Hamiltonian dynamics of 5D Kalb-Ramond theories with a compact dimension

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A detailed Hamiltonian analysis for a five-dimensional Kalb-Ramond, massive Kalb-Ramond and Stueckelberg Kalb-Ramond theories with a compact dimension is performed. We develop a complete constraint program, then we quantize the theory by constructing the Dirac brackets. From the gauge transformations of the theories, we fix a particular gauge and we find pseudo-Goldstone bosons in Kalb-Ramond and Stueckelberg Kalb-Ramond’s effective theories. Finally we discuss some remarks and prospects.

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I. INTRODUCTION

It is well-know that antisymmetric tensor fields have an important relevance in theoretical physics. In fact, the antisymmetric tensor fields has been used for describing mass zero spinless as well as vector particles [1–6]; in other cases, they appear in some formulations of supergravity theories [7–9] and as a way of gauging the apparent internal supersymmetry of the weak interactions [10]. In string theory, antisymmetric fields are mediators of the interaction between open strings with charged particles [11], and also they are a fundamental block for describing the unification of Yang-Mills and supergravity [12]. Moreover, they have also an important role characterizing defects in solid state physics [13].

For the reasons explained above, in this paper we analyze in the context of extra dimensions theories involving antisymmetric tensor fields. We study three models; 5D Kalb-Ramond, 5D Proca Kalb-Ramond and 5D Stueckelberg Kalb-Ramond theories [14]. We carryout the compactification process on a $S^1/Z_2$ orbifold obtaining an effective Lagrangian composed by a four-dimensional theory plus a tower of $kk$-excitations. We analize the effects of the compactification process on
the theory by performing a pure Dirac’s framework. We develop a complete constraints program, we find that 5D Kalb-Ramond and 5D Stueckelberg Kalb-Ramond theories present reducibility conditions among the constraints in both the zero modes and in the $kk$-excitations, while 5D Proca Kalb-Ramond is an irreducible system. We show that 5D Kalb-Ramond and 5D Stueckelberg Kalb-Ramond Lagrangians are gauge theories, from the gauge transformations we fixed the gauge and by using that gauge we obtain that there are present pseudo-Goldstone bosons in the theories. Respect to 5D Proca Kalb-Ramond Lagrangian, the theory is not a gauge theory and there are not present pseudo-Goldstone bosons. Because of 5D Kalb-Ramond and 5D Stueckelberg Kalb-Ramond theories are reducible systems, we use the phase space extension procedure for constructing the Dirac brackets and we calculate these brackets among the physical fields. All these ideas are clarified along the paper.

II. KALB-RAMOND ACTION IN 5D WITH A COMPACT DIMENSION

The notation that we will use along the paper is the following: the capital latin indices $M,N$ run over 0,1,2,3,5 here 5 label the extra compact dimension and these indices can be raised and lowered by the four-dimensional Minkowski metric $\eta_{MN} = (1, -1, -1, -1, -1)$; $y$ will represent the coordinate in the compact dimension and $\mu, \nu = 0, 1, 2, 3$ are spacetime indices, $x^\mu$ the coordinates that label the points for the four-dimensional manifold $M_4$; furthermore we will suppose that the compact dimension is a $S^1/\mathbb{Z}_2$ orbifold whose radius is $R$. Hence, let us study the five dimensional Kalb-Ramond action given by \[14\]

$$\mathcal{L} = \frac{1}{2 \times 3!} H_{MNL} H^{MNL},$$

where the strength fields $H_{MNL} = \partial_M B_{NL} + \partial_N B_{LM} + \partial_L B_{MN}$, with $B_{LM} = -B_{ML}$ is the Kalb-Ramond field. In this manner, for studying the theory in the context of Kaluza-Klein modes, we express any dynamical variable defined on $M_4 \times S^1/\mathbb{Z}_2$ as a complete set of harmonics \[15–18\]

\begin{align*}
B_{\mu\nu}(x,y) &= \frac{1}{\sqrt{2\pi R}} B_{\mu\nu}^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} B_{\mu\nu}^{(n)}(x) \cos \left(\frac{ny}{R}\right), \\
B_{\mu5}(x,y) &= \frac{1}{\sqrt{\pi R}} \sum_{n=1}^{\infty} B_{\mu5}^{(n)}(x) \sin \left(\frac{ny}{R}\right). \tag{2}
\end{align*}

For this theory, the dynamical variables for the zero mode are given by $B_{\mu\nu}^{(0)}$, $B_{ij}^{(0)}$ and for the $kk$-modes are $B_{0i}^{(n)}$, $B_{ij}^{(n)}$, $B_{05}^{(n)}$, $B_{i5}^{(n)}$ with $i,j = 1,2,3$. We shall suppose that the number of $kk$-excitations is $k$, and we will take the limit $k \to \infty$ at the end of the calculations, thus, $n = 1,2,3...k-1$.

By taking into account (2) in (1) and performing the integration over the fifth dimension, we obtain
the following effective Lagrangian given by

\[
\mathcal{L} = \frac{1}{2 \times 3!} H^{(0)}_{\mu \nu \lambda} H_{(0)}^{\mu \nu \lambda} + \sum_{n=1}^{\infty} \left[ \frac{1}{2 \times 3!} H^{(n)}_{\mu \nu \lambda} H_{(n)}^{\mu \nu \lambda} + \frac{1}{4} \left( \partial_\mu B^{(n)}_{\nu \lambda} + \partial_\nu B^{(n)}_{\lambda \mu} - \frac{n}{R} B^{(n)}_{\mu \nu} \right) \right].
\]  
(3)

In this manner, we can compute the following Hessian of the theory

\[
\frac{\partial^2 \mathcal{L}}{\partial (\partial_0 B{_{\alpha \beta}^{(0)}}) \partial (\partial_0 B{_{\alpha \beta}^{(0)}})} = \frac{1}{4} (g^{\alpha \lambda} g^{\beta \rho} - g^{\alpha \rho} g^{\beta \lambda}),
\]

\[
\frac{\partial^2 \mathcal{L}}{\partial (\partial_0 B{_{K M}^{(m)}}) \partial (\partial_0 B{_{L I}^{(l)}})} = \frac{1}{4} (g^{L K} g^{H M} - g^{L M} g^{H K}) + \frac{1}{4} \delta_i^H \delta_i^M g^{L K},
\]

(4)

it is straightforward to observe that the Hessian has a rank=4 and 4(k - 1) null vectors, thus, we expect 4(k - 1) primary constraints. Therefore, from the definition of the momenta \((\Pi^{0i}_{(0)}), \Pi^{ij}_{(0)}, \Pi^{0i}_{(n)}, \Pi^{ij}_{(n)}, \Pi^{05}_{(n)}, \Pi^{15}_{(n)}\) canonically conjugate to \((B^{(0)}_{0i}, B^{(0)}_{ij}, B^{(0)}_{0i}, B^{(1)}_{ij}, B^{(0)}_{0i}, B^{(0)}_{ij})\) given by

\[
\begin{align*}
\Pi^{0i}_{(0)} &= 0, & \Pi^{ij}_{(0)} &= \frac{1}{2} H^{0ij}_{(0)}, \\
\Pi^{0i}_{(n)} &= 0, & \Pi^{ij}_{(n)} &= \frac{1}{2} H^{0ij}_{(n)}, & \Pi^{05}_{(n)} &= 0, & \Pi^{15}_{(n)} &= \frac{1}{2} (\partial^0 B^{15}_{(n)} + \partial^j B^{50}_{(n)} - \frac{n}{R} B^{0i}_{(n)}),
\end{align*}
\]

(5)

we obtain the following 4(k - 1) primary constraints

\[
\begin{align*}
\phi^{0i}_{(0)} &= \Pi^{0i}_{(0)} \approx 0, \\
\phi^{0i}_{(n)} &= \Pi^{0i}_{(n)} \approx 0, & \phi^{05}_{(n)} &= \Pi^{05}_{(n)} \approx 0.
\end{align*}
\]

(6)

In this manner, by using the definition of the momenta, we obtain the following canonical Hamiltonian

\[
H_c = \int d^3 x \left[ 2 B^{(0)}_{0i} \partial_j \Pi^{ij}_{(0)} + \Pi^{ij}_{(0)} \Pi^{ij}_{(0)} - \frac{1}{2 \times 3!} H^{(0)}_{ij k} H_{(0)}^{ij k} + \sum_{n=1}^{\infty} \left[ 2 B^{(n)}_{0i} \partial_j \Pi^{ij}_{(n)} + \Pi^{ij}_{(n)} \Pi^{ij}_{(n)} - \frac{1}{2 \times 3!} H^{(n)}_{ij k} H_{(n)}^{ij k} + 2 B^{(n)}_{0i} \partial_j \Pi^{ij}_{(n)} + 2 B^{(n)}_{0i} \partial_j \Pi^{ij}_{(n)} - \frac{n}{R} B^{0i}_{(n)} \Pi^{ij}_{(n)} \right] \right],
\]

(7)

thus, the primary Hamiltonian takes the form

\[
H_1 = H_c + \int d^3 x \left[ a^{(0)}_{0i} \phi^{0i}_{(0)} + \sum_{n=1}^{k-1} \left( a^{(n)}_{0i} \phi^{0i}_{(n)} + a^{(n)}_{05} \phi^{05}_{(n)} \right) \right],
\]

