Virasoro Symmetries of Drinfeld-Sokolov Hierarchies and Equations of Painlevé Type

Si-Qi Liu¹, Chao-Zhong Wu², and Youjin Zhang¹

¹Department of Mathematics, Tsinghua University, Beijing 100084, P. R. China
Email: liusq@tsinghua.edu.cn; youjin@tsinghua.edu.cn
²School of Mathematics, Sun Yat-Sen University, Guangzhou 510275, P. R. China
Email: wuchaozhong@sysu.edu.cn

Abstract

We construct a tau cover of the generalized Drinfeld-Sokolov hierarchy associated to an arbitrary affine Kac-Moody algebra with gradations \( s \leq 1 \) and derive its Virasoro symmetries. By imposing the Virasoro constraints we obtain solutions of the Drinfeld-Sokolov hierarchy of Witten-Kontsevich and of Brezin-Gross-Witten types, and of those characterized by certain ordinary differential equations of Painlevé type. We also show the existence of affine Weyl group actions on solutions of such Painlevé type equations, which generalizes the theory of Noumi and Yamada on affine Weyl group symmetries of the Painlevé type equations.

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1 Introduction

In the seminal paper [8] of Drinfeld and Sokolov, an integrable hierarchy of Korteweg-de Vries (KdV) type was constructed from any given affine Kac-Moody algebra $g$ and a vertex of its Dynkin diagram. The construction and properties of these integrable hierarchies together with their generalizations [3, 13, 20] constitute an important part of the theory of integrable systems. They also have close relationships with several different research areas of mathematics and physics, such as conformal and cohomological field theories, see [12, 16, 15, 14, 28] and references therein. In particular, it was proved in [12] that the total descendant potential (or the partition function) of the FJRW invariants of ADE-singularities are tau functions of the Drinfeld-Sokolov hierarchies associated to the untwisted affine Kac-Moody algebras of ADE type, which generalizes the Witten-Kontsevich theorem on the relationship between the topological 2d gravity and the KdV hierarchy [41]. Such relationships were also studied for the FJRW theory and the Drinfeld-Sokolov hierarchies associated to the boundary singularities and untwisted affine Kac-Moody algebras of BCFG type respectively [28]. In establishing these relationships the Virasoro symmetries of the integrable hierarchies play an important role, they are used to select the solutions of the Drinfeld-Sokolov hierarchies whose tau functions coincide with the total descendant potential of the FJRW invariants.

The Virasoro symmetries of the Drinfeld-Sokolov hierarchies were studied in [23, 42] (see also references therein). In [23] Hollowood et al constructed the Virasoro symmetries of the generalized Drinfeld-Sokolov hierarchies associated to untwisted affine Kac-Moody algebras, based on a zero-curvature formalism of these hierarchies that involves certain functions taking values in the corresponding Lie groups. In particular, when the affine Kac-Moody algebra is of ADE type, such symmetries can be represented as infinitesimal transformations of the form

$$\tau \mapsto \tilde{\tau} = \tau + \epsilon L_k \tau, \quad k = -1, 0, 1, 2, \ldots.$$  

(1.1)

Here the tau function $\tau$ was introduced via the representation theory of affine Kac-Moody algebras [22] and the linear operators $L_k$, independent of $\tau$, obey the Virasoro commutation relations. For the Drinfeld-Sokolov hierarchy associated to an arbitrary affine Kac-Moody algebra and its zeroth vertex of its Dynkin diagram, the Virasoro symmetries acting on the tau function were studied in [22] by one of the authors of the present paper. In [12] the tau function was defined by choosing a special class of Hamiltonian densities, and the actions of the Virasoro symmetries on the tau function coincide with (1.1) for the Drinfeld-Sokolov hierarchies of ADE type, while such actions have the same form as (1.1) but with index $k = 0, 1, 2, \ldots$ for those hierarchies associated to twisted affine Kac-Moody algebras. Here we note that there is an error in the equation (4.25) of [12] for the twisted case, such that in this case the index should be $k \geq 0$ rather than $k \geq -1$; in fact, on the right hand side of the equation mentioned $[(B_k)_{\geq 0} + d_k, L_1]$ contains only terms with nonnegative
homogeneous degree when \( k \geq -1 \) for the untwisted case and when \( k \geq 0 \) for the twisted case. In contrast, it is founded in [12] that the actions of Virasoro symmetries on the tau function cannot be linearized as in (1.1) for all \( k \geq -1 \) whenever the affine algebra is of BCFG type.

In this paper we consider the generalized Drinfeld-Sokolov hierarchies associated to an arbitrary affine Kac-Moody algebra, of either untwisted or twisted type. Recall that in [3, 20], the construction of the generalized Drinfeld-Sokolov hierarchies depends on two gradations \( s \leq s' \). When \( s \) is the gradation \( s^0 \) and \( s' \) is the principal gradation \( \mathbb{E} = (1, 1, \ldots, 1) \) (see below), the corresponding generalized Drinfeld-Sokolov hierarchies coincide with the original Drinfeld-Sokolov hierarchies. The generalized Drinfeld-Sokolov hierarchies we consider here is for \( s' = \mathbb{E} \), and we will omit the word “generalized” henceforth. For such integrable hierarchies, we defined their tau functions and their solutions selected by certain Virasoro constraints.

In order to study the Virasoro symmetries of a Drinfeld-Sokolov hierarchy, we need to consider a certain extension, called the tau cover, of this integrable hierarchy to avoid nonlocal terms. Let \( \mathfrak{g} \) be an arbitrary affine Kac-Moody algebra of rank \( \ell \). Denote by \( s = (s_0, s_1, \ldots, s_\ell) \) an arbitrary gradation satisfying \( 0 \leq s_i \leq 1 \) (\( s \leq \mathbb{E} \) for short; see Subection 2.2 below). For instance, the gradation \( s^0 \) is defined as \( s_i = \delta_{im} \), and in particular \( s^0 = (1, 0, \ldots, 0) \) is called the homogeneous gradation. It is known that the Drinfeld-Sokolov hierarchy associated to the triple \((\mathfrak{g}, s, \mathbb{E})\) (see [8, 20, 29]) can be represented as the following system of evolutionary equations of an unknown vector function \( u = (u_1, u_2, \ldots, u_\ell) \) as

\[
\frac{\partial u_i}{\partial t_j} = X_j^i(u, u', u'', \ldots), \quad i = 1, 2, \ldots, \ell; \quad j \in J_+.
\]

Here \( J_+ \) stands for the set of positive exponents [25] of \( \mathfrak{g} \), and \( X_j^i \) are differential polynomials in \( u \) (note that \( u' = \partial u/\partial t_1 \)). Given a solution of the hierarchy (1.2), we define its tau function \( \tau^s \) such that [29]

\[
\frac{\partial \log \tau^s}{\partial t_j} = \omega_j, \quad \frac{\partial \omega_j}{\partial t_k} = \Omega_{kj}^s(u, u', u'', \ldots), \quad j, k \in J_+,
\]

where \( \Omega_{kj}^s \), symmetric with respect to the indices \( k \) and \( j \), are certain differential polynomials in \( u \) (see also [12] and [11, 30] for the cases \( s = s^0 \) and \( s = \mathbb{E} \) respectively). Following the notions in [10], the system consists of (1.2) and (1.3) is called the tau cover of the Drinfeld-Sokolov hierarchy (1.2). In fact, letting \( m_1, m_2, \ldots, m_\ell \in J_+ \) be the lowest \( \ell \) positive exponents, note that \( u \mapsto (\omega_{m_1}, \omega_{m_2}, \ldots, \omega_{m_\ell}) \) is up to a Miura-type transformation [8, 3], hence the Drinfeld-Sokolov hierarchy can be represented as a system of evolutionary equations of a single tau function.

One of the main results of the present paper is the following reformulation of the tau cover of the Drinfeld-Sokolov hierarchy, which enables us to define the Virasoro symmetries and to study their properties (see Theorem 3.11 below for a more precise statement).

**Theorem 1.1** The tau cover (1.2), (1.3) of the Drinfeld-Sokolov hierarchy is equivalent to the following system of evolutionary equations of an unknown function \( V \) taking value in \( \mathfrak{g}_{<0}[s] \):

\[
\sum_{m \geq 0} \frac{1}{(m+1)!} (\text{ad}_V)^m \frac{\partial V}{\partial t_j} = (e^{\text{ad}_V} \Lambda_j)_<, \quad j \in J_+.
\]

Here \( \Lambda_j \) are certain generators for the principal Heisenberg subalgebra of \( \mathfrak{g} \), and the subscript “\( <0 \)" means the projection to the negative component of the decomposition \( \mathfrak{g} = \mathfrak{g}_{<0}[s] \oplus \mathfrak{g}_{\geq0}[s] \).
Note that if we introduce the Kac-Moody group associated to \( g \) and the exponential map from \( g \) to this group, then \( \Theta = e^{V} \) has been introduced in [22] to study the tau functions of the generalized Drinfeld-Sokolov hierarchy. However, the notion of Kac-Moody group is sophisticated and unnecessary here. The above theorem shows that we can resolve the problem by using element in \( g \) only. On the other hand, we will show further that the components of \( V \) with respect to a certain basis can be represented as differential polynomials in \( u_i \) and \( \omega_j \), which ensure that the Virasoro symmetries for \( V \) (see (1.5)) are indeed local symmetries for the tau cover (1.2), (1.3), i.e. these symmetries can be represented via differential polynomials in \( u_i \) and \( \omega_j \).

Following the approach of [23, 42], in order to construct the Virasoro symmetries of the Drinfeld-Sokolov hierarchy (1.2), we need to extend \( g \) to a so-called Kac-Moody-Virasoro algebra \( d_s \ltimes g \), where \( d_s \) is a Virasoro operator generated by \( \{d^k_s | k \in \mathbb{Z}\} \) (see [40] for example). With the help of the unknown function in (1.4), we introduce the following evolutionary equations:

\[
\sum_{m \geq 0} \frac{1}{(m+1)!} (\text{ad}_V)^m \frac{\partial V}{\partial \beta_k} = - \left( e^{\text{ad}_V} e^{-\sum_{j \in J_+} t_j \text{ad}_{\lambda_j}} d^k_s - d^k_s \right)_{<0}, \tag{1.5}
\]

with

(I) \( k = -1, 0, 1, 2, \ldots \) for \( g \) being untwisted and \( s \) equal to \( s_0 \) up to a diagram automorphism of \( g \);

(II) \( k = 0, 1, 2, \ldots \) for any other case (to be emphasized, all cases that \( g \) is twisted are included).

The range of the index \( k \) will be explained in the proof of Lemma 4.1 (cf. [23] for the untwisted case and [42] for the case \( s = s_0 \)). We will show that the derivations (1.5) commute with \( \partial / \partial t_j \) for all \( j \in J_+ \) acting on \( V \), hence they lead to a series of symmetries on the tau cover of the Drinfeld-Sokolov hierarchy by using Theorem 1.1. Furthermore, we will represent these symmetries (1.5) via the tau function \( \tau^s \) in the following form (see Theorem 4.4 for the definition of the operators \( S_k \)):

\[
\frac{\partial \log \tau^s}{\partial \beta_k} = S_k(\log \tau^s), \tag{1.6}
\]

and prove the following Virasoro commutation relations

\[
\left[ \frac{\partial}{\partial \beta_l}, \frac{\partial}{\partial \beta_k} \right] \log \tau^s = (k - l) \frac{\partial \log \tau^s}{\partial \beta_{k+l}} \tag{1.7}
\]

with \( k \) and \( l \) given in the cases (I) or (II) above.

As an application of the tau cover above, we have an algorithm to calculate the formal power series solutions to the Cauchy problem of (the tau cover of) the Drinfeld-Sokolov hierarchy associated to \( (g, s, \mathbb{1}) \) with arbitrary initial values given (see Proposition 3.15 below). This algorithm will be applied to study the special solutions that fulfill certain Virasoro constraints.

More precisely, in the above cases (I) and (II), if the tau function satisfies the following equations respectively

\[
S_{-1}(\log \tau^s) - \sum_{p \in J_+} a_p \frac{\partial \log \tau^s}{\partial t_p} = 0, \tag{1.8}
\]

\[
S_0(\log \tau^s) - \sum_{p \in J_+} b_p \frac{\partial \log \tau^s}{\partial t_p} = 0 \tag{1.9}
\]
with \(a_p\) and \(b_p\) being constants that vanish except finitely many of them, then we will see that the tau function must satisfy a series of Virasoro constraints, indexed by \(k \geq -1\) and \(k \geq 0\) respectively (see Theorem 4.9).

The equation (1.8) is called the (generalized) string equation, which makes sense only for the untwisted affine Kac-Moody algebras. For instance, in Case (I) if we consider the string equation (1.8) with \(s = s^0\) and \(a_p = \delta_{p1}\), then it determines the topological solution of the Drinfeld-Sokolov hierarchy that plays a crucial role in cohomological field theories, as mentioned in the beginning. Such topological solutions can be computed by using Proposition 3.15 below. For example, the topological solution \(\tau^{s^0}\) for \(g = A_1^{(1)}\) is just the well-known Witten-Kontsevich tau function up to rescaling the time variables.

In contrast to the string equation, the equation (1.9) makes sense in both cases (I) and (II). We will call this equation the similarity equation, for it is related to the so-called similarity reductions of integrable hierarchies in the literature (see, for example, [6, 17, 18, 19]).

If we take \(b_p = \delta_{p1}\) in the similarity equation (1.9), then the solution \(\log \tau^s\) of the Drinfeld-Sokolov hierarchy is determined up to \(\ell - 1\) free parameters (see Theorem 4.11 below). For example, when \(g = A_1^{(1)}\) and \(s = s^0\), the solution \(\tau^{s^0}\) is called the Brezin-Gross-Witten tau function of the KdV hierarchy [4, 21]. This specified tau function gives (after rescaling the time variables) a generating function for the intersection numbers on the moduli spaces \(\mathcal{M}_{g,n}\) of stable curves with certain Theta cohomology classes involved [32], which is an analogue of the Witten-Kontsevich tau function selected by the string equation [27, 41]. For this reason, such kind of solutions of the Drinfeld-Sokolov hierarchy will be called of Brezin-Gross-Witten type.

In general, we will show the following result (see Theorems 5.1 and 5.6 below for more details).

**Theorem 1.2** For the Drinfeld-Sokolov hierarchy associated to \((g, s, 1)\) together with the similarity equation (1.9), the following assertions hold true:

(i) The solution space is characterized by a system of ODEs of Lax form (5.1), or equivalently, the compatibility condition of a system as follows (see (5.6) below):

\[
z \partial_z \Psi = M \Psi, \quad - \partial \zeta \Psi = L \Psi.\tag{1.10}
\]

(ii) When \(s = 1\), the Lax equation (5.1), i.e. the compatibility condition of (1.10), can be represented in the form:

\[
\varphi'_i + \theta_i \varphi_i + \chi_i = 0, \quad i = 0, 1, 2, \ldots, \ell,\tag{1.11}
\]

where \(\theta_i\) are unknown functions, \(\chi_i\) are constants, with the conditions (5.24) being fulfilled, and \(\varphi_i = \varphi_i(x, \theta_j, \theta'_j, \theta''_j, \ldots)\) are polynomials in their arguments. Moreover, the system of ODEs (1.11) admits a class of rational Bäcklund transformations \(R_j\) with \(j = 0, 1, 2, \ldots, \ell\), which give a realization of the affine Weyl group corresponding to \(g\). Namely, these Bäcklund transformations satisfy

\[
R_j^2 = \text{Id}, \quad (R_i R_j)^{m_{ij}} = \text{Id} \quad \text{for} \quad i \neq j,\tag{1.12}
\]

where \(m_{ij} = 2, 3, 4, 6\) or \(\infty\) when \(a_{ij}a_{j} = 0, 1, 2, 3\) or \(\geq 4\) respectively, with \(A = (a_{ij})_{0 \leq i,j \leq \ell}\) being the generalized Cartan matrix of affine type for \(g\).
Let us give some illustration to the theorem. When we take \( b_p \neq 0 \) for some \( p \in J_{>1} \) in the similarity equation (1.9), we will see that the ODEs given by the compatibility condition of (1.10) are of Painlevé type. For instance, if one take \( g = A_1^{(1)} \) and \( b_p = \delta_p \) for \( p \in J > 1 \), the similarity reduction gives the second Painlevé equation P2 for \( s = 1 \), and the thirty-fourth Painlevé equation P34 (or P4' in the appendix of [7]) for \( s = s^0 \). Some other examples will be given, including the ODEs for \( g = A_2^{(1)} \) and \( b_p = \delta_p \) that are related to the fourth Painlevé equation P4 (see also [6]).

A systematic investigation of the relation between (generalized) Drinfeld-Sokolov hierarchies and higher-order ODEs of Painlevé type may date back to Noumi and Yamada [33, 34, 36]. For such ODEs of Painlevé type, by representing them into a certain symmetric form, Noumi and Yamada constructed a class of birational Bäcklund transformations, whose commutation relations admit the generating relations for affine Weyl groups [35, 36]. This approach was developed by a series of work, for example, [17, 18, 19, 26, 31], most of which relied on matrix realizations of affine Kac-Moody algebras of some particular types. As it is hinted at the end of [36] (without a proof there), we obtain the second part of Theorem 1.2 aiming at a unified construction of the birational Bäcklund transformations related to the Drinfeld-Sokolov hierarchy associated to \((g, s, 1)\). In particular, for \( g = A_\ell^{(1)} \) with \( \ell \geq 2 \) and \( b_p = \delta_p \), we can verify the assertion straightforward with the help of the explicit formulae for \( R_j \), which coincides with the results in [34, 38] obtained in a different way. Note that our proof is independent of matrix realizations of \( g \). As it is mentioned above, the generalized Drinfeld-Sokolov hierarchies can be defined such that other gradations (namely, other Heisenberg subalgebras) are chosen [13, 20], and their relations to equations of Painlevé type were studied in [17, 18, 19, 26] e.t.c. for some particular cases. Such cases, which are not contained in the present paper, will be considered elsewhere.

This paper is arranged as follows. In Section 2 we lay out some properties of affine Kac-Moody algebra and Kac-Moody-Virasoro algebra. In Section 3 we firstly recall the definition of Drinfeld-Sokolov hierarchies and their tau covers, then prove Theorem 1.1 which leads to an algorithm to solve the Cauchy problem of Drinfeld-Sokolov hierarchies. In Section 4 we construct a series of Virasoro symmetries on the tau cover of each Drinfeld-Sokolov hierarchy, and study its solutions to the Virasoro constraints. In Sectoin 5, we derive ODEs of Painlevé type from similarity reductions of Drinfeld-Sokolov hierarchies, and study their discrete Bäcklund transformations. The final section is devoted to some concluding remarks.

2 Preliminaries

Let us first recall, mainly following [25, 40], some properties of affine Kac-Moody algebras.

2.1 Affine Kac-Moody algebras and their principal Heisenberg subalgebras

Let \( A = (a_{ij})_{0 \leq i, j \leq \ell} \) be a generalized Cartan matrix of affine type \( X^{(r)}_\ell \) with \( r = 1, 2, 3 \). The corresponding Kac labels and the dual Kac labels are denoted by \( \{k_i\}_i=0^\ell \) and \( \{k_i^\vee\}_i=0^\ell \) respectively, which satisfy the relations

\[
\sum_{m=0}^\ell a_{im}k_m = \sum_{m=0}^\ell k_m^\vee a_{mj} = 0, \quad k_i^\vee a_{ij}k_j = k_j^\vee a_{ji}k_i, \quad \forall i, j = 0, 1, \ldots, \ell.
\]  

(2.1)

Denote by \( g(A) \) the complex affine Kac-Moody algebra associated to \( A \). Let \( h \) be a fixed Cartan subalgebra of \( g(A) \), \( \Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\} \) and \( \Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \) be the corresponding sets of
simple roots and simple coroots respectively, and \( \Delta \) be the root system of \( \mathfrak{g}(A) \). Then the algebra \( \mathfrak{g}(A) \) admits the following root space decomposition:

\[
\mathfrak{g}(A) = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right). \tag{2.2}
\]

There is a set \( \{ e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i} \mid i = 0, 1, \ldots, \ell \} \) of Chevalley generators satisfying the following Serre relations:

\[
[e_i, f_j] = \delta_{ij} \alpha_i^\vee, \quad [\alpha_i^\vee, \alpha_j^\vee] = 0, \tag{2.3}
\]

\[
[\alpha_i^\vee, e_j] = a_{ij} e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij} f_j, \tag{2.4}
\]

\[
(ad_{e_i})^{1-a_{ij}} e_j = 0, \quad (ad_{f_i})^{1-a_{ij}} f_j = 0, \quad i \neq j, \tag{2.5}
\]

where \( 0 \leq i, j \leq \ell \), and \( \delta_{ij} \) is the Kronecker symbol. The Cartan subalgebra of \( \mathfrak{g}(A) \) can be written as

\[
\mathfrak{h} = \mathbb{C} \alpha_0^\vee \oplus \mathbb{C} \alpha_1^\vee \oplus \cdots \oplus \mathbb{C} \alpha_\ell^\vee \oplus \mathbb{C} d,
\]

where \( d \) is the scaling element and it satisfies the following conditions:

\[
[d, e_i] = e_i, \quad [d, f_i] = -f_i, \quad i = 0, 1, \ldots, \ell. \tag{2.6}
\]

On \( \mathfrak{h} \) there is a nondegenerate symmetric bilinear form defined by

\[
(\alpha_i^\vee | \alpha_j^\vee) = a_{ij} \frac{k_j}{k_i^\vee}, \quad (d | \alpha_j^\vee) = \frac{k_j}{k_j^\vee}, \quad (d | d) = 0, \tag{2.7}
\]

where \( i, j = 0, 1, \ldots, \ell \). The canonical central element of \( \mathfrak{g}(A) \) reads

\[
c = \sum_{i=0}^{\ell} k_i^\vee \alpha_i^\vee, \tag{2.8}
\]

which satisfies the relations

\[
(\alpha_i^\vee | c) = 0 \text{ for } 1 \leq i \leq \ell, \quad (c | c) = 0, \quad (d | c) = \sum_{i=0}^{\ell} k_i =: h. \tag{2.9}
\]

Here \( h \) is called the Coxeter number. The bilinear form on \( \mathfrak{h} \) can be uniquely extended to the normalized invariant symmetric bilinear form \( (\cdot | \cdot) \) on \( \mathfrak{g}(A) \).

