INTEGRABLE SYSTEMS AS QUANTUM MECHANICS

ROBERT CARROLL
UNIVERSITY OF ILLINOIS, URBANA, IL 61801

Abstract. This is a mainly expository sketch showing how some integrable systems (e.g. KP or KdV) can be viewed as quantum mechanical in nature.

1. INTRODUCTION

We review briefly some ideas and constructions for dKP (from KP) and its Hamiltonian dynamics, discuss its quantization via Moyal back to KP, and treat the KP hierarchy itself as a sequence of quantum Hamiltonian systems based on the dKP dynamics. Some connections to q-theories are also indicated. The constructions are all well known (cf. [6, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 23, 24, 29, 33, 36, 38, 41, 42, 54, 55, 56, 65]) but we try to organize matters and perhaps see them from a different perspective. We will use dKdV and KdV as a source of simple examples.

2. BACKGROUND

There is an enormous literature on KP and KdV which we do not try to reference (see e.g. [11, 22, 29]). For a brief sketch we follow [17] and simply write down the relevant formulas. Thus one begins with two pseudodifferential operators (A1) $L = \partial + \sum_{n=1}^{\infty} u_{n+1} \partial^{-n}$ and $W = 1 + \sum_{n=1}^{\infty} w_n \partial^{-n}$ (Lax and gauge operator respectively) with $L = W \partial W^{-1}$. Note here the generalized Leibniz rule

$$\partial^n f = \sum_{k=0}^{\infty} \binom{n}{k} (\partial^k f) \partial^{n-k}$$

The KP hierarchy is then determined by Lax equations (A2) $\partial_t L = [B_n, L] = B_n L - L B_n$ where $\partial_t \sim \partial/\partial t_n$ and $B_n = L^n_+$ is the differential part of $L^n = L^n_+ + L^n_-$. One can also express this via the Sato equation (A3) $\partial_t W W^{-1} = -L^n_+$. Now define wave functions via (A4) $\psi = W \exp(\xi) = w(t, \lambda) \exp(\xi)$ where $\xi = \sum_{n=1}^{\infty} t_n \lambda^n$ and $w(t, \lambda) = 1 + \sum_{n=1}^{\infty} w_n(t) \lambda^{-n}$ (with $t_1 = x$). There is also an adjoint wavefunction (A5) $\psi^* = W^{*-1} \exp(-\xi) = w^*(t, \lambda) \exp(-\xi)$ where $w^*(t, \lambda) = 1 + \sum_{n=1}^{\infty} w_n^* \lambda^{-n}$; there are then equations

$$L \psi = \lambda \psi; \quad \partial_t \psi = B_n \psi; \quad L^* \psi = \lambda \psi^*; \quad \partial_t \psi^* = -B_n^* \psi^*$$

Date: September, 2003.
email: rcarroll@math.uiuc.edu.

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Next one has the fundamental object, namely the tau function, which yields the wave functions via vertex operators $X, X^*$ in the form

$$\psi(t, \lambda) = \frac{X(\lambda) \tau(t)}{\tau(t)} = e^\xi \frac{\tau(t - [\lambda^{-1}])}{\tau(t)}; \quad \psi^*(t, \lambda) = \frac{X^*(\lambda) \tau(t)}{\tau(t)} = e^{-\xi} \frac{\tau(t + [\lambda^{-1}])}{\tau(t)}$$

We write $\tau_\pm(t) = \tau(t \pm [\lambda^{-1}]) = exp(\pm \xi (\tilde{\partial}, \lambda^{-1})$ where $\tilde{\partial} = (\partial_1, (1/2)\partial_2, (1/3)\partial_3, \ldots)$ and $t \pm [\lambda^{-1}] = (t_1 \pm \lambda^{-1}, t_2 \pm (1/2)\lambda^{-2}, \ldots)$ and we can also specify

$$e^\xi = exp \left( \sum_1^\infty \int_0^\infty p_j(t_1, \ldots, t_j) \lambda^j; \quad p_j(t) = \sum \left( \frac{t_1!}{k_1!} \right) \left( \frac{t_2!}{k_2!} \right) \ldots \right)$$

where the $p_j$ are elementary Schur polynomials (note $\sum jk_j = n$). One recalls also the famous Hirota bilinear identity (A6) $\oint_\infty \psi(t, \lambda) \psi^*(t', \lambda) d\lambda = 0$ (residue integral around $\infty$). Using (2.3) this can be written as (A7) $\oint_\infty (\tau - [\lambda^{-1}]) \tau(t' + [\lambda^{-1}]) exp(\xi(t, \lambda) - \xi(t', \lambda)) d\lambda = 0$ which leads to the characterization of the tau function via $(t \rightarrow t - y$ and $t' \rightarrow t' + y$)

$$\left( \sum_0^\infty \int_0^\infty p_n(-2y)p_{n+1}(\tilde{D}) \right) \tau \cdot \tau = 0$$

where $D_i$ is the Hirota derivative defined as (A8) $D_j^n a \cdot b = (\partial^n/\partial s_j^n) a(t_j + s_j)b(t_j - s_j)|_{s=0}$ and $\tilde{D} = (D_1, (1/2)D_2, (1/3)D_3, \ldots)$. In particular from the coefficient of the free parameter $y_n$ in (2.5) one obtains (A9) $D_1 D_0 \tau \cdot \tau = 2p_{n+1}(\tilde{D}) \tau \cdot \tau$ and these are called Hirota bilinear equations. Motivated via finite zone situations and Riemann surfaces (cf. [22]) where the tau function is intimately related to theta functions one can express the Fay trisecant identity in a form referred to as the Fay identity (A10) $\sum_{c.p.} (s_0 - s_1)(s_2 - s_3) \tau(t + [s_0] + [s_1]) \tau(t + [s_2] + [s_3]) = 0$ where c.p. means cyclic permutations (cf. [11][22]). Differentiating this in $s_0$, setting $s_0 = s_3 = 0$, dividing by $s_1s_2$, and shifting $t \rightarrow t - [s_2]$, leads to the differential Fay identity ($\partial = \partial_x$)

$$\tau(t) \partial \tau(t + [s_1] - [s_2]) - \tau(t + [s_1] - [s_2]) \partial \tau(t) = (s_1^{-1} - s_2^{-1}) \{ \tau(t + [s_1] - [s_2]) \tau(t) - \tau(t + [s_1]) \tau(t - [s_2]) \}$$

The Hirota equations (A9) can also be derived from (2.6) by taking the limit $s_1 \rightarrow s_2$.

For the dispersionless theory (dKP) one thinks of fast and slow variables for example taking $t_n \rightarrow \epsilon t_n = T_n$ and $t_1 = x \rightarrow \epsilon x = X$. Then letting $\epsilon \rightarrow 0$ the KP equation (A11) $u_t = (1/4) u_{xxx} + 3u_{ux} + (3/4) \partial^{-1} u_y (u \sim u_2)$ goes to $\partial T U = 2U_X + (3/4) \partial^{-1} U_{XY}$ where $\partial \sim \partial X$ now (for a discussion of the passage $u_n \rightarrow U_n$ we refer to [11][19] - generally we think here of $u_n(T/\epsilon) \sim U_n(T) + O(\epsilon)$ etc.). Now take a WKB form for the wavefunction of the form $\psi = exp(S(T, \lambda)/\epsilon)$ and define $P = \partial X S$; then as $\epsilon \rightarrow 0$ the equation $L \psi = \lambda \psi$ becomes (A13) $\lambda = P + \sum_0^\infty U_{n+1} P^{-n}$ with inverse $P = \lambda - \sum_1^\infty P_{n+1} \lambda^{-1}$. Further from $B_n \psi = \sum_0^n b_{nm}(\epsilon) m \psi$ one obtains (A14) $\partial_n S = \mathfrak{B}_n(P) = \lambda_n^n$ where $B_n = L_n^\tau \rightarrow \mathfrak{B}_n = \lambda_n^n = \sum_0^\infty b_{nm} M^n$. Consequently the KP hierarchy becomes (A15) $\partial_n P = \partial \mathfrak{B}_n$ (note $\partial_n S = \mathfrak{B}_n \Rightarrow \partial_n P = \partial \mathfrak{B}_n$ and $\partial_n \sim \partial T_n$ here). One can now write

$$S = \sum_1^\infty T_n \lambda^n - \sum_1^\infty \partial_m F/m \lambda^{-m}; \quad \tau = exp \left( \frac{1}{\epsilon^2} F(T) \right)$$
and there results $P = \partial S = B_1$ with

$$\mathcal{B}_n = \partial_n S = \lambda^n - \sum_{m}^{\infty} \frac{F_{mn}}{m} \lambda^{-m}$$

