On the first-visit-time problem for birth and death processes with catastrophes*

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Abstract

For a birth-death process subject to catastrophes, defined on the state-space $\mathcal{S} = \{r, r+1, r+2, \ldots\}$, with $r$ a positive integer or zero, the first-visit time to a state $k \in \mathcal{S}$ is considered and the Laplace transform of its probability density function is determined, use of which is then made to obtain mean and variance. The Laplace transform of the probability density function of the first effective catastrophe occurrence time and its expected value are also obtained. Some extensions to time-non-homogeneous processes are then provided. Finally, certain additional results concerning the determination of the steady-state distribution and the representation of the transition probabilities are worked out, while some applications to particular birth-death processes are shown in the Appendix.

Keywords: birth-death processes; catastrophes; first-visit times.
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1 Introduction

Great attention has been paid in the literature to the description of the evolution of systems modeled via discrete state-space random processes such as

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populations evolving in random environments or queueing and service systems under various operating protocols. However, with a few exceptions that will be indicated below, in these studies the systems under considerations are not subject to catastrophes, neither a target of investigation has been the time distribution when a preassigned state is reached. Among the relevant contributions to the area in which the present paper belongs, the following should be recalled: (i) The results concerning the distribution of the extinction time for a linear birth-death process subject to catastrophes and, in particular, the determination of necessary and sufficient conditions for population extinction to occur [3], [4]; (ii) the evaluation of the stationary probabilities of a simple immigration-birth-death process influenced by total catastrophes [12]; (iii) the studies on the transient and equilibrium behaviors of immigration-birth-death process with catastrophes [5], [16]; (iv) the analysis of linear birth-death processes under the influence of Poisson time-distributed catastrophes whose effect is to enhance the death probability [2], [14]; (v) the determination of the transient and the limiting distributions of continuous-time Markov chains subject to catastrophes occurring according to renewal processes [9]; (vi) analysis of the effect of catastrophes in the case of $M/M/1$ queueing systems [8] and when the number of initially present customers is random [11].

The determination of the distribution of the time when the zero-state is reached for the first time, studied in some of the above papers, is clearly of interest within the context of population dynamics (cf. extinction problems) or in operations research (cf. the running out of stored supplies, the disappearance of a queue at a service station, etc.). However, no mention appears to have been made so far on the determination of the distribution of the time when first a discrete continuous-time Markov process of birth-death type visits a preassigned non-zero state when subject to catastrophes, i.e. to the first-visit-time problem. This is the object of the present paper.

It should not pass unnoticed that the first-visit-time problem bears particular relevance in contexts such as populations for which extinction does not occur only when population size vanishes, but is an automatic consequence of its size reduction below some critical values (think of bisexual populations or
environmental effects by which very low densities random mating individuals automatically lead to sure extinction \cite{[15]}). Moreover, it is conceivable that in some instances, due to environmental regulation effects when a system reaches too large a size some new phenomena may arise whose effect may be determinant on the further evolution of the system, such as the stop of the access to a service provider if a waiting room fills up, the block of an information processing system as a consequence of excessive service demands, cannibalism phenomena among starving animal populations, etc., all this both in the presence and in the absence of catastrophes. In all such cases, the problem arises of determining the “statistics” of the time when a generally non-zero preassigned state is for the first time reached by the system under consideration.

In the sequel, we shall preliminarily establish certain relations among transition probabilities in the presence and in the absence of absorbing states, and then reduce the study of the dynamics of the process in the presence of catastrophes to that of a process insensitive to catastrophes, whose description is in principle more easily achieved. Some additional results and illustration examples to particular birth-death processes will be briefly outlined in the Appendix.

More specifically, in Section 2 we shall analyze the first-visit time of a birth-death process with catastrophes \( \{ N(t); t \geq 0 \} \) to any preassigned state \( k \in S = \{ r, r + 1, r + 2, \ldots \} \), with \( r \) a positive integer or zero, by considering separately the cases of first visit to state \( r \) and to state \( k \neq r \). We shall disclose various functional relations that allow to describe \( N(t) \) in terms of the corresponding birth-death process \( \{ \hat{N}(t); t \geq 0 \} \), defined on the same state-space \( S \) and characterized by the same birth and death rates as \( N(t) \), but for which catastrophes are absent. In particular, for \( N(t) \) we obtain the Laplace transform of the probability density function (pdf) of the first-visit time to state \( k \) and calculate its mean and variance.

We point out that since \( N(t) \) is skip-free to the right, all intermediate states are visited only when the first visit to state \( k \) occurs from a starting state \( j < k \). Note that, because of the assumed presence of catastrophes, this need not occur if \( j > k \).

The problem of the first occurrence of catastrophe is faced in Section 3,
where the Laplace transform of the catastrophe’s first-occurrence time pdf and its mean are obtained. Our approach is analogous to that exploited in [8] for a special birth-death process of interest in queueing theory. In Section 4 we shall obtain the expressions of the transition probabilities and of the density of the first-visit time of $N(t)$ to state $r$ in the special case when this process is time non-homogeneous and catastrophes occur with time-varying rates. In Section 5 we express the steady-state probabilities of $N(t)$ in terms of the Laplace transforms of the transient probabilities of $\hat{N}(t)$, and provide a probabilistic interpretation of the transition probabilities of $N(t)$. Indeed, we show that $N(t)$ has the same distribution as the minimum between the random variable that describes its steady state, and an independent birth-death process whose rates suitably depend on those of $N(t)$.

Some examples of applications of our results are finally given in the Appendix for birth process, immigration-emigration process, immigration-death process, and immigration-birth-death process in the presence of catastrophes.

### 2 First-visit-time problem

Let $\{N(t); t \geq 0\}$ be a birth-death process with catastrophes defined on the state-space $S = \{r, r+1, r+2, \ldots\}$, with $r$ zero or a positive integer, such that transitions occur according to the following scheme:

(i) $n \to n + 1$ with rate $\alpha_n$, for $n = r, r+1, \ldots,$
(ii) $n \to n - 1$ with rate $\beta_n$, for $n = r+2, r+3, \ldots,$
(iii) $r+1 \to r$ with rate $\beta_{r+1} + \xi$,
(iv) $n \to r$ with rate $\xi$, for $n = r+2, r+3, \ldots$.

Hence, births occur with rates $\alpha_n$, deaths with rates $\beta_n$, and catastrophes with rate $\xi$, the effect of each catastrophe being the instantaneous transition to the reflecting state $r$. For all $j, n \in S$ and $t > 0$ the transition probabilities

$$p_{j,n}(t) = P\{N(t) = n \mid N(0) = j\}$$

satisfy the following system of forward equations:

$$\frac{d}{dt} p_{j,r}(t) = -(\alpha_r + \xi) p_{j,r}(t) + \beta_{r+1} p_{j,r+1}(t) + \xi,$$
\[ \frac{d}{dt} p_{j,n}(t) = -(\alpha_n + \beta_n + \xi) p_{j,n}(t) + \alpha_{n-1} p_{j,n-1}(t) + \beta_{n+1} p_{j,n+1}(t), \quad n = r + 1, r + 2, \ldots, \]

with initial condition
\[ \lim_{t \downarrow 0} p_{j,n}(t) = \delta_{j,n} = \begin{cases} 1, & n = j \\ 0, & \text{otherwise}. \end{cases} \]

Denote by \( \hat{N}(t); t \geq 0 \) the time-homogeneous birth-death process obtained from \( N(t) \) by removing the possibility of catastrophes, i.e. by setting \( \xi = 0 \). Its transition probabilities \( \hat{p}_{j,n}(t) = P\{\hat{N}(t) = n | \hat{N}(0) = j\}, \quad j, n \in S, \quad t \geq 0 \)
then satisfy the system of forward equations obtained from (1) by setting \( \xi = 0 \), with initial condition \( \lim_{t \downarrow 0} \hat{p}_{j,n}(t) = \delta_{j,n} \).

Hereafter, we shall restrict our attention to non-explosive processes \( \hat{N}(t) \), i.e. we shall assume that \( \sum_{n=r}^{\infty} \hat{p}_{j,n}(t) = 1 \) for all \( j \in S \) and \( t \geq 0 \).

