Nonsmooth Herglotz variational principle

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Abstract—In this paper, the theory of smooth action-dependent Lagrangian mechanics (also known as contact Lagrangians) is extended to a non-smooth context appropriate for collision problems. In particular, we develop a Herglotz variational principle for non-smooth action-dependent Lagrangians which leads to the preservation of energy and momentum at impacts. By defining appropriately a Legendre transform, we can obtain the Hamilton equations of motion for the corresponding non-smooth Hamiltonian system. We apply the result to a billiard problem in the presence of dissipation.

I. INTRODUCTION

Lagrangian and Hamiltonian systems are well-known for representing a large class of mechanical systems which are conservative. However, many systems of interest in physics and engineering exhibit a dissipative, rather than a conservative, behaviour. As it is well-known, the equations of motion of a Lagrangian system (i.e., the Euler-Lagrange equations) can be obtained variationally from Hamilton’s principle of least action. A dissipative term may be added to the equations of motion by means of considering an external force besides the Lagrangian or the Hamiltonian function [1], [2], [3], [4]. Alternatively, in order to describe mechanical systems with a damping term one can make use of a generalization of Hamilton’s Principle: the so-called Herglotz Principle. Essentially, one considers Lagrangian functions that can depend explicitly on the action (the so-called action-dependent Lagrangians) and, taking variations of the action, one obtains equations of motion, called Herglotz equations [5], that are like ordinary Euler-Lagrange equations with an extra term accounting for dissipation. In recent years, action-dependent Lagrangians have gained popularity in the theoretical physics [6], [7], [8], geometric mechanics [9], [10], [11], [12], [13] and control [14], [15], [16] communities, since they can be used for modelling dissipative systems of particles and fields, as well as certain thermodynamical systems.

The possible trajectories considered both in Hamilton’s and Herglotz’s principles are usually smooth curves. Nevertheless, many mechanical systems of interest have non-smooth trajectories. As a matter of fact, the trajectories described by a system with impacts are not smooth, as it happens with the so-called hybrid systems. Mechanical systems with impacts are usually modeled as hybrid systems. Hybrid systems are dynamical systems with continuous-time and discrete-time components in their dynamics. This class of dynamical systems is capable of modeling several physical systems, such as UAVs (unmanned aerial vehicles) systems [17] and bipedal robots among many others [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]. The disadvantage of this approach is that the impact map –the map characterizing the change of velocity in the instant of the impact– has to be defined ad hoc or obtained in some phenomenological fashion, e.g., by the Newtonian impact law [28]. Another approach consists on considering an impulsive constraint acting on the instant of the impact [24], [29], [30], [31], [32]. Alternatively, in order to characterize the dynamics of a mechanical system with impacts, one can consider a variational principle for which the possible curves are not smooth at certain points. An extension of Hamilton’s Principle to a nonsmooth setting was developed by Fetecau, Marsden, Ortiz and West [33]. Inspired by their approach, in this paper we develop a non-smooth Herglotz principle for dissipative Lagrangian systems with impacts. The main advantage of this approach is that one can characterize the dynamics of the system with dissipation and impacts by means of just the variational principle, without the need of considering additional forces, maps or constraints.

The remainder of the paper is structured as follows. In Section II Herglotz variational principle for non-smooth Lagrangian and Hamiltonian systems is presented. Section III presents a case study: a billiard with dissipation. We finish the paper with some outlooks in Section IV.

II. Herglotz PRINCIPLE FOR NONSMOOTH ACTION-DEPENDENT LAGRANGIANS

A. Herglotz variational principle for smooth Lagrangians

Geometrically, the configuration space (the space of positions) Q of a mechanical system is a differentiable manifold of dimension n with local coordinates q = (q₁, ..., qₙ). At each point q ∈ Q, the tangent space to Q at q, denoted by Tₗ Q, is the vector space formed by the vectors which are tangent to the curves in Q passing by q. The union of the tangent spaces for every q ∈ Q is called the tangent bundle of Q and denoted by TQ. It represents the space of positions and velocities of the system. The tangent bundle has local coordinates v_q = (q^1, ..., q^n, q^1_t, ..., q^n_t) ∈ TQ with dim(TQ) = 2n and canonical projection τ_q: TQ → Q, v_q → q.

A Lagrangian L : TQ × ℝ → ℝ is said to be regular
if det $W \neq 0$, where $W = (W_{ij}) := \left( \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right)$ for all $i, j$ with $1 \leq i, j \leq n$.