Next following [15, 56] one can manipulate the differential Fay identity in various interesting ways and in [17] the author and Y. Kodama derived a dispersionless limit of the Hirota equations which provided polynomial identities among the coefficients $F_{mn} = \partial_n \partial_m F$. These dispersionless Hirota equations can be written in the form (A16) $F_{ij} = p_{j+1}(Z_1 = 0, Z_2, \cdots, Z_{j+1})$ where $p_n \sim$ Schur polynomial and $Z_j = \sum_{m+n=j} F_{mn}/mn$. One shows also that the $F_{mn}$ can be expressed as polynomials in the $P_{j+1} = F_{ij}/j$. There are many other results in [17] (including connections to D-bar techniques) and dKP has been developed and used in a number of fascinating contexts in recent years (we refer e.g. to [7, 9, 10, 43, 44, 45, 46, 47, 49, 51, 52, 63, 64]).

REMARK 2.1. Another important aspect of dKP is the Hamiltonian theory originating in [41] (cf. also [11, 15, 16, 17]). It is convenient here to rescale the variables via $t_n \rightarrow T_n = nT_n$ with $\Omega_n = (1/n)\mathcal{B}_n$. There results

$$P'_n = \frac{dP}{dT_n} = \partial \Omega_n; \quad X'_n = \frac{dX}{dT_n} = -\partial_p \Omega_n$$

One can also show that (cf. [17])

$$\frac{1}{P(\mu) - P(\lambda)} = \sum_{n}^{\infty} \partial_p \Omega_n(\lambda) \mu^{-n}$$

which is in fact equivalent to the dispersionless differential Fay identity

$$\sum_{m,n=1}^{\infty} \mu^{-m} \lambda^{-n} \frac{F_{mn}}{mn} = \log \left(1 - \sum_{m,n}^{\infty} \frac{\mu^{-m} \lambda^{-n} F_{1n}}{\mu - \lambda} \right)$$

The kernel in (2.10) represents a Cauchy type kernel and following Kodama has a version on Riemann surfaces related to the prime form. ■

3. INTEGRABLE SYSTEMS AND MOYAL

For background we follow [11, 12, 13] and one recalls for wave functions $\psi$ there are Wigner functions (WF) given via

$$f(x, p) = \frac{1}{2\pi} \int dy \psi^* \left(x - \frac{\hbar}{2}y\right) \exp(-iyp) \psi \left(x + \frac{\hbar}{2}y\right)$$

Then defining $f * g$ via

$$f * g = f \exp \left[ \frac{i\hbar}{2} \left( \overrightarrow{\partial}_x, \overrightarrow{\partial}_p - \overrightarrow{\partial}_p, \overrightarrow{\partial}_x \right) \right] g;$$

$$f(x, p) * g(x, p) = f \left(x + \frac{i\hbar}{2} \overrightarrow{\partial}_p, p - \frac{i\hbar}{2} \overrightarrow{\partial}_x \right) g(x, p)$$
time dependence of WF’s is given by \((H \sim \text{Hamiltonian})\)

\[
(3.3) \quad \partial_t f(x, p, t) = \frac{1}{i\hbar}(H \ast f(x, p, t) - f(x, p, t) \ast H) = \{H, f\}_M
\]

where \(\{f, g\}_M \sim \text{Moyal bracket.} \) As \(\hbar \to 0\) this reduces to \(\partial_t f - \{H, f\} = 0 \) (standard Poisson bracket). One can generalize and write out (3.2) in various ways. For example replacing \(ih/2\) by \(\kappa\) one obtains as in [36]

\[
(3.4) \quad f \ast g = \sum_{s=0}^{\infty} \frac{\kappa^s}{s!} \sum_{j=0}^{\infty} (-1)^j \binom{s}{j} (\partial_x^j \partial_p^{s-j} f)(\partial_x^{s-j} \partial_p^j g)
\]

leading to \(\{f, g\}_\kappa = (f \ast g - g \ast f)/2\kappa\)

\[
(3.5) \quad \{f, g\}_\kappa = \sum_{s=0}^{\infty} \frac{\kappa^{2s}}{(2s+1)!} \sum_{j=0}^{s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_p^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_p^j g)
\]

which can also be utilized in the form

\[
(3.6) \quad f \ast g = f e^{\kappa(\partial_x \partial_p - \partial_p \partial_x)} g = e^{\kappa(\partial_x \partial_p - \partial_p \partial_x)\frac{1}{2\kappa}} \sum_{s=0}^{\infty} \frac{\kappa^s}{s!} \sum_{j=0}^{s+1} (-1)^j \binom{s+1}{j} (\partial_x^j \partial_p^{s+1-j} f)(\partial_x^{s+1-j} \partial_p^j g)
\]

Note e.g.

\[
(3.7) \quad g \ast f = g(x + \kappa \partial_p, p - \kappa \partial_x) f = f(x - \kappa \partial_p, p + \kappa \partial_x) g
\]

The Moyal bracket can then be defined via

\[
(3.8) \quad \{f, g\}_M = \frac{1}{\kappa} \{f \sin[\kappa(\partial_x \partial_p - \partial_p \partial_x)] g\} = \frac{1}{2\kappa}(f \ast g - g \ast f) = \sum_{s=0}^{\infty} \frac{(-1)^s \kappa^{2s}}{(2s+1)!} \sum_{j=0}^{s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_p^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_p^j g)
\]

corresponding to \(\kappa \to \imath \kappa\) in (3.12) below.

Thus in [36] (cf. also [53]) one writes the Sato KP hierarchy via \((v_{-2} = 1, v_{-1} = 0)\)

\[
(3.9) \quad \partial_n L = [L^m_n, L] (m \geq 1); \quad L = \sum_{-2}^{\infty} v_n(\tilde{x}) \partial_{\tilde{x}}^{-n-1}
\]

for \(\tilde{x} = (x, t_2, \cdots)\) while the Moyal KP hierarchy is written via \((u_{-2} = 1, u_{-1} = 0)\)

\[
(3.10) \quad \lambda \sim \Lambda = \sum_{-2}^{\infty} u_n(\tilde{x}) \partial_{\tilde{x}}^{-n-1}; \quad \partial_m \Lambda = \{\Lambda^m_+, \Lambda\}_M (m \geq 1)
\]
where $\Lambda^m_+ \sim (\Lambda^m_\star)_+$ with

$$ f \star g = \sum_{j=0}^{\infty} \frac{\kappa^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (\partial_x^j \partial_y^{s-j} f)(\partial_x^{s-j} \partial_y^j g) $$

leading to

$$ \{ f, g \}_\kappa = \sum_{j=0}^{\infty} \frac{\kappa^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_y^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_y^j g) $$

Note $\lim_{\kappa \to 0} \{ f, g \}_\kappa = \{ f, g \} = f \lambda_g - f \lambda_x$ is so $(KP)_M \to dKP$ as $\kappa \to 0$, namely $\partial_m \Lambda = \{ \Lambda^m_+, \Lambda \} \sim \Lambda^{\star} \cdots \Lambda$. The isomorphism between $(KP)_{Sat} \to (KP)_M$ is then determined by relating $v_n$ and $u_n$ in the form $(\kappa = 1/2)$

$$ u_n = \sum_{j=0}^{n} 2^{-j} \binom{n}{j} v^n_{n-j} $$

where $n = 0, 1, \cdots$ and $v^j = \partial_x^j v_0$ (see Remark 3.1 for enhancement).

