On controllers of prime ideals in group algebras of torsion-free abelian groups of finite rank

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Abstract

Let $kA$ be the group algebra of an abelian group $A$ over a field $k$ and let $I$ be an ideal of $kA$. We say that a subgroup $B$ of the group $A$ controls the ideal $I$ if $I = (I \cap kB)kA$. The intersection $c(I)$ of all subgroups of the group $A$ controlling $I$ is said to be the controller of the ideal $I$. The ideal $I$ is said to be faithful if $I^\dagger = A \cap (1 + I) = 1$. In the presented paper we develop some methods which allow us to study controllers of prime faithful ideals in group algebras of abelian groups of finite rank. The main idea is that the quotient ring $kA/I$ by such an ideal $I$ can be embedded as a domain $k[A]$ in a field $F$ and the group $A$ becomes a subgroup of the multiplicative group of the field $F$. It allows us to apply for studying of $k[A]$ some methods of the theory of fields such as Kummer theory and Dirichlet Unit theorem. In its turn properties of $k[A] \cong kA/I$ strongly depend on the properties of the ideal $I$. Using these methods, in particular, we obtain an independent proof of a Brookes theorem on controllers of prime ideals in the case, where the field $k$ has characteristic zero.

Introduction

Let $R$ be a ring, let $G$ be a group and let $I$ be a right ideal of the group ring $RG$. The ideal $I$ is said to be faithful if $I^\dagger = G \cap (1 + I) = 1$. We say that a subgroup $H$ of the group $G$ controls the ideal $I$ if

$$I = (I \cap kH)kG. \quad (1)$$

The intersection $c(I)$ of all subgroups of the group $G$ controlling $I$ is said to be the controller of the ideal $I$. It is easy to note that a subgroup $B \leq G$ controls the ideal $I$ if and only if $c(I) \leq B$.

Let $H$ be a subgroup of the group $G$ and let $U$ be a right $RH$-module. Since the group ring $RG$ can be considered as a left $RH$-module, we can define the tensor product $U \otimes_{RH} RG$ (see [5, 5.1]), which, by [5, 5.1, proposition 3],

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is a right $RG$-module named the $RG$-module induced from the $RH$-module $U$. By [2, Chap. III, proposition 5.3], if $M$ is an $RG$-module and $U \leq M$, then

$$M = U \otimes_{RH} RG$$

if and only if

$$M = \oplus_{t \in T} Ut,$$

where $T$ is a right transversal to subgroup $H$ in $G$.

If a $kG$-module $M$ of some representation $\varphi$ of a group $G$ over a field $k$ is induced from some $kH$-module $U$, where $H$ is a subgroup of the group $G$, then we say that the representation $\varphi$ is induced from a representation of subgroup $H$, where $U$ is the module of the representation $\phi$.

Suppose that $M = aRG$ is a cyclic $RG$-module generated by a nonzero element $a \in M$. Put $I = \text{Ann}_{kG}(a)$ and let $U = akH$, where $H$ is a subgroup of the group $G$. It is not difficult to note that, in these denotations, the equation (1) holds if and only if the equation (3) holds. Thus, in this case, the equations (1), (2), (3) mean the same.

If $k$ is a subfield of a field $f$ and $A$ is a subset of $f$ then $[f : k]$ is the dimension of the field $f$ over $k$; by $k(A)$ we denote the field generated by $k$ and $A$; by $k[A]$ we denote the domain generated by $k$ and $A$. As usually, $f^*$ denotes the multiplicative group of the field $f$. Let $A$ be an abelian group then $t(A)$ denotes the torsion subgroup of $A$. If the group $A$ is torsion then $\pi(A)$ is the set of prime divisors of orders of elements of $A$.

Let $k$ be a field and let $A$ be an abelian group of finite rank. In the presented paper we consider properties of controllers of faithful prime ideals in the group algebra $kA$. The main idea of our studying is that in the case, where $P$ is a faithful prime ideal of $kA$, the quotient ring $kA/P$ can be embedded as a domain $k[A]$ in a field $F$ and, as the ideal $P$ is faithful, the group $A$ becomes a subgroup of the multiplicative group of the field $F$. It allows us to apply for studying of $k[A]$ some methods of the theory of fields such as Kummer theory and Dirichlet Unit theorem. In its turn properties of $k[A] \cong kA/P$ strongly depend on the properties of the ideal $P$. To prove equation (1) we prove equations (2), (3) for a $kA$-module $k[A]$.

In section 1 we consider relations between Kummer theory and equations (2) and (3) for modules over abelian groups. The main result of this section (theorem 1.3) can be considered as a generalization of [6, Chap. VIII, theorem 13] to the case of infinite dimensional extensions.

In section 2 we study properties of multiplicative groups of certain fields which are generated by a finite set and by roots from 1. The most important is that the multiplicative group of such a field is a direct product of Chernikov and free abelian groups (see proposition 2.3).

A combination of results of sections 1 and 2 allows us to study controllers of faithful prime ideals in group algebras of abelian groups of finite rank. For
the first time, such methods were applied in [9] (see [9, lemmas 2, 5]) where we proved the equation (3) for modules over abelian groups of finite rank. Then the methods were developed in [8, 10, 11, 12]. The main result (theorem 3.1) states that in the case where the field $k$ is finitely generated of characteristic zero, the controller of any faithful prime ideal $P$ of the group algebra $kA$ of an abelian group $A$ of finite rank with Chernikov torsion subgroup is finitely generated. In the case, where the group $A$ is minimax, such a result follows from a Segal theorem [8, theorem 1.1]. An abelian group is said to be minimax if it has a finite series each of whose factor is either cyclic or quasi-cyclic.

Moreover, there is an example which shows that in fact the theorem does not hold in the case of the field of characteristic zero. As it became known recently, the original proof of [1, theorem A] is incorrect. As a new independent proof of the Brookes theorem in the case of the field of characteristic zero (theorem 4.4). As from 1 of degree $p$ prime divisor $t$ between $|G/kG|$ as a $kG$-module and the field $k$ can be considered as a $k(G \cap k^*)$-module. Therefore, we can define the tensor product $k \otimes_{k(G \cap k^*)} kG$ and the equation $k(G) = k \otimes_{k(G \cap k^*)} kG$ means that $k(G) = \oplus_{t \in T} kt$, where $T$ is a transversal to the subgroup $G \cap k^*$ in the group $G$. If $|G/G \cap k^*| = m < \infty$ the equation $k(G) = k \otimes_{k(G \cap k^*)} kG$ holds if and only if $[k(G) : k] = m$. The relations between $|G/G \cap k^*|$ and $[k(G) : k]$ were considered in Kummer theory (see [6, Chap. VIII, theorem 13]). So, we can hope that there are relations between induced modules over abelian groups and Kummer theory. These relations are studied in this section.

