Cosmic Censorship in Quantum Gravity

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ABSTRACT

We study cosmic censorship in the Reissner-Nordstrom charged black hole by means of quantum gravity holding near the apparent horizons. We consider a gedanken experiment whether or not a black hole with the electric charge $Q$ less than the mass $M$ ($Q < M$) could increase its charge to go beyond the extremal limit $Q = M$ by absorbing the external charged matters. If the black hole charge could exceed the extremal value, a naked singularity would be liberated from the protection of the outer horizon and visible to distant observers, which means weak cosmic censorship to be violated in this process. It is shown that the black hole never exceeds the extremal black hole this way in quantum gravity as in classical general gravity. An increment of the trapped external charged matters by the black hole certainly causes the inner Cauchy horizon to approach the outer horizon, but its relative approaching speed gradually slows down and stops precisely at the extremal limit. It is quite remarkable that cosmic censorship remains true even in quantum gravity. This study is the first attempt of examining weak cosmic censorship beyond the classical analysis.

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1. Introduction

In recent years, we have had a recurrence of interests on a quantum theory of black holes. Many people have worked on the Hawking radiation and the fate of the endpoint of black hole evaporation [1], the information loss paradox [2], the black hole thermodynamics [3], in particular an understanding of the statistical-mechanical origin of black hole entropy e.t.c. by various different approaches.

Closely related to those problems, there exists one of the most important unsolved problems in classical general relativity, which is Penrose’s cosmic censorship conjecture [4]. Penrose has advocated a stronger form of cosmic censorship, what nowadays we call, strong cosmic censorship, which insists roughly that the whole spacetime should be globally hyperbolic. Saying more precisely, the globally hyperbolic development $D(\Sigma)$ is inextendible for generic initial data on compact or asymptotically flat partial Cauchy surface $\Sigma$. In other words, the Cauchy horizon $\partial D(\Sigma) = 0$. In previous works, we have studied this strong cosmic censorship by evaluating the quantum generalized affine parameter (QGAP) and shown that the spacetime is effectively globally hyperbolic due to the violent quantum fluctuation of the geometry [5, 6].

There is a weaker version of cosmic censorship, what is called, weak cosmic censorship, which states in physical terms that singularities arising in generic gravitational collapse are hidden behind the event horizon. In this paper, when referring to cosmic censorship merely we always have weak cosmic censorship in mind. Of course, cosmic censorship conjecture is unsolved but is widely believed to be true for suitably arranged general initial data and equations of state. This is because without validity of cosmic censorship not only one cannot prove the no-hair theorem of a black hole [7] but also the fact that a black hole can have intrinsic entropy in itself becomes dubious [8].

An early attempt of examining cosmic censorship has been performed in the Reissner-Nordstrom charged black hole [9]. Let us imagine a physical situation
where there is a static charged black hole which has the mass $M$ larger than the charge $Q$, that is, $M > Q$ in the geometrized units, for which we have two horizons, the inner Cauchy horizon and the outer event horizon. If one adds a stream of charged test particles with a large electric charge rather than its mass into this black hole, the initial black hole would turn to the black hole with the property of $M < Q$, where there is no horizon and a naked singularity is visible from external observers. If so, this would lead to a clean counter-example to cosmic censorship conjecture. But it was shown that one needs a lot of kinetic energy to exceed the extremal limit $M = Q$ by this method so that the net increase of black hole mass is always larger than that of black hole charge [9]. Thus this implies that one cannot violate cosmic censorship in this physical setting.

The main topic that we would like to address in this paper is that the above classical test of cosmic censorship remains true also in a quantum theory of general relativity. It is well known that while the outer horizon is a surface of infinite redshift, the inner Cauchy horizon is of infinite blueshift for our own asymptotically flat universe. Therefore as the infalling lightlike matters approach the inner Cauchy horizon the energy density of it will suffer an infinite blueshift, by which in the vicinity of the Cauchy horizon the curvature becomes extremely huge and quantum gravitational effects would play a dominant role. This is one of our motivations of attempting to analyze cosmic censorship by means of quantum gravity.

