The rationality problem for conic bundles

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Abstract. This expository paper is concerned with the rationality problem for three-dimensional algebraic varieties with a conic bundle structure. The main methods of this theory are discussed, proofs of certain principal results are sketched, and some recent achievements are presented. Many open problems are also stated.

Bibliography: 209 titles.

Keywords: algebraic threefolds, conic bundles, rationality, singularities, invariants, birational transformations, Sarkisov programme, intermediate Jacobian, Prym variety.

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1. Introduction

The ground field is supposed to be the field of complex numbers \( \mathbb{C} \) unless otherwise stated. In this paper we deal with algebraic varieties having the structure of conic bundles over surfaces. A motivation for the study of such varieties is that they naturally appear in the birational classification of threefolds of negative Kodaira dimension.

According to the minimal model programme (MMP) [123], [22] every uniruled algebraic variety \( Y \) is birationally equivalent to a projective variety \( X \) with at worst \( \mathbb{Q} \)-factorial terminal singularities that admits a contraction \( \pi: X \to S \) to a lower-dimensional normal projective variety \( S \) such that the anticanonical divisor \(-K_X\) is \( \pi \)-ample and

\[
\text{Pic}(X) = \pi^* \text{Pic}(S) \oplus \mathbb{Z}.
\]

1.1. In dimension 3 there are the following three possibilities:

- \( S \) is a point, and then \( X \) is called a \( \mathbb{Q} \)-Fano threefold;
- \( S \) is a smooth projective curve, and then \( \pi: X \to S \) is called a \( \mathbb{Q} \)-del Pezzo fibration;
- \( S \) is a normal surface, and then \( \pi: X \to S \) is called a \( \mathbb{Q} \)-conic bundle.

The \( \mathbb{Q} \)-Fano threefolds are bounded, that is, they lie in an algebraic family [91] (this is true even in arbitrary dimension [20]). However, an explicit classification is known only in the three-dimensional case for smooth Fano varieties [82]. There are a lot of partial results related to singular \( \mathbb{Q} \)-Fano threefolds (see, for example, [4], [28], [154], [150], [152], [153], and references there).

For \( \mathbb{Q} \)-del Pezzo fibrations there are partial results on construction of standard models [44], [98] and birational rigidity (see, for example, [162], [165], Chaps. 4 and 5, [31], [189], [57], [186]).

In this paper we concentrate on the last case of 1.1. We are mainly interested in rationality questions. It is known that any \( \mathbb{Q} \)-conic bundle has a standard model, that is, there exists a \( \mathbb{Q} \)-conic bundle \( \pi^* : X^* \to S^* \) such that the total space \( X^* \) and the surface \( S^* \) are smooth and there are birational maps \( X^* \dashrightarrow X \) and \( S^* \dashrightarrow S \) making the corresponding diagram commutative (see Theorem 3.12). The following conjecture is motivated by a rationality criterion for two-dimensional conic bundles (see §5).

1.2. Conjecture [180]. Let \( \pi: X \to S \) be a standard conic bundle over a rational surface \( S \) with discriminant curve \( \Delta \subset S \). In this case the variety \( X \) is rational if and only if the following holds:

\[
(*) \quad |2K_S + \Delta| = \emptyset \text{ and, in the case } p_a(\Delta) = 6, \text{ the Griffiths component } J_G(X) \text{ of the intermediate Jacobian } J(X) \text{ is trivial.}
\]

Note that in the case where \( |2K_S + \Delta| = \emptyset \) and \( p_a(\Delta) = 6 \) the variety \( X \) can be non-rational (that is, the condition \( J_G(X) = 0 \) here cannot be removed). In fact, in this case \( \pi: X \to S \) is fibrewise birational to a standard conic bundle \( \pi^\sharp: X^\sharp \to \mathbb{P}^2 \) with discriminant curve \( \Delta^\sharp \) of degree 5, so that the corresponding double cover \( \tilde{\Delta}^\sharp \to \Delta^\sharp \) is defined by an odd theta-characteristic (see Corollary 8.7.1). Moreover, such a variety \( X \) is birational to a three-dimensional cubic hypersurface (see Proposition 8.9).
1.3. Conjecture [73]. Let $\pi: X \to S$ be a standard conic bundle over a rational surface $S$ with discriminant curve $\Delta \subset S$. In this case $X$ is rational if and only if one of the following conditions is satisfied.

(i) There exists a commutative diagram

$$X' \xrightarrow{\Phi} X \xrightarrow{\pi} S' \xrightarrow{\alpha} S$$

where $\pi': X' \to S'$ is a standard conic bundle with discriminant curve $\Delta' \subset S'$, $\alpha$ is a birational morphism, $\Phi$ is a birational map, and a base point free pencil $L'$ of rational curves on $S'$ is such that $L' \cdot \Delta' \leq 3$.

(ii) There exists a commutative diagram

$$X \xrightarrow{\Phi} X' \xrightarrow{\pi'} S' \xrightarrow{\alpha} \mathbb{P}^2$$

where $\pi': X' \to S' = \mathbb{P}^2$ is a standard conic bundle with discriminant curve $\Delta' \subset \mathbb{P}^2$ of degree 5, $\alpha$ is a birational morphism, $\Phi$ is a birational map, and $X'$ is the blowup of $\mathbb{P}^3$ along a non-singular curve $\Gamma \subset \mathbb{P}^3$ of genus 5 and degree 7 (see Example 3.4.3).

It is not so difficult to show that Conjectures 1.3 and 1.2 are equivalent (see Corollary 8.7.2). The sufficiency of conditions (i) and (ii) in Conjecture 1.3 and condition ($\star$) in Conjecture 1.2 is also known (see Propositions 8.1 and 5.6 and Corollary 8.7.1). The hardest part of these conjectures is the necessity. The necessity of conditions 1.3, (i), (ii), and 1.2 ($\star$) was proved by Shokurov in the case of conic bundles over minimal rational surfaces.

1.4. Remark. Note that the condition (i) of Conjecture 1.3 can be replaced by the following one (see Proposition 8.6):

(i') $\pi$ is fibrewise birationally equivalent to a standard conic bundle

$$\pi^\sharp: X^\sharp \to S^\sharp,$$

where $S^\sharp = \mathbb{F}_n$ is a rational geometrically ruled surface and the discriminant curve $\Delta^\sharp$ meets a general fibre in at most three points.

The paper is organized as follows. Sections 2 and 3 are preliminary. There we fix the notation, give basic definitions, and collect general facts about varieties with conic bundle structures. In §4 we introduce the Sarkisov programme of decomposition of birational maps between Mori fibre spaces. The proof of the main theorem in the three-dimensional case is outlined. Section 5 is about surface conic bundles over algebraically non-closed fields. There we recall the classification of two-dimensional Sarkisov links and formulate a rationality criterion. This theory is a motivation for the main Conjectures 1.2 and 1.3 as well as one of the main tools in the proofs.
There are two important birational invariants of algebraic varieties that use transcendental and topological methods: torsions in the middle cohomology groups and the intermediate Jacobian. These notions are discussed in §§6 and 7, respectively. In §8 we collect examples of some special Sarkisov links in the category of conic bundles. They are used very frequently below. Following Shokurov [180], as the first application of the theory developed we reproduce in §9 the proof of 1.2 for conic bundles over minimal rational surfaces.

Although Conjectures 1.2 and 1.3 are formulated in terms of non-singular $X$ and $S$, in order to analyze the corresponding birational maps we use the techniques of factorization of birational maps between Mori fibre spaces, which in general admit terminal singularities. These techniques and results are discussed in §§10–13. Thus, §10 covers known results on local classification of $\mathbb{Q}$-conic bundles, and §11 contains numerous examples of Sarkisov links between them. In §§12 and 13 we apply the theory developed to rationality problems of conic bundles. In particular, we prove the Sarkisov Theorem 12.2 and Theorem 13.7, which is a very weak version of Conjecture 1.3. We also prove the equivalence of Conjecture 1.3 and a certain classical conjecture of projective geometry (see Proposition 13.6). In the last section, §14, we collect open problems and some results related to the birational geometry of conic bundles.

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2. Preliminaries

Unless otherwise specified, we shall always work over an algebraically closed field $k$ of characteristic 0. In certain situations we prefer to work over $\mathbb{C}$. We use the standard terminology and notation of the (Log) Minimal Model Programme [104]. For an introduction to rationality problems we refer to [105].

2.1. Notation. Throughout this paper

- the symbol $\equiv$ denotes the numerical equivalence of cycles,
- $K_X$ is the canonical (Weil) divisor of a normal variety $X$,
- $F_n = \mathbb{P}^1_p (\mathcal{O}_{\mathbb{P}^1_p} \oplus \mathcal{O}_{\mathbb{P}^1_p}(n))$ is the rational ruled (Hirzebruch) surface,
- $\rho(X) = \mathrm{rk} \ Pic(X)$ is the Picard number of a variety $X$,
- $\rho(X/S)$ is the relative Picard number.

A contraction is a surjective projective morphism $\pi: X \rightarrow S$ of normal varieties such that $\pi_*$ $\mathcal{O}_X = \mathcal{O}_S$. A contraction $\pi: X \rightarrow S$ is extremal if $\rho(X/S) = 1$.

A contraction $\pi: X \rightarrow S$ is called a Mori extremal contraction if $X$ is a variety with at most terminal $\mathbb{Q}$-factorial singularities, the divisor $-K_X$ is relatively ample, and the relative Picard number $\rho(X/S)$ equals 1. If additionally $\dim S < \dim X$, then $\pi$ is called a Mori fibre space or simply a Mori fibration. In dimension 3 for a Mori fibre space there are only the three possibilities listed in 1.1.

Let $\pi: X \rightarrow S$ and $\pi^\sharp: X^\sharp \rightarrow S^\sharp$ be Mori fibre spaces. A birational map $\Phi: X \dashrightarrow X^\sharp$ is said to be fibrewise if there exists a birational map $\alpha: S \dashrightarrow S^\sharp$ making the diagram
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\[ X - \Phi \rightarrow X^\sharp \]
\[ S - \alpha \rightarrow S^\sharp \]

commutative and inducing an isomorphism \( \Phi_\eta : X_\eta \rightarrow X^\sharp_\eta \) of generic fibres.

For example, \( X = \mathbb{P}^n \times \mathbb{P}^n \) has two Mori fibre space structures \( \pi_i : X \rightarrow \mathbb{P}^n \), the projections to factors. The identity map is not a fibrewise birational equivalence but the involution interchanging the factors is.

A Mori fibre space \( \pi : X \rightarrow S \) is said to be \textit{birationally rigid} if for any birational map \( \Phi : X \rightarrow X^\sharp \) to another Mori fibre space \( \pi^\sharp : X^\sharp \rightarrow S^\sharp \) there exists a birational selfmap \( \psi : X \rightarrow X \) such that the composition \( \Phi \circ \psi : X \rightarrow X^\sharp \) is fibrewise:

\[ X \xrightarrow{\Phi} X^\sharp \]
\[ S \xrightarrow{\alpha} S^\sharp \]

and induces an isomorphism \( \Phi_\eta : X_\eta \rightarrow X^\sharp_\eta \) of generic fibres. Examples will be given in Corollary 5.1.2 and Theorem 12.2. It is important to note that rigidity implies non-rationality but rigidity is much stronger.

3. Conic bundles

3.1. Definition. A \textit{conic bundle} is a proper flat morphism \( \pi : X \rightarrow S \) of non-singular varieties such that it is of relative dimension 1 and the anticanonical divisor \(-K_X\) is relatively ample. A conic bundle \( \pi : X \rightarrow S \) is said to be \textit{standard} if one of the following equivalent conditions holds:

(a) \( \text{Pic}(X) = \pi^* \text{Pic}(S) \oplus \mathbb{Z} \),

(b) \( \rho(X/S) = 1 \), that is, \( \pi \) is a Mori extremal contraction,

(c) for any prime divisor \( D \subset S \) its inverse image \( \pi^*(D) \) is irreducible.

If \( \pi : X \rightarrow S \) is any Mori extremal contraction from a smooth threefold to a surface, then \( \pi \) is a standard conic bundle [122]. There is a similar, but weaker, result in arbitrary dimension: if \( X \) is a non-singular variety and \( \pi : X \rightarrow S \) is an arbitrary Mori extremal contraction such that the dimension of any fibre equals 1, then \( \pi \) is a standard conic bundle [8].

The following facts are well known (see [13], Proposition 1.2, and [174], §1).

3.2. Theorem. Let \( \pi : X \rightarrow S \) be a conic bundle. Then the following statements hold.

(i) The anticanonical line bundle \( \omega_X^{-1} \) is relatively very ample and defines an embedding \( X \hookrightarrow \mathbb{P}(\mathcal{E}) \), where \( \mathcal{E} := \pi_* \omega_X^{-1} \) is a rank-3 locally free sheaf.

(ii) In the above embedding \( X \) is the zero locus of a section

\[ \sigma \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{L} \otimes p^*(\det \mathcal{E}^\vee \otimes \omega_S^{-1})) \]
where $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the tautological line bundle on $\mathbb{P}(\mathcal{E})$ and $p: \mathbb{P}(\mathcal{E}) \to S$ is the projection.

(iii) For any point $s \in S$, the scheme-theoretic fibre $X_s = \pi^{-1}(s)$ is isomorphic, over the residue field $\mathbb{k}(s)$, to a (possibly reducible or non-reduced) conic on the plane $\mathbb{P}(\mathcal{E})_s = \mathbb{P}_s^2$.

3.3. If $\pi: X \to S$ is a conic bundle, then a general fibre $X_s = \pi^{-1}(s)$ is a non-degenerate conic, that is, an irreducible smooth rational curve. Consider the sets

$$\Delta := \{s \in S \mid X_s \text{ is a degenerate conic}\}$$

and

$$\Delta_s := \{s \in S \mid X_s \text{ is a double line}\}.$$

Then $\Delta$ is a divisor on $S$. It is called the discriminant divisor or degeneration divisor. The set $\Delta_s$ coincides with the singular locus of $\Delta$ (see 3.3.2 below).

In a small affine neighbourhood $U \subset S$ of a point $s \in S$ one can write the equation of the variety $X \subset \mathbb{P}_s^2 \times X \times U$ as

$$\sum_{0 \leq i,j \leq 2} a_{i,j}x_i x_j = 0,$$

where $a_{i,j} \in \mathbb{C}[U]$. Then the divisor $\Delta$ is given by the determinant equation

$$\det \|a_{i,j}(s)\| = 0.$$

3.3.2. Moreover, $\text{rk} \|a_{i,j}(s)\| \neq 0$, and by using a $\mathbb{C}[U]$-linear coordinate change one can put (3.3.1) into one of the following forms:

$$b_0 x_0^2 + b_1 x_1^2 + b_2 x_2^2 = 0 \iff \text{rk} \|a_{i,j}(s)\| = 3 \iff s \notin \Delta,$$

$$b_0 x_0^2 + b_1 x_1^2 + c_2 x_2^2 = 0 \iff \text{rk} \|a_{i,j}(s)\| = 2 \iff s \in \Delta \setminus \Delta_s,$$

$$b_0 x_0^2 + c_1 x_1^2 + c_2 x_2^2 + c_3 x_1 x_2 = 0 \iff \text{rk} \|a_{i,j}(s)\| = 1 \iff s \in \Delta_s,$$

where

$$b_i(s) \neq 0, \quad c_i(s) = 0, \quad \text{and} \quad \text{mult}_s(c_1) = \text{mult}_s(c_2) = 1.$$

3.3.3. Corollary (see [174], the proof of Proposition 1.8.5). The discriminant divisor of a conic bundle $\pi: X \to S$ has only normal crossings in codimension two in $S$.

3.3.4. Remark. Let $\pi: X \to S$ be a contraction such that there exists a closed subset $Z \subset S$ of codimension $\geq 2$ such that the restriction $\pi^o: X^o \to S^o$ is a conic bundle, where

$$S^o := S \setminus Z \quad \text{and} \quad X^o := \pi^{-1}(S^o).$$

Then we can define the discriminant divisor $\Delta \subset S$ of the contraction $\pi$ as the closure of the discriminant divisor $\Delta^o$ of the conic bundle $\pi^o$. In general $\Delta$ is a (reduced) effective Weil divisor. Note, however, that in this case $\Delta$ is just the divisorial part of the maximal subset $R \subset S$ over which $\pi$ is not smooth. We do not assert that the morphism $\pi$ is smooth or flat over $S \setminus \Delta$ (see, for example, 10.3.1).
The above definition can be applied to arbitrary Mori extremal contractions $\pi: X \to S$ such that $\dim X = \dim S + 1$ and $X$ has at worst terminal singularities. Indeed, let $S_{\geq 2} \subset S$ be the set of points $s \in S$ such that $\dim \pi^{-1}(s) \geq 2$ and let

$$Z := S_{\geq 2} \cup \pi(\text{Sing}(X)).$$

The contraction $\pi$ is extremal, so $\pi^{-1}(S_{\geq 2})$ has no divisorial components. Since $X$ has at worst terminal singularities, $\text{codim}_X \text{Sing}(X) \geq 3$. Thus, $\text{codim}_S Z \geq 2$. Then the variety $S^\circ := S \setminus Z$ is non-singular and $\pi^\circ$ is a conic bundle over $S^\circ$ (see [122] and [8]).

3.3.5. Lemma. Let $\pi: X \to S$ and $\pi': X' \to S$ be standard conic bundles over a (not necessarily proper) variety $S$. Suppose that there is a fibrewise birational equivalence

$$\begin{array}{ccc}
X & \xrightarrow{\Phi} & X' \\
\pi \downarrow & & \pi' \downarrow \\
S & \xrightarrow{\alpha} & S
\end{array}$$

Then the discriminant divisors of $\pi$ and $\pi'$ coincide.

Proof. Let $\Delta$ and $\Delta'$ be the discriminant divisors of $\pi$ and $\pi'$, respectively. We may assume that $S = \text{Spec} A$ is a small affine neighbourhood of a given point. Then it is sufficient to consider the case where $\Delta = \emptyset$. Moreover, by shrinking $S$ we may assume that the vector bundles $\pi_* \mathcal{O}_X(-K_X)$ and $\pi'_* \mathcal{O}_{X'}(-K_{X'})$ are trivial. Then the anticanonical divisors define embeddings

$$X, X' \hookrightarrow \mathbb{P}^2 \times S = \mathbb{P}^2_A.$$

Thus, $X$ and $X'$ are given in $\mathbb{P}^2_A$ by the equations

$$\sum q_{i,j} x_i x_j = 0 \quad \text{and} \quad \sum q'_{i,j} x_i x_j = 0,$$

respectively, where $q_{i,j}, q'_{i,j} \in A$. By our assumption the matrix $\|q_{i,j}\|$ is non-singular and $\det \|q_{i,j}\| = 0$ is the equation of $\Delta'$. Since $X$ and $X'$ are birationally equivalent over $S$, the generic fibres $X_\eta$ and $X'_\eta$ are isomorphic over the field $K := \text{Frac}(A)$. Hence, there exists a projective transformation $\mathbb{P}^2_K \to \mathbb{P}^2_K$ that maps $X_\eta$ to $X'_\eta$. Let $T = \|t_{i,j}\|$, $t_{i,j} \in K$ be the corresponding matrix. Then the equation of $\Delta'$ is written in the form

$$\det \|q'_{i,j}\| = (\det \|t_{i,j}\|)^2 (\det \|q_{i,j}\|) = 0,$$

where $\det \|q_{i,j}\|$ is an invertible element of $A$. Since the discriminant curve of a standard conic bundle is reduced, the element $\det \|t_{i,j}\|$ must also be invertible. This means that $\Delta' = \emptyset = \Delta$ on $S$. ∎

3.3.6. Corollary. Assume a fibrewise birational equivalence of $\mathbb{Q}$-conic bundles

$$\begin{array}{ccc}
X & \xrightarrow{\Phi} & X' \\
\pi \downarrow & & \pi' \downarrow \\
S & \xrightarrow{\alpha} & S'
\end{array}$$
where $\alpha$ is a birational morphism. Then $\Delta' = \alpha(\Delta)$ for the corresponding discriminant curves $\Delta \subset S$ and $\Delta' \subset S'$.

The corollary shows that the discriminant curves of the fibrewise birational class of $X/S$ define a $b$-divisor on $S$ (see [183] and [79]). Note, however, that this is not a $b$-Cartier $b$-divisor (if it is non-trivial).

3.4. Lemma. Let $\pi : X \to S$ be a standard conic bundle. Then there is an isomorphism

$$\text{Pic}(X) \simeq \begin{cases} 
\pi^* \text{Pic}(S) \oplus \mathbb{Z} \cdot K_X & \text{if } \pi \text{ has no rational sections,} \\
\pi^* \text{Pic}(S) \oplus \mathbb{Z} \cdot D & \text{if } \pi \text{ has a rational section } D.
\end{cases}$$

From now on, we mainly concentrate on the study of conic bundles over rational surfaces. Let us consider several well-known examples.

3.4.1. Example. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth divisor of bidegree $(2, d)$, $d > 0$, given by the equation

$$f(x_0, x_1, x_2, y_0, y_1, y_2) = 0.$$ 

This equation is quadratic in $x_0, x_1, x_2$, so it can be seen as a symmetric $3 \times 3$ matrix $Q$ whose entries are homogeneous polynomials in $y_0, y_1, y_2$ of degree $d$. The projection $X \to \mathbb{P}^2$ to the second factor is a standard conic bundle whose discriminant curve is given by the equation $\det Q = 0$ of degree $3d$.

Starting with a well-known rationally connected variety, for example, a Fano threefold, one can in many cases construct a conic bundle using certain special birational transformations, so-called Sarkisov links (see §4). We give several such constructions below.

3.4.2. Example. Let $Y = Y_3 \subset \mathbb{P}^4$ be a smooth cubic hypersurface. It is well known (see, for example, [6]) that $Y$ contains a two-dimensional family of lines. Let $l \subset Y$ be a line and let $\sigma : X \to Y$ be the blowup of $l$. Let $E$ be the exceptional divisor and let $H^* = \sigma^* H$ be the pullback of a hyperplane section. Then the two-dimensional linear system $|H^* - E|$ is base point free and defines a conic bundle structure $\pi : X \to \mathbb{P}^2$. The discriminant curve $\Delta \subset \mathbb{P}^2$ is of degree 5. Indeed, a general member $F \in |H^* - E|$ is a cubic surface, and the restriction $\pi|_F$ is a conic bundle over the line $l \subset \mathbb{P}^2$. Degenerate fibres of $\pi|_F$ correspond to points of $\Delta \cap l$. By the Noether formula there are exactly 5 such fibres.

For a general choice of $l$ in the corresponding Hilbert scheme $\Sigma(Y)$, the curve $\Delta$ is smooth. However, $\Delta$ can be singular for some special choice of $l$. To illustrate this, we recall that the normal bundle $\mathcal{N}_{\Gamma/Y}$ of any line $\Gamma \subset Y$ has the form

$$\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a), \quad \text{where } a = 0 \text{ or } 1$$

([110], the proof of Proposition 2.2.8). The lines with $a = 1$ are said to be special. They are characterized by the property that there exists a plane $\mathbb{P}^2 \subset \mathbb{P}^4$ such that

$$Y \cap \mathbb{P}^2 = 2\Gamma + \Gamma',$$

where $\Gamma'$ is also a line, which is said to be complementary to $\Gamma$. The set of complementary lines is a closed one-dimensional subset of $\Sigma(Y)$. Now if in the above
construction we take the line \( l = \Gamma' \) to be complementary, then the proper transform of \( \Gamma \) on \( X \) will be a non-reduced fibre of the conic bundle \( \pi \), and so the discriminant curve \( \Delta \) will be singular at the corresponding point. But if \( l \) is not complementary, then the discriminant curve \( \Delta \) is smooth. Moreover, \( \Delta \) can be reducible: suppose that \( Y \) contains a cubic cone \( Z \). Take a line \( l \) so that \( l \subset Z \), and let \( Z_X \subset X \) be the proper transform of \( Z \). Then \( \Delta_1 := \pi(Z_X) \) is an irreducible component of \( \Delta \subset \mathbb{P}^2 \) of degree 1.

There is another type of conic bundles over \( \mathbb{P}^2 \) with degree-5 discriminant curve.

3.4.3. Example (cf. [135] and [23]). Let \( \Gamma \subset \mathbb{P}^3 \) be a smooth curve of degree 7 and genus 5. By the Riemann–Roch theorem the embedding \( \Gamma \subset \mathbb{P}^3 \) is given by the complete linear system \( |K \Gamma - P| \), where \( P \in \Gamma \) is a point. Let \( \sigma: X \rightarrow \mathbb{P}^3 \) be the blowup of \( \Gamma \) and let \( E \) be the exceptional divisor. Let \( H^* = \sigma^*H \) be the pullback of a hyperplane in \( \mathbb{P}^3 \). Then the linear system \( |3H^* - E| \) is base point free and defines a conic bundle structure \( \pi: X \rightarrow \mathbb{P}^2 \). The fibres of \( \pi \) are the proper transforms of conics in \( \mathbb{P}^3 \) meeting \( \Gamma \) at six points, and a general member of \( |3H^* - E| \) is a cubic surface. As above, the discriminant curve \( \Delta \subset \mathbb{P}^2 \) is of degree 5. Since \( \rho(X) = 2 \) and both contractions \( \pi \) and \( \sigma \) are \( K_X \)-negative, \( X \) is a Fano threefold with \( -K_X^3 = 16 \) (see [124], № 9). The morphism \( \sigma \times \pi: X \rightarrow \mathbb{P}^3 \times \mathbb{P}^2 \) is an embedding, and its image is an intersection of two divisors of bidegrees \((1, 1)\) and \((2, 1)\).

Since any smooth three-dimensional cubic hypersurface is non-rational [36], these two types of conic bundle (with discriminant curve of degree 5) cannot be birationally equivalent. They differ by the type of the corresponding double cover \( \tilde{\pi}: \tilde{\Delta} \rightarrow \Delta \) (see 3.9 below); for a conic bundle originating from a cubic hypersurface, the cover is defined by an odd theta-characteristic, whereas for the conic bundle constructed in Example 3.4.3 the theta-characteristic is even (see 3.9.2 below).

3.4.4. Example ([13], Example 1.4.4, [82], Theorem 4.3.3). Let \( Y = Y_{2,2,2} \subset \mathbb{P}^6 \) be a non-singular complete intersection of three quadrics. It is known that \( Y \) contains a one-dimensional family of lines (see, for example, [198], Lemma 5.2). Let \( l \subset Y \) be a line and \( \sigma: \tilde{Y} \rightarrow Y \) the blowup of \( l \). Let \( E \) be the exceptional divisor and \( H^* = \sigma^*H \) the pullback of a hyperplane section. The linear system \( |-K_{\tilde{Y}}| \) is base point free and defines a generically finite (but not finite) morphism. For a general choice of the line \( l \) this morphism is small, that is, it does not contract divisors. According to [96] there exists a flop \( \tilde{Y} \rightarrow X \), and by the cone theorem there exists a Mori extremal contraction on \( X \). It is not hard to show that the only possibility for this contraction is a conic bundle \( \pi: X \rightarrow \mathbb{P}^2 \) with discriminant curve \( \Delta \subset \mathbb{P}^2 \) of degree 7. Then the map \( \tilde{Y} \rightarrow \mathbb{P}^2 \) is given by the linear system \( |2H^* - 3E| \) (see [82], Theorem 4.3.3(ii)).

3.4.5. Example. Let \( Y = Y_{2,3} \subset \mathbb{P}^5 \) be an intersection of a quadric and a cubic. Suppose that \( Y \) contains a plane \( \Pi = \mathbb{P}^2 \). The projection from \( \Pi \) induces a rational curve fibration on \( Y \). If \( Y \) is sufficiently general, then by blowing up \( \Pi \) we obtain a standard conic bundle over \( \mathbb{P}^2 \) with discriminant curve of degree 7 (see, for example, [13], Example 1.4.6). In this case \( Y \) is the midpoint of a Sarkisov link ([86], Proposition 7.11).

More examples of conic bundles over \( \mathbb{P}^2 \) with discriminant curve of degree 7 can be found in [27].
3.4.6. Example ([13], Example 1.4.3, [40], [33]). Let $Y \subset \mathbb{P}^4$ be a quartic hypersurface which is singular along a line $l$. Suppose that $Y$ is general among the quartics satisfying this property. Then by blowing up $l$ we obtain a standard conic bundle over $\mathbb{P}^2$ with discriminant curve of degree $8$.

3.4.7. Example ([13], Example 1.4.5, [48], [33], [153]). Let $\varphi : Y \to \mathbb{P}^3$ be a double cover branched over a quartic surface $B \subset \mathbb{P}^3$. Assume that $B$ is singular and its singular locus consists of a unique node $Q$. Then $P := \varphi^{-1}(Q)$ is the only singular point of $Y$. In this case, the variety $Y$ is $\mathbb{Q}$-factorial (see, for example, [32]). Let $\sigma : X \to Y$ and $\lambda : \mathbb{P}^3 \to \mathbb{P}^3$ be blowups of $P$ and $Q$, respectively. Then $\varphi$ induces a double cover $\tilde{\varphi} : X \to \tilde{\mathbb{P}}^3$, and the projection from $Q$ induces a $\mathbb{P}^1$-bundle $\psi : \tilde{\mathbb{P}}^3 \to \mathbb{P}^2$. The composition $\pi : X \to \tilde{\mathbb{P}}^3 \to \mathbb{P}^2$ is a standard conic bundle whose fibres are double covers of the fibres of $\psi$. The discriminant curve is of degree $6$.

But if the point $P$ is not a unique singular point of $X$, then in the above construction $X$ is singular and $\pi : X \to \tilde{\mathbb{P}}^3 \to \mathbb{P}^2$ is a non-standard conic bundle. The standard forms for $X/\mathbb{P}^2$ were described in [33].

