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ON THE STEINBERG CHARACTER
OF A SEMISIMPLE $p$-ADIC GROUP

Ju-Lee Kim and George Lusztig
Dedicated to Robert Steinberg on the occasion of his 90-th birthday

1. Introduction

1.1. Let $K$ be a nonarchimedean local field and let $\overline{K}$ be a maximal unramified field extension of $K$. Let $\mathcal{O}$ (resp. $\overline{\mathcal{O}}$) be the ring of integers of $K$ (resp. $\overline{K}$) and let $p$ (resp. $\overline{p}$) be the maximal ideal of $\mathcal{O}$ (resp. $\overline{\mathcal{O}}$). Let $\mathbb{K}^* = \overline{K} - \{0\}$. We write $\mathcal{O}/p = \mathbb{F}_q$, a finite field with $q$ elements, of characteristic $p$.

Let $G$ be a semisimple almost simple algebraic group defined and split over $K$ with a given $\mathcal{O}$-structure compatible with the $K$-structure.

If $V$ is an admissible representation of $G(K)$ of finite length, we denote by $\phi_V$ the character of $V$ in the sense of Harish-Chandra, viewed as a $\mathbb{C}$-valued function on the set $G_{rs} := G_{rs} \cap G(K)$. (Here $G_{rs}$ is the set of regular semisimple elements of $G$ and $\mathbb{C}$ is the field of complex numbers.)

In this paper we study the restriction of the function $\phi_V$ to:

(a) a certain subset $G(K)_{vr}$ of $G(K)_{rs}$, that is to the set of very regular elements in $G(K)$ (see 1.2), in the case where $V$ is the Steinberg representation of $G(K)$ and

(b) a certain subset $G(K)_{svr}$ of $G(K)_{vr}$, that is to the set of split very regular elements in $G(K)$ (see 1.2), in the case where $V$ is an irreducible admissible representation of $G(K)$ with nonzero vectors fixed by an Iwahori subgroup.

In case (a) we show that $\phi_V(g)$ with $g \in G(K)_{rs}$ is of the form $\pm q^n$ with $n \in \{0, -1, -2, \ldots\}$ (see Corollary 3.4) with more precise information when $g \in G(K)_{svr}$ (see Theorem 2.2) or when $g \in G(K)_{cvr}$ (see Theorem 3.2); in case (b) we show (with some restriction on characteristic) that $\phi_V(g)$ with $G(K)_{svr}$ can be expressed as a trace of a certain element of an affine Hecke algebra in an irreducible module (see Theorem 4.3).

Note that the Steinberg representation $S$ is an irreducible admissible representation of $G(K)$ with a one dimensional subspace invariant under an Iwahori
subgroup on which the affine Hecke algebra acts through the “sign” representation, see [MA], [S]. This is a p-adic analogue of the Steinberg representation [St] of a reductive group over \( F_q \). In [R], it is proved that \( \phi_S(g) \neq 0 \) for any \( g \in G(K)_r \).

1.2. Let \( g \in G_{rs} \cap G(K) \). Let \( T' = T_g \) be the maximal torus of \( G \) that contains \( g \). We say that \( g \) is very regular (resp. compact very regular) if \( T' \) is split over \( K \) and for any root \( \alpha \) with respect to \( T' \) viewed as a homomorphism \( T'(K) \to K^* \) we have

\[
\alpha(g) \notin (1 + \mathfrak{p}) \quad (\text{resp. } \alpha(g) \notin (1 + \mathfrak{p})).
\]

Let \( G(K)_{vr} \) (resp. \( G(K)_{cvr} \)) be the set of elements in \( G(K) \) which are very regular (resp. compact very regular). We write \( G(K)_{vr} = G(K)_{vr} \cap G(K) \), \( G(K)_{cvr} = G(K)_{cvr} \cap G(K) \). Let \( G(K)_{svr} \) be the set of all \( g \in G(K)_{vr} \) such that \( T_g \) is split over \( K \).

1.3. Notation. Let \( K^* = K - \{0\} \) and let \( v : K^* \to \mathbb{Z} \) be the unique (surjective) homomorphism such that \( v(p^n - p^{n+1}) = n \) for any \( n \in \mathbb{N} \). For \( a \in K^* \) we set \( |a| = q^{-v(a)} \).

