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MANY-PARAMETER M-COMPLEMENTARY GOLAY SEQUENCES AND TRANSFORMS

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Abstract

In this paper, we develop the family of Golay–Rudin–Shapiro (GRS) m-complementary many-parameter sequences and many-parameter Golay transforms. The approach is based on a new generalized iteration generating construction, associated with n unitary many-parameter transforms and n arbitrary groups of given fixed order. We are going to use multi-parameter Golay transform in Intelligent-OFDM-TCS instead of discrete Fourier transform in order to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

Keywords: complementary sequences, many-parameter orthogonal transforms, fast algorithms, OFDM systems.

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Introduction

Binary ±1-valued Golay–Rudin–Shapiro sequences (2-GRSS) associated with the cyclic group \( Z_2 \) were introduced independently by Golay [1, 2, 3] in 1949-1951, Shapiro [4, 5] and Rudin [6] in 1951. M.J.E. Golay [2] introduced the general concept of "complementary pairs" of finite sequences all of whose entries are ±1. For building the classical FGRST in bases of classical 2-GRSS the following actors are used: 1) Abelian group \( Z_2 \), 2) 2-point Fourier transform \( F_2 \), and 3) complex field \( C \), i.e., these transforms are associated with the triple \((Z_2, F_2, C)\).

In previous papers [7, 8], we have shown a new unified approach to the \( GF(p) \), or Clifford-valued complementary sequences and Golay transforms. It was associated not with the triple \((Z_2, F_2, C)\), but with triples

\[
(Z_2, \{CS^k_1(\psi, \alpha, \gamma), CS^k_2(\phi, \alpha, \gamma), \ldots\}, A(\psi)) \text{ and } (Z_2, \{CS^k_1(\psi, \alpha, \gamma), CS^k_2(\phi, \alpha, \gamma), \ldots\}, A(\phi)),
\]

where \( CS^k_1(\psi, \alpha, \gamma) = \{\psi_k^{n_1}, \phi_k^{n_2}, \ldots\} \) is a set of arbitrary finite groups of given order \( m \) Here \( \{U_1^m, U_2^m, \ldots, U_m^m\} \) is a set of arbitrary unitary \((m \times m)\) -transforms represented in the many-parameter Jacobbi–Euler form [9–10]:

\[
U_1^m = U_1^m(\psi_1, \phi_1, \ldots, \psi_m, \phi_m) = \prod_{s=1}^m J(\psi_s, \phi_s),
\]

\[
U_2^m = U_2^m(\psi_1, \phi_1, \ldots, \psi_m, \phi_m) = \prod_{s=1}^m J(\psi_s, \phi_s),
\]

\[
\ldots,
\]

\[
U_n^m = U_n^m(\psi_1, \phi_1, \ldots, \psi_m, \phi_m) = \prod_{s=1}^m J(\psi_s, \phi_s),
\]

where

\[
J(\psi_s, \phi_s) = \begin{pmatrix}
1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & c(\psi_s) & \ldots & s(\psi_s) & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & s(\psi_s) & \ldots & -c(\psi_s) & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 1
\end{pmatrix}
\]

is the Jacobian orthonormal rotation with reflection, \( c(\psi_s) = \cos(\psi_s), s(\psi_s) = \sin(\psi_s) \) are the Jacoby parameters, \( m = m(m - 1)/2 \).
The object of the study.

**New iteration construction for original Golay sequences**

We begin by describing the original Golay $m$-complementary sequences.

**Definition 1.** A generalization of the Golay complementary pair, known as the Golay $m$-Complementary m-element Set (m-GCS) of complex-valued sequences [11]

\[
m\text{-GCS} = \{\text{com}_k(t) := (c_{0}(t), c_{1}(t), ..., c_{m-1}(t)) \mid k \in \mathbb{Z} \} = m\text{,}
\]

is defined by

\[
\sum_{k=0}^{m-1} \text{COR}_k(t) = m \cdot \delta(t), \quad \sum_{k=0}^{m-1} \text{COM}_k(z)^2 = m,
\]

where \( \text{COR}_k(t) \) is the periodic autocorrelation function of \( \text{com}_k(t) \) and \( \text{COM}_k(z) = \mathbb{Z} \{ \text{com}_k(t) \} \) are their $\mathbb{Z}$-transforms.

