Abelian Chern-Simons Vortices and Holomorphic Burgers’ Hierarchy

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Abstract

The Abelian Chern-Simons Gauge Field Theory in 2+1 dimensions and its relation with holomorphic Burgers’ Hierarchy is considered. It is shown that the relation between complex potential and the complex gauge field as in incompressible and irrotational hydrodynamics, has meaning of the analytic Cole-Hopf transformation, linearizing the Burgers Hierarchy in terms of the holomorphic Schrödinger Hierarchy. Then the motion of planar vortices in Chern-Simons theory, appearing as pole singularities of the gauge field, corresponds to motion of zeroes of the hierarchy. Using boost transformations of the complex Galilean group of the hierarchy, a rich set of exact solutions, describing integrable dynamics of planar vortices and vortex lattices in terms of the generalized Kampe de Feriet and Hermite polynomials is constructed. The results are applied to the holomorphic reduction of the Ishimori model and the corresponding hierarchy, describing dynamics of magnetic vortices and corresponding lattices in terms of complexified Calogero-Moser models. Corrections on two vortex dynamics from the Moyal space-time non-commutativity in terms of Airy functions are found.

1 Classical Ferromagnetic Model in Continuous Media

In order to solve the so-called momentum problem in planar ferromagnets a model of delocalized electrons has been introduced by Volovik, where the restoration of the linear momentum density has appeared by using hydrodynamical variables, the density and the normal velocity of the fermionic liquid. On this basis a simple model of ferromagnetic fluid or a spin-liquid, modifying the phenomenological Landau-Lifshitz equation, has been proposed in [1]. In the model,
the magnetic variable is described by the classical spin vector \( \vec{S} = \vec{S}(x, y, t) \) valued on two dimensional sphere: \( \vec{S}^2 = 1 \), and the hydrodynamic variable is velocity \( \vec{v}(x, y, t) \) of the incompressible flow. For particular anisotropic case of the space metric we have the system

\[
\begin{align*}
\partial_1 v_1 - \partial_2 v_2 &= 2 \vec{S} \cdot \nabla (\vec{S} \times \partial_1 \vec{S}) \\
\partial_2 v_1 - \partial_1 v_2 &= 2 \vec{S} \cdot \nabla (\vec{S} \times \partial_2 \vec{S})
\end{align*}
\]

(1)

The first equation is the Heisenberg model where time derivative \( \partial_t \) is replaced by the material derivative \( \partial_t = \partial_t + (\vec{v} \nabla) \), while the second one is relation between hydrodynamic and spin variables called the Mermin-Ho relation \( [2] \). It relates vorticity of the flow with topological charge density, or the winding number for spin. In this case the next theorem is valid \( [3] \).

**Theorem:** For the flow constrained by the incompressibility condition

\[
\partial_1 v_1 + \partial_2 v_2 = 0,
\]

(3)

the conservation law, \( \partial_t J_0 + \partial_1 J_2 - \partial_2 J_1 = 0 \), holds, where

\[
\begin{align*}
J_0 &= (\partial_1 \vec{S})^2 + (\partial_2 \vec{S})^2, \\
J_1 &= -2 \partial_1 \vec{S} \cdot \vec{S} \times (\vec{S} \cdot \partial_1 \vec{S}) + v_1 J_0 + 2 \vec{S} \cdot (\partial_1 \vec{S} \times \partial_2 \vec{S} - \partial_2 \vec{S} \times \partial_1 \vec{S}) \\
J_2 &= 2 \partial_2 \vec{S} \cdot \vec{S} \times (\vec{S} \cdot \partial_2 \vec{S}) + v_2 J_0 + 2 \vec{S} \cdot (\partial_2 \vec{S} \times \partial_1 \vec{S} - \partial_1 \vec{S} \times \partial_2 \vec{S}).
\end{align*}
\]

(4)

Due to the above theorem, for the incompressible flow \( [3] \) the functional \( [4] \) (the ”energy” functional) \( E = \int \int J_0 d^2 x \) is conserved quantity, bounded \( E \geq 8\pi |Q| \) (Bogomolnyi Inequality) by the topological charge \( Q \) of a spin configuration. This inequality is saturated for spin configurations satisfying the first order system, the Belavin-Polyakov self-duality equations

\[
\partial_1 \vec{S} \pm \epsilon_{ij} \vec{S} \times \partial_j \vec{S} = 0
\]

(5)

In contrast to the static case our inequalities are valid for time dependent fields. The stereographic projections of the spin phase space are given by formulas

\[
S_+ = S_1 + iS_2 = \frac{2\zeta}{1 + |\zeta|^2}, \quad S_3 = \frac{1 - |\zeta|^2}{1 + |\zeta|^2}
\]

