NONDISCRETENESS OF $F$-THRESHOLDS

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Abstract. We give examples of two dimensional normal $Q$-Gorenstein graded domains, where the set of $F$-thresholds of the maximal ideal is not discrete, thus answering a question by Mustaţă-Takagi-Watanabe.

We also prove that, for a two dimensional standard graded domain $(R, m)$ over a field of characteristic 0, with graded ideal $I$, if $(m_p, I_p)$ is a reduction mod $p$ of $(m, I)$ then $c^p(m_p) \neq c_{\infty}(m)$ implies $c^p(m_p)$ has $p$ in the denominator.

1. Introduction

Let $(R, m)$ be a Noetherian local ring of positive characteristic $p$. For an ideal $I$ of $R$, a set of invariants of singularities in positive characteristic, called $F$-thresholds, were introduced by [MTW] as follows

$$\{\text{$F$-thresholds of } I\} := \{c^J(I) \mid J \subseteq m \text{ such that } I \subseteq \text{Rad}(J)\},$$

where $c^J(I) := \lim_{e \to \infty} \max\{r \mid I^r \nsubseteq J^{[p^e]}\}/p^e$. In [MTW], it was shown that for regular local $F$-finite rings, the $F$-thresholds of an ideal coincide with the $F$-jumping numbers of the generalized test ideals of $I$ (introduced by [HY]), which are analogous to the jumping numbers of a multiplier ideal in characteristic 0. The first $F$-jumping number (introduced by [TaW] under the name $F$-pure threshold), denoted by $fpt$, corresponds to the first jumping number of the associated multiplier ideal and is called log canonical threshold of $I$. The set of the jumping numbers, for a given ideal, is known to be discrete and rational.

Here we consider the following question by Mustaţă-Takagi-Watanabe (Question 2.11 in [MTW]).

Question. Given an ideal $(0) \neq I \subseteq m$, could there exist finite accumulation points for the set of $F$-thresholds of $I$?

In the case of regular rings (with some additional mild conditions), the set of $F$-jumping numbers for $I$ is equal to the set of $F$-thresholds $\{c^J(I)\}_J$ of $I$ (Corollary 2.3 in [BMS2]). On the other hand, in such cases, it has been proven that the $F$-jumping numbers are discrete and rational (see [BMS1], [BMS2], [KLZ]) (in fact, as pointed out in [BMS2], the discreteness of the set of $F$-jumping numbers implies the rationality statement due to the fact that if $\lambda$ is an $F$-jumping number, then so are the fractional parts of $p^e\lambda$, for all $e \geq 1$).

Though the discreteness of the set of $F$-jumping numbers are known in some singular cases too e.g. when the ring is $F$-finite normal $Q$-Gorenstein domain ([GrS], [BSTZ], [KSSZ], [ST]), we cannot conclude the same for $F$-thresholds as they can be in general different from the $F$-jumping numbers, as shown by Example 2.5 in [TaW], where the ring $R = k[x, y, z]/(xy - z^2)$, and the first $F$-jumping number of $m = (x, y, z)$, $fpt_m(m) < c^m(m)$, the first $F$-threshold of $m$.

However when $R$ is a direct summand of a regular $F$-finite domain $S$, then the set $\{c^J(I)\}_J$ is known to be a discrete set of rational points (Proposition 4.17 in
Here the authors extend the theory of Bernstein-Sato polynomial to the direct summands of regular rings, while for regular rings the authors in [MTW] relate the Bernstein-Sato polynomials to the $F$-jumping numbers and the $F$-thresholds. Now in [HNMb], each $c^f(I)$ is identified with $c^{JS}(IS)$ and hence is an $F$-jumping number of $IS$.

In particular, in all of the above cases, the $F$-thresholds of an ideal have been studied by identifying them with the $F$-jumping numbers of some ideal in a regular ring where such set is discrete and consist of rational numbers.

In [TrW], using the theory of the Hilbert-Kunz density functions for graded rings and Frobenius semistability properties of vector bundles on projective curves, we had shown that in dimension two, the $F$-thresholds of the maximal ideal at graded ideals can be expressed in terms of the Harder-Narasimhan slopes of the associated syzygy bundles. As a result, we had deduced that the set $\{c^f(I) \mid I \text{ is graded}\}$ consists of rational points.