3.5. Let $\pi : X \to S$ be a conic bundle over a surface $S$ and let $\Delta \subset S$ be its discriminant curve. According to discussions in 3.3 the curve $\Delta$ is a reduced normal-crossing divisor (possibly $\Delta = \emptyset$). Moreover, a fibre $X_s$ over $s \in S$ is non-singular (respectively, is a pair of meeting lines, a double line) if $s \in S \setminus \Delta$ (respectively, $s \in \Delta \setminus \text{Sing}(\Delta)$, $s \in \text{Sing}(\Delta)$).

The following facts are easy and well known.

3.6. Lemma. Let $\pi : X \to S$ be a conic bundle over a (projective) surface $S$ and let $\Delta$ be the discriminant curve. Then

\begin{align*}
(3.6.1) \quad & \chi_{\text{top}}(X) = 2\chi_{\text{top}}(S) - 2p_a(\Delta) + 2, \\
(3.6.2) \quad & b_1(X) = b_1(S), \\
(3.6.3) \quad & b_3(X) = 2b_1(S) + 2b_2(X) - 2b_2(S) + 2p_a(\Delta) - 4
\end{align*}

(in the case $\Delta = \emptyset$ let $p_a(\emptyset) = 1$).

3.7. Lemma. Let $\pi : X \to S$ be a conic bundle over a surface and let $\Delta \subset S$ be its discriminant curve.

(i) If the variety $X$ is rationally connected, then the surface $S$ is rational. Conversely, if $\pi : X \to S$ is a conic bundle over a rational surface, then $X$ is rationally connected [102], [56].

(ii) If $\pi$ has a rational section, then $X$ is birationally equivalent to $S \times \mathbb{P}^1$.

(iii) Assume that the conic bundle is standard and the Brauer group of $S$ is trivial (for example, this holds if $S$ is rational). Then $\pi$ has a rational section if and only if $\Delta = \emptyset$.

3.8. Admissible double covers. Let $\tilde{\Delta}$ be a reduced connected curve with at worst nodal singularities and let $\tau : \tilde{\Delta} \to \Delta$ be an involution (an automorphism of order $2$). Let $\Delta := \tilde{\Delta}/\tau$ be the quotient and let $\tilde{\pi} : \Delta \to \Delta$ be the natural projection. It is easy to show that the singularities of $\Delta$ are also at worst nodes.
We also assume that the following condition is satisfied (the so-called Beauville condition):

\[(3.8.1) \quad \pi(\text{Sing}(\tilde{\Delta})) = \text{Sing}(\Delta), \quad \text{Sing}(\tilde{\Delta}) = \{x \in \tilde{\Delta} \mid \tau(x) = x\}.\]

Then the restriction of $\tilde{\pi}$ to each irreducible component of $\tilde{\Delta}$ does not split, and at a singular point of $\tilde{\Delta}$ the two branches are not switched by $\tau$. From (3.8.1) one can easily deduce the following:

\[(3.8.2) \quad \text{for every decomposition } \Delta = \Delta_1 + \Delta_2 \text{ such that } \Delta_i \geq 0, \quad \text{we have } #(\Delta_1 \cap \Delta_2) \equiv 0 \mod 2.\]

Note, however, that in our situation $\tilde{\pi}$ is not necessarily flat.

3.9. Now let $\pi : X \to S$ be a standard conic bundle over a projective surface $S$ and let $\Delta \subset S$ be its discriminant curve. Let $\tilde{\Delta}$ be the curve parameterizing components of fibres in the ruled surface $X_\Delta := \pi^{-1}(\Delta)$. The induced projection $\tilde{\pi} : \tilde{\Delta} \to \Delta$ is finite of degree 2 and satisfies the conditions of 3.8.

3.9.1. Corollary. Any connected component of the discriminant divisor of a standard conic bundle over a surface has arithmetic genus at least 1.

3.9.2. Recall that a theta-characteristic of a smooth curve $\Delta$ is the linear equivalence class of a divisor $D$ such that $2D \sim K_\Delta$. A theta-characteristic can be even or odd depending on the parity of the dimension of $H^0(\Delta, \mathcal{O}_\Delta(D))$ (see, for example, [133] or [52], Chap. 5). This definition can be naturally generalized to the case of reduced Gorenstein curves. A theta-characteristic of such a curve is a rank-1 torsion-free sheaf $\mathcal{F}$ such that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \omega_\Delta) \simeq \mathcal{F}$ [12], [139].

3.9.3. Example. Let $S = \mathbb{P}^2$, let the curve $\Delta$ be smooth, and let the degree of $\Delta$ be odd and $\geq 5$: $\deg \Delta = 2m + 3$. There is a one-to-one correspondence between étale double covers $\tilde{\pi} : \tilde{\Delta} \to \Delta$ and elements $\sigma$ of order 2 in the Jacobian $J(\Delta)$. The linear system $|mh|$, where $h$ is the class of a hyperplane section of $\Delta$, is a half-canonical linear system, that is, it is a theta-characteristic. Then $mh + \sigma$ is another theta-characteristic. In other words, the group of 2-torsion points $J_2(\Delta) \subset J(\Delta)$ acts on the set $\text{Th}(\Delta)$ of all theta-characteristics, making it a principal homogeneous space. The choice of a distinguished point $|mh| \in \text{Th}(\Delta)$ establishes an identification

$$\text{Th}(\Delta) \simeq J_2(\Delta).$$

We say that $mh + \sigma$ is the theta-characteristic corresponding to the cover $\tilde{\Delta} \to \Delta$. Thus, there is a one-to-one correspondence between étale double covers $\tilde{\pi} : \tilde{\Delta} \to \Delta$ and theta-characteristics of the curve $\Delta$.

It turns out that the double cover $\tilde{\Delta} \to \Delta$ is the most important invariant. In particular, this cover ‘almost determines’ our conic bundle.
3.10. Proposition. Let $S$ be a rational surface, $\Delta \subset S$ a reduced normal-crossing curve, and $\tilde{\pi}: \tilde{\Delta} \to \Delta$ a double cover satisfying the condition (3.8.1). Then there exists a standard conic bundle $\pi: X \to S$ with the given cover $\tilde{\pi}: \tilde{\Delta} \to \Delta$, and all such standard conic bundles are birationally equivalent over $S$.

Sketch of the proof. Consider a so-called Artin–Mumford exact sequence

$$0 \to \text{Br}(S) \to \text{Br}(k(S)) \xrightarrow{\alpha} \bigoplus \text{curves } C \subset S H^1_{\text{ét}}(C, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} \bigoplus \mu^{-1} \xrightarrow{\gamma} \mu^{-1} \to 0$$

(see [10]), where

$$\mu^{-1} = \bigcup_n \text{Hom}(\mu_n, \mathbb{Q}/\mathbb{Z})$$

and $\mu_n$ denotes the group of $n$th roots of unity. Then the double cover $\tilde{\pi}: \tilde{\Delta} \to \Delta$ determines an element

$$(3.10.1) \quad a \in \bigoplus_{C \subset \Delta} H^1_{\text{ét}}(C, \mathbb{Q}/\mathbb{Z})$$

of order 2 (a collection of local invariants).

From (3.8.1) it follows immediately that $\beta(a) = 0$, and hence there is an element $A \in \text{Br}(k(S))$ such that $\alpha(A) = a$.

Since $S$ is a rational surface, we have $\text{Br}(S) = 0$. Hence the element $A \in \text{Br}(k(S))$ has order 2. By virtue of the well-known Merkur’ev theorem [117], $A$ is a product of classes of quaternion algebras over the function field $k(S)$. Furthermore, since $k(S)$ is a $c_2$-field, by Albert’s theorem ([85], Theorem 2.10.9) a product of classes of quaternion algebras is also represented by a quaternion algebra. Thus, there exists a quaternion algebra $\mathcal{A}$ over $k(S)$ whose class in $\text{Br}(k(S))$ equals $A$. According to the classical theorem on central simple algebras, $\mathcal{A}$ is uniquely determined up to isomorphism. As in [10] (see also [174], Theorem 5.3), one proves that the maximal orders of $\mathcal{A}$ over $S$ are in bijection with standard conic bundles $\pi: X \to S$ with given local invariants (3.10.1) of the cover $\tilde{\pi}: \tilde{\Delta} \to \Delta$. All such conic bundles are birationally equivalent over $S$ since their generic fibres are isomorphic conics over $k(S)$ associated with the same quaternion algebra $\mathcal{A}$. $\square$

3.10.2. Example. Let $X \subset \mathbb{P}^2_{u_0, u_1, u_1} \times \mathbb{P}^2_{x_0, x_1, x_1}$ be given by the equation

$$f_0(u_0, u_1, u_1)x_0^2 + f_1(u_0, u_1, u_1)x_1^2 + f_2(u_0, u_1, u_1)x_2^2 = 0,$$

where the $f_i$ are sufficiently general homogeneous polynomials of degree $d \geq 1$. The projection $\pi: X \to \mathbb{P}^2$ to the first factor is a standard conic bundle whose discriminant curve is a reducible curve of degree $3d$ given by

$$f_0f_1f_2 = 0.$$

The corresponding quaternion algebra $\mathcal{A}$ over $k(\mathbb{P}^2)$ is generated by four vectors $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ with the standard relations

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{k} = -\mathbf{j} \cdot \mathbf{i}, \quad \mathbf{i}^2 = -\frac{f_0}{f_2}, \quad \mathbf{j}^2 = -\frac{f_1}{f_2}, \quad \mathbf{k}^2 = -\frac{f_0f_1}{f_2^2}. $$
Note, however, that the recipe given in Proposition 3.10 allows us to recover a conic bundle \( \pi: X \to S \) only up to birational equivalence over the base \( S \). There is no canonical way of reconstructing \( X/S \) from the double cover \( \tilde{\Delta} \to \Delta \).

3.11. Lemma (cf. [173]). Let \( \pi: X \to S \) be a contraction of projective varieties such that any fibre is one-dimensional, \( X \) has at worst terminal singularities, and \( -K_X \) is \( \pi \)-ample. Let \( \Delta \subset S \) be the discriminant divisor. Then the following numerical equivalence of cycles on \( S \) holds, where \( K_X^2 \) is regarded as a rational equivalence class of codimension-two cycles:

\[
-\pi_*K_X^2 \equiv 4K_S + \Delta.
\]

**Proof.** We prove (3.11.1) by induction on dimension. If \( \dim X = 2 \), then \( X \) is non-singular and (3.11.1) is an immediate consequence of the Noether formula. For \( \dim X \geq 3 \), let \( Z \subset S \) be an effective very ample divisor and let \( Y := \pi^{-1}(Z) \). Take \( Z \) to be general in the corresponding linear equivalence class. Then \( \pi_Y: Y \to Z \) will satisfy the same conditions as \( \pi: X \to S \). Let \( \Delta_Z = \Delta \cap Z \) be the corresponding discriminant divisor. It is sufficient to show that

\[
(\pi_*K_X^2 + 4K_S + \Delta) \cdot Z \equiv 0.
\]

Using the projection and adjunction formulae, we can write

\[
(\pi_*K_X^2 + 4K_S + \Delta) \cdot Z \equiv \pi_* (K_X^2 \cdot Y) + 4(K_Z - Z^2) + \Delta_Z
\]

\[
\equiv \pi_* K_Y^2 - 2\pi_* (K_Y \cdot Y^2) + 4K_Z - 4Z^2 + \Delta_Z
\]

\[
\equiv \pi_* K_Y^2 + 4K_Z + \Delta_Z.
\]

By the inductive hypothesis this proves (3.11.1). \( \square \)

3.12. Theorem. For any \( \Q \)-conic bundle \( \pi: X \to S \), there exist a birational contraction \( \alpha: S^\bullet \to S \) and a standard conic bundle \( \pi^\bullet: X \to S^\bullet \) that is fibrewise birationally equivalent to \( \pi: X \to S \). More precisely, there exists a commutative diagram

\[
\begin{array}{ccc}
X^\bullet & \xrightarrow{\psi} & X \\
\downarrow \pi^\bullet & & \downarrow \pi \\
S^\bullet & \xrightarrow{\alpha} & S
\end{array}
\]

where \( \psi \) is a birational map and \( \alpha \) is a birational morphism.

This theorem is a particular case of more general fact proved by Sarkisov: for any dominant rational map \( g: Y \to T \) of relative dimension 1 whose generic fibre is an irreducible rational curve there exists a standard conic bundle \( \tilde{\pi}^\bullet: X \to S^\bullet \) that is fibrewise birationally equivalent to \( g \). The proof can be found in [174], Theorem 1.13. The three-dimensional case was outlined earlier in [209] and [119]; see also [11] for the three-dimensional equivariant version.

4. Sarkisov category

In this section we describe the general structure of the Sarkisov programme. For details we refer to [43] (see also [115], Chap. 13, [78], [84], [186], and [60]).
4.1. First we recall the definition of a Sarkisov link ([43], Definition 3.4). Let \( \pi: X \to S \) and \( \pi_1: X_1 \to S_1 \) be Mori fibre spaces. A Sarkisov link \( \Phi: X_1 \dasharrow X \) between them is a transformation of one of four types:

\[
\begin{array}{cccc}
\text{I} & \text{II} & \text{III} & \text{IV} \\
\begin{array}{c}
Z - \chi \\
p \downarrow \\
X \\
\pi \\
S \\
\alpha \end{array} & \begin{array}{c}
Z - \chi \\
p \downarrow \\
X \\
\pi_1 \downarrow \\
X_1 \\
\pi \downarrow \\
S \\
\alpha \end{array} & \begin{array}{c}
X - \chi \\
qu \downarrow \\
X_1 \\
\pi_1 \downarrow \\
X_1 \\
\pi_1 \downarrow \\
S_1 \\
\alpha_1 \end{array} & \begin{array}{c}
X - \chi \\
\pi_1 \downarrow \\
X_1 \\
\pi \downarrow \\
S_1 \end{array}
\end{array}
\]

where \( p: Z \to X \) and \( q: Z_1 \to X_1 \) are divisorial Mori extremal contractions, \( \chi \) is a finite sequence of log-flips (in particular, \( \chi \) is an isomorphism in codimension one), and \( \alpha \) and \( \alpha_1 \) are extremal contractions in the log terminal category. A link of type III is a birational transformation that is inverse to a transformation corresponding to a type I link. For standard conic bundles, links of types I and III are exactly the elementary transformations described in Proposition 8.2. In dimension \( \leq 3 \), for a type IV link the contractions \( S \to T \) and \( S_1 \to T \) must be of fibre type (so the link switches the Mori fibre space structures). In higher dimensions the contractions \( S \to T \) and \( S_1 \to T \) can be small.

4.2. Theorem. Let \( \pi: X \to S \) and \( \pi^\#: X^\# \to S^\# \) be Mori fibre spaces and let

\[
\Phi: X \dasharrow X^\#
\]

be a (not necessarily fibrewise) birational map. Then \( \Phi \) is a composition of Sarkisov links.

The three-dimensional version of this programme was outlined in the preprints [175], [176], and [170], and a complete proof (including termination) was given in [43] (see also [84] and [186]). In higher dimensions the programme is established in a weaker form [60]. This approach uses a slightly different decomposition algorithm. We follow the ‘standard’ variant [43], which is more suitable for our applications. The process of decomposition whose existence is claimed in Theorem 4.2 is called the Sarkisov programme. Below we outline the general structure of this programme.

4.3. Let \( \mathcal{H}^\# \) be a very ample linear system on \( X^\# \) and let \( \mathcal{H} \) be its proper transform on \( X \). Clearly, we can write

\[
\mathcal{H}^\# \equiv -\mu^\# K_X + \pi^\#_* A^\#
\]

and

\[
\mathcal{H} \equiv -\mu K_X + \pi^* A,
\]

where \( \mu^\# \) and \( \mu \) are positive rational numbers and \( A^\# \) (respectively, \( A \)) is a \( \mathbb{Q} \)-divisor on \( S^\# \) (respectively, on \( S \)). We say that the pair \( (X, \mathcal{H}) \) (or the linear system \( \mathcal{H} \)) has maximal singularity if the pair \( (X, \frac{1}{\mu} \mathcal{H}) \) is not canonical.
The basic tool of the method of maximal singularities is the Noether–Fano inequality. Below is the version of it adapted for the Sarkisov programme.

**4.4. Theorem** (Noether–Fano inequalities; [43], Theorem 4.2, [80], and [165], Chap. 1, Def. 1.4). In the above notation the following hold.

(i) \( \mu > \mu^\sharp \), and equality implies that \( \Phi \) is fibrewise and induces an isomorphism \( \Phi : X_\eta \to X_\eta^\sharp \) of generic fibres.

(ii) If the pair \( (X, 1/\mu \mathcal{H}) \) is canonical and \( K_X + 1/\mu \mathcal{H} \) is nef, then \( \Phi \) is an isomorphism, and it also induces an isomorphism \( S \cong S^\sharp \). In particular, \( \mu = \mu^\sharp \).

**4.5. Case where \( \mathcal{H} \) has maximal singularity.** This step is often called untwisting maximal singularities. Since \( X \) has only terminal singularities, for some \( c \) with \( 0 < c < 1/\mu \) the pair \( (X, c\mathcal{H}) \) is canonical but not terminal. This number \( c \) is called the canonical threshold of the pair \( (X, \mathcal{H}) \) and denoted by \( c(X, \mathcal{H}) \). There exists a log-crepant extremal blowup

\[
p : (Z, c\mathcal{H}_Z) \to (X, c\mathcal{H}).
\]

Thus, we have \( \rho(Z/S) = 2 \) and

\[
K_Z + c\mathcal{H}_Z \equiv p^*(K_X + c\mathcal{H}).
\]

Moreover, \( p \) is a divisorial contraction and the variety \( Z \) has only terminal singularities and is \( \mathbb{Q} \)-factorial. Then we run the \( (K_Z + c\mathcal{H}_Z) \)-MMP over \( S \). Since \( \rho(Z/S) = 2 \), we obtain one of the links I or II. Then we replace \( X \) by \( X_1 \) and \( \mathcal{H} \) by its proper transform and continue the process.

**4.5.1.** Each step as above either decreases the number \( e(X, c\mathcal{H}) \) of crepant divisors or (if \( e(X, c\mathcal{H}) = 1 \)) increases the canonical threshold \( c(X, \mathcal{H}) \). The number \( e(X, c\mathcal{H}) \) is a positive integer, so it cannot decrease infinitely many times. Termination of an increasing sequence of thresholds \( c(X, \mathcal{H}) \) is a more delicate problem. At the moment it is not known that the set of all canonical thresholds satisfies the ascending chain condition (ACC), even in dimension 3. Particular cases of this problem were studied in [147], [190], and [188]. Termination of the Sarkisov programme in [43], Theorem 6.1, was proved by reduction to the log canonical case [3], [61]. The approach in [60] is different (see also [84] and [186]).

**4.6. Case where \( \mathcal{H} \) has no maximal singularities.** According to (4.3.2) we have

\[
\mathcal{H} + \mu K_X \equiv \pi^* A,
\]

where, by Theorem 4.4 (ii), the divisor \( A \) is not nef. If \( \rho(S) \leq 1 \), then the divisor \( -(\mathcal{H} + \mu K_X) \) is ample and we just run the \( (K_X + 1/\mu \mathcal{H}) \)-MMP and get a link of type IV. Assume that \( \rho(S) > 1 \). In particular, this implies that \( S \) is a surface. There is a natural identification of Mori cones

\[
\mathrm{NE}(S) = \pi_* \mathrm{NE}(X).
\]

By the cone theorem applied to \( X \), the cone

\[
\mathrm{NE}(S) \cap \{ z \mid A \cdot z < 0 \}
\]
is locally polyhedral. Hence, there exist extremal rays $R_S \subset \overline{\text{NE}}(S)$ and $R_X \subset \overline{\text{NE}}(X)$ such that

$$(\mathcal{H} + \mu K_X) \cdot R_X < 0, \quad A \cdot R_S < 0, \quad \text{and} \quad \pi_* R_X = R_S.$$ 

Since $\rho(S) > 1$, we have $R_S^2 \leq 0$, and we can show that the ray $R_S$ is contractible. Hence, there exists a contraction $S \to T$ of it with

$$\rho(S/T) = 1 \quad \text{and} \quad \dim T = 2 \text{ or } 1.$$ 

Then we run the $(K_X + \frac{1}{\mu} \mathcal{H})$-MMP over $T$. This produces a link of type $\text{III}$ with $S_1 = T$ or a link of type $\text{IV}$.

In all these cases, for links of type $\text{III}$ or $\text{IV}$ we get a new Mori fibre space $\pi_1: X_1 \to S_1$ such that

$$\mu_1 \leq \mu,$$

where the number $\mu_1$ is defined for the proper transform $\mathcal{H}_{X_1}$ of the linear system $\mathcal{H}$ in a way similar to (4.3.1) and (4.3.2). Since the numbers $\mu$ have bounded denominators (see [91]), this shows that the sequence of these links also terminates.

5. Surfaces over non-closed fields

In this section all the varieties (surfaces and curves) are assumed to be defined over a perfect field $k$ of arbitrary characteristic. If $\pi: X \to B$ is a two-dimensional conic bundle over a curve $B$, then, as in 3.1, we say that it is standard if $\rho(X/B) = 1$.

The discriminant locus in this case is a (reduced) zero-dimensional subscheme $\Delta \subset B$ (or empty). Every degenerate geometric fibre $X_{\bar{b}}, \bar{b} \in B \otimes \bar{k}$, is a union of two $(-1)$-curves meeting transversely at one point (see Fig. 1).

![Figure 1](image)

If the surface $X$ is geometrically rational, then by the Noether formula

$$K_X^2 + \deg \Delta = 8.$$ 

In particular, $K_X^2 \leq 8$ and the equality $K_X^2 = 8$ implies that $\pi$ is a smooth morphism. It is easy to show that for a standard conic bundle

$$K_X^2 \neq 7$$

(see, for example, [72]).
The Sarkisov programme works in the category of surfaces over \( k \). Moreover, in the two-dimensional case all the dashed arrows (log flips) in 4.1 must be isomorphisms. Thus, the Sarkisov links have the following simple form:

\[
\begin{array}{cccc}
\text{I} & \text{II} & \text{III} & \text{IV} \\
X & Z & X & X \\
\downarrow p & \downarrow q & \downarrow p & \downarrow q \\
B & X_1 & B & X_1 \\
\downarrow \pi & \downarrow \pi & \downarrow \pi & \downarrow \pi \\
\{\text{pt}\} & \{\text{pt}\} & \{\text{pt}\} & \{\text{pt}\}
\end{array}
\]

In the case \( \text{II} \) we distinguish two subcases:
- \( \text{II}_0 \) \( B \) is a point,
- \( \text{II}_1 \) \( B \) is a curve.

In Table 1 we give a short description of the links. In all cases the morphisms \( p \) and \( q \) are the blowups of closed points.

| type | \( X \) and \( \pi \) | \( X_1 \) and \( \pi_1 \) |
|------|-----------------|-----------------|
| I    | \( X \) is a del Pezzo surface with \( \rho(X) = 1 \) | \( X_1 \) is a del Pezzo surface with \( \rho(X_1) = 2 \) |
|      |                  | \( \pi_1 \) is a conic bundle |
|      | \( X \) is a del Pezzo surface with \( \rho(X) = 1 \) | \( X_1 \) is a del Pezzo surface with \( \rho(X_1) = 1 \) |
|      | \( Z \) is a del Pezzo surface with \( \rho(Z) = 2 \) |                 |
| II_0 | standard conic bundle | standard conic bundle |
| II_1 | standard conic bundle | standard conic bundle |
| III  | \( X \) is a del Pezzo surface with \( \rho(X) = 2 \) | \( X_1 \) is a del Pezzo surface with \( \rho(X_1) = 1 \) |
|      | \( \pi \) is a conic bundle |                 |
| IV   | standard conic bundle | standard conic bundle |
|      | \( X \) is a del Pezzo surface with \( \rho(X) = 2 \) |                 |

Iskovskikh [77] classified all links and relations between them in the category of surfaces over non-closed fields. Below we reproduce a part of this classification.

5.1. **Theorem** ([77], Theorem 2.6, [43], Theorem A.4, [151]). Let

\[
X/B \dashrightarrow X_1/B_1
\]

be a Sarkisov link in the category of rational surfaces. Assume that \( K_X^2 \leq 4 \). Then one of the following conditions holds:

(i) Type I: \( K_X^2 = 4 \), \( K_{X_1}^2 = 3 \), \( X_1 \) is a cubic surface containing a line \( l \) defined over \( k \), \( p \) is a contraction of \( l \), \( \pi_1 \) is the projection from \( l \);

(ii) Type II: \( B \) is a point, \( K_X^2 = 2 \), \( K_Z^2 = 1 \), \( p \) and \( q \) are the blowups of points of degree 1, the link is represented by the Bertini involution on \( Z \);
(iii) Type II: $B$ is a point, $K_X^2 = 3$, $K_Z^2 = 1$, $p$ and $q$ are the blowups of points of degree 2, the link is represented by the Bertini involution on $Z$;
(iv) Type II: $B$ is a point, $K_X^2 = 3$, $K_Z^2 = 2$, $p$ and $q$ are blowups of points of degree 1, the link is represented by the Geiser involution on $Z$;
(v) Type II: $B$ is a curve, the link consists of elementary transformations in non-degenerate geometric fibres which are conjugate over $k$;
(vi) Type III: inverse to (i);
(vii) Type IV: $K_X^2 = 1$, the link is represented by the (biregular) Bertini involution on $X$;
(viii) Type IV: $K_X^2 = 2$, the link is represented by the (biregular) Geiser involution on $X$;
(ix) Type IV: $K_X^2 = 4$, $X = X_4 \subset \mathbb{P}^4$ is an intersection of two quadrics containing two pencils of conics. Typically, this link is not represented by a biregular involution.

5.1.1. Corollary ([68]–[70], [77]). Let $\pi: X \to B$ and $\pi': X' \to B'$ be standard conic bundles over rational curves, and let $\Phi: X \dashrightarrow X'$ be a birational map.
1. If $K_X^2 \leq 0$, then the map $\Phi$ is fibrewise birational.
2. If $K_X^2 = 1, 2, \text{ or } 3$, then there exists a birational automorphism $\psi: X \dashrightarrow X$ such that the composition $\Phi \circ \psi$ is fibrewise birational.

5.1.2. Corollary. Let $\pi: X \to B$ be a standard conic bundle over a rational curve.
1. If $K_X^2 \leq 3$, then $\pi: X \to B$ is birationally rigid.
2. If $K_X^2 \leq 0$, then $\pi: X \to B$ is birationally superrigid.
3. If $K_X^2 = 4$, then the surface $X$ is not $k$-rational.

Note that the condition $K_X^2 \leq 0$ is equivalent to
$$|4K_B + \Delta| \neq \emptyset$$
(cf. Theorem 12.2).

It turns out that a surface with a standard conic bundle structure and positive self-intersection of the canonical class is ‘almost’ a del Pezzo surface, with a few exceptions.

5.2. Proposition ([101], Lemma 17; see also [69], [70], [72], and [151], §8). Let $\pi: X \to B$ be a standard conic bundle over a rational curve such that $0 < K_X^2 < 8$, and let $\Lambda$ be a geometric fibre. Then the divisor $-K_X$ is nef and big except in the following two cases:
(i) $K_X^2 = 1$ and there exists a geometrically irreducible rational curve $C \in |-K_X - \Lambda|$ such that $C^2 = -3$;
(ii) $K_X^2 = 2$ and there exists a curve $C \in |-K_X - 2\Lambda|$ which is a pair of conjugate disjoint sections $C_1$ and $C_2$ with $C_i^2 = -3$.
If furthermore $K_X^2 > 2$, then $-K_X$ is ample with one exception:
(iii) $K_X^2 = 4$ and there exists a curve $C \in |-K_X - 2\Lambda|$ which is a pair of conjugate disjoint sections $C_1$ and $C_2$ with $C_i^2 = -2$.

In the case (iii), by contracting the sections $C_1$ and $C_2$ we obtain a singular del Pezzo surface of degree 4. Its anticanonical image is a (quite special) intersection of two quadrics in $\mathbb{P}^4$. This surface is called the Iskovskikh surface [108], [42].
Proof. Put $d := K_X^2$. By the Riemann–Roch formula
\[ \dim |-K_X| \geq d > 0. \]
Assume that $K_X \cdot C > 0$ for some reduced irreducible curve $C$. Then $C$ is a fixed component of $|-K_X|$. Therefore, $C \sim -K_X - \beta \Lambda$ and hence $C \cdot \Lambda = 2$. In particular, this means that $C \otimes \overline{k}$ has at most two components. We have
\[ 0 > -K_X \cdot C = d - 2\beta, \quad 2\beta > d. \]
If the curve $C \otimes \overline{k}$ is connected, then
\[ -2 \leq 2p_a(C) - 2 = (K_X + C) \cdot C = -2\beta, \quad \beta = 1, \quad \text{and} \quad p_a(C) = 0. \]
In this case $d = 1$ and $C^2 = -3$. We obtain the case (i).
Assume that the curve $C \otimes \overline{k}$ has two connected components $C_1$ and $C_2$. In this case
\[ p_a(C_i) = 0, \quad C_i^2 = \frac{1}{2}d - 2\beta, \quad \text{and} \quad -K_X \cdot C_i = \frac{1}{2}d - \beta. \]
In particular, $d$ is even and $\beta \geq d/2$. Then
\[ -2 = 2p_a(C_1) - 2 = (K_X + C_1) \cdot C_1 = -\beta. \]
Thus, $\beta = 2 = d$. We obtain the case (ii).