(6)

where \( \zeta \) is complex valued function. By complex derivatives \( \partial_\zeta = \frac{1}{2}(\partial_1 - i\partial_2) \), \( \partial_\bar{\zeta} = \frac{1}{2}(\partial_1 + i\partial_2) \), the self-duality equations \( [5] \) in stereographic projection form become the analyticity \( \zeta_\bar{z}(z, t) = 0 \) or the anti-analyticity \( \zeta_z(\bar{z}, t) = 0 \) conditions. It is easy to show by direct calculations that for incompressible flow \( [3] \) the holomorphic constraint \( \zeta_\bar{z}(z, t) = 0 \) is compatible with the time evolution \( \frac{\partial}{\partial t} \zeta_\zeta = 0 \).
2 Holomorphic Reduction of Ishimori Model

From the theorem and the proposition presented above we have seen that incompressible flow admits existence of positive energy functional minimized by holomorphic reduction and this reduction is compatible with the time evolution. It suggests to solve the incompressibility conditions explicitly. So we consider the topological magnet model \(1,2\) with incompressibility condition solved in terms of real function \(\psi\), the stream function of the flow, \(v_1 = \partial_2 \psi, v_2 = -\partial_1 \psi\). Then we have analytic reduction of the Ishimori model [4]

\[
i\zeta_t - 2\psi\zeta + 2\zeta_{zz} - \frac{\zeta}{1 + |\zeta|^2}\zeta_z^2 = 0,
\]

\[
\psi_{z\bar{z}} = -\frac{2\zeta \bar{\zeta}}{(1 + |\zeta|^2)^2}.
\]

Choosing the stream function

\[
\psi = 2 \ln(1 + |\zeta|^2)
\]
equation (8) is satisfied automatically, while from (7) we have the holomorphic Schrödinger equation

\[
i\zeta_t + 2\zeta_{zz} = 0.
\]

Every zero of function \(\zeta\) in complex plane \(z\) determines magnetic vortex of the Ishimori model. The spin vector at center of the vortex is \(\vec{S} = (0,0,1)\) while at infinity \(\vec{S} = (0,0,-1)\). Then a motion of zeroes of equation (10) determines the motion of magnetic vortices in the plane. From another side, if we consider analytic function

\[
f(z,t) = \frac{\Gamma}{2\pi i} \log \zeta(z,t)
\]
as the complex potential of an effective flow [3], then every zero of function \(\zeta\) corresponds to hydrodynamical vortex of the flow with intensity \(\Gamma\), and to the simple pole singularity of complex velocity

\[
u(\bar{z},t) = \bar{f}_z = \frac{i\Gamma}{2\pi}(\log \bar{\zeta})\bar{z}.
\]

But the last relation has meaning of the holomorphic Cole-Hopf transformation, according to which the complex velocity is subject to the holomorphic Burgers’ equation

\[
iu_t + 8\pi i u_{\bar{z}z} = 2u_{\bar{z}\bar{z}}
\]

Thus, every magnetic vortex of the Ishimori model corresponds to hydrodynamical vortex of the anti-holomorphic Burgers’ equation. Moreover, relation (11) written in the form

\[
\zeta = e^{2\pi i f} = e^{2\pi i (\phi + i\chi)} = \sqrt{p} e^{2\pi i \phi}
\]
shows that the effective flow is just the Madelung representation for the linear holomorphic Schrödinger equation (10), where functions \(\phi\) and \(\chi\) are the velocity potential and the stream function correspondingly.

3
\section{N-Vortex System}

The system of N magnetic vortices is determined by N simple zeroes

\[ \zeta(z, t) = \prod_{k=1}^{N} (z - z_k(t)) \quad (15) \]

positions of which according to (10) are subject to the system

\[ \frac{dz_k}{dt} = \sum_{k(\neq l)=1}^{N} \frac{4i}{z_k - z_l}. \quad (16) \]

In one space dimension this system has been considered first in [6], (see also [7]) for moving poles of Burgers’ equation, determined by zeroes of the heat equation. However, complexification of the problem has several advantages. First of all the root problem of algebraic equation degree N is complete in the complex domain as well as the moving singularities analysis of differential equations. In contrast to one dimension, in this case the pole dynamics in the plane becomes time reversible (see below) and has interpretation of the vortex dynamics. Moreover generalization (non-integrable) of the system (16) to the case of three particles with different strength has been studied in [8] to explain the transition from regular to irregular motion as travel on the Riemann surfaces.

