Optimal Spend Rate Estimation and Pacing for Ad Campaigns with Budgets

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Abstract

Online ad platforms offer budget management tools for advertisers that aim to maximize the number of conversions given a budget constraint. As the volume of impressions, conversion rates and prices vary over time, these budget management systems learn a spend plan (to find the optimal distribution of budget over time) and run a pacing algorithm which follows the spend plan.

This paper considers two models for impressions and competition that varies with time: a) an episodic model which exhibits stationarity in each episode, but each episode can be arbitrarily different from the next, and b) a model where the distributions of prices and values change slowly over time. We present the first learning theoretic guarantees on both the accuracy of spend plans and the resulting end-to-end budget management system. We present four main results: 1) for the episodic setting we give sample complexity bounds for the spend rate prediction problem: given $n$ samples from each episode, with high probability we have $|\hat{\rho}_e - \rho_e| \leq \tilde{O}(\frac{1}{n^{1/3}})$ where $\rho_e$ is the optimal spend rate for the episode, $\hat{\rho}_e$ is the estimate from our algorithm, 2) we extend the algorithm of Balseiro and Gur [12] to operate on varying, approximate spend rates and show that the resulting combined system of optimal spend rate estimation and online pacing algorithm for episodic settings has regret that vanishes in number of historic samples $n$ and the number of rounds $T$, 3) for non-episodic but slowly-changing distributions we show that the same approach approximates the optimal bidding strategy up to a factor dependent on the rate-of-change of the distributions and 4) we provide experiments showing that our algorithm outperforms both static spend plans and non-pacing across a wide variety of settings.

1 Introduction

Online advertising is a massive industry worth around $140 billion dollars in 2020 in the United States alone [37]. Advertisers bidding within large online platforms are usually constrained by budget, and must decide how to distribute this budget over time as the supply and demand of impressions change. For example, there are more users online during the day than at night, leading to a variable density of impressions opportunities (see e.g. Figure 1 in Liu and Hill [29]). Furthermore, users may be more likely to interact with an ad outside of working hours, leading to those impressions generating more value for advertisers (e.g. Table 2 in Liu and Hill [29]). Finally, competition for impressions may vary over the course of the day, as other advertisers may allocate more budget to high-value periods (e.g. Figures 2 and 3 in Agarwal et al. [3]).

The temporal effects have led to a variety of work on constructing spend plans for a campaign which learn how to distribute a budget over time [30, 2, 3, 28]. Generally, the approach taken in these works is twofold: first, they use some model (e.g. a high dimensional time series model) to forecast the number of impression opportunities over the course of a day. This is taken as the spend plan. Secondly, they use a pacing algorithm, which tries to match the empirical spend rate to the spend plan. Lee et al. [28] modify the bid to control spend, while Agarwal et al. [3] modify their participation probability to control spend.

There are several limitations to the above approaches. First, they model the density of impression opportunities assuming that value per user and price per user is roughly constant. Considerable evidence [29, 3] refutes this assumption, suggesting that conversion rates and prices change over time. Second, their work focuses on empirical rather than theoretical results, limiting our understanding about which settings we can predict the resulting algorithms to have good performance. This motivates the problem that we study in this paper: Can we identify non-stationary settings for which we can provably learn a spend plan that approximates the optimal distribution of budget, and where the end-to-end system provably performs well?
More formally: We study the problem of computing optimal spend plans from a learning-theoretic perspective in two settings: an episodic model, and a model in which price and value distributions change smoothly over time. For the first, we consider an advertiser with budget $B$ that participates in a sequence of $T$ single-item second-price auctions, called rounds. These auctions are divided into $E$ episodes of $\tau = \frac{T}{E}$ round each. Each episode $e \in \{E\}$ has a fixed product distribution $Q_e = F_e \times D_e$, with values $v_t \sim F_e$ for $v_t \in [0,b]$ and independently prices $p_t \sim D_e$ in $p_t \in \mathbb{R}^+$. Prices and values within an episode are i.i.d., while prices and values across episodes are independently, but not identically, distributed. Let $\rho = \frac{B}{T}$ be the average spend per round of a strategy spending budget $B$ over $T$ rounds. For all $e$, $f_e$ and $d_e$ denote the probability density functions (pdf) of distributions $F_e$ and $D_e$ respectively. Second, we consider a non-episodic setting, where all distributions are guaranteed to change smoothly: each round has a product distribution $Q_t = F_t \times D_t$ with the property that $\|F_{t+1} - F_t\|_\infty \leq \zeta$, and $\|d_{t+1} - d_t\|_\infty \leq \theta$ for all $t \in [T]$.

For both settings, we ask: First, can we accurately estimate an optimal spend allocation? Second, given an (approximately) optimal spend plan, can we implement a pacing algorithm that satisfies the budget constraint and achieves vanishing regret compared to the ex-post optimal?

1.1 Main Contributions

Our main contributions are as follows.

- **Episodic Setting.** We propose a pair of algorithms 1) ApproxSpendRate, an offline algorithm that estimates the optimal spend plan on $n$ samples, and 2) EpisodicAdaptivePacing, an online algorithm that adaptively follows the spend plan over $T$ new auctions, that jointly have regret vanishing in $n$ and $T$, compared to the best bidding strategy in hindsight. The formal statement appears as Theorem 25 in Section 4 and relies on the following additional results:

  - **Estimating Optimal Spend Plan.** In Section 3 we bound the accuracy of constructing of an optimal spend plan. We give an algorithm ApproxSpendRate, that given $n$ samples from each episode, with probability at least $1 - \frac{2E}{n}$, produces a spend plan that satisfies $|\hat{\rho}_e - \rho_e| \leq (E + 1) \cdot \tilde{O} \left( \frac{1}{n^{1/3}} \right)$ where $\rho_e$ is the optimal spend rate for the episode, $\hat{\rho}_e$ is the estimate from our algorithm and $E$ is the number of episodes.

  - **Online Pacing Algorithm on Spend Plan.** In Section 4 we then give an adaptive pacing algorithm EpisodicAdaptivePacing that takes an (approximately accurate) spend plan, and implements a bidding strategy that follows this spend plan over $T$ new auctions. The regret of this algorithm vanishes in $n$ and $T$ with respect to the best bidding strategy in hindsight.

- **Slow-moving Distributions.** In Section 5 for slow-moving distributions we learn a spend plan as if the data came from an episodic model with number of episodes $E$. The end-to-end performance achieves a constant factor approximation of the to the best bidding strategy in hindsight, where the constant factor depends on the rate at which the distributions change.

- **Experimental Evidence.** Finally, in Section 6 we present experiments where we compare the performance of our method to the Balseiro and Gur [11] algorithm (which neither estimates nor uses a spend plan as it was designed for adversarial and stationary settings). Our method compares favorably to the ex-post optimal strategy and outperforms other methods in a wide variety of settings.

1.2 Related Work

**Optimal Spend Rate Estimation.** There are number of works that aim to estimate optimal spend rates for budget pacing [20, 2, 3, 28]. Ma et al. [30] and Agarwal et al. [2] primarily focus on the on the spend plan estimation. Both of these papers aim to forecast user visits, which correlates strongly with the number of impression opportunities. They do this using time series modeling techniques for users within the targeting criteria of a campaign. These works do not attempt to estimate how conversion rates or prices for ad opportunities change over time. Lee et al. [28] and Agarwal et al. [2] combine user visit estimates with

\[ \text{Experimental Evidence} \]

\[ \text{Optimal Spend Rate Estimation} \]

\[ \text{Related Work} \]
an online pacing algorithm to match the spend rate to the user visit rate. Similar to the approaches below, Lee et al. \cite{28} uses a multiplicative shading strategy (i.e. bidding $\alpha \cdot v_t$ instead of $v_t$) to control spend, while Agarwal et al. \cite{3} participate in each auction with a parameterized probability to control spend. None of these papers give formal guarantees on the performance of the end-to-end budget management system.

**Online Algorithms for Pacing.** Work on pacing algorithms has only focused on guarantees for pacing algorithms in absence of a spend plan. In many cases, for repeated second-price auctions, the optimal pacing strategy in hindsight is a multiplicative shading strategy (i.e. bidding $\alpha \cdot v_t$ for the auction at time $t$ for a fixed $\alpha \leq 1$ that does not vary over time) \cite{36,20,24,12,16}. Balseiro and Gur \cite{12} were the first to give online learning algorithms that approximate this best response. For i.i.d. value and price distributions, they give an online algorithm with regret $O(T^{1/2})$. Similar guarantees are also shown by Balseiro et al. \cite{10} who achieve $O(T^{1/2})$ regret for stationary value and price distributions setting without assuming independence between values and prices. There are few works that give provable guarantees for non-stationary competition and values. Balseiro and Gur \cite{12} consider the case of adversarial values and prices and show that no algorithm can achieve sub-linear regret with respect to any benchmark that obtains more than $\frac{B}{T h}$ fraction of the utility obtained by using the optimal strategy with the power of hindsight (where $h$ is an upper bound on the value). They also give an algorithm which obtains the $O(T^{1/2})$ upper bound on the regret with respect to $\frac{B}{T h}$ fraction of the optimal. Balseiro et al. \cite{10} considers both an ergodic setting and a periodic setting where regret grows as $O(T^{1/2})$. Their algorithms do not construct a spend plan and instead rely on the fact that at a macro-level the expected optimal spend rate is constant. By contrast, in our setting obtaining no-regret may depend on saving enough budget for the end of the campaign (for example to reach users on the weekend for a week-long campaign). Only by explicitly constructing an approximately optimal spend plan can one give guarantees for such campaigns.

Conitzer et al. \cite{17} show that for individual first-price single-item auctions, multiplicative shading yields the Eisenberg-Gale outcome of the corresponding Fisher market (though generally multiplicative shading is not a best response in this setting). Gao et al. \cite{22} give an online learning algorithm for this setting that results in this equilibrium and can be run in a decentralized way by each advertiser individually.

While bid modification yields the an optimal strategy for a bidder, an alternative way to respect a budget constraint is to limit the number of auctions a bidder participates in. Mehta et al. \cite{32} give revenue guarantees for the online matching problem where users (in this case, impressions for sale) arrive one at a time and the auction selects a winner who pays their bid; once a bidder has exhausted their budget they will no longer be selected as a winner. Subsequently bidder selection has been applied to more general settings \cite{1,6,23,27}. Since truthful bidding is not a best response for advertisers in bidder selection mechanisms, this line of work is less directly relevant to our work.