**REMARK 3.1.** This can be further clarified as follows. First we recall an important paper [33] where one considers star products of the form (B1) $f \star g = f g + \sum_{n \geq 1} h^n B_n(f, g)$ with bilinear differential operators $B_n$. In particular in [33] one shows that any bracket of the form

$$ \{ f, g \} = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \lambda^{r+s-2} \sum_{j=0}^{r} \sum_{k=0}^{s} b_{rj,sk}(\partial_x^j \partial_y^{r-j} f)(\partial_x^k \partial_y^{s-k} g) $$

may be transformed to one with $b_{00,10} = b_{00,11} = 0$ and any such bracket satisfying the Jacobi identity must be of the form

$$ \{ f, g \} = \sum_{r=1}^{\infty} \lambda^{r-1} \sum_{j=0}^{r} \sum_{k=0}^{s} b_{rj,k}(\partial_x^j \partial_y^{r-j} f)(\partial_x^k \partial_y^{s-k} g) $$

By suitable calculation one shows also that (3.15), plus Jacobi is equivalent to Moyal. Note that the Jacobi condition for $\{ f, g \} = (1/h)(f \star g - g \star f)$ can be proved directly via associativity of $\star$ (exercise). Thus (B2) $\{ \{ f, g \}, h \} + \{ \{ h, f \}, g \} + \{ \{ g, h \}, f \} = 0$. Now to connect the dKP theory with bracket (3.12) to a $\kappa - KP$ theory with PSDO bracket consider the PSDO symbol bracket (B3) $A \circ B = \sum (1/k!)(\partial_x^k A(x, \xi) \partial_x^{k} B(x, \xi)$ (cf. [11] [58]) where $A \sim \sum a_i(x) \xi^i$, $\partial_x^k A = \sum a_i(x) \partial_x^{k} \xi^i$, and $\partial_x^k A = \sum \partial_x^k a_i(x) \xi^i$. Note also

$$ A \circ_\kappa B = A e^{\kappa \tilde{\xi} \partial_x} B = \sum \frac{\kappa^n}{n!} \partial_x^n A \partial_x^n B $$

and the bracket based on this; thus

$$ A \circ_\kappa B = A(x, \xi + \kappa \partial_x) B(x, \xi); \quad B \circ_\kappa A = B(x, \xi + \kappa \partial_x) A(x, \xi) $$

and (B4) $(1/\kappa) A \circ_\kappa B - B \circ_\kappa A = \{ A, B \}_\kappa$ is of the form (3.16) with $b_{r0r} \neq 0$, $b_{0rr} \neq 0$, and all other coefficients equal 0. Also $b_{110} = -b_{101}$ and the Jacobi identity will follow from associativity so in fact any bracket such as (B4) is equivalent to Moyal in the symbols
involved ($\kappa$ is arbitrary). Note here that associativity is not obvious but is proved in \[\text{[15]}\] (note the $\kappa$ can be absorbed in $\xi$ by rescaling). The trick is to use the formula

\begin{equation}
(a\xi^n)(b\xi^r) = a \sum_{k \geq 0}^n \binom{n}{k} \partial^k_\xi b\xi^{n-k+r}
\end{equation}

which shows that (for $\kappa = 1$)

\begin{equation}
A \circ \kappa B = \sum \frac{1}{m!} \partial^m_\xi A \partial^m_\xi B = \sum \frac{1}{m!} \cdot \sum a_n n(n-1) \cdots (n-m+1) \xi^{n-m} \cdot \sum b_{jm}^{(m)} \xi^j = \sum a_n \binom{n}{m} b_{jm}^{(m)} \xi^{n-m+j}
\end{equation}

Now for associativity one checks that \([\xi^n(a\xi^r)]b = \xi^n[(a\xi^nb)]\) and we refer to \([\text{[15]}\) for further details.

We recall also the standard symbol calculus for PSDO following e.g. \([\text{[36]}\), \([\text{[50]}\), \([\text{[58]}\)]\) (cf. \([\text{[B3]}\), \([\text{[B0]}\), \([\text{[B1]}\), \([\text{[B2]}\)). First one recalls from \([\text{[36]}\) the ring or algebra $\mathfrak{A}$ of pseudodifferential operators (PSDO) via PSD symbols (cf. also \([\text{[58]}\) for a more mathematical discussion). Thus one looks at formal series \((\text{B6})\) $A(x, \xi) = \sum_{-\infty}^{\infty} a_i(x) \xi^i$ where $\xi$ is the symbol for $\partial_x$ and $a_i(x) \in \mathbb{C}^\infty$ (say on the line or circle). The multiplication law is given via the Leibnitz rule for symbols \((\text{B7})\) $A(x, \xi) \circ B(x, \xi) = \sum_{k \geq 0} (1/k!) A^k_\xi(x, \xi) B^{(k)}_\xi(x, \xi)$ where $A^k_\xi(x, \xi) = \sum_{-\infty}^{\infty} a_i(x) (\xi^i)^{(k)}$ and $B^{(k)}_\xi(x, \xi) = \sum_{-\infty}^{\infty} b_i^{(k)}(x) \xi^i$ with $b_i^{(k)}(x) = \partial^k_\xi b_i(x)$. This gives a Lie algebra structure on $\mathfrak{A}$ via \((\text{B8})\) $[A, B] = A \circ B - B \circ A$. Now let $A$ be a first order formal PSDO of the form \((\text{B9})\) $A = \partial_x + \sum_{-\infty}^{1} a_i(\bar{x}) \partial^i_\xi$ where $\bar{x} \sim (x, t_2, t_3, \cdots)$. Then the KP hierarchy can be written in the form \((\text{B10})\) $(\partial A/\partial t_m) = [(A^m)_+, A]$ which is equivalent to a system of evolution equations \((\text{B11})\) $(\partial a_i/\partial t_m) = f_i$ where the $f_i$ are certain universal differential polynomials in the $a_i$, homogeneous of weight $m + |i| + 1$ where $a^i_{-i}$ has weight $|i| + j + 1$ for $a^i_j \sim \partial^i_\xi a$. Somewhat more traditionally (following \([\text{[58]}\) - modulo notation and various necessary analytical details), one can write

\begin{equation}
Au(x) = (2\pi)^{-1} \int e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi
\end{equation}

where $\hat{u}(\xi) = \int \exp(-ix \cdot \xi) u(x) dx$. One takes $D = (1/i)\partial_x$ and writes $a = \text{sym}(A)$ with $A = op(a) \sim A$ where the $\cdot$ is to mod out $\mathfrak{S}^{-\infty}$ (we will not be fussy about this and will simply use $A$). The symbol of $A \circ B$ is then formally

\begin{equation}
(a \circ b)(x, \xi) = \sum \frac{1}{\alpha!} \partial^\alpha_\xi a(x, \xi) D^\alpha_x b(x, \xi)
\end{equation}

corresponding to \((\text{B7})\), while $[A, B] = AB - BA$ corresponds to the symbol \((\text{B12})\) \(\{a, b\} = (\partial a/\partial \xi)(\partial b/\partial x) - (\partial a/\partial x)(\partial b/\partial \xi)\) (note $\overline{P(D)T} = P(\xi)\overline{T}$). In any event it is clear that the algebra of differential operators on a manifold $M$ (which we have sometimes loosely referred to as quantum operators) may be considered as a noncommutative deformation of the algebra of functions on $T^*M$ and the extension to PSDO brings one into the arena of integrable systems.
4. QUANTUM MECHANICS

