Augmented analytic kludge waveform with quadrupole moment correction

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One of the most important sources for future space-borne gravitational wave (GW) detectors such as TianQin and LISA is extreme mass ratio inspiral (EMRI). It happens when a stellar origin compact object (CO) orbiting around a massive black hole (MBH) in the center of galaxies and has many benefits in the study of astrophysics and fundamental theories. One of the most important objectives is to test the no-hair theorem by measuring the quadrupole moment of the MBH. This requires us to estimate the parameters of an EMRI system accurately enough, which means we also need an accurate waveform templet for this process. Based on the fast and fiducial augmented analytic kludge (AAK) waveform for the standard Kerr black hole (BH), we develop a waveform model for a metric with non-Kerr quadrupole moment. We also analyze the accuracy of parameter estimation for different sources and detectors.

I. INTRODUCTION

The observation of gravitational wave has provided a new approach to probe the universe. By analyzing the signal of GW150914 [1] and all the GW events observed by ground-based observatory, the constraint for the theory of gravity has been enhanced to a higher [2, 3]. For example, the mass of graviton should be less than \( \sim 10^{-23} \, \text{eV} \). However, several space-borne GW detectors is proposed to be launched in the 2030s, such as the heliocentric detector LISA [4] and the geocentric detector TianQin [5].

Different from the ground-based GW detectors which are sensitive to the hundred Hz GW signals, space-borne detectors are sensitive to the micro Hz band. Among all the sources in this band, EMRI is one of the most important targets [6, 7]. It happens when a stellar mass compact object, which could be a neutron star or a black hole, captured by the MBH in the center of a galaxy, and orbiting around the MBH in the near horizon region for more than thousands cycles before plunges into the MBH. Since the CO will stay in the strong gravity region for a long time, so the GW radiated by EMRI will carry a wealth of information about the geometry and environment around the MBH. Then by analyzing the GW signal emitted by an EMRI system, one can verify the existence of different kinds of dark matters surrounding the MBH [8, 9] or test the “no-hair” nature [10, 11] of the MBH.

In this work, we will focus on the issue of no-hair theorem. More explicitly, one of the performance of no-hair theorem is that the multipole moments of a Kerr BH in GR is completely determined by its mass and spin as \( M_l + is_l = M(la)^l \) [12–14], where the mass multipole moments \( M_l \) and mass-current multipole moments \( S_l \) are real numbers. But in other theories of gravity or other BH solutions, it will also be influenced by the additional parameters. So by measuring the mass, spin, and quadrupole moment of a BH, and check whether they satisfy this relation within the range of error, we can judge whether it’s a Kerr BH or not. This is actually determined by the precision of parameter estimation (PE) for those parameters. Then an accurate waveform including this effect is needed to enhance the ability of testing no-hair theorem.

The waveform for an EMRI system is very complex, since many higher order perturbation effects will play important role in orbit evolution and GW generation. Due to the extreme mass ratio, inspiral will take a very long time, up to several years (\( \sim 10^5 \) seconds) or equivalently \( \sim 10^5 \) cycles. Then the longer the dephasing time for the waveform, the less the segments needed in the semi-coherent detection. And the critical signal to noise ratio (SNR) will be lower. On the other hand, in the semi-coherent search, we need to generate a large amount of waveform templates, so the time of generation is also very important.

In spite of the Teukolsky-based waveform based on the black hole perturbation theory which is computationally expensive, the kludge family is a class of very important and widely used methods which can be generated quickly and capture the main features of the true signals. The basic idea of kludge is to combine different features of orbital evolution and GW emission directly without the consideration of their coupling. Roughly speaking, there are three kludge models. The analytic kludge (AK) model [15] is constructed by calculating the orbit evolution with post newtonion (PN) expansion under the consideration of Lense-Thirring precession and pericenter precession. Then the waveform is generated with Peter-Mathews formula [16, 17] in the quadrupole approximation. AK generates the waveform very fast, but the accuracy is limited by the kludge method. However, we can improve its precision by simply adding higher order terms. So it’s still widely used in a lot of order of mag-
nitude analyses. The numerical kludge (NK) model [18] provides a more accurate waveform with a slightly expensive computational cost. It calculates the trajectory evolution first in the phase space defined by the constants, and integrates out the more reality trajectory in the coordinate space. Then the GW waveform can be calculated with the leading order quadrupole approximation. In recent years, the AAK model [19, 20] is also developed to combine the advantages of the previous models. It maps the parameters of AK waveform to match the frequencies of NK waveform, and then uses the new parameters to generate the waveform with AK. Generally, AAK shows an excellent overlap with NK, retaining the speed advantage of AK.

