Some universal relations between the gap and thermodynamic functions plausible for various models of superconductors

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It is proven that there exist some universal relations between the energy gap and the differences of the thermodynamic potential, entropy, specific heat and critical magnetic field for many two- and three-dimensional models of superconductivity with spin-singlet Cooper pairs forming $s$ or $d$ or $g$ etc. states. The obtained formulae make it possible to derive thermodynamic functions and, in particular, the superconducting specific heat and the critical magnetic induction in the whole temperature range $0 \leq T \leq T_c$ employing the form of $\Delta(T)$ only. The inverse formula allowing us to find $\Delta(T)$, when the temperature dependence of the specific heat difference is known, is also presented. The results are referred to some obtained within the McMillan formalism, and some remarks on an application of the present formulae to the $t$–$J$ model are given.

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I. INTRODUCTION

Nowadays, there exist several approaches which are able to explain properties of high-$T_c$ and other new-generation superconductors. In most of these approaches only one common microscopic mechanism responsible for superconductivity in all high-$T_c$ superconductors is presented. However, this mechanism has not been univocally identified so far, although particularly preferred is the coupling through electron-electron interaction $\Delta$, or spin exchange, as well as a modified phonon-mediated established by Anderson, based on the concept of the resonating valence bond states. On the other hand, in all these approaches the pairing interaction is modeled by means of the exchange-mediated coupling which leads to the formation of Cooper pairs. Then, in order to explain the enhancement of the transition temperature, one can employ the Van Hove scenario. Recently it has been shown that starting in the real space by setting a Hamiltonian, which contains atomic or itinerant electrons energy, all hopping and exchange integrals and all two-site interactions, and then solving the model Hamiltonian within the mean-field approximation, one can transfer the problem into the reciprocal space. Performing a specific conformal transformation of the reciprocal space, one can reduce the problem to the BCS-type one, where all elements of the symmetry and the dispersion relation are gathered in the Jacobian of the conformal transformation. Such Jacobian describes a specific scalar field of the density of states which is imposed on the BCS-type isotropic reciprocal space, and which can contain some singularities and nodes. The Jacobian appears in all equations and its mean value can be identified with the density of states. Therefore, the proposed formalism is an extension of the Van Hove scenario and can be applied for non-$s$-wave symmetry systems.

In the present paper we demonstrate that for a wide class of two- and three-dimensional models of superconductors with electrons or holes as carriers, and spin-singlet Cooper pairs forming a $s$ or $d$ or $g$ etc. state, there exist some universal relations between the energy gap and the thermodynamic potential, entropy and specific heat differences defined between the superconducting and the normal phase. Moreover, we show that the obtained formulae allow us to derive the critical magnetic induction and its derivatives. We also discuss the case when strong-coupling electron-phonon effects are included, and we give remarks on an application of the present formulae to a relevant model of superconductivity i.e. the $t$–$J$ model.

II. GAP EQUATION

Although the gap equation can be obtained in a self-consistent manner (e.g. in the Green function formalism) we emphasize that it defines the energy gap which minimizes the free energy, so it can be found from the free energy variation. Because in all known approaches the one-particle energy of eigenstates is modified by the term $E_k = \sqrt{(\epsilon_k - \mu)^2 + |\Delta_k|^2}$, we postulate that the gap equation, which can be obtained in various approaches, formulated for two- or three-dimensional systems can be always written in the following general form

$$\Delta_k = \frac{1}{N} \sum_{k'} V_{k,k'} \frac{\Delta_{k'}}{E_{k'}} f \left( \frac{\epsilon_{k'} - \mu}{T}, \frac{|\Delta_{k'}|}{T} \right),$$  \hspace{1cm} (1)
where $V_{k,k'}$ is a pairing interaction, $\mu$ denotes the chemical potential and $f$ is a certain analytic function properties of which are defined below. This equation should become complete with another self-consistent equation

$$2n = \frac{1}{N} \sum_k \left[ 1 - \frac{\epsilon_k - \mu}{E_k} f \left( \frac{\epsilon_k - \mu}{T}, \frac{|\Delta_k|}{T} \right) \right], \quad (2)$$

which gives the chemical potential $\mu$ as a function of the filling of the conduction band $n$, for the normal metallic phase. Eq. (1) after performing the conformal transformation of the momentum space \[17\] turns into the form

$$\Delta(\xi, \vartheta, \varphi) = \frac{1}{4\pi^{d-1}} \left\{ \int_{-\omega_d}^{\omega_u} \frac{V(\xi, \xi', \vartheta, \vartheta', \varphi, \varphi') J(\xi', \vartheta', \varphi') \Delta(\xi', \vartheta', \varphi')}{\sqrt{(\xi' - \mu^*)^2 + |\Delta(\xi', \vartheta', \varphi')|^2}} d\xi' \right\}, \quad (3)$$

where $d = 2$, or 3 is the dimension of the reciprocal space, $\mu^*$ defines the chemical potential measured from the Fermi level, so $\mu = \epsilon_F + \mu^*$ and $\xi$ measures the energy from the Fermi level. $\omega_u$ and $\omega_d$ mark the limits of integration which are the Debye energy $\omega_D$ for sufficiently wide conduction bands and one of them or both must be replaced by conduction band borders for partially-filled bands of width $2\omega_b = \omega_u + \omega_d$, when $\omega_b < \omega_D$. Then $\omega_u$ and $\omega_d$ can be identified with non-symmetric cut-off parameters of the pairing interaction, and for sufficiently narrow conduction bands all particles get paired. When the system is almost but less than half-filling and $n = 1 - \delta$, where $\delta$ is small, we have

$$\omega_d = (1 - \delta)\omega_b \quad \text{and} \quad \omega_u = (1 + \delta)\omega_b. \quad (4)$$

