On the number of nodal solutions for a nonlinear elliptic problem on symmetric Riemannian manifolds

M.Ghimenti, A.M.Micheletti

Abstract

We consider the problem

$$-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u$$

in $M$, where $(M,g)$ is a symmetric Riemannian manifold. We give a multiplicity result for antisymmetric changing sign solutions.

Keywords: Riemannian Manifolds, Nodal Solutions, Topological Methods

Mathematics Subject Classification: 35J60, 58G03

1 Introduction

Let $(M,g)$ be a smooth connected compact Riemannian manifold of finite dimension $n \geq 2$ embedded in $\mathbb{R}^N$. We consider the problem

$$-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u \quad \text{in } M, \quad u \in H^1_g(M) \quad (\mathcal{P})$$

where $2 < p < 2^* = \frac{2N}{N-2}$, if $N \geq 3$.

Here $H^1_g(M)$ is the completion of $C^\infty(M)$ with respect to

$$||u||^2 : g \equiv \int_M |
abla_g u|^2 + u^2 d\mu_g$$

It is well known that the problem $(\mathcal{P})$ has a mountain pass solution $u_\varepsilon$. In [3] the authors showed that $u_\varepsilon$ has a spike layer and its peak point converges to the maximum point of the scalar curvature of $M$ as $\varepsilon$ goes to 0.

Recently there have been some results on the influence of the topology and the geometry of $M$ on the number of solutions of the problem. In
the authors proved that, if $M$ has a rich topology, problem $(\mathcal{P})$ has multiple solutions. More precisely they show that problem $(\mathcal{P})$ has at least $\text{cat}(M) + 1$ positive nontrivial solutions for $\varepsilon$ small enough. Here $\text{cat}(M)$ is the Lusternik-Schnirelmann category of $M$. In [17] there is the same result for a more general nonlinearity. Furthermore in [9] it was shown that the number of solution is influenced by the topology of a suitable subset of $M$ depending on the geometry of $M$. To point out the role of the geometry in finding solutions of problem $(\mathcal{P})$, in [13] it was shown that for any stable critical point of the scalar curvature it is possible to build positive single peak solutions. The peak of these solutions approaches such a critical point as $\varepsilon$ goes to zero.

Successively in [6] the authors build positive $k$-peak solutions whose peaks collapse to an isolated local minimum point of the scalar curvature as $\varepsilon$ goes to zero.

The first result on sign changing solution is in [12] where it is showed the existence of a solution with one positive peak $\eta_1^\varepsilon$ and one negative peak $\eta_2^\varepsilon$ such that, as $\varepsilon$ goes to zero, the scalar curvature $S_g(\eta_1^\varepsilon)$ (respectively $S_g(\eta_2^\varepsilon)$) goes to the minimum (resp. maximum) of the scalar curvature when the scalar curvature of $(M, g)$ is non constant. Here we give a multiplicity result for changing sign solutions when the Riemannian manifold $(M, g)$ is symmetric.

We look for solutions of the problem

\[
\begin{align*}
-\varepsilon^2 \Delta_g u + u &= |u|^{p-2} u \quad u \in H^1_g(M); \\
u(\tau x) &= -u(x) \quad \forall x \in M,
\end{align*}
\]

where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal linear transformation such that $\tau \neq \text{Id}$, $\tau^2 = \text{Id}$, $\text{Id}$ being the identity of $\mathbb{R}^N$. Here $M$ is a compact connected Riemannian manifold of dimension $n \geq 2$ and $M$ is a regular submanifold of $\mathbb{R}^N$ which is invariant with respect to $\tau$. Let $M_\tau := \{x \in M : \tau x = x\}$ be the set of the fixed points with respect to the involution $\tau$; in the case $M_\tau \neq \emptyset$ we assume that $M_\tau$ is a regular submanifold of $M$.

We obtain the following result.

**Theorem 1.** The problem $(\mathcal{P})$ has at least $G_\tau - \text{cat}(M - M_\tau)$ pairs of solutions $(u, -u)$ which change sign (exactly once) for $\varepsilon$ small enough.

Here $G_\tau - \text{cat}$ is the $G_\tau$-equivariant Lusternik Schnirelmann category for the group $G_\tau = \{\text{Id, } \tau\}$.

