PARAMETER-FREE SUPERCONVERGENT $H(\text{div})$-CONFORMING HDG METHODS FOR THE BRINKMAN EQUATIONS.

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Abstract. In this paper, we present new parameter-free superconvergent $H(\text{div})$-conforming HDG methods for the Brinkman equations on both simplicial and rectangular meshes. The methods are based on a velocity gradient-velocity-pressure formulation, which can be considered as a natural extension of the $H(\text{div})$-conforming HDG method (defined on simplicial meshes) for the Stokes flow [Math. Comp. 83(2014), pp. 1571-1598].

We obtain optimal $L^2$-error estimate for the velocity in both the Stokes-dominated regime (high viscosity/permeability ratio) and Darcy-dominated regime (low viscosity/permeability ratio). We also obtain superconvergent $L^2$-estimate of one order higher for a suitable projection of the velocity error in the Stokes-dominated regime. Moreover, thanks to $H(\text{div})$-conformity of the velocity, our velocity error estimates are independent of the pressure regularity. Furthermore, we provide a discrete $H^1$-stability result of the velocity field, which is essential in the error analysis of the natural generalization of these new HDG methods to the incompressible Navier-Stokes equations.

Preliminary numerical results on both triangular and rectangular meshes in two dimensions confirm our theoretical predictions.

Key words. HDG, $H(\text{div})$-conforming, superconvergence, Brinkman

AMS subject classifications. 65N30, 65M60, 35L65

1. Introduction. In this paper, we devise superconvergent $H(\text{div})$-conforming hybridizable discontinuous Galerkin (HDG) method for the following Brinkman equations in velocity gradient-velocity-pressure formulation:

\[
\begin{align*}
L &= \nabla u \quad \text{in } \Omega, \quad (1.1a) \\
-\nu \nabla \cdot L + \gamma u + \nabla p &= f \quad \text{in } \Omega, \quad (1.1b) \\
\nabla \cdot u &= g \quad \text{in } \Omega, \quad (1.1c) \\
u(I_d - n \otimes n)u &= 0 \quad \text{on } \partial \Omega, \quad (1.1d) \\
u(I_d - n \otimes n)u &= 0 \quad \text{on } \partial \Omega, \quad (1.1e) \\
\int_{\Omega} p &= 0, \quad (1.1f)
\end{align*}
\]

where $L$ is the velocity gradient, $u$ is the velocity, $p$ is the pressure, $\nu$ is the effective viscosity constant, $\gamma \in L^\infty(\Omega)^{d \times d}$ is inverse of the permeability tensor, and $f \in L^2(\Omega)^d$ is the external body force. The domain $\Omega \subset \mathbb{R}^d$ is a polygon ($d = 2$) or polyhedron ($d = 3$).

One challenging aspect of numerical discretization of the Brinkman equations is the construction of stable finite element methods in both Stokes-dominated and Darcy-dominated regimes. We refer to such methods as uniformly stable methods. Uniformly stable methods for the Brinkman equations have been extensively studied.
for the classical velocity-pressure formulation, including the nonconforming methods with an \( H(\text{div}) \)-conforming velocity field \([26, 32, 34, 19]\), the conforming methods \([34, 21]\), the stabilized methods \([34, 3, 21]\), the \( H(\text{div}) \)-conforming discontinuous Galerkin method \([22]\), and the hybridized \( H(\text{div}) \)-conforming discontinuous Galerkin method \([23]\), and for other alternative formulations, including the vorticity-velocity-pressure formulation \([33, 1]\), the pseudostress-based formulation \([17]\), and a dual-mixed formulation \([20]\).

In this paper, we propose and study a class of high-order, parameter-free, \( H(\text{div}) \)-conforming HDG method for the Brinkman equations \((1.1)\) on both simplicial and rectangular meshes. This is the first HDG method for the Brinkman equations based on a velocity gradient-velocity-pressure formulation. Our method can be considered as a natural, stable extension to the Brinkman equations of the high-order, parameter-free, \( H(\text{div}) \)-conforming HDG method for the Stokes problem on simplicial meshes \([14]\). Three distinctive properties of the method make it attractive. Firstly, our method provides optimal error estimate in \( L^2 \)-norms for the velocity that is robust with respect to viscosity/permeability ratio \( \nu/\gamma \) (Theorem 2.3, Corollary 2.4), and superconvergent error estimate in the \( L^2 \)-norm of one order higher for a suitable projection of the velocity error (under a regularity assumption on the dual problem). To the best of our knowledge, this is the first superconvergent velocity estimate for the Brinkman equations. Secondly, thanks to \( H(\text{div}) \)-conformity of the velocity, our velocity error estimates are independent of the pressure regularity (see Corollary 2.4 and Theorem 2.5). Such pressure-robustness property is highly appreciated for incompressible flow problems \([24, 25]\). Finally, our error analysis, which is quite different from and more straightforward than that in \([14]\) for the Stokes flow, is based on a so-called discrete \( H^1 \)-stability result (see Theorem 2.1), which is the essential ingredient in the analysis of velocity gradient-velocity-pressure HDG formulation of the incompressible Navier-Stokes equations. We specifically remark that no stabilization parameter enters in our method, which has to be compared with the hybridized \( H(\text{div}) \)-conforming discontinuous Galerkin method \([23]\) in the classical velocity-pressure formulation, where Nitsche’s penalty method is used to impose tangential continuity of the velocity field and the stabilization parameter needs to be “sufficiently large”.

The organization of the paper is as follows. In Section 2, we introduce the parameter-free \( H(\text{div}) \)-conforming HDG method and give the main results on a priori error estimates. In Section 3, we prove our main results in Section 2. In Section 4, we discuss the hybridization of the \( H(\text{div}) \)-conforming HDG method. In Section 5, we provide preliminary two-dimensional numerical experiments on triangular and rectangular meshes to validate our theoretical results. We end in Section 6 with some concluding remarks.

2. Main results: Superconvergent \( H(\text{div}) \)-conforming HDG. In this section, we first introduce the notation that will be used throughout the paper, and then present the finite element spaces that define the \( H(\text{div}) \)-conforming HDG methods. We conclude with an a priori error estimates along with a key inequality that we call discrete \( H^1 \)-stability.

2.1. Meshes and trace operators. We denote by \( \mathcal{T}_h := \{ K \} \) (the mesh) a shape-regular conforming triangulation of the domain \( \Omega \subset \mathbb{R}^d \) into affine-mapped simplices (triangles if \( d = 2 \), tetrahedron if \( d = 3 \)) or hypercubes (squares if \( d = 2 \), cubes if \( d = 3 \)), and by \( \mathcal{E}_h \) (the mesh skeleton) the set of facets \( F \) (edges if \( d = 2 \), faces if \( d = 3 \)) of the elements \( K \in \mathcal{T}_h \). Let \( \mathcal{F}(K) \) denote the set of facets \( F \) of the
element $K$. We set $h_F := \text{diam}(F), h_K := \text{diam}(K)$ and $h := \max_{K \in \mathcal{T}_h} h_K$.

Let $K$ be the reference element ($d$-dimensional simplex or hypercube), and $F$ be the reference facet ($d-1$-dimensional simplex or hypercube). We denote $\Phi_K : K \to K$ and $\Phi_F : F \to F$ as the associated affine mappings.

For a $d$-dimensional vector-valued function $\mathbf{v}$ on an element $K \subset \mathbb{R}^d$ with sufficient regularity, we denote by

$$
\text{tr}_t^F(\mathbf{v}) := (\mathbf{v} - (\mathbf{v} \cdot \mathbf{n}_F) \mathbf{n}_F)|_F \quad \text{and} \quad \text{tr}_n^F(\mathbf{v}) := (\mathbf{v} \cdot \mathbf{n}_F) \mathbf{n}_F|_F
$$

(2.1)

the tangential and normal traces of $\mathbf{v}$ on the facet $F \in \mathcal{F}(K)$, where $\mathbf{n}_F$ is the unit normal vector to $F$. Note that the above trace operators are independent of the direction of the normal $\mathbf{n}_F$. Whenever there is no confusion, we suppress the superscript and denote $\text{tr}_t(\mathbf{v})$ and $\text{tr}_n(\mathbf{v})$ as the related tangential and normal traces, respectively. With an abuse of notation, we also denote

$$
\text{tr}_t(\mathbf{v}) := (\mathbf{v} - (\mathbf{v} \cdot \mathbf{n}_F) \mathbf{n}_F)|_F \quad \text{and} \quad \text{tr}_n(\mathbf{v}) := (\mathbf{v} \cdot \mathbf{n}_F) \mathbf{n}_F|_F
$$

for a $d$-dimensional vector-valued function $\mathbf{v}$ on a facet $F \subset \mathbb{R}^{d-1}$ with sufficient regularity.

2.2. The finite element spaces. Now, we define the finite element spaces associated with the mesh $\mathcal{T}_h$ and mesh skeleton $\mathcal{E}_h$ via appropriate mappings (cf. [6]) from (polynomial) spaces on the reference elements.

