xVIO:
A Range-Visual-Inertial Odometry Framework

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Abstract

xVIO is a range-visual-inertial odometry algorithm implemented at JPL. It has been demonstrated with closed-loop controls on-board unmanned rotorcraft equipped with off-the-shelf embedded computers and sensors. It can operate at daytime with visible-spectrum cameras, or at night time using thermal infrared cameras. This report is a complete technical description of xVIO. It includes an overview of the system architecture, the implementation of the navigation filter, along with the derivations of the Jacobian matrices which are not already published in the literature.

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Chapter 1

System Overview

Visual-Inertial Odometry (VIO) is the use of one camera and an Inertial Measurements Unit (IMU) to estimate the position and orientation of the sensing platform. Range-VIO adds a range sensor measurement to that sensor fusion problem.

Although xVIO makes no assumption about the platform itself, it is worth keeping in mind that it was developed for rotorcraft flight applications. The ‘x’ in xVIO has no particular meaning but one should feel free to see it either as a pair of propellers, a symbol for accuracy, or sensor fusion. xVIO is versatile and can run either range-VIO, VIO or simply inertial odometry.

On the date this report was last updated, xVIO had been tested on datasets both indoors and outdoors, at daytime with visible and at nighttime with long-wave infrared cameras, in flight and hand-held, on real and simulated data. It ran at 30 frames per second with modern off-the-shelf sensors and embedded computers\(^1\). That proved enough to meet the real-time constraints and close the control loop to enable autonomous flight.

Figure 1.1 illustrates the software architecture at system level. It can be used as a guide to read through the next sections, which will introduce the main software components in the figure: sensor interface, visual front end, state estimator, and VIO measurement construction.

\(^1\)Qualcomm Snapdragon Flight Pro
1.1 Sensor Interface

xVIO code is written in C++11 and compiles into an executable that has been tested in Linux Ubuntu 16.04, 18.04 and 20.04. It receives timestamped sensor through two call functions: one that reads an 8-bit grayscale image coming from the camera driver, another one that reads angular rates $\omega$ and specific force $f$ coming from the IMU driver.

xVIO’s callbacks are currently wrapped to interface with the ROS\textsuperscript{2} framework, which provides a multi-computer real-time communication network between the sensor drivers, xVIO, and the user interface. xVIO was tested with ROS Kinetic, Melodic and Noetic versions.

1.2 Visual Front End

The visual front end matches feature points between the current image and the previous one. Its output is a list of $N$ image coordinate pairs.

\textsuperscript{2}Robot Operating System: \url{http://www.ros.org}
On the first image, feature points are detected using the FAST algorithm (Rosten and Drummond, 2006). A neighborhood parameter can be set in pixels to ensure feature are not too close to each other. The image can be divided in tiles, and the maximum number of features per tile enforce to bound the computational cost of the visual front end if needed.

Subsequently, the features are tracked frame-to-frame with the Lucas-Kanade algorithm (Lucas and Kanade, 1981). When the number of feature matches reached below a user-specified threshold, FAST features are re-detected on the previous image and matched with the current one. Existing matches are preserved in this. The neighborhood parameter prevents a new feature to be extracted next to an existing match. Outlier matches are detected using the RANSAC algorithm with a fundamental matrix model (Fischler and Bolles, 1981).

The tracker class uses the OpenCV implementation of these algorithms. More details about the actual Lucas-Kanade implementation can be found in Bouguet (2001). xVIO, the tracker, feature, image and camera class are part of the X library.

1.3 State Estimator

xVIO uses an Extended Kalman Filter (EKF) for state estimation. The specific EKF implementation is called xEKF and is also part of the X library. It propagates the position, velocity, attitude and inertial biases states using IMU measurements in the dynamics model discussed in Subsection 2.4.1. It also implements a ring-buffer to accommodate sensor and processing delays in real-time operations. xVIO provides xEKF with an update through image and range measurement residuals $\tilde{z}$, Jacobian matrix $J$ and covariance $R$.

1.4 VIO Measurement Construction

The core capability of xVIO is the ability to construct visual measurements from image matches, in order to constrain the inertially-propagated EKF. This process is illustrated in the orange block in Figure 1.1.

Visual measurements can use either the Simultaneous Localization And Mapping (SLAM) or Multi-State Constraint Kalman Filter (MSCKF) paradigms.

\[3\text{https://opencv.org}\]
Both have pros and cons but Li and Mourikis (2012a) showed that they are complimentary and can be hybridized to get the most accuracy out of the computational resources available. xVIO follows their approach closely, although not identically. Our implementation of SLAM and MSCKF, along with a discussion of their pros and cons, is detailed in Section 2.5. Li’s PhD thesis (2014) is a good reference about this approach in general, along with the references cited locally in Chapter 2 and A. It is worth noting that even though EKF-based VIO is only optimal up to linearization error, Li and Mourikis (2013) showed that they could compete with approaches based on non-linear optimization in terms of accuracy and computational cost.

We defined a track as the list image observations of a specific feature. When feature matches come in from the visual front end, they are being appended to an existing track and a new one is created by the track manager. Tracks can be either of the SLAM or MSCKF type. An MSCKF track can become a SLAM after an MSCKF measurement through a process described in Subsection 2.7.2, or to distribute SLAM features among the image tiles. At the start, the best feature in each style are used for SLAM, since only this paradigm can provide constraints immediately. MSCKF measurements are converted to SLAM feature when there are slots available.

SLAM and MSCKF require the creation, destruction and reparametrization of states and their error covariance. A state manager handles the state queries, actions and responses between the track manager, SLAM and MSCKF modules and the EKF.
Chapter 2

Filter Implementation

2.1 Notations

A scalar is denoted by $x$, a vector by $\mathbf{x}$, and a matrix by $\mathbf{X}$. The estimate of a random variable is written with the hat operator, e.g. $\hat{x}$ for a vector.

The coordinates of vector $\mathbf{x}$ in reference frame $\{a\}$ are represented by $^a\mathbf{x}$. The time derivative of $^a\mathbf{x}$ is noted $^a\dot{x}$, such that

$$^a\dot{x} = \frac{d^a\mathbf{x}}{dt}.$$  \hfill (2.1)

The position, velocity and acceleration of frame $\{b\}$ with respect to frame $\{a\}$ are denoted by $^b\mathbf{p}_a$, $^b\mathbf{v}_a$ and $^b\mathbf{a}_a$, respectively. When their coordinates are expressed in the axes of $\{a\}$, velocity and acceleration are defined as time derivatives of position by

$$^a\mathbf{v}_a = ^a\dot{\mathbf{p}}_a,$$  \hfill (2.2)

$$^a\mathbf{a}_a = ^a\dot{\mathbf{v}}_a.$$  \hfill (2.3)

The angular velocity vector of $\{b\}$ with respect to $\{a\}$ is represented by $\boldsymbol{\omega}_b^a$.

Unless noted otherwise, in the rest of this report these vectors are represented by default by their coordinates in the origin frame $\{a\}$ to simplify the presentation, e.g. for position

$$^b\mathbf{p}_a = ^a\mathbf{p}_a.$$  \hfill (2.4)

The quaternion describing the rotation from $\{a\}$ to $\{b\}$ is denoted by $q_b^a$. $C(q_a^b)$ is the coordinate change matrix associated to that quaternion, such that

$$^b\mathbf{x} = C(q_a^b)^a\mathbf{x}.$$  \hfill (2.5)
2.2 Reference Frames

The following frames are used in the filter. They are all right-handed.

- World frame \{w\}. It is fixed with respect to the terrain, and assumed to be an inertial frame\(^1\). The \(z\)-axis points upwards along the local vertical, which is assumed to be constant within the area of operations\(^2\).

- IMU frame \{i\}. It is the reference body frame. It shares the same axes as the accelerometers and gyroscopes measurements.

- Camera frame \{c\}. Its origin is at the optical center of the lens. The \(z\)-axis is pointing outwards along the optical axis. The \(x\)-axis is in the sensor plane, pointing right in the image. The \(y\)-axis completes the right-handed frames.

2.3 State Space

2.3.1 States

The state vector \(x\) can be divided between the states related to the IMU \(x_I\), and those related to vision \(x_V\).

\[
x = [x_I^T \ x_V^T]^T \tag{2.6}
\]

The inertial states \(x_I \in \mathbb{R}^{16}\) include the position, velocity and orientation of the IMU frame \(\{i\}\) with respect to the world frame \(\{w\}\), the gyroscope biases \(b_g\) and the accelerometer biases \(b_a\).

\[
x_I = [p_i^w^T \ v_i^w^T \ q_i^w^T \ b_g^T \ b_a^T]^T \tag{2.7}
\]

The vision states \(x_V \in \mathbb{R}^{7M+3N}\) can be divided into two subsets

\[
x_V = [x_S^T \ x_F^T]^T \tag{2.8}
\]

\(^1\)This is equivalent to neglecting the rotation rate of the target celestial body. This assumption is usual for the typical mission duration of quadrotors

\(^2\)This is only valid over short distances.
where \( x_S \in \mathbb{R}^{7M} \) are the sliding window states given by
\[
x_S = \begin{bmatrix} p_{w}^{c_1} T & \cdots & p_{w}^{c_M} T & q_{w}^{c_1} T & \cdots & q_{w}^{c_M} T \end{bmatrix}^T
\] (2.9)
and \( x_F \in \mathbb{R}^{3N} \) are the feature states given by
\[
x_F = \begin{bmatrix} f_1 T & \cdots & f_N T \end{bmatrix}^T.
\] (2.10)

The sliding window states \( x_S \) include the positions \( \{ p_{w}^{c_i} \}_i \) and the orientations \( \{ q_{w}^{c_i} \}_i \) of the camera frame at the last \( M \) image time instances. They are involved in SLAM as well as in MSCKF updates. The feature states \( x_F \) includes the 3D coordinates of \( N \) features \( \{ f_j \}_j \). They are only involved in SLAM updates.

### 2.3.2 Error States

Similarly to the state vector \( x \), the error state vector \( \delta x \) can be broken into inertial and vision error states as
\[
\delta x = \begin{bmatrix} \delta x_I^T & \delta x_V^T \end{bmatrix}^T.
\] (2.11)

The inertial error states are
\[
\delta x_I = \begin{bmatrix} \delta p_w^i T & \delta v_w^i T & \delta q_w^i T & \delta b_g^i T & \delta b_a^i T \end{bmatrix}^T.
\] (2.12)

The position error is defined as the difference between the true and estimated vector \( \delta p_w^i = p^w_i - \hat{p}_w^i \in \mathbb{R}^3 \), and similarly for the velocity and inertial bias errors. The error quaternion \( \delta q \) is defined by \( q = \hat{q} \otimes \delta q \), where \( \otimes \) is the quaternion product. Using the small angle approximation, one can write \( \delta q \simeq [1 \quad 2\delta \theta^T]^T \), thus \( \delta \theta \in \mathbb{R}^3 \) is a correct minimal representation of the error quaternion. Hence, \( \delta x_I \in \mathbb{R}^{15} \) while \( x_I \in \mathbb{R}^{16} \).

Likewise, the vision error states are expressed as
\[
\delta x_V = \begin{bmatrix} \delta x_S^T & \delta x_F^T \end{bmatrix}^T,
\] (2.13)
where
\[
\delta x_S = \begin{bmatrix} \delta p_w^{c_1} T & \cdots & \delta p_w^{c_M} T & \delta \theta_w^{c_1} T & \cdots & \delta \theta_w^{c_M} T \end{bmatrix}^T
\] (2.14)
\[
\delta x_F = \begin{bmatrix} \delta f_1 T & \cdots & \delta f_N T \end{bmatrix}^T.
\] (2.15)

Because of the size reduction in rotation error states, \( \delta x_V \in \mathbb{R}^{6M+3N} \) while \( x_V \in \mathbb{R}^{7M+3N} \).
2.4 Inertial Propagation

This section describes the propagation model, which is identical to Weiss and Siegwart (2012) for the inertial part.

2.4.1 Continuous-Time Model

Position, velocity, and attitude vary according to

\[
\begin{align*}
\dot{p}_w^i &= v_w^i, \\
\dot{v}_w^i &= a_w^i, \\
\dot{q}_w^i &= \frac{1}{2} \Omega (i \omega_w^i) q_w^i.
\end{align*}
\]  

(2.16)

The operator \( \Omega \) is defined by

\[
\Omega(\omega) = \begin{bmatrix} 0 & -\omega^T \\ \omega & -[\omega \times] \end{bmatrix},
\]  

(2.17)

where \( [\omega \times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \).  

(2.18)

The IMU measurements are modeled as

\[
\begin{align*}
\omega_{IMU} &= i \omega_w^i + b_g + n_g, \\
a_{IMU} &= C(q_w^i)(a_w^i - w) + b_a + n_a
\end{align*}
\]  

(2.19, 2.20)

where \( g \) is the local gravity vector, \( n_g \) and \( n_a \) are zero-mean Gaussian white noises. The dynamics of the inertial biases \( b_g \) and \( b_a \) are modeled by random walks, driven by zero-mean Gaussian white noises \( n_{bg} \) and \( n_{ba} \), respectively.

\[
\begin{align*}
\dot{b}_g &= n_{bg} \\
\dot{b}_a &= n_{ba}
\end{align*}
\]  

(2.21)

By injecting Equations (2.19) and (2.20) in (2.16), and including Equations (2.21), we can now formulate the full inertial state propagation model.
Because the vision states refer either camera poses at given time instants, or 3D coordinates of terrain features which are assumed to be static, they have zero dynamics, i.e.

\[ \dot{x}_V = 0. \]  

\subsection*{2.4.2 State Propagation}

The dynamics of the inertial state estimate \( \hat{x}_I \) can be obtained by applying the expectation operator \( E \) to each of the terms of System (2.16)

\[
\begin{align*}
\dot{p}_w^i &= v_w^i \\
\dot{v}_w^i &= C(q_w^i)^T(a_{IMU} - b_a - n_a) + w_g \\
\dot{q}_w^i &= \frac{1}{2} \Omega(\dot{\omega}_{IMU} - b_g - n_g)q_w^i \\
\dot{b}_g &= n_{bg} \\
\dot{b}_a &= n_{ba}
\end{align*}
\]  

(2.22)

These equations are integrated at IMU rate at first order to propagate the inertial state estimation through time (Trawny and Roumeliotis, 2005). This process can be viewed as dead reckoning.

The vision state estimates do not change during inertial propagation

\[ \dot{x}_V = 0. \]  

(2.25)

\subsection*{2.4.3 Covariance Propagation}

System (2.16) can be reformulated with a nonlinear function \( f \) such that

\[ \dot{x}_I = f(x_I, n_{IMU}) \]  

(2.26)

with \( n_{IMU} = [n_a^T \quad n_{ba}^T \quad n_g^T \quad n_{bg}^T]^T \in \mathbb{R}^{12} \) the process noise coming from the IMU measurements. The function \( f \) is time-dependent since depends on inertial measurements that change with time.
\( f \) can be linearized with respect to the estimated inertial state vector \( \hat{x}_I \) so that the dynamics of the error state \( \delta x_I \) becomes

\[
\dot{\delta x}_I = F_d \delta x_I + G_d n_{IMU}.
\] (2.27)

In discrete time, Equation (2.27) can be written as

\[
\delta x_{I,k+1} = F_d \delta x_{I,k} + G_d n_k,
\] (2.28)

where \( n_k \) is a zero-mean white Gaussian noise vector with covariance matrix \( Q_d = \text{diag}(\sigma_{n_a}^2 I_3, \sigma_{n_b}^2 I_3, \sigma_{n_g}^2 I_3) \). We refer the reader to Weiss and Siegwart (2012) for the expressions of \( F_d \) and \( G_d \) implemented in xEKF.

