Sparse unitary operators and repeated entries in the process matrix

Vinayak Jagadish* and Anil Shaji
School of Physics, IISER TVM, CET Campus,
Thiruvananthapuram, Kerala, India 695016

We show that quantum processes of low rank are inexorably connected to the dimensions of the environmental state space that the system is in contact with. We also show that the process matrix representing such low rank processes will have only a few distinct entries which goes as \(O(r^2)\) (\(r\) is the rank) and hence the number of independent measurements needed to characterize the process is greatly reduced.

Understanding open quantum dynamics [1] better, developing strategies to control it [2, 3] and application to new technologies has gained significant interest. Quantum process tomography is the usual technique employed for tracking the unknown dynamics of quantum systems [4–7]. The prime difficulty in performing tomography tasks for large systems is the growth of resources needed for reconstructing the processes. For an \(n\) qubit system \(N = 16^n - 4^n\) independent measurements are required for process tomography when done in the conventional way [8]. In many situations, physical considerations regarding the process that is to be reconstructed and the structure of the problem at hand allows one to reduce the resources required to do process tomography. Indeed there has been considerable progress in reducing the number of independent measurements required for characterizing noisy processes [9–13]. In particular, extensions of the compressed sensing and matrix completion techniques [14–16] that have been used effectively for quantum state tomography when the states are described by sparse density matrices [17, 18] to process tomography [19] provide substantial reduction in the resources required under certain conditions.

The finite time open dynamics of a quantum system is described by a quantum process (or operation) represented by a map, \(\mathcal{E}\):

\[
\mathcal{E} : \rho \to \mathcal{E}(\rho),
\]

Since quantum operations take density matrices to density matrices, the map, in general, has to be convex-linear, trace preserving and completely positive on all the states in its domain. The Dynamical Matrix is a matrix representation of the map [20] that acts on a density matrix as

\[
\rho_{r's'} \to \mathcal{E}(\rho)_{rs} = \mathcal{B}_{rr',ss'} \rho_{r's'}.
\]

If \(\rho\) is \(d\) dimensional, then \(\mathcal{B}\) is a \(d^2\) dimensional Hermitian matrix that admits a spectral decomposition of the form:

\[
\mathcal{B} = \sum_{n=1}^{r} \lambda_n |e_n\rangle \langle e_n|
\]

The canonical Kraus operators [21] are the matricized [22] versions of the eigenvectors of \(\mathcal{B}\) such that \(K_n = \sqrt{\lambda_n} \text{mat}(e_n)\). The mat operation stacks the elements of a column matrix row by row.

* email: vinayak@iisertvm.ac.in
with rows of length $d$ to generate a $d \times d$ square matrix. The number of canonical Kraus operators is equal to the rank $r$ of the dynamical matrix. The map in the Kraus (operator-sum) form is:

$$
\mathcal{E}(\rho) = \sum_{n=1}^{r} K_n \rho K_n^\dagger \quad \text{where} \quad \sum_{n=1}^{r} K_n^\dagger K_n = 1
$$

(4)

The Kraus representation is not unique, since each $K_n$ can be multiplied on the left by a unitary matrix without violating the only constraint on the Kraus matrices given by the second equation in (4). Each of the Kraus operators for a map $\mathcal{E}$ can be expanded in a suitable operator basis $\{A_i\}$ as $K_n = \sum_i \alpha_i^{(n)} A_i$, with $\alpha_i^{(n)} \in \mathbb{C}$. The operator basis can be chosen to be orthonormal ($\text{tr}[A_i^\dagger A_j] = \delta_{ij}$) for convenience. Equation (4) can then be written as

$$
\mathcal{E}(\rho) = \sum_n \left( \sum_i \alpha_i^{(n)} A_i \right) \rho \left( \sum_j \alpha_j^{(n)*} A_j^\dagger \right) = \sum_{ij} \chi_{ij} A_i \rho A_j^\dagger,
$$

(5)

where

$$
\chi_{ij} = \sum_n \alpha_i^{(n)} \alpha_j^{(n)*}.
$$

(6)

So for a given basis set $\{A_i\}$ the matrix $\chi$ completely characterizes $\mathcal{E}$. The $\chi$ matrix is Hermitian and different Kraus representations of the same process $\mathcal{E}$ have the same $\chi$ matrix.

