Extracting the longitudinal structure function $F_L(x, Q^2)$ at small $x$
from a Froissart-bounded parametrization of $F_2(x, Q^2)$

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Abstract

We present a method to extract, in the leading and next-to-leading order approximations, the longitudinal deep-inelastic scattering structure function $F_L(x, Q^2)$ from the experimental data by relying on a Froissart-bounded parametrization of the transversal structure function $F_2(x, Q^2)$ and, partially, on the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations. Particular attention is paid on kinematics of low and ultra low values of the Bjorken variable $x$, $x \sim 10^{-5} \div 10^{-2}$. Analytical expressions for $F_L(x, Q^2)$ in terms of the effective parameters of the parametrization of $F_2(x, Q^2)$ are presented explicitly. We argue that the obtained structure functions $F_L(x, Q^2)$ within both, the leading and next-to-leading order approximations, manifestly obey the Froissart boundary conditions. Numerical calculations and comparison with available data from ZEUS and H1-Collaborations at HERA demonstrate that the suggested method provides reliable structure functions $F_L(x, Q^2)$ at low $x$ in a wide range of the momentum transfer ($1\text{ GeV}^2 < Q^2 < 3000\text{ GeV}^2$) and can be applied as well in analyses of ultra-high energy processes with cosmic neutrinos.

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I. INTRODUCTION

At small values of the Bjorken variable $x$, the nonperturbative effects in the deep inelastic structure functions (SF) were expected to play a decisive role in describing the corresponding cross sections. However, it has been observed, cf. Ref. [1], that even in the region of low momentum transfer $Q^2 \sim 1\text{ GeV}^2$, where traditionally the soft processes were considered to govern the cross sections, the perturbative QCD (pQCD) methods could be still adequate in description of high energy processes, in particular, at relatively low values of $x$: $10^{-5} \leq x \leq 10^{-2}$. It should also be noted that, at extremely low $x$, $x \to 0$, the pQCD evolution leads, nonetheless, to a rather singular behaviour of the parton distribution functions (PDF) (see e.g. Ref. [2] and references therein quoted), which is in a strong disagreement with the Froissart boundary conditions [3]. In Refs. [4–6] M. M. Block et al. have suggested a new parametrization of the SF $F_2(x, Q^2)$ which describes fairly well the available experimental data on the reduced cross sections and, at asymptotically low $x$, provides a behavior of the hadron-hadron cross sections $\sim \ln^2 s$ at large $s$, where $s$ is the Mandelstam variable denoting the square of the total invariant energy of the process, in a full accordance with the Froissart predictions [3]. The most recent parametrization suggested in Ref. [6] by M. M. Block, L. Durand and P. Ha, in what follows referred to as the BDH parametrization, is also pertinent in investigations of lepton-hadron processes at ultra-high energies, e.g. the scattering of cosmic neutrinos from hadrons [5–9]. Note that, in case of neutrino scattering other SF’s, such as the pure valence, $F_3(x, Q^2)$, and longitudinal, $F_L(x, Q^2)$, are relevant to describe the process. While at low values of $x$ the valence structure function $F_3(x, Q^2)$ vanishes, the longitudinal $F_L(x, Q^2)$ remains finite and can even be predominant in the cross section. Thus, a theoretical analysis of the longitudinal SF $F_L(x, Q^2)$ at low $x$, in context of fulfilment of the Froissart prescriptions, is of a great importance in treatments of ultra-high energy processes as well.

Hitherto, most theoretical analyses [8, 10] of neutrino processes have been performed in the leading order (LO) approximation, within which the Callan-Gross relation is assumed to be satisfied exactly, i.e. the longitudinal structure function $F_L = 0$. Beyond the LO the effects from $F_L$ can be sizable, hence it can not be longer neglected, cf. Ref. [8, 11].

In the present paper we present a method of extraction of the longitudinal SF, $F_L(x, Q^2)$ in the kinematical region of low values of the Bjorken variable $x$ from the known structure function $F_{2\text{BDH}}(x, Q^2)$ and known derivative $dF_{2\text{BDH}}/d\ln(Q^2)$ by relying, with some extent, on
the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) $Q^2$-evolution equations \[12\]. In our calculations we use the most recent version of the BDH-parametrization reported in Ref. \[6\]. In fact, the presented approach is a further development of the methods previously suggested in Refs. \[13, 14\] to extract some general characteristics of the gluon density and longitudinal SF at low $x$ from the experimentally known SF $F_2(x, Q^2)$ and logarithmic derivative $dF_2/d \ln Q^2$. The extraction procedure has been inspired by the Altarelli-Martinelli formula \[15\] suggested to determine the gluon density from $F_L(x, Q^2)$, and improved in Ref. \[16\].

In our case, the SF $F_2(x, Q^2)$ is considered experimentally known being defined by the $F_2^{BDH}(x, Q^2)$ parametrization \[6\], i.e. the $x$ and $Q^2$ dependencies of the transverse SF and the corresponding logarithmic derivative are supposed to be known. As a first step of the analysis, the method has been applied to extract $F_L(x, Q^2)$ in the LO. Results of such a procedure have been briefly reported in Ref. \[17\], where it has been demonstrated that the extracted structure function $F_L^{BDH}(x, Q^2)$ at moderate and low values of $x$ is in a reasonable good agreement with the available experimental data \[19\]. However, for ultra low values of $x$ the agreement becomes less satisfactorily and even rather poor in the limit $x \to 0$. This serves as a clear indication that the LO analysis is not sufficient in the region $x \to 0$ and the next-to-leading order (NLO) corrections become significant and are to be implemented in to the extraction procedure. Similar investigations of the longitudinal SF have been performed in Ref. \[18\].

In this paper, we present in some details the LO analysis \[17\], and provide further development of the method extending it beyond the LO approximation by considering and resumming the NLO corrections.

It is worth emphasizing that the NLO approximation for $F_L(x, Q^2)$, i.e. calculations up to $\alpha_s^2$-corrections, corresponds to the next-next-to-leading order (NNLO) approximation for $F_2(x, Q^2)$ which, in the LO, is $\propto \alpha_s^0$. Hence, in our approach it becomes possible to perform NLO and NNLO analyses of the ultra-high energy ($\sqrt{s} \sim 1$ TeV) neutrino cross-sections similar to existing NLO \[20\] and NNLO \[21\] investigations based on pQCD. Such analyses are rather important in view of the recently appeared possibility of a direct comparison with emerging data from the IceCube Collaboration \[22\] (cf. also Ref. \[23\]) and anticipated data from the IceCube-Gen2 \[24\], whose performance is much better and which is awaited to provide substantially more precise measurements of the neutrino-nucleon cross-section.

Our paper is organized as follows:

In Sec. II we present the basic formulae of the approach. The relevant system of equations to be
used in extraction of the longitudinal SF, together with the corresponding splitting functions and coefficient functions are displayed explicitly. In Subsections II A and II B we discuss the Mellin transforms of the transversal and longitudinal SF’s in the LO and NLO approximations for momenta corresponding to low $x$. Explicit expressions for the anomalous dimensions and Wilson coefficients in the LO for low $x$ are given as well.

In Sec. III we write down details of obtaining all the needed, in the subsequent calculations, quantities related to the BDH parametrization [6], such as the corresponding derivatives and Mellin transforms within the considered kinematics and approximations. Next two sections, Sec. IV and Sec. V are entirely devoted to description of the gist of the mathematic methods and manipulations used to calculate the Mellin transforms and their inverses to find the longitudinal SF in the LO, Sec. IV and NLO, Sec. V approximations. Numerical results for the extracted $F_L(x, Q^2)$ in the LO and NLO, together with comparisons with the experimental data from the H1-Collaboration, are presented in Sec. VI where we discuss the $Q^2$ and $x$-dependencies of the extracted SF $F_L(x, Q^2)$ and the ratio $R_L(x, Q^2)$ of the longitudinal to transversal cross sections within the LO and NLO approximations. Conclusions and summary are summarized on Sec. VII. Eventually, the most cumbersome expressions are relegated to Appendices A and B.

II. BASIC FORMULAE

In view of at low values of $x$ the non-singlet quark distributions become negligibly small in comparison with the singlet distributions, in the present analysis they are disregarded. Then, the transverse $F_2(x, Q^2)$ and longitudinal $F_L(x, Q^2)$ structure functions are expressed solely via the singlet quark and gluon densities $x f_a(x, Q^2)$ (hereafter $a = s, g$ and $k = 2, L$) as

$$F_k(x, Q^2) = e \sum_{a=s,g} \left[ B_{k,a}(x) \otimes x f_a(x, Q^2) \right], \quad (1)$$

where $e$ is the average charge squared, $e = \frac{1}{f} \sum_{i=1}^{f} \epsilon_i^2 \equiv \frac{e_f^2}{f}$ with $f$ as the number of considered flavors, $q^2 = -Q^2$ and $x = Q^2/2pq$ ($p$ being the momentum of the nucleon) denote the momentum transfer and the Bjorken scaling variable, respectively. The quantities $B_{k,a}(x)$ are the known Wilson coefficient functions. In Eq. (1) and throughout the rest of the paper, the symbol $\otimes$ is used for a shorthand notation of the convolution formula, i.e. $f_1(x) \otimes f_2(x) \equiv \int \frac{dy}{x} f_1(y) f_2 \left( \frac{x}{y} \right)$. 