Consider a Lagrangian function $L : TQ \times \mathbb{R} \rightarrow \mathbb{R}$ and fix two points $q_1, q_2 \in Q$ and an interval $[a, b] \subset \mathbb{R}$. Let us denote by $\Omega(q_1, q_2, [a, b]) \subseteq \mathcal{C}^\infty([a, b] \rightarrow Q)$ the space of smooth curves $\sigma$ such that $\sigma(a) = q_1$ and $\sigma(b) = q_2$. This space has the structure of an infinite dimensional smooth manifold whose tangent space at $\sigma$ is given by the set of vector fields over $\sigma$ that vanish at the endpoints [34, Proposition 3.8.2], that is,

$$T_\sigma \Omega(q_1, q_2, [a, b]) = \{ v_\sigma \in \mathcal{C}^\infty([a, b] \rightarrow TQ) \mid \tau_Q \circ v_\sigma = \sigma, \quad v_\sigma(a) = 0, \quad v_\sigma(b) = 0 \}.$$

The elements of $T_\sigma \Omega(q_1, q_2, [a, b])$ will be called infinitesimal variations of the curve $\sigma$. Let

$$Z : \mathcal{C}^\infty([a, b] \rightarrow Q) \rightarrow \mathcal{C}^\infty([a, b] \rightarrow \mathbb{R})$$

be the operator that assigns to each curve $\sigma$ the function $Z(\sigma)$ that solves the following ordinary differential equation (ODE):

$$\frac{dZ(\sigma)(t)}{dt} = L(\sigma(t), \dot{\sigma}(t), Z(\sigma)(t)), \quad Z(\sigma)(a) = 0.$$

Now we define the action functional $A$ as the map which assigns to each curve the solution to the previous ODE evaluated at the endpoint, namely,

$$A : \Omega(q_1, q_2, [a, b]) \rightarrow \mathbb{R} \quad \sigma \mapsto Z(\sigma)(b).$$

We will say that a path $\sigma \in \Omega(q_1, q_2, Q)$ satisfies the Herglotz variational principle if it is a critical point of $A$, i.e.,

$$T_\sigma A = 0.$$

These critical points are curves which satisfy the Herglotz equations [5], [35], [11]:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial \dot{z}} = 0, \quad i = 1, \ldots, n.$$

Observe that Herglotz equations are like Euler-Lagrange equations with an additional term due to the dependence of the Lagrangian on the action: $\frac{\partial L}{\partial z} \frac{dc}{dt}$. For instance, if the Lagrangian is of the form $L(q, \dot{q}, z) = \frac{1}{2} \dot{q}^T \dot{q} - V(q) - \gamma z$ for some $\gamma \in \mathbb{R}$, this extra term is $-\gamma \dot{q}^i$, corresponding to a dissipation linear in the velocities.

**B. Herglotz principle for nonsmooth Lagrangians**

Since trajectories with impacts are not smooth curves, the space of curves will no longer be a smooth manifold. Therefore, Herglotz variational principle cannot be generalized to a nonsmooth setting in a straightforward manner. In order to overcome this problem, we make use of the Fetecau, Marsden, Ortiz and West’s approach [33]: extend the problem to the nonautonomous case so that both position variables and time are functions of a parameter $\tau$. In this way, the impact can be fixed in $\tau$ space while remaining variable in both configuration and time spaces. Additionally, this allows to define a path space $\mathcal{M}$ which is indeed a smooth manifold. By taking variations on this submanifold, we shall obtain a nonsmooth Herglotz principle.

Consider a configuration manifold $Q$ and a submanifold with boundary $C \subset Q$, which represent the subset of admissible configurations. Let $L : TQ \times \mathbb{R} \rightarrow \mathbb{R}$ be a regular Lagrangian. Let us introduce the path space

$$\mathcal{M} := T \times Q([0, 1], \tau_i, \partial C, Q),$$

where

$$\mathcal{T} := \{ c_t \in \mathcal{C}^\infty([0, 1], \mathbb{R}) \mid c_t^i > 0 \text{ in } [0, 1] \},$$

$$\mathcal{Q}([0, 1], \tau_i, \partial C, Q) := \{ c_q : [0, 1] \rightarrow Q \mid c_q \text{ is a } C^0, \text{ piecewise } C^2 \text{ curve},$$

$$c_q(\tau) \text{ has only one singularity at } \tau_i, c_q(\tau_i) \in \partial C \}.$$

