Physics and differential geometry in modern control theory applied to a synchronous generator

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Abstract. This article will show how to address non-linear systems by means of state feedback linearization, through it we can have the stability of the system, which in most of the industry systems usually occurs. As the main contribution, tools of differential geometry will be presented, which is a recent branch of mathematics and physics which expands the concepts and definitions of classical mathematics, therefore, the model will be presented in terms of the definitions of space Tangent, derivative and Lie bracket all this allows us to understand and propose new solutions more efficiently since traditionally the techniques used revolve around the definitions of fields on finite spaces. The model on which the control techniques will be applied will be implemented on an electrical machine such as the synchronous generator, with the purpose of applying a control law that allows the outputs of the synchronous machine to adjust to the desired references, said the model is of great importance in the electricity sector industry because it can control the variables of the system.

1. Introduction

In a novel way, the theory of physics based on differential geometry is introduced to better describe a control system and have more solid mathematical tools that seek to describe it without many idealizations. The stability analysis of any dynamic system modeled by differential equations is necessary for the implementation of controllers that seek to bring the states of the system to a previously established reference. Therefore, in the process of designing different types of controllers for electrical power systems, numerous studies about the stability of these systems can be observed, such as [1] in which it is two types of external stability (input-output) and internal stability (associated with the initial conditions) [2].

There are different approaches for the analysis of stability and control of dynamic systems, one of them that has generated a high performance compared to classical approaches, are the analyzes based on differential geometry, which has allowed us to understand many phenomena that were previously neglected, or which limited their application to only small regions of operation. This document presents an observer-based control, which is responsible for estimating the state variables based on the measurements of the output and control variables. In this document we will work with related non-linear systems in the control, which are observable for all inputs, that is, systems where the observability property will not be affected by the type of inputs that are applied and allow modeling [3].

The organization of this document is as follows: the following section will deal with the mathematical concepts that will be used to determine the state feedback control law, describes the procedure to be
performed. To linearize non-linear dynamic systems and apply feedback control, shows the reduced model of the synchronous generator, finally, the results.

2. Methodology

In this section, some basic mathematical definitions will be given that will be used throughout the document [4]; linear and nonlinear systems can be described through the system of equations represented by Equation (1) [5].

\[ \begin{align*}
    \dot{x} &= f(x) + g(x)u, \\
    y &= h(x),
\end{align*} \tag{1} \]

where \( x = [x_1 \ x_2 \ \ldots \ x_n]^T \) is the vector of states, \( f(x) \) and \( g(x) \) are smooth vector functions and \( h(x) \) and \( u \) are scalar functions [5].

A vector field \( X \) over \( M \), where \( M \) is a differentiable manifold and \( TM \) is its tangent bundle Equation (2) [6].

\[ X: M \to TM : \quad p \to X(p) = X_p \in T_p M. \tag{2} \]

The field is said to be differentiable if the function \( X: M \to TM \) is differentiable. When considering a letter \((U, x)\) of \( M \), it is possible to write the field \( X \) is this letter Equation (3) [7].

\[ X(p) = \sum_{i=1}^{n} X^i_p \frac{\partial}{\partial x^i}, \tag{3} \]

where each \( X^i: U \to \mathbb{R} : p \to X^i(p) = X^i_p \) is a function in \( U \) and \( \frac{\partial}{\partial x^i} \) is the basis associated with \( X \). It is clear that \( X \) is differentiable if and only if the functions \( X^i \) are differentiable functions for some letter. It is useful to use the idea shown in Equation (2) and think of each field of vectors as a derivation \( X: \mathcal{D} \to \mathbb{R} \) from the set \( D \) of the differentiable functions in \( M \) to the set \( F \) of the functions in \( M \), defined by Equation (4) [8].

\[ (Xf)(p) = X_p(f) = \sum_{i=1}^{n} X^i_p \frac{\partial f}{\partial x^i}_{x(p)}, \tag{4} \]

where \( F = f \circ x^{-1} \) is the expression for \( f \) in the letter \((U, x)\).

