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Curve based approximation of measures on manifolds by discrepancy minimization. (English)

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Summary: The approximation of probability measures on compact metric spaces and in particular on Riemannian manifolds by atomic or empirical ones is a classical task in approximation and complexity theory with a wide range of applications. Instead of point measures we are concerned with the approximation by measures supported on Lipschitz curves. Special attention is paid to push-forward measures of Lebesgue measures on the unit interval by such curves. Using the discrepancy as distance between measures, we prove optimal approximation rates in terms of the curve’s length and Lipschitz constant. Having established the theoretical convergence rates, we are interested in the numerical minimization of the discrepancy between a given probability measure and the set of push-forward measures of Lebesgue measures on the unit interval by Lipschitz curves. We present numerical examples for measures on the 2- and 3-dimensional torus, the 2-sphere, the rotation group on $\mathbb{R}^3$ and the Grassmannian of all 2-dimensional linear subspaces of $\mathbb{R}^4$. Our algorithm of choice is a conjugate gradient method on these manifolds, which incorporates second-order information. For efficient gradient and Hessian evaluations within the algorithm, we approximate the given measures by truncated Fourier series and use fast Fourier transform techniques on these manifolds.

MSC:

65K10 Numerical optimization and variational techniques
90C26 Nonconvex programming, global optimization

Keywords:
approximation of measures; curves; Fourier methods; manifolds; non-convex optimization; quadrature rules; sampling theory

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NFFT3

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