In [4] the authors prove a result of this type for the Dirichlet problem...
\[
\begin{cases}
-\Delta u - \lambda u - |u|^{2^*-2}u = 0 & u \in H^1_0(\Omega); \\
u(\tau x) = -u(x).
\end{cases}
\]  

(\mathcal{P}_\lambda)

Here \(\Omega\) is a bounded smooth domain invariant with respect to \(\tau\) and \(\lambda\) is a positive parameter.

We point out that in the case of the unit sphere \(S^{N-1} \subset \mathbb{R}^N\) (with the metric \(g\) induced by the metric of \(\mathbb{R}^N\)) the theorem of existence of changing sign solutions of \([12]\) can not be used because it holds for manifold of non constant curvature. Instead, we can apply Theorem \(\Pi\) to obtain sign changing solutions because we can consider \(\tau = -\text{Id}\), and we have \(G_\tau - \text{cat} S^{N-1} = N\).

Equation like (\mathcal{P}) has been extensively studied in a flat bounded domain \(\Omega \subset \mathbb{R}^N\). In particular, we would like to compare problem (\mathcal{P}) with the following Neumann problem

\[
\begin{cases}
-\varepsilon^2\Delta u + u = |u|^{p-2}u & \text{in } \Omega; \\
\frac{\partial u}{\partial \nu} = 0 & \text{in } \partial \Omega.
\end{cases}
\]  

(\mathcal{P}_N)

Here \(\Omega\) is a smooth bounded domain of \(\mathbb{R}^N\) and \(\nu\) is the unit outer normal to \(\Omega\). Problems (\mathcal{P}) and (\mathcal{P}_N) present many similarities. We recall some classical results about the Neumann problem.

In the fundamental papers \([11, 14, 15]\), Lin, Ni and Takagi established the existence of least-energy solution to (\mathcal{P}_N) and showed that for \(\varepsilon\) small enough the least energy solution has a boundary spike, which approaches the maximum point of the mean curvature \(H\) of \(\partial \Omega\), as \(\varepsilon\) goes to zero. Later, in \([16, 18]\) it was proved that for any stable critical point of the mean curvature of the boundary it is possible to construct single boundary spike layer solutions, while in \([7, 19, 10]\) the authors construct multiple boundary spike solutions at multiple stable critical points of \(H\). Finally, in \([5, 8]\) the authors proved that for any integer \(K\) there exists a boundary \(K\)-peaks solutions, whose peaks collapse to a local minimum point of \(H\).

2 Setting

We consider the functional defined on \(H^1_g(M)\)

\[
J_\varepsilon(u) = \frac{1}{\varepsilon^N} \int_M \left( \frac{1}{2} \varepsilon^2 |\nabla_g u|^2 + \frac{1}{2} |u|^2 - \frac{1}{p} |u|^p \right) d\mu_g. 
\]  

(2)

It is well known that the critical points of \(J_\varepsilon(u)\) constrained on the Nehari manifold

\[
\mathcal{N}_\varepsilon = \{ u \in H^1_g \setminus \{0\} : J_\varepsilon(u)u = 0 \}
\]  

(3)
are non trivial solution of problem (\[\mathcal{D}\]).

The transformation \(\tau : M \rightarrow M\) induces a transformation on \(H^1_g\) we define the linear operator \(\tau^*\) as

\[
\tau^* : H^1_g(M) \rightarrow H^1_g(M)
\]

\[
\tau^*(u(x)) = -u(\tau(x))
\]

and \(\tau^*\) is a selfadjoint operator with respect to the scalar product on \(H^1_g(M)\)

\[
\langle u, v \rangle_\varepsilon = \frac{1}{\varepsilon^N} \int_M \left( \varepsilon^2 \nabla g u \cdot \nabla g v + u \cdot v \right) d\mu_g.
\]

Moreover \(\|\tau^* u\|_{L^p(M)} = \|u\|_{L^p(M)}\), and \(\|\tau^* u\|_\varepsilon = \|u\|_\varepsilon\), thus \(J_\varepsilon(\tau^* u) = J_\varepsilon(u)\).

