Tempered Mittag–Leffler Stability of Tempered Fractional Dynamical Systems

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Due to finite lifespan of the particles or boundedness of the physical space, tempered fractional calculus seems to be a more reasonable physical choice. Stability is a central issue for the tempered fractional system. This paper focuses on the tempered Mittag–Leffler stability for tempered fractional systems, being much different from the ones for pure fractional case. Some new lemmas for tempered fractional Caputo or Riemann–Liouville derivatives are established. Besides, tempered fractional comparison principle and extended Lyapunov direct method are used to construct stability for tempered fractional system. Finally, two examples are presented to illustrate the effectiveness of theoretical results.

1. Introduction

Fractional derivatives were first proposed by Leibnitz soon after the more familiar classic integer order derivatives. In recent decades, the study of fractional differential systems has attracted wide attention. Compared with the classical calculus, fractional calculus can better characterize memory and hereditary properties of processes and materials. They are now used to model the dynamical evolution in the fields of physics, chemistry, biology, and so on. Fractional calculus can be most easily understood in terms of probability. The relationships among random walks, Brownian motion, and diffusion processes were given in [1]. It is more reasonable to replace classic derivatives by fractional analogues in the diffusion equation [2].

Fractional calculus involves the operation of convolution with a power law function. Multiplying by an exponential factor results in tempered fractional derivatives and integrals [3], this exponential tempering has many merits both in mathematical and practical. A truncated Lévy flight was investigated to capture the natural cutoff in real physical systems [4]. Without a sharp cutoff, the tempered Lévy flight was studied as a smoother alternative [5]. Cartea and del-Castillo-Negrete [6] explored the tempered fractional diffusion equation by the tempered Lévy flight. In finance, the tempered stable process models describe price fluctuations with semiheavy tails [7–10]. Tempered fractional time derivatives can be also found in geophysics [11–13], Brownian motion [14], and so on.

As in classical calculus, stability analysis is still one of the most important tasks in fractional differential system [15–20]. It is a basic feature in fractional physical and biological systems, such as Duffing oscillator [21], neural networks [22–24], and predator-prey models [25]. At present, Lyapunov method has been applied to analyze Mittag–Leffler stability of different fractional systems [26–30]. Li et al. [26, 27] obtained a series of conclusions on the Mittag–Leffler stable for nonlinear fractional equations. In [28], Mittag–Leffler stability of multiple equilibrium points of fractional recurrent neural networks was considered. In [29], a convex and positive definite function was used to analyze Mittag–Leffler stable for fractional systems. In [30], the authors presented the Lyapunov stability analysis for fractional nonlinear systems with impulses.

As far as we know, no paper has discussed stability analysis for tempered fractional system. Motivated by this, we think it is very necessary and meaningful to study Mittag–Leffler stability of tempered fractional dynamical systems both in theoretical research and practical application. Because tempered fractional operators combine with nonlocal, weak singularity, and exponential factors [31–33], it has many differences to fractional case in stability analysis. In this paper, tempered Mittag–Leffler
stability is first proposed. It is a more appropriate concept for tempered fractional system. Tempered comparison principle and some inequalities are given for tempered fractional calculus or systems. Then, sufficient conditions for tempered Mittag–Leffler stability are provided and verified by the Lyapunov method. Finally, the theoretical results are applied to some examples.

This paper is organized as follows. In Section 2, some necessary definitions and lemmas are prepared. Section 3 mainly discusses the sufficient criterions ensuring Mittag–Leffler stability of the tempered fractional systems. In Section 4, two examples are presented to show the effectiveness of theoretical results. We conclude the paper with some discussions in Section 5.

2. Preliminaries

Tempered fractional calculus plays an important role in different fields [34, 35]. In practical application, many different tempered fractional derivatives are proposed, such as Caputo, Riemann–Liouville, and Riesz. Some definitions and lemmas are stated below, which will be used later.

**Definition 1** (see [13]). The tempered fractional integral of order $\alpha > 0$ and tempered parameter $\lambda \geq 0$ is defined as

$$\text{a}_t^\alpha \mu(t) = \frac{1}{\Gamma(\alpha)} \int_a^t e^{-\lambda(t-s)} (t-s)^{\alpha-1} \mu(s) \, ds,$$

where $\Gamma$ presents the Euler gamma function.

