EXACT SELF-INTERACTING SCALAR FIELD COSMOLOGIES

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We solve isotropic, homogeneous cosmological models containing a self-interacting scalar field. Calculations are performed in four and two-dimensional spacetimes. We find several exact solutions that have an inflationary regime or has a final Friedmann stage. Also their asymptotically stability is studied.

I. INTRODUCTION

A self-interacting scalar field has been introduced in cosmological models as a matter source to the Einstein equations because, when dominated by the potential energy, it violates the strong energy condition and drives the universe into an inflationary period [1]. Currently, there is no underlying principle that uniquely specifies the potential for the scalar field and many proposals have been considered. Some were based in new particle physics and gravitational theories, while others were postulated ad-hoc to obtain the desired evolution [2]. Also, a formalism has been proposed to reconstruct the potential from knowledge of tensor gravitational spectrum or the scalar density fluctuation spectrum [3].

Due to the non-linearity of the system of differential equations for the scalar and gravitational fields very little is known yet about exact solutions of these cosmological models. In section 2 of this paper we show a procedure to reduce to quadratures the Einstein and scalar field equations in a Robertson-Walker metric. Our procedure allows for an arbitrary potential and we show as an example new exact solutions. We study their asymptotic stability by means of the method of Lyapunov [4]. No a priori assumption like a slowly varying field is required to perform the calculations, and we check the validity of this assumption. Two-dimensional spacetimes are nowadays very useful for testing ideas on quantum gravity. For this reason, in section 3 we apply our procedure also in this case. The conclusions are stated in section 4.

II. THE EINSTEIN-SCALAR FIELD EQUATIONS

We wish to investigate the evolution of a universe with a scalar field $\phi$ which has a self-interaction potential $V(\phi)$ and is minimally coupled to gravity

$$\nabla^\mu \nabla_\mu \phi + \frac{dV}{d\phi} = 0$$ (1)

where $\nabla^\mu$ is the covariant derivative. Thus, we must solve Eq. (1) together with the Einstein equations

$$R_{ik} - \frac{1}{2} g_{ik} R = T^\phi_{ik}$$ (2)

We are using units such that $c = 8\pi G = 1$ and

$$T^\phi_{ik} = \phi, i \phi, k - g_{ik} \left( \frac{1}{2} \phi, m \phi, m - V(\phi) \right)$$ (3)

is the stress-energy tensor of the field. In a spatially flat Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left( dx_1^2 + dx_2^2 + dx_3^2 \right)$$ (4)

with scale factor $a(t)$, Eqs. (1) and (2) become

$$\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0$$ (5)
\[ 3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi) \]  
(6)

where the dot means \( d/dt \), \( H = \dot{a}/a \), and \( \phi = \phi(t) \). It becomes convenient to use the scale factor as the independent variable and write the potential in the following form:

\[ V[\phi(a)] = \frac{F(a)}{a^6} \]  
(7)

with a suitable function \( F(a) \). Thus we obtain a first integral of Eq. 5

\[ \frac{1}{2} \dot{\phi}^2 + V(\phi) - \frac{6}{a^6} \int da \frac{F}{a} = \frac{C}{a^6} \]  
(8)

where \( C \) is an arbitrary integration constant. Then, using Eqs. 6 and 8 we have reduced the problem to quadratures:

\[ \Delta t = \sqrt{3} \int \frac{da}{a} \left[ \frac{6}{a^6} \int da \frac{F}{a} + \frac{C}{a^6} \right]^{-1/2} \]  
(9)

\[ \Delta \phi = \sqrt{6} \int \frac{da}{a} \left[ -F + \frac{6}{a^6} \int da F/a + \frac{C}{a^6} \right]^{1/2} \]  
(10)

where \( \Delta \equiv t - t_0 \), \( \Delta \phi \equiv \phi - \phi_0 \) and \( t_0, \phi_0 \) are arbitrary integration constants.

