On a conjecture of Atiyah

Rostislav I. GRIGORCHUK
Steklov Mathematical Institute, Gubkina Str. 8 Moscow, 117966, Russia
e-mail: grigorch@mi.ras.ru

Peter LINNELL
Department of Mathematics, Virginia Polytechnic Institute and State University
Blacksburg, VA 24061, USA
e-mail: linnell@math.vt.edu

Thomas SCHICK
FB Mathematik, Universität Münster, Einsteinstr. 62, 48159 Münster, Germany
e-mail: thomas.schick@math.uni-muenster.de

Andrzej ŻUK
CNRS, Ecole Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées
46, Allée d’Italie, F-69364 Lyon cedex 07, France
e-mail: azuk@umpa.ens-lyon.fr

Abstract. In this note we explain how the computation of the spectrum of the lamplighter group from [GZ01] yields a counterexample to a strong version of the Atiyah conjectures about the range of $L^2$-Betti numbers of closed manifolds.

MSC-number: 57N65

---

Sur une conjecture d’Atiyah

Résumé. Dans cette note on montre comment le calcul du spectre du groupe de l’allumeur de réverbères fait dans [GZ01] donne un contre-exemple à une des conjectures d’Atiyah sur les nombres de Betti $L^2$ des variétés fermées.

Version française abrégée

Dans [Ati76] Atiyah a demandé si les nombres de Betti $L^2$ des revêtements universels des variétés fermées sont toujours rationnels. Plus tard cette question a été transformée en conjecture sous le nom d’Atiyah ([Coh79, Theorem 8], [LL95, Conjecture 7.1], [Lin98, Conjecture 9.8], [Lüc98a, Conjecture 5.1], [Rei99, Conjecture 5.1], [Lüc00, Conjecture 13], [DS00, Definition 1.2]). La réponse ne dépend que du groupe fondamental. La plus célèbre parmi ces conjectures est la suivante : les nombres de Betti $L^2$ d’un groupe de présentation finie sans torsion sont entiers. La conjecture d’Atiyah forte dit que les nombres de Betti $L^2$ d’un groupe de présentation finie $G$ sont des rationnels dont les dénominateurs sont déterminés par les ordres des sous-groupes finis de $G$. Il y a aussi des versions encore plus fortes de la conjecture d’Atiyah,
quand le groupe n’est pas supposé être de présentation finie [Luc01]. Il y a des reformulation
équivalente de cette conjecture en terme de CW complexes et en terme des variétés.

Plusieurs résultats confirment différentes formes de la conjecture d’Atiyah ([Coh79], [Lin93], [Sch99], [DS00]). Un de ces résultats est le théorème de Linnell [Lin93] qui affirme que la conjecture d’Atiyah forte est vraie pour les groupes moyennables élémentaires avec une borne uniforme sur l’ordre des sous-groupes finis et même pour une famille plus large de groupes contenant les groupes libres. D’autre part les résultats de [DS00] essentiellement montrent que la famille des groupes pour lesquels la conjecture d’Atiyah forte est vraie est fermée sous les HNN extensions et les produits amalgamés sous l’hypothèse que les ordres des sous-groupes finis de ces groupes sont uniformément bornés.

Le résultat principal de cet article montre que la conjecture d’Atiyah forte est fausse.

1. Théorème. Soit $G$ le groupe donné par la présentation

\[ G = \langle a, t, s \mid a^2 = 1, [t, s] = 1, [t^{-1}at, a] = 1, s^{-1}as = at^{-1}at \rangle. \]

Le groupe $G$ est metabelien et donc moyennable élémentaire. Chaque sous-groupe fini de $G$ est un 2-groupe abélien élémentaire, en particulier l’ordre de chaque sous-groupe fini de $G$ est une puissance de 2. Il existe une variété riemannienne fermée $(M, g)$ de dimension 7 telle que $\pi_1(M) = G$ pour laquelle le troisième nombre de Betti $L^2_2(M, g)$ est égal à

\[ b_3^{(2)}(M, g) = \frac{1}{3}. \]

Ce théorème montre qu’on ne peut pas généraliser les résultats de Linnell [Lin93] et Dick-Schick [DS00] aux groupes moyennables élémentaires sans une borne uniforme sur l’ordre des sous-groupes finis.

