Uniform semimodular lattice and valuated matroid

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February 27, 2018

Abstract

In this paper, we present a lattice-theoretic characterization for valuated matroids, which is an extension of the well-known cryptomorphic equivalence between matroids and geometric lattices (= atomistic semimodular lattices). We introduce a class of semimodular lattices, called uniform semimodular lattices, and establish a cryptomorphic equivalence between integer-valued valuated matroids and uniform semimodular lattices. Our result includes a coordinate-free lattice-theoretic characterization of integer points in tropical linear spaces, incorporates the Dress-Terhalle completion process of valuated matroids, and establishes a smooth connection with Euclidean buildings of type A.

Keywords: Valuated matroid, uniform semimodular lattice, geometric lattice, tropical linear space, tight span, Euclidean building.

1 Introduction

Matroids can be characterized by various cryptomorphically equivalent axioms; see e.g., [1]. Among them, a lattice-theoretic characterization by Birkhoff [4] is well-known: The lattice of flats of any matroid is a geometric lattice (= atomistic semimodular lattice), and any geometric lattice gives rise to a simple matroid.

The goal of the present article is to extend this classical equivalence to valued matroids (Dress-Wenzel [8, 9]). Valuated matroid is a quantitative generalization of matroid, which abstracts linear dependence structures of vector spaces over a field with a non-Archimedean valuation. A valuated matroid is defined as a function on the base family of a matroid satisfying a certain exchange axiom that originates from the Grassmann-Plücker identity. Just as matroids, valuated matroids enjoy nice optimization properties; they can be optimized by a greedy algorithm, and this property characterizes valuated matroids. In the literature of combinatorial optimization, theory of valuated matroids has evolved into discrete convex analysis [18], which is a framework of “convex” functions on discrete structures generalizing matroids and submodular functions. In tropical geometry (see e.g., [15]), a valuated matroid is called a tropical
Plücker vector. The space of valuated matroids is understood as a tropical analogue of the Grassmann variety in algebraic geometry; see [20] [21].

While Murota-Tamura [19] established cryptomorphically equivalent axioms for valuated matroids in terms of (analogous notions of) circuits, cocircuits, vectors, and covectors, a lattice-theoretic axiom has never been given in the literature. The goal of this paper is to establish a lattice-theoretic axiom for valuated matroids by introducing a new class of semimodular lattices, called uniform semimodular lattices. This class of lattices can be viewed as an affine-counterpart of geometric lattices, and is defined by a fairly simple axiom: They are semimodular lattices with the property that the operator \( x \mapsto \bigvee \{ \text{all elements covering } x \} \) is an automorphism. This operator was introduced in a companion paper [13] to characterize Euclidean buildings in a lattice-theoretic way.

The main result of this paper is a cryptomorphic equivalence between uniform semimodular lattices and integer-valued valuated matroids. The contents of this equivalence and its intriguing features are summarized as follows:

- A valuated matroid is constructed from a uniform semimodular lattice \( L \) as follows. We introduce the notion of a ray and end in \( L \). A ray is a chain of \( L \) with a certain straightness property, and an end is an equivalence class of the parallel relation on the set of rays. Ends will play the role of atoms in a geometric lattice. We introduce a matroid \( M^\infty \) on the set \( E \) of ends, called the matroid at infinity, which will be the underlying matroid of our valuated matroid. As expected from the name, this construction is inspired by the spherical building at infinity in a Euclidean building. A \( \mathbb{Z}^n \)-sublattice \( S(B) \) \((\simeq \mathbb{Z}^n)\) is naturally associated with each base \( B \) in \( M^\infty \), and plays the role of apartments in a Euclidean building. Then a valuated matroid \( \omega = \omega^L \cdot x \) on \( E \) is defined from apartments and any fixed \( x \in L \); the value \( \omega(B) \) is the negative of a “distance” between \( x \) and \( S(B) \). It should be emphasized that this construction is done purely in a coordinate-free lattice-theoretic manner.

- The reverse construction of a uniform semimodular lattice from a valuated matroid uses the concept of the tropical linear space. The tropical linear space \( T(\omega) \) is a polyhedral object in \( \mathbb{R}^E \) associated with a valuated matroid \( \omega \) on \( E \). This concept and the name were introduced by Speyer [20] in the literature of tropical geometry. Essentially equivalent concepts were earlier considered by Dress-Terhalle [6, 7] as the tight span and by Murota-Tamura [19] as the space of covectors. In the case of a matroid (i.e., \( \{0, -\infty\} \)-valued valuated matroid), the tropical linear space reduces to the Bergman fan of the matroid, which is viewed as a geometric realization of the order complex of the geometric lattice of flats [2]. We show that the set \( L(\omega) := T(\omega) \cap \mathbb{Z}^E \) of integer points in \( T(\omega) \) forms a uniform semimodular lattice. Then the original \( \omega \) is recovered by the above construction (up to the projective-equivalence), and \( T(\omega) \) is a geometric realization of a special subcomplex of the order complex of \( L(\omega) \). Thus our result establishes a coordinate-free lattice-theoretic characterization of tropical linear spaces.

- The above constructions incorporate, in a natural way, the completion process of valuated matroids by Dress-Terhalle [6], which is a combinatorial generalization of the \( p \)-adic completion. They introduced an ultrametric metrization of underlying set \( E \) by a valuated matroid \( \omega \), and a completeness concept for valuated matroids.
by the completeness of this metrization of $E$. They proved that any (simple) valuated matroid $(E, \omega)$ is (uniquely) extended to a complete valuated matroid $(\bar{E}, \bar{\omega})$, which is called a completion of $(E, \omega)$.

We show that the space $E$ of ends in a uniform semimodular lattice $\mathcal{L}$ admits an ultrametric metrization $d$, and it is complete in this metric, where $d$ coincides with the Dress-Terhalle metrization of the constructed valuated matroid $\omega^{\mathcal{L}(\omega)}$. Then the process $\omega \rightarrow \mathcal{L}(\omega) \rightarrow \omega^{\mathcal{L}(\omega)}$ coincides with the Dress-Terhalle completion.

- Our result sums up, from a lattice-theory side, connections between valuated matroids and Euclidean buildings (Bruhat and Tits [3]), pointed out by [5, 7, 14]. Let us recall the spherical situation, and recall modular matroid, which is a matroid whose lattice of flats is a modular lattice. We can say that a modular matroid is equivalent to a spherical building of type A [22]. Indeed, a classical result of Birkhoff [4] says that a modular geometric lattice is precisely the direct product of subspace lattices of projective geometries. Another classical result by Tits [22] says that a spherical building of type A is the order complex of the direct product of subspace lattices of projective geometries.

An analogous relation is naturally established for valuated matroids by introducing the notion of a modular valuated matroid, which is defined as a valuated matroid such that the corresponding uniform semimodular lattice is a modular lattice. The companion paper [13] showed that uniform modular lattices are cryptomorphically equivalent to Euclidean buildings of type A. Thus a modular valuated matroid $\omega$ is equivalent to a Euclidean building of type A, in which (the projection of) the tropical linear space $T(\omega)$ is a geometric realization of the Euclidean building. This generalizes a result by Dress and Terhalle [7] obtained for the Euclidean building of $\text{SL}(F^n)$ for a valued field $F$.

The rest of this paper is organized as follows. Sections 2 and 3 are preliminary sections on lattice, (valuated) matroids, and tropical linear spaces. Section 4 constitutes the main body of our results on uniform semimodular lattices. Section 5 discusses three representative examples of valuated matroids in terms of uniform semimodular lattices.

2 Preliminary

Let $\mathbb{R}$ denote the set of real numbers. Let $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the sets of integers and nonnegative integers, respectively. For a set $E$ (not necessarily finite), let $\mathbb{R}^E$, $\mathbb{Z}^E$, and $\mathbb{Z}_+^E$ denote the sets of all functions from $E$ to $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}_+$, respectively. A function $g : E \rightarrow \mathbb{Z}$ is said to be upper-bounded if there is $M \in \mathbb{Z}$ such that $g(e) \leq M$ for all $e \in E$. If $|g(e)| \leq M$ for all $e \in E$, then $g$ is said to be bounded. Let $\mathbf{1}$ denote the all-one vector in $\mathbb{R}^E$, i.e., $1(e) = 1$ ($e \in E$). For a subset $F \subseteq E$, let $1_F$ denote the incidence vector of $F$ in $\mathbb{R}^E$, i.e., $1_F(e) = 1$ if $e \in F$ and zero otherwise. Let $\mathbf{0}$ denote the zero vector. For $x, y \in \mathbb{R}^E$, let $\min(x, y)$ and $\max(x, y)$ denote the vectors in $\mathbb{R}^E$ obtained from $x, y$ by taking componentwise minimum and maximum, respectively; namely $\min(x, y)(e) = \min(x(e), y(e))$ and $\max(x, y)(e) = \max(x(e), y(e))$ for $e \in E$.

