An Extension on Neighbor Sum Distinguishing Total Coloring of Graphs

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Abstract: Let $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ be a non-proper total $k$-coloring of $G$. Define a weight function on total coloring as

$$\phi(x) = f(x) + \sum_{e \ni x} f(e) + \sum_{y \in N(x)} f(y),$$

where $N(x) = \{y \in V(G) | xy \in E(G)\}$. If $\phi(x) \neq \phi(y)$ for any edge $xy \in E(G)$, then $f$ is called a neighbor full sum distinguishing total $k$-coloring of $G$. The smallest value $k$ for which $G$ has such a coloring is called the neighbor full sum distinguishing total chromatic number of $G$ and denoted by $fgnd\sum(G)$. The coloring is an extension of neighbor sum distinguishing non-proper total coloring. In this paper we conjecture that $fgnd\sum(G) \leq 3$ for any connected graph $G$ of order at least three. We prove that the conjecture is true for (i) paths and cycles; (ii) 3-regular graphs and (iii) stars, complete graphs, trees, hypercubes, bipartite graphs and complete $r$-partite graphs. In particular, complete graphs can achieve the upper bound for the above conjecture.

Keywords: non-proper total coloring; neighbor full sum distinguishing total coloring; neighbor full sum distinguishing total chromatic number

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1 Introduction

All considered graphs are finite, undirected, simple and connected. Let $[s, t]$ denote the set of nonnegative integers $\{s, s+1, s+2, \ldots , t\}$ and $0 \leq s < t$. Let $d_G(v)$ and $\Delta(G)$ (or $\Delta$) denote the degree of vertex $v$ and the maximum degree of $G$, respectively. Let $d$-vertex denote the vertex of degree $d$, $1 \leq d \leq \Delta$. For general theoretic notations, we follow [3].

Graph coloring theory has a wide range of applications in many fields, such as computer science, physics, chemistry and network theory. Specifically related to time tabling and scheduling, frequency assignment problem, register allocation, computer security, coding theory, communication network and so on. Since customers have increased dramatically, it yields a confliction between the increasing customers and the limited expansion of communication network resource. Driven by this background, a class of distinguishing coloring on the sums of colors of vertices and edges has attracted extensive attention. Karoński et al. [6] firstly introduced and investigated neighbor sum distinguishing edge coloring of graphs.

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and they proposed a famous 1-2-3 Conjecture. Toward the 1-2-3 Conjecture, Karóński, Łuczak and Thomason [7] showed that if \( G \) is a \( k \)-colorable graph with \( k \) odd then \( G \) admits a vertex-coloring \( k \)-edge-weighting. So, for the class of 3-colorable graphs, including bipartite graphs, the answer is affirmative. However, in general, this question is still open. Addario-Berry et al. [1] showed that every graph without isolated edges has a proper \( k \)-weighting when \( k = 30 \). After improvements to \( k = 15 \) in [2] and \( k = 13 \) in [10], Kalkowski, Karóński, and Pfender [7] showed that every graph without isolated edges has a proper \( 5 \)-weighting. Przybyło [9] showed that every \( d \)-regular graph with \( d \geq 2 \) admits a vertex-coloring edge \( 4 \)-weighting and every \( d \)-regular graph with \( d \geq 10^8 \) admits a vertex-coloring edge \( 3 \)-weighting. Later, Przybyło and Wozniak [8] added the vertex coloring to the weight of edges, they gave the notation of neighbor sum distinguishing total coloring of graphs, meanwhile, they put forward to a 1-2 conjecture with respect to this definition. Thus far it is known that for every graph \( G \), \( \text{fgndi}_\Sigma(G) \leq 3 \) (see [5]), where \( \text{gndi}_\Sigma(G) \) is the neighbor sum distinguishing total chromatic number of \( G \). Recently, Flandrin et al. [4] considered the sum of the colors of neighbors of a vertex based on the neighbor sum distinguishing total coloring, they introduced a new coloring which is called the neighbor full sum distinguishing total coloring, while they didn’t give a depth study for this coloring, so we continue to study this type of coloring in this paper.

**Definition 1.** [4] Let \( f : V(G) \cup E(G) \to [1, k] \) be a non-proper \( k \)-total coloring of \( G \). Set \( \phi(x) = f(x) + \sum_{e \ni x} f(e) + \sum_{y \in N(x)} f(y) \), where \( N(x) = \{ y \in V(G) | xy \in E(G) \} \). For any edge \( xy \in E(G) \), if \( \phi(x) \neq \phi(y) \), then \( f \) is called a neighbor full sum distinguishing (NFSD) total \( k \)-coloring of \( G \). The smallest value \( k \) for which \( G \) has an NFSD-total coloring is called the neighbor full sum distinguishing total chromatic number of \( G \) and denoted by \( \text{fgndi}_\Sigma(G) \).