4
According to the DGLAP $Q^2$-evolution equations [12] the leading twist quark, $xf_s(x, Q^2)$, and gluon, $xf_g(x, Q^2)$, distributions obey the following system of integro-differential equations

$$\frac{d(xf_a(x, Q^2))}{d\ln Q^2} = -\frac{1}{2} \sum_{a,b=s,g} P_{ab}^{(0)}(x) \otimes xf_b(x, Q^2), \quad (2)$$

where $P_{ab}(x) \ (a, b = s, g)$ are the corresponding splitting functions.

Within the pQCD, and up to the NLO corrections, the coefficient functions $B_{k,a}(x)$ and the splitting functions $P_{ab}(x)$ read as

$$B_{2,s}(x) = \delta(1 - x) + a_s(Q^2) B_{2,s}^{(1)}(x), \quad (3)$$
$$B_{2,g}(x) = a_s(Q^2) B_{2,g}^{(1)}(x), \quad (4)$$
$$B_{L,a}(x) = a_s(Q^2) B_{L,a}^{(0)}(x) + a_s^2(Q^2) B_{L,a}^{(1)}(x), \quad (5)$$
$$P_{a,b}(x) = a_s(Q^2) P_{a,b}^{(0)}(x) + a_s^2(Q^2) P_{a,b}^{(1)}(x), \quad (6)$$

where $a_s(Q^2) = \alpha_s(Q^2)/4\pi$ is the QCD running coupling, which, for convenience, includes in to its definition an additional factor of $4\pi$ in comparison with the standard notation. In the above equations and hereafter the superscripts $(0, 1)$ mark the corresponding order of the perturbation theory: $(0)$ for LO and $(1)$ for NLO.

Inserting Eqs. (3)-(6) in to Eqs. (1)-(2) the final NLO system of equations for the sought PDF’s becomes

$$\frac{d(xf_g(x, Q^2))}{d\ln Q^2} = -\frac{a_s(Q^2)}{2} \left\{ \left( P_{gg}^{(0)}(x) + a_s(Q^2) \tilde{P}_{gg}^{(1)}(x) \right) \otimes xf_g(x, Q^2) + e^{-1} \left( P_{gs}^{(0)}(x) + a_s(Q^2) \tilde{P}_{gs}^{(1)}(x) \right) \otimes F_2(x, Q^2) + O(a_s^3) \right\}, \quad (7)$$

$$\frac{dF_2(x, Q^2)}{d\ln Q^2} = -\frac{a_s(Q^2)}{2} \left\{ e \left( P_{sg}^{(0)}(x) + a_s(Q^2) \tilde{P}_{sg}^{(1)}(x) \right) \otimes x f_g(x, Q^2) + \left( P_{ss}^{(0)}(x) + a_s(Q^2) \tilde{P}_{ss}^{(1)}(x) \right) \otimes F_2(x, Q^2) + O(a_s^3) \right\}, \quad (8)$$

$$F_L(x, Q^2) = a_s(Q^2) \left\{ e \left( B_{L,g}^{(0)}(x) + \tilde{B}_{L,g}^{(1)}(x) \right) \otimes x f_g(x, Q^2) + \left( B_{L,q}^{(0)}(x) + \tilde{B}_{L,q}^{(1)}(x) \right) \otimes F_2(x, Q^2) + O(a_s^3) \right\}, \quad (9)$$
where, for brevity, the following notations have been employed

\[
\tilde{P}_{sg}^{(1)}(x) = P_{sg}^{(1)}(x) + B_{2,s}^{(1)}(x) \otimes P_{sg}(x) + B_{2,g}^{(0)}(x) \otimes (2\beta_0 \delta(1-x) + P_{gg}^{(0)}(x) - P_{ss}^{(0)}(x)),
\]

\[
\tilde{P}_{ss}^{(1)}(x) = P_{ss}^{(1)}(x) + 2\beta_0 B_{2,s}^{(1)}(x) \otimes \delta(1-x) + B_{2,g}^{(1)}(x) \otimes P_{gg}^{(0)}(x),
\]

\[
\tilde{P}_{gs}^{(1)}(x) = P_{gs}^{(1)}(x) - B_{2,s}^{(1)}(x) \otimes P_{gs}(x), \quad \tilde{P}_{gg}^{(1)}(x) = P_{gg}^{(1)}(x) - B_{2,g}^{(1)}(x) \otimes P_{gs}^{(0)}(x)
\]

\[
\tilde{B}_{L,g}^{(1)}(x) = B_{L,g}^{(1)}(x) - B_{2,g}^{(1)}(x) \otimes B_{L,g}(x), \quad \tilde{B}_{L,s}^{(1)}(x) = B_{L,s}^{(1)}(x) - B_{2,s}^{(1)}(x) \otimes B_{L,s}(x)
\]

with \(\beta_0\) and \(\beta_1\) as the first two coefficients of the QCD \(\beta\)-function

\[
\beta_0 = \frac{1}{3} \left(11C_A - 2f\right), \quad \beta_1 = \frac{1}{3} \left(34C_A^2 - 2f(5C_A + 3C_F)\right).
\]

In Eq. (12) \(C_F = (N_c^2 - 1)/(2N_c)\) and \(C_A = N_c\) are the Casimir operators in the fundamental and adjoint representations of the \(SU(N_c)\) color group, respectively. Within QCD \(N_c = 3\), hence \(C_F = 4/3\) and \(C_A = 3\).

Few remarks are in order here. As known \([25, 26]\), equation (7) in its actual form leads to a too singular behaviour of the gluon distribution at small \(x\), violating the Froissart boundary restrictions. One can go beyond the perturbative theory and try to cure the problem by adding in the r.h.s. of Eq. (7) terms proportional to \((xf_g)^2\) which make the distribution (hence, the corresponding cross-sections \([27]\)) less singular at the origin and can, in principle, reconcile it with the Froissart requirements. A detailed inspection of Eq. (7) in context of implementation of additional modifications to fulfill the Froissart conditions is beyond the scope of the present paper and in what follows we omit it in our analysis. However, the gluon distribution originating from the omitted Eq. (7) and entering in to the remaining equations (8) and (9) is supposed to have the correct asymptotic behavior, i.e. to be of the same LO form as the BDH parametrization of the \(F_2^{BDH}(x, Q^2)\). This conjectures has been confirmed in previous analysis \([28]\) where the early parametrization of \(F_2(x, Q^2)\) \([5]\) has been employed to determine the gluon density within the LO. Then, Eq. (8) with the known \(F_2^{BDH}(x, Q^2)\), can be considered as the definition of the gluon density \(xf_g^{BDH}(x, Q^2)\) in the whole kinematical interval, cf. Ref. \([28]\).

Consequently, in the system (8)-(9) of two equations with two unknown distributions one can eliminate the gluon part and solve the remaining equation with respect to the longitudinal \(F_L(x, Q^2)\) and express it via the known paramterization of \(F_2(x, Q^2)\). With these statements, now we are in a position to solve Eqs. (8) and (9) and to extract the desired longitudinal SF. Notice that, albeit at the first glance the above equations are relatively simple, direct solving of (8)-(9) actually turns out to be a rather complicate and cumbersome procedure. One can
substantially simplify the calculations by considering Eqs. (8)-(9) in the space of Mellin momenta, and taking advantage of the fact the convolution form \( f_1(x) \otimes f_2(x) \) in \( x \) space becomes merely a product of individual Mellin transforms of the corresponding functions in the space of Mellin momenta. Consequently, all our further calculations we perform in Mellin space.

A. Mellin transforms

The Mellin transform of the PDF’s entering in to Eqs. (8)-(9) are defined as

\[
M_k(n, Q^2) = \int_0^1 dx x^{n-2} F_k(x, Q^2), \quad M_a(n, Q^2) = \int_0^1 dx x^{n-1} f_a(x, Q^2),
\]

(13)

\[
\gamma_{ab}^{(i)}(n) = \int_0^1 dx x^{n-2} P_{ab}^{(i)}(x), \quad B_{k,a}^{(i)}(n) = \int_0^1 dx x^{n-2} B_{k,a}^{(i)}(x),
\]

(14)

where, as before, \( a, b \) and \( k = 2, L \). Then, after some algebra, the Mellin transforms of Eqs. (8)-(9) read as

\[
\begin{align*}
\frac{d M_2(n, Q^2)}{d \ln Q^2} &= - \frac{a_s(Q^2)}{2} \left[ e \left( \gamma_{ss}^{(0)}(n) + a_s(Q^2) \tilde{\gamma}_{ss}^{(1)}(n) \right) M_g(n, Q^2) + \left( \gamma_{ss}^{(0)}(n) \right) M_2(x, Q^2) + O(a_s^3) \right], \\
M_L(n, Q^2) &= a_s(Q^2) \left[ e \left( B_{L,g}^{(0)}(n) + a_s(Q^2) \tilde{B}_{L,g}^{(1)}(n) \right) M_g(n, Q^2) \\
&\quad + \left( B_{L,s}^{(0)}(n) + a_s(Q^2) \tilde{B}_{L,s}^{(1)}(n) \right) M_2(x, Q^2) + O(a_s^3) \right],
\end{align*}
\]

(15)