Here $c_t^i(0) = \lim_{t \rightarrow 0^+} c_t^i(t)$ and $c_t^i(1) = \lim_{t \rightarrow 1^-} c_t^i(t)$ are understood, and similarly for $c_q^i$ and higher derivatives. A path $c \in \mathcal{M}$ is a pair $c = (c_t, c_q)$. Given a path, the associated curve $q : [c_t(0), c_t(1)] \rightarrow Q$ is given by $q(t) = c_q \circ c_t^{-1}$. Let $C$ denote the set of all paths $q(t) \in Q$.

The moment of impact $\tau_i \in (0, 1)$ is fixed in the $\tau$-space, but can vary in the $t$-space according to $t_i = c_t(\tau_i)$. One can show that $\mathcal{T}$ and $\mathcal{Q}([0, 1], \tau_i, \partial C, Q)$, and hence $\mathcal{M}$, are smooth manifolds [33]. The tangent space at $c_q \in \mathcal{Q}$ is given by

$$T_{c_q} \mathcal{Q} = \{ v : [0, 1] \rightarrow TQ \mid v \text{ is a } C^0 \text{ piecewise } C^2 \text{ map},$$

$$v(\tau_i) \in T_{c_q(\tau_i)} \partial C \}.$$

Let $\tilde{\Omega}(q_1, q_2, [0, 1]) \subset \mathcal{M}$ be the subset of curves such that $c_q(0) = q_1$ and $c_q(1) = q_2$. Consider the operator

$$\tilde{Z} : \tilde{\Omega}(q_1, q_2, [0, 1]) \rightarrow \mathcal{T}$$

that assigns to each $c_q \in \mathcal{M}$ the solution of the following ODE:

$$\frac{d\tilde{Z}}{d\tau} = L \left( c_q(\tau), c_q^i(\tau), \tilde{Z}(c_q, c_t)(\tau) \right) c_t^i(\tau),$$

$$\tilde{Z}(c_q, c_t)(0) = \tilde{z}_0,$$

and denote by $\tilde{A}$ the functional

$$\tilde{A} : \tilde{\Omega}(q_1, q_2, [0, 1]) \rightarrow \mathbb{R} \quad (c_q, c_t) \mapsto \tilde{Z}(c_q, c_t)(1).$$

**Theorem 1** (Nonsmooth Herglotz variational principle). Let $L : TQ \times \mathbb{R}$ be a smooth and regular Lagrangian function. Let $E_L = \dot{q}^T \frac{\partial L}{\partial \dot{q}} - L$ denote the energy. Let $c = (c_q, c_t)$ be a curve in $\tilde{\Omega}(q_1, q_2, [0, 1])$, and let $\chi(\tau) = \left( c_q(\tau), c_q^i(\tau), \tilde{Z}(c_q, c_t)(\tau) \right) \in TQ \times \mathbb{R}$. Then, $c$ is a critical point

1 Notice that, since $c_t^i > 0$, each $c_t \in \mathcal{T}$ is injective and thus invertible (if it is not surjective, it suffices to restrict the codomain), so $\mathcal{Q}$ is well-defined.
of $\hat{A}$ if and only if
\[
\frac{\partial L}{\partial q^i}(\chi(\tau)) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau)) + \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau)) \frac{\partial L}{\partial z}(\chi(\tau)) = 0, \quad (1a)
\]
\[
\frac{d}{dt} E_L(\chi(\tau)) = \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau)) E_L(\chi(\tau)), \quad (1b)
\]
for $\tau \in [0, \tau_1) \cup (\tau_1, 1]$, and
\[
\frac{\partial L}{\partial q^i}(\chi(\tau_i^-)) \nu_i = \frac{\partial L}{\partial q^i}(\chi(\tau_i^+)) \nu_i,
\]
\[
E_L(\chi(\tau_i^-)) = E_L(\chi(\tau_i^+)), \quad (2)
\]
where $\chi(\tau_i^\pm) = \lim_{\tau \to \tau_i^\pm} \chi(\tau)$, for any $v \in T_{c(s), \theta} \partial C$.

It is worth mentioning that by “nonsmooth” we refer to the fact that the curves in $\hat{O}(q_1, q_2, [0, 1])$ are not differentiable, even though the Lagrangian function $L$ is differentiable.

