OPTIMAL CONTROL INDICATORS
FOR THE ASSESSMENT OF THE INFLUENCE OF
GOVERNMENT POLICY TO BUSINESS CYCLE SHOCKS

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Abstract. We consider idealised dynamic models isolating the relationship
between GDP and government expenditures. In this setting we assess the
possibility of smoothing the effect of business cycle shocks via government
expenditure alone and propose optimal control indicators measuring the control
potential of this government action. This provides with new indicators and
indices refining the dynamic relationship obtained by ARMA or similar type
of macro - modeling.

1. Introduction. One of the simplest mathematical descriptions of business cycles
are the linear time – invariant second – order stochastic difference equations (see
Arnold, 2002 [13]). This model relates the value $y_t$ to two lagged values $y_{t-1}$
and $y_{t-2}$ as well as a random variable $\epsilon_t$. The solution of this equation can be
viewed as a disturbed sine wave or as a time series exhibiting variability, persistence
and reversion. The latter are key statistical properties of observed business cycles
whereas the former constitutes the traditional approach to business cycles. The
random variable $\epsilon_t$ is considered as an exogenous disturbance that initiates the
business cycle and the second order autoregressive part of the model the mechanism
that absorbs the disturbance. The appearance of business cycles is explained by the
interaction of a persistent random disturbance with the absorption AR mechanism.

These types of models appear in the majority of schools of economic thought
and can be derived based on the specific theoretical tools of each school. According
to the Keynesian school the multiplier – accelerator model (see e.g. Allen, 1968
[22]) explains the business cycles. Autonomous fluctuations in investment keep
the cycles from abating whilst induced investment is the variable that initiates
the turning points in economic activity. In contrast to the Keynesian multiplier –
accelerator model the monetarists’ model explains the business cycle based on the
relation between supply (Phillips curve) and demand (the quantity equation). The
appearance of the supply side in the monetarist model is the major accomplishment
and a key difference with the Keynesian multiplier – accelerator model. Laidler
(1976) [4] shows that the interplay between a Friedman accelerationist Phillips

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curve and the quantity equation, is sufficient to generate business cycles in the
sense of Frisch. The real business cycles theory (RBC) offers a pure supply – side
explanation for fluctuations of economic activity. This theory, originating from the
seminal contributions of Kydland and Prescott (1982) [6] and Long and Plosser
(1983) [8], explains business cycles within the framework of perfect markets and
rational expectations. The fluctuations of economic activity are regarded as the
efficient outcome of the interaction between maximizing agents. Extensions of RBC
theory studying the relationship between productivity, wages and hours worked
were made by Christiano and Eichenbaum (1992) [11], Bencivenga (1992) [24], Cho
and Cooley (1995) [7], Boldrin and Horvath (1995) [14]. Other extensions to the
basic RBC model include Burnside and Eichenbaum (1996) [2], Farmer and Guo
(1994) [20] and King and Rebelo (1999) [21]. The New – Keynesian approach uses
the postulates of rational expectations and optimizing behavior in order to come up
with Keynesian – style propositions emphasizing the role of the demand side in the
determination of aggregate output and its fluctuations. Mankiw and Romer (1991)
[15] is a collection of authoritative papers on the New – Keynesian approach.

The problem of smoothing the effects of business cycles via appropriate fiscal
or monetary policy can be classified as an optimal control problem. The optimal
control problem in its deterministic version consists in choosing time paths for the
control variables from a given control set where the variables describing the system
(state variables) are given by a set of difference or differential equations (equations of
motion). This choice has to be made in such a way that a given objective functional
which depends on the control and state variables is to be maximized or minimized.

One can apply optimal control theory to macroeconomic policy by utilizing an
econometric model interrelating the usual basic macroeconomic variables and con-
taining the policy variables as controls. This in turn may be used as a constraint
when minimizing a cost criterion. For the cost functional it may be assumed that the
primary aim of stabilization policy consists in preventing oscillations in economic
variables or in driving the variables along ideal paths, for instance, low unemploy-
ment and inflation.

According to Neck (2008) [23] the reason why only relatively little work was
done on directly applying traditional methods of control engineering to economics
may be found in the fact that the conditions of system construction differ between
the two cases. More specifically, despite the fact that the control goal in both
engineering and economics is to achieve a stable system with desired good behavior
or performance the ability to affect the internal relations of the system towards this
end is rarely available to the economist whereas it is practically always available to
the engineer.

