Recursive Importance Sketching for Rank Constrained Least Squares: Algorithms and High-order Convergence

Yuetian Luo, Wen Huang, Xudong Li, and Anru R. Zhang

Abstract

In this paper, we propose Recursive Importance Sketching algorithm for Rank constrained least squares optimization (RISRO). The key step of RISRO is recursive importance sketching, a new sketching framework based on deterministically designed recursive projections, which significantly differs from the randomized sketching in the literature (Mahoney, 2011; Woodruff, 2014). Several existing algorithms in the literature can be reinterpreted under this new sketching framework and RISRO offers clear advantages over them. RISRO is easy to implement and computationally efficient, where the core procedure in each iteration is to solve a dimension-reduced least squares problem. We establish the local quadratic-linear and quadratic rate of convergence for RISRO under some mild conditions. We also discover a deep connection of RISRO to the Riemannian Gauss-Newton algorithm on fixed rank matrices. The effectiveness of RISRO is demonstrated in two applications in machine learning and statistics: low-rank matrix trace regression and phase retrieval. Simulation studies demonstrate the superior numerical performance of RISRO.

Keywords: Rank constrained least squares, Sketching, Quadratic convergence, Riemannian manifold optimization, Low-rank matrix recovery, Non-convex optimization

1 Introduction

The focus of this paper is on the rank constrained least squares:

$$\min_{X \in \mathbb{R}^{p_1 \times p_2}} f(X) := \frac{1}{2} \|y - A(X)\|_2^2, \text{ subject to } \text{rank}(X) = r.$$  (1)

Here, $y \in \mathbb{R}^n$ is the given response and $A \in \mathbb{R}^{p_1 \times p_2} \to \mathbb{R}^n$ is a known linear map that can be explicitly represented as

$$A(X) = [\langle A_1, X \rangle, \ldots, \langle A_n, X \rangle]^T, \quad \langle A_i, X \rangle = \sum_{1 \leq j \leq p_1, 1 \leq k \leq p_2} (A_i)_{j,k} X_{j,k}$$  (2)

with given measurement matrices $A_i \in \mathbb{R}^{p_1 \times p_2}, i = 1, \ldots, n$.

The rank constrained least squares (1) is motivated by the widely studied low-rank matrix recovery problem, where the goal is to recover a low-rank matrix $X^*$ from the observation $y =$
\( A(X^*) + \epsilon \) (\( \epsilon \) is the noise). This problem is of fundamental importance in a variety of fields such as optimization, machine learning, signal processing, scientific computation, and statistics. With different realizations of \( A \), \( (1) \) covers many applications, such as matrix trace regression (Candes and Plan 2011; Davenport and Romberg 2016), matrix completion (Candes and Tao 2010; Keshavan et al. 2009; Koltchinskii et al. 2011; Miao et al. 2016), phase retrieval (Candes et al. 2013; Shechtman et al. 2015), blind deconvolution (Ahmed et al. 2013), and matrix recovery via rank-one projections (Cai and Zhang 2015; Chen et al. 2015). To overcome the non-convexity and NP-hardness of directly solving \( (1) \) (Recht et al. 2010), various computational feasible schemes have been developed in the past decade, including the prominent convex relaxation (Recht et al. 2010; Candes and Plan 2011):

\[
\min_{X \in \mathbb{R}^{p_1 \times p_2}} \frac{1}{2} \| y - A(X) \|^2_2 + \lambda \| X \|_* ,
\]

where \( \| X \|_* = \sum_{i=1}^{\min(p_1, p_2)} \sigma_i(X) \) is the nuclear norm of \( X \) and \( \lambda > 0 \) is a tuning parameter. Nevertheless, the convex relaxation technique has one well-documented limitation: the parameter space after relaxation is usually much larger than that of the target problem. Also, algorithms for solving the convex program often require the singular value decomposition as the stepping stone and can be prohibitively time-consuming for large-scale instances.

In addition, non-convex optimization, which directly enforces the rank \( r \) constraint on the iterates, renders another important class of algorithms for solving \( (1) \). Since each iterate lies in a low dimensional space, the computation cost of the non-convex approach can be much smaller than the convex regularized approach. Over the last a few years, there is a flurry of research on non-convex methods in solving \( (1) \) (Chen and Wainwright 2015; Hardt 2014; Jain et al. 2013; Miao et al. 2016; Sun and Luo 2016; Tran-Dinh 2015; Tu et al. 2016; Wen et al. 2012; Zhao et al. 2015; Zheng and Lafferty 2015), and many of the algorithms such as gradient descent and alternating minimization are shown to have nice convergence results under proper model assumptions (Hardt 2014; Jain et al. 2013; Sun and Luo 2016; Tran-Dinh 2015; Tu et al. 2016; Wen et al. 2012; Zhao et al. 2015). We refer readers to Section 1.2 for more review of recent works.

In the existing literature, many algorithms for solving \( (1) \) either require careful tuning of hyper-parameters or have a convergence rate no faster than linear. Thus, we raise the following question:

*Can we develop an easy-to-compute and efficient (hopefully has comparable per-iteration computational complexity as the first-order methods) algorithm with provable high-order convergence guarantees (possibly converge to a stationary point due to the non-convexity) for solving \( (1) \)?*

In this paper, we give an affirmative answer to this question by making contributions as outlined next.

### 1.1 Our Contributions

We introduce an easy-to-implement and computationally efficient algorithm, **Recursive Importance Sketching for Rank constrained least squares Optimization** (RISRO), for solving \( (1) \) in this paper. The proposed algorithm is tuning free and has the same per-iteration computational complexity as Alternating Minimization (Jain et al. 2013), as well as comparable complexity to many popular first-order methods such as iterative hard thresholding (Jain et al. 2010) and gradient descent (Tu et al. 2016) when \( r \ll p_1, p_2, n \). We then illustrate the key idea of RISRO under a general
framework of recursive importance sketching. This framework also renders a platform to compare RISRO and several existing algorithms for rank constrained least squares.

Assuming \( A \) satisfies the restricted isometry property (RIP), we prove RISRO is local quadratic-linearly convergent in general and quadratically convergent to a stationary point under some extra conditions. Figure 1 provides a numerical example of the performance of RISRO in the noiseless low-rank matrix trace regression (left panel) and phase retrieval (right panel). In both problems, RISRO converges to the underlying parameter quadratically and reaches a highly accurate solution within five iterations.

\[
\begin{align*}
(a) \text{Noiseless low-rank matrix trace regression. Here, } y_i &= \langle A_i, X^* \rangle \text{ for } 1 \leq i \leq n, X^* \in \mathbb{R}^{p \times p} \text{ with } p = 100, \sigma_1(X^*) = \cdots = \sigma_3(X^*) = 3, \sigma_k(X^*) = 0 \text{ for } 4 \leq k \leq 100 \text{ and } A_i \text{ has independently identically distributed (i.i.d.) standard Gaussian entries.}

(b) \text{Phase Retrieval. Here, } y_i &= \langle a_i a_i^T, x^* x^*^T \rangle \text{ for } 1 \leq i \leq n, x^* \in \mathbb{R}^p \text{ with } p = 1200, a_i \overset{i.i.d.}\sim \mathcal{N}(0, I_p) \end{align*}
\]

Figure 1: RISRO achieves a quadratic rate of convergence (spectral initialization is used in each setting and more details about the simulation setup are given in Section 7).

In addition, we discover a deep connection between RISRO and the Riemannian Gauss-Newton optimization algorithm on fixed rank matrices manifold. The least squares step in RISRO implicitly solves a Fisher Scoring or Riemannian Gauss-Newton equation on the Riemannian optimization of low-rank matrices and the updating rule in RISRO can be seen as a retraction map. With this connection, our theory on RISRO also improves the existing convergence results on the Riemannian Gauss-Newton method for the rank constrained least squares problem.

Next, we further apply RISRO to two prominent problems in machine learning and statistics: low-rank matrix trace regression and phase retrieval. In the noisy low-rank matrix trace regression, we prove the estimation error rate of RISRO converges quadratically to the information-theoretical limit with only a double-logarithmic number of iterations under the Gaussian ensemble design. To the best of our knowledge, RISRO is the first algorithm that provably achieves the minimax rate-optimal estimation error in matrix trace regression with only a double-logarithmic number of iterations, which offers an exponential improvement over the existing results of first-order methods [Jain et al., 2010, 2013; Chen and Wainwright, 2015]. We also discover a new “quadratic + one-iteration optimality” phenomenon for RISRO on low-rank matrix recovery (Remark 12). In phase retrieval, where \( A \) does not satisfy the RIP condition, we can still establish the local convergence of RISRO given a proper initialization. We also develop RISRO in the matrix completion and robust PCA applications, where the restricted isometry property completely fails. We find RISRO still has similar empirical performance as in the setting where the RIP condition holds.

Finally, we conduct simulation studies to support our theoretical findings and compare RISRO
with many existing algorithms. The numerical results show RISRO not only offers faster and more robust convergence but also requires a smaller sample size requirement for low-rank matrix recovery, compared to existing approaches.

1.2 Related Literature

This work is related to a range of literature on low-rank matrix recovery, convex/non-convex optimization, and sketching arising from several communities, including optimization, machine learning, statistics, and applied mathematics. We make an attempt to review the related literature without claiming the survey is exhaustive.

One class of the most popular approaches to solve \((1)\) is the nuclear norm minimization (NNM) \((3)\). Many algorithms have been proposed to solve NNM, such as proximal gradient descent (Toh and Yun, 2010), fixed-point continuation (FPC) (Goldfarb and Ma, 2011), and proximal point methods (Jiang et al., 2014). It has been shown that the solution of NNM has desirable properties under proper models, such as matrix trace regression and matrix completion (Cai and Zhang, 2013, 2014, 2015; Candès and Plan, 2011; Recht et al., 2010). In addition to NNM, the max norm minimization is another widely considered convex realization for the rank constrained optimization (Lee et al., 2010; Cai and Zhou, 2013). However, these convex programs are usually computationally intensive to solve, which motivates a line of work on using non-convex approaches. Since Burer and Monteiro (2003), one of the most popular non-convex methods for solving \((1)\) is to first factor the low-rank matrix \(X\) to \(RL^T\) with two factor matrices \(R \in \mathbb{R}^{p_1 \times r}, L \in \mathbb{R}^{p_2 \times r}\), then run either gradient descent or alternating minimization on \(R\) and \(L\) (Candès et al., 2015; Li et al., 2019b; Ma et al., 2019; Park et al., 2018; Sanghavi et al., 2017; Sun and Luo, 2016; Tu et al., 2016; Wang et al., 2017c; Zhao et al., 2015; Zheng and Lafferty, 2015; Tong et al., 2021b). Other methods, such as singular value projection or iterative hard thresholding (Goldfarb and Ma, 2011; Jain et al., 2010; Tanner and Wei, 2013), Grassmann manifold optimization (Boumal and Absil, 2011; Keshavan et al., 2009), Riemannian manifold optimization (Huang and Hand, 2018; Meyer et al., 2011; Mishra et al., 2014; Vandereycken, 2013; Wei et al., 2016) have also been proposed and studied. We refer readers to the recent survey paper Chi et al. (2019) for a comprehensive overview of existing literature on convex and non-convex approaches on solving \((1)\). Most of the convergence analyses in the literature were conducted under certain statistical models (e.g., noisy/noiseless matrix trace regression, matrix completion, and phase retrieval) and the goal was to recover the underlying parameter matrix. Here, we study \((1)\) from both an optimization perspective (how the algorithm converges to a stationary point) and a statistical perspective (how the iterates estimate the underlying true parameter). These two perspectives overlap in the noiseless settings as the parameter becomes a stationary point then, while disjoint in the more general noisy settings.

There are a few recent attempts in connecting the geometric structures of different approaches (Ha et al., 2020; Li et al., 2019b), and the landscape of problem \((1)\) has also been studied in various settings (Bhojanapalli et al., 2016; Ge et al., 2017; Uschmajew and Vandereycken, 2020; Zhang et al., 2019; Zhu et al., 2018).

Our work is also related to the idea of sketching in numerical linear algebra. Performing sketching to speed up the computation via dimension reduction has been explored extensively in recent years (Mahoney, 2011; Woodruff, 2014). Sketching methods have been applied to solve a number of problems including but not limited to matrix approximation (Song et al., 2017; Zheng et al., 2012; Drineas et al., 2012), linear regression (Clarkson and Woodruff, 2012), Dobriban and Liu, 2019; Pilanci and Wainwright, 2016; Raskutti and Mahoney, 2016), ridge regression (Wang et al., 2017b), etc. In most of the sketching literature, the sketching matrices are randomly constructed (Mahoney, 2011; Woodruff, 2014). Randomized sketching matrices are easy to generate and require
little storage for sparse sketching. However, randomized sketching can be suboptimal in statistical settings (Raskutti and Mahoney, 2016). To overcome this, Zhang et al. (2020) introduced an idea of importance sketching in the context of low-rank tensor regression. In contrast to the randomized sketching, importance sketching matrices are constructed deterministically with the supervision of the data and are shown capable of achieving better statistical efficiency. However, the method developed is Zhang et al. (2020) is essentially a “one-time” importance sketching, which yield a sub-optimal outcome when the noise level is small or moderate. This paper proposes a more powerful recursive importance sketching algorithm that iteratively refines the sketching matrices. We also provide a comprehensive convergence analysis for the proposed algorithm without the sample-splitting assumption used in Zhang et al. (2020); our theory demonstrates the optimality of the proposed algorithm at all different noise levels and advantages over other algorithms for the rank constrained least squares problem.

1.3 Organization of the Paper

The rest of this article is organized as follows. After a brief introduction of notation in Section 1.4, we present our main algorithm RISRO with an interpretation from the recursive importance sketching perspective in Section 2. The theoretical results of RISRO are given in Section 3. In Section 4, we present another interpretation for RISRO from Riemannian manifold optimization. The computational complexity of RISRO and its applications to low-rank matrix trace regression and phase retrieval are discussed in Sections 5 and 6, respectively. Numerical studies of RISRO and the comparison with existing algorithms in the literature are presented in Section 7. Conclusion and future work are given in Section 8.

1.4 Notation

The following notation will be used throughout this article. Upper and lowercase letters (e.g., $A, B, a, b$), lowercase boldface letters (e.g. $u, v$), uppercase boldface letters (e.g., $U, V$) are used to denote scalars, vectors, matrices, respectively. For any two series of numbers, say $\{a_n\}$ and $\{b_n\}$, denote $a = O(b)$ if there exists uniform constants $C > 0$ such that $a_n \leq C b_n, \forall n$. For any $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}, a \vee b := \max\{a, b\}$. For any matrix $X \in \mathbb{R}^{p_1 \times p_2}$ with singular value decomposition $\sum_{i=1}^{p_1 \wedge p_2} \sigma_i(X)u_i v_i^T$, where $\sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_{p_1 \wedge p_2}(X)$, let $X_{\max(r)} = \sum_{i=1}^{r} \sigma_i(X)u_i v_i^T$ be the best rank-$r$ approximation of $X$ and denote $\|X\|_F = \sqrt{\sum_i \sigma_i^2(X)}$ and $\|X\|_2 = \sigma_1(X)$ as the Frobenius norm and spectral norm, respectively. Let QR($X$) be the $Q$ part of the QR decomposition outcome of $X$. vec($X$) $\in \mathbb{R}^{p_1 p_2}$ represents the vectorization of $X$ by its columns. In addition, $I_r$ is the $r$-by-$r$ identity matrix. Let $\mathcal{O}_{p,r} = \{U : U^T U = I_r\}$ be the set of all $p$-by-$r$ matrices with orthonormal columns. For any $U \in \mathcal{O}_{p,r}$, $P_U = U U^T$ represents the orthogonal projector onto the column space of $U$; we also note $U_\perp \in \mathcal{O}_{p,p-r}$ as the orthonormal complement of $U$. We use bracket subscripts to denote sub-matrices. For example, $X_{[i_1, i_2]}$ is the entry of $X$ on the $i_1$-th row and $i_2$-th column; $X_{[(r+1)p_1, \cdot]}$ contains the $(r+1)$-th to the $p_1$-th rows of $X$. For any matrix $X$, we use $X^*$ to denote its Moore-Penrose inverse. For matrices $U \in \mathbb{R}^{p_1 \times p_2}, V \in \mathbb{R}^{m_1 \times m_2}$, let

$$U \otimes V = \begin{bmatrix} U_{[1,1]} \cdot V & \cdots & U_{[1,p_2]} \cdot V \\ \vdots & \ddots & \vdots \\ U_{[p_1,1]} \cdot V & \cdots & U_{[p_1,p_2]} \cdot V \end{bmatrix} \in \mathbb{R}^{(p_1 m_1) \times (p_2 m_2)}$$

be their Kronecker product. Finally, for any given linear operator $\mathcal{L}$, we use $\mathcal{L}^*$ to denote its adjoint, and use Ran($\mathcal{L}$) to denote its range space.
2 Recursive Importance Sketching for Rank Constrained Least Squares

In this section, we discuss the procedure and interpretations of RISRO, then compare it with existing algorithms from a sketching perspective. The pseudocode of RISRO is summarized in Algorithm 1.

2.1 RISRO Procedure and Recursive Importance Sketching

In each iteration $t = 1, 2, \ldots$, RISRO includes three steps.

Step 1 We sketch each $A_i$ ($i = 1, \ldots, n$) onto the subspace spanned by $[U^T \otimes V^T, U^T \otimes V^T, U^T \otimes V^T]$, where $U$ and $V$ span the column and row subspaces of $X$, respectively. This yields the sketched importance covariates $U^T A_i V^T, U^T A_i V^T, U^T A_i V^T$. See Figure 2 left panel for an illustration of the sketching scheme of RISRO. Then we construct the covariates maps $A_B : \mathbb{R}^{r \times r} \rightarrow \mathbb{R}^n$, $A_D_1 : \mathbb{R}^{(p_1 - r) \times r} \rightarrow \mathbb{R}^n$ and $A_D_2 : \mathbb{R}^{r \times (p_2 - r)} \rightarrow \mathbb{R}^n$; for matrix $\cdot$, let

$$[A_B(\cdot)]_i = \langle \cdot, U^T A_i V^T \rangle, \quad [A_D_1(\cdot)]_i = \langle \cdot, U^T A_i V^T \rangle, \quad [A_D_2(\cdot)]_i = \langle \cdot, U^T A_i V^T \rangle, \quad i = 1, \ldots, n.$$  \hspace{1cm} (4)

Step 2 We solve a dimension reduced least squares problem $[5]$ (provided in the box of Algorithm 1) where the number of parameters is reduced to $(p_1 + p_2 - r)r$ while the sample size remains $n$.

Step 3 We update the sketching matrices $U^{t+1}, V^{t+1}$ and $X^{t+1}$ in Steps 6 and 7. By construction, $U^{t+1}, V^{t+1}$ contain both the column and row spans of $X^{t+1}$.

Algorithm 1 Recursive Importance Sketching for Rank Constrained Least Squares (RISRO)

1: Input: $\mathcal{A}(\cdot) : \mathbb{R}^{p_1 \times p_2} \rightarrow \mathbb{R}^n$, $y \in \mathbb{R}^n$, rank $r$, initialization $X^0$ which admits singular value decomposition $U^0 \Sigma^0 V^0^T$, where $U^0 \in \mathbb{O}_{p_1, r}$, $V^0 \in \mathbb{O}_{p_2, r}$, $\Sigma^0 \in \mathbb{R}^{r \times r}$

2: for $t = 0, 1, \ldots$ do

3: Perform importance sketching on $\mathcal{A}$ and construct the covariates maps $A_B : \mathbb{R}^{r \times r} \rightarrow \mathbb{R}^n$, $A_D_1 : \mathbb{R}^{(p_1 - r) \times r} \rightarrow \mathbb{R}^n$ and $A_D_2 : \mathbb{R}^{r \times (p_2 - r)} \rightarrow \mathbb{R}^n$; for matrix $\cdot$, let

$$[A_B(\cdot)]_i = \langle \cdot, U^T A_i V^T \rangle, \quad [A_D_1(\cdot)]_i = \langle \cdot, U^T A_i V^T \rangle, \quad [A_D_2(\cdot)]_i = \langle \cdot, U^T A_i V^T \rangle, \quad i = 1, \ldots, n.$$  \hspace{1cm} (4)

4: Solve the unconstrained least squares problem

$$\left(B^{t+1}, D_1^{t+1}, D_2^{t+1}\right) = \arg \min_{B \in \mathbb{R}^{r \times r}, D_i \in \mathbb{R}^{(p_i - r) \times r}, i = 1, 2} \| y - A_B(B) - A_D_1(D_1) - A_D_2(D_2^T) \|_2^2$$  \hspace{1cm} (5)

5: Compute $X^{t+1}_U = (U^T B^{t+1} + U^T D_1^{t+1})$ and $X^{t+1}_V = (V^T B^{t+1} + V^T D_2^{t+1})$.

6: Perform QR orthogonalization: $U^{t+1} = \text{QR}(X^{t+1}_U)$, $V^{t+1} = \text{QR}(X^{t+1}_V)$.

7: Update $X^{t+1} = X^{t+1}_U (B^{t+1})^T X^{t+1}_V$.

8: end for

We give a high-level explanation of RISRO through a decomposition of $y_i$. Suppose $y_i = \langle A_i, X \rangle + \xi_i$ where $X$ is a rank $r$ target matrix with singular value decomposition $U \Sigma V^T$ with
Figure 2: Illustration of sketching strategies of RISRO (this work), Alter Mini [Hardt 2014, Jain et al. 2013], and R2RILS [Bauch et al. 2021]. Here, $A_i$ denotes the covariate matrix of the $i$th observation; $U^t$ and $V^t$ span the column and row subspaces of $X^t$, respectively. Covariate matrices colored in gray represent the sketching of $A_i$ onto the column and row subspaces of $X^t$, covariate matrices colored in green represent the sketching of $A_i$ onto the perpendicular column subspace and row subspace of $X^t$ and covariate matrices colored in blue represent the sketching of $A_i$ onto the column subspace and perpendicular row subspace of $X^t$. In Alter Mini and R2RILS, the sketched covariates colored in gray and blue (or green) are combined to represent the actual algorithmic implementation.

