LOCAL WELL-POSEDNESS AND A PRIORI BOUNDS FOR THE MODIFIED BENJAMIN-ONO EQUATION WITHOUT USING A GAUGE TRANSFORMATION

ZHUA GUO

Abstract. We prove that the complex-valued modified Benjamin-Ono (mBO) equation is locally wellposed if the initial data \( \phi \) belongs to \( H^s \) for \( s \geq 1/2 \) with \( \| \phi \|_{L^2} \) sufficiently small without performing a gauge transformation. Hence the real-valued mBO equation is globally wellposed for those initial datas, which is contained in the results of C. Kenig and H. Takaoka [25] where the smallness condition is not needed. We also prove that the real-valued \( H^{\infty} \) solutions to mBO equation satisfy a priori local in time \( H^s \) bounds in terms of the \( H^s \) size of the initial data for \( s > 1/4 \).

Contents

1. Introduction 1  
2. Notation and Definitions 5  
3. Properties of the Spaces \( Z_k \) 7  
4. A Symmetric Estimate 11  
5. Trilinear Estimates 16  
6. Proof of Theorem 1.1 24  
7. Short-time Trilinear Estimates 27  
8. Energy Estimates 31  
9. Proof of Theorem 1.3 37  
References 41

1. Introduction

In this paper, we study the initial value problem for the (defocusing) modified Benjamin-Ono equation of the form (also the equation with focusing nonlinearity of the form \(-u^2u_x\) can be treated by our method)

\[
\begin{align*}
&u_t + Hu_{xx} = u^2u_x, \quad (x,t) \in \mathbb{R}^2, \\
u(x,0) = \phi(x),
\end{align*}
\]

Key words and phrases. Modified Benjamin-Ono equation, Global wellposedness, A priori bounds.

This work is supported in part by RFDP of China No. 20060001010, the National Science Foundation of China, grant 10571004; and the 973 Project Foundation of China, grant 2006CB805902, and the Innovation Group Foundation of NSFC, grant 10621061.
where $u : \mathbb{R}^2 \to \mathbb{C}$ is a complex-valued function and $\mathcal{H}$ is the Hilbert transform which is defined as following

$$\mathcal{H}u(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{u(y)}{x-y} dy.$$  (1.2)

The equation with quadratic nonlinearity

$$u_t + \mathcal{H}u_{xx} = uu_x$$  (1.3)

was derived by Benjamin [2] and Ono [31] as a model for one-dimensional waves in deep water. On the other hand, the cubic nonlinearity is also of much interest for long wave models [1, 19].

The initial value problems for (1.1) and for the Benjamin-Ono equation (1.3) have been extensively studied [3, 7, 9, 10, 11, 13, 17, 18, 20, 21, 25, 27, 28, 29, 30, 32]. For instance, the energy method provides the wellposedness on the Sobolev space $H^s$ for $s > 3/2$ [17]. For the Benjamin-Ono equation (1.3), it has been know [18, 27] that this is locally wellposed for $s > 9/8$ by the refinement of the energy method and dispersive estimates. Tao [34] obtained the global wellposedness in $H^s$ for $s \geq 1$ by performing a gauge transformation as for the derivative Schrödinger equation and using the conservation law. This result was improved by Ionescu and Kenig [14] who obtained global wellposedness for $s \geq 0$, and also by Burq and Planchon [4] who obtained local wellposedness for $s > 1/4$.

For the modified Benjamin-Ono equation (1.1), Molinet and Ribaud [29] obtained the local wellposedness in Sobolev space $H^s$ for $s > 1/2$. Their proof is based on Tao’s gauge transformation [34]. The result for $s = 1/2$ has been obtained by Kenig and Takaoka [25] by using frequency dyadically localized gauge transformation. Their result was sharp in the sense that the solution map is not uniformly continuous in $H^s$ for $s < 1/2$. With the Sobolev space $H^s$ replaced by the Besov space $B^s_{2,1}$, the result has been obtained in [30] under a smallness condition on the data. To the author’s knowledge, these results are all restricted to the real-valued mBO equation where the gauge is easy to handle. Our method in this paper can also deal with the complex-valued mBO equation.

The mBO equation (1.1) has several symmetries. The first one is the scaling invariance

$$u(x, t) \rightarrow u_\lambda = \frac{1}{\lambda^{1/2}} u(x, t), \quad \phi_\lambda = \frac{1}{\lambda^{1/2}} \phi(x),$$  (1.4)

which leads to the constraint $s \geq 0$ on the wellposedness for (1.1). We see that the equation (1.1) is $L^2$ critical, hence the $L^2$ norm of the initial data is not automatically small by the scaling, which is the main reason for us assuming the initial data has a small $L^2$ norm. There are at least the following three conservation laws preserved under the flow of the real-valued mBO equation (1.1)

$$\frac{d}{dt} \int_\mathbb{R} u(x, t) dx = 0,$$  (1.5)

$$\frac{d}{dt} \int_\mathbb{R} u(x, t)^2 dx = 0,$$  (1.6)

$$\frac{d}{dt} \int_\mathbb{R} \frac{1}{2} u \mathcal{H}u_x - \frac{1}{12} \left| u(x, t) \right|^4 dx = 0.$$  (1.7)

These conservation laws provide a priori bounds on the solution. For instance, we get from (1.6) and (1.7) that the $H^{1/2}$ norm of the solution remains bounded for
finite time if the initial data belongs to $H^{1/2}$. Thus once one obtains a solution of existence interval with a length determined by the $H^{1/2}$ norm of the initial data, then the solution is automatically extended to a global one.

In the first part of this paper, we reprove the results of Kenig and Takaoka [25] without using a gauge transformation, but under an extra condition that the $L^2$ norm of the initial data is small. Since we don’t perform a gauge transformation, our proof also works for the complex-valued Cauchy problem (1.1). Our method is to use the standard fixed-point argument in some Banach space. Bourgain’s space $X^{s,b}$ defined as a closure of the following space

$$
\{ f \in \mathcal{S}(\mathbb{R}^2) : \| f \|_{X^{s,b}} = \| (\xi)^s (\tau - \omega(\xi))^b \hat{f}(\xi, \tau) \|_{L^2} \}
$$

is very useful in the study of the low regularity theory of the nonlinear dispersive equations [5, 22, 16]. One might try a direct perturbative approach in $X^{s,b}$ space as Kenig, Ponce and Vega [22] did for the KdV and modified KdV equations. However, one will find that the key trilinear estimate

$$(\partial_x(u^3)) \|_{X^{s,b-4}} \lesssim ||u||^3_{X^{s,b}}, \text{ for some } b \in [1/2, 1)$$

fails for any $s$ due to logarithmic divergences involving the modulation variable (see Proposition 5.7, 5.8 below). The key observation in this paper is that these logarithmic divergences can be removed by us using Banach spaces which combine $X^{s,b}$ structure with smoothing effect structure. The spaces of these structures were first found and used by Ionescu and Kenig [14] to remove some logarithmic divergence.

**Theorem 1.1.** Let $s \geq 1/2$. Assume $u_0 \in H^s$ with $\|u_0\|_{L^2} \ll 1$. Then

(a) **Existence.** There exists $T = T(\|u_0\|_{H^{1/2}}) > 0$ and a solution $u$ to the complex-valued mBO equation (1.1) (or its focusing version) satisfying

$$u \in F^s(T) \subset C([-T,T] : H^s),$$

where the function space $F^s(T)$ will be defined later (see section 2).

(b) **Uniqueness.** For the real-valued case, the solution mapping $u_0 \to u$ is the unique extension of the mapping $H^\infty \to C([-T,T] : H^\infty)$. For the complex-valued case, the solution $u$ is unique in $B(u_0)$ which is defined in (6.13).

(c) **Lipschitz continuity.** For any $R > 0$, the mapping $u_0 \to u$ is Lipschitz continuous from $\{ u_0 \in H^s : \|u_0\|_{H^s} < R, \|u_0\|_{L^2} \ll 1 \}$ to $C([-T,T] : H^s)$.

For the real-valued mBO equation, from the conservation laws (1.6), (1.7), and iterating Theorem 1.1, we obtain the following corollary.

**Corollary 1.2.** The Cauchy problem for the real-valued mBO equation (1.1) (or its focusing version) is globally wellposed if $\phi$ belongs to $H^s$ for $s \geq 1/2$ with $\|\phi\|_{L^2}$ sufficiently small.

In the second part, we study the low regularity problem of the real-valued mBO equation (1.1). From the ill-posedness result in [25], we see that for $s < 1/2$ one can’t use a direct contraction mapping method, but we expect some wellposedness results hold in the weak sense.

**Conjecture.** The solution map $S_T^\infty : H^\infty \to C([-T,T] : H^\infty)$ of the real-valued modified Benjamin-Ono equation (1.1) can be uniquely extended to a continuous map from $H^s$ to $C([-T,T] : H^s)$ for a small $T = T(||\phi||_{H^s}) > 0$ if $s > 1/4$. 


To prove this one would need to establish a priori $H^s$ bounds for the $H^\infty$ solutions and then prove continuous dependence on the initial data. We solve the easier half of this problem.

**Theorem 1.3.** Let $s > 1/4$. For any $M > 0$ there exists $T > 0$ and $C > 0$ so that for any initial data $u_0 \in H^\infty$ satisfying

$$\|u_0\|_{H^s} \leq M$$

then the solutions $u \in C([0, T] : H^\infty)$ to the real-valued mBO equation (1.1) satisfies

$$\|u\|_{H^s} \leq C\|u_0\|_{H^s}. \quad (1.10)$$

**Remark 1.4.** A very similar equation to (1.1) is the derivative nonlinear Schrödinger equation

$$u_t - iu_{xx} = |u|^2u_{x}, \quad (x, t) \in \mathbb{R}^2, \quad (1.11)$$

and local wellposedness was known for the equation in $H^s$ for $s \geq 1/2$ [33], where a fixed point argument is performed in an adapted Bourgain’s $X^{s,b}$ space after a gauge transformation on the equation. Our methods also give the same results in Theorem 1.1 and 1.3 for (1.11).

We discuss now some of the ingredients in the proof of Theorem 1.3. We will follow the method of Ionescu, Kenig and Tataru [16] which approaches the problem in a less perturbative way. It can be viewed as a combination of the energy method and the perturbative method. More precisely, we will define $F^{l,s}(T), N^{l,s}(T)$ and energy space $E^{l,s}(T)$ and show that if $u$ is a smooth solution of (1.1) on $\mathbb{R} \times [-T, T]$ with $\|u\|_{E^{l,s}(T)} \ll 1$, then

$$\begin{align*}
\left\{ \begin{array}{l}
\|u\|_{F^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)} + \|\partial_x (u^3)\|_{N^{l,s}(T)}; \\
\|\partial_x (u^3)\|_{N^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)}; \\
\|u\|_{E^{l,s}(T)}^2 \lesssim \|\phi\|_{H^{l,s}}^2 + \|u\|_{E^{l,s}(T)}^2.
\end{array} \right. \quad (1.12)
\end{align*}$$

The inequalities (1.12) and a simple continuity argument still suffice to control $\|u\|_{E^{l,s}(T)}$, provided that $\|\phi\|_{H^{l,s}} \ll 1$ (which can by arranged by rescaling if $l, s > 0$). The first inequality in (1.12) is the analogue of the linear estimate. The second inequality in (1.12) is the analogue of the trilinear estimate (1.8). The last inequality in (1.12) is an energy-type estimate.

We explain the strategies in [16] to define the main normed and semi-normed spaces. As was explained before, standard using of $X^{s,b}$ spaces for fixed-point argument will lead to a logarithmic divergence in the key trilinear estimate. But we use $X^{s,b}$-type structures only on small, frequency dependant time intervals. The high-low interaction can be controlled for short time. The second step is to define $\|u\|_{E^{l,s}(T)}$ sufficiently large to be able to still prove the linear estimate $\|u\|_{F^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)} + \|\partial_x (u^3)\|_{N^{l,s}(T)}$. Finally, we use frequency-localized energy estimates and the symmetries of the equation (1.1) to prove the energy estimate $\|u\|_{E^{l,s}(T)}^2 \lesssim \|\phi\|_{H^{l,s}}^2 + \|u\|_{E^{l,s}(T)}^2$.

The rest of the paper is organized as follows: In section 2 we present some notations and Banach function spaces. We summarize some properties of the spaces in section 3. A symmetric estimate will be given in section 4 which is used in section 5 to show dyadic trilinear estimate. The proof of Theorem 1.1 is given in section 6. In section 7 we prove short time dyadic trilinear estimate and in section 8 we prove an energy estimate. Finally in section 9 we prove Theorem 1.3.
2. Notation and Definitions

For $x, y \in \mathbb{R}^+$, $x \lesssim y$ means that there exists $C > 0$ such that $x \leq Cy$. By $x \sim y$ we mean $x \lesssim y$ and $y \lesssim x$. For $f \in \mathcal{S}'$ we denote by $\hat{f}$ or $\mathcal{F}(f)$ the Fourier transform of $f$ for both spatial and time variables,

$$\hat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-i\tau}\tau f(x, t)dxdt.$$ 

Besides, we use $\mathcal{F}_x$ and $\mathcal{F}_t$ to denote the Fourier transform with respect to space and time variable respectively. Let $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$. For $k \in \mathbb{Z}$ let

$$O_k = \{\xi : |\xi| \in [(3/4) \cdot 2^k, (3/2) \cdot 2^k]\} \text{ and } I_k = \{\xi : |\xi| \in [2^{k-1}, 2^{k+1}]\}.$$ 

For $k \in \mathbb{Z}_+$ let $\tilde{I}_k = [-2, 2]$ if $k = 0$ and $\tilde{I}_k = I_k$ if $k \geq 1$.

Let $\eta : \mathbb{R} \to [0, 1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $k \in \mathbb{Z}$ let $\chi_k = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$, $\chi_k$ supported in $\{\xi : |\xi| \in [(5/8) \cdot 2^k, (8/5) \cdot 2^k]\}$, and

$$\chi_{[k_1, k_2]} = \sum_{k = k_1}^{k_2} \chi_k \text{ for any } k_1 \leq k_2 \in \mathbb{Z}.$$ 

For simplicity of notation, let $\eta_k = \chi_k$ if $k \geq 1$ and $\eta_k = 0$ if $k \leq -1$. Also, for $k_1 \leq k_2 \in \mathbb{Z}$ let

$$\eta_{[k_1, k_2]} = \sum_{k = k_1}^{k_2} \eta_k \text{ and } \eta_{\leq k_2} = \sum_{k = -\infty}^{k_2} \eta_k.$$ 

Roughly speaking, $\{\chi_k\}_{k \in \mathbb{Z}}$ is the homogeneous decomposition function sequence and $\{\eta_k\}_{k \in \mathbb{Z}_+}$ is the non-homogeneous decomposition function sequence to the frequency space.

For $k \in \mathbb{Z}$ let $k_+ = \max(k, 0)$, and let $P_k$, $R_k$ denote the operators on $L^2(\mathbb{R})$ defined by

$$\tilde{P}_k u(\xi) = \chi_k(\xi)\hat{u}(\xi) \text{ and } \tilde{R}_k u(\xi) = 1_{O_k}(\xi)\hat{u}(\xi).$$

By a slight abuse of notation we also define the operators $P_k$, $R_k$ on $L^2(\mathbb{R} \times \mathbb{R})$ by formulæ $\mathcal{F}(P_k u)(\xi, \tau) = \chi_k(\xi)\mathcal{F}(u)(\xi, \tau)$ and $\mathcal{F}(R_k u)(\xi, \tau) = 1_{O_k}(\xi)\mathcal{F}(u)(\xi, \tau)$.

For $l \in \mathbb{Z}$ let

$$P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.$$ 

Similarly we also define the operators $R_{\leq l}$ and $R_{\geq l}$.

Let $a_1, a_2, a_3, a_4 \in \mathbb{R}$. It will be convenient to define the quantities $a_{\max} \geq a_{\text{sub}} \geq a_{\text{thd}} \geq a_{\text{min}}$ to be the maximum, sub-maximum, third-maximum, and minimum of $a_1, a_2, a_3, a_4$ respectively. We also denote sub$(a_1, a_2, a_3, a_4) = a_{\text{sub}}$ and thd$(a_1, a_2, a_3, a_4) = a_{\text{thd}}$. Usually we use $k_1, k_2, k_3, k_4$ and $j_1, j_2, j_3, j_4$ to denote integers, $N_i = 2^{k_i}$ and $L_i = 2^{j_i}$ for $i = 1, 2, 3, 4$ to denote dyadic numbers. For $a \in \mathbb{R}$ we define $a-$ to be the real number $a - \epsilon$ for some $0 < \epsilon \ll 1$. Similar we also define $a+$.

For $\xi \in \mathbb{R}$ let

$$\omega(\xi) = -\xi|\xi|,$$ 

which is the dispersion relation associated to the linear Benjamin-Ono equation. For $\phi \in L^2(\mathbb{R})$ let $W(t)\phi \in C(\mathbb{R} : L^2)$ denote the solution of the free Benjamin-Ono
evolution given by
\[ \mathcal{F}_x[W(t)\phi](\xi, t) = e^{it\omega(\xi)} \hat{\phi}(\xi), \]  
(2.2)
where \( \omega(\xi) \) is defined in (2.1). For \( k \in \mathbb{Z}_+ \) and \( j \geq 0 \) let \( D_{k,j} = \{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau - \omega(\xi) \in I_j \} \). For \( k \in \mathbb{Z} \) and \( j \geq 0 \) let \( \hat{D}_{k,j} = \{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : \xi \in I_k, \tau - \omega(\xi) \in I_j \} \). For \( k \in \mathbb{Z}_+ \) we define now the Banach spaces \( X_k(\mathbb{R} \times \mathbb{R}) \):
\[ X_k = \{ f \in L^2(\mathbb{R}^2) : f \text{ is supported in } I_k \times \mathbb{R} \text{ and} \|f\|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \|\eta_j(\tau - \omega(\xi)) \cdot f(\xi, \tau)\|_{L^2_{\tau} L^2_{\xi}} < \infty \}, \]  
(2.3)
where
\[ \beta_{k,j} = 1 + 2^{(j-2k)/2}. \]  
(2.4)
Here the spaces \( X_k \) is the same as those used by Ionescu and Kenig [14] for \( k > 0 \). \( X_0 \) is different, since we don’t have the special structures in the low frequency. The precise choice of the coefficients \( \beta_{k,j} \) is important in order for all the trilinear estimates in Section 5 to hold. We see that if \( k \) is small then \( \beta_{k,j} \approx 2^{j/2} \). This factor is particularly important in controlling the high-low interaction. From the technical level, we know from the K-Z method of Tao [35] that the worst interaction is that the low frequency component has a largest modulation. But the factor \( \beta_{k,j} \) will compensate for that. The logarithmic divergence caused by the other interaction, namely high frequency component with largest modulation, can be removed by using a smoothing effect structure.

