Interference in the radiation of two point-like charges

Yurij Yaremko*

Institute for Condensed Matter Physics, 1 Svientsitskii St., 79011 Lviv, Ukraine

Abstract

Energy-momentum and angular momentum carried by electromagnetic field of two point-like charged particles arbitrarily moving in flat spacetime are presented. Apart from usual contributions to the Noether quantities produced separately by particles 1 and 2, the conservation laws contain also joint contribution due to the fields of both particles. The mixed part of Maxwell energy-momentum density is decomposed into bound and radiative components which are separately conserved off the world lines of particles. The former describes the deformation of electromagnetic clouds of “bare” charges due to mutual interaction while the latter defines the radiation which escapes to infinity. The bound terms contribute to particles’ individual 4-momenta while the radiative ones exert the radiation reaction. Analysis of energy-momentum and angular momentum balance equations results the Lorentz-Dirac equation as an equation of motion for a pointed charge under the influence of its own electromagnetic field as well as field produced by another charge.

PACS numbers: 03.50.De, 11.10.Gh, 11.30.Cp

1 Introduction

The most natural and widely accepted equation of motion for a charge when radiation reaction is taken into account is the Lorentz-Dirac equation [1]. This equation has been discussed mainly for the case of one charge in an external electromagnetic field. In the present paper we consider an isolated system of two point electric charges and their electromagnetic field. We study the electromagnetic energy-momentum and angular momentum radiated by charges; the study of energy-momentum and angular momentum balance equations implies the Lorentz-Dirac equation for more than one charge.

*Electronic mail: yar@ph.icmp.lviv.ua
The dynamics of the entire system is governed by the action

\[ I = \sum_{a=1}^{2} \left( -m_a \int \sqrt{-\dot{z}_a^2} + e_a \int ds_a A_{a,\mu} \dot{z}_a^\mu \right) - \frac{1}{16\pi} \int \! d^4y f_{\mu\nu} f^{\mu\nu}, \]  

(1.1)

where \( f_{\mu\nu} = \sum_a (\partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu}) \) is the total field generated by two charges. Charge \( e_a \) moves on a world line \( \zeta_a \in M_4 \) described by functions \( z_a^\mu(s_a) \) which give the particle’s coordinates as functions of proper time \( s_a \); \( \dot{z}_a^\mu := dz_a^\mu/ds_a \) is the \( a \)-th four-velocity.

The action is invariant under the space-time translations and rotations which constitute the Poincaré group. This immediately implies conserved quantities which place stringent requirements on the dynamics of the system. They demand that the change in electromagnetic field momentum \[ p^\mu_{\text{em}} = \int_\Sigma \! d\sigma_\alpha T^{\alpha\mu}, \]  

(1.2)

and angular momentum \[ M^{\mu\nu}_{\text{em}} = \int_\Sigma \! d\sigma_\alpha (y^\mu T^{\alpha\nu} - y^\nu T^{\alpha\mu}), \]  

(1.3)

should be balanced by a corresponding change in the total momentum and total angular momentum of the particles. (By \( d\sigma_\alpha \) we denote the vectorial surface element on a space-like hypersurface \( \Sigma \).)

Since the Maxwell energy-momentum tensor density,

\[ 4\pi T^{\mu\nu} = f^{\mu\lambda}_\Lambda f^\nu_\Lambda - 1/4\eta^{\mu\nu} f^{\kappa\lambda}_\Lambda f_{\kappa\lambda}, \]  

(1.4)

is quadratic in the field and this field satisfies the superposition principle, the total electromagnetic field stress-energy tensor is

\[ T^{\mu\nu} = T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + T^{\mu\nu}_{\text{int}}, \]  

(1.5)

where \( a \)-th particle density \( T^{\mu\nu}_{(a)} \) is given by the expression (1.4) where “total” field strengths \( f^{\mu\nu} \) are substituted by “individual” ones \( f^{\mu\nu}_{(a)} \). The mixed term

\[ 4\pi T^{\mu\nu}_{\text{int}} = f^{\mu\lambda}_1 f^\nu_2 f^{\mu\lambda}_2 f^\nu_1 - 1/4\eta^{\mu\nu} \left( f^{\kappa\lambda}_1 f^\nu_2 f^{\mu\lambda}_2 f^\nu_1 + f^{\kappa\lambda}_2 f^\nu_1 f^{\mu\lambda}_1 f^\nu_2 \right) \]  

(1.6)

describes the joint contribution due to both fields.

In this paper we study radiation produced by an isolated system of two point electric charges and their electromagnetic field. Outgoing electromagnetic waves remove energy, momentum, and angular momentum from the sources which then undergo radiation reaction. The verification of conservation laws is not a trivial matter, since the interference contribution (1.8) involves divergent terms.

In the derivation of particle’s equation of motion, Dirac [1] evaluated the flux of electromagnetic energy-momentum over a narrow world tube surrounding the particle’s
world line. The author substituted the components of the retarded Liénard-Wiechert field in the stress-energy tensor (1.4) inside the narrow tube. In 1970 Teitelboim [3] splits the “retarded” stress-energy tensor into the “bound” and “emitted” parts which are separately conserved off the world line of the particle. The author calculates the flow of energy-momentum out of the portion of Bhabha world tube [4] bounded by tilted spacelike hypersurfaces which are orthogonal to particle’s four-velocity at instants $\tau$ and $\tau + d\tau$, respectively. Bound part, $\hat{T}_{\text{bnd}}$, describes a rigid electromagnetic “cloud” which are permanently attached to the source and carried along with it. “Bare” charge and “cloud” constitute new entity: dressed charged particle. $\hat{T}_{\text{bnd}}$ contributes into particle’s inertia: 4-momentum of dressed charge contains, apart from usual velocity term, also a term which is proportional to the square of charge $e_a$,

$$p_a^\mu = m_a u_a^\mu - \frac{2e_a^2}{3} a_a^\mu. \quad (1.7)$$

(Coulomb-like infinity stemming from the pointness of “bare” source is absorbed by the rest mass $m_a$ within the renormalization procedure.) Time derivative of the second term in eq.(1.7) is the well-known Schott term which describes a reversible form of emission and absorption of field energy, which never gets far from the point-like source. The radiative part, $\hat{T}_{\text{rad}}$, yields the Larmor relativistic rate of radiated energy-momentum. It detaches itself from the charge and leads an independent existence. This rate together with the Schott term constitutes the Abraham radiation reaction vector. López and Villarroel [5] split the torque of the stress-energy tensor into bound and emitted components which possess analogous properties.

The results can be applied to the first and the second terms in the total electromagnetic field stress-energy tensor (1.5) which describe “individual” radiation contributions due to particles 1 and 2, respectively. The question is what part of the mixed density should be taken instead of (1.6) to describe the radiation which reaches to a very distant sphere?\n
Aguirregabiria and Bel [6] studied the interference part of energy-momentum,

$$p_\text{int}^\nu = \int_\Sigma d\sigma \hat{T}_\text{int}^{\mu \nu}, \quad (1.8)$$

carried by electromagnetic field of two point charges. The authors prove the fundamental theorem that the mixed radiation rate does not depend on the shape of spacelike surface $\Sigma$ which is used to integrate the mixed part (1.6) of the Maxwell energy-momentum tensor density. For a prescribed plane motion of the charges the perturbation scheme is elaborated within the framework of predictive relativistic mechanics [7, 8]. The lowest approximation gives the well-known expression [9, p.214] for the dipole radiation of two point charges moving according to Coulomb’s law. In Ref. [10] the scheme is applied to the angular momentum carried by electromagnetic field of two point charges.\n
In the case of $N$ particles we would merely obtain as an obvious generalization of eq. (1.5) the sum of $N$ one-particle terms and the mixed contributions corresponding to $N(N-1)/2$ pairs of charges. Hence, the interference component dominates in the radiation from a bunch of identical charged particles (e.g., in free electron lasers). To evaluate the radiation of a relativistic $N$-body system, Klepikov [11] defines the center of a system of
radiation events which allows to synchronize the instants at which electromagnetic waves emitted by different charges combine on a very distant sphere. Fourier analysis is applied to calculate the time and angular distributions of energy-momentum flux. The radiation of a bunch of charged particles moving in a uniform magnetic field is considered in detail.

The only exact solution is obtained by Rivera and Villarroel Ref. [12, eq.(3.27)]. The authors calculate the rate of radiation (including interference part) generated by two identical point charges rotating uniformly at opposite ends of a diameter, in a fixed circle. External fields which govern the strictly prescribed motions are constructed. The rate of radiation is evaluated via the retarded Liénard-Wiechert fields produced by the charges. Further [13] more general case of circular motion of two unlike charges in two coplanar and concentric circumferences is considered.

Note that nine years before the radiation by a system of two uniformly circling charges has been evaluated by Hnizdo [14] (see also discussion [15, 16]). The author concludes that “the power radiated by such a system equals exactly the rate at which work is done on the system by external force”.

In this paper we study the interference part of energy-momentum and angular momentum of electromagnetic field generated by two arbitrarily moving charges\(^1\). We restrict ourselves to the retarded Liénard-Wiechert solutions; the advanced ones are rejected on the grounds of causality. In Section 2 we introduce coordinate system which makes relatively easy the calculation of the covariant 4-momentum radiated by interacting charges. The calculation is performed in Sections 3 and 4. We reveal divergence-free radiative part of the mixed density (1.6). It determines the radiation that escapes to infinity while the (short-range) bound part modifies individual 4-momenta (1.7) of dressed particles. The mixed part of radiated energy-momentum depends only on velocities, accelerations and the relative 4-position of the charges. In Section 5 we derive equations of motion of interacting charged particles. Analysis of energy-momentum and angular momentum balance equations results the well-known Lorentz-Dirac equation. In Section 6 we study symmetry properties of radiative energy-momentum and angular momentum which rely on invariance of action (1.1) under inversions of space and time axes. In Section 7 we discuss the results and implications.

2 “Interference” coordinate system

To perform the surface integration (1.8) of interference stress-energy tensor, an appropriate coordinate system is necessary. Such a coordinate system is introduced in Ref. [6]. It involves the evolution parameter \( \lambda \) associated with an inertial observer; the surface of integration is a surface of constant \( \lambda \). In Refs. [17] and [18] this coordinate system is adapted to the simplest hyperplane \( \Sigma_t = \{ y \in \mathbb{M}_4 : y^0 = t \} \) associated with an unmoving inertial observer. The “laboratory” time \( t \) is a single common parameter defined along all the world lines of the system.

\(^1\)The generalization of this work to \( N \) charges is an obvious one.
Figure 1: The interference picture in a plane $\Sigma_t$. Sphere $S_a$ with radius $k^0_a = t - t_a$ is centered at point $O_a$ with coordinates $z^i_a(t_a)$, $i' = 1, 2, 3$. Angle $\varphi$ distinguishes the points of intersection $S_1 \cap S_2 = C(O, h)$ which constitute support of integrals (2.1) and (2.2). $k^0_a$, $k^3_a$, and $h$ are the components of the future oriented null 4-vector $k^\alpha_a' = \Omega^\alpha a(y^\alpha - z^\alpha_a(t_a))$. Matrix $\hat{\Omega}$ determines transition to “momentarily rotating” Lorentz frame.

The mixed contributions to energy-momentum,

$$p^\nu_{\text{int}}(t) = \int_{\Sigma_t} \sigma_0 T^0\nu_{\text{int}},$$

and angular momentum,

$$M^\mu\nu_{\text{int}}(t) = \int_{\Sigma_t} \sigma_0 (y^\mu T^0\nu_{\text{int}} - y^\nu T^0\mu_{\text{int}}),$$

are due to interference of spherical wave fronts $S_1$ and $S_2$ in $\Sigma_t$ (see Fig. 1). Sphere,

$$S_a(z_a(t_a), t - t_a) = \{ y \in \mathbb{M}_4 : (y^0 - t_a)^2 = \sum_i (y^i - z^i_a(t_a))^2, y^0 = t, t - t_a > 0 \},$$

is the intersection of the future light cone with vertex at point $z_a(t_a) \in \zeta_a$ and $\Sigma_t$. This contribution is zero if the relative position 4-vector $q = z_1 - z_2$ is timelike. If $q$ is spacelike, the intersection $S_1 \cap S_2$ is the circle $C(O, h)$ with radius $h$; in “momentarily rotating” Lorentz frame the circle $C(O, h)$ lies in $Oxy$ plane and centered at the coordinate origin (see Fig. 1). If points $z_1$ and $z_2$ are related by a null ray, the intersection $S_1 \cap S_2$ contains the only point.

2.1 Local map

To find the local expressions for coordinate transformation $(y^\alpha) \mapsto (t, t_1, t_2, \varphi)$, we translate the origin of the laboratory Lorentz frame at the center $O$ of the circle $C(O, h) =$
Figure 2: The past light cone with vertex at point \( y \in \Sigma_t \) is punctured by the world lines of the 1-st particle and the 2-nd particle at points \( z_1(t_1) \) and \( z_2(t_2) \), respectively. The vector \( K_a \) is a null vector pointing from the emission point \( z_a(t_a) = (t_a, z^i_a(t_a)) \) to a field point \( y \). The relative position 4-vector \( q = z_1 - z_2 \) is equal to difference \( K_2 - K_1 \); its square \( (q \cdot q) = -2(K_2 \cdot K_1) \).

\[ S_1 \cap S_2 \] and then rotate space axes till a new \( z \)-axis be directed along 3-vector \( q := z_1 - z_2, \)

\[ g^a = z^a(t_a) + \Omega^a_{a'}(t_1, t_2)k^a_{a'}. \quad (2.4) \]

Here \( k_a, a = 1, 2 \) is the future oriented null-vector with components

\[ k^0_a = t - t_a, \quad k^1_a = h \sin \varphi, \quad k^2_a = h \cos \varphi, \quad k^3_a = (-1)^a \frac{q^1}{2} + \frac{(k^0_2)^2 - (k^0_1)^2}{2q}, \quad (2.5) \]

which arise from analysis of triangle \( O_1O_2H \) pictured in Fig. 1 (we denote \( q = |q| \)). Matrix space-time components are \( \Omega_{0\mu} = \Omega_{\mu 0} = \delta_{\mu 0} \). Its space components \( \Omega_{ij} \) constitute an orthogonal \( 3 \times 3 \) matrix (A.6) which determines the rotation described above (see Appendix A). It defines new orthonormal basis,

\[ n_\vartheta = \cos \varphi_q \cos \vartheta_q e_1 + \sin \varphi_q \cos \vartheta_q e_2 - \sin \vartheta_q e_3, \]

\[ n_\varphi = -\sin \varphi_q e_1 + \cos \varphi_q e_2, \]

\[ n_q = \cos \varphi_q \sin \vartheta_q e_1 + \sin \varphi_q \sin \vartheta_q e_2 + \cos \vartheta_q e_3, \quad (2.6) \]

which is constructed from components of the relative position 3-vector \( q \), e.g. \( \cos \varphi_q = q^1/\sqrt{(q^2)^2 + (q^1)^2}, \cos \vartheta_q = q^3/q \).

To find the Jacobian of coordinate transformation (2.4), we derive its differential chart. Setting \( \alpha = 0 \) in eq.(2.4) immediately follows \( y^0 = t \). Since \( t = y^0 \), then \( \partial t / \partial y^\alpha = \delta_{0\alpha} \).
Because \( y \) and \( z_a(t_a) \) lie on the light cone (see Fig. 2), a change field point \( y \) comes with a change in \( t_a \). Suppose that \( y \) is displaced to the new point \( y + \delta y \). The new intersection of the past light cone of this vertex with the \( a \)-th world line is then \( z_a(t_a + \delta t_a) \). These points are still related by the equation

\[
(y^0 + \delta y^0 - t_a - \delta t_a)^2 = \sum_i (y^i + \delta y^i - z^i_a(t_a + \delta t_a))^2. \tag{2.7}
\]

Expanding this to the first order in \( \delta y \) and \( \delta t_a \) and using the cone equation (2.3), we obtain

\[
\frac{\partial t_a}{\partial y^\alpha} = -\frac{K_{a,\alpha}}{r_a} = -\frac{\Omega_{\alpha\alpha'}k_{\alpha'}^a}{r_a}. \tag{2.8}
\]

Here \( K_a \) is \( a \)-th null vector pictured in figure 2 and symbol \( r_a \) denotes the scalar product \((v_a \cdot K_a)\), taken with opposite sign; noncovariant 4-velocity \( v_a := (1, \frac{dz^i_a}{dt_a}) \).

For the angular variable we have

\[
\frac{\partial \varphi}{\partial y^\alpha} = \Omega_{\alpha\alpha'}k_{\alpha'}^\varphi, \tag{2.9}
\]

where

\[
k_\varphi^0 = 0, \quad k_\varphi^1 = \frac{\cos \varphi}{h}, \quad k_\varphi^2 = -\frac{\sin \varphi}{h}, \quad k_\varphi^3 = 0. \tag{2.10}
\]

Recall that \( h \) is the radius of the circle \( C(O, h) = S_1 \cap S_2 \) pictured in Fig. 1.

Determinant of the matrix which defines this differential chart gives the inverse Jacobian: \( J^{-1} = q/(r_1r_2) \). The “interference” surface element,

\[
d\sigma_0 = \frac{r_1r_2}{q} dt_1 dt_2 d\varphi, \tag{2.11}
\]

is ill defined if and only if the particles are very close to each other.

### 2.2 Global mapping

Setting \( a = 1 \) in eq.(2.4) we obtain the coordinate system centered on an accelerated world line of the first particle. The flat spacetime \( \mathbb{M}_4 \) is a disjoint union of hyperplanes \( \Sigma_t = \{y \in \mathbb{M}_4 : y^0 = t \} \). An interference hyperplane \( \Sigma_t \) is a disjoint union of retarded spheres \( S_1(O_1, t - t_1) \) centered at points \( O_1 \in \Sigma_t \) with coordinates \( z_1(t_1) \). A sphere is covered by its intersections with spherical wave fronts of the second source. Each circle \( S_1 \cap S_2 \) can be labelled by the individual time \( t_2 \) of the second particle and each point on a given circle can be labelled by its polar angle \( \varphi \).