By adding the quadrupole moment term to the PN orbit evolution equation of A K, LISA’s ability of measuring the quadrupole of MBH has been studied [21]. This waveform model is denoted as quadrupole included analytic kludge (QAK) in this paper since it can produce the waveform for an EMRI system whose central massive object could possess arbitrary quadrupole moment. However, although the fisher information matrix (FIM) method is just an order of magnitude estimation, but with a more accurate waveform we can get a better estimation.

Beside considering to include the quadrupole moment corrections, there also exist many other alternative methods using the GW of EMRIs to test the nature of gravitational theory and black hole. In [22], by requiring the existence of a perturbative second-order Killing tensor for the bumpy black hole, the three constants for motion are still possessed in the parametric deformed Kerr metric for non-GR deviations. Then the leading order bump corrections to AK waveforms are obtained by [24]. This work applies the ppE framework into the EMRI waveform computations, and push forward a first attempt toward complete and model-independent tests of General Relativity with EMRI. The corresponding FIM analysis is also taken in [25]. Another very important progress is the development of a framework for testing GR with EMRI observations in [26]. The Bayesian method is used in the analysis using the bumpy AK waveform in [24].

In this paper, we describe a quadrupole included augmented analytic kludge (QAAK) waveform model based on the more accurate AAK model. (The code of QAAK is developed based on the AAK code from the EMRI Kludge Suite, which can be found from the following url: https://github.com/alvincjk/EMRI_Kludge_Suite.) We first update the NK waveform to include the quadrupole moment, and then map the parameters of QAK to match the frequencies. We also calculate the PE result based on the QAAK waveform.

A brief overview of the kludge waveforms is given in II. Then we review the quadrupole correction in the QAK waveform in III, and present the correction we used in the QAAK waveform model in IV. Finally we analyze the accuracy of parameter estimation for various sources and detectors in V. Then the paper ends with a conclusion in VI. We use the geometric unit with $G = c = 1$.

## II. A BRIEF REVIEW OF THE KLUDGE FAMILY

Currently, there are three members in the kludge family which is used for the generation of EMRI’s waveform, they are AK [15], NK [27] and AAK [19, 20]. However, there exist many other EMRI waveform models, such as [28, 29] and [30] which include the self-force correction, and so on. But we will not discuss these models here.

Generally, the kludge family described the inspiral waveform for a CO which is regarded as a point particle with mass $\mu$, in the background of a Kerr BH with mass $M$ and spin $a$. The metric is written in the Boyer-Lindquist coordinates:

$$
\begin{align*}
\text{d}s^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right)\text{d}t^2 - \frac{4aMr\sin^2\theta}{\Sigma}\text{d}t\text{d}\phi \\
&\quad + \left(\Delta + \frac{2Mr(r^2 + a^2)}{\Sigma}\right)\sin^2\theta\text{d}\phi^2 \\
&\quad + \frac{\Sigma}{\Delta}\text{d}r^2 + \Sigma\text{d}\theta^2,
\end{align*}
$$

with

$$
\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta.
$$

And according to the definition of EMRI, we have $M \gg \mu$. In fact, all the masses here are red-shifted mass $M(1 + z)$, but we will not use a subscript to distinguish these variables. The direction of the spin for the MBH is represented by the unit vector $\hat{S}$, or equivalently by $\theta_K$ and $\phi_K$.

In the kludge family, orbit is considered as an eccentric and non-equatorial one. We will use $e$ and $p$ to represent eccentricity and semi-latus rectum. The pericenter and apocenter distance is $r_p = p/(1 + e)$ and $r_a = p/(1 - e)$. The direction of the CO’s orbital angular momentum is represented by $\hat{L}$. Then the angle between $\hat{L}$ and $\hat{S}$ is $\iota$, the azimuthal of $\hat{L}$ is $\alpha$, and the angle between $\hat{L} \times \hat{S}$ and the pericenter is $\gamma$. For the CO’s motion, $\Phi$ is the mean anomaly, and $\nu$ is the radial frequency.