Now consider the case where the divisor $-K_X$ is nef (but not ample). Let $C$ be an irreducible curve such that $K_X \cdot C = 0$. Let $C \sim \alpha(-K_X) - \beta \Lambda$ and let $C^{(1)}, \ldots, C^{(k)}$ be the connected components of $C \otimes \overline{k}$. Since every component $C^{(i)}$ is a tree of smooth rational curves, we have $(C^{(i)})^2 = -2$ and $C^2 = -2k$. Further,
\begin{equation}
(5.2.1) \quad 0 = -K_X \cdot C = d\alpha - 2\beta \quad \text{and} \quad -2k = C^2 = \alpha(d\alpha - 4\beta).
\end{equation}
This gives us
\[ d\alpha = 2\beta, \quad -2k = \alpha(d\alpha - 4\beta), \quad \text{and} \quad k = \alpha\beta. \]
Let $m := C^{(i)} \cdot \Lambda$. Then $mk = C \cdot \Lambda = 2\alpha$. Therefore,
\[ m\beta = 2 \quad \text{and} \quad d\alpha = \frac{4}{m}. \]
Since $d > 2$ by our assumptions, the only possibility is
\[ d = 4, \quad m = \alpha = 1, \quad \text{and} \quad \beta = 2 = k. \]
Since $m = 1$, each connected component $C^{(i)}$ is (geometrically) irreducible. We obtain the case (iii). □

Thus, in the cases $K_X^2 = 3, 5, 6$ there always exists on $X$ a $K_X$-negative extremal ray whose contraction $q: X \to X'$ is different from $\pi$ [122]. It is not hard to find all the possibilities for $q$.

5.3. Theorem [72], [77]. Let $\pi: X \to B$ be a standard conic bundle over a rational curve $B$. If $K_X^2 \in \{3, 5, 6\}$, then there exists a birational contraction $q: X \to X'$, where $X'$ is a del Pezzo surface with $\rho(X') = 1$. More precisely, one of the following holds.
In this case the linear system \( (394 \text{ Yu.G. Prokhorov} \))

\[
-\Delta \colon \mathbb{P}^1 \to \mathbb{P}^4
\]

where \( C \) is the class of a fibre.

(ii) If \( K_X^2 = 5 \), then \( X \to \mathbb{P}^2 \) is a contraction of four conjugate \((-1)\)-curves. In this case \( B \simeq \mathbb{P}^2 \) and \( q \) is a contraction of the linear system \([-2K_X - C]\), where \( C \) is the class of a fibre.

(iii) If \( K_X^2 = 6 \), then \( K_X^2 \), \( K_X \), \( Q \) is a contraction of a pair of conjugate \((-1)\)-curves. The contraction \( q \) is given by the linear system \([-2K_X + \pi^* K_B]\).

5.4. Corollary. Let \( \pi : X \to B \) be a standard conic bundle, where \( B \) is a curve. Let \( \Delta \subset B \) be the discriminant divisor. Then the following hold.

(i) \( \deg \Delta \neq 1 \).

(ii) If \( \deg \Delta = 3 \), then the surface \( X \) is \( k \)-rational.

(iii) If \( X \) has a \( k \)-point and either \( \deg \Delta = 2 \) or \( \Delta = \emptyset \), then the surface \( X \) is \( k \)-rational.

Putting the above results together, one obtains the following rationality criterion.

5.5. Theorem ([77], Theorem 2.6). Let \( X \) be a minimal rational surface over \( k \). Then \( X \) is \( k \)-rational if and only if \( K_X^2 \geq 5 \) and \( X \) has a \( k \)-point.

As a consequence of Corollary 5.4, we have the following.

5.6. Proposition. Let the ground field \( k \) be algebraically closed. Let \( \pi : X \to S \) be a conic bundle over a rational surface with discriminant curve \( \Delta \subset S \). Assume that there exists a base point free pencil \( \mathcal{L} \) of rational curves on \( S \) such that \( \mathcal{L} \cdot \Delta \leq 3 \). Then the variety \( X \) is rational.

Proof. Consider the generic member \( L_\eta \) of the pencil \( \mathcal{L} \) over the residue field \( k(\eta) \) of the generic point \( \eta \in \mathbb{P}^1 = \mathcal{L} \). Since \( k(\eta) \) is a \( c_1 \)-field, by Tsen’s theorem the curve \( L_\eta \) is isomorphic to \( \mathbb{P}^1_{k(\eta)} \). Consider the generic surface \( F_\eta := \pi^{-1}(L_\eta) \) over \( k(\eta) \). Then the morphism \( \pi_\eta : F_\eta \to L_\eta \) defines a standard conic bundle structure on the surface \( F_\eta \) over the non-closed field \( k(\eta) \). The discriminant divisor of the fibration \( \pi_\eta \) is \( \Delta_\eta := \Delta \cdot L_\eta \), where \( \deg \Delta_\eta = \mathcal{L} \cdot \Delta \leq 3 \). Corollary 5.4 implies that the surface \( F_\eta \) is rational over \( k(\eta) \). This implies the rationality of \( X \) over \( k \). \( \square \)

From Proposition 5.6 we immediately obtain the following.

5.6.1. Corollary. Let \( \pi : X \to \mathbb{P}^2 \) be a conic bundle with discriminant curve \( \Delta \) of degree \( \deg \Delta \leq 4 \). Then the variety \( X \) is rational.

6. The Artin–Mumford invariant

One natural birational invariant of an algebraic variety is the cohomological Brauer–Grothendieck group

\[
\text{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m),
\]

where \( \mathbb{G}_m \) is the sheaf of invertible regular functions in the étale topology on \( X \). This a periodic Abelian group which can be expressed in the classical situation of smooth projective varieties over \( \mathbb{C} \) in terms of complex-analytic invariants of
a variety and the torsion in $H^3(X,\mathbb{Z})$. In particular, if $\dim X = 3$ and $p_g(X) = 0$, then
\[ \text{Br}(X) \simeq H^3(X,\mathbb{Z})_{\text{tors}}. \]
The birational invariance of the last group can be established immediately using the theorem on resolution of singularities by means of blowups with smooth centres.

6.1. Proposition [10]. The torsion subgroup $H^3(X,\mathbb{Z})_{\text{tors}}$ is a birational invariant of a complete non-singular complex variety $X$ of any dimension. In particular, $H^3(X,\mathbb{Z})_{\text{tors}} = 0$ if $X$ is rational.

There is a version of this statement which is valid in characteristic $p > 0$. In this case one has to replace $H^3(X,\mathbb{Z})$ by the étale $l$-adic cohomology group $H^3(X,\mathbb{Z}_l)$, where $l$ is not equal to $p$ [10].

6.2. Remark. It is easy to see that $H^3(X,\mathbb{Z})_{\text{tors}}$ is also a stably birational invariant: if $X_1 \times \mathbb{P}^{n_1}$ is birationally equivalent to $X_2 \times \mathbb{P}^{n_2}$, where $X_1$ and $X_2$ are smooth projective varieties, then
\[ H^3(X_1,\mathbb{Z})_{\text{tors}} \simeq H^3(X_2,\mathbb{Z})_{\text{tors}}. \]
Indeed, by Küneth’s formula
\[ H^3(X_i \times \mathbb{P}^{n_i},\mathbb{Z})_{\text{tors}} \simeq H^3(X_i,\mathbb{Z})_{\text{tors}}. \]
The birational invariance of $H^3(X,\mathbb{Z})_{\text{tors}}$ was used in [10] to construct counterexamples to the Lüroth problem in any dimension $\geq 3$. The authors first constructed examples of unirational but not rational three-dimensional varieties over $\mathbb{C}$ among conic bundles over rational surfaces with $H^3(X,\mathbb{Z})_{\text{tors}} \neq 0$.

It turns out that the group $H^3(X,\mathbb{Z})_{\text{tors}}$ is effectively computable in the case of conic bundles.

6.3. Theorem ([10], [209], Theorem 2). Let $\pi: X \to S$ be a standard conic bundle over a rational surface $S$ with discriminant curve $\Delta$. Then
\[ (6.3.1) \quad H^3(X,\mathbb{Z})_{\text{tors}} \simeq \text{Br}(X) \simeq (\mathbb{Z}/2\mathbb{Z})^{c-1}, \]
where $c$ is the number of connected components of the curve $\Delta$.

This assertion states, in particular, that the number of connected components of the discriminant curve is a birational invariant.

6.3.2. Corollary. Let $\pi: X \to S$ and $\pi: X' \to S'$ be standard conic bundles over rational surfaces, and let $\Delta$ and $\Delta'$ be the corresponding discriminant curves. If $X$ and $X'$ are birationally equivalent, then $\Delta$ and $\Delta'$ have the same number of connected components.

The above result shows that to construct examples of non-rational three-dimensional varieties it is sufficient to specify the local invariants (3.10.1) and a disconnected curve $\Delta$ on some rational surface $S$. 
6.3.3. Corollary. Let $\pi : X \to S$ be a standard conic bundle over a rational surface $S$ with discriminant curve $\Delta$. If $\Delta$ is disconnected, then $X$ is non-rational. Moreover, $X$ is not stably rational.

The collection of local invariants (3.10.1) can be chosen in such a way that the resulting conic bundle $X/S$ is unirational. This construction together with results in [174], Theorem 5.10, can also be used to produce examples of unirational non-rational conic bundles with trivial intermediate Jacobian and trivial group $H^3(X, \mathbb{Z})_{\text{tors}}$ (see Theorem 9.6).

6.4. Example. Let $C_1, C_2 \subset \mathbb{P}^2$ be smooth non-rational curves meeting each other transversely. Let $\sigma : S \to \mathbb{P}^2$ be the blowup of the intersection points and let $\Delta_i \subset S$ be the proper transform of $C_i$. Take étale double covers $\tilde{\Delta}_i \to \Delta_i$. According to Proposition 3.10 there exists a standard conic bundle $\pi : X \to S$ with discriminant curve $\Delta = \Delta_1 \cup \Delta_2$. By (6.3.1) the variety $X$ is not stably rational.

For smooth three-dimensional Fano varieties the Brauer group is trivial. This follows from the classification (see [82], Tables §12.2–12.7). However, for Fano varieties with singularities (more precisely, for their desingularizations) it can be non-trivial, as, for example, in [10], where the simplest examples of non-rational unirational three-dimensional varieties are given by double covers of $\mathbb{P}^3$ branched in singular quartics, that is, by three-dimensional Fano varieties with double singularities. This example is constructed as follows.

6.5. Example. Let $C = \{f(x_1, x_2, x_3) = 0\} \subset \mathbb{P}^2$ be a non-degenerate conic, and let $E_i = \{g_i(x_1, x_2, x_3) = 0\} \subset \mathbb{P}^2$, $i = 1, 2$, be smooth cubic curves such that

$E_i \cap C = \{P^{(i)}_1, P^{(i)}_2, P^{(i)}_3\}$,

$E_1 \cap E_2 = \{Q_1, \ldots, Q_9\}$,

where $P^{(1)}_1, \ldots, P^{(2)}_3, Q_1, \ldots, Q_9$ are mutually distinct points, $E_i$ meets $C$ tangentially at $P^{(i)}_1, P^{(i)}_2, P^{(i)}_3$, and $E_1$ meets $E_2$ transversely at $Q_1, \ldots, Q_9$ (see Fig. 2).

Since curves of degree 3 cut out a complete linear system on $C$, we have

$$\sum_{i,j} P^{(i)}_j = B|_C$$

for some third cubic curve $B$, that is, $(E_1 + E_2)|_C = 2B|_C$ as a cycle on $C$. Let $h(x_1, x_2, x_3) = 0$ be an equation of the curve $B$. For a suitable choice of $h$ (up to scalar multiplication) we can write

$$g_1 g_2 = h^2 - 4fs,$$

where $s = s(x_1, x_2, x_3)$ is some polynomial of degree 4.

Let $S \subset \mathbb{P}^3_{x_0, \ldots, x_3}$ be the quartic surface given by the homogeneous polynomial

$$f(x_1, x_2, x_3)x_0^2 + h(x_1, x_2, x_3)x_0 + s(x_1, x_2, x_3) = 0.$$ 

It is easy to see that $S$ has exactly 10 nodes and no other singularities. Next, let $Y$ be the double cover of $\mathbb{P}^3$ (the ‘double solid’) branched over $S$. Clearly, $Y$ can
be given in the weighted projective space \( \mathbb{P}(1,1,1,1,2) \) by the quasi-homogeneous polynomial

\[
x_4^2 + f x_0^2 + h x_0 + s = 0, \quad \deg x_0 = \cdots = \deg x_3 = 1, \quad \deg x_4 = 2.
\]

Let \( \tilde{Y} \to Y \) be the blowup of the point \( O := (0 : 0 : 0 : 1 : 0) \) and let \( \pi_Y : Y \to \mathbb{P}^2 \) be the morphism induced by the projection from \( O \). Then \( \pi_Y \) is a singular conic bundle whose discriminant divisor \( \Delta_Y \subset \mathbb{P}^2 \) coincides with \( E_1 \cup E_2 \). A standard form of \( \pi_Y \) is a conic bundle over \( S \) which is the blowup of \( \mathbb{P}^2 \) at \( Q_1, \ldots, Q_9 \), and the corresponding discriminant curve is a disjoint union of proper transforms of \( E_1 \) and \( E_2 \).

This example was generalized in [67] and [161] (see also [17]) to certain other types of singular Fano threefolds.

The birational invariance of the cohomological Brauer group was also used to construct counterexamples to the E. Noether problem on rationality of fields of invariants for finite linear groups acting on vector spaces over an algebraically closed field ([172], [24], [177]). This problem can be formulated in geometric terms as follows. Let \( G \) be a finite group acting by linear transformations on a projective space \( \mathbb{P}^n \). The problem is whether the quotient \( X = \mathbb{P}^n/G \) is rational. There are a lot of surveys on this subject (see, for example, [39], [148], [88]).

### 7. Intermediate Jacobians and Prym varieties

Let \( X \) be a smooth projective three-dimensional variety over \( \mathbb{C} \) such that

\[
H^1(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = 0.
\]
For example, this holds for rationally connected varieties. Then we have the Hodge decomposition in the following form:

\[ H^3(X, \mathbb{C}) = H^{2,1}(X) \oplus H^{1,2}(X). \]

Therefore, integration of \((2,1)\)-forms over 3-dimensional cycles defines an embedding

\[ H^3(X, \mathbb{Z})/(\text{torsion}) \hookrightarrow H^{2,1}(X)\end{equation} \]

of the free part of the homology group \( H^3(X, \mathbb{Z}) \), as a full-rank lattice, into the complex vector space \( H^{2,1}(X)\). The alternating integral-valued intersection form on \( H^3(X, \mathbb{Z}) \) is unimodular by Poincaré duality, and the corresponding Hermitian form on \( H^{2,1}(X)\) is positive definite. Consequently, by the Riemann–Frobenius criterion, the complex torus

\[ J(X) := H^{2,1}(X)\) / \text{image}(H^3(X, \mathbb{Z})) \]

is a principally polarized Abelian variety with polarization divisor \( \Theta \).

7.1. Definition. In the above notation, the pair \((J(X), \Theta)\) is called the intermediate Jacobian of the variety \( X \).

Applications of intermediate Jacobians to birational geometry are based on the following two observations.

7.2. Proposition. (i) Every principally polarized Abelian variety \((A, \Theta)\) can be decomposed in a unique way into a direct product of principally polarized simple Abelian varieties:

\[(7.2.1) \quad (A, \Theta) = \bigoplus_{i=1}^{m} (A_i, \Theta_i). \]

(ii) (See [36] and also [197], Lecture 1, [200], Proposition 1.2.8.) Let \( X' \to X \) be the blowup with smooth irreducible centre \( C \subset X \) (that is, \( C \) is either a curve or a point). Then the following isomorphism of principally polarized Abelian varieties holds:

\[ J(X') \cong \begin{cases} J(X) & \text{if } C \text{ is a point,} \\ J(X) \oplus J(C) & \text{if } C \text{ is a curve,} \end{cases} \]

where \( J(C) \) is the Jacobian of the curve \( C \) with the principal polarization determined by the Poincaré divisor \( \Theta \).

This proposition and Hironaka’s theorem on resolution of indeterminacies of rational maps immediately imply the birational invariance of that part (the so-called Griffiths component) of the decomposition (7.2.1) which is the product of those factors which are not Jacobians of curves. In other words, if one denotes by \( J_G(X) \) the product of the components in (7.2.1) which are not Jacobians of curves, then \( J_G(X) \) is a birational invariant. In particular, we have the following.

7.3. Corollary. If the variety \( X \) is rational, then

\[ J_G(X) = 0. \]
Thus, to prove the non-rationality of some three-dimensional varieties one should be able to distinguish principally polarized Abelian varieties from Jacobians of curves.

It turns out that the intermediate Jacobians of three-dimensional varieties having the structure of a standard conic bundle can be described as so-called Prym varieties. These varieties were studied first by Prym, Wirtinger, Schottky, and Jung in connection with the Schottky problem of distinguishing the Jacobians of curves in the moduli spaces of principally polarized Abelian varieties (see [12] and [34]). Mumford was the first who drew attention to possible applications of Prym varieties to the birational geometry of three-dimensional varieties in his appendix to [36]. He also studied double covers $\tilde{\Delta} \to \Delta$ of smooth curves $\Delta$ from the point of view of distinguishing their Prym varieties $\text{Pr}(\tilde{\Delta}, \Delta)$ from Jacobians of curves [134]. Mumford’s results were extended to singular curves with normal crossings in the subsequent papers [13], [12], [179], and [180].

7.4. Definition. Let $(\tilde{\Delta}, \tau)$ be the pair consisting of a complete reduced (possibly reducible) curve $\tilde{\Delta}$ with at worst ordinary double points and an involution $\tau$ on $\tilde{\Delta}$ (that is, $\tau^2 = \text{id}$) acting non-trivially on every irreducible component of the curve $\tilde{\Delta}$. Denote the quotient $\tilde{\Delta}/\tau$ by $\Delta$ and the corresponding double cover by $\tilde{\pi}: \tilde{\Delta} \to \Delta$.

The involution $\tau$ induces an involution $\tau^*$ on the Jacobian $J(\tilde{\Delta})$ and $\pi$ induces the norm map

$$\text{Nm}: J(\tilde{\Delta}) \to J(\Delta), \quad \text{where} \quad \pi^* \circ \text{Nm} := \text{id} + \tau^* \quad \text{and} \quad \tilde{\pi}^* \circ \text{Nm} = 2.$$ 

The connected commutative algebraic group

$$\text{Pr}(\tilde{\Delta}, \Delta) := \ker(\text{Nm})^0 = (\text{id} - \tau^*)J(\Delta)$$

is called the (generalized) Prym variety of the pair $(\tilde{\Delta}, \tau)$, where the superscript $^0$ denotes the identity connected component. It is easy to see that

$$\dim \text{Pr}(\tilde{\Delta}, \Delta) = p_a(\Delta) - 1.$$ 

7.4.1. The notion of polarization can be easily extended to generalized Prym varieties (being algebraic groups, they are not necessarily Abelian varieties when $\Delta$ is singular). Under certain conditions, $\text{Pr}(\tilde{\Delta}, \Delta)$ is an Abelian variety whose polarization is divisible by 2 and after division becomes principal. This is true, for example, if the Beauville conditions (3.8.1) are satisfied (see [12] or [180], Theorem 3.5). For applications one has to impose a stronger condition:

$$(7.4.2) \quad \text{for every decomposition } \Delta = \Delta_1 + \Delta_2 \text{ with } \Delta_i > 0, \quad \text{we have } \#(\Delta_1 \cap \Delta_2) \geq 4.$$ 

The condition $(7.4.2)$ implies that $\Delta$ is a stable curve. In particular, if a curve $\Delta$ satisfies $(7.4.2)$, then the canonical linear system $|\omega_\Delta|$ is base point free and defines a canonical morphism $\Delta \to \mathbb{P}^{p_a(\Delta) - 1}$ which is finite onto its image (see [29]).

The following theorem was proved by Shokurov in [180]. For weaker versions see [134], [12], and [47].
7.5. Theorem. Let $(\tilde{\Delta}, \tau)$ be a pair consisting of a curve $\tilde{\Delta}$ of arithmetic genus $2g - 1$ and an involution $\tau$ on $\tilde{\Delta}$ satisfying the Beauville conditions (3.8.1) as well as the condition (7.4.2). Let $\Delta = \tilde{\Delta}/\tau$ be the quotient curve and let $\text{Pr}(\tilde{\Delta}, \Delta)$ be the corresponding Prym variety with polarization divisor $\Xi$, so that $p_a(\Delta) = g$ and $\dim \text{Pr}(\tilde{\Delta}, \Delta) = g - 1$. Then $(\text{Pr}(\tilde{\Delta}, \Delta), \Xi)$ is the Jacobian of a curve or a product of Jacobians of curves only in the following cases:

(i) $\Delta$ is a hyperelliptic curve;
(ii) $\Delta$ is a trigonal curve (this case is considered in [167]);
(iii) $\Delta$ is a quasi-trigonal curve;
(iv) $\Delta$ is a plane quintic curve and the corresponding double cover is given by an even theta-characteristic.

Here, as usual, a (not necessarily smooth) curve $\Delta$ is said to be hyperelliptic (respectively, trigonal) if there exists a finite morphism $\Delta \to \mathbb{P}^1$ of degree 2 (respectively, 3). A curve $\Delta$ is said to be quasi-trigonal if it is obtained from a hyperelliptic curve by identifying two smooth points.

7.5.1. Remark. The canonical map $\phi_\Delta: \Delta \to \mathbb{P}^{g-1}$ of the curves listed in Theorem 7.5 has the following properties.

(i) If the curve $\Delta$ is hyperelliptic, then the map $\phi_\Delta$ is a finite morphism of degree 2.
(ii) If $\Delta$ is a trigonal non-hyperelliptic curve, then the map $\phi_\Delta$ is an embedding and its image has a one-dimensional family of 3-secant lines whose intersections with $\phi_\Delta(\Delta)$ generate the linear series $g_3^1$.
(iii) If $\Delta$ is a quasi-trigonal non-hyperelliptic curve, then the map $\phi_\Delta$ is an embedding and its image lies on a two-dimensional cone with vertex at a singular point. Here the generators of the cone are 3-secant lines of the curve $\phi_\Delta(\Delta)$.
(iv) If $\Delta$ is a plane quintic, then $\phi_\Delta$ is an embedding and its image lies on the Veronese surface.

7.5.2. Remark. Assume that $\tilde{\Delta}_1 \cap \tilde{\Delta}_2 = \{P_1, P_2\}$ and let $\tilde{\Delta}_i', i = 1, 2$, be the curves obtained from $\tilde{\Delta}_i$ by identifying the points $P_1$ and $P_2$ (see Fig. 3). Then

$$\text{Pr}(\tilde{\Delta}, \Delta) = \text{Pr}(\tilde{\Delta}_1', \Delta_1') \times \text{Pr}(\tilde{\Delta}_2', \Delta_2'),$$

where $\text{Pr}(\tilde{\Delta}_i', \Delta_i')$ are Prym varieties for $\tilde{\Delta}_i'$ with the induced involution (see [180], Corollary 3.16). This is the reason for the condition (7.4.2) to be included in the hypothesis of the theorem.

Prym varieties are used in three-dimensional birational geometry mainly in the following situation. Consider a standard conic bundle $\pi: X \to S$ over a smooth projective rational surface. Let $\Delta \subset S$ be the discriminant curve of $\pi$ and let $\Delta$ be the curve parameterizing irreducible components of degenerate conics over $\Delta$. If $\Delta \neq \emptyset$, then $\Delta$ is a reduced curve with normal crossings and $\pi$ induces a double cover $\tilde{\pi}: \tilde{\Delta} \to \Delta$ satisfying the Beauville conditions (3.8.1).

For the following theorem, see [13], Chap. 3, [19], or [15].

7.6. Theorem. Let $\pi: X \to S$ be a standard conic bundle over a rational surface $S$ with discriminant curve $\Delta \subset S$ and let $\tilde{\Delta} \to \Delta$ be the corresponding double cover. Then the intermediate Jacobian $J(X)$ is isomorphic to the Prym variety $\text{Pr}(\tilde{\Delta}, \Delta)$ as a principally polarized Abelian variety.
Beauville considered the case $S = \mathbb{P}^2$ and obtained the following statement as a consequence of Theorems 7.5 and 7.6 ([13], Theorem 4.9): if $S = \mathbb{P}^2$ and $\deg \Delta \geq 6$, then the variety $X$ is non-rational, since in this case $\text{Pr}(\tilde{\Delta}, \Delta)$ is neither the Jacobian of a curve nor a product of Jacobians of curves.

The intermediate Jacobian has been used to prove non-rationality of many types of (even singular) Fano threefolds (for example, see [13], Chap. 5, [35], [40], [41], [33], [138], [53], and [7]).

### 8. Birational transformations

The following result, based on the construction in [198], was obtained by Panin in [135]. It is related to the exceptional case (ii) of Conjecture 1.3.

#### 8.1. Proposition [135].

Let $\pi: X \to \mathbb{P}^2$ be a standard conic bundle with discriminant curve $\Delta \subset \mathbb{P}^2$ of degree 5. Assume that the corresponding double cover $\tilde{\pi}: \tilde{\Delta} \to \Delta$ associated with $\pi$ is given by an even theta-characteristic. Then the variety $X$ is rational. Moreover, $\pi: X \to \mathbb{P}^2$ is fibrewise birational to the conic bundle obtained by blowing up $\mathbb{P}^3$ along a smooth curve of degree 7 and genus 5 (see Example 3.4.3).

**Sketch of the proof.** According to Proposition 3.10 such a conic bundle exists, and any two of them are birationally equivalent over $\mathbb{P}^2$. Thus, it is sufficient to prove the rationality of a particular one. Since $\omega_\Delta = \mathcal{O}_\Delta(2)$, the curve $\Delta$ is neither hyperelliptic nor (quasi-)trigonal, because its canonical model $\phi_{K_\Delta}(\Delta) \subset \mathbb{P}^5$ lies in the Veronese surface. Since $\tilde{\pi}$ satisfies the conditions (3.8.1), the Prym variety $\text{Pr}(\tilde{\Delta}, \Delta)$ is a principally polarized Abelian variety (see 7.4.1). Let $\Xi$ be the polarization divisor. According to [134], §7, and [180], Proposition 5.4, we have

$$\dim \text{Sing}(\Xi) \leq 1.$$  

Therefore, $\text{Pr}(\tilde{\Delta}, \Delta)$ is not a product of two or more principally polarized Abelian varieties. Hence, the Prym variety $\text{Pr}(\tilde{\Delta}, \Delta)$ is (according to [114] and [198]) isomorphic as a principally polarized Abelian variety to the Jacobian $J(\Gamma)$ of a smooth curve of genus 5. One can show that the curve $\Gamma$ cannot be hyperelliptic or trigonal (see [180], [114], and [198]). Thus, the canonical image $\Gamma \subset \mathbb{P}^4$ is a complete intersection of three quadrics $Q_1 \cap Q_2 \cap Q_3$. In other words, $\Gamma$ is the base locus of a net of quadrics $|2H - \Gamma|$, where $H$ is a hyperplane in $\mathbb{P}^4$.

#### 8.1.1. Definition [198], [199].

Let $\mathcal{C}$ be a net of quadrics in $\mathbb{P}^{2m}$ such that the base locus $\text{Bs}(\mathcal{C})$ is a smooth variety. Then a general quadric in the net $\mathcal{C}$ is
smooth, and degenerate quadrics in \( \mathcal{C} \) are parametrized by a curve \( \Delta \subset \mathcal{C} \cong \mathbb{P}^2 \). A general quadric \( Q_s \) for \( s \in \Delta \) has two families of \( m \)-dimensional linear subspaces. This defines a double cover \( \hat{\Delta} \to \Delta \) branched exactly over singular points of \( \Delta \). This double cover is called the invariant of the net \( \mathcal{C} \).

8.1.2. Theorem ([198], [13], Theorem 6.3). Let \( \mathcal{C} \) be a net of quadrics in \( \mathbb{P}^4 \) such that the base locus \( \text{Bs}(\mathcal{C}) \) is a smooth curve \( \Gamma \). Let \( \hat{\Delta} \to \Delta \) be the invariant of the net \( \mathcal{C} \). Then the Prym variety \( \text{Pr}(\hat{\Delta}, \Delta) \) is isomorphic to the Jacobian \( J(\Gamma) \) of the curve \( \Gamma \) as a principally polarized Abelian variety.

Now starting from this net of quadrics \( |2H - \Gamma| \), we construct a standard conic bundle \( \pi: X \to \mathbb{P}^2 \) with the given double cover \( \hat{\Delta} \to \Delta \) and we prove the rationality of \( X \).

8.1.3. Lemma. Let \( \Gamma \subset \mathbb{P}^4 \) be a smooth canonical non-trigonal curve of genus 5 and let \( \mathcal{C} \) be the net of quadrics passing through \( \Gamma \). Let \( \hat{\Delta} \to \Delta \) be the invariant of the net \( \mathcal{C} \). Fix a point \( P \in \Gamma \). Let \( \Gamma_0 \subset \mathbb{P}^3 \) be the projection of \( \Gamma \) from \( P \), and let \( X \to \mathbb{P}^3 \) be the blowup of \( \Gamma_0 \). Then the discriminant curve of the standard conic bundle \( \pi: X \to \mathbb{P}^2 \) constructed in Example 3.4.3 is isomorphic to \( \Delta \), and the corresponding double cover coincides with \( \hat{\Delta} \to \Delta \).

