Robust and Sparse Estimation of Linear Regression Coefficients with Heavy-tailed Noises and Covariates

Takeyuki Sasai *

October 11, 2022

Abstract

Robust and sparse estimation of linear regression coefficients is investigated. The situation addressed by the present paper is that covariates and noises are sampled from heavy-tailed distributions, and the covariates and noises are contaminated by malicious outliers. Our estimator can be computed efficiently. Further, the error bound of the estimator is nearly optimal.

1 Introduction

Sparse estimation has been studied extensively over the past 20 years to handle modern high-dimensional data e.g., [61, 32, 65, 63, 31, 7, 56, 64, 59, 3, 39, 48, 57, 49, 6, 37, 38, 34]. Because the advancement of computer technology has made it possible to collect very high dimensional data efficiently, sparse estimation will continue to be an important and effective method for high dimensional data analysis in the future. On the other hand, in recent years, robust estimation methods for outliers or heavy-tailed distribution have been developed rapidly e.g., [52, 10, 41, 2, 8, 20, 21, 40, 22, 16, 11, 19, 14, 12, 30, 23, 27, 60, 43, 35, 15, 47, 26, 28, 13, 42, 36, 54, 44, 1, 25, 18, 17]. These studies dealt with estimating problems of mean, covariance, linear regression coefficients, half-spaces, parameters of Gaussian mixture moles, and so on. They are mainly interested in deriving sharp error bounds, deriving information-theoretical lower bounds of error bounds, and reducing computational complexity.

In the present paper, we consider sparse estimation of linear regression coefficients when covariates and noises are sampled from heavy-tailed distributions, and the samples are contaminated by malicious outliers. Define a normal sparse linear regression model as follows:

\[ y_i = x_i^T \beta^* + \xi_i, \quad i = 1, \ldots, n, \]  

where \( \{x_i\}_{i=1}^n \) is a sequence of independent and identically distributed (i.i.d.) random vectors, \( \beta^* \in \mathbb{R}^d \) is the true coefficient vector, and \( \{\xi_i\}_{i=1}^n \) is a sequence of i.i.d. random variables. We assume the number of non-zero elements of \( \beta^* \) is \( s (\leq d) \). When an adversary injects outliers to the normal sparse linear regression model, (1.1) changes as follows:

\[ y_i = X_i^T \beta^* + \xi_i + \sqrt{n}\theta_i, \quad i = 1, \ldots, n, \]  

where \( X_i = x_i + \varrho_i \) for \( i = 1, \ldots, n \) and \( \{\varrho_i\}_{i=1}^n \) and \( \{\theta_i\}_{i=1}^n \) are the outliers. We allow the adversary to inject arbitrary values into arbitrary samples of \( \{y_i, x_i\}_{i=1}^n \). Let \( O \) be the index set of the injected samples and \( I = (1, \ldots, n) \setminus O \). Therefore, \( \varrho_i = (0, \ldots, 0)^T \) and \( \theta_i = 0 \) hold for \( i \in I \). We note that \( \{\varrho_i\}_{i \in O} \) and \( \{\theta_i\}_{i \in O} \) can be arbitral values and they are allowed to correlate freely among them and correlate with \( \{x_i\}_{i=1}^n \) and \( \{\xi_i\}_{i=1}^n \). The difficulty is not only that \( \{\varrho_i\}_{i \in O} \)
and \( \{ \theta_i \}_{i \in \mathcal{O}} \) can take arbitrary values but also that \( \{ x_i \}_{i \in \mathcal{X}} \) and \( \{ \xi_i \}_{i \in \mathcal{X}} \) no longer follow sequences of i.i.d. random variables because we allow the adversary to freely select samples for injection. This kind of contamination by outliers is sometimes called strong contamination in contrast to the Huber contamination [24]. We note that the Huber contamination is more manageable because its outliers are not correlated to the inliers and do not restrict their independence.

Various studies [10, 55, 1, 53, 13] dealt with estimation of linear regression coefficients with samples drawn from some heavy-tailed distributions under the existence of outliers. Some [2, 65, 14, 50, 52, 10, 10, 35] considered sparse estimation of linear regression coefficients when samples are drawn from Gaussian or subGaussian distributions under the existence of outliers. However, to the best of our knowledge, none of the findings on robust estimation of linear regression coefficients provides the result when the covariates and noises are drawn from heavy-tailed distributions and contaminated by outliers, and where the true coefficient vector is sparse.

Our result is as follows: For the precise statement, see Theorem 2.1 in Section 2.5. For any vector \( v \), define the \( \ell_2 \) norm of \( v \) as \( \| v \|_2 \), and define \( x_i \) as the \( j \)-th element of \( x_i \). Define \( o = | \mathcal{O} | \), where \( | S | \) is the number of the elements of \( S \).

**Theorem 1.1.** Suppose that \( \{ x_i \}_{i=1}^{n} \) is a sequence of i.i.d. random vectors with zero mean and with finite kurtosis. Suppose that, for any \( 1 \leq j_1, j_2, j_3, j_4 \leq d \), \( \mathbb{E}(x_{ij_1}x_{ij_2}x_{ij_3}x_{ij_4})^2 \) exists. Suppose that \( \{ \xi_i \}_{i=1}^{n} \) is a sequence of i.i.d. random variables whose absolute moment is bounded, and that \( \{ \xi_i \}_{i=1}^{n} \) and \( \{ x_i \}_{i=1}^{n} \) are independent. Then, for a sufficiently large \( n \) such that \( C_1 \max (s^2, \| \beta^* \|_1^2, \| \beta \|_1^2/s^2) \log(d/\delta) \leq n \) and \( C_2 \left( \sqrt{\frac{\log(d/\delta)}{n}} + \sqrt{\frac{o}{n}} \right) \leq 1 \), we can efficiently construct \( \hat{\beta} \) such that

\[
\mathbb{P} \left\{ \| \hat{\beta} - \beta^* \|_2 \leq C_2 \left( \sqrt{\frac{\log(d/\delta)}{n}} + \sqrt{\frac{o}{n}} \right) \right\} \geq 1 - 3\delta,
\]

where \( C_1 \) and \( C_2 \) are some constants depending on the properties of limits of \( x_i \) and \( \xi_i \).

We see that, even when samples are contaminated by malicious outliers, paying only \( \sqrt{o/n} \) extra term is sufficient. Similar results showing that, with appropriate estimators, the impact of outliers can be reduced, have been revealed in many previous works. Some studies [1, 13] derived the information-theoretically optimal lower bound of estimating error of linear regression coefficients without sparsity when samples and noise are drawn from distributions with finite kurtosis and fourth moments, respectively and when the samples are contaminated by outliers. The optimal lower bound in [1] is \( \sqrt{o/n} \) (for sufficiently large \( n \)) and our estimation error bound coincides with the optimal one about the term involving \( \sqrt{o/n} \) up to constant factor. In our situation, we require not only finite kurtosis but also finite \( \mathbb{E}(x_{ij_1}x_{ij_2}x_{ij_3}x_{ij_4})^2 \) as an assumption for covariates. To remove the extra condition from the assumption is a future task.

Standard lasso requires \( n \) proportional to \( s \) ([50]), however our estimator requires \( n \) proportional to \( s^2 \). A similar phenomenon can be seen in [62, 34, 33, 2, 23] and so on. Our method relies on the techniques of [34, 62] and this is the cause for the stronger condition on the sample complexity of our estimator. [62] considered sparse principal component analysis (PCA), revealing that there is no randomized polynomial time algorithm to estimate the top eigenvector in a scheme where \( n \) is proportional to \( s \) (in [62], \( s \) is the number of non-zero elements of the top eigenvector of covariance matrices) under the assumptions of intractability of a variant of Planted Clique Problem. We leave the analysis in our situation for future work.

Finally, we note that the error bound or sample complexity of the estimators in [10, 2, 63, 34], that dealt with sparse estimation of linear regression coefficients where the case that both covariates and noises are sampled from some heavy-tailed distributions or both the covariates and noises are contaminated by outliers, depend on norms of \( \beta^* \). Our estimator requires sufficiently large \( n \) depending on \( \| \beta^* \|_1 \) because our estimator use the technique developed in [34] to tame heavy-tailed covariates. To remove the effects of the true coefficient vector would be important.
In Section 2, we describe our estimation method and state our main result. In Section 3, we state key propositions without proofs, and the proof of the main theorem. In Section 4, we provide the proofs that are omitted in Sections 2 and 3.

2 Method

To estimate $\beta^*$ in (1.2), we propose the following algorithm (ROBUST-SPARSE-ESTIMATION).

Algorithm 1 ROBUST-SPARSE-ESTIMATION

| Input: $\{y_i, X_i\}_{i=1}^n$ and the tuning parameters $\tau_x, \lambda_s, \tau_{suc}, \varepsilon, \lambda_o$ and $\lambda_s$ |
|-----------------------------------|
| Output: $\hat{\beta}$ |
| 1: $\{\tilde{X}_i\}_{i=1}^n \leftarrow$ PRUNING($\{X_i\}_{i=1}^n, \tau_x$) |
| 2: $\{\hat{w}_i\}_{i=1}^n \leftarrow$ COMPUTE-WEIGHT($\{\tilde{X}_i\}_{i=1}^n, \lambda_s, \tau_{suc}, \varepsilon$) |
| 3: $\{\hat{w}'_i\}_{i=1}^n \leftarrow$ ROUNDING($\{\hat{w}_i\}_{i=1}^n$) |
| 4: $\hat{\beta} \leftarrow$ WEIGHTED-PENALIZED-HUBER-REGRESSION ($\{y_i, \tilde{X}_i\}_{i=1}^n, \{\hat{w}'_i\}_{i=1}^n, \lambda_o, \lambda_s$) |

to make covariates bounded, which originated from [34], that deal with sparse estimations of vector/matrix when samples are drawn from a heavy-tailed distribution. COMPUTE-WEIGHT relies on the semi-definite programming developed by [2], which provides a method for sparse PCA to be robust to outliers. [2] considered a situation when samples that are drawn from Gaussian distribution and the samples are contaminated by outliers. PRUNING enables us to cast our heavy-tailed situation into the framework of [2].

In the following Sections 2.1, 2.2, 2.3 and 2.4 we describe the details of PRUNING, COMPUTE-WEIGHT, TRUNCATION and WEIGHTED-HUBER-REGRESSION, respectively. Define

$$r_o = \sqrt{\frac{\alpha}{n}}, \quad r_d = \sqrt{\frac{\log d}{n}}, \quad r_\delta = \sqrt{\frac{\log(1/\delta)}{n}}. \tag{2.1}$$

2.1 PRUNING

Define the $j$-th element of $X_i$ as $X_{ij}$. For the choice of $\tau_x$, see Remark 2.2.

Algorithm 2 PRUNING

| Input: data $\{X_i\}_{i=1}^n$, tuning parameter $\tau_x$. |
|-----------------------------------|
| Output: pruned data $\{\tilde{X}_i\}_{i=1}^n$. |
| For $i = 1 : n$ |
| For $j = 1 : d$ |
| $X_{ij} = \text{sgn}(X_{ij}) \times \min(X_{ij}, \tau_x)$ |
| return $\{\tilde{X}_i\}_{i=1}^n$. |

2.2 COMPUTE-WEIGHT

For any matrix $M \in \mathbb{R}^{d_1 \times d_2} = \{m_{ij}\}_{1 \leq i \leq d_1, 1 \leq j \leq d_2}$, define

$$\|M\|_1 = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} |m_{ij}|, \quad \|M\|_\infty = \max_{1 \leq i \leq d_1, 1 \leq j \leq d_2} |m_{ij}|. \tag{2.2}$$
For a symmetric matrix $M$, we write $M \succeq 0$ if $M$ is positive semidefinite. Define the following two convex sets: 

$$\mathcal{M}_r = \{ M \in \mathbb{R}^{d \times d} : \text{Tr}(M) \leq r^2, M \succeq 0 \}, \quad \mathcal{U}_\lambda = \{ U \in \mathbb{R}^{d \times d} : ||U||_\infty \leq \lambda, U \succeq 0 \},$$

where $\text{Tr}(M)$ for matrix $M$ is the trace of $M$. To reduce the effects of outliers of covariates, we require COMPUTE-WEIGHT to compute the weight vector $\hat{w} = (\hat{w}_1, \cdots, \hat{w}_n)$ such that the following quantity is sufficiently small:

$$\sup_{M \in \mathcal{M}_r} \left( \sum_{i=1}^{n} \hat{w}_i (\hat{X}_i \hat{X}_i^\top M) - \lambda_* ||M||_1 \right),$$

where $\lambda_*$ is a tuning parameter. Evaluation of (2.4) is required in the analysis of WEIGHTED-PENALIZED-HUBER-REGRESSION and the role of (2.4) is revealed in the proof of Proposition 3.3. For COMPUTE-WEIGHT, we use a variant of Algorithm 4 of [2]. For any vector $v$, define the $\ell_\infty$ norm of $v$ as $||v||_\infty$ and define the probability simplex $\Delta^{n-1}$ as

$$\Delta^{n-1} = \left\{ w \in [0,1]^n : \sum_{i=1}^{n} w_i = 1, \quad ||w||_\infty \leq \frac{1}{n(1 - \varepsilon)} \right\}.$$ 

COMPUTE-WEIGHT is as follows.