2.1. Lie’s derivative

Given the output function \( h(x) \) and the vector field \( f(x) \), we define a new scalar function called the Lie derivative, \( L_f h(x) \), of \( h \) with respect to \( f \) Equation (5), which is calculated as Equation (6) [9].

\[ L_f h(x) = \frac{\partial h}{\partial x} f = \nabla h \cdot f, \tag{5} \]

where \( \nabla h = \left( \frac{\partial h}{\partial x_1} \ \frac{\partial h}{\partial x_2} \ \ldots \ \frac{\partial h}{\partial x_n} \right). \tag{6} \]

The mixed derivative of Lie, that is, \( L_f L_g h(x) \) with respect to \( g(x) \) is obtained as Equation (7).

\[ L_g L_f h(x) = L_g (L_f h) = \nabla (L_f h) \cdot g. \tag{7} \]

2.2. Lie’s bracket

Given two vector fields of \( \mathbb{R}^n \), \( f(x) \) and \( g(x) \), the Lie bracket is defined as Equation (8).
\[(f, g) = \text{ad}_f g = \nabla g \cdot f = \nabla f \cdot g. \quad (8)\]

where the function \(\text{ad}_f g\), is known as the adjoint of a vector field, which is an operation that is defined to handle in a simpler way the reiterations of the Lie bracket and according to [10] it is defined as Equation (9).

\[
\begin{align*}
\text{ad}_f^0 g &= \text{gad}_f g = (f, g), \\
\text{ad}_f^r g &= (f, \text{ad}_f g) = (f, (f, \text{ad}_f g)), \\
\text{ad}_f^i g &= (f, \text{ad}_f^{i-1} g), \\
\text{with } i &= 1, 2, \ldots, n.
\end{align*}
\quad (9)
\]

### 2.3. Exact linearization by feedback

For partial linearization (input-output), the output of the system \(y = h(x)\), is derived with respect to time as many times as necessary to obtain a differential equation that relates to \(y(t)\) with the control signal \(u(t)\). The order of the equation obtained is known as the relative degree \(r\) and in well-defined cases it is less than or equal to the order of the physical system \((r \leq n)\) [11]. By deriving the output function and using the chain rule, Equation (10) is obtained.

\[
\dot{y} = \frac{dh}{dt} = \frac{\partial h}{\partial x} \cdot \frac{dx}{dt} = \nabla h \cdot \dot{x} = \nabla h (f(x) + g(x)u).
\quad (10)
\]

From the result obtained in the previous equation, you can separate the terms and use the Lie derivative notation, Equation (9) being expressed in the form Equation (11).

\[
\dot{y} = L_f h(x) + L_g h(x)u.
\quad (11)
\]

If \(L_g h(x) \neq 0\), then the relative degree of the system is \(r = 1\), otherwise, deriving the Equation (11) again results in Equation (12).

\[
\dot{y} = \frac{d}{dt} L_f h(x) = \frac{\partial L_f h(x)}{\partial x} \dot{x} \quad ; \quad \ddot{y} = L_f^2 h(x) + L_g L_f h(x)u.
\quad (12)
\]

In this case, if \(L_g L_f h(x) \neq 0\) the system has a relative degree \(r = 2\), in general the relative degree of a system is defined as the number of times the output must be derived to explicitly obtain the input, that is, the control signal Equation (13).

\[
y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x)u = v,
\quad (13)
\]

where the function \(v\) is a linear control signal (error regulator), which will allow to determine the control signal \(u\) in the following way Equation (14).

\[
u = \frac{1}{L_g L_f^{r-1} h(x)} \left( v - L_f^r h(x) \right).
\quad (14)
\]

The task will be to design a regulation signal in such a way that the output \(y(t)\) stores an established reference signal \(y_{\text{ref}}\) constant, this is achieved by Equation (15).

\[
v = -k_1 (y - y_{\text{ref}}) - k_2 \dot{y} - \cdots - k_r y^{r-1}.
\quad (15)
\]

By replacing Equation (14) in Equation (12), a linear differential equation is obtained that will describe the dynamics of the closed-loop system Equation (16) [12].
\[ y^{(r)} + k_1 y^{r-1} + \cdots + k_2 \dot{y} + k_1 y = k_1 y_{ref}, \]  

where the constants \(k_1, k_2, k_1\) should be selected in such a way that the characteristic polynomial associated with (3.7) has all its roots in the left complex half-plane.

**Theorem 1.** Let us be a nonlinear dynamic system with a single input, the system is said to be locally linearizable by state feedback if and only if in a location \(U\) of the origin:

- \(\text{span}\{g, adf g, \ldots, ad_{f}^{n-1}g\} = \mathbb{R}^n\).
- La distribution \(\text{span}\{g, adf g, \ldots, ad_{f}^{n-2}g\}\) es involutiva y de rango constante \(n - 1\).

