Notes on the Zeros of Riemann’s Zeta Function

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Abstract
The functional equation for Riemann’s Zeta function is studied, from which it is shown why all of the non-trivial, full-zeros of the Zeta function \( \zeta(s) \) will only occur on the critical line \( \sigma = 1/2 \) where \( s = \sigma + i \rho \), thereby establishing the truth of Riemann’s hypothesis. Further, two relatively simple transcendental equations are obtained; the numerical solution of these equations locates all of the zeros of \( \zeta(s) \) on the critical line.

1 Introduction
The study of the non-trivial zeros of \( \zeta(s) \) has been the subject of myriad investigations over the years and is of ongoing interest in number theory. It has also recently received attention from the physics community [11]. Strangely, the results being presented here cannot be found in any of the summaries (e.g. [1], [2], [3], [4], [9], [12], [13]) or primary research articles¹ (e.g. [5], [6], [7], [8]) that I have consulted. Although it seems inconceivable that they have escaped detection over the centuries, if such is the case, a possible explanation is that the analysis involves complicated manipulation of long expressions, a task best relegated to computer algebra, and only in the last few years have computer algebra codes reached a level of sophistication that allows such manipulation to proceed. In any case, since these results (perhaps buried, more likely new) impart significant insight into the nature and location of the zeros of \( \zeta(s) \), I am taking the opportunity to summarize here the results I have found.

On page 50 of Ivic’s book ([4]), it is written: ”The functional equation for \( \zeta(s) \) in a certain sense characterizes it completely”. Accepting the truth of that statement suggests that a study of the functional equation should yield insight into the nature of the zeros of \( \zeta(s) \). That is the path taken here.

2 The functional equation inside the critical strip \( 0 \leq \sigma \leq 1 \)

The functional equation for \( \zeta(s) \) is well-known (e.g. [4]):

\[
\zeta(1-s) = \frac{2 \Gamma(s) \cos (\frac{\pi s}{2}) \zeta(s)}{(2\pi)^{s}}
\]

(2.1)

and the existence of the trivial zeros of \( \zeta(-2n), n > 1 \) is immediately apparent due to the appearance of the cosine function on the right hand side. With reference to Appendix A, where an index of notation will be found, it is possible to break (1) into its real and imaginary parts, giving the functional equation in an equivalent form:

\[
\tilde{\zeta}_I(\sigma, \rho) = -Q \zeta_I(\sigma, \rho) - P \zeta_R(\sigma, \rho)
\]

(2.2)

¹A short, only representative list!
\[ \tilde{\zeta}_R(\sigma, \rho) = -P \zeta_I(\sigma, \rho) + Q \zeta_R(\sigma, \rho) \]  

(2.3)

where explicit expressions for the coefficient functions P and Q are presented in Appendix B and reference to dependence on the independent (real) variables \( \sigma \) and \( \rho \), where \( s = \sigma + I \rho \) have been omitted.

Instead of studying (2.1), being a functional equation between complex variables and functions, consider the equivalent forms (2.2) and (2.3), which can be interpreted as the statement of a coupling that exists among two independent functions \( \zeta_R(\sigma, \rho) \) and \( \zeta_I(\sigma, \rho) \) and two dependent functions \( \tilde{\zeta}_R(\sigma, \rho) \) and \( \tilde{\zeta}_I(\sigma, \rho) \) of two real variables \( \sigma \) and \( \rho \). All quantities are real and this is emphasized by writing \( \zeta_R(\sigma, \rho) \) to mean \( R(\zeta(\sigma + I \rho)) \) and similarly for \( \zeta_I(\sigma, \rho) \). The intent is to study (2.2) and (2.3) to determine if these two constraints can be used to specify a region(s) of the \( (\sigma, \rho) \) plane (corresponding to the complex \( s \) plane) where full-zeros of \( \zeta(s) \) may possibly be found. "Half-zero" refers to points (or continuous regions) of the \( (\sigma, \rho) \) plane where \( \zeta_R(\sigma, \rho) = 0 \) or \( \zeta_I(\sigma, \rho) = 0 \) but not both; "full-zero" refers to any of the set of points \( (\sigma_0, \rho_0) \) where \( \zeta_R(\sigma_0, \rho_0) = 0 \) and \( \zeta_I(\sigma_0, \rho_0) = 0 \) simultaneously. Because \( \zeta(s) \) is known to be meromorphic (no branch cuts) \[4\], the location of full-zeros of \( \zeta(s) \) must be isolated in the complex \( s \) plane, and this property will be reflected by a similar property of \( \zeta_R(\sigma, \rho) \) and \( \zeta_I(\sigma, \rho) \) in the \( (\sigma, \rho) \) plane.

