On the Cut-Off Prescriptions Associated with Power-Law Generalized Thermostatistics

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Abstract

We revisit the cut-off prescriptions which are needed in order to specify completely the form of Tsallis' maximum entropy distributions. For values of the Tsallis entropic parameter $q > 1$ we advance an alternative cut-off prescription and discuss some of its basic mathematical properties. As an illustration of the new cut-off prescription we consider in some detail the $q$-generalized quantum distributions which have recently been shown to reproduce various experimental results related to high $T_c$ superconductors.
I. INTRODUCTION

There is a growing body of evidence indicating that there are important systems and processes in physics, biology, economics, and other fields, which are described by statistical distributions of the Tsallis’ maximum entropy form. These distributions are obtained from the extremalization of Tsallis’ entropic functional under the constraints imposed by normalization and the mean values of a (in general small) set of relevant quantities which are regarded as input information. The Tsallis entropic measure is given by

\[ S_q = \frac{k}{q-1} \left( 1 - \sum_{i=1}^{w} p_i^q \right) \]  

where \( k \) is a positive constant (from here on set equal to 1), \( w \) is the total number of microstates in the system, \( \{p_i, \ i = 1, \ldots, w\} \) are the associated probabilities, and the Tsallis parameter \( q \) is any real number. It is straightforward to verify that the usual Boltzmann-Gibbs (BG) logarithmic entropy, \( S = -\sum p_i \ln p_i \), is recovered in the limit \( q \to 1 \). Tsallis’ maximum entropy distributions have been found to be relevant, for instance, in various astrophysical scenarios and in the study of the phenomenon of non-linear diffusion. The description of the behaviour of chaotic maps at the threshold of chaos constitutes another successful field of application for the Tsallis formalism. It is important to stress that some of the aforementioned developments involve a quantitative agreement between experimental data and theoretical models based on Tsallis maximum entropy distributions. For instance, it was experimentally found that pure electron plasmas in Penning traps relax to metastable states whose radial density profiles do not maximize the Boltzmann-Gibbs entropy. However, Boghosian showed that the observed profiles are well described by a Tsallis distribution with \( q \) close to \( 1/2 \). Beck’s recent investigations on fully developed turbulent flows constitute another interesting application. General overviews on Tsallis' formalism and its diverse applications can be found in. An updated bibliography is available in.

The formal solutions to Tsallis’ maximum entropy variational problem, the so called \( q \)-maxent distributions, are not always positive real numbers. In order to guarantee the real and positive character of these \( q \)-maxent distributions it is necessary to introduce appropriate cut-off prescriptions. The aim of the present work is to discuss one possible cut-off
prescription for the case of Tsallis parameter $q > 1$. Recent studies by various authors suggest that there are two classes of non-extensive descriptions, corresponding, respectively, to $q < 1$ and $q > 1$. For instance, earlier applications to low-dimensional dissipative maps at the edge of chaos yield $q$-values less then 1 [6], while it was latter realized that there is another class of $q$-values greater than 1 [8]. A similar situation occured in the case of turbulence [17, 18]. So far the $q < 1$ case has been studied in greater detail, and is consequently better understood, than the $q > 1$ case. We believe that the present work may contribute to a deeper understanding of this latter case.

The paper is organized as follows. In section II we review the Tsallis cut-off prescription. In section III we propose an alternative prescription for the case $q > 1$. In section IV we illustrate our prescription with a $q$-generalized Fermi-Dirac distribution and we prove that, in this case, the present prescription leads to a thermodynamically consistent formalism. Some conclusions are drawn in section V.

