Off-shell (4,4) supersymmetric sigma models with torsion in harmonic superspace

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Abstract

We present a manifestly supersymmetric off-shell formulation of a wide class of (4,4) 2D sigma models with torsion and non-commuting left and right complex structures in the harmonic superspace with a double set of SU(2) harmonic variables. The distinguishing features of the relevant superfield action are: (i) in general nonabelian and nonlinear gauge invariance ensuring a correct number of physical degrees of freedom; (ii) an infinite tower of auxiliary fields. This action is derived from the most general one by imposing the integrability condition which follows from the commutativity of the left and right analyticity-preserving harmonic derivatives. For a particular class of such models we explicitly demonstrate the non-commutativity of complex structures on the bosonic target.

1. Introduction. Remarkable target geometries of 2D sigma models with extended worldsheet SUSY are revealed most clearly within manifestly supersymmetric off-shell superfield formulations of these theories. For torsionless (2,2) and (4,4) sigma models the relevant superfield Lagrangians were found to coincide with (or to be directly related to) the fundamental objects underlying the given geometry: Kähler potential in the (2,2) case [1], hyper-Kähler or quaternionic-Kähler potentials in the flat or curved (4,4) cases [2 - 5]. One of the basic advantages of such a description is the possibility to explicitly compute the corresponding bosonic metrics (Kähler, hyper-Kähler, quaternionic ...) starting from an unconstrained superfield action [2, 6]. To have superfield off-shell formulations with all supersymmetries manifest is also highly desirable while quantizing these theories. For example, this simplifies proofs of the ultraviolet finiteness.

An important wide class of 2D supersymmetric sigma models is presented by (2,2) and (4,4) models with torsionful bosonic target manifolds and two independent left and right sets of complex structures (see, e.g. [7, 8]). These models and, in particular, their group manifold WZNW representatives [9] can provide non-trivial backgrounds for 4D superstrings (see, e.g., [10]) and be relevant to 2D black holes [11]. A manifestly supersymmetric formulation of (2,2) models with commuting left and right complex structures in terms of chiral and twisted chiral (2,2) superfields and an exhaustive discussion of their geometry have been given in [7]. For (4,4) models with commuting structures there exist manifestly supersymmetric off-shell formulations in the projective, ordinary and SU(2) × SU(2) harmonic (4,4) superspaces [11-13]. The appropriate superfields represent, in one or another way, the (4,4) 2D twisted multiplet [14, 7].
Much less is known about $(2,2)$ and $(4,4)$ sigma models with non-commuting complex structures, despite the fact that most of the corresponding group manifold WZNW sigma models \[9\] fall into this category \[11\]. In particular, it is unclear how to describe them off shell in general. As was argued in Refs. \[7, 15, 11\], twisted $(2,2)$ and $(4,4)$ multiplets are not suitable for this purpose. It has been then suggested to make use of some other off-shell representations of $(2,2)$ \[15, 16\] and $(4,4)$ \[15, 17\] worldsheet SUSY. However, it is an open question whether the relevant actions correspond to generic sigma models of this type.

In this talk we describe another approach to the off-shell description of general $(4,4)$ sigma models with torsion, exploiting an analogy with general torsionless hyper-Kähler $(4,4)$ sigma models in $SU(2)$ harmonic superspace \[2 - 4\]. The presentation is based upon two recent preprints of the author \[18, 19\] and his paper with A. Sutulin \[13\]. We start from a dual form of the general action of $(4,4)$ twisted superfields in $SU(2) \times SU(2)$ analytic harmonic superspace with two independent sets of harmonic variables \[13\] and construct a direct $SU(2) \times SU(2)$ harmonic analog of the hyper-Kähler $(4,4)$ action. The form of the action obtained, contrary to the torsionless case, proves to be severely constrained by the integrability conditions following from the commutativity of the left and right harmonic derivatives. While for four-dimensional bosonic manifolds the resulting action is reduced to that of twisted superfield, for manifolds of dimension $4n$, $n \geq 2$, the generic action cannot be written only in terms of twisted superfields. Its most characteristic features are (i) the unavoidable presence of infinite number of auxiliary fields and (ii) a nonabelian and in general nonlinear gauge symmetry which ensures the necessary number of propagating fields. These symmetry and action are harmonic analogs of the Poisson gauge symmetry and actions recently discussed in \[20, 21\]. For an interesting subclass of these actions, harmonic analogs of the Yang-Mills ones, we explicitly demonstrate that the left and right complex structures on the bosonic target do not commute.

2. $(4,4)$ twisted multiplet in $SU(2) \times SU(2)$ harmonic superspace. The $SU(2) \times SU(2)$ harmonic superspace is an extension of the standard real $(4,4)$ $2D$ superspace by two independent sets of harmonic variables $u^\pm\pm$ and $v^{\pm\pm\mp}$ ($u_{i}^{-1}u_{i}^{-1} = v_{a}^{-1}v_{a}^{-1} = 1$) associated with the automorphism groups $SU(2)_{L}$ and $SU(2)_{R}$ of the left and right sectors of $(4,4)$ supersymmetry \[13\]. The corresponding analytic subspace is spanned by the following set of coordinates

$$(\zeta, u, v) = (x^{++}, x^{--}, \theta^{1,0}, \theta^{0,1}, u^{\pm\mp}, v^{\pm\pm})$$

(1)

where we omitted the light-cone indices of odd coordinates. The superscript “$n, m$” stands for two independent harmonic $U(1)$ charges, left ($n$) and right ($m$) ones.

It was argued in \[13\] that this type of harmonic superspace is most appropriate for constructing off-shell formulations of $(4,4)$ sigma models with torsion. This hope mainly relied upon the fact that the twisted $(4,4)$ multiplet has a natural description as a real analytic $SU(2) \times SU(2)$ harmonic superfield $q^{1,1}(\zeta, u, v)$ (subjected to some harmonic constraints). The most general off-shell action of $n$ such multiplets is given by the following integral over the analytic superspace \[18, 19\]

$$S_{q,\omega} = \int \mu^{-2,-2}\{q^{1,1} M (D^{2,0} \omega^{-1,1} M + D^{0,2} \omega^{1,-1} M)h^{2,2}(q^{1,1}, u, v)\} (M = 1, ..., n).$$

(2)
where
\[
D^{2,0} = \partial^{2,0} + i\theta^{1,0}\sigma^1_\perp \partial_{++}, \quad
D^{0,2} = \partial^{0,2} + i\theta^{0,1}\sigma^2_\perp \partial_{--}
\]
(3)
\[
(\partial^{2,0} = u^1 i \frac{\partial}{\partial u^{-1}+}, \quad
\partial^{0,2} = v^1 a \frac{\partial}{\partial v^{-1} a})
\]
are the left and right analyticity-preserving harmonic derivatives and $\mu^{-2,-2}$ is the analytic superspace integration measure. In (2) the involved superfields are unconstrained analytic, so from the beginning the action (2) contains an infinite number of auxiliary fields coming from the double harmonic expansions with respect to the harmonics $u^{\pm 1}, v^{\pm 1}$. However, after varying with respect to the Lagrange multipliers $\omega^{1,-1}M, \omega^{-1,1}M$, one comes to the action written only in terms of $q^{1,1}N$ subjected to the harmonic constraints
\[
D^{2,0}q^{1,1}M = D^{0,2}q^{1,1}M = 0.
\]
(4)
For each value of $M$ these constraints define the $(4,4)$ twisted multiplet in the $SU(2) \times SU(2)$ harmonic superspace $(8 + 8$ components off-shell), so the action (2) is a dual form of the general off-shell action of $(4,4)$ twisted multiplets [13]
\[
S_q = \int \mu^{-2,-2} h^{2,2}(q, u, v).
\]
(5)
As an important particular example of such a $q^{1,1}$ action we give the action of $(4,4)$ extension of the group manifold $SU(2) \times U(1)$ WZNW sigma model
\[
S_{wzw} = -\frac{1}{4\kappa^2} \int \mu^{-2,-2} q^{1,1}(1,1) \left( \frac{1}{(1+X)} - \frac{\ln(1+X)}{X^2} \right).
\]
(6)
Here
\[
\dot{q}^{1,1} = q^{1,1} - c^{1,1}, \quad X = c^{-1,-1}q^{1,1}, \quad c^{\pm 1, \pm 1} = c^a u^{\pm 1}_i v^{\pm 1}_a, \quad c^a c_i^a = 2.
\]
(7)
Despite the presence of an extra quartet constant $c^a$ in the analytic superfield Lagrangian, the action (2) actually does not depend on $c^a$ [13] as it is invariant under arbitrary rescalings and $SU(2) \times SU(2)$ rotations of this constant.