Let \( \mathfrak{g} = [\mathfrak{g}(A), \mathfrak{g}(A)] \) be the derived algebra of \( \mathfrak{g}(A) \). Then \( \mathfrak{g} \) is generated by the Chevalley generators, and \( \mathfrak{g}(A) = \mathfrak{g} \oplus \mathbb{C} d \). We will also call \( \mathfrak{g} \) the affine Kac-Moody algebra associated to \( A \) below in case there is no confusion. According to (2.6), the adjoint action of \( d \) induces on \( \mathfrak{g} \) the principal gradation

\[
\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k, \quad \mathfrak{g}^k = \{ X \in \mathfrak{g} \mid [d, X] = kX \}. \tag{2.10}
\]

We fix a cyclic element

\[
\Lambda = \sum_{i=0}^{\ell} e_i \in \mathfrak{g}^1
\]
and consider its adjoin action on \( g \). It is known that
\[
g = \operatorname{Im} \operatorname{ad}_A + \mathcal{H}, \quad \operatorname{Im} \operatorname{ad}_A \cap \mathcal{H} = \mathbb{C} c
\] (2.11)
with \( \mathcal{H} = \{ X \in g \mid \operatorname{ad}_A X \in \mathbb{C} c \} \). The subalgebra \( \mathcal{H} \) is called the principal Heisenberg subalgebra of \( g \) which can be represented as
\[
\mathcal{H} = \bigoplus_{j \in J} \mathcal{C} \Lambda_j \oplus \mathbb{C} c,
\] (2.12)
where \( J \) is the set of exponents given by
\[
J = \{ m_1, m_2, \ldots, m_{\ell} \} + rh \mathbb{Z}, \quad 1 = m_1 < m_2 \leq m_3 \leq \cdots \leq m_{\ell-1} < m_{\ell} = rh - 1,
\] (2.13)
and the basis elements \( \Lambda_j \in g^j \) are chosen to satisfy the conditions
\[
[\Lambda_i, \Lambda_j] = i \delta_{i, -j} c, \quad i, j \in J.
\] (2.14)
In particular, since \( m_1 = 1 \in J \) is a single exponent, there is a constant \( \nu \) such that
\[
\Lambda_1 = \nu \Lambda.
\] (2.15)

### 2.2 The Kac-Moody-Virasoro algebras

Let us consider gradations on \( g \) that are indexed by integer vectors of the set
\[
S = \{ s = (s_0, s_1, \ldots, s_\ell) \in \mathbb{Z}^{\ell+1} \mid s_i \geq 0, s_0 + s_1 + \cdots + s_\ell > 0 \}.
\] (2.16)
For any given vector \( s = (s_0, s_1, \ldots, s_\ell) \in S \), there is an element \( d^s \in \mathfrak{h} \) defined by the conditions
\[
(d^s \mid \alpha_i^\vee) = \frac{k_i}{k_i^s} s_i \quad (0 \leq i \leq \ell), \quad (d^s \mid d^s) = 0.
\] (2.17)
Then we have
\[
[d^s, e_i] = s_i e_i, \quad [d^s, f_i] = -s_i f_i, \quad i = 0, 1, \ldots, \ell,
\]
so \( d^s \) induces the following gradation on \( g \):
\[
g = \bigoplus_{k \in \mathbb{Z}} g_k[s], \quad g_k[s] = \{ X \in g \mid [d^s, X] = k X \}.
\] (2.18)
For any \( X \in g \), suppose that its degree-zero part with respect to (2.10) reads \( X|_{g^0} = \sum_{i=0}^\ell x_i \alpha_i^\vee \), then
\[
(d^s \mid X) = \sum_{i=0}^\ell \frac{x_i k_i s_i}{k_i^s}.
\] (2.19)
In particular,
\[
(d^s \mid c) = k_0 s_0 + k_1 s_1 + \cdots + k_{\ell} s_{\ell} =: h^s,
\] (2.20)
\[
(d^s \mid [X_k, X_l]) = \delta_{k, -l} k(X_k \mid X_l), \quad X_k \in g_{k[s]}, \quad X_l \in g_{l[s]}.
\] (2.21)
We call \( h^s \) the Coxeter number of \( g \) with respect to the gradation \( s \).
Example 2.1 The vector $\mathbf{1} := (1, 1, \ldots, 1)$ gives the principal gradation (2.10) on $\mathfrak{g}$, with $d^1 = d$ and $h^1 = h$ given in (2.6) and (2.9) respectively. In contrast, the vector $s^0 := (1, 0, 0, \ldots, 0)$ induces the homogeneous gradation on $\mathfrak{g}$, with $h^{s^0} = k_0$ being the zero-th Kac label.

Given a gradation $\mathbf{s}$ on $\mathfrak{g}$, let us recall the realization of $\mathfrak{g}$ of type $\mathbf{s}$ (see §7 and §8 of [25]). We start with a simple Lie algebra $\mathcal{G}$ of type $X_r$, on which there is a diagram automorphism $\sigma$ of order $r$. Let $\{E_i, F_i, H_i \in \mathcal{G} \mid i = 0, 1, \ldots, \ell\}$ be a set of elements of $\mathcal{G}$ that is defined in §8.3 of [25]. It is known that $E_i \ (i = 0, 1, 2, \ldots, \ell)$ generate the Lie algebra $\mathcal{G}$, and so do $F_i \ (i = 0, 1, 2, \ldots, \ell)$. The assignment

$$\deg E_i = -\deg F_i = s_i, \quad i = 0, 1, \ldots, \ell$$

induces a $\mathbb{Z}/rh^\mathbf{s}\mathbb{Z}$-gradation of $\mathcal{G}$ as

$$\mathcal{G} = \bigoplus_{k=0}^{rh^\mathbf{s}-1} \mathcal{G}_k.$$

Then we have the following infinite dimensional Lie algebra:

$$\mathfrak{g}^\mathbf{s} = \bigoplus_{k \in \mathbb{Z}} \left( z^k \otimes \mathcal{G}_{k \text{mod} rh^\mathbf{s}} \right) \oplus \mathbb{C} c' \quad (2.22)$$

with $z$ being a parameter and $c'$ a central element. More precisely, if we denote by $X(k)$ an element $z^k \otimes X \in z^k \otimes \mathcal{G}_{k \text{mod} rh^\mathbf{s}}$, then the Lie bracket and the normalized invariant bilinear form on $\mathfrak{g}^\mathbf{s}$ are defined by

$$[X(k) + \xi c', Y(l) + \eta c'] = [X, Y](k + l) + \delta_{k, -l} \frac{k}{rh^\mathbf{s}}(X \mid Y)_G c', \quad (2.23)$$

$$(X(k) + \xi c' \mid Y(l) + \eta c') = \frac{\delta_{k, -l}}{r} (X \mid Y)_G, \quad (2.24)$$

where $\xi, \eta \in \mathbb{C}$ and $k, l \in \mathbb{Z}$, and $(\cdot \mid \cdot)_G$ is the normalized bilinear form on $\mathcal{G}$. As it is shown in §8.7 of [25], the Lie algebra $\mathfrak{g}^\mathbf{s}$ gives a faithful realization of $\mathfrak{g}$. In other words, there is an isomorphism

$$R^\mathbf{s} : \mathfrak{g}^\mathbf{s} \longrightarrow \mathfrak{g} \quad (2.25)$$

such that the following elements are mapped to the Chevalley generators and the simple coroots of $\mathfrak{g}$:

$$E_i(s_i) \mapsto e_i, \quad F_i(-s_i) \mapsto f_i, \quad H_i(0) + \frac{k_i s_i}{k_i \gamma h^s} \mapsto \alpha_i. \quad (2.26)$$

Clearly, one has $R^\mathbf{s}(c') = c$.

Lemma 2.2 For $i = 0, 1, 2, \ldots, \ell$, the following elements

$$R^\mathbf{s}(E_i(rh^s k + s_i)), \quad R^\mathbf{s}(F_i(rh^s k - s_i)), \quad k \in \mathbb{Z}; \quad R^\mathbf{s}(H_i(rh^s k)), \quad k \in \mathbb{Z} \setminus \{0\}$$

of $\mathfrak{g}$ are independent of the gradation $\mathbf{s}$.

Proof: The statement is trivial for $R^\mathbf{s}(E_i(s_i))$ and $R^\mathbf{s}(F_i(-s_i))$. According to the definition of $E_i$, $F_i$ and $H_i$ and the root-space decomposition of $\mathcal{G}$, one can represent $H_i$ in the form

$$H_i = \sum a_{i_1 i_2 \ldots i_m} [E_{i_1}, [E_{i_2}, \ldots, [E_{i_{m-1}}, E_{i_m}], \ldots]]$$

where $a_{i_1 i_2 \ldots i_m}$ are coefficients of $\mathfrak{g}$.
with \(i_1, i_2, \ldots, i_m\) contain exactly \(rk_j\) times of \(j \in \{0, 1, \ldots, \ell\}\). So

\[
R^s(H_i(rh^s)) = \sum a_{i_1i_2...i_m}[R^s(E_{i_1}(s_{i_1})), \ldots, [R^s(E_{i_m-1}(s_{i_m-1})), R^s(E_{i_m}(s_{i_m}))]]
\]

is independent of \(s\). For \(k \geq 2\), the independence of \(R^s(H_i(rh^s k))\) on \(s\) can be derived recursively by using the following relations:

\[
R^s(H_i(rh^s k)) = \frac{1}{2} \sum a_{i_1i_2...i_m}[R^s(E_{i_1}(s_{i_1})), \ldots, [R^s(H_{i_m}(rh^s (k-1))), R^s(E_{i_m}(s_{i_m}))]]
\]

In the same way, when \(k \leq -1\) we can show the validity of the statement for \(R^s(H_i(rh^s k))\) with \(E_i(s_i)\) replaced by \(F_i(-s_i)\). Finally, we complete the proof by using the relations:

\[
R^s(E_i(rh^s k + s_i)) = \frac{1}{2}[R^s(H_i(rh^s k)), R^s(E_i(s_i))];
\]

\[
R^s(F_i(rh^s k - s_i)) = -\frac{1}{2}[R^s(H_i(rh^s k)), R^s(F_i(-s_i))]
\]

for \(k \neq 0\).

Note that the isomorphism (2.25) induces an isomorphism, also denoted as \(R^s\),

\[
R^s : \text{Der}(\mathfrak{g}^s) \rightarrow \text{Der}(\mathfrak{g}).
\]

In particular, we denote

\[
d_k^s = R^s\left(-\frac{1}{rh^s} - rh^{s+1} \frac{d}{dz}\right), \quad k \in \mathbb{Z}.
\]

The action of \(d_k^s\) on an element \(Z \in \mathfrak{g}\) is written as \([d_k^s, Z]\), then these derivations satisfy the following relations:

\[
[d_k^s, R^s(X(l)) + \xi c] = -\frac{l}{rh^s} R^s(X(rh^s k + l)),
\]

\[
[d_k^s, d_l^s] = (k - l)d_{k+l}^s, \quad k, l \in \mathbb{Z}.
\]

The relations (2.26) show that \(\{d_k^s\}\) generate a Virasoro algebra (with trivial center), say, \(\mathfrak{d}^s\), hence we obtain a so-called Kac-Moody-Virasoro algebra \(\mathfrak{d}^s \times \mathfrak{g}\).

**Lemma 2.3** For any \(Z \in \mathfrak{g}\) and \(k \in \mathbb{Z}\), the element \(d_k^s\) defined in (2.17) satisfies the following identities:

\[
(d_k^s | [d_k^s, Z]) = 0, \quad k \in \mathbb{Z}.
\]

**Proof:** It follows from (2.19) and (2.28) that we only need to check \((d_k^s | R^s(H_i(0))) = 0\) for

\(i = 0, 1, 2, \ldots, \ell\), which can be easily verified by using (2.17), (2.20), and (2.26). Thus the lemma is proved. \(\square\)

In [10], Wakimoto studied the relations between the derivations \(d_k^s\) with different gradations \(s\). Let us review some results that will be applied in Section 4 below. Denote by \(\hat{A} = (a_{ij})_{1 \leq i, j \leq n}\) the Cartan matrix of finite type given by the affine Cartan matrix \(A\) with the zero-th row and the zero-th column deleted. Given a gradation \(s \in S\), we introduce a vector

\[
(r_1, r_2, \ldots, r_\ell) = \frac{1}{rh^s}(s_1, s_2, \ldots, s_\ell)\hat{A}^{-1}
\]
and an element
\[ \rho^s = \sum_{i=1}^{\ell} r_i a_i^\vee \in \mathfrak{h}. \] (2.32)

For any \( s' \in S \), let
\[ \rho^0_s(r h^s k; s') = \sum_{i=1}^{\ell} r_i R^s_H \left( H_i(r h^s k) \right) \in g_{r h^s k|s' \rangle}, \quad k \in \mathbb{Z}. \] (2.33)

According to Lemma 2.2, the elements \( \rho^0_s(r h^s k; s') \in g \) are independent of \( s' \) whenever \( k \neq 0 \). In terms of the above notations, let us present Wakimoto’s Lemma 2.4 in [40] in the following lemma.

Note that in Wakimoto’s original lemma, the central element term is missing.

Lemma 2.4 For \( s^0 = (1, 0, \ldots, 0) \) and any \( s \in S \), the following equalities hold true:
\[
d_k^s = \begin{cases} 
  d_0^{s^0} - \rho^s + \frac{r}{2} (\rho^s | \rho^s) c, & k = 0; \\
  d_k^{s^0} - \rho^s(r k_0 k; s^0), & k \neq 0.
\end{cases} \] (2.34)

Proof: Denote by \( d_k \) the right hand side of (2.34). We need to show that \( d_k \) with \( k \in \mathbb{Z} \) also satisfy the relations (2.28) and (2.29). By using the definition (2.31) of \( r_i \) and the properties (2.1) of \( k_i \), we obtain the identities
\[
\sum_{i=1}^{\ell} r_i a_{ij} = \frac{s_j}{r h^s}, \quad 1 \leq j \leq \ell; \\
\sum_{i=1}^{\ell} r_i a_{i0} = -\frac{1}{k_0} \sum_{i=1}^{\ell} r_i a_{ij} k_j = -\frac{1}{k_0 r h^s} \sum_{j=1}^{\ell} s_j k_j = -\frac{h^s - k_0 s_0}{k_0 r h^s} = -\frac{1}{r k_0} + \frac{s_0}{r h^s},
\]
from which it follows that
\[
[d_0, e_j] = \left[ d_0^{s^0} - \rho^s + \frac{r}{2} (\rho^s | \rho^s) c, e_j \right] = -\left( \frac{\delta j_0}{r k_0} + \sum_{i=1}^{\ell} r_i a_{ij} \right) e_j = -\frac{s_j}{r h^s} e_j, \quad j = 0, 1, 2, \ldots, \ell. \] (2.35)

Similarly, for \( k \neq 0 \) and \( 0 \leq j \leq \ell \) we have
\[
[d_k, R^{s^0}_H (E_j(\delta j_0))] = -\left( \frac{\delta j_0}{r k_0} + \sum_{i=1}^{\ell} r_i a_{ij} \right) R^{s^0}(E_j(r k_0 k + \delta j_0)) = -\frac{s_j}{r h^s} R^{s^0}_H (E_j(r k_0 k + \delta j_0)).
\]

Thus by using Lemma 2.2 we arrive at the relations
\[
[d_k, R^s_H (E_j(s_j))] = -\frac{s_j}{r h^s} R^s (E_j(r h^s k + s_j)).
\]

In the same way, we can prove the relations
\[
[d_k, R^s_H (F_j(-s_j))] = \frac{s_j}{r h^s} R^s (E_j(r h^s k - s_j)).
\]
Now by using Leibniz’s rule we arrive at
\[ [d_k, R^s(X(l))] = -\frac{s_i}{r h^s} R^s(X(r h^s k + l)), \quad k, l \in \mathbb{Z}. \]
Finally, we check the commutation relation (2.29) for \( d_k \) as follows:
\[ [d_k, d_l] = (k-l)d_{k+l}^0 + \frac{r k_0 (l-k)}{r k_0} \rho^s_0(r k_0 (k+l); s^0) \]
\[ + \delta_{k,-l} \frac{r k_0}{k_0} \left( \rho^s_0(0; s^0) \mid \rho^s_0(0; s^0) \right) c \]
\[ = (k-l) \left( d_{k+l}^0 - \rho^s_0(r k_0 (k+l); s^0) + \delta_{k,-l} \frac{r}{2} (\rho^s \mid \rho^s) c \right) \]
\[ = (k-l)d_{k+l}, \quad k, l \in \mathbb{Z}, \]
where we have used the fact that \( \rho^s_0(0; s^0) - \rho^s \in \mathbb{C} c \) and the properties (2.9). Therefore, the lemma is proved. □

The above lemma yields the following corollary.

**Corollary 2.5** Given any gradation \( s, s' \in S \) and any integers \( k, l \in \mathbb{Z} \), the element \( d_k^{s'} - d_k^s \in g \) satisfies the following relations:
\[ d_k^{s'} - d_k^s = \begin{cases} 
\rho^s - \rho^{s'} - \frac{r}{2} \left( (\rho^s \mid \rho^s) - (\rho^{s'} \mid \rho^{s'}) \right) c, & k = 0; \\
\rho^s_0(r h^s k; s) - \rho^{s'}_0(r h^{s'} k; s), & k \neq 0.
\end{cases} \] (2.36)

3 The tau covers of Drinfeld-Sokolov hierarchies and their solutions

In this section, we first recall the Drinfeld-Sokolov hierarchy associated to an affine Kac-Moody algebra and construct a tau cover of it, then we reformulate this tau cover in terms of the dressing operator of the Drinfeld-Sokolov hierarchy. This new formulation of the tau cover plays an important role in our study of the Virasoro symmetries of the Drinfeld-Sokolov hierarchies, which is given in the next section. Based on this tau cover, we also construct power series solutions of the initial value problem of the Drinfeld-Sokolov hierarchy.

### 3.1 Drinfeld-Sokolov hierarchies and their tau covers

Let \( g \) be an affine Kac-Moody algebra of rank \( \ell \). Apart from the principal gradation \( \mathbf{1} = (1,1,\ldots,1) \), we fix a gradation \( s = (s_0, s_1, s_2, \ldots, s_\ell) \in S \) with \( s_i \leq 1 \) (\( s \leq \mathbf{1} \) for short), and denote
\[ g^k = g_{k[1]}, \quad g_k = g_{k[s]}, \quad k \in \mathbb{Z}. \]
In what follows, we will use notations like \( g_{\geq l} = \bigoplus_{k \geq l} g_k, \ g_{< l} = \bigoplus_{k < l} g_k \) etc.

Let us briefly review the construction of the generalized Drinfeld-Sokolov hierarchy associated to the triple \((g, s, \mathbf{1})\) mainly following the notations used in [29] (cf. the original definition given in [8, 20]). We first introduce a Borel subalgebra
\[ \mathcal{B} = \{ X \in g_0 \cap g_{< 0} \mid (d^s \mid X) = 0 \}, \] (3.1)
and consider operators of the form

$$\mathcal{L} = \frac{\partial}{\partial x} + \Lambda_1 + Q, \quad Q \in C^\infty(\mathbb{R}, B) \quad (3.2)$$

with $x$ being the coordinate of $\mathbb{R}$. Note that the Lie bracket on $\mathfrak{g}$ can be extended naturally to $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}, \mathfrak{g})$, then we have the following dressing lemma.

