Topological of Asymptotic Cones and S-machines

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Abstract. Sapir, Birget and Rips showed how to construct groups from Turing machines. To achieve such a construction they introduced the notion of S-machine. Then considering a simplified S-machine Sapir and Olshanskii showed how to construct a group such that each of its asymptotic cone is non-simply connected. Still using the notion S-machine, they constructed a group with two asymptotic cone non-homeomorphic. In this paper we show that each asymptotic cone of a group constructed following the whole method of Sapir, Birget and Rips is not simply connected.

1 Introduction

Let \((X, d_X)\) a metric space \(s = (s_n)\) a sequence of points in \(X\), \(d = (d_n)\) an increasing sequences of numbers with \(\lim d_n = \infty\) and let \(\omega : P(\mathbb{N}) \rightarrow \{0, 1\}\) be a non-principal ultrafilter. An asymptotic cone of \(\text{Con}_\omega(X, s, d)\) of \((X, d_X)\) is the subset of the cartesian product \(X^\mathbb{N}\) consisting of sequences \((x_i)_{i \in \mathbb{N}}\) with \(\lim \omega \frac{d(x_i, x_j)}{d_i} < \infty\) where two sequences \((x_i)\) and \((y_i)\) are equivalent if and only if \(\lim \omega \frac{d(x_i, y_i)}{d_i} = 0\). The distance between two elements \((x_i), (y_i)\) in the asymptotic cone \(\text{Con}_\omega(X, s, d)\) is defined as \(\lim \omega \frac{d(x_i, y_i)}{d_i}\). Here \(\lim \omega\) is defined as follows. If \(a_n\) is a bounded sequence of real numbers then \(\lim \omega (a_n)\) is the unique number \(a\) such that for every \(\epsilon > 0\), \(\omega(\{n \mid |a_n - a| < \epsilon\}) = 1\). The asymptotic cones of a finitely generated group \(G\) are asymptotic cones of the Cayley graph of \(G\) and it well known that they do not depend on the choice of the sequence \(s\). It is then assumed that \(s = (1)\) where 1 is the identity. Given an ultrafilter \(\omega\) and an increasing sequence of numbers \(d\) the asymptotic cone of a finitely generated group \(G\) is then noted \(\text{Con}_\omega(X, d)\).

A function \(f : \mathbb{N} \rightarrow \mathbb{N}\) is an isoperimetric function of a finite presentation \((X, R)\) of a group \(G\) if every word \(w\) in \(X\), which is equal to 1 in \(G\), is freely equal to a product of conjugates \(\prod_{i=1}^m x_i^{-1} r_i x_i\) where \(r_i\) or \(r_i^{-1}\) is in \(R\), \(x_i\) is in \((X \cup X^{-1})^*\) and \(m \leq f(|w|)\). The Dehn function of a finite presentation \((X, R)\) is defined as the smallest isoperimetric function of the presentation.

Let \(f, g : \mathbb{N} \rightarrow \mathbb{N}\) be two functions, \(f \preceq g\) if there exists a positive constant \(c\) such that \(f(n) \leq cg(cn) + cn + c\). If all functions considered grow at least as fast as \(n f(n) \leq g(n)\) if and only if \(f(n) \leq cg(cn)\) for some positive constant \(c\). The results of Sapir, Birget, Rips and Olshanskii consider only the functions which grow at least as fast as \(n^k\). Two functions \(f, g\) are called equivalent if \(f \preceq g\) and \(g \preceq f\).

Since the results of [1, 3] it is well known that the Dehn function corresponding to different finite presentations of the same group are equivalent. Thus one can speak
about the Dehn function of a finitely presented group. In [2,3] it is shown that the Dehn function of a finitely presented group has a recursive upper bound if and only if the group has a decidable word problem.

In [4–6] the connections between Dehn functions, asymptotic geometry of groups and computational complexity of the word problem are discussed. In [5] Gromov showed that if all \textit{asymptotic cones} of a group \( G \) are simply connected then \( G \) is finitely presented, has polynomial isoperimetric function and linear isodiametric function. Papasoglu [7] proved that if a finitely presented group has quadratic isoperimetric function then all its asymptotic cones are simply connected. Sapir, Birget and Rips in [8] introduced the concept of \textit{S}-machines to show that the word problem of a finitely generated group is decidable in polynomial time if and only if this group can be embedded into a group with polynomial isoperimetric function. Olshanskii and Sapir in [9] constructed a group with polynomial isoperimetric function, linear isodiametric function and non-simply connected asymptotic cones, the group is roughly a \( S \)-machine introduced in [8]. In [10] they also constructed a group with two non-homeomorphic asymptotic cones using the concept of \( S \)-machine.