We fix a maximal torus \( T \) of \( G \) defined and split over \( K \). Let \( Y \) (resp. \( X \)) be the group of cocharacters (resp. characters) of the algebraic group \( T \). Let \( \langle , \rangle : Y \times X \to \mathbb{Z} \) be the obvious pairing. Let \( R \subseteq X \) be the set of roots of \( G \) with respect to \( T \), let \( R^+ \) be a set of positive roots for \( R \) and let \( \Pi \) be the set of simple roots of \( R \) determined by \( R^+ \). We write \( \Pi = \{ \alpha_i ; i \in I_0 \} \). Let \( R^- = R - R^+ \). Let \( Y^+ \) (resp. \( Y^{++} \)) be the set of all \( y \in Y \) such that \( \langle y, \alpha \rangle \geq 0 \) (resp. \( \langle y, \alpha \rangle > 0 \)) for all \( \alpha \in R^+ \). We define \( 2\rho \in X \) by \( 2\rho = \sum_{\alpha \in R^+} \alpha \).

We have canonically \( T(K) = K^* \otimes Y \); we define a homomorphism \( \chi : T(K) \to Y \) by \( \chi(\lambda \otimes y) = v(\lambda)y \) for any \( \lambda \in K^* \), \( y \in Y \). For any \( y \in Y \) we set \( T(K)_y = \chi^{-1}(y) \). For \( y \in Y \) let \( T(K)_{y} = T(K)_y \cap G(K)_{svr} \). Note that if \( y \in Y^{++} \) then \( T(K)_{y} = T(K)_y \).

For each \( \alpha \in R \) let \( U_{\alpha} \) be the corresponding root subgroup of \( G \).

Let \( G(K)' \) be the derived subgroup of \( G(K) \).

2. Calculation of \( \phi_S \) on \( G(K)_{svr} \)

2.1. Let \( \mathcal{W} \subseteq \text{Aut}(T) \) be the Weyl group of \( G \) regarded as a Coxeter group; for \( i \in I_0 \) let \( s_i \) be the simple reflection in \( \mathcal{W} \) determined by \( \alpha_i \). We can also view \( \mathcal{W} \) as a subgroup of \( \text{Aut}(Y) \) or \( \text{Aut}(X) \). Let \( w = w_0 \) be the longest element of \( \mathcal{W} \). For any \( J \subseteq I_0 \) let \( \mathcal{W}_J \) be the subgroup of \( \mathcal{W} \) generated by \( \{ s_i ; i \in J \} \) and let \( R_J \) be the set of \( \alpha \in R \) such that \( \alpha = w(\alpha_i) \) for some \( w \in \mathcal{W}_J, i \in J \). Let \( R_J^+ = R_J \cap R^+, R_J^- = R_J - R_J^+ \).

Let \( \mathfrak{g} \) be the Lie algebra of \( G \); let \( \mathfrak{t} \subseteq \mathfrak{g} \) be the Lie algebra of \( T \). For any \( J \subseteq I_0 \) let \( \mathfrak{n}_J \) be the Lie subalgebra of \( \mathfrak{g} \) spanned by \( \mathfrak{t} \) and by the root spaces corresponding to roots in \( R_J \); let \( \mathfrak{n}_J \) be the Lie subalgebra of \( \mathfrak{g} \) spanned by the root spaces corresponding to roots in \( R^+ - R_J^+ \).

According to [C1], \( \phi \) is an alternating sum of characters of representations induced from one dimensional representations of various parabolic subgroups of \( G \).
defined over $K$. From this one can deduce that, if $t \in T(K) \cap G(K)_{rs}$, then
\[
\phi_S(t) = \sum_{J \subseteq I} (-1)^{|J|} \sum_{w \in J W} \delta_J(w(t)) D_{I,J}(w(t))^{-1/2}
\]
where for any $J \subseteq I$ and $t' \in T(K) \cap G(K)_{rs}$ we set
\[
D_{I,J}(t') = |\det(1 - \operatorname{Ad}(t')|_{g/I_J})|,
\]
\[
\delta_J(t') = |\det(\operatorname{Ad}(t')|_{n_J})|,
\]
and $J W$ is a set of representatives for the cosets $WJ \setminus W$. (It will be convenient to assume that $J W$ is the set of representatives of minimal length for the cosets $WJ \setminus W$.) Here for a real number $a \geq 0$ we denote by $a^{1/2}$ or $\sqrt{a}$ the $\geq 0$ square root of $a$. We have the following result. (We write $\phi$ instead of $\phi_S$.)