**Algorithm 3 COMPUTE-WEIGHT**

**Input:** data $\{X_i\}_{i=1}^{n}$, tuning parameters $\lambda_*$, $\tau_{\text{suc}}$ and $\varepsilon$.

**Output:** weight estimate $\hat{w} = (\hat{w}_1, \cdots, \hat{w}_n)$.

Let $\tilde{w}$ be the solution to

$$\min_{w \in \Delta^{n-1}} \max_{M \in \mathcal{M}_r} \left( \sum_{i=1}^{n} w_i (\tilde{X}_i \tilde{X}_i^\top, M) - \lambda_* ||M||_1 \right),$$

if the optimal value of (2.6) $\leq \tau_{\text{suc}}$

return $\hat{w}$

else

return $\text{fail}$

We note that, from the arguments of [62, 50, 51], we have

$$\min_{w \in \Delta^{n-1}} \max_{M \in \mathcal{M}_r} \left( \sum_{i=1}^{n} w_i (\tilde{X}_i \tilde{X}_i^\top, M) - \lambda_* ||M||_1 \right) = \min_{w \in \Delta^{n-1}} \min_{U \in \mathcal{U}_\lambda} \max_{M \in \mathcal{M}_r} \left( \sum_{i=1}^{n} w_i \tilde{X}_i \tilde{X}_i^\top - U, M \right).$$

COMPUTE-WEIGHT and Algorithm 4 of [2] are very similar and the difference are the convex constraints and the values of parameters. For any fixed $w$, our objective function and the constraints are the same as the ones in Section 3 of [62] except for the values of the tuning parameters, and we can efficiently find the optimal $M \in \mathcal{M}_r$. Therefore, COMPUTE-WEIGHTS can be solved efficiently for the same reason as Algorithm 4 of [2].

To analyze COMPUTE-WIGHT, we introduce the following proposition. The proof of the following proposition is provided in Section 4. Define $\sigma_{\text{X},4} = \max_{1 \leq j \leq d} \mathbb{E}x_{ij}^4$ and $\rho_{\text{X},2} = \max_{1 \leq j \leq d} \mathbb{E}x_{ij}^2$. Define $\Sigma = \mathbb{E}x_{i}x_{i}^\top$, and $\hat{x}_{ij} = \text{sgn}(x_{ij}) \times \min (x_{ij}, \tau_x)$.

**Proposition 2.1.** Assume that $(x_i)_{i=1}^{n}$ is a sequence of i.i.d. $d (\geq 3)$-dimensional random vectors with zero mean and with finite $\sigma_{\text{X},4}$. For any matrix $M \in \mathcal{M}_r$, with probability at least $1 - \delta$, we have

$$\sum_{i=1}^{n} \frac{(\hat{x}_i \hat{x}_i^\top, M)}{n} \leq \sqrt{2\sigma_{\text{X},4}^2 (r_d + r_\delta) + \tau_x^2 (r_d^2 + r_\delta^2) + \frac{2\sigma_{\text{X},4}^4}{\tau_x^2}} ||M||_1 + ||\Sigma||_{\text{op}} r^2.$$
where \( \|\Sigma\|_{\text{op}} \) is the operator norm of \( \Sigma \).

Define \( \tau'_{\text{suc}} \), \( \lambda'_s \) and \( \{ w'^*_i \}_{i=1}^n \) as

\[
\tau'_{\text{suc}} = \frac{\|\Sigma\|_{\text{op}}}{1 - \varepsilon}, \quad \lambda'_s = \frac{1}{1 - \varepsilon} \left\{ \sqrt{2\sigma^2_{X,4}(r_d + r_{\delta})} + \sigma^2_{X,4}(r_d^2 + r_{\delta}^2) + 2 \frac{\sigma^4_{X,4}}{\tau_X} \right\}, \quad w'^*_i = \begin{cases} \frac{1}{n(1 - \varepsilon)} & i \in I \setminus \{ \tau \} \\ 0 & i \in \mathcal{O} \end{cases}.
\]

From Proposition \ref{prop:2.1} when \( \tau'_{\text{suc}} \leq \tau_{\text{suc}} \), \( \lambda'_s \leq \lambda_s \) and \( o/n \leq \varepsilon \) hold, we have, with probability at least \( 1 - \delta \),

\[
\max_{M \in \mathcal{M}_n} \left( \sum_{i=1}^n \hat{w}_i \langle \hat{X}_i, \hat{X}_i^\top, M \rangle - \lambda_s \|M\|_1 \right) \leq (a) \max_{M \in \mathcal{M}_n} \left( \sum_{i \in I} w'^*_i \langle \hat{X}_i, \hat{X}_i^\top, M \rangle - \lambda_s \|M\|_1 \right) = (b) \max_{M \in \mathcal{M}_n} \left( \sum_{i=1}^n \frac{1}{n(1 - \varepsilon)} \langle \hat{x}_i, \hat{x}_i^\top, M \rangle - \lambda_s \|M\|_1 \right) \leq (c) \max_{M \in \mathcal{M}_n} \left( \sum_{i=1}^n \frac{1}{n} \langle \hat{x}_i, \hat{x}_i^\top, M \rangle - \lambda'_s (1 - \varepsilon) \|M\|_1 \right) \times \frac{1}{1 - \varepsilon} \leq \tau'_{\text{suc}},
\]

where (a) follows from the optimality of \( \hat{w}_i \), \( o/n \leq \varepsilon \) and \( \{ w'^*_i \}_{i=1}^n \in \Delta^{n-1} \), (b) follows from positive semi-definiteness of \( M \), and (c) follows from \( \lambda'_s \leq \lambda_s \). Therefore, from \( \tau'_{\text{suc}} \leq \tau_{\text{suc}} \), we see that \textsc{Compute-Weight} succeed and return \( \hat{w} \) with probability at least \( 1 - \delta \). We note that (2.10) is used in the proof of Proposition \ref{prop:3.3}.

### 2.3 TRUNCATION

\textsc{Truncation} is a discretization of \( \{ \hat{w}_i \}_{i=1}^n \). \textsc{Truncation} makes it easy to analyze the estimator. We see that the number of \( \hat{w}_1, \ldots, \hat{w}_n \) rounded at zero is at most \( 2\varepsilon n \) from Proposition \ref{prop:3.6}.

**Algorithm 4 Truncation**

**Input:** weight vector \( w = \{ \hat{w}_i \}_{i=1}^n \).

**Output:** rounded weight vector \( \hat{w}' = \{ \hat{w}'_i \}_{i=1}^n \).

For \( i = 1 : n \)

- if \( \hat{w}_i \geq \frac{1}{n} \)
  - \( \hat{w}'_i = \frac{1}{n} \)
- else
  - \( \hat{w}'_i = 0 \)

return \( \hat{w}' \).

### 2.4 WEIGHTED-PENALIZED-HUBER-REGRESSION

\textsc{Weighted-Penalized-Huber-Regression} is a type of regression using the Huber loss with \( \ell_1 \) penalization. Define Huber loss function \( H(t) \),

\[
H(t) = \begin{cases} |t| - 1/2 & (|t| > 1) \\ t^2/2 & (|t| \leq 1) \end{cases}.
\]
and let
\[ h(t) = \frac{d}{dt} H(t) = \begin{cases} t & (|t| > 1) \\ \text{sgn}(t) & (|t| \leq 1) \end{cases}. \] (2.12)

We consider the following optimization problem. For any vector \( \mathbf{v} \), define the \( \ell_1 \) norm of \( \mathbf{v} \) as \( \| \mathbf{v} \|_1 \).

Algorithm 5 WEIGHTED-PENALIZED-HUBER-REGRESSION

Input: data \( \{(y_i, \tilde{X}_i)\}_{i=1}^n \), rounded weight vector \( \hat{w}' = \{\hat{w}'_i\}_{i=1}^n \) and tuning parameters \( \lambda_o, \lambda_s \).

Output: estimator \( \hat{\beta} \).

Let \( \hat{\beta} \) be the solution to
\[ \arg\min_{\beta \in \mathbb{R}^d} \sum_{i=1}^n \lambda_o^2 H \left( n\hat{w}'_i \frac{y_i - \tilde{X}_i^\top \beta}{\lambda_o \sqrt{n}} \right) + \lambda_s \| \beta \|_1, \] (2.13)
return \( \hat{\beta} \).

We note that many studies e.g., [52, 58, 16, 9, 15, 53] imply that the Huber loss is effective for linear regression under heavy-tailed noises or the existence of outliers.

2.5 Results

We introduce our assumption after first introducing the notion of finite kurtosis distribution.

Definition 2.1 (Finite kurtosis distribution). A random vector \( \mathbf{z} \in \mathbb{R}^d \) with zero mean is said to have finite kurtosis distribution if for every \( \mathbf{v} \in \mathbb{R}^d \),
\[ \mathbb{E}(\mathbf{v}^\top \mathbf{z})^4 \leq K^4 \{\mathbb{E}(\mathbf{v}^\top \mathbf{z})^2\}^2. \] (2.14)

We note that the finite kurtosis distribution sometimes referred as \( L_4-L_2 \) norm equivalence [46] or \( L_4-L_2 \) hyper-contractivity [13]. Define the minimum singular value of \( \Sigma^2 \) as \( \lambda_\Sigma \).

Assumption 2.1. Assume that

(i) \( \{\mathbf{x}_i\}_{i=1}^n \) is a sequence of i.i.d. \( d(\geq 3) \)-dimensional random vectors with zero mean, with finite kurtosis, \( \mathbb{E}(x_{ij_1} x_{ij_2} x_{ij_3} x_{ij_4})^2 \leq \sigma_{x,8}^8 \) for any \( 1 \leq j_1, j_2, j_3, j_4 \leq d \), and \( \lambda_\Sigma > 0 \), and for simplicity, assume \( 1 \geq \lambda_\Sigma \),

(ii) \( \{\xi_i\}_{i=1}^n \) is a sequence of i.i.d. random variables whose absolute moments are bounded by \( \sigma \),

(iii) \( \mathbb{E} h \left( \frac{\xi_i}{\lambda_\Sigma \sqrt{n}} \right) \times \mathbf{x}_i = 0 \).

Remark 2.1. The condition (iii) in Assumption 2.1 is a weaker condition than the independence between \( \{\xi_i\}_{i=1}^n \) and \( \{\mathbf{x}_i\}_{i=1}^n \).

Under Assumption 2.1, we have the following theorem. Define
\[ r_{x,d} = (\sigma_{x,2} + 1)r_d + \tau_{x,d}^2, \quad r_{x,\delta} = (\sigma_{x,2} + 1)r_\delta + \tau_{x,\delta}^2, \quad r_{d,\delta} = r_{x,d} + r_{x,\delta}. \] (2.15)
Theorem 2.1. Suppose that Assumption [A] holds. Suppose that parameters \( \tau_x, \lambda_x, \tau_{\text{suc}}, \varepsilon, \lambda_o, \lambda_s \) and \( r \) satisfy

\[
\tau_x^2 \geq \max \left\{ \frac{\|\beta^*\|_1^2 \sigma_{x,4}^8 \|\Sigma_x^2\|_{\text{op}}^2}{s \lambda_o^2 n}, \left( \frac{\|\beta^*\|_1}{\lambda_o \sqrt{n}} \right)^{\frac{3}{2}}, \frac{108 \sigma_{x,4}^4 s}{\lambda_o^2}, \left( \|\beta^*\|_1 \sigma_{x,4}^4 \right)^{\frac{5}{2}}, \frac{9 \sigma_{x,8}^4}{K^2}, \frac{9 \sigma_{x,8}^4}{K^2} \right\},
\]  
(2.16)

\[
\lambda_o \sqrt{n} \geq \max \left\{ \frac{16 K \|\Sigma_x^2\|_{\text{op}}}{\lambda_o^2}, \frac{300 K^4 \|\Sigma_x^2\|_{\text{op}} (\sigma + 1)}{\lambda_o^2}, \frac{4 K^2 \|\Sigma_x^2\|_{\text{op}}^2}{\lambda_o^2} \right\},
\]  
(2.18)

\[
\lambda_s \geq c_s \lambda_o \sqrt{n} \left\{ r_{d,\delta} + \frac{\sigma_{x,2} \sigma_{x,4}^2 + \sigma_{x,8}^8}{\tau_x^2} \right\} + (\sqrt{\lambda_s} + \sqrt{\lambda_o}) r_o + \sqrt{\lambda_s} \varepsilon + \frac{1}{\sqrt{s}} \|\Sigma_x^2\|_{\text{op}} \left( \sqrt{c_{\text{suc}}} r_o + \sqrt{\varepsilon} \right) + \frac{1}{\tau_x^2} \right\},
\]  
(2.19)

\[
n \geq C_x \sqrt{\lambda_s} + C_x \sqrt{n}
\]  

\[
\lambda_o \sqrt{n} \left\{ \sqrt{s} \left\{ r_{d,\delta} + \frac{\sigma_{x,2} \sigma_{x,4}^2 + \sigma_{x,8}^8}{\tau_x^2} \right\} + (\sqrt{\lambda_s} + \sqrt{\lambda_o}) r_o + \sqrt{\lambda_s} \varepsilon \right\} + \|\Sigma_x^2\|_{\text{op}} \left( \sqrt{c_{\text{suc}}} r_o + \sqrt{\varepsilon} \right) + \frac{1}{\tau_x^2} \right\},
\]  
(2.20)

and \( r \leq 1 \), where \( c_s, C_s \) and \( c_{\text{suc}} \) are sufficiently large numerical constants such that \( c_s \geq 6 \), \( C_s \geq 300 \) and \( c_{\text{suc}} \geq 1 \). Then, with probability at least \( 1 - 3 \delta \), the output of ROBUST-SPARSE-ESTIMATION \( \hat{\beta} \) satisfies

\[
\|\hat{\beta} - \beta^*\|_2 \leq r.
\]  
(2.21)

