A criterion for the existence of zero modes for
the Pauli operator with fastly decaying fields

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Abstract
We consider the Pauli operator in $\mathbb{R}^3$ for magnetic fields in $L^{3/2}$ that
decay at infinity as $|x|^{-2-\beta}$ with $\beta > 0$. In this case we are able to
prove that the existence of a zero mode for this operator is equivalent to
a quantity $\delta(B)$, defined below, being equal to zero. Complementing a
result from [4], this implies that for the class of magnetic fields considered,
Sobolev, Hardy and CLR inequalities hold whenever the magnetic field has
no zero mode.

1 Introduction
Consider the Pauli operator $P_A$ acting on $L^2(\mathbb{R}^3, \mathbb{C}^2) \equiv \mathcal{H}$, formally defined by

$$P_A = (p - A)^2 - \sigma \cdot B$$

where $B = \text{curl} A$. In appropriate units, this operator describes the kinetic
energy of a non-relativistic electron in the magnetic field $B$. We will also need
the Schrödinger operator $S_A = (p - A)^2$, which gives the kinetic energy of a
spinless particle in a magnetic field. An element of the kernel of $P_A$ is called a
zero mode for the corresponding Pauli operator.

The importance of zero modes for the Pauli operator was first pointed out in
[9], where the authors realized that their existence would imply a critical value of
the nuclear charge $Z$ in order to have a bounded ground state energy for a one-
electron atom in a magnetic field. In [12], the first examples of magnetic fields
producing zero modes were given. Further examples were given in [12 6 5, 2]
provides explicit examples of magnetic fields with an arbitrary number of zero
modes while in [6] a compactly supported magnetic field having a zero mode is
constructed. In [8] the authors use a geometrical approach which allows, for a
certain class of magnetic fields on $\mathbb{R}^3$, to relate the problem to the one on $S^2$,
which is better understood.

All of the above papers deal with the problem of describing the kernel of the
Pauli operator for fixed magnetic fields. A different point of view is adopted in [3]
and [7]. In these cases the authors describe the set of magnetic fields producing
zero modes, in [3] for $B \in L^{3/2}$ and in [6] for continuous $A$ decaying as $\alpha(|x|^{-1}).$
Both authors reach the conclusion that magnetic fields on $\mathbb{R}^3$ producing zero modes are rather rare which contrasts heavily with the situation in $\mathbb{R}^2$.

The existence of zero modes for the Pauli operator makes it impossible to use the kinetic energy of a wave function to control its potential energy as it is done for (magnetic) Schrödinger operators by Hardy’s inequality or the CLR-bound (5, 11, 13). However, in [4] it was shown that it is still possible to obtain this type of bounds for certain magnetic fields. Here, the goal is to give a more precise description of the class of magnetic fields for which this bound holds.

In order to make this statement precise, we first need to review some results of [3, 4].

If $|B| \in L^q$ for some $q \in [3^2, \infty]$, $S_A$ and $P_A$ have the same form domain $\mathcal{Q}(S_A)$. Both operators can be defined as Friedrich’s extensions of the respective quadratic forms. In addition, we will need the operator $\tilde{P}_A \equiv P_A + |B|$, with the same form domain. Since $\tilde{P}_A \geq S_A$, $\ker(\tilde{P}_A) = \{0\}$, so its range is dense in $\mathcal{H}$. The auxiliary Hilbert space $\tilde{\mathcal{H}}$ is defined as the completion of $\mathcal{Q}(S_A)$ with respect to the norm $\|u\|_{\tilde{\mathcal{H}}}^2 = (u, \tilde{P}_Au)$.

