Uniqueness in phaseless inverse scattering problems with known superposition of incident point sources

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Abstract
This paper is concerned with the uniqueness in inverse acoustic scattering problems with the modulus of the far-field patterns co-produced by the obstacle (resp. medium) and the point sources. Based on the superposition of point sources as the incident waves, we overcome the difficulty of translation invariance induced by a single incident plane wave, and rigorously prove that the location and shape of the obstacle as well as its boundary condition or the refractive index can be uniquely determined by the modulus of far-field patterns. This work is different from our previous work on phaseless inverse scattering problems (Zhang and Guo 2018 Inverse Problems 34 085002), in which the reference ball technique and the superposition of incident waves were used, and the phaseless far-field data generated only by the scatterer were considered. In this paper, the phaseless far-field data co-produced by the scatterer and the point sources are used, thus the configuration is practically more feasible. Moreover, since the reference ball is not needed, the justification of uniqueness is much more clear and concise.

Keywords: inverse scattering, phaseless, Helmholtz equation, far field, point source

(Some figures may appear in colour only in the online journal)
1. Introduction

The inverse scattering problems with the measurement of the far-field pattern is of significant importance in diverse areas of sciences and technology such as invasive testing, target identification, radar sensing, sonar detection and biomedical imaging (see, e.g. [9]). However, in many realistic situations, only intensity/modulus information of the far-field pattern might be measured, which leads to the study of inverse scattering problems with phaseless or intensity-only far-field data. Owing to the lack of phase information, the mathematical justifications and numerical reconstructions of phaseless inverse scattering problems are more challenging than the phased case [1]. In this study, we will deal with the uniqueness issue concerning the phaseless inverse acoustic scattering problems with incident point sources.

A well-known difficulty in phaseless inverse acoustic scattering with one plane wave as the incident field is the non-uniqueness. For example, the location of the obstacle (resp. medium) cannot be uniquely determined by the phaseless far-field data, which is due to the so-called translation invariance property, i.e. the modulus of the far field pattern is invariant under translations [22, 23, 25, 26, 31]. And even more notoriously, the invariance property of the phaseless far-field pattern cannot be remedied by using incident waves $\exp(i k_m x \cdot d_j)$ with finitely many different wave numbers $k_m, m = 1, 2, \cdots$ or different incident directions $d_j, j = 1, 2, \cdots$ (see, e.g. [26]). Consequently, some numerical algorithms for the shape reconstruction have been developed, such as the Newton method [26], the nonlinear integral equation method [13, 15–17], the fundamental solution method [21], the hybrid method [27], the reverse time migration method [6] and the direct imaging method [40]. We also refer to [2, 3, 24, 28, 29, 35] for the reconstruction of the shape of a polyhedral obstacle, a convex sound-soft scatterer, a periodic grating profile and multi-scale sound-soft rough surfaces from phaseless far-field or near-field data. For a small sound-soft ball centered at the origin, the uniqueness on radius of the ball from a single phaseless far-field datum was established in [32].

To break the translation invariance, the authors in [38] used the method of the superposition of two plane waves with different incident directions and all wave numbers in a finite interval, and proposed a recursive Newton-type iteration algorithm. Then, the translation invariance can be almost broken down (subject to a countably infinite number of cross points between the straight lines), see [38] for more details. This method was also developed in [39] to recover locally rough surfaces with phaseless far-field data. By the method of superposition of two incident plane waves, uniqueness results were proved in [36] under a priori assumption.