The crucial feature of the dual action (2) is the abelian gauge invariance
\[
\delta \omega^{1,-1}M = D^{2,0}\sigma^{-1,-1}M, \quad \delta \omega^{-1,1}M = -D^{0,2}\sigma^{-1,-1}M
\]
(8)
where $\sigma^{-1,-1}M$ are unconstrained analytic superfield parameters. This gauge freedom ensures the on-shell equivalence of the $q, \omega$ formulation of the twisted multiplet action to its original $q$ formulation [3] [13]: it neutralizes superfluous physical dimension component fields in the superfields $\omega^{1,-1}M$ and $\omega^{-1,1}M$ and thus equalizes the number of propagating fields in both formulations. It holds already at the free level, with $h^{2,2}$ quadratic in $q^{1,1}M$, so it is natural to expect that any reasonable generalization of the action (2) respects this symmetry or a generalization of it. We will see soon that this is indeed so.

After identifying harmonics $u$ and $v$ as well as two harmonic $U(1)$ charges, and defining
\[
D^{(+2)} = (D^{2,0} + D^{0,2})|_{u=v}, \quad
\omega^{N} = \omega^{1,-1}N|_{u=v} = \omega^{-1,1}N|_{u=v},
\]
\[
l^{(+2)}N = q^{1,1}N|_{u=v}, \quad L^{(+4)} = h^{2,2}|_{u=v}
\]
(9)
the action (2) is reduced to the dual action of tensor (4, 4) 2D multiplet in the $SU(2) 2D$ harmonic superspace [3]

$$S_t = \int \mu^{(-4)} \{-2\omega^N D^{(+2)} l^{(+2)} N + \tilde{L}^{(+4)}(l, u)\} .$$

(10)

Varying (10) with respect to $\omega^N$, we arrive at the action which contains only the $\tilde{L}^{(+4)}(l, u)$ part,

$$S_t = \int \mu^{(-4)} \tilde{L}^{(+4)}(l, u) ,$$

(11)

with the superfield $l^{(+)} N$ subjected to the constraint

$$D^{(+2)} l^{(+2)} N = 0 .$$

(12)

This is just the harmonic superspace action and constraint of $N = 2 \ 4D ((4, 4) \ 2D)$ tensor multiplet [3]. The action (10) is a particular case of the general action of torsionless hyper-Kähler (4, 4) sigma models in the $\omega, l^{(+2)}$ representation [3, 4]. Its specificity consists in that it respects $n$ independent abelian isometries realized as constant shifts of the Lagrange multipliers $\omega^N$.

3. More general (4,4) sigma models with torsion. The dual twisted multiplet action (2) is a good starting point for constructing more general actions which, as we will show, encompass sigma models with non-commuting left and right complex structures.

It is useful to apply to the suggestive analogy with the general action of hyper-Kähler (4, 4) sigma models in the $SU(2)$ harmonic superspace [22]. This action in the $\omega, l^{+2}$ representation [4] looks very similar to (2), the $SU(2)$ analytic superfield pair $\omega^M, l^{+2 M}$ being the clear analog of the $SU(2) \times SU(2)$ analytic superfield triple $\omega^{1,-1 M}, \omega^{-1,1 M}, q^{1,1 M}$ and the general hyper-Kähler potential being analogous to $h^{2,2}$. However, this analogy breaks in that the hyper-Kähler potential is in general an arbitrary function of all involved superfields and harmonics while $h^{2,2}$ in (2) depends only on $q^{1,1 M}$ and harmonics. As we just saw, (2) is the direct analog of the particular class of hyper-Kähler actions (11), with the hyper-Kähler potential displaying no dependence on $\omega^M$. The general hyper-Kähler (4, 4) action can be obtained from (2) by including an arbitrary dependence on $\omega^M$ in $L^{(+4)}$. Then an obvious way to generalize (2) to cover a wider class of torsionful (4,4) models is to allow for a dependence on $\omega^{1,-1 M}, \omega^{-1,1 M}$ in $h^{2,2}$.

With these reasonings in mind, we take as an ansatz for the general action the following one

$$S_{\text{gen}} = \int \mu^{-2,-2}\{q^{1,1 M} ( D^{2,0} \omega^{-1,1 M} + D^{0,2} \omega^{1,-1 M} ) + H^{2,2}(q^{1,1}, \omega^{1,-1}, \omega^{-1,1}, u, v)\} ,$$

(13)

where for the moment the $\omega$ dependence in $H^{2,2}$ is not fixed. In Sect.5 we will show that one can arrive at this action proceeding from the most general $q, \omega$ action containing first order harmonic derivatives. But, for the time being, it is convenient for us to rely on the analogy with the (4,4) sigma model action in the $SU(2)$ harmonic superspace.

Now we are approaching the most important point. Namely, we are going to show that, contrary to the case of $SU(2)$ harmonic action of torsionless (4,4) sigma models,
the \( \omega \) dependence of the potential \( H^{2,2} \) in (13) is completely specified by the integrability conditions following from the commutativity relation

\[
[D^{2,0}, D^{0,2}] = 0.
\]  

To this end, let us write the equations of motion corresponding to (13)

\[
D^{2,0} \omega^{-1,1} M + D^{0,2} \omega^{1,-1} M = -\frac{\partial H^{2,2}(q, \omega, u, v)}{\partial q^{1,1} M},
\]

\[
D^{2,0} q^{1,1} M = \frac{\partial H^{2,2}(q, \omega, u, v)}{\partial \omega^{-1,1} M},
\]

\[
D^{0,2} q^{1,1} M = \frac{\partial H^{2,2}(q, \omega, u, v)}{\partial \omega^{1,-1} M}.
\]

Applying the integrability condition (14) to the pair of equations (16) and imposing a natural requirement that it is satisfied as a consequence of the equations of motion (i.e. does not give rise to any new dynamical restrictions), after some algebra we arrive at the following set of self-consistency relations

\[
\frac{\partial^2 H^{2,2}}{\partial \omega^{-1,1} N \partial \omega^{-1,1} M} = \frac{\partial^2 H^{2,2}}{\partial \omega^{1,-1} N \partial \omega^{1,-1} M} = \frac{\partial^2 H^{2,2}}{\partial \omega^{1,-1} (N \partial \omega^{-1,1} M)} = 0,
\]

\[
\left(\frac{\partial^2}{\partial \omega^{1,-1} N} + \frac{\partial H^{2,2}}{\partial q^{1,1} N} \frac{\partial}{\partial q^{1,1} N} - \frac{1}{2} \frac{\partial H^{2,2}}{\partial q^{1,1} N} \frac{\partial}{\partial \omega^{1,-1} N} \right)^2 \frac{\partial H^{2,2}}{\partial \omega^{1,-1} M} = 0.
\]