So, for some superconducting systems the Debye energy $\omega_D$ must be replaced by the half band width $\omega_b$ after including additional factors connected with the filling of the band $n$. Therefore, we can state that the range of the pairing interaction is given by the energy $\omega$ which represents $\omega_D$ or $\omega_b$ according to Eq. (4) and hereafter we apply the following notation

$$\omega_d = n\omega \quad \text{and} \quad \omega_u = (2 - n)\omega, \quad (5)$$

where for the case of a symmetric pairing interaction as e.g. for the BCS-type models one has to put $n = 1$ and $\omega = \omega_D$. Moreover, $< \ldots >$ denotes the average over spherical angles $\vartheta'$ and $\varphi'$ for the three-dimensional case and over the angle $\varphi'$ only for the two-dimensional case, where $\vartheta' = \vartheta = \frac{\pi}{2}$ are consistently omitted. $J(\xi, \vartheta, \varphi)$ is the Jacobian of the employed conformal transformation, and the pairing interaction for pure $s, d, \ldots$ pairing ($l = 0, 2, \ldots$) can be taken in the form

$$V(\xi, \xi', \vartheta, \vartheta', \varphi, \varphi') = V_l(\xi, \xi') \sum_{m=-l}^{l} Y_{lm}(\vartheta, \varphi) Y_{lm}^{*}(\vartheta', \varphi'), \quad (6)$$

where $Y_{lm}(\vartheta, \varphi)$ are the spherical harmonics. Supposing that $V_l(\xi, \xi')$ contains only the attractive part of interaction, we can replace it by $g_l v_l(\xi) v_l(\xi')$. Then, for systems of strongly screened electrons we can assume $v_l(\xi) = 1$, whereas for a strong electronic repulsion $v_l(\xi)$ must be an odd function of quasiparticle energy \[18\, 19\, 20\]. $g_l$ is a coupling constant and its effective value should be obtained in a comprehensive renormalizing procedure including e.g. strong coupling effects and Coulomb repulsion of electrons forming a Cooper pair \[21\]. In order to be in accordance with the specified conditions we have to allow for the energy gap to be of the following form

$$\Delta(\xi, \vartheta, \varphi) = \Delta(T) \, D_l(\vartheta, \varphi) \, v_l(\xi), \quad (7)$$

where $\Delta(T)$ is the amplitude of the energy gap depending on temperature only, and

$$D_l(\vartheta, \varphi) = \sum_{m=-l}^{l} d_m(l) Y_{lm}(\vartheta, \varphi) \quad (8)$$

expresses the spatial structure of the energy gap on equi-energy surfaces. $d_m(l)$ are normalization constants chosen such, that the condition $< |D_l(\vartheta, \varphi)|^2 > = 1$ is satisfied. Moreover, since for the most models under consideration the function $f$ is of the form

$$f \left( \frac{\xi - \mu^*}{T}, \frac{|\Delta(\xi, \vartheta, \varphi)|}{T} \right) = \tanh \left( \frac{\sqrt{(\xi - \mu^*)^2 + |\Delta(\xi, \vartheta, \varphi)|^2}}{2T} \right), \quad (9)$$
as a consequence of the relation tanh(\(\frac{\xi}{2}\)) = 1 - 2n(x), where n(x) is the mean occupation number, we assume that in the limit \(T = 0\) it reduces to 1, whereas for \(\Delta(\xi, \vartheta, \varphi) = 0\) we have \(f(x, 0) \sim x\) when \(x \to 0\). After employing Eq. (7) and including Eq. (3) the gap equation (4) can be reduced to the form

\[
\frac{1}{g_{\mu}} \left( \frac{(2-n)\omega}{J[(2-n)\omega, \vartheta, \varphi]} \right) \ln[(2-n)\omega] + \frac{1}{4\pi^{d-1}} \left( \frac{J[(\xi, \vartheta, \varphi)]D_{l}(\vartheta, \varphi)}{(\xi - \mu^{*})^{2} + v_{l}^{2}(\xi)[D_{l}(\vartheta, \varphi)]^{2}D^{2}(T)} \right) d\xi.
\]  

(10)

The obtained formula allows us to derive \(\Delta(T)\) for a given structure of the energy gap when the Jacobian, \(v_{l}(\xi)\) and the function \(f\) are known.