**Lemma 1.1.** Let $k$ be a subfield of a field $f$ and let $G$ be a subgroup of the multiplicative group $f^*$ of the field $f$. Then the field $k(G)$ may be considered as a $kG$-module and the field $k$ can be considered as a $k(G \cap k^*)$-module. Therefore, we can define the tensor product $k \otimes_{k(G \cap k^*)} kG$ and the equation $k(G) = k \otimes_{k(G \cap k^*)} kG$ means that $k(G) = \oplus_{t \in T} kt$, where $T$ is a transversal to the subgroup $G \cap k^*$ in the group $G$. If $|G/G \cap k^*| = m < \infty$ the equation $k(G) = k \otimes_{k(G \cap k^*)} kG$ holds if and only if $[k(G) : k] = m$. The relations between $|G/G \cap k^*|$ and $[k(G) : k]$ were considered in Kummer theory (see [6, Chap. VIII, theorem 13]). So, we can hope that there are relations between induced modules over abelian groups and Kummer theory. These relations are studied in this section.

**Proof.** Let $g^t = a \in k^*$. Suppose that $a = 1$ then $g \in t(G)$ and hence all prime divisors of $t$ belong to the set $\pi(t(G)) \cap \pi(G/k^*)$. But then for any prime divisor $p$ of $t$ the field $k$ contains a primitive root from 1 of degree $p$, and it easily implies that $a \neq 1$.

Suppose that for some prime divisor $p$ of $t$ there is an element $b \in k$, such that $b^p = a$. Then $(g^mb^{-1})^p = 1$, where $m = t/p$. Evidently, $p \in (\pi(t(G)) \cap \pi(G/k^*))$ and hence $k$ contains a primitive root from 1 of degree $p$. Therefore, $g^mb^{-1} \in k^*$ and, as $b \in k^*$, we have $g^m \in k^*$. But it is impossible
because \(|\bar{g}| = t\).

Suppose that 4 divides \(t\) and \(a \in -4k^4\) then there is an element \(b \in k\) such that \(a = -4b^4\) and the quotient group \(G/k^*\) has an element of order 4. As the quotient group \(G/k^*\) has an element of order 4, the field \(k\) contains a primitive root from 1 of degree 4. Let \(m = t/4\) and \(h = g^m\) then \(h^4 = -4b^4\) and hence \(h^2 = (\pm 2i)b^2\), where \(i\) is a primitive root from 1 of degree 4. Since the field \(k\) contains \(i\), we have \(h^2 = 2g^m \in k^*\). But it is impossible because \(|\bar{g}| = t\).

Thus, \(a \not\in k^p\) for any prime divisor \(p\) of \(t\) and \(a \not\in -4k^k\) if 4 divides \(t\). Then it follows from [6, Chap. VIII, theorem 16] that the element \(g\) is a root of an irreducible polynomial \(X^t - a\) over the field \(k\) and, by [6, Chap. VII, proposition 3], \([d : k] = t\).

**Lemma 1.2.** Let \(k\) be a subfield of a field \(F\) and let \(G\) be a subgroup of \(F^*\) such that \(k^* \leq G\), \(k\) contains a primitive root from 1 of degree 4 if the quotient group \(G/k^*\) has an element of order 4, the quotient group \(G/k^*\) is torsion, for any \(p \in (\pi(t(G)) \cap \pi(G/k^*))\) the field \(k\) contains a primitive root from 1 of degree \(p\) and \(chark \not\in \pi(G/k^*)\). Suppose that \(|G/k^*| = n\), where \(n = p^m\) and \(p\) is a prime number. Then \([k(G) : k] = p^m = n\).

**Proof.** Suppose that the field \(k\) contains a primitive root \(\xi\) from 1 of degree \(p\). The proof is by induction on \(m\).

Let \(g\) be an element of \(G\) such that \(|\bar{g}| = p\), where \(\bar{g}\) is the image of \(g\) in the quotient group \(G/k^*\). Let \(d = k(g)\) then, by lemma 1.1, \([d : k] = p\). Suppose that there is an element \(h \in G \cap d^*\) such that \(|\bar{h}| = p^2\), where \(\bar{h}\) is the image of \(h\) in the quotient group \(G/k^*\).

By lemma 1.1, \([k(h) : k] = p^2\) and, as \(k(h) \leq k(g)\) and \([k(g) : k] = p\), we obtain a contradiction with [6, Chap. VII, proposition 2]. Thus, we can assume that the quotient group \((d^* \cap G)/k^*\) is elementary abelian. Put \(d^* \cap G = L\), then \(k^* \leq L\) and, evidently, \(d = k(L)\). By [6, Chap. VIII, theorem 13], \(p = [d : k] = |L^p/(k^*)^p|\) and, as the field \(k\) contains a primitive root from 1 of degree \(p\), it is not difficult to show that \(|L/k^*| = p\). Then \(G \cap d^* = k^* < g >\) and hence \(|G/d^*| = p^{m-1}\). Therefore, by the induction hypothesis, \([d(G) : d] = p^{m-1}\). Evidently, \(d(G) = k(G)\) and, as \([d : k] = p\), it follows from [6, Chap. VII, Proposition 2] that \([k(G) : k] = p^m = n\).

Suppose that the field \(k\) does not contain \(\xi\). Put \(d = k(\xi)\) then \([d : k] = r < p\) because the cyclotomic polynomial for \(\xi\) has degree \(p - 1\).

Now, we show that \(G \cap d^* = k^*\). Suppose that there is an element \(g \in (G \cap d^*) \setminus k^*\). Evidently, there is no harm in assuming that \(|\bar{g}| = p\), where \(\bar{g}\) is the image of \(g\) in the quotient group \((G \cap d^*)/k^*\). Then, by lemma 1.1, \([k(g) : k] = p\) but, as \(k(g) \leq d\), we obtain a contradiction with \([d : k] = r < p\). Thus, \(G \cap d^* = k^*\) and hence \(Gd^*/d^* \cong G/(G \cap d^*) = G/k^*\). Therefore, \(|G/(d^* \cap G)| = n = p^m\). Then, as it was shown above, \([d(G) : d] = p^m\) and hence, by [6, Chap. VII, Proposition 2], \([d(G) : k] = p^{mr}\), where \([d : k] = r < p\). Since \(d(G) = k(G)\), we have \([d(G) : k(G)] \leq r\). Then it follows from [6, Chap. VII, proposition 2] and the equation \([d(G) : k] = p^{mr}\)
that \([k(G) : k] \geq n\). But the relation \([k(G) : k] > n\) is impossible, because \([k(G) : k] \leq |G/k^*| = n\). Thus, \([k(G) : k] = n\).

