Lusztig’s conjecture for finite classical groups with even characteristic

Toshiaki Shoji

Abstract. The determination of scalars involved in Lusztig’s conjecture concerning the characters of finite reductive groups was achieved by Waldspurger in the case of finite classical groups $Sp_{2n}(F_q)$ or $O_n(F_q)$ when $p, q$ are large enough. Here $p$ is the characteristic of the finite field $F_q$. In this paper, we determine the scalars in the case of $Sp_{2n}(F_q)$ with $p = 2$, by applying the theory of symmetric spaces over a finite field due to Kawanaka and Lusztig. We also obtain a weaker result for $SO_{2n}(F_q)$ with $p = 2$, of split type.

0. Introduction

Let $G$ be a connected reductive group defined over a finite field $F_q$ of characteristic $p$ with Frobenius map $F$. Lusztig’s conjecture asserts that, under a suitable parametrization, almost characters of the finite reductive group $G^F$ coincide with the characteristic functions of character sheaves of $G$ up to scalar. Once Lusztig’s conjecture is settled, and the scalars involved there are determined, one obtains a uniform algorithm of computing irreducible characters of $G^F$. Lusztig’s conjecture was solved in [S1] in the case where the center of $G$ is connected. In [S2], the scalars in question were determined in the case where $G$ is a classical group with connected center, when $p$ is odd, and the scalars are related to the unipotent characters of $G^F$. By extending the method there, Waldspurger [W] proved Lusztig’s conjecture (or its appropriate generalization) for $Sp_{2n}$ and $O_n$ assuming that $p, q$ are large enough. He also determined the scalars involved in the conjecture. But these methods cannot be applied to the case of classical groups with even characteristic.

In this paper we take up the problem of determining the scalars in the case of classical groups with $p = 2$. We show that the scalars are determined explicitly in the case where $G = Sp_{2n}$ with $p = 2$. We also obtain a somewhat weaker result for the case $SO_{2n}$ of split type, when $p = 2$, containing the case related to the unipotent characters. The main ingredient for the proof is the theory of symmetric spaces over finite fields due to Kawanaka [K] and Lusztig [L4]. They determined the multiplicity of irreducible representations of $G^F$ occurring in the induced module $Ind_{G^F}^{G^F^2} 1$ in the case where the center of $G$ is connected (for arbitrary characteristic).

1991 Mathematics Subject Classification. Primary 20G40; Secondary 20G05.
Key words and phrases. finite classical groups, representation theory.
Using this, one can determine the scalars for $G^{F^2}$ in many cases for a connected classical group with connected center, with arbitrary characteristic.

On the other hand, it was shown in [S3] that there exists a good representatives of $G^F$ for a unipotent class $C$ in $Sp_{2n}$ or $SO_N$ for arbitrary characteristic. This implies that the generalized Green functions of $G^{F^m}$ turn out to be polynomials in $q$ (more precisely, rational functions in $q$ if $p = 2$) for various extension field $F_q$, and so certain values of almost characters are also rational functions in $q$. This makes it possible to apply some sort of specialization argument for the character values of $G^{F^m}$ for any $m \geq 1$, and one can determine the scalars of $G^F$ which are related to the unipotent characters, from the result for $G^{F^2}$. Thus we re-discover the results in [S2]. But this method works also for $p = 2$, and from this we can deduce the result for $G^F$.

The main result of this paper was announced in the 4th International Conference on Representation Theory, Lhasa, 2007.

1. Lusztig’s conjecture

1.1. Let $k$ be an algebraic closure of a finite field $F_q$ of characteristic $p$. Let $G$ be a connected reductive algebraic group defined over $k$. We fix a Borel subgroup $B$ of $G$, and a maximal torus $T$ contained in $B$, and a Weyl group $W = NG(T)/T$ of $G$ with respect to $T$. Let $DG$ be the bounded derived category of constructible $\mathcal{O}_T$-sheaves on $G$, and let $\mathcal{M}G$ be the full subcategory of $DG$ consisting of perverse sheaves. Let $S(T)$ be the set of isomorphism classes of tame local systems on $T$, i.e., the local systems $L$ of rank 1 such that $L^\otimes n \simeq \mathcal{O}_T$ for some integer $n \geq 1$, invertible in $k$. Take a local system $L \in S(T)$ such that $w^*L \simeq L$ for some $w \in W$. Then one can construct a complex $K_w \in DG$ as in [L2, III, 12.1]. For each $L \in S(T)$ we denote by $\widehat{G}_L$ the set of isomorphism classes of irreducible perverse sheaves $A$ on $G$ such that $A$ is a constituent of the $i$-th perverse cohomology sheaf $p^H(K_w^i)$ of $K_w^i$ for any $i, w$. The set $\widehat{G}$ of character sheaves on $G$ is defined as $\widehat{G} = \bigcup_{L \in S(T)} \widehat{G}_L$.

1.2. We consider the $F_q$-structure of $G$, and assume that $G$ is defined over $F_q$ with Frobenius map $F$. We assume that $B$ and $T$ are both $F$-stable. We assume further that the center of $G$ is connected. Let $G^T$ be the dual group of $G$ and $T^*$ a maximal torus of $G^*$ dual to $T$. By fixing an isomorphism $\iota : k^* \simeq Q^*/Z$ ($Q^*$ is the subring of $Q$ consisting of elements whose numerator is invertible in $k$), we have an isomorphism $f : T^* \simeq S(T)$ (see e.g., [S1, II, 1.4, 3.1]). Let $W^* = NG(T^*)/T^*$ be the Weyl group of $G^*$. Then $W^*$ may be identified with $W = NG(T)/T$, compatible with $f$. $F$ acts naturally on $S(T)$, via $F^{-1} : L \mapsto F^*L$, and the action of $F$ on $S(T)$ corresponds to the action of $F^{-1}$ on $T^*$ via $f$.

For each $s \in T^*$ such that the conjugacy class $\{s\}$ in $G^*$ is $F$-stable, we put

$$W_s = \{ w \in W^* \mid w(s) = s \},$$
$$Z_s = \{ w \in W^* \mid F(s) = w(s) \}.$$

Then $Z_s$ is non-empty, and one can write $Z_s = z_1W_s$ for some element $z_1 \in Z_s$. Since the center of $G$ is connected, $Z_{G^*}(s)$ is connected reductive, and $W_s$ is a Weyl group of $Z_{G^*}(s)$. We choose $z_1$ so that $\gamma = \gamma_s = z_1^{-1}F : W_s \mapsto W_s$ leaves invariant the set of simple roots of $Z_{G^*}(s)$ determined naturally from $B$ and $T$. 
Similarly, for any $\mathcal{L} \in \mathcal{S}(T)$ such that $F^*\mathcal{L} \simeq \mathcal{L}$, we define
\[ W_{\mathcal{L}} = \{ w \in W \mid w^*\mathcal{L} \simeq \mathcal{L} \}, \]
\[ Z_{\mathcal{L}} = \{ w \in W \mid F^*\mathcal{L} \simeq (w^{-1})^*\mathcal{L} \}. \]
Then $W_{\mathcal{L}}$ (resp. $Z_{\mathcal{L}}$) is naturally identified with $W_s$ (resp. $Z_s$).

1.3. Let $\text{Irr} \ G^F$ be the set of irreducible characters of $G^F$. Then $\text{Irr} \ G^F$ is partitioned into a disjoint union of subsets $\mathcal{E}(G^F, \{s\})$, where $s \in T^*$ and $\{s\}$ runs over all the $F$-stable semisimple classes in $G^*$. According to [L1], two parameter sets $X(W_s, \gamma)$ and $\overline{X}(W_s, \gamma)$ are attached to $\mathcal{E}(G^F, \{s\})$, and a non-degenerate pairing $\{ , \} : \overline{X}(W_s, \gamma) \times X(W_s, \gamma) \rightarrow \mathbb{Q}_l$ is defined. Here $\overline{X}(W_s, \gamma)$ is a finite set, and $X(W_s, \gamma)$ is an infinite set with a free action of the group $M$ of all roots of unity in $\mathbb{Q}_l$. More precisely, there exists a set $\overline{X}(W_s)$ with $\gamma$-action, and a natural map $X(W_s, \gamma) \rightarrow \overline{X}(W_s)$ whose image coincides with $\overline{X}(W_s)\gamma$, the set of $\gamma$-fixed points in $\overline{X}(W_s)$. In the case where $\gamma$ acts trivially on $W_s$, $X(W_s, \gamma)$ coincides with $\overline{X}(W_s)\gamma$.

Now the set $\mathcal{E}(G^F, \{s\})$ is parametrized by $\overline{X}(W_s, \gamma)$. We denote by $\rho_y$ the irreducible character in $\mathcal{E}(G^F, \{s\})$ corresponding to $y \in \overline{X}(W_s, \gamma)$. In turn, for each $x \in X(W_s, \gamma)$, an almost character $R_x$ is defined as
\begin{equation}
R_x = (-1)^{l(z_1)} \sum_{y \in \overline{X}(W_s, \gamma)} \{y, x\} \Delta(y) \rho_y,
\end{equation}
where $\Delta(y) = \pm 1$ is a certain adjustment in the case of exceptional groups $E_7, E_8$.

1.4. It is known by [L2, V], that the set $\tilde{G}_{\mathcal{L}}$ is parametrized by $\overline{X}(W_{\mathcal{L}}) = \overline{X}(W_s)$ under the identification $W_{\mathcal{L}} \simeq W_s$. For each $y \in \overline{X}(W_s)$, we denote by $A_y$ the corresponding character sheaf in $\tilde{G}_{\mathcal{L}}$. Let $\tilde{G}^F$ be the set of $F$-stable character sheaves, i.e., the set of $A \in \tilde{G}$ such that $F^*A \simeq A$. Then $\tilde{G}^F = \bigcup_{\mathcal{L} \in \mathcal{S}(T)} \tilde{G}_{\mathcal{L}}^F$, where $\mathcal{L}$ runs over the elements in $\mathcal{S}(T)$ such that $(Fw)^*\mathcal{L} \simeq \mathcal{L}$ for some $w \in W$. The set $\tilde{G}_{\mathcal{L}}^F$ is parametrized by $\overline{X}(W_{\mathcal{L}})\gamma$. For each $A \in \tilde{G}_{\mathcal{L}}^F$, we fix an isomorphism $\phi_A : F^*A \cong A$ as in [L2, V, 25.1]. Then $\phi_A$ is unique up to a root of unity multiple. We define a class function $\chi_A = \chi_{A, \phi_A}$ as the characteristic function $G^F \rightarrow \mathbb{Q}_l$ of $A$. In the case of classical groups, we have the following theorem, which is a (partial) solution to the Lusztig’ conjecture.

**Theorem 1.5** ([S1, II, Theorem 3.2]). Assume that $G$ is a (connected) classical group with connected center. Then for each $x \in X(W_s, \gamma)$, there exists an algebraic number $\zeta_x$ of absolute value 1 such that
\[ R_x = \zeta_x \chi_{A_x}, \]
where $\bar{x}$ is the image of $x$ under the map $X(W_s, \gamma) \rightarrow \overline{X}(W_s)\gamma$.

1.6. Assume that $G$ is a connected classical group with connected center. Let $P$ be an $F$-stable parabolic subgroup of $G$ containing $B$, and $L$ be an $F$-stable Levi subgroup of $P$ containing $T$, $U_P$ the unipotent radical of $P$. Then $W_L = N_L(T)/T$ is a Weyl subgroup of $W$, and $B_L = B \cap L$ is a Borel subgroup of $L$ containing $T$. Let $\tilde{L}$ be the set of character sheaves on $L$. We assume that $\tilde{L}_{\mathcal{L}}$ contains a cuspidal character sheaf $A_0$ for $\mathcal{L} \in \mathcal{S}(T)$, where $\mathcal{L}$ is $F^w$-stable for some $w \in W$. Then $A_0$
may be expressed by the intersection cohomology complex as \( A_0 = \mathrm{IC}(\Sigma, \mathcal{E})[\dim \Sigma] \), where \( \Sigma \) is the inverse image of a conjugacy class in \( \overline{G} = G/Z_0(G) \) under the natural map \( \pi : G \to \overline{G} \), and \( \mathcal{E} \) is a cuspidal local system on \( \Sigma \). The pair \((\Sigma, \mathcal{E})\), or its restriction on the conjugacy class, is called a cuspidal pair on \( G \). Then either \( L \) is the maximal torus or \( L \) has the same type as \( G \), and \( A_0 \) is a unique cuspidal character sheaf contained in \( \widehat{L}_L \). Consider the induced complex \( K = \text{ind}^G_L A_0 \) on \( G \). Then \( K \) is a semisimple perverse sheaf on \( G \) whose components are contained in \( \widehat{G} \). By Lemma 5.9 in [S1, I], the endomorphism algebra \( \text{End}_{\mathcal{M}G} K \) is isomorphic to the group algebra \( \mathbb{Q}_l[\mathcal{W}_G] \) of \( \mathcal{W}_G \), where

\[
\mathcal{W}_G = \{ n \in N_G(L) \mid n\Sigma n^{-1} = \Sigma, \text{ad}(n)\mathcal{E} \simeq \mathcal{E} \}/L,
\]

\[
\mathcal{Z}_G = \{ n \in N_G(L) \mid F(n\Sigma n^{-1}) = \Sigma, (F\text{ad}(n))\mathcal{E} \simeq \mathcal{E} \}/L.
\]

On the other hand, if we choose a positive integer \( r \) large enough, the set \( \mathcal{E}(L^{Fr}, \{ s \}) \) contains a unique cuspidal character \( \delta \) of \( L^{Fr} \), where \( s \in T^* \) corresponds to \( \mathcal{L} \) under \( f \). We define

\[
W_\delta = \{ w \in N_W(W_L) \mid wB_L w^{-1} = B_L, w \delta \simeq \delta \},
\]

\[
Z_\delta = \{ w \in N_W(W_L) \mid wB_L w^{-1} = B_L, Fw \delta \simeq \delta \}.
\]

Since \( \widehat{L}_L \) contains a unique cuspidal character sheaf, we have \( W_\delta \simeq \mathcal{W}_G, Z_\delta \simeq \mathcal{Z}_G \) by [S1, I, (5.16.1)]. Moreover there exists \( w_1 \in Z_\delta \) such that \( W_\delta = w_1 W_\delta \) and that \( \gamma_1 = Fw_1 : W_\delta \to W_\delta \) gives rise to an automorphism of the Coxeter group \( W_\delta \). We denote by \((W_\delta)\) the set of irreducible characters of \( W_\delta \), and \((W_\delta)_{\text{ex}}\), the subset of \((W_\delta)\) consisting of \( \gamma_1\)-stable characters. Then \( A_0 \) is \( Fw_1\)-stable, and for each \( E \in (W_\delta)_{\text{ex}} \), there exists \( x_E \in X(W_\gamma, \gamma) \) and \( \overline{x}_E \in \overline{X(W_\gamma)} \) such that \( A_E = A_{\overline{x}_E} \) is a \( F\)-stable character sheaf in \( \widehat{G}_L \) which is a simple component of \( K \) corresponding to \( E \in \text{End}_{\mathcal{M}G} K \). Moreover, \( \rho_{z_E} \in \mathcal{E}(G^{Fr}, \{ s \}) \) is an \( F\)-stable irreducible character which is a constituent of the Harish-Chandra induction \( \text{Ind}^{G^{Fr}}_{Fr} \delta \) corresponding to \( E \in (W_\delta)_{\text{ex}} \), and the image of the Shintani descent \( \text{Sh}_{Fr/E} \rho_{z_E} \) determines the almost character \( R_{z_E} \) of \( G^F \). (For the Shintani descent, see [S1]).

