 g(s)\varphi_R(s)\,ds + R \text{var}(A) \right), \]

whenever \( -r \leq t_1 < 0 < t_2 \leq 1 \) with \( |t_1 - t_2| < \delta \).

Therefore the sequence \( \{\mathcal{F}u^n\} \) is equicontinuous. The compactness of \( \mathcal{F} \) now follows from the Ascoli-Arzelà Theorem. \( \square \)

In the sequel, we give a condition that ensures that the index is 1 on \( K_{\psi, \rho} \) for a suitable \( \rho \) larger than the norm of \( \psi \). We stress that the assumption \( \rho > \|\psi\|_{[-r,0]} \) is needed in order to make the number \( F^{(-\rho, \rho)} \) (defined below) independent of the initial data.

**Lemma 3.3.** Assume that

\[(I^*_\rho) \text{ there exists } \rho > \|\psi\|_{[-r,0]} \text{ such that} \]

\[
\frac{F^{(-\rho, \rho)}}{m} < 1,
\]

where

\[
\frac{1}{m} := \sup_{t \in [0,1]} \left\{ \int_0^1 |k(t,s)|g(s)\,ds + \frac{|\gamma(t)|}{1 - \alpha[\gamma]} \int_0^1 \mathcal{K}_A(s)g(s)\,ds \right\}
\]
and
\[ F^{(-\rho,\rho)} = \sup \left\{ \frac{F(t, \phi)}{\rho} : t \in [0, 1], \phi \in C([-r, 0], \mathbb{R}) \text{ with } \|\phi\|_{[-r,0]} \leq \rho \right\}. \]

Then \( i_{K_\psi}(F, K_{\psi,\rho}) = 1 \).

**Proof.** We show that \( \mu(u - \psi) \neq F u - \psi \) for every \( u \in \partial K_{\psi,\rho} \) and for every \( \mu \geq 1 \).

In fact, if this does not happen, there exist \( \mu \geq 1 \) and \( u \in \partial K_{\psi,\rho} \) such that \( \mu(u - \psi) = F u - \psi \), that is
\[
\mu(u(t) - \psi(t)) = \int_0^1 k(t, s)g(s)F(s, u_s)\,ds + \gamma(t)\alpha[u], \quad \text{for every } t \in [-r, 1].
\]

Applying \( \alpha \) to both sides of the equation and noting that \( \alpha[\psi] = 0 \), we get
\[
\mu \alpha[u] = \int_0^1 K_A(s)g(s)F(s, u_s)\,ds + \alpha[\gamma]\alpha[u]
\]
thus, from (\( C_7 \)), \( \mu - \alpha[\gamma] \geq 1 - \alpha[\gamma] > 0 \), and we deduce that
\[
\alpha[u] = \frac{1}{\mu - \alpha[\gamma]} \int_0^1 K_A(s)g(s)F(s, u_s)\,ds
\]
and we get, by substitution,
\[
\mu(u(t) - \psi(t)) = \int_0^1 k(t, s)g(s)F(s, u_s)\,ds + \frac{\gamma(t)}{\mu - \alpha[\gamma]} \int_0^1 K_A(s)g(s)F(s, u_s)\,ds.
\]
Recall that \( u \in \partial K_{\psi,\rho} \) means that \( u = \psi + v \) with \( v \in \partial K_{0,\rho} \) and, in particular we have that \( \|v\|_{[0,1]} = \rho \). Now observe that \( F(s, u_s) \leq \rho F^{(-\rho,\rho)} \) for all \( s \in [0,1] \). This estimate follows from the definition of \( F^{(-\rho,\rho)} \) and the fact that, since \( \|\psi\|_{[-r,0]} < \rho \), we have that \( \|u_s\|_{[-r,0]} \leq \rho \) for all \( s \). Therefore, taking the absolute value and then the supremum for \( t \in [-r, 1] \) in the above equality, we get
\[
\mu \rho \leq \sup_{t \in [0,1]} \left\{ \int_0^1 |k(t, s)|g(s)F(s, u_s)\,ds + \frac{|\gamma(t)|}{\mu - \alpha[\gamma]} \int_0^1 K_A(s)g(s)F(s, u_s)\,ds \right\}
\leq \rho F^{(-\rho,\rho)} \cdot \sup_{t \in [0,1]} \left\{ \int_0^1 |k(t, s)|g(s)\,ds + \frac{|\gamma(t)|}{1 - \alpha[\gamma]} \int_0^1 K_A(s)g(s)\,ds \right\} < \rho.
\]
This contradicts the fact that \( \mu \geq 1 \) and proves the result. \( \square \)