$\tilde{U} \in \mathbb{O}_{p_1,r}$, $\tilde{\Sigma} \in \mathbb{R}^{r \times r}$ and $\tilde{V} \in \mathbb{O}_{p_2,r}$. Then

$$y_i = \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \epsilon_i$$

$$: = \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \epsilon_i^t. \tag{6}$$

Here, $\epsilon_i^t := \mathcal{A}(P_{U_i} \tilde{X} P_{V^t}) + \epsilon \in \mathbb{R}^n$ can be seen as the residual of the new regression model [6], and $U_i^T A_i V^t, U_i^T A_i V^t, U_i^T A_i V^t$ are exactly the importance covariates constructed in [4]. Let

$$\tilde{B}^t := U_i^T X V^t, \tilde{D}_1^t := U_i^T X V^t, \tilde{D}_2^t := U_i^T X V^t. \tag{7}$$

If $\epsilon_i^t = 0$, we have $(\tilde{B}^t, \tilde{D}_1^t, \tilde{D}_2^t)$ is a solution of the least squares in [5]. Hence, we could set $B_i^{t+1} = \tilde{B}^t, D_1^{t+1} = \tilde{D}_1^t, D_2^{t+1} = \tilde{D}_2^t$ and thus $X_i^{t+1} = X V^t, X_i^{t+1} = \tilde{X} U^t$. Furthermore, if $B_i^{t+1}$ is invertible, then it holds that

$$X_i^{t+1}(B_i^{t+1})^{-1}X_i^{t+1} = \tilde{X} V^t(U_i^T X V^t)^{-1}(U_i^T U^t)^T = \tilde{X}, \tag{8}$$

which means $\tilde{X}$ can be exactly recovered by one iteration of RISRO.

In general, $\epsilon_i^t \neq 0$. When the column spans of $U^t, V^t$ well approximate the ones of $\tilde{U}, \tilde{V}$, i.e., the column and row subspaces that the target parameter $\bar{X}$ lie on, we expect $U_i^T \tilde{X} V^t$ and $\epsilon_i^t = \langle U_i^T A_i V^t, U_i^T X V^t \rangle + \epsilon_i^t$ to have a small amplitude, then $B_i^{t+1}, D_1^{t+1}, D_2^{t+1}$, the outcome of the least squares problem [5], can well approximate $\tilde{B}^t, \tilde{D}_1^t, \tilde{D}_2^t$. In Lemma 1, we give a precise characterization for this approximation. Before that, let us introduce a convenient notation so that [5] can be written in a more compact way.

Define the linear operator $\mathcal{L}_t$ as

$$\mathcal{L}_t : W = \begin{bmatrix} W_0 \in \mathbb{R}^{r \times r} & W_2 \in \mathbb{R}^{r \times (p_2-r)} \\ W_1 \in \mathbb{R}^{(p_1-r) \times r} & 0_{(p_1-r) \times (p_2-r)} \end{bmatrix} \rightarrow [U^t, U_i^T] \begin{bmatrix} W_0 & W_2 \\ W_1 & 0 \end{bmatrix} [V^t, V_i^T]^T, \tag{9}$$
and it is easy to compute its adjoint $\mathcal{L}_t^* : \mathbf{M} \in \mathbb{R}^{n \times p_2} \rightarrow \begin{bmatrix} \mathbf{U}_t^T \mathbf{M} \mathbf{V}_t & \mathbf{U}_t^T \mathbf{M} \mathbf{V}_t^T \\ (\mathbf{U}_t^T)^T \mathbf{M} \mathbf{V}_t & 0 \end{bmatrix}$. Then, the least squares problem in (5) can be written as

$$(B^{t+1}, D_1^{t+1}, D_2^{t+1}) = \arg \min_{B \in \mathbb{R}^{r \times r}, D_1, D_2 \in \mathbb{R}^{(p_i - r) \times r}} \| y - A \mathcal{L}_t \begin{bmatrix} B & D_2^T \\ D_1 & 0 \end{bmatrix} \|^2_2. \quad (10)$$

**Lemma 1 (Iteration Error Analysis for RISRO)** Let $\hat{\mathbf{X}}$ be any given target matrix. Recall the definition of $\epsilon^t = \hat{\epsilon} + A(\mathbf{P}_{U_t} \hat{\mathbf{X}} / \mathbf{P}_{V_t})$ from (6). If the operator $\mathcal{L}_t^* A^* A \mathcal{L}_t$ is invertible over $\text{Ran}(\mathcal{L}_t^*)$, then $B^{t+1}, D_1^{t+1}, D_2^{t+1}$ in (5) satisfy

\[
\begin{bmatrix}
B^{t+1} - \hat{B}^t \\
D_1^{t+1} - \hat{D}_1^t
\end{bmatrix}
\begin{bmatrix}
D_2^{t+1T} - \hat{D}_2^T \\
0
\end{bmatrix}
= (\mathcal{L}_t^* A^* A \mathcal{L}_t)^{-1} \mathcal{L}_t^* A^* \epsilon^t, \quad (11)
\]

and

$$\|B^{t+1} - \hat{B}^t\|_F^2 + \sum_{k=1}^{2} \|D_k^{t+1} - \hat{D}_k^t\|_F^2 = \|\mathcal{L}_t^* A^* A \mathcal{L}_t\|_F^{-1} \|\mathcal{L}_t^* A^* \epsilon^t\|_F^2. \quad (12)$$

In view of Lemma 1, the approximation errors of $B^{t+1}, D_1^{t+1}, D_2^{t+1}$ to $\hat{B}^t, \hat{D}_1^t, \hat{D}_2^t$ are driven by the least squares residual $\|\mathcal{L}_t^* A^* A \mathcal{L}_t^{-1} \mathcal{L}_t^* A^* \epsilon^t\|_F^2$. This fact plays a key role in the proof for the high-order convergence theory of RISRO, see later in Remark 7.

**Remark 1 (Comparison with Randomized Sketching)** The importance sketching in RISRO is significantly different from the randomized sketching in the literature (see surveys Mahoney (2011); Woodruff (2014) and the references therein). The randomized sketching matrices are often randomly generated and reduce the sample size ($n$), the importance sketching matrices are deterministically constructed under the supervision of $\mathbf{y}$ and reduce the dimension of parameter space ($p_1p_2$). See (Zhang et al. 2020, Section 1.3 and 2) for more comparison of randomized and importance sketchings.

### 2.2 Comparison with More Algorithms in the View of Sketching

In addition to RISRO, several classic algorithms for rank constrained least squares can be interpreted from the recursive importance sketching perspective. Through the lens of the sketching, RISRO exhibits advantages over these existing algorithms.

We first focus on Alternating Minimization (Alter Mini) proposed and studied in Hardt (2014); Jain et al. (2013); Zhao et al. (2015). Suppose $\mathbf{U}^t$ is the left singular vectors of $\mathbf{X}^t$, the outcome of the $t$-th iteration, Alter Mini solves the following least squares problems to update $\mathbf{U}$ and $\mathbf{V}$,

$$\tilde{\mathbf{V}}^{t+1} = \arg \min_{\mathbf{V} \in \mathbb{R}^{p_2 \times r}} \sum_{i=1}^{n} (\mathbf{y}_i - \langle \mathbf{A}_i, \mathbf{U}^t \mathbf{V}^T \rangle)^2 = \arg \min_{\mathbf{V} \in \mathbb{R}^{p_2 \times r}} \sum_{i=1}^{n} (\mathbf{y}_i - \langle \mathbf{U}^T \mathbf{A}_i, \mathbf{V}^T \rangle)^2,$$

$$\hat{\mathbf{U}}^{t+1} = \arg \min_{\mathbf{U} \in \mathbb{R}^{p_1 \times r}} \sum_{i=1}^{n} (\mathbf{y} - \langle \mathbf{A}_i, \mathbf{U} (\mathbf{V}^{t+1})^T \rangle)^2 = \arg \min_{\mathbf{U} \in \mathbb{R}^{p_1 \times r}} \sum_{i=1}^{n} (\mathbf{y} - \langle \mathbf{A}_i \mathbf{V}^{t+1}, \mathbf{U} \rangle)^2, \quad (13)$$

$$\mathbf{V}^{t+1} = \text{QR}(\tilde{\mathbf{V}}^{t+1}), \quad \mathbf{U}^{t+1} = \text{QR}(\hat{\mathbf{U}}^{t+1}).$$

Then, Alter Mini essentially solves least squares problems with sketched covariates $\mathbf{U}^T \mathbf{A}_i, \mathbf{A}_i \mathbf{V}^{t+1}$ to update $\tilde{\mathbf{V}}^{t+1}, \hat{\mathbf{U}}^{t+1}$ alternatively and iteratively. The number of parameters of the least squares
Lemma 1 is proved, we can show that the approximation error of \( V \) is given by \( \| \tilde{V} - V \|_F = \| (\tilde{A}^T \tilde{A})^{-1} \tilde{A}^T \tilde{\epsilon} \|_F \), which implies the approximation error of \( V^{t+1} = QR(\tilde{V}^{t+1}) \) (i.e., the outcome of one iteration Alter Mini) to \( \tilde{V} \) (i.e., true row span of the target matrix \( \tilde{X} \)) is driven by \( \tilde{\epsilon} = A(P_{U^T} \tilde{X}) + \tilde{\epsilon} \), i.e., the residual of least squares problem (14). Recall for RISRO, Lemma 2 shows the approximation error of \( V^{t+1} \) is driven by \( \epsilon^t = A(P_{U^T} \tilde{X}P_{V^t}) + \tilde{\epsilon} \). Since \( \| P_{U^T} \tilde{X}P_{V^t} \|_F \leq \| P_{U^T} \tilde{X} \|_F \), the approximation error in per iteration of RISRO can be smaller than the one of Alter Mini. Such a difference between RISRO and Alter Mini is due to the following fact: in Alter Mini, the sketching captures the importance covariates correspond to only the row (or column) span of \( \tilde{X} \) in updating \( V^{t+1} \). Such a difference between RISRO and Alter Mini is due to the following: in Alter Mini, the sketching captures the importance covariates correspond to only the row (or column) span of \( \tilde{X} \). As a consequence, Alter Mini iterations yield first-order convergence while RISRO iterations render high-order convergence as will be established in Section 4.

Remark 2 Recently, Kümmerle and Sigl (2018) proposed a harmonic mean iterative reweighted least squares (HM-IRLS) method for low-rank matrix recovery: they specifically solve \( \min_{X \in \mathbb{R}^{r_1 \times r_2}} \| X \|_q \) subject to \( y = A(X) \), where \( \| X \|_q = \left( \sum \sigma_i^q(X) \right)^{1/q} \) is the Schatten-\( q \) norm of the matrix \( X \). Compared to the original iterative reweighted least squares (IRLS) (Fornasier et al., 2011; Mohan and Fazel, 2012), which only involves either the column span or the row span of \( X \) in constructing the reweighting matrix, HM-IRLS leverages both the column and row spans of \( X \) in constructing the reweighting matrix per-iteration and performs better. Such a comparison of HM-IRLS versus IRLS shares the same spirit as RISRO versus Alter Mini: the importance sketching of RISRO simultaneously captures the information of both column and row spans of \( X \) per iteration and achieves a better performance. Utilizing both row and column spans of \( X \) simultaneously is the key to achieve high-order convergence performance by RISRO.

Another example is the rank 2r iterative least squares (R2RLS) proposed by Bauch et al. (2021) for solving ill-conditioned matrix completion problems. In particular, at the \( t \)-th iteration, Step 1 of R2RLS solves the following least squares problem

\[
\min_{M \in \mathbb{R}^{r_1 \times r_1}, N \in \mathbb{R}^{r_2 \times r_2}} \sum_{(i,j) \in \Omega} \left( \langle U^T N^T + M V^T - X \rangle_{[i,j]} \right)^2,
\]

where \( \Omega \) is the set of index pairs of the observed entries. In the matrix completion setting, it turns out the following equivalence holds (proof given in Appendix)

\[
\arg \min_{M \in \mathbb{R}^{r_1 \times r_1}, N \in \mathbb{R}^{r_2 \times r_2}} \sum_{(i,j) \in \Omega} \left( \langle U^T N^T + M V^T - X \rangle_{[i,j]} \right)^2 = \arg \min_{M \in \mathbb{R}^{r_1 \times r_1}, N \in \mathbb{R}^{r_2 \times r_2}} \sum_{(i,j) \in \Omega} \left( \langle U^T A^{ij}, N^T \rangle + \langle M, A^{ij} V^T \rangle - X_{[i,j]} \right)^2,
\]

where \( A^{ij} \in \mathbb{R}^{r_1 \times r_2} \) is the special covariate in matrix completion satisfying \( A^{ij}_{[k,l]} = 1 \) if \( (i,j) = (k,l) \) and \( (A^{ij})_{[k,l]} = 0 \) otherwise. This equivalence reveals that the least squares step (15) in
R2RILS can be seen as an implicit sketched least squares problem similar to (5) and (13) with covariates $U^T A^{ij}$ and $A^{ij}V^t$ for $(i, j) \in \Omega$.

We give a pictorial illustration for the sketching interpretation of R2RILS on the bottom right part of Figure 2. Different from the sketching in RISRO, R2RILS incorporates the core sketch $U^T A, V^t$ twice, which results in the rank deficiency in the least squares problem (15) and brings difficulties in both implementation and theoretical analysis. RISRO overcomes this issue by performing a better-designed sketching and covers more general low-rank matrix recovery settings than R2RILS. With the new sketching scheme, we are able to give a new and solid theory for RISRO with high-order convergence.

3 Theoretical Analysis

In this section, we provide convergence analysis for the proposed algorithm. For technical convenience, we assume $\mathcal{A}$ satisfies the Restricted Isometry Property (RIP) (Candès, 2008). The RIP condition, first introduced in compressed sensing, has been widely used as one of the most standard assumptions in the low-rank matrix recovery literature (Cai and Zhang, 2013, 2014; Candès and Plan, 2011; Chen and Wainwright, 2015; Jain et al., 2010; Recht et al., 2010; Tu et al., 2016; Zhao et al., 2015). It also plays a critical role in analyzing the landscape of the rank constrained optimization problem (1) (Bhojanapalli et al., 2016; Ge et al., 2017; Uschmajew and Vandereycken, 2020; Zhang et al., 2019; Zhu et al., 2018). On the other hand, RIP is only a sufficient but not necessary condition for the convergence of RISRO. We will illustrate later in several examples that RISRO converges quadratically while RIP completely fails.

Definition 1 (Restricted Isometry Property (RIP)) Let $\mathcal{A} : \mathbb{R}^{p_1 \times p_2} \to \mathbb{R}^n$ be a linear map. For every integer $r$ with $1 \leq r \leq \min(p_1, p_2)$, define the $r$-restricted isometry constant to be the smallest number $R_r$ such that $(1 - R_r)\|Z\|_F^2 \leq \|\mathcal{A}(Z)\|_2^2 \leq (1 + R_r)\|Z\|_F^2$ holds for all $Z$ of rank at most $r$. And $\mathcal{A}$ is said to satisfy the $r$-restricted isometry property ($r$-RIP) if $0 \leq R_r < 1$.

The RIP condition provably holds when $\mathcal{A}$ has independent random sub-Gaussian design or $\mathcal{A}$ is a random projection (Candès and Plan, 2011; Recht et al., 2010). In addition, the definition of RIP above can be equivalently stated in a matrix format: define $\mathcal{A} = [\text{vec}(A_1), \cdots, \text{vec}(A_{n_1})]^T$ and $\mathcal{A}(Z) = \mathcal{A}\text{vec}(Z)$. Then $\mathcal{A}$ satisfies the RIP condition is equivalent to $(1 - R_r)\|\text{vec}(Z)\|_2^2 \leq \|\mathcal{A}(\text{vec}(Z))\|_2^2 \leq (1 + R_r)\|\text{vec}(Z)\|_2^2$ for all matrices $Z$ of rank at most $r$. By definition, $R_r \leq R_{r'}$ for any $r \leq r'$.

By assuming RIP for $\mathcal{A}$, we can show the linear operator $\mathcal{L}_t^* A^t \mathcal{A} \mathcal{L}_t$ mentioned in Lemma 1 is always invertible over $\text{Ran}(\mathcal{L}_t^*)$ (i.e. the least squares (5) has a unique solution). The following lemma gives explicit lower and upper bounds for the spectrum of this operator.

Lemma 2 (Bounds for Spectrum of $\mathcal{L}_t^* A^t \mathcal{A} \mathcal{L}_t$) Recall the definition of $\mathcal{L}_t$ in (9). It holds that

$$\|\mathcal{L}_t(M)\|_F = \|M\|_F, \quad \forall M \in \text{Ran}(\mathcal{L}_t^*).$$

Suppose the linear map $\mathcal{A}$ satisfies the $2r$-RIP. Then, it holds that for any matrix $M \in \text{Ran}(\mathcal{L}_t^*)$, $(1 - R_{2r})\|M\|_F \leq \|\mathcal{L}_t^* A^t \mathcal{A} \mathcal{L}_t(M)\|_F \leq (1 + R_{2r})\|M\|_F.$

Remark 3 (Bounds for spectrum of $(\mathcal{L}_t^* A^t \mathcal{A} \mathcal{L}_t)^{-1}$) By the relationship of the spectrum of an operator and its inverse, from Lemma 2 we also have the spectrum of $(\mathcal{L}_t^* A^t \mathcal{A} \mathcal{L}_t)^{-1}$ is lower and upper bounded by $\frac{1}{(1 + R_{2r})}$ and $\frac{1}{(1 - R_{2r})}$, respectively.
Proposition 1 (Upper Bound for Iteration Approximation Error) Let \( \bar{X} \) be a given target rank \( r \) matrix and \( \bar{e} = y - A(\bar{X}) \). Suppose that \( A \) satisfies the \( 2r\text{-RIP} \). Then at \( t \)-th iteration of RISRO, the approximation error \( \| A^*(\bar{e}) \|_F \) has the following upper bound:

\[
\| (L_t^* A^t A L_t)^{-1} L_t^* A^t \bar{e} \|_F^2 \leq \frac{R_2^* \| X^t - \bar{X} \|_F^2}{(1 - R_2)^2 \sigma_r^2(X)} + \frac{\| L_t^* A^t (\bar{e}) \|_F^2}{(1 - R_2)^2} + \frac{2R_3r \| X^t - \bar{X} \|_F \| X^t - \bar{X} \|_F}{\sigma_r(X)(1 - R_2)^2}.
\]

(18)

Note that Proposition [1] is rather general in the sense that it applies to any \( \bar{X} \) of rank \( r \) and we will pick different choices of \( \bar{X} \) depending on our purposes. For example, in studying the convergence of RISRO, e.g., the upcoming Theorem 1, we treat \( \bar{X} \) as a stationary point and in the setting of estimating the model parameter in matrix trace regression, we take \( \bar{X} \) to be the ground truth (see Theorem 3).

Now, we are ready to establish the deterministic convergence theory for RISRO. For problem (1), we use the following definition of stationary points: a rank \( r \) matrix \( \bar{X} \) is said to be a stationary point of (1) if \( \nabla f(\bar{X}) \tilde{U} = 0 \) and \( \nabla f(\bar{X}) \tilde{V} = 0 \) where \( \nabla f(\bar{X}) = A^*(A(\bar{X}) - y) \), and \( \tilde{U}, \tilde{V} \) are the left and right singular vectors of \( \bar{X} \). See also Ha et al. (2020). In Theorem 1, we show that given any target stationary point \( \bar{X} \) and proper initialization, RISRO has a local quadratic-linear convergence rate in general and quadratic convergence rate if \( y = A(\bar{X}) \).

Theorem 1 (Local Quadratic-Linear and Quadratic Convergence of RISRO) Let \( \bar{X} \) be a stationary point to problem (1) and \( \bar{e} = y - A(\bar{X}) \). Suppose that \( A \) satisfies the \( 2r\text{-RIP} \), and the initialization \( X^0 \) satisfies

\[
\| X^0 - \bar{X} \|_F \leq \left( \frac{1}{4} \wedge \frac{1 - R_2r}{4\sqrt{5}R_3r} \right) \sigma_r(\bar{X}),
\]

(19)

and \( \| A^*(\bar{e}) \|_F \leq \frac{1 - R_2r}{4\sqrt{5}} \sigma_r(\bar{X}) \). Then, we have \( \{X^t\} \), the sequence generated by RISRO (Algorithm 1), converges linearly to \( \bar{X} \): \( \| X^{t+1} - \bar{X} \|_F \leq \frac{3}{4} \| X^t - \bar{X} \|_F \), \( \forall t \geq 0 \).