At present, as is well known, we do not have a good grasp of a fully satisfactory and mathematically consistent theory of quantum gravity yet. However, it has been recently pointed out that one can construct models of quantum black holes in the spherically symmetric geometry at least near the horizons and/or the singularities, and then the models were applied fruitfully to clarify physically interesting problems such as the Hawking radiation [10, 11, 12], the mass inflation [13] and the quantum instability of the black hole singularity in three dimensions [14]. The key observation behind these works is that the problems associated with quantum black holes have at all events an intimate relationship with the quantum-mechanical be-
behavior of the horizons and/or the singularities and are largely irrelevant to the other regions of spacetime so that it might be sufficient to consider a quantum theory of black holes holding in their vicinities in order to answer the questions.

The article is organized as follows. In section 2, we review the canonical formalism of a spherically symmetric system with a black hole. This canonical formalism is used to construct the canonical formalism holding in the vicinity of the apparent horizons in section 3. In section 4, based on the formalism built in section 3, we analyse the Hawking radiation from purely quantum-mechanical viewpoint. Here it is shown that the mass loss rate is proportional to $M^{-2}$ whose result is identical to the result inferred by Hawking [1]. Section 5 is the main part of this article where it is shown that for the Reissner-Nordstrom black hole, cosmic censorship never be violated also even in quantum gravity as in classical general relativity. The last section is devoted to conclusion.

2. Review of Canonical Formalism

We start by reviewing a canonical formalism of a spherically symmetric system with a black hole. This canonical formalism was previously constructed by Hajicek et al. [15], and recently extended to the case of $\frac{\partial}{\partial r}$ being timelike in the interior of a black hole by us [11]. Our attention in this article lies in the spacetime regions where the Killing vector field $\frac{\partial}{\partial x^0}$ is timelike, thus we foliate the spacetime geometry by a family of spacelike hypersurfaces $x^0 = \text{const}$ according to the conventional procedure [16, 17].

The action that we consider is of the form

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{4\pi} g^{\mu\nu} (D\mu \Phi) \dag D\nu \Phi - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \right],$$  \hspace{1cm} (1)$$

where $\Phi$ is a complex scalar field with the covariant derivative

$$D\mu \Phi = \partial\mu \Phi + ieA\mu \Phi,$$  \hspace{1cm} (2)$$
$F_{\mu\nu}$ is $U(1)$ field strength as usual given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

(3)

and $e$ is a positive electric charge of $\Phi$. To show the four dimensional character explicitly we append the suffix $(4)$ in front of the metric tensor and the curvature scalar. We follow the conventions adopted in the MTW textbook [18] and use the natural units $G = \hbar = c = 1$. The Greek indices $\mu, \nu, ...$ take the values 0, 1, 2, and 3, on the other hand, the Latin indices $a, b, ...$ take the values 0 and 1. Of course, the inclusion of other matter fields, the cosmological constants and the surface terms in this formalism is straightforward even if we limit ourselves to the action (1) for simplicity.

Let us take the following spherically symmetric assumptions

$$ds^2 = (4) g_{\mu\nu} dx^\mu dx^\nu,$$

$$= g_{ab} dx^a dx^b + \phi^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

(4)

$$D_\theta \Phi = A_\theta = D_\varphi \Phi = A_\varphi = 0,$$

(5)

where the two dimensional metric $g_{ab}$ and the radial function $\phi$ are the functions of only the two dimensional coordinates $x^a$. After substituting (4) and (5) into (1) and then integrating over the angular coordinates $(\theta, \varphi)$, one obtains the following two dimensional effective action

$$S = \frac{1}{2} \int d^2x \sqrt{-g} \left[ 1 + g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} R \phi^2 \right]$$

$$- \int d^2x \sqrt{-g} \phi^2 g^{ab} (D_a \Phi) \dagger D_b \Phi - \frac{1}{4} \int d^2x \sqrt{-g} \phi^2 F_{ab} F^{ab}. $$

(6)

The suitable ADM splitting of (1+1)-dimensional spacetime is given by

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta^2 & \beta \\ \beta & \gamma \end{pmatrix},$$

(7)
and the normal unit vector $n^a$ orthogonal to the hypersurfaces $x^0 = \text{const}$ reads

$$n^a = \left( \frac{1}{\alpha}, -\frac{\beta}{\alpha \gamma}\right). \quad (8)$$