Let us see now a condition that guarantees the index is equal to zero on \( V_{\psi,\rho} \) for some appropriate \( \rho > 0 \).

**Lemma 3.4.** Assume that \( r < b - a \) and that

(\( P_\rho^b \)) there exist \( \rho > 0 \) such that such that
\[
\frac{F_{(\rho,\rho/\epsilon)}(a, b)}{M(a, b)} > 1,
\]
where
\[
\frac{1}{M(a,b)} := \inf_{t \in [a,b]} \left\{ \int_{a+r}^{b} k(t,s)g(s) \, ds + \frac{\gamma(t)}{1 - \alpha[\gamma]} \int_{a+r}^{b} K_A(s)g(s) \, ds \right\}
\]
and
\[
F_{(\rho,\rho/c)} = \inf \left\{ \frac{\int_{\rho}^{\rho/c} \Phi(t, \rho/c)}{\rho} : t \in [a,b], \Phi \in C([-r,0], \mathbb{R}) \text{ with } \Phi(\theta) \in [\rho, \rho/c] \text{ for all } \theta \in [-r,0] \right\}.
\]

Then \(i_{K_\psi}(\mathcal{F}, V_{\psi,\rho}) = 0\).

**Proof.** Since \(0 \neq \gamma \in K_0\) we can choose \(e = \gamma\) in Lemma 2.1. We now prove that
\[
u \neq \mathcal{F}u + \sigma \gamma \quad \text{for all } u \in \partial V_{\psi,\rho} \text{ and } \sigma \geq 0.
\]

In fact, if not, there exist \(u \in \partial V_{\psi,\rho}\) and \(\sigma \geq 0\) such that \(u = \mathcal{F}u + \sigma \gamma\). Then, in particular, we have
\[
u(t) = \int_{0}^{1} k(t,s)g(s)F(s,u_s) \, ds + \sigma \gamma(t) + \gamma(t)\alpha[u], \quad \text{for every } t \in [0,1]
\]
and
\[
\alpha[u] = \int_{0}^{1} K_A(s)g(s)F(s,u_s) \, ds + \sigma \alpha[\gamma] + \alpha[\gamma] \alpha[u].
\]

Therefore we have,
\[
\alpha[u] = \frac{1}{1 - \alpha[\gamma]} \int_{0}^{1} K_A(s)g(s)F(s,u_s) \, ds + \frac{\sigma \alpha[\gamma]}{1 - \alpha[\gamma]}
\]
and, by substitution, we obtain
\[
u(t) = \int_{0}^{1} k(t,s)g(s)F(s,u_s) \, ds + \sigma \gamma(t)
\]
\[
\quad + \frac{\gamma(t)}{1 - \alpha[\gamma]} \left( \int_{0}^{1} K_A(s)g(s)F(s,u_s) \, ds + \sigma \alpha[\gamma] \right).
\]

Now observe that, if \(s \in [a+r,b]\), then we have that \(u_s(\theta) = u(s+\theta) \in [\rho, \rho/c]\) for all \(\theta \in [-r,0]\). Hence, for all \(s \in [a+r,b]\) we have that \(F(s,u_s) \geq \rho F_{(\rho,\rho/c)}\).