Eqs. (17) imply

\[
H^{2,2} = h^{2,2}(q, u, v) + \omega^{-1,1} N h^{1,3,1} N (q, u, v) + \omega^{-1,1} N \omega^{1,-1} M h^{2,2}[N,M](q, u, v).
\]

Plugging this expression into the constraint (18), we finally deduce four independent constraints on the potentials \( h^{2,2}, h^{1,3,1} N, h^{3,1} N \) and \( h^{2,2}[N,M] \)

\[
\nabla^{2,0} h^{1,3,1} N - \nabla^{0,2} h^{3,1} N + h^{2,2}[N,M] \frac{\partial h^{2,2}}{\partial q^{1,1} M} = 0
\]

\[
\nabla^{2,0} h^{2,2}[N,M] - \frac{\partial h^{3,1} N}{\partial q^{1,1} T} h^{2,2}[T,M] + \frac{\partial h^{3,1} M}{\partial q^{1,1} T} h^{2,2}[T,N] = 0
\]

\[
\nabla^{0,2} h^{2,2}[N,M] - \frac{\partial h^{1,3,1} N}{\partial q^{1,1} T} h^{2,2}[T,M] + \frac{\partial h^{1,3} M}{\partial q^{1,1} T} h^{2,2}[T,N] = 0
\]

\[
h^{2,2}[N,T] \frac{\partial h^{2,2}[M,L]}{\partial q^{1,1} T} + h^{2,2}[L,T] \frac{\partial h^{2,2}[N,M]}{\partial q^{1,1} T} + h^{2,2}[M,T] \frac{\partial h^{2,2}[L,N]}{\partial q^{1,1} T} = 0
\]

where

\[
\nabla^{2,0} = \partial^{2,0} + h^{3,1} N \frac{\partial}{\partial q^{1,1} N}, \quad \nabla^{0,2} = \partial^{0,2} + h^{1,3} M \frac{\partial}{\partial q^{1,1} N}
\]

and \( \partial^{2,0}, \partial^{0,2} \) act only on the “target” harmonics, i.e. those appearing explicitly in the potentials.
Thus we have shown that the direct generalization of the generic hyper-Kähler (4, 4) sigma model action to the torsionful case is given by the action

$$S_{q,\omega} = \int \mu^{-2 - 2} \left\{ q^{1,1M} D^{0,2} \omega^{-1,-1M} + q^{1,1M} D^{2,0} \omega^{-1,1M} + \omega^{-1,1M} h^{1,3M} + \omega^{-1,1M} h^{3,1M} + \omega^{-1,1M} \omega^{-1,-1N} h^{2,2[M,N]} + h^{2,2} \right\},$$

where the involved potentials depend only on $q^{1,1M}$ and target harmonics and satisfy the target space constraints (20) - (23). These constraints certainly encode a nontrivial geometry which for the time being is unclear to us. To reveal it we need to solve the constraints, which is still to be done. At present we are only aware of their particular solution which will be discussed in the next section.

In the rest of this section we present a set of invariances of the action (25) and constraints (20) - (23) which can be useful for understanding the underlying geometry of the given class of sigma models.

One of these invariances is a mixture of reparameterizations in the target space (spanned by the involved superfields and target harmonics) and the transformations which are bi-harmonic analogs of hyper-Kähler transformations of Refs. [23, 4]. It is realized by

$$\delta q^{1,1N} = \lambda^{1,1N}, \quad \delta \omega^{-1,1N} = -\frac{\partial \lambda^{0,2}}{\partial q^{1,1N}} \sigma^{-1,-1M} \omega^{-1,-1M},$$

$$\delta \omega^{-1,-1N} = -\frac{\partial \lambda^{2,0}}{\partial q^{1,1N}} - \frac{\partial \lambda^{1,1M}}{\partial q^{1,1N}} \omega^{-1,-1M},$$

$$\delta h^{2,2} = \nabla^{2,0} \lambda^{0,2} + \nabla^{0,2} \lambda^{2,0},$$

$$\delta h^{3,1M} = \nabla^{2,0} \lambda^{1,1M} + h^{2,2} [M,N] \frac{\partial \lambda^{2,0}}{\partial q^{1,1N}} \sigma^{-1,-1M},$$

$$\delta h^{1,3M} = \nabla^{0,2} \lambda^{1,1M} - h^{2,2} [M,N] \frac{\partial \lambda^{0,2}}{\partial q^{1,1N}} \sigma^{-1,-1M},$$

$$\delta h^{2,2} [N,M] = \frac{\partial \lambda^{1,1N}}{\partial q^{1,1L}} h^{2,2} [L,M] - \frac{\partial \lambda^{1,1M}}{\partial q^{1,1L}} h^{2,2} [L,N],$$

all the involved transformation parameters being unconstrained functions of $(q^{1,1M}, u, v)$. This kind of invariance can be used to bring the potentials in (25) into a “normal” form similar to the normal gauge of the hyper-Kähler potential (see [4]).

Much more interesting is another invariance which has no analog in the hyper-Kähler case and is a nonabelain and in general nonlinear generalization of the abelian gauge invariance (8)

$$\delta \omega^{1,-1M} = \left( D^{2,0} \delta^{MN} + \frac{\partial h^{3,1N}}{\partial q^{1,1M}} \right) \sigma^{-1,-1N} \omega^{1,-1L} \frac{\partial h^{2,2} [L,N]}{\partial q^{1,1M}} \sigma^{-1,-1N},$$

$$\delta \omega^{-1,1M} = -\left( D^{0,2} \delta^{MN} + \frac{\partial h^{1,3N}}{\partial q^{1,1M}} \right) \sigma^{-1,-1N} \omega^{-1,1L} \frac{\partial h^{2,2} [L,N]}{\partial q^{1,1M}} \sigma^{-1,-1N},$$

$$\delta q^{1,1M} = \sigma^{-1,-1N} h^{2,2} [N,M].$$

As expected, the action is invariant only with taking account of the integrability conditions (20) - (23). In general, these gauge transformations close with a field-dependent Lie
bracket parameter. Indeed, commuting two such transformations, say, on \(q^{1,1,N}\), and using the cyclic constraint (23), we find

\[
\delta_{br} q^{1,1,M} = \sigma_{br}^{-1,-1,N} h^{2,2}[N,M], \quad \sigma_{br}^{-1,-1,N} = -\sigma_1^{-1,-1,L} \sigma_2^{-1,-1,T} \frac{\partial h^{2,2}[L,T]}{\partial q^{1,1,N}}.
\]  

(28)

We see that eq. (23) guarantees the nonlinear closure of the algebra of gauge transformations (27) and so it is a group condition similar to the Jacobi identities.

