Varieties defined by linear equations have the amalgamation property

PAOLO LIPPARINI

Abstract. A variety is a class of algebraic structures axiomatized by a set of equations. An equation is linear if there is at most one occurrence of an operation symbol on each side. We show that a variety axiomatized by linear equations has the strong amalgamation property.

Suppose further that the language has no constant symbol and, for each equation, either one side is operation-free, or exactly the same variables appear on both sides. Then also the joint embedding property holds.

Examples include most varieties defining classical Maltsev conditions. In a few special cases, the above properties are preserved when further unary operations appear in the equations.

1. Introduction

An equation is an atomic formula in a language with equality as the only relational symbol. We call an equation $\sigma$ linear if $\sigma$ has at most one occurrence of operation symbols on each side. Constants are not considered as operations. If in addition the set of variables appearing on the right is equal to the set of variables appearing on the left, we say that $\sigma$ is equilinear. We include in the class of equilinear equations also the equations in which only one operation occurs, that is, one side of the equation is a variable or a constant.

Equilinear equations are important in universal algebra since the great majority of classical Maltsev conditions are obtained by interpreting a variety axiomatizable by equilinear equations. Recall that a variety is a class axiomatized by a set of equations in a language with equality as the only relation. Every nontrivial idempotent Maltsev condition implies a nontrivial equilinear idempotent Maltsev condition. See [Ta], [HoMc Chapter 9] and [KK Section 2.4]. In passing, it is worth mentioning that there exists the weakest idempotent Maltsev condition [S, O], though this surprising recent result will play no role here.

2020 Mathematics Subject Classification: 03C05; 03C52; 08B05; 08B25.

Key words and phrases: Linear equation; equilinear equation; variety; amalgamation property; strong amalgamation property; joint embedding property; Maltsev condition; Fraissé limit.

December 7, 2021

Work performed under the auspices of G.N.S.A.G.A. Work partially supported by PRIN 2012 “Logica, Modelli e Insiemi”. The author acknowledges the MIUR Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.
As an argument more familiar to model theorists, the amalgamation property [F] is an important notion which has found many successful applications also in algebra, algebraic logic and category theory [E, GM, H, J1, KMPT, MMT]. From the perspective of universal algebra, the importance of the property stems to the fact that a variety $V$ has the amalgamation property if and only if the category of algebras in $V$ with embeddings has push outs [KMPT]. Here algebra is a shorthand, for algebraic structure, namely, a model in a language without relation symbols.

In Section 4, we show that a variety axiomatizable by linear equations, possibly containing constants, has the strong amalgamation property. The case of a variety axiomatizable by equilinear equations in a language without constants is much simpler and is treated in Section 3. In this case we also get the joint embedding property. At large, the above results provide a striking partial affirmative answer to an ancient problem by B. Jónsson, asking for general results that assert that if an elementary class is characterized by axioms of such and such a form, then this class has the amalgamation property [J1].

The above results apply to virtually all the varieties used to define classical Maltsev conditions, such as the variety with a Maltsev operation, the variety with $n$ Jónsson operations, the variety with $n$ Day operations, etc. Since we show that also the class of finite algebras in such varieties has both the amalgamation property and the joint embedding property, a classical result by Fraïssé [F] applies, to the effect that such classes have a Fraïssé limit, an ultrahomogeneous universal countable algebra. See [H, Theorem 7.1.2]. Such “generic” algebras probably deserve an accurate study, but we shall not deal with this here.

We conclude the note with a few remarks in Section 5 on adding unary operations. If $V$ is a variety in the language $L$, let $V_h$ be a variety in the language $L \cup \{h\}$, where $h$ is a new unary function symbol. It is almost trivial that if $V$ is a variety with the amalgamation property, then $V_h$ has the amalgamation property, if $h$ satisfies the equations asserting that $h$ is a homomorphism. A counterexample is provided showing that the amalgamation property is not necessarily preserved if $h$ is assumed to respect only a subset of the operations of $V$. However, the amalgamation property is preserved in $V_h$ when $h$ has an inverse and $V$ is axiomatized by linear equations, no matter which subset of operations is respected by $h$.

2. Preliminaries

We give the definition of the strong amalgamation property in the case of algebraic structures. The general model-theoretical definition presents no significant difference. For simplicity, we assume to deal with a class $K$ of algebras such that $K$ is closed under taking isomorphism. This is no big loss of generality; in fact, we shall almost invariably deal with varieties.
Definition 2.1. A class $\mathcal{K}$ closed under isomorphism is said to have the strong amalgamation property, SAP, for short, if the following holds. For every three algebras $A$, $B$ and $C$ in $\mathcal{K}$ such that

$$C \subseteq A, \quad C \subseteq B \quad \text{and} \quad A \cap B = C,$$

there is an amalgamating algebra, or an amalgam, for short, $D$ in $\mathcal{K}$ such that $A \subseteq D$ and $B \subseteq D$. Here, say, $C \subseteq A$ means that $C \subseteq A$ as sets, and $C$ is a substructure of $A$.

We shall usually prove the strong amalgamation property; however, in some comments we shall deal with the plain (=not strong) amalgamation property. A class $\mathcal{K}$ closed under isomorphism is said to have the amalgamation property, AP, for short, if, under the assumptions in (2.1), there are an algebra $D \in \mathcal{K}$ and embeddings $\iota : A \to D$ and $\kappa : B \to D$ which agree on $C$.

The point is that if the strong amalgamation property holds, then $D$ as above can be obtained in such a way that two elements $a \in A \setminus B$ and $b \in B \setminus A$ are never identified. On the other hand, it might happen that $\iota(a) = \kappa(b)$ for such elements, if only AP is assumed. This is the reason why in the definition of AP we need to deal with embeddings, not simply inclusions.

The class $\mathcal{K}$ has the joint embedding property, JEP, for short, if, for every $A, B \in \mathcal{K}$, there are an algebra $D \in \mathcal{K}$ and embeddings $\iota : A \to D$ and $\kappa : B \to D$.

We do not allow algebras with empty domain; hence, in general, JEP is not a consequence of AP.

In the present note we do not consider constants as operations, hence, in general, a linear equation is allowed to contain constants. This general case shall be treated in Section 4. We first deal with the simpler case when constants are not allowed.

Definition 2.2. A linear equation in a language without constants is an equation which has one of the following forms.

\begin{align*}
    f(x_1, x_2, \ldots, x_n) &= y_1, \quad \text{or} \quad (2.2) \\
    f(x_1, x_2, \ldots, x_n) &= g(y_1, y_2, \ldots, y_m), \quad (2.3)
\end{align*}

where $f$ and $g$ are not necessarily distinct operations and $x_1, x_2, \ldots, y_1, y_2, \ldots$ are not necessarily distinct variables. A more general definition of a linear equation for languages with constants shall be given in Definition 4.3.

Notice that the terminology is not uniform in the literature. The above terminology is the most common when dealing with Maltsev conditions.

We say that an equation $\sigma$ is equilinear if either $\sigma$ has the form (2.2), or $\sigma$ has the form (2.3) and the set of variables appearing on the left-hand side is equal to the set of variables appearing on the right-hand side, that is, $\{x_1, x_2, \ldots, x_n\} = \{y_1, y_2, \ldots, y_m\}$, with no consideration of multiplicities.
Notice that an equilinear equation may logically imply a linear but not equilinear equation. For example, the equilinear equation \( x = f(x, y, y) \) (relevant to the first discovered Maltsev condition, see the comments shortly after Corollary 3.2) logically implies the equation \( f(x, y, y) = f(x, z, z) \), which is not equilinear. Hence we shall always need to refer to varieties \( V \) axiomatized by equations of some particular form, rather than to varieties \( V \) such that all the equations valid in \( V \) have some particular form.

3. The strong amalgamation property for equilinear varieties

We first deal with languages without constants. This is the case of most interest; in this case proofs are much simpler and we get slightly stronger results.

**Theorem 3.1.** Suppose that \( V \) is a variety axiomatizable by equilinear equations and the language of \( V \) has no constant. Then \( V \) has both the strong amalgamation property and the joint embedding property.

**Proof.** Suppose that \( V \) is axiomatized by a set \( \Sigma \) of equilinear equations.

First, notice that there might be further equations of the form (2.2) which follow from the given set of equations \( \Sigma \). For example, if \( f(x, x, y, z) = g(x, x, y, z) \) and \( g(x, x, y, y) = y \) are among the equations defining \( V \), then \( f(x, x, y, y) = y \) holds in \( V \), as well, even if this equation is not explicitly listed in \( \Sigma \). However, we can suppose without loss of generality to have added to \( \Sigma \) all the equations of the form (2.2) which follow logically from the given equations. Henceforth we suppose to have extended the set \( \Sigma \) in this way.

Furthermore, we can suppose that, for every equation in \( \Sigma \) of the form (2.2), \( y_j \in \{x_1, x_2, \ldots\} \), since otherwise \( V \) is a trivial variety with only 1-element algebras, and in this case the conclusion is obvious.

