Fully developed mixed convection in vertical double passage porous annuli

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Abstract. In the present analysis, fully developed mixed convection in the vertical double-passage annuli filled with porous media is investigated both analytically and numerically by imposing asymmetric thermal conditions. Three vertical concentric cylinders are used to form double passage annuli of which thin and conductive middle cylinder is considered known as baffle. Using finite difference technique, the governing equations are solved numerically by considering viscous dissipation, whereas the closed form solution are obtained by neglecting viscous dissipation. Numerical solution matches with that of closed form solution in the absence of viscous dissipation. The results reveal that modified Grashof number, Brinkmann number, Darcy number, position of baffle has profound impacts on velocity profiles, temperature profiles and on heat transfer rate.

1. Introduction
The analysis of combined buoyancy and forced convection in vertical passages filled with porous materials has acquired substantial importance because of broad range of applications, namely geophysical problems, engineering fields, electronic equipment cooling and heating of the Trombe wall and many others. Jeng et al. [1] investigated mixed convection heat transfer numerically by fixing porous wall segments on the hot wall in a vertical parallel channel. Heat transfer improvement is due to increase in porosity, position of porous wall segment for lower values of Re. Kumar et al. [2] determined the analytical solution by perturbation method and semi analytical solution by differential transform method to study the influence of porosity, viscosity, mixed convection parameter on steady laminar mixed convection of permeable and viscous fluids in vertical channel for thermal boundary conditions of third kind. Considering the collective effects of thermal and mass diffusion, Umavathi [3] investigated natural convective flow of viscous fluid analytically by using perturbation method and numerically by using Runge-Kutta shooting method through vertical porous channel. Al-Nimr [4] obtained analytical solution by considering four fundamental boundary conditions for fully developed free convection in a vertical open-ended porous annulus between two concentric cylindrical tubes. For a partially heated open-ended annulus, Sankar [5] performed numerical simulation to estimate the thermal and solute transfer for different parametric regimes. Utilizing non-Darcian flow model, Kou and Huang [6] studied fully developed mixed convective flow and obtained analytical solution in
vertical porous annulus considering three (isothermal-isothermal, isothermal-isoflux and isoflux-isothermal) thermal conditions.

Paul and Singh [7] obtained analytical solutions for laminar fully developed natural convection using Brinkman-extended Darcy model in vertical porous annulus by maintaining walls of cylinders with different temperatures. Results revealed that Brinkman number, Darcy number as well as stress jump condition at the interface between fluid and porous layer significantly affected velocity profiles. Cheng [8] studied natural convective heat and mass transfer under fully developed situation. Considering asymmetric wall temperatures and concentrations author studied the consequences of modified Darcy number, inner gap-radius ratio and buoyancy ratio in vertical porous annulus on volume flow rate, total heat and species rate added to the fluid. Mixed convective heat transfer has been investigated, using Keller box method by Kaya [9] in vertical porous annulus to explore the effects of buoyancy and conjugate heat transfer.

Using a two-region model Kumar et al. [10] studied fully developed mixed convection to discover the influence of viscous dissipation in an infinite vertical porous channel by imposing different thermal boundary conditions. Gaikwad and Rahuldev[11] studied laminar mixed convection to discover the consequence of permeable viscous fluid in vertical double passage channel. They used regular perturbation method to obtain thermal and flow profiles in both streams of the channel. Results reveal that, irrespective of baffle position under consideration, the enhancement in velocity and temperature fields is due to mixed convection parameter whereas porous parameter enhances temperature but reduces the velocity. By taking in to account of viscous dissipation Umavathi et al. [12] systematically investigated mixed convection in a vertical double passage porous channel. Results show that porous and mixed convection parameters, viscosity ratio, Brinkman number, as well as position of the baffle influences velocity, temperature fields and heat transfer rate.

