SOLITONIC ASPECTS OF $q$-FIELD THEORIES

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Abstract. We have examined the deformation of a generic non-Abelian gauge theory obtained by replacing its Lie group by the corresponding quantum group. This deformed gauge theory has more degrees of freedom than the theory from which it is derived. By going over from point particles in the standard theory to solitonic particles in the deformed theory, it is proposed to interpret the new degrees of freedom as descriptive of a non-locality of the deformed theory. It also turns out that the original Lie algebra gets replaced by two dual algebras, one of which lies close to and approaches the original Lie algebra in a correspondence limit, while the second algebra is new and disappears in this same correspondence limit. The exotic field particles associated with the second algebra can be interpreted as quark-like constituents of the solitons, which are themselves described as point particles in the first algebra. These ideas are explored for $q$-deformed $SU(2)$ and $GL_q(3)$. 
1. Introduction.

The quantum groups made their original appearance in connection with the inverse scattering problem and Yang-Baxter physics.\(^1\) Later they became important in the theory of knots and in conformal field theory. Since the Lie groups may be regarded as degenerate forms of the quantum groups, it may also be of interest to replace the latter by the quantum groups in other physical contexts. For example, corresponding to quantum mechanical systems such as the harmonic oscillator and the hydrogen atom there are \(q\)-systems obtained by going over to \(q\)-groups. It is then found that the quantum mechanical \(q\)-systems have more degrees of freedom than the systems from which they are derived. When field theories based on gauged Lie groups are similarly deformed by replacing the Lie groups by the corresponding quantum groups, the new degrees of freedom may be interpreted as an expression of the non-locality exhibited by extended or solitonic particles.\(^2\) Here we explore some aspects of \(q\)-theories lying formally close to the standard model. It is proposed that the original Lie-based theory be replaced by a \(q\)-theory with two sectors: one describing the point particles representing the solitons, that may approximate the standard theory, while the second sector describes the dynamics of the fields comprising the solitons.

We first describe the simplest non-trivial quantum groups \(SL_q(2), SU_q(2)\) and the attached Hilbert space.

**Two-Dimensional Representation of \(SL_q(2)\)**

\[
T\epsilon T^t = T^t\epsilon T = \epsilon \quad T\epsilon SL_q(2)
\]

where \(t\) means transpose and

\[
\epsilon = \begin{pmatrix} 0 & q_1^{1/2} \\ -q_1^{1/2} & 0 \end{pmatrix}, \quad q_1 = q^{-1}
\]

Set

\[
T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

Then

(a) \(\alpha\beta = q\beta\alpha\) \quad (c) \(\alpha\gamma = q\gamma\alpha\) \quad (e) \(\alpha\delta - q\beta\gamma = 1\)

(b) \(\delta\beta = q_1\beta\delta\) \quad (d) \(\delta\gamma = q_1\gamma\delta\) \quad (f) \(\delta\alpha - q_1\beta\gamma = 1\)

(g) \(\beta\gamma = \gamma\beta\)

If \(q = 1\), the equations (1.4) are satisfied by complex numbers, and \(T\) is defined over a continuum; but if \(q \neq 1\), then \(T\) is defined only over the algebra—a non-commuting space.
We adopt a matrix representation of the algebra and impose \( \delta = \bar{\alpha} \) and \( \beta = \bar{\beta}, \bar{\gamma} = \gamma \) where the bar means Hermitian conjugate. There are no finite representations of this algebra unless \( q \) is a root of unity.

To define the state space attached to the algebra we interpret \( \bar{\alpha} \) and \( \alpha \) as raising and lowering operators respectively as follows.

**Ground state**

\[
\begin{align*}
\alpha |0\rangle &= 0 \\
\beta |0\rangle &= b |0\rangle \\
\gamma |0\rangle &= c |0\rangle
\end{align*}
\]

\( (1.5) \) \( (1.6a) \) \( (1.6b) \)

**Excited states**

\[
\begin{align*}
\bar{\alpha} |n\rangle &= \lambda_n |n+1\rangle
\end{align*}
\]

\( (1.7) \)

By iterating

\[
\begin{align*}
\beta \bar{\alpha} &= q \bar{\alpha} \beta \\
\gamma \bar{\alpha} &= q \bar{\alpha} \gamma
\end{align*}
\]

we find

\[
\begin{align*}
\beta |n\rangle &= q^n b |n\rangle \\
\gamma |m\rangle &= q^m c |m\rangle
\end{align*}
\]

\( (1.8) \) \( (1.9) \)

By (1.4f)

\[
(\bar{\alpha} \alpha - q_1 \beta \gamma) |0\rangle = |0\rangle
\]

\( (1.10) \)

Hence

\[
b c = -q
\]

\( (1.11) \)

Also by (1.4e)

\[
\langle n | n \rangle = 1 \rightarrow \lambda_n = (1 - q^{2n+2})^{1/2}
\]

\( (1.12) \)

and

\[
q < 1
\]
Two-Dimensional Representation of $SU_q(2)$

Introduce matrix representation of algebra (1.4) and set

$$\gamma = -q_1 \bar{\beta} \quad \delta = \bar{\alpha}$$

where bar means hermitian conjugate. Then

$$\alpha\beta = q\beta\alpha \quad \alpha\bar{\alpha} + \beta\bar{\beta} = 1$$
$$\alpha\bar{\beta} = q\bar{\beta}\alpha \quad \bar{\alpha}\alpha + q_1^2 \bar{\beta}\beta = 1$$

and

$$T \text{ is unitary: } T = T^{-1}$$

(1.14)

If $q = 1$, (A) may be satisfied by complex numbers and $T$ is the usual $U(2)$ unitary-symplectic matrix. If $q \neq 1$, there are no finite representations of (A) unless $q$ is a root of unity. Therefore, as before

If $q = 1$, $T$ is defined over a continuum.

if $|q| \neq 1$, $T$ is defined over algebra (A)-a non-commuting space.

We shall assume that $q$ is real.

The irreducible representations of $SU_q(2)$ are as follows:

$$D^j_{mm'}(\alpha, \bar{\alpha}, \beta, \bar{\beta}) = \Delta^j_{mm'} \sum_{s,t} \left\langle \frac{n_+}{s} \right\rangle_1 \left\langle \frac{n_-}{t} \right\rangle_1 q_1^{(n_1-s+1)t}(-)^t$$
$$\times \delta(s + t, n'_\pm) q^s \beta^{n_+ - s} \bar{\beta}^t \bar{\alpha}^{n_- - t}$$

(1.15)

where

$$n_\pm = j \pm m \quad \left\langle \frac{n}{s} \right\rangle_1 = \frac{\langle n \rangle_1!}{\langle s \rangle_1! \langle n-s \rangle_1!} \quad \langle n \rangle_1 = q^{2n-1}_1 \frac{q^2 - 1}{q^2 - 1}$$

(1.16)

$$q_1 = q^{-1} \quad \Delta^j_{mm'} = \left[ \frac{\langle n'_+ \rangle! \langle n'_- \rangle!}{\langle n_+ \rangle! \langle n_- \rangle!} \right]^{1/2}$$

(1.17)

In the limit $q = 1$ the $D^j_{mm'}$ become the Wigner functions, $D^j_{mm'}(\alpha, \beta, \gamma)$, the irreducible representations of $SU(2)$. The orthogonality relations may be expressed in the following way

$$\int_{SU_q(2)} D^j_{m\mu}(\alpha, \bar{\alpha}, \beta, \bar{\beta}) D^{j'}_{m'\mu'}(\alpha, \bar{\alpha}, \beta, \bar{\beta}) d\tau(\alpha, \bar{\alpha}, \beta, \bar{\beta}) = \frac{\delta^{jj'} \delta_{m'm'} \delta_{\mu\mu'}}{[2j + 1]_q} q^{2\mu}$$

(1.18)

$$[2j + 1]_q = \frac{q^{2j+1} - q_1^{2j+1}}{q - q_1}$$

(1.19)
where the integral means an integral over the algebra and should be understood in terms of the Haar measure as defined by Woronowicz.$^3$

We shall comment on the $q$-Lorentz group in the next section but the rest of the paper deals with the $q$-gauge group.