**Lemma 3.1** ([42, 29]) For an operator $\mathcal{L}$ of the form given in (3.2), there exists a unique function $U(Q) \in C^\infty(\mathbb{R}, \mathfrak{g}^{<0})$ satisfying the following two conditions:

(i) \quad $e^{-\text{ad}U(Q)}\mathcal{L} = \frac{\partial}{\partial x} + \Lambda_1 + H(Q), \quad H(Q) \in C^\infty(\mathbb{R}, \mathcal{H} \cap \mathfrak{g}^{<0}), \quad (3.3)$

(ii) \quad $(d^s | e^{\text{ad}U(Q)}\Lambda_j) = 0, \quad \forall j \in J_+.$ \quad (3.4)

Moreover, both $U(Q)$ and $H(Q)$ are $x$-differential polynomials with zero constant terms in the components of $Q$ w.r.t a basis of $B$ (differential polynomials in $Q$ for short).

The Borel subalgebra $\mathcal{B}$ contains a nilpotent subalgebra $\mathcal{N} := \mathfrak{g}_0 \cap \mathfrak{g}^{<0}$, that is, the subalgebra generated by the elements $f_i$ with $s_i = 0$. It follows from (2.15) and the Serre relations (2.3) that

$$\text{ad}_{\Lambda_1}\mathcal{N} = \text{ad}_{I}\mathcal{N} \subset \mathcal{B} \quad \text{with} \quad I := \sum_{i \mid s_i = 0} e_i. \quad (3.5)$$

Since $\mathcal{N} \cap \mathcal{H} = \{0\}$ (see (2.12) and § 14 of [25]), the map $\text{ad}_{\Lambda_1} : \mathcal{N} \rightarrow \mathcal{B}$ is an injection. Thus one can choose an $\ell$-dimensional subspace $\mathcal{V}$ of $\mathcal{B}$ such that

$$\mathcal{B} = \text{ad}_{\Lambda_1}\mathcal{N} \oplus \mathcal{V}. \quad (3.6)$$

Let us fix a complement subspace $\mathcal{V}$ in (3.6) henceforth, and consider operators of the form

$$\mathcal{L}^\mathcal{V} = \frac{\partial}{\partial x} + \Lambda_1 + Q^\mathcal{V}, \quad Q^\mathcal{V} \in C^\infty(\mathbb{R}, \mathcal{V}). \quad (3.7)$$

By using the method of [8, 20], we can prove the following results [29].

**Lemma 3.2** The following assertions hold true:

(i) For an operator $\mathcal{L}$ of the form given in (3.2), there exists a unique function $N \in C^\infty(\mathbb{R}, \mathcal{N})$ such that

$$\mathcal{L}^\mathcal{V} = e^{\text{ad}_N}\mathcal{L}, \quad (3.8)$$

takes the form of (3.7). Moreover, both $N$ and $Q^\mathcal{V}$ are differential polynomials in $Q$ with zero constant terms.

(ii) For an operator $\mathcal{L}^\mathcal{V}$ of the form given in (3.7), let $U(Q^\mathcal{V})$ be the function determined by Lemma 3.1, there is a unique function $R(Q^\mathcal{V}, \Lambda_j) \in C^\infty(\mathbb{R}, \mathcal{N})$ for any fixed $j \in J_+$ such that the commutator

$$\left[ -(e^{\text{ad}_{U(Q^\mathcal{V})}\Lambda_j})_{\geq 0} + R(Q^\mathcal{V}, \Lambda_j), \mathcal{L}^\mathcal{V} \right]$$

takes value in $\mathcal{V}$. Moreover, the components of $R(Q^\mathcal{V}, \Lambda_j)$ are differential polynomials in $Q^\mathcal{V}$ with zero constant terms.
Here and in what follows, the subscript “≥ 0” (respectively, “< 0”) of a \( g \)-valued function means its projection to \( g_{≥ 0} \) (respectively, to \( g_{< 0} \)) with respect to the gradation \( s \).

Due to the second assertion of the above lemma, one can formulate the Drinfeld-Sokolov hierarchy as follow.

**Definition 3.3** The Drinfeld-Sokolov hierarchy associated to the triple \((g, s, 1)\) is given by the following evolutionary equations:

\[
\frac{\partial \mathcal{L}^V}{\partial t_j} = \left[ -\left( e^{ad_{U(Q^V)}} \Lambda_j \right)_{≥ 0} + R(Q^V, \Lambda_j), \mathcal{L}^V \right], \quad j \in J_+.
\]  

(3.9)

It can be verified that the flows \((3.9)\) are compatible with each other (they are assumed to commute with \( \partial/\partial x \)). In particular, one has \( \partial/\partial t_1 = \partial/\partial x \), so from now on we identify \( t_1 \) with \( x \).

Let us choose a basis \( \eta_1, \eta_2, \ldots, \eta_\ell \) of the subspace \( V \), and represent \( Q^V \) in the form

\[
Q^V = \sum_{i=1}^\ell u_i \eta_i.
\]

(3.10)

Then the Drinfeld-Sokolov hierarchy \((3.9)\) can be represented in terms of the unknown function \( u := (u_1, u_2, \ldots, u_\ell) \) as follows:

\[
\frac{\partial u_i}{\partial t_j} = X^i_j(u, u', u'', \ldots), \quad i = 1, \ldots, \ell; \; j \in J_+.
\]

(3.11)

Here \( X^i_j \) are differential polynomials in \( u \) (the prime means to take the derivative with respect to \( x \)). In particular, one has \( X^i_1 = u'_i \).

Now let us define the differential polynomials

\[
\Omega^s_{kj}(u, u', u'', \ldots) = \frac{1}{h^s} \left( d^s \mid \left( e^{ad_{U(Q^V)}} \Lambda_k \right)_{≥ 0} - e^{ad_{U(Q^V)}} \Lambda_j \right), \quad k, j \in J_+.
\]

(3.12)

**Proposition 3.4** ([29]) \( The \) differential polynomials \( \Omega^s_{kj} = \Omega^s_{kj}(u, u', u'', \ldots) \) satisfy the relations

\[
\Omega^s_{kj} = \Omega^s_{j k}, \quad \frac{\partial \Omega^s_{kj}}{\partial t_l} = \frac{\partial \Omega^s_{lj}}{\partial t_k}, \quad j, k, l \in J_+.
\]

(3.13)

In particular,

\[
\Omega^s_{1 j} = \frac{j}{h} h_j, \quad j \in J_+.
\]

(3.14)

where \( h_j = -(\Lambda_j \mid H(Q^V)) \) with \( H(Q^V) \) being determined by Lemma [5.4] (recall that \( h \) is the Coxeter number of \( g \)).

Denote \( t = \{t_j \mid j \in J_+\} \). The first assertion of the proposition implies that, given a solution \( u(t) \) of the Drinfeld-Sokolov hierarchy \((3.9)\), there locally exists a function \( \tau^s = \tau^s(t) \), called the tau function, such that

\[
\frac{\partial^2 \log \tau^s}{\partial t_k \partial t_j} = \Omega^s_{kj}(u, u', u'', \ldots)|_{u=u(t)}, \quad j, k \in J_+.
\]

(3.15)

Note that \( \log \tau^s \) is determined by \( u(t) \) up to a linear function in \( t \).
Definition 3.5 The tau cover of the Drinfeld-Sokolov hierarchy associated to \((g, s, 1)\) is defined as the following hierarchies for the unknown functions \(f, \omega_j (j \in J_+), u_i (i = 1, 2, \ldots, \ell)\):

\[
\frac{\partial f}{\partial t_k} = \omega_k, \quad \frac{\partial \omega_j}{\partial t_k} = \Omega_{kj}^e(u, u', u'', \ldots), \quad \frac{\partial u_i}{\partial t_k} = X_i^k(u, u', u'', \ldots), \quad k \in J_+.
\]

(3.16)

Remark 3.6 When \(s = s^0\), the formulae (3.15) to define tau function are equivalent with those in [42], where

\[
\frac{\partial^2 \log \tau}{\partial t_k \partial t_j} = \frac{j}{\langle \Lambda_j | \Lambda_{-j} \rangle} \left( -\Lambda_j^{-1} \frac{\partial H(Q^V)}{\partial t_k} \right), \quad j, k \in J_+.
\]

(3.17)

Note that all components of \(\partial H(Q^V) / \partial t_k\) are total derivatives in \(u\) (see, for example, (3.14)). When \(s = 1\), the tau function in (3.15) coincides the one defined in [11, 30].

Remark 3.7 It is known that the Drinfeld-Sokolov hierarchy (3.18) has a Hamiltonian structure, and the functions \(h_j\) given in Proposition 3.4 are densities of the Hamiltonian functionals [8, 3, 42]. From Proposition 3.4 it follows that these densities of the Hamiltonians satisfy the tau-symmetry condition [10]. Moreover, if we denote \(h = (h_{m_1}, h_{m_2}, \ldots, h_{m_\ell})\) with \(m_1, m_2, \ldots, m_\ell\) being the first \(\ell\) positive exponents of \(g\) (see (2.13)), then it is known that \(u \mapsto h = h(u, u', \ldots)\) is a Miura-type transformation. Conversely, one can represent \(u = u(h, h', \ldots)\) as differential polynomials in \(h\) which may not be truncated unless \(s = s^0\).

The tau covers of the integrable hierarchies play crucial roles in the application of the Drinfeld-Sokolov hierarchies to the study of topological field theory. In fact, the tau function corresponds to the partition function of the topological field theory, and \(\omega_j, \Omega_{kj}\) are the one-point and the two-point correlators respectively.

3.2 A reformulation of the tau cover

In this subsection, we are to show that the tau cover (3.16) of the Drinfeld-Sokolov hierarchy can be reformulated as the following hierarchy of differential equations for an unknown function \(V\), depending on infinitely many variables \(t = \{t_j \mid j \in J_+\}\) and taking value in \(g_{<0}\):

\[
\nabla_{t_j, V} V = \left( e^{\mathrm{ad}_V \Lambda_j} \right)_{<0}, \quad j \in J_+.
\]

(3.18)

where

\[
\nabla_{t_j, V} V = \sum_{m \geq 0} \frac{1}{(m + 1)!} (\mathrm{ad}_V)^m \frac{\partial V}{\partial t_j}.
\]

(3.19)

Let us note that if a matrix realization of \(g\) is taken and its Lie group is generated by the exponential map, so that the adjoint actions can be represented by matrix multiplication, then one has an equivalent representation for the equations (3.18). Indeed, denote \(\Theta = e^V\), then one can write (3.18) in the form

\[
\frac{\partial \Theta}{\partial t_j} = (\Theta \Lambda_j \Theta^{-1})_{<0} \Theta, \quad j \in J_+.
\]

(3.20)

This hierarchy of differential equations for \(\Theta\) were used in the literature [30, 23, 42] to study the tau functions and Virasoro symmetries of the Drinfeld-Sokolov hierarchy. The relation of these
equations with the Drinfeld-Sokolov hierarchy is given by the following dressing procedure: if we define the Lax operator $\mathcal{L}$ (see (3.2)) by

$$\mathcal{L} = \Theta \left( \frac{\partial}{\partial x} + \Lambda_1 \right) \Theta^{-1} + \text{a central term},$$

then the equations (3.20) lead to the Drinfeld-Sokolov hierarchy (3.9) that consists of evolutionary equations for the unknown functions $u_1, \ldots, u_\ell$. To avoid the notion of Kac-Moody group, we would like to work with the function $V$ that takes value in the Kac-Moody algebra, instead of its exponential $\Theta$.

More exactly, if we expand the function $V$ with respect to the decomposition (2.18) in the form

$$V = \sum_{k \leq -1} V_k, \quad V_k = V|_{g_k},$$

then the equations (3.18) can be represented recursively as follows:

$$\frac{\partial V_{-1}}{\partial t_j} = \left( e^{\text{ad}_V \Lambda_j} \right)_{|_{g_{-1}}}, \quad (3.21)$$

$$\frac{\partial V_k}{\partial t_j} = \left( e^{\text{ad}_V \Lambda_j} \right)_{|_{g_k}} - T_{jk}, \quad k \leq -2, \quad (3.22)$$

where

$$T_{jk} = \sum_{m=1}^{k-1} \frac{1}{(m+1)!} \sum_{k_1 \leq -1, k_1 + \cdots + k_{m+1} = k} \text{ad}_{V_{k_1}} \cdots \text{ad}_{V_{k_m}} \frac{\partial V_{k_{m+1}}}{\partial t_j}.$$ 

**Lemma 3.8** For any solution $V$ of (3.18), the following equalities hold true:

$$\frac{\partial}{\partial t_j} \left( e^{\text{ad}_V \Lambda_i} \right) = \left[ \left( e^{\text{ad}_V \Lambda_j} \right)_{|_{g_{-1}}}, e^{\text{ad}_V \Lambda_i} \right], \quad (3.23)$$

$$\frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} V = \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_i} V. \quad (3.24)$$

Here $i, j \in J_+$. 

**Proof:** Denote $Y = \partial V / \partial t_j$, then the equalities (3.23) can be verified as follows:

$$\text{l.h.s.} = \sum_{k \geq 0} \frac{1}{(k+1)!} \sum_{l=0}^{k} (\text{ad}_V)^l Y (\text{ad}_V)^{k-l} \Lambda_i$$

$$= \sum_{k \geq 0} \frac{1}{(k+1)!} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{l!}{m!} \text{ad}_{(\text{ad}_V)^m Y} (\text{ad}_V)^{l-m+k-l} \Lambda_i$$

$$= \sum_{k \geq 0} \frac{1}{(k+1)!} \sum_{m=0}^{k} \frac{(k+1)!}{(m+1)!} \text{ad}_{(\text{ad}_V)^m Y} (\text{ad}_V)^{k-m} \Lambda_i$$

$$= \sum_{m=0}^{k} \sum_{k \geq m} \frac{1}{(k-m)!(m+1)!} \text{ad}_{(\text{ad}_V)^m Y} (\text{ad}_V)^{k-m} \Lambda_i$$
It implies that
\[ [A_i, A_j] = 0 \quad \text{for} \quad i, j \in J_+. \]
Then, for any \( k \in J_+ \), it follows from (3.23) that
\[
\frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \left( e^{adv} \Lambda_k \right)
= \left[ [(A_i)_{<0}, (A_j)_{<0}, A_k] + [(A_j)_{<0}, (A_i)_{<0}, A_k] \right.
- \left[ [(A_j)_{<0}, A_i]_{<0}, A_k] - [(A_i)_{<0}, (A_j)_{<0}, A_k] \right]
+ \left[(A_i)_{<0}, (A_j)_{<0}, A_k] + [(A_j)_{<0}, (A_i)_{<0}, A_k] \right] + \left[(A_j)_{<0}, A_i, (A_i)_{<0}] \right]
= 0.
\]

On the other hand, the left hand side of (3.26) is expanded to
\[
\sum_{m \geq 1} \frac{1}{m!} \sum_{p=0}^{m-1} (adv)^p ad Z (adv)^{m-1-p} \Lambda_k = 0,
\]
where \( Z = [\partial/\partial t_i, \partial/\partial t_j] V \) (the terms that contain the first-order derivatives with respect to the time variables are cancelled). Since \( s \leq 1 \), we have \( Z \in \mathfrak{g}_{<0} \subset \mathfrak{g}^{<0} \). Suppose that \( Z \neq 0 \), and let \( l < 0 \) be the largest integer such that \( Z_l := Z|_{\mathfrak{g}^{<0}} \neq 0 \) with respect to the decomposition (2.10). We take \( k = 1 \) in (3.27) and consider its highest degree term
\[ [Z_l, \Lambda_1] = 0. \]
It implies that \( Z_l \) lies in \( \mathcal{H} \cap \mathfrak{g}^{<0} \), and that \( l \) is in fact a negative exponent. Let us take \( k = -l \) in (3.27) and consider the highest degree term, then we arrive at \( Z_l = 0 \) due to (2.14), which is in contradiction with our assumption that \( Z_l \neq 0 \). So the equalities (3.24) hold true, and the lemma is proved. \( \Box \)

It follows from the equalities (3.24) that the flows (3.18) are compatible, so the systems of differential equations (3.18) form an integrable hierarchy. We proceed to introduce the tau function of the hierarchy (3.18), and then establish its relation with the tau cover (3.16) of the Drinfeld-Sokolov hierarchy.

Given a solution \( V \) of the equations (3.18), we introduce a collection of functions \( \{ \omega_j \mid j \in J_+ \} \) as follows:
\[
\omega_j = -\frac{1}{h^s} \left( d^s \mid e^{adv} \Lambda_j \right), \quad j \in J_+.
\]
Note that this notations (also \( \tau^s \) below) have been used in the last subsection as part of the unknown functions of the tau cover (3.16). We will show later that they actually coincide.

**Lemma 3.9** The functions \( \omega_j \) satisfy the following conditions:
\[
\frac{\partial \omega_j}{\partial t_k} = \frac{\partial \omega_k}{\partial t_j}, \quad j, k \in J_+.
\]
Proof: By using (2.19) and (3.23), we have, for \( j, k \in J_+ \),
\[
\frac{\partial \omega_j}{\partial t_k} = -\frac{1}{h^s} \left( d^s \mid \left( e^{ad_{V} \Lambda_k} \right)_{<0}, \left( e^{ad_{V} \Lambda_j} \right)_{>0} \right) = \frac{1}{h^s} \left( d^s \mid \left( e^{ad_{V} \Lambda_k} \right)_{>0}, \left( e^{ad_{V} \Lambda_j} \right)_{<0} \right) = \frac{\partial \omega_k}{\partial t_j}.
\]
(3.30)
Thus the lemma is proved. □

From the above lemma it follows the existence of a function \( \tau^{s} = \tau^{s}(t) \) such that
\[
\frac{\partial \log \tau^{s}}{\partial t_{j}} = \omega_{j}, \quad j \in J_+.
\]
(3.31)

Definition 3.10 The function \( \tau^{s} \) is called the tau function of the hierarchy of differential equations (3.18).

The following Theorem is the main result of this section, which shows the equivalence between the hierarchy of differential equations (3.18) and the tau cover (3.16) of the Drinfeld-Sokolov hierarchy (3.9). A proof of the theorem will be given in the next subsection.

Theorem 3.11 Let \( g \) be an affine Kac-Moody algebra of rank \( \ell \) with two gradations \( s \leq 1 \), and a decomposition (3.6) be fixed for the Borel subalgebra \( B \) defined in (3.1). The following two assertions hold true:

(i) For any solution \( V \in C^\infty(\mathbb{R}, g_{<0}) \) of the equations (3.18), there is an operator (recall \( x = t_1 \))
\[
\mathcal{L}^{V} = \frac{\partial}{\partial x} + \Lambda_1 + Q^{V}, \quad Q^{V} = \sum_{i=1}^{\ell} u_i \eta_i \in C^\infty(\mathbb{R}, V)
\]
(3.32)
of the form (3.7) and (3.10) such that the functions \( u_1, \ldots, u_\ell \) are differential polynomials of the components of \( V \) and they, together with \( \omega_j (j \in J_+) \) and \( f = \log \tau^{s} \) in (3.28) and in (3.31) respectively, give a solution of the tau cover (3.16) of the Drinfeld-Sokolov hierarchy (3.9). Moreover, the components of the function \( V \) can be represented uniquely as elements of the ring
\[
\mathcal{R} = \mathbb{C} \left[ u_i^{(k)}, \omega_j \mid 1 \leq i \leq \ell, k \in \mathbb{Z}_{\geq 0}, j \in J_+ \right]
\]
(3.33)
with zero constant terms, denoted as
\[
V = V(u_i^{(k)}, \omega_j \mid 1 \leq i \leq \ell, k \in \mathbb{Z}_{\geq 0}, j \in J_+).
\]
(3.34)
(ii) If the functions \( f = \log \tau^{s}, \omega_j (j \in J_+) \) and \( u_1, \ldots, u_\ell \) satisfy the tau cover (3.16) of the Drinfeld-Sokolov hierarchy (3.9), then the function \( V \) given by (3.34) solves the equations (3.18).
3.3 Proof of Theorem 3.11

Let us proceed to prove Theorem 3.11. Suppose that $V$ is a solution of the hierarchy of differential equations (3.18). We consider an operator as follows

$$L = \frac{\partial}{\partial t_1} + \Lambda_1 + Q$$

with $Q = \left( e^{\text{adv} \Lambda_1} \right)_{\geq 0} - \Lambda_1 + \omega_1 c$. (3.35)

Indeed, since $\Lambda_1 \in (g_0 \cup g_1) \cap g^{-1}$ and $V$ takes value in $g_{<0} \subset g^{<0}$, then we have $Q \in g_0 \cap g_{\leq 0}$; on the other hand, by using (3.28) we obtain

$$(d^s \mid Q) = \left( d^s \mid e^{\text{adv} \Lambda_1} \right) + \omega_1 (d^s \mid c) = -h^s \omega_1 + h^s \omega_1 = 0.$$

So the function $Q$ takes value in the Borel subalgebra $B$, and the operator $L$ is of the form (3.2). From the first assertion of Lemma 3.2, it follows the existence of a unique function $N$ taking value in $N$ such that

$$L^V := e^{\text{adv} N} L = \frac{\partial}{\partial t_1} + \Lambda_1 + Q^V$$

takes the form (3.32). Note that both $N$ and $Q^V$ can be represented as differential polynomials in $Q$, and so all these three functions can be represented as differential polynomials in $V$.

