Momentum-independent reflectionless transmission in the non-Hermitian time-reversal symmetric system

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We theoretically study the non-Hermitian systems, the non-Hermiticity of which arises from the unequal hopping amplitude (UHA) dimers. The distinguishing features of these models are that they have full real spectra if all of the eigenvectors are time-reversal (T) symmetric rather than parity-time-reversal (PT) symmetric, and that their Hermitian counterparts are shown to be an experimentally accessible system, which have the same topological structures as that of the original ones but modulated hopping amplitudes within the unbroken region. Under the reflectionless transmission condition, the scattering behavior of momentum-independent reflectionless transmission (RT) can be achieved in the concerned non-Hermitian system. This peculiar feature indicates that, for a certain class of non-Hermitian systems with a balanced combination of the RT dimers, the defects can appear fully invisible to an outside observer.

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I. INTRODUCTION

One of the fundamental characteristics of quantum mechanics is associated with the Hermiticity of physical observables. In the case of the Hamiltonian operator, this requirement not only implies real eigen energies but also guarantees probability conservation [1]. However, a decade ago it was observed that many non-Hermitian Hamiltonians possess real spectra [2] and a pseudo-Hermitian Hamiltonian connects with its equivalent Hermitian Hamiltonian via a similarity transformation [3, 4]. Thus quantum theory based on non-Hermitian Hamiltonian was established [4–13]. Most recently, the growth of interest in the theory of parity-time (PT) symmetric potentials was originated by suggestions of implementation of PT symmetry in a waveguide with gain and absorption [14], which was based on the analogy between quantum mechanics and paraxial optics where the refractive index plays the role of the potential in the Schrödinger equation [15–25]. By exploiting optical modulation of the refractive index in the complex dielectric permittivity plane and engineering both optical absorption and amplification, parity-time metamaterials can lead to a series of intriguing optical phenomena and devices, such as absorption enhanced transmission [21], double refraction, dynamic power oscillations and nonreciprocity of light propagation [10, 18, 20, 26–28], which can be used to realize a new generation of on-chip isolators and circulators [29]. Other intriguing results within the framework of PT optics include the study of Bloch oscillations [20], coherent perfect absorber-lasers [23, 30, 31], and nonlinear switching structures [32].

Over the last few years, PT-symmetric lattice models have become a focal point of research [33, 41]. Lattice models, in general, are popular in physics due to their versatility, availability of exact solutions [42], the absence of an ultraviolet divergence [43], the ability to capture counterintuitive phenomena that have no counterparts in the continuum theories [44] as well as the experimental accessibility. A general non-Hermitian tight-binding network is constructed topologically by the sites and the various non-Hermitian connections between them. There are three types of basic non-Hermitian clusters leading to the non-Hermiticity of a discrete non-hermitian system: i) complex on-site potential denoted as $|V|e^{i\phi}a_j^\dagger a_j$ with $a_j$ being boson (fermion) operator; ii) non-hermitian dimer denoted as $|J|e^{i\phi}(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j)$; iii) unequal hopping amplitude (UHA) dimer denoted as $\mu a_j^\dagger a_{j+1} + \nu a_{j+1}^\dagger a_j$ with $\mu \neq \nu$ being real numbers.

When $\mu = 1/\nu$, UHA dimer reduces to asymmetric hopping amplitude (AHA) dimer denoted as $e^h a_j^\dagger a_{j+1} + e^{-h} a_{j+1}^\dagger a_j$ with asymmetric parameter $h$ being real number, which is used in modeling a delocalization phenomenon relevant for the vortex pinning in superconductors [45]. The former two types of non-Hermitian clusters violate $T$ symmetry, while the last one does not. The systems containing the former two types of clusters have been mainly focused on lattices possessing $PT$ symmetry and were framed in the context of non-Hermitian quantum mechanics [34, 35, 38–41, 46, 50, 52]. However, much less attention has been paid to the property of the non-Hermitian system with the third type of the cluster.

It is the aim of this paper to study the characteristic of the non-Hermitian systems with UHA dimers, which possess the same properties with a pseudo-Hermitian Hamiltonian. We show that the system has full real spectrum if all of the eigenvectors have to respect $T$-rather than $PT$-reversal symmetry, and its Hermitian counterpart exhibits the same topological structure as that of the original one within the unbroken region. We also find out the behavior of the $k$-independent reflectionless transmission (RT) for the concerned non-Hermitian scattering center under the RT condition. This is made possible by considering the invisibility of a state in the system with a balanced combination of the reflectionless transmission...
This paper is organized as follows. Section II presents the exact analytical solution of the scattering problem for the concerned non-Hermitian scattering center. Section III consists of two exactly solvable examples to illustrate the connection with the pseudo-Hermiticity of the $PT$-symmetric non-Hermitian system. Section IV is devoted to the numerical simulation of the wave packet dynamics to demonstrate the phenomena of invisibility in two dimensional case. We conclude the paper with a brief discussion in Section V.