(8)

where \(a^{(0)}_{0i}, a^{(n)}_{0i}\) and \(a^{(n)}_{05}\) are Lagrange multipliers enforcing the constraints, and the fundamental Poisson brackets are

\[
\{ B^{(0)}_{\mu \nu}(x), \Pi^{\mu \nu}_{(0)}(z) \} = \frac{1}{2} (\delta_\mu^\nu \delta_\beta^\gamma - \delta_\mu^\gamma \delta_\beta^\nu) \delta^3(x - z),
\]

\[
\{ B^{(l)}_{H L}(x), \Pi^{MN}_{(n)}(z) \} = \frac{1}{2} \delta_l^M (\delta_H^N \delta_L^N - \delta_L^M \delta_H^N) \delta^3(x - z).
\]

(9)
Therefore, in order to determine if there are more constraints we calculate consistency relations among the constraints and we obtain the following secondary constraints

\[ \dot{\phi}^{0i}_{(0)}(x) = \{ \phi^{0i}_{(0)}(x), H_1(z) \} = \partial_j \Pi^{ij}_{(0)}(x) \approx 0, \]

\[ \dot{\phi}^{0i}_{(n)}(x) = \{ \phi^{0i}_{(n)}(x), H_1(z) \} = \partial_j \Pi^{ij}_{(n)}(x) + \frac{n}{R} \Pi^{5i}_{(n)}(x) \approx 0, \]

\[ \dot{\phi}^{05}_{(n)}(x) = \{ \phi^{05}_{(n)}(x), H_1(z) \} = \partial_j \Pi^{j5}_{(n)}(x) \approx 0. \] (10)

For this theory there are not third constraints. Therefore, we have obtained the following 8 constraints

\[ \phi^{0i}_{(0)} \equiv \Pi^{0i}_{(0)} \approx 0, \]

\[ \psi^{0i}_{(0)} \equiv \partial_i \Pi^{ij}_{(0)} \approx 0, \]

\[ \phi^{0i}_{(n)} \equiv \Pi^{0i}_{(n)} \approx 0, \]

\[ \phi^{05}_{(n)} \equiv \Pi^{05}_{(n)} \approx 0, \]

\[ \psi^{0i}_{(n)} \equiv \partial_i \Pi^{ij}_{(n)} + \frac{n}{R} \Pi^{5i}_{(n)} \approx 0, \]

\[ \psi^{05}_{(n)} \equiv \partial_j \Pi^{j5}_{(n)} \approx 0, \] (11)

we are able to observe that these constraints are all of first class. However, they are not all independent because there are reducibility conditions among the constraints in both, the zero mode and the \( kk \)-excitations. These conditions are given by the following \( k \) relations

\[ \partial_i \psi^{0i}_{(0)} = 0, \]

\[ \partial_i \psi^{0i}_{(n)} + \frac{n}{R} \psi^{05}_{(n)} = 0, \] (12)

thus, for the theory under study there are \([8k - 2] - k = 7k - 2\) independent first class constraints.

Therefore, the counting of degrees of freedom is performed as follows; there are \( 20k - 8 \) dynamical variables and \( 7k - 2 \) independent first class constraints, thus we obtain that the number of physical degrees of freedom is given by

\[ DF = \frac{1}{2}[20k - 8 - 2(7k - 2)] = 3k - 2, \] (13)

we observe if \( k = 1 \), then there is one degree of freedom, it is associated with the zero mode which correspond to 4D Kalb-Ramond theory without an extra dimension.

Because we have obtained a set of first class constraints, we can calculate the gauge transformations of the theory. For this aim, we define the following gauge generator of the theory

\[ G = \int \left[ \epsilon^{(0)}_0 \phi^{0i}_{(0)} + \epsilon^{(0)}_0 \psi^{0i}_{(0)} + \epsilon^{(n)}_0 \phi^{0i}_{(n)} + \epsilon^{(n)}_0 \psi^{0i}_{(n)} + \epsilon^{(n)}_{05} \phi^{05}_{(n)} + \epsilon^{(n)}_{05} \psi^{05}_{(n)} \right] d^3 z. \] (14)
In this manner, we obtain the gauge transformations of the theory given by

\[
\begin{align*}
B_{0i}^{(0)} & \rightarrow B_{0i}^{(0)} - \partial_0 \epsilon_i^{(0)}, \\
B_{ij}^{(0)} & \rightarrow B_{ij}^{(0)} + \partial_i \epsilon_j^{(0)} - \partial_j \epsilon_i^{(0)}, \\
B_{ij}^{(n)} & \rightarrow B_{ij}^{(n)} - \partial_0 \epsilon_j^{(n)}, \\
B_{05}^{(n)} & \rightarrow B_{05}^{(n)} + \partial_0 \epsilon_5^{(n)}, \\
B_{ij}^{(n)} & \rightarrow B_{ij}^{(n)} + \partial_i \epsilon_j^{(n)} - \partial_j \epsilon_i^{(n)}, \\
B_{i5}^{(n)} & \rightarrow B_{i5}^{(n)} + n \frac{R}{\epsilon_i^{(n)}} - \partial_5 \epsilon_5^{(n)},
\end{align*}
\]

(15)

however, they can be written as the following compact expressions

\[
\begin{align*}
\delta B_{0\mu}^{(0)} & = \partial_\mu \epsilon_0^{(0)} - \partial_\nu \epsilon_\nu^{(0)}, \\
\delta B_{\mu\nu}^{(n)} & = \partial_\mu \epsilon_\nu^{(n)} - \partial_\nu \epsilon_\mu^{(n)}, \\
\delta B_{\mu5}^{(n)} & = \frac{n}{R} \epsilon_\mu^{(n)} - \partial_\mu \epsilon_5^{(n)},
\end{align*}
\]

(16)

we can observe from (16) that by fixing the following gauge

\[
\epsilon_\mu^{(n)} = \frac{R}{n} (\partial_\mu \epsilon_5^{(n)} - B_{\mu5}^{(n)}),
\]

(17)

we find that the fields \(B_{\mu\nu}^{(n)}\) transforms as

\[
\delta B_{\mu\nu}^{(n)} = -\partial_\mu B_{\nu5}^{(n)} + \partial_\nu B_{\mu5}^{(n)}.
\]

(18)

Therefore, by taking into account (17) and (15) in the effective Lagrangian (3) we obtain

\[
\mathcal{L} = \frac{1}{2 \times 3!} H_{\mu\nu\lambda}^{(0)} H_{\mu\nu\lambda}^{(0)} + \sum_{n=1}^{\infty} \left[ \frac{1}{2 \times 3!} H_{\mu\nu\lambda}^{(n)} H_{\mu\nu\lambda}^{(n)} + \frac{1}{4} \left( \frac{n}{R} \right)^2 B_{\mu\nu}^{(n)} B_{\mu\nu}^{(n)} \right],
\]

(19)

where we can observe that the fields \(B_{\mu5}^{(n)}\) has been absorbed and therefore they are identified as a pseudo-Goldstone bosons. It is important to remark, that also there are present pseudo-Goldstone bosons in 5D-Maxwell and 5D-Stueckelberg theories with a compact dimension [16, 19]. This fact, show a close relation among Maxwell theory and Kalb-Ramond theory.

Now we will procedure to calculate the Dirac brackets among the physical fields. For this aim, we observe in the constraints that there are not mixed terms of the zero modes with the \(kk\)-excitations, thus, we can calculate the Dirac brackets independently for each case. First, we will calculate the Dirac brackets for the zero-mode, then for the \(kk\)-excitations. We need to remember that all the constraints are of first class, hence, we need to fix the gauge in order to obtain a set of second class constraints. Because the constraints are reducible, we introduce auxiliary variables by using the phase space extension procedure [14], thus we will work with the following set of constraints

\[
\begin{align*}
\chi_1^{(0)} & \equiv \Pi_{0i}^{(0)}, \quad \chi_2^{(0)} \equiv B_{0i}^{(0)}, \\
\chi_3^{(0)} & \equiv 2\partial_\mu \Pi_{ij}^{(0)} + \partial_\mu p_{ij}^{(0)}, \quad \chi_4^{(0)} \equiv \partial_\mu B_{ij}^{(0)} + \partial_\mu q_{ij}^{(0)},
\end{align*}
\]

(20)

where \(q_{ij}(0) p_{ij}(0)\) are auxiliary fields satisfying the following relations

\[
\{ q_{ij}^{(0)}(x), p_{ij}^{(0)}(z) \} = \delta^3(x - z).
\]

(21)
It is important to remark, that the introduction of the these auxiliary variables converts the constraints in a set of irreducible constraints, therefore it is possible to calculate the Dirac brackets of the theory. In this way, we obtain the following matrix whose entries are the Poisson brackets among the constraints \((20)\) given by

\[
(C^{(0)}_{\alpha\beta}) = 
\begin{pmatrix}
0 & -\frac{1}{2}\delta^i_j & 0 & 0 \\
\frac{1}{2}\delta^i_j & 0 & 0 & 0 \\
0 & 0 & 0 & -\delta^i_j \nabla^2 \\
0 & 0 & \delta^i_j \nabla^2 & 0
\end{pmatrix}
\delta^3(x - z),
\]