Evidently, when searching for the NFSD-total coloring it is sufficient to restrict our attention to connected graphs. Observe also, that \( G = K_2 \) does not have any NFSD-total coloring. So, we shall consider only connected graphs with at least three vertices. We propose the following conjecture.

**Conjecture 2.** For every connected graph \( G \) and \( G \) is not \( K_2 \), \( \text{fgndi}_\Sigma(G) \leq 3 \).

By Definition 1, the following result is easy to obtain.

**Lemma 3.** Let \( G \) be a connected simple graph of order at least three. Then (i) \( \text{fgndi}_\Sigma(G) = 1 \) if \( G \) contains no adjacent \( d \)-vertices; and (ii) \( \text{fgndi}_\Sigma(G) \geq 2 \) if \( G \) contains adjacent \( d \)-vertices.

**Proof** For any two adjacent vertices \( u \) and \( v \) of \( G \), (i) if \( G \) contains no adjacent \( d \)-vertices, namely \( d_G(u) \neq d_G(v) \), then we color all vertices and edges of \( G \) with 1, and it gets that \( \phi(u) = 2d_G(u) + 1 \neq 2d_G(v) + 1 = \phi(v) \); (ii) if \( d_G(u) = d_G(v) = d \), then \( \text{fgndi}_\Sigma(G) \geq 2 \), otherwise, \( u \) and \( v \) receive the same weight, a contradiction. \( \square \)

We organize the paper as follows. In Section 2, the neighbor full sum distinguishing total chromatic number of paths and cycles are determined. In Section 3, we offer an important structural lemma that every connected graph \( G \) contains a \( m \)-partite spanning subgraph \( H \) such that \( (1 - \frac{1}{m})d_G(v) \leq d_H(v) \). Therefore, every 3-regular graph \( G \) has a maximal bipartite spanning subgraph \( H \) such that \( G - E(H) \) is either isolated vertices or isolated edges. Via the structural between \( H \) and \( G - E(H) \) of 3-regular graphs \( G \) and combining with a coloring algorithm, we get that \( \text{fgndi}_\Sigma(G) \leq 3 \) for any 3-regular graph \( G \). In Section 4, we obtain the parameter \( \text{fgndi}_\Sigma(G) \) of several types of graphs with maximum degree \( \Delta \geq 4 \).
2 Graphs with $\Delta = 2$

**Proposition 4.** Let $P_n$ be a path of order $n$ ($\geq 3$). Then $\text{fgndi}_{\Sigma}(P_n) = 2$ if $n \geq 4$ and $\text{fgndi}_{\Sigma}(P_3) = 1$.

**Proof** Let $P_n = x_1x_2 \ldots x_n$. It is easy to verify that $\text{fgndi}_{\Sigma}(P_3) = 1$. By Lemma 3, $\text{fgndi}_{\Sigma}(P_n) \geq 2$ for $n \geq 4$. We define a total coloring $f$: $V(G) \cup E(G) \rightarrow \{1, 2\}$ as follows:

$$f(x_i) = \begin{cases} 1 & \text{if } i \equiv 1 \text{(mod 2)}, \\ 2 & \text{if } i \equiv 0 \text{(mod 2)}. \end{cases}$$

And all edges of $P_n$ are colored by 1.

Taking advantage of the above coloring $f$, we have $\phi(x_1) = \phi(x_n) = 4$. For any vertex $x_k$ ($2 \leq k \leq n - 1$), $\phi(x_k) = 6$ if $k$ is even and $\phi(x_k) = 7$ if $k$ is odd, which deduces that $f$ is an NFSD-total coloring of $P_n$. □

**Proposition 5.** Let $C_n$ be a cycle with order $n$ ($\geq 3$). Then

$$\text{fgndi}_{\Sigma}(C_n) = \begin{cases} 3 & \text{if } n = 3, \\ 2 & \text{if } n \geq 4. \end{cases}$$

**Proof** Let $C_n = x_1x_2 \ldots x_nx_1$. Clearly, $\text{fgndi}_{\Sigma}(C_3) = 3$. By Lemma 3, $\text{fgndi}_{\Sigma}(C_n) \geq 2$ for $n \geq 4$. The following two cases imply that $C_n$ has an NFSD-2-total coloring.