Below in Section 7 we show that solution of this system is determined by N complex constants of motion, this is why the dynamics of vortices in Ishimori model is integrable. In fact the system (16) admits mapping to the complexified Calogero-Moser (type I) N particle problem [9], [10]. For these we differentiate once and use the system again to have Newton’s equations

\[ \frac{d^2}{dt^2} z_k = \sum_{l=1(\neq k)}^{n} \frac{32}{(z_j - z_k)^3}. \quad (17) \]

These equations have Hamiltonian form with Hamiltonian function

\[ H = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \sum_{k<l}^{n} \frac{16}{(z_k - z_l)^2}. \quad (18) \]

and admit the Lax representation, from which follows the hierarchy of constants of motion in involution \( I_k = tr L^{k+1} \). Recently complexification of the classical Calogero-Moser model and holomorphic Hopf equation has been considered in connection with limit of an infinite number of particles, leading to quantum hydrodynamics and quantum Benjamin-Ono equation [11]. From another side holomorphic version of the Burgers equation is considered in [12] to prove existence and uniqueness of the non-linear diffusion process for the system of Brownian particles with electrostatic repulsion when the number of particles increases to infinity.
4 Integrable N-particle Problem for N-Vortex Lattices

The function $\zeta$ of the form

$$\zeta(z, t) = \sin(z - z_k(t)) = (z - z_k(t)) \prod_{n=1}^{\infty} \left(1 - \frac{(z - z_k(t))^2}{n^2\pi^2}\right)$$ (19)

has periodic infinite set of zeroes and determines the vortex lattice. First for \cite{10} we consider the system of N vortex chain lattices periodic in $x$

$$\zeta(z, t) = e^{-2iN^2t} \prod_{k=1}^{N} \sin(z - z_k(t))$$ (20)

so that positions are subject to the first order system

$$\dot{z}_k = 2i \sum_{l=1(l\neq k)}^{N} \cot(z_k - z_l).$$ (21)

Differentiating this system once in time we get the second order equations of motion in the Newton’s form

$$\ddot{z}_k = 32 \sum_{l} \frac{\cot(z_k - z_l)}{\sin^2(z_k - z_l)}$$ (22)

with the Hamiltonian function of the Calogero-Moser type II model \cite{9}

$$H = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \sum_{k<l} \frac{16}{\sin^2(z_k - z_l)}.$$ (23)

For periodic in $y$ lattices

$$\zeta(z, t) = e^{2iN^2t} \prod_{k=1}^{N} \sinh(z - z_k(t))$$ (24)

we get the Calogero-Moser type III Model

$$H = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \sum_{k<l} \frac{16}{\sinh^2(z_k - z_l)}.$$ (25)

5 Complex Galilean Group and Vortex Generations

Complex Galilean Group is generated by algebra

$$[P_0, P_z] = 0, \quad [P_0, K] = 4i P_z, \quad [P_z, K] = -i$$ (26)
where the energy and momentum operators are $P_0 = -i\partial_t$, $P_z = -i\partial_z$ correspondingly, while the Galilean Boost is operator

$$K = z + 4it\partial_z.$$  \hfill (27)

The Schrödinger operator

$$S = i\partial_t + 2\partial^2_z$$  \hfill (28)

corresponds to the dispersion relation $P_0 = -2P_z^2$ and is commuting with Galilean group

$$[P_0, S] = 0, \quad [P_z, S] = 0, \quad [K, S] = 0.$$  \hfill (29)

From the theory of dynamical symmetry, if exists operator $W$ such that

$$[S, W] = 0 \Rightarrow S(W\Phi) = W(S\Phi) = 0$$  \hfill (30)

then it transforms solution $\Phi$ of the Schrödinger equation to another solution $W\Phi$. It shows that Galilean generators provide dynamical symmetry for the equation. Two of them are evident: time translation $P_0 : e^{itP_0}\Phi(z, t) = \Phi(z, t + t_0)$ and the complex space translation $P_z : e^{itP_z}\Phi(z, t) = \Phi(z + z_0, t)$. While the Galilean boost creates new zero (new vortex in $C$)

$$\Psi(z, t) = K\Phi(z, t) = (z + 4it\partial_z)\Phi(z, t).$$  \hfill (31)

Starting from evident solution $\Phi = 1$ we have the chain of $n$-vortex solutions, $K \cdot 1 = z = H_1(z, 2it)$, $K^2 \cdot 1 = z^2 + 4it = H_2(z, 2it)$, $K^3 \cdot 1 = z^3 + 12it = H_3(z, 2it), ..., K^n \cdot 1 = H_n(z, 2it)$ in terms of the Kampe de Feriet Polynomials \cite{13}

$$H_n(z, it) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(it)^k z^{n-2k}}{k!(n-2k)!}.$$  \hfill (32)

