INTEGRABLE GEOMETRIES AND MONGE-AMPÈRE EQUATIONS

BERTRAND BANOS

Abstract. In this lecture delivered at the Integrable and Quantum Field Theory at Peyresq sixth meeting, we review the Lychagin’s Monge-Ampère operators theory and exhibit the link it establishes between the classical problem of local equivalence for non-linear partial differential equations and the problem of integrability of some geometrical structures.

Introduction

A general approach to the study of non-linear Partial Differential Equations, which goes back to Sophus Lie, is to see a k-order equation on a n-dimensional manifold \( N^n \) as a closed subset in the manifold of k-jets \( J^kN \). In particular, a second-order differential equation lives in the space \( J^2N \). Nevertheless, as it was noticed by Lychagin in his seminal paper "Contact geometry and non-linear second-order differential equations" (1979), it is possible to decrease one dimension and to work on the contact space \( J^1N \) for a large class of second order PDE’s, containing quasi linear PDE’s and Monge-Ampère equations.

Moreover, for a large class of operators (those which admit a symmetry), we can replace the 1-jet space by the cotangent space and contact geometry by symplectic geometry. This study of differential operators becomes then the study of differential forms in the presence of a symplectic form.

The aim of this review paper is to use this Lychagin’s correspondence to show that it is possible to reconstruct the geometrical background starting from such a partial differential equation and to interpret the integrability of this geometry in terms of "integrability" of the equation.

In the first part, a brief review on the Monge-Ampère operator theory is given. Concepts of generalized solution and local equivalence are presented in geometric terms.

In the second part, the geometry of differential forms is studied. A unified approach is given in any dimensions and classification results in dimensions 2 and 3 are presented then.

In the last part, a link between 2D-Monge-Ampère equations and generalized complex geometry is described. Conservation laws and Generating functions are presented as generalized complex objects.

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1. Monge-Ampère operators theory

Let $M^n$ be a $n$-dimensional manifold, $T^*N$ its cotangent bundle and $\Omega$ the canonical symplectic structure on it. Locally,

$$\Omega = \sum_{i=1}^{n} dq_i \wedge dp_i,$$

with $(q_1, \ldots, q_n)$ coordinates on $M$. We denote by $\Omega^*(T^*M)$ the space of differential forms on the $2n$-dimensional manifold $T^*M$. For example, $\Omega \in \Omega^2(T^*M)$.

1.1. Monge-Ampère operators.

**DEFINITION 1.** Let $\omega \in \Omega^n(T^*M)$. The Monge-Ampère operator associated with $\omega$ is the differential operator

$$\Delta_\omega : C^\infty(M) \rightarrow \Omega^n(M)$$

defined by

$$\Delta_\omega(f) = (df)^*(\omega),$$

with $df : M \rightarrow T^*M$ the natural section defined by $f$.

**EXAMPLE 1.** Consider on $T^*\mathbb{R}^2$, the 2-form $\omega = dq_1 \wedge dp_2 - dq_2 \wedge dp_1$. We get

$$\Delta_\omega(f) = (f_{q_1} + f_{q_2}) dq_1 \wedge dq_2.$$  

**EXAMPLE 2.** Consider on $T^*\mathbb{R}^3$, the 3-form

$$\omega = dp_1 \wedge dq_2 \wedge dq_3 + dq_1 \wedge dp_2 \wedge dq_3 + dq_1 \wedge dq_2 \wedge dp_3 - dp_1 \wedge dp_2 \wedge dp_3.$$  

We get

$$\Delta_\omega(f) = (\Delta f - \text{hess}(f)) dq_1 \wedge dq_2 \wedge dq_3.$$  

We obtain a large class of non linear partial differential equations, characterized by their "determinant like" nonlinearity. This class is called the class of symplectic Monge-Ampère equations (SMAE). The term symplectic means that the corresponding form lives on the cotangent bundle. The whole class of Monge-Ampère equations is obtained with forms on the contact manifold $J^1M$.

Using this correspondence between SMAE and differential forms, we will see now how one can describe in geometric terms two classical notions in the study of PDE’s: the notion of generalized solution and the notion of local equivalence.

1.2. Generalized solutions.

**DEFINITION 2.** A generalized solution of a SMAE $\Delta_\omega = 0$ is a lagrangian submanifold $L^n$ of the symplectic manifold $(T^*M, \Omega)$ on which vanishes $\omega$:

$$\omega|_L = 0.$$  

A lagrangian submanifold which is a graph is the graph of a closed form. Hence, a generalized solution can be thought as a smooth patching of regular solutions.