In this paper, we use this new viewpoint to show that such a set can have accumulation points. More precisely we prove the following

**Theorem 1.1.** Given a prime $p$ and an integer $g > 1$, there is a two-dimensional standard graded normal $\mathbb{Q}$-Gorenstein domain $(R, m)$ (a cone over a nonsingular curve of genus $g$) over an algebraically field of char $p > 0$ and a sequence of $m$-primary graded ideals $\{I_m\}_{m \geq 0}$ such that, the $F$-threshold of $m$ at $I_m$,

$$c^{I_m}(m) = \frac{3}{2} + \frac{(g - 1)p^{m+m_0d}}{p^{m+m_0d}}, \quad \text{for } m \geq 0,$$

where $d = e_0(R, m)$ and $m_0 \geq 0$ is an integer such that $p^{m_0} < g$. Moreover, each $I_m$ is generated by three elements, each of degree 1 in $R$.

This answers the above question (of [MTW]) affirmatively. In particular we have the following

**Corollary 1.2.** Given a prime $p$ and an integer $g > 1$, there exists a two dimensional standard graded normal $\mathbb{Q}$-Gorenstein domain $R$ with the graded maximal ideal $m$ such that the set of $F$-thresholds of $m$ has accumulation points, where $\text{Proj} R = X$ is a nonsingular projective curve of genus $g$ over a field of char $p$.

Moreover there is a strictly decreasing sequence consisting of $F$-thresholds of $m$; thus, the $F$-thresholds of an ideal need not satisfy the descending chain condition.

For the proof of Theorem 1.1 we crucially use the following construction by D. Gieseker in [G]. For a given $p$ and $g > 1$, there exists a family $X$ of stable curves of genus $g$ over $\text{Spec} k[[t]]$ ($k$ is an algebraically closed field of char $p$) with smooth generic fiber, and a closed fiber with particular singularities. By taking a specific representation of $G$ (analogous to the representation arising from a Schottky uniformization for a compact Riemann surface of genus $g$), where $G$ is the group of covering transformations of $Y_0$ (and where $Y_0$ is the universal cover over the special fiber $X_0$ of $X$), Gieseker constructed a rank 2 vector bundle $F_1$ on the generic fiber $X_K$ ($K = k((t))$) with an explicit Harder-Narasimhan filtration. Moreover the bundle $F_1$, associated to the representation of $G$, comes equipped with a sequence $\{F_k\}_{k \geq 1}$ of bundles such that $F_k \in F_{k+1} = F_k$.

From this sequence we construct a set of vector bundles with the similar properties such that the new set is also a bounded family of bundles on the curve $X_K$. By choosing $\mathcal{L} = \text{the power of the canonical bundle of the curve}$, we ensure that the coordinate ring (corresponding to the embedding of the curve by $\mathcal{L}$) is $\mathbb{Q}$-Gorenstein.
Next, we consider some behaviour of the $F$-thresholds of reductions mod $p$, as $p$ varies, from our view point (relating $F$-thresholds to vector bundles). We recall that Theorem 3.4 and Proposition 3.8 of [HY] imply that, for $R = \mathbb{Z}[X_1, \ldots, X_n]$ and $I \subseteq \mathfrak{m} = (X_1, \ldots, X_n)$, we have a formula for the log canonical threshold in terms of $F$-pure thresholds:

$$\lct(I) = \lim_{p \to \infty} \fpt_{(f_p)}(I_p),$$

where $\mathfrak{m}_p$ and $I_p$ are reductions mod $p$ of $\mathfrak{m}$ and $I$, respectively.

K.Schwee asked the following question. Assuming $\fpt_{\mathfrak{m}_p}(f_p) \neq \lct(I)$, is the denominator of $\fpt_{\mathfrak{m}_p}(f_p)$ (in its reduced form) a multiple of $p$?

In [CHSW] the authors explored the implication of the following two conditions: (1) the characteristic does not divide the denominator of the $F$-pure threshold and log canonical threshold coincide. Theorem A in [CHSW] and also the example 4.5 in [MTW] imply that for an explicit (nonhomogeneous) polynomial $f$ in a polynomial ring (note that here the above two conditions could be distinct.

On the other hand, there are examples (see [CHSW] for the references) of homogeneous polynomials $f$ of specific types where the two conditions are equivalent. In [BS] Proposition 5.4, it was shown that for a homogeneous polynomial $f$ of degree $d$ in $R = k[X_0, \ldots, X_n]$ (where $R/(f)$ is an isolated singularity), if $p \geq nd - \delta - n$ then either $\lct(f_p) = (n+1)/d$, or the denominator of $\lct(f_p)$ is a power of $p$. In other words

$$\lct(I) \neq \fpt_{\mathfrak{m}_p}(f_p) \implies \text{the denominator of } \lct(I) \text{ is a power of } p.$$

