Curvature Invariants for the Alcubierre and Natário Warp Drives

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Abstract: A process for using curvature invariants is applied to evaluate the metrics for the Alcubierre and the Natário warp drives at a constant velocity. Curvature invariants are independent of coordinate bases, so plotting these invariants will be free of coordinate mapping distortions. As a consequence, they provide a novel perspective into complex spacetimes such as warp drives. Warp drives are the theoretical solutions to Einstein’s field equations that allow the possibility for faster-than-light (FTL) travel. While their mathematics is well established, the visualisation of such spacetimes is unexplored. This paper uses the methods of computing and plotting the warp drive curvature invariants to reveal these spacetimes. The warp drive parameters of velocity, skin depth and radius are varied individually and then plotted to see each parameter’s unique effect on the surrounding curvature. For each warp drive, this research shows a safe harbor and how the shape function forms the warp bubble. The curvature plots for the constant velocity Natário warp drive do not contain a wake or a constant curvature indicating that these are unique features of the accelerating Natário warp drive. Keywords: Warp Drive, Curvature Invariant, General Relativity.

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1 Introduction

No particle may have a local velocity that exceeds the speed of light in a vacuum, \( c \), in Newtonian mechanics and special relativity. But, a particle’s global velocity may exceed \( c \) while its local velocity obeys the prior statement in general relativity. Alcubíerre noticed that spacetime itself may expand and contract at arbitrary rates [1]. He proposed pairing a local contraction of spacetime in front of the spaceship with a local expansion of spacetime behind it. While locally the spaceship remains within its own light cone and never exceeds \( c \), globally the relative velocity, which is defined as the proper spatial distance divided by proper time, may be much greater than \( c \) due to the contraction and expansion of spacetime. Distant observers will perceive the ship to be moving at a global velocity greater than \( c \), and the spaceship will be able to make a trip to a distant star in an arbitrarily short proper time. He named the faster-than-light (FTL) propulsion mechanism based on this principle a “warp drive.”

A spaceship using an FTL warp drive must obey eight prerequisites to carry a human to a distant star [3]. First, the rocket equation does not describe the portion of the flight undergoing FTL travel. Second, the duration of the trip to the distant star may be reduced to less than one year as seen both by the passengers in the warp and by stationary observers outside of the warp. Third, proper time measured by the passengers will not be dilated by any relativistic effects. Fourth, any tidal-gravity accelerations acting on any passengers needs to be less than \( g_{\oplus} \), which is the acceleration of gravity near the Earth’s surface. Fifth, the local speed of any passengers should be less than \( c \). Sixth, the matter of the passengers must not couple with any material used to generate the FTL space warp. Seventh, the FTL warp drives should not generate an event horizon. Eight, the passengers riding the FTL warp should not encounter a singularity inside or out of it.

The two most well known solutions to Einstein’s equations that obey these eight requirements are traversable wormholes and warp drives [1, 2, 4, 5, 6, 7, 8, 9, 10, 12]. While Einstein’s equations allow for their possibility, each solution remains at the theoretical level. Furthermore, constructing either solution in a lab is not readily accessible due to engineering constraints. In the present work, the authors will apply this procedure to analyze the Alcubierre and Natário Warp Drives at a constant velocity.

Research into FTL warp drives has advanced tremendously since Alcubierre’s original proposal. Kraskinov developed a non-tachyonic FTL warp bubble [4]. Van Den Broeck reduced the amount of energy required by Alcubierre’s warp drive by positing a warp bubble with a microscopic surface area and a macroscopic volume inside [5]. His modification reduced the energy requirements to form the warp bubble to only a few solar masses and his geometry has more lenient violation of the null-energy-conditions (NEC). Later, Natário presented a warp drive metric such that zero spacetime expansion occurs [2]. His warp drive “slides” through the exterior spacetime at a constant global velocity with the contracted spacetime in front of it balanced with the expansion of the spacetime behind it. Recently, Loup expanded Natário’s work to encompass a changing global velocity that would accelerate from rest to a multiple of \( c \) [9, 10]. Finally, recent research computed the complete Einstein
tensor $G_{\mu\nu}$ for the Alcubierre warp drive and derived a constraint on the trace of the energy momentum tensor that satisfied the weak, strong, null and dominant energy conditions from a dust matter distribution as it source [11].

While much progress has been made developing the physics of a warp drive spacetime, visualizing one is unexplored. The outside of the warp bubble is causally disconnected from the interior [1, 2, 12]. As a consequence, computer simulations of the spacetime surrounding the ship need to be developed to plot the flight and steer the warp bubble. To date, the only method to plot the surrounding spacetime is to compute the York time, which is defined as

$$\Theta = \frac{v}{c} \frac{x - x_s}{r_s} \frac{df}{dr},$$

to map the surrounding volume expansion [1, 12]. While the York time is appropriate when the 3-geometry of the hypersurfaces is flat, it will not contain all information about the surrounding spacetime in non-flat 3-geometries. Alternatively, curvature invariants are independent of coordinate bases, so plotting these invariants for a warp drive spacetime will be free of coordinate mapping distortions.

Curvature invariants are scalar products of the Riemann, Ricci or Weyl tensors, or their covariant derivatives. They are functions of the metric itself, the Riemann tensor, and its covariant derivatives as proven by Christoffel [13]. The Riemann invariants allow a manifestly coordinate invariant characterization of certain geometrical properties of spacetime [14]. When special non-degenerate cases are taken into account, a set of seventeen curvature invariants are required to completely describe a spacetime. The set of invariants proposed by Carminati and McLenaghan (CM) have the attractive properties of general independence, lowest possible degree, and contained a minimal independent set for any Petrov type and choice of Ricci Tensor [15]. For Class $B_1$ spacetimes, which include all hyperbolic spacetimes such as the general warp drive line element, only four CM invariants, $(R, r_1, r_2, \text{and } w_2)$, are necessitated to form a complete set [16].

Henry et al. [17] recently studied the hidden interiors of the Kerr-Newman black hole by computing and plotting all seventeen of its curvature invariants. They exposed surprisingly complex structures inside the interior of the Kerr-Newman black hole more so than what is normally suggested by textbook depictions using coordinate-dependent methods. In addition, curvature invariants has been calculated to study many other black hole metrics [18, 19, 20, 21]. Alternatively to the research in this paper, a spacetime has been investigated that admits closed timelike curves has been identified as cylindrically symmetric, Petrov type D and containing a naked curvature singularity [22]. Previously, the authors calculated the curvatures curvature invariants and plotting them to analyze several wormhole solutions and the accelerating Natário warp drive [23, 24]. All their work inspired the authors of this paper to investigate the curvature invariants of additional warped spacetimes such as the Alcubierre and Natário warp drives at a constant velocity that are considered in this paper.

2 Method to Compute the Invariants

The complete set of CM invariants requires the following: the metric $g_{ij}$ and a null tetrad $(l_i, k_i, m_i, \bar{m}_i)$ [25, 26]. The metric can be used to calculate the affine connection $\Gamma^{i}_{jk}$,
the Riemann tensor $R_{ijkl}$, the Ricci tensor $R_{ij}$, the Ricci scalar $R$, the trace free Ricci tensor $S_{ij}$ and the Weyl tensor $C_{ijkl}$. The indices $\{i, j, ...\}$ range from $\{0, 3\}$ in $(3+1)$ dimensions. The Newman-Penrose (NP) curvature components require specifically the null tetrad, the Ricci Tensor, and the Weyl Tensor. The NP components are presented in [26]. The thirteen different CM invariants are defined in [15]. Only four of these invariants are required by the syzygies for Class B spacetimes: the Ricci Scalar, the first two Ricci invariants, and the real component of the Weyl Invariant J [16]. In terms of the NP curvature coordinates, the complete set of the CM invariants for the Alcubierre and the Natário warp drive spacetimes are:

\[ R = g_{ij} R^{ij}, \]  
\[ r_1 = \frac{1}{4} S^i_i S^j_j = 2\Phi_{20}\Phi_{02} + 2\Phi_{22}\Phi_{00} - 4\Phi_{12}\Phi_{10} - 4\Phi_{21}\Phi_{01} + 4\Phi_{11}^2, \]  
\[ r_2 = -\frac{1}{8} S^i_i S^k_k S^j_j = 6\Phi_{02}\Phi_{21}\Phi_{10} - 6\Phi_{11}\Phi_{02}\Phi_{20} + 6\Phi_{01}\Phi_{12}\Phi_{20} - 6\Phi_{12}\Phi_{01}\Phi_{20} - 6\Phi_{22}\Phi_{01}\Phi_{10} + 6\Phi_{22}\Phi_{11}\Phi_{00}, \]  
\[ w_2 = -\frac{1}{8} C_{ijkl} C^{ijmn} C^{kl} = 6\Psi_4 \Psi_0 \Psi_2 - 6\Psi_2^3 - 6\Psi_1^2 \Psi_4 - 6\Psi_3^2 \Psi_0 + 12\Psi_2 \Psi_1 \Psi_3. \]  

The tetrad components of the traceless Ricci Tensor are $\Phi_{00}$ through $\Phi_{22}$ [26]. The complex tetrad components $\Psi_0$ to $\Psi_5$ are the six complex coefficients of the Weyl Tensor due to its tracelessness.

