Duursma’s reduced polynomial

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Abstract

The weight distribution \( \{W^{(w)}_C\}_{w=0}^n \) of a linear code \( C \subset \mathbb{F}_q^n \) is put in an explicit bijective correspondence with Duursma’s reduced polynomial \( D_C(t) \in \mathbb{Q}[t] \) of \( C \). We prove that the Riemann Hypothesis Analogue for a linear code \( C \) requires the formal self-duality of \( C \) and imposes an upper bound on the cardinality \( q \) of the basic field, depending on the dimension and the minimum distance of \( C \). Duursma’s reduced polynomial \( D_F(t) \in \mathbb{Z}[t] \) of the function field \( F = \mathbb{F}_q(X) \) of a curve of genus \( g \) over \( \mathbb{F}_q \) is shown to provide a generating function \( \frac{D_F(t)}{(1-q)(1-qt)} = \sum_{i=0}^{\infty} B_i t^i \) for the numbers \( B_i \) of the effective divisors of degree \( i \geq 0 \) of a virtual function field of a curve of genus \( g - 1 \) over \( \mathbb{F}_q \).

Let \( \overline{\mathbb{F}_q} = \bigcup_{m=1}^{\infty} \mathbb{F}_q^m \) be the algebraic closure of a finite field \( \mathbb{F}_q \) and \( X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q}) \) be a smooth irreducible projective curve of genus \( g \), defined over \( \mathbb{F}_q \). Denote by \( F = \mathbb{F}_q(X) \) the function field of \( X \) over \( \mathbb{F}_q \) and choose \( n \) different \( \mathbb{F}_q \)-rational points \( P_1, \ldots, P_n \in X(\mathbb{F}_q) := X \cap \mathbb{P}^N(\overline{\mathbb{F}_q}) \). Suppose that \( G \) is an effective divisor of \( F \) of degree \( 2g - 2 < \deg G = m < n \), whose support is disjoint from the support of \( D = P_1 + \ldots + P_n \). The space \( L(G) := H^0(X, \mathcal{O}_X(G)) \) of the global holomorphic sections of the line bundle, associated with \( G \) will be referred to as to the Riemann-Roch space of \( G \). We put \( l(G) := \dim_{\mathbb{F}_q} L(G) \) and observe that the evaluation map

\[ E_D : L(G) \to \mathbb{F}_q^n, \]

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\( \mathcal{E}_D(f) = (f(P_1), \ldots, f(P_n)) \) for \( \forall f \in L(G) \)
is an \( \mathbb{F}_q \)-linear embedding. Its image \( \text{im}(\mathcal{E}_D) = \mathcal{E}_D L(G) \) is known as an algebraic geometry code or Goppa code. The minimum distance of \( C \) is \( d(C) \geq n - m \). For an arbitrary \( s \in \mathbb{N} \) let \( N_s(F) := |X(\mathbb{F}_{q^s})| \) be the number of the \( \mathbb{F}_{q^s} \)-rational points of \( X \). Then the formal power series

\[
Z_F(t) := \exp \left( \sum_{s=1}^{\infty} \frac{N_s(F)}{s} t^s \right)
\]
is called the Hasse-Weil zeta function of \( F \). It is well known (cf. Theorem 4.1.11 from [8]) that

\[
Z_F(t) = \frac{L_F(t)}{(1-t)(1-q t)}
\]

for a polynomial \( L_F(t) \in \mathbb{Z}[t] \) of degree \( 2g \). We refer to \( L_F(t) \) as to the Hasse-Weil polynomial of \( F \).

In [2], [3] Duursma introduces the genus of a linear code \( C \subset \mathbb{F}_q^n \) as the deviation \( g := n + 1 - k - d \) of its dimension \( k := \text{dim}_\mathbb{F}_q C \) and minimum distance \( d \) from the equality in Singleton bound. Let \( W_C^{(w)} \) be the number of the codewords \( c \in C \) of weight \( d \leq w \leq n \). Then

\[
W_C(x, y) := x^n + \sum_{w=d(C)}^{n} W_C^{(w)} x^{n-w} y^w
\]
is called the homogeneous weight enumerator of \( C \). Denote by \( \mathcal{M}_{n,s}(x, y) \) the homogeneous weight enumerator of an MDS-code of length \( n \) and minimum distance \( s \). Put \( g^\perp := g + g^\perp \).

**Proposition 1.** (Duursma [3]) For an arbitrary \( \mathbb{F}_q \)-linear \( [n,k,d] \)-code \( C \), which is not contained in a coordinate hyperplane \( H_i := \{ x \in \mathbb{F}_q^n \mid x_i = 0 \} \) of \( \mathbb{F}_q^n \), there exist uniquely determined rational numbers \( a_0, \ldots, a_r \in \mathbb{Q} \), such that the homogeneous weight enumerator

\[
W_C(x, y) = a_0 \mathcal{M}_{n,d}(x, y) + a_1 \mathcal{M}_{n,d+1}(x, y) + \ldots + a_r \mathcal{M}_{n,d+r}(x, y)
\]
of \( C \) is the linear combination of the homogeneous weight enumerators \( \mathcal{M}_{n,d+i}(x, y) \) of MDS-codes of length \( n \) and minimum distance \( d + i \) with coefficients \( a_i \) and

\[
P_C(1) = \sum_{i=0}^{r} a_i = 1.
\]

The \( \zeta \)-polynomial \( P_C(t) := \sum_{i=0}^{r} a_i t^i \) of \( C \) is uniquely determined by

\[
\frac{W_C(x, y) - x^n}{q - 1} = \text{Coeff}_{n-d} \left( \frac{P_C(t)}{(1-t)(1-qt)}[y(1-t) + xt]^n \right),
\]

where \( \text{Coeff}_{n-d}(f(t)) \) stands for the coefficient of \( t^{n-d} \) in a formal power series \( f(t) \in \mathbb{C}[t] \).
Proposition 2. (Duursma’s considerations from [2]) Let \( X/F_q \subset \mathbb{P}^N(F_q) \) be a smooth irreducible curve of genus \( g \), defined over \( F_q \) and \( G_1, \ldots, G_h \) be a complete list of effective representatives of the linear equivalence classes of the divisors of \( F = F_q(X) \) of degree \( 2g - 2 < m < n \). Assume that there exist \( n \) different \( F_q \)-rational points \( P_1, \ldots, P_n \in X(F_q) \), such that \( D = P_1 + \ldots + P_n \in \text{Div}(F) \) has support \( \text{Supp}(D) \cap \text{Supp}(G_i) = \{ P_1, \ldots, P_n \} \cap \text{Supp}(G_i) = \emptyset \) for \( \forall 1 \leq i \leq h \). If
\[
\mathcal{L}(G_i) = H^0(X, \mathcal{O}_X(1)) := \{ f \in F^* \mid (f) + G_i \geq 0 \} \cup \{ 0 \}
\]
are the Riemann-Roch spaces of \( G_i \),
\[
\mathcal{E}_D(\mathcal{L}(G_i)) \rightarrow \mathbb{F}_q^n,
\]
\[
\mathcal{E}_D(f) = (f(P_1), \ldots, f(P_n)) \quad \forall f \in \mathcal{L}(G_i)
\]
are the evaluation maps at \( D \) and \( C_i := \mathcal{E}_D\mathcal{L}(G_i) \) are the corresponding Goppa codes with homogeneous weight enumerators \( W_{C_i}(x, y) \), then
\[
\sum_{i=1}^h W_{C_i}(x, y) - x^n = \text{Coeff}_{L_F(t)} \left( \frac{L_F(t)}{(1-t)(1-qt)}[y(1-t) + xt]^n \right)
\]
for the \( \zeta \)-polynomial \( L_F(t) \) of \( F \).

In particular,
\[
\sum_{i=1}^h t^{g - gn} C_i(t) = L_F(t)
\]
for the \( \zeta \)-polynomials \( C_i(t) \) of \( C_i = \mathcal{E}_D\mathcal{L}(G_i) \) and the Hasse-Weil polynomial \( L_F(T) \) of the function field \( F \).

Proof. Note that (4) is an equality of homogeneous polynomials of \( x \) and \( y \) of degree \( n \), whose monomials are of degree \( s \geq 1 \) with respect to \( y \). Therefore (4) is equivalent to
\[
\frac{h(F)}{q-1} \sum_{i=1}^{h(F)} W_{C_i}^{(s)}(x, y) = \text{Coeff}_{x^{-s}y^s} \left( (\zeta_F(t))[y(1-t) + xt]^n \right) = \text{Coeff}_{L_F(t)} \left( \binom{n}{s} t^{n-s}(1-t)^s \zeta_F(t) \right)
\]
for \( \forall s \in \mathbb{N} \). Note that \( C_i \) are of minimum distance \( d(C_i) \geq n - m \), so that \( W_{C_i}^{(s)} = 0 \) for \( 1 \leq s < n - m \). On the other hand,
\[
(1-t)^s \zeta_F(t) = \frac{(1-t)^{s-1}L_F(t)}{1-qt}
\]
has no pole at \( t = 0 \), so that \( \text{Coeff}_{t^{s-n+m}} ((1-t)^s \zeta_F(t)) = 0 \) for \( s - n + m < 0 \), \( s \in \mathbb{N} \). That is why it suffices to verify (6) for \( s \geq n - m \), \( s \in \mathbb{N} \).
Note that the number of the codewords \( c = (f(P_1), \ldots, f(P_n)) \in C_i, f \in L(G_i) \) of weight \( s \) equals the number of the rational functions \( f \in L(G_i) \setminus \{0\} \), vanishing at \( n-s \) of the points \( P_1, \ldots, P_n \). Bearing in mind that the projective space \( \mathbb{P}(L(G_i)) = \mathbb{P}^{m-g}(F_q) \) parameterizes the effective divisors, linearly equivalent to \( G_i \) and two rational functions \( f, f' \in F \setminus \{0\} \) have one and a same divisor exactly when they are on one and a same \( F_q^* \)-orbit, \( f' \in F_q^* f \), one concludes that \( \frac{W(s)}{q^s} \) is the number of the effective divisors \( E = (f) + G_i \), which are linearly equivalent to \( G_i \) with \( |\text{Supp}(E) \cap \text{Supp}(D)| = n-s \). Thus,

\[
e_{m,s} := \frac{h(F)}{q-1} \sum_{i=1}^{n} W_C^i
\]

equals the number of the effective divisors \( E \in \text{Div}(F)_{\geq 0} \) of degree \( \deg E = m \) with \( |\text{Supp}(E) \cap \text{Supp}(D)| = n-s \). For any \( s \)-tuple of indices \( i = \{i_1, \ldots, i_s\}, 1 \leq i_1 < \ldots < i_s \leq n \) let \( D_i := P_{i_1} + \ldots + P_{i_s} \) and \( e_m(i) \) be the number of the effective divisors \( E \in \text{Div}(F)_{\geq 0} \) of degree \( \deg E = m \) with \( \text{Supp}(E) \cap \text{Supp}(D) = \text{Supp}(D - D_i) \). Then \( e_{m,s} = \sum_i e_m(i) \) and it suffices to show that \( e_m(i) = \text{Coeff}_{m-n+i} ((1-t)^i \zeta_F(t)) \) for any \( i \), in order to justify (6) and (4).