**Lemma 3.12** For the operator $L^V$ defined in (3.36), the functions $U(Q^V)$ and $H(Q^V)$ given via Lemma 3.1 are uniquely determined by the following equations:

$$e^{\text{adv} U(Q^V)} = e^{\text{adv} N} e^{\text{adv} e^{-\text{adv} \Omega}}, \quad H(Q^V) = -\frac{\partial \Omega}{\partial t_1},$$

where

$$\Omega = \sum_{j \in J_+} \frac{\omega_j}{j} \Lambda_{-j}.$$ (3.38)

**Proof:** By using the Baker-Campbell-Hausdorff formula (see [24] for example), a $g^{<0}$-valued function $W$ can be uniquely determined by the relation

$$e^{\text{adv} W} = e^{\text{adv} N} e^{\text{adv} e^{-\text{adv} \Omega}}.$$

Clearly, we have $[\Omega, \Lambda_j] = -\omega_j c$ for any $j \in J_+$, so

$$L^V = e^{\text{adv} N} \left( \frac{\partial}{\partial t_1} + e^{\text{adv} \Lambda_1} - \nabla_{t_1} V \right) + \omega_1 c$$

$$= e^{\text{adv} N} e^{\text{adv} \Lambda_1} \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right) + \omega_1 c$$

$$= e^{\text{adv} N} e^{\text{adv} e^{-\text{adv} \Omega}} \left( \frac{\partial}{\partial t_1} + \Lambda_1 - \frac{\partial \Omega}{\partial t_1} \right) + [\Omega, \Lambda_1] + \omega_1 c$$

$$= e^{\text{adv} W} \left( \frac{\partial}{\partial t_1} + \Lambda_1 - \frac{\partial \Omega}{\partial t_1} \right).$$ (3.39)

By using the facts that $[d^s, N] = 0$ and the commutation relation (2.14), we obtain

$$(d^s \mid e^{\text{adv} \Lambda_j} \omega_1 c) = \left( e^{\text{adv} N} d^s \mid e^{\text{adv} e^{-\text{adv} \Omega}} \Lambda_j \right)$$
By using (3.23) and Lemma 3.12, the equations (3.41) can be proved as follows:

\[
\partial \omega_j = -\frac{1}{h_s} \left( d^s \left| \left( e^{\text{ad}_V \Lambda_k} \right)_{<0}, e^{\text{ad}_V \Lambda_j} \right| \right)
\]

(3.40)

Since \( \partial \Omega/\partial t_1 \in \mathcal{H} \cap g^{<0} \), it follows from (3.39), (3.40) and Lemma 3.1 that

\[
W = U(Q^V), \quad \frac{\partial \Omega}{\partial t_1} = H(Q^V).
\]

The lemma is proved.

**Lemma 3.13** The functions \( \omega_j \) and the operator \( \mathcal{L}^V \) defined by (3.28) and (3.36) respectively satisfy the following equations:

\[
\begin{align*}
\frac{\partial \omega_j}{\partial t_k} &= \frac{1}{h_s} \left( d^s \left| \left( e^{\text{ad}_V \Lambda_k} \right)_{<0}, e^{\text{ad}_V \Lambda_j} \right| \right) \\
\frac{\partial \mathcal{L}^V}{\partial t_j} &= \left[ -(e^{\text{ad}_V \Lambda_j})_{\geq 0} + R(Q^V, \Lambda_j), \mathcal{L}^V \right]
\end{align*}
\]

(3.41) (3.42)

with \( j, k \in J_+ \), where \( R(Q^V, \Lambda_j) \) is introduced in Lemma 3.2.

**Proof:** By using (3.23) and Lemma 3.12, the equations (3.41) can be proved as follows:

\[
\begin{align*}
\frac{\partial \omega_j}{\partial t_k} &= -\frac{1}{h_s} \left( d^s \left| \left( e^{\text{ad}_V \Lambda_k} \right)_{<0}, e^{\text{ad}_V \Lambda_j} \right| \right) \\
&= -\frac{1}{h_s} \left( d^s \left| \left( e^{\text{ad}_V \Lambda_k} \right)_{<0}, e^{\text{ad}_V \Lambda_j} - \omega_j \right| \right) \\
&= -\frac{1}{h_s} \left( d^s \left| \left( e^{\text{ad}_V \Lambda_k} \right)_{<0}, e^{\text{ad}_V \Lambda_j} \right| \right) \\
&= -\frac{1}{h_s} \left( d^s \left| \left( e^{\text{ad}_V \Lambda_k} \right)_{<0}, e^{\text{ad}_V \Lambda_j} \right| \right) \\
&= \frac{1}{h_s} \left( d^s \left| \left( e^{\text{ad}_V \Lambda_k} \right)_{\geq 0}, e^{\text{ad}_V \Lambda_j} \right| \right).
\end{align*}
\]

In order to prove the equations (3.42), by using (3.18) we rewrite the operator \( \mathcal{L} \) defined in (3.35) as follows:

\[
\mathcal{L} = e^{\text{ad}_V \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right)} + \omega_1 c,
\]

(3.43)

hence we have

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial t_j} &= \left[ \nabla_{t_j, V} V, e^{\text{ad}_V \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right)} \right] + \frac{\partial \omega_j}{\partial t_j} c \\
&= \left[ e^{\text{ad}_V \Lambda_j} \right]_{<0}, e^{\text{ad}_V \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right)} \right] + \frac{\partial \omega_j}{\partial t_j} c \\
&= \left[ -\left( e^{\text{ad}_V \Lambda_j} \right)_{\geq 0}, e^{\text{ad}_V \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right)} \right] + \frac{\partial \omega_j}{\partial t_j} c \\
&= \left[ -\left( e^{\text{ad}_V \Lambda_j} \right)_{\geq 0}, \mathcal{L} - \omega_j c \right] + \frac{\partial \omega_j}{\partial t_j} c \\
&= \left[ -\left( e^{\text{ad}_V \Lambda_j} \right)_{\geq 0}, \mathcal{L} \right] + \left( -\frac{\partial \omega_j}{\partial t_1} + \frac{\partial \omega_j}{\partial t_1} \right) c.
\end{align*}
\]

(3.44)
\[ \left[ -\left( e^{\text{ad}_V} e^{\text{ad}_\Omega} \Lambda_j \right)_0 \right] \geq 0, \quad (3.45) \]

So it follows from (3.36) that
\[
\frac{\partial \mathcal{L}^V}{\partial t_j} = \left[ \nabla_{t_j} N, \mathcal{L}^V \right] + e^{\text{ad}_N} \left[ -\left( e^{\text{ad}_V} e^{\text{ad}_\Omega} \Lambda_j \right)_0 \right] \geq 0, \quad (3.46)
\]

Note that the function \( \nabla_{t_j} N \) takes value in the nilpotent subalgebra \( \mathcal{N} \), and both sides of (3.46) take value in the subspace \( \mathcal{V} \). Hence, according to the second assertion of Lemma 3.2, we have
\[
\nabla_{t_j} N = R(Q^V, \Lambda_j),
\]
whose components are differential polynomials in \( Q^V \). Therefore, the equation (3.42) holds true. The lemma is proved. \( \square \)

Denote by \( \Omega_{ij}^s \) the right hand side of (3.41), then from the above lemma we know that the functions \( f = \log \tau^s, \omega_j (j \in J_+) \) defined by (3.28), (3.31) and the functions \( u_i, \ldots, u_\ell \) given in (3.32) satisfy the tau cover (3.16) of the Drinfeld-Sokolov hierarchy (3.9).

To finish the proof of the first assertion of Theorem 3.11, we still need to show that the function \( V \) can be represented via \( u_i \) with \( i = 1, 2, \ldots, \ell \) and \( \omega_j \) with \( j \in J_+ \). For this purpose let us first prove the following lemma.

**Lemma 3.14** Given any \( X \in \mathfrak{g}^{<0} \), there exists a unique element \( (M, Y) \) of \( \mathcal{N} \times \mathfrak{g}^{<0} \) such that
\[
e^{\text{ad}_Y} = e^{\text{ad}_M} e^{\text{ad}_X}.
\]
Moreover, both \( M \) and \( Y \) can be represented as polynomials in the components of \( X \).

**Proof:** According to the Baker-Campbell-Hausdorff formula and properties of the adjoint representation, the equation (3.47) can be represented as
\[
Y = M + X + \frac{1}{2} [M, X] + \frac{1}{12} ([M, [M, X]] + [X, [X, M]]) - \frac{1}{24} [X, [M, [M, X]]] + \ldots. \quad (3.48)
\]
Let us represent \( Z = M, X, Y \) in the form \( Z = \sum_{k \leq -1} Z_k \) with \( Z_k \in \mathfrak{g}^k \). By using the fact that \( \mathfrak{g}^{<0} = \mathcal{N} \oplus \mathfrak{g}^{<0} \), we can solve \( M_k \) and \( Y_k \) uniquely in a recursive way from (3.48), and they are clearly polynomials in \( X \). The lemma is proved. \( \square \)

Recall that the functions \( U(Q^V) \) and \( \Omega \) that are defined by (3.37) and (3.38) take values in \( \mathfrak{g}^{<0} \), hence they determine a \( \mathfrak{g}^{<0} \)-valued function \( \tilde{V} \) by the relation
\[
e^{\text{ad}_\psi} = e^{\text{ad}_{U(Q^V)}} e^{\text{ad}_\Lambda},
\]
which implies that
\[
e^{\text{ad}_V} = e^{-\text{ad}_N} e^{\text{ad}_\tilde{V}}.
\]
By using Lemma 3.14 we know that the \( \mathcal{N} \)-valued function \( N \) and \( \mathfrak{g}^{<0} \)-valued function \( V \) must be polynomial in \( \tilde{V} \), so they are polynomials in \( U(Q^V) \) and \( \Omega \). Thus, we arrive at the fact that the components of \( V \) are elements of the ring \( \mathcal{R} \) (recall (3.33)) with zero constant terms, so the first assertion of Theorem 3.11.

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In order to prove the second assertion of the theorem, let us assume that

\[ \{ f = \log \tau, \omega_j, u_i \mid j \in J_+, i = 1, 2, \ldots, \ell \} \]

is a solution of the tau cover (3.16) of the Drinfeld-Sokolov hierarchy, with \( V \) being the fixed complement subspace of \( B \) and \( Q^V = \sum_{i=1}^\ell u_i \eta_i \) taking value in \( V \). For the operator \( \mathcal{L}^V = \partial/\partial x + \Lambda_1 + Q^V \), let \( U(Q^V) \) and \( H(Q^V) \) be the functions determined via Lemma 3.1. Denote \( \Omega = \sum_{j \in J_+} \omega_j \Lambda_{-j} \), then it follows from (3.14) that

\[
H(Q^V) = \sum_{j \in J_+} \frac{(\Lambda_j | H(Q^V))}{h} \Lambda_{-j} = - \sum_{j \in J_+} \frac{\Omega^s_j}{j} \Lambda_{-j} = - \frac{\partial \Omega}{\partial x}. \tag{3.49}
\]

For any \( j \in J_+ \), the above operator \( \mathcal{L}^V \) satisfies (3.9), which can be recast to

\[
\frac{\partial \mathcal{L}^V}{\partial t_j} = \left[ (e^{ad_{u(Q^V)}} \Lambda_j)_{<0} + R(Q^V, \Lambda_j), \mathcal{L}^V \right] - [\Lambda_j, H(Q^V)]. \tag{3.50}
\]

Note that the second term on the right hand side is equal to \( \Omega^s_{ij} c \). On the other hand, by using the dressing formula (3.3) for \( \mathcal{L}^V \) again we have

\[
\frac{\partial \mathcal{L}^V}{\partial t_j} = \left[ \nabla_{t_j, U(Q^V)} U(Q^V), \mathcal{L}^V \right] + e^{ad_{u(Q^V)}} \frac{\partial H(Q^V)}{\partial t_j}. \tag{3.51}
\]

Denote

\[
G_j = e^{-ad_{u(Q^V)}} \left( \nabla_{t_j, U(Q^V)} U(Q^V) \right) - \left( e^{ad_{u(Q^V)}} \Lambda_j \right)_{<0} - R(Q^V, \Lambda_j), \tag{3.52}
\]

then it satisfies, by taking the equations (3.50) and (3.51) together, that

\[
\left[ G_j, \frac{\partial}{\partial x} + \Lambda_1 + H(Q^V) \right] + \frac{\partial H(Q^V)}{\partial t_j} - \Omega^s_{ij} c = 0. \tag{3.53}
\]

With the help of (2.11), by an induction on the principal gradation and the same arguments used in the proof of Lemma 3.4 in [42], we know that \( G_j \) takes values in \( \mathcal{H} \cap \mathfrak{g}_{<0} \) and that

\[
- \frac{\partial G_j}{\partial x} + \frac{\partial H(Q^V)}{\partial t_j} = 0, \quad [G_j, \Lambda_1] - \Omega^s_{ij} c = 0. \tag{3.54}
\]

Since both \( G_j \) and \( \partial \Omega / \partial t_j \) are differential polynomials in \( u_1, u_2, \ldots, u_\ell \) with zero constant terms, then the first equality together with (3.49) leads to

\[
G_j = - \frac{\partial \Omega}{\partial t_j}. \tag{3.55}
\]

Note that the second equality in (3.54) is also satisfied.

According to the Baker-CAMPBELL-Hausdorff formula and Lemma 3.14, there is a unique function \((N, V)\) lying in \( \mathcal{N} \times \mathfrak{g}_{<0} \) such that

\[
e^{ad_V} = e^{-ad_N} e^{ad_{U(Q^V)}} e^{ad_\Omega}. \tag{3.55}
\]
Moreover, both $V$ and $N$ are differential polynomials in the ring $\mathcal{R}$ with zero constant terms. For any $\mathfrak{g}$-valued function $X$, by using the notations $[X, \partial/\partial t_j] = -\partial X/\partial t_j$, we have

$$
\nabla_{t_j} V = \frac{\partial}{\partial t_j} - e^{ad_y} \frac{\partial}{\partial t_j} = \frac{\partial}{\partial t_j} - e^{-ad_N} e^{ad_{U(Q^j)}} \left( \frac{\partial}{\partial t_j} - \frac{\partial G_j}{\partial t_j} \right) = \frac{\partial}{\partial t_j} - e^{-ad_N} \left( \nabla_{t_j} U(Q^j) U(Q^j) + e^{ad_{U(Q^j)} G_j} \right) = \nabla_{t_j} (-N) - e^{-ad_N} \left( -e^{ad_{U(Q^j)} \Lambda_j} \right) < 0 + \nabla_{t_j} (-N) + e^{-ad_N} R(Q^j, \Lambda_j),
$$

where in the third equality we have used (3.52). Observe that $\nabla_{t_j} (-N) + e^{-ad_N} R(Q^j, \Lambda_j)$ lies in $\mathfrak{g}_0$, hence it must vanish. So we arrive at the equation

$$
\nabla_{t_j} V = \left( e^{-ad_N} e^{ad_{U(Q^j)} \Lambda_j} \right) < 0 = \left( e^{ad_y} \Lambda_j \right) < 0.
$$

Therefore, we complete the proof of Theorem 3.11.

### 3.4 Formal power series solutions of the tau cover

Let us consider the Cauchy problem of the system of equations (3.16) with initial values

$$
u^{(k)}_{i} \bigg|_{t=0} = U_{i}^{k}, \quad \omega_{j} \bigg|_{t=0} = W_{j}, \quad 1 \leq i \leq \ell, \quad k \in \mathbb{Z}_{\geq 0}, \quad j \in J_{+},
$$

where $U_{i}^{k}$ and $W_{j}$ are arbitrary constants. To solve this problem, we introduce the following notations: for $m \geq 2$,

$$
\omega_{j_{1}j_{2}...j_{m}} = \frac{\partial_{m} \log \tau^{\mathfrak{g}}}{\partial t_{j_{1}} \partial t_{j_{2}} \cdots \partial t_{j_{m}}}, \quad j_{1}, j_{2}, \ldots, j_{m} \in J_{+},
$$

then their initial values are given by

$$
W_{j_{1}j_{2}...j_{m}} := \omega_{j_{1}j_{2}...j_{m}} \bigg|_{t=0} = \frac{\partial_{m-2} \Omega_{j_{1}j_{2}...j_{m}} (u, u', \ldots)}{\partial t_{j_{1}} \partial t_{j_{2}} \cdots \partial t_{j_{m}}} \bigg|_{u^{(k)}_{i} \rightarrow U_{i}^{k}}.
$$

Thus, we can write down the formal power series solution to the system of equations (3.16) with the above initial data via the tau function given by

$$
\log \tau^{\mathfrak{g}}(t) = \sum_{m \geq 1} \sum_{j_{1}, j_{2}, \ldots, j_{m} \in J_{+}} \frac{W_{j_{1}j_{2}...j_{m}} t_{j_{1}} t_{j_{2}} \cdots t_{j_{m}}}{m!} + \text{const}.
$$

Without loss of generality, the constant term will be omitted below.

The solution (3.59) can be represented alternately as follows. We note that the initial conditions (3.56) are equivalent to the following data:

$$
\mu_{i}(x) := u_{i} \big|_{t_{p}=x \delta_{p_{i}}} \quad \omega_{j} \big|_{t=0} = W_{j}, \quad 1 \leq i \leq \ell, \quad j \in J_{+}.
$$
With the help of these data, the functions \( w_j(x) := \omega_j|_{\mu_p=x\delta p} \) can be solved from the equations:

\[
\begin{aligned}
  w'_j(x) &= \Omega^s_{ij} (u, u') |_{u_i \to \mu_j(x)}, \quad w_j(0) = W_j; \\
  \text{moreover, similar to (3.58), the following functions can be calculated:}
\end{aligned}
\]

\[
w_{j_1j_2\ldots j_m}(x) := \omega_{j_1j_2\ldots j_m}|_{\mu_p=x\delta p} = \frac{\partial^{m-2} \Omega^s_{j_1j_2} (u, u', \ldots)}{\partial t_{j_3} \ldots \partial t_{j_m}} |_{u_i \to \mu_j(x)}, \quad j_1, j_2, \ldots, j_m \in J_+, \ m \geq 2.
\]

Thus, the solution (3.59) can also be represented as follows:

\[
\log \tau^s = \int^{t_1} w_1(x) dx + \sum_{m \geq 1} \sum_{j_1, \ldots, j_m \in J_{>1}} \frac{w_{j_1\ldots j_m}(t_1)}{m!} t_{j_1} \ldots t_{j_m}.
\]

From another point of view, recall that the unknown functions \( u_1, \ldots, u_\ell \) of the system (3.16) can be represented, via Miura-type transformations, by \( \omega'_{m_1} \ldots \omega'_{m_\ell} \) (see the equalities (3.14) and Remark 3.7), the initial date (3.60) can also be replaced by

\[
w_{m_i}(x) := \omega_{m_i}|_{\mu_p=x\delta p}, \quad \omega_j|_{t=0} = W_j, \quad 1 \leq i \leq \ell, \ j \in J_{>m_\ell}.
\]

Therefore, we conclude the following result.

**Proposition 3.15** For the system of equations (3.16) with initial data given by any of (3.56), (3.60) or (3.64), there locally exists a unique formal power series solution (up to a constant term) given by (3.59) or (3.63).

**Remark 3.16** Theorem 3.11 implies that, the initial value \( V(0) := V|_{t=0} \) is determined by (3.56), and it provides an algebraic way to compute the initial values (3.58). More precisely, denote \( A_j(0) = e^{ad V(0)} \Lambda_j \), then by using (3.28) and (3.23) we have

\[
W_{jk} = -\frac{1}{h^s} \left( d^s \mid [A_j(0)_{<0}, A_k(0)] \right).
\]

By using the Leibniz rule, the values \( W_{j_1j_2\ldots j_m} \) can be computed recursively. For instance,

\[
W_{jkl} = -\frac{1}{h^s} \left( d^s \mid [[A_j(0)_{<0}, A_k(0)]_{<0}, A_l(0)] + [A_k(0)_{<0}, [A_j(0)_{<0}, A_l(0)]] \right).
\]

In fact, for such values \( W_{j_1j_2\ldots j_m} \) one can also derive certain generating functions that generalize those given in [5]. This will be considered in another occasion.

### 3.5 Examples

At the end of this section, let us illustrate the system of equations (3.18) and its solutions with some examples.