In the following, the intent is to search for zeros of \( \zeta_R(\sigma, \rho) \) and \( \zeta_I(\sigma, \rho) \) as a function of \( \sigma \) with the variable \( \rho \) being treated as a parameter (\( \rho = \rho_p \)). This corresponds to a search for full-zeros along horizontal lines of the \( (\sigma, \rho) \) plane within the critical strip, graphically corresponding to that same strip in the complex plane \( s = \sigma + I \rho \). Because P and Q have no singularities (poles)\(^2\), notice that if \( \zeta_R(\sigma_0, \rho_0) = 0 \) and \( \zeta_I(\sigma_0, \rho_0) = 0 \) at some point \( (\sigma_0, \rho_0) \) then (2.2) and (2.3) require that \( \tilde{\zeta}_R(\sigma_0, \rho_0) = 0 \) and \( \tilde{\zeta}_I(\sigma_0, \rho_0) = 0 \). So, any full-zero of \( \zeta(s) \) that lies in the range \( \sigma \leq 1/2 \) will be mirrored about the \( \sigma = 1/2 \) axis (the critical line) by a full-zero in the range \( \sigma \geq 1/2 \), on the horizontal line \( \rho = \rho_p \). This property is well-known and does not necessarily hold true for half-zeros.

With this result in mind, a search constraint will be applied that imposes a necessary, but not sufficient condition for a zero of \( \zeta(s) \) to exist. That is, the functions \( \zeta_R(\sigma, \rho) \) and \( \zeta_I(\sigma, \rho) \) and their respective functions reflected about the critical line will be required to be equal (but not necessarily zero). A full-zero of \( \zeta(s) \) represents a special case of this more general condition.

Specifically

\[ \tilde{\zeta}_R(\sigma, \rho_p) = \zeta_R(\sigma, \rho_p) \]  

(2.4)

and

\[ \tilde{\zeta}_I(\sigma, \rho_p) = \zeta_I(\sigma, \rho_p). \]  

(2.5)

Application of (2.4) and (2.5) to (2.2) and (2.3) yields a set of transcendental equations isolating correspondingly special values of \( \sigma \) and \( \rho_p \) through the following constraints:

\[ \zeta_R(\sigma, \rho_p) = -\frac{(1 + Q)}{P} \zeta_I(\sigma, \rho_p) \]  

(2.6)

\[ \zeta_I(\sigma, \rho_p) = -\frac{(1 - Q)}{P} \zeta_R(\sigma, \rho_p), \]  

(2.7)

giving a necessary condition on \( \sigma \) and \( \rho_p \) through the requirement that

\[ P^2 + Q^2 = 1 \]  

(2.8)

provided that

\[ \zeta_R(\sigma, \rho_p) \neq 0 \quad \text{and/or} \quad \zeta_I(\sigma, \rho_p) \neq 0. \]  

(2.9)

\(^2\)except on the negative real axis which is outside the region of interest
The cases corresponding to the failure of 2.9 will be discussed shortly.