3 Warp Drive Spacetimes

Alcubierre and Natário developed warp drive theory using $(3+1)$ ADM formalism. It decomposes spacetime into space-like hyper-surfaces parametrized by the value of an arbitrary time coordinate $dx^0$ [27, 28]. The proper time $d\tau = N(x^\alpha, x^0) \, dx^0$ separates two nearby hypersurfaces, $x^0$ and $x^0 + dx^0$. The ADM four-metric is

\[ g_{ij} = \begin{pmatrix} -N^2 - N_\alpha N_\beta g^{\alpha\beta} & N_\beta \\ N_\alpha & g_{\alpha\beta} \end{pmatrix}. \]  

$N$ is the lapse function between the hypersurfaces. $N_\alpha$ is the shift vector between each hypersurface’s internal 3-geometry. A spacetime that moves at an arbitrary but constant velocity corresponds to a choice of the lapse function $N$ equaling the identity and the shift vector $N_\alpha$ being a time dependent vector field. Since the velocity may exceed $c$, these choices form a warp drive space time traveling at a constant velocity.

A warp drive spacetime is defined as a globally hyperbolic spacetime $(M, g)$, where $M = \mathbb{R}^4$ and $g$ is given by the line element

\[ ds^2 = -dt^2 + \sum_{\alpha=1}^3 (dx^\alpha - X^\alpha dt)^2. \]
for three unspecified bounded smooth functions \((X^\alpha) = (X, Y, Z)\) in Cartesian coordinates \([1, 2]\). As it is a globally hyperbolic spacetime, it is classified as \(B_1\) \([16]\). Therefore, the invariants \((1)\) to \((4)\) for the complete set describing a warp drive spacetime. The functions form the time-dependent vector field in Euclidean 3-space given by the equation \([2]\):

\[
X = X^\alpha \frac{\partial}{\partial x^\alpha} = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z}.
\]  

(7)

Each warp drive spacetime considered in this article corresponds to specific choices of \((7)\). The future pointing normal covector to the Cauchy surface is

\[
n^i = -dt \Leftrightarrow n^i = \frac{\partial}{\partial t} + X^\alpha \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial t} + X.
\]

Any observer that travels along this covector is a Eulerian observer and a free-fall observer. Finally, a warp bubble with a constant velocity of \(v_s(t)\) results from the vector field if in the interior \(X = 0\) and in the exterior \(X = -v_s(t)\). The warp drive line elements considered in this paper fulfill these conditions.

### 3.1 Alcubierre’s Warp Drive with a Constant Velocity

The line element in the parallel covariant \((3 + 1)\) ADM for the Alcubierre warp drive is \([1]\)

\[
d s^2 = -dt^2 + (dx - v_s f(r_s) dt)^2 + dy^2 + dz^2.
\]  

(8)

It is in traditional Cartesian coordinates with \((-\infty < x, y, z < \infty)\) and has an origin at the beginning of the flight. The analysis in the following Section 4 relies on these choices made in the original paper. The ship is assumed to move only along the x-axis of a Cartesian coordinate system. As a consequence, the shift vector Eq.\((7)\) will only have an x-component i.e. \((X, Y, Z) = (-v_s(t) f(r_s(t)), 0, 0)\). The internal 3-geometry of the hypersurfaces is flat, so \(g_{\alpha\beta} = \delta_{\alpha\beta}\) in Eq. \((5)\). The velocity vector is given by \(v_s(t) = \frac{dx_s(t)}{dt}\) where the subscript “s” denotes the position of the spaceship. \(v_s(t)\) is the arbitrary speed Eulerian observers inside the warp bubble move in relation to Eulerian observers outside the warp bubble. The radial distance is \(r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}\). It is the path a Eulerian observer takes starting inside the warp bubble and traveling to the outside of the bubble. The Alcubierre warp drive continuous shape function, \(f(r_s)\), defines the shape of the warp bubble. It is

\[
f(r_s) = \frac{\tanh(\sigma (r_s + \rho) - \tanh \sigma (r_s - \rho))}{2 \tanh \sigma \rho}.
\]  

(9)

where \(\sigma\) is the skin depth of the warp bubble and \(\rho\) is the radius of the warp bubble. The parameters, \(\sigma > 0\) and \(\rho > 0\), are arbitrary apart from being positive. Appendix B describes how to find a comoving null tetrad from a line element. The comoving null tetrad describes light rays traveling parallel with the warp bubble. For Eq. \((8)\), it is

\[
\begin{align*}
  l_i &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + f(r_s) v_s & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \\
  k_i &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - f(r_s) v_s & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
  m_i &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & i \end{pmatrix}, & \\
  \bar{m}_i &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -i \end{pmatrix}.
\end{align*}
\]  

(10)
This choice for the null tetrad will position the origin of the invariants at the start of the flight. Any other appropriate choice will be related to what is derived in Section 5 by polynomial functions. Applying Eqs. (8) and (10) to the complete set of CM invariants Eqs. (1) to (4) gives the four CM invariants for the Alcubierre warp drive.

3.2 Natário Warp Drive at a Constant Velocity

Natário improved upon Alcubierre’s work by constructing a warp drive spacetime such that no net expansion occurred [2]. His warp drive spacetime chose a shift vector rotated around the $x$-axis in spherical polar coordinates. As a consequence, the line element is

$$ds^2 = (1 - X_{r_s}X^{r_s} - X_\theta X^\theta)dt^2 + 2(X_{r_s}dr_s + X_\theta d\theta)dt - dr_s^2 - r_s^2d\theta^2 - r_s^2\sin^2\theta d\phi^2. \quad (11)$$

The line element uses the standard spherical coordinates of $p = \rho \sin \theta; 0 \leq \theta \leq \pi$; $0 \leq \phi \leq 2\pi$). The analysis in Section 5 relies on the specific choices of Natário’s original paper. The vector field in (7) is set to

$$\mathbf{X} \sim -v_s(t)d[n(r_s)r_s^2\sin^2\theta d\phi] \sim -2v_sn(r_s)\cos\theta \mathbf{e}_{r_s} + v_s(2n(r_s) + r_sn'(r_s)) \sin\theta \mathbf{e}_\theta. \quad (12)$$

The internal 3-geometry is set to be flat, so $g_{\alpha\beta} = \delta_{\alpha\beta}$ in Eq. (5). Like the previous section, $v_s(t)$ is the constant speed for the Eulerian observers and $n(r_s)$ is the shape function of the warp bubble. The shape function is arbitrary other than the conditions $n(r_s) = \frac{1}{2}$ for large $r$ and $n(r_s) = 0$ for small $r$. The chosen shape function is

$$n(r_s) = \frac{1}{2}\left[1 - \frac{1}{2}\left(1 - \tanh[\sigma(r_s - \rho)]\right)\right], \quad (13)$$

where $\sigma$ is the skin depth of the bubble and $\rho$ is the radius of the bubble [10]. The front of the warp bubble corresponds to $\cos\theta > 0$, and a compression occurs there centered at a distance of $\rho$ along the radial direction. The back of the warp bubble corresponds to $\cos\theta < 0$, and an expansion occurs there centered at a distance of $\rho$ along the radial direction. The comoving null tetrad describes geodesics traveling parallel to the warp bubble. It is

$$l_i = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 + X_{r_s} \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad k_i = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 - X_{r_s} \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$m_i = \frac{1}{\sqrt{2}}\begin{pmatrix} X_\theta \\ 0 \\ -r \\ ir\sin\theta \end{pmatrix}, \quad \bar{m}_i = \frac{1}{\sqrt{2}}\begin{pmatrix} X_\theta \\ 0 \\ -r \\ -ir\sin\theta \end{pmatrix}. \quad (14)$$