Let us focus now on the cases when \(v_{l}(\xi) \equiv 1\) for all \(\xi\) or \(v_{l}(\xi) \equiv \text{const}\) for \(|\xi - \mu^{*}|\) less than a few \(T_{c}\), then we denote it by \(v_{l}\), and when the conduction band is wide enough to imply \(\Delta(0) \ll \omega_{b}\), and when the chemical potential \(\mu\) is independent of temperature and equal to its value for a normal metal at \(T = 0\), and \(\mu^{*}\) can be omitted \[22\]. Then the above conditions allow us to ignore changes of \(v_{l}(\xi)\) for other values of \(|\xi - \mu^{*}|\) and leave Eq. (2) out of account [23]. Hence, in the limit \(T = 0\) we have \(f = 1\) and integrating by parts over \(\xi\) the rhs of Eq. (10) we obtain

\[
\frac{4\pi^{d-1}}{g_{\mu}v_{l}^{2}} = \left[ J[(2-n)\omega, \vartheta, \varphi]D_{l}(\vartheta, \varphi) \right] \ln[(2-n)\omega] + \left[ J[-n\omega, \vartheta, \varphi]D_{l}(\vartheta, \varphi) \right] \ln[2(2-n)\omega] - 2 \left[ J[0, \vartheta, \varphi]D_{l}(\vartheta, \varphi) \right] \ln[v_{l}D_{l}(\vartheta, \varphi)] - \left[ J[(2-n)\omega, \vartheta, \varphi] + J[-n\omega, \vartheta, \varphi] \right] \ln \Delta(0) - \left[ D_{l}(\vartheta, \varphi) \right]^{2} \int \ln[u + \sqrt{u^{2} + v_{l}^{2}[D_{l}(\vartheta, \varphi)]^{2}]} \frac{\partial}{\partial u}J(u\Delta(0), \vartheta, \varphi) du.
\]  

(11)

If the last integral can be considered as constant to the postulated accuracy, Eq. (11) can be reduced to the form

\[
\frac{1}{g_{\mu}} = N(n) \ln \frac{A}{\Delta(0)},
\]  

(12)

where \(A\) is positive and may be considered as independent of \(\Delta(0)\), and

\[
N(n) = \frac{v_{l}^{2}}{4\pi^{d-1}} \left[ J[(2-n)\omega, \vartheta, \varphi] + J[-n\omega, \vartheta, \varphi] \right] \ln \Delta(0)
\]  

(13)

can be understood as a specifically defined mean value of the density of states [17]. Because the obtained relations hold for an arbitrary value \(\Delta(T)\), we have

\[
N(n) \ln \frac{A}{\Delta(0)} = \frac{1}{4\pi^{d-1}} \left( \int_{-n\omega}^{(2-n)\omega} \frac{J[(\xi, \vartheta, \varphi)]D_{l}(\vartheta, \varphi)\ln[(\xi - \mu)^{2} + v_{l}^{2}(\xi)[D_{l}(\vartheta, \varphi)]^{2}D^{2}(T)}{D^{2}(T)} d\xi \right).
\]  

(14)

Hence, subtracting Eq. (14) from Eq. (10) we can transform the gap equation to the form

\[
\frac{1}{\lambda} = \ln \frac{A}{\Delta(T)} + F \left( n, \omega \frac{T}{T_{c}}, \Delta(T), \frac{T}{T_{c}} \right),
\]  

(15)

where \(\lambda = g_{\mu}N(n)\) is the dimensionless coupling parameter, \(F\) is a functional of \(\Delta(T)\) obtained after integration over \(\xi\) and the angles \(\vartheta\) and \(\varphi\) for a given form of the Jacobian, the energy gap structure and the function \(f\).

Let us investigate the form of the gap equation (10) in the limit \(T = T_{c}\) when \(\Delta(T_{c}) = 0\). Integrating by parts over \(\xi\) and taking into account the above assumptions we can transform it to the following form

\[
\frac{4\pi^{d-1}N(n)}{\lambda v_{l}^{2}} = \left[ J[(2-n)\omega, \vartheta, \varphi]D_{l}(\vartheta, \varphi) \right] \ln[(2-n)\omega] + \left[ J[-n\omega, \vartheta, \varphi]D_{l}(\vartheta, \varphi) \right] \ln[n\omega] - \left[ f \left( \frac{(2-n)\omega}{T_{c}} \right) J[(2-n)\omega, \vartheta, \varphi] + f \left( -n\omega \frac{T}{T_{c}} \right) J[-n\omega, \vartheta, \varphi] \right] D_{l}(\vartheta, \varphi) \ln T_{c} - \int_{0}^{2\pi} \ln u \frac{\partial}{\partial u} \left[ f(u) J(uT_{c}, \vartheta, \varphi) D_{l}(\vartheta, \varphi) \right] du
\]  

(16)
which can be written as
\[ \frac{1}{\lambda} = \frac{1}{b} \ln \frac{B}{T_e}, \]
where \( B \) and \( b \) are positive and, due to the postulated accuracy, may be considered as independent of \( T_e \), and
\[ b = \frac{\langle [J((2 - n)\omega, \vartheta, \varphi) + J(-n\omega, \vartheta, \varphi)] |D_1(\vartheta, \varphi)\rangle^2}{\langle [f (2 - n)\omega, \vartheta, \varphi) + f (-n\omega, \vartheta, \varphi)] |D_1(\vartheta, \varphi)\rangle^2}. \]
is independent of \( v_1 \). Note that the characteristic ratio
\[ \frac{\Delta(0)}{T_e} = \frac{A}{B} \exp \left( \frac{b - 1}{\lambda} \right), \]
unlike in the BCS theory, is not a universal constant.