As in [14], the spaces \( X_k \) are not sufficient for our purpose, due to various logarithmic divergences involving the modulation variable (See Proposition 5.7 below). For \( k \geq 100 \) we also define the Banach spaces \( Y_k = Y_k(\mathbb{R}^2) \). For \( k \geq 100 \) we define
\[ Y_k = \{ f \in L^2(\mathbb{R}^2) : f \text{ is supported in } \bigcup_{j=0}^{k-1} D_{k,j} \text{ and} \|f\|_{Y_k} := 2^{-k/2} \|\mathcal{F}^{-1}[(\tau - \omega(\xi) + i)f(\xi, \tau)]\|_{L^1_{\tau} L^2_{\xi}} < \infty \}. \]  
(2.5)
Then for \( k \in \mathbb{Z}_+ \) we define
\[ Z_k := X_k \text{ if } k \leq 99 \text{ and } Z_k := X_k + Y_k \text{ if } k \geq 100. \]  
(2.6)
The spaces \( Z_k \) are our basic Banach spaces. For \( s \geq 0 \) we define the Banach spaces \( F^s = F^s(\mathbb{R} \times \mathbb{R}) \):
\[ F^s = \{ u \in S'(\mathbb{R} \times \mathbb{R}) : \|u\|_{F^s}^2 = \sum_{k=0}^{\infty} 2^{2sk} \|\eta_k(\xi) \mathcal{F}(u)\|_{L^2_{\xi}}^2 < \infty \}, \]  
(2.7)
and \( N^s = N^s(\mathbb{R} \times \mathbb{R}) \) which is used to measure the nonlinear term and can be viewed as an analogue of \( X^{s,b-1} \):
\[ N^s = \{ u \in S'(\mathbb{R} \times \mathbb{R}) : \|u\|_{N^s}^2 = \sum_{k=0}^{\infty} 2^{2sk} \|\eta_k(\xi)(\tau - \omega(\xi) + i)^{-1} \mathcal{F}(u)\|_{Z_k}^2 < \infty \}. \]  
(2.8)
The spaces \( F^s \) and \( N^s \) have the same structures in high frequency as those in [14], but with different structures in low frequency.
In order to prove a priori bounds, we will need another set of norms and semi-norms which were first used by Ionescu, Kenig and Tataru in [16] for the KP-I equation. Similar idea can be found in [26]. For $k \in \mathbb{Z}$ we define

$$B_k = \{ f \in L^2(\mathbb{R}^2) : f \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{B_k} := \sum_{j=0}^{\infty} 2^{j/2} \|\eta_j(\tau - w(\xi)) \cdot f(\xi, \tau)\|_{L^2_{\xi,\tau}} < \infty \}. \quad (2.9)$$

These $l^1$-type $X^{s,b}$ structures were first introduced and used in [37, 14, 16]. It is also useful in the study of uniform global wellposedness and inviscid limit for the KdV-Burgers equation [12].

At frequency $2^k$ we will use the $X^{s,b}$ structure given by the $B_k$ norm, uniformly on the $2^{-k^+}$ time scale. For $k \in \mathbb{Z}$ we define the normed spaces

$$F_k = \left\{ f \in L^2(\mathbb{R}^2) : \hat{f} \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{F_k} = \sup_{t \in [0,1]} \|F[f \cdot \eta_0(2^{k^+}(t-tk))]|_{B_k} < \infty \right\} ,$$

$$N_k = \left\{ f \in L^2(\mathbb{R}^2) : \hat{f} \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{N_k} = \sup_{t \in [0,1]} \|\tau - \omega(\xi) + ik^{2^k-1}f[f \cdot \eta_0(2^{k^+}(t-tk))]|_{B_k} < \infty \right\} .$$

The bounds we obtain for smooth solutions of the equation (1.1) are on a fixed time interval, while the above function spaces are not. Thus we define a local version of the spaces. For $T \in (0, 1]$ we define the normed spaces

$$F_k(T) = \{ f \in C([-T,T] : E_k) : \|f\|_{F_k(T)} = \inf_{\bar{f}} \|\bar{f}\|_{F_k} \} ;$$

$$N_k(T) = \{ f \in C([-T,T] : E_k) : \|f\|_{N_k(T)} = \inf_{\bar{f}} \|\bar{f}\|_{N_k} \} .$$

For $l, s \geq 0$ and $T \in (0, 1]$, we define the normed spaces

$$F^{l,s}(T) = \left\{ u \in C([-T,T] : H^\infty) : \|u\|_{F^{l,s}(T)} = \sum_{k=-\infty}^{0} 2^{2^k} \|R_k(u)|_{F^{l,s}(T)}^2 + \sum_{k=1}^{\infty} 2^{2^k} \|R_k(u)|_{N_k(T)}^2 < \infty \right\} ;$$

$$N^{l,s}(T) = \left\{ u \in C([-T,T] : H^\infty) : \|u\|_{N^{l,s}(T)} = \sum_{k=-\infty}^{0} 2^{2^k} \|R_k(u)|_{N_k(T)}^2 + \sum_{k=1}^{\infty} 2^{2^k} \|R_k(u)|_{N_k(T)}^2 < \infty \right\} .$$

We still need an energy space. For $l, s \geq 0$ and $u \in C([-T,T] : H^\infty)$ we define

$$\|u\|_{E^{l,s}_T}^2 = \|R_{l,0}(u(0))\|_{L^2}^2 + \sum_{k \geq 1} \sup_{t \in [-T,T]} 2^{2^k} \|R_k(u(t))|_{L^2}^2 .$$

3. Properties of the Spaces $Z_k$

In this section we devote to study the properties of the spaces $Z_k$. Using the definitions, if $k \in \mathbb{Z}_+$ and $f_k \in Z_k$ then $f_k$ can be written in the form

$$\begin{cases} f_k = \sum_{j=0}^{\infty} f_{k,j} + g_k ; \\ \sum_{j=0}^{\infty} 2^{j/2} \|f_{k,j}\|_{L^2} + \|g_k\|_{L^2} \leq 2 \|f_k\|_{Z_k} , \end{cases} \quad (3.1)$$

such that $f_{k,j}$ is supported in $D_{k,j}$ and $g_k$ is supported in $\cup_{j=0}^{k-1} D_{k,j}$ (if $k \leq 99$ then $g_k \equiv 0$). We start with the elementary properties.

**Lemma 3.1** (Lemma 4.1, [14]). (a) If $m, m' : \mathbb{R} \to \mathbb{C}$, $k \in \mathbb{Z}_+$, and $f_k \in Z_k$ then

$$\begin{cases} \|m(\xi)f_k(\xi, \tau)\|_{Z_k} \leq C\|\mathcal{F}^{-1}(m)\|_{L^1(\mathbb{R})}\|f_k\|_{Z_k} ; \\ \|m'(\tau)f_k(\xi, \tau)\|_{Z_k} \leq C\|m\|_{L^\infty(\mathbb{R})}\|f_k\|_{Z_k} . \end{cases} \quad (3.2)$$
(b) If \( k \in \mathbb{Z}_+ \), \( j \geq 0 \), and \( f_k \in Z_k \) then
\[
\| \eta_j (\tau - \omega(\xi)) f_k (\xi, \tau) \|_{X_k} \leq C \| f_k \|_{Z_k}.
\] (3.3)

(c) If \( k \geq 1 \), \( j \in [0, k] \), and \( f_k \) is supported in \( I_k \times \mathbb{R} \) then
\[
\| \mathcal{F}^{-1} [ \eta_j (\tau - \omega(\xi)) f_k (\xi, \tau) ] \|_{L^q_s L^r_t} \leq C \| \mathcal{F}^{-1} (f_k) \|_{L^q_s L^r_t}.
\] (3.4)

We study now the embedding properties of the spaces \( Z_k \). We will see that the spaces \( X_k \) and \( Y_k \) are very close.

**Lemma 3.2.** Let \( k \in \mathbb{Z}_+ \), \( s \in \mathbb{R} \) and \( I \subset \mathbb{R} \) be an interval. Let \( Y \) be \( L^q_t L^p_x \) or \( L^q_{t\in I} L^p_x \) for some \( 1 \leq p, q \leq \infty \) with the property that
\[
\| e^{-t\Delta_x^2} f \|_Y \lesssim 2^{ks} \| f \|_{L^2(\mathbb{R})}
\]
for all \( f \in L^2(\mathbb{R}) \) with \( \hat{f} \) supported in \( \tilde{I}_k \) and \( \tau_0 \in \mathbb{R} \). Then we have that if \( f_k \in Z_k \)
\[
\| \mathcal{F}^{-1} (f_k) \|_Y \lesssim 2^{ks} \| f_k \|_{Z_k}.
\] (3.5)

**Proof.** We assume first that \( f_k = f_{k,j} \) with \( \| f_k \|_{X_k} = 2^{j/2} \beta_{k,j} \| f_{k,j} \|_{L^2} \) and \( f_{k,j} \) is supported in \( D_{k,j} \) for some \( j \geq 0 \). Then we have
\[
\mathcal{F}^{-1} (f_k)(x, t) = \int f_{k,j}(\xi, \tau) e^{ix\xi} e^{it\tau} d\xi d\tau
\]
\[
= \int_{\tilde{I}_j} e^{it\tau} \int f_{k,j}(\xi, \tau + \omega(\xi)) e^{ix\xi} e^{it\omega(\xi)} d\xi d\tau.
\]

From the hypothesis on \( Y \), we obtain
\[
\| \mathcal{F}^{-1} (f_k)(x, t) \|_Y \lesssim \int \eta_j (\tau) \left\| e^{it\tau} \int f_{k,j}(\xi, \tau + \omega(\xi)) e^{ix\xi} e^{it\omega(\xi)} d\xi \right\|_Y d\tau
\]
\[
\lesssim 2^{ks} 2^{j/2} \| f_{k,j} \|_{L^2},
\]
which completes the proof in this case.

We assume now that \( k \geq 100 \) and \( f_k = g_k \in Y_k \). From definition \( g_k \) can be written in the form
\[
\begin{cases}
g_k(\xi, \tau) = 2^{k/2} \chi_{[k-1, k+1]}(\xi)(\tau - \omega(\xi) + i)^{-1} \eta_{\leq k}(\tau - \omega(\xi)) \mathcal{F} \varphi(\xi, \tau); \\
g_k \|_{Y_k} = C \| \varphi \|_{L^1_t L^2_x}.
\end{cases}
\] (3.6)

It suffices to prove that if
\[
f(\xi, \tau) = 2^{k/2} \chi_{[k-1, k+1]}(\xi)(\tau - \omega(\xi) + i)^{-1} \eta_{\leq k}(\tau - \omega(\xi)) \cdot \varphi(\tau)
\]
then
\[
\left\| \int_{\mathbb{R}^2} f(\xi, \tau) e^{ix\xi} e^{it\tau} d\xi d\tau \right\|_Y \lesssim 2^{ks} \| \varphi \|_{L^2},
\] (3.7)
which follows from the proof of Lemma 4.2 (b) in [14].

In order to obtain the more specific embedding properties of the spaces \( Z_k \), we need the estimate for the free Benjamin-Ono equation. We recall the Strichartz estimates, smoothing effects, and maximal function estimates (for the proof, see, e.g. [24, 23] and the reference therein).
Lemma 3.3. Let $k \in \mathbb{Z}_+$ and $I \subset \mathbb{R}$ be an interval with $|I| \leq 1$. Then for all $\phi \in L^2(\mathbb{R})$ with $\hat{\phi}$ supported in $I_k$, 
(a) Strichartz estimates
\[
\|e^{-i\xi^2/2}\hat{\phi}\|_{L^q(I) L^r(I)} \leq C\|\phi\|_{L^2(\mathbb{R})},
\]
where $(q, r)$ is admissible, namely $2 \leq q, r \leq \infty$ and $2/q = 1/2 - 1/r$.
(b) Smoothing effect
\[
\|e^{-i\xi^2/2}\hat{\phi}\|_{L^q(I) L^r(I)} \leq C2^{-k/2} \|\phi\|_{L^2(\mathbb{R})}.
\]
(c) Maximal function estimate
\[
\|e^{-i\xi^2/2}\hat{\phi}\|_{L^q(I) L^r(I)} \leq C2^{k/2} \|\phi\|_{L^2(\mathbb{R})},
\]
\[
\|e^{-i\xi^2/2}\hat{\phi}\|_{L^q(I) L^r(I)} \leq C2^{k/4} \|\phi\|_{L^2(\mathbb{R})}.
\]
In particular we note the case $(6, 6)$ is admissible which we will use in the sequel.
From Lemma 3.2, 3.3, we immediately get the following

Lemma 3.4. Let $k \in \mathbb{Z}_+$ and $I \subset \mathbb{R}$ be an interval with $|I| \leq 1$. Assume $(q, r)$ is admissible and $f_k \in Z_k$. Then
\[
\|\mathcal{F}^{-1}(f_k)\|_{L^q(I) L^r(I)} \leq C2^{k/2} \|f_k\|_{Z_k},
\]
\[
\|\mathcal{F}^{-1}(f_k)\|_{L^q(I) L^r(I)} \leq C2^{-k/2} \|f_k\|_{Z_k},
\]
\[
\|\mathcal{F}^{-1}(f_k)\|_{L^q(I) L^r(I)} \leq C2^{k/4} \|f_k\|_{Z_k},
\]
\[
\|\mathcal{F}^{-1}(f_k)\|_{L^q(I) L^r(I)} \leq C \|f_k\|_{Z_k}.
\]
As a consequence,
\[
F^s \subseteq C(\mathbb{R}; H^s) \text{ for any } s \geq 0.
\]

Now we turn to study the properties of the space $F^{l,s}$. The definition shows easily that if $k \in \mathbb{Z}$ and $f_k \in B_k$ then
\[
\left\|\int_{\mathbb{R}} |f_k(\xi, \tau')| d\tau' \right\|_{L^2(I)} \lesssim \|f_k\|_{B_k}.
\]
Moreover, if $k \in \mathbb{Z}$, $l \in \mathbb{Z}_+$, and $f_k \in B_k$ then
\[
\sum_{j=l+1}^{\infty} 2^{j/2} \left\| \eta_j(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2(I)}
\]
\[
+ 2^{j/2} \left\| \eta_{j-1}(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2(I)}
\]
\[
\lesssim \|f_k\|_{B_k}.
\]
In particular, if $k \in \mathbb{Z}$, $l \in \mathbb{Z}_+$, $t_0 \in \mathbb{R}$, $f_k \in B_k$, and $\gamma \in S(\mathbb{R})$, then
\[
\|\mathcal{F}[\gamma(2^l(t - t_0)) \cdot \mathcal{F}^{-1}(f_k)]\|_{B_k} \lesssim \|f_k\|_{B_k}.
\]
Indeed, to prove (3.10), first for the second term on the left-hand side of (3.10), we immediately get from Cauchy-Schwarz inequality and (3.9) that
\[
2^{j/2} \left\| \eta_{j-1}(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2(I)}
\]
\[
\lesssim \left\| \int_{\mathbb{R}} |f_k(\xi, \tau')| d\tau' \right\|_{L^2(I)} \lesssim \|f_k\|_{B_k}.
\]
For the first term on the left-hand side of (3.10), we decompose $f_k(\xi, \tau') = \sum_{j_1 \geq 0} f_{k,j_1}$, where $f_{k,j_1} = f_k(\xi, \tau') \eta_{j_1}(\tau' - \omega(\xi))$, and then we get

$$
\sum_{j=l+1}^{\infty} 2^{j/2} \left\| \eta_j(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_k(\xi, \tau')| \cdot 2^{-l}(1+2^{-l}|\tau - \tau'|)^{-4}d\tau' \right\|_{L^2}
\leq \sum_{j=l+1}^{\infty} \sum_{j_1=0}^{\infty} 2^{j/2} \left\| \eta_j(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_{k,j_1}(\xi, \tau')| \cdot 2^{-l}(1+2^{-l}|\tau - \tau'|)^{-4}d\tau' \right\|_{L^2}
= \sum_{j=l+1}^{\infty} \left( \sum_{j_1 > j + 5} + \sum_{j_1 < j - 5} + \sum_{j_1 \leq j + 5} \right) 2^{j/2} \left\| \eta_j(\tau - \omega(\xi)) \cdot \int_{\mathbb{R}} |f_{k,j_1}(\xi, \tau')| \cdot 2^{-l}(1+2^{-l}|\tau - \tau'|)^{-4}d\tau' \right\|_{L^2}
= : I + II + III.
$$

For the contribution of $I$, we first observe that $|\tau - \tau'| \sim 2^{j_1}$ in this case. Then we get that

$$
I \lesssim \sum_{j_1 \geq l} \sum_{j \leq j_1} 2^{j/2} 2^{3j_1/2 - 4j_1} \left\| \int_{\mathbb{R}} |f_{k,j_1}(\xi, \tau')|d\tau' \right\|_{L^2} \lesssim \|f_k\|_{B_k}.
$$

Similarly we can estimate the contribution of $II$. For the third term $III$, using Young’s inequality, then we get

$$
III \lesssim \sum_{j=l+1}^{\infty} \sum_{|j-j_1| \leq 5} 2^{j/2} \|f_{k,j_1}\|_{L^2} \lesssim \|f_k\|_{B_k}.
$$

As in [16], for any $k \in \mathbb{Z}$ we define the set $S_k$ of $k-$acceptable time multiplication factors

$$
S_k = \{ m_k : \mathbb{R} \to \mathbb{R} : \|m_k\|_{S_k} = \sum_{j=0}^{10} 2^{-jk_1} \|\partial^j m_k\|_{L^\infty} < \infty \}.
$$

For instance, $\eta(2^{k+1}t) \in S_k$ for any $\eta$ satisfies $\|\partial_x^j \eta\|_{L^\infty} \leq C$ for $j = 1, 2, \ldots, 10$. Direct estimates using the definitions and (3.10) show that for any $s \geq 0$ and $T \in (0,1]$

$$
\left\{ \begin{array}{l}
\|\sum_{k \in \mathbb{Z}} m_k(t) \cdot R_k(u)\|_{L^\infty} \lesssim \left( \sup_{k \in \mathbb{Z}} \|m_k\|_{S_k} \cdot \|u\|_{L^\infty} \right) \|F_{1,j}(T)\|; \\
\|\sum_{k \in \mathbb{Z}} m_k(t) \cdot R_k(u)\|_{L^\infty} \lesssim \left( \sup_{k \in \mathbb{Z}} \|m_k\|_{S_k} \cdot \|u\|_{L^\infty} \right) \|F_{1,j}(T)\|; \\
\|\sum_{k \in \mathbb{Z}} m_k(t) \cdot R_k(u)\|_{L^\infty} \lesssim \left( \sup_{k \in \mathbb{Z}} \|m_k\|_{S_k} \cdot \|u\|_{L^\infty} \right) \|F_{1,j}(T)\|.
\end{array} \right.
$$

Actually, for instance we show the first inequality in (3.13). In view of definition, it suffices to prove that if $u_k \in F_k$, then

$$
\|m_k(t)u_k\|_{F_k} \lesssim \|u_k\|_{F_k} \|m_k\|_{S_k}, \quad \forall \ k \in \mathbb{Z}.
$$

From (3.11) we see that we only need to prove that

$$
|F[m_k]\eta_0(2^{k+1}(\cdot - t_k))| \lesssim 2^{-k+1}(1+2^{-k+1}|\tau|)^{-4} \|m_k\|_{S_k},
$$

which follows from partial integration and the definition of $S_k$. 
4. A Symmetric Estimate

In this section we prove a symmetric estimate which will be used to prove a trilinear estimate. For \( \xi_1, \xi_2, \xi_3 \in \mathbb{R} \) and \( \omega : \mathbb{R} \rightarrow \mathbb{R} \) as in (2.1) let

\[
\Omega(\xi_1, \xi_2, \xi_3) = \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3). \tag{4.1}
\]

This is the resonance function that plays a crucial role in the trilinear estimate of the \( X^{s,b} \)-type space. See [35] for a perspective discussion. For compactly supported functions \( f, g, h, u \in L^2(\mathbb{R} \times \mathbb{R}) \) let

\[
J(f, g, h, u) = \int_{\mathbb{R}^2} f(\xi_1, \mu_1)g(\xi_2, \mu_2)h(\xi_3, \mu_3) u(\xi_1 + \xi_2 + \xi_3, \mu_1 + \mu_2 + \mu_3 + \Omega(\xi_1, \xi_2, \xi_3))d\xi_1d\xi_2d\xi_3d\mu_1d\mu_2d\mu_3.
\]

**Lemma 4.1.** Assume \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \) and \( k_1 \leq k_2 \leq k_3 \leq k_4, j_1, j_2, j_3, j_4 \in \mathbb{Z}_+ \), and \( f_{k_1,j_1} \in L^2(\mathbb{R} \times \mathbb{R}) \) are nonnegative functions supported in \( I_{k_1} \times \bigcup_{l=0}^4 I_l \), \( i = 1, 2, 3, 4 \). For simplicity we write \( J = |J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| \). Then