II. TSALLIS’ CUT-OFF PRESCRIPTION

When, in addition to normalization, one only has the mean energy constraint, Tsallis’ distribution can be parameterized as [27]

$$p_i = \frac{1}{Z_q} \left[ 1 - (1 - q)\beta \epsilon_i \right]^{\frac{1}{1-q}},$$

where $q$ is a real number (Tsallis’ parameter), $\epsilon_i$ is the energy of microstate $i$, and the partition function $Z_q$ is an appropriate normalization constant. This distribution can be regarded as the $q$-generalization of the Gibbs canonical distribution. In order to guarantee that the microstates probabilities $p_i$ are non-negative real numbers, it is necessary to supplement expression (2) with an appropriate prescription for treating negative values of the quantity under square brackets. That is, we need a prescription for the value of $p_i$ when

$$1 - (1 - q)\beta \epsilon_i < 0.$$  

The simplest possible prescription, and the one usually adopted, is to set $p_i = 0$ whenever inequality (3) holds [26, 27]. This rule, usually referred to as “Tsallis’ cut-off prescription”, may seem at first sight just an ad-hoc solution to the above “negativity issue”. However, in many specific scenarios, Tsallis’ cut-off prescription turns out to be a physically sensible
It is instructive to mention a few examples. In the polytropic models of self-gravitating \( N \)-body systems, which are described by Tsallis’ maximum entropy distributions, the cut-off corresponds to the escape velocity from the system \[28\]. Other interesting examples involve the so-called \( q \)-gaussian distributions,

\[
\frac{1}{Z_q} \left[ 1 - (1 - q)\beta x^2 \right]^{\frac{1}{1-q}},
\]

which reduce to the standard gaussian distribution in the limit \( q \to 1 \). For \( q < 1 \) there are important non-linear diffusion and Fokker-Planck equations, with multiple applications in diverse fields, which admit exact analytical solutions of the \( q \)-gaussian form with the Tsallis cut-off \[3, 4, 5\]. Finally, it is possible to construct one dimensional quantum mechanical potential functions, exhibiting interesting properties related to shape invariance and, again, admitting exact ground state wave functions of the \( q \)-gaussian form with the Tsallis cut-off \[29\]. It is clear from the above examples that Tsallis’ cut-off prescription is indeed a physically reasonable one. This prescription may not constitute, however, the complete and final answer to the negativity issue. In order to clarify this, we have first to realize that Tsallis’ prescription covers two very different situations:

- (a) When \( q < 1 \) there is a special (positive) value of the quantity \( \beta \epsilon, (\beta \epsilon)_c \), for which the probability distribution becomes zero. In this case the probability distribution is set equal to zero for \( \beta \epsilon > (\beta \epsilon)_c \). With this prescription the probability distribution remains a continuous function of \( \beta \epsilon \). Even the first derivative of \( p \) with respect to \( \beta \epsilon \) is, for a certain range of \( q \)-values, continuous at the cut-off point.

- (b) When \( q > 1 \) a completely different picture obtains. Now there is a particular (negative) value of \( \beta \epsilon, (\beta \epsilon)_c \), such that when \( \beta \epsilon \) approaches \( (\beta \epsilon)_c \) from the left \( p \to +\infty \). In this case Tsallis’ cut-off rule prescribes that \( p \) is to be set equal to zero for all \( \beta \epsilon < (\beta \epsilon)_c \).

Two comments are in order. First of all, most of the concrete physical realizations of Tsallis’ cut-off that have so far been studied (and, in particular, the three examples previously mentioned by us) correspond to the \( q < 1 \) (case (a)) instance. Secondly, the cut-off prescription is much less palatable in case (b) than in case (a). In the latter case, \( p(\beta \epsilon) \) is a continuous function while in the former case it jumps from \(+\infty\) to 0 at the cut-off.
point. The main aim of our present contribution is to discuss a possible alternative to the \( q > 1 \) case of the cut-off prescription.