III. ENERGY GAP AS A FUNCTION OF THE COUPLING PARAMETER

The difference between the thermodynamic potential of the superconducting and the normal phase, which is equal to the free energy difference when the chemical potential of both phases is identical, can be derived from the relation
\[ \Delta \Omega(T) = -N(n) \int_0^\lambda \frac{\Delta(T)}{(\lambda')^2} d\lambda', \]
where we have taken into account that \( q_1 = \lambda / N(n) \). The formulas (12) and (17) allow us to express \( \Delta(0) \) and \( T_e \) by means of an arbitrary \( \lambda' \) varying from 0 to \( \lambda \) in the following forms
\[ \Delta_\lambda(0) = A \exp \left( -\frac{1}{\lambda'} \right) \quad \text{and} \quad T_\lambda = B \exp \left( -\frac{b}{\lambda'} \right), \]
where, to the assumed accuracy, we can also put \( A = A_1 \omega \) and \( B = B_1 \omega \), where \( A_1 \) and \( B_1 \) are real numbers. In order to keep the standard notation, hereafter we assume that \( \Delta_\lambda(0) = \Delta(0) \) and \( T_\lambda = T_e \), and moreover \( \Delta_\lambda(T) = \Delta(T) \). Note that \( T_e \) is a decreasing function of \( b \), and for the BCS model \( b = 1 \). Moreover, after eliminating \( \lambda \) from these formulas we can find the characteristic ratio (19) as a function of \( \omega \). Then
\[ \frac{\Delta(0)}{T_e} = A_1 B_1^{-\frac{1}{b}} \left( \frac{\omega}{T_e} \right)^{\frac{b+1}{b}} \quad \text{or} \quad \frac{\Delta(0)}{T_e} = A_1^b B_1^{-1} \left( \frac{\omega}{\Delta(0)} \right)^{b-1}, \]
and it becomes an increasing or decreasing function of \( \omega \) when \( b > 1 \) or \( b < 1 \), respectively.

Employing the formulas (21) we can find the thermodynamic potential difference in the limit \( T = 0 \), and we state that it can be expressed in the following universal form
\[ \Delta \Omega(0) = -\frac{1}{2} N(n) \Delta^2(0). \]
In order to derive the thermodynamic potential difference for non-zero temperature we have to express \( \Delta_\lambda'(T) \), i.e. the solution of the gap equation (15) for \( \lambda' \) varying from 0 to \( \lambda \), by means of \( \Delta(T') \) where \( T' \) can change from 0 to \( T_e \). Let us suppose that we can find the energy gap for all values \( \lambda' \) as a function of reduced temperature \( T/\Delta_\lambda(0) \). Then introducing the reduced gap in the form
\[ \Lambda_\lambda'(\frac{T}{\Delta_\lambda'(0)}) = \frac{\Delta_\lambda'(\frac{T}{\Delta_\lambda'(0)})}{\Delta_\lambda'(0)}, \]
where, moreover, \( \Lambda_\lambda(\ldots) \equiv \Lambda(\ldots) \), and employing Eq. (21) we can transform Eq. (15) to the form
\[ \ln \Lambda_\lambda'(\frac{T}{\Delta_\lambda'(0)}) = F \left( n, \frac{\omega}{T}, \frac{\Lambda_\lambda'(\frac{T}{\Delta_\lambda'(0)})}{\frac{T}{\Delta_\lambda'(0)}} \right). \]
Since this equation has the same form for all values \( \lambda' \) from 0 to \( \lambda \) when \( T, \omega \) and \( n \) are fixed, all forms of \( \Lambda_{\lambda'} \left( T/ \Delta_{\lambda'}(0) \right) \) derived from Eq. (25) must be equivalent. They become identical for temperatures normalized by corresponding critical temperatures denoted as \( \tau \). This observation allows us to express the reduced gap derived for an arbitrary \( \lambda' \) by means of the reduced gap \( \Lambda \), which represents the real energy gap of a superconductor

\[
\Lambda_{\lambda'} \left( \tau \cdot \frac{T_{\lambda'}}{\Delta_{\lambda'}(0)} \right) = \Lambda \left( \tau \cdot \frac{T_c}{\Delta(0)} \right).
\]

Hence, after some transformations we find that the value of an arbitrary reduced gap \( \Lambda_{\lambda'} \) at \( T/T_{\lambda'} \) can be expressed by means of the function \( \Lambda \) as

\[
\Lambda_{\lambda'} \left( \frac{T}{T_{\lambda'}} \right) = \Lambda \left( \frac{T}{T_c} \cdot \left( \frac{T_c}{T_{\lambda'}} \right)^2 \cdot \frac{\Delta_{\lambda'}(0)}{\Delta(0)} \right),
\]

and the inclusion of Eqs. (24) and (25) results in the following fundamental relation

\[
\Delta_{\lambda'} \left( \frac{T}{T_{\lambda'}} \right) = \Delta \left( \frac{T}{T_c} \cdot \exp \left( (2b - 1) \frac{1}{\lambda} \right) \right) \cdot \exp \left[ - \left( \frac{1}{\lambda} - \frac{1}{\lambda'} \right) \right].
\]

Thus, in order to find the values of the energy gap for \( \lambda' < \lambda \) and for a fixed temperature \( T \), we have to know the real energy gap values for all higher temperature values if \( b > \frac{1}{2} \), and for all lower temperature values if \( b < \frac{1}{2} \). For \( b = \frac{1}{2} \) the temperature dependence coincides.

