Conditional Equi-concentration of Types

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To Mar, in memoriam

Abstract

Conditional Equi-concentration of Types on $I$-projections is presented. It provides an extension of Conditional Weak Law of Large Numbers to the case of several $I$-projections. Also a multiple $I$-projections extension of Gibbs Conditioning Principle is developed. $\mu$-projection variants of the probabilistic laws are stated. Implications of the results for Relative Entropy Maximization, Maximum Probability, Maximum Entropy in the Mean and Maximum Rényi-Tsallis Entropy methods are discussed.

Key Words and Phrases: multiple $I$-projections, Conditional Weak Law of Large Numbers, Gibbs Conditioning Principle, $\mu$-projection, Maximum Probability method, MaxProb/MaxEnt convergence, Maximum Rényi-Tsallis Entropy method, Maximum Entropy in the Mean, triple point

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1 Terminology and Notation

Let $\{X\}_{l=1}^{n}$ be a sequence of independently and identically distributed random variables with a common law (measure) on a measurable space. Let the measure be concentrated on finite number $m$ of atoms from the set $X = \{x_1, x_2, \ldots, x_m\}$ called support or alphabet. Let $q_i$ denote the probability (measure) of $i$-th element of $X$: $q$ will be assumed strictly positive and called source or generator. Let $P(X)$ be a set of all probability mass functions (pmf’s) on $X$.

A type (also called $n$-type, empirical measure, frequency distribution or occurrence vector) induced by a sequence $\{X\}_{l=1}^{n}$ is pmf $\nu^n \in P(X)$ whose $i$-th element $\nu^n_i$ is defined as: $\nu^n_i = n_i/n$ where $n_i = \sum_{l=1}^{n} I(X_l = x_i)$; there $I(\cdot)$ is the indicator function. Multiplicity $\Gamma(\nu^n)$ of type $\nu^n$ is: $\Gamma(\nu^n) = n! \prod_{i=1}^{m} n_i!$.

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Let $\Pi \subseteq \mathcal{P}(X)$. Let $P_n$ denote a subset of $\mathcal{P}(X)$ which consists of all $n$-types. Let $\Pi_n = \Pi \cap P_n$.

On $\mathcal{P}(X)$ topology induced by the standard topology on $\mathbb{R}^m$ is assumed.

$\mu$-projection $\hat{\nu}^n$ of $q$ on $\Pi_n \neq \emptyset$ is defined as: $\hat{\nu}^n = \arg \sup_{\nu^n \in \Pi_n} \pi(\nu^n; q)$, where $\pi(\nu^n; q) = \Gamma(\nu^n) \prod(q_i)^{\nu^n_i}$. Alternatively, the $\mu$-projection can be defined as $\hat{\nu}^n = \arg \sup_{\nu^n \in \Pi_n} \pi(\nu^n| \nu^n \in \Pi_n; q)$, where $\pi(\nu^n| \nu^n \in \Pi_n; q)$ denotes the conditional probability that if an $n$-type belongs to $\Pi_n$ then it is just the type $\nu^n$. $\mu$-projection can be also equivalently defined as the supremum of posterior probability, cf. [14].

$I$-projection $\hat{p}$ of $q$ on $\Pi$ is $\hat{p} = \inf_{p \in \Pi} I(p||q)$, where $I(p||q) = \sum_X p_i \log \frac{p_i}{q_i}$ is the $I$-divergence (also known as Kullback-Leibler distance, ± relative entropy).

$\pi(\nu^n \in A| \nu^n \in B; q)$ will denote the conditional probability that if a type drawn from $q \in \mathcal{P}(X)$ belongs to $B \subseteq \Pi$ then it belongs to $A \subseteq \Pi$.

2 Boltzmann Jaynes Inverse Problem and Conditional Law of Large Numbers

Having the terminology introduced, Boltzmann Jaynes Inverse Problem (BJIP) can be stated as follows: there is the source $q$ and a set $\Pi_n$ of $n$-types. It is necessary to select an $n$-type (one or more) from the set $\Pi_n$. To solve BJIP it is necessary to provide an algorithm for selection of type from $\Pi_n$ when the information-quadruple $\{X, n, q, \Pi_n\}$ and nothing else is supplied. Clearly, if $\Pi_n$ contains more than one type, BJIP becomes an under-determined and in this sense ill-posed problem.

Usually, BJIP is solved by means of the method of Relative Entropy Maximization (REM/MaxEnt). This is mostly done for $n \to \infty$. In this case the set of types $\Pi_n$ effectively turns into a set of probability mass functions $\Pi$.

Typically, $\Pi$ is defined by moment consistency constraints (mcc) of the following form: $\Pi_{mcc} = \{ p : \sum_{i=1}^m p_i u_i = a, \sum_{i=1}^m p_i = 1 \}$, where $a \in \mathbb{R}$ is a given number, $u$ is a given vector. The feasible set $\Pi_{mcc}$ which mcc define is convex and closed. $I$-projection $\hat{p}$ of $q$ on $\Pi_{mcc}$ is unique and belongs to the exponential family of distributions; $\hat{p}_i = k(\lambda) q_i e^{-\lambda u_i}$, where $k(\lambda) = 1/\sum_{i=1}^m q_i e^{-\lambda u_i}$, and $\lambda$ is such that $\hat{p}$ satisfies mcc.

In the case of BJIP with $\Pi_{mcc}$, or in general for any closed, convex, rare set $\Pi$, application of REM/MaxEnt method is justified by Conditional Weak Law of Large Numbers (CWLLN). CWLLN, in its textbook form, reads [4]:

**CWLLN.** Let $X$ be a finite set. Let $\Pi$ be a closed, convex set which does not contain $q$. Let $n \to \infty$. Then for $\epsilon > 0$ and $i = 1, 2, \ldots, m$,

$$
\lim_{n \to \infty} \pi(|\nu^n_i - \hat{p}_i| \leq \epsilon | \nu^n \in \Pi; q) = 1.
$$

CWLLN says that if types are confined to the set $\Pi$ then they asymptotically conditionally concentrate on the $I$-projection $\hat{p}$ of the source of types $q$ on the set $\Pi$. Stated, informally, from another side: if a source $q$ is confined to produce types from convex and closed $\Pi$ it is asymptotically conditionally ‘almost

\footnote{In the simplest case of single non-trivial constraint.}
impossible to find a type other than the one which has the highest/supremal value of relative entropy with respect to $q$.

Conditional Weak Law of Large Numbers emerged from a series of works which include [34, 1, 40, 28, 21, 39, 41, 38, 36, 5, 4, 8, 30, 29]. For new developments see [23].