More precisely, it holds that \( \forall t \geq 0 \):

\[
\| X^{t+1} - \bar{X} \|_F^2 \leq \frac{5 \| X^t - \bar{X} \|_F^2}{(1 - R_2)^2 \sigma_r^2(\bar{X})} \cdot \left( \frac{R_2^* \| X^t - \bar{X} \|_F^2}{(1 - R_2)^2 \sigma_r^2(\bar{X})} + \frac{2R_3r \| X^t - \bar{X} \|_F \| X^t - \bar{X} \|_F^2}{\sigma_r(\bar{X})(1 - R_2)^2} \right).
\]

(20)

In particular, if \( \bar{e} = 0 \), then \( \{X^t\} \) converges quadratically to \( \bar{X} \) as

\[
\| X^{t+1} - \bar{X} \|_F \leq \frac{\sqrt{5}R_3r}{(1 - R_2)\sigma_r(\bar{X})} \| X^t - \bar{X} \|_F^2, \quad \forall t \geq 0.
\]

Remark 4 (Quadratic-linear and Quadratic Convergence of RISRO) We call the convergence in (20) quadratic-linear since the sequence \( \{X^t\} \) generated by RISRO exhibits a phase transition from quadratic to linear convergence: when \( \| X^t - \bar{X} \|_F \gg \| A^*(\bar{e}) \|_F \), the algorithm has a quadratic convergence rate; when \( X^t \) becomes close to \( \bar{X} \) such that \( \| X^t - \bar{X} \|_F \leq c \| A^*(\bar{e}) \|_F \) for some \( c > 0 \), the convergence rate becomes linear. Even though the ultimate convergence of RISRO is linear to a stationary point in the noisy setting, we will show later in Section 6.1 that RISRO achieves quadratic convergence in estimating the underlying parameter matrix in statistical applications. Moreover, as \( \bar{e} \) becomes smaller, the stage of quadratic convergence becomes longer (see Section 7.1 for a numerical illustration of this convergence pattern). In the extreme case \( \bar{e} = 0 \),
Theorem 1 covers the widely studied matrix sensing problem under the RIP framework (Chen and Wainwright, 2015; Jain et al., 2010; Park et al., 2018; Recht et al., 2010; Tu et al., 2016; Zhao et al., 2015; Zheng and Lafferty, 2015). It shows as long as the initialization error is within a constant factor of $\sigma_r(\bar{X})$, RISRO enjoys quadratic convergence to the target matrix $\bar{X}$. To the best of our knowledge, we are among the first to give quadratic-linear algorithmic convergence guarantees for general rank constrained least squares and quadratic convergence for matrix sensing. Recently, Charisopoulos et al. (2021) formulated (1) as a non-convex composite optimization problem based on $X = RL^\top$ factorization and showed that the prox-linear algorithm (Burke, 1985; Lewis and Wright, 2016) achieves local quadratic convergence when $\bar{e} = 0$. In each iteration therein, a carefully tuned convex program needs to be solved exactly and the tuning parameter relies on the unknown weakly convexity parameter of the composite objective function. In contrast, the proposed RISRO is tuning-free, only solves a dimension-reduced least squares in each step, and can be as cheap as many first-order methods. See Section 3 for a detailed discussion on the computational complexity of RISRO.

Moreover, a quadratic-linear convergence rate also appears in several other methods under different settings: Pilanci and Wainwright (2017) studied the local convergence of the randomized Newton Sketch for objectives with strong convexity and smooth properties; Erdogdu and Montanari (2015) considered the sub-sampled Newton method to optimize an objective function in the form of a sum of convex functions and established their convergence theory with the well-conditioned sub-sampled Hessian. We consider the non-convex matrix optimization problem (1) and use the recursive importance sketching method. Our quadratic-linear convergence result can be boosted to quadratic when $\bar{e} = 0$.

Remark 5 (Initialization) The convergence theory in Theorem 1 requires a good initialization condition. Practically, the spectral method often provides a sufficiently good initialization that meets the requirement in (19) in many statistical applications. In Section 6 and 7, we will illustrate this point from two applications: matrix trace regression and phase retrieval.

Remark 6 (Small residual condition in Theorem 1) In addition to the initialization condition, the small residual condition $\|A^\ast(\bar{e})\|_F \leq \frac{1-R_{2r}}{4\sqrt{5}}\sigma_r(\bar{X})$ is also needed in Theorem 1. This condition essentially means that the signal strength at point $\bar{X}$ needs to dominate the noise. If $\bar{e} = y - A(\bar{X}) = 0$, then the aforementioned small residual condition holds automatically.

Remark 7 We provide a proof sketch of Theorem 1 and discuss our technical contributions therein.

Step 1. We bound $\|L_t^\ast A^\ast(\bar{e})\|_F \leq \frac{4|X^t - \bar{X}|^2}{\sigma_r^2(\bar{X})} \|A^\ast(\bar{e})\|_F$, and then apply Proposition 1 to obtain an upper bound for the approximation error in (12):

$$
\|L_t^\ast A^\ast(\bar{e})\|_F \leq \left\|\frac{X^t - \bar{X}}{1 - R_{2r}^2}\sigma_r^2(\bar{X})\right\|_F \\
\cdot \left(\frac{R_{3r}^2}{4} |X^t - \bar{X}|^2 + 4\|A^\ast(\bar{e})\|_F^2 + 4R_{3r}\|A^\ast(\bar{e})\|_F \|X^t - \bar{X}\|_F\right).
$$

Step 2. We use induction to show the following three claims,

1. $\max\{\sin \Theta(U^t, U), \sin \Theta(V^t, V)\} \leq \frac{1}{2}$; (C2) $B^{t+1}$ in (5) is invertible;

2. $\|X^{t+1} - \bar{X}\|_F \leq \frac{5|X^t - \bar{X}|^2}{1 - R_{2r}^2} \sigma_r^2(\bar{X}) \left(\frac{R_{3r}^2}{4} |X^t - \bar{X}|^2 + 4R_{3r}\|A^\ast(\bar{e})\|_F \|X^t - \bar{X}\|_F + 4\|A^\ast(\bar{e})\|_F^2\right)$

(C3) $\|X^{t+1} - \bar{X}\|_F \leq \frac{9}{16} \|X^t - \bar{X}\|_F$. 

12
the stepsize and $\eta$ notion of retraction is used, see e.g., Absil et al. (2008). Considering the manifold $X$, i.e., $x$ The Riemannian metric of $M$ and Step 7 in RISRO performs a type of Fisher Scoring as a Riemannian optimization algorithm on the manifold $M$. The superior performance of RISRO yields the following question:

4 A Riemannian Manifold Optimization Interpretation of RISRO

Here, (C2) means the iterates $X^t$ are always rank $r$. This fact is useful in Section 4 in connecting RISRO to Riemannian optimization on fixed rank matrix manifolds. (C2) is proved by (C1) and Lemma 1. In proving (C3), we introduce an intermediate quantity $p_{t+1} = \max\{\|D_1^{t+1}(B^{t+1})^{-1}\|, \|D_2^{t+1}D_2^{t+1T}\|\}$ and obtain

$$\|X^{t+1} - \hat{X}\|_F^2 = \left\| B^{t+1} - \hat{B}\right\|_F^2 + \left(D_1^{t+1} - \hat{D}_1\right)^T D_1^{t+1} (B^{t+1})^{-1} D_1^{t+1} - \hat{D}_1^{t+1} (B^{t+1})^{-1} \hat{D}_1^{t+1} \hat{D}_1^{t+1T} - \hat{D}_1^{t+1T} \right\|_F^2 \geq (a) 5 \left(\|\mathcal{L}_t^* A^* A \mathcal{L}_t\|_F - \|\mathcal{L}_t^* A^* e\|_F^2\right). \tag{22}$$

Here (a) is by the induction assumptions, Lemma 7 and Lemma 7. Finally, (C3) follows by plugging (21) into (22) and the induction assumptions and this proves the main result of Theorem 7.

The superior performance of RISRO yields the following question:

Is there a connection of RISRO to any class of optimization algorithms in the literature?

In this section, we give an affirmative answer to this question. We show RISRO can be viewed as a Riemannian optimization algorithm on the manifold $M_r := \{X \in \mathbb{R}^{p_1 \times p_2} | \text{rank}(X) = r\}$. We find the sketched least squares in (5) of RISRO actually solves the Fisher Scoring or Riemannian Gauss-Newton equation and Step 7 in RISRO performs a type of retraction under the framework of Riemannian optimization.

Riemannian optimization concerns optimizing a real-valued function $f$ defined on a Riemannian manifold $\mathcal{M}$. One commonly-encountered manifold is a submanifold of $\mathbb{R}^n$. Under such circumstances, a manifold can be viewed as a smooth subset of $\mathbb{R}^n$. When a smooth-varying inner product is further defined on the subset, the subset together with the inner product is called a Riemannian manifold. We refer to Absil et al. (2008) for the rigorous definition of Riemannian manifolds. Optimization on a Riemannian manifold often relies on the notion of Riemannian gradient/Riemannian Hessian (which are used to find a search direction) and the notion of retraction (which is defined for the motion of iterates on the manifold). The remainder of this section describes the required Riemannian optimization tools and the connection of RISRO to Riemannian optimization.

It has been shown in Lee (2013, Example 8.14) that the set $M_r$ is a smooth submanifold of $\mathbb{R}^{p_1 \times p_2}$ and the tangent space is also given therein. The result is given in Proposition 2 for completeness.

**Proposition 2** (Lee, 2013, Example 8.14) $M_r = \{X \in \mathbb{R}^{p_1 \times p_2} : \text{rank}(X) = r\}$ is a smooth embedded submanifold of dimension $(p_1 + p_2 - r)r$. Its tangent space $T_X M_r$ at $X \in M_r$ with the SVD decomposition $X = U \Sigma V^T$ $(U \in \mathbb{O}_{p_1, r}$ and $V \in \mathbb{O}_{p_2, r})$ is given by:

$$T_X M_r = \left\{ \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} \mathbb{R}^{r \times r} & \mathbb{R}^{r \times (p_2 - r)} \\ \mathbb{R}^{(p_1 - r) \times r} & 0_{(p_1 - r) \times (p_2 - r)} \end{bmatrix} \begin{bmatrix} V \\ V_\perp \end{bmatrix}^T \right\}. \tag{23}$$

The Riemannian metric of $M_r$ that we use throughout this paper is the Euclidean inner product, i.e., $\langle U, V \rangle = \text{trace}(U^T V)$.

In the Euclidean setting, the update formula in an iterative algorithm is $X^t + \alpha \eta^t$, where $\alpha$ is the stepsize and $\eta^t$ is a descent direction. However, in the framework of Riemannian optimization, $X^t + \alpha \eta^t$ is generally neither well-defined nor lying in the manifold. To overcome this difficulty, the notion of retraction is used, see e.g., Absil et al. (2008). Considering the manifold $M_r$, we have the
where \( s_2008, (3.37) \), we know in our problem, and here \( P \) is unique tangent vector \( \text{grad} \ f \), is the tangent bundle of \( \mathcal{M}_r \). The two conditions guarantee that \( R(X, \eta) \) stays in \( \mathcal{M}_r \) and \( R(X, t\eta) \) is a first-order approximation of \( X + t\eta \) at \( t = 0 \).

Next, we show that Step 7 in Algorithm 1 performs the orthographic retraction on the manifold of fixed-rank matrices given in [Absil and Malick, 2012]. Suppose at iteration \( t + 1 \), \( B^{t+1} \) is invertible (this is true under the RIP framework, see Remark 7 and Step 2 in the proof of Theorem 1). We can show by some algebraic calculations that the update \( X^{t+1} \) in Step 7 can be rewritten as

\[
X^{t+1} = X_t^{t+1} (B^{t+1})^{-1} X_t^{t+1\top} = [U^t \ U_{1\perp}] \begin{bmatrix} B^{t+1} & D_1^{t+1}D_2^{t+1\top} \\ D_1^{t+1} & D_1^{t+1}(B^{t+1})^{-1}D_2^{t+1\top} \end{bmatrix} [V^t \ V_{1\perp}]^\top. \tag{24}
\]

Let \( \eta^t \in T_{X^t}\mathcal{M}_r \) be the update direction and \( X^t + \eta^t \) has the following representation,

\[
X^t + \eta^t = [U^t \ U_{1\perp}] \begin{bmatrix} B^{t+1} & D_1^{t+1}D_2^{t+1\top} \\ D_1^{t+1} & D_1^{t+1}(B^{t+1})^{-1}D_2^{t+1\top} \end{bmatrix} [V^t \ V_{1\perp}]^\top. \tag{25}
\]

Comparing (24) and (25), we can view the update of \( X^{t+1} \) from \( X^t + \eta^t \) as simply completing the \( 0 \) matrix in

\[
\begin{bmatrix} B^{t+1} & D_1^{t+1}D_2^{t+1\top} \\ D_1^{t+1} & D_1^{t+1}(B^{t+1})^{-1}D_2^{t+1\top} \end{bmatrix} \]

by \( D_1^{t+1}(B^{t+1})^{-1}D_2^{t+1\top} \). This operation maps the tangent vector on \( T_{X^t}\mathcal{M}_r \) back to the manifold \( \mathcal{M}_r \) and it turns out that it coincides with the orthographic retraction \( R(X^t, \eta^t) \) on the set of fixed-rank matrices [Absil and Malick, 2012]. Therefore, we have \( X^{t+1} = R(X^t, \eta^t) \).

**Remark 8** Although the orthographic retraction defined in [Absil and Malick, 2012] requires that \( U^t \) and \( V^t \) are left and right singular vectors of \( X^t \), one can verify that even if the \( U^t \) and \( V^t \) are not exactly the left and right singular vectors but satisfy \( U^t = \hat{U}^tO \), \( V^t = \hat{V}^tQ \), then the mapping (26) is equivalent to the orthographic retraction in [Absil and Malick, 2012]. Here, \( O, Q \in \mathcal{O}_{r,r} \), and \( \hat{U}^t \) and \( \hat{V}^t \) are left and right singular vectors of \( X^t \).

The Riemannian gradient of a smooth function \( f : \mathcal{M}_r \to \mathbb{R} \) at \( X \in \mathcal{M}_r \) is defined as the unique tangent vector \( \text{grad} f(X) \in T_X\mathcal{M}_r \) such that \( \langle \text{grad} f(X), Z \rangle = D f(X)[Z], \forall Z \in T_X\mathcal{M}_r \), where \( D f(X)[Z] \) denotes the directional derivative of \( f \) at point \( X \) along the direction \( Z \). Since \( \mathcal{M}_r \) is an embedded submanifold of \( \mathbb{R}^{p_1 \times p_2} \) and the Euclidean metric is used, from [Absil et al., 2008 (3.37)], we know in our problem,

\[
\text{grad} f(X) = P_{T_X}(A^*(A(X) - y)), \tag{27}
\]

and here \( P_{T_X} \) is the orthogonal projector onto the tangent space at \( X \) defined as follows

\[
P_{T_X}(Z) = P_U Z P_V + P_{U\perp} Z P_{V\perp}, \quad \forall Z \in \mathbb{R}^{p_1 \times p_2}, \tag{28}
\]

where \( U \in \mathcal{O}_{p_1,r}, V \in \mathcal{O}_{p_2,r} \) are the left and right singular vectors of \( X \).

Next, we introduce the Riemannian Hessian. The Riemannian Hessian of \( f \) at \( X \in \mathcal{M}_r \) is the linear map \( \text{Hess} \ f(X) \) of \( T_X\mathcal{M}_r \) onto itself defined as \( \text{Hess} \ f(X)[Z] = \bar{\nabla} Z \text{grad} f, \quad \forall Z \in T_X\mathcal{M}_r \), where \( \bar{\nabla} \) is the Riemannian connection on \( \mathcal{M}_r \) [Absil et al., 2008, Section 5.3]. Lemma 3 gives an explicit formula for Riemannian Hessian in our problem.
Lemma 3 (Riemannian Hessian) Consider $f$ in (1). If $X \in \mathcal{M}_r$ has singular value decomposition $U \Sigma V^\top$ and $Z \in T_X \mathcal{M}_r$ has representation

$$Z = [U \quad U_\perp] \begin{bmatrix} Z_B & Z_{D_2}^\top \\ Z_{D_1} & 0 \end{bmatrix} [V \quad V_\perp]^\top,$$

then the Hessian operator in this setting satisfies

$$\text{Hess} f(X)[Z] = P_{T_X} (A^*(A(Z))) + P_{U_\perp} A^*(A(X) - y)V_p \Sigma^{-1}V^\top P_V$$

$$+ P_U U \Sigma U_\perp^{-1} U_p^\top A^*(A(X) - y)P_{V_\perp},$$

where $U_p = U_\perp Z_{D_1}$, $V_p = V_\perp Z_{D_2}$.

Next, we show that the update direction $\eta^t$, implicitly encoded in (25), finds the Riemannian Gauss-Newton direction in the manifold optimization of $\mathcal{M}_r$. Similar to the classic Newton’s method, at $t$-th iteration, the Riemannian Newton method aims to find the Riemannian Newton direction $\eta^t_{\text{Newton}}$ in $T_{X^t} \mathcal{M}_r$ that solves the following Newton equation

$$- \text{grad} f(X^t) = \text{Hess} f(X^t)[\eta^t_{\text{Newton}}].$$

If the residual $(y - A(X^t))$ is small, the last two terms in $\text{Hess} f(X^t)[\eta]$ of (29) are expected to be small, which means we can approximately solve the Riemannian Newton direction via

$$- \text{grad} f(X^t) = P_{T_{X^t}} (A^*(A(\eta))), \quad \eta \in T_{X^t} \mathcal{M}_r.$$

In fact, Equation (31) has an interpretation from the Fisher scoring algorithm. Consider the statistical setting $y = A(X) + \epsilon$, where $X$ is a fixed low-rank matrix and $\epsilon_{i,i\neq d} \sim N(0, \sigma^2)$. Then for any $\eta$,

$$\{E(\text{Hess} f(X)[\eta])\} |_{X=X^t} = P_{T_{X^t}} (A^*(A(\eta)),$$

where on the left hand side, the expression is evaluated at $X^t$ after taking expectation. In the literature, the Fisher Scoring algorithm computes the update direction via solving the modified Newton equation which replaces the Hessian with its expected value (Lange, 2010), i.e.,

$$\{E(\text{Hess} f(X)[\eta])\} |_{X=X^t} = -\text{grad} f(X^t), \quad \eta \in T_{X^t} \mathcal{M}_r,$$

which exactly becomes (31) in our setting. Meanwhile, it is not difficult to show that the Fisher Scoring algorithm here is equivalent to the Riemannian Gauss-Newton method for solving nonlinear least squares, see (Lange, 2010 Section 14.6) and (Absil et al., 2008 Section 8.4). Thus, $\eta$ that solves the equation (31) is also the Riemannian Gauss-Newton direction.

It turns out that the update direction $\eta^t$ (25) of RISRO solves the Fisher Scoring or Riemannian Gauss-Newton equation (31):

**Theorem 2** Let $\{X^t\}$ be the sequence generated by RISRO under the same assumptions as in Theorem 1. Then, for all $t \geq 0$, the implicitly encoded update direction $\eta^t$ in (25) solves the Riemannian Gauss-Newton equation (31).

Theorem 2 together with the retraction explanation in (26) establishes the connection of RISRO and Riemannian manifold optimization. Following this connection, we further show that each $\eta^t$ is always a decent direction in the next Proposition 3. This fact will be useful in boosting the local convergence of RISRO to the global convergence to be discussed in Remark 11.
Proposition 3. For all \( t \geq 0 \), the update direction \( \eta^t \in T_{X^t}M \) in (25) satisfies \( \langle \nabla f(X^t), \eta^t \rangle < 0 \), i.e., \( \eta^t \) is a descent direction. If \( A \) satisfies the 2r-RIP, then the direction sequence \( \{\eta^t\} \) is gradient related.

Remark 9. The convergence of Riemannian Gauss-Newton was studied in a recent work [Breiding and Vannieuwenhoven (2018)]. Our results are significantly different from and offer improvements to Breiding and Vannieuwenhoven (2018) in the following ways. First, Breiding and Vannieuwenhoven (2018) considered a more general Riemannian Gauss-Newton setting, while their convergence results are established for a local minimum, which is a stronger and less practical requirement than the stationary point assumption we need. Second, the convergence rate in Breiding and Vannieuwenhoven (2018) includes several unspecified constants while we manage to work out all constants explicitly in our statement. Third, the local convergence radius in Breiding and Vannieuwenhoven (2018) does not specify the dependence on the \( r \)-th singular value of the target matrix while our result does. Fourth, our recursive importance sketching framework provides new sketching interpretations for several classical algorithms for rank constrained least squares. Finally, in Section 6 we also apply RISRO in popular statistical models and show RISRO achieves quadratic convergence in terms of estimation. It is however not immediately clear how to utilize the results in Breiding and Vannieuwenhoven (2018) in these statistical settings.

Remark 10. In addition to providing an interpretation of the superiority of RISRO, the Riemannian Gauss-Newton perspective developed in this section can inspire algorithmic developments in more general settings. For example, consider a general constrained optimization programming: \( \min_{X \in M} f(X) \), where \( M \) is an embedded submanifold of \( \mathbb{R}^N \) and \( f \) is the restriction of a general twice differentiable objective in the ambient space to \( M \). Although importance sketching is hard to define for this setting, Riemannian Gauss-Newton equation inspires to compute \( \eta \in T_{X^t}M \) by solving \( P_{X^t} \nabla^2 f(X^t)[\eta] = -\nabla f(X^t) \), then update the iterate as \( X^{t+1} = R(X^t, \eta) \), where \( R(\cdot, \cdot) \) is a retraction operator onto \( M \). It is interesting to investigate the behavior of this algorithm from both optimization and statistical perspectives.

Meanwhile, the recursive sketching perspective also provides solutions to a wider range of constrained optimization problems. For example, one can replace the \( l_2 \) loss, i.e., the least squares in Eq. (5) by other loss functions, such as the \( l_1 \) loss, Huber loss, or logistic loss, to handle different types of error corruptions and develop more robust algorithms.

