Dynamical Lorentz symmetry breaking in a tensor bumblebee model

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Abstract

In this paper, we formulate a tensor bumblebee model describing the dynamics of the second-rank antisymmetric tensor interacting with a spinor field through a coupling involving the first derivative, calculate the one-loop effective potential of the tensor field in this model, and demonstrate explicitly that it is positively defined and possesses a set of minima. This allows us to conclude that our model displays the dynamical Lorentz symmetry breaking. We argue also that derivative-free couplings of the antisymmetric tensor field to a spinor do not generate the positively defined potential and thus do not allow for the dynamical Lorentz symmetry breaking.

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As it is well known, the Lorentz symmetry breaking can be introduced in two manners, the explicit one, where the constant vector or tensor introducing privileged spacetime direction is added from the very beginning, and the spontaneous one, where such constant vector or tensor emerges as a vacuum expectation of some vector or tensor field, respectively. While the first manner became paradigmatic, being used to formulate the Lorentz-breaking extension of the standard model [1, 2], the interest to the second, spontaneous manner, is based on the fact that this approach provides a mechanism for explaining the origin of Lorentz symmetry breaking. Namely, this way was originally proposed in [3] (see also [4]), where the Lorentz symmetry breaking was (first introduced and) suggested to arise as the low-energy limit of string theory.

The first vector field theory model, involving a potential allowing for spontaneous Lorentz symmetry breaking, was introduced in [5]. In [6], where this model was denominated as the bumblebee model for the first time, it was generalized to curved spacetime, and some solutions of gravitational field equations in the presence of spontaneous Lorentz symmetry breaking were obtained. Further, various issues related to the vector bumblebee model, including the case of curved background, were considered, see e.g. [8], where it has also been argued why spontaneous Lorentz symmetry breaking in curved space is the most appropriate way to introduce Lorentz-violating extension of gravity.

The aspects of dynamical Lorentz symmetry breaking, occurring due to perturbative corrections, have been treated in [9, 10]. In [9], by performing the fermion integration of a self-interacting massive theory, it was obtained the vector bumblebee model

\[
\mathcal{L}_B = -\frac{1}{12} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4} (B_\mu B^\mu - \beta^2)^2,
\]

however, with \(\lambda < 0\) (where \(F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\), and \(\beta^2 = \beta_\mu \beta^\mu\)). This situation was overcome in [10], where we considered a massless theory and the exact propagator allowing to take into account all orders in the constant \(\beta_\mu\), so that now \(\lambda > 0\), i.e., the potential is positive definite in this latter approach.

Thus, the spontaneous Lorentz symmetry breaking for vector field models has been relatively well studied. At the same time, it is interesting to carry out similar studies for more generic tensor field models. Although a systematic approach to the study of the spontaneous Lorentz symmetry breaking has been proposed already in [11], up to now there are very few studies on higher-rank Lorentz-breaking tensor field models, with only tree-level aspects being considered, see e.g. [12–14].

Therefore, it is natural to generalize the methodology developed for the vector bumblebee model to the analogous theory of the antisymmetric tensor field, which can display the sponta-
neous Lorentz symmetry breaking as well (for different issues related to the antisymmetric tensor field, without connection with Lorentz symmetry breaking, see [15, 16], and references therein).

Originally the bumblebee model on the base of the antisymmetric tensor field theory was introduced in [11]. Here we present its simplest version with the quartic potential looking like:

\[
\mathcal{L}_B = -\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \bar{\psi}(i\slashed{\partial} - ieB^{\mu\nu} \gamma_{[\mu} \partial_{\nu]} - m)\psi - \frac{\lambda}{4} (B_{\mu\nu} B^{\mu\nu} - \beta^2)^2,
\]

(2)

where \( H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]} + \partial_{[\nu} B_{\lambda\mu]} + \partial_{[\lambda} B_{\mu\nu]} \) is a stress tensor for \( B_{\mu\nu} \), and \( \gamma_{[\mu} \partial_{\nu]} = \frac{1}{2} (\gamma_{\mu} \partial_{\nu} - \gamma_{\nu} \partial_{\mu}) \). We note that this coupling differs from the spinor-tensor vertices introduced in [17, 18] and [19, 20], where the vertices look like \( i\bar{\psi} \gamma_5 \gamma^\rho \epsilon_{\mu\nu\lambda\rho} H^{\mu\nu\lambda} \psi \) and \( \bar{\psi} \gamma^\rho \epsilon_{\mu\nu\lambda\rho} H^{\mu\nu\lambda} \psi \), respectively, being proportional to the stress tensor \( H_{\mu\nu\lambda} \) rather than the \( B_{\mu\nu} \) itself. So, unlike these couplings, our interaction can be treated as a minimal one.