Then, for the Palais principle, the nontrivial solutions of (\[\mathcal{D}\]) are the critical points of the restriction of \(J_\varepsilon\) to the \(\tau\)-invariant Nehari manifold

\[
\mathcal{N}_\varepsilon^\tau = \{ u \in \mathcal{N}_\varepsilon : \tau^* u = u \} = \mathcal{N}_\varepsilon \cap H^\tau.
\]

Here \(H^\tau = \{ u \in H^1_g : \tau^* u = u \}\).

In fact, since \(J_\varepsilon(\tau^* u) = J_\varepsilon(u)\) and \(\tau^*\) is a selfadjoint operator we have

\[
\langle \nabla J_\varepsilon(\tau^* u), \tau^* \varphi \rangle_\varepsilon = \langle \nabla J_\varepsilon(u), \varphi \rangle_\varepsilon \quad \forall \varphi \in H^1_g(M).
\]

Then \(\nabla J_\varepsilon(u) = \tau^* \nabla J_\varepsilon(\tau^* u) = \tau^* \nabla J_\varepsilon(u)\) if \(\tau^* u = u\).

We set

\[
m_\infty = \inf_{f \in \mathcal{N}} |\nabla u|^2 + u^2 = \inf_{f \in \mathcal{N}} |u|^p;
\]

\[
m_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon;
\]

\[
m_\varepsilon^\tau = \inf_{u \in \mathcal{N}_\varepsilon^\tau} J_\varepsilon.
\]

**Remark 2.** It is easy to verify that \(J_\varepsilon\) satisfies the Palais Smale condition on \(\mathcal{N}_\varepsilon^\tau\). Then there exists \(v_\varepsilon\) minimizer of \(m_\varepsilon^\tau\) and \(v_\varepsilon\) is a critical point for \(J_\varepsilon\) on \(H^1_g(M)\). Thus \(v_\varepsilon^+\) and \(v_\varepsilon^-\) belong to \(\mathcal{N}_\varepsilon\), then \(J_\varepsilon(v_\varepsilon) \geq 2m_\varepsilon\).

We recall some facts about equivariant Lusternik-Schnirelmann theory. If \(G\) is a compact Lie group, then a \(G\)-space is a topological space \(X\) with a continuous \(G\)-action \(G \times X \rightarrow X, (g, x) \mapsto gx\). A \(G\)-map is a continuous function \(f : X \rightarrow Y\) between \(G\)-spaces \(X\) and \(Y\) which is compatible with the \(G\)-actions, i.e. \(f(gx) = gf(x)\) for all \(x \in X, g \in G\). Two \(G\)-maps \(f_0, f_1 : X \rightarrow Y\) are \(G\)-homotopic if there is a homotopy \(\theta : X \times [0, 1] \rightarrow Y\) such that \(\theta(x, 0) = f_0(x), \theta(x, 1) = f_1(x)\) and \(\theta(gx, t) = g\theta(x, t)\) for all \(x \in X, g \in G, t \in [0, 1]\). A subset \(A\) of \(X\) is \(G\)-invariant if \(ga \in A\) for every \(a \in A, g \in G\). The \(G\)-orbit of a point \(x \in X\) is the set \(Gx = \{gx : g \in G\}\).
Definition 3. The $G$-category of a $G$-map $f : X \to Y$ is the smallest number $k = G - \text{cat}(f)$ of open $G$-invariant subsets $X_1, \ldots, X_k$ of $X$ which cover $X$ and which have the property that, for each $i = 1, \ldots, k$, there is a point $y_i \in Y$ and a $G$-map $\alpha_i : X_i \to Gy_i \subset Y$ such that the restriction of $f$ to $X_i$ is $G$-homotopic to $\alpha_i$. If no such covering exists we define $G - \text{cat}(f) = \infty$.

In our applications, $G$ will be the group with two elements, acting as $G_\tau = \{\text{Id}, \tau\}$ on $\Omega$, and as $\mathbb{Z}/2 = \{1, -1\}$ by multiplication on the Nehari manifold $\mathcal{N}_\epsilon^\tau$. We remark the following result on the equivariant category.