In this paper we show that the whole machinery of Birget, Sapir and Rips leads groups with non-simply connected cones for every Turing machine considered. Indeed we show that the construction of [8] involves relations that totally break the topology of the asymptotic cones.

2 Preliminaries

This section introduces briefly the machinery introduced by Sapir, Birget and Rips in [8]. We need to explain, at least superficially, what is a \( S \)-machine, how it works and especially how it leads to the construction of groups.

2.1 \( S \)-machines

This section is closely modeled on [8], we recall the notion of \( S \)-machine defined in the work of Sapir, Birget and Rips in [8]. To begin let us present the initial assumptions needed for the construction. In [8] every Turing machine is modified according to the following lemma:

**Lemma 1** [8] For every Turing machine \( M \) recognizing a language \( L \) there exists a Turing machine \( M' \) with the following properties.

- The language recognized by \( M' \) is \( L \).
- \( M' \) is symmetric, that is, with every command \( U \rightarrow V \) it contains the inverse command \( V \rightarrow U \).
- The time, generalized time, space and generalized space functions of \( M' \) are equivalent to the time function of \( M \). The area function of \( M' \) is equivalent to the square of the time function of \( M \).
- The machine accepts only when all tapes are empty.
- Every command of \( M' \) or its inverse has one of the following forms for some \( i \)
  1. \( q_1 \omega \rightarrow q_1' \omega, \ldots, q_{i-1} \omega \rightarrow q_{i-1}' \omega, aq_i \omega \rightarrow q_i' \omega, q_{i+1} \omega \rightarrow q_{i+1}' \omega, \ldots \)
2. \( \{q_1\omega \rightarrow q'_1\omega, \ldots, q_{i-1}\omega \rightarrow q'_{i-1}\omega, \alpha q_i\omega \rightarrow \alpha q'_i\omega, q_{i+1}\omega \rightarrow q'_{i+1}\omega, \ldots \} \) where "\( \alpha \)" belongs to the tape alphabet of tape \( i \), and \( q_j, q'_j \) are state letters of tape \( j \).

3. The letters used on different tapes are from disjoint alphabets. This includes the state letters.

Let us present how Sapir, Birget and Rips define a \( S \)-machine in [8], roughly speaking it is defined as a rewriting system. A \( S \)-machine then comes with a hardware, a **language of admissible words**, and a set of rewriting rules. A hardware of a \( S \)-machine is a pair \((Y, Q)\) where \( Y \) is an \( n \)-vector of (not necessarily disjoint) sets \( Y_i, Q \) is an \((n+1)\)-vector of disjoints sets \( Q_i \) with \((\bigcup Y_i) \cap (\bigcup Q_i) = \emptyset \). The elements of \( \bigcup Y_i \) are called tape letters; the elements of \( \bigcup Q_i \) are called state letters. With every hardware \( S = (Y, Q) \) one can associate the **language of admissible words** \( L(S) = Q_1F(Y_1)Q_2 \cdots F(Y_n)Q_{n+1} \) where \( F(Y_i) \) is the language of all reduced group words in the alphabet \( Y_j \cup Y_i^{-1} \).

This language completely determines the hardware. One can then describe the language of admissible words instead of describing the hardware \( S \). If \( 1 \leq i < j \leq n \) and \( W = q_1u_1q_2 \cdots u_nq_{n+1} \) is an admissible word, \( q_i \in Q_i, u_i \in (Y_i \cup Y_i^{-1})^* \) then the subword \( q_1u_1 \cdots q_j \) of \( W \) is called the \((Q_i, Q_j)\)-subword of \( W \) \((i < j)\). The rewriting rules (\( S \)-rules) have the following form:

\[
[U_1 \rightarrow V_1, \ldots, U_m \rightarrow V_m]
\]

where the following conditions hold: each \( U_i \) is a subword of an admissible word starting with a \( Q_i \)-letter and ending with \( Q_j \)-letter. If \( i < j \) then \( r(i) < l(j) \), where \( r(i) \) is the end of \( U_i \) and \( l(j) \) the start of \( U_j \). Each \( V_i \) is a subword of an admissible word whose \( Q \)-letters belong to \( Q_{r(i)} \cup \cdots \cup Q_{l(j)} \). The machine applies a \( S \)-rule to a word \( W \) replacing simultaneously subword \( U_i \) by subword \( V_i, i = 1, \ldots, m \).