In Sections III-V it is shown how point particles are replaced by solitons if the field operator lies in the algebra of a $q$-global gauge group.

In Section VI the extension to a local gauge group is described.

Sections VII and VIII deal with two options for the vector connection, one lying in the $q$-algebra and the other in the dual algebra.

While III-VIII are based on $SU_q(2)$ and $SL_q(2)$, Sections IX-XII deal with $GL_q(3)$ and $SL_q(3)$.

Finally XIV describes a toy field.

II. $q$-Fields.

The structure of the standard field theories is determined by both the Lorentz group and the gauge group. Consider first the Lorentz group $SL(2,C)$. Replace $SL(2,C)$ by $SL_q(2,C)$. In the spin representation one has

$$
\text{Lorentz : } \epsilon = L^t \epsilon L \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.1)
$$

$$
\text{$q$-Lorentz: } \epsilon_q = L^t \epsilon_q L \quad \epsilon_q = \begin{pmatrix} 0 & q^{1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad (2.2)
$$

Since Pauli has shown that the Lorentz group establishes a connection between spin and statistics, one might expect this connection to be changed when one goes over to the $q$-Lorentz group. To illustrate this question let us consider a vector-spinor field with the conventional Lorentz invariant interaction

$$
\bar{\psi} A \psi
$$

and let us pursue conventional theory aside from the introduction of the following time ordered products:

$$
T_q(\psi(x)\psi(x')) = \psi(x)\psi(x') \quad t > t'
$$

$$
= q' \psi(x')\psi(x) \quad t < t' \quad \text{vector}
$$

$$
= q'' \psi(x')\psi(x) \quad t < t' \quad \text{spinor}
$$

Then the $q$-time-ordered $S$-matrix is

$$
S^{(q)} = T_q(e^{i \int \mathcal{L}(x)d^4x})
$$
where \((q) = (q', q'')\), and the Wick expansion leads to Feynman rules with the following \(q\)-propagators:

\[
D_{\mu\lambda}^{q'}(x) = \left( g_{\mu\lambda} - \frac{\partial_{\mu}\partial_{\lambda}}{m^2} \right) \left( \frac{1}{2\pi} \right)^4 \left( 1 + \frac{q'}{2} \right) \int e^{-ikx} \frac{1}{k^2 - m^2} \left[ 1 + \frac{1 - q' \, k_{\alpha}}{1 + q' \, \omega} \right] d^4k
\]

\[
S_{\alpha\beta}^{q''}(x) = (i\partial + m)_{\alpha\beta} \left( \frac{1}{2\pi} \right)^4 \left( 1 - \frac{q''}{2} \right) \int e^{-ikx} \frac{1}{k^2 - m^2} \left[ 1 + \frac{1 + q'' \, k_{\alpha}}{1 - q'' \, \omega} \right] d^4k.
\]

The difference between these two propagators comes from \((q', q'')\) and the sum over polarization and antiparticle states. The final theory may be tested for Lorentz invariance by calculating particle-particle scattering which depends on the photon propagator, and particle-antiparticle annihilation, which depends on the spinor propagator.

In both cases the result is frame dependent, i.e., Lorentz invariance is broken unless \(q' = 1\) for the vector and \(q'' = -1\) for the spinor propagator. This result is then a special case of the Pauli theorem that Lorentz invariance requires commutation and anticommutation rules for integer and half-integer spin respectively.

Our question is now: if the \(L\) group is replaced by \(L_q\), are \(q'\) and \(q''\) still constrained to be +1 and -1 or will one find two new functions: \(q' = q'(q)\) and \(q'' = q''(q)\)?

The answer to this question is that one does not find two new functions: it is still possible to require TCP and the usual connection between spin and statistics with \(L_q\).

If the \(q\)-Lorentz group is gauged, one obtains \(q\)-gravity.

We do not discuss the \(q\)-Lorentz group further. The remainder of this paper is devoted to the \(q\)-gauge groups.

### III. \(q\)-Scalar Theory

While retaining Lorentz invariance, we shall now examine a scalar theory invariant under global \(SL_q(2)\).

**Classical Hamiltonian:**

\[
H = \frac{1}{2} \int \left[ \vec{\pi}^2 + (\nabla\psi)^2 + m^2\psi^2 \right] d\vec{x}
\]

or

\[
H = \frac{1}{2} \int \left[ \sum_{k=0}^{3} (\partial_k\psi)(\partial_k\psi) + m^2\psi^2 \right] d\vec{x} \tag{3.1}
\]
**Hamiltonian Invariant under SL_q(2):**

\[ H^{(q)} = \frac{1}{2} \int \left[ \sum_{k=0}^{3} \partial_k \tilde{\psi} \epsilon \partial_k \psi + m^2 \tilde{\psi} \epsilon \psi \right] d\vec{x} \]  \hspace{1cm} (3.2)

\( H^{(q)} \) is invariant under

\[ \psi' = T \psi \quad (T^t \epsilon T = \epsilon) \]

\[ \tilde{\psi}' = \tilde{\psi} T^t \]  \hspace{1cm} (3.3)

Now \( \psi \) must lie in the \( q \)-algebra.

**Quantization of Scalar Field Operator:**

\[ \psi(x) = \left( \frac{1}{2\pi} \right)^{3/2} \int \frac{d\vec{p}}{2p_0} \frac{1}{\sqrt{2\pi}} \left( \sum_s u(s) \tau(s) \right) \times \left[ e^{-i\vec{p} \cdot \vec{x}} a(\vec{p}) + e^{i\vec{p} \cdot \vec{x}} \bar{a}(\vec{p}) \right] \]  \hspace{1cm} (3.4)

where we shall now assume

\[ \tau(s) = (\beta, \gamma) \]  \hspace{1cm} (3.5)

\[ \bar{u}(s) \epsilon u(s') = \delta(s, s') U(s) \]  \hspace{1cm} (3.6)

\[ (a(\vec{p}), \bar{a}(\vec{p}')) = \delta(\vec{p} - \vec{p}') \]  \hspace{1cm} (3.7)

**Expanded State Space:**

\[ |N(\vec{p})n_1n_2⟩ \sim \text{state of } N \text{ particles, each with} \]

momentum \( \vec{p} \) and quantum numbers

\( (n_1n_2) \) belonging to \( (\beta, \gamma) \)

The \( |N(\vec{p})n_1n_2⟩ \) are generated by application of raising operators \( a(\vec{p}) \) and \( \bar{a} \).

