Supporting Information:  
Non reciprocity and odd viscosity in chiral active fluids  

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I. TOTAL HYDRODYNAMIC MOMENTUM AND KINETIC ENERGY OF A FLUID OF DUMBBELLS

In this section we detail the derivation of Eq. (1) of the main text, and Eq. (18) in Materials and Methods. Considering a fluid of dumbbells as described in the main text, the momentum of the two point masses can be written in terms of the CM momentum, \(\mathbf{p}^\alpha\), and the unit vector \(\mathbf{\nu}^\alpha\) as  
\[
\mathbf{p}^\alpha_{1,2} = \frac{1}{2} \mathbf{p}^\alpha \pm m \dot{\mathbf{\nu}}^\alpha,
\]

such that the total hydrodynamic momentum density for the diatomic fluid is  
\[
\dot{g}_\mu(r) = \sum_{\alpha \mu} \mathbf{p}^\alpha_{\mu} \delta(r - r^{\alpha \mu})
\]

\[
= \sum_{\alpha} \left[ p^\alpha_{1,2} \delta(r - r^\alpha - a \mathbf{\nu}^\alpha) + p^\alpha_{1,2} \delta(r - r^\alpha + a \mathbf{\nu}^\alpha) \right]
\]

\[
\approx \sum_{\alpha} \left[ p^\alpha_1 \delta(r - r^\alpha) - I \dot{\mathbf{\nu}}^\alpha \mathbf{\nu}_j \nabla \delta(r - r^\alpha) \right], \quad (1)
\]

where we have used \(r^{\alpha,1} - r^{\alpha,2} = 2a \mathbf{\nu}^\alpha\), and the moment of inertia of each molecule is \(I = Ma^2\) with \(M = 2m\) being the molecule mass. As we are interested in the long wavelength limit, \(a \ll |r - r^\alpha|\), we only keep terms \(O(a^2)\). It is useful to write  
\[
\dot{g}_\mu(r) = -\frac{I}{2} \left[ \varepsilon_{ijk} \varepsilon_{klm} \sum_{\alpha} \dot{v}^\alpha_{ik} \nu^\alpha_m \nabla \delta(r - r^\alpha) + \sum_{\alpha} \dot{Q}^\alpha_{ij} \nabla \delta(r - r^\alpha) \right]. \quad (2)
\]

Here  
\[
Q^\alpha_{ij} = \nu^\alpha_i \nu^\alpha_j - \delta_{ij}/d
\]

is the alignment tensor and \(d\) is the number of dimensions. The first term of Eq. (2) includes the angular momentum of each molecule, \(\mathbf{\ell}^\alpha = I \mathbf{\nu}^\alpha \times \dot{\mathbf{\nu}}^\alpha\). By using the definition of the angular velocity of each molecule, \(\Omega^\alpha \equiv \mathbf{\nu}^\alpha \times \dot{\mathbf{\nu}}^\alpha\), one gets \(\mathbf{\ell}^\alpha = I \Omega^\alpha\). This is Eq. (1) of the main text.

Note that this equation (which is very similar to the one used in Ref. [1]), although written in terms of the microscopic fields, is already coarse-grained in a way. It accounts for the atoms positions within the molecule only in the long wavelength limit, but does not neglect it completely. We, therefore, refer to  
\[
\dot{g} = \dot{g}^c + \dot{g}^s
\]

as the hydrodynamic momentum. This is very similar to the classical treatment of rigid bodies [2], where the fast degrees-of-freedom are neglected and a rigid body is modeled as a point-like particle with CM and angular momenta (about the CM). Unlike the classical treatment, by maintaining some information about the atoms position, our coarse-graining (CG) of the total hydrodynamic momentum differ from the CM momentum. As this difference involves a gradient it is clearly related to a surface contribution. Indeed, its origin is in the fact that a molecule can be part within one CG volume and part in another, such that the angular momentum of the molecule will contribute to the total hydrodynamic momentum within the coarse-grained volume, see Fig. 1.

We continue by directly coarse-graining the kinetic Hamiltonian:

\[
H_k = \sum_{\alpha \mu} \frac{(\mathbf{P}^{\alpha \mu})^2}{2m_\alpha \nu^\alpha \cdot \mathbf{P}^{\alpha \mu}}. \quad (3)
\]

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Defining the kinetic energy density $H_k$ such that $H_k = \int \text{d}r H_k$ we have

$$H_k(r) = \frac{1}{2m} \sum_{\alpha} \left[ \left( \frac{1}{2} p^\alpha + m a \dot{\nu}^\alpha \right)^2 \delta(r - r^\alpha - a \nu^\alpha) + \left( \frac{1}{2} p^\alpha - m a \dot{\nu}^\alpha \right)^2 \delta(r - r^\alpha + a \nu^\alpha) \right]$$

$$\simeq \sum_{\alpha} \left[ \frac{p^\alpha_\alpha}{2M} + \frac{1}{2} I \dot{\nu}^\alpha_\alpha - \frac{I}{M} (\dot{\nu}^\alpha \cdot p^\alpha) (\nu^\alpha \cdot \nabla) \right] \delta(r - r^\alpha) , \quad (4)$$

where in the second line we have used the long-wavelength limit, which as explained above is already doing some kind of CG. This CG is consistent with the derivation of Eq. (2) of the main text and Ref. [1]. We now use the CG method described in Materials and Methods (which coarse-grain many molecules within some volume) to write Eq. (4) in terms of the various fields. Let us start with the first term of Eq. (4), which after coarse-graining is written as:

$$\sum_{\alpha} \frac{p^\alpha_\alpha}{2M} \delta(r - r^\alpha) = \frac{\langle \rho^\alpha \rangle^2}{2 \rho} , \quad (5)$$

where $\rho(r) = M n(r)$ and $\hat{n}(r) = \sum_{\alpha} \delta(r - r^\alpha)$. Noting that the relations $\Omega^\alpha = \nu^\alpha \times \dot{\nu}^\alpha$ and $\dot{\nu}^\alpha \cdot \nu^\alpha = 0$ implies that $(\dot{\nu}^\alpha)^2 = (\Omega^\alpha)^2$, coarse-graining the second term of Eq. (4) yields

$$\sum_{\alpha} \frac{1}{2} I \dot{\nu}^2_\alpha \delta(r - r^\alpha) = \sum_{\alpha} \frac{(E^\alpha)^2}{2I} \delta(r - r^\alpha) = \frac{\rho^2}{2I \rho} . \quad (6)$$