Likewise, from Equation (2.23), the vision error state dynamics is

\[
\dot{\delta x}_V = 0,
\] (2.29)

which can be written in discrete time as

\[
\delta x_{V,k+1} = \delta x_{V,k}.
\] (2.30)

The error covariance matrix can be divided in four blocks

\[
P_{k|k} = \begin{bmatrix} P_{II,k|k} & P_{IV,k|k} \\ P_{IV,k|k}^T & P_{VV,k|k} \end{bmatrix},
\] (2.31)

where \( P_{II,k|k} \in \mathbb{R}^{15 \times 15} \) is the covariance of the inertial error state vector, and \( P_{VV,k|k} \in \mathbb{R}^{(6M+3N) \times (6M+3N)} \) is that of the vision error states. \( P_{II} \) is propagated between \( t_k \) and \( t_{k+1} \) according to

\[
P_{II,k+1|k} = F_d P_{II,k|k} F_d^T + G_d Q_d G_d^T.
\] (2.32)

Covariance block \( P_{IV} \) propagates as

\[
P_{IV,k+1|k} = E[\delta x_{I,k+1} \delta x_{V,k+1}^T],
\] (2.33)

\[
= E[(F_d \delta x_{I,k} + G_d n_k) \delta x_{V,k}^T],
\] (2.34)

\[
= E[F_d \delta x_{I,k} \delta x_{V,k}^T],
\] (2.35)

\[
= F_d E[\delta x_{I,k} \delta x_{V,k}^T],
\] (2.36)

\[
= F_d P_{IV,k|k}.
\] (2.37)

Likewise for the vision error covariance block

\[
P_{VV,k+1|k} = E[\delta x_{V,k+1} \delta x_{V,k+1}^T],
\] (2.38)

\[
= E[\delta x_{V,k} \delta x_{V,k}^T],
\] (2.39)

\[
= P_{VV,k|k}.
\] (2.40)
2.5 Visual Update

This section describes the two paradigms which can be used to construct a filter update from an image feature with unknown 3D coordinates: SLAM and MSCKF. In both, the model for the image measurement $i \mathbf{z}_j \in \mathbb{R}^2$ of feature $\mathbf{p}_j$ observed in camera $\{c_i\}$ is the normalized projection in the plane $c_i z = 1$

$$i \mathbf{z}_j = \frac{1}{c_i z_j} \begin{bmatrix} c_i x_j \\ c_i y_j \\ c_i z_j \end{bmatrix} + i \mathbf{n}_j ,$$

(2.41)

where

$$c_i \mathbf{p}_j = \begin{bmatrix} c_i x_j \\ c_i y_j \\ c_i z_j \end{bmatrix}^T$$

(2.42)

$$= C(q_i^c) (w \mathbf{p}_j - p_i^c)$$

(2.43)

and $i \mathbf{n}_j$ is a zero-mean white Gaussian measurement noise with covariance matrix $i \mathbf{R}_j = \sigma_V^2 \mathbf{I}_2$. The standard deviation $\sigma_V$ of the normalized feature noise $i \mathbf{n}_j$ is function of the visual front-end performance. We assume it is uniform throughout the image, as well as identical in both image dimensions. This defines a nonlinear visual measurement function $\mathbf{h}$ such that

$$i \mathbf{z}_j = \mathbf{h}(p_i^w, q_i^c, w \mathbf{p}_j) + i \mathbf{n}_j .$$

(2.44)

In practice, $i \mathbf{z}_j$ can obtained for any image measurement coming from a camera calibrated for its pinhole model and distortions.

2.5.1 SLAM Update

The SLAM paradigm can provide a filter update from each observation of a visual feature $\mathbf{p}_j$ corresponding to one of the feature states $f_j$.

For a given reference frame $\{r\}$, the cartesian coordinates of $\mathbf{p}_j$ in world frame $\{w\}$ can be expressed as

$$w \mathbf{p}_j = \begin{bmatrix} w x_j \\ w y_j \\ w z_j \end{bmatrix}^T$$

(2.45)

$$= p_r^w + \frac{1}{\rho_j} C(q_r^w) \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix}$$

(2.46)

Before either of these updates is applied, we should note that the innovation is checked for outliers through a $\chi^2$ test with 95% confidence.
Figure 2.1: Geometry of SLAM visual measurement of feature $p_j$ observed in camera $\{c_i\}$, using inverse-depth parametrization with respect to anchor camera $\{c_{i_j}\}$ in the sliding window.

where $f_j = [\alpha_j \beta_j \rho_j]^T$ represents the inverse-depth parameters of $p_j$ with respect to the anchor pose $\{r\}$. The inverse-depth parametrization has been used to represent feature coordinates in SLAM due to its improved depth convergence properties (Civera et al., 2008).

In xVIO, the anchor is one of the sliding-window states $\{c_{i_j}\}$, i.e. the frame defined by the position $p_w^{c_{i_j}}$ and orientation $q_w^{c_{i_j}}$ at index $i_j$ in the sliding window. This defines the geometry of the SLAM measurement of feature $p_j$ in camera $\{c_i\}$ illustrated in Figure 2.1.

By letting $\{r\} = \{c_{i_j}\}$ in Equation (2.46) and injecting it into Equation (2.43), we can write

$$c_ip_j = C(q_w^{c_{i_j}}) \left( p_w^{c_{i_j}} + \frac{1}{\rho_j}C(q_w^{c_{i_j}})^T \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix} - p_w^{c_i} \right)$$

such that the visual measurement model of Equation (2.44) can be linearized as

$$i\delta z_j \simeq H_{p_j} \delta p_{w}^{c_{i_j}} + H_{p_j}^{\theta} \delta p_{w}^{c_i} + H_{\theta_j} \delta \theta_{w}^{c_{i_j}} + H_{\theta_j} \delta \theta_{w}^{c_i} + H_{f_j} \delta f_j + i n_j$$ (2.48)
with

\[ H_{p_{ij}} = \hat{J}_j C(\hat{q}_w^c) \quad (2.49) \]
\[ H_{p_i} = -\hat{J}_j C(\hat{q}_w^c) \quad (2.50) \]
\[ H_{\theta_{ij}} = -\frac{1}{\hat{\rho}_j} \hat{J}_j C(\hat{q}_w^c) C(\hat{q}_w^{c_j}) \left[ \begin{array}{c} \hat{\alpha}_j \\ \hat{\beta}_j \\ 1 \end{array} \right] \times \] (2.51)
\[ H_{\theta_i} = \hat{J}_j \left[ C(\hat{q}_w^c) \left( \hat{p}_w^{c_j} - \hat{p}_w^c + \frac{1}{\hat{\rho}_j} C(\hat{q}_w^{c_j}) \left[ \begin{array}{c} \hat{\alpha}_j \\ \hat{\beta}_j \\ 1 \end{array} \right] \right) \right] \times \] (2.52)
\[ H_{f_{ji}} = \frac{1}{\hat{\rho}_j} \hat{J}_j C(\hat{q}_w^c) C(\hat{q}_w^{c_j}) \left[ \begin{array}{ccc} 1 & 0 & -\frac{\hat{\alpha}_j}{\hat{\rho}_j} \\ 0 & 1 & -\frac{\hat{\beta}_j}{\hat{\rho}_j} \\ 0 & 0 & -\frac{1}{\hat{\rho}_j} \end{array} \right] \] (2.53)

and \[ \hat{J}_j = \frac{1}{\alpha_j} \left[ I_2 - \hat{\mathbf{z}}_j \right]. \] (2.54)

These Jacobians are derived in Section A.1.

Note that when \( i = j \), \( i^j \delta z_j = i^j \delta z_j = \left[ \begin{array}{c} \delta \alpha_j \\ \delta \beta_j \end{array} \right] \). Since SLAM measurements are applied at every frame, they correspond to the most recent pose in the sliding window, so \( i = 1 \).

### 2.5.2 MSCKF Update

MSCKF’s multi-state constraints can provide an update from a feature that has been observed over the last \( m \) images, as illustrated in Figure 2.2. The \( m \) measurements are processed as a batch when the track is lost, or its length exceeds the size of the sliding window. MSCKF was first proposed by Mourikis and Roumeliotis (2007) to reduce the computational cost per feature with respect to SLAM, since the features do not need to be included in the state any more. MSCKF requires the following conditions though:

1. \( m \leq M \): each image measurement must have a corresponding pose state in the sliding window.
2. \( m \geq 2 \): each feature must be triangulated from the prior poses, so it must be visible in at least two images.
3. $\Delta p^c \geq \Delta p^c_{min} > 0$: there must be a minimum camera translation between the images for feature triangulation\(^4\).

Overall, MSCKF vs. SLAM can be seen as a trade-off since the computational cost is still cubic in $M$, the size of the sliding window, in MSCKF; while it is cubic in $N$, the number of feature states in SLAM. Additionally, while MSCKF can process more features for a given computational cost, SLAM can provide a update at image rate on each of these features and does not have to wait until the corresponding tracks ends. Finally, SLAM does not have a minimum translation requirement, which means that it can constrain the state even if the sensor platform is not moving. For these reasons, xVIO uses both in a hybrid fashion, as proposed by Li and Mourikis (2012a).

Let us start by considering one visual measurement where prior estimates are available for the camera position, orientation, and the 3D coordinates of the feature. Using Equation (2.44), we can form the measurement prediction $i\hat{z}_j = h(\hat{p}_w^i, \hat{q}_w^i, w_p)$ . The measurement innovation $i\delta z_j = i z_j - i\hat{z}_j$ can

\(^4\)In presence of noise, $\Delta p^c_{min}$ is tuned for numerical stability.
then be linearized as

\[ i\delta z_j \simeq (i,j)H_p^i \delta p_{w}^i + (i,j)H_\theta^i \delta \theta_{w}^i + iH_p^i w \delta p_j + i n_j , \]  

(2.55)

with 

\[ (i,j)H_p = -iJ_j C(\hat{q}_w^i) , \]  

(2.56)

\[ (i,j)H_\theta = iJ_j C(\vec{p}_j - \hat{p}_w^i) \times ] , \]  

(2.57)

\[ iH_p^i = iJ_j C(\hat{q}_w^i) , \]  

(2.58)

and 

\[ iJ_j = \frac{1}{c_i \hat{z}_j} [I_2 - \hat{z}_j] . \]  

(2.59)

If \( \delta z_j \) is a MSCKF measurement, by definition \( \delta p_{w}^i \) and \( \delta \theta_{w}^i \) are part of the error state vector \( \delta x \) defined in Equation (2.11), but not \( w \delta p_j \). Thus Equation (2.55) can be written

\[ i\delta z_j \simeq iH_j \delta x + iH_p^i w \delta p_j + i n_j , \]  

(2.60)

with

\[ iH_j = [0_{2 \times 15} \ldots (i,j)H_p \ldots (i,j)H_\theta \ldots 0_{2 \times 3N}] . \]  

(2.61)

By stacking up the residuals \( i\delta z_j \) for each of the \( m_j \) observations of feature \( p_j \) \((m_j \leq M)\), one can form the overall residual \( \delta z_j \in \mathbb{R}^{2m_j} \)

\[ \delta z_j \simeq H_j \delta x + H_p^i w \delta p_j + n_j , \]  

(2.62)

where \( n_j \in \mathbb{R}^{2m_j \times 2m_j} \) has covariance \( R_j = \sigma_r^2 I_{2m_j} \).

Equation (2.62) cannot be applied as an EKF update because the \( w \delta p_j \) term can neither be formulated as a linear combination of the states, nor integrated in the noise. To correct for that, we can multiply on each side by the left nullspace of \( H_p \), denoted as \( A_j \), so that

\[ \delta z_{0j} = A_j^T \delta z_j \simeq A_j^T H_j \delta x + A_j^T n_j = H_{0j} \delta x + n_{0j} . \]  

(2.63)

From Equation (2.58), one can verify that \( H_p \in \mathbb{R}^{2m_j \times 3} \) has full column rank. Per the rank-nullity theorem, its left nullspace has dimension \( 2m_j - 3 \), thus \( A_j \in \mathbb{R}^{(2m_j-3) \times 2m_j} \) and \( \delta z_{0j} \in \mathbb{R}^{2m_j-3} \).

Note that to perform this update, an estimate of \( w p_j \) must be known beforehand and is triangulated\(^5\) using the sliding window state priors as described in Mourikis and Roumeliotis (2007). We also apply the QR decomposition of matrix \( H_0 \), resulting from stacking up all the MSCKF measurements per feature of Equation (2.63) together as described in Mourikis and Roumeliotis (2007).

\(^5\)This triangulation is the origin of MSCKF requirements 2 and 3, presented at the beginning of this subsection.
2.6 Range-Visual Update

Terrain range measurement models which are used alongside visual measurements typically assumed the ground is globally-flat (Bayard et al., 2019). Instead, we assume that the terrain is only locally-flat between three SLAM features surrounding the terrain point with respect to which the range is measured. We also assume zero translation between the optical center of the camera and the origin of the Laser Range Finder (LRF). Figure 2.3 illustrates these assumptions.

\( u_r \) is the unit vector oriented along the optical axis of the LRF. \( p_I \) is the intersection of this axis with the terrain. \( p_{j1}, p_{j2} \) and \( p_{j3} \) are SLAM features forming a triangle around \( p_I \) in image space. \( n \) is a normal vector to the plane containing \( p_{j1}, p_{j2}, p_{j3} \) and \( p_I \). If \( w u_{ri}^T w n \neq 0 \), we can express the

\footnote{This assumption can be removed by measuring the position of the LRF in camera frame.}
range measurement at time $i$ as

$$\begin{align*}
i_zr &= \frac{w_{u_i}^T w n}{w_{u_i}^T w n} \\
&= \frac{(p'_w - p^c_i)^T w n}{w_{u_i}^T w n} \\
&= \frac{(p'_w - p^{f_{j2}}_w + p^{f_{j2}}_w - p^c_w)^T w n}{w_{u_i}^T w n}
\end{align*}$$

(2.64)

where

$$w n = (p^{f_{j1}}_w - p^{f_{j2}}_w) \times (p^{f_{j3}}_w - p^{f_{j2}}_w).$$

(2.65)

(2.66)

Since $(p'_w - p^{f_{j2}}_w)^T w n = 0$, that means

$$\begin{align*}
i_zr &= \frac{(p^{f_{j2}}_w - p^c_i)^T w n}{w_{u_i}^T w n}
\end{align*}$$

(2.67)

(2.68)

Note that $w n$ is not a unit vector in general, and a Mahalanobis distance test is performed at the filter level to detect range outliers.

