Multiplex Networks with Intrinsic Fitness:
Modeling the Merit-Fame Interplay via Latent Layers

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We consider the problem of growing multiplex networks with intrinsic fitness and inter-layer coupling. The model comprises two layers; one that incorporates fitness and another in which attachments are preferential. In the first layer, attachment probabilities are proportional to fitness values, and in the second layer, proportional to the sum of degrees in both layers. We provide analytical closed-form solutions for the joint distributions of fitness and degrees. We also derive closed-form expressions for the expected value of the degree as a function of fitness. The model alleviates two shortcomings that are present in the current models of growing multiplex networks: homogeneity of connections, and homogeneity of fitness. In this paper, we posit and analyze a growth model that is heterogeneous in both senses.

\textit{keywords:} multiplex networks, intrinsic fitness, growing networks, preferential attachment, Stirling numbers.

I. INTRODUCTION

Multiplex networks are mathematical tools for modeling systems with multiple types of interaction. The system is conceptualized as being comprised of multiple layers, each hosting a distinct type of link (which corresponds to a type of interaction) between nodes. The set of nodes are the same for all layers. Many real systems have been modeled under the multiplex framework, such as citation networks \cite{1, 2}, online social media \cite{3}, airline networks \cite{4}, scientific collaboration networks \cite{5}, urban transportation networks \cite{6}, and online games \cite{7}. Theoretically, multiplex networks demonstrate how incorporating additional dimensions and types of interaction to simple one-layer systems can change their dynamics, add new properties and alter existing ones. Diverse processes have been studied theoretically on multiplex networks. Examples include epidemics \cite{8, 9}, pathogen-awareness interplay \cite{8}, percolation processes \cite{10, 11}, evolution of cooperation \cite{12, 13}, diffusion processes \cite{14} and social contagion \cite{15}. For thorough reviews, see \cite{16, 17}.

In the present paper we focus on the problem of growing multiplex networks with fitness. Previous studies on growing multiplex networks exhibit two main shortcomings: homogeneity of growth, and homogeneity (or absence) of nodal fitness. In \cite{18}, a growing two-layer network is studied, and various attachment kernels are envisaged. The number of links established by each newcomer is considered to be the same for both layers (In the Supplemental Material of \cite{18}, the possibility of heterogeneous growth rates is entertained in the asymptotic mean-field analysis of degrees of individual nodes within each layer. However, the effect of growth heterogeneity on the single-layer and inter-layer degree distributions remains unknown). Similarly, the model posited and thoroughly analyzed in \cite{19} exhibits homogeneity in the sense that, for a given node, the expected degree is the same across layers. In real systems, the nature of the connections in different layers differ, since they pertain to distinct types of interaction. For example, in \cite{6}, the interactions between the players of a massive online game is mapped onto six distinct layers, and their average degrees are different. It would be plausible to devise a growth model which incorporates heterogeneity explicitly. In \cite{20}, the problem of homogeneity is alleviated by considering heterogeneous link growth rates. In the present paper, we consider heterogeneous link growth rates.

Another unrealistic assumption that is made by the previous studies on growing multiplex networks is that the probability for each existing node to receive links from incoming nodes only depends on their degrees. For example, if we consider the network of citations between scientific papers, this assumption would mean that the inherent quality and novelty of the papers have no role in the future number of citations that they would receive. That is, only fame drives scientific success, not quality: when scholars cite a paper, they only take into account the number of citations that an existing paper has. This is obviously not the case. Similarly, consider the case of online social networks such as Twitter, Instagram, Pinterest, Google+ and Tumblr. In all these networks, each user can ‘follow’ other users. Assuming that links are established only based on existing degrees—and not incorporating any intrinsic fitness for the nodes—would be synonymous with disregarding the role of the quality of the content produced by each user on her/his popularity. True that after a user becomes famous, the fame on its own contributes to further accumulation of followers (which is the rationale behind all preferential at-
attachment models), but quality also has an undeniable role—especially, at the initial stages of the lifetime of each node (user). This motivates us to consider intrinsic fitness for nodes. In [21][23], intrinsic fitness is envisaged in the case of single-layer networks. To our knowledge, no fitness-based model on multiplex networks exists in the literature.