Sketch of the proof. Let \( T_{P, \mathbb{P}^4} \) be the tangent space to \( \mathbb{P}^4 \) at the point \( P \). We identify the target of the projection \( \mathbb{P}^4 \to \mathbb{P}^3 \) from \( P \) with the projectivization \( \mathbb{P}(T_{P, \mathbb{P}^4}) \). Consider the following family of conics in \( \mathbb{P}(T_{P, \mathbb{P}^4}) \). For each point \( s \in \mathcal{C} \cong \mathbb{P}^2 \) we consider the conic \( C_s \) which is the projectivized tangent cone \( \mathbb{P}(Q_s \cap T_{P, \mathbb{P}^4}) \) to the quadric \( Q_s \), \( s \in \mathcal{C} \), at \( P \). Since \( \Gamma \) is a smooth complete intersection of the members \( Q_s \), \( s \in \mathcal{C} \), none of the quadrics \( Q_s \) have singular points on \( \Gamma \) or, in particular, at \( P \). Therefore, we have one of the following possibilities:

\[
\begin{align*}
corank(Q_s) &= 0 & \iff & \text{\( C_s \) is non-singular,} \\
corank(Q_s) &= 1 & \iff & \text{\( C_s = \mathbb{P}^1 \vee \mathbb{P}^1 \),} \\
corank(Q_s) &= 2 & \iff & \text{\( C_s \) is a double line.}
\end{align*}
\]

Moreover, each \( C_s \) meets \( \Gamma_0 \) at \( 8 - 2 = 6 \) points.

Conversely, if \( C \subset T_{P, \mathbb{P}^4} \) is a (possibly degenerate) conic meeting \( \Gamma_0 \) at 6 points, then there is a quadric \( Q \in \mathcal{C} \) such that \( C \) is the projectivized tangent cone to \( Q \) at \( P \). As in Example 3.4.3, by blowing up \( \Gamma_0 \) we obtain a conic bundle \( \pi: X \to \mathbb{P}^2 \) whose fibres are the proper transforms of the conics \( C_s \). In particular, the degeneracies of \( \pi: X \to \mathbb{P}^2 \) correspond precisely to the given cover \( \hat{\pi}: \hat{\Delta} \to \Delta \) from which the net of quadrics was built. □

By Theorems 8.1.2 and 7.6 we have

\[ \text{Pr}(\hat{\Delta}, \Delta) \cong J(\Gamma) \cong J(X). \]

Since \( X \) is rational, this proves Proposition 8.1. □

8.2. Proposition ([173], Proposition 2.4). Let \( \pi: X \to S \) be a standard conic bundle. Let \( \alpha: S_1 \to S \) be the blowup of \( S \) with centre at a point \( s \in S \). Then there
exist a standard conic bundle \( \pi_1 : X_1 \to S_1 \) and a birational map \( \psi : X_1 \dashrightarrow X \) such that the diagram

\[
\begin{array}{ccc}
X & \xleftarrow{\psi} & X_1 \\
\downarrow{\pi} & & \downarrow{\pi_1} \\
S & \xleftarrow{\alpha} & S_1
\end{array}
\]

(8.2.1)

is commutative. More precisely, if \( \pi^{-1}(s) \) is a non-degenerate conic, then \( \psi \) is regular and is the blowup of \( \pi^{-1}(s) \). If \( \pi^{-1}(s) \) is a degenerate conic, then (8.2.1) can be completed to the following commutative diagram (cf. 4.1):

\[
\begin{array}{ccc}
Z & \xrightarrow{\chi} & X_1 \\
\downarrow{p} & & \downarrow{\pi_1} \\
X & \xleftarrow{\psi} & S_1 \\
\downarrow{\pi} & & \downarrow{\alpha} \\
S & &
\end{array}
\]

(8.2.2)

Here \( p \) is the blowup of a reduced irreducible component of \( \pi^{-1}(s) \) and \( \chi \) is a flop.

**Sketch of the proof.** Let \( \Delta \subset S \) be the discriminant curve and let \( l := \pi^{-1}(s)_\text{red} \) be the fibre over \( s \). There are three possibilities:

- **Case \( s \in S \setminus \Delta \).** Then we can take \( X_1 = X \times_S S_1 \). In other words, \( \psi : X_1 \to X \) is the blowup with centre \( l \). Since the curve \( l \) is non-singular, \( X_1 \) is a non-singular variety. Moreover, \( \psi \) is a morphism in this case.

- **Case \( s \in \Delta \setminus \text{Sing}(\Delta) \).** Then \( \pi^{-1}(s) = l = l' \lor l'' \). Let \( p : Z \to X \) be the blowup of \( l' \) and let \( \bar{l} \) be the proper transform of \( l'' \). For the normal bundle of \( \bar{l} \) in \( Z \) we have

\[
\mathcal{N}_{\bar{l}/Z} = \mathcal{O}_{\bar{p}^1}(1) \oplus \mathcal{O}_{\bar{p}^1}(1).
\]

Consequently, there exists an Atiyah–Kulikov flop \( Z \dashrightarrow X_1 \) with centre \( \bar{l} \). The anticanonical divisors of \( Z \) and \( X_1 \) are relatively nef over \( S \). According to the general theory [122] there exists a Mori extremal contraction \( \pi_1 : X_1 \to S_1 \) over \( S \) that can be completed to the diagram (8.2.2), where \( \pi_1 \) must be a standard conic bundle and \( \alpha \) must be the blowup of \( s \).

- **Case \( s \in \text{Sing}(\Delta) \).** Then \( \pi^{-1}(s) = 2l \) is a double line. Let \( p : Z \to X \) be the blowup of \( l \). As above, the anticanonical divisor \( -K_Z \) is nef over \( S \) and there exists a flop \( Z \dashrightarrow X_1 \) at the negative section of the exceptional divisor \( p^{-1}(l) \simeq \mathbb{F}_3 \). Thus, we obtain the diagram (8.2.2).

The transformations described in 8.2.3–8.2.5 are called *elementary transformations* of conic bundles. They are the simplest examples of Sarkisov links.

**8.3. Lemma [76].** Let \( S \) and \( S' \) be surfaces with rational singularities and let \( \alpha : S \to S' \) be a birational contraction. Let \( \Delta \subset S \) be a connected reduced curve and let \( \Delta' := \alpha(\Delta) \). Then \( p_\alpha(\Delta) \leq p_\alpha(\Delta') \).
8.4. Lemma ([74], Lemma 1). Let $\pi: X \to S$ and $\pi': X' \to S'$ be fibrewise birationally equivalent standard conic bundles over smooth rational surfaces and let $\Delta \subset S$ and $\Delta' \subset S'$ be the corresponding discriminant curves. Then

\begin{align}
\dim |mK_S + n\Delta| = \dim |mK_{S'} + n\Delta'| & \quad \forall m \geq n > 0, \\
p_a(\Delta) = p_a(\Delta') & , \\
b_3(X) = b_3(X').
\end{align}

Moreover, $\Delta$ is non-singular if and only if $\Delta'$ is.

Note that the assertions (8.4.1) and (8.4.2) fail in the category of $\mathbb{Q}$-conic bundles (see Example 11.4).

Proof. First we note that elementary birational transformations as in Proposition 8.2 do not change the values of $p_a(\Delta)$ and the linear system $|mK_S + n\Delta|$. Indeed, let $\alpha: S' \to S$ be the blowup of $s \in S$ and let $\pi': X' \to S'$ be a standard conic bundle obtained from $\pi$ by one of the elementary transformations 8.2.3–8.2.5. Then

$$\Delta' = \begin{cases} 
\alpha^*\Delta & \text{if } s \in S \setminus \Delta, \\
\alpha^*\Delta - E & \text{if } s \in \Delta,
\end{cases}$$

where $E = \alpha^{-1}(s)$ is the exceptional divisor. Since $K_{S'} = \alpha^*K_S + E$, we have

$$mK_{S'} + n\Delta' = \alpha^*(mK_X + n\Delta) + \begin{cases} 
ME & \text{if } s \in S \setminus \Delta, \\
(m-n)E & \text{if } s \in \Delta.
\end{cases}$$

This proves (8.4.1) in the case where $X/S \dashrightarrow X'/S'$ is an elementary transformation. The assertion (8.4.2) follows from Lemma 8.3 in this case.

In the general case, by applying transformations as in Proposition 8.2 we may replace $\pi$ and $\pi'$ by fibrewise birational conic bundles so that $S = S'$. In this case $\Delta = \Delta'$, because the discriminant curve depends only on the generic fibre (see Lemma 3.3.5). The last equality follows from (3.6.3). □

8.5. Proposition ([73], Remark 7, and [74], Lemma 4). Let $\pi: X \to S$ be a standard conic bundle with discriminant curve $\Delta$. Let $E \subset S$ be a $(-1)$-curve on $S$ such that one of the following three conditions is satisfied:

(i) $E \cdot \Delta = 0$, $E \not\subset \Delta$;
(ii) $E \cdot \Delta = 1$, $E \not\subset \Delta$;
(iii) $E \cdot \Delta = 1$, $E \subset \Delta$, and $E \cdot (\Delta - E) = 2$.

Let $\alpha: S \to S'$ be the contraction of $E$. Then there exists a standard conic bundle $\pi': X' \to S'$ with discriminant curve $\Delta' = \alpha(\Delta)$ that is fibrewise birationally equivalent to $\pi: X \to S$.

Proof. We show that $X'$ can be obtained from $X$ by means of explicit birational transformations. Let $F := \pi^{-1}(E)$. Since $\rho(X/S') = 2$, the relative Mori cone $\overline{NE}(X/S')$ is generated by two extremal rays. We need the following easy fact.

8.5.1. Lemma. Let $\pi: X \to S$ be a conic bundle and let $E \subset S$ be a complete smooth rational curve such that the inverse image $F := \pi^{-1}(E)$ is a non-singular surface. Let $C \subset F$ be a curve such that $\pi_C: C \to E$ is an isomorphism. Then

$$K_X \cdot C = -2 - (E)_S^2 - (C)_F^2, \quad \deg \mathcal{N}_{C/X} = (E)_S^2 + (C)_F^2,$$
where \((E)^2\) and \((C)^2\) are the self-intersection numbers of \(E\) and \(C\) on the surfaces \(S\) and \(F\), respectively.

Case (i). Then \(F\) is a geometrically ruled surface over \(E \simeq \mathbb{P}^1\). Let \(\Sigma \subset F\) be the exceptional section and let \(n := -(\Sigma)^2\). From (8.5.2) we get that \(K_X \cdot \Sigma = n - 1\) and the normal bundle \(N_{\Sigma/X}\) is isomorphic to \(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\). If \(n = 0\) on \(F\), then \(F \simeq \mathbb{P}^1 \times \mathbb{P}^1\) and \(\Sigma \cdot \tilde{F} = -1\) on \(X\), because \(E^2 = -1\) on \(S\). Therefore, the surface \(F \subset X\) can be contracted along the pencil \(\Sigma\) in the category of non-singular varieties, and the corresponding contraction is extremal and, in particular, projective. We obtain a standard conic bundle \(\pi' : X' \to S'\). If \(n \geq 1\), then by blowing \(\Sigma\) up and contracting the proper transform of \(F\) we obtain a standard conic bundle \(\pi_1 : X_1 \to S\) over \(S\), with geometrically ruled surface \(F_1 = \pi'^{-1}(E)\) and an exceptional section \(\Sigma_1 \subset F_1\) whose self-intersection number is equal to \(-(n - 1)\). The birational transformation \(X \dashrightarrow X_1\) is a Sarkisov link of type \(\Pi\). After \(n\) similar transformations, we arrive at the situation considered above. For details see [74], Lemma 4, or [146], Lemma 5.4.

Case (ii). Here \(F\) is a non-singular ruled surface with one blowup point. Let \(C \subset F\) be a section with minimal self-intersection number \(C^2 = -n\) on \(F\). It is clear that \(n \geq 1\) and that \(C\) intersects exactly one component of a unique degenerate fibre of the ruling \(F \to E\). We denote this component by \(\Gamma\). Again by (8.5.2) we have

\[
K_X \cdot C = n - 1 \quad \text{and} \quad N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).
\]

Suppose first that \(n = 1\). Since \(K_X \cdot C = 0\), the divisor \(-K_X\) is nef over \(S'\). Then we can make a flop \(X \dashrightarrow Z\) and then the proper transform of \(F\) becomes contractible to a curve, that is, there exists a Mori extremal contraction \(Z \to X'\) over \(S'\), and then \(\pi' : X' \to S'\) is a standard conic bundle. The transformation \(X \dashrightarrow X'\) is a type \(\Pi\) Sarkisov link inverse to the transformation described in 8.2.4.

If \(n \geq 2\), then we consider the blowup \(p : Z \to X\) of the curve \(C\) and complete it to a Sarkisov link of type \(\Pi\) (we use the notation of 4.1). Here \(\chi : Z \dashrightarrow Z_1\) is a flop in the proper transform of \(\Gamma\) and \(q : Z_1 \to X_1\) is the contraction of the proper transform of the surface \(F\) to a curve over \(S\). The exceptional divisor \(R := p^{-1}(C)\) over \(C\) is isomorphic to \(\mathbb{F}_{n-1}\) and its negative section is disjoint from the flopping curve. Thus, we obtain a standard conic bundle \(\pi_1 : X_1 \to S\). Moreover, the inverse image \(F_1 = \pi_1^{-1}(E)\) is obtained by blowing up \(\mathbb{F}_{n-1}\) at a point that does not lie on the negative section. In particular, \(F_1\) has a section \(C_1\) with \((C_1)^2 = -(n - 1)\). Therefore, after \(n - 1\) transformations of the type considered we arrive at the situation \(n = 1\).

Case (iii). Then the singular locus of the surface \(F\) is a curve \(\Sigma \subset F\) such that the restriction \(\pi_\Sigma : \Sigma \to \tilde{E}\) is an isomorphism. Let \(\nu : \tilde{F} \to F\) be the normalization and let \(\tilde{\Sigma} := \nu^{-1}(\Sigma)\). From the explicit equations in 3.3 one can see that the pair \((X, F)\) is log canonical, the surface \(F\) has generically normal crossings along \(\Sigma\), and \(\nu : \tilde{F} \to F\) induces a morphism \(\nu_\Sigma : \tilde{\Sigma} \to \Sigma\) of degree 2 branched at the two points \(\pi^{-1}(\Delta \setminus \tilde{E}) \cap \tilde{\Sigma}\). The surface \(\tilde{F}\) is a minimal rational ruled surface isomorphic to \(\mathbb{F}_e\) for some \(e \geq 0\) with ruling \(\tilde{F} \to \tilde{E}\), where \(\tilde{E}\) is a curve that parametrizes the irreducible components of the fibres of \(F \to E\) and \(\tilde{E} \to E\) is the double cover branched at the two points \(E \cap \Delta \setminus \tilde{E}\).
By the projection formula
\[ F \cdot \Sigma = \pi^* E \cdot \Sigma = E \cdot \pi_* \Sigma = E^2 = -1. \]
Consequently,
\[(8.5.3) \quad (K_X + F) \cdot \nu_* \tilde{\Sigma} = (K_X + F) \cdot 2\Sigma = 2K_X \cdot \Sigma - 2. \]
On the other hand, since \((X, F)\) is log canonical, by the adjunction formula ([182], 3.1) we have
\[ \nu^* (K_X + F)|_F = K_{\tilde{F}} + \tilde{\Sigma}. \]
Hence
\[(8.5.4) \quad (K_X + F) \cdot \nu_* \tilde{\Sigma} = (K_{\tilde{F}} + \tilde{\Sigma}) \cdot \tilde{\Sigma} = 2p_a(\tilde{\Sigma}) - 2 = -2. \]
Combining the above two relations \((8.5.3)\) and \((8.5.4)\), we obtain
\[(8.5.5) \quad K_X \cdot \Sigma = 0. \]
Let \(m := (\tilde{\Sigma})^2_{\tilde{F}}\). Let \(\Lambda\) be the ruling of \(\mathbb{F}_e = \tilde{F} \to \tilde{E}\). Since \(\pi : X \to S\) is a standard conic bundle, we have \(\nu^* K_X \cdot \Lambda = -1\). Therefore,
\[ \nu^* K_X = -\tilde{\Sigma} + a\Lambda \quad \text{for some} \quad a \in \mathbb{Z}. \]
The relation \((8.5.5)\) implies that \(a = m\), that is, \(\nu^* K_X = -\tilde{\Sigma} + m\Lambda\). Consequently,
\[ (\nu^* K_X)^2 = (-\tilde{\Sigma} + m\Lambda)^2 = m - 2m = -m. \]
On the other hand, by using \((3.11.1)\) we can compute that
\[ -m = (\nu^* K_X)^2 = \pi_* \nu_* \nu^* (K_X^2) = \pi_* (\nu^* E \cdot K_X^2) \]
\[ = E \cdot \pi_*(K_X^2) = E \cdot (-4K_S - \Delta) = 3. \]
Consequently, \(\tilde{F} \simeq \mathbb{F}_3\), and \(\tilde{\Sigma}\) is the negative section of \(\tilde{F} \to \tilde{E}\). Since any curve on \(\tilde{F}\) is linearly equivalent to a convex linear combination of \(\tilde{\Sigma}\) and \(\Lambda\), we conclude that the divisor \(-K_X\) is nef over \(S'\), and \(\Sigma\) is the only curve contracted by the linear system \(|-K_X|\) over \(S'\). Let \(X \dashrightarrow Z\) be the flop in \(\Sigma\). Then \(Z\) has a Mori extremal contraction \(Z \to X'\) over \(S'\), and the induced morphism \(\pi' : X' \to S'\) is a standard conic bundle. The transformation described above is a type \(\text{III}\) Sarkisov link inverse to the transformation described in 8.2.5.

The proof of Proposition 8.5 is complete. \(\square\)

8.6. Proposition [74]. Let \(\pi : X \to S\) be a standard conic bundle over a rational surface \(S\) and let \(\Delta \subset S\) be the discriminant curve. Assume that \(|2K_S + \Delta| = \emptyset\). Then \(2K_S + \Delta\) is not nef and there exist a standard conic bundle \(\pi^\sharp : X^\sharp \to S^\sharp\) with discriminant curve \(\Delta^\sharp\) and the following commutative diagram:

\[(8.6.1) \quad X \dashrightarrow X^\sharp \quad \quad \quad \pi \quad \quad \quad \quad \pi^\sharp \quad \quad \quad \quad \quad S \xrightarrow{\alpha} S^\sharp\]
where \( S^2 \simeq \mathbb{F}_n \) or \( \mathbb{P}^2 \), the morphism \( \alpha \) is birational, and \( X \dashrightarrow X^\sharp \) is a birational map. Moreover, \( \Delta^2 \cdot \Lambda^2 \leq 3 \) for a fibre \( \Lambda^2 \) of the surface \( \mathbb{F}_n \) (respectively, \( \deg \Delta^2 \leq 5 \)) if \( S^2 = \mathbb{F}_n \) (respectively, if \( S^2 = \mathbb{P}^2 \)). In particular, the discriminant curve \( \Delta \) is connected.

**Proof.** By the Riemann–Roch theorem

\[
h^0(S, K_S + \Delta) + h^0(S, -\Delta) \geq p_a(\Delta) \geq 1.
\]

Therefore, \( |K_S + \Delta| \neq \emptyset \). We apply Lemma 8.6.2 below with \( m = 2 \). Then the divisor \( 2K_S + \Delta \) is not nef. If \( C \) is a \((-1)\)-curve on \( S \) such that \( (2K_S + \Delta) \cdot C < 0 \), then \( \Delta \cdot C < 2 \). Thus, we can use Proposition 8.5 and obtain a new conic bundle \( \pi' : X' \to S' \) with \( |2K_{S'} + \Delta'| = \emptyset \) (see (8.4.1)). We can continue the process until we get a model with \( S^\sharp \simeq \mathbb{F}_n \) or \( \mathbb{P}^2 \). The connectedness of \( \Delta^2 \) is easy to check on \( S^2 = \mathbb{F}_n \) and \( \mathbb{P}^2 \). On the other hand, the number of connected components of the discriminant curve is preserved under our birational transformations. \( \square \)

**8.6.2. Lemma** [74]. Let \( S \) be a non-singular rational projective surface and let \( \Delta \) be a reduced effective divisor on \( S \) such that \( C \cdot (\Delta - C) \geq 2 \) for any smooth rational curve \( C \subset \Delta \). Let \( m \) and \( n \) be positive integers such that \( m > n \),

\[
|mK_S + n\Delta| = \emptyset \quad \text{and} \quad |(m - 1)K_S + n\Delta| \neq \emptyset.
\]

Then the divisor \( mK_S + n\Delta \) is not nef, and there exists an extremal ray \( R \subset \overline{\text{NE}}(S) \) such that

\[
K_S \cdot R < 0 \quad \text{and} \quad (mK_S + n\Delta) \cdot R < 0.
\]

Moreover, one of the following conditions holds:

(i) there exists a base point free pencil \( \mathcal{L} \) of rational curves on \( S \) such that

\[
2m - 2 \leq n \Delta \cdot \mathcal{L} < 2m;
\]

(ii) there exists a birational morphism \( \varphi : S \to \mathbb{P}^2 \) such that

\[
3m - 3 \leq n \deg \varphi(\Delta) < 3m.
\]

**Proof.** Put \( \Delta_{k,l} := kK_S + l\Delta \). We claim that the divisor \( \Delta_{m,n} \) is not nef. Indeed, otherwise by the Riemann–Roch formula

\[
h^0(S, \mathcal{O}_S(\Delta_{m,n})) + h^0(S, \mathcal{O}_S(-\Delta_{m-1,n})) \geq \frac{1}{2} \Delta_{m,n} \cdot \Delta_{m-1,n} + 1 > 0.
\]

By our assumption,

\[
H^0(S, \mathcal{O}_S(\Delta_{m,n})) = 0.
\]

Therefore, \( H^0(S, \mathcal{O}_S(-\Delta_{m-1,n})) \neq 0 \), and by our assumption we have \( \Delta_{m-1,n} = 0 \). In this case the divisor \( \Delta_{m,n} = K_S \) is not nef because \( S \) is a rational surface, a contradiction.

Thus, the divisor \( \Delta_{m,n} \) is not nef. We regard it as a linear function on the Mori cone \( \overline{\text{NE}}(S) \). Put

\[
\overline{\text{NE}^+}(S) = \{ z \in \overline{\text{NE}}(S) \mid z \cdot K_S \geq 0 \}.
\]
We claim that $\Delta_{m,n}$ is non-negative on $\overline{NE_+}(S)$. Indeed, assume that $C \cdot K_S := d > 0$ and $C \cdot \Delta_{m,n} < 0$ for some irreducible curve $C$. Then

$$nC \cdot \Delta = C \cdot \Delta_{m,n} - mC \cdot K_S < -dm.$$ 

Consequently, the curve $C$ is a component of $\Delta$ and $C^2 \leq C \cdot \Delta < -dm/n$. By the genus formula

$$2 p_a(C) - 2 = C^2 + C \cdot K_S < d - \frac{dm}{n} < 0.$$ 

Therefore, $p_a(C) = 0$. Then by our assumptions

$$-2 - d = -2 - C \cdot K_S = C^2 \leq C \cdot \Delta - 2 < -\frac{dm}{n} - 2.$$ 

This contradiction shows that the divisor $\Delta_{m,n}$ is non-negative on $\overline{NE_+}(S)$. Hence, $\Delta_{m,n}$ is negative on some $K_S$-negative extremal ray $R \subset \overline{NE}(S)$. According to [122] there are only three possibilities:

1) $R = \mathbb{R}_{>0}[E]$, where $E$ is a $(-1)$-curve;
2) $S \cong \mathbb{F}_e$ and $R = \mathbb{R}_{>0}[L]$, where $L$ is the ruling;
3) $S \cong \mathbb{P}^2$ and $R = \mathbb{R}_{>0}[L]$, where $L \subset \mathbb{P}^2$ is a line.

If 1) occurs, then let $\sigma : S \to S'$ be the contraction of $E$. Put

$$\Delta' := \sigma_* \Delta \quad \text{and} \quad \Delta'_{k,l} := kK_{S'} + l\Delta' = \sigma_* \Delta_{k,l}.$$ 

It is clear that the divisor $\Delta'$ is reduced, $\Delta' \geq 0$, and $|\Delta'_{m-1,n}| \neq \emptyset$. Moreover,

$$C' \cdot (\Delta' - C') \geq 2$$ 

for any irreducible rational component $C' \subset \Delta'$. Further,

$$\Delta_{m,n} = \sigma^* \Delta'_{m,n} - (\Delta_{m,n} \cdot E)E.$$ 

Since $\Delta_{m,n} \cdot E < 0$, it is easy to see that the divisor $\Delta'_{m,n}$ is not effective, and neither is $\Delta_{m,n}$. Consequently, all the conditions of our lemma are satisfied for $S'$ and $\Delta'$. Thus, it is sufficient to show that either (i) or (ii) holds on $S'$. Continuing the process of contraction of extremal $(-1)$-curves, we arrive at the situation of 2) or 3).

In the case 2) we have

$$-2 \leq \Delta_{m-1,n} \cdot L + K_S \cdot L = \Delta_{m,n} \cdot L = n\Delta \cdot L - 2m < 0.$$ 

Let $\mathcal{L} := |L|$. Then $2m - 2 \leq n\Delta \cdot \mathcal{L} < 2m$.

In the case 3) we have

$$-3 \leq \Delta_{m-1,n} \cdot L + K_S \cdot L = n\Delta \cdot L - 3m < 0.$$ 

Therefore, $3n - 3 \leq n \deg \varphi(\Delta) < 3m$. □

Using similar arguments, one can prove the following.
8.6.3. Lemma. Let $S$ be a non-singular rational projective surface and let $\Delta$ be a reduced effective divisor on $S$ such that $C \cdot (\Delta - C) \geq 2$ for any smooth rational curve $C \subset \Delta$. Let $\Delta_1$ be a connected component of $\Delta$. Then

\[ \dim |K_S + \Delta| \geq p_a(\Delta_1) - 1 \geq 0. \]  

Furthermore, if $(K_S + \Delta) \cdot E < 0$ for some irreducible curve $E$, then $E$ is a $(-1)$-curve disjoint from $\Delta$.

8.6.5. Corollary. Let $S$ be a smooth rational projective surface and let $\Delta$ be a reduced effective divisor on $S$ such that $C \cdot (\Delta - C) \geq 2$ for any smooth rational curve $C \subset \Delta$. Then there exists a sequence of contractions of $(-1)$-curves $f: S \to S'$ contained in $S \setminus \text{Supp}(\Delta)$ such that $K_{S'} + f_* \Delta$ is nef.

8.7. Recall that a $\mathbb{R}$-divisor is said to be pseudo-effective if its class is a limit of classes of effective $\mathbb{R}$-divisors. Now let $S$ be a projective variety and let $\Delta$ be an $\mathbb{R}$-divisor on $S$. Define the effective threshold (see [168] or [170]) as follows:

\[ \text{et}(S, \Delta) := \sup \{ t \in \mathbb{R} \mid tK_S + \Delta \text{ is pseudo-effective} \}. \]

Assume now that $S$ is a smooth projective rational surface and $\Delta$ is an effective reduced divisor on $S$ such that $K_S + \Delta$ is nef. To compute $\lambda := \text{et}(S, \Delta)$ in this situation we can run the $(K + \tau \Delta)$-MMP for the suitable sequence $\tau = \tau_i$ and reduce the problem to the case where $\lambda K_S + \Delta$ is nef (and then $S \simeq \mathbb{P}^2$ or $\mathbb{F}_n$).

Then it is easy to see that

\[ \lambda := \text{et}(S, \Delta) \in \frac{1}{2} \mathbb{Z} \cup \frac{1}{3} \mathbb{Z} \]

(see [168] for details). In particular, $\text{et}(S, \Delta)$ is a rational number. This consideration shows also $\lambda K_S + \Delta$ has a Zariski decomposition

\[ \lambda K_S + \Delta \equiv N + P \]

whose positive part $P$ is semi-ample and $P^2 = 0$. The negative part $N$ is contracted by the linear system $|nP|$, $n \gg 0$. Also, if $\text{et}(S, \Delta) = m/n$, then $|mK_S + n\Delta| \neq \emptyset$. Hence, the condition $|2K_S + \Delta| = \emptyset$ in Conjecture 1.2 is equivalent to the condition $\text{et}(S, \Delta) < 2$.

From Proposition 5.6 we obtain the following.

8.7.1. Corollary. Let $\pi: X \to S$ be a standard conic bundle over a rational surface $S$ and let $\Delta \subset S$ be the discriminant curve. Assume that $|2K_S + \Delta| = \emptyset$ and the variety $X$ is not rational. Then there exists a diagram of the form (8.6.1) with $S^2 \simeq \mathbb{P}^2$ and $\deg \Delta^2 = 5$. Moreover, the corresponding double cover $\Delta^2 \to \Delta^2$ is defined by an odd theta-characteristic (see Proposition 8.1).

8.7.2. Corollary [74]. Conjectures 1.2 and 1.3 are equivalent.

Proof. Assume that either the condition (i) or (ii) in Conjecture 1.3 holds. Then in the notation of Conjecture 1.3 we have

\[ |2K_{S'} + \Delta'| = \emptyset \text{ on } S' \text{ and } |2K_S + \Delta| = \emptyset \text{ on } S \]
by Lemma 8.4. On the other hand, the variety $X$ is rational (in the case (i) this follows from Proposition 5.6). Therefore, $J_G(X) = 0$.

