With the Lattice Boltzmann method (using the BGK approximation) we investigate the dynamics of Hele-Shaw flow under conditions corresponding to various experimental systems. We discuss the onset of the instability (dispersion relation), the static properties (characterization of the interface) and the dynamic properties (growth of the mixing zone) of simulated Hele-Shaw systems. We examine the role of reactive processes (between the two fluids) and we show that they have a sharpening effect on the interface similar to the effect of surface tension.

Viscous fingering occurs in the interfacial region between two fluids with different viscosities when a highly viscous fluid confined between two plates with a narrow gap (Hele-Shaw geometry) is displaced by a fluid with relatively low viscosity. Most theoretical and numerical studies on Hele-Shaw systems start with Darcy’s law assuming its validity under the simulation conditions, and thereby precluding discrimination between Navier-Stokes behavior or Darcy behavior of the system. With the Lattice Boltzmann (LB) approach the discrimination problem can be addressed because the LB method starts with a kinetic theoretical analysis where the macroscopic description is not pre-established.

While the LB method has now become a standard approach to investigate fingering processes, it has mostly been applied to the formation and growth of a single finger in a channel. Fingering in large systems have been studied and analyzed with Lattice Gas Automata constructed to simulate porous media. Here we consider fingering in spatially extended Hele-Shaw systems. We investigate the effects of surface tension and reactivity between the two fluids, and we demonstrate their...
role as determinant factors in the dynamics of the moving interface. The present paper gives a presentation of the essentials of the method and describes the main results of the work which will be discussed in detail elsewhere.

We use the LB equation with the Bhatnagar-Gross-Krook collision operator (LBGK method; see e.g. 3) in two-dimensional geometry emulating 3-D flows by introducing a Hele-Shaw drag term thereby simulating a system with a virtual cell gap in the third dimension. Using standard notation, the starting LBGK equation for multicomponent systems reads

$$f_{\sigma}^i(r + c_i, t + 1) - f_{\sigma}^i(r, t) = -\frac{1}{\tau^\sigma} (f_{\sigma}^i - f_{\sigma}^{eq}),$$

(1)

where $\sigma$ denotes the label of the component with density $\rho_{\sigma}(r) = \sum_i m_{\sigma} f_{\sigma}^i$ and velocity $u_{\sigma}(r) = m_{\sigma} \rho_{\sigma}(r) \sum_i f_{\sigma}^i c_i$. Taking the long wavelength limit of the LBGK equation, one recovers the phenomenological equations of Navier-Stokes hydrodynamics (see e.g. 5).

Additional features can be incorporated in the LBGK equation to account for the physics of the problem to be considered. Here we introduce (i) an interaction potential (initially used to simulate non-ideal fluids 7) for tunable miscibility (“surface tension”); (ii) an external force for the Hele-Shaw drag term (a term that can also be used to incorporate tunable gravitational effect in density driven fingering), and (iii) a reactive term which quantifies changes in the “chemical” nature of the species which are susceptible to react with each other.

(i) The interaction potential has the form $V(r, r') = G_{\sigma\sigma'}(r - r') \rho_{\sigma}(r) \rho_{\sigma'}(r')$ with $G_{\sigma\sigma'}(r - r') = G_{\sigma\sigma'}(r')$ for $|r - r'| = c$, and $G_{\sigma\sigma'}(r, r') = 0$ for $|r - r'| > c$, $c$ being the modulus of the relative velocity between particles. The potential enters the dynamics through the modified local velocity

$$\rho_{\sigma}(r) \tilde{u}_{\sigma}(r) = \rho_{\sigma}(r) u_{\sigma}(r) + \tau^\sigma F^\sigma(r, t),$$

(2)

with

$$F^\sigma(r, t) = -\rho^\sigma(r) \sum_{\sigma'} G_{\sigma\sigma'} \sum_i \rho^{\sigma'}(r + c_i) c_i.$$  

(3)

The amplitude ($G$) of the interaction term can be tuned from zero (miscible fluids) to practically total immiscibility.

