$p$-Adic open string amplitudes with Chan-Paton factors coupled to a constant $B$-field

H. García-Compeán and Edgar Y. López

Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Departamento de Física, P.O. Box 14-740, CP. 07000, México D.F., México

W. A. Zúñiga-Galindo

Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Departamento de Matemáticas, Unidad Querétaro, Libramiento Norponiente #2000, Fracc. Real de Juriquilla. Santiago de Querétaro, Qro. 76230, México

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Abstract

We establish rigorously the regularization of the $p$-adic open string amplitudes, with Chan-Paton rules and a constant $B$-field, introduced by Goshal and Kawano. In this study we use techniques of multivariate local zeta functions depending on multiplicative characters and a phase factor which involves an antisymmetric bilinear form. These local zeta functions are new mathematical objects. We attach to each amplitude a multivariate local zeta function depending on the kinematics parameters, the $B$-field and the Chan-Paton factors. We show that these integrals admit meromorphic continuations in the kinematic parameters, this result allows us to regularize the Goshal-Kawano amplitudes, the regularized amplitudes do not have ultraviolet divergencies. Due to the need of a certain symmetry, the theory works only for prime numbers which are congruent to 3 modulo 4. We also discuss the limit $p \rightarrow 1$ in the noncommutative effective field theory and in the Goshal-Kawano amplitudes. We show that in the case of four points, the limit $p \rightarrow 1$ of the regularized Goshal-Kawano amplitudes coincides with the Feynman amplitudes attached to the limit $p \rightarrow 1$ of the noncommutative Gerasimov-Shatashvili Lagrangian.

*Electronic address: compean@fis.cinvestav.mx, elopez@fis.cinvestav.mx. The work of E. López was supported by a CONACyT fellowship.

†Electronic address: wazuniga@math.cinvestav.edu.mx. This author was partially supported CONACyT grant 250845.
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I. INTRODUCTION

The deep connections between $p$-adic analysis and physics are a natural consequence of the emergence of ultrametricity in physics, which means the occurrence of ultrametric spaces in physical models, see e.g. [1–9] and the references therein. The existence of a Planck length implies that the spacetime considered as a topological space is completely disconnected, the points (which are the connected components) play the role of spacetime quanta. This is precisely the Volovich conjecture on the non-Archimedean nature of the spacetime below the Planck scale, [2, 3], [8, Chapter 6]. On the other hand, the paradigm in physics of complex systems (for instance proteins) asserting that the dynamics of a such system can be modeled as a random walk on the leaves on a rooted tree (a finite ultrametric space), which is constructed from the energy landscape. Mean-field approximations of these models drive naturally to models involving $p$-adic numbers, see e.g. [7, 10–12], and the references therein.

In the last forty years, the above mentioned ideas have motivated many developments in quantum field theory and string theory, see e.g. [1, 5, 13, 14], and more recently, [15–28].

In string theory, the scattering amplitudes are obtained integrating over the moduli space of Riemann surfaces. Even for tree-level amplitudes (on the sphere for closed strings and on
the disk for open strings) $N$-point amplitudes are difficult to compute beyond four points. Moreover, the convergence region of these integrals is not evident by itself [29, 30]. Recently, in [31] was established in a rigorous mathematical way that Koba-Nielsen amplitudes are bona fide integrals, which admit meromorphic continuations when considered as complex functions of the kinematic parameters.

String theory with a $p$-adic world-sheet was proposed and studied by the first time in [32]. Later this theory was formally known as $p$-adic string theory. The Adelic scattering amplitudes which are related with the Archimedean ones were studied in [33]. The tree-level string amplitudes were explicitly computed in the case of $p$-adic string world-sheet in [34] and [35]. These amplitudes can be formally obtained from a suitable action using general principles [36]. A general treatment starting by describing a discrete field theory on a Bruhat-Tits tree and obtaining the tree-level string amplitudes ([34]) was established in [37]. Similarly as in the standard string theory, in $p$-adic string theories it is difficult to determine the convergence region in the momentum space, however this was done precisely for the $N$-point tree amplitudes in [38]. In this article we show (in a rigorous mathematical way) that the $p$-adic open string $N$-point tree amplitudes are bona fide integrals that admit meromorphic continuations as rational functions, by relating them with multivariate local zeta functions (also called multivariate Igusa local zeta functions [39–41]).

In $p$-adic string theory the limit $p \to 1$ is very intriguing since it seems to be related to the real versions of these theories [36, 42]. This limit is special since the effective theory shows that it is related to physical string theories as a boundary string field theory [43]. Another interpretation of the limit $p \to 1$ was given in terms of the renormalization group scaling transformation in the Bruhat-Tits tree for some suitable $p$ [44]. In the worldsheet theory we cannot forget the nature of $p$ as a prime number, thus the analysis of the limit is more subtle. The correct way of taking the limit $p \to 1$ involves the introduction of finite extensions of the $p$-adic field $\mathbb{Q}_p$. In [45] the limit $p \to 1$ was discussed at the tree-level string amplitudes. We provided a rigorous definition of this limit using the theory of topological zeta functions due to Denef and Loeser [46, 47].

In ordinary string theory the effective action for bosonic open strings in gauge field backgrounds was discussed many years ago in [48]. The analysis incorporating a Neveu-Schwarz $B$ field in the target space leads to a noncommutative effective gauge theory on the world-volume of D-branes [49]. The study of the $p$-adic open string tree amplitudes
including Chan-Paton factors was started in [34]. However the incorporation of a $B$-field in the $p$-adic context and the computation of the tree level string amplitudes was discussed in [50, 51]. In these works was reported that the tree-level string amplitudes are affected by a noncommutative factor. In [50] Ghoshal and Kawano introduced new amplitudes involving multiplicative characters and a noncommutative factor, these amplitudes coincide with the ones obtained directly from the noncommutative effective action [52].

In the present article we study the $p$-adic string amplitudes, with Chan-Paton rules and a constant $B$-field, introduced in [50], by using techniques of multivariate local zeta functions. The $N$-point, $p$-adic, open string amplitudes, with Chan-Paton rules in a constant $B$-field, have the form

$$\int_{\mathbb{Q}_p^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|_p^{k_i k_j} H_\tau(x_i - x_j) \exp \left\{ -\frac{i}{2} \left( \sum_{1 \leq i < j \leq N} (k_i \theta k_j) \text{sgn}_\tau(x_i - x_j) \right) \right\} \prod_{i=1}^{N} dx_i,$$

where $N \geq 4$, $k = (k_1, \ldots, k_N)$, $k_i = (k_{0,i}, \ldots, k_{l_i})$, $i = 1, \ldots, N$, is the momentum vector of the $i$-th tachyon vertex operator (with Minkowski product $k_i k_j = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{l_i}k_{l,j}$) obeying

$$\sum_{i=1}^{N} k_i = 0, \quad k_i k_i = 2 \text{ for } i = 1, \ldots, N,$$ (2)

$H_\tau(x_i) = \frac{1}{2}(1 + \text{sgn}_\tau(x))$, $\text{sgn}_\tau(x)$ is a $p$-adic version of the sign function, $\theta$ is a fixed antisymmetric bilinear form, and $\prod_{i=2}^{N-2} dx_i$ is the normalized Haar measure of $\mathbb{Q}_p^{N-3}$.

Unfortunately, this theory is not invariant under projective Möbius transformations and consequently the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$ can not be carried out. This is a consequence of the fact that $\mathbb{Q}_p$ is not an ordered field. Anyway, in [50] the authors assumed a such normalization, which is equivalent to assume that the vertex operators are inserted in the boundary of the Bruhat-Tits tree at ‘non-generic points’, taking the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$, the amplitude takes the form

$$A^{(N)}(k, \theta, \tau) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|_p^{k_i k_i} |1 - x_i|_p^{k_{N-1} k_i} H_\tau(x_i) H_\tau(1 - x_i)$$

$$\times \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{k_i k_j} H_\tau(x_i - x_j)$$

$$\times \exp \left\{ -\frac{i}{2} \left( \sum_{1 \leq i < j \leq N-1} (k_i \theta k_j) \text{sgn}_\tau(x_i - x_j) \right) \right\} \prod_{i=2}^{N-2} dx_i.$$

(3)
We have called such integrals Ghoshal-Kawano amplitudes. The main goal of this article is the study of the amplitude $A^{(N)}(k, \theta, \tau)$ using twisted multivariate Igusa’s local zeta functions. We attach to $A^{(N)}(k, \theta, \tau)$ the following Igusa type integral:

$$Z^{(N)}(k, \theta, \tau) = \int_{\mathbb{Q}_p^{N-3}} \prod_{i=2}^{N-2} |x_i|^{|s_{ij}|} \prod_{2 \leq \iota < \iota \leq N-2} |x_i - x_j|^{|s_{ij}|} \prod_{i=2}^{N-2} dx_i,$$  

\begin{equation}
\times \prod_{2 \leq \iota < \iota \leq N-2} (1 - x_{\iota}^{|s_{ij}|}) H_\tau(x_{\iota}) H_\tau(1 - x_{\iota})
\end{equation}

where the $s_{ij}$ are complex symmetric variables and the $\tilde{s}_{ij}$ are real antisymmetric variables.

We have called integrals $Z^{(N)}(k, \theta, \tau)$ Ghoshal-Kawano local zeta functions. As a consequence on the presence of the Chan-Paton factors and the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$, the integration in (4) is actually on $Z_p^{N-3}$ (the $N-3$-dimensional unit ball). This fact implies that turning off the background $B$-field amplitude $A^{(N)}(k, \theta, \tau)$ does not reduce to the $p$-adic open string amplitude at the tree level. This fact was already noticed in [50] in the special case $N = 4$. We show that integrals $Z^{(N)}(s, \tilde{s}, \tau, \theta)$ can be expressed as finite sums of twisted multivariate Igusa’s local zeta functions, and by using the results of [39, 53], we establish that $Z^{(N)}(s, \tilde{s}, \tau, \theta)$ admits a meromorphic continuation as a rational function in the variables $p^{-s_{ij}}, p^{-(s_{ij}-1)}, p^{-s_{ij}}$. Furthermore, $Z^{(N)}(s, \tilde{s}, \tau, \theta)$ is holomorphic in

\begin{equation}
\bigcap_{k \in T} \left\{ s_{ij} \in \mathbb{C}_d; \sum_{i,j \in M} N_{i,j,k} \Re(s_{ij}) + \gamma_k > 0, \text{ for } k \in T \right\},
\end{equation}

where $N_{i,j,k} \in \mathbb{N}$, $\gamma_k \in \mathbb{N} \setminus \{0\}$, and $M, T$ are finite sets, and the real parts of its poles belong to the finite union of hyperplanes of type

$$\mathcal{H} = \left\{ s_{ij} \in \mathbb{C}_d; \sum_{i,j \in M} N_{i,j,k} \Re(s_{ij}) + \gamma_k = 0, \text{ for } k \in T \right\}.$$

We regularize the amplitude $A^{(N)}(k, \theta, \tau)$ by redefining it as

$$A^{(N)}(k, \theta, \tau) = Z^{(N)}(s, \tilde{s}, \tau, \theta) \bigg|_{s_{ij}=k, \tilde{s}_{ij}=k, \theta=k, k_j}$$

in this way $A^{(N)}(k, \theta, \tau)$ is a well-defined meromorphic function of the kinematics parameters $k, k_j$, which agrees with integral (3), if it exists. As a consequence of the description of the poles of $Z^{(N)}(s, \tilde{s}, \tau, \theta)$, $A^{(N)}(k, \theta, \tau)$ is defined for arbitrary large momenta, since in (5) the values $k, k_j$ can take arbitrarily large values. This fact is not valid for the $p$-adic
Koba-Nielsen amplitudes, see [38], and [31], since Ghoshal-Kawano amplitudes are supposed to be restrictions of the $p$-adic open amplitudes at the tree level, we conclude that the normalization $x_1 = 0$, $x_{N-1} = 1$, $x_N = \infty$ is not possible in the presence of a background $B$-field. In a forthcoming article we expect to study amplitudes (1). It is worth to mention here that the Ghoshal-Kawano local zeta functions are new Igusa-type integrals coming from $p$-adic string theory.

The construction of a physical theory over a $p$-adic spacetime (worldsheet in our case) raises the question about the physical meaning of the prime $p$. The spacetime is a quadratic space $(\mathbb{Q}_p^N, q)$, where $q$ is a quadratic form, and consequently, the spacetime depends on the pair $(p, q)$. In this article, we require $p \equiv 3 \text{ mod } 4$ in order to have the symmetry $\text{sgn}_\tau(-x) = -\text{sgn}_\tau(-x)$.

This article is organized as follows. In section II we study the limit $p \to 1$ in the noncommutative version of the effective action discussed in [52]. We describe the noncommutative version of the Gerasimov-Shatashvili action and found explicitly its four-point amplitudes. In section III we review the basic aspects of the twisted, multivariate Igusa’s local zeta functions. The local zeta functions required here are a variation of the ones considered in [53]. Sections IV-V are dedicated to establish the meromorphic continuation of Ghoshal-Kawano local zeta functions. Sections VI and VII are devoted to give the explicit calculation for the 4-point and 5-point amplitudes. The 4-point amplitude was already obtained by Ghoshal and Kawano in [50] under certain hypotheses and the 5-point amplitude is new. In section VIII we compute the $p \to 1$ limit of the $p$-adic 4-point and 5-point amplitudes. We verified that the $p \to 1$ limit of 4-point amplitude coincides with the Feynman amplitude computed from the noncommutative Gerasimov-Shatashvili action in section II. The final remarks are collected in section IX. Finally in the Appendix, we review the basic aspects of the $p$-adic analysis, and introduce some notation and conventions used along this article.