(a) For any \( k_1 \leq k_2 \leq k_3 \leq k_4 \) and \( j_1, j_2, j_3, j_4 \in \mathbb{Z}_+ \),

\[
J \leq C^2(2^{(j_1 + j_2 + j_3 + j_4)}/2)^{(k_{\min} + k_{\max})/2} \prod_{i=1}^4 \|f_{k_i,j_i}\|_{L^2}. \tag{4.2}
\]

(b) If \( k_2 \leq k_3 - 5 \) and \( j_2 \neq j_{\max} \),

\[
J \leq C^2(2^{(j_1 + j_2 + j_3 + j_4)}/2)^{(j_{\max} - 2j) + k_{\max}/2} \prod_{i=1}^4 \|f_{k_i,j_i}\|_{L^2}; \tag{4.3}
\]

if \( k_2 \leq k_3 - 5 \) and \( j_2 = j_{\max} \),

\[
J \leq C^2(2^{(j_1 + j_2 + j_3 + j_4)}/2)^{(j_{\max} - 2k_{\max}/2) + k_{\max}/2} \prod_{i=1}^4 \|f_{k_i,j_i}\|_{L^2}. \tag{4.4}
\]

(c) For any \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \) and \( j_1, j_2, j_3, j_4 \in \mathbb{Z}_+ \),

\[
J \leq C^2(2^{(j_1 + j_2 + j_3 + j_4)}/2)^{(j_{\max} - 2)}/2 \prod_{i=1}^4 \|f_{k_i,j_i}\|_{L^2}. \tag{4.5}
\]

(d) If \( k_{\min} \leq k_{\max} - 10 \), then

\[
J \leq C^2(2^{(j_1 + j_2 + j_3 + j_4)}/2)^{(k_{\max} - 2)}/2 \prod_{i=1}^4 \|f_{k_i,j_i}\|_{L^2}. \tag{4.6}
\]

**Proof.** Let \( A_k(\xi) = \int_{\mathbb{R}} |f_{k,j}(\xi, \mu)|^2 d\mu \) for \( i = 1, 2, 3, 4 \). Using the Cauchy-Schwarz inequality and the support properties of the functions \( f_{k_i,j_i} \),

\[
|J(f_{k_1,j_1}; f_{k_2,j_2}, f_{k_3,j_3}; f_{k_4,j_4})| \leq C^2(2^{(j_{\min} + j_{\max})}/2) \int_{\mathbb{R}^3} A_{k_1}(\xi_1)A_{k_2}(\xi_2)A_{k_3}(\xi_3)A_{k_4}(\xi_1 + \xi_2 + \xi_3)d\xi_1d\xi_2d\xi_3 \leq C^2(2^{(k_{\min} + k_{\max})}/2)^{(j_{\min} + j_{\max})/2} \prod_{i=1}^4 \|f_{k_i,j_i}\|_{L^2},
\]

which is part (a), as desired.
For part (b), in view of the support properties of the functions, it is easy to see that $J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4}) = 0$ unless

\[ k_4 \leq k_3 + 5. \quad (4.7) \]

Simple changes of variables in the integration and the observation that the function $\omega$ is odd show that

\[ |J(f, g, h, u)| = |J(g, f, h, u)| = |J(f, h, g, u)| = |J(f, g, u, h)|, \]

where $\tilde{f}(\xi, \mu) = f(-\xi, -\mu)$. We assume first that $j_2 \neq j_{max}$. Then we have several cases: if $j_4 = j_{max}$, then we will prove that if $g_i : \mathbb{R} \rightarrow \mathbb{R}_+$ are $L^2$ functions supported in $I_{k_i}$, $i = 1, 2, 3$, and $g : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is an $L^2$ function supported in $I_{k_4} \times I_{j_4}$, then

\[
\int_{\mathbb{R}^3} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3))d\xi_1d\xi_2d\xi_3 \\
\leq 2^{-k_{max}/2}2^{k_{min}/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}. \quad (4.8)
\]

This suffices for (4.3).

To prove (4.8), we first observe that since $k_2 \leq k_3 - 5$ then $|\xi_3 + \xi_2| \sim |\xi_3|$. By change of variable $\xi_1' = \xi_1, \xi_2' = \xi_2, \xi_3' = \xi_3 + \xi_2$, we get that the left side of (4.8) is bounded by

\[
\int_{|\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}, |\xi_3| \sim 2^{k_3}} g_1(\xi_1)g_2(\xi_2) \\
g_3(\xi_3 - \xi_2)g(\xi_1 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3 - \xi_2))d\xi_1d\xi_2d\xi_3. \quad (4.9)
\]

Note that in the integration area we have

\[
\frac{\partial}{\partial \xi_2} [\Omega(\xi_1, \xi_2, \xi_3 - \xi_2)] = |\omega'(\xi_2) - \omega'(\xi_3 - \xi_2)| \sim 2^{k_3},
\]

where we use the fact $\omega'(\xi) = |\xi|$ and $k_2 \leq k_3 - 5$. By change of variable $\mu_2 = \Omega(\xi_1, \xi_2, \xi_3 - \xi_2)$, we get that (4.9) is bounded by

\[
2^{-k_3/2} \int_{|\xi_1| \sim 2^{k_1}} g_1(\xi_1)\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}d\xi_1 \\
\leq 2^{-k_{max}/2}2^{k_{min}/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}. \quad (10.10)
\]

If $j_3 = j_{max}$, this case is identical to the case $j_4 = j_{max}$ in view of (4.7). If $j_1 = j_{max}$ it suffices to prove that if $g_i : \mathbb{R} \rightarrow \mathbb{R}_+$ are $L^2$ functions supported in $I_{k_i}$, $i = 2, 3, 4$, and $g : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is an $L^2$ function supported in $I_{k_4} \times I_{j_4}$, then

\[
\int_{\mathbb{R}^3} g_2(\xi_2)g_3(\xi_3)g_4(\xi_4)g(\xi_2 + \xi_3 + \xi_4, \Omega(\xi_2, \xi_3, \xi_4))d\xi_2d\xi_3d\xi_4 \\
\leq 2^{-k_{max}/2}2^{k_{min}/2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g_4\|_{L^2}\|g\|_{L^2}. \quad (11.11)
\]

Indeed, by change of variables $\xi_2' = \xi_2, \xi_3' = \xi_3, \xi_4' = \xi_2 + \xi_3 + \xi_4$ and noting that in the area $|\xi_2'| \sim 2^{k_2}, |\xi_3'| \sim 2^{k_3}, |\xi_4'| \sim 2^{k_4}$,

\[
\frac{\partial}{\partial \xi_2} [\Omega(\xi_2', \xi_3', \xi_4' - \xi_2' - \xi_3')] = |\omega'(\xi_2') - \omega'(\xi_4' - \xi_2' - \xi_3')| \sim 2^{k_3},
\]
we get from Cauchy-Schwarz inequality that
\[
\int_{\mathbb{R}^3} g_2(\xi_2)g_3(\xi_3)g_4(\xi_4)g(\xi_2 + \xi_3 + \Omega(\xi_2, \xi_3, \xi_4))d\xi_2d\xi_3d\xi_4
\]
\[\lesssim \int_{|\xi_2|<2^{k_2},|\xi_3|<2^{k_3},|\xi_4|<2^{k_4}} g_2(\xi_2')g_3(\xi_3')
\cdot g_4(\xi_4' - \xi_2' - \xi_3')g(\xi_2', \xi_3', \xi_4' - \xi_2' - \xi_3')d\xi_2'd\xi_3'd\xi_4'
\]
\[\lesssim 2^{-k_2/2} \int_{|\xi_2|<2^{k_2},|\xi_3|<2^{k_3},|\xi_4|<2^{k_4}} g_3(\xi_3')g_2(\xi_2')g_4(\xi_4' - \xi_2' - \xi_3')d\xi_2'd\xi_3'd\xi_4'
\]
\[\lesssim 2^{-k_{max}/2} 2^{k_{min}/2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g_4\|_{L^2} \|g\|_{L^2}.
\] (4.12)

We assume now that \(j_2 = j_{\text{max}}\). This case is identical to the case \(j_1 = j_{\text{max}}\). We note that we actually prove that if \(k_2 \leq k_3 - 5\) then

\[
J \leq C2^{(j_1+j_2+j_3+j_4)/2} 2^{-j_{\text{ub}}/2} 2^{-k_{\text{max}}/2} 2^{k_{\text{min}}/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.
\] (4.13)

Therefore, we complete the proof for part (b).

For part (c), setting \(f_{k_i,j_i}^3 = f_{k_i,j_i}(\xi, \tau - \omega(\xi))\), \(i = 1, 2, 3\), then we get

\[
|J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4})| = |\int f_{k_1,j_1}^3 \ast f_{k_2,j_2}^3 \ast f_{k_3,j_3}^3 \ast f_{k_4,j_4}^3|
\]
\[\lesssim \|f_{k_1,j_1}^3 \ast f_{k_2,j_2}^3 \ast f_{k_3,j_3}^3\|_{L^2} \|f_{k_4,j_4}^3\|_{L^2}
\]
\[\lesssim \prod_{i=1}^{3} \|\mathcal{F}^{-1}(f_{k_i,j_i}^3)\|_{L^6} \|f_{k_4,j_4}^3\|_{L^2}.
\]

On the other hand, from

\[
\mathcal{F}^{-1}(f_{k_1,j_1}^3) = \int_{\mathbb{R}^2} f_{k_1,j_1}(\xi, \tau - \omega(\xi))e^{ix\xi}e^{it\tau}d\xi d\tau
\]
\[= \int_{\mathbb{R}^2} f_{k_1,j_1}(\xi, \tau)e^{ix\xi}e^{it\omega(\xi)}e^{it\tau}d\xi d\tau,
\]
then it follows from Lemma 3.3 (a) that

\[
\|\mathcal{F}^{-1}(f_{k_1,j_1}^3)\|_{L^6} \lesssim \int_{\mathbb{R}} \left\|\int_{\mathbb{R}} f_{k_1,j_1}(\xi, \tau)e^{ix\xi}e^{it\omega(\xi)}e^{it\tau}d\tau\right\|_{L^6} d\xi \lesssim 2^{j_1/2} \|f_{k_1,j_1}\|_{L^2}.
\]

Thus part (c) forms the symmetry.

For part (d), we only need to consider the worst cases \(\xi_1 \cdot \xi_2 < 0\) and \(k_2 \leq k_3 - 5\). Indeed in the other cases we get (4.6) from the fact \(|\Omega(\xi_1, \xi_3, \xi_3)| \geq 2^{k_2 + k_3}\) which implies that \(j_{\text{max}} \geq k_2 + k_3 - 5\) by checking the support properties. Thus (d) follows from (b) and (c) in these cases. We assume now \(\xi_1 \cdot \xi_2 < 0\) and \(k_2 \leq k_3 - 5\). If \(j_4 = j_{\text{max}}\), it suffices to prove that if \(g_i\) is \(L^2\) nonnegative functions supported in \(I_{k_i}, i = 1, 2, 3\), and \(g\) is a \(L^2\) nonnegative function supported in \(I_{k_4} \times I_{j_4}\), then

\[
\int_{\mathbb{R}^3 \times \{\xi_1 \cdot \xi_2 < 0\}} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g(\xi_1 + \xi_2 + \xi_3, \Omega(\xi_1, \xi_2, \xi_3))d\xi_1d\xi_2d\xi_3
\]
\[\lesssim 2^{j_4/2} 2^{-k_3} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g_3\|_{L^2} \|g\|_{L^2}.
\] (4.14)
By localizing $|\xi_1 + \xi_2| \sim 2^l$ for $l \in \mathbb{Z}$, we get that the right-hand side of (4.14) is bounded by
\[
\sum_l \int_{\mathbb{R}^3} \chi_l(\xi_1 + \xi_2)g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)\gamma(\xi_1 + \xi_2, \xi_3)\Omega(\xi_1, \xi_2, \xi_3)d\xi_1d\xi_2d\xi_3. \tag{4.15}
\]
From the support properties of the functions $g_i$, $g$ and the fact that in the integration area
\[
|\Omega(\xi_1, \xi_2, \xi_3)| = (\xi_1 + \xi_2)(\xi_1 + \xi_3) \sim 2^{l+k_3},
\]
We get that
\[
j_{\max} \geq l + k_3 - 20. \tag{4.16}
\]
By changing variable of integration $\xi'_1 = \xi_1 + \xi_2$, $\xi'_2 = \xi_2$, $\xi'_3 = \xi_1 + \xi_3$, we obtain that (4.15) is bounded by
\[
\sum_l \int_{|\xi'_1| \sim 2^l} \chi_l(\xi'_1)g_1(\xi'_1 - \xi'_2)g_2(\xi'_2)g_3(\xi'_2 + \xi'_3 - \xi'_1)g(\xi'_2 + \xi'_3, \Omega(\xi'_1 - \xi'_2, \xi'_2 + \xi'_3 - \xi'_1))d\xi'_1d\xi'_2d\xi'_3d\xi'_4. \tag{4.17}
\]
Since in the integration area
\[
|\partial_{\xi'_1}\Omega(\xi'_1 - \xi'_2, \xi'_2 + \xi'_3 - \xi'_1)| = |\omega'(\xi'_1 - \xi'_2) - \omega'(\xi'_2 + \xi'_3 - \xi'_1)| \sim 2^{k_3}, \tag{4.18}
\]
then we get from (4.18) that (4.17) is bounded by
\[
\sum_l \int_{|\xi'_1| \sim 2^l} \chi_l(\xi'_1)\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}d\xi'_1d\xi'_2d\xi'_3d\xi'_4
\]
\[
\lesssim \sum_l 2^{l/2}2^{-k_3/2}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}
\]
\[
\lesssim 2^{l_{\max}/2}2^{-k_3}\|g_1\|_{L^2}\|g_2\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}, \tag{4.19}
\]
where we used (4.16) in the last inequality.

From symmetry we know the case $j_3 = j_{\max}$ is identical to the case $j_4 = j_{\max}$, and the case $j_1 = j_{\max}$ is identical to the case $j_2 = j_{\max}$, thus it reduces to prove the case $j_2 = j_{\max}$. It suffices to prove that if $g_i$ is $L^2$ nonnegative functions supported in $I_{k_2}$, $i = 1, 3, 4$, and $g$ is a $L^2$ nonnegative function supported in $I_{k_2} \times I_{j_2}$, then
\[
\int_{\mathbb{R}^3 \times (\xi_1, \xi_2 < 0)} g_1(\xi_1)g_3(\xi_3)g_4(\xi_4)\gamma(\xi_1 + \xi_3 + \xi_4, \Omega(\xi_1, \xi_3, \xi_4))d\xi_1d\xi_3d\xi_4
\]
\[
\lesssim 2^{j_2/2}2^{-k_3}\|g_1\|_{L^2}\|g_4\|_{L^2}\|g_3\|_{L^2}\|g\|_{L^2}. \tag{4.20}
\]
As the case $j_4 = j_{\max}$, we get that the right-hand side of (4.20) is bounded by
\[
\sum_l \int_{\mathbb{R}^3} \chi_l(\xi_3 + \xi_4)g_1(\xi_1)g_4(\xi_4)g_3(\xi_3)g(\xi_1 + \xi_4, \xi_3 + \xi_4, \Omega(\xi_1, \xi_3, \xi_4))d\xi_1d\xi_4d\xi_3. \tag{4.21}
\]
From the support properties of the functions $g_i$, $g$ and the fact that in the integration area
\[
|\Omega(\xi_1, \xi_2, \xi_3)| = |(\xi_1 + \xi_4)(\xi_4 + \xi_3)| \sim 2^{l+k_3},
\]
We get that
\[ j_{\text{max}} \geq l + k_3 - 20. \] (4.22)

By changing variable of integration \( \xi'_1 = \xi_1 + \xi_3, \xi'_3 = \xi_3 + \xi_4, \xi'_4 = \xi_1 + \xi_3 + \xi_4, \) we obtain that (4.21) is bounded by
\[
\sum_l \int_{|\xi| \sim 2^l} \chi_l(\xi_3') g_l(\xi'_1 - \xi'_3) g_3(\xi'_3 + \xi'_4 - \xi'_1) g_4(\xi'_4 - \xi'_1) \\
g(\xi'_1, \Omega(\xi'_4 - \xi'_3, \xi'_3 + \xi'_4 - \xi'_1, \xi'_4 - \xi'_1)) d\xi'_1 d\xi'_3 d\xi'_4.
\] (4.23)

Since in the integration area,
\[
|\frac{\partial}{\partial \xi_3'} \Omega(\xi'_1 - \xi'_3, \xi'_3 + \xi'_4 - \xi'_1, \xi'_4 - \xi'_1)| = | - \omega'(\xi'_3 - \xi'_3) + \omega'(\xi'_1 + \xi'_3 - \xi'_1) | \sim 2^{k_3},
\] (4.24)

then we get from (4.24) that (4.23) is bounded by
\[
\sum_l \int_{|\xi| \sim 2^l} \chi_l(\xi_3') \| g_l \|_{L^2} \| g_3 \|_{L^2} \| g_4 \|_{L^2} \\
\| g(\xi'_1 - \xi'_3) g(\xi'_3, \Omega(\xi'_4 - \xi'_3, \xi'_3 + \xi'_4 - \xi'_1, \xi'_4 - \xi'_1)) \|_{L^2_{\xi'_1, \xi'_3, \xi'_4} d\xi'_1 d\xi'_3 d\xi'_4} \\
\lesssim \sum_l 2^{l/2} 2^{-k_3/2} \| g_l \|_{L^2} \| g_3 \|_{L^2} \| g_4 \|_{L^2} \| g \|_{L^2},
\] (4.25)

where we used (4.22) in the last inequality. Therefore, we complete the proof for part (d).

We restate now Lemma 4.1 in a form that is suitable for the trilinear estimates in the next sections.

**Corollary 4.2.** Assume \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \), \( j_1, j_2, j_3, j_4 \in \mathbb{Z}^+, \) and \( f_{k_1, j_1} \in L^2(\mathbb{R} \times \mathbb{R}) \) are functions supported in \( D_{k_1, j_1}, \ i = 1, 2. \)

(a) For any \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \) and \( j_1, j_2, j_3, j_4 \in \mathbb{Z}^+, \)
\[
\| 1_{D_{k_4, j_4}}(\xi, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \\
\leq C 2^{(k_{\text{min}} + k_{\text{thd}})/2} 2^{(j_{\text{min}} + j_{\text{thd}})/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2}.
\] (4.26)

(b) For any \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \) with \( k_{\text{thd}} \leq k_{\text{max}} - 5, \) and \( j_1, j_2, j_3, j_4 \in \mathbb{Z}^+. \) If for some \( i \in \{1, 2, 3, 4\} \) such that \( (k_i, j_i) = (k_{\text{thd}}, j_{\text{max}}), \) then
\[
\| 1_{D_{k_4, j_4}}(\xi, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \\
\leq C 2^{-k_{\text{max}}/2} 2^{j_{\text{max}}/2} 2^{-j_{\text{max}}/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2},
\] (4.27)

else we have
\[
\| 1_{D_{k_4, j_4}}(\xi, \tau)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \\
\leq C 2^{-k_{\text{max}} + k_{\text{min}}}/2 2^{j_{\text{max}}/2} 2^{-j_{\text{max}}/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2}.
\] (4.28)
Then Corollary 4.2 follows from Lemma 4.1. 