**Case 1.** $n \equiv 1 \text{(mod 2)}$.

We define a total 2-coloring $f$ of $C_n$ as below.

$$f(x_1) = 2;$$

$$f(x_i) = \begin{cases} 1, & i \equiv 1 \text{(mod 2)} \\ 2, & i \equiv 0 \text{(mod 2)} \end{cases}, i \in [2, n];$$

$$f(x_ix_{i+1}) = 1, i \in [2, n - 1].$$

Then $\phi(x_1) = 9$, $\phi(x_2) = 8$, $\phi(x_n) = 8$, $\phi(x_i) = 7$ if $i$ is odd and $i \in [3, n - 1]$, $\phi(x_i) = 6$ if $i$ is even and $i \in [3, n - 1]$. Therefore, $\phi(u) \neq \phi(v)$ for any edge $uv \in E(C_n)$, namely, $f$ is an NFSD-total 2-coloring of $C_n$, and thus $\text{fgndi}_{\Sigma}(C_n) = 2$.

**Case 2.** $n \equiv 0 \text{(mod 2)}$.

We define a total 2-coloring $f$ of $C_n$ as below.

$$f(x_i) = \begin{cases} 1, & i \equiv 1 \text{(mod 2)} \\ 2, & i \equiv 0 \text{(mod 2)} \end{cases}, i \in [1, n];$$

Meanwhile, all edges of $C_n$ are colored by 1. Then $\phi(x_{2k}) = 6 \neq \phi(x_{2k+1}) = 7$. Therefore, $f$ is an NFSD-total 2-coloring of $C_n$, and hence $\text{fgndi}_{\Sigma}(C_n) = 2$. □

3 3-regular graphs

This section we investigate $\text{fgndi}_{\Sigma}(G)$ of 3-regular graphs. The following lemma is very crucial to the proof of the main theorem.

**Lemma 6.** Let $G$ be a graph on $n$ vertices. Then it exists a $m$-partite spanning subgraph $H$ such that $(1 - \frac{1}{m})d_G(v) \leq d_H(v)$ for all $v \in V(G)$, where $m$ is a positive integer and $m \leq n$.

**Proof** Let $H$ be a maximal $m$-partite spanning subgraph of $G$ with the greatest possible number of edges. Let $\{V_1, V_2, \ldots, V_m\}$ be the $m$-partition of $V(H)$ and let $v \in V_i$, $d_{V_i}(v) = |N_{V_i}(v)|$, $N_{V_i}(v) = \{u : u \in V_i, uv \in E(G)\}$, $i = 1, 2, \ldots, m$. Then $d_{V_i}(v) \leq d_{V_i}(v)$, $i = 1, 2, \ldots, m$. Otherwise, it exists
isolated vertices or isolated edges. We color all vertices in $X$ and all edges in $E$. For any $v \in V(G)$, select one endpoint $v_i$ of $v$, and let $v \in V(G)$. Then for any two vertices $v_x \in V_X$ and $v_y \in V_Y$, $\phi(v_x) = 10$ and $\phi(v_y) = 8$.

**Case 2.** $a_1 = b_2 = 0$ or $b_1 = a_2 = 0$.

Without loss of generality, assume that $a_1 = b_2 = 0$. We color all vertices in $X$ with 1, color all vertices in $Y$ with 2, color all edges in $E_H$ with 1 and color all edges in $E_Y$ with 1. Then for any two vertices $v_x \in V_X$ and $v_y \in V_Y$, $\phi(v_x) = 10$ and $\phi(v_y) = 9$. To assure that $\phi(v_{y_j}) \neq \phi(v'_{y_j})$, recolor an incident edge of $v_{y_j}$ (or $v'_{y_j}$) with 3. Then $\phi(v_x)$ belongs to $\{10, 12, 14, 16\}$ and $\phi(v_y)$ is equal to 9 or 11.

**Case 3.** $b_1 = b_2 = 0$.

By Lemma 6, $G$ contains a maximal bipartite spanning subgraph $H$ such that $G - E(H)$ is either isolated vertices or isolated edges. We color all vertices in $X$ with 1, color all vertices in $Y$ with 2, color all edges in $E_H$ and $E_X$ with 1, and color all edges in $E_Y$ with 2. For any edge $v_x, v'_x \in X$, select one edge $e_x$ from $E_H$ such that $v_x, v'_x$ is an endpoints of $e_x$ and recolor edge $e_x$ with 3, meanwhile, all incident edges (except for $e_x, v_x, v'_x$, $v_{y_1}, v'_{y_1}$) of $v_x, v'_x, v_{y_1}, v'_{y_1}$ keep the color 1 as before, and we call these edges being dominated, see Fig.1. Without loss of generality, assume that $v'_{x_1}$ and $v_{y_1}$ are connected.