II. THE LIMIT $p \to 1$ IN THE EFFECTIVE ACTION WITH A B-FIELD

A. The limit $p \to 1$ in the noncommutative effective action

In [52], it was considered a noncommutative action as the effective action of the theory of $p$-adic open strings with a $B$-field. The corresponding action in the $D$-dimensional spacetime
is given by

\[ S(\phi) = \frac{1}{g^2 p - 1} \int d^D x \left( -\frac{1}{2} \phi \star p^{-\frac{1}{2}} \Delta \phi + \frac{1}{p + 1} (\star \phi)^p \right), \]

(6)

where \( g \) and \( \Delta \) are the coupling constant and the Laplacian, respectively, and \((\star \phi)^p\) is defined by \( \phi \star \phi \star \cdots \star \phi \) \( p \)-times. Here \( \star \) is the Moyal star product, which is defined for any suitable pair of smooth functions \( f \) and \( g \) as

\[
(f \star g)(x) = \exp \left( \frac{i}{2} \theta^{\mu \nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu} \right) f(x + y) g(x + z) \bigg|_{y=z=0}.
\]

The corresponding equation of motion is given by

\[ p^{-\frac{1}{2}} \Delta \phi = (\star \phi)^p. \]

(7)

The solutions of this equation are defined in the target space \( \mathbb{R}^D \), where \( p \) plays the role of a real parameter. In particular the limit \( p \) approaches to one makes sense.

Now following [42], by considering the Taylor expansion of \( \exp(-\frac{1}{2} \Delta \log p) \) and \( \exp(p \log(\star \phi)) \) at \( p = 1 \), and keeping only the linear term, we get

\[ \Delta \phi = -2 \phi \star \log(\star \phi), \]

(8)

where \( \log(\star \phi) = \phi - \frac{1}{2} \phi \star \phi + \frac{1}{3} \phi \star \phi \star \phi - \cdots \). Thus the heuristic \( p \to 1 \) limit leads to a noncommutative version of the Gerasimov and Shatashvili Lagrangian:

\[
S(\phi) = \int d^D x \left( (\partial \phi)^2 - V(\star \phi) \right),
\]

(9)

where

\[ V(\phi) = (\star \phi)^2 \star \log \left[ \frac{(\star \phi)^2}{e} \right]. \]

In noncommutative field theory, it is well known that the nontrivial noncommutative effect comes from the potential energy of the Lagrangian. The propagators associated to the kinetic energy of the Lagrangian are the same as the ones of the commutative theory. Thus the free Lagrangian with an external source \( J(x) \) is

\[ S_0(\phi) = \int d^D x [(\partial \phi)^2 + \phi^2(x) + J(x) \phi(x)]. \]

The propagators are given by \( x_{ij} = \frac{1}{k_i \cdot k_{j+1}} \), where \( k_i \) with \( i = 1, \ldots, N \) are the external momenta of the particles. The Feynman rule for the interaction vertex can be obtained in the noncommutative theory by considering the cubic, quartic, etc. interaction terms and computing the correlation functions, see for instance, [54, 55].
B. Amplitudes from the noncommutative Gerasimov-Shatashvili Lagrangian

In this subsection we show how to extract the four-point amplitudes from the noncommutative Gerasimov-Shatashvili Lagrangian (9). In order to do that, we first require to study the interacting theory. The generating functional of the correlation function for the free theory is given by

\[ Z_0[J] = \mathcal{N} [\det(\Delta - 1)]^{-1/2} \exp \left\{ -\frac{i}{2\hbar} \int d^Dx \int d^Dx' J(x) G_F(x - x') J(x') \right\}, \]

where \( G_F(x - x') \) is the Green function of time-ordered product of two fields of the theory, \( \mathcal{N} \) is a normalization constant, \([\det(\Delta - 1)]^{-1/2}\) is a suitable regularization of the divergent determinant bosonic operator. The noncommutative action is given by

\[ S(\phi) = \int d^Dx \left[ (\partial \phi)^2 + 1 - U(\phi) \right], \quad (10) \]

where \( U(\phi) = 2(\phi)^2 \ast \log(\phi) \). We expand \( U(\phi) \) in Taylor series as follows:

\[ U(\phi) = A\phi \ast \phi + B\phi \ast \phi \ast \phi + C\phi \ast \phi \ast \phi \ast \phi + \cdots, \quad (11) \]

where \( A, B \) and \( C \) are certain real constants.

The generating \( Z[J] \) functional incorporating the interaction is given by

\[ Z[J] = \exp \left\{ -\frac{iB}{\hbar} \int d^Dx \left( -i\hbar \frac{\delta}{\delta J(x)} \right) \ast \left( -i\hbar \frac{\delta}{\delta J(x)} \right) \ast \left( -i\hbar \frac{\delta}{\delta J(x)} \right) \right\} Z_0[J]. \quad (12) \]

We are interested in checking whether connected tree-level scattering amplitudes of this theory match exactly with the corresponding \( p \)-adic amplitudes in the limit when \( p \) tends to one. The computation of the field theory performed here will be compared to the computation of the \( p \)-adic string amplitudes at section [VIII] (9).

C. Four-point amplitudes

In this subsection we consider the quartic term from the potential (11). The expansion of the exponential function of this term in the interacting generating functional is expressed as

\[ Z[J] = \cdots - i\hbar \delta^3 \int d^Dx \left\{ \left( \frac{\delta}{\delta J(x)} \right) \ast \left( \frac{\delta}{\delta J(x)} \right) \ast \left( \frac{\delta}{\delta J(x)} \right) \ast \left( \frac{\delta}{\delta J(x)} \right) \right\} Z_0[J] + \cdots \]
\[= \cdots - i \hbar^3 \lim_{x = y_1 = y_2 = y_3 = y_4} \lim_{w_1 = w_2 = w_3 = w_4 = 0} \int d^D y_1 d^D y_2 d^D y_3 d^D y_4 \]
\[\times \exp \left\{ \frac{i}{2} \frac{\partial w_{\mu_1}}{\partial w_1^{\mu_1}} \frac{\partial}{\partial w_2^{\nu_1}} \right\} \exp \left\{ \frac{i}{2} \frac{\partial w_{\mu_2}}{\partial w_3^{\mu_2}} \frac{\partial}{\partial w_4^{\nu_2}} \right\} \]
\[\times \left\{ \left( \frac{\delta}{\delta J(y_1 + w_1)} \right) \left( \frac{\delta}{\delta J(y_2 + w_2)} \right) \left( \frac{\delta}{\delta J(y_3 + w_3)} \right) \left( \frac{\delta}{\delta J(y_4 + w_4)} \right) \right\} Z_0[J]. \quad (13)\]

A straightforward computation of the 4-point vertex yields
\[\delta^4 Z[J] \left|_{J=0} \right. \]
\[= \cdots - 128 \times i \hbar^3 \int d^D x \left\{ \cos \left( \frac{\partial_1 \theta_2}{2} \right) \cos \left( \frac{\partial_3 \theta_4}{2} \right) + \cos \left( \frac{\partial_1 \theta_3}{2} \right) \cos \left( \frac{\partial_2 \theta_4}{2} \right) \right\} \left[ - \frac{i}{2\hbar} G_F(x - x_1) \right] \left[ - \frac{i}{2\hbar} G_F(x - x_2) \right] \]
\[\times \left[ - \frac{i}{2\hbar} G_F(x - x_3) \right] \left[ - \frac{i}{2\hbar} G_F(x - x_4) \right] + \cdots, \quad (14)\]

where \( G_F(x - y) \) is the propagator and \( \partial_{1,2,3,4} \) are the partial derivative with respect to the coordinates \( x_1, x_2, x_3 \) and \( x_4 \) respectively.

The interaction term \( B(\star \phi)^3 \) in the Lagrangian has also a non-vanishing contribution to the 4-points tree amplitudes at the second order in perturbation theory. They are described by Feynman diagrams with two vertices located at points \( y \) and \( z \) connected by a propagator \( G_F(y - z) \) and with two external legs attached to each vertex. In this case the amplitude is computed from the relevant part of the generating functional
\[ Z[J] = \cdots + B^2 \hbar^4 \int d^D y \int d^D z \left( \star \frac{\delta}{\delta J(y)} \right)^3 \left( \star \frac{\delta}{\delta J(z)} \right)^3 Z_0[J] + \cdots. \quad (15)\]

This expression can be written explicitly in terms of the Moyal product as
\[ Z[J] = \cdots + B^2 \hbar^4 \lim_{y = y_1 = y_2 = y_3} \lim_{z = z_1 = z_2 = z_3} \lim_{w_1 = w_2 = w_3 = w_4 = 0} \int d^D y_1 d^D y_2 d^D y_3 d^D z_1 d^D z_2 d^D z_3 \]
\[\times \exp \left\{ \frac{i}{2} \frac{\partial w_{\mu_1}}{\partial w_1^{\mu_1}} \frac{\partial}{\partial w_2^{\nu_1}} \right\} \exp \left\{ \frac{i}{2} \frac{\partial w_{\mu_2}}{\partial w_3^{\mu_2}} \frac{\partial}{\partial w_4^{\nu_2}} \right\} \]
\[\times \left\{ \left( \frac{\delta}{\delta J(y_1 + w_1)} \right) \left( \frac{\delta}{\delta J(y_2 + w_2)} \right) \left( \frac{\delta}{\delta J(y_3 + w_3)} \right) \right\} \left\{ \left( \frac{\delta}{\delta J(z_1 + w_3)} \right) \left( \frac{\delta}{\delta J(z_2 + w_4)} \right) \left( \frac{\delta}{\delta J(z_3 + w_4)} \right) \right\} Z_0[J] + \cdots. \]
The connected 4-point amplitudes at the second order from the cubic interaction $B\phi^3$ yields

$$\frac{\delta^4 Z[J]}{\delta J(x_1) \delta J(x_2) \delta J(x_3) \delta J(x_4)} \Bigr|_{J=0} = \cdots + 128 \times B^2 \hbar^4 \int d^D y \int d^D z \left[ -\frac{i}{2\hbar} G_F(y-z) \right]$$

$$\times \left\{ \cos \left( \frac{\partial_1 \partial_2}{2} \right) \cos \left( \frac{\partial_3 \partial_4}{2} \right) \right\}$$

$$\times \left\{ \left[ -\frac{i}{2\hbar} G_F(y-x_1) \right] \left[ -\frac{i}{2\hbar} G_F(y-x_2) \right] \left[ -\frac{i}{2\hbar} G_F(z-x_3) \right] \left[ -\frac{i}{2\hbar} G_F(z-x_4) \right] \right\}$$

$$+ \left\{ \left[ -\frac{i}{2\hbar} G_F(z-x_1) \right] \left[ -\frac{i}{2\hbar} G_F(z-x_2) \right] \left[ -\frac{i}{2\hbar} G_F(y-x_3) \right] \left[ -\frac{i}{2\hbar} G_F(y-x_4) \right] \right\}$$

$$+ \cos \left( \frac{\partial_1 \partial_3}{2} \right) \cos \left( \frac{\partial_2 \partial_4}{2} \right)$$

$$\times \left\{ \left[ -\frac{i}{2\hbar} G_F(z-x_1) \right] \left[ -\frac{i}{2\hbar} G_F(y-x_2) \right] \left[ -\frac{i}{2\hbar} G_F(y-x_3) \right] \left[ -\frac{i}{2\hbar} G_F(z-x_4) \right] \right\}$$

$$+ \left\{ \left[ -\frac{i}{2\hbar} G_F(y-x_1) \right] \left[ -\frac{i}{2\hbar} G_F(z-x_2) \right] \left[ -\frac{i}{2\hbar} G_F(y-x_3) \right] \left[ -\frac{i}{2\hbar} G_F(y-x_4) \right] \right\} \right\} + \cdots \quad (16)$$

This total amplitude corresponds exactly to the sum of the partial amplitudes associated to the channels $s, t$ and $u$. Expression (16) and the 4-point vertex (14) constitute the tree-level amplitudes arising in the 4-point $p$-adic amplitudes in the limit $p \to 1$. In section VIII we show that the limit $p \to 1$ in the $p$-adic Ghosal-Kawano amplitudes exists and it is given precisely by this heuristic limit. Moreover, five-point non-commutative amplitudes (and higher-order amplitudes) in the limit $p \to 1$, can be computed following a similar procedure but it will not be performed here.

III. MULTIVARIATE LOCAL ZETA FUNCTIONS

For the notation and the definition of basic objects such as multiplicative characters, sign functions, Haar measure, etc., the reader may consult the Appendix. In this section we review some basic aspects of the twisted multivariate local zeta functions. The meromorphic continuation of the local zeta functions play a central role in sections IV and V.
Let $f_1(x), \ldots, f_m(x) \in \mathbb{Q}_p[x_1, \ldots, x_n]$ be non-constant polynomials, we denote by $D := \bigcup_{i=1}^m f_i^{-1}(0)$ the divisor attached to them. Let $\chi_1, \ldots, \chi_m$ be multiplicative characters. We set $f := (f_1, \ldots, f_m)$, $\chi := (\chi_1, \ldots, \chi_m)$, and $s := (s_1, \ldots, s_m) \in \mathbb{C}^m$. The multivariate local zeta function attached to $(f, \chi, \Theta)$, with $\Theta$ a test function (i.e. a locally constant function with compact support), is defined as

$$Z_{\Theta}(f, \chi, s) = \int_{\mathbb{Q}_p^n \setminus D} \Theta(x) \prod_{i=1}^m \chi_i(ac(f_i(x))) |f_i(x)|_p^{s_i} \prod_{i=1}^n dx_i,$$  \hspace{1cm} (17)

with $\text{Re}(s_i) > 0$ for all $i$. Integrals of type (17) are analytic functions, and they admit meromorphic continuations to the whole $\mathbb{C}^m$, see also [39]. More precisely, the integral $Z_{\Theta}(f, \chi, s)$ admits a meromorphic continuation as a rational function in $p^{-s_1}, \ldots, p^{-s_m}$. Let us emphasize that the notation $\chi_i(ac(x))$, $x \neq 0$, means that character $\chi_i$ depends only on the angular component of $x$, see Appendix.