Remark 2.2. We consider the conditions \( \text{(2.10)} \) - \( \text{(2.20)} \) and the result \( \text{(2.21)} \) in detail. For simplicity, assume that \( \max \{o/n, 1/n\} = o/n \). Let \( \tau_x = 1/(r_d^2 + r_s^2) \) and assume that, for the tuning parameters, the lower bounds of the inequalities in \( \text{(2.10)} \) - \( \text{(2.20)} \) hold and \( \tau_{\text{suc}} = \tau'_{\text{suc}} \) with \( c_{\text{suc}} = 1 \) holds. Then,

\[
\tau_x^2 \geq \max \left\{ \frac{\|\beta^*\|_1^2 \sigma_{x,8}^8 \|\Sigma_x^2\|_{\text{op}}^2}{s \lambda_o^2 n}, \left( \frac{\|\beta^*\|_1}{\lambda_o \sqrt{n}} \right)^{\frac{3}{2}}, \frac{108 \sigma_{x,4}^4 s}{\lambda_o^2}, \left( \|\beta^*\|_1 \sigma_{x,4}^4 \right)^{\frac{5}{2}}, \frac{9 \sigma_{x,8}^4}{K^2}, \frac{9 \sigma_{x,8}^4}{K^2} \right\},
\]  
(2.22)

means that, \( n \) is sufficiently large so that

\[
\max \left\{ \frac{\|\beta^*\|_1^2 \sigma_{x,8}^8 \|\Sigma_x^2\|_{\text{op}}^2}{s \lambda_o^2 n}, \left( \frac{\|\beta^*\|_1}{\lambda_o \sqrt{n}} \right)^{\frac{3}{2}}, \frac{108 \sigma_{x,4}^4 s}{\lambda_o^2}, \left( \|\beta^*\|_1 \sigma_{x,4}^4 \right)^{\frac{5}{2}}, \frac{9 \sigma_{x,8}^4}{K^2}, \frac{9 \sigma_{x,8}^4}{K^2} \right\} \left\{ \log(d/\delta) \leq \sqrt{n} \right\}.
\]  
(2.23)

holds, where we use

\[
\left( \frac{\|\beta^*\|_1}{\lambda_o \sqrt{n}} \right)^{\frac{3}{2}} \leq \max \left\{ \frac{\|\beta^*\|_1}{\lambda_o \sqrt{n}}, 1 \right\}, \left( \|\beta^*\|_1 \sigma_{x,4}^4 \right)^{\frac{5}{2}} \leq \max \left\{ \|\beta^*\|_1 \sigma_{x,4}^4, 1 \right\}
\]  
(2.24)

from Young’s inequality. In addition, assume that

\[
(\sqrt{2} \sigma_{x,4}^2 + 1 + 2 \sigma_{x,4}^2) (r_d + r_s) \sqrt{s} < (1 - \varepsilon) \|\Sigma_x^2\|_{\text{op}}
\]  
(2.25)

holds and this means \( \sqrt{s \lambda_o} = \sqrt{s \lambda_o} < \|\Sigma_x^2\|_{\text{op}} \) holds. Then, from \( 1/(1 - \varepsilon) \leq 2 \), we see that \( \text{(2.21)} \) becomes

\[
\|\hat{\beta} - \beta^*\|_2 \leq C \lambda_o \sqrt{n} \left\{ \left( \sigma_{x,2} \sigma_{x,4}^2 + \sigma_{x,8}^8 + 1 \right) (r_d + r_s) + \sigma_{x,4}^4 (r_d^2 + r_s^2) \right\} \sqrt{s} + \|\Sigma_x^2\|_{\text{op}} r_o \right\},
\]  
(2.26)

where \( C \) is a numerical constant. We see that this gives the same result as in Theorem [1].

Remark 2.3. We do not optimize the numerical constants \( C_s \) and \( c_s \) in Theorem [2].
3 Key propositions

The proof of Propositions in this section are provided in Section 4. First, we introduce our main proposition (Proposition 3.1), that is stated in a deterministic form. We see that the conditions in Proposition 3.1 are satisfied with high probability by Proposition 3.2-3.6 under Assumption 2.1.

Let

\[ r_{v,i} = \frac{x'_{v,i}}{\lambda_0 \sqrt{n}} \]

and for \( \eta \in (0,1) \),

\[ \theta = \hat{\beta} - \beta^*, \quad \theta = (\hat{\beta} - \beta^*)\eta \]

**Proposition 3.1.** Suppose that \( \{\xi_i, \tilde{X}_i, \tilde{w}_{i}^*\}_{i=1}^{n} \) and \( \lambda_0 \) satisfies (3.3) and (3.4) for any \( \eta \in (0,1) \) such that \( \|\theta\|_2 = r \leq 1 \),

\[ b_1 \|\theta\|_2^2 - r_{b,2} \|\theta\|_2 - r_{b,1} \leq \sum_{i=1}^{n} \lambda_0 \sqrt{n} \tilde{w}_{i}^* - h(r_{\beta^*,i} + \eta) \right\} \tilde{X}_i^\top \theta \eta, \]

where \( b_1 > 0, r, r_{a,2}, r_{a,1}, r_{b,2}, r_{b,1} \geq 0 \) are some numbers. Suppose that \( \lambda_s \) satisfies

\[ \lambda_s - C_s > 0, \quad \frac{\lambda_s + C_s}{\lambda_s - C_s} \leq 2, \quad \text{where} \quad C_s = \frac{r_{a,2}}{3\sqrt{s}} + r_{a,1}. \]

Then, for \( r \) such that

\[ \frac{r_{a,2} + r_{b,2} + 3(r_{a,1} + \lambda_s)\sqrt{s} + \sqrt{b_1 r_{b,1}}}{b_1} < r, \]

the output of ROBUST-SPARSE-ESTIMATION \( \hat{\beta} \) satisfies \( \|\hat{\beta} - \beta^*\|_2 \leq r \).

In the remaining part of Section 3 we introduce some propositions to prove (3.8) and (3.4) are satisfied with high probability for appropriate values of \( b_1 > 0, r_{a,1}, r_{a,2}, r_{b,1}, r_{b,2} \) under the assumptions in Theorem 2.1.

**Proposition 3.2.** Suppose that Assumption 2.1 holds, and suppose that

\[ \tau_x^2 \geq \frac{\|\beta^*\|^2_1 + \sigma_{x,8}^2 + \sigma_{x,4}^2 + \sigma_{x,8}^2}{8\lambda_0^2 n}, \quad \left( \frac{\|\beta^*\|_1}{\lambda_0 \sqrt{n}} \right)^{\frac{1}{2}} \]

holds. Then, for any \( v \in \mathbb{R}^d \), we have

\[ \left| \sum_{i=1}^{n} \frac{1}{n} \left( \xi_{s,1} \right) (x_{i,1}, v) \right| \leq \|v\|_1 \left\{ \sqrt{2\tau_d,\delta} + \frac{\sigma_{x,2}^2 + \sigma_{x,4}^2 + \sigma_{x,8}^2}{\tau_x^2} \right\} + \sqrt{\frac{s}{\tau_x^2}} \|v\|_2 \]

with probability at least \( 1 - \delta \).

**Proposition 3.3.** Suppose that (2.8) holds and COMPUTE-WEIGHT returns \( \hat{w} \). For any \( \|u\| \in \mathbb{R}^n \) such that \( \|u\|_\infty \leq c \) for a numerical constant \( c \) and for any \( v \in \mathbb{R}^d \) such that \( \|v\|_2 = r \), we have

\[ \left| \sum_{i \in \mathcal{O}} \hat{w}_{i} u_{i}^T x_{i}^T v \right| \leq \sqrt{2cr_0 \sqrt{\tau_{\text{sec}}} + \sqrt{2}cr_0 \sqrt{\lambda_s} \|v\|_1}. \]
Define $I_m$ as the index set such that $|I_m| = m$.

**Proposition 3.4.** Suppose that (2.8) holds. For any $u \in \mathbb{R}^n$ such that $\|u\|_\infty \leq c$ for a numerical constant $c$ and for any $v \in \mathbb{R}^d$ such that $\|v\|_2 = r$, we have

$$\left| \frac{1}{n} \sum_{i \in I_m} u_i \hat{x}_i^\top v \right| \leq c \sqrt{\frac{m}{n}} \|\Sigma_half\|_{\text{op}} r + c \sqrt{\frac{m}{n}} \lambda_{\Sigma} v_1. \quad (3.10)$$

**Proposition 3.5.** Suppose that Assumption 2.1 holds. Let $w_i \in \mathbb{R}$.

The following proposition is also used to prove the main theorem.

**Proposition 3.6.** Suppose that $\|w_i\|_1 \leq 3\sqrt{s} \|v\|_2$.

$$R_v = \{v \in \mathbb{R}^d \mid \|v\|_2 = r, \|v\|_1 \leq 3\sqrt{s} \|v\|_2\}. \quad (3.11)$$

Suppose that

$$\lambda_n \sqrt{n} \geq \max \left\{ \frac{16K \|\Sigma_half\|_{\text{op}}}{\lambda_{\Sigma}^2}, \frac{300K^4 \|\Sigma^2\|_{\text{op}}(\sigma^2 + 1)}{\lambda_{\Sigma}^4}, 4K^2 \|\Sigma^2\|_{\text{op}}^2 \right\},$$

$$\tau_x^2 \geq \max \left\{ \frac{108\|\beta^*\|_1 \|\Sigma^4\|_{\text{op}}^4}{\lambda_{\Sigma}^2}, \left(\|\beta^*\|_1 \|\Sigma^4\|_{\text{op}}^4\right)^{\frac{5}{3}}, \frac{9\|\Sigma^4\|_{\text{op}}^4}{K^2} \right\}. \quad (3.12)$$

Then, for any $v \in R_v$, with probability at least $1 - \delta$, we have

$$\sum_{i=1}^n \frac{\lambda_n}{\sqrt{n}} - h(x_{\lambda_{\sum}}, i) + h(x_{\lambda_{\sum}}, i) \|x_i^\top v \| \geq \frac{\lambda_n^2}{6} \|v\|_2^2 - 24\lambda_n \sqrt{n} \sqrt{s} r_d, \|v\|_2 - 18\lambda_n^2 r_n^2. \quad (3.13)$$

Let $I_{\leq}$ and $I_{\geq}$ be the sets of the indices such that $w_i < 1/(2n)$ and $w_i \geq 1/(2n)$, respectively. The following proposition is also used to prove the main theorem.

**Proposition 3.6.** Suppose $0 < \varepsilon < 1$. Then, for any $w \in \Delta^{n-1}$, we have $|I_{\leq}| \leq 2n\varepsilon.$

### 3.1 Proof of the main theorem

We confirm (3.3) - (3.6) is satisfied with probability at least $1 - 3\delta$ under the assumptions in Theorem 2.1. First, from assumptions in Theorem 2.1, we see that, from union bound, Propositions 2.1 - 3.5 hold with probability at least $1 - 3\delta$. In the remaining part of Section 3.1, Propositions 2.1 - 3.5 are assumed to hold.

#### 3.1.1 Confirmation of (3.3)

We confirm (3.3). From triangular inequality, we have

$$\sum_{i=1}^n \frac{1}{n} h(x_{\beta_{\sum}}, i) x_i^\top \theta \eta \leq \sum_{i \in I} x_i^\top \theta \eta + \sum_{i \in O} x_i^\top \theta \eta - \sum_{i \in O \cup (I \setminus I_{\leq})} \frac{1}{n} h(x_{\lambda_{\sum}}, i) x_i^\top \theta \eta.$$

$$\leq \sum_{i=1}^n \frac{1}{n} h(x_{\lambda_{\sum}}, i) x_i^\top \theta \eta + \sum_{i \in O} h(x_{\beta_{\sum}}, i) x_i^\top \theta \eta + \sum_{i \in O \cup (I \setminus I_{\leq})} \frac{1}{n} h(x_{\lambda_{\sum}}, i) x_i^\top \theta \eta. \quad (3.14)$$
We note that $|h(\cdot)| \leq 1$ and from Proposition 3.10 $|O \cup (I \cap I_c)| \leq o + 2\varepsilon n$. Therefore, from Propositions 3.2, 3.4 with $c = 1$, we have

\[
\sum_{i=1}^{n} \lambda_i \nabla_i h(r_{\beta^* + \theta_{n,i}}) \hat{X}_i^\top \theta_n \leq \left\{ \begin{array}{l} 2r_{d,\delta} + \frac{2\sigma_{x,2}\sigma_{x,4}^2 + \sigma_{x,8}^2}{\tau_x^2} + \sqrt{2}r_o \sqrt{\lambda_x} + \sqrt{\frac{o + 2\varepsilon n}{n}} \sqrt{\lambda_x} \end{array} \right\} \|\theta_n\|_1 + \left\{ \begin{array}{l} \sqrt{2}r_o \sqrt{r_{\text{suc}}} + \sqrt{\frac{o + 2\varepsilon n}{n}} \|\Sigma_\theta\|_{\text{op}} + \frac{\sqrt{\varepsilon}}{\tau_x} \right\} \|\theta_n\|_2.
\]

where (a) follows from $\tau_{\text{suc}} = c_{\text{suc}}\|\Sigma\|_{\text{op}}r^2/(1-\varepsilon)$, $1 \leq c_{\text{suc}}/(1-\varepsilon)$, $1/(1-\varepsilon) \leq 2$, and $r = \|\theta_n\|_2$.

