LARGE VALUES OF $L$-FUNCTIONS ON 1-LINE

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Abstract. In this paper, we study lower bounds of a general family of $L$-functions on the 1-line. More precisely, we show that for any $F(s)$ in this family, there exists arbitrary large $t$ such that $F(1 + it) \geq e^{|t|}(\log_2 t + \log_3 t)^m + O(1)$, where $m$ is the order of the pole of $F(s)$ at $s = 1$. This is a generalization of the same result of Aistleitner, Munsch and the second author for the Riemann zeta-function. As a consequence, we get lower bounds for large values of Dedekind zeta-functions and Rankin-Selberg $L$-functions of the type $L(s, f \times f)$ on the 1-line.

1. Introduction

The growth of the Riemann zeta-function $\zeta(s)$ in the critical strip $1/2 < \Re(s) < 1$ has been of interest to number theorists for a long time. In this context, the upper bound is predicted by the Lindelöf hypothesis, which claims that $|\zeta(\sigma + it)| \ll |t|^\epsilon$ for any $\epsilon > 0$ and $1/2 < \sigma < 1$. This is, in fact a consequence of the famous Riemann hypothesis. Although there is significant progress towards this problem, no unconditional proof is known (see [22] for more details).

A more intricate question is to investigate how large $|\zeta(\sigma + it)|$ can be for a fixed $\sigma \in [1/2, 1)$ and $t \in [T, T + H]$, some interval. In this direction, Balasubramanian and Ramachandra [7] showed that

$$\max_{t \in [T, T + H]} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \exp \left( c \sqrt{\frac{\log H}{\log_2 H}} \right),$$

where $c$ is a positive constant, $H \ll \log_2 T$ and $\log_2 T$ denotes $\log \log T$. From now on, we will denote $\log \log \ldots \log T$ by $\log_k T$. This result was improved by Bondarenko and Seip [9] in a larger interval and was later optimized by Bretche and Tenenbaum [10], who showed that

$$\max_{t \in [0, T]} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \exp \left( (\sqrt{2} + o(1)) \sqrt{\frac{\log T \log_3 T}{\log_2 T}} \right).$$

For $\sigma \in (1/2, 1)$ and $c_\sigma = 0.18(2\sigma - 1)^{1-\sigma}$, Aistleitner [11] proved that

$$\max_{t \in [0, T]} \left| \zeta \left( \frac{1}{2} + it \right) \right| \geq \exp \left( c_\sigma \sqrt{\frac{\log T}{\log_2 T}} \right).$$

On the other hand, we expect much finer results for large values of $L$-functions on $\Re(s) = 1$. In [12], Granville and Soundararajan used techniques of diophantine approximation to show

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that
\[
\max_{t \in [0,T]} |\zeta(1 + it)| \geq e^{\gamma \left( \log_2 T + \log_3 T - \log_4 T + O(1) \right)}
\]
for arbitrarily large \( T \). This is an improvement on the previous bounds given by Levinson [14]. Granville and Soundararajan [12] conjectured that
\[
\max_{t \in [T,2T]} |\zeta(1 + it)| = e^{\gamma \left( \log_2 T + \log_3 T + C_1 + o(1) \right)},
\] (1)
where \( C_1 \) is an explicitly computable constant.

In 2017, Aistleitner, Munsch and the second author [2] used the resonance method to prove that there is a constant \( C \) such that
\[
\max_{t \in [\sqrt{T},T]} |\zeta(1 + it)| \geq e^{\gamma \left( \log_2 T + \log_3 T + C \right)}.
\] (2)
Note that this result essentially matches (1), however, the size of the interval here is much larger. Unfortunately, over shorter intervals \([T, T + H]\), very little seems to be known regarding large values of \( \zeta(1 + it) \) (see [5], [6] for further details).

In this paper, we generalize (2) to a large class of \( L \)-functions, namely \( G \), which conjecturally contains the Selberg class \( S \). We establish (2) for elements in \( G \) with non-negative Dirichlet coefficients. The key difference between \( G \) and \( S \) is that elements in \( G \) satisfy a polynomial Euler-product which is a more restrictive condition than that in \( S \). However, the functional equation in \( S \) is replaced by a weaker “growth condition” in \( G \). This is a significant generalization because most Euler products, which have an analytic continuation exhibit a growth condition, but perhaps not a functional equation. As applications, we prove the analogue of (2) for Dedekind zeta-functions \( \zeta_K(s) \) and Rankin-Selberg \( L \)-functions given by \( L(s,f \times f) \). We also prove a generalized Merten’s theorem for \( G \) as a precursor to the proof of our main theorem.