Going along the world line \( \zeta_1 \) we arrive unavoidably at the point \( t_1^{ret}(t) \), such that the future light cone of \( z_1[t_1^{ret}(t)] \) touches the 2-nd world line at point \( z_2(t) \in \Sigma_t \) (see Fig. 4). Light cone of upper vertices do not intersect the second world line at all.

In context with the principle of retarded causality, \( \Sigma_t \) is divided into two regions where outgoing waves sourced by charged particles combine in quite different manner.
Figure 3: For a given $t_1$ the wave front $S_1$ is covered by circles $S_1 \cap S_2$ if the parameter $t_2$ increases from $t_2^{ret}(t_1)$ to $t_2^{adv}(t_1)$. Minimal value labels the vertex of forward light cone which is punctured by $\zeta_1$ at $z_1(t_1)$. The largest sphere $S_2^{ret}$ touches the sphere $S_1$ at point $N$. World line $\zeta_2$ punctures the future light cone of $z_1(t_1)$ at point $z_2(t_2^{adv})$. Intersection of $S_1$ and the smallest sphere $S_2^{adv}$ contains the only point $S$.

(i) **Causal**, which is filled up by spheres $S_1$ of radii larger than or equal to $t - t_1^{ret}(t)$.

(ii) **Acausal**, where parameter $t_1$ increases from $t_1^{ret}(t)$ to the instant of observation $t$. It is the ball bounded by sphere of radius $t - t_1^{ret}(t)$ centered at point $z_1[t_1^{ret}(t)]$.

### 2.2.1 Causal region

Causal region is spanned by curvilinear coordinates (2.4) where $t_1$ increases from $-\infty$ to the instant $t_1^{ret}(t)$. To cover the sphere $S_1$ where $t_1$ is fixed we change the parameter $t_2$ which labels point $z_2(t_2) \in \zeta_2$. The starting point is the solution $t_2^{ret}(t_1)$ of algebraic equation $q^0 = q$ or

$$t_1 - t_2 = q(t_1, t_2),$$

where points $z_1(t_1) \in \zeta_1$ and $z_2(t_2^{ret}) \in \zeta_2$ are linked by a null ray. The largest sphere $S_2(O_2^{ret}, t - t_2^{ret})$ touches a given sphere $S_1(O_1, t - t_1)$ at only point N (see Fig. 3). If parameter $t_2$ increases to $t_2^{adv}(t_1)$ being the solution of algebraic equation $q^0 = -q$ or

$$t_2 - t_1 = q(t_1, t_2),$$

the intersection $S_1 \cap S_2^{adv}$ contains the only point $S$. If parameter $t_2$ changes from $t_2^{ret}(t_1)$ to $t_2^{adv}(t_1)$, the sphere $S_1$ is covered by circles $C(O, h) = S_1 \cap S_2$. 
2.2.2 Acausal region

Acausal region of an interference hyperplane $\Sigma_t$ corresponds to the fragments of the world lines which are not related to each other. (By this we mean that the radiation emitted by the first particle during the interval $[t^\text{ret}_1(t_1), t]$ does not come to the second one and vice versa.) Nevertheless, the outgoing waves of these portions of world lines combine in $\Sigma_t$.

Acausal region is filled up by spheres $S_1(O_1, t - t_1)$, where $t_1 \in [t^\text{ret}_1(t), t]$. The sphere $S_1$ with fixed $t_1$ is the disjoint union of circles $C(O_1, h) = S_1 \cap S_2$ if the parameter $t_2$ increases from $t^\text{ret}_2(t_1)$ to $t^\prime_2(t_1)$. The starting point of this interval is still the solution of eq.(2.12) while the maximal value of $t_2$ satisfies the algebraic equation $k_0^2 + k_1^0 = |q|$ or

$$2t - t_1 - t_2 = q(t_1, t_2). \quad (2.14)$$

A given sphere $S_1(O_1, t - t_1)$ touches $S_2(O_2, t - t^\prime_2)$ at only point $S$ (see Fig. 4).

2.2.3 Surface integration

In an analogous way we construct the coordinate system centered on the world line of the second particle. If $t_2 \in ]-\infty, t^\text{ret}_2(t)]$ then $t_1 \in [t^\text{ret}_1(t_2), t^\text{adv}_1(t_2)]$; if $t_2 \in [t^\text{ret}_2(t), t]$ then $t_1 \in [t^\text{ret}_1(t_2), t^\prime_1(t, t_2)]$, $\varphi \in [0, 2\pi[$. The ends of intervals are defined implicitly by algebraic equations (2.12), (2.13), and (2.14).

The surface integration (2.1) and (2.2) can be performed via the coordinate system.
centered on a world line either of the first particle,

\[
\begin{bmatrix}
  t_1^{et}(t) & t_1^{adv}(t_1) \\
  \int_{-\infty}^t dt_1 & \int_{t_1^{et}(t_1)}^t dt_2 + \\
  \int_{-\infty}^t dt_1 & \int_{t_1^{et}(t_1)}^t dt_2
\end{bmatrix}
\int_0^{2\pi} \frac{d\varphi}{q} r_1 r_2,
\]

or of the second particle,

\[
\begin{bmatrix}
  t_2^{et}(t) & t_2^{adv}(t_2) \\
  \int_{-\infty}^t dt_2 & \int_{t_2^{et}(t_2)}^t dt_1 + \\
  \int_{-\infty}^t dt_2 & \int_{t_2^{et}(t_2)}^t dt_1
\end{bmatrix}
\int_0^{2\pi} \frac{d\varphi}{q} r_1 r_2.
\]

To calculate the flows (2.1) of the mixed electromagnetic field energy and momentum which flow across the hyperplane \( \Sigma_t \), we should integrate the Maxwell energy-momentum tensor density (1.6) over angular variable \( \varphi \) and over time variables \( t_1 \) and \( t_2 \).

### 3 Angular integration of energy-momentum and angular momentum tensor densities

In this Section we trace a series of stages in integration of the mixed Maxwell energy-momentum tensor density over \( \varphi \). In Appendix A we derive some useful expressions.

In terms of Minkowski coordinates \((y^\alpha)\) the electromagnetic field generated by \( a \)-th particle is given by

\[
\hat{f}_{(a)} = \frac{e_a}{r_a^2} u_a \wedge k_a + \frac{e_a}{r_a} [a_a \wedge k_a + (k_a \cdot a_a) u_a \wedge k_a],
\]

where symbol \( \wedge \) denotes the wedge product. We use \texttt{sans-serif} symbols for the retarded distance\(^2\),

\[
r_a = -\eta_{\alpha\beta} (y^\alpha - z_a^\alpha(s_a)) u_a^\beta(s_a),
\]

and for the null vector \( K_a = y - z_a(s_a) \) rescaled by a factor \( r_a^{-1} \),

\[
k_a^\alpha = \frac{1}{r_a} (y^\alpha - z_a^\alpha(s_a)).
\]

To express field strengths in terms of curvilinear coordinates \((t, t_1, t_2, \varphi)\), it is advantageous to replace the retarded proper time \( s_a(y) \) by evolution parameter \( t_a \). The components of particles’ 4-velocities \( u_a \) and 4-accelerations \( a_a \), \( a = 1, 2 \), become [2]

\[
u_a^\mu = \gamma_a v_a^\mu(t_a), \quad a_a^\mu = \gamma_a^2 (v_a \cdot \dot{v}_a) v_a^\mu + \gamma_a^2 \dot{v}_a^\mu,
\]

where 4-vectors \( v_a^\mu = (1, v_1^\mu(t_a)), \dot{v}_a^\mu = (0, \dot{v}_1^\mu(t_a)) \) and factor \( \gamma_a := [1 - v_a^2]^{-1/2} \). Substituting these into eq.(3.1) and using the relation \( k_a^\mu = K_a^\mu / r_a \) yields

\[
\hat{f}_{(a)} = e_a \left( \frac{v_a \wedge K_a}{r_a^3} u_a + \frac{\dot{v}_a \wedge K_a}{r_a^2} \right),
\]

\(^2\)Because the speed of light is set to unity, \( r_a \) is equal to the spatial distance between \( z_a[s_a^{et}(y)] \) and \( y \) as measured in momentarily comoving Lorentz frame where \( u_a^\alpha = (1, 0, 0, 0) \).
where

\[ r_a = K_0^a - (K_a \nu_a), \quad c_a = \gamma_a^{-2} + (K_a \nu_a). \quad (3.6) \]

Note that \( r_a \) is the retarded distance (3.2) rescaled by a factor \( \gamma_a \), i.e. \( r_a = \gamma_a^{-1} r_a \). The separation vector \( K_a \) has the form \( \hat{\Omega} k_a \), where components of null vector \( k_a \) are given by eqs.

\( \text{(2.5)} \) and matrix \( \hat{\Omega} \) determines the transition to momentarily comoving Lorentz frame associated with basis \( \text{(2.6)} \).

It is straightforward to substitute the components of electromagnetic fields \( \text{(3.5)} \) in terms of “interference” coordinates \( (t, t_1, t_2, \varphi) \) into integrands of expressions \( \text{(2.1)} \) and \( \text{(2.2)} \) to calculate the interference part of radiated energy-momentum and angular momentum, respectively. Integration of the mixed stress-energy tensor over angular variable is the key to the problem. All \( \varphi \)-dependent terms are concentrated in the following constructions:

\[
D^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a}{q r_1 r_2}, \quad B^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a c_1}{q r_1^2 r_2^2}, \quad (3.7)
\]

\[
C^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a c_1}{q (r_1)^2 r_2}, \quad A^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a c_1 c_2}{q (r_1)^2 (r_2)^2}.
\]

They are labeled according to their dependence on the combination of components of the separation vectors \( K_1 \) and \( K_2 \): factor \( a \) is replaced by \( K_1^\mu K_2^\nu, K_1^\mu, K_2^\nu, \) or 1 for \( D_1^{\mu \nu}, D_1^\mu, D_2^\nu \) or \( D_0 \), respectively. (The others \( B^a, C^a, \) and \( A^a \) are marked analogously.)

The mixed part of the stress-energy tensor \( \text{(1.6)} \) is symmetric in indices 1 and 2. Substituting \( \text{(3.5)} \) into the first term of this expression and using the identities \( K_2 - K_1 = q \) and \( (K_2 \cdot K_1) = -1/2(q \cdot q) \) yields

\[
\frac{1}{4\pi} \int_0^{2\pi} d\varphi \, f_{(1)}^\mu f_{(2)}^\nu = \frac{e_1 e_2}{2} \left\{ T^{\mu \nu}_{12} \left( \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + T^\mu_1 \left( v_2^\nu \frac{\partial \sigma}{\partial t_1} \right) + T^\nu_2 \left( v_1^\mu \frac{\partial \sigma}{\partial t_2} \right) + T^0 (v_1^\mu v_2^\nu \sigma) \right. \\
- C_1^\mu v_2^\nu \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} - D_1^\mu v_2^\nu \frac{\partial^2 \sigma}{\partial t_2^2 \partial t_2} - B_2^\nu v_1^\mu \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} - D_2^\nu v_2^\mu \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \\
- \left. C_0^0 v_1^\mu v_2^\nu \frac{\partial \sigma}{\partial t_1} - D_0 \left( v_1^\mu v_2^\nu \frac{\partial \sigma}{\partial t_2} + v_1^\mu v_2^\nu \frac{\partial \sigma}{\partial t_1} + v_1^\mu v_2^\nu \frac{\partial \sigma}{\partial t_2} \right) \right\}.
\quad (3.8)
\]

after addition of similar terms and integration over \( \varphi \). World function \( \sigma(t_1, t_2) \) of two spacelike related points, \( z_1(t_1) \in \zeta_1 \) and \( z_2(t_2) \in \zeta_2 \), is equal to one-half of the square of vector \( q = z_1 - z_2 \), taken with opposite sign,

\[
\sigma(t_1, t_2) = -1/2(q \cdot q). \quad (3.9)
\]

Each second order differential operator,

\[
\hat{T}^a = D^a \frac{\partial^2}{\partial t_1 \partial t_2} + B^a \frac{\partial}{\partial t_1} + C^a \frac{\partial}{\partial t_2} + A^a, \quad (3.10)
\]
has been labeled according to dependence of coefficients (3.7) on the combination of vectors $K_1$ and $K_2$.

For the convolution $f^{(1)}_{\alpha\beta} f^{(2)}_{\alpha\beta}$, we obtain

$$
\frac{1}{4\pi} \int_0^{2\pi} d\varphi J f^{(1)}_{\alpha\beta} f^{(2)}_{\alpha\beta} = e_1 e_2 T^0(\lambda),
$$

(3.11)

where function

$$
\lambda = \sigma \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} - \frac{\partial \sigma}{\partial t_1} \frac{\partial \sigma}{\partial t_2}
$$

(3.12)

depends on two-point function (3.9) and its derivatives in time variables.

To distinguish the partial derivatives in time variables, we rewrite the operator (3.10) as the sum of the second-order differential operator,

$$
\hat{\Pi}^a = \frac{\partial^2}{\partial t_1 \partial t_2} D^a + \frac{\partial}{\partial t_1} \left( B^a - \frac{\partial D^a}{\partial t_2} \right) + \frac{\partial}{\partial t_2} \left( C^a - \frac{\partial D^a}{\partial t_1} \right),
$$

(3.13)

and the “tail”,

$$
\pi^a = \frac{\partial^2 D^a}{\partial t_1 \partial t_2} - \frac{\partial B^a}{\partial t_1} - \frac{\partial C^a}{\partial t_2} + A^a.
$$

(3.14)

For a smooth function $f(t_1, t_2)$ we have

$$
\hat{T}^a(f) = \hat{\Pi}^a(f) + f \pi^a.
$$

(3.15)

Cumbersome calculations which are presented in Appendix A give the relations

$$
\pi^0 = 0,
$$

(3.16)

$$
\pi^\mu_1 = v_1^\nu \left( B^\nu - \frac{\partial D^\nu}{\partial t_2} \right), \quad \pi^\nu_2 = v_2^\mu \left( C^\mu - \frac{\partial D^\mu}{\partial t_1} \right),
$$

$$
\pi^{\mu\nu}_{12} = v_1^\sigma \left( B^{\nu}_{2} - \frac{\partial D^{\nu}_{2}}{\partial t_2} \right) + v_2^\sigma \left( C^{\mu}_{1} - \frac{\partial D^{\mu}_{1}}{\partial t_1} \right) - v_1^\mu v_2^\nu D^0,
$$

which allow us to rewrite the sum of integrals (3.8) and (3.11) in terms of differential operators $\hat{\Pi}^a$ and partial derivatives in $t_1$ and $t_2$,

$$
\mathcal{P}^{\mu\nu}_{12} = \frac{1}{4\pi} \int_0^{2\pi} d\varphi J \left( f^{\mu\alpha}_{(1)} f^{\nu}_{(2)} - \frac{\eta^{\mu\nu}}{4} f^{\alpha\beta}_{(1)} f^{(2)}_{\alpha\beta} \right),
$$

(3.17)

$$
= \frac{e_1 e_2}{2} \left\{ \hat{\Pi}^\mu_{12} \left( \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \hat{\Pi}^\nu_{12} \left( \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \hat{\Pi}^{\mu\nu}_{12} \left( \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_1} \left( D^{\mu}_{1} v_2^\nu \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \frac{\partial}{\partial t_2} \left( D^{\nu}_{2} v_1^\mu \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_1} \left( D^0 v_1^\mu v_2^\nu \frac{\partial \sigma}{\partial t_2} \right) - \frac{\partial}{\partial t_2} \left( D^0 v_1^\mu v_2^\nu \frac{\partial \sigma}{\partial t_1} \right) - \frac{1}{2} \eta^{\mu\nu} \hat{T}^0(\lambda) \right\}.
$$
We denote $P_{21}^{\mu\nu}$ the integral over $\varphi$ of the remaining terms involved in tensor (1.6). It can be obtained by interchanging of indices 1 and 2.

Setting $\mu = 0$ and $\nu = i$ in eq.(3.17), we obtain the first term of the mixed space-time components of the stress-energy tensor (1.6). We add the term where indices 1 and 2 are interchanged. Since zeroth components $k_1^0$ and $k_2^0$ of the separation four-vectors $K_1$ and $K_2$ do not depend on $\varphi$, the final expression get simplified,

\[
P_{\text{int}}^{i} = P_{12}^{0i} + P_{21}^{0i}
\]

\[
= \frac{e_1 e_2}{2} \left[ \hat{\Pi}_2^i \left( \frac{\partial \lambda_1}{\partial t_2} \right) + \hat{\Pi}_1^i \left( \frac{\partial \lambda_2}{\partial t_1} \right) + \hat{\Pi}_0^0 \left( v_2^i \lambda_1 + v_1^i \lambda_2 \right) - \frac{\partial}{\partial t_1} \left( v_2^i \frac{\partial \lambda_1}{\partial t_2} \right) - \frac{\partial}{\partial t_2} \left( v_1^i \frac{\partial \lambda_2}{\partial t_1} \right) \right],
\]

where

\[
\lambda_1 = k_1^0 \frac{\partial \sigma}{\partial t_1} + \sigma, \quad \lambda_2 = k_2^0 \frac{\partial \sigma}{\partial t_2} + \sigma.
\]

Similarly we derive zeroth component $P_{\text{int}}^{0}$. Setting $\mu = 0$ and $\nu = 0$ in eq.(3.17), we obtain the first one-half of desired expression. The second one, $P_{21}^{00}$, can be derived via interchanging indices 1 and 2. The integral of energy density $T^{00}_{\text{int}}$ over the angular variable has the form

\[
P_{\text{int}}^{0} = P_{12}^{00} + P_{21}^{00}
\]

\[
= \frac{e_1 e_2}{2} \hat{\Pi}_0^0 \left( \Sigma \frac{\partial^2 \Sigma}{\partial t_1 \partial t_2} - \frac{\partial \Sigma}{\partial t_1} \frac{\partial \Sigma}{\partial t_2} \right),
\]

where three-point function,

\[
\Sigma(t, t_1, t_2) = 2k_1^0 k_2^0 + \sigma(t_1, t_2),
\]

depends on particles’ positions referred to the moments $t_1$ and $t_2$ before observation instant $t$ as well as on $t$ itself.