**IV. Reduction of Quantized Hamiltonian**

Assume normal ordering. Then by (3.2), (3.4) and (3.5)

\[ H^q = \frac{1}{2} \int \left[ \sum_{k=0}^{3} \partial_k \tilde{\psi} \epsilon \partial_k \psi + m^2 \tilde{\psi} \epsilon \psi \right] : d\vec{x} \]  \hspace{1cm} (4.1)

\[ = \left( \frac{1}{2\pi} \right)^{3} \int d\vec{x} \frac{d\vec{p} d\vec{p}'}{2(p_o p'_o)^{1/2}} \left[ \sum_{k=0}^{3} p_k p'_k + m^2 \right] \]

\[ \times e^{i(p-p') \cdot x} [\bar{U} \epsilon U] : \frac{1}{2} \left[ \bar{a}(p) a(p') + a(p') \bar{a}(p) \right] : \]  \hspace{1cm} (4.3)
where

$$[\tilde{U} \epsilon U] = \sum_{ss'} (\tau(s)\tilde{u}(s))\epsilon(u(s')\tau(s'))$$

(4.4)

$$= \sum_s U(s)\tau(s)^2$$

(4.5)

by (3.6). Then

$$H^q = \int \frac{d\vec{p}}{2p_o} \left[ \sum_0^3 p_k^2 + m^2 \right] \bar{a}(p)a(p)[\tilde{U}\epsilon U]$$

(4.6)

or

$$H^q = \int d\vec{p}_o(\tilde{U}\epsilon U)\bar{a}(p)a(p)$$

(4.7)

and

$$H^q|_{N_p n_1 n_2} = \int d\vec{p}_o N_p(\tilde{U}\epsilon U)|_{N_p n_1 n_2}$$

(4.8)

By (4.5) and (3.5)

$$(\tilde{U}\epsilon U)|_{N_p n_1 n_2} = [U(1)\beta^2 + U(2)\gamma^2]|_{N_p n_1 n_2}$$

(4.9)

By (1.8) and (1.9)

$$H^q|_{N_p n_1 n_2} = \int d\vec{p}_o N_p[U(1)q^{2n_1}b^2 + U(2)q^{2n_2}c^2]|_{N_p n_1 n_2}$$

(4.10)

Then we find the energy of a single particle:

$$p_o[U(1)q^{2n_1}b^2 + U(2)q^{2n_2}c^2]$$

(4.11)

Set $\vec{p} = 0$, then the mass of a single particle is by (1.11)

$$m(n_1, n_2) = m\left[U(1)q^{2n_1}b^2 + U(2)q^{2n_2+2} \frac{1}{b^2}\right]$$

(4.12)

where $U(i) = \tilde{u}(i)\epsilon u(i)$, $i = 1, 2$. Now introduce the length, $R$, by

$$U(1)b^2 = \frac{1}{R}$$

(4.13)

Then

$$m(n_1, n_2) = m\left[q^{2n_1} \frac{1}{R} + q^{2n_2+2}(U(1)U(2)R)\right]$$

(4.14)
This spectrum resembles the spectrum of a toroidally compacted string with an associated large-small (T) duality:\(^4\)

\[ R \rightarrow R' = (q^2U(1)U(2)R)^{-1} \quad n_1 \leftrightarrow n_2 \quad (4.15) \]

It is self-dual with the characteristic length

\[ \tilde{R} = (q^2U(1)U(2))^{-1/2} \quad \text{and} \quad n_1 = n_2 \quad (4.16) \]

**Remarks**

Since \( q < 1 \), the highest mass is \( m \). If \( \alpha \) and \( \bar{\alpha} \) are interchanged in (4), then \( q \rightarrow q_1 \) in (4.14) and the spectrum is inverted.

The point particles of the field theory with \( q = 1 \) have now become solitons since any kind of a mass spectrum must be interpreted to imply extension in spacetime. Assuming that the mass of a field quantum is \( \sim q^{2n} \), one may gain some idea of its spatial extension by noting that \( q^{2n} \) resembles the spectrum \( \sim \langle n \rangle q^2 \) of a \( q \)-harmonic oscillator. The wave functions of a \( q \)-oscillator are \( q \)-Hermite functions.\(^6\) A similar shape may be associated with the field soliton while the size may be related to the characteristic length \( (q^2U(1)U(2))^{-1/2} \).

If the scalar field is charged, one finds that the charge and mass of the soliton are proportional (as in BPS solitons).

**V. A General Gauge.**

The equations (3.4) and (3.5) may be regarded as defining a special gauge, according to which the field operator \( \psi \) lies in the \( \beta, \gamma \) subspace. More generally we may replace (3.4) by an expansion in the irreducible representations of \( SU_q(2) \)

\[ \psi(x) = u \sum_{jmn} \varphi^j_{mn}(x)D^j_{mn}(\alpha|q) \]

\[ \tilde{\psi}(x) = \sum_{jmn} \tilde{\varphi}^j_{mn}(x)\bar{D}^j_{mn}(\alpha|q)u^t \quad (5.1) \]

where we have also substituted \( SU_q(2) \) for \( SL_q(2) \). Here complex conjugation is denoted by a bar. We suppose that \( \varphi^i_{mn}(x) \) does not lie in the \( q \)-algebra and that \( u \) is a 2-rowed vector transforming as

\[ u' = Tu \]

\[ u'^t = u^tT^t \quad (5.2a) \]
and normalized according to
\[ u^t \epsilon u = 1 \] (5.2b)

The partial fields \( \varphi^j_{mn}(x) \) appearing in (5.1) may themselves be expanded in terms of annihilation and creation operators
\[
\varphi^j_{mn}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \left[ e^{-i\vec{p} \cdot \vec{x}} a^j_{mn}(\vec{p}) + e^{i\vec{p} \cdot \vec{x}} \bar{a}^j_{mn}(\vec{p}) \right]. \] (5.3)

Here \( a^j_{mn}(\vec{p}) \) is the absorption operator for a particle of momentum \( \vec{p} \) and additional quantum numbers \( (jmn) \). Instead of evaluating the Hamiltonian over a particular state we now average over the full algebra. We therefore replace (3.2) by
\[
H^q = \frac{1}{2} \hbar \int \left[ \sum_{k=0}^{3} \partial_k \bar{\psi} \epsilon \partial_k \psi + m_o^2 \bar{\psi} \epsilon \psi \right] : d\vec{x} \] (5.4)

where the symbol \( \hbar \) standing before the spatial integral denotes a Woronowicz integral over the \( SU_q(2) \) algebra. The evaluation of (5.4) with (5.1) and (5.3) gives
\[
H^q = \int d\vec{p} p_o \sum_{jmn} h(D^j_{mn} D^{j'}_{m'n'}) \frac{1}{2} : \left[ \bar{a}^j_{mn}(\vec{p}) a^{j'}_{m'n'}(\vec{p}) + a^{j'}_{m'n'}(\vec{p}) \bar{a}^j_{mn}(\vec{p}) \right] : \] (5.5)

The orthogonality of the \( q \)-irreducible representations is now expressed in terms of the Haar measure as in (1.18)
\[
h(D^j_{mn} D^{j'}_{m'n'}) = \delta^{jj'} \delta_{mm'} \delta_{nn'} \frac{q^{2n}}{[2j+1]_q}. \] (5.6)

Then
\[
H^q |N(p); jmn\rangle = \frac{p_o q^{2n}}{[2j+1]_q} |N(p); jmn\rangle \] (5.7)

Therefore the rest mass of a single particle with “internal” quantum numbers \( (jmn) \) is
\[
m_o q^{2n} \frac{1}{[2j+1]_q}. \] (5.8)

This spectrum resembles the square root of the spectrum of the \( q \)-H atom.\(^7\)

In the \( \beta\gamma \)-gauge we evaluated the rest mass in a particular state of expanded state space. In the more general expansion above we chose to average over the full state space. The \( \varphi^j_{mn}(x) \) may be regarded as constituent or preon fields.
**Spinor Case**

If $\psi$ is a Dirac spinor, the usual mass term is

$$M\psi^t C\psi$$

(5.9)

invariant under Lorentz transformations, $L$, since

$$L^t CL = LCL^t = C$$

(5.10)

where $C$ is the charge conjugation matrix.