It is emphasized at this moment that this choice for the null tetrad will center the invariants on the harbor as it it travels. Any other appropriate choice will be related to what is derived in Section 5 by polynomial functions. Plugging Eqs. (11) and (14) into the complete set of CM invariants Eqs. (1) to (4), the four CM invariants can be derived.
4 Invariants for the Alcubierre Warp Drive

The four curvature invariants in Eqs. (1) through (4) were computed and plotted in Mathematica® for the Alcubierre line element Eq. (8). Its invariant functions are

\[ R = \frac{1}{2} \sigma^2 v_s^2 \coth(\rho \sigma) \]

\[ \times \left( 4 \tanh \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \sech^2 \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \right. \]

\[ - 4 \tanh \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \sech^2 \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \]

\[ - 2 \sinh(\rho \sigma) \cosh^3(\rho \sigma) \sech^4 \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \sech^4 \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \]

\[ \times \left( \cosh \left( 2\sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) + \cosh \left( 2\sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) - 2 \cosh \left( 4\sigma \sqrt{(x - tv_s)^2} \right) + 4 \right) \]

\[ r_1 = \frac{1}{16} \sigma^4 v_s^4 \left( \cosh^4 (\rho \sigma) \sech^4 \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \sech^4 \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \right) \]

\[ \times \left( \cosh \left( 2\sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) + \cosh \left( 2\sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) - 2 \cosh \left( 4\sigma \sqrt{(x - tv_s)^2} \right) + 4 \right) \]

\[ + 2 \coth(\rho \sigma) \left( \tanh \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \sech^2 \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \right) \]

\[ \left. - \tanh \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \sech^2 \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \right) \right)^2 \]

\[ r_2 = 0, \]

\[ w_2 = -\frac{1}{288} \sigma^6 v_s^6 \left( 2 \coth(\rho \sigma) \left( \tanh \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \sech^2 \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \right) \right. \]

\[ \left. - \tanh \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \sech^2 \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \right) \right)^2 \]

\[ - \cosh^4 (\rho \sigma) \sech^4 \left( \sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) \sech^4 \left( \sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) \]

\[ \times \left( \cosh \left( 2\sigma \left( \sqrt{(x - tv_s)^2} - \rho \right) \right) + \cosh \left( 2\sigma \left( \rho + \sqrt{(x - tv_s)^2} \right) \right) - 2 \cosh \left( 4\sigma \sqrt{(x - tv_s)^2} \right) + 4 \right) \]

While Eqs. (15) through (18) are very complicated functions, several features are apparent from inspecting them directly. First, \( r_2 \) is zero. It will not be plotted in Appendix C as its plots will all be of a single smooth disc and reveal no curvature. Next, each non-zero invariant depends only on the tetrad elements \( t \) and \( x \). The axes of all plots are chosen to be these tetrad components. In addition, each of the non-zero invariants is proportional to both the skin depth \( \sigma^n \) and the velocity, \( v_s^n \). It should be expected that the magnitude of the invariants will then increase as these two parameters increase, with \( w_2 \) increasing the most. Finally, each invariant does not have any singularities inside the spacetime manifold.
In the next subsections, the non-zero CM invariants will be analyzed to see any individual effects of the parameters $v_s$, $\rho$ and $\sigma$.

The invariant plots for the Alcubierre warp drive have been included in Appendix C. They were plotted in Mathematica® using the Plot3D function. The $x$–coordinate, $t$–coordinate, and the magnitude of the invariants are along each axis. Since natural units were selected, the plots have been normalized such that $c = 1$ and a slope of 1 in the $x$ vs. $t$ plane corresponds with the warp bubble traveling at light speed. The presence of spacetime curvature may be detected on the plots by locating where the invariant functions have a non-zero magnitude. When the invariant functions have a positive magnitude, the spacetime has a positive curvature and vice versa.

The plots show a small range over the possible values of the parameters to demonstrate many of the basic features of each invariant. First, the shape of each invariant resembles the “top hat” function along each time slice [1]. However, there are some minor variations between each invariant. The Ricci scalar $R$ oscillates from a trough, to a peak, to a flat area, to a peak and back to a trough. The $r_1$ invariant simply has a peak with a flat area followed by another peak. The $w_2$ follows the reverse pattern as the $R$ wavering from a peak, into a trough, into a flat area, into a trough and returning back to a peak. In each invariant, a ship could safely surf along in the central flat area, which is named the harbor. The central harbor disappears in Figs. 1d through 1f because the plots lack precision. By plotting more points and consequently taking longer computational time, the central features will be recovered. The harbor’s width is less than the precision computed in these later plots. Inspecting the functions, the harbor remains, and choosing smaller time intervals allows it to reappear in the plots.

4.1 Invariant plots of velocity for the Alcubierre Warp Drive

The plots for variable velocities are Figs. 1 in this subsection and 5 and 6 in Appendix C. Varying the velocity has several effects. First, it linearly increases the slope of the peaks and troughs of the warp bubble in the $x$ vs. $t$ plane. This increase corresponds with a constant Newtonian velocity for the warp bubble. The warp bubble’s velocity acts exactly like $v = \frac{d}{t}$ and is a good check that the program has been encoded correctly. The warp bubble will cover an increasing amount of distance over time as observed by an Eulerian observer. Second, the velocity causes the magnitude of the invariants to decrease exponentially. This observation is in contrast to what was predicted by inspecting the leading terms of the invariants. It can be concluded that the additional terms overpower the leading term. Next, the shape of the warp bubble remains constant throughout the flight and the only effect of time is to increase the slope in the $x$ vs. $t$ plane. Finally, the plots have no discontinuities, agreeing with the previous statement that no singularities exist in the invariant functions. The invariant plots reveal nothing that will affect a spaceship inside the harbor.
Figure 1: Plots of the $R$ invariants for the Alcubierre warp drive while varying a velocity. $\sigma = 8 \, m^{-1}$ and $\rho = 1 \, m$ as Alcubierre originally suggested in his paper [1].
4.2 Invariant plots of skin depth for the Alcubierre Warp Drive

(a) Plot of Alcubierre $R$ with and $\sigma = 1 \text{ m}^{-1}$

(b) Plot of Alcubierre $R$ with and $\sigma = 2 \text{ m}^{-1}$

(c) Plot of Alcubierre $R$ with and $\sigma = 4 \text{ m}^{-1}$

(d) Plot of Alcubierre $R$ with $v_s = 6 \text{ m s}^{-1}$

(e) Plot of Alcubierre $R$ with $\sigma = 8 \text{ m}^{-1}$

(f) Plot of Alcubierre $R$ with $\sigma = 10 \text{ m}^{-1}$

Figure 2: Plots of the $R$ invariants for the Alcubierre warp drive while varying skin depth. The parameters were chosen as $v_s = 8 \text{ m s}^{-1}$ and $\rho = 1 \text{ m}$ to match the parameters Alcubierre originally suggested in his paper [1].

The plots for varying the skin depth are included in Fig. 2 in this subsection and Figs. 7 and 8 in Appendix C. Repeating the method of Section 4.1, the parameter $\sigma$ has been varied between values of 1 and 8 while maintaining the other parameters at the constant values of $\rho = 1 \text{ m}$ and $v_s = 1 \text{ m s}^{-1}$. Many of the features in these plots are the same as those discussed at the beginning of Section 4.1; thus, the skin depth variation reveals two additional features. First, the plots advance towards the “top hat” function by slowly straightening out any dips. This feature is most notable in the plots of $r_1$ in Figs. 2a and 7a. Multiple ripples occur in these two plots initially, but then gradually smooth out as $\sigma$ increases. These unforeseen
ripples could be the source of a rich internal structure inside the warp bubble similar to what
was observed in the accelerating Natário warp drive [24]. Second, the relative magnitude of
the Ricci scalar and $r_1$ is several orders of magnitude greater than that of $w_2$. This can be
seen as $\sigma \rightarrow 8 \text{ m}^{-1}$ the Ricci scalar goes to $10^{-9}$, $r_1$ goes to $10^{-11}$, and $w_2$ goes to the order
of $10^{-28}$. Consequently, the trace terms of the Riemann tensor that will have the greatest
effect on the curvature are the Ricci scalar and $r_1$, which are members of the Ricci invariants
in Eqs. (1) and (4). The terms of the Weyl tensor will have negligible effects since $w_2$ is
a member of the Weyl tensor in Eq. (4). The main effects of varying the skin depth is to
decrease the magnitude of the warp bubble’s curvature exponentially. This can be seen in
each of the invariants as the magnitude decreases from being on the order of $10^{-3}$ to $10^{-28}$.
The exponential decrease implies that thinner values for the warp bubble’s skin depth $\sigma$
would propel itself at greater velocities due to the greater amount of curvature.