**Example 3.17** Let \( g \) be of type \( A_1^{(1)} \) and \( s = s^0 = (1, 0) \). In this case the exponents are given by all the odd integers, and the elements \( \Lambda_j \) are chosen as in [8]. Denote

\[
f_{i_1i_2\ldots i_m} = [f_{i_1}, \ldots, [f_{i_{m-1}}, f_{i_m}], \ldots], \quad 0 \leq i_1, i_2, \ldots, i_m \leq \ell, \ m \geq 2.
\]
where \( f_0, f_1 \) are the Chevalley generators. We choose the subspace \( V = \mathbb{C}f_1 \), and represent the function \( Q^V \) in the form \( Q^V = -uf_1 \). Note that the functions \( U(Q^V) \) and \( V \) are calculated in [42], and the function \( V \) can be represented in terms of \( u, \omega_1, \omega_3, \ldots \) as follows:

\[
V = \gamma_1 f_0 + \gamma_2 f_{01} + \gamma_3 f_{001} + \kappa_3 f_{101} + \gamma_4 f_{0101} + \ldots,
\]

where

\[
\gamma_1 = \omega_1, \quad \gamma_2 = \frac{1}{4} (2\omega_1^2 + u), \\
\gamma_3 = -\frac{1}{24} (2\omega_1^3 + 4\omega_3 + 3\omega_1 u + u'), \quad \kappa_3 = -\frac{1}{12} (2\omega_1^3 - 2\omega_3 + 3\omega_1 u + u'), \\
\gamma_4 = \frac{1}{96} (4\omega_1^4 + 8\omega_1 \omega_3 + 12\omega_1^2 u + 8\omega_1 u' + 6u^2 + 3u'').
\]

In particular, the functions \( \omega_1, \omega_3 \) and \( u \) are related by

\[
\frac{\partial \omega_1}{\partial t_1} = \Omega^{s=0}_{11} = \frac{1}{2} u, \quad \frac{\partial \omega_3}{\partial t_1} = \Omega^{s=0}_{13} = \frac{1}{8} (u^2 + u''). \tag{3.66}
\]

Since \( \partial \omega_3 / \partial t_1 = \partial \omega_1 / \partial t_3 \), then one derives from (3.66) the KdV equation

\[
\frac{\partial u}{\partial t_3} = \frac{1}{2} uu' + \frac{1}{4} u'''.
\]

In this case, the Drinfeld-Sokolov hierarchy (3.42) is the KdV hierarchy.

Applying the approach in the previous subsection, we obtain the following tau function of the KdV hierarchy with the initial data \( u|_{t_j>1=0} = t_1 \):

\[
\log \tau^{s=0} = \frac{t_1^3}{12} + W_1 t_1 + \left( \frac{t_1^3}{8} + W_3 \right) t_3 + \left( \frac{5t_1^4}{64} + \frac{5t_1}{32} + W_5 \right) t_5 + \left( \frac{7t_1^5}{128} + \frac{35t_1^2}{128} + W_7 \right) t_7 + \ldots,
\]

where \( W_k \) are constants. In particular, if we take \( W_3 = \delta_3j/16 \), then this tau function corresponds to the well known Witten-Kontsevich tau function of the KdV hierarchy.

**Example 3.18** Let \( g \) be of type \( A_1^{(1)} \) and \( s = \mathbb{I} = (1, 1) \). In this case the nilpotent subalgebra \( \mathcal{N} \) is trivial, and the subspace \( V = B = \mathbb{C}(\alpha_1^\vee - \alpha_0^\vee) \). Let \( Q^V = \frac{1}{2}(\alpha_1^\vee - \alpha_0^\vee) \), then by using the relation

\[
e^{ad_V} = e^{ad_{U(Q^V)}} e^{ad_{\mathbb{I}}}
\]

we have

\[
V = \gamma_1 f_0 + \kappa_1 f_1 + \gamma_2 f_{01} + \gamma_3 f_{001} + \kappa_3 f_{101} + \gamma_4 f_{0101} + \ldots
\]

with

\[
\gamma_1 = \frac{1}{2} (2\omega_1 - v), \quad \kappa_1 = \frac{1}{2} (2\omega_1 + v), \quad \gamma_2 = \frac{1}{4} (2\omega_1 v + v'), \\
\gamma_3 = \frac{1}{48} (2\omega_1 v^2 - 6\omega_1 v' - 4\omega_1^2 v - 8\omega_3 + 4v^3 - 3v'' + vv'),
\]

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\[ \kappa_3 = \frac{1}{48} \left( -2\omega_1 v^2 - 6\omega_1 v' - 4\omega_1^2 v + 8\omega_3 + 4v^3 - 3v'' - vv' \right), \]
\[ \gamma_4 = \frac{1}{96} \left( -8\omega_1 v^3 + 6\omega_1 v'' + 4\omega_1^2 v' + 8\omega_3 v + 3v(3) - 17v^2 v' \right). \]

The functions \( \omega_1, \omega_3 \) and \( v \) satisfy
\[ \frac{\partial \omega_1}{\partial t_1} = \Omega_1^{\omega} = -\frac{1}{2} v^2, \quad \frac{\partial \omega_3}{\partial t_1} = \Omega_1^\omega = \frac{1}{8} (3v^4 + (v')^2 - 2vv''). \] (3.68)

Then the relation \( \partial \omega_3/\partial t_1 = \partial \omega_1/\partial t_3 \) leads to the modified KdV equation
\[ \frac{\partial v}{\partial t_3} = -\frac{3}{2} v'' v' + \frac{1}{4} v'''. \]

The Drinfeld-Sokolov hierarchy in this case is the modified KdV hierarchy, which is related to the KdV hierarchy via the Miura transformation \( u = -v^2 + v' \). It is easy to see that the tau functions \( \tau^4 \) and \( \tau^{s_0} \) of these two integrable hierarchies are related by the formula (see, for instance, Example A.1 in [42])
\[ v = 2 \frac{\partial}{\partial t_1} \log \frac{\tau^{s_0}}{\tau}. \]

Similarly to the above example, one can write down the formal power series expression of the tau function \( \log \tau^4 \) with any given initial data.

**Example 3.19** Let \( g \) be of type \( A_2^{(2)} \) and \( s = s_0 = (1, 0) \). The Coxeter number is \( h = 3 \), the set of exponents is given by \( J = 6\mathbb{Z} \pm 1 \), the constant in (2.15) is \( \nu = \sqrt{2} \), and the generators \( \Lambda_j \) are chosen as in [42]. We take the subspace \( \mathcal{V} = \mathbb{C} f_1 \), and the function \( Q^\mathcal{V} = -\sqrt{2} u f_1 \). Then, the function \( V \) in equation (3.18) reads
\[ V = \gamma_1 f_0 + \gamma_2 f_{01} + \gamma_3 f_{001} + \gamma_4 f_{0001} + \gamma_5 f_{00001} + \kappa_5 f_{10001} \ldots, \]
where
\[ \gamma_1 = \frac{\omega_1}{\sqrt{2}}, \quad \gamma_2 = \frac{1}{6} (3\omega_1^2 + u), \quad \gamma_3 = -\frac{1}{36\sqrt{2}} (3\omega_1 u + 3\omega_1^3 + u'), \]
\[ \gamma_4 = \frac{1}{432} (3\omega_1 u' + 3\omega_1^2 u + u'' + u^2), \]
\[ \gamma_5 = \frac{1}{8640\sqrt{2}} \left( 6\omega_1^5 + 36\omega_5 - 10\omega_1 u'' - 10\omega_1 u^2 - 10\omega_1^2 u' - 3u(3) - 10uu' \right), \]
\[ \kappa_5 = \frac{1}{2160\sqrt{2}} \left( 9\omega_1^5 - 36\omega_5 - 10\omega_1 u'' - 5\omega_1 u^2 - 15\omega_1^2 u' - 2u(3) - 5uu' \right). \]

The functions \( \omega_1, \omega_5 \) and \( u \) satisfy
\[ \frac{\partial \omega_1}{\partial t_1} = \Omega_1^{\omega} = \frac{1}{3} u, \quad \frac{\partial \omega_5}{\partial t_1} = \Omega_1^{\omega} = -\frac{1}{324} \left( 20u^3 + 30uu'' + 3u(4) \right). \] (3.69)

By using \( \partial \omega_5/\partial t_1 = \partial \omega_1/\partial t_5 \), one derives the Sawada-Kotera equation:
\[ \frac{\partial u}{\partial t_5} = -\frac{1}{108} \left( 20u^3 + 30uu'' + 3u(4) \right)'. \]
With the same method as before, the tau function for the initial data $u_{t^j=0} = 3t_1$ is given by

$$
\log \tau^{s_0} = \frac{t_1^3}{6} + t_1 W_1 + \left( W_5 - \frac{5t_1^4}{12} \right) t_5 + \left( -\frac{7t_1^5}{15} - \frac{7t_1^2}{12} + W_7 \right) t_7 
+ \left( \frac{5t_1^5}{2} + \frac{25t_1^2}{12} \right) t_5^2 + \ldots,
$$

(3.70)

where $W_j$ are constants.

4 Virasoro symmetries and Virasoro constraints for Drinfeld-Sokolov hierarchies

We want to consider in this section the Virasoro symmetries of the tau cover (3.16) of the Drinfeld-Sokolov hierarchy, and the corresponding Virasoro constraints to the tau function of the hierarchy.

4.1 Virasoro symmetries

Observe that the commutation relations (2.14) between the generators for the principal Heisenberg subalgebra $H$ are preserved by the scaling transformations: $\Lambda_{\pm j} \mapsto \lambda_{\pm j} \Lambda_{\pm j}$ for arbitrary nonzero constants $\lambda_j$ with $j \in J_+$, so we can adjust these generators so that they also satisfy the following commutation relations (recall (2.28)):

$$
[d^I_k, \Lambda_j] = -\frac{j}{r\hbar} \Lambda_{j+rhk}, \quad k \in \mathbb{Z}, \quad j \in J_+.
$$

(4.1)

Suppose that a function $V$ of $t = \{t_j \mid j \in J_+\}$ taking value in $g_<0$ solves the system of equations (3.18). Let $\Xi = \sum_{j \in J_+} t_j \Lambda_j$, and introduce

$$
B_k = e^{ad_V} e^{-ad_{\Xi}} d^I_k - d^S_k, \quad k \in \mathbb{Z}.
$$

(4.2)

Here to ensure that the right hand side of (4.2) makes sense, one may assume all but only finitely many $t_j$ vanish. From (2.36) and (2.29) it follows that $B_k$ takes value in $g$ and that they satisfy the commutation relations

$$
[B_k + d^S_k, B_l + d^S_l] = (k - l)(B_{k+l} - d^S_{k+l}), \quad k, l \in \mathbb{Z}.
$$

(4.3)

Similar to the system of equations (3.18), we introduce the following evolutionary differential equations for $V$:

$$
\nabla_{\beta_k, V} V = - (B_k)_{<0},
$$

(4.4)

where

(I) $k \geq -1$ whenever $r\hbar^S = 1$, namely, $g$ is of untwisted type $X^{(1)}_\ell$ and $s$ is equivalent to $s^0 = (1, 0, 0, \ldots, 0)$ via a diagram automorphism of $g$;

(II) $k \geq 0$ whenever $r\hbar^S > 1$, namely, all cases except those in class (I). In particular, it includes all cases that correspond to twisted affine Kac-Moody algebras.

The reason that we take such values of the indices $k$ will be explained in the proof of Lemma 4.2 (cf. [23] when $g$ is untwisted).

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Lemma 4.1 Any solution $V$ of the evolutionary equations (4.4) satisfies the following equations:

$$\frac{\partial B_k}{\partial t_j} = \left[ -(B_k)_{<0}, B_l + d_k^8 \right], \quad j \in J_+,$$

where the range of $k$ is specified in the above cases (I) or (II).

$$\frac{\partial}{\partial \beta_k} \left( e^{adv} \Lambda_j \right) = \left[ -(B_k)_{<0}, e^{adv} \Lambda_j \right], \quad j \in J_+,$$

Proof: By using (3.18) we check (4.7) as follows:

$$\frac{\partial B_k}{\partial t_j} = \left[ \nabla_{t_j}, V, B_k + d_k^8 \right] + e^{adv} \left[ \nabla_{t_j}, -\bar{\Xi}(\bar{\Xi}), e^{-adv} d_k^1 \right]$$

$$= \left[ \left( e^{adv} \Lambda_j \right)_{<0}, B_k + d_k^8 \right] + e^{adv} e^{-adv} \left[ -\Lambda_j, d_k^1 \right]$$

$$= \left[ \left( e^{adv} \Lambda_j \right)_{<0}, B_k + d_k^8 \right] - \left[ e^{adv} \Lambda_j, B_k + d_k^8 \right]$$

Let us proceed to prove the commutativity (4.8) of the flows. For the same reason as in proving (3.19), it suffices to verify that

$$\left[ \frac{\partial}{\partial t_j}, \frac{\partial}{\partial \beta_k} \right] \left( e^{adv} \Lambda_i \right) = 0, \quad \forall i \in J_+.$$

Let us denote $A_j = e^{adv} \Lambda_j$ as before, then the left hand side of the above equation equals

$$\left[ -(A_j)_{\geq 0}, -B_k - d_k^8 \right]_{<0}, A_i] + [-(B_k)_{<0}, [(A_j)_{<0}, A_i]]$$

$$= \left[ -(A_j)_{\geq 0}, A_i \right] + [-(B_k)_{<0}, -(A_j)_{<0}, A_i]$$

$$= \left[ (B_k)_{<0}, (A_j)_{\geq 0} \right]_{<0}, A_i] + [-(B_k)_{<0}, A_i]$$

$$= \left[ (B_k)_{<0}, (A_j)_{<0} \right]_{<0}, A_i] - \left[ (A_j)_{<0}, (B_k)_{<0} \right]_{<0}, A_i]$$

$$= 0.$$ (4.9)

Here in the second equality we have used the fact that $[(A_j)_{\geq 0}, d_k^8]_{<0} = 0$ whenever $k$ takes values specified in Cases (I) and (II) as given above. The lemma is proved.

It follows from the above lemma that the flows (4.4) are symmetries of the hierarchy of evolutionary differential equations (3.18).

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Proposition 4.3 The symmetries (3.4) of the system of differential equations (3.18) yield the following symmetries of the tau cover (3.16) of the Drinfeld-Sokolov hierarchy:

\[
\frac{\partial \log \tau^s}{\partial \beta_k} = \frac{1}{\hbar^s} (d^s \mid B_k),
\]
(4.10)
\[
\frac{\partial \omega_j}{\partial \beta_k} = \frac{1}{\hbar^s} \left( e^s \mid \left( (B_k)^{>0}, e^{adv} \Lambda_j \right) \right),
\]
(4.11)
\[
\frac{\partial \mathcal{L}^V}{\partial \beta_k} = \left[ \nabla_{\beta_k} N \cdot - (e^{adv} B_k)^{>0}, \mathcal{L}^V \right] + \frac{1}{\hbar^s} \left( d^s \mid \left( (B_k)^{>0}, e^{adv} \Lambda_1 \right) \right) c
\]
(4.12)

with \( k \) being given as in Cases (I) and (II). Here \( N \) is the \( N \)-valued function determined by (3.36), whose components are differential polynomials in \( V \).

Proof: The equations (4.11) follow directly from (4.6) and (3.28). Secondly, by using (3.36), (3.37) and (3.38) we obtain

\[
\frac{\partial \mathcal{L}^V}{\partial \beta_k} = \left[ \nabla_{\beta_k} N \cdot - (e^{adv} B_k)^{>0}, \mathcal{L}^V \right] + \frac{1}{\hbar^s} \left( d^s \mid \left( (B_k)^{>0}, e^{adv} \Lambda_1 \right) \right) c
\]

which, together with (4.11), yields (4.12). Finally, the equations (4.10) define symmetries since

\[
\frac{\partial}{\partial t_j} \frac{\partial \log \tau^s}{\partial \beta_k} = \frac{1}{\hbar^s} \left( d^s \mid \left( (B_k)^{>0}, B_k \right) \right) = -\frac{1}{\hbar^s} \left( d^s \mid \left( e^{adv} \Lambda_j, (B_k)^{>0} \right) \right) = \frac{\partial \omega_j}{\partial \beta_k}.
\]

Finally, The proposition is proved. \( \square \)

Theorem 4.4 The symmetries \( \partial/\partial \beta_k \) of the tau cover of the Drinfeld-Sokolov hierarchy satisfy the following Virasoro commutation relations:

\[
\left[ \frac{\partial}{\partial t_i}, \frac{\partial}{\partial \beta_k} \right] \log \tau^s = (k - l) \frac{\partial \log \tau^s}{\partial \beta_{k+l}}.
\]
(4.13)

Moreover, these symmetries can be represented in the form

\[
\frac{\partial \log \tau^s}{\partial \beta_k} = \begin{cases} 
\frac{1}{\hbar^s} \sum_{j \in J_+} (j + r \hbar) t_j + \frac{1}{2 \hbar} \sum_{j \in J_+} t_i t_j, & k = -1 \text{ in Case (I)}; \\
\frac{1}{\hbar^s} \sum_{j \in J_+} j t_j \omega_j + C^s, & k = 0; \\
\frac{1}{\hbar^s} \left( d^s \mid e^{adv(Q^V)} d_k^s - d_k^s \right) + \frac{1}{2 \hbar} \sum_{j \in J_+} \omega_i \omega_j + \frac{1}{\hbar} \sum_{j \in J_+} t_j \omega_{j+k}, & k = 1, 2, 3, \ldots,
\end{cases}
\]
(4.14)

where \( C^s \) is a constant given by (recall (2.32))

\[
C^s = \left( d^s \mid \rho^s - \rho^s \right) - \frac{r \hbar^s}{2} \left( \left( \rho^s \mid \rho^s \right) - \left( \rho^s \mid \rho^s \right) \right).
\]
(4.15)
Proof: By using (4.5) and (4.10) we obtain
\[
\left[ \frac{\partial}{\partial \beta_l} , \frac{\partial}{\partial \beta_k} \right] \log \tau^s = \frac{1}{h^s} \left( d^s \mid -(B_t)_{<0} , B_k + d_k^s \right) - \frac{1}{h^s} \left( d^s \mid - (B_k)_{<0} , B_l + d_l^s \right) = \frac{1}{h^s} \left( d^s \mid [(B_k)_{>0} , (B_l)_{<0}] + [(B_k)_{<0} , (B_l)_{>0}] \right) = \frac{1}{h^s} \left( d^s \mid [B_k , B_l] \right) \]
\[
= \frac{1}{h^s} \left( d^s \mid [B_k + d_k^s , B_l + d_l^s] - (k-l)d_{k+l}^s \right)
= \frac{1}{h^s} \left( d^s \mid (k-l)B_{k+l} \right) = (k-l) \frac{\partial \log \tau^s}{\partial \beta_{k+l}} .
\]

Here in the second and the fourth equalities we have used (2.30), and the fifth equality is due to (4.3). So the first assertion of the theorem holds true. By using (3.37) and the fact that \([d^s , N] = 0\) we have, with the indices \(i,j\) lying in \(J_+\),
\[
(d^s \mid B_k) \\
= (d^s \mid e^{-ad_N} e^{ad_{U(QV)}} e^{-ad_{\Omega}} e^{-ad_{\sum_{i \in J_+} t_i \Lambda_i}} d_k^s - d_k^s) \\
= (d^s \mid e^{ad_N} d^s \mid e^{ad_{U(QV)}} e^{ad_{\sum_{j \in J_+} -1 \omega_j \Lambda_j - j}} \left( d_k^s - \frac{\partial^s}{\partial \beta_{k+l}} \Lambda_{i+rhk} + \frac{\delta_{k,-1}}{2rh} \sum_{0<i<rh} i(rh - i)c \right) - d_k^s) \\
= \left( d^s \mid e^{ad_{U(QV)}} \left( d_k^s - \frac{\partial^s}{\partial \beta_{k+l}} \Lambda_{i+rhk} + \frac{1}{2rh} \sum_{0<j<rh} \omega_j \omega_{rhk-j}c - \sum_{i \in J_+} \frac{it_i}{rh} \Lambda_{i+rhk} \right) \\
+ \frac{\partial^s}{\partial \beta_{k+l}} \left( \omega_{i+rhk}c - \delta_{k,-1} \sum_{0<i<rh} i(rh - i)c \right) - d_k^s \right) \\
= \left( d^s \mid e^{ad_{U(QV)}} \frac{\partial^s}{\partial \beta_{k+l}} \Lambda_{i+rhk} + \frac{\delta_{k,-1}}{2rh} \sum_{0<i<rh} i(rh - i)c \right) \\
+ \frac{\partial^s}{\partial \beta_{k+l}} \sum_{i \in J_+} t_i \omega_{i+rhk} + \frac{\delta_{k,-1} \partial^s}{2rh} \sum_{0<i<rh} i(rh - i)c .
\]
\[(4.16)\]

Here in the second and the third equalities we have used the normalization equations (4.1) and (2.14), and in the last equality we have used the condition (3.3). Note that the first term on the right hand side of (4.16) vanishes when \(k = -1\) (only in Case (I)), and it is a constant when \(k = 0\), that is,
\[
C^n = \left( d^s \mid d_0^s - d_0^s \right) = \left( d^s \mid \rho^s - \rho^s \right) - \frac{r\hbar^s}{2} (\left( \rho^s \mid \rho^s \right) - \left( \rho^s \mid \rho^s \right))
\]
\[(4.17)\]
due to (2.36). Thus we have proved the theorem. \[\Box\]

Remark 4.5 From (3.14) and Remark 3.7 it follows that \(Q^V\) can be represented via second-order derivatives of \(\log \tau^s\) with respect to the time variables. Hence the right hand side of (4.14) can always be represented in terms of the functions \(\omega_j = \partial \log \tau^s/\partial t_j\) together with the time variables.
Example 4.6 When $s$ is the principal or the homogeneous gradation, we have respectively

$$C^I = 0, \quad C^{s^0} = \frac{rk_0}{2}(\rho^I | \rho^I) = \frac{k_0}{2rh^2}(1,1,\ldots,1)A^{-1}D(1,1,\ldots,1)^T$$

with $D = \text{diag}(k_1/k_1^\gamma, k_2/k_2^\gamma, \ldots, k_\ell/k_\ell^\gamma)$.