We note that some descriptors of \( N(t) \) can be expressed in terms of the corresponding ones of \( \hat{N}(t) \). Indeed, making use of the forward equations for probabilities \( p_{j,n}(t) \) and for \( \hat{p}_{j,n}(t) \), for all \( j, n \in S \) and \( t > 0 \) we have:
\[ p_{j,n}(t) = e^{-\xi t} \hat{p}_{j,n}(t) + \xi \int_0^t e^{-\xi \tau} \hat{p}_{r,n}(\tau) d\tau. \]

As an immediate consequence of Eq. (2), we can express the conditional moments of \( N(t) \) in terms of those of \( \hat{N}(t) \) as follows:
\[ E[N(t) | N(0) = j] = e^{-\xi t} E[\hat{N}(t) | \hat{N}(0) = j] + \xi \int_0^t e^{-\xi \tau} E[\hat{N}(\tau) | \hat{N}(0) = r] d\tau. \]

Moreover, by setting
\[ \pi_{j,n}(\lambda) := \int_0^{+\infty} e^{-\lambda t} p_{j,n}(t) dt, \quad \hat{\pi}_{j,n}(\lambda) := \int_0^{+\infty} e^{-\lambda t} \hat{p}_{j,n}(t) dt, \quad \lambda > 0, \]
from (2) it follows
\[ \pi_{j,n}(\lambda) = \hat{\pi}_{j,n}(\lambda + \xi) + \frac{\xi}{\lambda} \hat{\pi}_{r,n}(\lambda + \xi), \quad \lambda > 0. \]

Let us define the first-visit time of \( N(t) \) to state \( k \in S \) as
\[ T_{j,k} := \inf\{t \geq 0 : N(t) = k\}, \quad N(0) = j \in S, \quad j \neq k, \]
and denote its pdf by

\[ g_{j,k}(t) = \frac{d}{dt} P\{T_{j,k} \leq t\}. \]

Moreover, for the corresponding birth-death process \( \hat{N}(t) \) in the absence of catastrophes, we shall denote by \( \hat{T}_{j,k} \) the first-visit time, and by \( \hat{g}_{j,k}(t) \) its pdf.

We shall now analyze the first-visit-time problem. To this purpose, \( \gamma_{j,k}(\lambda) \) and \( \hat{\gamma}_{j,k}(\lambda) \) will denote the Laplace transforms of \( g_{j,k}(t) \) and \( \hat{g}_{j,k}(t) \), respectively.

Furthermore, for all \( t \geq 0 \) and \( k, j, n \in S \), with \( j \neq k \) and \( n \neq k \), let

\[ A_{j,n}^{(k)}(t) := P\{N(t) = n, N(\tau) \neq k \ \forall \tau \in (0, t) \mid N(0) = j\} \]

be the \( k \)-avoiding transition probability of \( N(t) \), and let \( \hat{A}_{j,n}^{(k)}(t) \) denote the corresponding probability for \( \hat{N}(t) \). For all \( t \geq 0 \) these functions are related as follows:

\[ \int_0^t \hat{g}_{j,k}(\tau) \, d\tau = \begin{cases} 1 - \sum_{n=k+1}^{+\infty} \hat{A}_{j,n}^{(k)}(t), & j > k \\ 1 - \sum_{n=r}^{k-1} \hat{A}_{j,n}^{(k)}(t), & j < k \end{cases} \]

\[ \int_0^t g_{j,k}(\tau) \, d\tau = \begin{cases} 1 - \sum_{n=r}^{+\infty} A_{j,n}^{(k)}(t), & j > k \\ 1 - \sum_{n=r}^{k-1} A_{j,n}^{(k)}(t), & j < k \end{cases} \]

where \( \int_0^t g_{j,k}(\tau) \, d\tau \) is the probability that \( N(t) \) enters state \( k \) before time \( t \), and \( \sum_n A_{j,n}^{(k)}(t) \) is the probability that \( N(t) \) does not enter state \( k \) up to time \( t \). Hereafter, \( A_{j,n}^{(k)}(\lambda) \) and \( \hat{A}_{j,n}^{(k)}(\lambda) \) will denote the Laplace transforms of \( A_{j,n}^{(k)}(t) \) and \( \hat{A}_{j,n}^{(k)}(t) \), respectively.

The first-visit-time problem will be discussed in three different cases: (i) first visit in state \( r \), (ii) first visit in state \( k > r \) starting from below, (iii) first visit in state \( k > r \) starting from above.

### 2.1 First visit in state \( r \)

Recalling (5), the \( r \)-avoiding transition probabilities of \( N(t) \) and \( \hat{N}(t) \) for all \( t \geq 0 \) are related as follows:

\[ A_{j,n}^{(r)}(t) = e^{-\xi t} \hat{A}_{j,n}^{(r)}(t), \quad j, n \in \{r + 1, r + 2, \ldots\}. \]
Hence, the probability of a transition of \( N(t) \) from \( j \) to \( n \) (\( n > r \)) at time \( t \) in the absence of visits to state \( r \) equals the probability of the same transition for \( \hat{N}(t) \) times the probability \( e^{-\xi t} \) that no catastrophe occurs in \((0, t)\).

The following proposition expresses \( g_{j,r}(t) \) and \( \gamma_{j,r}(\lambda) \) in terms of \( \hat{g}_{j,r}(t) \) and \( \hat{\gamma}_{j,r}(\lambda) \), respectively.

**Proposition 2.1** For \( j \in \{r+1, r+2, \ldots\} \) the following equations hold:

\[
g_{j,r}(t) = e^{-\xi t} \hat{g}_{j,r}(t) + \xi e^{-\xi t} \left[ 1 - \int_0^t \hat{g}_{j,r}(\tau) \, d\tau \right], \quad t > 0, \quad (9)
\]

\[
\gamma_{j,r}(\lambda) = \frac{\lambda}{\lambda + \xi} \hat{\gamma}_{j,r}(\lambda + \xi) + \frac{\xi}{\lambda + \xi}, \quad \lambda > 0. \quad (10)
\]

**Proof.** Making use of Eq. (6) and (7) for \( k = r \), from (8) we obtain:

\[
\int_0^t g_{j,r}(\tau) \, d\tau = 1 - e^{-\xi t} \sum_{n=r+1}^{+\infty} \hat{A}_{j,n}^{(r)}(t) = 1 - e^{-\xi t} \left[ 1 - \int_0^t \hat{g}_{j,r}(\tau) \, d\tau \right]. \quad (11)
\]

Differentiating both sides of (11) with respect to \( t \) we are led to (9). Taking the Laplace transform of both sides of (9) and making use of Fubini’s theorem we then have:

\[
\gamma_{j,r}(\lambda) = \hat{\gamma}_{j,r}(\lambda + \xi) + \frac{\xi}{\lambda + \xi}, \quad \lambda > 0,
\]

from which Eq. (10) follows. \( \diamond \)

In the limit as \( t \to +\infty \), from (11) we obtain \( \int_0^{+\infty} g_{j,r}(\tau) \, d\tau = 1 \), implying that the first visit of \( N(t) \) in state \( r \) occurs with probability 1, whereas for \( \hat{N}(t) \) such probability may be less than 1.

A characterization of the distribution of the first-visit time in state \( r \) is provided by the following

**Theorem 2.1** Let \( Z \) be an exponentially distributed random variable independent of \( \hat{T}_{j,r} \) with mean \( \xi^{-1} \). If \( P(\hat{T}_{j,r} < +\infty) = 1 \), then for \( j \in \{r+1, r+2, \ldots\} \) the r.v.

\[
\Theta_{j,r} := \min \{\hat{T}_{j,r}, Z\}
\]

has the same distribution as \( T_{j,r} \).

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**Proof.** From (12) we have:

\[ P(\Theta_{j,r} \leq t) = 1 - e^{-\xi t} + e^{-\xi t} P(\hat{T}_{j,r} \leq t) = P(T_{j,r} \leq t), \quad t \geq 0, \]

where the last equality follows from (11). This completes the proof. \(\diamond\)

### 2.2 First visit to state \(k\)

We shall investigate the first-visit-time problem to any state \(k \in S\). Note that the case \(j = r\) and \(k = r + 1\) is trivial and

\[ g_{r,r+1}(t) = \alpha_r e^{-\alpha_r t}, \quad A^{(r+1)}_r(t) = e^{-\alpha_r t}, \quad t > 0. \]

In the following theorem \(\gamma_{j,k}(\lambda)\) is expressed in terms of \(\hat{\gamma}_{j,k}(\lambda)\).