### III. ALTERNATIVE CUT-OFF PRESCRIPTION FOR \( q > 1 \).

Tsallis’ maximum entropy distributions can be conveniently written in terms of the \( q \)-generalized exponential function

\[
e_q(x) = \begin{cases} 
[1 + (1 - q)x]^{\frac{1}{1-q}}, & \text{if } [1 + (1 - q)x] > 0 \\
0, & \text{if } [1 + (1 - q)x] \leq 0.
\end{cases}
\] (5)

Notice that this last expression contains Tsallis’ cut-off condition, which is absorbed into the definition of the \( q \)-exponential function \( e_q(x) \). In terms of the \( q \)-generalized exponential, Tsallis’ distribution is

\[
p_i = e_q(-\beta \epsilon_i).
\] (6)

We are going to introduce now an alternative generalization \( \tilde{e}_q(x) \) of the exponential function, defined in the following way. For \( q < 1 \) we set \( \tilde{e}_q(x) = e_q(x) \). And for \( q > 1 \), we propose

\[
\tilde{e}_q(x) = \begin{cases} 
[1 + (q - 1)x]^{\frac{1}{q-1}}, & x > 0 \\
[1 + (1 - q)x]^{\frac{1}{1-q}}, & x \leq 0
\end{cases}
\] (7)

The function \( \tilde{e}_q(x) \) has, for \( q > 1 \), some desirable properties. First of all, \( \tilde{e}_q(x) \) complies with the “exponential-like” relation

\[
\tilde{e}_q(x) \cdot \tilde{e}_q(-x) = 1.
\] (8)

On the other hand, \( \tilde{e}_q(x) \) is clearly continuous at \( x = 0 \) (See Figure 1). Furthermore, we have

\[
\frac{d}{dx} \left( [1 + (q - 1)x]^{\frac{1}{q-1}} \right) = [1 + (q - 1)x]^{\frac{2}{q-1}} \quad \rightarrow 1 \quad \text{as} \quad x \rightarrow 0^+,
\]

and
\[
\frac{d}{dx} \left( \left[ 1 + (1 - q) x \right]^\frac{1}{1-q} \right) = \left[ 1 + (1 - q) x \right]^\frac{q}{1-q} \to 1 \text{ as } x \to 0^-,
\]

implying that \( \frac{d}{dx} \tilde{e}_q(x) \) is, for \( q > 1 \), continuous at \( x = 0 \).

Now, the alternative cut-off prescription that we want to consider is tantamount to adopting for the \( q \)-generalized Gibbs ensemble a distribution of the form,

\[
p_i = \tilde{e}_q(-\beta \epsilon_i).
\]

The generalized exponential function \( \tilde{e}_q(x) \) allows them to express our cut-off prescription, and the corresponding generalization of the canonical distribution in a compact form. Our aim in the present Letter is to explore some important physical consequences of our cut-off prescription. In particular, we want to address its thermodynamic consistency. It would be interesting to investigate in detail the mathematical properties of the function \( \tilde{e}_q(x) \) but, of course, that is not our aim here. For our present purposes we only need the basic features of \( \tilde{e}_q(x) \) already mentioned. A similar situation arose in the case of Tsallis’ generalized exponential \( e_q(x) \): this function was introduced as a compact and elegant notation for Tsallis’ maximum entropy distributions. The extensive literature on the purely mathematical properties of \( e_q(x) \) only appeared afterwards.

IV. THE GENERALIZED FERMI-DIRAC DISTRIBUTION.

A. The Standard Maximum Entropy Principle for Quantum Distributions

The quantum mechanical distributions can be obtained from a maximum entropy principle based on the entropic measure (the upper signs corresponding to bosons and lower one to fermions) \[30, 31, 32\]

\[
S = - \sum_i \left[ \bar{n}_i \ln \bar{n}_i \mp (1 \pm \bar{n}_i) \ln(1 \pm \bar{n}_i) \right],
\]

where \( \bar{n}_i \) denotes the number of particles in the \( i \)th energy level with energy \( \epsilon_i \). The extremalization of the above measure under the constraints imposed by the total number of particles,
\[ \sum_i \bar{n}_i = N, \quad (12) \]

and the total energy of the system,

\[ \sum_i \bar{n}_i \epsilon_i = E, \quad (13) \]

leads to the standard quantum distributions,

\[ \bar{n}_i = \frac{1}{\exp \beta (\epsilon_i - \mu) \mp 1}. \quad (14) \]

In the above equation the minus sign corresponds to the Bose-Einstein distribution and the plus sign corresponds to the Fermi-Dirac one.

