Novel Results on Global Robust Stability Analysis for Dynamical Delayed Neural Networks Under Parameter Uncertainties

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ABSTRACT In this paper, we focus on the global stability analysis with respect to dynamical delayed neural networks (NNs) that contain parameter uncertainties. Many investigations on the sufficient conditions utilizing different upper bounds for the norm of interconnection matrices pertaining to the global asymptotic robust stability of delayed NNs have been conducted. In this study, a new upper bound of the norm of connection weight matrices is derived for the delayed NNs under parameter uncertainties. The key focus is on how the new upper bound is able to yield minimum result with respect to some of the existing upper bounds. We demonstrate that the new upper bound can lead to some new sufficient conditions with respect to the global asymptotic robust stability of equilibrium point of the delayed NNs. The slope bounded activation functions and Lyapunov-Krasovskii functionals (LKFs) are employed for formulating the sufficient conditions of the equilibrium point of NNs. Moreover, the derived sufficient conditions are independent on the time delay parameter. Numerical examples are provided and the outcomes obtained are compared with those of the existing results subject to different network parameters.

INDEX TERMS Dynamical delayed neural networks, slope bounded activation function, interval matrices, parameter uncertainties, robust stability analysis.

I. INTRODUCTION

In recent years, the role of neural network (NN) has been significantly developed due to their successful applications to different areas. Indeed, many different types of neural networks (NNs), e.g. Hopfield, Cohen-Grossberg, Bidirectional Associative, and cellular NN models, have been utilized to solve various engineering problems pertaining to combinatorial optimization, pattern recognition, image and signal processing, etc. Recently, Amazon, Epinions, Facebook and Twitter are running in the field of data science and neural network science systems [1]–[4]. However, a common challenge of NN hardware design and implementation is that it is difficult to determine appropriate and accurate network parameters. The issue of parameter fluctuation of NN implementation on VLSI chips is also unavoidable. The designing process of NN includes numerous estimation errors in the measurement of important data such as synaptic interconnection weights, fire rates of neurons, and signal transmission delays. Nevertheless, it is possible to examine the range of network parameters even in the presence of incomplete information. In this regard, by using the interval theory of NN connection weight matrices, we can identify the upper bounds with respect to the norm of interval matrices. Recently, a number of studies on the derivation of the upper bounds of the norm of connection weight matrices have been conducted [5]–[10]. Specifically, the sufficient conditions pertaining to the NN global robust stability have been derived.

As reported in the literature, different kinds of NN stability analysis, such as global asymptotically robust stability (GARS), exponential stability and complete stability with time delays have been examined [7]–[14]. The Lyapunov stability theory, linear matrix inequalities, non-smooth analysis, M-matrix theory have been used in the stability analysis of delayed NN models. In this respect, the stability properties of equilibrium point play a vital role in dynamical delayed...
NN models. In other words, it is important to examine and understand the GARS of dynamical delayed NN models under parameter uncertainties, as reported in [15]–[28]. It is well-known that a delayed NN model usually includes a delay parameter in the state of a neuron. However, it is very interesting to add a delay to the neuron state and study their effects. Many different types of time delays can be used, e.g., constant time delay, discrete time delay, distributed time delay, neutral time delay, leakage time delay etc. In this paper, we concentrate on constant time delay NN models. We cover mathematical modelling of NN dynamics with time delays, in which the results have a wide range of practical engineering problems [29]–[32].

Motivated by the above account, we specifically examine the global robust stability of dynamical time-delayed NN models in this study. While several upper bounds with respect to the connection weight matrices of dynamical delayed NNs have been derived, we aim to obtain a new upper bound for the connection weight matrices of this class of NN models. Our study is significant because different upper bounds play a major role in the determination of the sufficient conditions pertaining to the global robust stability of dynamical delayed NN models. Through this new upper bound, we are able to formulate the sufficient conditions with respect to the GARS of delayed NN models. In our analysis, the activation functions are considered as unbounded, but as slope bounded functions.

This paper is organized in the following manner. The considered dynamical time delayed NN model is represented by a set of differential equations:

$$\frac{dw_i(t)}{dt} = -c_iw_i(t) + \sum_{j=1}^{n} d_{ij}f_j(w_j(t)) + \sum_{j=1}^{n} e_{ij}f_j(w_j(t - \tau)) + J_i, \quad i = 1, 2, \ldots, n,$$

where the total number of neurons is $n$ and the $i$th neuron state is $w_i(t)$. In addition, $e_{ij}$ and $d_{ij}$ are the connection weights between the $i$th and $j$th neurons with and without time delays respectively; $c_i$ indicates the rate of charge for the $i$th neuron; $f_j(\cdot)$ denotes the activation functions at time $t$ and $t - \tau$, with $\tau$ denotes the constant time delay. Besides that, $J_i$ represents the vector with constant input between the neurons. The matrix vector form of equation (1) is as follows:

$$\dot{w}(t) = -Cw(t) + Df(w(t)) + Eff(w(t - \tau) + J,$$

where $w(t) = [w_1(t), w_2(t), \ldots, w_n(t)]^T \in \mathbb{R}^n$, $C = \text{diag}(c_i > 0)$, $E = (e_{ij}) \in \mathbb{R}^{n \times n}$, $D = (d_{ij}) \in \mathbb{R}^{n \times n}$, $f(w(t)) = [f_1(w_1(t)), f_2(w_2(t)), \ldots, f_n(w_n(t))]^T \in \mathbb{R}^n$ and $J = [J_1, J_2, \ldots, J_n]^T \in \mathbb{R}^n$. The initial condition is $w(t) = \phi(t) \in C([-\tau, 0], \mathbb{R}^n)$. The most common approach for handling the delayed NN model is to make the connection weight matrices $D = (d_{ij})_{n \times n}, E = (e_{ij})_{n \times n}$ and $C = \text{diag}(c_i > 0)$ in an interval, i.e.,

$$\{C_1 = \{C = \text{diag}(c_i) : 0 < C \leq \bar{C}, \quad i.e., \quad 0 < c_i \leq \bar{C}, \quad i = 1, 2, \ldots, n\},$$

$$\{D_1 = \{D = (d_{ij}) : D \leq \bar{D}, \quad i.e., \quad d_{ij} \leq \bar{D}_{ij}, \quad i, j = 1, 2, \ldots, n\},$$

$$\{E_1 = \{E = (e_{ij}) : E \leq \bar{E}, \quad i.e., \quad e_{ij} \leq \bar{E}_{ij}, \quad i, j = 1, 2, \ldots, n\}.$$