**Theorem 7.** For any 3-regular graph $G$, $2 \leq \text{fgndi}_G(G) \leq 3$.

**Proof** Let $G$ be a 3-regular graph. Then $\text{fgndi}_G(G) \geq 2$ by Lemma 3.

**Case 1.** $a_1 = a_2 = 0$.

This case implies that $G$ is a 3-regular complete bipartite graph. We color all vertices in $X$ with 1, color all vertices in $Y$ with 2 and color all edges in $E_H$ with 1. Then for any two vertices $v_x \in V_X$ and $v_y \in V_Y$, $\phi(v_x) = 10$ and $\phi(v_y) = 8$.
by $e_z$. Then $\phi(v_{x1}) = 11$, $\phi(v'_{x1}) = 9$, $\phi(v_{y1}) = 12$, $\phi(v'_{y1}) = 10$. Continue this procedure $\frac{4k}{2}$ times until the weights of all adjacent vertices in $G$ are distinct. Now we prove it feasibility, namely, it verifies that there exists at least one edge in $E_H$ can not be dominated after $\frac{4k}{2} - 1$ operations. Suppose that all edges in $H$ are dominated after $\frac{4k}{2} - 1$ operations, and if there still exists a pair of adjacent vertices $v_{x_k}$ and $v'_{x_k}$ (or $v_{y_k}$ and $v'_{y_k}$) having the same weights, then the four incident edges of $v_{x_k}$ and $v'_{x_k}$ in $H$ receive the same color 1. By our coloring rule, it is impossible.

**Case 4.** $a_1 \neq a_2$, $b_1 \neq b_2$ and they are all positive integers.

We color all vertices in $X$ with 1, color all vertices in $Y$ with 3, color all edges in $E_H$ with 1, color all edges in $E_X$ with 2 and color all edges in $E_Y$ with 3. In the bipartite graph $H$, let $v_2(x)$ and $v_3(x)$ be the vertex with degree 2 and 3 in $X$, respectively. Similarly, $v_2(y)$ and $v_3(y)$ denote the vertex with degree 2 and 3 in $Y$, respectively. The edge between $v_2(x)$ and $v_2(y)$ is denoted by $\tilde{e}_2$, and the edge connects $v_2(x)$ (or $v_3(x)$) and $v_3(y)$ (or $v_2(y)$) is denoted by $\tilde{e}_2^{-3}$.

Using the technique in Case 3 to change the weight of the vertex which is connected only by $\tilde{e}_2$, we can distinguish all adjacent vertices which are connected by $\tilde{e}_2$. But we still need to distinguish adjacent vertices which are joined by $\tilde{e}_2^{-3}$. Select an incident edge from $\tilde{e}_2^{-3}$ and color it with 3, then it deduces that $\phi(v_{x_1}) \neq \phi(v'_{x_1})$ and $\phi(v_{y_1}) \neq \phi(v'_{y_1})$, meanwhile $\phi(v_2(x)) \in \{12, 14\}$, $\phi(v_2(y)) \in \{13, 15\}$, $\phi(v_3(x)) \in \{13, 15, 17, 19\}$ and $\phi(v_3(y)) \in \{9, 11, 13, 15\}$. Possibly, there are some cases that the weight of adjacent vertices can not distinguish. We deal with it as follows.

**Case 4.1.** $\phi(v_3(x)) = 13$. Let $N(v_3(x)) = \{v_{y1}, v_{y2}, v_{y3}\}$.

**Case 4.1.1.** All vertices in $N(v_3(x))$ have the same weight 13. Recolor $v_3(x)$ and its incident edges with 3. Then $\phi(v_3(x)) = 21$ and $\phi(v_{y1}) = \phi(v_{y2}) = \phi(v_{y3}) = 17$. If one of $v_{y1}$, $v_{y2}$, $v_{y3}$ has an adjacent vertex $v_{x1}$ with weight 17 in $X$, say $v_{y1}$, then recolor edges $v_3(x)v_{y1}$ and $v_{y1}v_{x1}$ with 2, and we have $\phi(v_3(x)) = 20$, $\phi(v_{y1}) = 17$ and $\phi(v_{x1}) = 18$.

**Case 4.1.2.** One of a vertex in $N(v_3(x))$ has weight 15, say $v_{y1}$. Recolor edge $v_3(x)v_{y1}$ with 2. Then $\phi(v_3(x)) = 14$ and $\phi(v_{y1}) = 16$.

**Case 4.1.3.** One of a vertex in $N(v_3(x))$ has weight 9, say $v_{y2}$. Recolor edge $v_3(x)v_{y2}$ with 2. Then $\phi(v_3(x)) = 14$ and $\phi(v_{y2}) = 10$.

**Case 4.1.4.** One of a vertex in $N(v_3(x))$ has weight 11, say $v_{y3}$. Recolor edge $v_3(x)v_{y3}$ with 3. Then $\phi(v_3(x)) = 15$ and $\phi(v_{y3}) = 13$. If there is a neighbor vertex (say $v_{x1}$) of $v_{y3}$ having weight 13 and a neighbor vertex of $v_{x1}$ having weight 11, then recolor $v_3(x)v_{y3}$ and $v_{y3}v_{x1}$ with 2, and we have $\phi(v_3(x)) = \phi(v_{x1}) = 14$ and $\phi(v_{y3}) = 13$. If Case 4.1.4 and Case 4.1.2 appear at the same time, use the method of Case 4.1.2.