**Definition 2** (see [3]). The tempered fractional Caputo derivative of tempered parameter $\lambda \geq 0$ is defined as

$$\text{a}_t^\alpha \mu(t) = e^{-\lambda t} \left( \int_a^t \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} (e^{\lambda t} \mu(t)) \, ds \right),$$

where $n-1 \leq \alpha < n, n \in \mathbb{N}$, and $\text{a}_t^\alpha \mu(t)$ is the Caputo fractional derivative.

**Definition 3** (see [9]). The tempered fractional Riemann–Liouville derivative of tempered parameter $\lambda \geq 0$ is defined as

$$\text{a}_t^\alpha \mu(t) = e^{-\lambda t} \left( \int_a^t \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} (e^{\lambda t} \mu(t)) \, ds \right),$$

where $n-1 \leq \alpha < n, n \in \mathbb{N}$, and $\text{a}_t^\alpha \mu(t)$ is the Riemann–Liouville fractional derivative operator.

In order to study the stability of tempered fractional systems, several lemmas are needed.

**Lemma 1** (see [36]). Let $0 < \alpha < 1, \lambda \geq 0$ and $t \geq a$, then

$$\frac{C}{\text{a}_t^\alpha \mu(t)} = \frac{e^{-\lambda t} \mu(t)}{\Gamma(1-\alpha)} \left[ e^{\lambda(t-a)} \right].$$

**Lemma 2** (see [36]). Let $0 < \alpha < 1, \lambda \geq 0$, then

(i) $\frac{C}{\text{a}_t^\alpha \mu(t)} = \frac{e^{-\lambda t} \mu(t)}{\Gamma(1-\alpha)} \left[ e^{\lambda(t-a)} \right].$

(ii) $\frac{C}{\text{a}_t^\alpha \mu(t)} = \frac{e^{-\lambda t} \mu(t)}{\Gamma(1-\alpha)} \left[ e^{\lambda(t-a)} \right].$

**Lemma 3** (see [36]). The Laplace transform of tempered fractional integral and Caputo derivative (2) are given as

(i) $\mathcal{L} \left[ \int_a^t \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dt^n} (e^{\lambda t} \mu(t)) \right] = (\lambda + s)^{-\alpha} \mathcal{L} \left[ \frac{\mu(t)}{\Gamma(1-\alpha)} \right].$

(ii) $\mathcal{L} \left[ \frac{C}{\text{a}_t^\alpha \mu(t)} \right] = (\lambda + s)^{-\alpha} \mathcal{L} \left[ \frac{\mu(t)}{\Gamma(1-\alpha)} \right].$

3. Main Results

In this section, tempered fractional comparison principles, some inequalities, and tempered Mittag–Leffler stability are derived.

3.1. Tempered Fractional Comparison Principles. In this section, we establish tempered fractional comparison principles.

**Lemma 4.** Assume that $\frac{C}{\text{a}_t^\alpha \mu(t)} \geq \frac{C}{\text{a}_t^\alpha \mu(t)} \mu(t), \mu(t) \neq 0, \alpha \in (0,1)$ and $\lambda \geq 0$, then $\mu(t) \leq \mu(t)$.

**Proof.** Following from $\frac{C}{\text{a}_t^\alpha \mu(t)} \geq \frac{C}{\text{a}_t^\alpha \mu(t)} \mu(t), \mu(t) \neq 0, \alpha \in (0,1)$ and $\lambda \geq 0$, then $\mu(t) \leq \mu(t)$.

By Lemma 3, equation (5) yields

$$\mathcal{L} \left[ \frac{C}{\text{a}_t^\alpha \mu(t)} \right] = (\lambda + s)^{-\alpha} \mathcal{L} \left[ \frac{\mu(t)}{\Gamma(1-\alpha)} \right].$$

According to $\mathcal{L} \left[ \frac{C}{\text{a}_t^\alpha \mu(t)} \right] \geq \mathcal{L} \left[ \frac{C}{\text{a}_t^\alpha \mu(t)} \right] \mu(t), \mu(t) \neq 0, \alpha \in (0,1)$ and $\lambda \geq 0$, we have

$$\mathcal{L} \left[ \frac{C}{\text{a}_t^\alpha \mu(t)} \right] = (\lambda + s)^{-\alpha} \mathcal{L} \left[ \frac{\mu(t)}{\Gamma(1-\alpha)} \right].$$