II. REFLECTIONLESS TRANSMISSION CONDITION

In order to understand the differences between the characters of an UHA dimer with other types of non-Hermitian clusters, it is appropriate to begin with the scattering problem for incident plane wave. As is normally done in previous works [17, 18], one can embed the scattering center in an infinite lead, which is illustrated schematically in Fig. 1 (a). The Hamiltonian of the concerned scattering tight-binding network has the form

$$H_{\text{Scatt}} = H_L + H_C + H_R,$$

where

$$H_L = -\kappa \sum_{j=-\infty}^{1} \left(a_j^\dagger a_{j+1} + \text{H.c.} \right), \quad (2)$$

$$H_R = -\kappa \sum_{j=1}^{\infty} \left(a_j^\dagger a_{j+1} + \text{H.c.} \right), \quad (3)$$

represents the left ($H_L$) and right ($H_R$) waveguides with real $\kappa$ and

$$H_C = - \left(\mu a_j^\dagger a_1 + \nu a_{-1}^\dagger a_0 \right). \quad (4)$$

describes an UHA dimer as a scattering center when $\mu \neq \nu$. Here $a_j^\dagger$ and $a_j$ denote the boson or fermion creation and annihilation operators on the $j$th site, respectively, satisfying the canonical commutation relations $[a_i, a_j^\dagger]_\pm = \delta_{ij}$ and $[a_i, a_j]_\pm = 0$. Here, for the sake of simplicity, we only concern the case with positive $\mu$ and $\nu$. We first deal with the single-particle case and will see that the generalization to the case of an arbitrary number of particles is straightforward.

In the single-particle subspace, for an incident plane wave with momentum $k$ incoming from the left direction with energy $E = -2\kappa \cos k$, the scattering wave function $|\psi_k\rangle$ can be obtained by the Bethe ansatz method. The wave function has the form

$$|\psi_k\rangle = \sum_{j=-\infty}^{\infty} f_k(j) a_j^\dagger |\text{Vac}\rangle \quad (5)$$

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where the scattering wave function $f_k(j)$ is in the form of

$$f_k(j) = \left\{ \begin{array}{ll} e^{ikj} & (j \leq 0) \\ r_k e^{-ikj} & (j > 0) \end{array} \right. \quad (6)$$

Here $r_k$, $t_k$ are the reflection and transmission amplitudes of the incident wave with momentum $k$, which are what we are concerned with most in this paper. By substituting the wave function $|\psi_k\rangle$ into the Schrödinger equation,

$$-\kappa f_k(j-1) - \kappa f_k(j+1) = E f_k(j), \quad (7)$$

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which lead to
\[
\begin{align*}
    r_k &= \frac{\kappa^2 - \mu \nu}{\mu \nu - \kappa^2 e^{-i2k}}, \\
    t_k &= \frac{\nu (1 - e^{-i2k})}{\mu \nu - \kappa^2 e^{-i2k}}.
\end{align*}
\]

According to traditional quantum mechanics, a scattering matrix (S-matrix), which relates to the initial state and the final state of a physical system undergoing a scattering process, reveals almost complete features of the scattering center. Then the corresponding S-matrix can be written as
\[
\tilde{S}_k = \left( \begin{array}{cc}
    r_k & \frac{\sqrt{\mu \nu} t_k}{r_k e^{-i2k}} \\
    t_k & r_k e^{-i2k}
\end{array} \right),
\]

where the tilt represents the dimer being non-Hermitian. To characterize the feature of \( \tilde{S}_k \), we consider the S-matrix at the Hermitian point with \( \mu = \nu = \sqrt{\mu \nu} \), which is unitary and has the form
\[
S_k = \left( \begin{array}{cc}
    r_k & \sqrt{\mu \nu} t_k \\
    \sqrt{\mu \nu} t_k & r_k e^{-i2k}
\end{array} \right).
\]

We notice that although \( \tilde{S}_k \) is non-Hermitian and not unitary, the reflection amplitudes are the same and the transmission coefficients are proportional to that of \( S_k \). These features show that the UHA dimer \( \{ \mu, \nu \} \) is familiar to a Hermitian dimer with hopping amplitude \( \sqrt{\mu \nu} \). Furthermore, in the case of \( \kappa^2 = \mu \nu \), we have \( r_k = 0 \) and \( t_k = \kappa / \mu \), which is termed as the reflectionless transmission condition. It shows that an incident plane wave can completely pass through the dimer and the amplitude of the transmitted wave is amplified by a factor of \( \kappa / \mu \). Inversely, an incident wave from right is diminished by a factor of \( \mu / \kappa \). This procedure is illustrated schematically in Fig. 1 (c). Before further discussion of application of the obtained result, two distinguishing features need to be mentioned. Firstly, the \( H_C \) is a non-Hermitian system can have full real spectrum. Secondly, the feature of RT is \( k \)-independent.