**Theorem 2.2** For all \(\lambda > 0\) and for \(j, k \in S, j \neq k\), there results:

\[ \gamma_{j,k}(\lambda) = \frac{\lambda \hat{\gamma}_{j,k}(\lambda + \xi) + \xi \hat{\gamma}_{r,k}(\lambda + \xi)}{\lambda + \xi \hat{\gamma}_{r,k}(\lambda + \xi)}. \quad (13) \]

**Proof.** For all \(t > 0\) and for \(j, k \in S, j \neq k\), the following renewal equation holds:

\[ p_{j,k}(t) = \int_0^t g_{j,k}(\tau) p_{k,k}(t - \tau) \, d\tau, \quad (14) \]

Taking the Laplace transform of (14) and recalling (4) we have:

\[ \gamma_{j,k}(\lambda) = \frac{\pi_{j,k}(\lambda)}{\pi_{k,k}(\lambda)} = \frac{\lambda \hat{\pi}_{j,k}(\lambda + \xi) + \xi \hat{\pi}_{r,k}(\lambda + \xi)}{\lambda \hat{\pi}_{k,k}(\lambda + \xi) + \xi \hat{\pi}_{r,k}(\lambda + \xi)}, \]

that leads one to (13) by virtue of the relation \(\hat{\gamma}_{j,k}(\lambda) = \hat{\pi}_{j,k}(\lambda)/\hat{\pi}_{k,k}(\lambda), \lambda > 0. \) \(\diamond\)

Setting \(\lambda = 0\) in Eq. (13) yields \(\gamma_{j,k}(0) = 1\), so that the first visit of \(N(t)\) to any state \(k\) occurs with unit probability, whereas for \(\hat{N}(t)\) this event may occur with probability less than 1. We note that Eq. (13) reduces to (10) if \(k = r\).

Making use of Eq. (13) we can now immediately express mean and variance of \(T_{j,k}\) in terms of the Laplace transform of \(g_{j,r}(t)\), for all \(j, k \in S, j \neq k\):

\[ \text{E}(T_{j,k}) = \frac{1 - \hat{\gamma}_{j,k}(\xi)}{\xi \hat{\gamma}_{r,k}(\xi)}, \quad (15) \]

\[ \text{Var}(T_{j,r}) = \frac{1}{\xi^2 \hat{\gamma}_{r,k}(\xi)} \left\{ 1 - \frac{\hat{\gamma}_{j,k}^2(\xi)}{\hat{\gamma}_{r,k}(\xi)} + 2\xi [1 - \hat{\gamma}_{j,k}(\xi)] \frac{d}{d\xi} \hat{\gamma}_{r,k}(\xi) \right\} + 2\xi \hat{\gamma}_{r,k}(\xi) \frac{d}{d\xi} \hat{\gamma}_{j,k}(\xi). \quad (16) \]
Eqs. (15) and (16) will be used in the Appendix for the analysis of an immigration-emigration process with catastrophes.

Next proposition yields the Laplace transform of the $k$-avoiding transition probability $A_{j,n}^{(k)}(t)$.

**Proposition 2.2** Under each of the following mutually exclusive assumptions:

(i) $k \in \{r+1, r+2, \ldots \}$ and $j, n \in \{r, r+1, \ldots, k-1\}$,

(ii) $n, k \in S$, $n \neq k$, and $j \in \{k+1, k+2, \ldots \}$,

for all $\lambda > 0$ we have:

$$A_{j,n}^{(k)}(\lambda) = \hat{\pi}_{j,n}(\lambda + \xi) - \hat{\gamma}_{j,k}(\lambda + \xi) \hat{\pi}_{k,n}(\lambda + \xi)$$

$$+ \xi \frac{1 - \hat{\gamma}_{j,k}(\lambda + \xi)}{\lambda + \xi \hat{\gamma}_{r,k}(\lambda + \xi)} [\hat{\pi}_{r,n}(\lambda + \xi) - \hat{\gamma}_{r,k}(\lambda + \xi) \hat{\pi}_{k,n}(\lambda + \xi)]. \ (17)$$

**Proof.** Under assumptions (i) or (ii) there holds:

$$A_{j,n}^{(r)}(t) = p_{j,n}(t) - \int_0^t g_{j,k}(\tau) p_{k,n}(t-\tau) \, d\tau, \quad t > 0.$$

Taking the Laplace transform of this equation and recalling (11) and (13) we obtain:

$$A_{j,n}^{(k)}(\lambda) = \hat{\pi}_{j,n}(\lambda + \xi) - \hat{\gamma}_{j,k}(\lambda + \xi) \hat{\pi}_{k,n}(\lambda + \xi)$$

$$+ \xi \frac{1 - \hat{\gamma}_{j,k}(\lambda + \xi)}{\lambda + \xi \hat{\gamma}_{r,k}(\lambda + \xi)} [\hat{\pi}_{r,n}(\lambda + \xi) - \hat{\gamma}_{r,k}(\lambda + \xi) \hat{\pi}_{k,n}(\lambda + \xi)],$$

that leads us to (17) after some calculations. \qed

We point out that under assumption (i) of Proposition 2.2 from the well-known relation

$$\hat{A}_{j,n}^{(k)}(t) = \hat{p}_{j,n}(t) - \int_0^t \hat{g}_{j,k}(\tau) \hat{p}_{k,n}(t-\tau) \, d\tau, \quad t > 0,$$

we obtain:

$$\hat{A}_{j,n}^{(k)}(\lambda) = \hat{\pi}_{j,n}(\lambda) - \hat{\gamma}_{j,k}(\lambda) \hat{\pi}_{k,n}(\lambda), \quad \lambda > 0.$$

Eq. (17) can thus be also expressed as

$$A_{j,n}^{(k)}(\lambda) = \hat{A}_{j,n}^{(k)}(\lambda + \xi) + \xi \frac{1 - \hat{\gamma}_{j,k}(\lambda + \xi)}{\lambda + \xi \hat{\gamma}_{r,k}(\lambda + \xi)} \hat{A}_{r,n}^{(k)}(\lambda + \xi), \quad j, n \in \{r, r+1, \ldots, k-1\}.$$
We remark that since $N(t)$ is not skip-free to the left, $A_{j,n}^{(k)}(\lambda)$ cannot be expressed in terms of $\hat{A}_{j,n}^{(k)}(\lambda)$ when $j > k$. Moreover, since $N(t)$ is skip-free to the right, for all $t > 0$ and $j \in \{ r, r+1, \ldots, k-1 \}$ the following equation holds:

$$g_{j,k}(t) = \int_0^t g_{j,n}(\tau) g_{n,k}(t-\tau) d\tau, \quad n = j+1, j+2, \ldots, k-1. \quad (18)$$

Instead, Eq. (18) does not hold when $j > k$ because $N(t)$ is not skip-free to the left.

### 3 First occurrence of effective catastrophe

This section focuses on “effective” catastrophes, i.e. on catastrophes that are able to change the state of $N(t)$. Therefore, the occurrence of catastrophes while the process is in state $r$ is not taken into account. Consequently, the catastrophes first-occurrence time is no longer exponentially distributed.

Let us denote by $C_{j,r}$ the first occurrence of an effective catastrophe, when $N(0) = j$, with $j \in \mathcal{S}$. An effective catastrophe, or shortly “a catastrophe” from now on, produces a transition, with rate $\xi$, from any state $n > r$ to the reflecting state $r$. Hence, certain transitions from $r+1$ to $r$ may be due to the occurrence of a catastrophe (with rate $\xi$), whereas the remaining transitions are due to a death (with rate $\beta_{r+1}$).

In order to investigate on the features of $C_{j,r}$, let us refer to a modified birth-death process with catastrophes that will be denoted as $\{M(t); t \geq 0\}$. This is assumed to be defined on the state-space $\{ r-1, r, r+1, \ldots \}$. Its behavior is identical to that of $N(t)$, the only difference being that the effect of a catastrophe from state $n > r$ is a jump from $n$ to the absorbing state $r-1$. In other words, the allowed transitions are the following:

(i) $n \to n + 1$ with rate $\alpha_n$, for $n = r, r+1, \ldots$,
(ii) $n \to n - 1$ with rate $\beta_n$, for $n = r+1, r+2, \ldots$,
(iii) $n \to r - 1$ with rate $\xi$, for $n = r+1, r+2, \ldots$.