In the previous two lines of work, advertisers know the value they have for an impression when they bid. A separate line of work considers a bandit setting, where the value is only revealed to advertisers after they win an auction. Amin et al. \cite{4} and Tran-Thanh et al. \cite{38} give theoretical guarantees for discrete value distributions. Flajolet and Jaillet \cite{21} extend these results to continuous distributions. Finally, Nuara et al. \cite{33} and Avadhanula et al. \cite{5} consider the problem of allocating budget across different channels. The different channels have different distributions and as such bear some similarity to the setting we consider. However, since all channels are simultaneous available and each channel is i.i.d., the spend rate remains constant over time (cf. our setting where spend rates change).

**Equilibrium Analysis** In addition to the work on online algorithms, there’s a growing body of work that analyzes the equilibria of pacing systems under the assumption that all advertisers use the same bid-shading approach, e.g. \cite{14,13,8,16,17,7,15}. The framework of Balseiro et al. \cite{9} studies stationary equilibria and characterize Bayesian optimal mechanisms that satisfy budget constraints.

## 2 Setting and Preliminaries

We study the problem of designing a bidding algorithm for budget-constrained advertisers in non-stationary settings. This bidding algorithm aims to maximize utility subject to a given budget constraint $B$. The
algorithm participates in a sequence of $T$ single-item second-price auctions.\footnote{Our model captures additional settings, including posted prices and second-price auctions with reserve prices, but for ease of exposition we consider second-price auctions throughout.} We refer to each auction as a round. In the following we present the notation for the episodic setting that we study, the non-episodic setting is formally introduced in Section 5.

In every round $t \in [T]$, the bidder observes a value $v_t$ for the impression opportunity and submits a bid $b_t$ to the auctioneer. Let $p_t$ be the highest competing bid for the impression opportunity. When $b_t \geq p_t$ the bidder wins, spends $p_t$, and gains utility $u_t = v_t - p_t$. Otherwise she loses, pays 0, and gains utility $u_t = 0$. The bidder’s goal is to maximize their utility subject to the sum of expenditures across $T$ rounds being at most $B$.

A strategy $\sigma$ of the bidder is a sequence of deterministic mappings $\sigma_1, \ldots, \sigma_T$ where $\sigma_t$ uses the information that is available to bidder in round $t$ to produce bid $b_t$. We focus attention on strategies that respect the budget constraint.

**Definition 1** (Budget-feasibly Strategy). $\sigma$ is budget feasible if $\sum_{t=1}^T \mathbb{1}\{b_t^\sigma \geq p_t\} p_t \leq B$ for any realized values $v = v_1, \ldots, v_T$ and prices $p = p_1, \ldots, p_T$.

A strategy’s utility is simply its total utility over $T$ rounds.

**Definition 2** (Performance of a Strategy). For a given budget feasible strategy $\sigma$, its performance on a realized sequence of values $v$ and prices $p$ is given by

$$\pi^\sigma (v; p) = \sum_{t=1}^T \mathbb{1}\{b_t^\sigma > p_t\} (v_t - p_t).$$

(1)

As benchmark, we consider the best (fractional) allocation in hindsight on the realized values $v$ and prices $p$. While the benchmark may appear to be strong, it is commonly used in the budget pacing literature.

**Definition 3** (Hindsight Strategy Benchmark). The performance of the hindsight strategy $H$ on a realized sequence of values $v$ and prices $p$ is given by

$$\pi^H (v; p) = \max_{x \in [0, 1]^T} \sum_{t=1}^T (v_t - p_t) x_t \text{ s.t. } \sum_{t=1}^T p_t x_t \leq B$$

Here $x_t \in [0, 1]$ represents a fractional allocation of impression opportunities. We measure the regret of a strategy compared to the benchmark in expectation over the values and prices. We use the notion of $\alpha$-regret proposed by \cite{CRS12}, a multiplicative notion:

**Definition 4** ($\alpha$-regret). For strategy $\sigma$ and $\alpha \in (0, 1]$, the $\alpha$-regret with respect to the hindsight strategy is:

$$\alpha\text{-REG}(\sigma; T) = \alpha\mathbb{E}[\pi^H (v, p)] - \mathbb{E}[\pi^\sigma (v, p)]$$

Where the expectation is over $(v, p)$ sampled from $\tilde{Q}$, that is, $(v_t, p_t)$ in episode $e$ is sampled from $Q_e = F_e \times D_e$. Our algorithm first constructs a spend plan prior to the $T$ auctions, using historical data. The accuracy of the spend plan will be a function of the sample size our algorithm is given.

**Definition 5** (Sample Complexity). The sample complexity of achieving a given approximation factor $1 - \epsilon$ is the minimum number of samples $m$ such that there exists an (offline) learning algorithm $A$ with the desired approximation.

Of particular interest are algorithms where both the $\alpha$-regret is sublinear in $T$, and additionally, $\alpha$ approaches 1 using a polynomial number of samples. We overload the term “vanishing regret” for such situations.

**Definition 6** (Vanishing Regret). A strategy $\sigma_m$ (which has access to $m$ samples from $\tilde{Q}$) has vanishing regret if $(1 - \epsilon)\text{-REG}(\sigma_m; T) = o(T) \text{ and } m \in O(\text{poly}(\epsilon^{-1})).$
2.1 Outline of the Solution

As mentioned previously, our algorithm first produces a spend plan from data, then uses a pacing algorithm to meet that spend plan. The former is an offline learning problem that happens before the campaign starts. The latter is an online algorithm that operates on the spend plan and realized expenditures. Before going into these components, it is informative to understand why this decomposition in a spend plan and pacing algorithm makes sense.

Why Historical Data is Needed. Balseiro et al. [9] have studied pacing for non-stationary distributions without using historical data. Could the episodic setting that we’re studying be amenable to positive results without historical data too? Unfortunately this is not the case. The following is an example with two episodes without using historical data. Could the episodic setting that we're studying be amenable to positive results?

Example 7. Consider two instances of the episodic setting characterized by the episodic distributions, \( I = (Q_1, Q_2) \) and \( I' = (Q_1', Q_2') \) (where \( Q = F \times D \)). All the distributions consist of a single atom: prices distributions \( D_1 = D_2 = D_2' \) and yield 1 with probability 1. The value generated by \( F_1 \) is 2, by \( Q_2 \) is 1 and by \( Q_2' \) is 3.

Consider a buyer with budget \( B = \frac{1}{2}T \), thus they can buy precisely half the impression opportunities. In both instances, the first episode yields utility \( 2 - 1 = 1 \) per round that is won, but the second episode differs for the two instances. For \( I \), the per-round utility when the bidder wins is \( 1 - 1 = 0 \), while for \( I' \) it is \( 3 - 1 = 2 \).

So if the bidder faces the first instance, she needs to win all but a sublinear (in \( T \)) number of rounds in episode 1 for vanishing regret, but if she faces the second instance she may win at most a sublinear number of rounds in the first episode. Since she doesn’t know which instance she faces until she enters episode 2, any strategy must incur \( \Omega(T) \) regret on at least one of \( I, I' \).

The example above can be generalized to a stronger result for instances with more episodes and that includes randomized algorithms.

Lemma 8. Any strategy \( \sigma \) that only depends on the history \( H_t = (v_i, b_i, p_i)_{i=1}^{t-1} \cup v^t \) in round \( t \), for a large enough \( T \), and budget \( B \) such that \( 0 < \rho = \frac{B}{T} < h \), for any number of episodes \( E \), for any \( \epsilon \) such that \( 1 - \epsilon > \max\{ \frac{\rho}{E}, \frac{1}{E} \} \), there exists an instance of the episodic setting with distributions \( \hat{Q} = (Q_1, \ldots, Q_E) \) such that

\[
(1 - \epsilon)\text{-REG}(\sigma; T) \geq \Omega(T).
\]

Proof. In an adversarial setting where the values and prices are arbitrary, in each round \( t \), Balseiro and Gur [12] Theorem 1 show that for any strategy \( \sigma \) such that \( \sigma_t \) depends only on the history \( H_t \), for a large enough \( T \), for any budget \( B \) satisfying \( \rho = \frac{B}{T} < h \) where \( h \) is the upper bound on the values, for any \( \epsilon \) such that \( (1 - \epsilon) > \frac{\rho}{E} \), there exists adversarial values \( u \) and \( p \) such that \((1 - \epsilon)\pi^u (u; p) - \pi^v (v; p) \geq \Omega(T) \).

Note that if \( \rho \geq h \) then truthful bidding is feasible and achieves the optimal utility. When \( \rho < h \), then the above result says that there exists a barrier of \( \frac{\rho}{E} \) such that for any strategy \( \sigma \) that only depends on the history, there always exists an instance where \( \sigma \) cannot obtain better than \( \frac{\rho}{E} \) fraction of the optimal utility. In other words, if the budget constraint is active, the lower the budget, the smaller the fraction of the optimal budget constrained utility the advertiser can hope to attain.

Since we don’t make any assumptions about how distributions \( Q_e \) are related across the episodes, we can extend the analysis and the adversarial case example of Balseiro and Gur [12] Theorem 1 to work in the episodic setting and show that similar lower bounds can be shown for such strategies in our setting as well. Specifically, in the proof of Balseiro and Gur [12] Theorem 1, the adversarial example has the value \( v_i \) fixed as \( h \) for all \( T \) rounds, and the price profile is samples from a distribution such that the \( T \) round auction is divided into \( m \) episodes. When \( \frac{1}{T} \leq \rho = \frac{B}{T} < h \), then by setting \( m = E \) in the proof for Balseiro and Gur [12] Theorem 1, we can recover the guarantee that there exists an instance such that \( \epsilon\text{-REG}(\sigma; T) \geq \Omega(T) \) for \( 1 - \epsilon > \frac{\rho}{E} \). If \( 0 < \rho < \frac{\rho}{E} \), then the complete example is scaled down by replacing \( h \) with \( h' = \rho E \). Note that since \( \rho < \frac{1}{E} \), we have that \( h' < h \), thus the example is valid. We also get that \( \frac{h'}{E} \leq \rho = \frac{B}{T} < h' \). Thus, we obtain that \( \epsilon\text{-REG}(\sigma; T) \geq \Omega(T) \) for \( 1 - \epsilon \geq \frac{\rho}{E} = \frac{1}{E} \).

Since algorithms in the episodic setting that only operate on the immediate history fail to have vanishing regret, we use access to historical data in the form of samples from the distribution.
**Why Spend Plans are Needed.** With access to samples from the distribution, one could still attempt to design an algorithm that does not involve a spend plan. Recall from the related work that ex-post the optimal bidding strategy is to bid $\beta^* \cdot v_t$ for some constant $\beta^*$. So what if we used samples to estimate this $\beta$ and used this directly? The following lemma shows that this yields linear regret with constant probability.

**Lemma 9.** There exists an instance of the episodic setting with distributions $\bar{Q} = (Q_1, Q_2)$ such that the ex-ante optimal pacing multiplier $\beta^*$ incurs $O(T)$ regret with probability at least $\frac{1}{5}$.