(16)

where the anomalous dimensions \( \gamma_{ab}^{(i)}(n) \) and the Wilson coefficients \( B_{L,a}^{(i)}(n) \) (\( i = 0, 1 \)) are

\[
\begin{align*}
\gamma_{ss}^{(1)}(n) &= \gamma_{ss}^{(1)}(n) + B_{2,s}^{(1)}(n) \gamma_{ss}^{(0)}(n) + B_{2,g}^{(0)}(n) \left( 2 \beta_0 + \gamma_{gg}^{(0)}(n) - \gamma_{ss}^{(0)}(n) \right), \\
\tilde{\gamma}_{ss}^{(1)}(n) &= \gamma_{ss}^{(1)}(x) + 2 \beta_0 B_{2,s}^{(1)}(n) + B_{2,g}^{(1)}(n) \gamma_{gg}^{(0)}(n), \\
\tilde{\gamma}_{gs}^{(1)}(n) &= P_{gs}^{(1)}(n) - B_{2,s}^{(1)}(n) \gamma_{gs}^{(0)}(n), \quad \tilde{\gamma}_{gg}^{(1)}(n) = \gamma_{gg}^{(1)}(n) - B_{2,g}^{(1)}(n) \gamma_{gg}^{(0)}(n) \\
\tilde{B}_{L,g}^{(1)}(n) &= B_{L,g}^{(1)}(n) - B_{2,g}^{(1)}(n) B_{L,g}^{(0)}(n), \quad \tilde{B}_{L,s}^{(1)}(n) = B_{L,s}^{(1)}(n) - B_{2,s}^{(1)}(n) B_{L,s}^{(0)}(n)
\end{align*}
\]

(17)

(18)

The explicit expressions for the anomalous dimensions \( \gamma_{ab}^{(i)}(n) \) (\( i = 0, 1 \)) and the LO Wilson coefficients \( B_{L,a}^{(0)}(n) \) can be found in Ref. 29, whereas the NLO part, \( B_{L,N}^{(1)}(n) \) and \( B_{L,a}^{(1)}(n) \), has been reported, for the first time, in Refs. 30, 31. Unfortunately, there are several misprints in the above mentioned references. Erratum for Refs. 30, 31 can be found in, e.g. Ref. 33. Yet, in Ref. 29, a factor two in the LO coefficients \( B_{L,a}^{(0)}(n) \) is missed. Observe that as mentioned...
above, the Mellin transform significantly simplifies our calculations for, the complicated integro-differential system of equations [3]-[9] in $x$-space, is translated in to a relatively simple system [15]-[16] of pure algebraic equations. Now, we solve the system [15]-[16] w.r.t the longitudinal Mellin momentum $M_L(n, Q^2)$ and express it through known momentum $M_2(x, Q^2)$ and the derivative $dM_2(n, Q^2)/dlnQ^2$

$$M_L(n, Q^2) = -2 \frac{B_{L,q}^{(1)}(n) + a_s(Q^2)\tilde{B}_{L,q}^{(1)}(n)}{\gamma_{sg}^{(1)}(n) + a_s(Q^2)\tilde{\gamma}_{sg}^{(1)}(n)} \frac{dM_2(n, Q^2)}{dlnQ^2} + \left[ \left( B_{L,s}^{(0)}(n) + a_s(Q^2)\tilde{B}_{L,s}^{(1)}(n) \right) - \left( B_{L,g}^{(0)}(n) + a_s(Q^2)\tilde{B}_{L,g}^{(1)}(n) \right) \right] M_2(x, Q^2) + O(a_s^3).$$  

(19)

B. Anomalous dimensions and coefficient functions

Here below we present the explicit expressions for the LO ingredients only. The corresponding expressions for the NLO corrections are rather cumbersome and, as already mentioned, can be found in Refs. [29, 31, 33]. Explicitly, the LO anomalous dimensions $\gamma_{ab}^{(0)}(n)$ and the Wilson coefficients $B_{L,a}^{(0)}(n)$, are as follows:

$$\gamma_{sg}^{(0)}(n) = -\frac{4f(n^2 + n + 2)}{n(n+1)(n+2)}, \quad \gamma_{ss}^{(0)}(n) = 8C_F \left[ S_1(n) - \frac{3}{4} - \frac{1}{2n(n+1)} \right],$$

$$\gamma_{gg}^{(0)}(n) = -\frac{4C_F(n^2 + n + 2)}{(n-1)n(n+1)}, \quad \gamma_{gs}^{(0)}(n) = 8C_A \left[ S_1(n) - \frac{1}{(n-1)n} - \frac{1}{(n+1)(n+2)} \right] + 2\beta_0,$$

$$B_{L,g}^{(0)}(n) = \frac{8f}{(n+1)(n+2)}, \quad B_{L,q}^{(0)}(n) = \frac{4C_F}{n+1}.$$  

(20)

In Eqs. (20)-(21) we introduce, and shall widely use throughout the rest of the paper, the notion of the so-called nested sums, defined as

$$S_{\pm i}(n) = \sum_{m=1}^{n} \frac{(-1)^m}{m^i}, \quad S_{\pm i,j}(n) = \sum_{m=1}^{n} \frac{(-1)^m}{m^i} S_j(m).$$

(22)

Note that in previous calculations [29, 31] the notion of nested sums [22] has not been yet incorporated. Instead, other notations related to the nested sums of negative indices have been used

$$S_m' \left( \frac{n}{2} \right) = 2^{m-1} \left( S_m(n) + S_{-m}(n) \right), \quad \tilde{S}_m(n) = S_{-1,n}(n) \quad \text{(Ref. [29])},$$

$$K_m(n) = -S_m(n), \quad Q(n) = -S_{-2,1}(n) \quad \text{(Refs. [30, 31])}.$$  

(23)

Coming back to the Mellin transforms [15], [16] and [19], we recall that we are interested in investigation of the PDF’s in the region of low $x$. In the Mellin space it corresponds to small
momenta, and at extremely low $x$, it suffices to restrict the analysis with the first momentum and to study the solutions of (15), (16) and (19) for $n = 1 + \omega$ at $\omega \to 0$. The nested sums $S_m(n)$ (here $n$ is not mandatorily an integer) at positive indices $m$ can be expressed via the familiar Riemann $\Psi$-functions $\Psi(n)$ as

$$S_1(n) = \Psi(n + 1) - \Psi(1), \quad S_m(n) = \zeta_m - \sum_{l=0}^{\infty} \frac{1}{(n + l + 1)^m}, \quad (m > 1),$$

where $\zeta_m$ are Euler constants and the last series in the r.h.s. of (24) is related with the $m$-th derivative of the $\Psi$-function.

A more complicated situation occurs for negative indices $m < 0$ for which the analytical continuation of the nested sums depends on the parity of the starting value of $n$. Since the anomalous dimensions (20) have been calculated for even $n$, in what follows we employ the analytical continuation of $S_{-m}(n)$ and $S_{-m,k}(n)$ starting from even values of $n$. The result is

$$S_1(n) = -\ln 2 - \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{n + l + 1}, \quad S_{-m}(n) = \zeta_m - \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{(n + l + 1)^m} \quad (m \geq 2), \quad \zeta_m = (1 - 2^{-m}) \zeta_m,$$

$$S_{-m,k}(n) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{(l + 1)^m} S_k(l + 1) - \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{(n + l + 1)^m} S_k(n + l + 1),$$

where the above series are well defined for any $n$ including noninteger values.

In what follows we are interested in quantities which contribute to NLO at low $x$, i.e. in the initial series $S_{-2}(n), S_{-3}(n)$ and $S_{-2,1}(n)$, anomalous dimensions and the coefficient functions, at $n = 1 + \omega$. In the vicinity of $\omega = 0$ we have

$$S_{-2}(1+\omega) = 1 - \zeta_2 + O(\omega), \quad S_{-3}(1+\omega) = 1 - \frac{3}{2} \zeta_3 + O(\omega), \quad S_{-2,1}(1+\omega) = 1 - \frac{5}{4} \zeta_3 + O(\omega).$$

III. BDH-LIKE RESULTS

We reiterate that, our analysis is based on Eqs. (8)-(9) or, equivalently, on Eqs. (15), (16) and (19), where the structure function $F_2(x, Q^2)$ is supposed to be known and determined from the existing experimental data. In the present paper we employ the BDH parametrization [6], obtained from a combined fit of the H1 and ZEUS collaborations data [19] in a range of the kinematical variables $x$ and $Q^2$, $x < 0.01$ and $0.15$ GeV$^2 < Q^2 < 3000$ GeV$^2$. The explicit
expression for the BDH parametrization reads as

\[ F_{BDH}(x, Q^2) = D(Q^2)(1 - x)^\nu \sum_{m=0}^{2} A_m(Q^2) L^m \]  

(27)

where the dependence on the effective parameters is encoded in \( D(Q^2) \) and \( A_m(Q^2) \)

\[ D(Q^2) = \frac{Q^2(Q^2 + \lambda M^2)}{(Q^2 + M^2)^2}, \quad A_0(Q^2) = a_{00} + a_{01} L_2, \quad A_i(Q^2) = \sum_{k=0}^{2} a_{ik} L_k, \quad i = (1, 2), \]  

(28)

with the logarithmic terms \( L \) as

\[ L = \ln \frac{1}{x} + L_1, \quad L_1 = \ln \left( \frac{Q^2}{Q^2 + \mu^2} \right), \quad L_2 = \ln \left( \frac{Q^2 + \mu^2}{\mu^2} \right). \]  

(29)

