THE SPECTRAL PROPERTIES OF THE STRONGLY COUPLED
STURM HAMILTONIAN OF CONSTANT TYPE

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ABSTRACT. We study the spectral properties of the Sturm Hamiltonian $H_{\alpha,V,\phi}$, where $\alpha$ is the frequency of constant type, i.e. $\alpha$ has the constant continued fraction expansion $[\kappa; \kappa, \kappa, \cdots]$, $V > 0$ is the coupling constant and $\phi \in [0,1)$ is the phase. Let $\Sigma_V = \Sigma_{\kappa,V}$ be the related spectrum and $N_V = N_{\kappa,V}$ be the related density of states measure. Let $s_V = s_V(\kappa)$ be the Hausdorff dimension of $\Sigma_{\kappa,V}$, $d_V = d_V(\kappa)$ be the Hausdorff dimension of $N_{\kappa,V}$ and $\gamma_V = \gamma_V(\kappa)$ be the optimal Hölder exponent of $N_{\kappa,V}$.

Assume $V > 20$. In this paper we show the following results: $H_{s_V} \mid \Sigma_V$ is a Gibbs type measure, consequently it is exact dimensional and $0 < H_{s_V}(\Sigma_V) < \infty$; $N_V$ is a Markov measure, consequently it is a Gibbs type measure and is exact dimensional; for each $\kappa \in \mathbb{N}$, there exists two constants $\hat{\kappa}_\kappa, \hat{\kappa}_\kappa > 0$ such that

\[
\lim_{V \to \infty} \gamma_V(\kappa) \ln V = \hat{\kappa}_\kappa \quad \text{and} \quad \lim_{V \to \infty} d_V(\kappa) \ln V = \hat{\kappa}_\kappa.
\]

It is known that (Liu, Qu and Wen, Adv. Math. 2014) there exists another constant $\rho_\kappa > 0$ such that

\[
\lim_{V \to \infty} s_V(\kappa) \ln V = \rho_\kappa \quad \text{and} \quad \lim_{V \to \infty} d_V(\kappa) \ln V = \rho_\kappa.
\]

We show that $\hat{\kappa}_\kappa^2 = \hat{\kappa}_\kappa^2 = \rho_\kappa^2 = \ln(1 + \sqrt{2})$ and $\hat{\kappa}_\kappa < \rho_\kappa < \rho_\kappa$ when $\kappa \neq 2$; as a consequence when $\kappa \neq 2$, there exists $V_0(\kappa) > 20$ such that $\gamma_V(\kappa) < d_V(\kappa) < s_V(\kappa)$ for all $V \geq V_0(\kappa)$.

We achieve them by introducing an auxiliary symbolic dynamical system and applying the thermodynamical and multifractal formalisms of almost additive potentials.

Keywords: Sturm Hamiltonian; constant type; spectral property; Hausdorff dimension; density of states measure; optimal Hölder exponent; Gibbs measure

1. INTRODUCTION

The Sturm Hamiltonian is a discrete Schrödinger operator

\[(H_{\alpha,V,\phi}\psi)_n := \psi_{n-1} + \psi_{n+1} + v_n\psi_n\]

on $l^2(\mathbb{Z})$, where the potential $(v_n)_{n \in \mathbb{Z}}$ is given by

\[v_n = V_{\chi_{[1-(\alpha),1)}}(n\alpha + \phi \mod 1), \quad \forall n \in \mathbb{Z},\]

where $\alpha > 0$ is irrational, and is called frequency ($\{\alpha\}$ is the fractional part of $\alpha$), $V > 0$ is called coupling, $\phi \in [0,1)$ is called phase. It is well-known that the spectrum of Sturm Hamiltonian is independent of $\phi$, which we denote by $\Sigma_{\alpha,V}$.

Sturm Hamiltonian is firstly introduced by physicists to model the quasicrystal system, see [4] and the references therein for a good introduction about the physical background. From the mathematical point of view, one is interested in its spectral properties.

For a discrete Schrödinger operator, two spectral objects are very important, one is the spectrum itself as a set, the other is the spectral measure. For a class of dynamical defined
Schrödinger operators, there is a third one, called the density of states measure, which is also very important. Let us recall the definition of the density of states measure in our setting. By the spectral theorem, there are Borel probability measures $\mu_{\alpha,V,\phi}$ on $\mathbb{R}$ such that
\[
\langle \delta_0, g(H_{\alpha,V,\phi})\delta_0 \rangle = \int_{\mathbb{R}} g(x) d\mu_{\alpha,V,\phi}(x)
\]
for all bounded measurable functions $g$, where $\delta_0$ is the element in $\ell^2(\mathbb{Z})$ which takes value 1 at site 0 and 0 elsewhere. The density of states measure $N_{\alpha,V}$ is given by the $\phi$-average of these measures with respect to Lebesgue measure, that is,
\[
\int T \langle \delta_0, g(H_{\alpha,V,\phi})\delta_0 \rangle d\phi = \int_{\mathbb{R}} g(x) dN_{\alpha,V}(x)
\]
for all bounded measurable functions $g$. It is well-known that the density of states measure $N_{\alpha,V}$ is continuous and supported on $\Sigma_{\alpha,V}$.

We will study the spectral properties of $\Sigma_{\alpha,V}$ and $N_{\alpha,V}$ for a special class of $\alpha$. Write $s_V(\alpha) := \dim_H \Sigma_{\alpha,V}$ and $d_V(\alpha) := \dim_H N_{\alpha,V}$ for the Hausdorff dimensions of $\Sigma_{\alpha,V}$ and $N_{\alpha,V}$ and denote by $\gamma_V(\alpha)$ the optimal H"older exponent of $N_{\alpha,V}$. Before introducing our results, let us give a brief survey about the known spectral results on Sturm Hamiltonian.

The most prominent model among the Sturm Hamiltonian is the Fibonacci Hamiltonian, for which the frequency is taken to be the golden number $\alpha_1 = (\sqrt{5} + 1)/2$. This model was introduced by physicists to model the quasicrystal system, see [24, 33]. Sütö [35] showed that the spectrum always has zero Lebesgue measure. Then it is natural to ask what is the fractal dimension of the spectrum. Raymond [34] first estimated the Hausdorff dimension, he showed that $\dim_H \Sigma_{\alpha_1,V} < 1$ for $V > 4$. Jitomirskaya and Last [22] showed that for any $V > 0$, the spectral measure of the operator has positive Hausdorff dimension, as a consequence $\dim_H \Sigma_{\alpha_1,V} > 0$. By using a dynamical method, Damanik et al. [8] showed that if $V \geq 16$ then $\dim_B \Sigma_{\alpha_1,V} = \dim_H \Sigma_{\alpha_1,V}$, where $\dim_B E$ denotes the Box-dimension of $E$. They also got lower and upper bounds for the dimensions. Due to these bounds they further showed that
\[
\lim_{V \to \infty} s_V(\alpha_1) \ln V = \ln(1 + \sqrt{2}).
\]
Cantat [5], Damanik and Gorodetski [9] showed that $s_V(\alpha_1)$ as a function of $V$ is analytic on $(0, \infty)$ and takes values in $(0,1)$. In [10], Damanik and Gorodetski further showed that $\lim_{V \to 0} s_V(\alpha_1) = 1$ and the speed is linear. In [14], Damanik and Gorodetski showed that $N_{\alpha_1,V}$ is exact dimensional and $d_V(\alpha_1) < s_V(\alpha_1)$ for small $V$. In [12], Damanik and Gorodetski showed that $\gamma_V(\alpha_1) \to 1/2$ as $V \to 0$ and
\[
\lim_{V \to \infty} \gamma_V(\alpha_1) \ln V = \frac{3}{2} \ln \alpha_1.
\]

Now we turn to the general Sturm Hamiltonian case. Fix an irrational $\alpha > 0$ with continued fraction expansion $[a_0; a_1, a_2, \cdots]$. Bellisard et al. [4] showed that $\Sigma_{\alpha,V}$ is a Cantor set of Lebesgue measure zero. Damanik, Killip and Lenz [14] showed that, if $\limsup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} a_i < \infty$, then $\dim_H \Sigma_{\alpha,V} > 0$. Based on the analysis of Raymond about the
structure of spectrum \cite{34}, the fractal dimensions of the spectrum of Sturm Hamiltonian were extensively studied in \cite{27,28,25,17,26}. The current picture is the following. Write

\[
K_s(\alpha) = \liminf_{k \to \infty} \left( \prod_{i=1}^{k} a_i \right)^{1/k} \quad \text{and} \quad K^* (\alpha) = \limsup_{k \to \infty} \left( \prod_{i=1}^{k} a_i \right)^{1/k}.
\]

Fix \( V \geq 24 \). Then it is proven in \cite{27,26} that

\[
\begin{align*}
\dim_H \Sigma_{\alpha,V} &\in (0,1) \quad \text{if and only if} \quad K_s(\alpha) < \infty \\
\dim_B \Sigma_{\alpha,V} &\in (0,1) \quad \text{if and only if} \quad K^*(\alpha) < \infty,
\end{align*}
\]

where \( \dim_B E \) denote the upper Box-dimension of \( E \). Raymond \cite{34}, Liu and Wen \cite{27} showed that the spectrum \( \Sigma_{\alpha,V} \) has a natural covering structure. This structure makes it possible to define the so-called pre-dimensions \( s_s(\alpha,V) \) and \( s^*(\alpha,V) \). Then it is proven in \cite{25,17,26} that

\[
\dim_H \Sigma_{\alpha,V} = s_s(\alpha,V) \quad \text{and} \quad \dim_B \Sigma_{\alpha,V} = s^*(\alpha,V).
\]

Moreover there exist two constants \( 0 < \rho_s(\alpha) \leq \rho^*(\alpha) \) such that

\[
\lim_{V \to \infty} s_s(\alpha,V) \ln V = \rho_s(\alpha) \quad \text{and} \quad \lim_{V \to \infty} s^*(\alpha,V) \ln V = \rho^*(\alpha).
\]

It is proven in \cite{26} that \( s_s(\alpha,V) \) and \( s^*(\alpha,V) \) are Lipschitz continuous on any bounded interval of \([24,\infty)\).

Recently several works deal with some sub-classes of Sturm Hamiltonian. Girand \cite{21} considered the frequency \( \alpha \) which is generated by a primitive invertible substitution. Mei \cite{30} considered the frequency \( \alpha \) which has eventually periodic continued fraction expansion, a strictly larger class than that generated by primitive invertible substitutions. In both papers they showed that \( \lim_{V \to \infty} d_V(\alpha) = \lim_{V \to \infty} s_V(\alpha) = 1 \), \( N_{\alpha,V} \) is exact dimensional and \( d_V(\alpha_1) < s_V(\alpha_1) \) for small \( V \). This generalizes the results in \cite{11}. Munger \cite{32} considered the frequency of constant type, i.e. \( \alpha_\kappa = [\kappa;\kappa,\kappa,\cdots] \). He gave estimations about the optimal Hölder exponent \( \gamma_V(\alpha_\kappa) \) and show the following asymptotic formula:

\[
(1.4) \quad \lim_{V \to \infty} \gamma_V(\alpha_\kappa) \ln V = \begin{cases}
\frac{3}{2} \ln \alpha_1 & \kappa = 1 \\
\frac{2}{\kappa} \ln \alpha_\kappa & \kappa \geq 2.
\end{cases}
\]

In this paper we will consider the frequency of constant type. We will study the spectral property of the related Sturm Hamiltonian when the coupling constant \( V \) is big. Remark that when \( \kappa = 1, \alpha_1 \) is the Golden number \( (\sqrt{5}+1)/2 \).

When \( \alpha = \alpha_\kappa \), we will write \( \Sigma_{\kappa,V} \) for \( \Sigma_{\alpha_\kappa,V} \) and \( N_{\kappa,V} \) for \( N_{\alpha_\kappa,V} \). We also write \( s_V(\kappa), d_V(\kappa), \gamma_V(\kappa) \) for \( s_V(\alpha_\kappa), d_V(\alpha_\kappa) \) and \( \gamma_V(\alpha_\kappa) \) respectively. When \( \kappa \) is clear from the context, we often suppress it from the index.

Given a probability measure \( \mu \) defined on a compact metric space \( X \). Fix \( x \in X \), we can define the local upper and lower dimensions of \( \mu \) at \( x \) as

\[
\overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\ln \mu(B(x,r))}{\ln r} \quad \text{and} \quad \underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\ln \mu(B(x,r))}{\ln r}.
\]
In the case $d_{\mu}(x) = d_{\mu}(x)$, we say that the local dimension of $\mu$ at $x$ exists and we denote it by $d_{\mu}(x)$. We define the optimal Hölder exponent $\gamma_{\mu}$ of $\mu$ as

$$\gamma_{\mu} := \inf \{ d_{\mu}(x) : x \in X \}.$$ 

If there exists a constant $d$ such that $d_{\mu}(x) = d$ for $\mu$ a.e. $x \in X$, then necessarily $\dim_{\mu} \mu = d$. In this case we call $\mu$ exact dimensional.

Our main result is the following.

**Theorem 1.** Fix $V > 20$ and $\kappa \in \mathbb{N}$. Take $\alpha = \alpha_{\kappa}$ in (1.1). Then the following assertions hold:

(i) $\mathcal{H}^{s_{V}}|_{\Sigma_{V}}$ is a Gibbs type measure. Consequently, $\mathcal{H}^{s_{V}}|_{\Sigma_{V}}$ is exact dimensional and $0 < \mathcal{H}^{s_{V}}(\Sigma_{V}) < \infty$.

(ii) $\mathcal{N}_{V}$ is a Markov measure. Consequently, it is a Gibbs type measure and exact dimensional.

(iii) For each $\kappa \in \mathbb{N}$, there exist three constants $0 < \hat{\eta}_{\kappa} \leq \eta_{\kappa} \leq \rho_{\kappa}$ such that

$$\lim_{V \to \infty} \gamma_{V}(\kappa) \ln V = \hat{\eta}_{\kappa}, \quad \lim_{V \to \infty} d_{V}(\kappa) \ln V = \eta_{\kappa} \quad \text{and} \quad \lim_{V \to \infty} s_{V}(\kappa) \ln V = \rho_{\kappa}.$$ 

Moreover, $\hat{\eta}_{2} = \eta_{2} = \rho_{2} = \ln(1 + \sqrt{2})$ and $\hat{\eta}_{\kappa} < \eta_{\kappa} < \rho_{\kappa}$ when $\kappa \neq 2$.

(iv) When $\kappa \neq 2$, there exists $V_{0}(\kappa) > 20$ such that for all $V \geq V_{0}(\kappa)$ we have

$$\gamma_{V}(\kappa) < d_{V}(\kappa) < s_{V}(\kappa).$$

**Remark 1.**

(1) The first equality in (iii) is known, see (1.4) ([32]); the third equality in (iii) is also known, see [25, 17, 26]. We state them here only for comparison. Nevertheless our method give a new proof for (1.4).

(2) See Theorem 5 for the notion of Gibbs measure. Here “Gibbs type measure” means that it is strongly equivalent to an image of a Gibbs measure under a bi-Lipschitz map, see Section 2.4 for the definition of strongly equivalence of measures. The assertions (i) and (ii) tell us that both measures $\mathcal{H}^{s_{V}}|_{\Sigma_{V}}$ and $\mathcal{N}_{V}$ have very good dynamical structure.