**Proof.** Let $Q_1 = F_1 \times D_1$ and $Q_2 = F_2 \times D_2$ such that the prices drawn from $D_1$ in episode 1 are equal to some positive constant $p_{\text{high}}$ and the prices drawn from $D_2$ in episode 2 is equal to $p_{\text{low}}$ such $p_{\text{low}} = \frac{p_{\text{high}}}{2}$. The values drawn from $F_1$ are 0 with probability $\frac{\tau}{\tau+1}$ and are equal to $p_{\text{high}} + v_{\text{low}}$ with the remaining probability $\frac{1}{\tau}$ for some constant $v_{\text{low}}$. The values drawn from $F_2$ in the second episode is always equal to $p_{\text{low}} + v_{\text{high}}$ such that $v_{\text{high}} > v_{\text{low}}$.

Consider the episodic setting defined by distribution $\bar{Q} = (Q_1, Q_2)$ and budget $B = 2p_{\text{high}}$. In episode 1, most of the impressions have no value but with a small probability, we might observe high value and expensive impressions. In the episode 2, every impression is cheap and high value.

The corresponding ex-ante optimal pacing multiplier $\beta^*$ in this case is $\frac{p_{\text{high}}}{p_{\text{high}} + v_{\text{low}}}$. The pacing strategy derived from $\beta^*$ always bids $b_t = \beta^* v_t$. In expectation, $\sigma^{\beta^*}$ wins all impressions in episode 2 each with utility $v_{\text{high}}$, and wins 1 impression of utility $v_{\text{high}}$ resulting in expected utility of $\frac{T}{2} v_{\text{high}} + v_{\text{low}}$.

Although the expected utility is high, but in the realized case, if more than one impression of price $p_{\text{high}} + v_{\text{low}}$ and price $p_{\text{high}}$ is present in episode 1, then the fixed pacing strategy wins both these impressions with depletes the complete budget, leaving no budget for episode 2 and a total utility $\pi^{\beta^*} (v, p) = 2v_{\text{low}}$.

When compared to the Hindsight strategy $H$, the hindsight strategy always ends up buying all the impressions in second episode and thus gets utility $\pi^H (v, p) \geq \frac{T}{2} v_{\text{high}}$.

Thus, if at least two impressions with positive utility appear in episode 1, the regret $\pi^H (v, p) - \pi^{\beta^*} (v, p)$ is equal to $\frac{T}{2} v_{\text{high}} - 2v_{\text{low}}$ which is $O(T)$. The probability that at least two positive utility impressions appear in episode 1 is $1 - 2(\frac{\tau}{\tau+1})^{\tau-1}$ which is greater than $\frac{1}{5}$ for $\tau \geq 3$.

This implies that with probability at least $\frac{1}{5}$, $\pi^H (v, p) - \pi^{\beta^*} (v, p) = O(T)$.

Instead we construct an intermediate spend plan and combine this with an adaptive pacing algorithm; an approach we outline next.

**Outline of the Solution using Spend Plans.** To understand the optimal spend plan, we first introduce the notation of spend functions which represent the expected expenditure of strategies that shade by a fixed shading multiplier.

**Definition 10 (Spend Function).** Consider a fixed pacing strategy $\sigma_\mu$ that always bids $b_t = \frac{1}{1+\mu} v_t$ and is not restricted by any budget constraints. The expected expenditure of $\sigma_\mu$ in a single round in episode $e$ is

$$\overline{G}_e(\mu) = E_{v \sim F_e, p \sim D_e} [\mathbb{1}\{v \geq (1+\mu)p\} \cdot p]$$

$$= \int_0^h (1 - F_e((1+\mu)p)) \cdot p \cdot d_e(p) dp.$$  

**Definition 11 (Optimal Spend Rates).** Given an episodic setting where the ex-post optimal bidding strategy is to bid $\beta^* \cdot v_t$, we define $\rho = \rho_1, \cdots, \rho_E$ as optimal spend rates if for all $e$, $\rho_e = \overline{G}_e(\mu^*)$, where $\frac{1}{1+\mu^*} = \beta^*$.

In simpler words, the optimal spend plan is characterized by the optimal spend rates $\rho = \rho_1, \cdots, \rho_E$, such that the $\rho_e$ is equal to expected expenditure of the ex post optimal bidding strategy in a single round in episode $e$. We define the dual of the expectation of the optimization problem in Definition 3 as

$$\Psi(\mu) = E_{v, p} \left[ \sum_{t=1}^T (v_t - (1+\mu)p_t)^+ + \mu B \right].$$
The ex-post optimal bidding strategy $\sigma_{\mu^*}$ bids $\frac{m}{1+\mu^*}$ where $\mu^*$ is the dual minimizer ($\Psi(\mu^*) = \inf_{\mu \geq 0} \Psi(\mu)$), which spends the complete budget in expectation:

$$
\tau \sum_{e=1}^{E} G_e(\mu^*) = \tau \sum_{e=1}^{E} \rho_e = B,
$$

where $\rho_1, \cdots, \rho_E$ are the optimal spend rates. Refer to Appendix [A] for a detailed exposition about the characterization of the optimal spend rates through the dual of the problem. Knowing the optimal spend rates can help decompose the entire campaign into smaller budget constrained campaigns for each episode where the distributions of values and prices remain stationary. The exact formulation of optimal spend rates requires complete knowledge of the distributions $\bar{Q} = (Q_1, \cdots, Q_E)$ which is not available, instead we have access to historic samples from the distribution $\hat{Q}$. This is reasonable to assume as we are designing this framework for large online ad exchanges which usually have access to a lot of historical data. Our solution is a two step pipeline:

1. **Approximate optimal spend rates**: Use historical samples from $\hat{Q}$ to approximate optimal spend rates $\rho = \rho_1, \cdots, \rho_E$ as $\hat{\rho} = \hat{\rho}_1, \cdots, \hat{\rho}_E$.

2. **Adaptive pacing with spend rates**: Use the approximate spend rates to construct an online pacing algorithm that runs on realized impressions.

### 2.2 Preliminaries

We will use some results on uniform convergence and pacing for i.i.d. settings in this paper.

#### 2.2.1 Dvoretzky-Kiefer-Wolfowitz (DKW) Inequality

The Dvoretzky-Kiefer-Wolfowitz (DKW) inequality [19, 31] gives a uniform convergence bound on the empirical cumulative distribution function.

**Lemma 12 (DKW Inequality).** Given $n$ samples $X_1, X_2, \ldots, X_n$ from a distribution $F$. The empirical cdf on the samples is given by $\hat{F}(x) = \frac{1}{n} \sum_{i=1}^{n} 1 \{X_i < x\}$. With probability at least $1 - \delta$

$$
\|F - \hat{F}\|_{\infty} \leq \sqrt{\frac{\log \frac{2}{\delta}}{2n}}.
$$

#### 2.2.2 Kernel Density Estimation

While the DKW inequality [12] gives strong uniform convergence bounds on the cdf of a distribution, bounding the probability density function (pdf) of a distribution is more challenging. A common approach to do this is to use Kernel Density Estimation (KDE) [18, 35]. Let $D$ be the distribution with pdf $d$ that we want to estimate as $\hat{d}$. Formally the Kernel Density Estimation is defined below for scalar distributions which we use in our setting.

**Definition 13 (Kernel Density Estimation).** Given a kernel $K$, scalar $s$, and $n$ samples $X_1, \cdots, X_n$ from the distribution $D$, the KDE is given by

$$
\hat{d}(x) = \frac{1}{n \cdot s} \sum_{i=1}^{n} K\left(\frac{x - X_i}{s}\right).
$$

While most results on KDE are for bounding the mean squared error ($\mathbf{E}_d[(\hat{d} - d)^2]$), recent work by Jiang [25] gives a uniform convergence guarantee for KDE. We present a simplified version of their result below.

**Lemma 14 ([25]).** If $d$ is Lipschitz and bounded i.e. there exists a constant $C_1$ such that $|d(x) - d(x')| \leq C_1 |x - x'|$ for $x, x' \in \mathbb{R}$ and $\|d\|_{\infty} \leq C_2$ for some constant $C_2$, then there exists a constant $C''$ (that depends
on $K$, $C_1$, $C_2$, and some other constants), such that with probability at least $1 - 1/n$ and by setting $s = n^{-1/3}$, the kernel density estimate $\hat{d}$ satisfies that

$$\|\hat{d} - d\|_\infty \leq C' \left(s + \sqrt{\log n / ns}\right) = \widetilde{O}\left(\frac{1}{n^{1/3}}\right).$$

provided that $K$ is spherically symmetric, non-increasing, and has exponential decay (i.e. $K(x) = k(|x|)$ where $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a non decreasing function s.t. for all $u > u_\eta$, $k(u) \leq C_\eta \exp(-u^\eta)$ for some fixed $\eta$, $C_\eta$, and $u_\eta$)

A number of popular kernel choices such that Gaussian, exponential, uniform, and many more satisfy the requirements of lemma 14. While the Lipschitz requirement appears strong, a large number of common distributions such as the normal distribution, Cauchy distribution, exponential distributions and lognormal distributions have Lipschitz and bounded pdfs.

### 2.2.3 Balseiro-Gur Pacing Algorithm

Consider a setting with just one episode such that the values and prices in every round $t$ are sampled from fixed stationary distributions $F$ and $D$. Balseiro and Gur [12] give an adaptive pacing algorithm based on minimizing the dual $\Psi(\mu)$. In every round $t$, the algorithm bids $\frac{n}{t^{1/2}}$ and the pacing parameter $\mu_t$ is updated using a projected gradient descent style update in the direction that minimizes the dual.

**Lemma 15 ([12]).** If the value and prices in each round are samples from a stationary distribution such that $\Psi(\mu)$ is thrice differentiable in $\mu$ with bounded gradients and is strongly convex, then using the Adaptive Pacing algorithm from Balseiro and Gur [12] with $\eta = O(T^{-1/2})$ results in strategy $A$ with

$$E[\pi^H(v,p)] - E[\pi^\sigma(v;p)] \leq O(\sqrt{T}).$$

### 3 Approximating Optimal Spend Rates

We first turn our attention to estimating optimal spend plans in the episodic setting. Given $n$ samples from $\tilde{Q}$, we will divide our budget $B$ across $E$ episodes by estimating target spend rates ($\rho_1, \ldots, \rho_E$) that approximate the optimal spend rates $\rho_1, \ldots, \rho_E$ additively. The main theorem we’ll prove in this section is the following.

