ON A CONJECTURE OF DAO-KURANO

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ABSTRACT. We prove a special case of a conjecture of Dao-Kurano concerning the vanishing of Hochster’s theta pairing. The proof uses Adams operations on both topological $K$-theory and perfect complexes with support.

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References

1. Introduction

Let $A$ be a local hypersurface ring with maximal ideal $\eta$. Assume $A$ has an isolated singularity; that is, assume $A_p$ is a regular local ring for all $p \in \text{Spec}(A) \setminus \{\eta\}$. If $M$ and $N$ are finitely generated $A$-modules, $l(\text{Tor}_i^A(M,N)) < \infty$ for $i \gg 0$, where $l(-)$ denotes length as an $A$-module. Further, since minimal free resolutions of finitely generated $A$-modules are eventually 2-periodic, $\text{Tor}_i^A(M,N) = \text{Tor}_{i+2}^A(M,N)$ for $i \gg 0$. This motivates the following definition:

**Definition 1.1.** Let $M$ and $N$ be finitely generated $A$-modules. The *Hochster theta pairing* applied to $M$ and $N$ is given by

$$\theta(M,N) = l(\text{Tor}_{2i}^A(M,N)) - l(\text{Tor}_{2i+1}^A(M,N)), \quad i \gg 0.$$ 

The Hochster theta pairing was introduced by Hochster in [Hoc81]. Various conjectures concerning the vanishing of $\theta$ have received a good deal of attention lately: see, for instance, work of Buchweitz-van Straten ([BvS12]), Dao ([Dao13]), Moore-Piepmeyer-Spiroff-Walker ([MPSW11]), and Polishchuk-Vaintrob ([PV12]). For a more detailed history of the Hochster theta pairing, we refer the reader to Section 3 of Dao-Kurano’s article [DK14].

Conjecture 3.1 of [DK14] lists several open questions regarding the vanishing of $\theta$. In particular, Dao-Kurano conjecture the following:

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**Conjecture 1.2** (Dao-Kurano, [DK14] Conjecture 3.1 (3)). Let $A$ be a local hypersurface of Krull dimension $d$ with isolated singularity, and let $M$ and $N$ be finitely generated $A$-modules. If $\dim(M) \leq \frac{d}{2}$, $\theta(M, N) = 0$.

Dao-Kurano themselves prove Conjecture 1.2 in the following cases:

- $A = S_\eta$, where $S$ is a positively graded algebra over a field $k$ such that $\text{Proj}(S)$ is smooth over $k$, and $\eta$ is the homogeneous maximal ideal of $S$
- $A$ is excellent, $A$ contains a field of characteristic 0, and $d \leq 6$

The goal of this paper is to prove Conjecture 1.2 in the following additional special case:

**Theorem 1.3.** Let $Q := \mathbb{C}[x_1, \ldots , x_n]$, let $m$ denote the maximal ideal $(x_1, \ldots , x_n) \subseteq Q$, let $f \in m$, and let $R := Q/(f)$. Assume the local hypersurface $R_m$ has an isolated singularity. Let $M$ and $N$ be finitely generated $R_m$-modules. If $\dim(M) \leq \frac{n-1}{2}$, $\theta(M, N) = 0$.

**Remark 1.4.** When $n$ is odd, Theorem 1.3 follows immediately from a theorem of Buchweitz-van Straten which implies that $\theta(M, N) = 0$ for all finitely generated $R_m$-modules $M, N$ (see the Main Theorem on page 245 of [BvS12]).

Our proof of the theorem uses Adams operations on both topological $K$-theory and perfect complexes with support in $V(f)$; the latter are introduced by Gillet-Soulé in [GS87]. It is convenient here to use constructions of Adams operations involving tensor powers of complexes of vector bundles and $Q$-modules, respectively. In topology, this approach is due to Atiyah in [Ati66], and in [BMTW16], the authors use Atiyah’s ideas to carry out a similar construction in the setting of perfect complexes with support; see also Hautoj’s Ph.D. thesis [Hau09].

In Sections 2 and 3, we discuss Adams operations on topological $K$-theory and perfect complexes with support, respectively. In Section 4, we discuss a sense in which these operations are compatible; see Proposition 4.3 for the precise statement. In Section 5, we prove Theorem 1.3.