Invariance under independent $L$ and $T$ transformations requires

$$M\tilde{\psi}C\epsilon\psi$$

(5.11)

Then the spinor solitons would have mass $\sim MQ^n$.

(If Lorentz symmetry is broken, one possibility is the replacement of $L$ by $L_q$ and $C$ by $\epsilon$. Then the term invariant under $L_q$ is

$$M\tilde{\psi}\epsilon\psi$$

(5.12)

All of the usual bilinear covariants as well as the relation between particle and antiparticle would then depend on $q$ when $C$ is replaced by $\epsilon$.)

**VI. Local Theory.**

We assume that $T$ is position dependent as in a Yang-Mills theory. Then

$$T(x)^t \epsilon T(x) = T(x) \epsilon T^t(x) = \epsilon$$

(6.1)

**Mass Terms:**

$$M_1\tilde{\psi}C\epsilon\psi + M_2\tilde{\varphi}\epsilon\varphi$$

(6.2)

where

$$\psi' = T\psi \quad \varphi' = T\varphi$$

$$\tilde{\psi}' = \tilde{\psi}T^t \quad \tilde{\varphi}' = \tilde{\varphi}T^t$$

(6.3)

**Kinetic Terms:**

$$\tilde{\psi}C\epsilon^\mu \vec{\nabla}_\mu \psi + \frac{1}{2}(\tilde{\varphi} \vec{\nabla}_\mu \epsilon(\vec{\nabla}^\mu \varphi))$$

(6.4)

where $\vec{\nabla}_\mu$ and $\vec{\nabla}_\mu$ are covariant derivatives.
Transformation Rules:

\[
\begin{align*}
\vec{\nabla}'_\mu &= T \vec{\nabla}_\mu T^{-1} \\
\vec{\nabla}_\mu &= \partial_\mu + \vec{A}_\mu \\
\vec{\nabla}'_\mu &= (T^t)^{-1} \vec{\nabla}_\mu T^t \\
\vec{\nabla}_\mu &= \partial_\mu + \vec{A}_\mu
\end{align*}
\]  

(6.5)

\[
\begin{align*}
\vec{A}'_\mu &= T \vec{A}_\mu T^{-1} + T \vec{\partial}_\mu T^{-1} \\
\vec{A}'_\mu &= (T^t)^{-1} \vec{A}_\mu T^t + (T^t)^{-1} \vec{\partial}_\mu T^t \\
\vec{A}_\mu &= \epsilon \vec{A}_\mu \epsilon
\end{align*}
\]  

(6.6)

There is now a left as well as a right covariant derivative (\(\vec{\nabla}\) and \(\vec{\nabla}'\)) and corresponding vector connections (\(\vec{A}_\mu\) and \(\vec{A}'_\mu\)).

Left and Right Field Strengths:

\[
\begin{align*}
\vec{F}_{\mu\nu} &= (\vec{\nabla}_\mu, \vec{\nabla}_\nu) \\
\vec{F}'_{\mu\nu} &= T \vec{F}_{\mu\nu} T^{-1}
\end{align*}
\]  

(6.7)

\[
\begin{align*}
\vec{F}'_{\mu\nu} &= (\vec{\nabla}_\mu, \vec{\nabla}_\nu) \\
\vec{F}_\mu T^t = (T^t)^{-1} \vec{F}_\mu T^t
\end{align*}
\]  

For the corresponding field invariants one may choose terms like

\[
\begin{align*}
\vec{\varphi}_\ell \vec{F}'_{\mu\nu} \vec{\varphi}_r \\
\vec{\varphi}_\ell \vec{F}_{\mu\nu} \vec{F}'_{\mu\nu} \vec{\varphi}_r
\end{align*}
\]  

(6.8)

where

\[
\begin{align*}
\vec{\varphi}_\ell &= \varphi_\ell T^{-1} \\
\vec{\varphi}_r &= T \varphi_r \\
\vec{\varphi}'_\ell &= \varphi_\ell T^t \\
\vec{\varphi}'_r &= (T^t)^{-1} \varphi_r
\end{align*}
\]  

(6.9)

The usual trace \(\text{Tr} F_{\mu\nu} F_{\mu\nu}\) is not invariant because

\[
(T_{ij}, (F_{\mu\nu})_{kl}) \neq 0.
\]

(6.10)

VII. Representations of the Free Fields.

The scalar and spinor expansions are simple extensions of the usual expressions but there are new options for the vector.

scalar:

\[
\varphi = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2\rho_o)^{1/2}} \sum_s u(p, s) \left[ e^{-ipx} a(p) + e^{ipx} \bar{a}(p) \right] \tau_s
\]

(7.1)
spinor:

\[
\psi = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \sum_{r,s} [u(p, r, s)e^{-i p x} a(p, r) + v(p, r, s)e^{i p x} \bar{b}(p, r)] \tau_s
\]  

(7.2)

where \(\tau_s\) lies in \(q\)-algebra.

vector:

In the standard \(SU(2)\) theory

\[
W_\mu = W_\mu(+) \tau(-) + W_\mu(-) \tau(+) + W_\mu(3) \tau_3
\]  

(7.3)

In the \(SU_q(2)\) theory one option is based on the following correspondence between the \(q\)-algebra and the Cartan algebra of \(SU(2)\):

\[
\bar{\alpha} \sim E_+ \quad \alpha \sim E_- \quad \text{and} \quad (\beta, \gamma) \sim H
\]  

(7.4)

Then one may propose for the vector field lying in the \(q\)-algebra:

\[
A^{(q)}_\mu = A_\mu(+) \alpha + A_\mu(-) \bar{\alpha} + A_\mu(\beta) \beta + A_\mu(\gamma) \gamma
\]  

(7.5)

There is a second option for the vector field, namely

\[
W_{\mu}^{(q)} = W_\mu(+) J^q(-) + W_\mu(-) J^q(+) + W_\mu(3) J_3^q
\]  

(7.6)

based on the deformation of the Lie algebra where

\[
(J_3^q, J_3^q) = J_+^q \quad (J_3^q, J_3^q) = -J_-^q
\]  

(7.7)

\[
(J_+^q, J_-^q) = \frac{1}{2} [2J_3^q]_q
\]  

(7.8)

Here

\[
[2J_3]_q = \frac{q^{2J_3} - q^{-2J_3}}{q - q^{-1}} = q^{-2J_3+1} \langle 2J_3 \rangle_q^2
\]  

(7.9)

The first option is based on a deformation of the Lie group and the second option is based on a deformation of the Lie algebra of the same group.

Therefore there appear to be two sectors of the \(q\)-theory: one describing particles lying in the deformed group and the other describing particles lying in the deformed algebra and suggesting that both \(W_{\mu}^{(q)}\) and \(A^{(q)}_\mu\) should be included in the \(q\)-theory.
VIII. The Dual Algebras.

The \( q \)-deformation of the Lie algebra, shown in (7.7)-(7.9), may be obtained in the following way.