The vector order $\leq$ on $\mathbb{R}^E$ is defined by $x \leq y$ if $x(e) \leq y(e)$ for $e \in E$. For $e \in E$ and $B \subseteq E$, we denote $B \cup \{e\}$ and $B \setminus \{e\}$ by $B + e$ and $B - e$, respectively.
2.1 Lattice

We use the standard terminology on posets (partially ordered sets) and lattices (see, e.g., \[\text{[1]}\]), where \(\leq\) means a partial order relation, and \(x < y\) means \(x \leq y\) and \(x \neq y\). A lattice is a poset \(L\) such that every pair \(x, y\) has the greatest common lower bound \(x \land y\) and the least common upper bound \(x \lor y\); the former is called the meet of \(x, y\), and the latter is called the join of \(x, y\). For a subset \(S \subseteq L\), the greatest lower bound of \(S\) (the meet of \(S\)) is denoted by \(\bigwedge S\) (if it exists), and the least upper bound of \(S\) (the join of \(S\)) is denoted by \(\bigvee S\) (if it exists). For elements \(x, y\) with \(x \leq y\), the interval \([x, y]\) of \(x, y\) is the set of elements \(z\) with \(x \leq z \leq y\). If \([x, y] = \{x, y\}\) and \(x \neq y\), then we say that \(y\) covers \(x\) and write \(x \lessdot_1 y\). A chain is a totally ordered subset \(C\) of \(L\); a chain will be written, say, as \(x_0 \lessdot_1 x_1 \lessdot_1 \cdots \lessdot_1 x_m \lessdot_1 \cdots\). The length of a chain \(C\) is defined as its cardinality \(|C|\) minus one. In this paper, we deal with lattices satisfying the following finiteness assumption:

(F) No interval \([x, y]\) has a chain of infinite length.

An order-preserving bijection \(\varphi : L \rightarrow L'\) is called an isomorphism. If \(L = L'\), then isomorphism \(\varphi\) is called an automorphism on \(L\). A sublattice of a lattice \(L\) is a subset \(L' \subseteq L\) with the property that \(x, y \in L'\) imply \(x \land y, x \lor y \in L'\). Intervals are sublattices.

An atom is an element that covers the minimum \(\overline{0} = \bigwedge L\). The rank of \(L\) (having the minimum and maximum) is defined as the maximum length of a maximal chain of \(L\).

A height function of a lattice \(L\) is an integer-valued function \(r : L \rightarrow \mathbb{Z}\) such that \(r(y) = r(x) + 1\) for any \(x, y \in L\) with \(x \lessdot_1 y\).

A lattice \(L\) is said to be semimodular if \(x \land a \lessdot_1 a\) implies \(x \lessdot_1 x \lor a\) for any \(x, a \in L\). From the definition, we easily see that a semimodular lattice satisfies the Jordan-Dedekind chain condition:

(JD) For any interval \([x, y]\), all maximal chains in \([x, y]\) have the same length.

We denote this length by \(r[x, y]\), which is finite by (F).

**Lemma 2.1.** For a lattice \(L\), the following conditions are equivalent:

1. \(L\) is semimodular.
2. For \(a, b \in L\), if \(a, b\) cover \(a \land b\), then \(a \lor b\) covers \(a, b\).
3. \(L\) admits a height function \(r\) satisfying
   \[r(x) + r(y) \geq r(x \land y) + r(x \lor y) \quad (x, y \in L).\]  

**Sketch of proof.** We verify (1) \(\Rightarrow\) (3); other directions are easy or obvious. Fix \(z \in L\), define \(r : L \rightarrow \mathbb{Z}\) by \(r(x) := r[z, x \lor z] - r[x, x \lor z]\). Consider an element \(y\) that covers \(x\). If \(y \leq x \lor z\), then \(x \lor z = y \lor z\) and \(r[y, y \lor z] = r[x, x \lor z] - 1\). If \(y \not\leq x \lor z\), then by semimodularity, \(y \lor z\) covers \(x \lor z\), and hence \(r[y, y \lor z] = r[x, x \lor z]\) and \(r[z, y \lor z] = r[z, x \lor z] + 1\). Thus \(r\) is a height function.

We show (2.1). Consider a maximal chain \(x \land y = z_0 \lessdot_1 z_1 \lessdot_1 \cdots \lessdot_1 z_k = y\), where \(k = r[x \land y, y]\) by (JD). Consider a chain \(x = x \lor z_0 \leq x \lor z_1 \leq \cdots \leq x \lor z_k = x \lor y\), which contains a maximal chain in \([x, y]\) by the semimodularity. This implies \(r(x \lor y) - r(x) = r[x, x \lor y] \leq k = r[x \land y, y] = r(y) - r(x \land y)\), and (2.1).
A modular pair is a pair $x, y \in \mathcal{L}$ satisfying (2.1) in equality. A geometric lattice is a semimodular lattice such that it has the minimum and maximum, and every element can be represented as the join of atoms. A hyperplane in a geometric lattice is an element that is covered by the maximum element. The following is well-known.

**Lemma 2.2** (See, e.g., [1, Section 2.3]). Let $\mathcal{L}$ be a geometric lattice.

1. Every element in $\mathcal{L}$ is written as the meet of hyperplanes.
2. Every interval in $\mathcal{L}$ is a geometric lattice.

A modular lattice is a lattice $\mathcal{L}$ such that for every triple $x, y, z \in \mathcal{L}$ with $x \leq y$ it holds $x \wedge (y \vee z) = x \vee (y \wedge z)$. A modular lattice is precisely a semimodular lattice in which every pair is modular.

### 2.2 Matroid

Here we introduce matroids on a possibly infinite ground set, where our treatment follows [1, Chapter VI]. A matroid $M = (E, \mathcal{I})$ is a pair of a set $E$ and a family $\mathcal{I}$ of subsets of $E$ such that $\emptyset \in \mathcal{I}$, $I' \subseteq I \in \mathcal{I}$ implies $I' \in \mathcal{I}$, and for $I, I' \in \mathcal{I}$ with $|I| < |I'|$ there is $e \in I' \setminus I$ such that $I + e \in \mathcal{I}$, and $\max_{I \in \mathcal{I}} |I| < +\infty$. A member of $\mathcal{I}$ is called an independent set. A maximal independent set is called a base. The set of all bases is denoted by $B$. A loop is an element $e \in E$ such that no base contains $e$. Non-loop elements $e, f \in E$ are said to be parallel if no base contains both $e$ and $f$. The parallel relation gives rise to an equivalence relation on the set of non-loop elements, and an equivalence class is called a parallel class. If matroid $M$ has no loop and no parallel pair, then $M$ is called simple.

**Theorem 2.3** ([4]; see [1, Chapter VI]). (1) For a geometric lattice $\mathcal{L}$ with rank $n$, the pair $M_{\mathcal{L}}$ of the set of atoms and the family of independent atoms is a simple matroid with rank $n$.

(2) The family of flats of a matroid $M$ with rank $n$ is a geometric lattice $\mathcal{L}$ with rank $n$, where $M_{\mathcal{L}}$ is a simplification of $M$.

### 3 Valuated matroid and tropical linear space

Let $M = (E, \mathcal{B})$ be a matroid with rank $n$. A valuated matroid on $M$ is a function $\omega : \mathcal{B} \to \mathbb{R}$ satisfying:
(EXC) For \( B, B' \in \mathcal{B} \) and \( e \in B \setminus B' \) there is \( e' \in B' \setminus B \) such that
\[
\omega(B) + \omega(B') \leq \omega(B + e' - e) + \omega(B' + e - e').
\] (3.1)

A valuated matroid \( \omega \) is viewed as a function on the set of all \( n \)-element subsets of \( E \) by defining \( \omega(B) = -\infty \) for \( B \notin \mathcal{B} \). A valuated matroid is also written as a pair \((E, \omega)\).

