On the Zero-Slope Limit of the Compactified Closed Bosonic String

R. Marotta\textsuperscript{a,b} and F. Pezzella\textsuperscript{b}

\textsuperscript{a} Dipartimento di Scienze Fisiche, Università di Napoli
Mostra d’Oltremare, Pad. 19, I-80125 Napoli, Italy
\textsuperscript{b} I.N.F.N. - Sezione di Napoli
Mostra d’Oltremare, Pad. 20, I-80125 Napoli, Italy

Abstract

In the framework of the compactified closed bosonic string theory with the extra spatial coordinates being circular with radius $R$, we perform both the zero-slope limit and the $R \to 0$ limit of the tree scattering amplitude of four massless scalar particles. We explicitly show that this double limit leads to amplitudes involving scalars which interact through the exchange of a scalar, spin 1 and spin 2 particle. In particular, this latter case reproduces the same result obtained in linearized quantum gravity.

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1. String theories have to reproduce, at the “low-energy” limit in which the slope $\alpha'$ goes to zero, ordinary field theories formulated in a number of space-time dimensions, which coincides with the dimensionality $D$ of the space-time in which the string is embedded ($D = 26$ for the bosonic string). In order to reduce this dimension to $D = 4$ a compactification scheme must be adopted. Our work just goes in the direction to show explicitly how, with a suitable compactification procedure, string amplitudes lead to the ones of four-dimensional field theories in the above mentioned limit.

We will consider here the so-called “toroidal compactification” of the bosonic closed string, in which twenty-two spatial coordinates are compactified into circles with radius $R$ and the compactified space becomes a lattice which is required to be Lorentzian, self-dual and even in order to have consistency with the properties of the bosonic string theory.

Constructing such a lattice yields the introduction of Lie algebra lattices where the massless states which arise from the toroidal compactification lie on the root lattice and belong to the adjoint representation of the gauge group relative to the algebra.

After having endowed the theory with the above compactification procedure, the limit $\alpha' \to 0$ makes all the massive modes uncouple giving rise to a description in terms of only massless states.

In particular we consider tree scattering amplitudes of massless scalar particles in the compactified bosonic closed string theory. We perform the double limit $\alpha' \to 0$ and $R \to 0$, keeping the ratio $a = R/\sqrt{\alpha'}$ fixed; in so doing, we obtain amplitudes corresponding to the exchange of a scalar, spin 1 and spin 2 particle. In particular this latter amplitude is coincident with the one obtained in the framework of linearized quantum gravity. An analogous computation was performed in the context of the generalized dual Virasoro model.

2. The toroidal compactification consists in associating the $d$ internal extra-coordinates of the string to $d$ circles having radius $R_i$ with $i = 1, \ldots, d$; this can be done by identifying the points of the internal space as follows:

$$X^I \equiv X^I + 2\pi L^I$$

where $I = 1, \ldots, d$ and

$$L^I = \sqrt{\frac{1}{2} \sum_{i=1}^{d} n_i R_i e_i^I}$$

being $n_i \in Z$ a so-called “winding number”. The vectors $\hat{e}_i \equiv (e_i^1, \ldots, e_i^d)$ are
linearly independent and normalized as follows:

\[ \hat{e}_i \cdot \hat{e}_i = 2. \]

The quantities \( L^I \)'s can be thought as the components of a vector defined on a \( d \)-dimensional lattice \( \Lambda^d \) which admits as a basis the set of vectors \( \{ \sqrt{\frac{2}{R_i}} \hat{e}_i \} \) with \( i = 1, \ldots, d \). It follows that the torus on which we compactify is the quotient space:

\[ T^d = \frac{\mathbb{R}^d}{2\pi \Lambda^d}. \]

One gets the following mode expansion for the compactified string field \( X^I \) [3]:

\[ X^I(z, \bar{z}) = X^I_L(z) + X^I_R(\bar{z}) \]

with

\[ X^I_L(z) = x^I_L - \frac{\alpha'}{2} p^I_L \log z + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n z^{-n} \]  

(1)

\[ X^I_R(\bar{z}) = x^I_R - \frac{\alpha'}{2} p^I_R \log \bar{z} + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n \bar{z}^{-n} \]  