La preuve du Théorème 1 est basée sur les résultats de [GZ01] sur le spectre et la mesure spectrale de l’opérateur de Markov $A$ de la marche aléatoire simple sur le groupe de l’allumeur de réverbères pour lequel $G$ est une HNN extension. Ce résultat affirme que la mesure spectrale de cet opérateur est discrète et concentrée sur un sous ensemble dense de $[−1, 1]$ avec des sauts de valeurs $1/(2^q − 1)$, $q \in \mathbb{N}$. Les résultats impliquent que $\dim \ker A = \frac{1}{3}$. Mais le dénominateur 3 ne divise pas les puissances de 2 qui sont les ordres des sous-groupes finis du groupe de l’allumeur de réverbères.

Atiyah [Ati76] introduced for a closed Riemannian manifold $(M, g)$ with universal covering $\tilde{\Phi}$ the analytic $L^2$-Betti numbers $b^p_{(2)}(M, g)$ which measure the size of the space of harmonic square-integrable $p$-forms on $\tilde{\Phi}$. Let $k_p(x, y)$ be the (smooth) integral kernel of the orthogonal projection of all square integrable forms onto this subspace. On the diagonal, the fiber-wise trace $\text{tr}_x k_p(x, x)$ is defined and is invariant under deck transformations. It therefore defines a smooth function on $M$, and Atiyah sets $b^p_{(2)}(M, g) := \int_M \text{tr}_x k_p(x, x) \, dx$. By a result of Dodziuk [Dod77] this does not depend on the metric.

A priori, the $L^2$-Betti numbers are non-negative real numbers. However, we can express the Euler characteristic $\chi(M)$, an integer, in terms of the $L^2$-Betti numbers in the usual way:

\[ \chi(M) = \sum_{p=0}^{\infty} (-1)^p b^p_{(2)}(M). \]
If \( \pi = \pi_1(M) \) is a finite group, then the \( L^2 \)-Betti numbers can be expressed in terms of ordinary Betti numbers as follows: \( b^p_{(2)}(M) = \frac{1}{|\pi|} b^p(\tilde{M}) \).

This note deals with the following conjecture. Let \( \pi \) be a discrete group. Denote with \( \text{fin}^{-1}(\pi) \) the additive subgroup of \( \mathbb{Q} \) generated by the inverses of the orders of the finite subgroups of \( \pi \). Note that \( \text{fin}^{-1}(\pi) = \mathbb{Z} \) if and only if \( \pi \) is torsion free.

1. **The Strong Atiyah Conjecture.** If \( M \) is a closed Riemannian manifold with fundamental group \( \pi \), then \( b^p_{(2)}(M) \in \text{fin}^{-1}(\pi) \). If \( \pi \) is torsion free, this specializes to \( b^p_{(2)}(M) \in \mathbb{Z} \).

In [Ati76] it is only asked whether the \( L^2 \)-Betti numbers are always rationals, and integers if the fundamental group is torsion free. Later, this question was popularized as the Atiyah conjecture, and also gradually was made precise in the way we formulate it in Conjecture 1, compare [L95, Conjecture 7.1], [Lin98, Conjecture 9.8], [Lüch98a, Conjecture 5.1], [Rei99, Conjecture 5.1], [Lüch00, Conjecture 13], [DS00, Definition 1.2], and talks of many mathematicians. Conjecture 1 is also suggested by [Coh79, Theorem 8], where it is checked for \( \pi \) abelian.

The conjecture is proved in many important cases, starting with the class \( C \) of Linnell [Lin93] which includes extensions of free groups with elementary amenable quotients, for residually torsion-free elementary amenable groups and poly-free groups. In [DS00] it is proved that the class of groups for which the Atiyah conjecture holds is closed under HNN-extensions, as long as \( \text{fin}^{-1}(\pi) \) is discrete. It follows that it holds for all subgroups of one-relator groups, and for all subgroups of right-angled Coxeter groups. Dicks-Schick [DS00] also prove that the class of all torsion-free groups for which the Atiyah conjecture holds is closed under taking extension by groups in a certain large class, namely the smallest class which contains all the torsion-free, elementary amenable groups, and contains all the free groups, and is closed under taking subgroups, extensions, directed unions, amalgamated free products, and HNN-extensions.