(22)

where its inverse is given by

\[
(C^{\alpha\beta})^{-1} = 
\begin{pmatrix}
0 & 2\delta_{ij} & 0 & 0 \\
-2\delta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & \delta^i_j \nabla^2 \\
0 & 0 & -\delta^i_j \nabla^2 & 0
\end{pmatrix}
\delta^3(x - z).
\]

(23)

In this manner, the Dirac brackets of two functionals \(A, B\) defined on the phase space, are expressed by

\[
\{F(x), G(z)\}_D \equiv \{F(x), G(z)\} + \int d^2u d^2w \{F(x), \xi_\alpha(u)\} C^{\alpha\beta} \{\xi_\beta(w), G(z)\},
\]

where \(\{F(x), G(z)\}\) is the Poisson bracket between two functionals \(F, G\), and \(\xi_\alpha = (\chi_1, \chi_2, \chi_3, \chi_4)\) represent the second class constraints. By using this fact, we obtain the following nonzero Dirac’s brackets for the zero-mode

\[
\{B_{ij}^{(0)}(x), \Pi_{kl}^{ij}(z)\}_D = \frac{1}{2}[\delta^i_k \delta^j_l - \delta^i_l \delta^j_k] - \frac{1}{\nabla^2}(\delta^i_k \partial^j \partial^l - \delta^i_l \partial^j \partial^k - \delta^i_j \partial^k \partial^l + \delta^i_j \partial^k \partial^l)] \delta^3(x - z).
\]

(24)

Furthermore, the Dirac brackets among physical and auxiliary variables vanish

\[
\{q^{(0)}(x), p_{(0)}(z)\}_D = 0,
\]

\[
\{q^{(0)}(x), \Pi_{ij}^{(0)}(z)\}_D = 0,
\]

\[
\{q^{(0)}(x), B_{ij}^{(0)}(z)\}_D = 0,
\]

\[
\{B_{kl}^{(0)}(x), p_{(0)}(z)\}_D = 0,
\]

\[
\{\Pi_{ij}^{(0)}(x), p_{(0)}(z)\}_D = 0.
\]

(25)

We are able to observe that the Dirac brackets are independent of the auxiliary variables \([14]\).

Now, we will compute the Dirac brackets for the \(kk\)-excitations. Just as it was performed above, we fix the gauge and also we will introduce auxiliary variables; we need to remember that for the constraints of the \(kk\)-excitations there are reducibility conditions as well. In this manner, we will
work with the following set of independent second class constraints

\[
\begin{align*}
\chi^1_{(n)} &= \Pi^0_{(n)}, & \chi^2_{(n)} &= B_{(n)}^0, \\
\chi^3_{(n)} &= \Pi_{(n)}^{05}, & \chi^4_{(n)} &= B_{(n)}^{05}, \\
\chi^5_{(n)} &= 2\partial_j\Pi^j_{(n)} + \frac{n}{R}\Pi^{5j}_0 + \partial^ip_{(n)}, & \chi^6_{(n)} &= \partial^iB^{ij}_{(n)} + \partial_iq^{(n)}, \\
\chi^7_{(n)} &= 2\partial_j\Pi^{5j}_0, & \chi^8_{(n)} &= \partial^iB_{(n)}^{ij},
\end{align*}
\]

just as above, the auxiliary fields \(q_{(n)}\) and \(p_{(n)}\) satisfy

\[
\{q^{(n)}(x), p_{(n)}(z)\} = \delta^3(x - z).
\]

Therefore, the non-zero Poisson brackets among the constraints are given by

\[
\begin{align*}
\{\chi^1_{(n)}(x), \chi^2_{(n)}(z)\} &= -\frac{1}{2}\delta^i\delta^3(x - z), \\
\{\chi^3_{(n)}(x), \chi^4_{(n)}(z)\} &= -\frac{1}{2}\delta^3(x - z), \\
\{\chi^5_{(n)}(x), \chi^6_{(n)}(z)\} &= -\delta^i\partial_j\partial^i\delta^3(x - z), \\
\{\chi^6_{(n)}(x), \chi^8_{(n)}(z)\} &= \frac{n}{R}\partial^3(x - z), \\
\{\chi^7_{(n)}(x), \chi^8_{(n)}(z)\} &= -\partial_i\partial^3(x - z),
\end{align*}
\]

thus, we obtain the following matrix

\[
\left( C^{(n)}_{(n)} \right) = \begin{pmatrix}
0 & -\frac{1}{2}\delta^i_j & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\delta^i_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta^i_j\nabla^2 & 0 & \frac{n}{R}\partial^i \\
0 & 0 & 0 & 0 & \delta^i_j\nabla^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nabla^2 \\
0 & 0 & 0 & 0 & 0 & -\frac{n}{R}\partial^i & 0 & \nabla^2 & 0
\end{pmatrix}
\]

where its inverse is given by

\[
\left( C^{-1}_{(n)} \right) = \begin{pmatrix}
0 & 2\delta^i_j & 0 & 0 & 0 & 0 & 0 & 0 \\
-2\delta^i_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \delta^i_j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \delta^i_j & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{n}{R}\partial^i & 0 \frac{1}{\nabla^2} \\
0 & 0 & 0 & 0 & 0 & \frac{n}{R}\partial^i & 0 & \frac{1}{\nabla^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\nabla^2} & 0
\end{pmatrix}
\]

In this way, we obtain the following non-zero Dirac brackets

\[
\begin{align*}
\{B^0_{(n)}(x), \Pi^p_{(n)}(z)\}_D &= \delta^i\partial^3(x - z), \\
\{B^i_{(n)}(x), \Pi^k_{(n)}(z)\}_D &= \frac{1}{2}\delta^k_i\delta^l_j - \delta^k_i\delta^l_j - \frac{1}{\nabla^2}(\delta^k_i\partial^j\partial_j - \delta^k_i\partial^j\partial_j - \delta^k_i\partial^j\partial_i + \delta^k_i\partial^j\partial_i))\delta^3(x - z).
\end{align*}
\]
and the Dirac brackets between physical and auxiliary variables vanish as expected, this is

\[ \{ q^{(n)}(x), p_{(n)}(z) \}_D = 0, \]
\[ \{ q^{(n)}(x), \Pi^{ij}_{(n)}(z) \}_D = 0, \]
\[ \{ q^{(n)}(x), B^{(n)}_{ij}(z) \}_D = 0, \]
\[ \{ B^{(n)}_{ki}(x), p_{(n)}(z) \}_D = 0, \]
\[ \{ \Pi^{ij}_{(n)}(x), p_{(n)}(z) \}_D = 0. \] (30)

### III. 5D PROCA KALB-RAMOND THEORY WITH A COMPACT DIMENSION

In this section we shall analyze the following action

\[ L = \frac{1}{2 \times 3!} H_{MN} H^{MN} - \frac{1}{4} m^2 B_{MN} B^{MN}, \] (31)

where the fields \( B_{MN} \) and \( H^{MNK} \) are defined as above. By performing the 4+1 decomposition, the Lagrangian \((31)\) takes the form

\[ L = \frac{1}{2 \times 3!} H^{(0)}_{\mu\nu\lambda} H^{(0)}_{\mu\nu\lambda} - \frac{1}{4} m^2 B^{(0)}_{\mu\nu} B^{(0)}_{\mu\nu} - \frac{1}{4} m^2 B^{(n)}_{\mu\nu} B^{(n)}_{\mu\nu} - \frac{1}{2} m^2 B^{(0)}_{5\mu} B^{(0)}_{5\mu}, \] (32)

thus, by taking into account the expansion \((2)\) and integrating over the compact dimension we obtain the following effective Lagrangian

\[ L = \frac{1}{2 \times 3!} H^{(0)}_{\mu\nu\lambda} H^{(0)}_{\mu\nu\lambda} - \frac{1}{4} m^2 B^{(0)}_{\mu\nu} B^{(0)}_{\mu\nu} + \sum_{n=1}^{\infty} \left[ \frac{1}{2 \times 3!} H^{(n)}_{\mu\nu\lambda} H^{(n)}_{\mu\nu\lambda} - \frac{1}{4} m^2 B^{(n)}_{\mu\nu} B^{(n)}_{\mu\nu} - \frac{1}{2} m^2 B^{(n)}_{5\mu} B^{(n)}_{5\mu} \right]. \] (33)