They satisfy the recursion relations

$$H_{n+1}(z, it) = \left(z + 2it \frac{\partial}{\partial z}\right) H_n(z, it),$$  \hfill (33)

$$\frac{\partial}{\partial z} H_n(z, it) = nH_{n-1}(z, it)$$  \hfill (34)

and can be written in terms of Hermite polynomials

$$H_n(z, 2it) = (-2it)^{n/2} H_n \left( \frac{z}{2\sqrt{-2it}} \right).$$  \hfill (35)

Let $w_n^{(k)}$ is the $k$-th zero of the Hermite polynomial, $H_n(w_n^{(k)}) = 0$, then evolution of corresponding vortex is given by

$$z_k(t) = 2w_n^{(k)}\sqrt{-2it}.$$  \hfill (36)
Under the time reflection $t \rightarrow -t$ position of the vortex rotates on 90 degrees $z_k \rightarrow z_k e^{i\pi/2}$. The last one is also symmetry of the vortex equations (16). Using formula

$$H_n(z, 2it) = \exp \left( it \frac{\partial^2}{\partial z^2} \right) z^n \quad (37)$$

and superposition principle we have solution

$$\Phi(z, t) = \sum_{n=0}^{\infty} a_n H_n(z, 2it) = \sum_{n=0}^{\infty} a_n \exp \left( 2it \frac{\partial^2}{\partial z^2} \right) z^n = \exp \left( 2it \frac{\partial^2}{\partial z^2} \right) \sum_{n=0}^{\infty} a_n z^n$$

So if $\chi(z) = \sum_{n=0}^{\infty} a_n z^n$ is any analytic function, then $\Phi(z, t) = \exp \left( 2it \frac{\partial^2}{\partial z^2} \right) \chi(z)$ is solution, determined by integrals of motion $a_0, a_1, ...$. Therefore for polynomial degree $n$ describing evolution of $n$ vortices, we have $n$ complex integrals of motion.

The generating function of the Kampe de Feriet Polynomials

$$\sum_{n=0}^{\infty} \frac{k^n}{n!} H_n(z, it) = e^{kz + ik^2 t} \quad (38)$$

is also solution of the plane wave type. If we exponentiate the Galilean boost $e^{i\lambda K} = e^{i\lambda (z + 4it\partial_z)}$ and factorize it by Baker-Hausdorff formula $e^{A+B} = e^B e^A e^{[A,B]/2}$, so that $e^{i\lambda K} = e^{i\lambda z + 2i\lambda^2 t} e^{-4\lambda t \partial_z}$, then applied on a solution $\Phi(z, t)$ it gives

$$e^{i\lambda K} \Phi(z, t) = e^{i\lambda z + 2i\lambda^2 t} \Phi(z - 4\lambda t, t) \quad (39)$$

the Galilean boost with velocity $4\lambda$, where the generating function of vortices (38) appears as the 1-cocycle.

The Galilean boost (31) $\Psi(z, t) = (z + 4it\partial_z) \Phi$ connecting two solutions of the holomorphic Schrödinger equation (10) generates the auto-Bäcklund transformation:

$$v = u + \frac{i\Gamma}{2\pi} \partial_{\bar{z}} \ln(\bar{z} - \frac{8\pi t}{\Gamma} u) \quad (40)$$

between two solutions

$$u(\bar{z}, t) = \frac{i\Gamma}{2\pi} \frac{\Phi_{\bar{z}}}{\Phi}, \quad v(\bar{z}, t) = \frac{i\Gamma}{2\pi} \frac{\Psi_{\bar{z}}}{\Psi} \quad (41)$$

of the anti-holomorphic Burgers equation (13).

As an example we consider double lattice solution

$$\zeta(z, t) = e^{-8it} \sin(z - z_1(t)) \sin(z + z_1(t)) \quad (42)$$

where $\cos 2z_1 = re^{8it}$, $r$ is a constant. Applying the boost transformation (31) we have solution describing collision of a vortex with the double lattice

$$\Psi(z, t) = \left( z + 4it \frac{\partial}{\partial z} \right) \zeta(z, t) \quad (43)$$
Generalizing we have $N$-vortices interacting with $M$-vortex lattices

$$
\Psi(z, t) = e^{iMt} \left( z + 4it \frac{\partial}{\partial z} \right)^N \prod_{k=1}^M \sin(z - z_k(t)) \tag{44}
$$

where $z_1, ..., z_k$ are subject to the system \[21\].