However, control theory has been used informally by economists in actual policy
selection. Policymakers have looked at the current state of the economy and used
an implicit law of motion to evaluate the consequences of alternative decisions upon
output, employment and prices. However to use control theory in a rigorous way,
one has to have a detailed and tested theory that predicts the timing and magnitude
of the effects of macroeconomic policy. A more fundamental attack upon the use
of macro - models and control theory to stabilize the economy is the Lucas critique
(1976) [19]. He argues that the structure of the econometric model describing the
motion of the economy is not invariant to the policy rule and that policy simulations
using the macro models are worthless in assessing alternative policy rules.
Here we will not attempt to use control theory to enhance or fine tune macroeconomic policy but instead, we will utilize optimal control tools to explore dynamic relationships between macro - variables. Specifically we will consider second order control systems of ARMA type, modeling the relationship between Gross Domestic Product (GDP) and government expenditure, according to the Frisch proposal for the explanation and modeling of business cycles. The first naive attempt to assess the dynamic relationship and influence between the two macro - variables are via the coefficients of ARMA regression and their statistical properties. However these coefficients do not reveal the ability of one variable (input) to control the other (output). This information is hidden within the coefficients of regression and can be recovered through the study of the dynamic relationship as an input – output control system. To this end after estimating the ARMA model of the detrended time series we solve the optimal control problem of minimizing the effect of business cycle shocks via shaping the government expenditure. The solution of this optimization problem provides with new indicators characterizing the relationship of the two macroeconomic variables. This relationship is examined for a sample of eight countries namely Bulgaria, Poland, Romania, Slovakia, Slovenia, Sweden, the United Kingdom and Lithuania. A series of other countries were also examined but not included due to unexpected statistical results and feedback optimal control features. These had to do with the fact that the gain turned out to be positive meaning fiscal policy is procyclical instead of countercyclical. This is in contrast to conventional wisdom and also to the Keynesian perspective on public expenditure according to which public expenditure must act as a stabilizing force and thus move in a countercyclical direction. The procyclicality of fiscal policy can be explained based on political economy factors (see e.g. Lane, 2003 [17]).

The poles of the ARMA model estimated for each country lie inside the unit circle. The largest open – loop pole magnitude is found in the case of Romania which also exhibits the most negative shock. Lithuania exhibits the best open – loop dynamics having the smallest open – loop pole magnitude. The solution of the optimal control problem is based on the following two assumptions: (1) A large negative shock equal to twice the minimum value of the residuals distribution and (2) an inequality constraint of the form \( g_t \leq y_{t-1} + c \) where the constant \( c \) is taken to be three times the standard deviation of the series \( g_t - y_{t-1} \). By \( g_t \) we denote the detrended logarithm of government spending whereas \( y_t \) denotes the same for the GDP. Solving the optimal control problem we found that Sweden exhibited the smallest open – loop error with Slovenia following in second place. Moving on we constructed a feedback optimal control of the form \( g_t = ky_{t-1} \).

The application of feedback reduced the magnitude of the open – loop poles. The reduction was greatest for Slovakia, Lithuania and Sweden which also exhibited the smallest closed – loop error. Furthermore the error reduction, compared to the uncontrolled ARMA model, was the second largest for the case of the Swedish economy in spite of the fact that it already had the lowest error for the uncontrolled case (\( g_t \equiv 0 \)). Upon estimating an appropriate regression model it was found that the variables which affected significantly the error reduction, in percentage terms, from the uncontrolled case to the feedback control case were the size of the shock, the magnitude of the open – loop poles and the government’s budget constraint. The first two factors are also significant in the interpretation of the size of the closed – loop error.
In the literature there are studies modeling the behavior of cyclical output of an economy as an ARMA model. Blanchard and Fischer (1989) \cite{16} find that cyclical US GNP from 1973 to 1987 is related to two lagged values of itself as well as a second – order moving average part. Upon estimating the ARMA model for the differenced series they come to the conclusion that GNP growth is close to a random walk with drift. Nelson and Plosser (1982) \cite{3} and many subsequent authors including Stock and Watson (1986) \cite{9}, Campbell and Mankiw (1987) \cite{10}, Malliaris and Urrutia (1990) \cite{1} demonstrated the same result. Other authors however have expressed serious disagreements about the degree of persistence in US GNP time series as well as the inherent unreliability of such estimations (see e.g. Christiano and Eichenbaum, 1990 \cite{12}).

All these studies examine only statistical properties of estimated models and do not address any control issue since the models considered are uncontrolled in the first place. Hence they are associated with only a few of the results obtained here and are not comprehensively related to the topics in question. Here we construct an optimal control model in order not just to examine statistical properties but to extract dynamic relationships between cyclical output and government expenditure and evaluate the economies of certain countries using a set of optimal control indicators.

2. **Problem definition.** Consider a discrete time dynamic equations modeling the dynamics of a macroeconomic system

\[ x_{k+1} = f(x_k; u_k; \epsilon_k); x(0) = x_0, \]

where \( x_k \) is the vector of macroeconomic variables, \( u_k \) is the vector of policy variables and \( \epsilon_k \) is a vector of shocks or disturbances. We may also have state - input constraints of the type:

\[ g(x_k; u_k) \leq c. \]

If the objective is to minimize oscillation or the deviation from an expected value or path then we may consider the minimization problem

\[
\text{Minimize } u_i \sum_{i=1}^{n-1} L(u_i, x_i),
\]

where \( L \) is some cost or penalty function that penalizes the deviation of the target variable from its desired trajectory. This function can be selected to be an \( l_1 \) or \( l_2 \) norm of the deviations or the expected regrets of the deviations (penalize only the negative deviations) or negative overshoot (lowest negative deviation) or settling time (time that returns back to the desired trajectory). The solution of the above optimization problem will be the optimal inputs and states that minimize the objective functional. The minimum value of the objective functional will be a measure of how well the system can be controlled. This value depends on the parameters and the nature of the system, the size and the nature of the constraints, the size and the type of the disturbances and finally on the nature of the objective functional. It is evident that this value incorporates a lot of information about the potential controllability of the system.