Now we investigate the S-matrix of multi-RT-dimer case. Consider a scattering center consisting of \( n \) RT dimers with hopping amplitude \( \{ \mu_i, \kappa^2 / \mu_i \}, i \in [1, n] \). Then the S-matrix of \( i \)-th dimer is
\[
M_i = \left( \begin{array}{cc}
    0 & \frac{\mu_i / \kappa}{\kappa / \mu_i} \\
    \frac{\kappa / \mu_i}{\mu_i / \kappa} & 0
\end{array} \right),
\]

where we use \( M \) to denote the matrix under the RT condition. One can go further, by direct analysis, to yield the S-matrix of the whole center
\[
M^{(n)} = \left( \begin{array}{cc}
    0 & \prod_{i=1}^{n} \left( \frac{\mu_i / \kappa}{\kappa / \mu_i} \right) \\
    \prod_{i=1}^{n} \left( \frac{\kappa / \mu_i}{\mu_i / \kappa} \right) & 0
\end{array} \right).
\]

Remarkably, in the case of unitary transmission, i.e.,
\[
\prod_{i=1}^{n} \mu_i = \kappa^n,
\]
we have
\[
M^{(n)} \left( M^{(n)} \right)\dagger = 1
\]
which is the characteristic of a Hermitian scattering center. It indicates that a multi-dimer \( (n > 1) \) can be equivalent to a Hermitian system. The physics of such a phenomenon is clear: It is the result of the balance between amplification and diminishment dimers. Nevertheless, unlike the system with imaginary potentials, such a balance does not require the symmetry of the dimer. This point can be seen from the condition in Eq. (14), which indicates that the balance obey the commutative law.

In the following, we will further extend the discussion in two aspects. Firstly, the obtained result from the single-particle scattering problem can be extended to many-particle case and exploited to construct the finite-size non-Hermitian system with fully real spectrum, which is beyond the framework of the \( \mathcal{PT} \)-symmetric quantum mechanics. Secondly, the \( k \)-independent Hermitian dynamical behavior of the RT non-Hermitian scattering center can be applicable to the realization of invisibility in higher dimensional lattice.
It is presumable that the analysis for a $\mathcal{PT}$-symmetric non-Hermitian system can be applied to the one with only $\mathcal{T}$ symmetry. In the following we will demonstrate this point by solving concrete examples explicitly. We will focus the investigation on three characteristic features: i) critical behavior near the exceptional point; ii) symmetry breaking; iii) biorthogonal bases.

It is natural to connect the Hermiticity of a non-Hermitian system to the reality of the spectrum of a finite non-Hermitian system which contains the center as building blocks. Before we start with our investigation, we first introduce the method of operator transformation. Consider a generalized Hamiltonian, which is written as,

$$H_{\text{Gen}} = H_{\text{Scatt}} + \sum_{j=-\infty}^{\infty} V_j a_j^+ a_j, \quad (17)$$

where $V_j$ is an arbitrary real on-site potential. Inspired by the solution of single-particle in Eqs. [8] and [9], one can rewrite the Hamiltonian $H_{\text{Gen}}$ as

$$h_{\text{Gen}} = -\kappa \sum_{j=-\infty}^{\infty} (b_j b_{j+1} + \bar{b}_{j+1} b_j + V_j b_j b_{\bar{j}}) + (\kappa - \sqrt{\mu^2}) (b_0 b_1 + \bar{b}_1 b_0),$$

by taking the transformation

$$b_j (b_{\bar{j}}) = \begin{cases} \sqrt{\mu} a_j \left( \frac{a^\dagger_{\bar{j}}}{\mu} \right) & (j > 0) \\ a_j (a^\dagger_{\bar{j}}) & (j \leq 0) \end{cases}. \quad (19)$$

Associating with the canonical commutation relations of operators $b_i$ and $\bar{b}_j$, i.e.,

$$[b_i, b_j]_+ = \delta_{ij}; \ [b_i, \bar{b}_j]_+ = 0,$$

the Hamiltonian $h_{\text{Gen}}$ can act as a Hermitian one under the biorthogonal basis $\{ |n_1, n_2, ..., n_N\rangle, \ |n_1, n_2, ..., n_N\rangle \}$. Here the occupation number basis is defined as

$$|n_1, n_2, ..., n_N\rangle = \prod_{i=1}^{N} \frac{(a^\dagger_{n_i})^{n_i}}{\sqrt{n_i!}} |0\rangle, \quad (20)$$

$$\bar{|n_1, n_2, ..., n_N\rangle} = \prod_{i=1}^{N} \frac{(b^\dagger_{n_i})^{n_i}}{\sqrt{n_i!}} |0\rangle.$$

Thus, the solution of the Hamiltonian $h_{\text{Gen}}$ can be obtained as an ordinary Hermitian tight-binding model, which has the unchanged topological structure. It is worthy pointing out that the Hamiltonian $H_{\text{Gen}}$ with a complete set of biorthonormal eigenvectors is pseudo-Hermitian $[52]$. Here we only consider the simplest case with $V_j = 0$ and $\kappa = \sqrt{\mu^2}$ (RT condition). One can see that the Hamiltonian $h_{\text{Gen}}$ reduces to an infinite uniform chain. Then a plane wave of many particles still
transmits completely through the scattering center without any reflection. The reflectionless transmission phenomenon can be well understood in this sense.