For general values of \( \sigma \) and \( \rho \), a lengthy calculation using (B.1) and (B.2) shows that \( P \) and \( Q \) have the general property that

\[
P^2 + Q^2 = (2\pi)^{1-2\sigma} \frac{\cosh(\pi \rho)}{\pi} |\Gamma(\sigma + I\rho)|^2 (1 + \frac{\cos(\pi \sigma)}{\cosh(\pi \rho)})
\]

(2.10)

from which (2.8) imposes the following constraint on \( \sigma \) and \( \rho \) after some rearrangement and the use of (A.7):

\[
(4\pi^2)^{\sigma-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + I\rho)\Gamma(\frac{1}{2} - I\rho)}{\Gamma(\sigma + I\rho)\Gamma(\sigma - I\rho)} - 1 = \frac{\cos(\pi \sigma)}{\cosh(\pi \rho)}
\]

(2.11)

for which the main solution is

\[
\sigma = \frac{1}{2}, \quad \rho \text{ arbitrary},
\]

(2.12)

consistent with what Riemann famously hypothesized. See Appendix C where a second possibility is isolated and discarded.

The converse is also true. That is, (2.12) trivially implies the truth of (2.4) and (2.5), but (2.8) doesn’t. But, with the exception of the case discussed in Appendix C, (2.12) implies (2.8) uniquely, so (2.12) is a necessary and sufficient condition for all of (2.4), (2.5) and (2.8), which themselves are prerequisites (necessary) for the presence of a zero of \( \zeta(s) \). So, with the exception of the pathology discussed in Appendix C, (2.4) and (2.5) can only occur, and hence a full-zero of \( \zeta(s) \) can only be found, when (2.12) is satisfied, subject to (2.9), whose failure unfortunately corresponds to exactly those special values of \( \sigma \) and \( \rho \) of specific interest.

There are two cases where (2.9) fails - half-zeros and full-zeros. The case of half-zeros is easily dealt with, since it is clear that (2.2) and (2.3) are incompatible with (2.4) and (2.5) at a half-zero unless \( P = 0 \) and \( Q = \pm 1 \), thereby satisfying (2.8) spontaneously. Thus there is no expectation that a half-zero will satisfy (2.6) and (2.7) in general, although the sieves (2.4) and/or (2.5) may occasionally catch some half-zeros, so this case is a subset of the general result, and (2.12) does not necessarily apply.

As noted before, all full-zeros of \( \zeta(s) \) are distinct, meaning that it is possible to expand \( \zeta(s) \) in a Taylor series in a neighbourhood of the full-zero. Furthermore, the imaginary and real parts of a meromorphic function at a full-zero must be of the same degree, so for a full-zero of degree \( m \) in the neighbourhood of a solution to (2.6) and (2.7) where it happens that \( \zeta_R(\sigma_0, \rho_p) = 0 \) and \( \zeta_I(\sigma_0, \rho_p) = 0 \), one can write

\[
\zeta_R(\sigma, \rho_p) = (\sigma - \sigma_0)^m \zeta_R^{(m)}(\sigma_0, \rho_p)/m!
\]

(2.13)

and

\[
\zeta_I(\sigma, \rho_p) = (\sigma - \sigma_0)^m \zeta_I^{(m)}(\sigma_0, \rho_p)/m!
\]

(2.14)

where

\[
\zeta_R^{(m)}(\sigma_0, \rho_p) = \frac{\partial^m}{\partial \sigma^m} \zeta_R(\sigma, \rho_p)|_{\sigma=\sigma_0}
\]

(2.15)

and

\[
\zeta_I^{(m)}(\sigma_0, \rho_p) = \frac{\partial^m}{\partial \sigma^m} \zeta_I(\sigma, \rho_p)|_{\sigma=\sigma_0}
\]

(2.16)

It is emphasized that the partial derivative is taken with respect to \( \sigma \) because the search for a full-zero is being conducted along a horizontal line in the \( (\sigma, \rho) \) plane. Substitution of (2.13) and (2.14) into (2.6) and (2.7) yields (2.8) and then (2.12), the same result as before, except that the equivalent of (2.9) is always true, because \( \zeta_R^{(m)}(\sigma_0, \rho_p) \) and \( \zeta_I^{(m)}(\sigma_0, \rho_p) \) are non-zero by the definition of ”a zero of degree \( m \).”