**Theorem 16.** Given $B$, $T$, $E$, sample oracles $F_e$ and $D_e$, where $d_e$ is Lipschitz and bounded, sampling budget $n$, $K$, setting $s = O(n^{-1/3})$, w.p. $1 - \frac{2E}{n}$, for each episode $e$, $\text{ApproxSpendRate}$ returns $\hat{\rho}_e$ s.t.

$$|\hat{\rho}_e - \rho_e| \leq (E + 1) \cdot \widetilde{O}\left(\frac{1}{n^{1/3}}\right).$$

$\text{ApproxSpendRate}$ (Algorithm 1) is based on the fact that the ex-post optimal bidding strategy spends the complete budget in expectation. The resulting algorithm consists of three main steps: i) use historical samples to approximate spend functions $\overline{G}_e(\mu)$ for each episode as $\overline{G}_e(\mu)$, ii) use eq. 5 to approximate $\mu^*$ as $\hat{\mu}$, and iii) estimate the expected spend per round for each episode using the approximate spend functions and $\hat{\mu}$.

Before we discuss how to approximate the spend functions $\overline{G}_e(\mu)$, in Lemma 17 we show that a good approximation of $\overline{G}_e(\mu)$ allows for a good approximation of the optimal spend rates $\rho_1, \ldots, \rho_E$.

**Lemma 17.** In Algorithm 1, for each episode $e$ if the estimated episodic spend function $\overline{G}_e$ obtained the end of Line 15 satisfies $\|\overline{G}_e - G_e\|_\infty \leq \gamma$, then for each $e$, the algorithm returns spend rate $\hat{\rho}_e$ such that $|\hat{\rho}_e - \rho_e| \leq (E + 1)\gamma$.

**Proof.** The proof progresses in two steps: First, we show that the episodic guarantee in the premise of the lemma yields a bound for the overall spend function. Next, we show that the approximate spend functions when evaluated on $\hat{\mu}^*$ yield provable bounds on the resulting episodic spend rates, where $\hat{\mu}^*$ is the optimal pacing parameter learned using the estimated overall spend function.
Where eq. (10) and eq. (11) follow from the monotonicity of $G$ and $\bar{G}_e$. Similarly, using the other direction for the case $\mu^* \geq \hat{\mu}$, we get that for every $e$,

$$|\bar{G}_e(\hat{\mu}) - \bar{G}_e(\mu^*)| \leq E \cdot |\bar{G}(\hat{\mu}) - \bar{G}(\mu^*)| \leq E \cdot \gamma$$
Now consider \( |\hat{\rho}_e - \rho_e| \) for some \( e \),
\[
|\hat{\rho}_e - \rho_e| = \left| \overline{G}_e(\hat{\mu}) - \overline{G}_e(\mu^*) \right|
= \left| \overline{G}_e(\mu^*) + \overline{G}_e(\mu) - \overline{G}_e(\mu^*) \right|
\leq \left| \overline{G}_e(\mu^*) - \overline{G}_e(\mu) \right| + \left| \overline{G}_e(\mu) - \overline{G}_e(\mu^*) \right|
\leq \gamma + E \cdot \gamma
\]

Thus, for all \( e \), it holds that \( |\hat{\rho}_e - \rho_e| \leq (E + 1)\gamma \).

### 3.1 Approximating spend functions

Recall from Definition 10 that for an episode \( e \) with value distribution \( F_e \) and price distribution \( D_e \),
\[
\overline{G}_e(\mu) = \int_0^h (1 - F_e((1 + \mu)p)) \cdot p \cdot d_e(p) \ dp.
\]

This implies that if we can approximate \( F_e \) and \( d_e \), we can use eq. (3) to approximate \( \overline{G}_e(\mu) \). In algorithm 2, we use the empirical estimate \( \hat{F}_e \) of \( F_e \), and use Kernel Density Estimation to approximate \( d_e \) as \( \hat{d}_e \).

**Algorithm 2: ApproxSpendSP:** Stochastic prices.

1. **Input:** \((V_1, \ldots, V_n)\): values samples, \((P_1, \ldots, P_n)\): price samples, Kernel function \( K \), scalar \( s \)
2. \( \hat{d}(p) \leftarrow \frac{1}{n \cdot s} \sum_{i=1}^n K \left( \frac{p - P_i}{s} \right) \)
3. \( \hat{F}(v) \leftarrow \frac{1}{n} \sum_{i=1}^n \mathbb{1} \{ V_i < v \} \)
4. \( \hat{G}(\mu) \leftarrow \int_0^h p(1 - \hat{F}((1 + \mu)p))\hat{d}(p) \)
5. return \( \hat{G}(\mu) \)

The estimate of the spend function satisfies the following uniform convergence bound.

**Lemma 18.** Given \( n \) samples from \( F_e \) and \( D_e \) (where \( d_e \) is Lipschitz and bounded), setting \( s = O(n^{-1/3}) \)

\( \text{ApproxSpendSP} \) (Algorithm 2) returns the approximate episodic spend function \( \hat{G}_e \) such that with probability at least \( 1 - 2/n \) it holds that

\[
\| \hat{G}_e - \overline{G}_e \|_{\infty} \leq O \left( \frac{1}{n^{1/3}} \right).
\]

**Proof.** Consider any \( \mu \geq 0 \), with probability at least \( 1 - \frac{2E}{n} \),
\[
\left| \hat{G}_e(\mu) - \overline{G}_e(\mu) \right| = \left| \int_0^h (1 - F_e((1 + \mu)p)) \cdot p \cdot d_e(p) \ dp - \int_0^h (1 - \hat{F}_e((1 + \mu)p)) \cdot p \cdot \hat{d}(p) \ dp \right|
\leq \int_0^h \left| (1 - F_e((1 + \mu)p)) \cdot d_e(p) - (1 - \hat{F}_e((1 + \mu)p)) \hat{d}_e(p) \right| \cdot p \ dp
\leq \int_0^h \left( |d_e(p) - \hat{d}_e(p)| + \left| \hat{F}_e((1 + \mu)p)\hat{d}_e(p) - F_e((1 + \mu)p)d_e(p) \right| \right) \cdot p \ dp
\leq \int_0^h \left( \hat{O} \left( \frac{1}{n^{1/3}} \right) + \hat{O} \left( \frac{1}{n^{1/3}} \right) + \hat{O} \left( \frac{1}{n^{1/3}} \right) + \hat{O} \left( \frac{1}{n^{1/3}} \right) \right) \cdot p \ dp
= h \cdot \hat{O} \left( \frac{1}{n^{1/3}} \right)
\]

where the first and second inequality follow from triangle inequality. For the third step, we use the PDF and CDF concentration bounds, and the fact for any \( 0 \leq a, b, c, d \leq 1 \), \( |ab - cd| \leq |a - c| + |d - b| + |(c - a)(d - b)| \).

Here \( \hat{O} \) notation hides the polylog\((n)\) terms along with constants like \( h, C' \) from lemma 14 and \( \|d_e\|_{\infty} \).
Combining the results of results of Lemma 17 and Lemma 18 completes the proof of Theorem 16. 

Theorem 16 implies that using \( n \) historical samples, we can approximate the optimal spend rates up to an additive factor that goes down at the rate of \( \tilde{O}(n^{-1/3}) \). In section 3.2, we show that in a simpler setting with constant prices, we can obtain a tighter error bound that goes down at the rate of \( \tilde{O}(n^{-1/2}) \).

3.2 Tighter results for Constant Prices

We consider a simpler setting where within an episode the price per impression is fixed as \( p \) and only the value is sampled from distribution \( F_e \). For the setting where all prices in episode \( e \) are \( p \), the spend function (Definition 10) simplifies to:

\[
G_e(\mu) = (1 - F_e((1 + \mu)p)) \cdot p.
\]

To estimate \( G_e(\mu) \) we only need to estimate \( F_e((1 + \mu)p) \); we give the procedure \texttt{ApproxSpendFP} in Algorithm 3.

The concentration guarantees for \( G_e(\mu) \) follow from a straightforward application of the DKW inequality (Lemma 12).

Algorithm 3: \texttt{ApproxSpendFP}: Approximate spend for constant prices.

1. **Input:** Number of samples \( n \), \((V_1, \ldots, V_n)\): values samples, \( p \): price of each impression
2. **Goal:** Estimate \( G(\mu) = \mathbb{E}_{v \sim F_e}[1 \{ v \geq (1 + \mu)p \}] \)
3. \( \hat{F}(v) = \frac{1}{n} \sum_{i=1}^{n} 1 \{ V_n < v \} \quad \) // Empirical cdf estimate for values
4. \( \hat{G}(\mu) = p(1 - \hat{F}((1 + \mu)p)) \quad \) // Estimate spend function
5. return \( \hat{G}(\mu) \)

**Lemma 19.** Given \( n \) value samples from \( F_e \) and price \( p \), \texttt{ApproxSpendFP} (Algorithm 3) returns the approximate episodic spend function \( \tilde{G}_e \) such that with probability at least \( 1 - \alpha \) it holds that

\[
\| \hat{G}_e - \tilde{G}_e \|_\infty \leq p \sqrt{\frac{\log 2}{2n}}.
\]

**Proof.** Using the DKW inequality (Lemma 12), with probability at least \( 1 - \alpha \), we have \( \| \hat{F}_e - F_e \|_\infty \leq \sqrt{\frac{\log 2}{2n}}. \)

Consider any \( \mu \geq 0 \), we have

\[
| \hat{G}_e(\mu) - \tilde{G}_e(\mu) | = | (1 - \hat{F}_e((1 + \mu)p)) - 1 + F_e((1 + \mu)p)) \cdot p |
= p \cdot | \hat{F}_e((1 + \mu)p) - F_e((1 + \mu)p) |
\leq p \cdot \sqrt{\frac{\log 2}{2n}}.
\]

Combining the results of results of lemma 19 and Lemma 17 we can show a tighter analogue of theorem 16 for the constant price setting.

**Theorem 20.** Given an episodic setting with fixed prices \( p \) and parameters \( B, T, E \), sampling oracles \( F_e \), sampling budget \( n \), \( K \), with probability at least \( 1 - \delta \), for each episode \( e \), by replacing \texttt{ApproxSpendSP} (algorithm 2) with \texttt{ApproxSpendFP} (algorithm 3) (at Line 7), \texttt{ApproxSpendRate} (algorithm 1) returns spend rate \( \hat{\rho}_e \) such that:

\[
| \hat{\rho}_e - \rho_e | \leq (E + 1)p \cdot \sqrt{\frac{2E}{2n}}.
\]
4 Pacing using Approximate Spend Rates

Now that we have learned the spend rates, in this section we show how we can adapt the Adaptive Pacing Algorithm of [12] to work with changing spend rates \( \rho'_1, \ldots, \rho'_E \) which approximate the optimal spend rates. The main idea is that using our learned spend rates, we can efficiently divide the budget across the episodes and then within each episode, we work with the budget assigned to us, and use the adaptive pacing algorithm of Balseiro and Gur [12] as subroutine. We present this algorithm as EpisodicAdaptivePacing (Algorithm 4), a detailed version of which appears as Algorithm 7 in Appendix B.

