The approximation of Laplace-Stieltjes transforms with finite order

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Abstract

In this paper, we study the irregular growth of an entire function defined by the Laplace-Stieltjes transform of finite order convergent in the whole complex plane and obtain some results about \( \lambda \)-lower type. In addition, we also investigate the problem on the error in approximating entire functions defined by the Laplace-Stieltjes transforms. Some results about the irregular growth, the error, and the coefficients of Laplace-Stieltjes transforms are obtained; they are generalization and improvement of the previous conclusions given by Luo and Kong, Singh and Srivastava.

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1 Introduction

Dirichlet series

\[
f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it, \quad (1)
\]

where

\[
0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad \lambda_n \to \infty \quad \text{as} \; n \to \infty; \quad (2)
\]

\(s = \sigma + it\) (\(\sigma, t\) are real variables), \(a_n\) are nonzero complex numbers. When \(a_n, \lambda_n, n\) satisfy some conditions, the series (1) is convergent in the whole plane or the half-plane, that is, \(f(s)\) is an analytic function or entire function in the whole plane or the half-plane. In the past few decades, many mathematicians studied the growth and value distribution of the analytic (entire) function defined by Dirichlet series and obtained lots of interesting results (see [1–9]).

As we know, Dirichlet series is regarded as a special example of the Laplace-Stieltjes transform. The Laplace-Stieltjes transform, named for Pierre-Simon Laplace and Thomas Joannes Stieltjes, is an integral transform similar to the Laplace transform. For real-valued functions, it is the Laplace transform of a Stieltjes measure, however it is often defined for functions with values in a Banach space. It can be used in many fields of mathematics, such as functional analysis, and certain areas of theoretical and applied probability.
For the Laplace-Stieltjes transforms,
\[ G(s) = \int_0^{+\infty} e^{-sx} d\alpha(x), \quad s = \sigma + it, \]
where \( \alpha(x) \) is a bounded variation on any finite interval \([0, Y]\) \((0 < Y < +\infty)\), and \( \sigma \) and \( t \) are real variables. Let
\[ B_n^* = \sup_{\lambda_n < \xi \leq \lambda_n + 1} \left| \int_{\lambda_n}^x e^{-ity} d\alpha(y) \right|, \]
where the sequence \( \{\lambda_n\}_{n=1}^{\infty} \) satisfies (2) and
\[ \limsup_{n \to +\infty} (\lambda_{n+1} - \lambda_n) = h < +\infty. \]

In 1963, Yu [10] proved the Valiron-Knopp-Bohr formula of the associated abscissas of bounded convergence, absolute convergence, and uniform convergence of Laplace-Stieltjes.

**Theorem A** Suppose that Laplace-Stieltjes transforms (3) satisfy (2), (4) and \( \limsup_{n \to +\infty} \frac{\log n}{\lambda_n} < +\infty \), then
\[ \limsup_{n \to +\infty} \frac{\log B_n^*}{\lambda_n} \leq \sigma_u^G \leq \limsup_{n \to +\infty} \frac{\log B_n^*}{\lambda_n} + \limsup_{n \to +\infty} \frac{\log n}{\lambda_n}, \]
where \( \sigma_u^G \) is called the abscissa of uniform convergence of \( F(s) \).

Moreover, Yu [10] first introduced the maximal molecule \( M_n(\sigma, G) \), the maximal term \( \mu(\sigma, G) \) and the Borel line, and the order of analytic functions represented by Laplace-Stieltjes transforms convergent in the complex plane. After his works, considerable attention has been paid to the growth and value distribution of the functions represented by the Laplace-Stieltjes transform convergent in the half-plane or the whole complex plane in the field of complex analysis (see [11–15]).

In 2012, Luo and Kong [16] studied the following form of Laplace-Stieltjes transform:
\[ F(s) = \int_0^{+\infty} e^{sx} d\alpha(x), \quad s = \sigma + it, \]
where \( \alpha(x) \) is stated as in (3), and \( \{\lambda_n\} \) satisfies (2),(4). Set
\[ A_n^* = \sup_{\lambda_n < \xi \leq \lambda_n + 1} \left| \int_{\lambda_n}^x e^{ity} d\alpha(y) \right|. \]
By using the same argument as in [10], we can get a similar result about the abscissa of uniform convergence of \( F(s) \) easily. If
\[ \limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D < \infty, \quad \limsup_{n \to +\infty} \frac{\log A_n^*}{\lambda_n} = -\infty, \]
by (2), (4) and Theorem 1, one can get that \( \sigma_u^F = +\infty \), i.e., \( F(s) \) is an entire function.
Set
\[ M(\sigma, F) = \sup_{-\infty < t < +\infty} |F(\sigma + it)|, \quad M_\mu(\sigma, F) = \sup_{0 < x < +\infty, -\infty < t < +\infty} \left| \int_0^x e^{(\sigma + it)y} \, d\alpha(y) \right| \]
and
\[ \mu(\sigma, F) = \max_{n\in\mathbb{N}} \{ A_n^* e^{\lambda_n \sigma} \} (\sigma < +\infty), \quad N(\sigma, F) = \max \{ \lambda_n : A_n^* e^{\lambda_n \sigma} = \mu(\sigma, F) \}. \]

Since \( M(\sigma, F) \) and \( M_\mu(\sigma, F) \) tend to +\( \infty \) as \( \sigma \to +\infty \), in order to estimate the growth of \( F(s) \) more precisely, we will adapt some concepts of order, lower order, type, lower type as follows.