Moreover, such kind of transformation can be applied to other kinds of finite one-dimensional system containing the UHA dimers to construct the biorthogonal bases and reveal the exceptional point. To illustrate this point, we will investigate two typical systems, ring and chain respectively.

A. Finite chain system

Now we consider a chain system which contains only a single UHA dimer. The obtained conclusion can be extended to the case with multi-UHA dimer. Without losing generality, we focus on the dimer with arbitrary \( \{ \mu, \nu \} \) rather than the RT dimer. In the following we consider the case with fixed \( \nu \) but variable \( \mu \). It allows us to simplify the problem of the phase transition associated with the symmetry breaking.

For simplicity, the UHA dimer is embedded in the middle of chain with even sites, which has the Hamiltonian

\[
H_{Cs} = -\kappa \sum_{j=1,j \neq N/2}^{N-1} (a_j^\dagger a_{j+1} + H.c.) - \mu a_{N/2}^\dagger a_{N/2+1} - \nu a_{N/2+1}^\dagger a_{N/2}.
\]

In the single-particle invariant subspace, the Bethe ansatz eigenwave function can be expressed as

\[
f^k(j) = \begin{cases} 
A e^{ikj} + B e^{-ikj}, & j \in [1, N/2] \\
C e^{ikj} + D e^{-ikj}, & j \in [N/2+1, N]
\end{cases}
\]

By carrying out the similar procedure as above, we get the critical equation

\[
\Gamma_{Cs}(k) = \kappa^2 \sin^2 \left( \frac{N}{2} + 1 \right) k - \mu \nu \sin^2 \left( \frac{N}{2} \right) k = 0,
\]

which determines the solution of the Hamiltonian. Here we concern the critical behavior when the complex level appears. When the transition occurs at energy level \( k_c \), the following condition should be satisfied \[38, 49\]

\[
\Gamma_{Cs}(k_c) = 0 \quad \text{and} \quad \frac{\partial \Gamma_{Cs}(k)}{\partial k} \bigg|_{k=k_c} = 0.
\]

which yields the exceptional point \( \mu = 0 \) and \( k_c = n \pi/(N/2 + 1), \ n \in [1, N/2] \). It indicates that the system has fully real spectrum in the region \( \mu > 0 \), and all the \( N/2 \) pairs of energy levels coalesce at the exceptional point simultaneously. It can also be demonstrated by the following critical behavior. Taking the derivative with respect to the parameter \( k \) on the Eq. \[29\], we have

\[
\lim_{k \to k_c} \left( \frac{\partial \mu}{\partial k} \right) = 0
\]

which leads to

\[
\lim_{\mu \to 0} \left[ \frac{\partial E(k)}{\partial \mu} \right] = -2\kappa \lim_{k \to k_c} \left[ \sin(k) \frac{\partial k}{\partial \mu} \right] = \infty.
\]

It shows that the energy level exhibits the repulsive behavior when the system tends to the exceptional point, which is in accordance with the traditional non-Hermitian quantum mechanics.

This peculiar behavior cannot occur in a general Hermitian tight-binding model. However, there should be a Hermitian model, referred to physical counterpart, which has the identical real spectrum. It is interesting to investigate what happens on the equivalent Hermitian model of the non-Hermitian chain \[21\] when the parameter \( \mu \) approaches to the exceptional point. To this end, one can apply the transformation

\[
b_j (\bar{b}_j) = \begin{cases} 
\sqrt{\frac{\mu}{\nu}} a_j (a_j^\dagger) & (N/2 < j) \\
\sqrt{\frac{\nu}{\mu}} a_j^\dagger (a_j) & (j \leq N/2)
\end{cases}
\]

on the Hamiltonian \[21\] and yields

\[
h_{Cs} = -\kappa \sum_{j=1,j \neq N/2}^{N-1} (\bar{b}_j b_{j+1} + \bar{b}_{j+1} b_j) - \sqrt{\mu \nu} (b_{N/2} b_{N/2+1} + \bar{b}_{N/2} \bar{b}_{N/2+1}).
\]

As pointed above, under the biorthogonal bases in the form of Eq. \[20\], \( h_{Cs} \) is Hermitian, which can be employ to investigate the critical behavior. We note that such a mapping \( H_{Cs} \to h_{Cs} \) is always available except the exceptional point \( \mu = 0 \).