Given a triple \( A, B, C \) of algebras in \( V \) as in (2.1), choose an arbitrary element \( \bar{d} \in A \cup B \). For each operation \( f \) in the language of \( V \), define \( f \) on \( A \cup B \) by

\[
\begin{align*}
  f(d_1, d_2, \ldots) &= f_A(d_1, d_2, \ldots) & \text{if } d_1, d_2, \ldots \in A, \\
  f(d_1, d_2, \ldots) &= f_B(d_1, d_2, \ldots) & \text{if } d_1, d_2, \ldots \in B, \\
  f(d_1, d_2, \ldots) &= d_j & \text{if this is forced by some equation in } \Sigma \text{ of the form (2.2), and} \\
  f(d_1, d_2, \ldots) &= \bar{d} & \text{if none of the above clauses apply,}
\end{align*}
\]

for \( d_1, d_2, \ldots \in A \cup B \) and where, in more detail, we apply the third clause if there is an equation in \( \Sigma \) of the form \( f(x_1, x_2, \ldots) = x_j \) and the expression \( f(d_1, d_2, \ldots) \) is obtained by substituting equal elements for equal variables, that is, \( d_h = d_k \), whenever \( x_h = x_k \). Notice that, as we mentioned, \( x_j \in \{x_1, x_2, \ldots\} \), since otherwise we are in a trivial variety. In particular, also \( d_j \in \{d_1, d_2, \ldots\} \), so that the definition makes sense. Moreover, the definition is
well-posed, since if we have both \( f(x_1, x_2, \ldots, x_n) = y_1 \) and \( f(x_1, x_2, \ldots, x_n) = y_2 \) in \( \Sigma \), for \( y_1 \) and \( y_2 \) distinct variables, then we are again in a trivial variety.

With the above definition, \( D = A \cup B \) becomes an algebra \( D \) for the language of \( \mathcal{V} \). Notice that the first three clauses possibly overlap, but they agree in any overlapping case, since \( C \) is a subalgebra of both \( A \) and \( B \); moreover, the equations in \( \Sigma \) of the form (2.2) hold in the three algebras to be amalgamated.

By the first two clauses in (3.1), \( D \) extends both \( A \) and \( B \).

It remains to show that the defining equations (2.2) and (2.3) are valid in \( D \). Since, by assumption, the equations are valid both in \( A \) and in \( B \), the conclusion is obvious if some identity involves only elements from \( A \) or only elements from \( B \). By the third clause in (3.1), \( D \) satisfies all the equations in \( \Sigma \) of the form (2.2). Hence it remains to consider an equation of the form

\[
  f(x_1, x_2, \ldots) = g(y_1, y_2, \ldots) \tag{3.2}
\]

from \( \Sigma \), that is, we have to prove that \( f(d_1, d_2, \ldots) = g(e_1, e_2, \ldots) \), when \( d_1, d_2, \ldots, e_1, e_2, \ldots \in A \cup B \) and equal elements are substituted for equal variables in (3.2). If the value of, say, \( f(d_1, d_2, \ldots) \) is forced by some equation in \( \Sigma \) of the form (2.2), then the value of \( g(e_1, e_2, \ldots) \) is forced, too, and the resulting value is the same. The reason for this fact is that we have completed (2.2) with all the equations logically following from all the equations defining the variety.

As side remark, notice that the possibility of using the third clause in combination with some equation of the form (3.2) does not necessarily trivialize the equation (3.2). For example, suppose that (3.2) is

\[
  f(x, x, y, y, z) = g(x, x, y, y, z) \tag{3.3}
\]

and we need to check the instance \( f(d, d, d, d, e) = g(d, d, d, d, e) \) of (3.3). It might happen that \( f(x, x, x, y) = y \) is an equation in \( \Sigma \), so that \( f(d, d, d, d, e) = e \) is “forced”. Then (3.3) and \( f(x, x, x, y) = y \) imply that \( g(x, x, x, y) = y \) is an equation in the “extended” \( \Sigma \), hence \( g(d, d, d, d, e) = e \) is forced, as well. However it might happen that, say, \( f(d, d, d', d', e) \) is not forced, hence neither \( g(d, d, d', d', e) \) is forced. If this is the case, the definition in (3.1) provides \( f(d, d, d', d', e) = d = g(d, d, d', d', e) \), a fact we are now going to exploit in general.

So far, we have seen that all the equations in \( \Sigma \) of the form (2.2) are valid in \( D \) and that, given an instance of (3.2), the value of the left-hand side is forced by the third clause in (3.1) if and only if the right-hand side is forced; in this case, the two forced values are the same. Hence we need to deal with the case when no side is “forced”. Since, by assumption, we are dealing with an equilinear equation, we have \( \{d_1, d_2, \ldots\} = \{e_1, e_2, \ldots\} \) in any instance of (3.2), hence \( \{d_1, d_2, \ldots\} \subseteq A \) if and only if \( \{e_1, e_2, \ldots\} \subseteq A \). If this is the case, \( f(d_1, d_2, \ldots) \) and \( g(e_1, e_2, \ldots) \) are given by the first clause in (3.1), and they are equal since (3.2) is valid in \( A \). A similar argument applies when \( \{d_1, d_2, \ldots\} = \{e_1, e_2, \ldots\} \subseteq B \). In all the other cases \( f(d_1, d_2, \ldots) \) and \( g(e_1, e_2, \ldots) \) are both
given by the fourth clause in (3.1), hence \( f(d_1, d_2, \ldots) = \bar{d} = g(e_1, e_2, \ldots) \).

We have checked that all the instances of (3.2) hold in \( D \), hence \( D \) is an amalgamating algebra in \( V \). The amalgamation property follows.

To prove the joint embedding property assume, without loss of generality, that \( A \cap B = \emptyset \) and repeat the above arguments considering \( C = \emptyset \).

The assumption that at most one operation symbol occurs on each side is obviously necessary in Theorem 3.1. The variety of semigroups is axiomatized by a single equation such that the same variables appear on both sides; however, the variety of semigroups has not AP, a result credited to Kimura [K] in [J1, KMPT].

As we mentioned, we do not allow algebras with empty domain. Were we allowing empty domains, the strong amalgamation property would fail in the case of a trivial variety, taking \( C = \emptyset \) and \( A \) and \( B \) 1-element algebras. Obviously, the amalgamation property holds even in this extraordinary situation.

The algebra \( D \) in the proof of Theorem 3.1 has been constructed on the union of \( A \) and \( B \). As we have discussed at length in [SAPU], this fact has interesting consequences, frequently with easy proofs. For example, items (2) and (4) in the following corollary follow immediately from the above observation.

**Corollary 3.2.** If \( V \) is a variety axiomatizable by a set of linear equations, then the following statements hold.

1. If \( V \) is not trivial, then, for every nonzero \( n \in \mathbb{N} \), \( V \) has an algebra with exactly \( n \) elements.
2. The class of finite algebras in \( V \) has the strong amalgamation property.
3. If \( V \) is axiomatized by a set of equilinear equations and the language of \( V \) has no constant, then the class of finite algebras in \( V \) has the joint embedding property.
4. Let \( \Gamma \) be a set of (possibly infinitary) universal-existential sentences for the language of \( V \) and such that in each sentence at most one variable is bounded by the universal quantifier.
   Then the class consisting of the algebras in \( V \) satisfying all the sentences in \( \Gamma \) has the strong amalgamation property.

**Proof.** We first give the proof for varieties satisfying the assumptions in Theorem 3.1. The proof for the general case shall be postponed to Section 4.

If \( V \) is not trivial, the proof of Theorem 3.1 shows that an algebra \( D \) witnessing JEP can be obtained over the disjoint union of \( A \) and \( B \). Then (1) follows by induction on \( n \), since every variety has an algebra with one element. The statement in (3), too, follows from the above observation. As we mentioned, all the rest follows from the fact that the proof of 3.1 provides strong amalgamation “into union”.

□

Starting from A. Maltsev breakthrough characterization of congruence permutable varieties [M], Maltsev conditions have since played a predominant role...
Varieties defined by linear equations have the amalgamation property

in universal algebra. Informally, a Maltsev condition asserts the existence of a finite set of terms satisfying a given finite set of equations. More precisely, a Maltsev condition is determined by some finitely presented variety $W$; some variety $V$ satisfies the Maltsev condition associated to $W$ if $W$ is interpretable in $V$. We shall not need the exact details here, this discussion serves only as a motivation; henceforth, we shall only briefly hint to the simplest example, the one discovered by Maltsev himself. Here the name of Maltsev is attributed both to the general notion and to the specific example.

The variety $W_M$ with a Maltsev operation is the variety in the language with a ternary operation $f$ such that $W_M$ is axiomatized by the equations

$$x = f(x, y, y), \quad f(x, x, y) = y.$$ 

A variety $V$ satisfies the Maltsev condition $M$ associated to $W_M$, namely, if $V$ has a ternary term $t$ satisfying the equations $x = t(x, y, y)$, $t(x, x, y) = y$.

It is customary to deal with Maltsev conditions rather informally, expressing them, as above, by means of the existence of appropriate terms. However, here we need to deal with varieties. Quite informally, one simply should replace the word “term” (sometimes, “polynomial” or “expression” in the classical literature) by “operation” in, say, Maltsev [M], Pixley [P], Day [Da], Jónsson [J2], Hagemann and Mitschke [HaMi].

Corollary 3.3. The conclusions of Theorem 3.1 and Corollary 3.3 hold for the following varieties.

1. The variety with a Maltsev operation $[M]$.
2. The variety with a Pixley operation $[P]$; and, for every $n$:
3. The variety with an $n$-ary near unanimity operation.
4. The variety with $n$ Day operations $[Da]$.
5. The variety with $n$ Jónsson operations $[J2]$.
6. The variety with $n$ Hagemann-Mitschke operations $[HaMi]$.

