WHEN ARE CROSSED PRODUCTS BY MINIMAL DIFEOMORPHISMS ISOMORPHIC?

N. CHRISTOPHER PHILLIPS

ABSTRACT. We discuss the isomorphism problem for both C* and smooth crossed products by minimal diffeomorphisms. For C* crossed products, examples demonstrate the failure of the obvious analog of the Giordano-Putnam-Skau Theorem on minimal homeomorphisms of the Cantor set. For smooth crossed products, there are many open problems.

0. INTRODUCTION

A remarkable theorem of Giordano, Putnam, and Skau (12; see Theorem 1.1 below) gives a dynamical characterization of isomorphism of the transformation group C*-algebras of minimal homeomorphisms of the Cantor set. The proof depends, among other things, on the fact [7] that simple direct limits of circle algebras with real rank zero and with the same scaled ordered K-theory are necessarily isomorphic. Recent progress in the classification of simple C*-algebras [20] and the structure of crossed products [24] (see the survey [23]) has made it possible in some cases to prove the isomorphism of crossed products by minimal diffeomorphisms with the same Elliott invariants. Some examples have been constructed [32], which we survey in this paper. The examples show that the analog of the condition of [12] is far too strong to correspond to isomorphism of the C*-algebras. There are no clear candidates for the correct condition, but the examples rule out a number of possibilities.

For a minimal diffeomorphism satisfying an additional technical condition, it is possible to construct a smooth crossed product, which is a locally multiplicatively convex Fréchet algebra. The smooth irrational rotation algebra is a well known example of this construction. The smooth crossed product presumably preserves much more information about the dynamics than the C* crossed product, although so far essentially no theorems to this effect are known. It turns out that very little is known about smooth crossed products. In the second half of this survey, we discuss conditions for the existence of smooth crossed products, the isomorphism problem, and some of the other interesting open questions.

This paper has four sections. In the first, which may serve as a more extended introduction, we discuss what is known in the case of minimal homeomorphisms of the Cantor set and of the circle, another low dimensional case where the situation can be completely described. We then discuss four previously known examples which suggest, but do not conclusively demonstrate, that the general case is rather...
different. In the next section, we present four examples of pairs of minimal diffeomorphisms which do not satisfy the condition of [12], but for which it has recently become possible to prove that the crossed product $C^*$-algebras are in fact isomorphic. The methods used to distinguish the homeomorphisms are different in each case. We raise a few specific questions about these examples, and give a brief discussion of the problem of relating isomorphism of the crossed product $C^*$-algebras to the dynamics. The third section describes a general sufficient condition for the existence of a smooth crossed product, and shows that the diffeomorphisms in at least some of our examples satisfy this condition. The problem, then, is to find a dynamical characterization of isomorphism of smooth crossed products by minimal diffeomorphisms. Smooth crossed products are also natural examples for Connes’s noncommutative geometry. In the last section, we discuss some more elementary questions about smooth crossed products which are still open, mostly about the analogs of stable and real rank. These arose when thinking about the isomorphism question for smooth crossed products, and realizing how little is in fact known. We give a recent example of Schweitzer which shows that at least one of these questions can’t be answered using the general theory of smooth subalgebras of $C^*$-algebras.

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1. Flip conjugacy, orbit equivalence, and crossed product $C^*$-algebras

The theorem of Giordano, Putnam, and Skau is as follows:

**Theorem 1.1** (Theorem 2.1 of [12]). Let $X$ be the Cantor set, and let $h_1, h_2 : X \to X$ be minimal homeomorphisms. Then $C^*(\mathbb{Z}, X, h_1) \cong C^*(\mathbb{Z}, X, h_2)$ if and only if $h_1$ and $h_2$ are strongly orbit equivalent.

The precise definition of strong orbit equivalence is given in Definition 1.3 of [12]. Since it is slightly technical and will not be needed, we do not reproduce it here. Rather, we define several related conditions, one stronger than strong orbit equivalence and one weaker. The first is very close to the obvious notion of isomorphism of homeomorphisms.

**Definition 1.2.** Let $X_1$ and $X_2$ be topological spaces, and let $h_1 : X_1 \to X_1$ and $h_2 : X_2 \to X_2$ be homeomorphisms. We say that $h_1$ and $h_2$ are *conjugate* if there is a homeomorphism $g : X_1 \to X_2$ such that $g \circ h_1 \circ g^{-1} = h_2$. We say that $h_1$ and $h_2$ are *flip conjugate* if $h_1$ is conjugate to either $h_2$ or $h_2^{-1}$.

If $h_1$ and $h_2$ are flip conjugate and $X_1$ and $X_2$ are locally compact, it is immediate that $C^*(\mathbb{Z}, X_1, h_1) \cong C^*(\mathbb{Z}, X_2, h_2)$.

**Definition 1.3.** Let $X_1$ and $X_2$ be topological spaces, and let $h_1 : X_1 \to X_1$ and $h_2 : X_2 \to X_2$ be homeomorphisms. We say that $h_1$ and $h_2$ are *topologically orbit equivalent* if there is a homeomorphism $g : X_1 \to X_2$ such that, for all $x \in X_1$,

$$g(\{h_1^n(x) : n \in \mathbb{Z}\}) = \{h_2^n(g(x)) : n \in \mathbb{Z}\}.$$
That is, \( g \) is required to map the orbits of the homeomorphism \( h_1 \) exactly to the orbits of the homeomorphism \( h_2 \). This definition is adapted from a similar definition in measurable dynamics.

For our purposes, the important facts about strong orbit equivalence are that flip conjugacy implies strong orbit equivalence and that strong orbit equivalence implies topological orbit equivalence. There are many examples of minimal homeomorphisms of the Cantor set which are strong orbit equivalent but not flip conjugate. For example, flip conjugacy preserves topological entropy on compact spaces (Theorems 7.2 and 7.3 of [46]), but all entropies in \([0, \infty)\) occur in every strong orbit equivalence class (Theorem 6.1 of [43], Theorem 7.1 of [44], and Theorem 7.1 of [45]). However, even topological orbit equivalence preserves the space of invariant probability measures. See the proof of (i) implies (iii) in Theorem 2.2 of [12], at the beginning of Section 5 there.

In the most interesting higher dimensional cases, the distinction between the equivalence relations disappears. Two orbit equivalent minimal homeomorphisms of a connected compact metric space are necessarily flip conjugate. See Theorem 3.1 and Remark 3.4 of [4], or Proposition 5.5 of [22]. Nevertheless, for minimal homeomorphisms of the circle \( S^1 \), it is true that isomorphism of the transformation group \( C^* \)-algebras implies flip conjugacy. This follows from the fact that every minimal homeomorphism of \( S^1 \) is conjugate to an irrational rotation (Proposition 11.1.4 and Theorem 11.2.7(1) of [17]), and the computation of the scaled ordered \( K \)-theory of the irrational rotation \( C^* \)-algebras in the Appendix of [35], which shows that \( K_0(A_{\theta_1}) \not\cong K_0(A_{\theta_2}) \) unless \( \theta_1 \) has the same image as \( \pm \theta_2 \) in \( \mathbb{R}/\mathbb{Z} \).

Evidence has accumulated that strong orbit equivalence (or flip conjugacy) is not the relation on general minimal homeomorphisms which corresponds to isomorphism of the transformation group \( C^* \)-algebras. We describe four known suggestive examples. (We should also note that flip conjugacy has for some time been known to fail in Examples 2.1 and 2.2 in the next section.) Since it plays a crucial role in the discussion, we give a formal definition of the Elliott invariant of a unital \( C^* \)-algebra. See, for example, [8].

**Definition 1.4.** Let \( A \) be a separable unital \( C^* \)-algebra. Its **Elliott invariant** consists of:

- The abelian group \( K_1(A) \).
- The scaled ordered abelian group \( K_0(A) \), in which the scale is the distinguished element \([1]\) and the order is defined by \( \eta > 0 \) if and only if there are an integer \( n \) and a projection \( p \in M_n(A) \) such that \( \eta = [p] \).
- The simplex \( T(A) \) of tracial states on \( A \), equipped with the weak* topology.
- The pairing \( T(A) \times K_0(A) \to \mathbb{R} \) determined by \((\tau, [p]) \mapsto \tau(p)\).

An **isomorphism** from the Elliott invariant of \( A \) to that of \( B \) consists of a group isomorphism \( \varphi_1 : K_1(A) \to K_1(B) \), an isomorphism \( \varphi_0 : K_0(A) \to K_0(B) \) of scaled ordered groups, and an isomorphism \( f : T(B) \to T(A) \) of simplexes, such that the pairs \((\tau, \varphi_0(\eta))\) and \((f(\tau), \eta)\) have the same image in \( \mathbb{R} \) for all \( \tau \in T(B) \) and \( \eta \in K_0(A) \).

The unital case of the Elliott conjecture asserts that if two simple separable nuclear (but not type I) \( C^* \)-algebras have isomorphic Elliott invariants, then the \( C^* \)-algebras are isomorphic. There is good evidence for this conjecture in “low rank” cases. It holds for the purely infinite case under the single additional assumption
that the algebras satisfy the Universal Coefficient Theorem [18], [29]. Among the many results in the stably finite case, we cite [9] and [20].

We now describe the examples. The first is about 15 years old.