The idea of resorting to the reference ball technique (see, e.g. [7, 8, 30]) in phaseless inverse scattering problems was proposed by Zhang and Guo in [41]. For a general setting of the underlying scatterer, the uniqueness results were established in [41] by utilizing the reference ball technique in conjunction with the superposition of a fixed plane wave and some point sources as the incident wave. With the aid of the reference ball technique, the a priori assumptions in [36] can be removed as well, see [37] for the details. In addition to the theoretical interests in phaseless inverse scattering, the strategy of adding reference objects/sources to the scattering system has also received an increasing attention in devising effective numerical inversion schemes. For instance, the reference ball based iterative methods have been developed in [11] to reconstruct both the location and shape of a scattering acoustic obstacle. A reference point scatterer based direct sampling method can be found in [19]. We would like to point out that the idea of using an interfering term (reference source) for the phaseless inverse source problem was proposed in [14]. Here we refer to [20, 42] for some recent investigations on this topic. We also refer to [12, 18] for some recent studies on utilizing artificial reference objects in phaseless inverse elastic scattering problems.
It should be remarked that the measured phaseless far-field data in [41] are only generated by the obstacle (resp. medium). However, in realistic applications, the far-field patterns are co-produced by the obstacle (resp. medium) and the point sources, and the corresponding phaseless data cannot be decoupled and measured separately. In this paper, based on the superposition of point sources as the incident field and the measurements of the phaseless far-field patterns co-produced by the obstacle (resp. medium) and the point sources, we rigorously prove that the location and shape of the obstacle as well as its boundary condition or the refractive index can be uniquely determined by the modulus of far-field patterns. We would like to emphasize that the co-produced phaseless far-field patterns provide more information and overcome the indispensability of incorporating a reference ball in the previous works on establishing the uniqueness.

In comparison with the previous reference ball technique in [41], the treatment in this study exhibits the superiorities in the following aspects: first, this configuration enables us to provide a unified framework for the justification of uniqueness in both the cases of penetrable and impenetrable scatterers. Second, the proofs are significantly simplified since the essential requirement of the reference ball turns out to be completely unnecessary. Finally, from the practical point of view, to access the modulus of the far-field patterns concurrently produced by the underlying scatterer and the point sources should be relatively easier than capturing the modulus of far-field due solely to the scatterer.

The rest of this paper is arranged as follows. In the next section, we present an introduction to the model problem. Section 3 is devoted to the uniqueness results on phaseless inverse scattering problem.

2. Problem setting

We begin this section with the acoustic scattering problems for an incident plane wave. Assume $D \subset \mathbb{R}^3$ is an open and simply connected domain with $C^2$ boundary $\partial D$. Denote by $\nu$ be the unit outward normal to $\partial D$ and by $S^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \}$ the unit sphere in $\mathbb{R}^3$. Let $u'(x, d) = e^{ik|x|} \delta d$ be a given incident plane wave, where $d \in S^2$ and $k > 0$ are the incident direction and wave number, respectively. Then, the obstacle scattering problem can be formulated as: to find the total field $u = u' + u$ which satisfies the following boundary value problem (see [9]):

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus D,$$

$$\mathcal{B} u = 0 \quad \text{on } \partial D,$$

$$\lim_{r = |x| \to \infty} r \left( \frac{\partial u'}{\partial r} - iku' \right) = 0,$$

where $u'$ denotes the scattered field and (3) is the Sommerfeld radiation condition. Here $\mathcal{B}$ in (2) is the boundary operator defined by

$$\mathcal{B} u := \begin{cases} u & \text{on } \partial D_D, \\ \frac{\partial u}{\partial n} + \lambda u & \text{on } \partial D_I, \end{cases}$$

where $\partial D_D \cup \partial D_I = \partial D$, $\partial D_D \cap \partial D_I = \emptyset$, $\lambda \in C(\partial D_I)$ and $\text{Im}\lambda \geq 0$. This mixed boundary condition (4) covers the Dirichlet/sound-soft boundary condition ($\partial D_D = \emptyset$), the Neumann/sound-hard boundary condition ($\partial D_D = \emptyset$ and $\lambda = 0$), and the impedance boundary condition ($\partial D_D = \emptyset$ and $\lambda \neq 0$).
The medium scattering problem is to find the total field \( u = u' + u' \) that fulfills
\[ \Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^3, \]

\[ \lim_{r = |x| \to \infty} r \left( \frac{\partial u'}{\partial r} - i k u' \right) = 0, \]
where the refractive index \( n(x) \) of the inhomogeneous medium is piecewise continuous such that \( \text{Re}(n) > 0, \text{Im}(n) \geq 0 \) and \( 1 - n(x) \) is supported in \( D \).