4.3 Invariant plots of radius for the Alcubierre Warp Drive

The plots for varying the radius $\rho$ of the Alcubierre warp bubble are included in Fig. 3 in
this subsection and Figs. 9, and 10 in Appendix C. Following the method in Section 4.1, the
parameter $\rho$ has been varied between values of 0.1 m and 5 m while maintaining the other
parameters at the constant values of $\sigma = 8 \text{ m}^{-1}$ and $v_s = 1 \text{ m s}^{-1}$. Many of the features in
these plots are the same as those discussed at the beginning of Section 4.1, but the variation
of the radius does reveal an additional feature. The spatial size of the harbor inside the warp
bubble is directly affected by the value of $\rho$. By inspecting the $x$-axis of each plot, the size
of the harbor is of the same value as that of $\rho$. This behavior is as expected of the radius $\rho$, which confirms that the program is encoded correctly. Of greater interest, the magnitude
of the invariants does not have a clear correlation with $\rho$. As an example, consider the $r_1$
plots in Fig. 9. When $\rho = 0$ m, the $r_1$ invariants has its lowest magnitude of the order of $10^{-13}$. As the radius increases in the next four plots, the invariant increases to an order of $10^{-8}$. At the largest value $\rho = 5$ m, the invariant decreases to an order of $10^{-9}$. Inspecting
the invariant function itself in Eq. (16), $\rho$ does not seem to have a noticeable relationship
that explains this behavior. In conclusion, the radius $\rho$ defines the size of the harbor and
the warp bubble. It must always be chosen large enough for the ship to be unaffected by the
curvature of the warp bubble itself.
Figure 3: Plots of the $R$ invariants for the Alcubierre warp drive while varying radius. The other parameters were chosen as $\sigma = 8 \text{ m}^{-1}$ and $v_s = 1 \text{ m s}^{-1}$ to match the parameters Alcubierre originally suggested in his paper [1].
5 Invariants for the Natário Warp Drive at a constant velocity

The four curvature invariants in Eqs. (1) through (4) were computed and plotted in Mathematica® for the Natário line element Eq. (11). Its Ricci scalar is

\[
R = -\frac{1}{8}\sigma^2 v_s^2 \text{sech}^4(\sigma(r - \rho))
\times \left( \cos(2\theta) + r^2 \sigma^2 \sin^2(\theta) \tanh^2(\sigma(r - \rho)) - 2r\sigma \sin^2(\theta) \tanh(\sigma(r - \rho)) + 2 \right).
\]

The Ricci scalar is included alone in this section as a demonstrative example due to its simplicity. The remaining three are significantly more complicated and are included in Appendix B.

Like the Alcubierre invariants, the Natário invariants are exceptionally complicated, but some features may be observed by inspecting its functions. The first significant difference between the two line elements is that the Natário invariants do not depend on time, but instead on the parameters \( r \) and \( \theta \). As a consequence, the coordinates for the plots are chosen to be \( r \) and \( \theta \). Since the warp bubble skims along the comoving null tetrad in Eq. (14), they will show the shape of the bubble around the ship during flight. Similar to the Alcubierre invariants, each invariant is proportional to both \( v_s \) and \( \sigma \). The magnitude of the bubble’s curvature will then increase exponentially with both velocity and skin depth. In addition, the Natário invariants are proportional to \( \cos^n(\frac{\theta}{2}) \) and \( \text{sech}^n(\sigma(r - \rho)) \). The warp bubble is shaped such that the curvature is at a maximum in front of the ship around \( \theta = 0 \) and a minimum behind the ship around \( \theta = \frac{\pi}{2} \). It is at a maximum for \( r = \rho \) along the center of the warp bubble, since there \( \text{sech}(0) = 1 \). Outside these values, the curvature should then fall off and go asymptotically to 0. These features show that the Natário warp bubble also uses a “top-hat” function described in [1]. Finally, there are no intrinsic singularities. The manifold is asymptotically flat and completely connected. The flight of such a warp bubble should be significantly less affected by any gravitational tidal forces as compared to the Alcubierre metric in the previous section. The CM curvature invariants confirm that the Natário warp drive is a more realistic alternative to Alcubierre’s.

The invariant plots for the Natário warp drive may be found in Appendix D. They were plotted in Mathematica® using the RevolutionPlot3D function. The \( r \)-coordinate, \( \theta \)-coordinate, and the magnitude of the invariants are along each axis. The presence of spacetime curvature may be detected in a similar manner to the Alcubierre plots. When the invariant functions have a positive magnitude, the spacetime has a positive curvature and vice versa. For the Natário invariants, the spacetime curvature lies in the area around \( r = \rho \).

Despite the complexity of the invariants, the shape of the invariant plots is simple. It forms a very narrow and jagged ring as in Fig. 4. Precisely at the warp bubble’s boundary located at \( r = \rho \), the CM curvature invariants spike to non-zero magnitudes depending on the invariant. The Ricci scalar \( R \) takes the form of a smooth disc outside the warp bubble. The shape of the \( r_1 \) invariant is that of a jagged disc at \( r = \rho \). The disc has jagged edges in the negative direction, with sharp spikes at radial values \( r = \rho \) and at polar angle values of \( \theta = 0 \) and \( \theta = \pi \). The shape of the \( r_2 \) invariant is that of a jagged disc at \( r = \rho \). Its edges
vary between positive and negative values depending on the polar angle $\theta$. Similarly, the shape of the $w_2$ invariant is that of a jagged disc at $r = \rho$. In front of the harbor ($\theta > 0$), the invariant has rapidly changing negative values between $-1$ and 0. Behind the harbor ($\theta < 0$), the invariant has rapidly changing positive values between 0 and 1. The jagged edges of the plots must mean that the $r_1$, $r_2$ and $w_2$ invariants oscillate rapidly between values of $-1$ and 1 along the circumference of the warp bubble. These oscillations must be occurring more rapidly than the program can plot. Outside of the warp bubble, the spacetime is more well behaved. For values of $r \gg \rho$, the magnitude of each invariant is zero and the spacetime is asymptotically flat. For values of $r < \rho$ inside the warp bubble, the invariants’ magnitude is also zero. Like the Alcubierre warp drive, this implies that there is a harbor unaffected by the curvature of the warp bubble.

In contrast to the accelerating Natário warp drive, the constant velocity Natário warp drive does not feature either a wake or a constant non-zero curvature outside of the warp bubble for the Ricci Scalar [24]. It can be concluded that these features are due to the acceleration of the warp drive. On the other hand, the invariant plots for the accelerating Natário line element contain features of the constant velocity plots. Since the constant velocity plots are zero everywhere except at $r = \rho$, their impact is along the warp bubble’s edge for the accelerating invariants. In the remainder of this section, the effect of each of the parameters: velocity $v_s$, skin depth $\sigma$, and radius $\rho$ is analyzed individually.

### 5.1 Invariant plots of velocity for the Natário Warp Drive at a constant velocity

Fig. 4 in this subsection and Figs. 11 to 13 in Appendix D plot the Natário invariants while varying the velocity. The plots reveal several new aspects of the invariant functions. First, the manifold is completely flat when $v_s = 0 \text{ m s}^{-1}$ for each invariant as expected. For the Ricci scalar, a non-zero velocity causes the invariant’s magnitude to jump to a small negative value along the warp bubble’s circumference at $r = \rho$. For $r_1$, an increase in velocity causes the magnitude of the invariant to swap from negative values to positive values as the velocity increases along the circumference of the warp bubble. For $r_2$, an increase in velocity causes the magnitude of the invariant to swap from positive values to negative values along the circumference of the warp bubble. For $w_2$, an increase in velocity changes the magnitude of the invariant between positive values to negative values as the velocity increases along the semicircle of the warp bubble behind the harbor. In front of the harbor, the $w_2$ invariant function remains negative regardless of the velocity along the warp bubble’s circumference.