The resonance method with a similar resonator was used by Aistleitner, Munsch, Peyrot and the second author [3] to establish large values of Dirichlet \( L \)-functions \( L(s,\chi) \) with a given conductor \( q \) at \( s = 1 \). Perhaps, a similar method can also be used to establish large values over more general orthogonal families of \( L \)-functions in \( G \).

1.1. The class \( G \). In 1989, Selberg [20] introduced a class of \( L \)-functions \( \mathcal{S} \), which is expected to encapsulate all naturally occurring \( L \)-functions arising from arithmetic and geometry.

**Definition 1.1 (The Selberg class).** The Selberg class \( \mathcal{S} \) consists of meromorphic functions \( F(s) \) satisfying the following properties.

(i) **Dirichlet series** - It can be expressed as a Dirichlet series
\[
F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},
\]
which is absolutely convergent in the region \( \Re(s) > 1 \). We also normalize the leading coefficient as \( a_F(1) = 1 \).

(ii) **Analytic continuation** - There exists a non-negative integer \( k \), such that \((s-1)^kF(s)\) is an entire function of finite order.
(iii) **Functional equation** - There exist real numbers $Q > 0$ and $\alpha_i \geq 0$, complex numbers $\beta_i$ and $w \in \mathbb{C}$, with $\Re(\beta_i) \geq 0$ and $|w| = 1$, such that

$$\Phi(s) := Q^s \prod_i \Gamma(\alpha_i s + \beta_i) F(s)$$

satisfies the functional equation

$$\Phi(s) = w \Phi(1 - \bar{s})$$

(iv) **Euler product** - There is an Euler product of the form

$$F(s) = \prod_{p \text{ prime}} F_p(s),$$

where

$$\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}$$

with $b_{p^k} = O(p^{k\theta})$ for some $\theta < 1/2$.

(v) **Ramanujan hypothesis** - For any $\epsilon > 0$,

$$|a_F(n)| = O(n^{\epsilon}).$$

The Euler product implies that the coefficients $a_F(n)$ are multiplicative, i.e., $a_F(mn) = a_F(m)a_F(n)$ when $(m, n) = 1$. Moreover, each Euler factor also has a Dirichlet series representation

$$F_p(s) = \sum_{k=0}^{\infty} \frac{a_F(p^k)}{p^{ks}},$$

which is absolutely convergent on $\Re(s) > 0$ and non-vanishing on $\Re(s) > \theta$, where $\theta$ is as in (iv).

For the purpose of this paper, we need a stronger Euler-product to ensure that the Euler factors factorize completely and further require a zero free region near 1-line, similar to what we notice in the proof of prime number theorem. However, we can replace the functional equation with a weaker condition on the growth of $L$-functions on vertical lines. This leads to the definition of the class $G$.

**Definition 1.2** (The class $G$). The class $G$ consists of meromorphic functions $F(s)$ satisfying (i), (ii) as in the above definition and further satisfies

(a) **Complete Euler product decomposition** - The Euler product in (i) factorizes completely, i.e.,

$$F(s) := \prod_p \prod_{j=1}^k \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}$$

with $|\alpha_j| \leq 1$ and $\Re(s) > 1$.

(b) **Zero-free region** - There exists a positive constant $c_F$, depending on $F$, such that $F(s)$ has no zeros in the region

$$\Re(s) \geq 1 - \frac{c_F}{\log(|F(s)| + 2)},$$

except the possible Siegel-zero of $F(s)$. 

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(c) **Growth condition** - For \( s = \sigma + it \), define
\[
\mu_F^*(\sigma) := \inf \{ \lambda > 0 : |F(s)| \ll (|t| + 2)^\lambda \}.
\]
Then,
\[
\frac{\mu_F^*(\sigma)}{1 - 2\sigma}
\]
is bounded for \( \sigma < 0 \).

One expects \( S \) to satisfy conditions (a) and (b). In fact, the Riemann zeta-function, the Dirichlet \( L \)-functions, the Dedekind zeta-functions and the Rankin-Selberg \( L \)-functions can be all shown to satisfy conditions (a) and (b). Furthermore, for elements in \( S \) the growth condition (c) is a consequence of the functional equation \( \text{[3]} \). However, it is possible to have \( L \)-functions not obeying a functional equation to satisfy the growth condition. One can consider linear combination of elements in \( S \) to see this. A family of \( L \)-functions based on growth condition was introduced by V. K. Murty in \( \text{[17]} \) and the reader may refer to \( \text{[11]} \) for more details on this family. Also the Igusa zeta-function, and the zeta function of groups have Euler products but may not have functional equation, which is discussed in \( \text{[19]} \).