We see that (3.10) holds with

\[
r_{a,1} = \frac{\lambda_o}{\sqrt{n}} \sqrt{\left\{ \begin{array}{l} 2r_{d,\delta} + \frac{2\sigma_{x,2}\sigma_{x,4}^2 + \sigma_{x,8}^2}{\tau_x^2} + \left( \sqrt{\lambda_x^2 + \lambda_x^2} \right) + \left( \sqrt{\lambda_x^2 + \lambda_x^2} \right) \varepsilon \end{array} \right\}},
\]

\[
r_{a,2} = \frac{\lambda_o}{\sqrt{n}} \left\{ \begin{array}{l} 3\sqrt{c_{\text{suc}}r_o + \sqrt{\varepsilon}} \right\} \|\Sigma_\theta^2\|_{\text{op}} + \frac{\sqrt{\varepsilon}}{\tau_x^2}.
\]

(3.16)

3.1.2 Confirmation of (3.16)

From (3.10), we see

\[
C_s = \frac{r_{a,2}}{3\sqrt{\delta}} + r_{a,1}
\]

\[
\leq \frac{\lambda_o}{\sqrt{n}} \sqrt{\left\{ \begin{array}{l} 3\sqrt{c_{\text{suc}}r_o + \sqrt{\varepsilon}} \right\} \|\Sigma_\theta^2\|_{\text{op}} + \frac{\sqrt{\varepsilon}}{\tau_x^2} + \left( \sqrt{\lambda_x^2 + \lambda_x^2} \right) + \left( \sqrt{\lambda_x^2 + \lambda_x^2} \right) \varepsilon \end{array} \right\}},
\]

\[
\leq 2\lambda_o \sqrt{\left\{ \begin{array}{l} 3\sqrt{c_{\text{suc}}r_o + \sqrt{\varepsilon}} \right\} \|\Sigma_\theta^2\|_{\text{op}} + \frac{\sqrt{\varepsilon}}{\tau_x^2} + \left( \sqrt{\lambda_x^2 + \lambda_x^2} \right) + \left( \sqrt{\lambda_x^2 + \lambda_x^2} \right) \varepsilon \end{array} \right\}}.
\]

(3.17)

Therefore, we see, for a sufficiently large constant $c_s$ such that $c_s \geq 6$, (3.10) holds. Then we have $\|\theta_n\|_1 \leq 3\sqrt{\delta}\|\theta_n\|_2$, that is proved by Proposition 4.2.

3.1.3 Confirmation of (3.14)

We confirm (3.14). From a similar calculation in Section 3.1.1, we have

\[
\sum_{i=1}^{n} \lambda_i \nabla_i h(r_{\beta^* + \theta_{n,i}}) \hat{X}_i^\top \theta_n \leq \left\{ \begin{array}{l} \sum_{i=1}^{n} \lambda_i \nabla_i \left\{ -h(r_{\beta^* + \theta_{n,i}}) + h(r_{\beta^* + \theta_{n,i}}) \right\} \hat{X}_i^\top \theta_n \right\} \|\theta_n\|_1 + \left\{ \begin{array}{l} \sum_{i=1}^{n} \lambda_i \nabla_i \left\{ -h(r_{\beta^* + \theta_{n,i}}) + h(r_{\beta^* + \theta_{n,i}}) \right\} \hat{X}_i^\top \theta_n \right\} \|\theta_n\|_2.
\]

(3.18)
Again, we note that \(|h(\cdot)| \leq 1\) and from Proposition 3.6 |O \cup (I \cap I_c)| \leq o + 2\varepsilon n, and we remember \(\| \theta \|_1 \leq 3\sqrt{s} \| \theta \|_2\) holds. Therefore, from Proposition 3.3-3.5 with \(c = 2\), we see that 3.24 holds with
\[
\begin{align*}
\nonumber
b_1 &= \frac{\lambda_2^2}{6}, \\
\nonumber
r_{b,1} &= 18\lambda_2^2 n^2, \\
\nonumber
r_{b,2} &= 24\lambda_0 \sqrt{\frac{\tau}{n}} \left\{ \sqrt{r_d, \delta} + \left( \sqrt{c_{suc}} \| \Sigma \|_{op} + \sqrt{s\lambda_s} \right) r_o + \left( \sqrt{s\lambda_s} + \| \Sigma \|_{op} \right) \sqrt{\varepsilon} + \sqrt{\frac{\tau}{\tau^2}} \right\}.
\end{align*}
\]

3.1.4 Confirmation of 3.16

From 3.16, 3.19, and from the fact that \(\lambda_0 \sqrt{n} \geq 1\), we see that
\[
\begin{align*}
\nonumber
r_{b,2} + C_{\lambda_0} + \sqrt{b_1 r_{b,1}} \\
\leq \frac{6}{\lambda_2^2} \left( r_{b,2} + r_{a,2} + 3(r_{a,1} + \lambda_0) \sqrt{s} + \sqrt{b_1 r_{b,1}} \right) \\
< 300 \lambda_2^2 \times \nonumber
\lambda_0 \sqrt{n} \left[ \sqrt{s} \left( r_{d, \delta} + \frac{\sigma_{x, \delta} \sigma_{\delta, 4} + \sigma_{x, \delta}}{r_s^4} \right) + \left( \sqrt{s\lambda_s} + \sqrt{s\lambda_s} r_o + \sqrt{s\lambda_s} \varepsilon \right) + \| \Sigma \|_{op} \left( \sqrt{c_{suc} r_o + \varepsilon} \right) + \frac{\sqrt{s}}{\tau^2} + \sqrt{s\lambda_s} \right],
\end{align*}
\]
and 3.16 holds for a sufficiently large constant \(C_s\) such that \(C_s \geq 300\), and the proof is complete.

4 Proofs

4.1 Proof of Proposition 2.1

Proof. We note that this proof is almost the same one of Lemma 2 of [94]. For any \(M \in \mathfrak{M}_r\), we have
\[
\begin{align*}
\frac{1}{n} \sum_{i=1}^n \langle \hat{x}_i \hat{x}_i^T, M \rangle = \frac{1}{n} \sum_{i=1}^n \langle \hat{x}_i \hat{x}_i^T, M \rangle - \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \langle \hat{x}_i \hat{x}_i^T, M \rangle \right] + \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \langle \hat{x}_i \hat{x}_i^T, M \rangle \right].
\end{align*}
\]

First, we evaluate \(T_1\). We note that, for any \(1 \leq j_1, j_2 \leq d\),
\[
\begin{align*}
\nonumber
\mathbb{E} \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2}^2 + \mathbb{E} \hat{x}_{i_2 j_2} \hat{x}_{i_1 j_1}^2 \leq \mathbb{E} \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2}^2 \leq \mathbb{E} \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2}^2 \leq \tau_s^{2(p-2)} \mathbb{E} \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2}^2 \leq \tau_s^{2(p-2)} \sigma_{x, 4}^4.
\end{align*}
\]

From Bernstein’s inequality (Lemma 5.1 of [94]), we have
\[
\begin{align*}
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2} - \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2} \right] \geq \sigma_{x, 4} \sqrt{\frac{2}{n} + \frac{\tau_s^2}{n}} \right\} \leq e^{-t}.
\end{align*}
\]

From the union bound, we have
\[
\begin{align*}
\mathbb{P} \left\{ \left\| \frac{1}{n} \sum_{i=1}^n \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2} - \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \hat{x}_{i_1 j_1} \hat{x}_{i_2 j_2} \right] \right\|_{\infty} \leq \sqrt{2\sigma_{x, 4}^2 (r_d + r_3)} + \frac{\tau_s^2 (r_d + r_3)}{n} \right\} \geq 1 - \delta.
\end{align*}
\]
From Hölder’s inequality, we have
\[
\mathbb{P}\left\{ T_1 \leq \sqrt{2} \sigma_{x,4}^2 (r_d + r_3) + \tau_x^2 (r_d^2 + r_3^2) \|M\|_1 \right\} \geq 1 - \delta. \tag{4.5}
\]
Next, we evaluate \( \mathbb{E}\langle \tilde{x}, \tilde{x}^\top, M \rangle \). We have
\[
\mathbb{E}\langle \tilde{x}, \tilde{x}^\top, M \rangle = \mathbb{E}\langle \tilde{x}, \tilde{x}^\top - \Sigma, M \rangle + \mathbb{E}\langle \Sigma, M \rangle. \tag{4.6}
\]
From Hölder’s inequality and the positive semi-definiteness of \( M \), we have
\[
\mathbb{E}\langle \Sigma, M \rangle \leq \|\Sigma\|_{op}\|M\|_* = \|\Sigma\|_{op}\text{Tr}(M). \tag{4.7}
\]
For any \( 1 \leq j_1, j_2 \leq d \), we have
\[
\mathbb{E}\hat{x}_{ij_1} \hat{x}_{ij_2} - \mathbb{E}x_{ij_1}x_{ij_2} \leq \mathbb{E}|x_{ij_1}x_{ij_2} (I_{(x_{ij_1} \geq \tau_x)} + I_{(x_{ij_2} \geq \tau_x)})|
\leq \sqrt{\mathbb{E}x_{ij_1}^2 x_{ij_2}^2} \left\{ \sqrt{\mathbb{P}(|x_{ij_1}| \geq \tau_x)} + \sqrt{\mathbb{P}(|x_{ij_2}| \geq \tau_x)} \right\}
\leq \sqrt{\mathbb{E}x_{ij_1}^2 x_{ij_2}^2} \left\{ \sqrt{\frac{x_{ij_1}}{\tau_x^2}} + \sqrt{\frac{x_{ij_2}}{\tau_x^2}} \right\}
\leq 2\frac{\sigma_{x,4}^2}{\tau_x^2} \tag{4.8}
\]
and from Hölder’s inequality, we have
\[
\mathbb{E}\langle \tilde{x}, \tilde{x}^\top - \Sigma, M \rangle \leq 2\frac{\sigma_{x,4}^2}{\tau_x^2} \|M\|_1. \tag{4.9}
\]
Finally, combining the arguments above, with probability at least \( 1 - \delta \), we have
\[
\left| \frac{1}{n} \sum_{i=1}^n \langle \tilde{x}_i, \tilde{x}_i^\top, M \rangle \right| \leq \sqrt{2\sigma_{x,4}^2 (r_d + r_3) + \tau_x^2 (r_d^2 + r_3^2) + 2\frac{\sigma_{x,4}^2}{\tau_x^2}} \|M\|_1 + \|\Sigma\|_{op} r^2 \tag{4.10}
\]
and the proof is complete. \( \square \)

### 4.2 Proof of Proposition 3.1

We show that \( \|\theta\|_1 \leq 3\sqrt{s}\|\theta\|_2 \) holds under the assumptions of Proposition 3.1. First, we prove the following proposition.

**Proposition 4.1.** Consider the output of Algorithm 7. Suppose that, for any \( \eta \in (0, 1) \),
\[
\left| \lambda_0 \sqrt{n} \sum_{i=1}^n \tilde{w}_i^\top h(r_{J_0, i}) \tilde{X}_i^\top \theta_\eta \right| \leq r_{a, 2} \|\theta_\eta\|_2 + r_{a, 1} \|\theta_\eta\|_1, \tag{4.11}
\]
where \( r_{a, 2}, r_{a, 1} \geq 0 \) are some numbers. Suppose that \( \lambda_a \) satisfy
\[
\lambda_a - C_s > 0, \quad \frac{\lambda_a + C_s}{\lambda_a - C_s} \leq 2, \quad \text{where } C_s = r_{a, 2}/\sqrt{s} + r_{a, 1}. \tag{4.12}
\]
Suppose that \( \|\theta_\eta\|_2 \leq \|\theta_\eta\|_1/\sqrt{s} \). Then, we have
\[
\|\theta_\eta, J_0^*\|_1 \leq \frac{\lambda_s + C_s}{\lambda_s - C_s} \|\theta_\eta, J_0^*\|_1 \left( \leq 2\|\theta_\eta, J_0^*\|_1 \right), \tag{4.13}
\]
where \( J_0^* \) is the index set of the non-zero entries of \( a \).
Proof. Let

\[ Q'(\eta) = \lambda_0 \sqrt{n} \omega_i' \sum_{i=1}^{n} \{ -h(r_{\beta^* + \theta_{\eta,.i}}) + h(r_{\beta_{\eta,.i}}) \} (X_i, \theta). \]  

(4.14)

From the proof of Lemma F.2. of [33], we have \( \eta Q'(\eta) \leq \eta Q'(1) \) and this means

\[ \sum_{i=1}^{n} \lambda_0 \sqrt{n} \omega_i' \{ -h(r_{\beta^* + \theta_{\eta,.i}}) + h(r_{\beta_{\eta,.i}}) \} \hat{X}_i^\top \theta_{\eta} \leq \sum_{i=1}^{n} \lambda_0 \sqrt{n} \omega_i' \eta \left\{ -h(r_{\beta_{\eta,.i}}) + h(r_{\beta_{\eta,.i}}) \right\} \hat{X}_i^\top \theta. \]  

(4.15)

Let \( \partial \nu \) be the sub-differential of \( ||v||_1 \). Adding \( \eta \lambda_s(\|\hat{\beta}\|_1 - \|\beta^*\|_1) \) to both sides of (4.15), we have

\[ \sum_{i=1}^{n} \lambda_0 \sqrt{n} \omega_i' \{ -h(r_{\beta^* + \theta_{\eta,.i}}) + h(r_{\beta_{\eta,.i}}) \} \hat{X}_i^\top \theta_{\eta} + \eta \lambda_s(\|\hat{\beta}\|_1 - \|\beta^*\|_1) \leq \sum_{i=1}^{n} \lambda_0 \sqrt{n} \omega_i' h(r_{\beta_{\eta,.i}}) \tilde{X}_i^\top \theta_{\eta}. \]  

(4.16)

where (a) follows from \( \|\hat{\beta}\|_1 - \|\beta^*\|_1 \leq \langle \partial \hat{\beta}, \theta \rangle \), which is the definition of the sub-differential, and (b) follows from the optimality of \( \hat{\beta} \).