In a typical process tomography experiment on a system made of $n$ qubits, the $\{A_i\}$’s are usually taken to be the $n$-fold tensor products of Pauli operators. To characterize an unknown operation, we prepare a complete set of linearly independent input states, subject them to the quantum operation $\mathcal{E}$ and determine the output states corresponding to each input using quantum state tomography. The details of performing a Standard Quantum Process Tomography (SQPT) can be found in [7]. We assume that the input states are initialized such that they are uncorrelated with the environment and hence the map that is reconstructed is completely positive.

We focus our attention to quantum processes which are sparse and low rank. We demonstrate mathematically, how processes of low rank arise. We also show that the number of distinct elements (in absolute value) in the $\chi$ matrix representing such processes goes as $O(r^2)$. The rank of a quantum process is intimately tied to the environmental degrees of freedom, which we show below. Quantum processes of low rank arise when the system of interest is interacting with an effectively low dimensional environment. Let $S$ be the system of interest (of dimensions $M$) and let $R$ represent a ‘reservoir’ (of dimension $N$) with which $S$ interacts. Choosing a product state as the initial state of $S$ and $R$, the dynamics of the coupled system is

$$
\rho_S \times \eta_R \rightarrow U \rho_S \times \eta_R U^\dagger
$$

where $U$ is the total unitary acting in $\mathcal{H}_S \times \mathcal{H}_R$. In index notation:

$$
\rho_{ra,s;\alpha'} \rho_{r's';\beta'} \rightarrow U_{ra;\alpha' \beta'} \rho_{r's';\alpha' \beta'} U_{sr,\beta'}^s
$$

The evolution of $S$ is obtained by partially tracing out the environmental degrees of freedom:

$$
\rho_{r,s} = \text{tr}_R(U_{ra;\alpha' \beta'} \rho_{r's';\alpha' \beta'} U_{sr,\beta'}^s) = U_{rn;\alpha' \beta'} \rho_{r'n's';\alpha' \beta'} U_{sn;\beta'}^s
$$

(7)
Assuming that $\eta$ can be made diagonal in $\mathcal{H}_R$ by a suitable unitary transformation, with eigenvalues $\eta(1), \eta(2), ..., \eta(n)$, the map is

$$\rho_{r,s} = \sum_{\nu,n,n',s'} \eta(\nu)U_{rr'}(n,\nu) \rho_{r's'}U_{s's}(n,\nu)$$

Since the nature of the reservoir $R$ is completely unknown, we can set $\eta$ to be a pure state with only one eigenvalue without loss of generality and one obtains

$$\rho_{r,s} = \sum_{n=1}^{N} U^{(n)}_{rr} \rho_{r's'}U^{(n)*}_{ss'} = \sum_{n=1}^{N} K_n \rho K_n^†$$

(8)

where the $\{K_n\}$ are the Kraus operators. The number of Kraus operators is set here by the index $n$ which runs over the dimensionality of the state of the reservoir. If the system is effectively interacting with only a few degrees of freedom of the reservoir then the number of Kraus operators and consequently the rank of the dynamical matrix is small as may be seen from Eq. (3). A low rank process refers to the fact that only a few eigenvectors of the dynamical matrix are relevant in describing the quantum process.