We need a slightly variation of the Loeser result [53, Théorème 1.1.4.], more precisely, when each $\chi_i \circ ac$ is the trivial character $\chi_{\text{triv}}(x)$ or $\text{sgn}_r(x)$. We denote by $\chi_i$ one of these characters. This last function is a multiplicative character on $\mathbb{Q}_p^\times$, but it depends on the angular component of $x$ and on the order of $x$. By using Hironaka’s resolution of singularities theorem, $Z_{\Theta}(f, \chi, s)$ can be written as as linear combination of integrals of type

$$\int_{c+p^e\mathbb{Z}_p} \prod_{j=1}^m \left\{ |y_j|_{p}^{\sum_{i=1}^m N_{f_i,j} s_i + v_j - 1} \chi_i^{N_{f_i,j}}(y_j) \right\} dy_j,$$

where $c = (c_1, \ldots, c_n) \in \mathbb{Q}_p^n$, $(N_{f_1,j}, \ldots, N_{f_m,j})$ is an $m$-tuple of nonnegative integers, $v_j$ a positive integer, for $j$ running through a finite set $T$, see proof of [39, Theorem 8.2.1] and [53, Théorème 1.1.4.].

Then, we have to study the meromorphic continuation of an integral of type

$$I(s) := \int_{c_j+p^e\mathbb{Z}_p} |y_j|_{p}^{\sum_{i=1}^m N_{f_i,j} s_i + v_j - 1} \text{sgn}_{\tau}^{N_{f_i,j}}(y_j) dy_j,$$

since the one corresponding to the trivial character is already known, see e.g. [39, Lemma 8.2.1]. Several cases occur. If $c_j \notin p^e\mathbb{Z}_p$, by using the fact that $|\cdot|_p$ and $\text{sgn}_r(\cdot)$ are locally constant functions we get that

$$I(s) = p^{-e} |c_j|_{p}^{\sum_{i=1}^m N_{f_i,j} s_i + v_j - 1} \text{sgn}_{\tau}^{N_{f_i,j}}(c_j).$$
In the case \( c_j \in p^e \mathbb{Z}_p \), we have

\[
I(s) = \sum_{l=e}^{\infty} \int_{p^l \mathbb{Z}_p^\times} |y_j| p^{\sum_{i=1}^m N_{f_i,j} s_i + v_j - 1} \text{sgn}_r^{N_{f_i,j}} (y_j) \, dy_j
\]

\[
= \left\{ \sum_{l=e}^{\infty} p^{-l(\sum_{i=1}^m N_{f_i,j} s_i + v_j)} \text{sgn}_r^{N_{f_i,j}} (p^l) \right\} \int_{\mathbb{Z}_p^\times} \text{sgn}_r^{N_{f_i,j}} (u) \, du
\]

\[
= : J(s) \int_{\mathbb{Z}_p^\times} \text{sgn}_r^{N_{f_i,j}} (u) \, du,
\]

where \( y_j = p^l u \).

Now if \( \tau = \varepsilon \), \( \text{sgn}_r (u) = (-1)^{ord(u)} \equiv 1 \) for any \( u \in \mathbb{Z}_p^\times \), then \( \int_{\mathbb{Z}_p^\times} \text{sgn}_r^{N_{f_i,j}} (u) \, du = 1 - p^{-1} \).

In the case \( \tau \neq \varepsilon \),

\[
\int_{\mathbb{Z}_p^\times} \text{sgn}_r^{N_{f_i,j}} (u) \, du = \begin{cases} 
1 - p^{-1} & \text{if } N_{f_i,j} \text{ is even} \\
0 & \text{if } N_{f_i,j} \text{ is odd}.
\end{cases}
\]

By using that

\[
\text{sgn}_r^{N_{f_i,j}} (p^l) = \text{sgn}_r^{1N_{f_i,j}} (p) = \begin{cases} 
1 & \text{if } l \text{ is even} \\
\text{sgn}_r^{N_{f_i,j}} (p) & \text{if } l \text{ is odd},
\end{cases}
\]

we have

\[
J(s) = \sum_{l=e}^{\infty} p^{-l(\sum_{i=1}^m N_{f_i,j} s_i + v_j)} \text{sgn}_r^{1N_{f_i,j}} (p)
\]

\[
= \sum_{k=0}^{\infty} p^{-k(\sum_{i=1}^m N_{f_i,j} s_i + v_j)} \text{sgn}_r^{N_{f_i,j}} (p^{k+e})
\]

\[
= p^{-e(\sum_{i=1}^m N_{f_i,j} s_i + v_j)} \text{sgn}_r^{N_{f_i,j}} (p^e) \sum_{k=0}^{\infty} p^{-k(\sum_{i=1}^m N_{f_i,j} s_i + v_j)} \text{sgn}_r^{kN_{f_i,j}} (p).
\]

If \( N_{f_i,j} \) is even

\[
J(s) = p^{-e(\sum_{i=1}^m N_{f_i,j} s_i + v_j)} \sum_{k=0}^{\infty} p^{-k(\sum_{i=1}^m N_{f_i,j} s_i + v_j)}
\]

\[
= \frac{p^{-e(\sum_{i=1}^m N_{f_i,j} s_i + v_j)}}{1 - p^{\sum_{i=1}^m N_{f_i,j} s_i - v_j}}.
\]
If $N_{f_{i,j}}$ is odd,

$$J(s) = p^{-e\left(\sum_{i=1}^{m} N_{f_{i,j}} s_i + v_j\right)} \text{sgn}_r (p^e) \sum_{k=0}^{\infty} p^{-k\left(\sum_{i=1}^{m} N_{f_{i,j}} s_i + v_j\right)} \text{sgn}_r (p^k)$$

$$= p^{-e\left(\sum_{i=1}^{m} N_{f_{i,j}} s_i + v_j\right)} \text{sgn}_r (p^e) \left\{ \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} p^{-2k\left(\sum_{i=1}^{m} N_{f_{i,j}} s_i + v_j\right)} \text{sgn}_r (p) \right\}$$

$$= p^{-e\left(\sum_{i=1}^{m} N_{f_{i,j}} s_i + v_j\right)} \text{sgn}_r (p^e) \left\{ \frac{1 + \text{sgn}_r (p) p^{-\sum_{i=1}^{m} N_{f_{i,j}} s_i - v_j}}{1 - p^{-2\left(\sum_{i=1}^{m} N_{f_{i,j}} s_i + v_j\right)}} \right\}$$

$$= p^{-e\left(\sum_{i=1}^{m} N_{f_{i,j}} s_i + v_j\right)} \text{sgn}_r (p^e) \left\{ \frac{1 + \text{sgn}_r (p) p^{-\sum_{i=1}^{m} N_{f_{i,j}} s_i - v_j}}{1 - p^{-\sum_{i=1}^{m} N_{f_{i,j}} s_i - v_j}} \right\}.$$ 

In conclusion, since $Z_{\Theta} (f, \chi, s)$ is a finite linear combination of products of integrals of type $I(s)$, then $Z_{\Theta} (f, \chi, s)$ admits a meromorphic continuation as a rational function in the variables $p^{-s_1}, \ldots, p^{-s_m}$. More precisely,

$$Z_{\Theta} (f, \chi, s) = \frac{L_{\Theta, \chi} (s)}{\prod_{i \in T} \left(1 - p^{-\sum_{i=1}^{m} N_{f_{i,j}} s_i - v_j}\right)}, \quad (18)$$

where $L_{\Theta, \chi} (s)$ is a polynomial in the variables $p^{-s_1}, \ldots, p^{-s_m}$, and the real parts of its poles belong to the finite union of hyperplanes

$$\sum_{i=1}^{m} N_{f_{i,j}} \text{Re} (s_i) + v_j = 0, \quad \text{for } j \in T.$$

This result is a slightly variation of [53, Théorème 1.1.4].

IV. THE GHOSHAL-KAWANO LOCAL ZETA FUNCTION

From now on, we use $\theta$ to denote a fixed antisymmetric bilinear form. In [50] Ghoshal and Kawano proposed the following amplitude (for the $N$-point tree-level, $p$-adic open string...
amplitude, with Chan-Paton rules in a constant $B$-field):

$$A^{(N)}(k, \theta, \tau) := \int_{\mathbb{Q}_{p}^{N-3} \setminus D} \prod_{i=2}^{N-2} |x_{i}|_{p}^{k_{i}k_{i}} |1 - x_{i}|_{p}^{k_{N-1}k_{i}} H_{\tau}(x_{i}) H_{\tau}(1 - x_{i})$$

\[
\times \prod_{2 \leq i < j \leq N-2} |x_{i} - x_{j}|_{p}^{k_{i}k_{j}} H_{\tau}(x_{i} - x_{j})
\]

\[
\times \exp \left\{ -\frac{i}{2} \sum_{1 \leq i < j \leq N-1} (k_{i}\theta k_{j}) \text{sgn}_{\tau}(x_{i} - x_{j}) \right\} \prod_{i=2}^{N-2} dx_{i},
\]

where $N \geq 4$, $k = (k_{1}, \ldots, k_{N})$, $k_{i} = (k_{0,i}, \ldots, k_{l,i})$, $i = 1, \ldots, N$, is the momentum vector of the $i$-th tachyon (with Minkowski product $k_{i}k_{j} = -k_{0,i}k_{0,j} + k_{1,i}k_{1,j} + \cdots + k_{l,i}k_{l,j}$) obeying to (2) and $\prod_{i=2}^{N-2} dx_{i}$ is the normalized Haar measure of $\mathbb{Q}_{p}^{N-3}$, and

$$D := \left\{ (x_{2}, \ldots, x_{N-2}) \in \mathbb{Q}_{p}^{N-3} ; \prod_{i=2}^{N-2} x_{i} (1 - x_{i}) \prod_{2 \leq i < j \leq N-2} (x_{i} - x_{j}) = 0 \right\}.$$

In the bosonic string theory $l = 26$, however, this dimension does not play any role in our calculations.

In order to study amplitude $A^{(N)}(k, \theta, \tau)$, we introduce

$$s^{i} = (s_{ij}) = \cup_{i=2}^{N-2} \left\{ s_{1i}, s_{(N-1)i} \right\} \cup \cup_{2 \leq i < j \leq N-2} \left\{ s_{ij} \right\} \in \mathbb{C}^{d}$$

a list consisting of $d$ complex variables, where

$$d := \begin{cases} 
2(N - 3) + \binom{N - 3}{2} & \text{if } N \geq 5 \\
2 & \text{if } N = 4 
\end{cases} = \frac{N(N - 3)}{2}.$$

We assume that $s_{ij} = k_{i}k_{j}$ in $\mathbb{R}$, and $s_{ij} = s_{ji}$ for any $i$ and $j$. Furthermore, we set

$$\tilde{s}_{ij} = k_{i}\theta k_{j} \in \mathbb{R},$$

for $1 \leq i < j \leq N - 1$. We denote by $\tilde{s} = (\tilde{s}_{ij})$ for $1 \leq i < j \leq N - 1$. We also set

$$F(x, s, \tau) := \prod_{i=2}^{N-2} |x_{i}|_{p}^{s_{ii}} |1 - x_{i}|_{p}^{s_{(N-1)i}} H_{\tau}(x_{i}) H_{\tau}(1 - x_{i}) \prod_{2 \leq i < j \leq N-2} |x_{i} - x_{j}|_{p}^{s_{ij}} H_{\tau}(x_{i} - x_{j})$$.
and
\[
E(x, \tilde{s}, \tau) := \exp \left\{ -\frac{i}{2} \left( \sum_{2 \leq j \leq N-1} \tilde{s}_{ij} \text{sgn}_\tau (x_1 - x_j) \right) \right\} \\
\times \exp \left\{ -\frac{i}{2} \left( \sum_{2 \leq i \leq N-2} \tilde{s}_{i(N-1)} \text{sgn}_\tau (x_i - x_{N-1}) \right) \right\} \exp \left\{ -\frac{i}{2} \left( \sum_{2 \leq i < j \leq N-2} \tilde{s}_{ij} \text{sgn}_\tau (x_i - x_j) \right) \right\}.
\]

(19)

Later on, we will use the convention \(x_1 = 0, x_{N-1} = 1\) and \(x_N = \infty\). Now, we define the Ghoshal-Kawano local zeta function as
\[
Z^{(N)}(s, \tilde{s}, \tau, \theta) = \int_{\mathbb{Q}^{N-3}_p \setminus D} F(x, s, \tau) E(x, \tilde{s}, \tau) \prod_{i=2}^{N-2} dx_i.
\]

(20)

For the sake of simplicity, from now on, we will use \(\mathbb{Q}^{N-3}_p\) as domain of integration in (20).

