Analytically stable Higgs bundles on some non-Kähler manifolds

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Abstract

In this paper, we study Higgs bundles on non-compact Hermitian manifolds. Under some assumptions for the underlying Hermitian manifolds which are not necessarily Kähler, we solve the Hermitian–Einstein equation on analytically stable Higgs bundles.

Keywords Higgs bundles · Gauduchon manifold · Hermitian–Einstein equation · Non-compact

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1 Introduction

Let \((M, \omega)\) be an \(n\)-dimensional Hermitian manifold and \((E, \overline{\partial}_E)\) a \(r\)-rank holomorphic vector bundle over \(M\). A Hermitian metric \(H\) on the bundle \(E\) is called \(\omega\)-Hermitian–Einstein if it satisfies the following Hermitian–Einstein equation on \(M\), i.e.,

\[
\sqrt{-1} \Lambda_\omega \left( F_H - \frac{1}{r} \text{tr} F_H \text{Id}_E \right) = 0,
\]

(1.1)

where \(F_H\) is the curvature tensor of Chern connection \(D_H\) with respect to \(H\) and \(\Lambda_\omega\) denotes the contraction with the Hermitian metric \(\omega\). When \((M, \omega)\) is a compact Kähler manifold, by the famous Hitchin–Kobayashi correspondence or the Donaldson–Uhlenbeck–Yau theorem [9, 16, 21, 30, 33], we know the holomorphic vector bundle \((E, \overline{\partial}_E)\) admits an \(\omega\)-Hermitian–Einstein metric if and only if \((E, \overline{\partial}_E)\) is \(\omega\)-poly-stable in the sense of Mumford–Takemoto. This classical result has a lot of interesting and important generalizations and extensions (see [1, 2, 5–7, 11, 13–15, 17–19, 24, 25, 28, 31], etc.).
A Higgs bundle \((E, \tilde{\partial}_E, \theta)\) over \(M\) is a holomorphic bundle \((E, \overline{\partial}_E)\) coupled with a Higgs field \(\theta \in \Omega^1,0_X(\text{End}(E))\) such that \(\tilde{\partial}_E \theta = 0\) and \(\theta \wedge \theta = 0\). Higgs bundles first emerged thirty years ago in Hitchin’s [13] reduction of self-dual equation on \(\mathbb{R}^4\) to Riemann surface and in Simpson’s [31] work on nonabelian Hodge theory. They have rich structures and play an important role in many different areas including gauge theory, Kähler and hyperkähler geometry, group representations and nonabelian Hodge theory. Letting \(H\) be a Hermitian metric on the bundle \(E\), we consider the Hitchin–Simpson connection:

\[
D_H, \theta = D_H + \theta + \theta^* H,
\]

where \(\theta^* H\) is the adjoint of \(\theta\) with respect to the metric \(H\). The curvature of this connection is

\[
F_{H, \theta} = F_H + [\theta, \theta^* H] + \partial_H \theta + \tilde{\partial}_E \theta^* H,
\]

where \(\partial_H\) is the \((1, 0)\)-part of \(D_H\).

A Hermitian metric \(H\) is said to be a Hermitian–Einstein metric on Higgs bundle \((E, \tilde{\partial}_E, \theta)\) if it satisfies

\[
\sqrt{-1} \Lambda^1 \omega F^1_{H, \theta} = 0. \tag{1.2}
\]

where \(F^1_{H, \theta}\) is the trace-free part of the curvature of the Hitchin–Simpson connection.

The Donaldson–Uhlenbeck–Yau theorem was generalized to the Higgs bundles case by Hitchin [13] and Simpson [31, 32]. Simpson [31] even studied some non-compact Kähler manifolds case. Under some assumption for the base manifold, he proved that the analytic stability implies the existence of Hermitian–Einstein metric. The Donaldson–Uhlenbeck–Yau theorem for the non-compact base manifold case is important and interesting [17, 24–26]. Recently, Mochizuki [27] made an important progress in this direction. He weakened the assumption in Simpson’s result such that the volume of base manifold may not be finite and he also studied the curvature decay of the Hermitian–Einstein metrics.

In this paper, we study the non-Kähler case. A Hermitian metric \(\omega\) is called to be Gauduchon if it satisfies \(\partial \overline{\partial} \omega^{n-1} = 0\). If \(M\) is compact, it has been proved by Gauduchon [12] that there exists a Gauduchon metric in the conformal class of every Hermitian metric \(\omega\). When the base Hermitian manifold is compact and Gauduchon, the Donaldson–Uhlenbeck–Yau theorem is also valid (see [3, 4, 8, 19, 22, 23]). Inspired by Mochizuki’s result [27], we consider the case that the base manifold \((M, \omega)\) is non-compact Gauduchon and satisfies the following assumption.

**Assumption 1** Let \(\varphi\) be a nonnegative function on \((M, \omega)\) with \(\int_M \varphi \frac{\omega^n}{n!} < +\infty\). For any nonnegative bounded function \(f\) satisfying

\[
\sqrt{-1} \Lambda \omega \partial \overline{\partial} f \geq -B \varphi
\]

in weak sense (see Definition 2.4 for details) for a positive number \(B\), we have

\[
\sup_{x \in M} f(x) \leq C_1 + C_2 \int_M f \varphi \frac{\omega^n}{n!},
\]

where \(C_1\) and \(C_2\) are positive constants depending only on \(B\). Moreover, if the function \(f\) satisfies \(\sqrt{-1} \Lambda \omega \partial \overline{\partial} f = 0\) on \(M\), then we have \(\sqrt{-1} \Lambda \omega \partial \overline{\partial} f \equiv 0\).

Let the background metric \(H_0\) be a Hermitian metric on \(E\) such that

\[
|\sqrt{-1} \Lambda \omega F_{H_0, \theta}|_{H_0} \leq \hat{B} \varphi
\]

where \(\hat{B}\) is a positive constant.
for some constant $\hat{B} > 0$. Define the analytic degree of $E$ to be the real number

$$deg_{\omega}(E, H_0) = \sqrt{-1} \int_M tr(\Lambda_{\omega} F_{H_0}) \frac{\omega^n}{n!}.$$ 

As in [31], we define the analytic degree of any saturated sub-Higgs sheaf $S$ of $(E, \bar{\partial}_E, \theta)$ by

$$deg_{\omega}(S, H_0) = \int_M \left( \sqrt{-1} tr(\pi_S \Lambda_{\omega} F_{H_0}) - |\bar{\partial}_\theta \pi_S|^2_{H_0} \right) \frac{\omega^n}{n!}, \quad \text{(1.6)}$$

where $\Sigma_S$ denotes the set of singularities where $S$ is not locally free, $\bar{\partial}_\theta := \bar{\partial}_E + \theta$ and $\pi_S$ denotes the projection onto $S$ with respect to the metric $H_0$ outside $\Sigma_S$. When the base Gauduchon manifold $(M, \omega)$ is compact, it is easy to see that the analytic degree $deg_{\omega}(S, H_0)$ is independent of the choice of the background metric $H_0$.

Following [31], we say that the Higgs bundle $(E, \bar{\partial}_E, \theta)$ is $H_0$-analytically stable (semi-stable) if for every proper saturated sub-Higgs sheaf $S \subset E$, it holds

$$\frac{deg_{\omega}(S, H_0)}{\text{rank}(S)} < \left( \leq \right) \frac{deg_{\omega}(E, H_0)}{\text{rank}(E)}. \quad \text{(1.7)}$$

Now we give our main theorem as follows.

**Theorem 1.1** Let $(M, \omega)$ be a non-compact Gauduchon manifold satisfying the Assumption 1. $(E, \bar{\partial}_E, \theta)$ a Higgs bundle over $M$ and $H_0$ a background Hermitian metric on $E$ satisfying the condition (1.5). If $(E, \bar{\partial}_E, \theta)$ is $H_0$-analytically stable, then there exists a Hermitian–Einstein metric $H$ satisfying the below conditions:

(i) $\det(H) = \det(H_0)$;

(ii) Set $h = H_0^{-1} H$. Then $|h|_{H_0}$ and $|h^{-1}|_{H_0}$ are bounded, and $\int_M (\bar{\partial}_E h)^2_{H_0} + \|[\theta, h]||^2_{H_0} \frac{\omega^n}{n!} < +\infty$.

The above theorem can be seen as a generalization of Mochizuki’s result [27] to the non-Kähler case. Mochizuki [27] proved the existence of an exhaustion function $\phi$ on $M$. Fix a number $a_i$ and let $M_i$ denote the compact space $\phi(x) \leq a_i$ with boundary $\partial M_i$, so we can take a sequence of exhaustion compact subsets $M_i$ in $M$ with $\bigcup M_i = M$. Let’s consider the Dirichlet problem on $M_i$:

$$\left\{ \begin{array}{l} \sqrt{-1} \Lambda_{\omega} F_{H_i}^{\perp} = 0, \\
H_i|_{\partial M_i} = H_0. \end{array} \right. \quad \text{(1.8)}$$

According to the results of Donaldson (see [10, 35] for the Hermitian manifold case), we know that there exists a unique Hermitian metric $H_i$ satisfying the above Dirichlet problem (1.8) and $\det(H_i) = \det(H_0)$ on $M_i$. Following the idea of [27], one can take $a_i \to +\infty$ and then obtain the limit $H_\infty$ of a subsequence of $H_i$ on any compact subset, with the property that $\sqrt{-1} \Lambda_{\omega} F_{H_\infty}^{\perp} = 0$ on the whole $M$. Now the key is to obtain a $C^0$-bound. When the base manifold is Kähler, Mochizuki [27] introduced the Donaldson’s functional on the space of Hermitian metrics satisfying the Dirichlet boundary condition. The Donaldson’s functional played a key role in Mochizuki’s proof of the uniform $C^0$-bound. However, in the non-Kähler case, the Donaldson functional may not be well-defined. So we need new arguments in our case. In fact, our argument relies on the following identity:
\[
\int_M \text{tr}(\Phi(H_0, \theta)s) \frac{\omega^n}{n!} + \int_M \langle \Psi(s) (\overline{\partial}_\theta s), \overline{\partial}_\theta s \rangle_{H_0} \frac{\omega^n}{n!} = \int_M \text{tr}(\Phi(H, \theta)s) \frac{\omega^n}{n!},
\]
(1.9)

where \( s = \log(H_0^{-1}H) \).

\[ \Phi(H, \theta) = \sqrt{-1} \Lambda_\omega (F_H + [\theta, \theta^* H]) \]
(1.10)

and

\[ \Psi(x, y) = \begin{cases} 
\frac{e^{y-x} - 1}{y - x}, & x \neq y; \\
1, & x = y.
\end{cases} 
\]
(1.11)

The above identity (1.9) was proved in [29] for the closed Gauduchon manifold case, and in [34] for the compact Gauduchon manifold with non-empty boundary and some non-compact case.