The performed fit of the experimental data [19] provided the following values of the effective parameters [6]:

\[ \mu^2 = 2.82 \pm 0.29 \text{ GeV}^2, \quad M^2 = 0.753 \pm 0.008 \text{ GeV}^2, \quad \nu = 11.49 \pm 0.99, \quad \lambda = 2.430 \pm 0.153, \]  

(30)

and

\[ a_{00} = 0.255 \pm 0.016, \quad a_{01} \cdot 10^4 = 1.475 \pm 0.3025, \]

\[ a_{10} \cdot 10^4 = 8.205 \pm 4.620, \quad a_{11} \cdot 10^2 = -5.148 \pm 0.819, \quad a_{12} \cdot 10^3 = -4.725 \pm 1.010, \]

\[ a_{20} \cdot 10^3 = 2.217 \pm 0.142, \quad a_{21} \cdot 10^2 = 1.244 \pm 0.0860, \quad a_{22} \cdot 10^4 = 5.958 \pm 2.320. \]  

(31)

Notice that, the BDH parametrization (27) is written in the \((x - Q^2)\) space, whereas the equation (19) is in space of Mellin momenta. Hence, before proceeding with consideration of Eq. (19), we transform the BDH parametrization in to Mellin space. Then, in the transformed parametrization \( M_2(n, Q^2) \) we consider the first momenta \( n = 1 + \omega \) and take the limit \( \omega \to 0 \), which corresponds to low \( x \), and obtain

\[ M_{2BDH}(n, Q^2) = D(Q^2) \sum_{m=0}^{2} A_m(Q^2) P_m(\omega, \nu, L_1) + O(\omega) \equiv D(Q^2) \tilde{M}_{2BDH}(n, Q^2), \]  

(32)

where the quantity \( P_m(\omega, \nu, L_1) \) stands for the approximate expression of the integral

\[ \int_0^1 dx x^{\omega-1} (1 - x)^\nu L^k(x) = P_k(\omega, \nu, L_1) + O(\omega). \]  

(33)

The explicit expression for the integral (33) is relegated in Appendix A. It should be noted that \( P_k(\omega, \nu, L_1) \), besides the finite part at \( \omega \to 0 \), contains also negative powers of \( \omega \), i.e.
is a singular function at $\omega = 0$. Namely these singularities are of our interests in further procedure of solving Eqs. (8)-(19). The strategy is as follow: we disregard the finite part of $M_2^{BDH}(n, Q^2)$ keeping only the singular terms. Then the same procedure we repeat for the longitudinal momentum $M_L^{BDH}(n, Q^2)$, Eq. (19), and equate the coefficients in front of each singularity. In such a way we obtain the representation for the longitudinal SF. Actually, in our calculations, we analyse the singularities in a slightly different way by using some specific properties of the expansions over $\omega$, avoiding direct comparison of the singular coefficients, see below. At $\omega \to 0$ the integral (33) becomes independent up on $\nu$ and the singular part of $\tilde{M}_2^{BDH}(n, Q^2)$ can written as

$$\tilde{M}_2^{BDH}(n, Q^2) = \sum_{m=0}^{2} A_m(Q^2) P_m^{\text{sing}}(\omega, L_1) + O(\omega^0),$$

where

$$P_0^{\text{sing}}(\omega) = \frac{1}{\omega}, \quad P_1^{\text{sing}}(\omega, L_1) = \frac{1}{\omega^2} + \frac{L_1}{\omega}, \quad P_2^{\text{sing}}(\omega, L_1) = \frac{2}{\omega^3} + \frac{2L_1}{\omega^2} + \frac{L_1^2}{\omega}.$$

Note that $P_i^{\text{sing}}(\omega) (i = 1, 2, 3)$ in the limit $\omega \to 0$ satisfy the following useful recurrent relations

$$\omega P_0^{\text{sing}}(\omega) = O(\omega^0), \quad \omega P_1^{\text{sing}}(\omega) = P_0^{\text{sing}}(\omega) + O(\omega^0),$$

$$\omega P_2^{\text{sing}}(\omega) = 2P_1^{\text{sing}}(\omega) + O(\omega^0), \quad \omega^2 P_2^{\text{sing}}(\omega) = 2P_0^{\text{sing}}(\omega) + O(\omega^0),$$

which are widely used subsequently.

A. Derivation of $M_2^{BDH}(n, Q^2)$

To conclude the low-x analysis one needs the explicit expressions for $dM_2^{BDH}(n, Q^2)/d \ln Q^2$ in Eq. (19), i.e. the derivatives of the corresponding ingredients

$$\frac{d}{d \ln Q^2} M_2^{BDH}(n, Q^2) = \frac{dD(Q^2)}{d \ln Q^2} \sum_{m=0}^{2} A_m(Q^2) P_m^{\text{sing}}(\omega, L_1)$$

$$+ D(Q^2) \sum_{m=0}^{2} \frac{dA_m(Q^2)}{d \ln Q^2} P_m^{\text{sing}}(\omega, L_1) + D(Q^2) \sum_{m=0}^{2} A_m(Q^2) \frac{dP_m^{\text{sing}}(\omega, L_1)}{d \ln Q^2} + O(\omega^0).$$

Noticing that

$$\frac{d}{d \ln Q^2} L_2 = \frac{Q^2}{Q^2 + \mu^2}, \quad \frac{d}{d \ln Q^2} L_1 = \frac{\mu^2}{Q^2 + \mu^2}, \quad \frac{d}{d \ln Q^2} D = \frac{M^2 Q^2((2 - \lambda)Q^2 + \lambda M^2)}{(Q^2 + M^2)^3} \equiv \tilde{D},$$

(38)
we can write
\[
\frac{d}{d \ln Q^2} A_m = \frac{Q^2}{Q^2 + \mu^2} \tilde{A}_m, \quad \tilde{A}_m = a_{m1} + 2a_{m2}L_2, \quad a_{02} = 0,
\] (39)
\[
\frac{d}{d \ln Q^2} P_m^{\text{sing}}(\omega, L_1) = \frac{Q^2}{Q^2 + \mu^2} P_m^{\text{sing}}(\omega, L_1), \quad P_m^{\text{sing}}(\omega, L_1) = (m-1)P_{m-1}^{\text{sing}}(\omega, L_1).
\] (40)

Collecting all results together, we have
\[
\frac{d}{d \ln Q^2} M_{BDH}^2(n, Q^2) = \sum_{m=0}^{2} \hat{A}_m(Q^2) P_m^{\text{sing}}(\omega, L_1) + O(\omega^0),
\] (41)
where
\[
\hat{A}_2 = \tilde{A}_2, \quad \hat{A}_1 = \tilde{A}_1 + 2DA_2 \frac{\mu^2}{Q^2 + \mu^2}, \quad \hat{A}_0 = \tilde{A}_0 + DA_1 \frac{\mu^2}{Q^2 + \mu^2}
\] (42)
and
\[
\hat{A}_i = \tilde{D}A_i + D\tilde{A}_i \frac{Q^2}{Q^2 + \mu^2}
\] (43)

**IV. LO ANALYSIS**

Herein bellow we present in details the extraction of the longitudinal SF, \(F_L^{\text{BDH}}(x, Q^2)\), in the LO approximation. In the LO Eq. (19) reads as
\[
M_{L,LO}(n, Q^2) = -2B_{L,g}^{(0)}(n) \frac{dM_2(n, Q^2)}{d\ln Q^2} + a_s(Q^2)\tilde{B}_{L,s}^{(0)}(n)M_2(x, Q^2),
\] (44)
where
\[
\tilde{B}_{L,s}^{(0)}(n) = B_{L,s}^{(0)}(n) - B_{L,g}^{(0)}(n) \frac{\gamma_{ss}^{(0)}(n)}{\gamma_{sg}^{(0)}(n)}.
\] (45)

The next step is to consider Eq. (44) at \(n = 1 + \omega\) and to perform the series expansion of the anomalous dimensions and the coefficient functions about \(\omega = 0\). Restricting the expansion up to terms \(\propto \omega^2\) we obtain
\[
\frac{B_{L,g}^{(0)}(1 + \omega)}{e^{\gamma_{sg}^{(0)}(1 + \omega)}} = -\frac{1}{2} \left(1 + \frac{\omega}{4} - \frac{7}{16} \omega^2\right); \quad B_{L,s}^{(0)}(1 + \omega) = 2C_F \left(1 - \frac{\omega}{2} + \frac{\omega^2}{4}\right);
\]
\[
\gamma_{ss}^{(0)}(1 + \omega) = 8C_F \omega \left(\zeta(2) - \frac{5}{8} + \left(\frac{9}{16} - \zeta(3)\right) \omega\right).
\] (46)

Observe that, in the above equation (44) both, the momentum \(M_2(x, Q^2)\) and the derivative \(dM_2(n, Q^2)/d\ln Q^2\) are of a similar form, cf. Eqs. (34) and (41), so that the r.h.s. of (44) with the expansions (46) can be represented as
\[
\sum_{m=0}^{2} A_m P_m^{\text{sing}} \times \left(\text{a cubic polynomial in } \omega\right).
\]
This substantially simplifies the calculations since, in such a case, we can apply the recurrent relations (36) to find the desired coefficients. For instance, for any (known) series of the form

$$T(\omega) = T_0 + T_1\omega + T_2\omega^2 + O(\omega^3)$$  \hspace{1cm} (47)$$

one easily obtains the result

$$T(\omega) \sum_{m=0}^{2} A_m P_m^{\text{sing}} = T_0 \sum_{m=0}^{2} A_m P_m^{\text{sing}} + T_1 \left( 2A_2 P_2^{\text{sing}} + A_1 P_1^{\text{sing}} \right) + 2T_2 A_2 P_1^{\text{sing}} + O(\omega^3)$$

$$= \sum_{m=0}^{2} \overline{A}_m P_m^{\text{sing}} + O(\omega^3),$$ \hspace{1cm} (48)$$

where

$$\overline{A}_2 = T_0 A_2, \quad \overline{A}_1 = T_0 A_1 + 2T_1 A_2, \quad \overline{A}_0 = T_0 A_0 + T_1 A_1 + 2T_2 A_2.$$ \hspace{1cm} (49)$$