1.2. **The Main Theorem.** In this paper, we produce a lower bound for large values of \( L \)-functions in \( \mathbb{G} \) on the 1-line. For a meromorphic function \( F(s) \) having a pole of order \( m \) at \( s = 1 \), define
\[
c_m(F) = \lim_{s \to 1} (s - 1)^m F(s).
\]

**Theorem 1.3.** Let \( F \in \mathbb{G} \) have non-negative Dirichlet coefficients \( a_F(n) \) and a pole of order \( m \) at \( s = 1 \). Then, there exists a constant \( C_F > 0 \) depending on \( F(s) \) such that
\[
\max_{t \in [\sqrt{T}, T]} |F(1 + it)| \geq e^{\gamma_F} ((\log_2 T + \log_3 T)^m - C_F),
\]
where \( \gamma_F = m \gamma + \log c_m(F) \) and \( \gamma \) is the Euler-Mascheroni constant.

In the above theorem, since \( a_F(n) \geq 0 \), we clearly have \( m \geq 1 \). This is important because if \( F \) has no pole at \( s = 1 \), it is possible for \( F(s) \) to grow very slowly on the 1-line.

As an immediate corollary, we get the following result for Dedekind zeta-functions \( \zeta_K(s) \). Let \( K/\mathbb{Q} \) be a number field. The Dedekind zeta-function \( \zeta_K(s) \) is defined on \( \Re(s) > 1 \) as
\[
\zeta_K(s) := \sum_{0 \neq a \in \mathcal{O}_K} \frac{1}{|Na|^s} = \prod_p \left(1 - \frac{1}{(Np)^s}\right)^{-1},
\]
where \( a \) runs over all non-zero integral ideals and \( p \) runs over all non-zero prime ideals of \( K \). The function \( \zeta_K(s) \) has an analytic continuation to the complex plane except for a simple pole at \( s = 1 \). Furthermore, \( \zeta_K \) satisfies properties (a), (b), (c) and therefore \( \zeta_K \in \mathbb{G} \). Thus, by Theorem \text{[1.3]} we have

**Corollary 1.** For a number field \( K \), there exists a constant \( C_K > 0 \) depending on \( K \) such that
\[
\max_{t \in [\sqrt{T}, T]} |\zeta_K(1 + it)| \geq e^{\gamma_K} ((\log_2 T + \log_3 T - C_K),
\]
where \( \gamma_K = \gamma + \log \rho_K \), with \( \rho_K \) being the residue of \( \zeta_K(s) \) at \( s = 1 \).
The $L$-function associated to the Rankin-Selberg convolution of any two holomorphic newforms $f$ and $g$, denoted by $L(s, f \times g)$, is in the Selberg class. Moreover, it can also be shown that $L(s, f \times g) \in \mathbb{G}$. Here $f$ and $g$ are normalized Hecke eigenforms of weight $k$. It is known that if $L(s, f \times g)$ has a pole at $s = 1$, then $f = g$. Hence, from Theorem 1.3, we have the following.

**Corollary 2.** For a normalized Hecke eigenform $f$, there exists a constant $C_f > 0$ such that

$$\max_{t \in [\sqrt{T}, T]} |L(1 + it, f \times f)| \geq e^{\gamma_f} \left( \log_2 T + \log_3 T - C_f \right),$$

where $\gamma_f = \gamma + \log \rho_f$, with $\rho_f$ being the residue of $L(s, f \times f)$ at $s = 1$.

The result obtained in Theorem 1.3 is a refined version of the bound established by Aistleitner-Pankowski [4], which states that if $F(s)$ is in the Selberg class and satisfies the prime number theorem, namely,

$$\sum_{p \leq x} |a_F(p)| = \kappa \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

then for large $T$,

$$\max_{t \in [T; 2T]} |F(1 + it)| = \Omega\left((\log \log T)^\kappa\right).$$

Furthermore, since we are assuming the zero-free region in $\mathbb{G}$, using [13, Theorem 1], we have $\kappa = m$. Hence, we get a slightly more refined result than (8), but on a larger interval $[\sqrt{T}, T]$.

The poles of any element $F$ in the Selberg class $\mathbb{S}$ are expected to arise from the Riemann zeta-function. More precisely, if $F(s)$ has a pole of order $m$ at $s = 1$, then $F(s)/\zeta(s)^m$ is expected to be entire and in $\mathbb{S}$. Thus, it is not surprising to expect the lower bound in Theorem 1.3 to be of the order $(\log \log T)^m$.