In terms of this parametrization, the action (6) can be written as

$$S = \int d^2x L = \int d^2x \left[ \frac{1}{2} \sqrt{\gamma} \left\{ 1 - (n^a \partial_a \phi)^2 + \frac{1}{\gamma} (\phi')^2 - Kn^a \partial_a (\phi^2) \right\} + \frac{\alpha'}{\alpha \gamma} \partial_1 (\phi^2) + \alpha \sqrt{\gamma} \phi^2 \right\} + \frac{1}{\gamma} |D_1 \Phi|^2 \right\} + \frac{1}{2} \gamma \sqrt{\gamma} \phi^2 E^2], \quad (9)$$

where

$$K = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} n^a) = \frac{\dot{\gamma}}{2 \alpha \gamma} - \frac{\beta'}{\alpha \gamma} + \frac{\beta}{2 \alpha \gamma^2} \gamma', \quad (10)$$

$$E = \frac{1}{\sqrt{-g}} F_{01} = \frac{1}{\alpha \sqrt{\gamma}} (\ddot{A}_1 - A_0'), \quad (11)$$

and $\frac{\partial}{\partial x^0} = \partial_0$ and $\frac{\partial}{\partial x^1} = \partial_1$ are also denoted by an overdot and a prime, respectively.

Now the canonical conjugate momenta can be read off from the action (9). They are

$$p_\Phi = \sqrt{\gamma} \phi^2 (n^a D_a \Phi)^\dagger, \quad (12)$$

$$p_\Phi^\dagger = \sqrt{\gamma} \phi^2 (n^a D_a \Phi), \quad (13)$$

$$p_\phi = -\sqrt{\gamma} n^a \partial_a \phi - \sqrt{\gamma} K \phi, \quad (14)$$

$$p_\gamma = -\frac{1}{4 \sqrt{\gamma}} n^a \partial_a (\phi^2), \quad (15)$$

$$p_A = \phi^2 E. \quad (16)$$
Then the Hamiltonian can be calculated to be

\[ H = \int dx (p_\Phi \dot{\Phi} + p_{\Phi^\dagger} \dot{\Phi}^\dagger + p_\phi \dot{\phi} + p_\gamma \dot{\gamma} + p_A \dot{A} + L), \]

\[ = \int dx (\alpha H_0 + \beta H_1 + A_0 H_2), \]

where the constraints are explicitly given by

\[ H_0 = \frac{1}{\sqrt{\gamma} \phi^2} p_\Phi p_{\Phi^\dagger} - \frac{\sqrt{\gamma}}{2} \frac{(\phi')^2}{\phi} + \frac{\partial_1}{2} \left( \frac{\partial_1 (\phi^2)}{\phi} \right) + \frac{\phi^2}{\sqrt{\gamma}} |D_1 \Phi|^2 - \frac{2 \sqrt{\gamma}}{\phi} p_{\phi} p_{\gamma} + \frac{2 \gamma \sqrt{\gamma}}{\phi^2} p_{\gamma}^2 + \frac{2 \gamma}{\phi^2} p_A^2, \]

\[ H_1 = \frac{1}{\gamma} [p_\phi D_1 \Phi + p_{\Phi^\dagger} (D_1 \Phi)^\dagger] + \frac{1}{\gamma} p_{\phi} \phi' - \frac{1}{\gamma} p_{\gamma} \gamma', \]

\[ H_2 = -ie (p_\Phi \dot{\Phi} - p_{\Phi^\dagger} \dot{\Phi}^\dagger) - p_A. \]

