STRUCTURAL STABILITY OF THE RIEMANN SOLUTION FOR A STRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS WITH FLUX APPROXIMATION

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(Communicated by Rafael Ortega)

Abstract. In this article, we study the Riemann problem for a strictly hyperbolic system of conservation laws under the linear approximation of flux functions with three parameters. The approximation does not affect the structure of Riemann problem. Furthermore, we prove that the Riemann solution to the approximated system converges to the original system as the perturbation parameter tends to zero.

1. Introduction. In this article we are concerned with the strictly hyperbolic system of conservation laws that arises in nonlinear elasticity and gasdynamics [15]

\[
\begin{align*}
&u_t + (u^2)_x = 0, \\
&v_t + ((2u + 1)v)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0,
\end{align*}
\]

where \(x\) and \(t\) represent space and time coordinate, respectively and \(u\) is the background flow field which carries certain dust particles with density \(v\). The first equation in (1) is just the inviscid Burgers equation and its solution to the Riemann problem is the classical entropy solution. The Dirac function is introduced as a part for \(v\) when the characteristic velocity \(u\) is discontinuous. Tan et al. [15] considered the Riemann problem of (1) and they proved the form of the standard Dirac delta function supported on a shock wave that was used as a part in their Riemann solution for certain initial data. Wave interactions and stability of the Riemann solution for the system (1) have been discussed by Anupam et al. [11]. A delta shock wave is a generalization of an ordinary shock wave; speaking informally, it consists of a discontinuity line \(x = x(t)\) plus a distributed Dirac delta function with the discontinuity line as its support. A delta shock wave is over-compressive in the sense that the number of characteristics entering the discontinuity line of the delta shock wave is more than in the case of an ordinary shock wave. Several approaches for constructing \(\delta\)-shock type solutions are known. In [1, 2], Chen and

2000 Mathematics Subject Classification. Primary: 35L45, 35L65, 58K25; Secondary: 35Q35, 35L67.

Key words and phrases. Delta shock wave, Riemann problem, strictly hyperbolic system, flux approximation.

The first author is supported by University Grant Commission, Government of India (Sr. No. 2121540947, Ref No. 20/12/2015(ii)EU-V). The second author is supported by Science and Engineering Research Board, Department of Science and Technology, Government of India (Ref No: SB/FTP/MS-047/2013).

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Liu identified and analyzed the phenomena of concentration, cavitation and the formation of delta shocks and vacuum states in solutions to the Euler equation for isentropic and nonisentropic fluids as the pressure vanishes. The weak asymptotic method was used by Danilov et al. [4] in their study of delta shock wave-type solution. In [5, 10, 6], it was shown that for some cases the hyperbolic system

\[ u_t + (F(u, v))_x = 0, \quad v_t + (G(u, v))_x = 0, \tag{2} \]

or

\[ v_t + (G(u, v))_x = 0, \quad (uv)_t + (H(u, v))_x = 0, \tag{3} \]

where \( F(u, v), G(u, v) \) and \( H(u, v) \) are smooth functions and are linear with respect to \( v \), admits “nonclassical” situations [7], when the Riemann problem does not possess a weak \( L^\infty \)-solution except for some particular initial data. In contrast to the standard results of existence of weak solutions to strictly hyperbolic systems, here the linear component \( v \) of the solution may contain Dirac measures and must be sought in the space of measures, while the nonlinear component \( u \) of the solution has bounded variation. In order to solve the Cauchy problem in this nonclassical situation, it is necessary to introduce \( \delta \)-shock type singularities, which are solutions of the system of conservation laws. The formation of delta shock wave has been extensively studied by using the vanishing pressure approximation for the system of pressureless gas dynamics [1, 2, 17] and Chaplygin gas dynamics [3], which is a particular case of flux approximation. Recently, the flux function approximation with two parameters [12] and three parameters [16] have been studied for the system of pressureless gas dynamics. Sun [14] proved that Riemann solution are stable for non strictly hyperbolic system of conservation laws under the linear approximation of flux functions with three parameters.

In this paper, we consider the linear approximation of flux function in (1) as follows:

\[ \{ \begin{array}{l}
    u_t + (u^2 + \varepsilon \alpha u)_x = 0, \\
    v_t + ((2u + 1)v + \varepsilon \beta u + \varepsilon \gamma v)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0,
\end{array} \tag{4} \]

where \( \alpha, \beta \) and \( \gamma \) are arbitrary real numbers such that \( \gamma > \alpha \) and \( \varepsilon \) is sufficiently small positive number.

Now, we consider the Riemann problem of (4) with initial data

\[ (u, v)(x, 0) = \begin{cases} 
    (u_-, v_-), & x < 0, \\
    (u_+, v_+), & x > 0,
\end{cases} \tag{5} \]

where \( u_\pm \) and \( v_\pm \) are given positive constants.