We now perform simple changes of variables, we get functions

Clearly, we have

\[ \text{Proof.} \]

We first divide it into three parts, according to the modulation.

\[ 2^{k_i} \| \eta_{k_i}(\xi - \omega(\xi) + i)^{-1} f_{k_i} * f_{k_2} * f_{k_3} \|_{L^2} \lesssim 2^{k_i + k_2 + k_3} \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}}. \]

\[ (5.1) \]

**Proof.** We first divide it into three parts, according to the modulation.

\[ 2^{k_i} \| \chi_{k_i}(\xi - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_i}} \]

\[ \leq 2^{k_i} \| \chi_{k_i}(\xi) \eta_{k_i}^{-1}(\tau - \omega(\xi)) \|_{L^1} \| (\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_i}} \]

\[ + 2^{k_i} \| \chi_{k_i}(\xi) \eta_{k_i,2k_i}(\tau - \omega(\xi)) \|_{L^1} \| (\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_i}} \]

\[ + 2^{k_i} \| \chi_{k_i}(\xi) \eta_{k_i,2k_i} \|_{L^1} \| (\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_i}} \]

\[ = I + II + III. \]
We consider first the contribution of $I$. Using $Y_k$ norm, then we get from Lemma 3.1 (a), (c) and Lemma 3.4 that

\[
I \leq 2^{k_4} \| \chi_{k_4}(\xi) \eta_{k_4-1}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Y_{k_4}} \\
\leq 2^{k_4/2} \| F^{-1}(f_{k_1} * f_{k_2} * f_{k_3}) \|_{L^1_t L^2_x} \\
\leq 2^{k_4/2} \| F^{-1}(f_{k_3}) \|_{L^\infty_t L^2_x} \| F^{-1}(f_{k_2}) \|_{L^\infty_t L^2_x} \| F^{-1}(f_{k_1}) \|_{L^2_t L^\infty_x} \\
\leq 2^{(k_1+k_2)/2} \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}},
\]

which is (5.1) as desired.

For the contribution of $II$, we use $X_k$ norm. Then we get from Lemma 3.4 that

\[
II \leq 2^{k_4} \| \chi_{k_4}(\xi) \eta_{k_4,2k_4}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
\leq \sum_{k_4 \leq j_2 \leq 2k_4} 2^{k_4} 2^{-j_2/2} \| 1_{D_{k_4,j_2}}(\xi,\tau) f_{k_1} * f_{k_2} * f_{k_3} \|_{L^2_x} \\
\leq \sum_{k_4 \leq j_2 \leq 2k_4} 2^{k_4} 2^{-j_2/2} \| F^{-1}(f_{k_3}) \|_{L^\infty_t L^2_x} \| F^{-1}(f_{k_2}) \|_{L^\infty_t L^2_x} \| F^{-1}(f_{k_1}) \|_{L^2_t L^\infty_x} \\
\leq 2^{(k_1+k_2)/4} \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}},
\]

which is acceptable.

Finally we consider the contribution of $III$. For $j_i \geq 0$, $i = 1, 2, 3$, let $f_{k_i,j_i}(\xi,\tau) = f_{k_i}(\xi,\tau) \eta_{j_i}(\tau - \omega(\xi))$. Using $X_k$ norm, we get

\[
III \leq \sum_{j_3 \geq 2k_4+1} \sum_{j_2,j_1 \geq 0} \| 1_{D_{k_4,j_3}}(\xi,\tau) f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} \|_{L^2_x}.
\]

Since in the area $\{ |\xi| \in \tilde{I}_{k_4}, i = 1, 2, 3 \}$, we have $|\Omega(\xi_1,\xi_2,\xi_3)| \ll 2^{k_4}$. By checking the support properties of $f_{k_i,j_i}$, we get $|j_{\max} - j_{\min}| \leq 5$. We consider only the worst case $|j_3 - j_2| \leq 5$, since the other cases are better. It follows from Corollary 4.2 and Lemma 3.1 (b) that

\[
III \lesssim \sum_{j_3 \geq 2k_4+1} \sum_{j_2,j_1 \geq 0} 2^{(j_1+j_2)/2} 2^{(k_1+k_2)/2} \| f_{k_1,j_1} \|_{L^2_x} \| f_{k_2,j_2} \|_{L^2_x} \| f_{k_3,j_3} \|_{L^2_x} \\
\lesssim \sum_{j_3 \geq 2k_4+1} 2^{j_3/2} 2^{k_3} \| f_{k_1} \|_{Z_{k_1}} \| f_{k_2} \|_{Z_{k_2}} \| f_{k_3} \|_{Z_{k_3}} \\
\leq \sum_{j_3 \geq 2k_4+1} 2^{k_4 - j_3} 2^{(k_1+k_2)/2} \| f_{k_1} \|_{Z_{k_1}} \| f_{k_2} \|_{Z_{k_2}} \| f_{k_3} \|_{Z_{k_3}} \\
\lesssim 2^{(k_1+k_2)/4} \| f_{k_1} \|_{Z_{k_1}} \| f_{k_2} \|_{Z_{k_2}} \| f_{k_3} \|_{Z_{k_3}}.
\]

Therefore, we complete the proof of the proposition.

This proposition suffices to control high × low interaction in the case that the two low frequencies are comparable. However, for the case that the two low frequencies are not comparable, we will need an improvement.

**Proposition 5.2.** Assume $k_3 \geq 110$, $|k_4 - k_3| \leq 5$, $0 \leq k_1,k_2 \leq k_3 - 10$, $k_2 \geq 10$, $k_1 \leq k_2 - 5$ and $f_{k_i} \in Z_{k_i}$ with $F^{-1}(f_{k_i})$ compactly supported (in time) in $J_0$,
\[ |J_0| \leq 1, \ i = 1, 2, 3. \] Then
\[ 2^{k_i} \| \chi_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \leq C 2^{(k_1 + k_2)/4} \prod_{i=1}^{3} \| f_{k_i} \|_{Z_{k_i}}. \] (5.5)

**Proof.** We first observe that in this case we have
\[ |\Omega(\xi_1, \xi_2, \xi_3)| \sim 2^{k_3 + k_2}, \] (5.6)
which follows from the fact that \( \xi_1 + \xi_2 + \xi_3 \) has the same sign as \( \xi_3, k_1 \leq k_2 - 10 \) and in the area \( \{ \xi_i \in I_{k_i}, i = 1, 2, 3 \} \)
\[ |\omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3)| \sim 2^{k_3 + k_2}. \] (5.7)

Dividing it into three parts as before, we obtain
\[
2^{k_i} \| \chi_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
\leq 2^{k_4} \| \chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau - \omega(\xi))((\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
+ 2^{k_i} \| \chi_{k_4}(\xi)\eta_{= 2k_4}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
+ 2^{k_i} \| \chi_{k_4}(\xi)\eta_{= 2k_4+1}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \\
= I + II + III.
\]

For the last two terms \( II, III \), we can use the same argument as for \( II, III \) in the proof of Proposition 5.1. We consider now the first term \( I \).
\[
I \leq 2^{k_4} \| \chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}^h \|_{Z_{k_4}} \\
+ 2^{k_4} \| \chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}^i \|_{Z_{k_4}} \\
= I_1 + I_2,
\]
where
\[
f_{k_3}^h = f_{k_3}(\xi, \tau)\eta_{\geq k+k_2-10}(\tau - \omega(\xi)), \quad f_{k_3}^i = f_{k_3}(\xi, \tau)\eta_{\leq k+k_2-9}(\tau - \omega(\xi)).
\]

For the contribution of \( I_1 \), we observe first that from the support of \( f_{k_3}^h \) and the definition of \( Y_{k_4} \), one easily get that
\[ \| f_{k_3}^h \|_{X_{k_4}} \lesssim \| f_{k_3} \|_{Z_{k_3}}. \] (5.8)

Thus from the definition of \( Y_{k_4} \), and from Hölder’s inequality, Lemma 3.1 (a), (c) and Lemma 3.4, we get
\[
I_1 \lesssim 2^{k_4} \| \chi_{k_4}(\xi)\eta_{\leq k_4-1}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}^h \|_{Y_{k_4}} \\
\lesssim 2^{k_4/2} \| \mathcal{F}^{-1}[f_{k_1} * f_{k_2} * f_{k_3}^h] \|_{L^2 \rightarrow L^2} \\
\lesssim 2^{k_4/2} \| \mathcal{F}^{-1}(f_{k_3}^h) \|_{L^2 \rightarrow L^2} \| \mathcal{F}^{-1}(f_{k_1}) \|_{L^2 \rightarrow L^\infty} \| \mathcal{F}^{-1}(f_{k_2}) \|_{L^2 \rightarrow L^\infty} \\
\lesssim 2^{k_4/2} 2^{(k_1 + k_2)/4} \| f_{k_3}^h \|_{L^2} \| f_{k_1} \|_{Z_{k_1}} \| f_{k_2} \|_{Z_{k_2}}.
\]

Then from the fact that
\[
2^{k_4/2} \| f_{k_3}^h \|_{L^2} \lesssim \sum_{j \geq k_4 + k_2 - 10} 2^{k_2/2} \| f_{k_2} \eta_j(\tau - \omega(\xi)) \|_{L^2} \\
\lesssim \| f_{k_3} \|_{X_{k_3}} \lesssim \| f_{k_3} \|_{Z_{k_3}}
\] (5.9)
we conclude the proof for \( I_1 \).
We consider now the contribution of $I_2$. Let $f_{k_i,j_i}((\xi, \tau)) = f_{k_i}((\xi, \tau)) \eta_{j_i}(\tau - \omega(\xi))$, $j_i \geq 0$, $i = 1, 2, 3$. Using $X_k$ norm, we get

$$I_2 \leq \sum_{j_4 \leq k - 1} \sum_{j_3 \leq k_3 + k_2 - 9} \sum_{j_2 \geq 0} 2^{k_4} 2^{-j_4/2} \|1_{D_{k_4,j_4}}((\xi, \tau)) f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3} \|_{L^2}.$$

By checking the support properties of the functions $f_{k_i,j_i}$, $(i = 1, 2, 3)$ and from (5.6), we easily get that $1_{D_{k_4,j_4}}((\xi, \tau)) f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3} \equiv 0$ unless

$$\begin{cases} j_1, j_2 \geq k_3 + k_2 - 10, |j_1 - j_2| \leq 5; \\
|j_1 - k_3 - k_2| \leq 5, j_2 \leq k_3 + k_2 - 10; \\
|j_2 - k_3 - k_2| \leq 5, j_1 \leq k_3 + k_2 - 10. 
\end{cases}$$

**Case 1.** $j_1, j_2 \geq k_3 + k_2 - 10, |j_1 - j_2| \leq 5$.

It follows from Corollary 4.2 (b) and Lemma 3.1 (b) that

$$I_2 \leq \sum_{j_1, j_2 \geq 0} 2^{k_4} 2^{-j_4/2} (2^{j_1 + j_2 + j_3} + 2^{j_1 + j_2 - k_3 - 2k_4 + 1/2}) \sum_{i=1}^3 \|f_{k_i,j_i}\|_2 \|f_{k_3}\|_{Z_{k_3}} \leq 2(k_1 + k_2)/4 \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}},$$

which is acceptable.

**Case 2.** $|j_1 - k_3 - k_2| \leq 5, j_2 \leq k_3 + k_2 - 10$; or $|j_2 - k_3 - k_2| \leq 5, j_1 \leq k_3 + k_2 - 10$.

We consider only the worse case $|j_2 - k_3 - k_2| \leq 5, j_1 \leq k_3 + k_2 - 10$. It follows from Corollary (b) that

$$I_2 \leq \sum_{j_1, j_3, j_4 \leq 2k_4} \sum_{j_2 \geq 0} 2^{k_4} 2^{j_1 + j_3} 2^{j_2 - k_3 - 2k_4 + 1/2} \prod_{i=1}^3 \|f_{k_i,j_i}\|_2 \leq k_4^3 2^{k_4} 2^{j_2 - k_3 - 2k_4 + 1/2} \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}} \leq 2(k_1 + k_2)/4 \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}}.$$

Therefore, we complete the proof of the proposition.

**Proposition 5.3.** Assume $k_3 \geq 110$, $|k_4 - k_3| \leq 5$, $k_3 - 10 \leq k_2 \leq k_3$, $0 \leq k_1 \leq k_2 - 10$ and $f_{k_i} \in Z_{k_i}$ with $\mathcal{F}^{-1}(f_{k_i})$ compactly supported (in time) in $J_0$, $|J_0| \lesssim 1$, $i = 1, 2, 3$. Then

$$2^{k_4} \|\chi_{k_4}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} \ast f_{k_2} \ast f_{k_3}\|_{Z_{k_4}} \lesssim 2(k_1 + k_2)/4 \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}}.$$

**Proof.** We first observe that this case corresponds to an integration in the area $\{\xi_i \in I_{k_i}, i = 1, 2, 3\} \cap \{\xi_1 + \xi_2 + \xi_3 \in I_{k_3}\}$, where we have

$$|\Omega(\xi_1, \xi_2, \xi_3)| \sim 2^{2k_3}.$$
Let $f_{k_i,j_i}(\xi, \tau) = f_k(\xi, \tau)\eta_{j_i}(\tau - \omega(\xi))$, $j_i \geq 0$ and $i = 1, 2, 3$. Using $X_k$ norm, we get

$$2^{k_i}\|\chi_k(\xi)(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}\|_{L^2_k} \lesssim \sum_{j_1, j_2, j_3, j_4 \geq 0} 2^{k_1}2^{-j_3/2}(1 + 2^{(j_4 - 2k_3)/2})\|1_{D_{k_1,j_1}}(\xi, \tau)f_{k_{1,j_1}} * f_{k_{2,j_2}} * f_{k_{3,j_3}}\|_{L^2_k}.$$  

From the support properties of the functions $f_{k_i,j_i}$, $i = 1, 2, 3$, it is easy to see that $1_{D_{k_1,j_1}}(\xi, \tau)f_{k_{1,j_1}} * f_{k_{2,j_2}} * f_{k_{3,j_3}} = 0$ unless

$$\begin{cases} j_{\text{max}}, j_{\text{sub}} \geq 2k_3 - 10, |j_{\text{max}} - j_{\text{sub}}| \leq 5; \\
|j_{\text{max}} - 2k_3| \leq 5, j_{\text{sub}} \leq 2k_3 - 10. \end{cases}$$

**Case 1.** $j_{\text{max}}, j_{\text{sub}} \geq 2k_3 - 10, |j_{\text{max}} - j_{\text{sub}}| \leq 5$.

It follows from Corollary 4.2 (a) that the right-hand side of (5.13) is bounded by

$$\sum_{j_1, j_2, j_3, j_4 \geq 0} 2^{k_1+2(j_1+j_2+j_3)/2}(1 + 2^{(j_4 - 2k_3)/2})2^{-j_{\text{sub}}/2}2^{(k_1+k_2)/2}2^{k_3/4} \prod_{i=1}^3 f_{k_{i,j_i}} \|_{L^2_k},$$

where we used Lemma 3.1 (b). Thus, we complete the proof of the proposition. 

**Proposition 5.4.** Assume $k_3 \geq 110$, $|k_1 - k_3| \leq 5$, $k_3 - 30 \leq k_1, k_2 \leq k_3$, and $f_{k_i} \in Z_{k_i}$ with $F^{-1}(f_{k_i})$ compactly supported (in time) in $J_0$, $|J_0| \leq 1$, $i = 1, 2, 3$. Then

$$2^{k_3}\|\chi_{k_3}(\xi)(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_3}} \leq C2^{k_3} \prod_{i=1}^3 f_{k_i} \|_{Z_{k_i}}.$$  

**Proof.** First we divide it into two parts.

$$2^{k_3}\|\chi_{k_3}(\xi)(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_3}} \lesssim 2^{k_3}\|\chi_{k_3}(\xi)\eta_{\leq 2k_3+20}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_3}} + 2^{k_3}\|\chi_{k_3}(\xi)\eta_{\geq 2k_3+21}(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}f_{k_1} * f_{k_2} * f_{k_3}\|_{Z_{k_3}} = I + II.$$
We consider first the contribution of the first term $I$. Using the $X_k$ norm and Lemma 3.4, then we get

$$I \lesssim 2^{k_i} \sum_{j_4 \geq 0} 2^{-j_4/2} \|1_{D_{k_4,j_4}}(\xi, \tau) f_{k_1} \ast f_{k_2} \ast f_{k_3}\|_{L^2}$$

$$\lesssim 2^{k_i} \prod_{i=1}^3 \|F^{-1}(f_{k_i})\|_{L^6} \lesssim 2^{k_i} \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}}. \quad (5.17)$$

We consider now the contribution of the second term $II$. Let $f_{k_i,j_i}(\xi, \tau) = f_{k_i}(\xi, \tau)\eta_{j_i}(\tau - \omega(\xi))$, $j_i \geq 0$, $i = 1, 2, 3$. Using the $X_k$ norm, we get

$$II \lesssim \sum_{j_4 \geq 2k_4 + 20} \sum_{j_1, j_2, j_3 \geq 0} \|1_{D_{k_4,j_4}}(\xi, \tau) f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3}\|_{L^2}. \quad (5.18)$$

Since in the area $\{ |\xi| \in k_i, i = 1, 2, 3 \}$ we have $|\Omega(\xi_j, \xi_1, \xi_2)| \lesssim 2^{2k_3}$, by checking the support properties of the functions $f_{k_i,j_i}$, $i = 1, 2, 3$, we get $|j_{\text{max}} - j_{\text{sub}}| \leq 5$ and $j_{\text{sub}} \geq 2k_3 + 10$. From symmetry, we assume $j_3, j_4 = j_{\text{max}}, j_{\text{sub}}$, then we get

$$II \lesssim \sum_{j_3 \geq 2k_3 + 20} \sum_{j_1, j_2, j_3 \geq 0} 2^{(j_1+j_2)/2}2^{k_3}2^{j_3-k_3}2^{j_3-k_3} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{2}$$

$$\lesssim \sum_{j_3 \geq 2k_3 + 20} 2^{2k_3-2j_3/2} \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}} \lesssim 2^{k_i} \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}}. \quad (5.19)$$

Therefore we complete the proof of the proposition.

We consider now the case which corresponds to high × high interactions. This case is better than high × low interaction case.

**Proposition 5.5.** Assume $k_1 \geq 110$, $|k_1 - k_2| \leq 5$, $0 \leq k_3 \leq k_1 + 10$, $0 \leq k_4 \leq k_1 - 10$, and $f_{k_i} \in Z_{k_i}$ with $F^{-1}(f_{k_i})$ compactly supported (in time) in $J_0$, $|J_0| \lesssim 1$, $i = 1, 2, 3$. Then

$$2^{k_4} \|\eta_{k_i}(\tau - \omega(\xi) + i)^{-1} f_{k_1} \ast f_{k_2} \ast f_{k_3}\|_{Z_{k_i}} \leq Ck_4^3 \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}} \quad (5.20)$$

**Proof.** We divide the argument into two cases. Let $f_{k_i,j_i}(\xi, \tau) = f_{k_i}(\xi, \tau)\eta_{j_i}(\tau - \omega(\xi))$, $j_i \geq 0$, $i = 1, 2, 3$. Using $X_k$ norm, then we get

$$2^{k_4} \|\chi_{k_i}(\tau - \omega(\xi) + i)^{-1} f_{k_1} \ast f_{k_2} \ast f_{k_3}\|_{Z_{k_i}}$$

$$\lesssim \sum_{j_i \geq 0} 2^{k_4}2^{-j_i/2}(1 + 2^{(j_j - 2k_4)/2})\|1_{D_{k_4,j_4}}(\xi, \tau) f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3}\|_{L^2}. \quad (5.21)$$

**Case 1.** $\max(j_1, j_2, j_3, j_4) \leq 2k_1 + 20$.

It follows from Corollary 4.2 (d) that the right-hand side of (5.21) is bounded by

$$\sum_{j_i \geq 0} 2^{k_4}(1 + 2^{(j_4 - 2k_4)/2})2^{(j_1+j_2+j_3)/2}2^{-k_i} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2} \lesssim k_4^4 \prod_{i=1}^3 \|f_{k_i}\|_{Z_{k_i}}. \quad (5.22)$$

where we used Lemma 3.1 (b).

**Case 2.** $\max(j_1, j_2, j_3, j_4) \geq 2k_1 + 20$. 


By checking the support properties, we get $|j_{\text{max}} - j_{\text{sub}}| \leq 5$. We consider only the worst case $j_1, j_4 = j_{\text{max}}, j_{\text{sub}}$. It follows from Corollary 4.2 (a) and Lemma 3.1 (b) that the right side of (5.21) is bounded by

$$
\sum_{j_i \geq 0} 2^{k_i} 2^{-j_i/2} (1 + 2^{(j_i - 2k_i)/2}) 2^{(j_2 + j_3)/2} 2^{2k_4} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2} \lesssim \prod_{i=1}^3 \| f_{k_i} \|_{Z_{k_i}}. \quad (5.23)
$$

Therefore, we complete the proof of the proposition. ■

The next proposition is used to control low $\times$ low interactions. This interaction is easy to control.