**Definition 1.1** If Laplace-Stieltjes transform (5) satisfies \( \sigma_F^+ = +\infty \) (the sequence \( \{\lambda_n\} \) satisfies (2), (4), and (6)) and
\[ \limsup_{\sigma \to +\infty} \frac{\log^+ \log^+ M_\mu(\sigma, F)}{\sigma} = \rho, \]
we call \( F(s) \) of order \( \rho \) in the whole plane, where \( \log^+ x = \max\{\log x, 0\} \). If \( \rho \in (0, +\infty) \), we say that \( F(s) \) is an entire function of finite order in the whole plane. Moreover, the lower order of \( F(s) \) is defined by
\[ \liminf_{\sigma \to +\infty} \frac{\log^+ \log^+ M_\mu(\sigma, F)}{\sigma} = \lambda. \]

**Remark 1.1** We say that \( F(s) \) is of the regular growth, when \( \rho = \lambda \), and \( F(s) \) is of the irregular growth, when \( \rho \neq \lambda \).

**Definition 1.2** If Laplace-Stieltjes transform (5) satisfies \( \sigma_F^+ = +\infty \) (the sequence \( \{\lambda_n\} \) satisfies (2), (4), and (6)) and is of order \( \rho (0 < \rho < \infty) \), then we define the type and lower type of L-S transform \( F(s) \) as follows:
\[ \limsup_{\sigma \to +\infty} \frac{\log^+ M_\mu(\sigma, F)}{e^{\sigma \rho}} = T, \quad \liminf_{\sigma \to +\infty} \frac{\log^+ M_\mu(\sigma, F)}{e^{\sigma \rho}} = \tau. \]

**Remark 1.2** The purpose of the definition of type is to compare the growth of class functions which all have the same order. For example, let \( f(s) = e^{s^2}, g(s) = e^{s^3} \), by a simple computation, we have \( \rho(f) = 1 = \rho(g) \), but \( T(f) = 1 \) and \( T(g) = \infty \). Thus, we can see that the growth of \( g(s) \) is faster than \( f(s) \) as \( |s| \to +\infty \).

## 2 Results and discussion
Recently, many people studied some problems on analytic functions defined by the Laplace-Stieltjes transforms and obtained a number of interesting results. Kong, Sun, Huo and Xu investigated the growth of analytic functions with kinds of order defined by the Laplace-Stieltjes transforms (see [16–22]), and Shang, Gao, and Sun investigated the value distribution of such functions (see [23–26]). From these references, we get the following results.
Theorem 2.1 If Laplace-Stieltjes transform (5) satisfies $\sigma_\nu^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6)), and is of order $\rho$ ($0 < \rho < \infty$) and of type $T$, then

$$
\rho = \limsup_{n \to +\infty} \frac{\lambda_n \log \lambda_n}{-\log A_n^*}, \quad T = \limsup_{n \to +\infty} \frac{\lambda_n t}{\rho (A_n^*)^\rho}.
$$

Furthermore, if $F(s)$ is of the lower order $\lambda$ and the lower type $\tau$, and $\lambda_n \sim \lambda_{n+1}$ and the function

$$
\psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n}
$$

forms a non-decreasing function of $n$ for $n > n_0$, then we have

$$
\lambda = \liminf_{n \to +\infty} \frac{\lambda_n \log \lambda_n}{-\log A_n^*}, \quad \tau = \liminf_{n \to +\infty} \frac{\lambda_n}{\rho e(A_n^*)^{\rho}}.
$$

From Definition 1.2, a natural question to ask is: What happened if $e^{\sigma \rho}$ is replaced by $e^{\lambda \sigma}$ in the definition of lower type when $\rho \neq \lambda$? We are going to consider this question.

Definition 2.1 If Laplace-Stieltjes transform (5) satisfies $\sigma_\nu^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6)), and is of order $\rho$ ($0 < \rho < \infty$) and of the lower order $\lambda$ ($0 < \lambda < \infty$), if $\lambda \neq \rho$, and

$$
\liminf_{\sigma \to +\infty} \frac{\log^+ M_\lambda(\sigma, F)}{e^{\sigma \lambda}} = \tau_\lambda,
$$

we say that $\tau_\lambda$ is the $\lambda$-type of $F(s)$.

Remark 2.1 Obviously, $\tau_\lambda \geq \tau$ and $\tau_\lambda = \tau$ as $\rho = \lambda$. But we cannot confirm whether $\tau_\lambda \geq T$ or $\tau_\lambda \leq T$.

The following results are the main theorems of this paper.