**Theorem 2.2.** Let $y \in Y^+$ and let $t \in T(K)_y^\bullet$. Then $\phi(t) = q^{-(y,2p)}$.

2.3. More generally let $t \in T(K)_y^\bullet$ where $y \in Y$. By a standard property of Weyl chambers there exists $w \in W$ such that $w(y) \in Y^+$. Let $t_1 = w(t)$. Then the theorem is applicable to $t_1$ and we have $\phi(t) = \phi(t_1) = q^{-(w(y),2p)}$.

2.4. Let $y' = w_0(y), t' = w_0(t)$. We have $\phi_S(t) = \phi_S(t'), t' \in T(K)_{y'}, -y' \in Y^+$. We show:

(a) if $\beta \in R^+$ then $v(1 - \beta(t'))) = v((\beta(t'))); if \beta \in R^-$ then $v(1 - \beta(t'))) = 0$.

Assume first that $\beta \in R^+$. If $v(\beta(t')) \neq 0$ then $\beta(t') < 0$ (since $\langle y', \beta \rangle \neq 0$, $\langle y', \beta \rangle \leq 0$) hence $v(1 - \beta(t'))) = v((\beta(t'))). If v(\beta(t')) = 0 then $\beta(t') - 1 \in O - p$ hence $v(1 - \beta(t'))) = 0 = v((\beta(t')))$ as required.

Assume next that $\beta \in R^-$. If $v(\beta(t')) \neq 0$ then $v(\beta(t')) > 0$ (since $\langle y', \beta \rangle \neq 0$, $\langle y', \beta \rangle \geq 0$) hence $v(1 - \beta(t'))) = 0$. If $v(\beta(t')) = 0$ then $\beta(t') - 1 \in O - p$ hence $v(1 - \beta(t'))) = 0$ as required.

For any $w \in W, J \subseteq I$ we have:

\[
D_{I,J}(w(t')) = \prod_{\alpha \in R - R_J} q^{-v(1 - \alpha(w(t')))}
\]
\[
= \prod_{\alpha \in R - R_J; w^{-1} \alpha \in R^+} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R - R_J; w^{-1} \alpha \in R^+} q^{-\langle y', w^{-1} \alpha \rangle},
\]
\[
\delta_J(w(t')) = \prod_{\alpha \in R^+ - R_J} q^{-v(\alpha(w(t')))} = \prod_{\alpha \in R^+ - R_J} q^{-\langle y', w^{-1} \alpha \rangle},
\]
\[
D_I(t') = \prod_{\alpha \in R^+} q^{-\langle y', \alpha \rangle}.
\]
(We have used (a) with $\beta = w^{-1}(\alpha)$.) We see that

$$\phi(t) = \phi(t') = \sum_{J \subseteq I} (-1)^{\sharp J} \sum_{w \in J \mathcal{W}} \sqrt{q}^{-\langle y', x_w, J \rangle}$$

where for $w \in J \mathcal{W}$ we have

$$x_{w, J} = \sum_{\alpha \in R^+ - R^+_j} w^{-1} \alpha - \sum_{\alpha \in R^-; w^{-1} \alpha \in R^+} w^{-1} \alpha$$

$$= \sum_{\alpha \in R^+ - R^+_j; w^{-1}(\alpha) \in R^-} w^{-1} \alpha - \sum_{\alpha \in R^- - R^-_j; w^{-1}(\alpha) \in R^+} w^{-1} \alpha$$

$$= 2 \sum_{\alpha \in R^+ - R^+_j; w^{-1} \alpha \in R^-} w^{-1} \alpha \in X.$$  