It is seen that it is sufficient to determine the coefficients $T_0$, $T_1$ and $T_2$ to find the solution of the system, avoiding in such a way, lengthy and cumbersome calculations. In our case the explicit expressions for $T_i$ can be inferred directly from Eq. (46). Then the coefficients $C_m$ of the LO expansions of the longitudinal Mellin momenta

$$M_{L,LO}(n, Q^2) = \sum_{m=0}^{2} C_m P_m^{\text{sing}} + O(\omega^3),$$ \hspace{1cm} (50)$$

explicitly read as

$$C_2 = \hat{A}_2 + \frac{8}{3} a_s D A_2, \quad C_1 = \hat{A}_1 + \frac{1}{2} \hat{A}_2 + \frac{8}{3} a_s D \left( A_1 + \left( 4\zeta_2 - \frac{7}{2} \right) A_2 \right),$$

$$C_0 = \hat{A}_0 + \frac{1}{4} \hat{A}_2 - \frac{7}{8} \hat{A}_2 + \frac{8}{3} a_s D \left( A_0 + \left( 2\zeta_2 - \frac{7}{2} \right) A_1 + \left( \zeta_2 - 4\zeta_3 + \frac{17}{8} \right) A_2 \right).$$ \hspace{1cm} (51)$$

It is worth mentioning once more that, the above results for $C_i$ ($i = 0, 1, 2$) have been obtained in a straight way avoiding direct comparisons of singularities in (44), as previously reported in [17, 28]. Another important result is that the adopted BDH parametrization for the $F_2(x, Q^2)$ led directly to the same BDH-like form of the longitudinal structure functions $F_{L,LO}^{BDH}(x, Q^2)$ which, consequently, also obey the Froissart boundary condition

$$F_{L,LO}^{BDH}(x, Q^2) = (1 - x)^{\nu_L} \sum_{m=0}^{2} C_m(Q^2) L^m,$$ \hspace{1cm} (52)$$

where $\nu_L = \nu$ and, according to the quark counting rules [35].
V. NLO ANALYSIS

In this section we discuss the longitudinal structure function within the NLO approximation. As before, particular attention is paid to the region of asymptotically small $x \to 0$ in context with the Froissart boundary.

Multiplying both sides of Eq. (19) by the factor $\left(1 + a_s(Q^2)\left[\tilde{\delta}_{sg}^{(1)}(n) - R_{L,g}^{(1)}(n)\right]\right)$, we have

$$\left(1 + a_s(Q^2)\left[\tilde{\delta}_{sg}^{(1)}(n) - R_{L,g}^{(1)}(n)\right]\right) M_L(n, Q^2) = -2 \frac{B_{L,g}^{(0)}(n)}{\gamma_{sg}^{(0)}(n)} \frac{dM_2(n, Q^2)}{dlnQ^2}$$

$$+ a_s(Q^2) \left(\tilde{B}_{L,s}^{(0)}(n) + a_s(Q^2)\tilde{B}_{L,s}^{(1)}(n)\right) M_2(n, Q^2) + O(a_s^3), \quad (53)$$

where

$$\tilde{\delta}_{sa}^{(1)}(n) = \frac{\tilde{\gamma}_{sa}^{(1)}(n)}{\gamma_{sg}^{(0)}(n)}, \quad R_{L,a}^{(1)}(n) = \frac{\tilde{R}_{L,a}^{(1)}(n)}{\tilde{B}_{L,a}^{(0)}(n)} \quad (54)$$

and

$$\tilde{B}_{L,s}^{(1)}(n) = B_{L,s}^{(0)}(n) \left(R_{L,s}^{(1)}(n) + \tilde{\delta}_{sg}^{(1)}(n) - R_{L,g}^{(1)}(n)\right) - B_{L,a}^{(0)}(n)\tilde{\delta}_{ss}^{(1)}(n). \quad (55)$$

Using equations (44) and (53) together, it is straightforward to obtain

$$\left(1 + a_s(Q^2)\left[\tilde{\delta}_{sg}^{(1)}(n) - R_{L,g}^{(1)}(n)\right]\right) M_L(n, Q^2) = M_{L,LO}(n, Q^2) + a_s^2(Q^2)\tilde{B}_{L,s}^{(1)}(n) M_2(n, Q^2). \quad (56)$$

When computing the inverse Mellin transform of (56) we employ the fact that in the l.h.s. of (56), up to $O(a_s^3)$ corrections, we can write

$$a_s(Q^2)\left[\tilde{\delta}_{sg}^{(1)}(n) - R_{L,g}^{(1)}(n)\right] M_L(n, Q^2) = a_s(Q^2)\left[\tilde{\delta}_{sg}^{(1)}(n) - R_{L,g}^{(1)}(n)\right] M_{L,LO}(n, Q^2) + O(a_s^3), \quad (57)$$

where the LO momentum $M_{L,LO}(n, Q^2)$ has already been calculated in the previous section, cf. Eq. (50).

Prior to proceed with the inverse Mellin transforms, it is convenient to extract the singular structure of the NLO coefficients $\tilde{\delta}_{sg}^{(1)}(n)$, $R_{L,g}^{(1)}(n)$ and $\tilde{B}_{L,s}^{(1)}(n)$. We have

$$\tilde{\delta}_{sg}^{(1)}(n) = \tilde{\delta}_{sa}^{(1)}(1 + \omega), \quad R_{L,g}^{(1)}(n) = \frac{\tilde{R}_{L,g}^{(1)}}{\omega} + R_{L,g}^{(1)}(1 + \omega), \quad \tilde{B}_{L,s}^{(1)}(n) = \frac{\tilde{B}_{L,s}^{(1)}}{\omega} + B_{L,s}^{(1)}(1 + \omega), \quad (58)$$

with $(\omega \to 0)$

$$\tilde{B}_{L,s}^{(1)} = \frac{20}{3} C_F \left(3C_A - 2f\right), \quad \tilde{B}_{L,s}^{(1)}(1) = 8 C_F \left[\frac{25}{9} f - \frac{449}{72} C_F + (2C_F - C_A) \left(\zeta_3 + 2\zeta_2 - \frac{59}{72}\right)\right],$$

$$\tilde{\delta}_{sg}^{(1)} = \frac{26}{3} C_A, \quad \tilde{\delta}_{sg}^{(1)}(1) = 3 C_F - \frac{347}{18} C_A, \quad \tilde{R}_{L,g}^{(1)} = -\frac{4}{3} C_A, \quad \tilde{R}_{L,g}^{(1)}(1) = -5 C_F - \frac{4}{9} C_A. \quad (59)$$
Then, the equation (56) can be rewritten in the following form

\[
M_L(n, Q^2) + \frac{a_s(Q^2)}{\omega} \left[ \tilde{\delta}_{sg}(n) - \tilde{R}_{L,g}(n) \right] M_{L,LO}(n, Q^2) = \left( 1 - a_s(Q^2) \left[ \tilde{\delta}_{sg}(n) - \tilde{R}_{L,g}(n) \right] \right) + \times M_{L,LO}(n, Q^2) + a_s^2(Q^2) \left[ \frac{\hat{B}_{L,s}^{(1)}}{\omega} + \overline{B}_{L,s}^{(1)}(n) \right] M_2(n, Q^2) + O(a_s^3).
\] (60)

Now the inverse Mellin transforms of the last equations can be easily performed (see also Appendix B). The result is

\[
F_{BDH,L}(x, Q^2) + a_s(Q^2) \left[ \frac{L}{3} \left( \frac{\hat{B}_{L,s}^{(1)}}{\omega} + \overline{B}_{L,s}^{(1)}(1) \right) \right] M_{2,BDH}(x, Q^2) = \left[ 1 - a_s(Q^2) \left( \tilde{\delta}_{sg}(1) - \tilde{R}_{L,g}(1) \right) \right] F_{L,LO,L}(x, Q^2) - a_s^2(Q^2) \left[ \frac{\hat{B}_{L,s}^{(1)}}{3}L_A + \overline{B}_{L,s}^{(1)}(1) \right] M_{2,BDH}(x, Q^2), (61)
\]

where

\[
L_A = L + \frac{A_1}{2A_2}; \quad L_C = L + \frac{C_1}{2C_2}. \tag{62}
\]

With the considered accuracy the obtained equation (61) can be rewritten as

\[
\left[ 1 + a_s(Q^2) \frac{L}{3} \left( \frac{\hat{B}_{L,s}^{(1)}}{\omega} + \overline{B}_{L,s}^{(1)}(1) \right) \right] F_{L,LO,L}(x, Q^2) = \left[ 1 - a_s(Q^2) \left( \tilde{\delta}_{sg}(1) - \tilde{R}_{L,g}(1) \right) \right] F_{L,LO,L}(x, Q^2) \]

\[-a_s^2(Q^2) \left[ \frac{\hat{B}_{L,s}^{(1)}}{3}L_A + \overline{B}_{L,s}^{(1)}(1) \right] M_{2,BDH}(x, Q^2) + O(a_s^3). \tag{63}
\]

Eventually, the final expression for the longitudinal SF \( \hat{F}_{L,BDH}^{BDH}(x, Q^2) \) reads as

\[
F_{L,BDH}^{BDH}(x, Q^2) = \frac{1}{\left[ 1 + a_s(Q^2) \frac{L}{3} \left( \tilde{\delta}_{sg}(1) - \tilde{R}_{L,g}(1) \right) \right]} \left\{ \left[ 1 - a_s(Q^2) \left( \tilde{\delta}_{sg}(1) - \tilde{R}_{L,g}(1) \right) \right] F_{L,LO,L}(x, Q^2) \right. \\
\left. - a_s^2(Q^2) \left[ \frac{\hat{B}_{L,s}^{(1)}}{3}L_A + \overline{B}_{L,s}^{(1)}(1) \right] F_{2,BDH}^{BDH}(x, Q^2) \right\}. \tag{64}
\]

This is our final expression for the longitudinal SF \( \hat{F}_{L,BDH}^{BDH}(x, Q^2) \) within the NLO approximation for low values of \( x \).

