Large N limit of Calogero–Moser models and Conformal Field Theories

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We discuss the large $N$ limit of Calogero–Moser models for the classical infinite families of simple Lie algebras $A_N$, $B_N$, $C_N$ and $D_N$. We show that the limit defines two different Conformal Field Theories with central charge $c > 1$. The value of $c$ and the dimension of the primary field are dictated by the underlying algebraic symmetries of the model.

The classical and quantum integrable model of $N$ interacting bodies in one spatial dimension introduced by Calogero has been studied intensively and still attracts much attention. Surprisingly, the model and its various generalizations have been found relevant to describe properties of theories ranging from spin chains, quantum Hall effect, two dimensional gravity and random matrices. Recently, in the high energy and gravity context the interest was renewed since the proposal of the classical infinite families of simple Lie algebras $A_N$, $B_N$, $C_N$ and $D_N$, corresponding, respectively, to the simple Lie algebras $su(N + 1)$, $so(N + 1)$, $sp(N)$ and $so(2N)$. The only, nontrivial, nonreduced case, $BC_N$, is equivalent to $A_{2N}$ with proper symmetry conditions preserved by the time evolution. Every other reducible case corresponds to the union of noninteracting systems.

We aim to study the $N \to \infty$ limit of the model pointing out a precise, yet very simple, relation with Conformal Field Theory (CFT) universality classes, or, more exactly, with, say, the holomorphic sector of them. We consider a CFT in $D = 1 + 1$ defined on a strip or on a cylinder. We will show that in the large $N$ limit the CM model is equivalent to two different CFTs, depending on the root system considered.

We stress that the large $N$ limit we consider here is not the thermodynamic limit, where the particle density is kept fixed. The latter limit was considered in previous papers dealing with similar models. In Ref. 3 for instance, using finite–size scaling relations, it was found that the low–energy physics of a Calogero–Sutherland (CS) model (see Eq. 13 below) is described by a $c = 1$ CFT. In Ref. 1, a second quantized ($D = 1 + 1$) Hamiltonian has been studied, which corresponds to the CS mechanical model. It was shown that that if one takes into account the leading ($1/N$) and subleading ($1/N^2$) terms in the thermodynamic limit, the dynamics of small fluctuations around the Fermi surface is described by an effective field theory endowed with an extended $W_{1+\infty}$ conformal symmetry. Our approach is rather different: we do not consider a thermodynamic limit and will not eventually write down an effective field theory; on the contrary we use the entire spectrum, not only the lowest–lying exited states, to reconstruct a CFT2 via a large $N$ limit. Needless to say that we rely on the exact integrability of the CM model, while, for instance, in Ref. 3 no use is made of the exact form of the spectrum.

The starting point is the very simple form of the spectrum of the CM model. It is well known that the spectrum is generated by $N$ decoupled harmonic oscillators. Moreover, it is possible to define similarity or unitary transformations that map the rational CM model into a set of decoupled harmonic oscillators. In this sense the equivalence of the CM model with a CFT in the large $N$ limit is self-evident. We want to identify

$\alpha$ being the length of the root $\alpha$. Since we are interested in the large $N$ limit we consider only the infinite (classical) root systems $A_N$, $B_N$, $C_N$ and $D_N$, corresponding, respectively, to the simple Lie algebras $su(N + 1)$, $so(N + 1)$, $sp(N)$ and $so(2N)$. The only, nontrivial, nonreduced case, $BC_N$, is equivalent to $A_{2N}$ with proper symmetry conditions preserved by the time evolution. Every other reducible case corresponds to the union of noninteracting systems.
the CFT precisely.

Let us briefly recall the spectral properties of the CM model (see Ref. 4 for a review). Whenever the stability condition \( \tilde{A} \) is satisfied the Hamiltonian \( H \) is self-adjoint and the energy level \( E_k = H \psi_k \) is characterized by \( N \) integers. If the couplings \( g_\alpha \) are written as usual as \( g_\alpha^2 = \mu_\alpha (\mu_\alpha - 1) |\alpha|^2 / 2 \), with \( \tilde{\mu}_\alpha = \mu_\alpha + \mu_2 \), the ground state energy \( E_0 \) is given by

\[
E_0 = \omega \left( \frac{N}{2} + \mu \right), \quad \mu = \sum_{\alpha \in R_+} \mu_\alpha. \tag{3}
\]

The energies of the excited states read

\[
E_{n_1, \ldots, n_N} = E_0 + \omega (\nu_1 n_1 + \cdots + \nu_N n_N), \tag{4}
\]

where the positive integers \( \nu_1, \ldots, \nu_N \) are the degrees of the \( N \) polynomial invariants of the Coxeter group, i.e. the orders of the Casimir operators of the Lie algebra under consideration. Given the order of the invariants, it is immediate to compute the quantum partition function \( Z(\beta) \):

\[
Z(\beta) = e^{\beta E_0 \left[ (1 - e^{-\beta \nu_1})(1 - e^{-\beta \nu_2}) \cdots (1 - e^{-\beta \nu_N}) \right]}^{-1}. \tag{5}
\]

The orders of the invariants for the classical root systems are known from the classic work of Racah 8, and are quoted in table I. Neglecting as usual the zero point energy, which is divergent in the large \( N \) limit, see Eq. 3, the partition functions have a very simple form in all cases.