A valuated matroid is called simple if the underlying matroid is a simple matroid.

Lemma 3.1 ([6]). Let \((E, \omega)\) be a valuated matroid. If \( e, f \in E \) are parallel in the underlying matroid, then there is \( \alpha \in \mathbb{R} \) such that \( \omega(K + e) = \omega(K + f) + \alpha \) for every \((n - 1)\)-element subset \( K \subseteq E \setminus \{e, f\} \).

Therefore no essential information is lost when a valuated matroid \((E, \omega)\) is restricted to a simplification of the underlying matroid. The obtained simple valuated matroid \((\tilde{E}, \tilde{\omega})\) is called a simplification of \((E, \omega)\).

For \( \omega : \mathcal{B} \to \mathbb{R} \) and \( x \in \mathbb{R}^E \), define \( \omega + x : \mathcal{B} \to \mathbb{R} \) by
\[
(\omega + x)(B) := \omega(B) + \sum_{e \in B} x(e) \quad (B \in \mathcal{B}).
\]

It is easy to see from (EXC) that \( \omega + x \) is a valuated matroid if \( \omega \) is a valuated matroid.

Two valuated matroids \( \omega \) and \( \omega' \) are said to be projectively equivalent if \( \omega' = \omega + x \) for some \( x \in \mathbb{R}^E \).

For \( \omega : \mathcal{B} \to \mathbb{R} \), let \( \mathcal{B}_\omega \) be the set of all bases \( B \) that attain \( \max_{B \in \mathcal{B}} \omega(B) \). A direct consequence of (EXC) is as follows.

Lemma 3.2 ([8]; see [17, Theorem 5.2.7]). Let \( \omega \) be a valuated matroid on \((E, \mathcal{B})\). A base \( B \in \mathcal{B} \) belongs to \( \mathcal{B}_\omega \) if and only if
\[
\omega(B - e + f) \leq \omega(B)
\]
for all \( e \in B \) and \( f \in E \setminus B \) with \( B - e + f \in \mathcal{B} \).

One can also observe from (EXC) that for a valuated matroid \( \omega \), the maximizer family \( \mathcal{B}_\omega \) is the base family of a matroid. Murota [16] proved that this property characterizes valuated matroids when \( E \) is finite.

Lemma 3.3 ([16]; see [17, Theorem 5.2.26]). Let \( M = (E, \mathcal{B}) \) be a matroid. An upper-bounded integer-valued function \( \omega : \mathcal{B} \to \mathbb{Z} \) is a valuated matroid if and only if \((E, \mathcal{B}_{\omega + x})\) is a matroid for every bounded integer vector \( x \in \mathbb{Z}^E \).

Proof: Reduction to finite case. We reduce the proof of the if-part to finite case. Consider bases \( B, B' \in \mathcal{B} \). Let \( E' := B \cup B' \), and let \((E', \omega')\) be the restriction of \((E, \omega)\). By the upper-boundedness of \( \omega \), for \( x' \in \mathbb{Z}^{E'} \), by choosing a large positive integer \( M \) and defining \( x(e) := -M \) (\( e \in E \setminus E' \)), we can extend \( x' \) to bounded vector \( x \in \mathbb{Z}^E \) so that \( \mathcal{B}_{\omega + x} = \mathcal{B}_{\omega' + x'} \subseteq 2^{E'} \). Thus the exchange property (EXC) of \( \omega \) on \( B \) and \( B' \) follows from that of \( \omega' \). \( \square \)

Next we introduce the tropical linear space [19, 20] of valuated matroid. Let \( \omega \) be an integer-valued valuated matroid on \((E, \mathcal{B})\). To deal with a possible infiniteness of \( E \),
we here employ the following definition. The tropical linear space $T(\omega)$ of $\omega$ is defined as the set of all vectors $x \in \mathbb{R}^E$ such that matroid $M_{\omega+x} = (E, B_{\omega+x})$ has no loop, i.e.,

$$T(\omega) := \{ x \in \mathbb{R}^E \mid M_{\omega+x} \text{ has no loop} \}.$$ 

This definition tacitly imposes that the maximum of $\omega + x$ for $x \in T(\omega)$ is attained by some $B \in \mathcal{B}$. According to the definition in [19, 20], the tropical linear space is the set of all points $x \in \mathbb{R}^E$ satisfying:

(TW) For any $(n+1)$-element subset $C \subseteq E$, the maximum of $\omega(C - f) - x(f)$ over all $f \in C$ with $C - f \in \mathcal{B}$ is attained at least twice.

(In the definition of [19], the sign of $x$ is opposite.) Speyer [20, Proposition 2.3] proved that the two definitions are equivalent when $E$ is finite. Our infinite setting needs a little care; we prove a slightly modified equivalence in Lemma 3.8 below.

In the literature, the tropical linear space is referred to as its projection $T(\omega)/\mathbb{R}1$, since $x \in T(\omega)$ implies $x + \mathbb{R}1 \subseteq T(\omega)$. Earlier than [19, 20], Dress and Terhalle [6, 7] introduced the concept of the tight span $TS(\omega)$ of $\omega$, which is defined by

$$TS(\omega) := \left\{ p \in \mathbb{R}^E \mid p(e) = \max_{B \in \mathcal{B}: e \in B} \{ \omega(B) - \sum_{f \in B \setminus \{e\}} p(f) \} \quad (e \in E) \right\}.$$ 

Observe that $TS(\omega)$ is the set of representatives of the negative of $T(\omega)/\mathbb{R}1$. More precisely, it holds

$$TS(\omega) = -\{ x \in T(\omega) \mid \max_{B \in \mathcal{B}} (\omega + x)(B) = 0 \}. \quad (3.2)$$

Dress and Terhalle [6, 7] introduced an ultrametric metrization of the ground set $E$ of valuated matroid $\omega$, which we explain below. Let us recall the notion of ultrametric. An ultrametric on a set $X$ is a metric $d : X \times X \to \mathbb{R}_+$ satisfying the ultrametric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad (x, y, z \in X). \quad (3.3)$$

For $p \in TS(\omega)$, define $D_p : E \times E \to \mathbb{R}$ by

$$D_p(e, f) := \begin{cases} \exp(\max\{(\omega - p)(B) \mid B \in \mathcal{B} : \{e, f\} \subseteq B\}) & \text{if } e \neq f, \\ 0 & \text{if } e = f \end{cases} \quad (e, f \in E).$$

**Proposition 3.4** ([6]). Let $(E, \omega)$ be a simple valuated matroid. For $p \in TS(\omega)$, we have the following:

1. $D_p$ is an ultrametric.
2. For $q \in TS(\omega)$, it holds $\alpha D_p \leq D_q \leq \beta D_p$ for some $\alpha, \beta > 0$.

A simple valuated matroid $(E, \omega)$ is called complete if the metric space $(E, D_p)$ is a complete metric space. By the property (2) the convergence property is independent of the choice of $p \in TS(\omega)$. A completion of a valuated matroid $(E, \omega)$ is a complete valuated matroid $(\bar{E}, \bar{\omega})$ with the properties that $\bar{E}$ contains $E$ as a dense subset, and $\omega$ is equal to the restriction of $\bar{\omega}$ to $n$-element subsets in $E$. 

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Theorem 3.5 ([6]). For a simple valuated matroid \((E, \omega)\), there is a (unique) completion \((\tilde{E}, \tilde{\omega})\) of \((E, \omega)\).

The construction of a completion of valuated matroid \((E, \omega)\) is analogous to (and generalizes) that of \(p\)-adic numbers from rational numbers: Consider the set \(\tilde{E}\) of all Cauchy sequences \((x_i)\), relative to \(D_p\), modulo the equivalence relation \(\sim\) defined by \((x_i) \sim (y_i) \iff \lim_{i \to \infty} D_p(x_i, y_i) = 0\). Regard \(E\) as a subset of \(\tilde{E}\) by associating \(x \in E\) with a Cauchy sequence converging to \(x\), and extend \(D_p\) to \(\tilde{E} \times \tilde{E} \to \mathbb{R}\) by \(D_p((x_i), (y_i)) := \lim_{i \to \infty} D_p(x_i, y_i)\). Then \(\tilde{E}\) is a complete metric space containing \(E\) as a dense subset. Accordingly, \(\omega\) is extended to \(\tilde{\omega}\) by

\[
\tilde{\omega}(B) := \lim_{i \to \infty} \omega(B_i),
\]

where \(B_i \subseteq E\) consists of \(n\) elements each converging to an element of \(B \subseteq \tilde{E}\). By a completion of nonsimple valuated matroid \((E, \omega)\) we mean a completion of a simplification of \((E, \omega)\). In Section 4.3, we give a natural interpretation of this completion process via a uniform semimodular lattice.