(3) The case $\kappa = 1$ corresponds to the Fibonacci Hamiltonian. By Remark 6, 7 and 8 in Section 8 we have

$$\hat{\eta}_{1} = \frac{3}{2} \ln \alpha_{1}, \quad \eta_{1} = \frac{5 + \sqrt{5}}{4} \ln \alpha_{1} \quad \text{and} \quad \rho_{1} = \ln(1 + \sqrt{2}).$$

The first and the third equalities in (1.5) are known, which are (1.3) and (1.2) respectively, see [8, 12]. But the second one in (1.5) is new.

(4) Recall that in [11], $d_{V}(1) < s_{V}(1)$ is proven for $V$ small. Here for any $\kappa \neq 2$ and $V$ is large, we show that $d_{V}(\kappa) < s_{V}(\kappa)$. As explained in [11], $\mathcal{N}_{\kappa,V}$ is the harmonic measure determined by $\Sigma_{\kappa,V}$. It is a general rule that if a planar set $E$ is dynamically defined, then the Hausdorff dimension of the harmonic measure determined by $E$ is strictly less than the Hausdorff dimension of $E$ (see for example the survey paper [29] or the book [19]). Our results verify this rule in this spectral setting.
In the case $\kappa = 2$, $\alpha_2$ is sometimes called silver number. By the explanation above, we still expect $d_V(2) < s_V(2)$. However, since $\varrho_2 = \rho_2$, we can not achieve this by the asymptotic formulas anymore. Finer estimations are needed. This makes the silver number case an interesting object to study. We note that in [15], the trace map related to silver number has been studied. They showed that the non-wandering set of this map is hyperbolic if the coupling is sufficiently large.

Finally let us say a few words about the proof. Based on the analysis in [34, 27, 17] about the nested structure of the spectrum, we can introduce an auxiliary symbolic dynamical system which codes the spectrum. By introducing two potentials (one is additive, which is related to the density of states measure $N_V$; the other is almost additive, which is related to the Hausdorff measure restricted to the spectrum $\Sigma_V$), we can use the thermodynamical and multifractal formalisms to analyze the spectrum and $N_V$ respectively. We show that both measures have the Gibbs property, moreover $H^s|_{\Sigma_V}$ is kind of a measure of maximal dimension (see Section 4, Remark 3 and Theorem 9) and $N_V$ is kind of a measure of maximal entropy (see Section 5, Theorem 10 and Theorem 11). These good structures in turn give very exact and explicit informations about the dimensions of the spectrum and the density of states measure.

Shortly after we finished our paper, we saw the impressive paper [13]. In [13], Damanik, Gorodetski and Yessen completed the picture for the Fibonacci Hamiltonian (the case $\kappa = 1$) by getting rid of the smallness and largeness restriction on the coupling constant $V$, giving the explicit formulas for $\gamma_V(\alpha_1), d_V(\alpha_1), s_V(\alpha_1)$ (and another important quantity—transport exponent) and the exact asymptotic behaviors of these quantities(among many other things). It seems interesting to make some comments on their methods and point out some connections of their work with ours. In their paper, they consider the so-called Fibonacci trace map $T$ and the related maximal hyperbolic invariant set $\Lambda_V$. It is well known that the spectrum is obtained by intersecting a special line $\ell_V$ with $\Lambda_V$. Through the previous work of the authors [9, 10, 11], to obtain the spectral properties of the spectrum, a crucial step is to show that $\ell_V$ intersects transversally with the stable manifolds of points in $\Lambda_V$ for any $V > 0$. In [13], they made a decisive progress and succeeded to achieve this step. Then by using powerful tools from hyperbolic dynamical system, they derived the complete picture for Fibonacci Hamiltonian. Now let us point out some connections of their work with ours. In [13] Theorem 1.6, they got dimension formulas for $\Sigma_{\alpha_1,V}$ (formula (10)) and $N_{\alpha_1,V}$ (formula (11))and formula for the optimal Hölder exponent of $N_{\alpha_1,V}$ (formula (12)). In our setting the counterparts are (4.7), (5.4) and (6.10). In [13] Theorem 1.10, they got exact asymptotic formulas for $\gamma_V(\alpha_1), d_V(\alpha_1), s_V(\alpha_1)$. In our case this is related to Remark 1 (3).

The rest of the paper are organized as follows. In Section 2 we introduce the notations and summarize some known results and connections which we will use. In Section 3 we prove a geometric lemma, which will be used to establish a bi-Lipschitz equivalence between the dynamical subset of the spectrum and the symbolic space. In Section 4 we study the Hausdorff dimension and Hausdorff measure of a dynamical subset of the spectrum. In Section 5 we study the dimension properties of the density of states measure.
\(N_V\) restricted to a dynamical subset. In Section 6 we do the multifractal analysis of \(N_V\), in particular, we get an expression for the optimal Hölder exponent of \(N_V\). In Section 7 by comparing two arbitrary dynamical subsets, we obtain the global picture. In particular, we prove Theorem \[\text{II}\] (i) and (ii). In Section 8 we study the asymptotic properties and prove Theorem \[\text{III}\](iii) and (iv). Finally in Section 9 we give another proof of the fact that \(d_V(\kappa) < s_V(\kappa)\) when \(\kappa \neq 2\).

2. Preliminaries

In this section we summarize some known results and connections which we will use.

At first we discuss the structure of the spectrum and give a coding for it. Next we collect some useful facts about Sturm Hamiltonian. Finally we recall the thermodynamical and multifractal formalisms for the almost additive potentials.

2.1. The structure of the spectrum.

We describe the structure of the spectrum \(\Sigma_V = \Sigma_{\kappa,V}\) for some fixed \(\kappa\) and \(V\). We also collect some known facts that will be used later, for more details, we refer to [4, 34, 35, 36].

Since \(\Sigma_V\) is independent with the phase \(\phi\), in the rest of the paper we can and will take \(\phi = 0\).

Let \(p_n/q_n (n > 0)\) be the \(n\)-th partial quotient of \(\{\alpha_\kappa\}\), the fractional part of \(\alpha_\kappa\), given by:

\[
\begin{align*}
p_{-1} &= 1, & p_0 &= 0, & p_{n+1} &= \kappa p_n + p_{n-1}, & n \geq 0, \\
q_{-1} &= 0, & q_0 &= 1, & q_{n+1} &= \kappa q_n + q_{n-1}, & n \geq 0.
\end{align*}
\]

It is seen that \(\alpha_\kappa > 1\) is the root of \(x^2 - \kappa x - 1 = 0\) and we have

\[
(2.1) \quad q_n = c_\kappa \alpha_\kappa^n + (1 - c_\kappa) (-\alpha_\kappa)^{-n} \quad (n \geq -1)
\]

with \(c_\kappa = \alpha_\kappa^2/(1 + \alpha_\kappa^2)\).

Let \(n \geq 1\) and \(x \in \mathbb{R}\), the transfer matrix \(M_n(x)\) over \(q_n\) sites is defined by

\[
M_n(x) := \begin{bmatrix}
x - v_{q_n} & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x - v_{q_n-1} & -1 \\
1 & 0
\end{bmatrix} \cdots \begin{bmatrix}
x - v_1 & -1 \\
1 & 0
\end{bmatrix},
\]

where \(v_n\) is defined in (\[\text{I}\]). By convention we take

\[
M_{-1}(x) = \begin{bmatrix}
1 & -V \\
0 & 1
\end{bmatrix} \quad \text{and} \quad M_0(x) = \begin{bmatrix}
x & -1 \\
1 & 0
\end{bmatrix}.
\]

For \(n \geq 0, p \geq -1\), let \(t_{(n,p)}(x) = \text{tr} M_{n-1}(x) M_0^n(x)\) and

\[
\sigma_{(n,p)} = \{x \in \mathbb{R} : |t_{(n,p)}(x)| \leq 2\},
\]

where \(\text{tr} M\) stands for the trace of the matrix \(M\). Then

\[
(\sigma_{(n+2,0)} \cup \sigma_{(n+1,0)}) \subset (\sigma_{(n+1,0)} \cup \sigma_{(n,0)}).
\]

Moreover

\[
\Sigma_V = \bigcap_{n \geq 0} (\sigma_{(n+1,0)} \cup \sigma_{(n,0)}).
\]
The intervals in $\sigma_{(n,p)}$ are called the bands. Take some band $B \in \sigma_{(n,p)}$, then $t_{(n,p)}(x)$ is monotone on $B$ and $t_{(n,p)}(B) = [-2, 2]$. We call $t_{(n,p)}$ the generating polynomial of $B$ and denote it by $h_B := t_{(n,p)}$.

\{\sigma_{(n+1,0)} \cup \sigma_{(n,0)} : n \geq 0\} form a covering of $\Sigma_V$. However there are some repetitions between $\sigma_{(n,0)} \cup \sigma_{(n-1,0)}$ and $\sigma_{(n,0)} \cup \sigma_{(n,0)}$. It is possible to choose a covering of $\Sigma_V$ elaborately such that we can get rid of these repetitions, as we will describe in the follows:

**Definition 1.** ([34, 27]) For $V > 4$, $n \geq 0$, we define three types of bands as follows:

- $(n,I)$-type band: a band of $\sigma_{(n,1)}$ contained in a band of $\sigma_{(n,0)}$;
- $(n,II)$-type band: a band of $\sigma_{(n+1,0)}$ contained in a band of $\sigma_{(n,-1)}$;
- $(n,III)$-type band: a band of $\sigma_{(n+1,0)}$ contained in a band of $\sigma_{(n,0)}$.

All three types of bands are well defined, and we call these bands spectral generating bands of order $n$. Note that for order $0$, there is only one $(0, I)$-type band $\sigma_{(0,1)} = [V - 2, V + 2]$ (the corresponding generating polynomial is $t_{(0,1)} = x - V$), and only one $(0, III)$-type band $\sigma_{(1,0)} = [-2, 2]$ (the corresponding generating polynomial is $t_{(1,0)} = x$). They are contained in $\sigma_{(0,0)} = (-\infty, +\infty)$ with corresponding generating polynomial $t_{(0,0)} \equiv 2$. For the convenience, we call $\sigma_{(0,0)}$ the spectral generating band of order $-1$.

For any $n \geq -1$, denote by $B_n$ the set of all spectral generating bands of order $n$, then the intervals in $B_n$ are disjoint. Moreover ([34, 27])

- $(\sigma_{(n+2,0)} \cup \sigma_{(n+1,0)}) \subset \bigcup_{B \in B_n} B \subset (\sigma_{(n+1,0)} \cup \sigma_{(n,0)})$, thus

  $$\Sigma_V = \bigcap_{n \geq 0} \bigcup_{B \in B_n} B;$$

- any $(n, I)$-type band contains only one band in $B_{n+1}$, which is of $(n + 1, II)$-type.
- any $(n, II)$-type band contains $2\kappa + 1$ bands in $B_{n+1}, \kappa + 1$ of which are of $(n + 1, I)$-type and $\kappa$ of which are of $(n + 1, III)$-type.
- any $(n, III)$-type band contains $2\kappa - 1$ bands in $B_{n+1}, \kappa$ of which are of $(n + 1, I)$-type and $\kappa - 1$ of which are of $(n + 1, III)$-type.

Thus $\{B_n\}_{n \geq 0}$ forms a natural covering([28, 25]) of the spectrum $\Sigma_V$.

**2.2. The coding of the spectrum.**

Now we will give a coding for the spectrum $\Sigma_V$. We define the alphabet as

$$\mathcal{A}_\kappa := \{(I,j) : j = 1, \cdots, \kappa + 1\} \cup \{(II, 1)\} \cup \{(III,j) : j = 1, \cdots, \kappa\}.$$  

Then $|\mathcal{A}_\kappa| = 2\kappa + 2$. We order the elements in $\mathcal{A}_\kappa$ as

$$(I,1) < \cdots < (I,\kappa + 1) < (II,1) < (III,1) < \cdots < (III,\kappa).$$

To simplify the notation we rename the above line as

$$e_1 < e_2 < \cdots < e_{2\kappa + 2}.$$  

Given $e_i, e_j \in \mathcal{A}_\kappa$, we call $e_i e_j$ admissible, denote by $e_i \rightarrow e_j$, if

$$(e_i, e_j) \in \{(I,j), (II,1) : 1 \leq j \leq \kappa + 1\} \cup$$
Thus the Perron-Frobenius eigenvalue of \( \hat{\alpha} \) is \( \alpha \). It is ready to show that the graph is aperiodic. Consequently, we define the incidence matrix

\[
A_k = (a_{ij})
\]

We also define a related matrix as follows:

\[
\hat{A}_k = \begin{bmatrix} 0 & 1 & 0 \\ \kappa + 1 & 0 & \kappa \\ \kappa & 0 & \kappa - 1 \end{bmatrix}
\]

**Lemma 1.** For each \( \kappa \in \mathbb{N} \), \( A_k, \hat{A}_k \) are primitive and have the same Perron-Frobenius eigenvalue \( \alpha_k \). Moreover, \( \hat{A}_k \) has three different eigenvalues \( \alpha, -1, -1/\alpha \), consequently \( \hat{A}_k \) is diagonalizable.

Recall that a nonnegative square matrix \( B \) is called **primitive** if there exists some \( k \in \mathbb{N} \) such that all the entries of \( B^k \) are positive.

**Proof.** It is straightforward to check that \( \hat{A}_k \) is primitive. We also have

\[
\det(\lambda I_3 - \hat{A}_k) = (\lambda^2 - \kappa \lambda - 1)(\lambda + 1).
\]

Thus the Perron-Frobenius eigenvalue of \( \hat{A}_k \) is \( \alpha_k \) and \( \hat{A}_k \) has three different eigenvalues \( \alpha_k, -1, -1/\alpha_k \).

On the other hand if we consider the graph related to the incidence matrix \( A_k \), then it is ready to show that the graph is aperiodic. Consequently \( A_k \) is primitive. By direct computation we get that

\[
\det(\lambda I_{2k+2} - A_k) = \lambda^{2k-1}(\lambda^2 - \kappa \lambda - 1)(\lambda + 1).
\]

Thus the Perron-Frobenius eigenvalue of \( A_k \) is also \( \alpha_k \). \( \square \)

Let \( \Omega^{(\kappa)} \) be the subshift of finite type with alphabet \( A_k \) and incidence matrix \( A_k \), i.e.

\[
\Omega^{(\kappa)} = \{e_{i_0}e_{i_1}\ldots e_{i_j} \in A_k; e_{i_j} \to e_{i_{j+1}}\}.
\]

For \( \omega \in \Omega^{(\kappa)} \), we write \( \omega|_n = \omega_0 \cdots \omega_n \). More generally we write \( \omega[n, \cdots, n+k] \) for \( \omega_n \cdots \omega_{n+k} \). For \( \omega, \tilde{\omega} \in \Omega^{(\kappa)} \), we denote by \( \omega \wedge \tilde{\omega} \) the maximal common prefix of \( \omega \) and \( \tilde{\omega} \). Define \( \Omega^{(n)}_n := \{e_{i_0} \cdots e_{i_n} : e_{i_j} \in A_k; e_{i_j} \to e_{i_{j+1}}\} \) and \( \Omega^{(\kappa)} = \bigcup_n \Omega^{(n)}_n \). Given \( w = w_0 \cdots w_n \in \Omega^{(n)}_n \), denote by \( |w| \) the length of \( w \), then \( |w| = n+1 \). If \( w_n = (T, j) \), then we write \( t_w = T \), and call \( w \) has type \( T \). If \( w = uv' \), then we say \( u \) is a **prefix** of \( w \) and denote by \( u \prec w \). Given \( u \in \Omega^{(n)}_m \) and \( v \in \Omega^{(n)}_n \), if \( u_n = v_0 \), then we write \( u* v := u_0 \cdots u_nv_1 \cdots v_m \). Given \( w \in \Omega^{(n)}_n \), we define the cylinder

\[
[w] := \{\omega \in \Omega^{(n)} : \omega|_n = w\}.
\]
Define $\tilde{\Omega}(\kappa) = \{ \omega \in \Omega(\kappa) : \omega_0 = (I, 1) \text{ or } (III, 1) \} = [e_1] \cup [e_{\kappa+3}]$. Similarly define $\tilde{\Omega}_n(\kappa) = \{ w \in \Omega_n(\kappa) : w_0 = (I, 1) \text{ or } (III, 1) \}$ and $\tilde{\Omega}_*(\kappa) = \bigcup_n \tilde{\Omega}_n(\kappa)$.

