A NEW GENERALIZED PRIME RANDOM APPROXIMATION PROCEDURE AND SOME OF ITS APPLICATIONS

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Abstract. We present a new random approximation method that yields the existence of a discrete Beurling prime system \( \mathcal{P} = \{ p_1, p_2, \ldots \} \) which is very close in a certain precise sense to a given non-decreasing, right-continuous, nonnegative, and unbounded function \( F \). This discretization procedure improves an earlier discrete random approximation method due to H. Diamond, H. Montgomery, and U. Vorhauer [Math. Ann. 334 (2006), 1–36], and refined by W.-B. Zhang [Math. Ann. 337 (2007), 671–704].

We obtain several applications. Our new method is applied to a question posed by M. Balazard concerning Dirichlet series with a unique zero in their half plane of convergence, to construct examples of very well-behaved generalized number systems that solve a recent open question raised by T. Hilberdink and A. Neamah in [Int. J. Number Theory 16 05 (2020), 1005–1011], and to improve the main result from [Adv. Math. 370 (2020), Article 107240], where a Beurling prime system with regular primes but extremely irregular integers was constructed.

1. Introduction

In their seminal work [5], Diamond, Montgomery, and Vorhauer established the optimality of Landau’s abstract prime number theorem \(^1\) [10], partly solving so a long-standing conjecture of Bateman and Diamond [1, Conjecture 13B, p. 199]. One of the cornerstones in their arguments is a probabilistic construction, which they developed in order to produce discrete approximations to ‘continuous prime distribution functions’ by random generalized primes. Refinements were obtained by Zhang in [15] (cf. [6]). Their discrete random approximation result, from now on referred to as the DMVZ-method, may be summarized as follows.

Theorem 1.1 (Diamond, Montgomery, Vorhauer [5], Zhang [15]). Let \( f \) be a non-negative \( L^1_{\text{loc}} \)-function supported on \([1, \infty)\) satisfying

\[
(1.1) \quad f(u) \ll \frac{1}{\log u} \quad \text{and} \quad \int_1^\infty f(u) \, du = \infty.
\]

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\(^1\)Landau’s original statement is the well-known Prime Ideal Theorem, but his reasoning essentially leads to the first ever known abstract PNT [1, 9].
Then there exists an unbounded sequence of real numbers \(1 < p_1 < p_2 < \cdots < p_j < \cdots\) such that for any real \(t\) and any \(x \geq 1\)

\[
(1.2) \quad \left| \sum_{p_j \leq x} p_j^{-it} - \int_1^x u^{-it} f(u) \, du \right| \ll \sqrt{x} + \sqrt{\frac{x \log(|t| + 1)}{\log(x + 1)}}.
\]

The sequence arising from the DMVZ-method might be regarded as a Beurling prime system. Indeed, following Beurling [3] (cf. [1, 6]), a set of generalized prime numbers is simply an unbounded non-decreasing sequence of real numbers \(P = \{p_j\}_{j=1}^{\infty}\) subject to the only requirement \(p_1 > 1\). We denote as \(\pi_P(x)\) the function that counts the number of generalized primes not exceeding a given number \(x\). The function \(f\) can then be interpreted as a template ‘prime density measure’ \(dF(u) = f(u) \, du\), whose continuous ‘prime distribution function’ \(F(x) = \int_1^x f(u) \, du\) is unbounded and satisfies the Chebyshev upper bound \(\ll x/\log x\). The importance of the bound (1.2) lies in the fact that it is often strong enough for transferring many properties from \(\exp\left(\int_1^x x^{-s} \, dF(x)\right)\) into desired analytic properties of the Beurling zeta function associated to \(P\), that is,

\[
\zeta_P(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1}.
\]

We refer to the monograph [6] and the recent article [4] for relevant applications of the DMVZ-method.

The main goal of this paper is to establish a direct improvement to the DMVZ-method by obtaining a significantly stronger bound for the difference \(\pi_P - F\) than the one delivered by Theorem 1.1. In fact, setting \(t = 0\) in (1.2) yields \(\pi_P(x) - F(x) \ll \sqrt{x}\). We will show that it is possible to select the sequence \(P\) in such a way that the much better bound \(\pi_P(x) - F(x) \ll 1\) holds, as stated in the ensuing theorem, our main result. In addition, our discretization procedure can be applied to approximate measures \(dF\) that are not necessarily absolutely continuous with respect to the Lebesgue measure.

**Theorem 1.2.** Let \(F\) be a non-decreasing right-continuous function tending to \(\infty\) with \(F(1) = 0\) and satisfying the Chebyshev upper bound \(F(x) \ll x/\log x\). Then there exists a set of generalized primes \(P = \{p_j\}_{j=0}^{\infty}\) such that \(|\pi_P(x) - F(x)| \leq 2\) and such that for any real \(t\) and any \(x \geq 1\)

\[
(1.3) \quad \left| \sum_{p_j \leq x} p_j^{-it} - \int_1^x u^{-it} dF(u) \right| \ll \sqrt{x} + \sqrt{\frac{x \log(|t| + 1)}{\log(x + 1)}}.
\]

If in addition \(F\) is continuous, the sequence \(P\) can be chosen to be (strictly) increasing and such that \(|\pi_P(x) - F(x)| \leq 1\).

The proof of Theorem 1.2 will be given in Section 2. The essential difference between the DMVZ probabilistic scheme and our proof is that we make a completely different choice of how the generalized prime random variables are distributed in order to generate the discrete random approximations, allowing for a more accurate control on the size of the difference \(\pi_P(x) - F(x)\).

The rest of the article is devoted to illustrating the usefulness of Theorem 1.2 through three applications. In all these applications, the stronger bound \(\pi_P(x) - F(x) \ll 1\) instead
of \( \pi_p(x) - F(x) \ll \sqrt{x} \) plays a crucial role. Our first application concerns a question posed by M. Balazard (we consider a strengthened version of [12, Open Problem 24]):

**Question 1.3.** Does there exist a Dirichlet series \( \sum_{n=1}^{\infty} a_n n^{-s} \) which has exactly one zero in its half plane of convergence?

This question is motivated by the fact that if the Riemann hypothesis is true, the Dirichlet series

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},
\]

where \( \mu \) is the (classical) Möbius function, provides an example of such a Dirichlet series, since it would have a unique zero, namely at \( s = 1 \), in its half plane of convergence \( \{ s : \text{Re} s > 1/2 \} \). The idea is of course to find an unconditional example. We are not able to answer Question 1.3 here for Dirichlet series as in its statement, but, armed with Theorem 1.2, we will prove that Balazard’s question can be affirmatively answered for general Dirichlet series.