The system (1) is a particular example of strictly hyperbolic system. It can be seen that (4) is also a strictly hyperbolic system of conservation laws under triangular linear approximation of flux functions. We show that the delta shock wave also appears in the solution of Riemann problem (4) and (5) for some special initial data. Furthermore, we prove rigorously that the limits of the Riemann solution to the problem (4) and (5) converges to the solution of Riemann problem (1) and (5), when the perturbation parameter \( \varepsilon \) tends to zero. In other words, the Riemann solution of (1) and (5) is stable with respect to the triangular linear approximation of flux functions in the form of (4). Thus, this triangular linear approximations of flux functions in the form of (4) does not change the structure of solution to the Riemann problem (1) and (5).

The organization of this paper is as follows. In section 2, we present some preliminaries for the system (1) and (5) and display the Riemann solution of (1)
and (5) with different possible initial data. In section 3, the Riemann problem for the approximated system (4) is considered and Riemann solutions are constructed for three different cases. In section 4, the limit of Riemann solution to the approximated system (4) is taken by letting the perturbation parameter $\varepsilon$ tends to zero, which is identical with the corresponding ones to the original system (1). Finally, conclusions are drawn in section 5.

2. Preliminaries. In this section, we describe some results on the Riemann solution to (1) which were obtained by Tan et al [15]. The system (1) is one of the example of strictly hyperbolic system for which the Riemann problem cannot be solved for all initial data with classical elementary waves such as shock waves, rarefaction waves and contact discontinuities. Such systems arise in nonlinear elasticity and gas dynamics. The eigenvalues of the system (1) are $\lambda_1 = 2u$ and $\lambda_2 = (2u + 1)$ and the corresponding right eigenvectors are $\vec{r}_1 = (1, -2v)^t$ and $\vec{r}_2 = (0, 1)^t$, respectively. Since $\nabla \lambda_1 \cdot \vec{r}_1 = 2 \neq 0$ and $\nabla \lambda_2 \cdot \vec{r}_2 = 0$, the first characteristic field is genuinely nonlinear and second one is linearly degenerate. The Riemann invariants associated with the first and second characteristic fields are $z_1 = ve^{2u}$ and $z_2 = u$, respectively.

Let $(u_-, v_-)$ and $(u, v)$ denote, respectively, the left-hand and right-hand states of either classical elementary waves or delta shock wave. Let us fix $(u_-, v_-)$ in the domain of hyperbolicity and compute the state $(u, v)$ which is connected on the right by either classical elementary waves or delta shock wave as given below (see Figure 1).

The 1-rarefaction wave curves in the phase plane are:

$$R(u_-, v_-) = \begin{cases} \frac{dx}{dt} = \lambda_1 = 2u, \\ v = v_- e^{2(u_- - u)}, \\ u_- < u, \end{cases}$$

and 1-shock wave curves in the phase plane are:

$$S(u_-, v_-) = \begin{cases} C = u + u_-, \\ u - u_- = \frac{v_- - v}{v_- + v}, \\ u < u_- < u + 1, \end{cases}$$

where $C$ is the shock speed.

The possible states that can be connected to $(u_-, v_-)$ on the right by a contact discontinuity curve, which is given as follows:

$$J(u_-, v_-) = \begin{cases} \tau = 2u + 1 = 2u_- + 1, \\ u = u_-, \end{cases}$$

where $\tau$ is the speed of contact discontinuity.

For the case $u_- \geq u_+ + 1$, the Riemann solution contains delta measure supported on a line. In order to define the delta shock wave solution to the Riemann problem (1) and (5), let us define the following:

**Definition 2.1.** To define the measure solutions, the two-dimensional weighted $\delta$-measure $w(s)\delta_{\Gamma}$ that has support on a smooth curve $\Gamma = \{(x(s), t(s)) : a < s < b\}$ can be defined by:

$$\langle w(s)\delta_{\Gamma}, \phi(x(s), t(s)) \rangle = \int_a^b w(s)\phi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2} ds,$$
for every test function $\phi \in C^\infty_0(\mathbb{R} \times \mathbb{R}_+)$.

Now, let us use the definition of delta shock wave solution which was introduced by Danilov and Shelkovich [4, 5] and improved by Kalisch and Mitrovic [8, 9] below.

Suppose that $\Gamma = \{ \gamma_i | i \in I \}$ is a graph in the closed upper half-plane $\{(x,t) | x \in \mathbb{R}, t \geq 0\}$ which contains smooth arcs $\gamma_i$, where $i \in I$ and $I$ is the finite index set. Let $I_0$ be a subset of $I$ which contains all indices of arcs starting from the $x$–axis and $\Gamma_0 = \{ x^0_j : j \in I_0 \}$ is the set of initial points of the arcs $\gamma_j$ with $j \in I_0$.