The direct scattering problems (1)–(3) and (5), (6) admit a unique solution (see, e.g. \([5, 9, 33]\)), respectively, and the scattered wave \( u' \) has the following asymptotic behavior
\[ u'(x, d) = e^{ik|x|} \left\{ u^\infty(\hat{x}, d) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty \]
uniformly in all observation directions \( \hat{x} = x/|x| \in S^2 \). The analytic function \( u^\infty(\hat{x}, d) \) defined on the unit sphere \( S^2 \) is called the far field pattern or scattering amplitude (see [9]).

Now, the phaseless inverse scattering problem is to determine the location and shape \( \partial D \) as well as the boundary condition \( B \) for the obstacle or the refractive index \( n \) for the medium inclusion by the phaseless far field data \( |u^\infty(\hat{x}, d)| \) for \( \hat{x}, d \in S^2 \) and a fixed \( k > 0 \). As we mentioned, there is no uniqueness on the phaseless inverse scattering problem due to the translation invariance, and this ambiguity cannot be remedied by using finitely many incident waves with different wave numbers or different incident directions.

In the following, we will try to break the translation invariance by utilizing the superposition of incident point sources. To this end, we first introduce the following definition of admissible surfaces.

**Definition 2.1 (Admissible surface).** An open surface \( \Gamma \) is called admissible if there exists a simply-connected domain \( \Omega \) such that
(i) \( \Omega \subset \mathbb{R}^3 \setminus \overline{D} \) and \( \partial \Omega \) is analytic homeomorphic to \( S^2 \);
(ii) \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \);
(iii) \( \Gamma \subset \partial \Omega \) is a two-dimensional analytic manifold with nonvanishing measure.

**Remark 2.1.** We would like to point out that this requirement for the admissibility of \( \Gamma \) is quite mild and thus can be easily fulfilled. For instance, \( \Omega \) can be chosen as a ball whose radius is less than \( \pi/k \) and \( \Gamma \) is chosen as an arbitrary corresponding semisphere.

For a generic point \( z \in \mathbb{R}^3 \setminus \overline{D} \), the incident field due to the point source located at \( z \) is given by
\[ \Phi(x, z) := \frac{e^{ik|x-z|}}{4\pi|x-z|}, \quad x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{ z \}), \]
which is also known as the fundamental solution to the Helmholtz equation. Denote by \( v^\infty_D(\hat{x}, z) \) the far-field patterns generated by \( D \) corresponding to the incident field \( \Phi(x, z) \). Define
\[ v^\infty(\hat{x}, z) := v^\infty_D(\hat{x}, z) + \Phi^\infty(\hat{x}, z), \quad \hat{x} \in S^2, \]
where \( \Phi^\infty(\hat{x}, z) := e^{-ik\hat{x} \cdot z}/(4\pi) \) is the far-field pattern of \( \Phi(x, z) \).
Remark 2.2. As far as we know, the phaseless data $|v_0^\infty(\hat{x}, z) + \Phi(\hat{x}, z)|$ is usually obtainable in realistic scenarios, whereas the phaseless data $|v_0^\infty(\hat{x}, z)|$ cannot be measured directly. Hence, in our view, it would make more sense to use intensity-only data $|v_0^\infty(\hat{x}, z)| = |v_0^\infty(\hat{x}, z) + \Phi(\hat{x}, z)|$.

For two generic and distinct source points $z_1, z_2 \in \mathbb{R}^3 \setminus \mathcal{D}$, we denote by $v_i(x; z_1, z_2) := \Phi(x, z_1) + \Phi(x, z_2), x \in \mathbb{R}^3 \setminus (\mathcal{D} \cup \{z_1\} \cup \{z_2\})$.