For all $t \geq 0$ and $j \in \mathcal{S}$, $n \in \{ r-1, r, r+1, \ldots \}$, let us now consider the transition probabilities of the modified process

$$h_{j,n}(t) = P\{M(t) = n \mid M(0) = j\}.$$
The link between \( M(t) \) and \( C_{j,r} \) is evident by noting that the transitions of \( M(t) \) from \( n > r \) to \( r - 1 \) corresponds to the transitions of \( N(t) \) from \( n \) to \( r \) due to a catastrophe. Hence, denoting by \( d_{j,r}(t) \) the density of \( C_{j,r} \) for all \( t > 0 \) we have:

\[
P(C_{j,r} < t) \equiv \int_0^t d_{j,r}(\tau) \, d\tau = h_{j,r-1}(t), \quad j \in S.
\]  

(19)

Moreover, for all \( j \in S \) the following system of forward equations holds:

\[
\begin{align*}
\frac{d}{dt} h_{j,r-1}(t) &= \xi [1 - h_{j,r-1}(t) - h_{j,r}(t)], \\
\frac{d}{dt} h_{j,r}(t) &= -\alpha_r h_{j,r}(t) + \beta_{r+1} h_{j,r+1}(t), \\
\frac{d}{dt} h_{j,n}(t) &= -(\alpha_n + \beta_n + \xi) h_{j,n}(t) + \alpha_{n-1} h_{j,n-1}(t) + \beta_{n+1} h_{j,n+1}(t),
\end{align*}
\]

\( n = r + 1, r + 2, \ldots, \)  

(20)

with initial condition

\[
h_{j,n}(0) = \delta_{j,n}.
\]

Let us denote by \( \eta_{j,n}(\lambda) \) the Laplace transform of \( h_{j,n}(t) \). In the following theorem we shall express \( \eta_{j,n}(\lambda) \) in terms of \( \hat{\pi}_{j,n}(\lambda) \).

**Theorem 3.1** For all \( j \in S \) and \( \lambda > 0 \) we have:

\[
\begin{align*}
\eta_{j,r-1}(\lambda) &= \frac{\xi}{\lambda + \xi} \left[ \frac{1}{\lambda} - \frac{\hat{\pi}_{j,r}(\lambda + \xi)}{1 - \xi \hat{\pi}_{r,r}(\lambda + \xi)} \right], \\
\eta_{j,n}(\lambda) &= \hat{\pi}_{j,n}(\lambda + \xi) + \xi \hat{\pi}_{r,n}(\lambda + \xi) \frac{\hat{\pi}_{j,r}(\lambda + \xi)}{1 - \xi \hat{\pi}_{r,r}(\lambda + \xi)}, \quad n = r, r + 1, \ldots.
\end{align*}
\]

(21) (22)

**Proof.** We shall prove the theorem by considering separately the two cases \( (a) \) \( j = r \) and \( (b) \) \( j > r \).

(a) Let \( j = r \). Taking the Laplace transform of second and third equation in (20), we obtain:

\[
\begin{align*}
(\lambda + \alpha_r) \eta_{r,r}(\lambda) - 1 &= \beta_{r+1} \eta_{r,r+1}(\lambda), \\
(\lambda + \alpha_n + \beta_n + \xi) \eta_{r,n}(\lambda) &= \alpha_{n-1} \eta_{r,n-1}(\lambda) + \beta_{n+1} \eta_{r,n+1}(\lambda), \quad n \geq r
\end{align*}
\]

(23)

Recalling that \( \pi_{r,n}(\lambda) \) is the Laplace transform of \( p_{r,n}(t) \), we look for a solution of the form

\[
\eta_{r,n}(\lambda) = A(\lambda) \pi_{r,n}(\lambda), \quad n = r + 1, r + 2, \ldots,
\]

(24)
where $A(\lambda)$ does not depend on $n$. Substituting Eq. (24) in (23), we obtain:

\[
A(\lambda)(\lambda + \alpha_r)\pi_{r,r}(\lambda) - 1 = \beta_{r+1}A(\lambda)\pi_{r,r+1}(\lambda),
\]

\[
(\lambda + \alpha_n + \beta_n + \xi)\pi_{r,r}(\lambda) = \alpha_{n-1}\pi_{r,n-1}(\lambda) + \beta_{n+1}\pi_{r,n+1}(\lambda), \quad n > r. \tag{25}
\]

Moreover, taking the Laplace transform of (11) it follows:

\[
(\lambda + \alpha_n + \beta_n + \xi)\pi_{r,r}(\lambda) = \alpha_{n-1}\pi_{r,n-1}(\lambda) + \beta_{n+1}\pi_{r,n+1}(\lambda), \quad n > r. \tag{26}
\]

Comparing (26) with (25) we obtain:

\[
A(\lambda) = \frac{\lambda}{\lambda + \xi - \lambda \xi \pi_{r,r}(\lambda)},
\]

that does not depend on $n$. By virtue of (24) and (4), this yields Eq. (22) for $j = r$.

(b) Let $j > r$. Taking the Laplace transform of (20), we have:

\[
(\lambda + \alpha_r)\eta_{j,r}(\lambda) = \beta_{r+1}\eta_{j,r+1}(\lambda),
\]

\[
(\lambda + \alpha_n + \beta_n + \xi)\eta_{j,n}(\lambda) = \alpha_{n-1}\eta_{j,n-1}(\lambda) + \beta_{n+1}\eta_{j,n+1}(\lambda), \quad n > r, \quad n \neq j,
\]

\[
(\lambda + \alpha_j + \beta_j + \xi)\eta_{j,j}(\lambda) - 1 = \alpha_{j-1}\eta_{j,j-1}(\lambda) + \beta_{j+1}\eta_{j,j+1}(\lambda).
\]

Similarly to case (a) above there results:

\[
\eta_{j,n}(\lambda) = B(\lambda)\pi_{j,n}(\lambda) + C(\lambda)\pi_{r,n}(\lambda), \quad n = r - 1, r, \ldots, \tag{27}
\]

where

\[
B(\lambda) = 1, \quad C(\lambda) = \frac{\xi[\lambda\pi_{r,r}(\lambda) - 1]}{\lambda + \xi - \lambda \xi \pi_{r,r}(\lambda)}.
\]

Hence, making use of (27) and (4), some straightforward calculations led us to Eq. (24) for $j > r$. Finally, taking the Laplace transform of the first equation in (20) we obtain:

\[
\eta_{j,r-1}(t) = \frac{\xi}{\lambda + \xi} \left[ \frac{1}{\lambda} - \eta_{j,r}(t) \right].
\]

Making again use of (4) Eq. (24) then follows. \hfill \Box

Let us now denote by $\delta_{j,r}(\lambda)$ the Laplace transform of $d_{j,r}(t), \quad j \in S$. Taking the Laplace transform of both sides of (19) and making use of (21), for $j \in S$ we have:

\[
\delta_{j,r}(\lambda) = \lambda\eta_{j,r-1}(\lambda) = \frac{\xi}{\lambda + \xi} - \frac{\lambda}{\lambda + \xi} + \frac{\xi}{1 - \xi \pi_{r,r}(\lambda)}.
\]

\[12\]
The following renewal equation for \( \hat{N}(t) \)
\[
\hat{p}_{j,r}(t) = \int_0^t \hat{g}_{j,r}(\tau) \hat{p}_{r,r}(t - \tau) d\tau, \quad t > 0,
\]
holding for all \( j > r \), yields:
\[
\hat{\pi}_{j,r}(\lambda) = \hat{\gamma}_{j,r}(\lambda) \hat{\pi}_{r,r}(\lambda), \quad \lambda > 0.
\]
Hence, for \( j > r \), (28) can be re-written as
\[
\delta_{j,r}(\lambda) = \frac{\xi}{\lambda + \xi} + \frac{\hat{\gamma}_{j,r}(\lambda + \xi)}{\lambda + \xi} - \frac{\hat{\gamma}_{j,r}(\lambda + \xi)}{1 - \xi \hat{\pi}_{r,r}(\lambda + \xi)}, \quad \lambda > 0.
\]
Making use of (11) in the right-hand-side, we have an alternative expression for \( \delta_{j,r}(\lambda) \), for all \( \lambda > 0 \) and \( j \in \{r+1, r+2, \ldots\} \):
\[
\delta_{j,r}(\lambda) = \gamma_{j,r}(\lambda) - \frac{\lambda}{\lambda + \xi} \frac{\hat{\gamma}_{j,r}(\lambda + \xi)}{1 - \xi \hat{\pi}_{r,r}(\lambda + \xi)}. \tag{29}
\]