(ii) The external force (here the Hele-Shaw drag term) is introduced by a force term which modifies the momentum of species $\sigma$; the new momentum reads

$$\rho^\sigma(r) \tilde{u}^\sigma(r) = \rho^\sigma(r) u^\sigma(r) (1 - \beta_{\sigma} \tau^\sigma).$$  

(4)

For the classical situation where fluid 1 invades fluid 2, the tunable drag coefficient is the ratio $\beta_1/\beta_2$.

(iii) Reactive processes are taken into account by a formulation similar to that used in reaction-diffusion equations (see e.g. 8). Here we shall use a typical reactive term $\sim \kappa \rho_1 \rho_2 (\rho_1 - \rho_2)$ for species 1 (and similarly for species 2 with $1 \leftrightarrow 2$) where $\kappa$ is the kinetic constant. This form restricts the reactivity effects to the mixing zone.
The LBGK model sketched above was used for binary fluids (σ = 1, 2) on the 2−D square lattice with nine velocities oriented along the lattice axes and the bisecting directions plus a zero speed (D2Q9 model). The simulations were performed on a $L_x \times L_y = 1024 \times 512$ lattice with the following initial conditions: fluid 1 with lowest viscosity is injected from the upper boundary of the system filled with the highly viscous fluid 2; periodic boundary conditions are imposed on the vertical boundaries; the initial planar interface is perturbed with white noise to trigger the instability. As time evolves, the highly viscous fluid 2 is displaced by fluid 1. The color code used in Fig.1 is from dark gray to light gray for $\rho_2 = 0 \rightarrow 1$ so that the pictures show “dark” fluid invading from above the system initially filled with “light” fluid.

Preliminary simulations were performed with a single phase fluid subject to a density gradient ($\nabla \rho$) imposed along the vertical axis ($y$), and the average velocity $v_y$ was measured for various values of the gradient: $v_y$ was found to be a linear function of the pressure gradient ($\propto \nabla \rho$) according to Darcy’s law.

The top panel in Fig.1 illustrates the case of miscible fluids ($G = 0$ in Eq.(3)) where a distinguishable mixing layer can be quantified to measure the mixing length $L_{\text{mix}}$ as a function of time $t$ for increasing values of the Péclet number ($Pe = L_x v_y / D$ where $D$ is the diffusion coefficient). In agreement with experimental observation and numerical simulation, we observe a transition from the short time diffusive regime where $L_{\text{mix}}$ scales as $t^\alpha$ with $\alpha \simeq 0.5$ to $\alpha \simeq 1$ behavior characteristic of the non-linear regime.

The middle panel of Fig.1 shows the effect of surface tension on the finger topology (compare with upper panel). Increasing the amplitude ($G$) of the interaction term reduces drastically the width of the mixing layer which eventually vanishes, and the finger wavelength is modified correspondingly. A quantification of this effect is given in Fig.2 where we show typical dispersion curves.

Reactive processes between two moving miscible fluids are expected to modify the fingering mechanisms as shown by phenomenological approaches based on a reaction-diffusion scheme, and experimental studies have indeed demonstrated that the interface properties depend strongly on chemical reactions. We have implemented a reactive mechanism by incorporating in the LB scheme a reaction term as described in (iii) above. The resulting effects are illustrated in the lower panel of Fig.1 which shows (i) a sharpening of the interfacial zone, and (ii) a modification of the topology of the fingering (compare with top panel) as quantified by the dispersion curves shown in Fig.2.

The purpose of this short paper is to demonstrate the applicability of the lattice Boltzmann method to the investigation of viscous fingering phenomena in spatially extended systems. In particular we have shown how surface tension and reactive processes can modify the dynamics and the characteristics of the fingering pattern. A quantitative analysis will be presented in a forthcoming paper with a detailed description of the work.
Fig. 1. LB simulations of viscous fingering (after $t = 6000$ time steps) in miscible fluids, $G = 0$ (top), fluids with surface tension, $G = 0.15$ (middle), and miscible ($G = 0$) reactive fluids (bottom).
Viscous Fingering in miscible, immiscible and reactive fluids

Fig. 2. Dispersion curves for (a) miscible fluids ($G = 0$; dots), (b) fluids with surface tension ($G = .05$; circles), and (c) miscible reactive fluids ($\kappa = .001$; squares) showing the effect of surface tension and of reactive processes on the most unstable wavelength and on the cutoff wave number.

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