In this context, here we prove the following

**Theorem 1.3.** Let $R$ be a two dimensional standard graded domain over an algebraically closed field $k$ of char 0 and let $I \subset R$ be a graded ideal of finite colength. Let $R_p$, $I_p$ and $\mathfrak{m}_p$ denote a reduction mod $p$ of $R$, $I$ and $\mathfrak{m}$ respectively, where char $R_p = p$. Then, for $p \gg 0$,

$$c^{I_p}(\mathfrak{m}_p) \neq c^{I_p}(\mathfrak{m}) \implies c^{I_p}(\mathfrak{m}_p) = c^{I_p}(\mathfrak{m}) + \frac{a}{pb},$$

for some $a,b \in \mathbb{Z}_+$ such that $g.c.d.(a,p) = 1$. Moreover $0 < a/b \leq 4(g - 1)(r - 1)$, where $r + 1 = \text{the minimal generators of } I$ and $g = \text{the genus of Proj } R$. Therefore

$$c^{I_p}(\mathfrak{m}_p) \neq c^{I_p}(\mathfrak{m}) \implies c^{I_p}(\mathfrak{m}_p) = \frac{a_1}{pb_1},$$

where $a_1, b_1 \in \mathbb{Z}_+$ and $g.c.d.(a_1, p) = 1$.

However, there exist examples (Remark [11]) where the denominators (in its reduced form) of $c^{I_p}(\mathfrak{m}_p)$ is divisible by $p$ but is not a power of $p$.

We recall that the existence of $c^{I_p}(\mathfrak{m}) := \lim_{p \to \infty} c^{I_p}(\mathfrak{m}_p)$ was shown in Theorem 5.5 of [TrW], and $c^{I_p}(\mathfrak{m})$ and $c^{I_p}(\mathfrak{m}_p)$ were given, respectively, in terms of the minimal HN slope of a $\mu$-reduction bundle (which is a char 0 invariant of the pair $(R, I)$) and the minimal strong HN slope of a strong $\mu$-reduction bundle for $(R, I)$. For the proof of the above theorem we use the relation between these two bundles.

In Section 2 we recall the required basic theory of Harder-Narasimhan filtrations of vector bundles on curves, and also results from [TrW]. In Section 3 and Section 4 we prove Theorem [11] and Theorem [13] respectively.
2. PRELIMINARIES

We recall few generalities about Harder-Narasimhan filtration of vector bundles on curves.

**Definition/Notations.** Let $X$ be a nonsingular curve over a field $k$. For a vector bundle $V$ on $X$, the degree of $V$ is $\text{deg} \ V = \text{deg}(\wedge^\text{rank} V)$ and the slope of $V$ is $\mu(V) = \text{deg} \ V / \text{rank} V$. A vector bundle $V$ is semistable if for every subbundle $W \subseteq V$, we have $\mu(W) \leq \mu(V)$.

Every bundle has the unique $\text{HN}$ (Harder-Narasimhan) filtration, which is a filtration

\[
\begin{align*}
0 &= E_0 \subset E_1 \subset \cdots \subset E_n = V \\
\text{such that } \mu(E_1) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1}). \quad \text{We call } \mu(E_i/E_{i-1}) \text{ an HN slope of } V.
\end{align*}
\]

If $\text{char} \ k = p > 0$ then the HN filtration of $V$ is strong HN filtration if each $\mu(E_i/E_{i-1})$ is strongly semistable, i.e., $F^{m_0 s}(E_i/E_{i-1})$ is semistable for every $m_0$-iterated Frobenius map $F^m : X \to X$. For every vector bundle $V$ there exists $m \geq 0$ (Theorem 2.7 of [L]) such that $F^{m_0 s}V$ has strong HN filtration.

For the vector bundle $V$ with the HN filtration (2.1), we denote $\mu_{\text{min}}(V) = \mu(V/E_{n-1})$.

If $m$ is an integer such that $F^{m_0 s}V$ achieves the strong HN filtration then we denote $a_{\text{min}}(V) = \mu_{\text{min}}(F^{m_0 s}(V)/p^m)$.

We recall the following definitions and results from [TrW].

**Notations 2.1.** Let $(R, I)$ be a standard graded pair defined over an algebraically closed field, where $R$ is a two dimensional domain and $I$ is generated by homogeneous elements of degrees $d_1, \ldots, d_s$. Let $X = \text{Proj} \ S$, where $S$ is the normalization of $R$ in its quotient field. Let

\[
\begin{align*}
0 \to V_0 \to M_0 = \oplus_{i=1}^s \mathcal{O}_X(1 - d_i) \to \mathcal{O}_X(1) \to 0
\end{align*}
\]

be the canonical sequence of $\mathcal{O}_X$-modules.

If $M_0$ has the HN filtration, $0 = M_{l_1} \supset M_{l_1-1} \supset \cdots \supset M_0 = M$ then let $0 = V_{l_1} \subseteq V_{l_1-1} \subseteq \cdots \subseteq V_1 \subseteq V_0$ denote the induced filtration on $V_0$, where $V_i = M_i \cap V_0$. Note that this need not be the HN filtration of $V_0$.