Here for later convenience, let us derive equations of motion arising from the action (6):

\[-2\nabla_a \nabla_b \phi + \frac{2}{\phi} g_{ab} \nabla_c \phi + \frac{1}{\phi^2} g_{ab} \partial_c \phi \partial^c \phi - \frac{1}{\phi^2} g_{ab} \]

\[ = -g_{ab} E^2 + 2[(D_a \Phi)^\dagger D_b \Phi + (D_b \Phi)^\dagger D_a \Phi - g_{ab} (D_c \Phi)^\dagger D_c \Phi], \]

\[ \nabla_a \nabla_a \phi - \frac{1}{2} R \phi = \phi E^2 - 2\phi (D_a \Phi)^\dagger D_a \Phi, \]

\[ \nabla_a (\phi^2 E) = i e \phi^2 \sqrt{-g} \varepsilon_{ab} [\Phi^\dagger D^b \Phi - \Phi (D^b \Phi)^\dagger], \]

\[ D_a (\sqrt{-g} \phi^2 g^{ab} D_b \Phi) = 0, \]

where \( \varepsilon_{01} = -\varepsilon_{10} = +1. \)
3. Canonical Formalism near Apparent Horizons

Following the key observation mentioned in the introduction, we shall focus our thoughts on the canonical formalism in the vicinity of the horizons ¹. As a bonus of taking account of the spacetime region near the horizons, it turns out that the complicated constraints (18)-(20) can be cast into rather simple and technically tractable forms.

To fix the system of coordinates, let us introduce the two dimensional coordinates $x^a$ by

$$x^a = (x^0, x^1) = (v - r, r),$$  \hspace{1cm} (25)

where the advanced time coordinate is defined as $v = t + r^*$ with the tortoise coordinate $dr^* = \frac{dr}{-g_{00}}$. Since we wish to consider the Reissner-Nordstrom charged black hole, we shall fix the gauge freedoms corresponding to the two dimensional reparametrization invariances by the gauge conditions

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \frac{\beta^2}{\gamma} & \beta \\ \beta & \gamma \end{pmatrix},$$

$$= \begin{pmatrix} -(1 - \frac{2M}{r} + \frac{Q^2}{r^2}) & \frac{2M}{r} - \frac{Q^2}{r^2} \\ \frac{2M}{r} - \frac{Q^2}{r^2} & 1 + \frac{2M}{r} - \frac{Q^2}{r^2} \end{pmatrix},$$  \hspace{1cm} (26)

where the black hole mass $M$ and the electric charge $Q$ are now in general the function of the two dimensional coordinates $x^a$. Notice that we have not fixed the gauge symmetries completely because we wish to consider a dynamical black hole whose mass and charge change in time under an influence of motion of charged matter field. However, the remaining gauge freedoms are effectively fixed in making the assumptions of dynamical fields near the horizons later. Moreover, the $U(1)$

¹ P.Moniz has independently considered a similar model from a different motivation [19] (private communication).
gauge symmetries are fixed by the gauge conditions

\[ A_0 = A_1 = \frac{Q}{r}. \]  \hspace{1cm} (27)

From (25) and (26) the two dimensional line element takes a form

\[ ds^2 = g_{ab} dx^a dx^b, \]
\[ = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2}) dv^2 + 2 dv dr. \]  \hspace{1cm} (28)

This is precisely the charged Vaidya metric for the Reissner-Nordstrom black hole.

For a dynamical black hole, it is convenient to introduce the local horizon, i.e., the apparent horizon, instead of the event horizon since a genuine event horizon is a complicated global object. The apparent horizons are now defined as

\[ r_{\pm} = M \pm \sqrt{M^2 - Q^2}, \]  \hspace{1cm} (29)

where \( r_+ \) and \( r_- \) denote respectively the outer apparent horizon and the inner Cauchy horizon. We will assume that \( M > Q \) as an initial condition. As a familiar fact, in the extremal limit \( M = Q \) these two horizons coincide, and for \( M < Q \) they are absent and a naked singularity is visible to external observers.

Near the apparent horizons (29), (26) yields

\[ \alpha \approx \frac{1}{\sqrt{2}}, \quad \beta \approx 1, \quad \gamma = \frac{1}{\alpha^2} \approx 2, \]  \hspace{1cm} (30)

From now on we shall use \( \approx \) to indicate the equalities which hold approximately near the apparent horizons. Note that the dynamical degrees of freedom representing “graviton” \( \gamma \) are effectively fixed in (30). Also note that we are concerned with the spacetime regions where \( x^0 \) plays a role of time, so \( x^1 = r \) coordinate must take
the ranges of $0 < r < r_-$ and $r_+ < r < \infty$. An analogous formulation which holds in the range of $r_- < r < r_+$ can be easily built in a similar way to the reference [13]. Now let us make physically plausible assumptions near the apparent horizons

$$\Phi \approx \Phi(v), \ M \approx M(v), \ Q \approx Q(v), \ \phi \approx r. \quad (31)$$