For small \( |\mu| \ll \kappa \), Hamiltonian \( h_{Cs} \) represents two weakly coupled uniform identical chains of size \( N/2 \). Since the Hermiticity of \( h_{Cs} \) under the biorthogonal bases, one can employ the standard perturbation method to investigate the property of the system in the vicinity of \( \mu = 0 \). In the single-particle subspace, the eigenfunction and energy of first-order approximation can be expressed as

\[
|\psi^k_\pm \rangle = \sqrt{\frac{2}{N + 2}} \left\{ \sum_{j=1}^{N/2} \sin(kj) a_j^\dagger |\text{Vac} \rangle + \sqrt{\frac{\nu}{\mu}} \sum_{j=N/2+1}^{N} \sin \left( k \left( j - \frac{N}{2} \right) \right) a_j^\dagger |\text{Vac} \rangle \right\},
\]

\[
\varepsilon^k_\pm = -2\kappa \cos(k) \pm \sqrt{\frac{\nu}{\mu}} \sin(k \sin(Nk/2)),
\]

where \( k = 2n\pi/(N/2), \ n = 1, \ldots, N/2 \). It shows that eigenvalues are real in the region of \( \mu > 0 \), and come in complex conjugate pairs in the region of \( \mu < 0 \). It accords with the conclusion at which we have arrived. Furthermore, the eigenfunctions obey

\[
T \mid \psi^k_\pm \rangle = \begin{cases} 
\mid \psi^k_\pm \rangle, & \mu > 0 \\
\mid \psi^k_\pm \rangle, & \mu < 0
\end{cases}
\]
which shows that all eigenfunctions break the $\mathcal{T}$ symmetry, at which the reality of the eigenvalues are lost. Then we conclude that it possesses the same characteristic features with that of a $\mathcal{PT}$-pseudo-Hermitian system.

The results also provides an interpretation for the level repulsion in the Hermitian counterpart. We find that for the Hermitian counterpart, the non-analytic behavior at the exceptional point is not from the Jordan block, which appears in the non-Hermitian model, but from the divergence of derivatives of matrix elements. The similar situation also occurs in another models $[50, 51]$, which may imply a general conclusion.

B. Ring system

Through the above discussion, we can make a conclusion that, for the case of the above open boundary condition model, namely a chain system, which contains one or more UHA dimers as shown in Figs. 2(b) and 2(c). The self-consistent transformation $[14]$ always holds for $\mu\nu > 0$, thus the reality of the spectrum can be achieved. However, on a system with periodic boundary conditions, i.e., a ring system with the RT dimers, the self-consistent transformation implies the unitarity of the S-matrix of whole RT dimers, which are illustrated in Fig. 2(a). It is presumable that the complex level should appear if the S-matrix of whole RT dimers is not unitary as shown in Fig. 2(d). Moreover, such a RT dimer can act as an amplifier, which induces a persistent amplification. Thus one can infer that the existence of steady state is impossible. We exemplify this point by a ring system with a single RT non-Hermitian dimer as sketched in Fig. 2(d), and the Hamiltonian of the concerned system can be written as

$$H_{Rs} = -\kappa \sum_{j=1}^{N-1} (a_j^\dagger a_{j+1} + \text{H.c.}) - \mu a_N^\dagger a_1 - (\kappa^2/\mu) a_1^\dagger a_N. \quad (31)$$

In the single particle invariant subspace, the Bethe ansatz wave function has the form

$$|\psi_k\rangle = \sum_{j=1}^{N} f^k(j) a_j^\dagger |0\rangle, \quad (32)$$

where $f^k(j)$ is in the form of

$$f^k(j) = Ae^{ikj} + Be^{-ikj}, j \in [1, N] \quad (33)$$

The explicit form of the Schrödinger equation reads

$$-\kappa f^k(j-1) - \kappa f^k(j+1) = E f^k(j), \quad (34)$$

for $j \in [2, N-1]$, and

$$-\kappa f^k(N-1) - \mu f^k(1) = E f^k(N), \quad (35)$$

$$-\kappa f^k(2) - (\kappa^2/\mu) f^k(N) = E f^k(1), \quad (36)$$

with the energy spectrum

$$E = -\kappa \left( e^{ik} + e^{-ik} \right). \quad (36)$$

The solution of $k$ is determined by the critical equation

$$\Gamma_{R_S}(k) = \sin\left( (N + 1) k \right) - \sin\left( (N - 1) k \right) - \left( \frac{\mu}{\kappa} + \frac{\kappa}{\mu} \right) \sin (k) = 0, \quad (37)$$

which can be reduced as

$$k = \frac{2m\pi}{N} + \theta, \ m \in [0, N-1] \quad (38)$$

with

$$\theta = \frac{1}{N} \arccos \left( \frac{\mu}{2\kappa} + \frac{\kappa}{2\mu} \right). \quad (39)$$

One can see that $\theta$ is always a complex number except the point $\mu = \kappa$, which corresponds to the Hermitian uniform ring. We note that a complex $\theta$ can lead to a complex $k$, which results in a complex energy level. It is worthy pointing out that, the complex energy level becomes real in the case of Re($k$) = 0, which corresponds to a bound state. Then we conclude that a uniform ring containing a RT dimer does not have a fully real spectrum. This point can also be understood by the operator transformation. One can not find a set of self-consistent transformation as Eq. (19) to rewrite the original Hamiltonian in Eq. (31) as any form of Hermitian tight-binding model. This conclusion can be extended to the case of containing multi-RT dimers with a non-unitary S-matrix. As we mentioned above, however, things are different for a chain. We will demonstrate it in the two dimensional non-Hermitian system.