Corollary 3.3 is just illustrative. Other examples of Maltsev conditions such that Theorem 3.1 applies to their defining varieties can be found, among many others and with partial overlaps, in Taylor [Ta, Corollary 5.3], Gumm [G, Theorem 7.4(iv)], Tschantz [Ts, Lemmata 3 and 4], Hobby, McKenzie [HoMc, Lemmata 9.4(3), 9.5(3), Theorems 9.8(4), 9.11(4) and 9.15(3)], Siggers [S, Theorem 1.1, Section 3], Kearnes, Kiss [KK, Theorems 3.21(3), 4.7, 5.23(3), 5.28(3), 8.13(3) and 8.14(3)], Kearnes, Marković, McKenzie [KMM, Theorem 2.2, Corollaries 3.1, 3.2], Olšák [O, Theorems 3.3, 6.1, Definition 5.1], Kazda, Kozik, McKenzie, M. Moore [KKMM, Section 1], Kazda, Valeriote [KV, Sections 3.2 and 3.3], Lipparrini [DTS, Definitions 2.1, 2.7, 6.1, 7.1, 7.6, 8.1, Remarks 6.4, 8.16, 8.19, 10.11(c)].

Notice that in all the examples dealing with Maltsev conditions the defining variety is finitely presented, namely, a variety axiomatized by a finite set of
equations in a finite language. On the other hand, no finiteness assumption is necessary in Theorem 3.1.

Turning to another consequence of Theorem 3.1, we briefly recall an important model theoretical construction. R. Fraïssé [F] developed a by now classical method for obtaining some limit “random” or “generic” homogeneous and universal structure, starting from a countable set of finite or countable structures with AP and JEP. This Fraïssé limit is the “most general” homogeneous countable structure into which all the starting structures can be embedded.

The classical example is the ordered set of the rationals, which is, in a sense, the most general countable linearly ordered set, and is the Fraïssé limit of the class of finite linear orders. As another example, the countable random graph is the limit of the class of finite graphs. The random graph can be obtained in a number of different ways; quite astonishingly, Fraïssé could have explicitly introduced it (apparently, he did not) about a decade before it became a subject of deep studies. See [H, Chapter 7] for more details and precise definitions.

**Corollary 3.4.** Suppose that \( \mathcal{V} \) is a variety in a finite language without constants and \( \mathcal{V} \) is axiomatizable by equilinear equations. Then the class of finite algebras in \( \mathcal{V} \) has a Fraïssé limit.

In particular, for every variety listed in Corollary 3.3 and in the comment below it, finite algebras have a Fraïssé limit.

**Proof.** Immediate from Corollary 3.2(2)(3) and Fraïssé’s Theorem; see, e. g., [H, Theorem 7.1.2]. □

**Problem 3.5.** It is probably interesting to study the Fraïssé limits of the varieties listed in Corollary 3.3 as well as of the varieties referred to shortly after.

**Remark 3.6.** The assumption that the language of \( \mathcal{V} \) is finite in Corollary 3.4 can be weakened to the assumption that \( \mathcal{V} \) contains only a countable number of finite algebras modulo isomorphism.

As an example to which this more general version of Corollary 3.4 applies, consider the variety \( \mathcal{V} \) in a language with operations \((f_n)_{n \in \mathbb{N}^+}\), each \( f_n \) \( n \)-ary. The equations axiomatizing \( \mathcal{V} \) are

\[
f_{n+1}(x_1, x_2, \ldots, x_{n+1}) = f_n(y_1, y_2, \ldots, y_n),
\]

for every \( n > 0 \), where at least one variable is repeated in the sequence \((x_1, x_2, \ldots, x_{n+1})\) and the sequence \((y_1, y_2, \ldots, y_n)\) is obtained from \((x_1, x_2, \ldots, x_{n+1})\) by deleting the first occurrence of a repeated variable.

The equations (3.4) are equilinear and, for every \( n \), there is only a finite number of algebras in \( \mathcal{V} \) of cardinality \( n \) up to isomorphism. Indeed, in an algebra of cardinality \( n \), all the operations of arity \( > n \) are defined in function of \( f_n \).

Of course, the above remarks apply to any subvariety of \( \mathcal{V} \) axiomatized by equilinear equations.
Remark 3.7. (a) There are possible variations on the construction of \( D \) in the proof of Theorem 3.1. Such variations might be of some interest; in fact, we shall see an application in the proof of Proposition 3.5 below.

(b) First of all, as we mentioned, there are interesting consequences of the fact that we can choose \( D \) to be over \( D = A \cup B \). However, for nontrivial varieties, the proof of Theorem 3.1 works for every \( D \supseteq A \cup B \), fixing any \( \bar{d} \in D \).

(c) Moreover, there is no need to consider a fixed element \( \bar{d} \). We get more varied amalgamating algebras if we fix a function \( \eta : D^{\text{fin}} \to D \), where \( D^{\text{fin}} \) is the set of all finite subsets of \( D \) and \( D \supseteq A \cup B \). Then replace the identity in the fourth clause in (3.1) by \( f(d_1, d_2, \ldots , d_n) = \eta(\{d_1, d_2, \ldots , d_n\}) \). The proof of Theorem 3.1 works with this definition, too, since when we prove that an identity of the form (3.2) is valid in \( V \) and we apply the fourth clause in (3.1) to \( f(d_1, d_2, \ldots ) \) and \( g(e_1, e_2, \ldots ) \), we have \( \{d_1, d_2, \ldots \} = \{e_1, e_2, \ldots \} \).

(d) There are many ways to choose some function \( \eta \) as above. Of course, we can pick a fixed element \( \bar{d} \in D \) and let \( \eta \) be the constant function with value \( \bar{d} \). In this case, if \( D = A \cup B \), we get back the structure constructed in the proof of 3.1. As another possibility, linearly order \( D \) and let \( \eta(\{d_1, d_2, \ldots \}) = \max\{d_1, d_2, \ldots , d_n\} \).

(e) Considering the possibility \( D \supset A \cup B \) provides also a slightly different proof for item (1) in Corollary 3.2 (so far, limited to the equilinear case without constants). Take \( A = B = C \) 1-element algebras and apply the proof of Theorem 3.1 as modified above, by considering any set \( D \) of cardinality \( n \) and containing \( C \).

4. The general case

We now treat the general case when the language of \( V \) might contain constants. We also deal with linear not necessarily equilinear equations. We still get the strong amalgamation property, but the joint embedding property fails, in general, as shown by the following examples. See, however, Corollary 4.6 for some special cases in which JEP holds even in the presence of constants.

Remark 4.1. (a) The assumption that the language of \( V \) has no constant is necessary in Theorem 3.1 in order to get the joint embedding property. For example, let the language of \( V \) have two constants \( c_1 \) and \( c_2 \). If \( c_1 \) and \( c_2 \) are interpreted in such a way that \( c_1 = c_2 \) in \( A \) and \( c_1 \neq c_2 \) in \( B \), then \( A \) and \( B \) cannot be embedded in a common extension.

As another example, let the language of \( V \) contain a constant \( c \) and a unary operation \( f \). As far as the equations in \( V \) do not logically imply \( f(c) = c \), for example, if \( V \) is axiomatized by the empty set of equations, let \( f(c) = c \) hold.
in \( A \) and \( f(c) \neq c \) in \( B \). Then \( A \) and \( B \) cannot be embedded in a common extension.

(b) Again in order to get the joint embedding property, it is not enough to assume that the equations axiomatizing \( V \) are linear. It is necessary to assume that the equations in Theorem 3.1 are equilinear, namely, that in each equation the left-hand and right-hand sides contain exactly the same variables.

Let the language of \( V \) have two unary operations \( f, g \) and let \( V \) be axiomatized by the equation \( f(x) = f(y) \), which is linear but not equilinear. For every algebra \( A \) in \( V \), the set \( \{ a \in A \mid f(a') = a, \text{ for some } a' \in A \} \) has exactly one element, call it \( a \). Let \( B \) be another algebra in \( V \) and let \( b \in B \) be the unique element such that \( f(b') = b \), for every \( b' \in B \). If \( g(a) = a \) in \( A \) and \( g(b) \neq b \) in \( B \), then obviously \( A \) and \( B \) cannot be embedded in a common extension. The formal computation goes as follows. Were \( \iota \) and \( \kappa \) embeddings from \( A \) and \( B \), respectively, to some algebra \( D \), then

\[
\iota(a) = \iota(g(a)) = \iota(g(f(a))) = g(f(\iota(a))) = g(f(\kappa(b))) = \kappa(g(f(b))) = \kappa(g(b)) \\
\neq \kappa(b) = \kappa(f(b)) = f(\kappa(b)) = f(\iota(a)) = \iota(f(a)) = \iota(a),
\]

a contradiction.

Notice that, had \( A \) and \( B \) a common subalgebra \( C \), then the above argument could not be carried over, since then \( C \) has an element \( c \) such that \( f(c) = c \), hence \( a = c = b \). Of course, the arguments in (a) cannot be carried over, either, when \( A \) and \( B \) have a common subalgebra \( C \), since then constants are already interpreted in \( C \).

The above counterexamples are rather pathological. Indeed, as we mentioned, the counterexamples prevent the joint embedding property, but we are still able to get the strong amalgamation property. We first need to introduce some terminology.

**Definition 4.2.** Let \( V \) be a variety and \( t(z_1, \ldots, z_p) \) be a term such that the variables \( z_1, \ldots, z_p \) are pairwise distinct. More formally, the intended meaning of the above sentence is that we write \( t(z_1, \ldots, z_p) \) to mean that \( t \) is a term and \( \{ z_1, \ldots, z_p \} \) is the set of variables occurring in \( t \), enumerated without repetitions. Subsequently, we might substitute some variables in \( t \) for other variables: the above convention applies only to the original term (after a substitution, we formally get a different term).

Under the above convention, call a variable \( z_i \) *exceptional* (in \( t \) for \( V \)) if

\[
t(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_p) = t(z_1, \ldots, z_{i-1}, w, z_{i+1}, \ldots, z_p)
\]  

(4.1)
is an equation valid in \( V \), where \( w \) is a variable distinct from \( z_i \). Otherwise, the variable \( z_i \) is *ordinary*.