An information-theoretic proof (see [4]) of CWLLN utilizes so-called Pythagorean theorem (cf. [2]), Pinsker inequality and standard inequalities for factorial. The Pythagorean theorem is known to hold for closed convex sets.

Alternatively, CWLLN can be obtained as a consequence of Sanov's Theorem (ST). The ST-based proof of CWLLN will be recalled here. First, Sanov’s Theorem and its proof (adapted from [7], [4]).

**Sanov’s Theorem.** Let $X$ be finite. Let $A \subseteq \Pi$ be an open set. Then

$$\lim_{n \to \infty} \frac{1}{n} \log \pi(\nu^n \in A) = -I(\hat{p}||q),$$

where $\hat{p}$ is an I-projection of $q$ on $A$.

**Proof.** [1, 7] $\pi(\nu^n \in A) = \sum_{\nu^n \in A} \pi(\nu^n; q)$. Upper and lower bounds on $\pi(\nu^n; q)$ (recall proof of the Lemma at Appendix):

$$\left(\frac{m}{n}\right)^m \prod_{i=1}^m \left(\frac{q_i}{\nu_i^n}\right)^{\nu_i^n} < \pi(\nu^n; q) \leq \prod_{i=1}^m \left(\frac{q_i}{\nu_i^n}\right)^{\nu_i^n}.$$

$$\sum_{\nu^n \in A} \pi(\nu^n; q) < N \prod_{i=1}^m \left(\frac{q_i}{\nu_i^n}\right)^{\nu_i^n},$$

where $N$ stands for number of all $n$-types and $\nu_i^n$ is an I-projection of $q$ on $A_n = A \cap P_n$ (i.e., any of the $n$-types which attain supremal value of $\prod_{i=1}^m \left(\frac{q_i}{\nu_i^n}\right)^{\nu_i^n}$). $N$ is smaller than $(n+1)^m$.

Thus

$$\frac{1}{n} \left(n \sum_{i=1}^m \hat{\nu}_i^n \log \frac{q_i}{\hat{\nu}_i^n} + m(\log m - \log n)\right) < \frac{1}{n} \log \pi(\nu^n \in A)$$

$$< \frac{1}{n} \left(\sum_{i=1}^m \hat{\nu}_i^n \log \frac{q_i}{\hat{\nu}_i^n} + m \log(n+1)\right).$$

Since $A$ is by the assumption open and under the maintained assumption of strictly positive $q$ it is also continuous, $\lim_{n \to \infty} \sum_{i=1}^m \hat{\nu}_i^n \log \frac{q_i}{\hat{\nu}_i^n} = \sum_{i=1}^m \hat{\nu}_i \log \frac{q_i}{\hat{\nu}_i},$ where $\hat{p}$ is an I-projection of $q$ on $A$. Thus, for $n \to \infty$ the upper and lower bounds on $\frac{1}{n} \log \pi(\nu^n \in A)$ collapse into $\sum_{i=1}^m \hat{\nu}_i \log \frac{q_i}{\hat{\nu}_i}$. \qed

**A proof of CWLLN.** [2] Let $A = \{p : |p_i - \hat{p}_i| > \epsilon, i = 1, 2, \ldots, m\}$. Then ST can be applied to it, leading $\lim_{n \to \infty} \frac{1}{n} \log \pi(\nu^n \in A|\nu^n \in \Pi; q) = -I(\hat{p}_\Lambda||q) - I(\hat{p}_\Pi||q)$. Since $I(\hat{p}_\Lambda||q) - I(\hat{p}_\Pi||q) > 0$ and since the set $\Pi$ admits unique I-projection (the uniqueness arises from the fact that the set is convex and closed, and $I(\cdot||\cdot)$ is convex), the proof is complete. \qed

CWLLN can be viewed as a special case of a stronger result, which is commonly known as Gibbs Conditioning Principle (GCP), see Sect. 5.
3 Motivation and Programme

Frequency moment constraints considered by physicists (see for instance [33]) define a non-convex feasible set of probability distributions which in general can admit multiple $I$-projections. This work builds upon [12], [27], [13], [15], [16] and aims to develop an extension of CWLLN and Gibbs Conditioning Principle to the case of multiple $I$-projections.

It has also another goal: to introduce concept of $\mu$-projection and to formulate $\mu$-projection variants of the probabilistic laws. They, among other things, allow for a more elementary reading than their $I$-projection counterparts. At the same time they provide a probabilistic justification of Maximum Probability method [11].

The paper is organized as follows: in the next section some basic questions regarding asymptotic behavior of conditional probability are posed. Two illustrative examples are then used to introduce Conditional Equi-concentration of Types on $I$-projections. Next, an extension of Gibbs Conditioning Principle - the stronger form of CWLLN - is provided. Asymptotic identity of $I$-projections and $\mu$-projections is discussed in Section 6 and $\mu$-variants of the probabilistic laws are presented afterwards. Implications of the results for Maximum Entropy, Maximum Probability and Maximum R´enyi-Tsallis Entropy methods are drawn at Section 8. Section 9 mentions in passing other related results: $r$-tuple extension of CWLLN and Bayesian Conditional Law of Large Numbers. Section 10 summarizes the paper. Appendix contains a sketch of proof of ICET and of Extended GCP. It also shows that concentration of types can in some sense happen also on isolated $I$-projections, provided that they are rational.

4 Conditional Equi-concentration of Types

What happens when $\Pi$ admits multiple $I$-projections? Do the conditional concentration of types happen on them? If yes, do the type concentrate on each of them? If yes, what is the proportion? In order to address these questions, it is instrumental to consider a couple of examples.

Example 1. [13] Let $\Pi = \{ \nu : \sum_{i=1}^{m} p_i^n = a, \sum_{i=1}^{m} p_i - 1 = 0 \}$, where $\alpha, a \in \mathbb{R}$. Note that the first constraint, known as frequency constraint, is non-linear in $p$ and $\Pi$ is for $|\alpha| > 1$ non-convex.

Let $\alpha = 2$, $m = 3$ and $a = 0.42$ (the value was obtained for $p = [0.5 \ 0.4 \ 0.1]$). Then there are the following three $I$-projections of uniform distribution $q$ on $\Pi$: $\hat{p}_1 = [0.5737 \ 0.2131 \ 0.2131]$, $\hat{p}_2 = [0.2131 \ 0.5737 \ 0.2131]$ and $\hat{p}_3 = [0.2131 \ 0.2131 \ 0.5737]$ (see [13]). Note that they form a group of permutations. As it will become clear later, it suffices to investigate convergence to say $\hat{p}_1$.

For $n = 30$ there are only two groups of types in $\Pi$: $G_1$ comprises $[0.5666 \ 0.2666 \ 0.1666]$ and five other permutations; $G_2$ consists of $[0.5 \ 0.4 \ 0.1]$ and the other five permutations. So, together there are 12 types.