Let \( H_0 \) be a Hermitian metric on the Higgs bundle \((E, \overline{\partial}_E, \theta)\) with

\[ \int_M |F_{H_0, \theta}|^2 \frac{\omega^n}{n!} < \infty, \]
(1.12)

where \( F_{H_0, \theta} \) is the curvature of the Hitchin–Simpson connection \( D_{H_0, \theta} \) with respect to \( H_0 \). It is worthwhile to ask whether the following Bogomolov–Gieseker inequality

\[ \left( c_2(E, \overline{\partial}_E, \theta, H_0) - \frac{r - 1}{2} c_1^2(E, \overline{\partial}_E, \theta, H_0) \right) \cdot [\omega^{n-2}] \geq 0 \]
(1.13)

is true provided \((E, \overline{\partial}_E, \theta)\) is \( H_0 \)-analytically stable. Here \( c_i(E, \overline{\partial}_E, \theta, H_0) \) are the Chern forms with respect to the Hitchin–Simpson connection \( D_{H_0, \theta} \). By Theorem 1.1, we know that \( H_0 \)-analytic stability implies the existence of Hermitian–Einstein metric \( H \). The Riemann bilinear relations assert that

\[
8\pi^2 \left( c_2(E, \overline{\partial}_E, \theta, H) - \frac{r - 1}{2} c_1^2(E, \overline{\partial}_E, \theta, H) \right) \wedge \frac{\omega^{n-2}}{(n-2)!} \left| F_{H, \theta} \right|^2 \wedge \Lambda_\omega \left| F_{H, \theta} \right|^2 
\]
(1.14)

The Bogomolov–Gieseker inequality (1.13) follows if one can prove that

\[
\int_M \text{tr}(F_{H, \theta} \wedge F_{H, \theta}) \wedge \frac{\omega^{n-2}}{(n-2)!} \leq \int_M \text{tr}(F_{H_0, \theta} \wedge F_{H_0, \theta}) \wedge \frac{\omega^{n-2}}{(n-2)!}
\]
(1.15)

for the Hermitian–Einstein \( H \) constructed in Theorem 1.1. However, we are currently unable to deal with (1.15) and will return to this question in our future work.

This paper is organized as follows. In Sect. 2, we give some estimates and preliminaries for the Hermitian–Einstein equation (1.2). In Sect. 3, we give a proof of Theorem 1.1 by using the identity (1.9). In Sect. 4, we study the uniqueness of Hermitian–Einstein metric in Theorem 1.1. In the appendix, we give a class of non-compact Gauduchon manifolds satisfying the Assumption 1.
2 Preliminary results

Let $(M, \omega)$ be a Hermitian manifold and $(E, \overline{\partial}_E, \theta)$ a Higgs bundle over $M$. Letting $H_0$ and $H$ be two Hermitian metrics on the bundle $E$, we denote

$$S_{H_0}(E) = \{ \eta \in \Omega^0(M, \text{End}(E)) \mid \eta^{*H_0} = \eta \},$$

and set

$$h = H_0^{-1}H = \exp s,$$

where $s \in S_{H_0}(E) \cap S_H(E)$. It is straightforward to check the following identities

$$\log \left( \frac{1}{2r} (trh + trh^{-1}) \right) \leq |s|_{H_0} \leq r \frac{1}{2} \log (trh + trh^{-1});$$

$$\partial_H - \partial_{H_0} = h^{-1} \partial_{H_0} h;$$

$$F_H - F_{H_0} = \overline{\partial}_E (h^{-1} \partial_{H_0} h);$$

$$\theta^{*H} = h^{-1} \theta^{*H_0} h,$$

where $r = \text{rank}(E)$ and $\partial_H$ is the $(1, 0)$-part of the Chern connection $D_{H_0}$. Furthermore, we have the following estimates (Lemma 3 (d) in [31], Proposition 2.7 in [35])

$$\sqrt{-1} \Lambda_{\omega} \partial \overline{\partial} \log(trh) \geq -|\Phi(H_0, \theta)|_{H_0} - |\Phi(H, \theta)|_H$$

and

$$\sqrt{-1} \Lambda_{\omega} \partial \overline{\partial} \log(trh + trh^{-1}) \geq -|\Phi(H_0, \theta)|_{H_0} - |\Phi(H, \theta)|_H.$$  

The Dirichlet problem for the Hermitian–Einstein equation was first solved in [10] by Donaldson for the Kähler manifold case, in [35] for the general Hermitian manifold case. The following proposition was proved in [34].

**Proposition 2.1** (Theorem 5.1 in [34]) Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over a compact Hermitian manifold $(\bar{X}, \omega)$ with non-empty boundary $\partial X$ and $H_0$ a Hermitian metric on $E$. There is a unique Hermitian metric $H$ on $E$ such that

$$\begin{cases} 
\sqrt{-1} \Lambda_{\omega} (F_H + [\theta, \theta^{*H}]) = \lambda \text{Id}_E, \\
H|_{\partial X} = H_0, 
\end{cases}$$

where $\lambda$ is a constant.

Let $\tilde{H}$ be a solution of (2.6), $f = \log \det(H_0)/\det(H)$ and $H := e^{f} \tilde{H}$. It is not hard to find that

$$\sqrt{-1} \Lambda_{\omega} (F_H + [\theta, \theta^{*H}]) = \left( \lambda - \frac{1}{r} \Lambda_{\omega} \partial \overline{\partial} f \right) \text{Id}_E.$$  

Hence
Given \( \eta \in \mathcal{S}_{H_0}(E) \), we can choose a local unitary basis \( \{ e_a \}_{a=1}^r \) with respect to \( H_0 \) and local functions \( \{ \lambda_a \}_{a=1}^r \) such that

\[
\eta = \sum_{a=1}^r \lambda_a \cdot e_a \otimes e^a,
\]

where \( \{ e^a \}_{a=1}^r \) denotes the dual basis in \( E^* \). Let \( \Psi \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and 
\[
A = \sum_{a, \beta=1}^r A^a_{\beta} e_a \otimes e^\beta \in \text{End}(E). \]

We define:

\[
\Psi(\eta)(A) = \Psi(\lambda_\beta, \lambda_a) A^a_{\beta} e_a \otimes e^\beta.
\]

**Proposition 2.3** (Proposition 2.6 in [34]) Let \( (E, \tilde{\partial}_E, \theta) \) be a Higgs bundle with a fixed Hermitian metric \( H_0 \) over a compact Gauduchon manifold \( (\overline{X}, \omega) \) with non-empty smooth boundary \( \partial \overline{X} \). Let \( H \) be a Hermitian metric on \( E \) satisfying \( H|_{\partial \overline{X}} = H_0|_{\partial \overline{X}} \). Then we have the following identity:

\[
\int_{\overline{X}} \text{tr}(\Phi(H_0, \theta)s) \, \frac{\omega^n}{n!} + \int_{\overline{X}} \langle \Psi(s)(\tilde{\partial}_\theta s), \tilde{\partial}_\theta s \rangle_{\overline{H_0}} \, \frac{\omega^n}{n!} = \int_{\overline{X}} \text{tr}(\Phi(H, \theta)s) \, \frac{\omega^n}{n!},
\]

for \( s := \log(H_0^{-1} H) \) and \( \Psi \) is the function which is defined in (1.11).

In the following, we always assume that the compact manifold \( \overline{X} \) with non-empty smooth boundary \( \partial \overline{X} \) is a subset of the Hermitian manifold \( (M, \omega) \). Let \( H \) be the unique Hermitian metric in Corollary 2.2. Set \( \exp s = h = H_0^{-1} H \). By the condition \( \det(h) \equiv 1 \) on \( \overline{X} \) and the relationship between the geometric mean and arithmetic mean, one can get that

\[
\frac{1}{r} \text{tr}(\exp s) \geq (\det h)^{\frac{1}{r}} = 1
\]

on \( \overline{X} \). As in [27], we extend \( \log(\text{tr}(\exp s)/r) \) and \( \left( |\Lambda_\omega F_{h_0, \beta}|_{H_0} \right)_X \) to the functions \( \log(\text{tr}(\exp s)/r) \) and \( |\Lambda_\omega F_{h_0, \beta}|_X \) on the whole \( M \) by setting 0 outside \( \overline{X} \).

Let \( g \) be the Hermitian metric with respect to \( \omega \). As usual, we denote the Beltrami-Laplace operator on the Hermitian manifold \( (M, \omega) \) by \( \Delta_g \), and define the complex Laplace operator \( \Delta_\omega \) for functions as

\[
\Delta_\omega f = 2\sqrt{-1} \Lambda_\omega \partial \overline{\partial} f.
\]
It is well known that the difference of the two Laplacians is given by a first-order differential operator as follows
\[(\tilde{\Delta}_\omega - \Delta_h)f = \langle V, \nabla f \rangle_g,\]
where \(V\) is a smooth vector field on \(M\). Usually the complex Laplace operator is not a self-adjoint operator.