We now turn to the integration of the angular momentum tensor density (2.2) carried by the electromagnetic field due to two pointlike charges. We present the torque $m_{\text{int}}^{\mu\nu} = y^{\mu} T_{\text{int}}^{0\nu} - y^{\nu} T_{\text{int}}^{0\mu}$ in the following form:

\[
m_{\text{int}}^{\mu\nu} = m_{12}^{\mu\nu} + m_{21}^{\mu\nu} - m_{12}^{\nu\mu} - m_{21}^{\nu\mu},
\]

where

\[
m_{12}^{\mu\nu} = \left( z_1^\mu + K_1^{\mu} \right) \frac{1}{4\pi} \left[ f_{(1)}^{\lambda \rho} f_{(2)}^{\nu \rho} - \frac{1}{4} f_{(1)}^{\rho \beta} f_{(2)}^{\nu \alpha} \frac{f_{(2)}^{\alpha \beta}}{4} \right].
\]

It is straightforward to substitute the fields (3.5) into this expression to calculate the first term of expression (3.22). The others can be obtained by interchanging of the pair of indices (1, 2) and ($\mu, \nu$).
Having integrated expression $Jm_{\mu \nu}^{12}$ over $\varphi$ we obtain

$$\mathcal{M}^{\mu \nu}_{12} = \frac{e_1 e_2}{2} \left\{ \frac{\partial \lambda_1}{\partial t_2} + \partial \lambda_1 \right\} + \partial \lambda_1 \left( v_2^{\mu} \lambda_1 \right) - v_2^{\mu} \frac{\partial \lambda_1}{\partial t_2} D_1^{\mu}$$

$$+ \frac{\partial \lambda_1}{\partial t_2} D_1^{\mu} - v_1^{\mu} \frac{\partial \lambda_1}{\partial t_2} B_1^{\mu} - v_1^{\mu} \frac{\partial \lambda_1}{\partial t_2} B_2^{\mu} + \partial \lambda_1 \left( v_2^{\mu} \lambda_1 \right) - v_1^{\mu} v_2^{\mu} \lambda_1 B_0^{\mu}$$

$$- v_1^{\mu} \left( v_2^{\mu} \lambda_1 + v_2^{\mu} \lambda_1 D_0^{\mu} \right) - z_1^{\mu} v_2^{\mu} \frac{\partial \lambda_1}{\partial t_2} C_0^{\mu} - z_1^{\mu} v_2^{\mu} \frac{\partial \lambda_1}{\partial t_2} D_0^{\mu}$$

$$- \frac{\eta^{\lambda \mu}}{2} \left[ \partial \lambda_1 \right]$$

where functions $\lambda$ and $\lambda_a$ are given by eqs. (3.12) and (3.19), respectively.

Usage of the equalities in eq.(3.16) derived in Appendix A allows us to rewrite the integrand (3.24) as follows:

$$\mathcal{M}^{\mu \nu}_{12} = \frac{e_1 e_2}{2} \left\{ \frac{\partial \lambda_1}{\partial t_2} + \partial \lambda_1 \right\} + \partial \lambda_1 \left( v_2^{\mu} \lambda_1 \right) - v_2^{\mu} \frac{\partial \lambda_1}{\partial t_2} D_1^{\mu}$$

$$+ \frac{\partial \lambda_1}{\partial t_2} D_1^{\mu} - v_1^{\mu} \frac{\partial \lambda_1}{\partial t_2} B_1^{\mu} - v_1^{\mu} \frac{\partial \lambda_1}{\partial t_2} B_2^{\mu} + \partial \lambda_1 \left( v_2^{\mu} \lambda_1 \right) - v_1^{\mu} v_2^{\mu} \lambda_1 B_0^{\mu}$$

$$- v_1^{\mu} \left( v_2^{\mu} \lambda_1 + v_2^{\mu} \lambda_1 D_0^{\mu} \right) - z_1^{\mu} v_2^{\mu} \frac{\partial \lambda_1}{\partial t_2} C_0^{\mu} - z_1^{\mu} v_2^{\mu} \frac{\partial \lambda_1}{\partial t_2} D_0^{\mu}$$

$$- \frac{\eta^{\lambda \mu}}{2} \left[ \partial \lambda_1 \right]$$

The other terms of mixed angular momentum,

$$\mathcal{M}^{\mu \nu}_{\text{int}} = \mathcal{M}^{\mu \nu}_{12} + \mathcal{M}^{\mu \nu}_{21} - \mathcal{M}^{\mu \nu}_{12} - \mathcal{M}^{\mu \nu}_{21},$$

can be obtained via interchanging of indices $(1, 2)$ and $(\mu, \nu)$.

We see that the integration of the mixed stress-energy tensor (1.6) over $\varphi$ yields the combinations of partial derivatives in time variables. In the next Section we classify them and reveal the long-range terms which contribute into radiated energy-momentum.

### 4 Radiative parts of mixed energy-momentum and angular momentum

In previous Section we integrate the mixed part of the stress-energy tensor and its torque over polar angle. Resulted expressions describe contributions to electromagnetic field’s energy-momentum and angular momentum due to interference of spherical wave fronts of charges $e_1$ and $e_2$ placed at fixed points $z_1(t_1) \in \zeta_1$ and $z_2(t_2) \in \zeta_2$, respectively. The crucial issue is that the integrals (3.18), (3.20) and (3.26) have the remarkable property of being the sum of partial derivatives in time variables. This circumstance allows us to calculate how much electromagnetic field’s energy-momentum and angular momentum flow across a hyperplane $\Sigma_t$. 

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It is natural to integrate the expression being the time derivative with respect to \( t_2 \) according to the rule (2.15),

\[
G_2 = \left[ \int_{-\infty}^{t_1^{ret}(t)} \int_{t_2^{adv}(t)}^{t_1^{ret}(t)} dt_1 \right] \frac{\partial G_2(t_1, t_2)}{\partial t_2} \] (4.1)

\[
= \int_{-\infty}^{t_1^{ret}(t)} dt_1 G_2[t_1, t_2^{adv}(t_1)] - \int_{-\infty}^{t} dt_1 G_2[t_1, t_2^{ret}(t_1)] + \int_{t_1^{ret}(t)}^{t} dt_1 G_2[t_1, t_2^{ret}(t_1)].
\]

Having applied the rule (2.16) to the expression of type \( \partial G_1/\partial t_1 \), we obtain

\[
G_1 = \left[ \int_{-\infty}^{t_2^{adv}(t_2)} \int_{t_1^{ret}(t_2)}^{t_2^{adv}(t_2)} dt_2 \right] \frac{\partial G_1(t_1, t_2)}{\partial t_1} \] (4.2)

\[
= \int_{-\infty}^{t_2^{adv}(t_2)} dt_2 G_1[t_1^{adv}(t_2), t_2] - \int_{-\infty}^{t} dt_2 G_1[t_1^{ret}(t_2), t_2] + \int_{t_2^{adv}(t_2)}^{t} dt_2 G_1[t_1^{ret}(t_2), t_2].
\]

The end points are valuable only in the integration procedure. They are solutions of algebraic equations (2.12), (2.13) and (2.14). The retarded instants \( t_a^{ret}(t_b) \) and advanced ones \( t_b^{adv}(t_a) \) label the points \( N \) and \( S \) in which fronts of \emph{outgoing} electromagnetic waves produced by charges touch each other (see Fig. 3). All the moments are \emph{before} the observation instant \( t \), so that the retarded causality is not violated.

It is worth noting that the functions \( t_1^{ret}(t_2) \) and \( t_2^{adv}(t_1) \) are inverted to each other as well as the pair of functions \( t_1^{adv}(t_2) \) and \( t_2^{ret}(t_1) \). For a fixed laboratory time \( t \) the functions \( t_1^{ret}(t, t_2) \) and \( t_1^{ret}(t, t_1) \) are inverses too. These circumstances allow us to change the variables \( t_a \mapsto t_a^{ret}(t_b) \) in the “advanced” integrals in eqs. (4.1) and (4.2). Further we couple them with their “retarded” counterparts. Since

\[
\frac{dt_1^{ret}(t_2)}{dt_2} = \frac{1 - (v_1 n_q)}{1 - (v_2 n_q)}, \quad \frac{dt_2^{ret}(t_1)}{dt_1} = \frac{1 + (v_2 n_q)}{1 + (v_1 n_q)}, \] (4.3)

we obtain

\[
G_2 + G_1 = \int_{-\infty}^{t} dt_1 \left[ \frac{1 - (v_1 n_q)}{1 - (v_2 n_q)} G_1 - G_2 \right] \bigg|_{t_2 = t_2^{ret}(t_1)} \] (4.4)

\[
+ \int_{-\infty}^{t} dt_2 \left[ G_1 + \frac{1 + (v_2 n_q)}{1 + (v_1 n_q)} G_2 \right] \bigg|_{t_1 = t_1^{ret}(t_2)} \]

\[
+ \int_{t_2^{ret}(t_2)}^{t} dt_2 \left[ G_1 + \frac{1 - (v_2 n_q)}{1 + (v_1 n_q)} G_2 \right] \bigg|_{t_1 = t_1^{ret}(t, t_2)}.
\]
The unit vector $\mathbf{n}_q = \mathbf{q}/q$ is the third vector of orthonormal triad (2.6). The last integral is due to interference of outgoing electromagnetic waves radiated out by the particles within the acausal region (see Fig. 4). We take into account that

$$\frac{dt_2'(t, t_1)}{dt_1} = -\frac{1 + (v_1 n_q)}{1 - (v_2 n_q)}. \quad (4.5)$$

Of course, one can change the variables $t_b \mapsto t_b^{adv}(t_a)$ in the “retarded” integrals in eqs.(4.1) and (4.2) and add them to their “advanced” counterparts,

$$g_2 + g_1 = \int_{-\infty}^{t_2^{ret}(t)} \int_{t_1^{ret}(t)}^{t_2^{ret}(t)} dt_2 \left[ G_1 - \frac{1 - (v_2 n_q)}{1 - (v_1 n_q)} G_2 \right]_{t_1 = t_1^{adv}(t_2)}^{t_1 = t_1^{adv}(t_1)} + \int_{-\infty}^{t} \int_{t_1^{ret}(t)}^{t} dt_1 \left[ \frac{1 + (v_1 n_q)}{1 + (v_2 n_q)} G_1 + G_2 \right]_{t_2 = t_2^{adv}(t_1)}^{t_2 = t_2^{adv}(t_1)} + \int_{t_1^{ret}(t)}^{t} dt_1 \left[ \frac{1 + (v_1 n_q)}{1 - (v_2 n_q)} G_1 + G_2 \right]_{t_2 = t_2'(t, t_1)}^{t_2 = t_2'(t, t_1)}.$$

A combination of the “retarded” and the “advanced” terms is valuable too.

Integral of a mixed double derivative can be written in the form either

$$I_1 = \left[ \int_{-\infty}^{t_2^{ret}(t)} dt_2 \int_{t_1^{ret}(t)}^{t_2^{ret}(t)} dt_1 + \int_{-\infty}^{t_2^{ret}(t)} dt_2 \int_{t_1^{ret}(t)}^{t_2^{ret}(t)} dt_1 \right] \frac{\partial}{\partial t_1} \left[ \frac{\partial G(t_1, t_2)}{\partial t_2} \right]_{t_1 = t_1^{adv}(t_2)}^{t_1 = t_1^{adv}(t_1)} - \int_{-\infty}^{t} dt_2 \left[ \frac{\partial G(t_1, t_2)}{\partial t_2} \right]_{t_1 = t_1^{ret}(t_2)}^{t_1 = t_1^{ret}(t_1)} + \int_{t_1^{ret}(t)}^{t} dt_2 \left[ \frac{\partial G(t_1, t_2)}{\partial t_2} \right]_{t_1 = t_1'(t, t_2)}^{t_1 = t_1'(t, t_2)}.$$  \quad (4.7)

or

$$I_2 = \left[ \int_{-\infty}^{t_2^{ret}(t)} dt_1 \int_{t_1^{ret}(t)}^{t_2^{ret}(t)} dt_2 + \int_{-\infty}^{t_2^{ret}(t)} dt_1 \int_{t_1^{ret}(t)}^{t_2^{ret}(t)} dt_2 \right] \frac{\partial}{\partial t_2} \left[ \frac{\partial G(t_1, t_2)}{\partial t_1} \right]_{t_2 = t_2^{adv}(t_1)}^{t_2 = t_2^{adv}(t_1)} - \int_{-\infty}^{t} dt_1 \left[ \frac{\partial G(t_1, t_2)}{\partial t_1} \right]_{t_2 = t_2^{ret}(t_1)}^{t_2 = t_2^{ret}(t_1)} + \int_{t_1^{ret}(t)}^{t} dt_1 \left[ \frac{\partial G(t_1, t_2)}{\partial t_1} \right]_{t_2 = t_2'(t, t_1)}^{t_2 = t_2'(t, t_1)}.$$  \quad (4.8)
Figure 5: Function $G[t_1, t_2^{ret}(t_1)]$ is associated with the interference of wave fronts $S_1(O_1, t - t_1)$ and $S_2(O_2^{ret}, t - t_2^{ret})$ at point $N \in \Sigma_t$. Function $G[t_1, t_2^{ret}(t, t_1)]$ is connected with the combination of wave fronts $S_1(O_1, t - t_1)$ and $S_2(O_2^{ret}, t - t_2^{ret})$ at point $S \in \Sigma_t$. If $t_1 \to t$, both the point $N$ and the point $S$ tend to $z_1(t) = \zeta_1 \cap \Sigma_t$. If $\lim_{t_1 \to t} G[t_1, t_2^{ret}(t_1)] = \lim_{t_1 \to t} G[t_1, t_2^{ret}(t, t_1)]$ the value of integral of mixed derivative $\partial^2 G/\partial t_1 \partial t_2$ over time variables does not depend on the order of integration.

The question is what expression should be used.

To compare $\mathcal{I}_1$ and $\mathcal{I}_2$ we change the variables in the “advanced” integrals and subtract (4.8) and (4.7). We arrive at the integrals being functions of the end points only,

$$\mathcal{I}_1 - \mathcal{I}_2 = G[t_1, t_2^{ret}(t_1)] \bigg|_{t_1 \to \infty}^{t_1 \to t} - G[t_2^{ret}(t_2), t_2] \bigg|_{t_2 \to \infty}^{t_2 \to t}$$

$$+ \begin{cases} 
\text{either} & G[t_1^{ret}(t, t_2), t_2] \bigg|_{t_2 \to \infty}^{t_2 \to t} \\
\text{or} & - G[t_1, t_2^{ret}(t_1)] \bigg|_{t_1 \to \infty}^{t_1 \to t} 
\end{cases}.$$ (4.9)

It vanishes if and only if (i) limiting values of $G$ (that evaluated at the remote past) cancel each other, and (ii) function $G$ is smooth at points at which the world lines puncture $\Sigma_t$. Indeed, the part of difference (4.9) which depends on the momentary state of particles’ motion can be rewritten as follows:

$$\mathcal{I}_1 - \mathcal{I}_2 = \lim_{t_1 \to t} \{G[t_1, t_2^{ret}(t_1)] - G[t_1, t_2^{ret}(t_1)]\}$$

$$- \lim_{t_2 \to t} \{G[t_1^{ret}(t_2), t_2] - G[t_1^{ret}(t_2), t_2]\}.$$ (4.10)

The situation is illustrated in Fig. 5.
4.1 Criteria

The main task of the present paper is to decompose the interference part of the Maxwell energy-momentum tensor density into bound and radiative components. The former modifies individual 4-momenta (1.7) of dressed particles while the latter shows how a charge is influenced by radiation of another charge.

To reveal meaningful radiative part of mixed energy-momentum (2.1) and angular momentum (2.2), we apply the criteria which were first formulated in Ref. [3, Table 1].

- The bound part diverges while the radiative one is finite.
- The bound component depends on the momentary state of the particles’ motion while the radiative one is accumulated with time.
- The form of the bound terms heavily depends on choosing of an integration surface while the radiative terms are invariant.

There are, however, a several properties that the desired expressions have possess before they can be accepted. We list them below.

1. Radiative parts should be completely determined by particles’ motion; they can not depend on distance to a point of observation.

2. They should be produced by divergence-free expressions.

3. Non-accelerated charges do not radiate.

4. Balance of Noether conserved quantities yields the Lorentz-Dirac equation.

4.2 Radiative part of mixed momentum

To decompose the momentum (3.18) into bound and radiative components is a straightforward integration of all the terms over time variables. Scrupulous computations reveal candidate for radiative part,

\[
P^i_{\text{int,rad}} = \frac{e_1 e_2}{2} \left[ \hat{\Pi}^i_2 \left( k_1^0 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_1} \left( v_2^i k_1^0 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} D^0 \right) \right] + \hat{\Pi}^i_1 \left( k_2^0 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_2} \left( v_1^i k_2^0 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} D^0 \right) \right],
\]

which produce the terms that satisfy Teitelboim’s criteria.