We use the following mapped finite element spaces on the mapped element $K$ and facet $F$:

$$
\mathcal{S}^{\text{row}}(K) := \{ \mathbf{v} \in L^2(K)^d : \mathbf{v} = \frac{1}{\det \Phi_K} \Phi'_K \mathbf{v} \circ \Phi_K^{-1}, \mathbf{v} \in \mathcal{S}^{\text{row}}(K) \}, \quad (2.2a)
$$

$$
\mathcal{V}(K) := \{ \mathbf{v} \in L^2(K)^d : \mathbf{v} = \frac{1}{\det \Phi_F} \Phi'_F \mathbf{v} \circ \Phi_F^{-1}, \mathbf{v} \in \mathcal{V}(K) \}, \quad (2.2b)
$$

$$
\mathcal{Q}(K) := \{ q \in L^2(K) : q = q \circ \Phi_K^{-1}, q \in \mathcal{Q}(K) \}, \quad (2.2c)
$$

$$
\mathcal{M}(F) := \{ \mathbf{v} \in L^2(F)^d : \mathbf{v} = \mathbf{v} \circ \Phi_F^{-1}, \mathbf{v} \in \mathcal{M}(F) \}. \quad (2.2d)
$$

Here $\Phi_K$ and $\Phi_F$ are the affine mappings introduced above, and $\Phi'_K$ is the Jacobian matrix of the mapping $\Phi_K$. Note that the vector spaces in (2.2a) and (2.2b) are obtained from the well-known Piola transformation which preserve normal continuity (cf. [16]).

The polynomial spaces on the reference elements are given in Table 2.1.

| element | $\mathcal{S}^{\text{row}}(K)$ | $\mathcal{V}(K)$ | $\mathcal{Q}(K)$ | $\mathcal{M}(F)$ |
|---------|-------------------------------|------------------|------------------|------------------|
| simplex | $\mathcal{P}_k(K)^d$         | $\mathcal{RT}_k(K)$ | $\mathcal{P}_k(K)$ | $\mathcal{P}_k(F)^d$ |
| hypercube | $\mathcal{BDM}_k(K)$ | $\mathcal{BDFM}_k(K)$ | $\mathcal{P}_k(K)$ | $\mathcal{P}_k(F)^d$ |

Here we denote $\mathcal{P}_k(D)$ and $\mathcal{P}_k(D)$ as the polynomials of degree no greater than $k$, and homogeneous polynomials of degree $k$, respectively, on the domain $D$. The vector space $\mathcal{RT}_k(K)$ on the reference simplex is the following Raviart-Thomas-Nédélec space, see [28] [27],

$$
\mathcal{RT}_k(K) := \mathcal{P}_k(K)^d \oplus x \mathcal{P}_k(K).
$$
the vector space $BDM_k(K)$ on the reference hypercube is the following Brezzi-Douglas-Marini space, see \cite{BDM} \cite{Douglas} \cite{FortinMarini}.

$$BDM_k(K) := \left\{ \begin{array}{ll}
P_k(K)^d \oplus \nabla \times \{ x y^{k+1}, y x^{k+1} \} & \text{if } d = 2, \\
P_k(K)^d \oplus \nabla \times \left\{ \begin{array}{l}
x \bar{P}_k(y, z) (y \nabla z - z \nabla y), \\
y \bar{P}_k(z, x) (z \nabla x - x \nabla z), \\
z \bar{P}_k(x, y) (x \nabla y - y \nabla x) \end{array} \right\} & \text{if } d = 3, \\
\end{array} \right.$$ 

and the vector space $BDFM_k(K)$ on the reference hypercube is the following Brezzi-Douglas-Fortin-Marini space, see \cite{BDFM}.

$$BDFM_k(K) := \left\{ \begin{array}{ll}
P_k(K)^d \oplus \left[ \begin{array}{c}
x \bar{P}_k(K) \\
y \bar{P}_k(K) \\
z \bar{P}_k(K) \end{array} \right] & \text{if } d = 2, \\
P_k(K)^d \oplus \left[ \begin{array}{c}
x \bar{P}_k(K) \\
y \bar{P}_k(K) \\
z \bar{P}_k(K) \end{array} \right] & \text{if } d = 3. \\
\end{array} \right.$$ 

Next, for the vector-valued finite element space $\mathcal{S}^{row}(K)$ given in \cite{2.2a}, we denote

$$\mathcal{S}(K) := [\mathcal{S}^{row}(K)]^d$$

as the tensor-valued space such that each of whose row is the space $\mathcal{S}^{row}(K)$.

We use the following finite element spaces on the mesh $\mathcal{T}_h$ and mesh skeleton $\mathcal{E}_h$ to define the $H(\text{div})$-conforming HDG method in the next section.

$$\mathcal{S}_h := \{ g \in L^2(\mathcal{T}_h)^{d \times d} : g|_K \in \mathcal{S}(K), \ K \in \mathcal{T}_h \} \tag{2.4a}$$

$$V_h := \{ \mathbf{v} \in L^2(\mathcal{T}_h)^d : \mathbf{v}|_K \in \mathbf{V}(K), \ K \in \mathcal{T}_h \}, \tag{2.4b}$$

$$V^\text{div}_h := \{ \mathbf{v} \in V_h : \mathbf{v} \in H(\text{div}; \Omega) \}, \tag{2.4c}$$

$$V^\text{div}_h(0) := \{ \mathbf{v} \in V^\text{div}_h : \text{tr}_n(\mathbf{v})|_{\partial \Omega} = 0 \}, \tag{2.4d}$$

$$Q_h := \{ q \in L^2(\mathcal{T}_h) : q|_K \in \mathcal{Q}(K), \ K \in \mathcal{T}_h \}, \tag{2.4e}$$

$$Q_h := \{ q \in Q_h : (q, 1)|_{\mathcal{E}_h} = 0 \}, \tag{2.4f}$$

$$M_h := \{ \hat{\mathbf{v}} \in L^2(\mathcal{E}_h)^d : \hat{\mathbf{v}}|_F \in M(F), \ F \in \mathcal{E}_h \}, \tag{2.4g}$$

$$M_h(0) := \{ \hat{\mathbf{v}} \in M_h : \hat{\mathbf{v}}|_F = 0, \ F \in \mathcal{E}_h \}, \tag{2.4h}$$

$$M_h^1 := \{ \hat{\mathbf{v}} \in M_h : \text{tr}_n(\hat{\mathbf{v}})|_{\partial \Omega} = 0 \}. \tag{2.4i}$$

2.3. The $H(\text{div})$-conforming HDG method. Now, we are ready to present the $H(\text{div})$-conforming HDG method for the Brinkman equations \cite{14}.

It is defined as the unique element $(L^h, u^h, p^h, \nu g^h)^{\mathcal{T}_h} \in \mathcal{S}_h \times V^\text{div}_h(0) \times Q_h \times M_h^1(0)$ such that the following weak formulation holds:

$$\begin{align}
(L^h, \nu g^h)^{\mathcal{T}_h} - (\nabla u^h, \nu g^h)^{\mathcal{T}_h} + \langle \mathrm{tr}_t(u^h) - \hat{u}_t^h, \mathrm{tr}_t(\nu g^h n) \rangle_{\partial \mathcal{T}_h} &= 0, \\
(\nu L^h, \nabla v^h)^{\mathcal{T}_h} - \langle \mathrm{tr}_t(\nu L^h n), \mathrm{tr}_t(v^h) - \hat{v}_t^h \rangle_{\partial \mathcal{T}_h} &= 0, \\
-(p^h, \nabla \cdot v^h)^{\mathcal{T}_h} + (\gamma u^h, v^h)^{\mathcal{T}_h} = (f, v^h)^{\mathcal{T}_h}, & \text{if } \nabla \cdot u^h = (g, q^h)^{\mathcal{T}_h}, \tag{2.5c}
\end{align}$$
for all \((g^h, v^h, q^h, \tilde{u}^h) \in \mathcal{S}_h \times V_h^{\text{div}} \times \hat{Q}_h \times M^I_h(0)\). Here we write \((\eta, \zeta)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \eta, \zeta \rangle_K\), where \((\eta, \zeta)_K\) denotes the integral of \(\eta \zeta\) over the domain \(K \subset \mathbb{R}^n\). We also write \(\langle \eta, \zeta \rangle_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h} \langle \eta, \zeta \rangle_K\), where \(\langle \eta, \zeta \rangle_K := \sum_{F \in \mathcal{F}(K)} \langle \eta, \zeta \rangle_F\), and \(\langle \eta, \zeta \rangle_F\) denotes the integral of \(\eta \zeta\) over the facet \(F \subset \mathbb{R}^{n-1}\) and where \(\mathcal{F}_h := \{\partial K : K \subset \mathcal{T}_h\}\). When vector-valued or tensor-valued functions are involved, we use similar notation.

As mentioned in the Introduction, we postpone to Section 4 to discuss the efficient implementation of the above method via hybridization. Here we focus on the presentation of its (superconvergent) a priori error estimates.

### 2.3.1. Discrete \(H^1\)-stability

We first obtain a key result, which will be used to prove the error estimates presented in the next subsection, on the control of a \(H^1\) of the elements \(K\) projection, see [28, 27]; when \(\delta^h = \{\partial K : K \subset \mathcal{T}_h\}\). We compare the numerical solution against the exact solution for all \((g^h, v^h, q^h, \hat{u}^h) \in V_h^{\text{div}} \times M^I_h\), by the \(L^2\)-norm of a tensor field.