We use two symbols $\alpha_k \in [0, m^{m-1}-1] = \mathbb{Z}_m^{m\text{-th}}$ and $\mathbf{t}_n \in [0, m^{m-1}-1] = \mathbb{Z}_m^{m\text{-th}}$ for numeration of Golay sequences and discrete time, respectively. For integer $\alpha_k \in [0, m^{m-1}-1] = \mathbb{Z}_m^{m\text{-th}}$ we shall use m-ary codes $\mathbf{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\mathbf{t}_n = (t_1, t_2, ..., t_n)$, where $\alpha_k t_i \in \{0, 1, ..., m-1\} = \mathbb{Z}_m$, $i = 1, 2, ..., n$.

Let $\mathbf{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\mathbf{t}_n = (t_1, t_2, ..., t_n)$ be m-ary codes, then define

\[
\mathbf{a}_n = \sum_{i=0}^{n-1} \alpha_{n-i} \cdot m^{-i}, \quad \text{and} \quad \mathbf{t}_n = \sum_{i=0}^{n} t_{n-i} \cdot m^{-i},
\]

as integers whose m-ary codes are $\mathbf{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\mathbf{t}_n = (t_1, t_2, ..., t_n)$, respectively. Obviously,

\[
\mathbf{a}_n = (\alpha_1, \alpha_2, ..., \alpha_n), \quad \mathbf{t}_n = (t_1, t_2, ..., t_n).
\]

**Example 1.** For $n = 1$ and $n = 2$ we have, respectively,

\[
\begin{align*}
\text{com}_{[0]}^{[1]}(t_{1}) & = \begin{bmatrix}
\text{com}_{[0]}^{[1]}(t_{1}) \\
\text{com}_{[0,0]}^{[1]}(t_{1}) \\
\cdots \\
\text{com}_{[0,m-1]}^{[1]}(t_{1})
\end{bmatrix} \\
\text{com}_{[0]}^{[2]}(t_{2}) & = \begin{bmatrix}
\text{com}_{[0]}^{[2]}(t_{2}) \\
\text{com}_{[0,0]}^{[2]}(t_{2}) \\
\cdots \\
\text{com}_{[0,m-1]}^{[2]}(t_{2})
\end{bmatrix}
\end{align*}
\]
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The matrix $G_{m+1}^{[1]}$ is constructed by an iteration construction. The initial matrix $G_{[1]}^{[1]}$ is formed by starting with an arbitrary unitary $(m \times m)$-matrix (in many-parameter form or not)

$$G_{[1]}^{[1]} = \begin{bmatrix} \text{com}^{[1]}_{0}(t_1) \\ \text{com}^{[1]}_{1}(t_1) \\ \vdots \\ \text{com}^{[1]}_{m}(t_1) \end{bmatrix}$$

$$U_m = [A_m(t)] = \begin{bmatrix} A_m(0) & A_m(1) & A_m(2) & \cdots & A_m(m-1) \\ A_m(0) & A_m(1) & A_m(2) & \cdots & A_m(m-1) \\ A_m(0) & A_m(1) & A_m(2) & \cdots & A_m(m-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m-1}(0) & A_{m-1}(1) & A_{m-1}(2) & \cdots & A_{m-1}(m-1) \end{bmatrix},$$

where $A_m(t) \in \text{Alg},$

$\text{com}^{[1]}(t) = (A_m(0), A_m(1), \ldots, A_m(m-1)).$

Example 2. The initial matrix $G_{[m]}^{[1]}$ can be the Fourier transform on Abelian group $Z_m$:

$$G_{[m]}^{[1]} = \begin{bmatrix} \text{com}^{[1]}_{0}(t_1) \\ \text{com}^{[1]}_{1}(t_1) \\ \vdots \\ \text{com}^{[1]}_{m}(t_1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{1i} & e^{2i} & \cdots & e^{(m-1)i} \\ 1 & e^{2i} & e^{2i} & \cdots & e^{(m-2)i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{(m-1)i} & e^{(m-1)i} & \cdots & e^{(m-1)(m-1)i} \end{bmatrix},$$

where $m = \sqrt{\text{Alg}}, \text{com}^{[1]}(t) = (1, e^{1i}, e^{2i}, \ldots, e^{(m-1)i}),$