Experimental evidence of sparse process matrices are often seen in quantum process tomography experiments on NMR, ion traps, and linear optics [23–26] systems. In these cases there are often only a few dominant system-environment interactions, or else the system is weakly decohering. Also, for many engineered quantum systems, the total unitary operator is almost sparse. A similar scenario is also seen in many-body systems where the dominant interactions are nearest-neighbour ones. Such sparse Hamiltonians generate sparse process matrices [27] and they are also typically low rank. Posing quantum tomography as a semidefinite program and using techniques of convex optimization [28] and compressed sensing [16], recently, it has been shown [18] that $O(rd\log^2d)$ measurements are needed to characterize an unknown quantum process of rank $r$ acting on a system of dimension $d$ ($d = 2^n$ for a system of $n$ qubits). This result was obtained by extending the arguments used in counting the resources needed for performing quantum state tomography [17], via the Choi-Jamiołkowski isomorphism [29], to quantum processes. The Choi-Jamiołkowski isomorphism maps a quantum operation to a quantum state. In a different paper, again using compressed sensing, it was shown that [19] only $O(s\log d)$ measurements are sufficient, where $s$ is the sparsity of the associated process matrix (i.e., having $s$ non-zero matrix elements). The limitations of this method are reported in [18]. Both methods have attracted much attention [30, 31] because of the fact that the number of independent measurement settings needed to reconstruct a quantum process is very much reduced and moreover the reconstructions work with very high fidelity.

In what follows we do not use the machinery of compressed sensing and our main result is purely based on the properties of the process matrix. The experimental accessibility of the $\chi$ matrix is the main motivation for studying its properties in detail for the low rank quantum operations considered here. Since Pauli measurements are relatively easy to perform in real quantum process tomography experiments, our results are readily tested as well. From Eq. (8), we see that the Kraus matrices are partial matrix elements. Choosing an orthonormal basis, $\{|n_R\rangle\}$ for the reservoir, one can find that

$$K_n(t) = \langle n_R|U|0_R\rangle$$
where $U$ is as defined previously and $|0_R\rangle$ is some fiducial initial state of the reservoir. In [32, 33], it has been shown that sparse Hamiltonians generate sparse process matrices. When the number of interaction terms in the Hamiltonian coupling the system and its environment is less, then in the overall unitary time evolution operator will also be sparse. This, in turn, leads to a low rank process with typically sparse Kraus matrices as discussed earlier. For such processes let us focus on the number of distinct entries in the $\chi$ matrix. The $\chi$ matrix can be written as an outer product as follows

$$\chi = \sum_{i=1}^{r} \mathcal{L}_i\tilde{\mathcal{L}}_i$$

where

$$\mathcal{L}_i = [K_i^{(1)}, K_i^{(2)}, \ldots, K_i^{(d)}]^T,$$

with $K_i^{(d)} = \text{tr}(K_i\lambda_d)$,

and

$$\tilde{\mathcal{L}}_i = [k_i^{(1)}, k_i^{(2)}, \ldots, k_i^{(d)}],$$

with $k_i^{(d)} = \text{tr}(K_i^T\lambda_d)$.

Here $\{\lambda_i\}$ denotes the convenient operator basis used for defining the $\chi$ matrix and the superscript $T$ denotes the transpose operation. For the discussion that follows we take the basis to be made of the $n$-fold tensor products of Pauli matrices.

The product $\mathcal{L}_i\tilde{\mathcal{L}}_i$ is Hermitian and so is $\chi$. The elements of $\mathcal{L}_i$ are conjugates of $\tilde{\mathcal{L}}_i$. Let us assume that the Kraus matrices, $K_i$, each have only a maximum of $r$ non-zero entries. The $\mathcal{L}_i$, which is the the matrix constructed out of the trace with $\lambda_d$‘s of the Kraus matrices, will have $cr$ distinct entries where $c$ is a constant. If the $r$ non-zero entries of a Kraus matrix are located in such a way that only one element contributes towards its trace with a particular $\lambda_i$, then $4r$ different possibilities alone arise, since the trace with Pauli tensors produces $\pm 1$ or $\pm i$ alone [34]. This indicates that if more elements contribute to the trace with a particular $\lambda_i$, the number of distinct entries scales linearly with $r$. From Eq. (9), it is therefore clear that the number of distinct entries in the $\chi$ matrix is additive w.r.t to the Kraus matrices and since they are $r$ in number, the number goes as $O(r^2)$. It can be understood from this construction that if different Kraus matrices have the same matrix positions of non-zero entries the number of distinct entries will not increase and that their numerical value alone gets altered. It is worth noting that the $\chi$ matrix in the basis of SU(N) generators will also have the same number of distinct elements due to the fact that the trace of Kraus matrices with generators of SU(N) Lie algebra also produces $\pm 1$ or $\pm i$ alone.