For a pair \((u^h, \hat{u}^h) \in V_h^{\text{div}} \times M^I_h\), we denote its discrete \(H^1\)-norm as follows:

\[
\left\| (u^h, \hat{u}^h) \right\|_{1, \mathcal{T}_h} := \left( \sum_{K \in \mathcal{T}_h} \left\| \nabla u^h \right\|_K^2 + \sum_{F \in \mathcal{E}_h} h_F^{-1} \left\| \text{tr}_t(u^h) - \hat{u}^h_t \right\|_F^2 \right)^{1/2}
\]  

(2.6)

**Theorem 2.1** (Discrete \(H^1\)-stability). Let \((r, z^h, \hat{z}^h) \in L^2(\mathcal{T}_h)^d \times V_h^{\text{div}} \times M^I_h\) satisfy the following equation

\[
(r, g^h)_{\mathcal{T}_h} - (\nabla z^h, g^h)_{\mathcal{T}_h} + \langle \text{tr}_t(z^h) - \hat{z}^h_t, \text{tr}_t(g^h \cdot n) \rangle_{\partial \mathcal{T}_h} = 0
\]  

(2.7)

for all \(g^h \in \mathcal{S}_h\), then we have

\[
\left\| (z^h, \hat{z}^h) \right\|_{1, \mathcal{T}_h} \leq C \| r \|_{\mathcal{T}_h},
\]

(2.8)

with a constant \(C\) depends only on the polynomial degree \(k\) and the shape-regularity of the elements \(K \subset \mathcal{T}_h\).

### 2.3.2. A priori error estimates

We are now ready to present the a priori error estimates for the method (2.5). We compare the numerical solution against the exact solution for all \((g^h, v^h, q^h, \hat{u}^h) \in V_h^{\text{div}} \times M^I_h\), by the \(L^2\)-norm of a tensor field.

**The projections.** In the following, we denote \(P_3, P_V, P_Q, P_M\) to be the \(L^2\)-projections onto \(\mathcal{S}_h, V_h, \hat{Q}_h\), and \(M^I_h\) respectively. Moreover, we set

\[
e_L = P_3 L - I^h, \quad e_u = \Pi_V u - u^h, \quad e_p = P_Q p - p^h, \quad e_{\hat{u}} = P_M u - \hat{u}^h, \quad e_p = P_Q p - p^h, \quad e_{\hat{u}} = \text{tr}_t(u) - P_M u.
\]

(2.9a)

Here the projection \(\Pi_V u \in V_h\) whose restriction to an element \(K\) is the unique function in \(V(K)\) such that

\[
(\Pi_V u, v)_K = (u, v)_K, \quad \forall v \in \nabla \cdot \mathcal{S}(K),
\]

(2.9a)

\[
\langle \text{tr}_n(\Pi_V u), \text{tr}_n(\hat{v}) \rangle_F = \langle \text{tr}_n(u), \text{tr}_n(\hat{v}) \rangle_F, \quad \forall \hat{v} \in M(F), \quad \forall F \in \mathcal{T}(K).
\]

(2.9b)

Recall that the spaces \(V(K), M(F), \) and \(\mathcal{S}(K)\) are defined in (2.2), and (2.3), respectively.

When \(K\) is a simplex, the above projection is nothing but the Raviar-Thomas projection, see [28, 27]; when \(K\) is a hypercube, the above projection is nothing but the Brezzi-Douglas-Fortin-Marini projection, see [8].
The following approximation property of the above projection is well-known; see [5, Chapter 2].

**Lemma 2.2.** There exists a unique function \( \Pi V u \in V^{\text{div}}_h \) defined element-wise by the equations (2.13). Moreover, there exists a constant \( C \) only depending on the polynomial degree and shape-regularity of the elements \( K \in T_h \) such that

\[
\| \Pi V u - u \|_{T_h} \leq C \left( \| P V u - u \|_{T_h} + \sum_{K \in T_h} h_K^{1/2} \| P V u - u \|_{\partial K} \right). \tag{2.10}
\]

**The projection errors.** Now, we state our main results on the superconvergent error estimates.

**Theorem 2.3.** Let \( (L^h, u^h, p^h, \hat{u}_t^h) \in S_h \times V^{\text{div}}_h(0) \times Q_h \times M_h^t(0) \) be the numerical solution of (2.5), then there exists a constant \( C \), depending only on the polynomial degree \( k \), the shape-regularity of the mesh \( T_h \), and the domain \( \Omega \), such that

\[
\| e_u \|_{T_h} \leq C \| (e_u, e_t u) \|_{1, T_h}, \tag{2.11a}
\]

\[
\| (e_u, e_t u) \|_{1, T_h} \leq C \| e_L \|_{T_h}, \tag{2.11b}
\]

\[
\nu \| e_L \|^2_{T_h} + \| \gamma^{1/2} e_u \|^2_{T_h} \leq C \left( \sum_{F \in \mathcal{E}_h} h_F^2 \| \delta_F n \|^2_F + \| \gamma^{1/2} \delta u \|^2_{T_h} \right). \tag{2.11c}
\]

Combining this result with Lemma 2.2, we immediately obtain optimal convergence of \( L^2 \)-error for \( L^h \) and \( u^h \), and superconvergent discrete \( H^1 \)-error for the pair \((u^h, \hat{u}_t^h)\) comparing with the projection \((\Pi V u, P M^t u)\); see the following corollary. We omit the proof due to its simplicity. We specifically remark that the errors below are independent of the regularity of the pressure.

**Corollary 2.4.** Let \( (L^h, u^h, p^h, \hat{u}_t^h) \in S_h \times V^{\text{div}}_h(0) \times Q_h \times M_h^t(0) \) be the numerical solution of (2.5), then there exists a constant \( C \), depending only on the polynomial degree \( k \), the shape-regularity of the mesh \( T_h \), and the domain \( \Omega \), such that

\[
\nu^{1/2} \left( \| e_L \|_{T_h} + \| (e_u, e_t u) \|_{1, T_h} \right) + \max \{ \nu^{1/2} \| e_u \|_{T_h}, \gamma^{1/2} \| e_u \|_{T_h} \} \leq C \Theta h^{k+1},
\]

where \( \Theta := \nu^{1/2} \| L \|_{k+1, \Omega} + \gamma^{1/2} \| u \|_{k+1, \Omega} \).

and \( \gamma_{\text{max}} \) is the maximum eigenvalue of the inverse permeability tensor \( \gamma \), and \( \| \cdot \|_{m, \Omega} \) denotes the \( H^m \)-norm on \( \Omega \).

Next, we obtain optimal \( L^2 \)-estimates for pressure for \( k \geq 0 \) and superconvergent \( L^2 \)-estimates for the projection error \( e_u \) for \( k \geq 1 \) (with a \( H^2 \)-regularity assumption for the dual problem).

We assume that the following regularity estimate holds

\[
\| \Phi \|_{1, \Omega} + \| \phi \|_{2, \Omega} + \| \varphi \|_{1, \Omega} \leq C_r \| \theta \|_\Omega \tag{2.12}
\]

for the dual problem

\[
\Phi - \nabla \phi = 0 \quad \text{in} \quad \Omega, \tag{2.13a}
\]

\[-\nu \nabla \cdot \Phi + \gamma \phi - \nabla \varphi = \theta \quad \text{in} \quad \Omega, \tag{2.13b}
\]

\[\nabla \cdot \phi = 0 \quad \text{in} \quad \Omega, \tag{2.13c}\]

\[\phi = 0 \quad \text{on} \quad \partial \Omega. \tag{2.13d}\]
We notice that it is easy to see the dual problem (2.14) is well-posed with \( \|\phi\|_{1,\Omega} \leq C\|\theta\|_{\Omega} \). Obviously, \((\Phi, \phi, \varphi)\) is the solution of the Stokes problem with the source term \( \theta = -\gamma \phi \). So, the regularity estimate (2.12) comes from that of the Stokes problem (see \[18\]).

**Theorem 2.5.** Let \((L^h, u^h, p^h, \bar{u}_t^h) \in S_h \times V_h^{\text{div}}(0) \times Q_h \times M_t^h(0)\) be the numerical solution of (2.5). then there exists a constant \( C \), depending only on the polynomial degree \( k \), the shape-regularity of the mesh \( \mathcal{T}_h \), and the domain \( \Omega \), such that

\[
\|\epsilon_h\|_{\tau_n} \leq C (\nu^{1/2} + \gamma_{max}^{1/2}) \Theta h^{k+1},
\]

where \( \gamma_{\text{max}} \) and \( \Theta \) are defined in Corollary 2.4.

In addition, if \( k \geq 1 \), the regularity assumption (2.12) holds and \( \gamma \in W^{1,\infty}(\Omega)^d \times \Omega \), then we have

\[
\|\epsilon_u\|_{\tau_n} \leq C C_r \left( (\nu^{1/2} + \gamma_{\text{max}}^{1/2}) \Theta + \|\gamma\|_{1,\infty}\|u\|_{k+1}\right) h^{k+2}.
\]

### 3. Proof of Theorem 2.1, Theorem 2.3 and Theorem 2.5

In this section, we prove the main results in Section 2, namely, Theorem 2.1, Theorem 2.3 and Theorem 2.5.

The following result is a key ingredient to prove Theorem 2.1. We postpone its proof to Appendix.

**Lemma 3.1.** Given \((z^h, \bar{z}^h) \in V(K) \times M(\partial K)\) where

\[
M(\partial K) := \{\bar{v} \in L^2(\partial K)^d : \bar{v}|_F \in M(F) \ \forall F \in \mathcal{F}(K)\},
\]

there exists a unique function \( r^h \in \mathcal{S}(K) \) such that

\[
\begin{align}
\langle r^h, g^h \rangle_K &= \langle \nabla z^h, g^h \rangle_K \quad \forall g^h \in \nabla V(K) \oplus S_{\text{shb}}(K), \quad (3.1a) \\
\langle \text{tr}_t(r^h \mathbf{n}), \text{tr}_t(\bar{v}) \rangle_{\partial K} &= \langle \text{tr}_t(\bar{z}^h), \text{tr}_t(\bar{v}) \rangle_{\partial K} \quad \forall \bar{v}^h \in M(\partial K), \quad (3.1b)
\end{align}
\]

where \( S_{\text{shb}}(K) := \{g \in \mathcal{S}(K) : \nabla \cdot g = 0, \ \text{tr}_n^F(g \mathbf{n}) = 0 \ \forall F \in \mathcal{F}(K)\} \).

