We consider a $D$-dimensional cosmological model describing an evolution of $(n + 1)$ Einstein factor spaces ($n \geq 2$) in the theory with several dilatonic scalar fields and generalized electro-magnetic forms, admitting an interpretation in terms of intersecting $p$-branes. The equations of motion of the model are reduced to the Euler-Lagrange equations for the so called pseudo-Euclidean Toda-like system. We consider the case, when characteristic vectors of the model, related to $p$-branes configuration and their couplings to the dilatonic fields, may be interpreted as the root vectors of a Lie algebra of the type $A_m$. The model is reduced to the open Toda chain and integrated. The exact solution is presented in the Kasner-like form.

1 Introduction

Last years have witnessed a growth of interest to classical $p$-brane solutions of (bosonic sector of) supergravities in various dimensions. This interest is inspired by a conjecture that $D = 11$ supergravity is a low-energy effective field theory of eleven-dimensional fundamental $M$-theory, which (together with so called $F$-theory) is a candidate for unification of five known $D = 10$ superstring theories. Classical $p$-brane solutions may be considered as a method for investigation of interlinks between superstrings and $M$-theory.

In this paper we consider a generalized bosonic sector of early supergravity theories in the form of multidimensional gravitational model with several dilatonic fields and Maxwell-like forms of various ranks. When $D$-dimensional space-time manifold is a product of several Einstein factor spaces, the most convenient to use $\sigma$-model approach (see, for instance, for constructing exact solutions with $p$-branes. It was shown that for cosmological space-times the equations of motion to such $\sigma$-model are reduced to the Euler-Lagrange equations for a pseudo-Euclidean Toda-like Lagrange system. The methods for integrating of pseudo-Euclidean Toda-like systems were developed in the papers (see, for instance, and references therein) devoted to the multidimensional cosmologies with multicomponent perfect fluid. Here we integrate the intersecting $p$-branes model reducible to Toda chain associated with Lie algebra of the type $A_m$. 

1
2 The general model

Following the papers\cite{13,18,20} we study a classical model in $D$ dimensions described by the action

\begin{equation}
S = \int_{\mathcal{M}} d^Dz \sqrt{|g|} \{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta \}
\end{equation}

\begin{equation}
- \sum_{a \in \Delta} \frac{1}{n_a!} \exp[2\lambda_a(\varphi)] (F^a)^2_g,
\end{equation}

where $g = g_{MN} dz^M \otimes dz^N$ is the metric on $D$-dimensional manifold $\mathcal{M}$ ($M, N = 1, \ldots, D$), $|g| = |\det(g_{MN})|$. We denote by $\varphi = (\varphi^a) \in \mathbb{R}^l$ a vector from dilatonic scalar fields, $(h_{\alpha\beta})$ is a non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$) and $\lambda_a$ is a 1-form on $\mathbb{R}^l$: $\lambda_a(\varphi) = \lambda_{a\alpha} \varphi^\alpha$, $a \in \Delta$, $\alpha = 1, \ldots, l$.

\begin{equation}
F^a = dA^a = \frac{1}{n_a!} F^a_{M_1 \ldots M_{n_a}} dz^M_1 \wedge \ldots \wedge dz^M_{n_a}
\end{equation}

is a $n_a$-form ($n_a \geq 1$) on $\mathcal{M}$. In Eq. (1) we denoted

\begin{equation}
(F^a)^2_g = F^a_{M_1 \ldots M_{n_a}} F^a_{N_1 \ldots N_{n_a}} g^{M_1 N_1} \ldots g^{M_{n_a} N_{n_a}},
\end{equation}

$a \in \Delta$, where $\Delta$ is some finite set.