**Definition 2.4** A function \(f\) on the Hermitian manifold \((M, \omega)\) satisfying
\[
\sqrt{-1} \Lambda_\omega \partial \bar{\partial} f \geq \eta
\]
in weak sense means that, for any nonnegative compactly supported smooth function \(\psi\), there holds
\[
\int_M f \sqrt{-1} \partial \bar{\partial} \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) \geq \int_M \eta \psi \frac{\omega^n}{n!}.
\]

**Proposition 2.5** We have
\[
\sqrt{-1} \Lambda_\omega \partial \bar{\partial} \log(\text{tr}(\exp s)/r)^\sim \geq -|\Lambda_\omega F^\perp_{H_0, \theta}|_H^X
\]
in weak sense on \(M\).

**Proof** Due to the inequality (2.10) and the boundary condition of \(H\), we know
\[
\frac{\partial \text{tr}(\exp s)/r}{\partial \nu} \leq 0,
\]
where \(\nu\) is the outer normal vector field at \(\partial X\). Direct computations give us that
\[
\int_M \log(\text{tr}(\exp s)/r)^\sim \cdot \sqrt{-1} \partial \bar{\partial} \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) = \int_X \log(\text{tr}(\exp s)/r) \cdot \sqrt{-1} \partial \bar{\partial} \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) = \int_X \sqrt{-1} \partial \left( \log(\text{tr}(\exp s)/r) \cdot \partial \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) \right) - \int_X \sqrt{-1} \partial \log(\text{tr}(\exp s)/r) \wedge \partial \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) = \int_X \sqrt{-1} \partial \left( \partial \log(\text{tr}(\exp s)/r) \wedge \psi \frac{\omega^{n-1}}{(n-1)!} \right) - \int_X \sqrt{-1} \partial \log(\text{tr}(\exp s)/r) \wedge \psi \frac{\omega^{n-1}}{(n-1)!}.
\]
where \(\psi\) is a test function. This means that
Similarly, we have

\[
\int_X \sqrt{-1\partial} \left( \partial \log(\exp s/r) \wedge \psi \frac{\omega^{n-1}}{(n-1)!} \right) = \int_X \log(\exp s/r) \cdot \sqrt{-1\partial} \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) - \int_X \sqrt{-1\partial} \log(\exp s/r) \wedge \psi \frac{\omega^{n-1}}{(n-1)!}.
\]

(2.16)

Now we should use the following result.

Lemma 2.6 Let \((M, \omega)\) be a Hermitian manifold, \(g\) be the Riemannian metric with respect to \(\omega\). Then we have

\[
- \ast df = \sqrt{-1} (\partial f - \overline{\partial} f) \wedge \frac{\omega^{n-1}}{(n-1)!},
\]

(2.19)

where \(\ast\) is the Hodge star operator with respect to the Riemannian metric \(g\) and \(f\) is a differential function on \(M\).

Proof Let \(\theta\) be a 1-form. By a simple calculation, we obtain that

\[
\theta \wedge \ast df = \langle \theta, df \rangle_g \frac{\omega^n}{n!} = \langle \theta^{1,0}, \overline{\partial} f \rangle_g \frac{\omega^n}{n!} + \langle \theta^{0,1}, \partial f \rangle_g \frac{\omega^n}{n!}
\]

(2.20)

and
\[
\theta \wedge \sqrt{-1}(\partial f - \bar{\partial} f) \wedge \frac{\omega^{n-1}}{(n-1)!} = - \theta^{1,0} \wedge \sqrt{-1}\partial f \wedge \frac{\omega^{n-1}}{(n-1)!} + \theta^{0,1} \wedge \sqrt{-1}\partial f \wedge \frac{\omega^{n-1}}{(n-1)!} = - \langle \theta^{1,0}, \partial f \rangle \frac{\omega^n}{n!} - \langle \theta^{0,1}, \partial f \rangle \frac{\omega^n}{n!}.
\] (2.21)

Therefore, because of the arbitrary of \( \theta \), the above lemma follows.

Note that

\[
\text{div}(\nabla f) \frac{\omega^n}{n!} = d^* (df).
\] (2.22)

One can easily check that

\[
\text{div}(\psi \nabla f) \frac{\omega^n}{n!} = d^* (\psi df).
\] (2.23)

Hence we derive

\[
2 \int_X \log(\text{tr}(\exp s)/r) \cdot \sqrt{-1}\partial \bar{\partial} \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) - \int_X \sqrt{-1}\partial \bar{\partial} \log(\text{tr}(\exp s)/r) \wedge \psi \frac{\omega^{n-1}}{(n-1)!}
\]

\[
= - \int_X d^* (\psi d \log(\text{tr}(\exp s)/r))
\]

\[
= - \int_X \text{div}(\psi \nabla \log(\text{tr}(\exp s)/r)) \frac{\omega^n}{n!}
\]

\[
= - \int_X \psi \frac{\partial \log(\text{tr}(\exp s)/r)}{\partial v} \frac{\omega^n}{n!} \geq 0.
\] (2.24)

This implies

\[
\int_M \log(\text{tr}(\exp s)/r)^\sim \cdot \sqrt{-1}\partial \bar{\partial} \left( \psi \frac{\omega^{n-1}}{(n-1)!} \right) \geq \int_X \sqrt{-1}\partial \bar{\partial} \log(\text{tr}(\exp s)/r) \wedge \psi \frac{\omega^{n-1}}{(n-1)!}
\]

\[
\geq - \int_X |\Lambda_{\omega} F_{H_0,\theta}^\perp|_{H_0} \psi \frac{\omega^n}{n!}
\]

\[
= - \int_M |\Lambda_{\omega} F_{H_0,\theta}^\perp|_{H_0} \cdot \psi \frac{\omega^n}{n!}.
\] (2.25)

This concludes the proof of the proposition.

\[\square\]

3 A proof of Theorem 1.1

Let’s first recall the following lemmas which are proved by Mochizuki in [27].

**Lemma 3.1** (Lemma 2.30 in [27]) There exists an exhaustion function \( \phi \in C^\infty(M) \).
**Lemma 3.2** (Lemma 2.31 in [27]) There is a sequence of compact subsets $M_i \subset M_{i+1} \subset M(i = 1, 2, \ldots)$ with $\cup M_i = M$, where each $M_i$ is a submanifold with non-empty smooth boundary $\partial M_i$ such that $M_i \setminus \partial M_i$ is an open subset of $M$. Moreover, each connected component of $M_i$ has non-empty boundary.

Fix a number $a_i$ and let $M_i$ denote the compact space $\{x \in M | \phi(x) \leq a_i\}$ with boundary $\partial M_i$. By choosing a sequence $a_i \to +\infty$ such that each $a_i$ is not a critical value of $\phi$, we have a sequence of exhaustion compact subsets $M_i \subset M$ with non-empty smooth boundary. Let’s consider the Dirichlet problem on $M_i$. Then Corollary 2.2 tells us that there exists a Hermitian metric $H_i$ on $E|_{M_i}$ such that

$$\begin{cases}
-\Lambda_{\omega} \partial \bar{\partial} \log(\exp s_i)/r \geq -|\Lambda_{\omega} F_{H_{0}}^{\perp}|_{H_0}, \\
H_i|_{\partial M_i} = H_0, \\
\det(H_0^{-1}H_i) = 1.
\end{cases}$$

(3.1)

Set $H_0^{-1}H_i = h_i = \exp s_i$. We already know on $M_i$, it holds that

$$\sqrt{-1}\Lambda_{\omega} \partial \bar{\partial} \log(\exp s_i)/r \geq -|\Lambda_{\omega} F_{H_{0}}^{\perp}|_{H_0}. \quad (3.2)$$

Now extend $\log(\exp s_i)/r$ and $(|\Lambda_{\omega} F_{H_{0}, \theta}|_{H_0})_{M_i}$ to the functions $\log(\exp s_i)/r \tilde{\omega}$ and $|\Lambda_{\omega} F_{H_{0}, \theta}|_{H_0}^{\perp M_i}$ on $M$ by setting 0 outside $M_i$. According to Proposition 2.5, we obtain

$$\sqrt{-1}\Lambda_{\omega} \partial \bar{\partial} \log(\exp s_i)/r \tilde{\omega} \geq -|\Lambda_{\omega} F_{H_{0}, \theta}|_{H_0}^{\perp M_i} \quad (3.3)$$

in weak sense on $M$. From the assumptions in Theorem 1.1, it can be seen that there exists two positive constants $C_1, C_2$ such that for any $i$, we have

$$\sup_{M_i} \log(\exp s_i)/r \leq C_1 + C_2 \int_{M_i} \log(\exp s_i)/r \cdot \frac{\omega^n}{n!}. \quad (3.4)$$

Noting $\text{tr} s_i = 0$, one can find that

$$\log(\exp s_i)/r \leq |s_i|_{H_0} \leq (r - 1) r^2 \log(\exp s_i)). \quad (3.5)$$

So clearly it implies that

$$\sup_{M_i} |s_i|_{H_0} \leq C_3 + C_4 \int_{M_i} |s_i|_{H_0} \cdot \frac{\omega^n}{n!}, \quad (3.6)$$

where $C_3$ and $C_4$ are positive constants depending only on $C_1, C_2, \hat{B}$ and $r$.

**Proof of Theorem 1.1** When the Higgs bundle $(E, \tilde{\partial}_E, \theta)$ is $H_0$-analytically stable over $(M, \omega)$, we will show that, after going to a subsequence, $H_i$ converge to a Hermitian–Einstein metric $H_{\infty}$ in $C_{\tilde{\omega}}^{\infty}$-topology as $i \to +\infty$.

1. **Uniform $C^0$-estimate.** By (3.6), the key is to get a uniform estimate for $\int_{M_i} |s_i|_{H_0} \cdot \frac{\omega^n}{n!}$, i.e. there exists a constant $\tilde{C}$ independent of $i$, such that
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\[ l_i := \int_{M_i} |s_i|_{H_0} \cdot \varphi \frac{\omega^n}{n!} \leq \hat{C} \]  

(3.7)

for all \( i \).