Kumar et al. [13] investigated the consequence of first order chemical reaction by using regular perturbation with Brinkman number as perturbation parameter in vertical double passage porous channel. Results reveal that flow enhancement is due to Brinkman number and thermal as well as mass Grashof number whereas porous and chemical reaction parameter declines the flow. Sankar and collaborators [14]-[17] made detailed analysis on the impacts of porous media and magnetic field in a closed vertical annular space for vast parametric regimes. In the double passage porous annuli, Sankar et al. [18] investigated convective thermal transport with radial magnetic field and found that flow distributions are significantly reformed with the magnetic force, porous media and also the location of baffle. However, viscous dissipation has profound influence on thermal distribution. Developing free convection is numerically studied by Girish et al. [19]-[20] in double passage annuli in presence and absence of porous media. The results reveal that both geometrical and physical parameters under consideration has significant effect on development of velocity, thermal fields and also on rate of heat transfer. In particular, the baffle position played a vital role in controlling thermal transport in annuli. Sankar et al. [21] numerically investigated buoyancy driven convective thermal transport in vertical porous annulus, where the inner and outer cylinders are discretely heated and cooled by different permutation of heat source and sink. The results reveal that rate of thermal transport is strongly influenced by heat source and sink combination, Darcy and Rayleigh numbers. From the literature review, it has been noticed that the impacts of baffle and porosity is not analyzed in vertical double-passage annuli and this motivates the present investigation.

2. Mathematical Formulation
For the present analysis, the physical configuration is vertical double-passage cylindrical annuli formed by three co-axial cylinders, with inner radius $r_i$, middle radius $r_m$ and outer radius $r_o$, as depicted in Fig. 1. In the double-passage annuli, the middle cylinder is considered as a thin and absolutely conductive. The interior and exterior cylindrical boundaries are maintained at higher
and lower temperatures respectively, and the case of asymmetric heating has been considered. We consider a steady, laminar flow in the passage. Further, the constant fluid properties assumption is invoked excluding the body force term in the Navier-Stokes equation. Employing the above stated assumptions and taking viscous dissipation into account, the dimensionless governing equations for the present study are [18]:

\[
\frac{\partial^2 V_j}{\partial R^2} + \frac{1}{R} \frac{\partial V_j}{\partial R} - \frac{\Lambda}{Da} V_j = \frac{\partial P_j}{\partial Z} - \frac{Gr}{Re} \theta_j \quad (1)
\]

\[
\frac{d^2 \theta_j}{dR^2} + \frac{1}{R} \frac{d \theta_j}{dR} + Br \left( \frac{d V_j}{dR} \right)^2 = 0 \quad (2)
\]

Since \( V_j \) and \( \theta_j \) are function of \( R \) only and \( P_j = P_j(Z) \), the above model equations are reduced to the following ODEs.

\[
\frac{d^2 V_j}{dR^2} + \frac{1}{R} \frac{d V_j}{dR} - \frac{\Lambda}{Da} V_j = \frac{d P_j}{dZ} - \frac{Gr}{Re} \theta_j \quad (3)
\]

\[
\frac{d^2 \theta_j}{dR^2} + \frac{1}{R} \frac{d \theta_j}{dR} + Br \left( \frac{d V_j}{dR} \right)^2 = 0 \quad (4)
\]
The following are non-dimensional parameters employed in the present study.

\[ R = \frac{r}{r_0}, \quad V = \frac{\nu}{v_r}, \quad Z = \frac{z}{r_0 Re}, \quad P = \frac{p}{\rho_0 v_r^2}, \quad T_r = \frac{(T_c + T_h)}{2}, \quad \theta_j = \frac{T_j - T_r}{T_h - T_c}, \]

\[ \lambda = \frac{r_i}{r_0}, \quad N = \frac{r_m}{r_0}, \quad \Lambda = \frac{\mu}{\mu_e}, \quad \nu = \frac{\mu_e}{\rho_0}, \quad Re = \frac{v_r r_0}{\nu}, \quad Da = \frac{K}{r_0^2}, \]

\[ Gr = \frac{g^3(T_h - T_c) r_0^3}{\nu^2}, \quad Br = \frac{\mu \nu^2}{k(T_h - T_c)}, \quad GR = \frac{Gr}{Re}, \]

Here \( Br, Da, Gr, Re, \Lambda \) and \( N \) are respectively the Brinkmann, Darcy, Grashof and Reynolds numbers, viscosity ratio, baffle position. The non-dimensional boundary conditions in passages 1 and 2 are:

\[ R = \lambda, \quad V_1 = 0, \quad \theta_1 = 0.5 \]
\[ R = N, \quad V_1 = 0 = V_2, \quad \theta_1 = \theta_2 \]
\[ R = 1, \quad V_2 = 0, \quad \theta_2 = -0.5 \]

The dimensionless mass conservation at any cross section is given in the following form

\[ \int_{\lambda}^{N} RV_1 dR = \frac{1}{2}(N^2 - \lambda^2) \quad \text{and} \quad \int_{N}^{1} RV_2 dR = \frac{1}{2}(1 - N^2) \quad (6) \]