3
Thus, with the exception of the case discussed in Appendix C, (2.12) is the only solution to a necessary condition for locating a full-zero of ζ(s) in the finite (σ,ρ) plane (and hence the finite complex s plane by extension), explaining why non-trivial, full-zeros of ζ(s) have only ever been located on the critical line (2.12).

3 On the critical line σ = 1/2

For the totality of this section and the next, the variable σ = 1/2. With this understanding, the constraints (2.4) and (2.5) reduce to an identity and (2.6) and (2.7) can conveniently be written in the form

\[ ζ_R = \frac{N}{D_R}ζ_I \] (3.1)

and

\[ ζ_I = \frac{N}{D_I}ζ_R \] (3.2)

where expressions for N, DR and DI are given in Appendix B, yielding the further identities

\[ N^2 = D_R D_I \] (3.3)

and

\[ D_R + D_I = 1, \] (3.4)

from which it is clear that

\[ 0 ≤ D_R, D_I ≤ 1. \] (3.5)

since DR and DI must have the same sign. (3.3) demonstrates that N shares the zeros of both DR and DI. Since both of the latter cannot vanish simultaneously due to (3.4), the zeros of DR and DI will locate the half-zeros, but not the full-zeros, of ζ(s) along the critical line, because if a zero of N carried one of the full-zeros of ζ(s), (3.1) and (3.2) show that the order of the zeros of ζR and ζI would be inconsistent. Specifically

\[ D_R = 0 ⇒ D_I = 1, ζ_I = 0, ζ_R ≠ 0 \]

\[ D_I = 0 ⇒ D_R = 1, ζ_R = 0, ζ_I ≠ 0 \] (3.6)

The full-zeros of ζ(s) for a zero of degree m are obtained by applying l’Hôpital’s rule of differentiation with respect to ρ. Any solution of

\[ \frac{ζ_R}{ζ_I} ⇒ \frac{ζ_R^{(m)}}{ζ_I^{(m)}} = \frac{N}{D_R}. \] (3.7)

will thus isolate a potential full-zero of ζ(s), but as discussed previously, (3.7) is only a necessary condition for achieving this task. Thus a numerical solution does not guarantee that a full-zero has been found, although the set of all solutions will include all the full-zeros as a subset. Limited experimentation (see Section 4) indicates that, at least for m = 1, only the full-zeros of ζ(s) are ever located by (3.7); no solutions with m = 2 have been found.

4 Locating the Zeros

The various functions introduced can be used to locate both the half- and full-zeros by numerically solving transcendental equations. From (3.3) and (3.6), all solutions of N² = 0 will specify all the half-zeros of ζ(s) on the critical line. In the notation of Appendix B,

\[ C_m \cos(ρ_ε) - C_p \sin(ρ_ε) = 0 \] (4.1)
is a simple form of this constraint. Each successive solution with increasing values of $\rho$ will locate successive half-zeros of $\zeta_R$ and $\zeta_I$ alternately, as illustrated in Figure (1).

An interesting variant of (4.1) arises by re-writing the terms explicitly, giving

$$\frac{\Gamma_I}{\Gamma_R} = \frac{\tanh(\pi\rho/2) + \tan(\rho_\pi)}{1 - \tanh(\pi\rho/2)\tan(\rho_\pi)}$$

and, to the extent that $\tanh(\pi\rho/2) \approx 1$, (4.2) can be inverted to read

$$\tan(\rho_\pi) = \frac{\Gamma_I/\Gamma_R - 1}{\Gamma_I/\Gamma_R + 1}.$$  

If the first order Stirling’s approximation ([1]) for $\rho \rightarrow \infty$ is applied to the ratio $\Gamma_I/\Gamma_R$, a simple form emerges:

$$\frac{\Gamma_I}{\Gamma_R} \approx -\tan(\rho - \rho_L)$$

which can replace the left-hand side of (4.2). Alternatively, (4.3) becomes

$$\tan(\rho_\pi) \approx -\cos(2\rho + \sin(2\rho_L)$$

These forms contain numerous poles and zeros and appear to have little numerical use, but may possibly be of use in deducing the spacing between zeros [8], [10].