We observe that only our coupling \( i\bar{\psi} B^{\mu\nu} \gamma_{[\mu} \partial_{\nu]} \psi \) allows us for obtaining a potential term for \( B_{\mu\nu} \), while other ones yield contributions depending on stress tensor only, which justifies our choice of this coupling to study dynamical Lorentz symmetry breaking. Then, we introduce the spontaneous Lorentz symmetry breaking in the standard way, that is, by shifting the bumblebee field \( B_{\mu\nu} \) around its non-trivial vacuum expectation value (VEV) \( \langle B_{\mu\nu} \rangle = \beta_{\mu\nu} \), with \( \beta^2 = \beta_{\mu\nu} \beta^{\mu\nu} \), by the rule \( B_{\mu\nu} \to \beta_{\mu\nu} + B_{\mu\nu} \), so that the above Lagrangian becomes

\[
\mathcal{L}_B = -\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \bar{\psi}(i\slashed{\partial} - ib^{\mu\nu} \gamma_{[\mu} \partial_{\nu]} - ieB^{\mu\nu} \gamma_{[\mu} \partial_{\nu]} - m)\psi
- \frac{\lambda}{4} \left( B_{\mu\nu} b^{\mu\nu} + \frac{2}{e} B_{\mu\nu} b^{\mu\nu} \right)^2,
\]

(3)

where \( b_{\mu\nu} = e\beta_{\mu\nu} \). Thus, we see that the spontaneous Lorentz violation in (2) has generated the new term \( i\bar{\psi} b^{\mu\nu} \gamma_{[\mu} \partial_{\nu]} \psi \). This term is nothing more as a particular form of the Lorentz-breaking extension of free spinor action, introduced in [2], for the case when \( b^{\mu\nu} \) (denoted there as \( c^{\mu\nu} \)) is antisymmetric. Although in most studies this coefficient is assumed to be symmetric, see f.e. [21], there is no reason forbidding it to be antisymmetric.

As we already said, different issues related to the bumblebee model have been studied in a number of papers (besides of the works above cited, see also, e.g., Refs. [22–26]). In this work, we will follow the idea originally proposed in [27], where the quantum corrections can give origin to the spontaneous symmetry breaking, and show that the bumblebee potential for the tensor field can be dynamically induced through radiative corrections from a self-interacting fermion theory, given by the Lagrangian

\[
\mathcal{L}_0 = \bar{\psi}(i\slashed{\partial} - m)\psi - \frac{G}{2} J_{\mu\nu} J^{\mu\nu},
\]

(4)
where the current is $J_{\mu\nu} = i\bar{\psi}\gamma_{[\mu}\partial_{\nu]}\psi$, as follows from [3]. Indeed, it is convenient to introduce an auxiliary field $B_{\mu\nu}$, in order to eliminate the term $J_{\mu\nu}J^{\mu\nu}$, with $G = \frac{e^2}{g}$, so that the above expression can be rewritten as

$$
\mathcal{L} = \mathcal{L}_0 + \frac{g^2}{2} \left( B_{\mu\nu} - \frac{e}{g^2} J_{\mu\nu} \right)^2
= \frac{g^2}{2} B_{\mu\nu} B^{\mu\nu} + \bar{\psi}(i\partial - ieB_{\mu\nu}\gamma_{[\mu}\partial_{\nu]} - m)\psi.
$$

(5)

In this paper, we generate the bumblebee potential in a very simple way. In order to obtain the effective action, and consequently the bumblebee effective potential, we start with the generating functional

$$
Z(\bar{\eta}, \eta) = \int DB \mu D\psi D\bar{\psi} e^{i\int d^4x (L + \bar{\eta}\psi + \bar{\psi}\eta)} = \int DB \mu e^{i\int d^4x \frac{g^2}{2} B_{\mu\nu} B^{\mu\nu} \int D\psi D\bar{\psi} e^{i\int d^4x (\bar{\psi}S^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta)}}.
$$

(6)

where $S^{-1} = i\partial - ieB_{\mu\nu}\gamma_{[\mu}\partial_{\nu]} - m$, is the operator describing the quadratic action. Now, by performing the shift of the fermion fields, $\psi \to \psi - \eta S$ and $\bar{\psi} \to \bar{\psi} - \bar{\eta} S$, so that $\bar{\psi}S^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta \to \bar{\psi}S^{-1}\psi - \bar{\eta} S \eta$, we obtain

$$
Z(\bar{\eta}, \eta) = \int DB \mu e^{i\int d^4x \frac{g^2}{2} B_{\mu\nu} B^{\mu\nu} \int D\psi D\bar{\psi} e^{i\int d^4x (\bar{\psi}S^{-1}\psi - \bar{\eta} S \eta)}}.
$$

(7)