By using that \(|E(x, \tilde{s}, \tau)| = 1, |H_\tau(x_i)| \leq 1, |H_\tau(1 - x_i)| \leq 1\), for any \(i\), and that \(|H_\tau(x_i - x_j)| \leq 1\), for any \(i, j\), we have
\[
|Z^{(N)}(s, \tilde{s}, \tau, \theta)| \leq \int_{\mathbb{Q}^{N-3}_p} \prod_{i=2}^{N-2} |x_i|_p^{\text{Re}(s_i)} |1 - x_i|_p^{\text{Re}(s_{i-1})} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_p^{\text{Re}(s_{ij})} \prod_{i=2}^{N-2} dx_i = Z^{(N)}(\text{Re}(s)),
\]

where \(Z^{(N)}(s)\) is the Koba-Nielsen string amplitude studied in [38], see also [31]. Since this last integral is holomorphic in an open set \(K \subset \mathbb{C}^d\), we conclude that
\[
Z^{(N)}(s, \tilde{s}, \tau, \theta) \text{ is holomorphic in } s \in K \text{ for any } \tilde{s}, \tau, \theta.
\]

We set \(T := \{2, \ldots, N-2\}\), and define for \(I \subseteq T\), the sector attached to \(I\) as
\[
\text{Sect}(I) = \left\{ (x_2, \ldots, x_{N-2}) \in \mathbb{Q}^{N-3}_p ; |x_i|_p \leq 1 \Leftrightarrow i \in I \right\}.
\]

Then \(\mathbb{Q}^{N-3}_p = \bigsqcup_{I \subseteq T} \text{Sect}(I)\) and
\[
Z^{(N)}(s, \tilde{s}, \tau, \theta) = \sum_{I \subseteq T} Z^{(N)}_I(s, \tilde{s}, \tau, \theta),
\]

(21)

where
\[
Z^{(N)}_I(s, \tilde{s}, \tau, \theta) := \int_{\text{Sect}(I)} F(x, s, \tau) E(x, \tilde{s}, \tau) \prod_{i=2}^{N-2} dx_i.
\]
In the case in which \( I^c = T \setminus I \neq \emptyset \), by using that \( H_\tau(x)H_\tau(-x) = 0 \), we have

\[
F(x, s, \tau) = \prod_{i \in I} |x_i|_{p}^{s_i} |1 - x_i|_{p}^{s(N-1)_i} H_\tau(x_i) H_\tau(1 - x_i) \prod_{i \notin I} |x_i|_{p}^{s_i} \prod_{2 \leq i < j \leq N-2} |x_i - x_j|_{p}^{s_{ij}} H_\tau(x_i - x_j) 
\]

and consequently \( Z^{(N)}_I(s, \tilde{s}, \tau, \theta) \equiv 0 \) if \( I \neq T \). For this reason, we redefine the Ghoshal-Kawano local zeta function as

\[
Z^{(N)}(s, \tilde{s}, \tau, \theta) = \int_{\mathbb{Z}_p^{N-3}} F(x, s, \tau) E(x, \tilde{s}, \tau) \prod_{i=2}^{N-2} dx_i. \tag{22}
\]

V. MEROMORPHIC CONTINUATION OF GHOSHAL-KAWANO LOCAL ZETA FUNCTION

A. Some formulae

For \( \tilde{s} \in \mathbb{R} \) and \( x \in \mathbb{Q}_p \setminus \{0\} \),

\[
\exp \left\{ -\frac{i \tilde{s}}{2} \text{sgn}_\tau(x) \right\} = \cos \left( \frac{\tilde{s}}{2} \right) - i \text{sgn}_\tau(x) \sin \left( \frac{\tilde{s}}{2} \right). \tag{23}
\]

By using this formula, and the convention \( x_1 = 0, x_{N-1} = 1 \), we obtain that

\[
\exp \left\{ -\frac{i}{2} \left( \sum_{2 \leq j \leq N-1} \tilde{s}_{ij} \text{sgn}_\tau(-x_j) \right) \right\} = \sum_{I \subseteq \{2, \ldots, N-1\}} C_I(\tilde{s}) \prod_{j \in I} \text{sgn}_\tau(x_j);
\]

\[
\exp \left\{ -\frac{i}{2} \left( \sum_{3 \leq i \leq N-1} \tilde{s}_{i(N-1)} \text{sgn}_\tau(x_i - 1) \right) \right\} = \sum_{J \subseteq \{3, \ldots, N-1\}} D_J(\tilde{s}) \prod_{j \in J} \text{sgn}_\tau(1 - x_j);
\]

\[
\exp \left\{ -\frac{i}{2} \left( \sum_{2 \leq i < j \leq N-2} \tilde{s}_{ij} \text{sgn}_\tau(x_i - x_j) \right) \right\} = \sum_{K \subseteq \{2 \leq i < j \leq N-2\}} D_K(\tilde{s}) \prod_{i,j \in K} \text{sgn}_\tau(x_i - x_j),
\]

with the convention that \( \prod_{j \in \emptyset} = 1 \). In conclusion,

\[
E(x, \tilde{s}, \tau) := \sum_{I, J, K} E_{I, J, K}(\tilde{s}) \prod_{j \in I} \text{sgn}_\tau(x_j) \prod_{j \in J} \text{sgn}_\tau(1 - x_j) \prod_{i,j \in K} \text{sgn}_\tau(x_i - x_j). \tag{24}
\]
In a similar way, we obtain that
\[
\prod_{i=2}^{N-2} H_\tau(x_i) H_\tau(1-x_i) \prod_{2\leq i<j\leq N-2} H_\tau(x_i-x_j) = \sum_{I,J,K} e_{I,J,K} \prod_{j\in I} \text{sgn}_\tau(x_j) \prod_{j\in J} \text{sgn}_\tau(1-x_j) \prod_{i,j\in K} \text{sgn}_\tau(x_i-x_j),
\]
where the \(e_{I,J,K}\)'s are constants.

**B. Meromorphic continuation of \(Z^{(N)}(s, \bar{s}, \tau, \theta)\)**

By using formulae (23)-(25) and (22), \(Z^{(N)}(s, \bar{s}, \tau, \theta)\) is a finite sum of integrals of type
\[
C(\bar{s}) \int_{Z_p^{N-3}} \prod_{i=2}^{N-2} |x_i|^p \mid 1-x_i\mid_p^{s(N-1)i} \prod_{2\leq i<j\leq N-2} |x_i-x_j|^p \prod_{j\in I} \chi_\tau(x_j) \prod_{j\in J} \chi_\tau(1-x_j) \times \prod_{i,j\in K} \chi_\tau(x_i-x_j) \prod_{i=2}^{N-2} dx_i,
\]
where \(C(\bar{s})\) is an \(\mathbb{R}\)-analytic function, \(\chi_\tau\) denotes the trivial character or \(\text{sgn}_\tau\). This formula implies that \(Z^{(N)}(s, \bar{s}, \tau, \theta)\) is a linear combination of multivariate Igusa local zeta functions with coefficients in the ring of \(\mathbb{R}\)-analytic functions in the variables \(\bar{s}\). Consequently, by [158], \(Z^{(N)}(s, \bar{s}, \tau, \theta)\) admits a meromorphic continuation as a rational function in the variables \(p^{-s_i}, p^{-(N-1)s_i}, p^{-s_{ij}}\) and the real parts of its poles belong to the finite union of hyperplanes of type
\[
\mathcal{H} = \left\{ s_{ij} \in \mathbb{C}^d; \sum_{ij\in M} N_{ij,k} \text{Re}(s_{ij}) + \gamma_k = 0, \text{ for } k \in T \right\},
\]
where \(N_{ij,k} \in \mathbb{N}, \gamma_k \in \mathbb{N}\setminus\{0\}, \text{ and } M, T \text{ are finite sets. Furthermore, } Z^{(N)}(s, \bar{s}, \tau, \theta) \text{ is holomorphic in }
\]
\[
\bigcap_{\mathcal{H}} \left\{ s_{ij} \in \mathbb{C}^d; \sum_{ij\in M} N_{ij,k} \text{Re}(s_{ij}) + \gamma_k > 0, \text{ for } k \in T \right\}.
\]

**C. Further remarks**

The Ghoshal-Kawano local zeta function depends on \(x_1, x_{N-1}\), i.e. \(Z^{(N)}(s, \bar{s}, \tau, \theta, x_1, x_{N-1})\). In [50], the corresponding amplitude was considered in the
case \( x_1 = 0, \ x_{N-1} = 0, \ x_N = \infty \). Our result about the meromorphic continuation of
\( Z^{(N)}(s, \tilde{s}, \tau, \theta) \) is also valid for \( Z^{(N)}(s, \tilde{s}, \tau, \theta, x_1, x_{N-1}) \). Indeed, by using that
\[
\mathbb{Z}_p^{N-3} = \bigoplus_{i=1}^M a_i + p^L \mathbb{Z}_p^{N-3},
\]
where \( a_i \in \mathbb{Z}_p^{N-3} \) and \( L \) is a positive integer sufficiently large, we have
\[
Z^{(N)}(s, \tilde{s}, \tau, \theta, x_1, x_{N-1}) = \sum_{i=1}^M Z^{(N)}_{a_i}(s, \tilde{s}, \tau, \theta, x_1, x_{N-1}),
\]
where
\[
Z^{(N)}_{b}(s, \tilde{s}, \tau, \theta, x_1, x_{N-1}) := \int_{b+p^L \mathbb{Z}_p^{N-3}} F(\mathbf{x}, s, \tau) E(\mathbf{x}, \tilde{s}) \prod_{i=2}^{N-2} dx_i,
\]
and \( E(\mathbf{x}, \tilde{s}, \tau) \) depends on \( x_1, \ x_{N-1}, \ [19] \). The meromorphic continuation of
\( Z^{(N)}_{b}(s, \tilde{s}, \tau, \theta, x_1, x_{N-1}) \) can be obtained by the methods presented in Sections [V A]-[V B],
by giving a local description polynomial \( \prod_{i=2}^{N-2} x_i \ (1 - x_i) \prod_{2 \leq i<j \leq N-2} (x_i - x_j) \) near \( b \). We
illustrate the technique in a particular but relevant case. We consider the case, \( x_1, \ x_{N-1} \in \mathbb{Z}_p \setminus \{0, 1\} \), with \( x_1 \neq x_{N-1} \) and \( b = (b_2, \ldots, b_{N-2}) \in \mathbb{Z}_p^{N-3} \), with \( b_i = x_1 \) for
\( i = 2, \ldots, l \), and \( b_i = x_{N-1} \) for \( i = l+1, \ldots, N-2 \), for some \( 2 \leq l < N-2 \). Now, we change
variables as \( x_i = x_1 + p^L y_i, \ i = 2, \ldots, l, \) and \( x_i = x_{N-1} + p^L y_i, \) for \( i = l+1, \ldots, N-2 \), then
\[
\prod_{i=2}^{N-2} |x_i|_p^{s_{1i}} 1 - x_i|_p^{s(N-1)i} = \prod_{i=2}^{l+1} |x_i|_p^{s_{1i}} 1 - x_1|_p^{s(N-1)i} H_\tau(x_1) H_\tau(1 - x_1)
\]
\[
\times \prod_{i=l+1}^{N-2} |x_{N-1}|_p^{s_{1i}} 1 - x_{N-1}|_p^{s(N-1)i} H_\tau(x_{N-1}) H_\tau(1 - x_{N-1});
\]
and
\[
\prod_{2 \leq i<j \leq N-2} |x_i - x_j|_p^{s_{ij}} H_\tau(x_i - x_j) = \prod_{2 \leq i<j \leq l} p^{-L s_{ij}} |y_i - y_j|_p^{s_{ij}} H_\tau(p^L (y_i - y_j))
\]
\[
\times \prod_{l+1 \leq i<j \leq N-2} p^{-L s_{ij}} |y_i - y_j|_p^{s_{ij}} H_\tau(p^L (y_i - y_j)) \prod_{2 \leq i<j \leq N-2} \prod_{2 \leq i<j \leq l, \ l+1 \leq j \leq N-2} |x_1 - x_{N-1}|_p^{s_{ij}} H_\tau(x_1 - x_{N-1}).
\]
Consequently,
\[
F(\mathbf{x}, s, \tau) = A(s_{ij}, x_1, x_{N-1}) \prod_{2 \leq i<j \leq l} |y_i - y_j|_p^{s_{ij}} H_\tau(p^L (y_i - y_j))
\]
\[
\times \prod_{l+1 \leq i<j \leq N-2} |y_i - y_j|_p^{s_{ij}} H_\tau(p^L (y_i - y_j)), \quad (26)
\]
for $L$ sufficiently large. Now
\[
\exp \left\{ -\frac{i}{2} \sum_{2 \leq j \leq N-1} \tilde{s}_{ij} \text{sgn}_r(x_1 - x_j) \right\}
\]
\[
= \exp \left\{ -\frac{i}{2} \left( \tilde{s}_{1(N-1)} \text{sgn}_r(x_1 - x_{N-1}) + \sum_{2 \leq j \leq l} \tilde{s}_{ij} \text{sgn}_r(-p^j y_j) + \sum_{l+1 \leq j \leq N-2} \tilde{s}_{ij} \text{sgn}_r(x_1 - x_{N-1}) \right) \right\};
\]
\[
\exp \left\{ -\frac{i}{2} \sum_{2 \leq j \leq N-2} \tilde{s}_{i(N-1)} \text{sgn}_r(x_i - x_{N-1}) \right\}
\]
\[
= \exp \left\{ -\frac{i}{2} \left( \sum_{2 \leq j \leq l} \tilde{s}_{i(N-1)} \text{sgn}_r(x_i - x_{N-1}) + \sum_{l+1 \leq j \leq N-2} \tilde{s}_{i(N-1)} \text{sgn}_r(p^l y_j) \right) \right\};
\]
\[
\exp \left\{ -\frac{i}{2} \sum_{2 \leq i < j \leq N-2} \tilde{s}_{ij} \text{sgn}_r(x_i - x_j) \right\} = \exp \left\{ -\frac{i}{2} \sum_{2 \leq i < j \leq l} \tilde{s}_{ij} \text{sgn}_r(p^l(y_i - y_j)) \right\}
\]
\[
\times \exp \left\{ -\frac{i}{2} \sum_{l+1 \leq i < j \leq N-2} \tilde{s}_{ij} \text{sgn}_r(p^l(y_i - y_j)) \right\}
\]
\[
\times \exp \left\{ -\frac{i}{2} \sum_{2 \leq i < j \leq N-2} \tilde{s}_{ij} \text{sgn}_r(x_i - x_{N-1}) \right\} ,
\]
for $L$ sufficiently large. Consequently,
\[
E(x, \tilde{s}, \tau) = B(\tilde{s}, x_1, x_{N-1}, \tau) \exp \left\{ -\frac{i}{2} \sum_{2 \leq j \leq l} \tilde{s}_{ij} \text{sgn}_r(y_j) \right\}
\]
\[
\times \exp \left\{ -\frac{i}{2} \sum_{l+1 \leq j \leq N-2} \tilde{s}_{i(N-1)} \text{sgn}_r(y_j) \right\} \exp \left\{ -\frac{i}{2} \sum_{2 \leq i < j \leq l} \tilde{s}_{ij} \text{sgn}_r(y_i - y_j) \right\}
\]
\[
\times \exp \left\{ -\frac{i}{2} \sum_{l+1 \leq i < j \leq N-2} \tilde{s}_{ij} \text{sgn}_r(y_i - y_j) \right\} .
\]
Now, by using formulae (26)-(27), we can apply the reasoning given in the Sections (VA)-(VB) to obtain the meromorphic continuation of $Z_b^{(N)}(s, \tilde{s}, \tau, \theta, x_1, x_{N-1})$.