To this end, observe that \( E \in \text{Div}(F)_{\geq 0} \) is an effective divisor of degree \( \deg E = m \) with \( \text{Supp}(E) \cap \text{Supp}(D) = \text{Supp}(D - D_i) \) if and only if the difference \( E_i := E - (D - D_i) \in \text{Div}(F)_{\geq 0} \) is an effective divisor of degree \( \deg E_i = m - n + s \) with support \( \text{Supp}(E_i) \cap \text{Supp}(D_i) = \emptyset \). Now, \( e_m(i) \) equals the number of the effective divisors \( E_i \in \text{Div}(F)_{\geq 0} \) of degree \( \deg E_i = m - n + s \) with \( \text{Supp}(E_i) \cap \text{Supp}(D_i) = \emptyset \). Recall that the Hasse-Weil \( \zeta \)-function

\[
\zeta_F(t) = \prod_{\nu \in \mathcal{P}} \frac{1}{1 - t^{\deg \nu}} = \sum_{i=0}^{\infty} \mathcal{A}_i t^i
\]

is the generating function for the number \( \mathcal{A}_i \) of the effective divisors of \( F \) of degree \( i \). Bearing in mind that \( D_i = \nu_{i_1} + \ldots + \nu_{i_s} \) is a sum of \( s \) different places \( \nu_i \) of degree \( \deg \nu_i = 1 \), one observes that \((1-t)^i \zeta_F(t)\) is the generating function for the number of the effective divisors of \( F \) of degree \( i \), whose support is disjoint with \( \text{Supp}(D_i) \). In other words, \( e_m(i) = \text{Coeff}_{m-n+i} ((1-t)^i \zeta_F(t)) \).

The equality (5) is an immediate consequence of Proposition 1, (4) and the fact that \( C_i = \mathcal{L}_i \mathcal{L}(G_i) \) are of dimension \( \dim_{F_q} C_i = l(G_i) = m-g+1 \), minimum distance \( d_i \geq n-m \) and, therefore, of genus

\[
g_i = n + 1 - \dim_{F_q} C_i - d_i = n - m - d_i + g \leq g.
\]

\( \square \)

Proposition 2 motivates Duursma to refer to \( P_C(t) \) as to the zeta polynomial of an arbitrary linear code \( C \subset \mathbb{F}_q^n \). He establishes that \( P_C(t) \) and \( W_C(x,y) \) are in a bijective correspondence and Mac Williams identities, relating the weight distributions \( \{W_C^{(w)}\}_{w=d}^n \), \( \{W_C^{(w)}\}_{w=d}^n \) of a pair \( (C,C^\perp) \) of mutually dual linear codes are equivalent to the functional equation

\[
P_{C^\perp}(t) = P_C \left( \frac{1}{qt} \right) q^{g} t^{g/2} \tag{7}
\]
for the corresponding zeta polynomials $P_C(t), P_{C^\perp}(t)$.

In [2] and [4] Duursma observes the existence of a polynomial $D_C(t) = \sum_{i=0}^{r-2} c_i t^i \in \mathbb{Q}[t]$, defined by the identity

$$P_C(t) = (1-t)(1-qt)D_C(t) + t^g$$

of polynomials in $t$, but does not make use of $D_C(t)$ for the study of the homogeneous weight enumerator $W_C(x, y)$ of $C$. He mentions in [4] that the analogue $D_F(t)$ of $D_C(t)$ for a function field $F$ of one variable accounts for the contribution of the special divisors of $F$ to the zeta function $Z_F(t)$. From now on, we refer to $D_C(t)$ as to Duursma’s reduced polynomial of $C$.

The present note provides an explicit bijective correspondence between the weight distribution $\{W_C^{(w)}\}_{w=0}^n$ of an arbitrary linear code $C \subset \mathbb{F}^n_q$ and the coefficients $\{c_i\}_{i=0}^{r-2}$ of its Duursma’s reduced polynomial $D_C(t) = \sum_{i=0}^{r-2} c_i t^i$ (cf. Proposition 3).

The classical Hasse-Weil Theorem establishes that all the roots of the Hasse-Weil polynomial $L_F(t) \in \mathbb{Z}[t]$ of the function field $F = \mathbb{F}_q(X)$ of a curve $X$ of genus $g$ over $\mathbb{F}_q$ are on the circle $S \left( \frac{1}{\sqrt{q}} \right)$, $\{z \in C \mid |z| = \frac{1}{\sqrt{q}} \}$ (cf. Theorem 4.2.3 form [8]). Duursma says that a linear code $C \subset \mathbb{F}^n_q$ satisfies the Riemann Hypothesis Analogue if all the roots of its zeta polynomial $P_C(t) = \sum_{i=0}^r a_i t^i \in \mathbb{Q}[t]$ are on the circle $S \left( \frac{1}{\sqrt{q}} \right)$. Let $C$ be an $\mathbb{F}_q$-linear code of dimension $k$ and minimum distance $d$, which satisfies the Riemann Hypothesis Analogue. Proposition 4 shows that $C$ is formally self-dual, while Corollary 5 provides an explicit upper bound on the cardinality $q$ of the basic field, depending on $k$ and $d$. Let us recall that $C$ is formally self-dual if it has the same weight distribution $W_C^{(w)} = W_{C^\perp}^{(w)}$, $\forall 0 \leq w \leq n$ as its dual code $C^\perp \subset \mathbb{F}^n_q$. In the light of Duursma’s results and our Proposition 3, the formal self-duality of $C$ turns to be equivalent to the functional equation $P_C(t) = P_C \left( \frac{1}{qt} \right) q^{2g} t^{2g}$ for $P_C(t)$ and to the functional equation $D_C(t) = D_C \left( \frac{1}{qt} \right) q^{2g-1} t^{2g-2}$ for $D_C(t)$. Proposition 6 from the present note expresses explicitly the homogeneous weight enumerator $W_C(x, y)$ of a formally self-dual code $C \subset \mathbb{F}^n_q$ by the lowest half of the coefficients of $D_C(t)$ or by the numbers $W_C^{(d)}, \ldots, W_C^{(k)}$ of the codewords $c \in C$, whose weights are between the minimum distance $d$ of $C$ and the dimension $k$.

In [1] Dodunekov and Landgev introduce the near-MDS code $C \subset \mathbb{F}^n_q$ as the ones with quadratic zeta polynomial $P_C(t)$. Kim and Hyun’s article [7] provides a necessary and sufficient condition for a near-MDS code to satisfy the Riemann Hypothesis Analogue. Note that the zeta polynomials $P_C(t)$ and Duursma’s reduced polynomials $D_C(t)$ of formally self-dual codes $C \subset \mathbb{F}^n_q$ are of even degree. Our Proposition 7 is a necessary and sufficient condition for a formally self-dual code $C \subset \mathbb{F}^n_q$ with zeta polynomial $P_C(T)$ of $\deg P_C(t) = 4$ to be subject to the Riemann Hypothesis Analogue. Let $S_\nu, \nu \in \mathbb{N}$ be the uniquely determined logarithmic coefficients of $P_C(t)$, defined by the equality of formal power series $\log P_C(t) = \sum_{\nu=1}^\infty S_\nu t^\nu \in \mathbb{C}[t]$. Adapting Bombieri’s proof of the
Hasse-Weil Theorem, [5] shows that a linear code $C$ satisfies the Riemann Hypothesis Analogue exactly when the sequence $\{ S_{\nu} q^{-\nu} \}_{\nu=1}^{\infty} \subset \mathbb{C}$ is absolutely bounded.

The last, third section is devoted to Duursma’s reduced polynomial $D_F(t)$ of the function field $F = \mathbb{F}_q(X)$ of a curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\mathbb{F}_q)$ of genus $g$ over $\mathbb{F}_q$. It establishes that $D_F(t) \in \mathbb{Z}[t]$ is determined uniquely by its lowest $g$ coefficients, which equal the numbers $A_i$ of the effective divisors of $F$ of degree $0 \leq i \leq g - 1$. Our Proposition 9 shows that the zeta function

$$
\frac{D_F(t)}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} B_i t^i,
$$

associated with $D_F(t)$ has the properties of a generating function for the numbers $B_i$ of the effective divisors of degree $i \geq 0$ of a virtual function field of genus $g - 1$ over $\mathbb{F}_q$. There arises the following Open Problem:

To characterize the function fields $F = \mathbb{F}_q(X)$ of curves $X/\mathbb{F}_q \subset \mathbb{P}^N(\mathbb{F}_q)$ of genus $g$ over $\mathbb{F}_q$, for which there are curves $Y/\mathbb{F}_q \subset \mathbb{P}^M(\mathbb{F}_q)$ of genus $g - 1$, defined over $\mathbb{F}_q$ with Hasse-Weil zeta function

$$
Z_{\mathbb{F}_q(Y)}(t) = \frac{D_F(t)}{(1-t)(1-qt)}.
$$

1 The homogeneous weight enumerator of an arbitrary code

Proposition 3. Let $C \subset \mathbb{F}_q^n$ be a linear code of dimension $k = \text{dim}_{\mathbb{F}_q} C$, minimum distance $d$ and genus $g = n + 1 - k - d \geq 1$, whose dual $C^\perp \subset \mathbb{F}_q^n$ is of minimum distance $d^\perp$ and genus $g^\perp = k + 1 - d^\perp \geq 1$. If

$$
D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t]
$$

is Duursma’s reduced polynomial of $C$ and $M_{n,n+1-k}(x,y)$ is the homogeneous weight enumerator of an MDS-code of length $n$, dimension $k$ and minimum distance $n + 1 - k$, then the homogeneous weight enumerator of $C$ is

$$
W_C(x,y) = M_{n,n+1-k}(x,y) + (q - 1) \sum_{i=0}^{g+g^\perp-2} c_i \binom{n}{d+i} (x-y)^{n-d-i} y^{d+i}. \quad (8)
$$

More precisely, Duursma’s reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$ determines uniquely the weight distribution of $C$, according to

$$
W_C^{(w)} = (q - 1) \binom{n}{w} \sum_{i=0}^{w-d} (-1)^{w-d-i} \binom{w}{d+i} c_i \text{ for } d \leq w \leq d + g - 1, \quad (9)
$$

6
\begin{align*}
W_C^{(w)} &= (q - 1) \binom{n}{w} \min(w - d, n - d + 1) \sum_{i=0}^{\min(w - d, n - d + 1)} (-1)^{w - d - i} \binom{w}{d + i} c_i \\
&\quad + \binom{n}{w} \sum_{j=0}^{w - n - 1 + k} (-1)^j \binom{w}{j} (q^{w - n + k - j} - 1) \quad \text{for} \quad d + g \leq w \leq n. \tag{10}
\end{align*}

Conversely, for \( \forall 0 \leq i \leq g + g^\perp - 2 \) the numbers \( W_C^{(d)}, \ldots, W_C^{(d+g)} \) determine uniquely the coefficient \( c_i \) of Duursma’s reduced polynomial \( D_C(t) = \sum_{i=0}^{g + g^\perp - 2} c_i t^i \) by

\begin{align*}
c_i &= (q - 1)^{-1} \binom{n}{d + i} \sum_{w=d}^{d+i} \binom{n - w}{n - d - i} W_C^{(w)} \tag{11}
\end{align*}

for \( 0 \leq i \leq g - 1. \)

\begin{align*}
c_i &= (q - 1)^{-1} \binom{n}{d + i} \sum_{w=d}^{d+i} \binom{n - w}{n - d - i} W_C^{(w)} \\
&\quad + \sum_{w=d+g}^{d+i} \left[ W_C^{(w)} - \binom{n - w}{n - d - i} \sum_{j=0}^{w - n - 1 + k} (-1)^j \binom{w}{j} (q^{w - n + k - j} - 1) \right] \tag{12}
\end{align*}

for \( g \leq i \leq g + g^\perp - 2. \)