Equations (4.1) and (4.2) imply that its integral over time variables is completely determined by values of the arguments of time differential operators at the ends of integration intervals. When a consideration is restricted to the end points where radius \( h \) of intersection \( S_1 \cap S_2 \) vanishes, the coefficients (3.7) get simplified. Since eq. (2.5) the
The distances are as follows. Radiative momentum (4.11) contains also non-trivial constructions, and some basic functions at limiting points are collected in Table 1 (see Appendix B).

The sum of differential operators \( \hat{\Pi}^1_i \) and \( \hat{\Pi}^2_i \) contains mixed double derivative of the function,

\[
G^i = k^0_1 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} D^i_2 + k^0_2 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} D^i_1. \tag{4.13}
\]

A surprising feature of integration of expression (4.11) over time variables is that the result heavily depends on the order of differentiation in \( \partial^2 G^i / \partial t_1 \partial t_2 \). If one choose the rule (4.7) they obtain

\[
\rho^i_{\text{rad},21} = \int_{-\infty}^t dt_1 \gamma_1^{-1} G^i_{21}[t_1, t_2^{\text{ret}}(t_1)] + \int_{-\infty}^{t_2^{\text{ret}}} dt_1 \gamma_1^{-1} G^i_{21}[t_1, t_2^{\text{adv}}(t_1)]. \tag{4.14}
\]

Having applied the rule (4.8) we arrive at

\[
\rho^i_{\text{rad},12} = \int_{-\infty}^t dt_2 \gamma_2^{-1} G^i_{12}[t_1^{\text{ret}}(t_2), t_2] + \int_{-\infty}^{t_2^{\text{ret}}} dt_2 \gamma_2^{-1} G^i_{12}[t_1^{\text{adv}}(t_2), t_2]. \tag{4.15}
\]

Function under integral signs,

\[
\gamma_a^{-1} G^i_{ba} = c_a c_b \left[ \frac{q_{ab}(v_a \cdot v_b)}{r_b^3} \right] \gamma_a^{-2} + \frac{q_{ab}(v_a \cdot v_b)}{r_b^3} (q_{ab} \cdot \dot{v}_b) + \frac{q_{ab}(v_a \cdot v_b)}{r_b^2} - \frac{v_b^i (v_a \cdot v_b)}{r_b^2}, \tag{4.16}
\]

is referred to the retarded and the advanced instants. Relative distance 4-vector \( q_{ab} := z_a - z_b \); in our notation \( q_{12} = q \) while \( q_{21} = -q \). Denominator \( r_b = -(q_{ab} \cdot v_b) \) is the retarded distance of type (3.6) where both the field point and the point of emission are placed on particles’ world lines. So, in expression (4.14) the field point is \( z_1(t_1) \in \zeta_1 \) where \( e_1 \) is placed. Points of emission are \( z_2[t_2^{\text{ret}}(t_1)] \in \zeta_2 \) in the first path integral and \( z_2[t_2^{\text{adv}}(t_1)] \in \zeta_2 \) in the second one,

\[
r_2^{\text{ret}} = q \left[ 1 - (v_2 n_q) \right] \bigg|_{[t_1, t_2^{\text{ret}}(t_1)]}, \quad r_2^{\text{adv}} = -q \left[ 1 + (v_2 n_q) \right] \bigg|_{[t_1, t_2^{\text{adv}}(t_1)]}. \tag{4.17}
\]

In expression (4.15) the field point is in \( z_2(t_2) \in \zeta_2 \) where the second charge is placed. It is acted on by the first charge placed at points either \( z_1[t_1^{\text{ret}}(t_2)] \in \zeta_1 \) or \( z_1[t_1^{\text{adv}}(t_2)] \in \zeta_1 \). The distances are as follows:

\[
r_1^{\text{ret}} = q \left[ 1 + (v_1 n_q) \right] \bigg|_{[t_1^{\text{ret}}(t_2), t_2]}, \quad r_1^{\text{adv}} = -q \left[ 1 - (v_1 n_q) \right] \bigg|_{[t_1^{\text{adv}}(t_2), t_2]}. \tag{4.18}
\]
It is of great importance that the integrand (4.16) does not depend on the distances to points \( N \) or \( S \) on the observation hyperplane \( \Sigma_t \). It depends on the relative distance between charged particles, on their velocities, and on their accelerations. The situation looks like \( a \)-th charge is acted on by \( b \)-th one directly. The charges are connected by a null ray: in the first integral in eq.(4.14) or in eq.(4.15) the interaction be forward while in the second one backward in time. All the moments are before the instant of observation \( t \) which labels the upper limits of path integrals.

Since the function (4.13) is not smooth in neighborhoods of intersections \( \zeta_a \cap \Sigma_t \), the expressions (4.14) and eq.(4.15) are not equivalent. By virtue of the relation (4.9) we compute the difference:

\[
p^{\text{t rad}}_{21} - p^{\text{t rad}}_{12} = - q^i(A^\text{ret}_{21} \cdot A^\text{adv}_{12})\left|_{t_1, t_2^\text{ret}(t_1)}^{t_1=t} \right. - q^i(A^\text{ret}_{12} \cdot A^\text{adv}_{21})\left|_{t_1^\text{ret}(t_2), t_2}^{t_2=t} \right. .
\]

(4.19)

Symbolically we denote

\[
A^\text{ret}_{ba} = e_b \frac{v_b[p^\text{ret}_b(t_a)]}{r_b[t_a, t_b^\text{ret}(t_a)]}, \quad A^\text{adv}_{ba} = e_b \frac{v_b[p^\text{adv}_b(t_a)]}{r_b[t_a, t_b^\text{adv}(t_a)]}
\]

the Liénard-Wiechert vector potential of \( b \)-th charge at point at which \( a \)-th one is placed.

### 4.3 Radiative part of mixed stress-energy tensor

In this subsection we find the terms which produce the radiative part of mixed momentum, either (4.14) or (4.15). We start with \( \varphi \)-momentum (4.11) which is then nothing but the mixed space-time component of the following tensor:

\[
\begin{align*}
\mathcal{P}^{\mu\nu}_{\text{int rad}} &= \frac{e_1 e_2}{2} \left[ \hat{\Pi}^{\mu\nu}_{12} \left( \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_1} \left( D^\mu_1 v^\nu_2 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_2} \left( D^\nu_2 v^\mu_1 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) \right] \\
&+ \hat{\Pi}^{\mu\nu}_{21} \left( \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_2} \left( D^\nu_2 v^\mu_1 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) - \frac{\partial}{\partial t_1} \left( D^\mu_1 v^\nu_2 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right)
\end{align*}
\]

(4.21)

To restore \( \varphi \)-dependent terms leading to this expression, we insert

\[
\hat{\Pi}^{\mu\nu}_{ab}(f) = \hat{T}^{\mu\nu}_{ab}(f) - f \pi^{\mu\nu}_{ab}
\]

(4.22)

and substitute the right hand side of the third line of eqs.(3.16) for \( \pi^{\mu\nu}_{ab} \). (Operator \( \hat{T}^{\mu\nu}_{ab} \) is defined by eq.(3.10).) After cancellation of like terms we obtain a linear combination of coefficients (3.7). Further we omit the integration over polar angle and multiply the result on inverse Jacobian \( J^{-1} = q/(r_1 r_2) \). Finally, we obtain the tensor,

\[
4\pi t^{\mu\nu} = e_1 e_2 \left\{ \left( \mathcal{K}^\mu_1 \mathcal{K}^\nu_2 + \mathcal{K}^\mu_2 \mathcal{K}^\nu_1 \right) \left[ \frac{c_1 c_2}{r_1^3 r_2^3} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} + \frac{c_1}{r_1^3} \frac{\partial^3 \sigma}{\partial t_1^2 \partial t_2} + \frac{c_2}{r_2^3} \frac{\partial^3 \sigma}{\partial t_1 \partial t_2^2} + \frac{1}{r_1^2 r_2^2} \frac{\partial^4 \sigma}{\partial t_1 \partial t_2^3} \right] \\
- \left( v_1^\mu K_2^\nu + K_1^\mu v_2^\nu \right) \left[ \frac{c_2}{r_1^2 r_2^3} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} + \frac{1}{r_1^2 r_2^2} \frac{\partial^3 \sigma}{\partial t_1 \partial t_2^2} \right] \\
- \left( v_1^\mu v_2^\nu + v_2^\mu K_1^\nu \right) \left[ \frac{c_1}{r_1^3} \frac{\partial^3 \sigma}{\partial t_1 \partial t_2^2} + \frac{1}{r_1^2 r_2^2} \frac{\partial^4 \sigma}{\partial t_1^2 \partial t_2} \right] \\
+ \left( v_2^\mu v_1^\nu + v_2^\mu v_1^\nu \right) \frac{1}{r_1^2 r_2^2} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right\}
\]

(4.23)
which is symmetric in its indices.

Similarly to the mixed stress-energy tensor (1.6) itself, its radiative part,

\[ T_{\text{int, rad}}^{\mu\nu} = t^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} t^\alpha_\alpha, \]  

(4.24)
contains the convolution of tensor (4.23),

\[ 4\pi t^\alpha_\alpha = 2e_1 e_2 \left\{ \frac{c_1 c_2}{r_1 r_2^3} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} + \frac{c_1}{r_1 r_2^3} \frac{\partial}{\partial t_1} \left( \frac{\sigma}{\partial t_1 \partial t_2} \right) + \frac{c_2}{r_1 r_2^3} \frac{\partial}{\partial t_2} \left( \frac{\sigma}{\partial t_1 \partial t_2} \right) \right\} \]

\[ + \frac{1}{r_1^2 r_2^2} \frac{\partial}{\partial t_1 \partial t_2} \left( \frac{\sigma}{\partial t_1 \partial t_2} \right) + \frac{1}{r_1 r_2} \frac{\partial^3 \sigma}{\partial t_1 \partial t_2} \]  

(4.25)
Integration over \( \varphi \) results in the combination of partial derivatives in time variables,

\[ \int_0^{2\pi} d\varphi J^\alpha t^\alpha = e_1 e_2 \left\{ \hat{\Pi}^0 \left( \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \frac{\partial}{\partial t_1} \left( \frac{1}{q \parallel r_1 \parallel} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \frac{\partial}{\partial t_2} \left( \frac{1}{q \parallel r_2 \parallel} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) \right\}, \]  

(4.26)
where expressions

\[ \parallel r_a \parallel = \sqrt{[k^0_a - (v_a n_a) k^3_a]^2 - h^2 [v_a n_a]^2}, \]  

(4.27)
a = 1, 2, are introduced in Appendix A.

Finally, after a tedious calculations\(^3\) we derive the identity \( \partial_\nu T_{\text{int, rad}}^{\mu\nu} = 0 \). It means that the radiative part (4.24) of mixed energy-momentum tensor density (1.6) is conserved off particles’ world lines.

It is worth noting that the terms in the first line of expression (4.23) only belong to the mixed part (1.6) of the electromagnetic field stress-energy tensor. The others provide vanishing of divergence of \( \hat{T}_{\text{int, rad}} \) as well as reasonable expression for radiated energy-momentum which escapes to infinity. By means of the relations

\[ v^\mu_a = \gamma^{-1}_a u^\mu_a, \quad \dot{v}^\mu_a = \gamma^{-2}_a \left( a^\mu_a - \gamma^{-1}_a a^0_a u^\mu_a \right) \]

(4.28)
\[ r_a = \gamma^{-1}_a r_a, \quad K^\mu_a = r_a k^\mu_a, \]
\[ c_a = \gamma^{-2}_a \left[ 1 + r_a (k^0_a, a_a) + \gamma^{-1}_a a^0_a r_a \right], \]

(4.29)
the expression (4.23) can be easily rewritten in manifestly covariant notations.

### 4.4 Radiative part of mixed energy

Since \( \eta^{00} = -1 \), the convolution (4.26) contributes into zeroth component of radiative part of mixed energy-momentum,

\[ P_{\text{int, rad}}^0 = e_1 e_2 \left\{ \hat{\Pi}^0 \left[ \left( k^0_1 k^0_2 + \frac{1}{2} \sigma \right) \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right] \right\} \]

\[ + \frac{1}{2} \frac{\partial}{\partial t_1} \left( \frac{1}{q \parallel r_1 \parallel} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) + \frac{1}{2} \frac{\partial}{\partial t_2} \left( \frac{1}{q \parallel r_2 \parallel} \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} \right) \}

(4.30)
\(^3\)Differentiation of \( T_{\text{int, rad}} \) is straightforward: one can apply the rule (2.8) and combine like terms scaled as \( r_1^{-m} r_2^{-n} \); exponents \( m \) and \( n \) run from 1 to 4 and their sum \( 3 \leq m + n \leq 7 \).
As could be expected the argument of mixed double derivative,

\[ G^0 = \left( \frac{k_1^0 k_2^0}{k_1^2 k_2^2} + \frac{1}{2} \sigma \right) \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} D^0, \] (4.31)

is not smooth in neighborhoods of end points \( \zeta_a \cap \Sigma_t, \)

\[ \Delta D^0 = - q^0 (A_{21}^{ret} \cdot A_{12}^{adv}) |_{t_1, t_2} - q^0 (A_{12}^{ret} \cdot A_{21}^{adv}) |_{t_1, t_2} |_{t_2 = t} - q^0 (A_{12}^{ret} \cdot A_{21}^{adv}) |_{t_1, t_2} |_{t_2 = t}. \] (4.32)

Hence the interference rate of radiated energy heavily depends on order of differentiation, either \( \partial^2 G^0 / \partial t \partial t \) or \( \partial^2 G^0 / \partial t \partial t \). Having used the rule (4.7) we arrive at the expression of type (4.14). Choosing the rule (4.8) we obtain the expression of type (4.15). The results can be generalized as follows:

\[ p_{rad, ba}^\alpha = \int_{-\infty}^{t} \text{dt}_a G_{ba}^{-1} G_{ba} [t_a, t_b^{ret}(t)] + \int_{-\infty}^{t} \text{dt}_a G_{ba}^{-1} G_{ba} [t_a, t_b^{adv}(t_a)], \] (4.33)

where

\[ \gamma_a^{-1} G_{ba} = e_a e_b \left\{ \frac{q_{ab} (v_a \cdot v_b)}{r_b^3} \gamma^{-2} + \frac{q_{ab} (v_a \cdot v_b)}{r_b^3} (q_{ab} \cdot \dot{v}_b) \right. \]
\[ + \frac{q_{ab} (v_a \cdot \dot{v}_b)}{r_b^2} - \frac{v_b^0 (v_a \cdot v_b)}{r_b^2} \right\}. \] (4.34)

It is convenient to rewrite eq.(4.33) in a manifestly covariant fashion,

\[ p_{rad, ab}^\alpha = \int_{-\infty}^{\tau_1} \text{ds}_b G_{ab}^{-1} s_a^{ret}(s_b, s), \] (4.35)

The particles’ world lines \( \zeta_1 \) and \( \zeta_2 \) are parametrized by individual proper times \( s_1 \) and \( s_2 \), respectively. Upper limits \( \tau_1 \) and \( \tau_2 \) label the points at which \( \zeta_1 \) and \( \zeta_2 \) puncture the observation hyperplane \( \Sigma \). Two-point function,

\[ G_{ab}^\alpha = e_a e_b \left\{ \frac{q_{ba} (u_b \cdot u_a)}{r_a^3} [1 + (q_{b} \cdot a)] + \frac{q_{ba} (u_b \cdot a_a)}{r_a^2} - \frac{u_a^0 (u_b \cdot u_a)}{r_a^2} \right\}. \] (4.36)

is evaluated at points on the world lines of particle \( a \) and particle \( b \) which are linked by a null ray. \( u_a(s_a) \) denotes the normalized 4-velocity of \( a \)-th particle; 4-acceleration \( a_a^\alpha = du_a^\alpha / ds_a \).

In contrast with “one-particle” contributions to radiated energy-momentum, the mixed one is not uniquely defined. Having generalized the expressions (4.19) and (4.32), we obtain

\[ p_{rad, 21}^\alpha - p_{rad, 12}^\alpha = Q_{21}^{s_2 = \tau_2} - Q_{12}^{s_1 = \tau_1,} \] (4.37)

where

\[ Q_{ab}^\alpha = q_{ab} (A_{ab}^{adv} \cdot A_{ba}^{ret}) |_{s_a, b^{ret}(s_a)}. \] (4.38)

This \( q \)-directed null vector is proportional to the scalar product of the Liénard-Wiechert potentials (4.20).
The difference (4.37) depends on the state of particles’ motion at the end points of path integrals. Let us evaluate the net energy-momentum which escapes to infinity. The integrals over entire world lines should be substituted for the integrals over the past motion. We suppose that the particles are asymptotically free. Since Liénard-Wiechert potentials fall off at large distances inversely as the first power of the separation vector between the charges, the right-hand side of expression (4.37) vanishes. Hence the full amount of radiative energy-momentum emitted by interacting particles does not depend on the method of integration.

4.5 Radiative part of mixed angular momentum

When an one-particle problem is considered, bound components of the stress-energy tensor [3] and its torque [5] contribute into individual particle’s 4-momentum and angular momentum, respectively. The corrections arise from the regularisation procedure which involves the Taylor expansion of surface integrals in which the first two terms lead to the diverging Coulomb self-energy and the Abraham radiation reaction 4-vector, respectively. The finite terms depend on the form of the hole that is cut out from the integration hypersurface to ensure regularization. The best suited hole must be coordinate-free one. Teitelboim [3] integrate over hyperplane \( \sigma(\tau) = \{ y \in \mathbb{M}_4 : u_\mu(\tau)(y^\mu - z^\mu(\tau)) = 0 \} \) which is orthogonal to the 4-velocity of the charge at point \( z(\tau) \) at which particle’s world line punctures \( \sigma(\tau) \). It is evident that the tilted hyperplane together with the future light cone cutting out the coordinate-free hole (see Refs. [3, Fig.1], [5, Fig.1], [2, Fig.5-2]). The considerations lead to manifestly covariant and structure-independent finite terms of clear physical sense.