We consider a growing directed multiplex network that comprises two layers. Each node is assigned an intrinsic fitness, which models its quality. The fitness of a node never changes. Each node belongs to two layers: a merit layer and a fame layer. In the former, fitness values are the sole drivers of the growth mechanism. In the fame layer, attachment is preferential, that is, the probability that a node receives a link from a newcomer is proportional to the total degree of that node, i.e., the sum of its degrees in both layers. For example, in the case of citation networks, the interpretation of the model is as follows. Two distinct types of citations can be discerned. The first type—the meritocratic type—is when a scholar reads a paper, and cites it because of its content (a citation which would be given regardless of the number of citations that paper already has). Another type of citation is what we call fame-driven. A paper can become trendy, or well-known in some literature in inchoate stages, many papers will be remote from those seminal papers, but the sets of links differ. The system initially comprises \( N(0) \) nodes, each with two types of links. The links are assumed to be established on two separate layers, layer 1 (the merit layer) and layer 2 (the fame layer). Let us emphasize that layers embody the set of nodes, but the sets of links differ. Suppose that there are \( L_1(0) \) links in the first layer and \( L_2(0) \) links in the second layer at the outset.

The network grows by the successive addition of new nodes. Time increments in discreet steps, and at each timestep one new node is added to the network. Each incoming node establishes \( \beta_1 \) layer-1 links and \( \beta_2 \) layer-2 links to the existing nodes. Upon being born, the fitness of an incoming node is drawn from \( \rho(\theta) \) and stays the same thereafter. The mean value of the fitness distribution, that is, the expected value of the fitness of incoming nodes, is denoted by \( \mu \).

In the first layer, the probability of receiving a link from an incoming node for node \( x \) is proportional to \( \theta_x \), where \( \theta_x \) is the fitness of node \( x \). In the second layer, the probability of node \( x \) receiving a link from the newcomer is proportional to \( k_x + \ell_x \). The probability of receiving a link in layer 1 can be written as \( \frac{\theta_x}{\sum_x \theta_x} \). The sum in the denominator can be computed at time \( t \) as follows. If the sum of the fitness values of the nodes at time \( t = 0 \) is \( \Theta \), then as time progresses, the sum of the fitness values of nodes converges to \( \Theta + \mu t \), where \( \mu \) is the mean of the fitness distribution. Since we will eventually limit the analysis to the steady state, the error of this
FIG. 1: Schematic illustration of the growth mechanism. A set of \(7\) nodes exist in the network, and node \(8\) is being appended. In the merit layer, only the fitness values drive nodes’ chances of receiving links from node \(8\) (who establishes \(\beta_1\) links in this layer). In the fame layer, the probability of receiving a link is proportional to total degree. For example, node \(1\) will have the highest chance of receiving a link, because it has the greatest total degree (which is equal to six). Conversely, node \(7\) has the smallest total degree (which is zero), and will not get a link from node \(8\) in the fame layer.

approximation vanishes. For layer 2, the probability of receiving a link for node \(x\) is equal to \(\frac{k_x + \ell_x}{\sum_x (k_x + \ell_x)}\). The sum in the denominator at any time equals the total number of links in both layers.

IV. JOINT INTERLAYER DISTRIBUTION OF DEGREES AND FITNESS

At time \(t\), upon the addition of the new node, certain events can change the value of \(N_x(k, \ell, \theta)\). If a node of \((k - 1, \ell, \theta)\) receives a layer-1 link, then it becomes a \((k, \ell, \theta)\) node, and \(N_x(k, \ell, \theta)\) increments consequently. Similarly, if a node of \((k, \ell - 1, \theta)\) receives a layer-2 link, then it becomes a \((k, \ell, \theta)\) node, and \(N_x(k, \ell, \theta)\) increments consequently. On the other hand, if a node is already a \((k, \ell, \theta)\) node and it receives a link in either layer, it will no longer be a \((k, \ell, \theta)\) node, and \(N_x(k, \ell, \theta)\) decrements consequently. Also, note that the layer-1 degree and layer-2 degree of each incoming node is \(0\) upon introduction, and such a node has fitness \(\theta\) with probability \(\rho(\theta)\). The following rate equation summarizes these events with their respective probabilities of occurrence, where \(E\{\cdot\}\) denotes expected value:

\[
E\{N_{t+1}(k, \ell, \theta)\} = N_t(k, \ell, \theta)
+ \beta_1 \frac{\theta N_t(k-1, \ell, \theta) - \theta N_t(k, \ell, \theta)}{\Theta + \mu t}
+ \beta_2 \frac{(k + \ell - 1)N_t(k, \ell - 1, \theta) - (k + \ell)N_t(k, \ell, \theta)}{L_1(0) + L_2(0) + (\beta_1 + \beta_2)t}
+ \delta_{k0} \delta_{\ell0} \rho(\theta).
\]

