Controlled diffeomorphic extension of homeomorphisms

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Abstract

Let $\Omega$ be an internal chord-arc Jordan domain and $\varphi : \mathbb{S} \rightarrow \partial \Omega$ be a homeomorphism. We show that $\varphi$ has finite dyadic energy if and only if $\varphi$ has a diffeomorphic extension $h : \mathbb{D} \rightarrow \Omega$ which has finite energy.

Keywords: Poisson extension, diffeomorphism, chord-arc curve.

1 Introduction

Let $\Omega \subset \mathbb{C}$ be a bounded convex domain and suppose that $\varphi$ is a homeomorphism from the unit circle $\mathbb{S}$ onto $\partial \Omega$. Then, by [7], the complex-valued Poisson extension $h$ of $\varphi$ is a homeomorphism from $\mathbb{D}$ onto $\bar{\Omega}$. This harmonic map $h$ is a diffeomorphism in $\mathbb{D}$ but its derivatives are not necessarily uniformly bounded. In 2007, G. C. Verchota [10] proved that the derivatives of $h$ may fail to be square integrable but that they are necessarily $p$-integrable over $\mathbb{D}$ for any $p < 2$. In 2009, T. Iwaniec, G. Martin and C. Sbordone improved on [6] by showing that the derivatives belong to weak-$L^2$ with sharp estimates. In a related work [11] by K. Astala, T. Iwaniec, G. Martin and J. Onninen, it was shown that if additionally $\partial \Omega$ is a $C^1$-regular Jordan curve, the square integrability of the derivatives of $h$ is equivalent to the requirement that

$$\int_{\partial \Omega} \int_{\partial \Omega} |\log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)||d\xi||d\eta| < +\infty.$$ 

In this note we give a generalization of the aforementioned results. Towards this end, recall that the Poisson extension of a homeomorphism $\varphi : \mathbb{S} \rightarrow \partial \Omega$ may fail to be injective if $\Omega$ is not convex. Next, the boundary $\partial \Omega$ of a bounded convex domain $\Omega$ is a chord-arc Jordan curve: $\partial \Omega$ is a rectifiable Jordan curve and there is a constant $C$ such that for all $w_1, w_2 \in \partial \Omega$, $\ell(w_1, w_2) \leq C|w_1 - w_2|$, where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial \Omega$ joining $w_1$ to $w_2$. A domain whose boundary is a chord-arc Jordan curve is called a chord-arc Jordan domain.
Hence we are lead to ask for the optimal regularity of homeomorphic extensions $h : \mathbb{D} \to \Omega$ for a given homeomorphism $\varphi : \mathbb{S} \to \partial \Omega$ and a chord-arc Jordan domain $\Omega$.

Before introducing our first result, let us fix a dyadic decomposition $\{\Gamma_{j,k} : j \in \mathbb{N}, k = 1, \cdots , 2^j\}$ of $\mathbb{S}$, such that for a fixed $j \in \mathbb{N}$, $\{\Gamma_{j,k} : k = 1, \cdots , 2^j\}$ is a family of arcs of length $2\pi/2^j$ with $\bigcup_k \Gamma_{j,k} = \mathbb{S}$. The next generation is constructed in such a way that for each $k \in \{1, \cdots , 2^j+1\}$, there exists a unique number $k' \in \{1, \cdots , 2^j\}$, satisfying $\Gamma_{j+1,k} \subset \Gamma_{j,k'}$. Here, we call $\Gamma_{j,k'}$ the parent of $\Gamma_{j+1,k}$. Fix $\Gamma_{1,1}$ to be the image of $[0, \pi]$ under the map $\theta \mapsto e^{i\theta}$. We denoted by $\ell(\varphi(\Gamma_{j,k}))$ the arc length of the image arc of $\Gamma_{j,k}$ under the homeomorphism $\varphi$.

**Theorem 1.** Let $\Omega \subset \mathbb{C}$ be an internal chord-arc Jordan domain and suppose that $\varphi : \mathbb{S} \to \partial \Omega$ is a homeomorphism. Then for $\lambda \in (-1, +\infty)$, the following are equivalent:

(i) $\int_{\mathbb{D}} |D\varphi(z)|^2 \log^\lambda (e + |D\varphi(z)|) \, dz < +\infty$ for some diffeomorphic extension $h : \mathbb{D} \to \Omega$ of $\varphi$;

(ii) $\int_{\mathbb{D}} |D\varphi(z)|^2 \log\left(\frac{2}{1-|z|}\right) \, dz < +\infty$ for some diffeomorphic extension $h : \mathbb{D} \to \Omega$ of $\varphi$;

(iii) $\int_{\partial \Omega} \int_{\partial \Omega} |\log |\varphi^{-1}(\eta) - \varphi^{-1}(\xi)||^{\lambda+1} \, |d\eta| \, |d\xi| < +\infty$;

(iv) $\sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) < +\infty$;

(v) $\sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^\lambda \left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) < +\infty$.

A Jordan domain $\Omega \subset \mathbb{C}$ is an internal chord-arc Jordan domain if $\partial \Omega$ is rectifiable and there is a constant $C > 0$ such that for all $w_1, w_2 \in \partial \Omega$,

$$
(1.1) \quad \ell(w_1, w_2) \leq C \lambda_\Omega(w_1, w_2),
$$

where $\ell(w_1, w_2)$ is the arc length of the shorter arc of $\partial \Omega$ joining $w_1$ to $w_2$ and $\lambda_\Omega(w_1, w_2)$ is the internal distance between $w_1, w_2$, which is defined as

$$
\lambda_\Omega(w_1, w_2) = \inf_\alpha \ell(\alpha),
$$

where the infimum is taken over all rectifiable arcs $\alpha \subset \Omega$ joining $w_1$ and $w_2$; if there is no rectifiable curve joining $w_1$ and $w_2$, we let $\lambda_\Omega(w_1, w_2) = \infty$; cf. [9, Section 3.1] or [2], Section 2).

Naturally, every chord-arc Jordan domain is also an internal chord-arc domain, but there are internal chord-arc domains that fail to be chord-arc; e.g., the inward cusp domain:

$$
\Omega_c = \mathbb{D} \setminus \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, |y| \leq x^2\}.
$$

The statement of Theorem [1] does not allow for $\lambda \leq -1$. Actually, (i)-(v) all hold for $\lambda < -1$, independently of $\varphi$. When $\lambda = -1$, (iii) needs to be reformulated via a
double logarithm, after which one still has a list of mutually equivalent conditions that may or may not hold, see [\[1\].

In the case when $\Omega = \mathbb{D}$, Theorem 1 can be formulated via the harmonic Poisson extension.