The difficulties associated with the computation of the mixed contribution (1.6) are twofold — to perform the meaningful decomposition of \( \tilde{T}_{\text{int}} \) into bound and radiative parts and to choose an appropriate surface of integration. The tilted hyperplane which plays privileged role in the one-particle radiation reaction problem is not suitable whenever two-body one is considered. Indeed, there is no a hyperplane which is orthogonal to the world lines of both the particles at all events. Kosyakov [19] constructs a piecewise hypersurface where a small fragment of a spacelike hyperplane \( \Sigma \) is replaced by a fragment of an orthogonal hyperplane \( \sigma_a(\tau_a) = \{ y \in \mathbb{M}_4 : u_{a,\mu}(\tau_a)(y^\mu - z_a^\mu(\tau_a)) = 0 \} \) in the vicinity of every intersection point. The deformed hyperplane is called locally adjusted. But the problem arises how to sew these fragments with \( \Sigma \).

Expressions (B.4), (B.5), (B.7), and (B.8) show that the surface integrals of bound component of the mixed part of stress-energy tensor depend on the state of particles’ motion in vicinities of intersection points \( \zeta_a \cap \Sigma_t \). Unfortunately, the Taylor expansions of the divergent components of bound energy-momentum do not lead to reasonable finite covariant terms. It is because the integration surface \( \Sigma_t \) is tightly connected with the laboratory inertial frame. This choice yields the coordinate-dependent hole around \( a \)-th particle in the point of intersection \( \zeta_a \cap \Sigma_t \). For this reason we assume that an intrinsic structure of a charged particle is beyond the limits of classical theory [2]. We do not require any assumptions about the particle structure, its charge distribution and its size (except that its “radius” does not vanish, although it is too small to be observed). To
reconcile the theory with observation, we assume that a dressed charged particle possesses finite 4-momentum and angular momentum.

We face a problem of how the 4-momenta of interacting dressed charges depend on their individual characteristics such as their masses, charges, 4-velocities, etc. Valuable information can be extracted from the radiative part of electromagnetic field’s angular momentum. Indeed, conserved quantities place stringent requirements on the dynamics of our closed system. They demand that that the change in radiated momentum and angular momentum should be balanced by a corresponding change in the individual momenta and angular momenta of dressed particles. It is felt that analysis of balance equations yields reasonable expressions. Indeed, it is shown \cite{20} that the Lorentz-Dirac equation can be derived from the energy-momentum and angular momentum balance equations. In \cite{21} the analog of the Lorentz-Dirac equation in six dimensions is obtained via analysis of 21 conserved quantities which correspond to the Poincaré symmetry of an isolated point particle coupled with electromagnetic field.

Recall from Section 3 that the integral of mixed angular momentum tensor density over $\varphi$ is the combination of partial derivatives in time variables. To reveal meaningful radiative terms we integrate eq.\((3.26)\) over $t_1$ and $t_2$ and apply Teitelboim’s criteria.

The calculation is virtually identical to that presented above, and we shall not bother with details. The result depends on the method of integration of mixed double derivatives,

$$
\tilde{M}_{rad,ab} = \int_{-\infty}^{\tau_b} ds_b (z_b \wedge G_{ab}^{ret} - e_b u_b \wedge A_{ab}^{ret})
$$

(\text{Symbol } \wedge \text{ denotes the wedge product.}) If we choose the rule \((4.8)\) we obtain at $\tilde{M}_{rad,21}$. If one prefer the rule \((4.7)\) they derive $\tilde{M}_{rad,12}$. Direct calculations show that the difference between these expressions depends on the momentary state of particles’ motion,

$$
\tilde{M}_{rad,21} - \tilde{M}_{rad,12} = z_2 \wedge Q_{21|s_2=t_2}^{s_2=t_2} - z_1 \wedge Q_{12|s_1=t_1}^{s_1=t_1}. \tag{4.40}
$$

Expression \((4.39)\) for radiated angular momentum satisfies Teitelboim criteria. Indeed, the integrand is finite and covariant; the radiation is accumulated with time. Non-covariant bound terms which are presented in Appendix B are quite different: they depend on the state of particles’ motion at points $\zeta_a \cap \Sigma_t$ and they contain divergences. We assume that the expression \((4.39)\) is involved in the angular momentum balance equation explicitly.

### 4.6 Radiation of non-accelerated charges

Let us consider a specific case of very massive particles, such that $e_1/m_1 << 1$ and $e_2/m_2 << 1$. Electromagnetic field is too small to accelerate the charges so that the particles move with constant velocities. We place the coordinate origin of the Lorentz inertial frame at point at which the second charge is placed. If $v_2 = 0$, the zeroth
component $\gamma^{-1}_1 G^0_{21}$ of two-point function (4.34) is identically equal to zero. Hence the radiative energy,

$$p^0_{\text{rad},21} = \int_{-\infty}^{t} dt_1 \gamma^{-1}_1 G^0_{21}[t_1, t^\text{ret}_2(t_1)] + \int_{-\infty}^{t^\text{ret}_1(t)} dt_1 \gamma^{-1}_1 G^0_{21}[t_1, t^\text{adv}_2(t_1)],$$

vanishes.

If $v_2 = 0$, the space components of this function get simplified,

$$\gamma^{-1}_1 G^i_{21} = e_1 e_2 \frac{q^i}{q^0}.$$  \hspace{1cm} (4.42)

The function evaluated at advanced instant $t^\text{adv}_2(t_1)$ is just the function referred to the retarded instant $t^\text{ret}_2(t_1)$ taken with opposite sign: $G^i_{21}[t_1, t^\text{adv}_2(t_1)] = -G^i_{21}[t_1, t^\text{ret}_2(t_1)]$. Because of shift in limits of the retarded and the advanced integrals the mixed part of radiative 3-momentum is as follows:

$$p^i_{\text{rad},21} = \int_{-\infty}^{t} dt_1 \gamma^{-1}_1 G^i_{21}[t_1, t^\text{ret}_2(t_1)] + \int_{-\infty}^{t^\text{ret}_1(t)} dt_1 \gamma^{-1}_1 G^i_{21}[t_1, t^\text{adv}_2(t_1)]$$

$$= e_1 e_2 \int_{t^\text{ret}_1(t)}^{t} dt_1 \frac{q^i}{q^3}.$$ \hspace{1cm} (4.43)

This is vanishingly small quantity. It should be rejected if for no other reason than that the radiated 4-momentum cannot be spacelike 4-vector.

In a like manner we calculate the radiative angular momentum,

$$\hat{M}^{0i}_{\text{rad},21} = e_1 e_2 \int_{t^\text{ret}_1(t)}^{t} dt_1 \left( t_1 \frac{q^i}{q^3} + \frac{v^i_1}{q} \right),$$

$$\hat{M}^{ij}_{\text{rad},21} = 0.$$ \hspace{1cm} (4.44)

If the upper limit of integration $t \to +\infty$, $p^i_{\text{rad},21} = 0$. Indeed, in this case for each point on $\zeta_1$ there are points on $\zeta_2$ labeled by the retarded and the advanced instants. Corresponding contributions cancel each other.

Alternative expression for mixed contribution to radiated energy-momentum,

$$p^\alpha_{\text{rad},12} = \int_{-\infty}^{+\infty} dt_2 \gamma_1^{-1} G^\alpha_{12}[t^\text{ret}_1(t_2), t_2] + \int_{-\infty}^{+\infty} dt_2 \gamma_2^{-1} G^\alpha_{12}[t^\text{adv}_1(t_2), t_2],$$

differs from $p^\alpha_{\text{rad},21}$ on the sum of null vectors,

$$e_1 e_2 \frac{n_\alpha^q}{q[1 - (n_q v_1)]} \bigg|_{t_1 \to +\infty} + e_1 e_2 \frac{n_\alpha^q}{q[1 + (n_q v_1)]} \bigg|_{t_2 \to +\infty},$$ \hspace{1cm} (4.46)

estimated at limits of integrals. If the particles move with different velocities, they are asymptotically free. If both the particles are static, the difference is equal to zero because the distance $q$ between charges does not change.
5 Equations of motion of radiating charges

In this Section we study the energy-momentum and angular momentum balance equations. We calculate how much electromagnetic field energy, momentum, and angular momentum flow across hyperplane $\Sigma_t$. We can do it at a time $t + \Delta t$. We demand that the change in these quantities be balanced by a corresponding change in those of the particles, so that the total energy-momentum,

$$ P = p_1 + p_2 + \frac{2e_1^2}{3} \int_{-\infty}^{\tau_1} ds_1 a_1^2 u_1(s_1) + \frac{2e_2^2}{3} \int_{-\infty}^{\tau_2} ds_2 a_2^2 u_2(s_2) + p_{\text{int,rad}}, \tag{5.1} $$

and total angular momentum,

$$ \hat{M} = z_1 \wedge p_1 + z_2 \wedge p_2 $$

$$ + \frac{2e_1^2}{3} \int_{-\infty}^{\tau_1} ds_1 (a_1^2 z_1 \wedge u_1 + u_1 \wedge a_1) + \frac{2e_2^2}{3} \int_{-\infty}^{\tau_2} ds_2 (a_2^2 z_2 \wedge u_2 + u_2 \wedge a_2) + \hat{M}_{\text{int,rad}}, \tag{5.2} $$

are properly conserved.

We suppose that particles’ individual 4-momenta $p_1$ and $p_2$ are already renormalized. The words “already renormalized” mean that momenta contain contributions due to bound component of the stress-energy tensor density, including its mixed part. While the radiation which detaches itself from charges and leads an independent existence is involved explicitly.

In contrast with “one-particle” contributions to conserved quantities (5.1) and (5.2), the mixed ones, (4.35) and (4.39), are not uniquely defined. Our theory faces in the radiation problem a significant issue: non-uniqueness in determination of radiation sourced by the mixed part of Maxwell energy-momentum tensor density. To solve the problem we mix the obtained expressions

$$ p_{\text{int,rad}} = \kappa p_{\text{rad,21}} + (1 - \kappa) p_{\text{rad,12}}, \tag{5.3} $$

$$ \hat{M}_{\text{int,rad}} = \kappa \hat{M}_{\text{rad,21}} + (1 - \kappa) \hat{M}_{\text{rad,12}}, \tag{5.4} $$

and try to find out the value of constant $\kappa$ that accords with experience.

Expressions $p_{\text{rad,ab}}$ and $\hat{M}_{\text{rad,ab}}$ are based on two-point function (4.36) referred to the points on the world lines of particle $a$ and particle $b$ which are linked by a null ray,

$$ \sigma = -\frac{1}{2} (z_1 - z_2)^2 \tag{5.5} $$

$$ = 0. $$

This implies that a displacement of $z_1(s_1)$ typically induces a simultaneous displacement of $z_2(s_2)$ because new points $z_1(s_1 + \delta s_1)$ and $z_2(s_2 + \delta s_2)$ must also be linked by a null geodesic,

$$ (q_{21} \cdot u_1) ds_1 + (q_{12} \cdot u_2) ds_2 = 0. \tag{5.6} $$

This immediately gives

$$ \frac{ds_a}{ds_b} = -\frac{(q_{ba} \cdot u_b)}{r_a}, \tag{5.7} $$
where $r_a = -(q_{ba} \cdot u_a)$.

By virtue of this expression we compare the retarded and the advanced integrals involved in eqs. (4.35) and (4.39). With understanding that functions $s_a^{\text{adv}}(s_b)$ and $s_b^{\text{ret}}(s_a)$ are inverses, after some algebra we obtain
\[
\int_{-\infty}^{\tau_b} ds_b G_{ab} \left[ s_a^{\text{adv}}(s_b), s_b \right] = \int_{-\infty}^{\tau_a} ds_a G_{ba} \left[ s_a, s_b^{\text{ret}}(s_a) \right] + Q_{ab} \bigg|_{s_a = s_b = \tau_a}, \tag{5.8}
\]
\[
\int_{-\infty}^{\tau_b} ds_b \left( z_b \wedge G_{ab}^{\text{adv}} - e_b u_b \wedge A_{ab}^{\text{adv}} \right) = \int_{-\infty}^{\tau_a} ds_a \left( z_a \wedge G_{ba}^{\text{ret}} - e_a u_a \wedge A_{ba}^{\text{ret}} \right) + z_a \wedge Q_{ab} \bigg|_{s_a \rightarrow -\infty} \cdot \tag{5.9}
\]

The relations accord with the right-hand sides of eqs. (4.37) and (4.40).

Equipped with these relations we rewrite the mixed parts of radiated energy-momentum (4.35) and angular momentum (4.39) as follows:
\[
P_{\text{rad},ab} = \int_{-\infty}^{\tau_b} ds_b G_{ab} \left[ s_a^{\text{ret}}(s_b), s_b \right] + \int_{-\infty}^{\tau_a} ds_a G_{ba} \left[ s_a, s_b^{\text{ret}}(s_a) \right] + Q_{ab} \bigg|_{s_a = s_b = \tau_a}, \tag{5.10}
\]
\[
\dot{M}_{\text{rad},ab} = \int_{-\infty}^{\tau_b} ds_b \left( z_b \wedge G_{ab}^{\text{ret}} - e_b u_b \wedge A_{ab}^{\text{ret}} \right) + \int_{-\infty}^{\tau_a} ds_a \left( z_a \wedge G_{ba}^{\text{ret}} - e_a u_a \wedge A_{ba}^{\text{ret}} \right) \tag{5.11}
\]
Since $Q_{ab} \neq Q_{ba}$, the expressions are not symmetric in indices $a$ and $b$.

Substituting eqs. (5.3) and (5.4) into right-hand sides of eqs. (5.1) and (5.2), respectively, we obtain the total energy-momentum,
\[
P = \sum_{b=1}^{2} \left\{ \left. p_b + \frac{2e_b^2}{3} \int_{-\infty}^{\tau_b} ds_b a_b^2 u_b \right|_{s_b = \tau_b} + \frac{2e_b^2}{3} \int_{-\infty}^{\tau_b} ds_b G_{ab}^{\text{ret}} \right. \bigg|_{s_b \rightarrow -\infty} \right\}, \tag{5.12}
\]
and angular momentum,
\[
\dot{M} = \sum_{b=1}^{2} \left\{ \left. z_b \wedge p_b + \frac{2e_b^2}{3} \int_{-\infty}^{\tau_b} ds_b \left( a_b^2 z_b \wedge u_b + u_b \wedge a_b \right) \right. \bigg|_{s_b = \tau_b} \right. \right\} \tag{5.13}
\]
(In our notations $\kappa_1 := 1 - \kappa$ and $\kappa_2 := \kappa$) The balance equations are differential consequences of these conserved quantities. Since the action is not propagated instantaneously, the balance in a vicinity of the first charge as well as in a neighborhood of the second charge should be achieved separately,
\[
\dot{p}_a = -\frac{2e_a^2}{3} a_a^2 u_a - G_{ba}^{\text{ret}} - \kappa_a \dot{Q}_{ab}, \tag{5.14}
\]
\[
u_a \wedge \left( \left. p_a + \frac{2e_a^2}{3} a_a - e_a A_{ba}^{\text{ret}} + \kappa_a Q_{ab} \right|_{s_a \rightarrow -\infty} \right) = 0. \tag{5.15}
\]
(The overdot means the derivation with respect to individual proper time $\tau_a$.) Solution of six linear equations (5.15) in four components of $a$-th 4-momentum contains an arbitrary scalar function, say $m_a$,

$$p_a = m_a u_a - \frac{2e_a^2}{3} a_a + e_a A_{ba}^{ret} - \kappa_a Q_{ab}. \quad (5.16)$$

Since $(u_a \cdot a_a) = 0$, the scalar product of momentum of $a$-th particle on its 4-acceleration is as follows:

$$(p_a \cdot a_a) = -\frac{2e_a^2}{3} a_a^2 + e_a (A_{ba}^{ret} \cdot a_a) - \kappa_a (Q_{ab} \cdot a_a). \quad (5.17)$$

Similarly, the scalar product of particle’s 4-velocity on the first order time derivative of particle’s 4-momentum (5.14) is given by

$$(p_a \cdot u_a) = \frac{2e_a^2}{3} a_a^2 - (G_{ba}^{ret} \cdot u_a) - \kappa_a (\dot{Q}_{ab} \cdot u_a). \quad (5.18)$$

Equipped with the expression (5.7) one can derive the significant relation,

$$G_{ba}^\alpha = -F_{ba}^\alpha - e_a \frac{dA_{ba}^\alpha}{d\tau_a}; \quad (5.19)$$

where $F_{ba}^\alpha$ is the well-known Lorentz force of $b$-th charge acted on $a$-th one. Substituting this into eq.(5.18) and summing up (5.17) and modified (5.18), we obtain

$$(p_a \cdot u_a) = \frac{2e_a^2}{3} a_a^2 - (G_{ba}^{ret} \cdot u_a) - \kappa_a (\dot{Q}_{ab} \cdot u_a) \quad (5.20)$$

where dot means the $a$-th proper time derivative. On the other hand the scalar product of 4-momentum (5.16) on $a$-th 4-velocity is written as

$$(p_a \cdot u_a) = -m_a + e_a (A_{ba}^{ret} \cdot u_a) - \kappa_a (Q_{ab} \cdot u_a). \quad (5.21)$$

We see clearly that $m_a$ is of constant value. It can be interpreted as already renormalized mass of $a$-th charged particle.

Finally, we differentiate the expression (5.16) and substitute it for the left-hand side of eq.(5.14). After cancellation of like terms and taking into account eq.(5.19) we arrive at the Lorentz-Dirac equation,

$$m_a a_a = \frac{2e_a^2}{3} (\dot{a}_a - a_a^2 u_a) + F_{ba}^{ret}. \quad (5.22)$$

We see that particles’ equations of motion do not depend on mixing parameter $\kappa$.