Equations of motion corresponding to (1) have the following form

\begin{equation}
R_{MN} - \frac{1}{2} g_{MN} R = T_{MN},
\end{equation}

\begin{equation}
\Delta[g] \varphi^\alpha - \sum_{a \in \Delta} \frac{\lambda_a}{n_a!} e^{2\lambda_a(\varphi)} (F^a)^2_g = 0,
\end{equation}

\begin{equation}
\nabla_{M_1}[g](e^{2\lambda_a(\varphi)} F^a_{M_1 \ldots M_{n_a}}) = 0,
\end{equation}

$a \in \Delta; \alpha = 1, \ldots, l$. In Eq. (1) $\lambda_a = h^{\alpha\beta} \lambda_{a\beta}$, where $(h^{\alpha\beta})$ is a matrix inverse to $(h_{\alpha\beta})$. In (2) we denoted

\begin{equation}
T_{MN} = T_{MN}[\varphi, g] + \sum_{a \in \Delta} e^{2\lambda_a(\varphi)} T_{MN}[F^a, g],
\end{equation}

where

\begin{equation}
T_{MN}[\varphi, g] = h_{\alpha\beta} \left( \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} g_{MN} \partial_P \varphi^\alpha \partial^P \varphi^\beta \right),
\end{equation}

\begin{equation}
T_{MN}[F^a, g] = \frac{1}{n_a!} \left[ -\frac{1}{2} g_{MN} (F^a)^2_g + n_a F^a_{M_1 \ldots M_{n_a}} F^a_{N_1 \ldots N_{n_a}} \right].
\end{equation}
In Eqs. (5), (6) $\triangle [g]$ and $\nabla [g]$ are Laplace-Beltrami and covariant derivative operators corresponding to $g$.

We consider the manifold

$$\mathcal{M} = \mathbb{R} \times M_0 \times \ldots \times M_n$$

(10)

with the metric

$$g = -e^{2\gamma(t)} dt \otimes dt + \sum_{i=0}^{n} e^{2x^i(t)} g^i,$$

(11)

where $g^i = g_{m,n}(y_i) dy^m_i \otimes dy^n_i$ is a metric on $M_i$ satisfying the equation

$$R_{m,n}^i[g^i] = \xi_i g_{m,n}^i,$$

(12)

$m_i, n_i = 1, \ldots, d_i; d_i = \dim M_i, \xi_i = \text{const}, i = 0, \ldots, n; n \in \mathbb{N}$. So all $(M_i, g^i)$ are Einstein spaces.

Each manifold $M_i$ is supposed to be oriented and connected. Then the volume $d_i$-form

$$\tau_i = \sqrt{|g^i(y_i)|} \ dy^1_i \wedge \ldots \wedge dy^{d_i}_i,$$

(13)

and the signature

$$\text{sign det}(g_{m,n}^i) = 1$$

(14)

are correctly defined for all $i = 0, \ldots, n$.

Let

$$\Omega_0 = \{\emptyset, \{0\}, \{1\}, \ldots, \{n\}, \{0,1\}, \ldots, \{0,1,\ldots, n\}\}$$

(15)

be a set of all ordered subsets of $I_0 = \{0, \ldots, n\}$. For any $I = \{i_1, \ldots, i_k\} \in \Omega_0$, $i_1 < \ldots < i_k$, we define a volume form

$$\tau(I) = \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}$$

(16)

of rank

$$d(I) = \sum_{i \in I} d_i = d_{i_1} + \ldots + d_{i_k}$$

(17)

and a corresponding $p$-brane submanifold

$$M_I \equiv M_{i_1} \times \ldots \times M_{i_k},$$

(18)

where $p = d(I) - 1$ ($\dim M_I = d(I)$).

