Dual Methods for Lattice Field Theories at Finite Density

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Sign Problem, Complex Action Problem

Non-zero chemical Potential $\mu$ in lattice QCD $\rightarrow \det[D] \in \mathbb{C}$ \(\times\)

$\rightarrow$ No phase diagram with standard Monte Carlo techniques!

Similar problems for other (lattice) field theories

Complex action $S$ for $\mu \neq 0$ \(\times\)

$\rightarrow$ Boltzmann factor $e^{-S} \in \mathbb{C}$ $\rightarrow$ no probabilistic interpretation

Model dependent solution: reformulation in terms of dual variables

Start with simple models and try to generalize!
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Dual Formulation – General Idea

**Partition function** \( Z \in \mathbb{R} \)

\[
Z = \int \mathcal{D}[\phi] \ e^{-S[\phi]} \quad \text{with} \quad e^{-S} \in \mathbb{C} \quad \text{and} \quad \phi \ldots \text{“conventional fields”}
\]

→ try to find representation, such that

\[
Z = \sum_{\{l\}} W[l] \quad \text{with} \quad W \in \mathbb{R}^+ \quad \text{and} \quad l \ldots \text{new degrees of freedom}
\]

→ probabilistic interpretation

\[
P[l] \equiv \frac{W[l]}{Z} \quad \ldots \quad \text{probability weight of configuration } l
\]
$\phi^4$ Model on the Lattice

**Continuum action**

\[
S = \int d^4x \left[ |\partial_{\nu} \phi|^2 + (m^2 - \mu^2) |\phi|^2 + \lambda |\phi|^4 + \mu (\phi^* \partial_4 \phi - \partial_4 \phi^* \phi) \right]
\]

**Lattice action**

\[
S = \sum_x \left( \sum_{\nu=1}^{4} \left( e^{\mu \delta_{\nu,4}} \phi_x \phi^{*}_{x+\hat{\nu}} + e^{-\mu \delta_{\nu,4}} \phi^{*}_x \phi_{x+\hat{\nu}} \right) + \kappa |\phi_x|^2 + \lambda |\phi_x|^4 \right)
\]

with kinetic, mass and self-interaction terms.

Chemical potential $\mu$ couples only in 4-direction (time).

→ **Complex action problem** for $\mu \neq 0$. 


Dual Formulation of the $\phi^4$ Model – Sketch

**Partition function**

$$Z = \int \mathcal{D}[\phi] \ e^{-S} \sim \int \mathcal{D}[\phi] \ \prod_{x,\nu} e^{S_{x,\nu}} \sim \int \mathcal{D}[\phi] \ \prod_{x,\nu} \sum_{l_{x,\nu}=0}^{\infty} \frac{(S_{x,\nu})^{l_{x,\nu}}}{l_{x,\nu}!}$$

Integrating out the original fields $\phi_x, \phi_x^*$ in terms of radial and angular parts ($\phi_x = r_x \ e^{i \theta_x}$)

→ **Kronecker deltas** (constrain the summation variables)

→ **Weight factors** (numerical integrals)
→ Partition function in dual representation

\[
Z = \sum_{\{k,m\}} \prod_{x,\nu} \left( \frac{1}{(|k_{x,\nu}| + m_{x,\nu})! \cdot m_{x,\nu}!} \prod_{x} \delta \left( \sum_{\nu} [k_{x,\nu} - \hat{k}_{x-\hat{\nu},\nu}] \right) \right) \\
\times \prod_{x} e^{\mu k_{x,4}} \mathcal{W} \left( \sum_{\nu} \left[ |k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(m_{x,\nu} + m_{x-\hat{\nu},\nu}) \right] \right)
\]

- **New degrees of freedom** (dual variables, fluxes)
  \[k_{x,\nu} \in \mathbb{Z} \text{ (constrained)} \text{ and } m_{x,\nu} \in \mathbb{N}_0 \text{ (unconstrained)}\]