The thermal transport rate is estimated through the following Nusselt numbers at hot and cold boundaries

\[ Nu_h = \frac{2(1 - \lambda)}{(\theta_{b1} - 0.5)\lambda \ln(\lambda)} \quad \text{and} \quad Nu_c = \frac{-2(1 - \lambda)}{(0.5 + \theta_{b2})\ln(\lambda)} \quad (7) \]

Here, \( \theta_{b1} \) and \( \theta_{b2} \) are the bulk temperatures in passage 1 and 2 respectively in non-dimensional form and are given as:

\[ \theta_{b1} = \frac{\int_{\lambda}^{N} \theta_1 V_1 R dR}{\int_{\lambda}^{N} V_1 R dR} \quad \text{and} \quad \theta_{b2} = \frac{\int_{N}^{1} \theta_2 V_2 R dR}{\int_{N}^{1} V_2 R dR} \quad (8) \]

3. Solution Methodology

The governing model equations along with boundary conditions are solved analytically by neglecting viscous dissipation effects. In the presence of viscous dissipation, the governing equations are nonlinear and coupled; hence an implicit finite difference technique along with successive over-relaxation (SOR) methods are used to obtain the numerical solutions. Numerical results are validated with the analytical solution without viscous dissipation effects. The analytical solutions are as follows:
3.1. Analytical solution

\[ \theta = \frac{\ln R}{\ln \lambda} - \frac{1}{2} \]

\[ V_1 = \frac{1}{A_1 \alpha^2} \left[ \frac{1}{A_{13}} \left( \frac{N^2 - \lambda^2}{2} + \frac{A_{14} Gr}{Re \ln \lambda} \right) (A_2 I_0(\alpha R) + A_3 K_0(\alpha R) - A_1) \right] \]

\[ + \frac{1}{A_1 \alpha^2} \left[ \left( \frac{Gr}{Re \ln \lambda} \right) (A_1 \ln R - A_4 I_0(\alpha R) - A_5 K_0(\alpha R)) \right] \]

\[ V_2 = \frac{1}{A_8 \alpha^2} \left[ \frac{1}{A_{15}} \left( \frac{1 - N^2}{2} + \frac{A_{16} Gr}{Re \ln \lambda} \right) (A_6 I_0(\alpha R) + A_7 K_0(\alpha R) - A_8) \right] \]

\[ + \frac{1}{A_8 \alpha^2} \left[ \left( \frac{Gr}{Re \ln \lambda} \right) (A_8 \ln R - \ln N(I_0(\alpha R)K_0(\alpha) - K_0(\alpha R)I_0(\alpha))) \right] \]

The constants appear in the above solution are defined as below:

\[ A_1 = K_0(\alpha \lambda)I_0(\alpha N) - K_0(\alpha N)I_0(\alpha \lambda) \]

\[ A_2 = K_0(\alpha \lambda) - K_0(\alpha N) \]

\[ A_3 = I_0(\alpha N) - I_0(\alpha \lambda) \]

\[ A_4 = ln N K_0(\alpha \lambda) - ln \lambda K_0(\alpha N) \]

\[ A_5 = ln N I_0(\lambda N) - ln N I_0(\alpha N) \]

\[ A_6 = K_0(\alpha) - K_0(\alpha N) \]

\[ A_7 = I_0(\alpha N) - I_0(\alpha) \]

\[ A_8 = K_0(\alpha)I_0(\alpha N) - K_0(\alpha N)I_0(\alpha) \]

\[ A_9 = \alpha N (K_0(\alpha \lambda)I_I(\alpha N) + I_0(\alpha \lambda)K_1(\alpha N)) \]

\[ A_{10} = \alpha (K_0(\alpha N)I_I(\alpha \lambda) + I_0(\alpha N)K_1(\alpha \lambda)) \]

\[ A_{11} = \alpha (K_0(\alpha N)I_1(\alpha) + I_0(\alpha N)K_1(\alpha)) \]

\[ A_{12} = \alpha N (K_0(\alpha)I_1(\alpha N) + I_0(\alpha)K_1(\alpha N)) \]

\[ A_{13} = \frac{1}{A_1 \alpha^2} \left( -2 + A_9 + A_{10} - \frac{\alpha^2}{2} (N^2 - \lambda^2) \right) \]

\[ A_{14} = \frac{1}{A_1 \alpha^2} \left[ \frac{A_9 ln N + A_{10} ln \lambda - ln (N \lambda)}{2} - \frac{\alpha^2}{2} \left( N^2 ln N - \lambda^2 ln \lambda - \left( \frac{N^2 - \lambda^2}{2} \right) \right) \right] \]

\[ A_{15} = \frac{1}{\alpha^4} \left( \frac{1}{A_8} (2 - A_{11} - A_{12}) - \frac{\alpha^2}{4} (1 - N^2) \right) \]

\[ A_{16} = \frac{1}{4 \alpha^2} \left( 1 + N^2 (2 ln N - 1) \right) + \frac{ln N}{\alpha^2} \left( 1 - \frac{A_{12}}{A_8} \right) \]