We adopt the following "composite electro-magnetic" Ansatz for fields of forms

$$F^a = \sum_{I \in \Omega_{a,e}} F^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} F^{(a,m,J)},$$

(19)
where
\[ F^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I), \quad (20) \]
\[ F^{(a,m,J)} = e^{-2\lambda_a(\varphi)} \ast \left( d\Phi^{(a,m,J)} \wedge \tau(J) \right), \quad (21) \]
\[ a \in \Delta, \; I \in \Omega_{a,e}, \; J \in \Omega_{a,m} \text{ and} \]
\[ \Omega_{a,e}, \Omega_{a,m} \subset \Omega_0. \quad (22) \]
(For empty $\Omega_{a,v} = \emptyset$, $v = e, m$, we put $\sum_{\emptyset} = 0$ in Eq. (19)). In Eq. (21) $\ast = \ast[g]$ is the Hodge operator on $(M, g)$.

For the potentials in Eqs. (20), (21) we put
\[ \Phi^s = \Phi^s(t), \quad (23) \]
\[ s \in S, \text{ where} \]
\[ S = S_e \sqcup S_m, \quad S_v = \coprod_{a \in \Delta} \{ a \} \times \{ v \} \times \Omega_{a,v}, \quad (24) \]
v = e, m.

For dilatonic scalar fields we also put
\[ \varphi^\alpha = \varphi^\alpha(t), \quad (25) \]
\[ \alpha = 1, \ldots, l. \]

From Eqs. (20) and (21) we obtain the relations between dimensions of $p$-brane worldsheets and ranks of forms
\[ d(I) = n_a - 1, \quad I \in \Omega_{a,e}, \quad (26) \]
\[ d(J) = D - n_a - 1, \quad J \in \Omega_{a,m} \quad (27) \]
in electric and magnetic cases respectively.

We put the following restrictions on $\Omega_{a,v}$. Let
\[ w_1 \equiv \{ i \mid i \in \{1, \ldots, n\}, \; d_i = 1 \}. \quad (28) \]
The set $w_1$ describes all 1-dimensional manifolds among $M_i$ ($i \geq 0$). We impose the following restrictions on the sets $\Omega_{a,v}$ (22):
\[ W_{ij}(\Omega_{a,v}) = \emptyset, \quad (29) \]
a ∈ ∆; v = e, m; i, j ∈ w_1, i < j and

\[ W_j^{(1)}(Ω_{a,m}, Ω_{a,e}) = ∅, \]  \tag{30} \]

a ∈ ∆; j ∈ w_1. Here

\[ W_{ij}(Ω_a) ≡ \{(I, J)| I, J ∈ Ω_a, I = \{i\} ∪ (I ∩ J), J = \{j\} ∪ (I ∩ J)\}, \]  \tag{31} \]
i, j ∈ w_1, i ≠ j, Ω_a ⊂ Ω_0 and

\[ W_j^{(1)}(Ω_{a,m}, Ω_{a,e}) ≡ \{(I, J) ∈ Ω_{a,m} × Ω_{a,e}| \bar{I} = \{j\} ∪ J\}, \]  \tag{32} \]
j ∈ w_1. In (32)

\[ \bar{I} ≡ I_0 \setminus I, \]  \tag{33} \]
is a "dual" set. (The restrictions (29) and (30) are trivially satisfied when \( n_1 ≤ 1 \) and \( n_1 = 0 \), respectively, where \( n_1 = |w_1| \) is the number of 1-dimensional manifolds among \( M_i \).)

It was shown that after the following gauge fixing

\[ γ = γ_0 ≡ ∑_{i=0}^n d_i x^i, \]  \tag{34} \]

the Maxwell equations (6) for \( s ∈ S_e \) and Bianchi identities \( dF_s = 0 \) for \( s ∈ S_m \) look as follows

\[ \frac{d}{dt} \left( \exp[-2γ_0 + 2χ_s λ_a_s(\varphi)]\dot{Φ}^s \right) = 0, \]  \tag{35} \]

where

\[ χ_s = +1, \quad v_s = e; \]  \tag{36} \]
\[ χ_s = -1, \quad v_s = m. \]  \tag{37} \]