In particular,

\begin{align*}
(q - 1) \binom{n}{d + i} c_i \in \mathbb{Z}
\end{align*}

are integers for all \( 0 \leq i \leq g + g^\perp - 2. \)

The aforementioned formulae imply that \( W_C^{(d)}, \ldots, W_C^{(d+g+g^\perp - 2)} \) determine uniquely the homogeneous weight enumerator \( W_C(x,y) \) of \( C \) by the formula

\begin{align*}
W_C(x,y) &= \sum_{w=d}^{d+g+g^\perp - 2} W_C^{(w)} \lambda_w(x,y) + \Lambda(x,y), \tag{13}
\end{align*}

with explicit polynomials

\begin{align*}
\lambda_w(x,y) := \sum_{s=w}^{d+g+g^\perp - 2} \binom{n - w}{n - s} (x - y)^{n - s} y^s \quad \text{for} \quad d \leq w \leq d + g + g^\perp - 2 \tag{14}
\end{align*}

and

\begin{align*}
\Lambda(x,y) := \mathcal{M}_{n,n+1-k}(x,y) - \sum_{w=d+g}^{d+g+g^\perp - 2} \mathcal{M}_{n,n+1-k}^{(w)} \lambda_w(x,y). \tag{15}
\end{align*}
Proof. In the case of \( g = 0 \), note that \( C \) is an MDS-code and \( W_C(x, y) = M_{n,n+1-k}(x, y) \). Form now on, we assume that \( g > 0 \) and put \( r := g + g^+ \). Making use of \( d + g = n + 1 - k \), let us express

\[
W_C(x, y) = M_{n,d+g}(x, y) + \sum_{i=0}^{r} b_i M_{n,d+i}(x, y)
\]

by some rational numbers \( b_i \in \mathbb{Q} \). Then the seta polynomial \( P_C(t) = t^g + \sum_{i=0}^{r} b_i t^i \) and Duursma’s reduced polynomial \( D_C(t) = \sum_{i=0}^{r-2} c_i t^i \) of \( C \) are related by the equality

\[
P_C(t) - t^g = (1 - t)(1 - qt) D_C(t).
\] (16)

Let us introduce \( c_{-2} = c_{-1} = c_{r-1} = c_r = 0 \) and compare the coefficients of \( t^i \) from the left and right hand side of (16), in order to obtain

\[
b_i = c_i - (q + 1)c_{i-1} + qc_{i-2} \quad \text{for} \quad \forall 0 \leq i \leq r.
\]

Therefore

\[
W_C(x, y) = M_{n,d+g}(x, y) + \sum_{i=0}^{r} c_i M_{n,d+i}(x, y)
\]

\[
- (q + 1) \sum_{i=0}^{r-1} c_{i-1} M_{n,d+i}(x, y) + q \sum_{i=0}^{r-2} c_{i-2} M_{n,d+i}(x, y).
\]

Setting \( j = i - 1 \), respectively, \( j = i - 2 \) in the last two sums, one obtains

\[
W_C(x, y) = M_{n,d+g}(x, y) + \sum_{i=0}^{r} c_i M_{n,d+i}(x, y)
\]

\[
- (q + 1) \sum_{j=-1}^{r-1} c_j M_{n,d+j+1}(x, y) + q \sum_{j=-2}^{r-2} c_j M_{n,d+j+2}(x, y),
\]

whereas

\[
W_C(x, y) = M_{n,d+g}(x, y)
\]

\[
+ \sum_{j=0}^{r-2} c_j [M_{n,d+j}(x, y) - (q + 1)M_{n,d+j+1}(x, y) + qM_{n,d+j+2}(x, y)].
\] (17)

Let us put

\[
W_{n,d+j}(x, y) := M_{n,d+j}(x, y) - (q + 1)M_{n,d+j+1}(x, y) + qM_{n,d+j+2}(x, y)
\]

and recall that the homogeneous weight enumerator of an MDS-code of length \( n \) and minimum distance \( d + j \) is

\[
M_{n,d+j}(x, y) = x^n + \sum_{w=d+j}^{n} M^{(w)}_{n,d+j} x^{n-w} y^w
\]
with
\[ M_{n,d+j}^{(w)} = \binom{n}{w} \sum_{i=0}^{w-d-j} (-1)^i \binom{w}{i} (q^{w+1-d-j-i} - 1). \] (18)

Therefore
\[ W_{n,d+j}(x,y) = M_{n,d+j}^{(d+j)} x^{n-d-j} y^{d+j} + M_{n,d+j}^{(d+j+1)} (q+1) M_{n,d+j+1}^{(d+j+1)} x^{n-d-j-1} y^{d+j+1} \]
\[ + \sum_{w=d+j+2}^{n} [M_{n,d+j}^{(w)} - (q+1) M_{n,d+j+1}^{(w)} + q M_{n,d+j+2}^{(w)}] x^{n-w} y^w. \]

Making use of the weight distribution (18) of an MDS-code and introducing
\[ W_{n,d+j}^{(w)} := M_{n,d+j}^{(w)} - (q+1) M_{n,d+j+1}^{(w)} + q M_{n,d+j+2}^{(w)} \quad \text{for} \quad d+j+2 \leq w \leq n, \]

one expresses
\[ W_{n,d+j}(x,y) = \binom{n}{d+j} (q-1) x^{n-d-j} y^{d+j} \]
\[ - \binom{n}{d+j+1} (q-1)(d+j+1) x^{n-d-j-1} y^{d+j+1} + \sum_{w=d+j+2}^{n} W_{n,d+j}^{(w)} x^{n-w} y^w. \]

For any \( d+j+2 \leq w \leq n \) one has
\[ W_{n,d+j}^{(w)} = \binom{n}{w} \binom{w}{d+j} (q-1)(-1)^{w-d-j}. \]

Making use of
\[ \binom{n}{w} \binom{w}{d+j} = \binom{n-d-j}{w-d-j} \binom{n}{d+j}, \]

one obtains
\[ W_{n,d+j}(x,y) = \binom{n}{d+j} (q-1) x^{n-d-j} y^{d+j} - \binom{n}{d+j+1} (q-1)(d+j+1) x^{n-d-j-1} y^{d+j+1} + \]
\[ + \sum_{w=d+j+2}^{n} \binom{n}{d+j} \binom{n-d-j}{w-d-j} (q-1)(-1)^{w-d-j} x^{n-w} y^w. \]

Bearing in mind that
\[ (d+j+1) \binom{n}{d+j+1} = (n-d-j) \binom{n}{d+j}, \]

one derives that
\[ W_{n,d+j}(x,y) = \binom{n}{d+j} (q-1) \left[ x^{n-d-j} y^{d+j} - (n-d-j) x^{n-d-j-1} y^{d+j+1} + \right. \]
\[ + \sum_{w=d+j+2}^{n} (-1)^{w-d-j} \binom{n-d-j}{w-d-j} x^{n-w} y^w \right]. \]
Introducing $s := w - d - j$, one expresses
\[
\sum_{w=d+j+2}^{n} (-1)^{w-d-j} \binom{n-d-j}{w-d-j} x^{n-w} y^w = \sum_{s=2}^{n-d-j} (-1)^s \binom{n-d-j}{s} x^{n-d-j-s} y^{d+j+s}
\]
and concludes that
\[
\mathcal{V}_{n,d+j}(x, y) = \binom{n}{d+j} (q-1)(x-y)^{n-d-j} y^{d+j}.
\]

The equality $\mathcal{W}_{n,n-k}(x, y) = \binom{n}{k} (q-1)(x-y)^k y^{n-k}$ is exactly the claim (c) of Lemma 1 from Kim and Nyun’s work [7]. Plugging in (19) in (17) and bearing in mind that $d + g = n + 1 - k$, one obtains (8).

In order to prove (9) and (10), let us put
\[
\mathcal{V}_C(x, y) := \mathcal{W}_C(x, y) - \mathcal{M}_{n,n+1-k}(x, y)
\]
and note that $\mathcal{V}_C(x, y) = \sum_{w=d}^{n} \mathcal{V}_C^{(w)} x^{n-w} y^w$ with $\mathcal{V}_C^{(w)} = \mathcal{W}_C^{(w)} - \binom{n-w}{d} \sum_{i=0}^{w-n-1+k} (-1)^i \binom{w}{i} (q^{w-n+k-i} - 1)$ for $d + g = n + 1 - k \leq w \leq n$. Making use of (8), one expresses
\[
\mathcal{V}_C(x, y) = (q-1) \sum_{i=0}^{g+g^+ - 2} c_i \binom{n}{d+i} \sum_{s=0}^{n-d-i} \binom{n-d-i}{s} (-1)^{n-d-i-s} x^s y^{n-s}
\]
\[
= (q-1) \sum_{s=0}^{n-d} \left[ \min(n-d-s,g^+ - 2) \sum_{i=0}^{n-d-s} c_i \binom{n-d-i}{d+i} \binom{n-d-i}{n-w} (-1)^{n-d-i-s} x^s y^{n-s} \right]
\]
after changing the summation order. Setting $w := n - s$, one obtains
\[
\mathcal{V}_C(x, y) = (q-1) \sum_{w=d}^{n} \left[ \min(n-w,n-d-d^+) \sum_{i=0}^{n-w,n-d-d^+} c_i \binom{n-d-i}{d+i} \binom{n-d-i}{n-w} (-1)^{n-d-i} x^w y^w \right]
\]
Then
\[
\binom{n}{d+i} \binom{n-d-i}{n-w} = \binom{n}{w} \binom{w}{d+i},
\]
allows to concludes that
\[
\mathcal{V}_C^{(w)} = (q-1) \binom{n}{w} \sum_{i=0}^{\min(n-w,n-d-d^+)} c_i \binom{w}{d+i} (-1)^{w-d-i} \quad \forall d \leq w \leq n,
\]
which proves (9), (10).
Towards (11), (12), let us introduce \( z := x - y \) and express (8) in the form
\[
\mathcal{V}_C(y + z, y) = (q - 1) \sum_{i=0}^{g+g^+ - 2} c_i \binom{n}{d+i} z^{n-d+i} y^{d+i}. \tag{20}
\]

On the other hand,
\[
\mathcal{V}_C(y + z, y) = \sum_{w=d}^{n} \mathcal{V}_C^{(w)}(y + z)^{n-w} y^w
\]
\[
= \sum_{w=d}^{n} \sum_{s=0}^{n-w} \binom{n-w}{s} \mathcal{V}_C^{(w)} y^{n-s} z^s = \sum_{s=0}^{n-d} \sum_{w=d}^{n-s} \binom{n-w}{s} \mathcal{V}_C^{(w)} y^{n-s} z^s,
\]

after changing the summation order. Comparing the coefficients of \( y^{d+i} z^{n-d-i} \) in the left and right hand side of (20), one obtains
\[
\sum_{w=d}^{d+i} \binom{n-w}{n-d-i} \mathcal{V}_C^{(w)} = (q - 1) c_i \binom{n}{d+i},
\]
whereas
\[
c_i = (q - 1)^{-1} \binom{n}{d+i} \sum_{w=d}^{d+i} \binom{n-w}{n-d-i} \mathcal{V}_C^{(w)}.
\]

Combining with (18), one justifies (11) and (12). These formulae imply also the fact that \( (q - 1) \binom{n}{d+i} c_i \in \mathbb{Z} \) are integers for all \( 0 \leq i \leq g + g^+ - 2 \).