When $\kappa$ is clear from the context, we often suppress the index $\kappa$ and simply write $\Omega, \tilde{\Omega}, \Omega_0, \tilde{\Omega}_0, \Omega_*, \tilde{\Omega}_*.$

Now we define $B_{(I,1)}$ to be the unique $(0, I)$ type band and $B_{(III,1)}$ to be the unique $(0, III)$ type band. Then

$$B_0 = \{ B_w : w \in \tilde{\Omega}_0 \}.$$ 

Assume $B_w$ is defined for any $w \in \tilde{\Omega}_{n-1}$. Now for any $w \in \tilde{\Omega}_n$, write $w = w' e = w'(T, j)$. Then define $B_w$ to be the unique $j$-th $(n, T)$ type band in $B_n$ which is contained in $B_{w'}$. Then

$$B_n = \{ B_w : w \in \tilde{\Omega}_n \}.$$ 

We also denote by $h_w$ the generating polynomial of $B_w$.

There is a natural projection $\pi : \tilde{\Omega} \to \Sigma_V$ defined as

$$\pi(\omega) := \bigcap_{n \geq 0} B_{\omega|n}.$$ 

It is seen that $\pi$ is a homeomorphism. We write $X_w = \pi([w])$ for each $w \in \tilde{\Omega}_*$. 

2.3. Useful known results for Sturm Hamiltonian.

In this subsection we collect some useful known results for Sturm Hamiltonian.

We write $h_k(x) := t_{k+1,0}(x) = \text{tr} M_k(x)$ and write

$$\sigma_k := \sigma_{(k+1,0)} = \{ x \in \mathbb{R} : |h_k(x)| \leq 2 \}.$$ 

The following two lemmas are essentially proven in [34]:

**Lemma 2.** $\sigma_k = \{ B \in \mathcal{B}_k : B \text{ is of type } (k, II) \text{ or } (k, III) \}$.

Define $v^I = (1, 0, 0), v^{II} = (0, 1, 0), v^{III} = (0, 0, 1)$ and $v_* = (0, 1, 1)^T$.

**Lemma 3.** Given $w \in \Omega_*$, then

$$\# \{ u \in \Omega_* : |u| = |w| + n; w < u, t_u = II, III \} = v^t w \cdot \tilde{A}_n \cdot v_*.$$ 

The following theorem is [17] Theorem 2.1, see also [26] Theorem 3.1.

**Theorem 2** (Bounded variation). Let $V > 20$ and $\alpha = [a_0; a_1, a_2, \cdots] > 0$ be irrational with $a_n$ bounded by $M$. Then there exists a constant $\xi = \xi(V, M) > 1$ such that for any spectral generating band $B$ with generating polynomial $h$, 

$$\xi^{-1} \leq \left| \frac{h'(x_1)}{h'(x_2)} \right| < \xi, \quad \forall x_1, x_2 \in B.$$ 

The following theorem is [17] Theorem 5.1, see also [26] Theorem 3.3.
Theorem 3 (Bounded covariation). Let \(V > 20\) and \(\alpha = [a_0; a_1, a_2, \cdots] > 0\) be irrational with \(a_n\) bounded by \(M\). Then there exists constant \(\eta = \eta(V, M) > 1\) such that if \(w, \tilde{w}, wu, \tilde{w}u \in \hat{\Omega}_s\), then
\[
\eta^{-1} \frac{|B_{wu}|}{|B_w|} \leq \frac{|B_{\tilde{w}u}|}{|B_{\tilde{w}}|} \leq \eta \frac{|B_{wu}|}{|B_w|}.
\]

In the rest of this subsection we always assume \(V > 20\) and \(\alpha = \alpha_\kappa\).

The following lemma is a direct consequence of [17] Corollary 3.1:

Lemma 4. There exist constants \(0 < c_1 = c_1(V, \kappa) < c_2 = c_2(V, \kappa) < 1\) and \(n_0 = n_0(V, \kappa) \in \mathbb{N}\) such that for any \(n > n_0\) and any \(w \in \hat{\Omega}_n\)
\[
c_1 |B_w|_{n-n_0} \leq |B_w| \leq c_2 |B_w|_{n-n_0}.
\]

There exists constant \(0 < c_3 = c_3(V, \kappa) < 1\) such that for any \(w \in \hat{\Omega}_n\)
\[
c_3^n \leq |B_w| \leq 2^{2-n}.
(2.4)

The following lemma is [26] Lemma 3.7 by taking \(a_i = \kappa\).

Lemma 5. Write \(t_1 = (V - 8)/3\) and \(t_2 = 2(V + 5)\). Then for any \(w = w_0 \cdots w_n \in \hat{\Omega}_n\)
\[
\prod_{w_i = (1, 1)} \frac{1}{t_2^{\kappa - 1} \cdot \prod_{w_i \neq (1, 1)} \frac{1}{t_2^{\kappa - 3}}} \leq |B_w| \leq 4 \prod_{w_i = (1, 1)} \frac{1}{t_1^{\kappa - 1} \cdot \prod_{w_i \neq (1, 1)} \frac{1}{t_1^{\kappa - 1}}},
\]
where \(i\) ranges from 1 to \(n\).

2.4. Recall on thermodynamical formalism and multifractal analysis.

If \(X\) is a compact metric space and \(T : X \to X\) is continuous, then \((X, T)\) is called a topological dynamical system, TDS for short. \(\mathcal{M}(X)\) is the set of all probability measures supported on \(X\). \(\mathcal{M}(X,T)\) is the set of all \(T\)-invariant probability measures supported on \(X\). Given \(\mu, \nu \in \mathcal{M}(X)\), if there exists a constant \(C > 1\) such that \(C^{-1} \nu \leq \mu \leq C \nu\), then we say \(\mu\) and \(\nu\) are strongly equivalent and denote by \(\mu \asymp \nu\).

Assume \((X, T)\) is a TDS and \(\Phi = \{\phi_n : n \geq 0\}\) is a family of continuous functions from \(X\) to \(\mathbb{R}\). If there exists a constant \(C(\Phi) \geq 0\) such that for any \(n, k \geq 0\)
\[
|\phi_{n+k}(x) - \phi_n(x) - \phi_k(T^n x)| \leq C(\Phi),
\]
then \(\Phi\) is called a family of almost additive potentials. We use \(C_{aa}(X, T)\) to denote the set of all the almost addtive potentials defined on \(X\). When \(C(\Phi) = 0\), it is seen that \(\phi_n(x) = \sum_{j=0}^{n-1} \phi_1(T^j x) =: S_n \phi_1(x)\). That is, \(\phi_n\) is the ergodic sum of \(\phi_1\). In this case we say \(\Phi\) is a family of additive potentials.

Given \(\Phi \in C_{aa}(X, T)\), if there exists a constant \(c > 0\) such that \(\phi_n(x) \geq cn\) for any \(n \geq 0\), then we say \(\Phi\) is positive and write \(\Phi \in C_{aa}^+(X, T)\). We can define \(C_{aa}^-(X, T)\) similarly.

Given \(\Phi \in C_{aa}(X, T)\), by subadditivity, \(\Phi_*(\mu) := \lim_{n \to \infty} \int_X \frac{\phi_n}{n} d\mu\) exists for every \(\mu \in \mathcal{M}(X, T)\). Notice that if \(\Phi\) is positive(negative), then \(\Phi_*(\mu) > 0(< 0)\).
Given a subshift of finite type \((\Sigma_A,\sigma)\) and \(\Phi \in C_{aa}(\Sigma_A,\sigma)\). If there exists another constant \(D(\Phi) \geq 0\) such that for any \(n \geq 0\)
\[
\sup\{|\phi_n(x) - \phi_n(y)| : x|_n = y|_n\} \leq D(\Phi)
\]
Then we say \(\Phi\) has bounded variation property.

2.4.1. Thermodynamical formalism.

Given \(\Phi \in C_{aa}(\Sigma_A,\sigma)\), the topological pressure can be defined as
\[
P(\Phi) := \lim_{n \to \infty} \frac{1}{n} \ln \sum_{|w| = n} \exp(\sup_{x \in [w]} \phi_n(x)).
\]
The following extension of the classical variational principle holds:

**Theorem 4.**\([2, 31, 6, 3]\) Let \((\Sigma_A,\sigma)\) be a topologically mixing subshift of finite type. For any \(\Phi \in C_{aa}(\Sigma_A,\sigma)\) we have
\[(2.5)\]
\[
P(\Phi) = \sup\{h_\mu(T) + \Phi_\mu(\mu) : \mu \in \mathcal{M}(\Sigma_A,\sigma)\}.
\]
Combining with the monotonicity of pressure, this variational principle has the following useful consequence:

**Corollary 1.** For any \(\Phi,\Psi \in C_{aa}(\Sigma_A,\sigma)\), the function \(Q(s) := P(\Phi + s\Psi)\) is convex on \(\mathbb{R}\). Consequently \(Q\) is continuous. If moreover \(\Psi \in C_{aa}^-(\Sigma_A,\sigma)\), then \(Q(s)\) is strictly decreasing and
\[
\lim_{s \to -\infty} Q(s) = \infty \quad \text{and} \quad \lim_{s \to \infty} Q(s) = -\infty.
\]
Thus \(Q(s) = 0\) has a unique solution.

**Theorem 5.**\([2, 31]\) Assume \(\Phi \in C_{aa}(\Sigma_A,\sigma)\) has bounded variation property. Then
(i) there exists a unique invariant Gibbs measure \(\mu_\Phi\) related to \(\Phi\), which satisfies
\[
C^{-1} \leq \frac{\mu_\Phi([w])}{\exp(-nP(\Phi) + \phi_n(x))} \leq C \quad (\forall x \in [w]).
\]
(ii) If \(\tilde{\Phi} \in C_{aa}(\Sigma_A,\sigma)\) and \(D > 0\) is a constant such that \(\|\phi_n - \tilde{\phi}_n\| \leq D\) for any \(n \geq 0\), then \(\tilde{\Phi}\) also has bounded variation property. Moreover \(P(\Phi) = P(\tilde{\Phi})\) and \(\mu_\Phi = \mu_{\tilde{\Phi}}\).
(iii) \(\mu_\Phi\) is strongly mixing and is the unique invariant measure which attains the supremum of \((2.5)\).

If \(\mu \in \mathcal{M}(\Sigma_A,\sigma)\) such that \(P(\Phi) = h_\mu(\sigma) + \Phi_\mu(\mu)\), then \(\mu\) is called an equilibrium state of \(\Phi\). The above proposition shows that every \(\Phi \in C_{aa}(\Sigma_A,\sigma)\) with bounded variation property has a unique equilibrium state.
2.4.2. *Multifractal analysis.*

We recall some results proved in [1] (see also [3]), which we will need in this paper.

Given a $\Psi \in C_{aa}(\Sigma_A, \sigma)$ we can define on $(\Sigma_A, \sigma)$ a weak Gibbs metric $d_\Psi$ as

$$d_\Psi(x, y) = \sup_{z \in [x \wedge y]} \exp(\psi|_{x \wedge y}(z)).$$

This kind of metric is considered in [20, 23, 1]. In the following we will work on the metric space $(\Sigma_A, d_\Psi)$.

Let $\Phi \in C_{aa}(\Sigma_A, \sigma)$ and $\Theta \in C_{aa}(\Sigma_A, \sigma)$. For any $\beta \in \mathbb{R}$, define the level set

$$\Lambda_{\Phi/\Theta}(\beta) := \left\{ x \in \Sigma_A : \lim_{n \to \infty} \frac{\phi_n(x)}{\theta_n(x)} = \beta \right\}.$$

Since $\Theta$ is negative, $\Theta^*(\mu) < 0$ for any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$. Define

$$L_{\Phi/\Theta} := \left\{ \frac{\Phi^*(\mu)}{\Theta^*(\mu)} : \mu \in \mathcal{M}(\Sigma_A, \sigma) \right\}.$$

Then $L_{\Phi/\Theta}$ is an interval. For any $q, \beta \in \mathbb{R}$ we define $\mathcal{L}_{\Phi/\Theta}(q, \beta)$ to be the unique solution $t = t(q, \beta)$ of the equation $P(q(\Phi - \beta \Theta) + t \Psi) = 0$. For any $\beta \in L_{\Phi/\Theta}$, define

$$\mathcal{L}_{\Phi/\Theta}^*(\beta) = \inf_{q \in \mathbb{R}} \mathcal{L}_{\Phi/\Theta}(q, \beta).$$

**Theorem 6.** $\Lambda_{\Phi/\Theta}(\beta) \neq \emptyset \iff \beta \in L_{\Phi/\Theta}$. If $\beta \in L_{\Phi/\Theta}$, then

$$\dim H \Lambda_{\Phi/\Theta}(\beta) = \mathcal{L}_{\Phi/\Theta}^*(\beta).$$

The following dimension formula for Gibbs measure is also useful.

**Theorem 7.** [3] Given $\Phi \in C_{aa}(\Sigma_A, \sigma)$ with bounded variation property. Let $\mu_\Phi$ be the related Gibbs measure. Then $\mu_\Phi$ is exact dimensional and

$$\dim H \mu_\Phi = \frac{h_{\mu_\Phi}(\sigma)}{-\Psi^*(\mu)}.$$

Finally we say some words about notations. Given two positive sequences $\{a_n\}$ and $\{b_n\}$, $a_n \lesssim b_n$ means that there exists some constant $C > 0$ such that $a_n \leq Cb_n$ for all $n \in \mathbb{N}$. $a_n \sim b_n$ means that $a_n \lesssim b_n$ and $b_n \lesssim a_n$. $a_n \lesssim_{\gamma_1, \ldots, \gamma_k} b_n$ means that $a_n \lesssim b_n$ with the constant $C$ only depending on $\gamma_1, \ldots, \gamma_k$.

3. A geometric lemma

In this section we will prove a geometric lemma, which claims that the ratios of the lengths of a gap and the minimal band which contains it are bounded from below. This lemma is fundamental in our study on the metric property of the spectrum because through it we can establish a Lipschitz equivalence between the symbolic space and the spectrum. Thus all the dimension problems of the spectrum are completely converted to those of the symbolic space, where we have dynamical tools to use.