IV. UNIVERSAL RELATIONS

The derivation of the relation (25) creates new possibilities to find the thermodynamic potential difference and its derivatives with respect to temperature as functionals of \( \Delta(T) \). Substituting Eq. (25) into formula (20), after some calculations, we find the thermodynamic potential for \( b \neq \frac{1}{2} \) in the form

\[
\Delta \Omega(T) = -\frac{1}{2b-1} N(n)T^{\frac{2}{2b-1}} \int_T^{T^*} \frac{\Delta^2(T')}{(T')^{\frac{2b}{2b-1}+1}} dT',
\]

where \( T^* = T_c \) if \( b > \frac{1}{2} \) because \( \Delta(T) = 0 \) for \( T \geq T_c \), and \( T^* = 0 \) if \( b < \frac{1}{2} \), whereas for \( b = \frac{1}{2} \) we have

\[
\Delta \Omega(T) = -\frac{1}{2} N(n)\Delta^2(T),
\]

and hereafter we do not consider this case. The formula (29) can be also written in an another quite equivalent form

\[
\Delta \Omega(T) = -\frac{1}{2b-1} N(n)T^{\frac{2}{2b-1}} \int_T^{T^*} \frac{\Delta^2(T') - \Delta^2(T)}{(T')^{\frac{2b}{2b-1}+1}} dT' - \frac{1}{2} N(n)\Delta^2(T) \left[ 1 - \left( \frac{T}{T_c} \right)^{\frac{2}{2b-1}} \Theta \left( b - \frac{1}{2} \right) \right],
\]

which can turn out to be more convenient for investigations. \( \Theta(x) \) is the Heaviside step function. Since

\[
\Delta S(T) = - \frac{d}{dT} \Delta \Omega(T) \quad \text{and} \quad \Delta C(T) = -T \frac{d^2}{dT^2} \Delta \Omega(T),
\]

where \( \Delta S(T) \) and \( \Delta C(T) \) are differences of the entropy and the specific heat, respectively, differentiating the formula (29) we obtain

\[
\Delta S(T) = \frac{2}{(2b-1)^2} N(n)T^{\frac{2}{2b-1} - 1} \int_T^{T^*} \frac{\Delta^2(T')}{(T')^{\frac{2b}{2b-1}+1}} dT' - \frac{1}{2} N(n)\Delta^2(T) \left( \frac{T}{T_c} \right)^{\frac{2}{2b-1} - 1} \Theta \left( b - \frac{1}{2} \right).
\]

which can be also written in an another quite equivalent form

\[
\Delta S(T) = \frac{2}{(2b-1)^2} N(n)T^{\frac{2}{2b-1} - 1} \int_T^{T^*} \frac{\Delta^2(T') - \Delta^2(T)}{(T')^{\frac{2b}{2b-1}+1}} dT' - \frac{1}{2b-1} N(n)\Delta^2(T) \left( \frac{T}{T_c} \right)^{\frac{2}{2b-1} - 1} \Theta \left( b - \frac{1}{2} \right).
\]
Now, employing formula \(33\) we can find the specific heat difference in the form

\[
\Delta C(T) = \frac{2(3 - 2b)}{(2b - 1)^3} N(n) T \frac{\pi^2}{6} \frac{1}{T^2} \int_T^{T_c} \frac{\Delta^2(T')}{(T')^{\frac{3}{2} - 1}} dT' - \frac{3 - 2b}{(2b - 1)^2} N(n) \Delta^2(T) \frac{2}{2b - 1} N(n) \Delta(T) \frac{d}{dT} \Delta(T),
\]

which can be also written in another quite equivalent form

\[
\Delta C(T) = \frac{2(3 - 2b)}{(2b - 1)^3} N(n) T \frac{\pi^2}{6} \frac{1}{T^2} \int_T^{T_c} \frac{\Delta^2(T') - \Delta^2(T)}{(T')^{\frac{3}{2} - 1}} dT' - \frac{3 - 2b}{(2b - 1)^2} N(n) \Delta^2(T) \frac{2}{2b - 1} N(n) \Delta(T) \frac{d}{dT} \Delta(T).
\]