Algorithm 4: EpisodicAdaptivePacing

1. **Input:** Budget \( B \), rounds \( T \), episodes \( E \), spend plan \( (\rho'_1, \ldots, \rho'_E) \), step size \( \eta \), max shading \( \hat{\mu} \).
2. \( \mu_t \leftarrow (0, \hat{\mu}] \), BUDGET\(_1 \leftarrow B \), \( \tau \leftarrow \frac{T}{E} \), \( \hat{B}_1 \leftarrow \rho'_1 \cdot \tau \)
3. **for** \( t = 1, \ldots, T \) **do**
   4. \( e \leftarrow \lfloor t/E \rfloor \)
   5. Observe value \( v_t \)
   6. Post bid \( b_t \leftarrow \min \{ \frac{v_t}{1+\mu_t}, \hat{B}_e, \text{BUDGET}_t \} \)
   7. Observe expenditure \( z_t \)
   8. \( \mu_{t+1} \leftarrow \text{PROJ}_{[0,\hat{\mu}]}[\mu_t - \eta(\rho'_e - z_t)] \)
   9. \( \hat{B}_e \leftarrow \hat{B}_e - z_t \), BUDGET\(_{t+1} \leftarrow \text{BUDGET}_t - z_t \)
   10. **if** \( t \mod E = 0 \) **then**
      11. \( \hat{B}_{e+1} \leftarrow \rho'_{e+1} \cdot \tau + \hat{B}_e \)
   12. **end**
13. **end**

At the beginning of the campaign, we instantiate an overall budget \( \text{BUDGET} \) as the total budget of the campaign and an episodic budget \( \hat{B}_1 \) for the first episode. The budget for each episode is limited ahead of time and if algorithm runs out of the episodic budget \( \hat{B}_e \), then it cannot buy more item in this episode, even though it may have leftover budget for the whole campaign. The intuition behind this is that the budget assigned to each episode is based on the (approximation of) the optimal spend rate. If there is left over budget after an episode ends, then the budget is simply carried forward to the next episode.

In each episode, the adaptive pacing algorithm tries to match the spend in each round to target spend rate of that round. Intuitively the algorithm works by taking the equivalent of a Stochastic Gradient Descent step in the direction of the negative of the gradient of the Lagrangian of that episode. Note that here the Lagrangian dual /average Lagrangian dual for each episode is different as is characterised by the budget for that episode. We can now show that if the spend rate estimates are good, then the resulting strategy has vanishing regret.

Definition 21 (Admissible Distributions). Joint distribution \( \bar{Q} \) such that the dual function \( \Psi_e(\mu, B_e) = E_{(v,p) \sim \bar{Q}_e} \left[ \tau (v - (1 + \mu)p)^+ + \mu B_e \right] \) is thrice differentiable in \( \mu \) for all \( e \) and \( B_e \) with bounded gradients and is strongly convex, where price distribution \( D_e \) is atomic with all mass on \( p \), or \( d_e \) is Lipschitz and bounded.

Lemma 22. If the spend rates used by Algorithm [4] satisfy \( \rho_e \geq \rho'_e \geq (1 - \omega)\rho_e \), with parameters \( B, T, E \) resulting in strategy \( A \), and \( \bar{Q} \) satisfies Definition 21 where \( \rho_1, \ldots, \rho_E \) are the optimal spend rates, then setting \( \epsilon = \omega \), we have

\[
(1 - \epsilon) - \text{Reg}(A; T) \leq \tilde{O} \left( \sqrt{ET} \right).
\]

To prove the lemma, we need the following additional result:

Lemma 23. If \( \rho_e \geq \rho'_e \geq (1 - \omega)\rho_e \), it holds that

\[
\inf_{\mu \geq 0} \Psi_e(\mu, \tau \rho'_e) \geq (1 - \omega) \inf_{\mu \geq 0} \Psi_e(\mu, \tau \rho_e).
\]

where \( \Psi_e(\mu, B_e) = E_{(v,p) \sim \bar{Q}_e} \left[ \tau (v - (1 + \mu)p)^+ + \mu B_e \right] \).
Proof. Let the optimizer of \( \Psi_e(\mu, \tau \rho_e) \) be \( \mu' \). We know that the optimizer of \( \Psi_e(\mu, \tau \rho_e) \) is \( \mu^* \). Note that since \( \rho_e > \rho_e' \), using monotonicity of the spend functions, \( \mu^* < \mu' \). Consider

\[
\Psi_e(\mu', \tau \rho'_e) - (1 - \omega)\Psi_e(\mu^*, \tau \rho_e)
\]

\[
= \tau E_{(v,p)}(v - (1 + \mu')p)^+ + \mu' \rho_e' - (1 - \omega) (v - (1 + \mu^*)p)^+ - (1 - \omega) \mu^* \rho_e
\]

\[
= \tau E_{(v,p)}(v - (1 + \mu')p)^+ + (v - (1 + \mu')p)^+ - (v - (1 + \mu^*)p)^+ + \omega (v - (1 + \mu^*)p)^+
\]

\[
= \tau E_{(v,p)}(\mu' \rho_e' - \mu^* (1 - \omega) \rho_e) - \mathbb{1} \{ \mu^* p \leq v - p \leq \mu' p \} (v - (1 + \mu^*)p) + \omega (v - (1 + \mu^*)p)^+
\]

\[
\geq 0
\]

\( \square \)

Proof of Lemma 23. Let’s assume we divide the budget \( B \) into budgets \( B_e \) for all episodes \( e \in [E] \). This results in an online budget constraint bid pacing problem for each individual episode. Let \( \Psi_e(\mu, B_e) = E_{(v,p)}(v - (1 + \mu')p)^+ + \mu B_e \) denote the episodic dual function for episode \( e \) when the budget for episode \( e \) is \( B_e \). Similar to spend functions, if the budget allocation is optimal, that is \( B_e = \tau \rho_e \), we can decompose the dual \( \Psi(\mu) = E_{v,p} \psi(\mu) \) across episodes by using episodic dual functions \( \Psi_e(\mu, \tau \rho_e) \).

\[
\Psi(\mu) = E_{v,p} \left[ \sum_{t=1}^{T} (v_t - (1 + \mu)p_t)^+ \right] + \mu B
\]

\[
= E_{v,p} \sum_{e=1}^{E} \left( \sum_{t=e \tau}^{e \tau + 1} (v_t - (1 + \mu)p_t)^+ + \mu \rho_e \right)
\]

\[
= \tau \sum_{e=1}^{E} E_{(v,p)}(v - (1 + \mu)p)^+ + \mu \rho_e
\]

\[
= \sum_{e=1}^{E} E_{(v,p)}(v - (1 + \mu)p)^+ + \mu \tau \rho_e
\]

\[
= \sum_{e=1}^{E} \Psi_e(\mu, \tau \rho_e)
\]

Equation 13 follows from Equation 5 and Equation 16 follows from the definition of \( \Psi_e(\mu, B_e) \). Thus \( \Psi_e(\mu, \tau \rho_e) \) is the dual for the episode when the budget \( B_e \) for the episode is \( \tau \rho_e \).

Let \( \Psi_e(\mu_e^*) = \min_{\mu_e \geq 0} \Psi_e(\mu_e^*, \tau \rho_e) \). Using KKT conditions, similar to Equation 5, we can show that if \( \mu_e^* > 0 \) for all \( e \in [E] \), then for all \( e \in [E] \)

\[
\left( \frac{\partial \Psi_e(\mu, \tau \rho_e)}{\partial \mu} \right)_{\mu_e^*} = \tau \rho_e - \tau G_e(\mu_e^*) = 0
\]

\[
\Rightarrow \rho_e = G_e(\mu_e^*)
\]

\[
\Rightarrow G_e(\mu^*) = G_e(\mu_e^*)
\]

Thus \( \mu^* \) satisfies the KKT conditions for \( \Psi_e(\mu, \tau \rho_e) \) as well. This furthermore implies that \( \mu^* \) is an optimizer for \( \Psi_e(\mu, \tau \rho_e) \). This results in the following conclusion:

\[
\Psi(\mu^*) = \sum_{e=1}^{E} \Psi_e(\mu_e^*, \tau \rho_e) = \sum_{e=1}^{E} \Psi_e(\mu_e^*, \tau \rho_e)
\]

This implies that if the budget allocation across each episode is \( \tau \rho_e \), i.e., optimal, then the optimal value of the dual can be obtained by optimizing the dual of each of the episode.
Let the strategy obtained by using our techniques be called \( \mathcal{A} \). \( \mathcal{A} \) uses spend rates \( \rho'_e \) in each episode and assigns budget according to these rates. Once the budget has been divided, the behaviour of \( \mathcal{A} \) in each episodes is independent of the other episodes. Hence we can divide the utility obtained by \( \mathcal{A} \) across the episodes, i.e. \( \pi^A(v, p) = \sum_{e=1}^E \pi^A_e(v_e, p_e) \).

Where \( \pi^A_e(v_e, p_e) = \sum_{t=(e-1)T+1}^{eT} \mathbb{1}\{h^A > p_t\} (v_t - p_t) \) and \( \mathcal{A}_e \) is the strategy induced by \( \mathcal{A} \) on episode \( e \) by limiting the budget for \( \mathcal{A}_e \) as \( \rho'_e \).

Thus \( \mathcal{A}_e \) is just the adaptive pacing strategy given in Balseiro and Gur [12], being run for episode \( e \) with spend rate \( \rho'_e \). Since things are i.i.d within the episode, we can directly use the results of [12]. The corresponding dual induced by the episodic sub-problem with budget \( \tau \rho'_e \) is

\[
\Psi_e(\mu, \tau \rho'_e) = E_{(v, p) \sim Q_e} \left[ \tau (v - (1 + \mu)p)^+ + \mu \tau \rho'_e \right].
\]

The expected utility of \( \mathcal{A}_e \) in episode \( e \) is given by \( E_{(v, p) \sim Q_e} \left[ \pi^A_e(v_e, p_e) \right] \). We use a corollary of Lemma [15] which implies that by fixing the budget for episode \( e \) as \( \tau \rho'_e \), using \( \eta = O(\tau^{-1/2}) \), we have

\[
\inf_{\mu \geq 0} \Psi_e(\mu, \tau \rho'_e) - \pi^A_e(v_e, p_e) \leq O(\sqrt{\tau}).
\]

We know show that if the estimates \( \rho'_e \) are good, the optimal of the episodic dual with budget \( \tau \rho'_e \) is not too less compared to optimal episodic dual when the budget of the episode is \( \tau \rho_e \). Using Lemma [23] for an episode \( e \)

\[
(1 - \omega) \inf_{\mu \geq 0} \Psi_e(\mu, \tau \rho_e) - E_{(v, p) \sim Q_e} \left[ \pi^A_e(v_e, p_e) \right] = O(\sqrt{\tau}).
\]

Summing over all rounds and using Equation [17] we get,

\[
(1 - \omega) \inf_{\mu \geq 0} \Psi(\mu) - E_{(v, p)} \left[ \pi^A(v, p) \right] = \tilde{O} \left( \sqrt{ET} \right).
\]

Using weak duality (Equation [27]), we have

\[
(1 - \omega) E_{(v, p)} \left[ \pi^A(v, p) \right] - E_{(v, p)} \left[ \pi^H(v, p) \right] = \tilde{O} \left( \sqrt{ET} \right).
\]

**Putting Everything Together.** The final missing component is that the spend rate estimator yields an additive guarantee, while the pacing algorithm expects a multiplicative guarantee. We give a transformation for the the spend plan in Algorithm [5] Lemma [24] shows that this yields the multiplicative guarantee.