(7)

the superposition of these point sources. Then, by the linearity of direct scattering problem, the far-field pattern co-produced by $\mathcal{D}$ and the incident wave $v'(x; z_1, z_2)$ is given by

$v^\infty(\hat{x}; z_1, z_2) := v^\infty(\hat{x}, z_1) + v^\infty(\hat{x}, z_2), \quad \hat{x} \in S^2$.

Now we are in the position to introduce the phaseless sets of data.

Definition 2.2 (Phaseless datasets). For a fixed wavenumber $k > 0$ and a fixed $z_0 \in \mathbb{R}^3 \setminus (\mathcal{D} \cup \Gamma)$, let us define

$\mathcal{D}_1 := \{ |v^\infty(\hat{x}, z_0)| : \hat{x} \in S^2 \}$,

$\mathcal{D}_2 := \{ |v^\infty(\hat{x}, z)| : \hat{x} \in S^2, z \in \Gamma \}$,

$\mathcal{D}_3 := \{ |v^\infty(\hat{x}, z_0) + v^\infty(\hat{x}, z)| : \hat{x} \in S^2, z \in \Gamma \}$,

where $\Gamma$ is an admissible surface.

With all these preparations, we formulate the phaseless inverse scattering problems as follows.

Problem 2.1 (Phaseless inverse obstacle scattering). Let $\mathcal{D}$ be the impenetrable obstacle with boundary condition $\mathcal{B}$. Given the phaseless far-field datasets $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ due to the obstacle $\mathcal{D}$, determine the location and shape $\partial \mathcal{D}$ as well as the boundary condition $\mathcal{B}$ for the obstacle.

Problem 2.2 (Phaseless inverse medium scattering). Let $\mathcal{D}$ be the inhomogeneous medium with refractive index $n$. Given the phaseless far-field datasets $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ due to the inhomogeneity $\mathcal{D}$, determine the refractive index $n$ for the medium inclusion.

Remark 2.3. It could be possible to numerically reconstruct the scatterer using partial phaseless data (i.e. one or more of the datasets $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ are not included), but the uniqueness result based on such partial data is still mathematically open. Therefore, to justify the uniqueness of problems 2.1 and 2.2, the datasets $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ should be simultaneously required.

We refer to figure 1 for an illustration of the geometry setting of problems 2.1 and 2.2. The uniqueness of this problem will be analyzed in the next section.

3. Uniqueness for the phaseless inverse scattering

Now we present the uniqueness results on phaseless inverse scattering. The following theorem shows that problem 2.1 (resp. problem 2.2) admits a unique solution, namely, the geometrical and physical information of the scatterer boundary (resp. the refractive index for the medium) can be uniquely determined from the modulus of far-field patterns $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$. 

Theorem 3.1. For two scatterers $D_1$ and $D_2$, suppose that the corresponding far-field patterns satisfy that
\begin{align}
|\psi_\infty^1(\hat{x}, z_0)| &= |\psi_\infty^2(\hat{x}, z_0)|, \quad \forall \hat{x} \in S^2, \quad (8) \\
|\psi_\infty^1(\hat{x}, z)| &= |\psi_\infty^2(\hat{x}, z)|, \quad \forall (\hat{x}, z) \in S^2 \times \Gamma \quad (9)
\end{align}
and
\begin{align}
|\psi_\infty^1(\hat{x}, z_0) + \psi_\infty^1(\hat{x}, z)| &= |\psi_\infty^2(\hat{x}, z_0) + \psi_\infty^2(\hat{x}, z)|, \quad \forall (\hat{x}, z) \in S^2 \times \Gamma \quad (10)
\end{align}
for an admissible surface $\Gamma$ and an arbitrarily fixed $z_0 \in \mathbb{R}^3 \setminus (D \cup \Gamma)$. Then we have

(i) If $D_1$ and $D_2$ are two impenetrable obstacles with boundary conditions $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then $D_1 = D_2$ and $\mathcal{B}_1 = \mathcal{B}_2$.