Notice that it is no loss of generality to assume that \( w \) in (4.1) is distinct from all the \( z_j \)'s, \( j = 1, \ldots, p \). Indeed, if \( t(z_1, \ldots, z_i, \ldots, z_p) = t(z_1, \ldots, z_j, \ldots, z_p) \), with \( i \neq j \), is an equation valid in \( V \), then, applying twice the equation,
Varieties defined by linear equations have the amalgamation property  

we get
\[ t(z_1, \ldots, z_i, \ldots, z_p) = t(z_1, \ldots, z_j, \ldots, z_p) = t(z_1, \ldots, w, \ldots, z_p) \]  
(4.2)
since \( z_i \) occurs only once on the left-hand side and, symmetrically, \( w \) occurs only once on the right-hand side, since we have assumed that \( w \) is distinct from all the \( z_j \)'s. Here we have used the assumption that all the variables in the original term \( t \) on the left-hand side are distinct. The above notions are dependent on the variety \( \mathcal{V} \), but the variety \( \mathcal{V} \) shall be always clear from the context, henceforth we shall drop reference to \( \mathcal{V} \).

Of course, it might happen that some term has no exceptional variable and, less frequently, that it has no ordinary variable.

**Definition 4.3.** In our general definition of a linear equation we require that there is at most one occurrence of operations on each side. If the language under consideration has possibly constants and since we do not consider constants as operations, the general forms of a linear equation are as follows.

\[ f(w_1, w_2, \ldots, w_n) = v_1, \quad \text{or} \]
\[ f(w_1, w_2, \ldots, w_n) = g(v_1, v_2, \ldots, v_m), \]  
(4.3) \hspace{1cm} (4.4)
where \( f \) and \( g \) are operations and each one among \( w_i \) or \( v_j \) is either a variable or a constant.

Strictly speaking, also the equation \( w_1 = v_1 \) obeys our definition of a linear equation. However, if either \( w_1 \) or \( v_1 \) is a variable, then we are in a trivial variety, in which all the statements we consider are either trivially true or trivially false. If both \( w_1 \) and \( v_1 \) are constants, we can equivalently consider a variety without the constant \( v_1 \), substituting every occurrence of \( v_1 \) for \( w_1 \) in every equation. In this respect, compare also a remark at the beginning of the proof of the following theorem.

**Theorem 4.4.** If \( \mathcal{V} \) is a variety axiomatized by a set of linear equations, possibly with constants, then \( \mathcal{V} \) has the strong amalgamation property.

**Proof.** The proof is similar to the proof of Theorem 3.1 but in the present case there are more situations in which the value of some expression of the form \( f(d_1, d_2, \ldots) \) is “forced” by some condition.

In detail, let \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) be algebras to be amalgamated as in (2.1). We shall highlight the various steps of the proof.

A preliminary remark about constants. First of all, notice that if \( c \) is a constant, then \( c \) is interpreted by the same element in \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \), since \( \mathbf{C} \) is a subalgebra of both \( \mathbf{A} \) and \( \mathbf{B} \). Hence it is not necessary to explicitly state where \( c \) is interpreted and, as a usual abuse of notation, we might not graphically distinguish the constant symbol from its interpretation, though we shall actually do it in a paragraph below for the sake of clarity.

Moreover, if two constants \( c_1 \) and \( c_2 \) are interpreted by the same element, say, in \( \mathbf{A} \), then \( c_1 \) and \( c_2 \) are interpreted by the same element in \( \mathbf{C} \), hence...
in $B$, too. Thus $c_1$ and $c_2$ should be interpreted by the same element in any amalgamating algebra $D$, as well. Hence, in the above situation, it is no loss of generality to assume that $V$ satisfies the equation $c_1 = c_2$. Again without loss of generality, we can work in the variety obtained from $V$ by deleting the constant $c_2$ from the language and replacing $c_2$ by $c_1$ in every equation axiomatizing $V$. Thus if we prove the theorem in the special case of algebras in which no two distinct constants are interpreted by the same element, we get a proof for the full theorem. Henceforth, we shall assume that no pair of constants are interpreted by the same element.

A term associated to an expression $f(d_1, d_2, \ldots, d_n)$. We are now going to address the following problem. Given an expression $f(d_1, d_2, \ldots, d_n)$, with $f$ an operation in the language of $V$ and $d_1, d_2, \ldots, d_n \in A \cup B$, when is the value of $f(d_1, d_2, \ldots, d_n)$ forced by some condition? Of course, the first three clauses in (3.1) in the proof of Theorem 3.1 force the value of $f(d_1, d_2, \ldots, d_n)$, but when considering linear not equilinear equations there are more possibilities. For example, if $f(x, x, y) = f(x, x, z)$ holds in $V$, then the first three clauses in (3.1) do not necessarily force the value of $f(a, a, b)$, when $a \in A \setminus B$ and $b \in B \setminus A$; however, the equation $f(x, x, y) = f(x, x, z)$ implies $f(a, a, b) = f(a, a, a)$ and the latter expression can be evaluated in $A$. We now present the general form of the argument.

Consider again the expression $f(d_1, d_2, \ldots, d_n)$, with $f$ an operation in the language of $V$ and $d_1, d_2, \ldots, d_n \in A \cup B$. In the expression $f(d_1, d_2, \ldots, d_n)$ replace every element interpreting some constant by the corresponding constant symbol, and replace all the other elements by variables in a bijective way, namely, to distinct elements make correspond distinct variables and to equal elements make correspond equal variables. Consider the expression obtained in this way as a term in which no variable is repeated. We thus get

(*) a term $t(x_1, x_2, \ldots)$ such that $x_1, x_2, \ldots$ are pairwise distinct variables and, in any amalgamating algebra, $f(d_1, d_2, \ldots)$ is expressed as $t(d_{i_1}, d_{i_2}, \ldots)$, where $d_{i_1}, d_{i_2}, \ldots$ are pairwise distinct elements from the set $\{d_1, d_2, \ldots\}$ and no $d_{i_k}$ is the interpretation of some constant.

The intended meaning in the construction of $t$ is that we should have $f(d_1, d_2, \ldots) = t(d_{i_1}, d_{i_2}, \ldots)$ but, of course, neither $f(d_1, d_2, \ldots)$ nor $t(d_{i_1}, d_{i_2}, \ldots)$ are generally interpretable, until we have constructed some amalgamating algebra. On the other hand, we do have $f(d_1, d_2, \ldots) = t(d_{i_1}, d_{i_2}, \ldots)$ in $A$, in case all the elements $d_1, d_2, \ldots$ belong to $A$, and similarly for $B$.

More formally, the above term $t$ associated to the expression $f(d_1, d_2, \ldots, d_n)$ is introduced as follows. For every expression of the form $f(d_1, d_2, \ldots, d_n)$, let $I = \{1, 2, \ldots, n\}$ and

$I' = \{i \in I \mid d_i$ is not the interpretation of some constant $\}$.

Define an equivalence relation $\sim'$ on $I'$ by

\[ i \sim' j \quad \text{if} \quad d_i = d_j. \]
To distinct equivalence classes of \( \sim' \) associate distinct variables. Then \( t \) is defined by

\[
t(x_1, x_2, \ldots) = f(w_1, w_2, \ldots, w_n),
\]

where, for \( i = 1, \ldots, n \), if \( d_i \) is the interpretation of some constant, then \( w_i \) is the symbol for that constant; otherwise \( w_i \) is the variable associated to the \( \sim' \)-class of \( i \).

The term \( t \) is essentially unique, modulo the choice or rearrangements of variables. Notice that, by the assumption in the preliminary remark above, every element can be the interpretation of at most one constant, hence \( t \) is well-defined. For example, given the expression \( f(d, d, e, e, g, g, g) \), where \( g = c_C \) is the interpretation of some constant \( c \), we take \( t(x, y) \) to be the term \( f(x, x, y, y, c, c, c) \), thus \( f(d, d, e, e, g, g, g) = f(d, d, e, e, c_C, c_C, c_C) = t(d, e) \) in any amalgamating algebra. Had we taken \( t'(x, y, z) \) as \( f(x, x, y, y, z, z, z) \), we still have \( f(d, d, e, e, g, g, g) = t'(d, e, g) \), but in this case we do not meet the requirement that \( g \) is not the interpretation of some constant. Notice also that we want the elements \( d_1, d_2, \ldots \) appearing in the (possible) evaluation of \( t \) to be pairwise distinct. Hence, in the above situation, defining \( t(x, x', y, y, c, c, c) \) again would not meet our requirements, since then, in order to get \( f(d, d, e, e, c, c, c) \), we need to express it as \( t(d, d, e) \).

When is the value of \( f(d_1, d_2, \ldots) \) forced by some condition? Now we can answer the above question. For every operation \( f \), and every sequence \( d_1, d_2, \ldots \in A \cup B \) of appropriate length, we say that the value of \( f(d_1, d_2, \ldots) \) is forced if one of the following two eventualities occurs.

\( \text{(E1)} \) For the term \( t \) constructed above in (*) depending on \( f \) and \( d_1, d_2, \ldots \), the equation \( t(x_1, x_2, \ldots) = v \) is valid in \( V \), for some \( v \) either a constant \( c \) or some variable \( x_j \) among \( x_1, x_2, \ldots \). As in the proof of 3.1 if such a \( v \) exists, it is unique, unless we are in a trivial variety, in which case the theorem is obviously true. Notice that we have excluded in the preliminary remark the possibility that \( c_1 = c_2 \) for two distinct constants, hence we cannot have at the same time \( t(x_1, x_2, \ldots) = c_1 \) and \( t(x_1, x_2, \ldots) = c_2 \). Excluding the trivial variety, under the present eventuality the value forced for \( f(d_1, d_2, \ldots) \) is either the interpretation of the constant \( c \), or the element from \( \{d_1, d_2, \ldots\} \) the variable \( x_j \) has been substituted for in the construction of \( t \).