IV. WAVEPACKET DYNAMICS AND INVISIBILITY

Based on the above analysis, one can extend the conclusion to higher dimensional non-Hermitian systems. In this paper, we focus on the application of such an extension, rather than providing a universal proof. We consider a two-dimensional square lattice with $N$ sites along the horizontal direction and $M$ sites along the vertical direction. It contains two non-Hermitian layers at $m$th and $n$th columns ($1 < m < n < N$), governed by the Hamiltonian

$$H_{2D} = H_0$$

$$- (\mu - \kappa) \sum_{i=1}^{M} \left( a_{i,m}^\dagger a_{i,m+1} - \frac{\kappa}{\mu} a_{i,m+1}^\dagger a_{i,m} \right)$$

$$- (\mu - \kappa) \sum_{i=1}^{M} \left( - \frac{\kappa}{\mu} a_{i,n}^\dagger a_{i,n+1} + a_{i,n+1}^\dagger a_{i,n} \right). \quad (40)$$
the behavior of the total occupation probability two layers can be regarded as an invisible object. In (c) shows in regions I and III as if there were no region II present. Then not scatter and absorb the incident wave packet, but instead the defect-free lattice. It can be seen that the region II does lattice with two non-Hermitian layers at \( m \) and \( n \)th shown these predictions by direct numerical simulations of wave packet propagation in Hermitian and non-Hermitian tight-binding systems of \( 2D \) and \( h_{2D} \). Figure 4(a) shows that the propagation of an initial Gaussian-shaped wave packet \( |l,j| \psi(t) \rangle \) in a non-Hermitian system of \( h_{2D} \) and \( h_0 \) The initial state is a Gaussian-shaped wave packet in the region I, which has the form

\[
|\psi(0)\rangle = \frac{1}{\sqrt{\Omega_0}} \sum_{l,j} e^{i(k_x l + k_y j)} e^{-\frac{\alpha^2}{4}[(l-N_A)^2 + (j-N_B)^2]} |a_{l,j}^\dagger |0\rangle,
\]

It represents a wave packet located at \( l, j \) site, with momentum \( (k_x, k_y) \), where \( \Omega_0 = \sum_{l,j} \exp\{-\alpha^2 \left[(i-N_A)^2 + (j-N_B)^2\right] \} \) is the normalization factor and the half-width of the wave packet is \( 2\sqrt{\ln 2}/\alpha \). Figure 4(a) shows that the propagation of an initial Gaussian-shaped wave packet \( |\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle \) in a non-Hermitian system of \( H_{2D} \) with the parameter \( M = 60, N = 150, m = 50, n = 100, k_x = \pi/2, k_y = 0, N_A = 30, N_B = 20 \)

\[
H_0 = -\kappa \sum_{i=1,j=1}^{M-1,N} a_{i,j}^\dagger a_{i+1,j} - \kappa \sum_{i=1,j=1}^{M,N-1} a_{i,j}^\dagger a_{i,j+1} + \text{H.c.}
\]

The two non-Hermitian layers are balanced in the sense that the S-matrix of them is unitary. This can be shown from the following analysis. In Fig. 3 the structure of the system is illustrated schematically. Two layers separate the square into three regions. Performing the transformation of the operators in these regions

\[
b_{i,j} (\bar{b}_{i,j}) = \begin{cases} a_{i,j} (a_{i,j}^\dagger) & (j \leq n + 1) \\ \frac{b_{i,j}}{\kappa} (\frac{b_{i,j}^\dagger}{\kappa}) & (m < j < n + 1) \\ a_{i,j} (a_{i,j}^\dagger) & (j \leq m) \end{cases}
\]

the original Hamiltonian can be written as a simply form

\[
h_{TD} = -\kappa \sum_{i=1,j=1}^{M-1,N} (\bar{b}_{i,j} b_{i+1,j} + \bar{b}_{i+1,j} b_{i,j}) + \kappa \sum_{i=1,j=1}^{M,N-1} (\bar{b}_{i,j} b_{i,j+1} + \bar{b}_{i,j+1} b_{i,j})
\]