**Theorem 2.** Let $\varphi : \mathbb{S} \to \mathbb{S}$ be a homeomorphism and $h = P[\varphi] : \mathbb{D} \to \mathbb{D}$ be the harmonic Poisson extension of $\varphi$. Then for $\lambda \in (-1, +\infty)$ the following are equivalent:

(i) $\int_{\mathbb{D}} |Dh(z)|^2 \log^\lambda(e + |Dh(z)|) \, dz < +\infty$;

(ii) $\int_{\mathbb{S}} |Dh(z)|^2 \log^\lambda(\frac{2}{1+|z|}) \, dz < +\infty$;

(iii) $\sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 < +\infty$;

(iv) $\sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^\lambda\left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{2j}}\right) < +\infty$.

Our main task is actually to prove Theorem 2. Indeed, once we know that Theorem 2 holds, Theorem 1 is obtained via a suitable change of variable, relying on the fact that there is a bi-Lipschitz map from $\bar{\Omega}$ onto $(\Omega, \lambda_\Omega)$; see Section 3.2.

The paper is organized as follows. In Section 2 we give some relevant facts about the dyadic decomposition of $\mathbb{S}$ and some properties of the $N$-function $\Phi(t) = t^2 \log^\lambda(e + t)$ for $\lambda > -1$. Section 3 contains the full proofs of Theorem 1 and Theorem 2.

2 Preliminaries

2.1 Dyadic decomposition

Since our dyadic decomposition $\{\Gamma_{j,k} : j \in \mathbb{N}, k = 1, \ldots, 2^j\}$ of $\mathbb{S}$ satisfies that $\Gamma_{1,1}$ is the image of $[0, \pi]$ under the map $\theta \mapsto e^{i\theta}$, we may assume that the dyadic decomposition $\{I_{j,k} = [2\pi(k-1)/2^j, 2\pi k/2^j] : j \in \mathbb{N}, k = 1, \ldots, 2^j\}$ of the interval $[0, 2\pi]$ matches with $\{\Gamma_{j,k}\}$ in the sense that each $\Gamma_{j,k}$ is exactly the image of $I_{j,k}$ under the map $\theta \mapsto e^{i\theta}$. Moreover, the dyadic arc $\Gamma_{j,k}$ is called a $j$-level dyadic arc, and we denote the two end points of $\Gamma_{j,k}$ by $\xi_{j,k}$ and $\xi_{j,k+1}$. Two dyadic arcs which are $j$-level dyadic arcs for some $j \in \mathbb{N}$ are called brother dyadic arcs if they have a common parent.

For the dyadic decompositions $\{\Gamma_{j,k}\}$ of $\mathbb{S}$ and $\{I_{j,k}\}$ of $[0, 2\pi]$ as above, we have a decomposition of the unit disk $\mathbb{D}$ given by $\{Q_{j,k} : j \in \mathbb{N}, k = 1, \ldots, 2^j\}$ where

$$Q_{j,k} = \{re^{i\theta} : 1 - 1/2^{j-1} \leq r \leq 1 - 1/2^j, \theta \in I_{j,k}\}$$

for any $j \in \mathbb{N}$ and $k = 1, \ldots, 2^j$, see Figure 1. It is easy to see that $\text{dist}(Q_{j,k}, \mathbb{S}) = 2^{-j}$ and that if $P : \mathbb{D} \to \mathbb{S}$ is the radial projection map, then $P(Q_{j,k}) = \Gamma_{j,k}$ for all $j \in \mathbb{N}$.
and $k = 1, \ldots, 2^j$. There is an uniform constant $C > 0$ such that for any $Q_{j,k}$ with center $x_{j,k} = r_{j,k}e^{i\theta_{j,k}}$, $r_{j,k} = 1 - 3/2^{j+1}$ and $\theta_{j,k} = \pi(2k-1)/2^j$, we have

$$B(x_{j,k}, C^{-1} \text{diam}(Q_{j,k})) \subset Q_{j,k} \subset B(x_{j,k}, C \text{diam}(Q_{j,k})).$$

So for any $Q_{j,k}$, we can find a disk $B_{j,k}$ satisfying $B_{j,k} \subset Q_{j,k} \subset CB_{j,k}$, where $C$ is a constant independent of $Q_{j,k}$.

In order to later estimate the integral of $|Dh|$ over $Q_{j,k}$, we employ the following decomposition of $\mathcal{S}$ via the dyadic arcs $\Gamma_{i,l}$ with $i \leq j$. The idea is to build a dyadic-type annular decomposition around $\Gamma_{j,k}$.

We fix $\Gamma_{j,k}$ and construct the decomposition via the following steps. For simplicity, we assume that the unique brother arc of $\Gamma_{j,k}$ is $\Gamma_{j,k+1}$ and located at the clockwise side of $\Gamma_{j,k}$.

Step 1. On the anticlockwise side of $\Gamma_{j,k+1}$, we choose the unique arc $\Gamma_{j-1,m_{j-1}^1}$, so that $\Gamma_{j,k+1} \cap \Gamma_{j-1,m_{j-1}^1}$ is a singleton. If the arc $\Gamma_{j-1,m_{j-1}^1}$ shares the common parent with $\Gamma_{j-1,m_{j-1}^1}$, then set $(\Gamma_{j-1,m_{j-1}^1}+1) = \Gamma_{j-1,m_{j-1}^1}$. If not, set $(\Gamma_{j-1,m_{j-1}^1}+1) = \emptyset$. Then define $\Gamma_{j-1,m_{j-1}^1}$ and $(\Gamma_{j-1,m_{j-1}^1-1})$ analogously along the clockwise side of $\Gamma_{j,k}$.

Step 2. Repeat the process in Step 1 with $\Gamma_{j,k+1}$ replaced by $\Gamma_{j-1,m_{j-1}^1}$ if we have $(\Gamma_{j-1,m_{j-1}^1}+1) = \emptyset$ or $\Gamma_{j-1,m_{j-1}^1+1}$ otherwise, and with $\Gamma_{j,k}$ replaced by $\Gamma_{j-1,m_{j-1}^1}$ if $(\Gamma_{j-1,m_{j-1}^1-1}) = \emptyset$ or $\Gamma_{j-1,m_{j-1}^1-1}$ otherwise; unless we get $\Gamma_{j_0,m_{j_0}^1}$, $(\Gamma_{j_0,m_{j_0}^1}+1)$, $\Gamma_{j_0,m_{j_0}^2}$ and $(\Gamma_{j_0,m_{j_0}^2-1})$ such that