By using equation (3), we can define matrices $D^*, D_a, E^*$ and $E_a$:

$$D^* = \frac{1}{2}(\bar{D} + D), \quad D_a = \frac{1}{2}(\bar{D} - D).$$

$$E^* = \frac{1}{2}(\bar{E} + E), \quad E_a = \frac{1}{2}(\bar{E} - E).$$

Definition 1: The NN model given in (2) with the network parameters satisfying (3) is globally asymptotically robust stable if the unique equilibrium point $w^*(t) = [w_1^*(t), w_2^*(t), \ldots, w_n^*(t)]^T \in \mathbb{R}^n$ of the model is globally asymptotically stable for all $C \in C_1, D \in D_1, E \in E_1$. 

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Definition 2: A slope bounded function has some positive constants $k_i$ such that

$$0 \leq \frac{f_j(w) - f_i(v)}{w - v} \leq k_i, \quad \forall w, v \in \mathbb{R}, w \neq v, \ i = 1, 2, \ldots, n.$$ 

A slope-bounded activation function of $f_j$ is used in this study, in which the class of functions is denoted by $f \in \mathcal{K}$. Note that it is not necessary for this class of functions to be monotonically increasing, differentiable, and bounded. The upper bounds for the norm of the connection weight matrices $D$ and $E$ is not necessary for this class of functions to be monotonically increasing, differentiable, and bounded. The upper bounds for the norm of the connection weight matrices $D$ and $E$ are the matrices defined as in equation (5).

Lemma 1 ([7]–[10]): A matrix $E$ is defined by $E \in \mathcal{E}$ as in equation (3), $E^*$ and $E_a$ are the matrices defined as in equation (5).

Let $T_1(E) = \sqrt[\parallel (E^*)^T E^* + E_a^2 \parallel_2}$, $T_2(E) = \parallel E_a^2 \parallel_2$, $T_3(E) = \sqrt[\parallel E_a^2 \parallel_2 + \parallel E_a^2 \parallel_2 + 2 \parallel E_a^2 \parallel_2]$, and $T_4(E) = \parallel E_a^2 \parallel_2$, where $E = (e_{ij})$ with $e_{ij} = \max(|e_{ij}|, |x_{ij}|)$. Then, $\parallel E \parallel_2 \leq T_1(E)$, where $i = 1, 2, 3, 4$.

Lemma 2 [25]: Suppose $E \in \mathcal{E}$ is any matrix defined as in equation (3), $E^*$ and $E_a$ are defined as in equation (5), then

$$\parallel E \parallel_2 \leq T_3(E),$$

where $T_3(E) = \sqrt[\lambda_{\text{max}}(\parallel (E^*)^T E^* + E_a^2 \parallel_2)]$.

Our major contribution of our current study is to derive a new upper bound with respect to the norm of matrices $D$ and $E$. Specifically, we formulate the new upper bound with respect to the norm of interval connection weight matrices $D$ and $E$ in the following form.

Lemma 3 Suppose $E \in \mathcal{E}$ is any matrix defined as in equation (3), $E^*$ and $E_a$ are the matrices defined as in equation (5), then

$$\parallel E \parallel_2 \leq T_6(E),$$

where $T_6(E) = \sqrt[\lambda_{\text{max}}(\parallel (E^*)^T E^* + 2E_a^2 \parallel_2)]$.

Proof: If $E \in \mathcal{E}_I$, then $e_{ij}$ can be written as follows:

$$e_{ij} = \frac{1}{2}(e_{ij} + e_{ij}) + t_{ij} \frac{1}{2}(e_{ij} - e_{ij}), -1 \leq t_{ij} \leq 1,$$

or

$$\mathcal{E} = (e_{ij}) = \frac{1}{2}(E + E) + \frac{1}{2}(E - E) = E^* + \Delta E,$$

where $\Delta = (t_{ij})_{n \times n}, i, j = 1, 2, \ldots, n$. For any vector $w(i) = [w_1(t), w_2(t), \ldots, w_n(t)] \in \mathbb{R}^n$, we can write

$$w^T E^T E_w = w^T (E^* + \Delta E)^T \Delta E w + w^T E_a^2 \Delta E w = w^T (E^*)^T E^* w + 2w^T E_a^2 \Delta E w$$

and

$$w^T E_a^2 \Delta E w \leq \sum_{i=1}^{n} (\lambda_{\text{max}}(E_a^2)) \parallel w \parallel_2.$$
In addition, 

$$\lambda_{\text{max}} (| (E^*)^T E^* | + 2E^T E^* | E^* | + E^T E^*) \\ \leq \| (| (E^*)^T E^* | + 2 | (E^*)^T | E^* + E^T E^*) \|_2. $$

Hence $T_0(E) \leq T_1(E).$ □

Lemma 5 [14]: Suppose $w(t) = [w_1(t), w_2(t), \ldots, w_n(t)]^T \in \mathbb{R}^n,$ and $D \in \mathcal{D}_2$ is a matrix defined as in equation (3), then the following inequalities holds for any positive diagonal matrix $M:$$$
\begin{align*}
(w^T (MD + D^T M) w & \leq w^T (MD^* + (D^*)^T M) \\
& + \| MD^* + D^T M \|_2 I) w, $$
\end{align*}
$$
$$
$$
$D^*$ and $D^t$ are defined as in equation (4).

