Lattice Charge Overlap: Towards the Elastic Limit

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A numerical investigation of time-separated charge overlap measurements is carried out for the pion in the context of lattice QCD using smeared Wilson fermions. The evolution of the charge distribution function is examined and the expected asymptotic time behavior $\sim e^{-(E_q-m_\pi)t}$, where $t$ represents the charge density relative time separation, is clearly visible in the Fourier transform. Values of the pion form factor are extracted using point-to-smeared correlation functions and are seen to be consistent with the expected monopole form from vector dominance. The implications of these results for hadron structure calculations is briefly discussed.
The use of charge overlap distribution functions to examine qualitative features of lattice hadrons is now a well established technique [1-3]. However, there is a need to go beyond qualitative measurements to investigate whether such methods are capable of extracting physical observables in the electromagnetic sector. For this reason the Fourier transform of the distribution functions for \( \pi \) and \( \rho \) mesons were examined in Refs. [4] and [5]. It was found that the equal-time functions contain a substantial contamination of intermediate states, preventing a reliable extraction of electric and magnetic form factors at available lattice momenta.

Consider now the continuum Euclidean time-separated charge distribution for zero-momentum pions:

\[
Q^{du}_{44}(\vec{q}^2, t) \equiv \int d^3r e^{i\vec{q} \cdot \vec{r}} P^{du}_{44}(r, t),
\]

where

\[
P^{du}_{44}(r, t) \equiv \frac{1}{2m_\pi}(\pi^+(0)|T[-J^d_4(\vec{r}, t)J^u_4(0)]|\pi^+(0)),
\]

(2)

\( J^u,d_4 \) are the \( u, d \) flavor charge densities. and \( 'T' \) is time-ordering. Fig. 1 shows a symbolic representation of the measurement in Eq. (2). Assuming \( t > 0 \) and inserting a complete set of states, this can be shown to result in

\[
Q^{du}_{44}(\vec{q}^2, t) = \sum_X \frac{\langle \pi^+(0) - J^d_4(0)|X(\vec{q})\rangle\langle X(\vec{q})|J^u_4(0)|\pi^+(0)\rangle}{4E_X m_\pi} e^{-(E_{q} - m_\pi)t}.
\]

(3)

In the large Euclidean time limit, the sum reduces to a single term, and we find that

\[
Q^{du}_{44}(\vec{q}^2, t) \to \frac{(E_q + m_\pi)^2}{4E_q m_\pi} F^2_{\pi}(q^2) e^{-(E_q - m_\pi)t},
\]

(4)

where (working in the flavor SU(2) limit)

\[
F^2_{\pi}(q^2) = \frac{\pi^+(0)|(J^u_4, -J^d_4)|\pi^+(0)}{m_\pi + E_{q}}.
\]

(5)

Thus, by separating the currents in time it is in principle possible to damp out the intermediate state contributions and to measure form factors at arbitrary lattice momenta. On a finite space-time lattice, however, achieving this elastic limit is problematical. The exponential damping factor, \( e^{-(E_q - m_\pi)t} \), is
not large and the fixed time positions of the initial and final pion interpolating fields limit the possible charge density separation times. Nevertheless, this paper demonstrates that the elastic limit can essentially be achieved at low momentum values on present-sized lattices. This has important implications for hadron structure function calculations on the lattice.

We begin by examining the evolution of these distribution functions as the time separation, \( t \), between the charge densities \( J_d^4(\vec{x}, t) \) and \( J_u^4(0) \) increases. On the lattice, we measure

\[
Q_{44}^{du}(\vec{q}, t_2, t_1, t) \equiv \sum_{\vec{x}} e^{i\vec{q} \cdot \vec{x}} P_{44}^{du}(\vec{x}, t_2, t_1, t),
\]

(6)

\[
P_{44}^{du}(\vec{r}, t, t_1, t_2) \equiv \frac{\langle \text{vac} | T[-\phi^{sm}(t_2) \sum_{\vec{x}} J_d^4(\vec{x} + \vec{r}, t_1) J_u^4(\vec{x}, t) \phi^\dagger(0, 0)] | \text{vac} \rangle}{\langle \text{vac} | T[\phi^{sm}(t_2) \phi^\dagger(0, 0)] | \text{vac} \rangle},
\]

(7)

where we are now using lattice states and operators, and the fields \( \phi^{sm}(t_2) \) and \( \phi^\dagger(0, 0) \) are smeared and point pion interpolating fields, respectively. We use the exactly conserved lattice charge densities (which are non-local in time) for \( J_d^4(\vec{x} + \vec{r}, t_1) \), \( J_u^4(\vec{x}, t) \). Note that carrying out the sum on \( \vec{x} \) in (6) for all \( \vec{r} \) takes a nontrivial amount of computer time, scaling as \( N_s^2 \), where \( N_s \) is the number of spatial points in the lattice. One may show that for large time separations \( (t_2 - t_1, t_1 - t, t \gg 1) \) and in the continuum limit,

\[
Q_{44}^{du}(\vec{q}, t_1, t_2) \rightarrow Q_{44}^{du}(\vec{q}^2, t_1 - t), \quad P_{44}^{du}(\vec{r}, t_1, t_2) \rightarrow P_{44}^{du}(r, t_1 - t).
\]

(8)

The numerical results for this study are on 12 quenched \( \beta = 6.0 \) configurations of a \( 16^3 \times 24 \) lattice at \( \kappa = .154 \) (the largest \( \kappa \) value studied in Refs. [5] and [7].) The pion interpolating sources were located at time slices 4 and 21 and the time-separated current densities were positioned as symmetrically possible in time between these sources. Two quark propagators per configuration, with origins at the positions of the interpolating sources, were necessary to reconstruct these relative overlap functions. Charge density self-contractions have been neglected in forming these quantities.