**Proof.** Let $c = (c_q, c_i) \in \hat{O}(q_1, q_2, [0, 1])$ be a curve. Consider a smoothly parametrized family of curves $c^\nu = (c_q^\nu, c_i^\nu)$ in $\hat{O}(q_1, q_2, [0, 1])$ such that $c^0 = c$, $u = \frac{dc^\nu}{d\lambda}|_{\lambda = 0}$, and
\[
\theta = \frac{dc^\nu}{d\lambda}|_{\lambda = 0}.
\]
Let $\varphi = T_{c}(\bar{z}(u, \theta))$, so that $T_{c}(\bar{z}(u, \theta)) = \varphi(1)$. Observe that $\varphi(0) = 0$, since $\bar{z}(c^\nu(0)) = \bar{z}$ for every $\nu$. We have that
\[
\varphi(\tau) = \frac{d}{d\tau} \bar{z}(c^\nu(\tau))|_{\nu = 0} = \frac{d}{d\lambda} \bar{z}(c^\nu(\tau))|_{\nu = 0} = \frac{d}{d\lambda} \left( c_q^\nu(c^\nu(\tau)), c_i^\nu(c^\nu(\tau)) \right)\]
\[
= \left[ \frac{\partial L}{\partial q^i}(\chi(\tau)) u^i(\tau) + \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau)) \frac{1}{c_i^\nu}(\tau) u^i(\tau) \right.
\]
\[
- \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau)) \frac{c_i^\nu(\tau)}{c_q^\nu(\tau)} \theta(\tau) \left. + \frac{\partial L}{\partial z}(\chi(\tau)) \varphi(\tau) \right] c_i^\nu(\tau)
\]
\[
+ L \left( c_q^\nu(\tau), c_i^\nu(\tau), \bar{z}(c_q^\nu, c_i^\nu)(\tau) \right) \theta(\tau).
\]

An integrating factor for this ODE is $\mu(\tau) = \exp \left( -\int_0^\tau \frac{d}{d\sigma}(\chi(s)) c_i^\nu(s) ds \right)$, so
\[
\varphi(\tau) \mu(\tau) = \int_0^\tau \mu(s) c_i^\nu(s) \left[ \frac{\partial L}{\partial q^i}(\chi(s)) u^i(s) + \frac{\partial L}{\partial \dot{q}^i}(\chi(s)) \frac{1}{c_i^\nu(s)} u^i(s) - \frac{\partial L}{\partial \dot{q}^i}(\chi(s)) \frac{c_i^\nu(s)}{c_q^\nu(s)} \theta(s) \right] ds
\]
\[
+ \int_0^\tau \mu(s) L(\chi(s)) \theta(\tau) ds
\]
\[
= \int_0^\tau \mu(s) c_i^\nu(s) u^i(s) \left[ \frac{\partial L}{\partial \dot{q}^i}(\chi(s)) \right] ds
\]
\[
+ \int_0^\tau \mu(s) \left[ L(\chi(s)) - \frac{\partial L}{\partial q^i}(\chi(s)) \frac{c_i^\nu(s)}{c_q^\nu(s)} \right] ds.
\]

Integrating by parts and taking into account that $u(0) = u(1) = 0$ and $\theta(0) = \theta(1) = 0$, we obtain
\[
\varphi(1) \mu(1) = \int_0^{\tau_1} \mu(s) c_i^\nu(s) u^i(s) \left( \frac{\partial L}{\partial q^i}(\chi(s)) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(\chi(s)) \right)
\]
\[
+ \frac{\partial L}{\partial \dot{q}^i}(\chi(s)) \frac{\partial L}{\partial \dot{q}^i}(\chi(s)) \right] ds
\]
\[
- \int_0^{\tau_1} \mu(s) \theta(s) \frac{d}{ds} \left[ L(\chi(s)) - \frac{\partial L}{\partial q^i}(\chi(s)) \frac{c_i^\nu(s)}{c_q^\nu(s)} \right] ds
\]
\[
+ \int_0^{\tau_1} \mu(s) \theta(s) \left[ L(\chi(s)) - \frac{\partial L}{\partial q^i}(\chi(s)) \frac{c_i^\nu(s)}{c_q^\nu(s)} \right] ds
\]
\[
+ \int_0^{\tau_1} \mu(s) \theta(s) \left[ L(\chi(s)) - \frac{\partial L}{\partial q^i}(\chi(s)) \frac{c_i^\nu(s)}{c_q^\nu(s)} \right] ds.
\]