We refer now to [65] where a lovely survey appears (cf. also [11 26 27 28 32]). There are three logically autonomous alternative paths to quantization, namely Hilbert space operators, path integrals, and deformation quantization. In fact the Wigner-Weyl-Moyal formulation gives a complete coverage as follows. It is based on the Wigner function (WF) which is a quasi probability distribution in phase space defined via

\[ f(x, p) = \frac{1}{2\pi} \int dy \psi^*(x - \frac{\hbar}{2} y) e^{-ipy} \psi(x + \frac{\hbar}{2} y) \]

Here one has (C1) \( \int dp dx f(x, p) = 1 \) and in the classical limit \( \hbar \to 0 \) \( f \) reduces to the probability density in coordinate space \( x \) (usually highly localized) multiplied by delta functions in momentum; thus the classical limit is “spiked”. The WF is manifestly real and constrained by the Schwartz inequality to be bounded with \( -(2/\hbar) \leq f(x, p) \leq (2/\hbar) \) (the bound disappearing in the spikey classical limit). Projection in \( x \) or \( p \) leads to marginal probability densities, namely, a spacelike shadow \( \int dp f(x, p) \) or a momentum space shadow \( \int dx f(x, p) = \sigma(p) \). The WF can and most frequently does become negative in some regions of phase space. Nevertheless WF is a distribution function providing the integration measure in phase space which yields expectation values from phase space c-number functions. Such functions can be classical functions but in general are associated to suitably ordered operators via Weyl’s correspondence rule. Given such an operator ordered via

\[ \mathcal{G}(\mathbf{r}, \mathbf{p}) = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \, g(x, p) e^{i\tau(p-p) + i\sigma(x-x)} \]

the corresponding phase space function \( g(x, p) \) (classical kernel) is obtained via (C2) \( p \to p \) and \( \mathbf{r} \to x \). The operators expectation value is then a phase space average (C3) \( \langle \mathcal{G} \rangle = \int dx dp f(x, p) g(x, p) \). The classical kernel is often the unmodified classical expression such as \( H = \left( \frac{p^2}{2m} \right) + V(x) \) but it contains \( \hbar \) when there are ordering ambiguities (see below). This operation corresponds to tracing with a density matrix as indicated below.

The dynamical evolution is specified by Moyal’s equation which extends the Liouville theorem of classical mechanics (CM), namely \( \partial_t f + \{ f, H \} = 0 \), and is given by (C4) \( \partial_t f = (1/\hbar)(H \ast f - f \ast H) \) where (C5) \( \ast \sim \exp[(ih/2)(\partial_x \partial_p - \partial_p \partial_x)] \). The right side of (C4) is of course the Moyal bracket and is the essentially unique 1-parameter associative deformation of the Poisson bracket (cf. [33 53]). In practice evaluation can be expressed through Bopp operators in the form

\[ f(x, p) \ast g(x, p) = f(x + \frac{i\hbar}{2} \partial_x, p - \frac{i\hbar}{2} \partial_p) g(x, p) \]

The equivalent Fourier representation of the star product can be expressed via

\[ f \ast g = \frac{1}{(h\pi)^2} \int du dv dw dz f(x + u, p + v) g(x + w, p + z) e^{(2i/(\hbar))(uz - vw)} \]
which exhibits noncommutativity and associativity. There is a complete isomorphism between star multiplication and operator multiplication indicated in

\begin{equation}
\mathcal{A}\mathcal{B} = \frac{1}{(2\pi)^2} \int d\tau d\sigma dx dp \langle a \star b \rangle e^{i\tau(p-p)+i\sigma(x-x)}
\end{equation}

One sees also from (4.4) that (C6) \( \int dpdx f \star g = \int dpdx fg = \int dpdx g \star f \). Note that the Moyal equation is necessary but does not suffice to specify the WF for a system (e.g. \( f(H) \) commutes with \( H \)).

Static or stationary WF’s obey more powerful star-geval equations (cf. 26, 27, 32)

\begin{equation}
H(x,p) \star f(x,p) = H \left( x + \frac{ih}{2} \frac{\partial}{\partial p}, p - \frac{ih}{2} \frac{\partial}{\partial x} \right) f(x,p) = f(x,p) \star H(x,p) = Ef(x,p)
\end{equation}

where \( H\psi = E\psi \) and this amounts to a complete characterization of the WF’s. Indeed, using a simple Hamiltonian \( p^2/2m + V(x) \) (without essential loss of generality) one proves (cf. 65) that for real \( f(x,p) \) the Wigner formula (4.1) for pure stationary eigenstates is equivalent to the star-geval equations (4.6) along with \( f \star H = Ef \). Conversely the pair of star-geval equations for \( f \) and \( g \) by virtue of (C6) one gets then (C8) \( f \star g = 0 \). Moreover for \( f = g \) one gets then (C9) \( f \star H \star f = Ef \star f = H \star f \star f \) so (C10) \( f \star f \propto f \) and in fact (C11) \( f_a \star f_b = (1/\hbar)\delta_{ab}f_a \). Here the normalization is important since it prevents superposition (which is handled differently as in the density matrix formulation). Note also by virtue of (C6) for different starfunctions one has (C12) \( \int dpdx fg = 0 \) so one must go negative to offset positive overlap (a virtue of negativity). Further note that integrating (4.6) yields the expectation of the energy (C13) \( \int H(x,p)f(x,p)dxdp = E \int f dxdp = E \) and from (C11) we get (C14) \( \int f^2dxdp = 1/\hbar \).

Next note (C15) \( g \star g \geq 0 \) which leads to the uncertainty principle. Indeed

\begin{equation}
\int dpdx(g^\star g) = h \int dxdp(g^\star g)(f \star f) = h \int dxdp(f \star g^\star)(g \star f) = h \int dxdp(g \star f)^2
\end{equation}

To produce Heisenberg’s uncertainty principle one chooses (C16) \( g = a + bx + cp \) for arbitrary constants \( a, b, c \in \mathbb{C} \). The resulting positive semi-definite form is then (C17) \( a^\star a + b^\star b < x \star x > + c^\star c < p \star p > + (a^\star b + b^\star a) < x > + (a^\star c + b^\star c) < p > + c^\star b < p \star x > + b^\star c < x \star p > \geq 0 \). The eigenvalues of the corresponding matrix are then non-negative and so must be the determinant. Some calculation then leads to (C18) \( \Delta x \Delta p \geq \hbar/2 \) (cf. 65 and see 23 for details in calculation).

Going now to time evolution the Moyal equation (C4) is formally solved by virtue of associative combinatoric operations completely analogous to Hilbert space quantum mechanics (QM) through definition of a star-unitary evolution operator (star exponential) in the form

\begin{equation}
U_\star(x,p,t) = e^{it\hbar/\hbar} = 1 + (it/\hbar)H(x,p) + \frac{(it/\hbar)^2}{2!}H \star H + \frac{(it/\hbar)^3}{3!}H \star H \star H + \cdots
\end{equation}
Given the WF at $t = 0$ the solution to the Moyal equation is then \( (C19) \) $f(x, p, t) = U^{-1}_*(x, p, t) \ast f(x, p, 0) \ast U_*(x, p, t)$. For variables $x, p$ this collapses to classical trajectories

\[
\dot{x} = \frac{x \ast H - H \ast x}{i\hbar} = \partial_p H; \quad \dot{p} = \frac{p \ast H - H \ast p}{i\hbar} = -\partial_x H
\]

Thus for the harmonic oscillator \( (C20) \) $x(t) = x\cos(t) + p\sin(t)$ and $p(t) = p\cos(t) - x\sin(t)$ so the functional form of the WF is preserved along classical phase space trajectories via \( (C21) \) $f(x, p, t) = f(x\cos(t) - p\sin(t), p\cos(t) + x\sin(t), 0)$.

5. MOMENTUM CALCULI

We summarize first some features of the basic situation.