---

**Algorithm 5:** End-to-end algorithm

1. **Input:** Budget \( B \), rounds \( T \), episodes \( E \), sampling oracles \( F_e \) and \( D_e \), per-episode sampling budget \( n \), Kernel \( K \), scalar \( s \), step size \( \eta \), max shading param \( \bar{\mu} \)
2. \([\hat{\rho}_1, \ldots, \hat{\rho}_E] \leftarrow \text{ApproxSpendRate}(B, T, E, F_e, D_e, n, K, s)\)
3. \( \hat{\rho}'_e = \frac{(\hat{\rho}_e + \Delta)B}{\sum_e (\hat{\rho}_e + \Delta)T} \) for all \( e \in \{1, \ldots, E\} \).
4. **EpisodicAdaptivePacing** \((B, T, E, [\hat{\rho}'_1, \ldots, \hat{\rho}'_E], \eta, \bar{\mu})\)

---

**Lemma 24.** Given spend rates \( \hat{\rho}_e \) such that \( |\rho_e - \hat{\rho}_e| \leq \Delta \) for all \( e \), then \( \hat{\rho}'_e = \frac{(\hat{\rho}_e + \Delta)B}{\sum_e (\hat{\rho}_e + \Delta)T} \geq (1 - \frac{2\Delta T}{B}) \rho_e \) for all \( e \).

**Proof.** From the premise it follows that

\[
\rho_e \leq \hat{\rho}_e + \Delta \leq \rho_e + 2\Delta.
\]

Similarly, scaling all sides by the constant \( \frac{B}{\sum_e (\hat{\rho}_e + \Delta)T} \), the inequalities continue to hold:

\[
\frac{\rho_e B}{\sum_e (\hat{\rho}_e + \Delta)T} \leq \frac{(\hat{\rho}_e + \Delta) B}{\sum_e (\hat{\rho}_e + \Delta)T} \leq \frac{\rho_e + 2\Delta B}{\sum_e (\hat{\rho}_e + \Delta)T}.
\]
Using these observations we can derive the multiplicative lower bound:

\[
\hat{\rho}'_e = \frac{(\hat{\rho}_e + \Delta)B}{\sum_e(\hat{\rho}_e + \Delta)\tau} \geq \frac{\rho_e B}{\sum_e(\rho_e + 2\Delta)\tau} \geq \frac{\rho_e B}{(\sum_e \rho_e + 2\Delta)\tau} = \frac{\rho_e B}{1 + \frac{2\Delta T}{B}} \geq \left(1 - \frac{2\Delta T}{B}\right) \rho_e.
\]  

(\text{by definition})

(by Eq. 19)

(since \(\hat{\rho}_e + \Delta \leq \rho_e + 2\Delta\) by Eq. 18)

(\tau \sum_e \rho_e = B, Eq. 31)

We can now restate our main result formally, which follows from Lemma 19, Lemma 18, Lemma 22, and Lemma 24.

**Theorem 25 (Main Theorem).** Consider the episodic setting with parameters \(B, T, E, \) and \(n\) samples from \(\vec{Q}\) satisfying Definition 21. Setting \(s = O(n^{-1/3})\) and \(\eta = O(\tau^{-1/2})\), with probability at least \(1 - \delta\), Algorithm 5 has \((1 - \epsilon)\)-Reg\((A; T) \leq \tilde{O}\left(\sqrt{ET}\right)\) with

- \(\epsilon = \frac{(E+1)\mu T}{B} \sqrt{\frac{2\log 2E/\delta}{n}} = \tilde{O}\left(\frac{1}{n^{1/2}}\right)\) for the constant-price setting, yielding vanishing regret, and
- \(\epsilon = \tilde{O}\left(\frac{1}{n^{1/3}}\right)\) and \(\delta = \frac{2E}{n}\) for the stochastic-price setting, yielding vanishing regret.

5 Slow-moving Distributions

In this section, we consider a setting where the value and price distributions change at every time step. In this setting, we still consider an advertiser with budget \(B\) who participates in \(T\) auctions. Each round \(t\) has a product distribution \(Q_t = F_t \times D_t\), where \(F_t\) is the distribution over impression value \(v_t \in [0, h]\) and \(D_t\) over the highest competing bid \(p_t \in \mathbb{R}^+\). Thus, the \(T\) round setting is characterized by distribution \(\vec{Q} = (Q_1, \cdots, Q_T)\). We consider settings where this distribution changes slowly over time.

**Definition 26 ((\(\zeta, \theta\))—slow-moving distribution).** A \(T\) round campaign distribution \(\vec{Q} = (Q_1 \cdots, Q_T)\) is called \((\zeta, \theta)\)—slow moving if for all \(t = 1, \cdots, T - 1\), we have

\[
\|F_{t+1} - F_t\|_\infty \leq \zeta \quad \text{and} \quad \|d_{t+1} - d_t\|_\infty \leq \theta.
\]  

(20)

Even though the value and price distributions change in every round, since \(\vec{Q}\) is slow moving, we can generate approximately accurate spend plans by treating ranges of auctions as episodes. While the distribution in these episodes aren’t stationary, the learned spend plan is approximately accurate as the distribution is slow-moving.

**Definition 27 (Admissible Moving Distributions).** Joint distribution \(\vec{Q}\) s.t. it satisfies definition 21 and for any rounds \(i\) and \(j\) that fall in the same episode, the spend function is strongly monotone, i.e. \((\mu' - \mu)(\vec{G}_i(\mu) - \vec{G}_j(\mu')) > C(\mu' - \mu)^2\) for some constant \(C\).
Algorithm 6: Spend Prediction and Pacing for Slowly Changing Distribution

1. **Input:** Budget $B$, Total rounds $T$, Number of episodes to divide into $E$, Sampling oracles $F_t$ for values and $D_t$ for prices, sampling budget $n$, Kernel $K$, scalar $s$, step size $\eta$, max shading param $\bar{\mu}$

2. Divide $T$ into $E$ episodes of size $\tau = \frac{T}{E}$

3. Construct episodic sampling oracles $\bar{F}_e = \frac{1}{\tau} \sum_{t = e \cdot \tau + 1}^{(e+1)\tau} F_t$ and $\bar{D}_e = \frac{1}{\tau} \sum_{t = e \cdot \tau + 1}^{(e+1)\tau} D_t$

4. $(\hat{\rho}_1, \ldots, \hat{\rho}_E) \leftarrow \text{ApproxSpendRate}(B, T, E, \bar{F}_e, \bar{D}_e, n, K, s)$

5. $\hat{\rho}_e = \frac{\hat{\rho}_e + \Delta B}{\sum_i (\hat{\rho}_i + \Delta B)}$ for all $e \in 1, \ldots, E$

6. EpisodicAdaptivePacing($B, T, E, (\hat{\rho}_1', \ldots, \hat{\rho}_E'), \eta, \bar{\mu}$)

As per definition [10] the average spend function for rounds in episode $e$ can be given as $\bar{G}_e(\mu) = \frac{1}{\tau} \sum_{t = e \cdot \tau + 1}^{(e+1)\tau} E(v, p) \sim Q_1[\{v \geq (1 + \mu)p\} p]$ and for the ex-post optimal bidding strategy of bidding $\frac{v_i}{1 + \mu}$, we have

$$\tau \sum_{e=1}^E \bar{G}_e(\mu^*) = \tau \sum_{e=1}^E \rho_e = B. \tag{21}$$

where $\rho_1, \ldots, \rho_E$ are the optimal spend rates. The accuracy of the spend plan now depends on the choice for $E$ and parameters $\zeta$ and $\theta$ that capture how fast the distribution is changing.

Lemma 28. Given $B, T$, sampling oracles $F_t$ and $D_t$ such that $\hat{Q}$ is $(\zeta, \theta)$—slow moving, number of episodes to break into $E$, $n$, $K$, then with probability at least $1 - \frac{2E}{n}$, using algorithm [10] in Line 4, $\text{ApproxSpendRate}$ (algorithm [1]) returns spend rates $\hat{\rho}_e$ such that for every episode $e$,

$$|\hat{\rho}_e - \rho_e| \leq (E + 1) \cdot \tilde{O} \left( \frac{1}{n^{1/3}} + \frac{T(\zeta + \theta)}{E} \right)$$

provided $d_t$ is Lipschitz and bounded by setting $s = n^{-1/3}$.