Since $\mu(\tau)$ is nonzero, $\varphi(0)$ vanishes for every $(u, \theta)$ (i.e., $\tau$ is a critical point of $\hat{A}$ if and only if
\[
\frac{\partial L}{\partial q^i}(\chi(\tau)) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau)) \frac{\partial L}{\partial z}(\chi(\tau)) = 0,
\]
\[
\frac{d}{d\tau} \left[ L(\chi(\tau)) - \frac{\partial L}{\partial q^i}(\chi(\tau)) \frac{c_i^\nu(s)}{c_q^\nu(s)} \right]
\]
\[
= \left[ L(\chi(\tau)) - \frac{\partial L}{\partial q^i}(\chi(\tau)) \frac{c_i^\nu(s)}{c_q^\nu(s)} \right] \frac{\partial L}{\partial z}(\chi(\tau)) c_i^\nu(\tau),
\]
for $\tau \in [0, \tau_1) \cup (\tau_1, 1]$, and
\[
\frac{\partial L}{\partial q^i}(\chi(\tau_i^-)) \nu_i = \frac{\partial L}{\partial \dot{q}^i}(\chi(\tau_i^+)) \nu_i,
\]
\[
L(\chi(\tau_i^-)) - \frac{\partial L}{\partial q^i}(\chi(\tau_i^-)) \frac{c_i^\nu(\tau_i^-)}{c_q^\nu(\tau_i^-)}
\]
\[
= L(\chi(\tau_i^+)) - \frac{\partial L}{\partial q^i}(\chi(\tau_i^+)) \frac{c_i^\nu(\tau_i^+)}{c_q^\nu(\tau_i^+)}
\]
for any $v \in T_{c(s), \theta} \partial C$. The result follows from the chain rule and the definition of $E_L$.

**Remark 1.** Equation (1b) is redundant. Indeed, we have that
\[
\frac{dE_L}{dt} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \right)
\]
\[
= \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial \dot{z}} \frac{\partial \dot{z}}{\partial \dot{q}^i} \right) \dot{q}^i - \frac{\partial L}{\partial \dot{z}} \frac{\partial \dot{z}}{\partial \dot{q}^i}
\]
\[
= \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - \frac{\partial L}{\partial \dot{z}} \frac{\partial \dot{z}}{\partial \dot{q}^i} = E_L \frac{\partial L}{\partial \dot{z}},
\]
along solutions of the Herglotz equations (1a). This equation expressed the rate of dissipation of the energy.

As it is well-known, the conserved quantities of an action-independent Lagrangian can be used to integrate its equations.
of motion. Although action-dependent Lagrangians may have conserved quantities as well, it is more natural to consider the so-called dissipated quantities. They can be used to integrate the equations of motion of an action-dependent Lagrangian system (see Section III).

**Definition 1.** Given an action-dependent Lagrangian function \( L : TQ \times \mathbb{R} \to \mathbb{R} \), a **dissipated quantity** is a function \( f : TQ \times \mathbb{R} \to \mathbb{R} \) which is dissipated at the same rate as the energy, namely,

\[
\frac{d}{dt} f(q(t), \dot{q}(t), z(t)) = \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t), z(t)) f(q(t), \dot{q}(t), z(t)),
\]

where \((q(t), \dot{q}(t), z(t))\) is a curve on \( TQ \times \mathbb{R} \) that satisfies Herlitz equations.

**C. Nonsmooth Hamiltonian equations for contact Hamiltonian systems**

Given a Lagrangian function \( L : TQ \times \mathbb{R} \to \mathbb{R} \), we can define the Legendre transform \( \text{Leg} : TQ \times \mathbb{R} \to T^*Q \times \mathbb{R} \) by

\[
\text{Leg} : (q^i, \dot{q}^i, z) \mapsto \left( q^i, \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}, z), z \right).
\]

Hereafter, we will assume that the Lagrangian \( L \) is hyper-regular, i.e., the Legendre transform is a diffeomorphism. The Hamiltonian function \( H \) is then given by \( H = E_L \circ \text{Leg}^{-1} \).

The Hamiltonian counterpart of Theorem 1 is as follows.