As mentioned in [8] there exists a natural way to convert a Turing machine \( M \) into a \( S \)-machine \( S \); one can concatenate all tapes of the given machine \( M \) together and replace every command \( a\omega \rightarrow q'\omega \) by \( a^{-1}q'\omega \). Unfortunately the \( S \)-machine constructed following this natural way will not inherit most of the properties of the original machine \( M \), that is it will not satisfy anymore the properties of Lemma[1] According to [8] the main problem is that it is nontrivial to construct a \( S \)-machine which recognizes only positive powers of a letter. Thus in order to construct a \( S \)-machine \( S(M) \) which will inherit the desired properties of a Turing machine \( M \), Sapir, Birget and Rips in [8] constructed eleven \( S \)-machines and then used them to construct the final \( S \)-machine \( S(M) \) simulating \( M \). The construction is quite involved and nontrivial, one can see [8] for details.

Taking any Turing machine \( M = (X, Y, Q, \Theta, \bar{s_0}, \tilde{s_0}) \) and modifying it according to Lemma[1] [8] constructs a \( S \)-machine \( S(M) \) simulating \( M \). The \( S \)-machine constructed in [8] is quite long to define, next we explain briefly the main part of the construction, for proofs and deeper understanding of the whole machinery the reader can refer to [8].

The main idea of the construction is to simulate the initial machine \( M \) using eleven \( S \)-machines \( S_1, S_2, \ldots, S_9, S_{a}, S_{w} \). We will explain how the machines \( S_4, S_9, S_{a}, S_{w} \) are used in the construction of \( S(M) \). The others \( S \)-machines are used to construct \( S_4 \) and \( S_9 \) and are rather of technical importance. First we need to describe what is an admissible word of the \( S \)-machine \( S(M) \). For every \( q \in Q \) the word \( q\omega \) is denoted by \( F_q \), in every command of \( M \) the word \( q\omega \) is replaced by \( F_q \). Left marker on tape \( i \) is
denoted by $E_i$. This gives a Turing Machine $M'$ such that the configurations of each tape have the form $E_i u F_q$ where $u$ is a word in the alphabet of tape $i$ and every command or its inverse has one of the forms:

$$ [F_{q_1} \rightarrow F_{q'_1}, \ldots, a F_{q_i} \rightarrow F_{q'_i}, \ldots, F_{q_k} \rightarrow F_{q'_k}]$$

(1)

where $a \in Y$ or

$$ [F_{q_1} \rightarrow F_{q'_1}, \ldots, E_i F_{q_i} \rightarrow E_i F_{q'_i}, \ldots, F_{q_k} \rightarrow F_{q'_k}].$$

(2)

An admissible word of the considered $S(M)$ machine is a product of three parts. The first part has the form

$$ E(0) a^{\alpha_1} x(0) a^{\alpha_2} F(0).$$

The second part is a product of $k$ words of the form

$$ E(i) v_1 x(i) w_1 F(i) E'(i) p(i) q(i) r(i) s(i) t(i) u(i), \overline{P}(i), \overline{Q}(i), \overline{M}(i), \overline{N}(i), \overline{O}(i), F'(i), i = 1, \ldots, k$$