The two-dimensional representation, \( T \), may be Borel factored:

\[
T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = e^{B\sigma_+}e^{\lambda\theta\sigma_3}e^{C\sigma_+}.
\] (8.1)

The algebra of \( (\alpha, \beta, \gamma, \delta) \) is then inherited by \( (B, C, \theta) \) as

\[
(B, C) = 0 \quad (\theta, B) = B \quad (\theta, C) = C \quad \lambda = \ln q
\] (8.2)

The \( 2j + 1 \) dimensional irreducible representation of \( SU_q(2) \) shown in (1.15) may by (8.1) be rewritten in terms of \( (B, C, \theta) \). Then one has by expanding \( \mathcal{D}^j_{mm'}(B, C, \theta) \) to terms linear in \( (B, C, \theta) \)

\[
\mathcal{D}^j_{mm'}(B, C, \theta) = \mathcal{D}^j_{mm'}(0, 0, 0) + B(J^j_B)_{mm'} + C(J^j_C)_{mm'} + 2\lambda\theta(J^j_\theta)_{mm'} + \ldots
\] (8.3)

where the non-vanishing matrix coefficients \( (J^j_B)_{mm'}, (J^j_C)_{mm'} \) and \( (J^j_\theta)_{mm'} \) are

\[
\langle m - 1 | J^j_B | m \rangle = [\langle j + m | 1 \rangle \langle j - m + 1 | 1 \rangle]^{1/2}
\]

\[
\langle m + 1 | J^j_C | m \rangle = [\langle j - m | 1 \rangle \langle j + m + 1 | 1 \rangle]^{1/2}
\]

\[
\langle m | J^j_\theta | m \rangle = m
\] (8.4a)

where

\[
\langle n \rangle = \frac{q^{2n} - 1}{q^2 - 1}
\] (8.4b)

The \( (B, C, \theta) \) and \( (J_B, J_C, J_\theta) \) are generators of two dual algebras satisfying the following commutation rules:

\[
(J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = q^{2j-1}[2J_\theta]
\] (8.5a)

\[
(B, C) = 0 \quad (\theta, B) = B \quad (\theta, C) = C
\] (8.5b)

We shall suppose that the \( q \)-theory describing deformations \( (G_q \text{ and } g_q) \) of both the group \( G \) and the algebra \( g \) is composed of two sectors. The \( G_q \)-sector contains new particles lying in the algebra of \( (\alpha, \bar{\alpha}, \beta, \gamma) \). The \( g_q \)-sector lies close to the standard theory and should approach the standard theory in a \( q = 1 \) correspondence limit. In the soliton constructions the \( g_q \) particles are the solitons while the \( G_q \) particles are related to the constituent fields.
In the \( q = 1 \) limit the solitons and the \( G_q \) particles vanish and the point particle picture is restored.

**IX. Extension to Groups of Higher Rank.\(^8\),\(^9\),\(^10\)**

Although the standard theory is based on \( U(1) \times SU(2) \times SU(3) \) it is likely that it will ultimately be extended to embrace Lie groups of higher rank. We shall therefore briefly discuss the extension of \( SU_q(2) \) to \( q \)-groups of higher rank and in particular \( GL_q(3) \).

In general the quantum groups may be defined by the relations\(^8\)

\[
RT_1 T_2 = T_2 T_1 R
\]

where

\[
T_1 = T \otimes I
\]
\[
T_2 = I \otimes T
\]

and the \( R \)-matrix satisfies the Yang-Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

For the series \( GL_q(N) \) \(^6\)

\[
R = q \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i,j=1 \atop i \neq j}^{N} e_{ii} \otimes e_{jj} + (q - q_1) \sum_{i,j=1 \atop i > j}^{N} e_{ij} \otimes e_{ji}
\]

where

\[
[e_{mn}]_{ij} = \delta_{mi} \delta_{nj}.
\]

Solution of (9.1) with (9.4) leads to simple rules which replicate the rules for \( SL_q(2) \) as follows. Let

\[
T = \begin{pmatrix}
t_{11} & \ldots & t_{1N} \\
t_{N1} & \ldots & t_{NN}
\end{pmatrix}
\]

\( T \in GL_q(N) \)

and let

\[
\begin{pmatrix}
{ik} & {i\ell} \\
{jk} & {j\ell}
\end{pmatrix}
\]

be any rectangle of elements within \( T \). Then

\[
t_{ik} t_{i\ell} = qt_{i\ell} t_{ik}
\]
\[
t_{ik} t_{jk} = qt_{jk} t_{ik}
\]
\[
(t_{ik}, t_{j\ell}) = (q - q_1) t_{jk} t_{i\ell}
\]
\[
(t_{i\ell}, t_{jk}) = 0
\]
i.e. the 4 vertices exhibited in (11.6b) belong to the algebra of $GL_q(2)$. Therefore all the commuting elements lie on lines of positive slope and the maximum set of commuting elements, fixing the rank, lie on the minor diagonal.

There is also the quantum determinant

$$\det_q T = \sum_{\sigma} (-q)^{\ell(\sigma)} t_{1\sigma_1} \cdots t_{N\sigma_N} \quad \sigma \in \text{symm}(N)$$  \hspace{1cm} (9.8)

where $\ell(\sigma)$ is the number of inversions in going from the order $1 \ldots N$ to the permutation $\sigma$.

The quantum determinant commutes with all the elements of $T$

$$(\det_q T, t_{ij}) = 0$$  \hspace{1cm} (9.9)

X. The Quantum Group $GL_q(3)$

In a matrix realization of the algebra (9.7) the following structure is permitted if $N = 3$.

$$T = \begin{pmatrix} \bar{E}_1 & \bar{E}_2 & H_a \\ \bar{E}_3 & H_b & E_2 \\ H_c & E_3 & E_1 \end{pmatrix}$$  \hspace{1cm} (10.1)

where the bar means Hermitian conjugation and

$$H_i = \bar{H}_i \quad i = a, b, c$$  \hspace{1cm} (10.2)

The full set of commutation relations up to their hermitian conjugates follows:

a) $H_a E_2 = q E_2 H_a$  \quad b) $H_a E_1 = q E_1 H_a$  \quad c) $(H_a, E_3) = 0$

$$H_b E_3 = q E_3 H_b \quad H_c E_3 = q E_3 H_c \quad (H_c, E_2) = 0$$  \hspace{1cm} (10.3)

$$H_c E_1 = q E_1 H_c \quad H_b E_2 = q E_2 H_b \quad (H_a, H_b) = (H_b, H_c) = (H_a, H_c) = 0$$

a) $(\bar{E}_1, E_1) = \bar{q} H_a H_c$  \quad b) $(\bar{E}_1, H_b) = \bar{q} \bar{E}_2 \bar{E}_3$  \quad c) $\bar{E}_2 E_3 = q E_3 \bar{E}_2$

$$(E_2, E_2) = \bar{q} H_a H_b \quad (E_1, E_2) = \bar{q} H_a \bar{E}_3 \quad E_2 E_1 = q E_1 E_2$$

$$(\bar{E}_3, E_3) = \bar{q} H_b H_c \quad (\bar{E}_3, E_1) = \bar{q} H_c E_2 \quad E_3 E_1 = q E_1 E_3$$

$$E_2 E_3 = E_3 E_2$$  \hspace{1cm} (10.4)

$$\bar{q} = q - q_1$$
Let us introduce the following basis states:

\[ |n_1^i n_2^i n_3^i \rangle = \bar{E}_1^{n_1^i} E_2^{n_2^i} E_3^{n_3^i} |000\rangle \quad i = a, b, c \]  

(10.5)

where the \( \bar{E}_i \) are regarded as raising operators. Then

\[ H_a |n_1^a n_2^a n_3^a \rangle = q_1^{n_1^a + n_2^a} \alpha_a |n_1^a n_2^a n_3^a \rangle \quad n_3^a = 0 \]  

(10.6)

\[ H_b |n_1^b n_2^b n_3^b \rangle = q_1^{n_1^b + n_2^b} \alpha_b |n_1^b n_2^b n_3^b \rangle \quad n_1^b = 0 \]  

(10.7)

\[ H_c |n_1^c n_2^c n_3^c \rangle = q_1^{n_1^c + n_2^c} \alpha_c |n_1^c n_2^c n_3^c \rangle \quad n_2^c = 0 \]  

(10.8)

These eigenstates are restricted by the requirement that \( H_i \) and the associated \( E_j \) lie in the same row or column. e.g. \( H_a \) lies in the same column as \( E_2 \) and \( E_1 \).