This space is not a subspace of $\mathcal{H}$. Its definition ensures $\tilde{P}_A^{-1/2}$ considered as an operator from $\text{Ran}(\tilde{P}_A^{1/2})$ to $\tilde{\mathcal{H}}$ preserves norms. As previously remarked, its domain is dense in $\mathcal{H}$. On the other hand, $\text{Ran}(\tilde{P}_A^{-1/2}) = \mathcal{D}(\tilde{P}_A^{1/2}) = \mathcal{Q}(P_A) = \mathcal{Q}(S_A)$, which is dense in $\tilde{\mathcal{H}}$ by construction. This means $\tilde{P}_A^{-1/2}$ can be extended to a unitary operator $U$ from $\mathcal{H}$ to $\tilde{\mathcal{H}}$. Multiplication by $|B|^{1/2}$ is a bounded operator from $\tilde{\mathcal{H}}$ to $\mathcal{H}$. This allows us to define

$$S = |B|^{1/2}U : \mathcal{H} \to \mathcal{H},$$

$$S = |B|^{1/2}(P_A + |B|)^{-1/2} \text{ on } \text{Ran}(\tilde{P}_A^{1/2}).$$

Finally, define

$$\delta(B) = \inf_{\|f\|=1, f \in \mathcal{H}} \|(1 - S^*S)f\|.$$  \hspace{1cm} (1)

With these definitions, we can state the main result.

Theorem 1.1. If $B \in L^{3/2}$ is such that $\delta(B) = 0$ and there exists $\beta > 0$, $C \geq 0$ and $r_0 \geq 0$ such that

$$|B|(x) \leq C|x|^{-2-\beta}$$

for all $|x| \geq r_0$, then the associated Pauli operator $P_A$ has a zero mode.

We do not know whether the condition on the decay of $B$ is optimal. In any case it can be replaced by the condition on the vector potential $A$ in hypothesis of lemma [5, 2]. Our method does not work without this additional decay of $A$.

The quantity $\delta(B)$ was introduced in [4] were the following result was proven:

Theorem 1.2 (Balinsky, Evans, Lewis, [4]). If $B \in L^{3/2}$, then

$$P_A \geq \delta(B) S_A.$$  \hspace{1cm} (2)
If $\delta(B) > 0$, this result allows to deduce for instance a Hardy inequality for $P_A$. If the Pauli operator corresponding to the magnetic field $B$ has a zero mode, then $\delta(B) = 0$. The content of theorem 1.1 is precisely the converse of this. For magnetic fields that decrease sufficiently fast at infinity, $\delta(B) = 0$ implies the existence of a zero mode for the corresponding Pauli operator. Unfortunately, inequality (2) still contains the positive but unknown quantity $\delta(B)$.

The remainder of this paper contains the proof of theorem 1.1. The next section contains some preliminary lemmas while the third section concludes the proof.

2 Simplifying the problem

To prove theorem 1.1 we will first simplify the statement, by reducing the condition $\delta(B) = 0$ to a simpler one and changing the hypothesis on the decay of $B$ into a hypothesis on $A$. This is done in the following two lemmas.

Lemma 2.1. If $\delta(B) = 0$, then

$$\inf_{g \in \mathbb{C}(8\Lambda)} \frac{(g, P_A g)}{(g, |B| g)} = 0.$$  
(3)

Proof. First, observe that if $\inf_{\|f\| = 1, Uf \in \mathcal{H}} \| (1 - S^*S)f \| = 0$, then

$$\sup_{\|f\| = 1, Uf \in \mathcal{H}} \| Sf \| = 1.$$  

To see this, first notice that for any $f \in \mathcal{H}$, $\| Sf \| \leq \| f \|$, so the sup in the above expression is at most 1. Now if $f_n$ is a minimizing sequence for the first problem,

$$(1 - S^*S)f_n \to 0 \text{ in } L^2$$

so in particular

$$(f_n, (1 - S^*S)f_n) \to 0.$$  

This means $\| Sf_n \|^2 = (f_n, S^*Sf_n) \to 1$.