(2)

where

\[ x^I_L = \frac{1}{2} x^I + \frac{\alpha'}{2} Q^I \]

\[ x^I_R = \frac{1}{2} x^I - \frac{\alpha'}{2} Q^I \]

being \( Q^I \) the operator canonically conjugate to \( L^I \), here introduced in order to define completely independent left and right sectors [3]; furthermore,

\[ p^I_R = p^I - \frac{L^I}{\alpha'} ; \quad p^I_L = p^I + \frac{L^I}{\alpha'}. \]

The following commutation relations hold:

\[ [x^I_L, p^J_L] = [x^I_R, p^J_R] = i \delta^{IJ} \]

with all the other commutators vanishing. The compactification of the internal spatial coordinates implies that also the momenta \( p^I \)'s, which represent the translation operators of those coordinates, lie on a \( d \)-dimensional lattice that is the dual of the lattice \( \Lambda^d \) and it is denoted by \( (\Lambda^d)^* \), i.e.:

\[ p^I = \sqrt{2} \sum_{i=1}^{d} \frac{m_i}{R_i} \hat{e}^*_i \]  

(3)
being the vector $\hat{e}_i^*$ the dual of $\hat{e}_i$. A basis on such a lattice is given by the
vectors $\{\sqrt{\alpha'} R_i \hat{e}_i\}$. In this compactification scheme, in which we consider $R_i = R$
$\forall i = 1, \ldots, d$, the constraint conditions of the bosonic closed string become:

\[ L_0 - 1 = 0 \iff \frac{\alpha'}{4} m^2 = \frac{\alpha'}{4} p_R^2 + N - 1 \quad (4) \]
\[ \bar{L}_0 - 1 = 0 \iff \frac{\alpha'}{4} m^2 = \frac{\alpha'}{4} p_L^2 + \bar{N} - 1 \quad (5) \]

The conditions (4) and (5) can be rewritten as follows:

\[ \frac{\alpha'}{2} m^2 = \frac{\alpha'}{4} \left( p_R^2 + p_L^2 \right) + N + \bar{N} - 2 \quad (6) \]
\[ N - \bar{N} = \frac{\alpha'}{4} \left( p_L^2 - p_R^2 \right). \quad (7) \]

From here it is possible to observe that the bivector

\[ \hat{P} \equiv \left( \sqrt{\frac{\alpha'}{2} p_R}, \sqrt{\frac{\alpha'}{2} p_L} \right) \]

lies on an even lattice $\Gamma_{d,d}$, after having chosen the metric of the lattice to be of
the form $((+1)^d, (-1)^d)$ (Lorentzian lattice); furthermore the modular invariance
imposes that such a lattice must be self-dual too.

Another condition that must be imposed on the lattice comes out from the
following considerations.

By analyzing the lattice $\Gamma_{d,d}$ it turns out that the right and the left
components of the bivector $\hat{P}$ can be written as:

\[ \sqrt{\frac{\alpha'}{2} p_R} = \sum_{i=1}^{d} m_i \left( \frac{\sqrt{\alpha'}}{R} \right) \hat{e}_i^* - \frac{1}{2} \sum_{i=1}^{d} n_i \left( \frac{R}{\sqrt{\alpha'}} \right) \hat{e}_i, \quad (8) \]
\[ \sqrt{\frac{\alpha'}{2} p_L} = \sum_{i=1}^{d} m_i \left( \frac{\sqrt{\alpha'}}{R} \right) \hat{e}_i^* + \frac{1}{2} \sum_{i=1}^{d} n_i \left( \frac{R}{\sqrt{\alpha'}} \right) \hat{e}_i. \quad (9) \]

The expressions (8) and (9) can be generalized by adding a constant back-
ground anti-symmetric tensor field $B_{ij}$ to the usual action of the bosonic string;
this operation is necessary to get more general and larger gauge groups [2] [3].
Taking into account this generalization we can rewrite the components of the
bivector $\hat{P}$ as follows:

\[ \sqrt{\frac{\alpha'}{2} p_R} = \sum_{i=1}^{d} m_i \left( \frac{\sqrt{\alpha'}}{R} \right) e_i^{*i} - \frac{1}{2} \sum_{i=1}^{d} n_i \left( \frac{R}{\sqrt{\alpha'}} \right) e_i^i \]