The rest of this section is to give some basic properties of the tropical linear space \(\mathcal{T}(\omega)\). Let \((E, \omega)\) be an integer-valued valuated matroid on underlying matroid \(\mathbf{M} = (E, \mathcal{B})\) of rank \(n\). We suppose that \(\mathcal{T}(\omega)\) is endowed with vector order \(\leq\).

Lemma 3.6. Let \((\tilde{E}, \tilde{\omega})\) be a simplification of \((E, \omega)\). Then the projection \(x \mapsto x|\tilde{E}\) is an order-preserving bijection from \(\mathcal{T}(\omega)\) to \(\mathcal{T}(\tilde{\omega})\).

Proof. Let \(e, f \in E\) be a parallel pair with \(\omega(K + e) = \omega(K + f) + \alpha\) for every \((n - 1)\)-element subset \(K \subseteq E \setminus \{e, f\}\); see Lemma 3.1. Let \(x \in \mathcal{T}(\omega)\). If \(B \in \mathcal{B}_{\omega+e}\) contains \(e\), then \(B - e + f \in \mathcal{B}_{\omega+e}\). From this, we see that \(\mathcal{B}_{\omega+x|E}\) is a subset of \(\mathcal{B}_{\omega+e}\) and \(\mathcal{M}_{\omega+x|E}\) has no loop. Thus the projection \(x \mapsto x|\tilde{E}\) is an order-preserving map from \(\mathcal{T}(\omega)\) to \(\mathcal{T}(\tilde{\omega})\). Since \(e, f\) are also parallel in \(\mathcal{B}_{\omega+e}\), it must hold \((\omega + x)(B) = (\omega + x)(B - e + f)\) for \(B \in \mathcal{B}_{\omega+e}\) with \(e \in B\). By \(\omega(B) = \omega(B - e + f) + \alpha\), we obtain \(x(e) + \alpha = x(f)\), where \(\alpha\) is independent of \(x\). This means that the coordinate of \(e\) in \(\mathcal{T}(\omega)\) is recovered from that of \(f\). From this, we see that the projection is a bijection.\[\square\]

For \(x \in \mathbb{R}_E\), let \([x] \in \mathbb{Z}_E\) denote the vector obtained from \(x\) by rounding down each fractional component of \(x\), i.e. \([x](e) := [x(e)]\) for \(e \in E\).

Lemma 3.7. For \(x \in \mathcal{T}(\omega)\), we have the following:

1. \([x] \in \mathcal{T}(\omega)\).

2. There are a chain \(\emptyset \neq F_1 \subset F_2 \subset \cdots \subset F_n = E\) of flats in \(\mathbf{M}_{\omega+|x|}\) and coefficients \(\lambda_i \geq 0\) such that \(\sum_{i=1}^n \lambda_i < 1\) and \(x = [x] + \sum_{i=1}^n \lambda_i 1_{F_i}\).

Proof. (1). Let \(B \in \mathcal{B}_{\omega+e}\). By Lemma 3.2 we have \((\omega + x)(B + e - f) \leq (\omega + x)(B)\) for all \(e \in E \setminus B\) and \(f \in B\). From this, we have

\[
(\omega + [x])(B + e - f) + \Delta(e) - \Delta(f) \leq (\omega + [x])(B),
\]

where \(\Delta(g) := x(g) - [x(g)]\) for \(g \in E\). Since \(\Delta(e) - \Delta(f) > -1\) and \(\omega\) is integer-valued, we have \((\omega + [x])(B + e - f) \leq (\omega + [x])(B)\). By Lemma 3.2 again, we have \(B \in \mathcal{B}_{\omega+|x|}\). Hence \(\mathcal{B}_{\omega+e} \subseteq \mathcal{B}_{\omega+|x|}\). Therefore \(\mathbf{M}_{\omega+|x|}\) has no loop.
(2). It suffices to show: If \( x \in \mathcal{T}(\omega) \) and \( \alpha \in [0,1) \), then \( F_\alpha := \{ e \in E \mid x(e) - [x(e)] \geq \alpha \} \) is a flat of matroid \( M_{\omega+\alpha} \). By \( \mathcal{B}_{\omega+\alpha} \subseteq \mathcal{B}_{\omega+\alpha} \) shown above, one can see that \( \mathcal{B}_{\omega+\alpha} \) is the maximizer family of linear objective function \( B \mapsto (x - [x])(B) \) over \( \mathcal{B}_{\omega+\alpha} \). Suppose to the contrary that \( e \in \text{cl}(F_\alpha ) \setminus F_\alpha \) exists. There is a base \( B \in \mathcal{B}_{\omega+\alpha} \) containing \( e \). Then \( \text{cl}(B - e) \supseteq F_\alpha \) since otherwise \( e \not\in \text{cl}(B - e) = \text{cl}(\text{cl}(B - e)) \supseteq \text{cl}(F_\alpha ) \ni e \). Thus we can choose \( f \in F_\alpha \) such that \( B + f - e \in \mathcal{B}_{\omega+\alpha} \). But the above linear objective function increases strictly. This is a contradiction. \( \square \)

**Lemma 3.8.** A vector \( x \in \mathbb{R}^E \) belongs to \( \mathcal{T}(\omega) \) if and only if the maximum of \( \omega + x \) over \( B \) is attained and \( x \) satisfies (TW).

**Proof.** (If part). Consider \( B \in \mathcal{B}_{\omega+\alpha} \) and \( e \in E \setminus B \). Then \( \max_{f \in B + e; B + e - f \in \mathcal{B}} \omega(B + e - f) - x(f) \) is attained by \( f = e \) (Lemma 3.2), and \( f \neq e \) by (TW). This means that \( B + e - f \in \mathcal{B}_{\omega+\alpha} \). Thus \( M_{\omega+\alpha} \) is loop-free.

(Only if part). By Lemma 3.7 (2) and \( |F| \setminus B| \subseteq \{0, 1, 2, \ldots, n\} \), it holds \( \{(\omega + x)(B) \mid B \in \mathcal{B}\} \subseteq \mathbb{Z} + U \) for a finite set \( U = \{ \sum_{i=1}^n k_i \mid 0 \leq k_i \leq n \} \). Consequently the maximum of \( \omega + x + \alpha 1_F \) is attained for all \( \alpha \geq 0 \) and \( F \subseteq E \). The rest is precisely the same as in the proof of [20, Proposition 2.3]. Consider an arbitrary \( n + 1 \) element subset \( C \). As \( \alpha \geq 0 \) increases, the maximizer family \( \mathcal{B}_{\omega+\alpha+1C} \) changes finitely many times. Also, for large \( \alpha \geq 0 \), \( \mathcal{B}_{\omega+\alpha+1C} \) consists only of bases \( B \in \mathcal{B} \) with \( B \subseteq C \). We show that each \( e \in C \) is not a loop in \( \mathcal{B}_{\omega+\alpha+1C} \) for \( \alpha \geq 0 \). For small \( \epsilon > 0 \), any base \( B \) in \( \mathcal{B}_{\omega+\alpha+1C} \) with maximal \( C \cap B \) is also a base in \( \mathcal{B}_{\omega+\alpha+1C} \); see below. Since each \( e \in C \) is not a loop in \( M_{\omega+\alpha} \), so is in \( M_{\omega+\alpha+1C} \). Thus, for large \( \alpha > 0 \), the maximum of \( \omega + x + \alpha 1_C \) must be attained by at least two bases in \( C \), which implies (TW). \( \square \)

In the last step of the proof, we use the following lemma:

**Lemma 3.9** ([20]). Let \( x \in \mathcal{T}(\omega) \) and \( F \subseteq E \). Any base \( B \in \mathcal{B}_{\omega+\alpha} \) with maximal \( B \cap F \) belongs to \( \mathcal{B}_{\omega+\alpha+1F} \) for sufficiently small \( \alpha > 0 \). If \( \omega \) and \( x \) are integer-valued, then we can take \( \alpha = 1 \).