**Proposition 3.1** For all \( j \in S \) there holds:
\[
E(C_{j,r}) = \frac{1}{\xi} \left[ \frac{\hat{\gamma}_{j,r}(\xi)}{1 - \xi \hat{\pi}_{r,r}(\xi)} \right]. \tag{30}
\]

**Proof.** For \( j > r \), (30) follows by differentiating the right-hand-side of (29) with respect to \( \lambda \) and then setting \( \lambda = 0 \). The same procedure starting from (28) yields (30) in the case \( j = r \). \( \diamond \)

### 4 An extension to time-non-homogeneous processes

In order to extend some of the above results to the time-non-homogeneous case, in this section \( \{N(t); t \geq t_0\} \), \( t_0 \geq 0 \), will denote a time non-homogeneous birth-death process with catastrophes, defined on the state-space \( S = \{r, r + 1, r + 2, \ldots\} \), where births occur with rates \( \alpha_n(t) \) and deaths with rates \( \beta_n(t) \). Moreover, catastrophes are assumed to occur according to a non-homogeneous Poisson process characterized by intensity function \( \xi(t) \). We assume that \( \alpha_n(t), \beta_n(t) \) and \( \xi(t) \) are continuous bounded functions such that \( \alpha_n(t) > 0, \beta_n(t) \geq 0 \) and \( \xi(t) > 0 \) for all \( t \geq t_0 \), with \( \int_{t_0}^{+\infty} \xi(t) dt = +\infty \). Moreover, \( \{\hat{N}(t); t \geq t_0\} \) will denote the time non-homogeneous birth-death process obtained from \( \hat{N}(t) \)
by removing the possibility of catastrophes, i.e. by setting $\xi(t) = 0$ for all $t \geq t_0$.

For all $t \geq t_0$ and $j, n \in S$ the transition probabilities of $N(t)$ and $\hat{N}(t)$ will be denoted by

$$p_{j,n}(t \mid t_0) = \mathbb{P}\{N(t) = n \mid N(t_0) = j\}, \quad \hat{p}_{j,n}(t \mid t_0) = \mathbb{P}\{\hat{N}(t) = n \mid \hat{N}(t_0) = j\}. \quad (31)$$

Assuming that $\hat{N}(t)$ is non-explosive, i.e. such that $\sum_{n=0}^{+\infty} \hat{p}_{j,n}(t \mid t_0) = 1$ for all $j \in S$ and $t \geq t_0$, we shall extend Eqs. (2) and (3) to the time-non-homogeneous case. To this end, we note that by making use of the forward equations for probabilities (31), for all $j, n \in S$ and $t > t_0$ there hold:

$$p_{j,n}(t \mid t_0) = \exp\left\{ -\int_{t_0}^{t} \xi(u) \, du \right\} \hat{p}_{j,n}(t \mid t_0) + \int_{t_0}^{t} \xi(\tau) \exp\left\{ -\int_{\tau}^{t} \xi(u) \, du \right\} \hat{p}_{r,n}(t \mid \tau) \, d\tau, \quad (32)$$

and

$$\mathbb{E}[N(t) \mid N(t_0) = j] = \exp\left\{ -\int_{t_0}^{t} \xi(u) \, du \right\} \mathbb{E}[\hat{N}(t) \mid \hat{N}(t_0) = j]$$

$$+ \int_{t_0}^{t} \xi(\tau) \exp\left\{ -\int_{\tau}^{t} \xi(u) \, du \right\} \mathbb{E}[\hat{N}(t) \mid \hat{N}(\tau) = r] \, d\tau. \quad (33)$$

Moreover, similarly to the cases of Eqs. (2) and (3), it is not hard to prove that the $r$-avoiding transition probabilities and the pdf’s of the first-visit time to state $r$ in the time-non-homogeneous case are related as follows for all $t \geq t_0$ and $j \in \{r+1, r+2, \ldots\}$:

$$A^{(r)}_{j,n}(t \mid t_0) = \exp\left\{ -\int_{t_0}^{t} \xi(u) \, du \right\} \hat{A}^{(r)}_{j,n}(t \mid t_0), \quad n \in \{r+1, r+2, \ldots\},$$

$$g_{j,r}(t \mid t_0) = \exp\left\{ -\int_{t_0}^{t} \xi(u) \, du \right\} \hat{g}_{j,r}(t \mid t_0)$$

$$+ \xi(t) \exp\left\{ -\int_{t_0}^{t} \xi(u) \, du \right\} \left[ 1 - \int_{t_0}^{t} \hat{g}_{j,r}(\tau \mid t_0) \, d\tau \right]. \quad (33)$$

Let $\hat{T}_{j,r}$ denote the first-visit time in state $r$ for the time-non-homogeneous birth-death process $\hat{N}(t)$. The characterization given in Theorem 2.1 can be extended to this time-non-homogeneous case. Indeed, let $Z$ be a random variable with hazard function $\xi(t)$, $t \geq t_0$, i.e. with distribution function $F(t) = 1 - \exp\{-\int_{t_0}^{t} \xi(u) \, du\}$, $t \geq t_0$. If $Z$ is independent of $\hat{T}_{j,r}$, and if $\mathbb{P}(\hat{T}_{j,r} < +\infty) = 1$, then it is possible to prove that for $j \in \{r+1, r+2, \ldots\}$ the r.v. (24) has the same distribution as the first-visit time $T_{j,r}$. 

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5 Some additional results

Let us denote by $N$ the random variable that describes the steady state of $N(t)$ in the time-homogeneous case, and let

$$q_n := P(N = n) = \lim_{t \to +\infty} p_{j,n}(t), \quad j, n \in \mathcal{S}. \quad (34)$$

Then, from (1) we obtain:

$$-(\alpha_r + \xi) q_r + \beta_{r+1} q_{r+1} + \xi = 0,$$
$$-(\alpha_n + \beta_n + \xi) q_n + \alpha_{n-1} q_{n-1} + \beta_{n+1} q_{n+1} = 0, \quad n = r + 1, r + 2, \ldots. \quad (35)$$

Making use of a Tauberian theorem, from Eq. (4) there results:

$$q_n = \xi \hat{\pi}_{r,n}(\xi), \quad t \geq 0, \quad n \in \mathcal{S}. \quad (36)$$

From the assumed non explosivity of $\hat{N}(t)$ and from (36) it follows that $N(t)$ possesses a steady-state distribution, with $\sum_{n=r}^{+\infty} q_n = 1$. We stress that due to (36), such a distribution can be obtained from the Laplace transform of the transition probabilities in absence of catastrophes. Moreover, we underline that (36)’s satisfy system (35).

In Eq. (2) we have expressed the transition probability of $N(t)$ in terms of that of $\hat{N}(t)$. In order to provide a probabilistic interpretation of $p_{r,n}(t)$, let us now introduce a time-homogeneous birth-death process $\{N^*(t); t \geq 0\}$ characterized by birth and death rates

$$\alpha_n^* = \alpha_n \sum_{k=r}^{+\infty} q_k, \quad (n \geq r), \quad \beta_n^* = \beta_n \sum_{k=n+1}^{+\infty} q_k, \quad (n > r), \quad (37)$$

with state-space $\mathcal{S}$, where $r$ is a reflecting state. Assuming that $P\{N^*(0) = r\} = 1$, we denote by $p_{r,n}^*(t)$ the transition probabilities of $N^*(t)$.

**Theorem 5.1** Let $u_{r,n}(t)$ be the transition probabilities of

$$U(t) := \min\{N^*(t), N\}, \quad t \geq 0, \quad (38)$$

where $N$ is independent of $N^*(t)$ and distributed as in (34). Then,

$$u_{r,n}(t) = p_{r,n}(t) \quad (39)$$

for all $t \geq 0$ and $n \in \mathcal{S}$. 