Hereinafter, we drop the expected value operator, and all the \(P(k, \ell, \theta)\) numbers denote expected values. Using the relation \(N_t(k, \ell, \theta) = (N(0) + t)P_t(k, \ell, \theta)\), we can rewrite \(1\) to quantify the evolution of \(P_t(k, \ell, \theta)\) as follows:

\[
[N(0) + t] [P_{t+1}(k, \ell, \theta) - P_t(k, \ell, \theta)] + P_{t+1}(k, \ell, \theta) =
\frac{\beta_1}{\Theta + \mu t} \theta N_t(k-1, \ell, \theta) - \theta N_t(k, \ell, \theta)
+ \frac{\beta_2}{L_1(0) + L_2(0) + (\beta_1 + \beta_2)t} \frac{(k + \ell - 1)N_t(k, \ell - 1, \theta) - (k + \ell)N_t(k, \ell, \theta)}{L_1(0) + L_2(0) + (\beta_1 + \beta_2)t}
+ \delta_{k0} \delta_{\ell0} \rho(\theta).
\]

Note that negative \(k\) or \(\ell\) does not have a physical meaning, so \(k\) and \(\ell\) in equation \(2\), as well as every equation henceforth, are nonnegative integers.

Now we focus on the steady state, where by definition, the values of \(P_t(k, \ell, \theta)\) reach horizontal asymptotes and their variations vanish. Also we note that in the limit as \(t \to \infty\), we have

\[
\begin{align*}
\lim_{t \to \infty} \beta_1 \frac{N(0) + t}{\Theta + \mu t} &= \frac{\beta_1}{\Theta + \mu} \\
\lim_{t \to \infty} \beta_2 \frac{N(0) + t}{L_1(0) + L_2(0) + (\beta_1 + \beta_2)t} &= \frac{\beta_2}{\beta_1 + \beta_2}
\end{align*}
\]

Using these limits, we can rewrite \(2\) for the steady state as follows:

\[
P(k, \ell, \theta) = \frac{\beta_1}{\mu} \theta \left[ P(k - 1, \ell, \theta) - P(k, \ell, \theta) \right]
+ \frac{\beta_2}{\beta_1 + \beta_2} \left( (k + \ell - 1)P(k, \ell - 1, \theta) - (k + \ell)P(k, \ell, \theta) \right)
+ \delta_{k0} \delta_{\ell0} \rho(\theta).
\]

This can be rearranged and recast as

\[
P(k, \ell, \theta) \left[ 1 + \frac{\beta_1 \theta}{\mu} \left( \frac{\beta_1 + \beta_2}{\beta_2} \right) + k + \ell \right] =
\frac{\beta_1 (\beta_1 + \beta_2) \theta}{\beta_2 \mu} P(k - 1, \ell, \theta) + (k + \ell - 1)P(k, \ell - 1, \theta)
+ \frac{\beta_1 + \beta_2}{\beta_2} \delta_{k0} \delta_{\ell0} \rho(\theta).
\]
Dividing both sides by the factor on the left hand side, this transforms into

\[
P(k, \ell, \theta) = \left( \frac{\beta_1 (\beta_1 + \beta_2) \theta}{\beta_2 \mu} \right) \frac{P(k-1, \ell, \theta)}{\left( 1 + \frac{\beta_1 \theta}{\mu} \right) \left( \frac{\beta_1 + \beta_2}{\beta_2} \right) + k + \ell} + \left( k + \ell - 1 \right) P(k, \ell - 1, \theta) + \frac{\delta_{k0} \delta_{\ell0} \rho(\theta)}{G_\theta}.
\]

Note that \( k + \ell \) is an upper index, not a power. We can readily rewrite (10) in terms of \( \phi \). The difference equation reads

\[
\phi_k^{k+\ell}(\theta) = \phi_k^{k+\ell-1}(\theta) + (k + \ell - 1)\phi_k^{k+\ell-1}(\theta) + A\delta_{k0} \delta_{\ell0} \rho(\theta) \Gamma(A G_\theta).
\]