Example 1.5. In the discussion on Pages 506–507 of [3], it is shown that there are two nonisomorphic antisymmetric bicharacters \( \rho \) and \( \rho' \) on \( \mathbb{Z}^3 \) such that the Elliott invariants of the corresponding higher dimensional noncommutative toruses \( A_\rho \) and \( A'_\rho \) are isomorphic. It follows from more recent work ([21] and [7]) that \( A_\rho \cong A'_\rho \). However, Theorem 2 of [3] shows that the corresponding smooth subalgebras are not isomorphic.

This example, although suggestive, does not directly bear on the question, since these algebras are not obviously crossed products. Even if they were, nonisomorphism of smooth subalgebras would only obviously rule out smooth flip conjugacy.

In preparation for the next example, recall that an extension of a homeomorphism \( h: X \to X \) of a compact metric space \( X \) consists of a compact metric space \( Y \), a homeomorphism \( k: Y \to Y \), and a continuous surjective map \( \pi: Y \to X \) such that \( \pi \circ k = h \circ \pi \), and further recall (see, for example, page 157 of [1]) that the extension is almost one to one if there exists a point \( x \in X \) such that \( \pi^{-1}(x) \subset Y \) consists of just one point. (If \( k \) is minimal, then in fact \( \pi^{-1}(x) \) will be a one point set for “most” \( x \in X \).)

Example 1.6. It is shown in Theorem 4 of [13] that every minimal homeomorphism \( h \) of the Cantor set \( X \) has an almost one to one extension which is a minimal homeomorphism \( k \) of a nonhomogeneous space \( Y \) such that \( C^*(\mathbb{Z}, X, h) \) and \( C^*(\mathbb{Z}, Y, k) \) have isomorphic Elliott invariants. Since it is not homogeneous, the space \( Y \) is not homeomorphic to \( X \), and in particular \( h \) and \( k \) can’t possibly be strong orbit equivalent, or even topologically orbit equivalent.

The space \( Y \) has covering dimension \( \dim(X) = 1 \). Theorem 1.9 below does not apply, because \( X \) and \( Y \) are not manifolds. However, it is probably now easy to prove in this case that isomorphism of the Elliott invariants implies isomorphism of the \( C^* \)-algebras. See Remark 1.10 below.

One might argue that one should only consider minimal homeomorphisms of the same space, or at least of spaces of the same dimension. However, almost one to one extensions are a standard construction of closely related homeomorphisms in dynamics.

Next, we present an unpublished example of Ian Putnam. It is reproduced here with his permission.

Example 1.7 (Putnam). For any \( \theta \in \mathbb{R} \setminus \mathbb{Q} \) let \( r_\theta: S^1 \to S^1 \) be rotation by \( 2\pi \theta \). Further let \( g_\theta \) be a minimal homeomorphism of a Cantor set \( X_\theta \subset S^1 \) obtained from a Denjoy homeomorphism \( g_\theta^{(0)}: S^1 \to S^1 \) as follows [36]. Any Denjoy homeomorphism has a unique minimal set \( X \), which is homeomorphic to the Cantor set. Choose \( g_\theta^{(0)} \) to have rotation number \( \theta \) and such that the unique minimal set \( X_\theta \subset S^1 \) has the property that the image of \( S^1 \setminus X_\theta \) under the semiconjugation to \( r_\theta \) is a single orbit of \( r_\theta \). Let \( g_\theta = g_\theta^{(0)}|_{X_\theta} \). See Section 3 of [36] for details, particularly Corollary 3.2 and Definitions 3.3 and 3.5, noting that we are requiring the set \( Q \) there to consist of exactly one orbit.

Following Remark 1 in Section 3 of [36], we may consider the \( C^* \) subalgebra of the bounded Borel functions on \( S^1 \) generated by \( C(S^1) \) and the characteristic
functions of all sets \( \exp(2\pi in\theta, (n+1)\theta) \) for \( n \in \mathbb{Z} \). This C*-algebra has an automorphism \( \beta_\theta \) given by rotation by \( 2\pi\theta \), and we can take \( X_\theta \) to be its maximal ideal space and \( g_\theta \) to be the homeomorphism determined by \( \beta_\theta \).

Now let \( \theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q} \) be numbers such that 1, \( \theta_1 \), \( \theta_2 \) are rationally independent. Define homeomorphisms

\[
h_1 = r_{\theta_1} \times g_{\theta_2} : S^1 \times X_{\theta_2} \rightarrow S^1 \times X_{\theta_2}
\]

and

\[
h_2 = r_{\theta_2} \times g_{\theta_1} : S^1 \times X_{\theta_1} \rightarrow S^1 \times X_{\theta_1}.
\]

It follows from Proposition 1.12 below that the crossed products by these homeomorphisms are simple C*-algebras with unique traces, and have isomorphic Elliott invariants.

We show that the homeomorphisms \( h_1 \) and \( h_2 \) are not topologically orbit equivalent. Suppose we had a topological orbit equivalence \( f \). By the proof of (i) implies (iii) in Theorem 2.2 of [12], at the beginning of Section 5 there, \( f \) preserves the invariant measures. (The proof works without the restriction that the space be the Cantor set.) So the sets of possible measures of compact open subsets are the same for both systems. For \( S^1 \times X_{\theta_2} \) this set contains \( \theta_2 \) and is contained in \( \mathbb{Z} + \theta_2 \mathbb{Z} \), while for \( S^1 \times X_{\theta_1} \) this set contains \( \theta_1 \) and is contained in \( \mathbb{Z} + \theta_1 \mathbb{Z} \). This is a contradiction.

As in Example 1.6, Remark 1.10 suggests that it should be easy to adapt known results to prove that the crossed product C*-algebras are isomorphic.

Unlike in Example 1.6, in this example the spaces are homeomorphic.

The last example is from our earlier work with Qing Lin.

**Example 1.8.** For a minimal diffeomorphism of a sphere \( S^n \) with \( n \geq 3 \) odd, the Elliott invariant of the transformation group C*-algebras is known. (See Section 5 of [22] and Example 4.6 of [30]. In fact, the calculation works for minimal homeomorphisms, using Corollary VI.12 of [10] in place of Corollary 3 in Section 5 of [22].) It depends only on the simplex of invariant Borel probability measures, and in particular is independent of \( n \), as long as \( n \geq 3 \), and of other properties of the diffeomorphism. It follows from Theorem 3 of [11] that every odd sphere admits a uniquely ergodic minimal diffeomorphism, and from [17] that every odd sphere of dimension at least 3 admits a minimal diffeomorphism with any given finite number of ergodic measures. If the Elliott conjecture holds for the corresponding transformation group C*-algebras, then those minimal diffeomorphisms having a given finite number of ergodic measures all give isomorphic C*-algebras independent of \( n \), as long as \( n \geq 3 \). Moreover, it is likely [16], although it remains unproved, that an odd sphere in fact admits many nonconjugate uniquely ergodic minimal diffeomorphisms.

Currently known classification theorems are not adequate to prove isomorphism of these C*-algebras from isomorphism of their Elliott invariants. The theorem presently available requires real rank zero, but these C*-algebras have no nontrivial projections (Corollary 3 in Section 5 of [3]; Corollary 12 in Section 6 of [11]).

In the next section, we describe several examples of minimal diffeomorphisms of compact connected manifolds for which it is now possible to prove that the transformation group C*-algebras are isomorphic, while the diffeomorphisms are not flip
conjugate and hence not even topologically orbit equivalent. In several of our examples, the manifolds on which the diffeomorphisms act are identical, in another they are different but have the same dimension, and in another they have different dimensions. The variety of different examples gives a collection of properties of minimal diffeomorphisms which are not invariants of the transformation group $C^*$-algebras.

The following result is a special case of results of [24] and [20], and is what we use to establish isomorphisms of crossed products. All crossed products covered by it have real rank zero, by [24] and [31].

**Theorem 1.9.** For $j = 1, 2$ let $M_j$ be a compact smooth manifold, and let $h_j: M_j \to M_j$ be a minimal diffeomorphism. Assume that the maps $K_0(C^*(Z, M_j, h_j)) \to \text{Aff}(T(C^*(Z, M_j, h_j)))$, from the $K_0$-groups to the spaces of real valued affine continuous functions on the trace spaces, have dense range. Assume that the Elliott invariants (Definition 1.4) of $C^*(Z, M_1, h_1)$ and $C^*(Z, M_2, h_2)$ are isomorphic. Then $C^*(Z, M_1, h_1) \cong C^*(Z, M_2, h_2)$.

In the theorem, density of the image of $K_0$ in the affine function space is equivalent to the transformation group $C^*$-algebra having real rank zero. See [31].

**Remark 1.10.** The main, although not the only, use of smoothness in the proof of Theorem 1.9 is for an exponential length bound in [24]. However, when the covering dimension of the space is at most 2, exponential length bounds are easy. See Section 2 of [33]. So current methods can probably be easily modified to prove that Theorem 1.9 holds for minimal homeomorphisms of compact metric spaces with covering dimension at most 2, possibly under the restriction that there be at most countably many ergodic measures.

In our examples, we computed the K-theory using the Pimsner-Voiculescu exact sequence, Theorem 1.11 below. In most of them, we computed the order on $K_0$ using Exel’s rotation numbers for automorphisms [10]. A relatively easy case is carried out in Example 4.9 of [30].