**Proposition 5.6.** Assume $0 \leq k_1, k_2, k_3, k_4 \leq 120$, and $f_{k_i} \in Z_{k_i}$, $i = 1, 2, 3$. Then

$$
2^{k_i} \| \eta_{k_i}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}} \leq C \prod_{i=1}^3 \| f_{k_i} \|_{Z_{k_i}}. \quad (5.24)
$$

**Proof.** Let $f_{k_1, j_1}(\xi, \tau) = f_{k_1}(\xi, \tau) \eta_{j_1}(\tau - \omega(\xi))$, $j_i \geq 0$, $i = 1, 2, 3$. Using $X_k$ norm, Corollary 4.2 (a) and Lemma 3.1 (b), then we get

$$
2^{k_1} \| \chi_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_4}}
\lesssim \sum_{j_1, j_2, j_3, j_4 \geq 0} 2^{(j_{\text{min}} + j_{\text{thd}})/2} 2^{(k_{\text{min}} + k_{\text{thd}})/2} \prod_{i=1}^3 \| f_{k_i, j_i} \|_{L^2}
\lesssim 2^{(k_{\text{min}} + k_{\text{thd}})/2} \prod_{i=1}^3 \| f_{k_i} \|_{Z_{k_i}},
$$

since for the case $j_{\text{max}} \geq 200$ we have $|j_{\text{max}} - j_{\text{sub}}| \leq 5$ by checking the support properties of the functions $f_{k_i, j_i}$, $i = 1, 2, 3$. ■

Finally we present two counterexamples. The first one shows why we use a $l^2$-type $X^{s,b}$ structure. The other one shows a logarithmic divergence if we only use $X_k$ which is the reason for us applying $Y_k$ structure. See also the similar phenomenon in [15] for the complex-valued Benjamin-Ono equation.

**Proposition 5.7.** Assume $k \geq 200$. Then there exist $f_1 \in X_1$, $f_k \in X_k$ such that

$$
2^{k} \| \eta_k(\xi)(\tau - \omega(\xi) + i)^{-1} f_1 * f_1 * f_k \|_{X_k} \lesssim 2^{j} \| f_1 \|_{X_1} \| f_1 \|_{X_1} \| f_k \|_{X_k}. \quad (5.25)
$$

**Proof.** From the proof of Proposition 5.1, we easily see that the worst interaction comes from the case that largest frequency component has a largest modulation. So we construct this case explicitly. Let $I = [1/2, 1]$, and take

$$
f_1(\xi, \tau) = \chi_I(\xi) \eta_1(\tau - \omega(\xi)), \quad f_k(\xi, \tau) = \chi_k(\xi) \eta_k(\tau - \omega(\xi)).
$$

From definition, we easily get $\| f_1 \|_{X_1} \sim 1$ and $\| f_k \|_{X_k} \sim 2^{k/2}$ and

$$
2^{k} \| \eta_k(\xi)(\tau - \omega(\xi) + i)^{-1} f_1 * f_1 * f_k \|_{X_k} \gtrsim 2^{k} \sum_{j=0}^{k/2} 2^{-j/2} \| 1_{D_{k,j}} \cdot f_1 * f_1 * f_k \|_{L^2_{\xi, \tau}}.
$$
On the other hand, we have for \( j \leq k/2 \)
\[
1_{D_{k,j}}(\xi, \tau) \cdot f_1 * f_1 * f_k = \int f_1(\xi_1, \tau_1) f_2(\xi_2, \tau_2) f_k(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
= \int \chi(\xi) \chi(\xi_1) \eta(\tau_1) \eta(\tau_2) \chi_k(\xi - \xi_1 - \xi_2) \cdot \eta(\tau - \tau_1 - \tau_2 - \omega(\xi_1) - \omega(\xi_2) - \omega(\xi - \xi_1 - \xi_2)) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
\geq \chi_{[210,210]}(\xi) \eta_j(\tau - \omega(\xi)).
\]
Therefore, we get
\[
2^k \sum_{j=0}^{k/2} 2^{-j/2} \| 1_{D_{k,j}} \cdot f_1 * f_1 * f_k \|_{L^2_{\xi,\tau}} \gtrsim k 2^{3k/2}, \quad (5.26)
\]
which completes the proof of the proposition.

**Proposition 5.8.** For any \( s \in \mathbb{R} \), there doesn’t exist \( b \in \mathbb{R} \) such that
\[
\| \partial_x (uvw) \|_{X^{s,b,-1}} \lesssim \| u \|_{X^{s,b}} \| v \|_{X^{s,b}} \| w \|_{X^{s,b}}. \quad (5.27)
\]

**Proof.** It is easy to see that the counterexample in the proof of Proposition 5.7 shows that (5.27) doesn’t hold for \( b = 1/2 \) with a \( k^{1/2} \) divergence in (5.26). We assume now \( b \neq 1/2 \). By using Plancherel’s equality, we get that (5.27) is equivalent to
\[
\| \langle \xi \rangle^s \langle \tau - \omega(\xi) \rangle^{-b} \int \frac{u(\xi_1, \tau_1)}{\langle \xi_1 \rangle^s} \frac{v(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s} \frac{w(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2)}{\langle \xi - \xi_1 - \xi_2 \rangle^s} d\tau_1 d\tau_2 d\xi_1 d\xi_2 \|_{L^2_{\xi,\tau}} \lesssim \| u \|_{L^2} \| v \|_{L^2} \| w \|_{L^2}. \quad (5.28)
\]
Fix any dyadic number \( N \gg 1 \). Let
\[
A = \{ 1/2 \leq \xi \leq 10, \ |\tau| \leq 1 \} \text{ and } B = \{ N/2 \leq \xi \leq 2N, \ |\tau| \leq 2^{10} \}.
\]
Take
\[
u(\xi, \tau) = v(\xi, \tau) = \chi_A(\xi, \tau - \omega(\xi)), \ w(\xi, \tau) = \chi_B(\xi, \tau - \omega(\xi)).
\]
We easily see that \( \| u \|_{L^2} = \| v \|_{L^2} \sim 1 \) and \( \| w \|_{L^2} \sim N^{1/2} \). Denote \( f(\xi, \tau) = u * v * w(\xi, \tau + \omega(\xi)) \). Then we have
\[
f(\xi, \tau) = \int u(\xi_1, \tau_1) v(\xi_2, \tau_2) w(\xi - \xi_1 - \xi_2, \tau + \omega(\xi) - \tau_1 - \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
= \int \chi_{\leq 2^{10}}(\tau - \tau_1 - \tau_2 + \omega(\xi) - \omega(\xi - \xi_1 - \xi_2) - \omega(\xi_1) - \omega(\xi_2)) \chi_A(\xi_1, \tau_1) \chi_A(\xi_2, \tau_2) \chi_{[N/2,2N]}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2
\]
\[
= \int \chi_{\leq 2^{10}}(\tau - \tau_1 - \tau_2 + 2(\xi_1 + \xi_2) \xi + (\xi_1 - \xi_2)^2 - \omega(\xi_1) - \omega(\xi_2)) \chi_A(\xi_1, \tau_1) \chi_A(\xi_2, \tau_2) \chi_{[N/2,2N]}(\xi - \xi_1 - \xi_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\]
Therefore, fixing $M \gg 1$, we get for any $(\xi, \tau) \in [(M - 1)N/M, (M + 1)N/M] \times [-8N, -4N]$, then $\tau = -C_0 \xi$ for some $2 \leq C_0 \leq 9$ and

$$f(\xi, \tau) \geq \int \chi_A(\xi_1, \tau_1) \chi_A(\xi_2, \tau_2) \chi_{\xi_1 + \xi_2 - C_0} \leq N^{-1} d\xi_1 d\tau_1 d\xi_2 d\tau_2 \geq N^{-1}.$$

Thus we see that the left-hand side of (5.28) is larger than $N^b$, while the right-hand side is $N^{1/2}$, which implies $b < 1/2$.

Similarly, by taking $B' = \{N/2 \leq \xi \leq 2N, \, N \leq |\tau| \leq N\}$ as before, we obtain that $b > 1/2$. Therefore we complete the proof of the proposition. \hfill \Box

6. Proof of Theorem 1.1

In this section we devote to prove Theorem 1.1 by using the standard fixed-point machinery. From Duhamel’s principle, we get that the equation (1.1) is equivalent to the following integral equation:

$$u = W(t)\phi + \int_0^t W(t-t') (\partial_x (u^3) (t')) dt'. \tag{6.1}$$

We will mainly work on the following truncated version

$$u = \psi(t) W(t) \phi + \psi(t) \int_0^t W(t-t') (\partial_x [(\psi(t') u^3)] (t')) dt', \tag{6.2}$$

where $\psi(t) = \eta_0(t)$ is a smooth cut-off function. Then we easily see that if $u$ is a solution to (6.2) on $\mathbb{R}$, then $u$ solves (6.1) on $t \in [-1, 1]$. Our first lemma is on the estimate for the linear solution.

**Lemma 6.1.** If $s \geq 0$ and $\phi \in H^s$ then

$$\|\psi(t) \cdot (W(t) \phi)\|_{F^s} \leq C\|\phi\|_{H^s}. \tag{6.3}$$

**Proof.** A direct computation shows that

$$\mathcal{F}[\psi(t) \cdot (W(t) \phi)](\xi, \tau) = \hat{\phi}(\xi) \hat{\psi}(\tau - \omega(\xi)).$$

In view of definition, it suffices to prove that if $k \in \mathbb{Z}_+$ then

$$\|\eta_k(\xi) \hat{\phi}(\xi) \hat{\psi}(\tau - \omega(\xi))\|_{L^2_{\xi, \tau}} \leq C\|\eta_k(\xi) \hat{\phi}(\xi)\|_{L^2_{\xi, \tau}}. \tag{6.4}$$

Indeed, from definition we have

$$\|\eta_k(\xi) \hat{\phi}(\xi) \hat{\psi}(\tau - \omega(\xi))\|_{L^2_{\xi, \tau}} \leq \|\eta_k(\xi) \hat{\phi}(\xi) \hat{\psi}(\tau - \omega(\xi))\|_{X_k} \leq C \sum_{j=0}^{\infty} 2^j \|\eta_k(\xi) \hat{\phi}(\xi)\|_{L^2_{\xi, \tau}}\|\eta_j(\tau) \hat{\psi}(\tau)\|_{L^2_{\tau}} \leq C\|\eta_k(\xi) \hat{\phi}(\xi)\|_{L^2_{\xi, \tau}},$$

which is (6.4) as desired. \hfill \Box

Next lemma is on the estimate for the retarded linear term. These estimates were also used in [14]. The only difference is that here we don’t have special structure for the low frequency.

**Lemma 6.2.** If $l, s \geq 0$ and $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$ then

$$\left\|\psi(t) \cdot \int_0^t W(t-s) (u(s)) ds\right\|_{F^s} \leq C\|u\|_{N^s}. \tag{6.5}$$
Proof. A straightforward computation shows that
\[ F \left[ \psi(t) \cdot \int_0^t W(t-s)(u(s))ds \right] (\xi, \tau) = c \int_{\mathbb{R}} F(u)(\xi, \tau') \frac{\tilde{\psi}(\tau - \tau') - \tilde{\psi}(\tau - \omega(\xi))}{\tau' - \omega(\xi)} d\tau'. \]

For \( k \in \mathbb{Z}_+ \) let \( f_k(\xi, \tau') = F(u)(\xi, \tau') \chi_k(\xi)(\tau' - \omega(\xi) + i)^{-1} \). For \( f_k \in Z_k \) let
\[ T(f_k)(\xi, \tau) = \int_{\mathbb{R}} f_k(\xi, \tau') \frac{\tilde{\psi}(\tau - \tau') - \tilde{\psi}(\tau - \omega(\xi))}{\tau' - \omega(\xi)} (\tau' - \omega(\xi) + i) d\tau'. \]

In view of the definitions, it suffices to prove that
\[ \|T\|_{Z_k \rightarrow Z_k} \leq C \text{ uniformly in } k \in \mathbb{Z}_+, \quad (6.6) \]
which follows from the proof of Lemma 5.2 in [14].

We prove a trilinear estimate in the following proposition which is an important component for using fixed-point argument.

**Proposition 6.3.** Let \( s \geq 1/2 \). Then
\[ \| \partial_x (\psi(t)^3 uvw) \|_{N^s} \lesssim \| u \|_{F^{\frac{3}{2}}} \| v \|_{F^\frac{1}{2}} \| w \|_{F^\frac{1}{2}} + \| u \|_{F^\frac{1}{2}} \| v \|_{F^{\frac{3}{2}}} \| w \|_{F^{\frac{3}{2}}}. \quad (6.7) \]

**Proof.** For the simplicity of notation, we write \( u = \psi(t)u, \ v = \psi(t)v \) and \( w = \psi(t)w \). In view of definition, we get
\[ \| \partial_x (uvw) \|_{N^s}^2 = \sum_{k_1=0}^{\infty} 2^{2k_1} \| \eta_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1} F(\partial_x (uvw)) \|_{Z_{k_1}}^2. \]

Setting \( f_{k_1} = \eta_{k_1}(\xi) F(u)(\xi, \tau), \ f_{k_2} = \eta_{k_2}(\xi) F(v)(\xi, \tau), \) and \( f_{k_3} = \eta_{k_3}(\xi) F(w)(\xi, \tau), \) for \( k_1, k_2, k_3 \in \mathbb{Z}_+ \), then we get
\[ 2^{k_1} \| \eta_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1} F(uvw) \|_{Z_{k_1}} \lesssim \sum_{k_1, k_2, k_3 \in \mathbb{Z}_+} 2^{k_1} \| \eta_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_1}}. \]

From symmetry it suffices to bound
\[ \sum_{0 \leq k_1 \leq k_2 \leq k_3} 2^{k_1} \| \eta_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_1}}. \]

Dividing the summation into the several parts, we get
\[ \sum_{k_1 \leq k_2 \leq k_3} 2^{k_1} \| \eta_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_1}} \leq \sum_{j=1}^{6} \sum_{(k_1, k_2, k_3, k_4) \in A_j} 2^{k_1} \| \eta_{k_1}(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_{Z_{k_1}}, \quad (6.8) \]
where we denote

\[ A_1 = \{0 \leq k_1 \leq k_2 \leq k_3 - 10, k_3 \geq 110, |k_4 - k_3| \leq 5, |k_1 - k_2| \leq 10\}; \]
\[ A_2 = \{0 \leq k_1 \leq k_2 \leq k_3 - 10, k_3 \geq 110, |k_4 - k_3| \leq 5, k_1 \leq k_2 - 5\}; \]
\[ A_3 = \{0 \leq k_1 \leq k_2 \leq k_3, k_2 \geq k_3 - 10, k_3 \geq 110, |k_4 - k_3| \leq 5, k_1 \leq k_2 - 10\}; \]
\[ A_4 = \{0 \leq k_1 \leq k_2 \leq k_3, k_1 \geq k_3 - 30, k_3 \geq 110, |k_4 - k_3| \leq 5\}; \]
\[ A_5 = \{0 \leq k_1 \leq k_2 \leq k_3, k_4 \geq k_3 - 10, k_3 \geq 110, |k_2 - k_3| \leq 5\}; \]
\[ A_6 = \{0 \leq k_1 \leq k_2 \leq k_3, \max(k_3, k_4) \leq 120\}. \]

Noting that \( F^{-1}(f_{\lambda})(x, t) \) is supported in \( \mathbb{R} \times I \) with \(|I| \leq 1\), we will apply Proposition 5.1-5.6 obtained in the last section to bound the six terms in (6.8). For example, for the first term, from Proposition 5.1, we have

\[ \|2^{kJ_1} \sum_{k_i \in A_1} 2^{k_1} \|\eta_{k_1}(\xi)(\tau - \omega(\xi) + \iota)^{-1} f_{k_1} * f_{k_2} * f_{k_3} \|_2 \|
\leq C \|2^{kJ_1} \sum_{k_i \in A_1} 2^{(k_1+k_2)/2} \| f_{k_1} \| \| f_{k_2} \| \| f_{k_3} \| \| \| f_{k_3} \| \| Z_{k_4} \|_2 \|
\leq \| u \|_{F^{1/2}} \| v \|_{F^{1/2}} \| w \|_{F^{s}}. \]

For the other terms we can handle them in the similar ways. Therefore we complete the proof of the proposition.

Now we prove Theorem 1.1. To begin with, we renormalize the data a bit via scaling. By the scaling (1.4), we see that if \( s \geq 1/2 \)

\[ \|\phi_\lambda\|_{L^2} = \|\phi\|_{L^2}, \]
\[ \|\phi_\lambda\|_{H^s} = \lambda^{-s} \|\phi\|_{H^s}. \]

From the assumption \( \|\phi\|_{L^2} \ll 1 \), thus we can first restrict ourselves to considering (1.1) with data \( \phi \) satisfying

\[ \|\phi\|_{H^s} = \|\phi\|_{L^2} \ll 1. \] (6.9)

This indicates the reason why we assume that \( \|\phi\|_{L^2} \ll 1 \).

Define the operator

\[ \Phi_\phi(u) = \psi(t)W(t)\phi + \psi(t) \int_0^t W(t - t')(\partial_x((\psi(t')u)^3(t')))dt', \]

and we will prove that \( \Phi_\phi(\cdot) \) is a contraction mapping from

\[ \mathcal{B} = \{w \in F^s : \|w\|_{F^s} \leq 2c\} \] (6.10)

into itself. From Lemma 6.1, 6.2 and Proposition 6.3 we get if \( w \in \mathcal{B} \), then

\[ \|\Phi_\phi(w)\|_{F^s} \leq c\|\phi\|_{H^s} \leq c\|\phi\|_{H^s} + \|\partial_x(\psi(t)^3 w^3(\cdot, t))\|_{N^s} \leq cr + c\|w\|_{F^s}^3 \leq cr + c(2cr)^3 \leq 2cr, \] (6.11)

provided that \( r \) satisfies \( 8c^3r^2 \leq 1/2 \). Similarly, for \( w, h \in \mathcal{B} \)

\[ \|\Phi_\phi(w) - \Phi_\phi(h)\|_{F^s} \leq c\|L\partial_x(\psi^3(\tau)(w^3(\tau) - h^3(\tau)))\|_{F^s} \leq c(\|w\|_{F^s}^3 + \|h\|_{F^s}^3)\|w - h\|_{F^s} \leq 8c^3r^2\|w - h\|_{F^s} \leq \frac{1}{2}\|w - h\|_{F^s}. \] (6.12)
Thus $\Phi_\phi(\cdot)$ is a contraction. Therefore, there exists a unique $u \in \mathcal{B}$ such that

$$u = \psi(t)W(t)\phi + \psi(t)\int_0^t W(t-t')(\partial_x((\psi(t')u)^3)(t'))dt'.$$

Hence $u$ solves the integral equation (6.1) in the time interval $[-1,1]$.

Part (c) of Theorem 1.1 follows from the scaling (1.4), Lemma 3.4 and Proposition 6.3. We prove now part (b). For the real-valued case, according to Theorem 1.2 in [18], it suffices to prove that if $s > 1$ then

$$\partial_t u \in L^2_t L^\infty_x.$$

Indeed, this follows from the fact that $u \in F^s(T)$ and $(4, \infty)$ is an admissible pair and Lemma 3.4. For the complex-valued case, we have some weak uniqueness. From the proof we see $u$ is unique in the following set

$$B(u_0) = \left\{ u : u = \lambda^{1/2}\tilde{u}(\lambda^2 t, \lambda x) \text{ in } [-T, T] \text{ for some } \lambda = \lambda(\|\phi\|_{H^s}) \gg 1 \text{ and a solution } \tilde{u} \text{ to (6.2) satisfying } \|\tilde{u}\|_{L^\infty} \lesssim \lambda^{-s} \right\}$$

(6.13)

Therefore, we complete the proof of Theorem 1.1.