Due to the commutation relation (4.13), we have the following definition.

Definition 4.7 The flows defined by (4.10)–(4.12) are called the Virasoro symmetries of the tau cover of the Drinfeld-Sokolov hierarchy (3.9) (the words “on the tau cover” are usually omitted) associated to $(g,s,\mathbb{1})$.

Remark 4.8 When the affine Kac-Moody algebra $g$ is of untwisted type, the Virasoro symmetries $\partial/\partial \beta$ were constructed by Hollowood, Miramontes and Guillen [23] via a Lie group element $\Theta$ that is given in (3.20). In particular, if $g$ is of simply-lace type (type $A_\ell^{(1)}$, $D_\ell^{(1)}$ or $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$) and $s = s^0$, then the Virasoro symmetries can be represented via certain linear Virasoro operators acting on the tau function [23, 42]. More precisely, there are a series of Virasoro operators $L_k$ independent of $\tau_{s^0}$ such that the Virasoro symmetries can be represented by the action of these operators on the tau function:

$$\frac{\partial \tau_{s^0}}{\partial \beta_k} = L_k(\tau_{s^0}), \quad k = -1, 0, 1, 2, \ldots$$

It is known that the Virasoro symmetries for the Drinfeld-Sokolov hierarchies associated to the twisted affine Kac-Moody algebras with $s = s^0$ can also be represented as the actions of linear operators on the tau function; however, for the Drinfeld-Sokolov hierarchies associated to the affine Kac-Moody algebras of BCFG type, we do not have such property of linearization of the Virasoro symmetries (see [42] and references therein).

4.2 Virasoro constraints

Let us consider solutions of the Drinfeld-Sokolov hierarchy associated to $(g,s,\mathbb{1})$ that satisfy either of the following Virasoro constraints:

$$\sum_{p \in J_+} \left( \frac{p+h}{h} t_{p+h} - a_p \right) \frac{\partial \log \tau^s}{\partial t_p} + \frac{1}{2h} \sum_{i,j \in J_+ | i+j = h} ij t_i t_j = 0 \quad \text{for Case (I)}, \quad (4.20)$$

$$\sum_{p \in J_+} \left( \frac{p}{rh} t_p - b_p \right) \frac{\partial \log \tau^s}{\partial t_p} + C^s = 0 \quad \text{for Cases (I) and (II)}, \quad (4.21)$$

where $a_p$ and $b_p$ are constants which vanish except for finitely many of the exponents $p \in J_+$. We call equation (4.20) the (generalized) string equation, and equation (4.21) the similarity equation for it is related to the so-called similarity reductions of the Drinfeld-Sokolov hierarchy.

Theorem 4.9 The equations (4.20) and (4.21) respectively, lead to the following Virasoro constraints to the tau function of the Drinfeld-Sokolov hierarchy (3.9):

$$S_k(\log \tau^s) - \sum_{p \in J_+} a_p \frac{\partial \log \tau^s}{\partial t_{p+h(k+1)}} = 0, \quad k = -1, 0, 1, 2, \ldots \text{ for Case (I)}, \quad (4.22)$$
\[ S_k(\log \tau^s) - \sum_{p \in J_+} b_p \frac{\partial \log \tau^s}{\partial t_{p+rhk}} = 0, \quad k = 0, 1, 2, \ldots \text{ for Cases (I) and (II)}, \] (4.23)

where \( S_k(\log \tau^s) \) are the right hand side of (4.14) (recall Remark 4.5).

**Proof:** The constraints (4.22) have been proven in [42] for \( s = s^0 \) and \( a_p = \delta_{p1} \), and in general cases the proof is almost the same, so we omit it here. Let us check the validity of (4.23). Note that the similarity equation (4.21) is just the equation

\[
\frac{\partial \log \tau^s}{\partial \beta_0} = \sum_{p \in J_+} b_p \frac{\partial \log \tau^s}{\partial t_p}. \tag{4.24}
\]

Since \( \partial / \partial \beta_0 \) is a symmetry of the hierarchy (3.18), it follows that

\[
\nabla_{\beta_0} V - \sum_{p \in J_+} b_p \nabla_{t_p} V = 0.
\]

So by using (4.24) and (3.18), we have

\[
- \left( e^{adv} e^{-adv} d_k^{\beta_0} - d_0^{\beta_0} + \sum_{p \in J_+} b_p e^{adv} \Lambda_p \right)_{<0} = 0. \tag{4.25}
\]

To simplify the notations in this proof, let us identify \( g \) with its realization (2.22). By using (2.27), (2.36) and (4.1) we have

\[
d_k^s = z^{rhk} d_{k-1}^s, \quad d_k^\xi = z^{rhk} d_{k-1}^\xi, \quad \Lambda_{p+rh} = z^{rh} \Lambda_p.
\]

Then the subscript “\(<0\)” in the equality (4.25) can be understood as to take the negative part of the Laurent series in \( z \). For any \( k \geq 1 \), by multiplying \( z^{rhk} \) to the equality (4.25) we obtain

\[
- \left( e^{adv} e^{-adv} d_k^\beta - d_0^\beta + \sum_{p \in J_+} b_p e^{adv} \Lambda_{p+rhk} \right)_{\leq0} = 0. \tag{4.26}
\]

Thus, from (4.10) and (3.28), it follows that

\[
- \frac{\partial \log \tau^s}{\partial \beta_k} + \sum_{p \in J_+} b_p \frac{\partial \log \tau^s}{\partial t_{p+rhk}} = 0, \quad k = 1, 2, 3, \ldots.
\]

Therefore the theorem is proved. \( \Box \)

**Definition 4.10** We call the series of equations (4.22) Virasoro constraints of the first type, and call the series of equations (4.23) Virasoro constraints of the second type.

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4.3 Solutions of Witten-Kontsevich and of Brezin-Gross-Witten types

In this subsection, we illustrate how to obtain solutions of the Drinfeld-Sokolov hierarchy satisfying the Virasoro constraints (4.20) or (4.21).

We first consider the Virasoro constraint (4.20) with $s = s_0$ and $a_p = \delta_p 1$, which is just the string equation in the literature. In this case, by taking the derivatives of the string equation with respect to $t_m$ and $t_1$, we obtain (recall (2.13) and note $m_\ell = h - 1$ in the current case):

$$\frac{\partial^k \omega_m}{\partial t_1^k} \Bigg|_{t=0} = \delta_{i,\ell} \delta_{k,2} \frac{h-1}{h}, \quad 1 \leq i \leq \ell, \quad k \geq 1.$$ (4.27)

As mentioned in Remark 3.7, the functions $u_1, \ldots, u_\ell$ in the system (3.16) can be represent by $\omega'_m$, via a Miura-type transformation, hence the initial values $U^{(k)}_i = u^{(k)}_i \Big|_{t=0}$ are determined by (4.27). Furthermore, for any $j \in J_+$ we take the derivative of the string equation with respect to $t_{j+h}$ and then obtain

$$W_j := \omega_j \big|_{t=0} = \frac{h}{j + h} \Omega^s_{1,j+h} \Big|_{U^{(k)}_i \rightarrow U^{(k)}_i}.$$ (4.29)

According to Proposition 3.15 we know that $\log \tau^{s_0}$ is determined up to a constant term. Such a tau function, which fulfills the Virasoro constraints of the first type (with $s = s_0$ and $a_p = \delta_p 1$), is called the topological solution of the corresponding Drinfeld-Sokolov hierarchy. In particular, the topological solution for $A_1^{(1)}$ is the well-known Witten-Kontsevich tau function given in (3.67). We remark that another algebraic procedure was proposed in [5] to compute explicitly the topological solution via Hankel determinants.

In contrast to the string equation (4.20), the similarity equation (4.21) is weaker. Let us proceed to consider solutions of the Drinfeld-Sokolov hierarchy satisfying this constraint.

**Theorem 4.11** Suppose that to solutions of the Drinfeld-Sokolov hierarchy it is imposed the Virasoro constraint (4.21) with $b_p = \delta_p 1$, namely

$$\sum_{p \in J_+} \left( \frac{p}{r h} t_p - \delta_p 1 \right) \frac{\partial \log \tau^s}{\partial t_p} + C^s = 0.$$ (4.28)

Then its solution $\log \tau^s$ is determined by the following $\ell - 1$ parameters

$$W_{m_i} = \omega_{m_i} \big|_{t=0}, \quad i = 2, 3, \ldots, \ell.$$ (4.29)

**Proof:** Following Subsection 3.4 let us consider the initial data $w_j(x) = \omega_j \big|_{t_p = x \delta_p 1}$ with $j \in J_+$ for some solution of the system (3.16) satisfying the constraint (4.28). By taking $t_p = x \delta_p 1$ in (4.28), we have

$$w_1(x) = C^s \frac{r h}{r h - x}.$$ (4.30)

For $p \in J_{>1}$, by taking the derivative of (4.28) with respect to $t_j$, we obtain

$$\left( \left( \frac{t_1}{r h} - 1 \right) \Omega^s_{1,j} + \frac{j}{r h} \omega_j \right) \bigg|_{t_p = x \delta_p 1} = 0,$$ (4.31)
In particular, it follows from (4.29) that
\[
w_j(x) = W_j \left( \frac{r h}{r h - x} \right)^j, \quad j = m_2, m_3, \ldots, m_t. \tag{4.32}
\]

Then the functions \( \mu_i(x) = u_{i|t_p=x_{p}^1} \) are determined by (4.30) and (4.32) due to Remark 3.7. By using (4.31) again and the fact that \( \Omega_{j}^{s} \) are differential polynomials in \( u = (u_1, \ldots, u_t) \), we have
\[
w_j(x) = \left. \frac{r h - x}{j} \Omega_{ij}^{s} \right|_{u \mapsto \mu_i(x)}, \quad j \in J > m_t. \tag{4.33}
\]

Thus the conclusion follows from Proposition 3.15. \( \square \)

Example 4.12 Let us consider the solution of the Drinfeld-Sokolov hierarchy associated to (\( g = A_1^{(1)} \), \( s^0 \), \( 1 \)) of the form given in the above theorem. Note that in this case \( C^{s^0} = 1/16 \) (recall (4.18)), then by letting \( t_p = \delta_{p1} \) in (4.28), and with the help of the notations in Example 3.17, we have
\[
w_1(x) = \frac{1}{8(2 - x)}, \quad \mu_1(x) = 2w'_1(x) = \frac{1}{4(2 - x)^2}.
\]

According to Proposition 3.15, we can find the following solution
\[
\log \tau^{s^0} = - \frac{1}{8} \log \left( 1 - \frac{t_1}{2} \right) + \frac{9t_3}{128(2 - t_1)^3} + \frac{225t_5}{1024(2 - t_1)^5} + \frac{55125t_7}{32768(2 - t_1)^7} + \ldots
\]
\[
= \frac{t_1}{16} + \frac{t_1^2}{64} + \frac{t_1^3}{192} + \frac{9t_3}{1024} + \frac{27t_3t_1}{2048} + \frac{27t_3t_1^2}{4096} + \frac{45t_3t_1^3}{65536} + \frac{55125t_7}{17014118346046923173168730371585408} \ldots
\]
\[
= \frac{1}{36} t_1 + \frac{t_1^2}{432} + \frac{t_1^3}{3888} - \frac{91}{5038848} t_5 - \frac{2821 t_7}{30233088} - \frac{4551 t_7^2}{60466176} - \frac{55125 t_7^3}{1088391168} + \ldots
\]
\[
= \frac{1088391168}{104485552128} + \frac{54145 t_5^2}{313456656384} + \frac{29779751^2 t_5^2}{3761479876608} + \frac{29779751^3 t_5^2}{5642219814912} + \ldots
\tag{4.34}
\]

This tau function is the so-called Brezin-Gross-Witten tau function of the KdV hierarchy [4, 21], and it gives a generating function for certain intersection numbers on the moduli space of stable curves [32].

Example 4.13 Let \( g \) be of type \( A_2^{(2)} \) and \( s = s^0 \). In this case, we have \( C^{s^0} = 1/36 \) and \( \ell = 1 \). The unique tau function determined by (4.28) is given by
\[
\log \tau^{s^0} = - \frac{1}{6} \log \left( 1 - \frac{1}{6} t_1 \right) - \frac{91 t_5}{648(6 - t_1)^3} - \frac{2821 t_7}{3888(6 - t_1)^7} + \frac{54145 t_5^2}{1728(6 - t_1)^{11}} + \ldots
\]
\[
= \frac{1}{36} t_1 + \frac{t_1^2}{432} + \frac{t_1^3}{3888} - \frac{91}{5038848} t_5 - \frac{2821 t_7}{30233088} - \frac{4551 t_7^2}{60466176} - \frac{55125 t_7^3}{1088391168} + \ldots
\]
\[
= \frac{1088391168}{104485552128} + \frac{54145 t_5^2}{313456656384} + \frac{29779751^2 t_5^2}{3761479876608} + \frac{29779751^3 t_5^2}{5642219814912} + \ldots
\tag{4.35}
\]
Remark 4.14 Suppose that the flows (4.10) are replaced by
\[
\frac{\partial \log \tau^s}{\partial \beta_k} = \frac{1}{h^s} (d^s \mid B_k) + \delta_0 C \tag{4.36}
\]
with an arbitrary constant C, then together with (4.11)–(4.12) they still give a series of symmetries on the tau cover (3.16) of the Drinfeld-Sokolov hierarchy. It is easy to see that the constant C does not change the communication relations (4.13) for k,l \geq 0, and that the similarity equation (4.21) also induces the Virasoro constraints of the form (4.23). Under this setting, the conclusion of Theorem 4.11 is modified to the following: the solution \( \log \tau^s \) of the Drinfeld-Sokolov hierarchy (3.9) satisfying the similarity equation (4.21) with (4.36) is characterized by \( \ell \) parameters
\[
W_{m_i} := \omega_{m_i} \mid_{t=0}, \quad i = 1, 2, \ldots, \ell.
\]
In particular, one sees \( W_1 = C^s + C \). As an example, such a tau function of the KdV hierarchy that depends on one parameter was derived by Bertola and Ruzza [2] (cf. [9]), and they showed that this tau function satisfies the Virasoro constraints (4.23).

5 Similarity reductions of Drinfeld-Sokolov hierarchies

Recall in Theorem 4.11 that the similarity equation (4.21) with parameters \( b_p = \delta_{p1} \) leads to a system of linear ODEs (4.31), which can be solved to obtain the initial conditions for the solutions of the Drinfeld-Sokolov hierarchy. In what follows, we will choose \( b_p \) in other ways, say \( b_p = \delta_{pj} \) \((j \in J_{>1})\), then the above-mentioned ODE are nonlinear and of the Painlevé type. We show in this section that there exist affine Weyl group actions on the solution spaces of these Painlevé type ODEs. In the particular cases when the affine Kac-Moody algebra \( \mathfrak{g} \) is of type \( A_{\ell}^{(1)} \), \( C_{\ell}^{(1)} \) and \( D_{2n+2}^{(1)} \), such kind of affine Weyl group actions are given in [19, 17, 31, 34].

5.1 Similarity reductions

Let us study solutions of the Drinfeld-Sokolov hierarchies that are constrained by the similarity equations.

Theorem 5.1 Given a solution of the tau cover of the Drinfeld-Sokolov hierarchy associated to \((\mathfrak{g}, s, 1)\) that satisfies the similarity equation (4.21), then the following Lax equation holds true:
\[
\left[ \frac{\partial}{\partial x} + L, rh^s d_0^s + M \right] = 0, \tag{5.1}
\]
where
\[
L = (\Lambda_1 + Q^V) \mid_{t_p=x\delta_{p1}}, \tag{5.2}
\]
\[
M = rh^s \left( \sum_{p \in J_{>1}} \left( b_p - \frac{\delta_{p1}}{rh} \right) \left( e^{\text{ad}_{U(Q^V)} \Lambda_p} \right) + e^{\text{ad}_N (\rho^s - \rho^s)} \right) \mid_{t_p=x\delta_{p1}} \tag{5.3}
\]
with \( Q^V \) and \( N \) given in (3.7) and (3.8).
Proof: Recall that the similarity equation (4.21) leads to (4.25), from which it follows that

\[
\begin{align*}
&\left[e^{ad^v} \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right), \left( e^{ad^v} e^{-ad^z} q^1_0 - d^s_0 + \sum_{p \in J_+} b_p e^{ad^v} \Lambda_p \right) \right] \\
&= \left[e^{ad^v} \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right), e^{ad^v} e^{-ad^z} q^1_0 + \sum_{p \in J_+} b_p e^{ad^v} \Lambda_p \right] \\
&= e^{ad^v} e^{-ad^z} \left[ e^{ad^v} \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right), q^1_0 + \sum_{p \in J_+} b_p \Lambda_p \right] \\
&= e^{ad^v} e^{-ad^z} \left[ \frac{\partial}{\partial t_1}, q^1_0 + \sum_{p \in J_+} b_p \Lambda_p \right] = 0.
\end{align*}
\]

With the help of (2.36), (3.43) and (4.1), the above equation can be recast to

\[
\left[ \frac{\partial}{\partial t_1} + \Lambda_1 + Q, \sum_{p \in J_+} \left( b_p - \frac{p}{rh} t_p \right) \left( e^{ad^v} \Lambda_p \right) \right] \geq 0 + \rho^s - \rho^s + d^s_0 = 0. \tag{5.4}
\]

By using (4.21) again, we have

\[
\sum_{p \in J_+} \left( b_p - \frac{p}{rh} t_p \right) \left( e^{ad^v} \Lambda_p \right) \geq 0 = \sum_{p \in J_+} \left( b_p - \frac{p}{rh} t_p \right) \left( \left( e^{ad^v} \Lambda_p \right) \geq 0 + \omega_p \cdot e \right) - C^s c
\]

\[
= \sum_{p \in J_+} \left( b_p - \frac{p}{rh} t_p \right) \left( e^{ad^v} e^{-ad^z} \Lambda_p \right) \geq 0 - C^s c. \tag{5.5}
\]

Let $e^{ad^N}$ act on (5.4) (recall (3.36) and (3.37)), and take $t_p = x \delta_{p1}$, then we arrive at the Lax equation (5.1). The theorem is proved. \hfill \Box

Suppose that it is given a solution $(L, M)$ of the Lax equation (5.1), then we can recover the solution of the Drinfeld-Sokolov hierarchy together with the similarity equation (4.21). In fact, from $L$ we have the initial data $\mu_i(x) = u_i|_{t_p = x \delta_{p1}}$ with $i = 1, 2, \ldots, \ell$, then we can solve $w_j(x) = \omega_j |_{t_p = x \delta_{p1}}$ for all $j \in J_+$ from the following equations:

\[
\frac{1}{rh} x w_1(x) - \sum_{p \in J_+} b_p w_p(x) + C^s = 0;
\]

\[
\frac{j}{rh} w_j(x) - \sum_{p \in J_+} b_p \Omega^s_p |_{t_m = x \delta_{m1}} = 0, \quad j \in J_{>1}.
\]

Here the two equations are derived from the similarity equation (4.21) by taking $t_m = x \delta_{m1}$ and by taking the derivative with respect to $t_j$, respectively. So the solution $\log \tau^s$ of the Drinfeld-Sokolov hierarchy is determined by using Proposition 3.15.