Thus,

$$X(s) = Y(s) + (s + \lambda)^{-\alpha} M(s).$$

Taking the inverse Laplace transform on (8), solution of system (5) can be written as

$$\mu(t) = \mu(t) + \int_a^t \frac{1}{\Gamma(1-\alpha)} \left[ e^{\lambda(t-a)} \right] \mu(t) \, ds.$$
According to $m(t) \geq 0$, therefore we obtain $x(t) \geq y(t)$. □

**Lemma 5.** Assume that $\frac{\partial}{\partial t}^{\alpha} x(t) \geq \frac{\partial}{\partial t}^{\alpha} y(t)$, $x(0) = y(0)$, and $\alpha \in (0, 1)$, then $x(t) \geq y(t)$.

**Proof.** From Lemma 1 and $\frac{\partial}{\partial t}^{\alpha} x(t) \geq \frac{\partial}{\partial t}^{\alpha} y(t)$, we derive

$$
\frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{-\lambda t}}{1-\alpha} x(0) \geq \frac{C D_t^{\alpha}}{\Gamma(\alpha)} y(0).
$$

That is $\frac{\partial}{\partial t}^{\alpha} x(t) \geq \frac{\partial}{\partial t}^{\alpha} y(t)$. From Lemma 4, we obtain $x(t) \geq y(t)$. □

### 3.2. Some Inequalities

In this section, we construct some inequalities for tempered fractional derivatives or systems.

From Lemma 1, we could easily obtain the following lemma.

**Lemma 6.** The relationship between $\frac{\partial}{\partial t}^{\alpha} x(t)$ and $\frac{\partial}{\partial t}^{\alpha} y(t)$ is as follows:

$$
\begin{cases}
\frac{\partial}{\partial t}^{\alpha} x(t) \leq \frac{\partial}{\partial t}^{\alpha} y(t), & \text{if } x(0) \geq 0, \\
\frac{\partial}{\partial t}^{\alpha} x(t) \geq \frac{\partial}{\partial t}^{\alpha} y(t), & \text{if } x(0) \leq 0,
\end{cases}
$$

where $\alpha \in (0, 1)$.

**Lemma 7.** If $x(t) \in C^1([0, +\infty), \mathbb{R})$ is a continuously differentiable function, the following inequality holds:

$$
\frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{-\lambda t}}{1-\alpha} x(t) \leq \frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{-\lambda t}}{1-\alpha} y(t), \quad 0 < \alpha < 1, \lambda \geq 0,
$$

where $x(t^*) = \lim_{t \to t^*} x(t)$.

**Proof.** We take $y(t) = e^{\lambda t} x(t)$ into Theorem 2 in [22]

$$
\frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{\lambda t}}{1-\alpha} y(t) \leq \frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{\lambda t}}{1-\alpha} y(t), \quad 0 < \alpha < 1,
$$

for the Caputo fractional derivative. That is,

$$
\frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{\lambda t}}{1-\alpha} x(t^*) \leq \frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{\lambda t}}{1-\alpha} y(t^*).
$$

Multiplying both sides of equation (14) by $e^{-\lambda t}$, it gives

$$
\frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{\lambda t}}{1-\alpha} x(t^*) \leq \frac{C D_t^{\alpha}}{\Gamma(\alpha)} \frac{e^{\lambda t}}{1-\alpha} y(t^*).
$$

Using Definition 2, we obtain (12).

Consider the following tempered fractional system

$$
t_0 D_t^{\alpha} x(t) = f(t, x(t)), \quad 0 < t < T,
$$

subjects to the proper initial conditions, where $\alpha \in (0, 1)$, $\lambda \geq 0$, $D$ denotes either $\frac{\partial}{\partial t}^{\alpha}$ or $\frac{\partial}{\partial t}^{\alpha}$, $f : [t_0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $x$, and $\mathbb{R} \subset \mathbb{R}^n$ is a domain contain the origin. □

**Theorem 1.** For the real-valued continuous function $f(t, x)$ in (16), we have

$$
\left\| \frac{\partial}{\partial t}^{\alpha} f(t, x) \right\| \leq \frac{\partial}{\partial t}^{\alpha} \left\| f(t, x) \right\|
$$

where $\alpha > 0$, $\lambda > 0$, and $\| \|$ denotes an arbitrary norm.