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## 2. Adams operations on topological $K$-theory

We review some facts concerning Adams operations on topological $K$-theory. Everything in this section is classical and can be found in [Ati66], Chapter 3 of [Ati67], or [Seg68].

### 2.1. Cyclic Adams operations.

Let $G$ be a topological group. If $Y$ is a compact Hausdorff $G$-space, let $\text{Vect}_G(Y)$ denote the exact category of complex equivariant vector bundles on $Y$, and let $\text{KU}_G(Y)$ denote the Grothendieck group of $\text{Vect}_G(Y)$. $\text{KU}_G(Y)$ is the **complex equivariant $K$-theory group** of $Y$.

When $G = \{\ast\}$, we write $\text{KU}(Y) := \text{KU}_G(Y)$.

Fix a connected compact Hausdorff space $X$ that is homotopy equivalent to a finite CW complex. The **Adams operations** on $\text{KU}(X)$ are functions $\psi^k : \text{KU}(X) \to \text{KU}(X)$ for $k \geq 0$ with the following properties:

- The $\psi^k$ are additive
- The $\psi^k$ are natural with respect to pullback along continuous maps
- If $l$ is a line bundle on $X$, $\psi^k[l] = [l^\otimes k]$

These properties determine the operations $\psi^k$ uniquely, by the splitting principle. The eigenvalues of $\psi^k$ on $\text{KU}(X) \otimes \mathbb{Q}$ are all of the form $k^i$ for some $i \geq 0$; let $\text{KU}(X)^{(i)}$ denote the eigenspace corresponding to the eigenvalue $k^i$ in $\text{KU}(X) \otimes \mathbb{Q}$. The Chern character determines an isomorphism of graded $\mathbb{Q}$-vector spaces

$$\text{ch} : \bigoplus_i \text{KU}(X)^{(i)} \overset{\cong}{\to} \bigoplus_i H^{2i}(X; \mathbb{Q}).$$
Let $\widetilde{KU}(X)$ denote the reduced topological $K$-theory of $X$, defined to be the cokernel of the map $KU(*) \to KU(X)$ induced by pullback along $X \to *$. Let $[n]$ denote the class in $KU(X)$ represented by the trivial bundle of rank $n$. A splitting of the short exact sequence

$$0 \to KU(*) \to KU(X) \xrightarrow{i} \widetilde{KU}(X) \to 0$$

is given by the map $i: \widetilde{KU}(X) \to KU(X)$ defined by $[V] \mapsto [V] - [\text{rank}(V)]$. We define operations on $\widetilde{KU}(X)$ by $\pi \circ \psi^k \circ i$, and we denote them also by $\psi^k$.

Let $p$ be a prime. We recall a construction of $\psi^p: KU(X) \to KU(X)$ due to Atiyah in [Ati66]. Given a vector bundle $V$ over $X$, the $p$-th tensor power $V^\otimes p$ of $V$ may be equipped with a canonical action of the symmetric group on $p$ letters, and hence determines a class in $KU_{\Sigma_p}(X)$ (where $X$ is equipped with the trivial action of $\Sigma_p$). By Proposition 2.2 of [Ati66], there exists a map $KU(X) \to KU_{\Sigma_p}(X)$ sending a class $[V]$ represented by a bundle $V$ to $[V^\otimes p]$. Let $C_p$ denote the cyclic subgroup of $\Sigma_p$ generated by $\sigma_p := (12\ldots p)$; the inclusion $C_p \hookrightarrow \Sigma_p$ determines a map $KU_{\Sigma_p}(X) \to KU_{C_p}(X)$. Let $t^p: KU(X) \to KU_{C_p}(X)$ denote the composition of these two maps.

Since $C_p$ acts trivially on $X$, one has a canonical isomorphism

$$KU_{C_p}(X) \xrightarrow{\cong} KU(X) \otimes R(C_p),$$

where $R(C_p)$ denotes the representation ring of $C_p$ ([Seg68] Proposition 2.2). On classes of the form $[V]$, where $V$ is a vector bundle, the isomorphism is given by

$$[V] \mapsto \sum_{j=0}^{p-1} \text{Hom}_{C_p}(M_j, V) \otimes W_j,$$

where $W_j$ is the irreducible representation of $C_p$ corresponding to the character $\sigma_p \mapsto e^{2\pi ij/p}$, and $M_j$ is the $C_p$-bundle $W_j \times X$. By line (2.7) of [Ati66], if $V$ is a vector bundle on $X$, then