**Definition 2.2.** Let $(u,v)$ be a pair of distributions where $v$ has the form

$$v(x,t) = \hat{v}(x,t) + w(x,t)\delta(\Gamma),$$

in which $u, \hat{v} \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ and the singular part is given by

$$w(x,t)\delta(\Gamma) = \sum_{i \in I} w_i(x,t)\delta(\gamma_i).$$

Let us consider the delta shock wave type initial data

$$(u,v)(x,0) = \left( u_0(x), \hat{v}_0(x) + \sum_{j \in I_0} w_j(x^0_j,0)\delta(x-x^0_j) \right), \quad (6)$$

in which $u_0, \hat{v}_0 \in L^\infty(\mathbb{R})$, then the pair of distributions $(u,v)$ are called as generalized delta shock wave solution for (1) with the delta shock type initial data (6) if the following integral identities

$$\int_0^\infty \int_{-\infty}^\infty (u\phi_t + u^2\phi_x) \, dx dt + \int_{-\infty}^\infty u_0(x)\phi(x,0) \, dx = 0,$$

and

$$\int_0^\infty \int_{-\infty}^\infty (\hat{v}\phi_t + (2u+1)\hat{v}\phi_x) \, dx dt + \sum_{i \in I} \int_{\gamma_i} w_i(x,t)\frac{\partial \phi(x,t)}{\partial l} \, dl$$

$$+ \int_{-\infty}^\infty \hat{v}_0(x)\phi(x,0) \, dx + \sum_{k \in I_0} w_k(x^0_k,0)\phi(x^0_k,0) = 0,$$

hold for all test functions $\phi \in C^\infty_0(\mathbb{R} \times \mathbb{R}_+)$, where $\frac{\partial \phi(x,t)}{\partial l}$ is the tangential derivative of $\phi$ on the curve $\gamma_i$ and $\int_{\gamma_i} dl$ is the line integral over the arc $\gamma_i$. 

---

**Figure 1.** Riemann solution of (1) and (5) in the phase plane.
With the above definition, a piecewise smooth solution of (1) and (5) can be constructed for the case $u_- \geq u_+ + 1$ in the form

$$
  u(x, t) = \begin{cases} 
    u_-, x < \sigma t, \\
    u_+, x > \sigma t,
  \end{cases} \quad v(x, t) = \begin{cases} 
    v_-, x < \sigma t, \\
    v_+, x > \sigma t,
  \end{cases} + w(t)\delta(x - \sigma t),
$$

(7)

where

$$
  \sigma = (u_- + u_+), \quad w(t) = ((v_+ + v_-)^2(1 - u_+ - u_+) + (v_+ - v_-))t,
$$

(8)

are propagation speed and strength of delta shock wave, respectively.

The delta shock wave solution (7) and (8) satisfies the generalized Rankine-Hugoniot condition

$$
  \begin{align*}
    \frac{dx(t)}{dt} &= \sigma, \\
    \frac{dw(t)}{dt} &= (\sigma[v] - [(2u + 1)v]), \\
    \sigma[w] &= [v^2],
  \end{align*}
$$

where $[v] = v(x(t) + 0) - v(x(t) - 0)$ denotes the jump in $v$ across the discontinuity curve $x = x(t)$. In order to ensure the uniqueness, the Delta entropy conditions $\lambda_1(u_-) \leq \sigma \leq \lambda_1(u_-)$ and $\lambda_2(u_+) \leq \sigma \leq \lambda_2(u_-)$ are imposed, which means that all the characteristics on both sides of the $\delta$-shock wave are incoming.

The above constructed delta shock wave solution (7) with (8) must hold

$$
\begin{align*}
  \langle u, \phi_t \rangle + \langle u^2, \phi_x \rangle &= 0, \\
  \langle v, \phi_t \rangle + \langle (2u + 1)v, \phi_x \rangle &= 0,
\end{align*}
$$

(9)

for every test function $\phi \in C^\infty_0(\mathbb{R} \times \mathbb{R}_+)$. From (9), as in [15, 13], we have

$$
\begin{align*}
  \langle v, \phi \rangle &= \int_0^\infty \int_{-\infty}^\infty v_0\phi dx dt + (w(t)\delta_S, \phi), \\
  \langle (2u + 1)v, \phi \rangle &= \int_0^\infty \int_{-\infty}^\infty (2u_0 + 1)v_0\phi dx dt + (\sigma w(t)\delta_S, \phi),
\end{align*}
$$