In order to perform the Hamiltonian analysis, we observe that the Hessian

\[ \frac{\partial^2 L}{\partial (\partial_0 B^{(0)}_{K\lambda}) \partial (\partial_0 B^{(0)}_{\alpha\beta})} = \frac{1}{4} (g^{\alpha\lambda} g^{\beta\rho} - g^{\alpha\rho} g^{\beta\lambda}), \]
\[ \frac{\partial^2 L}{\partial (\partial_0 B^{(n)}_{K\lambda}) \partial (\partial_0 B^{(n)}_{L\beta})} = \frac{1}{4} (g^{LK} g^{HM} - g^{LM} g^{HK}) + \frac{1}{4} \delta_5^L \delta_5^M g^{LK}, \] (34)

has a rank=4 and 4\((k-1)\) null vectors, thus we expect 4\((k-1)\) primary constraints. Therefore, from the definition of the momenta \((\Pi^{0i}_{(0)}, \Pi^{ij}_{(0)}, \Pi^{0i}_{(n)}, \Pi^{ij}_{(n)}, \Pi^{05}_{(n)}, \Pi^{ij}_{(n)})\) canonically conjugate to \((B^{(0)}_{0i}, B^{(0)}_{ij}, B^{(n)}_{0i}, B^{(n)}_{ij}, B^{(n)}_{05}, B^{(n)}_{ij})\) we obtain

\[ \Pi^{0i}_{(0)} = 0, \quad \Pi^{ij}_{(0)} = \frac{1}{2} H^{0ij}_{(0)}, \]
\[ \Pi^{0i}_{(n)} = 0, \quad \Pi^{ij}_{(n)} = \frac{1}{2} H^{0ij}_{(n)}, \quad \Pi^{05}_{(n)} = 0, \quad \Pi^{ij}_{(n)} = \frac{1}{2} (\partial^0 B^{ij}_{(n)} + \partial^i B^{50}_{(n)} - \frac{n}{R} B^{0i}_{(n)}), \] (35)

thus, we identify the following 4\(k-1\) primary constraints

\[ \phi^{0i}_{(0)} \equiv \Pi^{0i}_{(0)} \approx 0, \]
\[ \phi^{0i}_{(n)} \equiv \Pi^{0i}_{(n)} \approx 0, \quad \phi^{05}_{(n)} \equiv \Pi^{05}_{(n)} \approx 0. \] (36)
By using the definition of the momenta, we obtain the canonical Hamiltonian

\[
H_c = \int d^3x \left[ 2B^{(0)}_{ij} \partial_j \Pi^{(0)}_{ij} + \Pi^{(0)}_{ij} \Pi^{(0)}_{ij} - \frac{1}{2 \times 3!} H^{(0)}_{ijk} H^{(0)}_{ijk} + \frac{1}{2} m^2 B^{(0)}_{ij} B^{(0)}_{ij} + \frac{1}{4} m^2 B^{(0)}_{ij} B^{(0)}_{ij} \right]
+ \sum_{n=1}^{\infty} \left[ 2B^{(n)}_{ij} \partial_j \Pi^{(n)}_{ij} + \Pi^{(n)}_{ij} \Pi^{(n)}_{ij} - \frac{1}{2 \times 3!} H^{(n)}_{ijk} H^{(n)}_{ijk} + \frac{1}{2} m^2 B^{(n)}_{ij} B^{(n)}_{ij} + \frac{1}{4} m^2 B^{(n)}_{ij} B^{(n)}_{ij} \right]
+ \frac{1}{2} m^2 B^{(n)}_{05} B^{(n)}_{05} + \frac{1}{2} m^2 B^{(n)}_{i5} B^{(n)}_{i5} + 2 \Pi^{(n)}_{ij} \Pi^{(n)}_{ij} + 2 B^{(n)}_{ij} \partial_i \Pi^{(n)}_{ij} + \frac{n}{R} 2 B^{(n)}_{ij} \Pi^{(n)}_{ij} \left[ 1 \right]
\]

and the primary Hamiltonian is given by

\[
H_1 = H_c + \int d^3x \left[ a^{(0)}_0 \phi^{(0)}_0 + \sum_{n=1}^{k-1} \left( a^{(n)}_0 \phi^{(n)}_0 + a^{(n)}_0 \phi^{(n)}_0 \right) \right]
\]

where \( a^{(0)}_0, a^{(n)}_0 \) and \( a^{(n)}_0 \) are Lagrange multipliers enforcing the constraints. The fundamental Poisson brackets of the theory are as usual

\[
\{ B^{(0)}_{\alpha \beta} (x), \Pi^{(0)}_{\mu \nu} (z) \} = \frac{1}{2} \left( \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} - \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha} \right) \delta^3 (x - z),
\]

\[
\{ B^{(n)}_{iL} (x), \Pi^{(n)}_{MN} (z) \} = \frac{1}{2} \left( \delta^{M}_{i} \delta^{N}_{L} - \delta^{M}_{L} \delta^{N}_{i} \right) \delta^3 (x - z).
\]

In order to observe if there are more constraints, we demand consistency conditions for the primary constraints and we obtain the following secondary constraints

\[
\dot{\phi}^{(0)}_0 (x) = \{ \phi^{(0)}_0 (x), H_1 (z) \} = 2 \partial_j \Pi^{(0)}_{ij} (x) + m^2 B^{(0)}_{ij} (x) \approx 0,
\]

\[
\dot{\phi}^{(n)}_0 (x) = \{ \phi^{(n)}_0 (x), H_1 (z) \} = 2 \partial_j \Pi^{(n)}_{ij} (x) + m^2 B^{(n)}_{ij} (x) + \frac{n}{R} 2 \Pi^{(n)}_{ij} (x) \approx 0,
\]

\[
\dot{\phi}^{(n)}_0 (x) = \{ \phi^{(n)}_0 (x), H_1 (z) \} = 2 \partial_j \Pi^{(n)}_{ij} (x) + m^2 B^{(n)}_{ij} (x) \approx 0,
\]

for this theory there are not third constraints. Therefore, the full set of constraints for the theory is given by

\[
\phi^{(0)}_0 = \Pi^{(0)}_0 \approx 0,
\]

\[
\psi^{(0)}_0 = 2 \partial_j \Pi^{(0)}_{ij} + m^2 B^{(0)}_{ij} \approx 0,
\]

\[
\phi^{(n)}_0 = \Pi^{(n)}_0 \approx 0,
\]

\[
\psi^{(n)}_0 = 2 \partial_j \Pi^{(n)}_{ij} + m^2 B^{(n)}_{ij} + \frac{n}{R} 2 \Pi^{(n)}_{ij} \approx 0,
\]

\[
\phi^{(n)}_0 = \Pi^{(n)}_0 \approx 0,
\]

\[
\psi^{(n)}_0 = 2 \partial_j \Pi^{(n)}_{ij} + m^2 B^{(n)}_{ij} \approx 0.
\]

We can observe that the constraints given above are of second class and there are not reducibility conditions. In fact, the term of mass breaks down both, the gauge invariance of the kinetic term and the reducibility conditions among the constraints. Therefore, the counting of physical degrees of freedom is carry out in the following form; there are \( 20k - 8 \) dynamical variables and \( 8k - 2 \) independent second class constraints, thus there are

\[
DF = \frac{1}{2} \left[ 20k - 8 - (8k - 2) \right] = 6k - 3
\]
degrees of freedom. We observe that if we take \( k = 1 \), then we obtain \( DF = 3 \) as expected. On the other hand, we can observe that each excitation contribute with 6 degrees of freedom.

Now we will calculate the Dirac brackets of the theory. For this aim, we rewrite the constraints in the following form

\[
\begin{align*}
\chi^1_{(0)} &= \Pi^0_{(0)}, \\
\chi^2_{(0)} &= 2\partial_i \Pi^i_{(0)} + m^2 B^0_{(0)}, \\
\chi^1_{(n)} &= \Pi^0_{(n)}, \\
\chi^2_{(n)} &= 2\partial_i \Pi^i_{(n)} + m^2 B^0_{(n)} + \frac{m}{R^2} 2\Pi^5_{(n)}, \\
\chi^3_{(n)} &= \Pi^0_{(n)}, \\
\chi^4_{(n)} &= 2\partial_i \Pi^5_{(n)} + m^2 B^0_{5(n)},
\end{align*}
\]

we observe that the zero-modes and the excited modes are not mixed in the constraints, hence, we will calculate the Dirac brackets independently as was performed in above section. For the zero-mode we obtain

\[
\{\chi^1_{(0)}(x), \chi^2_{(0)}(z)\} = \frac{1}{2} m^2 \delta^i_j \delta^3(x - z),
\]

thus, the matrix whose entries are the Poisson brackets among the second class constraints for the zero-mode take the form

\[
(C^{(0)}_{\alpha \beta}) = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \frac{1}{2} m^2 \delta^i_j \delta^3(x - y),
\]

and it has an inverse given by

\[
(C^{\alpha \beta}_{(0)}) = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} 2 m^2 \delta^i_j \delta^3(x - y).
\]

In this manner, the Dirac brackets of two functionals \( A, B \) defined on the phase space, is expressed by

\[
\{F(x), G(z)\}_D \equiv \{F(x), G(z)\} + \int d^2u d^2w \{F(x), \xi_{\alpha}(u)\} C^{\alpha \beta} \{\xi_{\beta}(w), G(z)\},
\]

where \( \{F(x), G(z)\} \) is the Poisson bracket between two functionals \( F, G \), and \( \xi_{\alpha} = (\chi_1, \chi_2) \) represent the second class constraints. By using this fact, we obtain the following nonzero Dirac’s brackets for the zero-mode

\[
\{B_{0i}^{(0)}(x), B_{0q}^{(0)}(z)\}_D = -\frac{1}{m^2} (\delta_{ip} \delta^j_q - \delta_{iq} \delta^j_p) \partial_j \delta^3(x - z)
\]

\[
\{B_{0i}^{(0)}(x), \Pi_{0q}^{(0)}(z)\}_D = \delta^j_i \delta^3(x - z).
\]