VI. EXPLICIT COMPUTATION OF $Z^{(4)}(s, \tilde{s}, \tau, \theta)$

In this section we compute the Ghoshal-Kawano local zeta function for four points:
\[
Z^{(4)}(s, \tilde{s}, \tau, \theta) = \exp \left\{ \frac{i}{2} \tilde{s}_{13} \int_{z_p} \left| x_2 \right|^{s_{13}} \left| 1 - x_2 \right|^{s_{32}} H_\tau(x_2) H_\tau(1 - x_2) E^{(4)}(x_2, \tilde{s}, \tau) \right\} .
\]
where

\[ E^{(4)}(x_2, \tilde{s}, \tau) := E^{(4)}(x_2, \tilde{s}_{12}, \tilde{s}_{32}, \tau) = \exp \left\{ \frac{i}{2} \left( \tilde{s}_{12} \text{sgn}_r(x_2) + \tilde{s}_{23} \text{sgn}_r(1 - x_2) \right) \right\}. \]

We recall that Ghoshal and Kawano take \( x_1 = 0, \ x_3 = 1, \ x_4 = \infty \). By using the fact that \( \text{sgn}_r(y) \in \{1, -1\} \) and \( H_\tau(y) \in \{0, 1\} \), one verifies that

\[ \exp \left\{ \frac{i}{2} \left( \tilde{s}_{12} \text{sgn}_r(x_2) \right) \right\} H_\tau(x_2) = \exp \left( \frac{i}{2} \tilde{s}_{12} \right) H_\tau(x_2), \]
\[ \exp \left\{ \frac{i}{2} \left( \tilde{s}_{23} \text{sgn}_r(1 - x_2) \right) \right\} H_\tau(1 - x_2) = \exp \left( \frac{i}{2} \tilde{s}_{23} \right) H_\tau(1 - x_2), \]

and consequently

\[ E^{(4)}(x_2, \tilde{s}_{12}, \tilde{s}_{32}, \tau) = \exp \left\{ \frac{i}{2} \left( \tilde{s}_{12} + \tilde{s}_{23} \right) \right\}, \]

and

\[ Z^{(4)}(s, \tilde{s}, \tau, \theta) = \exp \left\{ \frac{i}{2} \left( \tilde{s}_{13} + \tilde{s}_{12} + \tilde{s}_{23} \right) \right\} \int \left| x_2 \right|_{p}^{s_{12}} |1 - x_2|_{p}^{s_{32}} H_\tau(x_2) H_\tau(1 - x_2) \, dx_2. \]

We first compute some \( p \)-adic integrals needed in this section.

### A. Some \( p \)-adic integrals

1. **Formula 1**

Assume that \( S \subset Z_p \setminus \{0\} \) satisfies \(-S = S\). Then

\[ \int_S |x_2|_{p}^{s_{12}} \text{sgn}_r(x_2) \, dx_2 = 0. \]

This formula follows from changing variables as \( x_2 = -y \) and using the fact that \( \text{sgn}_r(-y) = -\text{sgn}_r(y) \).

2. **Formula 2**

If \( p \equiv 3 \) mod 4, then

\[ S(\tau, p) := \frac{1}{p} \sum_{j=2}^{p-1} H_\tau(j) H_\tau(1 - j) = \begin{cases} 0 & \text{if } \tau = \varepsilon \\ \frac{p-3}{4p} & \text{if } \tau \neq \varepsilon. \end{cases} \]
From table (60), for $j = 2, \ldots, p - 1$,

$$H_{\tau}(j)H_{\tau}(1 - j) = \begin{cases} 0 & \text{if } \tau = \varepsilon \\ \frac{1}{4} \left\{ 1 + \left( \frac{j}{p} \right) \right\} \left\{ 1 - \left( \frac{j - 1}{p} \right) \right\} & \text{if } \tau \neq \varepsilon, \end{cases}$$

and thus $S(\tau, p) = 0$ for $\tau = \varepsilon$, and for $\tau \neq \varepsilon$,

$$S(\tau, p) := \frac{1}{4p} \left\{ p - 2 + \sum_{j=2}^{p-1} \left( \frac{j}{p} \right) - \sum_{j=2}^{p-1} \left( \frac{j - 1}{p} \right) - \sum_{j=2}^{p-1} \left( \frac{j}{p} \right) \left( \frac{j - 1}{p} \right) \right\}. \tag{28}$$

Now by using that $\sum_{k=1}^{p-1} \left( \frac{k}{p} \right) = 0$, we get that

$$\sum_{j=2}^{p-1} \left( \frac{j}{p} \right) = -1 \text{ and } \sum_{j=2}^{p-1} \left( \frac{j - 1}{p} \right) = \sum_{k=1}^{p-2} \left( \frac{k}{p} \right) = - \left( \frac{p - 1}{p} \right) = 1,$$

and thus

$$S(\tau, p) = \frac{1}{4p} \left\{ p - 2 - \sum_{k=1}^{p-2} \left( \frac{k + 1}{p} \right) \left( \frac{k}{p} \right) \right\}.$$

To compute

$$L(\tau, p) := \sum_{k=1}^{p-2} \left( \frac{k + 1}{p} \right) \left( \frac{k}{p} \right),$$

we define

$$A_{ij} = \left\{ a \in \{1, \ldots, p - 2\} ; \left( \frac{a}{p} \right) = (-1)^i \text{ and } \left( \frac{a + 1}{p} \right) = (-1)^j \right\},$$

then $\{1, \ldots, p - 2\} = A_{00} \sqcup A_{01} \sqcup A_{10} \sqcup A_{11}$ and

$$L(\tau, p) = \#A_{00} - \#A_{01} - \#A_{10} + \#A_{11}.$$

Now, if $p \equiv 3 \mod 4$, then

$$\#A_{00} = \#A_{10} = \#A_{11} = \frac{p - 3}{4}, \text{ and } \#A_{01} = \frac{p + 1}{4}, \tag{28}$$

see e.g. [57, Chapter 9, Exercise 5 in p. 201], and therefore

$$L(\tau, p) = \#A_{00} - \#A_{01} = -1, \text{ and } S(\tau, p) = \frac{1}{4p}(p - 3).$$

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3. **Formula 3**

Set

\[ I(s, \tau) = \int_{Z_p} |x_2|^{s_{12}} p \big| 1 - x_2 \big|^{s_{32}} p H_\tau(x_2) H_\tau(1 - x_2) dx_2. \]

Then

\[
I(s, \tau) = \begin{cases} 
\frac{p^{-1-s_{12}}(1-p^{-1})}{2(1-p^{-1-s_{12}})} + \frac{p^{-1-s_{32}}(1-p^{-1})}{2(1-p^{-1-s_{32}})} & \text{if } \tau = \varepsilon \\
\frac{p^{-3}}{4p} + \frac{p^{-1-s_{12}}(1-p^{-1})}{2(1-p^{-1-s_{12}})} + \frac{p^{-1-s_{32}}(1-p^{-1})}{2(1-p^{-1-s_{32}})} & \text{if } \tau \neq \varepsilon.
\end{cases}
\] (29)

By using the partition

\[ Z_p = \bigsqcup_{j=0}^{p-1} j + pZ_p \] (30)

and the fact that

\[ H_\tau(x_2) \mid_{j+pZ_p} = H_\tau(j) \] for \( j \neq 0 \) and \( H_\tau(1 - x_2) \mid_{j+pZ_p} = H_\tau(1 - j) \) for \( j \neq 1 \),

we have

\[ I(s, \tau) = \sum_{j=0}^{p-1} I_j(s, \tau), \]

where

\[ I_j(s, \tau) = \int_{j+pZ_p} |x_2|^{s_{12}} p \big| 1 - x_2 \big|^{s_{32}} p H_\tau(x_2) H_\tau(1 - x_2) dx_2. \]

If \( j \neq 0, 1 \), then

\[ I_j(s, \tau) = p^{-1} H_\tau(j) H_\tau(1 - x_2). \] (33)

If \( j = 0 \), then by using Formula 1,

\[
I_0(s, \tau) = \int_{pZ_p} |x_2|^{s_{12}} p H_\tau(x_2) dx_2 = \frac{1}{2} \int_{pZ_p} |x_2|^{s_{12}} dx_2 + \frac{1}{2} \int_{pZ_p} |x_2|^{s_{12}} \sgn_\tau(x_2) dx_2
\]

\[ = \frac{1}{2} \int_{pZ_p} |x_2|^{s_{12}} dx_2 = \frac{1}{2} \frac{p^{-1-s_{12}}(1-p^{-1})}{1-p^{-1-s_{12}}}. \] (34)

The case \( j = 1 \) is similar to the case \( j = 0 \),

\[ I_1(s, \tau) = \int_{1+pZ_p} |1 - x_2|^{s_{32}} p H_\tau(1 - x_2) dx_2 = \frac{1}{2} \frac{p^{-1-s_{32}}(1-p^{-1})}{1-p^{-1-s_{32}}}. \] (35)

Formula (29) follows from (32) by using (33)-(35) and Formula 2.
B. Computation of $Z^{(4)}(s, \bar{s}, \tau, \theta)$

In conclusion,

$$Z^{(4)}(s, \bar{s}, \tau, \theta) = \exp \left\{ \frac{i}{2} (\bar{s}_{12} + \bar{s}_{23}) \right\} I(s, \tau)$$

$$= \exp \left\{ \frac{i}{2} (\bar{s}_{12} + \bar{s}_{23}) \right\} \begin{cases} \frac{p^{-1-s_{12}}(1-p^{-1})}{2(1-p^{-1-s_{12}})} + \frac{p^{-1-s_{32}}(1-p^{-1})}{2(1-p^{-1-s_{32}})} & \text{if } \tau = \varepsilon \\ \frac{p-3}{4p} + \frac{p^{-1-s_{12}}(1-p^{-1})}{2(1-p^{-1-s_{12}})} + \frac{p^{-1-s_{32}}(1-p^{-1})}{2(1-p^{-1-s_{32}})} & \text{if } \tau \neq \varepsilon. \end{cases}$$

(36)

is holomorphic in

$$\text{Re}(s_{12}) > -1 \text{ and } \text{Re}(s_{32}) > -1.$$

(37)

The above formula for $Z^{(4)}(s, \bar{s}, \tau, \theta)$ was also obtained in [50].

VII. EXPLICIT COMPUTATION OF $Z^{(5)}(s, \bar{s}, \tau, \theta)$

In this section we compute the amplitude for five points:

$$Z^{(5)}(s, \bar{s}, \tau, \theta) = \int_{\mathbb{P}^2} E^{(5)}(x_2, x_3, \bar{s}, \theta, \tau) F^{(5)}(x_2, x_3, s, \tau) dx_2 dx_3,$$

with

$$E^{(5)}(x_2, x_3, \bar{s}, \theta, \tau) = \exp \left\{ \frac{-i}{2} \left( \bar{s}_{14} \text{sgn}_\tau(-1) + \bar{s}_{12} \text{sgn}_\tau(-x_2) + \bar{s}_{13} \text{sgn}_\tau(-x_3) \right) \right\}$$

$$\times \exp \left\{ \frac{-i}{2} \left( \bar{s}_{42} \text{sgn}_\tau(1-x_2) + \bar{s}_{43} \text{sgn}_\tau(1-x_3) + \bar{s}_{23} \text{sgn}_\tau(x_2-x_3) \right) \right\}$$

and

$$F^{(5)}(x_2, x_3, s, \tau) = |x_2|^{s_{12}} |x_3|^{s_{13}} |1-x_2|^{s_{42}} |1-x_3|^{s_{43}} |x_2-x_3|^{s_{23}}$$

$$\times H_\tau(x_2) H_\tau(x_3) H_\tau(1-x_2) H_\tau(1-x_3) H_\tau(x_2-x_3).$$

Taking $x_1 = 0$, $x_4 = 1$ and $x_5 = \infty$, and using the reasoning given at the beginning of the previous section we have

$$E^{(5)}(\bar{s}, \theta) = \exp \left\{ \frac{-i}{2} \left( \bar{s}_{14} + \bar{s}_{12} + \bar{s}_{13} + \bar{s}_{42} + \bar{s}_{43} + \bar{s}_{23} \right) \right\}.$$
and then
\[ Z^{(5)}(s, \tilde{s}, \tau, \theta) = E^{(5)}(\tilde{s}, \theta)L(s, \tau), \]  
where
\[ L(s, \tau) = \int_{\mathbb{Z}_p^2} F^{(5)}(x_2, x_3, s, \tau)dx_2dx_3. \]

First we give some formulae needed in the following calculations.