Curiously enough, the gauge transformations (27) augmented with the group condition (23) are precise bi-harmonic counterparts of the two-dimensional version of basic relations of the Poisson nonlinear gauge theory which recently received some attention [20, 21] (with the evident correspondence \(D^{2,0}, D^{0,2} \leftrightarrow \partial_\mu; \omega^{1,-1,M}, -\omega^{-1,1,M} \leftrightarrow A_\mu^M, \mu = 1,2\)). The action (25) coincides in appearance with the general (non-topological) action of Poisson gauge theory [21]. The manifold \((q,u,v)\) can be interpreted as a kind of bi-harmonic extension of some Poisson manifold and the potential \(h^{2,2}[N,M]\) as a tensor field inducing the Poisson structure on this extension. We find it remarkable that the harmonic superspace action of torsionful \((4,4)\) sigma models deduced using an analogy with hyper-Kähler \((4,4)\) sigma models proved to be a direct harmonic counterpart of the nonlinear gauge theory action constructed in [21] by entirely different reasoning! We believe that this exciting analogy is a clue to the understanding of the intrinsic geometry of general \((4,4)\) sigma models with torsion.

To avoid a possible confusion, it is worth mentioning that the theory considered is not a supersymmetric extension of any genuine 2D gauge theory: there are no gauge fields in the multiplet of physical fields. The only role of gauge invariance (27) seems to consist in ensuring the correct number of the sigma model physical fields (4n bosonic and 8n fermionic ones).

It should be pointed out that it is the presence of the antisymmetric potential \(h^{2,2}[N,M]\) that makes the considered case nontrivial and, in particular, the gauge invariance (27) nonabelian. If \(h^{2,2}[N,M]\) is vanishing, the invariance gets abelian and the constraints (21) - (23) are identically satisfied, while (22) is solved by

\[
h^{1,3,M} = \nabla^{0,2}\Sigma^{1,1,M}(q,u,v), \quad h^{3,1,M} = \nabla^{2,0}\Sigma^{1,1,M}(q,u,v),
\]  

(29)

with \(\Sigma^{1,1,M}\) being an unconstrained prepotential. Then, using the target space gauge symmetry (24), one may entirely gauge away \(h^{1,3,M}, h^{3,1,M}\), thereby reducing (23) to the dual action of twisted \((4,4)\) multiplets (2). In the case of one triple \(q^{1,1}, \omega^{1,-1}, \omega^{-1,1}\) the potential \(h^{2,2}[N,M]\) vanishes identically, so the general action (13) for \(n = 1\) is actually equivalent to (2). Thus only for \(n \geq 2\) a new class of torsionful \((4,4)\) sigma models comes out. It is easy to see that the action (23) with non-zero \(h^{2,2}[N,M]\) does not admit any duality transformation to the form with the superfields \(q^{1,1,M}\) only, because it is impossible to remove the dependence on \(\omega^{1,-1,N}, \omega^{-1,1,N}\) from the equations for \(q^{1,1,M}\) by any local field redefinition with preserving harmonic analyticity. Moreover, in contradistinction to the constraints (1), these equations are compatible only with using the equation for \(\omega\)'s. So, the obtained system definitely does not admit in general any dual description in terms of twisted \((4,4)\) superfields. Hence, the left and right complex structures on the target space can be non-commuting. In the next section we will explicitly show this non-commutativity for a particular class of the models in question.
4. Harmonic Yang-Mills sigma models. Here we present a particular solution to the constraints (20)-(23). We believe that it shares many features of the general solution which is as yet unknown.

It is given by the following ansatz

\[
\begin{align*}
    h^{1,3, N} &= h^{3,1, N} = 0; \quad h^{2,2} = h^{2,2}(t, u, v), \quad t^{2,2} = q^{1,1 M} q^{1,1 M} ; \\
    h^{2,2} [N, M] &= b^{1,1} f^{NML} q^{1,1 L}, \quad b^{1,1} = b^i q^1 e_i^1, \quad b^i = \text{const},
\end{align*}
\]

where the real constants \( f^{NML} \) are totally antisymmetric. The constraints (20) - (22) are identically satisfied with this ansatz, while (23) is now none other than the Jacobi identity which tells us that the constants \( f^{NML} \) are structure constants of some real semi-simple Lie algebra (the minimal possibility is \( n = 3 \), the corresponding algebra being \( so(3) \)). Thus the (4, 4) sigma models associated with the above solution can be interpreted as a kind of Yang-Mills theories in the harmonic superspace. They provide the direct nonabelian generalization of the twisted multiplet sigma models with the action (2) which are thus analogs of two-dimensional abelian gauge theory. The action (25), related equations of motion and the gauge transformation laws (27) specialized to the case (30) are as follows

\[
S_{\sigma Y M} = \int \mu^{-2-2} \left\{ q^{1,1 M} \left( D^{0,2} \omega^{1,-1 M} + D^{2,0} \omega^{1,1 M} + b^{1,1} \omega^{1,1 L} \omega^{1,-1 N} f^{LNM} \right) \right. \\
\left. + h^{2,2}(q, u, v) \right\}
\]

\[
D^{2,0} \omega^{1,1 N} + D^{0,2} \omega^{1,-1 N} + b^{1,1} \omega^{1,1 S} \omega^{1,-1 T} f^{STN} \equiv B^{1,1 N} = -\frac{\partial h^{2,2}}{\partial q^{1,1 N}},
\]

\[
\begin{align*}
    D^{2,0} q^{1,1 M} + b^{1,1} \omega^{1,1 N} f^{NML} q^{1,1 L} &\equiv \Delta^{2,0} q^{1,1 M} = 0 \\
    D^{0,2} q^{1,1 M} - b^{1,1} \omega^{1,1 N} f^{NML} q^{1,1 L} &\equiv \Delta^{0,2} q^{1,1 M} = 0 \\
    \delta \omega^{1,-1 M} &= \Delta^{2,0} \sigma^{1,-1 M}, \quad \delta \omega^{1,1 M} = -\Delta^{0,2} \sigma^{1,-1 M}, \\
    \delta q^{1,1 M} &= b^{1,1} \sigma^{1,-1 N} f^{NML} q^{1,1 L}.
\end{align*}
\]

These formulas make the analogy with two-dimensional nonabelian gauge theory almost literal, especially for

\[
h^{2,2} = q^{1,1 M} q^{1,1 M}.
\]

Under this choice

\[
q^{1,1 N} = -\frac{1}{2} B^{1,1 N}
\]

by first of eqs. (32), then two remaining equations are direct analogs of two-dimensional Yang-Mills equations

\[
\Delta^{2,0} B^{1,1 N} = \Delta^{0,2} B^{1,1 N} = 0,
\]

and we recognize (31) and (32) as the harmonic counterpart of the first order formalism of two-dimensional Yang-Mills theory. In the general case \( q^{1,1 M} \) is a nonlinear function of \( B^{1,1 N} \), however for \( B^{1,1 N} \) one still has the same equations (35).

Now it is a simple exercise to see that in checking the integrability condition (14) one necessarily needs first of eqs. (32)

\[
[\Delta^{2,0}, \Delta^{0,2}] q^{1,1 M} = -b^{1,1} B^{1,1 N} f^{NML} q^{1,1 L} = 2b^{1,1} \frac{\partial h^{2,2}}{\partial t^{2,2}} q^{1,1 N} f^{NML} q^{1,1 L} \equiv 0.
\]
At the same time, in the abelian, twisted multiplet case this condition is satisfied without any help from the equation obtained by varying the action (3) with respect to $q^{1,1\ N}$. This property reflects the fact that the class of $(4,4)$ sigma models we have found cannot be described only in terms of twisted $(4,4)$ multiplets (of course, in general the gauge group has the structure of a direct product with abelian factors; the relevant $q^{1,1\ N}$s satisfy the linear twisted multiplet constraints (13)).