\( \text{(E2)} \) Let \( t \) be the term from (*) and suppose, without loss of generality, that the variables are rearranged in such a way that all the ordinary variables appear before the exceptional variables, as in \( t(x_1, x_2, \ldots, x_{\ell-1}; \; x_{\ell}, \ldots, x_r) \), where the semicolon separates the two kinds of variables. Let \( d_{i_1}, d_{i_2}, \ldots, d_{i_{\ell-1}} \) be the elements corresponding to the ordinary variables \( x_1, x_2, \ldots, x_{\ell-1} \). In this case the value of \( f(d_1, d_2, \ldots) \) is forced if either \( \{d_1, d_2, \ldots, d_{i_{\ell-1}}\} \subseteq A \) or \( \{d_1, d_2, \ldots, d_{i_{\ell-1}}\} \subseteq B \). If one of these two eventualities occurs, the value forced for \( f(d_1, d_2, \ldots) \) is
Thus obtaining an algebra $D_f$ and hence the definition is correct. Moreover, when $d_1, d_2, \ldots, d_{\ell-1}$ is variable, we get that the equations

\[ t(x_1, \ldots, x_{\ell-1}; x_\ell, \ldots, x_p) = t(x_1, \ldots, x_{\ell-1}; z, \ldots, z) = t(x_1, \ldots, x_{\ell-1}; z', \ldots, z') \]

hold in $V$, hence the definition of the value forced does not depend on the choice of $d$. Notice that it is important to have $C$ nonempty, in order for this value to be well-defined. We might be in the extreme case when all the variables in $t$ are exceptional; then, without a common subalgebra $C$ of both $A$ and $B$, the value of $t(a, a, \ldots)$ computed in $A$ might be different from the value of $t(b, b, \ldots)$ computed in $B$. Compare Remark (4.1b). Of course, it may happen that \( \{d_1, d_2, \ldots, d_{\ell-1}\} \subseteq C \) in (E2). If this is the case, then $t(d_1, d_2, \ldots, d_{\ell-1}; d, \ldots, d)$ is evaluated in the same way in $A$, $B$ and $C$, since $C$ is a subalgebra of both $A$ and $B$. Hence the definition is not ambiguous, in this case, too. Finally, notice that the first two clauses in (3.1) from the proof of Theorem 3.1 are actually a special case of the above eventuality (E2).

We now check that (E1) and (E2) force the same value for $f(d_1, d_2, \ldots)$ in any overlapping case. Indeed, if (E1) applies, then $v = t(x_1, \ldots, x_{\ell-1}; x_\ell, \ldots, x_p) = t(x_1, \ldots, x_{\ell-1}; z, \ldots, z)$ are equations valid in $V$. If $v$ is a constant $c$ and also (E2) applies, then the above equations imply $c = t(d_1, d_2, \ldots, d_{\ell-1}; d, \ldots, d)$, as evaluated either in $A$ or in $B$. The argument is similar if $v$ is variable among the ordinary variables $x_1, \ldots, x_{\ell-1}$ of $t$. Otherwise, $v$ is an exceptional variable, but then $v = t(x_1, \ldots, x_{\ell-1}; x_\ell, \ldots, x_p)$ together with (4.2) imply that we are in a trivial variety.

**Defining the amalgamating algebra.** Since all the above “forced” conditions follow from equations valid in $V$, it is evident that in any amalgamating algebra $D$ the expression $f(d_1, d_2, \ldots)$ should actually assume the forced value indicated in (E1) or (E2), if applicable. Now, as in the proof of (3.1) we pick an arbitrary element $\bar{d} \in A \cup B$ and let $f(d_1, d_2, \ldots) = \bar{d}$ if no value is forced by either (E1) or (E2). In detail, we let $D = A \cup B$ and, for $d_1, d_2, \ldots \in D$ and $f$ an operation in the language of $V$, we define $f(d_1, d_2, \ldots)$ by

\[ f(d_1, d_2, \ldots) = \begin{cases} \text{the value forced by (E1) or (E2)}, & \text{if either (E1) or (E2) applies,} \\ \bar{d}, & \text{otherwise,} \end{cases} \]

thus obtaining an algebra $D$ appropriate for the language of $V$.

As we mentioned, (E1) and (E2) force the same value in any overlapping case, hence the definition is correct. Moreover, when $d_1, d_2, \ldots \in A$, eventuality (E2) applies and gives the same value for $f(d_1, d_2, \ldots)$ as evaluated in $A$, by a comment shortly after the definition (*) of $t$. Hence $D$ extends $A$. Similarly, $D$ extends $B$.

**Checking that the equations are satisfied.** Case (4.3). It remains to show that the equations of the form (4.3) and (4.4) axiomatizing $V$ hold in $D$. 

\[ \text{Defining the amalgamating algebra.} \]
Eventuality (E1) takes care of equations of the form \((4.3)\). We first outline the argument. Suppose that some equation of the form \((4.3)\) is evaluated in \(\mathbf{D}\) modulo some assignment. Of course, when we evaluate \(f(w_1, w_2, \ldots, w_n)\) modulo an assignment \(y_1 \mapsto d_1, y_2 \mapsto d_2, \ldots\), where \(y_1, y_2, \ldots\) are the variables (enumerated without repetitions) in the set \(\{w_1, w_2, \ldots, w_n\}\), it might happen that distinct variables are assigned equal elements; moreover, some variable might be assigned the value of some constant, hence the term \(t\) from \((*)\) might contain a smaller number of variables, say, it has the form \(t(x_1, x_2, \ldots)\), where \(\{|x_1, x_2, \ldots|\} \leq \{|y_1, y_2, \ldots|\}\). However, if we substitute all the occurrences of some variable by the same constant symbol in \((4.3)\), we still get an equation valid in \(\mathcal{V}\). Similarly, we can substitute many variables for a single different variable in \((4.3)\), getting an equation valid in \(\mathcal{V}\). Thus, under substitutions as above, \(f(w'_1, w'_2, \ldots, w'_n) = v'_1\) is an equation valid in \(\mathcal{V}\).

More formally, suppose that \(f(d_1, d_2, \ldots, d_n)\) is the evaluation of \(f(w_1, w_2, \ldots, w_n)\) under the assignment \(\rho\) given by \(y_1 \mapsto d_1, y_2 \mapsto d_2, \ldots\), where \(y_1, y_2, \ldots\) are the variables in the set \(\{w_1, w_2, \ldots, w_n\}\). If \((4.3)\) \(f(w_1, w_2, \ldots, w_n) = v_1\) is an equation valid in \(\mathcal{V}\), we have to show that in \(\mathbf{D}\) \(f(d_1, d_2, \ldots, d_n) = \rho(v_1)\) if \(v_1\) is a variable, and that \(f(d_1, d_2, \ldots, d_n) = c\) if \(v_1\) is the constant \(c\). Let \(\tau\) be the function from \(\{d_1, d_2, \ldots, d_n\}\) which has been implicitly used in the definition of the term \(t\), namely, if \(d_i\) is the interpretation of some constant \(c\), then \(\tau(d_i)\) is the symbol for that constant; otherwise \(\tau(d_i)\) is the variable associated to the \(\sim'\)-class of \(i\). Then \((4.5)\) reads

\[
t(x_1, x_2, \ldots) = f(\tau(d_1), \tau(d_2), \ldots, \tau(d_n)).
\]

Extend \(\rho\) by sending a constant symbol to its interpretation. Since \(f(w_1, w_2, \ldots, w_n) = v_1\) is valid in \(\mathcal{V}\), then also

\[
f(\tau(\rho(w_1)), \tau(\rho(w_2)), \ldots, \tau(\rho(w_n))) = \tau(\rho(v_1))
\]

is valid in \(\mathcal{V}\). By \((4.7)\) and \((4.8)\), the equation

\[
t(x_1, x_2, \ldots) = \tau(\rho(v_1))
\]

is valid, as well, hence we can apply eventuality (E1) getting \(f(d_1, d_2, \ldots, d_n) = \rho(v_1)\), what we had to show.

Checking that the equations are satisfied. Case \((4.4)\). It remains to consider equations of the form \((4.4)\). We need to prove that \(f(d_1, d_2, \ldots) = g(e_1, e_2, \ldots)\), when \(d_1, d_2, \ldots, e_1, e_2, \ldots \in A \cup B\) and equal elements are substituted for equal variables in some equation from \((4.4)\) valid in \(\mathcal{V}\). We first sketch the argument. Associate to \(f(d_1, d_2, \ldots)\) and \(g(e_1, e_2, \ldots)\) corresponding terms \(t\) and \(s\) as in \((*)\), naming the variables in a consistent way, say, if \(d_j = e_k\), then the variable corresponding to \(d_j\) in the construction of \(t\) should be equal to the variable corresponding to \(e_k\) in the construction of \(s\). As in the previous paragraph, the validity of \((4.4)\) in \(\mathcal{V}\) implies the validity of an equation of the form \(t = s\). Thus the value of \(f(d_1, d_2, \ldots)\) is forced by (E1) if and only if the value of \(g(e_1, e_2, \ldots)\) is forced by (E1). If we check that
$t$ and $s$ have the same ordinary variables, then the corresponding statement holds with regards to (E2), hence the value of $f(d_1, d_2, \ldots)$ is forced if and only if the value of $g(e_1, e_2, \ldots)$ is forced. If neither value is forced, then both expressions are assigned the value $\bar{d}$. In all cases, they are equal.