From now on we fix $V > 20$ and $\kappa \in \mathbb{N}$. Write $\mathbb{R} \setminus \Sigma_V = \bigcup_i G_i$. Each $G_i$ is called a gap of the spectrum. A gap $G$ is called of order $n$, if $G$ is covered by some band in $\mathcal{B}_n$ but not
covered by any band in $B_{n+1}$. We use $\mathcal{G}_n$ to denote the set of all gaps of order $n$. For any $G \in \mathcal{G}_n$, let $B_G$ be the unique band in $B_n$ which contains $G$.

**Lemma 6.** There exists a constant $C = C(\kappa, V) \in (0, 1)$ such that for any gap $G \in \bigcup_{n \geq 0} \mathcal{G}_n$ we have $|G| \geq C|B_G|$.

**Proof.** Given $B_w \in \mathcal{G}_n$, we study the gaps of order $n$ contained in $B_w$. If $w$ has type $t_w = I$, then there exists a unique band $B_{w(I,1)} \in B_{n+1}$ which is contained in $B_w$, thus there is no gap of order $n$ in $B_w$.

Now assume $t_w = II$. Then by [34], there exist $2\kappa + 1$ bands of order $n + 1$, which are disjoint and ordered as follows:

$$B_{we_1} < B_{we_{n+1}} < B_{we_2} < B_{we_{n+4}} < B_{we_3} < \cdots < B_{we_{2n+2}} < B_{we_{n+1}}.$$  

Thus there are $2\kappa$ gaps of order $n$ in $B_w$. We list them as $\{G_1, \cdots, G_{2\kappa}\}$.

Let $S_p(x)$ be the Chebyshev polynomial defined by

$$S_0(x) \equiv 0, \quad S_1(x) \equiv 1, \quad S_{p+1}(x) = x S_p(x) - S_{p-1}(x) \quad (p \geq 1).$$

It is well known that

$$S_p(2\cos \theta) = \frac{\sin p \theta}{\sin \theta}, \quad \theta \in [0, \pi].$$

Following [17], for each $p \in \mathbb{N}$ and $1 \leq l \leq p$, we define

$$I_{p,l} := \{2\cos \frac{l + c}{p + 1} \pi : |c| \leq \frac{1}{10} \text{ and } |S_{p+1}(2\cos \frac{l + c}{p + 1} \pi)| \leq \frac{1}{4}\}.$$ 

It is seen that $I_{p,l}, l = 1, \cdots, p$ are disjoint.

We claim that $I_{p,l}$ and $I_{p-1,s}$ are disjoint. Fix any $x \in I_{p-1,s}$, write $x = 2\cos \theta$. Then

$$\frac{s - 1/10}{p} \pi \leq \theta \leq \frac{s + 1/10}{p} \pi \quad \text{and} \quad \left|\frac{\sin p \theta}{\sin \theta}\right| = |S_p(2\cos \theta)| \leq \frac{1}{4}.$$ 

We need to show that $x \not\in I_{p,l}$. If otherwise by the definition of $I_{p,l}$ and (3.2) we should have

$$\left|\cos \theta - \frac{\sin p \theta}{\sin \theta}\right| \leq \left|\frac{\sin(p + 1) \theta}{\sin \theta}\right| = |S_{p+1}(2\cos \theta)| \leq \frac{1}{4}.$$ 

Thus we have $|\cos \theta| \leq 1/2$. On the other hand still by (3.2) we have

$$(s - 1/10) \pi \leq p \theta \leq (s + 1/10) \pi.$$ 

Thus $|\cos p \theta| \geq \cos \frac{\pi}{10} > \frac{1}{2}$, which is a contradiction.

By (3.1) and the claim above, it is easy to check that the following intervals are disjoint and ordered as

$$I_{\kappa+1,1} < I_{\kappa,1} < I_{\kappa+1,2} < \cdots < I_{\kappa+1,\kappa} < I_{\kappa,\kappa} < I_{\kappa+1,\kappa+1} \subset [-2, 2].$$

There are $2\kappa$ gaps $\{\tilde{G}_1, \cdots, \tilde{G}_{2\kappa}\}$. Define $g = \min\{|\tilde{G}_j| : j = 1, \cdots, 2\kappa\}$, then $g > 0$ is a constant only depending on $\kappa$.

Assume $x_s \in B_w$ such that

$$|h'_w(x_s)||B_w| = 4.$$
By Proposition 3.1, we have
\[ h_w(B_{w_1}) \subset I_{\kappa+1,l} \quad (1 \leq l \leq \kappa + 1) \] and
\[ h_w(B_{w_{l+\kappa+2}}) \subset I_{\kappa,l} \quad (1 \leq l \leq \kappa). \]
Since \( h_w : B_w \to [-2,2] \) is a bijection, \( h_w(G_j) \supset \tilde{G}_j \) for each \( j \). Fix a gap \( G_j \) in \( B_w \), then there exists \( x_j \in G_j \) such that
\[ |h'_w(x_j)| |G_j| = |h_w(G_j)| \geq |\tilde{G}_j|. \]
By Theorem 2, (3.3), (3.4), there exist a constant \( c(\kappa,V) > 0 \) such that
\[ |G_j| |B_w| \geq g \cdot |h'_w(x^*_j)| |h'_w(x_j)| \geq cg =: C_1(\kappa,V) > 0. \]
If \( t_w = III \), the same proof as above shows that there exists a constant \( C_2(\kappa,V) > 0 \) such that for any gap \( G \) in \( B_w \), we have
\[ \frac{|G|}{|B_w|} \geq C_2(\kappa,V) > 0. \]
Let \( C = \min\{C_1, C_2\} \), the result follows. \( \square \)

4. Hausdorff Dimension of the Spectrum

In this section we will study the Hausdorff dimension and Hausdorff measure of \( \Sigma_V \). We will show that when restricted to suitable subset \( \Sigma_{\tilde{\alpha}} \), the Hausdorff measure \( \mathcal{H}^s_{\tilde{\alpha}} \) is a Gibbs type measure and in some sense a measure of maximal dimension.

Recall that in Section 2.2 we have defined the subshift \( \Omega = \Omega^{(\kappa)} \) with alphabet \( A = A_\kappa \) and incidence matrix \( A_\kappa \). Together with the shift map \( \sigma : \Omega \to \Omega \) defined as \( \sigma((\omega_n)_{n=0}^{\infty}) = (\omega_n)_{n=1}^{\infty} \), \( (\Omega, \sigma) \) becomes a topological dynamical system. By Lemma 1, \( A_\kappa \) is primitive, thus this system is mixing. We are going to use the machinery of thermodynamical formalism to study the Hausdorff dimensions of the spectrum, with the aid of this symbolic dynamical system.

Recall also that we have defined \( \tilde{\Omega} = \tilde{\Omega}^{(\kappa)} = [e_1] \cup [e_{\kappa+3}] \subset \Omega \) and a natural projection \( \pi : \tilde{\Omega} \to \Sigma_V \) and \( \pi \) is a homeomorphism. Thus \( \Sigma_V \) is coded by \( \tilde{\Omega} \), which is not invariant under \( \sigma \). To use the dynamical system \( (\Omega, \sigma) \) we need to do some extra work.

Our strategy is as follows: At first we introduce a family of subsets of \( \Sigma_V \), such that each subset in this family can be coded by \( (\Omega, \sigma) \). With the aid of this dynamic, we can obtain all the spectral properties of this subset. Finally we will show that all the properties keep unchanged when we exhaust all the possible choices of subsets. Thus we get a global result on the whole spectrum. We will do this final step in Section 7.

4.1. Dynamical subsets.

Fix some \( N = N_\kappa \geq 4 \) such that \( A_\kappa^N > 0 \). Then the set of the last letters of words in \( \tilde{\Omega}_N \) is \( A \). In other words,
\[ \{w_N : w \in \tilde{\Omega}_N\} = A. \]
Define a subsets of \( (\tilde{\Omega}_N)^{2\kappa+2} \) as
\[ D := \{(w^{e_1}, \cdots, w^{e_{2\kappa+2}}) : w_N^{e_i} = e_i, i = 1, \cdots, 2\kappa + 2\}. \]
We denote by \( \vec{w} := (w^e_1, \ldots, w^e_{2\kappa+2}) \) an element in \( D \). Recall that \( X_w = \pi([w]) \) for each \( w \in \tilde{\Omega} \). Given \( \vec{w} \in D \), define
\[
\Sigma_{\vec{w}} := \bigcup_{i=1}^{2\kappa+2} X_{w^e_i}.
\]

It is seen that \( \Sigma_{\vec{w}} \) is made of total number \( 2\kappa + 2 \) \( N \)-level basic sets of \( \Sigma_V \) and \( \Sigma_V = \bigcup_{\vec{w} \in D} \Sigma_{\vec{w}} \). Now we can define a projection \( \pi_{\vec{w}} : \Omega \to \Sigma_{\vec{w}} \) as
\[
(4.2)
\pi_{\vec{w}}(\omega) = \pi(w^o \sigma^\omega).
\]

It is ready to show that \( \pi_{\vec{w}} \) is a homeomorphism. We call \( \Sigma_{\vec{w}} \) a dynamical subset of \( \Sigma_V \).

We will study the dimension properties of \( \Sigma_{\vec{w}} \) at first, then by comparing two different dynamical subsets \( \Sigma_{\vec{w}} \) and \( \Sigma_{\vec{v}} \), we obtain a global picture (we will finish this in Section 7).

From now on until the end of this section we will fix some \( \vec{w} \in D \).

4.2. Almost additive potentials related to Lyapunov exponents.

We will define some \( \Psi \in C_{aa}^{-}(\Omega, \sigma) \), which captures the exponential rate of the length of the generating bands and can be viewed as Lyapunov exponent function. We will see that \( \Psi \) is intimately related to the Hausdorff dimension of \( \Sigma_{\vec{w}} \).

For each \( n \in \mathbb{N} \) and each \( \omega \in \Omega \), define
\[
(4.3)
\psi_n(\omega) := \ln |B_{w^0 \omega [1, \ldots, n]}| = \ln |B_{w^0 \omega^\omega |n}|,
\]
where \( |B_w| \) denote the length of \( B_w \) (see the definition of \( v \star w \) in Section 2.2).

**Lemma 7.** \( \Psi = \{\psi_n : n \geq 0\} \in C_{aa}^{-}(\Omega, \sigma) \) and \( \Psi \) has bounded variation property.

**Proof.** Given \( \omega \in \Omega \) we have \( \psi_n(\omega) = \ln |B_{w^0 \omega [1, \ldots, n]}| \), \( \psi_{n+k}(\omega) = \ln |B_{w^0 \omega [1, \ldots, n+k]}| \) and \( \psi_k(\sigma^n \omega) = \ln |B_{w^{n+1} \omega [n+1, n+k]}| \). By Theorem 3,
\[
\frac{|B_{w^0 \omega [1, \ldots, n+k]}|}{|B_{w^0 \omega [1, \ldots, n]}|} \sim_{V, \kappa} \frac{|B_{w^{n+1} \omega [n+1, n+k]}|}{|B_{w^{n} \omega}|}.
\]
Notice that there are only finitely many different bands \( B_w \) with \( |w| = N + 1 \) and \( N \) only depends on \( \kappa \), thus we conclude from the above equation that
\[
|\psi_{n+k}(\omega) - \psi_n(\omega) - \psi_k(\sigma^n \omega)| \leq C(V, \kappa) =: C(\Psi).
\]
Thus \( \Psi \) is almost additive.

Recall that \( N \geq 4 \). By (2.4) we have
\[
(4.4) \quad (n + N) \ln c_3 \leq \psi_n(\omega) \leq -n \ln 2.
\]
Thus \( \Psi \in C_{aa}^{-}(\Omega, \sigma) \).

By the definition, if \( u \in \Omega_n \) and \( \omega, \bar{\omega} \in [u] \), then
\[
\psi_n(\omega) = \ln |B_{w^0 u [1, \ldots, n]}| = \psi_n(\bar{\omega}).
\]
So \( \Psi \) has bounded variation property with constant \( D(\Psi) = 0 \).  \( \square \)
4.3. Weak-Gibbs metric on $\Omega$.

Since $\Psi \in C_{aa}(\Omega, \sigma)$, we can define the weak-Gibbs metric $d_\Psi$ on $\Omega$ according to (2.6) as follows. Given $\omega, \tilde{\omega} \in \Omega$. If $\omega_0 \neq \tilde{\omega}_0$, define $d_\Psi(\omega, \tilde{\omega}) := \text{diam}(\Sigma_V)$. If $\omega_0 = \tilde{\omega}_0$ define

$$d_\Psi(\omega, \tilde{\omega}) := \sup_{\omega' \in [\omega \land \tilde{\omega}]} \exp(\psi_{|\omega \land \tilde{\omega}|}(\omega')) = |B_{w^{\mathbf{a}_0 * \sigma}_n}|.$$ 

Denote by $| \cdot |$ the standard metric on $\mathbb{R}$. Then we have

**Proposition 1.** $\pi_\mathbf{w} : (\Omega, d_\Psi) \rightarrow (\Sigma_\mathbf{w}, | \cdot |)$ is a bi-Lipschitz homeomorphism.

**Proof.** Given $\omega, \tilde{\omega} \in \Omega$. Assume $\omega|_n = \tilde{\omega}|_n$ and $\omega_{n+1} \neq \tilde{\omega}_{n+1}$. Then we have $d_\Psi(\omega, \tilde{\omega}) = |B_{w^{\mathbf{a}_0 * \sigma}_n}|$. Write $x := \pi_\mathbf{w}(\omega)$ and $y := \pi_\mathbf{w}(\tilde{\omega})$. It is seen that $x, y \in B_{w^{\mathbf{a}_0 * \sigma}_n}$, consequently

$$|x - y| \leq |B_{w^{\mathbf{a}_0 * \sigma}_n}| = d_\Psi(\omega, \tilde{\omega}).$$

On the other hand, since $\omega_{n+1} \neq \tilde{\omega}_{n+1}$, there is a gap $G$ of order $n + N$ which is contained in $(x, y)$. By Lemma 6

$$|x - y| \geq |G| \geq C|B_G| = C|B_{w^{\mathbf{a}_0 * \sigma}_n}| = Cd_\Psi(\omega, \tilde{\omega}).$$

Thus $\pi_\mathbf{w}$ is a bi-Lipschitz homeomorphism. □

**Remark 2.** This proposition is crucial for studying the dimensional properties of the spectrum and the density of states measure. Because by this proposition, the metric property on $(\Omega, d_\Psi)$ is the same with that on $(\Sigma_\mathbf{w}, | \cdot |)$ (see for example [16] chapter 2). Thus we can convert the dimension problem on spectrum completely to that in the symbolic space. We will use this proposition repeatedly in what follows.

4.4. Bowen’s formula, Hausdorff dimension and Hausdorff measure of $\Sigma_\mathbf{w}$.

Since $\Psi \in C_{aa}(\Omega, \sigma)$, by Corollary 1, $P(s\Psi) = 0$ has a unique solution $\tilde{s}_V$. By Lemma 7 $\Psi$ has bounded variation, so does $\tilde{s}_V\Psi$. Let $m$ be the unique Gibbs measure related to $\tilde{s}_V\Psi$. Then

**Theorem 8.** $m \asymp \mathcal{H}_{\tilde{s}_V}|_{\Omega}$. Thus $0 < \mathcal{H}_{\tilde{s}_V}(\Omega) < \infty$. Consequently $\dim_H m = \dim_H \Omega = \tilde{s}_V$ and $\mathcal{H}_{\tilde{s}_V}|_{\Omega}$ is exact dimensional. Moreover

$$\tilde{s}_V = \frac{h_{\tilde{m}}(\sigma)}{-\Psi_*(m)}.$$  

**Proof.** By Theorem 5(i), there exists a constant $C > 1$ such that for any $u \in \Omega$

$$C^{-1} \leq \frac{m(|u|)}{\exp(\tilde{s}_V \psi_{m}(\omega))} \leq C \quad (\forall \omega \in [u]).$$

Together with the definition of weak-Gibbs metric, we conclude that

$$C^{-1}\text{diam}(|u|) \tilde{s}_V \leq m(|u|) \leq C\text{diam}(|u|) \tilde{s}_V.$$

Then by the definition of Hausdorff measure it is ready to show that $\mathcal{H}_{\tilde{s}_V}|_{\Omega} \asymp m$. Consequently $0 < \mathcal{H}_{\tilde{s}_V}(\Omega) < \infty$ and $\dim_H \Omega = \tilde{s}_V$. 