The third term of Eq. (4) can be written as

$$- \nabla_j \sum_{\alpha} \hat{I} \dot{\nu}^\alpha_i \nu^\alpha_j p^\alpha_i \delta(r - r^\alpha) = - \frac{1}{2} \nabla_j \sum_{\alpha} \hat{I} \left[ \left( \dot{\nu}^\alpha_i \nu^\alpha_j + \dot{\nu}^\alpha_j \nu^\alpha_i \right) + \varepsilon_{ijk} \varepsilon_{kmn} \nu^\alpha_m \nu^\alpha_n \right] p^\alpha_i \delta(r - r^\alpha)$$

$$= - \frac{1}{2} \nabla_j \sum_{\alpha} \left( \hat{I} Q^\alpha_{ij} - \frac{1}{M} \varepsilon_{ijk} \ell^\alpha_k \right) p^\alpha_i \delta(r - r^\alpha) , \quad (7)$$

which after coarse-graining reads

$$- \nabla_j \sum_{\alpha} \hat{I} \dot{\nu}^\alpha_i \nu^\alpha_j p^\alpha_i \delta(r - r^\alpha) = \frac{1}{2} \nabla \cdot (\ell \times \nu - p \cdot A) , \quad (8)$$
where \( A(r) = I \sum_a \dot{Q}^a \delta (r - r^a) \). Collecting the above, we find that the kinetic energy is the one of Eq. (18) of Materials and Methods:

\[
H_k = \int dr \left[ \frac{(\dot{r}^c)^2}{2r} + \ell^2 + \frac{1}{2} \nabla \cdot (\ell \times \mathbf{v} - \mathbf{v} \cdot \mathbf{A}) \right].
\] (9)

### II. FAST MODES IN A DUMBBELL

This section will show one approach to the culling of fast modes. To keep matters simple, we will use our dumbbell model in two-dimensions modified to have a fast mode. We begin by providing a description of the dynamics of the individual atoms of the dumbbell (total of four dynamical variables) that describe all motions of the dumbbell including stretching and compressing the dumbbell along its axes. Let \( R_1 \) be the position atom 1 and \( R_2 \) the position of atom 2. Then

\[
\begin{align*}
R_1 &= R_c + (a + r)\nu, \\
R_2 &= R_c - (a + r)\nu,
\end{align*}
\] (10)

where \( R_c \) is the center of mass, \( a \) is the rest length of the intra-molecular potential \( U(|R_1 - R_2|) \) and \( r \) is the deviation from the rest length. From this we calculate the time derivatives:

\[
\begin{align*}
\dot{R}_1 &= \dot{R}_c + \dot{r}\nu + (a + r)\nu, \\
\dot{R}_2 &= \dot{R}_c - \dot{r}\nu - (a + r)\nu.
\end{align*}
\] (11)

The full kinetic energy is then:

\[
H_k = \frac{1}{2} m \left[ \dot{\mathbf{R}}_1^2 + \dot{\mathbf{R}}_2^2 \right] = \frac{1}{2} M \left[ \dot{\mathbf{R}}_c^2 + \dot{r}^2 + (a + r)^2 \Omega^2 \right],
\] (12)

where \( M = 2m \) is the dumbbell mass, and \( \nu \cdot \dot{\nu} = 0 \) implies that \( \Omega^2 = \dot{\nu}^2 \). We have replaced the four dynamical variables \( R_1 \) and \( R_2 \) by four other variables: \( R_c \), the rotational frequency \( \Omega = \nu \times \dot{\nu} \) and the deviation \( r \).

The potential energy is expanded in a Taylor series to quadratic order around the rest length \( a \)

\[
U(|R_1 - R_2|) \approx U(a) + \frac{1}{2} \frac{d^2 U}{da^2} |R_1 - R_2| = a
\] (13)

where we have used \( |R_1 - R_2| = a + r \) and \( |r| \ll a \). Defining the effective spring constant \( k \equiv \frac{d^2 U}{da^2} (|R_1 - R_2| = a) \), the total energy of our dumbbell is written as

\[
H = H_k + U = \frac{1}{2} M \dot{\mathbf{v}}_c^2 + \frac{1}{2} I(r) \Omega^2 + \frac{1}{2} M \dot{r}^2 + \frac{1}{2} kr^2
\] (14)

where \( \mathbf{v}_c = \dot{R}_c \), \( I(r) = Ma^2 + I_0 \), and the constant contribution \( U(a) \) was ignored. This equation has four kinetic terms, one potential term, and one term that couples the rotational velocity \( \Omega \) with \( r \). Because \( r \) has a potential energy coupling, it is not a hydrodynamic variable, but it does couple to \( \Omega \) via the moment of inertia \( r \) dependence. In the absence of such coupling, \( r \) is an harmonic oscillator with characteristic frequency \( \omega = \sqrt{k/M} \). (Note that \( \omega \) in this section is not related to the vorticity as in the rest of the paper.) We therefore eliminate \( r \) by replacing it with its average value \( r_c \), which is obtained by minimizing \( H \) with respect to \( r \):

\[
\left. \frac{\partial H}{\partial r} \right|_{r=r_c} = M[\omega^2 r + (a + r)\Omega^2] = 0,
\] (15)

such that

\[
r_c = -\frac{\Omega^2}{\omega^2 + \frac{1}{2} \left( \frac{\Omega}{\omega} \right)^2} \approx -\frac{\Omega^2}{\omega^2} \left( 1 - \frac{\Omega^2}{\omega^2} \right).
\] (16)
This result implies that \( r_c = 0 \) when \( \omega \to \infty \), as it should. Note that we have assumed in Eq. (13) that \( |r| \ll a \), such that consistency requires that \( \Omega \ll \omega \). Substituting \( r = r_c \) back into the Hamiltonian of Eq. (14) we get

\[
H = \frac{1}{2} M \dot{v}_c^2 + \frac{1}{2} I_0 \Omega^2 \frac{\omega^2}{\omega^2 + \Omega^2} \simeq \frac{1}{2} M \dot{v}_c^2 + \frac{1}{2} I_0 \Omega^2 \left( 1 - \frac{\omega^2}{\omega^2} \right), \tag{17}
\]

where the first term is the CM kinetic energy and the second term is the rotational kinetic energy after including the effect of \( r \) on the moment of inertia. To leading order in \( \Omega/\omega \), the rotational kinetic energy assumes its normal form \( I_0 \Omega^2/2 \). Moreover, we find that the coupling to the non-hydrodynamic variable \( r \) does not modify the form of the kinetic energy associated with only the zero-modes (which are the rigid-body translations and rotations). It does, however, introduce non-linearities in \( \Omega \) which will modify the renormalized harmonic-limit. The variable \( r \) oscillates with frequency \( \sqrt{\omega^2 + \Omega^2} \simeq \omega \left[ 1 + (\Omega^2/\omega^2)/2 \right] \) about \( r_c \) and thus averages to \( r_c \) in the hydrodynamic limit where one is interested in the long-time behavior, i.e., \( t \gg 1/\omega \). When there is dissipation, \( r \) will decay to \( r_c \), while oscillating.