A consistency of the assumptions (31) with the field equations (21)-(24) will be discussed in the end of this section. Given the assumptions (31), the canonical conjugate momenta (12)-(16) become

$$p_\Phi \approx \phi^2 (\partial_v \Phi^\dagger - ieA_0 \Phi^\dagger),$$
$$p_{\Phi^\dagger} \approx \phi^2 (\partial_v \Phi + ieA_0 \Phi),$$
$$p_\phi \approx 1 + \frac{1}{2} (\partial_v M - \frac{Q}{r} \partial_v Q) + \frac{3}{2} (\frac{M}{r} + \frac{Q^2}{r^2}),$$
$$p_\gamma \approx \frac{1}{4} \phi,$$
$$p_A \approx Q. \quad (32)$$

It is remarkable that the Hamiltonian constraint $H_0 = 0$ becomes proportional to the supermomentum constraint $H_1 = 0$ in the vicinity of the apparent horizons

$$\sqrt{2} H_0 \approx 2 H_1,$$
$$\approx \frac{2}{\phi^2} p_\Phi p_{\Phi^\dagger} - 2 p_\phi - 2 + \frac{5}{\phi} - \frac{Q^2}{\phi^2}, \quad (33)$$

since the time translation is frozen at the apparent horizons in this coordinate system [12]. In addition, along with $p_A \approx Q$ in (32) the $U(1)$ constraint $H_2 = 0$ produces the equation

$$\partial_v Q \approx - ie (\Phi p_\Phi - \Phi^\dagger p_{\Phi^\dagger}). \quad (34)$$

This will be used when evaluating the expectation value of $\partial_v Q$ in section 5. In the formalism developed so far, the dynamical degrees of freedom are contained entirely
in the matter field, and the other fields are determined by solving the constraint equations by help of the gauge conditions and the assumptions (31). This reduction of the dynamical degrees of freedom comes from the gauge symmetries and the assumptions adopted near the apparent horizons.

Before closing this section, we would like to discuss the consistency of the assumptions (31) with equations of motion (21)-(24). After a tedious but straightforward calculation, if we assume (31) to hold true except $\Phi$, equations of motion (21)-(24) lead to the following equations:

$$\partial_r \Phi \approx 0,$$

$$\left(D_v \Phi \right)^\dagger D_v \Phi \approx \frac{1}{2r} \left(\frac{\partial_v M}{r} - \frac{Q \partial_v Q}{r^2}\right),$$

$$\partial_v Q \approx i e \phi^2 [\Phi^\dagger D_v \Phi - \Phi (D_v \Phi)^\dagger],$$

$$2r^2 \partial_r \partial_v \Phi + 2r \partial_v \Phi + i e Q \Phi \approx 0,$$

$$\partial_r \partial_v \Phi \approx \partial_v \partial_r \Phi.$$

As shown in the reference [11], at least in the case of the Schwarzschild black hole and the neutral scalar matter field, the consistency is explicitly proved by finding the solution

$$\Phi(v) \approx \int^v \frac{1}{2M(v)} \sqrt{\frac{\partial_v M}{2}},$$

which implies that the increase of the black hole mass, $\partial_v M > 0$, is classically allowed, but the loss of the black hole mass, $\partial_v M < 0$, that is, the Hawking radiation, is classically forbidden and occurs only through the quantum-mechanical tunneling effect. Unfortunately, in the Reissner-Nordstrom geometry with the
charged scalar matter field it seems to difficult to find such an analytic solution owing to the highly nonlinear structure of (36) and (37). However, we believe that our assumptions (31) are not in unjustified leap in the dark from physical viewpoint and that the physically plausible results which will be obtained in later sections would support the assumptions.

4. Hawking Radiation

In the previous section, we have constructed the canonical formalism holding near the apparent horizons. We are now ready to perform the canonical quantization of it. After completing this, as its concrete application we will investigate the Hawking radiation by purely quantum-mechanical treatment.