An interesting specific feature of this “harmonic Yang-Mills theory” is the presence of the doubly charged “coupling constant” $b^{1,1}$ in all formulas, which is necessary for the correct balance of the harmonic $U(1)$ charges. Since $b^{1,1} = b^{ia}u^a$, we conclude that in the geometry of the considered class of $(4,4)$ sigma models a very essential role is played by the quartet constant $b^{ia}$. When $b^{ia} \to 0$, the nonabelian structure contracts into the abelian one and we reproduce the twisted multiplet action (2). We shall see soon that $b^{ia}$ measures the “strength” of non-commutativity of the left and right complex structures.

Let us limit ourselves to the simplest case (34) and compute the relevant bosonic sigma model action and complex structures. We will do this to the first order in physical bosonic fields, which will be sufficient to show the non-commutativity of complex structures.

We first impose a kind of Wess-Zumino gauge with respect to the local symmetry (33). We choose it so as to gauge away from fields, which will be sufficient to show the non-commutativity of complex structures.

Then we substitute (36) into (31) with $h^{2,2}$ given by (34), integrate over $\theta$s and $u$s, eliminate infinite tails of decoupling auxiliary fields and, after this routine work, find the physical bosons part of the action (3) as the following integral over $x$ and harmonics $v$

$$S_{bos} = \int d^2x [dv] \left( \frac{i}{2} g^{0,-1 \ M}(x,v) \partial_- q^{0,1 \ M}(x,v) \right).$$  (37)

Here the fields $g$ and $q$ are subjected to the harmonic differential equations

$$\partial^{0,2} g^{0,-1 \ M} - 2(b^{ka}v^1_a) f^{MNL} q^{0,1 \ N} g^L_{k} = 4i \partial_+ q^{0,1 \ M}$$

$$\partial^{0,2} q^{0,1 \ M} - 2f^{MNL} (b^{ka}v^1_a) q^L_k = 0$$

and are related to the initial superfields as

$$q^{1,1 \ M}(\zeta, u, v) = q^{0,1 \ M}(x,v)u^i + \ldots, \quad g^{0,-1 \ N}(\zeta_R, v) = g^{0,-1 \ N}(x,v),$$

where $|$ means restriction to the $\theta$ independent parts.

To obtain the ultimate form of the action as an integral over $x^{++, x^{--}}$, we should solve eqs. (38), substitute the solution into (33) and do the $v$ integration. Here we solve (38) to the first non-vanishing order in the physical bosonic field $q^{ia \ M}(x)$ which appears as the first component in the $v$ expansion of $q^{0,1 \ M}$

$$q^{0,1 \ M}(x,v) = q^{ia \ M}(x)v^1_a + \ldots.$$
Representing (37) as

\[ S_{bos} = \int d^2x \left( G_{ia \, kb}^{\cal M \, L} \partial^+ q^{ia \, M} \partial^- q^{kb \, L} + B_{ia \, kb}^{\cal M \, L} \partial^+ q^{ia \, M} \partial^- q^{kb \, L} \right) \]  

(39)

where the metric \( G \) and the torsion potential \( B \) are, respectively, symmetric and skew-symmetric with respect to the simultaneous interchange of the left and right triples of their indices, we find that to the first order

\[ G_{ia \, kb}^{\cal M \, L} = \delta_{\cal M \, L}^{ia \, kb} \epsilon_{ik} \epsilon_{ab} - \frac{2}{3} \epsilon_{ik} f^{\cal M \, L} b_{(a} \tilde q_{b)}^N , \quad B_{ia \, kb}^{\cal M \, L} = \frac{2}{3} f^{\cal M \, L N} [b_{(ia} \tilde q_{b)}^N + b_{(ib} \tilde q_{a)}^N] . \]  

(40)

Note that an asymmetry between the indices \( ik \) and \( ab \) in the metric is an artefact of our choice of the WZ gauge in the form (36). One could choose another gauge so that a symmetry between the above pairs of \( SU(2) \) indices is restored. Metrics in different gauges are related via the target space \( q^{ia \, M} \) reparametrizations.

Finally, let us compute, to the first order in \( q^{ia \, M} \), the left and right complex structures associated with the sigma models at hand. Following the well-known strategy [8, 7, 16], we need: (i) to partially go on shell by eliminating the auxiliary fermionic fields; (ii) to divide four supersymmetries in every light-cone sector into a \( N = 1 \) one realized linearly and a triplet of nonlinearly realized extra supersymmetries; (iii) to consider the transformation laws of the physical bosonic fields \( q^{ia \, M} \) under extra supersymmetries. The complex structures can be read off from these transformation laws.

In our case at the step (i) we should solve some harmonic differential equations of motion to express an infinite tail of auxiliary fermionic fields in terms of the physical ones and the bosonic fields \( q^{ia \, M} \). The step (ii) amounts to the decomposition of the \((4, 0)\) and \((0, 4)\) supersymmetry parameters \( \epsilon^{ik} \) and \( \epsilon^{a2} \) as

\[ \epsilon^{ik} = \epsilon^{ik} \tilde \epsilon^{ik} + \frac{i}{2} \epsilon^{ik} \left( e^{ik} \right) ^{+} , \quad \epsilon^{a2} = \epsilon^{a2} \tilde \epsilon^{a2} + \frac{i}{2} \epsilon^{a2} \left( e^{a2} \right) ^{-} , \]

where we have kept a manifest symmetry only with respect to the diagonal \( SU(2) \) groups in the full left and right automorphism groups \( SO(4)_L \) and \( SO(4)_R \). At the step (iii) we should redefine the physical fermionic fields so that the singlet supersymmetries with the parameters \( \epsilon^{-} \) and \( \epsilon^{+} \) be realized linearly. We skip the details and present the final form of the on-shell supersymmetry transformations of \( q^{ia \, M}(x) \)

\[ \delta q^{ia \, M} = \epsilon^{+} \psi^{ia \, M} + i \epsilon^{(kj)} + \left( F_{(kj)} \right) ^{ia \, M} \psi_{+ L}^{kj} + \epsilon^{-} \chi^{ia \, M} + i \epsilon^{(cd)} - \left( \tilde F_{(cd)} \right) ^{ia \, M} \chi_{- L}^{cd} . \]  

(41)

Introducing the matrices

\[ F^{n}_{(+)} \equiv (\tau^n)^k \, F^{kj} \]  

\[ F^{m}_{(-)} \equiv (\tau^m)^q \, \tilde F^{qk} \]  

\( \tau^n \) being Pauli matrices, we find that in the first order in \( q^{ia \, M} \) and \( \tilde \tau^{ia} \)

\[ F^{n}_{(+)} = -i \tau^n \otimes I \otimes I + \frac{i}{3} [M_{(+)}, \tau^n \otimes I \otimes I] \]

\[ F^{n}_{(-)} = -i I \otimes \tau^n \otimes I + \frac{i}{3} [M_{(-)}, I \otimes \tau^n \otimes I] \]  

(42)
\[
\left( M(+) \right)_{ia M}^{kb N} = -2 f^{MLN} \left( b^{(i}_b q^{a L}_k \right) + b^{ia q}_L b^{a}_k \right) , \quad \left( M(-) \right)_{ia M}^{kb N} = 2 f^{MLN} b^{ia q}_L , \quad (43)
\]

where the matrix factors in the tensor products are arranged so that they act, respectively, on the subsets of indices \( i, j, k, ... \), \( a, b, c, ... \), \( M, N, L, ... \).