To exemplify the consequence of driving a finite-frequency mode, it is instructive to add to Eq. (14) a term \(-fr\), where \( f \) is an external force that we assume to be constant for simplicity. In this case \( r_c = \left( (f/M) - a\Omega^2 \right) / (\omega^2 + \Omega^2) \) and

\[
H = \frac{1}{2} M \dot{v}_c^2 + \frac{1}{2} I_0 \frac{\omega^2}{\omega^2 + \Omega^2} \left[ 1 + \frac{\alpha^2}{\omega^2} \left( 2 - \frac{\alpha^2}{\Omega^2} \right) \right] \Omega^2 
\simeq \frac{1}{2} M \dot{v}_c^2 + \frac{1}{2} I_0 \left[ \left( 1 + \frac{2\alpha^2}{\omega^2} \right) \left( 1 - \frac{\Omega^2}{\omega^2} \right) - \frac{\alpha^4}{\omega^4} \right] \Omega^2 , \tag{18}
\]

where \( \alpha^2 \equiv f/(Ma) \) and in the second line we have also dropped a constant term \( \sim f^2 \). We see that the Hamiltonian structure remains intact, and that the external driving of the finite-frequency mode only affects the moment of inertia. Notice that \( f \) changes the value of the moment of inertia in the hydrodynamic limit, i.e., it affects the harmonic limit directly.

This phenomena is in a sense universal. We know that frames with sites connected by inextensible rods have \( d(d+1)/2 \) zero energy modes consisting of \( d \) independent rigid translations and \( d(d-1)/2 \) rigid rotations, independent of what kind of “bulk” modes a molecule has [3]. What this example shows, is that in the process of CG, all of the vibrational modes of the molecules (e.g., \( r \) in this simple example) are averaged out, and only the molecules’ zero-modes may contribute to the hydrodynamics (finite-frequency modes can only alter molecular parameters). It is therefore important to identify the zero-modes of the molecules before CG.

A useful characterization of complex molecules is that introduced by James Clerk Maxwell [4], and extended by Calladine [5], to describe the strength of what Maxwell called frames consisting of \( N \) sites (atoms) connected by \( N_B \) stress-supporting bonds (central-force springs). These frames, or complex molecules, generally have \( N_0 = 0 \) zero-energy modes, and \( N_S \) states of self-stress. The latter are structures that can support stress without the exertion of a force on any sites (e.g., a two-dimensional square with four springs connecting nearest-neighbor sites and two others connecting diagonal sites has one state of self-stress). The Maxwell-Calladine theorem states that \( N_0 - N_S = dN - N_B \), where \( d \) is the system dimensions. For simple molecules \( N_s = 0 \) such that the zero-modes are those associated with rigid body translations and rotations of the molecule, which is what we considered in the main text. For example, a two-dimensional equilateral triangle has \( N = 3 \) sites, \( N_B = 3 \), and \( N_S = 0 \) and thus \( N_0 = 2 \times 3 - 3 = 3 \) corresponding to the two translational and one rotational zero mode that every two-dimensional frame has. Our model diatomic molecules consist of two atoms (\( N = 2 \)) in \( d = 3 \) connected by a rigid spring (\( N_B = 1 \)) with no states of self stress (\( N_S = 0 \)). It, therefore, has \( 3N = 6 \) degrees of freedom, and \( N_0 = 6 - 1 = 5 \), zero modes. This is precisely the number of zero modes for each molecule that appears in \( g \) of (2).

Note that this discussion implies that if the fluid molecules have states of self-stress, other “internal” zero-modes will have to be considered in the CG process. It is not clear if all zero-modes affect the hydrodynamic equations, or if there are other conditions on these modes that must be satisfied. We leave this question for future study. In this work we assume no such internal zero-modes exists in the molecules.

### III. COARSE-GRAINED KINETIC ENERGY FOR GENERAL COMPLEX MOLECULES

Here we extend the model described in the main text to account for a fluid of general complex (rigid) molecules (rather than dumbbells), each of which composed of multiple sub-particles (atoms) with mass \( m^{\mu} \) and momentum \( p^{\mu} \) located at \( r^{\mu} \). As already shown in the supplementary of [6] the coarse-grained hydrodynamic momentum for such fluid is \( \dot{g} = \dot{g}^c + \frac{1}{2} \nabla \times \ell + \nabla \cdot A \), where the definitions of \( \ell \) and \( A \) are similar to those in the main text of the
current paper. The angular momentum density is \( \mathbf{\ell}(r) = \sum_{\alpha} \ell_{\alpha} \delta(r - r^\alpha) \) with
\[
\ell_{\alpha}^\alpha = \xi_{ijk} \sum_{\mu} \Delta p_{k}^{\alpha \mu} \Delta r_{j}^{\alpha \mu},
\]
where \( \Delta r^{\alpha \mu} = r^{\alpha \mu} - r^\alpha \) and \( \Delta p^{\alpha \mu} = m^{\alpha \mu} \partial_t \Delta r^{\alpha \mu} \). The CM momentum \( p^\alpha = \sum_{\mu} p^{\alpha \mu} \) such that \( \dot{\mathbf{g}}^\alpha(r) = \sum_{\alpha} p^\alpha \delta(r - r^\alpha) \). We further define \( \mathbf{K}(r) = \sum_{\alpha} \mathbf{K}^\alpha \delta(r - r^\alpha) \), which is a generalization of the alignment tensor and should be thought of as an order parameter (see supplementary of [6]) with \( \mathbf{K}^\alpha = \sum_{\mu} m^{\alpha \mu} \Delta r^{\alpha \mu} \Delta r^{\alpha \mu} = \frac{1}{2} (\mathbf{R}^\alpha + \frac{1}{3} \mathbf{I}^\alpha) \) and [7]
\[
R_{ij}^\alpha = \sum_{\mu} m^{\alpha \mu} \left[ \Delta r_{i}^{\alpha \mu} \Delta r_{j}^{\alpha \mu} - \frac{1}{3} (\Delta r^{\alpha \mu})^2 \delta_{ij} \right],
\]
and
\[
I_{ij}^\alpha = \sum_{\mu} m^{\alpha \mu} \left[ (\Delta r^{\alpha \mu})^2 \delta_{ij} - \Delta r_{i}^{\alpha \mu} \Delta r_{j}^{\alpha \mu} \right].
\]