In all these positive results one has to make one additional crucial assumption: there is a bound on the orders of finite subgroups of \( \pi \), i.e. \( \text{fin}^{-1}(\pi) \) is a discrete subset of \( \mathbb{R} \).

The following theorem shows that this additional assumption is essential, and that without it, the Strong Atiyah Conjecture is wrong:

2. **Theorem.** Let the group \( G \) be given by the presentation

\[
G = \langle a, t, s \mid a^2 = 1, [t, s] = 1, [t^{-1}at, a] = 1, s^{-1}as = at^{-1}at \rangle.
\]

We use the notation \([g, h] = g^{-1}h^{-1}gh\) for the commutator of \( g \) and \( h \).

The group \( G \) is metabelian and in particular elementary amenable. Every finite subgroup of \( G \) is an elementary abelian 2-group, in particular the order of every finite subgroup of \( G \) is a power of 2. There exists a closed Riemannian manifold \( (M, g) \) of dimension 7 with \( \pi_1(M) = G \) such that the third \( L^2 \)-Betti number

\[
b^3_{(2)}(M, g) = \frac{1}{3}.
\]

Conjecture 1 predicts that the denominator is a power of 2 and thus the group \( G \) is a counterexample to the Strong Atiyah Conjecture.

Observe however, that still there is no example of an irrational \( L^2 \)-Betti number, and that the group \( \text{fin}^{-1}(G) \) is not discrete.
To prove Theorem 2 we first study the structure of the group $G$. Let $H$ denote the lamp-lighter group $(\oplus_{i \in \mathbb{Z}} \mathbb{Z}/2) \times \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\oplus_{i \in \mathbb{Z}} \mathbb{Z}/2$ by translation. $H$ is generated by $t \in \mathbb{Z}$ and by $a = (\ldots, 0, 1, 0, \ldots) \in \oplus_{i \in \mathbb{Z}} \mathbb{Z}/2$ and has the presentation

$$H = \langle a, t \mid a^2 = 1, [t^{-k}at^k, t^{-n}at^n] = 1 \forall k, n \in \mathbb{Z} \rangle.$$ 

3. Lemma. Let $\alpha : H \to H$ be given by $\alpha(t) = t$ and $\alpha(a) = at^{-1}at$. This defines an injective group homomorphism, and $G$ is the ascending HNN-extension of $H$ along $\alpha$. Moreover $G'$ is isomorphic to a countable direct sum of copies of $\mathbb{Z}/2$.

Proof. The first assertion can be easily checked. The second part follows from the computation in [Bau72]. For completeness sake we give the argument here:

Let $V$ be the HNN-extension of $H$ along $\alpha$. Then $V$ has the presentation

$$V = \langle a, t, s \mid a^2 = 1, [s, t] = 1, s^{-1}as = at^{-1}at = [a, t], [t^{-k}at^k, t^{-n}at^n] = 1 \forall k, n \in \mathbb{Z} \rangle.$$ 

Obviously, we have a epimorphism $G \to V$ mapping $a$ to $a$, $s$ to $s$, and $t$ to $t$. It only remains to show that every relation in the given presentation of $V$ follows from the relations of $G$. Observe first in $G$ that by conjugation with $t^{-n}$, $[t^{-k}at^{-n+1}t^k, a] = 1$ implies $[t^{-k}at^k, t^{-n}at^n] = 1$. Moreover, commutativity is commutative, i.e. $[t^{-k}at^k, t^{-n}at^n] = 1$ implies $[t^{-n}at^n, t^{-k}at^k] = 1$.

Hence, it remains to prove $[t^{-n}at^n, a] = 1$ in $G$ for $n > 1$. We will do this by induction on $n$. Assume therefore $t^{-j}at^j$ commutes with $t^{-l}at^l$ for $0 \leq j \leq l < n$. Conjugate the relation $[t^{-n}at^n, a] = 1$ with $a$. We obtain

$$1 = [t^{-n-1}at^{-n-1}, at^{-1}at] = [(t^{-n-1}at^{-n-1})(t^{-n}at^n), a(t^{-1}at)]. \quad (4)$$

Now observe that by induction $a$ commutes with $a_1 := t^{-1}at$ and with $a_{n-1}t^{-n}at^{n-1}$. This second relation also implies (by conjugation with $t^{-1}$) that moreover, $t^{-1}at$ commutes with $a_n := t^{-n}at^n$. Therefore, we can simplify the commutator in (4) to the desired

$$1 = (a_{n-1}^{-1}a_{n-1}^{-1})(a_1^{-1}a_1)(a_{n-1}a_n)(aa_1) = a_n^{-1}(a_{n-1}^{-1}a_{n-1})(a_1^{-1}a_1)a^{-1}a_na = [t^{-n}at^n, a].$$

By induction we therefore see that $V = G$.