1.7. Let \( L_{w_1} \) be an \( F\)-stable Levi subgroup twisted by \( F(w_1) \), i.e., \( L_{w_1} = \alpha L_{w_1}^{-1} \) for \( \alpha \in G \) such that \( \alpha^{-1} F(\alpha) = F(w_1) \) for a representative \( w_1 \in N_G(L) \) of \( w_1 \in Z_\delta \). Then by \( \text{ad}(\alpha^{-1}) : L_{w_1} \simeq L, \text{ad}(\alpha^{-1}) \cdot A_0 \) gives rise to an \( F\)-stable cuspidal character sheaf on \( L_{w_1} \), which we denote by \( A_0' \). If we fix an isomorphism \( \varphi_0 : (Fw_1)^* \mathcal{E} \simeq \mathcal{E}, \varphi_0 \) induces an isomorphism \( \varphi_0^{w_1} : F^* A_0' \simeq A_0' \) on \( L_{w_1} \). We choose \( \phi_{A_0} : F^* A_0' \simeq A_0' \) as \( \phi_{A_0} = \varphi_0^{w_1} \). Then by Theorem 1.5, we have

\[
R_{0w_1} = \zeta_0 \chi_{A_0'}
\]

for some \( \zeta_0 \in \mathbb{Q}_l^* \) of absolute value 1, where \( R_{0w_1} \) is the almost character of \( L_{w_1} \) corresponding to \( A_0' \in \widehat{L}_{w_1} \). The following result was proved in [S1] in the course of the proof of the main theorem. (Note that in [S1], the constants \( \varepsilon_0 \xi_{A_0} \) and \( \varepsilon_0 \xi_{AE} \) are used. But the proof shows that these constants are indeed given by \( \zeta_0 = \varepsilon_0 \xi_{A_0} \).

**Lemma 1.8 ([S1, II, Lemma 3.7]).** Let \( \zeta_0 \) be as in (1.7.1). Then we have

\[
R_{z_E} = (-1)^{\dim \Sigma} \zeta_0 \chi_{AE}
\]

for any \( E \in (W_\delta)_{\text{ex}} \).
1.9. Lemma 1.8 shows that the determination of the scalars \( \zeta_e \) appeared in Theorem 1.5 is reduced to the case of cuspidal character sheaves. We note that it is further reduced to the case of adjoint groups. In fact, let \( A_0 \) be an \( F \)-stable cuspidal character sheaf contained in \( \hat{G}_L \). Let \( \pi : G \to \hat{G} \) be as before. Then \( A_0 \) can be written as \( A_0 \simeq \mathcal{E}_0 \otimes \pi^* \hat{A}_0[I] \), where \( I = \dim Z(G) \), and \( \mathcal{E}_0 \) is a local system on \( G \) which is the inverse image of \( \mathcal{E}_0' \in \mathcal{S}(G/G_{\text{der}}) \) under the natural map \( G \to G_{\text{der}} \) (\( G_{\text{der}} \) is the derived subgroup of \( G \)), and \( \hat{A}_0 \) is a cuspidal character sheaf on \( \hat{G} \). Since \( \hat{A}_0 \) is a unique cuspidal character sheaf in \( \hat{G}_L \), \( \hat{A}_0 \) is \( F \)-stable. Then \( \pi^* \hat{A}_0 \) is \( F \)-stable and so \( \mathcal{E}_0 \) is also \( F \)-stable. Then \( \phi_{A_0} : F^* A_0 \simeq A_0 \) is given by \( \phi_{A_0} = \varphi_0 \otimes \pi^* \hat{A}_0 \), where \( \phi_{A_0} : F^* \hat{A}_0 \simeq \hat{A}_0 \) is the map chosen for \( \hat{A}_0 \), and \( \varphi_0 \) is the pull-back of the canonical isomorphism \( F^* \mathcal{E}_0' \simeq \mathcal{E}_0' \). Hence \( \chi_{A_0} \) is written as \( \chi_{A_0} = \theta_0 \otimes \pi^* \chi_{\hat{A}_0} \), where \( \pi^* \chi_{\hat{A}_0} \) is the pull-back of \( \chi_{\hat{A}_0} \) under the induced map \( \pi : G^F \to \hat{G}^F \), and \( \theta_0 \) is a linear character of \( G^F \) corresponding to \( \mathcal{E}_0 \). A similar description works also for almost characters. Let \( R_0 \) (resp. \( \hat{R}_0 \)) be the almost character of \( G^F \) (resp. \( \hat{G}^F \)) corresponding to \( A_0 \) (resp. \( \hat{A}_0 \)). Then we have \( R_0 = \theta_0 \otimes \pi^* \hat{R}_0 \). (This follows from the fact that if \( \delta \) is a cuspidal irreducible character of \( G^F \) corresponding to \( A_0 \) for sufficiently large \( r \), then \( \delta \) can be written as \( \delta = \theta \otimes \delta' \), where \( \delta' \) is a cuspidal irreducible character of \( \hat{G}^F \) corresponding to \( \hat{A}_0 \), and \( \theta \) is an \( F \)-stable linear character of \( G^F \), and by applying the Shintani descent on \( \delta' \).) Thus \( \zeta_0 \) for \( A_0 \) coincides with \( \zeta_0 \) for \( \hat{A}_0 \).

2. generalized Green functions

2.1. Under the setting in 1.6, we further assume that \( G^F \) is of split type. Let \( L \) be as before. Assume that \( A_0 \) is a cuspidal character sheaf on \( L \) of the form \( A_0 = \text{IC}(\Sigma, \mathcal{E})[\dim \Sigma] \), where \( \Sigma = Z^0(L) \times C \) with a unipotent class \( C \) in \( L \) and \( \mathcal{E} = \mathcal{Q}_0 \boxtimes \mathcal{E}' \) for a cuspidal local system \( \mathcal{E}' \) on \( C \). Then \( W_\mathcal{E} = W = N_G(L)/L \). For each \( w \in W \), let \( L_w \) be an \( F \)-stable Levi subgroup of \( G \) obtained from \( L \) by twisting \( w \) as in 1.7, i.e., \( L_w = \alpha L \alpha^{-1} \) with \( \alpha \in G \) such that \( \alpha^{-1} F(\alpha) = F(\hat{w}) \) for a representative \( \hat{w} \in N_G(L) \) of \( W \). Put \( \Sigma_w = \alpha \Sigma \alpha^{-1}, \mathcal{E}_w = \text{ad}(\alpha^{-1})^* \mathcal{E} \), a local system on \( \Sigma_w \). We assume that the pair \((C, \mathcal{E}')\) is \( F \)-stable, and fix an isomorphism \( \varphi_0 : F^* \mathcal{E}' \simeq \mathcal{E}'_0 \). Then one can construct an isomorphism \( \varphi_0 : F^* \mathcal{E}' \simeq \mathcal{E}'_0 \). Then one can construct an isomorphism \( \varphi_w : F^* K_w \simeq K_w \), where \( K_w \) is a complex induced from the pair \((\Sigma_w, \mathcal{E}_w)\). Note that \( K_w \) is isomorphic to \( \text{ind}_P^G A_0 \), with a specific mixed structure twisted by \( w \in W \). We denote by \( \chi_{K_w, \varphi_w} \) the characteristic function of \( K_w \) with respect to \( \varphi_w \).

Since \( L \) is of the same type as \( G \), and \( F \) is of split type, \( \gamma_1 : W \to W \) is identity. Let \( K = \text{ind}_P^G A_0 = \bigoplus_{E \in W^\Lambda} V_E \otimes A_E \) be the decomposition of \( K \) into simple components, where \( A_E \) is a character sheaf corresponding to \( E \in W^\Lambda \), and \( V_E \) is the multiplicity space of \( A_E \) which has a natural structure of irreducible \( W \)-module corresponding to \( E \). Then there exists a unique isomorphism \( \phi_{AE} : F^* A_E \simeq A_E \) for each \( E \in W^\Lambda \) such that

\[
\chi_{K_w, \varphi_w} = \sum_{E \in W^\Lambda} \text{Tr} (w, V_E) \chi_{A_E}.
\]

Let \( G_{\text{uni}} \) be the unipotent variety of \( G \), and \( G^F_{\text{uni}} \) be the set of \( F \)-fixed points in \( G_{\text{uni}} \). The restriction of \( \chi_{K_w, \varphi_w} \) on \( G^F_{\text{uni}} \) is the generalized Green function \( Q^F_{w, \varphi_w} \left( C_w, \mathcal{E}_w, (\varphi_w)_{\mathcal{E}_w} \right) \) ([L2, II, 8.3]), where \( C_w = \alpha C \alpha^{-1}, \mathcal{E}_w = \text{ad}(\alpha^{-1})^* \mathcal{E} \), local system on \( C_w \), and
(φ₀)₀ is the restriction of (φ₀)₀ on ℰ₀. On the other hand, by the generalized Springer correspondence, for each ℰ ∈ ℭ, there exists a pair (C₁, ℰ₁), where C₁ is a unipotent class in G and ℰ₁ is a G-equivariant simple local system on C₁, such that

\[(2.1.2) \quad A_E|_{G_{uni}} \simeq IC(C₁, ℰ₁)[\dim C₁ + \dim Z^0(L)].\]

Now the pair (C₁, ℰ₁) is F-stable, and φₐₖ : F*ₐₖA_E ∼ₖ A_E determines an isomorphism ψ_{ℰ₁} : F*ₐₖℰ₁ ∼ₖ ℰ₁ via (2.1.2) (cf. [L2, V, 24.2]). In other words, the choice of φ₀ : F*ℰ' ∼ₖ ℰ' determines ψ_{ℰ₁}. Let \(\chi_{(C₁, ℰ₁)} : C₁ \to Q₁\) be the characteristic function of ℰ₁ with respect to ψ_{ℰ₁}. Lusztig ([L2, V, 24]) gave an algorithm of computing \(\chi_{A_E} = \chi_{A_E, φₐₖ}\) on \(G_{uni}^p\). Here \(\chi_{A_E}|_{G_{uni}^p}\) is expressed in terms of a linear combination of various \(\chi_{(C₁', ℰ₁')}\), where \((C₁', ℰ₁')\) is a pair as above corresponding to some ℰ' ∈ ℭ. Let \(P_{E, E'}\) be the coefficient of \(\chi_{(C₁, ℰ₁)}\) in the expansion of \(\chi_{A_E}\). Then \(P_{E, E'}\) satisfies the following property: if we replace \(F\) by \(F^m\) for any integer \(m > 0\), we obtain a similar coefficient \(P_{E, E'}(q^m)\) starting from \(φ₀(q^m) : (F^m)*ℰ' ∼ₖ ℰ'\) induced naturally from \(φ₀\), which we denote by \(P_{E, E'}(q^m)\). Then there exists a rational function \(P_{E, E'}(x)\) such that \(P_{E, E'}(x) = P_{E, E'}(q^m)\). (Note: It is shown in [L2, V] that \(P_{E, E'}(x)\) turns out to be a polynomial if \(p\) is good.)

Now \(\chi_{(C₁, ℰ₁)}\) is described as follows: take \(u ∈ C₁\) and put \(A_G(u) = Z_G(u)/Z_G^0(u)\). Then \(F\) acts naturally on \(A_G(u)\), and in our setting \(F\) acts trivially on it. The set of \(G\)-conjugacy classes in \(C₁\) is in 1:1 correspondence with the set \(A_G(u)\) (note: \(A_G(u)\) is abelian). We denote by \(uₐ\) a representative of a \(G\)-class in \(C₁\) corresponding to \(a ∈ A_G(u)\). On the other hand, the set of \(G\)-equivariant simple local systems on \(C₁\) is in 1:1 correspondence with the set \(A_G(u)^\wedge\) of irreducible characters of \(A_G(u)\). For each \(ρ ∈ A_G(u)^\wedge\), we define a function \(f_ρ\) on \(G_{uni}^p\) by

\[(2.1.3) \quad f_ρ(v) = \begin{cases} ρ(a) & \text{if } v = uₐ ∈ C₁, \\ 0 & \text{if } v \notin C₁ \end{cases}
\]

for \(v ∈ C₁\). Let \(ρ ∈ A_G(u)^\wedge\) be the character corresponding to \(ℰ₁\). Then there exists \(η_E ∈ Q₁\) of absolute value 1 such that

\[(2.1.4) \quad \chi_{(C₁, ℰ₁)} = η_E f_ρ.
\]

Note that \(η_E\) depends on the choice of \(φ₀ : F*ℰ' ∼ₖ ℰ'\) and on the choice of \(u ∈ C₁\). We have the following theorem.

**Theorem 2.2 ([S3]).** Let \(G\) be a classical group, simple modulo center. Assume that the derived subgroup of \(G\) does not contain the Spin group. Further assume that \(G\) is of split type. Then for each unipotent class \(C₁\) in \(G\), there exists \(u ∈ C₁\) (called a split unipotent element) satisfying the following: Let \((C, ℰ')\) be the pair in \(L\) as in 2.1 and \(u₀ ∈ C₁\) be a split element. Choose \(φ₀ : F*ℰ' ∼ₖ ℰ'\) so that the isomorphism \((φ₀)₀ : ℰ'₀ → ℰ'₀\) induced on the stalk \(ℰ'₀\) of ℰ' at \(u₀\) is identity. Choose a split element \(u ∈ C₁\) for defining \(f_ρ\) in (2.1.3). Then \(η_E = 1\) for any \(E ∈ ℭ\).

### 2.3.

Returning to the setting in 2.1, we choose a split element \(u ∈ C₁\) for each unipotent class \(C₁\) of \(G\). Then \(u ∈ C₁^F^m\) for any integer \(m > 0\) (in fact, it is a split element with respect to \(G^F^m\)), and we choose \(uₐ^{(m)} ∈ C₁^F^m\), a representative of \(G^F^m\)-class in \(C₁^F^m\) for each \(a ∈ A_G(u)\). For later discussion, we prepare a notation.
Let $G$ be an adjoint simple group of classical type. We assume that $G$ is of split type over $F_q$. Let $\hat{G}$ be the set of cuspidal character sheaves on $G$. $A \in \hat{G}$ is given in the form $A = IC(\mathcal{C}, \mathcal{E})[\dim C]$, where $C$ is a conjugacy class in $G$, and $\mathcal{E}$ is a simple $G$-equivariant local system on $C$. We shall describe the cuspidal character sheaves on $G$ (cf. [L2, V, 22.2, 23.2], see also [S2, 6.6]) and their mixed structures.

(a) $G = PSp_{2n}$ ($n \geq 1$) with $p$ : odd. $\hat{G}$ is empty if $n$ is even. Assume that $n$ is odd. Then for each pair $(N_1, N_2)$ such that $N_i = d_i^2 + d_i$ for some integers $d_i \geq 0$ and that $n = N_1 + N_2$, one can associate cuspidal character sheaves on $G$ as follows. Let $C$ be a conjugacy class of $g = su = us$, where $s$ is a semisimple element of $G$ such that $Z_0^G(s)$ is isomorphic to $H = (Sp_{2N_1} \times Sp_{2N_2})/\{\pm 1\}$, and $u$ is a unipotent element of $Z_0^G(s)$ isomorphic to $H$ such that the unipotent class $C_0$ containing $u$ gives a unique cuspidal pair $(C_0, \mathcal{E}_0)$ with unipotent support of $H$. Here $(C_0, \mathcal{E}_0)$ is described as follows. There exists a cuspidal pair $(C_i, \mathcal{E}_i)$ for $Sp_{2N_i}$, such that $C_0 = C_1 \times C_2$ and $\mathcal{E}_0 = \mathcal{E}_1 \boxtimes \mathcal{E}_2$. Choose $u = (u_1, u_2) \in C_0$ such that $u_i \in C_i$. Let $\rho_i \in A_{H_i}(u_i)^{\wedge}$ corresponding to $\mathcal{E}_i$, where $H_i = Sp_{2N_i}$. Then $\rho_1 \boxtimes \rho_2 \in (A_{H_1}(u_1) \times A_{H_2}(u_2))^{\wedge}$ factors through $A_H(u)$ and defines an irreducible character $\rho_0$ of $A_H(u)$ corresponding to $\mathcal{E}_0$.

Now assume that $N_1 \neq N_2$. Then $Z_G(s)$ is connected, and so $A_G(g) = A_H(u)$, and $\rho_0$ gives an irreducible character $\rho \in A_G(g)^{\wedge}$ which determines a local system $\mathcal{E}$ on $C$, and we denote by $A_{N_1, N_2}$ the character sheaf corresponding to $(C, \mathcal{E})$. Next assume that $N_1 = N_2$. Then $A_G(s) \simeq \mathbf{Z}/2\mathbf{Z}$ and $A_H(u)$ is a subgroup of $A_G(g)$ of index 2. We have $\text{Ind}_{A_H(u)}^{A_G(g)} \rho_0 = \rho + \rho'$, where $\rho, \rho'$ are linear characters of $A_G(u)$. If we write $\mathcal{E}, \mathcal{E}'$ the simple local system on $C$ corresponding to $\rho, \rho'$, then the pairs $(C, \mathcal{E}), (C, \mathcal{E}')$ are both cuspidal pairs of $G$. We denote by $A_{N_1, N_2}, A'_{N_1, N_2}$ the cuspidal character sheaves on $G$ corresponding to $(C, \mathcal{E}), (C, \mathcal{E}')$, respectively. The set $\hat{G}$ consists of these elements.