Since the range of $\tilde{P}_A$ is dense in $\mathcal{H}$ and $S$ is bounded, nothing is lost by restricting the sup to functions $f \in \text{Ran}(\tilde{P}_A^{1/2})$. For these functions the condition $Uf \in \mathcal{H}$ is trivially satisfied. The problem can then be rewritten in terms of $g = Uf$:

$$1 = \sup_{\|f\| = 1, Uf \in \mathcal{H}} \| Sf \| = \sup_{f \in \text{Ran}(\tilde{P}_A^{1/2}) \setminus \{0\}} \frac{\| Sf \|}{\| f \|} = \sup_{g \in \mathcal{D}(\tilde{P}_A^{1/2}) \setminus \{0\}} \frac{\| B^{1/2}g \|}{\| \tilde{P}_A^{1/2} g \|}$$
The result is obtained by expanding \( \|\hat{P}_{A}^{1/2}g\|^2 = (g, P_A g) + (g, |B|g) \) and using \( D(\hat{P}_{A}^{1/2}) = Q(S_A) \):

\[
1 = \sup_{\|f\|=1, \forall f \in \mathcal{H}} \|Sf\|^2 = \sup_{g \in Q(S_A) \setminus \{0\}} \left( \frac{(g, P_A g)}{(g, |B|g) + 1} \right)^{-1},
\]

which is only possible if

\[
\inf_{g \in Q(S_A)} \frac{(g, P_A g)}{(g, |B|g) \neq 0} = 0. \tag*{\square}
\]

Then, we show that the imposed decay of \( B \) implies a good decay of \( A \) if we fix the gauge

\[
\frac{1}{4\pi} A(x) = \int \frac{x - y}{|x - y|^3} \times B(y) dy. \tag{4}
\]

Note that \( A \) as defined above is in \( L^3 \) by the weak Young inequality.

**Lemma 2.2.** If \( B \in L^{3/2} \) is such that there exists \( \beta > 0 \), \( C_B \geq 0 \) and \( r_0 \geq 0 \) such that

\[
|B|(x) \leq C_B|x|^{-2-\beta}
\]

for all \( |x| \geq r_0 \), then there exist \( r_1 \geq r_0 \) and \( C_A \) such that

\[
|A|(x) \equiv 4\pi \int \frac{x - y}{|x - y|^3} \times B(y) dy \leq C_A|x|^{-1-\alpha}
\]

for \( \alpha = \min(1/2, \beta/2) \) and all \( |x| \geq r_1 \).

**Proof.** Take \( r_1 = \max((2r_0)^2, 1) \). Take any \( x \) such that \( |x| \geq r_1 \) and define \( r_x = |x|^{1/2} \geq r_0 \). Split the domain of integration in the definition of \( A \) in two parts and apply Hölder's inequality to the first part to obtain

\[
|A|(x) \leq 4\pi \int_{B_{r_x}} |B(y)||x - y|^{-2} dy + 4\pi \int_{B_{r_x}} |B(y)||y - x|^{-2} dy
\]

\[
\leq 4\pi \|B\|_{3/2} \left( \int_{B_{r_x}} |x - y|^{-6} dy \right)^{1/3} + 4\pi C_B \int_{B_{r_x}} |y|^{-2-\beta} |x - y|^{-2} dy
\]

The integrand in the first term is bounded, so

\[
\int_{B_{r_x}} |x - y|^{-6} dy \leq \frac{4\pi}{3} r_x^3 (|x| - r_x)^{-6}
\]

\[
\leq \frac{2^3 \pi}{3} |x|^{-9/2}
\]

The second integral requires some more care:

\[
\int_{B_{r_x}} |y|^{-2-\beta} |x - y|^{-2} dy = 4\pi \int_{r_x}^{\infty} r^{-\beta} dr \int_{-1}^{1} dt (|x|^2 + r^2 - 2r|x|t)^{-1}
\]

\[
= 2\pi |x|^{-1} \int_{r_x}^{\infty} r^{-\beta - 1} \ln \left( \frac{|x| + r}{|x| - r} \right) dr
\]

\[
= 2\pi |x|^{-1-\beta} \int_{r_x/|x|}^{\infty} t^{-\beta - 1} \ln \left( \frac{1 + t}{|1 - t|} \right) dt.
\]
This last integral is finite since for large $t$, the integrand is bounded by a constant times $t^{-\beta-1}$, while for $t$ close to 1 it diverges only as a logarithm. Separating the range of integration in $r_x/x \leq t \leq 1/2$ and $t > 1/2$ we note that the first part gives a contribution that behaves as $C_1(r_x/|x|)^{-\beta}$ while the contribution of the second part can be bounded by a constant. This means

$$\int_{B_{r_x}} |y|^{-2-\beta} |x - y|^{-2} dy \leq |x|^{-1-\beta} \left( C_1 \left( \frac{r_x}{|x|} \right)^{-\beta} + C_2 \right) \leq C_1 2^\beta |x|^{-1-\beta/2} + C_2 |x|^{-1-\beta}.$$ 