(10)

in which $u_0 = u_- + [u]H(x - \sigma t), v_0 = v_- + [v]H(x - \sigma t)$ and $(2u_0 + 1)v_0 = (2u_- + 1)v_- + [(2u + 1)v]H(x - \sigma t)$. In order to obtain the solution (7) such that which satisfies (9) in the sense of distribution, it is necessary to specify the value of velocity $u$ along the trajectory of singularity. Thus, $u_\delta$ is introduced in the formula (10) which stands for the assignment of $u$ on this delta shock wave curve $x = \sigma t$. Then, the solution (7) can be rewritten as

$$
(u, v)(x, t) = \begin{cases} 
    (u_-, v_-), x < \sigma t, \\\n    (u_\delta, w(t)\delta(x - \sigma t)), x = \sigma t, \\
    (u_+, v_+), x > \sigma t,
  \end{cases}
$$

where $\sigma = (u_- + u_+)$ and $u_\delta = \frac{(u_- + u_+ - 1)}{2}$.

Depending upon the initial data, there exist three different wave patterns for the solution of the Riemann problem (1) and (5) which are as follows:

$$
R + J (u_- < u_+), \quad S + J (u_- < u_- < u_+ + 1), \quad \delta S (u_- \geq u_+ + 1).
$$
3. Perturbed Riemann problem. In this section, we consider the Riemann problem (4) and (5) for any given sufficiently small parameter \( \varepsilon > 0 \). The system (4) can be rewritten in quasilinear form as

\[
\begin{bmatrix}
  u \\
v
\end{bmatrix}_t + \begin{bmatrix}
  2u + \varepsilon \alpha & 0 \\
  2v + \varepsilon \beta & (2u + 1) + \varepsilon \gamma
\end{bmatrix} \begin{bmatrix}
  u \\
v
\end{bmatrix}_x = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

(11)

The eigenvalues of the system (11) are \( \lambda_1 = 2u + \varepsilon \alpha \) and \( \lambda_2 = (2u + 1) + \varepsilon \gamma \) and the corresponding right eigenvectors are \( \vec{r}_1 = (\varepsilon (\alpha - \gamma) - 1, 2v + \varepsilon \beta) \) and \( \vec{r}_2 = (0, 1)^t \), respectively. Since \( \nabla \lambda_1 \cdot \vec{r}_1 = 2\{\varepsilon (\alpha - \gamma) - 1\} \neq 0 \) and \( \nabla \lambda_2 \cdot \vec{r}_2 = 0 \), the first characteristic field is genuinely nonlinear and second one is linearly degenerate. The Riemann invariants associated with the first and second characteristic fields are \( z_1 = (2v + \varepsilon \beta) \exp \{\frac{2u}{1 + \varepsilon (\gamma - \alpha)}\} \) and \( z_2 = u \), respectively.

Since system (4) and Riemann initial data (5) are unchanged under the coordinate transformation: \((x, t) \rightarrow (kx, kt)\), where \( k \) is a non zero constant in \((x, t)\) plane. We want to look for the self-similar solution of the form

\((u, v)(x, t) = (u, v)(\xi)\),

where \( \xi = \frac{t}{k} \). By self similar transformation the Riemann problem (4) and (5) becomes boundary-value problem for system of ordinary differential equations

\[-\xi u_\xi + (u^2 + \varepsilon \alpha u)\xi = 0,\]

\[-\xi v_\xi + ((2u + 1)v + \varepsilon \beta u + \varepsilon \gamma v)\xi = 0,\]

(12)

with boundary condition \((u, v)(\pm \infty) = (u_\pm, v_\pm)\).

Let \( U = (u, v)^t \), then for smooth solutions, system (12) can be rewritten into the form

\[A(U)U_\xi = 0,\]

(13)

where

\[A(u, v) = \begin{bmatrix}
  -\xi + 2u + \varepsilon \alpha & 0 \\
  2v + \varepsilon \beta & -\xi + (2u + 1) + \varepsilon \gamma
\end{bmatrix}.
\]

Besides the constant state solution, it provides a rarefaction wave which is continuous solution of (13) in the form \((u, v)(\xi)\) which is a function of single variable. Let us fix \((u_-, v_-)\) in the domain of hyperbolicity and compute the state \((u, v)\) which is connected on the right by either classical elementary waves or delta shock wave as given below.