**EXAMPLE 3.** On $\mathbb{R}^2$, a regular solution of the Laplace equation is of course an harmonic function. A generalized solution is a surface of $\mathbb{C}^2$ on which vanish $\Omega = \text{Re}(dz_1 \wedge dz_2)$ and $\omega = \text{Im}(dz_1 \wedge dz_2)$, that is a complex curve of $\mathbb{C}^2$. 

EXAMPLE 4. On $\mathbb{R}^3$, a generalized solution of the SMAE
\[ \Delta f - \text{hess}(f) = 0, \]
is a submanifold of $T^*\mathbb{R}^3 = \mathbb{C}^3$ on which vanish the symplectic form
\[ \Omega = \frac{i}{2} \sum_{j=1}^{3} dz_j \wedge d\bar{z}_j \]
and the 3-form
\[ \omega = \text{Im}(dz_1 \wedge dz_2 \wedge dz_3), \]
that is, a special lagrangian submanifold of $\mathbb{C}^3$.

For any form $\theta \in \Omega^{n-2}(T^*M)$, the form $\theta \wedge \Omega$ vanished on all lagrangian submanifolds of $T^*M$. The correspondence between SMAE and differential forms is therefore well defined up these particular forms. We introduce then the notion of primitive form:

DEFINITION 3. A $n$-form $\omega \in \Omega^n(T^*M)$ is said to be primitive if
\[ \omega \wedge \Omega = 0. \]

THEOREM (Hodge-Lepage-Lychagin). (1) any differential form $\omega \in \Omega^n(T^*M)$
admits an unique decomposition
\[ \omega = \omega_0 + \omega_1 \wedge \Omega \]
with $\omega_0$ primitive.

(2) if two primitive $n$-forms vanish on the same lagrangian submanifolds, then they are proportional.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Generalized solution}
\end{figure}
1.3. **Local equivalence.** Roughly speaking, two PDE’s are locally equivalent if they have the same solutions, up a change of dependent and independent coordinates. In Lychagin’s formalism, this notion becomes clear:

**DEFINITION 4.** Two SMAE $\Delta_{\omega_1} = 0$ and $\Delta_{\omega_2} = 0$ are said to be (locally) equivalent if there exists a (local) diffeomorphism $F : T^*M \to T^*M$ preserving the symplectic form, that is

$$F^*\Omega = \Omega,$$

and exchanging the two forms, that is

$$F^*\omega_1 = \omega_2.$$

Note that $F$ sends a generalized solution of $\Delta_{\omega_2} = 0$ on a generalized solution of $\Delta_{\omega_1} = 0$, but not necessarily a regular solution on a regular solution.

**EXAMPLE 5.** Let us consider on $\mathbb{R}^2$, the Monge-Ampère equation

$$f_{q_1q_1}f_{q_2q_2} - f^2_{q_1q_2} = 1,$$

corresponding to the form

$$\omega = dp_1 \wedge dp_2 - dq_1 \wedge dq_2.$$

Let $\phi : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$ be the partial Legendre transform

$$\phi(q_1, q_2, p_1, p_2) = (q_1, p_2, p_1, -q_2).$$

Since

$$\phi^*(\omega) = dq_2 \wedge dp_1 - dq_1 \wedge dp_2,$$

we see that our Monge-Ampère equation is equivalent to the Laplace equation. In theory, we can construct solutions using harmonic functions on $\mathbb{C}$. For example, consider the harmonic function $f(q_1, q_2) = e^{q_1} \sin(q_2)$. One can check that the graph $L_f$ of $df$ is sent by $\phi$ on a submanifold which is itself a graph:

$$\phi(L_f) = \{(t_1, t_2, g_{t_1}, g_{t_2})\}$$

with

$$g(t_1, t_2) = q_2 \arcsin(q_2e^{-t_1}) + \sqrt{e^{2t_1} - q_2^2}.$$  

This function $g$ is a non trivial solution of our Monge-Ampère equation.

2. **Geometry of differential forms**

Hence, the classical problem of local equivalence for Monge-Ampère equations can be understood as a problem of the Geometric Invariant Theory: the idea is to construct invariant structures which will characterize each equivalent class.

The first step of this approach is pointwise: we study the action of the symplectic group $SP(n\mathbb{R})$ on the space of primitive forms $\Lambda^n_0(\mathbb{R}^n)$. We will see next how one can ”integrate” such study.

2.1. **The bracket.** Let $V^{2n}$ be a $2n$ dimensional real vector space. We fix a symplectic form $\Omega$ on $V$ and the volume form

$$\text{vol} = \frac{\Omega^n}{n!}.$$

We denote by $\Lambda^n(V^*)$ the space of $n$-forms on $V$ and by $\Lambda^n_0(V^*)$ the space of primitive $n$-forms, that is

$$\Lambda^n_0(V^*) = \{\omega \in \Lambda^n(V^*), \, \Omega \wedge \omega = 0\}.$$
We denote by $SL(2n)$ the group of automorphisms preserving the volume form $\text{vol}$ and by $SP(n, \mathbb{R})$ the group of automorphisms preserving the symplectic form $\Omega$. Their Lie algebras are denoted by $sl(2n)$ and $sp(n, \mathbb{R})$.