$$\psi^p([V]) = \text{Hom}_{C_p}(M_0, t^p[V]) - \text{Hom}_{C_p}(M_1, t^p[V]).$$

### 2.2. Variant for relative $K$-theory.

Let $X$ be a connected compact Hausdorff space, and let $Y$ be a closed subspace of $X$ such that $(X, Y)$ is homotopy equivalent to a finite CW pair. Let $G$ be a topological group, suppose $X$ is equipped with an action of $G$, and suppose $Y$ is a $G$-subspace of $X$. Let $C_G(X, Y)$ denote the exact category of bounded complexes of complex equivariant vector bundles on $X$ whose restrictions to $Y$ are exact.

**Definition 2.1.** Objects $C_0, C_1$ of $C_G(X, Y)$ are said to be homotopic if there exists an object $C$ of $C_G(X \times [0, 1], Y \times [0, 1])$ such that the restriction of $C$ to $X \times \{i\}$ is isomorphic to $C_i$ for $i = 0, 1$.

Set $\text{Iso}(C(X, Y))$ to be the monoid of isomorphism classes in $C_G(X, Y)$ with operation $\oplus$. If $[C_0], [C_1] \in \text{Iso}(C(X, Y))$, we say $[C_0] \sim [C_1]$ if and only if there exist exact complexes $E_0, E_1 \in C_G(X, Y)$ such that $C_0 \oplus E_0$ is homotopic to $C_1 \oplus E_1$. Define $L_G(X, Y)$ to be the monoid $\text{Iso}(C(X, Y))/\sim$.

Let $KU(G, X, Y)$ denote the relative equivariant $K$-theory group of the pair $(X, Y)$ ([Seg68] Definition 2.8).

**Theorem 2.2** ([Seg68] Section 3). There exists a natural isomorphism

$$\chi: L_G(X, Y) \xrightarrow{\cong} KU_G(X, Y).$$

In particular, $L_G(X, Y)$ is a group.

When $Y = \emptyset$, $\chi$ is given by the Euler characteristic. Note that $L_G(X, Y)$ has a product operation $\otimes$ induced by tensor product of complexes, giving it the structure of a non-unital ring, and $\chi(C_0 \otimes C_1) = \chi(C_0) \otimes \chi(C_1)$, where the product on the right-hand side is induced by tensor product of $G$-bundles.
If $G = \{ * \}$, we write $L(X,Y) := L_G(X,Y)$ and $KU(X,Y) := KU_G(X,Y)$. By definition, $KU(X,Y) = KU(X/Y)$. Fix a prime $p$. We now wish to provide an alternative construction, via $\chi$, of the operations $\psi^p$ on $KU(X,Y)$ defined above.

By Section 3 of [Ati66], the $p$-th tensor power of complexes induces a homomorphism

$$L(X,Y) \to L_{\Sigma_p}(X,Y),$$

where, in the target, $X$ is equipped with the trivial action of $\Sigma_p$. Restricting along the inclusion $C_p \hookrightarrow \Sigma_p$, we obtain a map

$$t^p : L(X,Y) \to L_{C_p}(X,Y).$$

Given complexes $C, C'$ of $C_p$-bundles on $X$, let $\text{Hom}_{C_p}(C,C')$ denote the morphism complex in the category of complexes of $C_p$-bundles on $X$. Let $M_0, \ldots, M_{p-1}$ be as defined above, considered as complexes of $C_p$-bundles concentrated in degree 0. For each $j$, $\text{Hom}_{C_p}(M_j, -)$ yields an exact functor $C_{C_p}(X,Y) \to \mathcal{C}(X,Y)$, and it preserves homotopy; thus, $\text{Hom}_{C_p}(M_j, -)$ induces a map $L_{C_p}(X,Y) \to L(X,Y)$. Define

$$\psi^p : L(X,Y) \to L(X,Y)$$

to be given by

$$\psi^p([C]) = [\text{Hom}_{C_p}(M_0, t^p[C])] - [\text{Hom}_{C_p}(M_1, t^p[C])],$$

Since $\chi$ is multiplicative, it is not hard to check that one has a commutative square

$$\begin{array}{ccc}
L(X,Y) & \longrightarrow & KU(X,Y) \\
\downarrow \psi^p & & \downarrow \psi^p \\
L(X,Y) & \longrightarrow & KU(X,Y)
\end{array}$$