A sparse structure for the Kraus matrix can also be viewed as a reflection of the fact that the underlying dynamical matrix is sparse and low rank. The eigenspace of the dynamical matrix can be decomposed as $V \oplus V^\perp$, where $V$ corresponds to the space of eigenvectors corresponding to non-zero eigenvalues and $V^\perp$ to the space of eigenvectors corresponding to the zero eigenvalues. For an $s$-sparse matrix, it’s rank satisfies the bound $O(1) \leq r \leq s$ and also a tighter bound $O(1) \leq r \leq \min(s,d)$ for a $d$ dimensional matrix. [35] Since the dynamical matrix is Hermitian, it admits a spectral decomposition which suggests that the eigenvectors will have $O(r)$ non-zero
entries which indicates that the matricized version of them (which are nothing but the Kraus operators) having a sparse structure.

Consider the set of Pauli matrices along with the $2 \times 2$ identity matrix (denoted as $\sigma_0$), $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$. We know that $\{\sigma_0, \sigma_3\}$ is diagonal and the other two are off-diagonal. If we form a tensor product among these matrices of the form $\sigma_i \otimes \sigma_j$, they fall into four different sets, in terms of the positions of the matrix elements, say $\sigma_1 \otimes \sigma_0, \sigma_1 \otimes \sigma_1, \sigma_3 \otimes \sigma_1$ and $\sigma_3 \otimes \sigma_3$. It can be clearly seen the difference between $\sigma_1 \otimes \sigma_0$ and $\sigma_2 \otimes \sigma_0$ is a difference in values of the matrix element, but the positions of the non-zero matrix elements are the same. A simple check clearly reveals that only 4 different families are possible. On forming tensor products of the form $\sigma_i \otimes \sigma_j \otimes \sigma_k$ shows that $4 \times 2 = 2^3$ different families are possible. Extending the same argument to $n$-fold tensor products among the Pauli matrices will therefore form $2^n = d$ different sets.

This means that only $O(r^2)$ measurement settings are needed to characterise the $\chi$ matrix and thereby the quantum process. The histograms shown below clearly indicates that the number of distinct entries in the $\chi$ matrix are peaked at low numbers which is less than $r^2$. The histograms were made by constructing $\chi$ matrices from sparse Kraus matrices and counting the number of distinct entries in absolute values.

![Histogram of 100,000 realizations showing the number of distinct matrix elements (in absolute values) of a 64 dimensional $\chi$ matrix corresponding to a dynamical matrix of rank 2.](image)

Spectacular advances are taking place in the quantum control of even large quantum systems, from the experimental front. Simplifying quantum process tomography goes a long way in designing improved methods of quantum control and manipulation. Moving on from the most general process tomography schemes, several authors have looked at specific scenarios that are physically relevant and important for which considerable reduction in the resources required for quantum process tomography can be achieved. We have shown that when the underlying process is represented by a sparse process matrix, then in the $\chi$ matrix form, a simplification obtains as a significant reduction in the number of distinct entries in the matrix. This, in principle, means that very few measurements - much fewer than the bound placed by compressed sensing techniques - are really needed to pin down the quantum process. Devising a systematic protocol for identifying this minimal set of measurements that will yield the numerical value of each of the distinct elements in the $\chi$-matrix so that the reduction in independent parameters due to the repeated entries in the
FIG. 2. (Color online) Histogram of 100,000 realizations showing the number of distinct matrix elements (in absolute values) of a 64 dimensional $\chi$ matrix corresponding to a dynamical matrix of rank 3.

$\chi$-matrix can be leveraged remains to be done.

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The probability that only one element contributes to trace is $d \binom{d}{1} \left(1 - \frac{r}{d^2}\right)^{d-1} \frac{r}{d^2}$. From the structure of tensor product of Pauli matrices, it is clear that only $d$ positions in the tensor product matrices are non zero which alone contributes to the trace.