$k = 0, 1, \ldots, m-1$ are characters $Z_m$.\[\square\]

It is easy to check that

$$\left[|\text{COM}_{a}(z)|^2 + |\text{COM}_{b}(z)|^2 + \cdots + |\text{COM}_{m_a}(z)|^2 \right]_{|z|=1} = m.$$

Indeed,

$$\sum_{k=1}^{n-1} \left| \text{COM}_{k}(z) \right|^2 = \sum_{k=1}^{n-1} \left| \text{COM}_{k}(z) \right| \text{COM}^\dagger_{k}(z) =$$

$$= \sum_{k=1}^{n-1} \left( \sum_{i=0}^{n-1} a_i(t)z^i \right) \left( \sum_{i=0}^{n-1} \bar{a}_i(t)z^i \right) =$$

$$= \sum_{i=0}^{n-1} \sum_{k=1}^{n-1} (\sum_{i=0}^{n-1} a_i(t)\bar{a}_i(t)) z^i z^{i} = \sum_{i=0}^{n-1} \delta_{i,0} z^i = \sum_{i=0}^{n-1} |z|^2,$$

since $\sum_{i=0}^{n-1} a_i(t)\bar{a}_i(t) = \delta_{i,0}$ is true for an arbitrary unitary (orthogonal) matrix. Hence,

$$\left( \sum_{i=0}^{n-1} \left| \text{COM}_{i}(z) \right|^2 \right)_{|z|=1} = \left( \sum_{i=0}^{n-1} |z|^2 \right)_{|z|=1} = m$$

and initial sequences in the form of rows of an unitary matrix (in particular case, in the form of characters $\text{com}(t) = (1, e^{1i}, e^{2i}, \ldots, e^{(m-1)i})$ of cyclic group $Z_m$) are the Golay $m$-complementary sequences.

Methods

The matrix $G_{[m+1]}^{[1]}$ is constructed by an iteration construction

$$G_{[m]}^{[1]}(U_m) \rightarrow G_{[m]}^{[2]}(U_{m}, U_j) \rightarrow \cdots \rightarrow G_{[m]}^{[n]}(U_{m}, \ldots, U_j, U_{m+1}),$$

where

$$U_{m+1} = \{U_{m}, \ldots, U_{m+1}\} = \{U_{m}, U_{m+1}\},$$

Here $U_{m}(\varphi_{m}) = [A_{m}^{s}(t) | \varphi_{m})]^{m-1}_{s=0} \in SU(\text{Alg}, m)$

$s = 1, 2, \ldots, n$ are a sequence of unitary many-parameter $(m \times m)$-transforms, belonging to the special unitary group $SU(\text{Alg}, m)$, where $s = 1, 2, \ldots, n+1$ and $A_{m}^{s}(t) | \varphi_{m})$ are $\text{Alg}$-valued many-parameter sequences.

Let us assume that we have $m$-Golay matrix $G_{[m]}^{[1]}(U_{m}, \ldots, U_{m}) = G_{[m]}^{[1]}(U_{m})$ (depending on $n$ previous transforms $U_{m}, \ldots, U_{m}$). We need to construct the next $m$-Golay matrix $G_{[m]}^{[n+1]}(U_{m}, \ldots, U_{m+1}) = G_{[m]}^{[n+1]}(U_{m+1})$ using only $G_{[m]}^{[n]}(U_{m}, \ldots, U_{m})$ and $U_{m+1}$. We are going to use for $m$-Golay matrix $G_{[m]}^{[n]}(U_{m})$ the same structure as in (1):

$$G_{[m]}^{[n]}(U_{m}) = \begin{bmatrix} \text{com}^{[n-1]}_{u_{n-1}=0}(t_u | U_{m}) \\ \text{com}^{[n-2]}_{u_{n-2}=0}(t_u | U_{m}) \\ \vdots \\ \text{com}^{[0]}_{u_{0}=0}(t_u | U_{m}) \end{bmatrix},$$