(ii) If $D_1$ and $D_2$ are two medium inclusions with refractive indices $n_1$ and $n_2$ respectively, then $n_1 = n_2$.

Proof. From (8)–(10), we have for all $\hat{x} \in S^2, z \in \Gamma$
\begin{align}
\text{Re} \left\{ \psi_\infty^1(\hat{x}, z_0)\overline{\psi_\infty^1(\hat{x}, z)} \right\} = \text{Re} \left\{ \psi_\infty^2(\hat{x}, z_0)\overline{\psi_\infty^2(\hat{x}, z)} \right\}, 
\end{align}
where the overline denotes the complex conjugate. According to (8) and (9), we denote
\begin{align}
\psi_\infty^j(\hat{x}, z_0) &= r(\hat{x}, z_0)e^{i\alpha_j(\hat{x}, z_0)}, \quad \psi_\infty^j(\hat{x}, z) &= s(\hat{x}, z)e^{i\beta_j(\hat{x}, z)}, \quad j = 1, 2,
\end{align}
where $r(\hat{x}, z_0) = |\psi_\infty^j(\hat{x}, z_0)|, s(\hat{x}, z) = |\psi_\infty^j(\hat{x}, z)|, \alpha_j(\hat{x}, z_0)$ and $\beta_j(\hat{x}, z)$ are real-valued functions, $j = 1, 2$.

From $s(\hat{x}, z) \neq 0$ for $\hat{x} \in S^2, z \in \Gamma$, and its continuity, one knows that there exists an open set $\hat{S} \subset S^2$ such that $s(\hat{x}, z) \neq 0, \forall (\hat{x}, z) \in \hat{S} \times \Gamma$. Since $\psi_\infty^j(\hat{x}, z_0)$ are analytic with respect to $\hat{x} \in S^2$, $\psi_\infty^j(\hat{x}, z_0) \neq 0$ for $\hat{x} \in S^2$, and $z_0$ is fixed, we have $r(\hat{x}, z_0) \neq 0$ on $\hat{S}$. Again,
the continuity leads to \( r(\hat{x}, z_0) \neq 0 \) on an open set \( S \subset \hat{S} \). Therefore, we have \( r(\hat{x}, z_0) \neq 0 \), \( s(\hat{x}, z) \neq 0 \), \( \forall (\hat{x}, z) \in S \times \Gamma \). In addition, taking (11) into account, we derive that

\[
\cos[\alpha_1(\hat{x}, z_0) - \beta_1(\hat{x}, z)] = \cos[\alpha_2(\hat{x}, z_0) - \beta_2(\hat{x}, z)], \quad \forall (\hat{x}, z) \in S \times \Gamma.
\]

Hence, either

\[
\alpha_1(\hat{x}, z_0) - \alpha_2(\hat{x}, z_0) = \beta_1(\hat{x}, z) - \beta_2(\hat{x}, z) + 2m\pi, \quad \forall (\hat{x}, z) \in S \times \Gamma
\]

or

\[
\alpha_1(\hat{x}, z_0) + \alpha_2(\hat{x}, z_0) = \beta_1(\hat{x}, z) + \beta_2(\hat{x}, z) + 2m\pi, \quad \forall (\hat{x}, z) \in S \times \Gamma
\]

holds with some \( m \in \mathbb{Z} \).

First, we shall consider the case (12). Since \( z_0 \) is fixed, let us define \( \gamma(\hat{x}) := \alpha_1(\hat{x}, z_0) - \alpha_2(\hat{x}, z_0) - 2m\pi \) for \( \hat{x} \in S \), and then, we deduce for all \( (\hat{x}, z) \in S \times \Gamma \)

\[
v_1^\infty(\hat{x}, z) = s(\hat{x}, z)e^{i\beta_1(\hat{x}, z)} = s(\hat{x}, z)e^{i\beta_1(\hat{x}, z)+i\gamma(\hat{x})} = v_2^\infty(\hat{x}, z)e^{i\gamma(\hat{x})}.
\]