It is possible to generalize Theorem 1.3 to the Beurling zeta-function [8] by constructing a suitable resonator over Beurling numbers instead of integers. However, this will carry us far afield from our current focus. Hence, we relegate it to future research.

2. Mertens’ theorem for the class $\mathbb{G}$

In 1874, Mertens [15] proved the following estimate for truncated Euler-product of $\zeta(s)$, which is also known as Mertens’ third theorem given by

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^\gamma \log x + O(1),$$

where $\gamma$ denotes the Euler-Mascheroni constant. The analogue of Mertens’ theorem for number fields was proved by Rosen [18], who showed that

$$\prod_{n \in \mathbb{N}} \left(1 - \frac{1}{n}\right)^{-1} = \rho_K e^\gamma \log x + O(1),$$

where $\rho_K$ denotes the residue of $\zeta_K(s)$ at $s = 1$. The Mertens theorem for the extended Selberg class satisfying conditions (a) and (b) was proved by Yashiro [23] in 2013. Following similar approach, one can establish Mertens’ theorem for $\mathbb{G}$, where we replace the functional equation by the growth condition. However, Yashiro’s paper [23] seems to be available only on arXiv. Hence, we include the proof for the sake of completeness.
**Theorem 2.1.** Let $F(s) \in \mathbb{G}$. Suppose that $F(s)$ has a pole of order $m$ at $s = 1$ and $c_{m}(F)$ be as in (7). Then, for a constant $0 < C_{F} \leq 1$,

$$\prod_{p \leq x} \prod_{j=1}^{k} \left(1 - \frac{\alpha_{j}(p)}{p}\right)^{-1} = c_{m}(F) e^{m} (\log x)^{m} \left(1 + O\left(e^{-C_{F}\sqrt{\log x}}\right)\right).$$

**Proof.** We closely follow the method of Yashiro [23]. Denote by

$$F(1; x) := \prod_{p \leq x} \prod_{j=1}^{k} \left(1 - \frac{\alpha_{j}(p)}{p}\right)^{-1}.$$

Let

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_{F}(n)}{n^{s}}.$$

By the complete Euler product (6), we have $b_{F}(n) = 0$ if $n \neq p^{r}$ and $b_{F}(n) \ll n^{\theta}$ for some $\theta < 1/2$. Since

$$b_{F}(p^{r}) = \frac{1}{r} \sum_{j=1}^{k} \alpha_{j}(p)^{r},$$

we have $|b_{F}(p^{r})| \leq k$. Write

$$\log F(1; x) = \sum_{p \leq x} \sum_{r=1}^{\infty} \frac{b_{F}(p^{r})}{p^{r}} = \sum_{n \leq x} \frac{b_{F}(n)}{n} + \sum_{\sqrt{x} < p \leq x} \sum_{r=1}^{\infty} \frac{b_{F}(p^{r})}{p^{r}} + \sum_{p \leq \sqrt{x}} \sum_{p^{r} > x} \frac{b_{F}(p^{r})}{p^{r}}.$$  \hspace{1cm} (9)

It is easy to estimate the second and third term above as follows.

$$\sum_{\sqrt{x} < p \leq x} \sum_{r=1}^{\infty} \frac{b_{F}(p^{r})}{p^{r}} \ll \sum_{\sqrt{x} < p \leq x} \frac{1}{p^{r/2}} \ll \sum_{\sqrt{x} < p \leq x} \frac{1}{p^{2}} \ll \frac{1}{\sqrt{x}}.$$  \hspace{1cm} (10)

Also,

$$\sum_{p \leq \sqrt{x}} \sum_{p^{r} > x} \frac{b_{F}(p^{r})}{p^{r}} \ll \sum_{p \leq \sqrt{x}} \frac{1}{p} \ll \frac{1}{\sqrt{x}}.$$  \hspace{1cm} (11)

From (9), we get

$$\log F(1; x) = \sum_{n \leq x} \frac{b_{F}(n)}{n} + O\left(\frac{1}{\sqrt{x}}\right).$$

Setting $(1/\log x) = u$ and $e^{\sqrt{\log x}} = T$ and using Perron’s formula, we get

$$\sum_{n \leq x} \frac{b_{F}(n)}{n} = \frac{1}{2\pi i} \int_{u-iT}^{u+iT} x^{s} \log F(1 + s) \frac{ds}{s} + O\left(e^{-C_{F}\sqrt{\log x}}\right).$$

Let $u' = C_{F}/\log T = C_{F}/\sqrt{\log x}$. Choosing $x$ sufficiently large, we can ensure that there are no Siegel zeros for $F(1+s)$ in the region $[-u', u]$. Hence from the condition (b), $F(1+s)$ has no zeros in the region $-u' \leq \Re(s) \leq u$ and $|\Im(s)| \leq T$ and has a pole of order $m$ at $s = 0$.