Let us now consider the CFT side. The Hilbert space of a generic two-dimensional conformal field theory is given by (see for instance Ref. 3)

\[
\sum_{h, \bar{h}} V(c, h) \otimes \bar{V}(c, \bar{h}), \tag{6}
\]

where the sum is taken over the conformal dimensions of the theory and \( c \) is the central charge. \( V(c, h) \) is the Verma module generated from a vacuum \( |h\rangle \) with conformal weight \( h \), \( L_0 |h\rangle = h |h\rangle \), by the action of the holomorphic Virasoro generators \( L_n \). \( \bar{V}(h, c) \) describes the antiholomorphic sector (the holomorphic and antiholomorphic weights are in general independent quantities). The Verma modules need not to be irreducible representations of the conformal group. Indeed in general they are reducible. That means that the states

\[
L_{-k_1} L_{-k_2} \cdots L_{-k_n} |h\rangle, \quad 1 \leq k_1 \leq \cdots \leq k_n, \tag{7}
\]

are not necessarily all independent. It may happen that a combination of the states 3 defines a null vector \( |\chi\rangle \), that is a vector annihilated by all of the generators \( L_\alpha \) with \( n > 0 \) (we are considering only the holomorphic sector. For the antiholomorphic one the same statements hold). A null vector is orthogonal to the whole Verma module and in particular to himself, \( \langle \chi | \chi \rangle = 0 \). This is of course also true for all of its descendants. The null submodule arising from a null state does not contribute to the partition function, which is computed tracing over an irreducible representation of the Virasoro algebra. Given the module \( V(c, h) \), an irreducible representation \( M(c, h) \) is obtained by quotienting out of it all the null modules, i.e. identifying two states differing by a null vector.

The null vectors are in correspondence with the zeros of the Kac determinant of the Gram matrix \( G \), defined as \( G_{ij} = \langle i | j \rangle \), where \( |i\rangle \) and \( |j\rangle \) are the states of 7 of the Verma module \( V(c, h) \). The Gram matrix is block diagonal, with blocks \( G^{(l)} \) corresponding to the level \( l \), i.e. the \( L_0 - h \) eigenvalue of the states 7. The expression for the Kac determinant is (see e.g. Ref. 4)

\[
\det G^{(l)} = \alpha_l \prod_{r, s \geq 1 \text{ and } rs \leq l} [h - h_{r, s}] p(l - rs), \tag{8}
\]

where \( p(l - rs) \) is the number of partitions of the integer \( l - rs \) (not to be confused with the product \( p \times (l - rs) \)), and \( \alpha_l \) denotes a positive constant depending on \( l \). \( h_{r, s} \) are functions of the central charge \( c \) and can be written as

\[
c = 13 - 6 \left( t + \frac{1}{t} \right), \quad h_{r, s} = \frac{1}{4}(r^2 - 1)t + \frac{1}{4}(s^2 - 1) \left( t - \frac{1}{t} \right) - \frac{1}{2}(rs - 1), \tag{9}
\]

where \( t \) is a function of \( c \). Once we have identified the null vectors and constructed the irreducible representation \( M(c, h) \), the partition function for the holomorphic sector with conformal weight \( h \) can be defined as

\[
Z_h(\beta) = \text{Tr}_{M(c, h)} q^{L_0 - h}, \tag{10}
\]

where \( q \equiv e^{-\beta} \). Notice that the trace is taken in \( M(c, h) \). We bear in mind that \( L_0 \) is, in proper units, the energy operator \( H \) for \( M(c, h) \). In our definition we have again neglected the zero point energy (basically the vacuum conformal weight). We note that \( Z_h(\beta) \) depends on \( h \) since \( M(c, h) \) does. Now the question is if one can define a CFT such that \( Z_h(\beta) \) defined by Eq. 10 equals \( Z(\beta) \) of Eq. 3 in the limit \( N \to \infty \). The answer is simple and affirmative for the four cases \( A_N, B_N, C_N \) and \( D_N \).