EXAMPLE 5.1. Thus go back to Remark 2.1 and the Hamiltonian theory for dKP in the form \( (2.9) \), written here as \( (D1) \) $\dot{P} = \partial\Omega_n$ and $\dot{X}_n = -\partial_p\Omega_n$. We know KP is equivalent to MdKP (Moyal dKP) and examine here the nature of quantizing \( (D1) \) via Moyal-Wigner-Weyl (MWW) formulas. In \( (D1) \) the $T$ variable is $\tau_n = T'_n = nT_n$ where $\epsilon T_n \sim T_n$ and we recall here the origin of \( (D1) \) from [10] [17] [11]. Thus as in Section 2 one arrives at $\partial_p S = \mathfrak{B}_n$ and we set $n\Omega_n = \mathfrak{B}_n$. Then rescaling $\tau_n = T'_n = nT_n$ and writing $\partial_n \sim \partial/\partial\tau_n$ now we have (for $P = \partial_X S$ and $\mathfrak{B}_n = \mathfrak{B}_n(P, X)$) \( (D2) \) $\partial_n P = \partial_X \Omega_n = \partial\Omega_n + \partial_p\Omega_n \partial P$ (in an obvious notation). Then thinking of $P = P(X, \tau)$ write

\[
\dot{P}_n = \partial_n P + \partial P \dot{X}_n = \partial\Omega_n + \partial_p\Omega_n \partial P + \partial P \dot{X}_n
\]

This is then incorporated into a HJ theory with $S$ as a generating function and equations \( (D1) \). Now the quantization of the dynamical system via classical MWW methods involves Hamiltonians $H(X, P) = \Omega_n(X, P)$ and equations (using $\tau$ as the evolution time and $(X, P) \sim (x, p)$) \( (D3) \) $i\hbar\dot{X}_n = \{X, \Omega_n\}_M$ and \( (D4) \) $i\hbar\dot{P}_n = \{P, \Omega_n\}_M$ where $\{f, g\}_M = f \ast g - g \ast f$ with $\ast$ as in \( (C5) \). Hence as in \( (1.3) \) we have \( (D5) \) $\dot{X}_n = \partial\Omega_n$ and $\dot{P}_n = -\partial_X \Omega_n$ repeated (i.e. the “motion” is along the “classical” trajectories). One can ask now what this means in the standard KP theory. Recall from Remark 3.1 how the symbol calculus for PSDO is equivalent to Moyal dKP so in some sense we will have a Hamiltonian theory in the symbols of KP operators (based perhaps on some sort of relation such as \( (3.12) \)).

Now we look at momentum calculi (cf. also [6] [29] [59]). The approach of [6] is based on Lie algebras, Poisson structures, and R-matrices; the demands of the general framework adopted however seem to restrict severely the range of applicability (e.g. KP is not included). Thus we mainly omit this. Another, more flexible, approach is developed in [29] (cf. also [59] [62]) One begins with the standard phase space star product

\[
A(x, p) \ast B(x, p) = e^{\kappa(\partial_x \partial_p - \partial_p \partial_x)} A(x, p) B(\tilde{x}, \tilde{p})|_{\tilde{x}, \tilde{p} = (x, p)}
\]

with conventional Moyal bracket \( (D6) \) $\{A(x, p), B(x, p)\}_\kappa = (1/2\kappa)(A \ast B - B \ast A)$. As usual one has $\lim_{\kappa \to 0} \{A, B\}_\kappa = \{A, B\}$ (Poisson bracket). The star product gives the momentum an operator character via (note $\hbar/2 \sim \kappa$ when comparing notations)

\[
p^n \ast p^m = p^{n+m}; \quad p^n \ast f(x) = \sum_{m=0}^{n} \binom{n}{m}(2\kappa)^m f^{(m)} \ast p^{n-m};
\]
\[
\binom{n}{m} = \frac{n(n-1) \cdots (n-m+1)}{m!}; \quad \binom{n}{0} = 1
\]

Up to normalization these are precisely the relations satisfied by the derivative operator. Let us check here some calculations based on (3.6) where

\[
(5.4) \quad f \ast g = fe^{\kappa(\partial_x \bar{\partial}_p - \bar{\partial}_p \partial_x)}g = \sum_0^\infty \sum_0^s \frac{\kappa^s}{s!} \sum_0^j (-1)^j (\partial^j_\partial \partial^s_\partial f)(\partial^s_\partial \partial^j_\partial p g)
\]

\[
(5.5) \quad p^m \ast g = \sum_0^m \frac{\kappa^s}{s!} \partial^s_\partial p^m \partial^s_\partial g = \sum_0^m \binom{m}{s} \kappa^s p^{m-s} g(s);
\]

\[
(5.6) \quad f \ast p^m = \sum_0^m \frac{\kappa^s}{s!} (-1)^s \partial^s_\partial f \partial^s_\partial p^m = \sum_0^m \binom{m}{s} (-\kappa)^s p^{m-s} f(s);
\]

\[
g(s) \ast p^{m-s} = \sum_0^{m-s} \binom{m-s}{j} (-\kappa)^j p^{m-s-j} f(s+j)
\]

Look at some low order terms 1 \ast f = f, \ p \ast f = pf - \kappa f', \ f \ast p = pf - \kappa f', \text{ and } f \ast 1 = f.

Then \[p \ast f = pf = \kappa f' = f \ast p + 2\kappa f' = f \ast p + (2\kappa)(f' \ast 1)\] as in (5.3). For \[p^2\] we have \[p^2 \ast f = p^2 f + \left(\frac{2}{1}\right) \kappa pf' + \kappa^2 f''\] with \[f \ast p^2 = p^2 f - \left(\frac{2}{1}\right) \kappa pf' + \kappa^2 f''\]. Hence

\[
p^2 \ast f = f \ast p^2 + 2\kappa \left(\frac{2}{1}\right) \kappa pf' = f \ast p^2 + 2\kappa \left(\frac{2}{1}\right) [f' \ast p + \kappa f''] = f \ast p^2 + (2\kappa) \left(\frac{2}{1}\right) (f' \ast p) + (2\kappa)^2 f'' \ast 1 = \sum_0^2 \binom{2}{m} (2\kappa)^m f(m) \ast p^{m-n}.
\]

Thus (5.3) seems reasonable and to compare with derivation operators note also \((\kappa \partial f)^2 f = \kappa^2 (f'' g + 2f' g' + fg'')\) and \((f' \kappa g) g = f' \kappa g'\) so \(4(\kappa \partial)^2 f \sim f (2\kappa \partial)^2 + 4\kappa f' (2\kappa \partial) + (2\kappa)^2 f''\). Thus one has

\[
(2\kappa \partial)^n f = \sum_0^n \binom{n}{k} (2\kappa)^k f^{(k)} (2\kappa \partial)^{n-k}
\]

upon extrapolation.