Apart from the geodesic parameters, the orbit can also be described by the conserved quantities: the orbital energy $E$, the angular momentum on the direction of $\hat{S}$ which is $L_z$, and the Carter constant $\bar{K}$. Equivalently, it can be described by the dimensionless fundamental angular frequencies corresponding to the three spacial coordinate: $\omega_r, \omega_\theta, \omega_\phi$.

The location of the source is defined by the angular position $(\theta_S, \phi_S)$ and the luminosity distance $D_L$. In fact, in this paper we consider a sky-averaged response by the GW observatory. So the position of the source will not appear in the following discussion, since it will only influence the antenna pattern function, and has nothing to do with the waveform generation.
Then by assuming the initial value and equation of motion of these parameters, the orbit evolution and then the GW waveform can be produced by using the following three different kludge models.

A. Analytic kludge

In AK model, the orbital evolution is given by five first order ordinary differential equations of \((\Phi, \nu, \gamma, e, \alpha)\). The equations are given by the PN method, and is presented by Eqs. (27)–(31) in [15]. In that paper, the equations of \(\nu\) and \(e\) are accurately through 3.5 PN order, and the equations of \(\gamma\) and \(\alpha\) are accurately through 2 PN order, while they are all accurately through order 1 for the spin \(a\). Obviously, higher order terms can be added directly into these equations if available.

Then by integrating out these geodesic parameters, we can obtain the waveform with \(n\)-harmonics by using the Peter-Matthews method in the quadrupole approximation, which is described by Eqs. (7)–(10) in [15].

As a result of computational efficiency, AK is used in various works of the science case study of EMRI in [6, 7, 21, 31]. It has also been used in the mock LISA data challenges for the generation of injected signals and templates for search [32–35]. However, the insufficient accuracy will reduce the performance of detection and PE if it’s applied to analyze the data sets containing realistic EMRI signals. But for a PE analyze based on FIM, it will be accurate enough for an order of magnitude estimation of the EMRI signals with sufficiently high SNR.

B. Numerical kludge

In the NK model, the orbit is given by integrating the geodesic equations:

\[
\begin{align*}
\Sigma \frac{dr}{d\tau} &= \pm \sqrt{V_r}, \\
\Sigma \frac{d\theta}{d\tau} &= \pm \sqrt{V_\theta}, \\
\Sigma \frac{d\phi}{d\tau} &= V_\phi, \\
\Sigma \frac{dt}{d\tau} &= V_t,
\end{align*}
\]

(3)

where \(\tau\) denotes the proper time, and the potentials \(V_{r,\theta,\phi,t}\) are functions of the constants \((E, L_z, K)\) and the coordinates \((r, \theta)\).

For a bound orbit, the trajectory is determined by \((r_a, r_p, \theta_{min})\), where \(\theta_{min}\) is the minimum value of \(\theta\). In fact, \(r_a\) and \(r_p\) are the roots of \(V_r\), while \(\theta_{min}\) is the smallest root of \(V_\theta\). Equivalently, we can also describe a trajectory with \(p = \frac{r_a r_p}{r_a - r_p}, e = \frac{r_a - r_p}{r_a + r_p}, \) and \(t = \frac{r}{2} - \theta_{min}\). According to the well-known PN result, the time derivatives of the constants \((E, L_z, K)\) are functions of \((M, a, \mu)\) and \((p, e, t)\) [36]. Then the evolution of the constants can be integrated out. And the trajectory of the CO can be calculated out afterwards. So the waveform can be obtained from the inspiral trajectory.

The accuracy of NK waveform is well enough to agree with the Teukolsky-based waveform, which is much better than AK. But the computation cost is also more expensive, since it needs to integrate the trajectory both in the phase space and the coordinate space elaborately.

C. Augmented analytic kludge

The AAK model possesses both the speed of AK and the accuracy of NK. It first generates a small section of trajectory with NK, and then maps the AK trajectory to the NK result and finds out the best-fit parameters. Then the waveform will be generated by AK with these new parameters.

