Abstract. Kleinian groups are discrete groups of isometries of hyperbolic space. Their actions give rise to intricate fractal limit sets on the boundary at infinity and there is great interest in estimating the “dimension” of these limit sets. As an invitation to this fascinating area, we provide a proof of the (well-known) result that the Poincaré exponent of a nonelementary Kleinian group is a lower bound for the upper box dimension of the limit set. Our proof uses only elementary hyperbolic and fractal geometry.

1. KLEINIAN GROUPS, LIMIT SETS, AND THE POINCARÉ EXPONENT.
For integers \( n \geq 2 \), \( n \)-dimensional hyperbolic space can be modeled by the Poincaré ball

\[
\mathbb{D}^n = \{ z \in \mathbb{R}^n : |z| < 1 \}
\]
equipped with the hyperbolic metric \( d_H \) given by

\[
|ds| = \frac{2|dz|}{1 - |z|^2}.
\]

The group of orientation preserving isometries of \( (\mathbb{D}^n, d_H) \) is the group of conformal automorphisms of \( \mathbb{D}^n \), which we denote by \( \text{con}^+(\mathbb{D}^n) \). A good way to get a handle on this group is to view it as the (orientation preserving) stabilizer of \( \mathbb{D}^n \) as a subgroup of the Möbius group acting on \( \mathbb{R}^n \cup \{ \infty \} \). This group consists of maps given by the composition of reflections in spheres.

A group \( \Gamma \leq \text{con}^+(\mathbb{D}^n) \) is called Kleinian if it is discrete. Kleinian groups generate beautiful fractal limit sets defined by

\[
L(\Gamma) = \overline{\Gamma(0)} \setminus \Gamma(0)
\]
where \( \Gamma(0) = \{ g(0) : g \in \Gamma \} \) is the orbit of 0 under \( \Gamma \) and the closure is the Euclidean closure. Discreteness of \( \Gamma \) implies that all \( \Gamma \)-orbits are locally finite in \( \mathbb{D}^n \) and this ensures that \( L(\Gamma) \subseteq S^{n-1} \). Here \( S^{n-1} \) is the “boundary at infinity” of hyperbolic space. A Kleinian group is called nonelementary if its limit set contains at least 3 points, in which case it is necessarily an uncountable perfect set.

The Poincaré exponent captures the coarse rate of accumulation to the limit set and is defined as the exponent of convergence of the Poincaré series

\[
P_\Gamma(s) = \sum_{g \in \Gamma} \exp(-sd_H(0, g(0))) = \sum_{g \in \Gamma} \left(\frac{1 - |g(0)|}{1 + |g(0)|}\right)^s
\]
for \( s \geq 0 \). The Poincaré exponent is therefore

\[
\delta(\Gamma) = \inf\{ s \geq 0 : P_\Gamma(s) < \infty \}.
\]
It is a simple exercise to show that the Poincaré series may be defined using the orbit of an arbitrary \( z \in \mathbb{D}^n \) at the expense of multiplicative constants depending only on \( z \). In particular, the exponent of convergence does not depend on the choice of \( z \). (The definition above uses \( z = 0 \).) For more background on hyperbolic geometry and Kleinian groups see [1, 10].

There has been a great deal of interest in computing or estimating the fractal dimension of the limit set \( L(\Gamma) \) (as a subset of Euclidean space \( \mathbb{R}^n \)) and the Poincaré exponent plays a central role in this analysis. We write \( \dim_H, \dim_B, \dim_A \) to denote the Hausdorff, upper box, and Assouad dimensions, respectively. These constitute three distinct and well-studied notions of fractal dimension. See [6] for more background on dimension theory and fractal geometry, especially the box and Hausdorff dimensions, and [9] for the Assouad dimension. For all nonempty bounded sets \( F \subseteq \mathbb{R}^n \),

\[
0 \leq \dim_H F \leq \overline{\dim}_B F \leq \dim_A F \leq n.
\]

For all nonelementary Kleinian groups,

\[
\delta(\Gamma) \leq \dim_H L(\Gamma)
\]

and for nonelementary geometrically finite Kleinian groups,

\[
\delta(\Gamma) = \dim_H L(\Gamma) = \overline{\dim}_B L(\Gamma).
\]

See [5] for more details on geometric finiteness. Roughly speaking it means that the Kleinian group admits a reasonable fundamental domain. The equality of the Hausdorff dimension and the Poincaré exponent in the geometrically finite setting goes back to Sullivan [14], see also Patterson [11]. The coincidence with the box dimension in this case was proved (rather later) independently by Bishop and Jones [4] and Stratmann and Urbanski [13]. The fact that the Poincaré exponent is always a lower bound for the Hausdorff dimension (without the assumption of geometric finiteness) is due to Bishop and Jones [4]. See the survey [12]. In the presence of parabolic elements, the Assouad dimension can be strictly greater than \( \delta(\Gamma) \), even in the geometrically finite situation, see [8].