Then, \( \mathbf{A}(r) = \frac{1}{2} \sum_{\alpha} \mathbf{K}^\alpha \delta(r - r^\alpha) \).

Notably, the fundamental PBs (see Eq. (25) in Materials and Methods) are not modified, hence, the only thing left to find is the form of the Hamiltonian, which we will show to have the same form as in Eq. (9) such that the dynamics we use in the main text applies also for a fluid composed of general complex molecules. We begin with the Hamiltonian of Eq. (3) where now the kinetic energy density \( H_k \), defined as before via \( H_k = \int d\mathbf{r} H_k \), is:
\[
H_k(r) = \sum_{\alpha} \frac{1}{2 m^{\alpha \mu}} \left( \frac{m^{\alpha \mu}}{M^\alpha} p^\alpha + \Delta p^{\alpha \mu} \right)^2 \delta(r - r^\alpha - \Delta r^{\alpha \mu}).
\]

Here \( p^{\alpha \mu} = p^\alpha (m^{\alpha \mu}/M^\alpha) + \Delta p^{\alpha \mu} \) and \( M^\alpha = \sum_{\mu} m^{\alpha \mu} \) is the molecule mass. Assuming the molecules are small compared to typical distances (long wavelength limit), \( |\Delta r^{\alpha \mu}| \ll |r - r^\alpha| \) we write \( \delta(r - r^\alpha - \Delta r^{\alpha \mu}) \approx \delta(r - r^\alpha) - \Delta r^{\alpha \mu} \cdot \nabla \delta(r - r^\alpha) \) such that Eq. (22) becomes
\[
H_k(r) = \sum_{\alpha} \left( \frac{p^\alpha}{2 M^\alpha} \right)^2 \delta(r - r^\alpha) + \sum_{\alpha \mu} \left[ \frac{(\Delta p^{\alpha \mu})^2}{2 m^{\alpha \mu}} - \frac{p^\alpha \cdot \Delta p^{\alpha \mu}}{M^\alpha} \Delta r^{\alpha \mu} \cdot \nabla \right] \delta(r - r^\alpha),
\]
where we have used \( \sum_{\mu} m^{\alpha \mu} \Delta r^{\alpha \mu} = \sum_{\mu} \Delta p^{\alpha \mu} = 0 \) and neglected a term \( \sim (\Delta p^{\alpha \mu})^2 \Delta r^{\alpha \mu} \) that is proportional to the cube of the molecule size (we only keep quadratic order). The second term in Eq. (23) can be written in more familiar terms:
\[
\sum_{\mu} \frac{(\Delta p^{\alpha \mu})^2}{2 m^{\alpha \mu}} = \frac{1}{2} \ell^\alpha \left( I_{ij}^\alpha \right)^{-1} \ell_j^\alpha,
\]
with \( \Omega^\alpha = (I_{ij}^\alpha)^{-1} \ell_j^\alpha \) being the angular velocity of a molecule around its CM which for rigid molecules obeys \( \partial_t \Delta r^{\alpha \mu} = -\Delta r^{\alpha \mu} \times \Omega^\alpha \). The third term of Eq. (23) can also be simplified with the help of Eqs. (19)-(21):
\[
\sum_{\mu} \frac{p^\alpha \cdot \Delta p^{\alpha \mu}}{M^\alpha} \Delta r^{\alpha \mu} = \frac{p^\alpha}{2 M^\alpha} \left( \dot{K}_{ij}^\alpha - \xi_{ijk} \ell_j^\alpha \right).
\]

Substituting Eqs. (24) and (25) into (23) we get
\[
H_k(r) = \sum_{\alpha} \left[ \left( \frac{p^\alpha}{2 M^\alpha} \right)^2 + \frac{1}{2} \ell^\alpha \left( I_{ij}^\alpha \right)^{-1} \ell_j^\alpha + \frac{1}{2 M^\alpha} \nabla \cdot \left( \ell^\alpha \cdot p^\alpha - p^\alpha \cdot \dot{K}^\alpha \right) \right] \delta(r - r^\alpha).
\]

The last step is to coarse-grain using the method described in Materials and Methods, which gives a very similar Hamiltonian to the one of Eq. (9),
\[
H_k = \int \frac{d\mathbf{r}}{2 \rho} \left( \frac{g^c(r)^2}{2 \rho} + \frac{1}{2} \ell \cdot I^{-1} \cdot \ell + \frac{1}{2} \nabla \cdot (\ell \times v - v \cdot A) \right),
\]
where \( A \) here is a generalization of the one in the main text (for diatomic molecules \( K^\alpha = Q^\alpha \)).
IV. HYDRODYNAMIC MOMENTUM DENSITY DYNAMICS

Deriving the dynamics of the hydrodynamic momentum density (Eq. (5) of the main text) from that of the CM momentum density (Eqs. (3)-(4) of the main text) requires some non-trivial manipulation that we details in this section.