Lemma 6 [9]: Suppose $w(t) = [w_1(t), w_2(t), \ldots, w_n(t)]^T \in \mathbb{R}^n,$ and $D \in \mathcal{D}_2$ is a matrix defined as in equation (4), then the following inequalities holds for any positive diagonal matrix $M:$$$
\begin{align*}
(w^T (MD + D^T M) w & \leq - | w^T | Z | w |, $$
\end{align*}
$$
$$
$Z = (z_{ij})_{n \times n}$ with $z_{ii} = -2m \delta_{ii}$ and $z_{ij} = -\max(| m \delta_{ij} + m \delta_{ji} |, | m \delta_{ij} + m \delta_{ji} |),$ for $i \neq j.$

III. STABILITY ANALYSIS

In this section, we find some new sufficient conditions with respect to the global robust stability of our model (1) which will be achieved with the help of Lemma 2 and 3 for the norm of delayed connection weight matrices. Further, we denote the equilibrium point of (1) by $w^*$ and use some proper transformation say $u_i(t) = w_i(t) - w^*, i = 1, 2, \ldots, n.$ After giving such transformation, the network model (1) can be put in the following form:

$$
\dot{u}_i(t) = -c_i u_i(t) + \sum_{j=1}^{n} d_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} e_{ij} g_j(u_j(t - \tau)),
$$

(6)

where $g_i(u_i(\cdot)) = f_i(u_i(\cdot) + w^*_i) - f_i(w^*_i),$ $i = 1, 2, \ldots, n.$ Moreover the functions $g_i$ will satisfy the Definition 2 of $f_i,$ i.e., $f \in \mathcal{K}$ implies that $g \in \mathcal{K}$ with $g(0) = 0,$ $i = 1, 2, \ldots, n.$ Also that this transformation shifts the equilibrium point $w^*$ of (1) to the origin of (6).

Now, our aim is to prove the stability of the origin of the transformed model (6) instead of considering the stability of $w^*.$

The matrix form of neural network model (6) can be written in the form:

$$
\dot{u}(t) = -Cu(t) + Dg(u(t)) + Eg(u(t - \tau)),
$$

(7)

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathbb{R}^n$ is the state vector, $g(u(t)) = (g_1(u_1(t)), g_2(u_2(t)), \ldots, g_n(u_n(t)))^T \in \mathbb{R}^n$ and $g(u(t - \tau)) = (g_1(u_1(t - \tau)), g_2(u_2(t - \tau)), \ldots, g_n(u_n(t - \tau)))^T \in \mathbb{R}^n.$

Theorem 1: Let the activation function $g \in \mathcal{K}.$ Then, the origin of NN model (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $M = \text{diag}(m_i > 0)$ and $K = \text{diag}(k_i > 0)$ such that

$$
\begin{align*}
\Omega_0 & = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} m_i k_j (\lambda_j (M^T D + D^T M) + \| MD^* + D^T M \|_2 I) \\
& - 2 \| M \|_2 T_0(E) I > 0.
\end{align*}
$$

Proof: Consider the following positive definite Lyapunov-functional:

$$
V(u(t)) = u^T(t) u(t) + 2 \delta \sum_{i=1}^{n} \int_{0}^{t} m_i g_i(\xi) d\xi \\
+ (\delta \mu + \eta) \int_{t-\tau}^{t} g_i^2(u_i(\xi)) d\xi,
$$

(8)

where the $m_i, \delta, \eta$ and $\mu$ are some positive constants to be determined later. The time derivative of the above Lyapunov-functional along the trajectories of the model (7) is obtained as follows:

$$
\dot{V}(u(t)) = -2u^T(t) Cu(t) + 2u^T(t) Dg(u(t)) \\
+ 2u^T(t) E\delta g(u(t - \tau)) - 2\delta g^T(u(t)) Mcu(t) \\
+ 2\delta g^T(u(t)) Md^T g(u(t)) \\
+ 2\delta g^T(u(t)) MEG(u(t - \tau)) \\
+ \delta \mu \| g(u(t)) \|_2^2 - \delta \mu \| g(u(t - \tau)) \|_2^2 \\
+ \eta \| g(u(t)) \|_2^2 - \eta \| g(u(t - \tau)) \|_2^2.
$$

(9)

We write the following inequalities:

$$
\begin{align*}
- u^T(t) Cu(t) + 2u^T(t) Dg(u(t)) & \leq g^T(u(t)) D^T C^{-1} Dg(u(t)) \\
& \leq \| D \|_2^2 C^{-1} \| g(u(t)) \|_2^2, \quad (10) \\
- u^T(t) Cu(t) + 2u^T(t) E\delta g(u(t - \tau)) & \leq g^T(u(t)) E^T C^{-1} E g(u(t - \tau)) \\
& \leq \| E \|_2^2 C^{-1} \| E g(u(t - \tau)) \|_2^2, \\
2\delta g^T(u(t)) MEG(u(t - \tau)) & \leq 2 \delta \| M \|_2 \| E \|_2 \| g(u(t)) \|_2 \\
& \| g(u(t - \tau)) \|_2, \quad (11)
\end{align*}
$$

The matrix form of neural network model (6) can be written in the form:

$$
\begin{align*}
\dot{u}(t) & = -Cu(t) + Dg(u(t)) + Eg(u(t - \tau)), \\
& \text{where} \ u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathbb{R}^n \text{ is the state vector,} \\
& g(u(t)) = (g_1(u_1(t)), g_2(u_2(t)), \ldots, g_n(u_n(t)))^T \in \mathbb{R}^n \text{ and} \\
& g(u(t - \tau)) = (g_1(u_1(t - \tau)), g_2(u_2(t - \tau)), \ldots, g_n(u_n(t - \tau)))^T \in \mathbb{R}^n.
\end{align*}
$$

(7)