**Remark 6.4.** For the real-valued case, the uniqueness actually holds in $F^s(T)$ by the uniqueness in [25] and Lemma 3.4. From the proof we see the $L^2$ norm smallness condition is due to the high $\times$ low $\rightarrow$ high interaction where both low frequency are around 0. It is also due to this interaction that one cannot apply the methods as in the second part. This bad interaction is removed via gauge transformation in the previous results. On the other hand, one may also remove the smallness condition by performing a gauge transformation as following

$$v(x,t) = e^{-(i/2)\int_0^t (\tilde{\mathcal{F}}^1 u(x,t))^3dy} P_\geq 1 u(x,t),$$

and using the similar methods in [14].

### 7. Short-time Trilinear Estimates

In this section we devote to prove some dyadic trilinear estimates in the spaces $F_k, N_k$. For $k \in \mathbb{Z}$ and $j \in \mathbb{Z}$ let

$$\mathcal{D}_{k,j} = \{ (\xi, \tau) : \xi \in I_{k,j+1} | \tau - \omega(\xi)| \leq 2^j \}.$$  

(7.1)

Note that $\mathcal{D}_{k,j} \subset \mathcal{D}_{k,j}$ for any $k, j \in \mathbb{Z}$.

**Proposition 7.1.** If $k_4 \geq 20$, $|k_3 - k_4| \leq 5$, $k_1 \leq k_2 \leq k_3 \leq 10$, then

$$||R_{k_4}\partial_x(u_{k_3}v_{k_2}w_{k_1})||_{N_{k_4}} \leq C \min(2^{k_1/2}, \langle k_2 \rangle)||u_{k_1}||_{F_{k_3}}||v_{k_2}||_{F_{k_2}}||w_{k_3}||_{F_{k_3}}.$$  

(7.2)

**Proof.** Using the definitions and (3.11), we get that the left-hand side of (7.2) is dominated by

$$C \sup_{t_k \in \mathbb{R}} \| \tau - \omega(\xi) + i2^{k_4} \| F[u_{k_1} \eta_0(2^{k_4-2}(t-t_k))]$$

$$\ast \| F[v_{k_2} \eta_0(2^{k_4-2}(t-t_k))] \ast \| F[w_{k_3} \eta_0(2^{k_4-2}(t-t_k))] \|_{L^{k_4}}.$$  

(7.3)

It suffices to prove that if $j_i \geq k_4$ and $f_{k_i, j_i} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ are supported in $\mathcal{D}_{k_i, j_i}$ for $i = 1, 2, 3$, then

$$2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| 1_{\mathcal{D}_{k_4, j_4}} \ast (f_{k_1, j_1} \ast f_{k_2, j_2} \ast f_{k_3, j_3}) \|_{L^2}$$

$$\leq C \min(2^{k_1/2}, \langle k_2 \rangle)2^{j_1/2}\|f_{k_1, j_1}\|_{L^2}2^{j_2/2}\|f_{k_2, j_2}\|_{L^2}2^{j_3/2}\|f_{k_3, j_3}\|_{L^2}.$$  

(7.4)
We assume first \((7.4)\). Let \(f_{k_1} = \mathcal{F}[u_{k_1} \cdot \eta_{0}(2^{k_1-2}(t-t_k))], f_{k_2} = \mathcal{F}[v_{k_2} \cdot 
abla_t \eta_{0}(2^{k_1-2}(t-t_k))]\) and \(f_{k_3} = \mathcal{F}[w_{k_3} \cdot \eta_{0}(2^{k_1-2}(t-t_k))]\). Then from the definition of \(B_k\) we get that \((7.3)\) is dominated by

\[
\sup_{t_k \in \mathbb{R}} 2^{k_4} \sum_{j_4=0}^{\infty} 2^{j_4/2} \sum_{j_1,j_2,j_3 \geq k_4} \| (2^{j_4} + i2^{k_4})^{-1} \hat{D}_{k_4,j_4} \cdot f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3} \|_{L^2},
\]

where \(f_{k_1,j_1} = f_k(\xi, \eta)\) for \(j_1 \geq k_4\) and \(f_{k_1,k_4} = f_k(\xi, \tau)\eta_{\leq k_4}(\tau - \omega(\xi)), i = 1, 2, 3.\) For the summation on the terms \((i,j)\), we obtain \((7.2)\), as desired.

Thus using \((3.10)\) and \((3.11)\) we obtain \((7.2)\), as desired.

To prove \((7.4)\), we consider first the case \(|k_1 - k_2| \leq 5\). If \(k_2 \geq 0\), it follows from Corollary 4.2 (b) and Remark 4.3 that

\[
2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| \hat{D}_{k_4,j_4} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2} \leq 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| \hat{D}_{k_4,j_4} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2}
\]

which is \((7.4)\) as desired. If \(k_2 < 0,\) then from Corollary 4.2 (a) and Remark 4.3 we get that

\[
2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| \hat{D}_{k_4,j_4} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2} \leq 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| \hat{D}_{k_4,j_4} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2}
\]

which is \((7.4)\) as desired. If \(k_2 < 0,\) then from Corollary 4.2 (a) and Remark 4.3 we get that

\[
2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| \hat{D}_{k_4,j_4} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2} \leq 2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| \hat{D}_{k_4,j_4} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \|_{L^2}
\]
which is (7.4) as desired. We assume now \( k_1 < k_2 - 5 \). If \( k_2 < 0 \), then arguing as in (7.8) we get (7.4) as desired. If \( k_2 > 0 \), then (7.7) also holds in this case. On the other hand, by checking the support properties of the function \( f_{k_i,j_i} \), \( i = 1, 2, 3 \), we get that \( 1_{D_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \equiv 0 \) unless \( j_{\max} \geq k_4 + k_2 - 20 \). For the summation on the terms \( j_4 > k_4 + k_2 - 30 \) in (7.4), we have

\[
2^{k_4} \sum_{j_4 \geq k_4 + k_2 - 30} 2^{-j_4/2} ||1_{\bar{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})||_{L^2}
\]

\[
\lesssim 2^{k_4} \sum_{j_4 \geq k_4 + k_2 - 30} 2^{-j_4/2} \|2^{(j_1+j_2+j_3)/2}-j_4/2\|2^{k_1/2}2^{k_2/2} \prod_{i=1}^{3} ||f_{k_i,j_i}||_{L^2}
\]

\[
\lesssim 2^{k_1/2}(j_1+j_2+j_3)/2 \prod_{i=1}^{3} ||f_{k_i,j_i}||_{L^2}.
\]  

(7.9)

For the summation on the terms \( j_4 < k_4 + k_2 - 30 \), we have \( j_4 \leq j_{\text{med}} \). Thus using Corollary 4.2 (a), then we get

\[
2^{k_4} \sum_{j_4 < k_4 + k_2 - 30} 2^{-j_4/2} ||1_{\bar{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})||_{L^2}
\]

\[
\lesssim 2^{k_4} \sum_{k_4 \leq j_4 < k_4 + k_2 - 30} 2^{-j_4/2} \|2^{(j_1+j_2+j_3)/2}-j_4/2\|2^{k_1/2}2^{k_2/2} \prod_{i=1}^{3} ||f_{k_i,j_i}||_{L^2}
\]

\[
\lesssim 2^{k_1/2}(j_1+j_2+j_3)/2 \prod_{i=1}^{3} ||f_{k_i,j_i}||_{L^2}.
\]  

(7.10)

Therefore, we complete the proof of the proposition. 

\[\Box\]

\textbf{Proposition 7.2.} \textit{If} \( k_4 \geq 20, \ |k_3 - k_4| \leq 5, \ k_3 - 10 \leq k_2 \leq k_3 \) \textit{and} \( k_1 \leq k_2 - 20 \), \textit{then we have}

\[
||R_{k_4} \partial_v (u_{k_1} v_{k_2} w_{k_3})||_{L^4_{k_4}} \leq C \min(2^{k_1/2}, 1) ||u_{k_1}||_{F_{k_1}} ||v_{k_2}||_{F_{k_2}} ||w_{k_3}||_{F_{k_3}}.
\]  

(7.11)

\textbf{Proof.} \textit{As in the proof of Proposition 7.1, using (3.10) and (3.11), we see that it suffices to prove that if} \( j_1, j_2, j_3 \geq k_4 \), \textit{and} \( f_{k_i,j_i} : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) \textit{are supported in} \( D_{k_1,j_1}, \ i = 1, 2, 3 \), \textit{then}

\[
2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} ||1_{\bar{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3})||_{L^2}
\]

\[
\leq C \min(2^{k_1/2}, 1) \cdot 2^{j_1/2} ||f_{k_1,j_1}||_{L^2} \cdot 2^{j_2/2} ||f_{k_2,j_2}||_{L^2} \cdot 2^{j_3/2} ||f_{k_3,j_3}||_{L^2}.
\]  

(7.12)

Since in the area \( \{|\xi_i| \in \bar{I}_{k_4}, \ i = 1, 2, 3 \} \cap \{|\xi_1 + \xi_2 + \xi_3| \in I_{k_4}\} \)

\[|\Omega(\xi_1, \xi_2, \xi_3)| \sim 2^{2k_4},\]

then by checking the support properties, we get \( 1_{\bar{D}_{k_4,j_4}} \cdot (f_{k_1,j_1} * f_{k_2,j_2} * f_{k_3,j_3}) \equiv 0 \) unless \( j_{\max} \geq 2k_4 - 30 \). It follows from Corollary 4.2 (a) that the left-hand side of (7.12) is bounded by

\[
2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \|2^{(j_1+j_2+j_3)/2}-j_4/2\|2^{k_1/2}2^{k_2/2} \prod_{i=1}^{3} ||f_{k_i,j_i}||_{L^2}.
\]  

(7.13)

Then we get the bound (7.12) by considering either \( j_4 = j_{\max} \) or \( j_4 \neq j_{\max} \).  

\[\Box\]
Proposition 7.3. If \( k_4 \geq 20, |k_3 - k_4| \leq 5, k_3 - 10 \leq k_2 \leq k_3 \) and \( k_2 - 30 \leq k_1 \leq k_2 \), then we have
\[
\| R_{k_4} \partial_x (u_{k_1} v_{k_2} w_{k_3}) \|_{N_{k_4}} \leq C 2^{k_3/2} \| u_{k_1} \|_{F_{k_1}} \| v_{k_2} \|_{F_{k_2}} \| w_{k_3} \|_{F_{k_3}}. \tag{7.14}
\]

Proof. As in the proof of Proposition 7.1, using (3.10) and (3.11), it suffices to prove that if \( j_1, j_2, j_3 \geq k_4 \), and \( f_{k_1, j_1}, f_{k_2, j_2} : \mathbb{R}^2 \to \mathbb{R}_+ \) are supported in \( D_{k_1, j_1}, i = 1, 2, 3 \), then
\[
2^{k_4} \sum_{j_4 \geq k_4} 2^{-j_4/2} \| 1_{D_{k_4, j_4}} \cdot (f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \leq C 2^{k_4/2} \cdot 2^{j_1/2} \| f_{k_1, j_1} \|_{L^2} \cdot 2^{j_2/2} \| f_{k_2, j_2} \|_{L^2} \cdot 2^{j_3/2} \| f_{k_3, j_3} \|_{L^2},
\]
which follows immediately from Corollary 4.2 (c).

Proposition 7.4. If \( k_3 \geq 20, |k_3 - k_2| \leq 5, k_2 - 10 \leq k_1 \leq k_2 \) and \( k_4 \leq k_1 - 30 \), then we have
\[
\| R_{k_4} \partial_x (u_{k_1} v_{k_2} w_{k_3}) \|_{N_{k_4}} \leq C \min(2^{k_4}, 1) |k_2| \| u_{k_1} \|_{F_{k_1}} \| v_{k_2} \|_{F_{k_2}} \| w_{k_3} \|_{F_{k_3}}. \tag{7.15}
\]

Proof. Let \( \gamma : \mathbb{R} \to [0, 1] \) denote a smooth function supported in \([-1, 1]\) with the property that
\[
\sum_{n \in \mathbb{Z}} \gamma^3(x - n) \equiv 1, \quad x \in \mathbb{R}.
\]
Using the definitions, the left-hand side of (7.15) is dominated by
\[
C \sup_{t \in \mathbb{R}} \| (\tau - \omega(\xi) + i 2^{k_4+})^{-1} \cdot 2^{k_4} 1_{D_{k_4, j_4}}(\xi) \cdot \sum_{|m| \leq C 2^{k_4 - k_4+} \cdot \mathcal{F}\{u_{k_1} \eta_0(2^{k_4+}(t - t_k)) \gamma(2^{k_2}(t - t_k) - m)\}* \mathcal{F}\{v_{k_2} \eta_0(2^{k_4+}(t - t_k)) \gamma(2^{k_2}(t - t_k) - m)\}* \mathcal{F}\{w_{k_2} \eta_0(2^{k_4+}(t - t_k)) \gamma(2^{k_2}(t - t_k) - m)\}\|_{B_{k_1}}.
\]
In view of the definitions, (3.10) and (3.11), it suffices to prove that if \( j_1, j_2, j_3 \geq k_2 \), and \( f_{k_1, j_1}, f_{k_2, j_2} : \mathbb{R}^3 \to \mathbb{R}_+ \) are supported in \( D_{k_1, j_1}, i = 1, 2, 3 \), then
\[
2^{k_4} 2^{k_2 - k_4+} \sum_{j_4 \geq k_4+} 2^{-j_4/2} \| 1_{D_{k_4, j_4}} \cdot (f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \leq C \min(2^{k_4}, 1) |k_2| \cdot 2^{j_1/2} \| f_{k_1, j_1} \|_{L^2} \cdot 2^{j_2/2} \| f_{k_2, j_2} \|_{L^2} \cdot 2^{j_3/2} \| f_{k_3, j_3} \|_{L^2}. \tag{7.16}
\]
From the same argument as in Proposition 7.2, we get \( j_{max} \geq 2k_2 - 30 \). Then (7.16) follows from Corollary 4.2.

Proposition 7.5. If \( k_3 \geq 20, |k_3 - k_2| \leq 5, \) and \( k_1, k_4 \leq k_2 - 10 \), then
\[
\| R_{k_4} \partial_x (u_{k_1} v_{k_2} w_{k_3}) \|_{N_{k_4}} \leq C \min(2^{k_4+}/2, 1) |k_2| \| u_{k_1} \|_{F_{k_1}} \| v_{k_2} \|_{F_{k_2}} \| w_{k_3} \|_{F_{k_3}}. \tag{7.17}
\]

Proof. As in the proof of Proposition 7.4, it suffices to prove that if \( j_1, j_2, j_3 \geq k_2 \), and \( f_{k_1, j_1}, f_{k_2, j_2} : \mathbb{R}^3 \to \mathbb{R}_+ \) are supported in \( D_{k_1, j_1}, i = 1, 2, 3 \), then
\[
2^{k_4} 2^{k_2 - k_4+} \sum_{j_4 \geq k_4+} 2^{-j_4/2} \| 1_{D_{k_4, j_4}} \cdot (f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \leq C \min(2^{k_4+}/2, 1) |k_2| \cdot 2^{j_1/2} \| f_{k_1, j_1} \|_{L^2} \cdot 2^{j_2/2} \| f_{k_2, j_2} \|_{L^2} \cdot 2^{j_3/2} \| f_{k_3, j_3} \|_{L^2}. \tag{7.18}
\]
In proving (7.18) we may assume \( j_k \leq 10k_2 \) in the summation of (7.18), otherwise we use Corollary 4.2 (a). Using Corollary 4.2 (a) for \( k_1 \leq 0 \), else using Corollary 4.2 (b), then we get
\[
2^{k_1+2k_2-k_4+} \sum_{j_i \geq k_4} 2^{-j_i/2} \|1_{\hat{D}^{k_4,j_4}} \cdot (f_{k_1,j_1} \ast f_{k_2,j_2} \ast f_{k_3,j_3})\|_{L^2} \\
\leq C \min(2^{k_1/2}, 1) \|f_{k_1,j_1}\|_{L^2} 2^{j_2/2} \|f_{k_2,j_2}\|_{L^2} 2^{j_3/2} \|f_{k_3,j_3}\|_{L^2}.
\]
Therefore, we complete the proof of the proposition.

**Proposition 7.6.** If \( k_1, k_2, k_3, k_4 \leq 200 \), then
\[
\|R_{k_4} \partial_x (u_k, v_k, w_k)\|_{\mathcal{N}_{k_4}} \leq C 2^{k_{min}/2} 2^{k_{max}/2} \|u_{k_1}\|_{\mathcal{F}_{k_1}} \|v_{k_2}\|_{\mathcal{F}_{k_2}} \|w_{k_3}\|_{\mathcal{F}_{k_3}}.
\] (7.19)

**Proof.** This follows immediately from the definitions, Corollary 4.2 (a), Remark 4.3 and (3.10) and (3.11).

## 8. Energy Estimates

In this section we prove an energy estimate by using I-method [8], following some ideas in [26]. For the difference equation of two modified Benjamin-Ono equations, we don’t know how to prove a similar energy estimate due to the lack of symmetry. That’s why we can only solve the half problem of Conjecture.

**Proposition 8.1.** Assume that \( T \in (0, 1] \) and \( u \in C([-T, T]: H^\infty) \) is a real-valued solution of the initial value problem
\[
\begin{cases}
  u_t + \mathcal{H} u_{xx} = u^2 u_x, & (x, t) \in \mathbb{R} \times (-T, T); \\
  u(x, 0) = \phi(x),
\end{cases}
\] (8.1)

Then, for \( 0 \leq l < 1/4 \) and \( s > 1/4 \), there exists \( \delta_0 > 0 \) such that if \( \|u\|_{E^{l, s}(T)} \leq \delta_0 \) then we have
\[
\|u\|_{\mathcal{E}^{l, s}(T)} \lesssim \|\phi\|_{H^l \cap H^s} + \|u\|_{F^{l, s - 1/2 +}((T)}^{4} \cdot \|u\|_{F^{l, s}(T)}^{2}.
\] (8.2)

The following definition was first introduced in [26].