Starting from Eqs. (3)-(4) of the main text and using Eq. (2) of the main text we can write the dynamics of the hydrodynamic momentum:

\[
\dot{g}_i = \dot{g}_i^\epsilon + \frac{1}{2} \varepsilon_{ijk} \nabla_j \dot{\ell}_k - \nabla_j (v_j g_i) + \nabla_j \left( (v_j g_i - v_j^\epsilon g_i^\epsilon) - \nabla_i P + \eta_{ijkl} \nabla_i v_k \right).
\]

(28)

Converting \(\nabla_j(g_i^\epsilon v_j^\epsilon)\) to \(\nabla_j(g_i v_j)\) with the help of Eq. (2) of the main text we arrive at

\[
\dot{g}_i + \nabla_j (g_i v_j) = f_i - \nabla_i P + \nabla_j \left( \left[ \eta_{ijkl} - \frac{1}{2} \ell_n (\varepsilon_{iln} \delta_{jk} + \varepsilon_{jln} \delta_{ik}) \right] \nabla_i v_k + \frac{1}{2} \varepsilon_{ij} \tau_i \right),
\]

(29)

where we have dropped a non hydrodynamic term \(\sim \nabla (\nabla \times \ell)^2\), and we have also took advantage of the identity \(\nabla_j \varepsilon_{jln} \nabla_i \ell_n v_i = \nabla_j (v_j (\nabla \times \ell)_j + \varepsilon_{jln} \ell_n \nabla v_i) = 0\). The final step is to separate the velocity gradient tensor to its symmetric and antisymmetric parts. Then, the symmetric part of the first term in the square brackets of Eq. (29) gives the odd viscosity \(- (\ell_n/4) \gamma_{ijkl}^{\rho} \), while its antisymmetric part can be written as

\[
-\frac{\ell_n}{4} \left[ \varepsilon_{i} \varepsilon_{jlk}^{\rho} + \varepsilon_{j} \varepsilon_{ikl}^{\rho} - \varepsilon_{ik} \varepsilon_{j} \delta_{l}^{\rho} - \varepsilon_{jk} \varepsilon_{i} \delta_{l}^{\rho} \right] \nabla_i v_k = -\frac{\ell_n}{4} \varepsilon_{ikm} \varepsilon_{jst} \left( \varepsilon_{i} \varepsilon_{jlk}^{\rho} + \varepsilon_{j} \varepsilon_{ikl}^{\rho} \right) \nabla_s v_t.
\]

(30)

We then use

\[
\varepsilon_{ikm} \varepsilon_{jst} \varepsilon_{i} \varepsilon_{jlk}^{\rho} \nabla_s v_t = (\varepsilon_{i} \varepsilon_{jlk}^{\rho} - \varepsilon_{i} \varepsilon_{lk}^{\rho}) \nabla_i v_k,
\]

(31)

to obtain

\[
-\frac{\ell_n}{4} \left[ \varepsilon_{i} \varepsilon_{jlk}^{\rho} + \varepsilon_{j} \varepsilon_{ikl}^{\rho} - \varepsilon_{ik} \varepsilon_{j} \delta_{l}^{\rho} - \varepsilon_{jk} \varepsilon_{i} \delta_{l}^{\rho} \right] \nabla_i v_k = -\frac{\ell_n}{4} \left( \varepsilon_{i} \varepsilon_{jlk}^{\rho} + \varepsilon_{j} \varepsilon_{ikl}^{\rho} - 2 \varepsilon_{ik} \varepsilon_{j} \delta_{l}^{\rho} \right) \nabla_i v_k,
\]

(32)

where we also utilized Eq. (31) with interchanging \(\iota \leftrightarrow \jmath\). This gives Eq. (5) of the main text.

V. ELIMINATION OF ANGULAR MOMENTUM IN THE CM DYNAMICS

In this section we eliminate the spin angular momentum (SAM) using Eq. (3) of the main text and write the CM dynamics of Eq. (4) of the main text after elimination of \(\ell\). We then obtain Eq. (5) of the main text for the hydrodynamic momentum dynamics with \(\ell \rightarrow \ell_0\). This procedure is equivalent to what we do in the main text, namely, writing the hydrodynamic momentum dynamics and then eliminate \(\ell\).

We start by solving for \(\Omega\) in the hydrodynamic limit from Eq. (3) of the main text:

\[
\dot{\ell}_i (\mathbf{r}) + \nabla_j (\dot{\ell}_i v_j^\epsilon) = -\Gamma (\Omega_i - \omega_i^\epsilon) + \tau_i,
\]

(33)

where \(\ell = \rho \dot{I} \Omega\). We write \(\Omega = \Omega_0 + \delta \Omega\) and \(\rho = \rho_0 + \delta \rho\) while assuming that \(\delta \Omega\) and \(\delta \rho\) are of the order of \(\nabla v^\epsilon\). Here we include also possible friction with the surface such that \(\tau = \tau^s - \Gamma^s \Omega\). With the help of the continuity equation we then have:

\[
(\partial_t + \Gamma^T / I_0) \Omega_0 = \dot{\tau}(\mathbf{r}, t) / I_0,
\]

(34)

\[
(\partial_t + \Gamma^T / I_0) \delta \Omega = \Phi(\mathbf{r}, t) / I_0,
\]

(35)

where \(\Phi(\mathbf{r}, t) \equiv \Gamma \omega - I_0 v^\epsilon \cdot \nabla \Omega_0\), \(I_0 = \dot{I} \rho_0\), and \(\Gamma^T = \Gamma + \Gamma^{ex}\). The solution to Eqs. (34)-(35) is

\[
\Omega_0(\mathbf{r}, t) = \Omega_0(t = 0) e^{-t} + \int_0^t \frac{\tau(\mathbf{r}, u)}{\Gamma^T} e^{-(t-u)} du,
\]

(36)

\[
\delta \Omega(\mathbf{r}, t) = \delta \Omega(t = 0) e^{-t} + \int_0^t \Phi(\mathbf{r}, u) / \Gamma^T e^{-(t-u)} du,
\]

(37)
with $T \equiv T^T/t/I_0$. In the hydrodynamic limit $T \gg 1$ (which is equivalent to the zero-frequency limit) we are left with
\begin{equation}
\Omega_0(r, t) \simeq \bar{\tau}(r, t)/T^T; \quad \delta\Omega(r, t) \simeq \Phi(r, t)/T^T,
\end{equation}
such that $\ell^0 = I_0\bar{\tau}(r, t)/T^T$. This solution is valid for any $\bar{\tau}$, even if it is inhomogeneous and not constant in time. It essentially sets $\Omega = 0$ such that with the aid of the continuity equation we find that
\begin{equation}
\dot{\ell} \simeq -j\Omega_0 \nabla \cdot (\rho_0 \mathbf{v}^c).
\end{equation}
This result suggests that a description in terms of the CM momentum alone will not be sufficient as it will not obey conservation of total angular momentum. This is because the SAM within a fluid parcel is not constant, $\dot{\ell}_i + \nabla_j (\ell_i v_j^c) = I_0\mathbf{v}^c \cdot \nabla \Omega_0$. Only when $\Omega_0$ is homogeneous is the SAM constant within a fluid parcel and the CM momentum description obeys the balance of angular momentum. We will see this explicitly below.