We finish this section by giving the computation relevant to Example 1.7, since it has not appeared elsewhere. The order computation is somewhat different, since Exel’s methods are easily applied only to homeomorphisms of connected spaces. For use here, and for later reference, we first state the Pimsner-Voiculescu exact sequence [35] for the special case of a crossed product of a compact space by a homeomorphism.

**Theorem 1.11 (Pimsner-Voiculescu exact sequence).** Let $X$ be a compact Hausdorff space, and let $h: X \to X$ be a homeomorphism. Then there is a natural six term exact sequence

$$
\begin{array}{cccccc}
K^0(X) & \xrightarrow{id-h^*} & K^0(X) & \longrightarrow & K_0(C^*(Z, X, h)) \\
\exp \uparrow & & & & \downarrow \partial \\
K_1(C^*(Z, X, h)) & \longleftarrow & K_1(X) & \xrightarrow{id-h^*} & K^1(X)
\end{array}
$$

The maps $K^i(X) \to K_i(C^*(Z, X, h))$ are the maps on K-theory induced by the inclusion $C(X) \to C^*(Z, X, h)$. 

Proposition 1.12. Let $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ be numbers such that $1, \alpha, \beta$ are rationally independent. Let $r_\alpha: S^1 \to S^1$ and $g_\beta: X_\beta \to X_\beta$ be as in Example 3.7. Let
\[ h = r_\alpha \times g_\beta: S^1 \times X_\beta \to S^1 \times X_\beta. \]

Then $h$ is minimal and uniquely ergodic, and the Elliott invariant of the crossed product $A = C^*(\mathbb{Z}, S^1 \times X_\beta, h)$ is given as follows: $K_1(A) \cong \mathbb{Z}^3$, there is a unique tracial state $\tau$, and $\tau_*$ induces an order isomorphism $K_0(A) \to \mathbb{Z} + \alpha \mathbb{Z} + \beta \mathbb{Z} \subset \mathbb{R}$ (with the order on the range given by restriction from $\mathbb{R}$) such that $\tau_*([1]) = 1$.

**Proof:** The restriction to $X_\beta$ of the semiconjugation map of Corollary 3.2 of [36] is a continuous surjective map $f: X_\beta \to S^1$ such that $f \circ g_\beta = r_\beta \circ f$, and there is a countable subset $T \subset X_\beta$ such that $f|_{X_\beta \setminus T}$ is injective. Therefore
\[ \text{id}_{S^1} \times f\colon S^1 \times X_\beta \to S^1 \times S^1 \]
is a continuous surjective map such that
\[ (\text{id}_{S^1} \times f) \circ h = (r_\alpha \times r_\beta) \circ (\text{id}_{S^1} \times f), \]
which is injective on the dense set $S^1 \times (X_\beta \setminus T)$. The diffeomorphism $r_\alpha \times r_\beta$ is known to be minimal (Proposition 1.4.1 of [17]) and uniquely ergodic (Theorem 6.20 of [46]), with the ergodic measure $\nu$ being the product of two copies of Lebesgue measure.

It follows that $h$ is minimal. Indeed, if $Z \subset S^1 \times X_\beta$ is a nonempty closed invariant subset, then $(\text{id}_{S^1} \times f)(Z)$ is a nonempty closed subset of $S^1 \times S^1$ which is invariant under $r_\alpha \times r_\beta$, whence $(\text{id}_{S^1} \times f)(Z) = S^1 \times S^1$. Therefore $Z$ contains $S^1 \times (X_\beta \setminus T)$, whence $Z = S^1 \times X_\beta$.

It also follows that $h$ is uniquely ergodic. Indeed, the existence of an invariant Borel probability measure $\mu$ follows from the existence of such measures for $r_\alpha$ and $g_\beta$ (see Section 3 of [36]), or by general results. Moreover, if $\mu_0$ is any other invariant Borel probability measure, then the measure on $S^1 \times S^1$ given by $E \mapsto \mu_0((\text{id}_{S^1} \times f)^{-1}(E))$ is a Borel probability measure which is invariant under $r_\alpha \times r_\beta$, and hence equal to $\nu$. The Lebesgue measure of $f(T)$ is zero because $f(T)$ is countable, so $\mu_0(S^1 \times T) = 0$. For any Borel set $E \subset S^1 \times X_\beta$, we therefore get
\[ \mu_0(E) = \mu_0(E \cap [S^1 \times (X_\beta \setminus T)]) = \nu((\text{id}_{S^1} \times f)(E \cap [S^1 \times (X_\beta \setminus T)])) \]
\[ = \mu(E \cap [S^1 \times (X_\beta \setminus T)]) = \mu(E). \]

We now know that $A$ is simple and has a unique trace, say $\tau$. We use Theorem 11.11 to compute $K_*(A)$. We have
\[ K^0(S^1 \times X_\beta) \cong K^1(S^1 \times X_\beta) \cong K^0(X_\beta), \]
and we can identify
\[ \text{id} - h^*: K^0(S^1 \times X_\beta) \to K^0(S^1 \times X_\beta), \]
for both $i = 0$ and $i = 1$, with
\[ \text{id} - g_\beta^\#: K^0(X_\beta) \to K^0(X_\beta). \]

The proof of Lemma 6.1 of [36] gives
\[ \text{Ker}(\text{id} - g_\beta^\#) \cong \mathbb{Z} \quad \text{and} \quad \text{Coker}(\text{id} - g_\beta^\#) \cong \mathbb{Z}^2. \]
Therefore the sequence of Theorem 11.11 breaks up into two short exact sequences
\[ 0 \to \mathbb{Z}^2 \to K_i(A) \to \mathbb{Z} \to 0 \]
for \(i = 0\) and \(i = 1\). Both sequences must split, giving \(K_0(A) \cong K_1(A) \cong \mathbb{Z}^2\).

We next determine what the trace does on \(K_0(A)\). Define \(B_1 = C^*(\mathbb{Z}, S^1, r_n)\) and \(B_2 = C^*(\mathbb{Z}, X_{\beta}, g_{\beta})\). These \(C^*\)-algebras are simple and have unique traces, say \(\tau_1\) and \(\tau_2\). The coordinate projections induce equivariant maps \(C(S^1) \to C(S^1 \times X_{\beta})\) and \(C(X_{\beta}) \to C(S^1 \times X_{\beta})\), and hence injective maps \(\varphi_1: B_1 \to A\) and \(\varphi_2: B_2 \to A\).

Naturality in the Pimsner-Voiculescu exact sequence gives a commutative diagram, in which the vertical maps are sums (so that \(s(\eta_1, \eta_2) = (\varphi_1)_*(\eta_1) + (\varphi_2)_*(\eta_2)\) etc.), as follows:

\[
\begin{array}{ccc}
K^0(S^1) \oplus K^0(X_{\beta}) & \overset{(\iota_1) \oplus (\iota_2)_*}{\longrightarrow} & K_0(B_1) \oplus K_0(B_2) \\
\downarrow s_0 & & \downarrow s \\
K^0(S^1 \times X_{\beta}) & \overset{\iota_*}{\longrightarrow} & K_0(A) \\
& \downarrow \vartheta & \\
& K^1(S^1) \oplus K^1(X_{\beta}) &
\end{array}
\]

We claim that \(s\) is surjective. Let \(\eta \in K_0(A)\). The proof of Lemma 6.1 of [36] shows that the kernel of \(\text{id} - g_{\beta}^* : K^0(X_{\beta}) \to K^0(X_{\beta})\) is generated by \([1] \in K_0(C(X_{\beta}))\). Therefore the kernel of \(\text{id} - h^* : K^1(S^1 \times X_{\beta}) \to K^1(S^1 \times X_{\beta})\) is generated by the class of the unitary \(v(\zeta, x) = \zeta\). It follows that \(\partial(\eta) = n[v]\) for some \(n \in \mathbb{Z}\). Moreover, with \(u \in K^1(S^1)\) given by \(u(\zeta) = \zeta\), we get \([v] = s_1([u], 0)\). Since \(\text{id} - r_n^* = 0\), there is \(\mu \in K_0(B_1)\) such that \(\partial_1(\mu) = [u]\). Now \(\partial(\eta - s(n\mu, 0)) = 0\) by commutativity, so \(\eta - s(n\mu, 0)\) is in the image of \(s_0\). It is easy to check that \(s_0\) is surjective, and it follows from commutativity of the left square that \(\eta - s(n\mu, 0)\) is in the image of \(s\). Therefore \(\eta\) is in the image of \(s\). This completes the proof of the surjectivity of \(s\).

Uniqueness of the traces gives

\[
\tau_* \circ s = \tau_* \circ (\varphi_1)_* + \tau_* \circ (\varphi_2)_* = (\tau_1)_* + (\tau_2)_*.
\]

Since the range of \((\tau_1)_*\) is \(\mathbb{Z} + \alpha\mathbb{Z}\) and, by Theorem 5.3 of [30], the range of \((\tau_2)_*\) is \(\mathbb{Z} + \beta\mathbb{Z}\), it follows that the range of \(\tau_*\) is exactly \(\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}\). Since \(K_0(A) \cong \mathbb{Z}^2\), it follows that \(\tau_*\) is an isomorphism onto its image.