Then

\[ \dot{Φ}^s = Q_s \exp[2γ_0 - 2χ_s λ_a_s(\varphi)], \]  \tag{38} \]

where \( Q_s \) are constants, \( s = (a_s, v_s, I_s) ∈ S \). Let

\[ Q_s ≠ 0, \quad s ∈ S_s; \]  \tag{39} \]
\[ Q_s = 0, \quad s ∈ S \setminus S_s, \]

where \( S_s ⊂ S \) is a non-empty subset of \( S \).
For fixed $Q = (Q_s, s \in S_*)$ the equations of motion (4), (5) are equivalent (after the gauge fixing) to the Euler-Lagrange equations following from the Lagrangian

$$L_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B - V_Q$$

under the zero-energy constraint

$$E_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B + V_Q = 0.$$  

We denoted

$$(x^A) = (x^i, \varphi^\alpha),$$

$$V_Q = -\frac{1}{2} \sum_{i=0}^{n} \xi_i d^i \exp[-2x^i + 2\gamma_0] + \frac{1}{2} \sum_{s \in S_*} Q_s^2 \exp[2\gamma_0 - 2\chi_s \lambda_{s_0}(\varphi)]$$

and

$$(\bar{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix},$$

$$G_{ij} = d_i \delta_{ij} - d_i d_j.$$  

A system with Lagrangian of the type (40) is called the pseudo-Euclidean Toda-like system.

In the next section we consider a special case, when pseudo-Euclidean Toda-like system is reducible to the Toda lattice associated with a Lie algebra of the type $A_m$.

## 3 Integration of the models reducible to a classical open Toda chain

Now we consider the general model under the following restrictions

1. The factor space $M_0$ has non-zero Ricci tensor and all remaining factor spaces $M_1, \ldots, M_n$ are Ricci flat, i.e.

$$\xi_0 \neq 0, \quad \xi_1 = \ldots = \xi_n = 0.$$  

2. All nonvanishing (with $Q_s \neq 0$) $p$-branes do not "live" in the factor space $M_0$, i.e.

$$0 \notin I_s, \quad \forall s \in S_*.$$  

6
We introduce \( (n + l + 1) \)-dimensional real vector space \( \mathbb{R}^{n+l+1} \) with a canonical basis \( \{ e_A \} \), \( A = 0, \ldots, n + l \), where \( e_0 = (1, 0, \ldots, 0) \) etc. Hereafter we use the following vectors

1. The vector \( x(t) \) we need to calculate

\[
x(t) = \sum_{A=0}^{n+l+1} x^A(t) e_A, \quad x^A(t) = (x^i(t), \varphi^\alpha(t)),
\]

\( i = 0, \ldots, n, \ \alpha = 1, \ldots, l. \) We remind that \( \exp[x^i(t)] \) is the scale factor of the space \( M_i \), \( \varphi^\alpha(t) \) is the \( \alpha \)-th dilatonic scalar field.

2. The vector \( U_0 \) induced by the curvature of the factor space \( M_0 \)

\[
U_0 = \sum_{A=0}^{n+l+1} U_0^A e_A, \quad U_0^A = \left( -\frac{\delta_0}{d_0}, 0, \ldots, 0 \right),
\]

\( i = 0, \ldots, n. \)

3. The vector \( U_s \) induced by the \( p \)-brane corresponding to \( s = (a_s, v_s, I_s) \in S_s \)

\[
U_s = \sum_{A=0}^{n+l+1} U_s^A e_A, \quad U_s^A = \left( \delta_i I_s - \frac{d(I_s)}{D-2}, -\chi_s \lambda^\alpha_{a_s} \right),
\]

\( i = 0, \ldots, n, \ \alpha = 1, \ldots, l. \)

Let \( (., .) \) be a symmetrical bilinear form defined in \( \mathbb{R}^{n+l+1} \) by the following manner

\[
(e_A, e_B) = \bar{G}_{AB}.
\]