Theorem 4. Let $\phi : M \to \mathbb{R}$ be an even $C^1$ functional on a complete $C^{1,1}$ submanifold $M$ of a Banach space which is symmetric with respect to the origin. Assume that $\phi$ is bounded below and satisfies the Palais Smale condition $(PS)_c$ for every $c \leq d$. Then $\phi$ has at least $\mathbb{Z}/2 - \text{cat}(\phi^d)$ antipodal pairs $\{u, -u\}$ of critical points with critical values $\phi(\pm u) \leq d$.

3 Sketch of the proof

In our case we consider the even positive $C^2$ functional $J_\epsilon$ on the $C^2$ Nehari manifold $\mathcal{N}_\epsilon^\tau$ which is symmetric with respect to the origin. As claimed in Remark 2, $J_\epsilon$ satisfies Palais Smale condition on $\mathcal{N}_\epsilon^\tau$. Then we can apply Theorem 4 and our aim is to get an estimate of this lower bound for the number of solutions. For $d > 0$ we consider

$$M_d = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq d\}; \quad (10)$$
$$M_d^- = \{x \in M : \text{dist}(x, M) \geq d\}. \quad (11)$$

We choose $d$ small enough such that

$$G_\tau - \text{cat}_{M_d} M_d = G_\tau - \text{cat}_M M \quad (12)$$
$$G_\tau - \text{cat}_M M_d^- = G_\tau - \text{cat}_M (M - M_\tau) \quad (13)$$

Now we build two continuous operator

$$\Phi_\epsilon^\tau : M_d^- \to \mathcal{N}_\epsilon^\tau \cap J_\epsilon^{2(m_\infty + \delta)}; \quad (14)$$
$$\beta : \mathcal{N}_\epsilon^\tau \cap J_\epsilon^{2(m_\infty + \delta)} \to M_d, \quad (15)$$

such that $\Phi_\epsilon^\tau(\tau q) = -\Phi_\epsilon^\tau(q)$, $\tau \beta(u) = \beta(-u)$ and $\beta \circ \Phi_\epsilon^\tau$ is $G_\tau$ homotopic to the inclusion $M_d^- \to M_d$.

By equivariant category theory we obtain

$$G_\tau - \text{cat}_M (M - M_\tau) = G_\tau - \text{cat}(M_d^- \hookrightarrow M_d) =$$

$$= G_\tau - \text{cat} \beta \circ \Phi_\epsilon^\tau \leq \mathbb{Z}_2 - \text{cat} \mathcal{N}_\epsilon^\tau \cap J_\epsilon^{2(m_\infty + \delta)} \quad (16)$$
4 Technical lemmas

First of all, we recall that there exists a unique positive spherically symmetric function $U \in H^1(\mathbb{R}^n)$ such that

$$-\Delta U + U = U^{p-1} \text{ in } \mathbb{R}^n$$

(17)

It is well known that $U_\varepsilon(x) = U\left(\frac{x}{\varepsilon}\right)$ is a solution of

$$-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{p-1} \text{ in } \mathbb{R}^n.$$  (18)

Secondly, let us introduce the exponential map $\exp : TM \to M$ defined on the tangent bundle $TM$ of $M$ which is a $C^\infty$ map. Then, for $\rho$ sufficiently small (smaller than the injectivity radius of $M$ and smaller than $d/2$), the Riemannian manifold $M$ has a special set of charts $\{\exp_x : B(0, \rho) \to M\}$.

Throughout the paper we will use the following notation: $B_g(x, \rho)$ is the open ball in $M$ centered in $x$ with radius $\rho$ with respect to the distance given by the metric $g$. Corresponding to this chart, by choosing an orthogonal coordinate system $(x_1, \ldots, x_n) \subset \mathbb{R}^n$ and identifying $T_xM$ with $\mathbb{R}^n$ for $x \in M$, we can define a system of coordinates called normal coordinates.