Finally, integrating over fermions, we get

$$
Z(\bar{\eta}, \eta) = \int DB \mu \exp\left( iS_{\text{eff}}[B] - i \int d^4x \bar{\eta} S \eta \right),
$$

(8)

where the effective action is given by

$$
S_{\text{eff}}[B] = \frac{g^2}{2} \int d^4x B_{\mu\nu} B^{\mu\nu} - iT \ln(p - eB_{\mu\nu}\gamma^{[\mu} p^{\nu]} - m).
$$

(9)

The Tr symbol stands for the trace over Dirac matrices as well as for the integration in momentum or coordinate spaces. The matrix trace can be readily calculated, so that for the effective potential, we have

$$
V_{\text{eff}} = -\frac{g^2}{2} B_{\mu\nu} B^{\mu\nu} + iT \int \frac{d^4p}{(2\pi)^4} \ln(p - eB_{\mu\nu}\gamma^{[\mu} p^{\nu]} - m).
$$

(10)

We note that, unlike the case of the vector field [14], here, we have no essential simplifications when we consider the massless case, since there is no convenient exact form for the massless propagator in the presence of the constant tensor.
The nontrivial minima of this potential can be obtained as usual, from the condition of vanishing the first derivative of the potential:

$$\left. \frac{dV_{\text{eff}}}{dB_{\mu\nu}} \right|_{B_{\mu\nu}=\beta_{\mu\nu}} = -\frac{g^2}{e} \gamma_{\mu\nu} - ie \Pi^{\mu\nu} = 0,$$

(11)

where, again, $b_{\mu\nu} = e\beta_{\mu\nu}$ and the one-loop tadpole amplitude is

$$\Pi^{\mu\nu} = \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 - \mu\nu \gamma_{[\alpha} p_{\beta]} - m} \gamma^\nu_{\mu p^\rho].}$$

(12)

Let us now calculate the above expression by expanding the propagator in terms of $b_{\alpha\beta}$. For this, we use the fact that

$$\frac{\partial + m}{p^2 - m^2} b_{\alpha\beta} \gamma_{[\alpha} p_{\beta]} \frac{\partial + m}{p^2 - m^2} = b_{\alpha\beta} \gamma_{[\alpha} p_{\beta]} \frac{(\partial + m)(\partial + m)}{(p^2 - m^2)^2} = \frac{b_{\alpha\beta} \gamma_{[\alpha} p_{\beta]} p^2}{p^2 - m^2},$$

(13)

so that we can write the tadpole amplitude as a series in $b_{\alpha\beta}$, i.e., $\Pi^{\mu\nu} = \sum_n \Pi^{(2n+1)}_{\mu\nu}$, where

$$\Pi^{(2n+1)}_{\mu\nu} = \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{(-1)^n}{(p^2 - m^2)^n} (b_{\alpha\beta} \gamma_{[\alpha} p_{\beta]})^{2n+1} \gamma_{\mu p^\rho].}$$

(14)

Note that only odd contributions of $b_{\alpha\beta}$ survive, since the trace of even number of Dirac matrices does not vanish, while in the opposite case the result is proportional to $\text{tr}(b_{\alpha\beta} \gamma_{[\alpha} p_{\beta]}^n (\partial + m) \gamma_{\mu p^\rho]} = (b_{\alpha\beta} p_{\rho^n})^n \text{tr}(\partial + m) \gamma_{\mu p^\rho]}$, which always vanishes.

Now, by writing $(b_{\alpha\beta} \gamma_{[\alpha} p_{\beta]}^n)^{2n+1} = (b_{\mu\sigma} p^\sigma)^{2n} b_{\alpha\beta} \gamma_{[\alpha} p_{\beta]}$, we can easily calculate the trace of (14). Then, we obtain

$$\Pi^{(2n+1)}_{\mu\nu} = 4 \int \frac{d^4p}{(2\pi)^4} \frac{(-1)^n}{(p^2 - m^2)^n} (b_{\mu\sigma} p^\sigma)^{2n} b_{\rho\epsilon} p^\epsilon p_{\mu\rho},$$

(15)

where $b_{[\mu\beta] p^\rho p_{\nu]} = \frac{1}{2}(b_{\mu\beta} p^\rho p_{\nu} - b_{\nu\beta} p^\rho p_{\mu})$ is the product antisymmetrized with respect to $\mu$ and $\nu$.