$$\left(\Gamma_{j_0,m_{j_0}^1} \cup (\Gamma_{j_0,m_{j_0}^1+1})\right) \cap \left(\Gamma_{j_0,m_{j_0}^2} \cup (\Gamma_{j_0,m_{j_0}^2-1})\right) \neq \emptyset.$$
This procedure decomposes $S$:

$$\mathcal{P}(\Gamma_{j,k}) = \{ \Gamma_{j,k}, \Gamma_{j,k+1}, \Gamma_{j-1,m_{j-1}^{1}}, (\Gamma_{j-1,m_{j-1}^{1}+1}), \Gamma_{j-1,m_{j-1}^{2}}, (\Gamma_{j-1,m_{j-1}^{2}+1}), $$

$$\ldots, \Gamma_{j,m^{1}_j}, (\Gamma_{j,m^{1}_j+1}), \Gamma_{j,m^{2}_j}, (\Gamma_{j,m^{2}_j+1}) \}.$$

where each $(\cdot)$ is either empty or a dyadic arc. Correspondingly, we obtain a decomposition of $[0, 2\pi]$ as

$$\mathcal{P}(I_{j,k}) = \{ I_{j,k}, I_{j,k+1}, I_{j-1,m_{j-1}^{1}}, (I_{j-1,m_{j-1}^{1}+1}), I_{j-1,m_{j-1}^{2}}, (I_{j-1,m_{j-1}^{2}+1}), $$

$$\ldots, I_{j,m^{1}_j}, (I_{j,m^{1}_j+1}), I_{j,m^{2}_j}, (I_{j,m^{2}_j+1}) \}.$$

Let us illustrate this procedure by an example, see Figure 2 below. We use a binary tree to represent the dyadic decomposition $\{\Gamma_{j,k}\}$ of $S$. Suppose that we are given a dyadic arc $\Gamma_{4,7}$. Then $\Gamma_{4,8}$ is the unique brother arc of $\Gamma_{4,7}$. In Step 1, we choose $\Gamma_{3,5}$ and $\Gamma_{3,6}$ on the anticlockwise side of $\Gamma_{4,8}$, and $\Gamma_{3,3}$ on the clockwise side of $\Gamma_{4,7}$. In Step 2, we choose $\Gamma_{2,4}$ on the anticlockwise side of $\Gamma_{3,6}$, and $\Gamma_{2,1}$ on the clockwise side of $\Gamma_{3,3}$. There will be no more steps because $\Gamma_{2,1} \cap \Gamma_{2,4} \neq \emptyset$.

![Figure 2: The decomposition of $S$ around $\Gamma_{4,7}$](image)

Suppose we are given an $n$-level arc $\Gamma_{n,m}$. We are interested in the number of $j$-level arcs, $j \geq n$, which can induce $\Gamma_{n,m}$ by the above decomposition method. Here, we say that a dyadic arc $\Gamma_{j,l}$ induces $\Gamma_{n,m}$ by the above decomposition method if and only if $\Gamma_{n,m}$ is in the set $\mathcal{P}(\Gamma_{j,l})$.

To begin, $\Gamma_{n,m}$ can be first induced by its brother (without loss of generality assume that it is $\Gamma_{n,m+1}$), and two couples of $(n+1)$-level arcs. Notice that among all $(n+1)$-level arcs, there is only one couple that can induce both $\Gamma_{n,m}$ and $\Gamma_{n,m+1}$. It follows that $\Gamma_{n,m}$ can be induced by three couples of $(n+1)$-level arcs. Choose one of these $(n+1)$-level arcs. There is only one couple of $(n+2)$-level arcs which induce both $\Gamma_{n,m}$ and the chosen $(n+1)$-level arcs. Then $\Gamma_{n,m}$ can be induced by six couples of $(n+2)$-level arcs. Generally, for $j \geq n$ we have

$$\sharp \{ \Gamma : \Gamma \text{ is a } j\text{-level dyadic arc and induce } \Gamma_{n,m} \} \leq 3 \cdot 2^{j-n}.$$
2.2 $N$-functions

A function $\Phi : [0, \infty) \to [0, \infty)$ is an $N$-function if it is a continuous, increasing and convex function satisfying $\Phi(0) = 0$,

$$
\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty.
$$

An $N$-function $\Phi$ can be expressed as

$$
\Phi(t) = \int_0^t \phi(s) \, ds,
$$

where $\phi : [0, \infty) \to [0, \infty)$ is an increasing, right-continuous function with $\phi(0) = 0$ and $\lim_{t \to +\infty} \phi(t) = +\infty$.

For each $N$-function $\Phi$ and $t \geq 0$, set

$$
\psi(t) = \sup \phi(s) \leq t \quad \text{and} \quad \Psi(t) = \int_0^t \psi(s) \, ds.
$$

Then we call $\Psi$ the complementary function of $\Phi$. The complementary function of an $N$-function is also an $N$-function. We call $\Psi, \Phi$ a pair of complementary $N$-functions.

An $N$-function $\Phi$ is said to satisfy the $\Delta_2$-condition if there is a constant $C_\Phi > 0$, called a doubling constant of $\Phi$, such that

$$
\Phi(2t) \leq C_\Phi \Phi(t), \quad \forall \ t \geq 0.
$$

**Proposition 1.** If an $N$-function $\Phi$ satisfies the $\Delta_2$-condition, then for any constant $c > 0$, there exist $c_1, c_2 > 0$ such that

$$
c_1 \Phi(t) \leq \Phi(ct) \leq c_2 \Phi(t) \quad \text{for all} \quad t \geq 0,
$$

where $c_1$ and $c_2$ depend only on $c$ and the doubling constant $C_\Phi$. Therefore, we obtain that if $A \approx B$, then $\Phi(A) \approx \Phi(B)$.

**Lemma 1.** (\cite[Theorem 4.2]{Lq}) The complementary function of an $N$-function $\Phi$ satisfies the $\Delta_2$-condition on $[0, +\infty)$ if there is a constant $l > 1$ such that

$$
\Phi(t) \leq \frac{1}{2l} \Phi(lt) \quad \text{for any} \ t \geq 0.
$$

Given an $N$-function $\Phi$, we denote by $L_\Phi^*$ the collection of all measurable functions $f$ such that $\int_{\mathbb{R}^n} \Phi(af) < \infty$ for some $a > 0$.

For a measurable function $f$ on $\mathbb{R}^2$, we define the Hardy-Littlewood maximal function of $f$ by setting

$$
M_f(x) = \sup_B \int_B |f(z)| \, dz = \sup_B \frac{1}{|B|} \int_B |f(z)| \, dz,
$$

where the supremum is taken over all open disks $B$ that contain $x$. 
Proposition 2. ([3 Theorem 2.1]) Let $\Phi$ and $\Psi$ be a pair of complementary $N$-functions. The following two conditions are equivalent:

(i) there exists positive constants $C$ and $b$ such that

$$\int_{\mathbb{R}^n} \Phi(bM_f)(z)dz \leq C \int_{\mathbb{R}^n} \Phi(|f|)(z)dz, \quad \forall f \in L^*_\Phi,$$

(ii) $\Psi$ satisfies the $\Delta_2$-condition on $[0, +\infty)$.