**Theorem 1.3.** Let \(k\) be a subfield of a field \(f\) and let \(G\) be a subgroup of \(f^*\) such that \(k^* \leq G\), \(k\) contains a primitive root from 1 of degree 4 if the quotient group \(G/k^*\) has an element of order 4, the quotient group \(G/k^*\) is torsion, for any \(p \in (\pi(t(G)) \cap \pi(G/k^*))\) the field \(k\) contains a primitive root from 1 of degree \(p\) and \(\text{chark} \notin \pi(G/k^*)\). Then \(k(G) = k \otimes_{k(k^*)} kG = \bigoplus_{t \in T} kt\), where \(T\) is a transversal to \(k^*\) in \(G\).

**Proof.** Since the quotient group \(G/k^*\) is locally finite, it is sufficient to show that for any subgroup \(H \leq G\) such that \(k^* \leq H\) and \(|H/k^*| < \infty\) the equation \(k(H) = k \otimes_{k(k^*)} kH\) holds. Thus, there is no harm in assuming that \(|G/k^*| = n < \infty\) and \(\text{chark}\) does not divide \(n\). Since the equation \(k(G) = k \otimes_{k(k^*)} kG\) means that \(k(G) = \bigoplus_{t \in T} kt\), where \(T\) is a transversal to \(k^*\) in the group \(G\), it is sufficient to show that \([k(G) : k] = n\). We use the induction on \(n\).

Let \(p\) be the smallest prime divisor of \(n\) and let \(N/k^*\) be the Sylow \(p\)-subgroup of the quotient group \(G/k^*\). Suppose that \([N/k^*] = t\). Let \(d = k(N)\), then by lemma 1.2, \([d : k] = t\).

Now, we show that \(d^* \cap G = N\). Suppose, that there is an element \(g \in (d^* \cap G) \setminus N\). Evidently, there is no harm in assuming that \(|g| = p^t\), where \(\bar{g}\) is the image of the element \(g\) in the quotient group \(G/(k^* \cap G)\) and \(p^t\) is a prime number such that \(p < p^t\). Then, by lemma 1.1, \([k(g) : k] = p^t\) and, as \(k(g) \leq d\), by [6, Chap. VII, proposition 2], \(p^t\) divides \(t\) but it is impossible because \(p \neq p^t\). Thus, \(d^* \cap G = N\) and hence \(Gd^*/d^* \cong G/(d^* \cap G) = G/N\). So, we can conclude that \(|Gd^*/d^*| = |G/N| = n/t\). Then by the induction hypothesis, \([d(G) : d] = n/t\) and, as \([d : k] = t\), by [6, Chap. VII, proposition 2], we have \([d(G) : k] = n\). Finally, it is easy to see that \(d(G) = k(G)\) and the assertion follows.

**Corollary 1.4.** Let \(k\) be a subfield of a field \(f\) and let \(G\) be a subgroup of \(f^*\) such that \(k\) contains a primitive root from 1 of degree 4 if the quotient group \(Gk^*/k^*\) has an element of order 4, the quotient group \(Gk^*/k^*\) is torsion, for any \(p \in (\pi(t(Gk^*)) \cap \pi(Gk^*/k^*))\) the field \(k\) contains a primitive root from 1 of degree \(p\) and \(\text{chark} \notin \pi(G/k^*)\). Then \(k(G) = k \otimes_{k(k^* \cap G)} kG = \bigoplus_{t \in T} kt\), where \(T\) is a transversal to \(k^* \cap G\) in \(G\).

**Proof.** To prove the corollary, it is sufficient to apply the above theorem to the field \(k\) and the group \(Gk^*\).

2. On multiplicative groups of certain fields.

We will say that a field \(k\) is regular if it is countable and the multiplicative group of the field \(k\) is a direct product of a torsion group and a free abelian group.

**Lemma 2.1.** Let \(f = k(S)\) be a transcendent extension of a field \(k\), where
$S$ is a finite set of elements of the field $f$. If the field $k$ is regular then so is the field $f$.

**Proof.** Since the set $S$ is finite, so is the transcendent degree of the field $f$ over the subfield $k$. Let $z \notin k$ and let $z, z_1, \ldots, z_n$ be a maximal system of algebraically independent over $k$ elements of the field $f$. Put $f_1 = k(z, z_1, \ldots, z_n)$ then it is not difficult to show that $f_1 = k^* \times N$, where $N$ is a countable free abelian group and hence, as the field $k$ is regular, so is the field $f_1$. As $f \geq f_1 \geq k$ and $f = k(S)$, we can conclude that $f = f_1(S)$. Then, as the field $f$ is an algebraic extension of the field $f_1$, we see that $[f : f_1]$ is finite and hence we can put $[f : f_1] = m < \infty$.

Let $\varphi_z : f^* \to f_1^*$, be a homomorphism which maps each element of the multiplicative group $f^*$ to its regular norm over the field $f_1$. Then, evidently, $\varphi_z(z) = z^m$.

Suppose that $z \in t(f^*/Ker\varphi_z)$ then there is a positive integer $r$ such that $z^r \in Ker\varphi_z$ and hence $1 = \varphi_z(z^r) = (\varphi_z(z))^r = z^{mr}$. Therefore, $z \in t(f^*)$. On the other hand, as the field $f$ is a transcendent extension of the field $k$, the quotient group $f^*/k^*$ is torsion-free and , as $z \notin k$, we see that $z \notin t(f^*)$. Thus, a contradiction is obtained and hence $z \notin t(f^*/Ker\varphi_z)$. Since $f^*/Ker\varphi_z \cong \varphi_z(f^*) \leq f_1^*$ and the field $f_1$ is regular, it is not difficult to show that the quotient group $f^*/f_z$ is free abelian, where $f_z$ is a subgroup of $f^*$ and $f_z/Ker\varphi_z = t(f^*/Ker\varphi_z)$.