**B. The Nonextensive Maximum Entropy Principle for Fermions**

In order to deal with non extensive scenarios (characterized by \( q \neq 1 \)) we propose the extended measure of entropy,

\[ S_q^{(F)}[\bar{n}] = \sum_i \left[ \left( \frac{\bar{n}_i - \bar{n}^q_i}{q - 1} \right) + \left( \frac{(1 - \bar{n}_i) - (1 - \bar{n}^q_i)}{q - 1} \right) \right]. \quad (15) \]

That the entropic functional (15) constitutes a natural generalization of (11) can be easily realized if we express it in terms of the so-called “\( q \)-logarithms”,

\[ S_q^{(F)}[\bar{n}] = -\sum_i \left[ \bar{n}^q_i \ln_q(\bar{n}) + (1 - \bar{n}_i)^q \ln_q(1 - \bar{n}_i) \right], \quad (16) \]

where the \( q \)-logarithm is defined as

\[ \ln_q(x) = (1 - q)^{-1} \left( x^{1-q} - 1 \right), \quad (x > 1). \quad (17) \]

In the limit \( q \to 1 \) (16) reduces to (11). However, the main physical motivation for introducing the measure (15) is that it leads, via the maximum entropy principle, to quantum distribution functions that have been very successful in the study of a concrete and important physical phenomena: high \( T_C \) superconductivity [21]. Furthermore, as we shall presently explain, the formulation of a variational principle in terms of (15) allows to prove the thermodynamical consistency of the cut-off prescription used in [21].
The relevant constraints leading to the $q$-generalized quantum distributions are the total number of particles,

$$\sum_i \bar{n}_i^q = N,$$

and the total energy,

$$\sum_i \bar{n}_i^q \epsilon_i = E.$$  \hspace{1cm} (19)

The extremalization of the entropic measure (15) under the constraints (18) and (19) leads to the variational problem

$$\delta \left\{ S^{(F)}_{q}[\bar{n}] + \alpha \left( N - \sum_i \bar{n}_i^q \right) + \beta \left( E - \sum_i \epsilon_i \bar{n}_i^q \right) \right\} = 0,$$  \hspace{1cm} (20)

whose formal solution is

$$\bar{n}_i = \frac{1}{1 + [1 + \left( \frac{q - 1}{\alpha + \beta \epsilon_i} \right)]^\frac{1}{q-1}},$$  \hspace{1cm} (21)

where $\alpha$ and $\beta$ are the Lagrange multipliers associated, respectively, with the total number of particles and the total energy. Notice that we are using here un-normalized $q$-constraints, instead of using the normalized $q$-constraints studied in [33]. Within the present context it is not necessary to use the normalized $q$-values, because our variational problem is formulated in terms of mean occupation numbers. These numbers are not probabilities and, consequently, do not need to be normalized. Furthermore, both constraints, the one associated with the total number of particles, and the one corresponding to the total energy, are formulated using the same kind of $q$-values. This allows to re-formulate the variational problem in terms of standard linear constraints and to prove that the invariance under uniform shifts of the (single-particle) energy spectrum still holds (see reference [32] for details).

Quantum distributions of the form (21) have been studied by many researchers in recent years [21, 34, 35, 36, 37, 38, 39, 40]. Now, the formal solution (21) needs to be supplemented with an appropriate cut-off prescription to deal with negative values of the quantity $\alpha + \beta \epsilon_i$. For $q > 1$, our cut-off prescription is tantamount to adopt in expression (21) a new value of $q$. [32]
\[ q \rightarrow q^* = 2 - q \quad \text{when} \quad \alpha + \beta \epsilon_i \leq 0. \] (22)

It is interesting to notice that the transformation (22) also arises in other contexts as, for instance, in the study of renormalization-group dynamics at the onset of chaos in logistic maps [10].