From the mixed reciprocity relation [9, theorem 3.16] for the obstacle or [34, theorem 2.2.4] for the inhomogeneous medium, we have

\[
4\pi v_2^\infty_D(\hat{x}, z) = u_j^D(z, -\hat{x}), \quad j = 1, 2, \quad \forall (\hat{x}, z) \in S \times \Gamma
\]

where \( u_j^D(z, -\hat{x}) \) denotes the scattered field induced by \( D_j \) and the incident plane wave with impinging direction \( -\hat{x} \). Thus,

\[
u_1^D(z, -\hat{x}) + u_1^D(z, -\hat{x}) = e^{i\gamma(\hat{x})} \left[u_2^D(z, -\hat{x}) + u_2^D(z, -\hat{x})\right], \quad \forall (\hat{x}, z) \in S \times \Gamma.
\]

Let \( u_j = u_j^D + u_j^I (j = 1, 2) \) be the total field. Then, for every \( -d \in S \), it holds that \( u_1(z, d) = e^{i\gamma(-d)}u_2(z, d) \) for \( z \in \Gamma \). By using the analyticity of \( u_j(z, d) (j = 1, 2) \), we have

\[
u_1(z, d) = e^{i\gamma(-d)}u_2(z, d) \quad \text{for} \quad z \in \partial\Omega.
\]

Let \( w(x, d) = u_1(x, d) - e^{i\gamma(-d)}u_2(x, d) \), then

\[
\begin{align*}
\Delta w + k^2 w &= 0 & \text{in} \ \Omega, \\
 w &= 0 & \text{on} \ \partial\Omega.
\end{align*}
\]

By the assumption of \( \Omega \) that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \), we find \( w = 0 \) in \( \Omega \).

Now, the analyticity of \( u_j(x, d) (j = 1, 2) \) with respect to \( x \) yields

\[
u_1(x, d) = e^{i\gamma(-d)}u_2(x, d), \quad \forall x \in \mathbb{R}^3 \setminus (\overline{D}_1 \cup D_2)
\]

i.e.

\[
u_1(x, d) + u_1(x, d) = e^{i\gamma(-d)} \left[u_2(x, d) + u(x, d)\right], \quad \forall x \in \mathbb{R}^3 \setminus (\overline{D}_1 \cup D_2).
\]

By the Green’s formula [9, theorem 2.5], one can readily deduce that

\[
\lim_{|x| \to \infty} u_j^D(|x|, x, d) = 0, \quad j = 1, 2,
\]

holds uniformly in all directions \( \hat{x} = x'/|x| \). Hence, by letting \( |x| \to \infty \) in (14), we obtain
\[
0 = \lim_{{|x|\to\infty}} \left[ u_1^\infty(x, d) - e^{i\gamma(-d)} u_2^\infty(x, d) \right] = \lim_{{|x|\to\infty}} (e^{i\gamma(-d)} - 1)e^{ikx \cdot d}
\]

uniformly in all directions \(\hat{x} = x/|x|\). Therefore, \(e^{i\gamma(-d)} \equiv 1\), and

\[
u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d), \quad \forall (\hat{x}, -d) \in S^2 \times S.
\]

Further, the reciprocity relation and the analyticity of \(u_j^\infty(-d, \hat{x}) (j = 1, 2)\) with respect to \(d \in S^2\) implies that the far field patterns coincide, i.e.

\[
u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d), \quad \forall \hat{x}, d \in S^2.
\]

(15)

Next we are going to show that the case (13) does not hold. Suppose that (13) is true, then following a similar argument, we see that there exists \(\eta(-d)\) such that

\[
u_1(x, d) = e^{i\eta(-d)}u_2(x, d)
\]

for \(-d \in S\) and \(x \in \mathbb{R}^3 \setminus (D_1 \cup D_2)\). Again, taking \(|x| \to \infty\), we have

\[
\lim_{{|x|\to\infty}} e^{2ikx \cdot d} = e^{i\eta(-d)},
\]

which is a contradiction. Therefore, the case (13) does not hold.