In this paper we will follow Frisch’s suggestion that business cycles are modeled by means of a linear time – invariant second – order stochastic difference equation i.e.

\[ y_t + a_1 y_{t-1} + a_2 y_{t-2} = \epsilon_t, \]
where $y_t$ is the deviation of GDP from the trend and $\epsilon_t$ is white noise. Depending upon the values of $a_1, a_2$, if the above system has poles within the unit circle then $\epsilon_t$ may be interpreted as a random mechanism creating the cycles and and the left side a mechanism that absorbs the shocks.

This type of consideration arises from all schools of economic thought (Keynesian, monetarist, RBC, New – Keynesian) starting from different assumptions. Also the coefficients $a_1, a_2$ of the model have different interpretations with respect to the various schools. Here we consider a similar type of dynamical system, which is however controlled i.e.

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = u_t + \epsilon_t .$$

The control here is $u_t$ and the objective is for $u_t$ to be designed in such a way that it minimizes the effect of the cycles. The $y_t$ variable is modeling the deviation from the trend:

$$\ln GDP_t = y_t + z_t ,$$

where $y_t$ is the cyclical part and $z_t$ is the trend. The above decomposition may be realized by a Hodrick - Prescott filter or by other detrending methods (see Canova, 1998 [5]). The problem then is reduced to minimizing $\sum y_t^2$ with respect to the control $u_t$. The definition of the problem depends on the assumptions on $\epsilon_t$. If $\epsilon_t$ is a negative shock, then we have to find the policy that minimizes the effect of this shock. If $\epsilon_t$ are periodic negative shocks then the policy $u_t$ has to be defined accordingly. Here we will consider the problem of minimizing the effect of a single strong negative shock. The horizon will be 15 - 20 quarters and the results will be in terms of open - loop and closed - loop policies. The new indicators will be the control policy indices evaluating the ability of government policies to minimize the effect of economic shocks or fluctuations. The new indices will be utilized to characterize and classify various European state economies. Before proceeding, let us note that Frisch’s suggestion for the generation of business cycles implies that cycles are exogenous since the shocks $\epsilon_t$ which give rise to them are exogenous themselves. If the dynamics were nonlinear, then business cycles could be modeled without the presence of outside shocks whether on the demand side or the supply side. In this case the business cycle would be endogenous in the sense that it is generated from the dynamics of the model itself without the presence of outside shocks (see e.g. Puu and Sushko, 2004 [26]).

3. Problem methodology. The control schemes considered will be both open - loop and closed - loop i.e.

(1) Calculate the open - loop policy $u_t$ that minimizes the objective functional.

(2) Calculate the closed - loop policy $u_t = f(y_t, y_{t-1}, y_{t-2})$ that minimizes the objective functional.

For the construction of the open - loop model the first step is to detrend the GDP and government spending. This step gives rise to $y_t, y_{t-1}$ which are the deviations of $\ln GDP_t$ and $\ln GovExp_t$ from the trend. Secondly, $y_t$ is regressed on $y_{t-1}, y_{t-2}, ..., u_t, u_{t-1}, ...$ until a reliable model is obtained. The model considered here was

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = b g_t + \epsilon_t .$$

The model

$$y_t + a_1 y_{t-1} = b g_t + \epsilon_t$$
was also considered for some countries due to the fact that it was better based on the usual statistical criteria $t$, $F$, $R^2$. Such a model explains how the $y_t$ deviations from the trend are related to those of $g_t$ as well as $\epsilon_t$ which are considered as exogenous shocks. Cyclical government spending is not free but is subject to constraints of the form:

$$\text{GovExp}_t \leq c\text{GDP}_{t-1}.$$  
Taking logarithms on both sides yields that $g_t + g_t^{\text{trend}} \leq \ln c + y_{t-1} + z_{t-1}.$

Equivalently,

$$g_t \leq \ln c + y_{t-1} + z_{t-1} - g_t^{\text{trend}}$$

or

$$g_t \leq \ln c + y_{t-1} + (z_{t-1} - z_t) + z_t - g_t^{\text{trend}}.$$  
The term $z_{t-1} - z_t$ reflects the growth trend whereas the term $z_t - g_t^{\text{trend}}$ defines the difference of the GDP and government expenditures trends. Hence we have a dynamic inequality constraint of the type:

$$g_t \leq y_{t-1} + Q.$$  
In this setting, we will consider two optimal control problems:

a) Open - loop optimal control, where we consider as a penalty function the $l_2$ norm of the deviation from the trend and as control variable the government policy. Then we have to solve the following optimization problem:

Minimize $\sum g_t^2$

subject to

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = b g_t + \epsilon_t$$

$$g_t \leq y_{t-1} + Q.$$  
The objective of the government control policy is to minimize the effect of the negative shock $\epsilon_t$ given the dynamics of the economy and the constraints it obeys.