The substitution by (11), (12), (18) in (8) yields
\[
\mathcal{W}_C(x, y) = \mathcal{M}_{n,n+1-k}(x, y) + \sum_{i=g}^{g+g^+ - 2} \sum_{w=d}^{d+i} \binom{n-w}{n-d-i} \mathcal{W}_C^{(w)}(x - y)^{n-d-i} y^{d+i}
\]
\[
- \sum_{i=g}^{g+g^+ - 2} \sum_{w=d+g}^{d+i} \binom{n-w}{n-d-i} \mathcal{M}_{n,n+1-k}(x - y)^{n-d-i} y^{d+i}.
\]

One exchanges the summation order in the double sums towards
\[
\mathcal{W}_C(x, y) = \mathcal{M}_{n,n+1-k}(x, y) + \sum_{w=d}^{d+g+g^+ - 2} \mathcal{W}_C^{(w)} \sum_{i=w-d}^{g+g^+ - 2} \binom{n-w}{n-d-i} (x - y)^{n-d-i} y^{d+i}
\]
\[
- \sum_{w=d+g}^{d+g+g^+ - 2} \mathcal{M}_{n,n+1-k} \sum_{i=w-d}^{g+g^+ - 2} \binom{n-w}{n-d-i} (x - y)^{n-d-i} y^{d+i}.
\]

Introducing \( s := d + i \), one obtains (13) with (14) and (15).
Comparing the coefficients of $x^{n-d}y^d$ in the left and right hand sides of (8), one obtains $W^{(d)}_C = (q-1)\binom{n}{d}c_0$ for a linear code $C$ of genus $g \geq 1$. We claim that $c_0 < 1$. To this end, note that for any $d$-tuple $\{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$, supporting a word $c \in C$ of weight $d$ there are exactly $q-1$ words $c' \in C$ with $\text{Supp}(c') = \text{Supp}(c) = \{i_1, \ldots, i_d\}$. That is due to the fact that the columns $H_{i_1}, \ldots, H_{i_d}$ of an arbitrary parity check matrix $H$ of $C$ are of rank $d-1$ and there are no words of weight $\leq d-1$ in the right null space of the matrix $(H_{i_1} \ldots H_{i_d})$. It is clear that $\nu \leq \binom{n}{d}$, so that

$$c_0 = \frac{\nu}{\binom{n}{d}} \leq 1.$$ 

If we assume that $c_0 = 1$ then any $d$-tuple of columns of $H$ is linearly dependent. Bearing in mind that $\text{rk}H = n - k$, one concludes that $d > n - k$. Combining with Singleton Bound $d \leq n - k + 1$, one obtains $d = n - k + 1$. That contradicts the assumption that $C$ is not an MDS-code and proves that $c_0 < 1$ for any $\mathbb{F}_q$-linear code $C \subset \mathbb{F}_q^n$ of genus $g \geq 1$. Note that $c_0$ can be interpreted as the probability for a $d$-tuple to support a word of weight $d$ from $C$.

### 2 The Riemann Hypothesis Analogue and the formal self-duality of a linear code

Recall that a linear code $C \subset \mathbb{F}_q^n$ with dual code $C^\perp \subset \mathbb{F}_q^n$ is formally self-dual if $C$ and $C^\perp$ have one and a same number $W^{(w)}_C = W^{(w)}_{C^\perp}$ of codewords of weight $0 \leq w \leq n$. Let us mention some trivial consequences of the formal self-duality of $C$. First of all, $C$ and $C^\perp$ have one and a same minimum distance $d = d(C) = d(C^\perp) = d^\perp$. Further, $C$ and $C^\perp$ have one and a same cardinality

$$q^{\dim C} = \sum_{w=0}^{n} W^{(w)}_C = \sum_{w=0}^{n} W^{(w)}_{C^\perp} = q^{\dim C^\perp},$$

so that $k = \dim C = \dim C^\perp = k^\perp$ and the length $n = k + k^\perp = 2k$ is an even integer. The genera $g = k + 1 - d = g^\perp$ also coincide. Let $P_C(t) = \sum_{i=0}^{2g} a_i t^i$ and $P_{C^\perp} = \sum_{i=0}^{2g} a_i^\perp t^i$ be the zeta polynomials of $C$, respectively, of $C^\perp$. The consecutive comparison of the coefficients of $x^{n-d}y^d, x^{n-d-1}y^{d+1}, \ldots, x^{n-2g}y^{d+2g}$ from the homogeneous polynomial

$$a_0 M_{2k,d}(x,y) + a_1 M_{2k,d+1}(x,y) + \ldots + a_{2g} M_{2k,d+2g}(x,y) = W_{C}(x,y)$$

$$= W_{C^\perp}(x,y) = a_0^\perp M_{2k,d}(x,y) + a_1^\perp M_{2k,d+1}(x,y) + \ldots + a_{2g}^\perp M_{2k,d+2g}(x,y)$$

in $x,y$ yields $a_i = a_i^\perp$ for $0 \leq i \leq 2g$. It is clear that $a_i = a_i^\perp$ for $0 \leq i \leq 2g$ suffices for $W_{C}(x,y) = W_{C^\perp}(x,y)$, so that the formal self-duality of $C$ is tantamount to the coincidence $P_C(t) = P_{C^\perp}(t)$ of the zeta polynomials of $C$ and $C^\perp$. Duursma has shown that MacWilliams identities for $W^{(w)}_C$ and $W^{(w)}_{C^\perp}$ are equivalent to the functional equation (7) for the zeta polynomials $P_C(t)$, $P_{C^\perp}(t)$ of $C, C^\perp \subset \mathbb{F}_q^n$ with genera $g, g^\perp$. 

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Thus, an $F_q$-linear code $C \subseteq F_q^n$ is formally self-dual if and only if its zeta polynomial $P_C(t)$ satisfies the functional equation

$$P_C(t) = P_C \left( \frac{1}{q^r} \right) q^g t^{2g}$$

of the Hasse-Weil polynomial of the function field of a curve of genus $g$ over $F_q$.

**Proposition 4.** If a linear code $C \subseteq F_q^n$ satisfies the Riemann Hypothesis Analogue then $C$ is formally self-dual, i.e., the zeta polynomial $P_C(t)$ of $C$ is subject to the functional equation (21) of the Hasse-Weil polynomial of the function field of a curve of genus $g$ over $F_q$.

**Proof.** Let us assume that $P_C(t)$ of degree $r := g + g^1$ satisfies the Riemann Hypothesis Analogue, i.e.,

$$P_C(t) = a_r \prod_{j=1}^{r} (t - \alpha_j) \in \mathbb{Q}[t]$$

for some $\alpha_j \in \mathbb{C}$ with $|\alpha_j| = \frac{1}{\sqrt{q}}$ for all $1 \leq j \leq r$. If $\alpha_j$ is a real root of $P_C(t)$ then $\alpha_j = \frac{\pm}{\sqrt{q}}$ with $\varepsilon = \pm 1$. We claim that in the case of an even degree $r = 2m$, the zeta polynomial $P_C(t)$ is of the form

$$P_C(t) = a_{2m} \prod_{i=1}^{m} (t - \alpha_i)(t - \overline{\alpha}_i)$$

(22)

or of the form

$$P_C(t) = a_{2m} \left( t^2 - \frac{1}{q} \right) \prod_{i=1}^{m-1} (t - \alpha_i)(t - \overline{\alpha}_i),$$

(23)

while for an odd degree $r = 2m + 1$ one has

$$P_C(t) = a_{2m+1} \left( t - \frac{\varepsilon}{\sqrt{q}} \right) \prod_{i=1}^{m} (t - \alpha_i)(t - \overline{\alpha}_i)$$

(24)

for some $\varepsilon \in \{\pm 1\}$. Indeed, if $\alpha_i \in \mathbb{C} \setminus \mathbb{R}$ is a complex, non-real root of $P_C(t) \in \mathbb{Q}[t] \subset \mathbb{R}[t]$ then $\overline{\alpha}_i \neq \alpha_i$ is also a root of $P_C(t)$ and $P_C(t)$ is divisible by $(t - \alpha_i)(t - \overline{\alpha}_i)$. If $P_C(t) = 0$ has three real roots $\alpha_1, \alpha_2, \alpha_3 \in \left\{ \frac{1}{\sqrt{q}}, -\frac{1}{\sqrt{q}} \right\}$, then at least two of them coincide. For $\alpha_1 = \alpha_2 = \frac{\varepsilon}{\sqrt{q}}$ one has $(t - \alpha_1)(t - \alpha_2) = (t - \alpha_1)(t - \overline{\alpha}_1)$. Thus, $P_C(t)$ has at most two real roots, which are not complex conjugate (or, equivalently, equal) to each other and $P_C(t)$ is of the form (22), (23) or (24).

If $P_C(t)$ is of the form (22), then $P_C(t) = a_{2m} \prod_{i=1}^{m} \left( t^2 - 2\text{Re}(\alpha_i) + \frac{1}{q} \right)$ and (7) reads as

$$P_{C+}(t) = a_{2m} \left[ \prod_{i=1}^{m} \left( \frac{1}{q} - 2\text{Re}(\alpha_i)t + t^2 \right) \right] q^{g-m} = P_C(t)q^{g-m},$$

(25)
after multiplying each of the factors \( \frac{1}{q^2} - \frac{2\text{Re}(\alpha)}{qt} + \frac{1}{q} \) by \( qt^2 \). If \( D_C(t) \) is Duursma’s reduced polynomial of \( C \) and \( D_{C\perp}(t) \) is Duursma’s reduced polynomial of \( C\perp \), then

\[
(1 - t)(1 - qt)D_{C\perp}(t) + t^{g\perp} = P_{C\perp}(t) = P_C(t)q^{g - m} = (1 - t)(1 - qt)q^{g - m}D_C(t) + q^{g - m}t^g
\]

implies that

\[
(1 - t)(1 - qt)D_{C\perp}(t) - q^{g - m}D_C(t) = q^{g - m}t^g - t^{g\perp}.
\]

Plugging in \( t = 1 \), one concludes that \( q^{g - m} = 1 \), whereas \( g = m \). As a result, \( g + g\perp = 2m = 2g \) specifies that \( g = g\perp \) and (25) yields \( P_C(t) = P_{C\perp}(t) \), which is equivalent to the formal self-duality of \( C \).

If \( P_C(t) \) is of the form (23) then (7) provides

\[
P_{C\perp}(t) = a_{2m} \left( \frac{1}{q} - t^2 \right) \left[ \prod_{i=1}^{m-1} \left( \frac{1}{q} - \frac{2\text{Re}(\alpha)}{qt} + t^2 \right) \right] q^{g - m} = -P_C(t)q^{g - m}.
\] (26)

Expressing by Duursma’s reduced polynomials \( D_C(t), D_{C\perp}(t) \), one obtains

\[
(1 - t)(1 - qt)D_{C\perp}(t) + t^{g\perp} = P_{C\perp}(t) = -P_C(t)q^{g - m} = -(1 - t)(1 - qt)q^{g - m}D_C(t) - q^{g - m}t^g,
\]

whereas

\[
(1 - t)(1 - qt)D_{C\perp}(t) + q^{g - m}D_C(t) = -t^{g\perp} - q^{g - m}t^g.
\]

The substitution \( t = 1 \) in the last equality of polynomials yields \(-1 - q^{g - m} = 0 \), which is an absurd, justifying that a zeta polynomial \( P_C(t) \), subject to the Riemann Hypothesis Analogue cannot be of the form (23).