\[ \sqrt{\frac{\alpha'}{2} p_L} = \sum_{i=1}^{d} m_i \left( \frac{\sqrt{\alpha'}}{R} \right) e_i^{*i} + \frac{1}{2} \sum_{i=1}^{d} n_i \left( \frac{R}{\sqrt{\alpha'}} \right) e_i^i \]

\[ - \sum_{i,j=1}^{d} B_{ij} n_j \left( \frac{\sqrt{\alpha'}}{R} \right) e_i^{*i} \]
\[
\sqrt{\alpha'} p_L' = \sum_{i=1}^{d} m_i \left( \frac{\sqrt{\alpha'}}{R} \right) \hat{e}^i_t + \frac{1}{2} \sum_{i=1}^{d} n_i \left( \frac{R}{\sqrt{\alpha'}} \right) \hat{e}^i_t - \sum_{i,j=1}^{d} B_{ij} n_j \left( \frac{\sqrt{\alpha'}}{R} \right) \hat{e}^i_t
\]

The vector basis in \( \Gamma_{d,d} \) are evidently \( \frac{R}{\sqrt{\alpha'}} \hat{e}^i_t \) and the dual ones are \( \frac{R}{\sqrt{\alpha'}} \hat{e}^i_t \).

In the double limit \( \alpha' \to 0, R \to 0 \), \( p_L \) and \( p_R \) will be well-defined quantities only if the ratio \( a = \frac{R}{\sqrt{\alpha'}} \) is kept fixed \([5]\); in particular we choose \( a = 1 \) \([1]\). This choice leads to a rational lattice.

In conclusion, the lattice on which we compactify must be Lorentzian, self-dual, even and rational.

It is known that a large class of such lattices can be constructed in \( \mathbb{R}^{d,d} \) considering the set of all vectors of the form \((v_1, v_2)\) so that \( v_1 \) and \( v_2 \) belong to the same conjugacy class of a semi-simple Lie algebra of rank \( d \) \([3]\).

By evaluating the double limit \( R = \sqrt{\alpha'} \to 0 \) on the equations \((5)\) and \((7)\) one has that the only particles with finite masses which survive are massless particles for which the norm of the components of \( \hat{P} \) is null or equal to 2. In particular, these latter are by definition lattice roots. The roots include in any case those of the Lie algebra used in constructing the lattice, but in some cases there may be additional norm 2 vectors in the other conjugacy classes.

We are going to compactify on a lattice where all the norm 2 vectors belong to the root lattice of a simply laced Lie algebra. On this kind of lattices the scalar product between the vectors, which survive after performing the double limit, takes integer values. This will greatly simplify our computation.

A possible lattice satisfying the above requirements is \( \Gamma_{d,d} = \Gamma_d \otimes \Gamma_d \) with \( \Gamma_d = E_8 \otimes E_8 \otimes E_6 \); however there exist a large class of Lorentzian, self-dual, even and rational lattices that well suite to our problem.

3. We are going to consider scattering amplitudes involving scalar particles. These come from the levels reported in table 1, together with the corresponding norms of \( p_R \) and \( p_L \).

In order to compute those scattering amplitudes, we are going to consider a compactified version of the vertex operators \( V_\psi(z, \bar{z}) \) providing the amplitude for the emission of a state \( \psi \) in the string spectrum.

The compactified vertices can be obtained from the ordinary ones through simple correspondences; for example, the compactified vertex relative to the scalar particles belonging to the level \( N = \bar{N} = 0 \) is the naive version of the compactified tachyon vertex:

\[
V_0(k, k_R, k_L; z, \bar{z}) =: e^{ikX(z, \bar{z})} e^{ik_R X_R(z)} e^{ik_L X_L(\bar{z})}. 
\]
where \( X^\mu(z, \bar{z}) \) is the usual string field, with \( \mu = 1, \ldots, 26 - d; X^I_R \) and \( X^I_L \), with \( I = 1, \ldots, d \) are the fields defined in (2). The state corresponding to \( V_0 \) is defined, as usual, through the following limit:

\[
\lim_{z, \bar{z} \to 0} V_0(k, k_R, k_L; z, \bar{z}) |\text{vacuum}\rangle.
\]

Conformal invariance requires \( V_0 \) to be a conformal field with dimensions \( \Delta = \bar{\Delta} = 1 \); since the following operator product expansion holds between \( V_0 \) and the stress energy tensor:

\[
T(z)V_0(w, \bar{w}) = \left[ \frac{1}{z-w} \partial_w + \frac{(k^2 + k^2_R) \alpha'}{4} \right] V_0(w, \bar{w})
\]

it follows that

\[
k^2 + k^2_R = \frac{4}{\alpha'}
\]

with a similar relation holding for the anti-holomorphic sector. This shows again that for massless scalar particles one has \( k^2 = 0 \) and \( k^2_R = k^2_L = \frac{4}{\alpha'} \).