A. More \( p \)-adic Sums and Integrals

1. Formula 4

For \( A \subset \{1, 2, \ldots, p-1\} \), by using that \( H_{\tau}(x) = \frac{1}{2}(1 + \text{sgn}_{\tau}(x)) \), where the sign function \( \text{sgn}_{\tau} \) is given in Table (60), and that \( \text{sgn}_{\tau}(-x) = -\text{sgn}_{\tau}(x) \), for \( p \equiv 3 \bmod 4 \), we have
\[ V(A, p, \tau) := \sum_{i,j \in A, i \neq j} H_{\tau}(i-j) = (\#A)(\#A-1)/2 = \left( \frac{\#A}{2} \right). \]

Indeed
\[ V(A, p, \tau) = \sum_{i,j \in A, j < i} H_{\tau}(i-j) + \sum_{i,j \in A, j < i} H_{\tau}(-(j-i)) = \sum_{i,j \in A, j < i} [H_{\tau}(i-j) + H_{\tau}(-(i-j))]
= \frac{1}{2} \sum_{i,j \in A, j < i} \left[ 1 + \left( \frac{i-j}{p} \right) + 1 - \left( \frac{i-j}{p} \right) \right] = \sum_{i,j \in A, j < i} 1 = \left( \frac{\#A}{2} \right). \]

2. Formula 5

If \( p \equiv 3 \bmod 4 \) and \( \tau \neq \varepsilon \), then
\[ T(p, \tau) := \frac{1}{p^2} \sum_{i,j=2, i \neq j}^{p-1} H_{\tau}(i)H_{\tau}(1-i)H_{\tau}(j)H_{\tau}(1-j)H_{\tau}(i-j)
= \frac{(p-3)(p-7)}{32p^2}. \]

We define \( B := \{k \in \{2, 3, \ldots, p-1\}; H_{\tau}(k)H_{\tau}(1-k) = 1\} \). Then, by using the results and notation given in the proof of Formula 2, we have \( \#B = \#A_{10} = \frac{p^3}{4} \), and
\[ T(p, \tau) = \frac{1}{p^2} \sum_{i,j \in B, i \neq j} H_{\tau}(i-j) = \frac{1}{p^2} \left( \frac{\#B}{2} \right) = \frac{1}{2p^2} \left( \frac{p^3}{4} \right) \left( \frac{p-7}{4} \right) = \frac{(p-3)(p-7)}{32p^2}. \]
3. Formula 6

We set for \( a, b, c \in \mathbb{C} \),

\[
L_{00}(a, b, c) := \frac{1}{8} \int_{(p\mathbb{Z}_p)^2} |x_2|^a |x_3|^b |x_2 - x_3|^c dx_2 dx_3.
\]

Then

\[
L_{00}(a, b, c) = \frac{1}{8} \frac{p^{-a-b-c-2} (1 - p^{-1})}{1 - p^{-a-b-c-2}} \left\{ p^{-1} (p - 2) + \frac{p^{-1-a} (1 - p^{-1})}{1 - p^{-a}} + \frac{p^{-1-b} (1 - p^{-1})}{1 - p^{-b}} + \frac{p^{-1-c} (1 - p^{-1})}{1 - p^{-c}} \right\}.
\]

In order to compute \( L_{00}(a, b, c) \), we introduce the following subsets:

\[
A := \left\{ (x_2, x_3) \in (p\mathbb{Z}_p)^2 \mid \frac{|x_2|}{|x_3|} \leq 1 \right\},
\]

\[
B := \left\{ (x_2, x_3) \in (p\mathbb{Z}_p)^2 \mid \frac{|x_3|}{|x_2|} < 1 \right\}.
\]

Then

\[
(p\mathbb{Z}_p)^2 \setminus \{(x_2, x_3) \in (p\mathbb{Z}_p)^2; x_2 x_3 = 0\} = A \sqcup B,
\]

and \( L_{00}(a, b, c) = L_{00}^{A}(a, b, c) + L_{00}^{B}(a, b, c) \), where

\[
L_{00}^{A}(a, b, c) := \frac{1}{8} \int_A |x_2|^a |x_3|^b |x_2 - x_3|^c dx_2 dx_3,
\]

and

\[
L_{00}^{B}(a, b, c) := \frac{1}{8} \int_B |x_2|^a |x_3|^b |x_2 - x_3|^c dx_2 dx_3.
\]

We compute first \( L_{00}^{A}(a, b, c) \), by using the following change of variables:

\[
x_2 = uv, \ x_3 = u.
\]

Then \( dx_2 dx_3 = |u|_p du dv \) and

\[
L_{00}^{A}(a, b, c) = \frac{1}{8} \int_{p\mathbb{Z}_p \times \mathbb{Z}_p} |u|^{a+b+c+1} |v|^{a} |v - 1|_p^c du dv
\]

\[
= \frac{1}{8} \left\{ \int_{p\mathbb{Z}_p} |u|^{a+b+c+1} du \right\} \left\{ \int_{\mathbb{Z}_p} |v|^{a} |v - 1|_p^c dv \right\}
\]

\[
= \frac{1}{8} \frac{p^{-a-b-c-2} (1 - p^{-1})}{1 - p^{-a-b-c-2}} J(a, c).
\]
By using partition (30),

\[ J(a, c) = \sum_{i=0}^{p-1} J_i(a, c). \]

For \( i \neq 0, 1 \),

\[ J_i(a, c) = \int_{i+p \mathbb{Z}_p} |v|^a_p |v - 1|^c_p dv = p^{-1}, \]

and the contribution of all these integrals is

\[ \sum_{i=2}^{p-1} J_i(a, c) = p^{-1} (p - 2). \] (43)

For \( i = 0 \),

\[ J_0(a, c) = \int_{p \mathbb{Z}_p} |v|^a_p dv = \frac{p^{-1-a} (1 - p^{-1})}{1 - p^{-1-a}}. \] (44)

For \( i = 1 \),

\[ J_1(a, c) = \int_{1+p \mathbb{Z}_p} |v - 1|^c_p dv = \frac{p^{-1-c} (1 - p^{-1})}{1 - p^{-1-c}}. \] (45)

Therefore, from (42)-(45),

\[ L_{00}^{(A)}(a, b, c) = \frac{1}{8} \frac{p^{-a-b-c-2} (1 - p^{-1})}{1 - p^{-a-b-c-2}} \left\{ p^{-1} (p - 2) + \frac{p^{-1-a} (1 - p^{-1})}{1 - p^{-1-a}} + \frac{p^{-1-c} (1 - p^{-1})}{1 - p^{-1-c}} \right\}. \]

Now we compute \( L_{00}^{(B)}(a, b, c) \), by using the following change of variables:

\[ x_2 = t, \quad x_3 = zt. \] (46)

Then \( dx_2dx_3 = |t|^a_p dzdt \) and

\[ L_{00}^{(B)}(a, b, c) = \frac{1}{8} \int_{(p \mathbb{Z}_p)^2} |t|^{a+b+c+1} |z|^{b-1} |z|^c_p dzdt = \frac{1}{8} \int_{(p \mathbb{Z}_p)^2} |t|^{a+b+c+1} |z|^b_p dzdt \]

\[ = \frac{1}{8} \frac{p^{-a-b-c-2} (1 - p^{-1}) p^{-1-b} (1 - p^{-1})}{1 - p^{-a-b-c-2}}. \]

4. Formula 7

For \( a, b, c \in \mathbb{C} \),

\[ L_{00}^{(1)}(a, b, c, \tau) := \frac{1}{8} \int_{(p \mathbb{Z}_p)^2} |x_2|^a_p |x_3|^b_p |x_2 - x_3|^c_p \text{sgn}_\tau(x_2) \text{sgn}_\tau(x_3) dx_2 dx_3 \]

\[ = \frac{1}{8} \frac{p^{-a-b-c-2} (1 - p^{-1})}{1 - p^{-a-b-c-2}} \left\{ -p^{-1} + \frac{p^{-1-c} (1 - p^{-1})}{1 - p^{-1-c}} \right\}. \]
By using partition (40), we get that $L^{(1)}_{00}(a, b, c, \tau) = L^{(1,A)}_{00}(a, b, c, \tau) + L^{(1,B)}_{00}(a, b, c, \tau)$. We compute integral $L^{(1,A)}_{00}(a, b, c, \tau)$, respectively $L^{(1,B)}_{00}(a, b, c, \tau)$, by using change of variables (41), respectively (46), as follows:

$$L^{(1,A)}_{00}(a, b, c, \tau) = \frac{1}{8} \left\{ \int_{\mathbb{Z}_p} |u|_{p}^{a+b+c+1} du \right\} \left\{ \int_{\mathbb{Z}_p} |v|_{p}^{a} |v - 1|_{p}^{c} \text{sgn}_\tau(v) dv \right\} = \frac{1}{8} \frac{p^{-a-b-c-2} (1 - p^{-1})}{1 - p^{-a-b-c-2}} I(a, b, c, \tau).$$

By using partition (30),

$$I(a, b, c, \tau) = \sum_{j=0}^{p-1} I_j(a, b, c, \tau).$$

For $j \neq 0, 1$, $I_j(a, b, c, \tau) = p^{-1} \text{sgn}_\tau(j)$, thus, the contribution of all these integrals is

$$\sum_{j=2}^{p-1} I_j(a, b, c, \tau) = p^{-1} \sum_{j=2}^{p-1} \text{sgn}_\tau(j) = p^{-1} \sum_{j=2}^{p-1} \left( \frac{j}{p} \right) = -p^{-1}.$$

For $j = 0$, by using the Formula 1, $I_0(a, b, c, \tau) = 0$. For $j = 1$,

$$I_1(a, b, c, \tau) = \int_{1+p\mathbb{Z}_p} |v - 1|_{p}^{c} \text{sgn}_\tau(v) dv = \int_{1+p\mathbb{Z}_p} |v - 1|_{p}^{c} dv$$

$$= \int_{p\mathbb{Z}_p} |v|_{p}^{c} dv = \frac{p^{-1-c} (1 - p^{-1})}{1 - p^{-1-c}}.$$

In conclusion,

$$L^{(1,A)}_{00}(a, b, c, \tau) = \frac{1}{8} \frac{p^{-a-b-c-2} (1 - p^{-1})}{1 - p^{-a-b-c-2}} \left\{ -p^{-1} + \frac{p^{-1-c} (1 - p^{-1})}{1 - p^{-1-c}} \right\}.$$

Now, by Formula 1,

$$L^{(1,B)}_{00}(a, b, c, \tau) = \left\{ \frac{1}{8} \int_{p\mathbb{Z}_p} |t|_{p}^{a+b+c+1} dt \right\} \left\{ \int_{p\mathbb{Z}_p} |z|_{p}^{b} \text{sgn}_\tau(z) dz \right\} = 0.$$

5. Formula 8

For $a, b, c \in \mathbb{C}$, we set

$$L^{(2)}_{00}(a, b, c, \tau) := \frac{1}{8} \int_{(p\mathbb{Z}_p)^2} |x_2|_{p}^{a} |x_3|_{p}^{b} |x_2 - x_3|_{p}^{c} \text{sgn}_\tau(x_2) \text{sgn}_\tau(x_2 - x_3) dx_2 dx_3.$$

Then

$$L^{(2)}_{00}(a, b, c, \tau) = L^{(1)}_{00}(a, c, b, \tau).$$

This identity is obtained by changing variables as $u = x_2, v = x_2 - x_3$, and using Formula 7.
6. Formula 9

For $a, b, c \in \mathbb{C}$, we set

$$L^{(3)}_{00}(a, b, c, \tau) := \frac{1}{8} \int_{(p\mathbb{Z}_p)^2} |x_2|_p^a |x_3|_p^b |x_2 - x_3|^c \text{sgn}_x(x_3) \text{sgn}_x(x_2 - x_3) \, dx_2 \, dx_3.$$ 

Then

$$L^{(3)}_{00}(a, b, c, \tau) = -L^{(2)}_{00}(b, a, c, \tau).$$

This formula follows from Formula 8 by changing variables as $(x_2, x_3) \rightarrow (x_3, x_2)$.

B. Computation of $Z^{(5)}(s, \bar{s}, \tau, \theta)$

The computation of $Z^{(5)}(s, \bar{s}, \tau, \theta)$ is reduced to the computation of integral $L(s, \tau)$, see (38)-(39). By using the partition

$$\mathbb{Z}_p^2 = \bigcup_{i,j=0}^{p-1} (i + p\mathbb{Z}_p) \times (j + p\mathbb{Z}_p),$$

We have

$$L(s, \tau) = \sum_{i,j=0}^{p-1} L_{ij}(s, \tau)$$

where

$$L_{ij}(s, \tau) = \int_{i+p\mathbb{Z}_p \times j+p\mathbb{Z}_p} F^{(5)}(x_2, x_3, s, \tau) \, dx_2 \, dx_3.$$

The calculation of these integrals is achieved by considering several cases.