\( A_N \) is the case of major interest, both from the physical and mathematical point of view. We set \( \omega = 1 \). From the table I it follows that all the modes but the first are present. In order to have the same for the CFT partition function, we simply need a null vector at the first level. This implies \( h = h_{1,1} = 0 \), because at level one the only state is \( L_{-1} |h\rangle \). The corresponding field \( \phi_{(1,1)}(z) \) is a constant since it satisfies the differential equation

\[
\frac{\partial}{\partial z} \langle \phi_{(1,1)}(z) X \rangle = 0,
\]
From Eq. (9) it follows that branches and it is given by

\[ h - \text{property of the well known minimal models whose Hilbert space have a structure. The condition that } h_{r,s} \neq 0 \text{ for } r \neq 1 \text{ and } s \neq 1 \text{ restricts the possible values of } c. \] A general elementary unitary bound on } c = c \geq 0 \text{ (we consider only unitary theories). From Eq. (9) we immediately see that } c \text{ cannot be equal to 1, since in this case } h_{r,r} = 0 \forall r > 0. \text{ On the other hand if } c < 1 \text{ the unitary theories are given by the well known minimal models whose Hilbert space have a more complicated structure. Indeed the unitary minimal models with } c < 1 \text{ can be labeled by } (p = m + 1, p' = m + 1) \text{ with } m = 2, 3, \ldots \text{ and satisfy the periodicity property } h_{r,s} = h_{r+p,s+p'} \text{ which is sufficient to exclude them. We conclude that } c > 1 \text{ is the only possibility. From Eq. (9) it follows that } t \text{ as a function of } c \text{ has two branches and it is given by}

\[ t = 1 + \frac{1}{12} \left[ 1 - c \pm \sqrt{(1-c)(25-c)} \right]. \] (11)

From the previous equation we see that } h_{r,s} \text{ never vanishes if } 1 < c < 25. \text{ If } c \geq 25 \text{ we can resort to another well-known expression of the roots of the Kac determinant, namely}

\[ c = 1 - \frac{6}{m(m+1)}, \]

\[ h_{r,s} = \frac{[(m+1)r-ms]^{2} - 1}{4m(m+1)}. \] (12)

If } c \geq 25 \text{ then } -1 < m < 0, \text{ but this implies that } h_{r,s} \text{ vanishes only for } r = s = 1. \text{ We conclude that the limit } N \to \infty \text{ of the } A_N \text{ systems defines a two-dimensional CFT with } c > 1, h = 0 \text{ and the conformal family given by the descendants of } \mathbb{I}.

The cases } B_N, C_N \text{ and } D_N \text{ are less interesting. Setting } \omega = 1/2 \text{ we see from table I that the all the modes } 1, 2, \ldots \text{ contribute to } Z(\beta) \text{ giving the same large } N \text{ limit for the three cases.}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Type of Group} & \textbf{Order of invariants} \\
\hline
1. A_N & \( n \geq 1 \) \\
2. B_N & \( n \geq 2 \) \\
3. C_N & \( n \geq 2 \) \\
4. D_N & \( n \geq 4 \) \\
\hline
\end{tabular}
\end{table}

For high temperatures } \beta \omega \ll 1, \text{ we obtain from the partition function } Z(\beta) \text{ the energy–temperature relation in the large } N \text{ limit,

\[ E = E_0 + \frac{\pi^2 T^2}{6\omega}. \] (13)

This can be compared with the generic CFT behavior } E - E_0 = \frac{\pi L^2}{2\omega}, \text{ which is valid if the characteristic size } L \text{ of the system is much larger than the thermal wavelength of the particles. In this way, we get the equation

\[ \frac{2\pi}{\omega} = cL, \] (14)

which relates the frequency } \omega \text{ of the CM model and the product } cL \text{ of the central charge and the characteristic size of our two-dimensional CFT on the strip or cylinder.}

As a final consideration we note that the procedure used in this letter cannot be employed for the CS model,

\[ H = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \gamma(\gamma - 1) \frac{\pi^2}{L^2} \sum_{i<j} \frac{1}{\sin^2(\pi(x_i - x_j)/L)}, \] (15)

where } \gamma \text{ is a positive constant, often taken to be an integer. Indeed in this case the difference between two successive energy levels is proportional to } N, \text{ so that a large } N \text{ limit is meaningless.

Summarizing, in this letter we have considered a large } N \text{ limit for the CM model in order to reconstruct a CFT from the exactly known spectrum of excited states of the model. We have found two different CFTs both with central charge } c > 1. \text{ The value of the central charge as well as the dimension of the primary field are dictated only by the algebraic underlying Lie symmetries of the CM model and are independent of the couplings strength (provided the stability condition } (2) \text{ is satisfied).}
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