Remark 11 (Global Convergence of RISRO). By the classic theory of Riemannian optimization, the established connection of RISRO and Riemannian Gauss-Newton implies that vanilla RISRO may not converge when the RIP or the initialization condition fails. On the other hand, (Absil et al., 2008, Section 8.4) suggested that by adding or modifying the algorithm with certain line search or trust-region schemes, global convergence of Riemannian Gauss-Newton from any initialization to a stationary point can be guaranteed under proper assumptions. To be more specific, based on the Riemannian Gauss-Newton equation in (31) and Theorem 3, the Riemannian Gauss-Newton direction at iteration \( t \) satisfies

\[
\eta^t = \arg \min_{\eta \in T_{X^t}M} \| A P_{X^t} (X^t + \eta) - y \|_2^2 = (P_{X^t} A^* A P_{X^t})^{-1} P_{X^t} A^* (y - A^*(X^t)).
\]

(32)

After calculating \( \eta^t \), we can update \( X^t \) to \( X^t + \eta^t \).

We can equip the algorithm with line search and update \( X^t \) to \( X^t + \alpha_t \eta^t \), where \( \alpha_t \) is determined by some line search scheme, such as the Armijo method (Absil et al., 2008, Section 4.3). Since the update direction \( \eta^t \) is gradient related as shown in Proposition 3 under RIP condition, this
We can also apply the trust region method to achieve global convergence. Specifically, we calculate the update direction as
\[
\hat{\eta}^t = \arg \min_{\eta \in \mathbb{T}_X : M_r, \eta \leq \Delta_t} \| A P T_{X^t} (X^t + \eta) - y \|_2^2
\]
for some radius \( \Delta_t > 0 \). Then if \( \Delta_t \) is properly chosen such that \( \hat{\eta}^t \) guarantees sufficient decrease, the global convergence of this trust region method can be achieved under proper assumptions (Absil et al., 2008, Theorem 7.4.2).

### 5 Computational Complexity of RISRO

In this section, we discuss the computational complexity of RISRO. Suppose \( p_1 = p_2 = p \), the computational complexity of RISRO per iteration is \( O(np^2r^2 + (pr)^3) \) in the general setting. A comparison of the computational complexity of RISRO and other common algorithms is provided in Table 1. Here the main complexity of RISRO and Alter Mini is from solving the least squares. The main complexity of the singular value projection (SVP) (Jain et al., 2010) and gradient descent (GD) (Tu et al., 2016) is from computing the gradient. From Table 1, we can see RISRO has the same per-iteration complexity as Alter Mini and comparable complexity with SVP and GD when \( n \geq pr \) and \( r \) is much less than \( n \) and \( p \). On the other hand, RISRO and Alter Mini are tuning-free, while a proper step size is crucial for SVP and GD to have fast convergence: the convergence theory of SVP and GD were often established when the step size is chosen to be smaller than a hard-to-find threshold; there are several practical ways to determine this step size and one needs to select the best one based on the data (Zheng and Lafferty, 2015), which may cost extra time. Finally, RISRO enjoys a high-order convergence as we have shown in Section 3 and the convergence rates of all other algorithms are limited to being linear.

The main computational bottleneck of RISRO is solving the least squares, which can be alleviated by using iterative linear system solvers, such as the (preconditioned) conjugate gradient method when the linear operator \( \mathcal{A} \) has special structures. Such special structures occur, for example, in matrix completion problem (\( \mathcal{A} \) is sparse) (Vandereycken, 2013), phase retrieval for X-ray crystallography imaging (\( \mathcal{A} \) involves fast Fourier transforms) (Huang et al., 2017b), and blind deconvolution for imaging deblurring (\( \mathcal{A} \) involves fast Fourier transforms and Haar wavelet transforms) (Huang and Hand, 2018).

To utilize these structures, we introduce an intrinsic representation of tangent vectors in \( \mathcal{M}_r \): if \( \mathbf{U}, \mathbf{V} \) are the left and right singular vectors of a rank-\( r \) matrix \( \mathbf{X} \), an orthonormal basis of \( T_X \mathcal{M}_r \) can be 

\[
\left\{ \begin{bmatrix} \mathbf{U} & \mathbf{U}_\perp \end{bmatrix} \begin{bmatrix} \mathbf{e}_j^T & 0_{r \times (p-r)} \\ 0_{(p-r) \times r} & 0_{(p-r) \times (p-r)} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \mathbf{V}_\perp \end{bmatrix}^\top, i = 1, \ldots, r, j = 1, \ldots, r \right\} \cup
\]

---

**Table 1: Computational complexity per iteration and convergence rate for Alternating Minimization (Alter Mini) (Jain et al., 2013), singular value projection (SVP) (Jain et al., 2010), gradient descent (GD) (Tu et al., 2016), and RISRO modified line search method has guaranteed global convergence property as shown in (Absil et al., 2008, Theorem 4.3.1).**
where $\mathbf{e}_i$ and $\hat{\mathbf{e}}_i$ denote the $i$-th canonical basis of $\mathbb{R}^r$ and $\mathbb{R}^{p-r}$, respectively. It follows that any tangent vector in $T_{\mathbf{X}}\mathcal{M}_r$ can be uniquely represented by a coefficient vector in $\mathbb{R}^{(2p-r)r}$ via the basis above. This representation is called the intrinsic representation [Huang et al., 2017a]. Computing the intrinsic representations of a Riemannian gradient can be computationally efficient. For example, the complexity of computing the Riemannian gradient in matrix completion is $O(n^2r+nr^2)$ [Huang and Hand, 2018]. The complexities of computing intrinsic representations of the Riemannian gradients of the phase retrieval and the blind deconvolution are both $O(n\log(n)r^2)$ [Huang et al., 2017b; Huang and Hand, 2018].

By Theorem 2, the least squares problem (5) of RISRO is equivalent to solve $\eta \in T_{\mathbf{X}}\mathcal{M}_r$ such that $P_{T_{\mathbf{X}}}A^*(A(\eta)) = -\text{grad}(f(\mathbf{X}))$. Reformulating this equation by intrinsic representation yields

$$-\text{grad}(f(\mathbf{X})) = P_{T_{\mathbf{X}}}A^*(A(\eta)) \implies -u = B_{\mathbf{X}}^*(A^*(A(B_{\mathbf{X}}v))),$$

where $u, v$ are the intrinsic representations of $\text{grad}(f(\mathbf{X}))$ and $\eta$, the mapping $B_{\mathbf{X}} : \mathbb{R}^{(2p-r)r} \rightarrow T_{\mathbf{X}}\mathcal{M}_r \subset \mathbb{R}^{p \times p}$ converts an intrinsic representation to the corresponding tangent vector, and $B_{\mathbf{X}}^* : \mathbb{R}^{p \times p} \rightarrow \mathbb{R}^{(2p-r)r}$ is the adjoint operator of $B_{\mathbf{X}}$. The computational complexity of using conjugate gradient method to solve (33) is determined by the complexity of evaluating the operator $B_{\mathbf{X}}^* \circ (A^*A) \circ B_{\mathbf{X}}$ on a given vector. With the intrinsic representation, it can be shown that this evaluation costs $O(nr^2)$ in matrix completion and $O(n\log(n)r^2)$ in the phase retrieval and blind deconvolution. Thus, when solving (33) via the conjugate gradient method, the complexity is $O(k(nr+pr^2))$ in the matrix completion and $O(k(n\log(n)r+pr^2))$ in the phase retrieval and blind deconvolution, where $k$ is the number of conjugate gradient iterations and is provably at most $(2p-r)r$. Hence, for special applications such as matrix completion, phase retrieval and blind deconvolution, by using the conjugate gradient method with the intrinsic representation, the per iteration complexity of RISRO can be greatly reduced. This point will be further exploited in our future research.

### 6 Recursive Importance Sketching under Statistical Models

In this section, we study the applications of RISRO in machine learning and statistics. We specifically investigate the low-rank matrix trace regression and phase retrieval, while our key ideas can be applied to more problems. For the execution of RISRO, we assume that some estimate for the rank of the target parameter matrix, denoted by $r$, is available. In many statistical applications such as phase retrieval and blind deconvolution, this assumption trivially holds as the parameter matrix is known to be rank-1. In other applications, while the rank of the parameter is unknown, it is generally not difficult to obtain a rough estimate given the domain knowledge. Then, we can optimize over the set of fixed rank matrices using the formulation of (1) and dynamically update the selected rank (see, e.g., Vandereycken and Vandewalle [2010]; Zhou et al., [2016]).
6.1 Low-Rank Matrix Trace Regression

Consider the low-rank matrix trace regression model:

$$y_i = \langle A_i, X^* \rangle + \epsilon_i, \quad \text{for } 1 \leq i \leq n,$$

where $X^* \in \mathbb{R}^{p_1 \times p_2}$ is the true model parameter to be estimated. We estimate $X^*$ by solving the minimization problem where $r$ in the rank constraint satisfies $r \leq \operatorname{rank}(X^*)$, i.e., $r$ is an estimate of $\operatorname{rank}(X^*)$.

The following Theorem 3 shows RISRO converges quadratically to the best rank $r$ approximation of $X^*$, i.e., $X^*_{\max(r)}$, up to some statistical error given a proper initialization. Under the Gaussian ensemble design, RISRO with spectral initialization achieves the minimax optimal estimation error rate.

**Theorem 3 (RISRO in Matrix Trace Regression)** Consider the low-rank matrix trace regression problem. Define $\tilde{\epsilon}_i := \epsilon_i + \langle A_i, X^* - X^*_{\max(r)} \rangle$ for $i = 1, \ldots, n$. Suppose that $A$ satisfies the $2r$-RIP, the initialization of RISRO satisfies

$$\|X^0 - X^*_{\max(r)}\|_F \leq \left( \frac{1}{4} \wedge \frac{1 - R_{2r}}{2\sqrt{r} R_{3r}} \right) \sigma_r(X^*),$$

and

$$\sigma_r(X^*) \geq \left( 16\sqrt{5} \sqrt{\frac{40\sqrt{2} R_{3r}}{1 - R_{2r}}} \right) \frac{\|(A^* \tilde{\epsilon})_{\max(r)}\|_F}{1 - R_{2r}}.$$

Then the iterations of RISRO converge as follows $\forall t \geq 0$:

$$\|X^{t+1} - X^*_{\max(r)}\|_F \leq 10 \frac{R_{2r}}{1 - R_{2r}} \|X^t - X^*_{\max(r)}\|_F^2 \left( 1 - R_{2r} \right)^2 \sigma_r^2(X^*) + 20 \left( \|(A^* \tilde{\epsilon})_{\max(r)}\|_F + \|(A^* A (X^* - X^*_{\max(r)}))_{\max(r)}\|_F \right)^2 \left( 1 - R_{2r} \right)^2.$$

The overall convergence of RISRO shows two phases:

- (Phase I) When $\|X^t - X^*_{\max(r)}\|_F^2 \geq \frac{\sqrt{2}}{R_{3r}} \|(A^* \tilde{\epsilon})_{\max(r)}\|_F \sigma_r(X^*)$,

$$\|X^{t+1} - X^*_{\max(r)}\|_F \leq 2\sqrt{5} \frac{R_{3r}}{1 - R_{2r}} \|X^t - X^*_{\max(r)}\|_F \left( 1 - R_{2r} \right) \sigma_r(X^*) \leq 2\sqrt{5} \frac{R_{3r}}{1 - R_{2r}} \|X^t - X^*_{\max(r)}\|_F + \|X^*_{\max(r)}\|_F,$$

where $X^*_{\max(r)} = X^* - X^*_{\max(r)}$.

- (Phase II) When $\|X^t - X^*_{\max(r)}\|_F^2 \leq \frac{\sqrt{2}}{R_{3r}} \|(A^* \tilde{\epsilon})_{\max(r)}\|_F \sigma_r(X^*)$,

$$\|X^{t+1} - X^*_{\max(r)}\|_F \leq 2\sqrt{10} \|X^{t+1} - X^*_{\max(r)}\|_F + \|X^*_{\max(r)}\|_F.$$

Moreover, we assume $\operatorname{rank}(X^*) = r$, $(A_i)_{[i,j]}$ are independent sub-Gaussian random variables with mean zero and variance $1/n$ and $\epsilon_i$ are independent sub-Gaussian random variables with mean zero and variance $\sigma^2/n$ (i.e., $\mathbb{E}(A_i)_{[i,j]} = \mathbb{E}(\epsilon_i) = 0$, $\operatorname{Var}(A_i)_{[i,j]} = 1/n$, $\operatorname{Var}(\epsilon_i) = \sigma^2/n$, $\sup_{q \geq 1} (n/q)^{1/2} \mathbb{E}(|A_i)_{[i,j]}|^q)^{1/q} \leq C$, $\sup_{q \geq 1} (n/(q\sigma^2))^{1/2} \mathbb{E}(|\epsilon_i|^q)^{1/q} \leq C$ for some fixed $C > 0$).

Then there exist universal constants $C_1, C_2, C'$, $c > 0$ such that as long as $n \geq C_1 (p_1 + p_2)r^2(\frac{\sigma^2}{\sqrt{\operatorname{rank}(X^*)}} \vee r^2)$ (here $\kappa = \frac{\sigma_1(X^*)}{\sigma_r(X^*)}$ is the condition number of $X^*$) and $t_{\max} \geq C_2 \log \log(\frac{\sigma^2}{\sigma^2/(r(p_1 + p_2)\sigma)}) \vee 1$, the output of RISRO with spectral initialization $X^0 = (A^* (y))_{\max(r)}$ satisfies $\|X_{t_{\max}} - X^*\|_F^2 \leq c \frac{r(p_1 + p_2)}{n} \sigma^2$ with probability at least $1 - \exp(-C'(p_1 + p_2))$.  

19
Remark 12 (Quadratic Convergence, Two-phases Convergence, Statistical Error, and Robustness) The upper bound of $\|X^t - X^*_\text{max(r)}\|_F^2$ in (37) includes two terms: the optimization error term $O(\|X^t - X^*_\text{max(r)}\|_F^2)$ quadratically decreases over iteration $t$, and the statistical error term $O((A^*(\hat{\epsilon}))_{\text{max(r)}}\|\hat{\epsilon}\|_F^2)$ is static through iterations. Moreover, RISRO includes two phases in its convergence. In Phase I with large $\|X^t - X^*_\text{max(r)}\|_F^2$, RISRO converges quadratically towards $X^*_\text{max(r)}$; in Phase II with moderate $\|X^t - X^*_\text{max(r)}\|_F^2$, the estimator returned by one more iteration of RISRO achieves the best possible statistical error rate $O((A^*(\hat{\epsilon}))_{\text{max(r)}}\|\hat{\epsilon}\|_F^2)$ as suggested by the $d = 2$ case in [Luo and Zhang, 2021, Theorem 2]. Therefore, although the convergence rate of RISRO may decelerate to be linear in Phase II, Theorem 3 suggests there is no need to run further iterations as the estimator is already statistically optimal after one additional iteration. Such performance of “quadratic convergence + one-iteration optimality” is unique, which does not appear in common first-order methods.

Finally, (37) shows the error contraction factor is independent of the condition number $\kappa$, which demonstrates the robustness of RISRO to the ill-conditioning of the underlying low-rank matrix. We will further demonstrate this point by simulation studies in Section 7.2.

Remark 13 (Optimal Statistical Error) Under the Gaussian ensemble design and when $\text{rank}(X^*) = r$, RISRO with spectral initialization achieves the rate of estimation error $cr(p_1 + p_2)\sigma^2/n$ after double-logarithmic number of iterations when $n \geq C_1(p_1 + p_2)^r(\sigma^2 / \sigma^2_{(X^*)}) \vee r^2 \kappa^2$. Compared with the lower bound of the estimation error

$$\min_{\hat{X}} \max_{\text{rank}(X^*) \leq r} \mathbb{E}\|\hat{X} - X^*\|_F^2 \geq c'^r(p_1 + p_2)\sigma^2/n$$

for some $c' > 0$ in [Candès and Plan, 2011], RISRO achieves the minimax optimal estimation error with near-optimal sample complexity. To the best of our knowledge, RISRO is the first provable algorithm that achieves the minimax rate-optimal estimation error with only a double-logarithmic number of iterations and this is an exponential improvement over common first-order methods where a logarithmic number of iterations are needed.

### 6.2 Phase Retrieval

In this section, we consider RISRO for solving the following quadratic equation system

$$y_i = |\langle a_i, x^* \rangle|^2 \quad \text{for} \quad 1 \leq i \leq n,$$

where $y \in \mathbb{R}^n$ and covariates $\{a_i\}_{i=1}^n \in \mathbb{R}^p$ (or $\mathbb{C}^p$) are known whereas $x^* \in \mathbb{R}^p$ (or $\mathbb{C}^p$) are unknown. The goal is to recover $x^*$ based on $\{y_i, a_i\}_{i=1}^n$. One important application is known as phase retrieval arising from physical science due to the nature of optical sensors [Fienup, 1982]. In the literature, various approaches have been proposed for phase retrieval with provable guarantees, such as convex relaxation (Candes et al., 2013; Huang et al., 2017) and non-convex approaches (Candes et al., 2015; Chen and Candès, 2017; Gao and Xu, 2017; Ma et al., 2019). [Netrapalli et al., 2013; Sanghavi et al., 2017; Wang et al., 2017; Duchi and Ruan, 2019].

For ease of exposition, we focus on the real-value model, i.e., $x^* \in \mathbb{R}^n$ and $a_i \in \mathbb{R}^n$, while a simple trick in Sanghavi et al. [2017] can recast the problem (38) in the complex model into a rank-2 real value matrix recovery problem, then our approach still applies. In the real-valued setting, we can rewrite model (38) into a low-rank matrix recovery model

$$y = A(X^*) \quad \text{with} \quad X^* = x^*x^{*\top} \text{ and } [A(X^*)]_i = \langle a_i a_i^\top, x^*x^{*\top} \rangle.$$
There are two challenges in phase retrieval compared to the low-rank matrix trace regression considered previously. First, due to the symmetry of sensing matrices \(a_i, a_j^T\) and \(x^*x^{*T}\) in phase retrieval, the importance covariates \(A_1\) and \(A_2\) in (4) are exactly the same and an adaptation of Algorithm 1 is thus needed. Second, in phase retrieval, the mapping \(\mathcal{A}\) no longer satisfies a proper RIP condition in general (Cai and Zhang [2015] Candès et al. [2013]), so a new theory is needed. To this end, we introduce a modified RISRO for phase retrieval in Algorithm 2. Particularly in Step 4 of Algorithm 2, we multiply the importance covariates \(A_2\) by an extra factor 2 to account for the duplicate importance covariates due to symmetry.

**Algorithm 2 RISRO for Phase Retrieval**

1: Input: design vectors \(\{a_i\}_{i=1}^n \in \mathbb{R}^p, y \in \mathbb{R}^n\), initialization \(X^0\) that admits eigenvalue decomposition \(a_0^Tu_0u_0^T\)
2: for \(t = 0, 1, \ldots, \) do
3: Perform importance sketching on \(a_i\) and construct the covariates \(A_1 \in \mathbb{R}^n, A_2 \in \mathbb{R}^{n \times (p-1)}\), where for \(1 \leq i \leq n\), \((A_1)_{i,:} = (a_i^T u)^2, (A_2)_{i,:} = u_1^T a_i a_i^T u^T\).
4: Solve the unconstrained least squares problem \((b^{t+1}, d^{t+1}) = \arg\min_{b \in \mathbb{R}, d \in \mathbb{R}^{(p-1)}} \|y - A_1 b - 2 A_2 d\|^2\).
5: Compute the eigenvalue decomposition of \([u_t^T u_1^T \mid d_{t+1}^T 0]^T\), and denote it as \([v_1 v_2]^T \lambda_1 0 \lambda_2 \) with \(\lambda_1 \geq \lambda_2\).
6: Update \(u_t^{t+1} = v_1\) and \(X^{t+1} = \lambda_1 u_t^{t+1} u_t^{t+1T}\).
7: end for

Next, we show under Gaussian ensemble design, given the sample number \(n = O(p \log p)\) and proper initialization, the sequence \(\{X^t\}\) generated by Algorithm 2 converges quadratically to \(X^*\).

**Theorem 4 (Local Quadratic Convergence of RISRO for Phase Retrieval)** In the phase retrieval problem (38), assume that \(\{a_i\}_{i=1}^n\) are independently generated from \(N(0, I_p)\). Then for any \(\delta_1, \delta_2 \in (0, 1)\), there exist \(c, C(\delta_1), C' > 0\) such that when \(p \geq c \log n, n \geq C(\delta_1)p \log p\), if \(\|X^0 - X^*\|_F \leq \frac{1 - \delta_1}{C'(1+\delta_2)p}\|X^*\|_F\), with probability at least \(1 - C_1 \exp(-C_2 (\delta_1, \delta_2)n) - C_3 n^{-p}\), the sequence \(\{X^t\}\) generated by Algorithm 2 satisfies

\[
\|X^{t+1} - X^*\|_F \leq \frac{C'(1 + \delta_2)p}{(1 - \delta_1)\|X^*\|_F} \|X^t - X^*\|_F^2, \quad \forall t \geq 0
\]  

for some \(C_1, C_2(\delta_1, \delta_2), C_3 > 0\).

To overcome the technical difficulties in establishing quadratic convergence without RIP for phase retrieval, Theorem 4 is established under the assumption \(\|X^0 - X^*\|_F \leq O(\|X^*\|_F/p)\). Although it is difficult to prove that the spectral initializer meets this assumption under the near-optimal sample size (e.g., \(n = Cp \log p\)), we find by simulation that the spectral initialization yields quadratic convergence for RISRO (Section 7). On the other hand, we can also run a few iterations of factorized gradient descent to achieve the initialization condition in Theorem 4 with near-optimal sample complexity guarantee (Candès et al. [2015] Chen and Candès [2017] Ma et al. [2019]) and then switch to RISRO. Specifically, the initialization algorithm for RISRO in phase retrieval via factorized gradient descent is provided in Algorithm 3 and its guarantee is given in Proposition 4.
of Theorem 4 holds with probability at least 1.

We specifically consider two settings:

In this section, we conduct simulation studies to investigate the numerical performance of RISRO.