These considerations suggest to introduce the following general vertex, which is nothing but a linear combination of the vertices relative to the massless scalar particles introduced in table 1:

\[
V_s =: e^{ik X(z, \bar{z})} \left[ e^{ik_R X_R(z)} + \xi \cdot \partial_z X_R(z) \right] \left[ e^{ik_L X_L(\bar{z})} + \bar{\xi} \cdot \partial_{\bar{z}} X_L(\bar{z}) \right] : \tag{10}
\]

where \( \xi \) and \( \bar{\xi} \) are polarization vectors defined in the compactified space.

By using the version (10) of the vertex operators it is straightforward to compute the tree scattering amplitude of four scalar particles in \( D \) dimensions, by using the operatorial formalism of the \( N \)-String Vertex [6] [see fig. 1]:

\[
A = N_0 \int dzd\bar{z} \ z^{\frac{\alpha'}{2}k_3k_4 + n_{34}} (1 - z)^{\frac{\alpha'}{2}k_2k_3 + n_{23}} \bar{z}^{\frac{\alpha'}{2}k_3k_4 + \bar{n}_{34}} (1 - \bar{z})^{\frac{\alpha'}{2}k_2k_3 + \bar{n}_{23}} \tag{11}
\]

where \( N_0 \) is a suitable normalization constant dictated by unitarity and given explicitly by:

\[
N_0 = \frac{2}{\pi} g^2_D
\]

with \( g_D \) being the \( D \)-dimensional coupling constant [8], [9], [10], related to the 26-dimensional one and to the compactification radius by:

\[
g^2_D = g^2_{26} (2\pi R)^{-26+D}.
\]

Furthermore:

\[
n_{ij} = \frac{\alpha'}{2} k_{R,i} \cdot k_{R,j}
\]
\[ \bar{n}_{ij} = \frac{\alpha'}{2} k_{L,i} \cdot k_{L,j}. \]

Eq. (11) has been obtained by performing an average on the polarizations and a sum on all the possible values of the roots of the Lie algebra is understood; both of these operations make the sum of all the terms involving \( \xi \) and \( \bar{\xi} \) null.

Eq. (11) can be considered as a generalization of the dual Virasoro amplitude; its dependence on the lattice is entirely contained in the variables \( n_{ij} \). These are characterized by taking integer values, as we have previously seen as a consequence of our choice of the lattice. On the other hand, the conservation of the winding numbers and the compactified momenta [7] imply the following constraints:

\[ \hat{k}_{R,1} + \hat{k}_{R,2} + \hat{k}_{R,3} + \hat{k}_{R,4} = 0 \]

from which

\[ n_{13} + n_{23} + n_{34} = -2 \] (12)

and

\[ n_{ij} = \frac{\alpha'}{2} \hat{k}_{R,i} \cdot \hat{k}_{R,j} \leq \frac{\alpha'}{2} |\hat{k}_{R,i}|^2 = 2 \]

i.e.

\[ -2 \leq n_{ij} \leq 2 \] (13)

Analogously \(-2 \leq \bar{n}_{ij} \leq 2\). Furthermore, since the lattice can be chosen in such a way that the vectors \( \hat{k}_{R,i} \) are roots of the Lie algebra used in constructing the lattice, the scalar product between two of them must be integer.