6 Abelian Chern-Simons Theory and Complex Burgers’ Hierarchy

Now we show as the anti-holomorphic Burgers hierarchy appears in the Chern-Simons gauge field theory. The Chern-Simons functional is defined as follows

$$
S(A) = \frac{\kappa}{4\pi} \int_M A \wedge dA = \frac{\kappa}{4\pi} \int \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} \tag{45}
$$

where $M$ is an oriented three-dimensional manifold, $A$ is U(1) gauge connection, $\kappa$ the coupling constant - the statistical parameter. In the canonical approach $M = \Sigma \times \mathbb{R}$, where $\mathbb{R}$ we interpret as a time. Then $A_i = (A_0, A_i)$, $(i = 1, 2)$, where $A_0$ is the time component and the action takes the form

$$
S = -\frac{\kappa}{4\pi} \int dt \int_{\Sigma} \varepsilon^{ij} \left( A_i \frac{d}{dt} A_j - A_0 F_{ij} \right) \tag{46}
$$

In the first order formalism it implies that the Poisson brackets

$$
\{A_i(x), A_j(y)\} = \frac{4\pi}{\kappa} \varepsilon_{ij} \delta(x - y) \tag{47}
$$

and the Hamiltonian $H = A_0 \varepsilon^{ij} F_{ij}$. The last one is weakly vanishing $H \approx 0$ due to the Chern-Simons Gauss law constraint

$$
\partial_1 A_2 - \partial_2 A_1 = 0 \implies F_{ij} = 0 \tag{48}
$$

Then the evolution is determined by Lagrange multipliers $A_0$: $\partial_0 A_1 = \partial_1 A_0$, $\partial_0 A_2 = \partial_2 A_0$. Due to the gauge invariance $A_\mu \to A_\mu + \partial_\mu \lambda$, to fix the gauge freedom we choose the Coulomb gauge condition: $\text{div} \vec{A} = 0$. In addition we have the Chern-Simons Gauss law \[45\]: $\text{rot} \vec{A} = 0$. These two equations are identical to the incompressible and irrotational hydrodynamics. Solving the first equation in terms of the velocity potential $\varphi$: $A_k = \partial_k \varphi$, $(k = 1, 2)$, and the second one in terms of the stream function $\psi$: $A_1 = \partial_2 \psi$ and $A_2 = -\partial_1 \psi$; we have the Cauchy- Riemann Equations: $\partial_1 \varphi = \partial_2 \psi$, $\partial_2 \varphi = -\partial_1 \psi$. So these two functions are harmonically conjugate and the complex potential $f(z) = \varphi(x, y) + i\psi(x, y)$ is analytic function of $z = x + iy$, $\partial f/\partial \bar{z} = 0$. Corresponding "the complex gauge potential" $A = A_1 + iA_2 =']'(z)$ is an anti-analytic function. In analogy with hydrodynamics, the logarithmic singularities of the complex potential

$$
f(z, t) = \frac{1}{2\pi i} \sum_{k=1}^N \Gamma_k \text{Log}(z - z_k(t)) \tag{49}
$$
determine poles of the complex gauge field

\[ A = \frac{i}{2\pi} \sum_{k=1}^{N} \frac{\Gamma_k}{\bar{z} - z_k(t)} \]  

(50)

describing point vortices in the plane. Then the corresponding "statistical" magnetic field

\[ B = \partial_1 A_2 - \partial_2 A_1 = -\Delta \psi = -\Delta \Im f(z) \]  

(51)

where \( \Delta \) is the Laplacian, determined by the stream function

\[ \psi = -\frac{1}{2\pi} \sum_{k=1}^{N} \Gamma_k \text{Log}|z - z_k(t)| \]  

(52)

is

\[ B = \frac{1}{2\pi} \sum_{k=1}^{N} \Gamma_k \Delta \text{Log}|z - z_k(t)| = \sum_{k=1}^{N} \Gamma_k \delta(\bar{r} - \bar{r}_k(t)). \]  

(53)

Corresponding total magnetic flux is

\[ \int_{R^2} \int B d^2x = \sum_{k=1}^{N} \int \int \Gamma_k \delta(\bar{r} - \bar{r}_k(t))d^2x = \Gamma_1 + \Gamma_2 + \ldots + \Gamma_N. \]  

(54)

The above relation (53) has interpretation as the Chern-Simons Gauss law

\[ B = \frac{1}{\kappa} \bar{\psi} \psi = \frac{1}{\kappa} \rho \]  

(55)

for point particles located at \( \bar{r}_k(t) \) with density

\[ \rho = \sum_{k=1}^{N} \Gamma_k \delta(\bar{r} - \bar{r}_k(t)) \]  

(56)