3. Adams operations on perfect complexes with support

Let $Q$ be a commutative Noetherian $\mathbb{C}$-algebra, let $Z \subseteq \text{Spec}(Q)$ be a closed subset, and let $G$ be a finite group. Let $\mathcal{P}^Z(Q;G)$ denote the category of bounded complexes of finitely generated projective $Q$-modules with support in $Z$ and equipped with a left $G$-action (with $G$ acting via chain maps). Let $K^Z_0(Q;G)$ denote the Grothendieck group of $\mathcal{P}^Z(Q;G)$, defined to be the group generated by isomorphism classes of objects modulo the relations

$$[X] = [X'] + [X''].$$

if there exists an (equivariant) short exact sequence $0 \to X' \to X \to X''$, and

$$[X] = [Y]$$

if there exists an (equivariant) quasi-isomorphism joining $X$ and $Y$; the group operation is given by direct sum. When $G = \{ * \}$, we write $K^Z_0(Q) := K^Z_0(Q;G)$.

In this section, we recall a construction of Adams operations on $K^Z_0(Q)$ involving cyclic actions on tensor powers of complexes, following Section 3 of [BMTW16]; see also Hau’ton’s Ph.D. thesis [Hau99] for a similar discussion. The construction is inspired by the cyclic Adams operations on topological $K$-theory, due to Atiyah, which we discuss in the previous section.

Adams operations on $K^Z_0(Q)$ were introduced by Gillet-Soulé in [GSS77] for the purpose of proving Serre’s Vanishing Conjecture. By Corollary 6.14 of [BMTW16], the operations we discuss here agree, in our setting, with those constructed by Gillet-Soulé.

Let $p$ be a prime. If $X$ is an object of $\mathcal{P}^Z(Q)$, the $p$-th tensor power $X^{\otimes p}$ has a canonical signed left action of $\Sigma_p$. By Theorem 2.2 of [BMTW16], there exists a map $K^Z_0(Q) \to K^Z_0(Q;\Sigma_p)$ that sends a class $[X]$ represented by an object $X$ in $\mathcal{P}^Z(Q)$ to $[X^{\otimes p}]$. Restricting along the inclusion $C_p \hookrightarrow \Sigma_p$ yields a map

$$t^p : K^Z_0(Q) \to K^Z_0(Q;C_p).$$
For $0 \leq j \leq p-1$, let $Q_j$ denote the projective $Q[C_p]$-module $Q$ with action $\sigma_p q = e^{2\pi i j/p} q$ (recall that $\sigma_p = (12\ldots p)$). Define the $p$-th Adams operation $\psi^p : K^Z_0(Q) \rightarrow K^Z_0(Q)$ by

$$Y \mapsto [\text{Hom}_{Q[C_p]}(Q_0, t^p(Y))] - [\text{Hom}_{Q[C_p]}(Q_1, t^p(Y))].$$

By Theorem 3.7 of [BMTW16], $\psi^p$ is a group endomorphism. We refer the reader to [BMTW16] for a thorough discussion of the properties of the Adams operations $\psi^p$. One such property we will need later on is:

**Theorem 3.1 (BMTW16 Corollary 3.12).** If $p$ is prime, $Q$ is regular of Krull dimension $d$, and $Z$ has codimension $c$ in $\text{Spec}(Q)$, there exists a direct sum decomposition

$$K^Z_0(Q) \otimes Q = \bigoplus_{i=0}^{d} K^Z_0(Q)^{(i)}(Q)$$

where $K^Z_0(Q)^{(i)}$ is the eigenspace of $\psi^p$ in $K^Z_0(Q) \otimes Q$ corresponding to the eigenvalue $p^i$. Moreover, if $M$ is a finitely generated $Q$-module supported on $Z$, and $X$ is a finite projective resolution of $M$, one has

$[X] \in \bigoplus_{i=\text{codim}_Q M}^{d} K^Z_0(Q)^{(i)}.$

**Remark 3.2.** The idea of the proof of this theorem is essentially due to Gillet-Soulé in [GS87]. They show that the above theorem holds for any family of operations on $K$-theory with supports satisfying conditions A1) through A4) in Section 4.11 of [GS87]. Thus, the authors of [BMTW16] need only show that the operations $\psi^k$ defined above satisfy these conditions, and they prove this in Theorem 3.7 of [BMTW16] (note that Theorem 3.7 of [BMTW16] is proven only in the setting of affine schemes, but this is enough to conclude the above theorem; see Remark 3.8 of [BMTW16]).