For $w \in J \mathcal{W}$ we have $\alpha \in R^+_j \implies w^{-1} \alpha \in R^+$ hence

$$\sum_{\alpha \in R^+ - R^+_j; w^{-1} \alpha \in R^-} w^{-1} \alpha = \sum_{\alpha \in R^+; w^{-1} \alpha \in R^-} w^{-1} \alpha$$

so that $x_{w, J} = x_w$ where

$$x_w = 2 \sum_{\alpha \in R^+; w^{-1} \alpha \in R^-} w^{-1} \alpha \in X.$$  

Thus we have

$$\phi(t) = \sum_{J \subseteq I} (-1)^{\sharp J} \sum_{w \in J \mathcal{W}} \sqrt{q}^{-\langle y', x_w \rangle} = \sum_{w \in \mathcal{W}} c_w \sqrt{q}^{-\langle y', x_w \rangle}$$

where for $w \in \mathcal{W}$ we set

$$c_w = \sum_{J \subseteq I; w \in J \mathcal{W}} (-1)^{\sharp J}.$$  

For $w \in \mathcal{W}$ let $\mathcal{L}(w) = \{ i \in I; s_i w > w \}$ where $<$ is the standard partial order on $\mathcal{W}$. For $J \subseteq I$ we have $w \in J \mathcal{W}$ if and only if $J \subseteq \mathcal{L}(w)$. Thus,

$$c_w = \sum_{J \subseteq \mathcal{L}(w)} (-1)^{\sharp J}$$

and this is 0 unless $\mathcal{L}(w) = \emptyset$ (that is $w = w_0$) when $c_w = 1$. Note also that $x_{w_0} = -4\rho$. Thus we have

$$\phi(t) = c_{w_0} \sqrt{q}^{-\langle y', x_{w_0} \rangle} = q^{\langle y', 2\rho \rangle} = q^{-\langle y, 2\rho \rangle}.$$  

Theorem 2.2 is proved.
2.5. Assume now that \( \tau \in T(K) \) satisfies the following condition: for any \( \alpha \in R \) we have \( \alpha(\tau) - 1 \in \mathfrak{p} - \{0\} \) so that \( \alpha(\tau) - 1 \in \mathfrak{p}^{n_\alpha} - \mathfrak{p}^{n_\alpha+1} \) for a well defined integer \( n_\alpha \geq 1 \). Note that \( n_{-\alpha} = n_\alpha \) and \( v(1 - \alpha(\tau)) = n_\alpha \geq 1 \) for all \( \alpha \in R \). Hence

\[
\phi(\tau) = \sum_{J \subset I} (-1)^{|J|} \sum_{w \in J} q^{\sum_{\alpha \in R} n_\alpha / 2 - \sum_{\alpha \in R, j} n_{w-1}(\alpha) / 2}.
\]

Thus,

\[
\phi(\tau) = \sharp(W) q^{\sum_{\alpha \in R} n_\alpha / 2} + \text{strictly smaller powers of } q.
\]

In the case where \( K \) is the field of power series over \( F_q \), the leading term \( \sharp(W) q^{\sum_{\alpha \in R} n_\alpha / 2} \) is equal to \( \sharp(W) q^m \) where \( m \) is the dimension of the “variety” of Iwahori subgroups of \( G(K) \) that contain the topologically unipotent element \( \tau \) (see [KL2]).

3. Calculation of \( \phi \) on \( G(K)_{cvr} \)

3.1. We will again write \( \phi \) instead of \( \phi \). In this section we assume that we are given \( \gamma \in G(K)_{cvr} \). Let \( T' = T_{\gamma} \). Note that \( T' \) is defined over \( K \); let \( A' \) be the largest \( K \)-split torus of \( T' \). For any parabolic subgroup \( P \) of \( G \) defined over \( K \) such that \( \gamma \in P \) we set \( \delta_P(\gamma) = |\det(\text{Ad}(\gamma)|_n)| \) where \( n \) is the Lie algebra of the unipotent radical of \( P \).

Let \( \mathcal{X} \) be the set of all pairs \((P, A)\) where \( P \) is a parabolic subgroup of \( G \) defined over \( K \) and \( A \) is the unique maximal \( K \)-split torus in the centre of some Levi subgroup of \( P \) defined over \( K \); then that Levi subgroup is uniquely determined by \( A \) and is denoted by \( M_A \). Let \( \mathcal{X}' = \{(P, A) \in \mathcal{X}; A \subset A'\} \). According to Harish-Chandra [H] we have

\[
(\alpha) \quad \phi(\gamma) = (-1)^{\dim T} \sum_{(P, A) \in \mathcal{X}'} (-1)^{\dim A} \delta_P(\gamma)^{1/2} D_{G/M_A}(\gamma)^{-1/2}
\]

where \( D_{G/M_A}(\gamma) = |\det(1 - \text{Ad}(\gamma)|_g)| \) (we denote by \( I \) the Lie algebra of \( M_A \)).