We shall fix a mixed structure on $(C, \mathcal{E})$. Since $s \in G^F$, $H$ is $F$-stable, and so $(C_0, \mathcal{E}_0)$ is also $F$-stable. Choose $u = (u_1, u_2) \in C_0^F$ such that $u_i$ are split unipotent elements of $G$, then we say that $h$ is a rational function in $q$ if there exists a rational function $H_{C_1, a}(x)$ for each pair $(C_1, a)$ such that $h^{(m)}(u^{(m)}_a) = H_{C_1, a}(q^m)$. For each $E \in W^\wedge$, we have an isomorphism $\phi^{(m)}_{AE} : (F^m)^*A_E \simeq A_E$, and one can define a function $\chi^{(m)}_{AE} = \chi_{AE} \circ \phi^{(m)}_{AE}$ on $G^{F^m}$. Then in view of Theorem 2.2, Lusztig’s algorithm implies that $\{\chi^{(m)}_{AE} \in G_{\text{uni}}^{F^m}\}_{m>0}$ is a rational function in $q$.

Thus, by (2.1.1) we have the following corollary.

**Corollary 2.4.** Assume that $G$ is as in Theorem 2.2, and that $G^F$ is of split type. The generalized Green function $Q_{L_w, C_w, \mathcal{E}_w, (\varphi_0)_w}$ can be expressed as a rational function in $q$.

### 2.5. More generally, if there exists a family of values $h = \{h^{(m)} \in \mathbf{Q}_1\}_{m>0}$ such that $h^{(m)} = H(q^m)$ for some rational function $H(x)$, we say that $h$ is a rational function in $q$.

### 3. Cuspidal character sheaves

#### 3.1.

Let $G$ be an adjoint simple group of classical type. We assume that $G$ is of split type over $F_q$. Let $\hat{G}$ be the set of cuspidal character sheaves on $G$. $A \in \hat{G}$ is given in the form $A = IC(\mathcal{C}, \mathcal{E})[\dim C]$, where $C$ is a conjugacy class in $G$, and $\mathcal{E}$ is a simple $G$-equivariant local system on $C$. We shall describe the cuspidal character sheaves on $G$ (cf. [L2, V, 22.2, 23.2], see also [S2, 6.6]) and their mixed structures.
elements in $Sp_{2N_1}$, and fix $s \in T^F$ appropriately. We choose $\varphi_0 : F^*E \simeq E$ so that the induced isomorphism $((\varphi_0)_g) : E_g \to E_g$ on the stalk $E_g$ at $g$ is identity. Then $\varphi_0$ induces an isomorphism $\varphi : F^*A_{N_1,N_2} \simeq A_{N_1,N_2}$. We define $\phi_A = \phi_{A_{N_1,N_2}}$ by $\phi_A = q^{(\dim G - \dim C)/2}\varphi$. A similar construction is applied also for $(C,E')$.

(b) $G = PSO_m$ ($m \geq 3$) with $p \equiv 0$. $\hat{G}^0$ is empty unless $m$ is either odd or divisible by 8. Note that $PSO_m = SO_m$ if $m$ is odd. To each pair $(N_1,N_2)$ such that $N_1 = d_1^2$ for some $d_i \geq 1$ and that $m = N_1 + N_2$, one can associate cuspidal character sheaves $A$ associated to $(C,E)$ as follows. Let $C$ be the conjugacy class of $G$ containing $g = su = us$, where $s$ is a semisimple element such that $H = Z^0_G(s)$ is isomorphic to $SO_{N_1} \times SO_{N_2}$ if $m$ is odd, and to $(SO_{N_1} \times SO_{N_2})/(\{\pm 1\})$ if $m$ is even, and $u$ is a unipotent element in $Z^0_G(s)$ such that the unipotent class $C_0$ containing $u$ is a unique cuspidal pair $(C_0,E_0)$ with unipotent support on $H$. Here $C_0 = C_1 \times C_2$, $E_0 \simeq E_1 \boxtimes E_2$ with the cuspidal pair $(C_1,E_1)$ on $SO_{N_1}$. Choose $u = (u_1,u_2) \in C_0$ such that $u_i \in C_i$. Let $\rho_i \in A_{H_i}(u_i)\wedge$ corresponding to $E_i$, where $H_i = SO_{N_i}$. Then $\rho_1 \boxtimes \rho_2 \in (A_{H_1}(u_1) \times A_{H_2}(u_2))\wedge$ factors through $A_H(u)$ and gives an irreducible character $\rho_0 \in A_H(u)\wedge$ corresponding to $E_0$. Depending on the structure of $A_G(s)$, the three cases occur.

(i) The case where $N_1 = 0$ or $N_2 = 0$. In this case, $Z_G(s)$ is connected and so $A_G(u) \simeq A_H(u)$. $\rho_0$ gives an index 2 subgroup of $A_G(g)^\wedge$, which determines a local system $\mathcal{E}$ on $C$, and $(C,\mathcal{E})$ corresponds to the cuspidal character sheaf $A_{N_1,N_2}$.

(ii) The case where $N_1 > 0, N_2 > 0, N_1 \neq N_2$. Then $A_G(s) \simeq Z/2Z$ and $A_H(u)$ is regarded as an index 2 subgroup of $A_G(g)$. We have $\text{Ind}_{A_H(u)}^{A_G(g)} \rho_0 = \rho + \rho'$ for $\rho, \rho' \in A_G(g)\wedge$. If we write $\mathcal{E}, \mathcal{E}'$ the simple local system corresponding to $\rho, \rho'$, $(C,\mathcal{E}), (C,\mathcal{E}')$ are both cuspidal pairs for $G$. We denote by $A_{N_1,N_2}, A'_{N_1,N_2}$ the cuspidal character sheaves corresponding to them.

(iii) The case where $N_1 = N_2$. In this case $A_G(s) \simeq Z/2Z \times Z/2Z$ and so $A_G(g)/A_H(u) \simeq Z/2Z \times Z/2Z$. $\text{Ind}_{A_H(u)}^{A_G(g)} \rho_0$ decomposes into 4 irreducible (linear) characters, $\rho, \rho', \rho'', \rho'''$ of $A_G(g)$. Correspondingly, we have simple local systems $\mathcal{E}, \mathcal{E}', \mathcal{E}'', \mathcal{E}'''$ on $C$, and all of them give cuspidal pairs on $G$. We denote by $A_{N_1,N_2}, A'_{N_1,N_2}, A''_{N_1,N_2}, A'''_{N_1,N_2}$ the cuspidal character sheaves corresponding to them.

All of the above three cases give the set $\hat{G}^0$. We shall fix a mixed structure on cuspidal character sheaves. Since $s \in G^F$, $H$ is $F$-stable, and so $(C_0,E_0)$ is $F$-stable. Take $u = (u_1,u_2) \in C_0^F$ such that $u_i$ are split elements in $SO_{N_i}$, and fix $s \in T^F$. We choose $\varphi_0 : F^*E \simeq E$ so that the induced isomorphism $((\varphi_0)_g) : E_g \to E_g$ on the stalk $E_g$ at $g$ is identity. $\varphi_0$ induces an isomorphism $\varphi : F^*A \simeq A$ for $A = A_{N_1,N_2}$. We define $\phi_A$ by $\phi_A = q^{(\dim G - \dim C)/2}\varphi$. We define similarly for $A'_{N_1,N_2}, A''_{N_1,N_2}, A'''_{N_1,N_2}$.

(c) $G = Sp_{2n}$ ($n \geq 1$) with $p = 2$. $\hat{G}^0$ is empty unless $n = d^2 + d$ for some $d \geq 1$. Assume that $n = d^2 + d$. Then $G$ contains a unique cuspidal pair $(C,E)$. The set $\hat{G}^0$ consists of a single character sheaf $A$ associated to $(C,E)$. We fix a mixed structure of $A$. $C$ is an $F$-stable unipotent class of $G$, and we take a split element $u \in C^F$. We fix an isomorphism $\varphi_0 : F^*E \simeq E$ so that the induced isomorphism $((\varphi_0)_u) : E_u \to E_u$ on the stalk $E_u$ of $E$ at $u$ is identity. $\varphi_0$ induces $\varphi : F^*A \simeq A$. We define $\phi_A$ by $\phi_A = q^{(\dim G - \dim C)/2}\varphi$. 
4. Then we have

\[ \mathcal{L} \text{USZTIG'S CONJECTURE FOR FINITE CLASSICAL GROUPS WITH EVEN CHARACTERISTIC} \]

where \( d \) is an integer such that \( d \geq 1 \). Assume that \( n = 4d^2 \). Then \( G \) contains a unique cuspidal pair \((C, \mathcal{E})\). The set \( \mathcal{E}^0 \) consists of a single character sheaf \( A \) associated to \((C, \mathcal{E})\). \( C \) is an \( F \)-stable unipotent class of \( G \), and we take a split element \( u \in C^F \). We define \( \phi_A \) in a similar way as in the case (c), by \( \phi_A = q^{(\dim G - \dim C)/2} \).

We show the following lemma.

**Lemma 3.2.** Let \( \rho \) be the irreducible character of \( A_G(g) \) corresponding to the local system \( \mathcal{E} \) on \( C \), as in 3.1. Then \( \rho \) is a linear character such that \( \rho^2 = 1 \). A similar fact holds also for \( \rho' \), \( \rho'' \), \( \rho''' \) if there exists any.

**Proof.** Let \( \rho \) be one of the characters \( \rho, \rho', \rho'', \rho''' \) if there exists any. It is enough to show that \( \rho(a^2) = 1 \) for any \( a \in A_G(g) \). By investigating the structure of \( A_G(g) \), we see that \( A_G(g) \) is an elementary abelian 2-group if \( N_1 \neq N_2 \). Thus in this case, \( \rho(a^2) = 1 \). We assume that \( N_1 = N_2 \). Then \( a \) is odd, and \( G \) is \( PSO_{2n} \) or \( PSO_{2n} \). Assume that \( G = PSO_{2n} \). We have \( A_G(g) \simeq \langle \sigma \rangle \rtimes A_H(u) \), where \( \sigma \) is an element of order 2 permuting two factors of \( A_H(u) \). In this case \( a \in A_G(g) \) is of order 2 or 4. If \( a \) has order 2, there is nothing to prove. Assume that \( a \) has order 4. Then we have \( a^2 \in A_H(u) \). Put \( \theta = \text{Ind}_{A_H(u)}^{A_G(g)} \rho_0 \). Since \( A_H(u) \) is an elementary abelian 2-group, and \( \rho_0 \) is \( \sigma \)-stable, we see that \( \theta(a^2) = |A_G(g)|/|A_H(u)| \). This shows that \( \rho(a^2) = 1 \) for any irreducible factor \( \rho \) of \( \theta \). Next assume that \( G = SO_{2n} \). In this case, \( A_G(g) \simeq \langle \sigma \rangle \rtimes A_H(u) \), where \( A_H(u) \) is an elementary abelian 2-group containing \( A_H(u) \) as an index 2 subgroup, and \( \sigma \) is an element of order 2 acting on \( A_H(u) \). \( \sigma \) stabilizes \( A_H(u) \) permuting their two factors, and \( \rho_0 \) is \( \sigma \)-stable. Thus a similar argument shows that \( \rho(a^2) = 1 \). The lemma is proved.

4. Symbols and unipotent characters

4.1. Irreducible characters contained in \( \mathcal{E}(G^F, \{1\}) \) are called unipotent characters. In the case of classical groups, unipotent characters are parametrized by a combinatorial object called symbols. In this section, we review unipotent characters of classical groups.

Let \( G \) be a classical group over \( \mathbb{F}_q \) of type \( B_n, C_n \) or \( D_n \). We assume that \( G^F \) is of split type if \( G \) is of type \( D_n \). The set of unipotent characters of \( G^F \) is parameterized by symbols. A symbol is an (unordered) pair \((S, T)\) of finite subsets of \( \{0, 1, 2, \ldots \} \) modulo the shift operation \((S, T) \sim (S', T') \) with \( S' = \{0\} \cup (S + 1) \), \( T' = \{0\} \cup (T + 1) \). The rank of a symbol \( A = (S, T) \) is defined by

\[
\rho(A) = \sum_{\lambda \in S} \lambda + \sum_{\mu \in T} \mu - \left[ \frac{|S| + |T| - 1}{2} \right]^2,
\]

where \([z]\) denotes the largest integer which does not exceed \( z \). The defect \( d(A) \) of \( A \) is defined by the absolute value of \(|S| - |T|\). The rank and the defect are independent of the shift operation.

For each integer \( d \geq 0 \), we denote by \( \Phi_n^d \) the set of symbols of rank \( n \) and defect \( d \). In the case where \( A = (S, T) \) is defect 0, \( A \) is said to be degenerate if \( S = T \), and is said to be non-degenerate otherwise. We denote by \( \Phi_n^0 \) the set of symbols of rank
Then the unipotent characters of $G^F$ of type $B_n$ or $C_n$ (resp. $D_n$ of split type) are parametrized by $\Phi_n$ (resp. $\Phi_n^+$). In the notation of 1.3, $W_s = W$ and $\gamma = 1$ since $F$ is of split type, and $\Phi_n$ or $\Phi_n^+$ is nothing but $\overline{\chi}(W_s, \gamma) = \overline{\chi}(W, 1)$. We denote by $\rho_A$ the unipotent character of $G^F$ corresponding to $A \in \Phi_n$ or $\Phi_n^+$. The unipotent cuspidal character exists if and only if $n = d^2 + d$ (resp. $n = 4d^2$) for some integer $d \geq 1$ if $G$ is of type $B_n$ or $C_n$ (resp. $D_n$). In these cases, the symbol $A_c$ (the cuspidal symbol) corresponding to the (unique) cuspidal unipotent character is given as follows.

$$A_c = \begin{cases} (0, 1, 2, \ldots , 2d) \in \Phi_n^{2d+1} & (G : \text{type } B_n \text{ or } C_n, n = d^2 + d), \\ (0, 1, 2, \ldots , 4d - 1) \in \Phi_n^{4d} & (G : \text{type } D_n, n = 4d^2). \end{cases}$$

4.2. We introduce a notion of families in $\Phi_n$ or $\Phi_n^+$. Two symbols $A, A'$ belong to the same family if $A, A'$ are represented by $\left(\frac{s}{T} \right)$, $\left(\frac{s'}{T'} \right)$ such that $S \cup T = S' \cup T'$ and that $S \cap T = S' \cap T'$. Families give a partition of $\Phi_n$ or $\Phi_n^+$. A symbol $A \in \Phi_n$ of defect 1 is called a special symbol if $A = \left(\frac{s}{T} \right)$ with $S = \{a_0, a_1, \ldots , a_m\}$ and $T = \{b_1, \ldots , b_m\}$ such that $a_0 \leq b_1 \leq a_1 \leq \cdots \leq b_m \leq a_m$. Similarly, a symbol $A \in \Phi_n^+$ of defect 0 is called a special symbol if $A = \left(\frac{s}{T} \right)$ with $S = \{a_1, \ldots , a_m\}$, $T = \{b_1, \ldots , b_m\}$ such that $a_1 \leq b_1 \leq \cdots \leq a_m \leq b_m$. Each family contains a unique special symbol. Let $\mathcal{F}$ be a (non-degenerate) family. Then any symbol $A \in \mathcal{F}$ can be expressed as

$$A = A_M = \begin{pmatrix} Z_2 \coprod (Z_1 - M) \\ Z_2 \coprod M \end{pmatrix},$$

for some $M$, where $Z_1, Z_2$ are determined by $\mathcal{F}$; $Z_2$ is the set of elements which appear in both rows of $A$, $Z_1$ is the set of singles in $A$, and $M$ is a subset of $Z_1$. The map $M \mapsto A_M$ gives a bijective correspondence between the set of subsets $M$ of $Z_1$ such that $|M| \equiv d_1 \pmod{2}$ and $\mathcal{F}$, where $|Z_1| = 2d_1 + 1$ (resp. $|Z_1| = 2d_1$) if $\mathcal{F} \subset \Phi_n$ (resp. $\mathcal{F} \subset \Phi_n^+$) for some integer $d_1 \geq 1$. (In the case of $\mathcal{F} \subset \Phi_n$, we further assume that the smallest element in $M$ is bigger that that of $Z_1 - M$.) In particular, the special symbol in $\mathcal{F}$ can be written as $A_{M_0}$ for some $M_0 \subset Z_1$ such that $|M_0| = d_1$. For $M \in Z_1$, put $M^2 = M_0 \cup M - M_0 \cap M$. We define a pairing

$$(A_M, A_{M'}) = \frac{1}{2^f} (-1)^{|M \cap M'|},$$

where $f = d_1$ (resp. $f = d_1 - 1$) if $\mathcal{F} \subset \Phi_n$ (resp. $\mathcal{F} \subset \Phi_n^+$). We extend this pairing to the pairing on $\Phi_n$ or $\Phi_n^+$ by requiring that $\mathcal{F}$ and $\mathcal{F}'$ are orthogonal if $\mathcal{F} \neq \mathcal{F}'$, which we denote by the same symbol. Note that the pairing $(\cdot , \cdot)$ on $\Phi_n$ or $\Phi_n^+$ coincides with the pairing $(\cdot , \cdot)$ on $\overline{\chi}(W, 1)$ given in 1.3. Hence, for each $A \in \mathcal{F}$, the almost character $R_A$ is given as

$$(4.2.2) \quad R_A = \sum_{A' \in \mathcal{X}_n} \{A, A'\} \rho_{A'},$$

where $\rho_A$ is the unipotent character of $A$. The symbol $\mathcal{X}_n$ is the set of defect 0, where the degenerate symbols are counted twice. We put

$$\Phi_n = \prod_{d \equiv 0 \pmod{4}} \Phi_n^d, \quad \Phi_n^+ = \prod_{d \equiv 0 \pmod{4}} \Phi_n^d^d.$$
where $X_n = \Phi_n$ or $\Phi_+^*$ according to the cases $G$ is of type $B_n$ or $C_n$, or $G$ is of type $D_n$. By the property of the pairing \{ , \}, one can also write, for each $A \in \mathcal{F}$,

$$
\rho_A = \sum_{A' \in X_n} \{A, A'\} R_{A'}.
$$

4.3. Assume that $G^F$ contains a cuspidal unipotent character, and denote by $\mathcal{F}_c$ the family containing the cuspidal symbol $A_c$. Then the special symbol $A_0$ contained in $\mathcal{F}_c$ is given as follows.