Equation (2.66) also shows the range is a nonlinear function of the state

$$\begin{align*}
i_zr &= h_r(x) + n_r
\end{align*}$$

(2.69)

which can be linearized to update an EKF as

$$\begin{align*}
i\delta z_r &\approx H_{p'_1} \delta p^c_{w1} + H_{p'_2} \delta p^c_{w2} + H_{p'_3} \delta p^c_{w3} + H p_{i} \delta p^c_{wj} \\
&+ H_{\theta'_1} \delta \theta^c_{w1} + H_{\theta'_2} \delta \theta^c_{w2} + H_{\theta'_3} \delta \theta^c_{w3} + H \theta_i \delta \theta^c_{wj} \\
&+ H_{f_{j1}} \delta f_{j1} + H_{f_{j2}} \delta f_{j2} + H_{f_{j3}} \delta f_{j3} \\
&+ n_r
\end{align*}$$

(2.70)
where

\[ H_{p_1} = H_{p_{j1}} \]  
\[ H_{p_2} = H_{p_{j2}} \]  
\[ H_{p_3} = H_{p_{j3}} \]  
\[ H_p = -\frac{1}{b} \hat{n}^T \]  
\[ H_{\theta_1} = -\frac{1}{\rho_{j1}} H_{p_{j1}} C(\hat{q}_w^{c_{ij1}})^T \begin{bmatrix} \hat{\alpha}_{j1} \\ \hat{\beta}_{j1} \\ 1 \end{bmatrix} \times \]  
\[ H_{\theta_2} = -\frac{1}{\rho_{j2}} H_{p_{j2}} C(\hat{q}_w^{c_{ij2}})^T \begin{bmatrix} \hat{\alpha}_{j2} \\ \hat{\beta}_{j2} \\ 1 \end{bmatrix} \times \]  
\[ H_{\theta_3} = -\frac{1}{\rho_{j3}} H_{p_{j3}} C(\hat{q}_w^{c_{ij3}})^T \begin{bmatrix} \hat{\alpha}_{j3} \\ \hat{\beta}_{j3} \\ 1 \end{bmatrix} \times \]  
\[ H_{\theta} = -\frac{\hat{\alpha}}{b^2} ([\hat{c}_u \times] C(\hat{q}_w^{c_i}) \hat{n})^T \]  
\[ H_{f_{j1}} = \frac{1}{\rho_{j1}} H_{p_{j1}} C(\hat{q}_w^{c_{ij1}})^T \begin{bmatrix} 1 & 0 & -\alpha_{j1}^{c_{ij1}} \\ 0 & 1 & -\beta_{j1}^{c_{ij1}} \\ 0 & 0 & -1 \end{bmatrix} \]  
\[ H_{f_{j2}} = \frac{1}{\rho_{j2}} H_{p_{j2}} C(\hat{q}_w^{c_{ij2}})^T \begin{bmatrix} 1 & 0 & -\alpha_{j2}^{c_{ij2}} \\ 0 & 1 & -\beta_{j2}^{c_{ij2}} \\ 0 & 0 & -1 \end{bmatrix} \]  
\[ H_{f_{j3}} = \frac{1}{\rho_{j3}} H_{p_{j3}} C(\hat{q}_w^{c_{ij3}})^T \begin{bmatrix} 1 & 0 & -\alpha_{j3}^{c_{ij3}} \\ 0 & 1 & -\beta_{j3}^{c_{ij3}} \\ 0 & 0 & -1 \end{bmatrix} \]
and

\[ H_{p_{j1}} = \frac{1}{b} \left( \left[ (\hat{p}_{w}^{f_{j1}} - \hat{p}_{w}^{f_{j2}}) \times (\hat{x}_{w}^{f_{j2}} - \hat{x}_{w}^{f_{j1}}) \right] \right)^T \] (2.82)

\[ H_{p_{j2}} = \frac{1}{b} \left( w \hat{n} + \left[ (\hat{p}_{w}^{f_{j1}} - \hat{p}_{w}^{f_{j3}}) \times (\hat{x}_{w}^{f_{j3}} - \hat{x}_{w}^{f_{j1}}) \right] \right)^T \] (2.83)

\[ H_{p_{j3}} = \frac{1}{b} \left( \left[ (\hat{p}_{w}^{f_{j2}} - \hat{p}_{w}^{f_{j1}}) \times (\hat{x}_{w}^{f_{j1}} - \hat{x}_{w}^{f_{j2}}) \right] \right)^T \] (2.84)

\[ a = (p_{w}^{f_{j2}} - p_{w}^{f_{j1}})^T n \] (2.85)

\[ b = w u_{r_i}^T n \] (2.86)

To construct the range update in practice, we perform a Delaunay triangulation in image space over the SLAM features, and select the triangle in which the intersection of the LRF beam with the scene is located. We opted for the Delaunay triangulation since it maximizes the smallest angle of all possible triangulations (Gärtner and Hoffmann, 2013). This property avoids “long and skinny” triangles that do not provide strong local planar constraints.

Figure 2.4 shows the Delaunay triangulation, and the triangle selected as a ranged facet, over a sample image from our outdoor test sequence. It also illustrates the partitioning of the scene into triangular facets, with SLAM features at their corners. Note that that if the state estimator uses only 3 SLAM features in a lightweight fashion, then this is equivalent to a globally-flat world assumption. Conversely, when the density of SLAM features increases in the image, the limit of the area of the facets tends to zero and the scene assumption virtually disappears.

### 2.7 State Management

The inertial state vector \( x_I \) and its error covariance matrix \( P_{II} \) are initialized from values provided by the user. After that, they both follow the propagation-update cycle common to all Kalman filters. The management of the vision state vector \( x_V \) and the rest of the error covariance matrix is less straightforward though, and is the focus of this section.
Figure 2.4: Delaunay triangulation between image features tracked in the outdoor flight dataset. The features used as filter states for SLAM form the corners of the triangles. The red dot represents the intersection point of the LRF beam with the scene. The surrounding red triangle is the ranged facet.

2.7.1 Sliding Window States

The $M$ poses in the sliding window states $x_S$ are managed in a first-in first-out fashion at image rate. Each time an image is acquired with the camera, the following sliding window cycle happens in $x_S$:

1. the oldest pose is removed;
2. the remaining poses are shifted one slot, making the first slot empty;
3. the current camera pose is inserted in the first slot.
Equations (2.87) and (2.88) illustrate this sliding window process between two successive images $l$ and $l+1$

$$x_{S_l} = \begin{bmatrix} p_{l+1}^T & \ldots & p_{M+1}^T \\ q_{l+1}^T & \ldots & q_{M+1}^T \end{bmatrix}^T$$ (2.87)

$$x_{S_{l+1}} = \begin{bmatrix} p_{l+2}^T & \ldots & p_{M+2}^T \\ q_{l+2}^T & \ldots & q_{M+2}^T \end{bmatrix}^T$$ (2.88)

The estimates for the newest camera position $q_{l+1}^w$ and orientation $q_{l+1}^w$ in the sliding window is computed using the pose of the camera with respect to the IMU\footnote{The pose of the camera with respect to the IMU is assumed to be known through calibration.}\{i^c, q_i^c\} and the current pose estimate of the IMU at time $t_{l+1}$\{$\hat{p}_w^i(t_{l+1}), \hat{q}_w^i(t_{l+1})$\}.

$$\hat{p}_{l+1}^w = \hat{p}_w^i(t_{l+1}) + C(q_i^c(t_{l+1}))^T p_i^c$$ (2.89)

$$\hat{q}_{l+1}^w = \hat{q}_w^i(t_{l+1}) \otimes q_i^c$$ (2.90)

The error covariance matrix also needs to be modified to reflect the sliding window changes. To achieve that, we can form the new error states in the sliding window (Delaune, 2013),

$$\delta p_{l+1}^w = \delta p_w^i(t_{l+1}) - C(q_i^c(t_{l+1}))^T p_i^c$$ (2.91)

$$\delta \theta_{l+1}^w = C(q_i^c) \delta \theta_w^i(t_{l+1})$$ (2.92)

which can be written as

$$\delta p_{w_{l+1}} = \begin{bmatrix} I_3 & 0_3 & -C(q_i^c)^T [p_i^c \times] \end{bmatrix} \delta \mathbf{x}(t_{l+1})$$ (2.93)

$$\delta \theta_{w_{l+1}} = \begin{bmatrix} 0_{3 \times 6} & C(q_i^c) \end{bmatrix} \delta \mathbf{x}(t_{l+1})$$. (2.94)

Thus we can write the structure change of the error state at time $t_{l+1}$ as

$$\delta \mathbf{x}(t_{l+1}) \leftarrow J \delta \mathbf{x}(t_{l+1})$$ (2.95)
with

\[
J = \begin{bmatrix}
I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\
I_3 & 0 & -C(\hat{q}^i_w)^T [p^f_i \times] & 0 & 0 & 0_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{3(M-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & C(q^f_i) & 0 & 0 & 0 & 0 & 0_3 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{3(M-1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{3N_\theta}
\end{bmatrix}
\]

(2.96)

Since \( P(t_{t+1}) = E[\delta x(t_{t+1}) \delta x(t_{t+1})^T] \), we modify the error covariance matrix according to

\[
P(t_{t+1}) \leftarrow J P(t_{t+1}) J^T
\]

(2.97)

2.7.2 Feature States

A feature state can be initialized in two ways:

- from a MSCKF measurement, which means it has already been observed \( m \) times and a depth prior is available;
- from its first observation, which means its depth is unknown.

Li and Mourikis (2012a) show that it is most efficient to initialize a feature after a MSCKF update. However, it may not be always possible to do so because of the constraints listed at the beginning of Section 2.5.2, for instance if the sensor platform is not moving in translation.

**Initialization from MSCKF feature**

Li and Mourikis (2012b) proposed a method to initialize the state and error covariance of a SLAM feature \( p^j \) from its MSCKF measurements. We detail the equations below for completeness but the reader should refer to their paper for the full demonstration.

Before a MSCKF update happens, the state vector can be augmented with the feature coordinates \( f^j \) triangulated from the pose priors in MSCKF, and
the covariance matrix with infinite uncertainty on these coordinates.

\[
\hat{x}_{aug_{k+1}|k} \leftarrow \begin{bmatrix} \hat{x}_{k+1|k} \\ \hat{f}_j \end{bmatrix}
\]

(2.98)

\[
P_{aug_{k+1}|k} \leftarrow \begin{bmatrix} P_{k+1|k} & 0 \\ 0 & \mu I \end{bmatrix}
\]

(2.99)

with \( \mu \to \infty \).

Equation (2.62) can be rewritten as

\[
\delta z_j \approx \begin{bmatrix} H_j & H_{f_j} \end{bmatrix} \delta x_{aug_{k+1}|k} + n_j.
\]

(2.100)

and left-multiplied by the full-rank matrix \( W = [A \ B] \) where \( A \) is the left nullspace of \( H_{f_j} \) and \( B \) its column space, such that

\[
\delta z_{cj} = W^T \delta z_j
\]

(2.101)

\[
\delta z_{cj} = \begin{bmatrix} A^T \delta z_j \\ B^T \delta z_j \end{bmatrix}
\]

(2.102)

\[
\approx \begin{bmatrix} A^T H_j & A^T H_{f_j} \\ B^T H_j & B^T H_{f_j} \end{bmatrix} \delta x_{aug_{k+1}|k} + W^T n_j
\]

(2.103)

\[
\approx \begin{bmatrix} H_{0j} & 0 \\ H_{1j} & H_{2j} \end{bmatrix} \delta x_{aug_{k+1}|k} + W^T n_j
\]

(2.104)

Li and Mourikis (2012b) show that the state vector and error covariance matrix resulting from this update are

\[
\hat{x}_{aug_{k+1|k+1}} = \begin{bmatrix} \hat{x}_{k+1|k} + \Delta x_{k+1} \\ \hat{f}_j - H_{2j}^{-1} H_{1j} \Delta x_{k+1} + H_{2j}^{-1} B^T \delta z_j \end{bmatrix}
\]

(2.105)

\[
P_{aug_{k+1|k+1}} = \begin{bmatrix} P_{k+1|k+1} & -(H_{2j}^{-1} H_{1j} P_{k+1|k+1})^T \\ -(H_{2j}^{-1} H_{1j} P_{k+1|k+1}) & P_{22k+1|k+1} \end{bmatrix}
\]

(2.106)

where \( \Delta x_{k+1} \) and \( P_{k+1|k+1} \) are respectively the state correction and error covariance matrix resulting from the standard MSCKF update and

\[
P_{22k+1|k+1} = H_{2j}^{-1} H_{1j} P_{k+1|k+1} (H_{2j}^{-1} H_{1j})^T + \sigma_v^2 H_{2j}^{-1} H_{2j}^T
\]

(2.107)

The above process can be applied independently for each feature to initialize, or as batch like in xVIO by stacking the matrices \( H_{s_j} \).
Unknown-Depth Initialization

When MSCKF cannot be applied, we use Montiel et al. (2006) to initialize a SLAM feature \( p_j \) from its first observation. The 95% acceptance region for the depth is assumed to be between \( d_{\text{min}} \) and \( \infty \), which provides the initial inverse-depth estimate \( \hat{\rho}_0 = \frac{1}{2d_{\text{min}}} \) and uncertainty \( \sigma_{\rho_0} = \frac{1}{4d_{\text{min}}} \). The anchor for the inverse-depth parametrization of the new feature is the camera frame \( \{c_i\} \) where its first observation took place, such that \( i_j z_j = [\hat{\alpha}_0, \hat{\beta}_0]^T \).

The state estimate and error covariance matrix are augmented with this new SLAM feature as

\[
\hat{x}_{\text{aug}k|k} \leftarrow \begin{bmatrix}
\hat{x}_{k|k} \\
\hat{\alpha}_0_j \\
\hat{\beta}_0_j \\
\hat{\rho}_0_j
\end{bmatrix}.
\]

\[
P_{\text{aug}k|k} \leftarrow \begin{bmatrix}
P_{k|k} & 0 & 0 \\
0 & \sigma^2_{\rho_0} I_2 & 0 \\
0 & 0 & \sigma^2_{\rho_0}
\end{bmatrix}.
\]

2.7.3 Feature States Reparametrization

The inverse-depth parametrization \( f_{j_1} = [\alpha_{j_1}, \beta_{j_1}, \rho_{j_1}]^T \) of a SLAM feature \( p_j \) depends on its anchor pose \( \{c_{i_1}\} \). If a different anchor \( \{c_{i_2}\} \) is considered, the relationship

\[
w_p = p_{c_{i_1}} + \frac{1}{\rho_{j_1}} C(q^{c_{i_2}}) \begin{bmatrix}
\alpha_{j_1} \\
\beta_{j_1} \\
1
\end{bmatrix} = p_{c_{i_2}} + \frac{1}{\rho_{j_2}} C(q^{c_{i_2}}) \begin{bmatrix}
\alpha_{j_2} \\
\beta_{j_2} \\
1
\end{bmatrix}
\]

provides the conversion equations

\[
\frac{1}{\rho_{j_2}} \begin{bmatrix}
\alpha_{j_2} \\
\beta_{j_2} \\
1
\end{bmatrix} = C(q^{c_{i_2}}) \left( -p_{c_{i_2}} + p_{c_{i_1}} + \frac{1}{\rho_{j_1}} C(q^{c_{i_1}}) \begin{bmatrix}
\alpha_{j_1} \\
\beta_{j_1} \\
1
\end{bmatrix} \right). \tag{2.111}
\]

In xVIO, the anchor is one of the sliding window poses. When this anchor is about to leave the window, that feature state estimate \( \hat{f}_j = [\hat{\alpha}_j, \hat{\beta}_j, \hat{\rho}_j]^T \) is reparametrized with the most recent pose in the sliding window as an anchor. This is done using Equation (2.111). Likewise, the rows and
columns associated to that feature in the error covariance matrix are modified using

\[ P_{k|k} \leftarrow J_j P_{k|k} J_j^T \quad (2.112) \]

where

\[ J_j = \begin{bmatrix} I_{15+6M+3(j-1)} & 0 \\ J_{1j} & H_{1j} \\ 0 & I_{3(N-j)} \end{bmatrix}, \quad (2.113) \]

with

\[ J_{1j} = \hat{\rho}_{j2} \begin{bmatrix} 1 & 0 & \hat{\alpha}_{j2} \\ 0 & 1 & \hat{\beta}_{j2} \\ 0 & 0 & -\hat{\rho}_{j2} \end{bmatrix} \quad (2.114) \]

and

\[ H_{1j} = \begin{bmatrix} 0_{3 \times 15} & jH_{p1} & 0_{3 \times (M-6)} & jH_{pM} & jH_{\theta1} & jH_{\theta1} & 0_{3 \times (j-1)} & H_{fj} & 0_{3 \times (N-j)} \end{bmatrix}. \quad (2.115) \]

Section A.4 demonstrates the following expressions for the Jacobian matrices:

\[ jH_{p1} = C(\hat{q}_{w}^{CM}) \quad (2.116) \]

\[ jH_{pM} = -C(\hat{q}_{w}^{CM}) \quad (2.117) \]

\[ jH_{\theta1} = -\frac{1}{\hat{\rho}_{j1}} C(\hat{q}_{w}^{CM}) C(\hat{q}_{w}^{cl})^T \begin{bmatrix} \hat{\alpha}_{j1} \\ \hat{\beta}_{j1} \\ 1 \end{bmatrix} \times \quad (2.118) \]

\[ jH_{\theta1} = \begin{bmatrix} C(\hat{q}_{w}^{CM}) \left( -\hat{p}_{w}^{CM} + \hat{p}_{w}^{cl} + \frac{1}{\hat{\rho}_{j1}} C(\hat{q}_{w}^{cl})^T \begin{bmatrix} \hat{\alpha}_{j1} \\ \hat{\beta}_{j1} \\ 1 \end{bmatrix} \right) \times \end{bmatrix} \quad (2.119) \]

\[ H_{fj} = \frac{1}{\hat{\rho}_{j1}} C(\hat{q}_{w}^{CM}) C(\hat{q}_{w}^{cl})^T \begin{bmatrix} 1 & 0 & -\hat{\alpha}_{j1} \\ 0 & 1 & -\hat{\beta}_{j1} \\ 0 & 0 & -\hat{\rho}_{j1} \end{bmatrix} \quad (2.120) \]
Chapter 3

Observability Analysis

The observability analysis of the linearized range-VIO system is necessary since our system is based on an EKF. We note that the observability analysis of the nonlinear system would also be required for completeness but is out of the scope of this paper.