(with masses \( \Gamma_1, \Gamma_2, \ldots, \Gamma_N \)). Then magnetic fluxes are superimposed on particles and has meaning of anyons. As a result, an integrable evolution of the complex gauge field singularities (vortices) would lead to the integrable evolution of anyons. Evolution of the anti-holomorphic complex gauge potential is determined by equation, \( \partial_0 A = 2\partial_\bar{z} A_0 \), where as follows function \( A_0 \) is harmonic \( \Delta A_0 = 0 \), and is given by \( A_0 = \frac{1}{2} [F_0(\bar{z}, t) + \bar{F}_0(z, t)] \). Then the evolution equation is

\[ \partial_0 A = \partial_\bar{z} F_0. \]  

(57)

Let

\[ F_0 = \sum_{n=0}^{\infty} c_n F_0^{(n)}(\bar{z}, t) \]  

(58)

where

\[ F_0^{(n)}(\bar{z}, t) = (\partial_\bar{z} + A(\bar{z}, t))^n \cdot 1 \]  

(59)
then for arbitrary positive integer $n$ we have the anti-holomorphic Burgers Hierarchy
\[ \partial_{t_n} A(\bar{z}, t) = \partial_{\bar{z}}[(\partial_{\bar{z}} + A(\bar{z}, t))^n \cdot 1]. \tag{60} \]
Using the recursion operator $R = \partial_{\bar{z}} + \partial_{\bar{z}} A \partial_{\bar{z}}^{-1}$ we write it in the form
\[ \partial_{t_n} A = R^{n-1} \partial_{\bar{z}} A. \tag{61} \]

The above hierarchy can be linearized by anti-holomorphic Cole-Hopf transformation for the complex gauge field $A = \bar{\Phi}_{\bar{z}} \bar{\Phi} = (\ln \bar{\Phi})_{\bar{z}} = (f(z, t))_{\bar{z}} \tag{62}$

in terms of the holomorphic Schrödinger (Heat) Hierarchy
\[ \partial_{t_n} \Phi = \partial^n_{\bar{z}} \Phi. \tag{63} \]

For $n = 2$ the second member of the hierarchy is just \[10\] and zeroes of this equation corresponds to magnetic vortices of the Ishimori model. The relation between $\Phi$ and complex potential $f$ has meaning of the Madelung representation for the hierarchy
\[ \Phi(z, t) = e^{f(z, t)} = e^{e^+ i\psi} = (e^\varphi)(e^{i\psi} = \sqrt{\rho} e^{i\psi}. \tag{64} \]

Therefore hierarchy of equations for $f$ is the Madelung form of the holomorphic Schrödinger hierarchy\[ \partial_{t_n} f = (\partial_{\bar{z}} + \partial_{\bar{z}} f)^n \cdot 1 = e^{-f} \partial_{\bar{z}}^n e^{-f} \tag{65} \]
or
\[ \partial_{t_n} (e^f) = \partial_{\bar{z}}^n (e^f) \tag{66} \]
which is the potential Burgers’ hierarchy. We have the next Linear Problem for the Burgers hierarchy
\[ \Phi_{\bar{z}} = \bar{A} \Phi, \quad \Phi_{t_n} = \partial_{\bar{z}}^n \Phi. \tag{67} \]

It can be written as the Abelian zero-curvature representation for the holomorphic Burgers hierarchy, $\partial_{t_n} U - \partial_{\bar{z}} V_n = 0$, where $U = A, V_n = (\partial_{\bar{z}} + A)^n \cdot 1$. For the $N$-vortices of equal strength
\[ \Phi(z, t) = e^{f} = \prod_{k=1}^{N} (z - z_k(t)) \tag{68} \]
positions of the vortices correspond to zeroes of $\Phi(z, t)$. As a result the vortex dynamics, leading to integrable anyon dynamics, is related to motion of zeroes subject to the vortex equations \[10\] for $n = 2$ case and for arbitrary $n$ to equation
\[ -\frac{dz_k(t_n)}{dt_n} = Res_{z=z_k} \left( \partial_{\bar{z}} + \sum_{l=1}^{N} \frac{1}{z - z_l(t_n)} \right)^n \cdot 1, \quad (k = 1, ..., N). \tag{69} \]
7 Galilean Group Hierarchy and Vortex Solutions

Now we consider complex Galilean Group hierarchy

\[
\begin{align*}
[P_0, P_z] &= 0, \quad [P_0, K_n] = i^n n P^{n-1}_z, \quad [P_z, K_n] = -i
\end{align*}
\]  

(70)

where hierarchy of the boost transformations is generated by

\[
K_n = z + nt \partial^{n-1}_z
\]

(71)

is commuting with the holomorphic \( n \)-Schrödinger equation

\[
S_n = \partial_t - \partial^n_z.
\]

(72)

As a result, application of \( K_n \) to solution \( \Phi \) creates solution with additional vortex

\[
\Psi(z, t) = K_n \Phi(z, t) = (z + nt \partial^{n-1}_z) \Phi(z, t).
\]

(73)