Now one defines two classes of Lax operators on the phase space via

\[
(5.7) \quad L_n = p^n + u_1(x) \ast p^{n-1} + u_2(x) \ast p^{n-2} + \cdots + u_n(x);
\]

\[
\Lambda_n = p^n + u_1(x) \ast p^{n-1} + \cdots + u_n(x) + u_{n+1}(x) \ast p^{n-1} + \cdots
\]

Thus one has replaced the space of pseudodifferential operators by that of polynomials in momentum which inherits an operator structure through the star product and defines an algebra. This will be called the momentum algebra \(M_n\) and one notices that this is different from the concept of pseudodifferential operators (PSDO) with the coefficients taken from the Moyal algebra of [57]. All of the properties of PSDO carry through with suitable redefinitions. In particular thinking of the residue as the coefficient of the \(p^{-1}\) term one gets (D7) \(Res\{A, B\}_\kappa = (\partial_x C)\) exhibiting the residue as a total derivative. Consequently one can define (D8) \(Tr(A) = \int dx \ Res(A)\) which is unique (with the usual assumptions
of asymptotic decrease) and satisfies cyclicity. For a general Lax operator \( \Lambda_n \) one checks immediately that

\[
\frac{\partial \Lambda_n}{\partial t_k} = \left\{ \Lambda_n, \left( \Lambda_k^{k/n} \right)^{m}_{\kappa} \right\}; \quad (k \neq \ell n)
\]

defines a consistent Lax equation provided \( m = 0, 1, 2 \) and the projectors are defined with respect to the star product (the difference in ordering here can be adjusted via \( t_k \to -t_k \) if desired). Note here \( \Lambda^{k/n} = \Lambda^{1/n} * \cdots * \Lambda^{1/n} \) with \( k \) factors with the \( n^\text{th} \) root determined recursively. The projection with \( m = 0 \) is denoted by \( (\ )_+ \) and will be referred to as the standard Moyal-Lax representation (the others are called nonstandard and are not considered here). Note that \( (D9) \lim_{\kappa \to 0}(\Lambda * \Lambda')_+ = (\Lambda \Lambda')_+ \) where the factors on the right are phase space functions (not operators). Thus one can go the Lax representation of the dispersionless limit in a natural manner (cf. [16, 17, 59]).

In [29] the KdV hierarchy is developed via the Lax operator \( L = p^2 + u(x) \) where \( (D12) (L^{3/2})_+ = p^2 + (3/2)u + (3\kappa/2)u^{(1)} \) (we have changed the coefficient of \( u^{(1)} \) to agree with calculations below in \( (D25) \)). One gets then

\[
\frac{\partial t}{L} = \left\{ L, \left( L^{3/2} \right)^{3/2}_{\kappa} \right\} \Rightarrow \partial_t u = - \left( \kappa u^2 + \frac{3}{2} uu^{(1)} \right)
\]

The first few conserved quantities are (not checked)

\[
H_1 = Tr(L^{1/2}) = \int dx(u^{(2)}/2); \quad H_2 = Tr(L^{3/2}) = \int dx(u^{2}/4);
\]

\[
H_3 = Tr(L^{5/2}) = \int dx(4\kappa u^{(2)}u + u^3)
\]

The commutativity of flows follows directly from the Moyal-Lax representation.

**Example 5.3.** The conventional Lax equation in standard representation \( (D13) \partial_t \Lambda = [((L^{k/n})_+, L] \) (L a PSDO) resembles a Hamiltonian equation with \( (L^{k/n})_+ \) as a Hamiltonian. However such a relation cannot be further developed in the language of PSDO. In contrast consider the Moyal-Lax representation with an arbitrary flow in the KdV hierarchy described by \( (D14) \partial_t L = \left\{ L, (L^{(2n+1)/2})_+ \right\}_\kappa \) (L as in \( (D4) \)). Then consider an action of the form \( (D15) S = \int dt(p \dot{x} - (L^{(2n+1)/2})_+) \). It is important to remember here that \( L = L(p, x) \) but does not depend on time explicitly. Thus one can think of \( (L^{(2n+1)/2})_+ \) as the Hamiltonian on the phase space. That this is valid follows from the Euler-Lagrange equations \( ((2n + 1)/2 = \alpha - \text{ cf. also Example 4.1) }\)

\[
\dot{x} = \frac{\partial (L^\alpha)_+}{\partial p} = \{x, (L^\alpha)_+\}_\kappa; \quad \dot{p} = - \frac{\partial (L^\alpha)_+}{\partial x} = \{p, (L^\alpha)_+\}_\kappa
\]

Further since L is a function on the phase space one has \( (D16) \partial_t L = \left\{ L, (L^{(2n+1)/2})_+ \right\}_\kappa \) so that the Moyal-Lax equation is indeed a Hamiltonian equation with \( L^\alpha_+ \) playing the role of Hamiltonian. This procedure also goes through for the nonstandard representations.
The Moyal-Lax representation has now an advantage in that one can go to the dispersionless limit of an integrable system by simply taking the limit $\kappa \to 0$ and, while the Lax representations for various dispersionless integrable models are known (cf. [11, 16, 17]), the determination of Hamiltonian structures (at least the second) from such a Lax representation has often been open. The Moyal-Lax representation provides a solution to this problem in a natural way (cf. [12, 29] for details).

**REMARK 5.1.** An approach similar to [20] was developed in [59] (cf. also [62]) and we sketch a few points here. One considers an algebra of Laurent series of the form $\Lambda = \{ A; A = \sum_{-\infty}^{\infty} a_i p^i \}$ with coefficients depending on $t_1 = x, t_2, \cdots$. $\Lambda$ can be decomposed as $\Lambda = \Lambda_{\geq k} \oplus \Lambda_{< k}$ for $k = 0, 1, \cdots$ where e.g. $\Lambda_{\geq k} = \{ A = \sum_{i \geq k} a_i p^i \}$; the notation $\Lambda_{+} = \Lambda_{\geq 0}$ is used as before. Evidently $\Lambda$ is an associative but noncommutative algebra under the Moyal star product and one defines $Res(A) = a_{-1}$ with trace as $Tr(A) = \int Res(A)$. There results

$$\int Res(A \ast B) = \int \sum_{i,j} \frac{\kappa^{i+j+1} i!}{(i-j+1)!(j-1)!} (a_i b_j)^{(i-j+1)} = \sum_i \int a_i b_i^{+1}$$

One notes that this is the same as in the dispersionless limit $\kappa \to 0$ and because of this the Hamiltonian formulation for the Moyal KdV (for example) becomes possible. Indeed using (D12) one shows that $Tr\{ A, B \}_\kappa = 0$ and $Tr\{ A \ast B, C \}_\kappa = Tr\{ A, B \}_\kappa \ast C$. To see this use (D12) to replace the Moyal star product within the trace by the ordinary multiplication. Now given a functional $F(A) = \int f(a)$ one defines a gradient via (D17) $d_A F = \sum_i (\partial f/\partial a_i) p^{-1}$ where the variational derivative is defined via (D18) $\partial f/\partial a_k = \sum_i (-1)^i (\partial^i (\partial f/\partial a_k))$ where $a_k^{(i)} = (\partial^i \cdot a_k)$. The Moyal-Lax representation is defined by the Lax equations

$$\frac{\partial L}{\partial t_k} = \{(L^{1/n})_+^k, L\}_{\kappa} = \{L, (L^{1/n})_-^k\}_{\kappa}$$

where $(L^{1/n})_+^k = L^{1/n} \ast \cdots \ast L^{1/n}$ (k times) and here $L = p^n + \sum_{i=0}^{n-1} u_i p^i$ with $L^{1/n} = p + \sum a_i p^{-i}$ is the $n^{th}$ root (D19) $L = L^{1/n} \ast \cdots \ast L^{1/n}$ (n times). By definition of the Moyal bracket the highest order in $p$ on the right side of (5.10) is $n-2$ so $u_{n-1}$ is trivial in evolution equations and can be dropped in the Lax formulation (this changes in the Hamiltonian formulation however). The simplest example is written out in [59], namely $n=2$ with $L = p + u$ and $L^{1/2} = p + \sum a_i p^{-i}$ and (D20) $a_1 = u/2$; $a_3 = -u^2/8$; $a_5 = u^3/16 + (\kappa^2/8)(u_x^2 - 2u u_x); \cdots$ and the first few Lax flows are (D21) $u_{t_1} = u_x$; $u_{t_3} = (3/2) u u_x + \kappa^2 u_{xxx}; \cdots$. This set of equations forms a Moyal KdV hierarchy which can also be obtained from reduction of the Moyal KP hierarchy (cf. [11, 12] or noncommutative zero curvature equations (cf. [82]). When $\kappa = 0$ all higher order derivative terms disappear and the Moyal KdV reduces to the dispersionless KdV hierarchy (cf. [11]). In this sense the Moyal parameter $\kappa$ characterizes the dispersion effect. On the other hand when $\kappa = 1/2$ the Moyal KdV hierarchy is the ordinary KdV hierarchy and this is due to an isomorphism of Moyal KP to ordinary KP at $\kappa = 1/2$ (cf. [11, 36, 55] and Section 3).