To simplify the equations, our analysis assumes a state vector $\mathbf{x}^0 = [x_I^T \ x_P^T]^T$, where $x_I$ was defined in Equation (2.7) and $x_P = [w_{P1}^T \ldots w_{PN}^T]^T$ includes the cartesian coordinates of the $N$ SLAM features, $N \geq 3$. Li and Mourikis (2013) proved that observability analysis for the linearized system based on $\mathbf{x}^0$ is equivalent to observability analysis for the linearized system defined with $\mathbf{x}$ in the previous chapter.

3.1 Observability Matrix

For $k \geq 1$, $M_k = H_k \Phi_{k,1}$ is the $k$-th block row of observability matrix $M$. $H_k$ is the Jacobian of the range measurement in Equation (2.68) at time $k$ with respect to $\mathbf{x}^0$, which is derived in Equations (A.75-A.80). $\Phi_{k,1}$ is the state transition matrix from time 1 to time $k$ (Hesch et al., 2012).

Without loss of generality, we can assume the ranged facet is constructed from the first 3 features in $\mathbf{x}_P$. Then we derive the following expression for $M_k$ in Appendix B.1.

$$M_k = \frac{1}{b} \begin{bmatrix} M_{k,p} & M_{k,v} & M_{k,q} & M_{k,b_g} & M_{k,b_a} \\ M_{k,p_1} & M_{k,p_2} & M_{k,p_3} & 0_{1 \times 3(N-3)} \end{bmatrix}$$

(3.1)
where

\[ M_{k,p} = -^w n^T \]  \hspace{1cm} (3.2)
\[ M_{k,v} = -(k - 1) \delta t^w n^T \]  \hspace{1cm} (3.3)
\[ M_{k,\theta} = ^w n \left( -\frac{a}{b} C (q_w^c)^T \left[ ^c u_r \times \right] C (q_w^i) \right. \]
\[ \left. - \left[ p_w^i \right. - v_w^i (k - 1) \delta t - \frac{1}{2} g (k - 1)^2 \delta t^2 \right. \]
\[ \left. - p_w^k \times \right] C (q_w^i) \right) \]  \hspace{1cm} (3.4)
\[ M_{k,b} = -\frac{a}{b} n^T C (q_w^c)^T \left[ ^c u_r \times \right] \phi_{12} - ^w n \phi_{52} \]  \hspace{1cm} (3.5)
\[ M_{k,a} = -^w n^T \phi_{54} \]  \hspace{1cm} (3.6)
\[ M_{k,p1} = \left( (p_w^F_3 - p_w^F_2) \times \right) \left( p_w^F_2 - p_w^I \right)^T \]  \hspace{1cm} (3.7)
\[ M_{k,p2} = \left( ^w n + \left[ (p_w^F_1 - p_w^F_2) \times \right] \left( p_w^F_2 - p_w^I \right) \right)^T \]  \hspace{1cm} (3.8)
\[ M_{k,p3} = \left[ (p_w^F_2 - p_w^F_1) \times \right] \left( p_w^F_2 - p_w^I \right)^T \]  \hspace{1cm} (3.9)

and \( a \) and \( b \) are defined in Equations (A.37) and (A.38), respectively. \( \phi_* \) are integral terms defined in Hesch et al. (2012).

### 3.2 Unobservable Directions

One can verify that the vectors spanning a global position or a rotation about the gravity vector still belong to the right nullspace of \( M_k \). Thus, the ranged facet update does not improve the observability over VIO under generic motion Li and Mourikis (2013), which is intuitive. Likewise, in the absence of rotation, the global orientation is still not observable (Wu and Roumeliotis, 2016).

In the constant acceleration case though, unlike VIO (Wu and Roumeliotis, 2016), one can demonstrate that the vector

\[
N_s = \begin{bmatrix}
p_w^i & v_w^i & 0_{6 \times 1} & -^i a_w^T & p_w^F_1 & \ldots & p_w^F_N \end{bmatrix}^T
\]  \hspace{1cm} (3.10)
which spans the scale dimension, does not belong the right nullspace of $M_k$. The demonstration is provided in Appendix B.2. $i^a_{w}$ is the constant acceleration of the IMU frame in world frame, resolved in the IMU frame. Unlike VIO, range-VIO thus enables scale convergence even in the absence of acceleration excitations. Note that when the velocity is null, i.e. in hover, the following unobservable direction appears instead, as discussed in Appendix B.2.

$$N_h = \begin{bmatrix} 0_{24}^T & p_{w}^{F_1} & \cdots & p_{w}^{F_N} \end{bmatrix}^T$$

(3.11)

It corresponds to the depth of the SLAM features not included in the facet. This result means that in the absence of translation, i.e. when feature depths are unknown and uncorrelated from each other, the ranged facet provides no constrain on the features outside the facet. As soon as the platform starts moving, visual measurements begin to correlated all feature depths, and the depths of all features become observable from a single ranged facet.
Appendices
Appendix A

Jacobians Derivations

In this chapter, some expressions from Chapter 2 are demonstrated. We focus on the expressions which are not already demonstrated in the literature. For the others, a reference was provided next to the expression.

In this section, since all derivatives are evaluated using the available state estimates we simplify the notation $\frac{\partial \hat{a}}{\partial \hat{b}} \equiv \frac{\partial a}{\partial b}|_{\hat{x}=\hat{x}}$.

A.1 SLAM Measurement

Equations (2.49)–(2.53) provide the expressions of the measurement Jacobians $H_{p_{ij}}$, $H_{p_{i}}$, $H_{\theta_{ij}}$, $H_{\theta_{i}}$, $H_{f_{j}}$ for SLAM feature $p_{j}$ observed in camera $\{c_{i}\}$ with respect to the anchor camera position $p_{w}^{c_{ij}}$, the current camera position $p_{w}^{c_{i}}$, the anchor camera orientation $q_{w}^{c_{ij}}$, the current camera orientation $q_{w}^{c_{i}}$ and the feature coordinates $f_{j}$, respectively. Before proceeding to their derivation, we first recall the normalized vision measurement model from Equation (2.41)

$$iz_{j} = \frac{1}{c_{i}z_{j}} \begin{bmatrix} c_{i}x_{j} \\ c_{i}y_{j} \end{bmatrix}. \tag{A.1}$$

At first order, using small angle approximation about $\hat{q}_{w}^{c_{i}}$, one can write

$$C(q_{w}^{c_{i}}) = C(\hat{q}_{w}^{c_{i}} \otimes \delta q_{w}^{c_{i}}) \tag{A.2}$$

$$= C(\delta q_{w}^{c_{i}})C(\hat{q}_{w}^{c_{i}}) \tag{A.3}$$

$$\simeq (I_{3} - [\delta \theta_{w}^{c_{i}} \times])C(\hat{q}_{w}^{c_{i}}). \tag{A.4}$$
Similarly about $\hat{q}^{c_i^j}_w$,

$$C(\hat{q}^{c_i^j}_w) \simeq (I_3 - [\delta \theta_w^{c_i^j} \times ])C(\hat{q}^{c_i^j}_w).$$  \hspace{1cm} (A.5)

Hence at first order, we can rewrite Equation (2.47) about $\hat{x}$ as

$$\begin{align*}
(I_3 - [\delta \theta_w^{c_i^j} \times ])C(\hat{q}^{c_i^j}_w) \left( p^{c_i^j}_w \right) \\
\approx \left( \frac{1}{\rho_j} \left( (I_3 - [\delta \theta_w^{c_i^j} \times ])C(\hat{q}^{c_i^j}_w) \right)^T \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix} - p^{c_i^j}_w \right) \left( p^{c_i^j}_w \right).
\end{align*}$$  \hspace{1cm} (A.6)

which will be useful in the rest of this section.

A.1.1 Jacobian Prefix

The derivation of the Jacobian matrices in this section is made easier if we use the derivative chain rule to express each Jacobian with respect to $c_i^j p_j = [c_i x_j \\ c_i y_j \\ c_i z_j]^T$ first. For instance,

$$\frac{\partial^z z_j}{\partial p^{c_i^j}_w} = \frac{\partial^z z_j}{\partial c_i^j p_j} \frac{\partial c_i^j p_j}{\partial p^{c_i^j}_w},$$  \hspace{1cm} (A.8)

where

$$\frac{\partial^z z_j}{\partial c_i^j p_j} = \begin{bmatrix} \frac{\partial (c_i x_j)}{\partial (c_i x_j)} & \frac{\partial (c_i x_j)}{\partial (c_i y_j)} & \frac{\partial (c_i x_j)}{\partial (c_i z_j)} \\ \frac{\partial (c_i y_j)}{\partial (c_i x_j)} & \frac{\partial (c_i y_j)}{\partial (c_i y_j)} & \frac{\partial (c_i y_j)}{\partial (c_i z_j)} \\ \frac{\partial (c_i z_j)}{\partial (c_i x_j)} & \frac{\partial (c_i z_j)}{\partial (c_i y_j)} & \frac{\partial (c_i z_j)}{\partial (c_i z_j)} \end{bmatrix}$$  \hspace{1cm} (A.9)

$$= \begin{bmatrix} 1 & 0 & -\frac{c_i \dot{z}_j}{c_i z_j} \\ 0 & 1 & -\frac{c_i \dot{y}_j}{c_i y_j} \\ -\frac{c_i \dot{x}_j}{c_i x_j} & -\frac{c_i \dot{y}_j}{c_i y_j} & 1 \end{bmatrix}$$  \hspace{1cm} (A.10)

$$= \frac{1}{c_i \dot{z}_j} \begin{bmatrix} I_2 & -i \dot{z}_j \end{bmatrix}$$  \hspace{1cm} (A.11)

$$= i J_j.$$  \hspace{1cm} (A.12)

33
A.1.2 Anchor Camera Position

By injecting Equation (A.12) in Equation (A.8), and then using Equation (A.7), we get

\[
\frac{\partial i z_j}{\partial p_{w_i}} = i J_j \frac{\partial c_i p_j}{\partial p_{w_i}}
\]
\[
= i J_j C(\hat{q}_i)
\]
\[
= H_{p_i}.
\]

(A.12)  

A.1.3 Current Camera Position

\[
\frac{\partial i z_j}{\partial p_{w_i}} = i J_j \frac{\partial c_i p_j}{\partial p_{w_i}}
\]
\[
= -i J_j C(\hat{q}_i)
\]
\[
= H_{p_i}.
\]

(A.16)  

A.1.4 Anchor Camera Orientation

\[
\frac{\partial i z_j}{\partial \delta \theta_{w_i}} = i J_j \frac{\partial c_i p_j}{\partial \delta \theta_{w_i}}
\]
\[
\approx i J_j \frac{\partial}{\partial \delta \theta_{w_i}} \left( \frac{1}{\rho_j} (I_3 - [\delta \theta_w \times ] \right)
\]
\[
\approx C(\hat{q}_w)C(\hat{q}_w)^T [\delta \theta_w \times ] \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix}
\]

(A.19)  

\[
\approx -\frac{1}{\rho_j} (I_3 - [\delta \theta_w \times ]
\]
\[
\approx C(\hat{q}_w)C(\hat{q}_w)^T \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix} \times 
\]

(A.20)  

\[
\approx -\frac{1}{\rho_j} C(\hat{q}_w)C(\hat{q}_w)^T \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix}
\]
\[
\approx H_{\theta_{ij}}
\]

(A.21)  

(A.22)  

(A.23)
A.1.5 Current Camera Orientation

\[
\frac{\partial^i z_j}{\partial \delta \theta^c_w} = i J_j \frac{\partial p_j}{\partial \delta \theta^c_w} \quad (A.24)
\]

\[
i J_j \frac{\partial}{\partial \delta \theta^c_w} \left( - [\delta \theta^c_w \times] C(\hat{q}^c_w) \left( p^c_i \right) \right)
\sim
\frac{1}{\rho_j} C(\hat{q}^c_i)^T (I_3 + [\delta \theta^c_w \times]) \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix} - p^c_i \quad (A.25)
\]

\[
i J_j \frac{\partial}{\partial \delta \theta^c_w} \left( C(\hat{q}^c_i) \left( p^c_i \right) \right)
\sim
\frac{1}{\rho_j} C(\hat{q}^c_i)^T (I_3 + [\delta \theta^c_w \times]) \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix} - p^c_i \times \delta \theta^c_w \quad (A.26)
\]

\[
i J_j \left| C(\hat{q}^c_w) \left( \hat{p}^c_i - \hat{p}^c_i + \frac{1}{\hat{\rho}_j} C(\hat{q}^c_w)^T \begin{bmatrix} \hat{\alpha}_j \\ \hat{\beta}_j \\ 1 \end{bmatrix} \right) \times \delta \theta^c_w \right| \quad (A.27)
\]

\[
\approx H_{\theta_i} \quad (A.28)
\]

A.1.6 Feature Coordinates

To demonstrate the expression for \( H_{f_j} \), we first compute the Jacobians for each of its components.
\[
\frac{\partial^i z_j}{\partial \alpha_j} = i J_j \frac{\partial^i p_j}{\partial \alpha_j} \\
= i J_j \frac{\partial}{\partial \alpha_j} \left( C(q_w^c) \frac{1}{\hat{\rho}_j} C(q_w^{c_j})^T \begin{bmatrix} \alpha_j \\ \beta_j \\ 1 \end{bmatrix} \right) \\
= \frac{1}{\hat{\rho}_j} i J_j C(\hat{q}_w^c) C(\hat{q}_w^{c_j})^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{A.30}
\]

\[
\frac{\partial^i z_j}{\partial \beta_j} = \frac{1}{\hat{\rho}_j} i J_j C(\hat{q}_w^c) C(\hat{q}_w^{c_j})^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{A.31}
\]

\[
\frac{\partial^i z_j}{\partial \rho_j} = -\frac{1}{\hat{\rho}_j^2} i J_j C(\hat{q}_w^c) C(\hat{q}_w^{c_j})^T \begin{bmatrix} \hat{\alpha}_j \\ \hat{\beta}_j \\ 1 \end{bmatrix} \tag{A.32}
\]

Stacking the expressions above column-wise leads to

\[
\frac{\partial^i z_j}{\partial \mathbf{f}_j} = \frac{1}{\hat{\rho}_j} i J_j C(\hat{q}_w^c) C(\hat{q}_w^{c_j})^T \begin{bmatrix} 1 & 0 & -\hat{\alpha}_j \\ 0 & 1 & -\hat{\beta}_j \\ 0 & 0 & -\frac{1}{\hat{\rho}_j} \end{bmatrix} \tag{A.33}
\]

\[
= H_{f_j} \tag{A.35}
\]

### A.2 Range-Visual Update

In this section, we will show that the measurement Jacobian of the range-visual update can be decomposed into simpler Jacobians, derive preliminary expressions for these with respect to a simpler alternative state vector \( \mathbf{x}' \), derive the expression of the full Jacobian with respect to \( \mathbf{x}' \), and finally the Jacobian with respect to the actual filter state \( \mathbf{x} \).