It remains only to show that \(\tau_* : K_0(A) \to \mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}\) is an order isomorphism. This follows from Theorem 4.5(1) of [30].

2. Examples

We describe here four examples of pairs of different minimal diffeomorphisms giving isomorphic crossed products. The minimal diffeomorphisms in the pairs are distinguished in a variety of ways: the property of having topologically quasidiscrete spectrum, acting on manifolds of different dimensions or on nonhomeomorphic manifolds of the same dimension, and inducing automorphisms of singular cohomology which are not conjugate. Details of these examples will appear in [32]. We also describe an example which has not yet been proved to exist but whose existence seems likely. In this case, the diffeomorphisms are distinguished by the behavior of \(\lim_{n \to \infty} d(h^n(x), h^n(y))\) for distinct points \(x\) and \(y\) in the manifold. At the end of the section, we list several obvious questions related to our examples, and give a brief discussion of the problem of finding a dynamical condition for isomorphism of the crossed product \(C^*\)-algebras.

**Example 2.1** (Furstenberg transformations on \((S^1)^2\)). Rouhani has in [37] exhibited a Furstenberg transformation on the 2-torus \(S^1 \times S^1\) which does not have topologically quasidiscrete spectrum. (A homeomorphism \(h : X \to X\) is said to have topologically quasidiscrete spectrum if the linear map \(C(X) \to C(X)\), given
by $f \mapsto f \circ h$, has sufficiently many “quasieigenfunctions”, a kind of generalized eigenvector. See Section 1 of [37].)

Let $\theta \in [0, 1) \setminus \mathbb{Q}$ be an irrational number, to be chosen below, and let $r : S^1 \to \mathbb{R}$ be a smooth function, also to be chosen below. Define $h_1, h_2 : S^1 \times S^1 \to S^1 \times S^1$ by

$$h_1(\zeta_1, \zeta_2) = (e^{2\pi i \theta} \zeta_1, \zeta_1 \zeta_2) \quad \text{and} \quad h_2(\zeta_1, \zeta_2) = (e^{2\pi i \theta} \zeta_1, e^{2\pi i r(\zeta_1)} \zeta_1 \zeta_2)$$

for $(\zeta_1, \zeta_2) \in S^1 \times S^1$. (The only difference is the extra factor $e^{2\pi i r(\zeta_1)}$ in the definition of $h_2$.)

It is observed in [37] that the affine Furstenberg transformation $h_1$ is always minimal and uniquely ergodic and always has topologically quasidiscrete spectrum, and it is also shown how to choose $\theta$ and $r$ so that $h_2$ is minimal and uniquely ergodic but does not have topologically quasidiscrete spectrum. It is only proved in [37] that $r$ is continuous, but in fact the choices made there give a smooth function $r$, so that $h_1$ and $h_2$ are both diffeomorphisms. The Elliott invariants of the transformation group C*-algebras of this type are computed in [13]: a much faster calculation using more machinery is given in Example 4.9 of [30]. They turn out to depend only on $\theta$ and the space of invariant measures. Moreover, the dense range hypothesis in Theorem 1.9 is satisfied, and it follows that the transformation group C*-algebras are isomorphic.

However, the property of having topologically quasidiscrete spectrum is preserved by flip conjugacy. So $h_1$ and $h_2$ are not flip conjugate, hence not topologically orbit equivalent.

**Example 2.2 (Affine Furstenberg transformations on $(S^1)^3$).** In Section 6.1 of the unpublished thesis of R. Ji [13], two affine Furstenberg transformations on $(S^1)^3$ were given which have the same Elliott invariant but are not flip conjugate, and the question was raised whether they have isomorphic transformation group C*-algebras. Ji was not able to compute the order on $K_0$; he only computed the map on $K_0$ determined by the (unique) trace. However, Theorem 4.5(1) of [30] implies that the order on $K_0$ is that determined by the trace. The calculation is more complicated than for Example 2.1 because of the presence of torsion and more Bott elements.

Fix $\theta \in [0, 1) \setminus \mathbb{Q}$ and $m$, $n \in \mathbb{Z}$ with $0 < m < n$. Then the two affine Furstenberg transformations on $(S^1)^3$, given by

$$(\zeta_1, \zeta_2, \zeta_3) \mapsto (\exp(2\pi i \theta) \zeta_1, \zeta_1^m \zeta_2, \zeta_2^n \zeta_3)$$

and

$$(\zeta_1, \zeta_2, \zeta_3) \mapsto (\exp(2\pi i \theta) \zeta_1, \zeta_1^n \zeta_2, \zeta_2^m \zeta_3)$$

(the difference is that $m$ and $n$ have been exchanged), are not topologically orbit equivalent but have isomorphic crossed product C*-algebras.

The isomorphism of the C*-algebras is obtained from Theorem 1.3 and the calculation of the ordered K-theory, because any affine Furstenberg transformation is minimal and uniquely ergodic. If the first diffeomorphism above is called $h$, then for any nonzero values of $m$ and $n$ it turns out that $K_0(C^*(\mathbb{Z}, M, h))$ and $K_1(C^*(\mathbb{Z}, M, h))$ are both isomorphic to $\mathbb{Z}^4 \oplus \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, and that the isomorphism of $K_0(C^*(\mathbb{Z}, M, h))$ with this group can be chosen in such a way that the
unique trace $\tau$ induces the map

$$\tau_*(r_1, r_2, r_3, s_1, s_2) = r_1 + \theta r_3$$

and $K_0(C^*(Z, M, h))_+$ is identified with

$$\{(r_1, r_2, r_3, s_1, s_2) \in Z^4 \oplus Z/mZ \oplus Z/nZ : r_1 + r_3\theta > 0\} \cup \{0\}.$$  

There is in $[12]$ no indication of the proof that these two diffeomorphisms are not flip conjugate. This, however, can be obtained by examining their effect on singular cohomology with integer coefficients. We have $H^1((S^1)^3; Z) \cong Z^3$, and with respect to suitable bases the two diffeomorphisms induce maps with matrices

$$\begin{pmatrix} 1 & m & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}.$$  

Over $Z$, the first of these matrices is similar to neither the second nor its inverse, which rules out flip conjugacy. (However, the two matrices are similar over $Q$.)

If we further choose $m$ and $n$ to be relatively prime and with $|m|, |n| \geq 2$, then one can exhibit yet a third affine Furstenberg transformation on $(S^1)^3$ which gives the same transformation group $C^*$-algebra yet is not flip conjugate to either of the first two, namely

$$(\zeta_1, \zeta_2, \zeta_3) \mapsto (\exp(2\pi i\theta)\zeta_1, \zeta_1^{mn}\zeta_2, \zeta_2\zeta_3).$$

The point is that $Z/mZ \oplus Z/nZ \cong Z/mnZ$, but again the actions on $H^1((S^1)^3; Z)$ rule out flip conjugacy.

**Example 2.3** (Minimal diffeomorphisms on $S^2 \times S^1$ and $(S^1)^3$). Let $M_1 = S^2 \times S^1$, and let $u \in U(C(M_1))$ be given by $u(x, \zeta) = \zeta$. Adapting methods of $[11]$, we can prove that there exists a uniquely ergodic minimal diffeomorphism $h: M_1 \to M_1$, with unique invariant Borel probability measure $\mu$, which is homotopic to the identity map and such that the rotation number of $[u]$ with respect to $h$ and $\mu$ (in the sense of $[10]$) has the form $\exp(2\pi i\theta)$ for some $\theta \in [0, 1] \setminus Q$.

Let $M_2 = (S^1)^3$, and define $h_2: M_2 \to M_2$ by

$$h_2(\zeta_1, \zeta_2, \zeta_3) = (\exp(2\pi i\theta)\zeta_1, \zeta_1\zeta_2, \zeta_2\zeta_3)$$

for $(\zeta_1, \zeta_2, \zeta_3) \in (S^1)^3$.

The same methods as used for Example 2.2 enable one to compute the Elliott invariant of $C^*(Z, M_2, h_2)$. A similar calculation, but with different algebraic topology, computes the Elliott invariant of $C^*(Z, M_1, h_1)$. One gets

$$K_0(C^*(Z, M_1, h_1)) \cong K_0(C^*(Z, M_2, h_2)) \cong Z^4$$

and

$$K_1(C^*(Z, M_1, h_1)) \cong K_1(C^*(Z, M_2, h_2)) \cong Z^4.$$  

Moreover, the orders on both $K_0$ groups turn out to be described as follows: there are generators $\eta_1, \eta_2, \nu_1$, and $\nu_2$ such that the unique trace $\tau$ on the algebra satisfies

$$\tau_*(\eta_1) = 1, \quad \tau_*(\nu_1) = \theta, \quad \text{and} \quad \tau_*(\eta_2) = \tau_*(\nu_2) = 0,$$

and the positive cone of $K_0$ consists exactly of $0$ together with the elements $\eta$ such that $\tau_*(\eta) > 0$. In particular, the two Elliott invariants are isomorphic.
CROSSED PRODUCTS BY MINIMAL Diffeomorphisms 11

It follows from Theorem 1.3 that the two crossed product C*-algebras are isomorphic. However, the diffeomorphisms can’t possibly be topologically orbit equivalent because the spaces on which they act are not homeomorphic.