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On the other hand (4.6) obviously implies that \( d_m(x) = \tilde{s}_V \) for all \( \omega \in \Omega \). Thus \( m \) and \( H^n_{\tilde{s}_V|\Omega} \) are exact dimensional. Finally by (2.7) we get
\[
\tilde{s}_V = \dim_H m = \frac{h_m(\sigma)}{-\Psi^*(m)}.
\]
\[\Box\]

**Remark 3.** The equality \( \tilde{s}_V = \dim_H \Omega \) is known as Bowen’s formula and (4.5) is kind of Young’s formula. Both are very famous in the classical theories. Since \( \dim_H m \) reach the dimension of the whole space \( \Omega \), so we can say that \( m \) is the measure of maximal dimension.

By using Proposition 1 we have the following consequence.

**Theorem 9.** \( H^n_{\tilde{s}_V|\Sigma_{\vec{w}}} \) is a Gibbs type measure, consequently \( H^n_{\tilde{s}_V|\Sigma_{\vec{w}}} \) is exact dimensional and \( 0 < H^n_{\tilde{s}_V|\Omega} \) is finite. Thus \( \dim_H \Sigma_{\vec{w}} = \tilde{s}_V \) and
\[
\tilde{s}_V = \frac{h_m(\sigma)}{-\Psi^*(m)}.
\]

*Proof.* By Proposition 1, \( \pi_{\vec{w}} \) is a bi-Lipschitz homeomorphism. By general principle on Hausdorff measure (see for example [16]), Theorem 8 implies Theorem 9. \[\Box\]

5. The density of states measure of the operator

In this section we study \( \mathcal{N}_V \). We will show that \( \mathcal{N}_V \) is a Markov measure and in some sense the measure of maximal entropy.

5.1. \( \mathcal{N}_V \) is a Markov measure.

Let \( H_n \) be the restriction of \( H_{\alpha_n, V, 0} \) to the box \([1, q_n]\) with periodic boundary condition. Let \( \mathcal{X}_n = \{x_{n,1}, \cdots, x_{n,q_n}\} \) be the eigenvalues of \( H_n \). Recall that \( \sigma_n \) is defined in (2.2).

**Lemma 8.** Each band in \( \sigma_n \) contains exactly one value in \( \mathcal{X}_n \).

*Proof.* This comes from the Bloch theory. Write \( u^n = (v_1 \cdots v_{q_n})^Z \) and define \( H^{(n)} = H_{u^n} \), then \( \sigma_n = \sigma(H_{u^n}) \) is made of \( q_n \) disjoint bands. There is another way to represent the spectrum by using the Bloch wave. Consider the solution of \( H^{(n)} \psi(\theta) = x \psi(\theta) \) of the Bloch type, i.e. \( \psi(\theta, u) = e^{im\theta}u(m) \) with \( u(m) = u(m+q_n) \) and some \( \theta \in [0, 2\pi] \). When \( \theta \in [0, 2\pi] \) is fixed, \( H^{(n)} \psi(\theta) = x \psi(\theta) \) has exact \( q_n \) solutions. We denote the set of eigenvalues as \( E_\theta \).

Then each band in \( \sigma_n \) contains exactly one point of \( E_\theta \), moreover
\[
\sigma_n = \bigcup_{\theta \in [0, 2\pi]} E_\theta.
\]
Now it is direct to check that \( \mathcal{X}_n = E_0 \) is the set of endpoint of each band which is related to the phase \( \theta = 0 \). \[\Box\]
Define
\[ \nu_n = \frac{1}{q_n} \sum_{i=1}^{q_n} \delta_{x_{n,i}}. \]
It is well known that \( \nu_n \to \mathcal{N} \) weakly (see for example [7]).

**Lemma 9.** There exist constants \( C_I, C_{II}, C_{III} > 0 \) such that for any \( B_w \in B_n \) we have
\[ \mathcal{N}_V(B_w) = C_{t_w} \alpha_n^{-n}. \]

**Proof.** By Lemma 1, \( \hat{A}_n \) has eigenvalues \( \alpha_n, -1 \) and \( -1/\alpha_n \), then there exists invertible matrix \( P \) such that \( \hat{A}_n = P \cdot \text{diag}(\alpha_n, -1, -1/\alpha_n) \cdot P^{-1} \). For any \( l > n \), by Lemma 2 and Lemma 8
\[ \nu_l(B_w) = \sum_{|u|=l+1, w \prec u} v_l(B_u) \]
\[ = \#\{u : |u| = l + 1, w \prec u, t_u = II, III\} \]
\[ = \frac{\nu^n w \cdot \hat{A}_n^{l-n} \cdot v_s}{c_n \alpha_n^l + (1 - c_n)(-\alpha_n)^{l-t}} \quad \text{(by (2.1) and (2.3))} \]
\[ = \frac{\nu^n w \cdot P \cdot \text{diag}(\alpha_n^{l-n}, -1^{l-n}, -\alpha_n^{n-l}) \cdot P^{-1} \cdot v_s}{c_n \alpha_n^l + (1 - c_n)(-\alpha_n)^{l-t}} \]
\[ = \frac{C_{t_w,1} \alpha_n^{l-n} + C_{t_w,2} (-1)^{l-n} + C_{t_w,3} (-\alpha_n)^{n-l}}{c_n \alpha_n^l + (1 - c_n)(-\alpha_n)^{l-t}}. \]
Since \( B_w \cap \Sigma_V \) is open and closed in \( \Sigma_V \) and \( \nu_t \to \mathcal{N}_V \) weakly, by taking a limit we get
\[ \mathcal{N}_V(B_w) = \frac{C_{t_w,1}}{c_n} \alpha_n^{-n} =: C_{t_w} \alpha_n^{-n}. \]

Define a matrix \( Q = (q_{e_i e_j}) \) of order \( 2\kappa + 2 \) as
\[ (5.1) \]
\[ q_{e_i e_j} = \begin{cases} \frac{C_{t_w}}{C_{t_w,1}} \alpha_n^{-1} & e_i \to e_j \\ 0 & \text{otherwise} \end{cases} \]

**Proposition 2.** \( Q \) is a stochastic matrix. \( \mathcal{N}_V \) is a Markov measure on \( \Sigma_V \) with transition matrix \( Q \) and initial distribution
\[ \mathcal{N}_V(X_{e_1}) = C_I \quad \text{and} \quad \mathcal{N}_V(X_{e_{\kappa+3}}) = C_{III}. \]

**Proof.** At first we show that \( Q \) is a stochastic matrix. Fix some \( e_i \in \mathcal{A} \), choose \( w \in \tilde{\Omega}_* \) such that the last letter of \( w \) is \( e_i \). By the structure of the spectrum, we have
\[ X_w = \Sigma_V \cap B_w = \bigcup_{j : e_i \to e_j} \Sigma_V \cap B_{we_j}. \]
By Lemma 9 we get
\[ C_{t_{e_i}} \alpha_n^{-|w|+1} = \mathcal{N}_V(X_w) = \sum_{j : e_i \to e_j} \mathcal{N}_V(X_{we_j}) = \sum_{j : e_i \to e_j} C_{t_{e_j}} \alpha_n^{-|w|}. \]
This obviously implies that
\[ \sum_{j=1}^{2\kappa+2} q_{e_i e_j} = \sum_{j:e_i \to e_j} q_{e_i e_j} = \sum_{j:e_i \to e_j} \frac{C_{e_i}}{C_{e_i}} \cdot \alpha^{-1} = 1. \]

Thus \( Q \) is a stochastic matrix.

Now for any \( w \in \tilde{\Omega}_n \), by repeatedly using Lemma 9 we have
\[ \mathcal{N}_V(X_w) = \mathcal{N}_V(X_{u_0}) \cdot \frac{\mathcal{N}_V(X_{u_0u_1})}{\mathcal{N}_V(X_{u_0})} \cdots \frac{\mathcal{N}_V(X_{u_0\ldots u_{n-1}u_n})}{\mathcal{N}_V(X_{u_0\ldots u_{n-1}})} = \mathcal{N}_V(X_{u_0}) q_{u_0u_1} q_{u_1u_2} \cdots q_{u_{n-1}u_n}. \]

Then the result follows. \( \square \)

5.2. \( \mathcal{N}_V \) and the measure of maximal entropy.

To study the dimension property of \( \mathcal{N}_V \), we need to go back to the TDS \((\Omega, \sigma)\) again. We will establish the relation between \( \mathcal{N}_V \) and the measure of maximal entropy and obtain the dimension formula of \( \mathcal{N}_V \).

Let Stochastic matrix \( Q \) be defined according to (5.1). Let \( \mu_Q \) be the unique invariant Markov measure on the subshift \((\Omega, \sigma)\) with transition matrix \( Q \). It is well known that \( \mu_Q \) is a Gibbs measure on \( \Omega \) with additive potential \( \phi : \Omega \to \mathbb{R} \) defined by
\[ \phi(\omega) := \ln q_{\omega 01}. \]

Note that \( \phi \leq 0 \). Write \( \Phi = (\phi_n)_{n=0}^{\infty} \) with \( \phi_n = S_n \phi \), then \( \Phi \) is a family of additive potentials. By the definition of Gibbs measure it is not hard to compute that \( P(\Phi) = 0 \).

**Theorem 10.** \( \mu_Q \) is the measure of maximal entropy and the Parry measure of the subshift \((\Omega, \sigma)\). Moreover \( \mu_Q \) is exact dimensional with
\[ \dim_H \mu_Q = \frac{h_{\mu_Q}(\sigma)}{-\Psi_s(\mu_Q)} = \frac{-\ln \alpha}{-\Psi_s(\mu_Q)}. \]

**Proof.** At first since \((\Omega, \sigma)\) is a topologically mixing subshift and the incidence matrix \( A_\kappa \) has Perron-Frobenius eigenvalue \( \alpha_\kappa \), we have \( h_{\text{top}}(\sigma) = \ln \alpha_\kappa \) (see for example [37]).

Assume \( p = (p_{e_1}, \ldots, p_{e_{2\kappa+2}}) \) is the unique probability vector such that \( pQ = p \). Then for any \( u \in \Omega_n \), we have \( \mu_Q([u]) = p_{u_0} q_{u_0u_1} \cdots q_{u_{n-1}u_n} \). By Proposition 2 we have
\[ \mu_Q([u]) = p_{u_0} q_{u_0u_1} \cdots q_{u_{n-1}u_n} \frac{p_{u_0}}{\mathcal{N}_V(X_{u_0})} \mathcal{N}_V(X_{u_0\ldots u_{n-1}u_n}). \]

Then by Lemma 9 for any \( \omega \in \Omega \),
\[ \lim_{n \to \infty} -\frac{1}{n} \ln \mu_Q([\omega | u]) = \lim_{n \to \infty} -\frac{1}{n} \mathcal{N}_V(X_{u_0\ldots u_{[\omega | u]}[\ldots; u_n]}). \]

By Shannon-McMillan-Breiman Theorem, we conclude that \( h_{\mu_Q}(\sigma) = h_{\text{top}}(\sigma) \). Since the measure of maximal entropy is unique, which is the so called Parry measure (see for example [37] chapter 8), we conclude that \( \mu_Q \) is the measure of maximal entropy and the Parry measure of the system \((\Omega, \sigma)\).
Since $\mu_Q$ is a Gibbs measure, by Theorem $\Box$ $\mu_Q$ is exact dimensional and the Hausdorff dimension of $\mu_Q$ is given by (5.2). \hfill \Box

**Remark 4.** The proof indeed gives that for any $u \in \Omega_n$

(5.3) \[ \mu_Q([u]) \sim \alpha^{-n}. \]

We fix again a $\bar{w} \in \mathcal{D}$ and let $\pi_{\bar{w}}$ be defined as in (4.2).

**Theorem 11.** $\mathcal{N}_V|_{\Sigma_{\bar{w}}}$ is a Markov measure and Gibbs type measure. It is exact dimensional and

(5.4) \[ \dim_H \mathcal{N}_V|_{\Sigma_{\bar{w}}} = \frac{\ln \alpha_k}{\Psi_*(\mu_Q)}. \]

**Proof.** We already show that $\mathcal{N}_V$ is a Markov measure, so is $\mathcal{N}_V|_{\Sigma_{\bar{w}}}$. Write $\nu_Q := (\pi_{\bar{w}})_*(\mu_Q)$. By Proposition $\Box$ we have

\[ \nu_Q(X_{w^{u_0u[1,\ldots,n]}}) = \mu_Q([u]) = p_{u_0}q_{u_0u_1}\cdots q_{u_{n-1}u_n} = \frac{p_{u_0}}{\mathcal{N}_V(X_{w^{u_0u[1,\ldots,n]}})}\mathcal{N}_V(X_{w^{u_0u[1,\ldots,n]}}, \mathcal{N}_V|_{\Sigma_{\bar{w}}}). \]

Consequently $\nu_Q \sim \mathcal{N}_V|_{\Sigma_{\bar{w}}}$. Since $\mu_Q$ is a Gibbs measure, we conclude that $\mathcal{N}_V|_{\Sigma_{\bar{w}}}$ is a Gibbs type measure. Since $\pi_{\bar{w}} : \Omega \to \Sigma_{\bar{w}}$ is bi-Lipschitz and $\nu_Q \sim \mathcal{N}_V|_{\Sigma_{\bar{w}}}$, we conclude that both measures $\nu_Q$ and $\mathcal{N}_V|_{\Sigma_{\bar{w}}}$ are exact dimensional and has the same Hausdorff dimension with $\mu_Q$. \hfill \Box

5.3. Stationary distribution.

For later use, we need to study the stationary distribution $p = (p_1, \cdots, p_{2\kappa+2})$ of $Q$, i.e. the unique probability vector $p$ such that $pQ = p$.