Moreover, there exists a constant \( C \) only depending on the shape-regularity of the element \( K \) such that

\[
\|r^h\|_K \leq C \left( \|\nabla z^h\|_K + \sum_{F \in \mathcal{F}(K)} h_F \|\text{tr}_t(\bar{z}^h)\|_F^2 \right)^{1/2}.
\]

Now, we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.**

By Lemma 3.1, for any \( z^h \in V(K) \) and \( \bar{z}^h \in \{\bar{v} \in M(\partial K) : \text{tr}_n(\bar{v}) = 0\} \), there exists \( g^h \in \mathcal{S}(K) \) such that

\[
\langle \nabla z^h, g^h \rangle_K - \langle \text{tr}_t(z^h) - \bar{z}^h, \text{tr}_t(g^h \mathbf{n}) \rangle_{\partial K} = \|\nabla z^h\|_K^2 + \sum_{F \in \mathcal{F}(K)} h_F^{-1} \|P_M(\text{tr}_t(z^h)) - \bar{z}^h\|_F^2
\]
and \( \| g^h \| K \leq C (\| \nabla z^h \|_K^2 + \sum_{F \in \mathcal{T}(K)} h_F^{-1} \| P_{M'}(\text{tr}_i(z^h)) - \zeta^h_i \|_F^2)^{1/2} \). Taking such \( g^h \) in (2.7), we get
\[
\| \nabla z^h \|_K^2 + \sum_{F \in \mathcal{T}(K)} h_F^{-1} \| P_{M'}(\text{tr}_i(z^h)) - \zeta^h_i \|_F^2 = (r, g^h)_K
\]
\[
\leq C \left( \| \nabla z^h \|_K^2 + \sum_{F \in \mathcal{T}(K)} h_F^{-1} \| P_{M'}(\text{tr}_i(z^h)) - \zeta^h_i \|_F^2 \right)^{1/2} \| r \|_K.
\]
Hence,
\[
\left( \| \nabla z^h \|_K^2 + \sum_{F \in \mathcal{T}(K)} h_F^{-1} \| P_{M'}(\text{tr}_i(z^h)) - \zeta^h_i \|_F^2 \right)^{1/2} \leq C \| r \|_K. \quad (3.3)
\]
Moreover, on each facet \( F \in \mathcal{T}(K) \), we have
\[
\| \text{tr}_i(z^h) - P_{M'}(\text{tr}_i(z^h)) \|_F = \| z^h - P_{M}(z^h) \|_F \leq \| z^h - \overline{z^h} \|_F \leq C h_K^{1/2} \| \nabla z^h \|_K,
\]
where \( \overline{z^h} \) is the average of \( z^h \) in the element \( K \) and the last inequality is the Poinc\’are inequality. Combining the above result with (3.3), we obtain
\[
\left\| \left( z^h, \zeta^h_i \right) \right\|_{1,K} \leq C \| r \|_K.
\]
The proof of Theorem 2.1 is completed by summing the above estimate over all the elements \( K \in \mathcal{T}_h \).

We use the following error equation to prove Theorem 2.3. To simplify notation, we denote
\[
B_h(L, u, p, \hat{u}_i; g, v, q, \hat{v}_i) := (L, \nu g)_{\mathcal{T}_h} - (\nabla u, \nu g)_{\mathcal{T}_h}
\]
\[
+ \langle \text{tr}_i(u) - \hat{u}_i, \text{tr}_i(\nu g n) \rangle_{\mathcal{T}_h}
\]
\[
+ \langle \nu L, \nabla v \rangle_{\mathcal{T}_h} - \langle \text{tr}_i(\nu L n), \text{tr}_i(v) - \hat{v}_i \rangle_{\mathcal{T}_h}
\]
\[
- \langle p, \nabla v \rangle_{\mathcal{T}_h} + \langle \gamma u, v \rangle_{\mathcal{T}_h}
\]
\[
+ \langle \nabla u, q \rangle_{\mathcal{T}_h}.
\]

**Lemma 3.2.** Let \( (L, u, p) \) be the solution to (1.1), and \( (L^h, u^h, p^h, \hat{u}_i^h) \) be the numerical solution to (2.5). Then, we have
\[
B_h(e_L, e_u, e_p, e_{\hat{u}_i}; g^h, v^h, q^h, \hat{v}_i^h) = \langle \text{tr}_i(\nu \delta_L n), \text{tr}_i(v^h) - \hat{v}_i^h \rangle_{\mathcal{T}_h}
\]
\[
- \langle \gamma \delta_u, v^h \rangle_{\mathcal{T}_h}. \quad (3.5)
\]
for all \( (g^h, v^h, q^h, \hat{v}_i^h) \in \mathcal{S}_h \times V_h^{\text{div}}(0) \times \hat{Q}_h \times M_h(0) \).

**Proof.** By (1.1), (2.5), and (3.3), we have
\[
B_h(L^h, u^h, p^h, \hat{u}_i^h; g^h, v^h, q^h, \hat{v}_i^h) = (f, v^h)_{\mathcal{T}_h} + (g, q^h)_{\mathcal{T}_h}
\]
\[
B_h(L, u, p, \text{tr}_i(u); g^h, v^h, q^h, \hat{v}_i^h) = (f, v^h)_{\mathcal{T}_h} + (g, q^h)_{\mathcal{T}_h}
\]
for all \( (g^h, v^h, q^h, \hat{v}_i^h) \in \mathcal{S}_h \times V_h^{\text{div}}(0) \times \hat{Q}_h \times M_h(0) \). Hence,
\[
B_h(e_L, e_u, e_p, e_{\hat{u}_i}; g^h, v^h, q^h, \hat{v}_i^h) = -B_h(\delta_L, \delta_u, \delta_p, \delta_{\hat{u}_i}; g^h, v^h, q^h, \hat{v}_i^h).
\]
Using orthogonality properties of the projections, we easily obtain
\[ B_h(\delta_L, \delta_u, \delta_u, g^h, v^h, q^h, v^h_i) = -\langle \text{tr}(\nu \delta_L), \text{tr}(v^h) - \hat{v}^h_t \rangle_{\partial \Omega} + (\gamma \delta_u, v^h, v^h)_{\Omega_h}. \]

This completes the proof. \( \Box \)

**Proof of Theorem 2.3.** By [15] Theorem 2.1, we have
\[ \|\mathbf{e}_u\|_{\Omega_h} \leq C \left( \|\nabla \mathbf{e}_u\|_{\Omega_h} + \sum_{F \in \mathcal{E}(K)} h_F^{-1} \|\mathbf{e}_u\|_F \right)^{1/2}. \]

Here \([\mathbf{e}_u] := e_u^+ - e_u^-\) denotes the jump of \(e_u \in V^\text{div}(0)\) on an interior facet \(F := K^+ \cap K^-\), and \([\mathbf{e}_u] := e_u\) on a boundary facet \(F \subset \partial \Omega\), where \(e_u^\pm = e_u|_{K^\pm}\). Since \(e_u\) is \(H(\text{div})\)-conforming and has vanishing normal trace on the boundary, we have \(\text{tr}_n([\mathbf{e}_u]) = 0\) for all facets \(F \in \mathcal{E}_h\). Hence,
\[ \|\mathbf{e}_u\| = \text{tr}(e_u). \]

By triangle inequality, we have
\[ \|\text{tr}(e_u)\|_F \leq \|\text{tr}(e_u^+) - e_{u_i}\|_F + \|\text{tr}(e_u^-) - e_{u_i}\|_F. \]

Combining the above estimates, we finish the proof of the first error estimate \(2.11a\).

The second error estimate \(2.11b\) comes directly from Theorem 2.1.

Now, let us prove the last error estimate \(2.11c\). Taking \((g^h, v^h, q^h, \hat{v}^h_i) := (e_L, e_u, e_p, e_{u_i})\), we obtain
\[
\nu \|e_L\|^2_{\Omega_h} + \|\gamma^{1/2} e_u\|^2_{\Omega_h} \leq -\langle \text{tr}(\nu \delta_L), \text{tr}(e_u) - e_{u_i}, \nu \delta_u \rangle_{\partial \Omega} + (\gamma \delta_u, e_u)_{\Omega_h}
\leq \sum_{F \in \mathcal{E}_h} \left( h_F^{1/2} \|\text{tr}(\nu \delta_L)\|_F h_F^{-1/2} \|\text{tr}(e_u) - e_{u_i}\|_F \right)
+ \|\gamma^{1/2} \delta_u\|_{\Omega_h} \|\gamma^{1/2} e_u\|_{\Omega_h}
\leq C \left( \sum_{F \in \mathcal{E}_h} \nu h_F \|\delta_L\|_2 \|e_u\|^2_{\Omega_h} + \|\gamma^{1/2} \delta_u\|^2_{\Omega_h} \right)^{1/2} \left( \nu \|e_L\|^2_{\Omega_h} + \|\gamma^{1/2} e_u\|^2_{\Omega_h} \right)^{1/2}
\]

by the Cauchy-Schwartz inequality.