The 1-rarefaction wave curves in the phase plane are:

\[R(u_-, v_-) = \left\{ \begin{array}{l}
\xi = \lambda_1(u, v) = 2u + \varepsilon \alpha, \\
(2v + \varepsilon \beta) \exp \{\frac{2u}{1 + \varepsilon (\gamma - \alpha)}\} = (2v_- + \varepsilon \beta) \exp \{\frac{2u_-}{1 + \varepsilon (\gamma - \alpha)}\},
\end{array} \right.\]

\[u_- < u.
\]

For a bounded discontinuity at \( \xi = C \), the Rankine-Hugoniot condition can be expressed as:

\[C[u] = [u^2 + \varepsilon \alpha u],\]

\[C[v] = [(2u + 1)v + \varepsilon \beta u + \varepsilon \gamma v],\]

(14)

where \( C = \frac{d\xi}{dt} \) and \([u] = u_r - u_l\) with \( u_l = u(x(t) - 0) \) and \( u_r = u(x(t) + 0)\).

It follows from the first equation in (14) that

\[(C - u_l - u_r - \varepsilon \alpha)(u_r - u_l) = 0.\]
If \( u_1 = u_r \), then it follows from the second equation in (14) that
\[
C = 2u_1 + 1 + \varepsilon\gamma = 2u_r + 1 + \varepsilon\gamma,
\]
which exactly correspond to the contact discontinuity. The possible states that can be connected to \((u_-, v_-)\) on the right by contact discontinuity, which is given as follows
\[
J(u_-, v_-) = \begin{cases} 
\tau = 2u + 1 + \varepsilon\gamma = 2u_r + 1 + \varepsilon\gamma, \\
u = u_-.
\end{cases}
\]
On the other hand, the 1-shock wave curves in the phase plane are:
\[
S(u_-, v_-) = \begin{cases} 
C = u + u_- + \varepsilon\alpha, \\
u = u_- + \frac{(1 + \varepsilon\gamma - \varepsilon\alpha)(v_- - v)}{v + v_- + \varepsilon\beta}, \\
u < u_- < u + 1.
\end{cases}
\]
When \( u_- \geq u_+ + 1 \), then nonclassical situation appears where the Riemann problem (4) and (5) cannot be solved by a combination of shock waves, rarefaction waves and contact discontinuities. In such situation, it was motivated in [15] that a solution containing a weighted \( \delta \)-measure supported on a curve should be introduced.

For the case \( u_- \geq u_+ + 1 \), from Definitions 2.1 and 2.2, we can notice that the delta shock wave solution to the Riemann problem (4) and (5) can be constructed in the following form
\[
(u, v)(x, t) = \begin{cases} 
(u_-, v_-), & x < \sigma t, \\
(u_\delta, w(t)\delta(x - \sigma t)), & x = \sigma t, \\
(u_+, v_+), & x > \sigma t,
\end{cases}
\]
where \( w(t) \) and \( \sigma \) denote the strength and propagation speed of delta shock wave and \( u_\delta \) indicates the assignment of \( u \) on the delta shock wave curve, respectively.

Depending on the choice of initial data, there are three possible wave patterns for the solution of Riemann problem (4) and (5) which are described below.

**Case i:** If \( u_- < u_+ \), the Riemann solution consists of a rarefaction wave \( R \) followed by a contact discontinuity \( J \) (see Figure 2)

\[
\begin{align*}
(u, v)(x, t) = &\begin{cases} 
(u_-, v_-), & x < (2u_- + \varepsilon\alpha)t, \\
R, & (2u_- + \varepsilon\alpha)t \leq x \leq (2u_+ + \varepsilon\alpha)t, \\
(u_+, v_+), & (2u_+ + \varepsilon\alpha)t < x < (2u_+ + 1 + \varepsilon\gamma)t, \\
(u_+, v_+), & x > (2u_+ + 1 + \varepsilon\gamma)t.
\end{cases}
\end{align*}
\]
in which the state \((u,v)\) in \(R\) is given by
\[
(u,v) = \left( \frac{x - \varepsilon \alpha t}{2}, (v_+ + \varepsilon \beta) \exp\left\{ \frac{2(u_- - (x - \varepsilon \alpha t))}{1 + \varepsilon(\gamma - \alpha)} - \frac{\varepsilon \beta}{2} \right\} \right).
\]
and intermediate state between \(R\) and \(J\) is
\[
(u_*, v_*) = \left( u_+, (v_+ + \varepsilon \beta) \exp\left\{ \frac{2(u_- - u_+)}{1 + \varepsilon(\gamma - \alpha)} \right\} - \frac{\varepsilon \beta}{2} \right).
\]

Case ii: If \(u_- < u_- < u_+ + 1\), the Riemann problem consists of a shock wave \(S\) followed by a contact discontinuity \(J\) (see Figure 3)

![Figure 3](image)

**Figure 3.** The Riemann solution of (4) and (5) is S+J when \(u_+ < u_- < u_+ + 1\).

\[
(u,v)(x,t) = \begin{cases}
(u_-, v_-), & x < (u_- + u_+ + \varepsilon \alpha)t, \\
(u_*, v_*), & (u_- + u_+ + \varepsilon \alpha)t < x < (2u_+ + 1 + \varepsilon \gamma)t, \\
(u_+, v_+), & x > (2u_+ + 1 + \varepsilon \gamma)t,
\end{cases}
\] (16)

where \((u_*, v_*) = \left( u_+, \frac{(v_+ + \varepsilon \beta)(1+u_- - u_+ + \varepsilon \gamma - \varepsilon \alpha)}{1+u_- - u_+ + \varepsilon \gamma - \varepsilon \alpha} \right) - \frac{\varepsilon \beta}{2}\) is intermediate state between \(S\) and \(J\).

Case iii: When \(u_- \geq u_+ + 1\), the solution of the Riemann problem consists of a delta shock wave \(\delta S\) which is as follows (see Figure 4):

![Figure 4](image)

**Figure 4.** The Riemann solution of (4) and (5) is \(\delta S\) when \(u_- \geq u_+ + 1\).

\[
(u,v)(x,t) = \begin{cases}
(u_-, v_-), & x < \sigma t, \\
(u_+, w(t)\delta(x - \sigma t)), & x = \sigma t, \\
(u_+, v_+), & x > \sigma t,
\end{cases}
\] (17)

where \(w(t)\) and \(\sigma\) denote the strength and propagation speed of delta shock wave and \(u_\delta\) indicates the assignment of \(u\) on the delta shock wave curve, respectively.
In fact, the delta shock wave solution in the form (17) to the Riemann problem (4) and (5) must hold
\[
\langle u, \phi_t \rangle + \langle u^2 + \varepsilon \alpha u, \phi_x \rangle = 0,
\]
\[
\langle v, \phi_t \rangle + \langle (2u + 1)v + \varepsilon \beta u + \varepsilon \gamma v, \phi_x \rangle = 0,
\]
for every test function \( \phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+) \). Actually, we have the following theorem to describe completely the delta shock wave solution to the Riemann problem (4) and (5) for the case \( u_- \geq u_+ + 1 \).

**Theorem 3.1.** If \( u_- \geq u_+ + 1 \), then the delta shock wave solution to the Riemann problem (4) and (5) can be expressed in the form (17) where
\[
u_d = \frac{1}{2} (u_- + u_+ - 1 + \varepsilon(\alpha - \gamma)),
\]
\[
\sigma = u_- + u_+ + \varepsilon \alpha,
\]
\[
w(t) = ((v_+ + v_)(u_- - u_+ + (\varepsilon(\alpha - \gamma) - 1)(v_+ - v_-) - \varepsilon \beta(u_+ - u_-)) t.
\]

Furthermore, the delta shock wave solution (17) and (18) must satisfy the generalized Rankine-Hugoniot jump conditions
\[
\begin{cases}
\frac{dx(t)}{dt} = \sigma, \\
\frac{dw(t)}{dt} = (\sigma[v] - [2u + 1)v + \varepsilon \beta u + \varepsilon \gamma v], \\
\sigma[u] = [u^2 + \varepsilon \alpha u].
\end{cases}
\]

In order to ensure the uniqueness, the Delta entropy conditions \( \lambda_1(u_+) \leq \sigma \leq \lambda_1(u_-) \) and \( \lambda_2(u_+) \leq \sigma \leq \lambda_2(u_-) \) are imposed, which mean that all the characteristics on both sides of the \( \delta \)-shock wave are incoming.

**Proof.** In order to prove (17) is the solution of (4) and (5) in the sense of distribution, it is enough to prove the following integral identities
\[
0 = \int_0^\infty \int_{-\infty}^{\infty} (u\phi_t + (u^2 + \varepsilon \alpha u)\phi_x) dx dt,
\]
\[
0 = \int_0^\infty \int_{-\infty}^{\infty} (v\phi_t + ((2u + 1)v + \varepsilon \beta u + \varepsilon \gamma v)\phi_x) dx dt,
\]
hold for every test function \( \phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+) \).

Without loss of generality, let us assume \( \sigma > 0 \), then we have
\[
I = \int_0^\infty \int_{-\infty}^{\infty} (u\phi_t + (u^2 + \varepsilon \alpha u)\phi_x) dx dt
\]
\[
-\int_0^\infty \int_{-\infty}^{\infty} (u_-\phi_t + (u_-^2 + \varepsilon \alpha u_-)\phi_x) dx dt
\]
\[
+ \int_0^\infty \int_{-\infty}^{\infty} (u_+\phi_t + (u_+^2 + \varepsilon \alpha u_+)\phi_x) dx dt
\]
\[
= \int_0^\infty \int_{-\infty}^{\infty} u_-\phi_t dx + \int_0^\infty \int_{-\infty}^{\infty} u_+\phi_t dx
\]
\[
+ \int_0^\infty (u_-^2 + \varepsilon \alpha u_- - u_+^2 - \varepsilon \alpha u_+) \phi(\sigma t) dt
\]
\[
= \int_0^\infty (\sigma(u_+ - u_-) - ((u_+^2 + \varepsilon \alpha u_+) - (u_-^2 + \varepsilon \alpha u_-)) \phi(\sigma t) dt = 0.
\]
Similarly,

\[ II = \int_{0}^{\infty} \int_{-\infty}^{\infty} (v\phi + ((2u + 1)v + \varepsilon u + \varepsilon v)\phi_x) \, dx \, dt \]

\[ = \int_{0}^{\infty} \int_{-\infty}^{\infty} (v_\phi + ((2u + 1)v_\phi + \varepsilon u_\phi + \varepsilon v_\phi)) \, dx \, dt \]

\[ + \int_{0}^{\infty} \int_{\sigma}^{\infty} (v_\phi + ((2u + 1)v_\phi + \varepsilon u_\phi + \varepsilon v_\phi)) \, dx \, dt \]

\[ + \int_{0}^{\infty} w(t)(\phi_\sigma(\sigma, t) + (2u_\delta + 1 + \varepsilon)\phi_x(\sigma, t)) \, dt \]

\[ = \int_{0}^{\infty} \int_{-\infty}^{\infty} v_\phi \, dx \, dt + \int_{0}^{\infty} \int_{-\infty}^{\infty} ((2u + 1)v_\phi + \varepsilon u_\phi + \varepsilon v_\phi) \phi_x \, dx \, dt \]

\[ + \int_{0}^{\infty} \int_{\sigma}^{\infty} (v_\phi + ((2u + 1)v_\phi + \varepsilon u_\phi + \varepsilon v_\phi)) \, dx \, dt \]

\[ + \int_{0}^{\infty} w(t)(\phi_\sigma(\sigma, t) + (2u_\delta + 1 + \varepsilon)\phi_x(\sigma, t)) \, dt \]

\[ = -\int_{0}^{\infty} v_\phi(x, \frac{t}{\sigma}) \, dx + \int_{0}^{\infty} v_\phi(x, \frac{t}{\sigma}) \, dx \]

\[ - \int_{0}^{\infty} ((2u + 1)v_\phi - (2u + 1)v_\phi + \varepsilon u_\phi + \varepsilon v_\phi)) \phi(\sigma, t) \, dt \]

\[ + \int_{0}^{\infty} w(t) \phi(\sigma, t) \, dt \]

\[ = \int_{0}^{\infty} (\phi(v_\phi - v_\phi) + (2u + 1)v_\phi + (2u + 1)v_\phi) \phi(\sigma, t) \, dt \]

\[ + \int_{0}^{\infty} (\varepsilon u_\phi - u_\phi + \varepsilon v_\phi - v_\phi - w'(t)) \phi(\sigma, t) \, dt = 0, \]

for every test function \( \phi \in C^\infty_0(\mathbb{R} \times \mathbb{R}^+), \) in which \( \sigma = 2u_\delta + 1 + \varepsilon \gamma. \)

\[ \square \]

4. **Limits of Riemann solutions as \( \varepsilon \to 0. \)** In this section, we prove the limit of the solution of the Riemann problem (4) and (5) converges to the corresponding solution of the Riemann problem (1) and (5), when the perturbation parameter \( \varepsilon \) tends to zero.

**Theorem 4.1.** For any given Riemann initial data \( (u_\pm, v_\pm) \), the limit of the Riemann solution of (4) as \( \varepsilon \to 0 \) is exactly same as the Riemann solution of (1).

**Proof.** We divide the proof into three cases depending upon the initial data.

Case (I): If \( u_- < u_+ \), the solution of the Riemann problem (4) and (5) consists of
rarefaction wave followed by contact discontinuity. From (15), we take limit $\varepsilon$ tends to zero, we obtain

$$\lim_{\varepsilon \to 0} (u,v)(x,t) = \begin{cases} (u_-, v_-), & x < 2u_-t, \\ \left(\frac{2}{\varepsilon^2}, v_+ e^{2(u_- - u_+)}\right), & 2u_-t \leq x \leq 2u_+t, \\ (u_+, v_+), & 2u_+t < x < (2u_+ + 1)t, \\ (u_+, v_+), & x > (2u_+ + 1)t, \end{cases}$$

which is exactly the Riemann solution of (1) and (5) when $u_- < u_+$. Case (II): If $u_+ < u_- < u_+ + 1$, from (16) the Riemann solution of (4) and (5) is S+J. As $\varepsilon$ tends to zero, we get

$$\lim_{\varepsilon \to 0} (u,v)(x,t) = \begin{cases} (u_-, v_-), & x < (u_- + u_+)t, \\ (u_+, v_-\left(1 + u_--u_+\right)), & (u_- + u_+)t < x < (2u_+ + 1)t, \\ (u_+, v_+), & x > (2u_+ + 1)t, \end{cases}$$

which is exactly the Riemann solution of (1) and (5) when $u_+ < u_- < u_+ + 1$. Case (III): If $u_- \geq u_+ + 1$, from (17) and (18) the Riemann solution of (4) and (5) is delta shock wave $\delta S$. Now take limit $\varepsilon$ tends to zero, we obtain

$$\lim_{\varepsilon \to 0} (u,v)(x,t) = \begin{cases} (u_-, v_-), & x < (u_- + u_+)t, \\ (u_1, v_1), & x = (u_- + u_+)t, \\ (u_+, v_+), & x > (u_- + u_+)t, \end{cases}$$

where $(u_1, v_1) = \left(\frac{u_- - u_+ - 1}{2}, (v_+ + v_-)(u_- - u_+) + (v_- - v_+)t\delta(x - (u_- + u_+)t)\right)$, which is exactly the Riemann solution of (1) and (5) when $u_- \geq u_+ + 1$.

5. **Conclusions.** We noticed from the above discussion that the limits of solution to the Riemann problem (4) and (5) converge to the corresponding solution of the Riemann problem (1) and (5) as $\varepsilon \to 0$. We proved that the perturbation in flux functions does not change the structure of Riemann solution. We concluded that the Riemann problem (1) and (5) is stable under triangular linear approximation of flux functions.

### REFERENCES

[1] G. Q. Chen and H. Liu, Formation of $\delta$-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids, *SIAM J. Math. Anal.*, **34** (2003), 925–938.

[2] G. Q. Chen and H. Liu, Concentration and cavitation in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids, *Physica D: Nonlinear Phenomena*, **189** (2004), 141–165.

[3] H. Cheng and H. Yang, Approaching Chaplygin pressure limit of solutions to the Aw-Rascle model, *J. Math. Anal. and Appl.*, **416** (2014), 839–854.

[4] V. G. Danilov and V. M. Shelkovich, Dynamics of Propagation and interaction of $\delta$-shock waves in conservation law systems, *J. Differential Equations*, **211** (2005), 333–381.

[5] V. G. Danilov and V. M. Shelkovich, Delta-shock wave type solution of hyperbolic systems of conservation laws, *Quart. Appl. Math.*, **63** (2005), 401–427.

[6] G. Ercole, Delta-shock waves as self-similar viscosity limits, *Quart. Appl. Math.*, **58** (2000), 177–199.

[7] K. T. Joseph, A Riemann problem whose viscosity solutions contain $\delta$-measures, *Asymptotic Anal.*, **7** (1993), 105–120.

[8] H. Kalisch and D. Mitrovic, Singular solutions for the shallow-water equations, *The IMA Journal of Applied Mathematics*, **73** (2012), 340–350.

[9] H. Kalisch and D. Mitrovic, Singular solutions of a fully nonlinear $2 \times 2$ system of conservation laws, *Proceedings of the Edinburgh Mathematical Society*, **55** (2012), 711–729.
[10] E. Y. Panov and V. M. Shelkovich, \(\delta^\prime\)-Shock waves as a new type of solutions to systems of conservation laws, *J. Differential Equations*, **228** (2006), 49–86.

[11] A. Sen, T. Raja Sekhar and V. D. Sharma, Wave interactions and stability of the Riemann solution for a strictly hyperbolic system of conservation laws, *Quart. Appl. Math.*, **75** (2017), 539–554.

[12] C. Shen, The limits of Riemann solutions to the isentropic magnetogasdynamics, *Appl. Math. Lett.*, **24** (2011), 1124–1129.

[13] W. Sheng and T. Zhang, The Riemann problem for the transportation equations in gas dynamics, *American Mathematical Soc.*, **654**, 1999.

[14] M. Sun, Structural stability of solutions to the Riemann problem for a non-strictly hyperbolic system with flux approximation, *Electronic Journal of Differential Equations*, **2016** (2016), 1–16.

[15] D. Tan, T. Zhang and Y. Zheng, Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws, *J. Differential Equations*, **112** (1994), 1–32.

[16] H. Yang and J. Liu, Delta-shocks and vacuums in zero-pressure gas dynamics by the flux approximation, *Science China Mathematics*, **58** (2015), 2329–2346.

[17] G. Yin and W. Sheng, Delta-shocks and vacuum states in vanishing pressure limits of solutions to the relativistic Euler equations for polytropic gases, *J. Math. Anal. and Appl.*, **355** (2009), 594–605.

Received March 2018; revised May 2018.

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