Now, we will calculate the Dirac brackets for the \( kk \)-excitations. For this aim, we calculate the Poisson brackets among the second class constraints of the \( kk \)-excitations. The nonzero brackets are given by

\[
\{\chi^1_{(n)}, \chi^2_{(n)}\} = \frac{1}{2} m^2 \delta^i_j \delta^3(x - z),
\]

\[
\{\chi^3_{(n)}, \chi^4_{(n)}\} = \frac{1}{2} m^2 \delta^3(x - z),
\]

\[
\{\chi^1_{(n)}, \chi^3_{(n)}\} = \frac{1}{2} m^2 \delta^3(x - z),
\]

\[
\{\chi^2_{(n)}, \chi^4_{(n)}\} = \frac{1}{2} m^2 \delta^3(x - z),
\]

\[
\{\chi^1_{(n)}, \chi^4_{(n)}\} = \frac{1}{2} m^2 \delta^3(x - z),
\]

\[
\{\chi^2_{(n)}, \chi^3_{(n)}\} = \frac{1}{2} m^2 \delta^3(x - z).
\]
thus, the matrix whose entries are the poisson brackets among the second class constraints is given by

\[
(C_{\alpha \beta})^{(n)} = \begin{pmatrix}
0 & \delta^i_j & 0 & 0 \\
-\delta^i_j & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix} \frac{1}{2} m^2 \delta^3(x - z),
\]

this matrix has as inverse

\[
(C^{\alpha \beta})^{(n)} = \begin{pmatrix}
0 & -\delta^i_j & 0 & 0 \\
\delta^i_j & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix} \frac{2}{m^2} \delta^3(x - z).
\]

In this manner, we obtain the following nonzero Dirac brackets for the $kk$-excitations

\[
\{ B_0^{(n)}(x), B_{pq}^{(n)}(z) \}_D = -\frac{1}{m^2} (\delta_{ij} \delta_{pq} - \delta_{iq} \delta_{pj}) \partial_j \delta^3(x - z),
\]

\[
\{ B_0^{(n)}(x), \Pi^{pq}_{(n)}(z) \}_D = \delta_{ij} \delta^3(x - z),
\]

\[
\{ B_0^{(n)}(x), B_{5q}^{(n)}(z) \}_D = \frac{n}{R m^2} \delta_{ij} \delta^3(x - z).
\]

(46)

Therefore, we have computed the Dirac brackets of the theory and we can perform its canonical quantization.

\section*{IV. 5D STÜECKELBERG KALB-RAMOND THEORY WITH A COMPACT DIMENSION}

Now, we will study the following action

\[
\mathcal{L} = \frac{1}{2 \times 3!} H_{MNL} H^{MNL} - \frac{1}{4} (m B_{MN} - \Phi_{MN})(m B^{MN} - \Phi^{MN}),
\]

(47)

where the field strength $H_{MNL}$ are defined as above, $\Phi_N$ is the Stüeckelberg field and $\Phi_{MN} = \partial_M \Phi_N - \partial_N \Phi_M$. Just as in above sections, we can expand the fields in terms of the following series

\[
\Phi_\mu(x, y) = \frac{1}{\sqrt{2 \pi R}} \phi_\mu^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^\infty \phi^{(n)}_\mu(x) \cos \left( \frac{n y}{R} \right),
\]

\[
\Phi_5(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^\infty \phi_5^{(n)}(x) \sin \left( \frac{n y}{R} \right),
\]

\[
B_{\mu \nu}(x, y) = \frac{1}{\sqrt{2 \pi R}} B_{\mu \nu}^{(0)}(x) + \frac{1}{\sqrt{\pi R}} \sum_{n=1}^\infty B^{(n)}_{\mu \nu}(x) \cos \left( \frac{n y}{R} \right),
\]

\[
B_{\mu 5}(x, y) = \frac{1}{\sqrt{\pi R}} \sum_{n=1}^\infty B^{(n)}_{\mu 5}(x) \sin \left( \frac{n y}{R} \right).
\]

(48)
By performing the 4+1 decomposition in the Lagrangian \[^{17}\] and integrating over the fifth dimension, we obtain the following effective Lagrangian

\[
\mathcal{L} = \frac{1}{2 \times 3!} H^{(0)}_{\mu \nu \lambda} H^{\mu \nu \lambda}_{(0)} - \frac{1}{4} (m B^{(0)}_{\mu \nu} - \Phi^{(0)}_{\mu \nu}) (m B^{\mu \nu}_{(0)} - \Phi^{\mu \nu}_{(0)}) \\
+ \sum_{n=1}^{\infty} \left[ \frac{1}{2 \times 3!} H^{(n)}_{\mu \nu \lambda} H^{\mu \nu \lambda}_{(n)} - \frac{1}{4} (m B^{(n)}_{\mu \nu} - \Phi^{(n)}_{\mu \nu}) (m B^{\mu \nu}_{(n)} - \Phi^{\mu \nu}_{(n)}) \\
- \frac{1}{2} \left( m B^{(n)}_{\mu \nu} - \partial_{\mu} \Phi^{(n)}_{\nu} - \frac{n}{R} \Phi^{(n)}_{\mu} \right) \left( m B^{(n)}_{\mu \nu} - \partial_{\nu} \Phi^{(n)}_{\mu} + \frac{n}{R} \Phi^{(n)}_{\mu} \right) \\
+ \frac{1}{4} \left( \partial_{\mu} B^{(n)}_{\nu \rho} + \partial_{\nu} B^{(n)}_{\rho \mu} - \frac{n}{R} B^{(n)}_{\mu \nu} \right) \left( \partial^{\mu} B^{(n)}_{\nu \rho} + \partial^{\nu} B^{(n)}_{\rho \mu} - \frac{n}{R} B^{(n)}_{\mu \nu} \right) \right].
\]

We can observe that the effective Lagrangian describes a 4D Stueckelberg Kalb-Ramond theory plus a tower of \(k k\)-excitations. For this theory, the Hessian of the Lagrangian \[^{18}\] given by

\[
\frac{\partial^2 \mathcal{L}}{\partial (\partial_{\alpha} \Phi_{L}) \partial (\partial_{\beta} \Phi_{L})} = (g_{L0} g_{0 M} - g_{L M}) + \delta_{M}^{S} \delta_{5 L},
\]

\[
\frac{\partial^2 \mathcal{L}}{\partial (\partial_{\alpha} B^{(m)}_{M \nu}) \partial (\partial_{\beta} B^{(h)}_{L \mu})} = \frac{1}{4} (g_{L M} g_{M K} - g_{L K} g_{K M}) + \frac{1}{4} \delta_{M}^{S} \delta_{5 L} g_{L K},
\]

has rank \(8 k - 7\) and \(5 k - 1\) null vectors, which means that we expect \(5 k - 1\) primary constraints. Therefore, from the definition of the momenta \((\Pi_{(0)}^{0}, \Pi_{(0)}^{i}, \Pi_{(0)}^{ij}, \Pi_{(n)}^{n}, \Pi_{(n)}^{ij}, \Pi_{(n)}^{n \mu}, \Pi_{(n)}^{n \mu \lambda}, \Pi_{(n)}^{n i}, \Pi_{(n)}^{05}, \Pi_{(n)}^{55}, \Pi_{(n)}^{05}, \Pi_{(n)}^{55})\) canonically conjugate to \((\Phi^{0}_0, \Phi^{0}_i, B^{0}_0, B^{0}_i, B^{0}_0, B^{0}_i, B^{0}_0, B^{0}_i, B^{0}_0, B^{0}_i)\) we obtain

\[
\Pi_{(0)}^{0} = 0, \quad \Pi_{(0)}^{i} = m B_{0i}^{(0)} - \Phi_{0i}^{(0)}, \quad \Pi_{(0)}^{ij} = 0, \quad \Pi_{(n)}^{0} = 0, \quad \Pi_{(n)}^{ij} = m B_{0i}^{(n)} - \Phi_{0i}^{(n)}, \quad \Pi_{(n)}^{ij} = m B_{0i}^{(n)} - \Phi_{0i}^{(n)} - \frac{n}{R} \Phi_{0i}^{(n)},
\]

\[
\Pi_{(n)}^{0} = 0, \quad \Pi_{(n)}^{ij} = \frac{1}{2} H_{(n)}^{0ij}, \quad \Pi_{(n)}^{5} = \frac{1}{2} (\partial^{0} B_{5}^{(n)} + \partial^{0} B_{5}^{(n)} - \frac{n}{R} B_{0}^{(n)}), \quad \Pi_{(n)}^{5} = \frac{1}{2} (\partial^{0} B_{5}^{(n)} + \partial^{0} B_{5}^{(n)} - \frac{n}{R} B_{0}^{(n)}),
\]

thus, we identify the following \(5 k - 1\) primary constraints

\[
\phi_{(0)}^{0} \equiv \Pi_{(0)}^{0} \approx 0, \quad \phi_{0i}^{(0)} \equiv \Pi_{(0)}^{0i} \approx 0,
\]

\[
\phi_{(n)}^{0} \equiv \Pi_{(n)}^{0} \approx 0, \quad \phi_{0i}^{(n)} \equiv \Pi_{(n)}^{0i} \approx 0, \quad \phi_{(n)}^{05} \equiv \Pi_{(n)}^{05} \approx 0, \quad \phi_{(n)}^{55} \equiv \Pi_{(n)}^{55} \approx 0.
\]