Example 2.4 (Minimal diffeomorphisms on manifolds of different dimensions). As discussed in Example 1.8, it is expected that the transformation group C*-algebras of minimal homeomorphisms of odd spheres of dimension at least 3 depend up to isomorphism only on the space of invariant probability measures. Because these C*-algebras have no nontrivial projections, current machinery is not able to prove this. By forming the products of uniquely ergodic examples of this type with a suitable irrational rotation on the circle, we can produce examples to which current methods apply.

Use Theorem 3 (in Section 3.8) of [11] to find, for each odd n ≥ 3, a uniquely ergodic minimal diffeomorphism h_n^{(0)} : S^n → S^n. The Lefschetz fixed point theorem (Theorem 4.7.7 of [42]) can be used to show that an orientation reversing diffeomorphism of an odd sphere must have a fixed point. So our diffeomorphisms are all orientation preserving and therefore homotopic to the identity map.

Next, one proves using results from [28] that if h : X → X is a uniquely ergodic minimal homeomorphism of a connected compact metric space X, then there is a dense Gδ-set T ⊂ S^1 such that, for every λ ∈ T, the homeomorphism of X × S^1 given by (x, ζ) → (h(x), λζ) is minimal and uniquely ergodic. Let T_n be the dense Gδ-set obtained for h_n^{(0)}, and let T be the intersection of these sets, which is still a dense Gδ-subset of S^1. Choose θ ∈ [0, 1] \ Q such that exp(2πiθ) ∈ T. For odd n ≥ 3, define a uniquely ergodic minimal diffeomorphism h_n : S^n × S^1 → S^n × S^1 by h_n(x, ζ) = (h_n^{(0)}(x), exp(2πiθ)ζ).

Each h_n is homotopic to the identity map, and K-theory does not detect the difference between different odd spheres, so the Pimsner-Voiculescu exact sequence gives the same K-groups for all the crossed products C*(Z, S^n × S^1, h_n). In particular, the K0-groups are all Z^4. Using methods similar to those described in previous examples, one shows that there is a set of four generators of K0(C*(Z, S^n × S^1, h_n)) whose traces are 1, θ, 0, and 0. As before, Exel’s rotation numbers are used, and the fact that n ≥ 3 is used to show that there is essentially only one source for a noninteger trace, namely a class in K0(C*(Z, S^n × S^1, h_n)) whose image in K1(C(S^n × S^1)) is the class of the unitary (x, ζ) ↦ ζ.

It follows from Theorem 1.9 that the C*-algebras C*(Z, S^n × S^1, h_n) are all isomorphic. No two of these diffeomorphisms can be topologically orbit equivalent, since they act on manifolds of different dimensions.

To complete our collection of examples, we give a brief description of an example whose existence seems plausible but has not been proved.

Example 2.5 (Extensions of Furstenberg transformations). Let θ ∈ R \ Q, and let h_1 : S^1 × S^1 → S^1 × S^1 be given by

\[ h_1(ζ_1, ζ_2) = (e^{2πiθ}ζ_1, ζ_1ζ_2), \]

as in Example 2.1. Fix a point z_0 ∈ S^1 × S^1. We believe it should be possible to construct a surjective map f : S^1 × S^1 → S^1 × S^1 and a minimal diffeomorphism (or at least a minimal homeomorphism) h_2 : S^1 × S^1 → S^1 × S^1 with the following properties:

- \[ h_1 ∘ f = f ∘ h_2. \] (Thus, h_2 is an extension of h_1.)
• If we let $T$ denote the orbit of $z_0$ under $h_1$, then $f^{-1}(S^1 \times S^1 \setminus T)$ is dense in $S^1 \times S^1$, and the restriction of $f$ to this set is injective. (In particular, $h_2$ is an almost one to one extension of $h_1$, as defined before Example 1.6.)

• For $n \in \mathbb{Z}$, the set $I_n = f^{-1}(\{h_1^n(z_0)\})$ is homeomorphic to $[0,1]$.

• $\lim_{n \to -\infty} \text{diam}(I_n) = 0$ and $\lim_{n \to -\infty} \text{diam}(I_n) = 0$.

• $f$ commutes with the projection to the first coordinate.

• $f$ is a homotopy equivalence.

To construct $h_2$ and $f$, we replace each point in the orbit of $z_0$ under $h_1$ by a “vertical” interval, starting with $z_0$, then $h_1(z_0)$ and $h_1^{-1}(z_0)$, etc., with the lengths of the inserted intervals chosen to go to zero fast enough that the resulting space is still a torus. To see the possibility of replacing one point by an interval in a continuous manner, consider the map $f_0: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_0(x, y) = \begin{cases} (x, y) & |x| \geq 1 \\ (x, |x|y) & |x| \leq 1 \text{ and } |y| \leq 1 \\ (x, |x| + 2(y - 1)) & |x| \leq 1 \text{ and } 1 \leq y \leq 2 - |x| \\ (x, -|x| + 2(y + 1)) & |x| \leq 1 \text{ and } -1 \leq y \geq -(2 - |x|) \\ (x, y) & |x| \leq 1 \text{ and } |y| \geq 2 - |x| \end{cases}.$$ 

This map is the identity outside a compact set, injective off $\{0\} \times [-1, 1]$, and sends $\{0\} \times [-1, 1]$ to $(0, 0)$. Smooth versions exist, but are more complicated to describe. Then $h_2$ must be defined to carry $I_n$ homeomorphically to $I_{n+1}$ for each $n$. The details, especially for the smooth case, seem rather difficult to carry out, but we conjecture that this can be done.

If $f$ and $h_2$ exist, then the fact that $f$ is a homotopy equivalence can be used to show that the corresponding map from $C^*(\mathbb{Z}, S^1 \times S^1, h_1)$ to $C^*(\mathbb{Z}, S^1 \times S^1, h_2)$ induces an isomorphism of Elliott invariants. We show that $h_1$ and $h_2$ are not flip conjugate. Let $d$ be the metric on $S^1 \times S^1$ given by the maximum of the differences of the two coordinates. If $z_1, z_2 \in S^1 \times S^1$ are distinct, then

$$\lim_{n \to -\infty} \inf_{z_1} d(h_1^n(z_1), h_2^n(z_2)) > 0 \quad \text{and} \quad \lim_{n \to -\infty} \inf_{z_1} d(h_1^n(z_1), h_2^n(z_2)) > 0.$$ 

(Consider separately the cases in which $z_1$ and $z_2$ have different or equal first coordinates.) That is, $h_1$ is distal as defined at the beginning of Chapter 5 of [1]. However, if $z_1$ and $z_2$ are any two points in $f^{-1}(\{z_0\})$, then

$$\lim_{n \to -\infty} d(h_2^n(z_1), h_2^n(z_2)) = 0 \quad \text{and} \quad \lim_{n \to -\infty} d(h_2^n(z_1), h_2^n(z_2)) = 0.$$ 

These properties are unchanged if $d$ is replaced by an equivalent metric or one of the homeomorphisms is replaced by its inverse. Therefore $h_1$ and $h_2$ are not flip conjugate, and so can’t be topologically orbit equivalent either.

These examples leave open a number of interesting problems.

**Problem 2.6.** Make a more careful modification of the work of [1], so as to be able to construct examples like Example 2.3 and Example 2.4 for arbitrary irrational values of $\theta$.

**Problem 2.7.** Do there exist essentially different uniquely ergodic minimal diffeomorphisms of the same odd sphere? Can they be used to produce essentially different versions of Example 2.3?
Question 2.8. Is there a minimal diffeomorphism $h$ of a connected compact smooth manifold $M$ which is not uniquely ergodic but for which the map

$$K_0(C^*(\mathbb{Z}, M, h)) \to \text{Aff}(T(C^*(\mathbb{Z}, M, h)))$$

as in Theorem 1.3, still has dense range?

Following the methods of [10], one needs among other things a continuous function $u: M \to S^1$ which has different rotation numbers with respect to different invariant Borel probability measures on $M$.

Finally, we turn to the question which motivated the construction of these examples.

Problem 2.9. What dynamical relation on minimal homeomorphisms of compact metric spaces, or minimal diffeomorphisms of compact smooth manifolds, corresponds to isomorphism of the transformation group $C^*$-algebras?

At the moment, we don’t even have any plausible candidates. One might think of considering flow equivalence, as discussed, for example, at the beginning of Section 1 of [27] and in Definition 1.1 and the following discussion in [34]. (The right notion would actually be flip flow equivalence.) We do not know whether the minimal diffeomorphisms of Example 2.1 and of Example 2.2 are flow equivalent. Those in Example 2.3 and in Example 2.4 are not, because Theorem 2 of [38] implies that if $h_1$ and $h_1$ are flow equivalent diffeomorphisms on manifolds $M_1$ and $M_2$ then the universal covers of $M_1$ and $M_2$ are homeomorphic. Moreover, flow equivalence of minimal diffeomorphisms only implies stable isomorphism: see Section 2 of [27].

Problem 2.10. Are the minimal diffeomorphisms of Example 2.1 flow equivalent?

Are those of Example 2.2 are flow equivalent?

The following question arose in discussions of possible answers to Problem 2.9.