Using the presentation, we next check that the abelianization of $V$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and $s, t$ are mapped to two free generators, whereas $a$ is mapped to zero. Therefore, $G'$ is equal to the normal subgroup generated by $a$, which is generated by $s^{-l}t^{-k}at^l$, $k, l \in \mathbb{Z}$, $l < 0$. All these elements are of order 2, and by conjugation with sufficiently high powers of $s$ we see that they all commute. Therefore, $G'$ is a vector space over $\mathbb{Z}/2$ with countably many generators, and therefore isomorphic to a countable direct sum of copies of $\mathbb{Z}/2$. Observe, however, that $G'$ is quite different from the base of the HNN-extension $H$. The element $sas^k$ is a typical example which is not contained in $H$ but in $G'$.

Since, by Lemma 3, $G$ is a two-step HNN-extension of $\oplus_{i \in \mathbb{Z}} \mathbb{Z}/2$, it follows immediately that all finite subgroups of $G$ are elementary abelian 2-groups. To prove Theorem 3 we need to construct $M$.

Let $A \in CG$. Then left multiplication by $A$ on $CG$ induces a bounded linear operator on the Hilbert space $l^2(G)$; we shall also let $A$ indicate this operator. Let $pr_G : l^2(G) \to l^2(G)$ denote
the orthogonal projection onto $\ker(A)$ and let $e$ denote the identity element of $G$; we shall also consider $e$ as the corresponding element of $l^2(G)$. We now define $\dim_G(\ker A)$ according to the formula

$$\dim_G(\ker A) = \langle \text{pr}_G(e), e \rangle_{l^2(G)}.$$ 

We will crucially use the following result from [GZ01, Theorem 2 and Corollary 3]:

5. **Theorem.** Let $A := t + at + t^{-1} + (at)^{-1} \in \mathbb{Z}H$ be a multiple of the Markov operator of $H$. Then $A$, considered as an operator on $l^2(H)$, has eigenvalues

$$\{4 \cos \left( \frac{p}{q} \pi \right) \mid p \in \mathbb{Z}, q = 2, 3, \ldots \}.$$

The $L^2$-dimension of the corresponding eigenspaces is

$$\dim_H \ker \left( A - 4 \cos \left( \frac{p}{q} \pi \right) \right) = \frac{1}{2q-1} \quad \text{if } p, q \in \mathbb{Z}, \ q \geq 2, \ \text{with } (p, q) = 1.$$

Let us sketch the proof of this theorem. In [GZ01] the group $H$ is realized as a group defined by a two-state automaton (a general survey about automata and their groups can be found in [GNS00]). Correspondingly, $H$ acts on a binary tree and on the boundary of this tree. As to be expected for automata, this action shows a lot of self-similarity. This fractalness can be used to inductively compute the spectra (with multiplicity of the eigenvalues) of finite dimensional approximations $A_n$ of the operator $A$. The $A_n$ are obtained by restricting the action of $H$ to a finite subtree consisting of vertices up to the level $n$. It is important that there exists an infinite path in this tree with a trivial stabilizer. Similar computations have been carried out in [BG99]. Then by [GZ99] and using approximation results for $L^2$-Betti numbers [Far98, Luc94], the spectra (and multiplicities) of the $A_n$ converge, suitably normalized, to the $L^2$-dimensions of eigenspaces which have to be computed. Observe that in [GZ01, Corollary 3], the jump of the spectral measure at $4 \cos(\frac{p}{q} \pi)$ is computed to be $\frac{1}{2q-1}$, if $(p, q) = 1$. Since this jump is exactly the $L^2$-dimension of the corresponding eigenspace, the sketch of the proof of the Theorem is finished.