The assumptions of the model are the following:

1) We study the effect of a single negative shock (impulse). Therefore,

$$\epsilon_{t-1} = \epsilon_1 = \ldots = \epsilon_n = 0, \epsilon_0 = -\epsilon.$$  
2) Prior to the appearance of the shock the economy follows the trend i.e.

$$y_{t-1} = y_{t-2} = 0.$$  
3) The government policy is activated (in excess of the trend) one period after the appearance of the shock i.e.

$$g_{t-1} = 0.$$  
Under these assumptions the constraints may be rewritten as:

$$y_0 = -\epsilon$$

$$y_1 + a_1 y_0 = b g_1$$

$$y_2 + a_1 y_1 + a_2 y_0 = b g_2$$

$$\vdots$$

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} = b g_n$$

$$g_1 \leq y_0 + Q$$

$$g_2 \leq y_1 + Q$$

$$g_3 \leq y_1 + Q$$

$$\vdots$$

$$g_n \leq y_{n-1} + Q.$$
Hence the optimal control problem may be rewritten as:

\[
\begin{aligned}
\text{Minimize} & \quad \|y\|^2 \\
\text{subject to} & \quad Ty = bg + \epsilon a \\
& \quad y \leq S y + q ,
\end{aligned}
\]

where

\[
T = \begin{pmatrix}
1 & a_1 & a_2 & 0 \\
0 & 1 & a_1 & a_2 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & 1 & a_1 \\
0 & 0 & \ddots & 0 & 1 \\
\end{pmatrix},
S = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & 0 & 1 \\
0 & 0 & \ddots & 0 & 0 \\
\end{pmatrix},
\]

\[
y = (y_n, y_{n-1}, \ldots, y_1), g = (g_n, g_{n-1}, \ldots, g_1) \\
q = (Q, Q, \ldots, Q - \epsilon), a = (0, 0, \ldots, a_1, a_2).
\]

The above problem may be solved by (convex) quadratic programming methods (Boyd, 2004 [25]) and may be transformed to the simpler quadratic optimization problem:

\[
\begin{aligned}
\text{Minimize} & \quad \|A\tilde{z} + d\|^2 \\
\text{subject to} & \quad \tilde{z} \geq 0
\end{aligned}
\]

where

\[
A = (S - \frac{1}{b} T)^{-1} \\
d = -(S - \frac{1}{b} T)^{-1}(q + \frac{\epsilon}{b} a)
\]
or equivalently to the linear complimentarity problem (Cottle, 2009[18])

\[
0 \leq (A^T A\tilde{z} + A^T d) \perp \tilde{z} \geq 0 .
\]

b) Feedback optimal control problem, where we would like to minimize the deviation from the trend using a government policy in terms of an output feedback controller. In this setting, we may consider feedback controllers of the form:

\[
g_t = ky_{t-1} .
\]

We get the following dynamic equations:

\[
T(k)y = -\epsilon a(k) ,
\]

where

\[
T(k) = \begin{pmatrix}
1 & a_1 - bk & a_2 & 0 \\
0 & 1 & a_1 - bk & a_2 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & 1 & a_1 - bk \\
0 & 0 & \ddots & 0 & 1 \\
\end{pmatrix}
\]

and

\[
a(k) = (0, 0, \ldots, a_2, a_1 - bk)^T .
\]
This is subject to the constraints

\[(k-1)\epsilon \leq Q \ and \ (k-1)y_i \leq Q, \ i = 1, ..., n-1.\]

The linear dynamic equations may be easily solved as

\[y_i = f_i(k)\]

and the optimization problem becomes:

\[
\text{Minimize} \quad \sum_k f_i(k)^2 \\
\text{subject to} \quad (k-1)\epsilon \leq Q \\
\qquad \qquad (k-1)f_i(k) \leq Q.
\]

The above optimal control problems will provide us with two types of solutions. An open-loop optimal government policy and a feedback government policy both minimizing the deviation of GDP from the trend. Subsequently, indicators related to these optimal control problems will be calculated measuring the influence of government policy to business cycle shocks. These new indices calculated in this framework will provide additional macro-indicators for the examination and assessment of the economies of various countries.