**Case 4.2.** $\phi(v_3(x)) = 15$. Let $N(v_3(x)) = \{v'_{y1}, v'_{y2}, v'_{y3}\}$. Then $H$ contains a 2-vertex (vertex of degree 2) with weight 15 in $N(v_3(x))$, say $v'_{y1}$. Fig. 1: Edges labelled by solid line are dominated.
Case 4.2.1. If \( v_{y_2}' \) and \( v_{y_3}' \) have the same weight 15, recolor \( v_3(x)v_{y_2}' \) and \( v_3(x)v_{y_3}' \) with 2, then \( \phi(v_3(x)) = 17 \) and \( \phi(v_{y_2}') = \phi(v_{y_3}') = 16 \).

Case 4.2.2. If one of \( \{v_{y_2}', v_{y_3}'\} \) has weight 9, assume that \( \phi(v_{y_2}') = 9 \). Recolor edge \( v_3(x)v_{y_2}' \) with 2, then \( \phi(v_3(x)) = 16 \) and \( \phi(v_{y_2}') = 10 \).

Case 4.2.3. If one of \( \{v_{y_2}', v_{y_3}'\} \) has weight 11, assume that \( \phi(v_{y_2}') = 11 \). Recolor edge \( v_3(x)v_{y_2}' \) with 3, then \( \phi(v_3(x)) = 17 \) and \( \phi(v_{y_2}') = 13 \).

Case 4.2.4. If one of \( \{v_{y_2}', v_{y_3}'\} \) has weight 13, assume that \( \phi(v_{y_2}') = 13 \), then two cases appear as follows:

(i) \( d_H(v_{y_2}') = 3 \). Recolor edge \( v_3(x)v_{y_2}' \) with 3, then \( \phi(v_3(x)) = 17 \) and \( \phi(v_{y_2}') = 15 \).

(ii) \( d_H(v_{y_2}') = 2 \) and \( \phi(v_{y_3}') = 15 \). Recolor edge \( v_3(x)v_{y_2}' \) and vertex \( v_3(x) \) with 3, recolor \( v_3(x)v_{y_1}' \) and \( v_3(x)v_{y_3}' \) with 2, then \( \phi(v_3(x)) = 20 \) and \( \phi(v_{y_2}') = 16 \), \( \phi(v_{y_3}') = 17 \). If \( v_{y_3}' \) is adjacent to a vertex having weight 18 in \( X \), recolor \( v_3(x)v_{y_3}' \) with 1, then \( \phi(v_3(x)) = 19 \) and \( \phi(v_{y_3}') = 17 \). If \( v_{y_3}' \) is adjacent to a vertex having weight 17 in \( Y \), by our coloring rule, a vertex \( v_0 \) with weight 17 in \( Y \) must have an adjacent vertex \( v_0' \) with its weight not equal to 18 in \( X \). Then recolor edge \( v_0v_0' \) with 2. It deduces that \( \phi(v_0) = 16 \) and \( \phi(v_0') \) reduces 1 than before.

Case 4.2.5. If \( v_{y_2}' \) and \( v_{y_3}' \) have the same weight 13 and \( d_H(v_{y_2}') = d_H(v_{y_3}') = 2 \), we recolor edges \( v_3(x)v_{y_2}' \) and \( v_3(x)v_{y_3}' \) with 3, recolor edge \( v_3(x)v_{y_1}' \) and vertex \( v_3(x) \) with 2 and 3, respectively, then \( \phi(v_3(x)) = 20 \), \( \phi(v_{y_1}') = 16 \) and \( \phi(v_{y_3}') = \phi(v_{y_3}') = 17 \). If one of \( \{v_{y_2}', v_{y_3}'\} \) (say \( v_{y_2}' \)) is adjacent to a vertex \( v_{x_k}' \) having weight 17 in \( X \), then recolor edges \( v_3(x)v_{y_2}' \) and \( v_{y_2}'v_{x_k}' \) with 2, and it follows that \( \phi(v_3(x)) = 19 \), \( \phi(v_{y_2}') = 17 \) and \( \phi(v_{x_k}') = 18 \). □

4 Several types of graphs with \( \Delta \geq 4 \)

Proposition 8. Let \( S_n = K_{1,n-1} \) be a star of order \( n \). Then \( \text{fgnd}_2(S_n) = 1 \).