For constructing $G_{[m]}^{[n+1]}(U_{m+1})$ from $G_{[m]}^{[n]}(U_{m})$ we take each complementary set in the form

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\[ m \text{-GCS}^\alpha (\mathcal{U}_L) = \begin{bmatrix} \text{com}^{[\alpha]}_{(m,0)}(t_0 | \mathcal{U}_L) \\ \text{com}^{[\alpha]}_{(1,0)}(t_1 | \mathcal{U}_L) \\ \vdots \\ \text{com}^{[\alpha]}_{(m-1,0)}(t_{m-1} | \mathcal{U}_L) \end{bmatrix} \]

and construct \( m \) shifted versors of their components

\[ m \text{-GCS}^\alpha (\mathcal{U}_L) \rightarrow \begin{bmatrix} m \text{-GCS}^\alpha_{\alpha_0=0}(\mathcal{U}_{L+1}) \\ m \text{-GCS}^\alpha_{\alpha_1=1}(\mathcal{U}_{L+1}) \\ \vdots \\ m \text{-GCS}^\alpha_{\alpha_{m-1}=m-1}(\mathcal{U}_{L+1}) \end{bmatrix} \]

where

\[ m \text{-GCS}^\alpha_{\alpha_0}(\mathcal{U}_{L+1}) = U^\alpha_m \begin{bmatrix} T_{\alpha_0}^{m+1} \\ T_{\alpha_0}^{m} \\ \vdots \\ T_{\alpha_0}^{1} \end{bmatrix} \]

\[ \times \begin{bmatrix} \text{com}^{[\alpha]}_{(m,0)}(t_0 | \mathcal{U}_L) \\ \text{com}^{[\alpha]}_{(1,0)}(t_1 | \mathcal{U}_L) \\ \vdots \\ \text{com}^{[\alpha]}_{(m-1,0)}(t_{m-1} | \mathcal{U}_L) \end{bmatrix} \]

Here \( \alpha_n = 0, 1, \ldots, m-1 \), \( P_m^\alpha \) is the cyclic permutation operator on \( \alpha_0 \) positions (modulo \( m \)), \( T_{\alpha_0}^{m} \) is the shift operator on \( m \)'s positions \( T_{\alpha_0}^{m} f(t_x) = f(t_x + m^\alpha s) \), \( P_m^\alpha \) is transposed matrix of \( P_m^\alpha \).

According to (1) we obtain

\[ G_{m+1}^{\alpha+1}(\mathcal{U}_{L+1}) = \begin{bmatrix} \text{com}^{[\alpha+1]}_{(m,0)}(t_0 | \mathcal{U}_{L+1}) \\ \text{com}^{[\alpha+1]}_{(1,0)}(t_1 | \mathcal{U}_{L+1}) \\ \vdots \\ \text{com}^{[\alpha+1]}_{(m-1,0)}(t_{m-1} | \mathcal{U}_{L+1}) \end{bmatrix} = \begin{bmatrix} U_{\alpha+1}^m \begin{bmatrix} T_{\alpha_0}^{m+1} \\ T_{\alpha_0}^{m} \\ \vdots \\ T_{\alpha_0}^{1} \end{bmatrix} \end{bmatrix} \times (7) \]

and, consequently,

\[ \text{com}^{[\alpha+1]}_{(\alpha_0,\alpha_{m-1})}(t_{x} | \mathcal{U}_{L+1}) = \sum_{\beta_0=0}^{\alpha_0} a_{\alpha_0}^{m+1}(\beta_0) T_{\alpha_0}^{m+1}(\beta_0 | \alpha_{m-1}) \text{com}^{[\alpha]}_{(\beta_0,\beta_1)}(t_x | \mathcal{U}_L). \]