VI. RESULTS

With the explicit form of the basic expressions laid above, we can proceed to extract the longitudinal structure function \( F_L(x, Q^2) \) from data mediated by the BDH-parametrization of \( F_{2,BDH}^{BDH}(x, Q^2) \). In our calculations we employ the standard representation for QCD couplings in...
the LO and NLO (within the \(\overline{MS}\)-scheme) approximations

\[
\alpha_s(Q^2) = \frac{1}{\beta_0 \ln \left( \frac{Q^2}{\Lambda^2} \right)} \quad \text{(LO)},
\]

\[
\alpha_s(Q^2) = \frac{1}{\beta_0 \ln \left( \frac{Q^2}{\Lambda^2} \right)} - \frac{\beta_1 \ln \ln \left( \frac{Q^2}{\Lambda^2} \right)}{\beta_0 \left[ \beta_0 \ln \left( \frac{Q^2}{\Lambda^2} \right) \right]^2} \quad \text{(NLO)},
\]

\(65\)

The QCD parameter \(\Lambda\) has been extracted from the running coupling \(\alpha_s\) normalized at the Z-boson mass, \(\alpha_s(M_Z^2)\), using the \(b\) - and \(c\)-quarks thresholds according to Ref. [37]. Applying this procedure to ZEUS data, with \(\alpha_s(M_Z^2) = 0.1166\) [38], we obtain the following results for \(\Lambda\), cf. Ref. [36]

\[
\text{LO : } \Lambda(f = 5) = 80.80 \text{ MeV}, \quad \Lambda(f = 4) = 136.8 \text{ MeV}, \quad \Lambda(f = 3) = 136.8 \text{ MeV},
\]

\[
\text{NLO : } \Lambda(f = 5) = 195.7 \text{ MeV}, \quad \Lambda(f = 4) = 284.0 \text{ MeV}, \quad \Lambda(f = 3) = 347.2 \text{ MeV}.
\]

(66)

We have calculated the \(Q^2\)-dependence, at low \(x\), of the longitudinal structure function \(F_{BDH}^L(x, Q^2)\) as described above, in the LO, Eq. (52), and NLO, Eq. (64), approximations. Results of calculations and comparison with data of the H1-Collaboration [39] are presented in Fig. [4] where the dashed and solid lines correspond to the extracted SF in the LO and NLO approximations, respectively. Calculations have been performed at fixed value of the invariant mass \(W\), \(W = 230 \text{ GeV}\), allowing the Bjorken variable \(x\) to vary in the interval \((3 \cdot 10^{-5} < x < 7 \cdot 10^{-2})\) when \(Q^2\) varies in the interval \((1 \text{ GeV}^2 < Q^2 < 3000 \text{ GeV}^2)\).

Figure [4] clearly demonstrates that the extraction procedure provides correct behaviors of the extracted SF in both, LO and NLO approximations. At intermediate and high \(Q^2\) the extracted SF’s are in a good agreement with experimental data. In this region the NLO corrections are rather small and can be neglected. A different situation occurs at low \(Q^2 < 5 \text{ GeV}^2\), where the LO \(F_L(x, Q^2)\) substantially exceeds experimental data. The NLO corrections here are negative and result in a better agreement with data. However, at extremely low momenta, \(Q^2 < 1.5 \text{ GeV}^2\), the extracted SF within NLO is still above the experimental data. It should also be mentioned that our calculations are consistent with other theoretical results, obtained, e.g. in the framework of perturbation theory [40, 41] and/or in the Ref. [42] in the so-called \(k_t\)-factorization approach [43], both of which incorporate the Balitsky-Fadin-Kuraev-Lipatov (BFKL) resummation [44] at low \(x\) (for a review of low-x phenomenology see, e.g. cf. Ref. [45]).

Recall that inclusion of the BFKL resummation in study of PDF’s lead to an improvement of the description of data at small \(x\) and nowadays appears as an integral part in a bulk of approaches. The NNPDF Collaboration and the xFitter HERAPDF team [19, 47], whose
FIG. 1: (Color online) The extracted longitudinal structure function $F_L(x, Q^2)$ from the BDH-parametrization of $F_2(x, Q^2)$ at fixed value of the invariant mass $W = 230$ GeV. Dashed line - calculations within the LO approximation, Eq. (52), solid line - structure function within the NLO approximation, Eq. (64). Experimental data are from the H1-Collaboration, Ref. [39]. The Bjorken variable $x$ corresponding to the chosen kinematics lies in the interval $(3 \cdot 10^{-5} < x < 7 \cdot 10^{-2})$.

approaches are based on the DGLAP equations, recently included the BFKL resummation in to their analysis of the combined H1&ZEUS inclusive cross-section [48] achieving, in such a way, a much better description of data [40, 41]. Analogous studies have been performed in Refs. [40, 49–51]. This is in some contrast with results of the standard PDF sets [46, 47, 52] without the BFKL resummations.

A particular interests present the ratio of the longitudinal to transversal cross sections, defined as

$$R_L(x, Q^2) = \frac{F_L(x, Q^2)}{F_2(x, Q^2) - F_L(x, Q^2)}.$$  \hfill (67)

Recently, the H1-Collaboration has reported the ratio $R_L(x, Q^2)$ measured in several kinematical bins of averaged $Q^2$ and $x$, cf. Table 6 of Ref. [39]. Within such kinematics, the invariant mass $W$ changes from $W \sim 230$ GeV to $W \sim 184$ GeV with increase of $Q^2$ and $x$ in the selected bins. In Figure 2 we present the ratio (67), calculated with the extracted $F_L^{BDH}(x, Q^2)$ and parametrized $F_2^{BDH}(x, Q^2)$, in comparison with the mentioned H1-data. The open and full stars are results of calculations within the LO and NLO approximations, where $x$ and $Q^2$ correspond exactly to the experimental bins reported in Ref. [39]. The shaded areas are calculations for two fixed, minimal and maximal, values of the invariant mass $W$ within the chosen bins. From Figure 2 one can infer that the NLO results essentially improve the agreement with data in
FIG. 2: (Color online) The ratio of the longitudinal to transversal cross sections, Eq. (67), calculated with the extracted longitudinal SF within the leading and next-to-leading order approximations. The open and full stars are results of LO and NLO calculations, respectively, within the exact kinematical conditions reported in Ref. 39, i.e. for each experimental point the variable $x$ is taken from the corresponding $(Q^2, x)$-bin. The shaded areas are calculations with minimal and maximal values of $W$ from the Table 6 of Ref. 39, $W = 232$ GeV and $W = 184$ GeV for the upper and lower boundaries, respectively.

Now we proceed with an analysis of the $x$-evolution of the longitudinal SF at fixed $Q^2$. As mentioned above, investigation of $F_L(x, Q^2)$ as a function of $x$ is of interests in connection with theoretical investigations of ultra-high energy processes with cosmic neutrinos and also in the context of the Froissart restrictions at $x \rightarrow 0$. We have calculated the $x$-dependence of the longitudinal SF at several fixed values of $Q^2$ corresponding to H1-Collaboration data. Results are presented in Figure 3 where the $x$-evolution of $F_L(x, Q^2)$ is clearly exhibited. It is seen that, for all values of the presented $Q^2$, the extracted SF within the NLO approximation is in a much better agreement with data. This persuades us that the obtained SF in the NLO approximation can be pertinent in future analysis of the ultra-high energy neutrino data.

In Figure 4 we present the ratio (67) calculated for the same kinematics as in Figure 3. As in previous calculations, the NLO results are in a better agreement with data. It is also seen from Figure 4 that the NLO ratio $R_L(x, Q^2)$ exhibits a tendency to be almost independent on
FIG. 3: (Color online) The longitudinal structure function $F_L(x, Q^2)$ extracted from the BDH-parametrization of $F_2(x, Q^2)$ at fixed $Q^2$ as a function of the Bjorken variable $x$. The dashed lines represent results of calculations within the LO approximation, the solid lines represent the SF obtained within the NLO approximation. Experimental data are from the H1-Collaboration [39].