Hereinafter, for brevity of notation, we denote \( \frac{(\beta_1 + \beta_2)}{\beta_2} \) by \( A \), and we denote \( 1 + \frac{\beta_1 \theta}{\mu} \) by \( G_\theta \). Thus the difference equation we need to solve takes the following form

\[
P(k, \ell, \theta) = \frac{A \theta \beta_1}{\mu} P(k-1, \ell, \theta) \frac{\left( \frac{\beta_1 + \beta_2}{\beta_2} \right)}{\left( 1 + \frac{\beta_1 \theta}{\mu} \right) \left( \frac{\beta_1 + \beta_2}{\beta_2} \right) + k + \ell} + \left( k + \ell - 1 \right) \frac{A G_\theta}{k + \ell} + \frac{\delta_{k0} \delta_{\ell0} \rho(\theta)}{G_\theta}.
\]

(7)

Let us define

\[
\psi(k, \ell, \theta) \equiv \frac{\Gamma(A G_\theta + k + \ell + 1)}{\left( \frac{\beta_1 + \beta_2}{\beta_2} \right)^k} P(k, \ell, \theta).
\]

(8)

It is easy to verify the following relations using the properties of the Gamma function:

\[
\begin{cases}
A \theta \beta_1 P(k - 1, \ell, \theta) = \frac{\psi(k - 1, \ell, \theta)}{\left( \frac{\mu}{\mu A G_\theta} \right)^k \Gamma(A G_\theta + k + \ell + 1)} \\
P(k, \ell - 1, \theta) = \frac{\psi(k, \ell - 1, \theta)}{\left( \frac{\mu}{\mu A G_\theta} \right)^k \Gamma(A G_\theta + k + \ell + 1)}.
\end{cases}
\]

(9)

We substitute the first two terms on the right hand side of (8) with the expressions given in (9). Then we multiply both sides by the factor \( \left( \frac{\mu}{\mu A G_\theta} \right)^k \Gamma(A G_\theta + k + \ell + 1) \). We arrive at the following difference equation:

\[
\psi(k, \ell, \theta) = \psi(k - 1, \ell, \theta) + (k + \ell - 1) \psi(k, \ell - 1, \theta) + A \delta_{k0} \delta_{\ell0} \rho(\theta) \Gamma(A G_\theta).
\]

(10)

Without loss of generality, we can take \( k, k + \ell \) to be the arguments of the function instead of \( k, \ell \). Let us define the new auxiliary function:

\[
\phi_k^{k+\ell}(\theta) \equiv \psi(k, \ell, \theta).
\]

(11)

This is the recurrence relation which defines the unsigned Stirling numbers of the first kind. We denote the Stirling numbers by \( [\frac{n}{k}] \) in this paper. Incorporating the initial conditions, the solution to (13) is

\[
\phi_k^n = [\frac{n}{k}] \times A \Gamma(A G_\theta) \rho(\theta).
\]

(14)

Replacing \( n \) with \( k + \ell - 1 \), we have:

\[
\phi_k^{k+\ell-1}(\theta) = [\frac{k + \ell - 1}{k}] \times A \Gamma(A G_\theta) \rho(\theta).
\]

(15)

Comparing this with (11), we find the solution for \( \psi \) to be as follows:

\[
\psi(k, \ell, \theta) = [\frac{k + \ell}{k}] A \Gamma(A G_\theta) \rho(\theta).
\]

(16)

This readily yields \( P(k, \ell, \theta) \), using (8). We get

\[
P(k, \ell, \theta) = [\frac{k + \ell}{k}] A \Gamma(A G_\theta) \rho(\theta) \left( \frac{\left( \frac{\beta_1 + \beta_2}{\beta_2} \right)^k}{\Gamma(A G_\theta + k + \ell + 1)} \right).
\]

(17)

Plugging in the explicit expressions for \( A \) and \( G_\theta \), we arrive at the final solution:

\[
P(k, \ell, \theta) = \left[ \frac{k + \ell}{k} \right] \left( \frac{\beta_1 + \beta_2}{\beta_2} \right) \left( \frac{\left( \frac{\beta_1 + \beta_2}{\beta_2} \right)^k}{\Gamma(A G_\theta + k + \ell + 1)} \right) \frac{\Gamma\left( \frac{\beta_1 + \beta_2}{\beta_2} \left( 1 + \frac{\beta_1 \theta}{\mu} \right) + k + \ell + 1 \right)}{\rho(\theta)}.
\]