As that in [27], we prove (3.7) by contradiction. If not, there would exist a subsequence \( i \to +\infty \) such that \( l_i \to +\infty \). Set

\[ u_i = \frac{s_i}{l_i}. \]  

(3.8)

Then, we have

\[ \int_{M_i} |u_i|_{H_0} \cdot \varphi \frac{\omega^n}{n!} = 1, \]  

(3.9)

and

\[ \sup_{M_i} |u_i|_{H_0} \leq \frac{1}{l_i} (C_3 + C_4 l_i) < C_5 < +\infty. \]  

(3.10)

Now, we show that \( \|u_i\|_{L^2} \) are uniformly bounded on any compact subset of \( M \).

Based on Proposition 2.3, we deduce

\[ \int_{M_i} \text{tr} \left( \sqrt{-1} \Lambda_0 F^\perp_{H_0, \theta} u_i \right) \frac{\omega^n}{n!} + l_i \int_{M_i} \langle \Psi(l_i u_i) (\overline{\partial}_\theta u_i), (\overline{\partial}_\theta u_i) \rangle_{H_0} \frac{\omega^n}{n!} = 0. \]  

(3.11)

By the definition (1.11), it is easy to check that

\[ \Psi(lx, ly) \to \begin{cases} (x - y)^{-1}, & x > y; \\ +\infty, & x \leq y, \end{cases} \]  

(3.12)

increases monotonically as \( l \to +\infty \). Let \( \zeta \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+) \) satisfy \( \zeta(x, y) < (x - y)^{-1} \) whenever \( x > y \). Clearly Eqs. (3.11), (3.12) and the arguments in [31, Lemma 5.4] yield that

\[ \int_{M_i} \text{tr} \left( \sqrt{-1} \Lambda_0 F^\perp_{H_0, \theta} u_i \right) \frac{\omega^n}{n!} + \int_{M_i} \langle \zeta(u_i) (\overline{\partial}_\theta u_i), (\overline{\partial}_\theta u_i) \rangle_{H_0} \frac{\omega^n}{n!} \leq 0, \]  

(3.13)

From (3.10), we may assume that \( (x, y) \in (-C_5, C_5) \times (-C_5, C_5) \). Note that \( \frac{1}{2C_5} < \frac{1}{x - y} \) when \( x > y \). In particular, taking \( \zeta(x, y) = \frac{1}{2C_5} \) in (3.13), we immediately get

\[ \int_{M_i} \text{tr} \left( \sqrt{-1} \Lambda_0 F^\perp_{H_0, \theta} u_i \right) \frac{\omega^n}{n!} + \frac{1}{2C_5} \int_{M_i} |\overline{\partial}_\theta (u_i)|^2_{H_0} \frac{\omega^n}{n!} \leq 0, \]  

(3.14)

for \( i \gg 0 \), and then

\[ \int_{M_i} |\overline{\partial}_\theta (u_i)|^2_{H_0} \frac{\omega^n}{n!} \leq 2C_5^2 \int_{M_i} \left| \sqrt{-1} \Lambda_0 F^\perp_{H_0, \theta} \right|_{H_0} \frac{\omega^n}{n!} \leq C_6, \]  

(3.15)

where \( C_6 \) is a uniform constant. Thus, \( u_i \) are bounded in \( L^2 \) on any compact subset of \( M \). By choosing a subsequence, we have \( u_i \rightharpoonup u_{\infty} \) weakly in \( L^2_{1, \text{loc}} \). Of course \( \text{tr} u_i = 0 \) and (3.10) imply that
\[ \text{tr} u_\infty = 0, \quad \sup_M |u_\infty|_{L_0} \leq C_5 < +\infty. \] (3.16)

The condition \( \int_M \varphi \frac{\omega}{n!} < +\infty \) means that, for any \( \epsilon > 0 \), there exists \( i_0 \) such that

\[ 0 < \int_{M \setminus M_i} \varphi \frac{\omega}{n!} < \epsilon \] (3.17)

for all \( i \geq i_0 \). This together with (3.9) and (3.10) gives that

\[ 1 \geq \int_M |u_j|_{L_0} \varphi \frac{\omega}{n!} = \left( \int_M - \int_{M \setminus M_i} \right) |u_j|_{L_0} \varphi \frac{\omega}{n!} \geq 1 - C_5 \epsilon \] (3.18)

for any \( i_0 \leq i \leq j \). Noting that \( L^2 \hookrightarrow L^1 \) on any compact subset, we have

\[ 1 \geq \int_M |u_\infty|_{L_0} \varphi \frac{\omega}{n!} \geq 1 - C_5 \epsilon \] (3.19)

for all \( i_0 \leq i \), and then

\[ 1 \geq \int_M |u_\infty|_{L_0} \varphi \frac{\omega}{n!} \geq 1 - C_5 \epsilon. \] (3.20)

This indicates that

\[ \int_M |u_\infty|_{L_0} \varphi \frac{\omega}{n!} = 1 \] (3.21)

and \( u_\infty \) is non-trivial. If \( i_0 \leq i \leq j \), we derive

\[
\int_{M_i} \text{tr} \left( \sqrt{-1} \Lambda \omega F_{H_0, \theta} \frac{\omega}{n!} \right) + \int_{M_i} \langle \zeta(u_j), \tau \theta u_j, \bar{\tau} \bar{\theta} u_j \rangle \frac{\omega}{n!} \leq \left( \int_{M \setminus M_i} - \int_{M \setminus M_i} \right) \text{tr} \left( \sqrt{-1} \Lambda \omega F_{H_0, \theta} \frac{\omega}{n!} \right) + \int_{M_j} \langle \zeta(u_j), \tau \theta u_j, \bar{\tau} \bar{\theta} u_j \rangle \frac{\omega}{n!} \leq 2C_\star \hat{B} \epsilon, \]

(3.22)

where we have used (3.17), (3.13) and (1.5). Taking limits \( j \to \infty \) and \( i \to \infty \), one can obtain

\[
\int_M \text{tr} \left( \sqrt{-1} \Lambda \omega F_{H_0, \theta} u_\infty \right) \frac{\omega}{n!} + \int_M \langle \zeta(u_\infty), \tau \theta u_\infty, \bar{\tau} \bar{\theta} u_\infty \rangle \frac{\omega}{n!} \leq 2C_\star \hat{B} \epsilon. \]

(3.23)

The fact that \( \epsilon \) is arbitrary in the above inequality obviously implies

\[
\int_M \text{tr} \left( \sqrt{-1} \Lambda \omega F_{H_0, \theta} u_\infty \right) \frac{\omega}{n!} + \int_M \langle \zeta(u_\infty), \tau \theta u_\infty, \bar{\tau} \bar{\theta} u_\infty \rangle \frac{\omega}{n!} \leq 0. \]

(3.24)

Now following Simpson’s argument [31, Lemma 5.5], we conclude that the eigenvalues of \( u_\infty \) are constant almost everywhere. Let \( \mu_1 < \mu_2 < \cdots < \mu_l \) be the distinct eigenvalues of \( u_\infty \). Because \( \text{tr}(u_\infty) = 0 \) and \( u_\infty \neq 0 \), there must hold that \( 2 \leq l \leq r \). For each \( \mu_\alpha (1 \leq \alpha \leq l - 1) \), we construct a function \( P_\alpha : \mathbb{R} \to \mathbb{R} \) such that
\[ P_a = \begin{cases} 1, & x \leq \mu_a; \\ 0, & x \geq \mu_{a+1}. \end{cases} \]

Setting \( \pi_a = P_a(u_\infty) \), from [31, p. 887], we have: (1) \( \pi_a \in L^2_{1} \), \( \| \pi_a \|^2 = \pi_a H_0 \); (2) \( \Pi_a \) is a \( \omega \)-principal \( \kappa \)-subbundle of \( L^2_{1} \) and \( \| \pi_a \|_{\theta, \pi_a} = 0 \) and (4) \( \| \Pi_a \|_{\theta, \pi_a} = 0 \). By Uhlenbeck and Yau’s regularity statement of \( L^2_{1} \)-subbundle [33], \( \{ \pi_a \}_{a=1}^{\infty} \) determine \( l - 1 \) Higgs sub-sheaves \( \{ E_a \}_{a=1}^{l-1} \) of \( E \). Using the same argument as in [31, Proposition 5.5], we can prove that there must exist a Higgs sub-sheaf \( E_a \) which contradicts the stability of \( (E, \partial_E, \theta) \). This completes the proof of uniform \( C^1 \)-estimate.

(2) **Uniform local \( C^1 \)-estimate.** From the property that \( H_i \) satisfies the Hermitian–Einstein equation (1.2) and \( \det h_i = 1 \) on \( M_i \), it is easy to see that

\[ \text{tr} F_{h_i} = \text{tr} F_{h_0}, \]  

\[ \sqrt{-1} \Lambda_0 \partial E h_i = -\sqrt{-1} h_i \cdot \Lambda_0 F_{h_0, \theta} + \sqrt{-1} \Lambda_0 (\partial_E h_i \cdot h_i^{-1} \wedge \partial h_i h_i) + \sqrt{-1} \Lambda_0 \{ [\theta, h_i] \cdot h_i^{-1} \wedge [\theta^* h_0, h_i] - [\theta^* h_0, h_i], \theta \}, \]  

and then

\[ \sqrt{-1} \Lambda_0 \partial \theta \text{tr} h_i = -\sqrt{-1} \Lambda_0 \text{tr}(h_i \cdot F_{h_0, \theta}) - \left| h_i^{-1} \cdot \partial h_i h_i \right|^2_{H_0} - \left| [\theta, h_i] \cdot h_i^{-1} \right|^2_{H_0}. \]  