From the convexity of the Huber loss, the left-hand side (L.H.S) of (4.16) is positive and we have

\[ 0 \leq \sum_{i=1}^{n} \lambda_0 \sqrt{n} \omega_i' h(r_{\beta_{\eta,.i}}) \tilde{X}_i^\top \theta_{\eta} + \eta \lambda_s(\|\beta^*\|_1 - \|\hat{\beta}\|_1). \]  

(4.17)

From (4.11), the first term of the right-hand side (R.H.S) of (4.17) is evaluated as

\[ \sum_{i=1}^{n} \lambda_0 \sqrt{n} \omega_i' h(r_{\beta_{\eta,.i}}) \tilde{X}_i^\top \theta_{\eta} \leq r_{a,2} ||\theta_{\eta}||_2 + r_{a,1} ||\theta_{\eta}||_1. \]  

(4.18)

From (4.17), (4.18) and the assumption \( ||\theta_{\eta}||_2 \leq ||\theta_{\eta}||_1 / \sqrt{s} \), we have

\[ 0 \leq r_{a,2} ||\theta_{\eta}||_2 + r_{a,1} ||\theta_{\eta}||_1 + \eta \lambda_s(\|\beta^*\|_1 - \|\hat{\beta}\|_1) \leq C_s ||\theta_{\eta}||_1 + \eta \lambda_s(\|\beta^*\|_1 - \|\hat{\beta}\|_1). \]  

(4.19)

Furthermore, we see

\[ 0 \leq C_s ||\theta_{\eta}||_1 + \eta \lambda_s(\|\beta^*\|_1 - \|\hat{\beta}\|_1) \leq C_s( ||\theta_{\eta}, J_{\beta^*}, ||_1 + ||\theta_{\eta}, J_{\hat{\beta}}, ||_1 ) + \eta \lambda_s(\|\beta^*_{\hat{\beta}} - \hat{\beta}_{\hat{\beta}}\|_1 - \|\hat{\beta} - J_{\hat{\beta}}||_1) \]  

(4.20)

and the proof is complete. \( \square \)

From Proposition 4.1, we can easily prove the following proposition, which reveals a relation between \( ||\theta_{\eta}||_1 \) and \( ||\theta_{\eta}||_2 \).

**Proposition 4.2.** Suppose the conditions used in Proposition 4.1. Then, we have

\[ ||\theta_{\eta}||_1 \leq 3 \sqrt{s} ||\theta_{\eta}||_2. \]  

(4.21)

**Proof.** When \( ||\theta_{\eta}||_1 < \sqrt{s} ||\theta_{\eta}||_2 \), we obtain (4.21) immediately. When \( ||\theta_{\eta}||_1 \geq \sqrt{s} ||\theta_{\eta}||_2 \), from Proposition 4.1 we see that \( \theta_{\eta} \) satisfies \( ||\theta_{\eta}, J_{\beta^*}||_1 \leq 2 ||\theta_{\eta}, J_{\hat{\beta}}||_1 \leq 2 ||\theta_{\eta}, J_{\hat{\beta}}||_1 \). From this, we have

\[ ||\theta_{\eta}||_1 = ||\theta_{\eta}, J_{\beta^*}||_1 + ||\theta_{\eta}, J_{\hat{\beta}}||_1 \leq (2 + 1) ||\theta_{\eta}, J_{\hat{\beta}}||_1 \leq 3 \sqrt{s} ||\theta_{\eta}||_2, \]  

(4.22)

and the proof is complete. \( \square \)
4.2.1 Proving Proposition 4.3

In Section 4.2.1 we prove Proposition 4.3. We note that by combining Propositions 4.1, 4.2 and 4.3, we see the fact that Proposition 4.3 holds.

Proposition 4.3. Assume all the conditions used in Proposition 4.1. Suppose that, for any \( \eta \in (0, 1) \),

\[
b_1 \| \theta_{\eta} \|_2^2 - r_{b, 2} \| \theta_{\eta} \|_2 - r_{b, 1} \leq \sum_{i=1}^{n} \lambda_a \sqrt{n} \tilde{w}_i \{ -h(r_{\beta^* + \theta_{a, i}}) + h(r_{\beta^* + \theta_{a, i}}) \} \hat{X}_i^\top \theta_{\eta},
\]

(4.23)

where \( b_1 > 0 \), \( r_{b, c}, r_{b, 1} \geq 0 \) are some numbers. Suppose that

\[
\frac{r_{b, 2} + C_{\lambda_s} + \sqrt{b_1} r_{b, 1}}{b_1} < r, \text{ where } C_{\lambda_s} = r_{a, 2} + 3(r_{a, 1} + \lambda_s) \sqrt{s}.
\]

(4.24)

Then, the output of ROBUST-SPARSE-ESTIMATION \( \hat{\beta} \) satisfies \( \| \hat{\beta} - \beta^* \|_2 \leq r. \)

Proof. We prove Proposition 4.3 in a manner similar to the proof of Lemma B.7 in [33] and the proof of Theorem 2.1 in [9]. For fixed \( r > 0 \), define

\[
\mathbb{B}(r) := \{ \beta : \| \beta - \beta^* \|_2 \leq r \}.
\]

(4.25)

We prove \( \hat{\beta} \in \mathbb{B}(r) \) by assuming \( \hat{\beta} \notin \mathbb{B}(r) \) and deriving a contradiction. For \( \hat{\beta} \notin \mathbb{B}(r) \), we can find some \( \eta \in (0, 1) \) such that \( \| \theta_{\eta} \|_2 = r \). From (4.11), we have

\[
\sum_{i=1}^{n} \lambda_a \sqrt{n} \tilde{w}_i \{ -h(r_{\beta^* + \theta_{a, i}}) + h(r_{\beta^* + \theta_{a, i}}) \} \hat{X}_i \theta_{\eta} \leq \sum_{i=1}^{n} \lambda_a \sqrt{n} \tilde{w}_i h(r_{\beta^* + \theta_{a, i}}) \hat{X}_i \theta_{\eta} + \eta \lambda_s (\| \beta^* \|_1 - \| \hat{\beta} \|_1).
\]

(4.26)

We evaluate each term of (4.26). From (4.23), the L.H.S. of (4.26) is evaluated as

\[
b_1 \| \theta_{\eta} \|_2^2 - r_{b, 2} \| \theta_{\eta} \|_2 - r_{b, 1} \leq \sum_{i=1}^{n} \lambda_a \sqrt{n} \tilde{w}_i \{ -h(r_{\beta^* + \theta_{a, i}}) + h(r_{\beta^* + \theta_{a, i}}) \} \hat{X}_i \theta_{\eta}.
\]

(4.27)

From (4.11) and (4.21) and Proposition 4.2, the first term of the R.H.S. of (4.26) is evaluated as

\[
\sum_{i=1}^{n} \lambda_a \sqrt{n} \tilde{w}_i h(r_{\beta^* + \theta_{a, i}}) \hat{X}_i \theta_{\eta} \leq r_{a, 2} \| \theta_{\eta} \|_2 + 3 \sqrt{r_{a, 1}} \| \theta_{\eta} \|_2.
\]

(4.28)

From (4.21) and Proposition 4.2, the second term of the R.H.S. of (4.26) is evaluated as

\[
\eta \lambda_s (\| \beta^* \|_1 - \| \hat{\beta} \|_1) \leq \lambda_s \| \theta_{\eta} \|_1 \leq 3 \lambda_s \sqrt{s} \| \theta_{\eta} \|_2.
\]

(4.29)

Combining the two inequalities above with (4.26), we have

\[
b_1 \| \theta_{\eta} \|_2^2 - r_{b, 2} \| \theta_{\eta} \|_2 - r_{b, 1} \leq \{ r_{a, 2} + 3(r_{a, 1} + \lambda_s) \sqrt{s} \} \| \theta_{\eta} \|_2.
\]

(4.30)

From (4.30), \( \sqrt{A + B} \leq \sqrt{A} + \sqrt{B} \) for \( A, B > 0 \), we have

\[
\| \theta_{\eta} \|_2 \leq \frac{r_{a, 2} + r_{b, 2} + 3(r_{a, 1} + \lambda_s) \sqrt{s} + C_{\lambda_s} + \sqrt{b_1} r_{b, 1}}{b_1} < r.
\]

(4.31)

This is in contradiction to \( \| \theta_{\eta} \|_2 = r \). Consequently, we have \( \hat{\beta} \in \mathbb{B}(r_1) \) and \( \| \theta \|_2 < r. \)

\[\square\]
4.3 Proofs of the propositions in Section 3

4.3.1 Proof of proposition 3.6

Proof. We assume $|I_{<}| > 2\varepsilon n$, and then we derive a contradiction. From the constraint about $w_i$, we have $0 \leq w_i \leq \frac{1}{1 - \varepsilon n}$ for any $i \in \{1, \cdots, n\}$ and we have

\[
\sum_{i=1}^{n} w_i = \sum_{i \in I_{<}} w_i + \sum_{i \in I_{\geq}} w_i \\
\leq |I_{<}| \times \frac{1}{2n} + (n - |I_{<}|) \times \frac{1}{(1 - \varepsilon) n} \\
= 2\varepsilon n \times \frac{1}{2n} + (|I_{<}| - 2\varepsilon n) \times \frac{1 + \varepsilon}{2n} + (n - 2\varepsilon n) \times \frac{1}{(1 - \varepsilon) n} + (2\varepsilon n - |I_{<}|) \times \frac{1}{(1 - \varepsilon) n} \\
= \varepsilon + (n - 2\varepsilon n) \times \frac{1}{(1 - \varepsilon) n} + (|I_{<}| - 2\varepsilon n) \times \left( \frac{1}{2n} - \frac{1}{(1 - \varepsilon) n} \right) \\
< \varepsilon + \frac{n - 2\varepsilon n}{(1 - \varepsilon) n} \\
= \frac{1 - 2\varepsilon}{1 - \varepsilon} \\
\leq \frac{1 - \varepsilon - \varepsilon^2}{1 - \varepsilon} \\
< 1. \quad (4.32)
\]

This is in contradiction to $\sum_{i=1}^{n} w_i = 1$. Then, combining the assumption that, we have $|I_{<}| \leq 2\varepsilon n$. \hfill \square

4.3.2 Proof of proposition 3.2

Proof. This proof is similar to the proof of Lemma 1 of [34]. For any $v \in \mathbb{R}^d$, from (iii) of Assumption 2.1, we have

\[
\sum_{i=1}^{n} h(\xi_{\lambda, i}) \langle \tilde{x}_i, v \rangle = \sum_{i=1}^{n} \frac{h(\xi_{\lambda, i}) \langle \tilde{x}_i, v \rangle}{\lambda \sqrt{n}} - \mathbb{E} h(\xi_{\lambda, i}) \langle \tilde{x}_i, v \rangle \\
+ \mathbb{E} h(\xi_{\lambda, i}) \langle \tilde{x}_i, v \rangle - \mathbb{E} h \left( \frac{\xi_i}{\lambda \sqrt{n}} \right) \langle \tilde{x}_i, v \rangle \\
+ \mathbb{E} h \left( \frac{\xi_i}{\lambda \sqrt{n}} \right) \langle \tilde{x}_i, v \rangle - \mathbb{E} h \left( \frac{\xi_i}{\lambda \sqrt{n}} \right) \langle x_i, v \rangle. \quad (4.33)
\]

First, we evaluate $T_2$. We note that

\[
\mathbb{E} \left\{ h(\xi_{\lambda, i}) \tilde{x}_i \right\}^2 \leq \mathbb{E} h(\xi_{\lambda, i})^2 \tilde{x}_i^2 \leq \mathbb{E} x_i \leq \sigma^2_{\tilde{x}, 2},
\]

\[
\mathbb{E} \left\{ h(\xi_{\lambda, i}) \tilde{x}_i \right\}^p \leq \mathbb{E} h(\xi_{\lambda, i})^p \tilde{x}_i^p \leq \tau^p_{\tilde{x}} \sigma^2_{\tilde{x}, 2}. \quad (4.34)
\]