On the other hand, by using the definition of the momenta we identify the canonical Hamiltonian given by

\[
H_{c} = \int d^{3}x \left[ B_{0i}^{(0)} (m \Pi_{(0)}^{0} + 2 \partial_{i} \Pi_{(0)}^{ij}) - \Phi_{0i}^{(0)} \partial_{i} \Pi_{(0)}^{ij} - \frac{1}{2} \Pi_{(0)}^{0} \Pi_{(0)}^{0} + \Pi_{(0)}^{ij} \Pi_{(0)}^{ij} - \frac{1}{2 \times 3} H_{(0)}^{ij} H_{(0)}^{0ij} \\
+ \frac{1}{4} (m B_{ij}^{(0)} - \Phi_{ij}^{(0)}) (m B_{ij}^{(0)} - \Phi_{ij}^{(0)}) + \sum_{n=1}^{\infty} \left[ B_{0i}^{(n)} (m \Pi_{(n)}^{0} + 2 \partial_{i} \Pi_{(n)}^{ij}) - \Phi_{0i}^{(n)} \partial_{i} \Pi_{(n)}^{ij} \\
- \frac{1}{2} \Pi_{(n)}^{0} \Pi_{(n)}^{0} + \Pi_{(n)}^{ij} \Pi_{(n)}^{ij} - \frac{1}{2 \times 3} H_{(n)}^{ij} H_{(n)}^{0ij} + \frac{1}{4} (m B_{ij}^{(n)} - \Phi_{ij}^{(n)}) (m B_{ij}^{(n)} - \Phi_{ij}^{(n)}) \\
- \frac{1}{2} \Pi_{(n)}^{0} \Pi_{(n)}^{0} + \Pi_{(n)}^{ij} \Pi_{(n)}^{ij} + B_{0i}^{(n)} (m \Pi_{(n)}^{0} + 2 \partial_{i} \Pi_{(n)}^{ij}) + \frac{n}{R} (2 B_{ij}^{(n)} \Pi_{(n)}^{0} - \Phi_{0}^{(n)} \Pi_{(n)}^{0}) \\
+ \frac{1}{2} \left( m B_{ij}^{(n)} - \partial_{i} \Phi_{ij}^{(n)} - \frac{n}{R} \Phi_{ij}^{(n)} \right) \left( m B_{ij}^{(n)} - \partial_{j} \Phi_{ij}^{(n)} - \frac{n}{R} \Phi_{ij}^{(n)} \right) \\
+ \frac{1}{4} \left( \partial_{i} B_{5}^{(n)} + \partial_{j} B_{5}^{(n)} - \frac{n}{R} B_{ij}^{(n)} \right) \left( \partial_{i} B_{5}^{(n)} + \partial_{j} B_{5}^{(n)} - \frac{n}{R} B_{ij}^{(n)} \right) \right] \right],
\]

(53)
and the primary Hamiltonian takes the following form

\[ H_1 = H_c + \int d^4x \left[ a_0^{(0)} \phi_0^{(0)} + a_i^{(0)} \phi_i^{(0)} + \sum_{n=1}^{k-1} \left( a_n^{(n)} \phi_n^{(n)} + a_{0i}^{(n)} \phi_{0i}^{(n)} + a_{05}^{(n)} \phi_{05}^{(n)} \right) \right], \tag{54} \]

where \( a_0^{(0)}, a_i^{(0)}, a_0^{(n)}, a_i^{(n)}, \) and \( a_{05}^{(n)} \) are Lagrange multipliers enforcing the constraints. For this theory, the fundamental Poisson brackets are given by

\[
\begin{align*}
\{ \Phi^{(0)}_\nu(x), \Pi_{\nu(0)}^\mu(z) \} &= \delta_\nu^\mu \delta^3(x-z), \\
\{ B_{\alpha\beta}^{(0)}(x), \Pi_{(0)}^{\mu\nu}(z) \} &= \frac{1}{2} (\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu) \delta^3(x-z), \\
\{ \Phi^{(i)}_H(x), \Pi_{(n)}^L(z) \} &= \delta_\delta^L \delta^3(x-z), \\
\{ B_{H_L}^{(i)}(x), \Pi_{(n)}^{MN}(z) \} &= \frac{1}{2} \delta_\delta^I (\delta_H^M \delta_L^N - \delta_L^M \delta_H^N) \delta^3(x-z). \tag{55} \end{align*}
\]

On the other hand, by demanding consistency among the constraints, we find the following secondary constraints

\[
\begin{align*}
\psi_0^{(n)} &= \partial_i \Pi_{(n)}^i + \frac{n}{R} \Pi_{(n)}^5 \approx 0, \\
\psi_{0i}^{(n)} &= m \Pi_{(n)}^i + 2 \partial_j \Pi_{(n)}^{ij} + \frac{n}{R} 2 \Pi_{(n)}^5 \approx 0, \\
\psi_{05}^{(n)} &= m \Pi_{(n)}^5 + 2 \partial_j \Pi_{(n)}^{j5} \approx 0. \tag{56} \end{align*}
\]

For this theory, there are not third constraints. In this manner, we have found the following set of constraints

\[
\begin{align*}
\phi_0^{(0)} &= \Pi_{(0)}^0 \approx 0, \\
\phi_{0i}^{(0)} &= \Pi_{(0)}^{0i} \approx 0, \\
\psi_{0}^{(0)} &= \partial_i \Pi_{(0)}^i \approx 0, \\
\psi_{0}^{0} &= m \Pi_{(0)}^0 + 2 \partial_j \Pi_{(0)}^{0j} \approx 0, \\
\phi_{(n)}^0 &= \Pi_{(n)}^0 \approx 0, \\
\phi_{(n)}^{0i} &= \Pi_{(n)}^{0i} \approx 0, \\
\phi_{(n)}^{05} &= \Pi_{(n)}^{05} \approx 0, \\
\psi_{(n)}^0 &= \partial_i \Pi_{(n)}^i + \frac{n}{R} \Pi_{(n)}^5 \approx 0, \\
\psi_{(n)}^{0} &= m \Pi_{(n)}^0 + 2 \partial_j \Pi_{(n)}^{0j} + \frac{n}{R} 2 \Pi_{(n)}^5 \approx 0, \\
\psi_{(n)}^{05} &= m \Pi_{(n)}^5 + 2 \partial_j \Pi_{(n)}^{j5} \approx 0. \tag{57} \end{align*}
\]

we are able to observe that all these \(10k - 2\) constraints are of first class. It is important to comment that the Stueckelberg’s field convert to Proca Kalb-Ramond theory in a full gauge theory. In the Proca Kalb-Ramond model studied in above section, there are only second class constraints and there are not reducibility relations among the constraints. In this Stueckelberg Kalb-Ramond theory there are only first class constraints and there are reducibility among the constraints in both, zero modes and \(kk\)-excitations. These reducibility conditions are given by the following \(k\) relations

\[
\begin{align*}
\partial_i \psi_{(0)}^{0i} - m \psi_{(0)}^{0} &= 0, \\
\partial_i \psi_{(n)}^{0i} - m \psi_{(n)}^{0} + \frac{n}{R} \psi_{(n)}^{05} &= 0. \tag{58} \end{align*}
\]
we observe if \(k\) thus, the following gauge transformations of the theory are obtained

\[
DF = \frac{1}{2}[30k - 10 - 2(9k - 2)] = 6k - 3,
\]

we observe if \(k = 1\), then there are 3 degrees of freedom as expected. In fact, Stueckelberg Kalb-Ramond and Proca Kalb-Ramond have the same number of physical degrees of freedom, however, the former is a full gauge theory while the latter is not. We also can observe, that for each excitation there is a contribution of 6 degrees of freedom, just as it is present in the Kalb-Ramond theory.

We have observed that Stueckelberg Kalb-Ramond is a reducible system with only first class constraints, this means that the theory is a gauge theory. Hence, we shall calculate the gauge transformations of the theory. For this aim, we define the following gauge generator

\[
G = \int \left[ \epsilon_0^{(0)} \phi_0^{(0)} + \epsilon_0^{(0)} \phi_0^{(n)} + \epsilon_0^{(0)} \psi_0^{(0)} + \epsilon_0^{(0)} \psi_0^{(n)} + \epsilon_0^{(n)} \phi_0^{(n)} 
+ \epsilon_0^{(n)} \phi_0^{(n)} + \epsilon_0^{(n)} \psi_0^{(n)} + \epsilon_0^{(n)} \psi_0^{(n)} + \epsilon_0^{(n)} \phi_0^{(n)} \right] \, d^3z.
\]