It is easy to see that the matrices \( F_{(\pm)}^n \) to the first order in \( q, b \) possess all the standard properties of complex structures needed for on-shell (4, 4) SUSY \([7, 8]\). In particular, they form a quaternionic algebra

\[
F_{(\pm)}^n F_{(\pm)}^m = -\delta^{nm} + \epsilon^{nms} F_{(\pm)}^s ,
\]

and satisfy the covariant constancy conditions

\[
D_{lc K} \left( F_{(\pm)}^n \right)_{ia M}^{kb N} = \partial_{lc K} \left( F_{(\pm)}^n \right)_{ia M}^{kb N} - \Gamma_{(\pm) le}^{jd T} \left( F_{(\pm)}^n \right)_{ia M}^{jd T} + \Gamma_{(\pm) le}^{ia M} \left( F_{(\pm)}^n \right)_{jd T}^{jd T} = 0
\]

with

\[
\Gamma_{(\pm) le}^{jd T} \equiv \Gamma_{le}^{jd T} + T_{le}^{jd T},
\]

where \( \Gamma \) is the standard Riemann connection for the metric \((40)\) and \( T \) is the torsion

\[
T_{ia M}^{kb N} l d T = \frac{1}{2} \left( \partial_{ia M} B_{kb l d}^{N T} + \text{cyclic} \right).
\]

It is also straightforward to check two remaining conditions of the on-shell (4, 4) supersymmetry (the hermiticity of the metric with respect to both sets of complex structures and the vanishing of the related Nijenhuis tensors). In the present case all these conditions are guaranteed to be fulfilled because we proceeded from a manifestly (4, 4) supersymmetric off-shell superfield formulation.

It remains to find the commutator of complex structures. The straightforward computation (again, to the first order in fields) yields

\[
\left[ F_{(\pm)}^n, F_{(-)}^m \right] = (\tau^n \otimes I \otimes I) M_{(-)}(I \otimes \tau^m \otimes I) M_{(-)}(\tau^n \otimes I \otimes I)
- (\tau^n \otimes \tau^m \otimes I) M_{(-)} - M_{(-)}(\tau^n \otimes \tau^m \otimes I) \neq 0 .
\]

(44)

Thus in the present case in the bosonic sector we encounter a more general geometry compared to the one associated with twisted (4, 4) multiplets. The basic characteristic feature of this geometry is the non-commutativity of the left and right complex structures. It is easy to check this property also for general potentials \( h^{2,2}(q, u, v) \) in \((31)\). It seems obvious that the general case \((25), (20) - (23)\) reveals the same feature. Stress once more that this important property is related in a puzzling way to the nonabelian structure of the analytic superspace actions \([8]\), \([3]\): the “coupling constant” \( b^{1,1} \) (or the Poisson potential \( h^{2,2}[M,N] \) in the general case) measures the strength of the non-commutativity of complex structures.

The main purpose of this Section was to explicitly show that in the (4, 4) models we have constructed the left and right complex structures on the bosonic target do not

\footnote{This implies, in particular, that a subclass of metrics associated with twisted (4, 4) multiplets, for dimensions \( 4n, n \geq 3 \), admits a deformation which preserves (4, 4) SUSY but makes the left and right complex structures non-commuting.}
commute. For full understanding of the geometry of these models, at least in the particular case discussed in this Section, and for clarifying its relation to the known examples, e.g., to the group manifold ones \[4\], we need the explicit form of the metrics and torsion potentials in (39). This amounts to finding the complete (non-iterative) solution to eqs. (38) and their generalization to the case of non-trivial potentials \(h_{2,2}(t,u,v)\) in (31). A work along this line is now in progress. We wish to point out that one of the merits of the off-shell formulation proposed lies in the fact that, like in the case of (4,4) sigma models without torsion \[2\] or (4,0) models \[6\], we can explicitly compute the bosonic metrics starting from the unconstrained superfield action (31) (or its generalization corresponding to the general solution of constraints (20) - (23)). These metrics are guaranteed to satisfy all the conditions of on-shell (4,4) supersymmetry listed in refs. \[7, 8\]. It is worth mentioning that the latter conditions, in their own right, do not provide us with any explicit recipe for computing the metrics.

Though we are not yet aware of the detailed properties of the corresponding bosonic metrics (singularities, etc.), in the particular case (30) we know some of their isometries. Namely, the action (31) and its bosonic part (39) (for any choice of \(h_{2,2}(t,u,v)\) in (30)) respect invariance under the global transformations of the group with structure constants \(f^{MN}L\). This suggests a link with the group manifold (4,4) models \[9\].

Our last comment in this Section concerns the relation with the paper \[16\]. Its authors studied a superfield description of (2,2) sigma models with non-commuting structures and found a set of nonlinear constraints on the Lagrangian which somewhat resemble eqs. (20) - (23). An essential difference of their approach from ours seems to consist in that it does not allow a smooth limiting transition to the case with commuting structures.

5. The action (13) as a gauge-fixed form of general \(q,\omega\) action. Here we show that one can come to the ansatz (13) with constraints (14), (18) starting from the most general analytic harmonic superspace action of superfields \(q^{1,1, N}, \omega^{1,-1, N}\) and systematically using in this action consequences of the general integrability condition (14) combined with a freedom of target space reparametrizations.

The most general action linear in the harmonic derivatives of the involved superfields is given by (45)

\[
S_{q,\omega} = \int \mu^{-2,-2} \left\{ H^{2,2} + H^{-1,1, M} D^{2,0} q^{1,1, M} + H^{1,-1, M} D^{0,2} q^{1,1, M} + H^{1,1, M} D^{2,0} \omega^{1,-1, M} + \tilde{H}^{1,1, M} D^{0,2} \omega^{-1,1, M} + H^{-1,3, M} D^{2,0} \omega^{1,-1, M} + H^{3,-1, M} D^{0,2} \omega^{-1,1, M} \right\}
\equiv \int \mu^{-2,-2} L_{q,\omega}^{2,2}(q,\omega,u,v),
\]

where \textit{a priori} all the potentials \(H\) are arbitrary functions of the superfields \(q^{1,1, M}, \omega^{1,-1, M}, \omega^{-1,1, M}\) and harmonics \(u,v\).

We will try to use the set of target space gauge invariances of the type inherent to hyper-Kähler (4,4) actions \[23, 4\] in order to reduce the number of independent potentials as much as possible.

One type of such invariances of the action (43) is related to reparametrizations of the involved superfields

\[
\delta q^{1,1, M} = \Lambda^{1,1, M}(q,\omega,u,v), \quad \delta \omega^{1,-1, M} = \Lambda^{1,-1, M}(q,\omega,u,v),
\]
\( \delta \omega^{-1,1} \quad = \quad \Lambda^{-1,1}(q, \omega, u, v) \). \hspace{1cm} (46)

It is straightforward to find the transformations of the potentials such that the action is form-invariant. Their explicit structure is not too enlightening.