4. Compatibility of Adams operations

Let $Q := \mathbb{C}[x_1, \ldots, x_n]$, let $m := (x_1, \ldots, x_n)$, and let $f \in m \smallsetminus \{0\}$. The goal of this section is to exhibit a precise sense in which the Adams operations on $K^Y_0(f)(Q)$ are compatible with Adams operations on topological $K$-theory.

4.1. The Milnor fibration. By well-known theorems of Lê and Milnor ([Lê76], [Mil68]), there exist $\epsilon > 0$ and $0 < \delta' \ll \epsilon$ such that, if

- $B \subseteq \mathbb{C}^n$ denotes the closed ball centered at the origin of radius $\epsilon'$, where $0 < \epsilon' < \epsilon$, and
- $D^* \subseteq \mathbb{C}$ denotes the open disk of radius $\delta'$ centered at the origin with the origin removed, where $0 < \delta' < \delta$,

the map $\psi : B \cap f^{-1}(D^*) \rightarrow D^*$ given by $x \mapsto f(x)$ is a fibration, called the Milnor fibration of $f$. Choose such $\epsilon'$ and $\delta'$, choose $t \in D^*$, and set $F := \psi^{-1}(t)$. $F$ is called the Milnor fiber of $f$. $F$ is independent of the choices of $\epsilon'$, $\delta'$, and $t$ up to homotopy equivalence.

We refer the reader to Chapter 3 of Dimca’s text [Dim12] for a detailed discussion of the Milnor fibration. We point out a key property of the Milnor fiber which we will use later on. Set

$$\mu := \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]_m \frac{\partial f}{\partial x_1} \ldots \frac{\partial f}{\partial x_n},$$

the Milnor number of $f$ at the origin. The following is a famous theorem of Milnor:

**Theorem 4.1 (Mil68 Theorem 6.5).** If $\mu < \infty$, $F$ is homotopy equivalent to a wedge sum of $\mu$ copies of $S^{n-1}$. 
4.2. Matrix factorizations. We recall some background on matrix factorizations in commutative algebra. Let $S$ be a commutative ring, and let $w \in S$.

**Definition 4.2.** A *matrix factorization of $w$ over $S$* is a pair of finitely generated projective $S$-modules $F_0, F_1$ equipped with maps $$d_0 : F_0 \to F_1, \quad d_1 : F_1 \to F_0$$ such that $d_1 d_0 = w \cdot \text{id}_{F_0}$ and $d_0 d_1 = w \cdot \text{id}_{F_1}$. We denote matrix factorizations by $(F_0, F_1, d_0, d_1)$.

One may form the *homotopy category of matrix factorizations* $\mathcal{MF}(S, w)$ with objects given by matrix factorizations of $w$ over $S$; see Definition 2.1 of [Dyc11] for the definition of $\mathcal{MF}(S, w)$.

Assume $S$ is regular of finite Krull dimension and $w$ is a non-unit, non-zero-divisor of $S$. In this case, $\mathcal{MF}(S, w)$ may be equipped with a canonical triangulated structure; see Section 3.1 of [Orl03] for details. In fact, setting $T := S/(w)$, there exists an equivalence of triangulated categories $$\mathcal{MF}(S, w) \xrightarrow{\cong} D^b(T)/\text{Perf}(T),$$ where the right-hand side is the Verdier quotient of the bounded derived category of $T$ by the triangulated subcategory consisting of perfect complexes ([Orl03] Theorem 3.9). The equivalence sends a matrix factorization $(F_0, F_1, d_0, d_1)$ to the complex with $\text{coker}(d_1)$ concentrated in degree 0.