thus, the following gauge transformations of the theory are obtained

\[
\begin{align*}
\Phi_0^{(0)} & \rightarrow \Phi_0^{(0)} + \partial_0 \epsilon_0^{(0)}, \\
\Phi_i^{(0)} & \rightarrow \Phi_i^{(0)} - \partial_i \epsilon_0^{(0)} + m \epsilon_i^{(0)}, \\
B_{0i}^{(0)} & \rightarrow B_{0i}^{(0)} - \partial_0 \epsilon_i^{(0)}, \\
B_{ij}^{(0)} & \rightarrow B_{ij}^{(0)} + \partial_i \epsilon_j^{(0)} - \partial_j \epsilon_i^{(0)}, \\
\Phi_0^{(n)} & \rightarrow \Phi_0^{(n)} + \partial_0 \epsilon_0^{(n)}, \\
\Phi_i^{(n)} & \rightarrow \Phi_i^{(n)} - \partial_i \epsilon_0^{(n)} + m \epsilon_i^{(n)}, \\
\Phi_5^{(n)} & \rightarrow \Phi_5^{(n)} + \frac{n}{R} \epsilon_5^{(n)} - m \epsilon_5^{(n)}, \\
B_{0i}^{(n)} & \rightarrow B_{0i}^{(n)} - \partial_0 \epsilon_i^{(n)}, \\
B_{05}^{(n)} & \rightarrow B_{05}^{(n)} + \partial_0 \epsilon_5^{(n)}, \\
B_{ij}^{(n)} & \rightarrow B_{ij}^{(n)} + \partial_i \epsilon_j^{(n)} - \partial_j \epsilon_i^{(n)}, \\
B_{i5}^{(n)} & \rightarrow B_{i5}^{(n)} + \frac{n}{R} \epsilon_i^{(n)} - \partial_i \epsilon_5^{(n)},
\end{align*}
\]

we can write these gauge transformations in the following compact form

\[
\begin{align*}
\delta \Phi_0^{(0)} & = -\partial_0 \epsilon^{(0)} + m \epsilon_0^{(0)}, \\
\delta B_{0\nu}^{(0)} & = \partial_\nu \epsilon_0^{(0)} - \partial_\mu \epsilon_\mu^{(0)}, \\
\delta \Phi_0^{(n)} & = -\partial_0 \epsilon^{(n)} + m \epsilon_0^{(n)}, \\
\delta \Phi_5^{(n)} & = \frac{n}{R} \epsilon^{(n)} - m \epsilon_5^{(n)}, \\
\delta B_{0\nu}^{(n)} & = \partial_\nu \epsilon_0^{(n)} - \partial_\nu \epsilon_0^{(n)}, \\
\delta B_{05}^{(n)} & = \frac{n}{R} \epsilon^{(n)} - \partial_0 \epsilon_5^{(n)}. 
\end{align*}
\]
In this manner, the nonzero Poisson brackets among the constraints (66) are given by

\[ \epsilon^{(n)} = \frac{R}{n}(\epsilon_{5}^{(n)} - \Phi_{5}^{(n)}), \]
\[ \epsilon_{\mu}^{(n)} = \frac{R}{n}(\partial_{\mu} \epsilon_{5}^{(n)} - B_{\mu 5}^{(n)}), \]  
(63)
the fields transform like

\[ \delta \Phi_{\mu}^{(n)} = \frac{R}{n} \partial_{\mu} \Phi_{5} - B_{\mu 5}^{(n)}, \]
\[ \delta B_{\mu 5}^{(n)} = -\partial_{\mu} B_{\nu 5}^{(n)} + \partial_{\nu} B_{\mu 5}^{(n)}, \]  
(64)
under that fixed gauge the effective Lagrangian (44) is reduced to

\[ \mathcal{L} = \frac{1}{2 \times 3!} H_{\mu \nu \lambda}^{(0)} H_{(0)}^{\mu \nu \lambda} - \frac{1}{4} (m B_{\mu \nu}^{(0)} - \Phi_{\mu \nu}^{(0)}) (m B_{\mu \nu}^{\mu \nu} - \Phi_{\mu \nu}^{\mu \nu}) + \sum_{n=1}^{\infty} \left[ \frac{1}{2 \times 3!} H_{\mu \nu \lambda}^{(n)} H_{(n)}^{\mu \nu \lambda}ight. 
- \frac{1}{4} (m B_{\mu \nu}^{(n)} - \Phi_{\mu \nu}^{(n)}) (m B_{\mu \nu}^{\mu \nu} - \Phi_{\mu \nu}^{\mu \nu}) - \frac{1}{2} \left( \frac{n}{R} \right)^{2} \Phi_{\mu}^{(n)} \Phi_{\nu}^{(n)} + \frac{1}{4} \left( \frac{n}{R} \right)^{2} B_{\mu \nu}^{(n)} B_{\mu \nu}^{(n)} \right]. \]  
(65)
this means that the fields \( \Phi_{5}^{(n)} \) and \( B_{\mu 5}^{(n)} \) has been absorbed and they are identified as pseudo-Goldstone bosons, something similar is also present in the free 5D Stueckelberg theory [4].

On the other hand, because of the zero modes and the \( \text{kk}\)-excitations are not mixed in the constraints we procedure to calculate the Dirac brackets for all the modes. In fact, by fixing the gauge we have the following constraints for the zero mode

\[ \chi_{(0)}^{1} \equiv \Pi_{(0)}^{0}, \quad \chi_{(0)}^{2} \equiv \Phi_{(0)}^{0}, \]
\[ \chi_{(0)}^{3} \equiv \Pi_{(0)}^{0}, \quad \chi_{(0)}^{4} \equiv B_{(0)}^{0}, \]
\[ \chi_{(0)}^{5} \equiv \partial_{i} \Pi_{(0)}^{i}, \quad \chi_{(0)}^{6} \equiv \partial^{i} \Phi_{(0)}^{i}, \]
\[ \chi_{(0)}^{7} \equiv m \Pi_{(0)}^{i} + 2 \partial_{j} \Pi_{(0)}^{ij} + \partial^{i} p_{(0)}, \quad \chi_{(0)}^{8} \equiv \partial^{i} B_{(0)}^{ij} + \partial_{i} q_{(0)}, \]  
(66)
now the constraints are of second class and we have introduced the auxiliary variables \( q_{(0)} \) and \( p_{(0)} \) in order to have independent second class constraints. The auxiliary variables satisfy

\[ \{ q_{(0)}^{(x)}(z) \} = \delta^{3}(x - z). \]  
(67)
In this manner, the nonzero Poisson brackets among the constraints (66) are given by

\[ \{ \chi_{(0)}^{1}(x), \chi_{(0)}^{2}(z) \} = -\delta^{3}(x - z), \]
\[ \{ \chi_{(0)}^{3}(x), \chi_{(0)}^{4}(z) \} = -\frac{1}{2} \delta^{3}(x - z), \]
\[ \{ \chi_{(0)}^{5}(x), \chi_{(0)}^{6}(z) \} = -\partial_{i} \partial^{j} \delta^{3}(x - z), \]
\[ \{ \chi_{(0)}^{6}(x), \chi_{(0)}^{7}(z) \} = m \partial^{i} \delta^{3}(x - z), \]
\[ \{ \chi_{(0)}^{7}(x), \chi_{(0)}^{8}(z) \} = -\delta_{i} \partial_{j} \partial^{i} \delta^{3}(x - z), \]  
(68)
thus, we form the following matrix whose entries are given by the Poisson brackets

\[
\left( c^{(0)}_{\alpha\beta} \right) = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}\delta^i_j & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}\delta^i_j & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \nabla^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \nabla^2 & 0 & m\partial^i & 0 & 0 \\
0 & 0 & 0 & 0 & -m\partial^i & 0 & -\delta^i_j \nabla^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta^i_j \nabla^2 & 0 \\
\end{pmatrix} \delta^3(x - z),
\]

the inverse of this matrix is

\[
\left( c^{\alpha\beta}_{(0)} \right) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\delta_{ij} & 0 & 0 & 0 & 0 \\
0 & 0 & -2\delta_{ij} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{\nabla^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\nabla^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\nabla^2} & \delta^i_j & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\nabla^2} & 0 & \delta^i_j \\
\end{pmatrix} \delta^3(x - z).
\]

In this manner, we obtain the following nonzero Dirac brackets among the physical fields

\[
\{\Phi^0_0(x), \Pi^0_0(z)\}_D = 2\delta^3(x - z),
\]

\[
\{B^0_0(x), \Pi^0_0(z)\}_D = \delta^i_j \delta^3(x - z),
\]

\[
\{\Phi^i_0(x), \Pi^j_0(z)\}_D = [\delta^i_j - \frac{1}{\nabla^2} \partial_i \partial_j] \delta^3(x - z),
\]

\[
\{\Phi^i_0(x), \Pi^k_0(z)\}_D = \frac{n}{2\nabla^2} [\delta^i_j \partial^k - \delta^k_j \partial^i] \delta^3(x - z),
\]

\[
\{B^i_0(x), \Pi^k_i_0(z)\}_D = \frac{1}{2} [\delta^i_j \delta^k_j - \delta^i_k \delta^j_j] - \frac{1}{\nabla^2} (\delta^i_j \partial^k_j - \delta^k_j \partial^i_j - \delta^i_j \partial^k \partial_j - \delta^k_j \partial^i \partial_j) \delta^3(x - z). \tag{69}
\]

and we can observe that the Dirac brackets among physical and the auxiliary variables vanish as expected. In fact, the auxiliary fields do not contribute to the theory and they can be taken as zero at the end of the calculations \[14\].