**Theorem 3.2.** Assume in addition that \( \gamma \in G(K)_{cvr} \). Then

\[
\phi(\gamma) = (-1)^{\dim T - \dim A'}.\]

From our assumptions we see that for any \((P, A) \in \mathcal{X}'\) we have \( \delta_P(\gamma) = 1 = D_{G/M_A}(\gamma) \). Hence 3.1(a) becomes

\[
\phi(\gamma) = (-1)^{\dim T} \sum_{(P, A) \in \mathcal{X}'} (-1)^{\dim A}.
\]

Let \( \mathcal{Y} \) be the group of cocharacters of \( A' \) and let \( \hat{\mathcal{Y}} = \mathcal{Y} \otimes \mathbf{R} \). The real vector space \( \hat{\mathcal{Y}} \) can be partitioned into facets \( F_{P, A} \) indexed by \((P, A) \in \mathcal{X}'\) such that \( F_{P, A} \) is homeomorphic to \( \mathbf{R}^{\dim A} \). Note that the Euler characteristic with compact support of \( F_{P, A} \) is \((-1)^{\dim A} \) and the Euler characteristic with compact support of \( \hat{\mathcal{Y}} \) is \((-1)^{\dim A'} \). Using the additivity of the Euler characteristic with compact support we see that \( \sum (P, A) \in \mathcal{X}' (-1)^{\dim A} = (-1)^{\dim A'} \). Thus, \( \phi(\gamma) = (-1)^{\dim T - \dim A'} \), as required. \( \square \)
3.3. In the setup of 3.1 let $P_{\gamma}$ be the parabolic subgroup of $G$ associated to $\gamma$ as in [C2]. Note that $P_{\gamma}$ is defined over $K$. The following result can be deduced by combining Theorem 3.2 with the results in [C2] and with Proposition 2 of [R].

**Corollary 3.4.** We have $\phi(\gamma) = (-1)^{\dim T - \dim A'} \delta_{P_{\gamma}}(\gamma)$.

4. Iwahori Spherical Representations: Split Elements

4.1. Let $B$ be the subgroup of $G(K)$ generated by $U_{\alpha}(O), (\alpha \in R^+), U_{\alpha}(p), (\alpha \in R^-)$ and $T(K)_0$. (The subgroups $U_{\alpha}(O), U_{\alpha}(p)$ of $U_{\alpha}$ are defined by the $O$-structure of $G$. We have $B \in B$ where $B$ is the set of Iwahori subgroups of $G(K)$. Note that $B \subset G(K)'$. For any $\alpha \in R$ we choose an isomorphism $x_\alpha : K \sim \sim \sim U_{\alpha}(K)$ (the restriction of an isomorphism of algebraic groups from the additive group to $\alpha$) which carries $O$ onto $U_{\alpha}(O)$ and $p$ onto $U_{\alpha}(p)$. We set $W := Y \cdot W$ with $Y$ normal in $W$ (recall that $W$ acts naturally on $Y$). Let $Y'$ be the subgroup of $Y$ generated by the coroots. Then $W' := Y' \cdot W$ is naturally a subgroup of $W$. According to [IM], $W$ is an extended Coxeter group (the semidirect product of the Coxeter group $W'$ and the finite abelian group $Y/Y'$) with length function

$$l(yw) = \sum_{\alpha \in R^+; w^{-1}(\alpha) \in R^+} ||(y, \alpha)|| + \sum_{\alpha \in R^+; w^{-1}(\alpha) \in R^-} ||(y, \alpha) - 1||$$

where $||a|| = a$ if $a \geq 0$, $||a|| = -a$ if $a < 0$. According to [IM], the set of double cosets $B \backslash G(K)/B$ is in bijection with $W$; to $yw$ (where $y \in Y, w \in W$) corresponds the double coset $\Omega_{yw}$ containing $T(K)_y w$ (here $w$ is an element in $G(O)$ which normalizes $T(K)_0$ and acts on it in the same way as $w$); moreover, $z(\Omega_{yw}/B) = z(B/\Omega_{yw}) = q^{l(yw)}$ for any $y \in Y$, $w \in W$. For example, if $y \in Y^{++}$ then $l(y) = \langle y, 2\rho \rangle$.