Another type of invariance is similar to the hyper-Kähler one \[23, 4\] and is related to the freedom of adding full harmonic derivatives to the superfield Lagrangian in (45)

\[ \mathcal{L}_{q, \omega}^{2,2} \Rightarrow \mathcal{L}_{q, \omega}^{2,2} + D^{2,0} \Lambda^{0,2} + D^{0,2} \Lambda^{2,0}, \hspace{1cm} (47) \]

\[ \Lambda^{2,0} = \Lambda^{2,0}(q, \omega, u, v), \quad \Lambda^{0,2} = \Lambda^{0,2}(q, \omega, u, v). \]

Once again, it is easy to indicate how the potentials should transform to generate the shifts \[17\]. It will be important for our consideration that, assuming the existence of the flat limit (given by the action \[2\] with \( L^{2,2}(q, u, v) = q^{1,1} \omega^{1,1} \)), the full gauge freedom \[17\] can be fixed so that

\[ H^{-1,1} = \alpha \omega^{-1,1}, \quad H^{1,1} = \beta \omega^{1,1}, \quad H^{-1,1} = (1 + \alpha)q^{-1,1} + \tilde{H}^{1,1}, \quad (48) \]

\( \alpha, \beta \) being arbitrary parameters. In this gauge the action still contains four independent potentials, \( H^{2,2}, H^{-1,3}, H^{3,1}, \) and \( \tilde{H}^{1,1} \), and is invariant under the following target space gauge transformations which are a mixture of \[40\] and \[17\] (the unconstrained analytic parameters \( \Lambda^{2,0}, \Lambda^{0,2} \) below do not precisely coincide with those in eq. \[17\], but are related to them in a simple way)

\[ \delta \tilde{H}^{1,1} = -\Lambda^{1,1} + \partial \Lambda^{0,2} \partial \omega^{-1,1} + \Lambda^{1,1} \partial H^{-1,3} \partial \omega^{-1,1} + \Lambda^{1,1} \partial H^{1,1} \partial \omega^{-1,1} \]

\[ \delta H^{-1,3} = -\partial \Lambda^{0,2} \partial \omega^{-1,1} + \Lambda^{1,1} \partial H^{-1,3} \partial \omega^{-1,1} + \Lambda^{1,1} \partial H^{1,1} \partial \omega^{-1,1} \]

\[ \delta H^{3,1} = -\partial \Lambda^{0,2} \partial \omega^{-1,1} + \Lambda^{1,1} \partial H^{3,1} \partial \omega^{-1,1} \]

\[ \delta H^{2,2} = \partial \Lambda^{0,2} \partial \omega^{-1,1} + \Lambda^{1,1} \partial H^{2,2} \]

\[ + \Lambda^{1,1} \partial H^{-1,3} \partial H^3 \]

\[ \hspace{1cm} (50) \]

with

\[ \Lambda^{1,1} = \frac{\partial \Lambda^{2,0}}{\partial q^{1,1}} + \frac{\partial H^{3,1}}{\partial q^{1,1}} \Lambda^{-1,1} \]

\[ \Lambda^{1,1} = \frac{\partial \Lambda^{2,0}}{\partial q^{1,1}} - \frac{\partial H^{3,1}}{\partial q^{1,1}} \Lambda^{-1,1} \]

\[ \Lambda^{-1,1} = -(B^{-1})^{NM} \left\{ \frac{\partial \Lambda^{2,0}}{\partial q^{1,1}} - \frac{\partial \Lambda^{2,0}}{\partial q^{1,1}} \right\} \]

\[ B^{MN} = \delta^{MN} + \frac{\partial H^{1,1}}{\partial q^{1,1}} - \frac{\partial H^{3,1}}{\partial q^{1,1}} \frac{\partial H^{-1,3}}{\partial q^{1,1}} \], \( B^{MN}(B^{-1})^{NL} = \delta^{ML} \)
(one should add, of course, the coordinate transformations \([H]_0\) with the parameters \((\mathcal{H})\)). Note that in the case of general manifold \((M = 1, 2...n, n > 1)\) it is impossible to gauge away any of the surviving potentials with the help of this remaining gauge freedom, though one can still put them in the form similar to the normal gauge of the hyper-Kähler potential \(L^{(+4)} \) [1]. The fact that there remain three more potentials besides \(H^{2,2}\) (which is a direct analog of \(L^{(+4)}\)) is the essential difference of the considered case with torsion from the torsionless hyper-Kähler case. It is worth mentioning that upon the reduction to the \((4, 4)\) \(SU(2)\) harmonic superspace the superfields \(\omega^{1,-1}_N\) and \(\omega^{-1,1}_N\) in \([15]\) are identified with each other and recognized as the single superfield \(\omega\) to the \((4,4)\) actions. One recovers the \(\omega\) action of the general \((4,4)\) hyper-Kähler sigma model in some “flat” gauge. Note that the potentials in \([15], [19]\) will turn out to be severely constrained, so the reduction just mentioned actually produces some particular class of hyper-Kähler \((4,4)\) actions.

The equations of motion following from the action \([19]\) can be cast in the form

\[
D^{0,2}\omega^{1,-1}_M = -\frac{\partial H^{2,2}}{\partial q^{1,1}_M} - \left(\delta^{NM} + \frac{\partial H^{1,1}_N}{\partial q^{1,1}_M}\right)D^{2,0}\omega^{-1,1}_N - \frac{\partial H^{3,-1}_N}{\partial q^{1,1}_M}D^{2,0}\omega^{-1,1}_N - \frac{\partial H^{1,3}_N}{\partial q^{1,1}_M}D^{2,0}\omega^{1,-1}_N,
\]

\[
D^{0,2}q^{1,1}_M = T^{1,3}_M + T^{0,2}_M D^{2,0}\omega^{-1,1}_N + T^{2,0}_M D^{0,2}\omega^{-1,1}_N + T^{-2,4}_M D^{2,0}\omega^{-1,1}_N \equiv J^{1,3}_M.
\]

\[
D^{2,0}q^{1,1}_M = G^{3,1}_M + G^{2,0}_M D^{2,0}\omega^{-1,1}_N + G^{4,2}_M D^{0,2}\omega^{-1,1}_N + G^{0,2}_M D^{2,0}\omega^{1,-1}_N \equiv J^{3,1}_M.
\]

Here, the coefficient functions depend only on the potentials and their derivatives. It is a straightforward exercise to write down them explicitly. For simplicity, we do not give these expressions.

The commutativity condition \([14]\) in the present case gives rise to the following general integrability condition

\[
D^{2,0}J^{1,3}_M - D^{0,2}J^{3,1}_M = 0,
\]

which severely constrains the coefficient functions \(T\) and \(G\) in \(J^{1,3}_M, J^{3,1}_M\) and, further, the potentials through which these functions are expressed. By construction, this condition is covariant under the target space gauge group \([40], [41]\).

To extract the consequences of the integrability condition \([54]\), we should explicitly compute the action of harmonic derivatives on the potentials in \(J^{1,3}_N, J^{3,1}_N\), use once again the equations of motion \((52) - (54)\) to eliminate \(D^{0,2}q^{1,1}_M, D^{2,0}q^{1,1}_M\) and \(D^{0,2}\omega^{-1,1}_N\), and finally equate to zero the coefficients before independent structures in the obtained equality. These are the unity, the derivatives \(D^{2,0}\omega^{1,-1}_M, D^{0,2}\omega^{1,-1}_M, D^{2,0}\omega^{-1,1}_M, D^{0,2}\omega^{-1,1}_M\), all possible products of these derivatives, and the second-order derivatives \((D^{2,0})^2\omega^{1,-1}_M\), \((D^{0,2})^2\omega^{-1,1}_M\), \(D^{2,0}D^{0,2}\omega^{-1,1}_M\), \((D^{2,0})^2\omega^{-1,1}_M\). As a result we arrive at the set of constraints on the potentials \(H^{2,2}, H^{1,1}_N, H^{-1,3}_N, H^{3,-1}_N\). Since we started from the equations \((52) - (54)\) which respect the residual target space gauge freedom \((50), (51)\),
the set of integrability constraints is also covariant. Some of these constraints are covariant on their own, while others are mixed under (50). Instead of writing down the full set of constraints (it looks rather ugly), we will first discuss a few selected ones and show that they, being combined with the gauge freedom (50), (51), reduce the number of independent potentials to one $H^{2,2}$ and, respectively, the action (19) to (23).