**Proof.** Consider $\tilde{F}_e = \frac{1}{\tau} \sum_{t = e \cdot \tau + 1}^{(e+1)\tau} F_t$. For any round $t$ belonging to episode $e$, we have

$$\left\| F_t - \tilde{F}_e \right\|_{\infty} = \frac{1}{\tau} \left\| \frac{\tau}{\tau} F_t - \sum_{x = \lfloor t/\tau \rfloor \tau + 1}^{\lfloor (t/\tau) + 1 \rfloor \tau} F_x \right\|_{\infty} \leq \frac{1}{\tau} \sum_{x = \lfloor t/\tau \rfloor \tau + 1}^{\lfloor (t/\tau) + 1 \rfloor \tau} \| F_x - F_t \|_{\infty} \leq \frac{\zeta \tau}{2}. \tag{22}$$

Similarly, for any round $t$ belonging to episode $e$ we have

$$\left\| d_t - \tilde{d}_e \right\|_{\infty} \leq \frac{\theta \tau}{2}. \tag{23}$$

Let $\tilde{F}_e$ be the empirical cdf obtained using $n$ samples from $\tilde{F}_e$. Using the DKW inequality (Lemma [12]), with probability at least $1 - \alpha$, we have $\left\| \tilde{F}_e - \bar{F}_e \right\|_{\infty} \leq \sqrt{\frac{\log \frac{2}{\alpha}}{2n}}$. Similarly, let $\tilde{d}_e$ be the kernel density estimate of $d_e$ obtained using $n$ samples. Using Lemma [14] with probability at least $1 - \frac{1}{n}$, we have $\left\| \tilde{d}_e - \bar{d}_e \right\|_{\infty} = \tilde{O} \left( \frac{1}{n^{1/3}} \right)$. Combining the concentration results with eq. (22) and eq. (23), we that that with probability at least $1 - \frac{2E}{n}$, for all episodes $e$

$$\left\| F_t - \tilde{F}_e \right\|_{\infty} \leq \sqrt{\frac{\log \frac{2}{\alpha}}{2n}} + \frac{\zeta \tau}{2} \quad \text{and} \quad \left\| d_t - \tilde{d}_e \right\|_{\infty} \leq \tilde{O} \left( \frac{1}{n^{1/3}} \right) + \frac{\theta \tau}{2}. \tag{24}$$

Consider the episodic spend function induced by $\tilde{F}_e$ and $\tilde{d}_e$ as $\bar{G}_e(\mu)$. For all episodes $e$, with probability at least $1 - \frac{2E}{n}$, we have
\[ | \mathcal{G}_c(\mu) - \mathcal{G}_c(\mu) | \leq \frac{1}{\tau} \sum_{t=e+1}^{(c+1)\tau} \int_0^h (1 - F_t((1 + \mu)p)) \cdot p \cdot d_t(p) \, dp - \int_0^h (1 - \mathcal{F}_e((1 + \mu)p)) \cdot p \cdot \mathcal{A}_e(p) \, dp \]

Theorem 29. For the pacing setting with parameters B, T, \( \bar{Q} \), number of episodes to break into E, Kernel K, if \( \bar{Q} \) is \((\zeta, \theta)\)-slow moving and satisfies Definition 27 given n samples from \( \bar{Q} \), by setting s = \( n^{-1/3} \) and \( \eta = \tau^{-1/2} \), with probability at least \( 1 - \frac{2E}{n} \), Algorithm 6 resulting in strategy \( \mathcal{A} \) has \((1 - \epsilon)\)-REG(\( \mathcal{A}; T \)) \leq \( \tilde{O} \left( \sqrt{ET} \right) \) with \( \epsilon = \frac{2ET}{B} \cdot \tilde{O} \left( \frac{\zeta + \theta}{\sqrt{B}} \right) \).

\[ \tilde{O} \left( \frac{\zeta + 1}{n^{1/3}} + \zeta + \theta \right) \]

where the first, second, and third inequality follow from triangle inequality. For the fourth step, we use eq. 24 and the fact for any \( 0 \leq a, b, c, d \leq 1 \), \( |a - b| \leq |c - a| + |d - b| + |(c - a) \cdot (d - b)| \). Using Lemma 17, we get that for all \( \mu \), with probability at least \( 1 - \frac{2E}{n} \), we have \( |\hat{\rho}_e - \rho_e| \leq (E + 1) \cdot \tilde{O} \left( \frac{\zeta + \theta}{\sqrt{B}} \right) \).

Now that we have approximate spend rates, we can use the estimates to divide the budget across the smaller episodes and use EpisodicAdaptivePacing (Algorithm 4) to perform online pacing. Combining all the guarantees, we can show the following main result for this setting.

Theorem 29. For the pacing setting with parameters B, T, \( \bar{Q} \), number of episodes to break into E, Kernel K, if \( \bar{Q} \) is \((\zeta, \theta)\)-slow moving and satisfies Definition 27 given n samples from \( \bar{Q} \), by setting s = \( n^{-1/3} \) and \( \eta = \tau^{-1/2} \), with probability at least \( 1 - \frac{2E}{n} \), Algorithm 6 resulting in strategy \( \mathcal{A} \) has \((1 - \epsilon)\)-REG(\( \mathcal{A}; T \)) \leq \( \tilde{O} \left( \sqrt{ET} \right) \) with \( \epsilon = \frac{2ET}{B} \cdot \tilde{O} \left( \frac{\zeta + \theta}{\sqrt{B}} \right) \).

Theorem 29 is implied by combining lemma 25, lemma 24 and lemma 22. Theorem 29 implies our results for the episodic setting can be extended to obtain results for more general settings. We can also observe that in this case, the \( \epsilon \) in \((1 - \epsilon)\)-REG(\( \mathcal{A}; T \)) doesn’t converge to 0 as \( n \) grows, since the nonstationarity within an episode does not decrease with more samples.

6 Experiments

In this section we give empirical evidence that our end-to-end pipeline of optimal spend rate estimation and online pacing provides noticeable benefits over episode-blind pacing schedules and truthful bidding in a variety of synthetic environments.

Datasets. We create synthetic datasets to test the performance of the algorithms under consideration. For the values, we consider three distributions: uniform, normal, and lognormal. For the prices, we consider three settings: fixed prices (our analysis focuses on this setting before generalizing), normally distributed
Table 1: Descriptions of synthetic datasets.

| Data set          | Value distributions $F_e$                                      | Price distributions $D_e$                                      |
|-------------------|----------------------------------------------------------------|----------------------------------------------------------------|
| uniform_v_fix_p   | Uniform dist over $[l_e, r_e]$                                 | Fixed price $p$                                                |
| normal_v_fix_p    | $\mathcal{N}(\mu_v,e, \sigma^2_v,e)$                         | Fixed price $p$                                                |
| lognorm_v_fix_p   | Lognormal($\mu_v,e, \sigma^2_v,e$)                            | Fixed price $p$                                                |
| uniform_v_normal_p| Uniform dist over $[l_e, r_e]$                                 | $\mathcal{N}(\mu_p,e, \sigma^2_p,e)$                         |
| normal_v_normal_p | $\mathcal{N}(\mu_v,e, \sigma^2_v,e)$                         | $\mathcal{N}(\mu_p,e, \sigma^2_p,e)$                         |
| lognorm_v_maxlognorm_p | Lognormal($\mu_v,e, \sigma^2_v,e$) | $\max_{k \in [K]}$ Lognormal($\mu_k,e, \sigma^2_k,e$)     |

Figure 1: Performance of the end-to-end pacing system as a function of the size of training data. We plot the ratio of the optimal utility that our pacing system is able to obtain as a function of the number of training samples in the rate estimation phase. The budget is represented as $\text{budget\_frac}$, i.e. $B = \text{budget\_frac} \cdot \bar{C}$. We can see that 1) with increasing samples, the performance improves quickly 2) the budget level is important for the overall performance.

prices, and the max of multiple draws from a lognormal distribution\textsuperscript{5} We combine these into 6 synthetic datasets, see Table\textsuperscript{1}. We divide the time horizon into 10 episodes, i.e. $E = 10$ with differing parameters of distributions for each episode and use $T = 1000$.

6.1 Learning Accurate Spend Plans

To understand how the performance of our end-to-end pacing is dependent on the numbers of samples available in the spend plan estimation phase, we plot the ratio of the optimal utility that the pacing system is able to obtain as a function of the number of training samples in Figure\textsuperscript{1}. We consider 4 different budget levels: let $\bar{C}$ be the expenditure of the campaign that bids their value in each auction, we consider budgets $x\bar{C}$ for $x \in \{0.25, 0.5, 0.75, 1.0\}$.

Figure\textsuperscript{1} shows the effect of varying the training set size, where a sample in the training set corresponds to a value and price draw from each episode. Two things are clear from the results: 1) with increasing samples,

\textsuperscript{5}Prior work, e.g. [34], suggest that bids in ad auctions typically follow a lognormal distribution. The combination of lognormal values with max-of-lognormal-draws as prices is a realistic simulation of auction environment which is captured in the lognorm_v_maxlognorm_p dataset.
the performance improves quickly, and 2) the budget level is important for the overall performance as the optimal budget allocation problem gets harder for smaller budgets. This is in agreement with the theoretical results of the pacing system obtained in Theorem 25. Finally, it does appear that the performance hits a plateau. This is likely due to the online part of the algorithm which does not scale with increased offline learning sample size.

6.2 End-to-end Performance

**Algorithms.** We compare the performance of 1) our algorithm, 2) constant spend rate [12] (which uses a linear cumulative expenditure over the duration of the campaign, and gives us an understanding of the benefit of estimating spend rates when competition is time-varying), and 3) no pacing (which enters the advertiser’s value in each auction until they run out of budget).

**Results.** The values and prices were generated in the same way as above. For each dataset, we run simulations where we a budget for the campaign is drawn uniformly from $[0, \bar{C}]$, where $\bar{C}$ is the expenditure of the campaign that bids truthfully in each auction. Then we run all the pacing algorithms for this dataset sample and budget level. We repeat this process 150 times to get 150 data points per dataset for each algorithm. We plot the ratio of the optimal utility that the pacing system is able to obtain as a function of the budget level in Figure 2. Our algorithm outperforms both benchmarks almost everywhere. The only time where the “Truthful” benchmark performs better are in situations where the advertiser has enough budget to buy (almost) all impressions. There is one area where “Fixed spend (BG19)” outperforms our algorithm. It happens for the “normal_v_normal_p” dataset when $B \geq 0.8 \cdot \bar{C}$; we do not have an explanation why this particular range performs poorly.
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A  Characterizing the optimal pacing strategy and budget allocation in expectation

Recalling that the optimal strategy on the realized values and prices is obtained by the hindsight strategy $H$ (definition 3). The Lagrangian dual of the optimization problem in definition 3 is given by:

$$
\psi(\mu) = \left[ \sum_{t=1}^{T} (v_t - (1 + \mu)p_t) \right] + \mu B \quad \text{(L(KP))}
$$

Where we define $(z)^+$ to be max $\{z, 0\}$. The dual is obtained from the Lagrangian by setting $x_t = 1$ for all $t$ such that $v_t - (1 + \mu)p_t \geq 0$, i.e. winning all impressions with value greater than $(1 + \mu)$ times the price which can be done by bidding $b_t = v_t/(1 + \mu)$.

By weak duality, we have

$$
\pi^H(v, p) \leq \inf_{\mu \geq 0} \psi(\mu) \quad \text{(26)}
$$

Since $v$ and $p$ are being sampled from the fixed distribution defined by $Q$, taking expectation over equation 26 and using Jensen’s inequality, we get

$$
\mathbb{E}_{v, p}[\pi^H(v, p)] \leq \mathbb{E}_{v, p}\left[\inf_{\mu \geq 0} \psi(\mu)\right] = \inf_{\mu \geq 0} \mathbb{E}_{v, p}[\psi(\mu)] \quad \text{(27)}
$$

Let $\Psi(\mu) = \mathbb{E}_{v, p}[\psi(\mu)]$ and $\mu^*$ be the minimizer of $\Psi(\mu)$. Assuming $\Psi(\mu)$ to be differentiable, using Karush-Kuhn-Tucker conditions, we have $\mu^* \geq 0$, $\Psi'(\mu^*) \geq 0$, and $\mu^*\Psi'(\mu^*) = 0$. If $\mu^* = 0$, it implies that we are effectively not constrained by the budget and truthful bidding achieves the optimal utility in expectation as it wins all items with positive utility.