Value of the square of the Euclidean distance $\delta$ between $\nu$ and $\hat{p}_1$ attains its minimum $\delta_{G_1} = 0.0051$ within $G_1$ group for the following two types: $[0.5666 \ 0.2666 \ 0.1666]$, $[0.5666 \ 0.1666 \ 0.2666]$. Within $G_2$ the smallest $\delta_{G_2} = 0.0532$ is attained by $[0.5 \ 0.4 \ 0.1]$ and $[0.5 \ 0.1 \ 0.4]$.  

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Thus, if $\epsilon = \epsilon_1$ is chosen so that the $\epsilon$-ball $B(\hat{p}_1, \epsilon_1)$ centered at $\hat{p}_1$ contains only the two types from G1 (which at the same time guarantees that $\hat{p}_1$ is the only $I$-projection in the ball), then $\pi(\nu \in B(\hat{p}_1, \epsilon_1)|\nu \in \Pi) = 2 \cdot 0.1152 = 0.2304$. In words: probability that if $q$ generated a type from $\Pi$ than the type falls into the ball containing only types which are closest to the $I$-projection is 0.2304. If $\epsilon = \epsilon_2$ is chosen so that also the two types from G2 are included in the ball and also so that it is the only $I$-projection in the ball (any $\epsilon_2 \in (\sqrt{0.0532}, \sqrt{0.1253})$ satisfies both the requirements), then $\pi(\nu^\bullet \in B(\hat{p}_1, \epsilon_2)|\nu^\bullet \in \Pi) = \frac{1}{3}$.

For $n = 330$ there are four groups of types in $\Pi$: G1, G2 and a couple of new one: G3 consists of [0.4727 0.4333 0.0939] and all its permutations; G4 comprises the type [0.5727 0.2333 0.1939] and its permutations. Hence, the total number of types from $\Pi$ which are supported by random sequences of size $n = 330$ is 24.

$\delta_{G3}$ for the two types from G3 which are closest to $\hat{p}_1$ is 0.0729. The smallest $\delta_{G4} = 0.00077$ is attained by [0.5727 0.2333 0.1939] and by [0.5727 0.1939 0.2333]. Thus, clearly, the two types from G4 have the smallest Euclidean distance to $\hat{p}_1$ among all types from $\Pi$ which are based on samples of size $n = 330$. Again, setting $\epsilon$ such that the ball $B(\hat{p}_1, \epsilon)$ contains only the two types which are closest to $\hat{p}_1$ leads to the 0.261 value of the conditional probability. Note the important fact, that the probability has risen, as compared to the corresponding value 0.2304 for $n = 30$.

Moreover, if $\epsilon$ is set such that besides the two types from G4 also the second closest types (i.e. the two types from G1) are included in the ball then the conditional probability is indistinguishable from $\frac{1}{3}$. Hence, there is virtually no conditional chance of observing any of the remaining 4 types. The same holds for the types which concentrate around $\hat{p}_2$ or $\hat{p}_3$. Thus, in total, a half of the 24 types is almost impossible to observe.

The Example illustrates, that the conditional probability of finding a type which is close (in the Euclidean distance) to one of the three $I$-projections goes to $\frac{1}{3}$.

Example 2. Let $\Pi = \Pi_1 \cup \Pi_2$, where $\Pi_j = \{\nu: \sum_{i=1}^{m} p_i x_i = a_j; \sum_{i=1}^{m} p_i = 1\}, j = 1, 2$. Thus $\Pi$ is union of two sets, each of whose is given by the moment consistency constraint. If $q$ is chosen to be the uniform distribution, then values $a_1, a_2$ such that there will be two different $I$-projections of the uniform $q$ on $\Pi$ with the same value of $I$-divergence (as well as of the Shannon’s entropy) can be easily found. Indeed, for any $a_1 = \mu + \Delta$, $a_2 = \mu - \Delta$, where $\mu = E X$ and $\Delta \in (0, (X_{\text{max}} - X_{\text{min}})/2)$, $\hat{p}_1$ is just a permutation of $\hat{p}_2$, and as such attains the same value of Shannon’s entropy. To see that types which are based on random samples of size $n$ from $\Pi$ indeed concentrate on the $I$-projections with equal measure note, that for any $n$ to each type in $\Pi_1$ corresponds a unique permutation of the type in $\Pi_2$. Thus, types in $\epsilon$-ball with center at $\hat{p}_1$ have the same conditional probabilities $\pi$ as types in the $\epsilon$-ball centered at $\hat{p}_2$. This, together with convexity and closed-ness of both $\Pi_j$, for which the conditional concentration of types on the respective $I$-projection is established by CWLLN, directly implies that

$$\lim_{n \to \infty} \pi(\nu \in B(\hat{p}_j, \epsilon)|\nu^\bullet \in \Pi) = \frac{1}{2} \quad j = 1, 2.$$
Conditional Equi-concentration of Types on $I$-projections (ICET) attempts to capture behavior of the conditional measure which the above Examples illustrate. To this end, notion of the proper $I$-projection will be needed.

$I$-projection $\hat{p}$ of $q$ on $\Pi$ will be called proper if $\hat{p}$ is not isolated point of $\Pi$.

ICET. Let $X$ be finite. Let there be $k$ proper $I$-projections $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_k$ of $q$ on $\Pi$. Let $\epsilon > 0$ be such that for $j = 1, 2, \ldots, k$ $\hat{p}_j$ is the only proper $I$-projection of $q$ on $\Pi$ in the ball $B(\hat{p}_j, \epsilon)$. Let $n \to \infty$. Then for $j = 1, 2, \ldots, k$,

$$\pi(\nu^n \in B(\epsilon, \hat{p}_j) | \nu^n \in \Pi; q) = 1/k.$$  

ICET says, informally, that source/generator $q$, when confined to produce types from a set $\Pi$, - as $n$ gets large - hides itself behind any of the proper $I$-projections equally likely.

Expressed in Statistical Physics terminology ICET says that each of equilibrium points ($I$-projections) is asymptotically conditionally equally possible. The Conditional Equi-concentration of Types 'phenomenon' resembles the triple point phenomenon of Thermodynamics.

A sketch of proof of ICET is relegated to the Appendix.