Our prediction of an exponential increase in the invariants due to the velocity is not consistent with the invariants’ plots. A potential reason for this discrepancy is a dominant term inside each of the invariants that overcomes the exponential increase in the velocity. The dominant term in the invariant functions must either not depend on $v_s$ or the values for $\sigma$ are the dominant factor. The research in this paper may be extended to include either greater values of $v_s$ or lower values of the other parameters to further investigate this discrepancy.
Figure 4: The velocity evolution of R, the Ricci scalar for the Natário warp drive at a constant velocity. It is understood that $v_s$ is multiplied by $c$. The other parameters are set to $\sigma = 50,000 \text{ m}^{-1}$ and $\rho = 100 \text{ m}$.

### 5.2 Invariant plots of skin depth for the Natário Warp Drive at a constant velocity

Figs. 14 and 15 plot the Natário invariants while changing the skin depth from $\sigma = 500000 \text{ m}^{-1}$ to $\sigma = 100000 \text{ m}^{-1}$. Notably, the shape of the invariants remains the same. Since $\text{sech}(r - \rho) \to 1$ as $(r - \rho) \to 0$, the spike in the invariant functions match the limiting values of the sech function and Eq. (13). The dominant term(s) in the invariant function must then be proportional to $\text{sech}(r - \rho)$. The $\sigma$ plots add further evidence that the shape of the CM invariants is a consequence of the “top-hat” function selected for the shape function in Eq. (13). This conclusion indicates that the selection of the shape function will control the shape of the warp bubble. This analysis of the Natário skin depth reaches the same conclusion as discussed for the constant acceleration Natário invariants [24].
5.3 Invariant plots of radius for the Natário Warp Drive at a constant velocity

Like the Alcubierre plots in Section 4.3, the main effect of changing $\rho$ is that it increases the size of both the warp bubble and the Natário safe harbor. As $\rho$ increases from $\rho = 50 \, m$ to $\rho = 100 \, m$ in Figs. 16 through 17, the sizes of the safe harbor and the bubble double. The spacing between the fringes in $r_1$, $r_2$, and $w_2$ is not affected by changing $\rho$, nor is the shape of the bubble. The plots confirm that the parameter $\rho$ moderates the size of the Natário warp bubble in the same fashion as Section 4.3 and the constant acceleration Natário warp drive [24].

6 Conclusion

This paper demonstrates how computing and plotting the curvature invariants for various parameters of warp drive spacetimes can reveal features of the underlying curvature. While the individual invariant functions are complex and require significant computational time, their plots can be quickly scanned and understood. The plots give the magnitude of curvature at each point around the ship. Where the plots’ magnitudes are large, space is greatly curved and vice versa. Also by observing the changes in slopes of the plots, the rate at which spacetime is being folded can be approximated. This information can help in mapping the spacetime around the ship and aid potential navigation.

In this paper, the curvature invariant functions and plots were displayed for the Alcubierre and the Natário warp drives at constant velocity. The free parameters were varied to see the individual effect on each invariant. The curvature invariants reveal a safe harbor for a ship to travel inside the warp bubble and an asymptotically flat space outside the bubble in all cases. At the radial position $r = \rho$ of the warp bubble(s), the curvature invariants have local maxima or minima identifying that $\rho$ is the location of the warp bubble and where spacetime is curved the most. The sharp peaks around the radial position are due to the shape function converging on the “top-hat” function described by Alcubierre. For the Alcubierre warp drive, the warp bubbles resemble two troughs with simple internal structures. For the constant velocity Natário warp drive, the warp bubbles peak around $r = \rho$ while displaying rich internal structure. The internal structures of these warp bubbles are novel and require more diligent research to discover their effects on a warp drive’s flight.

Of particular interest is how the invariants of the constant velocity Natário warp drive compared to the invariants for the Natário warp drive at a constant acceleration [24]. The main difference between the two is the lack of a wake for each invariant and a constant non-zero curvature outside of the warp bubble for the Ricci scalar. It can be concluded that these two features will appear as the warp drive undergoes an acceleration to reach a velocity greater than $c$. More research is needed to explore these features and their impact on a warp drive and its surrounding spacetime. The other parameters of skin depth $\sigma$ and radius $\rho$ behave similarly for both the warp drives.
Computing and plotting the invariant functions has significant advantages for the inspection and potential navigation of warp drives. As mentioned previously, plotting the invariants has the advantages that they are free from coordinate mapping distortions, divergences, discontinuities or other artifacts of the chosen coordinates. Once the invariant plots reveal the location of any artifacts, their position can be related mathematically to the standard tensors, and their effect(s) on an object’s motion can be analyzed. The invariant plots properly illustrate the entire underlying spacetime independent of a chosen coordinate system. A second advantage is the relative ease with which the invariants can be plotted. Software packages exist or can be developed to calculate the standard tensors. The aforementioned tensors lead to a chosen basis of invariants. While the CM invariants were chosen in this paper, other sets of invariants exist such as the Cartan invariants and the Witten and Petrov invariants [21, 29]. It is an open challenge to inspect the curvature of the warp drive spacetimes in these invariant sets. For example, the Cartan invariants may be computed for the Alcubierre and Natário warp drives. Then, their invariants may be compared with the invariants of other spacetimes like wormholes for equivalence. It is expected that the main features identified in this paper will also hold in these different bases.

In addition to inspecting different invariant bases, further work can be done in mapping warp drive spacetimes such as Alcubierre’s at a constant acceleration, Krasnikov’s at either constant velocity or constant acceleration, or Van Den Broeck’s at either constant velocity or constant acceleration [4, 5, 10]. In addition, the lapse functions for the Krasnikov and Van Den Broeck’s warp drives would need to be identified and then their accelerating line elements could be derived. After plotting their line elements for the invariants, each proposed warp drive could be compared and contrasted to their corresponding invariants at a constant velocity as discussed in this paper.
7 Acknowledgements

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A Null Vectors of the Alcubierre and Natário Line Elements

A null tetrad contains two real null vectors, \( k \) and \( l \), and two complex conjugate null vectors, \( m \) and \( \bar{m} \) that satisfy the following algebraic relationships [26]:

\[
e_i = (m, \bar{m}, l, k),
\]

\[
g_{ij} = 2m_i m_j - 2k_i l_j = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]

If an orthonormal tetrad, \( E_a \), exists for a given metric, it can be related to a complex null tetrad (20) by:

\[
l_i = \frac{1}{\sqrt{2}}(E_1 + E_2), \quad k_i = \frac{1}{\sqrt{2}}(E_1 - E_2),
\]

\[
m_i = \frac{1}{\sqrt{2}}(E_1 + iE_2), \quad \bar{m}_i = \frac{1}{\sqrt{2}}(E_1 - iE_2).
\]

The orthonormal tetrad for the Alcubierre line element in (8) is:

\[
E_1 = (1 \ 0 \ 0 \ 0), \quad E_2 = (v_s f(r_s) - 1 \ 0 \ 0),
\]

\[
E_3 = (0 \ 0 \ 1 \ 0), \quad E_4 = (0 \ 0 \ 0 \ 1).
\]

The Natário line element in (11) has an orthonormal tetrad:

\[
E_1 = (1 \ 0 \ 0 \ 0), \quad E_2 = (X_{r_s} - 1 \ 0 \ 0),
\]

\[
E_3 = (X_\theta \ 0 \ - r \ 0), \quad E_4 = (0 \ 0 \ 0 \ r \sin \theta).
\]