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Theorem 5.2

Proof. From Eq. (38) it follows

\[ u_{r,n}(t) = q_n \sum_{k=n}^{+\infty} p_{r,k}^*(t) + p_{r,n}^*(t) \sum_{k=n+1}^{+\infty} q_k, \quad n \in S. \]  

(40)

Differentiating both sides of (40) and making use of the forward equations of \( p_{r,n}^*(t) \), we have:

\[
\frac{d}{dt} u_{r,n}(t) = -\alpha_r^* (1 - q_r) p_{r,r}^*(t) + \beta_{r+1}^* (1 - q_r) p_{r,r+1}^*(t),
\]

\[
\frac{d}{dt} u_{r,n}(t) = - \left( \alpha_n^* \sum_{k=n+1}^{+\infty} q_k + \beta_n^* \sum_{k=n}^{+\infty} q_k \right) p_{r,n}^*(t) + \alpha_{n-1}^* p_{r,n-1}^*(t) \sum_{k=n}^{+\infty} q_k
\]

\[ + \beta_{n+1}^* p_{r,n+1}^*(t) \sum_{k=n+1}^{+\infty} q_k, \quad n = r + 1, r + 2, \ldots. \]  

(41)

Moreover, from (40) we obtain:

\[ -(\alpha_r + \xi) u_{r,r}(t) + \beta_{r+1} u_{r,r+1}(t) + \xi = -\alpha_r^* p_{r,r}^*(t) + \beta_{r+1}^* (1 - q_r - q_{r+1}) p_{r,r+1}^*(t), \]

\[ -(\alpha_n + \beta_n + \xi) u_{r,n}(t) + \alpha_{n-1} u_{r,n-1}(t) + \beta_{n+1} u_{r,n+1}(t)
\]

\[ = \left[ -(\alpha_n + \beta_n + \xi) \sum_{k=n}^{+\infty} q_k + \alpha_{n-1} q_{n-1} \right] p_{r,n}^*(t) + \alpha_{n-1}^* p_{r,n-1}^*(t) \sum_{k=n}^{+\infty} q_k
\]

\[ + \beta_{n+1}^* p_{r,n+1}^*(t) \sum_{k=n+2}^{+\infty} q_k, \quad n = r + 1, r + 2, \ldots. \]  

(42)

Due to (35) and (37), the right-hand-sides of Eqs. (41) and (42) are identical, so that \( u_{r,n}(t) \) satisfy the system of forward equations (41). Furthermore, due to (38) or (40), initial condition \( u_{r,n}(0) = \delta_{r,n} \) holds. Hence, (39) follows. \( \diamond \)

An immediate consequence of Theorem 5.1 is that \( p_{r,n}(t) \) can be expressed in terms of probabilities \( p_{r,k}^*(t) \) and \( q_k \). Indeed, due to (39) and (40), for all \( t \geq 0 \) and \( n \in S \) we have:

\[ p_{r,n}(t) = q_n \sum_{k=n}^{+\infty} p_{r,k}^*(t) + p_{r,n}^*(t) \sum_{k=n+1}^{+\infty} q_k. \]  

(43)

Hereafter we show that this result can be extended to the more general case of arbitrary initial state.

Theorem 5.2 For all \( t \geq 0 \) and \( j, n \in S \) we have:

\[ p_{j,n}(t) = e^{-\xi t} \left[ \hat{p}_{j,n}(t) - \hat{p}_{r,n}(t) \right] + q_n \sum_{k=n}^{+\infty} p_{r,k}^*(t) + p_{r,n}^*(t) \sum_{k=n+1}^{+\infty} q_k. \]  

(44)
Proof. Comparing Eq. (2) written for \( j = r \) with (43) we obtain:

\[
\xi \int_0^t e^{-\xi \tau} \hat{p}_{r,n}(\tau) \, d\tau = p_{r,n}(t) - e^{-\xi t} \hat{p}_{r,n}(t)
\]

\[
= q_n \sum_{k=n}^{+\infty} p_{r,k}^*(t) + p_{r,n}^*(t) \sum_{k=n+1}^{+\infty} q_k - e^{-\xi t} \hat{p}_{r,n}(t).
\]

Making use of this equation in (2) we are finally led to (44). ⋄

Appendix

Hereafter we shall apply the results obtained above to some birth-death processes with catastrophes of interest in biological contexts.

A1. Birth process with catastrophes

Let \( N(t) \) denote the number of individuals present at time \( t \) in a birth process with catastrophes on \( S = \{r, r+1, \ldots\} \), with

\[
\alpha_n > 0 \quad (n = r, r+1, \ldots), \quad \beta_n = 0 \quad (n = r+1, r+2, \ldots).
\]

We assume that

\[
\sum_{n=r}^{+\infty} 1 = +\infty, \quad \sum_{n=r}^{+\infty} \frac{1}{\alpha_n} = +\infty,
\]

so that \( \sum_{n=r}^{+\infty} \hat{p}_{r,n}(t) = 1 \) for all \( t > 0 \) (cf. for instance Cox and Miller [7]). Since

\[
\hat{\pi}_{r,n}(\lambda) = \begin{cases} 
1 & \text{if } n = r, \\
\frac{\lambda + \alpha_r}{\lambda + \alpha_r + 1} & \text{if } n = r+1, r+2, \ldots,
\end{cases}
\]

we have:

\[
q_n = \xi \hat{\pi}_{r,n}(\xi) = \begin{cases} 
\frac{\xi}{\xi + \alpha_r} & \text{if } n = r, \\
\frac{\xi \alpha_r \alpha_{r+1} \cdots \alpha_{n-1}}{(\xi + \alpha_r)(\xi + \alpha_{r+1}) \cdots (\xi + \alpha_n)} & \text{if } n = r+1, r+2, \ldots,
\end{cases}
\]

Note that

(i) if \( \alpha_n = \alpha \), then the stationary distribution is geometric:

\[
q_n = \frac{\xi}{\xi + \alpha} \left( \frac{\alpha}{\xi + \alpha} \right)^{n-r}, \quad n = r, r+1, \ldots,
\]
(ii) if $\alpha_n = \xi (n + k)$, with $k$ a positive integer, then the stationary distribution is

$$q_n = \frac{r + k}{(n + k)(n + k + 1)}, \quad n = r, r + 1, \ldots.$$  

From (10) and (28) it follows $g_{j,r}(t) = d_{j,r}(t) = \xi e^{-\xi t}, \; t > 0.$

A2. Time non-homogeneous immigration-emigration process with catastrophes

Let $N(t)$ be the continuous-time Markov chains with state-space $\{0, 1, \ldots\}$ that describes the number of individuals in a population subject to an immigration-emigration process in the presence of catastrophes, with immigration rates $\alpha_n(t) = \alpha w(t)$, emigration rates $\beta_n(t) = \beta w(t)$ and catastrophe rate $\xi(t)$, with $\alpha > 0$, $\beta > 0$ and where $w(t)$ is a bounded, continuous and positive function such that $\int_{t_0}^{+\infty} w(t) \, dt = +\infty$. Making use of some well-known results (see, for instance, Medhi [13] and Bailey [1]), for $\hat{N}(t)$ in the absence of catastrophes we have:

$$\hat{p}_{j,n}(t \mid t_0) = \exp \left\{ - (\alpha + \beta) \int_{t_0}^{t} w(u) \, du \right\}$$

$$\times \left[ \rho^{(n-j)/2} I_{n-j} \left( 2\sqrt{\alpha \beta} \int_{t_0}^{t} w(u) \, du \right) + \rho^{(n-j-1)/2} I_{n+j+1} \left( 2\sqrt{\alpha \beta} \int_{t_0}^{t} w(u) \, du \right) 
+ (1 - \rho) \rho^j \sum_{k=n+j+2}^{+\infty} \rho^{-k/2} I_k(2\sqrt{\alpha \beta} \int_{t_0}^{t} w(u) \, du) \right]$$

(45)

and:

$$\hat{g}_{j,0}(t \mid t_0) = \frac{j w(t)}{\int_{t_0}^{t} w(u) \, du} \exp \left\{ - (\alpha + \beta) \int_{t_0}^{t} w(u) \, du \right\} \rho^{-j/2} I_j \left( 2\sqrt{\alpha \beta} \int_{t_0}^{t} w(u) \, du \right),$$

where $\rho = \alpha/\beta$. Hence, the transition probabilities and the first-visit-time density of $N(t)$ can be easily obtained by making use of (32) and (33).