The canonical commutation relations

\[
[b_{i,j}, b_{i',j'}^\dagger] = \delta_{ii'} \delta_{jj'}; [b_{i,j}, b_{i',j'}] = 0,
\]

ensure that the Hamiltonian \( h_{TD} \) can act as a defect-free lattice and has the same spectrum and structure with the Hamiltonian \( h_0 \). Furthermore, in regions I and III, we noticed that the particle operators remain unchanged. Then the defect layers cannot be detected (or invisible) for observers in regions I and III. We have checked these predictions by direct numerical simulations of wave packet propagation in Hermitian and non-Hermitian tight-binding systems of \( H_{2D} \) and \( h_0 \).
and $\alpha = 0.1$. For comparison, Fig. 3(b) shows the propagation of the same wave packet in the defect-free system of $H_0$. It can be seen that the total probability $P(t) = \sum_{i,j} |\langle i, j | \psi(t) \rangle|^2$ of the wave packet turns out to be amplified (diminished) during interaction with the first (second) defect layer. It is shown that the two non-Hermitian layers are balanced, which ensure the probability $P(t)$ conserved in regions I and III. The defect layers in addition of being reflectionless, are also invisible to the an outside observer requires that the phase $\varphi(k)$ of the transmission coefficient be flat, that is, $(d\varphi/dk) = 0$ almost everywhere. If this condition is not satisfied, the spectral components of a wave packet crossing the defect region of the partner lattice would acquire the additional phase contribution $\varphi(k)$, absent in the defect-free lattice, which would be responsible for a different time of flight and for a different distortion of the wave packet as compared to the same wave packet propagating in the ideal defect-free lattice. The advance in the time of flight experienced by the wave packet propagating in the lattice with defects can be readily calculated by standard methods of phase or group-delay time analysis and reads \[ \tau_g = \frac{1}{v_g} \left( \frac{d\varphi(k)}{dk} \right)_{k_x} = \frac{1}{2\kappa \sin(k_x)} \left( \frac{d\varphi(k)}{dk} \right)_{k_x} \] where $k_x = \pi/2$ is the carrier wave number of the wave packet and $v_g = 2\kappa \sin(k_x) > 0$ is its group velocity in the $x$-direction. The characteristic of reflectionless and $k$-independent transmission for the S-matrix of two layers leads to $\varphi(k) = 0$, which generates $\tau_g = 0$. Thus the wave packet is fully transmitted with no appreciable delay and/or distortion. An outside observer thus cannot distinguish whether the transmitted wave packet has been propagated in a defect-free system $H_0$ or in a non-Hermitian system $H_{2D}$ in regions I and III. It should be noted that the total probability $P(t)$ is not conserved in the non-Hermitian system $H_{2D}$, as shown in Fig. 3(c). Such the enhancement and diminishment of the probability $P(t)$, however, is not visible to the outside observer.

V. SUMMARY

In summary, we have investigated the non-Hermitian system which has $\mathcal{T}$ rather than $\mathcal{PT}$ symmetry, with the non-Hermiticity arising from the UHA dimers. Just like a $\mathcal{PT}$ symmetric Hamiltonian, the concerned system shares the same properties with that of a pseudo-Hermitian Hamiltonian. The eigenvalues of the Hamiltonian are either real or come in complex-conjugate pairs. We found that the corresponding Hermitian counterpart can be obtained by a similarity transformation, which is experimentally accessible system, with the unchanged topological structure but modulated hopping amplitude. Comparing with other two types of non-Hermitian clusters, complex on-site potential and non-hermitian dimer as the scattering centers, the UHA dimer exhibits the peculiar feature that the transmission coefficient of it is $k$-independent under the reflectionless transmission condition. This fact indicates that a balanced combination of the RT dimers can act as a Hermitian scattering center, leading to fully transmission of an arbitrary wave packet with no appreciable delay and/or distortion. The defects can appear fully invisible to an outside observer. This paves an alternative way for the design of invisible cloaking devices.