The first part of the theorem refers to the controllability condition. In particular, for a linear system represented by a model \(x = Ax + Bu\), where the vector functions \(f(x)\) and \(g(x)\) will be the matrices \(A\) and \(B\) respectively, it can be verified that Equation (17).

\[ [g \quad adf g \quad ... \quad ad_{f}^{n-1}g] = [A \quad AB \quad ... \quad A^{n-1}B] = C_0, \]  

where the matrix \(C_0\) is known as the controllability matrix.

If a nonlinear system satisfies theorem 1, it is possible to determine a transformation under an appropriate change of coordinates (diffeomorphism) that allows finding a system which is controllable by linearization by state feedback. This is achieved by changing the coordinates Equation (18).

\[ \Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_{n-1}(x) \\ \phi_n(x) \end{pmatrix} = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-2} h(x) \\ L_f^{n-1} h(x) \end{pmatrix}, \]  

therefore, the system in the new coordinate system will be expressed in the form Equation (19).

\[ z_i = \phi_i(x) = L_f^{i-1} h(x), \quad \forall \ i = 1, 2, ..., n. \]  

2.4. Synchronous generator model

The complete model of a synchronous generator has great complexity, so for studies of electrical power systems where a large number of these generators are found, it is not convenient since it would have great difficulty in its analysis, therefore it is necessary to reduce the order of the model. This is achieved through techniques based on the Integral Variety, where the effect of the damping flows is neglected, allowing to obtain a reduced model that contains only the variables of interest but that describes the behavior in a similar way to that of the complete model. Considering a synchronous generator connected to an infinite bus through a transmission line with a resistance \(R_e\) and an inductive reactance \(X_e\) as shown in Figure 1.
The mathematical model that describes the dynamics of the synchronous machine connected to an infinite bus is the following Equation (20).

\[ \dot{\delta} = \omega - \omega_0, \]
\[ M\dot{\omega} = T_m - \frac{V}{X'_d}E'_q \sin(\delta) - \frac{V^2(X'_d - X^*_q)}{2X'_dX^*_q} \sin(2\delta), \]
\[ T'_d E'_q = - \frac{X'_d}{X'_d}E'_q + \frac{V(X'_d - X^*_d)}{X'_d} \cos(\delta) + E_f, \]

(20)

where \( \omega_0 \) is the synchronous speed and \( H \) is the inertia constant, \( E_f \) is the field voltage and represents the control variable \( u(t) \). \( V \) is the voltage at terminals which remains constant and equal to; the reactance \( X'_d \), \( X'_d \) and \( X^*_q \) include the external effects of \( R_x \) y \( X_{eq} \). Now, defining new parameters Equation (21).

\[ m_1 = \frac{T_m}{M}, \quad m_2 = \frac{V}{MX'_d}, \quad m_3 = \frac{V^2(X'_d - X^*_q)}{2MX'_dX^*_q}, \quad m_4 = \frac{X'_d}{t'_dX'_d}, \quad m_5 = \frac{V(X'_d - X^*_d)}{t'_dX'_d}. \]

(21)

Furthermore, taking as Equation (22).

\[ x_1 = \delta, \quad x_2 = \omega, \quad x_3 = E'_q, \quad y = \frac{E_f}{t'_d}. \]

(22)

The system Equation (19) is described in a more compact way as Equation (23).

\[ \dot{x}_1 = x_2 - \omega_0, \]
\[ \dot{x}_2 = m_1 - m_2x_3 \sin(x_1) - m_3 \sin(2x_1), \]
\[ \dot{x}_3 = -m_4x_3 + m_5 \cos(x_1) + u, \]
\[ y = x_1. \]

(23)

The system Equation (22) can finally be rewritten in the form where the vector functions \( f(x) \) and \( g(x) \), and the scalar function \( h(x) \) will be defined as Equation (24).

\[ f(x) = \begin{pmatrix} x_2 - \omega_0 \\ m_1 - m_2x_3 \sin(x_1) - m_3 \sin(2x_1) \\ -m_4x_3 + m_5 \cos(x_1) \end{pmatrix}, \]
\[ g(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]
\[ h(x) = x_1. \]

(24)

Once these functions have been identified, the relative degree of the system must be determined, for this we have Equation (25).