By applying equations (10)-(13) in (9) results in:

$$
\begin{align*}
\dot{V}(u(t)) & \leq \| D \|_2^2 \| C^{-1} \|_2 \| g(u(t)) \|_2^2 \\
& + \| E \|_2 \| C^{-1} \|_2 \| g(u(t - \tau)) \|_2^2 \\
& - 2\delta g^T(u(t)) M \mathcal{K}^{-1} g(u(t)) \\
& + \delta \| M \|_2 \| T_0(E) \|_2 \| g(u(t)) \|_2^2 \\
& + \delta \| M \|_2 \| T_0(E) \|_2 \| g(u(t - \tau)) \|_2^2 \\
& + \delta \| M \|_2 \| E \|_2 \| g(u(t - \tau)) \|_2^2 \\
& \leq \| M \|_2 T_0(E) \| g(u(t)) \|_2^2 \\
& + \delta \| M \|_2 T_0(E) \| g(u(t - \tau)) \|_2^2 \\
& + \delta \| M \|_2 \| E \|_2 \| g(u(t - \tau)) \|_2^2 \quad (12)
\end{align*}
$$

$$
\begin{align*}
-2\delta g^T(u(t)) M Cu(t) & \leq -2\delta g^T(u(t)) M \mathcal{K}^{-1} g(u(t)).
\end{align*}
$$

(13)
Since \( \| C^{-1} \|_2 \leq \| (C^{-1})^T \|_2 \leq T_G(D) \) and 
\( \| E \|_2 \leq T_E(E) \), \( \dot{V}(u(t))) \) can be written as follows:

\[
\dot{V}(u(t)) \leq T_G(D) \| C^{-1} \|_2 \| g(u(t)) \|_2^2 \\
+ T_E(E) \| C^{-1} \|_2 \| g(u(t) - r) \|_2^2 \\
- 2\delta \| g(u(t)) \|_2 \| g(u(t) - r) \|_2^2 \\
+ \delta \| g(u(t)) \|_2 \| g(u(t) - r) \|_2^2 \\
= (T_G(D) + T_E(E)) \| C^{-1} \|_2 \| g(u(t)) \|_2^2 \\
- \delta \| g(u(t) - r) \|_2^2.
\]

(14)

By taking \( \eta = T_G(D) \| C^{-1} \|_2 \) and \( \mu = \| M \|_2 T_E(E) \), we can write \( \dot{V}(u(t)) \) in the form

\[
\dot{V}(u(t)) \leq (T_G(D) + T_E(E)) \| C^{-1} \|_2 \| g(u(t)) \|_2^2 \\
- \delta \| g(u(t)) \|_2 \| g(u(t) - r) \|_2^2 \\
= (T_G(D) + T_E(E)) \| C^{-1} \|_2 \| g(u(t)) \|_2^2 \\
- \delta \| g(u(t) - r) \|_2^2.
\]

(15)

Since \( \Omega_6 \) is a positive definite matrix, from (15) it follows that

\[
\dot{V}(u(t)) \leq (T_G(D) + T_E(E)) \| C^{-1} \|_2 \| g(u(t)) \|_2^2 \\
- \delta \| g(u(t)) \|_2 \| g(u(t) - r) \|_2^2.
\]

(16)

If we take \( \delta > \frac{T_G(D) + T_E(E)}{\lambda_{\min}(\Omega_6)} \| C^{-1} \|_2 \), then it follows that \( \dot{V}(u(t)) \) is negative definite for all \( g(u(t)) \neq 0 \). Since \( g(u(t)) \neq 0 \) implies that \( u(t) \neq 0 \). If \( g(u(t)) = 0 \) and \( u(t) \neq 0 \), then \( \dot{V}(u(t)) \) can be written in the following form:

\[
\dot{V}(u(t)) = -2u^T(t) \dot{C}u(t) + 2u^T(t) \dot{E}g(u(t) - r) \\
- \delta \| g(u(t)) \|_2 \| g(u(t) - r) \|_2^2 \\
\leq -2u^T(t) \dot{C}u(t) + 2u^T(t) \dot{E}g(u(t) - r) \\
\leq \dot{V}(u(t)) \leq -u^T(t) \dot{C}u(t).
\]

Since \( \dot{u}(t) \leq 0 \), we have \( \dot{V}(u(t)) = -u^T(t) \dot{C}u(t) \).

Therefore \( \dot{V}(u(t)) \) is negative definite for all \( u(t) \neq 0 \). Finally, consider \( g(u(t)) = 0 \) and \( u(t) = 0 \). Then, \( \dot{V}(u(t)) = \dot{V}(u(t)) = \dot{V}(u(t)) \).

It is obvious that \( \dot{V}(u(t)) \) is negative definite for all \( u(t) \neq 0 \). Hence, we have \( \dot{V}(u(t)) = 0 \) if and only if \( u(t) = 0 \). Otherwise \( \dot{V}(u(t)) \) is unbounded since \( V(u(t)) \rightarrow \infty \) as \( \| u \| \rightarrow \infty \). Hence, we conclude that the origin of system (7), or equivalently the equilibrium point of the neural system (2) is GARS.

**Theorem 2:** Let the activation function \( g \in k \). Then, the origin of NN model (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices \( M = \text{diag}(m_i > 0) \) and \( K = \text{diag}(k_i > 0) \) such that

\[
\Omega_5 = 2C^TMC^{-1} - (MD^S + (D^S)^T M + \| MMD_e + D^T(M + \| MMD_e + D^T(M + \| M \|_2 T_E(E)) \| g(u(t)) \|_2^2
\]

(14)

Now, we apply the results of Lemma 2, 3 and 6 we get some new sufficient conditions for the GARS of model (7).

**Theorem 3:** Let the activation function \( g \in k \). Then, the origin of NN model (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices \( M = \text{diag}(m_i > 0) \) and \( K = \text{diag}(k_i > 0) \) satisfying the following sufficient condition

\[
\Theta_6 = 2C^TMC^{-1} + \| E \|_2 T_E(E) > 0,
\]

where \( E = (z_{ij})_{n \times n} \) with \( z_{ii} = -2m_i \| d_i \|_2 \) and \( z_{ij} = -\max( \| m_i d_i \|, \| m_j d_j \|) \). for \( i \neq j \).