In terms of kinematical variables conserved quantities of our particles plus field system looks as follows:

$$P = \sum_{a=1}^{2} \left( m_a u_a - \frac{2e_a^2}{3} a_a + \frac{2e_a^2}{3} \int_{-\infty}^{\tau_a} ds_a a_a^2 u_a \right) - \sum_{a \neq b} \int_{-\infty}^{\tau_a} ds_a F_{ba}^{ret}, \quad (5.23)$$

$$\dot{M} = \sum_{a=1}^{2} \left[ z_a \wedge \left( m_a u_a - \frac{2e_a^2}{3} a_a \right) + \frac{2e_a^2}{3} \int_{-\infty}^{\tau_a} ds_a \left( a_a^2 z_a \wedge u_a + u_a \wedge a_a \right) \right]$$

$$- \sum_{a \neq b} \int_{-\infty}^{\tau_a} ds_a z_a \wedge F_{ba}^{ret}. \quad (5.24)$$
The work done by Lorentz forces of charges acting on one another exhausts the radiation reaction due to combination of fields.

Individual 4-momentum (5.16) of \(a\)-th dressed charged particle is modified comparing with the well-known Teitelboim’s expression (1.7). The bound component \(\hat{T}_{\text{int.bnd}}\) of mixed part of the electromagnetic field stress-energy tensor contributes two additional terms: “immovable core”, \(e_a A_{ba}^{\text{ret}}\), of clear physical sense and “changeable shell”, \(-\kappa_a Q_{ab}\), that heavily depends on mixing parameter \(\kappa\). The corrections are inspired by unavoidable deformation of bound electromagnetic “clouds” due to mutual interaction between the sources. In my opinion, the changeable term arises due to forced choice of non-covariant surface of integration (see Subsection 4.4). For this reason we proclaim the expression

\[
p_a = m_a u_a - \frac{2e_a^2}{3} a_a + e_a A_{ba}^{\text{ret}}
\]

(5.25)
as the only one of true physical meaning.

6 Discrete symmetries

In the past Sections we have emphasized the importance of the invariance of the action integral (1.1) under the continuous group of space-time translations and rotations. According to Noether theorem, these symmetry properties imply conservation laws, i.e., those quantities that do not change with time. In this Section we study symmetry properties of energy-momentum and angular momentum carried by electromagnetic field which rely on invariance of (1.1) under discrete transformation groups.

6.1 Time inversion

The transformation of the time inversion \(T\) is defined by [2]

\[
T : y^\mu \mapsto y'^\mu = y_\mu.
\]

(6.1)

It immediately gives

\[
T : z_a^\mu(s_a) \mapsto z'^\mu_a(s'_a) = z_{0,\mu}(s_a).
\]

(6.2)

The proper time possesses odd parity [2],

\[
T : s_a \mapsto s'_a = -s_a.
\]

(6.3)

The other kinematic quantities then follow easily,

\[
T : u_a^\mu(s_a) \mapsto u'^\mu_a(s'_a) = -u_{a,\mu}(s_a),
\]

(6.4)

\[
T : a_a^\mu(s_a) \mapsto a'^\mu_a(s'_a) = a_{a,\mu}(s_a),
\]

etc. The retarded and the advanced instants transform into each other,

\[
T s^\text{ret}_a(s_b) = s^\text{adv}_a(s_b), \quad T s^\text{adv}_a(s_b) = s^\text{ret}_a(s_b).
\]

(6.5)
Inserting eqs. (6.4) and (6.5) into two-point function (4.36) yields
\[ T G_{ab}^{\alpha}(s_a, s_b) = -G_{ab, \alpha}(s_a, s_b), \quad (6.6) \]
\[ T G_{ab}^{0}(s_a, s_b) = -G_{ab, 0}(s_a, s_b). \]

To establish the symmetry properties of \( P_{\text{rad}} \) and \( \dot{M}_{\text{rad}} \) with respect to time inversion, we locate the observation hyperplane at the distant future,
\[ P_{\text{rad}} = \frac{2e_1^2}{3} \int_{-\infty}^{+\infty} ds_1 a^2_1 u_1 + \frac{2e_2^2}{3} \int_{-\infty}^{+\infty} ds_2 a^2_2 u_2 \quad (6.7) \]
\[ \dot{M}_{\text{rad}} = \frac{2e_1^2}{3} \int_{-\infty}^{+\infty} ds_1 (a^2_1 z_1 \wedge u_1 + u_1 \wedge a_1) + \frac{2e_2^2}{3} \int_{-\infty}^{+\infty} ds_2 (a^2_2 z_2 \wedge u_2 + u_2 \wedge a_2) \]
\[ + \kappa \int_{-\infty}^{+\infty} ds_1 [z_1 \wedge (G_{21} + G_{21}) - e_1 u_1 \wedge (A_{21} + A_{21})] \]
\[ + (1 - \kappa) \int_{-\infty}^{+\infty} ds_2 [z_2 \wedge (G_{12} + G_{12}) - e_2 u_2 \wedge (A_{12} + A_{12})]. \quad (6.8) \]

These expressions give the full amount of radiation emitted by interacting particles.

The transformation of the retarded and the advanced functions \( G_{ab} \) into each other implies that the radiated energy-momentum and angular momentum are of odd time parity,
\[ TP^\mu_{\text{rad}} = -P_{\text{rad}, \mu} ; \quad TM^{\mu\nu}_{\text{rad}} = -\dot{M}_{\text{rad}, \mu\nu}. \quad (6.9) \]

### 6.2 Space inversion

This operation is defined by [2]
\[ \mathcal{P} : y^\mu \mapsto y'^\mu = -y_\mu. \quad (6.10) \]
Correspondingly,
\[ \mathcal{P} : z^\mu_a(s_a) \mapsto z'^\mu_a(s'_a) = -z_{a, \mu}(s_a). \quad (6.11) \]

The proper time \( s_a \) remains invariant. These transformation properties imply that that particles’ 4-velocities and 4-accelerations change as follows:
\[ \mathcal{P} : u^\mu_a(s_a) \mapsto u'^\mu_a(s'_a) = -u_{a, \mu}(s_a), \quad (6.12) \]
\[ \mathcal{P} : a^\mu_a(s_a) \mapsto a'^\mu_a(s'_a) = -a_{a, \mu}(s_a). \]

Since the retarded and the advanced instants remain invariant, the basic two-point function (4.36) transforms analogously,
\[ \mathcal{P} G_{ab}^{\alpha} = -G_{ab, \alpha}. \quad (6.13) \]
Substituting this into eqs. (6.7) and (6.9) and using the relations (6.11) and (6.12) yields
\[ \mathcal{P} P^\mu_{\text{rad}} = -P_{\text{rad}, \mu} ; \quad \mathcal{P} M^{\mu\nu}_{\text{rad}} = -\dot{M}_{\text{rad}, \mu\nu}. \quad (6.14) \]
6.3 Reciprocity of particles 1 and 2

When a closed system of two identical charges is considered, radiative conserved quantities should be symmetric in indices 1 and 2 that label the particles. Having interchanged these indices in function (4.36) we obtain

\[
G^{\alpha}_{12}[s_{1}^{\text{ret}}(s_{1}), s_{2}^{\text{adv}}(s_{2})]_{1\leftrightarrow 2} = G^{\alpha}_{12}[s_{1}^{\text{adv}}(s_{1}), s_{2}^{\text{ret}}(s_{2})].
\]

From the reciprocity relations we see clearly that mixed parameter \(\kappa\) in expressions (6.7) and (6.9) should be equal to 1/2. Choosing the linear superposition

\[
p_{\text{int,rad}} = \frac{1}{2} (p_{\text{rad},21} + p_{\text{rad},12}),
\]

\[
\tilde{M}_{\text{int,rad}} = \frac{1}{2} (\tilde{M}_{\text{rad},21} + \tilde{M}_{\text{rad},12}),
\]

we restore invariance of radiated energy-momentum and angular momentum with respect to reciprocity of particles 1 and 2.

7 Conclusions

The present paper is devoted to study of phenomena of emission and propagation of energy in classical electrodynamics. The field in action (1.1) has its own uncountably infinite degrees of freedom. Variation of (1.1) yields Maxwell’s equations with point-like sources and equations of motions of particles interacting through the medium of the field. The problem then becomes one of mutual determination: the field is determined by the charged particles and their motion, and the motion of the charges is determined by the field.

In this paper we study interference of outgoing electromagnetic waves in a hyperplane \(\Sigma_{t} = \{y \in M_{4} : y^{0} = t\}\) associated with an unmoving inertial observer. We calculate how much electromagnetic field energy, momentum, and angular momentum flow across this hyperplane. Surface integration of the stress-energy tensor (1.4) over \(\Sigma_{t}\) reduces field’s uncountably infinite degrees of freedom. After the renormalization procedure we arrive at the action at a distance theory [22, 23] where particles interact directly with one another. The fields in resulting expressions (5.23) and (5.24) do not have degrees of freedom of their own: they are functionals of particle paths. Following Ref. [24], we refer to them as direct particle fields.

Starting with the retarded Liénard-Wiechert fields, after integration we arrive at the retarded and the advanced direct particle fields. The retarded and the advanced instants arise naturally as the end points of interference integrals (2.15) and (2.16). But the retarded causality is not violated because the advanced instants as well as the retarded ones are before the fixed observation moment \(t\). Direct particle fields referred to advanced instants do not describe neither incoming radiation nor converging electromagnetic waves.

Nevertheless, the retarded and the advanced quantities transform into each other under the influence of inversion of time axes. To demonstrate the invariance of electromagnetic
field’s energy-momentum and angular momentum with respect to time inversion, we locate
the observation surface $\Sigma_t$ at distant future. We show that the full amount of radiation
emitted by a closed system of two interacting charged particles is invariant with respect
to past and future as well as with respect to inversion of space axes.

The part of Noether quantities which escapes to infinity is defined by basic two-point
function (4.36). It is equal to the sum of Lorentz force $F_{ab}^0$ of $a$-th charge acted on $b$-th
one and the total time derivative of direct Liénard-Wiechert potential $A_{ab}$, taken with
opposite sign (see eq.(5.19)). In the specific case of very massive particles, such that
$e_1/m_1 << 1$ and $e_2/m_2 << 1$, the mixed energy-momentum $p_{\text{int,rad}}$ vanishes as could be
expected for nonaccelerated charges. Indeed, let us consider static charge $b$ at a coordinate
origin. Zeroth component $F_{ba}^0$, either retarded or advanced, is potential one: its integral
over indicated portion of $\zeta$ cancels the change of only nontrivial $A_{ba}^0$. Space components
$F_{ba}^i$ of the retarded and the advanced Lorentz forces compensate each other. Having
performed a trivial Lorentz transformation we extend the statement on a motion with
constant velocity. Radiative angular momentum possesses analogous properties.

To derive the equations of motion of interacting particles we compare flows of energy-
momentum and angular momentum through very close hyperplanes $\Sigma_t$ and $\Sigma_{t+\Delta t}$. Having
balanced particles’ individual characteristics and corresponding quantities carried by elec-
 tromagnetic field, we obtain the well-known Lorentz-Dirac equation of motion of charged
particle in the retarded field of the other charge where the self-action is taken into ac-
count. Since the Lorentz-Dirac equation possesses pathological solutions, such as runaway
solutions or preaccelerations, the authors [25, 26, 27] propose the Landau-Lifshitz equa-
tion [9, §76] as more satisfactory alternative. Spohn [25] showed that a solution of the
Lorentz-Dirac equation which does not satisfy the Landau-Lifshitz equation is of the run-
away type. Moreover, Landau-Lifshitz equation does not permit runaway solutions or
preacceleration [27].

Obtained expressions for radiated energy-momentum and angular momentum (includ-
ing interference effects) are valuable also for a strictly prescribed motion of particles under
the influence of a very powerful external force. It is necessary to compare them with cor-
responding results for circling charges [12, 13]. It would be interesting to consider the
specific case when the charged particles are a very close to each other. Since the electro-
magnetic field satisfies the superposition principle, the models either an extended object
consisting of $N$ point charges or a continuous charge distribution are based on dynamics
of two-body system [28, 29].

Acknowledgments

I am grateful to V.I. Tretyak for continuous encouragement and for a helpful reading of
this manuscript. I would like to thank A. Duviryak for many useful discussions.

Appendix A Integration over angular variable

To calculate the mixed rates of energy-momentum (2.1) and angular momentum (2.2)
carried by the electromagnetic field, we should first perform the integration over angle.
When facing this problem it is convenient to mark out $\varphi$-dependent terms in expressions under the integral sign. In the Maxwell energy-momentum tensor density we distinguish the second-order differential operator (3.10) with $\varphi$-dependent coefficients (3.7). It can be decomposed into a combination of partial derivatives in time variables $\Pi_a$ given by eq.(3.13) and tail $\pi_a$ of the type in eq.(3.14).

This Appendix is concerned with the computation of the tails. Equipped with them we express the aforementioned integrand as a combination of partial derivatives in $t_1$ and $t_2$.

To implement this strategy we must first integrate the coefficients (3.7) over the angle variable. We start with the simplest one,

$$D^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a}{q r_1 r_2},$$

(A.1)

where numerator $a$ is equal to 1 or $K^a_i$. Our task is to rewrite the integrand as a sum of term with denominator $r_1$ and term with denominator $r_2$. To do it we introduce a new layer of mathematical formalism and develop convenient technique.

Let $V$ be the vector space such that $i_0, i_1, i_2$ is its linear basis. We shall use $\eta_{\alpha\beta} = \text{diag}(-1, 1, 1)$ and its inverse $\eta^{\alpha\beta} = \text{diag}(-1, 1, 1)$ to lower and raise indices, respectively. We introduce the pairing

$$(\cdot) : V \times V \to \mathbb{R},$$

$$\ (a \cdot b) \mapsto \eta_{\alpha\beta} a^{\alpha} b^{\beta},$$

(A.2)

which will be called the “scalar product”.

We introduce null vector $n^a = (1, \sin \varphi, \cos \varphi)$ which belongs to the vector space $V$. We express the $\varphi$-dependent constructions

$$r_a = -(K_a \cdot v_a), \quad c_a = \gamma^{-2} a + (K_a \cdot \dot{v}_a)$$

(A.3)

as the scalar products $-(r_a \cdot n)$ and $(c_a \cdot n)$, respectively. We shall use sans-serif letters for the components of timelike three-vectors $r_a \in V$ and $c_a \in V$,

$$r_a^0 = k_a^0 - (v_a n_q) k_a^3, \quad r_a^1 = h v_a^i \omega_1^1, \quad r_a^2 = h v_a^i \omega_1^2;$$

$$c_a^0 = -\gamma^{-2} - (v_a n_q) k_a^3, \quad c_a^1 = h \dot{v}_a^i \omega_1^1, \quad c_a^2 = h \dot{v}_a^i \omega_1^2,$$

(A.4)

where orthogonal matrix,

$$\hat{\omega} = \begin{pmatrix} \cos \varphi_q & -\sin \varphi_q & 0 \\ \sin \varphi_q & \cos \varphi_q & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \vartheta_q & 0 & \sin \vartheta_q \\ 0 & 1 & 0 \\ -\sin \vartheta_q & 0 & \cos \vartheta_q \end{pmatrix},$$

(A.6)

is constructed from components of the relative position 3-vector $q = z_1 - z_2$. The numerator in eq.(A.1) is the scalar product $(a \cdot n) = -a^0 + a^1 \sin \varphi + a^2 \cos \varphi$; it is equal to 1 if vector $a = (-1, 0, 0)$.
We introduce the dual space of one-forms, say $\hat{\omega}$, with basis $\hat{\omega}_0$, $\hat{\omega}_1$, and $\hat{\omega}_2$, such that $\hat{\omega}_\mu(i_\nu) = \delta_\mu^\nu$, where $i_0, i_1, i_2$ constitute the basis of $V$. The wedge product $\hat{L} = \hat{a} \wedge \hat{b}$ of two one forms $\hat{a}$ and $\hat{b}$ constitutes two-form,

$$\hat{L} = (a_0b_1 - a_1b_0) \hat{\omega}^0 \wedge \hat{\omega}^1 + (a_0b_2 - a_2b_0) \hat{\omega}^0 \wedge \hat{\omega}^2 + (a_1b_2 - a_2b_1) \hat{\omega}^1 \wedge \hat{\omega}^2. \quad (A.7)$$

We introduce dual three-vector $L = ^*\hat{L}$ with components

$$L_\alpha = \frac{1}{2!} \varepsilon^{\alpha\beta\gamma} L_{\beta\gamma} \quad (A.8)$$

$\varepsilon^{\alpha\beta\gamma}$ denotes the Ricci symbol in three dimensions,

$$\varepsilon^{\alpha\beta\gamma} = \begin{cases} 
1 & \text{when } \alpha\beta\gamma \text{ is an even permutation of } 0, 1, 2 \\
-1 & \text{when } \alpha\beta\gamma \text{ is an odd permutation of } 0, 1, 2 \\
0 & \text{otherwise.} 
\end{cases} \quad (A.9)$$

We raise indices in eq.(A.8) and define the vector product of two vectors, $a$ and $b$,

$$L_\alpha = \varepsilon^\alpha_{\mu\nu} a^\mu b^\nu. \quad (A.10)$$

Tensor,

$$\varepsilon^\alpha_{\mu\nu} = \varepsilon^{\alpha\beta\gamma} \eta_{\beta\mu} \eta_{\gamma\nu}, \quad (A.11)$$

has the components

$$\varepsilon^0_{\mu\nu} = \varepsilon^{0\mu\nu}, \quad \varepsilon^1_{\mu\nu} = -\varepsilon^{1\mu\nu}, \quad \varepsilon^2_{\mu\nu} = -\varepsilon^{2\mu\nu}. \quad (A.12)$$

Now we calculate the double vector product,

$$[A[BC]]^\alpha = \varepsilon^{\alpha}_{\beta\gamma} A^\beta \varepsilon^\gamma_{\mu\nu} B^\mu C^\nu. \quad (A.13)$$

Since

$$\varepsilon^{\alpha}_{\beta\gamma} \varepsilon^\gamma_{\mu\nu} = -\delta^\alpha_{\mu} \eta_{\beta\nu} + \delta^\alpha_{\nu} \eta_{\beta\mu}, \quad (A.14)$$

we arrive to the unusual rule,

$$[A[BC]] = -B(A \cdot C) + C(A \cdot B), \quad (A.15)$$

instead of the well-known law acting in space with Euclidean metric.