Using the exterior product, we define an isomorphism $A : \Lambda^{2n-1}(V^*) \to V$ by

$$<\alpha, A(\theta)> = \frac{\alpha \wedge \theta}{\text{vol}}, \quad \text{for } \alpha \in \Lambda^1(V^*) \text{ and } \theta \in \Lambda^{2n-1}(V^*).$$

**DEFINITION 5.** The bracket $\Phi : \Lambda^n(V^*) \times \Lambda^n(V^*) \to sl(V)$ is defined by

$$\Phi(\omega_1, \omega_2)(X) = A((iX \omega_1) \wedge (-1)^n \omega_1 \wedge (iX \omega_2)).$$

It is straightforward to check the two following lemmas:

**LEMMA 1.** This bracket is invariant under the action of $SL(2n)$, that is

$$\Phi(F^* \omega_1, F^* \omega_2) = F^{-1} \circ \Phi(\omega_1, \omega_2) \circ F$$

for any $F \in SL(2n)$.

**LEMMA 2.** Let $\tilde{\Phi}$ be the bracket defined for the $(2n + 2)$ dimensional vector space $\tilde{V} = V \times \mathbb{R}_{t_1} \times \mathbb{R}_{t_2}$ endowed with the volume form

$$\text{vol} = \text{vol} \wedge dt_1 \wedge dt_2.$$

Then the following relations hold

1. $\tilde{\Phi}(\omega_1 \wedge dt_1, \omega_2 \wedge dt_2)(\partial_t) = -\tilde{\Phi}(\omega_1 \wedge dt_1, \omega_2 \wedge dt_2)(\partial_t)$
2. $\tilde{\Phi}(\omega_1 \wedge dt_1, \omega_2 \wedge dt_2)(X) = \Phi(\omega_1, \omega_2)(X), \quad \forall X \in V$.

Note that this second lemma shows that $\Phi$ takes its values in $sl(2n)$.

In the case $n = 3$, the tensor $K_\omega = \frac{1}{2} \Phi(\omega, \omega)$ is the invariant constructed by Hitchin in [9], which can be easily extended to any odd $n$:

**PROPOSITION 1** (Hitchin). When $n$ is odd, the map $K : \Lambda^n(V^*) \to sl(2n)$, $\omega \mapsto \frac{1}{2} \Phi(\omega, \omega)$ is a moment map for the hamiltonian action of $SL(2n)$ on $\Lambda^n(V^*)$ endowed with the symplectic form

$$\Theta(\omega_1, \omega_2) = \frac{\omega_1 \wedge \omega_2}{\text{vol}}.$$

When $n$ is even, the bracket $\Phi$ is antisymmetric and the situation is completely different. The analog of [1] is the following, which is proved in [2] (in preparation):

**PROPOSITION 2.** We define on $\Lambda^n(V^*) \times sl(2n)$ the following bracket:

1. $[A_1, A_2] = A_1 A_2 - A_2 A_1$
2. $[A_1, \omega] = L_A(\omega)$
3. $[\omega_1, \omega_2] = \Phi(\omega_1, \omega_2)$

for $A, A_1$ and $A_2$ in $sl(2n)$ and $\omega, \omega_1$ and $\omega_2$ in $\Lambda^n(V^*)$.

Then $[\ , \ ]$ is a Lie bracket.

The $SP(n, \mathbb{R})$-version of these results is summed up in the following:

**PROPOSITION 3.**

1. If $\omega_1$ and $\omega_2$ are primitive then $\Phi(\omega_1, \omega_2) \in sp(n, \mathbb{R})$.
2. If $n$ is odd, then $K : \Lambda^n_0(V^*) \to sp(n, \mathbb{R})$, $\omega \mapsto \frac{1}{2} \Phi(\omega, \omega)$ is a moment map for the hamiltonian action of $SP(n, \mathbb{R})$ on the symplectic subspace $\Lambda^n_0(V^*)$ of $\Lambda^n(V^*)$.
3. If $n$ is even, the space $\Lambda^n_0(V^*) \oplus sp(n, \mathbb{R})$ is a Lie subalgebra of $\Lambda^n(V^*) \oplus sl(2n)$. 
Remark. When $n$ is odd, the tensor $K_\omega$ defines a family of scalar invariants
\[ a_k = \text{trace}(K_\omega^{2k}), \quad k \in \mathbb{N} \]
and a quadratic form called the Lychagin-Roubtsov quadratic form:
\[ q_\omega(X) = \Omega(K_\omega X, X). \]