**Proof:** From Lemma 6, we have

\[
g^T(\tau, u(t))(MD^S + (D^S)^T M)g(u(t)) \leq \| g^T(u(t)) \|_2^2.
\]

(14)

By applying the above inequality in (14) yields:

\[
\dot{V}(u(t)) \leq \| g(u(t)) \|_2^2 \\
- \delta \| g(u(t)) \|_2 \| g(u(t) - r) \|_2^2 \\
= (T_G(D) + T_E(E)) \| C^{-1} \|_2 \| g(u(t)) \|_2^2 \\
- \delta \| g(u(t) - r) \|_2^2.
\]

(15)

Since \( \Theta_6 \) is a positive definite matrix, (17) can be written as

\[
\dot{V}(u(t)) \leq \| g(u(t)) \|_2^2 \\
- \delta \| g(u(t)) \|_2 \| g(u(t) - r) \|_2^2.
\]

(18)

Note that (18) is exactly in the same form as (16) other than that \( \Omega_6 \) is replaced by \( \Theta_6 \). Hence, we conclude that \( \Theta_6 > 0 \) gives the sufficient condition for the GARS of the neural network model (7).
$M = \text{diag}(m_i > 0)$ and $K = \text{diag}(k_i > 0)$ satisfying the following sufficient condition

$$\Theta_0 = 2\mathcal{L}MK^{-1} + \mathcal{Z} - 2 \| M \|_2 T_5(\mathcal{E})I > 0,$$

where $\mathcal{Z} = (z_{ij})_{n \times n}$ with $z_{ii} = -2m_i d_{ii}$ and $z_{ij} = -\max(|m_i d_{ij} + m_j d_{ji}|, |m_j d_{ij} + m_i d_{ji}|)$, for $i \neq j$.

Proof: By utilizing the result in Lemma 2, we get similar to the arguments discussed as in Theorem 3. □

IV. COMPARISONS

In this section, we compare our new sufficient conditions with recent literature results. From Lemma 1 the different upper bounds $T_j(\mathcal{E})$, $j = 1, 2, 3, 4$ have been given. By using these different upper bounds, we get different sufficient conditions for the stability of equilibrium point which are discussed in [7]–[10]. The next Theorem clarifies these results.

Theorem 5 ([7]–[10]): Let the activation function $g \in \mathcal{K}$. Then, the origin of NN model (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $M = \text{diag}(m_i > 0)$ and $K = \text{diag}(k_i > 0)$ satisfying one of the following sufficient conditions:

$$\Omega_j = 2\mathcal{L}MK^{-1} - (MD^* + (D^*)^T)M + \| MD_a + D_a^T \|_2 T_j(\mathcal{E})I > 0,$$

where $j = 1, 2, 3, 4$, $D^*$, $D_a$ and $E^*$, $E_a$ are defined as in equations (4) and (5) respectively.

Remark 2: From the result in Lemma 4, we have $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ and $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$. Moreover, the sufficient conditions $\Omega_6$, $\Omega_5$ and $\Omega_1$ are derived from the upper bounds of $T_6(\mathcal{E})$, $T_5(\mathcal{E})$ and $T_1(\mathcal{E})$ respectively. The result $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Omega_6 \leq \Omega_1$ for all network parameters satisfying (3), while the result $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Omega_5 \leq \Omega_1$ for all network parameters satisfying (3). Hence, the new sufficient conditions of $\Omega_5$ and $\Omega_6$ always give the less conservative results than that of condition $\Omega_1$ in Theorem 5.

Theorem 6 ([7]–[10]): Let the activation function $g \in \mathcal{K}$. Then, the origin of NN model (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $M = \text{diag}(m_i > 0)$ and $K = \text{diag}(k_i > 0)$ satisfying one of the following sufficient conditions:

$$\Theta_j = 2\mathcal{L}MK^{-1} + \mathcal{Z} - 2 \| M \|_2 T_j(\mathcal{E})I > 0,$$

where $j = 1, 2, 3, 4$, $E^*$, $E_a$ are taken as in equation (5), $\mathcal{Z} = (z_{ij})_{n \times n}$ with $z_{ii} = -2m_i d_{ii}$ and $z_{ij} = -\max(|m_i d_{ij} + m_j d_{ji}|, |m_j d_{ij} + m_i d_{ji}|)$, for $i \neq j$.

Remark 3: From the result in Lemma 4, we have $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ and $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$. Moreover, the sufficient conditions $\Theta_6$, $\Theta_5$ and $\Theta_1$ are derived from the upper bounds of $T_6(\mathcal{E})$, $T_5(\mathcal{E})$ and $T_1(\mathcal{E})$ respectively. The result $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Theta_6 \leq \Theta_1$ for all network parameters satisfying (3), while the result $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Theta_5 \leq \Theta_1$ for all network parameters satisfying (3). Hence, the new sufficient conditions of $\Theta_5$ and $\Theta_6$ always give less conservative results than that of condition $\Theta_1$ in Theorem 6.

Remark 4: In this paper, the obtained sufficient conditions are valid for the time-varying delay. Since the new sufficient conditions of neural network model (2) are independent of the time delay parameter. So the obtained results are valid for time-varying delay.

The unified result of sufficient condition with respect to the GARS of the NN model (2) is as follows.

Theorem 7: Let the activation function $f \in \mathcal{K}$. For each input $i$, the NN model (2) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $M = \text{diag}(m_i > 0)$ and $K = \text{diag}(k_i > 0)$ such that

$$\Phi = 2\mathcal{L}MK^{-1} - \mathcal{Z} - 2 \| M \|_2 T_m(\mathcal{E})I > 0,$$

where $T_m(\mathcal{E}) = \min\{T_i(\mathcal{E}) : \| E \|_2 \leq T_i(\mathcal{E}), \forall i = 1, 2, 3, \ldots, n\}$, $D^*$, $D_a$ and $E^*$, $E_a$ are defined as in equations (4) and (5) respectively.

Remark 5: In this paper, the obtained sufficient conditions are always valid for the uniqueness and existence of an equilibrium point of the NN model (2). Moreover, the unified result given in Theorem 7 is also valid for the uniqueness and existence of an equilibrium point of the NN model (2).