For simplicity, in this section for simplicity let us consider the case of the neutral matter field $\Phi^\dagger = \Phi$, for which the constraint $H_2 = 0$ makes no sense since now the black hole charge $Q$ takes a fixed constant. By imposing the constraint (33) as an operator equation one obtains the Wheeler-DeWitt equation

$$\left(-\frac{1}{\phi^2} \frac{\partial^2}{\partial\phi^2} + 2i \frac{\partial}{\partial\phi} + \frac{M}{\phi} + \frac{Q^2}{\phi^2}\right)\Psi = 0.$$  \tag{41}

The simplest way to find a special solution of this Wheeler-DeWitt equation is to use the method of separation of variables. The result reads

$$\Psi = (Be^{-\sqrt{A}\phi} + Ce^{\sqrt{A}\phi})e^{\frac{i}{4}(\phi + \frac{2A-3Q^2}{\phi})},$$  \tag{42}

where $A$, $B$, and $C$ are integration constants. Without losing a generality, we shall choose the boundary condition $B = 0$. Here if one defines an expectation value

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1 We have changed the coefficient in front of the matter action from $\frac{1}{16\pi}$ to $\frac{1}{8\pi}$ in (1) in the case of the neutral scalar field, which gives rise to the modification from $\frac{1}{2}\sqrt{\rho \phi^2} \Phi^\dagger$ to $\frac{1}{8\pi}\rho \phi^2$ in (33).
\[ \langle \mathcal{O} \rangle = \frac{< \partial_v M >}{< r_\pm^2 >} \]

It is of interest to notice that different choices of the integration constant \( A \) lead to different physical pictures. Namely, a negative choice \( A = -k_1^2 \) gives us a physical picture of the absorption of the external matters by a black hole

\[ \langle \partial_v M \rangle = -\frac{k_1^2}{< r_\pm^2 >} \]  

with the wave function having a form of the scalar wave propagating into black hole in the superspace

\[ \Psi = C e^{-i|k_1|\Phi(v)+\frac{3}{4}(\phi-\frac{2k_1^2+3Q^2}{\phi})}. \]

On the other hand, a positive constant \( A = k_2^2 \) yields the Hawking radiation

\[ \langle \partial_v M \rangle = -\frac{k_2^2}{< r_\pm^2 >}. \]

for which this time the physical state has an exponentially damping-like form in the classically forbidden region showing the quantum tunneling

\[ \Psi = C e^{-i|k_2|\Phi(v)+\frac{1}{4}(\phi+\frac{2k_2^2-3Q^2}{\phi})}. \]

Moreover, if one assumes that the black hole charge is zero, the result for the outer apparent horizon exactly equals to the result calculated by Hawking in the

\[ \text{Here we have explicitly written the definition in the case of the complex scalar field for later convenience. Specification to the case of the real scalar field is obvious.} \]
semiclassical approach [1]

\[ < \partial_v M > = -\frac{k_2^2}{4 < M^2 >}. \] \hspace{1cm} (49)

It is worth pointing out the difference between the Hawking formulation and the present one. In the Hawking’s semiclassical approach the gravitational field is fixed as a classical background and only the matter field is treated to be quantum-mechanical. By contrast, our present formulation is purely quantum-mechanical in that both the gravitational field and the matter field are considered to be quantum fields.

5. Cosmic Censorship in Quantum Gravity

Being encouraged with success of deriving a physically interesting results with respect to the Hawking radiation by the quantum gravity in the previous section, we now turn our attention to the problem of cosmic censorship. To begin with, in order to clarify the contents of the problem which we wish to solve, let us remind you of the physical setting of the problem.

At the outset, suppose that there is a stable Reissner-Nordstrom electrically charged black hole satisfying the relation \( Q < M \) which means that this black hole has two event horizons at \( r = r_{\pm} = M \pm \sqrt{M^2 - Q^2} \). Then we would increase the black hole charge \( Q \) faster than its mass \( M \) by sending a lightlike flux of charged scalar matters with \( q > m \) to this black hole from the exterior. Here \( q \) and \( m \) denote respectively the charge and the energy of these scalar matters. As a result, if the black hole would become to have the charge larger than its mass, i.e., \( Q > M \), cosmic censorship would be violated in this process since for \( Q > M \) there is no more horizon so that a naked singularity is visible to external observers. Roughly speaking, this would be a test that one tries to destroy the outer horizon of a black hole by forcing it to meet the inner horizon. As mentioned in section 1, it was
proved that cosmic censorship could never be violated this way in classical theory of general relativity [9]. Our aim in this section is to examine whether quantum gravity would modify this classical result or not.