When $n$ is even, the adjoint operator $ad_\omega = [\omega, \cdot]$ defines an endomorphism
\[ ad_\omega^2 : sp(n, \mathbb{R}) \to sp(n, \mathbb{R}) \]
which gives also a family of scalar invariants
\[ a_k = \text{trace}(ad_\omega^{2k}), \quad k \in \mathbb{N} \]
and a symmetric polynomial of degree 4 defined by
\[ q_\omega(X) = \text{trace}([ad_\omega^2(X \otimes \iota X(\Omega))]^2). \]

2.2. Examples.

2.2.1. $n = 2$. The identity $\omega = \Omega(A_\omega \cdot, \cdot)$ gives an isomorphism between the space of 2-forms $\Lambda^2(\mathbb{R}^4)$ and the Jordan algebra $Jor(\Omega)$ defined by
\[ Jor(\Omega) = \{ A \in gl(4), \Omega(A \cdot, \cdot) = \Omega(\cdot, A \cdot) \}. \]

Our bracket $\Phi$ becomes then the usual bracket:
\[ \Phi(\omega_1, \omega_2) = A_{\omega_1}A_{\omega_2} - A_{\omega_2}A_{\omega_1}. \]

We easily see then the isomorphism of Lie algebras
\[ \Lambda^2_0(\mathbb{R}^4) \oplus sp(2, \mathbb{R}) = sl(4, \mathbb{R}). \]

Moreover, for $\omega \in \Lambda^2_0(\mathbb{R}^4)$, the endomorphism $ad_\omega^2 = ad_\omega^2 : sp(2, \mathbb{R}) \to sp(2, \mathbb{R})$ satisfies
\[ \text{trace}(ad_\omega^2) = 16 \text{pf}(\omega) \]
where the pfaffian of $\omega$ is the classical invariant
\[ \text{pf}(\omega) = \frac{\omega \wedge \omega}{\Omega \wedge \Omega}. \]

The polynomial $q_\omega$ is the null polynomial.

2.2.2. $n = 3$. It is proved in [9] that the action of $GL(6, \mathbb{R})$ on $\Lambda^3(\mathbb{R}^3)$ has two opened orbits separated by the hypersurface $\lambda = 0$ where
\[ \lambda(\omega) = \frac{1}{6} \text{trace}(K_\omega^2). \]

Note that, for any 3-form the following holds:
\[ K_\omega^2 = \lambda(\omega) \cdot Id. \]

By analogy with the 2-dimensional case, we call this invariant the Hitchin pfaffian. A 3-form with a non vanishing Hitchin pfaffian is said to be nondegenerate.

For a primitive form $\omega$, we get a triple $(g_\omega, K_\omega, \Omega)$ with $g_\omega = \Omega(K_\omega \cdot, \cdot)$ the Lychagin-Roubtsov metric (see [13], [1]). This triple defines a $\epsilon$-Kähler structure in the sense of [15], that is, the tensor $K_\omega$ satisfies, up a renormalization,
\[ K_\omega^2 = \epsilon, \quad \text{with } \epsilon = 0, 1, -1. \]

Note that the Lychagin-Roubtsov metric has signature.

Moreover, in the nondegenerate case the form $\omega$ admits an unique dual form $\hat{\omega}$, such that $\omega + \sqrt{\epsilon} \hat{\omega}$ and $\omega - \sqrt{\epsilon} \hat{\omega}$ are the volume forms of the two-eigenspaces of the Hitchin tensor $K_\omega$. Saying differently, to each nondegenerate primitive forms corresponds a $\epsilon$-Calabi-Yau structure.
2.2.3. $n = 4$. The Lie algebras $\Lambda^4(\mathbb{R}^8) \oplus sl(8, \mathbb{R})$ and $\Lambda^4_0(\mathbb{R}^8) \oplus sp(4, \mathbb{R})$ are known to be isomorphic to the exceptional Lie algebras $E_7$ and $E_6$ (see [16]). Moreover, it is proved in [11] that the family $\{a_k = \text{trace}(\text{ad}^2_k \omega)\}_{k \in \mathbb{N}}$ forms a complete family of invariants.

Nevertheless, computations in these dimensions are extremely complicated. The author plans to implement an algorithm which could give in a reasonable time these invariants $a_k$.

It is worth mentioning that, on many examples, the symmetric polynomial $q_\omega$ of degree 4 is the square of a quadratic form. Is it always true? A positive answer would be extremely useful to understand the geometry of PDE’s of Monge-Ampère type in 4 variables.