If \( P_C(t) \) is of odd degree \( 2m + 1 \), then (24) and (7) yield

\[
P_{C\perp}(t) = -\varepsilon \sqrt{q} a_{2m+1} \left( t - \frac{\varepsilon}{\sqrt{q}} \right) \left[ \prod_{i=1}^{m} \left( \frac{1}{q} - \frac{2\text{Re}(\alpha)}{qt} + t^2 \right) \right] q^{g - m - 1} = -\varepsilon \sqrt{q} P_C(t)q^{g - m - 1}
\]

after multiplying \( \frac{1}{q} - \frac{\varepsilon}{\sqrt{q}} \) by \( -\frac{\varepsilon}{\sqrt{q}}qt \) and each \( \frac{1}{q^2t^2} - \frac{2\text{Re}(\alpha)}{qt} + \frac{1}{q} \) by \( qt^2 \). Expressing by Duursma’s reduced polynomials

\[
(1 - t)(1 - qt)D_{C\perp}(t) + t^{g\perp} = P_{C\perp}(t) = -\varepsilon q^{g - m - \frac{1}{2}} P_C(t)
\]

\[
= -\varepsilon q^{g - m - \frac{1}{2}} (1 - t)(1 - qt)D_C(t) - \varepsilon q^{g - m - \frac{1}{2}} t^g,
\]

one obtains

\[
(1 - t)(1 - qt) \left[ D_{C\perp}(t) + \varepsilon q^{g - m - \frac{1}{2}} D_C(t) \right] = -t^{g\perp} - \varepsilon q^{g - m - \frac{1}{2}} t^g.
\]

The substitution \( t = 1 \) implies \(-1 - \varepsilon q^{g - m - \frac{1}{2}} = 0 \), which is an absurd, as far as \( q^x = 1 \) if and only if \( x = 0 \), while \( g - m - \frac{1}{2} \) cannot vanish for integers \( g, m \). Thus, none zeta polynomial of odd degree satisfies the Riemann Hypothesis Analogue.
Corollary 5. If an \( \mathbb{F}_q \)-linear code \( C \) of \( \dim_\mathbb{F}_q C = k \) and minimum distance \( d \) satisfies the Riemann Hypothesis Analogue then the cardinality \( q \) of the basic field satisfies the upper bound

\[
q \leq \left( 2^g \sqrt{\frac{2k}{d}} + 1 \right)^2.
\]

Proof. By Proposition 4, if \( C \) satisfies the Riemann Hypothesis Analogue then

\[
P_C(t) = a_{2g} \prod_{j=1}^{q} \left( t - \frac{e^{i\varphi_j}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi_j}}{\sqrt{q}} \right)
\]

for some \( \varphi_j \in [0, 2\pi) \). The formal self-duality of \( C \) is equivalent to the functional equation \( P_C(t) = P_C \left( \frac{1}{q} \right) q^g t^{2g} \) of the Hasse-Weil polynomial of a function field of genus \( g \) over \( \mathbb{F}_q \) and implies that \( a_{2g} = q^g a_0 \). Comparing the coefficients of \( t^{2k-d}y^d \) in the expression

\[
W_C(x, y) = a_0 M_{2k, d}(x, y) + a_1 M_{2k, d+1}(x, y) + \ldots + a_{2g} M_{2k, d+2g}(x, y)
\]

of the homogeneous weight enumerator \( W_C(x, y) \) of \( C \) by the homogeneous weight enumerators \( M_{2k, d+i}(x, y) \) of MDS-codes of length \( 2k \) and minimum distance \( d + i \), one concludes that \( W_C^{(d)} = a_0 M_{2k, d} = a_0(q-1)(\frac{k}{d}) \). Note that any word \( c \in C \) of weight \( d \) is a solution of a homogeneous linear system of rank \( d - 1 \) in \( d \) variables, as far as any \( d - 1 \) columns of a parity check matrix of \( C \) are linearly independent. Thus, there are exactly \( q - 1 \) words of weight \( d \) from \( C \) with the same support as \( c \). If \( \nu \) is the number of the \( d \)-tuples, supporting a word \( c \in C \) of weight \( d \) then \( W_C^{(d)} = (q-1)\nu \) and

\[
a_0 = \frac{\nu}{(\frac{k}{d})^2}
\]

is the probability for a \( d \)-tuple to support a word of weight \( d \) from \( C \). Altogether, one obtains that

\[
P_C(t) = \frac{q^g \nu}{(\frac{k}{d})^2} \prod_{j=1}^{q} \left( t - \frac{e^{i\varphi_j}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi_j}}{\sqrt{q}} \right) =
\]

\[
\frac{\nu}{(\frac{k}{d})^2} \prod_{j=1}^{q} (\sqrt{q}t - e^{i\varphi_j}) (\sqrt{q}t - e^{-i\varphi_j}) = \frac{\nu}{(\frac{k}{d})^2} \prod_{j=1}^{q} (\sqrt{q}t - 2 \sqrt{q} \cos \varphi_j t + 1).
\]

In particular,

\[
1 = P_C(1) = \frac{\nu}{(\frac{k}{d})^2} \prod_{j=1}^{q} (q - 2 \sqrt{q} \cos \varphi_j + 1).
\]

Bearing in mind that \( \cos \varphi_j \in [-1, 1] \), one estimates

\[
q - 2 \sqrt{q} \cos \varphi_j + 1 \geq (\sqrt{q} - 1)^2
\]

and concludes that

\[
1 = \frac{\nu}{(\frac{k}{d})^2} \prod_{j=1}^{q} (q - 2 \sqrt{q} \cos \varphi_j + 1) \geq \frac{\nu}{(\frac{k}{d})^2} (\sqrt{q} - 1)^2.
\]
As a result, there follows
\[ q \leq \left( 2q \sqrt{\frac{2k}{d}} + 1 \right)^2. \]

By assumption, \( C \) is of minimum distance \( d \), so that \( \nu \geq 1 \) and
\[ \left( 2q \sqrt{\frac{2k}{d}} + 1 \right)^2 \leq \left( 2q \sqrt{\frac{2k}{d}} + 1 \right)^2. \]

**Proposition 6.** The following conditions are equivalent for a linear code \( C \subset \mathbb{F}_q^n \):

(i) \( C \) is formally self-dual, i.e., the zeta polynomial \( P_C(t) \) of \( C \) satisfies the functional equation
\[ P_C(t) = P_C \left( \frac{1}{q^t} \right) q^{gt^2} \]
of the Hasse-Weil polynomial of the function field of a curve of genus \( g \) over \( \mathbb{F}_q \);

(ii) Duursma’s reduced polynomial \( D_C(t) = g + g^⊥ - 2 \sum_{i=0}^{g^⊥} c_i t^i \) satisfies the functional equation
\[ D_C(t) = D_C \left( \frac{1}{q^t} \right) q^{-1} t^{2g-2} \]
of the Hasse-Weil polynomial of the function field of a curve of genus \( g-1 \) over \( \mathbb{F}_q \);

(iii) the coefficients of Duursma’s reduced polynomial \( D_C(t) = g + g^⊥ - 2 \sum_{i=0}^{g^⊥} c_i t^i \) of \( C \) satisfy the equalities
\[ c_{g-1+i} = q^i c_{g-1-i} \quad \text{for} \quad \forall 1 \leq i \leq g-1; \]

(iv) the dual code \( C^⊥ \subset \mathbb{F}_q^n \) of \( C \) has dimension \( \dim_{\mathbb{F}_q} C^⊥ = \dim_{\mathbb{F}_q} C = k \), genus \( g(C^⊥) = g(C) = g \) and the homogeneous weight enumerator of \( C \) is
\[ W_C(x, y) = M_{2k, k+1}(x, y) + \sum_{j=0}^{g-1} c_{g-1-j} w_j(x, y), \]
where
\[ w_j(x, y) := (q - 1) \binom{2k}{k+j} \left[ (x - y)^{k+j} y^{k-j} + q^j (x - y)^{k-j} y^{k+j} \right] \]
for \( 1 \leq j \leq g-1 \).
\[ w_0(x, y) := (q - 1) \binom{2k}{k} (x - y)^k y^k. \]

(v) the dual code \( C^⊥ \subset \mathbb{F}_q^n \) of \( C \) has dimension \( \dim_{\mathbb{F}_q} C^⊥ = \dim_{\mathbb{F}_q} C = k \), genus \( g(C^⊥) = g(C) = g \) and the homogeneous weight enumerator
\[ W_C(x, y) = M_{2k, k+1}(x, y) + \sum_{w=d}^{k-1} W_{C_w}^{(w)} (x, y) + \sum_{w=d}^{k} W_{C_w}^{(w)} (x - y)^k y^k \]
with

\[ \varphi_w(x, y) := \sum_{s=w}^{k-1} \binom{2k-w}{s-w} [(x-y)2^{k-s}y^s + q^{k-s}(x-y)^s y^{2^{k-s}}] + \binom{2k-w}{k} (x-y)^k \]

(33)

for \( d \leq w \leq k-1 \), so that \( C \) can be obtained from an MDS-code of the same length \( 2k \) and dimension \( k \) by removing and adjoining appropriate words, depending explicitly on the numbers \( W_C^{(d)}, W_C^{(d+1)}, \ldots, W_C^{(k)} \) of the codeword of \( C \) of weight \( \leq k = \dim_q C \).

Proof. Towards \((i) \Rightarrow (ii)\), one substitutes by \( P_C(t) = (1-t)(1-qt)D_C(t) + t^g \) in (21), in order to obtain

\[ (1-t)(1-qt)D_C(t) + t^g = (qt-1)(t-1) \left[ D_C \left( \frac{1}{qt} \right) q^{g-1}t^{2g-2} \right] + t^g, \]

whereas (27).

Conversely, \((ii) \Rightarrow (i)\) is justified by

\[ P_C(t) = (1-t)(1-qt)D_C(t) + t^g =\]

\[ = (t-1)(qt-1) \left[ D_C \left( \frac{1}{qt} \right) q^{g-1}t^{2g-2} \right] + t^g \]

\[ = \left[ (1-\frac{1}{t}) t \right] \left[ (1-\frac{1}{qt}) qt \right] D_C \left( \frac{1}{qt} \right) q^{g-1}t^{2g-2} + \frac{q^g t^{2g+2}}{q^g t^g} \]

\[ = \left[ (1-\frac{g}{qt}) \right] \left( 1-\frac{1}{qt} \right) D_C \left( \frac{1}{qt} \right) + \frac{1}{(qt)^g} q^g t^{2g} = P_C \left( \frac{1}{qt} \right) q^g t^{2g}. \]

That proves the equivalence \((i) \Leftrightarrow (ii)\).