$$ E'(k + 1) a^{\beta_1} x'(k + 1) a^{\beta_2} F'(k + 1).$$

Here $v_1, w_1$ are group words in the alphabet $Y_i$ of tape $i$, and $A_{ij}$ is a power of $\delta$. The letters $E(i), x(i), F(i), E'(i), p(i), q(i), r(i), s(i), t(i), u(i), \overline{P}(i), \overline{Q}(i), \overline{M}(i)$ are standard and are included into the corresponding sets $X(i), P(i), Q(i), S(i), T(i), U(i), \overline{P}(i), \overline{Q}(i), \overline{M}(i)$ or $X(i), T(i), U(i)$, ($i = 1, \ldots, k$). Let $\tau$ be a command in $\Theta$ of the form (1) (command of the form (1) are called positive and their inverse negative). For every $\gamma \in \{4, 9, 10, \omega\}$ and for each component $V(i)$ of the vector of sets of state letters, the letters $V(i, \tau, \tau, \gamma) \in V(i)$ where $V \in \{P, Q, R, S, T, U, \overline{P}, \overline{Q}, \overline{R}, \overline{S}, \overline{T}, \overline{U}\}$. For each $S$-machine $S_\gamma, \gamma \in \{4, 9, 10, \omega\}$ a copy of $S_\gamma$ is considered where every state letter $z$ is replaced by $z(j, \tau, \tau, \gamma)$ where $j = i$ if $\gamma = 4, 9, j = 0$ if $\gamma = 4$ and $j = k + 1$ if $\gamma = \omega$. These state letters are included into the corresponding sets. The state letters we just described are all the state letters of $S(M)$. The rules of $S(M)$ are the rules of $S_4(\tau), S_9(\tau), S_{10}(\tau)$ for all $\tau \in \Theta$ of the form (1) plus the connecting rules. Basically the connecting rules allow to go from a machine to another one, there are five such rules: $R_4(\tau), R_{9,\omega}(\tau), R_{10,\omega}(\tau), R_{9,\omega}(\tau), R_{9,\omega}(\tau)$. They can be described informally as follows. $R_4(\tau)$ turns on the machine $S_4(\tau), R_{9,\omega}(\tau)$ turns on the machine $S_9(\tau)$ when $S_4(\tau)$ finishes its work, $R_{9,\omega}(\tau), R_{9,\omega}(\tau)$ do the same with the corresponding $S$-machines. $R_{9,\omega}(\tau)$ turns off $S_9(\tau)$ and gets the machine ready to simulate the next transition from $\Theta$. This machinery contains all the necessary steps to simulate a rule of the machine $M$.

Formally speaking, to every configuration $c = (E_1 v_1 F_{q_1}, \ldots, E_k v_k F_{q_k})$ of the machine $M$ is associated the following admissible word $\sigma(c)$ of $S(M)$:

$$ E(1) v_1 x(1) F_{q_1} (1) E'(1) p(1) q(1) r(1) s(1) t(1) u(1) $$
Let \( L \subseteq X^+ \) be a language accepted by a Turing machine \( M \) with a time function \( T(n) \) for which \( T(n)^4 \) is superadditive. Then there exists a finitely presented group \( G(M) = \langle A \rangle \) with Dehn’s function equivalent to \( T(n)^3 \), the smallest, isodiametric function equivalent to \( T^3(n) \), and there exists an injective map \( H : X^+ \to (A \cup A^{-1})^+ \) such that

1. \( u \in L \) if and only if \( H(u) = 1 \) in \( G \);
2. \( H(u) \) has length \( O(|u|)^2 \) and is computable in time \( O(|u|) \).
3 The machine $S_4$

As we already saw it, the construction of $S$-machine in [8] involves eleven others $S$-machine. We shall focus on four machines namely $S_1, S_2, S_3, S_4$ of [8]. As we will see the combination of some rule from these machines allows to construct words that deny the necessary condition the following statement:

**Statement 1** Suppose that an asymptotic cone $\text{Con}_{\omega}(G, d)$ is simply connected then for every $M > 1$ there exists a number $k$ such that for every constant $C \geq 1$, every loop $l$ in the Cayley graph of $G$ satisfying $\frac{1}{C}d_\infty \leq |l| \leq Cd_\infty$ for any sufficient large $m$, bounds a disc that can be subdivided into $k$ subdisc with perimeter at most $\frac{m}{M}$.

The reader will find a reference of this statement in [9]. Therefore such words will ensure that the asymptotic cones of the group $G_N(S)$ are not simply connected. The result is independent of the Turing machine considered and thus can be concluded for each group $G_N(S)$ constructed following [8]. Once the words are constructed, the proof works roughly as the one in [9]. First we need to explain how is constructed the machine $S_4$, this is a critical step in the proof. Formally the machine $S_4$ is constructed from $S_1, S_2, S_3$.