In order to calculate the integral, we use the Feynman formula

$$\int \frac{d^Dp}{(2\pi)^D} \frac{p_{\mu_1} \cdots p_{\mu_p}}{(p^2 - m^2)^\alpha} = \frac{i(-1)^{\frac{D}{2}}}{(4\pi)^{\frac{D}{2}}} \Gamma\left(\frac{\alpha - \frac{D}{2} - \frac{p}{2}}{2}\right) \left(-m^2\right)^{\frac{D}{2} - \alpha} \sum_{\text{perm}} g_{\mu_1\mu_2} g_{\mu_3\mu_4} \cdots g_{\mu_{p-1}\mu_p},$$

(16)

where the sum is taken over all permutations, with $\alpha = n + 1$ and $p = 2n + 2$, so that we get

$$\Pi^{(2n+1)}_{\mu\nu} = \frac{4i\mu^{4-D}(-1)^{n+1}(-1)^{-\frac{p}{2}}}{(4\pi)^{\frac{D}{2}}} \Gamma\left(-\frac{D}{2}\right) \left(-m^2\right)^{\frac{D}{2}} b_{\rho_1\mu_1} b^{\rho_1\mu_2} \cdots b_{\rho_n\mu_{2n-1}} b^{\rho_n\mu_{2n}}$$

$$\times b_{[\mu_{2n+1]} \sum_{\text{perm}} g^{\rho_{1\mu_2}} g^{\rho_{3\mu_4}} \cdots g^{\rho_{2n+1\nu}]}.}$$

(17)

To study the minima of effective potential, in the usual case of small field, where the potential is well described as a power series in the field, it is sufficient to consider only two lower terms in
the sum for \( \Pi_{\mu\nu} \), that is, those ones with \( n = 0 \) and \( n = 1 \), which are linear and cubic in the field \( b^{\mu\nu} \), respectively. So, for \( n = 0 \), we have

\[
(-ie)\Pi_{\mu\nu}^{(1)} = -\frac{em^4}{8\pi^2} \left( \frac{1}{\epsilon'} + \frac{3}{4} \right) b_{\mu\nu},
\]

whereas for \( n = 1 \), we obtain

\[
(-ie)\Pi_{\mu\nu}^{(3)} = \frac{em^4}{16\pi^2} \left( \frac{1}{\epsilon'} + \frac{3}{4} \right) (b_{\mu\nu} b_{\alpha\beta} b^{\alpha\beta} + 2b_{\mu\alpha} b_{\nu\beta} b^{\alpha\beta}),
\]

where \( \frac{1}{\epsilon'} = \frac{1}{\epsilon} - \ln \frac{2\mu}{\mu} \), with \( \epsilon = 4 - D \) and \( \mu^2 = 4\pi\mu^2 e^{-\gamma} \). This last contribution can be simplified by using the expression

\[
b_{\mu\alpha} b_{\nu\beta} b^{\alpha\beta} = \frac{1}{2} b_{\mu\nu} b_{\alpha\beta} b^{\alpha\beta} + \frac{1}{4} b_{\mu\nu} b_{\alpha\beta} \tilde{b}^{\alpha\beta},
\]

where \( \tilde{b}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\kappa\lambda} b_{\kappa\lambda} \).

Then, the gap equation (11) can be rewritten as

\[
\left. \frac{dV_{\text{eff}}}{dB_{\mu\nu}} \right|_{B_{\mu\nu}=\beta_{\mu\nu}} = \left[ -\frac{e}{G} - \frac{em^4}{8\pi^2} \left( \frac{1}{\epsilon'} + \frac{3}{4} \right) (1 - x_1) \right] b^{\mu\nu} + \frac{em^4}{32\pi^2} \left( \frac{1}{\epsilon'} + \frac{3}{4} \right) x_2 \tilde{b}^{\mu\nu} + \cdots = 0,
\]

where \( x_1 = b_{\alpha\beta} b^{\alpha\beta} \) and \( x_2 = b_{\alpha\beta} \tilde{b}^{\alpha\beta} \). Let us discuss the significance of this equation. First of all, we note that when we integrate it, we arrive at the result looking like a linear combination of \((B_{\mu\nu} B^{\mu\nu})^2\) and \((B_{\mu\nu} \tilde{B}^{\mu\nu})^2\), with both coefficients accompanying these terms being positive, hence, our effective potential is positively definite and thus displays minima (the explicit form of this potential will be given further). Second, while the first term possesses minimum at some definite value of \( x_1 \) different from zero, which gives the module of \( b_{\mu\nu} \) but not its direction, the second term displays minimum at \( x_2 = 0 \), which restricts the direction of the \( b_{\mu\nu} \). At the same time, it is interesting to note that if we take the second derivative of \( V_{\text{eff}} \), we find that its lower order in \( b_{\mu\nu} \) will be proportional to \( \eta_{\mu\nu} \), with the positive sign, i.e., we indeed have a minimum.