To simplify the denominator $r_1 r_2$ in the integrand of eq.(A.1) as much as possible, we rewrite $2\pi$-periodic functions $r_a = -r_{a,0} - r_{a,1} \sin \varphi - r_{a,2} \cos \varphi$ as follows:

$$r_a = -r_{a,0} - \rho_a \sin(\varphi + \phi_a), \quad \rho_a = \sqrt{r_{a,1}^2 + r_{a,2}^2}. \quad (A.16)$$

(We recall that $r_a$ is the scalar products $(r_a \cdot n)$ taken with opposite sign, components $r^a_\mu$ are given by eqs.(A.4).) Shift in argument of harmonic function is determined by the relations

$$r_{a,1} = \rho_a \cos \phi_a, \quad r_{a,2} = \rho_a \sin \phi_a. \quad (A.17)$$
After some algebra we rewrite the integrand of eq.(A.1) as the following sum:
\[
\frac{a}{r_1 r_2} = \frac{A_{12}^a + C_{12}^a \rho_1 \cos(\varphi + \phi_1)}{r_1} + \frac{A_{21}^a - C_{12}^a \rho_2 \cos(\varphi + \phi_2)}{r_2},
\]
(A.18)
where \(a = (a \cdot n)\). Coefficients \(A_{12}^a, A_{21}^a\) and \(C_{12}^a\) satisfy the vector equation
\[
- A_{12}^a r_2 - A_{21}^a r_1 + C_{12}^a L_{12} = a,
\]
(A.19)
where by \(r_1\) and \(r_2\) we mean three-vectors with components in eq.(A.4) and \(L_{12} = [r_1 \ r_2]\).

To solve equation (A.19) we postmultiply it on the vector product \([r_1 L_{12}]\), then on the vector product \([r_2 L_{21}]\), and, finally, on \(L_{12}\). After some algebra we obtain
\[
A_{12}^a = \frac{([ar_1] \cdot L_{12})}{D_{12}}, \quad A_{21}^a = \frac{([ar_2] \cdot L_{21})}{D_{21}}, \quad C_{12}^a = \frac{(a \cdot L_{12})}{D_{12}},
\]
(A.20)
where the denominator \(D_{12} = (L_{12} \cdot L_{12})\) is symmetric in its indices.

Substituting eq.(A.18) into eq.(A.1) and using the identities
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{r_0^a - \rho_a \sin(\varphi + \phi_a)} = \frac{1}{\sqrt{(r_0^a)^2 - \rho_a^2}} = \frac{1}{\|r_0^a\|},
\]
and
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{r_0^a - \rho_a \sin(\varphi + \phi_a)} = 0
\]
yields
\[
D^a = \frac{A_{12}^a}{q\|r_1\|} + \frac{A_{21}^a}{q\|r_2\|},
\]
(A.22)
after integration over \(\varphi\).

Now we turn to the calculation of the coefficient
\[
B^a = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{r_0^a - \rho_a \sin(\varphi + \phi_a)}\frac{ac_2}{q(r_2)^2}.
\]
(A.23)
Equipped with the relations in eq.(A.18) we rewrite the integrand as a sum of terms which are proportional to the \(1/r_1\), \(1/r_2\), and \(1/(r_2)^2\), respectively. Using the identities
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{[r_0^a - \rho_a \sin(\varphi + \phi_a)]^2} = \frac{r_0^a}{\|r_0^a\|^3},
\]
(A.24)
and taking into account the relations in eq.(A.21) gives
\[
B^a = - \frac{1}{q\|r_2\|^3} \frac{([ar_1] \cdot c_2 \cdot r_2)}{D_{21}} (r_2 \cdot r_1) + \frac{1}{q\|r_2\|} \left[ A_{12}^c A_{21}^a + A_{12}^a A_{21}^c - \frac{(a \cdot c_2)(r_1 \cdot r_2)}{D_{21}} \right]
\]
\[+ \frac{1}{q\|r_1\|} \left[ 2A_{12}^a A_{12}^c - \frac{([ar_1] \cdot [c_2 r_1])}{D_{12}} \right].
\]
(A.25)
The resulting expression for the term
\[ C^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a c_1}{q(r_1)^2 r_2} \] (A.26)
can be obtained by interchanging indices 1 and 2 in the right-hand side of eq.(A.25).

After a routine computation based on the repeated usage of relation (A.18), we find
the most complicate term,
\[ A^a = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{a c_1 c_2}{q(r_1)^2(r_2)^2} \] (A.27)
where
\[ J_1 = 2A^{a}_{12}A^{a}_{12} - \frac{[c_1 r_1 \cdot c_2 r_1]}{D_{12}}, \] (A.28)
\[ B_{12} = 3A^{a}_{12}A^{a}_{12}A^{a}_{21} + 3A^{a}_{12}A^{a}_{21}A^{a}_{12} + 2A^{c}_{12}A^{c}_{12}A^{c}_{21} + A^{c}_{12} \left\{ \frac{[c_1 r_1 \cdot c_2 r_2]}{D_{12}} + \frac{[c_1 r_2 \cdot c_2 r_1]}{D_{12}} \right\} \]
\[ - A^{c}_{21} \left( \frac{[c_2 r_1 \cdot a r_1]}{D_{12}} \right) - A^{c}_{21} \left( \frac{[c_1 r_1 \cdot a r_1]}{D_{12}} \right) + A^{c}_{12} \left( \frac{[c_2 r_1 \cdot a r_2]}{D_{12}} \right) + A^{c}_{12} \left( \frac{[c_1 r_1 \cdot a r_2]}{D_{12}} \right), \]
and the others, \( B_{21} \) and \( J_2 \), can be obtained via interchanging indices 1 and 2.

We now turn to the differentiation of coefficient (A.22) with respect to time variables
\( t_1 \) and \( t_2 \). Having substituted \( o = (-1, 0, 0) \) for \( a \) in the expressions (A.22), (A.25), and
(A.27) we obtain the terms \( \mathcal{D}^0, \mathcal{B}^0 \) and \( \mathcal{A}^0 \), respectively. The remaining term, \( \mathcal{C}^0 \), can be
obtained from \( \mathcal{B}^0 \) via reciprocity. The calculations are based on the relations obtained
via differentiation of third components \( k^3_a \) and \( h^2 \), i.e. the square of radius of the circle
\( C(O, h) = S_1 \cap S_2 \),
\[ \frac{\partial k^3_a}{\partial t_a} = -(-1)^a r^0_q + (n_q v_q), \quad \frac{\partial k^3_a}{\partial t_b} = (-1)^a r^0_q, \]
\[ \frac{\partial h^2}{\partial t_1} = -2k^3_2 T^1_1, \quad \frac{\partial h^2}{\partial t_2} = 2k^3_1 T^2_2. \] (A.29)
They immediately give
\[ \frac{\partial r^0_a}{\partial t_a} = c^0_a + (-1)^a \left( \frac{(v_a n_q) r^0_a + [v_a n_q]^2 k^3_b}{q} \right), \] (A.30)
\[ \frac{\partial r^0_b}{\partial t_b} = \frac{(-1)^b}{q} \left( (v_a n_q) r^0_b + ([v_a n_q][v_b n_q]) k^3_a \right), \]
and eventually give
\[ \frac{\partial (r_a \cdot r_a)}{\partial t_a} = \left. \frac{2}{q} \right| (r_a \cdot c_a) + (-1)^a \left( \frac{(v_a n_q) r_a}{q} \right)(r_a \cdot r_a), \] (A.31)
\[ \frac{\partial (r_a \cdot r_a)}{\partial t_b} = 2(-1)^b \left[ \frac{(v_a n_q)}{q} (r_a \cdot r_b) + \frac{k^3_3 r^0_b (r_a \cdot r_a) - r^0_a (r_a \cdot r_b)}{h^2} \right], \]
\[ \frac{\partial (r_b \cdot r_a)}{\partial t_b} = (r_a \cdot c_b) + (-1)^b \left[ \frac{(v_a n_q)}{q} (r_b \cdot r_a) + \frac{(v_a n_q)}{q} (r_b \cdot r_b) + \frac{k^3_3 r^0_b (r_b \cdot r_a) - r^0_a (r_b \cdot r_b)}{h^2} \right]. \]
We use Latin indices $a$ and $b$ which run from 1 to 2 ($a \neq b$). We use bold script for conventional three-velocities $v_a^q = dz_a^q/dt_q$ and unit three-vector $n_a^q = q^q/q$ in $q$-direction. By $(v_b n_a)$ and $[v_b n_a]$ we denote the conventional scalar product and cross product of these vectors, respectively.

Usage of these relations allows us to calculate the derivatives of coefficients $A_{ab}^0$:

\[
\frac{\partial A_{ab}^0}{\partial t_a} = A_{ab}^0 A_{ba}^{ca} + A_{ab}^c A_{ba}^c - \frac{([o_a \cdot [c_a r_b]])}{D_{ab}}, \tag{A.32}
\]

\[
\frac{\partial A_{ab}^0}{\partial t_b} = 2 A_{ab}^c A_{ba}^c - \frac{([o_a \cdot [c_a r_b]])}{D_{ab}} + (-1)^b \frac{k^3}{q h^2} (o^0 + r^0_2 A_{12}^0 + r^0_1 A_{21}^0) + (-1)^a \left( A_{ab}^0 (n_q v_b) - A_{ba}^0 (n_q v_a) \right). \tag{A.33}
\]

Substituting these into equality

\[
\frac{\partial D^0}{\partial t_a} = \frac{\partial}{\partial t_a} \left( A_{12}^0 \left( \frac{A_{21}^0}{q ||r_1||} + \frac{A_{21}^0}{q ||r_2||} \right) \right), \tag{A.34}
\]

and using the identities

\[
\frac{\partial}{\partial t_a} \left( \frac{1}{||r_a||} \right) = \frac{(c_a \cdot r_a)}{||r_a||^3} - (-1)^a \frac{1}{||r_a||} \left( \frac{n_q v_a}{q} \right), \tag{A.35}
\]

\[
\frac{\partial}{\partial t_b} \left( \frac{1}{||r_a||} \right) = (-1)^b \frac{(r_a \cdot r_b)}{||r_a||^3} \left( n_q v_a \right) + (-1)^a \frac{1}{q ||r_a||^3} \frac{k^3}{h^2} r_a^0 (r_a \cdot r_b) - r_b^0 (r_a \cdot r_a)
\]

yields

\[
\frac{\partial D^0}{\partial t_1} = C^0 + \frac{k_2^0 (n_q v_2)}{q^2 ||r_2||^3}, \quad \frac{\partial D^0}{\partial t_2} = B^0 - \frac{k_1^0 (n_q v_1)}{q^2 ||r_1||^3}. \tag{A.36}
\]

Further we calculate the partial derivative $\partial C^0/\partial t_2$, subtract it from $A^0$, and prove the identity

\[
A^0 - \frac{\partial C^0}{\partial t_2} = \frac{\partial}{\partial t_1} \left( B^0 - \frac{\partial D^0}{\partial t_2} \right) \text{ i.e. } \pi^0 = 0. \tag{A.37}
\]

(One can derive $\partial B^0/\partial t_1$, subtract it from $A^0$, and compare the result with $\partial/\partial t_2 (C^0 - \partial D^0/\partial t_1)$.)

Now, we calculate the tail

\[
\tau_a^a = A_a^a - \frac{\partial B_a^a}{\partial t_1} - \frac{\partial C_a^a}{\partial t_2} + \frac{\partial D_a^a}{\partial t_1 \partial t_2}, \tag{A.38}
\]

where

\[
D_a^a = \frac{1}{2 \pi} \int_0^{2 \pi} d \varphi \frac{K_a^a}{q r_1 r_2}, \quad B_a^a = \frac{1}{2 \pi} \int_0^{2 \pi} d \varphi \frac{K_a^a c_2}{q r_1 (r_2)^2}, \quad C_a^a = \frac{1}{2 \pi} \int_0^{2 \pi} d \varphi \frac{K_a^a c_1 c_2}{q (r_1)^2 (r_2)^2}. \tag{A.39}
\]

\[
A_a^a = \frac{1}{2 \pi} \int_0^{2 \pi} d \varphi \frac{K_a^a q_0}{q (r_1)^2 (r_2)^2}.
\]
The zeroth component, $K^0_a = k^0_a$, does not depend on $\phi$. Inserting relations $D^0_a = k^0_a D^0$, $B^0_a = k^0_a B^0$, $C^0_a = k^0_a C^0$, and $A^0_a = k^0_a A^0$ into eq.(A.37) and taking into account identity (A.36) yields

$$\pi^0_1 = B^0 - \frac{\partial D^0}{\partial t_2}, \quad \pi^0_2 = C^0 - \frac{\partial D^0}{\partial t_1}. \quad \text{(A.39)}$$

Space components, $K^i_a$, depend on $\phi$. They can be expressed as the scalar product $(K^i_a \cdot n)$ where components of three-vectors $K^i_a \in V$ are as follows:

$$K^i_{a,0} = n^i_a k^3_a, \quad K^i_{a,1} = h \omega^i_1, \quad K^i_{a,2} = h \omega^i_2. \quad \text{(A.40)}$$

Here $\omega^i_j$ are components of the orthogonal matrix (A.6). Having substituted $K^i_a$ for $a$ in expressions (A.22), (A.25), and (A.27) we obtain the terms $D^i_a$, $B^i_a$ and $A^i_a$, respectively. The last term, $C^i_a$, can be obtained from $B^i_a$ via reciprocity. To differentiate them we need the equalities

$$\frac{\partial (K^i_a \cdot r_i)}{\partial t_1} = (K^i_a \cdot c_i) - v^i_1 r^i_1 - \frac{n^i_a v_1}{q} (K^i_a \cdot r_i) - \frac{n^i_a}{q} (r_i \cdot r_i)$$

$$\frac{\partial (K^i_a \cdot r_i)}{\partial t_2} = \frac{\partial^2 (K^i_a \cdot r_i)}{\partial t^2} = \frac{\partial (K^i_a \cdot r_i)}{\partial t_1} - \frac{k^3_a}{q \hbar^2} \left[ K^i_{a,0} (r_i \cdot r_i) + r^i_0 (K^i_a \cdot r_i) \right],$$

$$\frac{\partial (K^i_a \cdot r_i)}{\partial t_2} = \frac{\partial (K^i_a \cdot r_i)}{\partial t_1} - \frac{k^3_a}{q \hbar^2} \left[ -r^i_1 (K^i_a \cdot r_i) + K^i_{a,0} (r_i \cdot r_i) + 2r^i_0 (K^i_a \cdot r_i) \right],$$

$$\frac{\partial (K^i_a \cdot r_i)}{\partial t_2} = \frac{\partial (K^i_a \cdot r_i)}{\partial t_1} - \frac{k^3_a}{q \hbar^2} \left[ K^i_{a,0} (r_i \cdot r_i) + r^i_0 (K^i_a \cdot r_i) \right],$$

in addition to eqs. (A.31) and (A.34).

The derivation of equalities

$$C^i_a - \frac{\partial D^i_a}{\partial t_1} = v^i_1 D^i_a - \frac{n^i_a}{q^2 \|r_i\|^2} - \frac{1}{q \|r_i\|^3} \frac{(n^i_a v_2)}{q} (K^i_a \cdot r_i)$$

$$\frac{\partial D^i_a}{\partial t_2} = \frac{n^i_a}{q^2 \|r_i\|^2} + \frac{1}{q \|r_i\|^3} \frac{(n^i_a v_1)}{q} (K^i_a \cdot r_i)$$

$$B^i_a - \frac{\partial D^i_a}{\partial t_2} = - \frac{1}{q \|r_i\|^3} \frac{k^3_a}{q} \left[ K^i_{a,0} (n^i_a v_1)^2 + r^i_0 (n^i_a v_1 n_i)^2 \right],$$

$$A^i_a - \frac{\partial D^i_a}{\partial t_2} = - \frac{1}{q \|r_i\|^3} \frac{k^3_a}{q} \left[ K^i_{a,0} (n^i_a v_2)^2 + r^i_0 (n^i_a v_2 n_i)^2 \right].$$
is virtually identical to that presented above, and we shall not bother with the details. By virtue of the relation $K_2^i - K_1^i = q^i$ the integrals (A.38) are related as follows:

$$D_2^i = D_1^i + q^i D^0, \quad B_2^i = B_1^i + q^i B^0, \quad C_2^i = C_1^i + q^i C^0, \quad A_2^i = A_1^i + q^i A^0.$$  \hspace{1cm} (A.42)

Equipped with these expressions we find

$$C_2^i - \frac{\partial D_2^i}{\partial t_1} = -\frac{n_q^i}{q^2 \| r_2 \|^2} - \frac{1}{q^2 \| r_2 \|^3} \left( n_q v_2 \right) \left( K_2^i \cdot r_2 \right) \hspace{1cm} (A.43)$$

$$+ \frac{1}{q^2 \| r_2 \|^3} \left( K_{2,0}^i [n_q v_2]^2 + r_2^0 [n_q v_2]^2 \right),$$

$$B_2^i - \frac{\partial D_2^i}{\partial t_2} = v_2^i D^0 + \frac{n_q^i}{q^2 \| r_1 \|^2} + \frac{1}{q^2 \| r_1 \|^3} \left( n_q v_1 \right) \left( K_1^i \cdot r_1 \right)$$

$$- \frac{1}{q^2 \| r_1 \|^3} \left( K_{1,0}^i [n_q v_1]^2 + r_1^0 [n_q v_1]^2 \right).$$

Finally, after a straightforward (but fairly lengthy) calculations we derive the following relations:

$$\pi_1^i = v_1^i \left( B^0 - \frac{\partial D^0}{\partial t_2} \right), \quad \pi_2^i = v_2^i \left( C^0 - \frac{\partial D^0}{\partial t_1} \right),$$  \hspace{1cm} (A.44)

which generalize eqs. (A.39).