Towards \((ii) \Leftrightarrow (iii)\), note that the functional equation of \( D_C(t) \) reads as

\[ \sum_{i=0}^{2g-2} c_i t^i = D_C(t) = D_C \left( \frac{1}{qt} \right) q^{g-1}t^{2g-2} = \left( \sum_{i=0}^{2g-2} \frac{c_i}{q^i t^i} \right) q^{g-1}t^{2g-2} \]

\[ = \sum_{i=0}^{2g-2} c_i q^{g-1-i}t^{2g-2-i} = \sum_{j=0}^{2g-2} c_{2g-2-j} q^{-g+1+j} t^j. \]

Comparing the coefficients of the left-most and the right-most side, one expresses the formal self-duality of \( C \) by the relations

\[ c_j = q^{-g+1+j}c_{2g-2-j} \quad \text{for} \quad \forall 0 \leq j \leq 2g-2. \]

Let \( i := g-1-j \), in order to express the above conditions in the form

\[ c_{g-1+i} = q^i c_{g-1-i} \quad \text{for} \quad \forall -g+1 \leq i \leq g-1. \]

(34)

For any \(-g+1 \leq i \leq -1\) note that \( c_{g-1+i} = q^i c_{g-1-i} \) is equivalent to \( c_{g-1-i} = q^{-i} c_{g-1+i} \) and follows from (34) with \( 1 \leq -i \leq g-1 \). In the case of \( i = 0 \), (34) holds trivially and (34) amounts to (28). That proves the equivalence of \((ii)\) with \((iii)\).
Towards \((iii) \Rightarrow (iv)\), one introduces a new variable \(z := x - y\) and expresses \((8)\) in the form

\[
\mathcal{V}_C(y + z, y) := \mathcal{W}_C(y + z, y) - M_{2k,k+1}(y + z, y) = (q - 1) \sum_{i=0}^{2q-2} c_i \left( \frac{2k}{d + i} \right) y^{d+i} z^{2k-d-i}
\]

\[
= (q - 1) \sum_{i=0}^{g-1} c_i \left( \frac{2k}{d + i} \right) y^{d+i} z^{2k-d-i} + (q - 1) \sum_{i=g}^{2q-2} c_i \left( \frac{2k}{d + i} \right) y^{d+i} z^{2k-d-i}.
\]

Let us change the summation index of the first sum to \(0 \leq j := g - 1 - i \leq g - 1\), put \(1 \leq j := i - g + 1 \leq g - 1\) in the second sum and make use of \(d + g = k + 1\), in order to obtain

\[
\mathcal{V}_C(y + z, y) = (q - 1) \sum_{j=0}^{g-1} c_{g-1-j} \left( \frac{2k}{k - j} \right) y^{k-j} z^{k+j} + (q - 1) \sum_{j=1}^{g-1} c_{j+g-1} \left( \frac{2k}{k + j} \right) y^{k+j} z^{k-j}.
\]

Extracting the term with \(j = 0\) from the first sum, one expresses

\[
\mathcal{V}_C(y + z, y) = (q - 1)c_{g-1} \left( \frac{2k}{k} \right) y^{k} z^{k} + \sum_{j=1}^{g-1} (q - 1) \left( \frac{2k}{k + j} \right) [c_{g-1-j} y^{k-j} z^{k+j} + c_{g-1+j} y^{k+j} z^{k-j}]
\]

for an arbitrary \(\mathbb{F}_q\)-linear code \(C \subset \mathbb{F}_{q}^n\). If \(C\) is formally self-dual, then plugging in by \((28)\) in \((36)\) and making use of \((30), (31)\), one gets

\[
\mathcal{V}_C(y + z, y) = \sum_{j=0}^{g-1} c_{g-1-j} w_j(y + z, y).
\]

Substituting \(z := x - y\) and \(\mathcal{V}_C(x, y) := \mathcal{W}_C(x, y) - M_{2k,k+1}(x, y)\), one derives the equality \((29)\) for the homogeneous weight enumerator of a formally self-dual linear code \(C \subset \mathbb{F}_{q}^n\).

In order to justify that \((iv)\) suffices for the formal self-duality of \(C\), we use that \((29)\) with \((30)\) and \((31)\) is equivalent to

\[
\mathcal{V}_C(y + z, y) = \sum_{j=1}^{g-1} c_{g-1-j} (q - 1) \left( \frac{2k}{k + j} \right) y^{k-j} z^{k+j} + c_{g-1} (q - 1) \left( \frac{2k}{k} \right) y^{k} z^{k} + \sum_{j=1}^{g-1} c_{g-1-j} (q - 1) \left( \frac{2k}{k + j} \right) y^{k+j} z^{k-j}
\]

Comparing the coefficients of \(y^{k+j} z^{k-j}\) with \(1 \leq j \leq g - 1\) from \((36)\) and \((37)\), one concludes that

\[
c_{g-1+j} = c_{g-1-j} q^j \quad \text{for} \quad \forall 1 \leq j \leq g - 1.
\]
These are exactly the relations (28) and imply the formal self-duality of $C$.

Towards $(iv) \iff (v)$, it suffices to put $E(x, y) := \sum_{j=0}^{q-1} c_{g-1-j} w_j(x, y)$ and to derive that

$$E(x, y) = \sum_{w=d}^{k-1} \sum_{i=w-d}^{g-2} \left( \frac{2k-w}{d+i-w} \right) W_C^{(w)} [(x-y)^{2k-d-i} y^{d+i} + q^{g-1-i}(x-y)^{d+i} y^{2k-d-i}]$$

$$+ \sum_{w=d}^{k} \left( \frac{2k-w}{k} \right) W_C^{(w)} (x-y) y^k.$$

Plugging in by (11) and exchanging the summation order, one gets

$$E(x, y) = \sum_{w=d}^{k-1} \sum_{i=w-d}^{g-2} \left( \frac{2k-w}{d+i-w} \right) W_C^{(w)} [(x-y)^{2k-d-i} y^{d+i} + q^{g-1-i}(x-y)^{d+i} y^{2k-d-i}]$$

$$+ \sum_{w=d}^{k} \left( \frac{2k-w}{k} \right) W_C^{(w)} (x-y) y^k.$$

Introducing $s := d + i$ and extracting $W_C^{(w)}$ as coefficients, one obtains

$$E(x, y) = \sum_{w=d}^{k-1} W_C^{(w)} \varphi_w(x, y) + W_C^{(k)} (x-y) y^k.$$

Let $C \subseteq \mathbb{F}_q^n$ be an $\mathbb{F}_q$-linear code of genus $g$, whose dual $C^\perp \subseteq \mathbb{F}_q^n$ is of genus $g^\perp$. In [1], Dodunekov and Landgev introduce the near-MDS linear codes $C$ as the ones with zeta polynomial $P_C(t) \in \mathbb{Q}[t]$ of degree $\text{deg} P_C(t) := g + g^\perp = 2$. Thus, $C$ is a near-MDS code if and only if it has constant Duursma’s reduced polynomial $D_C(t) = c_0 \in \mathbb{Q}$. Kim and Hyun prove in [7]) that a near-MDS code $C$ satisfies the Riemann Hypothesis Analogue exactly when

$$\frac{1}{(\sqrt{q}+1)^2} \leq c_0 \leq \frac{1}{(\sqrt{q}-1)^2}.$$

The next proposition characterizes the formally-self-dual codes $C \subseteq \mathbb{F}_q^n$ of genus 2, which satisfy the Riemann Hypothesis Analogue. By Proposition 6 (ii), $C$ is a formally self-dual linear code of genus 2 exactly when its Duursma’s reduced polynomial is

$$D_C(t) = c_0 + c_1 t + q c_0 t^2$$

for some $c_0, c_1 \in \mathbb{Q}$, $0 < c_0 < 1$.

**Proposition 7.** A formally self-dual linear code $C \subseteq \mathbb{F}_q^{2k}$ with a quadratic Duursma’s reduced polynomial $D_C(t) = c_0 + c_1 t + q c_0 t^2 \in \mathbb{Q}[t]$, $0 < c_0 < 1$ satisfies the Riemann Hypothesis Analogue if and only if

$$[(q+1)c_0 + c_1]^2 \geq 4c_0,$$

19
\[ q - 4\sqrt{q} + 1 \leq \frac{c_1}{c_0} \leq q + 4\sqrt{q} + 1, \quad (39) \]
\[ c_1 \leq \min \left( \frac{1}{(\sqrt{q} - 1)^2}, \quad \frac{1}{(\sqrt{q} + 1)^2} + 2\sqrt{qc_0} \right). \quad (40) \]

**Proof.** According to (22) from the proof of Proposition 4, the zeta polynomial

\[ P_C(t) = (1 - t)(1 - qt)(qc_0 t^2 + c_1 t + c_0) + t^2 \]

satisfies the Riemann Hypothesis Analogue if and only if there exist \( \varphi, \psi \in [0, 2\pi) \) with

\[ P_C(t) = q^2 c_0 \left( t - \frac{e^{i\varphi}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi}}{\sqrt{q}} \right) \left( t - \frac{e^{i\psi}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\psi}}{\sqrt{q}} \right). \]

Comparing the coefficients of \( t \) and \( t^2 \) from \( P_C(t) \), one expresses this condition by the equalities

\[ c_1 - (q + 1)c_0 = -2\sqrt{qc_0}[\cos(\varphi) + \cos(\psi)], \]
\[ 1 + 2qc_0 - (q + 1)c_1 = 2qc_0[1 + 2\cos(\varphi) \cos(\psi)]. \]

These are equivalent to

\[ \cos(\varphi) + \cos(\psi) = \frac{(q + 1)c_0 - c_1}{2\sqrt{qc_0}} \]
and

\[ \cos(\varphi) \cos(\psi) = \frac{1 - (q + 1)c_1}{4qc_0}. \]

In other words, the quadratic equation

\[ f(t) := t^2 + \frac{c_1 - (q + 1)c_0}{2\sqrt{qc_0}} t + \frac{1 - (q + 1)c_1}{4qc_0} \in \mathbb{Q}[t] \]

has roots \(-1 \leq t_1 = \cos(\varphi) \leq t_2 = \cos(\psi) \leq 1\). This, in turn, holds exactly when the discriminant

\[ D(f) = \left[ \frac{c_1 - (q + 1)c_0}{2\sqrt{qc_0}} \right]^2 - \frac{4[1 - (q + 1)c_1]}{4qc_0} \geq 0 \quad (41) \]

is non-negative, the vertex

\[ -1 \leq \frac{(q + 1)c_0 - c_1}{4\sqrt{qc_0}} \leq 1 \quad (42) \]

belongs to the segment \([-1, 1]\) and the values of \( f(t) \) at the ends of this segment are non-negative,

\[ f(1) \geq 0 , \quad f(-1) \geq 0. \quad (43) \]

The equivalence of (41) to (38) is straightforward. Since \( C \) is of minimum distance \( d = k - 1 \) and \( W_C^{(k-1)} = (q - 1)(k-1)c_0 \in \mathbb{N}, \) the constant term \( c_0 > 0 \) of \( D_C(t) \) is a
positive rational number and one can multiply (42) by \(-4\sqrt{q}c_0 < 0\), add \((q + 1)c_0\) to all the terms and rewrite it in the form

\[(q - 4\sqrt{q} + 1)c_0 \leq c_1 \leq (q + 4\sqrt{q} + 1)c_0.