**Proof.** We only show the case where \( \omega \) and \( x \) are integer-valued; the proof of non-integral case is essentially the same. We can assume that \( x = 0 \). Consider a base \( B \in \mathcal{B}_{\omega} \) with maximal \( B \cap F \). By Lemma 3.2 it suffices to show that \( \omega(B) + |B \cap F| \geq \omega(B - e + f) + |(B - e + f) \cap F| \) for \( e \in B \) and \( f \in E \setminus B \) with \( B - e + f \in \mathcal{B} \). If \( f \notin F \) or \( e \notin F \), then this obviously holds. Suppose \( f \in F \) and \( e \notin F \). Then \( |(B - e + f) \cap F| = 1 + |B \cap F| \).

By the maximality, \( B - e + f \) does not belongs to \( \mathcal{B}_{\omega} \), implying \( \omega(B - e + f) \leq \omega(B) - 1 \). Thus \( \omega(B) + |B \cap F| \geq \omega(B - e + f) + |(B - e + f) \cap F| \). \( \square \)

The tropical linear space enjoys a tropical version of convexity introduced by Develin-Sturmfels [10]. A subset \( Q \subseteq \mathbb{R}^E \) is said to be **tropically convex** [10] if \( \min(x + \alpha 1, y + \beta 1) \in Q \) for all \( x, y \in Q \) and \( \alpha, \beta \in \mathbb{R} \). An equivalent condition for the tropical convexity consists of (TC\(_\alpha\)) and (TC\(_{\alpha+1}\)) below:

\[
(\text{TC\(_\alpha\)}) \min(x, y) \in Q \text{ for all } x, y \in Q.
\]

\[
(\text{TC\(_{\alpha+1}\)}) x + \alpha 1 \in Q \text{ for all } x \in Q, \alpha \in \mathbb{R}.
\]

These two properties of \( \mathcal{T}(\omega) \) were recognized by Murota-Tamura [19] (in finite case).

**Lemma 3.10** ([19, Theorem 3.4]; see also [12, Proposition 2.14]). The tropical linear space \( \mathcal{T}(\omega) \) is tropically convex.
Lemma 4.2. Modular lattices.

Some of them were introduced and proved in [13] for uniform
some of basic properties, which will be a basis of our cryptomorphic equivalence to
In this section, we introduce basic concepts on uniform semimodular lattices and prove

4.1 Basic concepts and properties

Proof. We show that $T(\omega)$ satisfies (TC), while (TC$_{+1}$) is obvious. Let $x, y \in T(\omega)$. As in the proof of Lemma 3.8, we see from Lemma 3.7 (2) that the image $\{(\omega + x \land y)(B) \mid B \in B\}$ of $\omega + x \land y$ is discrete in $R$. Hence the maximum of $\omega + x \land y$ is attained by some base. Let $C$ be an $(n + 1)$-element subset of $E$. We may assume that

max$_f \omega(C - f) - x(f) \geq$ max$_f \omega(C - f) - y(f).$ Necessarily max$_f \omega(C - f) - (x \land y)(f) = max_f \omega(C - f) - x(f).$ By (TW) and Lemma 3.8, we can choose distinct $e, e' \in C$ that attain max$_f \omega(C - f) - x(f).$ Necessarily $x(e) = (x \land y)(e)$ and $x(e') = (x \land y)(e').$

Thus $e, e'$ attain max$_f \omega(C - f) - (x \land y)(f).$ By Lemma 3.8 we have $x \land y \in T(\omega).$ 

By this property, $T(\omega) \cap Z^E$ becomes a lattice with respect to vector order $\leq$. In
the next section, we characterize this lattice $T(\omega) \cap Z^E$.

4 Uniform semimodular lattice

The ascending operator of a lattice $L$ is a map $(\cdot)^+ : L \to L$ defined by

$$(x)^+ := \bigvee \{y \in L \mid y \text{ covers } x\}. $$

A uniform semimodular lattice $L$ is a uniform semimodular lattice such that the ascending operator $(\cdot)^+$ is defined, and is an automorphism on $L$. If, in addition, $L$ is a modular lattice, then $L$ is called a uniform modular lattice. The condition for $(\cdot)^+$ is viewed as a lattice-theoretic analogue of condition (TC$_{+1}$). The simplest but important example of a uniform (semi)modular lattice is $Z^m$:

Example 4.1. View $Z^m$ as a poset with vector order $\leq$. Then $Z^m$ is a lattice, where the
join $x \lor y$ and meet $x \land y$ are $\max(x, y)$ and $\min(x, y)$, respectively. The component sum
$x \mapsto \sum_{i=1}^m x_i$ is a height function satisfying semimodular inequality (2.1) (in equality). Therefore $Z^m$ is a (semi)modular lattice. Observe that the ascending operator is equal to $x \mapsto x + 1$, which is obviously an automorphism. Thus $Z^m$ is a uniform (semi)modular lattice.

4.1 Basic concepts and properties

In this section, we introduce basic concepts on uniform semimodular lattices and prove
some of basic properties, which will be a basis of our cryptomorphic equivalence to valuated matroids. Some of them were introduced and proved in [13] for uniform modular lattices.

Let $L$ be a uniform semimodular lattice.

Lemma 4.2. For $x, y \in L$, the intervals $[x, (x)^+]$ and $[y, (y)^+]$ are geometric lattices of
the same rank.

Proof. By definition, $(x)^+$ is the join of all atoms of $[x, (x)^+]$. Hence $[x, (x)^+]$ is a geometric lattice. We show that $[x, (x)^+]$ and $[y, (y)^+]$ have the same rank. It suffices to consider the case where $y$ covers $x$. Since $(\cdot)^+$ is an automorphism, $(y)^+$ covers $(x)^+$. Therefore we have $1 + r[y, (y)^+] = r[x, (x)^+] = r[x, (x)^+] + 1$ (by (JD)), which implies $r[x, (x)^+] = r[y, (y)^+]$. 

The uniform-rank of $L$ is defined as the rank $r[x, (x)^+]$ of interval $[x, (x)^+]$ for $x \in L$. We next study the inverse $(\cdot)^-$ of the ascending operator $(\cdot)^+$. 

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Lemma 4.3. The inverse \((\cdot)^-\) of \((\cdot)^+\) is given by

\[
(x)^- = \bigvee \{ y \in \mathcal{L} \mid y \text{ is covered by } x \} \quad (x \in \mathcal{L}).
\]  

(4.1)

Proof. Suppose that \(y \in \mathcal{L}\) is covered by \((x)^+\). Since \((\cdot)^+\) is an automorphism, there is \(y' \in \mathcal{L}\) such that \((y')^+ = y\). Also \(x\) covers \(y'\), which implies \(x \preceq (y')^+ = y\) by the definition of \((\cdot)^+\). Namely \(y\) belongs to \([x, (x)^+]\). Now \(x\) is also the meet of all hyperplanes of geometric lattice \([x, (x)^+]\) (Lemma 2.2 (1)). By the above argument, they are exactly elements covered by \((x)^+\) in \(\mathcal{L}\). This means that the right hand side of (4.1) exists, and equals \((x)^-\). \(\square\)

For \(x \in \mathcal{L}\) and \(k \in \mathbb{Z}\), define \((x)^{+k}\) by

\[
(x)^{+k} := \begin{cases} 
  x & \text{if } k = 0, \\
  ((x)^{+(k-1)})^+ & \text{if } k > 0, \\
  ((x)^{+(k+1)})^- & \text{if } k < 0.
\end{cases}
\]

For \(k > 0\), we denote \((x)^{+(-k)}\) by \((x)^{-k}\).

Lemma 4.4. For \(x, y \in \mathcal{L}\), there is \(k \geq 0\) such that \(x \preceq (y)^{+k}\).