5 Gibbs Conditioning Principle and its Extension

Gibbs conditioning principle (cf. [5], [8], [29]) - also known as the stronger form of CWLLN - complements CWLLN by stating that:

**GCP.** Let $X$ be a finite set. Let $\Pi$ be closed, convex set. Let $n \to \infty$. Then for a fixed $t$,

$$\lim_{n \to \infty} \pi(X_1 = x_1, \ldots, X_t = x_t | \nu^n \in \Pi; q) = \prod_{l=1}^{t} \hat{p}_{x_l}.$$  

GCP, says, very informally, that if the source $q$ is confined to produce sequences which lead to types in a set $\Pi$ then elements of any such sequence (of fixed length $t$) behave asymptotically conditionally as if they were drawn identically and independently from the $I$-projection of $q$ on $\Pi$ - provided that the last is unique (among other things).

GCP was developed at [5] under the name of conditional quasi-independence of outcomes. Later on, it was brought into more abstract form in large deviations literature, where it also obtained the GCP name (cf. [8], [29]). A simple proof of GCP can be found at [7]. GCP is proven also for continuous alphabet (cf. [21], [7], [8]).

The following theorem provides an extension of GCP to the case of multiple $I$-projections.

**EGCP.** Let there be $k$ proper $I$-projections $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_k$ of $q$ on $\Pi$. Then for a fixed $t$ and $n \to \infty$,

$$\pi(X_1 = x_1, \ldots, X_t = x_t | \nu^n \in \Pi; q) = \frac{1}{k} \sum_{j=1}^{k} \prod_{l=1}^{t} \hat{p}_{x_l}^j.$$
For $t = 1$ Extended Gibbs Conditioning Principle (EGCP) says that the conditional probability of a letter is asymptotically given by the equal-weight mixture of proper $I$-projection probabilities of the letter. For a general sequence, EGCP states that the conditional probability of a sequence is asymptotically equal to the mixture of joint probability distributions. Any $(j$-th) of the $k$ joint distributions is such as if the sequence was iid distributed according to a $(j$-th) proper $I$-projection.

A proof of EGCP is sketched at the Appendix.

6 Asymptotic Identity of $\mu$-Projections and $I$-Projections

At ([11], Thm 1 and its Corollary, aka MaxProb/MaxEnt Thm) it was shown that maximum probability type converges to $I$-projection; provided that $\Pi$ is defined by a differentiable constraints. A more general result which states asymptotic identity of $\mu$-projections and $I$-projections for general set $\Pi$ was presented at [16].

MaxProb/MaxEnt. Let $X$ be finite set. Let $M_n$ be set of all $\mu$-projections of $q$ on $\Pi_n$. Let $I$ be set of all $I$-projections of $q$ on $\Pi$. For $n \to \infty$, $M_n = I$.

Since $\pi(\nu^n; q)$ is defined for $\nu^n \in Q^m$, $\mu$-projection can be defined only for $\Pi_n$ when $n$ is finite. The Thm permits to define a $\mu$-projection $\hat{\nu}$ also on $\Pi$: $\hat{\nu} = \arg \sup_{\nu \in \Pi} \sum_{i=1}^{m} r_i \log \frac{\nu_i}{q_i}$. The $\mu$-projections of $q$ on $\Pi$ and $I$-projections of $q$ on the same set $\Pi$ are undistinguishable.

It is worth highlighting that for a finite $n$, $\mu$-projections and $I$-projections of $q$ on $\Pi_n$ are in general different. This explains why $\mu$-form of the probabilistic laws deserves to be stated separately of the $I$-form; though formally they are undistinguishable. Thus, the MaxProb/MaxEnt Thm (in its new and to a smaller extent also in its old version) permits directly to state $\mu$-projection variants of CWLLN, GCP, CET and EGCP: $\mu$CWLLN, $\mu$GCP, $\mu$CET and Boltzmann Conditioning Principle (BCP).

7 $\mu$-Variants of the Probabilistic Laws

$\mu$-variant of CWLLN reads:

$\mu$CWLLN. Let $X$ be a finite set. Let $\Pi$ be closed, convex set. Let $n \to \infty$. Then for $\epsilon > 0$ and $i = 1, 2, \ldots, m$,

$$
\lim_{n \to \infty} \pi(|\nu_i^n - \hat{\nu}_i| < \epsilon | \nu^n \in \Pi; q) = 1.
$$

Core of $\mu$CWLLN can be loosely expressed as: types, when confined to a set $\Pi$, conditionally concentrate on the asymptotically most probable type $\hat{\nu}$.

$\mu$-projection $\hat{\nu}$ of $q$ on $\Pi$ will be called proper if $\hat{\nu}$ is not isolated point of $\Pi$.

$\mu$CET. Let $X$ be finite. Let there be $k$ proper $\mu$-projections $\hat{\nu}^1, \hat{\nu}^2, \ldots, \hat{\nu}^k$ of $q$ on $\Pi$. Let $\epsilon > 0$ be such that for $j = 1, 2, \ldots, k$ $\hat{\nu}^j$ is the only proper $\mu$-projection of $q$ on $\Pi$ in the ball $B(\hat{\nu}^j, \epsilon)$. Let $n \to \infty$. Then for $j = 1, 2, \ldots, k$,

$$
\pi(\nu^n \in B(\epsilon, \hat{\nu}^j) | \nu^n \in \Pi; q) = 1/k.
$$
Core of µ-variant of the Conditional Equi-concentration of Types states, loosely, that types conditionally concentrate on each of the asymptotically most probable types in equal measure.

**µGCP.** Let \( X \) be a finite set. Let \( \Pi \) be closed, convex set. Let \( n \to \infty \). Then for a fixed \( t \),

\[
\lim_{n \to \infty} \pi(X_1 = x_1, \ldots, X_t = x_t | \nu^n \in \Pi; q) = \prod_{l=1}^{t} \hat{\nu}_{x_l}.
\]

µ-variant of EGCP deserves a special name. It will be called Boltzmann Conditioning Principle (BCP).

**BCP.** Let there be \( k \) proper µ-projections \( \hat{\nu}^1, \hat{\nu}^2, \ldots, \hat{\nu}^k \) of \( q \) on \( \Pi \). Then for a fixed \( t \) and \( n \to \infty \),

\[
\pi(X_1 = x_1, \ldots, X_t = x_t | \nu^n \in \Pi; q) = \frac{1}{k} \sum_{j=1}^{k} \prod_{l=1}^{t} \hat{\nu}^j_{x_l}.
\]

8 Implications

The results have some implications for application of REM, MaxProb and Maximum Rényi-Tsallis Entropy methods to Boltzmann Jaynes Inverse Problem.

8.1 I- or µ-Projection? MaxEnt or MaxProb?

With µ-projection Maximum Probability method (MaxProb, [11]) is associated. Given the BJIP information-quadruple \( \{X, n, q, \Pi_n\} \), MaxProb prescribes to select from \( \Pi_n \) type(s) which has the supremal/maximal probability \( \pi(\nu^n; q) \).