Conversely, assume that $|2K_S + \Delta| = \emptyset$, and apply Proposition 8.6. If $S^\sharp \simeq \mathbb{P}^n$, then the rulings of $\mathbb{F}_n$ give us the desired pencil on $S^\sharp$ and $S$ (because $\Delta^\sharp = \alpha(\Delta)$). Consequently, in this situation we have the case 1.3, (i). Assume that $S^\sharp \simeq \mathbb{P}^2$ and $\deg \Delta^\sharp \leq 5$. If $\deg \Delta^\sharp \leq 4$, then the pencil of lines on $S^\sharp \simeq \mathbb{P}^2$ passing through a smooth point $P^\sharp \in \Delta^\sharp$ again gives us a pencil as in the case 1.3, (i) (after blowing up $P^\sharp$; see Proposition 8.2). Let $\deg \Delta^\sharp = 5$. Then

$$p_a(\Delta^\sharp) = p_a(\Delta) = 6.$$ 

In this case $J_G(X) = 0$ (see the condition $(\ast)$ in Conjecture 1.2), so by Theorems 7.6 and 7.5 the corresponding double cover $\tilde{\Delta}^\sharp \to \Delta^\sharp$ is given by an even theta-characteristic. Applying Panin’s construction 8.1, we obtain the case 1.3, (ii).

The following assertion is a very weak version of Conjecture 1.3.

8.8. Proposition [74]. Let $\pi: X \to S$ be a standard conic bundle over a rational surface $S$ with discriminant curve $\Delta \subset S$. Assume that there exists a birational map $\Phi: X \dashrightarrow X^\sharp$ to another Mori fibre space $\pi^\sharp: X^\sharp \to S^\sharp$ such that $\Phi$ is not fibrewise. Then one of the following conditions holds:

(i) there exists a base point free pencil $\mathcal{L}$ of curves of genus zero on $S$ such that $\Delta \cdot L \leq 7$;

(ii) there exists a birational contraction $\varphi: S \to \mathbb{P}^2$ such that $\varphi(\Delta) = \Delta' \subset \mathbb{P}^2$ is a curve (not necessarily with normal crossings) of degree $\deg \Delta' \leq 11$.

Proof. It follows from Theorem 12.2 that $|4K_S + \Delta| = \emptyset$. In the case where $|2K_S + \Delta| = \emptyset$ the assertion follows from Proposition 8.6. Thus, we can take $m \in \{3, 4\}$, so that $|mK_S + \Delta| = \emptyset$ and $|(m-1)K_S + \Delta| \neq \emptyset$. Then by Lemma 8.6.2 we obtain either (i) or (ii). \qed

8.9. Proposition ([193], [16], Proposition 4.2b). Let $\pi: X \to \mathbb{P}^2$ be a standard conic bundle with smooth discriminant curve $\Delta$ of degree 5 such that the cover $\tilde{\Delta} \to \Delta$ is defined by an odd theta-characteristic. Then $\pi: X \to \mathbb{P}^2$ is fibrewise birationally equivalent to a standard conic bundle $\pi': X' \to \mathbb{P}^2$ obtained from the projection of a smooth cubic hypersurface $Y_3 \subset \mathbb{P}^4$ from a line (Example 3.4.2).

Note that the cubic $Y_3 \subset \mathbb{P}^4$ can be naturally reconstructed from the cover $\tilde{\Delta} \to \Delta$: there is an isomorphism of principally polarized Abelian varieties $\text{Pr}(\tilde{\Delta}, \Delta) \simeq J(X)$ and the theta-divisor $\Theta$ of the variety $J(X)$ has a unique singular point whose tangent cone is isomorphic to the affine cone over $Y_3 \subset \mathbb{P}^4$ ([196], [200], Proposition 2.1.5, [14]).

9. Conic bundles over minimal surfaces

In the case when the base of a conic bundle is a minimal rational surface, the criterion 1.2 was proved by Shokurov [180]. Below we reproduce his arguments.

9.1. Theorem. Let $\pi: X \to \mathbb{P}^2$ be a standard conic bundle and let $\Delta \subset \mathbb{P}^2$ be the discriminant curve. Then the following conditions are equivalent:
(i) the variety $X$ is rational;
(ii) either $\deg \Delta \leq 4$ or $\deg \Delta = 5$ and the corresponding double cover $\tilde{\Delta} \to \Delta$ is defined by an even theta-characteristic;
(iii) the intermediate Jacobian $J(X)$ is isomorphic as a principally polarized Abelian variety to a product of Jacobians of smooth curves, that is, its Griffiths component $J_G(X)$ is trivial.

In the case $\deg \Delta \geq 6$ this theorem was proved by Beauville ([13], Theorem 4.9). In the case $\deg \Delta \leq 4$ the variety $X$ is rational by Corollary 5.6.1. The case $\deg \Delta = 5$ follows from results in [197], §3, [198], [114], and [135].

9.2. Theorem [180]. Let $\pi: X \to S$ be a standard conic bundle over a minimal rational ruled surface $S$ (that is, $S \simeq \mathbb{F}_n$ with $n \neq 1$), and let $\Delta \subset S$ be the discriminant curve. Denote the ruling of $\mathbb{F}_n \to \mathbb{P}^1$ by $\Lambda$. Then the following conditions are equivalent:

(i) the variety $X$ is rational;
(ii) $\left|2K_S + \Delta\right| = \emptyset$;
(iii) $\Delta \cdot \Lambda \leq 3$ (for a suitable choice of the ruling $\mathbb{F}_0 \to \mathbb{P}^1$ if $n = 0$);
(iv) $J_G(X) = 0$.

9.2.1. Remark. In the case $S = \mathbb{F}_1$ one can show that the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) hold whenever $\Delta \cdot \Sigma > 1$. If $\Delta \cdot \Sigma \leq 1$, then we can apply Proposition 8.5 to reduce the problem to $\mathbb{P}^2$. Thus, for $S = \mathbb{P}^2$ and $\mathbb{F}_n$ with arbitrary $n$ the equivalence (i) $\Leftrightarrow$ (iv) holds.

9.3. Corollary. Conjectures 1.2 and 1.3 are true for $S = \mathbb{P}^2$ and $\mathbb{F}_n$, $n \geq 0$.

Proof of Theorem 9.2. The implication (i) $\Rightarrow$ (iv) is an immediate consequence of Corollary 7.3, and (iii) $\Rightarrow$ (i) is a consequence of Proposition 5.6. The implication (iii) $\Rightarrow$ (ii) is obvious.

We prove (ii) $\Rightarrow$ (iii). Denote a $(-n)$-section of the surface $S = \mathbb{F}_n$ by $\Sigma$. Let

$$\Delta \sim a\Sigma + b\Lambda.$$ 

Assume that $\left|2K_S + \Delta\right| = \emptyset$ and $a \geq 4$. If $\Sigma \nsubseteq \Delta$, then clearly $b \geq a$ (see, for example, [63], Chap. V, Theorem 2.17). If $\Sigma \subset \Delta$, then

$$\left(\Delta - \Sigma\right) \cdot \Sigma = b - an + n \geq 2$$

according to (3.8.2). Hence, in both cases

$$b \geq \min\{an, n(a - 1) + 2\}.$$ 

Further,

$$2K_S + \Delta \sim (a - 4)\Sigma + (b - 2n - 4)\Lambda,$$

where $b - 2n - 4 < (a - 4)n$, because $\left|2K_S + \Delta\right| = \emptyset$. This gives us that $n = 0$ and $b < 4$. Replacing the ruling of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ by another one, we get (iii).

It remains to prove the implication (iv) $\Rightarrow$ (iii). Assume to the contrary that $\Delta \cdot \Lambda \geq 4$.

9.4. Lemma. Under the above assumptions one of the following holds:
(i) the linear system $|K_S + \Delta|$ is very ample;  
(ii) the linear system $|K_S + \Delta|$ is base point free, but not very ample, the corresponding morphism is birational and contracts $\Sigma$, and  

$$ (\Delta - \Sigma) \cdot \Sigma = 2. $$

Proof. As above, we write  

$$ \Delta \sim a\Sigma + b\Lambda, \quad K_S + \Delta \sim (a - 2)\Sigma + (b - 2 - n)\Lambda. $$

By our assumption $a \geq 4$. If $n = 0$, then by symmetry we may assume that $b \geq 4$. The linear system $|K_S + \Delta|$ is base point free if and only if

$$(9.4.1) \quad b \geq an + 2 - n.$$  

This relation is obvious if $n = 0$. Thus, we assume that $n \geq 2$. If $b \geq an$, then $(9.4.1)$ holds. Consider the case $b < an$. Then $\Sigma$ is a component of $\Delta$. Put $\Delta' := \Delta - \Sigma$. Then  

$$ 2 \leq \Delta' \cdot \Sigma = b - an + n, $$

which is equivalent to $(9.4.1)$. Moreover, if the linear system $|K_S + \Delta|$ is not very ample, then equality holds in $(9.4.1)$ and we get (ii). \[ \square \]

The following assertion can be proved like Lemma 9.4.

9.5. Lemma. If the curve $\Delta$ does not satisfy the condition $(7.4.2)$, that is, there exists a decomposition  

$$ \Delta = \Delta_1 + \Delta_2 \quad \text{with} \quad \Delta_1 \cdot \Delta_2 = 2, $$

then up to a permutation of the $\Delta_i$ it can be assumed that  

$$ \Delta_2 = \Sigma \quad \text{and} \quad (K_S + \Delta) \cdot \Delta_2 = 0. $$

In particular, $n > 0$. Thus, $\Delta$ satisfies $(7.4.2)$ if and only if the linear system $|K_S + \Delta|$ is very ample.

Now we are in position to finish the proof of the implication $(iv) \Rightarrow (iii)$ in Theorem 9.2. Consider the morphism $\phi: S \to \mathbb{P}^N$ given by the linear system $|\omega_S(\Delta)|$ (see Lemma 9.4). From the exact sequence  

$$ 0 \longrightarrow \omega_S \longrightarrow \omega_S(\Delta) \longrightarrow \omega_\Delta \longrightarrow 0 $$

and the vanishing  

$$ H^0(S, \omega_S) = H^1(S, \omega_S) = 0 $$

it follows that the linear system $|\omega_S(\Delta)|$ on $S$ restricts isomorphically to the canonical linear system $|\omega_\Delta|$ on $\Delta$. Therefore, $\phi$ induces a canonical morphism on $\Delta$,  

$$ \phi_\Delta: \Delta \to \mathbb{P}^{\text{p}^N(\Delta)-1}. $$

According to Lemma 9.4 the morphism $\phi$ is birational. Consequently, the curve $\Delta$ is not hyperelliptic.
Assume that the divisor $K_S + \Delta$ is very ample. Then the morphism $\phi_\Delta$ is an embedding and $\Delta$ satisfies (7.4.2). According to Theorem 7.5 the curve $\Delta$ is either trigonal, quasi-trigonal, or isomorphic to a plane quintic. The surface $\phi(S)$ is an intersection of quadrics. Therefore, it contains all the 3-secants of the curve $\phi(\Delta)$. If $\Delta$ is trigonal or quasi-trigonal, then this implies that $\phi(S)$ is swept out by 3-secants (see Remark 7.5.1) which must be the images of the fibres $\Lambda$. Hence, in this case $\Delta \cdot \Lambda = 3$, which contradicts our assumption. If $\Delta$ is a plane quintic, then the intersection of quadrics passing through $\phi(\Delta)$ is the Veronese surface. But this surface cannot be an image of $\mathbb{F}_n$ for $n \neq 1$. Again we get a contradiction.

Finally, assume that the divisor $K_S + \Delta$ is not ample. By Lemma 9.5 we have

$$\Delta = \Sigma + \Delta_1, \quad \Sigma \cdot \Delta_1 = 2, \quad \text{and} \quad (K_S + \Delta) \cdot \Sigma = 0.$$ 

Moreover, the surface $\phi(S)$ is isomorphic to a cone over a rational normal curve. According to Remark 7.5.2

$$\text{Pr}(\tilde{\Delta}, \Delta) \simeq \text{Pr}(\tilde{\Delta}', \Delta'), \quad \text{where} \quad \Delta' = \phi(\Delta_1).$$

It is easy to see that the embedding $\Delta' = \phi(\Delta_1) \subset \mathbb{P}_{p_a}(\Delta)^{-1}$ is canonical. Now we can apply Theorem 7.5 and Remark 7.5.1 and obtain a contradiction as above. Theorem 9.2 is proved. □

In view of Theorems 9.1 and 9.2 one could expect that the rationality of a three-dimensional variety is equivalent to the triviality $J_G(X) = 0$ of the Griffiths component. However, this is wrong in general, as the following result of Sarkisov shows.

9.6. Theorem ([174], Theorem 5.10). There exists a smooth algebraic rationally connected non-rational threefold $X$ whose three-dimensional integral cohomology group $H^3(X, \mathbb{Z})$ is trivial.

Proof. Let $\Delta_0 \subset \mathbb{P}^2$ be an irreducible curve with at most ordinary double singular points whose normalization $\Delta$ has genus 1. Assume that $\deg \Delta_0 > 12$. Let $\sigma: (S, \Delta) \to (\mathbb{P}^2, \Delta_0)$ be the minimal embedded resolution of singularities. Finally, let $\tilde{\pi}: \tilde{\Delta} \to \Delta$ be a non-trivial étale double cover. Then it follows from Proposition 3.10 that there exists a standard conic bundle $\pi: X \to S$ with discriminant curve $\Delta$ such that the corresponding double cover is isomorphic to $\tilde{\pi}: \tilde{\Delta} \to \Delta$. It is easy to see that the curve $\Delta$ satisfies the condition $|4K_S + \Delta| \neq \emptyset$, and therefore the variety $X$ is non-rational according to Theorem 12.2 below. On the other hand, by (3.6.3) we have $H^3(X, \mathbb{C}) = 0$, and since the discriminant curve of $\pi: X \to S$ is connected, it follows that $H^3(X, \mathbb{Z})$ is torsion-free by (6.3.1). □

10. $\mathbb{Q}$-conic bundles

To study the local structure of $\mathbb{Q}$-conic bundles one needs the corresponding concept that works in the analytic category.

10.1. Definition. Let $(X, C)$ be an analytic germ of a threefold with terminal singularities along a reduced complete curve. We say that $(X, C)$ is a $\mathbb{Q}$-conic bundle germ if there exists a contraction

$$\pi: (X, C) \to (S, o)$$
to a normal surface germ \((S, o)\) such that \(C = \pi^{-1}(o)_{\text{red}}\) and the divisor \(-K_X\) is \(\pi\)-ample.

Note that in this definition we assume neither that \(X\) is (analytically) \(\mathbb{Q}\)-factorial nor that \(\rho_{\text{an}}(X/S) = 1\). This is because these properties are not stable under passage from the algebraic to the analytic category (cf. [90], §1). \(\mathbb{Q}\)-conic bundle germs were studied in the series of papers [142]–[145], [125]–[131]. These works essentially use the ideas and techniques introduced by Mori [123] and developed in [103]. If \(X\) has only Gorenstein (terminal) singularities along \(C\), then, as in the standard conic bundle case, the base surface \(S\) is smooth at \(o\) and \(\pi\) is a conic bundle, that is, \(X\) admits a local embedding into \(\mathbb{P}^2 \times S\) so that fibres of \(\pi\) are conics [46]. In this case, the singularities of \(X\) and the discriminant curve are partially described in [150], Proposition 5.2. From now on we assume that \(X\) is not Gorenstein.

Recall that the index of a \(\mathbb{Q}\)-Gorenstein singularity \((X, P)\) (or a \(\mathbb{Q}\)-Gorenstein variety \(X\)) is the smallest positive integer \(m\) such that \(mK_X\) is a Cartier divisor. For the classification of three-dimensional terminal singularities and the corresponding notation we refer the reader to [169], §6, and [104], Theorem 5.43.

If the total space \(X\) has only singularities of index 2 and the base is non-singular, then the description of \(\mathbb{Q}\)-conic bundle germs is very explicit.

10.2. Theorem [125], [142]. Let \(\pi: (X, C) \to (S, o)\) be a \(\mathbb{Q}\)-conic bundle germ. Assume that \(X\) is not Gorenstein, the base \((S, o)\) is non-singular, and the singularities of \(X\) have indices \(\leq 2\). Fix an isomorphism \((\mathbb{Z}, o) \simeq (\mathbb{C}^2, 0)\). Then there is an embedding

\[
X \hookrightarrow \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2
\]

such that \(X\) is given by two equations,

\[
q_1(y_1, y_2, y_3) = \psi_1(y_1, \ldots, y_4; u, v),
\]

\[
q_2(y_1, y_2, y_3) = \psi_2(y_1, \ldots, y_4; u, v),
\]

where the \(\psi_i\) and \(q_i\) are quasi-homogeneous polynomials in \(y_1, \ldots, y_4\) which are quadratic with respect to the weight \(\text{wt}(y_1, \ldots, y_4) = (1, 1, 1, 2)\) and

\[
\psi_i(y_1, \ldots, y_4; 0, 0) = 0.
\]

The only non-Gorenstein point of \(X\) coincides with \((0, 0, 0, 1; 0, 0)\).

In the case where the base of a \(\mathbb{Q}\)-conic bundle is singular, we also have a complete classification.

10.3. Theorem [125], [126]. Let \(\pi: (X, C) \to (S, o)\) be a \(\mathbb{Q}\)-conic bundle germ. Assume that the base \((S, o)\) is singular. Then one of the following holds.

10.3.1. The germ \((X, C)\) is biholomorphic to the quotient of \(\mathbb{P}^1_x \times \mathbb{C}^2_{u,v}\) by the \(\mu_m\)-action

\[
(x; u, v) \mapsto (\varepsilon x; \varepsilon^a u, \varepsilon^{-a} v),
\]
where $\varepsilon$ is a primitive $m$th root of unity and
\[ \gcd(m, a) = 1. \]
The singular locus of $X$ consists of two cyclic quotient singularities of types
\[ \frac{1}{m}(1, a, -a) \quad \text{and} \quad \frac{1}{m}(-1, a, -a). \]
The base surface $\mathbb{C}^2/\mu_m$ has a singularity of type $A_{m-1}$.

10.3.2. The germ $(X, C)$ is biholomorphic to the quotient of the non-singular $\mathbb{Q}$-conic bundle
\[ X' = \{ y_1^2 + uy_2^2 + vy_3^2 = 0 \} \subset \mathbb{P}_y^2 y_1, y_2, y_3 \times \mathbb{C}^2_{u, v} \longrightarrow \mathbb{C}^2_{u, v} \]
by the $\mu_m$-action
\[ (y_1, y_2, y_3, u, v) \longmapsto (\varepsilon^a y_1, \varepsilon^{-1} y_2, y_3, \varepsilon u, \varepsilon^{-1} v). \]
Here $m = 2a + 1$ is odd and $\varepsilon$ is a primitive $m$th root of unity. The singular locus of $X$ consists of two cyclic quotient singularities of types
\[ \frac{1}{m}(a, -1, 1) \quad \text{and} \quad \frac{1}{m}(a + 1, 1, -1). \]
The base surface $\mathbb{C}^2/\mu_m$ has a singularity of type $A_{m-1}$.

10.3.3. The germ $(X, C)$ is the quotient of the $\mathbb{Q}$-conic bundle of index 2 of the form (10.2.2) with
\[ q_1 = y_1^2 - y_2^2 \quad \text{and} \quad q_2 = y_1 y_2 - y_3^2 \]
by the $\mu_4$-action
\[ (y_1, y_2, y_3, y_4; u, v) \longmapsto (-i y_1, i y_2, -y_3, i y_4; i u, -i v). \]
The base surface $(S, o)$ is Du Val of type $A_3$, the variety $X$ has a cyclic quotient singularity $P$ of type $\frac{1}{8}(5, 1, 3)$, and it has no other singular points.

10.3.4. The germ $(X, C)$ is the quotient of the Gorenstein conic bundle given by the equation in $\mathbb{P}_y^2 y_1, y_2, y_3 \times \mathbb{C}^2_{u, v}$
\[ y_1^2 + y_2^2 + \psi(u, v)y_3^2 = 0, \quad \psi(u, v) \in \mathbb{C}\{u^2, v^2, uv\}, \]
by the $\mu_2$-action
\[ (y_1, y_2, y_3; u, v) \longmapsto (-y_1, y_2, y_3; -u, -v). \]
Here $\psi(u, v)$ has no multiple factors. In this case $X$ has a unique singular point and it is of type $cA/2$ or $cAx/2$, and the base surface $(S, o)$ is Du Val of type $A_1$. 
10.3.5. The germ \((X, C)\) is the quotient of the \(\mathbb{Q}\)-conic bundle of index 2 of the form (10.2.2) with
\[
q_1 = y_1^2 - y_2^2, \quad q_2 = y_3^2
\]
by the \(\mu_2\)-action
\[
(y_1, y_2, y_3, y_4; u, v) \mapsto (y_1, -y_2, y_3, -y_4; -u, -v).
\]
The base \((S, o)\) is Du Val of type \(A_1\), and \(X\) has a unique non-Gorenstein point which is either a cyclic quotient singularity of type \(\frac{1}{4}(1, 1, -1)\) or a singularity of type \(cAx/4\).

10.3.6. The germ \((X, C)\) is the quotient of the \(\mathbb{Q}\)-conic bundle of index 2 of the form (10.2.2) with
\[
q_1 = y_1^2 - y_3^2 \quad \text{and} \quad q_2 = y_2^2 - y_3^2
\]
by the \(\mu_2\)-action:
\[
(y_1, y_2, y_3, y_4; u, v) \mapsto (-y_1, -y_2, y_3, -y_4; -u, -v).
\]
The base \((S, o)\) is Du Val of type \(A_1\), and \(X\) has a unique non-Gorenstein point which is either a cyclic quotient singularity of type \(\frac{1}{4}(1, 1, -1)\) or a singularity of type \(cAx/4\).

The central curve \(C\) is irreducible except in the case 10.3.6, where \(C\) has two irreducible components meeting at the non-Gorenstein point.

The main step in the proof of Theorem 10.3 is the following version of Reid’s ‘general elephant conjecture’.

10.4. Theorem [125], [127]. Let \(\pi: (X, C) \to (S, o)\) be a \(\mathbb{Q}\)-conic bundle germ. Assume that the curve \(C\) is irreducible. Then a general member of the linear system \(|-K_X|\) has only Du Val singularities.

The proof uses the techniques developed by Mori in the study of flipping extremal curve germs [123], [103].

As an immediate consequence of Theorem 10.3 above we have the following.

10.5. Theorem [125]. Let \(\pi: X \to S\) be a \(\mathbb{Q}\)-conic bundle. Then the base \(S\) can have at worst Du Val singularities of type \(A\).

If \((S \ni o)\) is of type \(A_{m-1}\), then we say that the germ \((X, C)\) has topological index \(m\). In this case \((X, C)\) is the quotient of a \(\mathbb{Q}\)-conic bundle over a smooth base by \(\mu_m\) (see [125], (2.4), and [142], Construction 1.9).

10.6. The condition that \(X\) has terminal singularities is crucial in Theorem 10.5. In general (for contractions in the log terminal category), the singularities of the base are worse than canonical. It is known that the base \(S\) of an arbitrary Mori fibre space \(\pi: X \to S\) is a normal \(\mathbb{Q}\)-factorial variety with at worst log terminal singularities ([94], Lemma 5-1-5, and [55]).
10.6.1. Example. Consider the following action of $\mu_m$ on $\mathbb{P}^1 \times \mathbb{C}^2_{u,v}$:

$$(x; u, v) \longmapsto (\varepsilon x; \varepsilon u, \varepsilon^{m-2} v),$$

where $m$ is odd and $\varepsilon$ is a primitive $m$th root of unity. Let

$$X := \mathbb{P}^1 \times \mathbb{C}^2 / \mu_m, \quad S := \mathbb{C}^2 / \mu_m$$

and let $\pi: X \to S$ be the natural projection. Since $\mu_m$ acts freely in codimension one, the divisor $-K_X$ is $f$-ample. The two fixed points on $\mathbb{P}^1 \times \mathbb{C}^2$ give two quotient singularities of $X$ which are terminal of type $\frac{1}{m}(-1, 1, -2)$ and canonical Gorenstein of type $\frac{1}{m}(1, 1, -2)$. In this case $f$ is an extremal contraction in the category of threefolds with canonical singularities. The base surface $S$ has a quotient singularity of type $\frac{1}{m}(1, -2)$ which is not canonical but is $\delta$-log canonical with $\delta = (m+1)/(2m)$.

However, Shokurov conjectured that, for any extremal $K$-negative contraction from a threefold with only canonical singularities to a surface, the singularities of the base are $1/2$-log canonical. More generally, the following conjecture was posed by McKernan.

10.6.2. Conjecture. For a fixed integer $n > 0$ and a real number $\varepsilon > 0$ there exists a constant $\delta = \delta(n, \varepsilon) > 0$ such that the following holds: If $\pi: X \to S$ is an extremal contraction, where $X$ is a $\mathbb{Q}$-factorial $n$-dimensional variety with only $\varepsilon$-log terminal singularities, then the singularities of $S$ are $\delta$-log terminal.

This conjecture was proved in the toric case by Alexeev and Borisov [5]. Partial results in the general case were obtained in [21] and connected with log adjunction (see [184] and [185]). As a consequence of Theorem 10.3, we also have the following two facts.

10.7. Corollary. Let $\pi: (X, C) \to (S, o)$ be a $\mathbb{Q}$-conic bundle germ and let $\Delta \subset S$ be the discriminant curve. Then $\Delta$ is a Cartier divisor at $o$, except in the cases 10.3.3, 10.3.5, 10.3.6.

10.8. Corollary. Let $\pi: (X, C) \to (S, o)$ be a $\mathbb{Q}$-conic bundle germ and let $\Delta \subset S$ be the discriminant curve. Then the following hold.

(i) $\Delta \not\ni o$ if and only if $(X, C)$ is of type 10.3.1.

(ii) If $\Delta \ni o$ and the pair $(S, \Delta)$ is purely log terminal at $o$, then the variety $X$ is non-singular near $C$ (and the surface $S$ is non-singular at $o$).

10.9. The results in the case of a smooth base surface $S$ are not complete. However, this case is well understood under the additional assumption that the central curve $C$ is irreducible ([127]–[131]). The main strategy is as follows. Using a ‘good’ member $D \in |{-K_X}|$ (Theorem 10.4) it is possible to analyze a general hyperplane section $H \subset X$ passing through $C$. Then the threefold $X$ can be viewed as the total space of a one-parameter deformation of $H$. If $H$ is normal, then its singularities are rational.
10.9.1. Example [129]. Consider a normal surface germ \((H,C)\) along a curve \(C \cong \mathbb{P}^1\) whose minimal resolution \(H_{\text{min}}\) has the following dual graph:

\[
\begin{array}{cccccc}
-2 & -2 & -3 \\
\circ & \circ & \circ \\
-1 & -2 & -2 & -2 & -3 & -3 & -2 \\
\end{array}
\]

Here the white vertices correspond to exceptional rational curves and the black vertex corresponds to the curve \(C\). The number attached to a vertex is the self-intersection index of the corresponding curve. It is easy to see that the configuration of white vertices can be contracted to a rational singularity \(H \ni P\), and the whole configuration is a fibre of a rational curve fibration \(H_{\text{min}} \to \Gamma\). Detailed computations ([103], 10.8) show that, locally, \(H \ni P\) can be realized as a hyperplane section of a threefold terminal singularity \((X, P) \cong \mathbb{C}^3_{y_1, y_2, y_4}/\mu_5(2, 3, 1)\) in such a way that \(H \supset C\), where

\[C = \{y_1^3 - y_2^2 = y_4 = 0\}/\mu_5.\]

The deformations of \(H\) are unobstructed, so a general one-parameter deformation of \(H\) is a threefold \(X \supset H\) with a terminal singularity at \(P\). We obtain a \(\mathbb{Q}\)-conic bundle \(\pi: (X,C) \to (S,o)\) over a non-singular base \(S\).

The situation is more complicated in the case where \(H\) is not normal. We give the simplest example.

10.9.2. Example [128]. Let \(\lambda_1, \lambda_2 \in \mathbb{C}\) be some general constants, and let \(X\) be the threefold given in \(\mathbb{P}(1,a,ma,m) \times \mathbb{C}_t\) by the equation

\[x_1^{2m-2a}x_2^2 + x_1^{2a}x_3^2 + x_2x_3x_4 + (\lambda_1x_1^m - x_4)(\lambda_2x_1^t - x_4)t = 0.\]

Then a small analytic neighbourhood of the curve

\[C := \{x_2 = x_3 = t = 0\}\]

is a \(\mathbb{Q}\)-conic bundle germ. The singular locus of \(X\) near \(C\) consists of a cyclic quotient singularity of type \(\frac{1}{m}(1,a,m-a)\) and two (Gorenstein) ordinary double points. A general hyperplane section passing through \(C\) is not normal.

Below we need the following simple lemmas.

10.10. Lemma. Let \(\pi: (X,C) \to (S,o)\) be a \(\mathbb{Q}\)-conic bundle germ over a smooth base and let \(\Delta\) be the discriminant curve. Assume that \(X\) is singular and \(\text{mult}_o(\Delta) = 2\). Take a standard model \(\pi^*: X^* \to S^*\) fitting in the diagram (3.12.1), so that the relative Picard number \(\rho(S^*/S)\) is minimal. Let \(\Delta^* \subset S^*\) be the discriminant curve. Then the divisor \(\Delta^*\) and the exceptional divisor \(\alpha^{-1}(o)\) have no common components.
Proof. According to Corollary 3.3.6,
\[ \Delta^* \leq \alpha^* \Delta \quad \text{and} \quad \alpha(\Delta^*) = \Delta. \]
The curve germ \( o \in \Delta \) can be given locally by
\[ y^2 - x^n = 0, \]
and the inverse image \( \alpha^{-1}(o) \) is a tree of rational curves. Let \( \Delta' \subset S^* \) be the proper transform of \( \Delta \). It is clear that \( \Delta' \subset \Delta^* \). If the curve \( \Delta' \) is singular at \( P \in \alpha^{-1}(o) \), then \( \Delta' \cap \alpha^{-1}(o) = \{ P \} \), and none of the components of \( \alpha^{-1}(o) \) are contained in \( \Delta^* \) by Corollaries 3.3.3 and 3.9.1. From now on we assume that \( \Delta' \) is non-singular.