4.3. Compatibility. Set $R := Q/(f)$, where $Q$, $f$ are as in the beginning of this section. Our next goal is to define a group homomorphism $$\gamma : K_0^{V(f)}(Q) \to L(B, F),$$ that is compatible with Adams operations, where $B, F$ are as defined in Section 4.1. We proceed as follows:

- By Lemma 1.9 of [GS87], there exists an isomorphism $r : G_0(R) \xrightarrow{\cong} K_0^{V(f)}(Q)$ that sends a class represented by a module $M$ to the class represented by a finite $Q$-free resolution of $M$.
- We recall that the Grothendieck group of a triangulated category $\mathcal{T}$ is the free abelian group on isomorphism classes of objects in $\mathcal{T}$ modulo relations given by exact triangles. Let $K_0[\mathcal{MF}(Q, f)]$ denote the Grothendieck group of the triangulated category $[\mathcal{MF}(Q, f)]$. The equivalence $[\mathcal{MF}(Q, f)] \xrightarrow{\cong} D^b(R)/\text{Perf}(R)$ discussed in Section 4.2 yields an isomorphism $K_0[\mathcal{MF}(Q, f)] \xrightarrow{\cong} G_0(R)/\text{im}(K_0(R) \to G_0(R))$. In particular, we have a surjection $$s : G_0(R) \to K_0[\mathcal{MF}(Q, f)].$$
- Let $E = (F_0, F_1, d_0, d_1)$ be a matrix factorization of $f$ over $Q$. The following construction, introduced by Buchweitz-van Straten in [BvST12], associates a class in $L(B, F)$ to the matrix factorization $E$.

Denote by $C(B)$ the ring of $\mathbb{C}$-valued continuous functions on $B$. Applying extension of scalars along the inclusion $$Q \hookrightarrow C(B),$$ we obtain a map $$F_1 \otimes_Q C(B) \xrightarrow{d_1 \otimes \text{id}} F_0 \otimes_Q C(B)$$ of finitely generated free $C(B)$-modules. The category of complex vector bundles over $B$ is equivalent to the category of finitely generated free $C(B)$-modules; on objects, the equivalence sends a bundle to its space of sections. Let $$V_1 \xrightarrow{d_1} V_0$$
be a map of vector bundles over $B$ corresponding to the above map $d_1 \otimes \text{id}$ under this equivalence. Recall that $t \in \mathbb{C}$ is the value over which we defined the Milnor fiber $F$. Since $d_1 \circ d_0 = f \cdot \text{id}_F$, and $d_0 \circ d_1 = f \cdot \text{id}_F$, and since the restriction of the polynomial $f$, thought of as a map $\mathbb{C}^n \to \mathbb{C}$, to $F = B \cap f^{-1}(t)$ is constant with value $t \neq 0$, $d_1 \vert_F$ is an isomorphism of vector bundles on $F$. Its inverse is the restriction to $F$ of the map $V_0 \to V_1$ determined by

$$F_0 \otimes \mathbb{Q} C(B) \xrightarrow{\frac{1}{d_0} \cdot \text{id}} F_1 \otimes \mathbb{Q} C(B).$$

Define $\Phi(E)$ to be the class in $L(B, F)$ represented by the complex $0 \to V_1 \xrightarrow{d_1} V_0 \to 0$. By (the complex version of) Proposition 3.19 of [Bro15], Buchweitz-van Straten’s construction $E \mapsto \Phi(E)$ induces a group homomorphism

$$\phi : K_0[\text{MF}(Q, f)] \to L(B, F).$$

Finally, we define $\gamma : K_0^{(f)}(Q) \to L(B, F)$ to be the composition

$$K_0^{(f)}(Q) \xrightarrow{r^{-1}} G_0(R) \xrightarrow{\phi} K_0[\text{MF}(Q, f)] \xrightarrow{\phi} L(B, F).$$

Now, suppose $0 \to F_1 \xrightarrow{d_1} F_0 \to 0$ is a $Q$-projective resolution of an $R$-module $M$. Then $r^{-1}(0 \to F_1 \xrightarrow{d_1} F_0 \to 0) = [M]$, and $s([M])$ is of the form $[F_1, F_0, d_1, d_0]$ for some map $d_0 : F_0 \to F_1$. Let $V_1 \xrightarrow{d_1} V_0$ be a map of vector bundles over $B$ corresponding to the map

$$F_1 \otimes \mathbb{Q} C(B) \xrightarrow{d_1 \otimes \text{id}} F_0 \otimes \mathbb{Q} C(B)$$

of free $C(B)$-modules, as in the third bullet above. Then one has

$$\gamma([0 \to F_1 \xrightarrow{d_1} F_0 \to 0]) = [0 \to V_1 \xrightarrow{d_1} V_0 \to 0].$$