Let $\chi_\rho$ be a smooth cut off function such that

$$\begin{align*}
\chi_\rho(z) &= 1 \quad \text{if } z \in B(0, \rho/2); \\
\chi_\rho(z) &= 0 \quad \text{if } z \in \mathbb{R}^n \setminus B(0, \rho); \\
|\nabla \chi_\rho(z)| &\leq 2 \quad \text{for all } x.
\end{align*}$$

Fixed a point $q \in M$ and $\varepsilon > 0$, let us define the function $w_{\varepsilon,q}(x)$ on $M$ as

$$w_{\varepsilon,q}(x) = \begin{cases}
U_\varepsilon(\exp_q^{-1}(x))\chi_\rho(\exp_q^{-1}(x)) & \text{if } x \in B_g(q, \rho) \\
0 & \text{otherwise}
\end{cases}$$  

(19)

For each $\varepsilon > 0$ we can define a positive number $t(w_{\varepsilon,q})$ such that

$$\Phi_\varepsilon(q) = t(w_{\varepsilon,q})w_{\varepsilon,q} \in H^1_g(M) \cap \mathcal{N}_\varepsilon \text{ for } q \in M.$$  

(20)

Namely, $t(w_{\varepsilon,q})$ turns out to verify

$$t(w_{\varepsilon,q})^{p-2} = \frac{\int_M \varepsilon^2 |\nabla_g w_{\varepsilon,q}|^2 + |w_{\varepsilon,q}|^2 d\mu_g}{\int_M |w_{\varepsilon,q}|^p d\mu_g}$$  

(21)
Lemma 1. Given $\varepsilon > 0$ the application $\Phi_\varepsilon(q) : M \to H^1_g(M) \cap N_\varepsilon$ is continuous. Moreover, given $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that, if $\varepsilon < \varepsilon_0(\delta)$ then $\Phi_\varepsilon(q) \in N_\varepsilon \cap J^2_{\varepsilon}$. 

For the proof see [1, Proposition 4.2].

At this point, fixed a point $q \in M_d$, let us define the function

$$\Phi_\varepsilon^*(q) = t(w_{\varepsilon,q})w_{\varepsilon,q} - t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q} \quad (22)$$

Lemma 2. Given $\varepsilon > 0$ the application $\Phi_\varepsilon^*(q) : M_d^\varepsilon \to H^1_g(M) \cap N_\varepsilon^*$ is continuous. Moreover, given $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that, if $\varepsilon < \varepsilon_0(\delta)$ then $\Phi_\varepsilon^*(q) \in N_\varepsilon^* \cap J^2_{\varepsilon}$. 

Proof. Since $U_\varepsilon(z)\chi_\rho(z)$ is radially symmetric we set $U_\varepsilon(z)\chi_\rho(z) = \tilde{U}_\varepsilon(|z|)$. We recall that

$$\exp^{-1}_q \tau x = d_g(\tau x, \tau q) = d_g(x, q) = \exp^{-1}_q x;$$

$$\exp^{-1}_q \tau x = d_g(\tau x, q) = d_g(x, \tau q).$$

We have

$$\tau^* \Phi_\varepsilon^*(q)(x) = -t(w_{\varepsilon,q})w_{\varepsilon,q}(\tau x) + t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q}(\tau x) =$$

$$= -t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_q (\tau x)|) + t(w_{\varepsilon,\tau q})\tilde{U}_\varepsilon(|\exp^{-1}_q (\tau x)|) =$$

$$= t(w_{\varepsilon,\tau q})\tilde{U}_\varepsilon(|\exp^{-1}_q (x)|) - t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_q (x)|) =$$

$$= t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_q (x)|) - t(w_{\varepsilon,q})\tilde{U}_\varepsilon(|\exp^{-1}_q (x)|),$$

because by the definition we have $t(w_{\varepsilon,q}) = t(w_{\varepsilon,\tau q})$.

Moreover by definition the support of the function $\Phi_\varepsilon^*$ is $B_g(q, \rho) \cup B_g(\tau q, \rho)$, and $B_g(q, \rho) \cap B_g(\tau q, \rho) = \emptyset$ because $\rho < d/2$ and $q \in M_d$. Finally, because

$$\int_M |w_{\varepsilon,q}|^\alpha d\mu_g = \int_M |w_{\varepsilon,\tau q}|^\alpha d\mu_g \quad \text{for } \alpha = 2, p; \quad (23)$$

$$\int_M \nabla^2 w_{\varepsilon,q} d\mu_g = \int_M \nabla^2 w_{\varepsilon,\tau q} d\mu_g, \quad (24)$$

we have

$$J_\varepsilon(\Phi_\varepsilon^*(q)) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^{n}} \int_M |\Phi_\varepsilon^*(q)|^p d\mu_g = 2J_\varepsilon(\Phi_\varepsilon(q)). \quad (25)$$

Then by previous lemma we have the claim. 