Briefly speaking, given the orbit evolution in NK, by defining a timelike parameter \(\lambda = \int d\tau/\Sigma\), we can define the dimensionless fundamental frequencies \(\omega_{r,\theta,\phi}\) as

\[
\begin{align*}
\omega_r &= \frac{2\pi}{r M, \Gamma}, \\
\omega_\theta &= \frac{2\pi}{M a, \Gamma}, \\
\omega_\phi &= \frac{1}{M, \Lambda_\theta, \Gamma} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta V_\phi,
\end{align*}
\]

(4)

with \(\Lambda_r, \Lambda_\theta, \) and \(\Gamma\) are given by

\[
\begin{align*}
\Lambda_r &= 2 \int_{r_p}^{r_a} \frac{dr}{\sqrt{V_r}}, \\
\Lambda_\theta &= 2 \int_{\theta_{min}}^{\pi} \frac{d\theta}{\sqrt{V_\theta}}, \\
\Gamma &= \frac{1}{\Lambda_r \Lambda_\theta} \int_0^{\Lambda_r} d\lambda_r \int_0^{\Lambda_\theta} d\lambda_\theta V_t.
\end{align*}
\]

As a function of \((M, a, p)\), these fundamental frequencies can be related to the orbital frequencies as

\[
\begin{align*}
\dot{r}(M, a, \dot{p}) &= \omega_r(M, a, p), \\
\dot{\theta}(M, a, \dot{p}) &= \omega_\theta(M, a, p), \\
\dot{\phi}(M, a, \dot{p}) &= \omega_\phi(M, a, p).
\end{align*}
\]

(6)

The left hand side is given by the AK orbital equations. By solving these equations, we can get the unphysical parameters \((\dot{M}, \dot{a}, \dot{p})\). Next, the waveform can be generated by AK with the new parameters.

To reduce the computational cost, the map is done on a small section. Then, the correction along the local trajectory will be extrapolated to global inspiral as fitted polynomials. The details can be found in [19]. The AK part in AAK is replaced by a higher order equation in [37].

On the other hand, the last stable orbit (LSO) cutoff is also different from the one used in AK for Schwarzschild and Kerr. The plunge happens when

\[
\frac{\partial^2 V_{r}(r, a, E, L_z, K)}{\partial r^2} \leq \frac{\partial V_{r}(r, a, E, L_z, K)}{\partial r} = V_{r}(r, a, E, L_z, K) = 0
\]

(7)
In practice, AAK uses Kepler’s third law to estimate \( p \) by frequency roughly, then check the stability of \((c, \iota, p)\).

### III. QUADRUPOLE MOMENT AND ITS INFLUENCE ON THE EMRI WAVEFORM

The famous “no hair” theorem [11] of General Relativity has an important prediction: “settles down” to the Kerr solution almost immediately after its formation, and all of its property can be totally expressed in terms of two physical parameters alone: its mass \( M \) and spin parameter \( a \).

As a consequence of these theorem, the multipole moments of a Kerr BH is characterized by only \( M \) and \( a \) according to the neat relation [12–14]:
\[
M_l + i S_l = M (i a)^{\gamma_l},
\]
where \( M_l \) and \( S_l \) are the mass and mass-current multipole moments, respectively, and \( a = S/M \) is the spin parameter. For instance, the quadrupole moment \( Q \equiv M_2 \) of the pure Kerr geometry is given by
\[
Q = -S^2/M.
\]
But for other BH solutions in other theories of gravity, the relation may be modified, such as [38] in the scalar-tensor theory and [39] for bumpy black holes which we will discuss later. For simplify, we will use the dimensionless quadrupole as \( Q = \frac{Q}{M^3} \) in our calculation. So by measuring the value of the quadrupole moment, we can study the nature of the BH. Then we first need to construct a waveform model including the effect of quadrupole moment corrections.

The situation is more complicated if we consider a non-Kerr spacetime with quadrupole deviate from Kerr value. In the QAK waveform given by [21] with the lowest order corrections for \( Q \), the related terms in the equations for \( \tilde{\gamma} \) and \( \alpha \) is taken from [40], while the terms in the equation of \( \nu \) is obtained by replacing the terms quadratic in the spin parameter.