In fact, since \(u \in \partial V_{\psi,\rho}\), we have \(u = \psi + v\) with \(v \in \partial V_{0,\rho}\). Consequently, given \(s \in [a+r,b]\), we have that \(u(s+\theta) = v(s+\theta)\) for \(\theta \in [-r,0]\) due to the facts that \(s+\theta \in [a,b] \subseteq (0,1]\) for all \(\theta\) and that \(\psi\) vanishes on \([0,1]\). Furthermore, the function \(v \in \partial V_{0,\rho}\) is such that \(v(\tau) \in [\rho, \rho/c]\) for \(\tau \in [a,b]\). This follows from the definition of \(V_{0,\rho}\) and from the inclusion \(V_{0,\rho} \subseteq K_{0,\rho/c}\).

Hence we get, for \(t \in [a,b]\),
\[
u(t) \geq \int_{a+r}^{b} k(t,s)g(s)F(s,u_s) \, ds + \frac{\gamma(t)}{1 - \alpha[\gamma]} \int_{a+r}^{b} K_A(s)g(s)F(s,u_s) \, ds
\]
\[
\quad \geq \rho F_{(\rho,\rho/c)} \left( \int_{a+r}^{b} k(t,s)g(s) \, ds + \frac{\gamma(t)}{1 - \alpha[\gamma]} \int_{a+r}^{b} K_A(s)g(s) \, ds \right).
\]
Taking the minimum over \([a,b]\) gives \(\rho > \rho\), a contradiction. \(\square\)
The above Lemmas can be combined in order to prove the following Theorem. Here we deal with the existence of at least one, two or three nontrivial solutions, that is solutions that do not coincide with $\psi$. We stress that, by expanding the lists in conditions $(S_5), (S_6)$ below, it is possible to state results for four or more positive solutions, see for example the paper by Lan [15] for the type of results that might be stated. We omit the proof which follows directly from the properties of the fixed point index stated in Lemma 2.1.

**Theorem 3.5.** The integral equation (3.1) has at least one nontrivial solution in $K_\psi$ if one of the following conditions hold.

$(S_1)$ There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\|\psi\|_{[-r,0]} < \rho_2$ and $\rho_1/c < \rho_2$ such that $(I^0_{\rho_1})$ and $(I^1_{\rho_2})$ hold.

$(S_2)$ There exist $\rho_1, \rho_2 \in (0, \infty)$ with $\|\psi\|_{[-r,0]} < \rho_1 < \rho_2$ such that $(I^1_{\rho_1})$ and $(I^0_{\rho_2})$ hold.

The integral equation (3.1) has at least two nontrivial solutions in $K_\psi$ if one of the following conditions hold.

$(S_3)$ There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\|\psi\|_{[-r,0]} < \rho_2$ and $\rho_1/c < \rho_2 < \rho_3$ such that $(I^0_{\rho_1}), (I^1_{\rho_2})$ and $(I^0_{\rho_3})$ hold.

$(S_4)$ There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\|\psi\|_{[-r,0]} < \rho_1 < \rho_2$ and $\rho_2/c < \rho_3$ such that $(I^1_{\rho_1}), (I^0_{\rho_2})$ and $(I^1_{\rho_3})$ hold.

The integral equation (3.1) has at least three nontrivial solutions in $K_\psi$ if one of the following conditions hold.

$(S_5)$ There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$ with $\|\psi\|_{[-r,0]} < \rho_2$ and $\rho_1/c < \rho_2 < \rho_3$ and $\rho_3/c < \rho_4$ such that $(I^1_{\rho_1}), (I^0_{\rho_2}), (I^0_{\rho_3})$ and $(I^1_{\rho_4})$ hold.

$(S_6)$ There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0, \infty)$ with $\|\psi\|_{[-r,0]} < \rho_1 < \rho_2$ and $\rho_2/c < \rho_3 < \rho_4$ such that $(I^1_{\rho_1}), (I^0_{\rho_2}), (I^1_{\rho_3})$ and $(I^0_{\rho_4})$ hold.

**Remark 3.6.** Note that the solutions given by Theorem 3.5 are nontrivial in the sense that they do not coincide with $\psi$; nevertheless in $(S_1), (S_3), (S_5)$ one of the solutions could have the same norm as $\psi$.