From Bernstein’s inequality (Lemma 5.1 of [29]), we have

\[
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^{n} h(\xi_{\lambda, i}) \tilde{x}_{ij} - \mathbb{E} \sum_{i=1}^{n} \frac{1}{n} h(\xi_{\lambda, i}) \tilde{x}_{ij} \geq \sigma_{\tilde{x}, 2} \sqrt{2 \frac{t}{n} + \tau x t} \right\} \leq e^{-t}. \quad (4.35)
\]
From the union bound, we have
\[
\Pr\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} h(\xi_{\lambda, i}) \bar{x}_i - \mathbb{E} h(\xi_{\lambda, i}) \bar{x}_i \right\|_{\infty} \leq \sqrt{2} \sigma_{x, 2}(r_d + r_\delta) + \tau_x (r_d^2 + r_\delta^2) \right\} \geq 1 - \delta. \tag{4.36}
\]

From H"older’s inequality, we have
\[
\Pr \left[ T_2 \leq \left\{ \sqrt{2} \sigma_{x, 2}(r_d + r_\delta) + \tau_x (r_d^2 + r_\delta^2) \right\} \| v \|_1 \right] \geq 1 - \delta. \tag{4.37}
\]

Second, we evaluate \( T_3 \).
\[
T_3 = \mathbb{E} (\bar{x}_i, v) \left\{ h(\xi_{\lambda, i}) - h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\}
\overset{(a)}{=} \mathbb{E} (\bar{x}_i, v) \left\{ h' \left( t \xi_{\lambda, i} + (1 - t) \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) \right\} \times \frac{(\bar{x}_i - \xi_i)^\top \beta^*}{\lambda_0 \sqrt{n}}
\leq \frac{1}{\lambda_0 \sqrt{n}} \sqrt{\mathbb{E} \| (\bar{x}_i - \xi_i)^\top \beta^* \|_{2}^2} \sqrt{\| (\bar{x}_i, v) \|^2}, \tag{4.38}
\]

where (a) follows from the mean-valued theorem defining \( h' \) as the differential of \( h \) and \( t \in (0, 1) \).
We note that, for any \( 1 \leq j_1, j_2 \leq d \), we have
\[
\mathbb{E} (x_{ij_1} - \bar{x}_{ij_1}) (x_{ij_2} - \bar{x}_{ij_2}) \leq \mathbb{E} |x_{ij_1} \{ |x_{ij_1}| \geq \tau_x \}| |x_{ij_2} \{ |x_{ij_2}| \geq \tau_x \}| \leq (\mathbb{E} x_{ij_1}^4) \frac{1}{4} \mathbb{E} \{ I_{|x_{ij_1}| \geq \tau_x} \} \leq \sigma_{x, 8}^8 \tau_x^4, \tag{4.39}
\]

and we have
\[
\sqrt{\mathbb{E} \| (\bar{x}_i - \xi_i)^\top \beta^* \|_{2}^2} \leq \| \beta^* \|_{1} \frac{\sigma_{x, 8}^8 \tau_x^4}{\tau_x^4}. \tag{4.40}
\]

Additionally, we note that,
\[
\sqrt{\| (\bar{x}_i, v) \|^2} \leq \sqrt{\| (\bar{x}_i - \xi_i, v) \|^2} + \sqrt{\| (\bar{x}_i, v) \|^2} \leq \sqrt{\| (\bar{x}_i - \xi_i, v) \|^2} + \| \Sigma_{\beta}^2 \|_{op} \| v \|_2. \tag{4.41}
\]

From (4.38), (4.40) and (4.41), we have
\[
T_3 \leq \frac{1}{\lambda_0 \sqrt{n}} \| \beta^* \|_{1} \frac{\sigma_{x, 8}^8 \tau_x^4}{\tau_x^4} \left( \frac{\sigma_{x, 8}^4}{\tau_x^2} \| v \|_{1} + \| \Sigma_{\beta}^2 \|_{op} \| v \|_2 \right)
\overset{(a)}{=} \frac{\sigma_{x, 8}^8 \tau_x^4}{\tau_x^4} \| v \|_1 + \frac{\sqrt{\sigma_{x, 8}^8 \tau_x^4}}{\tau_x^2} \| v \|_2, \tag{4.42}
\]

where (a) follows from the assumption on \( \tau_x \). Lastly, we evaluate \( T_4 \). For \( 1 \leq j \leq d \), we have
\[
\mathbb{E} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) (\bar{x}_{ij} - x_{ij}) \leq \mathbb{E} h \left( \frac{\xi_i}{\lambda_0 \sqrt{n}} \right) |x_{ij} \cdot I_{|x_{ij}| \geq \tau_x}
\leq \sqrt{\mathbb{E} x_{ij}^2 \mathbb{E} I_{|x_{ij}| \geq \tau_x}}
\overset{(a)}{=} \sqrt{\mathbb{E} x_{ij}^2 \mathbb{P}(|x_{ij}| \geq \tau_x)}
\leq \frac{\sigma_{x, 2} \sigma_{x, 4}^2}{\tau_x^2}. \tag{4.43}
\]

From H"older’s inequality, we have
\[
T_4 \leq \frac{\sigma_{x, 2} \sigma_{x, 4}^2}{\tau_x^2} \| v \|_1. \tag{4.44}
\]
Combining the arguments above, we have
\[
\left| \sum_{i=1}^{n} \frac{h(\xi_{\lambda_{i}})(\mathbf{x}_i, \mathbf{v})}{n} \right| \leq \|\mathbf{v}\|_1 \left\{ \sqrt{2\sigma_{\mathbf{x},2}(r_{d} + r_{\delta})} + \tau_{\mathbf{x}}(r_{d} + r_{\delta}^2) + \frac{\sigma_{\mathbf{x},2}\sigma_{\mathbf{x},4}^2 + \sigma_{\mathbf{x},8}}{\tau_{\mathbf{x}}^2} \right\} + \sqrt{\frac{\nu}{\tau_{\mathbf{x}}}} \|\mathbf{v}\|_2
\]  
(4.45)
with probability at least 1 − δ, and the proof is complete. □

Define \( \mathcal{M}_{r,r} = \{M \in \mathbb{R}^{d \times d} : M = \mathbf{v}\mathbf{v}^\top, \|\mathbf{v}\|_2 = r\} \).

### 4.3.3 Proof of Proposition 3.3

**Proof.** We note that
\[
\left| \sum_{i \in \mathcal{O}} \hat{w}_i' u_i \mathbf{x}_i^\top \mathbf{v} \right| \leq c^2 \sum_{i \in \mathcal{O}} \hat{w}_i' |\mathbf{x}_i^\top \mathbf{v}|^2 \leq 2 \sum_{i \in \mathcal{O}} \hat{w}_i |\mathbf{x}_i^\top \mathbf{v}|^2,
\]  
(4.46)
where (a) follows from Hölder’s inequality and \( \|\mathbf{u}\|_{\infty} \leq c \) and \( |u_i'| \leq 1/n \), and (b) follows from the fact that \( \hat{w}_i' \leq 2\hat{w}_i \) for any \( i \in (1, \cdots, n) \). We focus on \( \sum_{i \in \mathcal{O}} \hat{w}_i |\mathbf{x}_i^\top \mathbf{v}|^2 \). For any \( \mathbf{v} \in \mathbb{R}^d \) such that \( \|\mathbf{v}\|_2 = r \),
\[
\sum_{i=1}^{n} \hat{w}_i (\mathbf{x}_i^\top \mathbf{v})^2 = \sum_{i=1}^{n} \hat{w}_i (\mathbf{x}_i^\top \mathbf{v})^2 - \lambda_* \|\mathbf{v}\|_1^2 + \lambda_* \|\mathbf{v}\|_1^2
\]  
\[
\leq \sup_{M \in \mathcal{M}_r} \left\{ \sum_{i=1}^{n} \hat{w}_i \left( \mathbf{x}_i^\top \mathbf{v}, M \right) - \lambda_* \|M\|_1 \right\} + \lambda_* \|\mathbf{v}\|_1^2
\]  
\[
\leq \tau_{\text{suc}} + \lambda_* \|\mathbf{v}\|_1^2,
\]  
(4.47)
where (a) follows from the fact that \( \mathcal{M}_{r,r} \subset \mathcal{M}_r \), and (b) follows from (4.10) and \( \tau'_{\text{suc}} \leq \tau_{\text{suc}} \). Combining the arguments above and from triangular inequality, we have
\[
\sum_{i \in \mathcal{O}} \hat{w}_i' u_i \mathbf{x}_i^\top \mathbf{v} \leq \sqrt{2} c r_o \sqrt{\tau_{\text{suc}}} + \sqrt{2} c r_o \sqrt{\lambda_*} \|\mathbf{v}\|_1,
\]  
(4.48)
and the proof is complete. □

### 4.3.4 Proof of Proposition 3.4

**Proof.** We note that, from Hölder’s inequality, for any \( \mathbf{v} \in \mathbb{R}^d \) such that \( \|\mathbf{v}\|_2 = r \), we have
\[
\left| \sum_{i \in \mathcal{I}_m} \frac{u_i \mathbf{x}_i^\top \mathbf{v}}{n} \right|^2 \leq \sum_{i \in \mathcal{I}_m} \frac{1}{n} u_i^2 \sum_{i \in \mathcal{I}_m} \frac{1}{n} (\mathbf{x}_i^\top \mathbf{v})^2 \leq c^2 n \sum_{i=1}^{n} \frac{1}{n} (\mathbf{x}_i^\top \mathbf{v})^2.
\]  
(4.49)
From the fact that \( \mathcal{M}_{r,r} \subset \mathcal{M}_r \), we have
\[
\sum_{i=1}^{n} (\mathbf{x}_i^\top \mathbf{v})^2 = \sum_{i=1}^{n} (\mathbf{x}_i^\top \mathbf{v})^2 - \lambda'_* \|\mathbf{v}\|_1^2 + \lambda'_* \|\mathbf{v}\|_1^2 \leq \sup_{M \in \mathcal{M}_r} \left( \sum_{i=1}^{n} \frac{\langle \mathbf{x}_i^\top \mathbf{x}_i^\top, M \rangle}{n} - \lambda'_* \|M\|_1 \right) + \lambda'_* \|\mathbf{v}\|_1^2.
\]  
(4.50)
From Proposition (2.11) and the definition of \( \lambda'_* \), we have
\[
\sup_{M \in \mathcal{M}_r} \left( \sum_{i=1}^{n} \frac{\langle \mathbf{x}_i^\top \mathbf{x}_i^\top, M \rangle}{n} - \lambda'_* \|M\|_1 \right) \leq \|\Sigma\|_{\text{op}}^2.
\]  
(4.51)
Combining the arguments above and from triangular inequality, we have
\[ \sum_{i \in I_m} \hat{w}_i u_i x_i ^\top v \leq c \sqrt{\frac{m}{n} \| \Sigma \|_{op}} + c \frac{m}{n} \sqrt{\lambda_v} \| v \|_1, \] (4.52)
and the proof is complete. \( \square \)

4.3.5 Proof of Proposition 3.5

Proof. This proposition is proved in a manner similar to the proof of Proposition B.1 of [9]. The L.H.S of (3.13) divided by \( \lambda_v \) can be expressed as
\[ \sum_{i=1}^{n} \{-h(\xi_{\lambda_v,i} - \bar{x}_{v,i}) + h(\xi_{\lambda_v,i})\} \bar{x}_{v,i}. \] (4.53)
From the convexity of the Huber loss, we have
\[ \sum_{i=1}^{n} \{-h(\xi_{\lambda_v,i} - \bar{x}_{v,i}) + h(\xi_{\lambda_v,i})\} \bar{x}_{v,i} \geq \sum_{i=1}^{n} \{-h(\xi_{\lambda_v,i} - \bar{x}_{v,i}) + h(\xi_{\lambda_v,i})\} \bar{x}_{v,i} 1_{E_i}, \] (4.54)
where \( 1_{E_i} \) is the indicator function of the event
\[ E_i := (|\xi_{\lambda_v,i}| \leq 1/2) \cap (|\bar{x}_{v,i}| \leq 1/2). \] (4.55)
Define the functions
\[ \varphi(x) = \begin{cases} x^2 & \text{if } |x| \leq 1/2 \\ (x - 1/2)^2 & \text{if } 1/2 \leq x \leq 1 \\ (x + 1/2)^2 & \text{if } -1 \leq x \leq -1/2 \\ 0 & \text{if } |x| > 1 \end{cases} \] (4.56)
Let \( f_i(v) = \varphi(\bar{x}_{v,i})\psi(\xi_{\lambda_v,i}) \) and we have
\[ \sum_{i=1}^{n} \{-h(\xi_{\lambda_v,i} - \bar{x}_{v,i}) + h(\xi_{\lambda_v,i})\} \bar{x}_{v,i} \geq \sum_{i=1}^{n} \bar{x}_{v,i}^2 1_{E_i} \geq \sum_{i=1}^{n} \varphi(\bar{x}_{v,i})\psi(\xi_{\lambda_v,i}) = \sum_{i=1}^{n} f_i(v), \] (4.57)
where (a) follows from \( \varphi(v) \geq v^2 \) for \( |v| \leq 1/2 \). We note that
\[ f_i(v) \leq \varphi(v_i) \leq \min(\bar{x}_{v,i}^2, 1). \] (4.58)
To bound \( \sum_{i=1}^{n} f_i(v) \) from below, for any fixed \( v \in R_v \), we have
\[ \sum_{i=1}^{n} f_i(v) \geq \mathbb{E} f(v) - \sup_{v' \in R_v} \left| \sum_{i=1}^{n} f_i(v') - \mathbb{E} \sum_{i=1}^{n} f_i(v') \right|. \] (4.59)
Define the supremum of a random process indexed by \( R_v \):
\[ \Delta := \sup_{v' \in R_v} \left| \sum_{i=1}^{n} f_i(v') - \mathbb{E} \sum_{i=1}^{n} f_i(v') \right|. \] (4.60)
From (4.57) and (4.58), we have
\[ \mathbb{E} \sum_{i=1}^{n} f_i(v) \geq \sum_{i=1}^{n} \mathbb{E} \bar{x}_{v,i}^2 - \sum_{i=1}^{n} \mathbb{E} \bar{x}_{v,i}^2 I(\bar{x}_{v,i} \geq 1/2) - \sum_{i=1}^{n} \mathbb{E} |\xi_{\lambda_v,i}| (|\xi_{\lambda_v,i}| \geq 1/2). \] (4.61)
We note that, from the definition of $\mathcal{R}_v$, we have
\[ \mathbb{E}(\hat{x}_i^T \nu)^2 = \mathbb{E}v^T (\hat{x}_i \hat{x}_i^T - x_i x_i^T + x_i x_i^T) v \geq -9\|\mathbb{E}(\hat{x}_i \hat{x}_i^T - x_i x_i^T)\|_\infty \|v\|_2^2 + \|\Sigma \hat{v}\|_2^2 \] (4.62)
and from (4.8) and $\|\Sigma \hat{v}\|_2^2 \geq \lambda_0^2 \|v\|_2^2$, we have
\[ -18 \frac{\sigma^2_s \|v\|_2^2}{\tau^2_x} + \lambda_0^2 \|v\|_2^2 \leq \mathbb{E}(\hat{x}_i^T \nu)^2. \] (4.63)