Problem 2.11. For $j = 1, 2$ let $X_j$ be a compact metric space, and let $h_j: X_j \to X_j$ be a minimal homeomorphism. Suppose that $C^*(\mathbb{Z}, X_1, h_1) \cong C^*(\mathbb{Z}, X_2, h_2)$. Does it follow that $h_1$ and $h_2$ have a common minimal almost one to one extension (as defined before Example 1.6)?

This is true in Example 1.6, since $k$ is already an almost one to one extension of $h$. It is true in Example 1.7, since $g_\theta_1 \times g_\theta_2$ is a common minimal almost one to one extension.

This condition certainly does not imply isomorphism of the transformation group $C^*$-algebras. In the notation of Example 1.7, the homeomorphism $g_\theta$ is a minimal almost one to one extension of $r_\theta$. However,

$$K_1(C^*(\mathbb{Z}, S^1, r_\theta)) \cong \mathbb{Z}^2 \quad \text{and} \quad K_1(C^*(\mathbb{Z}, X_\theta, g_\theta)) \cong \mathbb{Z}.$$

If one wants extensions to preserve $K$-theory of the crossed products, then some restriction on the space of the extension is necessary. The following seems like a good test case.

Problem 2.12. Let $h_1, h_2: S^1 \times S^1 \to S^1 \times S^1$ be Furstenberg transformations as in Example 2.1 with $\theta \in \mathbb{R} \setminus \mathbb{Q}$ arbitrary and $r: S^1 \to \mathbb{R}$ an arbitrary smooth function. Does there exist a common extension of $h_1$ and $h_2$ which is a minimal homeomorphism of $S^1 \times S^1$?
3. Smooth crossed products

The examples in Section 2 show that C*-algebra crossed products preserve little information about minimal homeomorphisms of connected compact metric spaces. For the case of a diffeomorphism satisfying an additional condition, one can construct instead a smooth crossed product. It is natural to hope, especially in view of Example 3, that the smooth crossed product might preserve more information. In fact, very little is known about smooth crossed products by minimal diffeomorphisms; even some very basic questions are open. In this section and the next, we discuss the smooth crossed products and raise some of these questions.

The foundations of the abstract theory of smooth crossed products of Banach and Fréchet algebras are laid in [41]. Here, we are looking at smooth crossed products and raise some of these questions. We can give a collection of seminorms on $C^\infty(M)$ which by Theorem 3.1.7 of [41] implies that $S(Z, C^\infty(M), h)$, with the topology of uniform convergence of all derivatives.

To formulate our condition in terms of the derivatives of $h$ should grow at most polynomially in $n$. The best estimate for general diffeomorphisms allows exponential growth of the derivatives of $h^n$; see Example 3.3 below. To formulate our condition in terms of the derivatives of $h^n$ requires setting up more notation than we want to introduce here. In all the explicit examples we actually discuss, $M$ will be $(S^1)^d \cong (\mathbb{R}/\mathbb{Z})^d$ for some $d$, and in this case, as we now explain, we can describe the situation in terms of ordinary derivatives on $\mathbb{R}^d$. For the case of a diffeomorphism satisfying an additional condition, one can construct instead a smooth crossed product. It is natural to hope, especially in view of Example 1.5, that the smooth crossed product might preserve more information.
Let $M = (\mathbb{R}/\mathbb{Z})^d$, and let $h: M \to M$ be a smooth function. Then $h$ has a universal cover $\tilde{h}: \mathbb{R}^d \to \mathbb{R}^d$. It is not unique, but every other choice has the form $x \mapsto \tilde{h}(x) + l$ for some $l \in \mathbb{Z}^d$. Moreover, for every $k \in \mathbb{Z}^d$ there is $l \in \mathbb{Z}^d$ such that $\tilde{h}(x + k) = \tilde{h}(x) + l$ for all $x \in \mathbb{R}^d$. (This is just continuity and the condition that $\tilde{h}$ gives a well defined function $(\mathbb{R}/\mathbb{Z})^d \to (\mathbb{R}/\mathbb{Z})^d$.) So the partial derivatives of $\tilde{h}$, of order at least 1, depend only on $h$ and are periodic with period 1 in each coordinate. In particular, they are bounded. Letting $\tilde{h}_i: \mathbb{R}^d \to \mathbb{R}$ be the $i$-th component of $\tilde{h}$, and letting $D_k$ denote partial differentiation on $\mathbb{R}^d$ with respect to the $k$-th coordinate, we can therefore define

$$\rho_m(h) = \max_{1 \leq i \leq d} \sum_{j_1 + \cdots + j_r = m} \|D_1^{j_1}D_2^{j_2} \cdots D_r^{j_r} \tilde{h}_i\|_{\infty} \in [0, \infty).$$

**Lemma 3.2.** Let $h: (\mathbb{R}/\mathbb{Z})^d \to (\mathbb{R}/\mathbb{Z})^d$ be a diffeomorphism. Let $\rho_m$ be as above. Suppose that for every $m \geq 1$ there is $C > 0$ and $r \in \mathbb{N}$ such that for every $n \in \mathbb{Z}$ we have $\rho_m(h^n) \leq C(1 + |n|)^r$. Then $h$ is tempered.

**Proof:** Let $M = (\mathbb{R}/\mathbb{Z})^d$. To every $f \in C^\infty(M)$ corresponds a function $\tilde{f} \in C^\infty(\mathbb{R}^d)$ which is periodic with period 1 in each coordinate. Moreover, $(f \circ \tilde{h}) = \tilde{f} \circ \tilde{h}$. With an obvious choice of vector fields on $M$, we get

$$\|f\|_m = \sum_{j_1 + \cdots + j_r = m} \|D_1^{j_1}D_2^{j_2} \cdots D_r^{j_r} \tilde{f}\|_{\infty}.$$

We now consider derivatives of $f \circ \tilde{h}$. Applying the chain rule and the product rule, we find that a derivative

$$D_1^{j_1}D_2^{j_2} \cdots D_r^{j_r}(f \circ \tilde{h})(x),$$

with $j_1 + j_2 + \cdots + j_r = m$, is a finite sum of products of at most $m + 1$ terms, one of which is a partial derivative of $f$ of order at most $m$ evaluated at $\tilde{h}(x)$ and the rest of which are partial derivatives of components of $\tilde{h}$ of order between 1 and $m$ evaluated at $x$. In particular, there is a constant $C_{m,d}$, not depending on $f \in C^\infty(M)$ or $h: M \to M$, such that

$$\|f \circ h\|_m \leq C_{m,d}(\|f\|_0 + \|f\|_1 + \cdots + \|f\|_m)[1 + \rho_1(h) + \rho_2(h) + \cdots + \rho_m(h)]^m.$$

Applying this to $h^n$ in place of $h$, and using the estimate on $\rho_m(h^n)$ in the hypotheses, we get a bound of the required sort for $n \geq 0$. For $n \leq 0$, apply the same argument to $h^{-1}$.

**Remark 3.3.** For $h: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and $n \geq 1$, we have

$$\langle \tilde{h}^n \rangle' = \tilde{h}'(\tilde{h}^{n-1}(t)) \cdot \tilde{h}'(\tilde{h}^{n-2}(t)) \cdots \tilde{h}'(h(t)) \cdot \tilde{h}'(t).$$

The naive estimate therefore gives $\rho_1(h^n) \leq \rho_1(h)^n$. If $h$ is a diffeomorphism, then it is possible to have $|\tilde{h}'(t)| \geq 1$ everywhere only if $|\tilde{h}'(t)| = 1$ everywhere. If furthermore $h$ is minimal, then one can hope that the iterates $x, h(x), h^2(x), \ldots$ are distributed well enough between places where the derivative is small and where it is large that the overall growth of the derivative is less than exponential. In fact, one expects $[13]$ that it is reasonably common for minimal diffeomorphisms of compact manifolds to be tempered.
Example 3.4. We show that rotations are tempered. Fix $\theta_1, \theta_2, \ldots, \theta_d \in \mathbb{R}$. Define $h: (S^1)^d \to (S^1)^d$ by
$$h(\xi_1, \xi_2, \ldots, \xi_d) = (e^{2\pi i \theta_1 \xi_1}, e^{2\pi i \theta_2 \xi_2}, \ldots, e^{2\pi i \theta_d \xi_d}).$$
Then we can take $\tilde{h}$ to be
$$\tilde{h}(x_1, x_2, \ldots, x_d) = (x_1 + \theta_1, x_2 + \theta_2, \ldots, x_d + \theta_d),$$
so
$$\tilde{h}^n(x_1, x_2, \ldots, x_d) = (x_1 + n\theta_1, x_2 + n\theta_2, \ldots, x_d + n\theta_d).$$
It follows that $\rho_1(h^n) = d$ and $\rho_m(h^n) = 0$ for all $m \geq 2$. So $h$ is tempered.

Example 3.5. Let $g: \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function with period 1, with $g'(t) > -1$ for all $t$, with $g(0) = 0$, and with $g'(0) = 1$. Let $h: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the diffeomorphism determined by $h(t) = t + g(t)$. Then for $n \geq 1$ we have $(h^n)'(0) = 2^n$. So $h$ does not satisfy the conditions of Lemma 3.2, and in fact it is easy to see that $h$ is not tempered.