The bilinear form \( (., .) \) for the vectors \( U_0, U_s \) reads

\[
(U_0, U_0) = \frac{1}{d_0} - 1, \quad (U_0, U_s) = 0, \ \forall s \in S_s,
\]

\[
(U_s, U_{s'}) = d(I_s \cap T_{s'}) - \frac{d(T_s) d(I_{s'})}{D-2} + \chi_s \chi_{s'} \sum_{\alpha, \beta=1}^l \lambda_{a_s, \alpha} \lambda_{a_{s'}, \beta} h^{\alpha \beta}.
\]

Using the notation \( (., .) \) and the vectors \( 46)-(48) \), we may present the Lagrangian and the zero energy constraint for this special type of the model in the form

\[
L_Q = \frac{1}{2} (\dot{x}, \dot{x}) - V_Q, \quad E_Q = \frac{1}{2} (\dot{x}, \dot{x}) + V_Q = 0,
\]
\[ V_Q = a^{(0)} e^{2(U_0 \cdot x)} + \sum_{s \in S_*} a^{(s)} e^{2(U_s \cdot x)}. \]  
\hfill (53)

We used the following constants in Eq. (53)
\[ a^{(0)} = -\frac{1}{2} \xi_0 d_0, \quad a^{(s)} = \frac{1}{2} Q_s^2. \]  
\hfill (54)

Let the vectors \( U_s \in \mathbb{R}^{n+l+1} \) satisfy the following conditions
\[ (U_s, U_s) = U^2 > 0, \; \forall s \in S_*, \]  
\hfill (55)

\[ (C_{ss'}) = \left( \frac{2(U_s, U_{s'})}{(U_s', U_{s'})} \right) = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}. \]  
\hfill (56)

where \((C_{ss'})\) is the Cartan matrix for the Lie algebra \( A_{|S_*|} = sl(|S_*|+1, C)\). We mention that the number of the vectors \( U_0, U_s \) should not exceed the dimension of \( \mathbb{R}^{n+l+1} \), i.e.
\[ n + l - |S_*| \equiv n' \geq 0. \]  
\hfill (57)

We choose in \( \mathbb{R}^{n+l+1} \) a basis \( \{ f_0, f_s, f_p \}, \; s \in S_*, \; p = |S_*|+1, \ldots, |S_*|+n' \), with the properties
\[ f_0 = \frac{U_0}{(U_0, U_0)}, \quad f_s = \frac{U_s}{(U_s, U_s)}, \; s \in S_*, \]  
\hfill (58)

\[ (f_0, f_p) = (f_s, f_p) = 0, \quad (f_p, f_{p'}) = 0, \; p \neq p'. \]  
\hfill (59)

\( p, p' = |S_*|+1, \ldots, |S_*|+n' \). It should be noted that if \( n' = 0 \), then the vectors \( f_p \) do not appear.

Using the decomposition
\[ x(t) = q^0(t)f_0 + \sum_{s \in S_*} q^s(t)f_s + \sum_{p=|S_*|+1}^{n'} q^p(t)f_p \]  
\hfill (60)

with respect to this basis, we present the Lagrangian and the zero-energy constraint (52) in the form
\[ L_Q = L_0 + L_{Q,T} + L_f, \quad E_Q = E_0 + E_{Q,T} + E_f = 0, \]  
\hfill (61)
where

\[ L_0 = \frac{(\dot{q}^0)^2}{2(U_0, U_0)} - V_0, \quad E_0 = \frac{(\dot{q}^0)^2}{2(U_0, U_0)} + V_0, \quad V_0 = a^{(0)} e^{2q^0}, \]  

\[ L_f = E_f = \frac{1}{2} \sum_{p=|S_s|+1}^{n'} (f_p, f_p)(\dot{q}^p)^2, \]  

\[ L_{Q,T} = \sum_{s,s' \in S_s} C_{ss'} \frac{e^{2q^s - V_{Q,T}}}{|S_s|+1} - V_{Q,T}, \quad E_{Q,T} = \sum_{s,s' \in S_s} C_{ss'} e^{2q^s} + V_{Q,T}, \]  

\[ V_{Q,T} = \sum_{s' \in S_s} a^{(s)} \exp \left[ \sum_{s' \in S_s} C_{ss'} q^{s'} \right]. \]