For particular values we have

\[
K_n \cdot 1 = z = H_1^{(n)}(z, t), \quad K_n^2 \cdot 1 = z^2 = H_2^{(n)}(z, t),
\]

..., \( K_n^{n-1} \cdot 1 = z^{n-1} = H_{n-1}^{(n)}(z, t), \quad K_n^n \cdot 1 = z^n + n! t = H_n^{(n)}(z, t), \quad ..., K_m^n \cdot 1 = H_m^{(n)}(z, t),
\]

where the generalized Kampe de Feriet polynomials \([15]\) are

\[
H_m^{(n)}(z, t) = m! \sum_{k=0}^{[m/n]} \frac{k^m z^{m-nk}}{k!(m-nk)!}.
\]

(74)

satisfy the holomorphic Schrödinger hierarchy \([63]\)

\[
\frac{\partial}{\partial t} H_m^{(n)}(z, t) = \partial^n_z H_m^{(n)}(z, t).
\]

(75)

The generating function is given by

\[
\sum_{m=0}^{\infty} \frac{k^m}{m!} H_m^{(n)}(z, t) = e^{kz + k^n t}.
\]

(76)

From operator representation

\[
H_n^{(N)}(z, t) = \exp \left( t \frac{\partial^N}{\partial z^N} \right) z^n \Rightarrow \Phi(z, t) = \exp \left( t \frac{\partial^N}{\partial z^N} \right) \psi(z)
\]

(77)

we have solution of \([63]\) in terms of arbitrary analytic function \( \psi \). Polynomials \( H_m^{(N)}(z, t) \) are connected with the generalized Hermite polynomials \([14]\)

\[
H_m^{(N)}(z, t) = t^{[m/N]} H_m^{(N)} \left( \frac{z}{\sqrt{t}} \right).
\]

(78)
Then the $k$-th zero $w_n^{(N)k}$ of generalized Hermite polynomial $H_n^{(N)}$ determine evolution of the corresponding vortex

$$H_n^{(N)}(w_n^{(N)k}) = 0 \Rightarrow z_k(t) = w_n^{(N)k} e^{Nt}.$$  \hspace{1cm} (79)

The zeroes are located on the circle in the plane with time dependent radius. When $t \to -t$ position of the vortex rotate on angle $z_k \to z_k e^{i\pi/N}$. The Galilean boost hierarchy (73) provides the Bäcklund transformation for n-th member of anti-holomorphic Burgers hierarchy (60)

$$v = u + \partial_z \ln[z + Nt(\partial_z + u)^{N-1} \cdot 1].$$  \hspace{1cm} (80)

8 The Negative Burgers’ Hierarchy

The holomorphic Schrödinger hierarchy and corresponding Burgers hierarchy can be analytically extended to negative values of $N$. Introducing negative derivative (pseudo-differential) operator $\partial^{-1} z$, so that, $\partial^{-m} z^n = \frac{z^n}{(n+1)\cdots(n+m)}$, we have the hierarchy

$$\partial_{t-n} \Phi = \partial^{-n} \Phi$$ \hspace{1cm} (81)

or differentiating $n$ times, in pure differential form $\partial_{t-n} \partial^n \Phi = \Phi$. In terms of $A$ defined by (62) we have the negative Burgers hierarchy

$$\partial_{t-n} A = \partial^{-1}_z \frac{1 - \partial_{t-n} A^n}{A^n}.$$ \hspace{1cm} (82)

For $n = 1$ we have equation $\partial_{t-1} \Phi = \partial^{-1} \Phi$ or the Helmholz equation $\partial_{t-1} \partial_z \Phi = \Phi$. Analytical continuation of the generalized Kampe de Feriet polynomials to $n = -1$ [15] is given by

$$H_M^{(-1)}(z, t) = M! \sum_{k=0}^{\infty} t^k z^{M+k}.$$ \hspace{1cm} (83)

Then

$$H_M^{(-1)}(z, t) = e^{t\partial^{-1}_z} H_M^{(-1)}(z, 0)$$ \hspace{1cm} (84)

$$H_M^{(-1)}(z, 0) = z^M.$$ \hspace{1cm} (85)

Moreover higher order functions are generated by the “negative Galilean boost”

$$H_M^{(-1)}(z, t) = (z - t\partial^{-2}_z)^M H_0^{(-1)}(z, t).$$ \hspace{1cm} (86)

Functions $H_M^{(-1)}(z, t)$ are related with Bessel functions [15]. First, they are directly related with the Tricomi functions

$$C_M(zt) = \frac{z^{-M}}{M!} H_M^{(-1)}(z, t)$$ \hspace{1cm} (87)

\[12\]
determined by the generating function

$$\sum_{M=-\infty}^{\infty} \lambda^M C_M(x) = e^{\lambda + x/\lambda}. \quad (88)$$