Let the space of states corresponding to \( H_i \) be \( \mathcal{H}_i \) \( (i = a, b, c) \). Let the states belonging to \( \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \) be denoted by

\[ |n_1^a n_2^a n_3^a ; n_1^b n_2^b n_3^b ; n_1^c n_2^c n_3^c \rangle \text{ or } \bar{n}^a \bar{n}^b \bar{n}^c \text{ or simply } |n\rangle . \]  

(10.9)

The matrix elements of \( E_i \) and \( \bar{E}_i \) in this basis are restricted by selection rules following from (10.3). For example

\[ E_2 H_a = q_1 H_a E_2 \]  

(10.10)

\[ \langle n'| E_2 H_a - q_1 H_a E_2 |n\rangle = 0 \]

in the notation of (10.9). By (10.6)

\[ (q_1^{n_1'} + n_2'^{+1} - q_1^{n_1^a + n_2^a} ) \langle n'| E_2 |n\rangle = 0 . \]

Then

\[ \langle n'| E_2 |n\rangle = 0 \quad \text{if} \quad n_1^a + n_2^a \neq n_1^a + n_2^a - 1 . \]  

(10.11)

We also have

\[ \langle n'| E_2 H_b - q_1 H_b E_2 |n\rangle = 0 \]  

(10.12)

implying

\[ \langle n'| E_2 |n\rangle = 0 \quad \text{if} \quad n_2^b + n_3^b \neq n_2^b + n_3^b - 1 \]  

(10.13)

In general we have

\[ \langle n - \Delta_i | E_i |n\rangle = 0 \quad i = 1, 2, 3 \]  

(10.14)
unless $\Delta_i$ is one of the following four possibilities:

| $\Delta_1$ | $\Delta_2$ | $\Delta_3$ |
|------------|------------|------------|
| a b c      | a b c      | a b c      |
| (100; 000; 100) | (100; 010; 000) | (000; 010; 100) |
| (100; 000; 001) | (010; 010; 000) | (000; 001; 100) |
| (010; 000; 100) | (100; 010; 000) | (000; 010; 001) |
| (010; 000; 001) | (010; 001; 000) | (000; 001; 001) |

(10.15)

where the entries in this table represent

$$\Delta_i(n_1^a n_2^a n_3^a; n_1^b n_2^b n_3^b; n_1^c n_2^c n_3^c) \quad i = 1, 2, 3$$

Among the equations of (10.4b) there are relations such as

$$(H_b, E_1) = \tilde{q} E_2 E_3$$

(10.16)

implying by (10.7) and (10.14)

$$\alpha_b(q_1^{n'_2+n'_3} - q_1^{n_2^b+n_3^b})\langle n'|E_1|n \rangle = \tilde{q} \sum_{\Delta_3} \langle n'|E_2|n - \Delta_3 \rangle \langle n - \Delta_3|E_3|n \rangle .$$

(10.17)

By the selection rules on $E_2$ and $E_3$ the right side of the preceding equation vanishes unless $n' = n - \Delta_2 - \Delta_3$; but the left side vanishes unless $n' = n - \Delta_1$ by the selection rules on $E_1$. Since $\Delta_1 \neq \Delta_2 + \Delta_3$ by (10.15), it follows that

$$\sum_{\Delta_3} \langle n - \Delta_2 - \Delta_3|E_2|n - \Delta_3 \rangle \langle n - \Delta_3|E_3|n \rangle = 0$$

(10.18)

or

$$\langle n'|E_2 E_3|n \rangle = 0$$

(10.19a)

and by (10.16)

$$\langle n'|(H_b, E_1)|n \rangle = 0$$

(10.19b)

Similar remarks hold for the other two equations of (10.4b), i.e.

$$\langle n'|(\bar{E}_1, E_2)|n \rangle = 0 \quad \text{since} \quad -\Delta_1 + \Delta_2 \neq -\Delta_3$$

(10.20)

$$\langle n'|(E_3, E_1)|n \rangle = 0 \quad \text{since} \quad -\Delta_3 + \Delta_1 \neq \Delta_2$$

(10.21)
XI. The Amplitudes of the Raising and Lowering Operators.

The $E_i$ and their hermitian conjugates are restricted by the following relations (10.4a)

$$(\bar{E}_1, E_1) = \tilde{q}H_a H_c$$

$$(\bar{E}_2, E_2) = \tilde{q}H_a H_b$$

$$(\bar{E}_3, E_3) = \tilde{q}H_b H_c$$

(11.1)

Consider for example

$$\langle n| \bar{E}_2 E_2 - E_2 \bar{E}_2|n\rangle = \tilde{q}\langle n|H_a H_b|n\rangle$$

(11.2)

or

$$\sum_p \langle n|\bar{E}_2|p\rangle \langle p|E_2|n\rangle - \sum_{p'} \langle n|E_2|p'\rangle \langle p'|\bar{E}_2|n\rangle = \tilde{q}\langle n|H_a H_b|n\rangle$$

in the notation of (10.9) for $|n\rangle$.

Since $E_2$ and $\bar{E}_2$ are hermitian conjugate, the preceding equation may be written as follows:

$$\sum_p |\langle p|E_2|n\rangle|^2 - \sum_{p'} |\langle n|E_2|p'\rangle|^2 = \tilde{q}\langle n|H_a H_b|n\rangle$$

(11.3)

or

$$\sum_{\Delta_2} |\langle n - \Delta_2|E_2|n\rangle|^2 - \sum_{\Delta_2} |\langle n|E_2|n + \Delta_2\rangle|^2 = \tilde{q}\alpha_a \alpha_b q_1^{a_1 + a_2 + b_2 + b_3}$$

(11.4)

where $\Delta_2$ is summed over the 4 possibilities shown in (10.15).

There are corresponding equations for $E_1$ and $E_3$.

In addition to the equations (10.4a) there are the equations (10.4b) and (10.4c) as well as (10.3) that the three $E_i$ matrices must satisfy.