In the geometrically infinite setting, \( \delta(\Gamma) < \dim_H L(\Gamma) < \overline{\dim}_B L(\Gamma) \) is possible, and it is an intriguing open problem to determine if \( \dim_H L(\Gamma) = \overline{\dim}_B L(\Gamma) \) for all finitely generated \( \Gamma \) for \( n \geq 4 \). For \( n = 3 \), Bishop and Jones prove that for finitely generated, geometrically infinite \( \Gamma \), \( \dim_H L(\Gamma) = \overline{\dim}_B L(\Gamma) = 2 \), see [4]. This result was extended by Bishop to analytically finite \( \Gamma \) [2, 3]. Falk and Matsuzaki characterized the upper box dimension of an arbitrary nonelementary Kleinian group in terms of the convex core entropy [7]. This can also be expressed as the exponent of convergence of an “extended Poincaré series”, but is more complicated to introduce.

Proving the general inequality \( \delta(\Gamma) \leq \dim_H L(\Gamma) \) involves carefully constructing a measure supported on the limit set and applying the mass distribution principle. Our investigation begins with the following question: since (upper) box dimension is a simpler concept than Hausdorff dimension, can we prove the weaker inequality \( \delta(\Gamma) \leq \overline{\dim}_B L(\Gamma) \) using only elementary methods? We provide a short and self-contained proof of this estimate in the sections which follow. It is instructive to think about why our proof fails to prove the equality \( \delta(\Gamma) = \overline{\dim}_B L(\Gamma) \) in general and what sort of extra assumptions on \( \Gamma \) would be needed to “upgrade” the proof to yield this stronger conclusion.

The (upper) box dimension of a nonempty bounded set \( F \subseteq \mathbb{R}^n \) can be defined in terms of the asymptotic behavior of the volume of the \( r \)-neighborhood of \( F \). Given \( r > 0 \) the \( r \)-neighborhood of \( F \) is denoted by \( F_r \) and consists of all points in \( \mathbb{R}^n \) which
are at Euclidean distance less than or equal to $r$ from a point in $F$. Write $V_E$ to denote the Euclidean volume, that is, $n$-dimensional Lebesgue measure. If $V_E(F) = 0$, then $V_E(F_r) \to 0$ as $r \to 0$. The upper box dimension of $F$ captures this rate of decay and is defined formally by

$$\overline{\dim}_B F = n - \liminf_{r \to 0} \frac{\log V_E(F_r)}{\log r}.$$ 

Another elementary proof of the estimate $\delta(\Gamma) \leq \overline{\dim}_B L(\Gamma)$, at least for $n = 2, 3$, can be found in [2, Lemmas 2.1 and 3.1]. Here the connection is made via “Whitney squares.”

2. PROOF OF DIMENSION ESTIMATE. We prove the following (well-known) result.

**Theorem 1.** Let $\Gamma$ be an arbitrary nonelementary Kleinian group acting on the Poincaré ball. Then

$$\delta(\Gamma) \leq \overline{\dim}_B L(\Gamma).$$

Throughout we write $A \lesssim B$ to mean there is a constant $c > 0$ such that $A \leq cB$. Similarly, we write $A \gtrsim B$ if $B \lesssim A$ and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. The implicit constants may depend on $\Gamma$ and other fixed parameters, but it will be crucial that they never depend on the scale $r > 0$ used to compute the box dimension or on a specific element $g \in \Gamma$.

**Elementary estimates from hyperbolic geometry.** Since $\Gamma$ is nonelementary, it is easy to see that it must contain a loxodromic element, $h$. Loxodromic elements have precisely two fixed points on the boundary at infinity. Let $z \in \mathbb{D}^n$ be a point lying on the (doubly infinite) geodesic ray joining the fixed points of $h$. We may assume $z$ is not fixed by any elliptic elements in $\Gamma$ since it is an elementary fact that the set of elliptic fixed points is discrete. Choose $a > 0$ such that the set