Thus, for any element $z \notin f\setminus k$ there exists a subgroup $f_z$ of the group $f^*$ such that $z \notin f_z$ and the quotient group $f^*/f_z$ is a free abelian. Put $T = \bigcap_{z \in f\setminus k}$, evidently $T \leq k^*$ and the quotient group $f^*/T$ is a residually free abelian and, as the group $f^*$ is countable, it implies that the quotient group $f^*/T$ is free abelian. Since $T \leq k^*$, we see that $T$ is a direct product of a torsion and a free abelian groups and hence the field $f$ is regular.

Let $k$ be a field, the quotient group $k = k^*/t(k^*)$ is said to be the reduced multiplicative group of the field $k$.

**Lemma 2.2.** Let $f$ be a finite extension of a field $k$ such that $[f : k] = n$. Suppose that the field $f$ is regular. Then $\tilde{f} = F \times (\tilde{f} \cap \overline{k}^{1/n})$, where $F$ is a countable free abelian group.

**Proof.** Let $\varphi : f^* \to k^*$ be a homomorphism given by $\varphi : x \mapsto |x|$, where $|x|$ is the regular norm of an element $x \in f$ over $k$. It is easy to note that $\varphi$ induces a homomorphism $\varphi : \tilde{f} \to \tilde{k}$. It follows from the definition of the homomorphism $\varphi$ that $\varphi(x) = x^n$ for any $x \in \tilde{k}$. Since the field $f$ is regular, we see that the groups $\tilde{k}$ and $Ker\varphi$ are free abelian. Put $A = Ker\varphi$, since $\varphi(x) = x^n$ for any $\tilde{x} \in k$, we can conclude that $1 = A \cap \tilde{k}$ and hence $1 = A \cap (\tilde{f} \cap \overline{k}^{1/n})$.

Let $x \in \tilde{f}$ and $\varphi(x) = y \in \tilde{k}$ then $\varphi(x^n y^{-1}) = \varphi(x^n) \varphi(y^{-1}) = \varphi(x)^n (y^{-1})^n = y^n (y^{-1})^n = 1$. Thus, for any element $x \in \tilde{f}$ there is an element $y \in \tilde{k}$ such that $x^n y^{-1} \in A$. Therefore, $(\tilde{f})^n \leq \tilde{k} \times A$ and hence the quotient group $R = \tilde{f}/(\tilde{f} \cap \overline{k}^{1/n})$ is torsion free. Since $1 = A \cap (\tilde{f} \cap \overline{k}^{1/n})$ and the subgroup $A$
is free abelian, we can conclude that the quotient group $R$ has a free abelian subgroup $T = (A(\bar{f} \cap \overline{k^1/n})/(\bar{f} \cap \overline{k^1/n})$ such that $R^n \leq T$. Then, as the group $R$ is torsion free, it easily implies that the group $R$ is free abelian. Thus, the quotient group $R = \bar{f}/(\bar{f} \cap \overline{k^1/n})$ is free abelian and hence there is a free abelian subgroup $F \leq \bar{f}$ such that $\bar{f} = F \times (\bar{f} \cap \overline{k^1/n})$.

**Proposition 2.3.** Let $f$ be a field of characteristic zero generated by a finite set $S$ and by all roots from 1 of degree $p^n$, where $n$ are positive integers, $p \in \pi$ and $\pi$ is a finite set of prime numbers. Then the set $\pi(t(f^*))$ is finite and the field $f$ is a locally cyclic Chernikov group.

**Proof.** Let $k$ be a subfield of $f$ generated by all roots from 1 of degree $p^n$, where $n$ are positive integer and $p \in \pi$, then $f = k(S)$. If $\pi = \emptyset$ then $k$ is the field of rational numbers. Let $k_1$ be the maximal algebraic extension of $k$ in $f$. Since $|S| < \infty$, we see that $[k_1 : k] < \infty$ and $f$ is a finitely generated transcendent extension of $k_1$. As $f$ is a finitely generated transcendent extension of $k_1$, we see that $t(f^*) = t(k_1^*)$ and it follows from lemma 2.1 that it is sufficient to show that the field $k_1$ is regular. So, we can assume that the elements of $S$ are algebraic over $k$ and hence $[f : k] = r < \infty$.

(i) In this case, $f$ is a finitely generated algebraic field of characteristic zero and hence $f$ is a finite extension of the rational number field. Then, be the Dirichlet Unit theorem (see [4, Chap. I, theorem 13.12]), the group $U$ of unities of the ring $R$ of algebraic integers of the field $f$ is finitely generated. There exists a homomorphism $\varphi$ of the group $f^*$ into the free abelian group $I(R)$ of fractional ideals of $R$ which maps each element $a \in f^*$ into the fractional ideal generated by the element $a$ (see [4, Chap. I, section 4]) and $Ker \varphi = U$. Then, by the theorem on homomorphism, the quotient group $f^*/U$ is free abelian and, as the group $U$ is finitely generated, it easily implies that $f^* = T \times F$, where $F$ is a free abelian group and $T$ is a finite group.

(ii) At first, we show that the set $\pi(t(f^*))$ is finite and hence the group $t(f^*)$ is Chernikov. Suppose that the field $f$ contains a primitive root $\xi$ from 1 of degree $q \notin \pi$. It is well known that the cyclotomic polynomial for $\xi$ is irreducible over $k$ and hence $[k(\xi) : k] = q - 1$. Then $q - 1 \leq r$ and evidently the set of all such $q$ is finite and, as the set $\pi$ is finite, so is $\pi(t(f^*))$. Therefore, $T = t(f^*)$ is a Chernikov locally cyclic group.

Now, we show that the field $f$ is regular. Let $D$ be the set of all roots from 1 of degree $p^n$, where $p \in \pi$, then the set $D$ is finite. Let $h$ be a subfield of $f$ generated by $(S \cup D)$. Then, by (i), $h^*$ is a direct product of a finite and a free abelian groups. Evidently, the field $f$ has an infinite series $\{h_i|i \in I\}$ of subfields such that $h = h_1$, $h_i \leq h_{i+1}$, $\cup_{i \in I}h_i = f$ and $h_{i+1} = h_i(\zeta_i)$, where $\zeta_i$ is a root from 1 of degree $p^n$ for some $p \in \pi$ and positive integer $n$. We also can assume that $\zeta_i^p \in h_i$ then it follows from lemma 1.1 that $[h_{i+1} : h_i] = p$ if $h_i \neq h_1$. Evidently, $h_i \leq \bar{h}_{i+1}$, $\cup_{i \in I}\bar{h_i} = \bar{f}$ and it follows from (i) that each group $\bar{h}_i$ is free abelian. Then it is sufficient to show that the quotient group
\(\bar{h}_{i+1}/\bar{h}_i\) is free abelian for each \(i\).