The location of the full-zeros of $\zeta(s)$ is specified indirectly in (3.7). For simple zeros ($m = 1$) the transcendental equation to be solved is

$$\frac{\zeta_I'}{\zeta_R} = -\frac{N}{D_R},$$

a more convenient form being

$$D_R \zeta_I' + N \zeta_R' = 0.$$ 

from which the full-zeros\(^3\) can be found by standard numerical techniques (see figure (1). Although it may possibly be useful for numerical work, this form is unsatisfying because it requires knowledge of the Zeta function derivatives, making it almost tautological. Unfortunately, a form for the full-zeros similar to (4.1), involving only the variable $\rho$ and transcendental functions of that variable, eludes me.

5 **Summary**

The functional equation for $\zeta(s)$ has been expressed in the form of a coupling between its real and imaginary components. It was shown that non-trivial, full zeros of $\zeta(s)$, if any exist, are only compatible with a solution to the functional coupling equations for special values of the underlying independent variable “$s$”. Two possible sets of values were located; one of those regions has been explored by others and no zeros have ever been found. The remaining region consists of the critical line $s = 1/2$. This establishes that Riemann’s hypothesis is true. Additionally, two relatively simple transcendental equations were isolated, the zeros of which coincide with all the zeros of $\zeta(s)$ on the critical line.

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\(^3\)and half-zeros belonging to $D_R = 0$.)
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A Appendix: Notation and identities

The Riemann Zeta function $\zeta(s)$ is written over the complex $s$ plane:

$$ s = \sigma + I \rho $$

as

$$ \zeta(s) = \Re(\zeta(\sigma + I \rho)) + I \Im(\zeta(\sigma + I \rho)) $$

$$ \equiv \zeta_R(\sigma, \rho) + I \zeta_I(\sigma, \rho) $$

$$ \rightarrow \zeta_R + I \zeta_I \text{ when } \sigma = \frac{1}{2}, \quad (A.1) $$

the latter for brevity. All variables in (A.1) are real. At reflected points, define

$$ \zeta(1 - s) = \Re(\zeta(1 - \sigma - I \rho)) + I \Im(\zeta(1 - \sigma - I \rho)) $$

$$ = \zeta_R(1 - \sigma, \rho) - I \zeta_I(1 - \sigma, \rho) $$

$$ \equiv \zeta_R(\sigma, \rho) - I \zeta_I(\sigma, \rho) $$

$$ \rightarrow \zeta_R - I \zeta_I \text{ when } \sigma = \frac{1}{2}. \quad (A.2) $$
With reference to (3.7), note that

\[
\begin{align*}
\zeta^{(m)}_R &\equiv \frac{\partial^m}{\partial \rho^m} \zeta_R \\
\zeta^{(m)}_I &\equiv \frac{\partial^m}{\partial \rho^m} \zeta_I
\end{align*}
\]  

(A.3)

and, for \( m = 1 \)

\[
\begin{align*}
(\zeta_R)' &\equiv \frac{\partial}{\partial \rho} (\zeta_R) = -\zeta_I' \\
(\zeta_I)' &\equiv \frac{\partial}{\partial \rho} (\zeta_I) = \zeta_R'
\end{align*}
\]  

(A.4)