Having verified (15), we shall complete our proof as the consequences of two existing uniqueness results. For the inverse obstacle scattering, by theorem 5.6 in [9], we have \(D_1 = D_2\) and \(B_1 = B_2\), and for inverse medium scattering, theorem 10.5 in [9] leads to \(n_1 = n_2\). □

Remark 3.1. We would like to point out that an analogous uniqueness result in two dimensions remains valid after appropriate modifications of the fundamental solution, the radiation condition and the admissible surface. So we omit the 2D details.

Remark 3.2. A similar result on uniqueness can be also obtained by using the superposition of a fixed plane wave and some point sources as the incident fields.

Remark 3.3. We would like to emphasize that the current study has the following limitations: (a) The amount of data which one has to collect is roughly the double than the ones used in many papers such as [4] and [10]. The uniqueness justification based on less phaseless data is still open and thus deserves further investigations. (b) All the analysis is based on the capability to get a perfect control of phases on the primary sources. Hence, the terminology ‘superposition’ throughout this paper should be understood as the weaker ‘phase controlled superposition’.

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References

[1] Ammari H, Chow Y T and Zou J 2016 Phased and phaseless domain reconstructions in the inverse scattering problem via scattering coefficients SIAM J. Appl. Math. 76 1000–30
[2] Bao G, Li P and Lv J 2013 Numerical solution of an inverse diffraction grating problem from phaseless data J. Opt. Soc. Am. A 30 293–9
[3] Bao G and Zhang L 2016 Shape reconstruction of the multi-scale rough surface from multi-frequency phaseless data Inverse Problems 32 085002
[4] Bucci O M, Crocco L, D’Urso M and Isernia T 2006 Inverse scattering from phaseless measurements of the total field on open lines J. Opt. Soc. Am. A 23 2566–77
[5] Cakoni F, Colton D and Monk P 2001 The direct and inverse scattering problem for partially coated obstacles Inverse Problems 17 1997–2015
[6] Chen Z and Huang G 2017 Phaseless imaging by reverse time migration: acoustic waves Numer. Math. Theor. Methods Appl. 10 1–21
[7] Colton D, Coyle J and Monk P 2000 Recent developments in inverse acoustic scattering theory SIAM Rev. 42 369–414
[8] Colton D, Giebermann K and Monk P 2000 A regularized sampling method for solving three-dimensional inverse scattering problems SIAM J. Sci. Comput. 21 2316–30
[9] Colton D and Kress R 2013 Inverse Acoustic and Electromagnetic Scattering Theory 3rd edn (New York: Springer)
[10] Crocco L, D’Urso M and Isernia T 2004 Inverse scattering from phaseless measurements of the total field on a closed curve J. Opt. Soc. Am. A 21 622–31
[11] Dong H, Zhang D and Guo Y 2018 A reference ball based iterative algorithm for imaging acoustic obstacle from phaseless far-field data Inverse Problems Imaging (http://dx.doi.org/10.3934/ipi.2019010)
[12] Dong H, Lai J and Li P 2019 Inverse obstacle scattering problem for elastic waves with phased or phaseless far-field data SIAM J. Imaging Sciences 12 809–38
[13] Gao P, Dong H and Ma F 2018 Inverse scattering via nonlinear integral equations method for a sound-soft crack from phaseless data Appl. Math. 63 149–65
[14] Isernia T, Leone G and Pierri R 1989 The phase retrieval by a reference source IEEE Digest on Antennas and Propagation Society Int. Symp. pp 64–7
[15] Ivanyshyn O 2007 Shape reconstruction of acoustic obstacles from the modulus of the far field pattern Inverse Problems Imaging 1 609–22