**Proposition 1.4.** There are an unbounded sequence \( 1 = n_0 < n_1 \leq n_2 \leq \cdots \leq n_k \leq \cdots \) and a general Dirichlet series of the form

\[
D(s) = \sum_{k=0}^{\infty} \frac{a_k}{n_k^s}, \quad \text{with } a_k \in \{-1, 0, 1\},
\]

such that \( D(s) \) has abscissa of convergence \( \sigma_c = 1/2 \) and has a unique zero on \( \{ s : \text{Re} s > 1/2 \} \), which is located at \( s = 1 \).

Our example of a general Dirichlet series satisfying the requirements of Proposition 1.4 arises from a Beurling prime system that we shall construct in Section 3. This example is actually the Beurling analog of the Dirichlet series (1.4). It turns out that the same constructed generalized primes yield a second application, as this generalized number system also provides a positive answer to a recent open problem raised by Hilberdink and Neamah (cf. [14, Section 4. Open Problem (1)]) on the existence of well-behaved Beurling number systems of a certain best possible type; see Section 3 for details.

As a third application, we conclude this article with an improvement to a result recently established by the authors together with G. Debruyne [4, Theorem 1.1]. In that paper, the authors showed the existence of a prime system with regular primes (i.e., satisfying the RH in the form of the PNT with remainder \( O(\sqrt{x}) \)) but with integers displaying extreme oscillation. Using Theorem 1.2, we will be able to improve in Section 4 the regularity of the primes from an \( O(\sqrt{x}) \) error term to one of order \( O(1) \), while still keeping the same irregularity of the integers.

### 2. The Main Result

This section is devoted to a proof of Theorem 1.2. Like in the DMVZ-method, our starting point is a probabilistic inequality for sums of random variables essentially due to Kolmogorov (see e.g. [11, Chapter V]), which we shall employ to bound the probability of certain events. The following inequality is a slight variant of [5, Lemma 8], and the proof given there can readily be adapted to yield the ensuing form of the lemma.
Lemma 2.1. For $1 \leq j \leq J$, let $X_j$ be independent random variables with $E(X_j) = 0$, $|X_j| \leq 2$, and $\text{Var}(X_j) = \sigma_j^2$. Let $S = \sum_{j=1}^{J} X_j$, and $\sigma^2 = \sum_{j=1}^{J} \sigma_j^2 = \text{Var}(S)$. Then
\[
P(S \geq v) \leq \begin{cases} 
\exp \left( -\frac{v^2}{4\sigma^2} \right) & \text{if } v \leq u_0\sigma^2; \\
\exp \left( -\frac{u_0 v}{4} \right) & \text{if } v > u_0\sigma^2.
\end{cases}
\]

Here $u_0$ is the positive solution of $e^u = 1 + u + u^2$, $u_0 \approx 1.79328$.

Proof of Theorem 1.2. Write $dF = dF_c + dF_d$, where $dF_c$ is a continuous measure, and $dF_d$ is purely discrete:
\[
dF_d = \sum_{n=1}^{\infty} \alpha_n \delta_{y_n}, \quad y_n > 1, \quad \alpha_n > 0,
\]
where $\delta_y$ denotes the Dirac measure concentrated at $y$ and the sequence $\{y_n\}_{n=1}^{\infty}$ consists of distinct points. We will discretize both measures separately\(^2\). Let us start with the continuous part.

Set $q_0 = 1$, $q_j = \min \{ x : F_c(x) = j \}$, for $j < j_{\text{max}}$, where $j_{\text{max}} = \infty$ if $F_c(\infty) = \infty$, and $j_{\text{max}} = \lceil F_c(\infty) \rceil$ if $F_c(\infty) < \infty$. Let $\{P_j\}_{1 \leq j < j_{\text{max}}}$ be a sequence of independent random variables, where $P_j$ is distributed on $(q_{j-1}, q_j]$ according to the probability measure $dF_c|_{(q_{j-1}, q_j]}$. Fix a number $t \in \mathbb{R}$ and set $X_{j,t} = \cos(t \log P_j)$. For such a fixed $t$, the $X_{j,t}$ are independent random variables with expectation
\[
E(X_{j,t}) = \int_{q_{j-1}}^{q_j} \cos(t \log u) \, dF_c(u)
\]
and variance $\text{Var}(X_{j,t}) \leq 1$. Let $C$ be a constant such that
\[
F_c(x) \leq C \frac{x}{\log(x+1)}.
\]

Let $J < j_{\text{max}}$ and suppose that $q_J / \log(q_J + 1) \geq \log(|t| + 1)$. Set $x = q_J$ and let $D = \max \{\sqrt{8C}, 8/u_0\}$, where $u_0$ is the number appearing in Lemma 2.1. Applying that lemma to the random variables $X_{j,t} - E(X_{j,t})$, with $v = D(\sqrt{x} + \sqrt{x \log(|t| + 1) / \log(x+1)})$, we get

\[
P \left( \sum_{j=1}^{J} \cos(t \log P_j) - \int_{1}^{x} \cos(t \log u) \, dF_c(u) \right) \geq D \left( \sqrt{x} + \sqrt{x \log(|t| + 1) / \log(x+1)} \right) \geq \max \left\{ \exp \left( -\frac{D^2}{4\sigma^2} \frac{x \log(|t| + 1)}{\log(x+1)} \right) : \exp \left( -\frac{u_0}{4} D \left( \sqrt{x} + \sqrt{x \log(|t| + 1) / \log(x+1)} \right) \right) \right\}.
\]

Here
\[
\sigma^2 = \sum_{j=1}^{J} \text{Var}(X_{j,t}) \leq J = F_c(x) \leq C \frac{x}{\log(x+1)} \quad \text{and} \quad \frac{x}{\log(x+1)} \geq \sqrt{\log(|t| + 1)}.
\]

\(^2\)d$F_d$ is already a purely discrete measure, but does not necessarily arise as the prime counting measure of a discrete Beurling prime system, since $\{y_n\}_{n=1}^{\infty}$ may have accumulation points, and since, even if this sequence happens to be discrete, we do not assume that the $\alpha_n$ are integers.
Hence the above probability is bounded by \((x + 1)^{-2}(|t| + 1)^{-2}\). Applying the same argument to the random variables \(-X_{j, t}, \pm Y_{j, t} = \pm \sin(t \log P_j)\), we get the same bounds for the corresponding probabilities.