This completes the proof of Theorem 2.3. \( \Box \)

The following result is used to prove the velocity estimate in Theorem 2.3.

**Lemma 3.3.** Let \((\Phi, \phi, \varphi)\) be the solution to the dual problem \(2.13\) for \(\theta \in L^2(\Omega)\). We have
\[
(e_u, \theta)_{\Omega_h} = \langle \nu e_L n, \varphi \rangle_{\partial \Omega} + \langle \text{tr}(\nu \delta_L n), \Pi \nu \phi - P_M \phi \rangle_{\partial \Omega} + (\gamma e_u, \delta_u)_{\Omega_h} - (\gamma u, \Pi \varphi)_{\Omega_h}
=: T_1 + T_2 + T_3 + T_4 + T_5,
\]

where \(\delta \Phi = \Phi - P_0 \Phi, \delta \varphi = \varphi - P_0 \varphi\).

**Proof.** By \(2.13\) and \(2.13c\), we have
\[
(e_u, \theta)_{\Omega_h} = -\langle e_u, \nu \nabla \Phi \rangle_{\Omega_h} + (e_u, \nu \phi)_{\Omega_h} - (e_u, \nabla \varphi)_{\Omega_h}
- \langle \nu e_L, \nabla \Phi \rangle_{\Omega_h} + (\nu e_L, \nabla \phi)_{\Omega_h} - (e_p, \nabla \phi)_{\Omega_h}
- \langle e_u, \nu \nabla P_0 \Phi \rangle_{\Omega_h} - (e_u, \nu \nabla \delta \Phi)_{\Omega_h} - (e_u, \nabla P_0 \varphi)_{\Omega_h} - (e_u, \nabla \delta \varphi)_{\Omega_h}
+ (e_u, \gamma \phi)_{\Omega_h} - (\nu e_L, P_0 \Phi)_{\Omega_h} + (\nu e_L, \nabla \phi)_{\Omega_h} - (e_p, \nabla \phi)_{\Omega_h}.
\]
This completes the proof of Lemma 3.3.

Taking \((g^h, v^h, q^h, \hat{v}^h) := (P_3\Phi, 0, -P_Q\varphi, 0)\) in the error equation \((3.5)\), putting the result identity into the above expression and simplifying, we have

\[
(e_u, \theta)_{\mathcal{T}_h} = -\langle e_u, \nu P_3\Phi n \rangle_{\partial \mathcal{T}_h} - \langle e_u, P_Q\varphi n \rangle_{\partial \mathcal{T}_h} + (\text{tr}_t(e_u) - e_{\hat{u}_i}, \nu P_3\Phi n)_{\partial \mathcal{T}_h} - \langle e_u, \nu \nabla \cdot \delta \Phi \rangle_{\mathcal{T}_h} - \langle e_u, \nabla \cdot \delta \Phi \rangle_{\mathcal{T}_h} + \langle e_u, \gamma \Phi \rangle_{\mathcal{T}_h} + (\nu e_L, \nabla \Phi)_{\mathcal{T}_h} - \langle e_p, \nabla \cdot \Phi \rangle_{\mathcal{T}_h}
\]

by inserting the zero term \(\langle e_u, \nu \Phi n \rangle_{\partial \mathcal{T}_h}\) and using the fact that \(\langle e_u, \nu \Phi n \rangle_{\partial \mathcal{T}_h} = \langle \text{tr}_t(e_u), \nu \Phi n \rangle_{\partial \mathcal{T}_h}\) and \(\langle e_u, \varphi n \rangle_{\partial \mathcal{T}_h} = 0\).

Take \((g^h, v^h, q^h, \hat{v}^h) := (0, \Pi \Phi, 0, PM_1\Phi)\) in the error equation \((3.5)\). Denoting by \(I := e_u, \gamma \Phi \rangle_{\mathcal{T}_h} + (\nu e_L, \nabla \Phi)_{\mathcal{T}_h} - \langle e_p, \nabla \cdot \Phi \rangle_{\mathcal{T}_h}\), we obtain

\[
I = (e_u, \gamma \delta \Phi)_{\mathcal{T}_h} + (\nu e_L, \nabla \delta \Phi)_{\mathcal{T}_h} - \langle e_p, \nabla \cdot \delta \Phi \rangle_{\mathcal{T}_h}
\]

This completes the proof of Lemma 3.3. □

Now we are ready to prove Theorem 2.5.

**Proof of Theorem 2.5** Proof. We first present the optimal error estimate for \(e_p\) by applying an \(inf-sup\) argument. It is well-known that the following \(inf-sup\) condition holds for a positive constant \(\kappa\), (cf. [18, Chapter 1, Corollary 2.4]),

\[
\sup_{\omega \in H^1_0(\Omega) \setminus \{0\}} \frac{\langle \nabla \omega, q \rangle_{\Omega}}{\|\omega\|_{1, \Omega}} \geq \kappa \|q\|_{\Omega}. \tag{3.7}
\]

Here \(\|\cdot\|_{1, \Omega}\) is the standard \(H^1\)-norm on \(\Omega\).

Since \(e_p \in L^2_0(\Omega)\), we have by \((3.7)\)

\[
\|e_p\|_{\Omega} \leq \frac{1}{\kappa} \sup_{\omega \in H^1_0(\Omega) \setminus \{0\}} \frac{\langle \nabla \omega, e_p \rangle_{\Omega}}{\|\omega\|_{1, \Omega}}. \tag{3.8}
\]

Taking \((g^h, v^h, q^h, \hat{v}^h) := (0, \Pi \Phi, 0, PM_1\Phi)\) in the error equation \((3.5)\) and applying
the integration by parts, we can rewrite the numerator as follows:

\((\nabla \cdot \omega, e_p)_{\mathcal{T}_h} = (\nabla \cdot \Pi \nabla \omega, e_p)_{\mathcal{T}_h} + (\nabla \cdot (\omega - \Pi \nabla \omega), e_p)_{\mathcal{T}_h} = (\nabla \cdot \Pi \nabla \omega, e_p)_{\mathcal{T}_h}
\)

\(= (\nu e_L, \nabla \Pi \nabla \omega)_{\mathcal{T}_h} - (\nabla \cdot (\nu e_L n) + \nabla \cdot (\nu \delta_u n))_{\mathcal{T}_h} + (\nu e_L, \Pi \nabla \omega)_{\mathcal{T}_h} + (\gamma e_u, \Pi \nabla \omega)_{\mathcal{T}_h} + (\gamma e_u, \Pi \nabla \omega)_{\mathcal{T}_h}
\)

\(= (\nu e_L, \nabla \Pi \nabla \omega)_{\mathcal{T}_h} - (\nabla \cdot (\nu e_L n) + \nabla \cdot (\nu \delta_u n), \Pi \nabla \omega)_{\mathcal{T}_h} + (\gamma e_u, \Pi \nabla \omega)_{\mathcal{T}_h}
\)

\(= I_1 + I_2 + I_3 + I_4.
\)

Then we will bound \(I_1 - I_4\) by Corollary 2.3 as follows.

\(I_1 \leq \nu \|e_L\|_{\mathcal{T}_h} \|\nabla \Pi \nabla \omega\|_{\mathcal{T}_h} \leq C \nu^{1/2} \Theta h^{k+1} \|\omega\|_{1,\Omega}.
\)

\(I_2 \leq \nu \|e_L n\|_{\mathcal{T}_h} + \|\delta_e n\|_{\mathcal{T}_h} \|\Pi \nabla \omega - P_M \omega\|_{\mathcal{T}_h}
\)

\(\leq C(\nu^{1/2} \Theta h^{k+1/2} + \nu \|L\|_{k+1,\Omega} h^{k+1/2} h^{1/2} \|\omega\|_{1,\Omega} \leq C \nu^{1/2} \Theta h^{k+1} \|\omega\|_{1,\Omega}.
\)

\(I_3 \leq C \gamma_{\max} \|\gamma^{1/2} e_u\|_{\mathcal{T}_h} \|\Pi \nabla \omega\|_{\mathcal{T}_h} \leq C \gamma_{\max} \|u\|_{k+1,\Omega} \|\omega\|_{1,\Omega}.
\)

\(I_4 \leq C \gamma_{\max} \|\delta_u\|_{\mathcal{T}_h} \|\Pi \nabla \omega\|_{\mathcal{T}_h} \leq C \gamma_{\max} \|u\|_{k+1,\Omega} \|\omega\|_{1,\Omega}
\)

\(\leq C \nu^{1/2} \Theta h^{k+1} \|\omega\|_{1,\Omega}.
\)

Then we have

\((\nabla \cdot \omega, e_p)_{\mathcal{T}_h} \leq C(\nu^{1/2} + \gamma_{\max}^{1/2}) \Theta h^{k+1} \|\omega\|_{1,\Omega}.
\)

By (38), we obtain the estimate for \(e_p\).

Now we give superconvergent estimate for \(e_u\). By (36), it suffices to estimate the terms \(T_1\) to \(T_5\). We apply Corollary 2.4 the regularity assumption 2.12 and the Poincaré inequality to bound these terms.