It is worth mentioning that if the currents are \( J_{\mu\nu} = \bar{\psi} \sigma_{\mu\nu} \psi \) and \( J_{\mu\nu} = i\bar{\psi} \sigma_{\mu\nu} \gamma_5 \psi \) (instead of \( J_{\mu\nu} = i\bar{\psi} \gamma_{[\mu} \partial_{\nu]} \psi \)) the result for (11) is

\[
\left. \frac{dV_{\text{eff}}}{dB_{\mu\nu}} \right|_{B_{\mu\nu}=\beta_{\mu\nu}} = \left[ -\frac{e}{G} + \frac{em^2}{\pi^2} \left( \frac{1}{\epsilon'} - \frac{1}{2} \right) - \frac{2ex_1}{3\pi^2 \epsilon' \epsilon} \right] b^{\mu\nu} - \frac{2ex_2}{3\pi^2 \epsilon' \epsilon} \tilde{b}^{\mu\nu} + \cdots = 0,
\]

for the two possibilities. In this expression, we can observe that the terms proportional to \( x_1 b_{\mu\nu} \) and \( x_2 \tilde{b}_{\mu\nu} \) (or, after we integrate (22), \((B_{\mu\nu} B^{\mu\nu})^2\) and \((B_{\mu\nu} \tilde{B}^{\mu\nu})^2\), respectively) have a negative sign, which indicates that the potential is not positive definite, and, hence, it does not display minima. This perception will be more clean, in the follows, when we will integrate the gap equation (21) to obtain the potential.
In order to get more information about the Eq. (21), let us try to evaluate the Eq. (12) in a general way, by writing it as

$$
\Pi^{\mu\nu} = \text{tr} \int \frac{d^4p}{(2\pi)^4} \frac{p' + m - \gamma[p^\nu]}{p'^2 - m^2} \gamma[p^\mu] = 2 \int \frac{d^4p}{(2\pi)^4} \frac{p'^\mu p'^\nu - p^\mu p^\nu}{p'^2 - m^2},
$$

(23)

where $p'_\alpha = M_{\alpha\beta} p^\beta$, with $M_{\alpha\beta} = g_{\alpha\beta} - b_{\alpha\beta}$. Thus, we have $d^4p' = \det (\frac{\partial p^\mu}{\partial p'^\mu}) d^4p$, i.e., $d^4p' = \det (M^{\mu\alpha} g_{\alpha\nu}) d^4p = -\det (M^{\mu\alpha}) d^4p$, so that

$$
d^4p = -\det^{-1} (M^{\mu\alpha}) d^4p'.
$$

(24)

Now, as $p_\alpha = (M^{-1})_{\alpha\beta} p^\beta$, we must calculate $(M^{-1})_{\alpha\beta}$, which is given by

$$
(M^{-1})_{\alpha\beta} = \left[ \left( 1 + \frac{x_1}{2} \right) g_{\alpha\beta} + b_{\alpha\beta} + b_{\alpha\gamma} b^{\gamma\beta} - \frac{x_2}{4} b_{\alpha\beta} \right] \left( 1 + \frac{x_1}{2} - \frac{x_2}{16} \right)^{-1},
$$

(25)

where we have used the expression (20). Then, the Eq. (23) can be rewritten as

$$
\Pi^{\mu\nu} = -\det^{-1} (M^{\kappa\lambda}) \left( g^{\mu\beta} (M^{-1})^{\nu\alpha} - (M^{-1})^{\mu\alpha} g^{\nu\beta} \right) \int \frac{d^4p'}{(2\pi)^4} \frac{p'_\alpha p'_\beta}{p'^2 - m^2}.
$$

(26)

Finally, by using the expression (16) and (25), as well as the fact that $\det (M^{\kappa\lambda}) = - \left( 1 + \frac{x_1}{2} \right)$, we obtain the gap equation

$$
\left. \frac{dV_{\text{eff}}}{dB_{\mu\nu}} \right|_{B_{\mu\nu} = \beta_{\mu\nu}} = -\frac{e}{G} b^{\mu\nu} - \frac{e m^4}{8 \pi^2} \left( \frac{1}{e'} + \frac{3}{4} \right) \left( 1 + \frac{x_1}{2} \right)^{-1} \left( 1 + \frac{x_1}{2} - \frac{x_2}{16} \right)^{-1} \left( b^{\mu\nu} - \frac{x_2}{4} b^{\mu\nu} \right) = 0.
$$

(27)

We can observe that this Eq. (27), up to first orders in $x_1$ and $x_2$, reproduces exactly the Eq. (21), as expected.