Lemma 3. We have $\lim_{\varepsilon \to 0} m_\varepsilon^* = 2m_{\infty}$
Proof. By the previous lemma and by Remark 2 we have that for any \( \delta > 0 \) there exists \( \varepsilon_0(\delta) \) such that, for \( \varepsilon < \varepsilon_0(\delta) \)

\[
2m_\varepsilon \leq m_\varepsilon^* \leq 2J_\varepsilon(\Phi_\varepsilon(q)) \leq 2(m_\infty + \delta).
\]

(26)

Since \( \lim_{\varepsilon \to 0} m_\varepsilon = m_\infty \) (see [1, Remark 5.9]) we get the claim.

For any function \( u \in N_\varepsilon^r \) we can define a point \( \beta(u) \in \mathbb{R}^N \) by

\[
\beta(u) = \frac{\int_M x|u^+(x)|^p d\mu_g}{\int_M |u^+(x)|^p d\mu_g} (27)
\]

Lemma 4. There exists \( \delta_0 \) such that, for any \( 0 < \delta < \delta_0 \) and any \( 0 < \varepsilon < \varepsilon_0(\delta) \) (as in Lemma 2) and for any function \( u \in N_\varepsilon^r \cap J_\varepsilon^{2(m_\infty + \delta)} \), it holds \( \beta(u) \in M_d \).

Proof. Since \( \tau^*u = u \) we set

\[
M^+ = \{ x \in M : u(x) > 0 \} \quad M^- = \{ x \in M : u(x) < 0 \}.
\]

It is easy to see that \( \tau M^+ = M^- \). Then we have

\[
J_\varepsilon(u) = \left( \frac{1}{2} - \frac{1}{p} \right) \frac{1}{\varepsilon^n} \int_M |u|^p d\mu_g =
\]

\[
= \left( \frac{1}{2} - \frac{1}{p} \right) \frac{1}{\varepsilon^n} \left[ \int_{M^+} |u^+|^p d\mu_g + \int_{M^-} |u^-|^p d\mu_g \right] = 2J_\varepsilon(u^+) \quad (28)
\]

By the assumption \( J_\varepsilon(u) \leq 2(m_\infty + \delta) \) we have \( J_\varepsilon(u^+) \leq m_\infty + \delta \) then by Proposition 5.10 of [1] we get the claim.

Lemma 5. There exists \( \varepsilon_0 > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) the composition

\[
I_\varepsilon = \beta \circ \Phi_\varepsilon^r : M_d^- \to M_d \subset \mathbb{R}^N
\]

is well defined, continuous, homotopic to the identity and \( I_\varepsilon(\tau q) = \tau I_\varepsilon(q) \).

Proof. It is easy to check that

\[
\Phi_\varepsilon^r(\tau q) = -\Phi_\varepsilon^r(q) \quad \beta(-u) = \tau \beta(u) \quad (30)
\]

Moreover, by Lemma 2 and by Lemma 4 for any \( q \in M_d^- \) we have \( \beta \circ \Phi_\varepsilon^r(q) = \beta(\Phi_\varepsilon(q)) \in M_d \), and \( I_\varepsilon \) is well defined.
In order to show that \( I_\varepsilon \) is homotopic to identity, we evaluate the difference between \( I_\varepsilon \) and the identity as follows.

\[
I_\varepsilon(q) - q = \frac{\int_M (x - q) |u^+|^p \, d\mu_g}{\int_M |u^+|^p \, d\mu_g} = \int_{B(0, \rho)} z \left| U \left( \frac{z}{\varepsilon} \right) \chi_\rho(|z|) \right|^p \left| g_q(z) \right|^\frac{1}{2} = \\
\varepsilon \int_{B(0, \rho)} z \left| U(z) \chi_\rho(|\varepsilon z|) \right|^p \left| g_q(\varepsilon z) \right|^\frac{1}{2} = \int_{B(0, \rho)} \left| U(z) \chi_\rho(|\varepsilon z|) \right|^p \left| g_q(\varepsilon z) \right|^\frac{1}{2},
\]

hence \( |I_\varepsilon(q) - q| < \varepsilon c(M) \) for a constant \( c(M) \) that does not depend on \( q \) \( \square \).