The procedure in proving Theorem 5.1 is called the similarity reduction of the Drinfeld-Sokolov hierarchy, and it leads to a system of ordinary differential equations (5.1) of $u_i|_{t_p = x \delta_{p1}}$ and $\omega_j|_{t_p = x \delta_{p1}}$ (note that the function $N$ in (5.3) may depend on $\omega_j$). From another point of view, if we take
a matrix realization of $G$ as in [8, 25] and identify $g$ with its realization (2.22), then the Lax equation (5.1) is just the compatibility condition of the following linear system of an unknown vector function $Ψ = Ψ(x; z)$:

$$z∂_zΨ = MΨ, \quad −∂_xΨ = LΨ. \quad (5.6)$$

**Remark 5.2** In [29] we have proved a general $Γ$-reduction theorem for the Drinfeld-Sokolov hierarchies. More exactly, suppose that $(g, s, 1)$ is a triple with $g$ possessing a diagram automorphism $σ$ given in Tables 1–3 of [29] and $s$ consistent with $σ$. We can choose a basis $Λ_j (j \in J)$ of the principal Heisenberg subalgebra $H$ to be eigenvectors of $σ$ with eigenvalues $ζ_j$. The $Γ$-reduction theorem shows that the diagram automorphism $σ$ induces an action on the flows $∂/∂t_j (j \in J_+)$ of the Drinfeld-Sokolov hierarchy, and the flow $∂/∂t_j$ is invariant under the action of $σ$ if and only if $ζ_j = 1$. Note that the folded Dynkin diagram of $g$ with respect to $σ$ corresponds to another affine Kac-Moody algebra, say, $\bar{g}$, on which there are gradations $\bar{s} ≤ 1$ induced by the gradation $s ≤ 1$ of $g$ respectively. From the reduction procedure in [29], we conclude:

- If $b_p = 0$ for any $p \in J_+$ with $ζ_p \neq 1$, then $σ$ induces an action on the Lax equation (5.1);
- If $b_p = 0$ unless $p$ is a positive exponent of $\bar{g}$, then any $σ$-invariant solution of the Lax equation (5.1) also solves the corresponding Lax equation for the Drinfeld-Sokolov hierarchy associated to $(\bar{g}, \bar{s}, 1)$.

### 5.2 ODEs of Painlevé type

Let us give some examples of ODEs that are induced by the similarity equation (4.21) with $b_p = δ_{pk}$ for some $k \in J_{>1}$.

**Example 5.3** Let $g$ be the affine Kac-Moody algebra of type $A_1^{(1)}$. We consider the similarity equation (4.21) with $b_p = δ_{p3}$, that is,

$$\sum_{p \in \mathbb{Z}_{\text{odd}}} \left( \frac{p}{2} t_p - δ_{p3} \right) \frac{1}{∂t_p} \log τ^s + C^s = 0. \quad (5.7)$$

By taking its second-order derivative with respect to $t_1$ we arrive at the equation

$$\left( -\frac{∂Ω_{13}^s}{∂t_1} + \frac{1}{2} t_1 \frac{∂Ω_{11}^s}{∂t_1} + Ω_{11}^s \right)_{t_p=xδ_{p1}} = 0. \quad (5.8)$$

We consider the cases when $s = 1$ and $s = s^0$ and follow the notations of Example 3.18 and Example 3.17 for these cases respectively. In order to write down explicitly the ODEs from the similarity reductions, let us denote

$$λ(x) := v|_{t_p=xδ_{p1}}, \quad μ(x) := u|_{t_p=xδ_{p1}}, \quad (5.9)$$

(i) When $s = 1$, by using (3.68) we can represent the equation (5.8) in the form

$$-\left( \frac{3λ^4}{8} - \frac{λλ''}{4} + \frac{(λ')^2}{8} \right)' - \frac{1}{2} x \left( \frac{λ^2}{2} \right)' - \frac{λ^2}{2} = 0.$$
Assume the function $\lambda$ is not identically zero, then the above equation leads to the second Painlevé equation (P2)

$$\lambda'' = 2\lambda^3 + 2x\lambda + \text{const.}$$  \hfill (5.10)

The formal power series solution of this equation has the form

$$\lambda(x) = a + bx + cx^2 + p_3(a, b, c)x^3 + p_4(a, b, c)x^4 + \ldots,$$

where $a$, $b$ and $c$ are arbitrary parameters, and $p_i$ are certain polynomials of these constant parameters. If a matrix realization of $g$ is taken as in [8], then we have the Lax equation (5.6) for P2 given by

$$L = \begin{pmatrix} -\lambda & z \\ z & \lambda \end{pmatrix}, \quad M = \begin{pmatrix} -2z^2\lambda + \lambda^3 + x\lambda - \frac{1}{2}\lambda'' & 2z^3 - z(\lambda^2 + x + \lambda') \\ 2z^3 - z(\lambda^2 + x - \lambda) & 2z^2\lambda - \lambda^3 - x\lambda + \frac{1}{2}\lambda'' \end{pmatrix}.$$

(ii) When $s = (1, 0)$, by using (3.66) we can rewrite the equation (5.8) in the form

$$-\left(\frac{3}{8}\mu^2 + \frac{1}{8}\mu''\right)' + \frac{1}{2}x\mu' + \frac{\mu}{2} = 0,$$

which is just the Painlevé equation (P34; see, for example, [2])

$$\mu''' = -6\mu\mu' + 2x\mu' + 4\mu.$$  \hfill (5.11)

This equation is related to P2 via the Miura transformation $\mu = -\lambda^2 + \lambda'$. The formal series solution of (5.11) has the expression

$$\mu(x) = a + bx + cx^2 + q_3(a, b, c)x^3 + q_4(a, b, c)x^4 + \ldots$$

with arbitrary parameters $a$, $b$ and $c$, and $q_i$ are certain polynomials of these parameters. It leads to solutions of the KdV hierarchy that satisfies the similarity equation. In particular, if we take $(a, b, c) = (0, 1, 0)$, then we obtain the solution of the KdV hierarchy corresponding to the Witten-Kontsevich tau function given by (3.67) with $W_j = \delta_{j3}/16$.

The Lax pair (5.6) for the Painlevé equation P34 is given by

$$L = \begin{pmatrix} 0 & z - \mu \\ 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} -\frac{1}{2}(\mu' - 1) & 2z^2 - z(\mu + x) - \frac{1}{2}(2\mu^2 - 2x\mu + \mu'' + 4w_1) \\ 2z + \mu - x & \frac{1}{2}(\mu' - 1) \end{pmatrix}.$$

where $w_1(x) = \frac{1}{2}\mu(x)$. We observe that when $\tilde{\mu}(x) := \mu(x) - x \neq 0$, the equation (5.11) can be rewritten as

$$\tilde{\mu}''' = \frac{(\tilde{\mu}')^2}{2\tilde{\mu}} - 2\tilde{\mu}^2 - 2x\tilde{\mu} + \text{const} \cdot \frac{1}{\tilde{\mu}},$$  \hfill (5.12)

which is the Painlevé equation P4' given in Appendix B of [7].

Given a solution of the Painlevé equation (5.10) or (5.11), we have

$$w_j(x) := \omega_j|_{t_p = x_{3p1}} = \left. \left(2\Omega^p_j - t_l\Omega^p_{lj}\right) \right|_{\text{or } u \Rightarrow \lambda(x), \text{ or } u \Rightarrow \mu(x)}$$  \hfill (5.13)

by taking derivatives on the left hand side of the equation (5.7) with respect to $t_j$ for $j \in J_+$. From (5.13) we obtain a solution of the Drinfeld-Sokolov hierarchy by using Proposition 3.15.
Example 5.4  Let \( \mathfrak{g} \) be of type \( A_2^{(1)} \) and \( s = s^0 = (1, 0, 0) \). The Coexter number is \( h = 3 \), the set of exponents is \( J = 3 \mathbb{Z} \pm 1 \), and the basis elements \( \Lambda_j \) of the principal Heisenberg subalgebra are chosen as in \([3]\). We take the subspace \( V = \mathbb{C}(f_1 + f_2) \oplus \mathbb{C}f_{12} \) and the function
\[
Q^V = -u f_2 - v f_{12},
\]
where \( f_{i_1 \ldots i_m} \) are defined as in \([3.65]\). Then we have
\[
V = \gamma_1 f_0 + \gamma_2 f_{01} + \kappa_2 f_{02} + \gamma_3 f_{102} + \kappa_3 f_{201} + \ldots
\]
with
\[
\gamma_1 = \omega_1, \quad \gamma_2 = \frac{1}{6} (3 \omega_1^2 + 3 \omega_2 + u), \quad \gamma_3 = \frac{1}{18} (-3 \omega_1 u - 3 \omega_1^3 + 9 \omega_2 \omega_1 - 4 u' + 6 v),
\]
\[
\kappa_2 = \frac{1}{6} (3 \omega_1^2 - 3 \omega_2 + u), \quad \kappa_3 = \frac{1}{18} (-3 \omega_1 u - 3 \omega_1^3 - 9 \omega_2 \omega_1 + 2 u' - 6 v).
\]
The second-order derivatives of \( \log \tau \) with respect to \( t_1 \) and \( t_2 \) are given by
\[
\Omega^s_{11} = \frac{u}{3}, \quad \Omega^s_{12} = \frac{2v}{3} - u', \quad \Omega^s_{22} = -\frac{2}{9} u^2 - \frac{u''}{9}, \quad \Omega^s_{21} = 0,
\]
which yields the Boussinesq equation
\[
\frac{\partial^2 u}{\partial t_2^2} = \frac{2}{3} \left( u^2 \right)' - \frac{u(4)}{3}.
\]
Let us consider the similarity equation
\[
\sum_{\mu \in J_+} \left( \frac{p}{3} t_p - \delta_{p2} \right) \frac{\partial \log s^0}{\partial t_p} + C s^0 = 0.
\]
Its solution is characterized by a system of ODEs
\[
\frac{\mu^{(4)}}{4} + \frac{x^2 \mu''}{12} + \mu \mu'' + \frac{7 x \mu'}{12} + (\mu')^2 + \frac{2 \mu}{3} = 0, \quad \lambda' = \frac{1}{6} \left( 2 \mu + 3 \mu'' + x \mu' \right)
\]
of the unknown functions
\[
\mu(x) = u|_{t_p = x \delta_p}, \quad \lambda(x) = v|_{t_p = x \delta_p},
\]
(cf. equation (2.9) given in \([3]\), which can be solved in terms of solutions of the Painlevé equation P4). Note that the system of ODEs \((5.17)\) is the compatibility condition of the Lax pair \((5.6)\) with
\[
L = \begin{pmatrix} 0 & 0 & z - \lambda \\ 1 & 0 & -\mu \\ 0 & 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 2\mu + 1 & M_{12} & M_{13} \\ -x & -\mu & -\lambda + \frac{2}{3} x \mu + 3z - w_1 \\ 3 & -x & -\mu - 1 \end{pmatrix},
\]
where
\[
M_{12} = \mu' - \frac{1}{3} x \mu - w_1 - \lambda + 3z, \quad M_{13} = \frac{1}{3} \mu'' - \frac{1}{3} x \mu' - \mu - \frac{1}{3} \mu^2 + \frac{2}{3} x \lambda - w_2 - xz.
\]
Here \( w_i(x) = \omega_i|_{t_p = x \delta_p} \) for \( i = 1, 2 \). What is more, it is implied by \((5.15)\) that
\[
w_1' = \frac{1}{3} \mu, \quad w_2' = \frac{2}{3} \lambda - \frac{1}{3} \mu',
\]
hence the equations \((5.17)\) can also be represented via the functions \( w_1 \) and \( w_2 \).
Example 5.5 Let \( g \) be of type \( A_2^{(2)} \) and \( s = s^0 = (1, 0) \). Let us consider the similarity equation

\[
\sum_{p \in J_+} \left( \frac{P}{6} t_p - \delta_{p7} \right) \frac{\partial \log \tau s^0}{\partial t_p} + C s^0 = 0.
\]  

(5.18)

Following the notations of Example 3.19 we have \( \Omega_{11}^s = u/3 \) and

\[
\Omega_{17}^s = -\frac{1}{1944} \left( 56u^4 + 84u \left( u' \right)^2 + 168u^2 u'' + 42 \left( u'' \right)^2 + 42u' u''' + 42uu^{(4)} + 3u^{(6)} \right).
\]

(5.19)

Denote \( \mu(x) = u|_{t_j = \delta_{j1}x} \) as before. By taking the second-order derivative of (5.18) with respect to \( t_1 \) and putting \( t_p = x\delta_{p1} \), we arrive at a seven-order nonlinear ODE

\[
\left( 56\mu^4 + 84\mu \left( \mu' \right)^2 + 168\mu^2 \mu'' + 42 \left( \mu'' \right)^2 + 42\mu' \mu''' + 42\mu(\mu^{(4)} + 3\mu^{(6)}) \right)' + 108x\mu' + 216\mu = 0.
\]

(5.20)

This ODE can be verified to pass the Painlevé test (see, e.g., Chapter 2 of [7]). More precisely, the leading behavior of its solution at movable singularities is \(-3(x - x_0)^{-2}\), and the Fuchs indices are \( j = -1, 2, 3, 4, 7, 8, 12 \) that correspond to seven arbitrary parameters.

The general formal power series solution to (5.20) has the form

\[
\mu(x) = \sum_{i=0}^6 a_i x^i + \sum_{i \geq 7} p_i(a_0, a_1, \ldots, a_6) x^i,
\]

(5.21)

with arbitrary parameters \( a_0, a_1, \ldots, a_6 \) and polynomials \( p_i \) of these parameters. We can also write down the Lax pair (5.21) for the ODE (5.20) as we have done in the previous examples.

5.3 Affine Weyl group actions

Motivated by a series of work of Noumi, Yamada et al (see [36] and references therein), let us study the affine Weyl group actions on the space of solutions of the Lax equation (5.1) with \( s = 1 \), that is

\[
\left[ \frac{\partial}{\partial x} + Q + \Lambda_1, -d + M \right] = 0.
\]

(5.22)

Note that in this case we have \( \mathcal{V} = \mathcal{B} \) and \( rhd^1 = -d^4 = -d \) (recall the notations for the affine Kac-Moody algebra \( g \) in Section 2). Here we denote \( Q|_{t_p = \delta_{p1}} \) by \( Q \) to simplify notations. Observe that the functions \( Q \) and \( M \) take values in \( g^0 \) and \( g^\geq0 \) respectively, and they satisfy

\[
(d \mid Q) = (d \mid M) = 0.
\]

Let us introduce the following scalar functions (recall the notations in Subsection 2.1):

\[
\theta_i = \frac{k_i^\vee}{k_i} (\alpha_i^\vee | Q), \quad \chi_i = \frac{k_i}{k_i^\vee} - (\alpha_i^\vee | M), \quad \varphi_i = \frac{1}{\nu} (f_i | M), \quad i = 0, 1, \ldots, \ell.
\]

(5.23)

Clearly, we have

\[
\sum_{i=0}^\ell k_i \theta_i = 0, \quad \sum_{i=0}^\ell k_i^\vee \chi_i = h.
\]

(5.24)

Let us note that \( \boldsymbol{\theta} := (\theta_1, \theta_2, \ldots, \theta_\ell) \) gives a coordinate system for the space \( \mathcal{V} = \mathcal{B} \), moreover, the functions \( \chi_i \) and \( \varphi_i \) are elements of the ring \( \mathbb{C}[x, \theta, \theta', \theta'', \ldots] \).
Theorem 5.6 The following assertions hold true:

(i) The Lax equation (5.22) implies that \( \chi_i \) must be constant functions.

(ii) For a fixed set of constants \( \{\chi_0, \ldots, \chi_\ell \mid \sum_{i=0}^\ell k_i^\vee \chi_i = h\} \), the Lax equation (5.22) can be represented as a system of ODEs of the unknown functions \( \theta_1, \ldots, \theta_\ell \) in the form

\[
\varphi_i' + \theta_i \varphi_i + \chi_i = 0, \quad i = 0, 1, 2, \ldots, \ell.
\]

(5.25)

Here only \( \ell \) equations are independent.

(iii) Denote \( \xi_j = \chi_j / (\nu \varphi_j) \) for \( j = 0, 1, \ldots, \ell \). Then the Lax equation (5.22) has the Bäcklund transformations

\[
R_j : (Q, M) \mapsto (\tilde{Q}, \tilde{M}), \quad j = 0, 1, \ldots, \ell
\]

(5.26)

defined by the following relations:

\[
\frac{\partial}{\partial x} + \tilde{Q} + \Lambda_1 = e^{-\xi_j \text{ad}_{f_j}} \left( \frac{\partial}{\partial x} + Q + \Lambda_1 \right) - \frac{\nu k_j}{h} \xi_j c,
\]

(5.27)

\[
-d + \tilde{M} = e^{-\xi_j \text{ad}_{f_j}} (-d + M) - \frac{\chi_j}{h} c.
\]

(5.28)

More explicitly, for \( i, j = 0, 1, 2, \ldots, \ell \), we have

\[
R_j(\chi_i) = \chi_i - a_{ij} \chi_j, \quad R_j(\theta_i) = \theta_i + a_{ji} \frac{\chi_j}{\varphi_j}, \quad R_j(\varphi_i) = \frac{1}{\nu} \left( e^{\xi_j \text{ad}_{f_j}} f_i | M \right).
\]

(5.29)

Recall that \( A = (a_{ij})_{0 \leq i,j \leq \ell} \) is the Cartan matrix of \( g \).

Proof: We represent \( M \) in the form \( \sum_{k=0}^m M_k \) with \( M_k \) taking value in \( g^k \). From (5.4) one can see that its left hand side takes value in \( B \), hence the Lax equation (5.22) can be represented as

\[
M_0' + [Q, M_0] + [Q, -d] = 0.
\]

(5.30)

Since \( Q \) and \( M_0 \) take value in \( g^0 \subset h \), we have \([Q, M_0] = [Q, -d] = 0\). Hence \( M_0' = 0 \), and we obtain the first assertion of the theorem.

In order to prove the second assertion of the theorem, we need to substitute the constants \( \chi_i \) yield by (5.30) into the \( g^1 \)-component of the Lax equation (5.22), namely

\[
M_1' + [Q, M_1] + [\Lambda_1, M_0] + [\Lambda_1, -d] = 0.
\]

(5.31)

Note that \( (e_i | f_j) = \delta_{ij} k_i / k_i^\vee \), then the definition of the functions \( \varphi_i \) given in (5.23) implies that

\[
M_1 = \sum_{i=0}^{\ell} \frac{\nu k_i^\vee}{k_i} \varphi_i e_i.
\]

Hence, the equation (5.31) is equivalent to the following ones:

\[
\frac{1}{\nu} (f_i | M_1') = -\frac{1}{\nu} (f_i | [Q, M_1]) - \frac{1}{\nu} (f_i | [\Lambda_1, M_0]) - \frac{1}{\nu} (f_i | [\Lambda_1, -d]), \quad i = 0, 1, 2, \ldots, \ell.
\]
Clearly, the left hand side is \( \varphi'_i \), while the right hand side is

\[
\text{r.h.s.} = -\frac{1}{\nu}([M_1, f_i] \mid Q) + \frac{1}{\nu}([\Lambda_1, f_i] \mid M_0) - \frac{1}{\nu}([-d, f_i] \mid \Lambda_1)
\]

\[
= -\frac{k^\nu_i}{k_i} \varphi_i (\alpha^\nu_i \mid Q) + (\alpha^\nu_i \mid M_0) - \frac{k_i}{k^\nu_i} = -\theta_i \varphi_i - \chi_i.
\]

Thus we arrive at the equations (5.25) and conclude this assertion.