Let $H$ be the algebra of $B$-biinvariant functions $G(K) \to \mathbb{C}$ with compact support with respect to convolution (we use the Haar measure $dg$ on $G(K)$ for which $\nu ol(B) = 1$). For $y, w$ as above let $\underline{\tau}_{yw} \in H$ be the characteristic function of $\Omega_{yw}$. Then the functions $\underline{\tau}_{yw}, w \in W$, form a $\mathbb{C}$-basis of $H$ and according to [IM] we have

$$\underline{\tau}_{yw} \underline{\tau}_{w'} = \underline{\tau}_{yw'w'} \text{ if } w, w' \in W \text{ satisfy } l(ww') = l(w) + l(w'),$$

$$l(\underline{\tau}_w + 1)(\underline{\tau}_w - q) = 0 \text{ if } w \in W', l(w) = 1.$$  

In other words, $H$ is what now one calls the Iwahori-Hecke algebra of the (extended) Coxeter group $W$ with parameter $q$.

4.2. Let $C_0^\infty(G(K))$ be the vector space of locally constant functions with compact support from $G(K)$ to $\mathbb{C}$. Let $(V, \sigma)$ be an irreducible admissible representation of $G(K)$ such that the space $V^B$ of $B$-invariant vectors in $V$ is nonzero. If $f \in C_0^\infty(G(K))$ then there is a well defined linear map $\sigma_f : V \to V$ such that for any $x \in V$ we have $\sigma_f(x) = \int_G f(g)\sigma(g)(x)dg$. This linear map has finite rank hence it has a well defined trace $\text{tr}(\sigma_f) \in \mathbb{C}$. From the definitions we see that for $f, f' \in C_0^\infty(G(K))$ we have $\sigma_{f^*f'} = \sigma_f \sigma_{f'} : V \to V$ where $*$ denotes convolution.
If \( f \in H \), then \( \sigma_f \) maps \( V \) into \( V^B \) and \( \text{tr}(\sigma_f) = \text{tr}(\sigma_f|_{V^B}) \). (Recall that \( \dim V^B < \infty \).) We see that the maps \( \sigma_f|_{V^B} \) define a (unital) \( H \)-module structure on \( V^B \). It is known [BO] that the \( H \)-module \( V^B \) is irreducible. Moreover for \( w \in W \) we have \( \text{tr}(\sigma_{\overline{w}}) = \text{tr}(\overline{\sigma_w}) \) where the trace in the right side is taken in the \( H \)-module \( V^B \). We have the following result.

**Theorem 4.3.** Assume that \( K \) has characteristic zero and that \( p \) is sufficiently large. Let \( y \in Y^+ \) and let \( t \in T(K)^\bullet \). We have

\[
\phi_V(t) = q^{-\langle y, 2\rho \rangle} \text{tr}(\Sigma_y)
\]

where the trace in the right side is taken in the irreducible \( H \)-module \( V^B \).

An equivalent statement is that

\[
\phi_V(t) = \frac{\text{tr}(\sigma_{\Sigma_y})}{\text{vol}(\Omega_y)}.
\]

(Recall that \( \Sigma_y \) in the right hand side is the characteristic function of \( \Omega_y = BT(K)_yB \).)

The assumption on characteristic in the theorem is needed only to be able to use a result from [AK], see 5.1(†). We expect that the theorem holds without that assumption.

In the case where \( y = 0 \) the theorem becomes:

(a) If \( t \in T(K) \cap G_{\text{cvr}} \) then \( \phi_V(t) = \dim(V^B) \).

As pointed out to us by R. Bezrukavnikov and S. Varma, in the special case where \( y \in Y^{++} \), Theorem 4.3 can be deduced from results in [C2].