Proof. We may assume that \(x \not\preceq y\). Hence \(x \not> x \wedge y\). Choose an atom \(a\) in \([x \wedge y, x]\). Then \(a \wedge y = x \wedge y\). By semimodularity, \(a \vee y\) is an atom in \([y, (y)^+]\), and \(x \wedge y \prec a \preceq x \wedge (y)^+\). Thus, for \(k \geq r[x \wedge y, x]\), it holds \(x \wedge (y)^{+k} = x\), implying \(x \preceq (y)^{+k}\). \(\square\)

4.1.1 Segment and ray

A segment is a chain \(e^0 \prec e^1 \prec \cdots \prec e^s\) such that \(e^\ell\) covers \(e^{\ell-1}\) for \(\ell = 1, 2, \ldots, s\), and \(e^{\ell+1} \not\in [e^{\ell-1}, (e^{\ell-1})^+](\ni e^{\ell})\) for \(\ell = 1, 2, \ldots, s-1\). A ray is an infinite chain \(e^0 \prec e^1 \prec \cdots\) such that \(e^0 \prec e^1 \prec \cdots \prec e^s\) is a segment for all \(\ell\). The following characterization of segments was suggested by K. Hayashi.

Lemma 4.5. A chain \(x = e^0 \prec e^1 \prec \cdots \prec e^s = y\) is a segment if and only if \([x, y] = \{e^0, e^1, \ldots, e^s\}\).

Proof. (If part). Suppose to the contrary that \(e^{\ell+1} \in [e^{\ell-1}, (e^{\ell-1})^+]\). Then there is an atom \(a\) in \([e^{\ell-1}, (e^{\ell-1})^+]\) such that \(e^{\ell+1} = a \vee e^{\ell}\) (by Lemma 2.2 (2)). This implies that \(a \in [e^{\ell-1}, e^{\ell+1}]\), which contradicts \([e^{\ell-1}, e^{\ell+1}] = \{e^{\ell-1}, e^{\ell}, e^{\ell+1}\}\).

(Only if part). We use the induction on the length \(s\); the case of \(s = 1\) is obvious. Suppose that \([x, e^{s-1}] = \{e^0, e^1, \ldots, e^{s-1}\}\), and suppose to the contrary that \([x, y]\) properly contains \(\{e^0, e^1, \ldots, e^s\}\). Then (by induction) there is an atom \(a\) of \([x, y]\) not belonging to \(\{e^0, e^1, \ldots, e^s\}\). In particular, \(a \not\in e^{s-1}\). By semimodularity, \(a \vee e^{s-1}\) covers \(e^{s-1}\), and is equal to \(e^s\). Consider \(e^{s-2} \vee a\), which covers \(e^{s-2}\) and is not equal to \(e^{s-1}\) (by \(a \not\in e^{s-1}\)). The join \((e^{s-2} \vee a) \vee e^{s-1}\) is equal to \(e^s\). However this contradicts \(e^s \in [e^{s-2}, (e^{s-2})^+]\). \(\square\)

A ray (or segment) \(e^0 \prec e^1 \prec \cdots\) with \(x = e^0\) is called an x-ray (or x-segment).

Lemma 4.6. Let \(x = e^0 \prec e^1 \prec \cdots \prec e^s\) be an x-segment. For \(p \succeq x\) with \(p \wedge e^1 = x\), chain \(p = p \vee e^0 \prec p \vee e^1 \prec \cdots \prec p \vee e^s\) is a p-segment.
Proof. It suffices to consider the case where \( p \) covers \( x \). By \( p \neq e^1 \) and \( [e^0, e^\ell] = \{e^0, e^1, \ldots, e^\ell\} \) by Lemma 4.5, it holds \( p \not\leq e^\ell \). Then, by semimodularity, \((p, e^\ell)\) is a modular pair. Consequently, \( p \lor e^{\ell+1} \) covers \( p \lor e^\ell \) and \( e^{\ell+1} \). Let \( f^\ell := p \lor e^\ell \). We show \((f^\ell)^+ = (f^{\ell-1})^+ \lor f^{\ell+1} \), which implies \( f^{\ell+1} \not\in [f^{\ell-1}, (f^{\ell-1})^+] \). By \( f^{\ell-1} \lor e^\ell = f^\ell \), we have \((f^{\ell-1})^+ \lor (e^\ell)^+ = (f^\ell)^+ \). By \((e^\ell)^+ = (e^{\ell-1})^+ \lor e^{\ell+1} \), we have \((f^\ell)^+ = (f^{\ell-1})^+ \lor (e^{\ell-1})^+ \lor e^{\ell+1} = (f^{\ell-1})^+ \lor e^{\ell+1} = (f^{\ell-1})^+ \lor f^{\ell+1} = (f^{\ell-1})^+ \lor f^{\ell+1} \), as required.

For \( x \in \mathcal{L} \), let \( r_x \) be a height function (on \([x, (x)^+]\)) defined by \( r_x(y) = r(y) - r(x) \). A set of \( x \)-rays \((e^i_j) \) \((i = 1, 2, \ldots, k)\) is said to be independent if \( r_x(e^1_j \lor e^2_j \lor \cdots \lor e^k_j) = k \), or equivalently if \( e^1_j \land (\lor_{i \neq j} e^i_j) = x \) for each \( i \).

**Proposition 4.7.** The sublattice generated by an independent set of \( k \) \( x \)-rays is isomorphic to \( \mathbb{Z}_+^k \), where the isomorphism is given by

\[
\mathbb{Z}_+^k \ni (z_1, z_2, \ldots, z_k) \mapsto e_1^{z_1} \lor e_2^{z_2} \lor \cdots \lor e_k^{z_k}.
\]

**Proof.** Suppose that \( x \)-rays \((e^i_j) \) \((i = 1, 2, \ldots, k)\) are independent. We first show:

**Claim.** For \( z \in \mathbb{Z}_+^k \), we have the following.

1. \( r_x(e_1^{z_1} \lor e_2^{z_2} \lor \cdots \lor e_k^{z_k}) = \sum_{i=1}^k z_i \).
2. \( e_j^{z_j} \land (\lor_{i \neq j} e_i^{z_i}) = x \) for \( j \in \{1, 2, \ldots, k\} \).

**Proof.** (1) We prove the claim by induction on \( k \); the case of \( k = 1 \) is obvious. From Lemma 4.5 and the independence of \((e^i_j)\), we have \( e_j^1 \land e_k^{z_k} = x \) for \( j = 1, 2, \ldots, k-1 \). By Lemma 4.6 \((e^i_j \lor e_k^{z_k}) \) \((j = 1, 2, \ldots, k-1)\) are \( e_k^{z_k} \)-segments. We next show that they are independent. Indeed, \( e_k^{z_k} \) covers \( e_k^1 \) and \( e_k^{z_k} \not\leq e_1^1 \lor e_2^1 \lor \cdots \lor e_k^1 \) (otherwise \( e_k^{z_k} \not\in [e_k^1, (e_k^1)^+] \)). Thus, by semimodularity, \( r_x(e_1^1 \lor e_2^1 \lor \cdots \lor e_k^{z_k}) = r_x(e_1^1 \lor e_2^1 \lor \cdots \lor e_k^{z_k}) = r_x(e_1^1 \lor e_2^1 \lor \cdots \lor e_k^1) = k-1 \), and \( e_k^1 \lor e_k^{z_k} \) \((j = 1, 2, \ldots, k-1)\) are independent in \([e_k^1, (e_k^1)^+]\). Repeating this, we see that \( e_1^{z_1} \lor e_2^{z_2} \lor \cdots \lor e_k^{z_k} \) are independent in \([e_1^{z_1}, (e_1^{z_1})^+]\). By induction, we have \( r_x(e_1^{z_1} \lor e_2^{z_2} \lor \cdots \lor e_k^{z_k}) = r_x(e_1^1 \lor e_2^1 \lor \cdots \lor e_k^{z_k}) + z_k = \sum_{i=1}^k z_i \), as required.