The most basic requirement of a thermostatistical formalism is to be thermodynamically consistent. That is, the thermostatistical formalism must lead to the standard thermodynamical relationships among thermodynamical variables such as entropy, energy, temperature, etc [33, 41, 42, 43]. We are now going to prove that this is indeed the case with our present formalism. Let us consider the entropic functional

\[ S^{(F)}_q = \sum_i C_q(\bar{n}_i), \] (23)

where the function \( C_q(x) \) is defined by

\[
C_q(x) = \begin{cases} 
  \left( \frac{x - x^q}{q-1} \right) + \left( \frac{(1-x)-(1-x)^q}{q-1} \right) & \text{if } x \leq \frac{1}{2} \\
  \left( \frac{x-x^{2-q}}{1-q} \right) + \left( \frac{(1-x)-(1-x)^{2-q}}{1-q} \right) & \text{if } x > \frac{1}{2}
\end{cases}
\] (24)

The function \( C_q(x) \) is discontinuous at \( x = 1/2 \) (See Figure 2). In fact, the limit value of \( C_q(x) \) when we approach \( x = 1/2 \) from the right is

\[ C_q(x \to 1/2) = \frac{1 - 2^{1-q}}{q - 1}, \] (25)

while the limit value when \( C_q(x) \) is approached from the left is,

\[ C_q(1/2 \leftarrow x) = 2^{q-1} \left( \frac{1 - 2^{1-q}}{q - 1} \right). \] (26)

However, the left and right limit values of the first derivative \( dC_q/dx \) at \( x = 1/2 \) are both equal to 0. That is, the first derivative \( dC_q/dx \) is continuous at the cut-off point (See Figure 3).

If we extremalize the entropic measure (23) under the constraints (18) and (19) we obtain the set of equations

\[ C_q'(\bar{n}_i) - \alpha q \bar{n}_i^{q-1} - \beta q \epsilon_i \bar{n}_i^{q-1} = 0, \] (27)
which can be solved for the occupation numbers, yielding

$$
\bar{n}_i = \frac{1}{1 + \left[1 + (\tilde{q} - 1)(\alpha + \beta \epsilon_i)\right]^\frac{1}{\tilde{q} - 1}},
$$

(28)

where $\mu = -\frac{\alpha}{\beta}$ is the chemical potential, and

$$
\tilde{q} = \begin{cases} 
q, & \text{if } \alpha + \beta \epsilon_i > 0 \\
2 - q, & \text{if } \alpha + \beta \epsilon_i \leq 0.
\end{cases}
$$

(29)

Notice that,

$$
\alpha + \beta \epsilon_i = 0 \implies \bar{n}_i = \frac{1}{2}.
$$

(30)

The $q$-generalized Fermi-Dirac distribution with our new cut-off prescription is depicted, for $q = 1, \frac{2}{3}$, and $\frac{5}{3}$ in Figure 4. For comparison purposes, the $q$-generalized Fermi-Dirac distribution with the standard cut-off rule is exhibited in Figure 5.

When the variational problem is formulated in terms of the entropic measure (23), the cut-off prescription need not be imposed on the maximum entropy distribution after we obtain the formal solution of the variational problem. On the contrary, the solution of the variational problem already contains the cut-off prescription. We can say that the prescription is incorporated into the definition (23) of the entropic functional itself. In terms of the generalized exponential function (7), our $q$-generalized quantum distributions for fermions can be written as

$$
\bar{n}_i = \frac{1}{\tilde{e}_q \left[\beta(\epsilon_i - \mu)\right] + 1}.
$$

(31)

Now, it can be shown [44, 45] that any thermostatistical formalism based upon the constrained extremalization of an entropic functional (that is, based upon Jaynes maximum entropy approach) complies with the thermodynamical relationships (which, in the context of Jaynes’ maxent formulation are often referred to as Jaynes’ relationships). This implies that the $q$-generalization of the Fermi-Dirac distribution considered in this work, which incorporates the cut-off rule we are here advocating, complies with the thermodynamical
relationships. Similar calculations as the ones we have performed here can be done for the case of bosons, leading to the $q$-deformed Bose-Einstein distribution,

$$\tilde{n}_i = \frac{1}{\tilde{e}_q [\beta(\epsilon_i - \mu)] - 1}. \quad (32)$$