Let us describe the machine $S_1$. Its hardware is:

- $Y(1) = ([\delta], [\delta], [\delta], [\delta], [\delta])$
- $Q(1) = ([p_1, p_2, p_3], \{q_1, q_2, q_3\}, \{r_1, r_2, r_3\}, \{s_1, s_2, s_3\}, \{t_1, t_2, t_3\}, \{u_1, u_2, u_3\})$.

The admissible words of $S_1$ have the following form:

$$p\delta^{n_1}q\delta^{n_2}r\delta^{n_3}s\delta^{n_4}t\delta^{n_5}u$$

where $p, q, r, s, t, u$ may have indices 1, 2, 3 and $n_i \in \mathbb{Z}, i \in \{1, \ldots, 5\}$. The program $P(1)$ of $S_1$ is constructed from the following rules and their inverses.

1. $[q_1 \rightarrow \delta^2 q_1 \delta^{-2}, r_1 \rightarrow \delta^{-1} r_1 \delta]$
2. $[p_1 q_1 \rightarrow p_2 q_2, r_1 \rightarrow r_2, s_1 \rightarrow s_2, t_1 \rightarrow t_2, u_1 \rightarrow \delta u_2]$
3. $[p_1 \delta q_1 \rightarrow p_2 \delta q_3, r_1 \rightarrow r_3, s_1 \rightarrow s_3, t_1 \rightarrow t_3, u_1 \rightarrow u_3]$

The hardware of $S_2$ is

- $Y(2) = Y(1)$,
- $Q(2) = ([p_1, p_2], \{q_1, q_2\}, \{r_1, r_2\}, \{s_1, s_2\}, \{t_1, t_2\}, \{u_1, u_2\})$.

The program $P(2)$ of $S_2$ consists of the following rules and their inverses:

1. $[q_2 \rightarrow \delta q_2 \delta^{-1}, s_2 \rightarrow \delta^{-1} s_2 \delta]$
2. $[p_2 \rightarrow p_1, q_2 r_2 s_2 \rightarrow q_1 r_1 s_1, t_2 \rightarrow t_1, u_2 \rightarrow u_1]$

The machine $S_3$ in [8] is defined as a cycle of machines $S_1, S_2$. Roughly speaking $S_3$ is obtained by taking the union of $S_1$ and $S_2$ and identifying two state vectors of $S_1$ with two state vector of $S_2$. The hardware of $S_3$ is $(Y(3), Q(3))$ is the same as the hardware of $S_1$. The program $P(3)$ is constructed from the following rules and their inverses.
We will construct the words that deny the statement it is useful to see the critical steps of the simulation. In the simulation of a Turing machine works as follows. Let $M$ be a Turing machine and $\tau$ a command of $M$ of the form 

$$\tau = \{F_{q_1} \rightarrow F'_{q_1}, \ldots, a F_{q_k} \rightarrow F'_{q_k}, \ldots, F_{q_h} \rightarrow F'_{q_h}\}.$$ 

Remember that $S_4(\tau)$ is a copy of machine $S_4$. The machine $S(M)$ simulates the command $\tau$ as follows. First, using $S_4(\tau)$, it is checked whether the word between $E'(i)$ and $F'_{q_1}(i)$ is empty. If it is empty, the execution cannot proceed to the next step. Otherwise the machine changes $q_i$ to $q'_i$ in the indices of the $F's$, inserts $a^{-1}$ next to the left of $x(i)$, removes one $\delta$ in the word between $E'(i)$ and $F'_{q_1}(i)$, removes one $\delta$ and removes one $\omega$. Using $S_6(\tau)$ it finally checks if after $a^{-1}$, the word between $E(i)$ and $F'_{q_1}(i)$ is positive. If it is the case the machine gets ready to execute the next transition.

The critical step for our work is the first one when the machine $S_4(\tau)$ is used. Indeed it means that for each command $\tau$ of the machine $M$ there exists a copy of the rules of $S_4$ in $S(M)$. The next section shall explain what are the consequences in the group $G_N(S)$.