$$
A_0 = \begin{cases}
(0, 2, 4, \ldots, 2d) & (G : \text{type } B_n \text{ or } C_n, n = d^2 + d), \\
(0, 2, \ldots, 4d - 2) & (G : \text{type } D_n, n = 4d^2).
\end{cases}
$$

In the case where $G$ is of type $B_n$ or $C_n$, we have $Z_1 = \{0, 1, \ldots, 2d\}$ and $M_0 = \{1, 3, \ldots, 2d - 1\}$. In the case where $G$ is of type $D_n$, we have $Z_1 = \{0, 1, \ldots, 4d - 1\}$ and $M_0 = \{1, 3, \ldots, 4d - 1\}$. In both cases, $A_0$ is given by $A_c = A_M$ with $M = \emptyset$. We denote by $R_0 = R_{A_0}$ the cuspidal almost character.

4.4. Let $A_0$ be the cuspidal character sheaf of $G$ corresponding to $R_0 = R_{A_0}$. Let $(C, \xi)$ be the cuspidal pair corresponding to $A_0$. Then it is contained in the list in 3.1. If $p = 2$, it is uniquely determined since $\widetilde{G}^0$ consists of a single element. In the case where $p \neq 2$, the explicit correspondence is known by [S2, Prop. 6.7], and see also [L3]. The conjugacy class is given as follows (though we don’t need it in later discussions). We use the notation in 3.1. Let $g = su = us \in C$ with $H = Z_2^{\xi}(s)$. Assume that $G = \text{PSp}_{2n}$ with $n = d^2 + d$. Then $H$ is isogeneous to $\text{Sp}_{d^2 + d} \times \text{Sp}_{d^2 + d}$. Assume that $G = \text{PSO}_{2n+1}$ with $n = d^2 + d$. Then $H$ is isogeneous to $\text{SO}_{(d+1)^2} \times \text{SO}_{d^2}$. Assume that $G = \text{PSO}_{2n}$ with $n = 4d^2$. Then $H$ is isogeneous to $\text{SO}_{4d^2} \times \text{SO}_{4d^2}$.

5. Symmetric space over finite fields

In this section, we apply the theory of symmetric space over finite fields to the problem of determining the scalars $\zeta_x$ occurring in Lusztig’s conjecture (Theorem 1.5)] for $G_{F^2}$.

5.1. Let $G$ be a connected reductive group over a finite field $\mathbf{F}_q$ with Frobenius map $F$. We consider the symmetric space $G_{F^2}/G^F$. For a class function $f$ on $G_{F^2}$, we define $m_2(f)$ by

$$
m_2(f) = \langle \text{Ind}_{G^F}^{G_{F^2}} 1, f\rangle_{G_{F^2}} = \frac{1}{|G^F|} \sum_{x \in G^F} f(x).
$$

In the case where $G$ has a connected center, $m_2(\rho)$ is determined by Kawanaka [K], Lusztig [L4] for any irreducible character $\rho$ of $G_{F^2}$.

5.2. Let $C$ be an $F$-stable conjugacy class in $G$. Take $x \in C^F$ and let $A_G(x)$ be the component group of $Z_G(x)$ as before. $F$ acts naturally on $A_G(x)$. We assume that $F$ acts trivially on $A_G(x)$, and then the set of $G^F$-conjugacy classes in $C^F$ is in bijection with the set $A_G(x)/\sim$ of conjugacy classes in $A_G(x)$. The correspondence is given as follows; for each $a \in A_G(x)$, take a representative $\hat{a} \in Z_G(x)$. There exists $h_a \in G$ such that $h_a^{-1}F(h_a) = \hat{a}$. Then $x_a = h_a x h_a^{-1}$ in contained in $C^F$,
and the set \( \{ x_a \mid a \in A_G(x) \} \) gives a complete set of representatives of the \( G^F \)-conjugacy classes in \( C^F \).

The above description works also for the case of \( C^{F^2} \). We denote by \( \{ y_a \mid a \in A_G(x) \} \) the set of \( G^{F^2} \)-conjugacy classes in \( C^{F^2} \). We define a class function \( f_\tau \) on \( G^{F^2} \) for each \( \tau \in A_G(x) \) as follows.

\[
(5.2.1) \quad f_\tau(g) = \begin{cases} \tau(a) & \text{if } g \text{ is } G^{F^2}\text{-conjugate to } y_a, \\ 0 & \text{if } g \notin C^{F^2}. \end{cases}
\]

We have the following lemma.

**Lemma 5.3.** Let \( \tau \in A_G(x) \) be a linear character such that \( \tau^2 = 1 \). Then we have

\[
m_2(f_\tau) = |C^F|/|G^F|.
\]

**Proof.** Take \( g \in C^F \). Then \( g \) is \( G^F \)-conjugate to an \( x_a \in C^F \) for some \( a \in A_G(x) \). Then there exists \( h \in G \) such that \( g = h x h^{-1} \) and that \( h^{-1} F(h) = \hat{a} \in Z_G(x) \). Since \( F \) acts trivially on \( A_G(x) \), we may choose \( \hat{a} \in Z_G(x)^F \). We have

\[
h^{-1} F^2(h) = h^{-1} F(h) \cdot F(h^{-1} F(h)) = \hat{a} F(\hat{a}) = \hat{a}^2.
\]

It follows that any element \( g \in C^F \) is \( G^{F^2} \)-conjugate to \( y_{\hat{a}^2} \) for some \( \hat{a} \in A_G(x) \). Hence we have \( m_2(g) = \tau(\hat{a}^2) = 1 \) by our assumption. We have

\[
m_2(f_\tau) = \frac{1}{|G^F|} \sum_{g \in G^F} f_\tau(g) = |C^F|/|G^F|
\]

as asserted. The lemma is proved. \( \square \)

**5.4.** Assume that \( G \) is as in 3.1, and we use the notation there. Let \( A = IC(\overline{C}, \mathcal{E})[\dim C] \) be the cuspidal character sheaf on \( G \). (Here \( \mathcal{E} \) represents one of the simple local systems \( \mathcal{E}, \mathcal{E}', \ldots \) on \( C \) if there exist more than one). Then \( A \) is \( F \)-stable. Let \( \rho \in A_G(g)^\wedge \) be the irreducible character corresponding to \( \mathcal{E} \). Note that \( F \) acts trivially on \( A_G(g) \). We consider the class function \( f_\rho \) on \( G_{uni}^{F^2} \) defined as in (5.2.1) for \( \tau = \rho \). We show that

\[
(5.4.1) \quad m_2(f_\rho) = q^{-(\dim G - \dim C)}.
\]

In fact thanks to Lemma 3.2, one can apply Lemma 5.3 and we have \( m_2(f_\rho) = |C^F|/|G^F| \). Since \( F \) acts trivially on \( A_G(g) \), \( C^F \) splits into several \( G^F \)-conjugacy classes, which are parametrized by \( A_G(g) \). Let \( a_1, \ldots, a_r \) be the representatives of the conjugacy classes in \( A_G(g) \), and let \( C_i \) the \( G^F \)-conjugacy classes in \( C^F \) corresponding to \( a_i \). We choose \( g_i \in C_i \). Since \( |Z_G(g_i)^F| = |A_A(a_i)| |Z_{G^F}(g_i)^F| \), where \( A = A_G(g) \), we have

\[
|C^F|/|G^F| = \sum_{i=1}^r |C_i|/|G^F| = \sum_{i=1}^r |Z_A(a_i)|^{-1} |Z_{G^F}(g_i)^F|^{-1}.
\]

Here \( Z_{G^F}(g_i)^F \cong Z_{H_i}^0(u_i)^F \), where \( g_i = s_i u_i = u_i s_i \) and \( H_i = Z_{G^F}(s_i) \). Note that \( Z_{H_i}^0(u_i) \cong Z_{H_i}^0(u) \), and since \( u \in C_0 \), where \( (C_0, \mathcal{E}_0) \) is a cuspidal pair with unipotent support, it is known by [L2,I, Prop.3.12] that \( Z_{H_i}^0(u) \) is a unipotent group. It follows that

\[
|Z_{G^F}(g_i)^F| = |Z_{H_i}^0(u_i)^F| = q^{\dim H - \dim C_0} = q^{\dim G - \dim C}.
\]
Hence we have
\[ |C^F|/|G^F| = q^{-(\dim G - \dim C)} \sum_{i=1}^{r} |Z_A(a_i)|^{-1} = q^{-(\dim G - \dim C)}. \]

This proves (5.4.1).

Let \( \varphi_0 : (F^2)^*E \cong E, \varphi : (F^2)^*A \cong A \) and \( \phi_A : (F^2)^*A \cong A \) be as in 3.1, but replacing \( F \) by \( F^2 \). Then the characteristic function \( \chi_{E, \varphi_0} \) on \( C^{F^2} \) coincides with \( f_{\rho |_{C^{F^2}}} \). Since \( A \) is clean, the function \( \chi_{A, \varphi} \) coincides with \( f_{\rho} |_{C^{F^2}} \). It follows that \( \chi_A \) coincides with \( q^{\dim G - \dim C} f_{\rho} \). This implies that

**Lemma 5.5.** Let \( A \) be a cuspidal character sheaf of \( G \). Then we have
\[ m_2(\chi_A) = 1. \]

**5.6.** Let \( s \in T^* \) be such that the class \( \{s\} \) is \( F^2 \)-stable. Then there exists \( s_0 \in \{s\} \) such that \( F^2(s_0) = s_0^{-1} \). Let \( H = Z^0_B(s_0) \). Then \( H \) is an \( F^2 \)-stable reductive subgroup of \( G \). It is known, since the center of \( G \) is connected, that there exists a natural bijection \( E(G^{F^2}, \{s\}) \leftrightarrow E(H^{F^2}, \{1\}) \), \( \rho \leftrightarrow \rho_{\text{uni}} \). Concerning the values of \( m_2(\rho) \) and \( m_2(\rho_{\text{uni}}) \), the following result is known. (For unipotent characters, we follow the notation in Section 4.)

**Theorem 5.7 ([K], [L4]).** Let \( G \) be a connected classical group with connected center. Then

(i) If there does not exist \( s_0 \in \{s\}^{F^2} \) such that \( F(s_0) = s_0^{-1} \), then \( m_2(\rho) = 0 \) for any \( \rho \in E(G^{F^2}, \{s\}) \). If there exists such \( s_0 \), then under the notation of 5.6, \( m_2(\rho) = m_2(\rho_{\text{uni}}) \) for any \( \rho \in E(G^{F^2}, \{s\}) \).

(ii) Assume that \( G \) is of type \( B_n \) or \( C_n \). Let \( F \) be a family in \( \Phi_n \) such that \( |Z_1| = 2d_1 + 1 \) (cf. 4.2). Then we have
\[ m_2(\rho_A) = \begin{cases} 2^{d_1} & \text{if } A \text{ is special,} \\ 0 & \text{otherwise.} \end{cases} \]

(iii) Assume that \( G \) is of type \( D_n \). Let \( F \) be a non-degenerate family in \( \Phi_n^+ \) such that \( |Z_1| = 2d_1 \) (cf. 4.2). Then we have
\[ m_2(\rho_A) = \begin{cases} 2^{d_1-1} & \text{if } A \text{ is special,} \\ 0 & \text{otherwise.} \end{cases} \]

If \( F = \{A, A'\} \) is a degenerate family, then we have
\[ m_2(\rho_A) = m_2(\rho_{A'}) = \begin{cases} 1 & \text{if } F \text{ is of split type,} \\ 0 & \text{otherwise.} \end{cases} \]

In view of (4.2.2), we have the following corollary.

**Corollary 5.8 ([L4]).** Let \( G \) be a classical group of split type. Assume that \( F \) is a non-degenerate family. Then for any \( A \in F \), we have \( m_2(R_A) = 1 \).

**5.9.** Let \( A \) be as in 5.4 and assume that \( A \in \hat{G}_L \). Let \( s \in T^* \) be such that the class \( \{s\} \) corresponds to \( \mathcal{L} \) via \( f \) in 1.2. Then \( s^2 = 1 \). Let \( R_0 \) be the almost character of \( G^F \) corresponding to \( A \) as given in Theorem 1.5. Then by Lemma 5.5, we have \( m_2(R_0) \neq 0 \). Note that \( R_0 \) is a linear combination of irreducible characters
contained in $\mathcal{E}(G_{F^2}, \{s\})$. Thus by Theorem 5.7, (i), there exists $s_0 \in \{s\}$ such that $s_0 \in G_{F^2}$ and that $F(s_0) = s_0^{-1} = s_0$. Then $H = Z_G^0(s_0)$ is an $F$-stable reductive subgroup of $G$, and so $H$ is split over $F_{q^2}$. Recall that the almost character $R_0$ is given as in (1.3.1). Then it is known that $(-1)^{l(z_1)} = \sigma(H)\sigma(G)$, where $\sigma(G)$ is a split rank of $G$ with respect to $F_{q^2}$, and vice versa for $H$. Since $G, H$ are split, we have $l(z_1) = \sigma(H)\sigma(G) = 1$. Let $R^H_0$ be the almost character of $H_{F^2}$ obtained from $R_0$ under the correspondence $\rho \leftrightarrow \rho_{uni}$. Then we see that

\[ m_2(R_0) = m_2(R^H_0) = 1. \]

In fact, the first equality follows from Theorem 5.7, (i), together with the fact that $l(z_1) = 1$. The second equality follows from Theorem 5.7, (ii), (iii).

Combining Lemma 5.5 with (5.9.1), we have the following theorem.

**Theorem 5.10.** Let $G$ be an adjoint simple group of classical type. Let $A$ be a cuspidal character sheaf, and $\chi_A = \chi_{A, \delta_A}$ be the characteristic function of $A$ on $G_{F^2}$ (defined as in 3.1). Let $R_0$ be the almost character of $G_{F^2}$ corresponding to $A$. Then we have

\[ R_0 = \chi_A. \]

As a corollary we have the following result, which holds without any restriction on $p$ nor $q$.

**Corollary 5.11.** Let $G$ be a connected classical group with connected center. Then the constants $\zeta_x$ appearing in Lusztig’s conjecture (Theorem 1.5) can be determined for $G_{F^2}$ in the following cases; under the notation of 1.6, assume that $W_\delta = Z_\delta$. Then we have

\[ R_{x_E} = (-1)^{\dim \Sigma} \chi_A \]

for any $E \in W^\wedge_\delta$. In other words, we have $\zeta_{x_E} = (-1)^{\dim \Sigma}$.