The obtained formulas include the parameter \(b\) which is an arbitrary positive number. Hence, if we could identify \(b\) we would be able to find the forms of the thermodynamic potential (free energy), the entropy and the specific heat for the superconducting phase based upon \(\Delta(T)\) shape. Note that for BCS-like models \(b = 1\), and this case has been considered by us in details before \[24, 25, 26, 27\]. On the other hand, if we knew \(\Delta(T)\) and \(\Delta C(T)\), then by virtue of the presented formalism we could estimate \(b\). We emphasize that in general \(N(n)\), i.e. the density of states defined above for the superconducting phase, is different from that one defined for the normal phase. This difference can be explained as a modification of the effective mass, and must be included in all normal-phase thermodynamic functions when we derive the thermodynamic functions for the superconducting phase from the above formulas. This procedure is necessary in some approaches \[24, 25\] to eliminate non-physical results, e.g. negative values of the superconducting specific heat. Note that the obtained Eqs. \[33\] and \[34\] imply a formal relation between \(\Delta S(T)\) and \(\Delta C(T)\), which can be written in the form

\[
\Delta C(T) = \frac{3 - 2b}{2b - 1} \Delta S(T) - \frac{1}{2b - 1} N(n) \frac{d}{dT} \Delta^2(T).
\]

In order to consider the specific heat in low temperature limit \((T \to 0)\), we have to take into account the normal specific heat \(C_N(T)\) which for this case in the frame of the presented formalism can be found in the form

\[
C_N(T) = T \int_0^\infty \tilde{N}(2Tu) u^2 \cosh^{-2}(u) du,
\]

where

\[
\tilde{N}(2Tu) = \frac{2}{\pi^{d-1}} (J(2Tu, \vartheta, \varphi) + J(-2Tu, \vartheta, \varphi))
\]

is another form of the density of states (cf. Eq. \[13\]). Since \(\tilde{N}(2Tu)\) depends on \(T\), \(C_N(T)\) it can become a complicated function of temperature. According to Eq. \[37\] and including Eq. \[33\] the superconducting specific heat can be expressed in the form

\[
C_S(T) = \frac{3 - 2b}{2b - 1} \Delta S(T) - \frac{2}{2b - 1} N(n) \Delta(0) \frac{d}{dT} \Delta(T) + C_N(T).
\]

Hence, we state that the formula for the superconducting specific heat can contain several different terms which determine the temperature dependence. For the BCS model with \(s\) - or \(d\)-pairing the superconducting specific heat is proportional to \(dT / dT\) only, since for \(b = 1\) the term \(\Delta S(T)\) is proportional to \(T\) and it is compensated by \(C_N(T)\) \[11, 24, 27\].

Let us now consider the specific heat jump at the critical temperature \(T = T_c\). In order to do that comprehensively, we have to assume that for \(T \to T_c\) the energy gap \(\Delta(T) \to \Delta(T_c)\) and \(\Delta(T_c) = 0\) for the second-order (continuous) transition or \(\Delta(T_c) > 0\) for the first-order (discontinuous) transition, and for \(T > T_c\) the energy gap vanishes, i.e. \(\Delta(T) \equiv 0\). Then, the specific heat jump can be found from Eq. \[37\] in the limit \(T \to T_c\) as

\[
\Delta C(T_c) = \frac{3 - 2b}{2b - 1} \Delta S(T_c) - \frac{2}{2b - 1} N(n) \lim_{T \to T_c} \Delta(T) \frac{d}{dT} \Delta(T).
\]

Since for continuous phase transitions \(\Delta S(T_c) = 0\), the obtained formula coincides with that known for BCS-type models \[15, 16, 24\].
V. RECONSTRUCTION OF THE ENERGY GAP

In the previous section we have shown that the thermodynamic potential, the entropy and the specific heat differences can be defined based upon the temperature dependence of the energy gap only. In this section we show that also the energy gap $\Delta(T)$ can be reconstructed from the specific heat difference if the values of $b$ and $\Delta S(0)$ are known.

Employing Eqs. (32) and (37), after some transformations, we can express the energy gap by means of $\Delta C(T)$ and $\Delta S(0)$ in the form

$$\Delta^2(T) = \Delta^2(0) + \frac{3 - 2b}{N(n)}T \Delta S(0) - \frac{3 - 2b}{N(n)}(2b - 1) \Delta C(T') dT' + \frac{1}{N(n)} \int_0^T \left[ (3 - 2b) \int_0^{T'} \frac{\Delta C(T'')}{T''} dT'' - (2b - 1) \Delta C(T') \right] dT'. \quad (42)$$

Hence, for the cases when $C_N(T) \sim T^q$, where according to the Nernst law $q > 0$, Eq. (42) can be reduced to the form

$$\Delta^2(T) = \Delta^2(0) + \frac{3 - 2b}{N(n)}T \Delta S(0) - \frac{3 - 2b}{N(n)}(2b - 1) \Delta C(T') dT' + \frac{1}{N(n)} \int_0^T \left[ (3 - 2b) \int_0^{T'} \frac{C_S(T'')}{T''} dT'' - (2b - 1) C_S(T') \right] dT', \quad (43)$$

which for BCS-type models, when $\Delta S(0) = 0$ and $b = q = 1$, includes $C_S(T)$ only [24].