### 4 Consequences in $G_N(S)$

As we saw previously the machine $S(M)$ contains copies of rules of the machine $S_4(\tau)$ where $\tau$ is a command of $M$. Remember that $G_N(S)$ contains in its presentation the transitions relations and that they correspond to the element of $\Theta$. Let $\tau \in \Theta$, $\tau = [U_1 \rightarrow V_1, \ldots, U_p \rightarrow V_p]$. Then relations $\tau^{-1} U_1 \tau = V_1, \ldots, \tau^{-1} U_p \tau = V_p$ are included into $P_N(S)$. If for some $j$ from 1 to $17k + 6$ the letters from $Q_j$ do not appear in any of the $U_i$ then the relations $\tau^{-1} q_i \tau = q_j$ for every $q_i \in Q_j$ are also included. Denote $\sigma_j(4, \tau)$ the copy of the rule $\sigma_j$ of $S_4$. Let $\sigma_1, \sigma_4$ be the following rule of $S_4$:

- $\sigma_1 : [q_1 \rightarrow \delta^{-2} q_1 \delta^2, r_1 \rightarrow \delta^{-1} r_1 \delta]$,
- $\sigma_4 : [q_2 \rightarrow \delta q_2 \delta^{-1}, s_2 \rightarrow \delta^{-1} s_2 \delta]$.

It means that in $G_N(S)$ the following relation exists:
\(- \sigma_1(4, \tau)^{-1}q_1\sigma_1(4, \tau) = \delta^2q_1\delta^2, \)
\(- \sigma_1(4, \tau)^{-1}r_1\sigma_1(4, \tau) = \delta^{-1}r_1\delta, \)
\(- \sigma_4(4, \tau)^{-1}q_2\sigma_4(4, \tau) = \delta q_2\delta^{-1}, \)
\(- \sigma_4(4, \tau)^{-1}s_2\sigma_4(4, \tau) = \delta^{-1}s_2\delta. \)

Moreover the following relations are also included in the presentation of \(G_N(S)\):
\(- \sigma_1(4, \tau)^{-1}s_2\sigma_1(4, \tau) = s_2, \)
\(- \sigma_4(4, \tau)^{-1}r_1\sigma_4(4, \tau) = r_1. \)

From now on and for the sake of simplicity we denote the rule \(\sigma_1(4, \tau)\) (resp. \(\sigma_4(4, \tau)\)) by \(\sigma_1\) (resp. \(\sigma_4\)). Let us study briefly the word \(\sigma_1^{-n}\sigma_4^n(s_2r_1)^n\sigma_4^n\).

**Lemma 2** In \(G_N(S)\) the word \(\sigma_1^{-n}\sigma_4^n(s_2r_1)^n\sigma_4^n\) is equal to \(\delta^{-n}(s_2r_1)^n\delta^n\).

**Proof.** First we prove by induction that
\[ \sigma_1^{-n}(s_2r_1)^n\sigma_4^n \]
is equal to
\[ (\delta^{-n}s_2\delta^n r_1)^n \]
If \(k = 0\) the equality is clear. Let \(n \in \mathbb{N}\) and assume \(\sigma_1^{-k}(s_2r_1)^k\sigma_4^k = (\delta^{-k}s_2\delta^k r_1)^k\) is true for \(k \leq n\). We shall show that
\[ \sigma_1^{-n+1}(s_2r_1)^{n+1}\sigma_4^{n+1} = (\delta^{-n}s_2\delta^{n+1} r_1)^{n+1} \]
inserting accordingly the word \(\sigma_1^2\sigma_4^{-n}\) we deduce from \(4\)
\[ \sigma_1^{-n}(s_2r_1)^n\sigma_4^n \]
applying the induction hypothesis on the word in the box we obtain
\[ \sigma_1^{-1}(\delta^{-n}s_2\delta^n r_1)^n\sigma_4^{-n}s_2\sigma_4^n\sigma_4^{-n}r_1\sigma_4^{n+1} \]
since \(\delta \in Y_i\) for some \(i \leq k\) then \(\sigma_4\delta = \delta\sigma_4\) is an auxiliary relation, then it comes from \(5\)
\[ \sigma_1^{-1}(\delta^{-n}s_2\delta^n r_1)^n\sigma_4^{-n}s_2\sigma_4^n\sigma_4^{-n}r_1\sigma_4 \]
combining the relations \(\sigma_4^{-1}s_2\sigma_4 = \delta^{-1}s_2\delta \) and \(\sigma_4\delta = \delta\sigma_4\) gives
\[ \delta^{-n}\sigma_4^{-1}s_2\delta^n r_1\delta^{-n}s_2\delta^n \ldots r_1\delta^{-n}s_2\delta^n r_1\sigma_4 \]
then inserting \(\sigma_4\sigma^{-1}\) we obtain
\[ \delta^{-n}\sigma_4^{-1}s_2\sigma_4^{-1}\delta^n r_1\sigma_4^{-1}\delta^{-n}s_2\sigma_4^{-1}\delta^n \ldots \sigma_4^{-1}\delta^n r_1\sigma_4^{-1}\delta^{-n}s_2\sigma_4^{-1}\delta^n r_1\sigma_4 \]
but since the letter $r_i$ is never involved in the rule $\sigma_i$ we have $\delta^{-1} r_i \sigma_i = r_1$ and thus combining it with the auxiliary rule, it comes