Using the isomorphisms

$$r : G_0(R) \xrightarrow{\cong} K_0^{(f)}(Q), \quad K_0[\text{MF}(Q, f)] \xrightarrow{\cong} G_0(R) / \text{im}(K_0(R) \to G_0(R)),$$

it is easy to see that classes of the form $[P]$, where $P$ is a two-term $Q$-free resolution of an $R$-module $M$, generate $K_0^{(f)}(Q)$ as an abelian group. Thus, the following is immediate from the constructions of the Adams operations on $K_0^{(f)}(Q)$ and $L(B, F)$ discussed above:

**Proposition 4.3.** If $p$ is prime and $X \in K_0^{(f)}(Q)$, $\gamma(\psi^p(X)) = \psi^p(\gamma(X))$.

**Remark 4.4.** Let $g$ be an element of $Q$ such that $g \notin \mathfrak{m}$. Suppose that, in the construction of the Milnor fiber, $\epsilon'$ is chosen to be so small that $B \cap g^{-1}(0) = \emptyset$. Then one may define maps

$$r_g : G_0(R_g) \xrightarrow{\cong} K_0^{(f)}(Q_g), \quad s_g : G_0(R_g) \to K_0[\text{MF}(Q_g, f)], \quad \phi_g : K_0[\text{MF}(Q_g, f)] \to L_1(B, F)$$

in exactly the same way as in the three bullets above. Set $\gamma_g := \phi_g s_g r_g^{-1}$. If $p$ is prime, one has $\gamma_g \psi^p = \psi^p \gamma_g$, by the same reasoning as above.

5. **Proof of Theorem 1.3**

**Proof of Theorem 1.3** Choose $g \notin \mathfrak{m}$ such that $R_g$ has an isolated singularity only at $\mathfrak{m}$; that is, such that $(R_g)_g$ is regular for all $p \in \text{Spec}(Q_g) \setminus \mathfrak{m}$. Without loss of generality, assume that, in the construction of the Milnor fiber $F$ in Section 4.1, $\epsilon'$ is chosen to be so small that $B \cap g^{-1}(0) = \emptyset$. Let $r_g, s_g, \phi_g$, and $\gamma_g$ be defined as in Remark 4.4. Also, let $s_m : G_0(R_m) \to K_0[\text{MF}(Q_m, f)]$ denote the surjection defined in the same way as the map $s$ in the second bullet of Section 4.3.

By Theorem 4.11 of [Dye11], the functor $[\text{MF}(Q_g, f)] \to [\text{MF}(Q_m, f)]$ induced by extension of scalars along the localization map $Q_g \to Q_m$ is an equivalence; let $l : K_0[\text{MF}(Q_g, f)] \xrightarrow{\cong} K_0[\text{MF}(Q_m, f)]$ denote the induced isomorphism on Grothendieck groups. By Propositions 4.1
and 4.2 of [BvS12], if \( \phi_g \circ l^{-1} \circ s_m \)(\([M]\)) = 0, then \( \theta(M, N) = 0 \). Let \( M' \) be an \( R_g \)-module such that \( s_g([M']) = (l^{-1} \circ s_m)([M]) \), and let \( P \) be a finite \( Q_g \)-free resolution of \( M' \). It suffices to show \( \gamma_g([P]) = 0 \).

Let \( p \) be a prime. Define

\[
m = \begin{cases} 
\frac{n}{2} + 1 & \text{if } n \text{ is even} \\
\frac{n+1}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

By Theorem 3.1, \([P] \in \bigoplus_{i=m}^{n} K_0^V(Q_g)^{(i)} \). Thus, Remark 4.4 and the commutativity of diagram (2.3) imply

\[
(ch \circ \chi \circ \gamma_g)([P]) \in \bigoplus_{i=m}^{n} H^{2i}(B/F; \mathbb{Q}) \cong \bigoplus_{i=m}^{n} H^{2i}((\Sigma F); \mathbb{Q}),
\]

where \( \Sigma F \) denotes the suspension of \( F \). By Theorem 4.1, \( H^i(\Sigma F; \mathbb{Q}) = 0 \) when \( i > n \); thus, \( (ch \circ \chi \circ \gamma_g)([P]) = 0 \), and so \( \gamma_g([P]) = 0 \).

\[\square\]

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