V. NUMERICAL EXAMPLE

In this section, we demonstrate the advantages of our results with an example as follows.

Example 8: Consider the following network parameters of the NN model (2).

\[ D = a \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \]

\[ D_a = a \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \]

\[ E = a \begin{bmatrix} -1 & 0 & -2 & 1 \\ 0 & -1 & -2 & 1 \\ -2 & 0 & -1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix} \]

\[ E_a = a \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix} \]

Let $k_1 = k_2 = k_3 = k_4 = 1$ and $z_1 = z_2 = z_3 = z_4 = 13.76$. From the above matrices, we get

\[ D^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_a = a \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \]

\[ E^* = a \begin{bmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \\ -2 & 0 & 0 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix} \]
Using the above parameters, we calculate the following upper bounds for matrix $\mathbf{E}$:

\[
T_1(\mathbf{E}) = \sqrt{\| (\mathbf{E}^*)^T \mathbf{E}^* \|_2 + 2 \| (\mathbf{E}^*)^T \|_{\mathbf{E}_0} + \mathbf{E}_0^T \mathbf{E}_0 \|_2} = 4.4232a, \\
T_2(\mathbf{E}) = \| \mathbf{E}^* \|_2 + \| \mathbf{E}_0 \|_2 = 4.7362a, \\
T_3(\mathbf{E}) = \sqrt{\| \mathbf{E}^* \|_2^2 + \| \mathbf{E}_0 \|_2^2 + 2 \| \mathbf{E}_0^T \|_2 \| \mathbf{E}^* \|_2} = 4.6260a, \\
T_4(\mathbf{E}) = \| \hat{\mathbf{E}} \|_2 = 4.3918a, \\
T_5(\mathbf{E}) = \sqrt{\max \{ (\mathbf{E}^*)^T \mathbf{E}^* + \mathbf{E}_0^T \|_2 \mathbf{E}^* + \mathbf{E}_0^T \mathbf{E}_0 \|_2 \} = 4.3536a. \\
\]

Here, $T_6(\mathbf{E}) \leq T_1(\mathbf{E})$ and $T_3(\mathbf{E}) \leq T_1(\mathbf{E})$. Moreover, based on the network parameters specified in the example, we have $T_m(\mathbf{E}) = \min(T_1(\mathbf{E}))$, where $i = 1, 2, 3, 4, 5, 6$, i.e., $T_m(\mathbf{E}) = 4.3536a = T_6(\mathbf{E})$.

The results of $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in Theorem 2 by taking $\mathbf{M}$ as an identity matrix. As such, $\Omega_6$ and $\Omega_5$ are calculated as follows.

\[
\Omega_6 = 2\mathbf{Z} \mathbf{M}^{-1} - (\mathbf{M}^*)^T \mathbf{M} \\
+ \| \mathbf{M} \mathbf{D}_0 + \mathbf{D}_0^T \mathbf{M} \|_2 (2(\mathbf{I} - 2 \| \mathbf{M} \|_2 T_1(\mathbf{E}) I) \\
= (27.52 - 16.7072a)I. \\
\]

Here $\Omega_6 > 0$, provided $a \leq 1.6471$. For the sufficient condition $\Omega_6 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6471$. Now, $\Omega_5$ is calculated as follows.

\[
\Omega_5 = 2\mathbf{Z} \mathbf{M}^{-1} - (\mathbf{M}^*)^T \mathbf{M} \\
+ \| \mathbf{M} \mathbf{D}_0 + \mathbf{D}_0^T \mathbf{M} \|_2 (2(\mathbf{I} - 2 \| \mathbf{M} \|_2 T_3(\mathbf{E}) I) \\
= (27.52 - 16.7838a)I. \\
\]

Here $\Omega_5 > 0$, provided $a \leq 1.6396$. For the sufficient condition $\Omega_5 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6396$. The computations of $\Omega_1, \Omega_2, \Omega_3$ and $\Omega_4$ are as follows:

\[
\Omega_1 = 2\mathbf{Z} \mathbf{M}^{-1} - (\mathbf{M}^*)^T \mathbf{M} \\
+ \| \mathbf{M} \mathbf{D}_0 + \mathbf{D}_0^T \mathbf{M} \|_2 (2(\mathbf{I} - 2 \| \mathbf{M} \|_2 T_1(\mathbf{E}) I) \\
= (27.52 - 16.8464a)I. \\
\]

\[
\Omega_1 > 0$, provided $a \leq 1.6335$. For the sufficient condition $\Omega_1 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6335.
\]

\[
\Omega_2 = 2\mathbf{Z} \mathbf{M}^{-1} - (\mathbf{M}^*)^T \mathbf{M} \\
+ \| \mathbf{M} \mathbf{D}_0 + \mathbf{D}_0^T \mathbf{M} \|_2 (2(\mathbf{I} - 2 \| \mathbf{M} \|_2 T_3(\mathbf{E}) I) \\
= (27.52 - 16.9472a)I + \mathbf{Z}. \\
\]

\[
\Omega_2 > 0$, provided $a \leq 1.5750$. For the sufficient condition $\Omega_2 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.5750.
\]

\[
T_3(\mathbf{E}) = 2\mathbf{Z} \mathbf{M}^{-1} - (\mathbf{M}^*)^T \mathbf{M} \\
+ \| \mathbf{M} \mathbf{D}_0 + \mathbf{D}_0^T \mathbf{M} \|_2 (2(\mathbf{I} - 2 \| \mathbf{M} \|_2 T_3(\mathbf{E}) I) \\
= (27.52 - 16.9472a)I + \mathbf{Z}. \\
\]

\[
\Omega_3 > 0$, provided $a \leq 1.5951$. For the sufficient condition $\Omega_3 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.5951.
\]

\[
\Omega_4 > 0$, provided $a \leq 1.6396. For the sufficient condition $\Omega_4 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6396.
\]