2.3. Classifications results.

2.3.1. Monge-Ampère equations in 2 and 3 variables. The action of the symplectic linear group on $2D$ and $3D$ symplectic Monge-Ampère equations with constant coefficients has a finite number of orbits and we know all of them as it is shown in tables 1 and 2 (see [13] and [2]).

\[ \Delta \omega = 0 \]
\[ \omega \]
\[ \text{pf}(\omega) \]

\begin{array}{|c|c|c|}
\hline
\Delta f = 0 & dq_1 \land dp_2 - dq_2 \land dp_1 & 1 \\
\Box f = 0 & dq_1 \land dp_2 + dq_2 \land dp_1 & -1 \\
\partial^2 f \partial q_2^2 = 0 & dq_1 \land dp_2 & 0 \\
\hline
\end{array}

Table 1. Classification of SMAE in 2 variables

\[ \Delta_\omega = 0 \]
\[ \text{signature}(q_\omega) \]
\[ \lambda(\omega) \]

\begin{array}{|c|c|c|}
\hline
1 & \text{hess}(f) = 1 & (3, 3) & 1 \\
2 & \Delta f - \text{hess}(f) = 0 & (0, 6) & -1 \\
3 & \Box f + \text{hess}(f) = 0 & (4, 2) & -1 \\
4 & \Delta f = 0 & (0, 3) & 0 \\
5 & \Box f = 0 & (2, 1) & 0 \\
6 & \Delta q_2,q_3 f = 0 & (0, 1) & 0 \\
7 & \Box q_2,q_3 f = 0 & (1, 0) & 0 \\
8 & \partial^2 f \partial q_2^2 = 0 & (0, 0) & 0 \\
\hline
\end{array}

Table 2. Classification of SMAE in 3 variables

Remark. (1) in two variables, any SMAE with constant coefficients is linearizable, that is equivalent to a linear PDE. Moreover, the pfaffian distinguishes the different orbits.

(2) in three variables, there exist nonlinearizable SMAE with constant coefficients and they correspond to nondegenerate primitive 3-forms. Moreover, the Hitchin pfaffian does not distinguish the different orbits but so does the signature of the Lychagin-Roubtsov metric.

In 4 variables, the action of the symplectic group is not discrete anymore and there is no hope to obtain an exhaustive list as in 2 or 3 variables. Moreover, it appears that on many interesting examples, the associated geometry is completely degenerated. Saying differently, in 4 variables appears the notion of non linear but degenerated
Monge-Ampère equation. In the table 3, we have computed the polynomial invariant $q_\omega$ for the following examples:

\[
\begin{align*}
\text{hess}(u) &= 1 \quad \text{(Usual Monge-Ampère equation)} \\
\text{hess}(u) - \left( \sum_{i<j} u_{q_i q_j} - u^2_{q_i} \right) + 1 &= 0 \quad \text{(4D SLAG equation)} \\
q_\omega &= 1 \quad \text{(Plebanski I equation)} \\
q_\omega &= 0 \quad \text{(Plebanski II equation)} \\
q_\omega &= 0 \quad \text{(Grant equation)}
\end{align*}
\]

| $\Delta_\omega = 0$ | $q_\omega$ |
|---------------------|-----------|
| usual Monge-Ampère | $(dq_1 dp_1 + dq_2 dp_2 + dq_3 dp_3 + dq_4 dp_4)^2$ |
| SLAG                | $(dq_1^2 + dq_2^2 + dq_3^2 + dp_1^2 + dp_2^2 + dp_3^2 + dp_4^2)^2$ |
| Plebanski I         | 0         |
| Plebanski II        | $dq_1^4$  |
| Grant               | 0         |

Table 3. Examples of SMAE in 4 variables

2.3.2. Note on ellipticity of Monge-Ampère equations. Recall that a second order linear partial differential equation

\[
\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0
\]

is said to be

(1) elliptic if the symmetric matrix $A$ has signature $(n, 0)$ or $(0, n)$,

(2) hyperbolic if the symmetric matrix $A$ has signature $(n-1, 1)$ or $(1, n-1)$,

(3) parabolic if the symmetric matrix $A$ is degenerate.

Following Harvey and Lawson ([HL]), we will say that a Monge-Ampère equation is elliptic, hyperbolic or parabolic if it is so in a first order approximation. More precisely, let $\Delta_\omega = 0$ be a Monge-Ampère equation and $\phi$ a solution. The linearization of $\Delta_\omega$ at $\phi$ is the linear differential operator

\[
D_\phi(\Delta_\omega)(u) = \frac{d}{dt} \big|_{t=0} \Delta_\omega(\phi + tu).
\]

The equation $\Delta_\omega = 0$ is said to be elliptic, hyperbolic or parabolic at the point $\phi$ if its linearization $D_\phi(\Delta_\omega) = 0$ is elliptic, hyperbolic or parabolic.