**Case $i, j \in \{2, 3, \ldots, p-1\}$ and $i \neq j$.**

In this case, by using that $H_x|_{i+p\mathbb{Z}_p} = H_x(i)$ for $i \in \{1, \ldots, p-1\}$,

$$L_{ij}(s, \tau) = p^{-2} H_x(i) H_x(1 - i) H_x(j) H_x(1 - i) H_x(i - j).$$

Now by using Formula 5, the contribution of all these integrals is

$$\sum_{i,j=2 \atop i \neq j}^{p-1} L_{ij}(s, \tau) = \frac{(p - 3)(p - 7)}{32p^2}. \tag{47}$$

**Case $i, j \in \{2, 3, \ldots, p-1\}$ and $i = j$.**
In this case, by using (34),

\[
L_{10}(s, \tau) = H_r^2(i)H_r^2(1-i) \int_{1+p\mathbb{Z}_p \times 1+p\mathbb{Z}_p} |x_2 - x_3|_p^{s_{23}} H_r(x_2 - x_3) dx_2 dx_3
\]

\[
= H_r(i)H_r(1-i) \int_{(p\mathbb{Z}_p)^2} |x_2 - x_3|_p^{s_{23}} H_r(x_2 - x_3) dx_2 dx_3
\]

\[
= p^{-1}H_r(i)H_r(1-i) \int_{p\mathbb{Z}_p} |x_2|_p^{s_{23}} H_r(x_2) dx_2 = p^{-1}H_r(i)H_r(1-i)I_0(s_{23}, \tau)
\]

\[
= p^{-1}H_r(i)H_r(1-i) \frac{p^{-1-s_{23}}(1-p^{-1})}{2(1-p^{-s_{23}})}.
\]

Now, by using that \( p \equiv 3 \mod 4, \tau \neq \varepsilon,\) and Formula 2, the contribution of all these integrals is

\[
\frac{p^{-1-s_{23}}(1-p^{-1})}{2(1-p^{-s_{23}})} \frac{1}{p} \sum_{j=2}^{p-1} H_r(j)H_r(1-j) = \left( \frac{p-3}{8p} \right) \frac{p^{-1-s_{23}}(1-p^{-1})}{(1-p^{-s_{23}})}. \tag{48}
\]

**Case \( i = 1 \) and \( j = 0 \).**

In this case by using Formula 1,

\[
L_{10}(s, \tau) = \int_{1+p\mathbb{Z}_p \times p\mathbb{Z}_p} |1 - x_2|_p^{8_{42}} |x_3|_p^{8_{13}} H_r(1-x_2)H_r(x_3) dx_2 dx_3
\]

\[
= \left\{ \int_{1+p\mathbb{Z}_p} |1 - x_2|_p^{8_{42}} H_r(1-x_2) dx_2 \right\} \left\{ \int_{p\mathbb{Z}_p} |x_3|_p^{8_{13}} H_r(x_3) dx_3 \right\}
\]

\[
= \left\{ \int_{p\mathbb{Z}_p} |x_2|_p^{8_{42}} H_r(-x_2) dx_2 \right\} \left\{ \int_{p\mathbb{Z}_p} |x_3|_p^{8_{13}} H_r(x_3) dx_3 \right\}
\]

\[
= \frac{1}{2} \int_{p\mathbb{Z}_p} |x_2|_p^{8_{42}} dx_2 \left\{ \frac{1}{2} \int_{p\mathbb{Z}_p} |x_3|_p^{8_{13}} dx_3 \right\} = \frac{(1-p^{-1})^2}{4} \frac{p^{-2-s_{42}-s_{13}}}{(1-p^{-s_{42}})(1-p^{-s_{13}})}. \tag{49}
\]

**Case \( i = 0 \) and \( j = 1 \).**

Since

\[
H_r(x_2 - x_3) \bigg|_{p\mathbb{Z}_p \times 1+p\mathbb{Z}_p} = H_r(-1) = 0,
\]

we have \( L_{01}(s, \tau) = 0.\)

**Case \( i = j = 0 \).**

In this case,

\[
L_{00}(s, \tau) = \int_{(p\mathbb{Z}_p)^2} |x_2|_p^{s_{12}} |x_3|_p^{s_{13}} |x_2 - x_3|_p^{s_{23}} H_r(x_2)H_r(x_3)H_r(x_2 - x_3) dx_2 dx_3.
\]
By using that
\[
H_r(x_2)H_r(x_3)H_r(x_2 - x_3) = \frac{1}{8} \{ 1 + \text{sgn}_r(x_2) + \text{sgn}_r(x_3) + \text{sgn}_r(x_2 - x_3) \\
+ \text{sgn}_r(x_2)\text{sgn}_r(x_3) + \text{sgn}_r(x_2)\text{sgn}_r(x_2 - x_3) + \text{sgn}_r(x_3)\text{sgn}_r(x_2 - x_3) \\
+ \text{sgn}_r(x_2)\text{sgn}_r(x_3)\text{sgn}_r(x_2 - x_3) \}
\]
and the notation introduced in Formulae 6 to 9, we have
\[
L_{00}(s, \tau) = L_{00}(s_{12}, s_{13}, s_{23}) + L_{00}^{(1)}(s_{12}, s_{13}, s_{23}, \tau) + L_{00}^{(2)}(s_{12}, s_{13}, s_{23}, \tau) + L_{00}^{(3)}(s_{12}, s_{13}, s_{23}, \tau) \\
= L_{00}(s_{12}, s_{13}, s_{23}) + L_{00}^{(1)}(s_{12}, s_{13}, s_{23}, \tau) + L_{00}^{(1)}(s_{12}, s_{23}, s_{13}, \tau) - L_{00}^{(1)}(s_{13}, s_{23}, s_{12}, \tau),
\]
the integrals involving an odd number of sign functions vanish. This fact can be established by a suitable change of variables as in Formula 1.

**Case** \( i = j = 1 \).

In this case,
\[
L_{11}(s, \tau) = \int_{(1+pZ_p)^2} |1 - x_2|^{s_{42}}|1 - x_3|^{s_{43}}|x_2 - x_3|^{s_{23}}H_r(1 - x_2)H_r(1 - x_3)H_r(x_2 - x_3)dx_2dx_3.
\]
Now by changing variables as \( u = 1 - x_2, v = 1 - x_3 \), we get
\[
L_{11}(s, \tau) = \int_{(pZ_p)^2} |u|^{s_{42}}|v|^{s_{43}}|u - v|^{s_{23}}H_r(u)H_r(v)H_r(v - u)dudv
= L_{00}(s_{42}, s_{43}, s_{23}) + L_{00}^{(1)}(s_{42}, s_{43}, s_{23}, \tau) - L_{00}^{(2)}(s_{42}, s_{43}, s_{23}, \tau) - L_{00}^{(3)}(s_{42}, s_{43}, s_{23}, \tau) \\
= L_{00}(s_{42}, s_{43}, s_{23}) + L_{00}^{(1)}(s_{42}, s_{43}, s_{23}, \tau) - L_{00}^{(1)}(s_{42}, s_{23}, s_{43}, \tau) + L_{00}^{(1)}(s_{43}, s_{23}, s_{42}, \tau)
\]

**Cases** \( i = 0 \) and \( j \in \{2, 3, \ldots, p - 1\} \) or \( i \in \{2, 3, \ldots, p - 1\} \) and \( j = 1 \).

In these cases,
\[
L_{0j}(s, \tau) = L_{11}(s, \tau) = 0.
\]

The vanishing of the integral \( L_{0j}(s, \tau) \) follows from
\[
H_r(x_3)H_r(x_2 - x_3)\bigg|_{pZ_p \times j + pZ_p} = H_r(j)H_r(-j) = 0.
\]
The other case is treated in a similar way.

**Case** \( i \in \{2, 3, \ldots, p - 1\} \) and \( j = 0 \).
By using (34),
\[
L_{i0}(s, \tau) = H_r^2(i)H_r(1-i)H_r(1)\int_{i+p\mathbb{Z}\times p\mathbb{Z}} |x_3|_{p}^{s_{13}} H_r(x_3)dx_2dx_3
\]
\[
= p^{-1}H_r(i)H_r(1-i)\int_{p\mathbb{Z}} |x_3|_{p}^{s_{13}} H_r(x_3)dx_3 = p^{-1}H_r(i)H_r(1-i)I_0(s_{13}, \tau)
\]
\[
= p^{-1}H_r(i)H_r(1-i)\frac{p^{-1-s_{13}}(1-p^{-1})}{2(1-p^{-1-s_{13}})}. \tag{53}
\]

Now, using Formula 2, the contribution of all these integrals is
\[
\sum_{i=2}^{p-1} L_{i0}(s, \tau) = \frac{p^{-1-s_{13}}(1-p^{-1})}{2(1-p^{-1-s_{13}})} \sum_{i=2}^{p-1} H_r(i)H_r(1-i) = \left(\frac{p-3}{8p}\right) \frac{p^{-1-s_{13}}(1-p^{-1})}{(1-p^{-1-s_{13}})}. \tag{54}
\]

Case \(i = 1\) and \(j \in \{2, 3, \ldots, p-1\}\).

This case is similar to the previous one,
\[
\sum_{j=2}^{p-1} L_{1j}(s, \tau) = p^{-1}I_0(s_{42}, \tau) \sum_{j=2}^{p-1} H_r(j)H_r(1-j) = \left(\frac{p-3}{8p}\right) \frac{p^{-1-s_{42}}(1-p^{-1})}{(1-p^{-1-s_{42}})}. \tag{55}
\]

In conclusion, from (38), (39), and (47)-(55), we have
\[
Z^{(5)}(s, \tilde{s}, \theta) = E^{(5)}(\tilde{s}, \theta) \left\{ \left(\frac{p-3}{8p}\right) \frac{p^{-1-s_{23}}(1-p^{-1})}{(1-p^{-1-s_{23}})} \right. \\
+ \left. \frac{p^{-1-s_{13}}(1-p^{-1})}{(1-p^{-1-s_{13}})} + \frac{p^{-1-s_{42}}(1-p^{-1})}{(1-p^{-1-s_{42}})} \right\} + \frac{(1-p^{-1})^2}{4} \frac{p^{-2-s_{13}-s_{42}}}{(1-p^{-1-s_{13}})(1-p^{-1-s_{42}})}
\]
\[
+ \frac{1}{4} \frac{p^{-s_{12}-s_{13}-s_{23}-2}(1-p^{-1})}{1-p^{-s_{12}-s_{13}-s_{23}-2}} \left[ \left(\frac{3}{2p}\right) - 1 \right]
\]
\[
+ \frac{1}{4} \frac{p^{-s_{42}-s_{43}-s_{23}-2}(1-p^{-1})}{1-p^{-s_{42}-s_{43}-s_{23}-2}} \left[ \left(\frac{3}{2p}\right) + 1 \right]
\]
\[
\left\{ \left(\frac{p-3}{8p}\right) \frac{p^{-1-s_{23}}(1-p^{-1})}{(1-p^{-1-s_{23}})} \right. \\
+ \left. \frac{p^{-1-s_{13}}(1-p^{-1})}{(1-p^{-1-s_{13}})} + \frac{p^{-1-s_{42}}(1-p^{-1})}{(1-p^{-1-s_{42}})} \right\}. \tag{56}
\]

\(Z^{(5)}(s, \tilde{s}, \theta)\) is a holomorphic function in
\[
\text{Re}(s_{13}) > -1; \quad \text{Re}(s_{23}) > -1; \quad \text{Re}(s_{42}) > -1;
\]
\[
\text{Re}(s_{12} + s_{13} + s_{23}) > -2; \quad \text{Re}(s_{42} + s_{43} + s_{23}) > -2.
\]

VIII. THE LIMIT \(p \to 1\) OF THE GHOSHAL-KAWANO AMPLITUDES

In [45] we established that the limit \(p\) approaches to one of \(p\)-adic open string amplitudes at the tree-level can be defined rigorously by using the Denef and Loeser theory of topological
zeta functions \[46\]. Notice that the calculations involving the limit \( p \to 1 \) in the case of the effective action are performed in \( \mathbb{R}^D \), meanwhile the calculations involving the limit \( p \to 1 \) in the case of \( p \)-adic string amplitudes are performed in \( \mathbb{Q}_p^D \), and in the \( p \)-adic topology the limit \( p \to 1 \) does not make sense. However, surprisingly, the computation of the limit \( p \to 1 \) (considering \( p \) as a real parameter) of the \( p \)-adic open string amplitudes gives the right answer! In this subsection we compute limit \( p \to 1 \) (considering \( p \) as a real parameter) in the cases \( N = 4, 5 \). The computation of the limit \( p \to 1 \) in the general case require the so called explicit formulas, see \[45\] for further details.

The limit \( p \to 1 \) of \( Z^{(N)}(s, \tilde{s}, \tau, \theta) \), \( N = 4, 5 \), with \( p \equiv 3 \mod 4 \), are given by

\[
\lim_{p \to 1} Z^{(4)}(s, \tilde{s}, \tau, \theta) = \exp \left\{ \frac{i}{2} (\tilde{s}_{12} + \tilde{s}_{23}) \right\} \left\{ -\frac{1}{2} + \frac{1}{2(s_{12} + 1)} + \frac{1}{2(s_{32} + 1)} \right\},
\]

for \( \tau \neq \varepsilon \), and

\[
\lim_{p \to 1} Z^{(5)}(s, \tilde{s}, \theta) = E^{(5)}(\tilde{s}, \theta) \left\{ \frac{3}{16} - \frac{1}{4(s_{23} + 1)} - \frac{1}{4(s_{13} + 1)} - \frac{1}{4(s_{42} + 1)} \right. \\
+ \frac{1}{4(s_{42} + 1)(s_{13} + 1)} + \frac{1}{4(s_{12} + s_{13} + s_{23} + 2)} \left[ -1 + \frac{1}{(s_{23} + 1)} + \frac{1}{(s_{13} + 1)} \right] \\
+ \frac{1}{4(s_{42} + s_{43} + s_{23} + 2)} \left[ -1 + \frac{1}{(s_{23} + 1)} + \frac{1}{(s_{42} + 1)} \right] \right. \\
\left. \right\}.
\]

In the case \( N = 4 \), the amplitude agrees with the amplitude the Feynman obtained from the noncommutative version of the Gerasimov-Shatashvili action with a logarithmic potential \( \mathfrak{g} \).