\(T_1 \leq \nu \|e_L\|_{\mathcal{T}_h} \|\nabla \omega\|_{\mathcal{T}_h} \leq C \nu h^{-1/2} \|e_L\|_{\mathcal{T}_h} h^{3/2} \|\phi\|_{2}
\)

\(\leq C \nu^{1/2} \Theta h^{k+2} \|\phi\|_{\mathcal{T}_h}.
\)

\(T_2 \leq \nu \|\nabla \delta_u\|_{\mathcal{T}_h} + \|e_L n\|_{\mathcal{T}_h} \|\Pi \nabla \phi - P_M \phi\|_{\mathcal{T}_h}
\)

\(\leq C(\nu \|L\|_{k+1,\Omega} h^{k+1/2} + \nu^{1/2} \Theta h^{k+1/2} h^{1/2} \|\phi\|_{2} \leq C \nu^{1/2} \Theta h^{k+2} \|\phi\|_{\mathcal{T}_h}.
\)

\(T_3 \leq \nu \|\nabla (e_u - e_u)\|_{\mathcal{T}_h} \leq \nu \|\nabla \delta_u\|_{\mathcal{T}_h} \|\phi\|_{\mathcal{T}_h}
\)

\(\leq C \nu^{1/2} \|\nabla (e_u - e_u)\|_{\mathcal{T}_h} \|\phi\|_{\mathcal{T}_h} \leq C \nu^{1/2} \Theta h^{k+2} \|\phi\|_{\mathcal{T}_h},
\)

\(T_4 \leq \gamma_{\max} \|\gamma^{1/2} e_u\|_{\mathcal{T}_h} \|\Pi \nabla \phi\|_{\mathcal{T}_h} \leq C \gamma_{\max} \|u\|_{k+1,\Omega} \|\phi\|_{\mathcal{T}_h}
\)

\(T_5 = ((\gamma - P_0, h) \gamma \delta_u, \Pi \nabla \phi)_{\mathcal{T}_h} + (P_0, h) \gamma \delta_u, \Pi \nabla \phi - \phi)_{\mathcal{T}_h}
\)

\(\leq \gamma \|P_0, h\|_{\infty} \|\delta_u\|_{\mathcal{T}_h} \|\Pi \nabla \phi\|_{\mathcal{T}_h} + (P_0, h) \gamma \|\delta_u\|_{\mathcal{T}_h} \|\Pi \nabla (\phi - \phi)\|_{\mathcal{T}_h}
\)

\(\leq C h \|\gamma\|_{1,\infty} \|u\|_{k+1,\Omega} \|\phi\|_{2} + C \gamma \|\Pi \nabla \phi\|_{\mathcal{T}_h} \|\phi\|_{\mathcal{T}_h}
\)

\(\leq C \gamma \|\gamma\|_{1,\infty} \|u\|_{k+1,\Omega} \|\phi\|_{2} + C \gamma \|u\|_{k+1,\Omega} \|\phi\|_{\mathcal{T}_h}.
\)

where \(P_0,h\) is \(L^2\) orthogonal projection onto \(P_0(T_h)^{d \times d}\) and \(\phi\) is defined as

\(\hat{\phi} = \frac{1}{|K|} (\phi, 1)_{K}, \quad \forall K \in \mathcal{T}_h.
\)

Combining all the above estimates, we have

\(\|e_u\|_{\mathcal{T}_h} \leq C(\nu^{1/2} + \gamma_{\max}^{1/2}) \Theta + \|\gamma\|_{1,\infty} \|u\|_{k+1,\Omega} h^{k+2}.
\)

This completes the proof of Theorem 2.5. \(\square\)
4. Hybridization. In this section, we hybridize the $H(\text{div})$-conforming HDG method (2.25) by relaxing the $H(\text{div})$-conformity of the velocity field via Lagrange multipliers; similar treatment was used in [14]. The resulting global linear system is a saddle point system for $(\mathbf{u}_h^h, \tilde{\mathbf{u}}_h^h, p^h) \in M_h^0(0) \times M_h^1(0) \times \mathcal{Q}_h$, where

\begin{equation}
M_h^1(0) := \{ \mathbf{v} \in M_h(0) : \text{tr}_i(\mathbf{v})|_F = 0, \ \forall F \in \mathcal{E}_h \},
\end{equation}

\begin{equation}
\mathcal{Q}_h := \{ q \in L^2(\mathcal{T}_h) : q|_K \in \mathcal{P}_0(K), \ \forall K \in \mathcal{T}_h \}.
\end{equation}

We show that $\tilde{\mathbf{u}}_h^h$ here is the same as that in (2.25), $\tilde{\mathbf{u}}_h^h = \text{tr}_n(\mathbf{u}^h)$ on $\mathcal{E}_h$, $p^h$ is equal to average of $p^h$ on each element of $\mathcal{T}_h$.

Here we first relax $H(\text{div})$-conformity of the velocity field in (2.25) to obtain the following result.

**Theorem 4.1.** There exists a unique element $(L^h, \mathbf{u}^h, p^h_\pm, p^h, \tilde{\mathbf{u}}_n^h, \tilde{\mathbf{u}}_i^h, \lambda^h) \in \mathcal{G}_h \times V_h \times \mathcal{Q}_h \times M_h^1(0) \times M_h^0$ such that the following weak formulation holds:

\begin{equation}
(L^h, \nu g^h)_{\mathcal{T}_h} + (\mathbf{u}^h, \nabla \cdot (\nu g^h))_{\mathcal{T}_h} - (\tilde{\mathbf{u}}_i^h + \tilde{\mathbf{u}}_n^h, \nu g^h \mathbf{n})_{\partial \mathcal{T}_h} = 0, \quad (4.1a)
\end{equation}

\begin{equation}
(\nu L^h - (p^h_\pm + p^h) I) \mathbf{u}^h + \nabla \mathbf{u}^h)_{\mathcal{T}_h} + (\gamma \mathbf{u}^h, \mathbf{v}^h)_{\mathcal{T}_h} - (\nu L^h \mathbf{n} - (p^h_\pm + p^h) \mathbf{n} + \lambda^h \mathbf{n}, \mathbf{v}^h)_{\partial \mathcal{T}_h} = (f, \mathbf{v}^h)_{\mathcal{T}_h}, \quad (4.1b)
\end{equation}

\begin{equation}
\langle (\mathbf{u}^h - \tilde{\mathbf{u}}_n^h) \cdot \mathbf{n}, \mu^h \rangle_{\partial \mathcal{T}_h} = 0, \quad (4.1c)
\end{equation}

\begin{equation}
(\tilde{\mathbf{u}}_i^h, \lambda^h)_{\mathcal{T}_h} = 0, \quad (4.1d)
\end{equation}

\begin{equation}
\langle \mathbf{Q}_h^1 \rangle := \{ q \in L^2(\mathcal{T}_h) : (q, 1)_K = 0, \ \forall K \in \mathcal{T}_h \},
\end{equation}

\begin{equation}
M_h^0 := \{ \mu \in L^2(\partial \mathcal{T}_h) : \mu|_{\partial K} \in \mathcal{P}_0(\partial K), \ \forall K \in \mathcal{T}_h \},
\end{equation}

\begin{equation}
\mathcal{P}_k(\partial \mathcal{T}) := \{ \mu \in L^2(\partial \mathcal{T}) : \mu|_F \in \mathcal{P}_k(F), \ \forall F \in \mathcal{F}(\partial \mathcal{T}) \}.
\end{equation}

Moreover, if $(L^h, \mathbf{u}^h, p^h_\pm, p^h, \tilde{\mathbf{u}}_n^h, \tilde{\mathbf{u}}_i^h, \lambda^h) \in \mathcal{G}_h \times V_h \times \mathcal{Q}_h \times M_h^1(0) \times M_h^0$ is the numerical solution to the above equations, then $(L^h, \mathbf{u}^h, p^h_\pm + p^h, \tilde{\mathbf{u}}_i^h) \in \mathcal{G}_h \times V_h^\text{div}(0) \times \mathcal{Q}_h \times M_h^0$ is the only solution to (2.25).

Note that $\lambda^h \in M_h^0$ is a quantity that approximates $0|_{\partial \mathcal{T}_h}$.

**Proof.** Let $(L^h, \mathbf{u}^h, p^h_\pm, p^h, \tilde{\mathbf{u}}_n^h, \tilde{\mathbf{u}}_i^h, \lambda^h) \in \mathcal{G}_h \times V_h \times \mathcal{Q}_h \times M_h^1(0) \times M_h^0$ be a numerical solution to equations (4.1). We prove such numerical solution is unique and $(L^h, \mathbf{u}^h, p^h_\pm + p^h, \tilde{\mathbf{u}}_i^h)$ is the unique solution to equations (2.25).

Since

\begin{equation}
(\mathbf{u}^h - \tilde{\mathbf{u}}_n^h) \cdot \mathbf{n}|_{\partial K} \in \mathcal{P}_k(\partial K) = M_h^0(K), \quad \forall K \in \mathcal{T}_h,
\end{equation}

we have $\text{tr}_n(\mathbf{u}^h) = \tilde{\mathbf{u}}_n^h$ on any facet $F \in \mathcal{E}_h$ by equations (4.1c). Hence, $\mathbf{u}^h \in V_h^\text{div}(0)$.

By equation (4.1b), we have $p^h_\pm + p^h \in Q_h$.

Then, taking $\mathbf{v}^h \in V_h^\text{div}(0)$ in (4.1b), $\mathbf{v}^h|_F = \text{tr}_n(\mathbf{u}^h)$ on any facet $F \in \mathcal{E}_h$ in (4.1c), and $q^h \in Q_h$ in (4.1d), we have

\begin{equation}
(L^h, \mathbf{u}^h, p^h_\pm + p^h, \tilde{\mathbf{u}}_i^h) \in \mathcal{G}_h \times V_h^\text{div}(0) \times \mathcal{Q}_h \times M_h^0
\end{equation}
is the unique solution to equations (2.6).