A3. Immigration-emigration process with catastrophes

Let $N(t)$ denote the number of individuals in a population subject to an immigration-emigration process with catastrophes and characterized by constant birth and death rates $\alpha_n = \alpha$ and $\beta_n = \beta$, with state-space $S = \{0, 1, \ldots\}$. Since (see, for instance, Section 3.1 of Conolly [3])

$$\hat{\pi}_{0,n}(\xi) = \frac{1}{\xi} (1 - q) q^n, \quad n = 0, 1, 2, \ldots,$$
with
\[ q = \frac{\alpha + \beta + \xi - \sqrt{(\alpha + \beta + \xi)^2 - 4\alpha\beta}}{2\beta}, \] (46)

Eq. (35) shows that the steady-state probabilities for the corresponding process \( N(t) \) are given by:
\[ q_n = (1 - q) q^n, \quad n = 0, 1, 2, \ldots. \] (47)

By virtue of (37) and (47), the birth-death process \( N^*(t) \) is characterized by the following rates:
\[ \alpha_n^* = \frac{\alpha}{q} \quad (n \geq 0), \quad \beta_n^* = \beta q \quad (n \geq 1). \]

Making use of (44) for all \( t \geq 0 \) and \( j, n \in \{0, 1, \ldots\} \), we have:
\[ p_{j,n}(t) = e^{-\xi t} \hat{p}_{j,n}(t) - \hat{p}_{0,n}(t)] + (1 - q)^n \sum_{k=n}^{+\infty} p_{0,k}^* + p_{0,n}^* q^{n+1}, \] (48)

where (see for instance Eq. (9.13) of Medhi [13], p. 120)
\[ p_{0,n}^* = \exp \left\{ - \left( \frac{\alpha}{q} + \beta q \right) t \right\} \left[ \left( \frac{\rho}{q^2} \right)^{n/2} I_n(2\sqrt{\alpha\beta} t) + \left( \frac{\rho}{q^2} \right)^{(n-1)/2} I_{n+1}(2\sqrt{\alpha\beta} t) \right. \]
\[ \left. + \left( 1 - \frac{\rho}{q^2} \right) \sum_{k=n+2}^{+\infty} \left( \frac{\rho}{q^2} \right)^{-k/2} I_k(2\sqrt{\alpha\beta} t) \right\}, \]

with \( \rho = \alpha/\beta \), and where \( \hat{p}_{j,n}(t) \) identifies with expression (45) upon setting \( w(t) = 1 \). We note that Eq. (48) extends the results obtained in Section 2 of [8] in which the case of initial state \( j = 0 \) is treated.

Let us discuss the first-passage time of \( N(t) \) in state \( r \). Since \( r = 0 \), in this case we have (see, for instance, Conolly [6]):
\[ \hat{\gamma}_{j,0}(\xi) = \left( \frac{\beta}{\alpha} q \right)^j \left( \frac{\xi + \alpha + \beta - \sqrt{(\xi + \alpha + \beta)^2 - 4\alpha\beta}}{2\alpha} \right)^j, \] (49)

where \( q \) is defined in (46). Making use of Eqs. (15) and (16) we thus obtain mean and variance of the first-visit time in state 0 in the presence of catastrophes:
\[ E(T_{j,0}) = \frac{1}{\xi} \left[ 1 - \left( \frac{\beta}{\alpha} q \right)^j \right], \] (50)
\[ \text{Var}(T_{j,0}) = \frac{1}{\xi^2} \left[ 1 - \left( \frac{\beta}{\alpha} q \right)^2 + \frac{1}{\alpha} \frac{2\xi q}{2\beta q - (\xi + \alpha + \beta)} \right]. \] (51)

Finally, recalling (49), from (40) we obtain the mean time of the first occurrence of an effective catastrophe:
\[ E(C_{j,0}) = \frac{1}{\xi} + \frac{1 - q}{q} \left( \frac{\beta}{\alpha} q \right)^j. \]
A4. Immigration-death process with catastrophes

Let \( N(t) \) denote the number of individuals present at time \( t \) in an immigration-death process with catastrophes. Its state-space is \( \{0, 1, \ldots\} \), and its birth and death rates are

\[
\alpha_n = \nu \quad (n = 0, 1, \ldots), \quad \beta_n = \beta n \quad (n = 1, 2, \ldots),
\]

with \( \nu > 0 \) and \( \beta > 0 \). Let us set \( \rho = \nu/\beta \). It is well-known (cf. Cox and Miller [7]) that

\[
\hat{p}_{0,n}(t) = \left[ \frac{\rho}{n!} \left( 1 - e^{-\beta t} \right) \right]^n \exp \left\{ -\rho \left( 1 - e^{-\beta t} \right) \right\}, \quad n = 0, 1, \ldots .
\]

Then (see Eq. n. 3.383 of Gradshteyn and Ryzhik [10], pag. 318),

\[
\hat{\pi}_{0,n}(\lambda) = \frac{\rho^n}{n!} e^{-\rho} \frac{1}{\beta} B \left( n + 1, \frac{\lambda}{\beta} \right) \Phi \left( \frac{\lambda}{\beta}, \frac{\lambda}{\beta} + n + 1; \rho \right), \quad n = 0, 1, \ldots ,
\]

where \( B(a, b) = \Gamma(a) \Gamma(b)/\Gamma(a + b) \) is the Beta function and

\[
\Phi(a, c; x) = 1 + \sum_{n=1}^{+\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}
\]

is the confluent hypergeometric function of first kind (Kummer function), with \( (a)_n = a(a + 1) \cdots (a + n - 1) \). Hence, from (52) we have:

\[
q_n = \xi \hat{\pi}_{0,n}(\xi) = \frac{\rho^n}{n!} e^{-\rho} \frac{\xi}{\beta} B \left( n + 1, \frac{\xi}{\beta} \right) \Phi \left( \frac{\xi}{\beta}, \frac{\xi}{\beta} + n + 1; \rho \right), \quad n = 0, 1, \ldots .
\]

In particular, the probability of having asymptotically zero individuals is:

\[
q_0 = e^{-\rho} \frac{\xi}{\beta} \sum_{k=0}^{+\infty} \frac{\rho^k}{k!} \frac{1}{k + \xi/\beta}.
\]

In the special case \( \beta = \xi \), from (52) we have:

\[
q_n = \frac{1}{\rho} e^{-\rho} \sum_{i=n+1}^{+\infty} \frac{\rho^i}{i!}, \quad n = 0, 1, \ldots .
\]

A5. Immigration-birth-death process with catastrophes

Let \( N(t) \) be the number of individuals present at time \( t \) in an immigration-birth-death process with catastrophes, characterized by state-space \( \{0, 1, \ldots\} \) and by birth and death rates

\[
\alpha_n = \alpha n + \nu \quad (n = 0, 1, \ldots), \quad \beta_n = \beta n \quad (n = 1, 2, \ldots),
\]

with \( \nu > 0 \) and \( \beta > 0 \). Let us set \( \rho = \nu/\beta \). It is well-known (cf. Cox and Miller [7]) that

\[
\hat{p}_{0,n}(t) = \left[ \frac{\rho}{n!} \left( 1 - e^{-\beta t} \right) \right]^n \exp \left\{ -\rho \left( 1 - e^{-\beta t} \right) \right\}, \quad n = 0, 1, \ldots .
\]