$x$ in each bin of $Q^2$, decreasing, however, with $Q^2$ increase, as it should be. We also mention that, as in the case of $Q^2$-dependence, our extracted longitudinal SF as a function of $x$ is in a reasonable good agreement with other theoretical predictions, see e.g. [40, 41] for pQCD results and/or Ref. [53] for results obtained within the $k_t$-factorization approach. Likewise our results are in a good agreement with the previous investigations reported in Ref. [14], were a similar analysis has been performed within the framework of pQCD, with the experimental data for the transverse SF $F_2(x, Q^2)$ and the logarithmic derivative $dF_2/d\ln(Q^2)$.

VII. CONCLUSIONS AND OUTLOOK

In this paper, we present a further development of the method of extraction of the longitudinal DIS structure function $F_L(x, Q^2)$ suggested in Refs. [13, 14, 17]. The method relies on the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations and on the Froissart-bounded parametrization of the DIS structure function $F_2(x, Q^2)$. We focus our attention on the kinematical region of low Bjorken variable ($10^{-5} \lesssim x \lesssim 0.1$) in a large interval of the momentum
transfer \((1 \, \text{GeV}^2 \lesssim Q^2 \lesssim 3 \cdot 10^3 \, \text{GeV}^2)\). The extraction procedure has been elaborated for an analysis of the SF \(F_L(x, Q^2)\) within the leading and next-to-leading order approximations. To this end, we consider the transversal SF \(F_2(x, Q^2)\) as known and use the DGLAP equations to relate it to the longitudinal SF. Then, in space of Mellin momenta we find, up to \(\alpha_s^2\) corrections, the corresponding Mellin transforms for the momenta corresponding to low and ultra-low values of \(x\). The inverse Mellin transform provides the south longitudinal SF in the usual \(x - Q^2\) representation. The obtained explicit expression for \(F_L(x, Q^2)\) is entirely determined by the effective parameters of the BDH parametrization \(27\) and is presented in Eq. \(64\). Some comments are in order here. Observe that, the final expression \(64\) contains the dominant logarithmic terms \(\sim \ln(1/x)\) in both the numerator and denominator parts. In principle, due to smallness of the running coupling \(\alpha_s(x, Q^2)\), the denominator part with \(L_C\) can be rewritten in the numerator as an alternate series leading to a behaviour similar to one known within the pQCD, where the NLO corrections, in the considered kinematical region, are negative and large, while NNLO contributions are positive and also large (see, e.g. Ref. \(54\)). A resummation of these contribution would allow to avoid such an alternate behaviour. In our case, this is achieved by keeping the logarithmic term in the denominator without expanding it in to series relative to \(\alpha_s(x, Q^2)\). With some extent, our representation of the basic NLO corrections, Eq. \(64\), can be considered as an effective resummation of the most important, at low \(x\), logarithmic terms in
each order of the perturbation theory. Another observation is that by keeping the logarithmic \( L_C \) terms in the denominator we also manifestly demonstrate that the SF \( F_L(x, Q^2) \) obeys the Froissart conditions. As shown in Appendix [11] the logarithmic part in Eq. (64) is a direct consequence of the inverse Mellin transforms of terms \( \sim 1/\omega \).

We have applied the developed method to extract the longitudinal SF within the kinematical conditions corresponding to that available at the HERA collider. It has been found that, at relatively large \( Q^2 > 10 \text{ GeV}^2 \) both, LO and NLO results reproduce fairly well the experimental data. At smaller \( Q^2 \) the LO approximation fails to describe data, being systematically well above. Accounting for NLO corrections, which at low \( x \) turn out to be negative, substantially improve the description of the SF and the ratio of the longitudinal to transversal cross sections. However, at extremely low momentum transfer \( Q^2 \lesssim 1 \text{ GeV}^2 \), the extracted SF still exceeds data.

We have performed an analysis of the \( x \)-evolution of the extracted SF. It has been demonstrated that the \( x \)-dependence of \( F_L(x, Q^2) \) also reproduces the behavior of the experimental data at low \( x \). Both, the extracted SF and the ratio \( R_L(x, Q^2) \) as functions of \( x \), are in a fairly good agreement with data, herewith the NLO results are in a much better agreement not only with data, but also with other existing theoretical investigations based on perturbative QCD, improved by the BFKL resummation (see Ref. [40, 41] and references therein quoted), as well as with the results [42] obtained in the framework of the \( k_t \)-factorization method [43], also based on the BFKL approach [44]. Calculations of the longitudinal SF based on the traditional pQCD without such improvements turn out to be rather unstable, due to the fact that the subsequent perturbative corrections can be even larger than the previous ones [54, 55]. Incorporation of corrections inspired by the BFKL resummation lead to a substantially more stable results for \( F_L(x, Q^2) \) [55] (see also similar investigations in Refs. [40, 49–51]), aimed to achieve a good description of the combined H1&ZEUS inclusive cross-sections [48].

Apart from the study of the Froissart boundary restrictions, the knowledge of the \( F_L(x, Q^2) \) at low \( x \) is of a great interest in connection with the theoretical treatments of the ultra-high energy processes with cosmic neutrinos. As already mentioned in Introduction, the NLO approximation for \( F_L(x, Q^2) \), i.e. calculations up to \( \alpha_s^2 \)-corrections, corresponds to next-next-to-leading order (NNLO) for \( F_2(x, Q^2) \) which, in the LO, is \( \propto \alpha_s^0 \). Consequently, with the NLO results [61], in our approach it becomes possible to perform NLO and NNLO analyses of the ultra-high energy (\( \sqrt{s} \sim 1 \text{ TeV} \)) neutrino cross-sections similar to NLO [20] and NNLO [21].
investigations based on pQCD. Such calculations are of a great importance in view of expected reliable cross sections from existing and forthcoming data at the IceCube and from the substantially improved IceCube-Gen2 Collaborations. Therefore, a direct comparison of the theoretical predictions with experimental data becomes feasible.

In the kinematical region where the gluon contributions are sizable the (large) corrections within the traditional pQCD can be strongly reduced by a proper change of the factorization and renormalization scales. An analysis of the precise H1&ZEUS combined data obtained within the kinematics near the limit of applicability of pQCD has shown that an employ of effective scales with large parameters provides much smaller high-order perturbative corrections. In such a case the strong couplings decrease as well and, as a rule, calculations with effective scales lead to a better agreement with data (cf. investigations of the NLO corrections in context with high-energy asymptotics of virtual photon-photon collision and studies of $F_L(x, Q^2)$ in the framework of the $k_t$-fragmentation approach). This encourages us to continue our low-$x$ analysis of the SF’s by implementing special change of the factorization and renormalization scales. This is the subject of our further investigations and results will be presented elsewhere.

Furthermore, we plan to improve our approach by accommodating the method to extract, in the LO and NLO approximations, also the gluon densities. We shall note that, the gluon distribution is by far less known, experimentally and theoretically. Even the shape of the gluon density are often taken quite different in different PDF sets, although considered within the same prerequisites of pQCD. However, the range of variation of gluon density strongly decreases when BFKL resummation is included in the analyses at low $x$ (see the most recent publication and discussion therein).

An extraction of the gluon distributions from experimental data, performed within an approach similar to the one suggested in the present paper, can provide valuable additional information on the problem. Such an analysis can be accomplished by employing the charm, $F_{2}^{cc}(x, Q^2)$, and beauty, $F_{2}^{bb}(x, Q^2)$, components of the SF $F_{2}(x, Q^2)$, which are directly related to the gluon density in the photo-gluon fusion reactions (see Ref. and discussion therein). The extracted SF’s can be compared with the recently obtained combined data from the H1&ZEUS-Collaboration for the $F_{2}^{cc}(x, Q^2)$ and $F_{2}^{bb}(x, Q^2)$ and with the theoretical predictions based on pQCD with BFKL corrections included, and also with the results obtained in the framework of the $k_t$-fragmentation.
Furthermore, the charmed parts of transverse $F_{c\bar{c}2}(x, Q^2)$ and longitudinal, $F_{c\bar{c}L}(x, Q^2)$, structure functions being calculated within our approach, can be used to predict the charmed part of the neutrino-nucleon cross-sections at ultra-high energy and to compare with other calculations \cite{21} based on pQCD with BFKL corrections. Yet, the BDH-gluon density itself can serve as a useful tool for estimations of the cross sections with cosmic rays, cf. Ref. \cite{63} (for a most recent review on the subject, see Ref. \cite{64} and references therein quoted). Investigations in this direction are in progress.

In summary, we present a theoretical method to extract, from the experimental data, the longitudinal DIS structure function $F_L(x, Q^2)$ at low $x$ within the leading and next-to-leading order approximations. Explicit, analytical expressions for the structure function in both, LO and NLO, approximations are obtained in terms of the effective parameters of the Froissart-bound parametrization of $F_2(x, Q^2)$ and results of numerical calculations as well as comparisons with available experimental are presented.