**Definition 2.2.** (1) The sequence (2.2) has the $\mu$-reduction at $t$ if there exists $0 \leq t < l_1$ such that

(a) for every $0 \leq i < t$, the canonical sequence $0 \to V_i \to M_i \to L \to 0$ is exact and $\mu_{\text{min}}(V_i) = \mu_{\text{min}}(M_i)$, and

(b) $\mu_{\text{min}}(V_t) < \mu_{\text{min}}(M_t)$.

We call $V_t$ the $\mu$-reduction bundle for the sequence (2.2) and a $\mu$-reduction bundle for the pair $(R, I)$.

(2) We say (provided $\text{char} \ k = p > 0$), the sequence (2.2) has the strong $\mu$-reduction at $t_0$, if for some choice of $m_1 > 0$ such that $F^{m_1 s}(V_0)$ has the strong HN filtration, the sequence

\[
\begin{align*}
0 \to F^{m_1 s}V_0 \to F^{m_1 s}M_0 \to F^{m_1 s}\mathcal{O}_X(1) \to 0,
\end{align*}
\]

has $\mu$-reduction sequence at $t_0$. Note that $F^{m_1 s}(V_i) = F^{m_1 s}M_i \cap F^{m_1 s}(V_0)$, for all $0 \leq i \leq l_1$. By Proposition 4.6 of [TrW], the sequence (2.2) does have the $\mu$-reduction for some $t < l_1$ and does have the strong $\mu$-reduction for some $t_0$. Moreover $t_0 \leq t$.

**Remark 2.3.** We recall Theorem 4.12, Remark 4.13 (1) and Lemma 5.2 from [TrW].
Nondiscreteness of F-thresholds

(1) If the sequence \((\textbf{2.2})\) has the strong \(\mu\)-reduction at \(t_0\) then the \(F\)-threshold of \(\textbf{m}\) at \(I\) is \(c_I^{\textbf{m}}(I) = 1 - \alpha_{\text{min}}(V_{t_0})/d\).

Moreover, if \(d_1 = \cdots = d_s\) (where \(d_i\) as in Notation \(\text{2.1}\)) then \(c_I^{\textbf{m}}(I) = 1 - \alpha_{\text{min}}(V_{t_0})/d\).

(2) If the sequence \((\textbf{2.2})\) has \(\mu\)-reduction as \(t\) (in characteristic 0) then, for \(p \gg 0\), where \((\textbf{m}_p, I_p)\) denote a reduction mod \(p\) of \((\textbf{m}, I)\) and \(V^p_I\) and \(V^p_{I-1}\) denote a reduction mod \(p\) of \(V_I\) and \(V_{I-1}\), respectively.

\[
either \quad c^{\textbf{m}_p}(\textbf{m}_p) = 1 - \alpha_{\text{min}}(V^p_I)/d \quad \text{and} \quad c_I^{\textbf{m}}(I) = 1 - \alpha_{\text{min}}(V^p_{I-1})/d,
\]
or

\[
\quad c^{\textbf{m}_p}(\textbf{m}_p) = 1 - \alpha_{\text{min}}(V^p_I)/d \quad \text{and} \quad c_{\text{min}}^{\textbf{m}_p}(\textbf{m}_p) = 1 - \alpha_{\text{min}}(V^p_{I-1})/d.
\]

In particular, for \(p \gg 0\), we have \(c_{\text{min}}^{\textbf{m}_p}(\textbf{m}_p) \leq c^{\textbf{m}_p}(\textbf{m}_p)\).

Though the above equalities hold, the strong \(\mu\)-reduction bundle may not be a reduction mod \(p\) of the \(\mu\)-reduction bundle. Also though \(\mu\)-reduction bundle \(V_I\) may not occur in the HN filtration of \(V_0\), the \(\alpha_{\text{min}}(V_I)\) is equal to one of the HN slopes of \(V_0\). Similarly, for the strong \(\mu\)-reduction bundle \(V_{t_0}, \alpha_{\text{min}}(F^m_0 V_{t_0})\) is equal to one of the HN slopes of \(F^m_0 V_{t_0}\).

3. Nondiscreteness of F-thresholds

We recall a result by Gieseker [G].

**Theorem 1** (Gieseker). For each prime \(p > 0\) and integer \(g > 1\), there is a nonsingular projective curve \(X\) of genus \(g\) over an algebraically closed field of char \(p\) and a semistable vector-bundle \(V\) of degree 0 such that \(F^*V\) is not semistable.