Theorem 2.2 If Laplace-Stieltjes transform (5) satisfies $\sigma_\nu^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6)), and is of order $\rho$ and of the lower order $\lambda$, $0 \leq \lambda \neq \rho < \infty$, then we have

$$
\liminf_{\sigma \to +\infty} \frac{\log M(\sigma, F)}{e^{\sigma \rho}} = \liminf_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{e^{\sigma \rho}} = 0,
$$

and

$$
\liminf_{\sigma \to +\infty} \frac{N(\sigma, F)}{e^{\sigma \rho}} = 0.
$$

Theorem 2.3 If Laplace-Stieltjes transform (5) satisfies $\sigma_\nu^F = +\infty$ (the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6)), and is of order $\rho$ and of the lower order $\lambda$, $0 < \lambda \neq \rho < \infty$, type $T$, $\lambda$-type $\tau_\lambda$,

$$
\limsup_{\sigma \to +\infty} \frac{N(\sigma, F)}{e^{\sigma \rho}} = H, \quad \liminf_{\sigma \to +\infty} \frac{N(\sigma, F)}{e^{\sigma \rho}} = h.
$$
and let

\[ T_\rho(\sigma, F) = \frac{\log \mu(\sigma, F)}{\exp(\rho \sigma)}, \quad T_\lambda(\sigma, F) = \frac{\log \mu(\sigma, F)}{\exp(\lambda \sigma)}, \]

then we have

\[ H - \rho T \leq \limsup_{\sigma \to +\infty} T_\rho'(\sigma, F) \leq H, \]
\[ -\infty \leq \liminf_{\sigma \to +\infty} T_\rho'(\sigma, F) \leq h - \lambda \tau_\lambda \]

for almost all values of \( \sigma > \sigma_0 \), where \( T_\rho'(\sigma, F) \) and \( T_\lambda'(\sigma, F) \) are the derivatives of \( T_\rho(\sigma, F) \) and \( T_\lambda(\sigma, F) \) with respect to \( \sigma \).

**Theorem 2.4** If Laplace-Stieltjes transform (5) satisfies \( \sigma_n^+ = +\infty \) (the sequence \( \{\lambda_n\} \) satisfies (2), (4), and (6)), and is of the lower order \( \lambda (0 \leq \lambda < \rho < \infty) \), if \( \lambda_n \sim \lambda_{n+1} \), then

\[ \tau_\lambda \geq \liminf_{n \to \infty} \left( \frac{\lambda_n}{e^{\lambda_n}} \right) (A_n^*)^{1/n} \quad (0 \leq \tau_\lambda \leq \infty). \] (11)

Furthermore, there exists a positive integer \( n_0 \) such that

\[ \psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n} \]

forms a non-decreasing function of \( n \) for \( n > n_0 \), then we have

\[ \tau_\lambda = \liminf_{n \to \infty} \left( \frac{\lambda_n}{e^{\lambda_n}} \right) (A_n^*)^{1/n} \quad (0 \leq \tau_\lambda \leq \infty). \] (12)

We denote by \( T_\beta \) the class of all the functions \( F(s) \) of the form (5) which are analytic in the half-plane \( \Re s < \beta \) \((-\infty < \beta < \infty)\) and the sequence \( \{\lambda_n\} \) satisfies (2) and (4); and we denote by \( L_\infty \) the class of all the functions \( F(s) \) of the form (5) which are analytic in the half-plane \( \Re s < +\infty \) and the sequence \( \{\lambda_n\} \) satisfies (2), (4), and (6). Thus, if \(-\infty < \beta < +\infty \) and \( F(s) \in T_\beta \), then \( F(s) \in L_\infty \). If Laplace-Stieltjes transform (5) \( A_n^* = 0 \) for \( n \geq k + 1 \) and \( A_n^* \neq 0 \), then \( F(s) \) will be called an exponential polynomial of degree \( k \) usually denoted by \( p_k \), i.e., \( p_k(s) = \int_0^{\beta} \exp(\alpha(y)) \) when we choose a suitable function \( \alpha(y) \), the function \( p_k(s) \) may be reduced to a polynomial in terms of \( \exp(\beta \lambda_i) \), that is, \( \sum_{i=1}^{k} b_i \exp(\beta \lambda_i) \).

For \( F(s) \in T_\beta \), \(-\infty < \beta < +\infty \), we denote by \( E_n(F, \beta) \) the error in approximating the function \( F(s) \) by exponential polynomials of degree \( n \) in uniform norm as

\[ E_n(F, \beta) = \inf_{p \in \Pi_n} \|F - p\|_\beta, \quad n = 1, 2, \ldots, \]

where

\[ \|F - p\|_\beta = \max_{-\infty < \beta < +\infty} |F(\beta + it) - p(\beta + it)|. \]

In this paper, we will further investigate the relation between \( E_n(F, \beta) \) and the growth of an entire function defined by the L-S transform with irregular growth. It seems that this problem has never been treated before. Our main result is as follows.
Theorem 2.5 If the Laplace-Stieltjes transform \( F(s) \in L_\infty \) and is of lower order \( \lambda \) \((0 \leq \lambda \neq \rho < \infty)\), if \( \lambda_n \sim \lambda_{n+1} \), then for any real number \(-\infty < \beta < +\infty\), we have

\[
\tau_\lambda \geq \lim_{n \to \infty} \left( \frac{\lambda_n}{e^{\lambda_n}} \right) \left( E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right)^{\frac{1}{\lambda_n}} \quad (0 \leq \tau_\lambda \leq \infty). \tag{13}
\]