We note that
\[ \frac{\mathbb{E}(\hat{x}_i^T \nu)^4}{8} \leq \mathbb{E}\{(\hat{x}_i - x_i + x_i)^T \nu\}^4, \]
\[ \leq \mathbb{E}\{(\hat{x}_i - x_i)^T \nu\}^4 + \mathbb{E}(\tilde{x}_i^T \nu)^4 \]
\[ \leq \mathbb{E}\{(\hat{x}_i - x_i)^T \nu\}^4 + K^4 \mathbb{E}(\tilde{x}_i^T \nu)^2 \]
\[ \leq \mathbb{E}\{(\hat{x}_i - x_i)^T \nu\}^4 + K^4 \mathbb{E}\Sigma \hat{v}\|_\text{op}\|v\|_2^2, \] (4.64)
and, for any $1 \leq j_1, j_2, j_3, j_4 \leq d$, we have
\[ \mathbb{E}(x_{ij_1} - \hat{x}_{ij_1})(x_{ij_2} - \hat{x}_{ij_2})(x_{ij_3} - \hat{x}_{ij_3})(x_{ij_4} - \hat{x}_{ij_4}) \]
\[ \leq \mathbb{E}|x_{ij_1}|_{x_{ij_1} \geq h} |x_{ij_2}|_{x_{ij_2} \geq h} |x_{ij_3}|_{x_{ij_3} \geq h} |x_{ij_4}|_{x_{ij_4} \geq h} \]
\[ \leq \{ \mathbb{E}|x_{ij_1}|_{x_{ij_1} \geq h} \}^2 \left( \mathbb{E}|x_{ij_1}|_{x_{ij_1} \geq h} \right)^\frac{3}{2} \] (4.65)
\[ \leq \frac{\sigma_{x, s}}{\tau_x^2}. \] (4.66)

From (4.64) and (4.65), we have
\[ \mathbb{E}(\tilde{x}_i^T \nu)^4 \leq 8 \left\{ 81 s^2 \frac{\sigma_{x, s}}{\tau_x^2} + K^4 \mathbb{E}\Sigma \hat{v}\|_\text{op}\|v\|_2^2 \right\} \|v\|_2^2 \leq 16 K^4 \mathbb{E}\Sigma \hat{v}\|_\text{op}\|v\|_2^2, \] (4.67)

We evaluate the right-hand side of (4.61) at each term. First, we have
\[ \sum_{i=1}^n \mathbb{E} \hat{x}_{\nu, i}^2 I(|\hat{x}_{\nu, i}| \geq 1/2) \leq \sum_{i=1}^n \sqrt{\mathbb{E} \hat{x}_{\nu, i}^4} \sqrt{\mathbb{E} I(|\hat{x}_{\nu, i}| \geq 1/2)} \]
\[ \leq \frac{\sum_{i=1}^n 4 \mathbb{E} \hat{x}_{\nu, i}}{\lambda_0^2} \mathbb{E}(\hat{x}_i, \nu)^4 \]
\[ \leq \frac{64}{\lambda_0^4 n} K^4 \mathbb{E}\Sigma \hat{v}\|_\text{op}\|v\|_2^2 \leq \frac{\lambda_0^2}{3 \lambda_0^2} \|v\|_2^2, \] (4.68)
where (a) follows from Hölder’s inequality, (b) follows from the relation between indicator function and expectation and Markov’s inequality, and (c) follows from (4.67), and (d) follows from the
definition of $\lambda_o$ and $\|v\|_2 \leq 1$. Second, we have

$$
\sum_{i=1}^{n} \mathbb{E} \tilde{x}_{i}^{4} f(|\xi_{\omega,i}| \geq 1/2) \overset{(a)}{\leq} \sum_{i=1}^{n} \sqrt{\mathbb{E} \tilde{x}_{i}^{4} \sqrt{\mathbb{E} I(|\xi_{\omega,i}| \geq 1/2)}
\overset{(b)}{\leq} \sum_{i=1}^{n} \frac{2}{\lambda_o \sqrt{n}} \sqrt{\mathbb{E} \tilde{x}_{i}^{4} \sqrt{\mathbb{E}|\xi| + \mathbb{E}[(\tilde{x}_i - x)^\top \beta^*]}}
\overset{(c)}{\leq} \sum_{i=1}^{n} \frac{2}{\lambda_o \sqrt{n}} \mathbb{E} \tilde{x}_{i}^{4} \sqrt{\sigma + \mathbb{E}|(\tilde{x}_i - x)^\top \beta^*|^2}
\overset{(d)}{=} \sum_{i=1}^{n} \frac{2}{\lambda_o \sqrt{n}} \mathbb{E} \tilde{x}_{i}^{4} \sqrt{\sigma + \|\beta^*\|_1 \frac{\sigma_{x,s}}{\tau_x}}
= \frac{1}{\lambda_o^2} \sum_{i=1}^{n} \frac{2}{\lambda_o \sqrt{n}} \mathbb{E}(\tilde{x}_i, v)^2 \sqrt{\sigma + \|\beta^*\|_1 \frac{\sigma_{x,s}}{\tau_x} \|v\|_2^2}
\overset{(e)}{\leq} \frac{4K\|\Sigma_{x}^2\|_{op}^2}{\lambda_o^2} \frac{2}{\lambda_o \sqrt{n}} \sqrt{\sigma + \|\beta^*\|_1 \frac{\sigma_{x,s}}{\tau_x} \|v\|_2^2}
\overset{(f)}{\leq} \frac{4\sqrt{2}K\|\Sigma_{x}^2\|_{op}^2}{\lambda_o^2} \sqrt{\frac{\sigma + 1}{\lambda_o \sqrt{n}}} \|v\|_2^2 \overset{(g)}{\leq} \frac{\lambda_o^2}{6\lambda_o^2} \|v\|_2^2,
$$
(4.69)

where (a) follows from Hölder’s inequality, (b) follows from relation between indicator function and expectation and Markov’s inequality, and (c) follows from the assumption on $\{\xi_i\}_{i=1}^n$ and Hölder’s inequality, (d) follows from (4.40), (e) follows from from (4.67), (f) follows from the assumption on $\tau_x$, and (g) follows from the definition of $\lambda_o$. Consequently, from (4.58), (4.59), (4.63), (4.68) and (4.69), we have

$$
\frac{\lambda_o^2}{6\lambda_o^2} \|v\|_2^2 - \Delta \leq \sum_{i=1}^{n} \{-h(\xi_{\omega,i} - \bar{x}_{\omega,i}) + h(\xi_{\omega,i})\} \tilde{x}_{i},
$$
(4.70)

where we use the assumption $9\sigma_x^2/\tau_x^2 \leq \lambda_o^2/12$. Next we evaluate the stochastic term $\Delta$ defined in (4.66). From (4.58) and Theorem 3 of [15], with probability at least $1 - \delta$, we have

$$
\Delta \leq 2\mathbb{E} \Delta + \sigma_f \sqrt{8 \log(1/\delta)} + 18 \log(1/\delta),
$$
(4.71)

where $\sigma_f^2 = \sup_{v \in \mathcal{K}_r} \sum_{i=1}^{n} \mathbb{E} \{f_i(v) - \mathbb{E} f_i(v)\}^2$. From (4.58), $r \leq 1$, (4.67) and the definition of $\lambda_o$, we have

$$
\mathbb{E} \{f_i(v) - \mathbb{E} f_i(v)\}^2 \leq \mathbb{E} f_i^2(v) \leq \mathbb{E} \tilde{x}_{i}^{4} \leq \frac{1}{\lambda_o \sqrt{n}} \|v\|_2^2.
$$
(4.72)

Combining this and (4.71), we have

$$
\Delta \leq 2\mathbb{E} \Delta + 4 \frac{\sqrt{\log(1/\delta)}}{\lambda_o \sqrt{n}} \|v\|_2 + 18 \log(1/\delta)
$$
(4.73)

with probability at least $1 - \delta$. From symmetrization inequality (Lemma 11.4 of [1]), we have $\mathbb{E} \Delta \leq 2 \mathbb{E} \sup_{v \in \mathcal{K}_r} |G_v|$, where

$$
G_v := \sum_{i=1}^{n} a_i \varphi(\tilde{x}_{i}) \psi(\xi_{\omega,i}),
$$
(4.74)
and \( \{a_i\}_{i=1}^n \) is a sequence of i.i.d. Rademacher random variables which are independent of \( \{\bar{X}_i, \xi_i\}_{i=1}^n \). We denote \( E^* \) as a conditional variance of \( \{a_i\}_{i=1}^n \) given \( \{\bar{X}_i, \xi_i\}_{i=1}^n \). From contraction principle (Theorem 11.5 of [3]), we have

\[
E^* \sup_{v \in \mathbb{R}_v} \left| \sum_{i=1}^n a_i \varphi(\bar{X}_{v,i}) \psi(\xi_{v,i}) \right| \leq E^* \sup_{v \in \mathbb{R}_v} \left| \sum_{i=1}^n a_i \varphi(\bar{X}_{v,i}) \right| \tag{4.75}
\]

and from the basic property of the expectation, we have

\[
E \sup_{v \in \mathbb{R}_v} \left| \sum_{i=1}^n a_i \varphi(\bar{X}_{v,i}) \psi(\xi_{v,i}) \right| \leq E \sup_{v \in \mathbb{R}_v} \left| \sum_{i=1}^n a_i \varphi(\bar{X}_{v,i}) \right|. \tag{4.76}
\]

Since \( \varphi \) is \( \frac{1}{2} \)-Lipschitz and \( \varphi(0) = 0 \), from contraction principle (Theorem 11.6 in [3]), we have

\[
E \sup_{v \in \mathbb{R}_v} \left| \sum_{i=1}^n a_i \varphi(\bar{X}_{v,i}) \right| \leq \frac{1}{\lambda_0 \sqrt{n}} E \sup_{v \in \mathbb{R}_v} \left| \sum_{i=1}^n a_i \bar{X}_{v,i} \right|. \tag{4.77}
\]

From Proposition 4.4 and Hölder’s inequality, we have

\[
\frac{1}{n} E \sup_{v \in \mathbb{R}_v} \left| \sum_{i=1}^n a_i \varphi(\bar{X}_{v,i}) \psi(\xi_{v,i}) \right| \leq \|v\|_1 (2\sigma_{x,2}r_d + 4\tau_x r_d^2) \leq 12\sqrt{s} (\sigma_{x,2}r_d + \tau_x r_d^2) \|v\|_2. \tag{4.78}
\]

Combining (4.78), (4.73), \( s \geq 0 \) and the definition of \( r_{d,\delta} \), we have

\[
\lambda_0^2 \Delta \leq 24\lambda_0 \sqrt{n} \sqrt{s} (\sigma_{x,2} r_d + \tau_x r_d^2) \|v\|_2 + 4\lambda_0 \sqrt{n} \|v\|_2 + 18\lambda_0^2 nr_d^2 \leq 24\lambda_0 \sqrt{n} \sqrt{s} r_{d,\delta} \|v\|_2 + 18\lambda_0^2 nr_d^2 \tag{4.79}
\]

and from (4.70), the proof is complete. \( \Box \)

The following proposition is used in the proof of Proposition 3.5.