By considering $x_2 = 0$, as we have discussed above, we get

$$
\left. \frac{dV_{\text{eff}}}{dB_{\mu\nu}} \right|_{B_{\mu\nu} = \beta_{\mu\nu}} = \left[ -\frac{e}{G} - \frac{e m^4}{8 \pi^2} \left( \frac{1}{e'} + \frac{3}{4} \right) \left( 1 + \frac{x_1}{2} \right)^{-2} \right] b^{\mu\nu} = 0,
$$

(28)

so that

$$
\frac{1}{G} = -m_R^{-4} \left( 1 + \frac{x_1}{2} \right)^{-2},
$$

(29)

where $m_R = Z_m^{-1/4} m$, with

$$
\frac{1}{Z_m} = \frac{1}{8 \pi^2} \left( \frac{1}{e'} + \frac{3}{4} \right).
$$

(30)

Note that, with this, $G < 0$, i.e., we can write $G = -|G|$.

Now, we can rewrite the expression (28) as

$$
\left. \frac{dV_{\text{eff}}}{dB_{\mu\nu}} \right|_{B_{\mu\nu} = \beta_{\mu\nu}} = \left[ -\frac{e}{G} - \frac{e m_R}{8 \pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k} (k + 1) x_1^k \right] b^{\mu\nu},
$$

(31)
so that, by integrating it, we have

$$V_{\text{eff}} = -\frac{1}{2G} X_1 - m_R^4 \frac{X_1}{2 + X_1} + \alpha,$$

(32)

where $X_1 = e^2 B_{\mu\nu} B^{\mu\nu}$, $\alpha$ is an integration constant, and we have employed the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} X_1^{k+1} = \frac{X_1}{2 + X_1}.$$

Finally, by using (29), we obtain

$$V_{\text{eff}} = \frac{m_R^4}{2} X_1 \left[ \left( 1 + \frac{x_1}{2} \right)^{-2} - \left( 1 + \frac{X_1}{2} \right)^{-1} \right] + \alpha.$$  

(33)

We see that this expression, first, involves arbitrary order in fields, being non-polynomial, second, includes terms with different signs. This means that our effective potential possesses a set of minima $\langle B_{\mu\nu} \rangle$ satisfying the condition $e^2 \langle B_{\mu\nu} \rangle \langle B^{\mu\nu} \rangle = b_{\mu\nu} b^{\mu\nu}$. Clearly, the most interesting situation is described by the approximation of small fields, which is more natural from the physical viewpoint, so that we can keep only some first orders in expansion of the effective potential in power series in $X_{1,2}$ and $x_{1,2}$. The lowest contribution to the effective potential is described by zero and first order in $x_1$ and $X_1$, and by choosing $\alpha = \frac{m_R^4}{4} x_1^2$, we get the simplest form of the effective potential

$$V_{\text{eff}} = \frac{m_R^4}{4} (e^2 B_{\mu\nu} B^{\mu\nu} - b_{\mu\nu} b^{\mu\nu})^2 + \ldots,$$

(34)

which is the bumblebee potential for the tensorial field $B_{\mu\nu}$, and dots are for higher-order terms.

The key feature of this potential we generated is its positive definiteness. Therefore, it indeed possesses a set of minima where $e^2 B_{\mu\nu} B^{\mu\nu} = b_{\mu\nu} b^{\mu\nu}$, so that a choice of one of these minima evidently generates a privileged spacetime direction and thus breaks the Lorentz symmetry in a spontaneous manner. Effectively we succeeded to generalize the methodology developed in \[10\] for an antisymmetric tensor field. In principle, it is natural to expect that these calculations can be generalized as well for the finite temperature regime, and the possibility of phase transitions can be studied.

After we have proved that the one-loop effective potential indeed displays the minima, let us find the explicit form of the one-loop effective action, where not only potential terms are taken into account, but the second derivative terms as well. To do this, we can rewrite the Eq. (9) as

$$S_{\text{eff}}[B] = \frac{g^2}{2} \int d^4 x B_{\mu\nu} B^{\mu\nu} + S_{\text{eff}}^{(n)}[B],$$

(35)

with

$$S_{\text{eff}}^{(n)}[B] = \mathcal{R} \text{Tr} \sum_{n=1}^{\infty} \frac{1}{n} \left[ S(p) e B_{\mu\nu} \gamma^{[\mu} p^{\nu]} \right]^n.$$  

(36)
and \( S(p) = (\not{p} - m)^{-1} \), where we have disregarded the field independent term \(-i \text{Tr} \ln(\not{p} - m)\).