**Proof.** Lemma 1.8 together with the argument in 1.9 shows that the determination of $\zeta_x$ is reduced to the case of $\zeta_0$ (the one corresponding to the cuspidal character sheaf) in the case of adjoint simple groups. We know that $\zeta_0 = 1$ by Theorem 5.10. Then the corollary follows from Lemma 1.8, together with 1.9 since $L_{F^2} = L_{F^2}^\wedge$. \(\square\)

**Remark 5.12.** In the case where $G = PSO_{2n}$ or $SO_{2n+1}$, Corollary 5.11 gives a complete answer for the determination of constants $\zeta_x$ for $G_{F^2}$ since we have always $W_\delta = Z_\delta$ in that case. In the case where $G = PSO_{2n}$, the corollary holds if $R_{x_E}$ is a linear combination of unipotent characters, i.e., if $A_{x_E} \in \widehat{G}_{Q_i}$. But it happens that $W_\delta \neq Z_\delta$ for some $\delta$.

**6. From $G_{F^2}$ to $G_{F}$**

**6.1.** The results Theorem 5.10 and Corollary 5.11 in the previous section are only valid for the group $G_{F^2}$. In this section, by using a certain specialization argument, we extend those results to the group $G_{F}$ as far as $\chi_A$ are concerned with unipotent characters. In the case of $p = 2$, this implies the extension of Theorem 5.10 and Corollary 5.11 for the group $G_{F}$ of split type. We have the following theorem.
Theorem 6.2. Let $G$ be a classical group of split type over $\mathbb{F}_q$. Let $A$ be a cuspidal character sheaf contained in $\hat{G}_{\mathbb{Q}}$, and $\chi_A = \chi_{A,\phi_A}$ be the characteristic function of $A$ over $G^F$. Let $R_0$ be the almost character of $G^F$ corresponding to $A$. Then we have

$$R_0 = \chi_A.$$  

As a corollary, we have

Corollary 6.3. Let $G = Sp_{2n}$ or $SO_{2n}$ with $p = 2$. Assume that $G^F$ is of split type. Then the constants $\zeta_E$ appearing in Lusztig’s conjecture (Theorem 1.5) can be determined completely for $Sp_{2n}$, and partly for $SO_{2n}$. More precisely, under the notation of 1.6, assume that $W_5 = Z_5$. Then we have

$$R_{x,E} = (-1)^{\dim E} \chi_{A_E}$$

for any $E \in W_5^\wedge$. In particular, (6.3.1) holds if $A_E \in \hat{G}_\mathbb{Q}$ in the case where $G = SO_{2n}$.

Proof. As in the proof of Corollary 5.11, the determination of $\zeta_E$ is reduced to that of $\zeta_0$. Assume that $G = Sp_{2n}$ or $SO_{2n}$ with $p = 2$. In this case, it is known by 3.1, (c), (d), that a unique cuspidal character sheaf (if it exists) is always contained in $\hat{G}_{\mathbb{Q}}$. Hence Theorem 6.2 can be applied, and the corollary follows from Lemma 1.8 (see also Remark 5.12). □

Remark 6.4. A similar argument as in Corollary 6.3 works also for the case where $p \neq 2$. In particular the formula (6.3.1) holds for the case where $A_E \in \hat{G}_\mathbb{Q}$. Thus we rediscover the results in Theorem 6.2 in [S2]. Although the argument given in [S2] can not be applied to the case where $p = 2$, the proof here works simultaneously for arbitrary $p$.

6.5. The remainder of this paper is devoted to the proof of Theorem 6.2. In what follows, we assume that $G$ is a classical group containing a cuspidal unipotent character. Hence $G$ is of type $B_n$ or $C_n$ with $n = d^2 + d$, or of type $D_n$ with $n = 4d^2$. We assume further that $G^F$ is of split type. We follow the notation in Section 4.

Let $\mathcal{I}_q$ be the $G^F$-module $Ind_{B^F}^{G^F} 1$ induced from $B^F$ to $G^F$. Then the irreducible component of $\mathcal{I}_q$ is in bijective correspondence with $W^\wedge$. We denote by $\rho_E$ the irreducible $G^F$-module occurring in $\mathcal{V}_q$ corresponding to $E \in W^\wedge$. $\rho_E$ gives a unipotent character, which we denote by $\rho_{A_E}$ with $A_E \in X_n$. Let $\mathcal{H}$ be the Iwahori-Hecke algebra over $\mathbb{Q}[u^{1/2}, u^{-1/2}]$ associated to the Coxeter system $(W,S)$ with generators $\{T_s \mid s \in S\}$. $\mathcal{H}$ has a basis $\{T_w \mid w \in W\}$, where $T_w$ is defined as $T_w = T_{s_{i_1}} \cdots T_{s_{i_k}}$ for a reduced expression $w = s_{i_1} \cdots s_{i_k}$. $\mathcal{H}$ is characterized by the following properties;

$$\begin{cases} (T_u - u)(T_u + 1) = 0, \\ T_uT_w = T_{uw} \text{ if } l(sw) = l(w) + 1, \end{cases}$$

where $l : W \rightarrow \mathbb{Z}_{>0}$ is the length function of $W$.

In the case of type $B_n$ or $D_n$, the generator set $S$ of $W$ is described as follows. Assume that $W$ is the Weyl group of type $B_n$. Then $W$ is realized as a group of signed permutations of $I = \{1, 1, 2, 2, \ldots, n, n\}$. The set $S$ of generators is given as $S = \{s_0, s_1, \ldots, s_{n-1}\}$ with $s_0 = (1,1), s_1 = (1,2), \ldots, s_{n-1} = (n-1,n)$, and we
denote by $T_i$ the generator of $\mathcal{H}$ corresponding to $s_i$. Note that the subalgebra of $\mathcal{H}$ generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra of type $A_{n-1}$. Next assume that $W$ is of type $D_n$. Then $W$ is a subgroup of the Weyl group of type $B_n$, generated by $s_i' = (1, \overline{i})(j, \overline{j}), s_1, \ldots, s_{n-1}$. We denote by $T'_0, T'_1, \ldots, T'_{n-1}$ the corresponding generators of $\mathcal{H}$.

The endomorphism algebra $\text{End}_{G^F} I_q$ is isomorphic to the specialized algebra $\mathbb{Q}_l \otimes \mathcal{H}$ via the algebra homomorphism $\mathbb{Q}[u^{1/2}, u^{-1/2}] \rightarrow \mathbb{Q}_l$ by $u \mapsto q$, which we denote by $\mathcal{H}_q$. We denote by $\mathcal{E}_q$ the irreducible representation of $\mathcal{H}_q$ corresponding to $E \in W^\wedge$. Now $I_q$ has a structure of $G^F \times \mathcal{H}_q$-module, and the trace for $g \in G^F, T_w \in \mathcal{H}_q$ is written as

$$\text{(6.5.1)} \quad \text{Tr} ((g, T_w), I_q) = \sum_{E \in W^\wedge} \text{Tr} (g, \rho_E) \text{Tr} (T_w, E_q).$$

By replacing $\rho_E = \rho_{\Lambda_E}$ by $R_A$ by using (4.2.3), we have

$$\text{(6.5.2)} \quad \text{Tr} ((g, T_w), I_q) = \sum_{\Lambda \in X_n} f_{\Lambda}(w) R_A(g),$$

where

$$\text{(6.5.3)} \quad f_{\Lambda}(w) = \sum_{E \in W^\wedge} \{A, \Lambda_E\} \text{Tr} (T_w, E_q).$$

It is known that $\text{Tr} (T_w, E_q)$ is a polynomial in $q$ in the sense of 2.3. Hence $f_{\Lambda}(w)$ is also a polynomial in $q$. We are interested in $f_{\Lambda}(w)$ in the case where $\Lambda$ is the cuspidal symbol $\Lambda_c$, and we want to find some special $w \in W$ such that $f_{\Lambda_c}(w) \neq 0$. Let $W$ be the Weyl group of type $B_n$ or $D_n$. Then any element of $W$ can be expressed as a product of positive cycles and negative cycles, where the number of negative cycles is even if $W$ is of type $D_n$. We have the following proposition.

**Proposition 6.6.** There exists an element $w \in W$ such that $f_{\Lambda_c}(w) \neq 0$, where either $w$ is a Coxeter element in $W$, or $w$ contains a positive cycle of length $\geq 2$.

**6.7.** The proof of the proposition will be given in Section 7. Here assuming the proposition, we continue the proof of the theorem. We prove the theorem by induction on the semisimple rank of $G$, and so we assume that the theorem holds for the classical groups of the smaller semisimple rank. Let $A$ be the cuspidal character sheaf on $G$ as in the theorem, and let $C$ be the conjugacy class which is the support of $A$. We choose $g = su = us \in C^F$ as in 3.1. We choose $w \in W$ as in Proposition 6.6. We consider the equation (6.5.2) simultaneously for the groups $G^{F^m}$ for any integer $m \geq 1$. Note that $g \in C^{F^m}$ and it has a uniform description for any $m \geq 1$ since the split unipotent element for $G^F$ is split for any extended group $G^{F^m}$. Then we can write (6.5.2) as

$$\text{(6.7.1)} \quad \text{Tr} ((g, T_w), I_{q^m}) = \sum_{\Lambda \neq \Lambda_c} f_{\Lambda}(w)(q^m) R^{(m)}_{\Lambda}(g) + f_{\Lambda_c}(w)(q^m) R^{(m)}_{\Lambda_c}(g),$$

where $T_w$ is an element of $\mathcal{H}_{q^m}$, and $R^{(m)}_{\Lambda}$ denotes the almost character of $G^{F^m}$. By induction hypothesis and by Remark 6.4, the formula (6.3.1) holds for $R^{(m)}_{\Lambda}$ if $\Lambda \neq \Lambda_c$. We show the following lemma.
LEMMA 6.8. Assume that $A \neq A_c$. Then under the induction hypothesis, $R_A^{(m)}(g)$ is a rational function in $q$.

PROOF. Let $A_A$ the character sheaf corresponding to $R_A$, and denote by $\chi_{A_A}^{(m)}$ the characteristic function on $G^{F_{um}}$ associated to $A_A$. In view of (6.3.1), it is enough to show that $\chi_{A_A}^{(m)}(g)$ is a rational function in $q$. Since $A \in \tilde{G}_{Q_l}$, by applying Corollary 4.10 in [S2] ($a_0 = 1$ in the notation there since $F$ is of split type), the computation of $\chi_{A_A}^{(m)}(g)$ is reduced to that of $\chi_{A_A}^{(m)}(u)$ for a subgroup $H^{F_{um}}$. So we consider $\chi_{A_A}^{(m)}$ on $G^{F_{um}}$. $A_A$ is a direct summand of a certain complex $K = \text{ind}_{\tilde{G}^0}^G A_0$, where $A_0 \in \tilde{L}_{Q_l}$ is a cuspidal character sheaf of a Levi subgroup $L$. Then the computation of $\chi_{A_A}^{(m)}|_{G^{F_{um}}}$ is reduced to the computation of the generalized Green functions of $G^{F_{um}}$ associated to $L^{F_{um}}$. Hence by Corollary 2.4, we obtain the lemma. \[\Box\]

6.9. Next we consider the left hand side of (6.7.1). The $G^F$-module $I_q$ is a permutation representation of $G^F$ on $G^F/B^F$. Since the action of $G^F$ on $G^F/B^F$ is independent of the isogeny, we may assume that $G = Sp_N$ or $SO_N$ and that $g \in G^F$. Let $V$ be the vector space over $F_q$ of dimension $V = N$, equipped with a non-degenerated alternating form (resp. symmetric bilinear form) $f$ on $V$ if $G = Sp_N$ (resp. $SO_N$). In the case where $G = SO_N$ with $p = 2$ and $N$ is even, we also consider the quadratic form $Q$ on $V$. Now the set $G^F/B^F$ may be identified with the set of flags $\mathcal{F}_q$ as follows; A flag $F = (V_0 \subset V_1 \subset \cdots \subset V_n)$ is a sequence of subspaces of $V$ such that $\dim V_i = i$ and that $V_i$ are isotropic with respect to $f$, where $N = 2n$, $2n+1$ or $2n$ according to the cases where $G = Sp_{2n}, SO_{2n+1}$ or $SO_{2n}$.

In the case where $p = 2$ and $G = SO_{2n}$, we assume further that the restriction of $Q$ on $V_n$ is zero. Now in the case where $G = Sp_{2n}$ or $SO_{2n+1}$, $\mathcal{F}_q$ consists of all flags on $V$. $G^F$ acts naturally on $\mathcal{F}_q$ via $x : (V_0 \subset V_1 \subset \cdots \subset V_n) \mapsto (xV_0 \subset \cdots \subset xV_n)$ for $x \in G^F$, and the $G^F$-set $\mathcal{F}_q$ is identified with the $G^F$-set $G^F/B^F$. In the case where $G = SO_{2n}$, we define $\tilde{\mathcal{F}}_q$ as the set of all flags on $V$ as above, then $G^F$ acts on $\tilde{\mathcal{F}}_q$ with two $G^F$-orbits, $\mathcal{F}_q$ and $\mathcal{F}_q'$. Either of them can be identified with $G^F/B^F$, and we have a natural bijection between $\mathcal{F}_q$ and $\mathcal{F}_q'$, which is given in the form $(V_0 \subset V_1 \subset \cdots \subset V_n) \mapsto (V_0 \subset V_1 \subset \cdots \subset V_n')$, (only the term $V_n$ is changed to $V_n'$).

We consider the vector space $\mathcal{F}_q$ over $\bar{Q}_l$ with basis $\mathcal{F}_q$, which is identified with $I_q$. By the identification $I_q \simeq \mathcal{F}_q$, $\mathcal{H}_q$ acts on $\mathcal{F}_q$, whose action is given as follows; let $F = (V_0 \subset \cdots \subset V_n)$. For $i = 1, \ldots, n-1$, we have

$$FT_{n-i} = \sum_{W \neq V_i} (V_0 \subset \cdots \subset V_i \subset W \subset V_{i+1} \subset \cdots \subset V_n),$$

where the sum is taken over all the isotopic subspaces $W$ such that $V_{i-1} \subset W \subset V_{i+1}$ and that $\dim W = i$, $W \neq V_i$. In the case of $B_n$, we have

$$FT_0 = \sum_{W \supset V_{n-1}} (V_0 \subset \cdots \subset V_{n-1} \subset W),$$
where the sum is taken over all the isotropic subspaces \( W \) such that \( \dim W = n \) and \( W \neq V_n \). In the case of type \( D_n, \) we have

\[
FT'_0 = \sum_{W \supset W' \supset V_{n-2}} (V_0 \subset \cdots \subset V_{n-2} \subset W' \subset W),
\]

where the sum is taken over the isotropic subspaces \( W \supset W' \) such that \( \dim W = n, \dim W' = n - 1 \) and \( W' \) contains \( V_{n-2} \) and some more conditions.

It follows from the description of the action of \( \mathcal{H} \) on \( I_q, \) we see that

(6.9.1) Assume that \( w \in W \) contains a positive cycle of length \( \geq 2 \). Then there exists \( k \geq 1 \) such that for any \( F = (V_0 \subset \cdots \subset V_n), \) \( FT_w \) is a linear combination of \( F' = (V'_0 \subset \cdots \subset V'_{n'}) \) such that \( V_k = V'_k. \)

We show the following lemma.