Let

\[ S(x, t) = \sum_{P_j \leq x} P_j^{-it}, \quad S_c(x, t) = \int_1^x u^{-it} dF_c(u). \]

Then for \(x = q_j\) with \(x/\log(x + 1) \geq \log(|t| + 1)\)

\[ P\left( |S(x, t) - S_c(x, t)| \geq \sqrt{2D}\left(\sqrt{x} + \sqrt{x \log(|t| + 1)/\log(x + 1)}\right) \right) \leq \frac{4}{(x + 1)^2(|t| + 1)^2}. \]

Let \(j_t = \min\{j < j_{\text{max}} : q_j/\log(q_j + 1) \geq \log(|t| + 1)\}\), where we set \(j_t = \infty\) when the set is empty (which may happen if \(j_{\text{max}} < \infty\) and \(t\) is sufficiently large). Let \(A_{k,j}\) denote the event \(|S(q_j, k) - S_c(q_j, k)| \geq \sqrt{2D}(\sqrt{q_j} + \sqrt{q_j \log(k + 1)/\log(q_j + 1)})\). Since\(^3\)

\[ \sum_{k=1}^{\infty} \sum_{j_k \leq j < j_{\text{max}}} P(A_{k,j}) \leq \sum_{k=1}^{\infty} \sum_{j_k \leq j < j_{\text{max}}} \frac{4}{(q_j + 1)^2(|k| + 1)^2} \ll \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j^2 k^2} < \infty, \]

the Borel-Cantelli lemma implies that the probability that infinitely many of the events \(A_{k,j}\), \(k \geq 1, j_k \leq j < j_{\text{max}}\), occur, is zero. Fix now a point \(\omega\) of the probability space which is only contained in finitely many \(A_{k,j}\) with \(k \geq 1\) and \(j_k \leq j < j_{\text{max}}\) and set \(p_j = P_j(\omega)\). Then there exists a \(k_0 \geq 1\) such that for every \(k \geq k_0\) and any \(j\) with \(j_k \leq j < j_{\text{max}}\) (with now \(S(x, k) = S(x, k)(\omega)\))

\[(2.1) \quad S(q_j, k) - S_c(q_j, k) \ll \sqrt{q_j} + \sqrt{\frac{q_j \log(k + 1)}{\log(q_j + 1)}}. \]

Also by construction of the random variables \(P_j\), we have

\[ |\pi_p(x) - F_c(x)| \leq 1, \quad \text{where} \quad \pi_p(x) = \sum_{p_j \leq x} 1. \]

Let now \(1 \leq k < k_0\) and \(j < j_{\text{max}}\) arbitrary. Integrating by parts,

\[ |S(q_j, k) - S_c(q_j, k)| = \left| \int_1^{q_j} u^{-ik} d(\pi_p(u) - F_c(u)) \right| \ll 1 + k_0 \int_1^{q_j} \frac{du}{u} \ll \log q_j. \]

If \(k \geq k_0\) and \(j < \min\{j_k, j_{\text{max}}\}\), then

\[ |S(q_j, k) - S_c(q_j, k)| = \left| \sum_{l=1}^{j} \int_{q_{l-1}}^{q_l} (p_l^{-ik} - u^{-ik}) dF_c(u) \right| \leq 2F_c(q_j) \ll \sqrt{\frac{q_j}{\log(q_j + 1)}} \sqrt{\log(k + 1)}. \]

We conclude that the bound (2.1) holds for any \(k \geq 0\) and any \(j < j_{\text{max}}\).

\(^3\)If \(j_k = \infty\), the inner sum is zero by convention.
Suppose now that $k \geq 1$ and that for some $1 \leq j < j_{\text{max}}$, $x \in (q_{j-1}, q_j]$. Then

$$S(x, k) = S(q_{j-1}, k) + O(1) = S_c(q_{j-1}, k) + O\left(\sqrt{q_{j-1}} + \frac{q_{j-1} \log(k+1)}{\log(q_{j-1}+1)}\right) + O(1)$$

$$= S_c(x, k) + O\left(\sqrt{x} + \frac{x \log(k+1)}{\log(x+1)}\right).$$

If $j_{\text{max}} < \infty$ and $x > q_{j_{\text{max}}}$, then

$$S(x, k) = S(q_{j_{\text{max}}}, k) = S_c(q_{j_{\text{max}}}, k) + O\left(\sqrt{q_{j_{\text{max}}}} + \frac{q_{j_{\text{max}}} \log(k+1)}{\log(q_{j_{\text{max}}}+1)}\right)$$

$$= S_c(x, k) + O\left(\sqrt{x} + \frac{x \log(k+1)}{\log(x+1)}\right).$$

If $t \in [k, k+1]$ for some $k \geq 0$, then by integration by parts,

$$S(x, t) = \int_1^x u^{-i(t-k)} dS(u, k) = S(x, k) x^{-i(t-k)} + i(t-k) \int_1^x S(u, k) u^{-i(t-k)-1} du$$

$$= S_c(x, k) x^{-i(t-k)} + i(t-k) \int_1^x S_c(u, k) u^{-i(t-k)-1} du + O\left(\sqrt{x} + \frac{x \log(t+1)}{\log(x+1)}\right)$$

$$= S_c(x, t) + O\left(\sqrt{x} + \frac{x \log(t+1)}{\log(x+1)}\right).$$

Finally for negative $t$ we obtain the same bounds by taking the complex conjugate.

In order to discretize $dF_d$, we can apply the same idea, but with a slight modification, since it may not be possible to partition $[1, \infty)$ into disjoint intervals each having total mass 1. We proceed as follows. Set $q_0 = 1$, $q_j = \min\{x : F_d(x) \geq j\}$, for $1 \leq j < j_{\text{max}}$, where again $j_{\text{max}} = \infty$ if $F_d(\infty) = \infty$ and $j_{\text{max}} = [F_d(\infty)]$ if $F_d(\infty) < \infty$. Note that it may occur that $q_j = q_{j+1} = \ldots = q_{j+k}$ for some $k \geq 1$; we have $q_j < q_{j+1} \iff [F_d(q_j)] = j$. We will distribute the masses $\alpha_n$ over the intervals $[q_{j-1}, q_j]$, $0 \leq j < j_{\text{max}}$ in such a way that each interval $[q_{j-1}, q_j]$ has mass 1. At points $q_j$, where $F_d$ “spills over” the next integer (or next $k+1$ integers), we divide the mass $\alpha$ of the point $q_j$ as $\alpha = \beta + k + \gamma$, where $\beta$ is “given” to the interval $[q_{j-1}, q_j]$, and $\gamma$ is “given” to $[q_{j+k}, q_{j+k+1}]$. Making this precise, set $\alpha_0 = 0$ and if $\gamma_{j-1}$ is defined with $j < j_{\text{max}}$, define numbers $\beta_j, \gamma_{j+k} \in [0, 1]$ as

$$\beta_j = 1 - \gamma_{j-1} - \sum_{q_{j-1} < y_n < q_j} \alpha_n, \quad \gamma_{j+k} = F_d(q_{j+k}) - [F_d(q_{j+k})] = F_d(q_{j+k}) - (j+k),$$

where $k$ is the largest number (possibly zero) such that $q_j = q_{j+1} = \ldots = q_{j+k}$. Note that the sum over $\alpha$ can be empty (hence zero), but may also consist of infinitely many terms.