7 Numerical Studies

7.1 Properties of RISRO

Algorithm 3 RISRO for Phase Retrieval with Gradient Descent Initialization

1: Input: design vectors \( \{a_i\}_{i=1}^n \in \mathbb{R}^p \) and \( y \in \mathbb{R}^n \).
2: Let \( \lambda_1(Y) \) and \( v_1 \) be the leading eigenvalue and eigenvector of \( Y = \frac{1}{n} \sum_{j=1}^n y_ja_ja_j^\top \), respectively, and set \( \hat{x}^0 = \sqrt{\lambda_1(Y)}/3v_1 \).
3: for \( t = 0, 1, \ldots, T_0 - 1 \) do
4: Update \( \hat{x}^{t+1} = \hat{x}^t - \eta_t \nabla g(\hat{x}^t) \), where \( g(x) = \frac{1}{4n} \sum_{j=1}^n \left( (a_j^\top x)^2 - y_j \right)^2 \).
5: end for
6: Apply Algorithm 2 with initialization \( \hat{x}^{T_0} \).

Proposition 4 In phase retrieval (38), suppose \( \{a_i\}_{i=1}^n \) are independently drawn from \( \mathcal{N}(0, I_p) \) and \( n \geq Cp \log p \) for some sufficient large constant \( C > 0 \). Assume the step size in Algorithm 3 obeys \( \eta_t = \eta = c_1/(\log p \cdot \|\hat{x}^0\|_2^2) \) for constant \( c_1 > 0 \), where \( \hat{x}^0 \) is given in the algorithm. Then there exist absolute constants \( c_2, c_3 > 0 \) such that when \( T_0 \geq c_2 \log p \cdot \log(\|x^*\|_2p) \), the initialization \( X^0 := \hat{x}^{T_0}\hat{x}^{T_0\top} \) in Algorithm 3 satisfies the initialization condition in Theorem 4 and the conclusion of Theorem 4 holds with probability at least \( 1 - c_3np^{-5} \).

7 Numerical Studies

In this section, we conduct simulation studies to investigate the numerical performance of RISRO. We specifically consider two settings:

- Matrix trace regression. Let \( p = p_1 = p_2 \) and \( y_i = \langle X^*, A_i \rangle + \epsilon_i \), where \( A_i \)'s are constructed with independent standard normal entries and \( \epsilon_i \sim \mathcal{N}(0, \sigma^2) \). Let \( U^*, V^* \in \mathbb{O}_{p,r} \) be randomly generated, \( \Sigma = \text{diag}(\lambda_1, \ldots, \lambda_r) \). Also, we set \( \lambda_1 = 3 \) and \( \lambda_i = \frac{\lambda_1}{\kappa^{i-1}} \) for \( i = 2, \ldots, r \), so the condition number of \( X^* \) is \( \kappa \). We initialize \( X^0 \) via \( (A^*(y))_{\text{max}(r)} \).

- Phase retrieval. Let \( y_i = \langle a_i, x^* \rangle^2 \), where \( x^* \in \mathbb{R}^p \) is a randomly generated unit vector, \( a_i \sim \mathcal{N}(0, I_p) \). We initialize \( X^0 \) via truncated spectral initialization (Chen and Candès, 2017).

Throughout the simulation studies, we consider errors in two metrics: (1) \( \|X^t - X^{t_{\text{max}}}\|_F/\|X^{t_{\text{max}}}\|_F \), which measures the convergence error; (2) \( \|X^t - X^*\|_F/\|X^*\|_F \), which is the relative root mean-squared error (Relative RMSE) that measures the estimation error for \( X^* \). The algorithm is terminated when it reaches the maximum number of iterations \( t_{\text{max}} = 300 \) or the corresponding error metric is less than \( 10^{-12} \). Unless otherwise noted, the reported results are based on the averages of 50 simulations and on a computer with Intel Xeon E5-2680 2.5GHz CPU. Additional development and simulation results for RISRO in matrix completion and robust PCA can be found in Appendix A.

7.1 Properties of RISRO

We first study the convergence rate of RISRO. Specifically, set \( p = 100, r = 3, n \in \{1200, 1500, 1800, 2100, 2400\} \), \( \kappa = 1, \sigma = 0 \) for low-rank matrix trace regression and \( p = 1200, n \in \{4800, 6000, 7200, 8400, 9600\} \) for phase retrieval. The convergence performance of RISRO (Algorithm 1 in low-rank matrix trace regression and Algorithm 2 in phase retrieval) is plotted in Figure 1. We can see RISRO with the (truncated) spectral initialization converges quadratically to the true parameter \( X^* \) in both problems, which is in line with the theory developed in previous sections. Although our theory
on phase retrieval in Theorem 4 is based on a stronger initialization assumption, the truncated spectral initialization achieves great empirical performance.

In another setting, we examine the quadratic-linear convergence for RISRO under the noisy setting. Consider the matrix trace regression problem, where $\sigma = 10^\alpha$, $\alpha \in \{0, -1, -2, -3, -5, -14\}$, $n = 1500$, and $p, r, \kappa$ are the same as the previous setting. The simulation results in Figure 3 show the gradient norm $\|\text{grad} f(X^t)\|$ of the iterates converges to zero, which demonstrates the convergence of the algorithm. Meanwhile, since the observations are noisy, RISRO exhibits the quadratic-linear convergence as we discussed in Remark 4: when $\alpha = 0$, i.e., $\sigma = 1$, RISRO converges quadratically in the first 2-3 steps and then reduces to linear convergence afterward; as $\sigma$ gets smaller, we can see RISRO enjoys a longer path of quadratic convergence, which matches our theoretical prediction in Remark 4.

Finally, we study the performance of RISRO under the large-scale setting of the matrix trace regression. Fix $n = 7000$, $r = 3$, $\kappa = 1$, $\sigma = 0$ and let dimension $p$ grow from 100 to 500. For the largest case, the space cost of storing $A$ reaches $7000 \cdot 500 \cdot 500 \cdot 8B = 13.04GB$. Figure 4 shows the relative RMSE of the output of RISRO and runtime versus the dimension. We can clearly see the relative RMSE of the output is stable and the runtime scales reasonably well as the dimension $p$ grows.

### Figure 3: Convergence plot of RISRO in matrix trace regression. $p = 100, r = 3, n = 1500, \kappa = 1, \sigma = 10^\alpha$ with varying $\alpha$

### Figure 4: Relative RMSE and runtime of RISRO in matrix trace regression. $p \in [100, 500], r = 3, n = 7000, \kappa = 1, \sigma = 0$

#### 7.2 Comparison of RISRO with Other Algorithms in Literature

In this subsection, we further compare RISRO with existing algorithms in the literature. In the matrix trace regression, we compare our algorithm with singular value projection (SVP) of Goldfarb.
and Ma, 2011; Jain et al., 2010), Alternating Minimization (Alter Mini) (Jain et al., 2013; Zhao et al., 2015), gradient descent (GD) (Park et al., 2018; Tu et al., 2016; Zheng and Lafferty, 2015), and convex nuclear norm minimization (NNM) (Toh and Yun, 2010). We consider the setting with $p = 100$, $r = 3$, $n = 1500$, $\kappa \in \{1, 50, 500\}$, $\sigma = 0$ (noiseless case) or $\sigma = 10^{-6}$ (noisy case). Following Zheng and Lafferty (2015), in the implementation of GD and SVP, we evaluate three choices of step size, $\{5 \times 10^{-3}, 10^{-3}, 5 \times 10^{-4}\}$, then choose the best one. In phase retrieval, we compare Algorithm 2 with Wirtinger Flow (WF) (Candès et al., 2015) and Truncated Wirtinger Flow (TWF) (Chen and Candès, 2017) with $p = 1200$, $n = 6000$. We use the codes of the accelerated proximal gradient for NNM, WF and TWF from the corresponding authors’ websites and implement the other algorithms by ourselves. The stopping criteria of all procedures are the same as RISRO mentioned in the previous simulation settings.

We compare the performance of various procedures on noiseless matrix trace regression in Figure 5. For all different choices of $\kappa$, RISRO converges quadratically to $X^*$ in 7 iterations with high accuracy, while the other baseline algorithms converge much slower at a linear rate. When $\kappa$ (condition number of $X^*$) increases from 1 to 50 and 500 so that the problem becomes more ill-conditioned, RISRO, Alter Mini, and SVP perform robustly, while GD converges more slowly. In Theorem 3, we have shown the quadratic convergence rate of RISRO is robust to the condition number (see Remark 12). As we expect, the non-convex optimization methods converge much faster than the convex relaxation method. Moreover, to achieve a relative RMSE of $10^{-10}$, RISRO only takes about five iterations and 1/5 runtime compared to other algorithms if $\kappa = 1$ and this factor is even smaller in the ill-conditioned cases that $\kappa = 50$ and 500.

The comparison of RISRO, WF, and TWF in phase retrieval is plotted in Figure 6. We can also see that RISRO can recover the underlying true signal with high accuracy in much less time than the other baseline methods.

Next, we compare the performance of RISRO with other algorithms in the noisy setting, $\sigma = 10^{-6}$, in the low-rank matrix trace regression. We can see from the results in Figure 7 that due to the noise, the estimation error first decreases and then stabilizes after reaching a certain level. Meanwhile, we can also find RISRO converges at a much faster quadratic rate before reaching the stable level compared to all other algorithms.

Finally, we study the required sample size to guarantee successful recovery by RISRO and other algorithms. We set $p = 100$, $r = 3$, $\kappa = 5$, $n \in [600, 1500]$ in the noiseless matrix trace regression and $p = 1200$, $n \in [2400, 6000]$ in phase retrieval. We say the algorithm achieves successful recovery if the relative RMSE is less than $10^{-2}$ when the algorithm terminates. The simulation results in Figure 8 show RISRO requires the minimum sample size to achieve a successful recovery in both matrix trace regression and phase retrieval; Alter Mini has similar performance to RISRO; and both RISRO and Alter Mini require smaller sample size than the rest of algorithms for successful recovery.

8 Conclusion and Discussion

In this paper, we propose a new algorithm, RISRO, for solving rank constrained least squares. RISRO is based on a novel algorithmic framework, recursive importance sketching, which also provides new sketching interpretations for several existing algorithms for rank constrained least squares. RISRO is easy to implement and computationally efficient. Under some reasonable assumptions, local quadratic-linear and quadratic convergence are established for RISRO. Simulation studies demonstrate the superior performance of RISRO.

The connection of recursive importance sketching and Riemannian Gauss-Newton discovered in
Figure 5: Relative RMSE of RISRO, singular value projection (SVP), Alternating Minimization (Alter Mini), gradient descent (GD), and Nuclear Norm Minimization (NNM) in low-rank matrix trace regression. Here, $p = 100, r = 3, n = 1500, \sigma = 0, \kappa \in \{1, 50, 500\}$.

This paper can be leveraged to other settings, such as in the low-rank tensor estimation problems (see a follow-up work in Luo and Zhang (2021) after the first preprint of this paper).

There are many interesting extensions to the results in this paper to be explored in the future. First, our current convergence theory on RISRO relies on the RIP assumption, which may not hold in many scenarios, such as phase retrieval, matrix completion, and robust PCA. In this paper, we give some theoretical guarantees of RISRO in phase retrieval with a strong initialization assumption. However, such an initialization requirement may be unnecessary and spectral initialization is good enough to guarantee quadratic convergence as we observe in the simulation studies. Empirically, we also observe RISRO achieves quadratic convergence in the matrix completion and robust PCA examples, see their development in Appendix A. To improve and establish theoretical guarantees for RISRO in phase retrieval and matrix completion or robust PCA, we think more sophisticated
Figure 6: Relative RMSE of RISRO, Wirtinger Flow (WF), Truncated Wirtinger Flow (TWF) in phase retrieval. Here, $p = 1200, n = 6000$

Figure 7: Relative RMSE of RISRO, singular value projection (SVP), Alternating Minimization (Alter Mini), gradient descent (GD), and Nuclear Norm Minimization (NNM) in low-rank matrix trace regression. Here, $p = 100, r = 3, n = 1500, \kappa = 5, \sigma = 10^{-6}$

analysis tools such as the “leave-one-out” method and some extra properties such as “implicit regularization” (Ma et al., 2019) need to be incorporated into the analysis and it will be interesting future work. Also, this paper focuses on the squared error loss in (1), while the other loss functions may be of interest in different settings, such as the $\ell_1$ loss in robust low-rank matrix recovery (Charisopoulos et al., 2021; Li et al., 2020a,b), which is worth exploring.

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Figure 8: Successful recovery rate comparison

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A  RISRO for Matrix Completion and Robust PCA

In this section, we provide implementation details and simulation results of RISRO in another two prominent examples: matrix completion (Candès and Tao, 2010) and robust PCA (Candès et al., 2011).

RISRO for Matrix Completion. In matrix completion, we observe a fraction of entries of an unknown low-rank matrix $X^*$ (denote the index set of the observed entries by $\Omega$) corrupted by measurement error, $Y_\Omega = X^*_{\Omega} + \varepsilon_\Omega$, and aim to recover $X^*$ from $Y_\Omega$. Suppose the number of observed entries is $n$. Let $\{(i_k, j_k)_{k=1}^n : (i_k, j_k) \in \Omega\}$ be the ordered index pairs of the observed entries with the same order as the one in the vectorization of $Y_\Omega$. Since sensing matrices in matrix completion are all binary with only one entry of 1, RISRO for matrix completion can be simplified to Algorithm 4.

Algorithm 4 RISRO for Matrix Completion
1: Input: $Y_\Omega$, ordered index pairs of the observed entries $\{(i_k, j_k)_{k=1}^n : (i_k, j_k) \in \Omega\}$, rank $r$, initialization $X^0$ that admits SVD $U^0 \Sigma^0 V^0^T$, where $U^0 \in \Omega_{p_1 \times r}$, $V^0 \in \Omega_{p_2 \times r}$, $\Sigma^0 \in \mathbb{R}^{r \times r}$
2: for $t = 0, 1, \ldots$ do
3: Construct the importance covariates matrix $A^t \in \mathbb{R}^{n \times (p_1 + p_2 - r) r}$:
4: Solve $\text{vec}(B^{t+1})^\top \text{vec}(D^{t+1}) = (A^t \Sigma^t A^t)^\top A^t \text{vec}(Y_\Omega)$.
5: Compute $X^{t+1}_U = (U^0 B^{t+1} + U^t D^{t+1})$ and $X^{t+1}_V = (V^0 B^{t+1} + V^t D^{t+1})$.
6: Perform QR orthogonalization: $U^{t+1} = QR(X^{t+1}_U)$, $V^{t+1} = QR(X^{t+1}_V)$.
7: Update $X^{t+1} = X^{t+1}_U (B^{t+1})^\top X^{t+1}_V$.
8: end for

RISRO for Robust Principal Component Analysis. The basic model of robust PCA is $Y = X^* + S^* \in \mathbb{R}^{p_1 \times p_2}$, where $X^*$ is an unknown low-rank matrix of interest and $S^*$ an unknown sparse corruption matrix. We consider the setting where a part of entries of $Y$, whose indices are denoted by $\Omega \subseteq \{(i, j) : 1 \leq i \leq p_1, 1 \leq j \leq p_2\}$, are observed. Our goal is to recover $X^*$ based on $Y_\Omega$. Define the following truncation operator $F : \mathbb{R}^{p_1 \times p_2} \rightarrow \mathbb{R}^{p_1 \times p_2}$ (Yi et al., 2016; Zhang and Yang, 2018),

\[
(F(A))_{[i,j]} = \begin{cases} 
0, & \text{if } |A_{[i,j]}[\gamma,\Omega]| > |A_{[i,j]}|[\gamma,\Omega] \text{ and } |A_{[i,j]}| > |A_{[i,j]}|[\gamma,\Omega]; \\
A_{[i,j]}, & \text{otherwise.}
\end{cases}
\]

Here, $|A_{[i,j]}|[\gamma,\Omega]$ and $|A_{[i,j]}|[\gamma,\Omega]$ represent the $(1-\gamma)$-th percentile of the absolute values of the observed entries of $A_{[i,j]}$ and $A_{[i,j]}$ of the matrix $A$, respectively. We provide an implementation of RISRO for robust PCA in Algorithm 5.

Next, we investigate the numerical performance of RISRO in matrix completion and robust PCA, i.e., Algorithms 4 and 5. In both settings, we generate $X^* \in \mathbb{R}^{p \times p}$ as a random rank-3 matrix with condition number $\kappa$ in the same way as the one in matrix trace regression described in Section 7. In matrix completion, we set $p = 500, \kappa = \{1, 50, 500\}$ and assume $n = 8p\kappa$ noiseless entries of $X^*$ are observed uniformly at random. In robust PCA, we set $p = 100, \kappa = \{1, 50, 100\}$. We observe the full matrix $Y = X^* + S^*$, where each entry of $S^*$ follows $N(0, 100)$ with probability
(b) Robust PCA

In the decomposition of (6), we have

**Proof of Lemma 1**

**Proof of the Main Results in the Paper**

**Proof of Lemma 1** First, by the decomposition of (6), we have

\[
 y = \mathcal{A}_t \left( \begin{bmatrix} \tilde{B}_t^\top & \tilde{D}_t^\top \end{bmatrix} \right) + \epsilon_t.
\]  

(42)
In view of (10), if the operator $L_t^* A^* A L_t$ is invertible, the output of least squares in (5) satisfies
\[
\begin{bmatrix}
B_{t+1} + D_{t+1}^{T+1} \\
D_{t+1}
\end{bmatrix} = \begin{bmatrix}
(L_t^* A^* A L_t)^{-1} L_t^* A^* y \\
\hat{B}_t^{T} \\
\hat{D}_t
\end{bmatrix} = \begin{bmatrix}
(B_t + D_t^{T}) \\
0
\end{bmatrix} + (L_t^* A^* A L_t)^{-1} L_t^* A^* e^t,
\]
where the second equality is due to (42). This finishes the proof. ■

**Proof of Lemma 2** Equation (17) can be directly verified from definitions of $L_t$ and $L_t^*$ in (9).

The second conclusion holds if $M$ is a zero matrix. When $M$ is not zero, to prove the claim it is equivalent to show the spectrum of $L_t^* A^* A L_t$ is upper and lower bounded by $1 + R_{2r}$ and $1 - R_{2r}$, respectively, in the range of $L_t^*$. Since $L_t^* A^* A L_t$ is a symmetric operator, the upper and lower bounds of its spectrum are given below
\[
\sup_{Z \in \text{Ran}(L_t^*): \|Z\|_F = 1} \langle Z, L_t^* A^* A L_t(Z) \rangle = \sup_{Z \in \text{Ran}(L_t^*): \|Z\|_F = 1} \|A L_t(Z)\|_2^2 \leq 1 + R_{2r}
\]
\[
\inf_{Z \in \text{Ran}(L_t^*): \|Z\|_F = 1} \langle Z, L_t^* A^* A L_t(Z) \rangle = \inf_{Z \in \text{Ran}(L_t^*): \|Z\|_F = 1} \|A L_t(Z)\|_2^2 \geq 1 - R_{2r}.
\]

Here (a) and (b) are due to the RIP condition of $A$, $L_t(Z)$ is a s at most rank $2r$ matrix by the definition of $L_t$ and (17). This finishes the proof. ■

**Proof of Proposition 1** Since $A$ satisfies $2r$-RIP, $R_{2r} < 1$. Then, by Lemma 2 $L_t^* A^* A L_t$ is invertible over $\text{Ran}(L_t^*)$. With a slight abuse of notation, define $P_{X^t}$ as
\[
P_{X^t}(Z) := L_t L_t^*(Z) = P_{U^t} Z P_{V_t} + P_{U^t_\perp} Z P_{V_t} + P_{U^t} Z P_{V_t_\perp}, \quad \forall Z \in \mathbb{R}^{p_1 \times p_2},
\]
where $U^t$, $V^t$ are the updated sketching matrices at iteration $t$ defined in Step 7 of Algorithm 1.

We can verify $P_{X^t}$ is an orthogonal projector. Let $P_{(X^t)_\perp}(Z) := Z - P_{X^t}(Z) = P_{U^t_\perp} Z P_{V_t_\perp}$. Recall $e^t = A(P_{U^t_\perp} \hat{X} P_{V_t_\perp}) + \bar{e} = AP_{(X^t)_\perp}(\hat{X}) + \bar{e}$ from (6), we have
\[
\|L_t^* A^* A L_t^{-1} L_t^* A^* e^t\|_F^2 \leq \frac{1}{(1 - R_{2r})^2} \|L_t^* A^* (A(P_{(X^t)_\perp} \hat{X}) + \bar{e})\|_F^2
\]
\[
= \frac{1}{(1 - R_{2r})^2} \left( \|L_t^* A^* A(P_{(X^t)_\perp} \hat{X})\|_F^2 + \|L_t^* A^* (\bar{e})\|_F^2 + 2 \langle L_t^* A^* A(P_{(X^t)_\perp} \hat{X}), L_t^* A^* (\bar{e}) \rangle \right),
\]

where (a) is due to Lemma 2 and the definition of $e^t$.

Notice term (2) is the target term we want, next, we bound (1) and (3) at the right-hand side of (44).