Consider the contour $C$ joining $u-iT, -u'-iT, -u'+iT$ and $u+iT$. By the residue theorem, we have

$$\text{Res}_{s=0} \left(\frac{x^{s}}{s} \log F(1+s)\right) = \frac{1}{2\pi i} \int_{C} \frac{x^{s}}{s} \log F(1+s) \frac{ds}{s}. \hspace{1cm} (10)$$
Hence, we have
\[ \mu \divides \text{supposing \( \mu \divides \)} \text{min} \text{Lemma 2.2.} \]

Using the zero-free region
\[ \log F(1 + \sigma + iT) \ll (\log T)^2. \]

Hence, we have
\[
\left| \frac{1}{2\pi i} \int_{u+iT}^{u+iT} \frac{x^s}{s} \log F(1+s) \, ds \right| \ll \left| \frac{(\log T)^2}{T} \int_0^T \frac{x^\sigma}{s} \, d\sigma \right|
\]
\[
\ll (\log x) e^{-\sqrt{\log x}}
\]
\[
\ll e^{-c'_F \sqrt{\log x}},
\]
for some \( 0 < c'_F < 1 \). Similarly, we also get
\[
\left| \frac{1}{2\pi i} \int_{u' + iT}^{u' + iT} \frac{x^s}{s} \log F(1+s) \, ds \right| \ll e^{-c'_F \sqrt{\log x}}.
\]

We use the following result due to Landau (see [16, p. 170, Lemma 6.3]) to estimate the other terms in (10).

**Lemma 2.2.** Let \( f(z) \) be an analytic function in the region containing the disc \( |z| \leq 1 \), supposing \( |f(z)| \leq M \) for \( |z| \leq 1 \) and \( f(0) \neq 0 \). Fix \( r \) and \( R \) such that \( 0 < r < R < 1 \). Then, for \( |z| \leq r \) we have
\[ \frac{f'(z)}{f(z)} = \sum_{|\rho| \leq R} \frac{1}{z - \rho} + O\left( \log \frac{M}{|f(0)|} \right), \]
where \( \rho \) is a zero of \( f(s) \).

Let \( f(z) = (z + 1/2 + it)^m F(1 + z + (1/2 + it)) \), \( R = 5/6 \) and \( r = 2/3 \) in the above Lemma 2.2. Using the zero-free region \( (b) \), we get
\[ |\log s^m F(1+s)| \ll \begin{cases} \log(|t|+4), & |t| \geq 7/8 \text{ and } \sigma \geq -u', \\ 1 & |t| \leq 7/8 \text{ and } \sigma \geq -u'. \end{cases} \]

We now have the estimate
\[
\left| \int_{-u'}^{-u' + iT} \frac{x^s}{s} \log F(1+s) \, ds \right| \ll \int_0^T \frac{x^{-u'}}{|s|} \left( |\log s^m| + |\log s^m F(1+s)| \right) \, dt
\]
\[
\ll e^{-c''_F \sqrt{\log x}},
\]
for some \( 0 < c''_F < 1 \). Similarly, we also have
\[
\left| \int_{-u' - iT}^{-u' - iT} \frac{x^s}{s} \log F(1+s) \, ds \right| \ll e^{-c''_F \sqrt{\log x}}.
\]

Using the estimates (11), (12), (13) and (14) in the Equation (10) and choosing \( C_F = \min(c_F, c'_F, c''_F) \), we get
\[
\frac{1}{2\pi i} \int_{u - iT}^{u + iT} \frac{x^s}{s} \log F(1+s) \, ds = \text{Res}_{s=0} \left( \frac{x^s}{s} \log F(1+s) \right) + O \left( e^{-C_F \sqrt{\log x}} \right)
\]