Let \( T_i = h_i^{-1} \partial h_i h_i \). A direct computation gives us that

\[ \sqrt{-1} \Lambda_0 \partial \text{tr} \left| T_i \right|^2_{H_0} \geq \frac{1}{2} \left| \nabla_{h_0} T_i \right|^2_{H_i} \]  

\[ - C_7 |F_{h_0}|_{H_i} + |\theta|^2_{H_i} + |\sqrt{-1} \Lambda_0 F_{h_0}|_{H_i} + |Rm(g)|_{g} + |\nabla g|) |T_i|^2_{H_i} \]  

\[ - C_8 |D_{h_0} (\Lambda_0 F_{h_0})|_{H_i} |T_i|_{H_i} \]  

\[ - C_9 |\nabla_{h_0} \theta|^2_{H_i}, \]  

where the constants \( C_7, C_8, C_9 \) depend only on the dimension \( n \) and the rank \( r \). We will follow the argument in [20, Lemma 2.4] to get local uniform \( C^1 \)-estimate. Let \( \Omega \) be a compact subset in \( M, d \) be a constant less than the distance of \( \Omega \) to \( \partial M_i \), where \( i_0 \) is large enough such that \( \Omega \subset M_i \). Set \( \Omega_1 = \{ x \in M \mid \text{dist}(x, \Omega) \leq \frac{1}{4} d \} \) and \( \Omega_2 = \{ x \in M \mid \text{dist}(x, \Omega) \leq \frac{1}{2} d \} \). Let’s choose two nonnegative cutoff functions \( \psi_1, \psi_2 \) such that

\[ \psi_1 = \begin{cases} 0, & x \in M \setminus \Omega_1; \\ 1, & x \in \Omega, \end{cases} \]

\[ \psi_2 = \begin{cases} 0, & x \in M \setminus \Omega_2; \\ 1, & x \in \Omega_1, \end{cases} \]

and

\[ |d \psi_a|^2 + |\Lambda_0 \partial \psi_a| \leq C_{10}, \quad a = 1, 2, \]

where \( C_{10} \) is a constant depending only on \( d^{-2} \) and the geometry of \( (\Omega_2, \omega) \). Consider the following test function

\[ \psi_a \]
\[ \eta_i = \psi_i^2 |T_i|_{H^1_i}^2 + \tilde{B} \psi_i^2 \text{tr} h_i, \tag{3.29} \]

where the constant \( \tilde{B} \) will be chosen large enough later and \( i \geq i_0 \).

It follows from (3.27) and (3.28) that

\[ \sqrt{-1} \Lambda \eta \bar{\partial} \eta_i \geq \psi_i^2 (\tilde{B} \eta - C_{12}) |T_i|_{H^1_i}^2 - C_{13}, \tag{3.30} \]

where \( C_{11} \) is a positive constant depending only on \( \sup_{\Omega} (\text{tr} h_i + 2 \text{tr} h_i^{-1}) \), \( C_{12} \) and \( C_{13} \) are positive constants depending only on \( \sup_{\Omega} (\text{tr} h_i + 2 \text{tr} h_i^{-1}) \), \( \sup_{\partial \Omega} |F_i|_{H^{1/2}} \), \( \sup_{\partial \Omega} |\theta|_{H^{1/2}} \), \( \sup_{\partial \Omega} |\sqrt{-1} \Lambda \eta \omega_{F_i}|_{H^{1/2}} \), \( \sup_{\partial \Omega} |D_{h_i} (\Lambda \omega_{F_i})|_{H^{1/2}} \), \( \sup_{\partial \Omega} |\nabla_{h_i} \theta|_{H^{1/2}} \), \( d^{-2} \) and the geometry of \( \Omega \). Choose \( \tilde{B} = e^{C_{11}} (C_{12} + 1) \), then

\[ \sqrt{-1} \Lambda \eta \bar{\partial} \eta_i \geq \psi_i^2 |T_i|_{H^1_i}^2 - C_{13} \tag{3.31} \]

on \( M \). Let \( \eta_i(P_0) = \sup_M \eta_i \). According to the definition of \( \psi_i \) and the uniform bound of \( \sup_{\partial \Omega} \text{tr} h_i \), we may assume that \( P_0 \in \Omega_i \). The inequality (3.31) and the maximum principle yield

\[ |T_i|_{H^1_i}^2 (P_0) \leq C_{13}, \]

and then there exists a positive constant \( C_{14} \) depending only on \( \sup_{\Omega_i} (\text{tr} h_i + 2 \text{tr} h_i^{-1}) \), \( \sup_{\partial \Omega_i} |F_i|_{H^{1/2}} \), \( \sup_{\partial \Omega_i} |\theta|_{H^{1/2}} \), \( \sup_{\partial \Omega_i} |\sqrt{-1} \Lambda \eta \omega_{F_i}|_{H^{1/2}} \), \( \sup_{\partial \Omega_i} |D_{h_i} (\Lambda \omega_{F_i})|_{H^{1/2}} \), \( \sup_{\partial \Omega_i} |\nabla_{h_i} \theta|_{H^{1/2}} \), \( d^{-2} \) and the geometry of \( \Omega \), such that

\[ \sup_{\Omega} |T_i|_{H^1_{p_0}}^2 \leq C_{14} \tag{3.32} \]

for all \( i \geq i_0 \). This concludes the proof of uniform local \( C^1 \)-estimate of \( H_i \).

We use Mochizuki’s argument [27] to give a uniform \( L^2 \)-bound of \( |\partial_{h_i} h_i|_{H^1} = |\bar{\partial}_{E} h_i|_{H^1} \). Applying (2.19) and the Stokes formula, one can deduce

\[
\int_{M} \sqrt{-1} \Lambda \eta \bar{\partial} \eta_i \left( \frac{\omega}{n!} \right) = \frac{1}{2} \int_{M} \sqrt{-1} (\bar{\partial} \eta - \bar{\partial} \eta_i) \nabla (\eta \omega_{n-1}) \left( \frac{\omega}{n!} \right) \\
= \frac{1}{2} \int_{M} \sqrt{-1} d(\bar{\partial} \eta) \nabla (\eta \omega_{n-1}) \left( \frac{\omega}{n!} \right) \\
= -\frac{1}{2} \int_{M} d(\star d \eta_i) = -\frac{1}{2} \int_{M} \text{div}(\nabla \eta_i) \left( \frac{\omega}{n!} \right) \\
= -\frac{1}{2} \int_{\partial M} \frac{\partial \eta_i}{\partial \nu_i} d\nu_{\partial M_i} \geq 0, \tag{3.33}
\]

where \( \nu_i \) is the outer normal vector field at \( \partial M_i \). This together with (3.27) and the uniform bound of \( \text{tr} h_i + 2 \text{tr} h_i^{-1} \) implies that there exists a uniform positive constant \( C_{15} \) such that

\[ \int_{M_i} \left( |\bar{\partial}_{E} h_i|^2_{H^1} + |\theta_i|^2_{H^1} \right) \left( \frac{\omega}{n!} \right) \leq C_{15}. \tag{3.34} \]

Since we have obtained a uniform \( C^0 \)-estimate and uniform local \( C^1 \)-estimate of \( h_i \), by Eq. (3.26) and the standard elliptic estimates, we can derive uniform local higher-order estimates of \( h_i \). So after going to a subsequence (which we also denote by \( H_i \), we know that \( H_i \)
converges to a Hermitian metric $H$ on the whole $M$ in $C^\infty$-topology as $i \to \infty$, and $H$ satisfies the Hermitian–Einstein equation (1.2). Furthermore, (3.34) implies $\int_M (|\overline{\partial}_E h|_{H_0}^2 + ||\theta, h||_{H_0}^2) \frac{\omega^n}{n!} \leq C_{15}$. This completes the proof of Theorem 1.1.

\[ \square \]

### 4 Hermitian–Einstein metrics

In this section, we follow Mochizuki’s arguments ([27], Proposition 2.4) to give a sufficient condition for the uniqueness of Hermitian–Einstein metric in Theorem 1.1.

**Proposition 4.1** Suppose $H_1$ and $H_2$ are two Hermitian–Einstein metrics on $(E, \overline{\partial}_E, \theta)$. Assume (1) $H_1$ and $H_2$ are mutually bounded, (2) $1 - \Lambda_\omega F_{H_i, \theta} = 1 - \Lambda_\omega F_{H_1, \theta}$. Then there exist a holomorphic decomposition $(E, \theta) = \bigoplus_{i=1}^m (E_i, \theta|_{E_i})$ and a tuple $(c_1, \ldots, c_m) \in \mathbb{R}_{>0}^m$ such that

(i) The decomposition $E = \bigoplus_{i=1}^m E_i$ is orthogonal with respect to both $H_i (i = 1, 2)$;

(ii) $H_1|_{E_i} = c_i H_2|_{E_i}$.

**Proof** Let $h = H_1^{-1} H_2$. A direct computation gives us that

\[
\sqrt{-1} \Lambda_\omega \overline{\partial}_E th = |h^{-\frac{1}{2}} \cdot \partial_{H_1} h|_{H_1}^2 + ||\theta, h|| \cdot h^{-\frac{1}{2}}|_{H_1}^2 \geq 0.
\]

(4.1)

From Assumption 1, it can be seen that $\sqrt{-1} \Lambda_\omega \overline{\partial}_E th = 0$. This means $|h^{-\frac{1}{2}} \cdot \partial_{H_1} h|_{H_1} = 0$ and $||\theta, h|| \cdot h^{-\frac{1}{2}}|_{H_1} = 0$. So $\partial_{H_1} h = 0$ and $||\theta, h|| = 0$. Obviously the fact that $h$ is self-adjoint with respect to $H_i (i = 1, 2)$ implies $\overline{\partial}_E h = 0$ and $\partial_{H_1} h = 0$. Then it follows that the eigenvalues of $h$ are constant. Let $E = \bigoplus_{i=1}^m E_i$ denote the eigendecomposition of $h$, which is the one we desired.

\[ \square \]

**Proposition 4.2** Let $(M, \omega)$ be a non-compact Gauduchon manifold satisfying the Assumption 1 and $|d\omega^{n-1}| \in L^2(M)$. Suppose that there is a positive exhaustion function $\phi_1 : M \to \mathbb{R}$ such that $|d log \phi_1| \in L^2(M)$. Let $(E, \overline{\partial}_E, \theta)$ be an $H_0$-analytically stable Higgs bundle over $M$. Assume that $H_i (i = 1, 2)$ are Hermitian–Einstein metrics such that (1) $\det(H_i) = \det(H_0)$; (2) $H_1$ and $H_0$ are mutually bounded. Then $H_1 = H_2$.