Integration of equations of motion for \( q^0 \) is straightforward and leads to the following result

\[ e^{-q^0(t)} = F(t - t_0), \]  

where \( t_0 \) is an arbitrary constant and the function \( F_0 \) has the form

\[ F_0(t) = \sqrt{-2(U_0, U_0)a^{(0)} t}, \quad \text{if} \quad a^{(0)} > 0, \quad E_0 = 0, \]  

\[ = \sqrt{|a^{(0)}/E_0| \cosh[\sqrt{2(U_0, U_0)E_0}]}, \quad \text{if} \quad a^{(0)} < 0, \quad E_0 < 0, \]  

\[ = \sqrt{|a^{(0)}/E_0| \sinh[\sqrt{2(U_0, U_0)E_0}]}, \quad \text{if} \quad a^{(0)} > 0, \quad E_0 < 0, \]  

\[ = \sqrt{|a^{(0)}/E_0| \sin[\sqrt{-2(U_0, U_0)E_0}]}, \quad \text{if} \quad a^{(0)} > 0, \quad E_0 > 0. \]

For \( q^p \) we get

\[ e^{-q^p(t)} = e^{A^p t + B^p}, \quad p = |S_s| + 1, \ldots, |S_s| + n', \]  

\[ E_f = \frac{1}{2} \sum_{p=|S_s|+1}^{n'} (f_p, f_p)(A^p)^2, \]

where \( A^p \) and \( B^p \) are arbitrary constants.

Using the following translation

\[ q^s \mapsto q^s - \ln C^s, \]  

where the constants \( C^s \) satisfy

\[ \sum_{s' \in S_s} C_{ss'} \ln C^{s'} = \ln[2U^2 a^{(s)}], s \in S_s, \]
we get $L_{Q,T}$ and $E_{Q,T}$ in the form

\[ L_{Q,T} = \frac{1}{4U^2} L_T, \quad E_{Q,T} = \frac{1}{4U^2} E_T, \quad (75) \]

where

\[ L_T = \frac{1}{2} \sum_{s,s' \in S_*} C_{ss'} \dot{q}^s \dot{q}^{s'} - V_T, \quad E_T = \frac{1}{2} \sum_{s,s' \in S_*} C_{ss'} \dot{q}^s \dot{q}^{s'} + V_T, \quad (76) \]

\[ V_T = \sum_{s' \in S_*} \exp \left[ \sum_{s' \in S_*} C_{ss'} q^{s'} \right]. \quad (77) \]

We notice that $L_T$ represents the Lagrangian of a Toda chain associated with the Lie algebra $A_{|S_*|} = sl(|S_*| + 1, C)$, when the root vectors are put into the Chevalley basis and the coordinate describing the motion of the mass center is separated out. Using the result of Anderson\[35\] we present the solution of the equations of motion following from $L_T$

\[ e^{-q^s(t)} = E_s(t) = \prod_{r_1 < \ldots < r_{m(s)}} v_{r_1} \ldots v_{r_{m(s)}} \Delta^2(r_1, \ldots, r_{m(s)}) e^{(w_{r_1} + \ldots + w_{r_{m(s)})} t}, \quad (78) \]

where we denoted by $m(s)$ the number of element $s \in S_*$. The $m(s)$ is assigned to $s$ in accordance with Eq. \[56\]. $\Delta^2(r_1, \ldots, r_{m(s)})$ denotes the square of the Vandermonde determinant

\[ \Delta^2(r_1, \ldots, r_m) = \prod_{r_i < r_j} (w_{r_i} - w_{r_j})^2. \quad (79) \]