2. (From (1) and semimodularity (2.1), we have \( \sum_{i \neq j} z_i + z_j = r_x(\lor_{i \neq j} e_i^{z_i}) + r_x(e_j^{z_j}) \geq r_x(e_1^{z_1} \lor e_2^{z_2} \lor \cdots \lor e_k^{z_k}) + r_x(e_j^{z_j} \land (\lor_{i \neq j} e_i^{z_i})) \geq \sum_{i=1}^k z_i \). Thus \( r_x(e_j^{z_j} \land (\lor_{i \neq j} e_i^{z_i})) = 0 \) must hold, implying \( x = e_j^{z_j} \land (\lor_{i \neq j} e_i^{z_i}) \).}

By (2) of the claim, any element \( y \) in the sublattice generated by \((e^i_j) \) \((i = 1, 2, \ldots, k, \ell = 0, 1, 2, \ldots)\) can be written as

\[
y = e_1^{z_1} \lor e_2^{z_2} \lor \cdots \lor e_k^{z_k} \tag{4.2}
\]

for \( z = (z_1, z_2, \ldots, z_k) \in \mathbb{Z}_+^k \). It suffices to show that the expression (4.2) is unique. For \( i = 1, 2, \ldots, k \), let \( z'_i := \max\{\ell \in \mathbb{Z}_+ \mid e_i^{z_i} \preceq y\} \). Then \( z_i \leq z'_i \) (since \( e_i^{z_i} \preceq y \)). Consider \( y' := e_1^{z'_1} \lor e_2^{z'_2} \lor \cdots \lor e_k^{z'_k} \). Then \( y' \preceq y \), which implies \( r_x(y') \leq r_x(y) \). On the other hand, \( r_x(y) = z_1 + z_2 + \cdots + z_k \leq z'_1 + z'_2 + \cdots + z'_k = r_x(y') \). Thus it must hold \( z_i = z'_i \) for \( i = 1, 2, \ldots, k \), and \( y = y' \).
4.1.2 Parallelism and end

Here we introduce a parallel relation for rays, and introduce the concept of an end as an equivalence class of this relation.

**Lemma 4.8.** Let \( x = e^0 \prec e^1 \prec \cdots \) be an \( x \)-ray. For \( y \geq x \), there is an index \( \ell \) such that \( e^\ell \leq y \) and \( e^{\ell+1} \not\leq y \). In particular, \( y = e^\ell \lor y \prec e^{\ell+1} \lor y \prec \cdots \) is a \( y \)-ray.

**Proof.** By (F), there is no infinite chain in any interval. Therefore \( e^\ell \leq y \) for all \( \ell \) is impossible. The latter statement follows from Lemma 4.6.

For an \( x \)-ray \( (e^\ell) = (x = e^0 \prec e^1 \prec \cdots) \) and \( y \geq x \), the \( y \)-ray in the above lemma is denoted by \( (e^\ell) \lor y \).

An \( x \)-ray \( (e^\ell) \) and \( y \)-ray \( (f^\ell) \) are said to be parallel if \( (e^\ell) \lor (x \lor y) = (f^\ell) \lor (x \lor y) \). We write \( (e^\ell) \approx (f^\ell) \) if they are parallel.

**Lemma 4.9.** The parallel relation \( \approx \) is an equivalence relation on the set of all rays.

**Proof.** We first show the following claim:

**Claim.** Let \( (e^\ell) \) and \( (f^\ell) \) be \( x \)-rays, and let \( y \geq x \). Then \( (e^\ell) = (f^\ell) \) if and only if \( (e^\ell) \lor y = (f^\ell) \lor y \).

**Proof.** The only if part is obvious. We prove the if part. Suppose that \( (e^\ell) \not= (f^\ell) \). We show that \( (e^\ell) \lor y \not= (f^\ell) \lor y \). We may assume that \( y \) covers \( x \). The above claim is clearly true when \( y = e^1 = f^1 \). Suppose that \( y = e^1 \not= f^1 \). By Proposition 4.7 applied to independent \( x \)-rays \( (e^\ell), (f^\ell) \), we have \( y \lor e^2 = e^2 \not= y \lor f^1 \), and \( (e^\ell) \lor y \not= (f^\ell) \lor y \).

Suppose that \( y \not= e^1 \) and \( y \not= f^1 \). For some \( k \geq 0 \), we have \( e^\ell = f^\ell \) for \( \ell \leq k \) and \( e^{k+1} \not= f^{k+1} \). It suffices to show that \( (y \lor e^k) \)-rays \( (y \lor e^k \prec y \lor e^{k+1} \prec \cdots) \) and \( (y \lor f^k \prec y \lor f^{k+1} \prec \cdots) \) are different. So we may consider the case \( k = 0 \). By the above argument, we can assume that \( y, e^1, \) and \( f^1 \) are different. If \( y, e^1, \) and \( f^1 \) are independent in \([x, x^+]\), then \( y \lor e^1 \) and \( y \lor f^1 \) are different, and \( (e^\ell) \lor y \not= (f^\ell) \lor y \), as required. Suppose that they are dependent; namely \( y \lor e^1 \lor f^1 = y \lor e^1 = y \lor f^1 = e^1 \lor f^1 =: z \). Then \( e^2 \not= z \) and \( f^2 \not= z \) (since \( e^0 \not< e^1 \not< e^2 \) is a segment). We show that \( y \lor e^2 \) and \( y \lor f^2 \) are different. By Lemma 4.6, \( e^1 \prec z = e^1 \lor f^1 \prec e^1 \lor f^2 = y \lor f^2 \) is a segment. If \( y \lor e^2 = y \lor f^2 \), then \( y \lor e^2 = z \lor e^2 \) implies that \( y \lor f^2 \) is the join of \( z \) and \( e^2 \), both covering \( e^1 \); this contradicts the fact that \( e^1 \not< z \not< y \lor f^2 \) is a segment. Thus \( (e^\ell) \lor y \not= (f^\ell) \lor y \).

It suffices to show that \( (e^\ell) \approx (f^\ell) \approx (g^\ell) \) imply \( (e^\ell) \approx (g^\ell) \). Suppose that \( (e^\ell), (f^\ell), \) and \( (g^\ell) \) are \( x \), \( y \), and \( z \)-rays, respectively. Then \( (e^\ell) \lor (x \lor y) = (f^\ell) \lor (x \lor y) \) and \( (f^\ell) \lor (y \lor z) = (g^\ell) \lor (y \lor z) \). This implies that \( (e^\ell) \lor (x \lor y \lor z) = (f^\ell) \lor (x \lor y \lor z) = (g^\ell) \lor (x \lor y \lor z) \). By the above claim, it must hold \( (e^\ell) \lor (x \lor z) = (g^\ell) \lor (x \lor z) \).

An equivalence class is called an end. Let \( E = E^C \) denote the set of all ends.

**Lemma 4.10.** For an \( x \)-ray \( (e^\ell) \) and \( y \in L \), there (uniquely) exists a \( y \)-ray that is parallel to \( (e^\ell) \).

**Proof.** Consider \( y' := (y)^{+k} \geq x \) (Lemma 4.4). Then \( ((e^\ell) \lor y')^{-k} \approx (e^\ell) \lor y' \approx (e^\ell) \), implying \( ((e^\ell) \lor y')^{-k} \approx (e^\ell) \), where \( ((e^\ell) \lor y')^{-k} \) is a \( y \)-ray.
Let $E_x$ denote the set of all $x$-rays. By the above lemma, for each end $e \in E$, there is an $x$-ray $e_x \in E_x$ that is a representative of $e$. In particular $E_x$ and $E$ are in one-to-one correspondence. For $e \in E$, the representative of $e$ in $E_x$ is denoted by $e_x = (x = e_x^0 < e_x^1 < e_x^2 < \cdots)$. In particular, $E_x = \{e_x \mid e \in E\}$.

### 4.1.3 Ultrametric on the space of ends

Let $x \in \mathcal{L}$. Define $\delta_x : E \times E \to \mathbb{Z}_+$ by

$$\delta_x(e, f) := \sup\{i \mid e_x^i = f_x^i\} \quad (e, f \in E),$$

and define $d_x : E \times E \to \mathbb{R}_+$ by

$$d_x(e, f) := \exp(-\delta_x(e, f)) \quad (e, f \in E).$$

Observe from Proposition 4.11 that two different $x$-rays $(e_x^i), (f_x^j)$ never meet again once they are separated, i.e., if $e_x^i \neq f_x^j$ then $e_x^j \neq f_x^i$ for $j > i$. In particular, all elements in $x$-rays in $E_x$ induce a rooted tree with root $x$ in the Hasse diagram of $\mathcal{L}$. From this view, $\delta_x(e, f)$ is the distance between the root $x$ and the lowest common ancestor (lca) of $e$ and $f$.