We will need also the tail

$$\pi_{12}^{\alpha \beta} = \frac{\partial^2 D_{12}^{\alpha \beta}}{\partial t_1 \partial t_2} - \frac{\partial B_{12}^{\alpha \beta}}{\partial t_1} - \frac{\partial C_{12}^{\alpha \beta}}{\partial t_2} + A_{12}^{\alpha \beta},$$  \hspace{1cm} (A.45)

where

$$D_{12}^{\alpha \beta} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta}{q r_1 r_2}, \quad B_{12}^{\alpha \beta} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta c_2}{q r_1 (r_2)^2},$$

$$C_{12}^{\alpha \beta} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta c_1}{q (r_1)^2 r_2}, \quad A_{12}^{\alpha \beta} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{K_1^\alpha K_2^\beta c_1 c_2}{q (r_1)^2 (r_2)^2}.$$  \hspace{1cm} (A.46)

It can be obtained by means of covariant generalization of previous relations. Setting $\alpha = 0$ and $\beta = 0$ and taking into account eq. (A.36), we obtain

$$\pi_{12}^{00} = k_1^0 \left( C^0 - \frac{\partial D^0}{\partial t_1} \right) + k_2^0 \left( B^0 - \frac{\partial D^0}{\partial t_2} \right) + D^0$$

$$= C_1^0 - \frac{\partial D_1^0}{\partial t_1} + B_2^0 - \frac{\partial D_2^0}{\partial t_2} - D^0,$$  \hspace{1cm} (A.47)

where relations $D_a^0 = k_a^0 D^0, B_a^0 = k_a^0 B^0,$ and $C_a^0 = k_a^0 C^0$ are taken into account. If $\alpha = i$ and $\beta = 0$, we have

$$\pi_{12}^{i0} = C_1^i - \frac{\partial D_1^i}{\partial t_1} + k_2^0 v_1^i \left( B_2^0 - \frac{\partial D_2^0}{\partial t_2} \right)$$

$$= C_1^i - \frac{\partial D_1^i}{\partial t_1} + v_1^i \left( B_2^0 - \frac{\partial D_2^0}{\partial t_2} \right) - v_1^i D^0.$$  \hspace{1cm} (A.48)
If $\alpha = 0$ and $\beta = j$, we arrive at

$$\pi_{12}^{ij} = k_1^{0} v_2^{ij} \left( c_0 \frac{\partial D_0}{\partial t_1} + B_2^j - \frac{\partial D_2^j}{\partial t_2} \right)$$

$$= v_2^{ij} \left( c_1^{0} \frac{\partial D_0}{\partial t_1} + B_2^j - \frac{\partial D_2^j}{\partial t_2} - v_2^{ij} D^0. \right) \quad \text{(A.49)}$$

An obvious generalization of expressions (A.47)-(A.49) is

$$\pi_{12}^{\alpha\beta} = v_1^{\alpha} \left( B_2^{\beta} - \frac{\partial D_2^{\beta}}{\partial t_2} \right) + v_2^{\beta} \left( C_1^{\alpha} - \frac{\partial D_1^{\alpha}}{\partial t_1} \right) - v_1^{\alpha} v_2^{\beta} D^0. \quad \text{(A.50)}$$

Since the angular integration leads to a combination of partial derivatives in time variables, the end points are valuable only. At these points the radius of circle $C(O, h) = S_1 \cap S_2$ vanishes. Hence we can restrict ourselves to calculation of expressions $B_{12}^{ij} - \frac{\partial D_{12}^{ij}}{\partial h}$ and $C_{12}^{ij} - \frac{\partial D_{12}^{ij}}{\partial h}$ at points where $h = 0$.

To simplify the calculations as much as possible we express the integrands of eqs.(A.46) in form of expansions in powers of $h$. It allows us to remove harmonic functions from denominators. Since the derivatives $\partial h^2 / \partial t_n$ do not vanish whenever $h^2 = 0$ (see eq.(A.29)), it is sufficient to expand $D_{12}^{ij}$ up to the first order of this parameter,

$$D_{12}^{ij} = \frac{k_1^2 k_2^3 n_q^i n_q^j}{q^2 r_2^0 r_2^0} \left[ 1 + \frac{h^2}{2} \left( \frac{[n_q v_1]}{(r_2^0)^2} + \frac{[n_q v_2]}{(r_2^0)^2} + \frac{[n_q v_1 v_2]}{(r_2^0)^2} \right) \right] \quad \text{(A.51)}$$

$$+ \frac{h^2 k_1^3 n_q^i [v_1 n_q]}{(r_2^0)^2} + \frac{h^2 k_1^3 n_q^j [v_2 n_q]}{(r_2^0)^2}$$

$$+ \frac{h^2 \delta^{ij} - n_q^i n_q^j}{2} \quad \text{(A.52)}$$

With degree of accuracy sufficient for our purposes,

$$B_{12}^{ij} = \frac{n_q^i k_1 n_q^j k_2 c_{2,0}}{q^2 (r_2^0)^2}, \quad C_{12}^{ij} = \frac{\delta^{ij} - n_q^i n_q^j}{q (r_2^0)^2 r_2^0}.$$
Appendix B  Bound parts of mixed energy-momentum and angular momentum

In this Appendix we consider the parts of mixed energy and mixed momentum which describe unavoidable deformations of electromagnetic “clouds” of charged particles due to mutual interaction. The short-range terms will be absorbed by four-momenta of “bare” particles within the renormalization procedure. They arise from the bound part of mixed momentum,

\[ \mathcal{P}_{\text{int, bnd}}^i = \mathcal{P}_{\text{int}}^i - \mathcal{P}_{\text{int, rad}}^i \]  

which is equal to the difference of the total interference momentum (3.18) and the radiative component (4.11).

Our next task is to integrate \( \mathcal{P}_{\text{int, bnd}}^i \) over time variables by means of expressions (2.15) and (2.16). Recall from Section 4 that it is sufficient to evaluate the arguments of time differential operators at the ends of integration intervals.

\[ \frac{\varepsilon_1 \varepsilon_2}{2} \left[ \tilde{\Pi}_2^i \left( \frac{\partial \sigma}{\partial t_2} \right) + \tilde{\Pi}_1^0 \left( v_2^i \lambda_1 \right) - \frac{\partial}{\partial t_1} \left( v_2^i \frac{\partial \sigma}{\partial t_2} D^0 \right) \right] + \tilde{\Pi}_1^i \left( \frac{\partial \sigma}{\partial t_1} \right) + \tilde{\Pi}_0^i \left( v_1^i \lambda_2 \right) - \frac{\partial}{\partial t_2} \left( v_1^i \frac{\partial \sigma}{\partial t_1} D^0 \right) \]

Figure 6: Boundary points of interference of spherical wave fronts \( S_1 \) and \( S_2 \) in a hyperplane \( \Sigma_t \). In momentarily comoving Lorentz frame pictured in Fig. 1 the wave fronts intersect at the coordinate origin. If \( h = 0 \), triangle \( O_1 O_2 H \) reduces to the line. Distance \( q \) between particles is equal to the difference of radii of spheres.

In Figs. 6 the boundary conditions of interference of spherical wave fronts \( S_1(O_1, k^0_1) \) and \( S_2(O_2, k^0_2) \) are presented. (Fig. 3 pictures the combination of waves in four-dimensional spacetime.) Distance \( q \) between their centers, \( O_1 \) and \( O_2 \), is equal to the difference of their...
Figure 7: Acausal interference in a hyperplane $\Sigma_t$. Distance $q$ between particles placed at centers of spheres is equal to the sum of their radii.

radii, $k_1^0$ and $k_2^0$. In the left figure $q = k_2^0 - k_1^0$ while in the right one $q = k_1^0 - k_2^0$. Inserting this into eq.(2.5) gives $k_3^0 = k_1^0$ and $k_3^0 = -k_1^0$ for the first and the second cases, respectively.

The retarded and the advanced instants label the points on particles’ world lines which are connected by a null ray. Since the relative position vector $q$ is of null length, the null vectors $K_1$ and $K_2$ are collinear: $(K_1 \cdot K_2) = 0$. Their space parts $K_1$ and $K_2$ are codirectional in this case.

Within acausal region wave fronts combine in quite different manner (see Fig. 7). In contrast to the pair of points $z_1(t_1)$ and $z_2[t_2^{ret}(t_1)]$, the vertices $z_1(t_1) \in \zeta_1$ and $z_2(t_1') \in \zeta_2$ are spacelike related. Indeed, the scalar product of the separation null vectors takes minimal value $-2k_1^0k_2^0$ if their space parts $K_1$ and $K_2$ are opposite directed. Hence $(q \cdot q) = 4k_1^0k_2^0$.

In Table 1 we collect the basic quantities and functions which are evaluated at the boundary points.

To integrate the expression (B.1) over time variables, we apply the scheme developed in Section 4. Usage of the rule (4.7) implies

$$p_{bnd,21}^i = \frac{e_1e_2}{2} \frac{v_i^1 + n_q^i}{k_2^0 [1 - (n_q v_2)]} [t_{2}^{ret}(t_1)]_{t_1 \to -\infty} + \frac{e_1e_2}{2} \frac{v_i^2 - n_q^i}{k_2^0 [1 + (n_q v_2)]} [t_{1}^{ret}(t_2)]_{t_1 \to -\infty}$$

$$+ \frac{e_1e_2}{2} \frac{v_i^1 - n_q^i}{k_2^0 [1 - (n_q v_2)]} [t_{2}^{ret}(t_1)]_{[t_1^{ret}(t_2),t_2]}$$

(B.2)
If one prefer another order of differentiation (4.8) they obtain

\[
p^{\text{i}}_{\text{bnd,12}} = e_1 e_2 \left[ \frac{v_i^j + n_i^j}{2 k_0^j [1 - (n_y v_1)]} \right]_{t_2 \rightarrow -\infty} + e_1 e_2 \left[ \frac{v_i^j - n_i^j}{2 k_0^j [1 + (n_y v_2)]} \right]_{t_2 \rightarrow -\infty} + e_1 e_2 \left[ \frac{v_i^j + n_i^j}{2 k_0^j [1 - (n_y v_2)]} \right]_{t \rightarrow t_2^{\text{ret}}(t)} \] (B.3)

The lower limits \( t_a \rightarrow -\infty \) vanish even if the motion is finite. Final expressions depend on particles’ positions and velocities referred to the moments \( t_1^{\text{ret}}(t) \) and \( t_2^{\text{ret}}(t) \) as well as on the laboratory time \( t \) itself,

\[
p^{\text{i}}_{\text{bnd,21}} = e_1 e_2 \left[ \frac{v_i^j}{k_0^j [1 - (n_y v_2)]} \right]_{t_1^{\text{ret}}(t_2) \rightarrow t_2} + e_1 e_2 \left[ \frac{v_i^j - n_i^j}{2 k_0^j [1 + (n_y v_2)]} \right]_{t \rightarrow t_2^{\text{ret}}(t_2)} \] (B.4)

\[
p^{\text{i}}_{\text{bnd,12}} = e_1 e_2 \left[ \frac{v_i^j}{k_0^j [1 + (n_y v_1)]} \right]_{t_1^{\text{ret}}(t_2) \rightarrow t_2^{\text{ret}}(t_1)} + e_1 e_2 \left[ \frac{v_i^j + n_i^j}{2 k_0^j [1 - (n_y v_1)]} \right]_{t \rightarrow t_2^{\text{ret}}(t_1)} \] (B.5)

In an analogous we find short-range contribution to the mixed energy due to time integration of the following expression:

\[
\mathcal{P}^{0}_{\text{int,bnd}} = \mathcal{P}^{0}_{\text{int}} - \mathcal{P}^{0}_{\text{int,rad}} \] (B.6)

\[
= e_1 e_2 \left[ \hat{\Pi}^0 \left( k_1^0 \frac{\partial \sigma}{\partial t_1} + k_2^0 \frac{\partial \sigma}{\partial t_2} + \sigma - \frac{1}{2} \frac{\partial \sigma}{\partial t_1} \frac{\partial \sigma}{\partial t_2} \right) \right]
\]
The calculation is virtually identical to that presented above, and we shall not bother with details. Finally, we obtain

\[
p_{\text{bnd.21}}^0 = \frac{e_1 e_2}{k_0^2 [1 - (n_q v_2)]} \left[ \frac{v_2^i - n_q^i}{k_0^2 [1 - (n_q v_2)]} - \frac{z_q^i}{k_0^2 [1 - (n_q v_2)]} - \frac{t_2 n_q^i - z_q^i n_q^0}{k_0^2 [1 - (n_q v_2)]} \right] \quad \text{[t_{\text{bnd.21}}^t]} \]

\[
p_{\text{bnd.12}}^0 = \frac{e_1 e_2}{k_1^0 [1 + (n_q v_1)]} \left[ \frac{v_2^i - n_q^i}{k_1^0 [1 + (n_q v_1)]} - \frac{z_q^i}{k_1^0 [1 + (n_q v_1)]} - \frac{t_2 n_q^i - z_q^i n_q^0}{k_1^0 [1 + (n_q v_1)]} \right] \quad \text{[t_{\text{bnd.12}}^t]} \]

As could be expected for bound terms, they (i) depend on the momentary state of particles' motion, (ii) contain divergent terms, and (iii) are non-covariant.

The bound components of angular momentum have similar structure,

\[
M_{\text{bnd.21}}^{iji} = \frac{e_1 e_2}{k_0^0 [1 - (n_q v_2)]} \left[ \frac{v_2^i - n_q^i}{k_0^0 [1 - (n_q v_2)]} - \frac{z_q^i}{k_0^0 [1 - (n_q v_2)]} - \frac{t_2 n_q^i - z_q^i n_q^0}{k_0^0 [1 - (n_q v_2)]} \right] \quad \text{[t_{\text{bnd.21}}^t]} \]

\[
M_{\text{bnd.11}}^{iji} = \frac{e_1 e_2}{k_1^0 [1 + (n_q v_1)]} \left[ \frac{v_2^i - n_q^i}{k_1^0 [1 + (n_q v_1)]} - \frac{z_q^i}{k_1^0 [1 + (n_q v_1)]} - \frac{t_2 n_q^i - z_q^i n_q^0}{k_1^0 [1 + (n_q v_1)]} \right] \quad \text{[t_{\text{bnd.11}}^t]} \]

Alternative expressions, \( M_{\text{bnd.21}}^{\mu\nu} \), can be obtained via reciprocity of indices 1 and 2.

In contrast to one-particle case, expanding of the expressions under limit signs in powers of \( \Delta a = t - t_a \) does not yield simple and manifestly covariant terms of clear physical sense. The “deformation” is due to the choice of the coordinate-dependent hole around the particle in the integration surface \( \Sigma_t \). We neglect these structureless terms.

References

[1] P. A. M. Dirac, Proc. R. Soc. London, Ser. A 167, 148 (1938).
[2] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Redwood, CA, 1990).

[3] C. Teitelboim, Phys. Rev. D 1, 1572 (1970).

[4] H. J. Bhabha, Proc. R. Soc. London, Ser. A 172, 384 (1939).

[5] C. A. López and D. Villarroel, Phys. Rev. D 11, 2724 (1975).

[6] J. M. Aguirregabiria and L. Bel, Phys. Rev. D 29, 1099 (1984).

[7] R. Lapiedra and A. Molina, J. Math. Phys. 20, 1308 (1979).

[8] R. Lapiedra, F. Marqués and A. Molina, J. Math. Phys. 20, 1316 (1979).

[9] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 2nd ed. (Pergamon, Oxford, 1962).

[10] J. M. Aguirregabiria and J. R. Etxebarria, J. Math. Phys. 29, 1832 (1988).

[11] N. P. Klepikov, Phys. Atomic Nuclei 58, 1227 (1995); Yad.Fiz. 58, 1307 (1995) (*in Russian*).

[12] R. Rivera and D. Villarroel, J. Math. Phys. 38, 5690 (1997).

[13] R. Rivera and D. Villarroel, J. Math. Phys. 43, 5026 (2002).

[14] V. Hnizdo, Phys. Lett. A 129, 426 (1988).

[15] V. Hnizdo, J. Math. Phys. 39, 5663 (1998).

[16] R. Rivera and D. Villarroel, J. Math. Phys. 39, 5664 (1998).

[17] Yu. Yaremko, J. Phys. A: Math. Gen. 37, L531 (2004).

[18] Yu. Yaremko, Int. J. Mod. Phys. A 20, 129 (2005).

[19] B. P. Kosyakov, Phys. Rev. D 57, 5032 (1998).

[20] Yu. Yaremko, J. Phys.A: Math.Gen. 36, 5149 (2003).

[21] Yu. Yaremko, J. Phys.A: Math. Gen. 37, 1079 (2004).

[22] J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys. 17, 157 (1945).

[23] J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys. 21, 425 (1949).

[24] F. Hoyle and J.V. Narlikar, Rev. Mod. Phys. 67, 113 (1995).

[25] H. Spohn, Europhys. Lett. 50, 287 (2000).

[26] F. Rohrlich, Phys. Lett. A 283, 276 (2001).

[27] G. Ares de Parga, Found. Phys. 36, 1474 (2006).
[28] A. Ori and E. Rosenthal, Phys. Rev. D 68, 041701(R) (2003).

[29] A. Ori and E. Rosenthal, J. Math. Phys. 44, 2347 (2004).