Furthermore, there exists a positive integer \( n_0 \) such that

\[
\psi_1(n) = \frac{\log A_n - \log A_{n+1}}{\lambda_{n+1} - \lambda_n}
\]

forms a non-decreasing function of \( n \) for \( n > n_0 \), then we have

\[
\tau_\lambda = \lim_{n \to \infty} \left( \frac{\lambda_n}{e^{\lambda_n}} \right) \left( E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right)^{\frac{1}{\lambda_n}} \quad (0 \leq \tau_\lambda \leq \infty), \]

i.e.,

\[
\exp(\beta \lambda) e^{\lambda \tau_\lambda} = \lim_{n \to \infty} \lambda_n \left( E_{n-1}(F, \beta) \right)^{\frac{1}{\lambda_n}}. \tag{14}
\]

3 Conclusions

From Theorems 2.2-2.5, we can see that the growth of Laplace-Stieltjes transforms is investigated under the assumption \( \rho \neq \lambda \), and that some theorems about the \( \lambda \)-lower type \( \tau_\lambda \), \( \lambda_n \), \( A_n^* \), and \( \lambda \) are obtained. In addition, we also study the problem on the error in approximating entire functions defined by the Laplace-Stieltjes transforms. This project is a new issue of Laplace-Stieltjes transforms in the field of complex analysis. Our results are generalization and improvement of the previous conclusions given by Luo and Kong [16, 27], Singhal and Srivastava [28].

4 Methods

4.1 Proofs of Theorems 2.2 and 2.3

To prove the above theorems, we require the following lemmas.

Lemma 4.1 (see [27], Lemma 2.1) If the L-S transform \( F(s) \in L_\infty \) for any \( \sigma (-\infty < \sigma < +\infty) \) and \( \epsilon > 0 \), we have

\[
\frac{1}{2} \mu(\sigma, F) \leq M_\sigma(\sigma, F) \leq C \mu((1 + 2\epsilon)\sigma, F),
\]

where \( C \) is a constant.

Lemma 4.2 (see [16], Lemma 2.2) If the L-S transform \( F(s) \in L_\infty \), then we have

\[
\log \mu(\sigma, F) = \log \mu(\sigma_0, F) + \int_{\sigma_0}^{\sigma} N(t, F) \, dt
\]

for \( \sigma_0 > 0 \).
4.1.1 The proof of Theorem 2.2
Since \( \rho > \lambda > 0 \) and \( F(s) \) is of the lower order \( \lambda \), that is,
\[
\lambda = \liminf_{\sigma \to +\infty} \frac{\log \log M_u(\sigma, F)}{\sigma},
\]
for any small \( \varepsilon (0 < \varepsilon < \rho - \lambda) \), it follows from (15) that there exists a constant \( \sigma_0 \) such that, for \( \sigma > \sigma_0 \),
\[
\log M_u(\sigma, F) > \exp\{\lambda - \varepsilon \sigma\},
\]
and there exists a sequence \( \{\sigma_k\} \) tending to \(+\infty\) such that
\[
\log M_u(\sigma_k, F) < \exp\{\lambda + \varepsilon \sigma_k\}.
\]
Since \( 0 < \varepsilon < \rho - \lambda \), it follows from (16) and (17) that
\[
\liminf_{\sigma \to +\infty} \frac{\log M_u(\sigma, F)}{\exp(\rho \sigma)} = 0.
\]
From Lemmas 4.1 and 4.2, we have
\[
\rho = \limsup_{\sigma \to +\infty} \frac{\log \log M_u(\sigma, F)}{\sigma} = \limsup_{\sigma \to +\infty} \frac{\log \log \mu(\sigma, F)}{\sigma} = \limsup_{\sigma \to +\infty} \frac{\log N(\sigma, F)}{\sigma}
\]
and
\[
\lambda = \liminf_{\sigma \to +\infty} \frac{\log \log M_u(\sigma, F)}{\sigma} = \liminf_{\sigma \to +\infty} \frac{\log \log \mu(\sigma, F)}{\sigma} = \liminf_{\sigma \to +\infty} \frac{\log N(\sigma, F)}{\sigma}.
\]
Thus, similar to the process of (18), we can easily prove
\[
\liminf_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{\exp(\rho \sigma)} = \liminf_{\sigma \to +\infty} \frac{N(\sigma, F)}{\exp(\rho \sigma)} = 0.
\]
Hence, this completes the proof of Theorem 2.2.

4.1.2 The proof of Theorem 2.3
From Lemma 4.2, it follows that
\[
\limsup_{\sigma \to +\infty} \frac{\int_{\sigma_0}^{\sigma} N(t, F) \, dt}{e^{\rho \sigma}} = \limsup_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{e^{\rho \sigma}} = \limsup_{\sigma \to +\infty} T_{\rho}(\sigma, F) = T
\]
and
\[
\liminf_{\sigma \to +\infty} \frac{\int_{\sigma_0}^{\sigma} N(t, F) \, dt}{e^{\lambda \sigma}} = \liminf_{\sigma \to +\infty} \frac{\log \mu(\sigma, F)}{e^{\lambda \sigma}} = \liminf_{\sigma \to +\infty} T_{\lambda}(\sigma, F) = \tau_{\lambda}.
\]
Dividing two sides of the equality in Lemma 4.2 by \( e^{\rho \sigma} \) and differentiating it with respect to \( \sigma \), for almost all values \( \sigma > \sigma_0 \), we have
\[
T'_{\rho}(\sigma, F) = -\rho \frac{\log \mu(\sigma_0, F)}{e^{\rho \sigma}} - \rho \frac{\int_{\sigma_0}^{\sigma} N(t, F) \, dt}{e^{\rho \sigma}} + \frac{N(\sigma, F)}{e^{\rho \sigma}}.
\]
On the basis of the assumptions of Theorem 2.3, taking \( \limsup \) in (21) when \( \sigma \rightarrow +\infty \), from Theorem 2.2 and (19), we get (9) easily.