**Bound for (1).**
\[
\|L_t^* A^* A(P_{(X^t)_\perp} \hat{X})\|_F^2 = \langle L_t^* A^* A(P_{(X^t)_\perp} \hat{X}), L_t^* A^* A(P_{(X^t)_\perp} \hat{X}) \rangle
\]
\[
= \langle A(P_{(X^t)_\perp} \hat{X}), AP_{X^t}(A(P_{(X^t)_\perp} \hat{X}) \rangle
\]
\[
\leq R_{3r} \|P_{(X^t)_\perp} \hat{X}\|_F \|P_{X^t} A^* A(P_{(X^t)_\perp} \hat{X})\|_F
\]
\[
= R_{3r} \|P_{(X^t)_\perp} \hat{X}\|_F \|L_t^* A^* A(P_{(X^t)_\perp} \hat{X})\|_F,
\]
where (a) is due to the Lemma 8 and the fact that $\langle P(X^t)\perp \tilde{X}, P_{X^t}A^*AP(X^t)\perp \tilde{X} \rangle = 0$, rank$(P(X^t)\perp \tilde{X}) \leq r$ and rank$(P_{X^t}A^*AP(X^t)\perp \tilde{X}) \leq 2r$. Note that

$$P(X^t)\perp \tilde{X} = \tilde{X} - P_{X^t}\tilde{X}$$

(a) is due to (46) and (b) is due to the singular subspace perturbation inequality when we set $P(X^t)\perp \tilde{X} = \tilde{X} - P_{X^t}\tilde{X}$.

$$P(X^t)\perp \tilde{X} = (P_U - P_{U^t})\tilde{X} + (P_{U^t}P_{V^t} - P_U\tilde{X}P_{V^t} + P_{U^t}\tilde{X}P_{V^t} - P_U\tilde{X}P_{V^t} + P_{U^t}\tilde{X}P_{V^t})$$

(b) is due to the fact that $(I - P_U)\tilde{X}(P_{V} - P_{V^t}) = 0$ and (c) is due to $\tilde{X}^t(I - P_{V^t}) = 0$. Now, from (45), we get

$$\|L^*_tA^*A(P(X^t)\perp \tilde{X})\|_F \leq R_3\|P(X^t)\perp \tilde{X}\|_F$$

(a) is due to (46) and (b) is due to the singular subspace perturbation inequality $\|P_U - P_{U^t}\| \leq \|X^t - \tilde{X}\|/\sigma_r(\tilde{X})$ obtained from Lemma 9.

**Bound for (3).**

$$2\langle L^*_tA^*AP(X^t)\perp \tilde{X}, L^*_tA^*(\tilde{e}) \rangle = 2\langle AP(X^t)\perp \tilde{X}, AP(X^t)(A^*(\tilde{e})) \rangle$$

(a) is due to $2R_3\|P(X^t)\perp \tilde{X}\|_F\|L^*_tA^*(\tilde{e})\|_F$

(b) is due to $2R_3\|X^t - \tilde{X}\|/\sigma_r(\tilde{X})\|L^*_tA^*(\tilde{e})\|_F$.

Plugging (48) and (47) into (44), we get (45). This finishes the proof of this Proposition. ■

**Proof of Theorem 1**. First notice that quadratic convergence result follows easily from (20) when we set $\tilde{e} = 0$. So the rest of the proof is devoted to proving (20) and we also prove the linear convergence along the way. The proof can be divided into four steps. In Step 1, we use Proposition 1 and give an upper bound for the approximation error in the case $\tilde{X}$ is a stationary point. Then, we use induction to show the main results in Step 2,3,4. To start, similar as (45), define

$$P_X(Z) = P_UZP_{V^t} + P_{U^t}ZP_{V^t} + P_{U^t}ZP_{V^t}, \quad \forall Z \in \mathbb{R}^{p_1 \times p_2},$$

where $U, V$ are left and right singular vectors of $\tilde{X}$.

**Step 1.** In this step, we apply Proposition 1 in the case $\tilde{X}$ is a stationary point. In view of (18), the term that we can simplify is $\|L^*_tA^*(\tilde{e})\|_F$. Since $\tilde{X}$ is a stationary point, we know $P_X(A^*(\tilde{e})) = P_X(A^*(y - A(\tilde{X}))) = 0$. Then

$$\|L^*_tA^*(\tilde{e})\|_F^2 = \|P_{X^t}A^*(\tilde{e})\|_F^2 = \|(P_{X^t} - P_X)A^*(\tilde{e})\|_F^2 \leq \|P_{X^t} - P_X\|^2\|A^*(\tilde{e})\|_F^2,$$

(49)
where the first equality is by the definition of $P_{X^t}$ in (43) and equation (17). Meanwhile, it holds that

$$ \|P_{X^t} - P_X\| = \sup_{\|z\|_F \leq 1} \|(P_{X^t} - P_X)(z)\|_F \leq \sup_{\|z\|_F \leq 1} \left( \|P_{U^t} - P_U\|_F \|z\|_F + \|(I - P_{U^t})z(P_{V^t} - P_V)\|_F \right) \leq \sup_{\|z\|_F \leq 1} \frac{2\|X^t - \bar{X}\|_F \|z\|_F}{\sigma_r(X)} \leq \frac{2\|X^t - \bar{X}\|_F}{\sigma_r(X)}, $$

here (a) is because $(P_{X^t} - P_X)(z) = (P_{U^t} - P_U)z(I - P_V) + (I - P_{U^t})z(P_{V^t} - P_V)$ by a similar argument in (16) and (b) is by Lemma [9]. Then, from (49), we see that

$$ \|L_t^s A^*(\epsilon)\|_F^2 \leq \frac{4\|X^t - \bar{X}\|_F^2}{\sigma_r^2(X)} \|A^*(\epsilon)\|_F^2, $$

which, together with (18), implies that

$$ (L_t^s A^* A L_t^s)^{-1} L_t^s A^* \epsilon_t^t F \leq \frac{\|X^t - \bar{X}\|_F^2}{(1 - R_{2\rho})^2 \sigma_r^2(X)} \cdot (R_{3\rho}^2 \|X^t - \bar{X}\|_F^2 + 4\|A^*(\epsilon)\|_F^2 + 4R_{3\rho} \|A^*(\epsilon)\|_F \|X^t - \bar{X}\|_F). $$

**Step 2.** In this step, we start using induction to show the convergence of $X^t$ in (20). First, we introduce $\theta_t$ to measure the goodness of the current iterate in terms of subspace estimation:

$$ \theta_t = \max\{\|\sin \Theta(U^t, U)\|, \|\sin \Theta(V^t, V)\|\}, $$

where $U, V$ are the left and right singular vectors of $\bar{X}$ and here $\|\sin \Theta(U^t, U)\| := \|\cos^{-1}(\sigma_r(U^t U))\|$ measures the largest angle between subspaces $U^t, U$.

Recall the definition of $\tilde{B}^t, \tilde{D}_1^t$ and $\tilde{D}_2^t$ in (7). The induction we want to show is: given $\theta_t \leq 1/2$, $\tilde{B}^t$ is invertible and $\|X^t - \bar{X}\|_F \leq \|X^0 - \bar{X}\|_F$, we prove $\theta_{t+1} \leq 1/2$, $\tilde{B}^{t+1}$ is invertible, $\|X^{t+1} - \bar{X}\|_F \leq \|X^0 - \bar{X}\|_F$ as well as $B^{t+1}$ is invertible and

$$ \|X^{t+1} - \bar{X}\|_F^2 \leq \|L_t^s A^* A L_t^s\|^{-1} L_t^s A^* \epsilon_t^t F \leq \frac{5\|X^t - \bar{X}\|_F^2}{(1 - R_{2\rho})^2 \sigma_r^2(X)} \cdot \left( R_{3\rho}^2 \|X^t - \bar{X}\|_F^2 + 4\|A^*(\epsilon)\|_F^2 + 4R_{3\rho} \|A^*(\epsilon)\|_F \|X^t - \bar{X}\|_F \right) \leq \frac{9}{16} \|X^t - \bar{X}\|_F^2. $$

For the rest of this step, we show when $t = 0$, the induction assumption holds, i.e. $\theta_0 \leq 1/2$ and $\tilde{B}^0$ is invertible. Since $\frac{\|X^0 - \bar{X}\|}{\sigma_r(X)} \leq \frac{1}{4}$, it holds by Lemma [9] that $\theta_0 = \max\{\|\sin \Theta(U^0, U)\|, \|\sin \Theta(V^0, V)\|\} \leq 2\frac{\|X^0 - \bar{X}\|}{\sigma_r(X)} \leq \frac{1}{2}$. Then, we have

$$ \sigma_r(\tilde{B}^0) \geq \sigma_r(U^{0\top} \bar{U}) \sigma_r(\bar{X}) \sigma_r(\bar{V}^\top V^0) = (1 - \theta_0^2) \sigma_r(\bar{X}) \geq \frac{3}{4} \cdot \sigma_r(\bar{X}) > 0, $$

where (a) is because $U^{0\top} \bar{U}, \bar{X}, \bar{V}^\top V^0$ are all rank $r$ matrices and (Luo et al., 2021, arXiv v1, Lemma 5) and (b) is by the definition of $\|\sin \Theta(U^t, U)\|$. 38
**Step 3.** In this step, we first show given \( \theta_t \leq 1/2 \) and \( \|X^t - \bar{X}\|_F \leq \|X^0 - \bar{X}\|_F \), \( B^{t+1} \) is invertible. Then we also do some preparation to show the main contraction \([52]\). Notice that

\[
\sigma_r(B^{t+1}) \geq \sigma_r(\hat{B}^t) - \|B^{t+1} - \hat{B}^t\| \geq \sigma_r(U^T \hat{U}) \sigma_r(\hat{X}) \sigma_r(V^T \hat{V}) - \|B^{t+1} - \hat{B}^t\|
\geq (1 - \theta_t^2) \sigma_r(\hat{X}) - \|B^{t+1} - \hat{B}^t\|,
\]

and

\[
\max\{\|B^{t+1} - \hat{B}^t\|, \|D_1^{t+1} - \hat{D}_1^t\|, \|D_2^{t+1} - \hat{D}_2^t\|\} \leq \frac{\|L_t^* A^* A L_t\|^{-1} L_t^* A^* e^t\|}{\|L_t^* A^* A L_t\|^{-1} L_t^* A^* e^t\|_F}, \tag{55}
\]

where (a) is due to \([11]\).

Under the induction assumption \( \|X^t - \bar{X}\|_F \leq \|X^0 - \bar{X}\|_F \), the initialization condition \([19]\) and \( \|A^*(\bar{e})\|_F \leq \frac{1 - R_2}{4\sqrt{5}} \sigma_r(\bar{X}) \), we have

\[
\frac{R_2^2 \|X^t - \bar{X}\|_F^2}{(1 - R_2^2)^2 \sigma_r^2(\bar{X})} \leq 1/80, \quad \frac{4 \|A^*(\bar{e})\|_F^2}{(1 - R_2^2)^2 \sigma_r^2(\bar{X})} \leq 1/20, \quad \frac{4R_2 \|X^t - \bar{X}\|_F \|A^*(\bar{e})\|_F}{(1 - R_2^2)^2 \sigma_r^2(\bar{X})} \leq 1/20.
\]

By combining these inequalities with \([50]\), we obtain

\[
\|L_t^* A^* A L_t\|^{-1} L_t^* A^* e^t\|_F \leq \frac{3}{4\sqrt{5}} \|X^0 - \bar{X}\| \leq \frac{3}{16} \sigma_r(\bar{X}).
\]

Thus, from \([55]\) we have

\[
\max\{\|B^{t+1} - \hat{B}^t\|, \|D_1^{t+1} - \hat{D}_1^t\|, \|D_2^{t+1} - \hat{D}_2^t\|\} \leq \frac{3}{16} \sigma_r(\bar{X}), \tag{56}
\]

and \( \sigma_r(B^{t+1}) \geq \frac{9}{16} \sigma_r(\bar{X}) > 0 \) because of \([54]\). This shows the invertibility of \( B^{t+1} \).

With the invertibility of \( B^{t+1} \), we also introduce \( \rho_{t+1} \) to measure the goodness of the current iterate in the following way

\[
\rho_{t+1} = \max\{\|D_1^{t+1}(B^{t+1})^{-1}\|, \|B^{t+1} - D_2^{t+1}U^T\|\}. \tag{57}
\]

Notice \( \hat{D}_1^t \hat{B}^t = U_1^T X V^t (U_1^T X V^t)^{-1} = U_1^T \bar{U}(U_1^T \bar{U})^{-1} \), so

\[
\|\hat{D}_1^t \hat{B}^t\| \leq \|U_1^T \bar{U}\| \|U_1^T \bar{U}\|^{-1} \| \sin \Theta(U_1^T \bar{U})\| \leq \frac{\| \sin \Theta(U_1^T \bar{U})\|}{\sqrt{1 - \| \sin \Theta(U_1^T \bar{U})\|^2}} \leq \frac{\theta_t}{\sqrt{1 - \theta_t^2}}, \tag{58}
\]

where (a) is due to the \( \sin \Theta \) property in Lemma 1 of \[Cai and Zhang (2018)\] and (b) is due to \([51]\).

The same bound also holds for \( \|\hat{B}^t\|^{-1} \hat{D}_2^t \| \|. \) Meanwhile, it holds that

\[
\|D_1^{t+1}(B^{t+1})^{-1}\| \leq \|D_1^{t+1} \hat{D}_1^t (B^{t+1})^{-1}\| + \|\hat{D}_1^t (B^{t+1})^{-1}\|
\leq \|D_1^{t+1} \hat{D}_1^t\| \sigma_r(B^{t+1})^{-1} + \|\hat{D}_1^t (B^{t+1})^{-1}\| (a) \leq \frac{\|D_1^{t+1} \hat{D}_1^t\|}{\sigma_r(B^{t+1})^{-1}} + \|\hat{D}_1^t (B^{t+1})^{-1}\| = \frac{\theta_t}{\sqrt{1 - \theta_t^2}} + \frac{\theta_t}{\sigma_r(B^{t+1})^{-1}} \|B^{t+1} - \hat{B}^t\| \|B^{t+1} - \hat{B}^t\| (b) \leq \frac{\|D_1^{t+1} \hat{D}_1^t\|}{\sigma_r(B^{t+1})^{-1}} + \frac{\theta_t}{\sqrt{1 - \theta_t^2}} + \frac{\theta_t}{\sqrt{1 - \theta_t^2}} \|B^{t+1} - \hat{B}^t\| \sigma_r(B^{t+1})^{-1},
\]

where (a) is because \( (B^{t+1})^{-1} = (\hat{B}^t)^{-1} - (\hat{B}^t)^{-1}(B^{t+1} - \hat{B}^t)(B^{t+1})^{-1} \) and (b) is due to the bound for \( \|\hat{D}_1^t (B^{t+1})^{-1}\| \) in \([58]\). We can also bound \( \|B^{t+1} - D_2^{t+1}U^T\| \) in a similar way.
Thus, by plugging $\sigma_r(B^{t+1}) \geq \frac{9}{16} \sigma_r(\tilde{X})$, $\theta_t \leq 1/2$ and the upper bound in (56) into (59), we have
\[
\rho_{t+1} \leq \frac{1}{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} \leq \frac{4 + \sqrt{3}}{3\sqrt{3}}.
\]

**Step 4.** In the last step, we show the contraction inequality (52). Note from Lemma 9 that $\theta_t$ decreases as $\|X^t - \tilde{X}\|_F$ decreases. So after we show the contraction of $\|X^{t+1} - \tilde{X}\|_F$, it automatically implies that $\theta_{t+1} \leq 1/2$, which, together with similar arguments in [53], further guarantees the invertibility of $B^{t+1}$.

Since $\tilde{X}$ is a rank $r$ matrix and $\tilde{B}^t, B^{t+1}$ are invertible, a quick calculation asserts $\tilde{D}_1^T(\tilde{B}^t)^{-1}\tilde{D}_2^T = U_1^T \tilde{X} V_1^T$. Therefore,
\[
\tilde{X} = [U^T \ U_1^T][U^T \ U_1^T]^T \tilde{X} [V^T \ V_1^T][V^T \ V_1^T]^T = [U^T \ U_1^T] \begin{bmatrix} B^{t+1} & D^{t+1}_1 \ \bar{D}^{t+1}_1 & \tilde{D}_1 \ \tilde{D}_1^T(\tilde{B}^t)^{-1}\tilde{D}_2^T \end{bmatrix} [V^T \ V_1^T]^T.
\]

At the same time, it is easy to check
\[
X^{t+1} = X_{U_1}^{t+1}(B^{t+1})^{-1}X_{V}^{t+1} = [U^T \ U_1^T] \begin{bmatrix} B^{t+1} & D^{t+1}_1 \ \bar{D}^{t+1}_1 & \tilde{D}_1 \ \tilde{D}_1^T(\tilde{B}^t)^{-1}\tilde{D}_2^T \end{bmatrix} [V^T \ V_1^T]^T.
\]

So
\[
\|X^{t+1} - \tilde{X}\|_F^2 = \begin{bmatrix} B^{t+1} - \tilde{B}^t & D^{t+1}_2 - \tilde{D}_2^T \end{bmatrix} \begin{bmatrix} B^{t+1} & D^{t+1}_1 \ \tilde{D}_1 & \tilde{D}_1^T(\tilde{B}^t)^{-1}\tilde{D}_2^T \end{bmatrix}^T \|V^T \ V_1^T\|^2_F,
\]

where $\Delta^{t+1} = D^{t+1}_1(B^{t+1})^{-1}D^{t+1}_2 - \tilde{D}_1(\tilde{B}^t)^{-1}\tilde{D}_2^T$. Recall that (12) gives a precise error characterization for $\|B^{t+1} - \tilde{B}^t\|^2_F + \sum_{k=1}^{q} \|D^k - \tilde{D}_k\|^2_F$. Hence, to bound $\|X^{t+1} - \tilde{X}\|_F^2$ from (60), we only need to obtain an upper bound of $\|\Delta^{t+1}\|_F^2$.

Combining (57), (58) and Lemma 3, we have
\[
\|\Delta^{t+1}\|_F \leq \rho_{t+1}\|D^{t+1}_1 - \tilde{D}_1\|_F + \frac{\theta_t}{\sqrt{1 - \theta_t^2}}\|D^{t+1}_2 - \tilde{D}_2\|_F + \frac{\theta_t\rho_{t+1}}{\sqrt{1 - \theta_t^2}}\|B^{t+1} - \tilde{B}^t\|_F.
\]

Moreover, since $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and (12), it holds that
\[
\|\Delta^{t+1}\|_F^2 \leq 3(\rho_{t+1} \sqrt{1 - \theta_t^2} \frac{\theta_t}{\sqrt{1 - \theta_t^2}} + \theta_t\rho_{t+1})^2 \|L^*_t A^* \! A L_t\|^{-1} \|L^*_t A^* \! A^t \! e^t\|_F^2.
\]

In summary, combining (60), (12) and (61), we get
\[
\|X^{t+1} - \tilde{X}\|_F^2 \leq (1 + 3(\rho_{t+1} \sqrt{1 - \theta_t^2} + \theta_t\rho_{t+1}))^2 \|L^*_t A^* \! A L_t\|^{-1} \|L^*_t A^* \! A^t \! e^t\|_F^2.
\]

Plugging in the upper bounds for $\theta_t$ and $\rho_{t+1}$, we have $1 + 3(\rho_{t+1} \sqrt{1 - \theta_t^2} + \theta_t\rho_{t+1})^2 \leq 5$, and thus
\[
\|X^{t+1} - \tilde{X}\|_F^2 \leq 5 \|L^*_t A^* \! A L_t\|^{-1} \|L^*_t A^* \! A^t \! e^t\|_F^2
\]
\[
\leq \frac{5}{(1 - R_{23})^2 \sigma^2_\delta(X)} \left( R_{32}^2 \|X^t - \tilde{X}\|_F^2 + 4\|A^*(\tilde{e})\|_F^2 + 4R_{32}\|A^*(\tilde{e})\|_F \|X^t - \tilde{X}\|_F \right)
\]
\[
\leq \frac{9}{16} \|X^t - \tilde{X}\|_F^2,
\]
where (a) is due to (50) and (b) is because under the initialization condition (19) and \( \|A^*(\epsilon)\|_F \leq \frac{1-R_{2r}}{4\sqrt{5}} \|\epsilon\|_F \), the following inequalities hold

\[
\frac{5R_{2r}^2\|X^t - \bar{X}\|_F^2}{(1 - R_{2r})^2\sigma_r^2(X)} \leq 1/16, \quad \frac{20\|A^*(\epsilon)\|_F^2}{(1 - R_{2r})^2\sigma_r^2(X)} \leq 1/4, \quad \frac{20R_{3r}\|X^t - \bar{X}\|_F\|A^*(\epsilon)\|_F}{(1 - R_{2r})^2\sigma_r^2(X)} \leq 1/4.
\]

This completes the induction and finishes the proof of this theorem. \( \blacksquare \)

**Proof of Theorem 2** In view of the Riemannian Gauss-Newton equation in (31), to prove the claim, we only need to show

\[
P_{T_{X^t}}(A^*(A(\eta^t + X^t) - y)) = 0.
\]

From the optimality condition of the least squares problem (10), we know that \( B^{t+1}, D_1^{t+1} \) and \( D_2^{t+1} \) obtained in (5) satisfy

\[
\mathcal{L}_t^* A^*(A\mathcal{L}_t \begin{bmatrix} B^{t+1} & (D_2^{t+1})^\top \\ D_1^{t+1} & 0 \end{bmatrix} - y) = 0.
\]

Then, the updating formula in (25), together with the definition of \( \mathcal{L}_t \), implies that

\[
\eta^t + X^t = \mathcal{L}_t \begin{bmatrix} B^{t+1} & (D_2^{t+1})^\top \\ D_1^{t+1} & 0 \end{bmatrix}.
\]

Hence, (64) implies \( \mathcal{L}_t^* A^*(A(\eta^t + X^t) - y) = 0 \). Note from the proof of Theorem 1 that for all \( t \geq 1 \), \( B^t \) is invertible. Then, it is not difficult to verify that \( U^t, V^t \) are orthonormal bases of the column and row spans of \( X^t \) and \( P_{T_{X^t}} = \mathcal{L}_t \mathcal{L}_t^* \) for all \( t \geq 0 \). We thus proved (63). \( \blacksquare \)

**Proof of Proposition 3** We compute the inner product between the update direction \( \eta^t \) in (25) and the Riemannian gradient:

\[
\langle \text{grad}_t f(X^t), \eta^t \rangle = \langle P_{T_{X^t}} A^*(A(X^t) - y), \eta^t \rangle
\]

\[
\overset{(a)}{=} \langle -P_{T_{X^t}} A^* A \eta^t, \eta^t \rangle
\]

\[
\overset{(b)}{=} -\langle A^* A \eta^t, \eta^t \rangle = -\|A(\eta^t)\|_2^2,
\]

here (a) is due to (63) and (b) is because \( \eta^t \) lies in \( T_{X^t} M_r \). With this, we conclude the update direction \( \eta^t \) has a negative inner product with the Riemannian gradient unless it is 0. Thus the update \( \eta^t \) is a descent direction.