Now, by previous lemma and by Theorem 4 we can prove Theorem 1.

In fact, we know that, if \( \varepsilon \) is small enough, there exist \( G_{\tau, cat}^\tau(M - M_\tau) \) minimizers which change sign, because they are antisymmetric. We have only to prove that any minimizer changes sign exactly once. Let us call \( \omega = \omega_\varepsilon \) one of these minimizers. Suppose that the set \( \{ x \in M : \omega_\varepsilon(x) > 0 \} \) has \( k \) connected components \( M_1, \ldots, M_k \). Set

\[
\omega_i = \begin{cases} 
\omega_\varepsilon(x) & x \in M_i \cup \tau M_i; \\
0 & \text{elsewhere}
\end{cases}
\]

For all \( i, \omega_i \in N_\varepsilon^\tau \). Furthermore we have

\[
J_\varepsilon(\omega) = \sum_i J_\varepsilon(\omega_i),
\]

thus

\[
m^\tau_\varepsilon = J_\varepsilon(\omega) = \sum_{i=1}^k J_\varepsilon(\omega_i) \geq k \cdot m^\tau_\varepsilon,
\]

so \( k = 1 \), that concludes the proof.

References

[1] V. Benci, C. Bonanno, and A.M. Micheletti, On the multiplicity of solutions of a nonlinear elliptic problem on Riemannian manifolds, J. Funct. Anal. 252 (2007), no. 2, 464–489.
[2] V. Benci and G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **114** (1991), no. 1, 79–93.

[3] J. Byeon and J. Park, *Singularly perturbed nonlinear elliptic problems on manifolds*, Calc. Var. Partial Differential Equations **24** (2005), no. 4, 459–477.

[4] A. Castro and M. Clapp, *The effect of the domain topology on the number of minimal nodal solutions of an elliptic equation at critical growth in a symmetric domain*, Nonlinearity **16** (2003), no. 2, 579–590.

[5] E. Dancer and S. Yan, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific J. Math **189** (1999), no. 2, 241–262.

[6] E. Dancer, A.M. Micheletti, and Angela Pistoia, *Multipeak solutions for some singularly perturbed nonlinear elliptic problems in a Riemannian manifold*, to appear on Manus. Math.

[7] C. Gui, *Multipeak solutions for a semilinear Neumann problem*, Duke Math J. **84** (1996), no. 3, 739–769.

[8] C. Gui, J. Wei, and M. Winter, *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), no. 1, 47–82.

[9] N. Hirano, *Multiple existence of solutions for a nonlinear elliptic problem on a Riemannian manifold*, Nonlinear Anal., **70** (2009), no. 2, 671–692.

[10] Y.Y. Li, *On a singularly perturbed equation with Neumann boundary condition*, Comm. Partial Differential Equations **23** (1998), no. 3-4, 487–545.

[11] C.S. Lin, W.M. Ni, and I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations **72** (1988), no. 1, 1–27.

[12] A.M. Micheletti and A. Pistoia, *Nodal solutions for a singularly perturbed nonlinear elliptic problem in a Riemannian manifold*, to appear on Adv. Nonlinear Stud.

[13] A.M. Micheletti and A. Pistoia, *The role of the scalar curvature in a nonlinear elliptic problem in a Riemannian manifold*, Calc. Var. Partial Differential Equation, **34** (2009), 233–265.
[14] W.M. Ni and I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), no. 7, 819–851.

[15] W.M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. **70** (1993), no. 2, 247–281.

[16] M. Del Pino, P. Felmer, and J. Wei, *On the role of mean curvature in some singularly perturbed Neumann problems*, SIAM J. Math. Anal. **31** (1999), no. 1, 63–79.

[17] D. Visetti, *Multiplicity of solutions of a zero-mass nonlinear equation in a Riemannian manifold*, J. Differential Equations, **245** (2008), no. 9, 2397–2439.

[18] J. Wei, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Differential Equations **134** (1997), no. 1, 104–133.

[19] J. Wei and M. Winter, *Multipeak solutions for a wide class of singular perturbation problems*, J. London Math. Soc. **59** (1999), no. 2, 585–606.