3.2. Numerical solution

The nondimensional governing equations are nonlinear and coupled when viscous dissipation effect is incorporated ie \((Br \neq 0)\) and are solved by an implicit finite difference technique along with successive over relaxation. The description of solution method is not discussed here for brevity and can be found in Sankar et al. [18].

4. Results and discussion

This section mainly focuses on the effects of modified Grashof, Darcy and Brinkmann numbers, radii ratio and baffle position on velocity and temperature profiles as well as on the heat transfer
rate. Figure 2 illustrate the velocity and temperature distributions for three different baffle positions and three values of $GR$. Due to enhanced convection at higher values of $GR$, it is found that the fluid velocity increases with $GR$. Also, the peak or maximum velocity occurs in the wider annular region. In the narrow region, the effect of $GR$ is not noticeable. The variation in velocity profiles can be observed in both the channels when baffle is placed in the middle of the fluid flow. As regards to the effect of middle cylinder (baffle) position on temperature profiles, it has been found that the thermal profiles are altered as the baffle location moves toward the inner wall. However, as the baffle is located near outer wall, the temperature profiles remain unaltered. This is expected due to the fact that energy equation is independent of velocity in the case of small Brinkman number. The influence of Darcy number on the fluid velocity and temperature profiles is illustrated in Fig.3. As expected, the magnitude of velocity decreases as Darcy number decreases. The maximum magnitude of the velocity appears in passage 1 for small values of Brinkman number but it appears in passage 2 when Brinkmann number is increased. The temperature profiles reveal significant variation at higher value of $Br$.

The effect of Darcy and modified Grashof numbers on heat transfer rate for $\lambda = 0.5, N = 0.75, Br = 0.01$ is portrayed in Fig. 4. The hot wall Nusselt number increases moderately and cold wall Nusselt number decreases sharply with an increase in $GR$. The Nusselt number on hot and cold walls shows decreasing and increasing trend respectively with a decrease in the values of porous parameter. The combined influence of modified Grashof and Brinkmann numbers on heat transfer rate is displayed in Fig. 5 by fixing $\lambda = 0.5, N = 0.75, Da = 0.1$. The figure reveals that for lower values $Br$, the Nusselt number on hot wall is invariant with $GR$, but for $Br = 0.05$, a decrease in hot wall Nusselt number is observed with an increase in $GR$. The cold wall Nusselt number decreases with increase in $GR$ but rises with an increase in $Br$. The effect of baffle location on heat transfer rate is depicted in Fig. 6. The Nusselt number on hot wall increases whereas cold wall Nusselt number decreases as the baffle moves towards hot wall, however the effect of modified Grashof number is insignificant.

5. Conclusions
In this paper, we have performed the combined numerical and analytical investigation of mixed convection in vertical double passage porous annuli utilizing the fully developed flow condition. From the results, it is observed that the baffle position, Darcy, modified Grashof and Brinkman numbers have significantly modifies the flow fields by large extent. However, the thermal fields are mainly altered by Brinkman and Darcy numbers as well as baffle position. The effect of baffle location, Brinkman and Darcy numbers on heat transfer rate is significant on both the walls.

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**Figure 2.** Effect of $GR$ and $N$ on velocity and temperature profiles for $\lambda = 0.5$, $Da = 0.01$, $Br = 0.025$
Figure 3. Effect of \( Da \) and \( Br \) on velocity and temperature profiles for \( \lambda = 0.5, N = 0.75, GR = 1000 \)
Figure 4. Effect of $Da$ on $Nu_h$ and $Nu_c$ for different $GR$, with $\lambda = 0.5$, $N = 0.75$, $Br = 0.01$

Figure 5. Effect of $Br$ on $Nu_h$ and $Nu_c$ for different $GR$, with $\lambda = 0.5$, $N = 0.75$, $Da = 0.1$

Figure 6. Effect of $N$ on $Nu_h$ and $Nu_c$ for different $GR$, with $\lambda = 0.5$, $Br = 0.01$, $Da = 0.1$