Fig. 2 shows a log_{10} plot of the spatial evolution of the charge overlap distribution function, \( P_{44}^{du} \), as the relative time separation between the \( d, u \) current densities is increased. It is seen that the distribution becomes flatter as the relative time is increased; there is also a substantial change in the shape. Points on the periodic lattice boundary are given by the filled-in
squares, which are seen to have their values raised. Note that these functions have been calculated using point-to-smeared correlation functions; the quarks are smeared over the entire volume of the lattice at time slice 21 using the lattice Coulomb gauge [8] to produce zero-momentum quark and hadron fields.

Fig. 3 represents a log$_{10}$-plot of the Fourier transform of these distribution functions, $Q^{du}_{44}$, at the two lowest lattice spatial momentum values, $|\vec{q}| = \pi/8$, $\sqrt{2}(\pi/8)$ as a function of relative time separation between current densities. The solid lines shown in this figure come from the expected asymptotic exponential falloff specified by Eq. (4), using the (smeared) $\kappa = .154$ data from Table 1 ($m_\pi = .369, m_\rho = .46$) of Ref. [5], assuming the vector dominance form for the pion form factor: $F_\pi(q^2) = 1/(1 + q^2/m_\pi^2)$. (We also assume the continuum relation $E_q = (m_\pi^2 + \vec{q}^2)^{1/2}$.) Actually shown in this figure are results for both point-to-smeared (•) as well as smeared-to-smeared (□) correlation functions. In all cases, the expected functional dependence $\sim e^{-(E_q-m_\pi)t}$ is present, indicating that by time step 7 or 8 single exponential behavior has emerged. This is similar to the number of time steps needed in hadron spectrum calculations. This behavior is remarkable because although we are damping out intermediate states as $t$ increases, we are also moving closer to possible contaminations from the fixed interpolating fields at either time end. In fact, we do not see any indications of such contamination in the data. These correlation functions are also unusual because the asymptotic approach is from below, indicating damping of negative terms in the $d, u$ correlation function. Ref. [5] indicates that these intermediate states, at least in the continuum limit, are primarily positive G-parity states [9].

Although the point-to-smeared and smeared-to-smeared results exhibit essentially overlapping error bars, the smeared-to-smeared values are systematically low compared to the point-to-smeared results, indicating a slight dependence on the form of the interpolating field used. However, the results of Ref. [5] indicate that such a dependence decreases as $\kappa \to \kappa_{\text{critical}}$, that is, as the physical regime is approached.

In looking back at Fig. 2, it is clear that the spatial distributions, $P^{du}_{44}(\vec{r})$, have substantial contaminations near the periodic lattice boundaries from the surrounding image sources. Why is it then that we have not attempted to do image corrections on the Fourier transform, $Q^{du}_{44}(\vec{q})$, of these distributions? Consider an infinite spatial grid and associate the value of a function, $F(\vec{x})$,
with each point $\vec{x} \in \{X\}$ of the grid. Then, by doing a spatial translation on points outside the given primitive cell (of size $L^3$) to equivalent points, associated with the same phase factor $e^{i\vec{q} \cdot \vec{x}}$, inside the cell $\vec{x} \in \{x\}$, it can be shown that

$$\sum_{\vec{x} \in \{X\}} e^{i\vec{q} \cdot \vec{x}} F(\vec{x}) = \sum_{\vec{x} \in \{x\}} e^{i\vec{q} \cdot \vec{x}} \hat{F}(\vec{x}),$$

(9)

where for $\vec{x} \in \{x\}$

$$\hat{F}(\vec{x}) \equiv \sum_{\vec{n}} F(\vec{x} + \vec{n}L).$$

(10)

($\vec{n} = (n_x, n_y, n_z)$, where $n_{x,y,z}$ are integers.) The terms with $\vec{n} \neq 0$ are the image contributions. Eq. (9) says that the discrete Fourier transform of $F(\vec{x})$ on the infinite grid is the same as the transform of $\hat{F}(\vec{x})$ in the primitive cell for $\vec{q}$ values allowed in the original primitive cell. In our case this means there are no image corrections to $Q_{44}(\vec{q})$, assuming the distributions $P_{44}(\vec{r})$ are strictly periodic.