Suppose that \( n \) is even. Then \( \Delta \) has two smooth analytic branches at the point \( o \), and their proper transforms \( \Delta^*_1 \) and \( \Delta^*_2 \) are disjoint. Furthermore, \( \Delta^*_1 \) and \( \Delta^*_2 \) are connected by a chain of smooth rational curves \( E_1, \ldots, E_n \subset \alpha^{-1}(o) \), and all components of \( \alpha^{-1}(o) \) other than \( E_1, \ldots, E_n \) are not contained in \( \Delta^* \) by Corollary 3.9.1 and (3.8.2). Thus, we may assume that \( \Delta^* \) contains some component, say \( E_i \). Again by (3.8.2),
\[ \Delta^* = \Delta^*_1 + E_1 + \cdots + E_n + \Delta^*_2, \quad E_i \subset \Delta^* \quad \forall i = 1, \ldots, n. \]
It is obvious that \( \alpha^{-1}(o) \) contains at least one \((-1)\)-curve, say \( E \), and the inequality \( \Delta^* \cdot E \leq 1 \) holds. Then by Proposition 8.5 we can blow down \( E \) and get a standard model \( X'/S' \) with smaller value \( \rho(S'/S) \). This contradicts our assumptions.

Assume that \( n \) is odd. Then the curve \( \Delta' \) is irreducible (and non-singular) and \( \Delta' \cap \alpha^{-1}(o) \) is a single point (say \( P \)). According to (3.8.2) the curves \( \Delta^* \) and \( \alpha^{-1}(o) \) have no common components. \( \square \)

10.11. Lemma. Let \( f: (X, C) \rightarrow (Z, o) \) be a \( \mathbb{Q} \)-conic bundle germ. Assume that there exists an effective Weil divisor \( H \) on \( X \) such that the pair \( (X, H) \) is canonical and the divisor \( K_X + H \) is numerically trivial. Then \( K_X + H \) is a Cartier divisor.

Proof. Assume the converse, that is, \( K_X + H \) is not Cartier at some point \( P \in C \subset X \) of index \( m \). Since the point \( P \in X \) is terminal, the divisor \( kK_X + H \) is Cartier in a neighbourhood of \( P \) for some \( k \) (see [90], Corollary 5.2). According to our assumption, \( k \neq 1 \) mod \( m \). In particular, \( m > 1 \). We may assume that \( 1 < k < m \).

Assume that \( H \ni P \). It follows from the main result of [92] that there exists an exceptional divisor \( E \) over \( P \in X \) with discrepancy
\[ a(E, X, 0) = \frac{1}{m}. \]
Since \( kK_X + H \) is a Cartier divisor at \( P \), we have
\[ -ka(E, X, 0) + \text{mult}_E(H) = -\frac{k}{m} + \text{mult}_E(H) \in \mathbb{Z}. \]
Hence, \( \text{mult}_E(H) \geq k/m > 1/m \) and
\[ a(E, X, H) = a(E, X, 0) - \text{mult}_E(H) \leq \frac{1}{m} - \frac{k}{m} < 0. \]
This contradicts the canonicity of the pair \((X, H)\). Thus, \(H\) does not contain \(P\).

Since \(\text{Pic}(X) \simeq H^2(C, \mathbb{Z})\) and \(K_X + H \equiv 0\), the divisor \(K_X + H\) is a torsion element in the Weil divisor class group and it defines a non-trivial, étale in codimension one, cover of the germ \((X, C)\) (see [125], Corollary 2.3.1 and 2.7.1). In particular, \((Z, o)\) is a singular point. For any component \(C_i \subset C\) we have

\[
H \cdot C_i = -K_X \cdot C_i < 1
\]

(see [125], Lemma 2.8). Consequently, \(H\) is not a Cartier divisor at some point \(P_1 \in C \subset X, P_1 \neq P\). Hence, \(X\) has at least two non-Gorenstein points. By Theorem 10.3 the germ of \((X, C)\) is of type 10.3.1 or 10.3.2. Then the curve \(C\) is irreducible and \(H \cap C = \{P_1\}\). The torsion subgroup \(\text{Cl}(X)_{\text{tors}}\) of the Weil divisor class group \(\text{Cl}(X)\) is cyclic ([125], Corollary 2.7.1). One can see from the constructions that in both cases 10.3.1 and 10.3.2 the restrictions

\[
\text{Cl}(X)_{\text{tors}} \to \text{Cl}(X, P) \quad \text{and} \quad \text{Cl}(X)_{\text{tors}} \to \text{Cl}(X, P_1)
\]

to the local class groups are isomorphisms. In particular, this means that \(K_X + H\) is not a Cartier divisor at \(P_1\). According to the discussions in the beginning of the proof we have \(H \not\ni P_1\), a contradiction. \(\square\)

11. Examples of Sarkisov links on \(\mathbb{Q}\)-conic bundles

In this section we consider Sarkisov links on non-Gorenstein \(\mathbb{Q}\)-conic bundles. Such links are elementary steps in the decomposition of birational transformations of conic bundles (see section 12). For applications it would be very useful to have a classification of them or at least substantive theorems like Theorem 5.1 describing their structure. Since very little is known in this direction, we will restrict ourselves to a number of examples.

We use the following natural construction.

11.1. Construction. Let \(\pi: (X, C) \to (S, o)\) be a \(\mathbb{Q}\)-conic bundle germ, let \(C_1, \ldots, C_r\) be the irreducible components of \(C\), and let \(P \in C \subset X\) be a point of index \(m_1 > 1\). We consider only the cases described in Theorems 10.2 or 10.3. In all these cases all the curves \(C_i\) pass through \(P\). Then for any \(i = 1, \ldots, r\) we have

\[
-K_X \cdot C_i = \begin{cases} 
1/2 & \text{in the cases 10.2, 10.3.3, 10.3.5, 10.3.6}, \\
1 & \text{in the case 10.3.4}, \\
2/m_1 & \text{in the case 10.3.1}, \\
1/m_1 & \text{in the case 10.3.2}.
\end{cases}
\]

(11.1.1)

These equalities can be checked directly or deduced from [125], Lemma 2.8, and [123], (2.3) and (4.9). According to [92], [65], and [66] there exists an extraction \(p: Z \to X\) in the Mori category with exceptional divisor \(E\) such that \(p(E) = P\) and the discrepancy of \(E\) is \(1/m_1\). Let \(\tilde{C}_i \subset Z\) be the proper transform of \(C_i\). Then

\[
K_Z \cdot \tilde{C}_i = K_X \cdot C_i + \frac{1}{m_1} E \cdot \tilde{C}_i.
\]

(11.1.2)

Assume that

\[
K_Z \cdot \tilde{C}_i \leq 0
\]

(11.1.3)
(thus, $Z$ is the central object of the corresponding link [186]). This inequality holds and can be checked directly in many cases. Then there exists a flop or a flip along $\tilde{C}_i$, and we can run the MMP on $Z$ over $S$ in a direction different from the contraction to $X$. After some number of flops and flips

$$\chi: Z \to Z_1$$

we obtain a non-small Mori extremal contraction. There are two possibilities: either this contraction $q: Z_1 \to X_1$ is divisorial or it is a $\mathbb{Q}$-conic bundle $\pi_1: Z_1 = X_1 \to S$. In other words, we obtain a link of type II or of type I, respectively (see 4.1). Moreover, for a type II link the morphism $q$ contracts the proper transform of the divisor $E$. Thus, $X$ and $X_1$ are isomorphic in codimension one. Since the divisors $-K_X$ and $-K_{X_1}$ are ample over $S$, the varieties $X$ and $X_1$ are in fact isomorphic. If in the case of a type I link the base $(S, o)$ is singular, then the morphism $\alpha: S_1 \to S$ must be a crepant contraction according to the following fact.

11.2. Proposition [132]. Let $S' \to S''$ be a proper birational contraction between surfaces with Du Val singularities. Then $\alpha$ can be decomposed as a sequence

$$\alpha_i: S_i \to S_{i+1},$$

where each morphism $\alpha_i$ is either a crepant contraction or a weighted blowup with weights $(1, n_i)$ of a non-singular point.

11.3. Recall that Shokurov’s difficulty $d_{\text{III}}(V)$ of a variety $V$ with terminal singularities is defined as the number of exceptional divisors on $V$ with discrepancy $< 1$ (see [181], Definition 2.15). It is known that this number is well defined and finite. Moreover, in the three-dimensional case it is strictly decreasing under flips ([181], Corollary 2.16). If $V \ni P$ is a terminal cyclic quotient of index $m$, then

$$d_{\text{III}}(V \ni P) = m - 1.$$ 

Thus, in our case (11.1.4) we have

$$d_{\text{III}}(Z_1) \geq d_{\text{III}}(Z),$$

and the inequality is strict if $\chi$ contains at least one flip.

Below we consider explicit examples. We use the notation of Construction 11.1.

11.4. Example. Let $(X, C) \subset \mathbb{P}(1,1,1,2)$ be the $\mathbb{Q}$-conic bundle of index 2 given by

$$\begin{cases}
y_1^2 - y_2^2 = uy_4,
\end{cases}$$

(see (10.2.2)), and let

$$\pi: (X, C) \to (S, o) = (\mathbb{C}^2, 0)$$

be the corresponding contraction. Then $X$ has a unique singular point $P = (0,0,0,1;0,0)$, which is a terminal quotient singularity of type $\frac{1}{2}(1,1,1)$. The central curve has four components $C_1, \ldots, C_4$. All of them pass through $P$, and they do not meet each other elsewhere. The discriminant curve is given by

$$(u^2 + 4v^2)u = 0.$$
Let \( p: Z \to X \) be the blowup of \( P \), let \( E \) be the exceptional divisor, and let \( \tilde{C}_i \subset Z \) be the proper transform of \( C_i \). Then the variety \( Z \) is non-singular, \( E \simeq \mathbb{P}^2 \), and \( \mathcal{O}_E(E) \simeq \mathcal{O}_{\mathbb{P}^2}(-2) \) (see [122]). Furthermore, the curves \( \tilde{C}_1, \ldots, \tilde{C}_4 \) are disjoint. By (11.1.1) and (11.1.2)

\[
K_Z \cdot \tilde{C}_i = 0.
\]

There exists a flop \( \chi: Z \to X_1 \) with centre \( \tilde{C} \) which is the simplest Atiyah–Kulikov flop along each curve \( \tilde{C}_i \). Since the divisor \(-K_X\), nef, there exists a Mori extremal contraction \( \pi_1: X_1 \to S_1 \) over \( S \). The restriction \( \chi_E: E \to E_1 \subset X_1 \) is inverse to the blowup of the four points \( E \cap \tilde{C}_i \). Therefore, the proper transform \( E_1 \subset X_1 \) of the divisor \( E \) is a del Pezzo surface of degree 5. Hence, \( \pi_1: X_1 \to S_1 \) is a standard conic bundle (see, for example, [122]). Then we obtain a type \( \text{I} \) link, where \( S_1 \) is a non-singular surface and \( \alpha: S_1 \to S \) is the blowup of \( o \). The discriminant curve \( \Delta_1 \) of \( \pi_1 \) is the proper transform of \( \Delta \) (and so \( \Delta_1 \) has three smooth irreducible branches near \( \alpha^{-1}(o) \)).

### 11.5. Example

Let \( \pi: (X, C \simeq \mathbb{P}^1) \to (S, o) \) be a toric \( \mathbb{Q} \)-conic bundle germ as in 10.3.1. In this case the discriminant curve is empty. Consider the Kawamata weighted blowup \( p: Z \to X \) of the point \( P \in X \) of type \( \frac{1}{m}(1, a, m - a) \) (see [93]). It is easy to see that \( Z \) has two terminal quotient singularities of types

\[
\text{(11.5.1)} \quad \frac{1}{a}(1, -m, m) \quad \text{and} \quad \frac{1}{m-a}(1, a, -m)
\]
on \( E \) and \( E \cap \tilde{C} \) is a non-singular point of \( Z \). According to (11.1.1) and (11.1.2),

\[
K_Z \cdot \tilde{C} = -\frac{1}{m}.
\]

Therefore, \( \tilde{C} \) is a flipping curve. Running the MMP on \( Z \) over \( S \) as in (11.1.4), we have

\[
\text{(11.5.2)} \quad d_{\text{III}}(Z) = 2m - 3 \quad \text{and} \quad d_{\text{III}}(Z_1) \leq 2m - 4.
\]

If we are in the situation of a type \( \text{II} \) link, then the contraction \( \pi_1: X_1 \to S \) must be a \( \mathbb{Q} \)-conic bundle of the same type as \( \pi \) (because its discriminant divisor is trivial). On the other hand,

\[
d_{\text{III}}(X_1) \leq d_{\text{III}}(Z_1) + 1 \leq 2m - 3.
\]

Consequently, \( d_{\text{III}}(X_1) < d_{\text{III}}(X) \), a contradiction. Thus, we have a type \( \text{I} \) link and \( \alpha: S_1 \to S \) is a crepant blowup of the point \( o \in S \) (see Proposition 11.2). Therefore, the surface \( S_1 \) has Du Val singularities of types \( A_{m_1-1} \) and \( A_{m_2-1} \) with \( m_1 + m_2 = m \). We may assume that \( m_1 \geq m_2 \) (it is possible that \( m_2 = 1 \)). The discriminant divisor of \( \pi_1 \) is trivial, so \( \pi_1 \) is also of type 10.3.1. More precisely, \( \pi_1 \) has two germs (or one) of type 10.3.1, which are quotients by \( \mu_{m_1} \) and \( \mu_{m_2} \). In this case

\[
d_{\text{III}}(X_1) = 2m_1 - 2 + 2m_2 - 2 = 2m - 4.
\]

Thus, equality holds in (11.5.2). This implies that \( \chi \) is a single flip, and hence \( X_1 \) has singularities of types (11.5.1). This means that up to a permutation we have

\[
m_1 = a \quad \text{and} \quad m_2 = m - a.
\]
11.6. Example. Let \( \pi: (X, C \simeq \mathbb{P}^1) \to (S, o) \) be a \( \mathbb{Q} \)-conic bundle germ, as in 10.3.2. Then the discriminant curve \( \Delta \) is given by \( \{uv = 0\}/\mu_m \). In particular, the pair \( (S, \Delta) \) is log canonical. Consider the Kawamata weighted blowup \( p: Z \to X \) of the point \( P \in X \) of type

\[
\frac{1}{m}(a + 1, 1, -1) = \frac{1}{m}(1, 2, m - 2).
\]

Then \( E \cap \tilde{C} \) is a singularity of type \( \frac{1}{2}(1, 1, 1) \) on \( Z \). By (11.1.1) and (11.1.2)

\[
K_Z \cdot \tilde{C} = -\frac{1}{2m}.
\]

Running the MMP as in Construction 11.1, we obtain a Sarkisov link of type I, where \( \pi_1 \) is a \( \mathbb{Q} \)-conic bundle and \( \alpha: S_1 \to S \) is a crepant contraction. For the discriminant curve \( \Delta_1 \subset S_1 \) we have

\[
\Delta_1 = \alpha^*(\Delta)_{\text{red}}
\]

(see Corollary 10.8), and the pair \( (S_1, \Delta_1) \) is log canonical. This means that \( \pi_1 \) cannot contain a germ of type 10.3.1. The same computations with difficulty as in Example 11.5 show that \( \chi \) is single flip, and thus \( X_1 \) contains a singularity of type \( \frac{1}{m-1}(1, 2, -2) \). Therefore, \( S_1 \) has two Du Val points of types \( A_{m-3} \) and \( A_1 \), and over \( A_{m-3} \) we have a germ of type 10.3.2 of index \( m - 2 \). Then by the classification in Theorem 10.3 there is only one possibility for the germ over \( A_1 \): the case 10.3.4 with \( \psi = uv \).

11.7. Example. Consider a \( \mathbb{Q} \)-conic bundle germ \( \pi: (X, C) \to (S, o) \) of type 10.3.3 which is the \( \mu_4 \)-quotient of \( X' \subset \mathbb{P}(1, 1, 1, 2) \) given by (11.4.1). The discriminant curve \( \Delta \) is given by

\[
\{(u^2 + 4v^2)u = 0\}/\mu_4 \quad \text{on} \quad S = \mathbb{C}^2/\mu_4(1, -1).
\]

One can see that the dual graph of the minimal resolution of \( (S \supset \Delta \ni o) \) has the following form:

where the vertices \( \circ \) correspond to (crepant) exceptional divisors over \( o \), and the vertices \( \bullet \) correspond to the components of \( \Delta \). Consider the Kawamata weighted blowup \( p: Z \to X \) of the point \( P \in X \) of type \( \frac{1}{5}(5, 1, 3) \). Then \( Z \) has two cyclic quotient singularities \( P_1 \) and \( P_2 \) of types \( \frac{1}{5}(-3, 1, 3) \) and \( \frac{1}{3}(2, 1, 1) \). The curve \( \tilde{C} \) passes through \( P_1 \). Using (11.1.1) and (11.1.2), we can compute

\[
K_Z \cdot \tilde{C} < 0.
\]

Thus, there exists a flip along \( \tilde{C} \). We have

\[
d_{\text{III}}(Z) = 6 \quad \text{and} \quad d_{\text{III}}(Z_1) \leq 5.
\]
As above, we obtain a type I link, where $\alpha: S_1 \to S$ is a crepant contraction. Let $\Delta_1 \subset S_1$ be the discriminant curve. If $\alpha$ is the blowup of $\Theta_2$, then $S_1$ has two points, say $o_1$ and $o_2$, of type $A_1$. From Corollary 10.8 we obtain $\alpha^{-1}(o) \subset \Delta_1$. But then the pair $(S_1, \Delta_1)$ is purely log terminal at one the points $o_i$. This is impossible, again by Corollary 10.8. Hence, $S_1$ has a point $o_1$ of type $A_2$. Then by the classification in Theorem 10.3 the germ over $o_2$ is of type 10.3.2, so the pair $(S_1, \Delta_1)$ is log canonical at $o_1$. Thus, $\alpha^{-1}(o) \not\subset \Delta_1$ and $\alpha$ is the blowup of $\Theta_3$.

11.8. Example [11]. Consider a $\mathbb{Q}$-conic bundle germ $\pi: (X, C) \to (S, o)$ such that the base $(S, o)$ is non-singular and the discriminant curve $\Delta$ has an ordinary double singularity at $o$. Let

$$U := S \setminus \{o\} \quad \text{and} \quad V := X \setminus C = \pi^{-1}(U).$$

Let $j: U \hookrightarrow S$ be the embedding. The sheaf $\mathcal{E}_0 := \pi_* \mathcal{O}_V(-K_X)$ is locally free on $U$. It has a unique extension to $S$ as a locally free sheaf. Indeed, the sheaf $(j_* \mathcal{E}_0)^{\vee}$ is reflexive and therefore is locally free. Clearly, $V$ is embedded in $\mathbb{P}_U(\mathcal{E}_0) \subset \mathbb{P}_S(\mathcal{E})$. Let $X' \subset \mathbb{P}_S(\mathcal{E})$ be the closure of $V$. We may assume that $(S, o)$ is an open (analytic) subset of $\mathbb{C}_{u,v}^2$ and $\Delta$ is given by $uv = 0$.

Assume that the variety $X$ is singular. Then easy computations show that $X'$ can be given locally by the equation

$$uvx_0^2 + x_1^2 + x_2^2 = 0$$

in $\mathbb{P}_S(\mathcal{E}) = \mathbb{P}_{x_0,x_1,x_2}^2 \times \mathbb{C}_{u,v}^2$. In particular, the projection $\pi': X' \to S$ is a flat morphism, and $X'$ has a unique singular point $P \in X'$ which is a node. On the other hand, the varieties $X'$ and $X$ are isomorphic in codimension one over $S$, and their anticanonical divisors are relatively ample. Hence $X'$ and $X$ are isomorphic over $S$ and we can identify them. In particular, $X$ can be embedded in $\mathbb{P}_S(\mathcal{E})$.

Let $p: Z \to X$ be the blowup of $P$. Then the variety $Z$ is non-singular and the divisor $-K_Z$ is nef and big over $S$. There exists a flop $Z \dashrightarrow X_1$ with centre the proper transform of $C$, and on $X_1$ there exists a $K$-negative extremal Mori contraction $\pi_1: X_1 \to S_1$ over $S$. Since the variety $X_1$ is non-singular, so is the surface $S_1$. Thus, we obtain a type I Sarkisov link, where $\alpha: S_1 \to S$ is the blowup of $o$. Moreover, the exceptional curve $\alpha^{-1}(o)$ is not contained in the discriminant divisor $\Delta_1 \subset S_1$ of $\pi_1$ (otherwise we are in the situation of Proposition 8.5, (iii), and $X$ is non-singular). Therefore, $\Delta_1$ is the proper transform of $\Delta$.

11.8.1. Corollary. Let $\pi: (X, C) \to (S, o)$ be a $\mathbb{Q}$-conic bundle and let $\Delta$ be the discriminant curve. Assume that the pair $(S, \Delta)$ is log canonical, but not purely log terminal at $o$. Then either $\pi$ is a standard conic bundle, or $\pi$ is as in Example 11.8, or as in 10.3.2, or as in 10.3.4 with $\psi = uv$.

Proof. By Theorem 10.5 the base $(S, o)$ is the quotient of a non-singular germ $(S', o')$ by the group $\mu_m$, $m \geq 1$. Let $X'$ be the normalization of the fibre product $X \times_S S'$ (see [125], (2.4), or [142], Construction 1.9). Then the projection $\pi': X' \to S'$ is a $\mathbb{Q}$-conic bundle over a smooth base. According to Example 11.8 the variety $X'$ is Gorenstein. Then the assertion follows from [142], Theorem 2.4. □
11.8.2. Corollary. Let $\pi: (X, C) \to (S, o)$ be a $\mathbb{Q}$-conic bundle germ such that its discriminant curve $\Delta$ is non-empty and non-singular at $o$. Then both varieties $X$ and $S$ are non-singular, and $\pi$ is a standard conic bundle.

**Proof.** By Theorem 10.3 the base $(S, o)$ is either smooth or Du Val of type $A_1$ or $A_3$. In these cases the pair $(S, \Delta)$ must be log canonical, because the curve $\Delta$ is smooth. Then the assertion follows from Corollaries 11.8.1 and 10.8. □

Avilov [11] used Sarkisov links to show the existence of standard models for three-dimensional $\mathbb{Q}$-conic bundles over a non-closed field of characteristic $0$ (cf. Theorem 3.12). His construction is canonical and works in the category of $G$-varieties. In particular, he used the following easy observation.

11.9. Lemma ([11], Lemma 3). Let $\pi: (X, C) \to (S, o)$ be a $\mathbb{Q}$-conic bundle germ over a singular base $(S, o)$. Then there exists a Sarkisov link of type $I$, where $\alpha: S_1 \to S$ is a crepant blowup of $o$.

**Sketch of the proof.** Take a linear system $\mathcal{L}$ of hyperplane sections of $S$ passing through $o$ and put $\mathcal{M} := \pi^* \mathcal{L}$. Take $c$ so that the pair $(X, c.\mathcal{M})$ is maximally canonical (in other words, $c = c(X, \mathcal{M})$ is the canonical threshold of $(X, \mathcal{M})$). Then, as in 4.5, there exists a log crepant extremal blowup $p: (Z, c.\mathcal{M}_Z) \to (X, c.\mathcal{M})$. Here $\rho(Z/S) = 2$ and

$$K_Z + c.\mathcal{M}_Z \equiv p^*(K_X + c.\mathcal{M}).$$

Moreover, $p$ is a divisorial contraction, and the variety $Z$ has terminal singularities and is $\mathbb{Q}$-factorial. Then we run the $(K_Z + c.\mathcal{M}_Z)$-MMP over $S$. Since $\dim S = 2$, we obtain a link of type $I$. By Proposition 11.2 the contraction $\alpha: S_1 \to S$ is a crepant blowup. □

Avilov’s proof of the existence of standard models is based on the (weak) Sarkisov programme and essentially uses Theorem 10.4. In fact, since the number of crepant divisors on $S$ is finite, any sequence of transformations as in Lemma 11.9 terminates. Consequently, by applying such transformations one can resolve the singularities of the base.

12. Birational transformations of $\mathbb{Q}$-conic bundles, I

Below we basically follow the paper [76] by Iskovskikh, with some improvements.

12.1. Theorem. Let $\pi: X \to S$ be a $\mathbb{Q}$-conic bundle over a projective rational surface $S$. Assume that there exist another Mori fibre space $\pi^\sharp: X^\sharp \to S^\sharp$ and a birational map

$$X \xrightarrow{\Phi} X^\sharp$$

(12.1.1)

$$\downarrow \quad \downarrow \pi^\sharp$$

$$S \quad S^\sharp$$
which cannot be completed to a commutative square. Then there exists the following commutative diagram:

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\pi}} & \hat{S} \\
\downarrow & & \downarrow \\
\hat{X} & \xrightarrow{\pi} & \hat{S} \\
\end{array}
\]

where \( \hat{X}/\hat{S} \) and \( \bar{X}/\bar{S} \) are \( \mathbb{Q} \)-conic bundles, \( \hat{X}/\hat{S} \rightarrow X/S \) is a sequence of Sarkisov links of types I and II, \( \hat{X}/\hat{S} \rightarrow \bar{X}/\bar{S} \) is a sequence of Sarkisov links of type III, and the contractions \( \hat{S} \rightarrow S \) and \( \bar{S} \rightarrow \bar{S} \) are described in Proposition 11.2. Moreover, let \( \Delta \subset S \) be the discriminant curve of the conic bundle \( \hat{\pi} : \hat{X} \rightarrow \hat{S} \). Then one of the following assertions holds:

(i) \( \rho(S) = 1 \) and the divisor \( -(4K_{\bar{S}} + \Delta) \) is ample;

(ii) \( \rho(S) = 2 \) and there exists a base point free pencil \( \bar{L} \) of rational curves such that

\[
2 \leq \bar{L} \cdot \Delta < -4 \bar{L} \cdot K_{\bar{S}} = 8.
\]

Moreover, if \( \pi \) is a standard conic bundle, then

\[
p_a(\hat{\Delta}) = p_a(\Delta) \leq p_a(\bar{\Delta}).
\]

**Proof.** Let us apply the Sarkisov programme (see Theorem 4.2). First we untwist the maximal singularities. This means that we apply a sequence of type I and II links, as explained in 4.5:

\[
\begin{array}{ccc}
X & \xleftarrow{\Phi_1} & X_1 \\
\downarrow & & \downarrow \\
S & \xleftarrow{\alpha_1} & S_1 \\
\end{array} \quad \begin{array}{ccc}
X_1 & \xleftarrow{\Phi_2} & \cdots & \xleftarrow{\Phi_{m-1}} & X_{m-1} & \xleftarrow{\Phi_m} & \hat{X} = X_m \\
\downarrow & & & & & \downarrow \\
S_1 & \xleftarrow{\alpha_2} & \cdots & \xleftarrow{\alpha_{m-1}} & S_{m-1} & \xleftarrow{\alpha_m} & \hat{S} = S_m \\
\end{array}
\]

Here the \( \alpha_i \) are birational contractions or isomorphisms, and each of the squares \( (\Phi_i, \alpha_i) \) is commutative. Moreover, the proper transforms \( \mathcal{H}_{X_i} \) of the linear system \( \mathcal{H}_\hat{X} \) have the form

\[
\mathcal{H}_{X_i} = -\mu K_{X_i} + \pi_i^* A_i,
\]

where \( A_i \) is a \( \mathbb{Q} \)-divisor on \( S_i \). In the last step the linear system \( \mathcal{H}_\hat{S} \) has no maximal singularities.

Next, we follow the step 4.6 of the Sarkisov programme. The divisor \( \hat{A} \) is not nef by the Noether–Fano inequality (Theorem 4.4), where \( \hat{A} \), as usual, satisfies the relation

\[
\mathcal{H}_\hat{X} \equiv -\mu K_\hat{X} + \hat{\pi}^* \hat{A}.
\]
Consequently, there exists a link $\Phi_{m+1}: \tilde{X}/\tilde{S} \to X_{m+1}/S_{m+1}$ of type III or IV. Assume that this link is of type III with birational contraction $\alpha_{m+1}: \tilde{S} \to S_{m+1}$. Then $\pi_{m+1}: X_{m+1} \to \tilde{S}_{m+1}$ satisfies the same conditions as $\tilde{\pi}: \tilde{X} \to \tilde{S}$, and the procedure may be repeated:

$$
X_m = \tilde{X} - \Phi_{m+1} \to X_{m+1} - \Phi_{m+2} \to \cdots - \Phi_n \to X = X_n
$$

(12.1.6)

$$
S_m = \tilde{S} - \alpha_{m+1} \to S_{m+1} - \alpha_{m+2} \to \cdots - \alpha_n \to \tilde{S} = S_n
$$

so the maps $\alpha_{m+i}, i = 1, \ldots, n - m$, are birational contractions, $\rho(S_{m+i}/S_{m+i+1}) = 1$, each square $(\Phi_{m+i}, \alpha_{m+i})$ is commutative, the proper transforms of $\mathcal{H}_\chi$ on $X_{m+i}$ have the form

$$
\mathcal{H}_{X_{m+i}} = -\mu K_{X_{m+i}} + \pi^{*}_{m+i} A_{m+i},
$$

where $A_{m+i}$ is some non-nef $\mathbb{Q}$-divisor on $S_{m+i}$, and the linear system $\mathcal{H}_{X_{m+i}}$ has no maximal singularities.