Similarly, dividing two sides of the equality in Lemma 4.2 by \( e^{\lambda \sigma} \) and differentiating it with respect to \( \sigma \), for almost all values \( \sigma > \sigma_0 \),

\[
T'_\sigma(\sigma, F) = -\lambda \frac{\log \mu(\sigma_0, F)}{e^{\lambda \sigma}} - \frac{\lambda}{e^{\lambda \sigma}} \int_{\sigma_0}^{\sigma} N(t, F) dt + \frac{N(\sigma, F)}{e^{\lambda \sigma}}. \tag{22}
\]

On the basis of the assumptions of Theorem 2.3, taking \( \liminf \) in (22) when \( \sigma \rightarrow +\infty \), from Theorem 2.1 and (20), we get (10) easily.

Thus, this completes the proof of Theorem 2.3.

### 4.2 The proof of Theorem 2.4

Let

\[
\vartheta = \liminf_{n \rightarrow +\infty} \frac{\lambda_n}{e^{\lambda \sigma}} (A_n^*)^{\frac{1}{\lambda_n}} \quad (0 < \vartheta < +\infty).
\]

Thus, for any \( \varepsilon > 0 \), there exists an integer \( n_0(\varepsilon) \) such that, for \( n > n_0(\varepsilon) \),

\[
\lambda_n (A_n^*)^{\frac{1}{\lambda_n}} > (\vartheta - \varepsilon) e^{\lambda \sigma}. \tag{23}
\]

By Lemma 4.1, it follows from (23) that for \( n > n_0(\varepsilon) \)

\[
\frac{\log M_n(\sigma, F)}{e^{\lambda \sigma}} \geq \frac{\log A_n^* + \lambda_n \sigma - \log 2}{e^{\lambda \sigma}} > e^{-\lambda \sigma} \left( \lambda_n \sigma + \frac{\lambda_n}{\lambda} \log((\vartheta - \varepsilon) e^{\lambda \sigma}) - \frac{\lambda_n}{\lambda} \log \lambda_n - \log 2 \right). \tag{24}
\]

Let

\[
\left( \frac{\lambda_n}{\lambda \vartheta} \right)^{\frac{1}{n}} \leq e^{\sigma} < \left( \frac{\lambda_{n+1}}{\lambda \vartheta} \right)^{\frac{1}{n}},
\]

and take

\[
\sigma = \frac{1}{\lambda} \log \left( \frac{\lambda_n}{\lambda \vartheta} \right) + o \left( \frac{1}{\lambda_n} \right).
\]

Then from (24) it follows

\[
\frac{\log M_n(\sigma, F)}{e^{\lambda \sigma}} \geq \frac{\lambda \vartheta}{\lambda_{n+1}} \left( \frac{\lambda_n}{\lambda} \log \frac{1}{\lambda \vartheta} + \frac{\lambda_n}{\lambda} \log((\vartheta - \varepsilon) e^{\lambda \sigma}) - \log 2 + o(1) \right). \tag{25}
\]

Since \( \lambda_n \sim \lambda_{n+1} \) and \( \lambda_n \rightarrow +\infty \) as \( n \rightarrow +\infty \), thus by a simple computation, from (25) we have \( \tau_2 \geq \vartheta \). When \( \vartheta = 0 \), \( \tau_2 \geq \vartheta \) is obvious; if \( \vartheta = \infty \), we also prove that \( \tau_2 \geq \vartheta \) by using the same argument as above. Hence we prove that (11) holds.

Let \( \mu(\sigma, F) \) denote the maximum term for \( \Re s = \sigma, -\infty < t < +\infty \). Since

\[
\psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n}.
\]
forms a non-decreasing function of $n$ for $n > n_0$, then for $\psi(n) - 1 \leq \sigma < \psi(n)$

$$\log \mu(\sigma, F) = \log A_n^* + \lambda_n \sigma.$$ 

Since $\tau_\lambda \leq \infty$, for any small $\varepsilon > 0$, it follows from (20) that

$$\log \mu(\sigma, F) = \log A_n^* + \lambda_n \sigma \geq (\tau_\lambda - \varepsilon) \exp(\lambda \sigma)$$

(26)

for all $\sigma > \sigma_0$ and all $n$ such that $\psi(n - 1) \leq \sigma < \psi(n)$.