By fitting time steps 7 through 10 of Fig. 3 with a single exponential of known slope and removing the kinematical factor in Eq. (4), a measurement of $F_\pi(q^2)$ can be made. The results for the two calculated $q^2$ values using point-to-smeared correlation functions are shown in Fig. 4. (The error bars in Figs. 3 and 4 were determined from first and second-order single elimination jackknives, respectively.) The values found are consistent within errors with vector dominance, shown as the solid line. Also shown in this figure are the results from a previous three-point-function simulation of the pion form factor \cite{11}. Comparison of the two types of results shows that the error bars of these different techniques are of the same order of magnitude for similar numbers of configurations. (Note the different $\beta$ values and lattice sizes of the two simulations, however.)

The significance of these results for hadron structure calculations goes beyond the specific findings discussed here for form factors. The ability to reach the elastic limit for hadron four-point-functions is crucial in attempting to perform direct simulations of hadron structure functions. Up to the present, attention has been focused on the moments of such functions, which are given by the operator product expansion. These expansions are based upon separation of the short-distance physics, calculated perturbatively, from the long-distance part, evaluated on the lattice (or by other techniques) in the form of certain operator expectation values. The presence on the lattice, in the continuum limit, of all appropriate physical scales, combined with the
ability to make contact with the elastic limit of charge overlap four-point-functions, demonstrates that direct simulations of structure functions are feasible with current technology, at least at low lattice momenta.

What additional quantities need to be calculated in order to initiate such a program? Clearly, the overlap function considered here, $Q_{44}^{du}(q^2, t)$ is a sub-dominant quantity for structure functions; the short-distance physics will mainly be contained in the same-flavor Fourier transform: $Q_{44}^{uu}(q^2, t)$ or $Q_{44}^{dd}(q^2, t)$. Calculating this piece involves putting both electromagnetic current densities on the same quark line. This type of zero-momentum four-point function is difficult to simulate on the lattice, but can be obtained by combining fixed-source and quark-smearing techniques and is currently under investigation. In addition, the the current-current overlap distribution function, $(\pi(0)| \sum \ell J_\ell(\vec{r}, t) J_\ell(0)|\pi(0))$, for both flavor-diagonal and nondiagonal currents is necessary in order to complete measurements of the two standard structure functions, $W_1$ and $W_2$. Of course it is the proton, not the pion, which is of most phenomenological interest. However, there is no barrier to applying these techniques to the proton as well, as long the lattices used are able to contain all the necessary length scales and the elastic limit in all these various four-point-functions can be demonstrated.

To summarize, this paper shows that the elastic limit of $d, u$ charge overlap distribution functions, in the context of the pion, can essentially be achieved at low momentum values on present-sized lattices. We do not claim that all the systematics of such measurements are yet completely understood, but the message that this limit is very close is extremely encouraging to attempts to provide direct calculations of structure functions from the fundamental QCD Lagrangian.

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References

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[6] In a high energy inelastic context, the discrete sum \( \sum_X \) in Eq. (1) would be replaced by an integral over the appropriate density of states.

[7] W. Wilcox, T. Draper and K. F. Liu, Chiral Limit of Nucleon Electromagnetic Form Factors, Phys. Rev. D (in press).

[8] These are the same Coulomb gauge-fixed configurations used in Refs. [4] and [7]. No attempt was made to eliminate possible Coulomb gauge Gribov copies.

[9] Note that an alternate, momentum-smeared (no \( \vec{x} \) sum in Eq. (7)) \( d, u \) correlation function discussed in Ref. [3] never exhibits the crucial exponential behavior \( \sim e^{-(E_q-m_\pi)t} \) as the charge densities are separated in time \( t \); it is no longer considered a viable measurement of the continuum distribution \( P^{du}_{4t}(r, t) \) by the present author.

[10] T. Draper, R. M. Woloshyn, W. Wilcox and K. F. Liu, Nucl. Phys. B318 (1989) 319.
Figure Captions

Fig. 1: Symbolic representation of the $d, u$ charge overlap measurement. The pion sources, shown as circles, are at fixed time positions in the lattice.

Fig. 2: Evolution of the lattice $d, u$ spatial charge density correlation function, $P_{44}^{du}$, for pions as the time separation, $t_1 - t$, between $J^d_4(\vec{r}, t_1)$, $J^u_4(\vec{0}, t)$ operators varies, as a function of non-equivalent lattice $r$ values. (a) $t_1 - t = 0$. (b) $t_1 - t = 3$. (c) $t_1 - t = 6$. (d) $t_1 - t = 10$. Filled-in squares identify points on the periodic lattice boundary.

Fig. 3: Log$_{10}$ plot of the Fourier transform of the spatial charge density correlation function, $Q_{44}^{du}$, as a function of time separation $t_1 - t$ between $J^d_4(\vec{r}, t_1)$, $J^u_4(\vec{0}, t)$ operators. The upper points give the $|\vec{q}| = \pi/8$ results, the lower points correspond to $|\vec{q}| = \sqrt{2}(\pi/8)$. Results are given for both point-to-smeared correlation functions (◇) as well as smeared-to-smeared correlation functions (□).

Fig. 4: Pion form factor, $F_\pi$, as a function of $q^2/m_\rho^2$. The solid line is the monopole form from vector dominance. The data shown are the present results from point-to-smeared correlation functions (◇) and previous results on the pion form factor from Ref. [10] (□).