**A. Example**

As an example of our procedure let us consider the function

\[ F(a) = Ba^s (b + a^s)^n \]  
(11)

This is interesting because it yields new exact solutions that exhibit stable exponential or Friedmann behavior. In Eq. 11 \( B > 0, b > 0, s \) and \( n \) are constants and we require that \( s(n + 1) = 6 \). Then, inserting Eq. 11 in Eqs. 10 and 7 and taking for simplicity \( C = 0 \) we get

\[ V(\phi) = B \left[ \cosh \left( \frac{s}{2 \sqrt{6}} \Delta \phi \right) \right]^{2n} \]  
(12)

This potential has a nonvanishing minimum at \( \Delta \phi = 0 \) for \( s > 0 \), which is equivalent to a effective cosmological constant. When \( s < 0 \), the origin becomes a maximum, and the potential vanishes exponentially for large \( \phi \). We can evaluate Eq. 9 for some values of \( s \), for instance:

\[ \Delta t = \sqrt{\frac{3}{B}} \left[ \arcsinh \left( \frac{a}{\sqrt{b}} \right) - \frac{a}{(b + a^2)^{1/2}} \right], \quad s = 2 \]  
(13)

\[ a = \left\{ b \left[ \exp \left( \sqrt{3B} \Delta t \right) - 1 \right] \right\}^{1/3}, \quad s = 3 \]  
(14)

For \( s > 0 \), the evolution begins from a singularity as \( \Delta t^{1/3} \) and is asymptotically de Sitter with \( \Delta \phi \to 0 \) for \( t \to \infty \). On the other hand, for \( s < 0 \) the evolution has a deflationary behaviour from a de Sitter era in the far past to a Friedmann behavior \( \Delta t^{1/3} \) when \( t \to \infty \).
B. Slow-Roll Approximation

A common framework to solve Eqs. 5, 6 in discussions of inflation is the "slow-roll" approximation [1]. To investigate its limitations we calculate the two slow-roll parameters [5]

\[ \epsilon \equiv \frac{\dot{\phi}^2}{V + \dot{\phi}^2/2} = 1 - \frac{F}{C + 6 \int daF/a} \] (15)

\[ \eta \equiv \frac{\ddot{\phi}}{H \dot{\phi}} = \frac{12F - aF' - 36 \int daF/a - 6C}{2(C + 6 \int daF/a + F)} \] (16)

The parameter \( \epsilon \) measures the relative contribution of the field’s kinetic energy to its total energy density and \( \eta \) measures the ratio of the field’s acceleration relative to the friction term. The slow-roll approximation is valid when \( |\epsilon| \ll 1 \) and \( |\eta| \ll 1 \). Using Eqs. 15 and 16, we find that they impose a constrain on the form of the potential and the value of the initial conditions. For instance, in our example they imply that \( b/a^s \ll 1 \). Thus, it is violated for short times if \( s > 0 \) or long times if \( s < 0 \).

C. Stability of the Solutions

For models (like our example for \( s > 0 \)) such that \( V(\phi) \) has a local minimum at \( \phi_m \) and \( V(\phi_m) \geq 0 \), we can study the stability of solutions with asymptotic behavior \( \phi(t) \to \phi_m \). First we note that the evolution \( a(t) \) is monotonic. Then, differentiating Eq. 6 and using Eq. 5, we find \( 2\dot{H} = -\dot{\phi}^2 < 0 \) so that these models have a sub-inflationary behavior. As \( H \geq 0 \) and \( dH/da \leq 0 \) in a neighborhood of \( \phi_m \), \( H(\phi, a) \) is a Lyapunov function for the system Eqs. 5 and 6 and any solution such that \( \phi \to \phi_m \) for \( a \to \infty \) (equivalently \( t \to \infty \)), is asymptotically stable.

III. TWO-DIMENSIONAL SPACETIME

In recent years, there have been a number of investigations into the structure of relativistic gravitational theories in two spacetime dimensions [6]−[8] mainly because they reduce the complexity of four-dimensional general relativity and constitute useful testing grounds for ideas on quantum gravity. Among them, the so called "\( R = T \)" theory [9] has attracted some interest due to the fact that it has many classical aspects which are similar to general relativity.