Since \( t_{x+1} = (t_x, t_{x+1}) \), then believing \( t_{x+1} = \alpha_n \oplus \beta_n \), we obtain:

\[ \text{com}^{[\alpha+1]}_{(\alpha_0,\alpha_{m-1})}(t_{x+1} | \mathcal{U}_{L+1}) = \sum_{\beta_0=0}^{\alpha_0} a_{\alpha_0}^{m+1}(\beta_0) T_{\alpha_0}^{m+1}(\beta_0 | \alpha_{m-1}) \text{com}^{[\alpha]}_{(\beta_0,\beta_1)}(t_x | \mathcal{U}_L). \]

It is finally recurrent relation between \( m \)-complementary sequences of \( G_{m+1}^{\alpha+1}[\mathcal{U}_{L+1}] \) and \( G_{m+1}^{\alpha}[\mathcal{U}_{L}]. \)

From (9) we obtain expression for \( \text{com}^{[\alpha+1]}_{(\alpha_0,\alpha_{m-1})}(t_{x+1} | \mathcal{U}_{L+1}) \):

\[ \text{com}^{[\alpha+1]}_{(\alpha_0,\alpha_{m-1})}(t_{x+1} | \mathcal{U}_{L+1}) = \sum_{\beta_0=0}^{\alpha_0} a_{\alpha_0}^{m+1}(\beta_0) T_{\alpha_0}^{m+1}(\beta_0 | \alpha_{m-1}) \text{com}^{[\alpha]}_{(\beta_0,\beta_1)}(t_x | \mathcal{U}_L). \]

Definition 2. One-to-one map from a set \( X \) to itself \( g: X \rightarrow X \), \( x' = g(x) = \sigma x \) is called a transformation of the set \( X \).

If \( X \) is finite and consists of \( m \) elements (for example, \( X = \{0, 1, 2, \ldots, m\} \) then a transformation of the set \( X \) is called a permutation. As is well known, the set of all permutations of \( X \) forms a group \( S_m = \text{Sym}(X) \) in which the product \( \sigma \pi \) of a pair of permutations \( \sigma, \pi \) is defined by \((\sigma \pi)x = (\sigma (\pi x)).\)

If \( X \) contains more than two elements, \( S_m \) is not commutative. Any subgroup of \( S_m \) is called a permutation group on \( X \), or a group of permutations of \( X \). We shall say that the permutations in \( \text{Sym}(X) \) act or operate on the elements of \( X \).

Definition 3. A homomorphism of a group on a set \( h: \text{Gr} \rightarrow \text{Sym}(X) \) is called a permutation representation (or realization) of \( \text{Gr} \).

The image \( h(\text{Gr}) \subseteq \text{Sym}(X) \) is a permutation group and the elements of \( X \) are represented as permutations of \( X \). A permutation representation is equivalent to an action of \( \text{Gr} \) on \( X \). To specify an action, we need to define for element \( g \in \text{Gr} \) the corresponding permutation \( h(g) \) of \( X \), that is, \( h(g)x \) for any \( x \in X \). We are going to write \( h(g)x \)
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in the short form $g \otimes x$ and to call the group of transformations of $x$. The pair $\langle \rangle$ is called a space with transformation group the elements $x \in X$ are called points of the space.

**Definition 4.** If is a permutation group of degree $m$, then the permutation representation of is the linear permutation representation of $P: Gr \rightarrow \text{GL}_{m}(\mathbb{F})$ which maps to the corresponding permutation matrix $P(g)$. That is, acts on by permuting the standard basis vectors $\{e_α\}_{α \in X} \in \text{Alg}^m$ such that

$$P(g)e_α = e_{gα} = e_α \in \{e_α\}_{α \in X},$$

where $P(g)$'s are the operators in $\text{Alg}^m$ which define the above mentioned linear representation.