Let \( C \) denote the circle of radius \( u' \) centered at 0. Then,
\[
\frac{1}{2\pi i} \int_C \frac{x^s}{s} \log F(1+s) \, ds = \text{Res}_{s=0} \left( \frac{x^s}{s} \log F(1+s) \right).
\]
Hence, it suffices to estimate the above integral. Since $F(s)$ has a pole of order $m$ at $s = 1$,

$$
c_{-m}(F) = \lim_{s \to 1} (s - 1)^m F(s) \neq 0.
$$

Writing $F(s + 1) = (s^{-m})(s^{m} F(s + 1))$, we get

$$
\frac{1}{2\pi i} \int_{C} \frac{x^s}{s} \log F(1 + s) \, ds = -\frac{m}{2\pi} \int_{C} \frac{x^s}{s} \log s \, ds + \log c_{-m}(F). \tag{15}
$$

The integral on the right hand side is

$$
\int_{C} \frac{x^s}{s} \log s \, ds = \int_{-\pi}^{\pi} \frac{x^{u' e^{i\theta}}}{u' e^{i\theta}} (\log u' e^{i\theta}) (iu' e^{i\theta}) \, d\theta \tag{16}
$$

$$
= i(\log u') \int_{-\pi}^{\pi} e^{u' e^{i\theta} \log x} \, d\theta - \int_{-\pi}^{\pi} \theta e^{u' e^{i\theta} \log x} \, d\theta. \tag{17}
$$

By the series expansion of exponential function, we have

$$
\int_{-\pi}^{\pi} e^{u' e^{i\theta} \log x} \, d\theta = \int_{-\pi}^{\pi} \theta \, d\theta + \sum_{r=1}^{\infty} \frac{(u' \log x)^r}{r!} \int_{-\pi}^{\pi} e^{i \theta r} \, d\theta = 2\pi
$$

Similarly,

$$
\int_{-\pi}^{\pi} \theta e^{u' e^{i\theta} \log x} \, d\theta = \int_{-\pi}^{\pi} \theta \, d\theta + \sum_{r=1}^{\infty} \frac{(u' \log x)^r}{r!} \int_{-\pi}^{\pi} \theta e^{i \theta r} \, d\theta
$$

$$
= \sum_{r=1}^{\infty} \frac{(u' \log x)^r}{r!} \left( \frac{(-1)^r 2\pi i}{ir} \right)
$$

$$
= \frac{2\pi}{i} \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \int_{0}^{\infty} u' \log x \frac{e^{-w} - 1}{w} \, dw
$$

But the Euler-Mascheroni constant $\gamma$ satisfies the identity

$$
\gamma = \int_{0}^{1} \frac{1 - e^{-w}}{w} \, dw - \int_{1}^{\infty} e^{-w} \, dw.
$$

Thus, we have

$$
\int_{0}^{u' \log x} \frac{e^{-w} - 1}{w} \, dw = \gamma + \int_{1}^{u' \log x} \frac{d}{w} \frac{e^{-w}}{w} \, dw - \int_{u' \log x}^{\infty} \frac{e^{-w}}{w} \, dw
$$

$$
= \gamma + \log \log x + \log u' + O \left( e^{-C_F \sqrt{\log x}} \right). \tag{18}
$$

Combining the estimates above (16)-(18), we get

$$
\log F(1; x) = \log c_{-m}(F) + m \gamma + m \log \log x + O \left( e^{-C_F \sqrt{\log x}} \right).
$$

Taking exponential on both sides and using the fact that $e^y = 1 + O(y)$ for $|y| < 1$, we are done. $\Box$
3. Proof of the main theorem

For $F \in \mathbb{G}$, define

$$F(s; Y) := \prod_{p \leq Y} \prod_{j=1}^{k} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}.$$ 

We use the following approximation lemma.

**Lemma 3.1.** For large $T$,

$$F(1 + it) = F(1 + it; Y) \left(1 + O\left(\frac{1}{(\log T)^{10}}\right)\right),$$

for $Y = \exp((\log T)^{10})$ and $T^{1/10} \leq |t| \leq T$.

**Proof.** From the Euler product of $F(s)$, we have for $\Re(s) > 1$,

$$\log F(s) = -\sum_{p} \sum_{j=1}^{k} \log \left(1 - \frac{\alpha_j(p)}{p^s}\right) = \sum_{p} \sum_{j=1}^{k} \sum_{l} \frac{\alpha_j(p)^{l}}{l p^{s}}.$$ 

Let $t_0 > 0$ and let $\alpha > 0$ be any sufficiently large constant. Define

$$\sigma_0 := \frac{1}{\alpha \log T}, \quad \sigma_1 := \frac{1}{(\log T)^{20}} \quad \text{and} \quad T_0 := \frac{T^{1/10}}{2}.$$ 