After a finite number of steps we arrive at a link $\Phi: \tilde{X}/\tilde{S} \to X^b/S^b$ of type III with non-birational contraction $\delta = \tilde{\alpha}: \tilde{S} \to T = \tilde{S}_1$ or at a link of type IV. As above, for the proper transform $\mathcal{H}_{\tilde{\chi}}$ of the linear system $\mathcal{H}$ we can write

$$
\mathcal{H}_{\tilde{\chi}} \equiv -\mu K_{\tilde{\chi}} + \tilde{\pi}^{*} \tilde{A},
$$

where $\tilde{A}$ is not nef according to the Noether–Fano inequality (Theorem 4.4). By the projection formula and (3.11.1) we have

$$
\Gamma := \pi_{*}(\mathcal{H}^{2}_{\tilde{\chi}}) \equiv 4\mu \tilde{A} - \mu^{2}(4K_{\tilde{S}} + \tilde{\Delta}),
$$

and this cycle is an effective non-zero integral Weil divisor on $\tilde{S}$. There are the following two possibilities.

12.1.9. The link $\tilde{\Phi}: \tilde{X}/\tilde{S} \to X^b/S^b$ is of type IV and $T$ is a point. Then $\rho(\tilde{S}) = 1$. In this case the divisors $-\tilde{A}, -K_{\tilde{S}}$, and $-(4K_{\tilde{S}} + \tilde{\Delta})$ are ample (the last divisor is ample according to (12.1.8)).

12.1.10. The link $\tilde{\Phi}: \tilde{X}/\tilde{S} \to X^b/S^b$ is of type III (respectively IV), where in the notation of 4.1 the variety $S_1$ (respectively, $T$) is a curve. Then this curve must be smooth and rational, and $\rho(\tilde{S}) = 2$. Thus, we have an extremal contraction $\delta: \tilde{S} \to \mathbb{P}^1$. Let $\mathcal{L}$ be the pencil of fibres. Then $\tilde{A} \cdot \mathcal{L} < 0$ by the construction of the link $\tilde{\Phi}$ (see 4.6). Further,

$$
(4K_{\tilde{S}} + \tilde{\Delta}) \cdot \mathcal{L} < 0
$$

by (12.1.8). This proves (12.1.3).

To prove the last equality, consider a standard conic bundle $\pi^*: X^* \to S^*$ that is birationally equivalent to $\tilde{\pi}: \tilde{X} \to \tilde{S}$, as in Theorem 3.12. Let $\Delta^* \subset S^*$ be the corresponding discriminant curve. Then by Lemma 8.3

$$
p_a(\Delta^*) \leq p_a(\tilde{\Delta}) \leq p_a(\Delta) \leq p_a(\tilde{\Delta}).
$$

But $p_a(\Delta) = p_a(\Delta^*)$ by Lemma 8.4. This completes the proof of the theorem. □
As an easy consequence of Theorem 12.1 we obtain the following.

12.2. Theorem Sarkisov [173]. Let $\pi: X \to S$ be a standard conic bundle over a rational surface $S$ and let $\Delta$ be the discriminant curve. Assume that 

$$\left| 4K_S + \Delta \right| \neq \emptyset.$$

Then any birational map $X \dashrightarrow X^{\sharp}$ to another Mori fibre space $\pi^{\sharp}: X^{\sharp} \to S^{\sharp}$ is fibrewise (in particular, $\pi: X \to S$ is birationally rigid).

We note that in terms of the effective threshold the condition $\left| 4K_S + \Delta \right| \neq \emptyset$ is equivalent to $et(S, \Delta) > 4$ (see 8.7).

Proof. Assume that the divisor $4K_S + \Delta$ is effective. Then by Lemma 8.4 the divisor $4K_S^{\bullet} + \Delta^{\bullet}$ on the standard model $\pi^{\bullet}: X^{\bullet} \to S^{\bullet}$ is also effective (see Theorem 3.12), and so is its image under the birational contraction $\hat{S} \to \bar{S}$. This contradicts Theorem 12.1, (i) and (ii).

Theorem 12.2 was proved by Sarkisov in [173]. It has a higher-dimensional generalization ([174], Theorem 4.1). Another proof can be found in [164] (see also [186], Corollary 7.3, for a more general approach).

12.3. Remark. Note that the condition $\left| 4K_S + \Delta \right| \neq \emptyset$ follows from the condition that the 1-cycle $-(K_X)^2$ is effective.

12.4. Corollary. With the notation and assumptions of Theorem 12.2 the variety $X$ is not birationally equivalent to a $\mathbb{Q}$-Fano threefold, and $X$ is not birational to a variety with the structure of a $\mathbb{Q}$-del Pezzo fibration. In particular, $X$ is not rational.

For a rationally connected variety $X$, it is an interesting problem to describe all birational structures of fibrations on $X$ into varieties with trivial canonical divisor. This is an old classical problem raised by Halphen [62], [51]. For conic bundles, the following necessary condition was obtained by Cheltsov:

12.5. Theorem [30]. Let $\pi: X \to S$ be a standard conic bundle (of any dimension) and let $\Delta$ be the discriminant divisor. If the divisor $4K_S + \Delta$ is big, then $X$ is not birationally equivalent to a fibration whose general fibre is a smooth variety with numerically trivial canonical divisor. In particular, $X$ does not have the birational structure of an elliptic fibration.

This can be proved in the same style as Theorem 12.2. One has to modify the Noether–Fano inequalities for a more general situation.

13. Birational transformations of $\mathbb{Q}$-conic bundles, II

In this section we study the transformation $X/S \dashrightarrow \bar{X}/\bar{S}$ in Theorem 12.1 in detail. We also discuss various applications to rationality problems. From Theorem 12.1 and the general structure of the Sarkisov programme one immediately obtains the following.

13.1. Corollary. In the notation of Theorem 12.1 there exists a link 

$$\bar{\Phi}: \bar{X}/\bar{S} \dashrightarrow X^{\flat}/S^{\flat}$$

of one of the following types.
(i) **Type III over a point.** Then $X^b$ is a $\mathbb{Q}$-Fano threefold with $\rho(X^b) = 1$:

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\bar{\pi}} & Z^b \\
\downarrow & & \downarrow \\
\bar{S} & \xrightarrow{\alpha} & \{\text{pt}\} \\
\end{array}
\]

\[(13.1.1)\]

(ii) **Type IV over a point.** Then either $\pi^b : X^b \to S^b$ is a $\mathbb{Q}$-conic bundle with $\rho(S^b) = 1$, or $S^b \cong \mathbb{P}^1$ and $\pi^b$ is a $\mathbb{Q}$-del Pezzo fibration:

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\bar{\pi}} & X^b \\
\downarrow & & \downarrow \\
\bar{S} & \xrightarrow{\alpha} & \{\text{pt}\} \\
\end{array}
\]

\[(13.1.2)\]

(iii) **Type III over a curve.** Then $\pi^b : X^b \to \mathbb{P}^1$ is a $\mathbb{Q}$-del Pezzo fibration:

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\bar{\pi}} & Z^b \\
\downarrow & & \downarrow \\
\bar{S} & \xrightarrow{\alpha} & \mathbb{P}^1 \\
\end{array}
\]

\[(13.1.3)\]

(iv) **Type IV over a curve.** Then $\pi^b : X^b \to S^b$ is a $\mathbb{Q}$-conic bundle:

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\bar{\pi}} & X^b \\
\downarrow & & \downarrow \\
\bar{S} & \xrightarrow{\alpha} & \mathbb{P}^1 \\
\end{array}
\]

\[(13.1.4)\]

Moreover, the link $\bar{\Phi} : \bar{X}/\bar{S} \dashrightarrow X^b/S^b$ decreases the coefficient $\mu$ (see (4.3.2)), that is,

\[(13.1.5)\]

$$\mu^b < \mu,$$

where $\mu^b$ is defined by the equality

$$\mathcal{H}_{X^b} \equiv -\mu^b K_{X^b} + \pi^b A^b.$$  

**13.2.** Assume that we are in the situation of (13.1.1) or (13.1.2). Then, in particular, the divisor $-K_\bar{S}$ is ample, that is, $\bar{S}$ is a del Pezzo surface with at worst Du Val singularities. All such surfaces have been classified (see, for example, [120]
and [52], Chap. 8). In our case we have additional restrictions: \( \rho(\bar{S}) = 1 \) and the singularities of \( \bar{S} \) are of type A. According to the classification, there are 28 combinatorial types of Du Val del Pezzo surfaces of Picard number one, and there are exactly 16 types among them with at worst type A singularities. In Table 2 we reproduce the complete list, where the notation \( nA_m \) means that \( n \) is the number of \( A_m \)-points. By Theorem 12.1, (i), the divisor \(- (4K_{\bar{S}} + \bar{\Delta})\) is ample. Clearly, there are only a limited number of possibilities for \( \bar{\Delta} \). Note, however, that \( \bar{\Delta} \) is not necessarily a Cartier divisor on \( \bar{S} \) (see Corollary 10.7). By the adjunction formula we have the following lemma.

\[
\begin{array}{c|c}
K^2_{\bar{S}} & \text{Sing}(S) \\
1 & 4A_2, 2A_1 + 2A_3, 2A_4, A_1 + A_2 + A_5, A_1 + A_7, A_8 \\
2 & A_1 + 2A_3, A_2 + A_5, A_7 \\
3 & 3A_2, A_1 + A_5 \\
4 & 2A_1 + A_3 \\
5 & A_4 \\
6 & A_1 + A_2 (S \simeq \mathbb{P}(1, 2, 3)) \\
8 & A_1 (S \simeq \mathbb{P}(1, 1, 2)) \\
9 & \emptyset (S \simeq \mathbb{P}^2)
\end{array}
\]

\[13.2.1. \text{Lemma.} \text{ Let } \bar{S} \text{ be a del Pezzo surface with at worst Du Val singularities and } \rho(\bar{S}) = 1. \text{ Let } d := K^2_{\bar{S}} \text{ and let } \bar{\Delta} \subset \bar{S} \text{ be a reduced curve. Let } J \text{ be the ample generator of the group } Cl(\bar{S})/\text{tors} \simeq \mathbb{Z} \text{ and let}
\]
\[-K_{\bar{S}} \equiv \iota J \quad \text{and} \quad \bar{\Delta} \equiv aJ \]

for some positive integers \( \iota \) and \( a \). Then one of the cases listed in Table 3 occurs.

\[
\begin{array}{c|c|c|c}
d & \bar{S} & \iota & p_a(\bar{\Delta}) \\
9 & \mathbb{P}^2 & 3 & \frac{(a - 3)a}{2} + 1 \\
8 & \mathbb{P}(1, 1, 2) & 4 & \leq \frac{(a - 4)a}{4} + 1 \\
\leq 6 & d & \leq \frac{(a - d)a}{2d} + 1
\end{array}
\]

\[13.2.2. \text{Corollary.} \text{ In the above notation assume that the divisor } - (4K_{\bar{S}} + \bar{\Delta}) \text{ is ample. Then}
\]
\[
a < 4\iota \quad \text{and} \quad p_a(\bar{\Delta}) \leq 45.
\]

\[13.2.3. \text{Remark.} \text{ In some cases the inequalities above can be improved significantly. For example, if in the above notation the divisor } - (4K_{\bar{S}} + \bar{\Delta}) \text{ is ample and } K^2_{\bar{S}} = 5,
\]
then by Corollary 10.7 $\Delta$ is a Cartier divisor on $\bar{S}$,
\[ a \leq 15, \quad \text{and} \quad p_a(\Delta) \leq 16. \]

13.3. Now assume that we are in the situation of (13.1.4) or (13.1.3). Then there exists a $K_\bar{S}$-negative extremal contraction
\[ \bar{\alpha}: \bar{S} \to \mathbb{P}^1 =: T. \]

It is easy to describe the degenerate fibres of this fibration. We leave the following statement as an exercise.

13.3.1. Lemma. Let $(F, C)$ be the germ of a surface with Du Val singularities of type A along an irreducible reduced curve $C$ such that there exists a $K_F$-negative contraction $\delta: (F, C) \to (T, o)$ to a curve. Then $C \simeq \mathbb{P}^1$, and for the dual graph of the minimal resolution of $(F, C)$ there are only two possibilities:

(13.3.2) \[ \circ - \bullet - \circ \]
(13.3.3) \[ \circ - \circ - \circ - \circ - \bullet \]

where the vertices $\circ$ correspond to exceptional $(-2)$-curves and the vertex $\bullet$ corresponds to $C$. In particular, the singular locus of $F$ consists of two points of type $A_1$ in the case (13.3.2) and one point of type $A_3$ in the case (13.3.3).

13.3.4. Lemma. Let $\varphi: Y \to Z$ be a $\mathbb{Q}$-conic bundle over a projective rational surface, and let $\mathcal{M}$ be a linear system on $Y$ without fixed components. Let $\mathcal{M} + \lambda K_Y \equiv \varphi^* B$, where $B$ is a $\mathbb{Q}$-divisor on $Z$ and $\lambda \in \mathbb{Q}$. Assume that there exists a base point free pencil $\mathcal{L}$ on $Z$ such that $B \cdot \mathcal{L} < 0$. Let $C \subset Y$ be an irreducible curve such that $K_Y \cdot C > 0$. Then one of the following holds:

(i) $C \subset \text{Bs}(\mathcal{M})$,
(ii) $\varphi(C) \cdot \mathcal{L} > 0$, or
(iii) $\varphi(C)$ is a component of a degenerate fibre of $\mathcal{L}$.

Proof. It is obvious that $C$ is not contained in the fibres of $\varphi$. We have
\[ 0 \leq \lambda K_Y \cdot C = -\mathcal{M} \cdot C + \varphi^* B \cdot C. \]

If $C \not\subset \text{Bs}(\mathcal{M})$, then $\mathcal{M} \cdot C \geq 0$ and so $\varphi^* B \cdot C > 0$. In this case $B \cdot \varphi(C) > 0$ by the projection formula. Therefore, the curve $\varphi(C)$ cannot coincide with a full fibre of the pencil $\mathcal{L}$. \(\square\)

13.3.5. Corollary. In the above notation, let $Y_\eta$ be the generic fibre of the composition
\[ Y \xrightarrow{\mathcal{L}} Z \xrightarrow{} \mathbb{P}^1. \]

Then the divisor $-K_{Y_\eta}$ is ample.

Let $\mathcal{L}$ be the base point free pencil of rational curves generated by the fibres of $\bar{\alpha}: \bar{S} \to \mathbb{P}^1$. Let $\bar{X}_\eta$ and $\bar{S}_\eta$ be generic fibres over $T$ of the morphisms $\bar{\alpha} \circ \bar{\pi}: \bar{X} \to T$ and $\bar{\alpha}: \bar{S} \to T$, respectively.
respectively. Then $\tilde{X}_\eta$ is a smooth surface over the non-closed field $k(T)$, and $\bar{S}_\eta \simeq \mathbb{P}^1_{k(T)}$ is a smooth rational curve. The contraction $\bar{\pi}$ induces a conic bundle structure $\bar{\pi}_\eta: \tilde{X}_\eta \to \bar{S}_\eta$ with

$$\text{Pic}(\tilde{X}_\eta) = \mathbb{Z} \cdot \Lambda_\eta \oplus \mathbb{Z} \cdot K_{X_\eta},$$

where $\Lambda_\eta$ is the class of the fibre of $\bar{\pi}_\eta$. According to Corollary 13.3.5, $\tilde{X}_\eta$ is a del Pezzo surface with $\rho(\tilde{X}_\eta) = 2$. Since the log flips $\bar{\chi}$ do not change the generic fibre, we also have the following.

13.3.6. Corollary. The Sarkisov link $\Phi: \tilde{X}/\bar{S} \dasharrow X^b/S^b$ induces a link on $\tilde{X}_\eta$ which is of type IV in the case (13.1.4) and of type III in the case (13.1.3).

13.3.7. Using the classification of two-dimensional Sarkisov links in Theorems 5.1 and 5.3, we obtain the following possibilities.

(i) $K^2_{X_\eta} = 1, \mathcal{L} \cdot \Delta = 7$, $\pi^b$ is a $\mathbb{Q}$-conic bundle, and the link is of type IV and is induced by the Bertini involution $\beta_\eta: \tilde{X}_\eta \to \tilde{X}_\eta$. Then in (13.1.4) we have $\pi^b = \lambda \circ \bar{\pi} \circ \beta$ for some birational maps $\beta: \tilde{X} \dasharrow \tilde{X}$ and $\lambda: S^b \dasharrow S^b$.

(ii) $K^2_{X_\eta} = 2, \mathcal{L} \cdot \Delta = 6$, $\pi^b$ is a $\mathbb{Q}$-conic bundle, and the link is of type IV and is induced by the Geiser involution $\gamma_\eta: \tilde{X}_\eta \to \tilde{X}_\eta$. As above, $\pi^b = \lambda \circ \bar{\pi} \circ \gamma$ for some birational maps $\gamma: \tilde{X} \dasharrow \tilde{X}$ and $\lambda: S^b \dasharrow S^b$.

(iii) $K^2_{X_\eta} = 3, \mathcal{L} \cdot \Delta = 5$, and the link is of type III and is induced by the contraction of a $(-1)$-curve on $\tilde{X}_\eta$. The contraction $\pi^b: X^b \to \mathbb{P}^1$ is a $\mathbb{Q}$-del Pezzo fibration of degree 4.

(iv) $K^2_{X_\eta} = 4, \mathcal{L} \cdot \Delta = 4$, $\pi^b$ is a $\mathbb{Q}$-conic bundle, and the link is of type IV. In general $\Phi$ is not induced by a birational automorphism of $\tilde{X}$.

(v) $K^2_{X_\eta} = 5, \mathcal{L} \cdot \Delta = 3$, and the link is of type III and is induced by the contraction of four conjugate $(-1)$-curves on $\tilde{X}_\eta$. Then $\pi^b: X^b \to \mathbb{P}^1$ is a generically $\mathbb{P}^2$-bundle.

(vi) $K^2_{X_\eta} = 6, \mathcal{L} \cdot \Delta = 2$, and the link is of type III and is induced by contraction of a pair of conjugate $(-1)$-curves on $\tilde{X}_\eta$. Then $\pi^b: X^b \to \mathbb{P}^1$ is a generically quadric bundle.

13.3.8. Remark. In the cases 13.3.7, (i) and (ii), we obtain a new conic bundle $\pi^b: X^b \to S^b$ which is fibrewise birationally equivalent the original one. On the other hand, by (13.1.5) we have $\lambda^b < \lambda$. In order to prove the hard part of Conjecture 1.3 we may assume that the map $X \dasharrow \mathbb{P}^3$ is chosen so that $\mu$ is minimal, and then the cases 13.3.7, (i) and (ii) do not occur.

13.3.9. Remark. By construction, in the cases 13.3.7, (i), (ii), and (iv), for the pencil $\mathcal{L}^b$ of fibres of $\alpha^b: S^b \to \mathbb{P}^1$ and for the discriminant curve $\Delta^b \subset S^b$ we have

$$\mathcal{L}^b \cdot \Delta^b = \mathcal{L} \cdot \Delta$$

(see Corollary 13.3.6).

Now we consider the easy case of Conjecture 1.3. More precisely, we assume that in the notation of the proof of Theorem 12.1 the constant $\mu$ equals 1. Note that in this notation we always have $\mu \in \frac{1}{2}\mathbb{Z}$ and $\mu > 0$, and the equality $\mu = \frac{1}{2}$ implies that $\Delta = \emptyset$ and $\pi: X \to S$ is a $\mathbb{P}^1$-bundle.
13.4. Proposition [73]. Let \( \pi: X \to S \) be a standard conic bundle with discriminant curve \( \Delta \subset S \). Assume that there exists a birational map \( \Phi: X \dashrightarrow \mathbb{P}^3 \) such that for the proper transform \( \mathcal{H} \) on \( X \) of the linear system of planes on \( \mathbb{P}^3 \) one has

\[
\mathcal{H} \equiv -K_X + \pi^* A.
\]

Then for \( \pi: X \to S \) the condition (i) or (ii) in Conjecture 1.3 is satisfied.

This was proved by Iskovskikh in [73], Theorem 2. We propose a slightly different proof below.

Proof. Clearly, we may assume that \( \Delta \neq \emptyset \). Applying Theorem 12.1, we thus have the diagram (12.1.2) and one of the diagrams in Corollary 13.1 with all the required properties. Therefore, there exists a \( \mathbb{Q} \)-conic bundle \( \tilde{\pi}: \tilde{X} \to \tilde{S} \) which is fibrewise birational to \( \pi \) and satisfies the properties in Theorem 12.1. In particular, for the proper transform \( \tilde{\mathcal{H}} \) of the linear system \( \mathcal{H} \) on \( \tilde{X} \) we have

\[
\tilde{\mathcal{H}} \equiv -K_{\tilde{X}} + \tilde{\pi}^* \tilde{A},
\]

and the pair \((\tilde{X}, \tilde{\mathcal{H}})\) is canonical (see the proof of Theorem 12.1).

Assume that we are in the situation (ii) of Theorem 12.1, that is, \( \rho(\tilde{S}) = 2 \). Hence, one of the possibilities 13.3.7, (i)–(vi) occurs. In the cases 13.3.7, (i), (ii), and (iv) the contraction \( \pi^b \) must be a \( \mathbb{Q} \)-conic bundle with non-trivial discriminant curve by Remark 13.3.9. Since \( \mu^b < \mu = 1 \), these cases do not occur. In the case 13.3.7, (iii), the contraction \( \pi^b \) must be a \( \mathbb{Q} \)-del Pezzo fibration of degree 4 with \( \mu^b < \mu = 1 \). Again, this is impossible. In the cases 13.3.7, (v) and (vi), the proper transform of \(|\tilde{\mathcal{L}}|\) on a good model is a desired pencil as required in Conjecture 1.3, (i).

Assume now that we are in the situation (i) of Theorem 12.1, that is, \( \rho(\tilde{S}) = 1 \). According to (12.1.8) we have

\[
4(-K_{\tilde{S}} + \tilde{A}) \equiv \tilde{\Delta} + \Gamma,
\]

where \( \Gamma = \tilde{\pi}_*(\mathcal{H}_{\tilde{X}})^2 \) is an effective non-zero divisor on \( \tilde{S} \), and \( \tilde{A} \) is a Cartier divisor by Lemma 10.11. Since \( \rho(\tilde{S}) = 1 \), by the Noether–Fano inequality (Theorem 4.4) the divisor \( -\tilde{A} \) is ample. If \( \tilde{S} \not\cong \mathbb{P}^2 \), \( \mathbb{P}(1,1,2) \), then \( \text{Pic}(\tilde{S}) = \mathbb{Z} \cdot K_{\tilde{S}} \) and we get a contradiction.

Consider the case \( \tilde{S} \cong \mathbb{P}(1,1,2) \). Then according to (13.4.2) the only possibility is

\[
\tilde{A} \equiv \frac{1}{2} K_{\tilde{S}} \quad \text{and} \quad \tilde{\Delta} < -2K_{\tilde{S}}.
\]

Consider the minimal resolution \( \mathbb{F}_2 \to \tilde{S} \). According to Lemma 11.9 there exists a \( \mathbb{Q} \)-conic bundle \( \pi': X' \to \mathbb{F}_2 \) which completes the map \( \mathbb{F}_2 \to \tilde{S} \) to a Sarkisov link of type I. Let \( \Delta' \subset \mathbb{F}_2 \) be the discriminant curve. We write

\[
\Delta' \sim a\Sigma + b\Lambda,
\]

where \( \Sigma \) and \( \Lambda \) are the negative section and a fibre of the ruling, respectively. Since \( \Delta < -2K_{\tilde{S}} \), we have \( b \leq 7 \). We may assume that \( a \geq 4 \) (otherwise \( |\Lambda| \) gives us the pencil desired in Conjecture 1.3, (i)). But then \( \Sigma \) is a rational component
of $\Delta'$ and $(\Delta' - \Sigma) \cdot \Sigma \leq 1$. This is impossible for a discriminant curve (see (3.8.2) and Corollary 3.9.1).

Finally, consider the case $\bar{S} \simeq \mathbb{P}^2$. We may assume that $\deg \bar{\Delta} \geq 5$. Again by (13.4.2) we have

$$\deg \bar{\Delta} + \deg \Gamma = 8$$

and $-\bar{A}$ is the class on a line $l \subset \mathbb{P}^2$. Let $\bar{H} \in \mathcal{H}_{\bar{X}}$ and let $\bar{M} \in |\bar{\pi}^*l|$ be general members of the corresponding linear systems. Thus, by (13.4.1)

$$-K_{\bar{X}} \sim \bar{H} + \bar{M}.$$ 

By the adjunction formula the divisor $-K_{\bar{M}} = \bar{H}|_{\bar{M}}$ is ample, that is, $\bar{M}$ is a (non-singular) del Pezzo surface. Since the restriction $\mathcal{H}_{\bar{X}}|_{\bar{M}}$ defines a birational map to a surface in $\mathbb{P}^3$, we have $K_{\bar{M}}^2 \geq 3$. Using the projection formula and (13.4.2), we obtain

$$3 \leq K_{\bar{M}}^2 = \bar{H}^2 \cdot \bar{M} = \deg \Gamma, \quad \deg \bar{\Delta} \leq 5.$$ 

According to our assumptions $\deg \bar{\Delta} = 5$, and then by the Noether formula

$$K_{\bar{M}}^2 = 3.$$ 

Assume that $\bar{\pi}$ is not a standard conic bundle. Then $\bar{X}$ is singular at some point of a fibre $\bar{\pi}^{-1}(o)$, $o \in \bar{S}$. Let $\pi^*: X^* \rightarrow S^*$ be a standard model of $\bar{\pi}: \bar{X} \rightarrow \bar{S}$, as in Theorem 3.12, and let $\Delta^*$ be the corresponding discriminant curve. Thus, we have a birational morphism $\alpha: S^* \rightarrow \bar{S}$ such that

$$\Delta^* \leq (\alpha^*\bar{\Delta})_{\text{red}} \quad \text{and} \quad \alpha(\Delta^*) = \bar{\Delta}$$

(see Corollary 3.3.6). Let $\mathcal{L}$ be the pencil of lines on $\bar{S} = \mathbb{P}^2$ passing through $o$ and let $\mathcal{L}^*$ be its proper transform on $S^*$. By the projection formula

$$\mathcal{L}^* \cdot \alpha^*\bar{\Delta} = \mathcal{L} \cdot \bar{\Delta} = 5.$$ 

According to Corollary 10.8 the point $o$ is singular on $\bar{\Delta}$. Let $m_o$ be the multiplicity of $\bar{\Delta}$ at $o$. Then a general member of $\mathcal{L}^*$ meets some component of $\alpha^*\bar{\Delta}$ whose coefficient equals $m_o$. Consequently,

$$\mathcal{L}^* \cdot \Delta^* \leq \mathcal{L}^* \cdot \alpha^*\bar{\Delta} - (m_o - 1) \leq 6 - m_o.$$ 

If $m_o \geq 3$, then $\mathcal{L}^* \cdot \Delta^* \leq 3$, and thus the condition (i) of Conjecture 1.3 is satisfied. Hence, we may assume that $m_o = 2$. Then by Lemma 10.10 the curves $\alpha^{-1}(o)$ and $\Delta^*$ have no common components. Again, we have

$$\mathcal{L}^* \cdot \Delta^* \leq \mathcal{L}^* \cdot \alpha^*\bar{\Delta} - m_o \leq 3,$$

and the condition (i) in Conjecture 1.3 is satisfied. Thus, we may assume that $\bar{\pi}$ is a standard conic bundle. If the corresponding double cover $\bar{\Delta}$ is defined by an odd theta-characteristic, then the variety $X$ is non-rational by Theorems 7.5 and 7.6. Then according to Proposition 8.1 we have the case (ii) of Conjecture 1.3. \qed
Proposition 13.4 shows that Conjecture 1.3 is equivalent to the following classical conjecture of Kantor. Recall that a congruence of curves in $\mathbb{P}^3$ is an irreducible two-dimensional family of curves that cover $\mathbb{P}^3$. The index of a congruence is the number of curves passing through a general point.

13.5. Conjecture (S. Kantor; cf. [89] and [118]). For any congruence of index 1 of rational curves $C$ in $\mathbb{P}^3$ of index 1 there exists a Cremona transformation $\tau: \mathbb{P}^3 \rightarrow \mathbb{P}^3$ that sends $C$ to a two-dimensional family of conics or lines.

In other words, Kantor’s conjecture can be formulated as follows.

13.5.1. Conjecture. Let $\pi: X \rightarrow S$ be a standard conic bundle. Suppose that there exists a birational map $\Phi: X \dashrightarrow \mathbb{P}^3$. Then there exists a Cremona transformation $\tau: \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that the composition $\tau \circ \Phi: X \dashrightarrow \mathbb{P}^3$ sends a general fibre either to a conic or to a line.

13.6. Proposition. Conjectures 1.3 and 13.5 are equivalent.

Proof. Let $\pi: X \rightarrow S$ be a standard conic bundle and let $\Phi: X \dashrightarrow \mathbb{P}^3$ be a birational map.

Assume that Conjecture 13.5 holds. By 13.5.1 we may replace $\Phi$ by another birational map $\Phi' = \tau \circ \Phi$ which sends a general fibre either to a conic or to a line. This means that $\mu \leq 1$ (in the notation of the proof of Theorem 12.1). Then by Proposition 13.4 the condition (i) or (ii) in Conjecture 1.3 is satisfied.