VI. CRITICAL MAGNETIC INDUCTION

The formulas obtained in Section IV allow us to find the critical magnetic field $H_c(T)$ or the critical magnetic induction $B_c(T) = \mu_0 H_c(T)$ and their derivatives for superconducting systems as functionals of $\Delta(T)$, only. In accordance to the thermodynamic relation

$$G_S(T, H = 0) - G_N(T, H = 0) = -\frac{1}{2} \mu_0 H_c^2(T) \quad (44)$$

after taking into account that $G_S(T, H = 0) - G_N(T, H = 0) = \Delta \Omega(T)$ when the chemical potential is independent of temperature and identical in both phases, the critical magnetic induction can be presented in the form

$$B_c(T) = B_c(0) \sqrt{\frac{-2 \Delta \Omega(T)}{N(n) \Delta^2(0)}}, \quad (45)$$

where

$$B_c(0) = \sqrt{2 \mu_0 N(n)} \Delta(0), \quad (46)$$

and $\Delta \Omega(T)$ should be taken in one of the forms (31)–(33). In a similar way employing standard relations and including forms of Eqs. (35)–(38) we can express derivatives of the critical magnetic induction in the form

$$\frac{dB_c(T)}{dT} = \sqrt{\frac{\mu_0}{2}} \frac{\Delta \Omega(T)}{\sqrt{-\Delta \Omega(T)}} \quad (47)$$

$$\frac{d^2 B_c(T)}{d^2 T} = \sqrt{\frac{\mu_0}{2}} \left[ \frac{\Delta C(T)}{T \sqrt{-\Delta \Omega(T)}} - \frac{(\Delta S(T))^2}{(-\Delta \Omega(T))^2} \right], \quad (48)$$

which should be useful to estimate values of the introduced parameter $b$ for some unconventional superconductors by means of experimental data.

VII. REMARKS ON MCMILLAN’S CASE

In the formalism developed by McMillan [21], where strong-coupling electron-phonon effects were included, the critical temperature was derived in the form

$$T_c \sim \omega_D \exp \left( -\frac{1 + \lambda}{\lambda - \mu^* (1 + \lambda)} \right), \quad (49)$$
where the parameter $\mu^*$ expresses the Coulomb repulsion between electrons forming a Cooper pair, and for known materials $\mu^* \simeq 0.13$, $\omega_D$ is the Debye energy which is the actual cut-off parameter for conventional superconductors. Note that the critical temperature $T_c = 0$ for $\lambda = -\frac{\mu^*}{1 - \mu^*} > 0$, so in obedience to the presented consideration we should rather use an effective coupling constant which varies form 0 to its real value when the coupling interaction is included. Such effective coupling constant is $\lambda_{\text{eff}} = a^{-1}[\lambda - \mu^*(1 + \lambda)]$. Hence, putting it into Eq. (18) we obtain

$$T_c \sim \omega_D \exp\left(-\frac{1}{1 - \mu^*}\right) \exp\left(-\frac{b}{\lambda_{\text{eff}}}\right),$$

where $b^{-1} = a(1 - \mu^*) > 0$ and $a$ is positive constant which ensures that $\Delta(0)$ keeps the form (21). Hence, the formalism developed by us can be applied to the McMillan-type superconductors.

### VIII. APPLICATION OF DERIVED RELATIONS TO HIGH-$T_c$ SUPERCONDUCTORS

The mechanism of superconductivity for HTSC based on antiferromagnetic correlations was firstly proposed to predict antiferromagnetic behaviour for pure $\text{La}_2\text{CuO}_4$. As soon as these materials are doped, giving a few per cent of holes in the CuO$_2$ plane, the antiferomagnetism is destroyed. As the number of carriers $\delta$ is always small, they cannot completely screen the Coulomb interaction on copper site. Thus the copper electrons can move but are always subject to strong on-site repulsion. The introduction of additional holes causes that in sufficiently low temperatures they are coupled by superexchange interaction, through the oxygen atoms, forming Cooper pairs [30, 31, 32, 33, 34, 35]. The similar situation occurs in the compound $\text{YBa}_2\text{Cu}_3\text{O}_6$ which is an insulating antiferromagnetic. In this so-called $t$–$J$ model we have to put $b = 1$. Note that their solution is valid only for the range of doping $\delta \geq 0.05$, when the order parameter has the $s$ symmetry, they obtained the critical temperature $T_c$ in the form

$$T_c = 1.14 W(\mu) \exp[-1/JN(\mu)],$$

where $JN(\mu)$ corresponds to the dimensionless coupling constant $\lambda$, and $W(\mu)$ is proportional to $\omega_c$ (cf. Eq. (17) and Eq. (21)). Moreover, they showed that at zero temperature $\gamma_k(\mu)J\Delta(0) = 1.76T_c$. So, the $t$–$J$ model within the developed approach [22] reveals BCS-like properties with $b = 1$.

In order to employ the found universal relations to the $t$–$J$ model we have to put $b = 1$, since the other quantities or representing them expressions are not involved in these relations. Moreover, let us emphasize that in our method the summation over $\mathbf{k}$ is replaced by the integration over $\xi = \epsilon - \mu$ for an in general non-symmetric pairing interaction by virtue of the conformal transformation, in which the Jacobian represents the scalar field of the density of states [17]. It causes that our method is more precise than the one discussed above, and it allows us to predict that $b$ can be different than 1. Note that the three substantial assumptions mentioned above imply the following facts: the constancy of the density of states corresponds to putting the Jacobian constant, the inclusion of a symmetric pairing interaction is equivalent to $n = 1$, and for $\omega \equiv \omega_D$, same as the hyperbolic tangent, the function $f$ can be replaced by 1. As a consequence the parameter $b$ given by Eq. (18) reduces to 1.