We conclude

$$|A|(x) \leq C_A(|x|^{-1-1/2} + |x|^{-1-\beta/2}) \leq 2C_A|x|^{-1-\alpha}.$$ 

### 3 Compactness and Integrability

Now we use a compactness-argument to find a candidate zero mode if the infimum in equation (3) equals zero.

**Lemma 3.1.** If $B \in L^{3/2}$, and $\delta(B) = 0$ then there exist $g \in W^{1,2}_{\text{loc}} \cap L^6$ such that

$$\sigma \cdot (p - A)g = 0$$

in the particular gauge for $A$ defined in (4).

**Proof.** Take $(g_n)$ a minimizing sequence for the problem (3) with $(g_n, |B|g_n) = 1$. Then $(g_n, P_A g_n)$ is bounded, which implies by the diamagnetic inequality that $(pg_n)$ is bounded in $L^2$ so $(g_n)$ is bounded in $L^6$. By the Banach-Alaoglu theorem, this guarantees the existence of a subsequence such that $pg_n$ converges weakly in $L^2$ to some $pg$ and $g_n \rightharpoonup g$ weakly in $L^6$. Since $|B| \in L^{3/2}$, this implies $(g, |B|g) = 1$, so $g \neq 0$. In addition, since $A \in L^3$, $(A g_n)$ is bounded in $L^2$ so we can assume $A g_n \rightharpoonup A g$ weakly in $L^2$. Using the fact that $L^p$-norms are weakly lower-semi-continuous, we obtain $\| \sigma \cdot (p - A) g \|_2 = 0$.

To conclude the proof of theorem 1.1 we only need to show that this candidate zero mode is in $L^2$. This is achieved by using the decay of $A$ given by lemma 2.2 in a bootstrap argument. The procedure is not that straightforward since the decay of $A g$ and the Pauli equation imply only a decay of $\sigma \cdot p g$, which does not directly imply the decay of $p g$.

**Lemma 3.2.** If there exist $\alpha > 0$ and $r_1 > 0$ such that $|A|(x) < C_A|x|^{-1-\alpha}$ for all $x \in \mathbb{R}^3$ with $|x| \geq r_1$ and $g \in W^{1,2}_{\text{loc}} \cap L^p$, with $p \geq 2$, is such that

$$\sigma \cdot (p - A)g = 0,$$

then $g \in L^2$.

In order to prove this lemma, one more technical lemma will be necessary. Its proof can be found in the appendix. The inner product in $L^2(S^2, C^2)$ will be denoted by $\langle \cdot , \cdot \rangle$. When $f$ and $g$ are defined on all of $\mathbb{R}^3$, we will abuse notation and write $\langle f, g \rangle (r) \equiv \langle f(r \omega), g(r \omega) \rangle$. We will also use the notation $\langle f \rangle (r) = (\langle f \rangle)^{1/2} (r)$.
Lemma 3.3. If \( f \in W^{1,2}_{\text{loc}}(\mathbb{R}^3) \) then \( f \in W^{1,2}([a,b]) \) for all \( b > a > 0 \), and its weak derivative equals

\[
h(r) = \begin{cases} 
(f^{-1}(r)\Re \langle f, \partial_r f \rangle & \text{if } \langle f \rangle(r) > 0 \\
0 & \text{else.}
\end{cases}
\]

In particular \( \langle f \rangle \) is continuous except maybe at 0.