Similarly, the Gamma function is written

\[
\begin{align*}
\Gamma(s) &= \Re(\Gamma(\sigma + I \rho)) + I \Im(\Gamma(\sigma + I \rho)) \\
&= \Gamma_R(\sigma, \rho) + I \Gamma_I(\sigma, \rho) \\
&\to \Gamma_R + I \Gamma_I \text{ when } \sigma = \frac{1}{2}.
\end{align*}
\]  

(A.5)

The following identities are noted [1]

\[
|\Gamma(I \rho)|^2 = \Gamma(I \rho) \Gamma(-I \rho) = \frac{\pi}{\rho \sinh(\pi \rho)}
\]  

(A.6)

\[
|\Gamma(\frac{1}{2} + I \rho)|^2 = (\Gamma_R + I \Gamma_I) (\Gamma_R - I \Gamma_I) = \frac{\pi}{\cosh(\pi \rho)}
\]  

(A.7)

\[
|\Gamma(1 + I \rho)|^2 = \Gamma(1 + I \rho) \Gamma(1 - I \rho) = \frac{\pi \rho}{\sinh(\pi \rho)},
\]  

(A.8)

the symbols \( \rho_\pi \) and \( \rho_L \) are used to decrease the printed size of some formulae:

\[
\rho_\pi \equiv \rho \log(2\pi)
\]  

(A.9)

\[
\rho_L \equiv \frac{\rho}{2} \log(1/4 + \rho^2),
\]  

(A.10)

and \( m \) and \( n \) are always positive integers.

**B Appendix: Formulae**

In (2.2) and (2.3) the following functions are used

\[
\begin{align*}
P &= 2[(-\Gamma_R(\sigma, \rho) \sin(\rho_\pi) + \Gamma_I(\sigma, \rho) \cos(\rho_\pi)) \cosh(\frac{\pi \rho}{2}) \cos(\frac{\pi \sigma}{2}) \\
&\quad + (-\Gamma_R(\sigma, \rho) \cos(\rho_\pi) - \Gamma_I(\sigma, \rho) \sin(\rho_\pi)) \sin(\frac{\pi \sigma}{2}) \sinh(\frac{\pi \rho}{2})] e^{-\sigma \ln(2\pi)}
\end{align*}
\]  

(B.1)

\[
\begin{align*}
Q &= 2[\Gamma_R(\sigma, \rho) \cos(\rho_\pi) + \Gamma_I(\sigma, \rho) \sin(\rho_\pi)) \cosh(\frac{\pi \rho}{2}) \cos(\frac{\pi \sigma}{2}) \\
&\quad + (-\Gamma_R(\sigma, \rho) \sin(\rho_\pi) + \Gamma_I(\sigma, \rho) \cos(\rho_\pi)) \sinh(\frac{\pi \rho}{2}) \sin(\frac{\pi \sigma}{2})] e^{-\sigma \ln(2\pi)}
\end{align*}
\]  

(B.2)

The following functions are introduced in (3.1) and (3.2):

\[
N = \frac{C_m \cos(\rho_\pi)}{\sqrt{\pi}} - \frac{C_p \sin(\rho_\pi)}{\sqrt{\pi}}
\]  

(B.3)
\[ D_R = \frac{1}{2} - \frac{1}{2} \frac{C_p \cos(\rho \pi) + C_m \sin(\rho \pi)}{\sqrt{\pi}} \]  

\( \text{(B.4)} \)

\[ D_I = 1 - D_R \]

where

\[ C_p = \cosh\left(\frac{\pi \rho}{2}\right) \Gamma_R + \sinh\left(\frac{\pi \rho}{2}\right) \Gamma_I \]  

\( \text{(B.5)} \)

\[ C_m = -\sinh\left(\frac{\pi \rho}{2}\right) \Gamma_R + \cosh\left(\frac{\pi \rho}{2}\right) \Gamma_I \]  

\( \text{(B.6)} \)

C Appendix: Another solution?