**Proposition 2 (Hamiltonian nonsmooth Herlitz principle).** Let \( H \) be a regular Hamiltonian function on \( T^*Q \times \mathbb{R} \). Let \( \xi = (q^i, p_i, z) \) be a continuous and piecewise \( C^2 \) curve on \( T^*Q \times \mathbb{R} \) whose only singularities occur at \( t_i \). Then,

\[
\frac{d\xi}{dt} = \frac{\partial H}{\partial p_i}(\xi(t)), \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}(\xi(t)) - p_i \frac{\partial H}{\partial z}(\xi(t)),
\]

for \( t \neq t_i \), and

\[
p_i(\xi(t_i^-)) = p_i(\xi(t_i^+)), \quad H(\xi(t_i^-)) = H(\xi(t_i^+)),
\]

where \( \xi(t_i^\pm) = \lim_{\tau \to t_i^\pm} \xi(\tau) \).

**III. CASE STUDY: BILLIARD WITH DISSIPATION**

Consider a particle moving in the plane confined to the surface \( C \subset \mathbb{R}^2 \) defined by \( x^2 + y^2 = 1 \). The Lagrangian \( L : T^2 \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is given by

\[
L(x, \dot{x}, \dot{y}, z) = \frac{1}{2}(x^2 + y^2) - \gamma z,
\]

where \( \gamma \) is a real constant. Herlitz equations thus yield

\[
\dot{x} = -\gamma \dot{x}, \quad \dot{y} = -\gamma \dot{y}, \quad \ddot{z} = L(x, \dot{x}, \dot{y}, z).
\]

Their solutions for the initial conditions \( x(0) = x_0, \ y(0) = y_0, \ \dot{x}(0) = \dot{x}_0, \ \dot{y}(0) = \dot{y}_0 \) and \( E_L(0) = E_0 \) are

\[
x(t) = x_0 + \frac{x_0}{\gamma} (1 - e^{-\gamma t}), \\
y(t) = y_0 + \frac{y_0}{\gamma} (1 - e^{-\gamma t}), \\
z(t) = -\frac{1}{\gamma} \left[ \frac{1}{2} (x_0^2 + y_0^2) e^{-2\gamma t} + E_0 e^{-\gamma t} \right].
\]

Observe that \( v \in T_{(x, y)} \partial C \), if and only if \( dh(v) = 0 \), where \( h(x, y) = 1 - x^2 - y^2 \) is the function characterizing the surface \( C \), so \( v = v_x \partial_x - \frac{\dot{y}}{v_x} \partial_y \). Conditions (2) can then be written as

\[
\dot{x} - \frac{x}{y} \dot{y} = \dot{x}^+ - \frac{x}{y} \dot{y}^+, \\
\frac{1}{2} ((\dot{x}^-)^2 + (\dot{y}^-)^2) + \gamma z|_{t_i^-} = \frac{1}{2} ((\dot{x}^+)^2 + (\dot{y}^+)^2) + \gamma z|_{t_i^+},
\]

where \( \dot{x}^\pm = \dot{x}(t_i^\pm) \) and \( \dot{y}^\pm = \dot{y}(t_i^\pm) \). Now, \( z|_{t_i^+} = z|_{t_i^-} \), so

\[
\dot{x}^+ = -\dot{x}^- x^2 + \dot{x}^- y^2 - 2\dot{y}^- xy, \\
\dot{y}^+ = -\frac{2\dot{x}^- xy + \dot{y}^- x^2 - \dot{y}^- y^2}{x^2 + y^2}.
\]

In polar coordinates,

\[
L(r, \theta, \dot{r}, \dot{\theta}, z) = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - \gamma z.
\]

One can check that \( \ell = r^2 \dot{\theta} \) is a dissipated quantity, namely,

\[
\frac{d\ell}{dt} = \frac{\partial L}{\partial z} \ell = -\gamma \ell,
\]

so we can write \( \ell = \ell_0 e^{-\gamma t} \), where \( \ell_0 = \ell(t = 0) \). The equations of motions outside the impact surface can thus be expressed as

\[
\dot{r} = -r \gamma \dot{\theta} + \ell_0 \frac{2}{r} e^{-\gamma t}, \\
\dot{\theta} = \ell_0 \frac{r^2}{2} e^{-\gamma t}.
\]