Using Mathematica®, it can be verified that

\[
g_{ij} = E_i \cdot E_j,
\]

and by applying the equations (22) to (24) and (25) the null vectors in (10) and (14) result respectively.
B Invariants for the Natário Line Element at Constant Velocity

\[
\begin{aligned}
    r_1 &= \frac{1}{1024}v_s^2\sigma^2 \cos^2\left(\frac{\theta}{2}\right) \text{sech}^4((r - \rho)\sigma) \left(32r^4v_s^2\sigma^4(\sin\left(\frac{\theta}{2}\right) \right) \\
    &\quad - \sin(3\theta/2))^2 \tanh^6((r - \rho)\sigma) + 64r^3v_s^2\sigma^3(r\sigma - 1)(\sin\left(\frac{\theta}{2}\right) \\
    &\quad - \sin(3\theta/2))^2 \tanh^5((r - \rho)\sigma) \\
    &\quad - 4r^2\sigma^2(-r^2\sigma^2v_s^2 + 4r\sigma v_s^2 + (r^2\sigma^2 - 4r\sigma - 4) \cos(4\theta)v_s^2 - 12v_s^2 + 16r^2\sigma^2 \\
    &\quad - 8(v_s^2 + 2r^2\sigma^2)\cos(2\theta)) \sec^2\left(\frac{\theta}{2}\right) \tanh^4((r - \rho)\sigma) + 2r\sigma(-4^2\sigma^2v_s^2 + 48r\sigma v_s^2 \\
    &\quad + (4^2\sigma^2 + 16r\sigma + 3) \cos(4\theta)v_s^2 - 7v_s^2 + 192r^2\sigma^2 \\
    &\quad + 4(v_s^2(8r\sigma + 1) - 48r^2\sigma^2) \cos(2\theta)) \sec^2\left(\frac{\theta}{2}\right) \tanh^3((r - \rho)\sigma \\
    &\quad + (48^2\sigma^2v_s^2 - 28r\sigma v_s^2 + (16r^2\sigma^2 + 12r\sigma - 3) \cos(4\theta)v_s^2 + 7v_s^2 - 768r^2\sigma^2 \\
    &\quad + 4((8^2\sigma^2 + 4r\sigma - 1)v_s^2 + 160r^2\sigma^2) \cos(2\theta)) \sec^2\left(\frac{\theta}{2}\right) \tanh^2((r - \rho)\sigma \\
    &\quad + 2(-7r\sigma v_s^2 + 3(r\sigma - 1) \cos(4\theta)v_s^2 + 7v_s^2 + 320r\sigma \\
    &\quad + 4(v_s^2(r\sigma - 1) - 16r\sigma) \cos(2\theta)) \sec^2\left(\frac{\theta}{2}\right) \tanh((r - \rho)\sigma \\
    &\quad - 2(3 \cos(2\theta) + 5)(\cos(2\theta)v_s^2 - v_s^2 + 32) \sec^2\left(\frac{\theta}{2}\right) \sec^6\left(\frac{\theta}{2}\right) \\
    &\quad - 16r\sigma \text{sech}^2((r - \rho)\sigma) \tan^2\left(\frac{\theta}{2}\right) (2^3v_s^2\sigma^3(3 \cos(2\theta) - 1) \tan^4((r - \rho)\sigma) \\
    &\quad + r^2v_s^2\sigma^2(2r\sigma + (10r\sigma - 9) \cos(2\theta) + 9) \tan^3((r - \rho)\sigma) \\
    &\quad + r\sigma(4^2\sigma^2 + 5r\sigma - 9)v_s^2 \\
    &\quad + (4^2\sigma^2 - 13r\sigma + 9) \cos(2\theta)v_s^2 - 32r^2\sigma^2) \tan^2((r - \rho)\sigma) \\
    &\quad + ((-4^2\sigma^2 - 5r\sigma + 9)v_s^2 - (4^2\sigma^2 - 13r\sigma - 7) \cos(2\theta)v_s^2 + 96r^2\sigma^2) \tanh((r - \rho)\sigma) \\
    &\quad + 9v_s^2 + 4rv_s^2\sigma - 32r\sigma + 7v_s^2 \cos(2\theta) + 4rv_s^2\sigma \cos(2\theta) \\
    &\quad \sec^4\left(\frac{\theta}{2}\right) + \frac{1}{4}r^2\sigma^2 \text{sech}^4((r - \rho)\sigma)(4^2\sigma^2v_s^2 + 16r\sigma v_s^2 - (4^2\sigma^2 + 16r\sigma - 3) \cos(4\theta)v_s^2 \\
    &\quad + 109v_s^2 - 64r^2\sigma^2 - 4(v_s^2 - 16r^2\sigma^2) \cos(2\theta)) \sec^8\left(\frac{\theta}{2}\right) \\
    &\quad - 416r^2v_s^2\sigma^2(\cos(2\theta) - 2) \tan^2\left(\frac{\theta}{2}\right) \tanh^2((r - \rho)\sigma) \sec^4\left(\frac{\theta}{2}\right) \\
    &\quad - 64rv_s^2\sigma(2r\sigma + 2r \cos(2\theta)\sigma + 17) \tan^2\left(\frac{\theta}{2}\right) \tanh((r - \rho)\sigma) \sec^4\left(\frac{\theta}{2}\right) \\
    &\quad + 704r^4v_s^2\sigma^4 \tan^4\left(\frac{\theta}{2}\right) \tan^4((r - \rho)\sigma) - 2816^3v_s^2\sigma^3 \tan^4\left(\frac{\theta}{2}\right) \tanh^3((r - \rho)\sigma)\right), \quad (B.1)
\end{aligned}