We concentrate now on Examples 4.1 and 4.3 and for simplicity begin with KdV. The
theme is somehow to exhibit the quantum mechanical nature of KdV via the dynamical system (5.1) or (5.11). The momentum algebra as in this section which leads to (5.11) and (D16) is very attractive and we spell out more details. We go first to (59) and extract some specific formulas to enable the study of (D22) $\partial_k L = \{L_{(2n+1)/2}, L\}_\kappa$ ($t_k \rightarrow -t_k$ in (D16)). We will not emphasize the bihamiltonian structure. Then for $L = p^2 + u$ one has (D23) $L^{1/2} = p + \sum_{i=1}^\infty a_i p^{-i}$ with (cf. (D12))

$$(5.14) \quad L^{3/2}_{1/2} = L \star L^{1/2} = (p^2 + u) \star (p + \sum_{i=1}^\infty a_i p^{-i}) = p^3 + p^2 \star a_1 p^{-1} + p^2 \star a_2 p^{-2} + \cdots + u \star p + u \star a_1 p^{-1} + u \star a_2 p^{-2} + \cdots$$

From [59] we have $a_{2k} = 0$ and (cf. (D20))

$$(5.15) \quad a_1 = \frac{1}{2} u; \quad a_3 = -\frac{1}{8} u^2; \quad a_5 = \frac{1}{16} u^3 + \frac{1}{8} \kappa^2 (u_x^2 - 2uu_{xxx});$$

$$a_7 = -\frac{5}{128} u^4 - \frac{5}{16} \kappa^2 (u_x^2 - 2uu_{xxx}) - \frac{1}{8} \kappa^2 (u_{xxx}^2 - 2u_x u_{xxxx} + 2uu^{(4)}); \cdots$$

The first few Lax flows are given by

$$(5.16) \quad u_{t_1} = u_x; \quad u_{t_3} = \frac{3}{2} uu_x + \kappa^2 u_{xxx}; \quad u_{t_5} = \frac{15}{8} u^2 u_x + \frac{5}{2} \kappa^2 (u_{xxxx} + 2u_x u_{xx}) + \kappa^4 u_x;$$

The comparison here is with (D12) (as changed from [29]). Thus the only plus terms involve the composition (since $a_2 = 0$)

$$(5.17) \quad p^2 \star a_1 p^{-1} = a_1 p + 2\kappa \partial_x a_1 + \kappa^2 p^{-1} \partial_x^2 a_1$$

so (D24) $L^{3/2}_{1/2} = p^3 + a_1 p + 2\kappa \partial_x a_1 + u \star p = p^3 + (1/2) u p + \kappa u_x + u \star p$. We know that $u \star p = pu - \kappa u_x$ from calculations after (5.3) so (D25) $L^{3/2} = p^3 + (1/2) [u \star p + \kappa u_x] + \kappa u_x + u \star p = p^3 + (3/2) u \star p + (3/2) \kappa u_x$.

**REMARK 5.2.** We see via Examples 4.1 and 4.3 that KP or KdV for example have a distinctly quantum mechanical flavor via the phase space dynamics (5.1) (note again that it doesn’t matter whether we take $\{L^+_k, L\}$ or $\{L, L^+\}$ to express the time variation since it is simply a matter of time reversal). Thinking of KdV for simplicity, as a result one say that all $\partial u / \partial t_{2n+1}$ arise from QM equations (D26) $\partial_{2n+1} = \{L, L_{(2n+1)/2}\}$. Further, given the dependence of the $a_i$ in (D20) or (5.15) on $u$ one has a direct computation for $\partial a_{2n+1} / \partial t_{2n+1}$ (this also results from (D26) upon writing out terms). The QM nature of the flows arises explicitly via the presence of $\kappa$ in the equations and this reduces to “classical” behavior when $\kappa \rightarrow 0$ (where classical here is unrealistic since it ignores dispersion however). Thus dispersion for example appears as a quantum phenomenon; QM represents a smoothing or calming factor which eliminates some wave caustics, breaking, etc. Since one knows also that $q \sim exp(\kappa)$ is characteristic of the emergence of $q$ in various $q$-versions of quantum phenomena (cf. [12]) we have here another physical motivation for $q$ which makes the study of $q$-QM seem more meaningful (one can ask here also about whether a $\kappa$ quantization as in Moyal is just a first order version of a $q = exp(\kappa)$ discrete quantum theory). On the other hand water waves for example are eminently classical macro-phenomena so one can ask why they should obey QM rules of behavior. As an aside we mention, following [29], that upon
constructing the bihamiltonian theory associated with (D26) one finds that \( \kappa \) is directly related to conformal field theory (CFT) and to the central charge of the second Hamiltonian structure of KdV (which in turn is related to a standard Virasoro algebra - not q-Virasoro - cf. [11, 21, 34]). If one thinks of KdV (or KP) then as a QM extension of dKdV (or dKP) then some real world macro-phenomena are essentially quantum mechanical in nature. This is perhaps a viewpoint to be further explored both philosophically and technically. Note that (elementary) particles have only one QM time evolution to drive them but in some sense "fluids" seem to have many, whose nature could be further examined with the study of higher order evolutions in KdV for example.

EXAMPLE 5.4. Following [29, 39] we compute the first terms of the “quantum” equation \( \partial_t L = \{ L_{3/2}^{1/2}, L \}_\kappa \) with \( L_{3/2} = p^3 + (3/2)u \ast p + (3/2) \kappa u' \) (cf. (D12) and (D25)). Thus first we get (since \( u \ast u' = uu' \))

\[
(5.18) \quad \Gamma = \left( p^3 + \frac{3}{2} u \ast p + \frac{3}{2} \kappa u' \right) \ast (p^2 + u) - (p^2 + u) \ast \left( p^3 + \frac{3}{2} u \ast p + \frac{3}{2} \kappa u' \right) = \\
= p^3 u + \frac{3}{2} u \ast p^2 + \frac{3}{2} u \ast p \ast u + \frac{3}{2} \kappa u' \ast p^2 - \frac{3}{2} p^2 \ast (u \ast p) + \frac{3}{2} p^2 \ast \kappa u' + u \ast p^3 + \frac{3}{2} u \ast u \ast p
\]

Note now from (5.5) that (D27) \( p^3 \ast u = 3u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' \) while (D28) \( u \ast p^3 = -\kappa u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' + 3 \kappa \kappa u'' \). Next consider \( \Xi = (3/2)(u \ast p^3 - p^2 \ast u + 2 \kappa u' \) so (D30) \( \Xi = (3/2)(u \ast p^3 - p^2 \ast u + 2 \kappa u' = -3/2)(2 \kappa u'' + 6 \kappa u'' + 6 \kappa u'' \) + (3/2)2 \kappa \kappa u' = -9 \kappa u'' + 3 \kappa(p^2 u' + 2 \kappa u' = -6 \kappa u'' + 6 \kappa u' u'\). Finally (D31) \( (3/2)(u \ast p^3 - p^2 \ast u + 2 \kappa u' \) = (3/2)(u \ast p^3 - p^2 \ast u + 2 \kappa u' \) = 3 \kappa u u' and (D32) \( (3/2)(u \ast p^3 - p^2 \ast u' = 6 \kappa u'' \) Combining then D27) we get (D33) \( \Gamma = 2 \kappa u'' + 3 \kappa u \). Combining then D27 - (D32) we get (D33) \( \Gamma \) = \( 2 \kappa u'' + 3 \kappa u \) so the Lax equation then decrees that (D34) \( u_t = (1/2 \kappa)(2 \kappa u'' + 3 \kappa u) = (3/2) uu' + \kappa u'' \) which is the standard KdV form with the dispersion coming from \( \kappa \).