$$
\delta^{-n} \sigma_4^{-1} s_2 \sigma_4 \delta^n r_1 \delta^{-n} \sigma_4^{-1} s_2 \sigma_4 \delta^n \sigma_4^{-1} r_1 \sigma_4 \delta^{-n} \sigma_4^{-1} s_2 \sigma_4 \delta^n r_1 = (\sigma_4 \delta \sigma_4^{-1} r_1 \sigma_4 \delta^{-n} \sigma_4^{-1} s_2)^n^{-1}
$$

(8)

now we can apply relation $\sigma_4^{-1} s_2 \sigma_4 = \delta^{-1} s_2 \delta$ and obtain

$$(\delta^{-(n+1)} s_2 \sigma_4)^n = (\delta^{-(n+1)} s_2 \delta^{n+1} r_1)^n
$$

(9)

and thus

$$
\sigma_4^{-(n+1)} (s_2 r_1)^{n+1} \sigma_4^{n+1} = (\delta^{-(n+1)} s_2 \delta^{n+1} r_1)^{n+1}
$$

(10)

Now we shall start from the second member of equation (10) and show the following

$$
\sigma_4^{-1} (s_2 r_1)^{n} \sigma_4 = \delta^{-1} (s_2 r_1)^{n} \delta^n.
$$

(11)

Inserting the word $\sigma_4^{-1} (s_2 r_1)^{n}$ accordingly and using auxiliary relation we obtain

$$
\sigma_4^{-1} \delta^{-n} s_2 (s_2 r_1)^{n} \sigma_4 = \sigma_4^{-1} \delta^{-n} s_2 (s_2 r_1)^{n+1} \sigma_4
$$

(12)

using the relations $\sigma_4^{-1} s_2 \sigma_4 = s_2$ and $\sigma_4 \delta = \delta \sigma_4$ leads to

$$
\delta^{-n} s_2 (s_2 r_1)^{n} \sigma_4 = \delta^{-n} s_2 (s_2 r_1)^{n+1} \sigma_4
$$

(13)

applying $\sigma_4^{-1} r_1 \sigma_4 = \delta^{-1} r_1 \delta$ and the auxiliary relation $\sigma_1 \delta = \delta \sigma_1$ gives

$$
\delta^{-n} (s_2 r_1)^{n} \delta^n.
$$

(14)

therefore the equality is proved.

Lemma $\mathbb{2}$ allows one to consider van Kampen diagram $A_n$ with a boundary labeled by the word

$$
\sigma_4^{-1} (s_2 r_2)^{n} \sigma_4 = \delta^{-n} (s_2 r_1)^{n} \delta^n.
$$

(15)

We shall use such van Kampen diagram to deny the necessary condition of Statement $\mathbb{1}$ That is we show that no loop corresponding to $A_n$ can bound a disc decomposed into at most $k \leq \sqrt{n}$. First let us recall what is an $x$-band, for every letter $x$. An $x$-edge in a van Kampen diagram is an edge labeled by $x^+$. An $x$-cell is a cell whose boundary contains an $x$-edge. An $x$-band in a diagram is a sequence of cells containing $x$-edges, such that every two consecutive cells share an $x$-edge. The boundary of the union of cells from an $x$-band $B$ has the form $s^{-1} peq^{-1}$ where $s, e$ are the only $x$-edges on the boundary representing respectively the start and the end of the band. The paths $p, q$ are called the sides of $B$.