Note that if $|n\rangle$ is a ground state, the first term in (11.4) vanishes. Then (11.4) implies

$$\tilde{q}\alpha_a \alpha_b < 0 .$$

(11.5)

There are similar equations for $E_1$ and $E_3$ and therefore

$$\tilde{q}\alpha_i \alpha_j < 0$$

(11.6)

for $(i, j) = (a, b, c)$. Hence the $\alpha_i$ are all of the same sign; therefore $\alpha_i \alpha_j$ are positive and

$$\tilde{q} = q - q_1 < 0$$

(11.7)

so that

$$q < 1 .$$

(11.8)
One may regard the three \( H_i \) operators as the formal Hamiltonians of three \( q \)-oscillators and the three \( \mathcal{H}_i \) as their state spaces. One may also regard these same oscillators as field oscillators associated with three distinct fields. Then the \( E_i \) and \( \bar{E}_i \) correspond to absorption and emission operators. In the same language the \( E_i \) and \( \bar{E}_i \) describe associated absorption and emission of two particles, since according to (10.15) two population numbers change in every case.

XII. The Quantum Determinant of \( SL_q(3) \).

Let
\[
\Delta = \det_q T
\]  
(12.1)

Then by (9.8)
\[
\Delta = \bar{E}_1 H_b E_1 - q[\bar{E}_2 \bar{E}_3 E_1 + \bar{E}_1 E_2 E_3]
- q^3 H_a H_b H_c + q^2 [\bar{E}_2 E_2 H_c + H_a \bar{E}_3 E_3].
\]  
(12.2)

Since \( \Delta \) belongs to the center of the algebra, we set
\[
\Delta = 1
\]  
(12.3)

Applied to the \( H \)-vacuum we have
\[
\Delta |0\rangle = |0\rangle
\]  
(12.4)

By (12.2), (12.3) and (10.6)-(10.8)
\[
\alpha_a \alpha_b \alpha_c = -q_1^3
\]  
(12.5)

We also have
\[
\langle \bar{n} | \Delta | \bar{p} \rangle = \langle \bar{n} | \Delta | \bar{n} \rangle \delta(\bar{n}, \bar{p})
\]  
(12.6)

and
\[
\langle n | \Delta | n \rangle = 1
\]  
(12.7)

By (12.2), (10.3) and (10.4)
\[
\langle n | \Delta | n \rangle = H_b(n) \langle n | \bar{E}_1 E_1 | n \rangle - q^3 \langle n | H_a H_b H_c | n \rangle
+ q^2[H_c(n) \langle n | \bar{E}_2 E_2 | n \rangle + H_a(n) \langle n | \bar{E}_3 E_3 | n \rangle]
\]  
(12.8)

or by (12.5) and (12.7)
\[
1 - q_1^{n^a + n^b + n^c} = \alpha_b q_1^{n^a} \langle n | \bar{E}_1 E_1 | n \rangle + q^2 \alpha_c q_1^{n^c} \langle n | \bar{E}_2 E_2 | n \rangle
+ q^2 \alpha_a q_1^{n^a} \langle n | \bar{E}_3 E_3 | n \rangle
\]  
(12.9)
where
\begin{align*}
n^a &= n_1^a + n_2^a \\
n^b &= n_2^b + n_3^b \\
n^c &= n_1^c + n_3^c
\end{align*}
(12.10)

Eq. (12.9) is consistent with our earlier conclusion that the \( \alpha_i \) are negative and \( q_1 > 1 \).

XIII. The Dual Algebras.

We have already described the \( q \)-algebra dual to \( SU_q(2) \). The corresponding \( q \)-algebra dual to the higher \( q \)-groups is described by the following equations as expressed in the Chevalley-Serre basis\(^8,^9,^10\)

\begin{align*}
(H_i, H_j) &= 0 \quad (13.1) \\
(H_i, X_j^\pm) &= \pm (\alpha_i, \alpha_j) X_j^\pm \quad i, j = 1 \ldots r \quad (13.2) \\
(X_i^+, X_j^-) &= \delta_{ij} \frac{\sinh h H_i}{\sinh h} = \delta_{ij} \frac{q^{H_i} - \hat{q}_1^{H_i}}{q - \hat{q}_1} \quad (13.3a)
\end{align*}

where \( r \) is the rank and
\[ q = e^h \quad (13.3b) \]

The Serre relations are
\[ \sum_{k=0}^{m} (-)^k \binom{m}{k} q \hat{q}_i^{-k(m-k)/2} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{m-k} = 0 \quad i \neq j \quad (13.4a) \]

where
\begin{align*}
m &= 1 - A_{ij} \\
\hat{q}_i &= e^{h(\alpha_i, \alpha_i)} \\
\binom{m}{k}_q &= \frac{\langle m \rangle_q!}{\langle k \rangle_q! \langle m-k \rangle_q!}
\end{align*}
(13.4b)

Here the \( \alpha_i \) are the simple roots and \( A_{ij} \) are the elements of the Cartan matrix for the simple Lie algebras.

XIV. Invariant Bilinears and Lagrangians of a Toy Theory.

Eqs. (9.1) may be rewritten as
\[ (T_2 T_1) R (T_2^{-1} T_1^{-1}) = (T_1^{-1} T_2^{-1}) R (T_1 T_2) = R \quad (14.1) \]
Set
\[ T = T_1 T_2 \]  
\[ \hat{T} = T_2 T_1 \]  \hspace{1cm} (14.2)

\[ \mathcal{T} \] lies in two spaces which are permuted in \( \hat{T} \). Then
\[ \hat{T} R T^{-1} = \hat{T}^{-1} R T = R \]  \hspace{1cm} (14.4)

When \( N = 2 \), one also has (1.1):
\[ T \epsilon T^t = T^t \epsilon T = \epsilon \]  \hspace{1cm} (14.5)

where \( \epsilon \) is given by (1.2) while
\[ R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \bar{q} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \]  \hspace{1cm} (14.6)

Let \( \psi \) and \( \hat{\psi} \) be matrix fields transforming as
\[ \psi' = \mathcal{T} \psi \]  
\[ \hat{\psi}' = \hat{\psi} \hat{T}^{-1} \] \hspace{1cm} or \hspace{1cm} \[ \psi'_{\ell k} = \sum (T_1)_{\ell s} (T_2)_{kt} \psi_{st} \]  
\[ \hat{\psi}'_{pm} = \sum \hat{\psi}_{ji} (T_1^{-1})_{jm} (T_2^{-1})_{im} \]  \hspace{1cm} (14.7)

Then by (14.4)
\[ (\hat{\psi} R \psi)' = \hat{\psi} R \psi \]  \hspace{1cm} (14.8)

is the bilinear invariant corresponding to \( \psi^t \epsilon \psi \).

To construct a toy theory one may distinguish between \( H \) and \( E \) fields by associating the \( H \) with a Lorentz spinor field and \( E \) with a Lorentz vector field in a particular gauge. This association leads to a supersymmetric feature since the spinor and vector then belong to the same multiplet and therefore transform into each other under a \( q \)-transformation. In this special gauge one may write the spinor field as
\[ \psi_{ij} = \sum_k \varphi_{ij}^k H(k) \]  
\[ k = a, b, c \]  \hspace{1cm} (14.9)

and a mass term as
\[ M \hat{\psi} CR \psi = M \sum_k H(k)^2 U(k) \]  \hspace{1cm} (14.10)
where $\psi_{ij}$ transforms as (14.7) and we orthonormalize as follows

$$\hat{\varphi}^j CR \varphi^k = \delta^{jk} U(k)$$

(14.11)

With respect to the Dirac algebra $\hat{\varphi}^j$ is the transposed spinor and $C$ is the charge conjugation matrix.