On the other hand, deriving \( r^2 = x^2 + y^2 \) we obtain \( 2r \dot{r} = 2(x \dot{x} + y \dot{y}) \), so

\[
r \dot{r}^+ = x \dot{x}^+ + y \dot{y}^+ = \frac{x (-\dot{x}^- x^2 + \dot{y}^- y^2 - 2\dot{y}^- xy)}{x^2 + y^2} + \frac{y (-2\dot{x}^- xy + \dot{y}^- x^2 - \dot{y}^- y^2)}{x^2 + y^2} = -x \dot{x}^- + y \dot{y}^- = -r \dot{r}^-,
\]

hence \( r \dot{r}^+ = -r \dot{r}^- \). Similarly, deriving \( \theta = \arctan(y/x) \) yields

\[
\dot{\theta}^+ = \frac{1}{1 + (y/x)^2} \left( \frac{x \dot{y}^- - y \dot{x}^-}{x^2} \right) = \frac{x (-2\dot{x}^- x^2 + \dot{y}^- x^2 - \dot{y}^- y^2)}{x^2 + y^2} \frac{y (-2\dot{x}^- xy + \dot{y}^- x^2 - \dot{y}^- y^2)(x^2 + y^2)^2}{(x^2 + y^2)^2} + \frac{y (-\dot{x}^- x^2 + \dot{y}^- y^2)}{x^2 + y^2} = \dot{\theta}^-.
\]
From Proposition 2 we can obtain the trajectories of the system:
\[
\frac{dx}{dt} = p_x, \quad \frac{dy}{dt} = p_y, \quad \frac{dp_x}{dt} = - p_x \gamma, \quad \frac{dp_y}{dt} = - p_y \gamma,
\]
for \( t \neq t_i \), and
\[
p_x^\pm = \frac{- p_x x^2 + p_y y^2 - 2 p_y x y}{x^2 + y^2},
\]
\[
p_y^\pm = \frac{- 2 p_x x y + p_y y^2 - p_y y^2}{x^2 + y^2},
\]
where \( p_x^\pm = p_x(t_i) \) and \( p_y^\pm = p_y(t_i) \).

### A. Elliptical billiard

Suppose now that the particle is confined to the surface
\[
C = \left\{(\frac{x}{a})^2 + (\frac{y}{b})^2 \leq 1\right\},
\]
where \( a \) and \( b \) are positive constants. Then, from conditions (2) we obtain
\[
\dot{x}^+ = \frac{a^4 x^2 - 2 a^2 b^2 y^2 x y - b^4 x^2}{a^4 y^2 + b^4 x^2},
\]
\[
\dot{y}^+ = \frac{- a^4 y^2 - 2 a^2 b^2 x^2 y + b^4 y^2}{a^4 y^2 + b^4 x^2},
\]
or, in polar coordinates,
\[
\dot{r}^+ = \frac{r}{4r (a^4 \sin^2(\theta) + b^4 \cos^2(\theta))} \left[ 2 r \dot{\theta}^+ (b^4 - a^4) \sin(2\theta) + 2 \dot{r}^+ (a^4 - b^4) \cos(2\theta) 
\right. \\
\left. + r \dot{\theta}^- (a^4 - b^4)^2 \sin(4\theta) - \dot{r}^- (a^4 - b^4)^2 \cos(4\theta) 
\right. \\
\left. - \dot{r}^- (a^4 + b^4)^2 \right],
\]
and
\[
\dot{\theta}^+ = \frac{r}{4r (a^4 \sin^2(\theta) + b^4 \cos^2(\theta))} \left[ 2 r \dot{\theta}^+ (b^4 - a^4) \sin(2\theta) + 2 \dot{r}^+ (a^4 - b^4) \cos(2\theta) 
\right. \\
\left. + r \dot{\theta}^- (a^4 - b^4)^2 \sin(4\theta) - \dot{r}^- (a^4 - b^4)^2 \cos(4\theta) 
\right. \\
\left. - \dot{r}^- (a^4 + b^4)^2 \right].
\]

### IV. Conclusions and Future work

In this paper we have developed a non-differentiable Her- glotz variational principle that allows us to deal with impact problems. This principle is connected with the formulation of hybrid contact systems, and also takes into account the inherent dissipation of these systems in addition to the impact problem.

In subsequent work we shall study a number of issues related to those discussed in this paper:

- The reduction of this type of systems when there are symmetries that leave the Lagrangian invariant.
- The construction of variational integrators that preserve the qualitative behaviour of the system.
- Additionally, we want to prove a Carnot Theorem that accounts for the energy lost or gained in these types of situations.

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