By lemma 2.2, \(\bar{h}_{i+1} = F_i \times (\bar{h}_{i+1} \cap h_{i+1}^{1/p})\), where \(F_i\) is a countable free abelian group and it is sufficient to show that \(\bar{h}_i = \bar{h}_{i+1} \cap h_{i+1}^{1/p}\). Suppose that \(\bar{h}_i \neq \bar{h}_{i+1} \cap h_{i+1}^{1/p}\) then there is an element \(a \in \bar{h}_{i+1} \setminus \bar{h}_i\) such that \(a^p \in \bar{h}_i\) and hence there is an element \(b \in h_{i+1} \setminus h_i\) such that \(b^p \in h_i\) and \(b \notin \langle \zeta \rangle\). It easily implies that the group \(h_{i+1}^{1/p}\) has a subgroup \(H\) such that \(h_i^{*} \leq H\) and \(G/h_i^{*}\) is an elementary abelian \(p\)-group of order \(p^2\). Therefore, as \(h\) contains a primitive root from 1 of degree \(p^2\), it follows from lemma 1.2 that \([h_i(G) : h_i] = p^2\) but it is impossible because \(h_i^{*}(G) \leq h_{i+1}^{*}\) and \([h_{i+1}^{*} : h_i] = p\).

Thus, the field \(f\) is regular and hence \(f^* = T \times F\), where \(T\) is a locally cyclic Chernikov group and \(F\) is a free abelian group.

3. Controllers in prime ideals of abelian groups of finite rank.

Let \(A\) be an abelian group and let \(B\) be a subgroup of \(A\). The set \(is_A(B)\) of elements \(a \in A\) such that \(a^n \in B\), for some positive integer \(n\), is a subgroup of \(A\) which is said to be the isolator of the subgroup \(B\) in the group \(A\). The subgroup \(B\) is said to be dense if \(is_A(B) = A\) and the subgroup \(B\) is said to be isolated if \(is_A(B) = B\).

**Theorem 3.1.** Let \(k\) be a finitely generated field of characteristic zero, let \(G\) be an abelian group of finite rank such that the torsion subgroup \(t(G)\) is Chernikov and let \(P\) be a prime faithful ideal of \(kG\). Then the controller of \(P\) is finitely generated.

**Proof.** Let \(M\) be the field of fraction the domain \(kG/P\), then \(M = k(G)\) and \(G\) is a subgroup of the multiplicative group of \(M\). Let \(h\) be the algebraic closure of the field \(M\). Let \(d = k(H,i)\), where \(H\) is a finitely generated dense subgroup of \(G\) which contains elements of order \(p\) for each \(p \in \tau = \pi(t(G))\), and \(i\) is a primitive root of degree 4 from 1. By proposition 2.3(i), \(d^* = T \times L\), where \(T\) is a finite group and \(L\) is a free abelian group. It implies, that \(G \cap d^*\) is a finitely generated dense subgroup of \(G\) and changing \(H\) by \(G \cap d^*\) we can assume that \(H = G \cap d^*\).

Let \(D = Gd^*\) then \(D/d^* = Gd^*/d^* = G/(G \cap d^*) = G/H\) and hence the quotient group \(D/d^*\) is torsion, besides \(d\) contains a primitive root of degree 4 from 1. So, it would be possible to apply theorem 1.3 but there may be a situation that not for any \(p \in (\pi(t(D)) \cap \pi(D/d^*))\) the field \(d\) contains a primitive root from 1 of degree \(p\). However, as \(D = Gd^*\), where \(\pi(t(D)) \subseteq \pi(t(d^*))\) and \(d^*\) is an almost free abelian group, the above situation may be possible only for \(p \in \pi = \pi((is_dH)/H)\), where \(\bar{d} = d^*/t(d^*)\) and \(\bar{H} = Ht(d^*)/t(d^*)\). Since the group \(\bar{d}\) is free abelian, it is easy to note that the set \(\pi\) is finite. Let \(p\) be the biggest prime from \(\pi\) and let \(\omega\) be the set of all primes \(q \leq p\). Let \(\xi\) be a primitive root from 1 of degree \(p\), let \(d_1 = d(\xi)\) and let \(H_1 = G \cap d_1^*\). By lemma 2.2, \(\bar{d}_1 = R \times (\bar{d}_1 \cap \bar{d}_1^{1/n})\), where \(R\) is a free abelian group and \(n = [d_1 : d] < p\), and hence \(\pi((is_{d_1}\bar{H}_1)/H_1) \subseteq \omega \setminus \{p\}\), where \(\bar{d}_1 = d_1^*/t(d_1^*)\) and \(\bar{H}_1 = (G \cap d_1^*)t(d_1^*)/t(d_1^*)\). Thus, after several steps,
adding primitive roots from 1, we obtain a field $f$ such that the quotient group $F/f^*$ is torsion, where $F = Gf^*$, $i \in f$ and for any $p \in \pi(t(F)) \cap \pi(F/f^*)$ the field $f$ contains a primitive root from 1 of degree $p$. Therefore, by theorem 1.3, $f(F) = \oplus_{t \in T} ft$, where $T$ is a transversal to $f^*$ in $F$. Since $F = Gf^*$ we see that $f(G) = \oplus_{t \in T} ft$, and $T$ can be chosen as a transversal to $L = f^* \cap G$ in $G$. Therefore, $k(G) = \oplus_{t \in T} k(L)t$ and hence $k[G] = \oplus_{t \in T} k[L]t$, where $T$ is a transversal to $L$ in $G$. Evidently, it implies that $P = (P \cap kL)kG$ and, as the field $f$ is finitely generated, it follows from proposition 1.3(i) that so is the subgroup $L$.

**Corollary 3.2.** Let $k$ be a finitely generated field of characteristic zero and let $G$ be an abelian group of finite rank such that the torsion subgroup $t(G)$ is Chernikov. Then any faithful irreducible representation of the group $G$ over the field $k$ is induced from some finitely generated subgroup of $G$.