Example 1. Denote by $\Phi$ the function $\Phi(t) = t^2 \log(e + t)$ for $\lambda > -1$. An elementary computation shows that $\Phi$ is increasing, continuous and convex on $[0, +\infty)$ with $\lim_{t \to 0^+} \frac{\Phi(t)}{t} = 0$ and $\lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty$. So $\Phi$ is an $N-$function. Moreover, both the function $\Phi$ and its complementary function satisfy the $\Delta_2$-condition. Hence we know that $\Phi$ satisfies Proposition 1 and 2. We will use Proposition 1 frequently in Section 3.1.

Actually, a direct computation shows that the above $\Phi$ satisfies the $\Delta_2$-condition on $[0, +\infty)$. In order to check that the complementary $N-$function of $\Phi$ satisfies the $\Delta_2$-condition, by Proposition 1, we only need to find a constant $l > 1$ so that

$$2 \log^\lambda(e + t) \leq l \log^\lambda(e + lt), \quad \forall t \geq 0.$$  \hfill (2.2)

In fact, if $\lambda \geq 0$ we can take $l = 2$. By monotonicity of $\log^\lambda(e + \cdot)$, we have that inequality (2.2) holds for any $t \geq 0$. If $-1 < \lambda < 0$, we take $l = 2^{1/(1+\lambda)}$. By monotonicity we have

$$\log(e + lt) \leq l \log(e + t), \quad \forall t \geq 0.$$  Together with $l = (2/l)^{1/\lambda}$, it follows that (2.2) holds for all $t \geq 0$.

3 Proofs of Theorem 1 and Theorem 2

In this section, the notation $A \lesssim B$ means that there is a constant $C > 0$ so that $A \leq C \cdot B$. Here and in this section, the notation $C$ denotes a positive constant which may differ from line to line. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

3.1 Proof of Theorem 2

Proof of (iv)$\iff$(v). Our claim is obvious when $\lambda = 0$.

Case $\lambda > 0$. Assume (iv) holds. Since $\ell(\varphi(\Gamma_{j,k})) \leq 2\pi$ for any $\Gamma_{j,k}$ and $\lambda > 0$, we have

$$\log^\lambda \left(e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}}\right) \leq \log^\lambda \left(e + 2^{j-1}\pi\right) \lesssim j^\lambda.$$
Therefore, we get
\[ \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^\lambda \left( e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}} \right) \lesssim \sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 < +\infty, \]
which gives the implication (iv)⇒(v). For the other direction, assume that (v) holds.

In order to estimate the logarithmic term from below, we set
\[ (3.1) \quad \chi(j, k) = \begin{cases} 1, & \text{if } \ell(\varphi(\Gamma_{j,k})) > 2^{\frac{3}{4}} - \frac{3}{2}j; \\ 0, & \text{otherwise}. \end{cases} \]

Then
\[ \sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 = \sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} \chi(j, k) \ell(\varphi(\Gamma_{j,k}))^2 + \sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} (1 - \chi(j, k)) \ell(\varphi(\Gamma_{j,k}))^2 =: P_1 + P_2. \]

If \( \ell(\varphi(\Gamma_{j,k})) > 2^{\frac{3}{4}} \), then we have
\[ \log^\lambda \left( e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}} \right) \geq \log^\lambda (e + 2^{2j}) \gg j^\lambda. \]

Hence, we obtain from (v) that
\[ P_1 \lesssim \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^\lambda \left( e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}} \right) < +\infty. \]

For \( P_2 \), we have
\[ P_2 \leq \sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} 2^{-\frac{3}{2}j} = \sum_{j=1}^{+\infty} 2^{-\frac{1}{2}j} j^\lambda < +\infty. \]

In conclusion, \( P_1 + P_2 < +\infty \), and (iv) follows.

Case \( \lambda < 0 \). Assume that (v) holds. Since \( \ell(\varphi(\Gamma_{j,k})) \leq 2\pi \) for any \( \Gamma_{j,k} \) and \( \lambda < 0 \), we have
\[ \log^\lambda \left( e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}} \right) \geq \log^\lambda (e + 2^{j-1} \pi) \gg j^\lambda. \]

Therefore, we get
\[ \sum_{j=1}^{\infty} j^\lambda \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \lesssim \sum_{j=1}^{\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^\lambda \left( e + \frac{\ell(\varphi(\Gamma_{j,k}))}{2^{-j}} \right) < +\infty, \]
which gives us the implication (v)⇒(iv). For the other direction, assume that (iv) holds. Using the equation (3.1), if \( \ell(\phi(\Gamma_{j,k})) > 2^{-3j/4} \), we have

\[
\log^\lambda \left( e + \frac{\ell(\phi(\Gamma_{j,k}))}{2^{-j}} \right) \leq \log e (e + 2^{1/4}j) \lesssim j^\lambda.
\]

Therefore, we obtain

\[
P'_1 := \sum_{j=1}^{\infty} 2^j \sum_{k=1}^{\infty} \chi(j,k) \ell(\phi(\Gamma_{j,k}))^2 \log^\lambda \left( e + \frac{\ell(\phi(\Gamma_{j,k}))}{2^{-j}} \right) \lesssim \sum_{j=1}^{\infty} 2^j \sum_{k=1}^{\infty} \ell(\phi(\Gamma_{j,k}))^2 < +\infty.
\]

Moreover, since \( \ell(\phi(\Gamma_{j,k})) \geq 0 \) and \( \lambda < 0 \), we always have

\[
\log^\lambda \left( e + \frac{\ell(\phi(\Gamma_{j,k}))}{2^{-j}} \right) \leq 1.
\]

Hence, we obtain

\[
P'_2 := \sum_{j=1}^{\infty} 2^j \sum_{k=1}^{\infty} (1 - \chi(j,k)) \ell(\phi(\Gamma_{j,k}))^2 \log^\lambda \left( e + \frac{\ell(\phi(\Gamma_{j,k}))}{2^{-j}} \right)
\leq \sum_{j=1}^{\infty} 2^{-3j} = \sum_{j=1}^{\infty} 2^{-j} < +\infty.
\]

Then \( P'_1 + P'_2 < +\infty \) which gives us (v).