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[1] R. Shankar, Principles of Quantum Mechanics (Plenum Press, 1994).
[2] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[3] A. Mostafazadeh, J. Math. Phys. 43, 205 (2002).
[4] A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. 37, 11645 (2004).
[5] Z. Ahmed, Phys. Lett. A 282, 343 (2001); 286, 30 (2001); Phys. Rev. A 64, 042716 (2001).
[6] M. V. Berry, J. Phys. A 31, 3493 (1998); Czech. J. Phys. 54, 1039 (2004).
[7] W. D. Heiss, Phys. Rep. 242, 443 (1994); J. Phys. A: Math. Gen. 37, 2455 (2004).
[8] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. 89, 270401 (2002).
[9] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[10] H. F. Jones, J. Phys. A: Math. Gen. 38, 1741 (2005); Phys. Rev. D 76, 125003 (2007); 78, 065032 (2008).
[11] J. G. Muga, J. P. Palao, B. Navarro, and I. L. Egusquiza, Phys. Rep. 395, 357 (2004).
[12] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Theor. 40, R205 (2007).
[13] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010).
[14] A. Ruschhaupt, F Delgado and J G Muga, J. Phys. A: Math. Gen. 38, L171 (2005).
[15] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. 32, 2632 (2007).
[16] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. 100, 103904 (2008).
[17] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. A 81, 063807 (2010).
[18] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, Phys. Rev. Lett. 100, 030402 (2008).
[19] S. Klaiman, U. Günther, and N. Moiseyev, Phys. Rev. Lett. 101, 080402 (2008).
[20] S. Longhi, Phys. Rev. Lett. 103, 123601 (2009).
[21] A. Guo et al., Phys. Rev. Lett. 103, 093902 (2009).
[22] H. Schomerus, Phys. Rev. Lett. 104, 233601 (2010).
[23] S. Longhi, Phys. Rev. A 82, 031801(R) (2010); Phys. Rev. Lett. 105, 013903 (2010).
[24] Y. D. Chong, Li Ge, Hui Cao, and A. D. Stone, Phys. Rev. Lett. 105, 053901 (2010).
[25] K. Zhou, Z. Guo, J. Wang and S. Liu, Opt. Lett. 35, 2928 (2010).
[26] M. C. Zheng, D. N. Christodoulides, R. Fleischmann, and T. Kottos, Phys. Rev. A 82, 010103(R) (2010).
[27] E.M. Graefe and H. F. Jones, Phys. Rev. A 84, 012123 (2011).
[28] A. E. Miroshnichenko, B. A. Malomed, and Y. S. Kivshar, Phys. Rev. A 84, 012123 (2011).
[29] H. Ramezani et al., Phys. Rev. A 82, 043818 (2011).
[30] Y. D. Chong, L. Ge, and A. D. Stone, Phys. Rev. A 84, 043820 (2011).
[31] A. A. Sukhorukov, Z. Xu, and Y. S. Kivshar, Phys. Rev. A 82, 043818 (2010).
[32] M. Znojil, Phys. Lett. B 650, 440 (2007); J. Phys. A: Math. Theor. 40, 13131 (2007); Phys. Rev. A 82, 052113 (2010); J. Phys. A: Math. Theor. 44, 075302 (2011); Phys. Lett. A 375, 2503 (2011); Int. J. Theor. Phys. 50, 1614 (2011).
[33] O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, Phys. Rev. Lett. 103, 030402 (2009).
[34] S. Longhi, Phys. Rev. Lett. 103, 123601 (2009); Phys. Rev. B 80, 235102 (2009); Phys. Rev. B 81, 195118 (2010); Phys. Rev. A 82, 032111 (2010); Phys. Rev. B 82, 041106(R) (2010).
[35] H. Zhong, W. Hai, G. Lu, and Z. Li, Phys. Rev. A 84, 013410 (2011).
[36] L. Drissi, E. H. Saidi, and M. Bousmina, J. Math. Phys. 52, 022306 (2011).
[37] L. Jin and Z. Song, Phys. Rev. A 80, 052107 (2009); Phys. Rev. A 81, 032109 (2010).
[38] Y. N. Joglekar, D. Scott, M. Babey, and A. Saxena, Phys. Rev. A 82, 030103(R) (2010).
[39] Y. N. Joglekar and A. Saxena, Phys. Rev. A 83, 050101(R) (2011); Y. N. Joglekar and A. Saxena, Phys. Rev. A 83, 050102(R) (2011); Y. N. Joglekar and J. L. Barnett, Phys. Rev. A 84, 024103 (2011).
[40] D. D. Scott and Y. N. Joglekar Phys. Rev. A 85, 062105 (2012); Phys. Rev. A 86, 043882 (2012); Phys. Rev. A 87, 012106 (2012).
[41] D. D. Scott and Y. N. Joglekar Phys. Rev. A 85, 062105 (2012); Phys. Rev. A 86, 043882 (2012); J. Phys. A: Math. Theor. 45, 444030 (2012).
[42] L. Onsager, Phys. Rev. 65, 117 (1944).
[43] J. B. Kogut, Rev. Mod. Phys. 51, 659 (1979); 55, 775 (1983).
[44] K. Winkler et al., Nature (London) 441, 853 (2006).
[45] N. Hatano and D. R. Nelson, Phys. Rev. Lett. 77, 570 (1996); Phys. Rev. B 56, 8651 (1997).
[46] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[47] L. Jin and Z. Song, Phys. Rev. A 84, 042116 (2011).
[48] L. Jin and Z. Song, Phys. Rev. A 85, 012111 (2012).
[49] L. Jin and Z. Song, Ann. Phys. (NY) 330, 142 (2013).
[50] X. Z. Zhang, L. Jin, and Z. Song, Phys. Rev. A 85, 012106 (2012).
[51] X. Z. Zhang and Z. Song, Phys. Rev. A 87, 012114 (2013).
[52] X. Z. Zhang, L. Jin, and Z. Song, Phys. Rev. A 87, 042118 (2013).
[53] A. Mostafazadeh, J. Math. Phys. 43, 2814 (2002).