To obtain a QAAK model, we need both an enhanced quadrupole included numerical kludge (QNK) model with arbitrary quadrupole moment, and an enhanced QAK model with higher order terms suitable with the one in AAK model. Then the following procedure will be done almost the same as the original AAK model. In the parameters’ mapping of QAAK, we didn’t include \( Q \) since it’s a higher order correction.

For the enhanced QAK model with higher order corrections, we choose a rough approach by replacing all the terms quadratic in spin with \(-Q\) in the higher order equations [37]. The same operation is applied to the evolution of the constants in the enhanced QNK model given by PN. So \((\tilde{E}, \tilde{L}_z, \tilde{K})\) are now functions of \((M, a, Q, \mu)\) and \((p, c, \iota)\). Then the final step is to obtain the fundamental frequencies for the metric with the corresponding quadrupole corrections. And then the unphysical parameters \((\tilde{M}, \tilde{a}, \tilde{p})\) can be obtained by a direct mapping.

We should notice that in the mapping of parameters for QAAK, we will keep \( Q \) fixed. So the solutions of the frequencies equations will have a slightly different with the AAK result, since the quadratic terms of \( S \) are now fixed terms of \( Q \).

Note that in QNK model, we merely evolve the \((E, L_z, K)\) and calculate the Kerr frequencies with them. The geodesic equation is not modified, since \( K \) is not well defined in a spacetime with arbitrary quadrupole moment. The quadrupole correction is not included in the convert \((e, i, p) \rightarrow (E, L_z, K)\) and the check of plunge either.

In general, the most important thing we need to do is to obtain the frequency correction corresponding to the variation of quadrupole moment. We find that this has been obtained for the bumpy Kerr black hole with a quadrupole bump [23].

### IV. THE CALCULATION OF FREQUENCY CORRECTION

Motivated by probing the multipole moments deviation, the bumpy black hole which can deviates in a small, controllable manner from the exact black holes of GR is introduced in [41] for the Schwarzschild case. Then the bumpy Kerr is first obtained by [42], and then obtained by [23] using the Newman-Janis algorithm[43]. The bumpy Kerr metric is given by \( g_{\mu\nu} = \hat{g}_{\mu\nu} + b_{\mu\nu} \), where the traditional Kerr part (1) is \( \hat{g}_{\alpha\beta} \), and the bumpy part \( b_{\mu\nu} \) is:

\[
\begin{align*}
\hat{b}_{tt} &= -2 \left( 1 - \frac{2Mr}{\Sigma} \right) \psi_1, \\
\hat{b}_{rr} &= 2 \left( \gamma_1 - \psi_1 \right) \frac{\Sigma}{\Delta}, \\
\hat{b}_{\varphi\varphi} &= \Delta \sin^2 \theta \left( \gamma_1 - \psi_1 \right) \frac{8a^2M^2r^2\sin^2 \theta}{\Delta \Sigma (\Sigma - 2Mr)} \\
&\quad - 2\psi_1 \left( 1 - \frac{2Mr}{\Sigma} \right)^{-1} \\
\hat{b}_{\theta\theta} &= 2 \left( \gamma_1 - \psi_1 \right) \frac{\Sigma}{\Delta}, \\
\hat{b}_{t\varphi} &= - \frac{2a^2Mr \sin^2 \theta}{\Delta \Sigma}, \\
\hat{b}_{\varphi\varphi} &= \gamma_1 a \sin^2 \theta \\
&\quad \left[ \left( 1 - \frac{2Mr}{\Sigma} \right)^{-1} - \frac{4a^2M^2r^2\sin^2 \theta}{\Delta \Sigma (\Sigma - 2Mr)} \right].
\end{align*}
\]

In the case of quadrupole bumps, the \( \psi_1 \) and \( \gamma_1 \) is given
Then the quadrupole moment is given by
\[ \psi_1^l=2(r, \theta) = \frac{B_2 M^3}{4} \sqrt{\frac{5}{\pi}} d(r, \theta, a)^3 \]
\[ \left[ 3L(r, \theta, a) \cos^2 \theta - \frac{c_20(a, r) + c_{22}(a, r) \cos^2 \theta + c_{24}(a, r) \cos^4 \theta}{d(r, \theta, a)^3} \right], \]  
\[ \gamma_1^l=2(r, \theta) = \frac{B_2}{\sqrt{\pi}} \left[ \frac{L(r, \theta, a)}{2} - \frac{\Omega^l}{\Omega^l} \right], \]  
where
\[ d(r, \theta, a) = \sqrt{r^2 - 2Mr + (M^2 + a^2) \cos^2 \theta}, \]
\[ L(r, \theta, a) = \sqrt{(r - M)^2 + a^2 \cos^2 \theta}, \]
\[ c_20(a, r) = 2(r - M)^4 - 5M^2(r - M)^2 + 3M^4, \]
\[ c_{22}(a, r) = 5M^2(r - M)^2 - 3M^4 + a^2 [4(r - M)^2 - 5M^2], \]
\[ c_{24}(a, r) = a^2(2a^2 + 5M^2). \]