**Proof.** It follows from (1) that

$$
\left\| \frac{\partial}{\partial t}^{\alpha} f(t, x(t)) \right\| = \left\| \frac{1}{\Gamma(\alpha)} \int_0^{\lambda t} e^{-\lambda (t-s)} (t-s)^{\alpha-1} f(s, x(s)) ds \right\|
$$

$$
\leq \frac{1}{\Gamma(\alpha)} \int_0^{\lambda t} e^{-\lambda (t-s)} (t-s)^{\alpha-1} \left\| f(s, x(s)) \right\| ds
$$

$$
= \frac{\partial}{\partial t}^{\alpha} \left\| f(t, x(t)) \right\|.
$$

□

**Theorem 2.** If $x = 0$ is an equilibrium point of system (16) with $t_0 D_t^{\alpha} = \frac{\partial}{\partial t}^{\alpha}$ and $f(t, x)$ is Lipschitz on $x$ with Lipschitz constant $l$ and piecewise continuous with respect to $t$, then, we have

$$
\left\| x(t) \right\| \leq \left\| x(0) \right\| e^{-\lambda t} E_\alpha (lt^\alpha).
$$

**Proof.** By applying the tempered fractional integral operator $\frac{\partial}{\partial t}^{\alpha}$ to system (16), it follows from Lemma 2 and Lipschitz condition in $f(t, x)$ that

$$
\left\| x(t) \right\| \left\| e^{-\lambda t} \right\| \left\| x(0) \right\| \leq \left\| x(t) \right\| - \left\| x(0) \right\|
$$

$$
= \left\| \frac{\partial}{\partial t}^{\alpha} \left( \frac{\partial}{\partial t}^{\alpha} x(t) \right) \right\| - \left\| \frac{\partial}{\partial t}^{\alpha} f(t, x(t)) \right\|
$$

$$
\leq \frac{\partial}{\partial t}^{\alpha} \left\| f(t, x(t)) \right\| \leq l \frac{\partial}{\partial t}^{\alpha} \left\| x(t) \right\|.
$$

There exists a function $M(t) \geq 0$ such that

$$
\left\| x(t) \right\| - \left\| e^{-\lambda t} \right\| \left\| x(0) \right\| \leq l_0 \frac{\partial}{\partial t}^{\alpha} \left\| x(t) \right\| - M(t).
$$

Combining with Lemma 3 and Laplace transform to (21), we obtain

$$
\left\| x(s) \right\| = \frac{(s + \lambda)^{\alpha-1} \left\| x(0) \right\| - (s + \lambda)^{\alpha} M(s)}{(s + \lambda)^\alpha - l}
$$

where $\left\| x(s) \right\| = \mathcal{L} \left\| x(t) \right\|$. Taking the inverse Laplace transform to (22) gives

$$
\left\| x(t) \right\| = \left\| x(0) \right\| e^{-\lambda t} E_\alpha (lt^\alpha) - M(t) \left[ e^{-\lambda t} E_\alpha (lt^\alpha) \right],
$$

where $\ast$ denotes the convolution operator. Obviously, $e^{-\lambda t} E_\alpha (lt^\alpha) \geq 0$, then inequality

$$
\left\| x(t) \right\| \leq \left\| x(0) \right\| e^{-\lambda t} E_\alpha (lt^\alpha)
$$

is obtained. □

### 3.3. Tempered Mittag–Leffler Stability

In this section, some sufficient conditions are established for the tempered Mittag–Leffler stability of system (16).

**Definition 4.** If and only if $f(t, x) = \frac{\partial}{\partial t}^{\alpha} x$, then $x \in \mathbb{R}^n$ is an equilibrium point of tempered fractional system (16).
Definition 5. Assume $x = 0$ is an equilibrium point of system (16), the solution of (16) is said to be tempered Mittag-Leffler stable if
\[
\|x(t)\| \leq \left[ m(x(0))e^{-\lambda t}E_{\alpha}(-\lambda(t-t_0)^{\alpha}) \right]^b,
\]
where $\alpha > 0$, $\lambda \geq 0$, $b > 0$, and $m(0) = 0$, $m(x) \geq 0$ satisfies locally Lipschitz condition.

**Remark 1.** Tempered Mittag-Leffler stability is a generalization of Mittag-Leffler stability. When $\lambda = 0$, tempered Mittag-Leffler stability can be reduced to Mittag-Leffler stability [26, 27].