Let $\{P_j\}_{1 \leq j < j_{\text{max}}}$ be a sequence of independent discrete random variables, where $P_j$ is distributed according to the probability law

$$P(P_j = y_n) = \begin{cases} 
\gamma_{j-1} & \text{if } y_n = q_{j-1}, \\
\alpha_n & \text{if } q_{j-1} < y_n < q_j, \\
\beta_j & \text{if } y_n = q_j;
\end{cases}$$
in the case that \(q_{j-1} < q_j\), and \(P_j\) is distributed according to the trivial law \(P(P_j = q_j) = 1\) in the case that \(q_{j-1} = q_j\). Note that when \(q_{j-1} < q_j\) it can happen that \(q_{j-1}\) or \(q_j\) do not occur in the sequence \(\{y_n\}_{n=1}^\infty\), however in these cases one sees that \(\gamma_{j-1} = 0\) and \(\beta_j = 0\) respectively. Again we consider for fixed \(t\) the independent random variables \(X_{j,t} = \cos(t \log P_j)\). Let \(J < j_{\text{max}}\) be such that \(q_J < q_{J+1}\) or \(J = j_{\text{max}} - 1\). Suppose also that \(q_J / \log(q_J + 1) \geq \log(|t| + 1)\), and set \(x = q_J\). We apply Lemma 2.1 to the random variables \(X_{j,t} - E(X_{j,t})\); however, in this case

\[
\sum_{j=1}^J E(X_{j,t}) = \sum_{y_n \leq x} \alpha_n \cos(t \log y_n) - \gamma_J \cos(t \log q_J).
\]

Nevertheless, we can absorb the last term in the error term by multiplying it by 2:

\[
P\left(\sum_{j=1}^J X_{j,t} - \sum_{y_n \leq x} \alpha_n \cos(t \log y_n) \geq 2v\right) \leq P\left(\sum_{j=1}^J (X_{j,t} - E(X_{j,t})) \geq v\right)
\]

for \(v \geq 1\). Applying Lemma 2.1 with \(v = D'(\sqrt{x} + \sqrt{x \log(|t| + 1)/\log(x + 1)})\), where \(D' = \max\{\sqrt{8C'}, 8/u_0\}\), with \(C'\) a constant such that \(F_d(x) \leq C'x/\log(x + 1)\), we obtain

\[
P\left(\sum_{j=1}^J X_{j,t} - \sum_{y_n \leq x} \alpha_n \cos(t \log y_n) \geq 2D'\left(\sqrt{x} + \sqrt{x \log(|t| + 1)/\log(x + 1)}\right)\right) \leq \frac{1}{(x + 1)^2(|t| + 1)^2}.
\]

The proof can now be completed, mutatis mutandis, as in the continuous case. \(\Box\)

We now show that under the assumption that \(F\) is absolutely continuous on any finite interval, we can ensure that the approximating discrete primes are supported on strictly increasing sequences which tend to \(\infty\) sufficiently slowly, while still having the bound \(\pi_{\mathcal{P}}(x) - F(x) \ll 1\) instead of the weaker \(\pi_{\mathcal{P}}(x) - F(x) \ll \sqrt{x}\) delivered by the DMVZ-method. The following corollary is a direct improvement to [15, Lemma 4].

**Corollary 2.2.** Suppose \(f\) is a non-negative \(L^1_{\text{loc}}\)-function supported on \([1, \infty)\) and satisfying the conditions \((1.1)\). Let

\[1 < v_1 < \ldots < v_k < v_{k+1} < \ldots, \quad v_k \to \infty,
\]

be a sequence such that \(v_{k+1} - v_k \ll \log v_k\) and such that for any \(t \geq 0\)

\[
\sum_{v_k \geq h(t)} \frac{(v_k - v_{k-1})^2}{v_k \log v_k} \leq \frac{\log(t + 1)}{t}, \quad \text{where} \quad h(t) = \log(t + 1) \log \log(t + e).
\]

Then there exists a generalized prime system \(\mathcal{P} = \{p_j\}_{j=1}^\infty\) supported\(^4\) on the sequence \(\{v_k\}_{k=1}^\infty\) such that for any \(x \geq 1\) and any \(t\)

\[
(2.2) \quad \left|\pi_{\mathcal{P}}(x) - \int_1^x f(u) \, du\right| \ll 1 \quad \text{and} \quad \left|\sum_{p_j \leq x} p_j^{-it} - \int_1^x u^{-it} f(u) \, du\right| \ll \sqrt{x} + \sqrt{x \log(|t| + 1)/\log(x + 1)}.
\]

Some examples of admissible sequences are \(v_k = (\log(k + k_0))^a(\log \log(k + k_0))^b\) with \(0 < a < 1\) and \(b \in \mathbb{R}\) and \(v_k = \log(k + k_0)(\log \log(k + k_0))^b\) with \(b \leq 1\).

\(^4\)Strictly speaking, \(\{p_j\}_{j=1}^\infty\) needs not be a subsequence of \(\{v_k\}_{k=1}^\infty\), since some primes \(p_j\) may be repeated.
Proof. Write $dF(u) = f(u)\, du$. The idea of the proof is to construct an ‘intermediate’ measure $dG$ which is close to $dF$ and supported on the sequence $\{v_k\}_{k=1}^\infty$. The primes $p_j$ will then be obtained discretizing $dG$ by using Theorem 1.2.