Conversely, assume that Conjecture 1.3 holds. If we are in the situation of 1.3, (ii), then in view of Proposition 8.1 and Example 3.4.3 modulo Cremona transformations we have $\mu = 1$. In the case 1.3, (i), by Theorem 5.3 we again see that the images of the fibres of $X'/S'$ are conics. □

13.7. Theorem [76]. Let $\pi: X \rightarrow S$ be a standard conic bundle with discriminant curve $\Delta \subset S$. Assume that $\Delta$ is irreducible and non-singular. If $X$ is rational, then one of the following assertions holds:

(i) $p_a(\Delta) \leq 45$;
(ii) the condition (i) in Conjecture 1.3 is satisfied;
(iii) the curve $\Delta$ is hyperelliptic, and $\pi: X \rightarrow S$ is fibrewise birationally equivalent to a standard conic bundle $\pi^*: X^* \rightarrow S^*$ with discriminant curve $\Delta^* \simeq \Delta$ such that there exists a base point free pencil $\mathcal{L}^*$ on $S^*$ with $\Delta^* \cdot \mathcal{L}^* = 4$.

Iskovskikh [76] proposed a stronger assertion. However, [76] contains serious errors and gaps in the proofs which we could not eliminate.

Proof. If $\Delta = 0$, then $\pi: X \rightarrow S$ is a locally trivial $\mathbb{P}^1$-bundle and there is nothing to prove. Assume now that $\Delta \neq \emptyset$ and let $\Phi: X \dashrightarrow \mathbb{P}^3$ be a birational map:

$$
\begin{align*}
X & \xrightarrow{-\Phi} \mathbb{P}^3 \\
\pi & \downarrow \\
S & \underset{\text{Spec } k}{\text{Spec } k}
\end{align*}
$$

(13.7.1)

Let $\mathcal{H}_X \equiv -\mu K_X + \pi^* A$ be the proper transform on $X$ of the linear system of planes on $\mathbb{P}^3$. Since $\Delta \neq \emptyset$, we have $\mu \in \mathbb{Z}$, $\mu \geq 1$, and the maps (13.7.1) cannot be
completed to a commutative diagram that induces an isomorphism of fibres over generic points. Hence the hypotheses of Theorem 12.1 are satisfied, and thus we have the diagram (12.1.2) and one of the diagrams in Corollary 13.1 with all the required properties.

The case $\mu = 1$ is a consequence of Proposition 13.4.

Assume now that the map $\Phi$ is chosen in such a way that $\mu$ has the least possible value. We can then formulate the following inductive hypothesis.

13.7.2. Conjecture 1.3 holds for all standard conic bundles over a rational surface for which there is a birational map onto $\mathbb{P}^3$ with $\mu' < \mu$.

We can make use of this hypothesis if we succeed in finding a birational selfmap of $X$ that transfers $\pi$ to another birational conic bundle structure on $X$ for which the map $\Phi$ has degree $\mu' < \mu$.

Assume that we are in the situation of Theorem 12.1, (ii). Let

$$\pi^*: X^* \to S^*$$

be a standard conic bundle that is fibrewise birational to

$$\bar{\pi}: \bar{X} \to \bar{S}$$

and such that $S^*$ dominates $\bar{S}$ (see Theorem 3.12). Let $\Delta^*$ be the corresponding discriminant curve and let $|L^*|$ be the inverse image of the pencil $\mathcal{L}$ on $S^*$. Then

$$L^* \cdot \Delta^* = \mathcal{L} \cdot \bar{\Delta}.$$ 

Let us consider the possibilities of 13.3.7 in succession.

In the cases 13.3.7, (v) and (vi), we have $L^* \cdot \Delta^* \leq 3$, so the condition (i) in Conjecture 1.3 is satisfied, as stated in the theorem.

In the cases 13.3.7, (i) and (ii), the $\mathbb{Q}$-conic bundle $\pi^b: X^b \to S^b$ is fibrewise birational to $\bar{\pi}: \bar{X} \to \bar{S}$, and for the corresponding map $X^b \dashrightarrow \mathbb{P}^3$ we have $\mu^b < \mu$ (see (13.1.5)). Therefore, by our assumption 13.7.2 Conjecture 1.3 is true for $\pi$.

It remains to consider the cases 13.3.7, (iii) and (iv). The curve $\Delta^*$ has a component whose normalization is isomorphic to $\Delta$. Since $\Delta^*$ satisfies (3.8.2) and $p_a(\Delta^*) = p_a(\Delta)$ according to (8.4.2), it must be isomorphic to $\Delta$. In the case 13.3.7, (iii) (respectively, the case 13.3.7, (iv)), on $\Delta$ and thus also on $\Delta^*$ there exists a linear series of degree 5 (respectively, 4) $q: \Delta^* \to \mathbb{P}^1$. Because of the rationality of $X^*$, we get by Theorem 7.5 that there exists a morphism $p: \Delta^* \to \mathbb{P}^1$ of degree 3 or 2.

We claim that the product morphism

$$p \times q: \Delta^* \to \Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1$$

is birational. Indeed, otherwise $q$ passes through $p$:

$$q: \Delta^* \overset{p}{\longrightarrow} \mathbb{P}^1 \overset{r}{\longrightarrow} \mathbb{P}^1$$

and $\deg q = (\deg p) \cdot (\deg r)$. This is possible only if

$$\deg q = 4, \quad \deg p = \deg r = 2.$$
In particular, the curve $\Delta^\bullet$ is hyperelliptic. Then we have the case (iii). Thus, the curve $\Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(n, m)$, where $n \leq 3$ and $m \leq 5$. Consequently,

$$p_a(\Delta^\bullet) \leq p_a(\Delta') = (n - 1)(m - 1) \leq 8,$$

that is, we are in the case (i).

Finally, if we are in the case (i) of Theorem 12.1, then according to Corollary 13.2.2 we have $p_a(\Delta) = p_a(\Delta^\bullet) \leq 45$, a contradiction. The proof of Theorem 13.7 is complete. $\Box$

13.8. We note that the inequality $p_a(\Delta) \leq 45$ in Theorem 13.7, (i), can be strengthened in many cases. For example, if $K_S^2 \leq 3$, then

$$p_a(\Delta) \leq 15$$

by Corollary 13.2.2. If $K_S^2 = 5$, then

$$p_a(\Delta) \leq 16$$

by Remark 13.2.3. In the case $K_S^2 = 9$ we have

$$\bar{S} \simeq \mathbb{P}^2 \quad \text{and} \quad \deg \bar{\Delta} \leq 11.$$

The pencil of lines passing through a sufficiently general point $s \in \bar{\Delta} \subset \mathbb{P}^2$ cuts out a one-dimensional linear series of degree $\geq 10$ on $\bar{\Delta}$. It determines a map $\bar{q}: \bar{\Delta} \to \mathbb{P}^1$ of degree $\leq 10$. Since the point $s$ is chosen sufficiently general, this linear series is not composed of a hyperelliptic nor trigonal pencil. The map $\bar{q}: \bar{\Delta} \to \mathbb{P}^1$ can be lifted to a finite map $q^*: \Delta^\bullet \to \mathbb{P}^1$ whose degree is also $\leq 10$. Hence there is a birational morphism

$$p^* \times q^*: \Delta^\bullet \to \bar{\Delta} \subset \mathbb{P}^1 \times \mathbb{P}^1,$$

where $\bar{\Delta}$ is of type $(d_1, d_2)$ with $d_1 \leq 3$ and $d_2 \leq 10$. The birationality follows from the fact that $q^*$ does not factor through a hyperelliptic nor trigonal series on $\Delta^\bullet$. Therefore,

$$p_a(\Delta^\bullet) \leq (d_1 - 1)(d_2 - 1) \leq 18.$$

14. Some related results and open problems

14.1. Birational rigidity. It is desirable to have a good criterion for rigidity in terms of the discriminant curve and the local invariants analogous to (3.10.1). Corti [45] suggested that a sufficient condition should be

$$|3K_S + \Delta| \neq \emptyset, \quad \text{that is,} \quad \text{et}(S, \Delta) \geq 3.$$

In particular, a standard conic bundle over $\mathbb{P}^2$ should be birationally rigid if the discriminant curve has degree $\geq 9$. See [27] for some supporting examples.

The proof of the following is similar to that of Theorem 13.7.

14.1.1. Proposition. Let $\pi: X \to S$ be a standard conic bundle over a rational surface with discriminant curve $\Delta \subset S$. Assume that $X$ is not birationally rigid. Then one of the following holds:

(i) $|3K_S + \Delta| = \emptyset$;

(ii) $X$ is birationally equivalent to a $\mathbb{Q}$-conic bundle $\bar{\pi}: \bar{X} \to \bar{S}$ over a Du Val del Pezzo surface, as in 13.2, and if $\bar{\Delta} \subset \bar{S}$ is the discriminant curve, then the divisor $-(4K_{\bar{S}} + \bar{\Delta})$ is ample.
14.2. Unirationality of conic bundles. Unirationality is the most complicated and delicate property of algebraic varieties. Of course one can show that a variety is unirational using an explicit geometric construction. However, at this time there are no techniques for proving non-unirationality (in a non-trivial situation).

First we discuss the unirationality of surfaces over a non-closed field $k$. A necessary condition for $k$-unirationality is the existence of a $k$-point ([105], Exercise 1.12). Any rational surface with a $k$-point and with $K_X^2 > 5$ is $k$-rational (see Theorem 5.5).

14.2.1. Observation. Let $\pi: X \to S$ be a rational curve fibration. Then $X$ is $k$-unirational if and only if $\pi$ has a rational multi-section.

Indeed, if $\sigma: S \dashrightarrow X$ is a rational multi-section and $C := \sigma(S)$, then the base change $X \times_S C \to C$ is a rational curve fibration admitting a section. Then $X \times_S C$ is $k$-rational and hence $X$ is $k$-unirational. The converse is an easy exercise, but there are some difficulties over finite fields (see [99], Lemma 2.3).

14.3. Corollary. Let $\pi: X \to B$ be a standard conic bundle over a curve. If $K_X^2 > 3$ and the surface $X$ has a $k$-point, then this surface is either $k$-rational or $k$-unirational of degree 2.

Sketch of the proof. If $K_X^2 = 3$, then according to Theorem 5.3, (i), and Observation there exists a $(-1)$-curve on $X$ which is defined over $k$ and is a double section.

Let $K_X^2 = 4$ and let $P \in X$ be a $k$-point. Let $\sigma: \tilde{X} \to X$ be its blowup and let $E$ be the exceptional divisor. The point $P$ cannot lie on a $(-2)$-curve according to Proposition 5.2, (iii). In this situation it is easy to show that the divisor $-K_{\tilde{X}}$ is nef and $(-K_{\tilde{X}})^2 = 3$. Moreover, the linear system $| -K_{\tilde{X}} - E|$ is base point free and defines a morphism $\tilde{X} \to \mathbb{P}^1$ whose generic fibre is a rational curve. Here again $E$ is a double section. \hfill $\square$

14.3.1. Theorem ([101], Theorem 7). Let $k$ be a field of characteristic $\neq 2$ and let $\pi: X \to B$ be a conic bundle over a curve. If $K_X^2 = 1$, then the surface $X$ is $k$-unirational.

The proof is based on the classification of surface conic bundles with $K_X^2 = 1$ (see Proposition 5.2). For example, if the divisor $-K_X$ is not nef, then the curve $C$ in Proposition 5.2, (i), is a rational double section. If $-K_X$ is nef (that is, $X$ is a weak del Pezzo surface) and the Bertini involution $\beta: X \to X$ does not preserve the conic bundle structure, then the image $\beta(F)$ of a fibre $F$ is a rational multisection. Thus, the only interesting case is where $-K_X$ is nef and the Bertini involution $\beta: X \to X$ acts fibrewise (see [101]).

14.3.2. Corollary. Let $k$ be a field of characteristic $\neq 2$ and let $\pi: X \to B$ be a conic bundle over a curve. Assume that $K_X^2 \geq 1$. Then the surface $X$ is $k$-unirational if and only if it has a $k$-point.

On the other hand, Iskovskikh [68] showed that in the case $k = \mathbb{R}$ any rational surface over $k$ with a conic bundle structure is $k$-unirational whenever it has a $k$-point. In particular, this implies that there are $k$-unirational conic bundles whose discriminant locus is arbitrarily large, and hence we cannot expect that
a unirationality criterion can be formulated in a form similar to Theorem 5.5. Unirationality of surface conic bundles over some ‘large’ fields was discussed in \[204\]–\[206\].

The unirationality of del Pezzo surfaces of degree \( \geq 3 \) with a \( k \)-point was proved in \[113\] and \[99\]. It is expected that a del Pezzo surface of degree 2 with a \( k \)-point is always unirational. This has been proved in many but not all cases (see \[113\], \[171\], and \[54\]). Very little is known about the unirationality of del Pezzo surfaces of degree 1.

Manin observed that the invariant

\[
H^1(\text{Gal}(\overline{k}/k), \text{Pic}(X \otimes \overline{k}))
\]

allows one to bound the degree of unirationality (\[113\], §IV.7).

We go back to our discussions of unirationality of three-dimensional varieties with conic bundle structures. As an immediate consequence of Observation , one can see also that a non-singular cubic hypersurface in \( \mathbb{P}^4 \) and a non-singular intersection of three quadrics in \( \mathbb{P}^6 \) over an algebraically closed field are unirational (because they have the structures of conic bundles with rational multisections; see Examples 3.4.2 and 3.4.4). From Corollary 14.3.2 we also obtain a sufficient condition for the unirationality of three-dimensional conic bundles over \( \mathbb{C} \).

**14.3.3. Corollary** (cf. \[116\]). Let \( \pi : X \to S \) be a standard conic bundle over a surface and let \( \Delta \subset S \) be the discriminant curve. Assume that there exists a base point free pencil \( \mathcal{L} \) of rational curves on \( S \) such that \( \Delta \cdot \mathcal{L} \leq 7 \). Then the variety \( X \) is unirational.

**14.3.4. Corollary.** Let \( \pi : X \to \mathbb{P}^2 \) be a standard conic bundle and let \( \Delta \) be the discriminant curve. If \( \deg \Delta \leq 8 \), then \( X \) is unirational.

It is likely that a conic bundle with a sufficiently large and general discriminant curve is not unirational.

**14.4. Stable rationality.** The authors of the paper \[18\] constructed a counterexample to the birational Zariski cancellation problem.

**14.4.1. Theorem** [18]. Let \( k \) be a field of characteristic \( \neq 2 \), and let \( p(x) \in k[x] \) be an irreducible separable polynomial of degree 3 with discriminant \( a \in k^* \setminus (k^*)^2 \). Then the surface \( X \) given by the affine equation

\[
y^2 - az^2 = p(x)
\]

has the property that the variety \( X \times \mathbb{P}^3 \) is \( k \)-rational but \( X \) itself is not.

The stable rationality of the surface (14.4.2) was established using the technique of torsors. The projection to \( \mathbb{A}^1_k \) induces a conic bundle structure on a suitable projective model of \( X \). Then the non-rationality of \( X \) follows from Theorem 5.5. The result in Theorem 14.4.1 was improved by Shepherd-Barron [178]: one can replace \( X \times \mathbb{P}^3 \) by \( X \times \mathbb{P}^2 \).

Over \( \mathbb{C} \), an example of a stably rational non-rational algebraic threefold \( X_0 \) can be given by the affine equation

\[
y^2 - a(t)z^2 = p(t, x)
\]
for a suitable polynomial \( p(t, x) \) and \( a(t) = \text{disc}_x p(t, x) \). The variety \( X_0 \) is birationally equivalent to a standard conic bundle \( \pi: X \to S \) over a rational surface. The corresponding discriminant curve \( \Delta \) is the union of a trigonal curve \( C \) of genus \( g \) and smooth rational curves \( L, F_1, \ldots, F_{2g+4} \), so that the singularities of \( \Delta \) are the double points
\[
\{ P_i \} = C \cap F_i \quad \text{and} \quad \{ R_i \} = L \cap F_i.
\]

We refer to the original paper [18], §3, for details.

On the other hand, it turns out that stably rational non-rational varieties are very rare. The authors of [64] applied the specialization technique of Voisin [203] developed by Totaro [191], Colliot-Thélené, and Pirutka [38], [37] to conic bundles.

**14.4.3. Theorem** ([64]; see also [26]). Let \( S \) be a non-singular projective rational surface. Let \( \mathcal{L} \) be a linear system on \( S \) whose general member is smooth and irreducible. Let \( \mathcal{M} \) be an irreducible component of the space of reduced nodal curves in \( \mathcal{L} \) together with a degree-2 étale cover. Assume that \( \mathcal{M} \) contains a cover which is non-trivial over every irreducible component of a reducible curve with smooth irreducible components. Then the standard conic bundle corresponding to a very general point of \( \mathcal{M} \) is not stably rational.

**14.4.4. Corollary.** A very general conic bundle \( \pi: X \to \mathbb{P}^2 \) with discriminant curve of degree \( \geq 6 \) is not stably rational.

For a higher-dimensional analog of this result see [1]. Stable rationality has an interesting analogue in the category of \( G \)-varieties [25], [149].

**14.5. Birationality to Calabi–Yau pairs.** We say that a pair \((Y, D)\) consisting of a projective variety \( X \) and an effective \( \mathbb{R} \)-divisor \( D \) is a log Fano pair if it has only Kawamata log terminal singularities and the divisor \(- (K_Y + D)\) is ample. If some variety \( X \) has an effective \( \mathbb{R} \)-divisor \( D \) such that \((Y, D)\) is a log Fano pair, then we say that \( Y \) is of Fano type [155]. The usual Fano varieties and toric varieties are standard examples of varieties of Fano type. It is known that any \( \mathbb{Q} \)-factorial variety of Fano type is a Mori dream space [22]. It turns out that Fano type varieties have many important properties: they are rationally connected, and this class is closed under the MMP as well as under arbitrary contractions [155].

We say that the pair \((Y, D)\), where \( D \) is an effective \( \mathbb{R} \)-divisor, is a log Calabi–Yau pair if \( K_Y + D \sim_0 0 \) and the singularities of \((Y, D)\) are log canonical. We say that \( Y \) is of Calabi–Yau type if \((Y, D)\) is a log Calabi–Yau pair for some \( D \). Finally, we say that \((Y, D)\) is a numerically log Calabi–Yau pair if \( K_X + D \equiv 0 \), where \( D \) is pseudo-effective (no restrictions are imposed on the singularities of \( Y \)). As above, one can define varieties of numerically Calabi–Yau type. According to [121], for any numerical log Calabi–Yau pair \((X, D)\) with \( D \neq 0 \) the underlying variety \( Y \) is uniruled.

It is known that log Fano varieties with bounded singularities form a bounded family [20]. Hence, the following question is natural.

**14.5.1. Question.** Under what conditions is a conic bundle \( \pi: X \to S \) of Fano type (respectively, Calabi–Yau type, numerically Calabi–Yau type)?

The following result of Kollár shows that the condition for a variety even to be of numerically Calabi–Yau type is very restrictive.
14.5.2. **Theorem** [100]. Let \( \pi: X \to \mathbb{P}^2 \) be a standard conic bundle with discriminant curve \( \Delta \subset \mathbb{P}^2 \) of degree \( \geq 19 \). Then the following are equivalent:

(i) \( X \) is birational to a variety of numerically Calabi–Yau type;

(ii) there exists a generically finite double section \( D \subset X \) with normalization \( \tau: \bar{D} \to D \) such that the branch curve of \( \pi \circ \tau: \bar{D} \to \mathbb{P}^2 \) has degree \( \leq 6 \);

(iii) \( X \) is birationally equivalent to a standard conic bundle \( \pi': X' \to \mathbb{P}^2 \) (with the same discriminant curve) such that \( |-K_{X'}| \neq 0 \).

As a consequence, one has the following theorem.

14.5.3. **Theorem** [100]. Let \( X_{d,2} \subset \mathbb{P}^2 \times \mathbb{P}^2 \) be a general hypersurface of bidegree \((d,2)\). Then \( X_{d,2} \) is not birational to a variety of numerically Calabi–Yau type for \( d > 7 \).

There are analogues of these results for surfaces over non-closed fields. A similar question for del Pezzo fibrations was studied by Krylov [107].

14.6. **Conic bundle structures on Fano threefolds.** The following question is natural.

14.6.1. **Question.** Which (smooth) Fano threefolds admit a birational conic bundle structure?

We recall ([82], §2.1) that the **Fano index** \( \iota = \iota(X) \) of an \( n \)-dimensional Fano variety \( X \) is the maximal integer such that

\[
-K_X = \iota A,
\]

where \( A \) is an (integral) divisor. The number

\[
d = d(X) := A^n
\]

is called the **degree** of \( X \). In the case where \( \iota(X) = n - 2 \) the degree is even, and hence the number

\[
g = g(X) := \frac{d(X)}{2} + 1
\]

is an integer. It is called the **genus** of \( X \). Let us summarize the known facts about conic bundle structures on Fano threefolds with Picard number \( \rho(X) = 1 \). We use the classification and terminology from [82], Table §12.2.

**Case \( \iota > 2 \).** If \( \iota = 4 \), then \( X \simeq \mathbb{P}^3 \); if \( \iota = 3 \), then \( X \simeq Q \subset \mathbb{P}^4 \) is a quadric. These varieties are rational and so have a lot of birational conic bundle structures.

**Case \( \iota = 2 \), \( d \geq 4 \).** It is known that \( d \leq 5 \). For \( d = 5 \) and 4 the Fano threefolds are rational.

**Case \( \iota = 2 \), \( d = 3 \).** Then \( X \simeq X_3 \subset \mathbb{P}^4 \) is a cubic threefold. It has a lot of conic bundle structures (see Example 3.4.2).

**Case \( \iota = 2 \), \( d = 2 \).** Then \( X = X_2 \) can be represented as a double cover \( X \to \mathbb{P}^3 \) branched over a quartic (it is a so-called **quartic double solid**). The existence of conic bundle structures on smooth varieties of this type is not known. However, if \( X \) has at least one ordinary double point, say \( P \), then the ‘projection’ from \( P \) gives a conic bundle (see Examples 3.4.7 and 6.5). Rationality questions for singular quartic double solids were studied in [35], [202], [201], [48], [33], and [203].
Case \( \iota = 2, \ d = 1 \). Then \( X = X_1 \) is a so-called double Veronese cone. The birational geometry of this variety is very rich. It was studied in a series of papers by Grinenko (see [58] and [59]). In particular, he proved that there are no conic bundle structures on \( X_1 \).

If \( \iota = 1 \), then \( g \in \{2, 3, \ldots, 10, 12\} \).

Case \( \iota = 1 \), \( g \in \{7, 9, 10, 12\} \). Then the variety \( X \) is rational.

Case \( \iota = 1 \), \( g = 8 \). The double projection from a line gives a conic bundle structure ([82], Theorem 4.3.3). Conic bundle structures exist also because \( X \) is birationally equivalent to a cubic threefold ([71], [192], [82], Theorem 4.5.8; see Example 3.4.2).

Case \( \iota = 1 \), \( g = 6 \). The existence of conic bundle structures is unknown for smooth varieties of this type. However, singular Fano threefolds with \( \iota = 1 \) and \( g = 6 \) may have conic bundle structures. For example, if such a variety \( X \) is \( \mathbb{Q} \)-factorial and has a unique node, then it can be birationally transformed to a standard conic bundle over \( \mathbb{P}^2 \) with discriminant curve a smooth sextic \( \Delta \subset \mathbb{P}^2 \) [49], [153]. If \( X \) is not \( \mathbb{Q} \)-factorial and has a unique node, then it can be transformed into a smooth quartic double solid (see above) by [50], Proposition 5.2, [160], Example 1.11, or [27], §4.4.1.

Case \( \iota = 1 \), \( g = 5 \). Then \( X \simeq X_{2,2,2} \subset \mathbb{P}^6 \) is an intersection of three quadrics. Conic bundle structures exist by Example 3.4.4.

Case \( \iota = 1 \), \( g = 4 \). Then \( X \simeq X_{2,3} \subset \mathbb{P}^5 \) is an intersection of a quadric and a cubic. A general member of the family is birationally rigid and has no conic bundle structures ([83], Chap. 3, §2, or [165], Chap. 2, §§5, 6). However, this is not known for an arbitrary smooth variety of this type.

Case \( \iota = 1 \), \( g = 3 \). Then \( X \) is either a quartic in \( \mathbb{P}^4 \) or a double cover of a quadric branched over a surface of degree 8 (\( X \) is called a double quadric). In both cases \( X \) is birationally rigid and has no conic bundle structures (see [81] and [71]).

Case \( \iota = 1 \), \( g = 2 \). Then \( X \) is a so-called sextic double solid. It is birationally superrigid and so has no conic bundle structures [71].

More examples of singular Fano threefolds with birational conic bundle structure can be found in [86], §§7.2, 7.5–7.7. In contrast to the case \( \rho(X) = 1 \) considered above, many Fano threefolds with \( \rho(X) > 1 \) have (even biregular) conic bundle structures [124]; higher-dimensional examples can be found in [109]. Note that Kollár’s method of reduction to positive characteristic [97] allows one to show that certain Fano hypersurfaces of high dimension do not admit birational conic bundle structures.

14.7. Del Pezzo fibrations. It is desirable to construct a good theory of three-dimensional del Pezzo fibrations. A certain generalization of Sarkisov’s theorem on standard models was obtained by Corti [44] (see also [98] and [111]). For del Pezzo fibrations of degree \( \geq 3 \) he proved that there exist a Gorenstein terminal model such that its fibres are reduced and irreducible. For del Pezzo fibrations of degree 2 and 1 the situation is more complicated: it turns out that a Gorenstein terminal model does not always exist, and in this case one should consider models with non-Gorenstein points of low indices.
A del Pezzo fibration of degree $\geq 5$ over $\mathbb{P}^1$ is always rational (see Theorem 5.5). Any del Pezzo fibration $\pi: X \to B$ of degree 4 has birationally a conic bundle structure (see Theorem 5.3, (i)). In the case of smooth $X$, a rationality criterion was obtained by Alexeev [2] (see also [187]). The case of del Pezzo fibrations of degree $\leq 3$ is much more complicated. There are some results on birational rigidity (we refer to [75], [162], [165], [31], [189], [57], [186], and [106]). A higher-dimensional generalization was discussed in the recent paper [166]. It would be interesting to reformulate these results in terms of degeneracy loci and local invariants. There are also results on local birational rigidity [136], [137], [163].

14.8. Extremal contractions of relative dimension one. Consider a Mori extremal contraction $\pi: X \to S$ from a smooth $n$-dimensional variety such that the generic fibre is of dimension 1. If $n \leq 3$, then $S$ is smooth and $\pi$ is a standard conic bundle [122]. The same holds in arbitrary dimension under the additional assumption that all the fibres are one-dimensional [8]. However, if we relax the equidimensionality condition, then the situation becomes more complicated even for $n = 4$.

14.9. Proposition. Let $\pi: X \to S$ be a Mori extremal contraction with $0 < \dim S < \dim X$. Assume that the variety $X$ is non-singular. Then the following hold.

(i) The variety $S$ has at worst locally factorial canonical singularities.
(ii) For any point $P \in S$ its local algebraic fundamental group is trivial.
(iii) $\text{codim} \text{Sing}(S) \geq 3$. In particular, $S$ is non-singular if $\dim S \leq 2$.
(iv) (Cf. [8], Proposition 1.3.) If $S$ has at worst quotient singularities, then it is smooth.
(v) If $\pi$ is equidimensional, then $S$ is smooth.

Proof. (i) We recall that in general the singularities of $S$ are rational ([95], Corollary 7.4). The factoriality of $S$ is proved in the same style as in [94], Lemma 5-1-5, or [104], Corollary 3.18. For the convenience of the reader we reproduce these arguments. Let $B$ be a prime Weil divisor on $S$ and let $D$ be the divisorial part of $\pi^{-1}(B)$. Then $D$ is Cartier divisor on $X$ and $\pi(D) = B$. Since $D$ does not meet a general fibre, $D = \pi^*B_0$ for some Cartier divisor on $S$ ([104], Corollary 3.17). It is clear that $B_0 \subset B$, and so $B_0 = B$. Thus, the singularities of $S$ are rational and $K_S$ is a Cartier divisor. In this case the singularities must be canonical (see [104], Corollary 5.24).

(ii) We regard $S$ as a sufficiently small neighbourhood of $o$. Assume that there is a Galois cover

$(S' \ni o') \to (S \ni o)$

which is étale over $S \setminus \text{Sing}(S)$. Thus,

$(S \ni o) = (S' \ni o')/G$

and $G$ is a finite group acting on $S'$ freely in codimension one. Consider the normalization $X'$ of the fibre product $X \times_S S'$. Then $f: X' \to X$ is a finite Galois morphism which is étale in codimension one. Hence, by purity of the branch locus $f$ is étale, and $X'$ is smooth. The projection $\pi': X' \to S'$ is a contraction such
that the divisor \(-K_{X'}\) is relatively ample. According to [156], Lemma 3.4, there exists an irreducible \(G\)-invariant rationally connected subvariety \(V \subset \pi'^{-1}(o')\). On the other hand, an action of a cyclic subgroup \(G_1 \subset G\) on a rationally connected variety always has a fixed point. Hence, the action of \(G\) on \(V\) and \(X'\) cannot be free, and thus \(G = \{1\}\).

The assertions (iii) and (iv) follow easily from (i) and (ii). Finally, for the proof of (v) we just have to note that in this case a general section \(H \subset X\) by a subspace of codimension \(\dim X - \dim S\) is smooth and the restriction \(\pi_H: H \to S\) is finite, hence \(S\) has at worst quotient singularities. □

Kachi [87] classified contractions from smooth fourfolds to threefolds. In this case the base variety is not necessarily smooth.

14.10. Equivariant conic bundles. It would be useful to develop the theory of conic bundles in the category of \(G\)-varieties [112], that is, for varieties over non-closed fields, as well as for varieties with a group action. The first step was done by Avilov [11]. He proved an analogue of Theorem 3.12. We hope that Theorems 12.2 and 12.5 and Conjectures 1.3 and 1.2 can be generalized to the equivariant case. This is important for applications to the study of Cremona groups and rationality problems (see [159], [157], [158], [140], [207], [208], [141], [161], [194], and [195]).

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