If \( A \) satisfies the 2r-RIP, by similar arguments as in Lemma 2, we see that \( P_{T_{X^t}} A^* A P_{T_{X^t}} \) is symmetric positive definite over \( T_{X^t} M_r \) for all \( t \geq 0 \). Since \( \eta^t \) solves the Riemannian Gauss-Newton equation (31), we know that \( \eta^t = -(P_{T_{X^t}} A^* A P_{T_{X^t}})^{-1} \text{grad}_t f(X^t) \) for all \( t \geq 0 \). For any subsequence \( \{X^t\}_{t \in \mathcal{K}} \) that converges to a nonstationary point \( \tilde{X} \), it is not difficult to show that

\[
\lim_{t \to \infty, t \in \mathcal{K}} P_{T_{X^t}} A^* A P_{T_{X^t}} = P_{T_{\tilde{X}}} A^* A P_{T_{\tilde{X}}} \quad \text{and} \quad \tilde{\eta} = \lim_{t \to \infty, t \in \mathcal{K}} \eta^t = -(P_{T_{\tilde{X}}} A^* A P_{T_{\tilde{X}}})^{-1} \text{grad}_t f(\tilde{X}).
\]

Hence, \( \{\eta^t\}_{t \in \mathcal{K}} \) is bounded, \( \tilde{\eta} \neq 0 \) and

\[
\lim_{t \to \infty, t \in \mathcal{K}} \langle \text{grad}_t f(X^t), \eta^t \rangle = \langle \text{grad}_t f(\tilde{X}), \tilde{\eta} \rangle = -\|A(\tilde{\eta})\|_2^2 \overset{(\text{RIP condition})}{\leq} -(1 - R_{2r})\|\tilde{\eta}\|_2^2 < 0,
\]

i.e., the direction sequence \( \{\eta^t\} \) is gradient related by (Absil et al. 2008 Definition 4.2.1). \( \blacksquare \)
Proof of Theorem 3 The proof of (37) shares many similar ideas to the proof of Theorem 1. Hence, we point out the main difference first and then give the complete proof. Compared to Theorem 1, where the target matrix is a stationary point, here the target matrix is \(X^*_{\text{max}(r)}\). So when we apply Proposition 1, \(X = X^*_{\text{max}(r)}, \varepsilon = \tilde{\varepsilon}\) and we no longer have (49). Due to this difference, here we have an unavoidable statistical error term in the upper bound.

We begin by proving (37). We first apply Proposition 1 to bound \(\|L^*_t A^* A L_t^{-1} L^*_t A^* \epsilon^t\|_F^2\) in this setting. Set \(X = X^*_{\text{max}(r)}\) and \(\varepsilon = \tilde{\varepsilon}\), by Proposition 1 we have

\[
\frac{\|L^*_t A^* A L_t^{-1} L^*_t A^* \epsilon^t\|_F^2}{\|X^t - X^*\|_F^2} \leq \frac{2 R_{3r}^2 \|X^t - X^*\|_F^2 \|X^t - X^*\|_F^2}{(1 - R_{2r})^2 \sigma_r^2(X^*)} + \frac{\|L^*_t A^* (\tilde{\varepsilon})\|_F^2}{(1 - R_{2r})^2} + \frac{\|L^*_t A^* (\tilde{\varepsilon})\|_F^2}{\sigma_r(X^*) (1 - R_{2r})^2}
\]

(66)

where (a) is by Cauchy-Schwarz inequality. Recall \(P_{X^1}\) in (43). \(P_{X^1}(A^* (\tilde{\varepsilon}))\) is a at most rank 2r matrix and \(\sigma_i(P_{X^1} A^* (\tilde{\varepsilon})) \leq \sigma_i(A^* (\tilde{\varepsilon}))\) for \(1 \leq i \leq p_1 \land p_2\) by the projection property of \(P_{X^1}\). Then we have

\[
\frac{\|L^*_t A^* (\tilde{\varepsilon})\|_F^2}{\sigma_r(A^*)^2} = \frac{\|P_{X^1} A^* (\tilde{\varepsilon})\|_F^2}{(A^* (\tilde{\varepsilon}))_{\text{max}(2r)}^2} \leq \|P_{X^1} A^* (\tilde{\varepsilon})\|_F^2 \leq 2 \|P_{X^1} A^* (\tilde{\varepsilon})\|_F^2 \leq 2 \|P_{X^1} A^* (\tilde{\varepsilon})\|_F^2.
\]

(67)

Recall in this setting, the target matrix is \(X^*_{\text{max}(r)}\). We replace \(X^*_{\text{max}(r)}\) by \(X^*_{\text{max}(r)}\) and obtain \(X^*_{\text{max}(r)} := U^T X^*_{\text{max}(r)} V^t, \tilde{D}_1 := U^T X^*_{\text{max}(r)} V^t, \tilde{D}_2 := U^T X^*_{\text{max}(r)} V^t\). Next, we use induction to prove the main results. Define \(\theta_t, \rho_{t+1}\) in the same way as in the proof of Theorem 1. We aim to show: given \(\theta_t \leq 1/2\), \(\tilde{B}^t\) is invertible, \(\|X^t - X^*_{\text{max}(r)}\|_F \leq \|X^t - X^*_{\text{max}(r)}\|_F \lor \frac{2 \sqrt{10}}{(1 - R_{2r})^2} \|A^* (\tilde{\varepsilon})\|_F\), then \(\theta_{t+1} \leq 1/2\), \(\tilde{B}^{t+1}\) is invertible, \(\|X^{t+1} - X^*_{\text{max}(r)}\|_F \leq \|X^t - X^*_{\text{max}(r)}\|_F \lor \frac{2 \sqrt{10}}{(1 - R_{2r})^2} \|A^* (\tilde{\varepsilon})\|_F\), as well as \(B^{t+1}\) is invertible and (37).

First, we can easily check the assumption is true when \(t = 0\) under the initialization condition. Now, suppose the induction assumption is true at iteration \(t\). Under the conditions (35), (36), from (66) and (67), we have

\[
\|L^*_t A^* A L_t^{-1} L^*_t A^* \epsilon^t\|_F^2 \leq \frac{3}{16} \sigma_r(X^*)^2.
\]

Then following the same proof as the Step 3,4 of Theorem 1, we have \(B^{t+1}\) is invertible, \(\rho_{t+1} \leq (4 + \sqrt{3})/3\sqrt{3}\) and

\[
\|X^{t+1} - X^*_{\text{max}(r)}\|_F^2 \leq (1 + 3(\rho_{t+1} \lor \sqrt{1 - \theta_t})^2) \|L^*_t A^* A L_t^{-1} L^*_t A^* \epsilon^t\|_F^2 \leq 5 \|L^*_t A^* A L_t^{-1} L^*_t A^* \epsilon^t\|_F^2.
\]

(68)

Plugging (67) and (66) into (68), we arrive at

\[
\|X^{t+1} - X^*_{\text{max}(r)}\|_F^2 \leq \frac{R_{3r}^2 \|X^t - X^*_{\text{max}(r)}\|_F^2 \|X^t - X^*_{\text{max}(r)}\|_F^2}{(1 - R_{2r})^2 \sigma_r^2(X^*)} + \frac{20 \|A^* (\tilde{\varepsilon})\|_F^2}{(1 - R_{2r})^2}
\]

\[
\leq \frac{1}{2} \|X^t - X^*_{\text{max}(r)}\|_F^2 + \frac{20 \|A^* (\tilde{\varepsilon})\|_F^2}{(1 - R_{2r})^2},
\]

(69)

where (a) is because under conditions (35), (36) and induction assumption at iteration \(t\), it holds that

\[
\frac{10 R_{3r}^2 \|X^t - X^*_{\text{max}(r)}\|_F^2}{(1 - R_{2r})^2 \sigma_r^2(X^*)} \leq 1/2.
\]

42
By (69), we get $\|X^{t+1} - X^*_{\text{max}(r)}\|_F \leq \|X^0 - X^*_{\text{max}(r)}\|_F \vee \frac{2\sqrt{10}}{(1 - R_{2r})} \| (A^*(\check{e}))_{\text{max}(r)} \|_F$. Under the initialization conditions, Lemma 9 also implies $\theta_{t+1} \leq \frac{1}{2}$ and $B_{t+1}$ is invertible. This finishes the proof of (37).

Next, we prove the guarantee of RISRO under the sub-Gaussian ensemble design with spectral initialization. Throughout the proof, we use various $c, C, C_1$ to denote constants and they may vary from line to line. Recall now $X^*$ is a rank $r$ matrix, so we have $\check{e} = e$. First we give the guarantee for the initialization $X^0 = (A^*(y))_{\text{max}(r)}$. Define $Q_0 \in \mathbb{R}^{p_1 \times 2r}$ as an orthogonal matrix which spans the column subspaces of $X^0$ and $X^*$. Let $Q_{0\perp}$ be the orthogonal complement of $Q_0$. Since

$$\|X^0 - A^*(y)\|_F^2 = \|X^0 - P_{Q_0}(A^*(y))\|_F^2 + \|P_{Q_{0\perp}}(A^*(y))\|_F^2$$

and

$$\|X^* - A^*(y)\|_F^2 = \|X^* - P_{Q_0}(A^*(y))\|_F^2 + \|P_{Q_{0\perp}}(A^*(y))\|_F^2,$$

the SVD property $\|X^0 - A^*(y)\|_F^2 \leq \|X^* - A^*(y)\|_F^2$ implies that

$$\|X^0 - P_{Q_0}(A^*(y))\|_F^2 \leq \|X^* - P_{Q_0}(A^*(y))\|_F^2.$$

Note that

$$\|P_{Q_0} - P_{Q_0}A^*AP_{Q_0}\| \overset{(a)}{=} \sup_{\|Z\|_F \leq 1} \langle (P_{Q_0} - P_{Q_0}A^*AP_{Q_0})Z, Z \rangle$$

$$= \sup_{\|Z\|_F \leq 1} \| P_{Q_0}Z \|_F^2 - \|AP_{Q_0}Z\|_F^2 \overset{(b)}{\leq} \sup_{\|Z\|_F \leq 1} R_{2r} \| P_{Q_0}Z \|_F^2 \leq R_{2r},$$

where (a) is because $P_{Q_0} - P_{Q_0}A^*AP_{Q_0}$ is symmetric and (b) is by the $2r$-RIP of $A$. Hence,

$$\|X^0 - X^*\|_F \leq \|X^0 - P_{Q_0}(A^*(y))\|_F + \|X^* - P_{Q_0}(A^*(y))\|_F$$

$$\leq 2 \|X^* - P_{Q_0}(A^*(y))\|_F \overset{(a)}{=} 2 \|X^* - P_{Q_0}(A^*(A^*(X^*) + e))\|_F$$

$$= 2 \|P_{Q_0}X^* - P_{Q_0}A^*(P_{Q_0}X^*) - P_{Q_0}(A^*(e))\|_F$$

$$\leq 2 (\|P_{Q_0} - P_{Q_0}A^*AP_{Q_0}\| \|X^*\|_F + \|P_{Q_0}(A^*(e))\|_F)$$

$$\overset{(b)}{\leq} 2R_{2r} \|X^*\|_F + 2\sqrt{2} \|(A^*(\check{e}))_{\text{max}(r)}\|_F$$

$$\leq 2R_{2r} \|X^*\|_F + 2\sqrt{2} \| (A^*(\check{e}))_{\text{max}(r)} \|_F,$$

where (a) is due to the model of $y$ and (b) is due to that $P_{Q_0}(A^*(e))$ is a at most rank $2r$ matrix and the spectral norm bound for the operator $(P_{Q_0} - P_{Q_0}A^*AP_{Q_0})$ in (70). Hence, there exists $c_1, c_2, C > 0$ such that when

$$R_{2r} \leq c_1 \frac{1}{\kappa \sqrt{r}}, R_{3r} \leq \frac{1}{2}, \text{ and } \sigma_r(X^*) \geq C \|(A^*(e))_{\text{max}(r)}\|_F,$$

we have $\|X^0 - X^*\|_F \leq c_2 \sigma_r(X^*)$ by (71) and the conditions in (35) and (36) are satisfied.

Next, we show under the sample complexity indicated in the Theorem, (72) are satisfied with high probability. First by a similar argument of (Zhang et al, 2020, Lemma 6), for the sub-Gaussian
ensemble design considered here, we have with probability at least \(1 - \exp(-c(p_1 + p_2))\) for some \(c > 0\) that \(\|A^*(\epsilon)\| \leq c' \sqrt{\frac{p_1 + p_2}{n}} \sigma\). So with the same high probability, we have

\[
\|(A^*(\epsilon))^{\text{max}(r)}\|_F \leq c' \sqrt{\frac{(p_1 + p_2)^r}{n}} \sigma,
\]

and when \(n \geq C(p_1 + p_2)^r \sigma^2 \sigma^2(X^*)\), we have \(\sigma_r(X^*) \geq C\|(A^*(\epsilon))^{\text{max}(r)}\|_F\). At the same time, by [Candes and Plan 2011, Theorem 2.3], there exists \(C > 0\) when \(n \geq C(p_1 + p_2)^r \sigma^2 \sigma^2(X^*)\) and \(R_2r < \frac{1}{2}\) are satisfied with probability at least \(1 - \exp(-c(p_1 + p_2))\) for some \(c > 0\).

In summary, there exists \(C > 0\) such that when \(n \geq C(p_1 + p_2)^r \sigma^2 \sigma^2(X^*)\), \((73)\) holds with probability at least \(1 - \exp(-c(p_1 + p_2))\) for some \(c > 0\). So by the first part of the Theorem, we have with the same high probability:

\[
\|X^{t+1} - X^*\|_F^2 \leq 10 \frac{R^2}{(1 - R_2r)^2} \frac{X^t - X^*}{F}^4 + \frac{20\|(A^*(\epsilon))^{\text{max}(r)}\|_F^2}{(1 - R_2r)^2}, \quad \forall t \geq 0.
\]

More specifically, the above convergence can be divided into two phases. Let

- (Phase I) When \(\|X^t - X^*\|_F^2 \geq \frac{\sigma^2}{R^2} \|(A^*(\epsilon))^{\text{max}(r)}\|_F \sigma_r(X^*)\),

\[
\|X^{t+1} - X^*\|_F \leq 2\sqrt{\frac{R^2}{(1 - R_2r)} \frac{X^t - X^*}{F}^2},
\]

- (Phase II) When \(\|X^t - X^*\|_F^2 \leq \frac{\sigma^2}{R^2} \|(A^*(\epsilon))^{\text{max}(r)}\|_F \sigma_r(X^*)\),

\[
\|X^{t+1} - X^*\|_F \leq \frac{2\sqrt{\|X^0 - X^*\|_F^2}}{1 - R_2r}.
\]

Combining Phase I, II and \((73)\), by induction we have \(\|X^t - X^*\|_F \leq 2^{-2t} \|X^0 - X^*\|_F + c\sqrt{\frac{r(p_1 + p_2)}{n}} \sigma^2\), and this implies the desired error bound for \(\|X^t - X^*\|_F\) after double-logarithmic number of iterations.

**Proof of Theorem** [4] In the phase retrieval example, the mapping \(A\) no longer satisfies a proper RIP condition and the strategy we use is to show the contraction of \(X^t - X^*\) in terms of its nuclear norm and then transform it back to Frobenius norm.

We also use induction to show the main results. Specifically, we show: given \(|u^*^T u^t| > 0\) where \(u^* = \frac{x^*}{\|x^*\|_2}\) and \(\|X^t - X^*\|_F \leq \|X^0 - X^*\|_F\), then \(|u^*_t^T u^t| > 0\), \(\|X^{t+1} - X^*\|_F \leq \|X^0 - X^*\|_F\) and \((40)\).

First, the induction assumption is true when \(t = 0\) by the initialization condition and the perturbation bound in Lemma [3]. Assume it is also correct at iteration \(t\). Let \(\tilde{b}^t = u^*_t^T X^* u^t\), \(\tilde{d}^t = (u^*_t)^T X^* u^t\). It is easy to verify \(\tilde{d}^t (\tilde{b}^t)^{-1} \tilde{d}^t = u^*_t^T X^* u^t\) and \(X^* = [u^*_t^T u^t]\left[\begin{array}{c} \tilde{b}^t \\ \tilde{d}^t \end{array}\right] \left[\begin{array}{c} \tilde{d}^t \tilde{b}^t^{-1} \tilde{d}^t \\ u^*_t^T u^t \end{array}\right] = [u^*_t^T u^t]^T\). Define the linear operator \(L_t\) similar as \((9)\) in this setting in the following way

\[
L_t : W = \left[\begin{array}{c} w_0 \\ w_1 \end{array}\right] \in \mathbb{R}^{(p-1) \times 1} \quad \rightarrow \quad [u^*_t^T u^t]\left[\begin{array}{c} w_0 \\ w_1 \end{array}\right] = [u^*_t^T u^t]^T.
\]
and it is easy to compute its adjoint \( L_t^*(M) = \begin{bmatrix} u^T M u^T & u^T M u_\perp^T \\ u^T M u^T & 0 \end{bmatrix} \), where \( M \) is a rank 2 symmetric matrix. Define operator \( P_{X_t} \) similar as [43] over the space of \( p \times p \) symmetric matrices

\[
P_{X_t}(W) := L_t L_t^*(W) = u^T u^T W u^T + u_\perp u_\perp^T W u_\perp^T + u^T u^T W u_\perp^T u_\perp^T.
\]

It is easy to verify that \( P_{X_t} \) is an orthogonal projector. Meanwhile, let \( P_{(X_t)^\perp}(W) = W - P_{X_t}(W) = u^T u_\perp^T W u_\perp^T \).

By using the operator \( L_t \), the least squares solution in Step 4 can be rewritten in the following way

\[
\begin{bmatrix} b^{t+1} \\ d^{t+1} \end{bmatrix} = \arg\min_{b \in \mathbb{R}, d \in \mathbb{R}^{p-1}} \| y - A L_t \begin{bmatrix} b \\ d \end{bmatrix} \|_2^2
\]
\[
= (L_t^* A^* A L_t)^{-1} L_t^* A^* y
\]
\[
= (L_t^* A^* A L_t)^{-1} L_t^* A^* (A(P_{X_t} X^*) + A(P_{(X_t)^\perp} X^*))
\]
\[
= \begin{bmatrix} \tilde{b}^T \\ \tilde{d}^T \end{bmatrix} + (L_t^* A^* A L_t)^{-1} L_t^* A^* (P_{(X_t)^\perp} X^*)).
\]

Here, \( L_t^* A^* A L_t \) is invertible due to the lower bound of the spectrum of \( L_t^* A^* A L_t \) in Lemma 4. So

\[
\| X^{t+1} - X^* \|_* \leq \| X^{t+1} - [u^T u_\perp] \begin{bmatrix} b^{t+1} \\ d^{t+1} \end{bmatrix} [u^T u_\perp]^T \|_*
\]
\[
+ \| [u^T u_\perp] \begin{bmatrix} b^{t+1} \\ d^{t+1} \end{bmatrix} [u^T u_\perp]^T - [u^T u_\perp] \begin{bmatrix} \tilde{b}^T \\ \tilde{d}^T \end{bmatrix} + (\tilde{b}^T - \tilde{d}^T) \|_*
\]
\[
\overset{(a)}{\leq} 2 \| [u^T u_\perp] \begin{bmatrix} b^{t+1} \\ d^{t+1} \end{bmatrix} [u^T u_\perp]^T - [u^T u_\perp] \begin{bmatrix} \tilde{b}^T \\ \tilde{d}^T \end{bmatrix} + (\tilde{b}^T - \tilde{d}^T) \|_*
\]
\[
\leq 2 \| \tilde{b}^{t+1} - \tilde{b}^t - \tilde{d}^{t+1} - \tilde{d}^t \|_*
\]
\[
\overset{(b)}{=} 2 \| (L_t^* A^* A L_t)^{-1} L_t^* A^* (P_{(X_t)^\perp} X^*) \|_* + 2 \| \tilde{d}^t (\tilde{b}^t)^{-1} \tilde{d}^t \|_1,
\]

(76)

here (a) is due to Lemma 4 and (b) is due to (75).

First notice \( \| \tilde{d}^t (\tilde{b}^t)^{-1} \tilde{d}^t \|_1 = \| P_{(X_t)^\perp} X^* \|_* \). Next we give bound for \( \| (L_t^* A^* A L_t)^{-1} L_t^* A^* (P_{(X_t)^\perp} X^*) \|_* \).