- **Weight factors** \(\mathcal{W}(n_x)\) (numerical integrals)

- **Complex action problem is gone**
  \[Z = \sum_{\{k,m\}} P[k,m] \quad \text{with} \quad P[k,m] \in \mathbb{R}^+ \sim \text{probability weight}\]

- **Exact rewriting**
Numerical Simulation

.constraint \iff flux conservation at each lattice site (closed loops)

→ Generalization of the Prokof’ev-Svistunov worm algorithm

• Standard Metropolis update sweep for the $m$-variables
• Worm update for the restricted $k$-variables
Results

Particle number density

\[ \langle n \rangle \propto \frac{\partial \ln Z}{\partial \mu} \propto \left\langle \sum_{k_x,4} k_x,4 \right\rangle \sim \text{winding number} \]

The Silver Blaze Phenomenon

At \( T = 0 \) physics is independent of \( \mu < \mu_{\text{crit}} \).

second order phase transition at

\[ \mu_{\text{crit}} = m_{\text{ren}} = 1.146(1) \]

\[ \kappa = 9, \ \lambda = 1, \ T = 0 \]
Analysis at different temperatures (and observables, parameter sets, ...)

→ **Phase diagram** and **relativistic Bose condensation**

![Graph showing phase diagram and relativistic Bose condensation](image)

- **Silver Blaze region**
  - $\kappa = 7.44$ ($\chi'|\Phi|^2$ data)
  - $\kappa = 7.44$ ($\chi'n$ data)
  - $\kappa = 9$

- **Strong $\mu$-dependence**

- **Pseudo Silver Blaze behavior**
2-Point Functions

Correlators

\[ \langle \phi_a \phi_b^* \rangle = \frac{1}{Z} \int D[\phi] \, e^{-S} \, \phi_a \phi_b^* \equiv \frac{1}{Z} \, Z_{a,b} \]

cannot be expressed as partial derivatives of \( Z \).

Dual representation of \( Z_{a,b} \)

\[ \sum_{\{k,m\}} \prod_x \delta \left( \sum_\nu \left[ \ldots \right] - \delta_{x,a} + \delta_{x,b} \right) \mathcal{W} \left( \sum_\nu \left[ \ldots \right] + \delta_{x,a} + \delta_{x,b} \right) \left( \ldots \right) \]

→ modified arguments of

- Kronecker deltas and
- Weightfactors at lattice sites \( a \) and \( b \).
Numerical Simulation

New Constraint for $Z_{a,b}$

\[ \mathcal{Z} \equiv \sum_{a,b} Z_{a,b} \sim \text{closed loops (} a = b \text{) } + \text{ open lines (} a \neq b \text{)} \]

Enlarged ensemble (Korzec et. al., Comput. Phys. Commun. 182 (2011))

\[ C(t) \propto \langle \delta_{t,a_4-b_4} \rangle \mathcal{Z} \propto e^{-mt} \]
\[ \mu_{\text{crit}} = 0.170(1) \]

\[ m_{\text{red}} \equiv E_{\pm} \pm \mu \]

\[ E_{+} = 0.1687(3) \]

\[ E_{-} = 0.1684(3) \]
Fits to the slopes behave exactly as expected!
Conclusion

- Different models can be mapped to a dual representation → **Complex action problem solved.**

- **Physical Observables** (e.g. particle number) and properties (e.g. phase diagram) can be studied **in terms of dual variables.**

- Generalization allows us to carry out **spectroscopy** calculations for **non-zero chemical potential.**

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Outlook

• Continue work on correlators.

• **Find dual formulation for models with non-abelian degrees of freedom.**

• At the moment: SU(2) spin model

\[
Z = \int \mathcal{D}[U] \prod_{x, \nu} e^{\beta \text{Tr}[U_x U_{x+\nu}^\dagger]} + h \text{Tr}[U_x] \quad U_x \in SU(2)
\]

successfully reformulated.