A comment concerning the status of the $q$-deformed quantum distributions (31,32) is in order here. This distribution were introduced by Buyukkilic and Demirhan (BD) [34]. BD attempted to derive these distributions in [34] from the full $N$-body, ($\Gamma$-space [31]) $q$-generalized grand canonical ensemble, in a manner similar to the standard, $q = 1$ (BG) case [31]. It is important to realize that BD’s derivation does not yield the exact quantum statistics associated with the full $N$-body, ($\Gamma$-space [31]) $q$-generalized grand canonical ensemble [35]. The BD distributions, which have been used by many people [36, 37, 38, 39, 40], can be regarded only as an approximation. Indeed, numerical evidence has been reported suggesting that, for fermions, the BD distributions constitute a reasonable approximation in some cases [39]. Furthermore, a recent BCS s-wave model for high $T_c$ superconductivity, which uses (31) for the quantum distribution functions of the independent quasi-particles [32], exhibits remarkable agreement with the available experimental data if a value of the Tsallis parameter $q \sim 1.6$ is adopted [21]. This suggests that, at least at some level, such quantum distributions might provide a reasonable ($\mu$-space [31]) description of some quantum many body systems. It is important to realize that a similar state of affairs occurs with many of the other experimental verifications of the $q$-nonextensive thermostatistics. In most cases a successful account of the experimental data has been achieved by recourse to a $\mu$-space theoretical model [2, 14, 23, 24]. To determine the detailed connection between these successful $\mu$-space descriptions of concrete physical phenomena, on the one hand, and the $\Gamma$-space formulation, on the other, still constitutes an open and formidable problem. It, therefore, seems prudent, for the time being, to pay serious attention to the $\mu$-space treatments, whenever they reproduce experimental data.

V. CONCLUSIONS

After considering in detail the usual cut-off prescriptions for the $q$-maxent probability distributions, and discussing their most important concrete physical realizations, the following conclusions are inescapable. First of all, the main physical realizations of the standard
cut-off rule correspond to the case \( q < 1 \). Secondly, the usual prescription for \( q > 1 \) is physically not as interesting classically as the usual cut-off rule for \( q < 1 \). In the present effort we proposed an alternative rule for the case \( q > 1 \). We illustrated our proposal with \( q \)-generalized quantum distributions that have already been successfully applied to the study of both high \( T_c \) superconductivity [21] and the formation of the quark-gluon plasma [40]. In the particular example of the \( q \)-generalized quantum distributions we have proved that our cut-off prescription leads to a thermodynamically consistent formalism. Even though we have focused here upon Tsallis’ generalized thermostatistics, it has not escaped our attention that the present considerations may be also relevant for other non-standard thermostatistical formalisms that have recently been proposed [46, 47].

Further developments and applications of our variable-\( q \) Tsallis formalism, and the exploration of its relationship with problems, will be greatly welcome.

Acknowledgments

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FIG. 1: The generalized exponential function $e^{x_q}(x)$ for $q = 1, 4/3, 5/3$. All depicted quantities are dimensionless.

FIG. 2: The function $C_q(x)$, appearing in the definition of the entropic functional, for $q = 1, 4/3$, and $5/3$. All depicted quantities are dimensionless.
FIG. 3: The derivative $dC_q(x)/dx$ of the function $C_q(x)$ appearing in the definition of the entropic functional, for $q = 1, 4/3, \text{ and } 5/3$. All depicted quantities are dimensionless.

FIG. 4: The average occupation number $n_i$ for the $q$-generalized Fermi-Dirac distribution, which incorporates our new cut-off prescription, as a function of $\beta(\epsilon_i - \mu)$ and for the same $q$-values as in Figures (1-3). All depicted quantities are dimensionless.
FIG. 5: The average occupation number for the \( q \)-generalized Fermi-Dirac distribution with the standard cut-off prescription, as a function of \( \beta(\epsilon_i - \mu) \) and for the same \( q \)-values as in Figures (1-3). All depicted quantities are dimensionless.