### IX. CONCLUSIONS

The presented formalism proves that there exist some universal relations between the free energy, entropy, specific heat differences, and the energy gap amplitude for a wide class of models of superconductivity. In our previous
papers [24, 26, 27] we have shown that applying the formulae (35) or (36) established for the case when \( b = 1 \) towards a general s-paired BCS case, when the magnetic field and superflow are taken into account, and for a pure \( d \)-paired BCS model, we obtained correct forms of the superconducting specific heat, where the linear (with respect to temperature) terms were entirely eliminated by \( C_N(T) \). Therefore we expect now that these relations will be verified experimentally, and they will turn out to be helpful in explanation of common elements of the mechanism of low- and high-\( T_c \) superconductivity. On the other hand, we remind that we imposed some restrictions on conditions which must be satisfied by the superconducting system. Hence, the presented formalism can be applied to superconducting systems when the cut-off parameter \( \omega_D \) exceeds \( \Delta(0) \) or \( v_1 \Delta(0) \) by far, the chemical potential is identical in both phases, and the pairing interaction can be taken in a form corresponding to BCS-type models, which can be always attained by performing a BCS linearisation of the appropriate interaction term.
[1] W. Kohn, J.M. Luttinger, Phys. Rev. Lett. 15, 524 (1965).
[2] Yu.A. Krotov, D.H. Lee, A.V. Balatsky, Phys. Rev. B 56, 8367 (1997).
[3] M. Granath, H. Johannesson, Phys. Rev. Lett. 83, 199 (1999).
[4] P.W. Anderson, Science 64, 188 (1986).
[5] P.W. Anderson, Science 235, 1996 (1987).
[6] P.W. Anderson, The Theory of High-T_c Superconductivity in the Cuprates, (Princeton Univ. Press, 1997).
[7] J. Labbe, J. Bok, Europhys. Lett. 3, 1225 (1987).
[8] R.S. Markiewicz, Physica C 168, 195 (1990); Physica C 183, 303 (1991).
[9] C.C. Tsuei, D.M. Newns, C.C. Chi, P.C. Pattnaik, Phys. Rev. Lett. 65, 2724 (1990); Phys. Rev. Lett. 68, 1091 (1992).
[10] J. Bok, Physica C 209, 107 (1993).
[11] J. Bouvier and J. Bok, in: The Gap Symmetry and Fluctuations in High-T_c Superconductors, Eds J. Bok, G. Deutscher, D. Pavuna, S. Wolf Plenum Press, New York 1998, p. 37.
[12] J.M. Getino et al., Phys. Rev. B 48, 597 (1993).
[13] X. Tiefeng, B. Guiru, Z. Phys. B 89, 35 (1992).
[14] J.E. Hirsch, D.J. Scalapino, Phys. Rev. Lett. 56, 2732 (1986).
[15] R. Gonczarek and M. Mulak, Phys. Lett. A 251, 262 (1999).
[16] M. Gladysiewicz, R. Gonczarek, M. Mulak, Acta Phys. Polon. A 97, 1039 (2000).
[17] R. Gonczarek, M. Gladysiewicz, M. Mulak, Int. J. Mod. Phys. B 15, 495 (2001).
[18] F. Mila, E. Abrahams, Phys. Rev. Lett. 67, 2379 (1991).
[19] A. Balatsky, E. Abrahams, Phys. Rev. B 45, 13125 (1992).
[20] M.H. Cohen, Phys. Rev. Lett. 12, 664 (1964).
[21] M.L. McMillan, Phys. Rev. 167, 331 (1968).
[22] M. Cyrot, D. Mayou, J.P. Julien, F. Cyrot-Lackmann, J. Phys.: Condens. Matter 1, 4359 (1989).
[23] R. Gonczarek, M. Krzyzosiak, submitted to Phys. Rev. B.
[24] M. Mulak, R. Gonczarek, Acta Phys. Polon. A 89, 689 (1996).
[25] R. Gonczarek, M. Mulak, Phys. Stat. Sol. (b) 208, 457 (1998).
[26] R. Gonczarek, M. Mulak, W. Kumala Acta Phys. Polon. A 95, 231 (1999).
[27] W. Kumala, R. Gonczarek, M. Mulak, Acta Phys. Polon. A 97, 693 (2000).
[28] S. Drobolo, M. Houssa, M. Ausloos, Physica C 267, 242 (1996).
[29] R. Baquero, D. Quesada, C. Trallero-Giner, Physica C 271, 122 (1996).
[30] F.C. Zhang, T.M. Rice, Phys. Rev. B 37, 3759 (1988).
[31] M. Cyrot, Solid State Comm. 62, 821 (1987).
[32] M. Cyrot, Solid State Comm. 63, 1015 (1987).
[33] M. Cyrot, Nature 330, 115 (1987).
[34] G. Baskaran, Z. Zou, P.W. Anderson, Solid State Comm. 63, 973 (1987).
[35] V.J. Emery, Phys. Rev. Lett. 58, 2794 (1987).