**Proof.** Let $M$ be a module of a faithful irreducible representation of the group $G$ over the field $k$. Then $M$ is a simple $kG$-module and hence $M \cong kG/P$, where $P = \text{Ann}_{kG}(a)$ for some nonzero element $a \in M$ is a faithful maximal ideal of $kG$. By the above theorem, the ideal $P$ is controlled by a finitely generated subgroup $H \leq G$. It means that $M = \oplus_{t \in T} Ut$, where $U = kH/\text{Ann}_{kH}(a)$ and $T$ is a transversal to $H$ in $G$, and hence $M = U \otimes_{kH} kG$.

**Theorem 3.3.** Let $k$ be a finitely generated field of characteristic zero, let $A$ be a torsion-free abelian minimax group and let $P$ be a faithful prime ideal of the group algebra $kA$. Let $R = kA/P$ and let $h$ be the field of fractions of $R$ then $A \leq h^*$. Let $C$ be a finitely generated dense subgroup of $A$ and let $\tau$ be the set of all roots from 1 of degree $p^n$ for all $p \in \pi(A/C)$ and all positive integer $n$ in addition with a primitive root of degree 4 from 1. Let $f$ be a field obtained by addition of all roots from the set $\tau$ to the field $h$ and let $s$ be a subfield of $f$ generated by all roots from $\tau$ and by the subgroup $C$ and let $B = A \cap s^*$. Let $D$ be a dense subgroup of $C$, let $d$ be a subfield of $f$ generated by $D$ and by $t(s^*)$ and let $H = A \cap d^*$. Then :

(i) $B$ and $H$ are finitely generated dense subgroups of $A$, besides $B \geq H$;

(ii) $B$ and $H$ control the ideal $P$.

**Proof.** (i) By proposition 2.3(ii), the field $s$ is regular and hence so is its subfield $d$. It easily implies that $B$ and $H$ are finitely generated dense subgroups of $A$ and the relation $B \geq H$ is evident.

(ii) Put $F = As^*$. Since $B$ is a dense subgroup of $A$, we see that the quotient group $F/s^*$ is torsion. By the definition of $\tau$, $s$ contains a primitive root from 1 of degree 4 and a primitive root of degree $p$ for any $p \in \pi(F/s^*)$. So, we can conclude that $F$ and $s$ meet all conditions of theorem 1.3 and hence $s(A) = s \otimes_{s(s^*)} sA$. Therefore, $k[A] = k[B] \otimes_{kB} kA$ and hence $P = (P \cap kB)kA$.

Put $L = Ad^*$. Since $H$ is a dense subgroup of $A$, we see that $L/d^*$ is a torsion group. As $d \leq s$ and $t(s^*) \leq d$, we can conclude that $t(s^*) = t(d^*)$ and hence $d$ contains a primitive root from 1 of degree 4.
Now, we show that for any \( p \in (\pi(t(L)) \cap \pi(L/d^*)) \) the field \( d \) contains a primitive root from 1 of degree \( p \). Evidently, \( L/d^* = Ad^*/d^* \cong A/(A \cap d^*) = A/H \) and the group \( A \) has a series of subgroups \( A \geq B \geq H \). Since \( \pi(A/B) \subseteq \tau \) and \( \tau \subseteq \pi(t(s^*)) = \pi(t(d^*)) \), it is sufficient to consider the case where \( p \in (\pi(t(Bd^*)) \cap \pi(B/H)) \). But \( Bd^* \leq s^* \) and, as \( t(s^*) = t(d^*) \), we can conclude that \( p \in \pi(t(d^*)) \). Thus, \( L \) and \( d \) meet all conditions of theorem 1.3 and the above arguments show that the subgroup \( H \) controls the ideal \( P \).

4. On controllers of standardized prime faithful ideals in group algebras of abelian groups of finite rank.

Let \( k \) be a field, let \( A \) be a torsion-free abelian group of finite rank acted by a group \( \Gamma \) and let \( I \) be an ideal of a group algebra \( kA \). The set \( \Delta_\Gamma(A) \) of elements of \( A \) which have finite orbits under action of the group \( \Gamma \) is a subgroup of the group \( A \). The subgroup \( N_\Gamma(I) \leq \Gamma \) of elements \( \gamma \in \Gamma \) such that \( I = I^\gamma \) is said to be the normalizer of the ideal \( I \) in the group \( \Gamma \). The subgroup \( S_\Gamma(I) \leq \Gamma \) of elements \( \gamma \in \Gamma \) such that \( I \cap kB = I^\gamma \cap kB \) for some finitely generated dense subgroup \( B \) of \( A \) is said to be the standardiser of the ideal \( I \) in the group \( \Gamma \). Evidently, \( N_\Gamma(I) \leq S_\Gamma(I) \). Theorem 4.4 states that if the field \( k \) has characteristic zero and \( P \) is a prime faithful ideal of the group algebra \( kA \) such that \( S_\Gamma(P) = \Gamma \) then \( P \) is controlled by \( \Delta_\Gamma(A) \). In the case, where the group \( A \) is finitely generated and \( |\Gamma : N_\Gamma(P)| < \infty \), such a result was proved by Roseblade in [7, theorem D]. As lemma 4.2 shows, it is sufficient to consider the situation where the field \( k \) is finitely generated, the group \( A \) is minimax and the group \( \Gamma \) is cyclic. Lemma 4.3 shows that it is sufficient to prove that the ideal \( P \) is controlled by a finitely generated \( \Gamma \)-invariant subgroup of \( A \). So, if the ideal \( P \) is \( \Gamma \)-invariant, the result would follow immediately from theorem 3.1 or from [8, theorem 1.1], because the controller of a \( \Gamma \)-invariant ideal is a \( \Gamma \)-invariant subgroup. However, the condition \( S_\Gamma(P) = \Gamma \) is much more general than \( N_\Gamma(P) = \Gamma \) and does not mean that the ideal \( P \) is \( \Gamma \)-invariant. This circumstance creates the main difficulties which are conquered in the proof of theorem 4.4.

**Lemma 4.1.** Let \( k \) be a field, let \( A \) be a torsion-free abelian group of finite rank acted by a group \( \Gamma \) and let \( P \) be a faithful prime ideal of the group algebra \( kA \). Then:

(i) \( r(c(P)) = r(c(kB \cap P)) \) for any dense subgroup \( B \) of \( A \);
(ii) if \( S_\Gamma(P) = \Gamma \) then \( is_{A\Gamma}(P) \) is a \( \Gamma \)-invariant subgroup of \( A \).