The last one is connected with Bessel functions according to

$$J_M(x) = \left(\frac{x}{2}\right)^M C_M\left(-\frac{x^2}{4}\right). \quad (89)$$

Then we have explicitly

$$H_M^{(-1)}(z,t) = M! \left(\frac{-z}{t}\right)^{M/2} J_M(2\sqrt{-zt}) \quad (90)$$

This provides solution of the negative (-1) flow Burgers equation

$$\partial_z A = \partial_x 1 - \partial_x A \quad (91)$$

in the form

$$A = \frac{(H_M^{(-1)}(\bar{z},t))_{\bar{z}}}{H_M^{(-1)}(\bar{z},t)} = \frac{M}{2\bar{z}} + \sqrt{\frac{t}{-\bar{z}}} \frac{J_M^2}{J_M} = \sqrt{\frac{t}{-\bar{z}}} \frac{J_{M-1}(2\sqrt{-zt})}{J_M(2\sqrt{-zt})}. \quad (92)$$

For arbitrary member of the negative hierarchy we have

$$H_M^{(-N)}(z,t) = e^{i\bar{z}^N} H_M^{(-N)}(z,0) \quad (93)$$

$$H_M^{(-N)}(z,0) = z^M \quad (94)$$

and relation

$$W_M^{(N)}(zt^{1/N}) = z^{-M} M! H_M^{(-N)}(z,t) \quad (95)$$

where the Wright-Bessel functions \[15\] \[ W_M^{(N)}(x) \] are given by generating function

$$\sum_{M=-\infty}^{\infty} \lambda^M W_M^{(N)}(x) = e^{\lambda + x/\lambda}. \quad (96)$$

9 Space-Time Noncommutativity

Now we consider influence of the space-time non-commutativity on the vortex dynamics. Remarkable is that for two vortices the problem can be solved explicitly. Non-commutative Burgers’ equation, its linearization by Cole-Hopf transformation and two-shock soliton collision has been considered in \[16\]. Here we consider the holomorphic heat equation

$$\partial_t \Phi = \nu \partial_z^2 \Phi \quad (97)$$
with two-vortex solution in the form
\[ \Phi(z, t) = (z - z_1(t)) \ast (z - z_2(t)) \] (98)
where the Moyal product is defined as
\[ f(t, z) \ast g(t, z) = e^{i\theta(\partial_t \partial_z - \partial_z \partial_t)} f(t, z) g(t', z') |_{z = z', t = t'} \] (99)
\[ = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \sum_{k=0}^{n} (-1)^k \left( \frac{n}{k} \right) (\partial_t^{n-k} \partial_z^k f)(\partial_t^k \partial_z^{n-k} g) \] (100)

Then we have \( \theta \) deformed vortex equations
\[ \dot{z}_1 = -2\nu z_1 - i\theta \ddot{z}_2 \]
\[ \dot{z}_2 = 2\nu z_1 + i\theta \ddot{z}_1 \] (101)

Adding we have the first integral of motion \( z_1 + z_2 = C \) - the center of mass. Choosing beginning of coordinates in the center of mass we have \( C = 0 \) and \( z_2 = -z_1 \). Integrating reduced equation for \( z_1 \) and substituting \( z_1 = 2i\theta Y(t) \) we obtain the Ricatti equation
\[ \dot{Y} + Y^2 = \frac{\nu}{2\theta^2}(t - t_0). \] (102)

This can be linearized by \( Y = \frac{\dot{\psi}}{\psi} \) in terms of the Airy Equation
\[ \ddot{\psi} = \frac{\nu}{2\theta^2}(t - t_0) \psi. \] (103)
The solution is
\[ z_1(t) = 2i\theta \frac{Ai'(\beta(t - t_0))}{Ai(\beta(t - t_0))} = -i \sqrt{2\nu(t - t_0)} \frac{K_{2/3}(\sqrt{2\nu(t - t_0)^{3/2}})}{K_{1/3}(\sqrt{2\nu(t - t_0)^{3/2}})} \] (104)
where \( \beta = (\nu/2\theta^2)^{1/3} \) and \( K_n \) are modified Bessel functions of fractional order. This solution should be compared with the undeformed one [36]. The noncommutative corrections are coming from the ratio of two Bessel functions depending on \( \theta \). Using asymptotic form of Airy function we have correction in the form
\[ z_1(t) = -z_2(t) \approx -i \sqrt{2\nu(t - t_0)} - \frac{i\theta}{2(t - t_0)} \] (105)
as \( t \to +\infty \). As easy to see the correction is independent of diffusion coefficient and has the global character.

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