**Remark 3.7.** A similar approach can be used, depending on the signs of $k$ and $\gamma$, to prove the existence of solutions that are non-negative on $[0, 1]$. See for example Remark 3.4 of [9] and also Sections 2 and 3 of [3].

4. Nontrivial solutions of some FBVP’s

In this Section we provide some applications of the results of Section 3. We turn our attention to the FBVP

$$-u''(t) = g(t)F(t, u), \; t \in [0, 1],$$

(4.1)

with initial conditions

$$u(t) = \psi(t), \; t \in [-r, 0]$$

(4.2)

and BCs

$$u(0) = 0, \; \beta u'(1) + u(\eta) = \alpha[u], \; \eta \in (0, 1),$$

(4.3)
The solution of the ODE \(-u'' = y\) under the BC (4.3) is

\[
u(t) = \frac{t}{\beta + \eta} \alpha[u] + \frac{\beta t}{\beta + \eta} \int_0^1 y(s) ds + \frac{t}{\beta + \eta} \int_0^\eta (\eta - s) y(s) ds - \int_0^t (t - s) y(s) ds.
\]

By a solution of the FBVP (4.1)–(4.2)–(4.3) we mean a solution \(u \in C[-r, 1]\) of the corresponding integral equation

\[
u(t) = \psi(t) + \int_0^1 k(t, s) g(s) F(s, u_s) ds + \gamma(t) \alpha[u], \quad t \in [-r, 1],
\]

where \(\gamma(t) = \frac{t}{\beta + \eta} H(t)\) and

\[
k(t, s) = \left[ \frac{\beta t}{\beta + \eta} + \frac{t}{\beta + \eta} (\eta - s) H(\eta - s) - (t - s) H(t - s) \right] H(t),
\]

with

\[
H(\tau) = \begin{cases} 1, & \tau \geq 0, \\ 0, & \tau < 0. \end{cases}
\]

When \(\beta \geq 0\), \(k(t, s)\) changes sign when \(0 < \beta + \eta < 1\), but is nonnegative on the strip \(0 \leq t \leq b, b < \beta + \eta\). This is the case for which we give details and apply the results of Section 3 taking \([a, b] = [\eta, b]\) for a suitable \(b\) with \(\eta < b < \beta + \eta\). Observe that \((C_7)\) holds with \(c_2 = \eta\). We want to find \(\Phi, c_1\) so that \((C_3)\) holds. For this purpose we follow the outline of [9]. First one can check that, setting

\[
\Phi(s) = \begin{cases} s, & \text{for } \beta + \eta \geq \frac{1}{2}, \\ \frac{1 - (\beta + \eta)}{\beta + \eta} s, & \text{for } \beta + \eta < \frac{1}{2}. \end{cases}
\]

the upper bound \(|k(t, s)| \leq \Phi(s)\) holds. Concerning the lower bounds, we have that if \(\beta + \eta \geq \frac{1}{2}\), we may choose

\[
c_1 = \min \left\{ \frac{\beta \eta}{\beta + \eta}, \frac{\beta + \eta - b}{\beta + \eta} \right\}. \tag{4.4}
\]

While if \(\beta + \eta < \frac{1}{2}\),

\[
c_1 = \min \left\{ \frac{\beta \eta}{1 - (\beta + \eta)}, \frac{\beta + \eta - b}{1 - (\beta + \eta)} \right\}. \tag{4.5}
\]

Here we state, for brevity, a result regarding the existence of one nontrivial solution, that is a direct consequence of Theorem 3.5. A similar result holds in the case of the multiplicity results.

**Theorem 4.1.** Let \([a, b] = [\eta, b]\), with \(\eta < b < \eta + \beta\), where \(\beta + \eta \geq \frac{1}{2}\), and let \(c_1\) as in (4.4), \(c_2 = \eta\) and \(\int_\eta^b \Phi(s) g(s) ds > 0\). Then the FBVP (4.1)–(4.2)–(4.3) has at least one nontrivial solution, strictly positive on \([\eta, b]\), if either (S_1) or (S_2) of Theorem 3.5 hold.
An analogous result holds for $\beta + \eta < \frac{1}{2}$ but with $c_1$ as in (4.5).