**Example 3.** Let $X = \{0,1,..., m-1\}$, $Gr = \{0,1,..., m-1\}$ be the cyclic group of order $m$. Then

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} = \begin{bmatrix} P(0) \\ P(1) \\ \vdots \\ P(m-1) \end{bmatrix}$$

In particular, for $m = 2$ and $m = 3$ we have

$$P(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad P(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

In expression (7) was used linear permutation representation $P(g)$ of only one group. However, we can use others finite groups of given order $m$. Let $Gr = \{g_{α}\}_{α = 0}^{m-1}$ be a group of given order $m$ and $\{P(α)\}_{α = 0}^{m-1}$. Then

$$G^{m-1}_{α_{0},m}(U_{α_0};Gr_{α}) = \begin{bmatrix} \text{com}^{[n]}_{α_{0},0}(t_{α} | U_{α};Gr_{α}) \\ \text{com}^{[n]}_{α_{0},1}(t_{α} | U_{α};Gr_{α}) \\ \vdots \\ \text{com}^{[n]}_{α_{0},m-1}(t_{α} | U_{α};Gr_{α}) \end{bmatrix}$$

$$= \begin{bmatrix} P_{α}(g_{α}) \\ \begin{bmatrix} I_{α} \\ T_{α}^{α_{0}} \\ \vdots \\ T_{α}^{α_{0}-1} \end{bmatrix} \end{bmatrix} \cdot \text{com}^{[n]}_{α_{0},0}(t_{α} | U_{α};Gr_{α})$$

$$= \begin{bmatrix} \text{com}^{[n]}_{α_{0},1}(t_{α} | U_{α};Gr_{α}) \\ \vdots \\ \text{com}^{[n]}_{α_{0},m-1}(t_{α} | U_{α};Gr_{α}) \end{bmatrix}$$

is the Golay matrix associated with triple $\{Gr_{α}, U_{α}, U^{n_{α}}_{α_{0}}\}$, $\text{Alg}$. 

**Example 4.** For $m = 4$ we have two groups: $Z_4 = \{0, 1, 2, 3\}$ and $Z_2 \times Z_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. For both groups we have the following permutation representations:

$$P(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad P(1) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad P(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad P(3) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, we can construct two different set of Golay matrices associated with two triples

1) $Z_4, \{U_{α}, U^{n_{α}}_{α_{0}}\}, \text{Alg}$,

2) $Z_2 \times Z_2, \{U_{α}, U^{n_{α}}_{α_{0}}\}, \text{Alg}$.

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respectively.

Let \( \mathcal{G}_{m} := \{ \mathcal{G}^{1 \alpha}_{m}, \mathcal{G}^{2 \alpha}_{m}, \ldots, \mathcal{G}^{n_{\alpha}}_{m}, \mathcal{G}^{n_{\alpha}+1}_{m} \} \) be a set of arbitrary groups of given order \( m : \mathcal{G}^{1 \alpha}_{m} = \{ g^{1 \alpha}_{u,0,0} \}, \ldots, \mathcal{G}^{n_{\alpha}+1}_{m} = \{ g^{n_{\alpha}+1 \alpha}_{u,0,0} \} \). Then we can use on each \( k \)-iteration permutation representations \( \{ P_{\alpha}^{k}(g_{u}) \}_{u,0}^{n_{\alpha}+1} \) for \( \mathcal{G}^{\alpha}_{m} \). In this case, we obtain the following Golay transform

\[
\mathcal{G}^{(n_{\alpha}+1)}_{m}(U_{\alpha}; \mathcal{G}^{n_{\alpha}+1}_{m}) = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | U_{\alpha}; \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | U_{\alpha}; \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | U_{\alpha}; \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | U_{\alpha}; \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

\[
= \mathcal{P}^{m}_{\alpha}(g_{u}) = \left[ \begin{array}{c} I_{t_{u}} \\ T_{t_{u}}^{n_{\alpha}+1} \\ \vdots \\ T_{t_{u}}^{n_{\alpha}+1} \\ \end{array} \right] \quad \text{for } \mathcal{G}^{n_{\alpha}+1}_{m}
\]

It is associated with triple \( \left( \{ \mathcal{G}^{1 \alpha}_{m}, \mathcal{G}^{2 \alpha}_{m}, \ldots, \mathcal{G}^{n_{\alpha}+1}_{m} \}, \{ U^{1}_{m}, U^{2}_{m}, \ldots, U^{n_{\alpha}+1}_{m} \}, \text{Alg} \) \).