**Remark 2.** Both Mittag-Leffler stability and tempered Mittag-Leffler stability imply asymptotic stability, that is, $\|x(t)\| \to 0$ as $t \to \infty$.

**Theorem 3.** Assume $x = 0$ be an equilibrium point for (16) and domain $\mathcal{D} \subset \mathbb{R}^n$ contains the origin. Let $V(t, x(t))$ be a continuously differentiable function and locally Lipschitz with respect to $x$, that such
\[
\alpha_i \|x(t)\|^a \leq V(t, x(t)) \leq \alpha_2 \|x(t)\|^b.
\]
\[
\underline{C} \frac{D^{\alpha}}{D^{\alpha}} V(t, x(t)) \leq -\alpha_3 \|x(t)\|^b,
\]
where $t \geq 0$, $x \in \mathcal{D}$, $\beta \in (0, 1)$, and $\alpha_1, \alpha_2, \alpha_3, a, b$ are given positive constants, then $x = 0$ is tempered Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally tempered Mittag-Leffler stable.

**Proof.** It follows from equations (25) and (26) that
\[
\underline{C} \frac{D^{\alpha}}{D^{\alpha}} V(t, x(t)) \leq -\alpha_3 \|x(t)\|^b.
\]
There exists a function $M(t) \geq 0$ such that
\[
\underline{C} \frac{D^{\alpha}}{D^{\alpha}} V(t, x(t)) + M(t) = -\frac{\alpha_3}{\alpha_2} V(t, x(t)).
\]
Taking the Laplace transform to (28) gives
\[
(s + \lambda)^\alpha V(s) = (s + \lambda)^{\beta - 1} V(0) + M(s) = -\frac{\alpha_3}{\alpha_2} V(s),
\]
where nonnegative constant $V(0) = V(x(0))$ and $V(s) = \mathcal{L}[V(t, x(t))]$. We rewrite this in the form
\[
V(s) = \frac{V(0)(s + \lambda)^{\beta - 1} - M(s)}{(s + \lambda)^\beta + \frac{\alpha_3}{\alpha_2} s}.
\]
Applying the inverse Laplace transform to (30), unique solution of (28) is
\[
V(t) = V(0)e^{-\lambda t}E_{\alpha}(-\lambda t^{\alpha}) - M(t) \left[ e^{-\lambda t^{\beta - 1}}E_{\alpha}(\frac{\alpha_3}{\alpha_2} t^{\beta}) \right].
\]
Because $t^{\beta - 1}$ and $E_{\alpha, \beta}(-\alpha_3/\alpha_2, t^\beta)$ are nonnegative functions, we obtain
\[
V(t) \leq V(0)e^{-\lambda t}E_{\alpha}(-\lambda t^{\alpha}).
\]
Substituting (32) into (25) satisfies
\[
\|x(t)\| \leq \left[ \frac{V(0)}{\alpha_1} e^{-\lambda t}E_{\alpha}(\frac{\alpha_3}{\alpha_2}) \right]^{1/\alpha},
\]
and $x(0) = 0$ if and only if $(V(0)/\alpha_1)e^{-\lambda t} = 0$.

Because $V(t, x)$ is local Lipschitz condition with respect to $x$ and $x(0) = 0$ if and only if $(V(0)/\alpha_1)e^{-\lambda t}$ satisfies local Lipschitz condition about $x(0)$. So, system (16) is tempered Mittag-Leffler stable.

**Theorem 4.** Assume all conditions in Theorem 3 are satisfied except replacing $\underline{C} \frac{D^{\alpha}}{D^{\alpha}}$ by $\underline{C} \frac{D^{\alpha}}{D^{\alpha}}$, then we have
\[
\|x(t)\| \leq \left[ \frac{V(0)}{\alpha_1} e^{-\lambda t}E_{\alpha}(\frac{\alpha_3}{\alpha_2}) \right]^{1/\alpha}.
\]

**Proof.** It follows from Lemma 6 and $V(t, x) \geq 0$ that
\[
\underline{C} \frac{D^{\alpha}}{D^{\alpha}} V(t, x) \leq \underline{C} \frac{D^{\alpha}}{D^{\alpha}} V(t, x) \leq -\alpha_3 \|x(t)\|^b.
\]
A similar proof method in Theorem 3 shows result (35).