**4.4.** In the case where \( V = S \), see 1.1, for any \( y \in Y^+ \), \( \Sigma_y \) acts on the one dimensional vector space \( V^B \) as the identity map so that \( \phi_V(t) = q^{-\langle y, 2\rho \rangle} \); we thus recover Theorem 2.2 (which holds without assumption on the characteristic).
$B_n/T_nB_{n+1}$ is abelian. Take $g = \prod_{\alpha \in \mathcal{Z}} x_\alpha((1 - \alpha(t' - 1))^{-1}a_\alpha)$. Then, we have $t' - 1 gt'g^{-1} \equiv z^{-1} \pmod{T_nB_{n+1}}$. Moreover, since $y \in Y^+$, we have $|1 - \alpha(t' - 1)| \geq 1$ and thus $g \in B_n$. (We argue as in 2.4(a).) Assume first that $\alpha \in R^+$. If $v(\alpha(t' - 1)) \neq 0$ then $v(\alpha(t' - 1)) < 0$ (since $\langle y, \alpha \rangle \neq 0$, $\langle y, \alpha \rangle \geq 0$) hence $v(1 - \alpha(t' - 1)) = v((\alpha(t' - 1)) < 0$ and $|1 - \alpha(t' - 1)| > 1$. If $v(\alpha(t' - 1)) = 0$ then $\alpha(t' - 1) - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1 - \alpha(t' - 1)) = 0$ and $|1 - \alpha(t' - 1)| = 1$ as required. Assume next that $\alpha \in R^-$. If $v(\alpha(t' - 1)) \neq 0$ then $v(\alpha(t' - 1)) > 0$ (since $\langle y, \alpha \rangle \neq 0$, $\langle y, \alpha \rangle \leq 0$) hence $v(1 - \alpha(t' - 1)) = 0$ and $|1 - \alpha(t' - 1)| = 1$ as required. If $v(\alpha(t' - 1)) = 0$ then $\alpha(t' - 1) - 1 \in \mathcal{O} - \mathfrak{p}$ hence $v(1 - \alpha(t' - 1)) = 0$ and $|1 - \alpha(t' - 1)| = 1$ as required.

Writing $\text{Ad}(g)(t'z) = t' \cdot (t' - 1 gt'g^{-1}) \cdot (gzg^{-1})$, we observe that $gzg^{-1} \equiv z \pmod{B_{n+1}}$ and $t' - 1 gt'g^{-1}z \in T_nB_{n+1}$. Hence $\text{Ad}(g)(t'z)$ can be written as $t't'z'$ with $t' \in T_n$ and $z' \in B_{n+1}$. □

**Lemma 5.3.** $B_1tB_1 \subseteq G(tT_1)$.

**Proof.** It is enough to show that $tB_1 \subseteq G(tT_1)$. Let $t_0z_1 \in tB_1$ with $t_0 = t$ and $z_1 \in B_1$. We will construct inductively sequences $g_1, g_2, \ldots, t_1, t_2 \cdots$ and $z_1, z_2, \cdots$ such that $\text{Ad}(g_k \cdots g_2g_1)(t_0z_1) = \text{Ad}(g_k)(t_0t_1 \cdots t_{k-1}z_k) = (t_0t_1 \cdots t_k)z_{k+1}$ with $g_i \in B_i$, $t_i \in T_i$ and $z_i \in B_i$.

Applying Lemma 5.2 to $n = 1$, $t' = t_0$ and $z = z_1$, we find $t_1 \in T_1$ and $z_2 \in B_2$ such that $g_1t_0z_1g_1^{-1} = t_0t_1z_2$ with $t_1 \in T_1$ and $z_2 \in B_2$. Suppose we found $g_i \in B_i$, $z_{i+1} \in B_{i+1}$ and $t_i \in T_i$ for $i = 1, \cdots k$ where $k \geq 1$. Applying Lemma 5.2 to $n = k + 1$, $t' = t_0t_1 \cdots t_k$ and $z = z_{k+1}$, we find $g_{k+1} \in B_{k+1}$, $t_{k+1} \in T_{k+1}$ and $z_{k+2} \in B_{k+2}$ so that $g_{k+1}t_0t_1 \cdots t_kz_{k+1}g_{k+1}^{-1} = \text{Ad}(g_{k+1} \cdots g_2g_1)(t_0z_1) = t_0t_1t_2 \cdots t_{k+1}z_{k+2}$. (To apply Lemma 5.2 we note that $t' \in T(K)_y^\bullet$ since $t_0 \in T(K)_y$ and $t_1 \cdots t_k \in T_1$ so that for any $\alpha \in R$ we have $\alpha(t_1 \cdots t_k) \in 1 + \mathfrak{p}$.) Taking $g \in B_1$ be the limit of $g_k \cdots g_2g_1$ as $k \to \infty$, we have $\text{Ad}(g)(t_0z_1) \in tT_1$. □