### A. CM stress tensor

Let us now write the CM dynamics after elimination of $\ell$. Before elimination of $\ell$ the CM dynamics follows Eq. (4) of the main text $\dot{\mathbf{g}}^c_i + \nabla_j (v_j^c g^c_i) = \nabla_j \sigma^c_{ij} + f_i$ where the CM stress tensor is:
\begin{equation}
\sigma^c_{ij} = -P\delta_{ij} + \eta^c_{ijkl}\nabla_i v^c_k + \frac{1}{2}\delta_{ijk}(\Omega_k - \omega_k).
\end{equation}
It is convenient to use Eq. (33) to simplify the antisymmetric part of the stress:
\begin{equation}
\sigma^c_{ij} = -P\delta_{ij} + \eta^c_{ijkl}\nabla_i v^c_k + \frac{1}{2}\right[i - \nabla_k (\ell_n v^c_k).\end{equation}
Then, by using Eq. (39) we find that
\begin{equation}
\sigma^c_{ij} = -P\delta_{ij} + \eta^c_{ijkl}\nabla_i v^c_k + \frac{1}{2}\left[i - \nabla_k (\ell_n v^c_k)\right] + \frac{1}{2}\nabla_i (\tau_n - \ell_n v^c_i).
\end{equation}
It is clear that even after elimination of $\ell$, odd viscosity does not appear in the CM stress tensor. Only when the active torques are inhomogeneous, the angular velocity affects the CM stress by adding an antisymmetric term to it. This term, not only seems to break Galilean invariance in the CM dynamics, but also breaks conservation of angular momentum (the only antisymmetric term allowed in the stress tensor is the external torque, $\frac{1}{2}\varepsilon_{ijn}\tau_n$). The latter explicitly shows that some angular momentum is “missing” in the CM description, which is also consistent with the fact that even after relaxation of $\ell$ the SAM of a fluid parcel is changing with time, see Eq. (39).

In the case of homogeneous $\bar{\tau}$ the CM stress tensor can be written in the regular form [8] with the addition of an antisymmetric stress directly due to the external torque:
\begin{equation}
\sigma^c_{ij} = -P\delta_{ij} + \eta^c_{ijkl}\nabla_i v^c_k + \frac{1}{2}\varepsilon_{ijn}\tau_n.
\end{equation}
As discussed in the previous subsection, only in this case does the CM momentum obey the balance of angular momentum. In the case that there is also no external friction $\Gamma^c_{\text{ex}} = 0$, the antisymmetric term vanishes in the bulk, such that in the bulk the CM stress is symmetric and therefore conserve angular momentum. Remarkably, as we shall see in the next subsection, even in this simplified case, the flux of hydrodynamic momentum (which is the hydrodynamic momentum stress tensor) is different from the flux of CM momentum (the CM stress tensor above). This is manifested in the odd terms that only emerge in the hydrodynamic momentum dynamics.

### B. Hydrodynamic momentum stress tensor

Following the previous subsection, we show here how to obtain Eq. (5) of the main text after elimination of $\ell$ in the CM dynamics. The derivation is similar to that in Sec. IV where the only difference is the change of $\sigma^c$ (see Eq. (42)) due to elimination of $\ell$. We start from writing a modified Eq. (28):
\begin{equation}
\dot{\mathbf{g}}^c_i + \nabla_j (v_j^c g^c_i) = \dot{\mathbf{g}}^c_i + \frac{1}{2}\varepsilon_{ijk}\nabla_j \ell_k + \nabla_j (v_j^c g^c_i) = \nabla_j (v_j^c g^c_i - v^c_j \dot{\mathbf{g}}^c_i) + \frac{1}{2}\varepsilon_{ijk}\nabla_j \ell_k + \nabla_j \sigma^c_{ij} + f_i.
\end{equation}
Then, following the same derivation of Eq. (29) and using Eq. (42) we have:

\[ \dot{g}_i + \nabla_j (g_i v_g) = f_i - \nabla_i P + \frac{1}{2} \varepsilon_{ijk} \nabla_j \dot{g}_k + \nabla_j \left[ \left( \eta_{ijkl} - \frac{1}{2} \delta^0_n \left[ \varepsilon_{ilk} \delta_{jk} + \varepsilon_{jln} \delta_{ik} \right] \right) \nabla_l v_k + \frac{1}{2} \delta_{ijn} \left( \tau_n + \bar{H}_n \nabla_k (\rho_0 v_k) \right) \right]. \]  

(45)

Substituting Eq. (39) we finally get

\[ \dot{g}_i + \nabla_j (g_i v_g) = f_i - \nabla_i P + \nabla_j \left[ \left( \eta_{ijkl} - \frac{1}{2} \delta^0_n \left[ \varepsilon_{ilk} \delta_{jk} + \varepsilon_{jln} \delta_{ik} \right] \right) \nabla_l v_k + \frac{1}{2} \delta_{ijn} \tau_n \right], \]  

(46)

which is the same as Eq. (29) with \( \ell \rightarrow \ell^0 = \bar{I}_0 \bar{T}/T^T \). The derivation of Eq. (5) of the main text from this point is the same as after Eq. (29).

VI. SIMPLE EXAMPLES FOR OUR COARSE-GRAINING PROCESS

To verify the validity of our coarse-graining approach let us apply it to the kinetic energy density of a monoatomic gas

\[ \mathcal{H}_k(r) = \sum_\alpha \left( \frac{p_\alpha^2}{2m} \right) \delta(r - r^\alpha). \]  

(47)

This is done straightforwardly by writing \( A^\alpha = p^\alpha \) and \( B^\alpha = p^\alpha/(2m) \). Then, using Eq. (22) in Materials and Methods the coarse-grained kinetic energy is

\[ \epsilon(r) = \frac{g(r)^2}{2\rho(r)}, \]  

(48)

as expected. Here \( \rho(r) = mn(r) \) and \( g(r) = \sum_\alpha p^\alpha W(r - r^\alpha) \).