Making use of \(c_0 > 0\), one observes that the above inequalities are tantamount to (39). Finally,

\[4qc_0 f(1) = 4qc_0 + 2\sqrt{q}[c_1 - (q + 1)c_0] + 1 - (q + 1)c_1 = (-c_1 - 2\sqrt{q}c_0)(\sqrt{q} - 1)^2 + 1 \geq 0\]

and

\[4qc_0 f(-1) = 4qc_0 - 2\sqrt{q}[c_1 - (q + 1)c_0] + 1 - (q + 1)c_1 = (2\sqrt{q}c_0 - c_1)(\sqrt{q} + 1)^2 + 1 \geq 0\]

can be expressed as (40).

\[\square\]

### 3 Duursma’s reduced polynomial of a function field

Let \(F = \mathbb{F}_q(X)\) be the function field of a curve \(X\) of genus \(g\) over \(\mathbb{F}_q\) and \(h_g := h(F)\) be the class number of \(F\), i.e., the number of the linear equivalence classes of the divisors of \(F\) of degree 0. The present section introduces an additive decomposition of the Hasse-Weil polynomial \(L_F(t) \in \mathbb{Z}[t]\) of \(F\), which associates to \(F\) a sequence \(\{h_i\}_{i=1}^{g-1}\) of virtual class numbers \(h_i\) of function fields of curves of genus \(i\) over \(\mathbb{F}_q\).

**Lemma 8.** The following conditions are equivalent for a polynomial \(L_g(t) \in \mathbb{Q}[t]\) of degree \(\deg L_g(t) = 2g\):

(i) \(L_g(t)\) satisfies the functional equation

\[L_g(t) = L_g\left(\frac{1}{qt}\right) q^{g^2}t^{2g}\]

of the Hasse-Weil polynomial of the function field of a curve of genus \(g\) over \(\mathbb{F}_q\);

(ii) \(L_{g-1}(t) := \frac{L_g(t) - L_g(1)t^g}{(1 - t)(1 - qt)}\)

is a polynomial with rational coefficients of degree \(2g - 2\), satisfying the functional equation

\[L_{g-1}(t) = L_{g-1}\left(\frac{1}{qt}\right) q^{g-1}t^{2g-2}\]

of the Hasse-Weil polynomial of the function field of a curve of genus \(g - 1\) over \(\mathbb{F}_q\);

(iii) \(L_g(t) = \sum_{i=0}^{g} h_i t^i(1 - t)^{g-i}(1 - qt)^{g-i}\)

for some rational numbers \(h_i \in \mathbb{Q}\).
Proof. Towards (i) \( \Rightarrow \) (ii), let us note that the polynomial \( M_g(t) := L_g(t) - L_g(1)t^g \) vanishes at \( t = 1 \), so that it is divisible by \( 1 - t \). Further,

\[
M_g(t) = L_g(t) - L_g(1)t^g = \left[ L_g \left( \frac{1}{qt} \right) - \frac{L_g(1)}{q^{2g}} \right] q^g t^{2g} = M_g \left( \frac{1}{qt} \right) q^g t^{2g}
\]

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus \( g \) over \( \mathbb{F}_q \). In particular, \( M_g \left( \frac{1}{q} \right) = M_g(1) \frac{q^g}{q^{2g}} = 0 \) and \( M_g(t) \) is divisible by the linear polynomial \( q \left( \frac{1}{q} - t \right) = 1 - qt \), which is relatively prime to \( 1 - t \) in \( \mathbb{Q}[t] \). As a result,

\[
L_{g-1}(t) := \frac{M_g(t)}{(1-t)(1-qt)} \in \mathbb{Q}[t]
\]

is a polynomial of degree \( \deg L_{g-1}(t) = 2g - 2 \). Straightforwardly,

\[
L_{g-1} \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} = \left[ M_g \left( \frac{1}{qt} \right) : \left( 1 - \frac{1}{qt} \right) \left( 1 - \frac{1}{t} \right) \right]
\]

\[
= \frac{M_g(t)}{qt^2} : \frac{(qt - 1)(t - 1)}{qt^2} = \frac{M_g(t)}{(1-t)(1-qt)} = L_{g-1}(t)
\]

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus \( g - 1 \) over \( \mathbb{F}_q \).

The implication (ii) \( \Rightarrow \) (i) follows from the functional equation of \( L_{g-1}(t) \), applied to \( L_g(t) = (1-t)(1-qt)L_{g-1}(t) + L_g(1)t^g \). Namely,

\[
L_g \left( \frac{1}{qt} \right) q^g t^{2g} = \left[ \left( 1 - \frac{1}{qt} \right) qt \right] \left[ \left( 1 - \frac{1}{t} \right) t \right] L_{g-1} \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} + \frac{L_g(1)}{q^{2g}} q^g t^{2g}
\]

\[
= (qt - 1)(t - 1)L_{g-1}(t) + L_g(1)t^g
\]

\[
= (1-t)(1-qt)L_{g-1}(t) + L_g(1)t^g = L_g(t).
\]

We derive (i) \( \Rightarrow \) (iii) by an induction on \( g \), making use of (ii). More precisely, for \( g = 1 \) one has \( L_0(t) := \frac{L_1(t) - L_1(1)t}{(1-t)(1-qt)} \in \mathbb{Q}[t] \) of degree \( \deg L_0(t) = 0 \) or \( L_0 \in \mathbb{Q} \). Then

\[
L_1(t) = (1-t)(1-qt)L_0 + L_1(1)t = \sum_{i=0}^{1} h_i t^i (1-t)^{1-i}(1-qt)^{1-i}
\]

with \( h_0 := L_0 \in \mathbb{Q} \) and \( h_1 := L_1(1) \in \mathbb{Q} \). In the general case, (ii) provides a polynomial

\[
L_{g-1}(t) := \frac{L_g(t) - L_g(1)t^g}{(1-t)(1-qt)},
\]

subject to the functional equation

\[
L_{g-1}(t) = L_{g-1} \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2}
\]
of the Hasse-Weil polynomial of the function field of a curve of genus $g - 1$ over $\mathbb{F}_q$. By the inductional hypothesis, there exist $h'_i \in \mathbb{Q}$, $0 \leq i \leq g - 1$ with

$$L_{g-1}(t) = \sum_{i=0}^{g-1} h'_i t^i (1 - t)^{g-1-i}(1 - qt)^{g-1-i}.$$ 

Then

$$L_g(t) = (1 - t)(1 - qt)L_{g-1}(t) + L_g(1)t^g = \sum_{i=0}^{g} h_i t^i (1 - t)^{g-i}(1 - qt)^{g-i}$$

with $h_i := h'_i \in \mathbb{Q}$ for $0 \leq i \leq g - 1$ and $h_g := L_g(1) \in \mathbb{Q}$ justifies $(i) \Rightarrow (iii)$.

Towards $(iii) \Rightarrow (i)$, let us assume that $L_g(t) = \sum_{i=0}^{g} h_i t^i (1 - t)^{g-i}(1 - qt)^{g-i}$. Then

$$L \left( \frac{1}{qt} \right) q^g t^{2g} = \left[ \sum_{i=0}^{g} \frac{h_i}{q^i t^i} \left( \frac{1}{1 - \frac{1}{qt}} \right)^{g-i} \left( \frac{1 - \frac{1}{t}}{1 - \frac{1}{qt}} \right)^{g-i} \right] t^{2g}$$

$$= \sum_{i=0}^{g} \left[ \frac{h_i}{q^i t^i q^i t^i} \right] \left[ \left( 1 - \frac{1}{qt} \right) \left( \frac{1 - \frac{1}{t}}{1 - \frac{1}{qt}} \right) \right] \sum_{i=0}^{g} h_i t^i (qt - 1)^{g-i}(t - 1)^{g-i} = L_g(t)$$

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus $g$ over $\mathbb{F}_q$.

**Proposition 9.** Let $F = \mathbb{F}_q(X)$ be the function field of a smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\mathbb{F}_q)$ of genus $g$, defined over $\mathbb{F}_q$, with $h(F)$ linear equivalence classes of divisors of degree $0$, $A_i$, effective divisors of degree $i \geq 0$, Hasse-Weil polynomial $L_F(t) \in \mathbb{Q}[t]$ and Duursma’s reduced polynomial $D_F(t) \in \mathbb{Q}[t]$, defined by the equality

$$L_F(t) = (1 - t)(1 - qt)D_F(t) + h(F)t^g.$$ 

Then:

(i) $D_F(t) = \sum_{i=0}^{g-2} A_i(t - q^{-1-i}q^{-2-i}) + A_{g-1}t^{g-1} \in \mathbb{Z}[t]$ is a polynomial with integral coefficients, which is uniquely determined by $A_0 = 1, A_1, \ldots, A_{g-1}$;

(ii) the equality

$$\frac{D_F(t)}{(1 - t)(1 - qt)} = \sum_{i=0}^{\infty} B_i t^i$$

(44)

of formal power series of $t$ holds for

$$B_i = \sum_{j=0}^{i} A_j \left( \frac{q^{i-j+1} - 1}{q - 1} \right)$$

(45)
for $0 \leq i \leq g - 1$,
\[
B_i = \sum_{j=0}^{g-1} A_j \left( \frac{q^{i-j+1} - 1}{q - 1} \right) + \sum_{j=g}^{i} A_{2g-2-j} \left( \frac{q^{i-g+2} - q^{j-g+1}}{q - 1} \right)
\]  
(46)

for $g \leq i \leq 2g - 3$,
\[
B_i = D_F(1) \left( \frac{q^{i-g+2} - 1}{q - 1} \right)
\]  
(47)

for $i \geq 2g - 2$;

(iii) the natural numbers $B_i$, $i \geq 0$ from (ii) satisfy the relations
\[
B_i = q^{i-g+2}B_{2g-4-i} + D_F(1) \left( \frac{q^{i-g+2} - 1}{q - 1} \right) \quad \text{for } \forall g - 1 \leq i \leq 2g - 4;
\]  
(48)
\[
B_i = D_F(1) \left( \frac{q^{i-g+2} - 1}{q - 1} \right) \quad \text{for } \forall i \geq 2g - 3.
\]  
(49)

(iv) the number $h(F)$ of the linear equivalence classes of the divisors of $F$ of degree 0 satisfies the inequalities
\[
(\sqrt{q} - 1)^{2g} \leq h(F) \leq (\sqrt{q} + 1)^{2g}
\]

Proof. (i) By Theorem 4.1.6. (ii) and Theorem 4.1.11 from [8], the Hasse-Weil zeta function of $F$ is the generating function
\[
Z_F(t) = \frac{L_F(t)}{(1-t)(1-qt)} = \sum_{j=0}^{\infty} A_j t^j
\]
of the sequence $\{A_i\}_{i=0}^{\infty}$. According to Lemma 8 and $L_F(1) = h(F)$,
\[
D_F(t) := \frac{L_F(t) - h(F)t^g}{(1-t)(1-qt)}
\]
is a polynomial of deg $D_F(t) = 2g - 2$, subject to the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus $g - 1$ over $\mathbb{F}_q$. Thus,
\[
Z_F(t) = D_F(t) + \frac{h(F)t^g}{(1-t)(1-qt)} = \sum_{j=0}^{\infty} A_j t^j.
\]  
(50)

Let $l(G)$ is the dimension of the space $H^0(X, \mathcal{O}_X(G))$ of the global holomorphic sections of the line bundle $\mathcal{O}_X(G) \rightarrow X$, associated with a divisor $G \in \text{Div}(F)$. Riemann-Roch Theorem asserts that
\[
l(G) = l(K_X - G) + \deg(G) - g + 1
\]
for a canonical divisor $K_X$ of $X$. For any $j \geq g - 1$, suppose that $G_1, \ldots, G_{h(F)} \in \text{Div}(F)$ is a complete set of representatives of the linear equivalence classes of the divisors of $F$ of degree $j$. Then
\[
A_j = \sum_{\nu=1}^{h(F)} \frac{q^{l(G_{\nu})} - 1}{q - 1} = q^{j-g+1} \sum_{\nu=1}^{h(F)} \left( \frac{q^{l(K_X - G_{\nu})} - 1}{q - 1} \right) + h(F) \left( \frac{q^{j-g+1} - 1}{q - 1} \right)
\]  
(51)
for $g \leq j \leq 2g - 2$ and

$$A_j = h(F) \left( \frac{q^{j-g+1} - 1}{q - 1} \right) \quad \text{for } \forall j \geq 2g - 1. \quad (52)$$