By combining the cases \( \lambda = 0, \lambda > 0 \) with \( \lambda < 0 \) above, we finish the proof of (iv)⇔(v).

\[ \square \]

**Proof of (iv)⇒(i).** The harmonic extension of \( \phi \) is given by the Poisson integral formula:

\[
h(z) = \frac{1}{2\pi} \int_{S} \frac{1 - |z|^2}{|z - \zeta|^2} \phi(\zeta) \, d\zeta.
\]

We can therefore compute the \( z \)-derivative by differentiating this kernel:

\[
(3.2) \quad h_z(z) = \frac{\partial h}{\partial z}(z) = \frac{1}{2\pi} \int_{S} \zeta \frac{\phi(\zeta)}{(z - \zeta)^2} \, d\zeta = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i\theta} \frac{\phi(e^{i\theta})}{(z - e^{i\theta})^2} \, d\theta.
\]

We may write

\[
\phi(e^{i\theta}) = \phi(1) \cdot e^{if(\theta)}
\]

where \( f : [0, 2\pi] \to [0, 2\pi] \) is continuous and increasing with \( f(0) = 0 \) and \( f(2\pi) = 2\pi \). This allows us to rewrite (3.2) as

\[
(3.3) \quad h_z(z) = \frac{\phi(1)}{2\pi i} \int_{0}^{2\pi} \frac{1}{(z - e^{i\theta})} \, d\theta = \frac{\phi(1)}{2\pi i} \int_{0}^{2\pi} e^{if(\theta)} \frac{1}{(z - e^{i\theta})} \, d\theta.
\]
Here we use the Lebesgue-Stieltjes integration on the right-hand side of (3.3). By integrating the right-hand side of (3.3) by parts, we obtain the estimate

\[ h_z(z) = \frac{\varphi(1)}{2\pi i} \int_{0}^{2\pi} \frac{1}{z - e^{i\theta}} d(e^{if(\theta)}) . \]

We therefore have

\[ |h_z(z)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{|z - e^{i\theta}|} |d(e^{if(\theta)})| . \]

Denote by \( \mu_f \) the Lebesgue-Stieltjes measure of the continuous increasing function \( f \). Then we have \( |d(e^{if(\theta)})| \leq d(f(\theta)) = d\mu_f(\theta) \). Therefore, we obtain the formula

(3.4) \[ |h_z(z)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{|z - e^{i\theta}|} d\mu_f(\theta). \]

Using our decomposition of the unit disk \( \mathbb{D} \) and the decomposition \( \mathcal{P}(I_{j,k}) \) of \([0, 2\pi]\) in Section 2.1, for \( \Phi(t) = t^2 \log^\lambda(e + t) \), we get

\[
\int_{\mathbb{D}} \Phi(|h_z(z)|) \, dz = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} \Phi(|h_z(z)|) \, dz \\
\leq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} \Phi \left( \frac{1}{2\pi} \sum_{I \in \mathcal{P}(I_{j,k})} \int_{I} \frac{1}{|z - e^{i\theta}|} d\mu_f(\theta) \right) \, dz \\
= \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} \Phi \left( \frac{1}{2\pi} \sum_{n \leq j} \sum_{m} \int_{I_{n,m}} \frac{1}{|z - e^{i\theta}|} d\mu_f(\theta) \right) \, dz ,
\]

(3.5) where we abuse the notation: sum over those \( m \) which belong to \( \{1, \cdots, 2^n\} \) and satisfy \( I_{n,m} \in \mathcal{P}(I_{j,k}) \). It is easy to check that \( \#\{m\} \leq 3 \). For any \( I_{n,m} \in \mathcal{P}(I_{j,k}) \), we know that \( |z - e^{i\theta}| \approx 2^{-n} \) for any \( z \in Q_{j,k} \) and \( \theta \in I_{n,m} \). Since \( \mu_f \) is the Lebesgue-Stieltjes measure of our continuous increasing function \( f \), for any interval \( I_{n,m} \), we get

\[
\int_{I_{n,m}} d\mu_f(\theta) = |f(I_{n,m})| = \ell(\varphi(\Gamma_{n,m})).
\]

Hence we obtain

(3.6) \[
\int_{\mathbb{D}} \Phi(|h_z(z)|) \, dz \leq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} \Phi \left( \sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right) \, dz \\
\leq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{-2j} \Phi \left( \sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right) .
\]
Since \( \ell(\varphi(\Gamma_{n,m})) \leq 2\pi \), we have
\[
\sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \leq 2\pi \sum_{n \leq j} \sum_{m} \frac{1}{2^{-n}} \lesssim 2^j.
\]

Using the same idea as in Proof of (iv)\( \Leftrightarrow \)(v), we have the estimate
\[
+\infty \sum_{j=1}^{2^j} 2^{-2j} \Phi \left( \sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right) \lesssim +\infty \sum_{j=1}^{2^j} 2^{-2j} j^\lambda \left( \sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right)^2 + C.
\]

This allows us to estimate (3.6) as
\[
(3.7) \quad \int_{D} \Phi(|h(z)|) \, dz \lesssim +\infty \sum_{j=1}^{2^j} 2^{-2j} j^\lambda \left( \sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right)^2 + C.
\]

For fixed \( \varphi(\Gamma_{n,m}) \) and fixed \( j \geq n \), the estimate (2.1) gives us \( \# \{ k \} \leq 3 \cdot 2^{j-n} \). By applying the Hölder inequality on the right-hand side of inequality (3.7) and Fubini’s theorem for series, we obtain
\[
\int_{D} \Phi(|h(z)|) \, dz \lesssim +\infty \sum_{n=1}^{2^n} \sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^2}{2^{-\frac{3}{2}n}} + C
\]
\[
\lesssim +\infty \sum_{n=1}^{2^n} \sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^2}{2^{-\frac{3}{2}n}} \left( \sum_{j \geq n} \sum_{k} 2^{-\frac{3}{2}j} j^\lambda \right) + C
\]
\[
\lesssim +\infty \sum_{n=1}^{2^n} \sum_{m=1}^{2^n} \frac{\ell(\varphi(\Gamma_{n,m}))^2}{2^{-\frac{3}{2}n}} \left( \sum_{j \geq n} 2^{-\frac{3}{2}j} j^\lambda \right) + C
\]
\[
\lesssim +\infty \sum_{n=1}^{2^n} \sum_{m=1}^{2^n} n^\lambda \ell(\varphi(\Gamma_{n,m}))^2 + C < +\infty.
\]

An analogous estimate follows for \( h_{\bar{z}} \) by the same reasons. Thus we finish the proof of (iv)\( \Rightarrow \)(i).