**Proof** Assume that $H_1$ satisfies Theorem 1.1 and let $h_1 = H_0^{-1} H_1$.

Firstly Proposition 4.1 gives us the decomposition $(E, \theta) = \bigoplus_{i=1}^m (E_i, \theta|_{E_i})$ such that (1) the decomposition is orthogonal with respect to $H_i (i = 1, 2)$; (2) $H_1|_{E_i} = c_j H_2|_{E_i}$ for some $c_j > 0$. Let $\pi_j$ denote the projection onto $E_j$ with respect to the decomposition and $\pi_j^{*H_0}$ denote the adjoint of $\pi_j$ with respect to $H_0$. Because $H_0$ and $H_i (i = 1, 2)$ are mutually bounded, one can immediately know that $\pi_j$ are bounded with respect to $H_0$.

Set $\partial_{H_0} = \partial_{H_0} + \theta^{*H_0}$. Noting that $\theta^{*H_1} = h_1^{-1} \theta^{*H_0} h_1$ and $\partial_{H_1} - \partial_{H_0} = h_1^{-1} \partial_{H_0} h_1$, we deduce

[Springer]
By Mochizuki’s arguments [27], we also consider the Hermitian metric $H_3 = \bigoplus_{j=1}^{m} H_0 |_{E_j}$. Then $H_3$ and $H_0$ are mutually bounded.

Set $h_3 = H_0^{-1} H_3$. It can be expressed as that

$$h_3 = \sum_{j=1}^{m} \pi_j^{*H_0} \circ \pi_j.$$ (4.3)

And there holds that $\bar{\partial}_E h_3 \in L^2(H_0)$ and $\partial_{H_0} h_3 \in L^2(H_0)$.

Denote $\bar{\partial}_{E,\theta} = \bar{\partial}_E + \theta$. Then we obtain

$$\begin{aligned}
\Lambda_w \bar{\partial}_{E,\theta} (h_3^{-1} \partial_{H_0, \theta} h_3) \\
= \Lambda_w \bar{\partial}_E (h_3^{-1} \partial_{H_0} h_3 + [\theta^{*H_0}, h_3])) + \Lambda_w [\theta, h_3^{-1} (\partial_{H_0} h_3 + [\theta^{*H_0}, h_3])] \\
= \Lambda_w \bar{\partial}_E (h_3^{-1} \partial_{H_0, \theta} h_3) + \Lambda_w \bar{\partial}_E (h_3^{-1} \theta^{*H_0} h_3 - \theta^{*H_0}) + \Lambda_w [\theta, h_3^{-1} \partial_{H_0} h_3] \\
+ \Lambda_w [\theta, h_3^{-1} \theta^{*H_0} h_3 - \theta^{*H_0}] \\
= \Lambda_w \bar{\partial}_E (h_3^{-1} \partial_{H_0, \theta} h_3) + \Lambda_w [\theta, \theta^{*H_0} h_3 - \theta^{*H_0}] \\
= \Lambda_w F_{H_3} - \Lambda_w F_{H_0} + \Lambda_w [\theta, \theta^{*H_0}] - \Lambda_w [\theta, \theta^{*H_0}].
\end{aligned}$$ (4.4)

According to the holomorphic decomposition $(E, \theta) = \bigoplus_{i=1}^{m} (E_i, \theta|_{E_i})$ with respect to $H_i (i = 1, 2)$, we have $\partial_{H_i, \theta} \pi_j = \partial_{H_0} \pi_j + [\theta^{*H_1}, \pi_j] = 0$ and $\bar{\partial}_{E,\theta} \pi_j = 0$. Together with (4.2), this means

$$\begin{aligned}
\partial_{H_0, \theta} \pi_j &= \partial_{H_0} \pi_j + [\theta^{*H_0}, \pi_j] \\
&= -[h_1^{-1} \partial_{H_0, \theta} h_1, \pi_j] - [\theta^{*H_0}, \pi_j] \\
&= -[h_1^{-1} \partial_{H_0} h_1, \pi_j].
\end{aligned}$$ (4.5)

Since $H_1$ is an Hermitian–Einstein metric on $(E, \theta)$, from Theorem 1.1, we can get that $\partial_{H_0} h_1$, $\partial_{H_0, \theta} h_1$, $\partial_{H_0} \pi_j$ and $\partial_{H_0, \theta} \pi_j$ are in $L^2(H_0)$. Then it follows that $[\theta^{*H_0}, \pi_j] \in L^2(H_0)$ and $[\theta, \pi_j^{*H_0}] \in L^2(H_0)$. So immediately we know $\bar{\partial}_{E,\theta} \pi_j^{*H_0} \in L^2(H_0)$.

One can easily check that
\[
[\theta, h_3] = \sum_{j=1}^{m} \left( \theta \circ \pi_j^{*H_0} \circ \pi_j - \pi_j^{*H_0} \circ \theta \circ \pi_j \right)
\]
\[
= \sum_{j=1}^{m} \left( \theta \circ \pi_j^{*H_0} \circ \pi_j - \pi_j^{*H_0} \circ \theta \circ \pi_j + \pi_j^{*H_0} \circ \theta \circ \pi_j - \pi_j^{*H_0} \circ \pi_j \circ \theta \right)
\]
\[
= \sum_{j=1}^{m} \left[ \theta, \pi_j^{*H_0} \right] \circ \pi_j.
\]

Then it can be seen that \([\theta, h_3] \in L^2(H_0)\) and \([\theta^{*H_0}, h_3] \in L^2(H_0)\). This implies that \(\partial_{E,0} h_3\) and \(\partial_{H_0,0} h_3\) are square integrable with respect to \(H_0\).

A straightforward computation yields
\[
\partial_{E,0}(h_3^{-1} \partial_{H_0,0} h_3) = -h_3^{-1}(\partial_{E,0} h_3) h_3^{-1} \partial_{H_0,0} h_3 + h_3^{-1} \partial_{E,0} \partial_{H_0,0} h_3
\]
(4.7)
and
\[
\partial_{E,0} \partial_{H_0,0} h_3 = \sum \partial_{E,0} \pi_j^{*H_0} \circ \partial_{H_0,0} \pi_j + \sum \pi_j^{*H_0} \circ [F_{H_0,0}, \pi_j].
\]
(4.8)

Due to the assumption \(|\Lambda_{\omega, F_{H_0,0}}| h_0 \leq \hat{B} \varphi, |\Lambda_{\omega, F_{H_0,0}}| H_0\) is \(L^1\). Then we obtain the following lemma.

**Lemma 4.3** \(\Lambda_{\omega, \text{tr} \partial_{E,0}}(h_3^{-1} \partial_{H_0,0} h_3)\) is \(L^1\).

Furthermore, we can derive

**Lemma 4.4** \(\int_M \text{tr}(\partial_{E,0}(h_3^{-1} \partial_{H_0,0} h_3)) \wedge \omega^{n-1} = 0\).

**Proof** Set \(\chi_N := \rho(N^{-1} \phi_1),\) where \(\rho\) is a nonnegative \(C^\infty\)-function such that \(\rho(t) = 0\) if \(t \geq 2\) and \(\rho(t) = 1\) if \(t \leq 1\). Because \(\text{tr}(\partial_{E,0}(h_3^{-1} \partial_{H_0,0} h_3))\) is \(L^1\), we just need to show
\[
\lim_{N \to \infty} \int_M \chi_N \cdot \text{tr}(\partial_{E,0}(h_3^{-1} \partial_{H_0,0} h_3)) \wedge \omega^{n-1} = 0.
\]
(4.9)

After calculating directly, one can find that
\[
\int_M \chi_N \cdot \text{tr} \left( \tilde{\partial}_{E, \theta} \left( h_3^{-1} \partial_{H_0, \theta} h_3 \right) \right) \wedge \omega^{n-1} \\
= \int_M \chi_N \cdot \text{tr} \left( h_3^{-1} \partial_{H_0, \theta} h_3 \right) \wedge \tilde{\partial} \omega^{n-1} \\
= \int_M \chi_N \cdot \text{tr} \left( h_3^{-1} \partial_{H_0, \theta} h_3 \right) \wedge \omega^{n-1} - \int_M \partial \chi_N \cdot \text{tr} \left( h_3^{-1} \partial_{H_0, \theta} h_3 \right) \wedge \omega^{n-1} \\
= \int_M \chi_N \cdot \partial \left( \log \det h_3 \right) \wedge \omega^{n-1} - \int_M \partial \chi_N \cdot \text{tr} \left( h_3^{-1} \partial_{H_0, \theta} h_3 \right) \wedge \omega^{n-1} \\
= - \int_M \partial \chi_N \cdot \log \det h_3 \wedge \omega^{n-1} - \int_M \chi_N \cdot \log \det h_3 \cdot \partial \omega^{n-1} \\
- \int_{\{N \leq \phi_1 \leq 2N\}} \rho' \left( \frac{\phi_1}{N} \right) \frac{\partial \phi_1}{\partial \omega} \cdot \text{tr} \left( h_3^{-1} \partial_{H_0, \theta} h_3 \right) \wedge \omega^{n-1} \\
= - \int_{\{N \leq \phi_1 \leq 2N\}} \rho' \left( \frac{\phi_1}{N} \right) \frac{\partial \phi_1}{\partial \omega} \log \det h_3 \wedge \omega^{n-1} \\
- \int_{\{N \leq \phi_1 \leq 2N\}} \rho' \left( \frac{\phi_1}{N} \right) \frac{\partial \phi_1}{\partial \omega} \cdot \text{tr} \left( h_3^{-1} \partial_{H_0, \theta} h_3 \right) \wedge \omega^{n-1}.
\]

(4.10)

Clearly \(|d \omega^{n-1}| \omega \in L^2(M)\) and \(|d \log \phi_1| \omega \in L^2(M)\) implies (4.9).