µ-projections and I-projections are asymptotically indistinguishable. In plain words: for \( n \to \infty \) the Relative Entropy Maximization method (REM/MaxEnt) (either in its Jaynes’ [24, 25] or Csiszár’s interpretation [6]) selects the same distribution(s) as MaxProb (in its more general form which instead of the maximum probable types selects supremum-probable µ-projections). This result (in the older form, [11]) was at [11] interpreted as saying that REM/MaxEnt can be viewed as an asymptotic instance of the simple and self-evident Maximum Probability method.

Alternatively, [35] suggests to view REM/MaxEnt as a separate method and hence to read the MaxProb/MaxEnt Thm as claiming that REM/MaxEnt asymptotically coincides with MaxProb. If one adopts this interesting and legitimate view then it is necessary to face the fact that if \( n \) is finite, the two methods in general differ. This would open new questions. Among them also: MaxEnt/REM or MaxProb? (i.e., I- or µ-projection?) This is too delicate a question to be answered by one sentence. Let us note, only, that unless \( n \to \infty \) entropy ignores multiplicity.

\[\text{A technique for determination of } \mu \text{-projections was suggested at [17].}\]
8.2 $I/\mu$- or $\tau$-projection? MaxEnt/MaxProb or MaxTent?

The previous question (i.e., MaxEnt or MaxProb?) is a problem of drawing interpretational consequences from two variants of the same probabilistic laws, and in this sense it can be viewed as an 'internal problem' of MaxEnt and MaxProb. From outside, from the point of view of the Maximum Rényi-Tsallis Entropy method (maxTent, [37], [26], [3]) MaxProb and MaxEnt can be viewed as 'twins'.

maxTent is to the best of our knowledge intended by its proponents for selection of probability distribution(s) under the setting of BJIP with $\Pi$ defined by $X$-frequency moment constraints (cf. [15]). It is not known whether such a feasible set $\Pi$ admits unique distribution with maximal value of Rényi-Tsallis entropy (called $\tau$-projection at [15]) as it is also not known whether $I$-projection on such a set is unique or not. The non-uniqueness makes it difficult to rely upon CWLLN when one wants to draw from an established non-identity of $\tau$ and $I$-projection conclusion that maxTent method violates CWLLN, cf. [27]. At [15] this difficulty has been avoided by considering an instance of the $X$-frequency constraints where the feasible set reduced into a convex set. Since $\tau$- and $I$-projection on the set were shown to be different, CWLLN directly implies that maxTent in this case selects asymptotically conditionally improbable distribution. The Example below (taken from [15]) illustrates the point.

Example 3. [15] Let $\Pi = \{p : \sum_{i=1}^{3} p_i^3 (x_i - b) = 0, \sum_{i=1}^{3} p_i^2 - 1 = 0\}$. Let $X = [-2 0 1]$ and let $b = 0$. Then $\Pi = \{p : p_2 = 2p_1^2, \sum p_i - 1 = 0\}$ which effectively reduces to $\Pi = \{p : p_2 = 1 - p_1(1 + \sqrt{2}), p_3 = \sqrt{2}p_1\}$. The source $q$ is assumed to be uniform.

The feasible set $\Pi$ is convex. Thus $I$-projection $\hat{p}$ of $u$ on $\Pi$ is unique, and can be found by direct analytic maximization to be $\hat{p} = [0.2748 0.3366 0.3886]$. Straightforward maximization of Rényi-Tsallis’ entropy leads to maxTent pmf $\hat{p}_T = [0.2735 0.3398 0.3867]$, which is different than $\hat{p}$.

Convexity of the feasible set guarantees uniqueness of the $I$-projection, and consequently allows to invoke CWLLN to claim that any pmf from $\Pi$ other than the $I$-projection has asymptotically zero conditional probability that it will be generated by $u$.

Obviously, ICET permits to show the fatal flow of maxTent in a more direct and more general way.

9 Further Results

Some further results related to asymptotic concentration of conditional probability are contained in this Section.

9.1 $r$-tuple ICET/CWLLN and MEM/GME Methods

Maximum Entropy in the Mean method (MEM), or its discrete-case relative, Generalized Maximum Entropy (GME) method, are interesting extensions of the standard REM/MaxEnt method. Though, usually a hierarchical structure

\[3\] For a tutorial on MEM see [22]. GME was introduced at [10], see also [31].
of the methods is highlighted, here a different feature of the method(s) will appear to be important.

First, Golan-Judge-Miller ill-posed inverse problem (GJMIP) has to be introduced. Its simple instance can be described as follows: Let there be two independent sources \( q^1, q^2 \) of sequences and hence types. Let \( X, Y \) be support of the first, second source, respectively. Let a set \( C_n \) comprise pairs of the types \([\nu^{n,1}, \nu^{n,2}]\) which were drawn at the same time. GJMIP amounts to selection of specific pair(s) of types from \( C_n \) when the information \( \{X, Y, n, q^1 \perp q^2, C_n\} \) is supplied.

**Example 4.** An example of GJMIP. Let \( X = Y = [1, 2, 3] \). Let \( q^1 = q^2 = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right] \); \( q^1 \perp q^2 \); \((q^1 \mapsto \nu^{n,1}) \land (q^2 \mapsto \nu^{n,2})\). Let \( n = 100, C_n = \{[\nu^{n,1}, \nu^{n,2}] : \sum_{i=1}^{3} \nu^{n,1}_i x_i + \nu^{n,2}_i y_i = 4; \sum_{i=1}^{3} \nu^{n,1}_i = 1; \sum_{i=1}^{3} \nu^{n,2}_i = 1\}\). Given this information, it is necessary to select a pair (one or more) of types from \( C_n \).

Since throughout the paper discrete and finite alphabet is assumed, GME will be considered instead of MEM, in what follows. The important feature of GME is that it selects jointly and independently drawn pairs (or \( r \)-tuples) of types/pmfs. Thus, it is suitable for application at the GJMIP context. An \( r \)-tuple extension of CWLLN (\( r \)-CWLLN) provides a probabilistic justification to the GME, at the GJMIP context.

Given GJMIP information, GME selects from the feasible set of the pairs of pmfs the one \([\hat{p}^1, \hat{p}^2]\) (or more) which maximizes sum of the relative entropies with respect to \( q^1, q^2 \); respectively.