Then the quadrupole moment is given by
\[ Q = -Ma^2 - B_2 M^3 \sqrt{5/4\pi} = Q_K + \Delta Q. \]  
or equivalently, \( \Delta Q = -B_2 \sqrt{5/4\pi}. \)

For a point particle moving on the bumpy Kerr with mass \( \mu \) and momentum \( p^\mu \), the Hamiltonian \( \mathcal{H} \) is given by:
\[ \mathcal{H} = \frac{1}{2} g^\alpha\beta p_\alpha p_\beta = -\frac{\mu^2}{2} = \mathcal{H} + \mathcal{H}_1, \]

where \( \mathcal{H} \) is the Hamiltonian corresponding to the Kerr background, and \( \mathcal{H}_1 \) represents the influence of the spacetime’s bumpiness.

Then the orbital frequencies of Kerr is given by
\[ \mu \tilde{\Omega}^\mu = \frac{\partial \mathcal{H}}{\partial \dot{J}_\mu} \]

where the derivatives are taken with respect to the action variables defined for the background motion:
\[ J_i = \frac{1}{2\pi} \int p_idx^i, \quad \dot{J}_i = -E. \]

And the frequencies’ shift can be expressed by averaged \( \mathcal{H}_1 \):
\[ \mu \delta \tilde{\Omega}^\mu = \frac{\partial \langle \mathcal{H}_1 \rangle}{\partial J_\mu}, \]

while the orbit averaged form of \( \mathcal{H}_1 \) can be defined as:
\[ \langle \mathcal{H}_1 \rangle = \frac{1}{\Gamma \Delta r \Delta \theta} \int_0^{\Delta r} d\lambda_r \int_0^{\Delta \theta} d\lambda_\theta \mathcal{H}_1 V, \]

Once the derivatives of \( \mathcal{H} \) and background frequencies are available, it is simple to compute the changes to the observable frequencies. Expanding
\[ \omega^i = \frac{\Omega^i}{\Omega^i} = \frac{\Omega^i + \delta \Omega^i}{\Omega^i + \delta \Omega^i} \equiv \tilde{\omega}^i + \delta \omega^i, \]

So the deviation of frequencies can be written as
\[ \delta \omega^i = \frac{\delta \Omega^i}{\Omega^i} - \frac{\tilde{\omega}^i \delta \Omega^i}{\tilde{\Omega}^i}, \]

Then by replacing all the parameter \( B_2 \) with \(-2Q\sqrt{\pi/5}\), we can get the fundamental frequencies corresponding to the spacetime with quadrupole moment deviation \( \Delta Q \). However, in the Newtonian limit, the deviation is given by
\[ \delta \omega^r = -\frac{3\Delta Q}{4M} \frac{1}{p^{7/2}} (1 - e^2)^2 (2\sin^2 \theta_m - 1), \]
\[ \delta \omega^\theta = -\frac{3\Delta Q}{4M} \frac{1}{p^{7/2}} (1 - e^2)^{3/2} \]
\[ [\sin^2 \theta_m(5 + 3 \sqrt{1 - e^2}) - \sqrt{1 - e^2} - 1], \]
\[ \delta \omega^\phi = -\frac{3\Delta Q}{4M} \frac{1}{p^{7/2}} (1 - e^2)^{3/2} [\sin^2 \theta_m \]
\[ (5 + 3 \sqrt{1 - e^2}) - 2\sin \theta_m - \sqrt{1 - e^2} - 1]. \]