\textbf{Proof of (iv)\( \Rightarrow \)(ii).} Recall the estimate (3.4) for \( |h_{z}| \) and formula (3.5). From (3.5) and the fact that \( \log^\lambda(\frac{2}{1-|z|}) \approx j^\lambda \) for any \( z \in Q_{j,k} \), we have
\[
\int_{D} |h_{z}(z)|^2 \log^\lambda(\frac{2}{1-|z|}) \, dz = \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} |h_{z}(z)|^2 \log^\lambda(\frac{2}{1-|z|}) \, dz
\]
\[ \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} j^2 \int_{Q_{j,k}} |h_z(z)|^2 \, dz. \]

By replacing \( \Phi(t) = t^2 \log \lambda(e + t) \) by \( t^2 \) in (3.6), we obtain
\[ \int_{Q_{j,k}} |h_z(z)|^2 \, dz \lesssim 2^{-2j} \left( \sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right)^2. \]

Hence we have
\[ \int_{D} |h_z(z)|^2 \log \left( \frac{2}{1 - |z|} \right) \, dz \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} 2^{-2j} j^2 \left( \sum_{n \leq j} \sum_{m} \frac{\ell(\varphi(\Gamma_{n,m}))}{2^{-n}} \right)^2. \]

Consequently, we may apply the very same arguments that we used above to estimate the right-hand side of (3.7) so as to arrive at
\[ \int_{D} |h_z(z)|^2 \log \left( \frac{2}{1 - |z|} \right) \, dz \lesssim \sum_{n=1}^{+\infty} \sum_{m=1}^{2^n} n^2 \ell(\varphi(\Gamma_{n,m})) < +\infty. \]

One may similarly deal with \( h_{\bar{z}} \) to finish the proof of (iv) \( \Rightarrow \) (ii).

\[ \square \]

**Proof of (iv) \( \Leftrightarrow \) (iii).** We first divide the double integral on \( S \) into two parts:
\[ \int_{\mathbb{S}} \int_{\mathbb{S}} |\log |x^{-1}(\xi) - y^{-1}(\eta)||^{\lambda+1} \, |d\eta|\, |d\xi| \]
\[ = \int_{\mathbb{S}} \int_{\mathbb{S} \setminus \{|x^{-1}(\xi) - y^{-1}(\eta)| \leq 2\}} + \int_{\mathbb{S} \setminus \{|x^{-1}(\xi) - y^{-1}(\eta)| < 1\}} = I + II. \]  

(3.8)

Since \( \lambda > -1 \), we have \( 0 \leq |\log t|^{\lambda+1} \leq (\log 2)^{\lambda+1} \) for \( 1 \leq t \leq 2 \). Therefore we obtain \( 0 \leq I \leq C \). So our job is to estimate \( II \).

Fubini’s theorem changes \( II \) to
\[ II = \int_{\mathbb{S}} \int_{\{|x^{-1}(\xi) - y^{-1}(\eta)| < 1\}} \int_{0}^{1} \Lambda(t) \chi_{\{|x^{-1}(\xi) - y^{-1}(\eta)| \leq t<1\}} dt \, |d\eta||d\xi| \]
\[ = \int_{\mathbb{S}} \int_{0}^{1} \ell(\{\eta \in \mathbb{S} : |x^{-1}(\xi) - y^{-1}(\eta)| \leq t\}) \Lambda(t) \, dt \, |d\xi|, \]

(3.9)

where \( \Lambda(t) = (\lambda + 1) \log^\lambda \left( \frac{1}{t} \right) / t \). Here we used the fact that
\[ |\log t|^{\lambda+1} = \log^{\lambda+1} \left( \frac{1}{t} \right) = \int_{t}^{1} \Lambda(t) \, dt \], for any \( 0 < t < 1 \).
Then we have

\[
II = \sum_{j=0}^{+\infty} \int_{S} \int_{2^{-j-1}}^{2^{-j}} \ell\left(\{\eta : |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)| \leq t\}\right) \Lambda(t) d\xi dt.
\]

(3.10) \[ \leq C + \sum_{j=1}^{+\infty} \int_{S} \ell\left(\{\eta : |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)| \leq 2^{-j}\}\right) d\xi \int_{2^{-j-1}}^{2^{-j}} \Lambda(t) dt. \]

If \( \xi \in \varphi(\Gamma_{j,k}) \) and \( |h^{-1}(\xi) - h^{-1}(\eta)| \leq 2^{-j} \), then the arc length of the shorter arc from \( h^{-1}(\xi) \) to \( h^{-1}(\eta) \) is at most \( 2^{-j} \pi \) and hence it follows that \( \eta \in \bigcup_{n=k-1}^{k+1} \varphi(\Gamma_{j,n}) \). This means that for \( j \geq 1 \), we have

\[
\sum_{k=1}^{2^j} \int_{\varphi(\Gamma_{j,k})} \ell\left(\{\eta : |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)| \leq 2^{-j}\}\right) d\xi \leq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) \leq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2.
\]

(3.11) \[ \leq \sum_{k=1}^{2^j} \ell\left(\bigcup_{n=k-1}^{k+1} \varphi(\Gamma_{j,n})\right) \leq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2. \]

By the mean value theorem, we have

\[
\int_{2^{-j-1}}^{2^{-j}} \Lambda(t) dt = \log^{\lambda+1}(2^{j+1}) - \log^{\lambda+1}(2^j) \approx j^\lambda, \quad j \geq 1.
\]

(3.12) \[ \int_{2^{-j-1}}^{2^{-j}} \Lambda(t) dt = \log^{\lambda+1}(2^{j+1}) - \log^{\lambda+1}(2^j) \approx j^\lambda, \quad j \geq 1. \]

Combining (3.8), (3.9), (3.10), (3.11) and (3.12), we therefore deduce

\[
\int_{S} \int_{S} |\log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)||^{\lambda+1} d\xi d\eta \leq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda + C.
\]

We are left to deal with the converse direction. We divide the interval \([0,1]\) in (3.9) into intervals \([\pi/8, 1]\) and \([2^{-j+1}\pi, 2^{-j+2}\pi]\) for \( j \geq 5 \). Then we have

\[
\int_{S} \int_{S} |\log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)||^{\lambda+1} d\xi d\eta \leq \sum_{j=5}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda + C.
\]

(3.13) \[ \int_{S} \int_{S} |\log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)||^{\lambda+1} d\xi d\eta \leq \sum_{j=5}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda + C. \]