Proof of lemma 3.2. Define

\[
K = -1 - \sigma \cdot L,
\]

which can be considered as a self-adjoint operator on \( L^2(S^2, \mathbb{C}^2) \) with eigenvalues \( \pm 1, \pm 2, \ldots \) (see for instance [10], section 1.5). Write \( g = g_+ + g_- \) where \( \langle g_+, Kg_+ \rangle > 0 \) and \( \langle g_-, Kg_- \rangle < 0 \). If \( g \in L^p(\mathbb{R}^3) \), there exists \( C > 0 \) such that

\[
\int_{S^2} |g|^p(r \omega) d\omega \leq Cr^{-3}.
\]

By Jensen’s inequality, this implies

\[
Cr^{-3} \geq \int_{S^2} |g|^p(r \omega) d\omega \geq (4\pi)^{1-p/2} \left( \int_{S^2} |g|^2(r \omega) d\omega \right)^{p/2} = (4\pi)^{1-p/2} \left( \langle g_+, g_+ \rangle + \langle g_-, g_- \rangle \right)^{p/2},
\]

so both \( \langle g_+ \rangle(r) \) and \( \langle g_- \rangle(r) \) decay as \( Cr^{-3/p} \).

At first, we will prove the theorem in the case that \( g_+ \) and \( g_- \) are \( C^2 \)-functions. The Pauli operator can be written conveniently as

\[
\sigma \cdot p = (\sigma \cdot \hat{x})^2 \sigma \cdot p = -i\sigma \cdot \hat{x} \left( \partial_r + \frac{K+1}{r} \right),
\]

where the operator inside the parenthesis commutes with \( K \). This allows to rewrite the equation for \( g \) as

\[
\partial_r g + \frac{K+1}{r} g = i\sigma \cdot \hat{x} \sigma \cdot A g.
\]

For shortness, define \( \sigma_A = \sigma \cdot \hat{x} \sigma \cdot A \). The only property of this matrix needed is \( \|\sigma_A(r \omega)\| \leq C_A r^{-1-\alpha} \) when \( r \geq r_1 \). Taking the \( C^2 \) product with \( g_+ \) and \( g_- \) and integrating over \( S^2 \), we obtain

\[
\begin{align*}
\langle g_+, \partial_r g_+ \rangle(r) &= -\frac{1}{r} \langle g_+, (K+1)g_+ \rangle(r) + i \langle g_+, \sigma_A (g_+ + g_-) \rangle \quad (5) \\
\langle g_-, \partial_r g_- \rangle(r) &= -\frac{1}{r} \langle g_-, (K+1)g_- \rangle(r) + i \langle g_-, \sigma_A (g_+ + g_-) \rangle.
\end{align*}
\]

By taking the real part of these equations, we obtain a differential equation for \( \langle g_+ \rangle \) and \( \langle g_- \rangle \):

\[
\frac{d}{dr} \langle g_+ \rangle^2 = -2\frac{1}{r} \langle g_+, (K+1)g_+ \rangle(r) - 2\Re \langle g_+, \sigma_A (g_+ + g_-) \rangle
\]

\[
\frac{d}{dr} \langle g_- \rangle^2 = -2\frac{1}{r} \langle g_-, (K+1)g_- \rangle(r) - 2\Re \langle g_-, \sigma_A (g_+ + g_-) \rangle.
\]

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Defining $\bar{g}_+ = g+r^2$ we get the system of equations
\[
\frac{d}{dr}(\bar{g}_+)^2 = -2\frac{1}{r} (\bar{g}_+, (K - 1)\bar{g}_+) (r) - 2\mathbb{A} (\bar{g}_+, \sigma \mathbb{A} (\bar{g}_+ + r^2 g_-))
\]
\[
\frac{d}{dr}(g_-)^2 = -2\frac{1}{r} (g_-, (K + 1)g_-) (r) - 2\mathbb{A} (g_-, \sigma \mathbb{A} (r^{-2} \bar{g}_+ + g_-))
\]

Fix $r \geq r_1$. We now use a bootstrap argument to obtain \( (g_\pm) (r) \leq C r^{-2} \).
As remarked previously \( (g_-)^2 (r) \leq C r^{-\epsilon} \) and \( (\bar{g}_+)^2 (r) \leq C r^{3 - \epsilon} \) with \( \epsilon = 3/p \).
We will see the equations imply \( (g_-)^2 (r) \leq C r^{-\epsilon - \alpha} \) and \( (\bar{g}_+)^2 (r) \leq C r^{4 - \epsilon} \),
where \( \epsilon_1 = \min (\epsilon + \alpha, 4) \).