In order to write explicitly the details, it is convenient to write (4.4) as

$$f(w_1, w_2, \ldots, w_n) = g(w_{n+1}, w_{n+2}, \ldots, w_p),$$

where each $w_i$ is either a variable or a constant and, similarly, express the identity to be proved as

$$f(d_1, d_2, \ldots, d_n) = g(d_{n+1}, d_{n+2}, \ldots, d_p).$$

As in the previous case, let (4.11) be obtained from (4.10) through some assignment $\sigma$. As in the definition (*) of the associated term, let $I = \{1, 2, \ldots, p\}$, $I' = \{i \in I \mid d_i$ is not the interpretation of some constant $\}$ and let $\sim'$ on $I'$ be defined by $i \sim' j$ if $d_i = d_j$. To distinct equivalence classes of $\sim'$ associate distinct variables, say, $\tau(d_i)$ is the variable associated to the class of $d_i$. If $i \in I \setminus I'$, let $\sigma(d_i)$ be the constant of which $d_i$ is the interpretation. Thus, as in (4.7), the term $t$ associated to the expression $f(d_1, d_2, \ldots, d_n)$ is $t(x_1, x_2, \ldots) = f(\tau(d_1), \tau(d_2), \ldots, \tau(d_n))$ and, similarly, the term $s$ associated to the expression $g(d_{n+1}, d_{n+2}, \ldots, d_p)$ is $s(y_1, y_2, \ldots) = g(\tau(d_{n+1}), \tau(d_{n+2}), \ldots, \tau(d_p))$. Of course, it might happen that the sets $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ are distinct; however, we shall see that their symmetrical difference consists entirely of exceptional variables. As we mentioned, the choice of variables for $s$ and $t$ here is not arbitrary, we have named variables in such a way that if $d_i = d_j$, then the variables associated to $d_i$ and $d_j$ are the same, even when $d_i$ is in the range of $f$ and $d_j$ is in the range of $g$. Henceforth,

$$t(x_1, x_2, \ldots) = s(y_1, y_2, \ldots)$$

is an equation valid in $V$, since (4.10) is valid in $V$. As in the previous case, (4.12) and (4.10) are not necessarily equivalent, since some variables from (4.10) might have been identified in (4.12); and moreover, some variables from (4.10) might have been replaced by constants. However, (4.10) does imply (4.12), and this is what we need.

From (4.12) we immediately get that eventuality (E1) holds for $f(d_1, d_2, \ldots, d_n)$ if and only if it holds for $g(d_{n+1}, d_{n+2}, \ldots, d_p)$ and, if this is the case, the forced values are equal.

To deal with (E2), we first need to observe that if some variable $x$ occurs in $t$ but not in $s$, then $x$ is exceptional in $t$ for $V$, by applying (4.12) twice as follows $t(x_1, x_2, \ldots, x, \ldots) = s(y_1, y_2, \ldots) = t(x_1, x_2, \ldots, z, \ldots)$. Symmetrically, a variable occurring in $s$ and not in $t$ is exceptional in $s$. It follows obviously from (4.12) that if some variable $x$ occurs both in $t$ and in $s$, then $x$ is exceptional in $t$ if and only if $x$ is exceptional in $s$. In conclusion, the sets of variables of $t$ and $s$ might be distinct, but the sets of ordinary variables in $t$ and $s$ are equal. In particular, the set $\{d_1, d_2, \ldots\}$ of elements corresponding to the ordinary
vars is the same both for \( f(d_1, d_2, \ldots, d_n) \) and for \( g(d_{n+1}, d_{n+2}, \ldots, d_p) \),
thus the value of the former is forced according to (E2) if and only if the value of the latter is forced. If this is the case, the two forced values are the same, again by (4.12), which holds in \( V \), hence both in \( A \) and in \( B \).

In conclusion, we have showed that the value of \( f(d_1, d_2, \ldots, d_n) \) is forced if and only if the value of \( g(d_{n+1}, d_{n+2}, \ldots, d_p) \) is forced and, if this is the case, the two values are equal. Otherwise, \( f(d_1, d_2, \ldots, d_n) = \bar{d} = g(d_{n+1}, d_{n+2}, \ldots, d_p) \).

We have proved equality in each case, hence \( D \) is indeed an algebra in \( V \), thus a desired amalgamating algebra.

---

Remark 4.5. (a) Remark 3.7 applies with no essential modification to the context in Theorem 4.4.

As far as part (c) in Remark 3.7 is concerned, let \( D \supseteq A \cup B \) and \( \eta \) be a function from \( D^{\text{fin}} \) to \( D \). If \( d_1, d_2, \ldots, d_n \in D \) and the value of \( f(d_1, d_2, \ldots, d_n) \) is neither forced by (E1) nor by (E2), set

\[
f(d_1, d_2, \ldots, d_n) = \eta(\{d_{i1}, d_{i2}, \ldots\}) \tag{4.13}
\]

in \( D \), where \( \{d_{i1}, d_{i2}, \ldots\} \) corresponds to the set of ordinary variables in the term associated to the expression \( f(d_1, d_2, \ldots, d_n) \). We have showed in the proof of Theorem 1.4 that if some equation of the form (4.10) is valid in \( V \), then, for any instance of (4.11), the sets of ordinary variables associated to \( f(d_1, d_2, \ldots, d_n) \) and to \( g(d_{n+1}, d_{n+2}, \ldots, d_p) \) coincide, hence, if these values are not forced, we get \( f(d_1, d_2, \ldots) = \eta(\{d_{i1}, d_{i2}, \ldots\}) = g(d_{n+1}, d_{n+2}, \ldots, d_p) \) from (4.13).

(b) The full version of Corollary 3.2 follows from Theorem 4.4. Only (1) needs a comment, since we cannot use JEP to prove it. However Theorem 4.4 can be applied, together with the methods indicated in Remark 3.7(e).

Remark 4.1 essentially provides all kinds of counterexamples to JEP. We obtain JEP when the equations axiomatizing some variety are linear, provided there is exactly one constant and this constant is always interpreted as a one-element subalgebra.

**Corollary 4.6.** Suppose that \( V \) is a variety axiomatized by a set of linear equations, that the language of \( V \) contains exactly one constant \( c \) and that \( f(c, c, c, \ldots) = c \) is valid in \( V \), for every operation \( f \) in the language.

Then \( V \) has the joint embedding property. The class of finite algebras in \( V \) has the joint embedding property. If the language of \( V \) is finite (or just if there are countably many finite algebras up to isomorphism), then the class of finite algebras in \( V \) has a Fraïssé limit.

**Proof.** Under the assumptions, the one-element algebra is (isomorphic to) a subalgebra of every algebra. Hence AP implies JEP, by taking \( C \) a one-element algebra.

The above results imply that many varieties corresponding to Maltsev conditions “localized” at some constant 0 have SAP, JEP and Fraïssé limits. In
particular, Theorem 4.4 and Corollary 4.6 apply to varieties with operations witnessing permutability at 0, arithmeticity at 0 [Du Theorems 1(2) and 4(2)], distributivity at 0 [C1 Theorem 1], 3-permutability at 0 [C2 Theorem 1], \(n\)-permutability at 0 together with distributivity at 0 [C2 Theorem 2(3)], as well as varieties with a near 0-unanimity operation [Du Definition 4]. In this respect, compare also the example after Definition 11.1 in [G].

Remark 4.7. We cannot generalize Theorem 4.4 to quasiequations (= Horn sentences in a language without relations), namely, if some quasivariety \(Q\) is axiomatized by means of quasidentities constructed using linear identities, it is not necessarily the case that \(Q\) has AP.

For example, let \(Q\) be the quasivariety in the language with two unary operations \(f\) and \(g\) such that \(Q\) is axiomatized by the quasiequation

\[
f(x) = f(y) \implies x = y. \tag{4.14}
\]

Then \(Q\) has not AP.

Indeed, let \(C\) be \(\mathbb{N}^+\) with both \(f\) and \(g\) interpreted as the successor function. Extend \(C\) to \(A\) and \(B\) by adding two copies of 0. In \(A\) we let \(f\) and \(g\) be again the successor function. On the other hand, in \(B\), we let \(f\) be the successor function, while we set \(g(0') = 0'\). If by contradiction there is an amalgamating structure, we have \(f(0) = f(0')\), hence 0 and 0' should be identified by (4.14), but this is impossible, since \(g(0) \neq g(0')\).

Of course, it might happen that some analogue of Theorem 4.4 holds for quasiequations of some special form, but we have not investigated in depth this possibility.

5. Adding unary operations

It is easy to see that if some variety has the amalgamation property, then the amalgamation property is preserved by adding a new unary operation \(h\) with equations asserting that \(h\) is an endomorphism. See the following observation.

In the next observation \(\mathcal{V}\) is an arbitrary variety with the amalgamation property. We are not assuming that \(\mathcal{V}\) is axiomatized by linear equations. The observation is probably folklore, but we know no reference for it.

Observation 5.1. Suppose that \(\mathcal{V}\) is a variety with the (strong) amalgamation property. Let \(\mathcal{V}_h\) be the variety in a language with a new unary operation \(h\) added; the equations axiomatizing \(\mathcal{V}_h\) are the equations axiomatizing \(\mathcal{V}\) plus equations asserting that \(h\) is an endomorphism of \(\mathcal{V}\), namely,

\[
h(f(x_1, x_2, \ldots)) = f(h(x_1), h(x_2), \ldots) \tag{5.1}
\]

for every operation \(f\) in the language of \(\mathcal{V}\) and where \(x_1, x_2, \ldots\) are distinct variables.