Given any \( j \geq 5 \) and \( 1 \leq k \leq 2^j \), the inequality

\[
|\varphi^{-1}(\xi) - \varphi^{-1}(\eta)| \leq \ell(\Gamma_{j,k}) \leq \pi 2^{1-j}
\]

holds for all \( \eta, \xi \in \varphi(\Gamma_{j,k}) \). Thus we have

\[
\sum_{k=1}^{2^j} \int_{\varphi(\Gamma_{j,k})} \ell\left(\{\eta : |h^{-1}(\xi) - h^{-1}(\eta)| \leq \pi 2^{1-j}\}\right) d\xi \geq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2.
\]

(3.14) \[ \sum_{k=1}^{2^j} \int_{\varphi(\Gamma_{j,k})} \ell\left(\{\eta : |h^{-1}(\xi) - h^{-1}(\eta)| \leq \pi 2^{1-j}\}\right) d\xi \geq \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2. \]
Finally, the estimates (3.8), (3.12), (3.13) and (3.14) yield
\[
\int_{S} \int_{S} |\log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)||^\lambda |d\xi||d\eta| + C \geq \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda.
\]

Proof of (ii)⇒(iv). Since \( \varphi \) is uniformly continuous, there exists \( j_0 \geq 3 \) such that
\( \ell(\varphi(\Gamma_{j,k})) \leq \pi/3 \) whenever \( j \geq j_0 \) and \( k = 1, \ldots, 2^j \). Fix such \( j, k \) and let \( \xi \in \Gamma_{j,k-1} \), \( \eta \in \Gamma_{j,k+1} \) (\( \Gamma_{j,0} = \Gamma_{j,2^j}, \Gamma_{j,2^j+1} = \Gamma_{j,1} \)). Since \( h \in C(\overline{\mathbb{D}}) \) is a diffeomorphism, we have that
\[
|h(\xi) - h(\eta)| \leq \int_{\gamma} |Dh| \, ds.
\]
for any rectifiable curve \( \gamma \subset \mathbb{D} \) joining \( \xi \) to \( \eta \). Moreover, \( |h(\xi_{j,k}) - h(\xi_{j,k+1})| \leq |h(\xi) - h(\eta)| \) because \( \varphi \) is homeomorphic and \( \ell(\varphi(\Gamma_{j,k-1} \cup \Gamma_{j,k} \cup \Gamma_{j,k+1})) \leq \pi \). Denote by \( \tilde{\xi} \) the midpoint of the arc \( \Gamma_{j,k} \). Let \( t_1 > 0 \) be the smallest \( t \) for which \( \partial B(\tilde{\xi}, t) \cap \Gamma_{j,k+1} \neq \emptyset \) and let \( t_2 \) be correspondingly the largest such \( t \). Write \( \gamma_t = \partial B(\tilde{\xi}, t) \cap \mathbb{D} \) for \( t_1 \leq t \leq t_2 \). Then there exists an absolute constant \( C \) so that
\[
\bigcup_{t \in [t_1, t_2]} \gamma_t \subset C Q_{j,k} \cap \mathbb{D}.
\]
Now
\[
|h(\xi_{j,k}) - h(\xi_{j,k+1})| \leq \int_{\gamma_t} |Dh| \, ds
\]
for \( t \in [t_1, t_2] \) and by integrating with respect to \( t \) we obtain
\[
(3.15) \quad (t_2 - t_1)|h(\xi_{j,k}) - h(\xi_{j,k+1})| \leq \int_{t_1}^{t_2} \int_{\gamma_t} |Dh| \, ds \, dt \leq \int_{C Q_{j,k} \cap \mathbb{D}} |Dh(z)| \, dz.
\]
Notice that \( t_2 - t_1 \) is uniformly comparable to \( 2^{-j} \approx \ell(\Gamma_{j,k}) \) when \( j \geq j_0 \) and \( k \in \{1, \ldots, 2^j\} \). It follows from (3.15) that
\[
(3.16) \quad |h(\xi_{j,k}) - h(\xi_{j,k+1})| \lesssim \frac{1}{\ell(\Gamma_{j,k})} \int_{C Q_{j,k} \cap \mathbb{D}} |Dh(z)| \, dz.
\]
Recall that we have \( \ell(\varphi(\Gamma_{j,k})) \leq \pi/3 \). Thus \( \ell(\varphi(\Gamma_{j,k})) \) is also uniformly comparable to \( |\varphi(\xi_{j,k}) - \varphi(\xi_{j,k+1})| \). We can therefore conclude from (3.16) that for any \( j \geq j_0 \) and \( 1 \leq k \leq 2^j \), the following inequality holds:
\[
(3.17) \quad \ell(\varphi(\Gamma_{j,k})) \lesssim \frac{1}{\ell(\Gamma_{j,k})} \int_{C Q_{j,k} \cap \mathbb{D}} |Dh(z)| \, dz.
\]
Fix $p \in (1, 2)$ and $q \in (2, +\infty)$ with $1/p + 1/q = 1$. By applying the Hölder inequality to (3.17), we have

$$\ell(\varphi(\Gamma_{j,k}))^p \lesssim \frac{1}{\ell(\Gamma_{j,k})^p} \int_{CQ_{j,k}} H(z) dz \left( \int_{CQ_{j,k} \cap D} \log^{-\frac{\lambda q}{2}} \left( \frac{1}{1 - |z|} \right) dz \right)^{p/q},$$

where $H(z) = |Dh(z)|^p \log^{\lambda p/2} \left( \frac{1}{1 - |z|} \right) \chi_D(z)$. Moreover by changing to polar coordinates we have the estimate

$$\int_{CQ_{j,k} \cap D} \log^{-\frac{\lambda q}{2}} \left( \frac{1}{1 - |z|} \right) dz \lesssim \ell(\Gamma_{j,k}) \int_0^{\ell(\Gamma_{j,k})} \log^{-\frac{\lambda q}{2}} \left( \frac{1}{s} \right) ds$$

$$\lesssim \ell(\Gamma_{j,k})^2 \log^{-\frac{\lambda q}{2}} \left( \frac{1}{\ell(\Gamma_{j,k})} \right)$$

$$\approx \ell(\Gamma_{j,k})^2 \frac{j^{-\lambda q/2}}{2}.$$

It follows that

$$(3.18) \quad \ell(\varphi(\Gamma_{j,k}))^p j^{\lambda p/2} \lesssim \ell(\Gamma_{j,k})^p \int_{CQ_{j,k} \cap D} H(z) dz.$$

Next by the inclusion relationship between $B_{j,k}$ and $Q_{j,k}$ and the definition of Hardy-Littlewood maximal function we have

$$(3.19) \quad \int_{CQ_{j,k}} H(z) dz \leq \int_{C B_{j,k}} H(z) dz \leq \int_{B_{j,k}} M_H(z) dz \leq \int_{Q_{j,k}} M_H(z) dz.$$