Now, we only need to show the uniqueness of \( \lambda^h \). If there are two \( \lambda^h \), then by equation (4.2b), their difference which we still call \( \lambda^h \) satisfies

\[
(\lambda^h, \nu^h \cdot n)_{\partial \Omega_h} = 0, \quad \forall \nu^h \in V_h.
\]

Since \( M_0^h(K) = \text{tr}_n(V_h(K)) \) for any \( K \in \Omega_h \), we have \( \lambda^h = 0|_{\partial \Omega_h} \). So, \( \lambda^h \) is also unique. This completes the proof. \( \square \)

Then, we identify local and global solvers.

Because of the lack of uniqueness of pressure in the Brinkman equations, we will keep \( \tilde{p}_h \in \mathcal{Q}_h \) as a separate unknown.

Given \( (\tilde{u}_h, \tilde{u}_n) \in M_h^l(0) \times M_h^u(0), f \in L^2(\Omega_h)^d, \) and \( g \in L^2(\Omega_h) \), we consider the solution to the set of local problems in each element \( K \in \Omega_h \): find

\[
(L^h, u^h, p^h, \lambda^h) \in S(K) \times V(K) \times Q^+(K) \times M_0^h(K)
\]

such that

\[
(\nu^h L^h + \nabla \cdot (\nu^h g^h))_{\partial \Omega_h} = (\tilde{u}_h + \tilde{u}_n, \nu^h g^h, n)_{\partial \Omega_h},
\]

\[
(\nabla \cdot (\nu^h L^h) - \nabla p^h - \gamma u^h, v^h)_{\partial \Omega_h} = (f, v^h)_{\Omega_h} - (\lambda^h n, v^h)_{\partial \Omega_h},
\]

\[
(\nabla \cdot u^h, q^h)_{\partial \Omega_h} = (g, q^h)_{\partial \Omega_h},
\]

\[
(\nu^h L^h, \nu^h g^h)_{\partial \Omega_h} = 0
\]

for all \( (g^h, w^h, q^h, \mu^h) \in S(K) \times V(K) \times Q^{+}(K) \times M_0^h(K) \).

Unique solvability of this problem is a simple consequence of unique solvability of the equations (4.2).

The solution to (4.3) can be written as

\[
(L^h, u^h, p^h, \lambda^h) = (L^h, \tilde{u}_h, \tilde{u}_n, p^h_{\perp} + \lambda^h n, \lambda^h) + (L^h, f, g, p^h_{\perp} + \lambda^h) \]

by considering separately the influence of \( (\tilde{u}_h, \tilde{u}_n) \) and \( (f, g) \) in the solution. For example, \( (L^h, \tilde{u}_h, \tilde{u}_n, p^h_{\perp} + \lambda^h n, \lambda^h) \) is the solution of (4.3) when \( (f, g) = (0, 0) \).

According to equations (4.2c, 4.2d, 4.2b), the global (hybrid) problem is to find \( (\tilde{u}_h^l, \tilde{u}_n^l, \tilde{q}^h) \in M_h^l(0) \times M_h^u(0) \times \mathcal{Q}_h \) such that

\[
(\nu L^h_{(\tilde{u}_h^l, \tilde{u}_n^l)} n - p^h_{\perp, l} + \lambda^h n, n)_{\partial \Omega_h} + \lambda^h (\tilde{u}^h + \tilde{v}^h_{\perp})_{\partial \Omega_h} = 0
\]

\[
(\nabla \cdot (\nu L^h_{(f, g)} n - p^h_{\perp, l} + \lambda^h n, n)_{\partial \Omega_h} + \lambda^h (\tilde{u}^h + \tilde{v}^h_{\perp})_{\partial \Omega_h} = 0
\]

\[
(p^h, 1)_{\partial \Omega_h} = 0
\]

for all \( (\tilde{u}^h, \tilde{u}^h_{\perp}, \tilde{q}^h) \in M_h^l(0) \times M_h^u(0) \times \mathcal{Q}_h \). Again, unique solvability of this problem is a simple consequence of that for equations (4.2). Moreover, we have the following characterization of the equations (4.4). Its proof is trivial; see, e.g., [14].
Proposition 4.2. The equations (4.4) can be rewritten as
\[ A_h(\hat{\mathbf{u}}^h, \hat{\mathbf{u}}^h; \hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h) + B_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h) = F_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h), \]
\[ B_h(\hat{\mathbf{u}}^h, \hat{\mathbf{p}}^h) = 0, \]
\[ (\hat{\mathbf{p}}^h, 1)_\mathcal{T}_h = 0, \]
where
\[ A_h(\hat{\mathbf{u}}^h, \hat{\mathbf{u}}^h; \hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h) := (vL_h(\hat{\mathbf{u}}^h, \hat{\mathbf{u}}^h), L_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h))_{\mathcal{T}_h} + (\gamma u_h(\hat{\mathbf{u}}^h, \hat{\mathbf{u}}^h), u_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h))_{\mathcal{T}_h}, \]
\[ B_h(\hat{\mathbf{v}}^h, \hat{\mathbf{p}}^h) := - (\hat{\mathbf{p}}^h, \hat{\mathbf{v}}^h ; \mathbf{n})_{\partial \mathcal{T}_h}, \]
\[ F_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h) := (f, u_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h))_{\mathcal{T}_h} - (\nu L_h(\hat{f}, g), L_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h))_{\mathcal{T}_h} - (\gamma u_h(\hat{f}, g), u_h(\hat{\mathbf{v}}^h, \hat{\mathbf{v}}^h))_{\mathcal{T}_h}. \]

5. Numerical results. In this section, we present two-dimensional numerical studies on both rectangular and triangular meshes to validate the theoretic results in Section 2.

We use the Deal.II [4] software to implement the HDG method (2.5) on rectangular meshes, and NGSolve [29, 30] on triangular meshes. Recall that our approximation spaces are given in Table 2.1.

The implementation on rectangular meshes use the hybridization discussed in Section 3, while the implementation on triangular meshes use NGSolve’s built-in static condensation approach, see [30].

We present three numerical tests with a manufactured solution to validate our theoretic results in Section 2. For all the tests, the body forces \( f \) and \( g \) are chosen such that the exact solution \( (\mathbf{u}, p) \) takes the following form:
\[ \mathbf{u} = (\sin(2 \pi x) \sin(2 \pi y), \sin(2 \pi x) \sin(2 \pi y))^T, \]
\[ p = \sin(m \pi x) \sin(m \pi y), \] where \( m \) is a fixed number.

We take \( \nu = 1, \gamma = 1, \) and \( m = 2 \) for the first test, \( \nu = 1, \gamma = 1, \) and \( m = 20 \) for the second test, and \( \nu = 0.0001, \gamma = 1, \) and \( m = 2 \) for the third test. The first two tests are in the Stokes-dominated regime, while the last test is in the Darcy-dominated regime. The second test examine the effect of pressure regularity on the convergence of the velocity field.

In Table 5.1 we present the \( L^2 \)-convergence rates for \( L^h, \mathbf{u}^h, p^h, \) and \( \mathbf{u}^{r,h} \) for the HDG method (2.5) with polynomial degree varying from \( k = 0 \) to \( k = 3 \) on rectangular meshes. The first level mesh consists of \( 8 \times 8 \) congruent squares, and the consequent meshes are obtained by uniform refinements.

In Table 5.2 we present the same convergence study with polynomial degree varying from \( k = 1 \) to \( k = 3 \) on triangular meshes. The first level mesh consists of \( 2 \times 4 \times 4 \) congruent triangles, and the consequent meshes are obtained by uniform refinements.

In both tables, \( N_{ele} \) denotes the number of elements, \( N_{global} \) denotes the number of globally coupled degrees of freedom and \( N_{local} \) denotes the number of local (static-condensed) degrees of freedom.
Here, the local postprocessing $u^{*,h} \in \mathcal{P}_{k+1}(K)$ is defined element-wise by the following set of equations:

\[
(\nabla u^{*,h}, \nabla v)_K = (L^h, \nabla v)_K \quad \forall v \in \mathcal{P}_{k+1}(K),
\]
\[
(u^{*,h}, w)_K = (u^h, w)_K \quad \forall w \in \mathcal{P}_0(K).
\]

It is quite easy to show (c.f. [31, 13]) that $u^{*,h}$ convergence with an order of $k+2-\delta_{0,k}$. From the results for the first test in Table 5.1, we observe optimal convergence order of $k+1$ for all the three variables $L^h$, $u^h$, and $p^h$, and superconvergence order of $k+2$ for the postprocessing $u^{*,h}$. The convergence results for $L^h$, $u^h$, and $p^h$ are in full agreements with the theoretic predictions in Corollary 2.4 and Theorem 2.5. The superconvergence for $u^{*,h}$ is in agreement with the theoretic predictions in Theorem 2.5 for $k \geq 1$, while the superconvergence of $u^{*,h}$ for $k = 0$ is not covered by our analysis in Theorem 2.5.

From the results for the second test in Table 5.1, we observe the same $L^2$-errors in $L^h$, $u^h$, and $u^{*,h}$ as the corresponding ones in the first test. This indicates velocity error is independent of the pressure, in full agreement with the estimates in Corollary 2.4. We also observe the $L^2$-error for $p^h$ is significantly larger than that for the first test. It is clear that, in this test, convergence for pressure is not in the asymptotic regime yet.

From the results for the third test in Table 5.1, we observe similar convergence rates for all the variables as the first test. This indicates uniform stability of the proposed HDG method.

The convergence results on triangular meshes in Table 5.2 are similar to that on rectangular meshes in Table 5.1.