V. PARAMETER ESTIMATION RESULT

Given the quadrupole moment included waveform and a specific detector, we can get the expected accuracy of PE with the FIM method. The inner product is defined as \( [44] \):
\[ \langle a | b \rangle = 2 \int_0^\infty df \frac{\tilde{a}^*(f) \tilde{b}(f) + \tilde{a}(f) \tilde{b}^*(f)}{S_n(f)}. \]  

The SNR is defined as \( \rho = \sqrt{\langle h | h \rangle} \), and the FIM is
\[ \Gamma_{ij} = \left[ \frac{\partial h_i}{\partial \lambda_j} \frac{\partial h_j}{\partial \lambda_i} \right], \]

where \( \lambda_i \) are parameters which are used to generate the waveform. When the SNR of the signal is high enough, then the PE accuracy is given by:
\[ \delta \lambda_i \approx \sqrt{(\Gamma^{-1})_{ii}}. \]  

In this paper, we considered both LISA and TianQin since their sensitivity band is a little bit different. We compared the results of QAK and QAAK in FIG. 1 for LISA, and in FIG. 2 for TianQin.

The power spectral density \( S_n(f) \) is chosen to be the sky averaged one for LISA [45] and TianQin [46]. The length of the signal is chosen to be 1 yr. To have a fair comparison, we normalize the SNR to \( \rho = 100 \). The red shifted mass of the CO is fixed to be \( \mu = 10M_\odot \), and the
inclination angle $\iota$ is chosen to be $\pi/3$. The parameters $(M, a, e)$ are chosen to be different values for analysis. The initial value of $p$ is chosen to be $6.5 M$: for the events which plunge less than 1 yr, we will calculate backwards until the length reaches 1 yr, for the events which will not plunge after evolve for 1 yr, we will not evolve it after that.

We can find that the QAK model is not sensitive to the eccentricity, while in the QAAK model, the PE accuracy for sources with different eccentricity will vary by several times. Another interesting feature is that for lower spin, the accuracy of QAAK is higher for more massive source. But for higher spin, this difference is not very obvious. The reason of these feature should be the frequency evolution is different in these two waveforms.

Since we have normalized the SNR of all the signals to be the same value, the PE accuracy for both detectors seems to be the same. But if we fix the distance of each source, TianQin will have higher SNR for sources with lower mass, and vice versa. So for the same source, the accuracy is better for TianQin in the lower mass part, and it’s better for LISA in the higher mass part. This meets our expectations since the band of TianQin is higher than LISA. The MBHs with mass in the range of $10^5 M_\odot \sim 10^7 M_\odot$ can be measured with very high accuracy. For the systems with mass larger than $10^6 M_\odot$, the accuracy will be worse and worse.

We also listed part of the results in TABLE. I for LISA and TABLE. II for TQ. The first line in boldface of each mass corresponding to the result given by QAAK waveform, while the second line in plain face corresponding to the result given by QAK waveform. We also listed the result without including the estimation of $Q$ in the bracket.

We can find that the result of QAK and QAAK almost agree with each other at the same order of magnitude. But since QAAK is developed based on the AAK waveform, the waveform should be more reliable in the realistic matched filtering.

VI. CONCLUSION

Based on the AAK model, we considered the quadrupole moment corrections due to the off-Kerr deviations in the geometry of massive black holes. We modified the evolution equations of the orbital frequencies in the AK side, as well as the constants’ evolution equations in the NK side, by a simple substitution of $a^2/M^2$ with $-Q/M^3$. The fundamental frequencies are obtained in the bumpy Kerr BH spacetime with arbitrary quadrupole moment. The definition of the constants and the geodesic equations are still the one for Kerr BH.

We also compare QAAK and QAK’s ability of PE for various sources with LISA and TianQin. We find that although these waveforms are expected to have a different
TABLE I. The result of QAAK and QAK for LISA for $e = 0.3$, $a = 0.9M$ and the mass of MBH is chosen to be different values. The boldfaced data in the first line for each mass corresponding to the PE accuracy given by the QAAK, while the plain-faced in the second line for each mass is given by QAK. The data in the bracket is the PE result without including the estimation of $Q$.