Proof This conclusion is easily proved by using 1 to color all vertices and edges of \( S_n \). □

Proposition 9. For any complete bipartite graph \( K_{m,n} \), \( \text{fgnd}_2(K_{m,n}) = 1 \) if \( m \neq n \) and \( \text{fgnd}_2(K_{m,n}) = 2 \) if \( m = n \).

Proof Suppose that \( K_{m,n} = (X,Y,E) \) is a complete bipartite graph with bipartition classes \( X \) and \( Y \). Let \( |X| = m \) and \( |Y| = n \). If \( m \neq n \), then use 1 to color all vertices and edges of \( K_{m,n} \), and it follows that \( \phi(x) = 2n+1 \neq 2m+1 = \phi(y) \), where \( x \in X, y \in Y \). For \( m = n \), if we use 1 to color all vertices and edges of \( K_{m,n} \), then \( \phi(x) = 2n+1 = 2m+1 = \phi(y) \) for any edge \( xy \in K_{m,n} \), a contradiction, which deduces that \( \text{fgnd}_2(K_{m,n}) \geq 2 \). We define a total 2-coloring of \( f \) of \( K_{n,n} \) as follows: using 2 to color each vertex of \( Y \) and the remaining vertices and edges are colored by 1. Then we have \( \phi(x) = 3n+1 \neq 2n+2 = \phi(y) \) for any edge \( xy \in K_{n,n} \). Namely, \( f \) is an NFSD-total 2-coloring of \( K_{m,n} \).

Theorem 10. For any complete graph \( K_n \) (\( n \geq 3 \)), \( \text{fgnd}_2(K_n) = 3 \).

Proof It is well known that all vertices are neighbors in \( K_n \), so the neighbor full sum distinguishing total coloring is actually a neighbor sum distinguishing edge coloring of \( K_n \). We need only to consider a neighbor sum distinguishing edge coloring of \( K_n \). Suppose that \( f \) is a neighbor sum distinguishing edge 2-coloring of \( K_n \) and all vertices of \( K_n \) are colored by 1. For each vertex of \( K_n \), its \( (\Delta - 1) \) incident edges are colored by 1 and 2, there exists two vertices \( u \) and \( v \) such that all incident edges of \( u \) are colored by 1 and all incident edges of \( u \) are colored by 2, a contradiction. Therefore, \( \text{fgnd}_2(K_n) \geq 3 \). We offer a method to give a neighbor full sum distinguishing total 3-coloring of \( K_n \).

Let \( f_3 \) be the total coloring of \( K_3 \) defined as follows: \( f_3(x_i) = 1 \) for \( x_i \in V(K_3), i \in \{1,2,3\} \),
Theorem 12. For any tree, (i) $\text{fgndi}_T(T) = 1$ if $T$ contains no adjacent $d$-vertices; and (ii)
Theorem 13. \( \text{fgndi}_\Sigma (T) = 2 \) if \( T \) contains adjacent \( d \)-vertices.

**Proof** By Lemma 3, conclusion (i) is obvious. Next we consider the case that \( T \) has adjacent \( d \)-vertices. The proof is by induction on order \( n \). By Proposition 8, the theorem is trivial if \( T \) is a star \( S_n \), hence, in particular, for every tree of order \( n = 3 \).

Suppose that our assertion is true for all trees of order \( n - 1 \) \((n \geq 4)\) and let \( T \) be a tree of order \( n \). We may assume that \( T \) is not isomorphic to \( S_n \). Let \( x \) be an end vertex of a longest path \( P = xyz \ldots \) in \( T \) and let \( T' \) denote the tree \( T - \{ x \} \). By the choice of \( x \) and \( T \), \( z \) is the only neighbor of \( y \) having the degree \( \geq 2 \) in \( T \). Let \( d_T(t) \) for any vertex \( t \in V(T') \). The degree in \( T' \) of any vertex \( t \) in \( T' \) is the same as in \( T \), except for \( t = y \) for which \( d_T(y) = 2 \).

By induction hypothesis, there is an NSFSD-total 2-coloring \( f' \) of \( T' \). We will color the edge \( xy \) and the vertex \( x \) by \( a \) and \( b \), resp., \( a, b \in \{ 1, 2 \} \), so that the new coloring \( f \) of \( T \) defined as follows:

\[
 f(\theta) = \begin{cases} 
 f'(\theta) & \text{if } \theta \in V(T') \cup E(T'), \\
 a & \text{if } \theta = xy, \\
 b & \text{if } \theta = x,
\end{cases}
\]

would be an NSFSD-total 2-coloring of \( T \). We prove that this is always possible. Let \( \phi'(v) \) denote the expanded sum at \( v \in V(T') \) with respect to the coloring \( f' \).