Bundles of positive degree with such properties have been constructed by J.P. Serre and H. Tango. But for our result we use the other properties of this bundle, which were proved by Gieseker. We recall the relevant results from [G]:

For each \(g > 1\) and each algebraically closed field \(k\) of char \(p\), there is a family of stable curves \(X\) of genus \(g\) over \(k[[\ell]]\), such that the special fiber \(X_0\) is a rational curve over \(k\) with \(g\) nodes and is \(k\)-split degenerate, and the generic fiber \(X_K\) is smooth and geometrically connected, where \(K\) is the quotient field of \(k[[\ell]]\). Now if \(Y_0\) is the universal covering space of the special fiber \(X_0\) and \(G\) is the group of the covering transformations of \(Y_0\) over \(X_0\), then (Proposition 2, [G]) any representation \(\rho\) of \(G\) on \(K^n\) gives a rank \(n\) bundle \(F_\rho\) on \(X\) such that the pull back bundle \(F_1\) on \(X_K\) comes with a sequence of bundles \(F_1, F_2, F_3, \ldots\) such that \(F^k F_{k+1} = F_k\). Now, by making a specific choice of a representation \(\rho\) (attributed to Mumford by [G]) of the group \(G\) on \(K^2\), Gieseker derives (Lemma 4, [G]) a rank 2 bundle \(F_\rho\) of degree 0 on \(X\) and an exact sequence

\[
0 \rightarrow L \rightarrow F_\rho \rightarrow L^{-1} \rightarrow 0,
\]

where \(\deg L = g - 1\). Now pull back of \(L\) to \(X_K\) gives the HN filtration \(L \subset F_1\) and also a sequence of bundles \(F_1, F_2, F_3, \ldots\) such that \(F^k F_{k+1} = F_k\). By a simple argument it follows (Lemma 5, [G]) that if \(g \leq p^{k-1}\) then \(F_k\) is semistable. Hence one can choose a (unique) bundle \(V\) from the set \(\{F_k\}_{k \geq 1}\) such that \(V\) is semistable and \(F^*V\) is not semistable.

In the following lemma we consider a modified version of such a family \(\{F_m\}_m\) of bundles.

**Lemma 3.1.** Given an integer \(g > 1\) and a prime \(p\), there is a nonsingular curve \(X\) of genus \(g\) over a field of characteristic \(p\) and a family of bundles \(\{F_m\}_{m \geq 0}\) such that

1. \(\text{rank } E_m = 2\) and \(\det(E_m) = O_X\), for \(m \geq 0\) and
(2) for each $E_m$, the number $m \geq 0$ is the least integer such that the bundle $F^{m*}E_m$ is not semistable. Moreover the HN filtration of $F^{m*}E_m$ is

$$L_m \subset F^{m*}E_m, \text{ where } \deg(L_m) = (g-1)/p^{m_0},$$

for some $m_0 \geq 0$ where $p^{m_0} < g$.

(3) There exists a very ample line bundle $L$ on $X$, such that for every $m \geq 0$, the bundle $E_m \otimes L$ is generated by its global sections.

In particular $\{E_m \otimes L\}_{m \geq 0}$ is a bounded family.

Proof. The results in [G] (see the above discussion) give the following: for given $g > 1$ and $p$, there is a nonsingular curve $X$ of genus $g$ over an algebraically closed field of char $p$ and a family of bundles $\{F_m\}_{m \geq 1}$ such that

1. $F_m$ is of rank 2 and of degree 0, for $m \geq 1$ and
2. $F^*F_{m+1} = F_m$, and $F_m$ semistable if $g \leq p^{m-1},$
3. $F_1$ has the HN filtration $L \subset L_1$, where $\deg L = g-1$ and $\deg F_1 = 0$.

Hence, for some $m_0 \geq 0$, there is a (unique) bundle $F_{m_0+2} \in \{F_k\}_{k \geq 1}$ which is semistable and $F^*F_{m_0+2} = F_{m_0+1}$ is not semistable. Since $\text{Pic}^0(X)$ (the set of degree 0 line bundles on $X$) is an abelian variety, (Application 2, page 59 in [Mu1]) the map $n_X : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$, given by $L \mapsto L^n$ is surjective,

where we denote $L^n = L^\otimes n$ (and $L^{-n} = (L^{-1})^\otimes n$). Therefore, for each $m$, we can choose $L_m \in \text{Pic}^0(X)$ such that $\text{det}(F_m) = L_m^2$ (recall that $\deg(F_m) = \deg(\text{det}(F_m)) = 0$).

We define $E_m = F_{m+m_0+1} \otimes L_{m+m_0+1}^{-1}$, for $m \geq 0$.