Then \(\mathcal{V}_f\) has the (strong) amalgamation property.
Proof. The argument is purely categorical. If $A, B, C \in \mathcal{V}$ are as in the assumptions of (S)AP and have an amalgam in $\mathcal{V}$, then their push out $D$ (in the category of algebras in $\mathcal{V}$ with $\mathcal{V}$-homomorphisms) is still an amalgam $\text{[KMPT]}$.

Now suppose that $A_h, B_h, C_h$ are in $\mathcal{V}_h$, let $A, B, C$ be their reducts to the language of $\mathcal{V}$ and let $D$ be as above. The unary operation $h$ satisfies (5.1) on $A_h, B_h$ and $C_h$, hence we can think of $h_A, h_B$ and $h_C$ as $\mathcal{V}$-endomorphisms of $A, B, C$, respectively. Composing these endomorphisms with the appropriate embeddings, we get a commuting diagram of homomorphisms from $A, B$ and $C$ to $D$. By the push out property of $D$, such homomorphisms can be extended to an endomorphism $h_D$ of $D$. Now turn back and think of $h_D$ as a unary operation on $D$. Expanding $D$ by adding $h_D$ we get an algebra in $\mathcal{V}_f$ which amalgamates $A_h, B_h, C_h$. □

Remark 5.2. (a) Of course, Observation 5.1 holds in the case we add a family of unary operations which are endomorphisms.

(b) Observation 5.1 applies also to an arbitrary class $\mathcal{K}$ of structures (not necessarily a variety), provided $\mathcal{K}$ has push outs. In the general case, condition 5.1 should be replaced by a condition asserting that $h$ is an endomorphism.

(c) We can also add unary operations which are automorphisms. The property that some $h$ is bijective cannot be expressed equationally without expanding the language; however, we might assume that there is one more unary operation $k$ and that $k(h(x)) = h(k(x)) = x$ are equations valid in the variety under consideration.

The assumption that 5.1 holds for every operation $f$ in the language of $\mathcal{V}$ is necessary in Observation 5.1. Let $+$ and juxtaposition denote lattice operations.

Example 5.3. The variety of distributive lattices with a unary operation $h$ satisfying

$$h(x + y) = h(x) + h(y)$$

(5.2)

has not AP.

Proof. Let $C$ be the three-elements chain with base set $\{0, 1, 2\}$, and let $h$ be the identity on $C$.

Expand $C$ in two ways. Let $A = \{0, 1, 2, a\}$, with $a$ a complement of 1 in $A$ and, again, let $h$ be the identity on $A$. Let $B = \{0, 1, 2, b\}$, with $b$ a complement of 1 in $B$ and let $h$ be the identity on $C$ and $h(b) = 2$.

Then $C \subseteq A, B$ and all algebras satisfy (5.2). However, in any amalgamating structure, $a$ and $b$ should be identified, since complements are unique in distributive lattices. This contradicts $h(a) = a$ and $h(b) = 2$, hence $A, B$ and $C$ cannot be amalgamated. □

The variety of distributive lattices has the amalgamation property, but not the strong amalgamation property $\text{[FG]}$. In fact, Example 5.3 is just a small elaboration on this result.
Problem 5.4. Is there an example similar to Example 5.3 in which $V$ has the strong amalgamation property?

Is there an example similar to Example 5.3 in which $h$ is bijective? (We can express the assumption that $h$ is bijective by introducing a further operation $k$, as in Remark 5.2(c).)

We have seen in Example 5.3 that Observation 5.1 does not generalize when we assume that $h$ preserves only some operation in $V$. However, the observation does generalize when $V$ is axiomatized by linear equations and $h$ is assumed to be bijective.

Proposition 5.5. Suppose that $V$ is a variety axiomatized by a set of linear equations and let $V_{h,k}$ be a variety with two new unary operations $h$ and $k$ added. The variety $V_{h,k}$ is supposed to satisfy all the equations of $V$, the equations

\[ k(h(x)) = h(k(x)) = x, \quad h(c) = c, \quad (5.3) \]

for every constant $c$ in the language of $V$, as well as possibly some equations of the form (5.1), $f$ varying on a set of operations of $V$.

Then $V_{h,k}$ has the strong amalgamation property.

If either $V$ is axiomatized by a set of equilinear equations in a language without constants, or the language of $V$ has only one constant which represents a subalgebra, then $V_{h,k}$ has the joint embedding property.

Proof. Given $A, B, C \in V_{h,k}$ as in the assumptions (2.1) of SAP, first construct an amalgamating structure $D$ for the reducts in the language of $V$ as in the proof of Theorem 4.4, but here take $d \notin A \cup B$, a possibility mentioned in Remarks 3.7(b) and 4.5(a). Thus here we are taking $D = A \cup B \cup \{d\}$.

The structure $D$ expands in a unique way to a structure $D$ for the language of $V_{h,k}$, under the assumption that $D$ extends $A$ and $B$. The values of $h$ and $k$ on $A \cup B$ are determined by the above request. Then set $h(d) = \bar{d}$ and $k(d) = \bar{d}$. Such identities should be satisfied if we want $h$ and $k$ to be one the inverse of the other. Thus we have a structure $D$ which extends both $A$ and $B$ and $k(h(x)) = h(k(x)) = x$ are satisfied in $D$, since they are satisfied in both $A$ and $B$.

It remains to show that all the requested equations of the form (5.1) are satisfied in $D$. To this end, we first observe that, for every operation $f$ and every sequence $d_1, d_2, \ldots \in A \cup B \cup \bar{d}$, the value of $f(d_1, d_2, \ldots)$ is forced by one of the conditions (E1), (E2) in the proof of Theorem 4.4 if and only if the value of $f(e_1, e_2, \ldots)$ is forced by the same condition, where $e_1 = h(f_1)$, $e_2 = h(f_2)$ \ldots Indeed, the terms associated to $f(d_1, d_2, \ldots)$ and to $f(e_1, e_2, \ldots)$ according to (*) are identical (modulo the ordering of the variables), since, say, $d_i$ is the interpretation of some constant if and only if $h(d_i)$ is the interpretation of (the same) constant, because of the second condition in (5.3). Moreover, since $h$ is bijective, then $d_i = d_j$ if and only if $e_i = e_j$. Thus if the value of $f(d_1, d_2, \ldots)$ is forced by condition (E1), then the value of $f(e_1, e_2, \ldots)$ is forced by (E1),
and conversely. Moreover, if the value forced for \( f(d_1, d_2, \ldots) \) is \( d_i \), then the value forced for \( f(e_1, e_2, \ldots) \) is \( e_i = h(d_i) \).

On the other hand, the value of \( f(d_1, d_2, \ldots) \) is forced by (E2) if and only if either \( \{ d_{i_1}, d_{i_2}, \ldots, d_{i_{\ell -1}} \} \subseteq A \) or \( \{ d_{i_1}, d_{i_2}, \ldots, d_{i_{\ell -1}} \} \subseteq B \), under the conventions in (E2). Since the image of \( A \) under \( h \) is \( A \) itself, then \( \{ d_{i_1}, d_{i_2}, \ldots, d_{i_{\ell -1}} \} \subseteq A \) if and only if \( \{ e_{i_1}, e_{i_2}, \ldots, e_{i_{\ell -1}} \} = \{ h(d_{i_1}), h(d_{i_2}), \ldots, h(d_{i_{\ell -1}}) \} \subseteq A \) and similarly for \( B \). Hence the value of \( f(d_1, d_2, \ldots) \) is forced by (E2) if and only if the value of \( f(e_1, e_2, \ldots) \) is forced by (E2).

In the remaining case, neither the value of \( f(d_1, d_2, \ldots) \) nor the value of \( f(e_1, e_2, \ldots) \) are forced, hence the proof of \([4.4] \) gives \( f(d_1, d_2, \ldots) = f(e_1, e_2, \ldots) = \hat{d} = h(\hat{d}) \) in \( D \).

The proof is almost complete. We need to show that, limited to those operations \( f \) for which \([5.1] \) is assumed in \( \mathcal{V}_{h,k} \), the very same equation \([5.1] \) holds in \( D \). That is, if \( d^* = f(d_1, d_2, \ldots) \), then \( h(d^*) = f(e_1, e_2, \ldots) \) in \( D \), under the above conventions. We have already shown in the course of the above arguments that this identity holds both when the values are forced by condition (E1) and when the values are not forced.

It remains to treat the case when the values are forced by (E2). In this case the values forced for \( f(d_1, d_2, \ldots) \) and \( f(e_1, e_2, \ldots) \) are, respectively, \( t(d_{i_1}, d_{i_2}, \ldots, d_{i_{\ell -1}}; d, \ldots, d) \) and \( t(e_{i_1}, e_{i_2}, \ldots, e_{i_{\ell -1}}; d, \ldots, d) \), evaluated in \( A \) or in \( B \). For the sake of brevity, suppose from now on that everything is evaluated in \( A \). Let \( \hat{d}_i = d_i \), \( \hat{e}_i = e_i \) if \( d_i \) corresponds to an ordinary variable of \( t \) and let \( \hat{d}_i = d \), \( \hat{e}_i = h(d) \) if \( d_i \) corresponds to an exceptional variable of \( t \). Then in \( A \) we have

\[
\begin{align*}
t(e_1, e_2, \ldots, e_{\ell -1}; d_1, \ldots, d) &= t(e_1, e_2, \ldots, e_{\ell -1}; h(d), \ldots, h(d)) = \text{def} \quad f(\hat{e}_1, \hat{e}_2, \ldots) = f(h(\hat{d}_1), h(\hat{d}_2), \ldots) = \text{def} \quad h(t(d_{i_1}, d_{i_2}, \ldots, d_{i_{\ell -1}}; d, \ldots, d)),
\end{align*}
\]

where the first equality follows from the fact that the variables after the semicolon are exceptional in the enumeration of the arguments of \( t \), the equalities labeled with \( \text{def} \) follow from the definitions of the term \( t \) and of \( \hat{d}_i \), \( \hat{e}_i \) and, finally, we can apply \([5.1] \) since \( \hat{d}_1, \hat{d}_2, \ldots \) all belong to \( A \) and \( A \) satisfies \([5.1] \) by assumption. We have showed that if equation \([5.1] \) holds in \( \mathcal{V} \) with regard to \( f \) and \( d^* = f(d_1, d_2, \ldots) \), then \( h(d^*) = f(e_1, e_2, \ldots) = f(h(d_1), h(d_2), \ldots) \), where \( f(d_1, d_2, \ldots) \) and \( f(e_1, e_2, \ldots) \) are the values forced in \( D \) by (E2). In conclusion equation \([5.1] \) holds in \( D \), hence \( D \) witnesses SAP.