Now let $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be a given Carathéodory map, and consider the following delay differential equation

$$- u''(t) = g(t)f(t, u(t), u(t - r)), \ t \in [0, 1],$$

(4.6)

with $g$ nonnegative and measurable. We can apply the techniques developed in this paper to study the nontrivial solution of (1.6) with BCs (4.2)–(4.3), in the following way. Define the operator $F : [0, 1] \times C([-r, 0], \mathbb{R}) \to [0, \infty)$ by

$$F(t, \phi) = f(t, \phi(0), \phi(-r))$$

Note that the operator $F$ is Carathéodory-type, that is, $F$ verifies condition $(C_5)$ provided the map $f$ satisfies the following assumption:

$$[(C'_5)]$$

For each $R > 0$, there exists $\varphi^*_R \in L^\infty[0, 1]$ such that

$$f(t, u, v) \leq \varphi^*_R(t) \text{ for all } u, v \in \mathbb{R} \text{ with } |u| \leq R, |v| \leq R, \text{ and a.e. } t \in [0, 1].$$

Observe that, in order to obtain from Theorem 3.5 existence and multiplicity results for the FBVP (4.6)–(4.2)–(4.3), it is sufficient to consider the following numbers:

$$f^{(-p, \rho)} = \sup \left\{ \frac{f(t, u, v)}{\rho} : t \in [0, 1], |u|, |v| \leq \rho \right\},$$

$$f_{(\rho, \rho/c)} = \inf \left\{ \frac{f(t, u, v)}{\rho} : t \in [a, b], \rho \leq u, v \leq \rho/c \right\}.$$

These numbers are easier to compute than the analogous ones in the general case of a functional differential equation.

Let us consider, for illustrative purposes, the following autonomous equation depending on a positive parameter $\lambda$:

$$- u''(t) = \lambda |u(t)|^{p-1}|u(t - r)|, \ t \in [0, 1],$$

(4.7)

where $p \geq 1$, with the initial conditions (4.2) and the boundary conditions

$$u(0) = 0, \ \frac{1}{4} u'(1) + u \left(\frac{1}{4}\right) = 0.$$

(4.8)

Here we have $f(u, v) = |u|^{p-1}|v|$ so that $f^{(-p, \rho)} = \rho^{p-1} = f_{(\rho, \rho/c)}$. Moreover we have

$$k(t, s) = \left[ \frac{1}{2} t + 2t \left(\frac{1}{4} - s\right) H \left(\frac{1}{4} - s\right) - (t - s)H(t - s) \right] H(t).$$

A direct calculation shows that

$$\sup_{t \in [0, 1]} \int_0^1 |k(t, s)| \, ds = \frac{17}{16}$$

and therefore, in this case, $m = 16/17$.

As a consequence of Theorem 4.1 (using $(S_2)$ of Theorem 3.5) we get the following.
Corollary 4.2. Let $[a, b] = [1/4, 7/16]$, and let $c_2 = 1/4$ and $c_1 = 1/8$. Assume that $r < 3/16$ and let $\psi$ with $\|\psi\|_{[-r,0]} < 1$ be given. Then, for every $0 < \lambda < 16/17$ the FBVP (4.7)–(4.2)–(4.8) has at least one nontrivial solution $u_\lambda$, strictly positive on $[1/4, 7/16]$, with $\|u_\lambda\|_{[-r,1]} > 1$.

Proof. Take $\rho_1 = 1$ and observe that $(I_{\rho_1}^1)$ holds since $\lambda < 16/17 = m$. Moreover, for $\rho_2$ large enough (precisely $\rho_2 > \frac{M(a, b)}{\lambda}$) condition $(I_{\rho_2}^0)$ holds as well. Thus, Theorem 3.5 (S2) can be applied, yielding at least one solution $u_\lambda$, positive on $[1/4, 7/16]$, with $1 < \|u_\lambda\|_{[-r,1]} < \rho_2$. □

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