**IX. FINAL REMARKS**

In the present article, we study the Ghoshal-Kawano amplitudes for \( p \)-adic open strings at tree-level level \[50\]. These amplitudes include Chan-Paton factors and an external \( B \)-field.

In section \[III\] starting from the noncommutative effective action \( \mathfrak{g} \) discussed in \[50, 52\], in the present article, we obtain the corresponding tree-level four-point amplitudes \( \mathcal{A} \) in the limit \( p \to 1 \). This result was achieved by adapting the heuristic approach given in \[42\] for the noncommutative case. By an explicit computation using the noncommutative field theory \[34, 55\], we determine the four-point amplitude at the tree level coming from the noncommutative Gerasimov-Shatashvili Lagrangian. This amplitude is the sum of the expressions \( \mathcal{A} \) and \( \mathcal{B} \). The first one represents the noncommutative vertex four-point
function and the second one is the superposition of the amplitudes corresponding to the noncommutative channels \( s, t \) and \( u \). The calculated tree-level amplitude is completely described by planar Feynman diagrams and consequently the noncommutativity effect arises as a global phase factor in front of the amplitude. Five-point amplitudes (or higher-order amplitudes) can be also computed in a straightforward way following the same procedure.

The study of the \( p \)-adic Ghoshal-Kawano amplitudes requires the use of multivariate local zeta functions involving multiplicative characters and a phase factor including the noncommutative parameter \( \theta \). These are new mathematical objects. We call these objects Ghoshal-Kawano zeta functions. In sections \( \text{IV} \) and \( \text{V} \) by using Hironaka’s resolution of singularities theorem, we prove that these integrals admit meromorphic continuation as complex functions in the external momenta of the \( N \) external particles.

Four and five point amplitudes were computed explicitly in sections \( \text{VI} \) and \( \text{VII} \) see (36) and (56). The four-point amplitude (36) coincides with the one obtained in [50]. The five-point amplitude was not obtained previously.

In section \( \text{VIII} \) we study again the amplitudes from the worldsheet viewpoint. We compute the limit \( p \to 1 \) limit for four and five point amplitudes resulting in the formulae (57) and (58), respectively. The four-point amplitude (57) agrees with the heuristic computation given by the superposition of formulae (14) and (16).

As we mentioned before, in the computation of Ghoshal-Kawano amplitudes at the tree-level, the noncommutative effect coming from the constant \( B \)-field arises only as a global phase factor because only planar diagrams are involved. In the computation of amplitudes at one-loop or multi-loops non-planar diagrams systematically arise. It would be very interesting to study the possibility of finding a non-trivial noncommutative effect as the IR/UV mixing as a result of the contribution of one-loop non-planar diagrams. Probably the multiloop analysis of the \( p \)-adic string theory studied in [56], will play an important role for the analysis of the IR/UV mixing and other interesting effects of the \( B \)-field in \( p \)-adic string theory amplitudes.

On the other hand, we think that the study of the amplitudes \( (1) \) without the ad hoc normalization \( x_1 = 0, x_{N-1} = 1, x_N = \infty \) may provide new insights on the effects of the \( B \)-field in \( p \)-adic string theory amplitudes. However, the study of these amplitudes is more involved than the one done here. Some of this work is in progress and will be reported elsewhere.
X. APPENDIX: BASIC ASPECTS OF THE $p$-ADIC ANALYSIS

In this appendix we collect some basic results about $p$-adic analysis that will be used in
the article. For an in-depth review of the $p$-adic analysis the reader may consult [5, 58, 59].

A. The field of $p$-adic numbers

Along this article $p$ will denote a prime number different from 2. The field of $p$−
adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$−adic norm $| \cdot |_p$, which is defined as

$$ |x|_p = \begin{cases} 
0 & \text{if } x = 0 \\
 p^{-\gamma} \text{ if } x = p^{\gamma} \frac{a}{b}, 
\end{cases} $$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the $p$−adic order of $x$.

Any $p$−adic number $x \neq 0$ has a unique expansion of the form

$$ x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j, $$

where $x_j \in \{0, \ldots, p - 1\}$ and $x_0 \neq 0$. In addition, any non-zero $p$−adic number can be represented uniquely as $x = p^{\text{ord}(x)} \text{ac}(x)$ where $\text{ac}(x) = \sum_{j=0}^{\infty} x_j p^j$, $x_0 \neq 0$, is called the angular component of $x$. Notice that $|\text{ac}(x)|_p = 1$.

We extend the $p$−adic norm to $\mathbb{Q}_p^n$ by taking

$$ ||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \text{ for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n. $$

We define $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$, then $||x||_p = p^{-\text{ord}(x)}$. The metric space $(\mathbb{Q}_p^n, || \cdot ||_p)$ is a separable complete ultrametric space. For $r \in \mathbb{Z}$, denote by $B_r^n(a) = \{x \in \mathbb{Q}_p^n; ||x-a||_p \leq p^r\}$ the ball of radius $p^r$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_r^n(0) := B_r$. Note that $B_r^n(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius $p^r$ with center at $a_i \in \mathbb{Q}_p$. The ball $B_0^n$ equals the product of $n$ copies of $B_0 = \mathbb{Z}_p$, the ring of $p$−adic integers of $\mathbb{Q}_p$. We also denote by $S_r^n(a) = \{x \in Q_p^n; ||x-a||_p = p^r\}$ the sphere of radius $p^r$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$. 

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and take $S^n_r(0) := S^n_r$. We notice that $S^n_0 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $(\mathbb{Z}_p^\times)^n \subsetneq S^n_0$.

The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p^n, || \cdot ||_p)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$; see e.g. [5, Section 1.3], or [58, Section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^n, || \cdot ||_p)$ is a locally compact topological space.

**B. Integration**

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, there exists a measure $dx$, which is invariant under translations, i.e. $d(x + a) = dx$. If we normalize this measure by the condition $\int_{\mathbb{Q}_p} dx = 1$, then $dx$ is unique. A such measure is called the Haar measure of $(\mathbb{Q}_p, +)$. In the $n$-dimensional case, $(\mathbb{Q}_p^n, +)$ is also locally compact topological group. We denote by $d^n(x)$ the Haar measure normalized by the condition $\int_{\mathbb{Q}_p^n} d^n(x) = 1$. This measure agrees with the product measure $dx_1 \cdots dx_n$, and it also satisfies that $d^n(x + a) = d^n x$, for $a \in \mathbb{Q}_p^n$.

A function $h : U \to \mathbb{Q}_p$ is said to be analytic on an open subset $U \subseteq \mathbb{Q}_p^n$, if for every $b = (b_1, \ldots, b_n) \in U$, there exists an open subset $\tilde{U} \subset U$, with $b \in \tilde{U}$, and a convergent power series $\sum_{i \in \mathbb{N}_n} a_i (x - b)^i$ for $x = (x_1, \ldots, x_n) \in \tilde{U}$, such that $h(x) = \sum_{i \in \mathbb{N}_n} a_i (x - b)^i$ for $x \in \tilde{U}$, with $i = (i_1, \ldots, i_n)$ and $(x - b)^i = \prod_{j=1}^n (x_j - b_j)^{i_j}$. In this case, $\frac{\partial}{\partial x_i} h(x) = \sum_{i \in \mathbb{N}_n} a_i \frac{\partial}{\partial x_i} (x - b)^i$ is a convergent power series.

Let $U, V$ be open subsets in $\mathbb{Q}_p^n$. A mapping $H : U \to V$, $H = (H_1, \ldots, H_n)$ is called analytic if each $H_i$ is analytic. The mapping $H$ is said to be bi-analytic if $H$ and $H^{-1}$ are analytic.

1. **Change of variables formula**

Let $K_0, K_1 \subset \mathbb{Q}_p^n$ be open compact subsets, and let $H = (H_1, \ldots, H_n) : K_1 \to K_0$ be a bi-analytic map such that

$$\det \left[ \frac{\partial H_i}{\partial y_j} (x) \right] \neq 0, \text{ for } x \in K_1.$$
If \( f \) is a continuous function on \( K_0 \), then
\[
\int_{K_0} f(x) \, d^n x = \int_{K_1} f(\sigma(y)) \left| \det \left[ \frac{\partial H_i}{\partial y_j}(y) \right] \right| \, d^n y, \quad (x = H(y)).
\]

For the proof of this theorem the reader may consult [39, Prop. 7.4.1] or [60, Section 10.1.2].

C. Some arithmetic functions

In this section we review some arithmetic functions that we shall use throughout this article.

1. Multiplicative characters

A multiplicative character (or quasi-character) of the group \((\mathbb{Q}_p^\times, \cdot)\) is a continuous homomorphism \( \chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times \) satisfying \( \chi(xy) = \chi(x) \chi(y) \). Every multiplicative character has the form
\[
\chi(x) = |x|_p^s \chi_0(ac(x)), \text{ for some } s \in \mathbb{C},
\]
where \( \chi_0 \) is the restriction of \( \chi \) to \( \mathbb{Z}_p^\times \), which is a continuous multiplicative character of \( (\mathbb{Z}_p^\times, \cdot) \) into the complex unit circle.

2. The Legendre symbol

For \( a \) an integer number and \( p \) a prime number, the Legendre symbol is defined as
\[
\left( \frac{a}{p} \right) = \begin{cases} 
1 & \text{if } x^2 \equiv a \mod p \text{ has a solution} \\
-1 & \text{otherwise.}
\end{cases}
\]

The following formulas are used in several calculations in this article:
\[
\left( \frac{1}{p} \right) = 1; \quad \left( \frac{-1}{p} \right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 
1 & \text{if } p \equiv 1 \mod 4 \\
-1 & \text{if } p \equiv 3 \mod 4.
\end{cases}
\]

Take for \( x \in \mathbb{Q}_p^\times \), \( ac(x) = x_0 + x_1 p + \ldots \in \mathbb{Z}_p^\times \), then
\[
\mathbb{Q}_p^\times \to \{\pm1\}
\]
\[
x \to \left( \frac{x_0}{p} \right)
\]
is a unitary multiplicative character.

3. The sign function

We review $p$-adic sign function, which is a multiplicative character with values in $\{\pm 1\}$. We denote $\left[Q_p^\times\right]^2$ the multiplicative subgroup of squares in $Q_p^\times$, i.e.

$$\left[Q_p^\times\right]^2 = \{a \in Q_p; a = b^2 \text{ for some } b \in Q_p^\times\}.$$ 

If $p \neq 2$, where $\varepsilon \in \{1, \ldots, p-1\}$ satisfying $\left(\frac{\varepsilon}{p}\right) = -1$. This means that any nonzero $p$-adic number can be written in either of four ways

$$x = \begin{cases} 
   a^2 & \text{for some } a \in Q_p^\times \\
   \varepsilon a^2 \\
   p a^2 \\
   \varepsilon p a^2 
\end{cases}$$

For a fixed $\tau \in \{\varepsilon, p, \varepsilon \cdot p\}$, and $x \in Q_p^\times$, we set

$$\text{sgn}_\tau(x) := \begin{cases} 
   1 & \text{if } x = a^2 - \tau b^2 \text{ for } a, b \in Q_p \\
   -1 & \text{otherwise.} 
\end{cases} \quad (59)$$

The following is the list of all the possible $p$-adic sign functions:

| $p \equiv 1 \mod 4$ | $p \equiv 3 \mod 4$ |
|---------------------|---------------------|
| $\text{sgn}_\varepsilon(x) = (-1)^{\text{ord}(x)}$ | $\text{sgn}_\varepsilon(x) = (-1)^{\text{ord}(x)}$ |
| $\text{sgn}_p(x) = \left(\frac{x_0}{p}\right)$ | $\text{sgn}_p(x) = (-1)^{\text{ord}(x)} \left(\frac{x_0}{p}\right)$ |
| $\text{sgn}_\varepsilon p(x) = (-1)^{\text{ord}(x)} \left(\frac{x_0}{p}\right)$ | $\text{sgn}_\varepsilon p(x) = \left(\frac{x_0}{p}\right)$ |

see [23]. Then $\text{sgn}_\tau$ is a multiplicative character, and a locally constant function in $Q_p^\times$, more precisely, $\text{sgn}_\tau(x - y) = \text{sgn}_\tau(x)$ if $\text{ord}(y) > \text{ord}(x)$.

We take $p \equiv 3 \mod 4$, to have $\text{sgn}_\tau(-y) = -\text{sgn}_\tau(y)$, for any $y \in Q_p^\times$. In all the calculations involving $\text{sgn}_\tau$ we assume that $p \equiv 3 \mod 4$. We define the Heaviside step...
function as

\[ H_\tau(x) := H_\tau^+ = \frac{1}{2}(1 + \text{sgn}_\tau(x)) = \begin{cases} 1 & \text{if } \text{sgn}_\tau(x) = 1 \\ 0 & \text{if } \text{sgn}_\tau(x) = 1, \end{cases} \]

for any \( x \in \mathbb{Q}_p^\times \). It is convenient to set

\[ H_\tau^\pm(x) := \frac{1}{2}(1 \pm \text{sgn}_\tau(x)). \]

The following properties are useful:

\[ H_\tau(x)H_\tau(x) = H_\tau(x); \quad H_\tau^+(x) + H_\tau^-(x) = 1; \quad H_\tau^+(x)H_\tau^-(x) = 0; \]

\[ H_\tau(xy) = H_\tau(x)H_\tau(y) + H_\tau^-(x)H_\tau^-(y). \]

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