**Proposition 3.** We have

(5.5) \[ \mu_Q([e_{\kappa+2}]) = p_{\kappa+2} = \frac{\alpha_k}{\kappa\alpha_k + 2}. \]

**Proof.** Denote the $i$-th row of $Q$ by $q^i$. By a direct computation we have

(5.6) \[ q^i = \begin{cases} 
\left(0,\cdots,0,1,0,\cdots,0\right) & i = 1,\cdots,\kappa + 1 \\
\left(\frac{\alpha_k^{\kappa+1}}{\alpha_k^{\kappa}},\cdots,\frac{1}{\alpha_k^2},\frac{\alpha_k-1}{\alpha_k^2},\cdots,\frac{\alpha_k-1}{\alpha_k^2}\right) & i = \kappa + 2 \\
\left(\frac{\alpha_k(\alpha_k-1)}{\alpha_k^{\kappa-1}},\cdots,\frac{1}{\alpha_k},\frac{\alpha_k-1}{\alpha_k},\cdots,\frac{1}{\alpha_k},0,0,\frac{1}{\alpha_k},\cdots,\frac{1}{\alpha_k}\right) & i = \kappa + 3,\cdots,2\kappa+2.
\end{cases} \]

Write $\delta := \alpha^{-2}_k$ and $\beta := (\alpha_k(\alpha_k-1))^{-1}$. Since $p = pQ$, we have

\[ \begin{cases} 
\delta p_{\kappa+2} + \beta (p_{\kappa+3} + \cdots + p_{2\kappa+2}) = p_i & (i = 1,\cdots,\kappa) \\
\delta p_{\kappa+2} = p_{\kappa+1} \\
p_1 + \cdots + p_{\kappa+1} = p_{\kappa+2}.
\end{cases} \]

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Then we have
\[ p_{k+2} = p_1 + \cdots + p_{k+1} = (k + 1)\delta p_{k+2} + \kappa \beta (p_{k+3} + \cdots + p_{2k+2}). \]
Notice that \( p \) is a probability vector, thus we have
\[ 1 = (p_1 + \cdots + p_{k+1}) + p_{k+2} + (p_{k+3} + \cdots + p_{2k+2}) = (2 + \frac{(\alpha_\kappa - 1)^2}{\alpha_\kappa})p_{k+2}. \]
Then we get \( p_{k+2} = \alpha_\kappa/(\kappa \alpha_\kappa + 2). \)

6. Multifractal analysis and optimal Hölder exponent of \( N_V \)

In this section we will study the optimal Hölder exponent of \( N_V \) restricted to the dynamical subset \( \Sigma_{\bar{w}} \). We will see that the exponent can be obtained from the multifractal analysis of \( N_V|\Sigma_{\bar{w}} \).

At first we will do the multifractal analysis of \( \mu_Q \), then by the bi-Lipschitz homeomorphism \( \pi_{\bar{w}} \), the result is converted to that of \( N_V|\Sigma_{\bar{w}} \).

6.1. Multifractal analysis of \( \mu_Q \).

We begin with two useful lemmas:

**Lemma 10.** For any \( \omega \in \Omega \), we have
\[
\underline{d}_{\mu_Q}(\omega) = \liminf_{n \to \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)} \quad \text{and} \quad \overline{d}_{\mu_Q}(\omega) = \limsup_{n \to \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)}.
\]

**Proof.** Fix \( \omega \in \Omega \) and \( r > 0 \) very small. Assume \( n \) is the unique number such that
\[
\exp(\psi_{n0}(\omega)) = |B_{w^{-0}\omega[1, \ldots, n0]}| < r \leq |B_{w^{-0}\omega[1, \ldots, (n-1)n0]}| = \exp(\psi_{(n-1)n0}(\omega)),
\]
where \( n0 \) is given in Lemma 4. Thus \( [\omega|n0] \subset B(\omega, r) \subset [\omega|(n-1)n0] \). Consequently
\[
\mu_Q([\omega|n0]) \leq \mu_Q(B(\omega, r)) \leq \mu_Q([\omega|(n-1)n0]).
\]
Notice that for any \( u \in \Omega_n \) and \( \omega' \in [u] \), we have
\[
\mu_Q([u]) = p_{u0}q_{u0u1} \cdots q_{un-1un} = p_{u0} \exp(S_n\phi(\omega')) = p_{u0} \exp(\phi_n(\omega')).
\]
Combine (6.2) and (6.3) we get
\[
\frac{\phi_{(n-1)n0}(\omega) + \ln p_{u0}}{\psi_{n0}(\omega)} + \frac{\ln \mu_Q(B(x, r))}{\ln r} \leq \frac{\phi_{nn0}(\omega) + \ln p_{u0}}{\psi_{(n-1)n0}(\omega)}.
\]
By (2.4), \( c_3 \left< N^{-n_00} \leq r \leq 2^{-N-n_00} \right. \), thus \( n \to \infty \) when \( r \to 0 \). By (3.3), \( \phi_n(\omega) \leq -n \ln \alpha_\kappa + c \). By (4.4), \( \psi_n(\omega) \leq -n \ln 2 \). Now by using the definition of almost additive potential and taking the upper and lower limits we get the result. \( \square \)

**Lemma 11.** There exists \( d_1 = d_1(\kappa) < d_2 = d_2(\kappa) < 0 \) such that for any \( \mu \in \mathcal{M}(\Omega, \sigma) \), we have
\[
d_1 \leq \Phi_*(\mu) \leq d_2.
\]
Proof. Since $\Phi$ is additive and $\mu$ is invariant, we have

$$
\Phi_*(\mu) = \int_\Omega \phi d\mu = \int_\Omega S_3 \phi d\mu.
$$

By the definition we get

$$
S_3 \phi(\omega) = \ln q_{\omega_0 \omega_1} q_{\omega_1 \omega_2} q_{\omega_2 \omega_3}.
$$

We discuss two cases. At first we assume $\kappa = 1$. By (\ref{5.6}) we know that

$$
q_{e_i e_j} \begin{cases} 
1 & (i, j) = (1, 3); (2, 3); (4, 1) \\
0 & (i, j) = (3, 1); (3, 2); (3, 4) \\
= 0 & \text{otherwise}
\end{cases}
$$

It is seen that if $e_{i_0} e_{i_1} e_{i_2}$ is admissible, then there exists at least one $j \in \{0, 1, 2\}$ such that $i_j = 3$. Write $q_{\min} = \min\{q_{e_3 e_1}, q_{e_3 e_2}, q_{e_3 e_4}\}$ and $q_{\max} = \max\{q_{e_3 e_1}, q_{e_3 e_2}, q_{e_3 e_4}\}$ then $0 < q_{\min} \leq q_{\max} < 1$. Thus

$$
3 \ln q_{\min} \leq S_3 \phi(\omega) \leq \ln q_{\max}.
$$

Next we assume $\kappa \geq 2$. By (\ref{5.6}), $q_{e_i e_j} = 1$ if and only if $i \leq \kappa + 1$ and $j = \kappa + 2$. Write $q_{\min} := \min\{q_{e_i e_j} : e_i \rightarrow e_j; q_{e_i e_j} \neq 1\}$ and $q_{\max} := \max\{q_{e_i e_j} : e_i \rightarrow e_j; q_{e_i e_j} \neq 1\}$. Then $0 < q_{\min} \leq q_{\max} < 1$. From the structure of $A_\kappa$, it is ready to see that if $e_{i_0} e_{i_1} e_{i_2}$ is admissible, then $i_1 \neq \kappa + 2$ or $i_2 \neq \kappa + 2$. Thus we still have

$$
3 \ln q_{\min} \leq S_3 \phi(\omega) \leq \ln q_{\max}.
$$

Take $d_1 = \ln q_{\min}$ and $d_2 = \ln q_{\max}/3$ we get the result. \hfill \Box

Consider the function $Q(q, t) := P(q \Phi + t \Psi)$. By Corollary 1 since $\Psi \in C_{aa}^-(\Omega, \sigma)$, for each $q \in \mathbb{R}$ fixed, there exists a unique number $\tau(q)$ such that $Q(q, \tau(q)) = 0$. Define

$$
\mathcal{B} := \{ \frac{\Phi_*(\mu)}{\Psi_*(\mu)} : \mu \in \mathcal{M}(\Omega, \sigma) \}.
$$

Then $\mathcal{B} = [\beta_*, \beta^*]$ is an interval.

**Theorem 12.** (i) Define $\Lambda_{\beta} := \{ \omega \in \Omega : d_{\mu_Q}(\omega) = \beta \}$. Then $\Lambda_{\beta} \neq \emptyset$ if and only if $\beta \in \mathcal{B}$. For any $\beta \in \mathcal{B}$

$$
\dim_H \Lambda_{\beta} = \tau^*(\beta) := \inf_{q \in \mathbb{R}} (\tau(q) + \beta q).
$$

(ii) There exist two constants $0 < C_1(\kappa, V) \leq C_2(\kappa, V)$ such that $C_1 \leq \beta_* \leq \beta^* \leq C_2$. Moreover

$$
\beta_* = \inf_{\omega \in \Omega} d_{\mu_Q}(\omega) \quad \text{and} \quad \beta^* = \sup_{\omega \in \Omega} d_{\mu_Q}(\omega).
$$

Thus $\beta_*$ is the optimal Hölder exponent of $\mu_Q$. 

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Proof. (i) Recall the definition in Section 2.4.2. By (6.1) we have $\Lambda_\beta = \Lambda_\Phi/\Psi(\beta)$ and $\tau(q) = \mathcal{L}_\Phi/\Psi(q, \beta) - q\beta$. Thus by Theorem 6 we have $\Lambda_\beta \neq \emptyset$ if and only if $\beta \in \mathcal{B}$ and if $\beta \in \mathcal{B}$ we have

$$\dim H \Lambda_\beta = \mathcal{L}_\Phi^*(\beta) = \inf_{q \in \mathbb{R}} (\tau(q) + \beta q).$$

(ii) By (4.4), for any invariant measure $\mu$ we have

$$\ln c_3 \leq \Psi^*(\mu) \leq -\ln 2 < 0.$$

By (6.4), for any invariant measure $\mu$ we have

$$d_1 \leq \Phi^*(\mu) \leq d_2 < 0.$$

From this we conclude that

$$0 < C_1 := \frac{d_2}{\ln c_3} \leq \beta_* \leq \frac{\Phi^*(\mu)}{\Psi^*(\mu)} \leq \beta^* \leq \frac{d_1}{-\ln 2} =: C_2.$$

Now fix any $\omega \in \Omega$. By Lemma 10 we have

$$d_{\mu_Q}(\omega) = \liminf_{n \to \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)} = \lim_{k \to \infty} \frac{\phi_{n_k}(\omega)}{\psi_{n_k}(\omega)}.$$

By choosing a further subsequence we can further assume that $(\sum_{j=0}^{n_k-1} \delta_{\sigma^j\omega})/n_k \to \mu$. Then $\mu$ is invariant and by [18] Lemma A.4(ii), we have

$$\lim_{k \to \infty} \frac{\phi_{n_k}(\omega)}{n_k} = \Phi^*(\mu) \quad \text{and} \quad \lim_{k \to \infty} \frac{\psi_{n_k}(\omega)}{n_k} = \Psi^*(\mu).$$

Thus $d_{\mu_Q}(\omega) \in \mathcal{B}$. Similarly we can show that $\overline{d}_{\mu_Q}(\omega) \in \mathcal{B}$. On the other hand, since $\Lambda_{\beta_*}$ and $\Lambda_{\beta^*}$ are all nonempty, there exist $\omega_*$ and $\omega^*$ such that $d_{\mu_Q}(\omega_*) = \beta_*$ and $d_{\mu_Q}(\omega^*) = \beta^*$. Thus (6.5) holds.

In the following we will give another expression for the optimal Hölder exponent of $\mu_Q$, which is more convenient when we study the asymptotic property.

Write $\psi_{n,\min} := \inf\{\psi_n(\omega) : \omega \in \Omega\}$. By the almost additivity of $\Psi$ we have

$$\psi_{n+k,\min} \geq \psi_{n,\min} + \psi_{k,\min} - C(\Psi),$$

Thus $\{C(\Psi) - \psi_{n,\min} : n \geq 0\}$ form a sub-additive sequence. Then it is well known that the following limit exists

$$\lim_{n \to \infty} \frac{\psi_{n,\min}}{n} = \sup_{n \geq 0} \frac{\psi_{n,\min} - C(\Psi)}{n} =: \Psi_{\min}.$$

Notice that by (4.4), we have $\Psi_{\min} \leq -\ln 2 < 0$.

**Proposition 4.** We have

$$\gamma_{\mu_Q} = \frac{\ln \alpha_k}{-\Psi_{\min}}.$$
Proof. At first we show $\gamma_{\mu Q} \geq -\ln \alpha_\kappa/\Psi_{\text{min}}$. It is sufficient to show that $d_{\mu Q}(\omega) \geq -\ln \alpha_\kappa/\Psi_{\text{min}}$ for any $\omega \in \Omega$. By the definition of $\phi_n$ and $\mu Q$ we have

$$\phi_n(\omega) = \ln q_{\omega_0\omega_1} \cdots q_{\omega_n-1\omega_n} = \ln \frac{\mu Q(\omega|n)}{p_{\omega_0}q_{\omega_n\omega_{n+1}}}.$$  

By (5.3) we have

$$\lim_{n \to \infty} \frac{\phi_n(\omega)}{n} = \lim_{n \to \infty} \frac{\ln \mu Q(\omega|n)}{n} = -\ln \alpha_\kappa.$$

On the other hand we have

$$\lim_{n \to \infty} \psi_n(\omega) = \lim_{n \to \infty} \psi_{\text{n min}} = \Psi_{\text{min}}.$$

Combining (6.6), (6.7) and (6.11) we get

$$d_{\mu Q}(\omega) = \liminf_{n \to \infty} \frac{\phi_n(\omega)}{\psi_n(\omega)} \geq \frac{\ln \alpha_\kappa}{-\Psi_{\text{min}}}.$$

Next we show $\gamma_{\mu Q} \leq -\ln \alpha_\kappa/\Psi_{\text{min}}$. It is sufficient to show that for any $\epsilon > 0$ small $d_{\mu Q}(\omega^\epsilon) < \ln \alpha_\kappa/(-\Psi_{\text{min}} - \epsilon)$ for some $\omega^\epsilon \in \Omega$. By (6.6), it is sufficient to show that

$$\liminf_{n \to \infty} \psi_n(\omega^\epsilon)/n \leq \Psi_{\text{min}} + \epsilon$$

for some $\omega^\epsilon \in \Omega$. We find such a $\omega^\epsilon$ as follows. Recall that the incidence matrix $A = A_\kappa$ is primitive, thus there exists $N = N_\kappa$ such that $A^{N-2}$ are positive. At first take $n_0$ big enough such that

$$\|\psi_N\|_\infty/n_0 \leq \frac{\epsilon}{8}, \quad C(\Psi)/n_0 \leq \frac{\epsilon}{8}, \quad -4N\Psi_{\text{min}} \leq n_0\epsilon \quad \text{and} \quad \psi_{n_0,\text{min}} \leq n_0(\Psi_{\text{min}} + \epsilon/4).$$

Let $\tilde{\omega} \in \Omega$ such that $\psi_{n_0}(\tilde{\omega}) = \psi_{n_0,\text{min}}$. Since $A^{N-2}$ is positive, we can find $w \in \Omega_{N-2}$ such that both $u := \tilde{\omega}|_{n_0} w$ and $w\tilde{\omega}|_{n_0}$ are admissible. Thus $\omega^\epsilon := u^\infty \in \Omega$. Notice that $|u| = |\tilde{\omega}|_{n_0} + |w| = n_0 + 1 + N - 1 = n_0 + N$. Let $n_1 = n_0 + N$. By the definition of $\Psi$ we have $\psi_{n_0}(\tilde{\omega}) = \psi_{n_0}(\omega^\epsilon)$. By almost additivity and (6.8) we have

$$\psi_{n_1}(\omega^\epsilon) \leq \psi_{n_0}(\omega^\epsilon) + \psi_N(\sigma^{n_0}\omega^\epsilon) + C(\Psi) \leq \psi_{n_0}(\tilde{\omega}) + \frac{n_0\epsilon}{8} + \frac{n_0\epsilon}{4} \leq n_1(\Psi_{\text{min}} + 3\epsilon/4).$$

Notice that by the definition of $\omega^\epsilon$ we have $\sigma^{j\epsilon}\omega^\epsilon = \omega^\epsilon$ for any $j \geq 0$. Again by almost additivity we get

$$\psi_{kn_1}(\omega^\epsilon) \leq \sum_{j=0}^{k-1} \psi_{n_1}(\sigma^{j\epsilon}\omega^\epsilon) + (k - 1)C(\Psi) \leq k\psi_{n_1}(\omega^\epsilon) + kC(\Psi).$$

Combining with (6.9) and (6.8) we conclude that

$$\psi_{kn_1}(\omega^\epsilon) \leq kn_1(\Psi_{\text{min}} + \epsilon).$$

Consequently we have

$$\liminf_{n \to \infty} \frac{\psi_n(\omega^\epsilon)}{n} \leq \liminf_{k \to \infty} \frac{\psi_{kn_1}(\omega^\epsilon)}{kn_1} \leq \Psi_{\text{min}} + \epsilon.$$

□
6.2. Multifractal analysis of $\mathcal{N}_V|_{\Sigma_{\bar{\omega}}}$.