Combining (4.4) and Lemma 4.4, we obtain

\[
\int_M \Lambda_0 \text{tr} F_{H_0} = \int_M \Lambda_0 \text{tr} F_{H_1} = \sum_{i=1}^m \int_M \Lambda_0 \text{tr} F_{H_0|E_i}.
\]

(4.11)

From \(\text{rank}(E) = \sum_{i=1}^m \text{rank}(E_i)\), one can see that there exists \(j_0\), such that \(\mu(E, H_0) \leq \mu(E_{j_0}, H_0|E_{j_0})\). This contradicts with the analytic stability of \((E, \tilde{\partial}_E, H_0)\). Thus \(H_1 = H_2\).

\[ \square \]

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Appendix

Let \(\mathbb{C}\) be the plane, \(\omega_\mathbb{C}\) be the Euclidean metric and \(\varphi_\mathbb{C}(w) = (1 + |w|^2)^{-1-\delta}\), where \(w\) is the complex coordinate of \(\mathbb{C}\) and \(\delta > 0\). By using Simpson’s result ([31], Proposition 2.4), Mochizuki showed that \((\mathbb{C}, \omega_\mathbb{C}, \varphi_\mathbb{C})\) meets the Assumption 1, i.e., we have the following lemma.

Lemma 5.1 ([27], Lemma 3.5) Let \((\mathbb{C}, \omega_\mathbb{C}, \varphi_\mathbb{C})\) be defined as above. For any nonnegative bounded function \(f\) satisfying

\[ \square \]
we have
\[ \sup_{w \in \mathbb{C}} f(w) \leq C(B)a \left( \int_{\mathbb{C}} f \cdot \varphi_{C}^{\omega_{C}} \right), \tag{5.2} \]
where \( a : [0, \infty) \to [0, \infty) \) is an increasing function with \( a(0) = 0 \) and \( a(x) = x \) (\( x \geq 1 \)). \( C(B) \) is a positive constant depending only on \( B \). Moreover, if \( \sqrt{-1} \Lambda_{\omega_{C}} \partial \overline{\partial} f \geq 0 \), then \( f \) is a constant.

Set
\[ (M, \omega_{M}) = (\mathbb{C}, \omega_{C}) \times (Y, \omega_{Y}), \tag{5.3} \]
\( \varphi_{M}(w, \cdot) = \varphi_{C}(w) \), where \((Y, \omega_{Y})\) is any compact Gauduchon manifold without boundary. Since \( \omega_{C} \) is Kähler and \( \omega_{Y} \) is Gauduchon, the product metric \( \omega_{M} \) is also Gauduchon. Following Mochizuki’s argument [27] and running Moser’s iteration procedure, we can prove the following proposition.

**Proposition 5.2** \((M, \omega_{M}, \varphi_{M})\) satisfies the Assumption 1.

**Proof** For any nonnegative bounded function \( f : M \to \mathbb{R} \) satisfying
\[ \sqrt{-1} \Lambda_{\omega_{M}} \partial \overline{\partial} f \geq -B \varphi_{M}, \tag{5.4} \]
we set
\[ \tilde{f}(w) = \int_{Y} f(w, \cdot) \frac{\omega_{Y}^{n}}{n!}, \tag{5.5} \]
where \( n \) is the complex dimension of \( Y \). By direct calculation, we derive
\[ \sqrt{-1} \Lambda_{\omega_{C}} \partial \overline{\partial} \tilde{f}(w) = \int_{Y} \sqrt{-1} \Lambda_{\omega_{C}} \partial \overline{\partial} f(w, \cdot) \frac{\omega_{Y}^{n}}{n!} \]
\[ = \int_{Y} \sqrt{-1} \left( \Lambda_{\omega_{M}} \partial \overline{\partial} f(w, \cdot) - \Lambda_{\omega_{Y}} \partial \overline{\partial} f(w, \cdot) \right) \frac{\omega_{Y}^{n}}{n!} \]
\[ = \int_{Y} \sqrt{-1} \Lambda_{\omega_{M}} \partial \overline{\partial} f(w, \cdot) \frac{\omega_{Y}^{n}}{n!} \]
\[ \geq -B \cdot \varphi_{C}(w) \cdot \text{Vol}(Y, \omega_{Y}), \tag{5.6} \]
where we have used the condition that \((Y, \omega_{Y})\) is a compact Gauduchon manifold. Then (5.2) implies
\[ \sup_{w \in \mathbb{C}} \tilde{f}(w) \leq C(B \cdot \text{Vol}(Y, \omega_{Y}))a \left( \int_{M} f \cdot \varphi_{M}^{n+1} \frac{\omega_{M}^{n+1}}{(n+1)!} \right), \tag{5.7} \]
For any point \( w_0 \in \mathbb{C} \), set \( B_r(w_0) := \{ w \in \mathbb{C} \mid |w - w_0| < r \} \). Then \( (X = B_2(w_0) \times Y, \omega_M) \) is a compact Riemannian manifold with smooth boundary (it doesn’t depend on \( w_0 \)) and we have the following Sobolev inequality

\[
\left( \int_X |v|^{\frac{2(n+1)}{n}} \frac{\omega_M^{n+1}}{(n+1)!} \right)^{\frac{n}{n+1}} \leq C_S \left[ \left( \int_X |d\nu|^2 \frac{\omega_M^{n+1}}{(n+1)!} \right)^{\frac{1}{2}} + \left( \int_X |v|^2 \frac{\omega_M^{n+1}}{(n+1)!} \right)^{\frac{1}{2}} \right]
\]

(5.8)

for any compactly supported function \( v \in L^2(X) \). On the other hand, there holds

\[
\sqrt{-1} \Lambda_{\omega_M} \partial \tilde{\phi} \geq -B \varphi_M \geq -B
\]

(5.9)

on \( X \). In the following, by using the Sobolev inequality (5.8) and the inequality (5.9), we will run Moser’s iteration procedure to obtain a mean value inequality.

Take \( 1 \leq r \leq r_2 < r_1 \leq R \leq 2 \) and let \( \psi_1 \in C^\infty_0(X) \) be the cutoff function such that

\[
\psi_1(x) = \begin{cases} 1, & x \in B_{r_2}(w_0) \times Y, \\ 0, & x \in X \setminus B_{r_1}(w_0) \times Y, \end{cases}
\]

(5.10)

\[0 \leq \psi_1(x) \leq 1 \text{ and } |d\psi_1|_{\omega_M} \leq 4(r_1 - r_2).\] Set \( \hat{f} = f + 1 \). Of course (5.9) implies

\[
\sqrt{-1} \Lambda_{\omega_M} \partial \tilde{\phi} \geq -B \cdot \hat{f}.
\]

(5.11)

Multiplying \( \hat{f}^{q-1} \psi_1^2 \) on both sides of the inequality (5.11) \((q \geq 2)\), and integrating it over \( X \), we know

\[
-B \int_X \hat{f}^{q-1} \psi_1^2 \frac{\omega_M^{n+1}}{(n+1)!} \leq \int_X \sqrt{-1} \hat{f}^{q-1} \cdot \psi_1^2 \partial \tilde{\phi} \wedge \frac{\omega_M^n}{n!}
\]

\[= \int_X \sqrt{-1} \partial (\hat{f}^{q-1} \cdot \psi_1^2 \partial \tilde{\phi} \wedge \frac{\omega_M^n}{n!}) - \int_X \sqrt{-1} \partial (\hat{f}^{q-1} \cdot \psi_1^2) \wedge \partial \tilde{\phi} \wedge \frac{\omega_M^n}{n!}
\]

\[+ \int_X \sqrt{-1} \hat{f}^{q-1} \cdot \psi_1^2 \partial \tilde{\phi} \wedge \partial \left( \frac{\omega_M^n}{n!} \right)
\]

\[= -\int_X \sqrt{-1} (q-1) \hat{f}^{q-2} \cdot \psi_1^2 \partial \tilde{\phi} \wedge \partial \tilde{\phi} \wedge \frac{\omega_M^n}{n!} - \int_X 2 \sqrt{-1} \hat{f}^{q-1} \cdot \psi_1 \partial \psi_1 \wedge \partial \tilde{\phi} \wedge \frac{\omega_M^n}{n!}
\]

\[+ \int_X \sqrt{-1} \hat{f}^{q-1} \cdot \psi_1^2 \partial \tilde{\phi} \wedge \partial \omega_M \wedge \frac{\omega_M^{n-1}}{(n-1)!}.
\]

(5.12)

Because the manifold \( Y \) is compact, there is a uniform constant \( C_1 \) such that \( \sup_X |\partial \omega_M|_{\omega_M} \leq C_1 \), and then

\[
\left| \partial \tilde{f} \wedge \partial \omega_M \wedge \left( \frac{\omega_M^{n-1}}{(n-1)!} \right) \right| \leq C_2 |\partial \phi|_{\omega_M} |\partial \omega_M|_{\omega_M} \leq C_3 |\partial f|_{\omega_M}.
\]

On the other hand, applying Cauchy’s inequality, we have
\[
|\overline{\partial f}|_{\omega_M} \hat{\psi}_1^{q-1} \psi_1^2 \leq \varepsilon |\overline{\partial f}|_{\omega_M} \hat{\psi}_1^{q-2} \psi_1^2 + \frac{1}{4\varepsilon} \hat{\psi}_1^2
\]
(5.14)

and
\[
2|\overline{\partial f}|_{\omega_M} |\partial \psi_1|_{\omega_M} \hat{\psi}_1^{q-1} \leq \varepsilon |\overline{\partial f}|_{\omega_M} \hat{\psi}_1^{q-2} \psi_1^2 + \frac{1}{4\varepsilon} |\partial \psi_1|_{\omega_M}.
\]
(5.15)

where \(\varepsilon\) is a positive constant which will be chosen later. By (5.12), (5.13), (5.14) and (5.15), we conclude that
\[
\int_X |\overline{\partial f}|^2 \cdot \psi_1^2 \frac{\alpha_{n+1}^M}{(n+1)!} \leq \frac{q^2}{4(q-1)} \left( B + \frac{C_3}{4\varepsilon} \right) \int_X \hat{\psi}_1^2 \frac{\alpha_{n+1}^M}{(n+1)!} + \frac{q^2}{4\varepsilon(q-1)} \int_X \hat{\psi}_1^2 |\partial \psi_1|^2 \frac{\alpha_{n+1}^M}{(n+1)!}.
\]
(5.16)