In this paper, we confine our attention to classical cosmological properties of an two-dimensional universe based on the gravitational field equations:

\[ R = 8\pi GT \quad \nabla^\mu T_{\mu\nu} = 0 \] (17)

where \( R \) is the curvature scalar and \( T \) is the trace of the energy-momentum tensor. We consider the two-dimensional form of the Robertson-Walker metric filled with a minimally coupled scalar field \( \phi \) with a self interacting potential \( V(\phi) \) which obeys the Klein-Gordon Eq. 1. Inserting \( T = 2V(\phi) \) and \( R = -2\dot{a}/a \) in Eq. 17, the system of Eqs. 1 and 17 become:

\[ \ddot{\phi} + H \dot{\phi} + \frac{dV}{d\phi} = 0 \] (18)

\[ \dot{a} = -aV(\phi) \] (19)

Following the steps of the previous section we write the potential in the form

\[ V[\phi(a)] = \frac{F(a)}{a^2} \] (20)

and we obtain two first integrals of Eqs. 17 and 1.
\[
\frac{1}{2} \dot{\phi}^2 + V(\phi) - \frac{2}{a^2} \int da \frac{F}{a} = \frac{C_1}{a^2}
\]

(21)

\[
\frac{\dot{a}^2}{2} + \int da \frac{F}{a} = C_2
\]

(22)

where \( C_1 \) and \( C_2 \) are arbitrary integration constants. Then the problem have been reduced to quadratures:

\[
\Delta t = \int da \left[ 2C_2 - 2 \int da \frac{F}{a} \right]^{-1/2}
\]

(23)

\[
\Delta \phi = \int \frac{da}{a} \left[ \frac{C_1 - F + 2 \int da F/a}{C_2 - \int da F/a} \right]^{1/2}
\]

(24)

Eqs. 23 and 24 show two important differences with respect to their four dimensional counterparts. The first one is the appearance of two different independent constants \( C_1 \) and \( C_2 \) while in four dimensions we have the constrain \( C_1 = C_2 \). The second one is the sign of the terms proportional to \( \int da F/a \).

A. Example

In the case of Eq. 11 we find an exact solution taking \( s(n + 1) = 2, s > 0 \) and \( C_1 = C_2 = 0 \) in Eqs. 23 and 24. Taking into account that the Eqs. 21 and 22 lead to the constrains \( B < 0, b < 0 \), the potential results

\[
V(\phi) = B \sin^{2n} \left( \frac{s}{2\sqrt{2}} \Delta \phi \right)
\]

(25)

Contrary to the results obtained in the four dimensional case, we obtained a negative periodic potential which oscillates between the values \( V_{\text{min}} = B < 0 \) and \( V_{\text{max}} = 0 \). This is related to the change in sign we pointed out in the previous subsection. We can evaluate Eq. 23 for some particular values of \( s \):

\[
a(t) = b \left\{ \exp \left( \sqrt{-B} \Delta t \right) - 1 \right\}, \quad s = 1
\]

(26)

\[
\Delta t = \frac{3}{\sqrt{-B}} \left[ \text{arccosh} \left( \frac{a^{1/3}}{\sqrt{b}} \right) - \left( \frac{a^{2/3}}{b + a^{2/3}} \right)^{1/2} \right], \quad s = 2/3
\]

(27)

Thus, in two dimensions the evolution behaves like \( \Delta t \) near the singularity and, notwithstanding the fact that \( V(\phi) < 0 \), it has an asymptotically de Sitter behavior for \( t \to \infty \) as in the four-dimensional case.

IV. CONCLUSIONS

We present a procedure which reduces to quadratures the gravitational field equations with a classical self-interacting scalar field as a matter source in a Robertson-Walker spacetime. The freedom to choose the potential of the scalar field is expressed in terms of the function \( F(a) \), which we can select at will. We take a simple example of this function and we show that it yields new stable exact solutions in the four-dimensional spacetime. We analyse the restrictions imposed by the "slow-roll" approximation and we verify in our example that they may very easily violated. In the two-dimensional case, we show that the same function \( F(a) \) leads to a negative oscillating potential. The evolution changes its behavior near the singularity but remains the same in the far future.

In a future paper we will extend our procedure to more general models with curvature term, a cosmological constant and a perfect fluid source.
V. REFERENCES

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