Theorem 2 Let $\omega$ be a non-principal ultrafilter and $(d_i), i \in \mathbb{N}$ an increasing sequence of numbers with $\lim d_i = \infty$. Let $\text{Con}_\omega(G_N(S), (d_i))$ be an asymptotic cone of $G_N(S)$. Then $\text{Con}_\omega(G_N(S), (d_i))$ is not simply connected.
Proof. Let $\text{Con}_m(G_N(S), (d_i))$ be an asymptotic cone of $G_N(S)$. Fix $n = d_m$ for a large $m$. Let $u_n = \sigma_1^\gamma \sigma_2^\gamma (s_2 r_1)^\gamma \sigma_2^\gamma \sigma_2^\gamma (s_2 r_1)^\gamma \delta^n$ and $A_n$ the corresponding van Kampen diagram. The top path $t$ of $A_n$ is labeled by $(s_2 r_1)^\gamma$, the bottom path $b$ is labeled by $\delta^n(s_2 r_1)^\gamma \delta^n$. The left and right sides, $l, r$ are labeled by $\sigma_2^\gamma \sigma_2^\gamma$. The perimeter of $A_n$ is $|A_n| = 8n$. We shall show that a loop in the Cayley graph of $G_N(S)$ corresponding to $u_n$ cannot bound a disc decomposed into at most $k \leq \sqrt{n}$ subdiscs of perimeter $n$. Assume that such a decomposition exists, there is a van Kampen diagram $A$ with boundary label $u_n$ composed of $k$ subdiagrams $A_1, \ldots, A_k$ with perimeter at most $n$. Consider any $\sigma_4$-edges of the path $l$. Since there is no $\sigma_4$-edges on path $t$, any $\sigma_4$-band in $A$ that starts at $e$ cannot end on $t, b$. Therefore it finishes on $r$. Moreover the $\sigma_4$-bands do not intersect and thus they connected corresponding $\sigma_4$-edges. Let $e$ be the $\sigma_4$-edges number $n$ on $l$ and $B$ be the maximal $\sigma_4$-band starting at $e$. Let $r$ be the top side of $B$. The label $\text{Lab}(r)$ of path $r$ belongs to the free group $\langle \delta, s_2, r_1 \rangle$ and it is equal to $\sigma_4^{-n}(s_2 r_1)\sigma_4^{n}$ in $G_N(S)$, thus it can be written $(\delta^n s_2 \delta^n r_1)^n$. Since the number of diagrams $A_1$ is less than $\frac{\sqrt{n}}{n}$ there is a subpath $w$ of $r$ such that the initial and the terminal vertices of $w$ belong to the boundary of one of the $A_i$. $w$ contains the subword $s_2 \delta^n r_1$. It means that there exists in $G_N(S)$ a word $u$ such that $u = w$ and $|u| \leq \frac{\sqrt{n}}{4}$ since the perimeter of $A_1$ does not exceed $n$. Therefore we can consider a reduced diagram $\Gamma$ with boundary $p_1 p_2^{-1}$ where $\text{Lab}(p_1) = w, \text{Lab}(p_2) = u$. We look at the subword $t_1 = s_2 \delta^n r_1$. Let $B_1$ the maximal $s_2$-band starting on $t_1$ and $B_2$ the maximal $r_1$-band starting on $t_1$. Denote by $\Gamma_1$ the subdiagram of $\Gamma$ bounded by $q_1 = t_1 \setminus \{s_2, r_1\}$, the sides of $B_1$ and $B_2$ and a part $q_2$ of $p_2$. Let $\partial \Gamma_1 = q_1 q_2'$ the boundary of $\Gamma_1$. We shall bound the length of $q_2'$. One can remark that a cell appearing in a $s_2$-band in $\Gamma$ can be written in the form $s_2 = \delta \sigma_4^{-1} s_2 \sigma_4 \delta^{-1}$. That is the length of a side of $B_1$ is at most twice the number of $\sigma_4$-edges in it. The number of $\sigma_4$-edges on $B_1$ and $B_2$ is at most $|p_2| - |q_2| - 2$. Thus we have $|q_2'| \leq 2|p_2| - 2|q_2| + 2 - 4$ and then it comes $|q_2'| < 2|p_2| - |q_2|$. It is not difficult to see that the diagram $\Gamma_1$ can be chosen such that its top path $q_2 = s_2 \delta^n r_1$. Moreover in such a diagram there are no $s_2$-cells and thus every $\delta$-band starting on $q_1$ must end on $q_2'$. But this is a contradiction since $q_2' < n$.

Remark 1 The proof works roughly as the proof in [9].

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