As a first step we write down the constraint following from nullifying the coefficient before $(D^{0,2})^{2} \omega^{-1,1} M$

$$F^{4,-2} [M,N] = \frac{\partial H^{3,-1} M}{\partial \omega^{-1,1} N} + \frac{\partial H^{3,-1} M}{\partial q^{1,1} S} \frac{\partial H^{3,-1} N}{\partial \omega^{1,1} S} - (M \leftrightarrow N) = 0 . \quad (56)$$

It is not difficult to verify that this constraint is covariant with respect to (50), (51)

$$\delta F^{4,-2} [M,N] = \left( \frac{\partial \Lambda^{-1,1} T}{\partial \omega^{-1,1} M} + \frac{\partial \Lambda^{-1,1} T}{\partial q^{1,1} S} \frac{\partial H^{3,-1} M}{\partial \omega^{1,1} S} \right) F^{4,-2} [T,N] - (M \leftrightarrow N) . \quad (57)$$

Then it immediately follows that $H^{3,-1} M$ can be completely eliminated. Indeed, using gauge freedom (51), one can gauge away the totally symmetric parts of all the coefficients in the Taylor expansion of $H^{3,-1}$ in $\omega^{-1,1} N$. The remaining parts with mixed symmetry are zero in virtue of (56). Thus

$$H^{3,-1} M = 0 , \quad (58)$$

and the gauge function $\Lambda^{2,0}$ in (50), (51) gets restricted in the following way

$$\frac{\partial \Lambda^{2,0}}{\partial \omega^{-1,1} M} = 0 \Rightarrow \Lambda^{2,0} = \Lambda^{2,0}(q^{1,1}, \omega^{1,1}, u, v) . \quad (59)$$

With taking account of (58), the constraints which follow from vanishing of the coefficients before $D^{0,2} D^{2,0} \omega^{-1,1} N$, $(D^{2,0})^{2} \omega^{-1,1} N$ and $(D^{2,0})^{2} \omega^{1,-1} N$ in (54) are, respectively, of the form

$$F^{2,0} [M,N] \equiv \frac{\partial \hat{H}^{1,1} M}{\partial \omega^{-1,1} N} - \frac{\partial \hat{H}^{1,1} N}{\partial \omega^{-1,1} M} = 0 \quad (60)$$

$$F^{0,2} [M,N] \equiv (B^{-1})^{MS} \left( \frac{\partial \hat{H}^{1,1} S}{\partial \omega^{1,1} N} - \frac{\partial H^{-1,3} N}{\partial \omega^{1,1} S} \right) - (M \leftrightarrow N) = 0 \quad (61)$$

$$F^{-2,4} [M,N] \equiv \frac{\partial H^{-1,3} M}{\partial \omega^{1,1} N} - \frac{\partial H^{-1,3} N}{\partial \omega^{1,1} M} = 0 . \quad (62)$$

We will also need the constraint which comes from putting to zero the coefficient in front of the product $\left( D^{2,0} \omega^{1,-1} N \right) \left( D^{0,2} \omega^{-1,1} K \right)$

$$\frac{\partial}{\partial \omega^{-1,1} K} \left\{ (B^{-1})^{ML} \left( \frac{\partial \hat{H}^{1,1 L}}{\partial \omega^{1,1} N} - \frac{\partial H^{-1,3} N}{\partial \omega^{1,1} L} \right) \right\} = 0 . \quad (63)$$

The constraint (12) together with the gauge freedom associated with the parameter $\Lambda^{0,2}$ (still unrestricted) allow one to fully eliminate $H^{-1,3} M$

$$H^{-1,3} M = 0 . \quad (64)$$
Since the expression in the curly brackets in (63) does not depend on $\omega^{1,1-M}$, and its transformation law starts with the symmetric inhomogeneous term

$$-\frac{\partial^2 A^{2,0}}{\partial \omega^{1,-1-M} \partial \omega^{1,-1-N}},$$

the part of this expression which is symmetric in the indices $M,N$ can be gauged away. Then the constraint (61) requires the antisymmetric part also to vanish, whence

$$\frac{\partial \hat{H}^{1,1-M}}{\partial \omega^{1,-1-N}} = 0. \quad (65)$$

Finally, since $\hat{H}^{1,1-M}$ does not depend on $\omega^{1,-1-N}$, the residual target space gauge freedom supplemented with the constraint (60) is still capable to completely gauge away $\hat{H}^{1,1-M}$

$$\hat{H}^{1,1-M} = 0. \quad (66)$$

As the result of gauge fixings (58), (64) and (66), the general action (49) is reduced to (13). The remainder of consequences of the integrability condition (55) is reduced to eqs. (17), (18) already explored.

6. Conclusion. To summarize, proceeding from an analogy with the $SU(2)$ harmonic superspace description of $(4,4)$ hyper-Kähler sigma models, we have constructed off-shell $SU(2) \times SU(2)$ harmonic superspace actions for a new wide class of $(4,4)$ sigma models with torsion and non-commuting left and right complex structures on the bosonic target. The generality of this class has been proven by starting from the most general analytic superspace action of the analytic superfield triple $q^{1,1-N}, \omega^{1,-1-N}, \omega^{-1,1-N}$ which is the true analog of the pair $\omega^N, l^{(+2)}N$ of the hyper-Kähler case, and using the target space gauge invariance together with some consequences of the integrability condition (55).

The non-commutativity of target complex structures is directly related to the remarkable non-abelian Poisson gauge structure of the actions constructed. One of the most characteristic features of the general action is the presence of an infinite number of auxiliary fields and the lacking of dual-equivalent formulations in terms of $(4,4)$ superfields with finite sets of auxiliary fields. It would be interesting to see whether such formulations exist for some particular cases, e.g., those corresponding to the bosonic manifolds with isometries. An example of $(4,4)$ sigma model with non-commuting structures which admits such a formulation has been given in [17].

The obvious problems for further study are to compute the relevant metrics and torsions in a closed form and to try to utilize the corresponding manifolds as backgrounds for some superstrings. An interesting question is as to whether the constraints (20) - (23) admit solutions corresponding to the $(4,4)$ supersymmetric group manifold WZNW sigma models. The list of appropriate group manifolds has been given in [9]. The lowest dimension manifold with non-commuting left and right structures [17] is that of $SU(3)$. Its dimension 8 coincides with the minimal bosonic manifold dimension at which a non-trivial $h^{2,2} [M,N]$ in (23) can appear.

It still remains to prove that the action (25) indeed describes most general $(4,4)$ models with torsion. One way to do this is to start, like in the hyper-Kähler and quaternionic cases...
with the constrained formulation of the relevant geometry in a real $4n$ dimensional manifold and to reproduce the potentials in (25) as some fundamental objects which solve the initial constraints.

We note that the constrained superfield $q^{1,1;M}$ the dual action of which was a starting point of our construction, actually comprises only one type of $(4,4)$ twisted multiplet \cite{[14]}. There exist other types which differ in the $SU(2)_L \times SU(2)_R$ assignment of their components \cite{[1], [12]}. At present it is unclear how to simultaneously describe all of them in the framework of the $SU(2) \times SU(2)$ analytic harmonic superspace. Perhaps, their actions are related to those of $q^{1,1}$ by a kind of duality transformation. It may happen, however, that for their self-consistent description one will need a more general type of $(4,4)$ harmonic superspace, with the whole $SO(4)_L \times SO(4)_R$ automorphism group of $(4,4)$ 2D SUSY harmonized. The relevant actions will be certainly more general than those constructed in \cite{[13], [14]}

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