**Lemma 6.10.** Under the induction hypothesis, \( \text{Tr}((g,T_w), I_{q^n}) \) is a rational function in \( q. \)

**Proof.** The following argument was inspired by \([HR2]\), where the combinatorial properties of \( \text{Tr}((u,T_w), I_q) \) is discussed in the case of \( GL_n(F_q) \) with a unipotent element \( u. \) First assume that \( w \) contains a positive cycle of length \( \geq 2 \). Then by (6.9.1), for any flag \( F = (V_0 \subset \cdots \subset V_n), \) \( FT_w \) is a linear combination of \( F' = (V'_0 \subset \cdots \subset V'_{n'}) \) such that \( V_k = V'_k. \) We now prepare a notation. If \( v \in \mathcal{J}_q, \) and \( F \in \mathcal{F}_q, \) we denote by \( w|_F \) the coefficient of \( F \) in the expression of \( v \) as a linear combination of base vectors. Let \( F = (V_0 \subset \cdots \subset V_n) \in \mathcal{F}_q, \) and assume that \( gFT_w|_F \neq 0. \) Since \( gFT_w|_F = FT_w|_F', g^{-1}F = (V_0' \subset \cdots \subset V_n') \) is of the form that \( V_k = V'_k = g^{-1}V_k. \) It follows that \( V_k \) is stabilized by \( g. \) Thus we have

\[
\text{Tr}((g,T_w), I_q) = \sum_{F \in \mathcal{F}_q} gFT_w|_F
\]

\[
= \sum_{W, F=(V_0 \subset \cdots \subset V_0 \subset V_n)} gFT_w|_F
\]

\[
= \sum_{W, F=(V_0 \subset \cdots \subset V_n)} gFT_w|_F \sum_{F''=(W \subset \cdots)} gF''T_w|_{F''}
\]

where \( W \) runs over all the isotropic subspaces in \( V \) such that \( \dim W = k \) and that \( gW = W. \) Let \( H' = GL(W), \) and \( H'' \) be the group of isometries \( Sp(W) \) or \( SO(W) \)
for \( W = W^*/W. \) Let \( I_q^W \simeq \mathcal{J}_q^W \) be the corresponding induced modules for \( H'^W, \) and similarly define \( I_q^W \simeq \mathcal{J}_q^W \) for \( H''^W. \) \( g \) acts naturally on \( W \) (resp. on \( W^* \)), and we denote by \( g_W \in H'^W \) (resp. \( g_W^* \in H''^W \)) the corresponding elements. Also the action of \( T_w \) on \( I_q \) induces an action on \( \mathcal{J}_q^W \) (resp. on \( \mathcal{J}_q^W \)) which is given by \( T_{w'} \) (resp. \( T_{w''} \)) with an element \( w' \) (resp. \( w'' \)) in the Weyl group of \( H' \) (resp. \( H'' \)). Then the last sum can be written as

(6.10.1) \( \text{Tr}((g,T_w), I_q) = \sum_W \text{Tr}((g_W,T_{w'}), I_q^W) \text{Tr}((g_W^*,T_{w''}), I_q^W). \)

Let \( F_0 = (V_0^0 \subset \cdots \subset V_{n}^0) \) be the standard flag whose stabilizer in \( G^F \) is \( B^F, \) and put \( W_k = V_k^0. \) Then there exists an \( F \)-stable maximal parabolic subgroup \( P \) of \( G \) containing \( B \) such that \( P^F \) is the stabilizer of \( W_k \) in \( G^F. \) Let \( L \) be an \( F \)-stable Levi subgroup of \( P \) containing \( T. \) Then \( L \) is isomorphic to \( L' \times L'', \) where \( L' = GL_k \) and \( L'' \) is a similar group as \( G \) of rank \( n - k. \) Let \( g'_1, \ldots, g'_r \) (resp. \( g''_1, \ldots, g''_s \))
such that \( g \in G^F \) is conjugate to \( g \) under \( G^F \). We denote by \( X_q^{ij} \) the set of \( W \) such that \( W = xW_k \) and that \( x^{-1}gx \) is conjugate to \( g_{ij} \) in \( L^F \). Then (6.10.1) implies that

\[
(6.10.2) \quad \text{Tr} ((g, T_w), I_q) = \sum_{s=1}^{r} \sum_{j=1}^{s} X_q^{ij} \text{Tr} ((g', T_{w'}), I_q^{L^F}) \text{Tr} ((g''_q, T_{w''}), I_q^{L''F}),
\]

where \( I_q^{L^F}, I_q^{L''F} \) are corresponding induced modules for \( L^F, L''F \). Choose \( x_{ij} \in G^F \) such that \( g = x_{ij}g_{ij}x_{ij}^{-1} \) for \( i, j \). Then \( X_q^{ij} \) is in bijection with the set \( Z_G(g)^F x_{ij}L^F / L^F \), hence \( |X_q^{ij}| = |Z_G(g)^F| / |Z_L(g_{ij})^F| \).

We now consider (6.10.2) for any \( G^{Fm} \). Then the choice of representatives \( g_{ij} \in G^{Fm} \) does not depend on \( m \), and we see that \( |X_q^{ij}| \) is a rational function in \( q \). On the other hand, one can write as in (6.5.1)

\[
\text{Tr} ((g'_{q}, T_{w''}), I_q^{L''}) = \sum_{E \in W_{L''}} \text{Tr} (g''_q, \rho_E) \text{Tr} (T_{w''}, E_q),
\]

where \( W_{L''} \) is the Weyl group of \( L'' \). By induction hypothesis, \( R_A^{(m)}(g')_q \) is a rational function in \( q \) for any almost character \( R_A \) of \( L''F \). Hence \( \text{Tr} (g''_q, \rho_E) \) is a rational function in \( q \). It follows that \( \text{Tr} ((g''_q, T_{w''}), I_q^{L''}) \) is a rational function in \( q \). Similarly, and as it is known since \( L' = GL_k \), \( \text{Tr} ((g'_{q}, T_{w''}), I_q^{L''}) \) is a rational function in \( q \). Thus we conclude that \( \text{Tr} ((g, T_w), I_q^{m}) \) is a rational function in \( q \) as asserted.

Next assume that \( w \) is a Coxeter element of \( W \). We note that

\[
(6.10.3) \quad \text{Tr} ((x, T_w), I_q) = \begin{cases} 
q^r & \text{if } x_u \text{ is regular unipotent}, \\
0 & \text{otherwise},
\end{cases}
\]

for \( x \in G^F \), where \( x_u \) is the unipotent part of \( x \) and \( r \) is the semisimple rank of \( G \). In fact (6.10.3) is discussed in [HR2, Prop. 3.2] in the case where \( G = GL_n \).

The argument there works in general if we notice that \( Z_G(v) = Z_G(\rho) \) for a regular unipotent element \( v \in U^F \) and that \( |Z_{\rho}(v)| = q^r \), where \( U \) is the unipotent radical of \( B \). (6.10.3) implies that \( \text{Tr} ((g, T_w), I_q^{m}) \) is a polynomial in \( q \). Hence the lemma holds.

\[ \square \]

6.11 We now prove the theorem. In the formula (6.7.1) the left hand side is a rational function in \( q \) by Lemma 6.10. For for \( A \neq A_e, R_A^{(m)}(g) \) is a rational function in \( q \) by Lemma 6.8. Since \( f_A(w) \) is a polynomial in \( q \), and \( f_A(w) \neq 0 \) by Proposition 6.6, we see that \( R_A^{(m)}(g) \) is a rational function in \( q \). By Theorem 1.5, one can write \( R_A^{(m)}(g) = \zeta^{(m)}(\chi_{A_0}^{(m)}(g) \gamma_1(\chi_{A_0}^{(m)}(g) \gamma_1)^{-1} \zeta^{(m)} \in \mathbb{Q}_1^* \) of absolute value 1, where \( A_0 = A_{A_e} \) is the cuspidal character sheaf. We know that \( \chi_{A_0}^{(m)}(g) \) is a non-zero polynomial in \( q \). We also know by Theorem 5.10 that \( \zeta^{(m)} = 1 \) for even \( m \). It follows that \( R_A^{(m)}(g)/\chi_{A_0}^{(m)}(g) \) is a rational function in \( q \), and takes the value 1 for any power of \( q^r \). Hence \( R_A^{(m)}(g) = \chi_{A_0}^{(m)}(g) \) for any \( m \), and we have \( \zeta^{(m)} = 1 \). This shows that \( R_A = \chi_{A_0} \), and the theorem is proved (modulo Proposition 6.6).
7. Proof of Proposition 6.6

7.1. Recall that $\Phi^1_n$ is the set of symbols of rank $n$ and defect 1 as in 4.1. In the case where $W$ is the Weyl group of type $B_n$, the set $W^\wedge$ is in bijection with $\Phi^1_n$. The correspondence is given as follows; let $P_n$ be the set of of double partitions $(\lambda, \mu)$ such that $|\lambda| + |\mu| = n$. Then $W^\wedge$ is parametrized by $P_n$. For a double partition $(\lambda, \mu) \in P_n$, we write $\lambda = (\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_m)$ and $\mu = (\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m)$ with $\lambda_i, \mu_i \geq 0$ for some integer $m$. We put $a_i = \lambda_i + i, b_i = \mu_i + (i-1)$, and define the sets $S, T$ by $S = \{a_0, a_1, \ldots, a_m\}$ and $T = \{b_1, \ldots, b_m\}$. Then $\Lambda = (\frac{\delta}{T}) \in \Phi^1_n$, and this gives a bijective correspondence between $\Phi^1_n$ and $P_n$, and so gives a bijection between $\Phi^1_n$ and $W^\wedge$. As in 6.5, we denote by $A_E$ the symbol in $\Phi^1_n$ corresponding to $E \in W^\wedge$. Assume that $n = d^2 + d$, and let $A_0$ be the special symbol in $F_c$. Then $A_0 = A_{E_0}$, where $E_0$ corresponds to $(\lambda, \mu) \in P_n$ such that $\lambda = (0 \leq 1 \leq \cdots \leq d), \mu = (1 \leq 2 \leq \cdots \leq d)$.

Recall that $\Phi^0_n$ is the set of symbols of rank $n$ and defect 0 as in 4.1. In the case where $W$ is the Weyl group of type $D_n$, $W^\wedge$ is in bijection with $\Phi^0_n$, which is given as follows; let $\mathcal{P}_n$ be the set of unordered partitions $(\lambda, \mu)$ such that $|\lambda| + |\mu| = n$, where $(\lambda, \lambda)$ is counted twice. Then $W^\wedge$ is parametrized by $\mathcal{P}_n$. For $(\lambda, \mu) \in \mathcal{P}_n$, we write $\lambda = (\lambda_1 \leq \cdots \leq \lambda_m), \mu = (\mu_1 \leq \cdots \leq \mu_m)$ with $\lambda_i, \mu_i \geq 0$ for some integer $m$. We put $a_i = \lambda_i + (i-1), b_i = \mu_i + (i-1)$, and define the sets $S, T$ by $S = \{a_1, \ldots, a_m\}, T = \{b_1, \ldots, b_m\}$. Then $\Lambda = (\frac{\delta}{T}) \in \Phi^0_n$, and this gives a bijective correspondence between $\Phi^0_n$ and $W^\wedge$. As in 6.5, we denote by $A_E$ the symbol in $\Phi^0_n$ corresponding to $E \in W^\wedge$. Assume that $n = 4d^2$ and let $A_0$ be the special symbol in $F_c$. Then $A_0 = A_{E_0}$, where $E_0$ corresponds to $(\lambda, \mu) \in \mathcal{P}_n$ such that $\lambda = (0 \leq 1 \leq \cdots \leq 2d - 1), \mu = (1 \leq 2 \leq \cdots \leq 2d)$.

First we show the following lemma.

**Lemma 7.2.** Assume that $E \in W^\wedge$ is such that $A_E \in F_c$. If $E$ corresponds to $(\lambda, \mu) \in \mathcal{P}_n$ (resp. $(\lambda, \mu) \in \mathcal{P}_n$), in the case where $W$ is of type $B_n$ (resp. $D_n$), then we have

$$\{A_c, A_E\} = \begin{cases} \frac{1}{2^d}(-1)^{|\lambda| + d(d+1)/2} & \text{if } G \text{ is of type } B_n, \\ \frac{1}{2^{2d-1}}(-1)^{|\lambda| + d(2d-1)} & \text{if } G \text{ is of type } D_n. \end{cases}$$

**Proof.** In the case where $\mathcal{F} = F_c$ is the cuspidal family, $Z_1 = \{0, 1, \ldots, 2d\}$ (resp. $Z_1 = \{0, 1, \ldots, 4d-1\}$ and $M_0 = \{1, 3, \ldots, 2d-1\}$ (resp. $M_0 = \{1, 3, \ldots, 4d-1\}$) if $G$ is of type $B_n$ (resp. $D_n$) by 4.3. Moreover $A_E = \Lambda_M$ with $M = \emptyset$. Then $M^2 = M_0$, and for any $M' \subset Z_1$ such that $|M'| = d$ (resp. $|M'| = 2d$), we have $M'^2 \cap M^2 = M_0 - M'$. Now $M'$ is written as $M' = M'_{\text{odd}} \bigsqcup M'_{\text{ev}}$, where $M'_{\text{odd}}$ (resp. $M'_{\text{ev}}$) is the subset of $M'$ consisting of odd numbers (resp. even numbers). We have $M' \cap M_0 = M'_{\text{odd}}$ and so $M_0 - M' = M_0 - M'_{\text{odd}}$. Moreover we have $Z_1 - M' = (Z_1_{\text{ev}} - M'_{\text{ev}}) \bigsqcup (M_0 - M'_{\text{odd}})$, where $(Z_1)_{\text{ev}}$ is defined similarly. Assume that $M'$ corresponds to $(\lambda, \mu) \in \mathcal{P}_n$ (resp. $(\lambda, \mu) \in \mathcal{P}_n$). Let $\gamma = \sum_{i=1}^{d} i = d(d+1)/2$.
We use the notation in 6.5. For each \( f \) by making use of the Murnaghan-Nakayama formula for elements of \( I \) signed permutation, then \( T \) we also use the following cycle type expression of (7.3.1) (7.3.2) (7.3.3) as follows;

\[
\alpha, \beta
\]

This proves the lemma.

7.3. We consider \( f_{A_1}(w) \in \mathbb{Z}[u^{1/2}, u^{-1/2}] \) for some \( w \in W \), and compute it by making use of the Murnaghan-Nakayama formula for \( H \) due to Halverson and Ram [HR1]. We use the notation in 6.5.

First assume that \( W \) is of type \( B_n \) and \( H \) is the corresponding Hecke algebra. For each 1 \( \leq k \leq n \), we define \( L_k \in H \) inductively as \( L_1 = T_0 \) and \( L_k = T_{k-1}L_{k-1}T_{k-1} \) for \( k = 2, \ldots, n \). For 1 \( \leq k < l \leq n \), we define

\[
R_{kl} = T_kT_{k+1} \cdots T_{l-1}, \quad R_{kl} = L_kT_kT_{k+1} \cdots T_{l-1}.
\]

For an element \( i \) or \( \bar{i} \) in \( I \), we put \( |i| = |ar{i}| = i \). For a sequence \( r = (r_1, \ldots, r_k) \) of elements of \( I \) such that \( |r_1| < |r_2| < \cdots < |r_k| \), we define \( T_r \in H \) by

\[
T_r = R_{r_1, r_2} \cdots R_{r_{k-1}+1, r_k}.
\]

Then \( T_r \) coincides with \( T_{w_r} \), where \( w_r \in W \) is given by a cyclic notation of the signed permutation,

\[
w_r = (1, 2, \ldots, |r_1| - 1, r_1)(|r_1| + 1, |r_1| + 2, \ldots, |r_2| - 1, r_2)
\]

\[
\cdots (|r_{k-1}| + 1, \ldots, |r_k| - 1, r_k).
\]

We also use the following cycle type expression of \( w_r \)

\[
w_r = [l_1, \ldots, l_r],
\]

where \( l_i \in I \) is such that \( |l_i| = |r_i| - |r_{i-1}| \) and \( l_i \) is barred if \( r_i \) is barred. For example, if \( r = (1, 4, 7, 12) \), then \( w_r = [1, 3, 3, 5] \).

We now prepare some notation related to the skew diagram. Let \( \lambda \) be a double partition of size \( n \). Apart from the notation in 7.1, we express it as \( \lambda = (\lambda^\alpha, \lambda^\beta) \), where \( \lambda^\alpha, \lambda^\beta \) are partitions. For \( \mu \in \mathcal{P}_n \), we write \( \mu \subseteq \lambda \) if \( \mu^\alpha \subseteq \lambda^\alpha \) and \( \mu^\beta \subseteq \lambda^\beta \).

The Young diagram of \( \lambda \) is defined as a pair of Young diagrams of \( \lambda^\alpha \) and \( \lambda^\beta \). We often identify the double partition and the corresponding Young diagram. For double partitions \( \mu \subseteq \lambda \), the skew diagram \( \lambda/\mu = ((\lambda/\mu)^\alpha, (\lambda/\mu)^\beta) \) is defined naturally. For each node \( x \) in the skew diagram \( \lambda/\mu \), the content \( ct(x) \) is defined as follows;

\[
ct(x) = \begin{cases} 
  u^{j-i+1} & \text{if } x \text{ is in position } (i, j) \text{ in } (\lambda/\mu)^\alpha, \\
  -u^{j-i} & \text{if } x \text{ is in position } (i, j) \text{ in } (\lambda/\mu)^\beta.
\end{cases}
\]

The skew diagram \( X \) is called a border strip if it is connected and does not contain any 2×2 block of nodes (“connected” means that two nodes are connected horizontally or vertically). The skew diagram \( X \) is called a broken border strip if its connected components are border strip. Note that a double partition \( (\alpha, \beta) \) with both \( \alpha, \beta \) non-empty consists of two connected components. For a border strip \( X \), a sharp corner is a node with no node above it and no node to its left. A dull corner
in a border strip is a node which has a node to its left and a node above it (and so has no node directly northwest of it).