Suppose now that the degree \( d_T(y) \) of \( y \) in \( T \) is at least 3 and observe that for any total 2-coloring \( f \) of \( T \) and for any \( t \in N_T(y) - \{ z \} \) we have \( \phi(t) = f(t) + f(y) + f(yt) \leq 6 \) and \( \phi(y) \geq 7 \), so the vertices \( t \) and \( y \) are distinguished. Therefore, we can choose \( a \) and \( b \) such that \( \phi(z) = \phi'(z) \neq \phi'(y) + a + b = \phi(y) \) and the new total coloring \( f \) of \( T \) will distinguish all vertices of \( T \).

If \( d_T(y) = 2 \), we can also choose \( a \) and \( b \) such that \( \phi(x) = a + f'(y) \neq \phi'(y) + a + b = \phi(y) \) and \( \phi(z) = \phi'(z) \neq \phi'(y) + a + b = \phi(y) \), so the total coloring \( f \) distinguishes all adjacent vertices of \( T \).

**Theorem 13.** For any hypercube \( Q_n \) \((n \geq 3)\), \( \text{fgndi}_\Sigma (Q_n) = 2 \).

**Proof** Observe that hypercube \( Q_n \) is a \( n \)-regular graph. By Lemma 3, \( \text{fgndi}_\Sigma (Q_n) \geq 2 \). Next we will prove that \( Q_n \) has an NSFSD-total 2-coloring.

For \( n = 3 \) let

\[
 V(Q_3) = \{ u_i^{(1)}, v_i^{(1)} \mid i \in [1, 4] \},
\]

\[
 E(Q_3) = \{ u_i^{(1)} u_{i+1}^{(1)}, v_i^{(1)} v_{i+1}^{(1)}, u_i^{(1)} v_i^{(1)} \mid i \in [1, 4] \},
\]

where subscripts are taken modulo 4.

Let \( f_3 \) be a total coloring of \( Q_3 \), and \( f_3 \) is defined as follows:

\[
 f_3(u_1^{(1)} u_2^{(1)}) = 2, f_3(u_3^{(1)} u_4^{(1)}) = 2,
\]

\[
 f_3(u_2^{(1)} v_2^{(1)}) = 2, f_3(u_4^{(1)} v_4^{(1)}) = 2,
\]

\[
 f(\alpha) = 1, \alpha \in Q_3 - \{ u_1^{(1)} u_2^{(1)}, u_3^{(1)} u_4^{(1)}, u_2^{(1)} v_2^{(1)}, u_4^{(1)} v_4^{(1)} \}.
\]

Then \( f_3 \) is an NSFSD-total 2-coloring of \( Q_3 \).

Suppose that \( \overline{Q_{n-1}} \) is a copy of \( Q_{n-1} \). Let

\[
 V(Q_{n-1}) = \{ u_i^{(k)}, u_2^{(k)}, u_3^{(k)}, u_4^{(k)}, v_1^{(k)}, v_2^{(k)}, v_3^{(k)}, v_4^{(k)} \mid k \in [1, 2^{n-4}] \},
\]

\[
 V(\overline{Q_{n-1}}) = \{ u_i^{(k)}, u_2^{(k)}, u_3^{(k)}, u_4^{(k)}, v_1^{(k)}, v_2^{(k)}, v_3^{(k)}, v_4^{(k)} \mid k \in [1, 2^{n-4}] \}.
\]

Observe that each \( Q_{n-1} \) and \( \overline{Q_{n-1}} \) include \( 2^{n-4} \) numbers of \( Q_3 \), respectively. Meanwhile, \( Q_n \) is constructed in the following procedure, \( Q_n = Q_{n-1} \cup \overline{Q_{n-1}} \cup \{ u_i^{(k)} u_i^{(k)}, v_i^{(k)} v_i^{(k)} \mid i \in [1, 4], k \in [1, 2^{n-4}] \} \).
Let \( Q_n \) be a total coloring of \( Q_n \) and be defined recursively by \( f \) as follows:

\[
f(e) = 1, \quad i \in [1, 2^{n-1}].
\]

The total coloring of \( Q_{n-1} \) is obtained by exchanging the color 2 of each \( Q_3 \) in \( Q_{n-1} \) (see Fig.3) and use 1 to color the remaining vertices and edges. □

![Fig.3: Diagram of the exchanging color of a cube.](image)

**Theorem 14.** Let \( G = (X, Y, E) \) be a bipartite graph with bipartition classes \( X \) and \( Y \). Then \( f_{\text{ndi}}(G) \leq 3 \).