#### A.2.1 Decomposition of the Jacobian

We can rewrite the range-visual update model of Equation (2.66) as

\[
i z_r = \frac{a}{b} \tag{A.36}
\]
with

\[ a = (p^f_j - p^c_i)^T \! w = a^Tw n \]
\[ b = wu^Tw n = b^0Tw n . \]

(A.37)

(A.38)

The measurement Jacobian can be expressed as

\[ H = \frac{\partial^j z_r}{\partial x} \]
\[ = \frac{1}{b^2} \left( \frac{\partial a}{\partial x} \dot{b} - \dot{a} \frac{\partial b}{\partial x} \right) \]
\[ = \frac{1}{b^2} \left( \frac{\partial (a^Tw n)}{\partial x} \dot{b} - \dot{a} \frac{\partial (b^Tw n)}{\partial x} \right) \]

(A.39)

(A.40)

(A.41)

The partial derivative of the cross product of two vectors of identical size \( u \) and \( v \), with respect to a third vector \( w \) can be demonstrated to be

\[ \frac{\partial (u^Tv)}{\partial w} = v^T \frac{\partial u}{\partial w} + u^T \frac{\partial v}{\partial w} . \]

(A.42)

Then we can write \( H \) as

\[ H = \frac{1}{b^2} \left( w \dot{n}^T \frac{\partial a}{\partial x} + a^T \frac{\partial w n}{\partial x} \right) \dot{b} - \dot{a} \left( w \dot{n}^T \frac{\partial b}{\partial x} + b^T \frac{\partial w n}{\partial x} \right) \]
\[ = \frac{1}{b} \left( w \dot{n}^T \frac{\partial a}{\partial x} + \dot{a}^T \frac{\partial w n}{\partial x} \right) - \frac{\dot{a}}{b^2} \left( w \dot{n}^T \frac{\partial b}{\partial x} + \dot{b}^T \frac{\partial w n}{\partial x} \right) . \]

(A.43)

(A.44)

A.2.2 Preliminary Derivations

Recalling Equations (2.11)-(A.47), we break down the state vector \( x \) into its inertial states \( x_I \in \mathbb{R}^{16} \), sliding window states \( x_S \in \mathbb{R}^{7M} \), and feature states \( x_F \in \mathbb{R}^{3N} \).

\[ x = [x_I^T\ x_S^T\ x_F^T]^T \]

(A.45)

From Equation (2.46), we also know the feature states are expressed in inverse-depth coordinates \( f_j = [\alpha_j\ \beta_j\ \rho_j]^T \) using anchor poses selected among the sliding window states.
At this point, we introduce an alternative state vector \( \mathbf{x}' \) that retains the same inertial and sliding window states as \( \mathbf{x} \), but the feature states are expressed by their cartesian coordinates in world frame \( w \mathbf{p}_j = [w_{x_j} \ w_{y_j} \ w_{z_j}]^T \), i.e.
\[
\mathbf{x}' = \begin{bmatrix} \mathbf{x}_I^T & \mathbf{x}_S^T & \mathbf{x}_F^T \end{bmatrix}^T
\]
with
\[
\mathbf{x}_F = \begin{bmatrix} w_{p_1}^T & \ldots & w_{p_N}^T \end{bmatrix}^T.
\]
It is worth noting that \( \mathbf{x}' \in \mathbb{R}^{16+7M+3N} \), i.e. maintains the same size as \( \mathbf{x} \).

Equation (A.44) gives measurement Jacobian from the partial derivatives \( w_{n}, a_0 \) and \( b_0 \) with respect to the state vector \( \mathbf{x} \). To make the derivation easier, we will consider the partials with respect to \( \mathbf{x}' \) first, before going back to \( \mathbf{x} \) in the last subsection.

Before the first derivation, we remind the reader that for two vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \), and a vector \( \mathbf{w} \) of any dimension, we can prove that
\[
\frac{\partial (\mathbf{u} \times \mathbf{v})}{\partial \mathbf{w}} = \frac{\partial \mathbf{u}}{\partial \mathbf{w}} \times \mathbf{v} + \mathbf{u} \times \frac{\partial \mathbf{v}}{\partial \mathbf{w}}
\]
\[
(A.48)
\]

Now let us derive the partials of interest.
\[
\frac{\partial w_{n}}{\partial \mathbf{x}'} = \frac{\partial}{\partial \mathbf{x}'} \left( \left( p_{w_1}^{f_1} - p_{w_2}^{f_2} \right) \times \left( p_{w_3}^{f_3} - p_{w_2}^{f_2} \right) \right)
\]
\[
= \frac{\partial (p_{w_1}^{f_1} - p_{w_2}^{f_2})}{\partial \mathbf{x}'} \times \left( p_{w_3}^{f_3} - p_{w_2}^{f_2} \right) + \left( p_{w_3}^{f_3} - p_{w_2}^{f_2} \right) \times \frac{\partial (p_{w_1}^{f_1} - p_{w_2}^{f_2})}{\partial \mathbf{x}'}
\]
\[
= \left( \mathbf{p}_{w_1}^{f_1} - \mathbf{p}_{w_2}^{f_2} \right) \times \frac{\partial p_{w_3}^{f_3}}{\partial \mathbf{x}'} + \left( \mathbf{p}_{w_3}^{f_3} - \mathbf{p}_{w_2}^{f_2} \right) \times \frac{\partial p_{w_1}^{f_1}}{\partial \mathbf{x}'}
\]
\[
+ \left( \mathbf{p}_{w_1}^{f_1} - \mathbf{p}_{w_2}^{f_2} \right) \times \frac{\partial p_{w_3}^{f_3}}{\partial \mathbf{x}'}
\]
\[
(A.49)
\]

\[
= \begin{bmatrix} 0_{3 \times (15+6M+3(j_1-1))} & \left( \mathbf{p}_{w_1}^{f_1} - \mathbf{p}_{w_3}^{f_3} \right) \times \mathbf{0}_{3 \times (j_2-j_1-1)} \\
\left( \mathbf{p}_{w_1}^{f_1} - \mathbf{p}_{w_3}^{f_3} \right) \times & 0_{3 \times (j_3-j_2-1)} \\
\left( \mathbf{p}_{w_1}^{f_1} - \mathbf{p}_{w_3}^{f_3} \right) \times & \mathbf{0}_{3 \times (N-j_3)} \end{bmatrix}
\]
\[
(A.50)
\]
\[
\frac{\partial a_0}{\partial x'} = \frac{\partial}{\partial x'} \left( p_{f_{j_2}} - p_{w}^c \right) 
= \left[ \begin{array}{c|c|c|c|c|c|c}
0_{3 \times (15+3(i-1))} & -I_3 & 0_{3 \times (15+6M+3(j_2-i-1))} & I_3 & 0_{3 \times (3N-j_2)} 
\end{array} \right] 
\] (A.54)

\[
\frac{\partial b_0}{\partial x'} = \frac{\partial^u u_r}{\partial x'} 
= \frac{\partial}{\partial x'} \left( C(q_w^c)^T c u_r \right) 
\approx \frac{\partial}{\partial x'} \left( (I_3 - [\delta \theta w^c \times]) C(q_w^c)^T c u_r \right) 
\approx \frac{\partial}{\partial x'} \left( C(q_w^c)^T (I_3 + [\delta \theta w^c \times])^T c u_r \right) 
\approx \frac{\partial}{\partial x'} \left( C(q_w^c)^T (I_3 - [c u_r \times]) \delta \theta w^c \right) 
\approx -C(q_w^c)^T [c u_r \times] \left[ \begin{array}{c|c|c|c|c|c|c}
0_{3 \times (15+3M+3(i-1))} & I_3 & 0_{3 \times (3N+3M-i)} 
\end{array} \right] 
\] (A.55)

\section*{A.2.3 Measurement Jacobian with respect to \(x'\)}

We can use Equation (A.44) to formulate the measurement Jacobian with respect to \(x'\) as

\[
H' = \frac{\partial^i z_r}{\partial x'} = \frac{1}{b} \hat{\alpha} - \frac{\hat{a}}{b^2} \hat{\beta} 
\] (A.63)

with

\[
\alpha = w n^T \frac{\partial a_0}{\partial x'} + a_0^T \frac{\partial^w n}{\partial x'} 
\] (A.65)

\[
\beta = w n^T \frac{\partial b_0}{\partial x'} + b_0^T \frac{\partial^w n}{\partial x'} 
\] (A.66)

Using the expressions derived in the previous section, we now can move forward and derive \(\alpha, \beta, \) and then \(H'\).
\[
\hat{\alpha} = \begin{bmatrix}
0_{3 \times (15+3(i-1))} & -w \hat{n}^T & 0_{3 \times (6M+3(j_1-i-1))} \\
\end{bmatrix}
\]

\[
\hat{\alpha} = \begin{bmatrix}
\left( \hat{p}_{w}^{f_{j_2}} - \hat{p}_{w}^{c_{i}} \right)^T \left( \hat{p}_{w}^{f_{j_2}} - \hat{p}_{w}^{f_{j_3}} \right) \times & 0_{3 \times 3(j_2-j_1-1)} \\
\left( \hat{p}_{w}^{f_{j_3}} - \hat{p}_{w}^{f_{j_1}} \right)^T \left( \hat{p}_{w}^{f_{j_3}} - \hat{p}_{w}^{f_{j_2}} \right) \times & 0_{3 \times 3(j_3-j_2-1)} \\
\left( \hat{p}_{w}^{f_{j_2}} - \hat{p}_{w}^{c_{i}} \right)^T \left( \hat{p}_{w}^{f_{j_1}} - \hat{p}_{w}^{f_{j_2}} \right) \times & 0_{3 \times 3(N-j_3)} \\
\end{bmatrix}
\]

\[
\hat{\beta} = \begin{bmatrix}
\begin{bmatrix}
c_{u_r}^T C \left( \hat{q}_{w}^{c_{i}} \right)^T [c_{u_r} \times] \\
0_{3 \times 3(M+3(j_1-i-1))} \\
\end{bmatrix} \\
\end{bmatrix}
\]

\[
\hat{\beta} = \begin{bmatrix}
\begin{bmatrix}
c_{u_r}^T C \left( \hat{q}_{w}^{c_{i}} \right)^T [c_{u_r} \times] \\
0_{3 \times 3(j_2-j_1-1)} \\
\end{bmatrix} \\
\end{bmatrix}
\]

\[
\begin{align*}
H' & = \begin{bmatrix}
0_{3 \times (15+3(i-1))} & -\frac{1}{b} w \hat{n}^T & 0_{3 \times (3M-1)} \\
\hat{\alpha}^T w \hat{n} C \left( \hat{q}_{w}^{c_{i}} \right)^T [c_{u_r} \times] & 0_{3 \times (3M+3(j_1-i-1))} \\
\frac{1}{b} \left( \hat{p}_{w}^{f_{j_2}} - \hat{p}_{w}^{c_{i}} - \frac{\hat{\alpha}}{b} \left( c_{u_r}^T C \left( \hat{q}_{w}^{c_{i}} \right)^T \right) \right)^T \left( \hat{p}_{w}^{f_{j_2}} - \hat{p}_{w}^{f_{j_3}} \right) \times & 0_{3 \times 3(j_2-j_1-1)} \\
\frac{1}{b} \left( \hat{p}_{w}^{f_{j_3}} - \hat{p}_{w}^{c_{i}} - \frac{\hat{\alpha}}{b} \left( c_{u_r}^T C \left( \hat{q}_{w}^{c_{i}} \right)^T \right) \right)^T \left( \hat{p}_{w}^{f_{j_3}} - \hat{p}_{w}^{f_{j_1}} \right) \times & 0_{3 \times 3(j_3-j_2-1)} \\
\frac{1}{b} \left( \hat{p}_{w}^{f_{j_2}} - \hat{p}_{w}^{c_{i}} - \frac{\hat{\alpha}}{b} \left( c_{u_r}^T C \left( \hat{q}_{w}^{c_{i}} \right)^T \right) \right)^T \left( \hat{p}_{w}^{f_{j_1}} - \hat{p}_{w}^{f_{j_2}} \right) \times & 0_{3 \times 3(N-j_3)} \\
\end{bmatrix}
\end{align*}
\]

(A.67)
We can rewrite

\[
\left( \frac{\hat{a}}{b} c_r^T C \left( \hat{q}_w^c \right) \right)^T = \frac{\hat{a}}{b} C \left( \hat{q}_w^c \right)^T c_r^T u_r, \tag{A.70}
\]

\[
= \hat{p}_w^c + \hat{i} \hat{z}_w^u u_{ri} - \hat{p}_w^c, \tag{A.71}
\]

\[
= \hat{p}_w^I - \hat{p}_w^c, \tag{A.72}
\]

where \( \hat{p}_w^I \) are the 3D cartesian coordinates of the impact point of the LRF.
on the terrain at time instant $i$, so that we can simplify Equation (A.69) as

$$H' = \begin{bmatrix} 0_{3 \times (15+3(i-1))} & -\frac{1}{b} \hat{n}^T & 0_{3 \times (3(M-1))} \\ \frac{\hat{a}^w}{b^2} \hat{n}^T C (\hat{q}_w^c)^T \left[ -u_r \times \right] & 0_{3 \times (3(M+3(i-1)))} \\ \frac{1}{b} \left( \hat{p}_{f_2}^w - \hat{p}_{f_1}^w \right)^T \left[ (\hat{p}_{f_3}^w - \hat{p}_{f_2}^w) \times \right] & \end{bmatrix}$$

$$= \begin{bmatrix} 0_{3 \times (15+3(i-1))} & -\frac{1}{b} \hat{n}^T & 0_{3 \times (3(M-1))} \\ -\frac{\hat{a}^w}{b^2} \left[ -u_r \times \right] C (\hat{q}_w^c)^w \hat{n}^T & 0_{3 \times (3(M+3(j_1-i-1)))} \\ \frac{1}{b} \left[ \left( \hat{p}_{f_3}^w - \hat{p}_{f_2}^w \right) \times \right] \left[ (\hat{p}_{f_1}^w - \hat{p}_{f_2}^w) \times \right] & \end{bmatrix}$$

(A.73)

Or equivalently,

$$H_{p_1} \delta p_{w_1} + H_{\theta_1} \delta \theta_{w_1}$$

$$i \delta z_r \simeq + H_{p_{j_1}} \delta p_{w_{j_1}} + H_{p_{j_2}} \delta p_{w_{j_2}} + H_{p_{j_3}} \delta p_{w_{j_3}}$$

(A.75)

$$42$$
where

\[ H_{p_i} = -\frac{1}{b} \hat{v}^T \]  
(A.76)

\[ H_{\theta_i} = -\frac{\hat{\alpha}}{b^2} ([c u_r] \times C(\hat{q}_w^c) \hat{v})^T \]  
(A.77)

\[ H_{p_{j1}} = \frac{1}{b} \left( \left( p_{w}^{f_{j2}} - p_{w}^{f_{j1}} \right) \times \left( p_{w}^{f_{j2}} - p_{w}^{f_{j1}} \right) \right)^T \]  
(A.78)

\[ H_{p_{j2}} = \frac{1}{b} \left( \left( p_{w}^{f_{j2}} - p_{w}^{f_{j1}} \right) \times \left( p_{w}^{f_{j2}} - p_{w}^{f_{j1}} \right) \right)^T \]  
(A.79)

\[ H_{p_{j3}} = \frac{1}{b} \left( \left( p_{w}^{f_{j2}} - p_{w}^{f_{j1}} \right) \times \left( p_{w}^{f_{j2}} - p_{w}^{f_{j1}} \right) \right)^T \]  
(A.80)

A.2.4 Measurement Jacobian with respect to $x$

In this last subsection, we recover the measurement Jacobian with respect to the actual state $x$. We remind the reader that $x'$ includes the cartesian coordinates of feature $j$, $w p_j = [w x_j \ w y_j \ w z_j]^T$, whereas $x$ includes its inverse-depth coordinates $f_j = [\alpha_j \ \beta_j \ \rho_j]^T$ and an anchor pose corresponding to the camera frame at index $i_j$ in the sliding window. The other are related according to

\[ w p_j = p_j^{ci_j} + \frac{1}{\rho_j} C(q_w^{ci_j} \ [\alpha_j \ \beta_j \ 1]^T \]  
(A.81)