Then (see Eq. n. 3.383 of Gradshteyn and Ryzhik [10], pag. 318),

\[
\hat{\pi}_{0,n}(\lambda) = \frac{\rho^n}{n!} e^{-\rho} \frac{1}{\beta} B \left( n + 1, \frac{\lambda}{\beta} \right) \Phi \left( \frac{\lambda}{\beta}, \frac{\lambda}{\beta} + n + 1; \rho \right), \quad n = 0, 1, \ldots ,
\]

where \( B(a, b) = \Gamma(a) \Gamma(b)/\Gamma(a + b) \) is the Beta function and

\[
\Phi(a, c; x) = 1 + \sum_{n=1}^{+\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!}
\]

is the confluent hypergeometric function of first kind (Kummer function), with \( (a)_n = a(a + 1) \cdots (a + n - 1) \). Hence, from (52) we have:

\[
q_n = \xi \hat{\pi}_{0,n}(\xi) = \frac{\rho^n}{n!} e^{-\rho} \frac{\xi}{\beta} B \left( n + 1, \frac{\xi}{\beta} \right) \Phi \left( \frac{\xi}{\beta}, \frac{\xi}{\beta} + n + 1; \rho \right), \quad n = 0, 1, \ldots .
\]

In particular, the probability of having asymptotically zero individuals is:

\[
q_0 = e^{-\rho} \frac{\xi}{\beta} \sum_{k=0}^{+\infty} \frac{\rho^k}{k!} \frac{1}{k + \xi/\beta}.
\]

In the special case \( \beta = \xi \), from (52) we have:

\[
q_n = \frac{1}{\rho} e^{-\rho} \sum_{i=n+1}^{+\infty} \frac{\rho^i}{i!}, \quad n = 0, 1, \ldots .
\]
with $\alpha > 0$, $\nu > 0$ and $\beta > 0$. As is well-known, in absence of catastrophes for $\nu = 0, 1, \ldots$ there results:

$$\hat{p}_{0,n}(t) = \begin{cases} \left( \frac{1}{1+\alpha t} \right)^\frac{\nu}{\alpha} \frac{1}{n!} \left( \frac{\nu}{\alpha} \right)_n \left( \frac{\alpha t}{1+\alpha t} \right)^n, & \alpha = \beta \\ \left[ \frac{\alpha - \beta}{\alpha e^{(\alpha-\beta)t} - \beta} \right]^\frac{\nu}{\alpha} \frac{1}{n!} \left( \frac{\nu}{\alpha} \right)_n \left\{ \frac{\alpha \left[ e^{(\alpha-\beta)t} - 1 \right]}{\alpha e^{(\alpha-\beta)t} - \beta} \right\}^n, & \alpha \neq \beta, \end{cases}$$

with

$$E[\hat{N}(t) | \hat{N}(0) = j] = \begin{cases} j + \nu t, & \alpha = \beta \\ j e^{(\alpha-\beta)t} + \frac{\nu \left[ e^{(\alpha-\beta)\xi} - 1 \right]}{(\alpha - \beta - \xi)}, & \alpha \neq \beta + \xi. \end{cases} \tag{53}$$

Let us now consider the process in the presence of catastrophes, whose transient probabilities have been studied in [16] by making use of a generating function approach. From (3) it follows:

$$E[N(t) | N(0) = j] = \begin{cases} j + \nu t, & \alpha = \beta + \xi \\ j e^{(\alpha-\beta-\xi)t} + \frac{\nu \left[ e^{(\alpha-\beta)\xi} - 1 \right]}{(\alpha - \beta - \xi)}, & \alpha \neq \beta + \xi. \end{cases} \tag{53}$$

It is interesting to note that the mean of $N(t)$ identifies with that of $\hat{N}(t)$ if in Eq. 53 we substitute $\beta + \xi$ with $\beta$. Moreover, from (53) we have

$$\lim_{t \to +\infty} E[N(t) | N(0) = j] = \begin{cases} \frac{\nu}{\beta + \xi - \alpha}, & \alpha < \beta + \xi \\ +\infty, & \alpha \geq \beta + \xi. \end{cases}$$

We shall now obtain the steady-state distribution of $N(t)$. We recall that in absence of catastrophes the steady-state distribution exists only if $\alpha < \beta$.

(i) Let $\alpha < \beta$. Then,

$$\pi_{0,n}(\lambda) = \frac{1}{\alpha} \left( \frac{\alpha}{\beta} \right)^{n+1} \frac{1}{n!} \left( \frac{\nu}{\alpha} \right)_n \left( 1 - \frac{\alpha}{\beta} \right)^{\frac{\nu}{\alpha} - 1} B \left( \frac{\lambda}{\beta - \alpha}, n + 1 \right) \times F \left( \frac{\nu}{\alpha} + n, \frac{\lambda}{\beta - \alpha}; n + 1 + \frac{\lambda}{\beta - \alpha}; \frac{\alpha}{\beta} \right), \quad n = 0, 1, \ldots,$$

where:

$$F(a, b; c; x) = 1 + \sum_{n=1}^{+\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!}$$
Hence, from (36) it follows
\[ q_n = \xi \hat{\pi}_{0,n}(\xi) = \frac{\xi}{\alpha} \left( \frac{\alpha}{\beta} \right)^{n+1} \frac{1}{n!} \left( \frac{\nu}{\alpha} \right)_n \left( 1 - \frac{\alpha}{\beta} \right)^{\frac{n}{\alpha}-1} B \left( \frac{\xi}{\beta - \alpha}, n + 1 \right) \]
\[ \times F \left( \frac{\nu}{\alpha} + n, \frac{\xi}{\beta - \alpha}; n + 1 + \frac{\xi}{\beta - \alpha}; \frac{\alpha}{\beta} \right), \quad n = 0, 1, \ldots. \]

(ii) If \( \alpha = \beta \), then
\[ \hat{\pi}_{0,n}(\lambda) = \frac{1}{\alpha} \left( \frac{\nu}{\alpha} \right)_n \left( \frac{\lambda}{\alpha} \right)^{\frac{n}{\alpha}-1} \Psi \left( \frac{\nu}{\alpha} + n, \frac{\nu}{\alpha}; \frac{\lambda}{\alpha} \right), \]
where
\[ \Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{+\infty} e^{-xt} t^{a-1} (1 + t)^{c-1} \, dt, \quad \Re(a) > 0 \]
is the confluent hypergeometric function of the second kind. Since \( \Psi(a-c+1, 2-c; x) = x^{c-1} \Psi(a, c; x) \), from (36) we thus have:
\[ q_n = \xi \hat{\pi}_{0,n}(\xi) = \left( \frac{\xi}{\alpha} \right)^{\frac{n}{\alpha}} \left( \frac{\nu}{\alpha} \right)_n \Psi \left( \frac{\nu}{\alpha} + n, \frac{\nu}{\alpha}; \frac{\xi}{\alpha} \right), \quad n = 0, 1, \ldots. \]

In particular, from identity \( \Psi(a, a; x) = e^x \Gamma(1-a, x) \), we obtain:
\[ q_0 = \left( \frac{\xi}{\alpha} \right)^{\frac{n}{\alpha}} e^{x/\alpha} \Gamma \left( 1 - \frac{\nu}{\alpha}, \frac{\xi}{\alpha} \right). \]

(iii) Finally, if \( \alpha > \beta \) there results:
\[ \hat{\pi}_{0,n}(\lambda) = \frac{1}{\alpha} \left( \frac{\nu}{\alpha} \right)_n \left( 1 - \frac{\beta}{\alpha} \right)^{\frac{n}{\alpha}-1} B \left( \frac{\lambda}{\alpha - \beta} + \frac{\nu}{\alpha}, n + 1 \right) \]
\[ \times F \left( \frac{\nu}{\alpha} + n, \frac{\lambda}{\alpha - \beta} + \frac{\nu}{\alpha}; \frac{\lambda}{\alpha - \beta} + \frac{\nu}{\alpha} + n + 1; \frac{\beta}{\alpha} \right), \quad n = 0, 1, \ldots, \]
that by virtue of (36) implies:
\[ q_n = \xi \hat{\pi}_{0,n}(\xi) = \frac{\xi}{\alpha} \left( \frac{\nu}{\alpha} \right)_n \left( 1 - \frac{\beta}{\alpha} \right)^{\frac{n}{\alpha}-1} B \left( \frac{\xi}{\alpha - \beta} + \frac{\nu}{\alpha}, n + 1 \right) \]
\[ \times F \left( \frac{\nu}{\alpha} + n, \frac{\xi}{\alpha - \beta} + \frac{\nu}{\alpha}; \frac{\xi}{\alpha - \beta} + \frac{\nu}{\alpha} + n + 1; \frac{\beta}{\alpha} \right), \quad n = 0, 1, \ldots. \]

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