**EXAMPLE 6.** Let us consider a generic 2D SMAE

\[
A + B \psi_{xx} + 2C \psi_{xy} + D \psi_{yy} + E(\psi_{xx} \psi_{yy} - \psi_{xy}^2) = 0.
\]

Its linearization at $\phi$ is

\[
(B + E \phi_{yy}) u_{xx} + 2(C - E \phi_{xy}) u_{xy} + (D + E \phi_{xx}) u_{yy} = 0.
\]

and since

\[
\begin{align*}
&B + E \phi_{yy} \\
&C - E \phi_{xy} \\
&D + E \phi_{xx}
\end{align*}
\]

\[
= BD - C^2 + E(B \phi_{xx} + 2C \phi_{xy} + D \phi_{yy} + E(\phi_{xx} \phi_{yy} - \phi_{xy}^2))
\]

\[
= BD - C^2 - AE,
\]
we deduce that our 2D-SMAE is elliptic for instance if and only if and only if its corresponding primitive form has positive pfaffian everywhere.

**EXAMPLE 7.** Let us consider now the 3D special lagrangian equation

$$\Delta \psi - \det(\text{Hess} \, \psi) = 0.$$ 

Its linearization at a point \( \phi \) is

$$\sum_{i,j=1}^{3} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

with

$$A = I_3 - \Phi^*$$

where \( I_3 \) denotes the matrix identity, \( \Phi \) denotes the hessian matrix of \( \phi \) and \( \Phi^* \) denotes its comatrix (the matrix of cofactors).

Choose a basis in which \( \Phi = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \) with \( \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 \lambda_2 \lambda_3 \). You get

$$A = \begin{pmatrix} 1 - \lambda_2 \lambda_3 & 0 & 0 \\ 0 & 1 - \lambda_1 \lambda_3 & 0 \\ 0 & 0 & 1 - \lambda_1 \lambda_2 \end{pmatrix}$$

$$= (1 - \lambda_1 \lambda_2 - \lambda_1 \lambda_3 - \lambda_2 \lambda_3) \begin{pmatrix} \frac{1}{1+\lambda_1} & 0 & 0 \\ 0 & \frac{1}{1+\lambda_2} & 0 \\ 0 & 0 & \frac{1}{1+\lambda_3} \end{pmatrix}$$

The 3D special lagrangian equation is therefore elliptic everywhere.

The following formula generalizes this last example and gives a relation between the linearization of a 3D-SMAE, the Lychagin-Roubtsov metric and, surprisingly, its dual equation:

**PROPOSITION 4.** The linearisation at a point \( \phi \) of a 3D-SMAE \( \Delta \omega = 0 \) is

$$D_{\phi}(\Delta \omega)(u) = \sum_{i,j=1}^{3} B_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

with

$$B = \Delta \omega(\phi) \cdot (g_{\omega}^{-1})|_{L_0}.$$ 

2.4. Integration of the classification. After this linear approach, the next step is to try to integrate the classification: when is a given MAE \( \Delta \omega = 0 \) equivalent to a MAE with constant coefficient \( \Delta \omega_c = 0 \)? One can try to understand this classical problem of integrability in terms of integrability of a certain geometric structure. The idea is that a MAE contains information on its underlying geometry. For example, 2D-Laplace equation contains information about the complex structure of \( \mathbb{R}^4 \) and 3D-special lagrangian equation contains information about the Calabi-Yau structure of \( \mathbb{C}^3 \). Using the Lychagin-Roubtsov metric \( g_{\omega} \) and the Hitchin tensor \( K_{\omega} \), one can actually define for any symplectic 3D Monge-Ampère equation some geometrical structure of Calabi-Yau type. The following result is proved in \([1]\):

**THEOREM.** A symplectic 3D MAE is locally equivalent to one of the following equations

$$\text{hess}(f) = 1$$

$$\Delta f - \text{hess}(f) = 0$$

$$\Box f + \text{hess}(f) = 0$$

if and only if the structure of Calabi-Yau type it defines is nondegenerate, flat and integrable.
This result has to be compared with its 2D analog obtained in \[14\]:

**THEOREM.** A symplectic 2D MAE is locally equivalent to one of the following equations

\[
\Delta f = 0 \\
\Box f = 0
\]

if and only the almost complex structure or almost product structure it defines is integrable.

3. **2D Monge-Ampère equations of divergent type and generalized complex geometry**

These results are quite frustrating: which kind of integrable geometries could we define for more general MAE? One answer could be: generalized complex geometry.