Our aim is to study the expression (36) up to the fourth order in fields, in order to obtain lower terms of the derivative and field expansion of the effective action. First, for \( n = 1 \) and \( n = 3 \), evidently, \( S_{\text{eff}}^{(1)}[B] \) and \( S_{\text{eff}}^{(3)}[B] \) vanish. Then, let us focus our attention on contributions with \( n = 2 \) and \( n = 4 \). First, for \( n = 2 \), we have

\[
S_{\text{eff}}^{(2)}[B] = \frac{i}{2} \text{Tr} \frac{e B_{\kappa \lambda} \gamma^{[\kappa} p^{\lambda]} S(p) e B_{\mu \nu} \gamma^{[\mu} p^{\nu]} = \frac{i e^2}{2} \int d^4 x \Pi^{\kappa \lambda \mu \nu} B_{\kappa \lambda} B_{\mu \nu},
\]

where

\[
\Pi^{\kappa \lambda \mu \nu} = \text{tr} \int \frac{d^4 p}{(2\pi)^4} S(p) \gamma^{[\kappa} p^{\lambda]} S(p - i \partial) \gamma^{[\mu} (p - i \partial)^{\nu]}.
\]

In order to calculate the above integral, we use the Feynman parametrization, so that we obtain

\[
\mathcal{L}_{\text{eff}}^{(2)} = -\frac{e^2 m^2}{16 \pi^2} B_{\mu \nu}(2 k^2 B^{\mu \nu} - 2 k^\nu k_\alpha B^{\alpha \mu}) + \frac{e^2 m^4}{16 \pi^2} B_{\mu \nu} B^{\mu \nu} + \frac{e^2}{96 \pi^2} B_{\mu \nu} k^4 B^{\mu \nu}
\]

\[+ \frac{e^2}{576 \pi^2} B_{\mu \nu} \left[ (27 m^4 - 42 m^2 k^2 + 8 k^4) + 6 k^2 \left( \frac{4 m^2}{k^2} - 1 \right) \frac{1}{2} \csc^{-1} \left( \frac{2 m}{\sqrt{k^2}} \right) \right] k^\nu k_\alpha B^{\alpha \mu},
\]

with (the external momentum being related with the derivative of the field through the relation) \( k_\mu = i \partial_\mu \), where we have taken into account that the effective action is the integral from the effective Lagrangian over the spacetime, \( S_{\text{eff}}^{(2)} = \int d^4 x \mathcal{L}_{\text{eff}}^{(2)} \).

Now, by imposing the limit of slowly varying fields, which is formally written as \( \partial^2 \ll m^2 \) (while \( m \neq 0 \)), we get

\[
\mathcal{L}_{\text{eff}}^{(2)} = \frac{e^2 m^2}{16 \pi^2} \left( \frac{1}{e^2} + \frac{1}{2} \right) B_{\mu \nu}(\partial^2 B^{\mu \nu} - 2 \partial^\nu \partial_\alpha B^{\alpha \mu}) + \frac{e^2 m^4}{16 \pi^2} \left( \frac{1}{e^2} + \frac{3}{4} \right) B_{\mu \nu} B^{\mu \nu}
\]

\[+ \frac{e^2}{96 \pi^2} B_{\mu \nu} \partial^4 B^{\mu \nu} + O \left( \frac{\partial^2}{m^2} \right).
\]

We can rewrite the above expression as

\[
\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{4 Z_3} B_{\mu \nu}(\partial^2 B^{\mu \nu} - 2 \partial^\nu \partial_\alpha B^{\alpha \mu}) + \frac{e^2 m^4}{2} B_{\mu \nu} B^{\mu \nu} + O \left( \frac{\partial^2}{m^2} \right),
\]

where

\[
\frac{1}{Z_3} = \frac{e^2 m^2}{4 \pi^2} \left( \frac{1}{e^2} + \frac{1}{2} \right).
\]

By defining the renormalized field \( B_{\mu \nu}^{\text{ren}} = Z_3^{-1/2} B_{\mu \nu} \), as well as the renormalized coupling constant \( e_R = Z_3^{1/2} e \), we obtain

\[
\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{12} H_{\rho \mu \nu} H_{\rho \mu \nu}^{\text{\(\text{ren}\)}} + \frac{e^2 m^4}{2} B_{\mu \nu} B_{\mu \nu}^{\text{\(\text{ren}\)}}.
\]
where we have disregarded the terms contributing to higher orders of the derivative expansion.

Finally, for \( n = 4 \), we have

\[
S_{\text{eff}}^{(4)}[B] = \frac{i}{4} \text{Tr} S(p) e B_{\alpha \beta} \gamma^{|\alpha} p^{|\beta} S(p) e B_{\gamma \delta} \gamma^{|\gamma} p^{|\delta} S(p) e B_{\kappa \lambda} \gamma^{|\kappa} p^{|\lambda} S(p) e B_{\mu \nu} \gamma^{[|\mu} p^{|\nu]} \\
= \frac{i e^4}{4} \int d^4x \Pi^{\alpha \beta \gamma \delta \kappa \lambda \mu \nu} B_{\alpha \beta} B_{\gamma \delta} B_{\kappa \lambda} B_{\mu \nu},
\]

where

\[
\Pi^{\alpha \beta \gamma \delta \kappa \lambda \mu \nu} = \text{tr} \int \frac{d^4p}{(2\pi)^4} S(p) \gamma^{|\alpha} p^{|\beta} S(p) \gamma^{|\gamma} p^{|\delta} S(p) \gamma^{|\kappa} p^{|\lambda} S(p) \gamma^{[|\mu} p^{|\nu]} + \mathcal{O}(\partial^4).
\]