\((r = 2)\)-tuple CWLLN. Assume a GJMIP. Let \( C \) be convex, closed set. Let \( B([\hat{p}^1, \hat{p}^2], \epsilon) \) be an \( \epsilon \)-ball centered at the pair

\[ [\hat{p}^1, \hat{p}^2] = \arg \sup_{[p^1, p^2] \in C} \sum_{i=1}^{m^1} p^1_i \log \frac{p^1_i}{q^1_i} + \sum_{j=1}^{m^2} p^2_j \log \frac{p^2_j}{q^2_j}. \]

Let \( n \to \infty \). Then

\[ \pi([\nu^{n,1}, \nu^{n,2}] \in B([\hat{p}^1, \hat{p}^2], \epsilon) | [\nu^{n,1}, \nu^{n,2}] \in C; (q^1 \mapsto \nu^{n,1}) \land (q^2 \mapsto \nu^{n,2}); q^1 \perp q^2) = 1 \]

Proof of \( r \)-CWLLN can be constructed along the same lines as the proof of CWLLN; the assumption that the pairs of sequences/types are drawn at the same time and from independent sources is crucial for establishing the result. Similarly, \( r \)-generalization of JCET can be formulated and proven; and obviously \( \mu \)-variants of the results hold true.

\( r \)-CWLLN permits to rise the same objections to application of Rényi entropy based variant of GME in the GJMIP context, as those that were risen to maxTent in the BJIP context.

Needless to say, \( \mu \)-variant of \( r \)-CWLLN provides a probabilistic justification to MaxProb variant of GME.

### 9.2 Bayesian Conditional Law of Large Numbers

It is worth a brief mentioning, that there is an inverse problem which is in a sense antipodal to Boltzmann Jaynes Inverse Problem. Let us call it the \( \beta \)-problem, after [19].
One form of the β-problem can be formulated as follows: let there be a set Q of sources over which a prior distribution π(·) is specified. Let \( \nu^n \) be an \( n \)-type drawn from a source \( r \), not necessarily in Q. It is necessary to select a source \( q \in Q \), given the information-pentad \{n, X, \nu^n, Q, \pi(\cdot)\}.

Conditional Law of Large Number for Sources [15, 20] is concerned about the asymptotic behavior of posterior probability \( \pi(q \in B(q \in Q) \land \nu^n) \). It states that, under certain conditions, the posterior probability asymptotically piles up on the \( L \)-projection of \( r \) on Q. Hence, the particular β-problem has to be solved by \( L \)-divergence maximization method.

An application of Conditional Limit Theorem for Sources to criterion choice problem associated with the Empirical Estimation [31, 32] as well as further discussion, can be found at [19].

### 10 Summary

Conditional Equi-concentration of Types on \( I \)-projections – an extension of CWLLN to the case of non-unique \( I \)-projection – was presented. ICET states that the conditional concentration of types happens on each of the proper \( I \)-projections in equal measure. Also, Gibbs Conditioning Principle was enhanced to capture multiple \( I \)-projections. Extended GCP says (when \( t = 1 \)) that conditional probability of a letter is asymptotically given by the equal-weight mixture of proper \( I \)-projection probabilities of the letter. The conditional equi-concentration/equi-probability ‘phenomenon’ is in our view an interesting feature of ‘randomness’. It might be of some interest also for Statistical Mechanics as it resembles phase coexistence of Thermodynamics (e.g. triple point of water, vapor and ice).

A general form of MaxProb/MaxEnt Thm, which states asymptotic identity of \( I \)- and \( \mu \)-projections, was recalled. It permits to formulate \( \mu \)-projection variants of the corresponding \( I \)-projection laws: CWLLN/GCP/ICET/EGCP. In our view, the \( \mu \)-variants allow for a deeper reading than their \( I \)-projection counterparts – since the \( \mu \)-laws express the asymptotic conditional behavior of types in terms of the most probable types. For instance, \( \mu \)-projection variant of CWLLN says that types conditionally concentrate on the asymptotically most probable one. This is, in our view, more obvious statement than that made by \( I \)-variant of CWLLN. MaxProb/MaxEnt Theorem is also instrumental for establishing of ICET.

The main results – Conditional Equi-concentration of Types (CET) in both its \( I \)- and \( \mu \)-projection form as well as Extended Gibbs Conditioning and Boltzmann Conditioning – were supplemented also by further considerations. They are summarized below.

Though \( \mu \)-projections and \( I \)-projections asymptotically coincide, for a finite \( n \) they are, in general, different. In light of this fact, the asymptotic identity of \( \mu \)- and \( I \)-projections can be viewed in two ways. Either as saying that 1) \( I \)-projection of \( q \) on \( \Pi \) is the asymptotic form of \( \mu \)-projection of \( q \) on \( \Pi_n \) or that 2) \( \mu \)-projections on \( \Pi_n \) and \( I \)-projections on \( \Pi_n \) asymptotically coincide. Regardless of the preferred view, the \( \mu \)-variants of the laws provide a probabilistic justification of Maximum Probability method (MaxProb, cf. [11]) (at least in the area of Boltzmann-Jaynes inverse problem). If the second view is adopted, then, when \( n \) is finite, it is necessary to face the challenge of selecting between
The results have a relevance also for Maximum Rényi-Tsallis Entropy method (maxTent), which is over the last years in vogue in Statistical Physics. maxTent is to the best of our knowledge proposed as a method for solving BJIP, albeit with the feasible set \( \Pi \) defined by non-linear moment constraints. Since, in general, maxTent distributions (\( \tau \)-projections on \( \Pi \)) are different than \( \nu/\mu \)-projections on \( \Pi \), ICET implies that the maxTent method selects asymptotically conditionally improbable/impossible distributions.

A straightforward extension of CWLLN/CET for \( \tau \)-tuples of types was also mentioned. It was noted that the extension provides a justification to the Generalized Maximum Entropy method in the area of Golan-Judge-Miller Inverse Problem.

Conditional Law of Large Numbers for Sources and its implications for the \( \beta \)-problem were also mentioned, in passing.

11 Appendix

11.1 MaxProb/MaxEnt

**MaxProb/MaxEnt.** Let \( X \) be finite set. Let \( M_n \) be set of all \( \mu \)-projections of \( q \) on \( \Pi_n \). Let \( I \) be set of all \( I \)-projections of \( q \) on \( \Pi \). For \( n \to \infty \), \( M_n = I \).