$$\{B_H(g(z), a)\}_{g \in \Gamma}$$

is pairwise disjoint, where $B_H(g(z), a)$ denotes the closed hyperbolic ball centered at $g(z)$ with radius $a$. To see that such an $a$ exists recall that the orbit $\Gamma(z)$ is locally finite. As such, $a$ can be chosen such that $B_H(z, 2a)$ contains only one point from the orbit $\Gamma(z)$, namely $z$ itself. Then the pairwise disjointness of the collection $\{B_H(g(z), a)\}_{g \in \Gamma}$ is guaranteed since if $y \in B_H(g_1(z), a) \cap B_H(g_2(z), a)$ for distinct $g_1, g_2 \in \Gamma$, then

$$d_H(z, g_1^{-1}g_2(z)) = d_H(g_1(z), g_2(z)) \leq d_H(g_1(z), y) + d_H(y, g_2(z)) \leq 2a$$

which gives $g_1^{-1}g_2(z) \in B_H(z, 2a)$, a contradiction.

We will use the simple volume estimate

$$V_E(B_H(g(z), a)) \approx (1 - |g(z)|)^n$$

for all $g \in \text{con}^+(\mathbb{D}^n)$, where the implicit constants are independent of $g$ and $z$, but depend on $a$ and $n$. This follows since $B_H(g(z), a)$ is an Euclidean ball with diameter comparable to $1 - |g(z)|$ (most likely not centered at $g(z)$). To derive this explicitly it is useful to recall the (well-known and easily derived) formula for hyperbolic distance

$$d_H(0, w) = \log \frac{1 + |w|}{1 - |w|}, \quad (w \in \mathbb{D}^n).$$
The next result says that if $1 - |g(z)|$ is small, then the image of a fixed set under $g$ must be contained in a comparably small neighborhood of the limit set. This is the only point in the proof where the fact that the group is nonelementary is used. It is instructive to find an example of an elementary group where the conclusion fails.

**Lemma 2.** Let $a, z$ be as above. There exists a constant $c > 0$ depending only on $\Gamma$, $a$ and $z$ such that if $g \in \Gamma$ satisfies $1 - |g(z)| < 2^{-k+1}$ for a positive integer $k$, then

$$B_H(g(z), a) \subseteq L(\Gamma)c_{2-k}.$$ 

**Proof.** The idea is that there must be a loxodromic fixed point close to $g(z)$ and loxodromic fixed points are necessarily in the limit set. Indeed, $g(z)$ lies on the geodesic ray joining the fixed points of the loxodromic map $ghg^{-1}$. These fixed points are the images of the fixed points of $h$ under $g$ and at least one of them must lie in the smallest Euclidean sphere passing through $g(z)$ and intersecting the boundary $S^{n-1}$ at right angles. This uses the fact that geodesic rays are orthogonal to the boundary and $g$ is conformal. The diameter of this sphere is

$$\lesssim 1 - |g(z)| < 2^{-k+1}$$

and the result follows, recalling that the Euclidean diameter of the ball $B_H(g(z), a)$ is $\approx 1 - |g(z)|$. $\blacksquare$

**Estimating the Poincaré series using the limit set.** Let $s > t > \dim_B L(\Gamma)$. Then by definition

$$V_E(L(\Gamma)_r) \lesssim r^{n-t}$$

for all $0 < r < c/2$ with implicit constant independent of $r$ but dependent on $t$ and where $c$ is the constant from Lemma 2. Then, with $z$ fixed as above,

$$P_{\Gamma}(s) \approx \sum_{g \in \Gamma} \left( \frac{1 - |g(z)|}{1 + |g(z)|} \right)^s \approx \sum_{k=1}^{\infty} \sum_{g \in \Gamma: 2^{-k} \leq 1 - |g(z)| < 2^{-k+1}} (1 - |g(z)|)^s \approx \sum_{k=1}^{\infty} 2^{-k(s-n)} \sum_{g \in \Gamma: 2^{-k} \leq 1 - |g(z)| < 2^{-k+1}} (1 - |g(z)|)^n$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k(s-n)} \sum_{g \in \Gamma: 2^{-k} \leq 1 - |g(z)| < 2^{-k+1}} V_E(B_H(g(z), a)) \quad \text{(by (1))}$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k(s-n)} V_E(L(\Gamma)c_{2-k}) \quad \text{(by Lemma 2 and choice of } a)$$

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\[ \sum_{k=1}^{\infty} 2^{-k(s-n)} 2^{-k(n-t)} \quad \text{(by (2))} \]

\[ \sum_{k=1}^{\infty} 2^{-k(s-t)} < \infty. \]

Therefore \( \delta(\Gamma) \leq s \) and letting \( s \to \dim_{\text{B}} L(\Gamma) \) proves Theorem 1.

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