Since we have specified the contents of the problem, we challenge to solve the problem by applying the quantum gravity developed in previous sections. Since the charged matter field plays an essential role here, one has to solve the Wheeler-DeWitt equation arising from (33) which is given by

$$\left(-\frac{2}{\phi^2} \frac{\partial^2}{\partial \phi \partial \Phi^\dagger} + 2i \frac{\partial}{\partial \phi} - 2 + \frac{5M}{\phi} - \frac{Q^2}{\phi^2}\right) \Psi = 0. \quad (50)$$

Again a use of the method of separation of variables yields a special solution

$$\Psi = C e^{A \Phi^\dagger + B \Phi + i\left(\frac{AB + B^2}{\phi} - \phi + \frac{5}{2} M \log \phi\right)}. \quad (51)$$

As a simple check, it is valuable to calculate $< p_A >$ by inserting the physical state (51) into the definition of an expectation value (43). The result is $< p_A > = Q$, which is consistent with the last equation in (32).

Next let us evaluate the expectation value of the change rate of charge $< \partial_v Q >$. After a simple calculation, we obtain

$$< \partial_v Q > = e (|A|^2 - |B|^2). \quad (52)$$

First of all, this equation indicates that if $\Phi^\dagger$ creates the particle with a charge $e$, $\Phi$ creates the particle with the opposite charge $-e$ in a language of the field theory. The more important point is that the right-hand side of (52) is a certain constant so the black hole charge $Q$ increases or decreases at the constant rate according

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1 Even if we introduce the weight factor $e^{-K|\Phi|^2}$ instead of $e^{-|\Phi|^2}$ in the definition of an expectation value (43), the coefficient $K$ can be “renormalized” into the electric charge by $e_R = \frac{e}{K}$. 

to the motion of external charged matters across the apparent horizons. At this point, we assume that \( e \gg m \) and \(|A| > |B|\), which corresponds to the physical setting described above that charged matters with a large \( \frac{q}{m} \) ratio flow in a black hole to urge the black hole charge rather than its mass to increase.

In addition, from (32) or (33), we obtain the equations

\[
< \partial_v r_\pm > = -\frac{4AB}{< r^2_\pm >}.
\]

Then adding or subtracting these equations each other gives

\[
< \partial_v M > = -4AB \frac{2M^2 - Q^2}{Q^4},
\]

\[
< \partial_v (r_+ - r_-) > = 16AB \frac{M \sqrt{M^2 - Q^2}}{Q^4},
\]

where \( r_\pm = M \pm \sqrt{M^2 - Q^2} \) is used. As told in the above, since we have in mind a physical situation where external charged matters come in a black hole across the apparent horizons, the black hole mass must increase as time passes by. This demands us to choose the product of integration constants \( AB \) to be a negative constant, e.g., \(-\frac{k^2}{16}\) in (54). Then (55) becomes

\[
< \partial_v (r_+ - r_-) > = -k_1^2 \frac{M \sqrt{M^2 - Q^2}}{Q^4}.
\]

Eq.(56) gives very important informations about the behavior of the outer apparent horizon and the inner Cauchy horizon. Namely, as external matters are accumulated into a black hole, the inner Cauchy horizon approaches the outer apparent horizon but the relative approaching speed of the horizons, which is quantitatively controlled by the right-hand side of this equation, decreases gradually. Once the
extremal limit $Q = M$ is realized, its relative speed vanishes so that both horizons coalesce, and afterward expand together. The reason why the expansion of the coalesced horizons does not stop is that in this model external charged matters constantly flow into a black hole as understood from (52). Note that the above picture is consistent with (54) where even in the extremal limit the increase rate of the black hole mass is positive. One of the most important points here is that the inner Cauchy horizon never goes beyond the outer apparent horizon, which means that cosmic censorship remains true also in quantum gravity as in classical gravity.