Again we compare our results $\Theta_3$ and $\Theta_6$ with $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ in Theorem (3) by taking $\mathbf{M}$ as an identity matrix and $\mathbf{Z}$ as in the form:

\[
\mathbf{Z} = a \begin{bmatrix}
-2 & -2 & -2 & -2 \\
-2 & -2 & -2 & -2 \\
-2 & -2 & -2 & -2 \\
-2 & -2 & -2 & -2 \\
\end{bmatrix}
\]

Now, $\Theta_6$ and $\Theta_5$ are calculated as follows:

\[
\Theta_6 = 2\mathbf{Z} \mathbf{M}^{-1} + \mathbf{Z} - 2 \| \mathbf{M} \|_2 T_6(\mathbf{E}) I \\
= (27.52 - 8.7072a)I + \mathbf{Z}. \\
\]

Here $\Theta_6 > 0$, provided $a \leq 1.6471. For the sufficient condition $\Theta_6 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6471.
\]

\[
\Theta_5 = 2\mathbf{Z} \mathbf{M}^{-1} + \mathbf{Z} - 2 \| \mathbf{M} \|_2 T_5(\mathbf{E}) I \\
= (27.52 - 8.7836a)I + \mathbf{Z}. \\
\]

Here $\Theta_5 > 0$, provided $a \leq 1.6396. For the sufficient condition $\Theta_5 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6396.
\]

\[
\Theta_1 = 2\mathbf{Z} \mathbf{M}^{-1} + \mathbf{Z} - 2 \| \mathbf{M} \|_2 T_1(\mathbf{E}) I \\
= (27.52 - 8.8464a)I + \mathbf{Z}. \\
\]

Here $\Theta_1 > 0$, provided $a \leq 1.6335. For the sufficient condition $\Theta_1 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6335.
\]

\[
\Theta_2 = 2\mathbf{Z} \mathbf{M}^{-1} + \mathbf{Z} - 2 \| \mathbf{M} \|_2 T_2(\mathbf{E}) I \\
= (27.52 - 9.4724a)I + \mathbf{Z}. \\
\]

Here $\Theta_2 > 0$, provided $a \leq 1.5750. For the sufficient condition $\Theta_2 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.5750.
\]

\[
\Theta_3 = 2\mathbf{Z} \mathbf{M}^{-1} + \mathbf{Z} - 2 \| \mathbf{M} \|_2 T_3(\mathbf{E}) I \\
= (27.52 - 9.2520a)I + \mathbf{Z}. \\
\]
Here $\Theta_3 > 0$, provided $a \leq 1.5951$. For the sufficient condition $\Theta_3 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.5951$.

$$\Theta_4 = 2E/MK^{-1} + Z - 2 || M ||_2 T_d(\mathcal{E})I, = (27.52 - 8.7836a)I + Z.$$ 

Here $\Theta_4 > 0$, provided $a \leq 1.6369$. For the sufficient condition $\Theta_4 > 0$, the NN model (2) is robust and stable whenever $a \leq 1.6369$.

We will give simulation figure to verify the utilization of our results. For this, we consider the following fixed NN parameters:

$$C = \begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix}.$$ 

Let the activation function $g(u(t)) = e^{u(t)} - 1$, and constant time delay $\tau = 0.5$, the state response is given in Figure 1.

![Figure 1: System solution for the initial states $u(0) = [-0.2, 0.42, 0.2, 0.5]$.](image)

From this example, our sufficient conditions $\Omega_5$, $\Omega_6$, and $\Theta_5, \Theta_6$ are less conservative than those imposed by the earlier results of $\Omega_1$ and $\Theta_1$, where $i = 1, 2, 3, 4$, respectively. We have proved that the obtained upper bound $T_3(\mathcal{E})$ is the minimum as compared with $T_1(\mathcal{E})$ and also the upper bound $T_0(\mathcal{E})$ is the minimum as compared with $T_1(\mathcal{E})$. Based on the illustrative example, it is evident that our results are more beneficial as compared with those in previous studies. While our sufficient conditions may have less advantage than the existing stability conditions for different sets of network parameters, all such results provide the required sufficient conditions. Therefore, a unified condition is given in Theorem 7 which is less conservative than the previous results.

**VI. CONCLUSION**

A new upper bound has been derived with respect to the norm of interval connection weight matrices of dynamical delayed NN models in this study. We have shown that our upper bound gives the minimum result as compared with those of some existing upper bounds with respect to the norm of interval connection weight matrices. Based on the result, we are able to derive the new sufficient conditions pertaining to the GARS of the NN model (2). The unification of our current result as compared with the previous robust stability results has clearly demonstrated that it is a generalization of robust stability results. Finally, we have presented a numerical example satisfying our requirements, which clearly ascertains the advantages of our finding. In future, this work can be extended to stochastic NN under parameter uncertainties.

**REFERENCES**

[1] H.-J. Li and L. Wang, “Multi-scale asynchronous belief percolation model on multiplex networks,” New J. Phys., vol. 21, no. 1, Jan. 2019, Art. no. 015005.

[2] H.-J. Li, Z. Bu, Z. Wang, and J. Cao, “Dynamical clustering in electronic commerce systems via optimization and leadership expansion,” IEEE Trans. Ind. Informat., vol. 16, no. 8, pp. 5327–5334, Aug. 2020.

[3] H.-J. Li and J. J. Daniels, “Social significance of community structure: Statistical view,” Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top., vol. 91, no. 1, Jan. 2015, Art. no. 012801.

[4] H.-J. Li, Q. Wang, S. Liu, and J. Hu, “Exploring the trust management mechanism in self-organizing complex network based on game theory,” Phys. A, Stat. Mech. Appl., vol. 542, Mar. 2020, Art. no. 123514.

[5] B. Yang, J. Cao, M. Hao, and X. Pan, “Further stability analysis of generalized neural networks with time-varying delays based on a novel Lyapunov–Krasovskii functional,” IEEE Access, vol. 7, pp. 91253–91264, 2019.