For $\bar{g}_+$, we can use $\langle \bar{g}_+, K \bar{g}_+ \rangle \geq \langle \bar{g}_+ \rangle^2$ in order to obtain
\[
\langle \bar{g}_+ \rangle^2 = \int_{r_1}^r -2s^{-1} (\bar{g}_+, (K - 1)\bar{g}_+) (s) - 2\mathbb{A} (\bar{g}_+, \sigma \mathbb{A} (\bar{g}_+ + s^2 g_-)) \, ds + C_1
\]
\[
\leq 2 \int_{r_1}^r |\bar{g}_+, \sigma \mathbb{A} \bar{g}_+ \rangle (s) |s^{-2}| \rangle (\bar{g}_+, \sigma \mathbb{A} g_- \rangle (s) |ds + C_1
\]
\[
\leq 4CC_A \int_{r_1}^r s^{4 - \epsilon - 1 - \alpha} + C_1
\]
\[
= \frac{4CC_A}{\epsilon - \alpha} (r^{4 - \epsilon - \alpha} - 1) + C_1.
\]

For $g_-$, we can use the fact $\langle g_- \rangle$ tends to zero as $r \to \infty$ and $\langle g_-, Kg_- \rangle \leq -\langle g_- \rangle^2$
to write
\[
\langle g_- \rangle^2 = \int_r^\infty 2s^{-1} (g_-, (K + 1)g_-) (s) + 2\mathbb{A} (g_-, \sigma \mathbb{A} (s^{-2} \bar{g}_+ + g_-)) \, ds
\]
\[
\leq \int_r^\infty 2 |\langle g_-, \sigma \mathbb{A} (s^{-2} \bar{g}_+ + g_-) \rangle |ds
\]
\[
\leq 2CC_A \int_r^\infty 2s^{-\epsilon - 1 - \alpha} ds
\]
\[
= \frac{2CC_A}{\epsilon + \alpha} r^{-\epsilon - \alpha}.
\]

By iterating this procedure a finite number of times we reach the conclusion
\( \langle g_\pm \rangle (r) \leq C r^{-2} \) and \( (g_-) (r) \leq C r^{-\epsilon} \), so $g \in L^2 (\mathbb{R}^3)$.
This concludes the proof of the lemma when $g_+$ and $g_-$ are $C^2$-functions.

In the general case, $g$ has a decomposition in a series of spherical spinors (see
for example [10], section 1.5) where the coefficients are functions of $r$ belonging
to $W^{1,2}_{loc} (\mathbb{R}^3, r^2 dr)$. By taking the projections on the positive and negative
eigenspaces of $K$ and using dominated convergence, we conclude $g_+$ and $g_-$ are
in $W^{1,2}_{loc} (\mathbb{R}^3)$. Thus, by Fubini’s theorem, $g_\pm$, and $\partial_r g_\pm$ are
in $L^2 (S^2 (r))$ for almost every $r > 0$. This justifies the integration over $S^2$ used to obtain [5].

By lemma 4, $(g_+)$ and $(g_-)$ are in $W^{1,2} ([a, b])$ for any $b > a > 0$ and thus continuous. The use of the fundamental theorem of calculus in [3]
can be justified by applying it to a sequence of $C^\infty$-functions converging to $(g_+)$
pointwise and in $W^{1,2} ([r_1, r])$. In the same way we can obtain
\( (g_-) = \frac{d}{dr} (g_-) (r) dr + (g_-) (r_2) \) for any $r_2 > r > r_1$.
Since $\frac{d}{dr} (g_-) (r)$ is in
$L^1 ([r_1, +\infty))$, we can let $r_2 \to \infty$ in order to obtain [4].

\[\Box\]
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Appendix A: proof of lemma [3.3]

By Fubini’s theorem, $f$ and $\partial_r f$ are in $L^2(S^2(r))$ for almost every $r > 0$ and

$(f), (\partial_r f)$ are in $L^2_{\text{loc}}(\mathbb{R}^3, r^2 dr)$. Fix $b > a > 0$ and define the annulus $A = \{x \in \mathbb{R}^3 | a \leq |x| \leq b\}$.