This very rich concept defined recently by Hitchin (\[10\]) and developed by Gualtieri (\[8\]), which interpolates between complex and symplectic geometry, is very popular since it seems to provide a well-adapted geometric framework for different models in string theory.

3.1. Monge-Ampère equations of divergent type. Let us introduce first the Euler operator and the notion of Monge-Ampère equation of divergent type (see \[13\]).

**DEFINITION 6.** The Euler operator is the second order differential operator $\mathcal{E} : \Omega^2(M) \to \Omega^2(M)$ defined by

\[
\mathcal{E}(\omega) = d \perp d\omega.
\]

A Monge-Ampère equation $\Delta_\omega = 0$ is said to be of divergent type if $\mathcal{E}(\omega) = 0$.

**EXAMPLE 8** (Born-Infeld Equation). The Born-Infeld equation is

\[
(1 - f_t)^2 f_{xx} + 2 f_t f_x f_{tx} - (1 + f_x^2) f_{tt} = 0.
\]

The corresponding primitive form is

\[
\omega_0 = (1 - p_1^2) dq_1 \wedge dp_2 + p_1 p_2 (dq_1 \wedge dp_1) + (1 + p_2^2) dq_2 \wedge dp_1,
\]

with $q_1 = t$ and $q_2 = x$. A direct computation gives

\[
d\omega_0 = 3 (p_1 dp_2 - p_2 dp_1) \wedge \Omega,
\]

and then the Born-Infeld equation is not of divergent type.

**EXAMPLE 9** (Tricomi equation). The Tricomi equation is

\[
v_{xx} x v_{yy} + \alpha v_x + \beta v_y + \gamma(x, y).
\]

The corresponding primitive form is

\[
\omega_0 = (\alpha p_1 + \beta p_2 + \gamma(q)) dq_1 \wedge dq_2 + dq_1 \wedge dp_2 - q_2 dq_2 \wedge dp_1,
\]

with $x = q_1$ and $y = q_2$. Since

\[
d\omega_0 = (-\alpha dq_2 + \beta dq_1) \wedge \Omega,
\]

we conclude that the Tricomi equation is of divergent type.

**LEMMA 3.** A Monge-Ampère equation $\Delta_\omega = 0$ is of divergent type if and only if it exists a function $\mu$ on $M$ such that the form $\omega + \mu \Omega$ is closed.
Proof. Since the exterior product by $\Omega$ is an isomorphism from $\Omega^1(M)$ to $\Omega^3(M)$, for any 2-form $\omega$, there exists a 1-form $\alpha_\omega$ such that
\[d\omega = \alpha_\omega \wedge \Omega.\]
Since $\perp(\alpha_\omega \wedge \Omega) = \alpha_\omega$, we deduce that $\mathcal{E}(\omega) = 0$ if and only if $d\alpha_\omega = 0$, that is $d(\omega + \mu \Omega) = 0$ with $d\mu = -\alpha_\omega$. \qed

Hence, if $\Delta_\omega = 0$ is of divergent type, one can choose $\omega$ being closed. The point is that it is not primitive in general.

3.2. Hitchin pairs. Let us denote by $T$ the tangent bundle of $M$ and by $T^*$ its cotangent bundle. The natural indefinite interior product on $T \oplus T^*$ is
\[(X + \xi, Y + \eta) = \frac{1}{2}(\xi(Y) + \eta(X)),\]
and the Courant bracket on sections of $T \oplus T^*$ is
\[[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi).\]

**DEFINITION 7** (Hitchin [9]). An almost generalized complex structure is a bundle map $J : T \oplus T^* \to T \oplus T^*$ satisfying
\[J^2 = -1,\]
and
\[(J', \cdot) = -(\cdot, J').\]
Such an almost generalized complex structure is said to be integrable if the spaces of sections of its two eigenspaces are closed under the Courant bracket.

The standard examples are
\[J_1 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}\]
and
\[J_2 = \begin{pmatrix} 0 & \Omega^{-1} \\ -\Omega & 0 \end{pmatrix}\]
with $J$ a complex structure and $\Omega$ a symplectic form.

**LEMMA 4** (Crainic [7]). Let $\Omega$ be a symplectic form and $\omega$ any 2-form. Define the tensor $A$ by $\omega = \Omega(A\cdot, \cdot)$ and the form $\tilde{\omega}$ by $\tilde{\omega} = -\Omega(1 + A^2, \cdot)$.

The almost generalized complex structure
\[J = \begin{pmatrix} A & \Omega^{-1} \\ \tilde{\omega} & -A^* \end{pmatrix}\]
is integrable if and only if $\omega$ is closed. Such a pair $(\omega, \Omega)$ with $d\omega = 0$ is called a Hitchin pair.