Choose \(\varepsilon = \frac{q-1}{2(C_3+1)}\), then
\[
\int_X |\overline{\partial f}|^2 \cdot \psi_1^2 \frac{\alpha_{n+1}^M}{(n+1)!} \leq \frac{q^2}{2(q-1)} \left( B + \frac{C_3}{2\varepsilon} \right) \int_X \hat{\psi}_1^2 \frac{\alpha_{n+1}^M}{(n+1)!} + \frac{2q^2(C_3+1)}{(q-1)^2} \int_X \hat{\psi}_1^2 |\partial \psi_1|^2 \frac{\alpha_{n+1}^M}{(n+1)!}.
\]
(5.17)

and
\[
\int_X |\partial (\hat{\psi}_1 \cdot \psi_1)|^2 \frac{\alpha_{n+1}^M}{(n+1)!} \leq 2 \int_X |\overline{\partial f}|^2 \cdot \psi_1^2 \frac{\alpha_{n+1}^M}{(n+1)!} + 2 \int_X \hat{\psi}_1^2 |\partial \psi_1|^2 \frac{\alpha_{n+1}^M}{(n+1)!}.
\]
(5.18)

Using (5.18) and the Sobolev inequality (5.8) \((v = \hat{\psi}_1 \cdot \psi_1)\), one can get
\[
\left( \int_{B_{r_2}(w_i) \times Y} \hat{\psi}_1^{q+1} \frac{\alpha_{n+1}^M}{(n+1)!} \right)^{\frac{1}{n+1}} \leq \left( \int_X (\hat{\psi}_1)^2 \frac{\alpha_{n+1}^M}{(n+1)!} \right)^{\frac{1}{n+1}}
\]
\[
\leq 2C_3 \left( \int_M 2|\partial (\hat{\psi}_1 \cdot \psi_1)|^2 \frac{\alpha_{n+1}^M}{(n+1)!} \right)^{\frac{1}{n+1}} + \int_X \hat{\psi}_1^2 \frac{\alpha_{n+1}^M}{(n+1)!}
\]
\[
\leq 2C_3 \left( \frac{q^2}{q-1} \left( B + \frac{C_3}{2\varepsilon} \right) + 1 \right) \int_X \hat{\psi}_1^2 \frac{\alpha_{n+1}^M}{(n+1)!} + 2C_3 \frac{4q^2(C_3+1)}{(q-1)^2} \int_X \hat{\psi}_1^2 |\partial \psi_1|^2 \frac{\alpha_{n+1}^M}{(n+1)!}.
\]
(5.19)

and then
\[
\left( \int_{B_{r_{i}}(w_{0}) \times Y} \hat{f}_{i}^{q_{i}+1} \frac{\partial^{q_{i}+1} M}{(n+1)!} \right)^{\frac{1}{q_{i}+1}} \leq \left( \int_{B_{r_{i+1}}(w_{0}) \times Y} \hat{f}_{i+1}^{q_{i+1}} \frac{\partial^{q_{i+1}} M}{(n+1)!} \right)^{\frac{1}{q_{i+1}}}, \tag{5.20}
\]

where \( C_{4} \) is a positive constant depending only on \( B, C_{3} \) and \( \sup_{Y} |\partial \omega_{Y}| \). Let \( R_{i} = r + 2^{-i} \cdot (R - r) \), \( q_{i} = 2^{i} \frac{n+1}{n} \). Substituting \( r_{2} = R_{i+1}, r_{1} = R_{i} \) and \( q = q_{i} \) into (5.20), we obtain

\[
\left( \int_{B_{R_{i+1}}(w_{0}) \times Y} \frac{\partial^{q_{i+1}} M}{n+1} \right)^{\frac{1}{q_{i+1}}} \leq C_{5} \left( \frac{2n+2}{n} \right)^{i} \left( \frac{n+1}{n} \right) \left( \int_{B_{R_{i}}(w_{0}) \times Y} \frac{\partial^{q_{i+1}} M}{n+1} \right)^{\frac{1}{q_{i+1}}}. \tag{5.21}
\]

Iterating the inequality (5.21) and using \( \sum_{i=0}^{\infty} k^{-i} = \frac{k}{k-1}, \sum_{i=0}^{\infty} (i+1)k^{-i} = \frac{k^{2}}{(k-1)^{2}} \), we conclude that

\[
\sup_{B_{R_{i}}(w_{0}) \times Y} \hat{f} \leq \lim_{i \to +\infty} \left( \int_{B_{R_{i+1}}(w_{0}) \times Y} \frac{\partial^{q_{i+1}} M}{n+1} \right)^{\frac{1}{q_{i+1}}} \leq C_{6} ((R - r)^{2} + 1)^{\frac{n+1}{2}} \left( \int_{B_{R_{i}}(w_{0}) \times Y} \hat{f}^{2} \frac{\partial^{q_{i+1}} M}{n+1} \right)^{\frac{1}{2}}. \tag{5.22}
\]

where \( C_{i} \) is a positive constant depending only on \( B, n, C_{S} \) and \( \sup_{Y} |\partial \omega_{Y}| \). For any \( 0 < \tilde{r} \leq 2 \) and \( 0 < \delta < 1 \), let \( h_{0} = \delta \tilde{r}, h_{i} = h_{i-1} + 2^{-i}(1-\delta)\tilde{r} \) for each \( i = 1, 2, 3, \ldots \). Putting \( r = h_{i} \) and \( R = h_{i+1} \) into (5.22), we have

\[
\sup_{B_{h_{i}}(w_{0}) \times Y} \hat{f} \leq C_{6} ((1-\delta)^{-2}\tilde{r}^{-2} + 1)^{\frac{n+1}{2}} \left( \int_{B_{h_{i+1}}(w_{0}) \times Y} \hat{f}^{2} \frac{\partial^{q_{i+1}} M}{n+1} \right)^{\frac{1}{2}} \left( \sup_{B_{h_{i+1}}(w_{0}) \times Y} \hat{f} \right)^{\frac{1}{2}}. \tag{5.23}
\]

Denote \( A(i) := \sup_{B_{h_{i}}(w_{0}) \times Y} \hat{f} \). Then (5.23) yields

\[
A(0) \leq \prod_{i=0}^{j-1} C_{6} ((1-\delta)^{-2}\tilde{r}^{-2} + 1)^{\frac{n+1}{2}} \left( \int_{B_{h_{i}}(w_{0}) \times Y} \hat{f}^{2} \frac{\partial^{q_{i+1}} M}{n+1} \right)^{\frac{1}{2}} A(j)^{2^{-i}} \leq C_{7} ((1-\delta)^{-2}\tilde{r}^{-2} + 1)^{n+1} \int_{B_{h_{j}}(w_{0}) \times Y} \hat{f} \frac{\partial^{q_{j+1}} M}{n+1}, \tag{5.24}
\]

where \( C_{7} \) is a positive constant depending only on \( B, n, C_{S} \) and \( \sup_{Y} |\partial \omega_{Y}| \). Take \( \tilde{r} = 2 \) and \( \delta = \frac{1}{2} \). Clearly (5.24) gives us that
and

\[
\sup_{B_1(w_0) \times Y} f \leq C_8 \left( \int_{B_1(w_0) \times Y} f \frac{\alpha_M^{n+1}}{(n+1)!} + \text{Vol}(B_2(w_0)) \cdot \text{Vol}(Y, \omega_Y) \right)
\]

\[
\leq C_8 \left( \int_{B_2(w_0)} \tilde{f} \omega_C + \text{Vol}(B_2(w_0)) \cdot \text{Vol}(Y, \omega_Y) \right)
\]

\[
\leq C_9 \left( a \left( \int_M f \cdot \varphi_M \frac{\alpha_M^{n+1}}{(n+1)!} \right) + 1 \right).
\]

where \(C_8\) and \(C_9\) are positive constants depending only on \(B, n, C_5\) and \(\sup_Y |\partial \omega_Y|\). Since \(w_0\) is arbitrary, we have

\[
\sup_M f \leq C_9 \left( a \left( \int_M f \cdot \varphi_M \frac{\alpha_M^{n+1}}{(n+1)!} \right) + 1 \right).
\]

Suppose that \(\sqrt{-1} \Lambda_{\omega_C} \bar{\partial} \bar{\partial} f \geq 0\) on \(M\). From the definition of \(\tilde{f}\) and the condition \(\omega_Y\) is Gauduchon, it is easy to see that \(\tilde{f}\) is a bounded function on \((\mathbb{C}, \omega_C)\) and satisfies \(\sqrt{-1} \Lambda_{\omega_C} \bar{\partial} \bar{\partial} \tilde{f} \geq 0\). According to Lemma 5.1, we know that \(\tilde{f} \equiv \tilde{C}\). Set

\[
f_1(w, \cdot) = f(w, \cdot) - \frac{\tilde{C}}{\text{Vol}(Y, \omega_Y)}.
\]

Then

\[
\int_{[w] \times Y} f_1 \frac{\alpha_Y^n}{n!} = 0
\]

and

\[
\sqrt{-1} \Lambda_{\omega_C} \bar{\partial} \bar{\partial} \int_{[w] \times Y} |f_1|^2 \frac{\alpha_Y^n}{n!} = \sqrt{-1} \Lambda_{\omega_C} \bar{\partial} \bar{\partial} \int_{[w] \times Y} |f|^2 \frac{\alpha_Y^n}{n!}
\]

\[
\geq \int_{[w] \times Y} \sqrt{-1} \Lambda_{\omega_M} \bar{\partial} \bar{\partial} M \cdot [f]^2 \frac{\alpha_Y^n}{n!}
\]

\[
\geq \int_{[w] \times Y} 2|\partial f|^2 \frac{\alpha_Y^n}{n!}
\]

\[
\geq C_p \int_{[w] \times Y} |f|^2 \frac{\alpha_Y^n}{n!},
\]

where we have used the Poincaré inequality on the compact Riemannian manifold \((Y, \omega_Y)\). Applying Lemma 5.1 again, we have \(f_1 \equiv 0\) and then \(f\) is constant.

\[\square\]
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