Let $\beta = \sigma > \sigma_0$ and $A_n^{*1} \exp(\lambda \sigma)_{n_1}$ and $A_n^{*2} \exp(\sigma \lambda_{n_2})_{n_1 > n_0, \psi(n-1) > \sigma_0}$ be two consecutive maximum terms such that $n_2 - 1 \geq n_1$, it follows from (26) that

$$\log A_n^{*2} + \lambda_{n_2} \sigma \geq (\tau_\lambda - \varepsilon) \exp(\lambda \sigma)$$

for all $\sigma > \sigma_0$ satisfying $\psi(n_2 - 1) \leq \sigma < \psi(n_2)$. Let $n_1 \leq n \leq n_2 - 1$, then

$$\psi(n_1) = \psi(n_1 + 1) = \cdots = \psi(n) = \cdots = \psi(n_2 - 1)$$

and $A_n^{*1} \exp(\lambda_n \sigma) = A_n^{*2} \exp(\lambda_{n_2} \sigma)$ for $\sigma = \psi(n)$. Then there exists a positive integer $n_1$ such that, for $n > n_1$ and $\sigma > \sigma_0$,

$$\log A_n^{*1} > (\tau_\lambda - \varepsilon)e^{\lambda \sigma} - \lambda_n \sigma.$$ 

Since $e^x \geq e^x$ for any $x$, so it follows

$$\lambda_n(A_n^{*1})^{\frac{1}{\lambda_n}} > \frac{\lambda_n}{e^{\lambda \sigma}} \exp\left(\frac{\lambda(\tau_\lambda - \varepsilon)}{\lambda_n}e^{\lambda \sigma}\right) > \frac{\lambda_n e^{(\tau_\lambda - \varepsilon)} \lambda_n}{e^{\lambda \sigma}} = e^{(\tau_\lambda - \varepsilon)\lambda}. (27)$$

Thus, for $\varepsilon \to 0$ and $n \to +\infty$, from (27) it follows

$$\vartheta = \liminf_{n \to +\infty} \frac{\lambda_n}{e^{\lambda \sigma}} (A_n^{*1})^{\frac{1}{\lambda_n}} \geq \tau_\lambda. (28)$$

Hence, this proves that (12) holds.

### 4.3 The proof of Theorem 2.5

To prove this theorem, we require the following lemma.

**Lemma 4.3** If the abscissa $\sigma^F_u = +\infty$ of uniform convergence of the Laplace-Stieltjes transformation $F(s)$ and sequence (2) satisfies (4), (6), then for any real number $\beta$, we have

$$\int_{-\infty}^{\infty} \exp\{(\beta + it)y\} d\sigma(y) \leq 2 \sum_{n=k}^{+\infty} A_n^* \exp[\beta \lambda_{n+1}],$$

where

$$A_n^* \sup_{\lambda_n < x < \lambda_{n+1}, \alpha > x > \infty} \int_{-\infty}^{\infty} e^{ity} d\sigma(y).$$
Proof. Set

\[ I(x; it) = \int_0^x \exp(ity) \, d\alpha(y). \]

For any real number \( \beta \), since

\[
\left| \int_{\lambda_k}^{\infty} \exp((\beta + it)y) \, d\alpha(y) \right| = \lim_{b \to \infty} \left| \int_{\lambda_k}^{b} \exp((\beta + it)y) \, d\alpha(y) \right|.
\]

Set \( I_{j,k}(b; it) = \int_{\lambda_{j,k}}^{b} \exp(ity) \, d\alpha(y), (\lambda_{j,k} < b \leq \lambda_{j,k+1}) \), then we have \( |I_{j,k}(b; it)| \leq A^*_{j,k} \). Thus, it follows

\[
\left| \int_{\lambda_k}^{b} \exp((\beta + it)y) \, d\alpha(y) \right| = \left| \sum_{j=k}^{n+k-1} \int_{\lambda_j}^{\lambda_{j+1}} \exp(\beta y) J_j(y; it) \right| + \left| \int_{\lambda_{n+k}}^{b} \exp(\beta y) J_{n+k}(y; it) \right|
\]

\[
= \left[ \sum_{j=k}^{n+k-1} e^{\lambda_{j+1} \beta} I_{j,j+1}(\lambda_{j+1}; it) - \beta \int_{\lambda_j}^{\lambda_{j+1}} e^{\beta y} J_j(y; it) \right] + \left| e^{\beta b} I_{n+k}(b; it) - \beta \int_{\lambda_{n+k}}^{b} e^{\beta y} J_{n+k}(y; it) \right|
\]

\[
\leq \sum_{j=k}^{n+k-1} \left[ A^*_j e^{\lambda_{j+1} \beta} + A^*_j \left( e^{\lambda_{j+1} \beta} - e^{\beta \lambda_{j+1}} \right) \right] + 2 e^{\beta \lambda_{n+k+1}} A^*_{n+k} - e^{\beta \lambda_{n+k}} A^*_{n+k}
\]

\[
\leq 2 \sum_{j=k}^{n+k} A^*_j e^{\lambda_{j+1} \beta}.
\]

When \( n \to +\infty \), we have \( b \to +\infty \), thus we have

\[
\left| \int_{\lambda_k}^{\infty} \exp((\beta + it)y) \, d\alpha(y) \right| \leq \sum_{n=k}^{+\infty} A^*_n \exp(\beta \lambda_{n+1}). \quad \square
\]

Now, we are going to prove Theorem 2.5.