Then $\text{det}(E_m) = \text{det}(F_{m+m_0+1}) \otimes L_{m+m_0+1}^2 = \mathcal{O}_X$. This proves Assertion (1).

Note that

$$F^{k*}E_m = F^{k*}F_{m+m_0+1} \otimes L_{m+m_0+1}^{-p^k} = F_{m-k+m_0+1} \otimes L_{m+m_0+1}^{-p^k},$$

hence for any $m \geq 0$, the bundles $E_0, F^*E_m, \ldots, F^{m-1*}E_m$ are semistable. Since $F^{m*}E_m = F_{m_0+1} \otimes L_{m+m_0+1}^{-p^m}$, it has the HN filtration $L_m \subset F^{m*}E_m$ if and only if $F^{m*}F_{m_0+1} \subset F^{m*}F_{m_0+1}$ is the the HN filtration of $F_1$. Therefore, by the uniqueness of the HN filtration we have $\deg L_m = (g-1)/p^{m_0}$. Moreover $p^{m_0} < g$ as $F_{m_0+1}$ is not semistable. This proves Assertion (2).

Let $\mathcal{O}_X(1) = K_X^3$, where $K_X$ is the canonical line bundle on $X$. Since $g \geq 2$, the line bundle $\mathcal{O}_X(1)$ is very ample on $X$ (Chap IV, [H]). Then (this is a standard argument in the literature) we

**Claim.** For $m \geq 1$, the bundle $E_m$ is 3-regular, i.e., $H^1(X, E_m(n-1)) = 0$, for $n \geq 3$.

**Proof of the claim:** By Serre duality $H^1(X, E_m(n-1)) = \text{Hom}(E, \omega_X(1-n))^\vee$. If $E_m \longrightarrow \omega_X(1-n)$ is a nonzero map then the semistability property of the sheaf $E_m$ implies $\mu(E_m) \leq \mu(\omega_X(1-n))$. Therefore $0 \leq (2g-2) + (1-n) \deg \mathcal{O}_X(1)$. This proves the claim.

Hence (Chapter 14, [Mu2]), for $m \geq 1$, every $E_m(3)$ is generated by its global sections. Moreover, we can choose $n_0 \geq 3$ (Theorem 5.17, [H]) such that $E_0(n_0)$ is generated by its global sections. Hence Assertion (3) follows by taking $\mathcal{L} = \mathcal{O}_X(n_0) = K_X^{3n_0}$.

Moreover each $E_m \otimes \mathcal{L}$ has the same Hilbert polynomial with respect to $\mathcal{O}_X(1)$ (as each $E_m$ has the same rank and degree). Therefore the family $\{E_m \otimes \mathcal{L}\}_{m \geq 0}$ is a bounded family. \[\square\]
1.1: For given curve and a bounded family \( F \)

**Remark 3.2.** Lemma 3.1 implies that, for any prime \( p \) and \( g > 1 \), there is a nonsingular curve and a bounded family \( \{ E_m \}_{m \geq 0} \) of bundles and a line bundle \( L = K_X^{3n_0} \), for some \( n_0 \geq 3 \). in Lemma 3.1. Since \( E_m \) is a vector bundle of rank two over a curve, the (globally generated) bundle \( E_m \otimes L \) is generated by 3 global sections (Ex. 8.2, Chap II, [H]). Hence there is a short exact sequence of \( \mathcal{O}_X \)-modules

\[
0 \to M_m \to \mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{O}_X \to E_m \otimes L \to 0.
\]

Now \( M_m = \text{det}(E_m \otimes L)^\vee = (L^\otimes 2)^\vee \). Dualizing the above short exact sequence we get

\[
(3.1) \quad 0 \to (E_m \otimes L)^\vee \to \mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{O}_X \xrightarrow{\eta} L^\otimes 2 \to 0.
\]

Let

\[
R = \oplus_{n \geq 0} R_n = \oplus_{n \geq 0} H^0(X, \mathcal{L}^\otimes 2n)
\]

and \( I_m = h_{m1}R + h_{m2}R + h_{m3}R \), where the map \( \eta \) is given by \((s_1, s_2, s_3) \mapsto h_{m1}s_1 + h_{m2}s_2 + h_{m3}s_3 \). Since \( L^\otimes 2 = K_X^{6n_0} \), for some integer \( n_0 \), the ring \( R \) is a normal \( \mathbb{Q} \)-Gorenstein domain. Let \( m \) be the graded maximal ideal of \( R \). Note that \( h_{m1}, h_{m2}, h_{m3} \in R_1 \) and \( \deg X = e_0(R, m) = \deg L^\otimes 2 \).