Note that we can apply our prescription also for functions incorporating more microscopic variables. For example, if the particles’ masses are not equal,

\[ \mathcal{H}_k(r) = \sum_\alpha \left( \frac{p_\alpha^2}{2m_\alpha^2} \right) \delta(r - r^\alpha). \]  

(49)

Let us write in general

\[ \hat{O}(r) = \sum_\alpha A^\alpha (C^\alpha)^{-1} B^\alpha \delta(r - r^\alpha). \]  

(50)

Then, after using \( B(r) = B(r)/n(r) \) as we do in Materials and Methods, and replacing \( (C^\alpha)^{-1} \) with \( (C^\alpha)^{-1} \) within the CG volume:

\[ \left( C^\alpha \right)^{-1} = \frac{\int_{\Delta V} du \sum_\alpha (C^\alpha)^{-1} W(r - u - r^\alpha)}{\int_{\Delta V} du \sum_\alpha W(r - u - r^\alpha)} \simeq \left[ \hat{O}(r) \right]^{-1}, \]  

(51)

we find that

\[ \hat{O}(r) \simeq A(r) \left[ \hat{C}(r) \right]^{-1} B(r). \]  

(52)

For the example of the kinetic energy above this clearly gives \( \mathcal{H}_k \simeq g^2/(2\rho) \) as desired. Note that in Eq. (51) we have used

\[ \frac{1}{\Delta V} \int_{\Delta V} du \sum_\alpha (C^\alpha)^{-1} W(r - u - r^\alpha) \simeq n(r) \left[ \hat{C}(r) \right]^{-1}. \]  

(53)

Next we want to examine coarse-graining of functions that include gradients such as

\[ \hat{L}(r) = \sum_\alpha A^\alpha B^\alpha \nabla \delta(r - r^\alpha). \]  

(54)
Clearly the integrated value of $\hat{L}(r)$ must give a boundary term because:

$$L \equiv \int \text{d}r \hat{L}(r) = \int \text{d}r \nabla \left[ \sum_{\alpha} A^\alpha B^\alpha \delta(r - r^\alpha) \right] = \int_S \text{d}S \sum_{\alpha} A^\alpha B^\alpha \delta(s - s^\alpha). \quad (55)$$

Within our coarse-graining scheme we have

$$\hat{L}(r) \simeq \nabla \left[ \frac{A(r)B(r)}{n(r)} \right], \quad (56)$$

which gives a boundary term as it should.

**VII. EXCEPTIONAL POINTS**

Our analysis showed that the non-Hermitian matrices characterizing the excitations of active systems of rotating objects exhibit an exceptional point at which two eigenvalues coalesce and the system appears to “lose” an eigenvalue. In terms of the differential equation we are solving this means we lost a fundamental solution, which is of course not true. In much of the literature (see e.g., Refs. [9–11]) the discussion on exceptional points (EPs) was focused on the EPs marking a breaking of the (generalized) $\mathcal{P}\mathcal{T}$ symmetry and non-reciprocal phase transition [9]. Here follow work that explains EPs through the textbook example of a damped harmonic oscillator (Ref. [12] is the most elaborated, but see also the SI of Ref. [9] and Ref. [10]) and explain where is the “lost” fundamental solution.

**A. Exceptional points in a damped harmonic oscillator**

Let us start with the equation of a one-dimensional damped harmonic oscillator:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0. \quad (57)$$

Here, $\omega_0$ is the natural frequency of the undamped oscillator and $\gamma = \Gamma/m$, the inverse decay time, is the friction coefficient $\Gamma$ divided by particle mass $m$. Following standard procedure, we assume a solution of the form $x(t) \propto e^{-\alpha t}$ which leads to

$$\alpha^2 + \gamma \alpha + \omega_0^2 = 0 \quad (58)$$

with two solutions for $\alpha$,

$$\alpha_{\pm} = \frac{1}{2} \left( \gamma \pm \sqrt{\gamma^2 - 4\omega_0^2} \right), \quad (59)$$

and associated eigenfunctions, $\exp(-\alpha_{\pm} t)$. When $\gamma^2 > 4\omega_0^2$, the over-damped regime, the two modes decay exponentially, and when $\gamma^2 < 4\omega_0^2$, both modes oscillate and decay (underdamped regime). When $\gamma^2 = 4\omega_0^2$, there is only one solution, $\alpha = \gamma/2$, and the system is at the exceptional point. Of course even when $\gamma^2 = 4\omega_0^2$, Eq. (57) must have two solutions, the first of which is the simple exponential one just discussed and the second of which can be found by considering the limit as the EP is approached. To this end, we set

$$\alpha_{\pm} = \frac{1}{2} \gamma \pm \sqrt{\Delta}, \quad (60)$$

where $\Delta = (\gamma^2/4) - \omega_0^2$, and we construct two fundamental solutions using the sum and difference of the two eigenfunctions:

$$\psi_1(t) = \frac{1}{2} \left( e^{-\gamma t/2 + \sqrt{\Delta} t} + e^{-\gamma t/2 - \sqrt{\Delta} t} \right),$$

$$\psi_2(t) = \frac{1}{2\sqrt{\Delta}} \left( e^{-\gamma t/2 + \sqrt{\Delta} t} - e^{-\gamma t/2 - \sqrt{\Delta} t} \right). \quad (61)$$

The first fundamental solution, $\psi_1$ has a well-defined $\Delta \to 0$ limit that is $e^{-\gamma t/2}$. The second fundamental solution, $\psi_2$, is divided by $\sqrt{\Delta}$ to give a proper limit as $\Delta \to 0$, which is $te^{-\gamma t/2}$. Thus we have two linearly independent solutions at the EP:

$$\psi_1(t) = e^{-\gamma t/2}, \quad \text{and} \quad \psi_2(t) = te^{-\gamma t/2}. \quad (62)$$
We can then write a general solution to Eq. (57) (at the EP) for some initial condition \( x(t = 0) = x_0 \) as \( x(t) = a\psi_1(t) + b\psi_2(t) \), where \( a \) and \( b \) are determined from the initial condition.

Though linearly independent, these \( \psi_1 \) and \( \psi_2 \) are not orthogonal because \( \int_0^\infty dt\psi_1(t)\psi_2(t) = \gamma^{-2} \). Although not necessary, one can create an orthogonal pair by adding to \( \psi_2(t) \) a term proportional to \( \psi_1 \). A trivial exercise establishes that the function \( \tilde{\psi}_2 = (1 - \gamma t)e^{-\gamma t/2} = \psi_1 - \gamma \psi_2 \) is orthogonal to \( \psi_1 \).