Note that $K_Y - G_1, \ldots, K_Y - G_{h(F)}$ is a complete set of representatives of the linear equivalence classes of the divisors of $F$ of degree $2g - 2 - j$, so that

$$A_{2g-2-j} = \sum_{\nu=1}^{h(F)} \frac{q^{i(K_Y - G_{\nu})} - 1}{q - 1}. \quad (53)$$

Plugging in by (53) in (51), one obtains

$$A_j = q^{j-g+1}A_{2g-2-j} + h(F) \left( \frac{q^{j-g+1} - 1}{q - 1} \right) \quad \text{for } g \leq j \leq 2g - 2, \quad (54)$$

whereas

$$Z_F(t) = \sum_{j=0}^{2g-2} A_j t^j + \sum_{j=g}^{2g-2} q^{j-g+1}A_{2g-2-j} t^j + h(F) \sum_{j=g}^{\infty} \left( \frac{q^{j-g+1} - 1}{q - 1} \right) t^j,$$

Putting $i := 2g - 2 - j$ in the second sum and $i := j - g$ in the third sum, one expresses

$$Z_F(t) = \sum_{i=0}^{g-2} A_i (t^i + q^{g-1-i}t^{2g-2-i}) + A_{g-1}t^{g-1} + h(F) \left[ \frac{q^g}{q - 1} \left( \sum_{i=0}^{\infty} q^i t^i \right) - \frac{t^g}{q - 1} \left( \sum_{i=0}^{\infty} t^i \right) \right],$$

Summing up the geometric progressions

$$\sum_{i=0}^{\infty} q^i t^i = \frac{1}{1 - qt}, \quad \sum_{i=0}^{\infty} t^i = \frac{1}{1 - t},$$

one derives

$$Z_F(t) = \sum_{i=0}^{g-2} A_i (t^i + q^{g-1-i}t^{2g-2-i}) + A_{g-1}t^{g-1} + h(F) \frac{t^g}{(1-t)(1-qt)},$$

whereas

$$D_F(t) = \sum_{i=0}^{g-2} A_i (t^i + q^{g-1-i}t^{2g-2-i}) + A_{g-1}t^{g-1}.$$

In particular, $D_F(t) \in \mathbb{Z}[t]$ has integral coefficients.

(ii) Let us expand

$$\frac{1}{1 - t} = \sum_{i=0}^{\infty} t^i, \quad \frac{1}{1 - qt} = \sum_{i=0}^{\infty} q^i t^i.$$
as sums of geometric progressions and note that
\[
\frac{1}{(1 - t)(1 - qt)} = \sum_{i=0}^{\infty} (1 + q + \ldots + q^i)t^i = \sum_{i=0}^{\infty} \left( \frac{q^{i+1} - 1}{q - 1} \right) t^i.
\]

Then represent Duursma’s reduced polynomial in the form
\[
D_F(t) = \sum_{j=0}^{g-1} A_j t^j + \sum_{j=g}^{2g-2} A_{2g-2-j} q^{j-g+1} t^j. \tag{55}
\]

Now, the comparison of the coefficients of \(t^i, i \geq 0\) from the left hand side and the right hand side of (44) provides (45), (46) and
\[
B_i = \sum_{j=0}^{g-1} A_j \left( \frac{q^{i+j+1} - 1}{q - 1} \right) + \sum_{j=g}^{2g-2} A_{2g-2-j} q^{j-g+1} \left( \frac{q^{i+j+1} - 1}{q - 1} \right) \quad \text{for } i \geq 2g - 2.
\]

The last formula can be expressed in the form
\[
B_i = \frac{q^{i+1}}{q - 1} \left( \sum_{j=0}^{g-1} A_j q^{-j} + \sum_{j=g}^{2g-2} A_{2g-2-j} q^{j-g+1} q^{-j} \right) - \frac{1}{q - 1} \left( \sum_{j=0}^{g-1} A_j + \sum_{j=g}^{2g-2} A_{2g-2-j} q^{j-g+1} \right)
\]
\[
= \frac{q^{i+1}}{q - 1} D_F \left( \frac{1}{q} \right) - \frac{1}{q - 1} D_F(1).
\]

According to Lemma 8 (i) \(\Rightarrow\) (iii), Duursma’s reduced polynomial of \(F\) satisfies the functional equation \(D_F(t) = D_F \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2}\). In particular, \(D_F(1) = D_F \left( \frac{1}{q} \right) q^{g-1}\) and there follows (47).

(iii) Due to \(A_i \geq 0\) for \(\forall i \geq 0\), \(B_i\) are sums of non-negative integers. Moreover, \(B_i \geq A_i \left( \frac{q^{i+1}}{q-1} \right) \geq A_0 = 1 > 0\) for \(\forall i \geq 0\) reveals that all \(B_i\) are natural numbers.

Towards (48), let us introduce the polynomial \(\psi(t) := \sum_{j=0}^{g-2} A_j t^j \in \mathbb{Z}[t]\) and express
\[
D_F(t) = \sum_{j=0}^{g-2} A_j t^j + q^{g-1} t^{2g-2} \left[ \sum_{j=0}^{g-2} A_j (qt)^{-j} \right] + A_{g-1} t^{g-1}
\]
\[
= \psi(t) + \psi \left( \frac{1}{qt} \right) q^{g-1} t^{2g-2} + A_{g-1} t^{g-1}.
\]

In particular,
\[
D_F(1) = \psi(1) + \psi \left( \frac{1}{q} \right) q^{g-1} + A_{g-1}. \tag{56}
\]
Straightforwardly,
\[
\begin{align*}
\sum_{j=0}^{g-2} A_j q^{-j} &- \frac{1}{q-1} \left( \sum_{j=0}^{g-2} A_j \right) + A_{g-1} - \\
- \frac{q^g-1}{q-1} \left( \sum_{j=0}^{g-2} A_j q^{-j} \right) &+ \frac{q}{q-1} \left( \sum_{j=0}^{g-2} A_j \right) \\
= \psi \left( \frac{1}{q} \right) q^{g-1} + \psi(1) + A_{g-1} = D_F(1).
\end{align*}
\]
That proves (48) for \( i = g-1 \). In the case of \( g \leq i \leq 2g-4 \) note that \( 0 \leq 2g-4-i \leq g-4 \) and
\[
\begin{align*}
\sum_{j=0}^{g-1} A_j (q^{i-j}+1-1) + \sum_{j=g}^{i} A_{2g-2-j} (q^{i-g+2} - q^{i-g+1}) - \sum_{j=0}^{2g-4-i} A_j (q^{g-1-j} - q^{i-g+2}).
\end{align*}
\]
Changing the summation index of the second sum to \( s := 2g - 2 - j \), one obtains
\[
\begin{align*}
(q-1)(B_i - q^{i-g+2} B_{2g-4-i}) \\
= q^{i+1} \left( \sum_{j=0}^{g-1} A_j q^{-j} \right) - \left( \sum_{j=0}^{g-1} A_j \right) + q^{i-g+2} \left( \sum_{s=2g-2-i}^{g-2} A_s \right) \\
- q^{g-1} \left( \sum_{s=2g-2-i}^{g-2} A_s q^{-s} \right) - q^{g-1} \left( \sum_{j=0}^{2g-4-i} A_j q^{-j} \right) + q^{i-g+2} \left( \sum_{j=0}^{2g-4-i} A_j \right).
\end{align*}
\]
An appropriate grouping of the sums yields
\[
\begin{align*}
(q-1)(B_i - q^{i-g+2} B_{2g-4-i}) \\
= \psi \left( \frac{1}{q} \right) q^{i+1} + A_{g-1} q^{i-g+2} - \psi(1) - A_{g-1} + \psi(1) q^{i-g+2} - \psi \left( \frac{1}{q} \right) q^{g-1} \\
= (q^{i-g+2} - 1) \left[ \psi(1) + \psi \left( \frac{1}{q} \right) q^{g-1} + A_{g-1} \right] = D_F(1)(q^{i-g+2} - 1).
\end{align*}
\]
That justifies (48).
Note that (49) with \( i \geq 2g - 2 \) coincides with (47). In the case of \( i = 2g - 3 \),
\[
\begin{align*}
(q-1)B_{2g-3} = \sum_{j=0}^{g-1} A_j (q^{2g-2-j} - 1) + \sum_{s=1}^{g-2} A_s (q^{g-1} - q^{g-1-s}),
\end{align*}
\]
after changing the summation index of the second sum to \( s := 2g - 2 - j \). Then

\[
(q - 1)B_{2g-3} = q^{2g-2} \left( \sum_{j=0}^{2g-2} A_j q^{-j} \right) - \left( \sum_{j=0}^{2g-2} A_j \right) + A_{g-1}(q^{g-1} - 1) + q^{g-1} \left( \sum_{j=0}^{g-1} A_j \right) - q^{g-1} \left( \sum_{j=0}^{g-2} A_j q^{-j} \right)
\]

\[
= (q^{g-1} - 1) \left[ \psi(1) + \psi \left( \frac{1}{q} \right) q^{g-1} + A_{g-1} \right] = D_F(1)(q^{g-1} - 1),
\]

which is tantamount to (49) with \( i = 2g - 3 \).

(iv) By the Hasse-Weil Theorem, all the roots of \( L_F(t) \) belong to the circle \( S \left( \frac{1}{\sqrt{q}} \right) = \{ z \in \mathbb{C} \mid |z| = \frac{1}{\sqrt{q}} \} \). The proof of Proposition 4 specifies that

\[
L_F(t) = a_{2g} \prod_{j=1}^{g} \left( t - \frac{e^{i\varphi_j}}{\sqrt{q}} \right) \left( t - \frac{e^{-i\varphi_j}}{\sqrt{q}} \right)
\]

for some \( \varphi_j \in [0, 2\pi) \). The functional equation \( L_F(t) = L_F \left( \frac{1}{qt} \right) q^{gt^2} \) implies that \( a_{2g} = q^g a_0 \). Combining with \( a_0 = L_F(0) = 1 \), one gets

\[
L_F(t) = \prod_{j=1}^{g} (\sqrt{q} t - e^{i\varphi_j})(\sqrt{q} t - e^{-i\varphi_j}) = \prod_{j=1}^{g} (qt^2 - 2\sqrt{q} \cos \varphi_j t + 1).
\]

The substitution \( t = 1 \) provides

\[
h(F) = L_F(1) = \prod_{j=1}^{g} (q - 2\sqrt{q} \cos \varphi_j + 1).
\]

However, \( \cos \varphi_j \in [-1, 1] \) requires

\[
(\sqrt{q} - 1)^2 \leq q - 2\sqrt{q} \cos \varphi_j + 1 \leq (\sqrt{q} + 1)^2,
\]

whereas

\[
(\sqrt{q} - 1)^{2g} \leq h(F) = L_F(1) = \prod_{j=1}^{g} (q - 2\sqrt{q} \cos \varphi_j + 1) \leq (\sqrt{q} + 1)^{2g}.
\]
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