Combining (3.18) with (3.19) and then applying Jensen’s inequality, we arrive at

$$\ell(\varphi(\Gamma_{j,k}))^2 j^\lambda \lesssim \int_{Q_{j,k}} M_{H}^{2/p}(z) dz, \quad \forall j \geq j_0 \text{ and } 1 \leq k \leq 2^j.$$

Then the $L^{2/p}$-boundedness of the Hardy-Littlewood maximal operator implies

$$\sum_{j=j_0}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} M_{H}^{2/p}(z) dz \leq \int_{\mathbb{R}^2} M_{H}^{2/p}(z) dz$$

$$\lesssim \int_D |Dh(z)|^2 \log \left( \frac{1}{1 - |z|} \right) dz.$$

Moreover, we have that

$$\sum_{j=1}^{j_0-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda \leq \sum_{j=1}^{j_0-1} j^\lambda \left( \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k})) \right)^2 \lesssim \sum_{j=1}^{j_0-1} j^\lambda \leq C.$$
Hence we conclude
\[
\sum_{j=0}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda = \sum_{j=1}^{j_0-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda + \sum_{j=j_0}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda
\]
\[
\lesssim C + \int_\mathbb{D} |Dh(z)|^2 \log \left( \frac{1}{1 - |z|} \right) \, dz.
\]
\[
\approx C + \int_\mathbb{D} |Dh(z)|^2 \log \left( \frac{1}{1 - |z|} \right) \, dz.
\]

\[\square\]

**Proof of (i)⇒(v).** Set \(K(z) = |Dh(z)|\chi_D(z).\) As in the proof of (ii) ⇒ (iv), we have
\[
\frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \lesssim \int_{CQ_{j,k}} K(z) \, dz \quad \text{for any } j \geq j_0 \text{ and } 1 \leq k \leq 2^j.
\]
Thus Jensen’s inequality for the function \(\Phi(t) = t^2 \log^\lambda(e + t)\) and inequality (3.19) with \(H\) replaced by \(K\) imply that
\[
\sum_{j=j_0}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^2 \Phi \left( \frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\Gamma_{j,k})^2 \int_{Q_{j,k}} \Phi(M_K)(z) \, dz
\]
\[
\lesssim \sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \int_{Q_{j,k}} \Phi(M_K)(z) \, dz \leq \int_{\mathbb{R}^2} \Phi(M_K)(z) \, dz.
\]
(3.20)

Here in the second inequality, we used the fact that \(\ell(\Gamma_{j,k})^2 \approx |Q_{j,k}|.\) Therefore Proposition 2 gives
\[
\int_{\mathbb{R}^2} \Phi(M_K)(z) \, dz \lesssim \int_{\mathbb{R}^2} \Phi(K)(z) \, dz.
\]
(3.21)

Moreover, we have that
\[
\sum_{j=1}^{j_0-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^\lambda \left( e + \frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \lesssim \sum_{j=1}^{j_0-1} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 j^\lambda \lesssim \sum_{j=1}^{j_0-1} j^\lambda \lesssim C.
\]

Combining (3.20), (3.21) with the above inequality gives
\[
\sum_{j=1}^{+\infty} \sum_{k=1}^{2^j} \ell(\varphi(\Gamma_{j,k}))^2 \log^\lambda \left( e + \frac{\ell(\varphi(\Gamma_{j,k}))}{\ell(\Gamma_{j,k})} \right) \lesssim C + \int_{\mathbb{D}} |Dh(z)|^2 \log^\lambda(e + |Dh(z)|) \, dz.
\]
\[\square\]
3.2 Proof of Theorem 1

Proof. From [2, Lemma 3.2], we know that any internal chord-arc Jordan domain \( \Omega \subset \mathbb{C} \) is a bounded John disk whose boundary \( \partial \Omega \) satisfies (1.1). Note also that in bounded John disks, the internal distance of any two boundary points is finite, [9, Remark 6.6]. By using the arc length parametrization of \( \partial \Omega \) and the property (1.1), we see that there is a bi-Lipschitz map \( g : \mathbb{S} \to (\partial \Omega, \lambda_\Omega) \). Then applying [2, Theorem 4.7] and the fact that the internal distance in the unit disk is the same as the Euclidean distance, we know that \( g \) extends to a bi-Lipschitz map \( \tilde{g} : \overline{D} \to (\overline{\Omega}, \lambda_\Omega) \). Moreover, the bi-Lipschitz map \( \tilde{g} \) is a diffeomorphism in \( D \).

In the statements (i) and (ii) in Theorem 1, we let \( h = \tilde{g} \circ P[g^{-1} \circ \varphi] \), where \( P[g^{-1} \circ \varphi] \) is the Poisson extension of \( g^{-1} \circ \varphi \). Hence \( h \) is a diffeomorphism. Since \( \tilde{g} \) is bi-Lipschitz with respect to the internal distance and the internal distance \( \lambda_\Omega \) is the same as the Euclidean distance locally in \( \Omega \), we obtain that there is a constant \( C > 0 \) so that \( 1/C \leq |D\tilde{g}(z)| \leq C \) for any \( z \in \overline{D} \). Hence the convergence of the integrals of \( |Dh| \) in (i) and (ii) in Theorem 1 is equivalent to the statements (i) and (ii) for \( P[g^{-1} \circ \varphi] \) in Theorem 2.

By [2, Lemma 3.2, Lemma 3.4 and Lemma 3.6], we know that for any \( z_1, z_2 \in \partial \Omega \), the shorter arc \( \gamma_{z_1,z_2} \) of \( \partial \Omega \) joining \( z_1 \) and \( z_2 \) satisfies

\[ \text{diam}(\gamma_{z_1,z_2}) \approx \lambda_\Omega(z_1, z_2). \]

Hence the arc length \( \ell(\cdot) \) on \( \partial \Omega \) with respect to the Euclidean distance is comparable to the arc length \( \ell_{\lambda_\Omega}(\cdot) \) on \( \partial \Omega \) with respect to the internal distance \( \lambda_\Omega \). Combining with the bi-Lipschitz property of \( g \), we obtain that \( \ell(\varphi(\Gamma_{j,k}))/C \leq \ell(g^{-1} \circ \varphi(\Gamma_{j,k})) \leq C\ell(\varphi(\Gamma_{j,k})) \) for any \( \Gamma_{j,k} \) and that we can use a change of variable via \( g \) to conclude the equivalence of (iii) for \( \varphi^{-1} \) and \( \varphi^{-1} \circ g \) in Theorem 1 and 2. Hence the statements (iii)-(v) for \( \varphi \) in Theorem 1 is equivalent to the statements (iii)-(v) for \( g^{-1} \circ \varphi \) in Theorem 2.

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