Fix $\epsilon > 0$. As a first step, we will prove $f_\epsilon \equiv (|f|^2 + \epsilon)^{1/2} \in W^{1,2}([a, b])$. Define $h_\epsilon = \langle f_\epsilon \rangle^{-1} R \langle f, \partial_r f \rangle$. By the Cauchy-Schwarz inequality $h_\epsilon$ is in $L^2([a, b])$. It remains to check whether $h_\epsilon$ is the distributional derivative of $f_\epsilon$.

To this end, take a sequence $(f_n) \subset C^1(A, \mathbb{C}^2)$ approaching $f$ in $W^{1,2}(A)$ and pointwise almost everywhere in $A$. This means $(f_n) \to (f)$ in $L^2([a, b])$ so by extracting a subsequence we may assume $(f_n)(r) \to (f)(r)$ for almost every $r \in [a, b]$. Define $h_n \equiv \partial_r ((|f_n|^2 + \epsilon)^{1/2})$. We have $h_n = (|f_n|^2 + \epsilon)^{-1/2} R \langle f_n, \partial_r f_n \rangle (r)$. In order to conclude, we should prove that, for any test function $\phi \in C^\infty_0([a, b])$,

\[ \int_a^b \phi(r) h_n(r) dr \to \int_a^b \phi(r) h_\epsilon(r) dr \quad \text{as } n \to \infty. \]

To achieve this, fix $\phi \in C^\infty_0([a, b])$ and define $\Phi_n = (|f_n|^2 + \epsilon)^{-1/2} \phi f_n$ and $\Phi_\epsilon = f_\epsilon^{-1} \phi f_\epsilon$. $\Phi_n(x)$ converges to $\Phi_\epsilon(x)$ when $x \in A$ is such that $\langle f_n \rangle(|x|)$ converges to $\langle f \rangle(|x|)$ and $f_n(x) \to f(x)$, which holds for almost every $x$ in $A$. Since $(\Phi_n)$ is bounded in $L^2(A)$, by dominated convergence $\Phi_n \to \Phi_\epsilon$ in $L^2(A[a, b])$. This allows us to obtain

\[ \int_a^b |\phi(h_n - h_\epsilon)| \leq \int_a^b |(\Phi_n, \partial_r f_n) - (\Phi_\epsilon, \partial_r f_\epsilon)| \]

\[ \leq \int_a^b |(\Phi_n - \Phi_\epsilon, \partial_r f_n)| + \int_a^b |(\Phi_\epsilon, \partial_r f_n - \partial_r f_\epsilon)| \]

\[ \leq a^{-2} |\Phi_n - \Phi_\epsilon|_{2, A} \|\partial_r f_n\|_{2, A} + a^{-2} |\Phi_\epsilon|_{2, A} \|\partial_r f_n - \partial_r f_\epsilon\|_{2, A}. \]

In the last line, we used $1 \leq a^{-2} r^2$ in the domain of integration to transform the integral over an interval in an integral over $A$. Since $f_n$ tends to $f$ in $W^{1,2}(A)$, the second term tends to zero and the second factor of the first term is bounded. As previously remarked, $\Phi_n - \Phi_\epsilon$ tends to zero in $L^2(A)$ so the first term goes to zero too. This means $f_\epsilon \in W^{1,2}(A[a, b])$ and its distributional derivative equals $h_\epsilon$.

Now, we can let $\epsilon$ tend to zero. Then $f_\epsilon \to (f)$ and $h_\epsilon(r) \to h(r)$ in $L^2([a, b])$. We conclude $(f) \in W^{1,2}(a, b)$ and $h = \frac{d}{dr} (f)$. \hfill \Box

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