(18)
V. COLLAPSED JOINT DISTRIBUTION OF DEGREE AND FITNESS

In real settings, the total number of links received by nodes are observed. For example, in the network of citations between scientific papers, what is observed and documented is \(k + \ell\), that is, the total number of links (citations) received by papers. One cannot observe the number of citations that a paper receives purely based on its merit (meritocratic attachment), or the number of citations it receives due to its popularity (fame-driven attachment). This motivates us to derive the distribution of the total number of links, that is, \(k + \ell\). Let us denote it by \(q\). The joint distribution of \(q, k\) is simply \(P(k, q - k, \theta)\). If we sum over all possible values of \(k\), we get:

\[
P(q, \theta) = \frac{\left(\frac{\beta_1 + \beta_2}{\beta_2}\right) \Gamma \left[\frac{\beta_1 + \beta_2}{\beta_2} \left(1 + \frac{\beta_1 \theta}{\mu}\right)\right]}{\Gamma \left[\frac{\beta_1 + \beta_2}{\beta_2} \left(1 + \frac{\beta_1 \theta}{\mu}\right) + q + 1\right]} \rho(\theta) \times \sum_{k=0}^{q} \binom{q}{k} \left(\frac{(\beta_1 + \beta_2)\beta_1}{\beta_2} \frac{\theta}{\mu}\right)^k.
\]

(19)

From the properties of the unsigned Stirling numbers of the first kind, the sum on the right hand side of (19) can be readily evaluated. We have:

\[
P(q, \theta) = \frac{\left(\frac{\beta_1 + \beta_2}{\beta_2}\right) \Gamma \left[\frac{\beta_1 + \beta_2}{\beta_2} \left(1 + \frac{\beta_1 \theta}{\mu}\right)\right]}{\Gamma \left[\frac{\beta_1 + \beta_2}{\beta_2} \left(1 + \frac{\beta_1 \theta}{\mu}\right) + q + 1\right]} \rho(\theta) \times \frac{\Gamma \left[q + \frac{(\beta_1 + \beta_2)\beta_1}{\beta_2} \frac{\theta}{\mu}\right]}{\Gamma \left[q + 1 + \frac{\beta_1 + \beta_2}{\beta_2} \left(1 + \frac{\beta_1 \theta}{\mu}\right)\right]}.
\]

(20)

This is depicted in Figure 2.

If we use the generalization of binomial coefficients to non-integers, we can express (20) more concisely. Let us use the following notation for binomial coefficients:

\[
\binom{a}{b} = \frac{\Gamma(a + 1)}{\Gamma(b + 1)\Gamma(a - b + 1)},
\]

(21)

where \(a, b\) need not be integers. Using this notation, we can express (20) equivalently as follows:

\[
P(q, \theta) = \frac{\beta_1 + \beta_2}{\beta_1 + 2\beta_2} \left(\frac{1 + \frac{\beta_1 \theta}{\mu}}{1 + \frac{\beta_1 \theta}{\mu}} - 1\right) \times \frac{\beta_1 + \beta_2}{\beta_1 + \beta_2} \left(1 + \frac{\beta_1 \theta}{\mu}\right) + q\right) \rho(\theta).
\]

(22)

Now let us investigate the asymptotic behavior of \(P(q, \theta)\) for large values of \(q\). From (20), we observe that only two terms have \(q\). Using the Stirling approximation, we have

\[
P(q, \theta) \sim \frac{(q + AG_\theta)^{q + AG_\theta - \frac{1}{2} e^{-(q + AG_\theta)}}}{(q + AG_\theta + A + 1)^{q + AG_\theta + A + 1} e^{-(q + AG_\theta + A + 1)}}.
\]

(23)

So we arrive at the following asymptotic relation

\[
P(q, \theta) \sim q^{2 - \frac{\beta_1}{\beta_2}} \forall \theta.
\]

(24)

Note that the exponent does not depend on \(\theta\). This means that the degree distribution of the subpopulations with any fitness value follows the same exponent. In other words, the rate at which the degree distribution vanishes is the same for all fitness values. The relative chances of different nodes attaining extremely large degrees depend only on their fitness values, and not the degree itself, because if we divide the respective...
probabilities, only the fitness-dependent multiplicative factors would determine the ratio, as the $q$-dependent parts cancel out. This is illustrated in Figure 3. Another implication of (24) is that the total degree distribution of the network, i.e., $P(q)$, has a power-law tail with exponent $2 + \frac{\beta_1}{\beta_2}$.