\[
\begin{align*}
  r_2 &= -\frac{1}{32768}\frac{3v_s^4\sigma^3}{\tau^3} \cos^2 (\theta) \tan^2 \left( \frac{\theta}{2} \right) \tanh^6 \left( 32r \sigma \tan \left( \frac{\theta}{2} \right) \right) \left( \frac{1}{2} \tan \left( \frac{\theta}{2} \right) + 1 \right) \\
  &\times (512r^5 v_s^2 \sigma^5 \cos (\theta) \tan^2 \left( \frac{\theta}{2} \right) \tanh^7 \left( \frac{r - \sigma}{r} \sigma \right) + 32r^4 v_s^2 \sigma^4 (4r \sigma + \\
  &\times (4r \sigma - 3) \cos (2\theta) - 5) \sec^2 \left( \frac{\theta}{2} \right) \left( \sin \left( \frac{3\theta}{2} \right) - \sin \left( \frac{\theta}{2} \right) \right) \tanh^6 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ 4r^3 \sigma^3 (2r^2 \sigma^2 v_s^2 - 10r \sigma v_s^2 - 2r^2 \sigma^2 - 5r \sigma - 4) \cos (4\theta) \sigma v_s^2 - r^2 \sigma^2 \cos (6\theta) v_s^2 \\
  &+ 3r \sigma \cos (6\theta) v_s^2 + 4 \cos (6\theta) v_s^2 + 24 v_s^2 - 64 r^2 \sigma^2 \\
  &+ ((r^2 \sigma^2 - 3r \sigma + 28) v_s^2 + 64r^2 \sigma^2) \cos (2\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh^5 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ 2r^2 \sigma^2 (-10r^2 \sigma^2 v_s^2 + 96r \sigma v_s^2 + 3r^2 \sigma^2 \cos (6\theta) v_s^2 + 16r \sigma \cos (6\theta) v_s^2 \\
  &+ 4 \cos (6\theta) v_s^2 - 20v_s^2 + 816r^2 \sigma^2 + (v_s^2 (-3r^2 \sigma^2 + 112r \sigma + 12) - 832r^2 \sigma^2) \cos (2\theta) \\
  &+ 2((5r^2 \sigma^2 + 16r \sigma + 2) v_s^2 + 8r^2 \sigma^2) \cos (4\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh^4 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ r \sigma (96r^2 \sigma^2 v_s^2 - 80r \sigma v_s^2 + 16r^2 \sigma^2 \cos (6\theta) v_s^2 + 16r \sigma \cos (6\theta) v_s^2 - 3 \cos (6\theta) v_s^2 \\
  &+ 22v_s^2 - 3968r^2 \sigma^2 + ((112r^2 \sigma^2 + 48r \sigma - 13)v_s^2 + 3072r^2 \sigma^2) \cos (2\theta) \\
  &+ 2(16r^2 \sigma^2 + 8r \sigma - 3) \cos (4\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh^3 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ 2(-12r^2 \sigma^2 v_s^2 + 22r \sigma v_s^2 + 4r^2 \sigma^2 \cos (6\theta) v_s^2 + 3r \sigma \cos (6\theta) v_s^2 - \cos (6\theta) v_s^2 \\
  &- 2v_s^2 + 2448r^2 \sigma^2 + (v_s^2 (12r^2 \sigma^2 - 13r \sigma + 1) - 448r^2 \sigma^2) \cos (2\theta) \\
  &+ 2((r^2 \sigma^2 - 3r \sigma + 1) v_s^2 + 24r^2 \sigma^2) \cos (4\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh^2 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ (-22r \sigma v_s^2 + 3r \sigma \cos (6\theta) v_s^2 + 4 \cos (6\theta) v_s^2 + 8v_s^2 + 313r \sigma + (13r \sigma - 4)v_s^2 \\
  &+ 1024r \sigma) \cos (2\theta) + (v_s^2 (6r \sigma - 8) - 64r \sigma) \cos (4\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ 32(\cos (4\theta) v_s^2 - v_s^2 + 112 \cos (2\theta) + 144) \tan^2 \left( \frac{\theta}{2} \right) \sec^8 \left( \frac{\theta}{2} \right) \\
  &+ 2r^2 \sigma^2 \sec^4 ((r - \sigma) \sigma) \tan^2 \left( \frac{\theta}{2} \right) (-32r^5 v_s^2 \sigma^5 \cos (2\theta) - 3) \tan^2 \left( \frac{\theta}{2} \right) \tanh^6 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &- 16r^4 v_s^2 \sigma^4 (-2r \sigma + (6r \sigma - 7) \cos (2\theta) + 31) \tan^2 \left( \frac{\theta}{2} \right) \tanh^5 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ r^3 \sigma^3 (-16r^2 \sigma^2 v_s^2 - 57r \sigma v_s^2 + (4r^2 \sigma^2 - 19r \sigma - 9) \cos (4\theta) v_s^2 + 141v_s^2 + 64r^2 \sigma^2 \\
  &+ 4(v_s^2 (19r \sigma - 37) - 16r^2 \sigma^2) \cos (2\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh^4 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &- r^2 \sigma^2 (-12r^2 \sigma^2 v_s^2 - 69r \sigma v_s^2 + (12r^2 \sigma^2 + 17r \sigma - 15) \cos (4\theta) v_s^2 + 215v_s^2 + 32r^2 \sigma^2 \\
  &+ 4(3v_s^2 (19r \sigma - 6) - 80r^2 \sigma^2) \cos (2\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh^3 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &- r \sigma (60r^2 \sigma^2 v_s^2 + 119r \sigma v_s^2 + (4r^2 \sigma^2 - 31r \sigma + 29) \cos (4\theta) v_s^2 - 157v_s^2 - 512r^2 \sigma^2 \\
  &+ 8r \sigma ((8r \sigma - 23) v_s^2 + 64r \sigma) \cos (2\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh^2 \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ (-128r^3 \sigma^3 + 12r^3 \sigma^3 v_s^2 \sigma^3 - 416r^2 \sigma^2 + 94r^2 \sigma^2 v_s^2 \sigma^2 + 113r \sigma v_s^2 \sigma^2 - 98v_s^2 \\
  &+ 4((4r^3 \sigma^3 + 28r^2 \sigma^2 - 12r \sigma - 19) v_s^2 + 136r^2 \sigma^2) \cos (2\theta) \\
  &+ v_s^2 (64r^2 \sigma^2 + 18r^2 \sigma^2 - 33r \sigma - 18) \cos (4\theta)) \sec^4 \left( \frac{\theta}{2} \right) \tanh \left( \frac{r - \sigma}{r} \sigma \right) \\
  &+ 2(-r^2 \sigma^2 v_s^2 - 22r \sigma v_s^2 + (r^2 \sigma^2 - 2r \sigma - 9) \cos (4\theta) v_s^2 - 49v_s^2 + 144r^2 \sigma^2 + 224r \sigma \\
  &+ (16r \sigma (7r \sigma + 6) - 2v_s^2 (12r \sigma + 19)) \cos (2\theta)) \sec^4 \left( \frac{\theta}{2} \right) \sec^4 \left( \frac{\theta}{2} \right) \\
  &+ 1)
\end{align*}
\]
\[ w_2 = \frac{1}{2451919104} v_s^4 \sigma^3 \text{sech}^6((r - \rho)\sigma) (-32v_s^2 \sigma^3 (rv_s \sigma \sin(\theta) \text{sech}^2((r - \rho)\sigma) \\
+ 2(v_s \cos(\theta) + v_s \sin(\theta) + v_s(\cos(\theta) + \sin(\theta)) \tanh((r - \rho)\sigma) + 4))^3 \\
\times (\text{sech}^2((r - \rho)\sigma)(6 \cos^2(\theta) + 7 \sin^2(\theta) + 4r^2 \sigma^2 \sin^2(\theta) \tanh^2((r - \rho)\sigma) \\
- 8r \sigma^2 \sin^2(\theta) \tanh((r - \rho)\sigma)) \\
- 3(3 \cos(2\theta) + 1) \tanh((r - \rho)\sigma)(\tanh((r - \rho)\sigma) + 1)^3 \\
+ \frac{27i}{\sqrt{3}} \text{sech}^6((r - \rho)\sigma) \sin^4(\theta)(\cosh(2(r - \rho)\sigma)(4 + 4i) + (4 + 4i) + (1 + i)v_s \cos(\theta) \\
+ i\nu \cos(\theta + 2i(r - \rho)\sigma) + v_s \cos(\theta - 2i(r - \rho)\sigma) + (1 + i)v_s \sin(\theta) \\
+ (1 + i)rv_s \sigma \sin(\theta) + v_s \sin(\theta + 2i(r - \rho)\sigma) \\
+ i\nu \sin(\theta - 2i(r - \rho)\sigma))^2(-r^2 \sigma^2 v_s^2 - 2r \sigma v_s^2 + r^2 \sigma^2 \cos(2\theta)v_s^2 \\
+ 2r \sigma^2 \cos(2\theta)v_s^2 - 2i \cos(2\theta + 2i(r - \rho)\sigma)v_s^2 + (1 - i)r \sigma \cos(2\theta + 2i(r - \rho)\sigma)v_s^2 \\
- i \cos(2\theta + 4i(r - \rho)\sigma)v_s^2 + 2i \cos(2\theta - 2i(r - \rho)\sigma)v_s^2 \\
+ (1 + i)r \sigma \cos(2\theta - 2i(r - \rho)\sigma)v_s^2 + i \cos(2\theta - 4i(r - \rho)\sigma)v_s^2 - 2r \sigma \sin(2\theta)v_s^2 \\
- 2 \sin(2\theta)v_s^2 - (1 + i)r \sigma \sin(2\theta + 2i(r - \rho)\sigma)v_s^2 - 2 \sin(2\theta + 2i(r - \rho)\sigma)v_s^2 \\
- \sin(2\theta + 4i(r - \rho)\sigma)v_s^2 - (1 - i)r \sigma \sin(2\theta - 2i(r - \rho)\sigma)v_s^2 \\
- 2 \sin(2\theta - 2i(r - \rho)\sigma)v_s^2 - \sin(2\theta - 4i(r - \rho)\sigma)v_s^2 - 2r \sigma \sinh(2(r - \rho)\sigma)v_s^2 \\
- 4 \sinh(2(r - \rho)\sigma)v_s^2 - 2 \sinh(4(r - \rho)\sigma)v_s^2 - 2v_s^2 - 12 \cos(\theta)v_s \\
- (8 + 4i) \cos(\theta + 2i(r - \rho)\sigma)v_s - (2 + 2i) \cos(\theta + 4i(r - \rho)\sigma)v_s \\
- (8 - 4i) \cos(\theta - 2i(r - \rho)\sigma)v_s - (2 - 2i) \cos(\theta - 4i(r - \rho)\sigma)v_s - 8r \sigma \sin(\theta)v_s \\
- 12 \sin(\theta)v_s - (8 - 4i) \sin(\theta + 2i(r - \rho)\sigma)v_s - 4r \sigma \sin(\theta + 2i(r - \rho)\sigma)v_s \\
- (2 - 2i) \sin(\theta + 4i(r - \rho)\sigma)v_s - (8 + 4i) \sin(\theta - 2i(r - \rho)\sigma)v_s \\
- 4r \sigma \sin(\theta - 2i(r - \rho)\sigma)v_s - (2 + 2i) \sin(\theta - 4i(r - \rho)\sigma)v_s \\
- 2((r \sigma + 2)v_s^2 + 16) \cosh(2(r - \rho)\sigma) \\
- 2(v_s^2 + 4) \cosh(4(r - \rho)\sigma) - 24)(\tanh((r - \rho)\sigma) + 1) \\
\times (r \sigma \tanh((r - \rho)\sigma) - 1)((4r - v_s(r^2 \sigma^2 - r^2 + 1) \cos(\theta)) \tanh^3((r - \rho)\sigma) \\
- 3(2r^2 \sigma^2 + v_s(r^2 \sigma^2 - r^2 + 1) \cos(\theta) - 2) \tanh^2((r - \rho)\sigma) \\
- 3(v_s(r^2 \sigma^2 - r + 1) \cos(\theta) - 4r \sigma) \tanh((r - \rho)\sigma) - 2r^2 \sigma^2 - r^2 v_s \sigma^2 \cos(\theta) \\
- v_s \cos(\theta) + rv_s \sigma \cos(\theta) + \text{sech}^2((r - \rho)\sigma)(6r^2 \sigma^2 + v_s(3r^2 \sigma^2 + r^2 - 3) \cos(\theta) \\
+ (4r \sigma + v_s(3r^2 \sigma^2 + r^2 - 1) \cos(\theta)) \tanh((r - \rho)\sigma) + 6) + 2)^2 \\
+ \frac{2304}{r^2} v_s^2(\sigma \sin^4(\theta)(\tanh((r - \rho)\sigma) + 1)^2(r \sigma \tanh((r - \rho)\sigma) - 1)^2 \\
\times (rv_s \sigma \sin(\theta) \text{sech}^2((r - \rho)\sigma) \\
+ 2(v_s \cos(\theta) + v_s \sin(\theta) + v_s(\cos(\theta) + \sin(\theta)) \tanh((r - \rho)\sigma) + 4)) \\
\times (\text{sech}^2((r - \rho)\sigma)(6 \cos^2(\theta) + 7 \sin^2(\theta) + 4r^2 \sigma^2 \sin^2(\theta) \tanh^2((r - \rho)\sigma) \\
- 8r \sigma^2 \sin^2(\theta) \tanh((r - \rho)\sigma)) \\
- 3(3 \cos(2\theta) + 1) \tanh((r - \rho)\sigma)(\tanh((r - \rho)\sigma) + 1) \\
\times (r^2 v_s^2 \sigma^2 \sin^4(\theta) \theta) \text{sech}^4((r - \rho)\sigma) \\
+ 2rv_s \sigma(2 \sin(\theta)(v_s \cos(\theta) + v_s \sin(\theta) + 2) + v_s(2 \sin^2(\theta) + \sin(2\theta)) \\
\times \tanh((r - \rho)\sigma)) \text{sech}^2((r - \rho)\sigma) \\
+ 2( - \text{sech}^2((r - \rho)\sigma)(\cos(\theta) + \sin(\theta))^2 v_s^2 + (\cos(\theta) + \sin(\theta))^2 \tanh^2((r - \rho)\sigma) v_s^2 \\
+ 6 \cos(\theta) \sin(\theta)v_s^2 + 2v_s^2 + 8 \cos(\theta)v_s + 2\theta \sin(\theta)v_s \\
+ 4(\cos(\theta) + \sin(\theta))(v_s \cos(\theta) + v_s \sin(\theta) + 2) \tanh((r - \rho)\sigma)v_s + 16))}}
C Invariant Plots for the Alcubierre Warp Drive