For a skew diagram $X$, let $\mathcal{C}$ be the set of connected components of $X$, and put $m = |\mathcal{C}|$, the number of connected components of $X$. We define $\Delta(X), \overline{\Delta}(X) \in \mathbb{Z}[u^{1/2}, u^{-1/2}]$ as follows:

$$
\Delta(X) = \begin{cases} 
(u^{1/2} - u^{-1/2})^{m-1} \prod_{Y \in \mathcal{C}} (u^{1/2})^{c(Y)}(-u^{-1/2})^{r(Y)} & \text{if $X$ is a broken border strip}, \\
0 & \text{otherwise}, 
\end{cases}
$$

$$
\overline{\Delta}(X) = \begin{cases} 
(u^{1/2})^{c(X)}(-u^{-1/2})^{r(X)} \prod_{y \in DC} ct(y)^{-1} \prod_{z \in SC} ct(z) & \text{if $X$ is a (connected) border strip}, \\
0 & \text{otherwise}, 
\end{cases}
$$

where $SC$ and $DC$ denote the set of sharp corners and dull corners in a border strip, and $r(X)$ (resp. $c(X)$) is the number of rows (resp. columns) in the border strip $X$.

The Murnaghan-Nakayam formula for $\mathcal{H}$ by Halverson-Ram is given as follows. Note that in the formula below, $l'(w)$ denotes the number of $s_1, \ldots, s_{n-1}$ (excluding $s_0$) occurring in the reduced expression of $w \in W$.

**Theorem 7.4** ([HR1, Theorem 2.20]). Assume that $W$ is of type $B_n$. Let $E^\lambda_w$ be the irreducible representation of $\mathcal{H}$ associated to $\lambda \in P_n$. Then

$$
\text{Tr}(T_{w^r}, E^\lambda) = u^{l'(w^r)}/2 \sum_{\mu=w^{(0)} \subseteq \mu^{(1)} \subseteq \cdots \subseteq \mu^{(k)}=\lambda} \Delta(\mu^{(1)}) \Delta(\mu^{(2)}/\mu^{(1)}) \cdots \Delta(\mu^{(k)}/\mu^{(k-1)}),
$$

where the sum is taken over all the sequences $\emptyset = \mu^{(0)} \subseteq \mu^{(1)} \subseteq \cdots \subseteq \mu^{(k)}=\lambda$ such that $|\mu^{(k)}/\mu^{(k-1)}| = |r_k| - |r_{k-1}|$ and the factor $\Delta(\mu^{(k)}/\mu^{(k-1)})$ is barred if $r_k$ in $r$ is barred.

**7.5.** Next assume that $W$ is the Weyl group of type $D_n$ and $\mathcal{H}$ is the corresponding Hecke algebra. Then under the notation of 6.5, we define $L'_k$ by $L'_1 = 1, L'_2 = T'_0T_1$, and $L'_k = T_{k-1}L'_{k-1}T_{k-1}$ for $k = 3, \ldots, n$. For $1 \leq k < l \leq n$, we define

$$
R_{kl} = T_kT_{k+1} \cdots T_{l-1}, \quad R_{k\bar{l}} = L'_kT_kT_{k+1} \cdots T_{l-1}.
$$

For a sequence $r = (r_1, \ldots, r_k)$ of elements $I$ such that $|r_1| < |r_2| < \cdots < |r_k|$ and that the even numbers of $r_i$ are barred, we define $T_r \in \mathcal{H}$ by

$$
T_r = R_{1, r_1}R_{|r_1|+1, r_2} \cdots R_{|r_{k-1}|+1, r_k}. 
$$

Note that $T_r$ does not always correspond to $T_{w^r}$ for some $w$, but it corresponds to $T_{w^r}$ for $w \in W$ as given in (7.3.1) in the following two cases,

(i) $r_i > 0$ for $i = 1, \ldots, k$,

(ii) $r_1 = -1, r_2 < 0$ and $r_i > 0$ for $i = 3, \ldots, k$.

In what follows, we only consider $r$ as above, and so assume that $T_r = T_{w^r}$. Note that $w_r$ is regarded as an element of the Weyl group of type $B_n$ in the notation of 7.3.
As in 7.3, we consider the skew diagram $\lambda/\mu$, and define $c'(x)$ by modifying (7.3.2),

$$c'(x) = \begin{cases} u^{j-i} & \text{if } x \text{ is in position } (i,j) \text{ in } (\lambda/\mu)_{\alpha}, \\ -u^{j-i} & \text{if } x \text{ is in position } (i,j) \text{ in } (\lambda/\mu)_{\beta}. \end{cases}$$

For a skew diagram $X$, we define $\Delta(X)$, $\underline{\Delta}(X) \in \mathbb{Z}[u^{1/2}, u^{-1/2}]$ by the formula in 7.3, but for $\underline{\Delta}(X)$, we modify the definition of $\underline{\Delta}(X)$ by replacing $c(x)$ by $c'(x)$. Concerning the irreducible characters of $H$, we have the following result.

**Theorem 7.6** ([HR1, Theorem 4.21]). Assume that $W$ is of type $D_n$. Let $E_{n}^{\lambda}$ be the irreducible representation of $H$ associated to $\lambda \in \overline{P}_n$ such that $\lambda^\alpha \neq \lambda^\beta$. Then $\text{Tr}(T_{w}, E_{n}^{\lambda})$ can be computed by the formula in Theorem 7.4 for type $B_n$, by replacing $\underline{\Delta}(X)$ by $\underline{\Delta}(X)$.

**7.7.** Let $F_c$ be the cuspidal family as in 4.3, where $F_c$ is a subset of $\Phi_n$ or $\Phi_d^\circ$. Let $F_c^1$ (resp. $F_c^0$) be the set of symbols of defect 1 (resp. defect 0) contained in $F_c$. First assume that $F_c \subset \Phi_n$. Let $P_n$ be the set of double partitions $\lambda$ such that $L_{E_{\lambda}} \in F_c$. Then $\lambda \in P_n$ can be written as $\lambda = (\alpha, \beta)$ with $\alpha : \lambda_1 \geq \cdots \geq \lambda_d \geq \alpha_{d+1} \geq 0$, and $\beta : \beta_1 \geq \cdots \geq \beta_d \geq 0$. Let $\beta^* : \beta_1^* \geq \cdots \geq \beta_d^* \geq 0$ be the dual partition of $\beta$. The following fact is easily checked.

(7.7.1) \[ F_c^1 \simeq P_n = \{ \lambda = (\alpha, \beta) \in P_n | \alpha_i + \beta^*_i = d \text{ for } 1 \leq i \leq d+1 \}. \]

Next assume that $F_c \subset \Phi_n^\circ$. Let $P_n$ be the set of double partitions $\lambda \in \overline{P}_n$ such that $L_{E_{\lambda}} \in F_c$, where $\lambda = (\alpha, \beta)$ with $\alpha : \lambda_1 \geq \cdots \geq \lambda_d \geq 0$, $\beta : \beta_1 \geq \cdots \geq \beta_d \geq 0$ (in this case always $\alpha \neq \beta$). Let $\beta^*$ be the dual partition of $\beta$. Then we have

(7.7.2) \[ F_c^0 \simeq \overline{P}_n = \{ \lambda = (\alpha, \beta) \in \overline{P}_n | \alpha_i + \beta^*_i = 2d \text{ for } 1 \leq i \leq d \}. \]

(7.7.1) shows that $\lambda = (\alpha, \beta) \in P_n$ is obtained from a diagram $\gamma = (d^{d+1})$ of rectangular shape as follows; take any partition $\alpha \supseteq \gamma$, and let $\beta$ be the dual of the partition obtained by rearranging the skew diagram $\gamma/\alpha$. Similarly, (7.7.2) shows that $\lambda = (\alpha, \beta) \in \overline{P}_n$ is obtained from the diagram $\gamma = ((2d)^{2d})$ of rectangular shape by a similar process as above.

For example, in the case of type $B_n$ with $d = 2$, we have $n = d(d+1) = 6$ and

$$P_6 = \{(21; 21), (2^2; 1^2), (1^2; 31), (2^3; -), (2^21; 1), (2^2; 2), (1^3; 3), (2; 2^2), (1; 32), (-; 3^2)\}.$$

In the case of type $D_n$ with $d = 1$, we have $n = 4d^2 = 4$ and

$$\overline{P}_4 = \{(2^2; -), (21^2; 1), (2; 1^2)\}.$$

**7.8.** Let $w_r$ be an element in $W$ associated to some $r = (r_1, \ldots, r_k)$ as in (7.3.1). By Lemma 7.2 and by (6.5.3), we have

(7.8.1) \[ f_{A_n}(w_r) = \delta \sum_{(\alpha, \beta)} (-1)^{|\alpha|} \text{Tr}(T_{w_r}, E_{n}^{(\alpha, \beta)}), \]

where the sum is taken over all $\lambda = (\alpha, \beta) \in P_n$ (resp. in $\overline{P}_n$), and the constant $\delta$ is given as $\delta = (-1)^{d(d+1)/2}2^{-d}$ (resp. $(-1)^{d(2d-1)/2}2^{-2d+1}$) if $G$ is of type $B_n$ (resp. $D_n$).
We shall compute this sum for some specific choice of \( r \) by applying the Murnaghan-Nakayama formula (Theorem 7.4 or Theorem 7.6). In order to discuss the case \( B_n \) and \( D_n \) simultaneously, we consider the following setting. Let \( \mathcal{P}^{a,b}_n \) be the set of double partitions \( \lambda = (\alpha, \beta) \in \mathcal{P}_n \), where \( \alpha : \alpha_1 \geq \cdots \geq \alpha_a \geq 0, \beta : \beta_1 \geq \cdots \geq \beta_b \geq 0 \) and \( n = ab \), such that

\[
(7.8.2) \quad \mathcal{P}^{a,b}_n = \{ \lambda = (\alpha, \beta) \in \mathcal{P}_n \mid \alpha_i + \beta_{a-i+1} = b \text{ for } 1 \leq i \leq a \}.
\]

Hence \( \mathcal{P}^{a,b}_n \) is the set of \( \lambda \) contained in the Young diagram \( \gamma = (b^n) \) of rectangular shape in the above sense. In particular, \( \mathcal{P}^{a,b}_n \) coincides with \( \mathcal{P}_n \), and \( \mathcal{P}^{2d,2d}_n \) (under the identification \( (\alpha, \beta) = (\beta, \alpha) \)) coincides with \( \mathcal{P}_n^d \).

Put

\[
(7.8.3) \quad f_{a,b}(w_r) = \sum_{(\alpha, \beta) \in \mathcal{P}^{a,b}_n} (-1)^{|\alpha|} \text{Tr} (T_{w_r} E^{(\alpha, \beta)}).
\]

We take \( r_k = n, r_{k-1} = n - (2a + 2b - 6) \) so that in applying Theorem 7.4 or Theorem 7.6, \( \mu^{(k)}/\mu^{(k-1)} \) is a broken border strip of length \( 2a + 2b - 6 \). Let \( X \) be a broken border strip of length \( 2a + 2b - 6 \) contained in \( \lambda = (\alpha, \beta) \in \mathcal{P}^{a,b}_n \). We can write \( X = Y \sqcup Z \) with \( Y \subset \alpha, Z \subset \beta \), broken border strips. Since the maximum length of a border strip is \( a + b - 1 \), we have only to consider the following 5 cases.

- Case I. \( |Y| = a + b - 1, |Z| = a + b - 5 \),
- Case II. \( |Y| = a + b - 2, |Z| = a + b - 4 \),
- Case III. \( |Y| = a + b - 3, |Z| = a + b - 3 \),
- Case IV. \( |Y| = a + b - 4, |Z| = a + b - 2 \),
- Case V. \( |Y| = a + b - 5, |Z| = a + b - 1 \).

We consider the diagram \( \gamma = (b^n) \) of rectangular shape so that \( \alpha \cup \beta^* = \gamma \). In the following discussion, we regard \( Y \) and \( Z \) as paths in \( \gamma \), instead of considering \( \alpha \) and \( \beta \) separately. For example, the following figure explains an example of the case I, where \( \lambda = (\alpha, \beta) = (42^21^2, 4^221) \) with \( n = 20 \), \( Y \) is a border strip of length 8 and \( Z \) is a border strip of length 4. In the figure, \( \bullet \) (resp. \( \times \)) denotes the starting point and the ending point of \( Y \) (resp. \( Z \)).

\[
\begin{align*}
\gamma &= (4^5) \\
\alpha &= (42^21^2) \\
\beta &= (4^22)
\end{align*}
\]

In computing \( f_{a,b}(w_r) \), we use the following cancellation property.

**Lemma 7.9.** Let \( x \) be the top rightmost node of \( \gamma \), and \( y \) the west of \( x \), \( z \) the south of \( x \). Assume that \( x, y \in Y \) and that \( z \in Z \). Then \( \alpha : \alpha_1 \geq \alpha_2 \geq \cdots, \beta : \beta_1 \geq \beta_2 \geq \cdots \) with \( \alpha_1 = \alpha, \beta_1 = b - 1 \). Let \( X' = (\alpha', \beta') \in \mathcal{P}_n \) be defined by \( \alpha'_1 = \alpha_1 - 1, \beta'_1 = \beta_1 + 1 \) and \( \alpha'_j = \alpha_j, \beta'_j = \beta_j \) for \( j \neq 1 \). Then \( X' \in \mathcal{P}^{a,b}_n \). Put \( Y' = Y \setminus \{x\} \) and \( Z' = Z + \{x\} \). Then \( Y' \) (resp. \( Z' \)) is a broken border strip of \( \alpha' \) (resp. \( \beta' \)). Let \( X'' = Y' \sqcup Z' \). Then we have \( \Delta(X) = \Delta(X') \).
Moreover, the double partition \((\alpha - Y, \beta - Z)\) coincides with \((\alpha' - Y', \beta - Z')\). Hence in the computation of \(f_{a,b}(w_r)\), the broken border strip \(X\) of this type may be ignored. Similar situations occur also for the cases, such as when \(r(\lambda) = r(Y), r(Z') = r(Z)\), we see that \(\Delta(X) = \Delta(X')\). Clearly, we have \((\alpha - Y, \beta - Z) = (\alpha' - Y', \beta' - Z')\). It follows that in the sequence \(\emptyset \subset \ldots \subset \mu^{(1)} \subset \ldots \subset \mu^{(k)} = \lambda\) in Theorem 7.4 or Theorem 7.6, \(\emptyset \subset \ldots \subset \mu^{(k-1)}\) is common for \(X\) if \(\mu^{(k)}/\mu^{(k-1)} = X\).

Proof. Since \(y \in Y\) and \(z \in Z\), the number of border strips in \(X\) is the same as the numbers in \(X'\). Since \(c(\gamma') - c(Y) - 1, c(Z) = c(Z) + 1\), and \(r(Y') = r(Y), r(Z') = r(Z)\), we see that \(\Delta(X) = \Delta(X')\). Clearly, we have \((\alpha - Y, \beta - Z) = (\alpha' - Y', \beta' - Z')\). It follows that in the sequence \(\emptyset \subset \ldots \subset \mu^{(1)} \subset \ldots \subset \mu^{(k)} = \lambda\) in Theorem 7.4 or Theorem 7.6, \(\emptyset \subset \ldots \subset \mu^{(k-1)}\) is common for \(X\) if \(\mu^{(k)}/\mu^{(k-1)} = X\). Thus Case I is divided into 4 classes, and in each case, \(y_t\) or \(y_b\), the starting node and the ending node of the border strip \(Z\), is a unique border strip connecting the node west of \(y_b\) and the north of \(y_t\). Next assume that \(\alpha_2 = b\) and \(\alpha_a = 1\). If \(y_b \in Z\), then Lemma 7.9 can be applied. So we may assume that \(y_b \notin Z\). In this case \(Z\) is a unique border strip connecting the node north of \(y_b\) and \(y_t\) which is the south of \(x_t\). Next assume that \(\alpha_2 < b\) and \(\alpha_a = 2\). Then \(y_t\) is the south of \(x_t\), and if \(y_t \in Z\), the lemma can be applied. So we may assume that \(y_t \notin Z\) and \(Z\) is a unique border strip connecting the node west of \(y_b\) and \(y_b\). Finally, assume that \(\alpha_2 = b\) and \(\alpha_a = 2\). In this case, \(Z\) is a unique border strip connecting \(y_t\) and \(y_b\).