We set $v_0 = 1$ and define the measure $dG$ as

$$dG = \sum_{k=1}^\infty \alpha_k \delta_{v_k}, \quad \text{where} \quad \alpha_k = \int_{v_{k-1}}^{v_k} dF.$$ 

By the first requirement on the sequence $\{v_k\}_{k=1}^\infty$ and the bound $dF(u) \ll du/\log u$, we have $G(x) - F(x) \ll 1$. Let now $t$ be arbitrary, and let $x$ be such that $x/\log(x + 1) < \log(|t| + 1)$. Then trivially

$$\left| \sum_{v_k \leq x} \alpha_k v_k^{-it} - \int_1^x u^{-it} \, dF(u) \right| \leq 2F(x) \ll \frac{x}{\log(x + 1)} \sqrt{\frac{x \log(|t| + 1)}{\log(x + 1)}}.$$

If on the other hand $x/\log(x + 1) \geq \log(|t| + 1)$, we proceed as follows:

$$\left| \sum_{v_k \leq x} \alpha_k v_k^{-it} - \int_1^x u^{-it} \, dF(u) \right| \ll 1 + \int_1^{v_L} dF(u) + \sum_{k=L+1}^{K} \int_{v_{k-1}}^{v_k} |v_k^{-it} - u^{-it}| \, dF(u).$$

Here $K$ is such that $v_K \leq x < v_{K+1}$, and $L$ is the largest integer $\leq K$ such that $v_L < h(|t|) = \log(|t| + 1) \log \log(|t| + \epsilon)$. Bounding $|v_k^{-it} - u^{-it}|$ by $|t| (v_k - v_{k-1})/v_k$ (note that $v_k/v_{k-1} \ll 1$) and using the bound $dF(u) \ll du/\log u$, we get

$$\left| \sum_{v_k \leq x} \alpha_k v_k^{-it} - \int_1^x u^{-it} \, dF(u) \right| \ll \frac{v_L}{\log(v_L + 1)} + |t| \sum_{v_k \geq h(|t|)} \frac{(v_k - v_{k-1})^2}{v_k \log v_k} \ll \log(|t| + 1) \leq \sqrt{\frac{x \log(|t| + 1)}{\log(x + 1)}},$$

where we used the second property of the sequence $\{v_k\}_{k=1}^\infty$ and $\log(|t| + 1) \leq x/\log(x + 1)$. Applying Theorem 1.2 to $G$ yields a sequence $\{p_j\}_{j=1}^\infty$ of primes satisfying (2.2) (by comparing with $dG$ via the triangle inequality). By construction of the discrete random variables in the proof of Theorem 1.2, the primes $p_j$ are contained in the support of $dG$, that is the sequence $\{v_k\}_{k=1}^\infty$. □

Remark 2.3. It is possible to generalize Theorem 1.2 to functions $F$ with different growth. Indeed, suppose that $F(x) \ll A(x)$, where $A$ is non-decreasing, has tempered growth, namely, $A(x) \ll x^n$ for some $n$, and satisfies

$$\int_1^x \frac{\sqrt{A(u)}}{u} \, du \ll \sqrt{A(x)}$$

(which implies $x^\delta \ll A(x)$ for some $\delta > 0$ depending on the implicit constant above). Then the conclusion of theorem holds if we replace the bound (1.3) by

$$\left| \sum_{p_j \leq x} p_j^{-it} - \int_1^x u^{-it} \, dF(u) \right| \ll \sqrt{A(x)} \left( \sqrt{\log(x + 1)} + \sqrt{\log(|t| + 1)} \right).$$

We remark that (2.3) is satisfied whenever $A$ is of positive increase (see [2, Theorem 2.6.1(b) and Definition of PI in p. 71]).
3. Balazard’s question for general Dirichlet series and the existence of well-behaved Beurling numbers of type \([0, 1/2, 1/2]\)

In this section we simultaneously give a proof of Proposition 1.4 and address an open question from [14]. In preparation, we need to introduce some concepts.

Let \(\mathcal{P} = \{p_j\}_{j=0}^{\infty}\) be a Beurling generalized prime system. The associated (multi)set of Beurling generalized integers \(\mathcal{N}\) is the semi-group generated by 1 and the numbers \(p_j\), which we arrange in a non-decreasing fashion (taking multiplicities into account): \(1 = n_0 < n_1 \leq n_2 \leq \ldots \leq n_k \leq \ldots\). Besides \(\pi_{\mathcal{P}}\) and \(\zeta_{\mathcal{P}}\), we can associate to the number system familiar number theoretic functions [6]. The counting function of the generalized integers is denoted as \(N_{\mathcal{P}}(x) = \sum_{n_k \leq x} 1\). As in classical number theory, one defines the Riemann prime counting function as

\[
\Pi_{\mathcal{P}}(x) = \sum_{\nu p^\nu \leq x} \frac{1}{\nu} = \sum_{\nu = 1}^{\infty} \frac{\pi_{\mathcal{P}}(x^{1/\nu})}{\nu}.
\]

The functions \(N_{\mathcal{P}}\) and \(\Pi_{\mathcal{P}}\) are then linked via the zeta function identity

\[
(3.1) \quad \zeta_{\mathcal{P}}(s) = \int_{1-}^{\infty} x^{-s} \, dN_{\mathcal{P}}(x) = \exp \left( \int_{1-}^{\infty} x^{-s} \, d\Pi_{\mathcal{P}}(x) \right).
\]

The Möbius function of the generalized number system is determined by its sum function \(M(x) = \sum_{n_k \leq x} \mu(n_k)\), where \(dM\) is defined as the (multiplicative) convolution inverse of \(dN\); equivalently,

\[
(3.2) \quad \sum_{k=1}^{\infty} \frac{\mu(n_k)}{n_k^s} = \frac{1}{\zeta_{\mathcal{P}}(s)}.
\]

Let us assume that \(\zeta_{\mathcal{P}}\) has abscissa of convergence 1. Following [5] Hilberdink and Neamah (cf. [14]), we define the three numbers \(\alpha, \beta, \gamma\) as the unique exponents (necessarily elements of \([0, 1]\)) for which the relations

\[
\Pi(x) = \text{Li}(x) + O(x^{\alpha+\varepsilon}),
\]

\[
N(x) = \rho x + O(x^{\beta+\varepsilon})
\]

\[
M(x) = O(x^{\gamma+\varepsilon})
\]

hold for some \(\rho > 0\) and for any \(\varepsilon > 0\), but no \(\varepsilon < 0\). Here we choose to normalize the logarithmic integral as

\[
\text{Li}(x) := \int_{1}^{x} \frac{1 - u^{-1}}{\log u} \, du.
\]

We then call such a Beurling generalized number system an \([\alpha, \beta, \gamma]\)-system. The main result [6] of [14] (see also [7]) tells us that \(\Theta = \max\{\alpha, \beta, \gamma\}\) is at least 1/2 and that at least two of these numbers must be equal to \(\Theta\). Hilberdink and Lapidus [8] call a Beurling number system well-behaved [7] if \(\Theta < 1\).

---

5 We count the primes using Riemann’s counting function \(\Pi\) instead of Chebyshev’s \(\psi\). An error term for \(\Pi\) can be transported to one for \(\psi\) at just the cost of an additional log-factor.

6 For this result it is imperative to consider discrete number systems, since it is obviously false for continuous ones: consider for example \(\Pi_0 = \text{Li}\), for which [6] \(N_0(x) = x\) and \(M_0(x) = 1 - \log x\), for an easy counterexample.