This means that to form $H$, we need an expression of $\frac{\partial z_r}{\partial p_{wj}}$, $\frac{\partial z_r}{\partial \theta_{wj}}$, $\frac{\partial z_r}{\partial \alpha_j}$, $\frac{\partial z_r}{\partial \beta_j}$, and $\frac{\partial z_r}{\partial \rho_j}$ for each feature $j$ in the LRF triangle.

\[ \frac{\partial z_r}{\partial p_{wj}} = \frac{\partial z_r}{\partial p_{wj}} \frac{\partial p_{wj}}{\partial p_{wj}} \]  
(A.82)

\[ = \frac{\partial z_r}{\partial p_{wj}} \]  
(A.83)

\[ = H_{p_j} \]  
(A.84)
\[
\frac{\partial^i z_r}{\partial \delta \theta_w^{c_i j}} = \frac{\partial^i z_r}{\partial w^p_j} \frac{\partial w^p_j}{\partial \delta \theta_w^{c_i j}}
\tag{A.85}
\]

\[
\simeq \frac{\partial^i z_r}{\partial w^p_j} \frac{\partial}{\partial \delta \theta_w^{c_i j}} \left( \frac{1}{\rho_j} \left( (I_3 - [\delta \theta_w^{c_i j} \times]) C'(\hat{q}_w^{c_i j}) \right)^T \left[ \begin{array}{c} \alpha_j \\ \beta_j \\ 1 \end{array} \right] \right)
\tag{A.86}
\]

\[
\simeq \frac{\partial^i z_r}{\partial w^p_j} \frac{\partial}{\partial \delta \theta_w^{c_i j}} \left( \frac{1}{\rho_j} C'(\hat{q}_w^{c_i j})^T (I_3 - [\delta \theta_w^{c_i j} \times])^T \left[ \begin{array}{c} \alpha_j \\ \beta_j \\ 1 \end{array} \right] \right)
\tag{A.87}
\]

\[
\simeq \frac{\partial^i z_r}{\partial w^p_j} \frac{\partial}{\partial \delta \theta_w^{c_i j}} \left( \frac{1}{\rho_j} C'(\hat{q}_w^{c_i j})^T (I_3 + [\delta \theta_w^{c_i j} \times]) \left[ \begin{array}{c} \alpha_j \\ \beta_j \\ 1 \end{array} \right] \right)
\tag{A.88}
\]

\[
\simeq \frac{\partial^i z_r}{\partial w^p_j} \frac{\partial}{\partial \delta \theta_w^{c_i j}} \left( \frac{1}{\rho_j} C'(\hat{q}_w^{c_i j})^T [\delta \theta_w^{c_i j} \times] \left[ \begin{array}{c} \alpha_j \\ \beta_j \\ 1 \end{array} \right] \right)
\tag{A.89}
\]

\[
\simeq -1 \frac{\partial^i z_r}{\partial w^p_j} C'(\hat{q}_w^{c_i j}) \left[ \begin{array}{c} \alpha_j \\ \beta_j \\ 1 \end{array} \right] \times \delta \theta_w^{c_i j}
\tag{A.90}
\]

\[
\simeq -1 \frac{\partial^i z_r}{\partial w^p_j} C'(\hat{q}_w^{c_i j}) \left[ \begin{array}{c} \alpha_j \\ \beta_j \\ 1 \end{array} \right] \times \delta \theta_w^{c_i j}
\tag{A.91}
\]

\[
\simeq -1 \frac{\partial^i z_r}{\partial w^p_j} H_{p_j} C'(\hat{q}_w^{c_i j}) \left[ \begin{array}{c} \alpha_j \\ \beta_j \\ 1 \end{array} \right] \times \delta \theta_w^{c_i j}
\tag{A.92}
\]

\[
\frac{\partial^i z_r}{\partial \alpha_j} = \frac{\partial^i z_r}{\partial w^p_j} \frac{\partial w^p_j}{\partial \alpha_j}
\tag{A.93}
\]

\[
= \frac{1}{\rho_j} \frac{\partial^i z_r}{\partial w^p_j} C'(\hat{q}_w^{c_i j}) \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]
\tag{A.94}
\]

\[
= \frac{1}{\rho_j} H_{p_j} C'(\hat{q}_w^{c_i j}) \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]
\tag{A.95}
\]
\[
\frac{\partial^i z_r}{\partial \beta_j} = \frac{\partial^i z_r}{\partial w_p} \frac{\partial w_p}{\partial \beta_j} \\
= \frac{1}{\hat{\rho}_j} \frac{\partial^i z_r}{\partial w_p} C(\hat{q}_w^{c_j})^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} 
\]  
(A.96)

\[
= \frac{1}{\hat{\rho}_j} H_{p_j} C(\hat{q}_w^{c_j})^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} 
\]  
(A.97)

\[
\frac{\partial^i z_r}{\partial \rho_j} = \frac{\partial^i z_r}{\partial w_p} \frac{\partial w_p}{\partial \rho_j} \\
= -\frac{1}{\hat{\rho}_j^2} \frac{\partial^i z_r}{\partial w_p} C(\hat{q}_w^{c_j})^T \begin{bmatrix} \hat{\alpha}_j \\ \hat{\beta}_j \\ 1 \end{bmatrix} 
\]  
(A.98)

\[
= -\frac{1}{\hat{\rho}_j^2} H_{p_j} C(\hat{q}_w^{c_j})^T \begin{bmatrix} \hat{\alpha}_j \\ \hat{\beta}_j \\ 1 \end{bmatrix} 
\]  
(A.99)

Thus we can extend Equation (A.75) to

\[
\begin{align*}
i \delta z_r &\sim H_{p_{i_1}} \delta p_{w_1} + H_{p_{i_2}} \delta p_{w_2} + H_{p_{i_3}} \delta p_{w_3} + H_{p_i} \delta p_w \\
&+ H_{\theta_{i_1}} \delta \theta_{w_1} + H_{\theta_{i_2}} \delta \theta_{w_2} + H_{\theta_{i_3}} \delta \theta_{w_3} + H_{\theta_i} \delta \theta_w \\
&+ H_{f_{j_1}} \delta f_{j_1} + H_{f_{j_2}} \delta f_{j_2} + H_{f_{j_3}} \delta f_{j_3} \\
&+ i n_r 
\end{align*} 
\]  
(A.100)
where

\[
H_{p_1} = H_{p_{j_1}} \quad \text{(A.103)}
\]
\[
H_{p_2} = H_{p_{j_2}} \quad \text{(A.104)}
\]
\[
H_{p_3} = H_{p_{j_3}} \quad \text{(A.105)}
\]
\[
H_{p_i} = -\frac{1^w}{b} \hat{n}^T \quad \text{(A.106)}
\]
\[
H_{\theta_1} = -\frac{1}{\rho_{j_1}} H_{p_{j_1}} C(\hat{q}_{w}^{c_{ij_1}})^T \begin{bmatrix} \hat{\alpha}_{j_1} \\ \hat{\beta}_{j_1} \\ 1 \end{bmatrix} \times \quad \text{(A.107)}
\]
\[
H_{\theta_2} = -\frac{1}{\rho_{j_2}} H_{p_{j_2}} C(\hat{q}_{w}^{c_{ij_2}})^T \begin{bmatrix} \hat{\alpha}_{j_2} \\ \hat{\beta}_{j_2} \\ 1 \end{bmatrix} \times \quad \text{(A.108)}
\]
\[
H_{\theta_3} = -\frac{1}{\rho_{j_3}} H_{p_{j_3}} C(\hat{q}_{w}^{c_{ij_3}})^T \begin{bmatrix} \hat{\alpha}_{j_3} \\ \hat{\beta}_{j_3} \\ 1 \end{bmatrix} \times \quad \text{(A.109)}
\]
\[
H_{\theta_i} = -\frac{\hat{a}}{b^2} \left( [^c u_r \times] C (\hat{q}_{w}^{c_i}) w \hat{n} \right)^T \quad \text{(A.110)}
\]
\[
H_{f_{j_1}} = \frac{1}{\rho_{j_1}} H_{p_{j_1}} C(\hat{q}_{w}^{c_{ij_1}})^T \begin{bmatrix} 1 & 0 & -\frac{\hat{\alpha}_{j_1}}{\rho_{j_1}} \\ 0 & 1 & -\frac{\hat{\beta}_{j_1}}{\rho_{j_1}} \\ 0 & 0 & -\frac{1}{\rho_{j_1}} \end{bmatrix} \quad \text{(A.111)}
\]
\[
H_{f_{j_2}} = \frac{1}{\rho_{j_2}} H_{p_{j_2}} C(\hat{q}_{w}^{c_{ij_2}})^T \begin{bmatrix} 1 & 0 & -\frac{\hat{\alpha}_{j_2}}{\rho_{j_2}} \\ 0 & 1 & -\frac{\hat{\beta}_{j_2}}{\rho_{j_2}} \\ 0 & 0 & -\frac{1}{\rho_{j_2}} \end{bmatrix} \quad \text{(A.112)}
\]
\[
H_{f_{j_3}} = \frac{1}{\rho_{j_3}} H_{p_{j_3}} C(\hat{q}_{w}^{c_{ij_3}})^T \begin{bmatrix} 1 & 0 & -\frac{\hat{\alpha}_{j_3}}{\rho_{j_3}} \\ 0 & 1 & -\frac{\hat{\beta}_{j_3}}{\rho_{j_3}} \\ 0 & 0 & -\frac{1}{\rho_{j_3}} \end{bmatrix} \quad \text{(A.113)}
\]
and
\[ H_p_{j1} = \frac{1}{b} \left( \left| (\hat{p}_{f_{j3}} - \hat{p}_{f_{j3}}) \times (\hat{p}_{f_{j2}} - \hat{p}_{f_{j1}}) \right| \right)^T \] (A.114)
\[ H_p_{j2} = \frac{1}{b} \left( w \hat{n} + \left| (\hat{p}_{w_{f_{j1}}} - \hat{p}_{w_{f_{j3}}}) \times (\hat{p}_{w_{f_{j2}}} - \hat{p}_{w_{f_{j1}}}) \right| \right)^T \] (A.115)
\[ H_p_{j3} = \frac{1}{b} \left( \left| (\hat{p}_{w_{f_{j2}}} - \hat{p}_{w_{f_{j1}}}) \times (\hat{p}_{w_{f_{j2}}} - \hat{p}_{w_{f_{j3}}}) \right| \right)^T \] (A.116)

### A.3 Feature State Initialization with MSCKF

The equations presented in Subsection 2.7.2 to initialize a feature state from MSCKF measurements are already demonstrated in Li and Mourikis (2012b). Some parts of this demonstration are not trivial and this intends to fill these local gaps. Since the reader is intended to follow up the demonstration in Li and Mourikis (2012b), we follow their notation and the numbering of the equations in their paper, which differs from that in Subsection 2.7.2.

Even though Li and Mourikis (2012b) and this report only provide the demonstration for the initialization, one can verify that matrices $H_o$, $H_1$ and $H_2$ can be of any size. So the demonstration still holds for the batch initialization of multiple features, by stacking up these matrices.

#### A.3.1 Covariance of the Innovation

Starting from Equation (16) in Li and Mourikis (2012b),

\[ S = \begin{bmatrix} H_o & 0 \\ H_1 & H_2 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & \mu I \end{bmatrix} \begin{bmatrix} H_o & 0 \\ H_1 & H_2 \end{bmatrix}^T + \sigma^2 I \] (A.117)
\[ = \begin{bmatrix} H_o & 0 \\ H_1 & H_2 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & \mu I \end{bmatrix} \begin{bmatrix} H_o^T & H_1^T \\ 0 & H_2^T \end{bmatrix}^T + \sigma^2 I \] (A.118)
\[ = \begin{bmatrix} H_o & 0 \\ H_1 & H_2 \end{bmatrix} \begin{bmatrix} PH_o^T & PH_1^T \\ 0 & \mu H_2^T \end{bmatrix} + \sigma^2 I \] (A.119)
\[ = \begin{bmatrix} H_o PH_o^T & H_o PH_1^T \\ H_1 PH_o^T + \mu H_2 H_2^T & H_1 PH_1^T + \mu H_2 H_2^T \end{bmatrix} + \sigma^2 I \] (A.120)
\[ = \begin{bmatrix} H_o PH_o^T + \sigma^2 I & H_o PH_1^T \\ H_1 PH_o^T & H_1 PH_1^T + \mu H_2 H_2^T + \sigma^2 I \end{bmatrix} \] (A.121)
which is Equation (17) in Li and Mourikis (2012b).

If we pose

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$  \hspace{1cm} (A.122)

with

$$A = H_o PH_o^T + \sigma^2 I$$  \hspace{1cm} (A.123)

$$B = H_o PH_1^T$$  \hspace{1cm} (A.124)

$$C = H_1 PH_o^T$$  \hspace{1cm} (A.125)

$$D = H_1 PH_1^T + \mu H_2 H_2^T + \sigma^2 I$$  \hspace{1cm} (A.126)

then if $D$ and $A - BD^{-1}C$ are non-singular, using block-wise inversion we get

$$S^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -D^{-1}C(A - BD^{-1}C)^{-1} \\ -D^{-1}(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}$$  \hspace{1cm} (A.127)

$$= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$  \hspace{1cm} (A.128)

with

$$F_{11} = \left( H_o PH_o^T + \sigma^2 I \right)$$  \hspace{1cm} (A.130)

$$-H_o PH_1^T \left( H_1 PH_1^T + \mu H_2 H_2^T + \sigma^2 I \right)^{-1} H_1 PH_o^T$$

$$F_{12} = -F_{11} H_o PH_1^T \left( H_1 PH_1^T + \mu H_2 H_2^T + \sigma^2 I \right)^{-1}$$  \hspace{1cm} (A.131)

$$F_{21} = -\left( H_1 PH_1^T + \mu H_2 H_2^T + \sigma^2 I \right)^{-1} H_1 PH_1^T F_{11}$$  \hspace{1cm} (A.132)

Since for a non-singular matrix $M$, $(M^T)^{-1} = (M^{-1})^T$, we also get $F_{12} = F_{21}^T$. 

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\( F_{22} \) is easier to derive if we consider the equivalent block-wise inversion
\[
S^{-1} = \begin{bmatrix}
A^{-1} + A^{-1}B \left( D - CA^{-1}B \right)^{-1} CA^{-1} \\
-(D - CA^{-1}B)^{-1} CA^{-1} \\
-A^{-1}B \left( D - CA^{-1}B \right)^{-1} \\
(D - CA^{-1}B)^{-1}
\end{bmatrix}
\] (A.133)
so that
\[
F_{22} = \begin{bmatrix}
H_1 PH_1^T + \mu H_2 H_2^T + \sigma^2 I \\
-H_1 PH_o^T \left( H_o PH_o^T + \sigma^2 I \right)^{-1} H_o PH_1^T
\end{bmatrix}^{-1}
\] (A.134)

### A.3.2 Updated Error Covariance

Let us define \( N \) from Equation (26) in Li and Mourikis (2012b)
\[
N = \begin{bmatrix}
P & 0 \\
0 & \mu I
\end{bmatrix}
\begin{bmatrix}
H_o & 0 \\
H_1 & H_2
\end{bmatrix}^T
\begin{bmatrix}
F_{11} & F_{21}^T \\
F_{21} & F_{22}
\end{bmatrix}
\begin{bmatrix}
H_o & 0 \\
H_1 & H_2
\end{bmatrix}
\begin{bmatrix}
P & 0 \\
0 & \mu I
\end{bmatrix}
\] (A.135)
\[
= \begin{bmatrix}
P & 0 \\
0 & \mu I
\end{bmatrix}
\begin{bmatrix}
H_o^T & H_1^T \\
0 & H_2^T
\end{bmatrix}
\begin{bmatrix}
F_{11} & F_{21}^T \\
F_{21} & F_{22}
\end{bmatrix}
\begin{bmatrix}
H_o P & 0 \\
H_1 P & \mu H_2
\end{bmatrix}
\] (A.136)
\[
= \begin{bmatrix}
PH_o^T & PH_1^T \\
0 & \mu H_2^T
\end{bmatrix}
\begin{bmatrix}
F_{11} H_o P + F_{21}^T H_1 P & \mu F_{21}^T H_2 \\
F_{21} H_o P + F_{22} H_1 P & \mu F_{22} H_2
\end{bmatrix}
\] (A.137)
\[
= \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}
\] (A.138)
with
\[
N_{11} = PH_o^T F_{11} H_o P + PH_1^T F_{21} H_1 P + PH_1^T F_{22} H_o P
\] (A.139)
\[
N_{12} = \mu PH_o^T F_{21} H_2 + \mu PH_1^T F_{22} H_2
\] (A.140)
\[
N_{21} = \mu H_2^T F_{21} H_o P + \mu H_2^T F_{22} H_1 P
\] (A.141)
\[
= N_{12}^T
\] (A.142)
\[
N_{22} = \mu^2 H_2^T F_{22} H_2
\] (A.143)
which demonstrate Equation (27).