Thus Case I is divided into 4 classes, and in each case, \(Y\) and \(Z\) are determined uniquely by \(\lambda = (\alpha, \beta)\). In a similar way, one can classify all the possible broken border strips \(X\) for Case II and Case III. Case IV is symmetric to Case II, and each class is obtained from the class in Case II, by rotating the diagram \(\gamma\) by the angle 180°, and then replacing \(Y\) and \(Z\). Similarly, Case V is obtained from Case I. We shall list up all the possible cases for the cases I, II, III, in the list below, assuming that \(a, b \geq 4\). Case I is divided into 4 classes, Case II into 6 classes, and Case III into 12 classes, (Case IV : 6 classes, Case V : 4 classes). Here \(\bullet\) (resp. \(\times\)) denotes the starting node and the ending node of the border strip \(Y\) (resp. \(Z\)). In each case, \(Y\) and \(Z\) are determined uniquely by \(\lambda = (\alpha, \beta)\). Or alternately, if we draw in the diagram \(\gamma\) the path connecting nodes marked by \(\bullet\), so that it is compatible with the path connecting boxes marked by \(\times\), then it determines \(\lambda = (\alpha, \beta)\) uniquely.

Note that in each case, \(\Delta(X)\) does not depend on \(\lambda\) belonging to the class, and has a common value. We have listed those \(\Delta(X)\) for each class, where \(U = (u^{1/2} - u^{-1/2})\). For Case IV or Case V, \(\Delta(X)\) is obtained from the corresponding \(\Delta(X)\) for Case II or Case I, by replacing \(u^{1/2} \leftrightarrow -u^{-1/2}\).
Case I. $|Y| = a + b - 1, |Z| = a + b - 5.$

(1) $-U$

(2) $u^{-1}U$

(3) $uU$

(4) $-U$

Case II. $|Y| = a + b - 2, |Z| = a + b - 4.$

(1) $u^{1/2}U^2$

(2) $-u^{1/2}U^2$

(3) $u^{1/2}U^2$

(4) $-u^{-1/2}U^2$

(5) $-u^{-1/2}U^2$

(6) $u^{1/2}U^2$

Case III. $|Y| = a + b - 3, |Z| = a + b - 3.$

(1) $U^3$

(2) $-U$

(3) $U^3$

(4) $uU$
7.11. In each case listed in 7.10, one obtains a unique $\lambda' \in \mathcal{P}_{n'}$ ($n' = n - (2a + 2b - 6)$) from $\lambda$ by removing the broken border strip $X = Y \sqcup Z$, where $\lambda' = (\alpha', \beta')$ with $\alpha' = \alpha - Y, \beta' = \beta - Z$. Let $J$ be the set of classes in the list in 7.10, and let $Q_j$ be the set of $\lambda' \in \mathcal{P}_{n'}$ satisfying the condition in each case $j$. For example, we consider the case $j = I(1)$. Then $Q = Q_{I(1)}$ is the set of $\lambda' = (\alpha', \beta')$ satisfying the following conditions.

\[
\begin{cases}
\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_{a-3}), \\
\beta'^* = (\beta'^*_1, \beta'^*_2, \ldots, \beta'^*_{a-1}) \quad \text{with} \quad \beta'^*_{a-1} = 1, \\
\alpha_i + \beta'^*_{a-1-i} = b - 2 \text{ for } i = 1, \ldots, a - 3.
\end{cases}
\]

For each $\lambda' = (\alpha', \beta') \in Q$, we consider a broken border strip $X'$ of length $2a + 2b - 10$. Then $X'$ is unique, and is given as $X' = Y' \sqcup Z'$, where $Y'$ is a unique border strip in $\alpha'$ of length $a + b - 7$, and $Z'$ is a unique border strip in $\beta'$ of length $a + b - 3$. By removing $X'$ from $\lambda'$, one obtains $\lambda'' \in \mathcal{P}_{n''}$, where $n'' = n' - (2a + 2b - 10) = (a - 4)(b - 4)$. Then it is easy to check that

\[(\lambda'' \in \mathcal{P}_{n''} \mid \lambda' \in Q) = \mathcal{P}_n^{a-4,b-4},\]

and one recovers the original set $\mathcal{P}_n^{a-4,b-4}$ by replacing $a, b$ by $a - 4, b - 4$. We can compute that $\Delta(X') = -(u^{1/2} - u^{-1/2})$, which is independent from $\lambda' \in Q$. For example, let $a = 7, b = 6$ with $n = 42$. Take $\lambda = (6432^21^2, 6^5542) \in \mathcal{P}_4^{a,b}$. Then $\lambda' = (\alpha', \beta') = (321^2, 6431^2)$, and under an appropriate rearrangement, $\gamma' = \alpha' \cup \beta'^*$ can be drawn as in the following figure. Here $Y'$ (resp. $Z'$) is a unique border strip of length 6, (resp. length 10), and $\bullet$ (resp. $\times$) denotes the starting point and the ending point of $Y'$ (resp. $Z'$). From this, we obtain $\gamma'' = (2^3)$, and $\lambda'' = (\alpha'', \beta'') = (1, 32) \in \mathcal{P}_6^{a,b}$. 

\[
\begin{array}{cccc}
(5) U^3 & (6) u^{-1}U & (7) -U & (8) U^3 \\
(9) -U & (10) u^{-1}U & (11) uU & (12) -U
\end{array}
\]
In fact, a similar kind of arguments work for all other cases, and the set \( Q_j \) is described in a similar way. In particular, \( \gamma' = \alpha' \cup \beta'^* \) is of the shape obtained from a rectangle by attaching two nodes, one on the above or right of the northeast corner, and the other on the below or left of the southwest corner of the rectangle. In all the cases, the broken border strip \( X' = Y' \coprod Z' \) of length \( 2a + 2b - 10 \) is determined uniquely, and we always find the set \( \mathcal{P}_{n''-4,b-4}^{a-4,b-4} \) after removing the border strips \( X' \). Moreover, \( \Delta(X') \) takes the common value \( -(u^{1/2} - u^{-1/2}) \) for all the cases through Case I \( \sim \) Case V.

Recall that \( w_r = (r_1, \ldots, r_k) \) with \( r_k = n \). Assume that

\[
r_k - r_{k-1} = 2a + 2b - 6, \quad r_{k-1} - r_{k-2} = 2a + 2b - 10.
\]

Thus \( r_{k-2} = n'' \) with \( n'' = (a - 4)(b - 4) \). We put \( r'' = (r_1, \ldots, r_{k-2}) \) and consider \( w_{r''} \in W_{n''} \). By investigating the above list, we have the following lemma.

**Lemma 7.12.** Under the notation above, there exists a non-zero (Laurent) polynomial \( h(u) \) such that

\[
f_{a,b}(w_r) = h(u)f_{a-4,b-4}(w_{r''}).
\]

**Proof.** For each \( j \in J \), we denote by \( \Delta_j(u) \) the Laurent polynomial \( \Delta(X) \) attached to the broken border strip \( X \) for \( Q_j \) as in the list. We put \( \varepsilon_j = (-1)^{|\lambda|} \) for \( \lambda = (\alpha, \beta) \in Q_j \). (Note that \( \varepsilon_j \) is independent of the choice of \( \lambda \in Q_j \).) By (7.11.1), the cardinality of \( Q_j \) coincides with the cardinality of \( \mathcal{P}_{n''-4,b-4}^{a-4,b-4} \), hence is independent of \( j \in J \). Then the investigation in 7.11 shows that

\[
\text{(7.12.1)} \quad f_{a,b}(w_r) = \left\{(u^{1/2} - u^{-1/2})[\mathcal{P}_{n''-4,b-4}^{a-4,b-4}] \sum_{j \in J} \varepsilon_j \Delta_j(u) \right\}f_{a-4,b-4}(w_{r''}).
\]

Thus in order to show the lemma, it is enough to see that \( \sum_j \varepsilon_j \Delta_j(u) \neq 0 \). For this we compare the highest degree term \( u^{3/2} \) in \( \Delta_j \). It follows from the list in 7.10, \( \Delta_j \) contains the term \( u^{2/3} \) in the following cases, where the coefficients are always 1.

- Case I. (3),
- Case II. (1), (3), (6),
- Case III. (1), (3), (4), (5), (8), (11),
- Case IV. (2), (4), (5),
- Case V. (2),

where the numbering in Case IV and V is given by the bijective correspondence with Case II and Case I through the rotation of \( \gamma \). Note that \( \varepsilon_j \) takes the constant value for each case, \( I \sim V \). They have the common value for Case I, III, or V, and
have a different common value for Case II or IV. This shows that the coefficient of
$u^{2/3}$ in $\sum_j \varepsilon_j \Delta_j \neq 0$. Hence $h(u) \neq 0$ as asserted. □

Returning to the original setting, we show the following two propositions, which
give the proof of Proposition 6.6.

**PROPOSITION 7.13.** Assume that $W$ is of type $B_n$ with $n = d^2 + d$. We define
an element $w_r \in W$ by the cycle type expression given in (7.3.2) as follows.
\[
[2, 12, 16, 16 + 16, 16 + 16, \ldots, 16 + 16k, 16 + 16k], \quad \text{if } d \equiv 1 \pmod{4},
\]
\[
[6, 16, 20, 16 + 16, 20 + 16, \ldots, 16 + 16k, 20 + 16k], \quad \text{if } d \equiv 2 \pmod{4},
\]
\[
[4, 8, 4 + 16, 8 + 16, \ldots, 4 + 16k, 8 + 16k], \quad \text{if } d \equiv 3 \pmod{4},
\]
\[
[8, 12, 8 + 16, 12 + 16, \ldots, 8 + 16k, 12 + 16k], \quad \text{if } d \equiv 0 \pmod{4},
\]
for some $k \geq 1$, where the last term is equal to $4d - 4$, and the next term is equal
to $4d - 8$, and so on. Then we have $f_{\Lambda_r}(w_r) \neq 0$.

**PROOF.** We apply Lemma 7.12 with $a = d + 1, b = d$. Then $f_{\Lambda_n}(w_r) =
h(u)f_{\Lambda_n}(w_{r''})$ for some non-zero $h(u)$, where $r = (r_1, \ldots, r_k)$ and $r'' = (r_1, \ldots, r_{k-2})$
with $r_k = n = (d+1), r_k - r_{k-1} = 4d - 4, r_{k-1} - r_{k-2} = 4d - 8, r_{k-2} = (d-4)(d-3)$. Thus
the computation of $f_{\Lambda_n}(w_r)$ is reduced to the case where $d = 1, 2, 3$. Assume
that $d = 1$, then $n = 2$. One can check by using the formula for $\overline{\Delta}(X)$ that
$f_{\Lambda_n}(w) \neq 0$ for $w = (\bar{2})$. (Note that $f_{\Lambda_n}(w) = 0$ for $w = (2)$). Next assume that
$d = 2$, then $n = 6$. The direct computation shows that $f_{\Lambda_n}(w) \neq 0$ for $w = (6)$. Finally
assume that $d = 3$, then $n = 12$. By using a similar method as in the proof
of Lemma 7.12, one can show that $f_{\Lambda_n}(w) \neq 0$ for $w = (4, 8)$ (we obtain a similar
list, but some classes in the list in 7.10 don’t appear for this case). This proves the
pros position. □

**PROPOSITION 7.14.** Assume that $W$ is of type $D_n$ with $n = 4d^2$. We define an
element $w_r \in W$ as follows.
\[
[3, 14, 18, 14 + 16, 18 + 16, \ldots, 14 + 16k, 18 + 16k], \quad \text{if } d \equiv 1 \pmod{2},
\]
\[
[6, 10, 6 + 16, 10 + 16, \ldots, 6 + 16k, 10 + 16k], \quad \text{if } d \equiv 0 \pmod{2},
\]
for some $k \geq 1$, where the last term is equal to $4d - 6$ and the next term is equal
to $4d - 10$, and so on. Then we have $f_{\Lambda_n}(w_r) \neq 0$.

**PROOF.** We apply Lemma 7.12 with $a = b = 2d$. Note that since $\alpha \neq \beta$ for
any $\lambda = (\alpha, \beta)$ in $\mathcal{P}_n^{2d, 2d}$, the argument for type $B_n$ can be applied without change.
Then $f_{\Lambda_n}(w_r) = h(u)f_{\Lambda_n}(w_{r''})$ for some non-zero $h(u)$, where $r = (r_1, \ldots, r_k)$ and $r'' = (r_1, \ldots, r_{k-2})$
with $r_k = n = 4d^2, r_k - r_{k-1} = 4d - 6, r_{k-1} - r_{k-2} = 4d - 10, r_{k-2} = 4(d-2)^2$. Thus
the computation of $f_{\Lambda_n}(w_r)$ is reduced to the case where $d = 1, 2$. Assume that $d = 2$. Then $n = 16$. Since $a = b = 4$, Lemma 7.12 can
be applied, and we see that $f_{\Lambda_n}(w) \neq 0$ for $w = (6, 10)$. Next assume that $d = 1$. Then $n = 4$. One can check by using $\overline{\Delta}(X)$ that
$f_{\Lambda_n}(w) \neq 0$ for $w = (3, 3)$. This proves the proposition. □

**REMARK 7.15.** As the formula (7.12.1) shows, our element $f_{\Lambda_n}(w_r)$ turns out
to be 0 if $u \mapsto 1$. So our computation cannot be performed in the level of Weyl
groups. On the other hand, Lusztig showed in [L5] that there exists an element
$w \in W$ such that $f_{\Lambda_n}(w)|_{w=1} \neq 0$. Thus $f_{\Lambda_n}(w) \neq 0$ as a polynomial. This $w$ has
a simpler form than ours, but since it is a product of negative cycles of the full length
(i.e., the sum of the lengths of negative cycles is equal to $n$), it is not appropriate for the computation of the bitrace on the flags (cf. Lemma 6.10).

References

[HR1] T. Halverson and A. Ram; Murnaghan-Nakayama rules for characters of Iwahori-Hecke algebras of classical type, Trans. Amer. Math. Soc. 348 (1996), 3976 - 3995.

[HR2] T. Halverson and A. Ram; Bitraces for $GL_n(F_q)$ and the Iwahori-Hecke algebra of type $A_{n-1}$, Indag. Math. New Series 10 (1999), 247 - 268.

[K] N. Kawanaka; On subfield symmetric spaces over a finite field, Osaka J. Math. 28 (1991) 759 - 791.

[L1] G. Lusztig; “Characters of Reductive groups over a finite field”, Ann. of Math. Studies, Vol. 107, Princeton Univ. Press, Princeton, 1984.

[L2] G. Lusztig; Character sheaves, I Adv. in Math. 56 (1985), 193–237, II Adv. in Math. 57 (1985), 226–265, III, Adv. in Math. 57 (1985), 266–315, IV, Adv. in Math. 59 (1986), 1–63, V, Adv. in Math. 61 (1986), 103–155.

[L3] G. Lusztig; Remarks on computing irreducible characters, J. of Amer. Math. Soc., 5 (1992), 971–986.

[L4] G. Lusztig; $G(F_q)$-invariants in irreducible $G(F_{q^2})$-modules, Represent. Theory, 4 (2000), 446 - 465.

[L5] G. Lusztig; Rationality properties of unipotent representations, J. Algebra 258 (2002), 1 - 22.

[S1] T. Shoji; Character sheaves and almost characters of reductive groups, Adv. in Math. 111 (1995), 244 - 313, II, Adv. in Math. 111 (1995), 314 - 354.

[S2] T. Shoji; Unipotent characters of finite classical groups, in “Finite reductive groups; related structures and representations,” Progress in Math., Vol.141 (1997), 373 - 413.

[S3] T. Shoji; Generalized Green functions and unipotent classes for finite reductive groups, II. To appear in Nagoya Math. Journal.

[W] J.-L. Waldspurger; “Une conjecture de Lusztig pour les groupes classiques”, Mémoires de la Soc. Math. France, No. 96, Soc. Math. Fracne, 2004.

Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan