Fractional coloring of planar graphs of girth five

Zdeněk Dvořák*         Xiaolan Hu†

Abstract

A graph \( G \) is \((a : b)\)-colorable if there exists an assignment of \( b \)-element subsets of \( \{1, \ldots, a\} \) to vertices of \( G \) such that sets assigned to adjacent vertices are disjoint. We first show that for every triangle-free planar graph \( G \) and a vertex \( x \in V(G) \), the graph \( G \) has a set coloring \( \varphi \) by subsets of \( \{1, \ldots, 6\} \) such that \( |\varphi(v)| \geq 2 \) for \( v \in V(G) \) and \( |\varphi(x)| = 3 \). As a corollary, every triangle-free planar graph on \( n \) vertices is \((6n : 2n + 1)\)-colorable. We further use this result to prove that for every \( \Delta \), there exists a constant \( M_\Delta \) such that every planar graph \( G \) of girth at least five and maximum degree \( \Delta \) is \((6M_\Delta : 2M_\Delta + 1)\)-colorable. Consequently, planar graphs of girth at least five with bounded maximum degree \( \Delta \) have fractional chromatic number at most \( 3 - \frac{3}{2M_\Delta + 1} \).

Keywords: planar graph; fractional coloring; triangle-free; girth

1 Introduction

A function that assigns sets to all vertices of a graph is a set coloring if the sets assigned to adjacent vertices are disjoint. For positive integers \( a \) and \( b \leq a \), an \((a : b)\)-coloring of a graph \( G \) is a set coloring with range \( \binom{\{1, \ldots, a\}}{b} \), i.e., a set coloring that to each vertex assigns a \( b \)-element subset of \( \{1, \ldots, a\} \). The concept of \((a : b)\)-coloring is a generalization of the conventional vertex coloring. In fact, an \((a : 1)\)-coloring is exactly an ordinary proper \( a \)-coloring. The fractional chromatic number of \( G \), denoted by \( \chi_f(G) \), is the infimum of the fractions \( a/b \) such that \( G \) admits an \((a : b)\)-coloring. Note that \( \chi_f(G) \leq \chi(G) \) for any graph \( G \), where \( \chi(G) \) is the chromatic number of \( G \).

Much of the interest in the chromatic properties of triangle-free planar graphs stems from Grötzsch’s theorem [4], stating that such graphs are 3-colorable. Even in the fractional coloring setting, it is not possible to significantly improve Grötzsch’s theorem. For any positive integer \( n \) such that \( n \equiv 2 \pmod{3} \), Jones [5] constructed a triangle-free planar graph on \( n \) vertices with independence number \( \frac{n+1}{3} \). Since \( \alpha(G) \geq |V(G)|/\chi_f(G) \), these graphs have fractional chromatic number at least \( \frac{3n}{n+1} = 3 - \frac{3}{n+1} \) (in fact, they are \((3n : n + 1)\)-colorable). Thus, there exist triangle-free planar graphs with fractional chromatic number arbitrarily close to 3. On the other hand, Dvořák,

---

*Computer Science Institute (CSI) of Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: rakhver@iuuk.mff.cuni.cz Supported by project 17-04611S (Ramsey-like aspects of graph coloring) of Czech Science Foundation.

†School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, PR China. Partially supported by NSFC under grant number 11601176 and NSF of Hubei Province under grant number 2016CFB146.
Sereni and Volec [3] showed that there does not exist a triangle-free planar graph with fractional chromatic number exactly 3 by establishing the following upper bound.

**Theorem 1** (Dvořák, Sereni and Volec [3]). *Every planar triangle-free graph on \( n \) vertices is \((9n : 3n + 1)\)-colorable, and thus it has fractional chromatic number at most \( 3 - \frac{3}{3n + 1} \).*

Note that the graphs built by Jones [5] contain a large number of separating 4-cycles. Motivated by this observation, Dvořák, Sereni and Volec [3] conjectured that triangle-free plane graphs without separating 4-cycles cannot have fractional chromatic number arbitrarily close to 3, and proved this is the case under an additional assumption that the maximum degree is at most 4. They also remarked that since faces of length four are usually easy to deal with in the proofs by collapsing, a key step would be to prove this conjecture for planar graphs of girth at least five (this special case was previously conjectured by Dvořák and Mnich [2]).

**Conjecture 2.** *There exists a real number \( c < 3 \) such that every planar graph of girth at least five has fractional chromatic number at most \( c \).*

The purpose of this work is to establish the following upper bound on the fractional chromatic number of planar graphs of girth at least five with maximum degree \( \Delta \), proving Conjecture 2 for graphs with bounded maximum degree.

**Theorem 3.** *For every positive integer \( \Delta \), there exists a positive integer \( M_\Delta \) as follows. If \( G \) is a planar graph of girth at least five and maximum degree at most \( \Delta \), then \( G \) is \((6M_\Delta : 2M_\Delta + 1)\)-colorable, and thus \( \chi_f(G) \leq 3 - \frac{3}{2M_\Delta + 1} \).*

Theorem 3 is an easy corollary of the following result on special set colorings of planar graph of girth at least five.

**Theorem 4.** *For every positive integer \( k \), there exists a positive integer \( s \) such that the following holds. Let \( G \) be a planar graph of girth at least five and let \( X \) be a set of vertices of \( G \) of degree at most \( k \). If the distance between vertices of \( X \) is at least \( s \), then \( G \) has a set coloring \( \varphi \) by subsets of \( \{1, \ldots, 6\} \) such that \( |\varphi(v)| \geq 2 \) for \( v \in V(G) \) and \( |\varphi(x)| = 3 \) for \( x \in X \).*

Using standard techniques, we can argue that it suffices to prove Theorem 4 in the special case \( |X| = 1 \). In this special case, we only need to assume that the graph is triangle-free (rather than having girth at least five).

**Theorem 5.** *Let \( G \) be a triangle-free planar graph. For any vertex \( x \in V(G) \), the graph \( G \) has a set coloring \( \varphi \) by subsets of \( \{1, \ldots, 6\} \) such that \( |\varphi(v)| \geq 2 \) for \( v \in V(G) \) and \( |\varphi(x)| = 3 \).*

Let us remark that in Theorem 5, it does not suffice to forbid triangles: It is easy to see that any graph \( G \) satisfying the outcome of the theorem has an independent set of size at least \( \frac{2n + |X|}{6} \), implying that for the graphs constructed by Jones [5] (which have unbounded diameter), the outcome cannot be true for any set \( X \) of size at least three. It might be possible to improve the ratio of extra colors assigned to the vertex \( x \) in Theorem 5 a bit; e.g., it could be true that there exists a coloring by subsets of \( \{1, \ldots, 9\} \) such that all vertices get at least three colors and \( x \) gets five. However, when \( G \) is the graph obtained from the wheel with five spokes by subdividing each of the spokes once, \( x \)
is the center of the wheel, and \( \varphi \) is a coloring by subsets of \( \{1, \ldots, k\} \) and each vertex has at least \( \frac{k}{2} \) colors, then \( |\varphi(x)| \leq \frac{k}{2} + \frac{3k}{5} \).

Before proceeding with the proofs, let us mention another consequence of Theorem 5. Consider a triangle-free planar graph \( G \) with \( |V(G)| = \{v_1, v_2, \ldots, v_n\} \). For each vertex \( v_i \in V(G) \), the graph \( G \) has a set coloring \( \varphi_i \) by subsets of \( \{6i - 5, 6i - 4, 6i - 3, 6i - 2, 6i - 1, 6i\} \) such that \( |\varphi(v_i)| \geq 2 \) for \( 1 \leq j \leq n \) and \( |\varphi(v_i)| = 3 \). Let us set \( \varphi(v) = \bigcup_{i=1}^{n} \varphi_i(v) \) for each \( v \in V(G) \). Then \( \varphi \) is a set coloring of \( G \) by \( (2n + 1) \)-element subsets of \( \{1, \ldots, 6n\} \). Hence, we have the following corollary, which improves upon Theorem 5.

Corollary 6. Every triangle-free planar graph on \( n \) vertices is \((6n : 2n + 1)\)-colorable, and thus its fractional chromatic number is at most \( 3 - \frac{3}{2n+1} \).

## 2 Set coloring of triangle-free planar graphs

In this section, we give a proof of Theorem 5. Let \( G \) be a graph and let \( X \) be a set of vertices of \( G \). An \( X \)-enhanced coloring of \( G \) is a set coloring \( \varphi \) of \( G \) by subsets of \( \{1, \ldots, 6\} \) such that \( |\varphi(v)| \geq 2 \) for all \( v \in V(G) \) and \( |\varphi(x)| = 3 \) for \( x \in X \). We are going to prove a mild strengthening of Theorem 5 where the outer face is precolored.

Theorem 7. Let \( G \) be a triangle-free plane graph whose outer face is bounded by a cycle \( C \) of length at most 5, and let \( X \) be a subset of \( V(C) \) of size at most one. Then any \( X \)-enhanced coloring of \( C \) can be extended to an \( X \)-enhanced coloring of \( G \).

Theorem 5 follows from Theorem 7 by redrawing the graph so that \( x \) is incident with the outer face, adding three new vertices \( v_1, v_2, \) and \( v_3 \) and the edges of the 4-cycle \( C = xv_1v_2v_3 \) bounding the outer face of the resulting graph, letting \( X = \{x\} \) and choosing an \( X \)-enhanced coloring of \( C \) arbitrarily.

A (hypothetical) counterexample to Theorem 7 is a triple \((G, X, \varphi)\), where \( G \) is a triangle-free plane graph whose outer face is bounded by a cycle \( C \) of length at most 5, \( X \) is a subset of \( V(C) \) with \( |X| \leq 1 \), and \( \varphi \) is an \( X \)-enhanced coloring of \( C \) such that \( \varphi \) does not extend to an \( X \)-enhanced coloring of \( G \). The counterexample \((G, X, \varphi)\) is minimal if there is no counterexample \((G', X', \varphi')\) such that either \( |V(G')| < |V(G)| \), or \( |V(G')| = |V(G)| \) and \( |E(G')| > |E(G)| \); i.e., \( G \) has the minimum number of vertices among all counterexamples, and the maximum number of edges among all counterexamples with the minimum number of vertices.

### 2.1 Properties of a minimal counterexample

Let us start with some observations on vertex degrees and face lengths in a minimal counterexample.

**Lemma 8.** If \((G, X, \varphi)\) is a minimal counterexample, then \( G \) is 2-connected, all vertices of degree two are incident with the outer face or adjacent to a vertex in \( X \), and every \((\leq 5)\)-cycle in \( G \) bounds a face.

**Proof.** Let \( v \) be a vertex of \( G \) of degree at most two, not contained in the cycle \( C \) bounding the outer face of \( G \). Since \((G, X, \varphi)\) is a minimal counterexample, the coloring \( \varphi \) extends to an \( X \)-enhanced
Suppose for a contradiction that $k$ is a face of length at least 3. Since the outer face of $G$ is triangle-free and has more edges than $G$, we conclude by the minimality of $(G, X, \varphi)$, there exists an $X$-enhanced coloring of $G + v_1v_2$ extending $\varphi$. This also gives an $X$-enhanced coloring of $G$, which is a contradiction. Hence, $G$ is 2-connected.

**Lemma 9.** If $(G, X, \varphi)$ is a minimal counterexample with the outer face bounded by a cycle $C$, then $G$ contains no 4-cycle other than $C$.

**Proof.** Suppose that $G$ contains a 4-cycle $K = v_1v_2v_3v_4$ distinct from $C$. By Lemma 8 $K$ bounds a face. Since $K \neq C$, we can assume that $v_3 \notin V(C)$. Let $G'$ be the graph obtained from $G$ by identifying $v_1$ with $v_3$. Note that each $X$-enhanced coloring of $G'$ corresponds to an $X$-enhanced coloring of $G$, and thus $\varphi$ does not extend to an $X$-enhanced coloring of $G'$. Since $|V(G')| < |V(G)|$, we conclude by the minimality of $(G, X, \varphi)$ that $G'$ contains a triangle. Hence, $G$ contains a 5-cycle $Q = v_1v_2v_3uw$. By Lemma 8 the 5-cycles $Q$ and $Q' = v_1v_2v_3uw$ bound faces. We conclude that $G$ has only three faces, bounded by the cycles $K$, $Q$, and $Q'$. However, $v_3 \in V(K \cap Q \cap Q')$, but we chose $v_3$ not to be incident with the outer face of $G$, which is a contradiction.

A $k$-face is a face of length exactly $k$, and a $k$-vertex is a vertex of degree exactly $k$. A $k^+$-face is a face of length at least $k$, and a $k^+$-vertex is a vertex of degree at least $k$.

**Lemma 10.** If $(G, X, \varphi)$ is a minimal counterexample, then $G$ contains no $6^+\text{-faces}$. 

**Proof.** Suppose for a contradiction that $G$ contains a $6^+$-face bounded by a cycle $K = v_1 \ldots v_k$, where $k \geq 6$. Since the outer face of $G$ is bounded by a cycle $C$ of length at most five, we can choose the labeling of vertices of $K$ so that $v_1 \notin V(C)$. By Lemma 9 $v_1v_4 \notin E(G)$. Let $G' = G + v_1v_4$. If $G'$ contained a triangle, then $G$ would contain a 5-cycle $Q = v_1v_2v_3v_4u$, which would bound a face by Lemma 8. Hence, the path $v_1v_2v_3v_4$ would be contained in boundaries of two distinct faces of $G$, and thus $v_2$ and $v_3$ would have degree two. Since $v_1 \notin V(C)$, we would also have $v_2, v_3 \notin V(C)$, and thus $v_2$ would be a vertex of degree two not contained in $C$ and not adjacent to $X$, contradicting Lemma 8. Hence, $G'$ is triangle-free, and $(G', X, \varphi)$ is a counterexample contradicting the minimality of $(G, X, \varphi)$.

By Lemmas 9 and 10 we have the following corollary.
Corollary 11. If $(G, X, \varphi)$ is a minimal counterexample, then every face other than the outer one is a 5-face.

Next, we prove two claims restricting the 5-faces.

Lemma 12. Let $(G, X, \varphi)$ be a minimal counterexample with the outer face bounded by a cycle $C$. Let $K = v_1v_2v_3v_4v_5$ be a cycle bounding a 5-face in $G$ such that $v_1$, $v_2$, $v_3$ and $v_4$ have degree three and do not belong to $V(C)$. For $i \in \{1, 2, 3, 4\}$, let $u_i$ denote the neighbor of $v_i$ not belonging to $V(K)$. Then either $\{u_1, u_2, u_3\} \cap X \neq \emptyset$ or $\{|u_1, u_4, v_5\} \cap V(C) \geq 2$.

Proof. Suppose for a contradiction that $u_1, u_2, u_3 \notin X$ and at most one of the vertices $u_1, ..., u_4$, and $v_5$ belongs to $V(C)$. If $u_i = u_j$ for distinct $i, j \in \{1, 2, 3, 4\}$, then $v_i$ and $v_j$ are contained in a triangle or a 4-cycle. The former is not possible, since $G$ is triangle-free. In the latter case, Lemma 9 implies this 4-cycle is $C$, contradicting the assumption that $v_i \notin V(C)$. Therefore, the vertices $u_1, ..., u_4$ are pairwise distinct.

Suppose that $G$ contains an edge $u_iu_j$ for distinct $i, j \in \{1, 2, 3, 4\}$. Analogously to the previous paragraph, this is not possible when $|i - j| = 1$. If $|i - j| = 2$, then let $k = (i + j)/2$, otherwise (when $\{i, j\} = \{1, 4\}$), let $k = 5$. Then $G$ contains a 5-cycle $u_iu_jv_kv_{j+1}u_j$, and by Lemma 8 this 5-cycle bounds a face, implying that $v_k$ has degree two. Since $v_i, v_j \notin X$ and $|V(K) \cap V(C)| \leq 1$, this contradicts Lemma 8. Therefore, the vertices $u_1, ..., u_4$ are pairwise non-adjacent.

Next, we show that for $i \in \{1, 2, 3\}$, the graph obtained from $G - \{v_i, u_{i+1}\}$ by identifying $u_i$ and $u_{i+1}$ is triangle-free. Otherwise, $G$ contains a 6-cycle $Q = v_iu_{i+1}w_{i+1}w_iu_i$. By Corollary 11 since $\deg(v_i) = \deg(v_{i+1}) = 3$, $G$ has a 5-face bounded by a 5-cycle $u_iu_iv_{i+1}u_{i+1}v_i$, and by Lemma 8 the 5-cycle $u_iw_{i+1}w_{i+1}w_iu_i$ also bounds a face. Consequently, $y_i$ has degree two, and by Lemma 8 we conclude that either $y_i$ has a neighbor in $X$ or $y_i \in V(C)$. However, then either $\{u_i, u_{i+1}\} \cap X \neq \emptyset$ or $u_i, u_{i+1} \in V(C)$, which is a contradiction.

Let $G'$ be the graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by adding the edge $u_1u_4$ and by identifying $u_2$ with $u_3$. If $G'$ is triangle-free, then by the minimality of $(G, X, \varphi)$, there exists an $X$-enhanced coloring $\psi$ of $G'$ extending $\varphi$. Note that $\psi(u_1) \cap \psi(u_4) = \emptyset$ and we can assume that $|\psi(u_1)| = |\psi(u_4)| = |\psi(v_5)| = 2$. Hence, we can let $\psi(v_1)$ be a 2-element subset of $\{1, 2, 3\} \setminus (\psi(u_1) \cup \psi(v_5))$ and $\psi(v_4)$ a 2-element subset of $\{4, 5\} \setminus (\psi(u_4) \cup \psi(v_5))$ such that $\psi(v_1) \cap \psi(v_4) = \emptyset$. Since $\psi(v_2) = \psi(u_2)$, $\psi$ can be extended to $v_2$ and $v_3$. This gives an $X$-enhanced coloring of $G$ extending $\varphi$, which is a contradiction.

So $G'$ has a triangle, necessarily containing the edge $u_1u_4$. Since $u_1u_2, u_3u_4 \notin E(G)$, the vertex obtained by identifying $u_2$ with $u_3$ is not contained in the triangle. Hence, $u_1$ and $u_4$ have a common neighbor $w$ in $G$.

Let $G''$ be the graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by identifying $u_1$ with $u_2$, and $u_3$ with $v_5$. If $G''$ is triangle-free, then there exists an $X$-enhanced coloring $\psi$ of $G'$ extending $\varphi$ by the minimality of $(G, X, \varphi)$. We can assume $|\psi(u_1)| = |\psi(u_2)| = |\psi(u_3)| = |\psi(v_5)| = 2$, and thus $\psi$ can be extended to $v_4$ and $v_3$. Note that $\psi(v_5) = \psi(u_3)$, and thus $\psi(v_4) \cap \psi(v_5) = \emptyset$, enabling us to extend $\psi$ to $v_1$ and $v_2$. This gives an $X$-enhanced coloring of $G$ extending $\varphi$, which is a contradiction.

Therefore, $G''$ has a triangle, necessarily containing the vertex obtained by the identification of $u_3$ with $v_5$. Since $u_1u_3, u_1v_5 \notin E(G)$, we conclude that the triangle does not contain the vertex obtained by the identification of $u_1$ with $u_2$, and thus $G - \{v_1, v_2, v_3, v_4\}$ contains a path $u_3xyv_5$. Note that $u_1, u_4 \notin \{x, y\}$, since $G$ is triangle-free and $u_1u_3, u_3u_4 \notin E(G)$. Since $v_1, v_4 \notin V(C)$,
Lemma 9 implies $u_1 y, u_4 y \notin E(G)$. Since $w$ is a common neighbor of $u_1$ and $u_4$, by planarity we conclude that $w = x$, and thus $w$ is adjacent to $u_3$. By a symmetric argument applied to the graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by identifying $u_3$ with $u_4$, and $u_2$ with $v_5$, we conclude that $w$ is also adjacent to $u_2$. However, then Lemma 8 implies that $G$ has exactly 6 faces, bounded by $K, wyv_5 u_1, wyv_5 u_4$, and $wu_4 v_i + u_{i+1}$ for $i \in \{1, 2, 3\}$. One of these 5-cycles is $C$, implying that $\{u_1, \ldots, u_4, v_5\} \cap V(C) \geq 2$, which is a contradiction.

**Corollary 13.** Let $(G, X, \varphi)$ be a minimal counterexample with the outer face bounded by a cycle $C$ and with $X = \{x\}$. Let $K = v_1 v_2 v_3 v_4 v_5$ be a cycle in $G$ vertex-disjoint from $C$ such that $\deg(v_5) = 3$ and $x v_5 \in E(G)$. Then at least one of vertices $v_1, \ldots, v_4$ has degree at least four.

**Proof.** By Lemma 8, $K$ bounds a face. For $i \in \{1, \ldots, 4\}$, Lemma 9 and the assumption that $G$ is triangle-free implies $x v_i \notin E(G)$, and thus $\deg(v_i) \geq 3$ by Lemma 8. Suppose for a contradiction that $\deg(v_i) = 3$ for $i \in \{1, \ldots, 4\}$. Let $u_i$ denote the neighbor of $v_i$ not in $V(K)$. As in the proof of Lemma 12, we argue that the vertices $u_1, \ldots, u_4$ are pairwise disjoint and non-adjacent.

By Lemma 12 two of the vertices $u_1, \ldots, u_4$ belong to $V(C)$. Consequently, at least one of them is adjacent to $x$. By symmetry, we can assume that there exists $i \in \{1, 2\}$ such that $u_i \in V(C)$ and $u_i$ is adjacent to $x$. If $i = 1$, then Lemma 9 applied to the 4-cycle $v_1 u_1 v_5 x$ implies $v_1 \in V(C)$, which is a contradiction. If $i = 2$, then the 5-cycle $u_2 v_2 v_1 v_5 x$ bounds a face by Lemma 8 and thus $\deg(v_1) = 2$, which is again a contradiction.

### 2.2 Reducible configurations

Let us now derive further properties of special configurations in a minimal counterexample. We will often need the following observation.

**Observation 14.** If $\psi$ is an $\{x\}$-enhanced coloring of a path $xuv$, then $\psi(x) \cap \psi(v) \neq \emptyset$. Conversely, any precoloring $\psi'$ of $x$ and $v$ such that $|\psi'(x)| = 3$, $|\psi'(v)| = 2$, and $\psi'(x) \cap \psi'(v) \neq \emptyset$ extends to an $\{x\}$-enhanced coloring of the path.

Next, we restrict degrees of vertices near to $X$.

**Lemma 15.** Let $(G, X, \varphi)$ be a minimal counterexample with the outer face bounded by a cycle $C$ and with $X = \{x\}$. Suppose a cycle $x v_1 u_1 u_2 v_2$ bounds a 5-face in $G$. If $\deg(v_1) = 2$, $\deg(u_1) = 3$, and $v_1, u_1, u_2 \notin V(C)$, then $\deg(v_2) = 2$ and $v_2 \notin V(C)$.

**Proof.** Let $G'$ be the graph obtained from $G - \{v_1, u_1\}$ by identifying $x$ and $u_2$. Suppose first that $G'$ has an $X$-enhanced coloring $\psi'$ extending $\varphi$. We may assume without lose of generality that $\psi'(x) = \{1, 2, 3\}$. Let $u_0$ denote the neighbor of $u_1$ distinct from $v_1$ and $u_2$. By Corollary 11, $u_0$ has a common neighbor with $x$, and by Observation 14 we can without lose of generality assume $1 \in \psi'(u_0)$. Furthermore, by symmetry between the colors 2 and 3, we can assume $3 \notin \psi'(u_0)$. Let $\psi(v) = \psi'(v)$ for $v \in V(G) \setminus \{v_1, u_1, u_2\}$. Let $\psi(u_2) = \{1, 2\}$ (this is a subset of $\psi'(u_2) = \psi'(x) = \{1, 2, 3\}$), let $\psi(u_1)$ be a 2-element subset of $\{3, \ldots, 6\} \setminus \psi'(u_0)$ containing 3, and extend $\psi$ to $v_1$ by Observation 14. Then $\psi$ is an $X$-enhanced coloring of $G$ extending $\varphi$, which is a contradiction.

Consequently, $G'$ does not have an $X$-enhanced coloring extending $\varphi$, and by the minimality of $(G, X, \varphi)$, we conclude $G'$ contains a triangle. Hence, $G$ contains a 5-cycle $x v_2 u_2 u_2 w_1$ disjoint from
\( \{u_1, v_1\} \), and by Lemma 5 this 5-cycle bounds a face. Hence, \( v_2 \) has degree two. Since \( u_2 \not\in V(C) \), we conclude \( v_2 \not\in V(C) \).

**Lemma 16.** Let \((G, X, \varphi)\) be a minimal counterexample with the outer face bounded by a cycle \( C \) and with \( X = \{x\} \). Let \( xv_1u_1v_2v_3 \) and \( xv_2u_2v_3v_1 \) be distinct cycles bounding 5-faces in \( G \) such that \( u_1, u_2, u_3 \not\in V(C) \). If \( \deg(u_1) = \deg(u_2) = 3 \), then \( \deg(u_3) \geq 5 \).

**Proof.** Note that \( \deg(v_2) = 2 \) and \( v_2 \not\in V(C) \). By Lemma 15 we have \( \deg(v_1) = \deg(v_3) = 2 \) and \( v_1, v_3 \not\in V(C) \). By Lemma 8 \( \deg(u_3) \geq 3 \). Suppose for a contradiction that \( \deg(u_3) \leq 4 \). By Corollary 11 there exist paths \( u_1u_0v_0x \) and \( u_3u_4v_4x \) in \( G \) with \( u_0 \neq u_2 \neq u_4 \).

First consider the case \( \deg(u_3) = 3 \). By the minimality of \((G, X, \varphi)\), the graph \( G' = G - \{v_2, u_2\} \) has an \( X \)-enhanced coloring \( \psi' \) extending \( \varphi \). We may assume without lose of generality that \( \psi'(x) = \{1, 2, 3\} \). If \( (\psi'(u_1) \cup \psi'(u_3)) \cap \{1, 2, 3\} \leq 2 \), then \( \psi' \) can be extended to \( u_2 \) and \( v_2 \) by Observation 14. This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction. Therefore, we can assume \( \psi'(u_1) = \{1, 2\} \) and \( 3 \in \psi'(u_3) \). By Observation 14 we can assume \( \psi'(u_0) = \{3, 4\} \). Furthermore, by symmetry between colors 1 and 2, and between colors 5 and 6, we can assume \( 1, 6 \notin \psi'(u_3) \). Let \( \psi(v) = \psi'(v) \) for \( v \in V(G) \setminus \{v_1, u_1, v_2, u_2\} \). Set \( \psi(u_1) = \{2, 5\}, \psi(v_1) = \{4, 6\}, \psi(u_2) = \{1, 6\} \) and \( \psi(v_2) = \{4, 5\} \). Then \( \psi \) is an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction.

Now we assume \( \deg(u_3) = 4 \). Let \( w \) be the neighbor of \( u_3 \) distinct from \( u_2, v_3 \), and \( u_4 \). By Lemma 9 \( xw \notin E(G) \), and in particular \( w \neq v_0 \). By Corollary 11 we have \( wu_0 \in E(G) \), and in particular \( \deg(u_0) \geq 3 \). Let \( G' \) be the graph obtained from \( G - \{v_2, u_2\} \) by identifying \( u_1 \) and \( w \). By Lemma 8 since \( \deg(u_0) \geq 3 \), \( G - u_2 \) does not contain a path of length three between \( u_1 \) and \( w \), and thus \( G' \) is triangle-free. By the minimality of \((G, X, \varphi)\), there exists an \( X \)-enhanced coloring \( \psi' \) of \( G' \) extending \( \varphi \). We may assume without lose of generality that \( \psi'(x) = \{1, 2, 3\} \). If \( (\psi'(u_1) \cup \psi'(u_3)) \cap \{1, 2, 3\} \leq 2 \), then \( \psi' \) extends to \( u_2 \) and \( v_2 \) by Observation 14. This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction. Therefore \( \{1, 2, 3\} \subseteq \psi'(u_1) \cup \psi'(u_3) \).

If \( \{\psi'(u_3) \cap \{1, 2, 3\} \} = 2 \), we can assume \( \psi'(u_3) = \{1, 2\} \) and \( 3 \notin \psi'(u_1) = \psi'(w) \). By Observation 14 we can also assume \( \psi'(u_4) = \{3, 4\} \). By symmetry between the colors 1 and 2, and between the colors 5 and 6, we can assume \( 1, 6 \notin \psi'(u_1) = \psi'(w) \). Let \( \psi(v) = \psi'(v) \) for \( v \in V(G) \setminus \{v_2, u_2, v_3, u_3\}, \psi(u_3) = \{2, 6\}, \psi(v_3) = \{4, 5\}, \psi(u_2) = \{1, \alpha\} \) for a color \( \alpha \in \{4, 5\} \setminus \psi'(u_1) \), and \( \psi(v_2) = \{9-\alpha, 6\} \). This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction. Hence \( \{\psi'(u_3) \cap \{1, 2, 3\} \} = 1 \), and we can assume \( \psi'(u_3) = \{3, 4\} \) and \( \psi'(u_1) = \psi'(w) = \{1, 2\} \). By Observation 14 we have \( 3 \in \psi'(u_0) \), and thus \( \{5, 6\} \cap \psi'(u_0) \leq 1 \) and by symmetry between the colors 5 and 6, we can assume that \( 6 \notin \psi'(u_0) \). Let \( \psi(v) = \psi'(v) \) for \( v \in V(G) \setminus \{v_2, u_2, v_1, u_1\}, \psi(u_1) = \{2, 6\}, \psi(v_1) = \{4, 5\}, \psi(u_2) = \{1, 5\} \), and \( \psi(v_2) = \{4, 6\} \). This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction.

**Lemma 17.** Let \((G, X, \varphi)\) be a minimal counterexample with the outer face bounded by a cycle \( C \) and with \( X = \{x\} \). Let \( xv_1u_1v_2v_3 \) and \( xv_2u_2v_3v_1 \) be cycles bounding 5-faces in \( G \), such that \( \deg(v_1) = \deg(v_2) = \deg(v_3) = 2 \) and \( \deg(u_1) = \deg(u_3) = 3 \). If \( u_1, u_2, u_3 \not\in V(C) \), then \( \deg(u_2) \geq 5 \).

**Proof.** By Corollary 11 there exist paths \( u_1u_0v_0x \) and \( u_3u_4v_4x \) in \( G \) with \( u_0 \neq u_2 \neq u_4 \). By Lemma 8 \( \deg(u_2) \geq 3 \), and by Lemma 16 \( \deg(u_2) \geq 4 \). Suppose for a contradiction that \( \deg(u_2) = \)
4, and let $w$ be the neighbor of $u_2$ distinct from $u_1$, $v_2$, and $u_3$. By Lemma 8, $xw \notin E(G)$, and $w$ and $x$ have no common neighbor. Let $G' = G - \{v_2, u_2\} + xw$. Then $G'$ is triangle-free. By the minimality of $(G, X, \varphi)$, there exists an $X$-enhanced coloring $\psi'$ of $G'$ extending $\varphi$. We may assume without lose of generality that $\psi'(x) = \{1, 2, 3\}$ and $\psi'(w) = \{4, 5\}$.

Let $\psi$ be the restriction of $\psi'$ to $G - \{v_1, u_1, v_2, u_2, v_3, u_3\}$. By Observation 14, we can assume $1 \in \psi(u_0)$, and thus by symmetry between the colors 2 and 3, and between the colors 4 and 5, we can assume that $\{2, 5\} \cap \psi(u_0) = \emptyset$. Set $\psi(u_1) = \{2, 5\}$ and $\psi(v_1) = \{4, 6\}$. By a symmetric argument, there exist $\alpha \in \{1, 2, 3\}$ and $\beta \in \{4, 5\}$ such that $\{\alpha, \beta\} \cap \psi(u_3) = \emptyset$. Set $\psi(u_3) = \{\alpha, \beta\}$ and $\psi(v_3) = \{9 - \beta, 6\}$. Let $\gamma$ be a color in $\{1, 3\} \setminus \{\alpha\}$ and set $\psi(u_2) = \{\gamma, 6\}$ and $\psi(v_2) = \{4, 5\}$. This gives an $X$-enhanced coloring of $G$ extending $\varphi$, which is a contradiction. \qed

### 2.3 More reducible configurations

Before we proceed with our analysis of configurations in a minimal counterexample, let us establish an auxiliary result on colorings of the graph depicted in Figure 1.

![Graph H](image)

**Figure 1:** The graph $H$ from the statement of Lemma 18

**Lemma 18.** Let $H$ be the graph shown in Figure 1 and let $L$ be an assignment of subsets of $\{1, \ldots, 6\}$ to vertices of $H$ satisfying the following conditions: $L(x) = \{1, 2, 3\}$, $|L(u_1)| = 3$ and $L(u_1) \cap \{1, 2, 3\} = \emptyset$, $L(u_3) = \{1, 2, 5, 6\}$, $|L(w_1)| = 4$, $|L(y_1)| = 2$ and $L(y_1) \subseteq \{3, 4, 5, 6\}$, and $L(v_1) = L(v_2) = L(v_3) = L(u_2) = L(u_1) = \{1, \ldots, 6\}$. There exists a 2-element set $S \subseteq L(u_1)$ such that $3 \in S$ and $S \cap L(y_1) \neq \emptyset$, and for any such set $S$, the graph $H$ has an $\{x\}$-enhanced coloring $\varphi$ such that $\varphi(v) \subseteq L(v)$ for all $v \in V(H)$ and $\varphi(u_1) = S$.

**Proof.** Since $|L(u_1)| = 3$, $|L(y_1)| = 2$ and $L(u_1), L(y_1) \subseteq \{3, 4, 5, 6\}$, we have $L(u_1) \cap L(y_1) \neq \emptyset$. Hence, there exists a 2-element set $S \subseteq L(u_1)$ such that $3 \in S$ and $S \cap L(y_1) \neq \emptyset$.

Consider any such set $S$, and let $\varphi(u_1) = S$, $\varphi(x) = \{1, 2, 3\}$ and $\varphi(y_1) = L(y_1)$. Then $\varphi(u_1) \cup \varphi(y_1) \subseteq \{3, 4, 5, 6\}$ and $|\varphi(u_1) \cup \varphi(y_1)| \leq 3$. Let $\varphi(v_1)$ be a 2-element subset of $\{4, 5, 6\} \setminus S$.

If there exists a color $\alpha \in L(w_3) \setminus L(u_3)$, then let $\varphi(w_3)$ be a 2-element subset of $L(w_3) \setminus (\varphi(u_1) \cup \varphi(y_1) \cup \{\alpha\})$. Let $\varphi(w_3)$ be a 2-element subset of $L(w_3) \setminus \varphi(w_3)$ such that $\alpha \in L(w_3)$. Then $|L(w_3) \setminus \varphi(w_3)| \geq 3$. Thus we can choose a 2-element subset $\varphi(u_3)$ of $L(u_3) \setminus \varphi(w_3)$ such that $|\varphi(u_3) \cap \{1, 2\}| = 1$; by symmetry, we can assume that $\varphi(u_3) = \{1, 5\}$. Since $3 \in \psi(u_1)$, we have
Note that $\deg(4) = 2, 3, 4, 5, 6$; we can assume $6 \in \psi(u_1)$ and the coloring of $H$. By symmetry, we can assume $\psi(u_3) = \{2, 5\}$ and $\psi(v_3) = \{4, 5\}$. Let $\varphi(\psi)$ be a 2-element subset of $\{2, 3, 4, 5\} \setminus \varphi(u_1)$ containing the color 2, and let $\varphi(v_2)$ be a 2-element subset of $\{4, 5, 6\} \setminus \varphi(u_2)$. This again gives a set coloring of $H$ as required.

Lemma 19. Let $(G, X, \varphi)$ be a minimal counterexample with the outer face bounded by a cycle $C$ and with $X = \{x\}$. Let $xv_1u_1v_2v_3, xv_2u_2v_3$ and $u_1u_2u_3w_3w_1$ be cycles bounding distinct 5-faces in $G$. If $u_1, u_2, u_3 \notin V(C)$ and $\deg(u_1) = \deg(u_3) = 4$, then $w_1, w_2 \notin V(C)$ and $\max(\deg(w_1), \deg(w_3)) \geq 4$.

Proof. Note that $\deg(u_2) = 3$, $\deg(v_2) = 2$, and $v_2 \notin V(C)$, and thus $\deg(v_1) = \deg(v_3) = 2$ and $v_1, v_3 \notin V(C)$ by Lemma 15. By Corollary 11 there exist paths $u_1u_0v_1x$ and $u_3u_4v_3x$ in $G$ with $u_0 \neq u_2 \neq u_4$. If $w_i \in V(C)$ for some $i \in \{1, 3\}$, then by Lemma 8 the cycle formed by the path $xu_iu_iw_i$ together with a path of length at most two between $x$ and $w_i$ in $C$ would bound a face, contradicting the assumption $\deg(u_i) = 4$. Hence, $w_1, w_3 \notin V(C)$. Furthermore, $w_1x, w_2x \notin E(G)$ by Lemma 9. Hence, Lemma 8 implies $\deg(w_1), \deg(w_3) \geq 3$.

Suppose for a contradiction that $\deg(u_1) = \deg(u_3) = 4$. For $i \in \{1, 3\}$, let $y_i$ be the neighbor of $w_i$ distinct from $u_i$ and $w_{4-i}$. Let $G'$ be the graph obtained from $G - \{v_2, u_2\}$ by identifying $u_3$ and $w_1$. Since $\deg(u_3) = 3$, Lemma 8 implies that $u_3$ and $w_1$ are not joined by a path of length three in $G - u_2$, and thus $G'$ is triangle-free. By the minimality of $(G, X, \varphi)$, there exists an $X$-enhanced coloring $\psi'$ of $G'$ extending $\varphi$. We may assume without lose of generality that $\psi'(x) = \{1, 2, 3\}$. If $|(\psi'(u_1) \cup \psi'(u_3)) \cap \{1, 2, 3\}| \leq 2$, then $\psi'$ can be extended to $u_2$ and $v_2$ by Observation 14. This gives an $X$-enhanced coloring of $G$ extending $\varphi$, which is a contradiction. Therefore, $\{1, 2, 3\} \subseteq \psi'(u_1) \cup \psi'(u_3)$.

Suppose first $|\psi'(u_1) \cap \{1, 2, 3\}| = 2$, and thus we can assume $\psi'(u_1) = \{1, 2\}$ and $3 \in \psi'(u_3) = \psi'(w_1)$. By Observation 14 we can assume $\psi'(u_0) = \{3, 4\}$. By symmetry between the colors 5 and 6, we can assume $6 \notin \psi'(u_4) = \psi'(w_1)$. Let $\psi(v) = \psi'(v)$ for $v \in V(G) \setminus \{u_1, v_1, u_2, v_2\}$, $\psi(u_1) = \{2, 6\}, \psi(v_1) = \{4, 5\}, \psi(u_2) = \{1, \alpha\}$ for a color $\alpha \in \{4, 5\} \setminus \psi'(u_3)$, and $\psi(v_2) = \{9 - \alpha, 6\}$. This gives an $X$-enhanced coloring of $G$ extending $\varphi$, which is a contradiction.

Therefore, $|\psi'(u_1) \cap \{1, 2, 3\}| = 1$, and we can assume $\psi'(u_1) \cap \{1, 2, 3\} = \{3\}$ and $\psi(u_3) = \psi'(w_1) = \{1, 2\}$. By Observation 14 we can assume $\psi'(u_4) = \{3, 4\}$ and $\psi'(u_0) \cap \{4, 5, 6\} \leq 1$. Let $L(x) = \{1, 2, 3\}, L(v_1) = L(v_2) = L(v_3) = L(u_2) = L(w_1) = \{1, \ldots, 6\}, L(u_3) = \{1, 2, 5, 6\}$, $L(w_3) = \{1, \ldots, 6\} \setminus \psi'(u_3)$, $\psi'(y_3), \psi'(y_1)$, and let $L(u_1)$ be a 3-element subset of $\{3, 4, 5, 6\} \setminus \psi'(u_0)$ containing the color 3. Let $R = \{x, v_1, u_1, v_2, u_2, v_3, u_3, w_3, y_3\}$, and observe that $G[R]$ is isomorphic to the graph depicted in Figure 3. Let $\psi$ be the union of the restriction of $\psi'$ to $G - R$ and the coloring of $G[R]$ obtained by Lemma 15 for the list assignment $L$. Then $\psi$ is an $X$-enhanced coloring of $G$ extending $\varphi$, which is a contradiction.

Suppose $u_2u_3u_4v_4v_2$ is a cycle bounding a face in a plane graph $G$, where $u_2, u_3$, and $u_4$ are not incident with the outer face, $\deg(u_3) = 3$ and $\deg(u_4) = 5$. We say that the cycle is $(u_4, u_3)$-
dangerous if either \( \deg(u_2) = 3 \), or \( \deg(u_2) = 4 \) and \( \deg(w_4) = \deg(w_2) = 3 \). We now exclude the situations in Figure 2 involving dangerous faces.

**Lemma 20.** Let \((G, X, \varphi)\) be a minimal counterexample with the outer face bounded by a cycle \(C\) and with \(X = \{x\}\). Let \(xv_2uv_3, xv_2uv_4, xv_4uv_5, \) and \(xv_5uv_6\) be distinct cycles bounding 5-faces in \(G\), where \(u_2, \ldots, u_6 \notin V(C)\) and \(\deg(u_3) = \deg(u_5) = 3\). Let \(K_1 = u_2u_3u_4w_4w_2\) and \(K_2 = u_6u_5u_4w_4w_6\) be 5-cycles bounding faces. If \(\deg(u_4) = 5\), then \(K_1\) is not \((u_4, u_3)\)-dangerous or \(K_2\) is not \((u_4, u_5)\)-dangerous.

**Proof.** Note that \(\deg(u_3) = \deg(u_5) = 3\), \(\deg(v_3) = \deg(v_5) = 2\), and \(u_2, u_6 \notin V(C)\), and thus \(\deg(v_2) = \deg(v_6) = 2\) and \(v_2, v_6 \notin V(C)\) by Lemma 15. By Corollary 11, there exist paths \(u_2v_1x\) and \(u_6u_7v_7x\) in \(G\) with \(u_1 \neq u_3\) and \(u_7 \neq u_5\). Suppose for a contradiction that \(K_1\) is \((u_4, u_3)\)-dangerous and \(K_2\) is \((u_4, u_5)\)-dangerous. By Corollary 11, Lemmas 8 and symmetry, we can assume that \(G\) contains one of the subgraphs \(H_1, H_2, \) or \(H_3\) depicted in Figure 2 (up to possible identification of vertices \(u_1\) and \(u_7\) in the graph \(H_1\); all other identifications can be excluded using Lemma 8). Let \(G'\) be the graph obtained from \(G - \{v_3, u_3, v_5, u_3\}\) by identifying \(u_2\) and \(w_4\), and identifying \(u_6\) and \(w_5\). Using Lemma 8, observe \(G'\) is triangle-free. By the minimality of \(G\), there exists an \(X\)-enhanced coloring \(\psi'\) of \(G'\) extending \(\varphi\). We may assume without lose of generality that \(\psi'(x) = \{1, 2, 3\}\). Suppose first that \(\psi'(u_4) \subset \{1, 2, 3\}\), say \(\psi'(u_4) = \{1, 2\}\). For \(i \in \{2, 6\}\), by Observation 14, we have \(\psi'(u_i) \cap \{1, 2, 3\} \neq \emptyset\). Hence, we can assume \(\psi'(u_2) = \psi'(w_4) = \{3, \alpha\}\) and \(\psi'(w_6) = \psi'(w_5) = \{3, \beta\}\) for some \(\alpha, \beta \in \{4, 5\}\). Let \(\psi(v) = \psi'(v)\) for \(v \in V(G) - \{v_3, u_3, v_4, u_4, v_5, u_5\}\). Set \(\psi(u_4) = \{2, 6\}, \psi(v_4) = \{4, 5\}, \psi(u_3) = \{1, 9 - \alpha\}, \psi(v_3) = \{\alpha, 6\}, \psi(u_5) = \{1, 9 - \beta\}, \) and \(\psi(v_5) = \{\beta, 6\}\). Then \(\psi\) is an \(X\)-enhanced coloring of \(G\) extending \(\varphi\), which is a contradiction.

Therefore, by Observation 14, we can assume \(\psi'(u_4) \cap \{1, 2, 3\} = \{3\}\). Let us now discuss the cases regarding the ways \(K_1\) and \(K_2\) could be dangerous.

1. Suppose first that \(\deg(u_2) = \deg(u_6) = 3\), and thus \(w_2 = u_1\) and \(w_6 = u_7\), see the subgraph \(H_1\) in Figure 2. Let \(\psi\) be the restriction of \(\psi'\) to \(G' - \{v_2, u_2, v_6, u_6\}\). If \(\psi'(u_2) \neq \{1, 2\}\), then we can set \(\psi(u_2) = \psi'(u_2), \psi(v_2) = \psi'(u_2), \) choose \(\psi(u_3)\) as a 2-element subset of \(\{1, \ldots, 6\} \setminus (\psi'(u_2) \cup \psi'(u_4))\) containing color 1 or 2, and choose \(\psi(v_3)\) as a 2-element subset of \(\{4, 5, 6\} \setminus \psi(u_3)\). If \(\psi'(u_2) = \{1, 2\}\), then by Observation 14 and symmetry, we can assume...
\( \psi'(u_1) = \{3,4\} \) and \( 6 \notin \psi'(u_4) \). We set \( \psi(u_2) = \{1,5\} \), \( \psi(v_2) = \{4,6\} \), \( \psi(u_3) = \{2,6\} \) and \( \psi(v_3) = \{4,5\} \). Symmetrically, we extend \( \psi \) to \( u_5, v_5, u_6, \) and \( v_6 \). This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction.

(ii) Hence, we can by symmetry assume that \( \deg(u_6) = 4 \) and \( w'_4 \) and \( w_6 \) are vertices of degree three. By Lemma 8 we have \( w'_4, w_6 \notin V(C) \). Suppose that \( \deg(u_2) = 3 \), see the subgraph \( H_2 \) in Figure 2. If \( \psi(u_6) \neq \{1,2\} \), then by Observation 14 we can assume that \( \psi'(u_6) = \{2,4\} \) and \( \psi'(u_4) \subseteq \{3,4,5\} \). Let \( \psi(v) = \psi'(v) \) for \( v \in V(G) \setminus \{u_2, v_2, v_3, u_3, v_5, u_5\} \). Then \( \psi \) extends to \( u_2, v_2, u_3, v_3 \) as in the previous case, and we can choose \( \psi(u_6) = \{1,6\} \) and \( \psi(v_5) = \{4,5\} \). This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction.

Therefore, \( \psi'(u_6) = \{1,2\} \). By Observation 14 we can assume \( \psi'(u_7) = \{3,4\} \). Let \( R = \{x, v_4, u_4, v_5, u_5, v_6, w'_4, y'_4, w_6\} \), where \( y'_4 \) is the neighbor of \( w'_4 \) distinct from \( u_4 \) and \( w_6 \). Note that \( G[R] \) is isomorphic to the graph depicted in Figure 1. Since \( 3 \in \psi'(u_4) \) and \( \psi'(u_6) = \psi'(w'_4) \), by Observation 14 we have \( \psi'(u_4) \cap \{1,2\} \neq \emptyset \) and thus there exists a 3-element set \( L(u_4) \subseteq \{3,\ldots,6\} \setminus \psi'(w'_4) \) containing the color 3. Let \( \psi \) be the \( X \)-enhanced coloring of \( G - \{v_3, v_5\} \) obtained from the restriction of \( \psi' \) to \( G - (R \cup \{v_3, v_5\}) \) by extending it to \( G[R] \) using Lemma 18. Note that \( \psi(u_4) \subseteq \{3,4,5,6\} \) and \( 3 \in \psi(u_4) \).

By Observation 14 we have \( \psi(u_2) \cap \{1,2\} \neq \emptyset \). If \( \psi(u_2) \neq \{1,2\} \), then \( \psi \) can be extended to \( u_3 \) and \( v_3 \) by Observation 14. This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction. If \( \psi(u_2) = \{1,2\} \), then by Observation 14 \( \psi(u_1) = \{3,\alpha\} \) for some \( \alpha \in \{4,5,6\} \). Let \( \psi_0(v) = \psi(v) \) for \( v \in V(G) \setminus \{v_3, u_3, v_2, u_2\} \), \( \psi_0(u_2) = \{1,\beta\} \), and \( \psi_0(u_3) = \{2,\gamma\} \) for \( \beta \in \{4,5,6\} \setminus \{\alpha\} \) and \( \gamma \in \{4,5,6\} \setminus (\psi(u_4) \cup \{\beta\}) \). Then \( \psi_0 \) can be extended to \( v_2 \) and \( v_3 \) by Observation 14 giving an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction.

(iii) Therefore, \( \deg(u_2) = \deg(u_6) = 4 \) and \( w_2, w_4, w'_4 \) and \( w_6 \) are vertices of degree three, see the subgraph \( H_3 \) in Figure 2. By Lemma 8 we have \( w_2, w_4, w'_4, w_6 \notin V(C) \). If \( \psi'(u_2) \neq \{1,2\} \) and \( \psi'(u_6) \neq \{1,2\} \), then \( \psi' \) extends to \( u_2, v_2, u_3, v_3 \) and \( v_5, u_5 \) by Observation 14. This gives an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction.

Hence, we can by symmetry assume \( \psi'(u_6) = \{1,2\} \). By Observation 14 we can assume \( \psi'(u_7) = \{3,4\} \). Let \( R' = \{x, v_2, v_3, u_3, v_4, u_4, w_4, y_4, w_2\} \), where \( y_4 \) is the neighbor of \( u_4 \) distinct from \( u_4 \) and \( w_2 \). If \( \psi'(u_2) \neq \{1,2\} \), then color \( G[R] \) by Lemma 18 and then extend the coloring to \( u_3 \) and \( v_3 \) as in the case (ii). Hence, we can also assume that \( \psi'(u_2) = \{1,2\} \), and \( \psi'(u_1) = \{3,\alpha\} \) for some \( \alpha \in \{4,5\} \). Let \( R' = \{x, v_2, v_3, u_3, v_4, u_4, w_4, y_4, w_2\} \), where \( y_4 \) is the neighbor of \( u_4 \) distinct from \( u_4 \) and \( w_2 \). Since \( \psi'(u_4) = \psi'(w'_4) = \{1,2\} \), we have \( \psi'(y_4), \psi'(y'_4) \subseteq \{3,4,5,6\} \), and if \( \psi'(y_4) \cap \psi'(y'_4) = \emptyset \), then \( \psi'(y_4) \cup \psi'(y'_4) = \{3,4,5,6\} \). Hence, there exists a 2-element set \( S \subseteq \{3,4,5,6\} \) such that \( 3 \in S \) and \( S \cap \psi'(y_4) \neq \emptyset = S \cap \psi'(y'_4) \).

Let \( \psi \) be the restriction of \( \psi' \) to \( G - (R \cup R') \). By Lemma 18 \( \psi \) extends to colorings \( \psi_1 \) of \( G[R] \) and \( \psi_2 \) of \( G[R'] \) such that \( \psi_1(x) = \psi_2(x) = \{1,2,3\} \) and \( \psi_1(u_4) = \psi_2(u_4) = S \). Note also \( \psi_1(v_4) = \psi_2(v_4) = \{3,4,5,6\} \setminus S \). Then \( \psi \cup \psi_1 \cup \psi_2 \) is an \( X \)-enhanced coloring of \( G \) extending \( \varphi \), which is a contradiction. 

\[ \square \]
2.4 Discharging

2.4.1 Notation

Consider a minimal counterexample \((G, X, \varphi)\) with the outer face bounded by a cycle \(C\). By Corollary 11, every face other than the outer one is a 5-face. If \(X = \{x\}\), consider a cycle \(xv_1u_1u_2v_2\) such that \(u_1, v_1 \not\in V(C)\) bounding a 5-face \(f\). If \(\deg(v_1) = 2\), \(\deg(u_1) = 3\) and \(u_2 \not\in V(C)\), then \(\deg(v_2) = 2\) and \(v_2 \not\in V(C)\) by Lemma 15 and we say \(f\) is a type-A face. By Lemma 8, we have \(\deg(u_2) \geq 3\). If \(\deg(u_2) = 3\), then we say \(f\) is a type-A-1 face incident with \(x\). If \(\deg(u_2) = 4\), then we say \(f\) is a type-A-2 face incident with \(x\). If \(\deg(u_2) \geq 5\), then we say \(f\) is a type-A-3 face incident with \(x\). For \(i \in \{0, 3\}\), if \(u_i \in V(C)\) or \(\deg(u_i) \geq 5\), and \(f\) is a type-A-1 face or type-A-2 face, then we say \(u_i\) is connected to \(f\).

Suppose \(f\) is a 5-face bounded by a 5-cycle \(xv_1u_1u_2v_2\) satisfying \(u_1, v_1 \not\in V(C)\), \(\deg(v_1) = 2\), \(\deg(u_1) = 4\), and \(\deg(v_2) \geq 3\) \((v_2\) may or may not belong to \(V(C)\)). In this case we say \(f\) is a type-B face.

Suppose now a cycle \(xv_1u_1u_2v_2\) bounds a type-A-2 face \(f_1\) incident with \(x\), where \(\deg(u_1) = 4\) and \(\deg(u_2) = 3\). Since \(\deg(u_2) = 2\), there exists a cycle \(xv_2u_2u_3v_3\) bounding a face \(f_2\) distinct from \(f_1\). Suppose furthermore \(u_3 \not\in V(C)\); then \(\deg(v_3) = 2\) by Lemma 15 and \(\deg(u_3) \geq 4\) by Lemma 16.

Let us consider the case that \(\deg(u_3) = 4\), and let \(u_1u_2u_3w_3v_1\) be the cycle bounding the 5-face \(g\) incident with \(u_2\) distinct from \(f_1\) and \(f_2\). Note that \(\deg(u_1) \geq 4\) or \(\deg(u_3) \geq 4\) by Lemma 19. We say \(g\) is a type-C face, and for \(i \in \{1, 2\}\) we say \(g\) is connected to \(f_i\) if \(\deg(v_{2i-1}) = 3\). Note that a type-C face is connected to at most one type-A-2 face and is incident with at least three \(4^+\)-vertices.

A type-A-2 face is tight if no vertex or type-C face is connected to it.

Continuing in the situation of the previous paragraph, suppose that \(\deg(u_3) \geq 4\). Since \(\deg(v_3) = 2\), there exists a cycle \(xv_3u_3w_4v_4\) bounding a 5-face \(f_3\) distinct from \(f_2\). Suppose that \(u_4 \not\in V(C)\).

- If \(\deg(v_4) = 2\), then \(\deg(u_4) \geq 4\) by Lemma 17. If \(\deg(u_4) = 4\), then let \(w_3u_3u_4w_4y_4\) be a cycle bounding the 5-face \(h\) incident with \(u_3\) distinct from \(f_2\), \(f_3\), and \(g\), see the left graph in Figure 3 for an illustration. We say \(h\) is a type-D face connected to \(f_2\).

- Suppose now \(\deg(v_4) = \deg(u_4) = 3\) (so \(f_3\) is a type-B face) and a cycle \(v_4u_4w_4y_4z_4\) bounds a 5-face \(k \neq f_3\), where \(\deg(y_4) = \deg(z_4) = 3, v_4, w_4, y_4, z_4 \not\in V(C)\) and \(\deg(w_4) \geq 4\). Let \(q\) be the face incident with \(u_4\) distinct from \(f_3\) and \(k\), bounded by the cycle \(q = w_4u_4w_3v_3y_3\), see the right graph in Figure 3 for an illustration. We say \(q\) is a type-E face connected to \(k\). Note that each type-E face is incident with at least three \(4^+\)-vertices.

By Lemma 8, the distance of \(w_3\) and \(w_4\) from \(x\) is three, and thus a type-D or type-E face cannot also be a type-A, type-B, or type-C face, and a type-D face cannot also be a type-E face. Furthermore, each type-D face is connected to \(p \leq 2\) type-A-2 faces and is incident with at least \((p + 2)\) \(4^+\)-vertices, and each type-E face is connected to a unique face.

Suppose now cycles \(xv_1u_1u_2v_2\) and \(xv_2u_2u_3v_3\) bound distinct 5-faces \(f_1\) and \(f_2\), where \(u_1, u_2, u_3 \not\in V(C)\), \(\deg(v_1) = \deg(v_2) = \deg(v_3) = 2, \deg(u_1) = 5, \deg(u_2) = 3,\) and \(\deg(u_3) = 4\). Let \(g\) be the face incident with \(u_2\) distinct from \(f_1\) and \(f_2\), bounded by the cycle \(u_1w_1u_3v_3w_2\). If for some \(i \in \{1, 3\}\), the vertex \(w_i\) has degree at least four, we say \(g\) is a type-F face connected to \(u_1\). Note
that each type-F face is incident with at most two vertices of degree three not belonging to $V(C)$. By Lemma $\text{[5]}$ the distance of $w_3$ and $w_4$ from $x$ is three, and thus a type-F face cannot also be a type-A, . . . , or type-E face, and each type-F face is connected to a unique vertex.

Let $Q$ be a 5-cycle in $G$ vertex-disjoint from $X$ and intersecting $C$ in at most one vertex. We say the face bounded by $Q$ is tied to a vertex $z \in V(G)$ if $z \notin V(Q)$ and $z$ has a neighbor in $V(Q) \setminus V(C)$ of degree three. Suppose $X = \{x\}$ and $x$ is tied to a 5-face $f$ not incident with $x$ bounded by the cycle $v_5v_1v_2v_3v_4$ via an edge $xv_5$. By Lemmas $\text{[8]}$ and $\text{[9]}$ no vertex of $C$ is incident with $f$. By Corollary $\text{[13]}$ a vertex incident with $f$ has degree at least four, without loss of generality $v_1$ or $v_2$. If four vertices of $Q$ have degree three, then let $g$ be the face whose boundary contains the path $xv_5v_1$; in this situation, we say that $f$ is a special 5-face tied to $x$ and connected to $g$.

2.4.2 Initial charge and discharging rules

Now we proceed by the discharging method. Consider a minimal counterexample $(G, X, \varphi)$ with the outer face bounded by the cycle $C$. Set the initial charge of every vertex $v$ of $G$ to be $\text{ch}_0(v) = \deg(v) - 4$, and the initial charge of every face $f$ of $G$ to be $\text{ch}_0(f) = |f| - 4$. By Euler’s formula,

$$
\sum_{v \in V(G)} \text{ch}_0(v) + \sum_{f \in F(G)} \text{ch}_0(f) = \sum_{v \in V(G)} (\deg(v) - 4) + \sum_{f \in F(G)} (|f| - 4) = 4(|E(G)| - |V(G)| - |F(G)|) = -8. \quad (1)
$$

We can without loss of generality assume that $X \neq \emptyset$ (and thus $|X| = 1$), as otherwise we observe that the cycle $C$ bounding the outer face contains a subpath $uxv$ such that $|\varphi(u) \cup \varphi(v) \cup \varphi(x)| \leq 5$, and we can set $X = \{x\}$ and add a color to $\varphi(x)$. Let $x$ denote the unique vertex in $X$. We redistribute the charges according to the following rules.

R1 Each face other than the outer one sends $\frac{1}{3}$ to each incident vertex that either has degree two and belongs to $V(C)$, or has degree three and does not belong to $V(C)$.

R2 Each face sends 1 to each incident vertex that has degree two and does not belong to $V(C)$.

R3 The vertex $x$ sends 1 to each incident face other than the outer one.

R4 Each $5^+$-vertex other than $x$ sends $\frac{1}{3}$ to each incident type-A-3 face.
R5 If \( v \neq x \) is a 5*-vertex or belongs to \( V(C) \), then \( v \) sends \( \frac{1}{3} \) to each connected type-A-1 face or type-A-2 face.

R6 Each type-B face sends \( \frac{1}{3} \) to each tight type-A-2 face with which it shares an edge incident with \( x \).

R7 Each type-C face sends \( \frac{1}{3} \) to each connected type-A-2 face.

R8 Each type-D face sends \( \frac{1}{4} \) to each connected type-A-2 face.

R9 Each type-F face sends \( \frac{1}{3} \) to each connected 5-face.

R10 Suppose \( f \) is a special 5-face tied to \( x \) and connected to a face \( g \). If a type-E face \( h \) is connected to \( f \), then \( h \) sends \( \frac{1}{2} \) to \( f \), otherwise \( g \) sends \( \frac{1}{3} \) to \( f \).

R11 Each vertex on the outer face other than \( x \) sends \( \frac{1}{3} \) to each 5-face tied to it.

Let the charge obtained by these rules be called final and denoted by \( ch \). Note that the redistribution does not change the total amount of charge, and thus the sum of the final charges assigned to vertices and faces of \( G \) is \(-8\) by (1).

2.4.3 Final charges of vertices

Lemma 21. Let \( (G, \{x\}, \varphi) \) be a minimal counterexample with the outer face bounded by a cycle \( C \). Then each vertex \( v \in V(G) \setminus V(C) \) satisfies \( ch(v) \geq 0 \).

Proof. By Lemma \( 8 \), \( v \) has degree at least two. If \( v \) has degree two, then \( v \) receives 1 from each incident face by \( \text{R2} \), and thus \( ch(v) = ch_0(v) + 2 \times 1 = 0 \). If \( v \) has degree three, then it receives \( \frac{1}{3} \) from each incident face by \( \text{R1} \) and thus \( ch(v) = ch_0(v) + 3 \times \frac{1}{3} = 0 \). If \( v \) has degree four, then \( ch(v) = ch_0(v) = 0 \).

If \( v \) has degree five, then \( v \) sends \( \frac{1}{3} \) to each incident type-A-3 face by \( \text{R4} \) and each connected type-A-1 face or type-A-2 face by \( \text{R5} \). Let \( k \) be the number of faces to that \( v \) sends charge. By Lemma \( 9 \), there exists at most one path of length two between \( v \) and \( x \), and thus \( v \) is incident with at most two type-A-3 faces, and connected to at most two type-A-1 or type-A-2 faces, implying that \( k \leq 4 \). If \( k \leq 3 \), then \( ch(v) \geq ch_0(v) - 3 \times \frac{1}{3} = 0 \). Hence, we can assume \( k = 4 \). By Lemma \( 20 \), \( v \) is incident with at least one type-F face, from which it receives \( 1/3 \) by \( \text{R9} \). Therefore, \( ch(v) \geq ch_0(v) - 4 \times \frac{1}{3} = 0 \).

If \( \deg(v) \geq 6 \), then similarly \( v \) sends charge to at most four faces by \( \text{R4} \) and \( \text{R5} \) and \( ch(v) \geq ch_0(v) - 4 \times \frac{1}{3} > 0 \).

\( \square \)

Lemma 22. Let \( (G, \{x\}, \varphi) \) be a minimal counterexample with the outer face bounded by a cycle \( C \). Then \( ch(x) = -3 \), and for any vertex \( v \in V(G) \setminus \{x\} \), \( ch(v) = -\frac{2}{3} \) if \( \deg(v) = 2 \) and \( ch(v) \geq \frac{2}{3}(deg(v) - 5) \) if \( deg(v) \geq 3 \).

Proof. Note that \( x \) sends 1 to each incident face other than the outer one by \( \text{R3} \) and thus \( ch(x) = ch_0(x) - (deg(v) - 1) = -3 \). Consider a vertex \( v \in V(G) \setminus \{x\} \). If \( deg(v) = 2 \), then \( v \) receives \( \frac{1}{3} \) from the incident non-outer face by \( \text{R1} \) and \( ch(v) = ch_0(v) + \frac{1}{3} = -\frac{5}{3} \). If \( deg(v) \geq 3 \), then \( v \) sends \( \frac{1}{3} \) to at most \( deg(v) - 2 \) faces tied or connected to it, and thus \( ch(v) \geq ch_0(v) - (deg(v) - 2) \times \frac{1}{3} = \frac{2}{3}(deg(v) - 5) \).

\( \square \)
2.4.4 Final charges of faces

**Lemma 23.** Let \((G, \{x\}, \varphi)\) be a minimal counterexample with the outer face bounded by a cycle \(C\). Every face \(f\) not incident with \(x\) satisfies \(\text{ch}(f) \geq 0\).

*Proof.* By Corollary 11 we have \(|f| = 5\) and \(\text{ch}_0(f) = 1\). Since \(f\) is not incident with \(x\), Lemma 8 implies that every vertex of degree two incident with \(f\) belongs to \(V(C)\), and thus \(f\) does not send charge by R2. By R1, \(f\) sends at most \(\frac{1}{3}\) to each incident vertex.

If \(f\) is a type-C face, then \(f\) sends \(\frac{1}{3}\) to each connected type-A-2 face by R7. Recall that \(f\) is connected to at most one type-A-2 face and \(f\) is incident with at least three \(4^+\)-vertices, i.e., the number of vertices to that \(f\) sends charge is at most 2. Then \(\text{ch}(v) \geq \text{ch}_0(v) - \frac{1}{3} - 2 \times \frac{1}{3} = 0\).

If \(f\) is a type-D face, then \(f\) sends \(\frac{1}{3}\) to each connected type-A-2 face by R8. Suppose that \(f\) is connected to \(p\) type-A-2 faces. Recall that \(p \leq 2\) and \(f\) is incident with at least \((p+2) 4^+\)-vertices, and thus the number of vertices to that \(f\) sends charge is at most \(5 - (2 + p) = 3 - p\). Hence, \(\text{ch}(v) \geq \text{ch}_0(v) - \frac{1}{3} - 2 \times \frac{1}{3} = 0\).

If \(f\) is a type-E face, then \(f\) is connected to exactly one special 5-face \(g\) tied to \(x\), and \(f\) sends \(\frac{1}{3}\) to \(g\) by R10. Recall that \(f\) is incident with at least three \(4^+\)-vertices, and thus the number of vertices to that \(f\) sends charge is at most 2. Hence, \(\text{ch}(v) \geq \text{ch}_0(v) - \frac{1}{3} - 2 \times \frac{1}{3} = 0\).

If \(f\) is a type-F face, then \(f\) is connected to exactly one 5-vertex \(v\), and \(f\) sends \(\frac{1}{3}\) to \(v\) by R9. Recall that the number of vertices to that \(f\) sends charge is at most 2. Then \(\text{ch}(v) \geq \text{ch}_0(v) - \frac{1}{3} - 2 \times \frac{1}{3} = 0\).

Therefore, \(f\) is not a type-C, type-D, type-E, or type-F face. Hence, \(f\) only sends \(\frac{1}{3}\) to each incident 2-vertex in \(C\) or 3-vertex not in \(C\) by R1. Let \(k\) be the number of vertices to that \(f\) sends charge. If \(k \leq 3\), then \(\text{ch}(f) \geq 0\), and thus we can assume that \(k \geq 4\). If \(f\) is incident with a vertex \(v\) of degree two, then note that \(v \in V(C)\) by Lemma 8. Furthermore, since \(G\) is 2-connected and \(G \neq C\), we conclude that \(f\) is incident with at least two \(3^+\)-vertices belonging to \(V(C)\), to which \(f\) does not send charge. This contradicts the assumption that \(k \geq 4\). Hence, no vertex of degree two is incident with \(f\), and thus \(k\) is the number of incident vertices of degree three not belonging to \(V(C)\). If \(f\) is tied to \(x\), then \(f\) is incident with exactly four \(3^+\)-vertices by Corollary 13. By R10, \(f\) receives \(\frac{1}{3}\) from some face, and thus \(\text{ch}(f) = \text{ch}_0(f) - 4 \times \frac{1}{3} + \frac{1}{3} = 0\). If \(f\) is not tied to \(x\), then \(f\) is tied to at least \(k - 3\) vertices of \(C\) by Lemma 12 and \(f\) receives \(\frac{1}{3}\) from each of them by R11 and \(\text{ch}(f) \geq \text{ch}_0(f) - k \times \frac{1}{3} + (k - 3) \times \frac{1}{3} = 0\).

**Lemma 24.** Let \((G, \{x\}, \varphi)\) be a minimal counterexample with the outer face bounded by a cycle \(C\). Any face \(f\) incident with \(x\) other than the outer one satisfies \(\text{ch}(f) \geq 0\).

*Proof.* By Corollary 11, we have \(|f| = 5\) and \(\text{ch}_0(f) = 1\). Note that \(f\) receives 1 from \(x\) by R3 and sends charge only by R1, R2, R6, and R10. Let \(xv_3u_3u_4v_4\) denote the cycle bounding \(f\).

Consider first the case that neither \(v_3\) nor \(v_4\) is a vertex of degree two not belonging to \(C\). Then \(f\) sends at most \(4 \times \frac{1}{3}\) by R1 and at most \(2 \times \frac{1}{3}\) by R10, implying \(\text{ch}(f) \geq \text{ch}_0(f) + 1 - 4 \times \frac{1}{3} - 2 \times \frac{1}{3} = 0\).

Hence, we can assume \(\text{deg}(v_3) = 2\) and \(v_3 \notin V(C)\), and thus \(f\) sends 1 to \(v_3\) by R2. By Lemma 9, we have \(u_3 \notin V(C)\), and thus \(\text{deg}(u_3) \geq 3\) by Lemma 8. Let \(f_2 \neq f\) be the other 5-face incident with \(xv_3\), bounded by a cycle \(xv_3u_3u_2v_2\). By R1 and R6, \(f\) sends at most \(\frac{1}{3}\) to \(v_3\) and \(f_2\) in total. We now discuss the case that \(v_4\) is not a vertex of degree two not belonging to \(C\).
• If \( v_4 \in V(C) \), then \( f \) does not send charge by \( \text{R10} \) and sends at most \( \frac{1}{3} \) to \( v_4 \) and \( u_4 \) in total by \( \text{R11} \) (if \( \deg(v_4) = 2 \), then \( u_4 \in V(C) \), and since \( u_3 \notin V(C) \), we have \( \deg(u_4) \geq 3 \)). Hence, \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - \frac{1}{3} - \frac{1}{3} > 0 \). Therefore, we can assume \( v_4 \notin V(C) \), and thus \( \deg(v_4) \geq 3 \). By Lemma 15, we have \( u_4 \notin V(C) \). Then \( \deg(u_3) \geq 4 \) by Lemma 15 and \( f \) does not send charge to \( u_3 \) by \( \text{R11} \).

• If \( f \) sends at most \( \frac{1}{3} \) by \( \text{R6} \) and \( \text{R10} \) in total, or \( f \) does not send charge by \( \text{R1} \) to at least one of \( v_4 \) and \( u_4 \), then \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 1 - 3 \times \frac{1}{3} = 0 \).

• Hence, we can assume that \( f \) sends charge by both \( \text{R6} \) and \( \text{R10} \) and \( f \) sends charge to both \( v_3 \) and \( u_4 \) by \( \text{R1} \). Consequently, \( \deg(u_3) = 4 \), \( \deg(v_4) = \deg(u_4) = 3 \), and the neighbor \( w_4 \) of \( u_4 \) distinct from \( v_3 \) and \( v_4 \) is a 4+-vertex. Since \( f \) sends charge by \( \text{R6} \) we conclude that \( f_2 \) is a tight type-A-2 face, and thus \( \deg(v_2) = 2 \), \( \deg(u_4) = 3 \) and \( v_2, u_2 \notin V(C) \). Let \( f_1 \neq f_2 \) be the other 5-face incident with \( x_{v_2} \), bounded by a cycle \( x_{v_1}u_1u_2v_2 \). Since \( f_2 \) is tight, we have \( u_1 \notin V(C) \) and \( \deg(u_1) \leq 4 \), and thus \( v_1 \notin V(C) \) and \( \deg(v_1) = 2 \) by Lemma 15. By Lemma 16 we have \( \deg(u_1) = 4 \). Let \( u_3u_4u_3w_3 \in 2 \) the cycle bounding the face \( g \) incident with \( u_3 \) distinct from \( f_1 \) and \( f_2 \). Since \( f_2 \) is tight, we conclude that \( u_3 \) is a 4+-vertex, and thus the face incident with \( u_3 \) distinct from \( f, f_2 \), and \( g \) is a type-E face, contradicting the assumption that \( f \) sends charge by \( \text{R10} \).

Finally, let us consider the case that both \( v_3 \) and \( v_4 \) are vertices of degree two not belonging to \( V(C) \). Then \( f \) sends charge only by \( \text{R11} \) and \( \text{R2} \). By Lemma 8, we have \( u_3, u_4 \notin V(C) \). If both \( u_3 \) and \( u_4 \) are 4+-vertices, then \( f \) only sends charge to \( v_3 \) and \( v_4 \) and \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 2 \times 1 = 0 \). Therefore, we can assume \( \deg(u_3) = 3 \). If \( u_4 \) is a 5+-vertex, then \( f \) receives \( \frac{1}{3} \) from \( u_4 \) by \( \text{R4} \) and \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 2 \times 1 - \frac{1}{3} + \frac{1}{3} = 0 \), and thus we can assume \( \deg(u_4) \leq 4 \). Let \( f_2 \neq f \) be the face incident with \( x_{v_2} \), bounded by a cycle \( x_{v_2}u_2u_3u_1 \), and let \( f_4 \neq f \) be the face incident with \( x_{v_4} \), bounded by a cycle \( x_{v_4}u_4u_5v_5 \).

If \( \deg(u_4) = 3 \), then by Lemma 16 we have either \( u_i \in V(C) \) or \( \deg(u_i) \geq 5 \) for \( i \in \{2, 5\} \), and \( f \) receives \( 2 \times \frac{1}{3} \) by \( \text{R5} \) and \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 2 \times 1 - 2 \times \frac{1}{3} + 2 \times \frac{1}{3} = 0 \). Therefore, we can assume \( \deg(u_4) = 3 \). If \( u_2 \notin V(C) \) or \( \deg(u_2) \geq 5 \), then \( f \) receives \( \frac{1}{3} \) by \( \text{R5} \) and \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 2 \times 1 - \frac{1}{3} + \frac{1}{3} = 0 \). Hence, we can assume \( u_2 \notin V(C) \) and \( \deg(u_2) \leq 4 \), and analogously, \( u_5 \notin V(C) \) and \( \deg(u_5) \leq 4 \). By Lemma 15 \( \deg(v_2) = 2 \), and \( \deg(u_2) = 4 \) by Lemma 16. Let \( u_2u_3u_4u_1w_1w_2 \) be the cycle bounding the face \( g \) incident with \( u_3 \) distinct from \( f_2 \) and \( f \). If \( \deg(u_3) = 3 \), then \( f \) receives \( \frac{1}{3} \) by \( \text{R7} \) and \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 2 \times 1 - \frac{1}{3} + \frac{1}{3} \geq 0 \). Hence, we can assume \( u_3 \) is a 4+-vertex, which implies \( f \) is a tight type-A-2 face. If \( \deg(v_5) \geq 3 \), then \( f_4 \) is a type-B face. By \( \text{R6} \) \( f \) receives \( \frac{1}{3} \) from \( f_4 \), and \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 2 \times 1 - \frac{1}{3} + \frac{1}{3} = 0 \). Therefore, \( \deg(v_5) \geq 2 \), and since \( u_5 \notin V(C) \), we have \( v_2 \notin V(C) \). By Lemma 17 we have \( \deg(u_5) = 4 \). However, then \( f \) receives \( \frac{1}{3} \) by \( \text{R8} \) and \( \text{ch}(f) \geq \text{ch}_0(f) + 1 - 2 \times 1 - \frac{1}{3} + \frac{1}{3} = 0 \).

2.4.5 Proof of Theorem 7

Proof of Theorem 7 Suppose for a contradiction there exists a minimal counterexample \((G, X, \varphi)\), with the outer face bounded by a cycle \( C \). As we argued before, we can assume \( X \neq \emptyset \); let \( X = \{x\} \). By Lemma 21 \( \text{ch}(v) \geq 0 \) for \( v \in V(G) \setminus V(C) \). By Lemmas 23 and 24 \( \text{ch}(f) \geq 0 \) for every non-outer face \( f \) of \( G \).
The final charge of the outer face is $|C| - 4$. Consider a vertex $v \in V(C)$. By Lemma 22, $\text{ch}(x) = -3$, $\text{ch}(v) = -\frac{3}{2}$ if $v \neq x$ and $\text{deg}(v) = 2$, and $\text{ch}(v) \geq \frac{3}{2}(\text{deg}(v) - 5) \geq -\frac{4}{3}$ if $v \neq x$ and $\text{deg}(v) \geq 3$. If $|C| = 4$, then by Lemma 8 and Corollary 11 all vertices of $C$ have degree at least three, and thus the sum of the final charges is at least $(|C| - 4) - 3 - 3 \times \frac{3}{2} = -7$, a contradiction to [I].

Therefore, $|C| = 5$; let $C = x_1v_1v_2v_3v_4$. If $V(C) \setminus \{x\}$ contains at most one vertex of degree two, then the sum of the final charges is at least $(|C| - 4) - 3 - \frac{3}{2} - 3 \times \frac{3}{4} = -\frac{23}{4} > -8$, a contradiction to [I]. By Lemma 10 and Corollary 11 no two vertices of degree two in $V(C) \setminus \{x\}$ are adjacent. Hence, exactly two vertices of $V(C) \setminus \{x\}$ have degree two. If a vertex of $V(C) \setminus \{x\}$ has degree at least 4, then the sum of the final charges is at least $(|C| - 4) - 3 - 2 \times \frac{2}{3} - \frac{3}{2} - \frac{3}{4} = -\frac{22}{3} > -8$, a contradiction to [I]. Hence, we can by symmetry assume that $\text{deg}(v_1) = 2$ and either $\text{deg}(v_3) = 2$ or $\text{deg}(v_4) = 2$, and all other vertices of $V(C) \setminus \{x\}$ have degree exactly three.

If $\text{deg}(v_1) = \text{deg}(v_3) = 2$, then by Corollary 11 $G$ contains cycles $v_2y_2y_4v_4v_3$ and $v_2y_2y_4'v_1$ bounding 5-faces. However, then Lemma 8 applied to the cycle $y_2y_4x_1y_4'$ implies $\text{deg}(y_4) = 2$, which is a contradiction.

If $\text{deg}(v_1) = \text{deg}(v_4) = 2$, then by Lemma 8 and Corollary 11 $G$ contains cycles $v_1v_2y_2x_2x$, $v_3y_3x_3x$, and $v_2y_2y_4y_4$ bounding 5-faces $f_1$, $f_3$, and $f$. If $\text{deg}(x_2) \geq 3$, then $f_1$ sends at most $3 \times \frac{1}{4}$ to $v_1$, $y_2$, and $x_2$ by R1 and at most $\frac{1}{4}$ by R10 and receives 1 from $x$ by R3 implying $\text{ch}(f_1) \geq \text{cho}(f_1) + 1 - 3 \times \frac{1}{4} - \frac{1}{2} = \frac{2}{3}$. It follows that the sum of the final charges is at least $(|C| - 4) - 3 - 2 \times \frac{3}{2} - 2 \times \frac{1}{4} + \frac{3}{2} = -\frac{22}{3} > -8$, a contradiction. Consequently, $\text{deg}(x_2) = 2$. Then $f_1$ sends 1 to $x_2$ by R2 and at most $2 \times \frac{1}{2}$ to $v_1$ and $y_2$ by R1 and does not send anything by R10 implying $\text{ch}(f_1) \geq \text{cho}(f_1) + 1 - 1 - 2 \times \frac{1}{4} = \frac{1}{2}$. Then, the sum of the final charges is at least $(|C| - 4) - 3 - 2 \times \frac{3}{2} - 2 \times \frac{1}{4} + \frac{1}{2} = -\frac{22}{3} > -8$, which is again a contradiction.

We conclude there exists no counterexample to Theorem 7.

### 3 Set coloring of planar graphs of girth at least 5

#### 3.1 Strong hyperbolic property

A class $G$ of graphs embedded in closed surfaces (which possibly can have a boundary) is hyperbolic if there exists a constant $c_G$ such that for each graph $G \in G$ embedded in a surface $\Sigma$ and each open disk $\Delta \subset \Sigma$ whose boundary $\partial \Delta$ intersects $G$ only in vertices, the number of vertices of $G$ in $\Delta$ is at most $c_G(|\partial \Delta \cap |G| - 1)$. The class is strongly hyperbolic if the same holds for all sets $\Delta \subset \Sigma$ homeomorphic to an open cylinder (sphere with two holes).

Let $G$ be a graph and let $S$ be a proper subgraph of $G$. We say $G$ is $S$-critical for $(6 : 2)$-coloring if for every proper subgraph $H \subset G$ such that $S \subseteq H$, there exists a $(6 : 2)$-coloring of $S$ that extends to a $(6 : 2)$-coloring of $H$, but not to a $(6 : 2)$-coloring of $G$.

In [I], we proved a strengthening of the following claim.

**Theorem 25** (Dvořák and Hu [I]). Let $G$ be the class of graphs of girth at least five embedded in surfaces such that if $G \in G$ is embedded in $\Sigma$ and $S$ is the subgraph of $G$ drawn in the boundary of $\Sigma$, then $G$ is $S$-critical for $(6 : 2)$-coloring. Then $G$ is strongly hyperbolic.

By Theorem 7.11 in [I], we have the following result.
Let $d$ be a graph of girth at least five and let $C$ be cycles bounding faces of $G$. If $G$ is $(C_1 \cup C_2 \cup \cdots \cup C_k)$-critical for $(6 : 2)$-coloring, then $|V(G)| \leq \lambda \sum_{i=1}^{k} |C_i|$. 

3.2 Proof of Theorem 4

Let $G$ be a plane graph of girth at least 5 and let $x$ be a vertex of $G$ with neighbors $y_1, \ldots, y_d$ in order. For $d \geq 3$, to split $x$ is to replace $x$ by $d$ independent vertices $y_1', y_2', \ldots, y_d'$, and to replace each edge $xy_i$ by edges $y_iy_j$ and $y_jy_{i+1}$ (where $y_{d+1} = y_1$). Then $C_x = y_1'y_2'y_3' \cdots y_d'y_{d-1}'y_{d-2}' \cdots y_2'y_1'y_2'$. For $d = 2$, $x$ is replaced by three independent vertices $y_1', y_2', y_3'$, the edge $xy_1$ is replaced by edges $y_1'y_2'$, $y_2'y_3'$, and the edge $xy_2$ is replaced by edges $y_2'y_3'$ and $y_3'y_1'y_2'$. In this case, $C_x$ is the 5-cycle $y_1'y_2'y_3'y_2'y_3'y_2'$. Note that the girth of the graph obtained from $G$ by splitting $x$ is also at least five.

Proof of Theorem 4. Let $\lambda$ be the constant from Theorem 26 and let $s = 4\lambda k + 5$.

Let $G$ be a plane graph of girth at least 5 and let $X$ be a set of vertices of $G$ of degree at most $k$, such that the distance between vertices of $X$ is at least $s$. Let $X' \subseteq X$ consist of all vertices in $X$ of degree at least two.

For each $x \in X'$, by Theorem 3 there exists an $\{x\}$-enhanced coloring $\psi_x$ of $G$. Let $G'$ be the graph obtained from $G$ by splitting every vertex in $X'$. Let $\varphi$ be a $(6 : 2)$-coloring of $S = \bigcup_{x \in X'} C_x$ defined as follows. For each $x \in X'$ and each vertex $y \in V(C_x)$ corresponding to a neighbor of $x$ in $G$, we let $\varphi(y) = \psi_x(y)$. To other vertices of $S$, we extend the coloring arbitrarily (this is possible, since they have degree two).

We claim that $\varphi$ extends to a $(6 : 2)$-coloring of $G'$; suppose for a contradiction this is not the case. Let $G''$ be a minimal subgraph of $G'$ such that $S \subseteq G''$ and $\varphi$ does not extend to a $(6 : 2)$-coloring of $G''$. Clearly, $G'' \neq S$; let $G''_0$ be a connected component of $G''$ such that $E(G''_0) \not\subseteq E(S)$, let $S'' = S \cap G''_0$, and let $\varphi''$ be the restriction of $\varphi$ to $S''$. By the minimality of $G''$, observe that $G''_0$ is $S''$-critical for $(6 : 2)$-coloring and $\varphi''$ does not extend to a $(6 : 2)$-coloring of $G''_0$. Let $X'' = \{x \in X' : C_x \subseteq S''\}$. If $|X''| \leq 1$, and thus $X'' \subseteq \{x\}$ for some $x \in X'$, then note that $\psi_x$ would give an extension of $\varphi''$ to a $(6 : 2)$-coloring of $G''_0$, which is a contradiction. Therefore, $|X''| \geq 2$. By Theorem 26 we have $|V(G''_0)| \leq \lambda \sum_{x \in X''} |C_x| \leq 2K\lambda |X''|$. 

On the other hand, for each $x \in X''$, let $N_x$ denote the set of vertices of $G''_0$ at distance at most $(s - 3)/2 = 2K\lambda + 1$ from $C_x$. Since the distance between vertices of $X$ in $G$ is at least $s$, the distance between $C_x$ and $C_x'$ in $G''_0$ for distinct $x, x' \in X''$ is at least $s - 2$, and thus $N_x \cap N_x' = \emptyset$. Furthermore, since $G''_0$ is connected and $|X''| \geq 2$, $N_x$ contains at least $2K\lambda + 1$ vertices on a path from $x$ to $C_x'$. Consequently, $|V(G''_0)| \geq \sum_{x \in X''} |N_x| \geq (2K\lambda + 1)|X''|$, which is a contradiction.

Therefore, $\varphi$ indeed extends to a $(6 : 2)$-coloring of $G'$. Then the restriction of $\varphi$ to $G - X$ extends to an $X$-enhanced coloring of $G$ (for each $x \in X'$ we set $\varphi(x) = \psi_x(x)$, and for each $x \in X \setminus X'$ we choose $\varphi(x)$ as a 3-element subset of $\{1, \ldots, 6\}$ disjoint from the color set of the neighbor of $x$, if any).
4 Fractional coloring of planar graphs of girth at least 5

We are now ready to prove our main result.

Proof of Theorem 3. Let $s$ be the constant of Theorem 4 for $k = \Delta$, and let $M_{\Delta} = \Delta^s$.

Let $G$ be a planar graph of girth at least five with maximum degree at most $\Delta$. Let $G'$ be the graph obtained from $G$ by adding edges between all pairs of vertices at distance at most $s - 1$. The maximum degree of $G'$ is less than $\Delta^s = M_{\Delta}$, and thus $G'$ has a coloring by at most $M_{\Delta}$ colors. Let $V_1, V_2, \ldots, V_{M_{\Delta}}$ be the color classes of this coloring. Then the distance in $G$ between any two vertices of the same color class is at least $s$. By Theorem 3 for $i \in \{1, \ldots, M_{\Delta}\}$, $G$ has a $V_i$-enhanced set coloring $\varphi_i$ by subsets of $\{6i - 5, 6i - 4, 6i - 3, 6i - 2, 6i - 1, 6i\}$. Then $\varphi = \bigcup_{i=1}^{M_{\Delta}} \varphi_i$ is a set coloring of $G$ by subsets of $\{1, \ldots, 6M_{\Delta}\}$ such that $|\varphi(v)| \geq 2(M_{\Delta} - 1) + 3 = 2M_{\Delta} + 1$ for every $v \in V(G)$. Therefore, $G$ has a $(6M_{\Delta} : 2M_{\Delta} + 1)$-coloring, and $\chi_f(G) \leq \frac{6M_{\Delta}}{2M_{\Delta} + 1}$. \qed

References

[1] Z. Dvořák and X. Hu, (3a : a)-list-colorability of embedded graphs of girth at least five, ArXiv, 1805.11507 (2018).

[2] Z. Dvořák and M. Mnich, Large independent sets in triangle-free planar graphs, SIAM J. Discrete Math., 31 (2017), pp. 1355–1373.

[3] Z. Dvořák, J.-S. Sereni, and J. Volec, Fractional coloring of triangle-free planar graphs, Electronic Journal of Combinatorics, 22 (2015), p. P4.11.

[4] H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Math.-Natur. Reihe, 8 (1959), pp. 109–120.

[5] K. Jones, Independence in graphs with maximum degree four, J. Combin. Theory Ser. B, 37 (1984), pp. 254–269.

[6] L. Postle and R. Thomas, Hyperbolic families and coloring graphs on surfaces, arXiv, 1609.06749 (2013).