6. REMARKS ON Q-THEORIES

QKP (and qKdV) can be developed in a hierarchy form following [2, 3, 19, 35, 37, 39, 40, 41, 4]. This is pursued in connection with Hirota formulas at some length in [19] and we remark here only that the resulting QKP or qKdV equations are very complicated due to formulas of the form

(6.1) \( u = a_1 = (1 - D) \left( \frac{1}{2} \left( \partial_2 \tau q - \partial_2 q \tau q \right) \right) - \left( \frac{\partial_1 q \tau q}{\tau q} \right)^2 + \frac{\partial_1 q \tau q}{\tau q} D \left( \frac{\partial_1 q \tau q}{\tau q} \right) + D q \frac{\partial_1 q \tau q}{\tau q}
\]

instead of the classical \( u = \partial^2 \log(\tau) \). Similarly for qKdV one has a difficult formula (A31) \( u = \partial_q \partial_1 \log(\tau(x, t) D \tau(x, t) \).

REMARK 6.1. In this direction if one actually writes out a qKdV equation for example from the hierarchy picture it will have the form (cf. [19])

(6.2) \( \partial_t u = (\partial_q^2 u) + w_2(\partial_q^2 u) + w_1(\partial_q u) - [(\partial_q^2 w_0) + u_1(\partial_q w_0)] \)

where (setting (E1) \( u_1 = (q + 1) xu = (1 + D) s_0 \) and \( u = s_1 + D s_1 + s_0^2 + \partial_q s_0 \))

(6.3) \( w_2 = D^2 s_0 + u_1 = D^2 s_0 + D s_0 + s_0; \)

\( w_1 = (q + 1)(D \partial_q s_0) + D^2 s_1 + [(D s_0) + s_0(D s_0) + u; \)
\[ w_0 = \partial_q^2 s_0 + (q + 1)(D\partial_q s_1) + u_1\partial_q s_0 + u_1(D s_1) + u s_0 + D^2 s_2 \]

cf. \[12, 20, 30\] for other forms of q-equations or noncommutative integrable equations). It appears therefore that after expressing e.g. \( s_0 = (1 + D)^{-1}(q - 1)xu \) with \( (E2) \) \((1 + D)^{-1} \sim \sum_0^\infty (-D)^n \) formally the qKdV equation was derived in \[21\] (cf. also \[24\]), by use of a version of q-Virasoro, in the form \((E3)\) \( u_t + c\partial_q^2(D + D^{-1})^{-1}\partial_q u + \partial_q(uD u) + D^{-1}u\partial_q D^{-1}u = 0 \)

where \( \partial_q f = [f(qx) - f(q^{-1}x)]/(q - q^{-1})x \).

One recalls here that \( \partial_q f(z) = [f(qz) - f(z)]/(q - 1)z \) with \((E4)\) \( \partial_q(f g) = \partial_q(f)g + \tau(f)\partial_q g \) where \( \tau(f)(z) = f(qz) \) (we use \( \tau \) and \( D \) interchangably now and note that \( \partial_q \tau = \tau \partial_q \)).

PSDO are defined via an equation \((E5)\) \( A(x, \partial_q) = \sum_{-\infty}^n u_i(x)\partial_q^i \) with \( \partial_q u = (\partial_q u) + (\tau(u)\partial_q) \) and one has \((D_q \sim \partial_q)\)

\[(6.4) \partial_q^{-1} u = \sum_{k=0}^n (-1)^k q^{-k(k+1)/2}(\tau^{-k-1}(\partial_q^k u))\partial_q^{-k-1}; \partial_q^n u = \sum_{k=0}^n \left[ \begin{array}{c} n \\ k \end{array} \right] (\tau^{n-k}(\partial_q^k u))\partial_q^{-k} \]

Recall also

\[(6.5) \quad (n)_q = \frac{q^n - 1}{q - 1}; \quad \left[ \begin{array}{c} m \\ k \end{array} \right]_q = \frac{(m)_q(m-1)_q\cdots(m-k+1)_q}{(1)_q(2)_q\cdots(k)_q} \]

There are then q-analogues of the Leibniz rule etc. and for \( L_q = \partial_q + u_1(z) + u_2(z)\partial_q^{-1} + u_3(z)\partial_q^{-2} + \cdots \) and one has q-KP via \((\partial L_q/\partial t_m) = [(L^n_q)_+ \cdot L_q]\) where \( u_1(z) \) has a nontrivial evolution. In accord with Section 3 we should now represent the ring or algebra \( \mathfrak{A}_q \) of qPSDO symbols via a product as in say \((B7)\) and thence provide expressions for deformation thereof. The \( X \) and \( P \) variables should come from the phase space for dKP. Evidently the qPSDO symbols will involve a variation on \( \mathfrak{C}21 \) and one can utilize techniques of \[2, 3, 4, 12, 50, 40\] (cf. also \[13, 19, 20\] for q-formulas) for calculations. In this direction one finds (heuristically) that the ring or algebra calculi of \( \mathfrak{A} \sim \) PSDO and \( \mathfrak{A}_q \) correspond symbolically via \( \partial \sim D_q = \partial_q \) and suitable insertion of \( D = \partial_q \) factors along with q-subscripts; in particular \((E6)\) \( (\partial_q)^k_i \xi^j = i_q \cdots (i - k + 1)_q(\tau \xi)^j \) is needed. When commutators are also envisioned individual terms may differ because e.g. brackets \([\ , \ ]\) have different degrees, etc. but one notes that e.g.

\[(6.6) \quad [a\partial, b\partial] = (ab' - ba')\partial; [a\partial_q, b\partial_q] = (ab_q' - ba_q')\partial_q + (a\tau b - b\tau a)\partial_q^2; (ab' - ba')\partial \to (a\partial_q b - b\partial_q a)\partial_q = [a(b_q' + \tau b\partial q) - b(a_q' + \tau a\partial q)]\partial_q \]

**REMARK 6.2.** Given \( L_q = D + q + b_0(t) + \sum_1^\infty b_{-i} D_q^{-i} \) for qKP and \( L = \partial + \sum_1^\infty \beta_{-i} \partial^{-i} \) for KP, with nontrivial evolution of \( b_0 \) one cannot perhaps expect KP and qKP to be “isomorphic” but as indicated above they do correspond symbolically so there seems to be no reason to regard one as more “quantum” than the other (except perhaps the fact that the q-theory represents a discretized version of the other and such discretization may be an essential quantum signature - not yet authenticated in conventional QM such as the Moyal quantization). There is also another way to view qKP due to E. Frenkel (cf. \[35\]) via \((E7)\) \( qKP : L_q = D + a_0(t) + \sum_1^\infty a_{-i}(t) D_q^{-i} \) where \( Df(t) = f(qt) \) (recall here that
\((q-1)xD_qf = (D-1)f\). It is shown in e.g. [2, 3] (cf. also [12]) that there is an isomorphism mapping qKP or \(qKP\) into the discrete KP hierarchy which exhibits their equivalence via

\[
(6.7) \quad a_i(y) = \sum_{0 \leq k \leq n-i} \frac{\binom{k+i}{k}}{(-y(q-1)q^i)^k} b_{k+i}(y)
\]

We refer to [2, 3] for details on discrete KP which in fact is equivalent to the 1-Toda lattice (cf. [2, 2]). The correspondence to qKP can be seen best through the notation of [56] where one works with Toda Lax operators \(L = \Lambda + \sum_0^\infty u_{n+1}(\varepsilon,t,s)\Lambda^{-n}\) with \(\Lambda \sim \exp(\varepsilon\partial_s)\) (cf. also [11]). We note also connections of discrete KP to algebraic equations over finite fields in [5] and an interesting paper [31] on formulating quantum mechanics with difference operators.

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