**Fast Golay transforms**

Let us consider expressions (8) and (9) for \( m = 2 \) (i.e., expressions (6) and (7) from our work [7]): and find matrix representations of these expressions. We introduce the following \( \sigma \)-parametrized \((2^{n_{\alpha}} \times 2^{n_{\alpha}})\)-matrix:

\[
\mathcal{G}^{(n_{\alpha}+1)}_{2^{\sigma}} := \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

\[
= \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

and construct the direct sum of introduced matrices

\[
\mathcal{G}^{(n_{\alpha}+1)}_{2^{\sigma}} = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

\[
= \left( \mathcal{G}^{(n_{\alpha}+1)}_{2^{\sigma}} \right) = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

\[
= \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

From (16) we see that \( \mathcal{G}^{(n_{\alpha}+1)}_{2^{\sigma}} \) represents \( \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \) in (14). It is easy to see, that

\[
\mathcal{G}^{(n_{\alpha}+1)}_{2^{\sigma}} = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] = \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

\[
= \left[ \begin{array}{c} \text{com}^{(\alpha,0)}_{u,0}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \\ \vdots \\ \text{com}^{(\alpha,1)}_{u,1}(t_{u}) | \mathcal{G}^{n_{\alpha}+1}_{m} \end{array} \right] 
\]

where
is the permutation matrix with controlling digit \( \{ t_{n+1} \} \). According to (15) the Golay matrix \( G_{2m}^{[n+1]} \) is the product of three matrices

\[
G_{2m}^{[n+1]} = \Delta \{ (-1)^{t_{n+1}} \} \left[ \delta_{n,t_{n+1}}^{(2)} \right] \left[ I_{2m} \otimes G_{2m}^{[n]} \right] = \Delta \{ (-1)^{t_{n+1}} \} \left[ \delta_{n,t_{n+1}}^{(2)} \right] \left[ I_{2m} \otimes P_{2m}^{[n+1]} \right].
\]

Where \( \Delta \{ (-1)^{t_{n+1}} \} = \text{diag} \{ (-1)^{t_{n+1}} \} \) is diagonal matrix, and \( \left[ \delta_{n,t_{n+1}}^{(2)} \right] \) has the following structure

\[
\left[ \delta_{n,t_{n+1}}^{(2)} \right] = \left[ \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{array} \right] \otimes \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{array} \right] \right] = \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{array} \right] \otimes \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{array} \right] = N_{2m}.
\]

Here \( \hat{\otimes} \) is new tensor product:

\[
\left[ I_{2m} \right] \left[ I_{2m} \right] \otimes \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{array} \right] = \left[ I_{2m} \otimes \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{array} \right] \right] = \left[ I_{2m} \otimes \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{array} \right] \right].
\]

From recurrent relation (17) we obtain

\[
G_{2m}^{[n]} = \left[ \prod_{k=2}^{m} \left[ I_{2m} \otimes \Delta \{ (-1)^{t_{n+1}} \} \cdot N_{2m} \cdot P_{2m}^{[n+1]} \right] \right] = \left[ I_{2m} \otimes \Delta \{ (-1)^{t_{n+1}} \} \right] \left[ \delta_{n,t_{n+1}}^{(2)} \right] \left[ I_{2m} \otimes G_{2m}^{[n+1]} \right].
\]

This expression represents the fast algorithm for the Golay transform.