### B. Exceptional points using matrix representation

The damped oscillator equation can be couched in terms of a matrix operating on the vector \( \mathbf{u}(t) = [x(t), v(t) = \dot{x}(t)]^T \), which may be more familiar to those who study exceptional points. The equation of motion for \( \mathbf{u}(t) \) is

\[
\dot{\mathbf{u}} = \begin{pmatrix}
0 & 1 \\
-\omega_0^2 & -\gamma
\end{pmatrix} \mathbf{u} \equiv \mathbf{M} \mathbf{u} .
\]  

(64)

\( \mathbf{M} \) has two eigenvectors, \( \mathbf{u}^\pm(t) = [1, -\alpha^\pm]^T \), associated with the two eigenvalues \( \alpha^\pm \). At the EP, where \( \Delta = 0 \), the matrix \( \mathbf{M} \) is defective (not diagonalizable) and has only one eigenvector \( \mathbf{u}_1 = [1, -\gamma/2]^T \). It does not have a set of linearly independent eigenvectors that form a basis.

Nevertheless, in general, for any \( n \times n \) matrix one can form a basis using generalized eigenvectors generated by Jordan chains. For a generic matrix \( \mathbf{M} \) and eigenvalue \( \lambda \) a generalized eigenvector of rank \( m \) is defined as

\[
[\mathbf{M} - \lambda \mathbf{I}]^m \mathbf{v}_m = 0 ; \quad [\mathbf{M} - \lambda \mathbf{I}]^{m-1} \mathbf{v}_m \neq 0 ,
\]

(65)

where \( \mathbf{v}_1 \) (a generalized eigenvector of rank 1) is clearly an ordinary eigenvector. Once one knows a generalized eigenvector \( \mathbf{v}_m \), the Jordan chain of generalized eigenvectors can be found using \( [\mathbf{M} - \lambda \mathbf{I}] \mathbf{v}_{m+1} = \mathbf{v}_m \). The generalized eigenvectors can be used as a generalized modal matrix \( \mathbf{P} \) (a matrix in which the columns are the generalized eigenvectors) to find its Jordan normal form \([12]\) using \( \mathbf{J} = \mathbf{P}^{-1} \mathbf{M} \mathbf{P} \). The Jordan normal form is a block diagonal matrix, where each block is a Jordan block – it has the eigenvalues on the diagonal, ones on the superdiagonal and zeros elsewhere. This form simplifies matrix functions (similarly to matrix functions on diagonal matrices).

In our example we solve \( [\mathbf{M} - \lambda \mathbf{I}] \mathbf{u}_2 = \mathbf{u}_1 \) for \( \lambda = \gamma/2 \) and find the generalized eigenvector \( \mathbf{u}_2 = [0, 1]^T \). With the help of the modal matrix

\[
\mathbf{P} = \begin{pmatrix}
1 & 0 \\
-\gamma/2 & 1
\end{pmatrix},
\]

(66)

we can write \( \mathbf{M} \) in Jordan normal form

\[
\mathbf{J} = \mathbf{P}^{-1} \mathbf{M} \mathbf{P} = \begin{pmatrix}
-\gamma/2 & 1 \\
0 & -\gamma/2
\end{pmatrix}.
\]

(67)

This matrix is a Jordan block of size two, hence, the EP is of order two (in general a Jordan block of size \( m \) is related to an EP of order \( m \) [9]).

Notice that the generalized eigenvectors we have found \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are not orthogonal. It is, however, straightforward to find an orthogonal generalized eigenvector because \( \bar{\mathbf{u}}_2 = \mathbf{u}_2 + \mathbf{c}\mathbf{u}_1 \) (\( \mathbf{c} \) being a scalar) is also a generalized eigenvector, and for \( \mathbf{c} = -\mathbf{u}_1 \cdot \mathbf{u}_2/\mathbf{u}_1^2 \) we get that \( \mathbf{u}_1 \cdot \bar{\mathbf{u}}_2 = 0 \). Now that we know the generalized eigenvectors, we can write the formal solution to Eq. (64):

\[
\mathbf{u}(t) = \exp(\mathbf{M}t) \mathbf{u}_0 ,
\]

(68)

where \( \mathbf{u}_0 = \mathbf{u}(t = 0) \), and since \( \mathbf{u}_1 \) and \( \bar{\mathbf{u}}_2 \) span the space we can write \( \mathbf{u}_0 = a\mathbf{u}_1 + b\bar{\mathbf{u}}_2 \). Next, we wish to be able to write the general solution to Eq. (64) as in Sec. VIIA. To do so we calculate

\[
\exp(\mathbf{M}t) \mathbf{u}_1 = \sum_{n=0}^{\infty} \frac{\mathbf{M}^n}{n!} t^n \mathbf{u}_1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} t^n \mathbf{u}_1 = e^{\lambda t} \mathbf{u}_1 ,
\]

\[
\exp(\mathbf{M}t) \mathbf{u}_2 = \mathbf{u}_2 + \sum_{n=1}^{\infty} \frac{\mathbf{M}^{n-1}}{n!} t^n (\mathbf{u}_1 + \lambda \mathbf{u}_2) = e^{\lambda t} \mathbf{u}_2 + \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} t^n n \mathbf{u}_1 = e^{\lambda t} (\mathbf{u}_2 + t \mathbf{u}_1) ,
\]

(69)

and find that \( \mathbf{u}(t) = a\psi_1(t)\mathbf{u}_1 + b[\psi_1(t)\bar{\mathbf{u}}_2 + \psi_2(t)\mathbf{u}_1] \), with \( \psi_1(t)\mathbf{u}_1 \) being an eigenfunction (the usual normal mode) and \( \psi(t) = \psi_1(t)\bar{\mathbf{u}}_2 + \psi_2(t)\mathbf{u}_1 \) is a generalized eigenfunction. This solution is very similar to what we found in Sec. VIIA, where the generalized eigenfunctions are composed of \( \psi_1 \) and \( \psi_2 \), the two linearly independent solutions.
To conclude, the generalized eigenvectors generate the linearly independent solutions of the differential equations as they should. Notably, without using the generalized eigenvectors one cannot construct a general solution for the differential equation (in our case, Eq. (64)).

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