By using standard techniques it is possible to write the eq. (11) in the following form:

\[
A_{\text{tree}}(s, t, u) = N_0 \pi \frac{\Gamma \left[ -\frac{\alpha'}{4} s + n_{23} + 1 \right] \Gamma \left[ -\frac{\alpha'}{4} t + n_{34} + 1 \right] \Gamma \left[ -\frac{\alpha'}{4} u + \bar{n}_{13} + 1 \right]}{\Gamma \left[ \frac{\alpha'}{4} s - \bar{n}_{23} \right] \Gamma \left[ \frac{\alpha'}{4} t - \bar{n}_{34} \right] \Gamma \left[ \frac{\alpha'}{4} u - n_{13} \right]}. \]

(14)

Eq. (14) holds if at least two of the differences \( n_{ij} - \bar{n}_{ij} \) are non negative. Otherwise, one gets:

\[
A_{\text{tree}}(s, t, u) = N_0 \pi \frac{\Gamma \left[ -\frac{\alpha'}{4} s + \bar{n}_{23} + 1 \right] \Gamma \left[ -\frac{\alpha'}{4} t + \bar{n}_{34} + 1 \right] \Gamma \left[ -\frac{\alpha'}{4} u + n_{13} + 1 \right]}{\Gamma \left[ \frac{\alpha'}{4} s - n_{23} \right] \Gamma \left[ \frac{\alpha'}{4} t - n_{34} \right] \Gamma \left[ \frac{\alpha'}{4} u - \bar{n}_{13} \right]}. \]

(15)

The amplitude (15) corresponds to (14) in which the variables \( n_{ij} \) and \( \bar{n}_{ij} \) are interchanged.

The amplitude (14) has poles compatible with the mass formula (4). Analogously the amplitude (15) has poles consistent with the constraint (5).
We are now interested in performing the limit $\alpha' \to 0$ of the scattering amplitudes (14) [or, equivalently, of (15)]. Taking into account the possible values of the variables $n_{ij}$ and by using the analytic properties of the $\Gamma$-function, it is straightforward to obtain the following result in the limit $\alpha' \to 0$:

$$A_{\text{tree}}(s, t, u) = A_{\text{tree}}^{(\lambda\phi^4)} + A_{\text{tree}}^{(\lambda\phi^3)} + A_{\text{tree}}^{(\text{spin }1)} + A_{\text{tree}}^{(\text{spin }2)}$$

with

$$A_{\text{tree}}^{(\lambda\phi^4)} \sim g_D^2$$

$$A_{\text{tree}}^{(\lambda\phi^3)} = -2 \frac{4}{\alpha'} g_D^2 \left[ \frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right]$$

$$A_{\text{tree}}^{(\text{spin }1)} = 2 g_D^2 \left[ \frac{u + t}{s} + \frac{s + u}{t} + \frac{s + t}{u} \right]$$

$$A_{\text{tree}}^{(\text{spin }2)} = 2 \frac{\alpha'}{4} g_D^2 \left[ \frac{tu}{s} + \frac{su}{t} + \frac{st}{u} \right]$$

The amplitudes (17), (18) and (19) represent scalars interacting with the exchange respectively of a scalar ($\lambda\phi^3$ theory, with $\lambda^2 \equiv 2\frac{4}{3} g_D^2$), spin 1 (“scalar electrodynamics”) and spin 2 particle (quantum gravity). The constant value (16) can be interpreted as a tree diagram of an “effective” $\lambda\phi^4$ theory, coming from a tree diagram of a $\lambda\phi^3$ theory on which the double limit produces an ultralocal limit of the propagator.

From the comparison of the amplitude (19) with the analogous one computed in quantum gravity, we can obtain a relationship between the string coupling constant $g_D$ and the gravitational coupling constant $G_N$ in $D = 4$ dimensions:

$$g_D^2 = \frac{16\pi G_N}{\alpha'}.$$

This expression coincides with the one already known in literature [10].

In conclusion we have explicitly shown that the compactification of the closed bosonic string theory reproduces only in the double limit $\alpha' \to 0$ and $R \to 0$, at the tree level, the ordinary field theories. We would like to stress here that in our work specifying the lattice has resulted to be unnecessary: hence, at least at that level, compactification does not influence the low-energy limit.

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Figure Captions.

FIG. 1 Scattering of four scalar particles.
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| $N$ | $\bar{N}$ | $p_R^2$ | $p_L^2$ |
|-----|---------|--------|--------|
| 0   | 0       | $4/\alpha'$ | $4/\alpha'$ |
| 0   | 1       | $4/\alpha'$ | 0      |
| 1   | 0       | 0       | $4/\alpha'$ |
| 1   | 1       | 0       | 0      |

Tab. 1
This figure "fig1-1.png" is available in "png" format from:

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