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Appendix A: Details of evaluation of some relevant integrals

Herebelow we present the evaluation of the integrals $P_k(\omega, \nu)$ \((k = 0, 1, 2)\) appearing in Eq. \cite{33}. It can be rewritten in a more general form

$$
\hat{P}_k(\omega, \nu) = \int_0^1 dx x^{-1}(1 - x)\nu \left( \ln \frac{1}{x} \right)^k = \left( -\frac{d}{d\omega} \right)^k \int_0^1 dx x^{-1}(1 - x)\nu, \quad (k = 0, 1, 2). \quad (A1)
$$
1. \( k = 0 \)

\[
\hat{P}_0(\omega, \nu) = \int_0^1 dx x^{\omega - 1} (1 - x)^{\nu} = \frac{\Gamma(\omega)\Gamma(\nu + 1)}{\Gamma(\omega + \nu + 1)} = \frac{1}{\omega} \frac{\Gamma(\omega + 1)\Gamma(\nu + 1)}{\Gamma(\omega + \nu + 1)}. 
\] (A2)

The last results in the r.h.s. can be represented as (cf. also [65])

\[
\frac{\Gamma(\omega + 1)\Gamma(\nu + 1)}{\Gamma(\omega + \nu + 1)} = \exp \left[ - \sum_{i=1}^{\infty} S_i(\nu) \omega^i \right], 
\] (A3)

where the nested sums \( S_i(\nu) \) are defined by Eqs. (24) and (25).

Expanding r.h.s. of (A2) in \( \omega \) series, we have

\[
\hat{P}_0(\omega, \nu) = \frac{1}{\omega} - S_1(\nu). 
\] (A4)

2. \( k = 1, 2 \)

For the next basic integral \( \hat{P}_1(\omega) \) we have

\[
\hat{P}_1(\omega, \nu) = \left( -\frac{d}{d\omega} \right) \hat{P}_0(\omega, \nu) = \left( -\frac{d}{d\omega} \right) \frac{1}{\omega} \frac{\Gamma(\omega + 1)\Gamma(\nu + 1)}{\Gamma(\omega + \nu + 1)}. 
\] (A5)

Expanding r.h.s. of (A5) in series w.r.t. \( \omega \), we have

\[
\hat{P}_1(\omega, \nu) = \frac{1}{\omega^2} - Z_2(\nu), 
\] (A6)

where (see, for example, [66])

\[
Z_1(\nu) = S_1(\nu), \quad Z_2(\nu) = \frac{1}{2} S_1^2(\nu) - \frac{1}{2} S_2(\nu), \quad Z_3(\nu) = \frac{1}{6} S_1^3(\nu) - \frac{1}{2} S_1(\nu) S_2(\nu) + S_3(\nu), 
\] (A7)

where \( S_i(\nu) \), for integer \( \nu \), are the known harmonic numbers \( S_i(\nu) = \sum_{k=1}^{\nu} 1/k^i \). For arbitrary arguments \( \nu \), these coefficients are related to the Euler \( \Psi(1 + \nu) \)-function and its derivatives \( \Psi^{(m)}(1 + \nu) = d/(d\nu) \Psi(1 + \nu) \) as

\[
S_1(\nu) = \Psi(1 + \nu) + \gamma_E, \quad S_2(\nu) = \zeta_2 - \Psi^{(1)}(1 + \nu), \quad S_2(\nu) = \frac{1}{2} \left( \Psi^{(2)}(1 + \nu) - \zeta_3 \right), 
\] (A8)

where \( \gamma_E \) is Euler constant and \( \zeta_i \) are Euler \( \zeta \)-functions.

Analogous calculations for \( \hat{P}_2(\omega) \) provide

\[
\hat{P}_2(\omega, \nu) = \left( -\frac{d}{d\omega} \right)^2 \hat{P}_0(\omega, \nu) = \left( -\frac{d}{d\omega} \right)^2 \frac{1}{\omega} \frac{\Gamma(\omega + 1)\Gamma(\nu + 1)}{\Gamma(\omega + \nu + 1)}, 
\] (A9)
which, being expanded in to series about $\omega$, results in

$$\hat{P}_2(\omega, \nu) = 2 \left( \frac{1}{\omega^3} - Z_3(\nu) \right). \tag{A10}$$

Equations (A4), (A6) and (A10) allow to write the considered integral (34) as

$$\int_0^1 dx x^{\omega-1}(1 - x)^\nu L^k(x) = P_k(\omega, \nu) + O(\omega), \tag{A11}$$

where

$$P_0(\omega, \nu) = \frac{1}{\omega} - Z_1(\nu), \quad P_1(\omega, \nu, L_1) = \frac{1}{\omega^2} - Z_2(\nu) + L_1 P_0(\omega, \nu),$$

$$P_2(\omega, \nu, L_1) = 2 \left( \frac{1}{\omega^3} - Z_3(\nu) \right) + 2L_1 P_1(\omega, \nu) + L_1^2 P_0(\omega, \nu). \tag{A12}$$

In Eq. (A12) the finite part of the integral is encoded in functions $Z(\nu)$, Eqs. (A7), while the singular one is given by Eqs. (35).

3. $k = 3$

Consider now the integral

$$\int_0^1 dx x^{\omega-1}(1 - x)^\nu \left( \ln \frac{1}{x} \right)^3 = P_3(\omega, \nu) + O(\omega), \tag{A13}$$

which is the third derivative of $\hat{P}_3(\omega)$ with respect to $\omega$

$$\hat{P}_3(\omega, \nu) = \left( -\frac{d}{d\omega} \right)^3 \hat{P}_0(\omega, \nu) = \left( -\frac{d}{d\omega} \right)^2 \left( \frac{1}{\omega} \frac{\Gamma(\omega + 1)\Gamma(\nu + 1)}{\Gamma(\omega + \nu + 1)} \right). \tag{A14}$$

Eventually, at $\omega \to 0$ we obtain

$$\hat{P}_3(\omega, \nu) = 6 \left( \frac{1}{\omega^4} + O(\omega^0) \right). \tag{A15}$$

Appendix B: Inverse Mellin transforms at low $x$

In this section we present some details of calculations of the inverse Mellin transform of the longitudinal momentum $M_L(n, Q^2)$, Eq. (56). Observe, that $M_L(n, Q^2)$ is expressed via $M_L^{LO}(n, Q^2)$ and $M_2^{BDH}(n, Q^2)$. Hence, it is sufficient to determine the inverse Mellin transforms of $M_2^{BDH}(n, Q^2)$ augmented with some coefficients depending on $\omega$ to find the desired longitudinal SF $F_L(x, Q^2)$. To facilitate the calculations, consider the following auxiliary integral
\[ I(\omega, Q^2) = \int_0^1 dx x^{\omega - 1} \left( \ln \frac{1}{x} \right) F_2^{BDH}(x, Q^2). \]  

(B1)

It is obvious that this integral is proportional to the Mellin transform \( M_2(\omega, Q^2) \) with some coefficients of proportionality as functions of \( \omega \),

\[ I(\omega, Q^2) \propto \left( \frac{K_{-1}}{\omega} + K_0 + K_1 \omega + K_2 \omega^2 + \cdots \right) M_2^{BDH}(\omega, Q^2). \]  

(B2)

If so, we can avoid direct calculations of the inverse Mellin transform of \( M_2^{BDH}(\omega, Q^2) \). Instead, for any constants \( \hat{F}_1 \) and \( \hat{F}_2 \) and vanishing \( \omega \), we can use the obvious relation

\[ \left( \frac{\hat{F}_1}{\omega} + \hat{F}_2 \right) M_2^{BDH}(\omega, Q^2) \xrightarrow{\text{Inverse Mellin}} \left( \frac{\hat{F}_1}{3} L_A + \hat{F}_2 \right) F_2^{BDH}(x, Q^2), \]  

(B3)

where \( L_A = \ln \frac{1}{x} + L_1 + \frac{A_1}{2 A_2} \), cf. Eqs. (62) and (29). The integral \( I(\omega, Q^2) \) in (B1) can be calculated directly by sing Eqs. (A4), (A6), (A10) and (A15). Up to \( \mathcal{O}(\omega^0) \), we have

\[ \int_0^1 dx x^{\omega - 1} \left( \ln \frac{1}{x} \right) F_2^{BDH}(x, Q^2) = D \left[ A_0 \frac{1}{\omega^2} + A_1 \left( \frac{2}{\omega^3} + \frac{L_1}{\omega^2} \right) + A_2 \left( \frac{6}{\omega^4} + \frac{4 L_1}{\omega^3} + \frac{L_2}{\omega^2} \right) \right]. \]  

(B4)

Expression (B4) must be compared with Eq. (B2), which, after insertion of (34)-(35) reads as

\[ \left( \frac{K_{-1}}{\omega} + K_0 + K_1 \omega + K_2 \omega^2 \right) M_2^{BDH}(\omega, Q^2) \xrightarrow{\text{Inverse Mellin}} \left( \frac{K_{-1}}{3} + K_0 + K_1 \omega + K_2 \omega^2 \right) D \left[ A_0 \frac{1}{\omega^2} + A_1 \left( \frac{1}{\omega^2} + \frac{L_1}{\omega} \right) + A_2 \left( \frac{2}{\omega^3} + \frac{2 L_1}{\omega^2} + \frac{L_2}{\omega} \right) \right]. \]  

(B5)

Now, equating in (B4) and (B5) the corresponding coefficients in front of \( \omega^{-k} \) we obtain

\[ K_{-1} = 3, \quad K_0 = -L_1 - \frac{A_1}{2 A_2}, \quad K_1 = -\frac{A_0}{A_2} + \frac{A_1^2}{4 A_2^2}, \]

\[ K_2 = \frac{1}{2} L_1^2 + \frac{3 A_1}{4 A_2} L_1 + \frac{3 A_0}{2 A_2} L_1 + \frac{3 A_0 A_1}{4 A_2^2} - \frac{A_1^3}{8 A_2^2}. \]  

(B6)

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