Now, we will calculate the Dirac brackets for the \(kk\)-excitations. In fact, by fixing the gauge and introducing auxiliary variables we obtain the following set of second class constraints

\[
\lambda^1_{(n)} \equiv \Pi^0_0(z), \quad \lambda^2_{(n)} \equiv \Phi^0_0(z),
\]
\[
\lambda^3_{(n)} \equiv \Pi^0_i(z), \quad \lambda^4_{(n)} \equiv B^0_0(z),
\]
\[
\lambda^5_{(n)} \equiv \Pi^0_0_{(n)}, \quad \lambda^6_{(n)} \equiv B_{0_{(n)}},
\]
\[
\lambda^7_{(n)} \equiv \partial_i \Pi^i_{(n)} + \frac{n}{R} \Pi^5_{(n)}, \quad \lambda^8_{(n)} \equiv \partial^i \Phi^i_{(n)},
\]
\[
\lambda^9_{(n)} \equiv m \Pi^i_{(n)} + 2\partial_i \Pi^j_{(n)} + \frac{n}{R} 2\Pi^5_{(n)} + \partial^i p_{(n)}, \quad \lambda^{10}_{(n)} \equiv \partial^i B^i_{(n)} + \partial_i q_{(n)},
\]
\[
\lambda^{11}_{(n)} \equiv m \Pi^5_{(n)} + 2\partial_i \Pi^5_{(n)}, \quad \lambda^{12}_{(n)} \equiv \partial^i B^5_{(n)}.
\tag{70}
\]
where the auxiliary variables $q_{(n)}$ y $p_{(n)}$ satisfy the brackets

$$\{q_{(n)}^{(n)}(x), p_{(n)}^{(n)}(z)\} = \delta^3(x - z).$$  \tag{71}$$

The nonzero Poisson brackets among the constraints [70] are given by

$$\begin{align*}
\{\chi_{(n)}^1(x), \chi_{(n)}^2(z)\} &= -\delta^3(x - z), \\
\{\chi_{(n)}^3(x), \chi_{(n)}^4(z)\} &= -\frac{1}{2} \delta^3_i \delta^3_j (x - z), \\
\{\chi_{(n)}^5(x), \chi_{(n)}^6(z)\} &= -\frac{1}{2} \delta^3 (x - z), \\
\{\chi_{(n)}^7(x), \chi_{(n)}^8(z)\} &= -\partial_i \partial^i \delta^3 (x - z), \\
\{\chi_{(n)}^9(x), \chi_{(n)}^9(z)\} &= m \partial^i \delta^3 (x - z), \\
\{\chi_{(n)}^9(x), \chi_{(n)}^{10}(z)\} &= -\delta^i_j \partial^j \partial^i \delta^3 (x - z), \\
\{\chi_{(n)}^9(x), \chi_{(n)}^{12}(z)\} &= \frac{n}{R} \partial^i \delta^3 (x - z), \\
\{\chi_{(n)}^{11}(x), \chi_{(n)}^{12}(z)\} &= -\partial_i \partial^i \delta^3 (x - z),
\end{align*}$$

by using these brackets we construct the matrix

$$
\left( C_{\alpha\beta}^{(n)} \right) =
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} \delta^i_j & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} \delta^i_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\nabla^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \nabla^2 & 0 & m \partial^i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m \partial^i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^i_j \nabla^2 & 0 & \frac{n}{R} \partial^i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nabla^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nabla^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nabla^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\nabla^2 \\
\end{pmatrix} \delta^3 (x - z).
$$
whose inverse is given by

\[
(C^{\alpha\beta}_{(n)}) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2\delta_i^j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{m\partial^\mu}{(\nabla^2)^2} & 0 & -\frac{m\partial^\mu}{(\nabla^2)^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{m\partial^\mu}{(\nabla^2)^2} & 0 & -\frac{m\partial^\mu}{(\nabla^2)^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \delta^3(x-z).
\]

By using this matrix, we obtain the following nonzero Dirac brackets

\[
\{\Phi_0^{(n)}(x), \Pi_0^{(n)}(z)\}_D = 2\delta^3(x-z),
\]

\[
\{B_{0i}^{(n)}(x), \Pi_0^{(n)}(z)\}_D = \delta_i^j\delta^3(x-z),
\]

\[
\{\Phi_i^{(n)}(x), \Pi_j^{(n)}(z)\}_D = [\delta_i^j - \frac{1}{\sqrt{2}}\partial_1\partial^j]\delta^3(x-z),
\]

\[
\{\Phi_i^{(n)}(x), \Pi_k^{(n)}(z)\}_D = \frac{m}{2\sqrt{2}}[\delta_i^j\partial^k - \delta_i^k\partial^j]\delta^3(x-z),
\]

\[
\{B_{ij}^{(n)}(x), \Pi_k^{(n)}(z)\}_D = \frac{1}{2}[\delta_i^k\delta_j^l - \delta_i^j\delta_k^l - \frac{1}{\sqrt{2}}(\delta_i^k\partial^l\partial_j - \delta_i^j\partial^l\partial_k - \delta_k^l\partial^j\partial_i + \delta_j^l\partial^k\partial_i)]\delta^3(x-z),
\]

and the brackets among physical and auxiliary variables vanish. It is important to remark, that all results of this work are absent in the literature.

V. CONCLUSIONS AND PROSPECTS

In this paper, the Hamiltonian analysis for a 5D Kalb-Ramond, 5D Proca Kalb-Ramond and St"ueckelberg’s Kalb-Ramond theories with a compact dimension has been performed. Respect to 5D Kalb-Ramond theory, we obtained the complete canonical description of the theory. After performing the compactification of the fifth dimension on a $S^1/Z_2$ orbifold, we found that the effective theory is composed by a 4D Kalb-Ramond theory identified with the zero-mode plus a tower of kk-excitations. We reported the complete constraints program, we found that the constraints of the theory are of first class and reducible. From the gauge transformations of the theory and by fixing a particular gauge, we identified a tower of massive fields and the fields $B_{5\mu}^{(n)}$ are identified as pseudo-Goldston bosons. Furthermore, in order to obtain an irredicible set of constraints we introduced auxiliary variables and we calculate the fundamental Dirac’s brackets for the zero modes and the kk-excitations.
On the other hand, for the Proca Kalb-Ramond theory we observed that the theory is not a
gauge theory as expected. In fact, for the mode zero and for the $kk$-excitations we found that
there are only second class constraints, there are not reducibility conditions and there are not
present pseudo-Goldstone bosons. We constructed the Dirac brackets for the zero mode and the
$kk$-excitations.

Furthermore, we performed the Hamiltonian analysis for Stueckelberg Kalb-Ramond theory. We
found that the theory have only first class constraints; there are reducibility conditions among
the constraints of the zero mode and reducibility conditions for the $kk$-excitations. By fixing the
gauge parameters we can observe that the fields $\Phi^{(n)}_5$ and $B^{(n)}_{5\mu}$ are identified as pseudo-Goldstone
bosons, thus, the theory describes a 4D Stueckelberg Kalb-Ramond fields plus a tower of massive
$kk$-excitations. In order to construct the Dirac brackets, we used the phase space extension
procedure for obtaining a irreducible set of second class constraints and we could construct the
Dirac brackets for the zero mode and for the $kk$-excitations. In this manner, we have all tools for
performing the quantization of the theories under study. In fact, we can calculate the propagators
among the physical fields by using the Dirac brackets. In this respect, we would like to comment
that the quantization of the theories by using the results of this work and by using the symplectic
method is already in progress, and all these ideas will be the subject of forthcoming works 20.

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[1] V.I. Ogievetsky and I.V. Polubarinov, Sov. J. Nucl. Phys. 4 (1967) 156.
[2] S. Deser, Phys. Rev. 187 (1969) 1931.
[3] Y. Nambu, Phys. Reports 23 (1976) 250.
[4] D.Z. Freedman and P.K. Townsend, Nucl. Phys. B177 (1981) 282.
[5] S. Deser and E. Witten, Nucl. Phys. B178 (1981) 491.
[6] S. Deser, P.K. Townsend and W. Siegel, Nucl. Phys. B184 (1981) 333
[7] W. Siegel, Phys. Lett. B, 85, (1979) 333.
[8] E.S. Fradkln and M.A. Vasillev, Phys. Lett. B, 85 (1979) 47.
[9] E. Cremmer and B. Julia, Nucl. Phys. B, 159 (1979) 141.
[10] J. Thierry-Mieg, Y. Ne'eman, Proc. Nat. Acad. Sc. USA 79 (1982) 7068
[11] M. Kalb and P. Ramond, Phys. Rev. D9 (1974) 2273.
[12] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B(1982) 97; G. F. Chapline
    and N. S. Manton, Phys. Lett. B 120 (1983) 105.
[13] B. Julia and G. Toulouse, J. de Phys. 16 (1979) 395.
[14] E. Harikumar, M. Sivakumar, Mod.Phys.Lett. A15 (2000) 121-132.
[15] A. Perez-Lorenzana, J. Phys. Conf. Ser. 18, 224 (2005).
[16] A. Muck, A. Pilaftsis and R. Ruckl, Phys. Rev. D 65, 085037 (2002).
[17] H. Novales-Sanchez and J. J. Toscano, Phys. Rev. D, 82, 116012 (2010).
[18] E. Stueckelberg, Helv. Phys. Acta 11, 299-312. (1938).
[19] Alberto Escalante and Moisés Zárate, Dirac and Faddeev-Jackiw quantization of a 5D Stueckelberg theory with a compact dimension, submitted to Physical Review D, (2014).
[20] Alberto Escalante, Faddeev-Jackiw quantization of reducible gauge systems, in preparation, (2014).