**Theorem 5.** For the tempered fractional system (10), where $\underline{C} \frac{D^{\alpha}}{D^{\alpha}}$ is Lipschitz on $x$ with constant $l > 0$, and $f(t, 0) = 0$, if there exists a Lyapunov candidate $V(t, x)$ yielding
\[
\alpha_i \|x(t)\|^a \leq V(t, x(t)) \leq \alpha_2 \|x(t)\|^b,
\]
then $\dot{V}(t, x(t)) \leq -\alpha_3 \|x(t)\|^b$.

A similar proof method in Theorem 3 shows result (35).
Using (36) and (37) and Lipschitz condition on \( f(t, x) \), we obtain
\[
C_0 D_t^{\alpha \lambda} V(t, x) \leq \lambda \alpha_2 \alpha t^\lambda \|x(t)\| - \alpha_3 \theta t^\lambda \|x(t)\| = -(\alpha_3 - \lambda \alpha_2) t^\lambda \|x(t)\| \leq -(\alpha_3 - \lambda \alpha_2) \Gamma^{\alpha \lambda} \|f(t, x(t))\|.
\]
(40)

We can use Lemmas 7 and 2 to write
\[
C_0 D_t^{\alpha \lambda} V(t, x) \leq -(\alpha_3 - \lambda \alpha_2) \Gamma^{\alpha \lambda} \|f(t, x(t))\| = -(\alpha_3 - \lambda \alpha_2) \Gamma^{\alpha \lambda} \|x(t)\|,
\]
(41)

where \( \alpha t^\lambda \|e^\lambda u(t)\| \|0 \). By the same proof in Theorem 3, conclusion (38) holds.

\[\square\]

4. Applications

In this section, we will give three examples to demonstrate theoretical analysis. The Adams–Bashforth–Moulton method [37] is employed for solving tempered fractional differential equations in the simulations.

Example 1. Consider the tempered fractional Riemann–Liouville system:
\[
_0 D^{\alpha \lambda}_t x(t) = -|x(t)|, \quad 0 < \alpha < 1, \lambda \geq 0,
\]
where \( x(0) > 0 \). The Lyapunov function candidate is chosen as \( V(t, x) = |x| \). From Lemma 1, we obtain
\[
C_0 D_t^{\alpha \lambda} V = C_0 D_t^{\alpha \lambda} |x| \leq C_0 D_t^{\alpha \lambda} |x| = -|V|.
\]
(43)

By Theorem 3, we have
\[
|\dot{x}(t)| \leq |x(0)| e^{-\lambda t} E_{\alpha}(-t^\alpha).
\]
(44)

Then, \( x = 0 \) is tempered Mittag–Leffler stable. To verify the result, we choose parameters as \( \alpha = 0.95 \) and \( x(0) = 4 \) and the tempered parameters as \( \lambda = 2, 4, 6, 8 \), respectively. The time evolution of the system states (42) is shown in Figure 1. It is presented that the solution of system (42) converges to the equilibrium point \( x = 0 \). The larger the tempered parameter \( \lambda \) is, the faster the convergence speed becomes.

Example 2. Consider the tempered fractional Caputo Hopfield neural networks:
\[
_0 D_t^{\alpha \lambda} x_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + I_i,
\]
where \( 0 < \alpha < 1, \lambda \geq 0, i = 1, 2, \ldots, n \) and \( n \) is number of units. \( x_i(t) \) is the \( i \)th state, \( f_j \) is the \( j \)th activation function, \( b_{ij} \) is constant connection weight of the \( j \)th neuron on the \( i \)th neuron, \( a_i > 0 \) denotes the resting rate when the \( i \)th neuron disconnected, and \( I_i \) is the external inputs. Under the conditions

\[
[f_j(x)] < l_j |x|, \quad l_j > 0, \quad j = 1, 2, \ldots, n,
\]
(46)

\[
c_i = a_i - \sum_{j=1}^{n} |b_{ij}| l_j > 0, \quad i = 1, 2, \ldots, n,
\]
(47)

system (45) is globally tempered Mittag–Leffler stable. Let \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \) be any solution of system (45). We choose Lyapunov function as follows:
\[
V(t, x(t)) = \sum_{i=1}^{n} |x_i(t)|.
\]
(48)