VI. EXPECTED DEGREE DISTRIBUTION AS A FUNCTION OF FITNESS

It is straightforward to compute the expected value of the degree distribution (20). We have

$$\langle q \rangle_\theta = \sum_{q=0}^{\infty} q P(q|\theta) = \sum_{q=0}^{\infty} q \frac{P(q, \theta)}{\rho(\theta)}$$

$$= \left( \frac{\beta_1 + \beta_2}{\beta_2} \right) \Gamma \left[ \frac{\beta_1 + \beta_2}{\beta_2} (1 + \frac{\beta_1}{\theta}) \right]$$

$$\Gamma \left( \frac{\beta_1 + \beta_2}{\beta_2} \frac{\theta}{\mu} \right) \times \sum_{q=0}^{\infty} q \frac{\Gamma \left( q + \left( \frac{\beta_1 + \beta_2}{\beta_2} \frac{\theta}{\mu} \right) \right)}{\Gamma \left( q + 1 + \frac{\beta_1 + \beta_2}{\beta_2} (1 + \frac{\beta_1}{\theta}) \right)}.$$  \hspace{1cm} (25)

We now perform the following summation:

$$S = \sum_{q=0}^{\infty} \frac{\Gamma \left( q + \left( \frac{\beta_1 + \beta_2}{\beta_2} \frac{\theta}{\mu} \right) \right)}{\Gamma \left( q + 1 + \frac{\beta_1 + \beta_2}{\beta_2} (1 + \frac{\beta_1}{\theta}) \right)}.$$ \hspace{1cm} (26)

In Appendix A, we prove the following identity for general real positive numbers $y, x$:

$$\sum_{q=0}^{\infty} \frac{\Gamma(q + y)}{\Gamma(q + x + y + 1)} = \frac{\Gamma(y)}{(x - 1)\Gamma(y - 1 + x)}.$$ \hspace{1cm} (27)

If we use $x + 1$ instead of $x$ in (27), we get

$$\sum_{q=0}^{\infty} \frac{\Gamma(q + y)}{\Gamma(q + x + y + 1)} = \frac{\Gamma(y)}{x\Gamma(y + x)}.$$ \hspace{1cm} (28)

Now note that, using the basic properties of the Gamma function, we can rewrite (27) equivalently as follows

$$\sum_{q=0}^{\infty} \frac{q \Gamma(q + y)}{\Gamma(q + x + y + 1)} = \frac{\Gamma(y)}{(x - 1)\Gamma(y - 1 + x)}.$$ \hspace{1cm} (29)

Expanding the left hand side, we have

$$\sum_{q=0}^{\infty} \frac{q \Gamma(q + y)}{\Gamma(q + x + y + 1)} + (x + y) \sum_{q=0}^{\infty} \frac{\Gamma(q + y)}{\Gamma(q + x + y + 1)}$$

$$= \frac{\Gamma(y)}{(x - 1)\Gamma(y - 1 + x)} + (x + y) \frac{\Gamma(y)}{(x - 1)\Gamma(y - 1 + x)}.$$ \hspace{1cm} (30)

Combining this with (28), we arrive at

$$\sum_{q=0}^{\infty} \frac{q \Gamma(q + y)}{\Gamma(q + x + y + 1)} = \frac{\Gamma(y)}{(x - 1)\Gamma(y - 1 + x)} - \frac{\Gamma(y)}{x\Gamma(y + x)}.$$ \hspace{1cm} (31)

Using the basic properties of the Gamma function and algebraic simplifications, we can express this in the following form:

$$\sum_{q=0}^{\infty} \frac{q \Gamma(q + y)}{\Gamma(q + x + y + 1)} = \frac{\Gamma(y)}{(x - 1)\Gamma(y - 1 + x)} - \frac{\Gamma(y)}{x\Gamma(y + x)}.$$ \hspace{1cm} (32)