**Definition 8.2.** Let \( s \in \mathbb{R} \) and \( \epsilon > 0 \). Then \( S^{s}_\epsilon \) is the class of spherical symmetric symbols with the following properties:

(i) symbol regularity,
\[
|\partial^\alpha a(\xi)| \lesssim a(\xi)(1 + \xi^2)^{-\alpha/2}
\]

(ii) decay at infinity, for \( |\xi| \gg 1 \),
\[
s \leq \frac{\log a(\xi)}{\log(1 + \xi^2)} \leq s + \epsilon, \quad s - \epsilon \leq \frac{d \log a(\xi)}{d \log(1 + \xi^2)} \leq s + \epsilon.
\]

Assume \( u \in C([-T, T]: H^\infty) \) solves (8.1) and \( a \in S^{s}_\epsilon \). Denote \( A(D) = \mathcal{F}^{-1}a(\xi)\mathcal{F} \). We first set
\[
E_0(u) = (A(D)u, u) = \int_{\xi_1 + \xi_2 = 0} a(\xi_1)\hat{u}(\xi_1)\hat{u}(\xi_2).
\]
Using the equation (8.1) and noting that \(a(\xi)\) is even while \(\omega(\xi)\) is odd, then we easily get that
\[
\frac{d}{dt} E_0(u) = R_4(u) = \frac{1}{6} \int_{\Gamma_4} i[\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4),
\]
where for \(k \in \mathbb{N}\), we denote
\[
\Gamma_k = \{\xi_1 + \xi_2 + \ldots + \xi_k = 0\}.
\]
Following the idea of I-method, we define a multi-linear correction term to achieve a cancelation
\[
E_1(u) = \int_{\Gamma_4} b_4(\xi_1, \xi_2, \xi_3, \xi_4) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4),
\]
where \(b_4\) will be determined soon. Again using the equation (1.1), we get
\[
\frac{d}{dt} E_1(u) = R_6(u) + \int_{\Gamma_4} ib_4(\xi_1, \xi_2, \xi_3, \xi_4)[\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4)] \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) \hat{u}(\xi_4),
\]
where
\[
R_6(u) = C \int_{\Gamma_6} b_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6) \prod_{j=1}^{6} \hat{u}(\xi_j).
\]
To achieve the cancelation of the quadrilinear form we define \(b_4\) on \(\Gamma_4\) by
\[
b_4(\xi_1, \xi_2, \xi_3, \xi_4) = C \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4)}.
\]
Thus we get
\[
\frac{d}{dt}(E_0(u) + E_1(u)) = R_6(u).
\]

**Proposition 8.3.** Assume that \(a \in S^*_\mu\). Then for each dyadic \(\lambda \leq \alpha \leq \mu\) there is an extension of \(b_4\) from the diagonal set
\[
\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \Gamma_4, |\xi_1| |\xi_2| |\xi_3| |\xi_4| \sim \alpha\}
\]
to the full dyadic set
\[
\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}_4, |\xi_1| |\xi_2| |\xi_3| |\xi_4| \sim \mu\}
\]
which satisfies
\[
|b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\mu) \mu^{-1}
\]
and
\[
\sum_{j=1}^{4} |\partial_j b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim a(\alpha) \mu^{-1} + a(\mu) \mu^{-2}.
\]
Proof. From symmetry we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3| \leq |\xi_4|$ and $\xi_3 > 0, \xi_4 < 0$. We first consider the case that $\xi_1 \xi_2 > 0$, say $\xi_1, \xi_2 > 0$. Then $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = -2(\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_4).$ Thus in $\Gamma_4$ we have

$$Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 \xi_2 + (\xi_1 + \xi_2) \xi_3} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3 \xi_4 + (\xi_3 + \xi_4) \xi_1}.$$  

Using $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ we get

$$\frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_4} = \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3 \xi_4 + \xi_3 \xi_4 + \xi_4 \xi_4}.$$ 

Therefore, we extend $b_4$ by setting

$$Cb_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 \xi_2 + \xi_2 \xi_3 + \xi_3 \xi_4} + \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{\xi_3 \xi_4 + \xi_3 \xi_4 + \xi_4 \xi_4}. \quad (8.5)$$ 

It is easy to see from the properties of $a(\xi)$ that

$$|b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim |\xi| a(\mu) \mu^{-1}.$$ 

It remains to check the derivatives. We only consider $|\partial_1 b_4|$, since the others can be handled in similar ways. For $|\partial_1 b_4|$ it suffices to consider the first term on the right-hand side of (8.5). Direct computations show that

$$\partial_1 \left( \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 \xi_2 + (\xi_1 + \xi_2) \xi_3} \right) = \frac{|a(\xi_1) - a(\xi_2)|^2 |\xi_2 \xi_3 - \xi_2 a(\xi_2)|}{(\xi_1 \xi_2 + (\xi_1 + \xi_2) \xi_3)^2} + \frac{a'(\xi_1) \xi_1}{\xi_1 \xi_2 + (\xi_1 + \xi_2) \xi_3},$$ 

which satisfies (8.4) as desired.

We consider now $\xi_1 \xi_2 < 0$, say $\xi_1 < 0, \xi_2 > 0$. Thus we get $\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) + \omega(\xi_4) = \xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = (\xi_1 + \xi_2)(\xi_1 + \xi_3)$. We will extend $b_4$ in the following cases.

(a) $\lambda \ll \mu, \alpha \leq \mu$. Then the extension of $b_4$ is defined using the formula

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)} - \frac{\xi_3 a(\xi_3) + \xi_4 a(\xi_4)}{(\xi_3 + \xi_1)(\xi_1 + \xi_3)}.$$ 

Since $\lambda \ll \mu$, we see that $|\xi_1 + \xi_3| \sim \mu$. By using the properties of $a(\xi)$ we see (8.3) and (8.4) are satisfied as desired.

(b) $\lambda \sim \mu$. Then the extension of $b_4$ is defined using the formula

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2) + \xi_3 a(\xi_3) - (\xi_1 + \xi_2 + \xi_3) a(\xi_1 + \xi_2 + \xi_3)}{(\xi_1 + \xi_2)(\xi_1 + \xi_3)}.$$ 

To check the properties, setting

$$q(\xi_1, \xi_2) = \frac{\xi_1 a(\xi_1) + \xi_2 a(\xi_2)}{\xi_1 + \xi_2},$$

then we get that

$$b_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{q(\xi_1, \xi_2) - q(\xi_1 - (\xi_1 + \xi_3), \xi_2 + (\xi_1 + \xi_3))}{\xi_1 + \xi_3},$$

from which we easily verify (8.3) and (8.4). \hfill \blacksquare
In view of the definition, for Proposition 8.1 we are mainly concerned with the control of the energy in high frequency. From the definition we see that if \( a \in S^s_\epsilon \) then \( a(\xi)\eta_{\geq 1}(\xi) \in S^s_\epsilon \). Let
\[
A^s_\epsilon = \{ a(\xi)\eta_{\geq 1}(\xi) : a \in S^s_\epsilon \}. \tag{8.6}
\]

**Proposition 8.4.** Assume \( a \in A^s_\epsilon \) and \( s - \epsilon \geq 0 \), then we have
\[
|E_1(u)| \lesssim \|u\|_{H^{1/4-\epsilon} \cap H^{1/4}}^2 E_0(u).
\]

**Proof.** Using the definition, we get
\[
|E_1(u)| \leq \int_{\Gamma^j_4} |b_4(\xi_1, \xi_2, \xi_3, \xi_4)| \cdot |\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\hat{u}(\xi_4)|.
\]
From symmetries, we may assume that \( |\xi_1| \leq |\xi_2| \leq |\xi_3| \leq |\xi_4| \). Localizing \( |\xi_j| \sim N_j \) for \( N_j \) dyadic number, we may assume \( N_3 \sim N_4 \). Then it follows from Proposition 8.3 that
\[
|E_1(u)| \lesssim \sum_{N_j} \int_{\Gamma^j_4, |\xi| \sim N_j} N_4^{-1} |\hat{u}(\xi_1)\hat{u}(\xi_2)a(N_4)^{1/2}\hat{u}(\xi_3)a(N_4)^{1/2}\hat{u}(\xi_4)|
\]
\[
\lesssim \sup_{N_4 \geq 1} \sum_{N_1 \leq N_2 \leq N_4} N_4^{-1/2} \{R_1 u\}^2 \|R_{N_1} u\|_{L^2} E_0(u)
\]
\[
\lesssim \|u\|_{H^{1/4-\epsilon} \cap H^{1/4}}^2 E_0(u).
\]
Therefore we complete the proof of the proposition. \( \blacksquare \)

**Proposition 8.5.** Assume \( a \in A^s_\epsilon \), \( s - \epsilon \geq 0 \) and \( T \in (0, 1] \). Then
\[
\left| \int_{-T}^T R_6(u) dt \right| \lesssim \|u\|_{F^{1/4-\epsilon} \cap F^{1/4}}^2 \|u\|_{F^{1, \epsilon}(T)}.
\tag{8.7}
\]

**Proof.** We first fix extension \( \tilde{u} \in C_0(\mathbb{R} : H^{\infty}) \) of \( u \) such that \( \|R_k(\tilde{u})\|_{F^k} \leq 2\|R_k(u)\|_{F^k(T)}, k \in \mathbb{Z} \). It suffices to prove that
\[
\left| \int_0^T R_6(\tilde{u}) dt \right| \lesssim \|\tilde{u}\|_{F^{1/4-\epsilon} \cap F^{1/4}}^2 \|\tilde{u}\|_{F^{1, \epsilon}}^2.
\tag{8.8}
\]

For simplicity of the notations we still write \( \tilde{u} = u \). From symmetry, we get
\[
CR_6(u) = \int_{\Gamma^6_0} [b_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6)
\]
\[
- b_4(-\xi_1, -\xi_5, -\xi_6, \xi_4 + \xi_5 + \xi_6)(\xi_4 + \xi_5 + \xi_6)] \prod_{j=1}^6 \hat{u}_j(\xi_j).
\]
Localizing \( |\xi_j| \sim N_j = 2^{k_0} \) and using symmetry, we may assume \( N_1 \leq N_2 \leq N_3, N_4 \leq N_5 \leq N_6 \) and \( \max(N_j) \sim \text{sub}(N_j) \geq 1 \) where \( \max(N_j) \) and \( \text{sub}(N_j) \) are the maximum and second-maximum of \( N_j, j = 1, 2, \ldots, 6 \). Let \( u_k = R_k(u) \) and \( \xi_{456} = \xi_4 + \xi_5 + \xi_6 \). Thus
\[
\left| \int_0^T R_6(u) dt \right| \lesssim \sum_{N_j} \int_0^T \int_{\Gamma^6_0} [b_4(\xi_1, \xi_2, \xi_3, \xi_{456})(\xi_{456})
\]
\[
- b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})(\xi_{456})] \prod_{j=1}^6 \hat{u}_{\xi_j}(\xi_j) dt].
\tag{8.9}
\]
Let \( \gamma : \mathbb{R} \to [0,1] \) denote a positive smooth function supported in \([-1,1]\) with the property that
\[
\sum_{n \in \mathbb{Z}} \gamma^n(x-n) \equiv 1, \quad x \in \mathbb{R}.
\]

We will bound (8.8) in several cases.

**Case 1.** \( N_4 \leq N_5, N_6 \) and \( N_5 \sim N_6 \geq 1 \). Then we get that the right-hand side of (8.9) is bounded by
\[
\sum_{N_j} \sum_{|n| \leq 2^k_b} \left| \int_{\Gamma_6} \int_{\Gamma_5} [b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})] \right|
\]
\[
\xi_{456}[\gamma(2^{k_a} t - n)1_{[0,T]}(t)\tilde{u}_k](\xi_1) \prod_{j=2}^6 [\gamma(2^{k_a} t - n)\tilde{u}_k](\xi_j)|dt|.
\] (8.10)

We observe first that
\[
|A| = |\{n : \gamma(2^{k_a} t - n)1_{[0,T]}(t) \neq \gamma(2^{k_a} t - n), 0, \forall t \in \mathbb{R} \}| \leq 4.
\]

Let \( f_{k,}(\xi, \tau) = \mathcal{F}[\gamma(2^{k_a} t - n)R_k u] \), \( j = 1, 2, \ldots, 6 \). Using proposition 8.3 and Plancherel's theorem, we easily get that the summation for \( n \in A^c \) of (8.10) is bounded by
\[
\sum_{N_j} \sum_{|n| \leq 2^k_b, n \in A^c} (a(N_3)N_3^{-1} + a(N_6)N_6^{-1})N_3 \int_{\Gamma_6(\mathbb{R}^2)} \prod_{j=1}^6 f_{k,}(\xi_j, \tau_j) \left| \prod_{j=1}^6 f_{k,}||B_k\right|.
\] (8.11)

Using Hölder's inequality and the embedding properties of \( B_k \), we get
\[
\left| \int_{\Gamma_6(\mathbb{R}^2)} \prod_{j=1}^6 f_{k,}(\xi_j, \tau_j) \right| \lesssim \prod_{j=1}^6 ||\mathcal{F}^{-1}(f_{k,})||_{L^2_x L^\infty_t}
\]
\[
||\mathcal{F}^{-1}(f_{k,})||_{L^\infty_x L^2_t}||\mathcal{F}^{-1}(f_{k,})||_{L^\infty_x L^2_t}
\]
\[
\lesssim N_3^{-1} \prod_{j=1}^4 N_{j}^{1/4} \prod_{j=1}^6 ||f_{k,}||_{B_k}.
\]

Then we can bound (8.11) by
\[
\sum_{N_j} (a(N_3)N_3^{-1} + a(N_6)N_6^{-1})N_3 \prod_{j=1}^4 N_{j}^{1/4} \prod_{j=1}^6 ||f_{k,}||_{B_k}
\]
\[
\lesssim ||u||_{\mathbb{H}^{1/4-1/4,1/4}}^4 ||u||_{\mathbb{H}^{1/4}}^4,
\]
which is (8.8) as desired.

For the summation of \( n \in A \), we observe that if \( I \subset \mathbb{R} \) is an interval, \( k \in \mathbb{Z} \), \( f_k \in B_k \), and \( f_k^t = \mathcal{F}(1f(t) \cdot \mathcal{F}^{-1}(f_k)) \) then
\[
\sup_{j \in \mathbb{Z}_+} 2^{j/2} ||\eta_j(\tau - \omega(\xi)) \cdot f_k^t||_{L^2} \lesssim ||f_k||_{B_k}.
\]

Let \( f_{k,i,j} = \eta_i(\tau - \omega(\xi))\mathcal{F}[\gamma(2^{k_a} t - n)R_k u] \), \( i = 1, 2, \ldots, 5 \), and \( f_{k_a,j_6} = \eta_{j_6}(\tau - \omega(\xi))\mathcal{F}[\gamma(2^{k_a} t - n)1_{[0,T]}(t)R_k u] \). If \( j_6 \geq 100k_6 \), then by checking the support properties we get
\[
\int_{\Gamma_6(\mathbb{R}^2)} \prod_{j=1}^6 f_{k,i,j,}(\xi_j, \tau_j) \equiv 0 \text{ unless } |j_{max} - j_{sub}| \leq 10 \text{ and } j_{max} \geq 100k_6,
\]
where \( j_{\text{max}} \) and \( j_{\text{sub}} \) are the maximum and sub-maximum of \( j_1, j_2, \ldots, j_6 \). By using Cauchy-Schwarz inequality, we get that

\[
\sum_{j_i} \left| \int_{\Gamma_6(R^2)} \prod_{i=1}^6 f_{k_i, j_i}(\xi_j, \tau_j) \right| \lesssim 2^{(k_1 + \ldots + k_6)/2} 2^{j_1 + \ldots + j_6)/2} - (j_{\text{max}}) \prod_{i=1}^6 \| f_{k_i, j_i} \|_{L^2} \]
\[
\lesssim N_5^{-1} \prod_{j=1}^4 N_j^{1/4} \prod_{j=1}^6 \| f_{k_j} \|_{F_k},
\]

which is acceptable. If \( j_6 \leq 100k_6 \) then we argue as before for \( n \in A^c \), hence we get that

\[
\sum_{j_i} \left| \int_{\Gamma_6(R^2)} \prod_{i=1}^6 f_{k_i, j_i}(\xi_j, \tau_j) \right| \lesssim k_6 N_5^{-1} \prod_{j=1}^4 N_j^{1/4} \prod_{j=1}^6 \| f_{k_j} \|_{F_k},
\]

which combined with (8.11) gives (8.8).

**Case 2.** \( N_6 \leq N_2, N_3 \) and \( N_2 \sim N_3 \geq 1 \). From symmetry, this case is identical to Case 1. We omit the details.

**Case 3.** \( N_2, N_5 \ll N_3, N_6 \) and \( N_6 \sim N_3 \geq 1 \). From the proof of Proposition 8.3, we get that

\[
|b_4(\xi_1, \xi_2, \xi_3, \xi_{456}) - b_4(-\xi_4, -\xi_5, -\xi_6, \xi_{456})| \cdot |\xi_{456}| \lesssim a(N_3) + a(N_6) N_6^{-1}.
\]

Then following the same argument as in Case 1, we obtain (8.8) as desired.  

**Lemma 8.6 (Lemma 5.5, [26]).** There is a sequence \( \{\beta_\lambda\} \) with the following properties:

(a) \( \lambda^2 \| P_\lambda(u_0) \|_{L^2}^2 \leq \beta_\lambda \| u_0 \|_{L^*}^2 \),

(b) \( \sum \beta_\lambda \leq 1 \),

(c) \( \beta_\lambda \) is slowly varying in the sense that \( |\log_2 \beta_\lambda - \log_2 \beta_\mu| \leq \frac{\epsilon}{2} |\log_2 \lambda - \log_2 \mu| \).

**Proof of Proposition 8.1.** In view of the definition, we get

\[
\| u \|_{L^t(\cdot, T)}^2 \lesssim \| P_{\leq 0} u_0 \|_{H^1}^2 + \sum_{k \geq 1} \sup_{t \in [-T, T]} 2^{2k}\| P_k(u(t)) \|_{L^2}^2.
\] (8.12)

We will prove that if \( k \geq 1 \) then

\[
\sup_{t \in [-T, T]} 2^{2k}\| P_k(u(t)) \|_{L^2}^2 \lesssim \beta_k \| P_{\geq 1} u_0 \|_{L^2}^2 + \| u \|_{L^t(\cdot, T)}^4 \| u \|_{L^4(\cdot, T)}^4 \cdot \| u \|_{L^2(\cdot, T)}^4,
\] (8.13)

which suffices to prove Proposition 8.1 in view of Lemma 8.6 (b). In order to prove (8.13) for some fixed \( k_0 \) we define the sequence

\[
a_k = 2^{2k} \max(1, \beta_{k_0}^{-1} 2^{-\epsilon |k-k_0|}).
\]
Using the slowly varying condition (iii), then we get
\[ \sum_{k \geq 1} a_k \| P_k(u_0) \|_{L^2}^2 \lesssim \sum_k 2^{2k} \| P_k(u_0) \|_{L^2}^2 \]
\[ + 2^{-c_k-k_0/2} \beta_k^{-1} \| P_k(u_0) \|_{L^2}^2 \]
\[ \lesssim \| P_{\geq 1}(u_0) \|_{H^s}^2. \]
We may assume that \( \beta_0 = 1 \). Then we see that \( \max(|\beta_k|, |\beta_k^{-1}|) \leq 2^{k/2} \). Correspondingly we find a function \( a(\xi) \in S^s \) so that
\[ a(\xi) \sim a_k, \quad |\xi| \sim 2^k. \]
Thus we apply Proposition 8.4, 8.5 for \( a(\xi) \eta_{\geq 1}(\xi) \), then we get
\[ \sup_{t \in [-T, T]} |E_0(u(t) + E_1(u(t))| \leq |E_0(u_0) + E_1(u_0)| + \left| \int_{-T}^T \mathcal{R}_0(u) \, dt \right|, \]
from which we see that
\[ \sup_{t \in [-T, T]} |E_0(u(t)| \leq |E_0(u_0)| + \| u \|^4_{F^4/4^{-1/2}} \| u \|^2_{F^{2,s}(T)}. \]
Therefore, we get
\[ \left( \sum_{k \geq 1} a_k \| P_k(u(t)) \|_{L^2}^2 \right) \lesssim \| P_{\geq 1}u_0 \|_{H^s}^2 + \| u \|^4_{F^4/4^{-1/2}} \| u \|^2_{F^{2,s}(T)}, \]
which at \( k = k_0 \) gives (8.13) as desired.

9. PROOF OF THEOREM 1.3

In this section we devote to prove Theorem 1.3. The main ingredients are energy estimates and short-time trilinear estimates. The idea is due to Ionescu, Kenig and Tataru [16].