The diffeomorphism of Example 3.5 is not minimal. However, minimal examples are known to exist. We use the minimal real analytic diffeomorphism $F_n$ with nonzero topological entropy constructed in [14]. For us, the relevant properties are in Proposition 5.1 and Theorem 5.3 of [14]. In particular, the proof of Proposition 5.1 of [14] depends on the inequality, for a suitable invariant probability measure $\mu$,
$$\inf_{n \in \mathbb{N}} \frac{1}{n} \int_M \log(\|TF^n(x)\|) \, d\mu(x) > 0,$$
in which $TF^n$ is the tangent map of $F_n^0$. This inequality is incompatible with a polynomial bound on the growth of the derivatives of powers of $F_n$.

Example 3.6. The affine Furstenberg transformations of Example 2.2 are tempered. To prove this, take
$$\tilde{h}(x_1, x_2, x_3) = (x_1 + \theta, x_2 + mx_1, x_3 + nx_2).$$
We can write this in the more general form $\tilde{h}(x) = t + x + N(x)$, where $t \in \mathbb{R}^d$ is fixed and $N \in M_d(\mathbb{R})$ is nilpotent with $N^d = 0$. (Here $d = 3$.) By induction, we find that if $n \geq 0$ then
$$\tilde{h}^n(x) = t + (1 + N)t + \cdots + (1 + N)^{n-1}t + (1 + N)^nx.$$
For suitable $t_n \in \mathbb{R}^d$, using $N^d = 0$, and with $\binom{n}{k}$ denoting the binomial coefficient,
$$\tilde{h}^n(x) = t_n + \sum_{k=0}^{\min(n, d-1)} \binom{n}{k} N^k x.$$
All partial derivatives of all components $(\tilde{h}^n)_i$ of order 1 are constant, and all partial derivatives of higher order are zero. Since the binomial coefficients $\binom{n}{k}$, with $k \leq d - 1$, are polynomials in $n$ of degree at most $d - 1$, it follows that there is a constant $C$ such that
$$|D_j(\tilde{h}^n)_i(x)| \leq C(1 + n)^{d-1}$$
for all $i$ and $j$, all $n > 0$, and all $x \in \mathbb{R}^d$. The diffeomorphism $\tilde{h}^{-1}$ has the same form, so the same method applies. Therefore $h$ is tempered by Lemma 3.3.
The argument of Example 3.6 actually applies to all of the affine Furstenberg transformations of [15].

Example 3.7. The Furstenberg transformations of Example 2.1 are tempered. For $h_1$, this is the same as Example 3.6. For $h = h_2$, we take

$$\tilde{h}(x, y) = (x + \theta, y + x + \tilde{r}(x)),$$

where $\tilde{r}(t) = r(e^{2\pi i t})$. By induction on $n$, we get

$$\tilde{h}^n(x, y) = \left(x + n\theta, y + nx + \frac{n}{2}n(n - 1)\theta + \sum_{k=0}^{n-1} \tilde{r}(x + k\theta)\right)$$

for $n \geq 0$. The first component has $D_1(\tilde{h}^n)_1(x, y) = 1$, while all other partial derivatives, of all orders, are zero. The second component has

$$D_2(\tilde{h}^n)_2(x, y) = \frac{n}{2}n(n - 1)\theta + \sum_{k=0}^{n-1} \tilde{r}'(x + k\theta),$$

while

$$D_1^m(\tilde{h}^n)_2(x, y) = \sum_{k=0}^{n-1} \tilde{r}^{(m)}(x + k\theta)$$

for $m \geq 2$, and all other partial derivatives are zero. It follows that

$$\|D_1(\tilde{h}^n)_2\|_\infty \leq n + n\|\tilde{r}'\|_\infty \quad \text{and} \quad \|D_1^m(\tilde{h}^n)_2\|_\infty \leq n\|\tilde{r}^{(m)}\|_\infty,$$

while all other partial derivatives are bounded by constants independent of $n$. So $h$ is tempered by Lemma 3.2.

We do not know whether the minimal diffeomorphisms of odd spheres constructed in [11] and [47], the minimal diffeomorphism of $S^2 \times S^1$ of Example 2.3 or the minimal diffeomorphisms of $S^n \times S^1$ for odd $n$ of Example 2.4, can be chosen to be tempered, although it seems reasonable to expect that they can be [16].

Question 3.8. Let $M_1$ and $M_2$ be compact smooth manifolds, and let $h_1: M_1 \to M_1$ and $h_2: M_2 \to M_2$ be tempered minimal diffeomorphisms. Suppose that the smooth crossed products $S(Z, C^\infty(M_1), h_1)$ and $S(Z, C^\infty(M_2), h_2)$ are isomorphic. Does it follow that $h_1$ and $h_2$ are flip conjugate?

We have left one point ambiguous in this question: should the homeomorphism implementing the flip conjugacy be required to be smooth? This makes a difference, even on $S^1$. See Theorem 12.5.1 and the preceding discussion in [17], and for further examples also Theorem 12.6.1 in [17]. (We do not know if this kind of behavior can occur for tempered minimal diffeomorphisms.)

Even if the flip conjugacy is merely required to be continuous, we suspect that in general isomorphism does not imply flip conjugacy. On the other hand, isomorphism of the smooth crossed products is probably a much more restrictive condition than isomorphism of the transformation group C*-algebras. As specific evidence that this might be the case, we offer Example 1.5. One way to extract extra information is via the computation of cyclic cohomology. This is how the nonisomorphism in Example 1.7 was proved in [3]. Cyclic cohomology for crossed products by $Z$ has been studied in [25].
4. Ranks and Schweitzer’s example

Our consideration of Question 3.8 made us realize just how little is known about smooth crossed products by tempered diffeomorphisms. It is known that the smooth crossed product is spectrally invariant in the transformation group C*-algebra, by Corollary 7.16 of [40]. For the significance of this, see the introduction to [40] and Section 1 of [39]. In particular, $S(\mathbb{Z}, C^\infty(M), h)$ is closed under holomorphic functional calculus evaluated in $C^*(\mathbb{Z}, M, h)$, and the inclusion induces an isomorphism on K-theory. (See [34] for more on the K-theory of smooth crossed products by $\mathbb{Z}$.) These seem to be the basic properties wanted for noncommutative differential geometry as in [6]. Indeed, if it is in fact true that isomorphism of smooth crossed products is much less common than isomorphism of the C* crossed products, then the examples in Section 2 provide examples of C*-algebras which have quite different natural smooth structures.

However, as far as we can tell, the following questions all remain open, even for the smooth irrational rotation algebras. They are motivated by the importance of the stable and real ranks as invariants for C*-algebras, and by the role they play in the classification of crossed product C*-algebras.

**Question 4.1.** Let $M$ be a compact smooth manifold, and let $h: M \to M$ be a tempered minimal diffeomorphism. Does it follow that the smooth crossed product $S(\mathbb{Z}, C^\infty(M), h)$ has stable rank one, that is, that the invertible elements are dense?

It is known that $C^*(\mathbb{Z}, M, h)$ has stable rank one ([2]; Corollary 1.2 of [23]), but it is not known whether stable rank one passes to a spectrally invariant subalgebra, or even a strongly spectrally invariant subalgebra. The invertible elements of $S(\mathbb{Z}, C^\infty(M), h)$ are of course dense in the topology of $C^*(\mathbb{Z}, M, h)$, but this is not what is being asked for.

**Question 4.2.** Let $M$ be a compact smooth manifold, and let $h: M \to M$ be a tempered minimal diffeomorphism. Suppose $C^*(\mathbb{Z}, M, h)$ has real rank zero. Does it follow that the selfadjoint invertible elements in $S(\mathbb{Z}, C^\infty(M), h)$ are dense in the set of all selfadjoint elements?

**Question 4.3.** Let $M$, $h$, and $C^*(\mathbb{Z}, M, h)$ be as in Question 4.2. Does it follow that the selfadjoint elements in $S(\mathbb{Z}, C^\infty(M), h)$ with finite spectrum are dense in the set of all selfadjoint elements?

For C*-algebras, the properties asked for in Question 4.2 and Question 4.3 are equivalent—both are real rank zero. We give a dense selfadjoint subalgebra of $C_0(\mathbb{N})$ which is a Banach algebra in its own topology, is strongly spectrally invariant in $C_0(\mathbb{N})^+$ (even satisfying the Blackadar-Cuntz differential seminorm conditions [2]), and is closed under $C^\infty$ functional calculus (even $C^1$ functional calculus) in $C_0(\mathbb{N})^+$ for selfadjoint elements, but for which conclusion in Question 4.3 does not hold. The conclusion in Question 4.2 does hold, so this example shows two things: that these conditions are not equivalent for Banach *-algebras, and that the conclusion in Question 4.3 does not pass to strongly spectrally invariant subalgebras. This example was constructed by Larry Schweitzer, and is reproduced here with his permission.