We get then immediatly the following:

**PROPOSITION 5.** To any 2-dimensional symplectic Monge-Ampère equation of divergent type $\Delta_\omega = 0$ corresponds a Hitchin pair $(\omega, \Omega)$ and therefore a 4-dimensional generalized complex structure.

**Remark.** Let $L^2 \subset M^4$ be a 2-dimensional submanifold. Let $T_L \subset T$ be its tangent bundle and $T^*_L \subset T^*$ its annihilator. $L$ is a generalized complex submanifold (according to the terminology of [8]) or a generalized lagrangian submanifold (according to the terminology of [5]) if $T_L \oplus T^*_L$ is closed under $J$. When $J$ is defined by (1), this is equivalent to saying that $L$ is lagrangian with respect to $\Omega$ and closed under $A$, that is, $L$ is a generalized solution of $\Delta_\omega = 0$. 


3.3. Conservation laws and Generating functions. The notion of conservation laws is a natural generalization to partial differential equations of the notion of first integrals (see [12] for more details).

A 1-form $\alpha$ is a conservation law for the equation $\Delta \omega = 0$ if the restriction of $\alpha$ to any generalized solution is closed. Note that conservations laws are actually well defined up closed forms.

**EXAMPLE 10.** Let us consider the Laplace equation and the complex structure $J$ associated with. The 2-form $\alpha$ vanish on any complex curve if and only if $[d\alpha]_{1,1} = 0,$ that is

\[ \overline{\partial} \alpha_{1,0} + \partial \alpha_{0,1} = 0 \]

or equivalently

\[ \overline{\partial} \alpha_{1,0} = \overline{\partial} \psi \]

for some real function $\psi$. (Here $\overline{\partial}$ is the usual Dolbeault operator defined by the integrable complex structure $J$.) We deduce that $\alpha - d\psi = \beta_{1,0} + \beta_{0,1}$ with $\beta_{1,0} = \alpha_{1,0} - \partial \psi$ is a holomorphic $(1,0)$-form.

Hence, the conservation laws of the $2D$-Laplace equation are (up exact forms) real part of $(1,0)$-holomorphic forms.

According to the Hodge-Lepage-Lychagin theorem, $\alpha$ is a conservation law if and only if there exist two functions $f$ and $g$ such that $d\alpha = f\omega + g\Omega$. The function $f$ is called a generating function of the Monge-Ampère equation $\Delta \omega = 0$. By analogy with the Laplace equation, we will say that the function $g$ is the conjugate function to the generating function $f$. We show in [3] that these generating functions can be understood as “generalized harmonic functions” for $2D$ Monge-Ampère equations of divergent type.

The tensor $\mathbb{J}$ lives in $so(n, n)$, which can be identified with the space of 2-forms on $T \oplus T^*$, using the inner product. Moreover the space of forms, is isomorphic as a Clifford algebra to the space of endomorphisms on $T$, and therefore $\mathbb{J}$ acts on the tangent bundle. One obtains then a differential operator $\overline{\partial}$, we can write as follows:

\[ \overline{\partial} = d + i\mathbb{J} \circ d \circ \mathbb{J}. \]

This operator is the exact analog to the Dolbeault operator in complex geometry. M. Gualtieri actually proves in [8] that an almost complex structure is integrable if and only if $\overline{\partial}^2 = 0$.

Using this Gualtieri operator, we get the following characterization for generating functions of symplectic $2D$ MAE of divergent type:

**THEOREM.** A function $f$ is a generating function of the symplectic $2D$ MAE of divergent type $\Delta \omega = 0$ if and only if

\[ \partial_\omega \overline{\partial} f = 0. \]

**Remark** (Reduction of the special lagrangian equation). Only few explicit examples of special lagrangian submanifolds of $\mathbb{C}^n$ are known and almost none in compact Calabi-Yau manifolds (see papers of R. Bryant and D. Joyce on the subject) and any explicit new example would be of great interest.

Historical first examples were constructed by R. Harvey and B. Lawson in [HL]. They considered invariant solutions with respect to $SO(n)$ or a maximal torus. I propose to keep this approach, considering action groups on the generalized complex manifold $\mathbb{C}^n$. We know after [6] that the reduce space can admit also a generalized complex structure and when it is 4 dimensional, its generalized lagrangian submanifolds are solutions of $2D$-Monge-Ampère equations of divergent type. The idea would be the to construct global solutions as fibrations over these Monge-Ampère solutions.
If this approach is efficient, it should be generalized to other classical examples of Calabi-Yau manifolds as $T^*S^n$ or degree $n+2$ hypersurfaces in $P^{n+1}(C)$. This would be a great motivations to investigate the geometry of 2D-Monge-Ampère equations on $T^*S^2$ or $CP^2$.

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E-mail address: bertrand.banos@wanadoo.fr