With probability at least \( 1 - C_1 \exp(-C_2 (\delta_1, \delta_2) p) - C_3 n^{-p} (C_1, C_2, C_3 > 0) \), we have

\[
\| (L_t^* A^* A L_t)^{-1} L_t^* A^* (P_{(X_t)^\perp} X^*) \|_*
\]
\[
\overset{(a)}{\leq} \frac{4}{(1 - \delta_1)n} \| L_t^* A^* A P_{(X_t)^\perp} (X^*) \|_*
\]
\[
\overset{(b)}{\leq} C \frac{4p}{(1 - \delta_1)n} \| A P_{(X_t)^\perp} (X^*) \|_1
\]
\[
\overset{(c)}{\leq} C' \frac{1 + \delta_2}{1 - \delta_1} n \| P_{(X_t)^\perp} (X^*) \|_*
\]

for some \( C, C' > 0 \). Here \( \| \cdot \|_1 \) denotes the \( \ell_1 \) norm of a vector, (a) is due to Lemma 5, (b) is due to Lemma 6 and (c) is due to [Candes et al., 2013] Lemma 3.1 and \( P_{(X_t)^\perp} (X^*) \) is a symmetric matrix.
Putting above results into (76), we have with probability at least $1 - C_1 \exp(-C_2(\delta_1, \delta_2)p) - C_3n^{-p}$,

$$\|X^{t+1} - X^*\|_F \leq \|X^t - X^*\|_F \leq C \frac{1 + \delta_2}{1 - \delta_1} p \|P_{X^t}^\bot(X^*)\|_F \leq C \frac{1 + \delta_2}{1 - \delta_1} p \|P_{X^t}^\bot(X^*)\|_F \leq C \frac{1 + \delta_2}{1 - \delta_1} \sigma_1(X^*),$$

where (a) is because $P_{X^t}^\bot(X^*)$ is a symmetric rank 1 matrix, (b) is due to the same argument as (47). Since $\sigma_1(X^*) = \|X^*\|_F$, when $\|X^0 - X^*\|_F \leq \frac{(1 - \delta_1)}{C(1 + \delta_2)p} \|X^*\|_F$ for some large enough $C$, we have $\|X^{t+1} - X^*\|_F \leq \|X^t - X^*\|_F \leq \|X^0 - X^*\|_F$. Also $\|u^{t+1}\| > 0$ as it is a non-decreasing function of $\|X^{t+1} - X^*\|_F$ by Lemma 9. This finishes the induction and the proof. ■

**Proof of Proposition 4.** First, under the assumptions assumed in the proposition and (Ma et al., 2019) Theorem 1), we have

$$\min(\|\hat{x}^t - x^*\|_2, \|\hat{x}^t + x^*\|_2) \leq (1 - \eta)\|x^*\|_2^2 / 2\|x^*\|_2, \quad \forall t \geq 0,$$

holds for some $\epsilon \in (0, 1)$ with probability at least $1 - c_3np^{-5}$. So when $T_0 \geq c_2 \log p \cdot \log(\|x^*\|_2p)$, we have

$$\|\hat{x}^T_0\hat{x}^T_0 - X^*\|_F \leq \frac{9}{4} \|x^*\|_F \min(\|\hat{x}^T_0 - x^*\|_2, \|\hat{x}^T_0 + x^*\|_2) \leq c\|x^*\|_F / p.$$

Here (a) is by (Tu et al., 2016) Lemma 5.3). This finishes the proof of initialization and the rest of the proof follows from Theorem 4. ■

## C Additional Proofs and Technical Lemmas

We collect the additional proofs and technical lemmas that support the main technical results in this section.

**Proof of Equation (16).** First, we denote

$$U^t = [u_1, \ldots, u_{p_1}]^T, \ V^t = [v_1, \ldots, v_{p_2}]^T, \ M = [m_1, \ldots, m_{p_1}]^T, \ N = [n_1, \ldots, n_{p_2}]^T.$$

Then

$$\arg\min_{M \in \mathbb{R}^{p_1 \times r}, N \in \mathbb{R}^{p_2 \times r}} \sum_{(i,j) \in \Omega} \left\{ (U^T N^T + M V^T)^T - X_{i,j} \right\}^2 = \arg\min_{M \in \mathbb{R}^{p_1 \times r}, N \in \mathbb{R}^{p_2 \times r}} \sum_{(i,j) \in \Omega} (u_i^T n_j + m_j^T v_j - X_{i,j})^2. \quad (77)$$

If $(i,j) \in \Omega$, then the corresponding design matrix $A^{ij}$ has 1 at location $(i,j)$ and 0 at the rest of locations. Then

$$A^{ij} V^t = [0, \ldots, \frac{i^{th}}{v_j}, \ldots, 0]^T, \quad A^{ij} U^t = [0, \ldots, \frac{j^{th}}{u_i}, \ldots, 0]^T.$$

So on the sketching perspective of R2RILS, we have

$$\arg\min_{M \in \mathbb{R}^{p_1 \times r}, N \in \mathbb{R}^{p_2 \times r}} \sum_{(i,j) \in \Omega} \left( \langle U^T A^{ij}, N^T \rangle + \langle M, A^{ij} V^t \rangle - X_{i,j} \right)^2$$

$$= \arg\min_{M \in \mathbb{R}^{p_1 \times r}, N \in \mathbb{R}^{p_2 \times r}} \sum_{(i,j) \in \Omega} \left( \langle N, [0, \ldots, \frac{i^{th}}{u_i}, \ldots, 0]^T \rangle + \langle M, [0, \ldots, \frac{j^{th}}{v_j}, \ldots, 0]^T \rangle - X_{ij} \right)^2$$

$$= \arg\min_{M \in \mathbb{R}^{p_1 \times r}, N \in \mathbb{R}^{p_2 \times r}} \sum_{(i,j) \in \Omega} (u_i^T n_j + m_j^T v_j - X_{i,j})^2,$$
which is exactly the same as (77) and this finishes the proof. ■

**Proof of Lemma 3** The proof is the same as the proof of Proposition 2.3 [Vandereycken (2013)], except here we need to replace the gradient in the matrix completion setting to the gradient $\mathcal{A}^*(\mathcal{A}(X) - y)$ in our setting. ■

**Lemma 4 (Projection onto the Positive Semidefinite Cone in the Nuclear norm)** Given any symmetric matrix $A \in \mathbb{R}^{p \times p}$, and denotes its eigenvalue decomposition as $\sum_{i=1}^{p} \lambda_i v_i v_i^T$ with $\lambda_1 \geq \cdots \geq \lambda_p$. Let $A_0 = \sum_{i=1}^{p}(\lambda_i \wedge 0)v_i v_i^T$, then

$$A_0 = \arg\min_{X \in S^+_p} \|A - X\|_*,$$

here $S^+_p$ is the set of $p \times p$ positive semidefinite (PSD) matrices.

**Proof of Lemma 4** Here the main property we use is the variational representation of nuclear norm. Let $m = \max\{i : \lambda_i \geq 0\}$. For any PSD matrix $X$,

$$\|X - A\|_* \geq \sum_{i=1}^{m} \sigma_i(X - A) = \sup_{U \in \mathbb{C}^{p \times (p-m)}, V \in \mathbb{C}^{p \times (p-m)}} \text{tr}(U^T(X - A)V) \geq \sup_{U \in \mathbb{C}^{p \times (p-m)}} \text{tr}(U^T(X - A)U) \geq 0 - \inf_{U \in \mathbb{C}^{p \times (p-m)}} \text{tr}(U^T A U) \geq - (\sum_{i=m+1}^{p} \lambda_i).$$

On the other hand, $\|A_0 - A\|_* = - (\sum_{i=m+1}^{p} \lambda_i)$ and this finishes the proof. ■

**Lemma 5 (Bounds for spectrum of $\mathcal{L}^* \mathcal{A}^* \mathcal{A} \mathcal{C}$ in Phase Retrieval)** For any given unit vector $u \in \mathbb{R}^p$, define the linear map

$$\mathcal{L} : W = \left[ \begin{array}{c} w_0 \in \mathbb{R} \\ w_1 \in \mathbb{R}^{(p-1) \times 1} \end{array} \right] \rightarrow \left[ \begin{array}{c} u \{u_u\} \\ w_0 \ w_1 \end{array} \right] \left[ \begin{array}{c} u \{u_u\} \\ w_0 \ w_1 \end{array} \right]^T.$$

It is easy to compute $\mathcal{L}^*(M) = \left[ \begin{array}{cc} u^T Mu & u^T M u_\perp \\ u_\perp^T M u_\perp & 0 \end{array} \right]$, where $M \in \mathbb{R}^{p \times p}$ is a symmetric matrix.

Suppose $\{a_i\}_{i=1}^{n} \sim i.i.d. N(0, I_p)$. Then $\forall \delta \in (0, 1)$, $\exists C(\delta) > 0$ such that when $n \geq C(\delta) p \log p$, with probability at least $1 - c_1 \exp(-c_2(\delta)p) - c_3 n^{-p}$, we have for any $u$ and $M \in \text{Ran}(\mathcal{L}^*)$

$$\|\mathcal{L}^* \mathcal{A}^* \mathcal{A} \mathcal{C}(M)\|_F \geq \frac{1 - \delta}{2} n \|M\|_F. \quad (78)$$

where $\mathcal{A}$ is the linear map in (39) generated by $\{a_i\}_{i=1}^{n}$. Also for any $u$ and matrix $M \in \text{Ran}(\mathcal{L}^*)$, with the same high probability, we have

$$\| (\mathcal{L}^* \mathcal{A}^* \mathcal{A} \mathcal{C})^{-1}(M)\|_* \leq \frac{4}{(1 - \delta)n} \|M\|_* \quad (79)$$

**Proof of Lemma 5** Note that (78) is true when $M$ is a zero matrix. When $M$ is non-zero, $\mathcal{L}(M) = um^T + mu^T$ for some $m$. Then $\frac{1}{n}\|\mathcal{A}(M)\|_F^2 = \frac{1}{n}\sum_{i=1}^{n}|a_i|^2|a_i^T m|^2$ and $\|M\|_F^2 = \|\mathcal{L}(M)\|_F^2 = 2(|m^T u|^2 + |m|^2_2 |u|^2_2)$. For any $\mathcal{L}$ and $M$, with probability $1 - c_1 \exp(-c_2(\delta)p) - c_3 n^{-p}$, we have

$$\|\mathcal{L}^* \mathcal{A}^* \mathcal{A} \mathcal{C}(M)\|_F = \|\mathcal{A}(M)\|_F \|M\|_F \geq \frac{1 - \delta}{2} n \|\mathcal{L}(M)\|_F \|M\|_F = \frac{1 - \delta}{2} n \|\mathcal{L}(M)\|_F. \quad (80)$$
where (a) is due to the [Sun et al., 2018, Lemma 6.4].

Next, we prove (79). First suppose \( W \in \text{Ran}(\mathcal{L}^*) \) satisfies \( (\mathcal{L}^* A^* A \mathcal{L})(W) = M \), then we have

\[
\| (\mathcal{L}^* A^* A \mathcal{L})^{-1} M \|_* = \frac{\| W \|_*}{\| \mathcal{L}^* A^* A \mathcal{L}(W) \|_*}.
\]

Hence, to prove the desired result, we only need to obtain a lower bound of \( \| \mathcal{L}^* A^* A \mathcal{L}(W) \|_* \):

\[
\| \mathcal{L}^* A^* A \mathcal{L}(W) \|_* = \sup_{|z| \leq 1} \langle A \mathcal{L}(W), A \mathcal{L}(z) \rangle \geq \langle A \mathcal{L}(W), A \mathcal{L}( \frac{W}{\| W \|} ) \rangle
\]

\[
= \| A \mathcal{L}(W) \|_2^2 / \| W \| \geq (a) \frac{1 - \delta}{2} n \| A \mathcal{L}(W) \|_2^2 / \| W \| \geq (b) \frac{1 - \delta}{4} n \| W \|_*,
\]

where (a) holds for any \( \mathcal{L}, W \) with probability \( 1 - c_1 \exp(-c_2(\delta)p) - c_3 n^{-p} \) by the same reason as (a) in [80]; (b) is true because \( W \in \text{Ran}(\mathcal{L}^*) \) and \( \| \mathcal{L}(W) \|_F = \| W \|_F \geq \| W \|_* / \sqrt{2} \). □

**Lemma 6 (Upper Bound for \( \| \mathcal{L}^* A^*(z) \|_* \) in Phase Retrieval)** Consider the same linear operator \( \mathcal{L} \) as in Lemma 3. Suppose \( a_i \overset{i.i.d.}{\sim} N(0, I_p) \) and \( A \) is the linear map in (39) generated by \( \{a_i\}_{i=1}^n \). Then there exists \( C, c_1, c_2 > 0 \) such that when \( n \geq C p \), with probability at least \( 1 - c_1 \exp(-c_2 p) \), for any \( \mathcal{L} \) and \( z \), we have \( \| \mathcal{L}^* A^*(z) \|_* \leq c p \| z \|_1 \) for some \( c > 0 \), where \( \| z \|_1 \) denotes the \( \ell_1 \) norm of \( z \).

**Proof of Lemma 6** The proof is based on the concentration of sub-exponential random variables. First, for fixed \( \mathcal{L} \), we have

\[
\sup_{|z| \leq 1} \| \mathcal{L}^* A^*(z) \|_* = \sup_{|z| \leq 1} \sup_{w \in S} \langle \mathcal{L}^* A^*(z), W \rangle = \sup_{|z| \leq 1} \sup_{w \in S, \| w \| \leq 1} \langle z, A \mathcal{L}(W) \rangle = \sup_{w \in S, \| w \| \leq 1} \| A \mathcal{L}(w) \|_\infty,
\]

here \( S \) is the set of symmetric matrices and \( \| A \mathcal{L}(w) \|_\infty \) denotes the largest absolute value in the vector \( A \mathcal{L}(W) \).

Notice \( \mathcal{L}(W) \) is a symmetric rank-2 matrix with spectral norm bounded by 1, without loss of generality, we can consider the bound for \( \| A(M) \|_\infty \) for fixed rank 2 matrix \( M \) with eigenvalue decomposition \( u_1 u_1^T - t u_2 u_2^T \) and \( t \in [-1, 1] \). In this case \( \| A(M) \|_\infty = \max_i |a_i^T u_1|^2 - t |a_i^T u_2|^2 | \) and \( |a_i^T u_1|^2 - t |a_i^T u_2|^2 | \) is a subexponential random variable. By the concentration of subexponential random variable [Vershynin, 2010], we have

\[
P(\| |a_i^T u_1|^2 - t |a_i^T u_2|^2 | - \xi > x) \leq \exp(-c \min(x^2, x)),
\]

where \( \xi = E |a_i^T u_1|^2 - t |a_i^T u_2|^2 | \). And a union bound yields

\[
P(\max_i (|a_i^T u_1|^2 - t |a_i^T u_2|^2 | - \xi > x) \leq \eta \exp(-c \min(x^2, x)),
\]

Next, we use the \( \epsilon \)-net argument to extend the bound to hold for any symmetric rank 2 matrices \( M \) with spectral norm bounded by 1. Notice that by proving that, we also prove the desired inequality for any \( \mathcal{L} \). Let \( S_\epsilon \) be an \( \epsilon \)-net on the unit sphere, \( T_\epsilon \) be an \( \epsilon \) net on \( [-1, 1] \) and set

\[
N_\epsilon = \{ M = u_1 u_1^T - t u_2 u_2^T : (u_1, u_2, t) \in S_\epsilon \times S_\epsilon \times T_\epsilon \}.
\]

48
Since \(|S_\epsilon| \leq (3/\epsilon)^p\), we have \(|N_\epsilon| \leq (3/\epsilon)^{2p+1}\). A union bound yields
\[
P(\forall \epsilon \in N_\epsilon, \max_i \{||\mathbf{a}_i^T \mathbf{u}_1||^2 - t ||\mathbf{a}_i^T \mathbf{u}_2||^2 - \xi > x\} \leq n \exp(-c \min(x^2, x) + (2p + 1) \log(3/\epsilon)).
\] (84)

Now suppose \((\mathbf{u}_1^*, \mathbf{u}_2^*, t^*) = \arg \max_{\mathbf{u}_1, \mathbf{u}_2 \in [-1, 1]} ||\mathbf{A}(\mathbf{M})||_x\) and denote \(\mathbf{M}^* = \mathbf{u}_1^* \mathbf{u}_1^T - t^* \mathbf{u}_2^* \mathbf{u}_2^T\), \(\mu = ||\mathbf{A}(\mathbf{M}^*)||_x\). Then find the approximation \(\mathbf{M}_0 = \mathbf{u}_0 \mathbf{u}_0^T - t_0 \mathbf{v}_0 \mathbf{v}_0^T \in N_\epsilon\) such that \(||\mathbf{u}_0 - \mathbf{u}^*||_2, ||\mathbf{v}^* - \mathbf{v}_0||_2, ||t^* - t_0||\) are each at most \(\epsilon\). First notice
\[
||\mathbf{u}^* \mathbf{u}^T - \mathbf{u}_0 \mathbf{u}_0^T|| = \sup_{\mathbf{x}, ||\mathbf{x}||_2 = 1} ||\mathbf{u}^* \mathbf{x}||^2 - ||\mathbf{u}_0 \mathbf{x}||^2
\]
\[
= \sup_{\mathbf{x}, ||\mathbf{x}||_2 = 1} ||(\mathbf{u}_0 - \mathbf{u}^*) \mathbf{x}|| ||(\mathbf{u}_0 + \mathbf{u}^*) \mathbf{x}||
\]
\[
\leq ||\mathbf{u}^* - \mathbf{u}_0||_2 ||\mathbf{u}^* + \mathbf{u}_0||_2 \leq 2 ||\mathbf{u}^* - \mathbf{u}_0||_2 \leq 2 \epsilon.
\]
Using the above bound, we have
\[
||\mathbf{M}^* - \mathbf{M}_0|| \leq ||\mathbf{u}^* \mathbf{u}^T - \mathbf{u}_0 \mathbf{u}_0^T|| + ||t^* - t_0|| ||\mathbf{v}^* \mathbf{v}^T - \mathbf{v}_0 \mathbf{v}_0^T|| \leq 5 \epsilon.
\] (85)
Now take \(x \geq cp\) for some \(c > 0\), then from (84), we have the event \(\{||\mathbf{A}(\mathbf{M}_0)||_x \leq cp\}\) happens with probability at least \(1 - \exp(-CP)\). And on this event, we have
\[
\mu \leq ||\mathbf{A}(\mathbf{M}^* - \mathbf{M}_0)||_x + ||\mathbf{A}(\mathbf{M}_0)||_x \overset{(a)}{\leq} 5 \epsilon \cdot 2 \mu + cp \implies \mu \leq \frac{cp}{(1 - 10 \epsilon)},
\]
where (a) is by triangle inequality and the fact that \(\mathbf{M}^* - \mathbf{M}_0\) can be decomposed into the sum of two rank 2 symmetric matrices with spectral norm bounded by \(5 \epsilon\) \(\Box\). Take \(\epsilon < 1/10\), we get \(||\mathbf{A}(\mathbf{M}^*)||_x < cp\) for some \(c > 0\) with probability at least \(1 - \exp(-CP)\). This finishes the proof.

\textbf{Lemma 7 (Zhang et al. 2020, Lemma 3)} Suppose \(\mathbf{F}, \mathbf{\hat{F}}, \mathbf{G}, \mathbf{\hat{G}} \in \mathbb{R}^{p_1 \times r}, \mathbf{H}, \mathbf{\hat{H}} \in \mathbb{R}^{r \times p_2}\). If \(\mathbf{G}\) and \(\mathbf{\hat{G}}\) are invertible, \(\|\mathbf{FG}^{-1}\| \leq \lambda_1\), and \(\|\mathbf{\hat{G}}^{-1}\mathbf{\hat{H}}\| \leq \lambda_2\), we have
\[
\|\mathbf{\hat{F}}\mathbf{\hat{G}}^{-1}\mathbf{\hat{H}} - \mathbf{F}\mathbf{G}^{-1}\mathbf{H}\|_F \leq \lambda_2 \|\mathbf{\hat{F}} - \mathbf{F}\|_F + \lambda_1 \|\mathbf{\hat{H}} - \mathbf{H}\|_F + \lambda_1 \lambda_2 \|\mathbf{\hat{G}} - \mathbf{G}\|_F.
\] (86)

\textbf{Lemma 8 (Candès and Plan 2011, Lemma 3.3)} Let \(\mathbf{Z}_1, \mathbf{Z}_2 \in \mathbb{R}^{p_1 \times p_2}\) be two low rank matrices with \(r_1 = \text{rank}(\mathbf{Z}_1), r_2 = \text{rank}(\mathbf{Z}_2)\). Suppose \(\langle \mathbf{Z}_1, \mathbf{Z}_2 \rangle = 0\) and \(r_1 + r_2 \leq \min(p_1, p_2)\). Then
\[
\|\langle \mathbf{A}(\mathbf{Z}_1), \mathbf{A}(\mathbf{Z}_2) \rangle\|_F \leq R_{r_1 + r_2} \|\mathbf{Z}_1\|_F \|\mathbf{Z}_2\|_F.
\]

\textbf{Lemma 9} Let \(\mathbf{X}_1 = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^T\) and \(\mathbf{X}_2 = \mathbf{U}_2 \mathbf{\Sigma}_2 \mathbf{V}_2^T\) be two rank \(r\) matrices with corresponding singular value decompositions. Then
\[
\begin{align*}
\|\mathbf{U}_1 \mathbf{U}_1^T - \mathbf{U}_2 \mathbf{U}_2^T\| &\leq \frac{\|\mathbf{X}_1 - \mathbf{X}_2\|}{\sigma_r(\mathbf{X}_1) \vee \sigma_r(\mathbf{X}_2)}, \\
\max\{||\sin \Theta(\mathbf{U}_1, \mathbf{U}_2)||, ||\sin \Theta(\mathbf{V}_1, \mathbf{V}_2)||\} &\leq \frac{2\|\mathbf{X}_1 - \mathbf{X}_2\|}{\sigma_r(\mathbf{X}_1) \vee \sigma_r(\mathbf{X}_2)}.
\end{align*}
\]

\textbf{Proof.} See Lemma 4.2 of [Wei et al. 2016] and Theorem 5 of [Luo et al. 2021]. \(\Box\)