7 To ensure this it suffices to know that just two of the numbers are < 1, as we can deduce from [8, Theorem 2.3] and (the proof of) [14, Theorem 2.1].
The best possible types of well-behaved generalized numbers are then of type \([0, 1/2, 1/2], [1/2, 0, 1/2], [1/2, 1/2, 0]\). If the RH holds, then the rational integers are a \([1/2, 0, 1/2]\)-system, so that we have a candidate example of this instance. It is conjectured in [14] that there are no \([1/2, 1/2, 0]\)-systems. The following open question is also posed in [14, Section 4]: Does there exist a \([0, \beta, \beta]\) system with \(\beta < 1\)? The following theorem answers this question positively; we actually establish the existence of \([0, 1/2, 1/2]\)-systems.

**Theorem 3.1.** There is a discrete Beurling generalized prime system \(\mathcal{P}\) such that

\[
\Pi_\mathcal{P}(x) = \text{li}(x) + O(\log \log x),
\]

\[
N_\mathcal{P}(x) = x + O\left(x^{1/2}\exp(c(\log x)^{2/3})\right), \quad M_\mathcal{P}(x) = O\left(x^{1/2}\exp(c(\log x)^{2/3})\right),
\]

for some \(c > 0\), and

\[
N_\mathcal{P}(x) = x + \Omega_\varepsilon\left(x^{1/2-\varepsilon}\right), \quad M_\mathcal{P}(x) = \Omega_\varepsilon\left(x^{1/2-\varepsilon}\right),
\]

for any \(\varepsilon > 0\).

It follows at once that (3.2) for the Beurling number system from Theorem 3.1 furnishes an example of a general Dirichlet series having abscissa of convergence \(\sigma_\varepsilon = 1/2\) and with a unique zero in its half plane of convergence, namely, at \(s = 1\), which proves Proposition 1.4.

**Proof.** We apply Theorem 1.2 to \(F(x) = \text{li}(x)\), where \(\text{li}\) is such that \(\text{li}(x) = \sum_{\nu \geq 1} \text{li}(x^{1/\nu})/\nu\). A small computation shows that

\[
\text{li}(x) = \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \text{li}(x^{1/\nu}) = \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n(\zeta(n + 1)).}
\]

Here \(\zeta\) and \(\mu(\nu)\) are the classical Riemann-zeta and Möbius functions, respectively. The Chebyshev bound holds since

\[
\text{li}(x) \leq \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n} = \text{Li}(x) \leq 2\frac{x}{\log x}, \quad \text{if } x \gg 1.
\]

We thus find generalized primes \(\mathcal{P} : 1 < p_1 < p_2 < \ldots\) with \(\pi_{\mathcal{P}}(x) = \sum_{p_j \leq x} 1 = \text{li}(x) + O(1)\) and satisfying (1.3). To simplify the notation, we drop the subscript \(\mathcal{P}\) from all counting functions associated to this generalized prime system, but we make an exception with \(\zeta_{\mathcal{P}}(s)\) for which the subscript is kept in order to distinguish it from the Riemann zeta function \(\zeta(s)\). The Riemann prime counting function \(\Pi\) of \(\mathcal{P}\) satisfies

\[
\Pi(x) = \sum_{\nu=1}^{\log x} \frac{1}{\nu} \pi(x^{1/\nu}) = \sum_{\nu=1}^{\log x} \frac{1}{\nu} \left(\text{li}(x^{1/\nu}) + O(1)\right)
\]

\[
= \sum_{\nu=1}^{\infty} \sum_{n=1}^{\log x} \frac{(\log x)^n}{n!n\zeta(n + 1)} \nu^{-n} - \sum_{\nu=1}^{\log x} \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n\zeta(n + 1)} \nu^{n+1} + O(\log \log x)
\]

\[
= \text{Li}(x) + O(\log \log x).
\]

Also

\[
\text{Li}(x) = \text{li}(x) + O\left(\frac{\sqrt{x}}{\log x}\right), \quad \text{so } \Pi(x) = \pi(x) + O\left(\frac{\sqrt{x}}{\log x}\right).
\]
The bound $\Pi(x) - \text{Li}(x) \ll \log \log x$ implies that $Z(s) := \log \zeta_p(s) - \log(s/(s-1))$ has analytic continuation to the half plane $\sigma > 0$. By changing a finite number of primes, we may assume that $Z(1) = 0$, so that the corresponding integers have density 1. Using the bound (1.3) we can deduce good bounds for $Z$ in the half plane $\sigma > 1/2$, which allows one to deduce the asymptotic relations (3.5) via Perron inversion. The proof is essentially the same as that of Zhang’s theorem [15, Theorem 1], but we will repeat it for convenience of the reader.

We shift the contour to the line $\sigma = \sigma_x$. The bound $\Pi(\sigma_x) - \text{Li}(\sigma_x)$ may assume that

$$\Pi(x) - \text{Li}(x) \ll (\log x)^{1/3} \sqrt{\log(|t| + 1)}.$$

Let now $x$ be large but fixed. We want to derive an estimate for $N(x)$ by Perron inversion. Actually we will apply the Perron formula to $N_1(x) := \int_1^x N(u) \, du$, because then the Perron integral is absolutely convergent. Indeed, we have for any $\kappa > 1$ that

$$N_1(x) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{x^{s+1} \zeta_p(s)}{s(s+1)} \, ds.$$

One then uses the fact that $N$ is non-decreasing, so that

$$N_1(x) - N_1(x-1) \leq N(x) \leq N_1(x+1) - N_1(x).$$

Set $\sigma_x = 1/2 + (\log x)^{-1/3}$. Then uniformly for $\sigma \geq \sigma_x$,

$$|Z(s)| \leq C \left( (\log x)^{1/3} + (\log x)^{1/6} \sqrt{\log(|t| + 1)} \right).$$

We shift the contour to the line $\sigma = \sigma_x$. By the residue theorem (recall that $Z(1) = 0$):

$$N(x) \leq \frac{(x+1)^2}{2} - \frac{x^2}{2} + \frac{1}{2\pi i} \int_{\sigma_x-i\infty}^{\sigma_x+i\infty} \frac{(x+1)^{s+1} - x^{s+1})e^{Z(s)}}{(s-1)(s+1)} \, ds$$

$$= x + \frac{1}{2} + \frac{1}{2\pi i} \int_{\sigma_x-i\infty}^{\sigma_x+i\infty} \frac{(x+1)^{s+1} - x^{s+1})e^{Z(s)}}{(s-1)(s+1)} \, ds.$$
We split the range of integration into two pieces: $|t| \leq x$ and $|t| > x$. In the first piece we bound $(x + 1)^{s+1} - x^{s+1}$ by $|s + 1| x^s$, whereas in the second one by $x^s$. This gives

$$N(x) \leq x + \frac{1}{2} + O\left\{x^{1/2} \exp\left((\log x)^{2/3}\right) \int_0^x \frac{\exp\left(2C(\log x)^{2/3}\right)}{t + 1} \, dt + x^{3/2} \exp\left((\log x)^{2/3}\right) \int_x^\infty \exp\left(2C(\log x)^{1/6}\right) \frac{\exp(t^{1/2})}{t^{1/2}} \, dt \right\}$$