### 4.4 The proof of Theorem 2.5

Let

\[
\vartheta_1 = \liminf_{n \to +\infty} \left( \frac{\lambda_n}{\epsilon \lambda} \right) (E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \frac{\lambda_n}{\epsilon \lambda} \right) \quad (0 < \vartheta_1 < +\infty).
\]

Then, for any small \( \epsilon > 0 \), there exists an integer \( n_0(\epsilon) \) such that, for any \( n > n_0(\epsilon) \),

\[
\log(E_{n-1}(F, \beta) \exp(-\beta \lambda_n)) > \frac{\lambda_n}{\lambda} \log \left( \frac{\vartheta_1 - \epsilon}{\epsilon \lambda} \right). \tag{29}
\]
Since $F(s) \in L_\infty$, thus for any constant $\beta (-\infty < \beta < +\infty)$, we have $F(s) \in \mathcal{L}_\beta$. For $\beta < \sigma < +\infty$. It follows from the definitions of $E_n(F, \beta)$ and $p_n$ that

$$E_n(F, \beta) \leq \|F - p_n\|_\beta \leq \left|F(\beta + it) - p_n(\beta + it)\right|$$

$$\leq \left|\int_0^{\infty} \exp\{(\beta + it)y\} \, d\alpha(y) - \int_0^{\lambda_n} \exp\{(\beta + it)y\} \, d\alpha(y)\right|$$

$$= \left|\int_0^{\lambda_n} \exp\{(\beta + it)y\} \, d\alpha(y)\right|. \quad (30)$$

Thus, from the definition of $A_n^*$ and $M_n(\sigma, F)$, and by Lemma 4.1, we have $A_n^* \leq 2M_n(\sigma, F)e^{-\sigma \lambda_n}$ for any $\sigma (\beta < \sigma < +\infty)$. It follows from (30) and Lemma 4.3 that

$$E_n(F, \beta) \leq 2 \sum_{k=n+1}^{\infty} A_k \exp\{\beta \lambda_k\} \leq 4M_n(\sigma, F) \sum_{k=n+1}^{\infty} \exp\{(\beta - \sigma) \lambda_k\}. \quad (31)$$

From (4), take $h' (0 < h' < h)$ such that $(\lambda_{n+1} - \lambda_n) \geq h'$ for $n \geq 0$. Then, for $\sigma \geq \frac{\beta}{2}$, it follows from (31) that

$$E_n(F, \beta) \leq 4M_n(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \sum_{k=n+1}^{\infty} \exp\{(\lambda_k - \lambda_{n+1})(\beta - \sigma)\}$$

$$\leq 4M_n(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \exp\left\{-\frac{\beta}{2} h'(n + 1) \right\} \sum_{k=n+1}^{\infty} \exp\left\{\frac{\beta}{2} h' k\right\}$$

$$= 4M_n(\sigma, F) \exp\{\lambda_{n+1}(\beta - \sigma)\} \left(1 - \exp\left\{\frac{\beta}{2} h'\right\}\right)^{-1},$$

that is,

$$E_{n-1}(F, \beta) \leq KM_n(\sigma, F) \exp\{\lambda_n(\beta - \sigma)\}, \quad (32)$$

where $K$ is a constant. Let

$$\gamma_n = E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \quad (n = 1, 2, \ldots).$$

Thus, from (29) and (32), it follows that for $n > n_0(\varepsilon)$

$$\frac{\log M_n(\sigma, F)}{e^{\lambda_n \sigma}} \geq \frac{\log \gamma_n + \lambda_n \sigma - \log K}{e^{\lambda_n \sigma}}$$

$$> e^{\lambda_n \sigma} \left(\frac{\lambda_n \sigma + \frac{\lambda_n \sigma}{\lambda}}{\log(\vartheta_n - \varepsilon) e \lambda} - \frac{\lambda_n}{\lambda} \log \lambda_n - \log K\right). \quad (33)$$

By using the same argument as in Theorem 2.4, we can easily prove that $\tau_\lambda \geq \vartheta_1$.

From the proof of Theorem 2.4, we have that there exists a positive integer $n_1$ such that

$$\log A_n^* > (\tau_\lambda - \varepsilon) e^{\lambda \sigma} - \lambda_n \sigma.$$
for \( n > n_1 \) and \( \sigma > \sigma_0 \). Since for any \( \beta < +\infty \), from the definition of \( E_{k}(F, \beta) \), there exists \( p_1 \in \Pi_{n-1} \) such that

\[
\|F - p_1\| \leq 2E_{n-1}(F, \beta).
\]

(34)

And since

\[
A_{n}^* \exp(\beta \lambda_{n}) = \sup_{\lambda_{n} < \lambda_{n+1} - \infty < \xi < \infty} \left| \int_{\lambda_{n}}^{\xi} \exp(ity) \, d\alpha(y) \exp(\beta \lambda_{n}) \right|
\]

\[
\leq \sup_{\lambda_{n} < \lambda_{n+1} - \infty < \xi < \infty} \left| \int_{\lambda_{n}}^{\xi} \exp((\beta + it)y) \, d\alpha(y) \right|
\]

\[
\leq \sup_{-\infty < \xi < \infty} \left| \int_{\lambda_{n}}^{\infty} \exp((\beta + it)y) \, d\alpha(y) \right|
\]

thus for any \( p \in \Pi_{n-1} \), it follows

\[
A_{n}^* \exp(\beta \lambda_{n}) \leq \left| F(\beta + it) - p(\beta + it) \right| \leq \|F - p\|_{\beta}.
\]

(35)