[6] L. Wan and Q. Zhou, “Stability analysis of neutral-type Cohen–Grossberg neural networks with multiple time-varying delays,” IEEE Access, vol. 8, pp. 27618–27623, 2020.

[7] O. Faydasio and S. Arik, “A new upper bound for the norm of interval matrices with application to robust stability analysis of delayed neural networks,” Neural Netw., vol. 44, pp. 64–71, Aug. 2013.

[8] J. Cao, D.-S. Huang, and Y. Qu, “Global robust stability of delayed recurrent neural networks,” Chaos, Solitons Fractals, vol. 23, no. 1, pp. 221–229, Jan. 2005.

[9] T. Ensari and S. Arik, “New results for robust stability of dynamical neural networks with discrete time delays,” Expert Syst. Appl., vol. 37, no. 8, pp. 5925–5930, Aug. 2010.

[10] V. Singh, “Global robust stability of delayed neural networks: Estimating upper limit of norm of delayed connection weight matrix,” Chaos, Solitons Fractals, vol. 32, no. 1, pp. 259–263, Apr. 2007.

[11] S. Arik, “New criteria for stability of neutral-type neural networks with multiple time delays,” IEEE Trans. Neural Netw. Learn. Syst., vol. 31, no. 5, pp. 1504–1513, May 2020.

[12] S. Arik, “A modified Lyapunov functional with application to stability of neutral-type neural networks with time delays,” J. Franklin Inst., vol. 356, no. 1, pp. 276–291, Jan. 2019.

[13] M. Syed Ali, N. Gunasekaran, and Q. Zhu, “State estimation of T–S fuzzy delayed neural networks with Markovian jumping parameters using sampled-data control,” Fuzzy Sets Syst., vol. 306, pp. 87–104, Jan. 2017.

[14] H. Qi, “New sufficient conditions for global robust stability of delayed neural networks,” IEEE Trans. Circuits Syst. I, Reg. Papers, vol. 54, no. 5, pp. 1131–1141, May 2007.

[15] O. Faydasio and S. Arik, “Robust stability analysis of a class of neural networks with discrete time delays,” Neural Netw., vol. 29–30, pp. 52–59, May 2012.

[16] J.-L. Shao, T.-Z. Huang, and S. Zhou, “Some improved criteria for global robust exponential stability of neural networks with time-varying delays,” Commun. Nonlinear Sci. Numer. Simul., vol. 15, no. 12, pp. 3782–3794, Dec. 2010.

[17] N. Ozcan and S. Arik, “Global robust stability analysis of neural networks with multiple time delays,” IEEE Trans. Circuits Syst. I, Reg. Papers, vol. 53, no. 1, pp. 166–176, Jan. 2006.
G. Nagamani, C. Karthik, and G. Soundararajan, “Observer-based exponential stabilization for time-delay systems via augmented weighted L-K functional,” *Neurocomputing*, vol. 234, pp. 198–204, Apr. 2017.

N. Gunasekaran, M. S. Ali, and S. Pavithra, “Finite-time $L_{\infty}$ performance state estimation of recurrent neural networks with sampled-data signals,” in *Proc. Neural Process. Lett.*, 2019, pp. 1–14.

M. S. Ali and N. Gunasekaran, “State estimation of static neural networks with interval time-varying delays and sampled-data control,” *Comput. Appl. Math.*, vol. 37, no. S1, pp. 183–201, Dec. 2018.

J. Cao and D. W. C. Ho, “A general framework for global asymptotic stability analysis of delayed neural networks based on LMI approach,” *Chaos, Solitons Fractals*, vol. 24, no. 5, pp. 1317–1329, Jun. 2005.

M. Syed Ali and N. Gunasekaran, “Sampled-data state estimation of Markovian jump static neural networks with interval time-varying delays,” *J. Comput. Appl. Math.*, vol. 343, pp. 217–229, Dec. 2018.

X. Liao, K.-W. Wong, Z. Wu, and G. Chen, “Novel robust stability criteria for interval-delayed Hopfield neural networks,” *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 11, pp. 1355–1359, Nov. 2001.

S. Arik, “New criteria for global robust stability of delayed neural networks with norm-bounded uncertainties,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 25, no. 6, pp. 1045–1052, Jun. 2014.

X. Huang, J. Jia, Y. Fan, Z. Wang, and J. Xia, “Interval matrix method based synchronization criteria for fractional-order memristive neural networks with multiple time-varying delays,” *J. Franklin Inst.*, vol. 357, no. 3, pp. 1707–1733, Feb. 2020.

B. Hu, Q. Song, and Z. Zhao, “Robust state estimation for fractional-order complex-valued delayed neural networks with interval parameter uncertainties: LMI approach,” *Appl. Math. Comput.*, vol. 373, May 2020, Art. no. 125033.

E. Yucel, M. Syed Ali, N. Gunasekaran, and S. Arik, “Sampled-data filtering of Takagi–Sugeno fuzzy neural networks with interval time-varying delays,” *Fuzzy Sets Syst.*, vol. 316, pp. 69–81, Jun. 2017.

G. Nagamani, C. Karthik, and G. Soundararajan, “Observer-based exponential stabilization for time-delay systems via augmented weighted integral inequality,” *J. Franklin Inst.*, vol. 356, no. 16, pp. 9023–9042, Nov. 2019.

S. Arik, “Dynamical analysis of uncertain neural networks with multiple time delays,” *Int. J. Syst. Sci.*, vol. 47, no. 3, pp. 730–739, Feb. 2016.

N. Gunasekaran and G. Zhai, “Stability analysis for uncertain switched delayed complex-valued neural networks,” *Neurocomputing*, vol. 367, pp. 198–206, Nov. 2019.

Y. Shen and J. Wang, “Robustness analysis of global exponential stability of recurrent neural networks in the presence of time delays and random disturbances,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 23, no. 1, pp. 87–96, Jan. 2012.

N. Gunasekaran and G. Zhai, “Sampled-data state-estimation of delayed complex-valued neural networks,” *Int. J. Syst. Sci.*, vol. 51, no. 2, pp. 303–312, Jan. 2020.

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