Figure 5: Plots of the $r_1$ invariants for the Alcubierre warp drive while varying velocity. The other parameters were chosen as $\sigma = 8 \, m^{-1}$ and $\rho = 1 \, m$ to match the parameters Alcubierre originally suggested in his paper [1].
Figure 6: Plots of the $w_2$ invariants for the Alcubierre warp drive while varying velocity. The radius was chosen as $\rho = 1 \text{ m}$ to match the parameters Alcubierre originally suggested in his paper [1]. The skin depth was chosen as $\sigma = 2 \text{ m}^{-1}$ to keep the plots as machine size numbers. Both velocity and skin depth have an exponential affect on the magnitude of the invariants.
Figure 7: Plots of the $r_1$ invariants for the Alcubierre warp drive while varying skin depth. The other parameters were chosen as $v_s = 1 \text{ m s}^{-1}$ and $\rho = 1 \text{ m}$ to match the parameters Alcubierre originally suggested in his paper [1].
Figure 8: Plots of the $w_2$ invariants for the Alcubierre warp drive while varying skin-depth. The radius and velocity were chosen as $\rho = 1 \text{ m}$ and $v_s = 1 \text{ m s}^{-1}$ to match the parameters Alcubierre originally suggested in his paper [1].
Figure 9: Plots of the $r_1$ invariants for the Alcubierre warp drive while varying the radius. The other parameters were chosen as $\sigma = 8 \text{ m}^{-1}$ and $v_s = 1 \text{ m s}^{-1}$ to match the parameters Alcubierre originally suggested in his paper [1].
Figure 10: Plots of the $w^2$ invariants for the Alcubierre warp drive while varying radius. The other parameters were chosen as $\sigma = 8 \, m^{-1}$ and $v_s = 1 \, m \, s^{-1}$ to match the parameters Alcubierre originally suggested in his paper [1].
D Invariant Plots for the Natário Warp Drive

Figure 11: The velocity evolution of the $r_1$ invariant for the Natário warp drive at a constant velocity. It is understood that $v_s$ is multiplied by $c$. The other parameters are set to $\sigma = 50,000 \, \frac{1}{m}$ and $\rho = 100 \, m$. 
Figure 12: The velocity Evolution of the $r_2$ invariant for the Natário warp drive at a constant velocity. It is understood that $v_s$ is multiplied by $c$. The other parameters are set to $\sigma = 50,000 \frac{1}{m}$ and $\rho = 100 \text{ m}$.
Figure 13: The velocity evolution of the $w_2$ invariant for the Natário warp drive at a constant velocity. The other parameters are set to $\sigma = 50,000 \frac{1}{\text{m}}$ and $\rho = 100 \text{ m.}$
(a) The invariant $R$ with $\sigma = 50,000 \frac{1}{m}$

(b) The invariant $R$ with $\sigma = 100,000 \frac{1}{m}$

(c) The invariant $r_1$ with $\sigma = 50,000 \frac{1}{m}$

(d) The invariant $r_1$ with $\sigma = 100,000 \frac{1}{m}$

Figure 14: The warp bubble skin depth for the Ricci scalar and $r_1$ for the Natário warp drive at a constant velocity. The other parameters were chosen to be $v = 1\,\text{m/s}$, and $\rho = 100\,\text{m}$ in natural units.
(a) The invariant $r_2$ with $\sigma = 50,000 \ \frac{1}{m}$
(b) The invariant $r_2$ with $\sigma = 100,000 \ \frac{1}{m}$

(c) The invariant $w_2$ with $\sigma = 50,000 \ \frac{1}{m}$
(d) The invariant $w_2$ with $\sigma = 100,000 \ \frac{1}{m}$

Figure 15: The warp bubble skin depth for $r_2$ and $w_2$ for the Natário warp drive at a constant velocity. The other parameters were chosen to be $v = 1 \ \frac{m}{s}$, and $\rho = 100 \ m$ in natural units.
(a) The invariant $w_2$ with $\rho = 50$ m
(b) The invariant $R$ with $\rho = 100$ m
(c) The invariant $r_1$ with $\rho = 50$ m
(d) The invariant $r_1$ with $\rho = 100$ m

Figure 16: The warp bubble radius for the Ricci scalar and $r_1$ for the Natário warp drive at a constant velocity. The other parameters were chosen to be $v = 1\ m\ s^{-1}$, and $\sigma = 50,000\ m^{-1}$ in natural units.

(a) The invariant $r_2$ with $\rho = 50$ m
(b) The invariant $r_2$ with $\rho = 100$ m
(c) The invariant $w_2$ with $\rho = 50$ m
(d) The invariant $w_2$ with $\rho = 100$ m

Figure 17: The warp bubble radius for the $r_2$, and $w_2$ for the Natário warp drive at a constant velocity. The other parameters were chosen to be $v = 1\ m\ s^{-1}$, and $\sigma = 50,000\ m^{-1}$ in natural units.
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