**Example 5.**

\[
G_{2}^{[2]} = \left[ \delta_{n,t_{2}}^{(2)} \right] \left[ I_{2} \otimes \Delta \cdot N \cdot P^{[2]} \right],
\]

\[
G_{2}^{[3]} = \left[ \delta_{n,t_{3}}^{(3)} \right] \left[ I_{2} \otimes \Delta \cdot N \cdot P^{[3]} \right],
\]

\[
G_{2}^{[3]} = \left[ \delta_{n,t_{3}}^{(3)} \right] \left[ I_{2} \otimes \Delta \cdot N \cdot P^{[3]} \right].
\]
Many-parameter $m$-complementary Golay sequences and transforms
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**Conclusion and future researches**

In this paper, we have shown a new unified approach to the so-called generalized multi-parameter $m$-complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbb{Z}_2, \mathbb{F}_2, C)$, but with:

1) $(\mathbb{Z}_m, U_m, A_l)$,
2) $(\mathbb{Z}_m, \{U_{m^1}, U_{m^2}, ..., U_{m^n}\}, A_l)$,
3) $(\mathbb{Z}_m, \mathbb{G}_m, A_l)$,
4) $(\mathbb{Z}_m, \mathbb{G}_m, \mathbb{G}_m^r, \mathbb{G}_m^s, \mathbb{G}_m^t, \mathbb{U}_m, \mathbb{U}_m^r, \mathbb{U}_m^s, \mathbb{U}_m^t, A_l)$,

where $\{U_{m^1}, U_{m^2}, ..., U_{m^n}\}$ is a set of arbitrary unitary $(m \times m)$-transforms and $\{\mathbb{G}_m, \mathbb{G}_m^r, ..., \mathbb{G}_m^t\}$ is a set of arbitrary groups of given order $m$. Furthermore, we have derived demonstrated fast algorithms for Golay transforms.

We are going to use generalized multi-parameter $m$-complementary sequences as subcarriers of Intelligent OFDM telecommunication system. Most of the data transmission systems nowadays use orthogonal frequency division multiplexing telecommunication system (OFDM-TCS) based on the discrete Fourier transform (DFT) $\mathcal{F}$. The conventional OFDM will be denoted by $\mathcal{F}_N$-OFDM. Conventional OFDM-TCS makes use of signal orthogonality of the multiple sub-carriers $e^{j2\pi kn/N}$ (complex exponential harmonics). Sub-carriers $U_{k,0}^{N-1}$ form matrix of DFT $\mathcal{F} = \sum_{k=0}^{N-1} U_{k,0}^{N-1} = \sum_{k=0}^{N-1} e^{j2\pi kn/N}$.

At the time, the idea of using the fast algorithm of different orthogonal transforms $U_{k,0}^{N-1} = \sum_{k=0}^{N-1} e^{j2\pi kn/N}$ for a software-based implementation of the OFDM’s modulator and demodulator, transformed this technique from an attractive, but difficult to implement idea, into an incredibly successful story of the data transmission. OFDM-TCS, based on arbitrary orthogonal (unitary) transform $U_{k,0}^{N-1}$ will be denoted as $U_{k,0}^{N-1}$-OFDM. The idea which links $\mathcal{F}^N$-OFDM and $U_{k,0}^{N-1}$-OFDM is that, in the same manner that the complex exponentials $e^{j2\pi kn/N}$ are orthogonal to each-other, the members of a family of $U_{k,0}^{N-1}$-sub-carriers $\{U_{k,0}^{N-1}\}_{k=0}^{N-1}$ (rows of the matrix $U_{k,0}^{N-1}$) will satisfy the same property. The $U_{k,0}^{N-1}$-OFDM reshapes the multi-carrier transmission concept, by using carriers $\{U_{k,0}^{N-1}\}_{k=0}^{N-1}$ in-
Instead of OFDM’s complex exponentials \( \left\{ e^{ \frac{2 \pi i n k}{N}} \right\} \). In this paper, we propose a simple and effective anti-eavesdropping and anti-jamming Intelligent OFDM system, based on MPTs. In our Intelligent-OFDM-TCS we are going to use multi-parameter Golay transform \( G_{2q}(\varphi_1, \varphi_2, \ldots, \varphi_q) \) at the place of DFT \( F_N \). We are going to study of Intelligent-\( G_{2q}(\varphi_1, \varphi_2, \ldots, \varphi_q) \)-OFDM-TCS to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

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