By inequalities (46) and (47) and Lemma 7, tempered fractional Caputo derivative on \( V(t, x(t)) \) can be written as
\[
_0 D_t^{\alpha \lambda} V(t, x(t)) = \sum_{i=1}^{n} _0 D_t^{\alpha \lambda} |x_i(t)| \leq \sum_{i=1}^{n} |\text{sgn}(x_i(t))|_0 C_0 D_t^{\alpha \lambda} x_i(t)
\]
\[
\leq -a_i \sum_{i=1}^{n} |x_i(t)| + \sum_{i=1}^{n} \sum_{j=1}^{n} l_j |b_{ij}| |x_j(t)|
\]
\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} l_j |b_{ij}| |x_i(t)|
\]
\[
\leq -c \|x(t)\|,
\]
(49)

where \( c = \min\{c_1, c_2, \ldots, c_n\} \). From (49) and Theorem 3, system (45) is globally tempered Mittag–Leffler stable.

To illustrate the effectiveness of Example 2, in system (45), we let \( x = (x_1, x_2, x_3)^T, \alpha = 0.98, x_1(0) = 5, x_2(0) = 0, x_3(0) = 3, f_j(x_j) = \tanh(x_j) \), and \( c_j = 6 \) for \( j = 1, 2, 3 \) and
\[
A = (a_{ij})_{3 \times 3} = \begin{bmatrix} 2 & -1.2 & 0 \\ 1.8 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{bmatrix}
\]
(50)

It is obvious that condition (47) is satisfied. Let tempered parameters \( \lambda = 0, 2, 4, 6 \), respectively. As shown in Figure 2,
the equilibrium point $x = 0$ is tempered Mittag–Leffler stable and the solution of system (45) converges to $x = 0$.

**Example 3.** Consider the following tempered fractional system:

$$
0_D^\alpha_1 x_1 = -2x_1 + \frac{\sin(x_3)}{1 + t^2}x_1,
$$

$$
0_D^\alpha_1 x_2 = -2x_2 + \cos(x_1)x_2,
$$

$$
0_D^\alpha_1 x_3 = x_3,
$$

(51)

where $0 < \alpha < 1, \lambda \geq 0$ and $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3$.

Let the Lyapunov function $V(t, x) = |x_1| + |x_2|$. By Lemma 7, we obtain

$$
0_D^\alpha_1 V(t, x) \leq \text{sgn}(x_1(t))0_D^\alpha_1 x_1(t) + \text{sgn}(x_2(t))0_D^\alpha_1 x_2(t) + \text{sgn}(x_3(t))0_D^\alpha_1 x_3(t)
$$

$$
= \text{sgn}(x_1(t))\left(-2x_1(t) + \frac{\sin(x_3(t))}{1 + t^2}x_1(t)\right) + \text{sgn}(x_2(t))\left(-2x_2(t) + \cos(x_1(t))x_2(t)\right)
$$

$$
\leq -2|x_1(t)| + \frac{\sin(x_3(t))}{1 + t^2}|x_1(t)| - 2|x_2(t)|
$$

$$
+ |\cos(x_1(t))||x_2(t)|
$$

$$
\leq -\left(|x_1(t)| + |x_2(t)|\right).
$$

(52)
Then, the conditions of Theorem 3 are satisfied. Hence, 
$x = 0$ is globally tempered Mittag–Leffler stable with respect to 
$(x_1, x_2)$. Take $\alpha = 0.9$, $x_1 (0) = 10$, $x_2 (0) = -5$, and $x_3 (0) = 5$. The numerical simulation is shown as Figure 3 with different 
tempered parameters $\lambda = 0, 0.5, 1, 1.5$. It is obvious $x_1 (t)$ and 
$x_2 (t)$ converge to 0. When tempered parameter of the system 
increase, the part solution of system converges faster.

5. Conclusions

In this paper, we present some stability results for the 
tempered fractional systems. Based on the Laplace transform, we obtain the comparison principle for the tempered fractional systems. Some theorems about tempered Mittag–Leffler stability are derived, which enrich the knowledge of the system theory and the tempered fractional calculus and are helpful in characterizing the tempered fractional system models. Furthermore, we will study stability of 
tempered fractional systems with time-varying delays in future work.

Data Availability

The authors affirm that all data necessary for confirming the conclusions of the article are present in the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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