The gradient of $\Psi(\mu)$ can be written as

$$
\Psi'(\mu) = B - G(\mu)
$$

where

$$
G(\mu) = \mathbb{E}_{v, p}\left[\sum_{t=1}^{T} \mathbb{I}\{v_t \geq (1 + \mu)p_t\} p_t\right]
$$

We call $G(\mu)$ the overall spend function. By definition, $G(\mu)$ is the expected expenditure over all the $T$ rounds when buying all items such that $v_t \geq (1 + \mu)p_t$, obtained by bidding $b_t = v_t/(1 + \mu)$. The KKT complementary slackness condition implies that if $\mu^* > 0$, then $\Psi'(\mu^*) = 0$ i.e.

$$
G(\mu^*) = B \quad \text{(28)}
$$

This implies that the strategy with a fixed pacing multiplier that bids $b_t = v_t/(1 + \mu^*)$ achieves better expected utility than the expected utility of the hindsight strategy $H$. If truthful bidding is not optimal (i.e $\mu^* > 0$ ), then the expected expenditure of this strategy is $B$. Not that the expenditure guarantee for the optimal fixed shading strategy is only satisfied in expectation, i.e. it spends budget $B$ in expectation.

For the rest of the theoretical claims, we restrict ourselves to the case that $\mu^* > 0$. The case $\mu^* = 0$ implies that the budget constraint is not binding, so truthful bidding is the optimal strategy. Our algorithm will naturally adapt to this setting as well.

Let $\overline{G}(\mu) = \mathbb{E}_{(v, p) \sim Q_e}[\mathbb{I}\{v \geq (1 + \mu)p\} p]$ be the average spend in each round (over the whole campaign) if we buy all items such that $v_t \geq (1 + \mu)p_t$. Equation 28 implies that for the optimal dual variable $\mu^* > 0$,

$$
\overline{G}(\mu^*) = \frac{B}{T}. \quad \text{(29)}
$$

Using Definition 10, $\overline{G}_c(\mu) = \mathbb{E}_{(v, p) \sim Q_c}[\mathbb{I}\{v \geq (1 + \mu)p\} p]$ is the expected spend per round in an episode $e$ if the dual variable is $\mu$. Corresponding to the strategy which bids by multiplicatively shading the value $v_t$ by a factor of $1/(1+\mu)$ and ends up buying all the impressions in the episode with value per unit spent at

23
least $(1 + \mu)$. In our framework, the spend function can be decomposed across the episodes by introducing episodic spend functions. We show this decomposition below:

$$G(\mu) = \mathbb{E}_{v,p \sim \vec{Q}} \left[ \sum_{t=1}^{T} \mathbb{1} \{ v_t \geq (1 + \mu)p_t \} p_t \right]$$

$$= \mathbb{E}_{v,p \sim \vec{Q}} \left[ \sum_{e=1}^{E} \sum_{t=(e-1)\tau+1}^{e\tau} \mathbb{1} \{ v_t \geq (1 + \mu)p_t \} p_t \right]$$

$$= \sum_{e=1}^{E} \sum_{t=(e-1)\tau+1}^{e\tau} \mathbb{E}_{v,p_t \sim \vec{Q}_e} [\mathbb{1} \{ v_t \geq (1 + \mu)p_t \} p_t]$$

$$= \tau \sum_{e=1}^{E} \mathbb{E}_{v,p \sim \vec{Q}_e} [\mathbb{1} \{ v \geq (1 + \mu)p \} p]$$

$$= \tau \sum_{e=1}^{E} \mathcal{G}_e(\mu)$$

$$\implies \frac{G(\mu)}{T} = \frac{1}{E} \sum_{e=1}^{E} \mathcal{G}_e(\mu)$$

$$\implies \mathcal{G}(\mu) = \frac{1}{E} \sum_{e=1}^{E} \mathcal{G}_e(\mu)$$

where $\mathcal{G}_e(\mu) = \mathbb{E}_{v,p \sim \vec{Q}_e} [\mathbb{1} \{ v \geq (1 + \mu)p \} p]$ is the episodic spend function. This definition helps us to define the optimal budget allocation as $B_e = \tau \mathcal{G}_e(\mu^*) = \tau \rho_e$, where $\rho_e$ is the optimal spend rate for episode $e$ given by $\rho_e = \mathcal{G}_e(\mu^*)$. Note that if $\mu^* > 0$, using Equation 5, we have

$$\sum_{e=1}^{E} \rho_e = \tau \sum_{e=1}^{E} \mathcal{G}_e(\mu^*) = G(\mu^*) = B$$

### B Detailed Algorithms

#### B.1 EpisodicAdaptivePacing: Adaptive pacing using a spend plan

We present ApproxSpendRate (Algorithm 1) which uses historical data to compute approximately optimal spend rates $(\hat{\rho}_1, \ldots, \hat{\rho}_E)$ from samples.

For each episode $e$, the subroutine ApproxSpendSP (Algorithm 2) estimates the episodic spend function $\mathcal{G}_e(\mu)$ as a function of $\mu$ using the historic samples $\vec{V}$ and $\vec{P}$. For fixed prices, we use a simpler episodic spend prediction function estimate ApproxSpendFP (Algorithm 3). Both functions try to estimate $\mathcal{G}_e(\mu)$ and return an empirical approximate of $\mathcal{G}_e(\mu)$ we denote as $\hat{\mathcal{G}}_e(\mu)$.

Then using the structure of overall average spend function $\mathcal{G}(\mu)$, (Equation 30), we can construct an approximation of the overall average spend function $\hat{\mathcal{G}}(\mu)$ as $\frac{\sum_{e=1}^{E} \hat{\mathcal{G}}_e(\mu)}{E}$ Based on our discussion about the optimal structure of the problem, for optimal dual variable $\mu^*$, we know that $\mathcal{G}(\mu^*) = \frac{B}{\tau}$ (Equation 5). Using our empirical estimate $\hat{\mathcal{G}}(\mu)$, we compute $\hat{\mu}$, an empirical estimate of $\mu^*$. We can compose our approximations to form $\hat{\rho}_e = \hat{\mathcal{G}}_e(\hat{\mu})$, an approximation to $\mathcal{G}_e(\mu^*) = \rho_e$.
Algorithm 7: EpisodicAdaptivePacing: Adaptive pacing using a spend plan.

1 Input: Budget $B$, rounds $T$, episodes $E$, spend plan $(\rho_1', \ldots, \rho_E')$, step size $\eta$, max shading param $\bar{\mu}$
2 $\mu_t \leftarrow [0, \bar{\mu}]$ \hspace{1em} // Initialize shading multiplier
3 $B_{\text{Budget}_1} \leftarrow B$ \hspace{1em} // Overall remaining budget left for campaign
4 $\tau \leftarrow T/E$ \hspace{1em} // Impressions in each episode
5 $B_1 \leftarrow \rho_1' \cdot \tau$ \hspace{1em} // Remaining budget for episode 1
6 for $t = 1, \ldots, T$ do
7 \hspace{1em} $e \leftarrow \lceil t / E \rceil$ \hspace{1em} // Current episode
8 \hspace{1em} Observe value $v_t$
9 \hspace{1em} Post bid $b_t \leftarrow \min \{v_t + \mu_t, \hat{B}_e, \text{Budget}_t\}$
10 \hspace{1em} Observe expenditure $z_t$
11 \hspace{1em} $\mu_{t+1} \leftarrow \text{Proj}_{[0, \bar{\mu}]}[\mu_t - \eta(\rho_e' - z_t)]$ \hspace{1em} // Update shading parameter
12 \hspace{1em} $\hat{B}_{e+1} \leftarrow \hat{B}_e - z_t$ \hspace{1em} // Update remaining budget
13 \hspace{1em} $\text{Budget}_{t+1} \leftarrow \text{Budget}_t - z_t$
14 \hspace{1em} if $t \mod E = 0$ then
15 \hspace{1em} \hspace{1em} $\hat{B}_{e+1} \leftarrow \rho_{e+1}' \cdot \tau + \hat{B}_e$ \hspace{1em} // Carry over left-over budget
16 \hspace{1em} end
17 end

Algorithm 8: ApproxSpendRate: Approximate optimal spend rates

1 Input: Budget $B$, Total rounds $T$, Number of episodes $E$, Episodic sampling oracles $F_e$ for values and $D_e$ for prices, Per episode sampling budget $n$, Kernel $K$, scalar $s$
2 Goal: Estimate optimal spend rates $(\rho_1', \ldots, \rho_E')$
3 if in the constant-price setting then
4 \hspace{1em} for $e = 1, \ldots, E$ do
5 \hspace{2em} Samples $n$ values $\tilde{V} = (V_1, V_2, \ldots, V_n) \sim F_e$
6 \hspace{2em} Set price $p \sim D_e$
7 \hspace{2em} $\hat{G}_e(\mu) = \text{ApproxSpendFP}(n, \tilde{V}, p)$ \hspace{1em} // Estimate episodic spend function
8 \hspace{1em} end
9 else
10 \hspace{1em} for $e = 1, \ldots, E$ do
11 \hspace{2em} Samples $n$ values $\tilde{V} = (V_1, V_2, \ldots, V_n) \sim F_e$
12 \hspace{2em} Samples $n$ prices $\tilde{P} = (P_1, P_2, \ldots, P_n) \sim D_e$
13 \hspace{2em} $\hat{G}(\mu) = \text{ApproxSpendSP}(n, \tilde{V}, \tilde{P}, K, s)$ \hspace{1em} // Estimate episodic spend function
14 \hspace{1em} end
15 \hspace{1em} $G_e(\mu) = \frac{\sum_{e=1}^E \hat{G}_e(\mu)}{E}$ \hspace{1em} // Construct overall average spend function
16 \hspace{1em} $\hat{\mu} = \min \mu \text{ s.t. } G_e(\mu) \leq \frac{B}{T}$ \hspace{1em} // Estimating the optimal dual variable
17 \hspace{1em} for $e = 1, \ldots, E$ do
18 \hspace{2em} $\hat{\rho}_e = \hat{G}_e(\hat{\mu})$ \hspace{1em} // Expected spend rate in episode for the estimated dual variable
19 \hspace{1em} end
20 return $(\hat{\rho}_1', \ldots, \hat{\rho}_E')$