**Proof.** [10] Necessary and sufficient conditions for \( \hat{\nu}^n \) to be a \( \mu \)-projection of \( q \) on \( \Pi_n \) are: a) \( \pi(\nu^n; q) \geq \pi(\nu^n; q), \forall \nu^n \in \Pi_n \); b) whenever \( \hat{\nu}^n \) has the property a) then \( \pi(\hat{\nu}^n; q) \leq \pi(\hat{\nu}^n; q) \). Requirement a) can be equivalently stated as:

\[
\left( \prod \frac{n_i}{\hat{n}_i!} \right)^{1/n} \geq \left( \prod q_{i}^{n_i-\hat{n}_i} \right)^{1/n} \quad (1)
\]

and b) similarly. Standard inequality \((n/e)^n < n! < n(n/e)^n \) (valid for \( n > 6 \)) allows to bind the LHS of \( (1) \):

\[
\frac{n^{m/n} \prod (\nu^n)^{\nu^n}}{n^{m/n} \prod (\nu^n)^{\nu^n} (\nu^n)^{1/n}} \quad \text{LHS} < \frac{n^{m/n} \prod (\nu^n)^{\nu^n} (\prod \nu^n)^{1/n}}{n^{m/n} \prod (\nu^n)^{\nu^n}} \quad (2)
\]

and similar bounds can be stated in the case of the requirement b). Since \( n \) is by assumption finite, as \( n \to \infty \) the lower and upper bounds at \( (2) \) collapse into the ratio \( \frac{\prod (\nu^n)^{\nu^n}/(\nu^n)^{\nu^n}}{\prod (\nu^n)^{\nu^n}} \). Consequently, the necessary and sufficient conditions a), b) for \( \mu \)-projection turn as \( n \to \infty \) into (expressed in an equivalent log-form: i) \( \sum (\nu_i^n \log \nu_i^n - \hat{\nu}_i^n \log \hat{\nu}_i^n) \geq \sum (\hat{\nu}_i^n - \hat{\nu}_i^n) \log \hat{\nu}_i^n \) for all \( \nu^n \in \Pi_n \); and ii) whenever \( \hat{\nu}^n \) has the property i) then \( \sum \hat{\nu}_i^n \log \hat{\nu}_i^n - \hat{\nu}_i^n \log \hat{\nu}_i^n \geq \sum (\hat{\nu}_i^n - \hat{\nu}_i^n) \log \hat{\nu}_i^n \).

Necessary and sufficient conditions for \( \hat{\nu} \) to be an \( I \)-projection of \( q \) on \( \Pi \) are the following: I) \( \sum (p_i \log p_i - \hat{p}_i \log \hat{p}_i) \geq \sum (p_i - \hat{p}_i) \log q_i \) for all \( p \in \Pi \); and II) whenever \( \hat{\nu} \) has the property I) then \( \sum \hat{p}_i \log \hat{p}_i - \hat{p}_i \log \hat{p}_i \geq \sum (\hat{p}_i - \hat{p}_i) \log q_i \).

Comparison of i), ii) and I), II) then completes the proof. \( \square \)

\[ \text{Note that if an } i \text{-th component } \nu^n_i \text{ of a type is zero then it can be effectively omitted from calculations of } \pi(\nu^n; q). \text{ Thus, it is assumed that product operations at (1), (2) are performed on non-zero components only.} \]
11.2 \textit{ICET}

The conditional equi-concentration of types can be seen as a consequence of Sanov’s Theorem and MaxProb/MaxEnt Theorem. Indeed, Sanov’s Theorem implies that the probability \( \pi(\nu^n \in C; q) \) decays to zero for any open set \( C \) which excludes all of the \( I \)-projections. The asymptotic identity of \( I \)- and \( \mu \)-projections shows that for \( n \to \infty \), the \( I \)-projections have the same value of the probability \( \pi(\nu^n; q) \).

The following is a rough attempt to make the argument a bit more formal. It relays upon MaxProb/MaxEnt Thm and the Lemma, which states a standard inequality for ratio of probabilities:

\textbf{Lemma.} Let \( \nu^n, \hat{\nu}^n \) be two types from \( \Pi_n \). Then

\[
\frac{\pi(\nu^n; q)}{\pi(\hat{\nu}^n; q)} < \left( \frac{n}{m} \right)^m \prod_{i=1}^{m} \left( \frac{\hat{\nu}^n_i}{\nu^n_i} \right)^{\nu^n_i}
\]

\textbf{Proof.} \( \pi(\nu^n; q) \leq \prod_{i=1}^{m} \left( \frac{\hat{\nu}^n_i}{\nu^n_i} \right)^{\nu^n_i} \). Since for \( n > 6 \), \( (n/e)^n < n! < n(n/e)^n \), it follows that \( \pi(\nu^n; q) > \frac{1}{n_1 \ldots n_m} \prod_{i=1}^{m} \left( \frac{\hat{\nu}^n_i}{\nu^n_i} \right)^{\nu^n_i} \). \( \hat{n}_1 \ldots \hat{n}_m < \left( \frac{n}{m} \right)^m \). \qed

\textbf{ICET.} Let \( X \) be finite. Let there be \( k \) proper \( I \)-projections \( \hat{p}^1, \hat{p}^2, \ldots, \hat{p}^k \) of \( q \) on \( \Pi \). Let \( \epsilon > 0 \) be such that for \( j = 1, 2, \ldots, k \) \( \hat{p}^j \) is the only proper \( I \)-projection of \( q \) on \( \Pi \) in the ball \( B(\hat{p}^j, \epsilon) \). Let \( n \to \infty \). Then for \( j = 1, 2, \ldots, k \),

\[
\pi(\nu^n \in B(\epsilon, \hat{p}^j)|\nu^n \in \Pi; q) = 1/k.
\]

\textbf{Proof.} Clearly,

\[
\pi(\nu^n \in B(\epsilon, \hat{p}^j)|\nu^n \in \Pi; q \to \nu^n) \leq \frac{\sum_{\nu^n \in B} \pi(\nu^n; q)}{\sum_{\nu^n \in \Pi} \pi(\nu^n; q)}
\]

Let \( B_n(\epsilon, \hat{p}^j) \triangleq B(\epsilon, \hat{p}^j) \cap \Pi_n \).

Without loss of generality, let there be unique \( I \)-projection \( \hat{p}^j_B \) of \( q \) on the ball \( B_n(\hat{p}^j, \epsilon) \). (Sequence of the \( I \)-projections on \( \Pi_n \) converges to a proper \( I \)-projection of \( q \) on \( \Pi \). To an \( I \)-projection on \( \Pi \) which is not proper, no sequence of \( I \)-projections converges.) Also, without loss of generality let there be \( k \) \( I \)-projections \( \hat{p}^j_{\Pi_n} \), \( j = 1, 2, \ldots, k \) of \( q \) on \( \Pi_n \).

Let \( A \triangleq B_n \setminus \{ \hat{p}^j_B \} \). B \triangleq \Pi_n \setminus \{ \hat{p}^j_B \}, j \neq 1 \), C \triangleq \Pi_n \setminus B.