Repeating the calculations of the integrals over momenta carried out above, we arrive at the following result for the fourth-order contribution to the effective Lagrangian:

\[
\mathcal{L}_{\text{eff}}^{(4)} = -\frac{e^4 m^4}{64 \pi^2} \left( \frac{1}{e^4} + \frac{3}{4} \right) (B_{\alpha \lambda} B^{\alpha \lambda} B_{\mu \nu} B^{\mu \nu} + 2 B_{\kappa \lambda} B^{\lambda \mu} B_{\mu \nu} B^{\nu \kappa}).
\]

where we have disregarded the derivative terms, which contribute only to higher orders of the expansion. Now, by using (20), we can rewrite the above expression as

\[
\mathcal{L}_{\text{eff}}^{(4)} = -\frac{e^4 m^4}{4} B_{\alpha \lambda} B^{\alpha \lambda} B_{\mu \nu} B_{\mu \nu} - \frac{e^4 m^4}{16} B_{\kappa \lambda} B^{\kappa \lambda} B_{\mu \nu} B_{\mu \nu}.
\]

Therefore, by considering (31), (35), and (47), we have the complete expression for the low-energy tensorial bumblebee Lagrangian, given by

\[
\mathcal{L}_B = -\frac{1}{12} H_{R \mu \nu \lambda} H^{\mu \nu \lambda}_R + \frac{e^2 m^4}{2} B_{R \mu \nu} B^{\mu \nu}_R + \frac{e^2}{2 G} B_{R \mu \nu} B^{\mu \nu}_R - \frac{e^4 m^4}{4} B_{R \alpha \lambda} B^{\alpha \lambda}_R B_{R \mu \nu} B^{\mu \nu}_R \\
- \frac{e^4 m^4}{16} B_{R \kappa \lambda} B^{\kappa \lambda}_R B_{R \mu \nu} B^{\mu \nu}_R.
\]

To simplify this expression, we can now use (29) taken up to first order in \( x_1 \), i.e., \( \frac{1}{G} = -m_R^2(1 - b_{\mu \nu} b^{\mu \nu}) \), so that we obtain

\[
\mathcal{L}_B = -\frac{1}{12} H_{R \mu \nu \lambda} H^{\mu \nu \lambda}_R - \frac{m_R^4}{4} (e^2 B_{R \mu \nu} B^{\mu \nu}_R - b_{\mu \nu} b^{\mu \nu})^2 - \frac{e^4 m^4}{16} B_{R \alpha \lambda} B^{\alpha \lambda}_R B_{R \mu \nu} B^{\mu \nu}_R,
\]

where we have added the constant \(-m_R^2 x_1^2\). Therefore, we conclude that we have succeeded to generate the low-energy effective action for the tensor bumblebee field, which includes the usual kinetic term, for the second-rank antisymmetric tensor field, the positively defined potential displaying the set of minima \( x_1 = 2(1 - \sqrt{G|m_R^4|}) \), allowing for spontaneous Lorentz symmetry breaking, and another one with the trivial minimum \( x_2 = 0 \) (see Eqs. 21 and 27). We note that other derivative-free forms of the second-rank tensor current, such as \( J_{\mu \nu} = \bar{\psi} \sigma_{\mu \nu} \psi \) and \( J_{\mu \nu} = i \bar{\psi} \sigma_{\mu \gamma} \gamma_5 \psi \), do not possess this feature, i.e., the one-loop effective potentials generated with their use do not display minima (see Eq. 22), hence, the spontaneous Lorentz symmetry breaking cannot occur.
Now, let us discuss our results. Within our paper, we have successfully generalized the mechanism of the dynamical Lorentz symmetry breaking for the case of the second-rank antisymmetric tensor field, i.e., for the first time, we generated the tensor bumblebee action through the perturbative methodology. The approach we have used continues the line of our earlier paper [10] and guarantees that our effective potential indeed possesses minima, which justifies the consistency of our results. We note that actually our paper represents itself as one of the first studies of quantum aspects of Lorentz symmetry breaking for the higher-rank tensor field models. It is natural to expect that this methodology can be generalized to more sophisticated models, for example, those applied within the string context. Another possible generalization of our study could consist in introducing the finite temperature with the subsequent study of the possibility of phase transitions, as well as in introducing of a curved background. We expect to carry out these generalizations in forthcoming papers.

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