Hence from (34) and (35), for any \( \beta < +\infty \) and \( F(s) \in L_{\infty} \), we have

\[
A_{n}^* \exp(\beta \lambda_{n}) \leq 2E_{n-1}(F, \beta).
\]

Since \( e^{x} \geq ex \) for any \( x \), so it follows

\[
\lambda_{n}(\gamma_{n})^{\frac{1}{\sigma}} \geq \frac{\lambda_{n}}{e^{\sigma}} \exp \left\{ \frac{\lambda(\gamma_{n} - \epsilon)}{\lambda_{n}} e^{\frac{\lambda \log 2}{\lambda_{n}}} \right\}
\]

\[
> \frac{\lambda_{n}}{e^{\sigma}} \left( \frac{e(\gamma_{n} - \epsilon \lambda)}{\lambda_{n}} e^{\frac{\lambda \log 2(\gamma_{n})}{\lambda_{n}}} \right) = e(\gamma_{n} - \epsilon \lambda).
\]

(36)

Thus, for \( \epsilon \to 0 \) and \( n \to +\infty \), from (36) it follows

\[
\vartheta_{1} = \lim inf_{n \to \infty} \frac{\lambda_{n}}{e^{\sigma}}(\gamma_{n})^{\frac{1}{\sigma}} \geq \tau_{\lambda}.
\]

Since \( [E_{n-1}(F, \beta) \exp(-\beta \lambda_{n})]^{\frac{1}{\sigma}} = [E_{n-1}(F, \beta)]^{\frac{1}{\sigma}} \exp(-\beta \lambda) \), then (14) follows.

Therefore, we complete the proof of Theorem 2.5.

### 4.5 Remarks

From the proof of Theorem 2.5, and combining those results of the Laplace-Stieltjes transforms in Ref. [14, 16, 27], we can obtain the following results on the approximation of Laplace-Stieltjes transforms, which can be found partly in [28].

**Theorem 4.1** If the L-S transform \( F(s) \in L_{\infty} \) and is of order \( \rho \) \((0 < \rho < \infty)\) and of type \( T \), then for any real number \(-\infty < \beta < +\infty \), we have

\[
\rho = \lim sup_{r \to +\infty} \frac{\lambda_{n} \log \lambda_{n}}{-\log E_{n-1}(F, \beta) \exp(-\beta \lambda_{n})} = \lim sup_{r \to +\infty} \frac{\lambda_{n} \log \lambda_{n}}{-\log E_{n-1}(F, \beta)}
\]
and

\[ T = \limsup_{n \to +\infty} \frac{\lambda_n}{\rho} \left( E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right)^{\frac{\rho}{\rho+1}} \]

\[ = \limsup_{n \to +\infty} \frac{\lambda_n}{\rho \exp(\rho \beta + 1)} \left( E_{n-1}(F, \beta) \right)^{\frac{\rho}{\rho+1}}. \]

Furthermore, if \( F(s) \) is of the lower order \( \lambda \) and the lower type \( \tau \), and \( \lambda_n \sim \lambda_{n+1} \) and the function

\[ \psi(n) = \frac{\log A_n^* - \log A_{n+1}^*}{\lambda_{n+1} - \lambda_n} \]

forms a non-decreasing function of \( n \) for \( n > n_0 \), then we have

\[ \lambda = \liminf_{n \to +\infty} \frac{\lambda_n \log \lambda_n}{-\log E_{n-1}(F, \beta)}, \quad \tau = \liminf_{n \to +\infty} \frac{\lambda_n}{\rho \exp(\rho \beta + 1)} \left( E_{n-1}(F, \beta) \right)^{\frac{\rho}{\rho+1}}. \]

**Theorem 4.2** If the L-S transform \( F(s) \in L_\infty \), then for any real number \(-\infty < \beta < +\infty\). For \( p = 1 \), we have

\[ \limsup_{\sigma \to +\infty} \frac{h(\log M_\sigma(\sigma, F))}{h(\sigma)} - 1 = \limsup_{n \to +\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n} \log \left[ E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right])}, \]

and for \( p = 2, 3, \ldots \), we have

\[ \limsup_{n \to +\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n} \log \left[ E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right])} \leq \limsup_{\sigma \to +\infty} \frac{h(\log M_\sigma(\sigma, F))}{h(\sigma)} \]

\[ \leq \limsup_{n \to +\infty} \frac{h(\lambda_n)}{h(-\frac{1}{\lambda_n} \log \left[ E_{n-1}(F, \beta) \exp(-\beta \lambda_n) \right])} + 1, \]

where \( h(x) \) satisfies the following conditions:

(i) \( h(x) \) is defined on \([a, +\infty)\) and is positive, strictly increasing, differentiable and tends to \(+\infty\) as \( x \to +\infty \);

(ii) \( \lim_{x \to +\infty} \frac{d (h(x))}{d (\log^{[p]} x)} = k \in (0, +\infty), p \geq 1, p \in \mathbb{N}^+ \), where \( \log^{[0]} x = x, \log^{[1]} x = \log x \) and \( \log^{[p]} x = \log (\log^{[p-1]} x) \).

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**Competing interests**

The authors declare that none of the authors has any competing interests in the manuscript.

**Authors’ contributions**

HYX and SYL completed the main part of this article, HYX and SYL corrected the main theorems. All authors read and approved the final manuscript.
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