If \( \mathcal{V} \) is axiomatized by a set of equilinear equations in a language without constants, then the simpler arguments used in the proof of Theorem \([3.4] \) apply. In this case we do not need to use elements of \( C \); compare the comment shortly after the introduction of eventuality (E2) in the proof of Theorem \([4.4] \). Hence we can repeat the argument taking \( C = \emptyset \) and we get JEP. As in the proof of Corollary \([4.6] \) in the presence of a one-element subalgebra of every algebra AP implies JEP.
As a final remark, the arguments from the present note are likely to be generalizable to a broader setting, but we do not know how far. The main open problem is to ascertain which arguments can be extended to languages with further relations besides equality. In this sense, the main obstacle to generalizations appears to be the possible mutual incompatibility of AP with conditions corresponding to (4.3) and (4.4) or, put in another way, the fact that the outcomes given by conditions similar to (E1) and (E2) might not agree in all the overlapping cases.

We present a simple example in which we renounce to the analogue of (4.3) but we can work with an equilinear analogue of (4.4). The example might be a starting point for possible generalizations.

In the following proposition \( L \) is an arbitrary language, possibly containing relation symbols.

**Proposition 5.6.** Suppose that \( T \) is a theory in the language \( L \) and \( T \) has the (strong) amalgamation property. Suppose further that every model \( D_1 \) of \( T \) can be extended to some model \( D \) with an element \( \bar{d} \in D \) such that \( R(\bar{d}, \bar{d}, \ldots, \bar{d}) \) holds in \( D \), for every relation symbol \( R \in L \).

Let \( T' \supseteq T \) be a theory in a language \( L' \supseteq L \) such that \( L' \setminus L \) consists only of function symbols. Suppose that \( T' \setminus T \) consists only of axioms of the form

\[
R(f_1(x_1,1, x_1,2, \ldots, x_1,n_1), f_2(x_2,1, x_2,2, \ldots, x_2,n_2), \ldots, f_m(x_m,1, x_m,2, \ldots, x_m,n_m)),
\]

(5.5)

for \( R \in L \), \( f_1, f_2, \ldots, f_m \in L' \setminus L \) and where \( \{x_1,1, x_1,2, \ldots, x_1,n_1\} = \{x_2,1, x_2,2, \ldots, x_2,n_2\} = \cdots = \{x_m,1, x_m,2, \ldots, x_m,n_m\} \)

(we are not considering multiplicities.)

Then \( T' \) has the (strong) amalgamation property.

**Proof.** Given \( A', B' \) and \( C' \) models of \( T' \) in \( L' \) as in (2.1), their reducts to \( L \) can be amalgamated to a model \( D \) of \( T \), since \( T \) satisfies AP. By the additional assumption on \( T \), it is no loss of generality to assume that there is \( \bar{d} \) in \( D \) such that \( R(\bar{d}, \bar{d}, \ldots, \bar{d}) \) holds in \( D \), for every relation symbol \( R \in L \).

Expand \( D \) to a model \( D' \) for \( L' \) by setting, for every \( f \in L' \setminus L \) and \( d_1, d_2, \ldots \in D \):

\[
\begin{align*}
  f(d_1, d_2, \ldots) &= f_A(d_1, d_2, \ldots) & \text{if } d_1, d_2, \ldots \in A, \\
  f(d_1, d_2, \ldots) &= f_B(d_1, d_2, \ldots) & \text{if } d_1, d_2, \ldots \in B, \\
  f(d_1, d_2, \ldots) &= \bar{d} & \text{otherwise.}
\end{align*}
\]

As standard by now, the first two clauses assure that \( D' \) extends both \( A' \) and \( B' \). The third clause and the properties of \( \bar{d} \) imply that each instance of (5.5) is satisfied in \( D' \), provided it is satisfied in \( A', B' \) and \( C' \). Just notice that, for any evaluation of (5.5) in \( D' \), the assumption on the occurrences of the variables implies that \( f_1 \) obeys one of the above clauses if and only if each \( f_1 \) obeys the very same clause.
Hence $D'$ is an amalgamating structure for $A'$, $B'$ and $C'$.

REFERENCES

[C1] I. Chajda, Congruence distributivity in varieties with constants, Arch. Math. (Brno) 22, 121–124 (1986)
[C2] I. Chajda, On n-permutable and distributive at 0 varieties, Acta Math. Univ. Comenian. (N.S.) 68, 271–277 (1999)
[Da] A. Day, A characterization of modularity for congruence lattices of algebras, Can. Math. Bull. 12, 167–173 (1969)
[Du] J. Duda, Arithmeticty at 0, Czechoslovak Math. J. 37(112), 197–206 (1987)
[E] D. M. Evans, Examples of $\aleph_0$-categorical structures, in Automorphisms of first-order structures, R. Kaye, D. Macpherson (eds.), Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 33–72 (1994)
[F] R. Fraïssé, Sur l'extension aux relations de quelques propriétés des ordres, Ann. Sci. Ecole Norm. Sup. (3) 71, 363–388 (1954)
[FG] E. Fried, G. Grätzer, Strong Amalgamation of Distributive Lattices, J. Algebra 128, 446–455 (1990)
[GM] D. M. Gabbay, L. Maksimova, Interpolation and definability. Modal and intuitionistic logics, Oxford Logic Guides 46, The Clarendon Press, Oxford University Press, Oxford (2005)
[G] H.-P. Gumm, Geometrical methods in congruence modular algebras, Mem. Am. Math. Soc. 45 (1983)
[HaMi] J. Hagemann, A. Mitschke, On n-permutable congruences, Algebra Universalis 3, 8–12 (1973)
[HoMc] D. Hobby, R. McKenzie, The structure of finite algebras, Contemp. Math. 76 (1988)
[H] W. Hodges, Model theory, Encyclopedia of Mathematics and its Applications 42, Cambridge University Press, Cambridge (1993)
[J1] B. Jonsson, Extensions of relational structures, in Theory of Models (Proc. Internat. AP Sympos. Berkeley, 1963), North-Holland, Amsterdam, 146–157 (1965)
[J2] B. Jonsson, Algebras whose congruence lattices are distributive, Math. Scand. 21, 110–121 (1967)
[KKMM] A. Kazda, M. Kozik, R. McKenzie, M. Moore, Absorption and directed Jonsson terms, in: J. Czelakowski (ed.), Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science, Outstanding Contributions to Logic 16, Springer, Cham, 203–220 (2018)
[KV] A. Kazda, M. Valeriote, Deciding some Mal’tsev conditions in finite idempotent algebras, J. Symb. Logic 85, 539–562 (2020)
[KK] K. A. Kearnes, E. W. Kiss, The shape of congruence lattices, Mem. Amer. Math. Soc. 222, (2013)
[KKM] K. A. Kearnes, P. Marković, R. McKenzie, Optimal strong Mal’cev conditions for omitting type 1 in locally finite varieties, Algebra Universalis 72, 91–100 (2014)
[K] N. Kimura, On semigroups, Doctoral dissertation, Tulane University, New Orleans (1957)
[KMPT] E. W. Kiss, L. Márki, P. Pröhle, W. Tholen, Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity, Studia Sci. Math. Hungar. 18, 79–140 (1982)
[DTS] P. Lipparini, Day’s Theorem is sharp for n even, arXiv:1902.05993, 1–60 (2019/2021)
[SAPU] P. Lipparini, The strong amalgamation property into union, arXiv:2103.00563, 1–16 (2021)
[M] A. I. Mal’tsev, On the general theory of algebraic systems (in Russian), Mat. Sb. N.S. 35 (77), 3–20 (1954); translated in Amer. Math. Soc. Transl. (2) 27, 125–142 (1963)
[MMT] G. Metcalfe, F. Montagna, C. Tsinakis, Amalgamation and interpolation in ordered algebras, J. Algebra 402, 21–82 (2014)
[O] M. Olšák, *The weakest nontrivial idempotent equations* Bull. Lond. Math. Soc., 49, 1028–1047 (2017)

[P] A. F. Pixley, *Distributivity and permutability of congruence relations in equational classes of algebras*, Proc. Amer. Math. Soc. 14, 105–109 (1963)

[S] M. H. Siggers, *A strong Mal’cev condition for locally finite varieties omitting the unary type*, Algebra Universalis 64, 15–20 (2010)

[Ta] W. Taylor, *Varieties obeying homotopy laws*, Canadian J. Math. 29, 498–527 (1977)

[Ts] S. T. Tschantz, *More conditions equivalent to congruence modularity*, In: Comer, S.D. (ed) *Universal Algebra and Lattice Theory (Charleston, S.C., 1984)*, Lecture Notes in Math., vol. 1149, pp. 270–282. Springer, Berlin (1985)

### Paolo Lipparini

Dipartimento di Matematica, Varietà della Ricerca Scientifica, Università di Roma “Tor Vergata”, I-00133 ROME ITALY