| $M(M_\odot)$ | $\Delta(\ln \mu)$ | $\Delta(\ln M)$ | $\Delta(a/M)$ | $\Delta Q$ |
|--------------|-------------------|-----------------|--------------|---------|
| $10^5$       | $7.2 \times 10^{-7}(6.5 \times 10^{-7})$ | $1.2 \times 10^{-6}(1.0 \times 10^{-6})$ | $3.5 \times 10^{-6}(1.4 \times 10^{-6})$ | $2.1 \times 10^{-5}(\_\_)$ |
|              | $8.7 \times 10^{-7}(6.6 \times 10^{-7})$ | $1.1 \times 10^{-6}(1.0 \times 10^{-6})$ | $1.3 \times 10^{-6}(1.3 \times 10^{-6})$ | $1.5 \times 10^{-5}(\_\_)$ |
| $10^6$       | $2.3 \times 10^{-6}(1.6 \times 10^{-6})$ | $6.9 \times 10^{-7}(6.5 \times 10^{-7})$ | $4.3 \times 10^{-6}(8.2 \times 10^{-7}$) | $2.2 \times 10^{-5}(\_\_)$ |
|              | $3.2 \times 10^{-6}(4.8 \times 10^{-7})$ | $1.0 \times 10^{-6}(3.1 \times 10^{-7})$ | $5.4 \times 10^{-6}(9.0 \times 10^{-7})$ | $3.3 \times 10^{-5}(\_\_)$ |
| $10^7$       | $6.9 \times 10^{-4}(4.5 \times 10^{-4})$ | $2.4 \times 10^{-4}(2.3 \times 10^{-4})$ | $2.4 \times 10^{-3}(1.1 \times 10^{-4})$ | $1.2 \times 10^{-2}(\_\_)$ |
|              | $2.2 \times 10^{-3}(2.5 \times 10^{-4})$ | $1.7 \times 10^{-4}(1.7 \times 10^{-4})$ | $4.3 \times 10^{-3}(3.8 \times 10^{-5})$ | $2.5 \times 10^{-2}(\_\_)$ |

TABLE II. The result of QAAK and QAK for TianQin for $e = 0.3$, $a = 0.9M$ and the mass of MBH is chosen to be different values. The boldfaced data in the first line for each mass corresponding to the PE accuracy given by the QAAK, while the plain-faced in the second line for each mass is given by QAK. The data in the bracket is the PE result without including the estimation of $Q$.

| $M(M_\odot)$ | $\Delta(\ln \mu)$ | $\Delta(\ln M)$ | $\Delta(a/M)$ | $\Delta Q$ |
|--------------|-------------------|-----------------|--------------|---------|
| $10^5$       | $1.0 \times 10^{-6}(8.5 \times 10^{-7})$ | $1.5 \times 10^{-6}(1.1 \times 10^{-6})$ | $5.4 \times 10^{-6}(1.9 \times 10^{-6})$ | $3.0 \times 10^{-5}(\_\_)$ |
|              | $1.0 \times 10^{-6}(6.6 \times 10^{-7})$ | $1.2 \times 10^{-6}(1.1 \times 10^{-6})$ | $1.7 \times 10^{-6}(1.6 \times 10^{-6})$ | $1.6 \times 10^{-5}(\_\_)$ |
| $10^6$       | $2.3 \times 10^{-6}(1.5 \times 10^{-6})$ | $6.0 \times 10^{-7}(5.6 \times 10^{-7})$ | $4.0 \times 10^{-6}(8.0 \times 10^{-7})$ | $2.0 \times 10^{-5}(\_\_)$ |
|              | $2.6 \times 10^{-6}(5.0 \times 10^{-7})$ | $8.1 \times 10^{-7}(2.6 \times 10^{-7})$ | $4.3 \times 10^{-6}(9.3 \times 10^{-7})$ | $2.6 \times 10^{-5}(\_\_)$ |
| $10^7$       | $6.7 \times 10^{-4}(4.2 \times 10^{-4})$ | $2.4 \times 10^{-4}(2.1 \times 10^{-4})$ | $2.1 \times 10^{-3}(1.1 \times 10^{-4})$ | $1.1 \times 10^{-2}(\_\_)$ |
|              | $2.4 \times 10^{-3}(2.2 \times 10^{-4})$ | $1.4 \times 10^{-4}(1.4 \times 10^{-4})$ | $4.4 \times 10^{-3}(3.2 \times 10^{-5})$ | $2.4 \times 10^{-2}(\_\_)$ |

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