**Example 4.4** (Schweitzer). By convention, we take $\mathbb{N} = \{1, 2, \ldots\}$. Set $$B = C_0(\mathbb{N})^+ = \{a \in l^\infty(\mathbb{N}) : \lim_{n \to \infty} a(n) \text{ exists}\}.$$
Write $|| \cdot ||_\infty$ for its norm. Let $\lambda: B \to \mathbb{C}$ be evaluation at $\infty$, that is, $\lambda(a) = \lim_{n \to \infty} a(n)$. Define

$$B_0 = \{ a \in B : \lim_{n \to \infty} n[a(n) - \lambda(a)] \text{ exists} \},$$

and define $\omega: B_0 \to \mathbb{C}$ by $\omega(a) = \lim_{n \to \infty} n[a(n) - \lambda(a)]$. Then for $a \in B_0$ define

$$\|a\|_\omega = \sup_{n \in \mathbb{N}} |n[a(n) - \lambda(a)]| \quad \text{and} \quad \|a\| = \|a\|_\infty + \|a\|_\omega.$$

We establish the properties of this example in a sequence of lemmas.

**Lemma 4.5.** Let $a, b \in B_0$. Then $\|ab\|_\omega \leq \|a\|_\infty \|b\|_\omega + \|a\|_\omega \|b\|_\infty$.

**Proof:** It is obvious that $\lambda(ab) = \lambda(a)\lambda(b)$ and $|\lambda(a)| \leq ||a||_\infty$. Therefore

$$\|ab\|_\omega = \sup_{n \in \mathbb{N}} |n[a(n)b(n) - \lambda(a)\lambda(b)]|$$

$$\leq \sup_{n \in \mathbb{N}} |n[a(n) - \lambda(a)]| \cdot |b(n)| + \sup_{n \in \mathbb{N}} n|\lambda(a)| \cdot |b(n) - \lambda(b)|$$

$$\leq \|a\|_\omega \|b\|_\infty + \|a\|_\infty \|b\|_\omega.$$

This is the result.

**Corollary 4.6.** The definitions $T_0(a) = \|a\|_\infty$, $T_1(a) = \|a\|_\omega$, and $T_n(a) = 0$ for $n \geq 2$, give a differential seminorm on $B_0$ in the sense of Definition 3.1 of [2].

**Proof:** The required inequalities are

$$\|ab\|_\infty \leq \|a\|_\infty \|b\|_\infty \quad \text{and} \quad \|ab\|_\omega \leq \|a\|_\infty \|b\|_\omega + \|a\|_\omega \|b\|_\infty.$$

The first is known and the second is Lemma 4.5.

**Proposition 4.7.** The norm $|| \cdot ||$ is submultiplicative on $B_0$, satisfies $||a^*|| = ||a||$ for all $a \in B_0$, and satisfies $||1|| = 1$.

**Proof:** For the first part, we estimate:

$$\|ab\| = \|ab\|_\infty + \|ab\|_\omega \leq \|a\|_\infty \|b\|_\infty + \|a\|_\infty \|b\|_\omega + \|a\|_\omega \|b\|_\infty$$

$$\leq (\|a\|_\infty + \|a\|_\omega)(\|b\|_\infty + \|b\|_\omega).$$

The other two parts are obvious.

**Lemma 4.8.** The algebra $B_0$ is a Banach $^*$-algebra in $|| \cdot ||$.

**Proof:** It only remains to prove that $B_0$ is complete. Let $(a_k)$ be a Cauchy sequence in $B_0$. Then, using $||a||_\infty \leq ||a||$ and $|\omega(a)| \leq ||a||$, there are $a \in B$ and $\alpha \in \mathbb{C}$ such that

$$\lim_{k \to \infty} ||a_k - a||_\infty = 0 \quad \text{and} \quad \lim_{k \to \infty} |\omega(a_k) - \alpha| = 0.$$

We first show that $a \in B_0$, by proving that $\lim_{n \to \infty} n[a(n) - \lambda(a)] = \alpha$.

Let $\varepsilon > 0$. Choose $N_0$ so large that if $k, l \geq N_0$ then $||a_k - a_l|| < \frac{1}{3}\varepsilon$. For such $k$ and $l$, we have in particular

$$\sup_{n \in \mathbb{N}} |n[a_k(n) - a_l(n) - \lambda(a_k - a_l)]| < \frac{1}{3}\varepsilon.$$

Letting $l \to \infty$, we get

$$\sup_{n \in \mathbb{N}} |n[a_k(n) - a(n) - \lambda(a_k - a)]| \leq \frac{1}{3}\varepsilon.$$
for all $k \geq N_0$. Now choose $k \geq N_0$ and also so large that $|\omega(a_k) - \alpha| < \frac{1}{2^2} \varepsilon$. For this $k$, choose $N$ so large that if $n \geq N$ then

$$|n[a_k(n) - \lambda(a_k)] - \omega(a_k)| < \frac{1}{3} \varepsilon.$$ 

For all $n \geq N$ we then have

$$|n[a(n) - \lambda(a)] - \alpha| 
\leq n[a_k(n) - a(n) - \lambda(a_k - a)] + |n[a_k(n) - \lambda(a_k)] - \omega(a_k)| + |\omega(a_k) - \alpha| 
\leq \frac{1}{3} \varepsilon + \frac{1}{3} \varepsilon + \frac{1}{3} \varepsilon = \varepsilon.$$ 

Now we prove that $\|a_k - a\| \to 0$. Let $\varepsilon > 0$, and choose $N_0$ as before. As there, for $k \geq N_0$ we get

$$\sup_{n \in \mathbb{N}} n[a_k(n) - a(n) - \lambda(a_k - a)] \leq \frac{1}{3} \varepsilon \quad \text{and} \quad \sup_{n \in \mathbb{N}} |a_k(n) - a(n)| \leq \frac{1}{3} \varepsilon,$$

whence $\|a_k - a\| \leq \frac{2}{3} \varepsilon < \varepsilon$. 

**Corollary 4.9.** The algebra $B_0$ is closed under holomorphic functional calculus in $B$. That is, if $a \in B_0$ and $f$ is a holomorphic function defined on a neighborhood of $\text{sp}_B(a)$, then $f(a) \in B_0$.

**Proof:** Use Theorem 1.17 of [40] and Lemma 1.2 of [39].

**Proposition 4.10.** The algebra $B_0$ is closed under $C^1$ functional calculus for self-adjoint elements in $B$. That is, if $a \in B_0$ satisfies $a^* = a$ and $f$ is a $C^1$ function defined on a neighborhood of $\text{sp}_B(a)$, then $f(a) \in B_0$.

**Proof:** The element $f(a) \in B$ is given by $f(a)(n) = f(a(n))$ for all $n \in \mathbb{N}$. We must prove that this element is in $B_0$, which we do by showing that

$$\lim_{n \to \infty} n[f(a(n)) - \lambda(a(n))] = f'(\lambda(a))\omega(a).$$

Let $\varepsilon > 0$. Choose $\delta > 0$ such that whenever $|t - \lambda(a)| < \delta$ then

$$|f(t) - f(\lambda(a)) - f'(\lambda(a))(t - \lambda(a))| \leq \left(\frac{\varepsilon}{2|\omega(a)| + 2}\right) |t - \lambda(a)|.$$ 

Choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|a(n) - \lambda(a)| < \delta \quad \text{and} \quad |n[a(n) - \lambda(a)] - \omega(a)| < \min\left(1, \frac{\varepsilon}{2|f'(\lambda(a))| + 2}\right).$$

For such $n$, we then use $\lambda(f(a)) = f(\lambda(a))$ to get

$$|n[f(a(n)) - \lambda(f(a))] - f'(\lambda(a))\omega(a)| 
\leq \left(\frac{\varepsilon}{2|\omega(a)| + 2}\right) \cdot n[a(n) - \lambda(a)] + |f'(\lambda(a))| n[a(n) - \lambda(a)] - f'(\lambda(a))\omega(a)| 
\leq \left(\frac{\varepsilon}{2|\omega(a)| + 2}\right) (|\omega(a)| + 1) + |f'(\lambda(a))| \cdot |n[a(n) - \lambda(a)] - \omega(a)| 
< \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = \varepsilon.$$ 

The algebra $B_0$ is not closed under continuous functional calculus for selfadjoint elements in $B$. Take $f(x) = \sqrt{x}$ for $x \geq 0$. Define $a \in B_0$ by $a(n) = \frac{1}{n}$. Then $a \in B_0$ but $f(a) \notin B_0$. 

Proposition 4.11. The invertible elements of $B_0$ are dense in $B_0$, and the invertible selfadjoint elements in $B_0$ are dense in the set of all selfadjoint elements in $B_0$.

Proof: It follows from Corollary 4.9 that every element of $B_0$ which is invertible in $B$ is also invertible in $B_0$. So let $a \in B_0$ and let $\varepsilon > 0$. Choose any real number $\alpha \notin \{ \lambda(a) \} \cup \{ a(n) : n \in \mathbb{N} \}$ with $|\alpha| < \varepsilon$. Then $b = a - \alpha \cdot 1$ is invertible in $B$ and satisfies $\| b - a \| < \varepsilon$. Moreover, if $a$ is selfadjoint, then so is $b$. □

Proposition 4.12. The selfadjoint elements in $B_0$ which have finite spectrum are not dense in the set of all selfadjoint elements in $B_0$.

Proof: Define $a \in B_0$ by $a(n) = \frac{1}{n}$. Let $b \in B_0$ have finite spectrum. Then the range of $b$ is finite, whence $b(n) = \lambda(b)$ for all sufficiently large $n$. Therefore $\omega(b) = 0$. Since $|\omega(b - a)| \leq \| b - a \|_\omega$ and $\omega(a) = 1$, it follows that $\| b - a \| \geq 1$. □

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