Through the bi-Lipschitz homeomorphism $\pi_{\bar{\omega}}$ we have the following

**Theorem 13.** Let $\mathcal{B} = [\beta_*, \beta^*]$ and $\tau$ be defined as above. Define $\Lambda_\beta := \{x \in \Sigma_{\bar{\omega}} : d_X(x) = \beta\}$, then $\Lambda_\beta \neq \emptyset$ if and only if $\beta \in \mathcal{B}$. For any $\beta \in \mathcal{B}$

$$\dim_H \Lambda_\beta = \tau^*(\beta) := \inf_{q \in \mathbb{R}}(\tau(q) + \beta q).$$

Moreover $\beta_*$ is the optimal Hölder exponent of $\mathcal{N}_V|_{\Sigma_{\bar{\omega}}}$ and

$$(6.10) \beta_* = \gamma_{\mathcal{N}_V|_{\Sigma_{\bar{\omega}}}} = \frac{\ln \alpha_\kappa}{-\Psi_{\min}}.$$ 

**Proof.** Given $\omega \in \Omega$, write $x = \pi_{\bar{\omega}}(\omega)$. Since $\pi_{\bar{\omega}}$ is a bi-Lipschitz homeomorphism, we have $d_{\mu_Q}(\omega) = d_{\mathcal{N}_V}(x)$ and $\bar{d}_{\mu_Q}(\omega) = \bar{d}_{\mathcal{N}_V}(x)$. Then Theorem 13 follows from Theorem 12 and Proposition 11. \hfill \Box

7. Global picture

In this section we will obtain the global picture and prove Theorem 1 (i) and (ii) by comparing any two different dynamical subsets $\Sigma_{\bar{v}}$ and $\Sigma_{\bar{w}}$.

Recall the definition of $\mathcal{D}$ in (1.1). For some $\vec{w} \in \mathcal{D}$ fixed, we defined the potential $\Psi = \Psi_{\bar{\omega}}$. Let $\bar{s}_{V,\bar{w}}$ be the root of $P(s\Psi_{\bar{w}}) = 0$. Let $d_{\bar{w}} = d_{\Psi_{\bar{w}}}$ be the weak Gibbs metric on $\Omega$. Let $m_{\bar{w}}$ be the Gibbs measure with potential $\bar{s}_{V,\bar{w}}\Psi_{\bar{w}}$. Let $\tilde{d}_{V,\bar{w}} = \dim_H \mu_Q$ be the Hausdorff dimension of $\mu_Q$ on the metric space $(\Omega, d_{\bar{w}})$. Let $\tilde{\gamma}_{V,\bar{w}}$ be the optimal Hölder exponent of $\mu_Q$ on the metric space $(\Omega, d_{\bar{w}})$.

Now we fix any two $\vec{v}, \vec{w} \in \mathcal{D}$ and compare all the quantities mentioned above.

**Theorem 14.** Fix any two $\vec{v}, \vec{w} \in \mathcal{D}$. Then $d_{\vec{v}}$ is equivalent to $d_{\bar{w}}$, $m_{\vec{v}} = m_{\bar{w}}$ and

$$\tilde{s}_{V,\vec{v}} = \tilde{s}_{V,\bar{w}}, \quad \tilde{d}_{V,\vec{v}} = \tilde{d}_{V,\bar{w}} \quad \text{and} \quad \tilde{\gamma}_{V,\vec{v}} = \tilde{\gamma}_{V,\bar{w}}.$$ 

**Proof.** At first we show that $d_{\vec{v}}$ is equivalent to $d_{\bar{w}}$. Given $\omega, \bar{\omega} \in \Omega$ and $\omega \neq \bar{\omega}$. If $\omega_0 \neq \bar{\omega}_0$, then

$$(7.1) d_{\vec{v}}(\omega, \bar{\omega}) = \text{diam}(\Sigma_V) = d_{\bar{w}}(\omega, \bar{\omega}).$$

Now assume $\omega_0 = \bar{\omega}_0 = e$. Then

$$d_{\vec{v}}(\omega, \bar{\omega}) = |B_{\vec{v}^* \omega \wedge \bar{\omega}}| \quad \text{and} \quad d_{\bar{w}}(\omega, \bar{\omega}) = |B_{\bar{w}^* \omega \wedge \bar{\omega}}|.$$ 

Notice that $\#\mathcal{D}$ is bounded by a constant only depending on $\kappa$ and $N$. By Theorem 3 we have

$$\frac{|B_{\vec{v}^* \omega \wedge \bar{\omega}}|}{|B_{\bar{w}^* \omega \wedge \bar{\omega}}|} \sim^{(\kappa, N)} \frac{|B_{\vec{v}^*}|}{|B_{\bar{w}^*}|} \sim^{(\kappa, V, N)} 1.$$ 

Together with (7.1) we conclude that $d_{\vec{v}}(\omega, \bar{\omega})/d_{\bar{w}}(\omega, \bar{\omega}) \sim 1$. That is, $d_{\vec{v}}$ and $d_{\bar{w}}$ are equivalent. As a consequence the Hausdorff dimensions and the optimal Hölder exponents of $\mu_Q$ on $(\Omega, d_{\vec{v}})$ and $(\Omega, d_{\bar{w}})$ are equal. That is, $\tilde{d}_{V,\vec{v}} = \tilde{d}_{V,\bar{w}}$ and $\tilde{\gamma}_{V,\vec{v}} = \tilde{\gamma}_{V,\bar{w}}$. 

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Now we show that $\tilde{s}_{V,\vec{v}} = \tilde{s}_{V,\vec{w}}$ and $m_{\vec{v}} = m_{\vec{w}}$. Fix $\omega \in \Omega$, By (4.3) we have

$$\psi^{(#)}_n(\omega) = \ln |B_{\omega_{[1,\cdots,n]}}| \quad \text{and} \quad \psi^{(\vec{w})}_n(\omega) = \ln |B_{\omega_{[1,\cdots,n]}}|.$$

Still by Theorem \ref{thm:4.3} we have

$$\frac{|B_{\omega_{[1,\cdots,n]}}|}{|B_{\omega_{[1,\cdots,n]}}|} \sim (\kappa, V) \frac{|B_{\omega_{0}}|}{|B_{\omega_{0}}|} \sim (\kappa, V, N) 1.$$ 

From this we conclude that

$$|\psi^{(\vec{v})}_n(\omega) - \psi^{(\vec{w})}_n(\omega)| \lesssim (\kappa, V, N) 1.$$

Now by Theorem \ref{thm:5} (ii), we conclude that $P(s\Psi_{\vec{v}}) = P(s\Psi_{\vec{w}})$ for any $s \in \mathbb{R}$, consequently they have the same zeros. That is, $\tilde{s}_{V,\vec{v}} = \tilde{s}_{V,\vec{w}} =: \tilde{s}_V$. Since we also have

$$|\tilde{s}_V \psi^{(#)}_n(\omega) - \tilde{s}_V \psi^{(\vec{v})}_n(\omega)| \lesssim (\kappa, V, N) 1.$$

Still by Theorem \ref{thm:5} (ii), we have $m_{\vec{v}} = m_{\vec{w}}$. $\square$

Fix any $\vec{w} \in D$ define

$$\tilde{s}_V := \tilde{s}_{V,\vec{w}}, \quad \tilde{d}_V := \tilde{d}_{V,\vec{w}} \quad \text{and} \quad \tilde{\gamma}_V := \tilde{\gamma}_{V,\vec{w}}.$$

**Proof of Theorem \ref{thm:1} (i) and (ii).** Recall that $\Sigma_V = \bigcup_{\vec{w} \in D} \Sigma_{\vec{w}}$.

(i) By Theorem \ref{thm:14} and Theorem \ref{thm:9} for any $\vec{v}, \vec{w} \in D$ we have

$$\dim_H \Sigma_{\vec{v}} = \tilde{s}_{V,\vec{v}} = \tilde{s}_V = \tilde{s}_{V,\vec{w}} = \dim_H \Sigma_{\vec{w}}.$$ 

Consequently $s_V = \dim_H \Sigma_V = \dim_H \bigcup_{\vec{w} \in D} \Sigma_{\vec{w}} = \tilde{s}_V$. Now (i) follows from Theorem \ref{thm:9}

(ii) By Theorem \ref{thm:14}, Theorem \ref{thm:10} and Theorem \ref{thm:11} for any $\vec{v}, \vec{w} \in D$ we have

$$\dim_H \mathcal{N}_V|_{\Sigma_{\vec{v}}} = \tilde{d}_{V,\vec{v}} = \tilde{d}_V = \tilde{d}_{V,\vec{w}} = \dim_H \mathcal{N}_V|_{\Sigma_{\vec{w}}}.$$ 

Consequently $d_V = \dim_H \mathcal{N}_V = \tilde{d}_V$. Now (ii) follows from Theorem \ref{thm:11} $\square$

**Remark 5.** Recall that we have defined $\gamma_V$ to be the optimal Hölder exponent of $\mathcal{N}_V$. Since $\Sigma_V = \bigcup_{\vec{w} \in D} \Sigma_{\vec{w}}$, we have

$$\gamma_V = \inf_{\vec{w}} \gamma_{\mathcal{N}_V|_{\Sigma_{\vec{w}}}}.$$ 

On the other hand by Theorem \ref{thm:14}, Theorem \ref{thm:12} (ii) and Theorem \ref{thm:13} we have

$$\gamma_{\mathcal{N}_V|_{\Sigma_{\vec{v}}}} = \tilde{\gamma}_{V,\vec{v}} = \tilde{\gamma}_V = \tilde{\gamma}_{V,\vec{w}} = \gamma_{\mathcal{N}_V|_{\Sigma_{\vec{w}}}}.$$

we conclude that $\gamma_V = \tilde{\gamma}_V$ and consequently $\gamma_V$ has an formula given by (6.10).

8. Asymptotic properties and the consequences

In this section we discuss the asymptotic properties of $\gamma_V(\kappa)$, $s_V(\kappa)$ and $d_V(\kappa)$ when $V \to \infty$. In particular, we finish the proof of Theorem \ref{thm:1} (iii) and (iv).
8.1. Asymptotic property of $\gamma_V(\kappa)$.

Recall that $e_{\kappa+2} = (II, 1)$. For any $w \in \Omega_n$, we write $|w|_{e_{\kappa+2}} := \#\{1 \leq i \leq n : w_i = e_{\kappa+2}\}$. At first we note that Lemma 5 implies the following useful fact: There exists a constant $c = c_\kappa > 1$ such that for any $w \in \Omega_n$

$$c^{-n}V^{-(\kappa-2)|w|_{e_{\kappa+2}}-n} \leq |B_w| \leq c^nV^{-(\kappa-2)|w|_{e_{\kappa+2}}-n}. \tag{8.1}$$

The proof is a direct computation by noticing that $c^{8.1}$. Asymptotic property of $\gamma_V(\kappa)$.

Proposition 5.

$$\lim_{V \to \infty} \gamma_V(1) \ln V = \frac{3}{2} \ln \alpha_1 =: \hat{\gamma}_1 \quad \text{and} \quad \lim_{V \to \infty} \gamma_V(\kappa) \ln V = \frac{2}{\kappa} \ln \alpha_\kappa =: \hat{\gamma}_\kappa \quad (\kappa \geq 2). \tag{8.2}$$

Proof. Fix some $\bar{w} \in D$ and define $\Psi$ according to (4.3). By Remark 5, $\gamma_V$ has the formula:

$$\gamma_V(\kappa) = \frac{\ln \alpha_\kappa}{-\Psi^{\min}}. \tag{8.2}$$

Thus we only need to estimate $\Psi^{\min}$. Recall that $\psi_{n}(\omega) = \ln |B_{w^{-\omega}[1,\ldots,n]}| = \ln |B_{w^{-\omega}[1,\ldots,n]}|$ and $\psi_{n,\min} = \min\{\psi_{n}(\omega) : \omega \in \Omega\}$. Thus $\exp(\psi_{n,\min})$ is just the minimal length of the bands $\{B_u : |u| = n + N + 1, w^j \prec u \text{ for some } j\}$.

At first we assume $\kappa = 1$. Assume $u = w^j v$ with $|v| = n$. Then by (8.1)

$$c^{-n}V^{\vert v\vert_{e_3}-n} \leq |B_u| \leq c^nV^{\vert v\vert_{e_3}-n} \tag{8.3}$$

Notice that $e_1e_3e_4e_1$ is admissible. Take $\tilde{u} = w^{\kappa} \tilde{v}$ such that $|\tilde{v}| = n$ and $\tilde{v} \prec (e_1e_3e_4)\infty$. Then $|\tilde{v}|_{e_3} \leq n/3 + 1$. Then (8.3) implies that there exists some constant $C \quad (\text{depending on } \kappa, N, V)$ such that

$$\psi_{n,\min} \leq \ln |B_{\tilde{u}}| \leq C + n \ln c - \frac{2n}{3} \ln V. \tag{8.3}$$

On the other hand for any $u = w^j v$ with $|v| = n$, by the definition of the incidence matrix $A_1$, it is ready to show that $|v|_{e_3} \geq n/3 - 1$. Then (8.3) implies that there exists some constant $C^\prime \quad (\text{depending on } \kappa, N, V)$ such that

$$\psi_{n,\min} \geq \min_u \ln |B_u| \geq C^\prime - n \ln c - \frac{2n}{3} \ln V. \tag{8.3}$$

Consequently

$$- \ln c - \frac{2n}{3} \ln V \leq \Psi_{\min} \leq - \frac{2n}{3} \ln V. \tag{8.3}$$

Now by (8.2) we conclude that $\hat{\gamma}_1 = \frac{3}{2} \ln \alpha_1$.