**Proposition 9.1.** Let \( l, s \geq 0, T \in (0, 1] \), and \( u \in F^{l,s}(T) \), then
\[ \sup_{t \in [-T, T]} \| u(t) \|_{H^l \cap H^s} \lesssim \| u \|_{F^{l,s}(T)}. \] (9.1)

**Proof.** In view of the definitions, it suffices to prove that if \( k \in \mathbb{Z}, t_k \in [-1, 1] \), and \( \tilde{u}_k \in F_k \) then
\[ \| \mathcal{F}[\tilde{u}_k(t_k)] \|_{L^2} \lesssim \| \mathcal{F}[\tilde{u}_k \cdot \eta_0(2^{k^*}(t - t_k))] \|_{B_k}. \] (9.2)
Let \( f_k = \mathcal{F} [\tilde{u}_k \cdot \eta_0(2^{k^*}(t - t_k))] \), so
\[ \mathcal{F} [\tilde{u}_k(t_k)](\xi) = c \int_R f_k(\xi, \tau)e^{it_k\tau} \, d\tau. \]
From the definition of \( B_k \), we get that
\[ \| \mathcal{F}[\tilde{u}_k(t_k)] \|_{L^2} \lesssim \int_R |f_k(\xi, \tau)| \, d\tau \lesssim \| f_k \|_{B_k}, \]
which completes the proof of the proposition. 

\[ \blacksquare \]
Proposition 9.2. Assume $T \in (0, 1)$, $u, v \in C([-T, T] : H^\infty)$ and

$$u_t + \mathcal{H}u_{xx} = v \text{ on } \mathbb{R}^2 \times (-T, T).$$

Then for any $l, s \geq 0$,

$$\|u\|_{F^{l,s}(T)} \lesssim \|u\|_{E^{l,s}(T)} + \|v\|_{N^{l,s}(T)}.$$  \hspace{1cm} (9.4)

Proof. In view of the definitions, we see that the square of the right-hand side of (9.4) is equivalent to

$$\sum_{k \leq 0} \left( 2^{2lk} \|R_k(u(0))\|^2_{L^2} + 2^{2lk} \|R_k(v)\|^2_{N_k(T)} \right)
+ \sum_{k \geq 1} \left( \sup_{t_k \in [-T, T]} 2^{2sk} \|R_k(u(t_k))\|^2_{L^2} + 2^{2sk} \|R_k(v)\|^2_{N_k(T)} \right).$$  \hspace{1cm} (9.5)

Thus, from definitions, it suffices to prove that if $k \in \mathbb{Z}$ and $u, v \in C([-T, T] : H^\infty)$ solve (9.3), then

$$\|R_k(u)\|_{F_k(T)} \lesssim \|R_k(u(0))\|_{L^2} + \|R_k(v)\|_{N_k(T)} \text{ if } k \leq 0;
\|R_k(u)\|_{F_k(T)} \lesssim \sup_{t_k \in [-T, T]} \|R_k(u(t_k))\|_{L^2} + \|R_k(v)\|_{N_k(T)} \text{ if } k \geq 1.$$  \hspace{1cm} (9.6)

Let $\widetilde{v}$ denote an extension of $R_k(u)$ such that $\|\widetilde{v}\|_{N_k} \leq C\|v\|_{N_k(T)}$. Using (3.13), we may assume that $\widetilde{v}$ is supported in $\mathbb{R} \times [-T - 2^{-k+10}, T + 2^{-k+10}]$, $k \in \mathbb{Z}$. Indeed, let $\beta(t)$ be a smooth function such that

$$\beta(t) = 1, \text{ if } t \geq 1; \quad \beta(t) = 0, \text{ if } t \leq 0.$$  \hspace{1cm}

Thus $\beta(2^{k+10}(t + T + 2^{-k+10}))$, $\beta(-2^{k+10}(t - T - 2^{-k+10})) \in S_k$. Then we see that $\beta(2^{k+10}(t + T + 2^{-k+10}))\beta(-2^{k+10}(t - T - 2^{-k+10}))$ is supported in $[-T - 2^{-k+10}, T + 2^{-k+10}]$, and equal to 1 in $[-T, T]$. For $t \geq T$ we define

$$\widetilde{u}(t) = \eta_0(2^{k+5}(t - T))|W(t - T)R_k(u(T)) + \int_T^t W(t - s)(R_k(\widetilde{v}(s)))ds].$$

For $t \leq -T$ we define

$$\widetilde{u}(t) = \eta_0(2^{k+5}(t + T))|W(t + T)R_k(u(-T)) + \int_{-T}^t W(t - s)(R_k(\widetilde{v}(s)))ds].$$

For $t \in [-T, T]$ we define $\widetilde{u}(t) = u(t)$. It is clear that $\widetilde{u}$ is an extension of $u$. Also, using (3.13), we get

$$\|u\|_{F_k(T)} \lesssim \sup_{t_k \in [-T, T]} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{k+5}(t - t_k))]\|_{B_k}.$$  \hspace{1cm} (9.7)

Indeed, to prove (9.7), it suffices to prove that

$$\sup_{t_k \in \mathbb{R}} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{k+5}(t - t_k))]\|_{B_k} \lesssim \sup_{t_k \in [-T, T]} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{k+5}(t - t_k))]\|_{B_k}.$$  \hspace{1cm} (9.8)

For $t_k > T$, since $\tilde{u}$ is supported in $[-T - 2^{-k+5}, T + 2^{-k+5}]$, it is easy to see that

$$\tilde{u}\eta_0(2^{k+5}(t - t_k)) = \tilde{u}\eta_0(2^{k+5}(t - T))\eta_0(2^{k+5}(t - t_k)).$$

Therefore, we get from (3.11) that

$$\sup_{t_k > T} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{k+5}(t - t_k))]\|_{B_k} \lesssim \sup_{t_k \in [-T, T]} \|\mathcal{F}[\tilde{u} \cdot \eta_0(2^{k+5}(t - t_k))]\|_{B_k}.$$  \hspace{1cm}

Using the same way for $t_k < -T$, we obtain (9.7) as desired.
It remains to prove (9.6). In view of the definitions, (9.7) and (3.11), it suffices to prove that if \( k \in \mathbb{Z} \), \( \phi_k \in L^2 \) with \( \hat{\phi_k} \) supported in \( I_k \), and \( v_k \in N_k \) then
\[
\| \mathcal{F}[u_k \cdot \eta_0(2^k t)] \|_{B_k} \lesssim \| \phi \|_{L^2} + \| (\tau - \omega(\xi) + i 2^{k+1})^{-1} \cdot \mathcal{F}(v_k) \|_{B_k},
\]
where
\[
u_k(t) = W(t)(\phi_k) + \int_0^t W(t - s)(v_k(s))ds.
\]

Straightforward computations show that
\[
\mathcal{F}[u_k \cdot \eta_0(2^k t)](\xi, \tau) = \hat{\phi_k}(\xi) \cdot 2^{-k} \hat{\eta_0}(2^{-k} (\tau - \omega(\xi)))
\]
\[
+ C \int_{\mathbb{R}} \mathcal{F}(v_k)(\xi, \tau') \cdot \frac{2^{-k} \hat{\eta_0}(2^{-k} (\tau - \tau')) - 2^{-k} \hat{\eta_0}(2^{-k} (\tau - \omega(\xi)))}{\tau' - \omega(\xi)} d\tau'.
\]

We observe now that
\[
\left| \frac{2^{-k} \hat{\eta_0}(2^{-k} (\tau - \tau')) - 2^{-k} \hat{\eta_0}(2^{-k} (\tau - \omega(\xi)))}{\tau' - \omega(\xi)} \right| \lesssim 2^{-k}(1 + 2^{-k} |\tau - \tau'|)^{-4} + 2^{-k}(1 + 2^{-k} |\tau - \omega(\xi)|)^{-4}.
\]

Using (3.9) and (3.10), we complete the proof of the proposition. \( \square \)

We prove a crucial trilinear estimates in the following proposition.

**Proposition 9.3.** Let \( 0 \leq l \leq 1/4 \) and \( s \geq 1/4 \). Then
\[
\| \partial_x (uvw) \|_{N^{l,s}(T)} \lesssim \| u \|_{F^{l,s}(T)} \| v \|_{F^{l,s}(T)} \| w \|_{F^{l,s}(T)} + \sum_{k \in \mathbb{N}} \sum_{k_1, k_2} 2^{2k_1} \| R_{k_1}(\partial_x (uvw)) \|_{N_{k_1}}.
\]

**Proof.** Since \( R_k R_j = 0 \) if \( k \neq j \), then we can fix extensions \( \tilde{u}, \tilde{v}, \tilde{w} \) of \( u, v, w \) such that \( \| R_k(\tilde{u}) \|_{F_k(T)} \leq 2 \| R_k(u) \|_{F_k(T)} \), \( \| R_k(\tilde{v}) \|_{F_k(T)} \leq 2 \| R_k(v) \|_{F_k(T)} \) and \( \| R_k(\tilde{w}) \|_{F_k(T)} \leq 2 \| R_k(w) \|_{F_k(T)} \) for any \( k \in \mathbb{N} \). In view of definition, we get
\[
\| \partial_x (\tilde{u} \tilde{v} \tilde{w}) \|_{N^{l,s}} \lesssim \sum_{k_1 = -\infty}^{\infty} 2^{2k_1} \| R_{k_1}(\partial_x (\tilde{u} \tilde{v} \tilde{w})) \|_{N_{k_1}}.
\]

Let \( \tilde{u}_k = R_k(\tilde{u}), \tilde{v}_k = R_k(\tilde{v}) \) and \( \tilde{w}_k = R_k(\tilde{w}) \). Then we get
\[
\| R_{k_1}(\partial_x (\tilde{u} \tilde{v} \tilde{w})) \|_{N_{k_1}} \lesssim \sum_{k_{1, k_2, k_3} \in \mathbb{Z}} \| R_{k_1}(\partial_x (\tilde{u}_k \tilde{v}_k \tilde{w}_k)) \|_{N_{k_1}}.
\]

From symmetry we may assume \( k_1 \leq k_2 \leq k_3 \). Dividing the summation into several parts, we get
\[
\sum_{k_{1, k_2, k_3} \in \mathbb{Z}} \| R_{k_1}(\partial_x (\tilde{u}_k \tilde{v}_k \tilde{w}_k)) \|_{N_{k_1}} \lesssim \sum_{j=1}^{6} \sum_{A_j} \| R_{k_1}(\partial_x (\tilde{u}_k \tilde{v}_k \tilde{w}_k)) \|_{N_{k_1}},
\]
where \( A_j, j = 1, 2, \ldots, 6 \), are as in the proof of Proposition 6.3. We will apply Proposition 7.1-7.6 obtained in the Section 7 to bound the six terms in (9.12). For
example, for the first term, from Proposition 7.1, we have
\[
\left\|2^{sk_4} \sum_{(k_1) \in A_1} \left\| R_{k_4} \left( \partial_x (\tilde{u}_{k_1} \tilde{v}_{k_2} \tilde{w}_{k_3}) \right) \right\|_{\dot{H}^{s+4}} \right\|
\]
\[
\leq C \left( 2^{sk_4} \sum_{k_1 \in A_1} \min(\phi^{k_1/2}, |k_2| + 1) \|\tilde{u}_{k_1}\|_{\dot{H}^{s}} \|\tilde{v}_{k_2}\|_{\dot{H}^{s}} \|\tilde{w}_{k_3}\|_{\dot{H}^{s}} \right) \leq \|	ilde{u}\|_{\dot{F}^{1/4,1/4}} \|	ilde{v}\|_{\dot{F}^{1/4,1/4}} \|	ilde{w}\|_{\dot{F}^{1,s}}.
\]
For the other terms we can handle them in the similar ways. Therefore we complete the proof of the proposition.

We prove now Theorem 1.3. Fix \(0 < l < 1/4\) and \(s > 1/4\). By the scaling (1.4) we may assume that
\[
\|\phi\|_{\dot{H}^{l} \cap \dot{H}^{s}} \leq \delta_0 / M,
\]
(9.13)
where \(M \gg 1\) and \(\delta_0\) is given as in Proposition 8.1. For any \(T' \in [0,1]\), we denote \(X(T') = \|u\|_{E^{l,s}(T')} + \|\partial_x (u^3)\|_{N^{l,s}(T')}\). We assume first that \(X(T')\) is continuous and satisfies
\[
\lim_{T' \to 0} X(T') \lesssim \|\phi\|_{\dot{H}^{l} \cap \dot{H}^{s}}.
\]
(9.14)
We will prove that
\[
X(T') \leq 2c\delta_0 / M, \quad \text{for any } T' \in [0,1].
\]
(9.15)
By using bootstrap (e.g. see [36]), we may assume that \(X(T') \leq 3c\delta_0 / M\) for any \(T' \in [0,1]\). It follows from Proposition 9.2, 9.3 that for any \(T' \in [0,1]\) we have
\[
\left\{ \begin{array}{l}
\|u\|_{E^{l,s}(T')} \lesssim \|u\|_{E^{l,s}(T')} + \|\partial_x (u^3)\|_{N^{l,s}(T')};

\|\partial_x (u^3)\|_{N^{l,s}(T')} \lesssim \|u\|_{E^{l,s}(T')}^3;

\|u\|_{E^{l,s}(T')}^2 \lesssim \|\phi\|_{\dot{H}^{l} \cap \dot{H}^{s}}^2 + \|u\|_{E^{l,s}(T')}^6.
\end{array} \right.
\]
(9.16)
Thus we get
\[
X(T')^2 \lesssim \|\phi\|_{\dot{H}^{l} \cap \dot{H}^{s}}^2 + X(T')^6,
\]
(9.17)
from which and (9.13) and the assumption \(X(T') \leq 3c\delta_0 / M\), we obtain (9.15) as desired. Then, using (9.16), (9.15) and Proposition 9.1, we have
\[
\|u\|_{C([-1,1];\dot{H}^{l} \cap \dot{H}^{s})} \lesssim \|\phi\|_{\dot{H}^{l} \cap \dot{H}^{s}}.
\]
(9.18)
For general \(\phi\) and the \(L^2\) norm of the solution, we just use the scaling (1.4) and the \(L^2\) conservation law (1.6).

It remains to prove that \(X(T)\) is continuous and (9.14). Obviously, for \(u \in C([-T,T];H^\infty)\) the first component \(T'\) is increasing and continuous on \([-T,T]\) and
\[
\lim_{T' \to 0} \|u\|_{E^{l,s}(T')} \lesssim \|\phi\|_{\dot{H}^{l} \cap \dot{H}^{s}}.
\]
For the second component it follows from the similar argument as in the proof of Lemma 4.2 in [16]. We omit the details.

Remark 9.4. From the proof we see that we actually prove a stronger result than that stated in Theorem 1.3. We expect that some wellposedness results hold for the initial data in \(\dot{H}^{l} \cap \dot{H}^{s}\).
Acknowledgment. This work was finished under the patient advising of Prof. Carlos E. Kenig while the author was visiting the Department of Mathematics at the University of Chicago under the auspices of China Scholarship Council. The author is deeply indebted to Prof. Kenig for the many encouragements and precious advices. The author is also very grateful to Prof. Heping Liu and Prof. Jianmin Liu, and especially to his advisors Prof. Lizhong Peng and Prof. Baoxiang Wang for their encouragements, supports and advisings.

References

[1] L. Abdelouhab, J. L. Bona, M. Felland, J.-C. Saut, Nonlocal models for nonlinear, dispersive waves, Physica D. Nonlinear Phenomena 40 (1989), no. 3, 360-392.
[2] T. B. Benjamin, Internal waves of permanent form in fluids of great depth, The Journal of Fluid Mechanics 29 (1967), 559-592.
[3] H. A. Biagioni and F. Linares, Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, Transactions of the American Mathematical Society 353 (2001), no. 10, 3649-3659.
[4] N. Burq, F. Planchon, On well-posedness for the Benjamin-Ono equation, preprint, 2005.
[5] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, II, Geom. Funct. Anal., 3:107-156, 209-262, 1993.
[6] M. Christ, J. Colliander, T. Tao, A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev spaces of negative order, Preprint (2006).
[7] J. Colliander, C. E. Kenig, and G. Staffilani, Local well-posedness for dispersion-generalized Benjamin-Ono equations, Differential and Integral Equations 16 (2003), no. 12, 1441-1472.
[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on R and T. J. Amer. Math. Soc., 16(3),705-749, 2003.
[9] J. Ginibre and G. Velo, Commutator expansions and smoothing properties of generalized Benjamin-Ono equations, Annales de l’Institut Henri Poincaré. Physique Théorique 51 (1989), no. 2, 221-229.
[10] J. Ginibre and G. Velo, Propriétés de lissage et existence de solutions pour l’équation de Benjamin-Ono généralisée, Comptes Rendus des Séances de l’Académie des Sciences. Série I. Mathématique 308 (1989), no. 11, 309-314.
[11] J. Ginibre and G. Velo, Smoothing properties and existence of solutions for the generalized Benjamin-Ono equation, Journal of Differential Equations 93 (1991), no. 1, 150-212.
[12] Z. Guo, B. Wang, Global well posedness and inviscid limit for the Korteweg-de Vries-Burgers equation, Preprint
[13] N. Hayashi and P. I. Naumkin, On the reduction of the modified Benjamin-Ono equation to the cubic derivative nonlinear Schrödinger equation, Discrete and Continuous Dynamical Systems 8 (2002), no. 1, 237-255.
[14] A. D. Ionescu, C. E. Kenig, Global well-posedness of the Benjamin-Ono equation in low-regularity spaces, J. Amer. Math. Soc., 20 (2007), no. 3, 753-798.
[15] A. D. Ionescu, C. E. Kenig, Well-posedness of the complex-valued Benjamin-Ono equation.
[16] A. D. Ionescu, C. E. Kenig, D. Tataru, Global well-posedness of KP-I initial-value problem in the energy space, arXiv:0705.4239v1.
[17] R. J. Jório Jr., On the Cauchy problem for the Benjamin-Ono equation, Communications in Partial Differential Equations 11 (1986), no. 10, 1031-1081.
[18] C. E. Kenig, K. D. Koenig, On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations, Mathematical Research Letters 10 (2003), no. 5-6, 879-895.
[19] C. E. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Communications on Pure and Applied Mathematics 46 (1993), no. 4, 527-620.
[20] C. E. Kenig, G. Ponce, L. Vega, On the generalized Benjamin-Ono equation, Transactions of the American Mathematical Society 342 (1994), no. 1, 155-172.
[21] C. E. Kenig, G. Ponce, L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Mathematical Journal 106 (2001), no. 3, 617-633.
[22] C. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation. J. Amer. Math. Soc., 9:573-603, 1996. MR 96k:35159.
[23] C. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schrödinger equations, Annales de l’Institut Henri Poincaré. Analyse Non Linéaire 10 (1993), no. 3, 255-288.
[24] M. Keel, T. Tao, Endpoint Strichartz estimates, Amer. J. of Math., 120 (1998), 360-413.
[25] C. E. Kenig, H. Takaoka, Global wellposedness of the Modified Benjamin-Ono equation with initial data in $H^{1/2}$, International Mathematics Research Notices 2006 (2006), no. 1, 1-44.
[26] H. Koch, D. Tataru, A Priori bounds for the 1-d cubic NLS in negative Sobolev spaces, International Mathematical Research Notices
[27] H. Koch, N. Tzvetkov, On the local well-posedness of the Benjamin-Ono equation in $H^4(\mathbb{R})$, International Mathematics Research Notices 2003 (2003), no. 26, 1449-1464.
[28] H. Koch, N. Tzvetkov, Nonlinear wave interactions for the Benjamin-Ono equation, International Mathematics Research Notices 2005 (2005), no. 30, 1833-1847.
[29] L. Molinet and F. Ribaud, Well-posedness results for the generalized Benjamin-Ono equation with arbitrary large initial data, International Mathematics Research Notices 2004 (2004), no. 70, 3757-3795.
[30] L. Molinet and F. Ribaud, Well-posedness results for the generalized Benjamin-Ono equation with small initial data, Journal de Mathématiques Pures et Appliquées. Neuvième Série 83 (2004), no. 2, 277-311.
[31] H. Ono, Algebraic solitary waves in stratified fluids, Journal of the Physical Society of Japan 39 (1975), no. 4, 1082-1091.
[32] G. Ponce, On the global well-posedness of the Benjamin-Ono equation, Differential and Integral Equations 4 (1991), no. 3, 527-542.
[33] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity, Advances in Differential Equations 4 (1999), no. 4, 561-580.
[34] T. Tao, Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$, Journal of Hyperbolic Differential Equations 4 (2007), no. 1, 1-27.
[35] T. Tao, Multiplier weighted convolution of $L^2$ functions and applications to nonlinear dispersive equations. Amer. J. Math., 123(5):839-908, 2001. MR 2002k:35283
[36] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis. CBMS Regional Conference Series in Mathematics 106.
[37] D. Tataru, Local and global results for wave maps I, Comm. Partial Differential Equations 23 (1998), 1781-1793.

LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
E-mail address: zihuaguo@math.pku.edu.cn
URL: http://www.math.pku.edu.cn:8000/blog/gallery/167/