The first integral is bounded by $\exp\left(2C(\log x)^{2/3}\right) \log x$ and the second one by the term $x^{-1} \exp\left(2C(\log x)^{2/3}\right)$. A similar reasoning applies for a lower bound for $N$, and one sees that the asymptotic relation in (3.5) for $N$ holds with any $C > 2C + 1$.

To obtain the asymptotic behavior of $M$, we apply the same reasoning to $N(x) + M(x)$, which is also non-decreasing, and which has Mellin transform

$$\zeta_P(s) + \frac{1}{\zeta_P(s)} = \frac{s}{s - 1} \zeta(s) + \frac{s - 1}{s} e^{-Z(s)},$$

to show that $N(x) + M(x) = x + O\left(x^{1/2} \exp\left(c(\log x)^{2/3}\right)\right)$. The bound for $M$ in (3.5) then follows by combining this asymptotic estimate with that we have already obtained for $N$.

Finally, the oscillation estimates (3.6) follow at once from (3.5) and the result of Hilberdink and Neamah from [14] quoted above.

\[\square\]

**Remark 3.2.** We stress that the strong bound $\Pi_P(x) - \text{Li}(x) \ll \log \log x$ is crucial in the above arguments to generate the oscillation estimates (3.6). In particular, if only the weaker bound $\Pi_P(x) - \text{Li}(x) \ll \sqrt{x}$ had been known (like in Zhang’s generalized number system from [15, Theorem 1], whose construction is based upon application of the DMVZ-method), the Hilberdink and Neamah theorem could not have been used to exclude the possibility that the abscissa of convergence $\sigma_c$ of $\sum_{k=1}^\infty \mu(n_k)n_k^{-s}$ satisfies $\sigma_c < 1/2$ and that $1/\zeta_P(s)$ has additional zeros $s = \sigma + it$ with $\sigma_c < \sigma \leq 1/2$.

**Remark 3.3.** Let $\mathcal{P}$ be a generalized prime number system like in Theorem 3.1. Another example of a general Dirichlet series with abscissa of convergence $1/2$ and with a unique zero in the half plane $\sigma > 1/2$ is that of the Liouville function associated with the generalized number system. Its Liouville function, with sum function $L_\mathcal{P}(x) = \sum_{n_k \leq x} \lambda(n_k)$, can be defined via the identity

$$\sum_{k=0}^\infty \lambda(n_k) n_k^s = \frac{\zeta_P(2s)}{\zeta_P(s)},$$

so that its Dirichlet series has a zero at $s = 1$. Clearly, we have

$$L_\mathcal{P}(x) = \sum_{n_k^2 \leq x} M_\mathcal{P}(x/n_k^2) \ll x^{1/2} \exp(c(\log x)^{2/3}) \sum_{n_k \leq \sqrt{x}} \frac{1}{n_k} \ll x^{1/2} \exp(c(\log x)^{2/3}) \log x.$$  

Furthermore, the estimate (3.4) and (the proof of) [13, Proposition 19] imply

$$L_\mathcal{P}(x) = \Omega(\sqrt{x}),$$

which completes the proof of our claim.
Remark 3.4. The bound $\Pi_P(x) - \log x < \log \log x$ implies that $\zeta_P$ has meromorphic
continuation to $\sigma > 0$, and that it has one simple pole at $s = 1$ and no other zeros there.
The equality $\beta = \gamma = 1/2$ implies that both $\zeta_P$ and $1/\zeta_P$ must have infinite order in the
strip $0 < \sigma < 1/2$. (However, using convexity arguments one might show that $\zeta_P$ and $1/\zeta_P$
are of polynomial growth in the region $\sigma > 1/2 - 1/\log(|t| + 2)$.)

4. A Beurling number system with highly regular primes but integers with
large oscillation

In [4], the authors showed the existence of a Beurling prime system $P$ for which

\[ \pi_P(x) = \log(x) + O(\sqrt{x}) \quad \text{and} \quad N_P(x) = \rho x + \Omega(x \exp(-c \sqrt{\log x \log \log x})) \]

for any $c > 2\sqrt{2}$, where $\rho > 0$ is the asymptotic density of $N_P$. This was done by first
considering a continuous number system $(\Pi, N_c)$, for which $\Pi$ and $N_c$ have the desired
asymptotic behavior, and then discretizing this continuous system with the aid of a variant
of the DMVZ-method, supplemented with a specific technique to control the argument of
the zeta function. The continuous prime system is given\(^8\) by

\[ \Pi_c(x) = \log(x) + \sum_{k=0}^{\infty} R_k(x), \quad R_k(x) = \begin{cases} \sin(\tau_k \log x) & \text{for } \tau_k^{1+\delta_k} < x \leq \tau_k^{\nu_k}, \\ 0 & \text{otherwise.} \end{cases} \]

Here $\tau_k^{\infty}_{k=0}$ is a rapidly increasing sequence, $\delta_k = (\log \log \tau_k + a_k)/\log \tau_k$, and $\{a_k\}^{\infty}_{k=0}$ and
$\{\nu_k\}^{\infty}_{k=0} \subset (2, 3)$ are bounded sequences chosen such that $\Pi_c$ is (absolutely) continuous. For
a detailed analysis of this continuous example, we refer to [4], where additional technical as-
sumptions are imposed on the sequences $\{\tau_k\}^{\infty}_{k=0}$, $\{a_k\}^{\infty}_{k=0}$, and $\{\nu_k\}^{\infty}_{k=0}$ in order to achieve the
needed extremal behavior of its zeta function $\zeta_c(s) = \int_{1-}^{\infty} x^{-s} dN_c(x) = \exp \left( \int_{1-}^{\infty} x^{-s} d\Pi_c(x) \right)$.