6. **Remark.** An independent proof of the result of Grigorchuk-Zuk [3] which does not use automata, is given in [Sch].

As a corollary of Theorem 5, we obtain:

7. **Corollary.** There is an $A \in \mathbb{Z}G$ such that

$$\dim_G(\ker A) = \frac{1}{3}.$$

**Proof.** Observe that if $A$ is induced from $H$, i.e. $A \in \mathbb{C}H$ (so we can view $A$ also as an operator on $l^2(H)$), then essentially $\text{pr}_H = \text{pr}_G$ and we deduce that $\dim_H(\ker A) = \dim_G(\ker A)$ (cf. [Sch98, Proposition 3.1]). Therefore, it will be sufficient to find $A \in \mathbb{Z}H$ such that $\dim_H(\ker A) = 1/3$.

Take $A$ of Theorem 5. Choosing $p = 1$ and $q = 2$, we see that 0 is in the spectrum of $A$, and that $\dim_H(\ker A) = 1/3$. 

\[ \square \]
8. Proposition. There is a 3-dimensional finite CW-complex $X$ with $\pi_1(X) = G$ and with $b_3^{(2)}(X) = \frac{1}{3}$.

Proof. We perform a standard construction where one attaching map will be given by the $A$ of Corollary \[Lüc98b\, Lemma 2.2\].

Let $X'$ be a finite 2-dimensional CW-complex with $\pi_1(X') = G$, e.g. the 2-complex of the finite presentation given above. Let $X''$ be the wedge product of $X'$ and $S^2$. The corresponding map $\alpha : S^2 \to X''$ generates a free copy of $\mathbb{Z}\pi_1(X'') = \mathbb{Z}G$ inside $\pi_2(X'')$. Define now $X := X'' \cup_f D^3$, where $(f : S^2 \to X'') \in \pi_2(X'')$ is given by $A \in \mathbb{Z}G$ of Corollary \[Lüc98b\, Lemma 2.2\], and where $\mathbb{Z}G \to \pi_2(X'')$ is given using $\alpha$. Choosing an appropriate basis of cells, it follows that on the cellular $L^2$-chain complex $C_*(\tilde{X}) = C_*(\tilde{X}) \otimes \mathbb{Z}G L^2(G)$ of the universal covering $\tilde{X}$ of $X$, the differential $d_3$

$$l^2(G) \cong C_2^{(2)}(\tilde{X}) \xrightarrow{d_3} C_3^{(2)}(\tilde{X}) \cong (l^2(G))^n$$

is given by the matrix $(A, 0, \ldots, 0)^t$, where $t$ denotes transpose and $n$ is the number of 2-cells in $X$. Since there are no 4-cells, $d_4$ is zero. Consequently,

$$b_3^{(2)}(X) = \dim_G(\ker d_3) = \dim_G(\ker A) = \frac{1}{3}.$$ 

We now can finish the proof of Theorem \[Lüc98b\, Theorem 2\] in a standard way (compare e.g. \[Lüc01\, Lemma 2.2\]):

9. Theorem. There is a 7-dimensional smooth Riemannian manifold $(M, g)$ with

$$b_3^{(2)}(M, g) = b_3^{(2)}(M) = \frac{1}{3}$$

and with $\pi_1(M) = G$. Here, $b_3^{(2)}(M)$ denotes the combinatorial $L^2$-Betti number of a triangulation of $M$.

Proof. Choose a finite 3-dimensional simplicial complex $Y$ homotopy equivalent to the CW-complex $X$ of Proposition \[Lüc98b\]. Then embed $Y$ into $\mathbb{R}^8$ \[Pon52\, Theorem 5\] and thicken $Y$ to a homotopy equivalent 8-dimensional compact smooth manifold $W$ with boundary $M$, such that moreover the inclusion of $M$ into $W - Y$ is a homotopy equivalence \[RSS2\, Chapter 3\]. Recall that a map $f : V \to V'$ between two CW-complexes is called an $r$-equivalence ($r \in \mathbb{N}$), if $\pi_j(f) : \pi_j(V) \to \pi_j(V')$ is an isomorphism for $0 \leq j < r$, and an epimorphism for $j = r$. By transversality \(\text{[RSS2, 5.3 and 5.4]},\) every map of $S^j$ or $D^j$ to $W$ with $j \leq 4$ is homotopic to a map into $W - Y$. It follows that the inclusion $M \hookrightarrow W$ is a 4-equivalence. Consequently, by \[Lüc01\, Theorem 1.7\], $b_3^{(2)}(M) = b_3^{(2)}(W) = b_3^{(2)}(X) = \frac{1}{4}$ and $\pi_1(M) = \pi_1(X) = G$. If we choose a smooth Riemannian metric $g$ on $M$, then by the $L^2$-Hodge de Rham theorem \[Dod74\, Theorem 1\] we also obtain $b_3^{(2)}(M, g) = \frac{1}{3}$. 