\[ L_y h(x) = 0 \]
\[ L_x h(x) = x_2 - \omega_0 \]
\[ L_y L_x h(x) = 0; \]
\[ L^2_y h(x) = m_1 - m_2x_3 \sin(x_1) - m_3 \sin(2x_1), \]
\[ L^2_g L^2_y h(x) = -m_2 \sin(x_1). \]

(25)

So, the relative degree is \( r = 3 \). This fact will allow to transform the transformation Equation (26).
\[ z(x) = \begin{pmatrix} h(x) \\ L_h h(x) \\ L_h^2 h(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - \omega_0 \\ m_1 - m_2 x_3 \sin(x_1) - m_3 \sin(2x_1) \end{pmatrix}, \] (26)

3. Results
Using the parameters proposed in [12], which are summarized in the following Table 1. It should be noted that the synchronous generator does not have its equilibrium points at the origin, therefore, it is necessary to carry out a change of coordinates and transfer this equilibrium point to the origin. This is achieved by defining \( x = x' - x_e \) where \( x_e \) is the equilibrium point. In addition, the values of the constants \( K_1 \) during the simulation were \( K_1 = 350 \), \( K_2 = 155 \), and \( K_3 = 22 \). Taking an arbitrary reference output of \( y_{ref} = 0.5 \); the results obtained can be viewed in Figure 2.

| Parameter | Units | Value |
|-----------|-------|-------|
| \( X_{fd} \) | p. u. | 1.0000 |
| \( X_{fd} \) | p. u. | 0.2000 |
| \( X_{q} \) | p. u. | 0.6000 |
| \( T_{do} \) | seg | 2.6530 |
| \( H \) | seg | 0.1326 |
| \( T_m \) | p. u. | 1.0000 |
| \( M \) | rad/seg\(^2\) | 7.034x10\(^{-3}\) |
| \( \omega_0 \) | rad/seg\(^2\) | 377.9000 |

Figure 2. (a) Rotor angle; (b) synchronous speed.

4. Conclusions
It can be observed from the simulations obtained, that the control law applied to the reduced model of the synchronous generator, effectively manages to stabilize the system, and also bring the rotor angle to the reference desired this through the proposed control algorithm. Furthermore, the application of these and other control techniques to electrical power systems continues to be an area of great interest within the electrical sector. Using techniques based on differential geometry are very useful since they allow us a generalization and better understanding of the problem.

References
[1] Perera P C, Blaabjerg F, Pedersen J K, Thogersen P 2003 A sensorless, stable V/f control method for permanent-magnet synchronous motor drives IEEE Transactions on Industry Applications 39(3) 783
[2] Souza C L, Neto L M, Guimaraes G C, Morales A J 2001 Power system transient stability analysis including synchronous and induction generators Porto Power Tech. Proceedings (Oporto: IEEE) p.6
[3] Zou Y, Elbuluk M, Sozer Y 2013 Stability analysis of maximum power point tracking (MPPT) method in wind power systems Industry Applications Society Annual Meeting (Orlando: IEEE) p 1129
[4] Achilles S, Poller M 2003 Direct drive synchronous machine models for stability assessment of wind farms 4th International Workshop on Large-scale Integration of Wind Power and Transmission Networks for Offshore Wind Farms (Netherlands: Enerynautics) p 1

[5] Do Carmo M P 2016 Differential Geometry of Curves and Surfaces: Revised and Updated, Second Edition (Rio de Janeiro: Dover Publications)

[6] Wells R O N, García Prada O 1980 Differential Analysis on Complex Manifolds (New York: Springer)

[7] Dubrovin B A E, Novikov S P 1989 Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory Russian Mathematical Surveys 44(6) 35

[8] Jeon I, Lee K, Park J H 2011 Differential geometry with a projection: application to double field theory Journal of High Energy Physics 4 14

[9] Delgado A 2000 Linealización entrada/salida de sistemas no lineales afines utilizando un filtro Ingeniería e Investigación 45 62

[10] Del Castillo G T 1988 Potenciales de Debye mediante el método de operadores adjuntos Revista Mexicana de Física 35(2) 282

[11] Li P Y 2002 Feedback Linearization and Robust Sliding Mode Control (Minesota: University of Minesota)

[12] Morales L, Acha Daza S 1998 Estabilización de una clase de sistemas no lineales: aplicación a un generador síncrono Ingenierías 1(2) 31