Next we assume $\kappa \geq 2$. Assume $u = w^j v$ with $|v| = n$. Then by (8.1)

$$c^{-n}V^{-(\kappa-2)|w|_{e_{\kappa+2}}-n} \leq |B_u| \leq c^nV^{-(\kappa-2)|w|_{e_{\kappa+2}}-n}. \tag{8.4}$$

Notice that $\kappa - 2 \geq 0$ and $e_1e_{\kappa+2}e_1$ is admissible. Take $\tilde{u} = w^{\kappa+2} \tilde{v}$ such that $|\tilde{v}| = n$ and $\tilde{v} \prec (e_1e_{\kappa+2})\infty$. Then $|\tilde{v}|_{e_{\kappa+2}} \geq n/2 - 1$. Then (8.4) implies that there exists some constant $\tilde{C} \quad (\text{depending on } \kappa, N, V)$ such that

$$\psi_{n,\min} \leq \ln |B_{\tilde{u}}| \leq \tilde{C} + n \ln c - \frac{\kappa n}{2} \ln V. \tag{8.4}$$

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On the other hand for any \( u = w^v \) with \( |v| = n \), since \( e_{\kappa+2}^v e_{\kappa+2} \) is not admissible, we have \( |v|_{e_{\kappa+2}} \leq n/2 + 1 \). Then (8.4) implies that there exists some constant \( \tilde{C}' \) such that
\[
\psi_{n, \min} \geq \min_u \ln |B_u| \geq \tilde{C}' - n \ln c - \frac{\kappa n}{2} \ln V.
\]
Consequently
\[
- \ln c - \frac{\kappa}{2} \ln V \leq \Psi_{\min} \leq - \ln c - \frac{\kappa}{2} \ln V.
\]
Now by (8.2) we conclude that \( \hat{\varrho}_\kappa = \frac{2}{\kappa} \ln \alpha_\kappa \).

Remark 6. When \( \kappa = 1, 2 \), we have
\[
\hat{\varrho}_1 = \frac{5 + \sqrt{5}}{2} \ln \frac{\sqrt{5} + 1}{2} \quad \text{and} \quad \hat{\varrho}_2 = \ln(\sqrt{2} + 1).
\]

8.2. Asymptotic property of \( d_V(\kappa) \).

Proposition 6.
\[
\lim_{V \to \infty} d_V(\kappa) \ln V = \frac{\kappa \alpha_\kappa + 2}{2 \alpha_\kappa (\alpha_\kappa - 1)} \ln \alpha_\kappa =: \varrho_\kappa.
\]

Proof. By (5.2) we have
\[
d_V(\kappa) = \dim_H \mu_Q = \frac{\ln \alpha_\kappa}{-\Psi_*(\mu_Q)}.
\]
Now we study \( \Psi_*(\mu_Q) \). Since \( \mu_Q \) is ergodic, by Kingman’s sub-additive ergodic theorem, for \( \mu_Q \) a.e. \( \omega \in \Omega \), we have
\[
- \frac{\psi_n(\omega)}{n} \to - \Psi_*(\mu_Q).
\]
Recall that by the definition (4.3), \( \psi_n(\omega) = \ln |B_{w^\omega}[1, \ldots, n]| \). By (8.1) we have
\[
(8.5) \quad e^{-n V^{-n} |\omega|_{e_{\kappa+2}} - n} \lesssim |B_{w^\omega}[1, \ldots, n]| \lesssim e^{n V^{-n} |\omega|_{e_{\kappa+2}} - n}.
\]
To get the value \( - \Psi_*(\mu_Q) \), we need to know the frequency of \( e_{\kappa+2} \) in a \( \mu_Q \) typical point \( \omega \). Define \( \varphi(\omega) = \chi_{[e_{\kappa+2}]}(\omega) \). Since \( \mu_Q \) is ergodic, by (5.5) for \( \mu_Q \) a.e. \( \omega \in \Omega \) we have
\[
\frac{\# \{1 \leq j \leq n : \omega_j = e_{\kappa+2} \}}{n} = \frac{S_n \varphi(\omega)}{n} \to \int_\Omega \varphi d\mu_Q = \mu_Q([e_{\kappa+2}]) = \frac{\alpha_\kappa}{\kappa \alpha_\kappa + 2}.
\]
Now combining with (8.5) we conclude that
\[
\frac{2 \alpha_\kappa (\alpha_\kappa - 1)}{\kappa \alpha_\kappa + 2} \ln V - \ln c \leq - \Psi_*(\mu_Q) \leq \frac{2 \alpha_\kappa (\alpha_\kappa - 1)}{\kappa \alpha_\kappa + 2} \ln V + \ln c.
\]
Now combining with the dimension formula we get the result. \( \square \)

Remark 7. When \( \kappa = 1, 2 \), we have
\[
\varrho_1 = \frac{5 + \sqrt{5}}{4} \ln \frac{\sqrt{5} + 1}{2} \quad \text{and} \quad \varrho_2 = \ln(\sqrt{2} + 1).
\]
Asymptotic property of $s_V(\kappa)$.

The asymptotic properties of $s_V(\kappa)$ has been studied in [25, 17, 26]. Let us recall the result.

For any $0 \leq x \leq 1$ define

$$R(x) := \begin{pmatrix} 0 & x^{(\kappa - 1)} & 0 \\ (\kappa + 1)x & 0 & \kappa x \\ \kappa x & 0 & (\kappa - 1)x \end{pmatrix}$$

Let $\psi(x)$ be the spectral radius of $R(x)$. Then it is seen that $\psi(0) = 0$, $\psi(1) = \alpha_\kappa$ and $\psi(x)$ is continuous and strictly increasing. Assume $x_\kappa$ is the unique number such that $\psi(x_\kappa) = 1$.

**Theorem 15** ([25, 17, 26]).

$$\lim_{V \to \infty} s_V(\kappa) \ln V = - \ln x_\kappa =: \rho_\kappa.$$

In the following we will make $x_\kappa$ explicit. It is seen that for any $x > 0$, the matrix $R(x)$ is primitive, thus the spectral radius $\psi(x)$ of $R(x)$ is the largest positive eigenvalue of $R(x)$. By a direct computation, we have

$$\det(\lambda I_3 - R(x)) = \lambda^3 - (\kappa - 1)x\lambda^2 - (\kappa + 1)x^\kappa \lambda - x^{\kappa + 1}.$$  

Thus $x_\kappa$ is the unique number in $(0, 1)$ such that

$$1 - (\kappa - 1)x_\kappa - (\kappa + 1)x_\kappa^\kappa - x_\kappa^{\kappa + 1} = 0.$$

Write $y_\kappa = 1/x_\kappa$, then $y_\kappa$ is the unique number in $(1, \infty)$ such that

$$y_\kappa^{\kappa + 1} - (\kappa - 1)y_\kappa^\kappa - (\kappa + 1)y_\kappa - 1 = 0.$$

We claim that $\kappa - 1 < y_\kappa < \kappa$ when $\kappa \geq 3$. Indeed define

$$F(y) := y^{\kappa + 1} - (\kappa - 1)y^\kappa - (\kappa + 1)y - 1,$$

then $F(\kappa - 1) = -\kappa^2 < 0$ and $F(\kappa) = \kappa^\kappa - \kappa(\kappa + 1) - 1 > 0$. Thus $F(y) = 0$ has a root in $(\kappa - 1, \kappa)$. On the other hand we know that $F(y) = 0$ has only one root $y_\kappa$ in $(1, \infty)$, thus $\kappa - 1 < y_\kappa < \kappa$.

**Remark 8.** When $\kappa = 1, 2$, by direct computation we have

$$\rho_1 = \rho_2 = \ln(\sqrt{2} + 1).$$

Proof of Theorem 1 (iii) and (iv).

At first we show (iii). The three asymptotic properties have been established by Proposition 5, Proposition 6 and Theorem 15. By Remark 6, 7 and 8 we have $\hat{\rho}_2 = \rho_2 = \rho_2 = \ln(1 + \sqrt{2})$. When $\kappa \neq 2$, it is direct to verify that $\hat{\rho}_\kappa < \rho_\kappa$.

Now we show that $\hat{\rho}_\kappa < \rho_\kappa$ for any $\kappa \neq 2$.

At first we claim that $\kappa < \alpha_\kappa < \kappa + 1$. Indeed define $G(x) = x^2 - \kappa x - 1$, then $G(\kappa) = -1$ and $G(\kappa + 1) = \kappa > 0$, thus $G(x) = 0$ has a root in $(\kappa, \kappa + 1)$. On the other hand $G(x)$ has a unique positive root, which is $\alpha_\kappa$, thus we conclude that $\kappa < \alpha_\kappa < \kappa + 1$. 
Write $\delta_\kappa := \frac{\kappa \alpha_\kappa + 2}{2\alpha_\kappa(\alpha_\kappa - 1)}$. We claim that $\delta_\kappa \leq 2/3$ when $\kappa \geq 8$. Indeed for $\kappa \geq 2$ we have

$$\delta_\kappa = \frac{\kappa \alpha_\kappa + 2}{2\alpha_\kappa(\alpha_\kappa - 1)} \leq \frac{\kappa(\kappa + 1) + 2}{2\kappa(\kappa - 1)}.$$ 

By a simple computation we get $\delta_\kappa \leq 2/3$ for $\kappa \geq 8$. As a result for $\kappa \geq 8$ we have

$$e^{\delta_\kappa} = \alpha_\kappa^{\delta_\kappa} \leq (\kappa + 1)^{2/3}.$$ 

On the other hand

$$e^{\rho_\kappa} = y_\kappa > \kappa - 1.$$ 

Thus for $\kappa \geq 8$ we have

$$e^{\delta_\kappa} \leq (\kappa + 1)^{2/3} < \kappa - 1 < e^{\rho_\kappa}.$$ 

That is, $\delta_\kappa < \rho_\kappa$ for $\kappa \geq 8$.

By direct computation we get $\delta_\kappa < \rho_\kappa$ for $1 \leq \kappa < 8$ and $\kappa \neq 2$. Thus (iii) follows.

Now we show (iv). Assume $\kappa \neq 2$, then $\delta_\kappa < \rho_\kappa < \rho_\kappa$. By the definition we have

$$\lim_{V \to \infty} \gamma_V(\kappa) \ln V < \lim_{V \to \infty} d_V(\kappa) \ln V < \lim_{V \to \infty} s_V(\kappa) \ln V.$$ 

Consequently there exists $V_0(\kappa) > 0$ such that for any $V \geq V_0(\kappa)$,

$$\gamma_V(\kappa) \ln V < d_V(\kappa) \ln V < s_V(\kappa) \ln V.$$ 

That is, $\gamma_V(\kappa) < d_V(\kappa) < s_V(\kappa)$ for $V \geq V_0(\kappa).$

\[\square\]

9. APPENDIX

In this appendix, we give another proof of the fact that $d_V(\kappa) < s_V(\kappa)$ for $V \geq V_0(\kappa)$ when $\kappa \neq 2$. This proof is more elementary and has the advantage that the constant $V_0(\kappa)$ can be estimated explicitly.

By (4.17) and (5.2) we have

$$d_V(\kappa) = \dim_H \mu_Q = \frac{h_{\mu_Q}(\sigma)}{-\Psi_s(\mu_Q)} \quad \text{and} \quad s_V(\kappa) = \dim_H m = \frac{h_m(\sigma)}{-\Psi_s(m)}.$$ 

At first we claim that if $\mu_Q \neq m$, then $d_V(\kappa) < s_V(\kappa)$. Indeed by Theorem 5 (iii), the unique equilibrium state of $s_V \Psi$ is $m$. Since $\mu_Q \neq m$ we have

$$h_{\mu_Q}(\sigma) + s_V \Psi_s(\mu_Q) < h_m(\sigma) + s_V \Psi_s(m) = P(s_V \Psi) = 0.$$ 

Since $\Psi_s(\mu_Q) < 0$ we conclude that

$$d_V(\kappa) = \dim_H \mu_Q = \frac{h_{\mu_Q}(\sigma)}{-\Psi_s(\mu_Q)} < s_V.$$ 

Thus we only need to study when $\mu_Q \neq m$. For this purpose we consider two words $u^n = ((II, 1)(I, 1))^3 = (e_{n+2}e_1)^3$ and $\tilde{u}^n = ((II, 1)(III, 1)(I, 1))^2 = (e_{n+2}e_{n+3}e_1)^2$. We will estimate respectively the following

$$\mu_Q([e_1 u^n]), \mu_Q([e_1 \tilde{u}^n]), \frac{m([e_1 u^n])}{30} \quad \text{and} \quad \frac{m([e_1 \tilde{u}^n])}{30}.$$
At first by \((5.3)\) we have
\[
\mu_Q([e_1u^n]) \sim \alpha^{-6n-1}_\kappa \quad \text{and} \quad \mu_Q([e_1\tilde{u}^n]) \sim \alpha^{-6n-1}_\kappa.
\]
Consequently
\[
(9.1) \quad \frac{\mu_Q([e_1u^n])}{\mu_Q([e_1u^n])} \sim 1.
\]

Next we estimate \(m([e_1u^n])\) and \(m([e_1\tilde{u}^n])\). Since \(m\) is the Gibbs measure with potential \(s_V\Psi\), we have
\[
m([e_1u^n]) \sim |B_{w^{e_1}u^n}|^s_V \quad \text{and} \quad m([e_1\tilde{u}^n]) \sim |B_{w^{e_1}\tilde{u}}|^s_V.
\]
Now we estimate \(|B_{w^{e_1}u^n}|\) and \(|B_{w^{e_1}\tilde{u}}|\). At first we have
\[
3n \leq |w^{e_1}u^n|_{\epsilon_{n+2}} \leq N + 3n \quad \text{and} \quad 2n \leq |w^{e_1}\tilde{u}^n|_{\epsilon_{n+2}} \leq N + 2n.
\]
By \((8.1)\) we have
\[
\left\{\begin{array}{ll}
c^{-6n}V^{-3\kappa n} & \leq |B_{w^{e_1}u^n}| \leq c^{6n}V^{-3\kappa n} \\
c^{-6n}V^{-2(\kappa+1)n} & \leq |B_{w^{e_1}\tilde{u}}|^s_V \leq c^{6n}V^{-2(\kappa+1)n}
\end{array}\right.
\]
As a consequence we get
\[
(9.2) \quad C_{V,\kappa}^n := (c^{-12}V^{2-\kappa})^n \leq \frac{|B_{w^{e_1}u^n}|}{|B_{w^{e_1}\tilde{u}}|^s_V} \leq (c^{12}V^{2-\kappa})^n =: D_{V,\kappa}^n.
\]
Note that \(c = c_\kappa\) is a constant only depending on \(\kappa\). Define \(V_0(\kappa) := c_\kappa^{12}\). By \((9.2)\), it is direct to check that for \(\kappa = 1\), if \(V > V_0(1)\), then \(C_{V,1} > 1\); for \(\kappa \geq 3\), if \(V > V_0(\kappa)\), then \(D_{V,\kappa} < 1\). Consequently if \(V > V_0(\kappa)\), then
\[
\frac{m([e_1u^n])}{m([e_1\tilde{u}^n])} \sim \left(\frac{|B_{w^{e_1}u^n}|}{|B_{w^{e_1}\tilde{u}}|^s_V}\right)^{s_V} \gtrsim (C_{V,1})^{n s_V} \to \infty, \quad (n \to \infty) \quad \kappa = 1
\]
\[
\frac{m([e_1u^n])}{m([e_1\tilde{u}^n])} \sim \left(\frac{|B_{w^{e_1}u^n}|}{|B_{w^{e_1}\tilde{u}}|^s_V}\right)^{s_V} \gtrsim (D_{V,\kappa})^{n s_V} \to 0, \quad (n \to \infty) \quad \kappa \geq 3.
\]
Combine with \((9.1)\) we conclude that \(\mu_Q \neq m\). Then the result follows. \(\square\)

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