By Remark 2.3, we have

\[
e^I_m(m) = 1 - a_{\min}((E_m \otimes L)^\vee) / \deg(L^\otimes 2).
\]

Now the exact sequence \( 0 \to L_m \to F^{m*}E_m \to L_m^{-1} \to 0 \) gives

\[
0 \to L_m \otimes F^{m*}(L^\vee) \to F^{m*}((E_m \otimes L)^\vee) \to L_m^{-1} \otimes F^{m*}(L^\vee) \to 0.
\]

and also the strong HN filtration \( 0 \subset L_m \otimes F^{m*}(L^\vee) \subset F^{m*}(E_m \otimes L)^\vee \) of \( F^{m*}(E_m \otimes L)^\vee \).

Hence

\[
a_{\min}(E_m \otimes L)^\vee = \mu_{\min}(F^{m*}(E_m \otimes L)^\vee) = -\deg(L) - \frac{\deg(L_m)}{p^m} = -\deg(L) - \frac{(g-1)}{p^{m+m_0}}.
\]

Therefore

\[
e^I_m(m) = 1 + \frac{1}{2 \deg(L)} \left[ \deg L + \frac{g-1}{p^{m+m_0}} \right] = \frac{3}{2} + \frac{(g-1)}{d^{m+m_0}},
\]

where \( d = e_0(R, m) = \deg L^\otimes 2 \). This proves the theorem.

**Remark 3.3.** We recall that when \( R \) is a regular local ring, then, apart from the set of F-thresholds of an ideal, being discrete and rational, there can be no strictly decreasing sequence of F-thresholds of an ideal \( I \) (Remark 2.9, [MTW]). This is because in the regular case there is a bijection between the set of F-thresholds of \( I \) and the set of test ideals of \( I \), given by \( c \mapsto \tau(I^c) \) such that if \( c_1 \) and \( c_2 \) are two F-thresholds of \( I \) then \( c_1 < c_2 \) if and only if \( \tau(I^{c_2}) \subset \tau(I^{c_1}) \).

Hence the above example in Theorem 1.1 shows that any “order reversing” such bijective correspondence between the set of F-thresholds and a set of ideals of some kind, would not hold.
4. $F$-thresholds reduction mod $p$

We follow notations and definitions as given in Section 2.

Proof of Theorem 1.3: First we prove the following claim.

Claim. Let $V$ be a vector bundle of rank $r$ on a nonsingular curve of genus $g$ over a field of char $p > 0$. If $p > \max\{4(g-1)r^3, r!\}$ then

$$a_{\min}(V) < \mu_{\min}(V) \implies \mu_{\min}(V) = a_{\min}(V) + a/pb,$$

where $a, b$ are positive integers and $\gcd(a, p) = 1$ and $0 < a/b < 4(g-1)(r-1)$.

Proof of the claim: Let $m$ be an integer such that $F^{m*}V$ achieves the strong HN filtration. Note that, by the hypothesis $m \geq 1$ and, by definition $a_{\min}(V) = \mu_{\min}(F^{m*}V)/p^m$. By Lemma 1.14 of [T],

$$\mu_{\min}(F^{m*}V)/p^m + C/p = \mu_{\min}(V), \text{ where } 0 < C \leq 4(g-1)(r-1).$$

This implies $Cp^{m-1}(r!) \in \mathbb{N}$ and we can write

$$\mu_{\min}(V) = a_{\min}(V) + \frac{Cp^{m-1}(r!)}{p^m(r!)} = a_{\min}(V) + \frac{a}{pb},$$

where $a$ and $b$ are positive integers such that $\gcd(a, p) = 1$. This proves the claim.

By Remark 2.3 if $c^{I_p}({\mathfrak{m}}_p) \neq c^{I_{\infty}}({\mathfrak{m}})$ then $c^{I_p}({\mathfrak{m}}_p) > c^{I_{\infty}}({\mathfrak{m}})$. Moreover there exists a vector bundle $W$ on $X = \text{Proj } S$, where $\pi : R \rightarrow S$ is the normalization of $R$ and $X$ is a nonsingular projective curve of degree $d$ such that, for $p \gg 0$,

$$c^{I_{\infty}}({\mathfrak{m}}) = 1 - \mu_{\min}(W)/d \text{ and } c^{I_p}({\mathfrak{m}}_p) = 1 - a_{\min}(W_p)/d,$$

where $W_p$ denotes the bundle $W$ reduction mod $p$, (similarly for $I$ and $\mathfrak{m}$) and where char $R_p = p$. The openness property of the semistable bundle ([Ma]) imply that for $p \gg 0$, $\mu_{\min}(W_p) = \mu_{\min}(W)$. Therefore we can write

$$c^{I_p}({\mathfrak{m}}_p) = 1 - \frac{\mu_{\min}(W_p)}{d} + \frac{a}{pb}.$$

Since $-\mu_{\min}(W_p) = d_1/r_1$, where $d_1, r_1 \in \mathbb{Z}^+$ such that $r_1 < \mu(I)$, the theorem follows for $p \gg 0$. \hfill \square