6. Conclusion

In this article, we have investigated weak cosmic censorship from the viewpoint of quantum gravity, and reached the result that cosmic censorship is not violated as in classical theory of general relativity. Although we have obtained the same conclusion as the classical gravity, the intermediate way of calculations are quite different between them. We should clarify their relationship in the future. It is true that cosmic censorship cannot be proved in general by studying a specific model, but we believe that the physical setting that we have examined is so fundamental that the results obtained here would further the more advanced study.

In previous works [11-14], we have so far applied the quantum gravity holding near the horizons and/or the singularities in black holes to several problems of quantum black holes, by which we have gained the physically interesting results. Since the modern unsolved problems associated with quantum black holes such as the black hole entropy and the fate of the endpoint of black hole evaporation e.t.c. are all related to the properties of the horizons and the singularities, it seems to be natural that our formalism has had a wide applications toward these problems. In other words, there should be a hopeful possibility that the quantum physics in the vicinity of the horizons and the singularities determine the physical properties of quantum black holes essentially. Incidentally, the idea that the (stretched) horizon
would play an important role in understanding the phenomena about black holes was previously advocated as the Membrane Paradigm [20-22]. The results of this paper might partly support this conjecture.

References

[1] S.W.Hawking, Comm. Math. Phys. 43 (1975) 199.

[2] D.Page, in Proceedings of the Fifth Canadian Conference on General Relativity and Relativistic Astrophysics, edited by R.B.Mann et al. (World Scientific, Singapore, 1994).

[3] J.D.Bekenstein, Phys. Rev. D7 (1973) 2333.

[4] R.Penrose, Rev. del Nuovo Cimento 1 (1969) 252.

[5] A.Hosoya, Class. Quant. Grav. 12 (1995) 2967.

[6] A.Hosoya and I.Oda, “Black Hole Singularity and Generalized Quantum Affine Parameter”, TIT/HEP-334/COSMO-73, EDO-EP-5, gr-qc/9605069.

[7] R.Penrose, in Theoretical Principles in Astrophysics and General Relativity, edited by N.R.Lebovitz et al. (University of Chicago Press, Chicago, 1978); W.Israel, Can. J. Phys. 64 (1986) 120.

[8] S.W.Hawking, “Nature of Space and Time”, hep-th/9409193.

[9] R.Wald, Ann. Phys. 82 (1974) 548.

[10] A.Tomimatsu, Phys. Lett. B289 (1992) 283.

[11] A.Hosoya and I.Oda, Prog. Theor. Phys. 97 (1997) 233.

[12] I.Oda, “Evaporation of Three Dimensional Black Hole in Quantum Gravity”, EDO-EP-9, gr-qc/9703053.

[13] I.Oda, “Mass Inflation in Quantum Gravity”, EDO-EP-8, gr-qc/9701058.
[14] I.Oda, “Quantum Instability of Black Hole Singularity in Three Dimensions”, EDO-EP-10, gr-qc/9703050.

[15] P.Hajicek, Phys. Rev. D30 (1984) 1178; P.Thomi, B.Isaak, and P.Hajicek, Phys. Rev. D30 (1984) 1168.

[16] P.A.M.Dirac, Lectures on Quantum Mechanics (Yeshiva University, 1964).

[17] P.Arnowitt, S.Deser, and C.W.Misner, in Gravitation: An Introduction to Current Research, edited by L.Witten (Wiley, New York, 1962).

[18] C.W.Misner, K.S.Thorne, and J.A.Wheeler, Gravitation (Freeman, 1973).

[19] P.Moniz, preprint to appear.

[20] K.S.Thorne, R.H.Price and D.A.Macdonald, Black Hole: The Membrane Paradigm (Yale University Press, 1986).

[21] G.’t Hooft, Nucl. Phys. B335 (1990) 138; Phys. Scripta T15 (1987) 143; ibid. T36 (1991) 247.

[22] I.Oda, Int. J. Mod. Phys. D1 (1992) 355; Phys. Lett. B338 (1994) 165; Mod. Phys. Lett. A10 (1995) 2775.