10. Remark. The dimension of the manifold which is a counterexample to the strong Atiyah conjecture can be reduced to 6 as follows:

By \[Lüc98b\, Theorem 3.3\] $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2$, $H, G$, and the direct limit of $H \twoheadrightarrow H \twoheadrightarrow \cdots$ all have vanishing $L^2$-Betti numbers in all degrees. Moreover, the zero-th and first $L^2$-Betti number of
On a conjecture of Atiyah

a space is equal to the first $L^2$-Betti number of its fundamental group \[\text{Theorem 1.7},\]
i.e. $b_1^{(2)}(G) = b_1^{(2)}(X) = b_1^{(2)}(M) = 0 = b_0^{(2)}(X) = b_0^{(2)}(M)$.

The CW-complex $X$ has 1 zero-cell, 1 three-cell, and 3 one-cells and 5 two-cells (using the presentation of $G$ given in Theorem 2). Consequently, $\chi(X) = 2$. Since

$$2 = \chi(X) = \sum_{k=0}^{3} (-1)^k b_k^{(2)}(X) = b_2^{(2)}(X) - b_3^{(2)}(X) - b_1^{(2)}(X) - 1/3,$$

we have $b_2^{(2)}(X) = 7/3$.

Now we can do the same construction as in the proof of Theorem 3 but embed $Y$ into $\mathbb{R}^7$ instead of $\mathbb{R}^8$. The inclusion of the boundary $M'$ of the regular neighborhood $W'$ into $W'$ will now only be a 3-equivalence, but this is enough to conclude that $b_2^{(2)}(M') = b_2^{(2)}(W') = b_2^{(2)}(X) = 7/3$, and the denominator still is not a power of 2, giving the desired counterexample.

Using the Künneth formula and Poincaré duality \[\text{Theorem 1.7}\] for $L^2$-cohomology, one can on the other hand easily arrange that the dimension of a counterexample, as well as the degree of the Betti number which contradicts the strong Atiyah conjecture, is arbitrarily high.

We conclude this paper with a list of open questions regarding the Atiyah conjecture:

• Is there an example of an $L^2$-Betti number of a closed manifold which is not rational?

• Is there an example of a closed manifold with a fundamental group $\pi$ with $\text{fin}^{-1}(\pi)$ discrete in $\mathbb{R}$ which provides a counterexample to Conjecture 1?

• Is there even a counterexample to the Atiyah conjecture with torsion free fundamental group? It is well known that, for a torsion free group $\pi$, the Atiyah conjecture implies that there are no non-trivial zero-divisors in $\mathbb{Q}[\pi]$, even stronger, that $\mathbb{Q}[\pi]$ embeds into a skew field (compare e.g. \[\text{Lemma 4.4}\]). A torsion-free counterexample hence would be particularly interesting in view of this zero-divisor conjecture.

In the construction of the present counterexample an important role was played by the spectral properties of a Markov operator. We would like to formulate the following open problems:

• Is there a torsion free group with a Markov operator which has a gap in its spectrum?

• Is there a torsion free group with a Markov operator whose spectral measure is not absolutely continuous with respect to the Lebesgue measure or even is not a continuous measure?

These questions are also interesting for the operators given by any self-adjoint element in a group ring.

References

[Ati76] M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In Colloque “Analyse et Topologie” en l’honneur de Henri Cartan (Orsay, 1974), pages 43–72. Astérisque, No. 32–33. Soc. Math. France, Paris, 1976.

[Bau72] G. Baumslag. A finitely presented metabelian group with a free abelian derived group of infinite rank. Proc. Amer. Math. Soc., 35:61–62, 1972.
Grigorchuk, R.I., Linnell, P.A., Schick, T. and Žuk, A.