Remark 4.1. We recall Example 6.9 from [TrW]. Let $R_p = k[x, y, z]/(h)$, where $h = x^{d-1}y + y^{d-1}z + z^{d-1}x$ and $d \geq 7$ is an odd integer. Let char $R_p = p$ where $p \geq d^2$ such that $p \equiv \pm 2 \pmod{(d^2 - 3d + 3)}$. Then $c^{\mathfrak{m}}(\mathfrak{m}) = (3pd + d^2 - 9d + 15)/2pd$. Therefore if $c^{I_p}({\mathfrak{m}}_p) \neq c^{I_{\infty}}({\mathfrak{m}})$ then though $p$ divides the denominator of $c^{m_p}({\mathfrak{m}}_p)$, the denominator need not be a power of $p$.

References

[BMS1] Blickle, M., Mustaţă, M., Smith, K., $F$-thresholds of hypersurfaces, Trans. Amer. Math. Soc. 361 (2009), no. 12, 6549-6565.
[BMS2] Blickle, M., Mustaţă, M., Smith, K., Discreteness and rationality of $F$-thresholds, Michigan Math. J., 57 (2008), pp. 43-61 (Special volume in honor of Melvin Hochster).
[BS] Bhargav, B., Singh, A., The $F$-pure threshold of a Calabi-Yau hypersurface. Math. Ann. (2015) 362, 551-567.
[BSTZ] Blickle, M., Schwede, K., Takagi, S., Zhang, W., Discreteness and rationality of $F$-jumping numbers on singular varieties. Math. Ann. 347 (2010), no. 4, 917-949.
[CHSW] Canton, E., Herndez, D., Schwede, K., Witt, E., On the behavior of singularities at the $F$-pure threshold, Illinois J. Math. 60 (2016), no. 3-4, 669-685.
[G] Gieseker, D., Stable vector bundles and the Frobenius morphism, Ann. Sci. cole Norm. Sup. (4) 6 (1973), 95-101.
[GrS] Graf, P., Schwede, K., Discreteness of F-jumping numbers at isolated non-Q-Gorenstein points. Proc. Amer. Math. Soc. 146 (2018), no. 2, 473-487.
[HY] Hara, N., Yoshida, K., A generalization of tight closure and multiplier ideals Trans. Am. Math. Soc. 355, 3143-3174 (2003).
[H] Hartshorne, R., Algebraic geometry, Springer-Verlag NY (1977).
[HMNb] Huneke, C., Montanera, J.A., Núñez-Betancourt, L., D-modules, Bernstein-Sato polynomials and F-invariants of direct summands, Advances in Mathematics Volume 321, 1 December 2017, Pages 298-325.
[KSSZ] Katzman, M., Schewde, K., Singh, A., Zhang, W., Rings of Frobenius operators. Math. Proc. Cambridge Philos. Soc. 157 (2014), no. 1, 151167.
[KLZ] Katzman, M., Lyubeznik, G., Zhang, W., On the discreteness and rationality of F-jumping coefficients, J. Algebra 322 (2009), no. 9, 3238-3247.
[Ma] Maruyama, M., Openness of a family of torsion free sheaves. J. Math. Kyoto Univ. 16-3 (1976), 627-637.
[Mu1] Mumford, D., Abelian varieties, Tata Institute of Fundamental Research, Studies in Mathematics, No 5, corrected reprint, Hindustan Book Agency, New Delhi (2012).
[Mu2] Mumford, D., Lectures on curves on an Algebraic Surface, Annals of Math. studies 59, Princeton University Press, Princeton, NJ (1966).
[MTW] Mustat˘ a, M., Takagi, S., Watanabe, K.I., F-thresholds and Bernstein-Sato polynomials, European congress of mathematics, 341-364, Eur. Math. Soc., Zurich, 2005.
[ST] Schwede, K., Tucker, T., Test ideals of non-principal ideals: Computations, Jumping Numbers, Alterations and Division Theorems, J. Math., Pure Appl. (9) 102 (2014), no. 5, 891-929.
[TaW] Takagi, S., Watanabe, K.I., On F-pure thresholds, J. Algebra 282 (2004), 278-297.
[T] Trivedi, V., Hilbert-Kunz multiplicity and reduction mod p, Nagoya Math. Journal 185 (2007), 123-141.
[TrW] Trivedi, V., Watanabe, K., Hilbert-Kunz density functions and F-threshold. [arXiv:1808.04093v1].

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