Then the Right-Hand Side of (3) can be rewritten as:

\[
\frac{\pi(\hat{p}^j_B)}{\pi(\hat{p}^j_B^{1})} = \frac{1 + \sum_{\nu^n \in A} \pi(\nu^n)}{\pi(\hat{p}^j_B^{1})} \frac{1 + \sum_{\nu^n \in B} \pi(\nu^n)}{\pi(\hat{p}^j_B^{1})} \frac{1 + \sum_{\nu^n \in C} \pi(\nu^n)}{\pi(\hat{p}^j_B^{1})}
\]

By MaxProb/MaxEnt Thm \( I \)-projections have for \( n \to \infty \) the same and supremal value of \( \pi(\cdot) \). This implies that \( \pi(\hat{p}^j_B)/\pi(\hat{p}^j_B^{1}) \) converges to 1 (the case of 0/0 limit is excluded by the supremity of \( \pi(\cdot) \)). The same argument implies that the first ratio in the denominator converges to \( k-1 \). The Lemma implies that the ratio in the nominator as well as the second ratio in the denominator converge to zero. \qed
11.3 Extended GCP

**EGCP.** Let \(X\) be a finite set. Let \(\Pi\) be such that it admits \(k\) proper \(I\)-projections \(\hat{p}^1, \hat{p}^2, \ldots, \hat{p}^k\) of \(q\) on \(\Pi\). Then for a fixed \(t\),

\[
\lim_{n \to \infty} \pi(X_1 = x_1, \ldots, X_t = x_t | \nu^n \in \Pi; q \mapsto \nu^n) = 1/k \sum_{j=1}^{k} \prod_{l=1}^{t} \hat{p}^j_{x_l}.
\]

Proof. Clearly,

\[
\pi(X_1 = x_1, \ldots, X_t = x_t | \nu^n \in \Pi; q \mapsto \nu^n) = \frac{\sum_{\nu^n \in \Pi} \pi(X_1 = x_1, \ldots, X_t = x_t, \nu^n)}{\sum_{\nu^n \in \Pi} \pi(\nu^n; q)}
\]

Let, in addition to partitioning used in proof of ICET, \(D \triangleq \bigcup_{j=1}^{k} \{\hat{p}^j_{\Pi_n}\}\). Then the RHS of (5) can be rewritten as:

\[
\frac{\sum_{\nu^n \in D} \pi(X_1 = x_1, \ldots, X_t = x_t, \nu^n) + \sum_{\nu^n \in \Pi_n \setminus D} \pi(X_1 = x_1, \ldots, X_t = x_t, \nu^n)}{\pi(\hat{p}^j_{\Pi_n})(1 + \sum_{\nu^n \in \Pi_n} \pi(\nu^n) + \sum_{\nu^n \in \Pi_n \setminus D} \pi(\nu^n))}
\]

MaxProb/MaxEnt Thm implies that the first ratio in the denominator converges to \(k - 1\). By the Lemma, the second ratio in the denominator of (6) converges to zero as \(n\) goes to infinity. The second term in the nominator as well goes to zero as \(n \to \infty\) (to see this, express the joint probability \(\pi(X_1 = x_1, \ldots, X_t = x_t, \nu^n)\) as \(\pi(X_1 = x_1, \ldots, X_t = x_t, \nu^n)\) and employ the Lemma).

Then, MaxProb/MaxEnt Thm implies, that for \(n \to \infty\) the RHS of (6) becomes equal to \(1/k \sum_{j=1}^{k} \pi(X_1 = x_1, \ldots, X_t = x_t, \nu^n)\). Finally, invoke Csiszár’s ‘urn argument’ (cf. [7]) to conclude that the asymptotic form of the RHS of (6) is \(1/k \sum_{j=1}^{k} \prod_{l=1}^{t} \hat{p}^j_{x_l} \).

11.4 Rational I-projections

Types can concentrate, in some sense, on rational \(I\)-projection \(\hat{p} \in \mathbb{Q}^m\) even though the \(I\)-projection is isolated point of the set \(\Pi\). The following Example illustrates the concentration.

**Example 5.** Consider \(\Pi = \{p, \hat{p}\}\), where \(p = [n_1/n_0, \ldots, n_m/n_0]\) and \(\hat{p} = [\hat{n}_1/n_0, \ldots, \hat{n}_m/n_0], n_0 \in \mathbb{N}\). For \(n \neq kn_0, k \in \mathbb{N}\) the set \(\Pi_n\) is empty; otherwise it contains \(p\) and \(\hat{p}\). In this case, concentration of types on \(\mu\)-projection is a direct consequence of the next two Lemmas. The \(I\)-variant of the concentration then arises from MaxProb/MaxEnt Thm.

**Lemma 1.** Let \(\nu^n, \hat{\nu}^n\) be two \(n\)-types. Let \(\delta = \nu^n - \hat{\nu}^n\). Let \(K\) denote the non-negative elements of \(n\delta\), \(L\) the absolute value of negative elements of \(n\delta\). Let \(c = \|n\delta_i^L - n\delta_i^K\|, i \neq +, +\) indicates that the index \(i\) goes through the elements of \(K, L\), respectively. Then \(\Gamma(k\nu^n) < c^k\), for any \(k \in \mathbb{N}\).

Proof. \(\frac{\Gamma(k\nu^n)}{\Gamma(k\hat{\nu}^n)} = \prod_{i}(k(n_i^L + 1) - k(n_i^K + 1)) \prod_{i}(n_i^L - n_i^K)\). So \(\frac{\Gamma(k\nu^n)}{\Gamma(k\hat{\nu}^n)} \leq \prod_{i} \frac{n_i^L}{n_i^K}, \) which is just \(c^k\). \(\square\)
Lemma 2. Let types $\nu^n$, $\dot{\nu}^n$ be such that $\pi(\dot{\nu}^n ; q) < \pi(\nu^n ; q)$. Then $\frac{\pi(k\nu^n ; q)}{\pi(k\dot{\nu}^n ; q)} \to 0$ as $k \to \infty$.

Proof. By the assumption, $\prod q_i^{n_i - \dot{n}_i} < \Gamma(\dot{\nu}_i) / \Gamma(\nu_i)$. The gamma-ratio is, by the Lemma 1, smaller or equal to $c$, as defined at the Lemma. Thus, $\prod q_i^{n_i - \dot{n}_i} = \gamma c$, where $\gamma \in [0, 1)$, $\gamma \in \mathbb{R}$. By Lemma 1, for any $k \in \mathbb{Z}$, $\frac{\pi(k\nu^n ; q)}{\pi(k\dot{\nu}^n ; q)} \leq (1/c)^k \prod q_i^{k(n_i - \dot{n}_i)}$. The RHS of the inequality, $\gamma^k$, goes for $k \to \infty$ to zero, which completes the proof. \qed

In this case, if $\Pi$ admits several rational $I/\mu$-projections, then clearly, types equi-concentrate on them.

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