**Proof** Let \( G = (X, Y, E) \) be a connected bipartite graph with bipartition classes \( X \) and \( Y \). If \( G \) is a star, then by Proposition 3, the conclusion holds. Now we consider the case that \(|X| \geq 2 \) and \(|Y| \geq 2 \). We define a non-proper total coloring \( f \) of \( G \) with the following properties: (1) \( f(x) = 1 \) for any vertex \( x \) of \( X \); (2) \( f(y) = 2 \) for any vertex \( y \) of \( Y \); (3) Edges between \( X \) and \( Y \) are colored by 1. For an edge \( xy \) and \( x \in X, y \in Y \), it deduces that \( \phi(x) = 3d_G(x) + 1 \) and \( \phi(y) = 2d_G(y) + 2 \). Hence there may appear a case that \( \phi(x) = \phi(y) = 6k + 4 \) if \( d_G(x) = 2k + 1 \) and \( d_G(y) = 3k + 1 \), where \( k \) is a positive integer. Let \( N(y) = \{x, v_{x_1}, v_{x_2}, \ldots, v_{x_{2k}}\} \). Then two cases appear as follows:

**Case 1.** Not all vertices in \( N(y) \) have the same weight \( 6k + 4 \). Let \( v_{x_p} \) be the vertex whose weight is not equal to \( 6k + 4 \).

If \( \phi(v_{x_p}) \) is even, recolor \( v_{y}v_{x_p} \) with 2, then the weights of vertices of \( G \) keep unchanged as before except for vertices \( y \) and \( v_{x_p} \). Let \( \phi'(y) \) and \( \phi'(v_{x_p}) \) be the new weights of \( y \) and \( v_{x_p} \), respectively. Then \( \phi'(y) = \phi(y) + 1 = 6k + 5 \) and \( \phi'(v_{x_p}) = \phi(v_{x_p}) + 1 = 3d_G(v_{x_p}) + 2 \). Thus the weights of \( y \) and \( v_{x_p} \) are natural distinct. It is easy to verify that \( \phi'(v_{x_p}) = 3d_G(v_{x_p}) + 2 \neq 2d_G(v_{y_0}) + 2 = \phi(y_0), \ y_0 \in N(v_{x_p}) \setminus \{y\} \). By our coloring rule, for two distinct vertices \( v_{x_i} \) and \( v_{x_j} \), \( |\phi(v_{x_i}) - \phi(v_{x_j})| = 6 \), so \( \phi'(y) \neq \phi(v_{x_0}), v_{x_0} \in N(y) \setminus \{v_{x_p}\} \).

If \( \phi(v_{x_p}) \) is odd, recolor \( v_{y_0}v_{x_p} \) with 3, then \( \phi'(y) = \phi(y) + 2 = 6k + 6 \) and \( \phi'(v_{x_p}) = \phi(v_{x_p}) + 2 = 3d_G(v_{x_p}) + 3 \). Thus the weights of \( y \) and \( v_{x_p} \) are natural distinct. It is easy to verify that \( \phi'(v_{x_p}) = 3d_G(v_{x_p}) + 3 \neq 2d_G(v_{y_0}) + 2 = \phi(y_0), y_0 \in N(v_{x_p}) \setminus \{y\} \). Similar to the above discussion, \( |\phi(v_{x_i}) - \phi(v_{x_j})| = 6 \) for two distinct vertices \( v_{x_i} \) and \( v_{x_j} \), so \( \phi'(y) \neq \phi(v_{x_0}), v_{x_0} \in N(y) \setminus \{v_{x_p}\} \).

**Case 2.** All vertices in \( N(y) \) have the same weight \( 6k + 4 \).

Recolor edges \( xy \) and \( yv_{x_p} (1 \leq p \leq 3k) \) with 2 and recolor vertex \( y \) with 1. Then the weight of vertices in \( N(y) \) keep the same as before and the weight of \( y \) is added to \( d_G(y) - 1 \) than before. Therefore, the weights between vertex \( y \) and its neighbors are distinguished. □

From Theorem 14, the following two results are obvious.

**Corollary 15.** Let \( G = (X, Y, E) \) be a bipartite graph with bipartition classes \( X \) and \( Y \) such that the degree of all vertices of \( X \) are even. Then \( f_{\text{ndi}}(G) \leq 2 \).
Corollary 16. Let $G$ be a bipartite graph with $\Delta = 3$. Then $fgndi_{\Sigma}(G) \leq 2$.

5 Future Works

Problem 1. Whether $fgndi_{\Sigma}(G) \leq 3$ holds for every connected graph $G$ with $\Delta = 3$ ?

Problem 2. Let $K_{n_1,n_2,...,n_r}$ be a complete $r$-partite graph with $r$ vertex sets $X_i$ ($i \in [1,r]$) and $|X_i| = n_i$, $\sum_{i=1}^{r} n_i = n$. Besides (i) and (ii) in Theorem 8, $fgndi_{\Sigma}(K_{n_1,n_2,...,n_r}) \leq 3$ ?

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