Applying Perron’s summation formula as in [21, Theorem II.2.2], we get

$$\int_{\sigma_1 - iT_0}^{\sigma_1 + iT_0} \log F(1 + it + s) \frac{Y^s}{s} ds = -\sum_{p} \sum_{j=1}^{k} \log \left(1 - \frac{\alpha_j(p)}{p^{1+it_0}}\right) + O\left(\frac{1}{(\log T)^{10}}\right) + \int_{C} \log F(1 + it_0 + s) \frac{Y^s}{s} ds,$$

where $C$ is the contour joining $\sigma_0 - iT_0, \sigma_1 - iT_0, \sigma_1 + iT_0$, and $\sigma_0 + iT_0$. Since, $|\log F(\sigma + it)| \ll \log t$ on $C$, we get

$$\int_{\sigma_1 - iT_0}^{\sigma_1 + iT_0} \log F(1 + it + s) \frac{Y^s}{s} ds \ll \frac{\log T}{T_0^{1/10}}, \quad \int_{-\sigma_0 - iT_0}^{\sigma_1 + iT_0} \log F(1 + it_0 + s) \frac{Y^s}{s} ds \ll \frac{\log T}{T_0^{1/10}},$$

and

$$\int_{-\sigma_0 - iT_0}^{-\sigma_0 + iT_0} \log F(1 + it_0 + s) \frac{Y^s}{s} ds \ll \frac{(\log T)^{2}}{\exp\left(\frac{1}{\alpha (\log T)^{10}}\right)},$$

where all implied constants are absolute. Substituting the bounds from [20] and [21] in [19], for $T^{1/10} \leq t_0 \leq T$, we obtain

$$\log F(1 + it_0) = -\sum_{p} \sum_{j=1}^{k} \log \left(1 - \frac{\alpha_j(p)}{p^{1+it_0}}\right) + O\left(\frac{1}{(\log T)^{10}}\right).$$

Similarly we may argue when $t_0$ is negative. \qed
By Lemma 3.1 it suffices to show Theorem 1.3 for $F(1 + it; Y)$. We closely follow the argument in [2]. Set

$$X = \frac{1}{6} (\log T)(\log_2 T)$$

and for primes $p \leq X$ set

$$q_p = \left(1 - \frac{p}{X}\right).$$

Also set $q_1 = 1$ and $q_p = 0$ for $p > X$. Extend the definition completely multiplicatively to define $q_n$ for all integers $n \geq 1$. Now define

$$R(t) = \prod_{p \leq X} \left(1 - q_p p^{-it}\right)^{-1}.$$

Then we have

$$\log(|R(t)|) \leq \sum_{p \leq X} (\log X - \log p)$$

$$= \pi(X) \log X - \vartheta(X),$$

where $\pi(X)$ is the prime counting function and $\vartheta(X)$ is the first Chebyshev function. By partial summation, we know that

$$\pi(X) \log X - \vartheta(X) = \int_2^X \frac{\pi(t)}{t} dt = (1 + o(1)) \frac{X}{\log X}. $$

By our choice of $X$, we get

$$|R(t)|^2 \leq T^{1/3 + o(1)}.$$  \hspace{1cm} (22)

From the Euler product, $R(t)$ has the following series representation

$$R(t) = \sum_{n=1}^{\infty} q_n n^{-it},$$

and hence we get

$$|R(t)|^2 = (\sum_{n=1}^{\infty} q_n n^{-it})(\sum_{n=1}^{\infty} q_n n^{-it}) = \sum_{m,n=1}^{\infty} q_m q_n \left(\frac{m}{n}\right)^{-it}.$$

We have

$$F(1 + it; Y) = \prod_{p \leq Y} \prod_{j=1}^{k} \left(1 - \frac{\alpha_j(p) p^{-it}}{p}\right)^{-1}$$

Since $|\alpha_j(p)| \leq 1$, we get

$$|F(1 + it; Y)| \ll (\log Y)^k \ll (\log T)^{10k}.$$  

Set $\Phi(t) = e^{-t^2}$ and recall that its Fourier transform is positive. Using (22), we have

$$\left| \int_{|t| \geq T} F(1 + it; Y)|R(t)|^2 \Phi \left(\frac{\log T}{T} t\right) dt \right| \ll 1,$$

and

$$\left| \int_{|t| \leq \sqrt{T}} F(1 + it; Y)|R(t)|^2 \Phi \left(\frac{\log T}{T} t\right) dt \right| \ll T^{5/6 + o(1)}.$$  