We now show,

**Theorem 4.1.** There exist discrete Beurling prime systems $P_1$ and $P_2$ which satisfy

\[ \pi_{P_j}(x) = \log(x) + O(1) \quad \text{and} \quad \Pi_{P_j}(x) = \rho_j x + \Omega(x \exp(-c \sqrt{\log x \log \log x})), \]

and for any $c > 2\sqrt{2}$, and $j = 1, 2$,

\[ N_{P_j}(x) = \rho_j x + \Omega(x \exp(-c \sqrt{\log x \log \log x})), \]

where $\rho_j > 0$ is the asymptotic density of the generalized integer counting function $N_{P_j}$.

**Remark 4.2.** In view of its closeness to the ‘most natural’ continuous number system
$\Pi_0(x) = \log(x)$ and $N_0(x) = x$, the primes of the system $P_2$ might be considered to be more
regular than those of $P_1$. Note that the zeta function associated to $P_2$ has no zeros in the
right half plane $\{s : \Re s > 0\}$, as is also the case for the number system from Theorem 3.1.

**Proof.** The prime systems $P_1$ and $P_2$ will be obtained by applying (a variant of) Theorem
1.2 to $F = \Pi_c$ and $F = \pi_c$, respectively, where $\pi_c$ is such that $\Pi_c(x) = \sum_{n=1}^{\infty} \pi_c(x^{1/n})/n$. We have the formula

\[ \pi_c(x) = \log(x) + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} r_{k,n}(x), \]

\(^8\)An integer distribution function $N$ is uniquely determined by $\Pi$. Explicitly one has $dN = \exp^*(d\Pi)$, where the exponential is with respect to the multiplicative convolution of measures (see e.g. [6, Chapter 3]).
where (cf. Section 3)
\[
\text{li}(x) = \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!n\zeta(n+1)} \quad \text{and} \quad r_{k,n}(x) = \frac{\mu(n)}{n} R_k(x^{1/n}) = \begin{cases} \frac{\mu(n)}{n} \sin((\tau_k/n) \log x) & \text{for } (\tau_k^{1+\delta_k})^n < x \leq (\tau_k^n)^n, \\
0 & \text{otherwise.} \end{cases}
\]

Let us first verify that \( \Pi_c \) satisfies the requirements of Theorem 1.2. We only have to show that it is non-decreasing, the rest is clear. Set \( I_{k,n} = [ (\tau_k^{1+\delta_k})^n, (\tau_k^n)^n ] \), and let \( x \geq 1 \) be fixed. We have the following elementary observations:

a) for every \( n \geq 1 \) there exists at most one \( k \) such that \( x \in I_{k,n} \);

b) if \( n_1 \leq n_2 \) and \( k_1, k_2 \) are such that \( x \in I_{k_1,n_1} \cap I_{k_2,n_2} \), then \( k_1 \geq k_2 \).

The first observation is a consequence of \( \Pi \) that \( \mu(n) \) are non-decreasing, the rest is clear. Set \( I_{k,1} = \left[ (\tau_k^{1+\delta_k})^n, (\tau_k^n)^n \right] \), and let \( x \geq 1 \) be fixed. We have the following elementary estimates:

- if \( k \in I_{k,1} \) and \( k \geq 1 \), then \( \tau_k^{(1+\delta_k)\nu_k} \leq x \).
- if \( k \notin I_{k,1} \) and \( k \geq 2 \), then \( \tau_k^{(1+\delta_k)^n} \leq x \).

Now \( \tau_k \) is absolutely continuous, so it will follow that it is non-decreasing if we show that \( \tau_k \) is non-negative. We have

\[
\left| \left( \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} r_{k,n}(x) \right) \right| \leq \frac{1}{x} \sum_{k,n} \frac{\tau_k}{n^2}.
\]

Let \( m \) be the smallest integer \( \geq 1 \) such that \( x \in I_{k,m} \) for some \( k \), and let \( K \) be such that \( x \in I_{K,m} \). By a) and b), the above quantity is bounded by

\[
\frac{1}{x} \sum_{n=m}^{\infty} \frac{\tau_K}{n^2} \leq \frac{2\tau_K}{mx} \leq \frac{2}{m} \tau_{K^{-\delta_k}} = \frac{2}{me^{\nu_k \log \tau_K}},
\]

where the last inequality follows from the fact that \( x \in I_{K,m} \). Also,

\[
\text{li}'(x) \geq \frac{1}{\zeta(2)} \frac{1 - x^{-1}}{\log x} \geq \frac{1}{2\zeta(2) \log x} \geq \frac{1}{2m\zeta(2)^{\nu_k} \log \tau_K},
\]

if \( x \in I_{K,m} \). Hence \( \text{li}'(x) \geq 0 \) if we impose that \( a_k \geq \log(12\zeta(2)) \) say (recall that \( \nu_k \leq 3 \)).

If we now apply Theorem 1.2 with \( F = \Pi_c \) and \( F = \pi_c \), we obtain two prime systems \( P_1 \) and \( P_2 \) with \( \tau_{P_1}(x) = \Pi_c(x) + O(1) = \text{Li}(x) + O(1) \) and \( \tau_{P_2}(x) = \pi_c(x) + O(1) \), so that as in the proof of Theorem 3.1, \( \Pi_{P_2}(x) = \Pi_c(x) + O(\log \log x) = \text{Li}(x) + O(\log \log x) \). To deduce the oscillation estimate (4.1) from the behavior of the zeta functions \( \zeta_{P_1} \) and \( \zeta_{P_2} \), two additional properties alongside (1.3) were required, namely that

\[
S(x,t) - S_c(x,t) \ll \sqrt{x} (\log \tau_k)^{1/4}, \quad \text{uniformly for } |t - \tau_k| \leq \exp \left( d \left( \frac{\log \tau_k}{\log \log \tau_k} \right)^{1/3} \right),
\]

for a certain specified constant \( d > 0 \) and \( x \) and \( k \) large enough, and that

\[
\left| \arg \frac{\zeta_{P_1}(1 + it_{\tau_k})}{\zeta_{P_2}(1 + it_{\tau_k})} \right| < \frac{\pi}{80}, \quad \text{on an (infinite) subsequence } \{ \tau_k \} \text{ of } \{ \tau_k \}.
\]

Both of these additional properties can be obtained as in [4]. For the first one, one modifies the proof of Theorem 1.2 and also considers the events \( B_{k,j} \), corresponding to the violation of the above bound for \( x = q_j \) and \( t \) close to \( \tau_k \). Using the rapid growth of the sequence
\{\tau_k\}_{k=0}^\infty$, one then shows that the sum of the probabilities of the events $B_{k,j}$ is also finite. For the second property, one adds a finite number of well chosen primes around $80/\pi$, which shift the phase of the zeta function at the points $1 + i\tau_k$, to the desired range. We choose to omit further details and instead refer to [4, Section 6] for an account on both methods. □

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