This has the same form as (26). We can use identity (26) with $y = \frac{(\beta_1 + \beta_2)\theta}{\mu}$ and $x = \frac{\beta_1 + \beta_2}{\beta_2}$ to calculate $S$ as follows:

$$S = \frac{\Gamma \left( \frac{(\beta_1 + \beta_2)\theta}{\mu} \right)}{\Gamma \left( \frac{\beta_1 + \beta_2}{\beta_2} \frac{\theta}{\mu} \right)} \times \frac{\left( \frac{\beta_1 + \beta_2}{\beta_2} \frac{\theta}{\mu} \right)}{\left( \frac{\beta_1 + \beta_2}{\beta_2} + \frac{(\beta_1 + \beta_2)\theta}{\mu} \right)}.$$ \hspace{1cm} (33)

Plugging this into (25), we get

$$\langle q \rangle_\theta = \frac{(\beta_1 + \beta_2)\theta}{\mu}.$$ \hspace{1cm} (34)

This is a linear relationship (see Figure 4). If we take the average degree over all nodes, we need to sum up (34) over all possible values of $\theta$. In the numerator, $\mu$ is created, which cancels out the $\mu$ in the denominator and we get

$$\langle q \rangle = (\beta_1 + \beta_2).$$ \hspace{1cm} (35)

We know this result is true, because by construction, the total number of links created in the system (which is always equal to the sum of in-degrees of all nodes) is $t$ at large times (when the effects of the initial conditions vanish), and the total number of nodes is $t$, which means that their ratio (which yields the average degree) is equal to $\beta_1 + \beta_2$.

VII. SUMMARY AND OPEN PROBLEMS

This paper extends the literature of multiplex networks by introducing a simple model which incorporates intrinsic fitness and preferential attachment. The merit layer is latent, yet drives the growth mechanism and the degree dynamics. We obtained closed-form expressions
for the joint interlayer distribution of degrees and fitness, as well as that of the total degrees. We observed that the expected value of the total degree linearly increases with fitness.

An immediate generalization of the present problem would be its extension to an arbitrary number of layers. Also, we have disregarded the temporal dynamics of the system and its transients in favor of the steady state. This loses valuable information about the transient state and the effects of initial conditions on the evolution of the network. In other words, for a given initial network (not necessarily small), one can study the evolution of the system in arbitrary time regimes, and investigate how the properties of the initial network affect the equilibration of the system, and how they affect the asymptotic properties of the network.

Another immediate step to augment the present model is to devise statistical recipes for inference. Since fitness is a latent variable and only $q$ can be observed, one can use (20) (or its time-dependent version) to devise maximum likelihood techniques to infer the distribution of fitness (merit) of scientific publications, blog posts, etc., by observing the distribution (or evolution) of degrees.

Appendix A: Proof of Identity [27]

We need to prove the following identity which concludes the proof.

\[
\sum_{q=0}^{\infty} \frac{\Gamma(q+y)}{\Gamma(q+x+y)} = \frac{\Gamma(y)}{(x-1)\Gamma(y-1+x)}. \quad (A1)
\]

The definition of the Beta function for positive real values $x, y$ is

\[
B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (A2)
\]

We can rewrite the summand of (A1) as follows:

\[
\frac{\Gamma(q+y)}{\Gamma(q+x+y)} = \int_0^1 t^{q+y-1}(1-t)^{x-1}dt = \frac{\Gamma(q+y)}{\Gamma(x)} \frac{1}{\Gamma(y)}. \quad (A3)
\]

We now have

\[
\sum_{q=0}^{\infty} \frac{\Gamma(q+y)}{\Gamma(q+x+y)} = \frac{1}{\Gamma(x)} \sum_{q=0}^{\infty} \int_0^1 dt(1-t)^x y^{-1} \int_0^1 dt(1-t)^x y^{-1} \int_0^1 dt(1-t)^x y^{-1} \frac{1}{1-t} = \frac{1}{\Gamma(x)} B(x-1, y) = \frac{1}{\Gamma(x)} \frac{\Gamma(x-1)\Gamma(y)}{\Gamma(x+y-1)} = \frac{1}{(x-1)\Gamma(y-1+\mu)} \frac{\Gamma(y)}{\Gamma(y-1+\mu)}.
\]

\[
\sum_{q=0}^{\infty} \frac{\Gamma(q+y)}{\Gamma(q+x+y)} = \frac{\Gamma(y)}{(x-1)\Gamma(y-1+\mu)}. \quad (A4)
\]
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