6. Conclusion. We present and analyze a class of parameter-free superconvergent $H(\text{div})$-conforming HDG method on both simplicial and rectangular meshes for the Brinkman equations. Numerical results in two dimensions are presented to validate the theoretic findings.

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Appendix: Proof of Lemma 3.1. In this Appendix, we prove Lemma 3.1. We use the following result, whose proof comes directly from Lemma 2.2 and the usual scaling argument.

Lemma 6.1. Given $(r^h, z^h) \in \mathcal{S}(K) \times \mathcal{M}(\partial K)$ where

\[
\mathcal{M}(\partial K) := \{ \tilde{w} \in L^2(\partial K)^d : \tilde{w}|_F \in \mathcal{M}(F) \ \forall F \in \mathcal{F}(K) \},
\]

there exists a unique function $w^h \in \mathcal{V}(K)$ such that

\[
(w^h, v^h)_K = (\nabla \cdot r^h, v^h)_K \quad \forall v^h \in \mathcal{V}(K),
\]
\[
\langle \text{tr}_n(w^h), \text{tr}_n(\tilde{w}) \rangle_{\partial K} = \langle \text{tr}_n(z^h), \text{tr}_n(\tilde{w}) \rangle_{\partial K} \quad \forall \tilde{w}^h \in \mathcal{M}(\partial K).
\]
Moreover, there exists a constant $C$ only depending on the shape-regularity of the element $K$ such that

$$
\|u^h\|_K \leq C \left( \|\nabla \cdot r^h\|_K^2 + \sum_{F \in \mathcal{F}(K)} h_F \|\text{tr}_n(z^h)\|_F^2 \right)^{1/2}
$$

(6.1)
Table 5.2

| k | mesh | $N_{sec}$ | $N_{local}$ | \(D.O.F.\) | $\|L - L^h\|_{\gamma_c}$ error order | $\|u - u^h\|_{\gamma_c}$ error order | $\|p - p^h\|_{\gamma_c}$ error order | $\|u^h - u\|_{\gamma_c}$ error order |
|---|---|---|---|---|---|---|---|---|
| 32 | 256 | 555 | 1.563e+00 | 1.102e+00 | 8.376e-02 | 6.086e-02 | 1.615e-02 | 7.768e-04 |
| 128 | 960 | 2203 | 3.378e-01 | 1.167e+00 | 3.220e-02 | 3.156e-03 | 1.518e-02 | 6.449e-03 |
| 512 | 3712 | 8763 | 9.967e-01 | 3.190e+00 | 2.378e-02 | 9.081e-03 | 1.713e-02 | 3.219e-03 |
| 2048 | 14592 | 34939 | 1.993e+00 | 6.380e+00 | 1.912e-02 | 2.820e-03 | 2.167e-02 | 6.329e-03 |
| 8192 | 57856 | 130513 | 3.896e+00 | 1.276e+01 | 1.550e-02 | 7.095e-03 | 2.134e-02 | 1.074e-02 |

First test, \(\nu = 1, \gamma = 1, m = 2\).

| k | mesh | $N_{sec}$ | $N_{local}$ | \(D.O.F.\) | $\|L - L^h\|_{\gamma_c}$ error order | $\|u - u^h\|_{\gamma_c}$ error order | $\|p - p^h\|_{\gamma_c}$ error order | $\|u^h - u\|_{\gamma_c}$ error order |
|---|---|---|---|---|---|---|---|---|
| 32 | 368 | 1163 | 9.679e-02 | 1.024e+00 | 3.555e-02 | 4.949e-02 | 2.407e-03 | 7.698e-04 |
| 128 | 1376 | 4635 | 3.471e+00 | 1.172e+00 | 3.555e-02 | 4.949e-02 | 2.407e-03 | 7.698e-04 |
| 512 | 5312 | 18491 | 4.381e-03 | 1.172e+00 | 3.555e-02 | 4.949e-02 | 2.407e-03 | 7.698e-04 |
| 2048 | 20864 | 73853 | 5.488e+00 | 1.172e+00 | 3.555e-02 | 4.949e-02 | 2.407e-03 | 7.698e-04 |
| 8192 | 82688 | 295163 | 6.848e+00 | 1.172e+00 | 3.555e-02 | 4.949e-02 | 2.407e-03 | 7.698e-04 |

Second test, \(\nu = 1, \gamma = 1, m = 20\).

| k | mesh | $N_{sec}$ | $N_{local}$ | \(D.O.F.\) | $\|L - L^h\|_{\gamma_c}$ error order | $\|u - u^h\|_{\gamma_c}$ error order | $\|p - p^h\|_{\gamma_c}$ error order | $\|u^h - u\|_{\gamma_c}$ error order |
|---|---|---|---|---|---|---|---|---|
| 32 | 480 | 1995 | 3.551e-02 | 1.295e+00 | 2.155e-02 | 1.760e-03 | 6.029e-03 | 4.028e-04 |
| 128 | 1792 | 7963 | 1.815e+00 | 1.295e+00 | 2.155e-02 | 1.760e-03 | 6.029e-03 | 4.028e-04 |
| 512 | 6912 | 31803 | 1.172e+00 | 1.295e+00 | 2.155e-02 | 1.760e-03 | 6.029e-03 | 4.028e-04 |
| 2048 | 27136 | 127099 | 7.398e-03 | 1.295e+00 | 2.155e-02 | 1.760e-03 | 6.029e-03 | 4.028e-04 |
| 8192 | 107520 | 508155 | 4.638e-07 | 1.295e+00 | 2.155e-02 | 1.760e-03 | 6.029e-03 | 4.028e-04 |

Third test, \(\nu = 0.0001, \gamma = 1, m = 2\).

Proof of Lemma 3.1. We only prove the existence and uniqueness of the function \(r^h \in \mathcal{G}(K)\) satisfying equations (3.1) on the reference element \(K = K\), the result on an affine-mapped element \(K\) can be easily obtained from that on the reference element (cf. [5 Chapter 2]), and the estimate (3.2) is a direct consequence of the usual scaling argument and equivalence of norms on finite-dimensional spaces.

We first show that (3.1) define a square system. We use the concept of an M-
decomposition \cite{12,10,11} to prove it.

By the choice of $G^{\text{row}}(K)$ in Table 2.1 we have the pair $G^{\text{row}}(K) \times P_k(K)$ admits an $M$-decomposition with the trace space

$$M(\partial K) := \{ \hat{w} \in L^2(\partial K) : \hat{w}|_F \in P_k(F) \ \forall F \in \mathcal{F}(K) \}.$$ 

Hence,

$$\dim G^{\text{row}}(K) + \dim P_k(K) = \dim G^{\text{row}}sbb(K) + \dim \nabla \cdot G^{\text{row}}(K)$$

$$+ \dim \nabla P_k(K) + \dim M(\partial K).$$

Here $G^{\text{row}}sbb(K) := \{ v \in G^{\text{row}}(K) : \nabla \cdot v = 0, \text{tr}_n(v) = 0 \text{ on } \partial K \}$. This immediately implies that

$$\dim \mathcal{G}(K) + \dim P_k(K)^d = \dim \mathcal{G}_{sbb}(K) + \dim \nabla \cdot \mathcal{G}(K)$$

$$+ \dim \nabla P_k(K)^d + \dim M(\partial K).$$

By Lemma 2.2, we have

$$\dim V(K) = \dim \nabla \cdot \mathcal{G}(K) + \dim \text{tr}_n(M(\partial K)).$$

Combing the above equality with (6.2) and reordering the terms, we get

$$\dim \mathcal{G}(K) = \dim \mathcal{G}_{sbb}(K) + \dim \text{tr}_n(M(\partial K))$$

$$+ \dim V(K) - \dim P_k(K)^d + \dim \nabla P_k(K)^d.\tag{6.3}$$

Since it is trivial to prove that

$$\dim V(K) - \dim P_k(K)^d + \dim \nabla P_k(K)^d = \dim \nabla V(K)$$

for the vector space $V(K)$ in Table 2.1 we conclude that equations (3.1) is indeed a square system. Hence, we are left to prove the uniqueness.

To this end, we take $z^h = 0, \hat{z}^h = 0$ in (3.1). By (3.1a), we have

$$\text{tr}_t(r^h n) = 0. \tag{6.4}$$

By (3.1a), we have, for all $v \in V(K)$,

$$0 = (r^h, \nabla v)_K = - (\nabla \cdot r^h, v)_K + (\text{tr}_n(r^h n), \text{tr}_n(v))_{\partial K} + (\text{tr}_t(r^h n), \text{tr}_t(v))_{\partial K}$$

$$= -(\nabla \cdot r^h, v)_K + (\text{tr}_n(r^h n), \text{tr}_n(v))_{\partial K}.$$ 

Then, by Lemma 6.1 there exists a function $v \in V(K)$ such that

$$-(\nabla \cdot r^h, v)_K + (\text{tr}_n(r^h n), \text{tr}_n(v))_{\partial K} = (\nabla \cdot r^h, \nabla r^h)_K + (\text{tr}_n(r^h n), \text{tr}_n(r^h n))_{\partial K}.$$ 

Hence, $\nabla \cdot r^h = 0$ and $\text{tr}_n(r^h n) = 0$. This implies that $r^h \in \mathcal{G}_{sbb}(K)$. Then, taking $g^h := r^h \in \mathcal{G}_{sbb}(K)$ in (3.1a), we conclude that $r^h = 0$.

This conclude the proof of Lemma 3.1.

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18
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