**Proof.** (i) Since \( c(kB \cap P) \leq c(P) \cap B \), there is no harm in assuming that \( is_{A\Gamma}(P) = A \). Suppose that \( r(c(P)) > r(c(kB \cap P)) \) then there is an isolated subgroup \( D \) of \( A \) such that \( c(P \cap kB) \leq D \cap B \) and \( r(A/D) = 1 \). Let \( C \) be a maximal isolated subgroup of \( D \) such that \( P \cap kC = 0 \). Since \( c(P) \) is a dense subgroup of \( A \), we can conclude that \( C \) is a maximal subgroup of \( A \) such that \( kC \cap P = 0 \). Then the transcendence degree over \( k \) of the field of fraction \( k_1 \) of the domain \( kA/P \) is \( r(C) \). On the other hand, as
$k(C \cap B) \cap P = 0$ and $c(P \cap kB) \leq D \cap B$, we see that the transcendence degree of the field of fractions $k_2$ of the domain $kB/(kB \cap P)$ is at least $r(C) + 1$. But it is impossible because $A/B$ is a torsion group and hence $k_1$ is an algebraic extension of $k_2$.

(ii) By the definition of $S_T(P)$, for any element $\gamma \in \Gamma$ there is a finitely generated dense subgroup $B$ of $A$ such that $kB \cap P = kB \cap P^\gamma$. Therefore $c(kB \cap P) \leq c(P) \cap (c(P))^\gamma$, and by (i), $r(c(P)) = r(c(kB \cap P)) = r(c(P)^\gamma)$. It easily implies that $c(P) \cap c(P)^\gamma$ is a dense subgroup in $c(P)$ and $c(P)^\gamma$. Therefore, $is_{AC}(P) = is_{AC}(P)^\gamma$.

**Lemma 4.2.** Suppose that there exist a field $k$, a torsion-free abelian group $A$ of finite rank acted by a group $\Gamma$ and a faithful prime ideal $P$ of $kA$ which is not controlled by $\Delta_T(A)$ and such that $S_T(P) = \Gamma$. Then there exist a finitely generated subfield $k_1 \leq k$, a minimax subgroup $A_1 \leq A$ acted by a cyclic subgroup $\langle \gamma \rangle \leq \Gamma$ and a faithful prime ideal $P_1 = P \cap k_1A_1$ of $k_1A_1$ which is not controlled by $\Delta_T(A_1)$ and such that $S_{\langle \gamma \rangle}(P_1) = \langle \gamma \rangle$.

**Proof.** At first, we show that there is no harm in assuming that the field $K$ is finitely generated. Suppose that $P$ is not controlled by $\Delta_T(A)$ then there is an element $\alpha \in P_1 \setminus ((kA_1)(\Delta_T(A))kA)$. Let $k_1$ be a subfield of $k$ generated by coefficients of $\alpha$ and let $P_1 = P \cap k_1A$ then $\alpha \in P_1 \setminus ((k_1 \Delta_T(A))k_1A)$ and hence $P_1$ is not controlled by $\Delta_T(A)$. It is easy to note that $P_1$ is a prime faithful ideal of $k_1A$ and $S_T(P_1) = \Gamma$. Thus, it is sufficient to consider the case where the field $k$ is finitely generated.

Let $c(P)$ be the controller of $P$, by lemma 4.1(ii), $is_Ac(P)$ is a $\Gamma$-invariant subgroup of $A$. Then there is no harm in assuming that $is_Ac(P) = A$. We also assume that $\Gamma$ acts on the group $A$ faithfully, that is $C_T(A) = 1$.

Suppose that $c(P)$ is not contained in $\Delta_T(A)$. Let $F$ be a free dense subgroup of $c(P)$ with free generators $\{a_i|i = 1, ..., n\}$.

Suppose that for any element $\gamma \in \Gamma$ we have $|\langle \gamma \rangle/C_{\langle \gamma \rangle}(a_i)| < \infty$. Let $C_1 = \cap_{i=1}^n C_{\langle \gamma \rangle}(a_i)$ then $|\langle \gamma \rangle/C_{\gamma}| < \infty$ and it easy to see that $C_{\gamma}$ centralizes $A$. Since $\Gamma$ acts on the group $A$ faithfully, it implies that the group $\Gamma$ is torsion. But $\Gamma$ is a linear group over the field of rational numbers and hence the group $\Gamma$ is finite. Then, evidently, $\Delta_T(A) = A$ and a contradiction is obtained.

Thus, if $c(P)$ is not contained in $\Delta_T(A)$, then there are an element $a \in c(P)$ and an element $\gamma \in \Gamma$ such that the group $\langle \gamma \rangle$ is infinite cyclic and $C_{\langle \gamma \rangle}(a) = 1$. Then we can replace $\Gamma$ by $\langle \gamma \rangle$ because $c(P)$ is not contained in $\Delta_{\langle \gamma \rangle}(A)$. Let $A_1$ be a dense subgroup of $A$ which is finitely generated as a $\langle \gamma \rangle$-module and which contains the element $a$. Then, by [3, lemma 5.2], the subgroup $A_1$ is minimax.

Put $P_1 = P \cap kA_1$, by lemma 4.1(i), $c(P_1)$ is a dense subgroup of $A_1$. If $c(P_1)$ is contained in $\Delta_T(A_1)$ then $\Delta_T(A_1)$ is a dense subgroup of $A_1$ and hence $\Delta_T(A_1) = A_1$. But it is impossible because $a \notin \Delta_T(A_1)$. Thus, $c(P_1)$ is not contained in $\Delta_T(A_1)$. 

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Lemma 4.3. Let $k$ be a field of characteristic zero, let $A$ be a torsion-free finitely generated abelian group acted by a cyclic group $\Gamma = \langle g \rangle$ and let $P$ be a faithful prime ideal of the group algebra $kA$. Suppose that $S_\Gamma(P) = \Gamma$ then $P$ is controlled by $\Delta_\Gamma(A)$.

Proof. Since $S_\Gamma(P) = \Gamma$, there is a subgroup $B$ of finite index in $A$ such that $P \cap kB = P^\gamma \cap kB$. As $A^n \leq B$ fore some positive integer $n$, we can assume that the subgroup $B$ is $\Gamma$-invariant and it implies that the ideal $P_1 = P \cap kB$ is $\langle \gamma \rangle$-invariant. Since $|A : B| < \infty$, there is only finite set of ideals $X$ of $kA$ such that $P_1 = X \cap kB$. Then, as the ideal $P_1$ is $\langle \gamma \rangle$-invariant and $\Gamma = \langle g \rangle$, it easily implies that $|\Gamma : N_\Gamma(P)| < \infty$ and the assertion follows from [7, theorem D].