\( (2.12) \) is the obvious solution to \( (2.11) \). Are there more? To answer this question note that the magnitude of the right-hand side of \( (2.11) \) is strictly less than one, so any new solution with \( \sigma \neq \frac{1}{2} \) must occur when the left-hand side is in that range. Consider \( L(\sigma, \rho) \), the left-hand side of \( (2.11) \) as a function of \( \rho \) at its endpoints \( \sigma = 0 \) and \( \sigma = 1 \). From \( (A.6) \) one gets

\[ L(0, \rho) = \frac{\rho}{2\pi} \tanh(\pi \rho) - 1 \]  

\( \text{(C.1)} \)

\[ \rightarrow \infty \text{ as } \rho \rightarrow \infty \]

\[ \rightarrow -1 \text{ as } \rho \rightarrow 0 \]  

\( \text{(C.2)} \)

and from \( (A.8) \) one finds

\[ L(1, \rho) = \frac{2\pi}{\rho} \tanh(\pi \rho) - 1 \]  

\( \text{(C.3)} \)

\[ \rightarrow -1 \text{ as } \rho \rightarrow \infty \]

\[ \rightarrow 2\pi^2 - 1 \text{ as } \rho \rightarrow 0 \]  

\( \text{(C.4)} \)

Clearly \( L(\sigma, \rho) \) changes sign for at least one value of \( \rho = \rho_s \) and \( \sigma \neq \frac{1}{2} \), in the neighbourhood of which \( (2.11) \) could possibly be satisfied. Numerically, \( \rho_s = 6.283185307 \); figure 2 demonstrates that the slope of \( L(\sigma, \rho) \) changes sign near \( \rho = \rho_s \) suggesting that a numerical solution to \( (2.11) \) lies close by. To locate that neighbourhood precisely, consider

\[ \frac{\partial}{\partial \sigma} L(\sigma, \rho) = (L(\sigma, \rho) + 1)(4\pi^2 - 2\Re(\psi(\sigma + I\rho))). \]  

\( \text{(C.5)} \)

The sign of the left-hand side of \( (C.5) \) will be determined by the factor

\[ B(\sigma, \rho) = (4\pi^2 - 2\Re(\psi(\sigma + I\rho))) \]  

\( \text{(C.6)} \)

since the factor \( (L(\sigma, \rho) + 1) \) is always positive. A change in the sign of \( B(\sigma, \rho) \) is consistent with the possibility of a numerical solution to \( (2.11) \). Figure 3 shows that the sign of \( B(\sigma, \rho) \) changes for various values of \( \sigma \) and \( \rho \) near \( \rho_s \) with \( 0 \leq \sigma \leq 1 \), \( \sigma \neq \frac{1}{2} \), thereby isolating a second solution to \( (2.11) \), and a potential location to uncover a non-trivial, full-zero of \( \zeta(s) \) off the critical line.

Others [7] have carefully searched this neighbourhood, and found no indication of such a zero. Since \( B(\sigma, \rho) \) is monotonic with increasing(decreasing) values of \( \rho \), there are no other possibilities. Thus \( (2.12) \) defines the sole remaining range of possible solutions to \( (2.11) \).
Figure 1: Numerical solution of (4.1) and (4.7) in the range $50 \leq \rho \leq 57$ showing coincidence with the half- and full-zeros of $\zeta_R$ and $\zeta_I$ respectively. The vertical dotted lines denote known full-zeros at $\rho = 52.9703$ and $\rho = 56.4462$. 
Figure 2: A plot of $L(\sigma, \rho)$ at three different values of $\rho$, bracketing $\rho_s$. The inset contains a $10^5$ magnification of the right-hand side of (2.11), and the circles indicate intersection points of the two curves with $\sigma \neq 1/2$, yielding a numerical solution to (2.11) and the location of a potential zero of $\zeta(s)$ off the critical line.
Figure 3: A parametric scan of the function $B(\sigma, \rho)$ near $\rho_s$ as a function of $0 \leq \sigma \leq 1$