The updated covariance matrix $P_{aug}^+$ is written as

$$P_{aug}^+ = \lim_{\mu \to \infty} \begin{bmatrix} P - N_{11} & -N_{21}^T \\ -N_{21} & \mu I - N_{22} \end{bmatrix} \quad (A.144)$$

$$= \begin{bmatrix} P^+ & P_{21}^+ \\ P_{21} & P_{22}^+ \end{bmatrix}. \quad (A.145)$$

We will now demonstrate the expressions of the various matrix blocks.

Expression of $P^+$

$$P^+ = \lim_{\mu \to \infty} \left( P - \left( PH_o^T F_{11} H_o P + PH_o^T F_{21}^T H_1 P \right. \right.$$  

$$+ \left. PH_1^T F_{21} H_o P + PH_1^T F_{22} H_1 P \right) \right)$$  

$$= P - PH_o^T \left( H_o \left( PH_o^T + \sigma^2 I \right)^{-1} \right) H_o P \quad (A.147)$$

which comes trivial from Equations (22), (23) and (24).

Expression of $P_{21}^+$

Since $\lim_{\mu \to \infty} F_{21} = \lim_{\mu \to \infty} F_{22} = 0$, the terms $\mu F_{21}$ and $\mu F_{22}$ in $N_{21}$ raise the limit indeterminate form $0 \times \infty$. To solve that, let us note that for a non-singular matrix $M$ and a scalar $\mu$, $\mu M^{-1} = \left( \frac{1}{\mu} M \right)^{-1}$. Then

$$\lim_{\mu \to \infty} \mu F_{21} = \frac{\lim_{\mu \to \infty} \left( -\mu \left( H_1 PH_1^T + \mu H_2 H_2^T + \sigma^2 I \right)^{-1} \right)}{H_1 PH_1^T F_{11}} \quad (A.148)$$

$$= \lim_{\mu \to \infty} \left( \frac{1}{\mu} H_1 PH_1^T + H_2 H_2^T + \frac{1}{\mu} \sigma^2 I \right)^{-1} \quad (A.149)$$

$$= \frac{H_1 PH_1^T F_{11}}{H_1 PH_1^T F_{11}}$$

$$= -\left( H_2 H_2^T \right)^{-1} H_1 PH_1^T \left( H_o \left( PH_o^T + \sigma^2 I \right)^{-1} \right) \quad (A.150)$$
and

$$\lim_{\mu \to \infty} \mu F_{22} = \lim_{\mu \to \infty} \left(\mu \left( H_1 P H_1^T + \mu H_2 H_2^T + \sigma^2 I \right)
$$

$$- H_1 P H_1^T (H_o P H_o^T + \sigma^2 I)^{-1} H_o P H_1^T \right)^{-1} \right)$$

$$= \lim_{\mu \to \infty} \left( \left( \frac{1}{\mu} H_1 P H_1^T + H_2 H_2^T + \frac{1}{\mu} \sigma^2 I \right)
$$

$$- \frac{1}{\mu} H_1 P H_1^T (H_o P H_o^T + \sigma^2 I)^{-1} H_o P H_1^T \right)^{-1} \right)$$

$$= (H_2 H_2^T)^{-1}$$

(A.153)

Hence

$$P_{21}^+ = \lim_{\mu \to \infty} - N_{21}$$

(A.154)

$$= H_2^T (H_2 H_2^T)^{-1} H_1 P H_1^T (H_o P H_o^T + \sigma^2 I)^{-1} H_o P$$

$$- H_2^T (H_2 H_2^T)^{-1} H_1 P$$

(A.155)

$$= H_2^{-1} H_1 P H_o^T (H_o P H_o^T + \sigma^2 I)^{-1} H_o P - H_2^{-1} H_1 P$$

(A.156)

$$= H_2^{-1} H_1 \left( P H_o^T (H_o P H_o^T + \sigma^2 I)^{-1} H_o P - P \right)$$

(A.157)

$$= -H_2^{-1} H_1 P^+$$

(A.158)

Expression of $P_{22}^+$

$$P_{22}^+ = \lim_{\mu \to \infty} (\mu I - N_{22})$$

(A.159)
where

\[ \mu I - N_{22} = \mu I - \mu^2 H_2^T F_{22} H_2 \]

(A.160)

\[ = \mu H_2^T F_{22} H_2 \left( ( H_2^T F_{22} H_2 )^{-1} - \mu I \right) \]

(A.161)

\[ = \mu H_2^T F_{22} H_2 \left( H_2^{-1} F_{22}^{-1} H_2^{-T} - \mu I \right) \]

(A.162)

\[ = \mu H_2^T F_{22} H_2 \left( H_2^{-1} ( F_{22}^{-1} - \mu H_2 H_2^T ) H_2^{-T} \right) \]

(A.163)

\[ = \mu H_2^T F_{22} H_2 \left( H_2^{-1} \left( H_1 PH_1^T + \sigma^2 I \right) \right. \]

(A.164)

\[ = -H_1 PH_o^T \left( H_o PH_o^T + \sigma^2 I \right)^{-1} H_o PH_1^T H_2^{-T} \]

Using Equation (A.153) in the current demonstration, one can then write

\[ P_{22}^+ = H_2^T ( H_2 H_2^T )^{-1} H_2 \left( H_2^{-1} \left( H_1 PH_1^T + \sigma^2 I \right) \right. \]

(A.165)

\[ \left. -H_1 PH_o^T \left( H_o PH_o^T + \sigma^2 I \right)^{-1} H_o PH_1^T \right) H_2^{-T} \]

\[ = H_2^T H_2^{-T} H_2^{-1} H_2 \left( H_2^{-1} \left( H_1 PH_1^T + \sigma^2 I \right) \right. \]

(A.166)

\[ \left. -H_1 PH_o^T \left( H_o PH_o^T + \sigma^2 I \right)^{-1} H_o PH_1^T \right) H_2^{-T} \]

\[ = H_2^{-1} \left( H_1 PH_1^T + \sigma^2 I \right) \]

(A.167)

\[ -H_1 PH_o^T \left( H_o PH_o^T + \sigma^2 I \right)^{-1} H_o PH_1^T H_2^{-T} \]

\[ = H_2^{-1} \left( H_1 PH_1^T \right) \]

(A.168)

\[ -H_1 PH_o^T \left( H_o PH_o^T + \sigma^2 I \right)^{-1} H_o PH_1^T H_2^{-T} + \sigma^2 H_2^{-1} H_2^{-T} \]

\[ = H_2^{-1} \left( P - PH_o^T \left( H_o PH_o^T + \sigma^2 I \right)^{-1} H_o P \right) H_1^T H_2^{-T} + \sigma^2 H_2^{-1} H_2^{-T} \]

(A.169)

\[ = H_2^{-1} H_1 P^+ H_1^T H_2^{-T} + \sigma^2 H_2^{-1} H_2^{-T} \]

(A.170)
A.3.3 State Update

Let us define $O$ from Equation (32) in Li and Mourikis (2012b)

\[
O = \begin{bmatrix}
P & 0 \\
0 & \mu I
\end{bmatrix}
\begin{bmatrix}
H_o & 0 \\
H_1 & H_2
\end{bmatrix}^T
\begin{bmatrix}
F_{11} & F_{12}^T \\
F_{21} & F_{22}
\end{bmatrix}
\]

(A.171)

\[
= \begin{bmatrix}
PH_o^T & PH_1^T \\
0 & \mu H_2^T
\end{bmatrix}
\begin{bmatrix}
F_{11} & F_{12}^T \\
F_{21} & F_{22}
\end{bmatrix}
\]

(A.172)

\[
= \begin{bmatrix}
PH_o^T F_{11} + PH_1^T F_{21} & PH_o^T F_{21} + PH_1^T F_{22} \\
\mu H_2^T F_{21} & \mu H_2^T F_{22}
\end{bmatrix}
\]

(A.173)

Using Equations (22)-(24) in Li and Mourikis (2012b), along with (A.150) and (A.153) in this demonstration, one gets

\[
\lim_{\mu \to \infty} O = \lim_{\mu \to \infty} \begin{bmatrix}
PH_o^T F_{11} \\
0
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{H_2^T (H_2 H_2^T)^{-1}} H_1 PH_o^T (H_o PH_o^T + \sigma^2 I)^{-1} \\
0
\end{bmatrix}
\]

(A.174)

\[
= \lim_{\mu \to \infty} \begin{bmatrix}
PH_o^T F_{11} & 0 \\
-H_2^{-1} H_1 PH_o^T F_{11} & H_2^{-1}
\end{bmatrix}
\]

(A.175)

which brings Equation (33) in Li and Mourikis (2012b).

A.4 Feature State Reparametrization

Equations (2.112)–(2.120) provide the expression of the error covariance when SLAM feature $p_j$ is reparametrized from anchor pose state $\{c_{i_1}\}$ to $\{c_{i_2}\}$ in the sliding window.

If $P_2$ and $\delta x_2$ respectively denote the error covariance and error state after the reparametrization, then by definition

\[
P_2 = E \begin{bmatrix}
\delta x_2 \delta x_2^T
\end{bmatrix}
\]

(A.176)

where the operator $E$ is the expected value. If we further denote $J_j = \frac{\partial x_2}{\partial x_1}$ the Jacobian of the reparametrization, then at first order

\[
\delta x_2 \simeq J_j \delta x_1
\]

(A.177)
\[ P_2 \simeq E [J_j \delta x_1 \delta x_1^T J_j^T] \quad \text{(A.178)} \]
\[ \simeq J_j E [\delta x_1 \delta x_1^T] J_j^T \quad \text{(A.179)} \]
\[ \simeq J_j P_1 J_j^T, \quad \text{(A.180)} \]

which demonstrates Equation (2.112).

We define the vector \( g_j \) from Equation (2.111)
\[
g_j = \frac{1}{\rho_{j2}} \begin{bmatrix} \alpha_{j2} \\ \beta_{j2} \\ 1 \end{bmatrix} = C(q_{c1}^{c}w) \left( -p_{c1}^{c} + p_{c2}^{c} + \rho_{j1}^{c} C(q_{c1}^{c}w) \right)
\]
\[
+ \frac{1}{\rho_{j1}} C(q_{c1}^{c}w)^T \begin{bmatrix} \alpha_{j1} \\ \beta_{j1} \\ 1 \end{bmatrix}
\]  

(A.181)

At first order, by differentiating on both sides, we get
\[
\delta g_j \simeq \begin{bmatrix}
\frac{\partial (\alpha_{j2})}{\partial \alpha_{j2}} & \frac{\partial (\alpha_{j2})}{\partial \beta_{j2}} & \frac{\partial (\alpha_{j2})}{\partial \rho_{j2}} \\
\frac{\partial (\beta_{j2})}{\partial \alpha_{j2}} & \frac{\partial (\beta_{j2})}{\partial \beta_{j2}} & \frac{\partial (\beta_{j2})}{\partial \rho_{j2}} \\
\frac{1}{\rho_{j2}} & \frac{1}{\rho_{j2}} & \frac{1}{\rho_{j2}}
\end{bmatrix}
\begin{bmatrix}
\delta \alpha_{j2} \\
\delta \beta_{j2} \\
\delta \rho_{j2}
\end{bmatrix}
\simeq \frac{\partial g_j}{\partial x_1} \delta x_1
\]

(A.182)

\[
\begin{bmatrix}
\frac{1}{\rho_{j2}} & 0 & -\frac{\dot{\alpha}_{j2}}{\rho_{j2}^2} \\
0 & \frac{1}{\rho_{j2}} & -\frac{\dot{\beta}_{j2}}{\rho_{j2}^2} \\
0 & 0 & -\frac{1}{\rho_{j2}}
\end{bmatrix}
\begin{bmatrix}
\delta \alpha_{j2} \\
\delta \beta_{j2} \\
\delta \rho_{j2}
\end{bmatrix}
\simeq H_{1j} \delta x_1
\]

(A.183)

\[
\begin{bmatrix}
\frac{1}{\rho_{j2}} \delta \alpha_{j2} - \frac{\dot{\alpha}_{j2}}{\rho_{j2}^2} \delta \rho_{j2} \\
\frac{1}{\rho_{j2}} \delta \beta_{j2} - \frac{\dot{\beta}_{j2}}{\rho_{j2}^2} \delta \rho_{j2} \\
-\frac{1}{\rho_{j2}} \delta \rho_{j2}
\end{bmatrix}
\simeq H_{1j} \delta x_1
\]

(A.184)

The third row gives
\[
\delta \rho_{j2} \simeq -\rho_{j2}^2 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} H_{1j} \delta x_1
\]

(A.185)
which can be injected the first two rows, so that

\[
\begin{aligned}
\frac{1}{\rho_{j_2}} \delta \alpha_{j_2} + \dot{\alpha}_{j_2} 
&\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
H_{1j} \delta x_1 
\simeq
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
H_{1j} \delta x_1 \\
\frac{1}{\rho_{j_2}} \delta \beta_{j_2} + \dot{\alpha}_{j_2} 
&\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
H_{1j} \delta x_1 
\simeq
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
H_{1j} \delta x_1 \\
\delta \alpha_{j_2} \simeq \dot{\rho}_{j_2} 
&\begin{bmatrix} 1 & 0 \end{bmatrix}
H_{1j} \delta x_1 \\
\delta \beta_{j_2} \simeq \dot{\rho}_{j_2} 
&\begin{bmatrix} 0 & 1 \end{bmatrix}
H_{1j} \delta x_1
\end{aligned}
\tag{A.186}
\]

By identifying Equations (A.185) and (A.187) in Equation (A.177), we can write

\[
J_{j} = 
\begin{bmatrix}
I_{15+6M+3(j-1)} & 0 \\
J_{1j} & H_{1j} \\
0 & I_{3(N-j)}
\end{bmatrix},
\tag{A.188}
\]

with

\[
J_{1j} =
\begin{bmatrix} 1 & 0 & -\dot{\alpha}_{j_2} \\ 0 & 1 & -\dot{\beta}_{j_2} \\ 0 & 0 & -\dot{\rho}_{j_2} \end{bmatrix},
\tag{A.189}
\]

which are Equations (2.113) and (2.114), respectively.