4. **Quantitative results.** The following results where obtained using a sample of 36 quarterly observations for the period 1999:1 until 2007:4 for the GDP and total government expenditure for eight countries: Bulgaria, Poland, Romania, Slovakia, Slovenia, Sweden, the United Kingdom and Lithuania. Both GDP and government expenditure were initially deseasonalized and then detrended using a Hodrick–Prescott filter in order to capture the cyclical component. Grouping the countries together was done using a variety of control indicators for the open-loop and closed-loop systems such as size of error after the application of optimal policy, percentage error reduction between the uncontrolled and the feedback optimal control case and largest open-loop pole magnitude reduction when feedback is applied.

![Figure 1. Table of quantitative results.](image)

4.1. **Summary of results.** The poles of the open-loop and closed-loop systems lie inside the unit circle in all cases. Hence these are stable dynamical systems. The application of feedback improved the response of the open-loop system by reducing the largest pole magnitude and hence making shock absorption faster. The reduction is almost 100% for the cases of Slovakia, Sweden and Lithuania. Sweden has the lowest error for all three cases examined namely the feedback control, the
open-loop and the uncontrolled system. It has nearly 100% reduction of the largest open-loop pole magnitude when linear feedback is applied and the second largest percentage error reduction between the uncontrolled and feedback control case. Furthermore, it has the lowest shock in absolute value terms and the lowest standard deviation for the series $g_t - y_{t-1}$. Hence the Swedish economy may be ranked in first place according to these statistical and optimal control indicators.

In order to interpret the percentage error reduction between the uncontrolled and closed-loop systems we regressed the relevant variable on the magnitude of the largest open-loop pole, the shock and the constraint constant $c$ (calculated from the distribution of the series $g_t - y_{t-1}$). The results obtained are given in Figure 13 of the appendix. The sign of the estimated coefficients is negative in all cases with low p-values. Hence as the magnitude of the largest open-loop pole increases the percentage error reduction is larger in absolute value and the same is true for an increase in the magnitude of the shock and the constraint constant. Furthermore the magnitude of the largest closed-loop pole was regressed on the shock and the results obtained are given in Figure 14 of the appendix. As the size of the shock increases (in magnitude) the largest closed-loop pole magnitude increases as indicated by the negative sign of the coefficient (as the shock is negative) and the result is significant at the 5% level. Finally the closed-loop error was regressed on the magnitude of the largest open-loop pole as well as the shock with the results given in Figure 15. The closed-loop error and the shock are strongly negatively correlated. The value of the relevant linear correlation coefficient is equal to -0.9833. The scatter diagram for the closed-loop error and shock is given below:

![Scatter diagram: Closed-loop error and shock.](scatter_diagram.png)

The ARMA control model along with the impulse response function for the uncontrolled system, the feedback control system and the open-loop system (after the application of optimal policy) are given below for each country. The feedback optimal control is also provided. In the appendix are given the statistical tables of the estimations of the model for each country along with the distributions of the shocks and the series $g_t - y_{t-1}$ which forms the government budget constraint.
4.2. **Analysis of Results.**

4.2.1. **Case I – Bulgaria.** Fitting the data for Bulgaria to the model yielded

\[ y_t - 0.468697y_{t-1} + 0.157351y_{t-2} = 0.075097g_t + e_t. \]

The above is a stable dynamical system having poles $0.234 \pm 0.32i$. The full estimation results are given in Figure 16. The size of the negative shock considered in the solution to the optimal control problem is equal to -0.08 which is twice the minimum value of the relative distribution given in Figure 17 of the appendix. The budget constraint of the government is taken to be $g_t \leq y_{t-1} + 0.24$ where the constant 0.24 equals three times the standard deviation of the distribution of the series $g_t - y_{t-1}$ given in Figure 18. The impulse response of the uncontrolled system ($g_t \equiv 0$) as well as the open-loop and closed-loop impulse response after the application of optimal policy is given in Figure 3.

![Figure 3. Impulse response in the case of Bulgaria.](image)

The error for the feedback optimal control case is equal to 0.0071 whereas for the open-loop and uncontrolled cases the error is 0.0070 and 0.00785 respectively. The feedback optimal control found is $g_t = -2y_{t-1}$. Hence feedback is negative which means that fiscal policy is countercyclical as expected. The magnitude of the largest closed-pole as a function of the feedback gain $k$ is given in Figure 4.

The gain is selected in order to minimize $\sum y_t^2$ subject to closed-loop dynamics and the government budget constraint.

4.2.2. **Case II – Poland.** Fitting the data for Poland to the model yielded

\[ y_t - 0.420572y_{t-1} + 0.293990y_{t-2} = 0.761364g_t + e_t. \]

This is a stable dynamical system having poles $0.21 \pm 0.5i$. All the coefficients are statistically significant even at the 1% level and the coefficient of determination is equal to $R^2 = 0.91$ which is very satisfactory. The value of the DW statistic is also satisfactory. The relevant estimation is presented in Figure 19 of the appendix. The size of the negative shock considered is equal to -0.09. The distribution of the shocks is provided in Figure 20 of the appendix. The budget constraint is given by
$g_t \leq y_{t-1} + 0.12$ where the constant 0.12 is three times the standard deviation of the series $g_t - y_{t-1}$ who’s distribution is given in Figure 21. The following graph shows the impulse response for the uncontrolled system ($g_t \equiv 0$) as well as the open-loop and closed-loop impulse response after the application of optimal policy:

The error for the case of the feedback optimal control system equals 0.009016. The open-loop optimal control as well as the uncontrolled error is 0.008325 and 0.009914 respectively. The feedback rule found is $g_t = -0.333y_{t-1}$. The graph of the magnitude of the largest closed-loop pole as a function of the feedback gain is given below (Figure 6).