Theorem 4.4. Let $k$ be a field of characteristic zero, let $A$ be a torsion-free abelian group of finite rank acted by a group $\Gamma$ and let $P$ be a faithful prime ideal of the group algebra $kA$. Suppose that $S_\Gamma(P) = \Gamma$ then $P$ is controlled by $\Delta_\Gamma(A)$.

Proof. By lemma 4.2, it is sufficient to consider the case where the field $k$ is finitely generated, the group $A$ is minimax and the group $\Gamma$ is cyclic.

By lemma 4.1(i), $is_{AC}(P)$ is a $\Gamma$-invariant subgroup of $A$ and changing $A$ by $is_{AC}(P)$, we can assume that $c(P)$ is a dense subgroup of $A$. Then it follows from theorem 3.1 that $c(P)$ is a finitely generated dense subgroup of $A$.

Since the group $A$ is minimax, the set $\pi(A/c(P))$ is finite. Let $\tau$ be the set of all roots from 1 of degree $p^n$ for all $p \in \pi(A/c(P))$ and all positive integer $n$ in addition with a primitive root from 1 of degree 4.

Put $C = c(P)$. Since $S_\Gamma(P) = \Gamma$, for any $\gamma \in \Gamma$ there is a finitely generated dense subgroup $D_\gamma$ of $A$ such that $kD_\gamma \cap P = kD_\gamma \cap P^\gamma$. Evidently, we can assume that $D_\gamma \leq C \cap C^\gamma$.

Let $R = kA/P$ and let $h$ be the field of fractions of $R$, then $A \leq h^*$. Let $f$ be a field obtained by addition to the field $h$ of all roots from the set $\tau$ and let $s$ be a subfield of $f$ generated by all roots from the set $\tau$ and the subgroup $C$. Let $B = A \cap s^*$ then, by theorem 3.3, $B$ is a finitely generated dense subgroup of $A$ which controls $P$. Let $d$ be a subfield of $f$ generated by $D_\gamma$ and $t(s^*)$ and let $H = A \cap d^*$ then, by theorem 3.3, $c(P) \leq H \leq B$.

Let $R_\gamma = kA/P^\gamma$ and let $h_\gamma$ be the field of fractions of $R_\gamma$. Let $f_\gamma$ be a field obtained by addition to the field $h_\gamma$ all roots from the set $\tau$ and let $s_\gamma$ be a subfield of $f_\gamma$ generated by all roots from the set $\tau$ and the subgroup $C^\gamma$. Let $d_\gamma$ be a subfield of $f_\gamma$ generated by $D_\gamma$ and $t(s_\gamma^*)$ and let $H_\gamma = A \cap (d_\gamma^*)$ then, by theorem 3.3, $C^\gamma = c(P^\gamma) \leq H_\gamma$.

Since $(kA/P)^\gamma = kA/P^\gamma$, we can conclude that the fields $f$ and $f_\gamma$ are isomorphic under the isomorphism $\varphi$ induced by action of $\gamma$. Since $\varphi(C) = C^\gamma$, we see that the fields $s$ and $s_\gamma$ are isomorphic and hence $t(s^*) \cong t(s_\gamma^*)$.

Let $K_1$ be a subring of $f$ generated by $k$ and $D_\gamma$, and let $K_2$ be a subring of $f_\gamma$ generated by $k$ and $D_\gamma$. Since $kD_\gamma \cap P = kD_\gamma \cap P^\gamma$, we can conclude that $K_1 = K_2 = K$. Then, as $t(s^*) \cong t(s_\gamma^*)$, the fields $d$ and $d_\gamma$ are the fields
of decomposition of the same set $\Omega$ of polynomials over the field of fractions of the domain $K$. More precisely, $\Omega$ is the set of cyclotomic polynomials for roots from $1$ which belong to the set $t(s^{*}) \cong t(s^{*})$. Therefore, by [6, Chap. VII, theorem 3], there is an isomorphism $\psi$ between the fields $d$ and $d_\gamma$ which centralizes the elements of $K$ and we can assume that $K \leq d \cap d_\gamma$.

Put $\pi = \pi(A/C)$ then $\pi = \pi(A/C^\gamma)$. As $C \leq H$ and $C^\gamma \leq H_\gamma$, we can conclude that $A/H$ and $A/H_\gamma$ are $\pi$-groups.

Suppose that $H \neq H_\gamma$ then either $H \setminus H_\gamma \neq \emptyset$ or $H_\gamma \setminus H \neq \emptyset$. Suppose that $H \setminus H_\gamma \neq \emptyset$ then $H/(H \cap H_\gamma)$ is a nontrivial $\pi$-group and hence the Hall $\pi$-subgroup of the quotient group $H/D_\gamma$ is not contained in $(H \cap H_\gamma)/D_\gamma$. Therefore, there exists an element $a \in H \setminus H_\gamma$ such that $|\bar{a}| = m$ is a $\pi$-number, that is all prime divisors of $m$ belong to $\pi$, where $\bar{a}$ is the image of $a$ in the quotient group $H/D_\gamma$. Thus, $m$ is the smallest integer such that $a^m = b \in D_\gamma$. Since $a \in H = A \cap d$ and $d$ contains all roots from the set $\tau$, we can conclude that all roots of the polynomial $X^m - b$ are contained in $d$. On the other hand, since $a \notin H_\gamma = A \cap d_\gamma$ and $a$ is a root of polynomial $X^m - b$ in $f_\gamma$, we can conclude that $d_\gamma$ does not contain all roots of the polynomial $X^m - b$. Since the fields $d$ and $d_\gamma$ are isomorphic and $b \in K \leq d \cap d_\gamma$, it leads to a contradiction. If $H_\gamma \setminus H \neq \emptyset$ then the same arguments lead to the same contradiction. Thus, $H = H_\gamma$.

Evidently, $c(P^\gamma) = (c(P))^\gamma$. It follows from theorem 3.3 that $(c(P))^\gamma \leq H_\gamma$ and, as $H_\gamma = H \leq B$, we can conclude that $(c(P))^\gamma \leq B$ for each $\gamma \in \Gamma$. Then $S = \langle (c(P))^\gamma \mid \gamma \in \Gamma \rangle \leq B$ and hence $S$ is a finitely generated $\Gamma$-invariant subgroup of $A$ which controls $P$. Then the assertion follows from lemma 4.3.

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