**Proposition 4.4.** Suppose that Assumption 2.7 holds. Then, we have

\[
E \sup_{v \in \mathbb{R}_v} \left| \frac{1}{n} \sum_{i=1}^n a_i \bar{X}_{v,i}^T v \right| \leq 12\sqrt{s} (\sigma_{x,2}r_d + \tau_x r_d^2) \|v\|_2. \tag{4.80}
\]

**Proof.** From Hölder’s inequality, we have

\[
E \sup_{v \in \mathbb{R}_v} \left| \frac{1}{n} \sum_{i=1}^n a_i \bar{X}_{v,i} \right| \leq E\|v\|_1 \left| \left| \frac{1}{n} \sum_{i=1}^n a_i \bar{X}_{v,i} \right| \right| \leq 3\sqrt{s} \|v\|_2 E \left| \left| \frac{1}{n} \sum_{i=1}^n a_i \bar{X}_{v,i} \right| \right|_\infty. \tag{4.81}
\]

We note that,

\[
E a_i^2 \bar{x}_{ij}^2 \leq E \bar{x}_{ij}^2 \leq \sigma_{x,2}^2, \quad E a_i^p \bar{x}_{ij}^p \leq \tau_x^{p-2} E \bar{x}_{ij} \leq \tau_x^{p-2} \sigma_{x,2}^2. \tag{4.82}
\]

From Lemma 14.12 of [3] and \( d \geq 3 \), we have

\[
E \left| \frac{1}{n} \sum_{i=1}^n \frac{\bar{X}_{v,i}}{n} \right|_{\infty} \leq \sqrt{2 \frac{\sigma_{x,2}^2 \log(d+1)}{n} + 2\tau_x \frac{\log(d+1)}{n}} \leq 2\sigma_{x,2}r_d + 4\tau_x r_d^2. \tag{4.83}
\]

Combining the arguments above, the proof is complete. \( \Box \)
References

[1] Ainesh Bakshi and Adarsh Prasad. Robust linear regression: Optimal rates in polynomial time. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 102–115, 2021.

[2] Sivaraman Balakrishnan, Simon S Du, Jerry Li, and Aarti Singh. Computationally efficient robust sparse estimation in high dimensions. In *Conference on Learning Theory*, pages 169–212. PMLR, 2017.

[3] Pierre C Bellec, Guillaume Lecué, and Alexandre B Tsybakov. Slope meets lasso: improved oracle bounds and optimality. *The Annals of Statistics*, 46(6B):3603–3642, 2018.

[4] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press, 2013.

[5] Peter Bühlmann and Sara Van De Geer. *Statistics for high-dimensional data: methods, theory and applications*. Springer Science & Business Media, 2011.

[6] T Tony Cai and Anru Zhang. Sparse representation of a polytope and recovery of sparse signals and low-rank matrices. *IEEE transactions on information theory*, 60(1):122–132, 2013.

[7] Emmanuel Candes and Terence Tao. The dantzig selector: Statistical estimation when p is much larger than n. *The annals of Statistics*, 35(6):2313–2351, 2007.

[8] Mengjie Chen, Chao Gao, and Zhao Ren. Robust covariance and scatter matrix estimation under huber’s contamination model. *The Annals of Statistics*, 46(5):1932–1960, 2018.

[9] Xi Chen and Wen-Xin Zhou. Robust inference via multiplier bootstrap. *Annals of Statistics*, 48(3):1665–1691, 2020.

[10] Yudong Chen, Constantine Caramanis, and Shie Mannor. Robust sparse regression under adversarial corruption. In *International Conference on Machine Learning*, pages 774–782. PMLR, 2013.

[11] Yu Cheng, Ilias Diakonikolas, and Rong Ge. High-dimensional robust mean estimation in nearly-linear time. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2755–2771. SIAM, 2019.

[12] Yu Cheng, Ilias Diakonikolas, Rong Ge, and David Woodruff. Faster algorithms for high-dimensional robust covariance estimation. *arXiv preprint arXiv:1906.04661*, 2019.

[13] Yeshwanth Cherapanamjeri, Efe Aras, Nilesh Tripuraneni, Michael I Jordan, Nicolas Flammarion, and Peter L Bartlett. Optimal robust linear regression in nearly linear time. *arXiv preprint arXiv:2007.08137*, 2020.

[14] Yeshwanth Cherapanamjeri, Nicolas Flammarion, and Peter L Bartlett. Fast mean estimation with sub-gaussian rates. In *Conference on Learning Theory*, pages 786–806. PMLR, 2019.

[15] Geoffrey Chinot. Erm and rerm are optimal estimators for regression problems when malicious outliers corrupt the labels. *Electronic Journal of Statistics*, 14(2):3563–3605, 2020.

[16] Arnak Dalalyan and Philip Thompson. Outlier-robust estimation of a sparse linear model using $\ell_1$-penalized huber’s m-estimator. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems 32*, pages 13188–13198. Curran Associates, Inc., 2019.

[17] Arnak S Dalalyan and Arshak Minasyan. All-in-one robust estimator of the gaussian mean. *The Annals of Statistics*, 50(2):1193–1219, 2022.
[18] Jules Depersin and Guillaume Lecué. Robust sub-gaussian estimation of a mean vector in nearly linear time. The Annals of Statistics, 50(1):511–536, 2022.

[19] Ilias Diakonikolas, Themis Gouleakis, and Christos Tzamos. Distribution-independent pac learning of halfspaces with massart noise. Advances in Neural Information Processing Systems, 32, 2019.

[20] Ilias Diakonikolas, Gautam Kamath, Daniel Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Robust estimators in high-dimensions without the computational intractability. SIAM Journal on Computing, 48(2):742–864, 2019.

[21] Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Being robust (in high dimensions) can be practical. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 999–1008. JMLR. org, 2017.

[22] Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Robustly learning a gaussian: Getting optimal error, efficiently. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 2683–2702. Society for Industrial and Applied Mathematics, 2018.

[23] Ilias Diakonikolas, Daniel Kane, Sushrut Karmalkar, Eric Price, and Alistair Stewart. Outlier-robust high-dimensional sparse estimation via iterative filtering. In Advances in Neural Information Processing Systems, pages 10689–10700, 2019.

[24] Ilias Diakonikolas and Daniel M Kane. Recent advances in algorithmic high-dimensional robust statistics. arXiv preprint arXiv:1911.05911, 2019.

[25] Ilias Diakonikolas, Daniel M Kane, Vasilis Kontonis, Christos Tzamos, and Nikos Zarifis. Efficiently learning halfspaces with tsybakov noise. In Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing, pages 88–101, 2021.

[26] Ilias Diakonikolas, Daniel M Kane, and Pasin Manurangsi. The complexity of adversarially robust proper learning of halfspaces with agnostic noise. Advances in Neural Information Processing Systems, 33:20449–20461, 2020.

[27] Ilias Diakonikolas, Weihao Kong, and Alistair Stewart. Efficient algorithms and lower bounds for robust linear regression. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 2745–2754. SIAM, 2019.

[28] Ilias Diakonikolas, Vasilis Kontonis, Christos Tzamos, and Nikos Zarifis. Learning halfspaces with massart noise under structured distributions. In Conference on Learning Theory, pages 1486–1513. PMLR, 2020.

[29] Sjoerd Dirksen. Tail bounds via generic chaining. Electronic Journal of Probability, 20:1–29, 2015.

[30] Yihe Dong, Samuel Hopkins, and Jerry Li. Quantum entropy scoring for fast robust mean estimation and improved outlier detection. In Advances in Neural Information Processing Systems, pages 6067–6077, 2019.

[31] David L Donoho. Compressed sensing. IEEE Transactions on information theory, 52(4):1289–1306, 2006.

[32] Jianqing Fan and Runze Li. Variable selection via nonconcave penalized likelihood and its oracle properties. Journal of the American statistical Association, 96(456):1348–1360, 2001.

[33] Jianqing Fan, Han Liu, Qiang Sun, and Tong Zhang. I-lamm for sparse learning: Simultaneous control of algorithmic complexity and statistical error. Annals of statistics, 46(2):814, 2018.
[34] Jianqing Fan, Weichen Wang, and Ziwei Zhu. A shrinkage principle for heavy-tailed data: High-dimensional robust low-rank matrix recovery. *Annals of statistics*, 49(3):1239, 2021.

[35] Chao Gao. Robust regression via multivariate regression depth. *Bernoulli*, 26(2):1139–1170, 2020.

[36] Samuel B Hopkins. Mean estimation with sub-gaussian rates in polynomial time. *The Annals of Statistics*, 48(2):1193–1213, 2020.

[37] Olga Klopp. Noisy low-rank matrix completion with general sampling distribution. *Bernoulli*, 20(1):282–303, 2014.

[38] Olga Klopp, Karim Lounici, and Alexandre B Tsybakov. Robust matrix completion. *Probability Theory and Related Fields*, 169(1-2):523–564, 2017.

[39] Vladimir Koltchinskii, Karim Lounici, and Alexandre B Tsybakov. Nuclear-norm penalization and optimal rates for noisy low-rank matrix completion. *The Annals of Statistics*, 39(5):2302–2329, 2011.

[40] Pravesh K Kothari, Jacob Steinhardt, and David Steurer. Robust moment estimation and improved clustering via sum of squares. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1035–1046. ACM, 2018.

[41] Kevin A Lai, Anup B Rao, and Santosh Vempala. Agnostic estimation of mean and covariance. In *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*, pages 665–674. IEEE, 2016.

[42] Zhixian Lei, Kyle Luh, Prayaag Venkat, and Fred Zhang. A fast spectral algorithm for mean estimation with sub-gaussian rates. In *Conference on Learning Theory*, pages 2598–2612, 2020.

[43] Liu Liu, Yanyao Shen, Tianyang Li, and Constantine Caramanis. High dimensional robust sparse regression. In *International Conference on Artificial Intelligence and Statistics*, pages 411–421. PMLR, 2020.

[44] Gabor Lugosi and Shahar Mendelson. Robust multivariate mean estimation: the optimality of trimmed mean. *The Annals of Statistics*, 49(1):393–410, 2021.

[45] Pascal Massart. About the constants in talagrand’s concentration inequalities for empirical processes. *The Annals of Probability*, 28(2):863–884, 2000.

[46] Shahar Mendelson and Nikita Zhivotovskiy. Robust covariance estimation under $L_4-L_2$ norm equivalence. *The Annals of Statistics*, 48(3):1648–1664, 2020.

[47] Omar Montasser, Surbhi Goel, Ilias Diakonikolas, and Nathan Srebro. Efficiently learning adversarially robust halfspaces with noise. In *International Conference on Machine Learning*, pages 7010–7021. PMLR, 2020.

[48] Sahand Negahban and Martin J Wainwright. Estimation of (near) low-rank matrices with noise and high-dimensional scaling. *The Annals of Statistics*, pages 1069–1097, 2011.

[49] Sahand Negahban and Martin J Wainwright. Restricted strong convexity and weighted matrix completion: Optimal bounds with noise. *The Journal of Machine Learning Research*, 13(1):1665–1697, 2012.

[50] Arkadi Nemirovski. Prox-method with rate of convergence o (1/t) for variational inequalities with lipschitz continuous monotone operators and smooth convex-concave saddle point problems. *SIAM Journal on Optimization*, 15(1):229–251, 2004.
[51] Yu Nesterov. Smooth minimization of non-smooth functions. *Mathematical programming, 103*(1):127–152, 2005.

[52] Nam H Nguyen and Trac D Tran. Robust lasso with missing and grossly corrupted observations. *IEEE transactions on information theory, 59*(4):2036–2058, 2012.

[53] Ankit Pensia, Varun Jog, and Po-Ling Loh. Robust regression with covariate filtering: Heavy tails and adversarial contamination. *arXiv preprint arXiv:2009.12976*, 2020.

[54] Adarsh Prasad, Sivaraman Balakrishnan, and Pradeep Ravikumar. A robust univariate mean estimator is all you need. In *International Conference on Artificial Intelligence and Statistics*, pages 4034–4044. PMLR, 2020.

[55] Adarsh Prasad, Arun Sai Suggala, Sivaraman Balakrishnan, Pradeep Ravikumar, et al. Robust estimation via robust gradient estimation. *Journal of the Royal Statistical Society Series B*, 82(3):601–627, 2020.

[56] Garvesh Raskutti, Martin J Wainwright, and Bin Yu. Restricted eigenvalue properties for correlated gaussian designs. *The Journal of Machine Learning Research*, 11:2241–2259, 2010.

[57] Angelika Rohde and Alexandre B Tsybakov. Estimation of high-dimensional low-rank matrices. *The Annals of Statistics*, 39(2):887–930, 2011.

[58] Yiyuan She and Art B Owen. Outlier detection using nonconvex penalized regression. *Journal of the American Statistical Association*, 106(494):626–639, 2011.

[59] Weijie Su and Emmanuel Candes. Slope is adaptive to unknown sparsity and asymptotically minimax. *The Annals of Statistics*, 44(3):1038–1068, 2016.

[60] Philip Thompson. Outlier-robust sparse/low-rank least-squares regression and robust matrix completion. *arXiv preprint arXiv:2012.06750*, 2020.

[61] Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society: Series B*, 58(1):267–288, 1996.

[62] Tengyao Wang, Quentin Berthet, and Richard J Samworth. Statistical and computational trade-offs in estimation of sparse principal components. *The Annals of Statistics*, 44(5):1896–1930, 2016.

[63] Ming Yuan and Yi Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society: Series B*, 68(1):49–67, 2006.

[64] Cun-Hui Zhang. Nearly unbiased variable selection under minimax concave penalty. *The Annals of statistics*, 38(2):894–942, 2010.

[65] Hui Zou and Trevor Hastie. Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society: Series B*, 67(2):301–320, 2005.