The gain is chosen, as before, in order to minimize $\sum y_t^2$ subject to closed-loop dynamics and the budget constraint of the government.
4.2.3. Case III - Romania. Fitting the data for Romania to the model yielded
\[ y_t - 0.639156y_{t-1} = 0.170296g_t + e_t. \]

This is a stable dynamical system having pole 0.639156. Both coefficients are statistically significant with very low p-values and high $R^2$. The addition of $y_{t-2}$ was not chosen because the relative coefficient turned out to be statistically insignificant with a very high p-value. The full estimation table is given in the appendix [22]. The size of the negative shock considered is twice the minimum value of the relative distribution [23]. The budget constraint of the government is given by $g_t \leq y_{t-1} + 0.28$ where the constant is equal to three times the standard deviation of the distribution of the series $g_t - y_{t-1}$ [24]. The impulse response for the uncontrolled model as well as the feedback and open-loop models when the optimal policy is applied is given below (Figure 7).

The error for the feedback optimal control case is equal to 0.011244. The open-loop error equals 0.01106 while the error for the uncontrolled system is 0.016907. The feedback rule found is $g_t = -1.8y_{t-1}$. The gain is chosen, as in all cases, in order to minimize $\sum y_t^2$ subject to closed-loop dynamics and the budget constraint of the government.

4.2.4. Case IV - Slovakia. Fitting the data for Slovakia to the model yielded
\[ y_t - 0.43244y_{t-1} = 0.22181g_t + e_t. \]

This is a stable dynamical system with a single pole at 0.43244. The coefficients of the model are both statistically significant. The full estimation results are provided in Figure 25. The size of the shock considered is -0.06 (two times the minimum value of the distribution). The graph of the relative distribution is given by Figure 26 in the appendix. The distribution of the series $g_t - y_{t-1}$ is also given in the appendix [27]. Hence the budget constraint is $g_t \leq y_{t-1} + 0.2$ where the constant is three times the standard deviation of the series $g_t - y_{t-1}$. The impulse response for the feedback control case and the open-loop system after the application of
optimal policy is given in Figure 8. In the graph the response of the uncontrolled system is also given.

The errors for the feedback optimal control case and open – loop optimal control are both equal to 0.0036. The error for the uncontrolled system equals 0.004428. The feedback rule found is $g_t = -1.9496y_{t-1}$ where the gain is determined from minimizing the functional $\sum y_t^2$ subject to closed – loop dynamics and the government’s budget constraint $g_t \leq y_{t-1} + 0.2$.

4.2.5. Case V – Slovenia. Fitting the data for Slovenia to the model yielded

$$y_t - 0.319139y_{t-1} = 0.509202g_t + e_t .$$
This dynamical system has a stable pole equal to 0.319139. The estimations of the coefficients are statistically significant at the 5% level of statistical significance [28]. The shock considered is equal to -0.05. The distribution of the series $g_t - y_{t-1}$ is given in the appendix [30]. The budget constraint considered for the Slovenian economy is $g_t \leq y_{t-1} + 0.06$. The uncontrolled, feedback and open – loop impulse response (after the application of optimal policy) is given as follows:

![Figure 9. Impulse response in the case of Slovenia.](image)

The feedback optimal control, open – loop optimal control and uncontrolled errors are 0.002624, 0.002618 and 0.002783 respectively. The feedback rule determined from the solution of the closed – loop optimal control problem is $g_t = -0.2y_{t-1}$.

4.2.6. Case VI - Sweden. Fitting the data for Sweden to the model yielded

$$y_t - 0.538109y_{t-1} = 0.511353g_t + e_t .$$

This dynamical system has a stable pole equal to 0.538109. The estimations of the coefficients are statistically significant in every case [31]. The size of the shock considered is twice the minimum value of the relative distribution i.e. -0.04 [32]. The distribution of the series $g_t - y_{t-1}$ is given in Figure 33. The government’s budget constraint is $g_t \leq y_{t-1} + 0.09$. The graph of the impulse response for the feedback and open – loop systems, after optimal policy is applied, as well as the uncontrolled case is given below (Figure 10).

The feedback, open – loop and uncontrolled errors are equal to 0.0016, 0.0016 and 0.002252 respectively. The feedback optimal control is given by $g_t = -1.05232y_{t-1}$ where the value of the feedback gain minimizes $\sum y_t^2$ subject to closed – loop dynamics and government’s budget constraint.