The constants $v_r$ and $w_r$ have to satisfy the relations

\[ \prod_{r=1}^{\mid S_* \mid + 1} v_r = \Delta^{-2}(1, \ldots, \mid S_* \mid + 1), \quad \sum_{r=1}^{\mid S_* \mid + 1} w_r = 0. \quad (80) \]

The energy of the Toda chain described by this solution is given by

\[ E_T = \frac{1}{2} \sum_{r=1}^{\mid S_* \mid + 1} w_r^2. \quad (81) \]
Taking into account the transformation (73), we get
\[ e^{-\eta'(t)} \equiv \tilde{F}_s(t) = \frac{F_s(t)}{C_s}, \quad s \in S, \]
(82)
\[ E_{Q,T} = \frac{1}{4U^2} \sum_{r=1}^{|S|+1} w_r^2. \]
(83)

Finally, we obtain the following decomposition of the vector \( x(t) \)
\[ x(t) = \ln[F_0(t-t_0)]\frac{U_0}{1-1/d_0} - \sum_{s \in S_s} \ln[\tilde{F}_s(t)]U_s/\tilde{U}/2 - \sum_{p=|S_s|+1}^{|S_s|+n'} [A_p t + B_p] f_p. \]
(84)

We introduce the following vectors in \( \mathbb{R}^{n+l+1} \)
\[ a \equiv \sum_{A=0}^{n+l} a^A e_A = \sum_{p=|S_s|+1}^{n+l} A^p f_p, \quad b \equiv \sum_{A=0}^{n+l} b^A e_A = \sum_{p=|S_s|+1}^{n+l} B^p f_p. \]
(85)

Their coordinates \( a^A \) and \( b^A \) with respect to the canonical basis satisfy the relations
\[ (a, U_0) = \sum_{A,B=0}^{n+l} G_{AB} a^A U_0^B = 0, \quad (a, U_s) = \sum_{A,B=0}^{n+l} G_{AB} a^A U_s^B = 0, \]
(86)
\[ (b, U_0) = \sum_{A,B=0}^{n+l} G_{AB} b^A U_0^B = 0, \quad (b, U_s) = \sum_{A,B=0}^{n+l} G_{AB} b^A U_s^B = 0, \]
(87)
\[ (a, a) = \sum_{A,B=0}^{n+l} G_{AB} a^A a^B = 2E_f, \]
(88)

\( s \in S_s \). We note that, if \( n' = n+l-|S_s| = 0 \), we must put \( a = b = 0 \).

Using Eq. (84) with Eq. (85), the definitions (46)-(48), we obtain the scale factors \( \exp[x^i(t)] \) and the dilatonic fields \( \phi^\alpha(t) \)
\[ e^{x^0(t)} = [F_0(t-t_0)]^{-\frac{1}{2d_0}} \prod_{s \in S_s} [\tilde{F}_s(t)]^{-\frac{d_l}{(d-2)^2/2}} \exp [a^0 t + b^0], \]
(89)
\[ e^{x^i(t)} = \prod_{s \in S_s} [\tilde{F}_s(t)]^{-\frac{d_l}{(d-2)^2/2} - \delta_{ij}} \exp [a^i t + b^i], \quad i = 1, \ldots, n, \]
(90)
\[ \phi^\alpha(t) = \prod_{s \in S_s} [\tilde{F}_s(t)]^{-\frac{d_l}{(d-2)^2/2} - \delta_{ij}} \exp [a^{n+1+\alpha} t + b^{n+1+\alpha}], \quad \alpha = 1, \ldots, l. \]
(91)
For the potentials $\Phi^s$ we obtain
\begin{equation}
\dot{\Phi}^s(t) = Q_s \prod_{s' \in S} [\tilde{F}_{ss'}(t)]^{-C_{ss'}}.
\end{equation}

We remind that within this model $t$ is the time in the harmonic gauge.

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