To prove the third assertion of the theorem, we first note that the action \( R_j \) yields

\[
\tilde{\mathcal{Q}} = \xi'_j f_j + Q + [Q, \xi_j f_j] + [\Lambda_1, \xi_j f_j] + \frac{1}{2}[[\Lambda_1, \xi_j f_j], \xi_j f_j] - \frac{\nu k_j}{\hbar k^\nu_j} \xi_j c
\]

\[
= Q + \nu \xi_j \left( \frac{\alpha^\nu_j}{\hbar k^\nu_j} + \theta_j f_j, \right)
\]

(5.32)

where

\[
\theta_j = \frac{k^\nu_j}{k_j} \left( e_j \mid \xi'_j f_j + [Q, \xi_j f_j] + \frac{1}{2}[[\Lambda_1, \xi_j f_j], \xi_j f_j] \right)
\]

\[
= \xi'_j - \frac{k^\nu_j}{k_j} \xi_j \left( [e_j, f_j] \mid Q \right) - \frac{1}{2} \xi_j^2 \left( [e_j, f_j] \mid [\Lambda_1, f_j] \right) \frac{k^\nu_j}{k_j}
\]

\[
= \xi'_j - \frac{k^\nu_j}{k_j} \xi_j \left( \alpha^\nu_j \mid Q \right) - \frac{1}{2} \xi_j^2 \left( \alpha^\nu_j \mid \nu \alpha^\nu_j \right) \frac{k^\nu_j}{k_j}
\]

\[
= \xi'_j - \theta_j \xi_j - \nu \xi_j^2 = -\nu \xi_j^2 \left( \left( \frac{1}{\nu \xi_j} \right) + \theta_j \frac{1}{\nu \xi_j} + 1 \right)
\]

\[
= -\nu \xi_j^2 \left( \frac{\varphi'_j}{\chi_j} + \theta_j \varphi_j \chi_j + 1 \right).
\]

From (5.25) we obtain \( \theta_j = 0 \). Secondly, since

\[
(\text{ad}_{f_j})[e_k, e_i] = -\delta_{jk} [\alpha^\nu_j, e_i] - \delta_{ji} [e_k, \alpha^\nu_j] = -\delta_{jk} a_{ji} e_l + \delta_{ji} a_{jk} e_k,
\]

(5.33)

\[
(\text{ad}_{f_j})^2[e_k, e_i] = \delta_{jk} a_{ji} \delta_{jl} \alpha^\nu_j - \delta_{ji} a_{jk} \delta_{jl} \alpha^\nu_j = 0,
\]

(5.34)

the negative part of \( \tilde{M} \) in (5.25) reads

\[
\tilde{M}^{<0} = [-d, \xi_j f_j] + [M_0, \xi_j f_j] + \frac{1}{2}[[\Lambda_1, \xi_j f_j], \xi_j f_j],
\]

(5.35)

which takes value in \( g^{-1} \) and satisfies the relations: for any \( i = 0, 1, 2, \ldots, \ell \),

\[
(e_i \mid \tilde{M}^{<0}) = \xi_j ([d, e_i] \mid f_j) - \xi_j ([e_i, f_j] \mid M_0) + \frac{1}{2} \xi_j^2 ([e_i, f_j], f_j) \mid M_1)
\]

\[
= \delta_{ij} \xi_j \left( \frac{k_i}{k^\nu_i} - (\alpha^\nu_j \mid M_0) + \frac{1}{2} \xi_j^2 (-2f_j \mid M_1) \right)
\]

\[
= \delta_{ij} \xi_j \left( \frac{k_i}{k^\nu_i} + (\chi_j - \frac{k_i}{k^\nu_i}) - \xi_j \nu \varphi_j \right)
\]

\[
= \delta_{ij} \xi_j \left( \chi_j - \xi_j \frac{\varphi_j}{\xi_j} \right) = 0.
\]
It follows that $\tilde{M} < 0$ indeed vanishes, and hence we can expand $\tilde{M} = \sum_{k=0}^m \tilde{M}_k$ with $\tilde{M}_k$ lying in $g^k$. In particular, we have

$$\tilde{M}_0 = M_0 + [M_1, \xi_j f_j] - \frac{\chi_j}{h} c = M_0 + \frac{k_j}{k_j} \nu \varphi_j \xi_j \alpha_j^\vee - \frac{\chi_j}{h} c = M_0 + \chi_j \left( \frac{k_j}{k_j} \alpha_j^\vee \right), \quad (5.36)$$

It is easy to check that both functions $\tilde{Q}$ and $\tilde{M}$ satisfy the relations

$$(d | \tilde{Q}) = (d | \tilde{M}) = 0,$$

and that

$$\left[ \frac{\partial}{\partial x} + \tilde{Q} + \Lambda_1, -d + \tilde{M} \right] = 0.$$

Thus $\mathcal{R}_i$ is a Bäcklund transformations of the Lax equation (5.22).

Finally, by using (5.36), (5.32) and (5.28), it is straightforward to verify

$$\mathcal{R}_j(\chi_i) = \frac{k_j}{k_i} (\alpha_i^\vee | \tilde{M}_0) = \frac{k_j}{k_i} (\alpha_i^\vee | M_0) - \chi_j \left( \frac{k_j}{k_j} (\alpha_i^\vee | \alpha_j^\vee) \right) = \chi_i - a_{ij} \chi_j, \quad (5.37)$$

$$\mathcal{R}_j(\theta_i) = \frac{k_j}{k_i} (\alpha_i^\vee | \tilde{Q}) = \frac{k_j}{k_i} (\alpha_i^\vee | Q) + \frac{k_j}{k_i} \nu \varphi_j (\alpha_i^\vee | \alpha_j^\vee) = \theta_i + a_j \chi_i, \quad (5.38)$$

$$\mathcal{R}_j(\varphi_i) = \frac{1}{\nu} (f_i | \tilde{M}) = \frac{1}{\nu} (f_i | e^{-\xi_j \text{adj}_j} M) = \frac{1}{\nu} (e^{\xi_j \text{adj}_j} f_i | M). \quad (5.39)$$

The theorem is proved. \hfill \Box

The above theorem shows that the actions of $\mathcal{R}_j$ on the constants $\chi_i$ generate an affine Weyl group associated to the Cartan matrix $A = (a_{ij})_{0 \leq i, j \leq \ell}$. Moreover, by using a general result of [36], we have the following proposition.

**Proposition 5.7** The actions $\mathcal{R}_j$, with $j = 0, 1, 2, \ldots, \ell$, on the space of solutions of the Lax equation (5.22) satisfy the following relations:

$$\mathcal{R}_j^2 = \text{Id}, \quad (\mathcal{R}_i \mathcal{R}_j)^{m_{ij}} = \text{Id} \quad \text{for} \quad i \neq j, \quad (5.40)$$

where $m_{ij} = 2, 3, 4, 6$ or $\infty$ when $a_{ij} a_{ji} = 0, 1, 2, 3$ or $\geq 4$ respectively.

**Proof:** For the Cartan matrix $A = (a_{ij})_{0 \leq i, j \leq \ell}$ of the affine Kac-Moody algebra $g$, a certain nilpotent Poisson algebra $\mathcal{K}$ was constructed in [36] (in fact, an even more general setting has been considered there, but here only the case of affine type is concerned). The Poisson algebra $\mathcal{K}$ is generated by $\phi_i \in g^*$ together with a set of parameters $\lambda_i$ with $i = 0, 1, 2, \ldots, \ell$, say,

$$\mathcal{K} = \mathbb{C}(\lambda_i, \phi_i, \{\phi_i, \phi_j\}, \{\phi_i, \{\phi_j, \phi_k\}\}, \ldots). \quad (5.41)$$

The Poisson bracket satisfies $\{\lambda_i, \phi_j\} = 0$ and the following locally nilpotent conditions

$$(\text{ad}_{\phi_j})^{1-a_{ij}} \phi_i = 0, \quad i \neq j. \quad (5.42)$$

For any $j = 0, 1, 2, \ldots, \ell$, let $\sigma_j$ be an automorphism of $\mathcal{K}$ such that

$$\sigma_j(\lambda_i) = \lambda_i - a_{ij} \lambda_j, \quad \sigma_j(\phi_i) = \phi_i. \quad (5.43)$$

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Then, on $\mathcal{K}$ there is a class of automorphisms given by

$$R_j = \exp \left( \frac{\lambda_j}{\phi_j} \right) \circ \sigma_j, \quad j = 0, 1, \ldots, \ell. \quad (5.44)$$

It is shown by Noumi and Yamada [36] that such automorphisms $R_j$ satisfy the relations (5.40), namely, they give a realization of the affine Weyl group for the Cartan matrix $A = (a_{ij})_{0 \leq i, j \leq \ell}$.

Noumi and Yamada also explained a Lie theoretic background for the above nilpotent Poisson algebra. More exactly, one can choose (see § 4.1 in [36])

$$\phi_i(X) = (f_i | X), \quad \left\{ \phi_j, \phi_i \right\}(X) = -([f_j, f_i] | X), \quad X \in \mathfrak{g}, \quad (5.45)$$

such that the nilpotent conditions (5.42) are satisfied due to the Serre relations (2.5). In terms of our notations, if we take

$$\phi_i \left( \frac{1}{\nu} M \right) = \varphi_i, \quad \lambda_i = -\frac{\chi_i}{\nu}, \quad i, j = 0, 1, \ldots, \ell, \quad (5.46)$$

then the isomorphisms (5.44) coincide with those given in Theorem 5.6 (note that the Lax equation (5.22) can be represented in the variables $\varphi_i$ and parameters $\chi_i$ due to (5.25)). Thus the relations (5.40) for $R_j$ defined by (5.26) are verified, and the proposition is proved. \qed

### 5.4 Examples

Let us give more details of the ODEs (5.25) and their discrete symmetries for some examples of affine Kac-Moody algebras.

**Example 5.8** Assume $\mathfrak{g}$ to be of type $A_\ell^{(1)}$ with $\ell \geq 2$. Then the components of the Cartan matrix are given by

$$a_{ij} = a_{ji} = \begin{cases} 2, & i = j; \\ -1, & i - j = \pm 1; \\ 0, & \text{else}. \end{cases}$$

Throughout the present example, by convention the indices $i, j, k \in \mathbb{Z}/(\ell + 1)\mathbb{Z}$. Note that the Kac labels and their duals are given by $k_i = k_i^\vee = 1$, the Coxeter number is $h = \ell + 1$ and the constant in (2.15) reads $\nu = 1$. Let us consider the Lax equation (5.22) induced by the similarity equation (4.21) with $b_p = \delta_{p2}$, namely,

$$M = h \left( e^{\text{ad}U(\varphi)} \Lambda_2 \right) \geq 0 - x \left( e^{\text{ad}U(\varphi)} \Lambda_1 \right) \geq 0 = M_0 + M_1 + h\Lambda_2 \quad (5.47)$$

with $\Lambda_2 = \sum_{k \in \mathbb{Z}/(\ell + 1)\mathbb{Z}} [e_{k+1}, e_k]$ (see, for example, [8]). According to Theorem 5.6, we have

$$R_j(\varphi_i) = \left( e^{\xi_j \text{ad}f_j} f_i | M_1 + h\Lambda_2 \right) = (f_i | M_1) + h\xi_j ([f_j, f_i] | \Lambda_2) = \varphi_i + hb_{ij} \xi_j, \quad (5.48)$$

where

$$b_{ij} = \sum_{k \in \mathbb{Z}/(\ell + 1)\mathbb{Z}} ([f_j, f_i] | [e_{k+1}, e_k]) = (f_i | [e_{k+1}, e_k], f_j)$$
\[
= \sum_{k \in \mathbb{Z}/(t+1)\mathbb{Z}} (f_i \mid \delta_{j,k+1}a_{jk}e_k - \delta_{jk}a_{j,k+1}e_{k+1}) \\
= \sum_{k \in \mathbb{Z}/(t+1)\mathbb{Z}} (\delta_{ik}\delta_{j,k+1} - \delta_{i,k+1}\delta_{jk})a_{ji}.
\]

Indeed, the numbers \(b_{ij}\) indicate a direction on the Dynkin diagram of type \(A_k^{(1)}\):

\[
b_{ij} = \begin{cases} 
1, & j = i - 1; \\
-1, & j = i + 1; \\
0, & \text{else.}
\end{cases} \tag{5.49}
\]

By using (5.48) we also have

\[
\mathcal{R}_j(\xi_i) = \frac{\chi_i - a_{ij}\chi_j}{\varphi_i + h\beta_{ij}\xi_j} = \begin{cases} 
-\xi_i, & j = i; \\
\frac{\chi_i + \chi_{i+1}}{\varphi_i - h\xi_{i+1}}, & j = i + 1; \\
\frac{\chi_i + \chi_{i-1}}{\varphi_i + h\xi_{i-1}}, & j = i - 1; \\
\xi_i, & \text{else.}
\end{cases} \tag{5.50}
\]

From (5.48)–(5.50) it follows that

\[
\mathcal{R}_k\mathcal{R}_j(\varphi_i) = \varphi_i + h\beta_{ik}\xi_k + h\beta_{ij}\mathcal{R}_k(\xi_j).
\]

It is straight forward to verify that

- If \(k = j\), then \(R_j(\xi_j) = -\xi_j\), and hence \(R_j^2(\varphi_i) = \varphi_i\);
- If \(a_{jk} = 0\), then \(R_k(\xi_j) = \xi_j\), and hence \(\mathcal{R}_k\mathcal{R}_j(\varphi_i) = \mathcal{R}_k(\varphi_i)\);
- If \(a_{jk} = -1\), namely \(k - j = \pm 1\), then

\[
\mathcal{R}_j\mathcal{R}_{j+1}\mathcal{R}_j(\varphi_i) = \mathcal{R}_j(\varphi_i + h\beta_{i,j+1}\xi_{j+1} + h\beta_{ij}\mathcal{R}_{j+1}(\xi_j)) \\
= \varphi_i + h\beta_{ij}(\xi_j + h\mathcal{R}_{j+1}(\xi_j)) + h\beta_{i,j+1}\mathcal{R}_j(\xi_{j+1}) \\
= \varphi_i + h\beta_{ij}\mathcal{R}_{j+1}(\xi_j) + h\beta_{i,j+1}\mathcal{R}_j(\xi_{j+1}) = \mathcal{R}_{j+1}\mathcal{R}_j\mathcal{R}_{j+1}(\varphi_i), \tag{5.51}
\]

in which the third equality holds true since

\[
\xi_j + \mathcal{R}_j\mathcal{R}_{j+1}(\xi_j) = \xi_j + \mathcal{R}_j \left( \frac{\chi_j + \chi_{j+1}}{\varphi_j - h\xi_{j+1}} \right) = \xi_j + \frac{-\chi_j + \chi_{j+1} + \chi_j}{\varphi_j - h\frac{\chi_{j+1} + \chi_j}{\varphi_j + h\xi_j}} \\
= \xi_j + \frac{\chi_{j+1}(\varphi_{j+1} + h\xi_j)}{\varphi_j\varphi_{j+1} - h\chi_{j+1}} = \xi_j + \frac{\chi_{j+1}(\varphi_{j+1} + h\xi_j)}{\varphi_j - h\xi_{j+1}} \\
= \frac{\xi_j\varphi_j + \xi_{j+1}\varphi_{j+1}}{\varphi_j - h\xi_{j+1}} = \frac{\chi_j + \chi_{j+1}}{\varphi_j - h\xi_{j+1}} = \mathcal{R}_{j+1}(\xi_j).
\]
Thus we have derived the relations (5.40) based on the explicit representation (5.48) of $\mathcal{R}_j$. The result agrees with that achieved in [33] (see also [38]), where a matrix realization of $\mathfrak{g}$ was used.

In the current case the system of ODEs (5.25) can be represented in an alternative form as follows. Let us expand $U(Q) = \sum_{m<0} U_m$ with $U_m \in \mathfrak{g}^m$, then by using Lemma 3.11 we have

$$[U_{-1}, A_1] = Q, \quad (d | [U_{-1}, A_1]) = 0.$$  

Let us introduce the notation

$$\tilde{\psi}_i = (e_i | U_{-1}), \quad \psi_i = \tilde{\psi}_i - \tilde{\psi}_{i-1} - \frac{x}{2(\ell+1)}, \quad i \in \mathbb{Z}/(\ell+1)\mathbb{Z}. \quad (5.52)$$

Clearly, we have $\sum_{i=0}^\ell \psi_i = -x/2$, and we can represent $\theta_i$ and $\varphi_i$ as follows:

$$\theta_i = (\alpha_i^\vee | [U_{-1}, A_1]) = -([\alpha_i^\vee, A_1] | U_{-1}) = -\sum_{j=0}^\ell a_{ij}(e_j | U_{-1}) \quad (5.53)$$

$$= \tilde{\psi}_{i-1} - 2\tilde{\psi}_i + \tilde{\psi}_{i+1} = \psi_{i+1} - \psi_i, \quad (5.54)$$

$$\varphi_i = (f_i | M) = (f_i | [U_{-1}, A_2] - xA_1) = h([A_2, f_i] | U_{-1}) - x(f_i | A_1)$$

$$= h([\alpha_i^\vee, e_{i-1}] + [e_{i+1}, \alpha_i^\vee] | U_{-1}) - x = h(\psi_{i-1} + \psi_{i+1}) - x = h(\psi_{i+1} + \psi_i). \quad (5.55)$$

Here we have used the representation of $A_2$ as mentioned above. So the system (5.25) can be represented as

$$\psi_{i+1}' + \psi_i' + \psi_{i+1}^2 - \psi_i^2 + \frac{\chi_i}{\ell+1} = 0, \quad i \in \mathbb{Z}/(\ell+1)\mathbb{Z}. \quad (5.56)$$

Note that the system (5.56) is the so-called nonlinear chain studied in [39, 1], which is related to the forth and the fifth Painlevé equations (P4 and P5) whenever $\ell = 2$ and $3$, respectively.

In the general case the system (5.25) looks more complicated than (5.56). The reason is that $\varphi_i$ are no longer linear functions in $\theta_i$. Let us illustrate this fact by the following examples, which also illustrate Remark 5.2.

**Example 5.9** Let $\mathfrak{g}$ be of type $A_3^{(1)}$ and $b_p = \delta_{p3}$. We have

$$\varphi_i = \frac{1}{2} \theta_{i-1}' + \frac{1}{2} \theta_i' + 2\theta_i'^2 + 3\theta_i'^2 - \frac{3}{4} \theta_i'^2 - \frac{1}{8} (\theta_{i-1} + \theta_{i+1} - 2\theta_i)^2 - x \quad (5.57)$$

with $i \in \mathbb{Z}/4\mathbb{Z}$. Observe that this system is invariant with respect to the rotation $\pi : i \mapsto i + 1$ or the reflection $\sigma : (0, 1, 2, 3) \mapsto (0, 3, 2, 1)$.

**Example 5.10** Let $\mathfrak{g}$ be of type $C_2^{(1)}$ and $b_p = \delta_{p3}$. Note that $(k_0, k_1, k_2) = (1, 2, 1), (k_0^\vee, k_1^\vee, k_2^\vee) = (1, 1, 1)$ and $\nu = 1$ (see [25] [3]), so we have

$$\varphi_i = \begin{cases} 
\theta_i' + 2\theta_{i-2}' + \theta_i^2 + \theta_{i-1} \theta_{i-2} - \frac{1}{2} \theta_{i-2}^2 - x, & i = 0; \\
-\theta_0' - \theta_2' + \frac{1}{2} \theta_0^2 - 2\theta_0 \theta_2 + \frac{1}{2} \theta_2^2 - 2x, & i = 1.
\end{cases} \quad (5.58)$$

Observe that this system can be obtained from (5.57) by letting $\theta_3 = \theta_1$ and $(\varphi_0, \varphi_1, \varphi_2) \mapsto (\varphi_0, 2\varphi_1, \varphi_2).$
Example 5.11 Let \( g \) be of type \( D_4^{(1)} \) and \( b_p = \delta_{p,3} \). Note that \( k_i = k_{i'} = 1 \) for all \( i \in \{0, 1, 2, 3, 4\} \). With \( \nu = \sqrt{2} \) and \( \Lambda_3 \) being normalized as in [42], we have

\[
\varphi_i = \begin{cases}
-3\theta'_{1-i} + \frac{3}{2}\theta'_3 + \frac{3}{2}\theta'_4 + \frac{3}{2}\theta_{1-i}^2 - \frac{3}{4}\theta_3^2 - \frac{3}{4}\theta_4^2 - x, & i = 0, 1; \\
\frac{3}{2}(\theta_0\theta_1 + \theta_3\theta_4) - \frac{3}{4}(\theta_0 + \theta_1)(\theta_3 + \theta_4) - x, & i = 2; \\
-3\theta'_{7-i} + \frac{3}{2}\theta'_6 + \frac{3}{2}\theta'_1 + \frac{3}{2}\theta_{7-i}^2 - \frac{3}{4}\theta_0^2 - \frac{3}{4}\theta_1^2 - x, & i = 3, 4.
\end{cases}
\]

This system is invariant with respect to the following reflections

\[
\sigma_1 : (0, 1, 2, 3, 4) \mapsto (0, 1, 2, 4, 3), \quad \sigma_2 : (0, 1, 2, 3, 4) \mapsto (1, 0, 2, 3, 4).
\]

Thus by letting \( \theta_3 = \theta_4 \) we obtain the system of ODEs (5.25) for \( D_3^{(1)} \), while by letting \( \theta_0 = \theta_1 \) and \( \theta_3 = \theta_4 \) we obtain the system of ODEs (5.25) for \( D_3^{(2)} \).

6 Concluding remarks

In this paper we present a tau cover for the Drinfeld-Sokolov hierarchy associated to any affine Kac-Moody algebra \( g \) with two gradations \( s \leq 1 \), and construct a series of Virasoro symmetries for the tau cover of the integrable hierarchy. This tau cover leads to an algorithm to construct formal power series solution of the Cauchy problem of the Drinfeld-Sokolov hierarchy with arbitrary initial values. By using this algorithm, we compute solutions of the Drinfeld-Sokolov hierarchy that satisfy two types of Virasoro constraints which are given by the string equation and the similarity equation respectively. In particular, the Virasoro constraints given by the similarity equation lead to a system of ODEs of Painlevé type. When \( s = 1 \), the solution space of such ODEs admit an affine Weyl group actions, which generalizes the theory of Noumi, Yamada et al on the affine Weyl group symmetries in Painlevé type equations. Moreover, for the affine Kac-Moody algebras that are listed in Tables 1–3 of [29], the associated ODEs of Painlevé type also possess discrete symmetries that are induced from the automorphisms of the Dynkin diagrams of the affine Kac-Moody algebras. We hope that these results would help us to have a better understanding of properties of higher order Painlevé-type equations and their relations to the Drinfeld-Sokolov hierarchies.

The Drinfeld-Sokolov hierarchies we consider in this paper are associated to the principal Heisenberg subalgebra of \( g \). There are generalized Drinfeld-Sokolov hierarchies that are associated to other Heisenberg subalgebras of \( g \), see for example, [13, 20, 25], and they can also yields ODEs of Painlevé type [17, 18, 19, 26]. It is natural to ask how the similarity reductions of the Drinfeld-Sokolov hierarchies corresponding to different Heisenberg subalgebras are related to each other. We will study this question elsewhere.

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