Now we proceed to deriving the non-null components of \( H_{1j} \) when \( i_1 = 1 \) and \( i_2 = M \).

\[
\frac{\partial g_j}{\partial p_{cM}^{c_t}} = 
\frac{\partial}{\partial p_{cM}^{c_t}} 
\left( C(q_c^{cM}) \left(-p_{cM}^{c_t} + p_{c1}^{c_t}\right) \right) \\
= C(q_c^{cM}) \tag{A.190}
\]

\[
= \gamma H_{p_1} \tag{A.191}
\]

\[
\frac{\partial g_j}{\partial p_{cM}^{c_t}} = -C(q_c^{cM}) \tag{A.192}
\]

\[
= \gamma H_{p_M} \tag{A.193}
\]
Using the small angle assumption like in Equation (A.7), we can write

\[
\frac{\partial g_i}{\partial \delta \theta^c_w} \approx \left( I_3 - [\delta \theta^c_w \times] \right) C(\hat{q}^c_w) \left( -p^c_w + p^c_w \right)
\]

\[
+ \frac{1}{\rho_{j1}} C(\hat{q}^c_{j1})^T (I_3 + [\delta \theta^c_w \times]) \left[ \begin{array}{c} \alpha_{j1} \\ \beta_{j1} \\ 1 \end{array} \right]
\]

(A.195)

\[
\frac{\partial g_i}{\partial \delta \theta^c_{w1}} \approx \left( I_3 - [\delta \theta^c_{w} \times] \right) C(\hat{q}^c_{w1})
\]

A.196

\[
\frac{\partial}{\partial \delta \theta^c_{w1}} \left( I_3 - [\delta \theta^c_{w} \times] \right) C(\hat{q}^c_{w1})
\]

\[
\frac{1}{\rho_{j1}} C(\hat{q}^c_{w1})^T [\delta \theta^c_{w1} \times] \left[ \begin{array}{c} \alpha_{j1} \\ \beta_{j1} \\ 1 \end{array} \right]
\]

(A.197)

\[
- \frac{1}{\hat{\rho}_{j1}} C(\hat{q}^c_{w1}) \hat{C}(\hat{q}^c_{w1})^T \left[ \begin{array}{c} \hat{\alpha}_{j1} \\ \hat{\beta}_{j1} \\ 1 \end{array} \right] \times \delta \theta^c_{w1}
\]

(A.198)

\[
\simeq jH_{\theta_1}
\]

(A.199)
\[
\frac{\partial g_j}{\partial \delta \theta^c_{w}} \approx \frac{\partial}{\partial \delta \theta^c_{w}} \left( -| \delta \theta^c_{w} \times | C(\hat{q}^c_{w}) \left( -p^c_{w} + p^c_{1w} \right) \right) \\
+ \frac{1}{\rho_{j1}} C(\hat{q}^c_{w})^T (I_3 + | \delta \theta^c_{w} \times |) \left[ \begin{array}{c} \alpha_{j1} \\ \beta_{j1} \\ 1 \end{array} \right] \right) \right) \\
\frac{\partial}{\partial \delta \theta^c_{w}} \left( C(\hat{q}^c_{w}) \left( -p^c_{w} + p^c_{1w} \right) \right) \\
+ \frac{1}{\rho_{j1}} C(\hat{q}^c_{w})^T (I_3 + | \delta \theta^c_{w} \times |) \left[ \begin{array}{c} \alpha_{j1} \\ \beta_{j1} \\ 1 \end{array} \right] \times | \delta \theta^c_{w} \right) \right)
\]
\[
\rho \left[ C(\hat{q}^c_{w}) \left( -p^c_{w} + p^c_{1w} + \frac{1}{\rho_{j1}} C(\hat{q}^c_{w})^T \left[ \hat{\alpha}_{j1} \hat{\beta}_{j1} \right] \right) \right] \times | \delta \theta^c_{w} \right) \right)
\]
\[
\approx H_{\theta_M} 
\]
\[
\frac{\partial g_j}{\partial \alpha_{j1}} = \frac{\partial}{\partial \alpha_{j1}} \left( \frac{1}{\rho_{j1}} C(\hat{q}^c_{w}) C(\hat{q}^c_{1w})^T \left[ \begin{array}{c} \alpha_{j1} \\ \beta_{j1} \\ 1 \end{array} \right] \right) \\
= \frac{1}{\rho_{j1}} C(\hat{q}^c_{w}) C(\hat{q}^c_{1w})^T \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \right)
\]
\[
\frac{\partial g_j}{\partial \beta_{j1}} = \frac{1}{\rho_{j1}} C(\hat{q}^c_{w}) C(\hat{q}^c_{1w})^T \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] \right)
\]
\[
\frac{\partial g_j}{\partial \rho_{j1}} = -\frac{1}{\rho_{j1}^2} C(\hat{q}^c_{w}) C(\hat{q}^c_{1w})^T \left[ \begin{array}{c} \hat{\alpha}_{j1} \\ \hat{\beta}_{j1} \\ 1 \end{array} \right] \right)
\]

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Hence

\[
\frac{\partial g_j}{\partial f_{j_1}} = \left[ \frac{\partial g_j}{\partial \alpha_{j_1}}, \frac{\partial g_j}{\partial \beta_{j_1}}, \frac{\partial g_j}{\partial \rho_{j_1}} \right] \tag{A.208}
\]

\[
= \frac{1}{\hat{\rho}_{j_1}} C(\hat{q}_w^{cm}) C(\hat{q}_w^{c1})^T \begin{bmatrix} 1 & 0 & -\hat{\alpha}_{j_1} \\ 0 & 1 & -\hat{\beta}_{j_1} \\ 0 & 0 & -\frac{1}{\hat{\rho}_{j_1}} \end{bmatrix} \tag{A.209}
\]

\[
= H_{f_j} \tag{A.210}
\]
Appendix B

Range-Visual-Inertial Odometry Observability

B.1 Derivation of the Observability Matrix

In this subsection, we demonstrate the expressions of the k-th block row of observability matrix $M$ in Equations (3.1-3.9).

To simplify the equations, our analysis assumes the state vector $x^0 = \begin{bmatrix} x_I^T & x_P^T \end{bmatrix}^T$ (B.1)

where $x_I' = \begin{bmatrix} q_i^T & b_g^T & v_i^T & b_a^T & p_i^T \end{bmatrix}^T$ (B.2)

and $x_P$ was defined in Equation (A.47) with the cartesian coordinates of the $N$ SLAM features, $N \geq 3$. $x_I'$ includes the same states as $x_I$ defined in Equation (1) of the paper, but in a different order so we can refer the reader to Hesch et al. (2012) for the expression and derivation of the state transition matrix $\Phi_{k,1}$ from time 1 to time $k$.

The block row associated to the ranged facet update is defined as

$M_k = H_k \Phi_{k,1}$ (B.3)

where $H_k$ is the Jacobian of the ranged facet measurement at time $k$ with respect to $x^0$, from Equations (A.75-A.80).
Without loss of generality, we can assume the ranged facet is constructed from the first three SLAM features in \( x_P \), then

\[
M_k = \begin{bmatrix}
H_{\theta_1} & 0_{1 \times 9} & H_{P_1} & H_{P_3} & H_{P_2} & H_{P_3} & 0_{1 \times 3(N-3)}
\end{bmatrix} \Phi_{k,1} \tag{B.4}
\]

\[
= \frac{1}{b} \left[ -\frac{a}{b} w^T C (q_w^{c_k})^T [c u_r \times]^T \phi_{11} - w n^T \phi_{51} \right]
\]

\[
- \frac{a}{b} w^T C (q_w^{c_k})^T [c u_r \times]^T \phi_{12} - w n^T \phi_{52} \right]
\]

\[
- w^T \phi_{53} \right]
\]

\[
= \frac{1}{b} \left[ M_{k,q} M_{k,bq} M_{k,v} M_{k,ba} M_{k,p} M_{k,p1} M_{k,p2} M_{k,p3} \right] \tag{B.7}
\]

with

\[
M_{k,\theta} = -\frac{a}{b} w^T C (q_w^{c_k})^T [c u_r \times]^T \phi_{11} - w n^T \phi_{51} \right] \tag{B.8}
\]

\[
M_{k,bq} = -\frac{a}{b} w^T C (q_w^{c_k})^T [c u_r \times]^T \phi_{12} - w n^T \phi_{52} \right] \tag{B.9}
\]

\[
M_{k,v} = -w^T \phi_{53} \right] \tag{B.10}
\]

\[
M_{k,ba} = -w^T \phi_{54} \right] \tag{B.11}
\]

\[
M_{k,p} = -w^T n^T \right] \tag{B.12}
\]

\[
M_{k,p1} = \left( (p_{w}^{F_3} - p_{w}^{F_1}) \times \right) (p_{w}^{F_2} - p_{w}^{I_k})^T \right] \tag{B.13}
\]

\[
M_{k,p2} = \left( w n \right) + \left( (p_{w}^{F_3} - p_{w}^{F_1}) \times \right) (p_{w}^{F_2} - p_{w}^{I_k})^T \right] \tag{B.14}
\]

\[
M_{k,p3} = \left( (p_{w}^{F_2} - p_{w}^{F_1}) \times \right) (p_{w}^{F_2} - p_{w}^{I_k})^T \right] \tag{B.15}
\]

where \( \phi_k \) are integral terms defined in Hesch et al. (2012).
Since Hesch et al. (2012) showed that $\phi_{53} = (k - 1)\delta t$, we can further expand

$$M_{k,v} = -(k - 1)\delta t^w n^T$$  \hspace{1cm} (B.16)

Similarly, since

$$\phi_{11} = C(q^{i_k}_{11})$$  \hspace{1cm} (B.17)

and

$$\phi_{51} = \left[ p^{i_k}_w - v^{i_k}_w(k - 1)\delta t - \frac{1}{2} w g(k - 1)^2 \delta t^2 - p^{i_k}_w \times \right] C(q^{w}_{i_1})$$  \hspace{1cm} (B.18)

We can write

$$M_{k,\theta} = -\frac{a}{b} n^T C\left(q^{c_k}_{w}\right)^T [c_{u_r} \times] C(q^{i_k}_{11})$$

$$- w n^T \left[ p^{i_k}_w - v^{i_k}_w(k - 1)\delta t - \frac{1}{2} w g(k - 1)^2 \delta t^2 - p^{i_k}_w \times \right] C(q^{w}_{i_1})$$

$$= w n \left(- \frac{a}{b} C\left(q^{c_k}_{w}\right)^T [c_{u_r} \times] C(q^{i_k}_{w})ight)$$

$$- \left[ p^{i_k}_w - v^{i_k}_w(k - 1)\delta t - \frac{1}{2} w g(k - 1)^2 \delta t^2 - p^{i_k}_w \times \right] C(q^{w}_{i_1})$$  \hspace{1cm} (B.19)

We can write

$$M_k N_s = - w n^T p^{i_k}_w - (k - 1)\delta t^w n^T v^{i_k}_w + w n^T \phi_{54} i_{aw}$$

$$+ \left( [p^{F_3}_w - p^{F_2}_w] \times (p^{F_2}_w - p^{l_1}_w) \right)^T p^{F_1}_w$$

$$+ \left( w n + [p^{F_3}_w - p^{F_2}_w] \times (p^{F_2}_w - p^{l_1}_w) \right)^T p^{F_2}_w$$

$$+ \left( [p^{F_2}_w - p^{F_1}_w] \times (p^{F_1}_w - p^{l_1}_w) \right)^T p^{F_3}_w$$  \hspace{1cm} (B.20)

$$= w n \left(- \frac{a}{b} C\left(q^{c_k}_{w}\right)^T [c_{u_r} \times] C(q^{i_k}_{w})ight)$$

$$- \left[ p^{i_k}_w - v^{i_k}_w(k - 1)\delta t - \frac{1}{2} w g(k - 1)^2 \delta t^2 - p^{i_k}_w \times \right] C(q^{w}_{i_1})$$

End of the proof.

### B.2 Observability Under Constant Acceleration

In this subsection, we want to prove that under a constant acceleration $a^i_{aw}$, the vector

$$N_s = \begin{bmatrix} p^{i_k}_w & v^{i_k}_w & 0_{6 \times 1}^T & -i a^i_{aw}^T & p^{F_3}_w & \ldots & p^{F_N}_w \end{bmatrix}^T$$  \hspace{1cm} (B.21)

is not in the right nullspace of $M_k$. Hence, we need to demonstrate that $M_k N_s \neq 0$.

$$M_k N_s = - w n^T p^{i_k}_w - (k - 1)\delta t^w n^T v^{i_k}_w + w n^T \phi_{54} i_{aw}$$

$$+ \left( [p^{F_3}_w - p^{F_2}_w] \times (p^{F_2}_w - p^{l_1}_w) \right)^T p^{F_1}_w$$

$$+ \left( w n + [p^{F_3}_w - p^{F_2}_w] \times (p^{F_2}_w - p^{l_1}_w) \right)^T p^{F_2}_w$$

$$+ \left( [p^{F_2}_w - p^{F_1}_w] \times (p^{F_1}_w - p^{l_1}_w) \right)^T p^{F_3}_w$$  \hspace{1cm} (B.22)
Wu and Roumeliotis (2016) show that, under constant acceleration,

\[ \phi_{54}^i a_i^w = - (p_{w}^{i_k} - p_{w}^{i_j} - (k - 1) \delta t v_{w}^{i_j}) \]  

so

\[ \begin{align*}
M_k N_s &= - w n^T p_{w}^{i_k} - (k - 1) \delta t w n^T v_{w}^{i_k} - w n^T (p_{w}^{i_k} - p_{w}^{i_j} - (k - 1) \delta t v_{w}^{i_j}) \\
&+ \left[ (p_{w}^{F_3} - p_{w}^{F_2}) \times \right] T p_{w}^{F_1} \\
&+ \left[ (w n + (p_{w}^{F_3} - p_{w}^{F_2}) \times \right] (p_{w}^{F_2} - p_{w}^{I_1}) T p_{w}^{F_2} \\
&+ \left[ (p_{w}^{F_2} - p_{w}^{F_1}) \times \right] (p_{w}^{F_1} - p_{w}^{I_1}) T p_{w}^{F_1} \\
&= w n^T (p_{w}^{F_2} - p_{w}^{I_1}) + (p_{w}^{F_2} - p_{w}^{I_1}) T \left[ (p_{w}^{F_3} - p_{w}^{F_2}) \times \right] T p_{w}^{F_1} \\
&+ \left[ (p_{w}^{F_2} - p_{w}^{F_3}) \times \right] T p_{w}^{F_2} + \left[ (p_{w}^{F_2} - p_{w}^{F_1}) \times \right] T p_{w}^{F_1}
\end{align*} \]  

The cross product of the first term can be modified such that

\[ \begin{align*}
(p_{w}^{F_2} - p_{w}^{I_1})^T &\left[ (p_{w}^{F_3} - p_{w}^{F_2}) \times \right] T p_{w}^{F_1} \\
&= (p_{w}^{F_2} - p_{w}^{I_1})^T \left[ (p_{w}^{F_3} - p_{w}^{F_2}) \times \right] T (p_{w}^{F_3} - p_{w}^{I_1} + p_{w}^{I_1}) \\
&= (p_{w}^{F_2} - p_{w}^{I_1})^T \left[ (p_{w}^{F_3} - p_{w}^{F_2}) \times \right] T p_{w}^{I_1}
\end{align*} \]

By definition of the cross product,

\[ \exists \lambda \in \mathbb{R}, \left[ (p_{w}^{F_3} - p_{w}^{F_2}) \times \right] T (p_{w}^{F_3} - p_{w}^{I_1}) = \lambda w n \]  

and

\[ \lambda (p_{w}^{F_2} - p_{w}^{I_1})^T w n = 0 \]  

Thus, by applying this to all cross-product terms,

\[ \begin{align*}
M_k N_s &= w n^T (p_{w}^{F_2} - p_{w}^{I_1}) \\
&+ (p_{w}^{F_2} - p_{w}^{I_1})^T \left[ (p_{w}^{F_3} - p_{w}^{F_2} + p_{w}^{F_1} - p_{w}^{F_3} + p_{w}^{F_2} - p_{w}^{F_1}) \times \right] T p_{w}^{I_1} \\
&= w n^T (p_{w}^{F_2} - p_{w}^{I_1})
\end{align*} \]

By definition, if the three features of the facet are not aligned in the image,

\[ w n^T (p_{w}^{F_2} - p_{w}^{I_1}) \neq 0. \]

End of proof.
As a note in the hover case, looking back at Equation (B.22), we can note that when \( \mathbf{v}_{i_1}^w = \mathbf{0} \), then \( \mathbf{p}_{i_1}^w = \mathbf{p}_w^k \) and
\[
\mathbf{N}_h = \begin{bmatrix} 
\mathbf{0}_{24 \times 1}^T & \mathbf{p}_w^{F_1} & \cdots & \mathbf{p}_w^{F_N} \end{bmatrix}^T
\]
becomes a vector of the right nullspace, which spans the depth of the SLAM features not included in the facet, as discussed in Section 3.2.
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