The eigenvalues of the mass term in the state $|n^a n^b n^c\rangle$ are

$$M \sum_k q_1^{2n_k} \alpha_{k}^2 U(k) \quad k = a, b, c$$

(14.12)

where by (12.5)

$$\alpha_a \alpha_b \alpha_c = -q_1^3.$$  

(14.13)

One may compare (14.12) and (14.13) with (4.9) and (1.11). In (14.12) there is a rising rather than an inverted mass spectrum since $q$ is there replaced by $q_1$ and in both cases $q < 1$. (The difference between the two cases comes from an arbitrary switch of the positions of the Hermitian conjugate operators with respect to the minor diagonal of $T$.)

Both spectra and also the spectrum (5.8) imply an underlying soliton structure with constituent fields associated with the elements of the $q$-algebra. These three spinor fields may interact with the three vector fields through terms like

$$\hat{\psi} CR \nabla \psi$$

(14.14)

where

$$\nabla_\mu = \partial_\mu + A_\mu, \quad \nabla = \gamma^\mu \nabla_\mu.$$  

(14.15)

Here the vector field is

$$A = \sum_{\alpha=1}^{3} (\tilde{A}(\alpha)E(\alpha) + \tilde{A}(\alpha)\tilde{E}(\alpha)).$$

(14.16)

The $q$-invariance of the interaction term requires the following $q$-transformation laws for the covariant derivative

$$\nabla' = \mathcal{T} \nabla \mathcal{T}^{-1}$$

(14.17)

and the $q$-vector connection

$$A' = \mathcal{T} A \mathcal{T}^{-1} + \mathcal{T} \varphi \mathcal{T}^{-1}.$$  

(14.18)

The general relations shown in Section 6 will continue to hold here if $\mathcal{T}$ is substituted for $T$. In particular the form (6.8) will replace the usual trace

$$\text{Tr} \, F_{\mu\nu} F^{\mu\nu}$$

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even though
\[ F'_{\mu \nu} = T F_{\mu \nu} T^{-1} \]
because the matrix elements of \( F_{\mu \nu} \) lie in the \( q \)-algebra.

The preceding discussion can be generalized to any \( N \), including \( N = 2 \), by representing the fermion field as
\[ \psi = \sum_{i=1}^{N} \phi^i H_i . \]  
(14.19)

In the \( N = 2 \) case we have put \( H_1 = \beta, \ H_2 = \gamma \) in Section 3. The interacting boson field may be written as
\[ A = \sum_{i<j} (\bar{A}_{ij} E_{ij} + A_{ji} E_{ji}) \quad i, j = 1 \ldots N \] 
(14.20)

The general fermionic mass term is
\[ M \hat{\psi} C R \psi = M \sum_{k=1}^{N} q_1^{2n_k} \alpha_k^2 U(k) \] 
(14.21)

where we have again normalized as follows
\[ \hat{\phi}^k C R \phi^j = \delta^{kj} U(k) . \] 
(14.22)

Here, by the argument leading to (14.13)
\[ \prod_{i}^{N} \alpha_k = -q_1^{N} . \] 
(14.23)

The lowest order interaction term is
\[ \hat{\psi} C R A \hat{\psi} . \] 
(14.24)

The matrix element between arbitrary states is
\[ \langle n' | (\sum_{k} \hat{\phi}_k H_k) C R (\sum_{j>i} A_{ij} E_{ij}) (\sum_{j} \phi_{i} H_{i}) | n \rangle + \sum_{i<j} \bar{A}_{ij} \bar{E}_{ij} \text{ terms} \] 
(14.25)
\[ = \sum_{j>i} \sum_{k, \ell} (\hat{\phi}_k C R A_{ij} \phi_{\ell}) \langle n' | H_k E_{ij} H_{\ell} | n \rangle + \sum_{i<j} \cdots \] 
\[ = \sum_{j>i} \sum_{k, \ell} (\hat{\phi}_k C R A_{ij} \phi_{\ell}) q_1^{n_k} q_1^{n_{\ell}} \alpha_k \alpha_{\ell} \langle n' | E_{ij} | n \rangle + \sum_{i<j} \cdots \] 
(14.26)
i.e., the \( q \)-Lorentz invariant form \( \hat{\phi}_k C R A_{ij} \phi_{\ell} \) is averaged over the algebra in the way shown.
In this way and in other procedures that generalize naturally from usual field theory one could in principle construct a quantum mechanical description of the interaction of these constituent fields.

Finally we return to the picture in which the particles of the theory are solitons composed of the constituent fields. We shall work at the level of $SU_q(2)$. In this picture the fields may be represented by the expansions (5.1). Then the mass of a single soliton depends on expressions like (5.5) and

$$\ldots h[(\tilde{\varphi}^j_{m_p} \tilde{\mathcal{D}}^j_{mp})(\varphi^j_{m_p} \mathcal{D}^j_{mp})]$$

which leads to (5.8).

Similarly the lowest order interaction term in the same model is

$$\ldots h \left[ (\psi^j_{p_{m_1}p_1})^t \mathcal{D}^j_{m_1p_1} \epsilon C (A^j_{p_{2m_2}p_2} \mathcal{D}^j_{m_2p_2}) (\psi^j_{m_3p_3} \mathcal{D}^j_{m_3p_3}) \right]$$

The preceding expression reduces to the following:

$$\left[ (\psi^j)^{j_1}_{m_1p_1} \epsilon C A^j_{m_2p_2} \psi^j_{m_3p_3} \right] h (\mathcal{D}^j_{m_1p_1} \mathcal{D}^j_{m_2p_2} \mathcal{D}^j_{m_3p_3})$$

$$= \left[ (\psi^j)^{j_1}_{m_1p_1} \epsilon C A^j_{m_2p_2} \psi^j_{m_3p_3} \right] \left[ j_1j_2j_3 \right]_{m_1m_2m_3} q \left[ j_1j_2j_3 \right]_{p_1p_2p_3} q$$

in terms of $q$-Clebsch-Gordan coefficients $^{11}$

In the limit $q = 1$, there is no internal structure ($j_1 = j_2 = j_3 = 0$) and (14.29) reduces to

$$\psi^t C A \psi$$

XV. Summary.

There are two $q$-algebras, $G_q$ and $g_q$, that are respective deformations of the Lie group ($G$) and its algebra ($g$).

In a matrix representation the matrix elements of $G, g$, and $g_q$ all commute, but the matrix elements of $G_q$ form a non-commuting algebra.

One ordinarily associates a vector connection with the Lie algebra of $SU(2)$ to obtain the electroweak vectors and similarly one associates a vector connection with the Lie algebra of $SU(3)$ to obtain the gluons of the standard model.

If one proceeds in the same way with the deformed Lie algebras $g_q$ of $SU_q(2)$ and $SU_q(3)$ one obtains a hypothetical theory lying close to the standard theory but with
differences that can in principle be computed by a perturbation expansion in $q$. In the limit $q = 1$ one ought to recover the standard theory and when $q$ is near unity, the $q$-theory may have the good formal properties of the standard theory. Here, however, we have only succeeded in discussing a toy theory based on $SL_q(3)$ and not on $SU_q(3)$.

In the hypothetical $q$-theory there is also the dual algebra $G_q$ with which one may also associate a different set of hypothetical dual fields that play the role of quark-like and gluon-like constituent fields.

The two sets of fields, associated with the two dual algebras, do not represent independent sectors of the full theory. They should rather be regarded as complementary descriptions; one picture is microscopic and the second picture, approximating the standard theory, is phenomenological.

Since the Lie groups used in particle physics are, unlike the Poincaré group, only phenomenological, they have no greater a priori claim than the $q$-groups. Therefore the $q$-theories must be judged, just as the Lie theories, by their phenomenological usefulness.

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