[BG99]  L. Bartholdi and R.I. Grigorchuk. On the spectrum of Hecke type operators related to some fractal groups. preprint of Max-Planck Institut Bonn, 1999.

[Coh79]  J. M. Cohen. Von Neumann dimension and the homology of covering spaces. Quart. J. Math. Oxford Ser. (2), 30(118):133–142, 1979.

[Dod77]  J. Dodziuk. deRham-Hodge theory for $L^2$-cohomology of infinite coverings. Topology, 16:157–165, 1977.

[DS00]  W. Dicks and T. Schick. Graphs of groups and the Atiyah conjecture for one-relator groups. Preprintreihe SFB 478 - Münster, no. 106, 2000.

[Far98]  M. Farber. Geometry of growth: approximation theorems for $L^2$ invariants. Math. Ann., 311(2):335–375, 1998.

[GNS00]  R. Grigorchuk, V.V. Nekrashevych, and V.I. Sushchansky. Automata, dynamical systems and groups. to appear in Proceedings fo the Steklov Institute of Mathematics, no. 4, 2000.

[GZ99]  R. Grigorchuk and A. Žuk. On the asymptotic spectrum of random walks on infinite families of graphs. In M. Picardello and W. Woess, editors, Proceedings of the Cortona Conference on Random Walks and Discrete Potential Theory, pages 188–204. Cambridge Univ. Press, 1999.

[GZ01]  R. Grigorchuk and A. Žuk. The lamplighter group as a group generated by a 2-state automaton and its spectrum. Preprint of FIM ETH Zürich, 1999, to appear in Geometriae Dedicata, 2001.

[Lin93]  P. A. Linnell. Division rings and group von Neumann algebras. Forum Math., 5(6):561–576, 1993.

[Lin98]  P. A. Linnell. Analytic versions of the zero divisor conjecture. In Geometry and cohomology in group theory (Durham, 1994), pages 209–248. Cambridge Univ. Press, Cambridge, 1998.

[LL95]  J. Lott and W. Lück. $L^2$-topological invariants of 3-manifolds. Invent. Math., 120(1):15–60, 1995.

[Lück94]  W. Lück. Approximating $L^2$-invariants by their finite-dimensional analogues. Geom. Funct. Anal., 4(4):455–481, 1994.

[Lück98a]  W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and $L^2$-Betti numbers. II. Applications to Grothendieck groups, $L^2$-Euler characteristics and Burnside groups. J. Reine Angew. Math., 496:213–236, 1998.

[Lück98b]  W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and $L^2$-Betti numbers ii: Applications to Grothendieck groups. J. für Reine und Angewandte Mathematik, 496:213–236, 1998.

[Lück00]  W. Lück. $L^2$-invariants and their applications to geometry, group theory and spectral theory. to appear in “Mathematics Unlimited - 2001 and Beyond” Springer, 2000.

[Lück01]  W. Lück. $L^2$-invariants of regular coverings of compact manifolds and CW-complexes. to appear in “Handbook of Geometry”, Elsevier; available at http://wwwmath.uni-muenster.de/u/lueck/publ/lueck/015hand.html, 2001.

[Pon52]  L.S. Pontryagin. Foundations of combinatorial topology. Graylock Press, Rochester, N.Y., 1952.

[Reich99]  H. Reich. Group von Neumann algebras and related algebras. Dissertation, Universität Göttingen, 1999. available at http://wwwmath.uni-muenster.de/u/lueck/publ/diplome/reich.dvi.

[RS82]  C.P. Rourke and B.J. Sanderson. Introduction to piecewise-linear topology. Springer study edition. Springer, 1982.

[Sch]  T. Schick. $l^2$-eigenspaces and multiplicities of markov operators on lamplighter-like groups. in preparation.

[Sch98]  T. Schick. $L^2$-determinant class and approximation of $L^2$-Betti numbers. preprint, SFB 418 Münster, to appear in Transactions of the AMS, 1998.

[Sch99]  T. Schick. Integrality of $L^2$-Betti numbers. Mathematische Annalen, 317:727–750, 1999.

Preprints of SFB 478, Münster are available via http://wwwmath.uni-muenster.de/sfb/about/publ/index.htm