**5.4.** Continuing with the proof of Theorem 4.3, we note that by Lemma 5.3 and 5.1(†), for the characteristic function $f_t$ of $B_1tB_1$ we have

\[(*) \quad \text{tr}(\sigma_{f_t}) = \int_G f_t(g)\phi_V(g) \, dg = \int_{B_1tB_1} \phi_V(t) \, dg = \text{vol}(B_1tB_1)\phi_V(t).\]

Thus it remains to show that

\[\text{tr}(\sigma_{f_t})/\text{vol}(B_1tB_1) = \text{tr}(\sigma_{T_y^\mathbb{C}})/\text{vol}(BtB).\]

Since $B_1$ is normalized by $B$, $B$ acts on $V^{B_1}$. Moreover, since $V$ is irreducible and $V^B \neq 0$, $B$ acts trivially on $V^{B_1}$ (otherwise, there would exist a nonzero subspace of $V$ on which $B$ acts through a nontrivial character of $B/B_1$; since $V^B \neq 0$ we see that $(V, \sigma)$ would have two distinct cuspidal supports, a contradiction). Thus
we have $V^{B_1} = V^B$. Since $\sigma_{f_t}$ and $\sigma_{\Xi_y}$ have image contained in $V^{B_1} = V^B$, it is enough to show that

\[(a) \quad \text{tr}(\sigma_{f_t}|_{V_B})/\text{vol}(B_1 t B_1) = \text{tr}(\sigma_{\Xi_y}|_{V_B})/\text{vol}(B t B).
\]

We can find a finite subset $L$ of $T(K)_0$ such that $B t B = \sqcup_{\tau \in L} B_1 t B_1 \tau$. It follows that

\[(b) \quad \text{vol}(B t B) = \text{vol}(B_1 t B_1)^\sharp(L)
\]
and $\sigma_{\Xi_y} = \sum_{\tau \in L} \sigma_{f_t} \sigma(\tau)$ as linear maps $V \to V$. Restricting this equality to $V^B$ and using the fact that $\sigma(\tau)$ acts as identity on $V^B$ we obtain

\[(c) \quad \sigma_{\Xi_y}|_{V_B} = \sharp(L) \sigma_{f_t}|_{V_B}
\]
as linear maps $V^B \to V^B$. Clearly, (a) follows from (b) and (c). This completes the proof of Theorem 4.3.

The following result will not be used in the rest of the paper.

**Proposition 5.5.** If $y \in Y^{++}$ and $t \in T(K)_y$ then $B t B \subset G^\sigma(T(K)_y)$.

**Proof.** It is enough to show that $t z \subset G^\sigma(T(K)_y)$ for any $z \in B$. We can write $z = t_0 z'$ where $t_0 \in T(K)_0, z' \in B_1$. We have $t z = t t_0 z'$ where $t t_0 \in T(K)_y = T(K)^y$ (here we use that $y \in Y^{++}$). Using Lemma 5.3 we have $t t_0 z' \in G(t t_0 T_1) \subset G^\sigma(T(K)_y)$.

This completes the proof. $\square$

**5.6.** In the remainder of this section we assume that $G$ is adjoint. In this case the irreducible representations $(V, \sigma)$ as in 4.2 (up to isomorphism) are known to be in bijection with the irreducible finite dimensional representations of the Hecke algebra $H$ (see [BO]) by $(V, \sigma) \mapsto V^B$. The irreducible finite dimensional representations of $H$ have been classified in [KL1] in terms of geometric data. Moreover in [L] an algorithm to compute the dimensions of the (generalized) weight spaces of the action of the commutative semigroup $\{\Xi_y; y \in Y^{++}\}$ on any tempered $H$ module is given. In particular the right hand side of the equality in Theorem 4.3 (hence also $\phi_V(t)$ in that Theorem) is computable when $V$ is tempered.

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