4.2.7. Case VII - United Kingdom. Fitting the data for the United Kingdom to the model yielded

$$y_t - 0.539207y_{t-1} = 0.277484g_t + e_t .$$

This dynamical system has a stable pole equal to 0.539207. The estimations of the coefficients are statistically significant in every case [34]. The size of the shock considered is equal to -0.09 i.e. twice the minimum value of the relative residuals
distribution given in Figure 35. The budget constraint considered is $g_t \leq y_{t-1} + 0.12$ where the distribution of $g_t - y_{t-1}$ is given in Figure 36 of the appendix. The impulse response is given below:

The error in the case of the feedback optimal control system is 0.010119. For the open-loop system this error is equal to 0.009716 while for the uncontrolled case this error equals 0.01142. The feedback optimal control is $g_t = -0.333y_{t-1}$ where the feedback gain is chosen such that the objective functional is minimized subject to closed-loop dynamics and the government’s budget constraint.

4.2.8. Case VIII - Lithuania. Fitting the data for Lithuania to the model yielded

$$y_t - 0.121988y_{t-1} = 0.289782g_t + e_t.$$
This dynamical system has a stable pole equal to 0.121988 [37]. The size of the considered shock to the GDP is equal to -0.07 [38]. The government’s budget constraint is \( g_t \leq y_{t-1} + 0.17 \) where the distribution of the series \( y_t - y_{t-1} \) is given in Figure 39 of the appendix. The impulse response for the case \( g_t \equiv 0 \) as well as the open-loop and feedback optimal control case is given in Figure 12.

![Impulse response in the case of Lithuania.](image)

**Figure 12.** Impulse response in the case of Lithuania.

The error in the case of the feedback optimal control system as well as the open-loop optimal control system is equal to 0.0049. For the uncontrolled system the error equals 0.004974. The feedback optimal control found is \( g_t = -0.421y_{t-1} \).

4.3. Discussion – Conclusions. We applied optimal control methods to examine the dynamic relationship and the control potential of a simple macro dynamic model arising in business cycle theory. The control problem makes sense under large negative shocks where the impulse response was examined with or without government intervention. The dynamic relation can be assessed by examining various control and performance indicators such as overshoot, settling time or mean square deviation. Government intervention apparently improves the dynamic performance, a fact that is reflected on the indicators. These indicators may be used as a measure of the control ability of the dynamical system and may refine the information obtained by ARMA modeling. The new indicators proposed in the paper are: a) the minimized deviation from the trend obtained by the optimal control scheme (both open and closed-loop), b) the magnitudes of the worst open or closed-loop pole. Various regressions relating the sizes of the shocks, the open-loop poles and the closed-loop errors produced the expected results. Finally we calculated these indicators for the economies of eight European countries. Using these proposed indicators we classified them according to the ability to control business fluctuations via government actions.

Appendix. The Tables of estimation results and distribution graphs.
**Figure 13.** Estimation results from the regression of the percentage error reduction on the largest open-loop pole magnitude, the shock and the constraint constant.

| Variable     | Coefficient | Std. Error | t-Score | Prob.  |
|--------------|-------------|------------|---------|--------|
| C            | 0.009488    | 0.076372   | 0.124210| 0.9071 |
| MAGNITUDECL  | -0.623991   | 0.130566   | -4.778235| 0.0088 |
| SHOCK        | -5.44191    | 1.213366   | -4.529558| 0.0012 |
| CONSTRAINT   | -0.86796    | 0.317556   | -2.733061| 0.042  |

R-squared: 0.87324
Adjusted R-squared: 0.745317
S.E. of regression: 0.025230
Sum squared resid: 0.010912
Log likelihood: 15.03751
Durbin-Watson stat: 1.735325

**Figure 14.** Estimation results from the regression of the largest closed-loop pole magnitude on the shock.

| Variable | Coefficient | Std. Error | t-Score | Prob.  |
|----------|-------------|------------|---------|--------|
| C        | -0.322847   | 0.211268   | -1.528140| 0.2173 |
| SHOCK    | -7.79016    | 2.31063    | -3.371647| 0.0224 |

R-squared: 0.361469
Adjusted R-squared: 0.298020
S.E. of regression: 0.157748
Sum squared resid: 0.145037
Log likelihood: 4.573265
Durbin-Watson stat: 1.804191

**Figure 15.** Estimation results from the regression of the closed-loop error on the largest open-loop pole magnitude and the shock.

| Variable     | Coefficient | Std. Error | t-Score | Prob.  |
|--------------|-------------|------------|---------|--------|
| C            | -0.006660   | 0.000748   | -8.908737| 0.0003 |
| MAGNITUDECL  | 0.000378    | 0.001173   | 2.556588 | 0.0506 |
| SHOCK        | -0.134040   | 0.009350   | -14.06288| 0.0000 |