Using the fact that $q_1 = 1$ and the positivity of the Fourier coefficients of $\Phi$, we also have the following lower bound
\[
\int_{-T}^{T} |R(t)|^2 \Phi \left( \frac{\log T}{T} t \right) \, dt \gg T^{1+o(1)}.
\]

By a similar argument, again using the positivity of the Fourier coefficients, we have
\[
\int_{-\infty}^{\infty} F(1 + it; Y)|R(t)|^2 \Phi \left( \frac{\log T}{T} t \right) \, dt \geq \int_{-\infty}^{\infty} F(1 + it; X)|R(t)|^2 \Phi \left( \frac{\log T}{T} t \right) \, dt.
\]

So, we restrict ourselves to primes \( p \leq X \) in the truncated Euler-product. This is to ensure both \( R(t) \) and \( F(1 + it; X) \) have the terms with same \( q \)'s.

Write \( F(1 + it; X) \) as
\[
F(1 + it; X) := \sum_{n=1}^{\infty} a_k k^{-it},
\]
where \( a_k \geq 0 \). This is because the Dirichlet coefficients of \( F(s) \) are non-negative. Now define
\[
I_1 := \int_{-\infty}^{\infty} F(1 + it; X)|R(t)|^2 \Phi \left( \frac{\log T}{T} t \right) \, dt
\]
\[
= \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} k^{-it} q_m q_n \left( \frac{m}{n} \right) \, t \Phi \left( \frac{\log T}{T} t \right) \, dt.
\]

We also define
\[
I_2 := \int_{-\infty}^{\infty} |R(t)|^2 \Phi \left( \frac{\log T}{T} t \right) \, dt.
\]

Notice that since we are working with truncated Euler-products, everything is absolutely convergent. Now, using the fact that the Fourier coefficients of \( \Phi \) are positive and that \( q_n \) are completely multiplicative, we get the inner sum of \( I_1 \) as
\[
\sum_{m,n=1}^{\infty} \int_{-\infty}^{\infty} k^{-it} q_m q_n \left( \frac{m}{n} \right) \, t \Phi \left( \frac{\log T}{T} t \right) \, dt \geq \sum_{n=1}^{\infty} \sum_{k \mid m} \int_{-\infty}^{\infty} k^{-it} q_m q_n \left( \frac{m}{n} \right) \, t \Phi \left( \frac{\log T}{T} t \right) \, dt
\]
\[
= q_k \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \int_{-\infty}^{\infty} q_r q_n \left( \frac{r}{n} \right) \, t \Phi \left( \frac{\log T}{T} t \right) \, dt.
\]

Thus, we have
\[
\frac{I_1}{I_2} \geq \sum_{k=1}^{\infty} a_k q_k = \prod_{p \leq X} \prod_{j=1}^{k} \left( 1 - \frac{\alpha_j(p)}{p} q_p \right)^{-1}
\]
\[
= \left( \prod_{p \leq X} \prod_{j=1}^{k} \left( 1 - \frac{\alpha_j(p)}{p} \right)^{-1} \right) \left( \prod_{p \leq X} \prod_{j=1}^{k} \left( \frac{p - \alpha_j(p)}{p} q_p \right) \right).
\]

Using the generalized Merten's Theorem \[2.1\] we have
\[
\prod_{p \leq X} \prod_{j=1}^{k} \left( 1 - \frac{\alpha_j(p)}{p} \right)^{-1} = e^{\gamma \nu} (\log X)^m + O(1)
\]
\[
= e^{\gamma \nu} (\log_2 T + \log_3 T)^m + O(1)
\]

(24)
The second product in (23) can be bounded as follows.

\[-\log \left( \prod_{p \leq X} \prod_{j=1}^{k} \left( \frac{p - \alpha_j(p)}{p - \alpha_j(p)q_p} \right) \right) = -\sum_{p \leq X} \sum_{j=1}^{k} \log \left( \frac{p - \alpha_j(p)}{p - \alpha_j(p)q_p} \right) \ll \frac{1}{X} \quad \text{(25)}\]

\[\ll \frac{1}{\log X}. \quad \text{(26)}\]

From (23), (24) and (25), we get

\[I_1 \geq e^{\gamma_F (\log_2 T + \log_3 T)^m + O(1)}. \]

In other words, we have

\[\frac{I_1}{I_2} \geq e^{\gamma_F (\log_2 T + \log_3 T)^m + O(1)}. \]

Hence, we conclude

\[\max_{t \in [\sqrt{T}, T]} |F(1 + it)| \geq e^{\gamma_F ((\log_2 T + \log_3 T)^m - C_F)}. \]

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