[16] Ivanyshyn O and Kress R 2010 Identification of sound-soft 3D obstacles from phaseless data Inverse Problems Imaging 4 131–49
[17] Ivanyshyn O and Kress R 2011 Inverse scattering for surface impedance from phaseless far field data J. Comput. Phys. 230 3443–52
[18] Ji X and Liu X 2018 Inverse elastic scattering problems with phaseless far field data (arXiv:1812.02359v1)
[19] Ji X, Liu X and Zhang B 2018 Target reconstruction with a reference point scatterer using phaseless far field patterns (arXiv:1805.08035v3)
[20] Ji X, Liu X and Zhang B 2018 Phaseless inverse source scattering problem: phase retrieval, uniqueness and direct sampling methods (arXiv:1808.02385v1)
[21] Karageorghis A, Johansson B T and Lesnic D 2012 The method of fundamental solutions for the identification of a sound-soft obstacle in inverse acoustic scattering Appl. Numer. Math. 62 1767–80
[22] Klibanov M V 2014 Phaseless inverse scattering problems in three dimensions SIAM J. Appl. Math. 74 392–410
[23] Klibanov M V 2017 A phaseless inverse scattering problem for the 3D Helmholtz equation Inverse Problems Imaging 11 263–76
[24] Klibanov M V and Romanov V G 2016 Reconstruction procedures for two inverse scattering problems without the phase information SIAM J. Appl. Math. 76 178–96
[25] Klibanov M V and Romanov V G 2017 Uniqueness of a 3D coefficient inverse scattering problem without the phase information Inverse Problems 33 095007
[26] Kress R and Rundell W 1997 Inverse obstacle scattering with modulus of the far field pattern as data Inverse Problems in Medical Imaging and Nondestructive Testing Engl H W, Louis A K and Rundell W (eds) (Vienna: Springer) pp 75–92
[27] Lee K M 2016 Shape reconstructions from phaseless data Eng. Anal. Bound. Elem. 71 174–8
[28] Li J and Liu H 2015 Recovering a polyhedral obstacle by a few backscattering measurements J. Differ. Equ. 259 2101–20
[29] Li J, Liu H and Wang Y 2017 Recovering an electromagnetic obstacle by a few phaseless backscattering measurements Inverse Problems 33 035001
[30] Li J, Liu H and Zou J 2009 Strengthened linear sampling method with a reference ball SIAM J. Sci. Comput. 31 4013–40
[31] Liu J and Seo J 2004 On stability for a translated obstacle with impedance boundary condition Nonlinear Anal. 59 731–44
[32] Liu X and Zhang B 2009 Unique determination of a sound soft ball by the modulus of a single far field datum J. Math. Anal. Appl. 365 619–24
[33] McLean W 2000 Strongly Elliptic Systems and Boundary Integral Equations (Cambridge: Cambridge University Press)
[34] Potthast R 2001 Point Sources and Multipoles in Inverse Scattering Theory (London: Chapman and Hall)
[35] Shin J 2016 Inverse obstacle backscattering problems with phaseless data Euro. J. Appl. Math. 27 111–30
[36] Xu X, Zhang B and Zhang H 2018 Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency SIAM J. Appl. Math. 78 1737–53
[37] Xu X, Zhang B and Zhang H 2018 Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency II SIAM J. Appl. Math. 78 3024–39
[38] Zhang B and Zhang H 2017 Recovering scattering obstacles by multi-frequency phaseless far-field data J. Comput. Phys. 345 58–73
[39] Zhang B and Zhang H 2017 Imaging of locally rough surfaces from intensity-only far-field or near-field data Inverse Problems 33 055001
[40] Zhang B and Zhang H 2018 Fast imaging of scattering obstacles from phaseless far-field measurements at a fixed frequency Inverse Problems 34 104005
[41] Zhang D and Guo Y 2018 Uniqueness results on phaseless inverse scattering with a reference ball Inverse Problems 34 085002
[42] Zhang D, Guo Y, Li J and Liu H 2018 Retrieval of acoustic sources from multi-frequency phaseless data Inverse Problems 34 094001