R-squared: 0.985704
Adjusted R-squared: 0.983085
S.E. of regression: 0.000513
Sum squared resid: 1.31E-06
Log likelihood: 31.13678
Durbin-Watson stat: 2.266090
| Variable      | Coefficient | Std. Error | t-Statistic | Prob.   |
|---------------|-------------|------------|-------------|---------|
| GDPBULGACY(-1) | 0.468697    | 0.159547   | 2.937681    | 0.0052  |
| GDPBULGACY(-2) | -0.137314   | 0.164011   | -0.859393   | 0.3948  |
| TEBULGACY     | 0.085607    | 0.034406   | 2.515644    | 0.0506  |

R-squared: 0.307039  Mean dependent var: 0.022313
Adjusted R-squared: 0.263232  S.D. dependent var: 0.019420
S.E. of regression: 0.016679  Akaike info criterion: -5.265186
Sum squared resid: 0.008624  Schwarz criterion: -5.130307
Log Likelihood: 92.08416  Durbin-Watson stat: 1.21309

**Figure 16.** Estimation results for the case of Bulgaria.

**Figure 17.** Distribution of the shocks in the case of Bulgaria.

**Figure 18.** Distribution of the series $g_t - y_{t-1}$ in the case of Bulgaria.
Figure 19. Estimation results for the case of Poland.

Figure 20. Distribution of the shocks in the case of Poland.

Figure 21. Distribution of the series $g_t - y_{t-1}$ in the case of Poland.
Figure 22. Estimation results for the case of Romania.

Figure 23. Distribution of the shocks in the case of Romania.

Figure 24. Distribution of the series $g_t - y_{t-1}$ in the case of Romania.
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Figure 25. Estimation results for the case of Slovakia.

Figure 26. Distribution of the shocks in the case of Slovakia.

Figure 27. Distribution of the series \( g_t - y_{t-1} \) in the case of Slovakia.
Figure 28. Estimation results for the case of Slovenia.

| Variable      | Coefficient | Std. Error | t-Statistic | Prob.  |
|---------------|-------------|------------|-------------|--------|
| GDPSLOVACY(1) | 0.319139    | 0.145039   | 2.200519    | 0.0349 |
| TESLOVACY     | 0.306902    | 0.172017   | 1.819655    | 0.0756 |

R-squared: 0.333653  Mean dependent var: -0.00237
Adjusted R-squared: 0.313800  S.D. dependent var: 0.01627
S.E. of regression: 0.013443  Akaike info criterion: -5.510313
S. squared resid: 0.005099  Schwarz criterion: -5.630436
Log likelihood: 1020880  Durbin-Watson stat: 1.971876

Figure 29. Distribution of the shocks in the case of Slovenia.

Figure 30. Distribution of the series $g_t - y_{t-1}$ in the case of Slovenia.
Figure 31. Estimation results for the case of Sweden.

Figure 32. Distributions of the shocks in the case of Sweden.

Figure 33. Distribution of the series $g_t - y_{t-1}$ in the case of Sweden.
Figure 34. Estimation results for the case of the United Kingdom.

| Variable          | Coefficient | Std. Error | t-Statistic | Prob.  |
|-------------------|-------------|------------|-------------|--------|
| GDPUNITSA CY(-1)  | 0.530207    | 0.066042   | 5.362150    | 0.0000 |
| TRUNITSA CY       | 0.277484    | 0.091405   | 3.083763    | 0.0047 |
| R-squared         | 0.603917    |            |             | 0.0000 |
| Adj. R-squared    | 0.591915    |            |             | 0.0000 |
| S.E. of regression| 0.015241    |            |             | -1.154709 |
| S.E. of residuals | 0.010981    |            |             | -5.326022 |
| Log Likelihood    | 0.558828    |            |             | 1.236464 |

Figure 35. Distribution of the shocks in the case of the United Kingdom.

Figure 36. Distribution of the series $g_t - y_{t-1}$ in the case of the United Kingdom.
DEPENDENT VARIABLE: GDP/LITHE5ACY
Method: Least Squares
Date: 04/03/12 Time: 15:46
Sample: 1999.2 2007.4
Included observations: 35 after dropping endpoints

| Variable          | Coefficient | Std. Error | t-Statistic | Prob.  |
|-------------------|-------------|------------|-------------|--------|
| GDP/LITHE5ACY(-1) | 0.121988    | 0.127309   | 0.897322    | 0.3814 |
| TELITHE5ACY       | 0.287822    | 0.004414   | 4.517853    | 0.0001 |
| R-squared         | 0.504555    | 0.001520   |             |        |
| Adjusted R-squared| 0.469541    | 0.039671   |             |        |
| S.E. of regression| 0.020342    | 0.001503   |             |        |
| Sum squared resid | 0.015030    | 0.0017193  |             |        |
| Log likelihood    | 86.01522    | 2.197488   |             |        |

Figure 37. Estimation results for the case of Lithuania.

Figure 38. Distribution of the shocks in the case of Lithuania.

Figure 39. Distribution of the series $gt - gt-1$ in the case of Lithuania.
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