**Proposition 4.11.** For $x \in \mathcal{L}$, we have the following:

1. $d_x$ is an ultrametric on $E$.
2. The metric space $(E, d_x)$ is complete.
3. For $y \in \mathcal{L}$, it holds $\alpha d_y \leq d_x \leq \beta d_y$ for positive constants $\alpha := \exp(-r[x, x \lor y])$ and $\beta := \exp(r[y, x \lor y])$.

**Proof.** (1) From the view of rooted tree, one can easily see that $\delta_x$ satisfies the anti-ultrametric inequality:

$$\delta_x(e, f) \geq \min(\delta_x(e, g), \delta_x(g, f)) \quad (e, f, g \in E).$$

Hence $d_x$ satisfies ultrametric inequality (3.3). If $e \neq f$ then $\delta_x(e, f)$ is finite, and $d_x(e, f)$ is nonzero. This means that $d_x$ is an ultrametric.

(2) Consider a Cauchy sequence $(e_i)_{i=1,2,\ldots}$ in $E$ relative to $d_x$. We construct $e \in E$ such that $\lim_{i \to \infty} d_x(e, e_i) = 0$. Let $a^0 := x$. For $\ell \in \mathbb{Z}_+$, there is $n_\ell \in \mathbb{Z}_+$ such that $\delta_x(e_i, e_i') \geq \ell$ for $i, i' \geq n_\ell$. Let $a^\ell := f_x^i$ for $f := e_{n_\ell}$. Then all $(e_i)_x$ for $i \geq n_\ell$ contain $a^\ell$. Hence $(a^\ell)$ is an $x$-ray such that $(e_i)$ converges to the end $e$ of $x$-ray $(a^\ell)$.

(3) For $z \geq x$, it holds $\delta_x(e, f) \leq \delta_x(e, f) \leq \delta_x(e, f) + r[x, z]$. Therefore $\delta_x(e, f) \leq \delta_x(e, f) + r[x, x \lor y] \leq \delta_y(e, f) + r[x, x \lor y]$, which implies $d_x(e, f) \geq \exp(-r[x, x \lor y])d_y(e, f)$. Similarly $d_y(e, f) \leq \exp(r[y, x \lor y])d_x(e, f)$. □

Thus $\mathcal{E}_x$ is endowed with the topology induced by ultrametric $d_x$, which is independent of the choice of $x \in \mathcal{L}$ by (3). We will see in Section 4.3 that $\mathcal{E}_x$ coincides with the Dress-Terhalle completion when $\mathcal{L}$ comes from a valuated matroid $(E, \omega)$. 
4.1.4 Realization in $\mathbb{Z}^E$

Here we show that $\mathcal{L}$ can be realized as a subset of $\mathbb{Z}^E$, which will be the set of integer points of a tropical linear space. Let $x \in \mathcal{L}$. For $y \succeq x$, the $x$-coordinate of $y$ is an integer vector $y_x \in \mathbb{Z}^E$ defined by

$$y_x(e) := \max\{ \ell \in \mathbb{Z}_+ \mid e_x^\ell \preceq y \} \quad (e \in E).$$

**Lemma 4.12.** For $x \preceq y \preceq z$, we have the following:

1. $z_x = z_y + y_x$.

2. $(y)_x^+ = y_x + 1 = y(x)^-.$

3. $y = \bigvee_{e \in E} e_y^x(e)$.

**Proof.** (1). It suffices to consider the case where $z$ covers $y$. Consider $e \in E$. By semimodularity, $y \vee e_y^x(e) + 1$ covers $y$. If $z = e_y^x(e) + 1$, then $z = e_y^1$ and $z_y(e) = 1$, and $z_x(e) = y_x(e) + 1$, where $z_x(e) > y_x(e) + 1$ is impossible by Lemma 4.16. If $z \neq e_y^x(e) + 1$, then $z_y(e) = 0$ and $z_x(e) = y_x(e)$ (since $z \vee e_y^x(e) + 1 = z \vee (y \vee e_y^x(e) + 1)$ covers $z$).

(2). It is easy to see $(x)_x^+ = 1$. By (1), we obtain (2).

(3). Observe from $y \succeq e_y^x(e)$ that $(\succeq)$ holds; in particular, the right hand side of (3) actually exists. We show the equality ($=$). Let $u(\preceq y)$ denote the right hand side of (3). Then $u_x = y_x$. From $y_x = u + u_x$ by (1), we have $y_x = 0$. Here $y \succeq u$ is impossible, otherwise $y_x \neq 0$.

For general $x, y \in \mathcal{L}$, the $x$-coordinate $y_x$ of $y$ is defined by

$$y_x := (y)_x^+ - k1$$

for an integer $k$ with $y_x^+ \succeq x$. This is well-defined by Lemma 4.12 (2). Then it is easy to see that Lemma 4.12 (1) and (2) also hold for general $x, y, z$. By $0 = x_x = x_y + y_x$, we have:

**Lemma 4.13.** For $x, y \in \mathcal{L}$, it holds $y_x = -x_y$.

For $x \in \mathcal{L}$, define $\mathcal{Z}(\mathcal{L}, x) \subseteq \mathbb{Z}^E$ by

$$\mathcal{Z}(\mathcal{L}, x) := \{ y_x \mid y \in \mathcal{L} \}. \quad (4.3)$$

The partial order on $\mathcal{Z}(\mathcal{L}, x)$ is induced by vector order $\leq$ in $\mathbb{Z}^E$

**Proposition 4.14.** Let $x \in \mathcal{L}$. Then $\mathcal{L}$ is isomorphic to $\mathcal{Z}(\mathcal{L}, x)$ by $y \mapsto y_x$.

**Proof.** By Lemma 4.12 (3), the map $y \mapsto y_x$ is injective on $\{ y \in \mathcal{L} \mid y \succeq x \}$. Consequently it is bijective. We show that the order is preserved. Suppose that $y \preceq x$. For some $k$, we have $x \preceq y_x + k \succeq z^+ k$. By Lemma 4.12 we have $(z)_x^+ k = (z)_x^+ k + (y)_x^+ k$, and $z_x = z_y + y_x$. By $z_y \geq 0$, we have $z_x \geq y_x$. \qed
4.1.5 Matroid at infinity

Here we introduce matroid structures on the set $E$ of ends. Suppose that $\mathcal{L}$ has uniform-rank $n$. For $x \in \mathcal{L}$, a subset $I \subseteq E$ of ends is called independent at $x$ or $x$-independent if $\{e_1 \mid e \in I\}$ is independent in $[x, (x)^+]$. Let $\mathcal{I}^x = \mathcal{I}^{\mathcal{L} - x}$ denote the family of all $x$-independent subsets in $E$.

Lemma 4.15. $(E, \mathcal{I}^x)$ is a matroid with rank $n$.

Indeed, $M^x = M^{\mathcal{L} - x} := (E, \mathcal{I}^x)$ is obtained by adding parallel elements to the simple matroid corresponding to geometric lattice $[x, (x)^+]$ whose rank is equal to uniform rank $n$ of $\mathcal{L}$. We call $M^x$ the matroid at $x$. Its base family is denoted by $B^x$. Let $\mathcal{I}^\infty := \bigcup_{x \in \mathcal{L}} \mathcal{I}^x$ be the union of all $x$-independent subsets over all $x \in \mathcal{L}$. The goal here is to show the following.

Proposition 4.16. $(E, \mathcal{I}^\infty)$ is a simple matroid with rank $n$.

We call $M^\infty := (E, \mathcal{I}^\infty)$ the matroid at infinity. The base family $B^\infty$ of $M^\infty$ is given by $B^\infty = \bigcup_{x \in \mathcal{L}} B^x$. We see in Section 4.2 that $B^\infty$ is the domain of the valued matroid corresponding to $\mathcal{L}$.

We are going to prove Proposition 4.16.

Lemma 4.17. For $K \subseteq E$ and $x \in \mathcal{L}$, we have the following:

1. For any $z \in \mathcal{L}$ with $z \geq x$ and $z \not\in e^1_z \ (e \in K)$, if $K \in \mathcal{I}^x$, then $K \in \mathcal{I}^z$.

2. For any $z \in [x, (x)^+]$ with $z \geq \bigvee_{e \in K} e^1_z$, it holds $r[z, \bigvee_{e \in K} e^1_z] \geq r[x, \bigvee_{e \in K} e^1_z]$; in particular, if $K \in \mathcal{I}^x$, then $K \in \mathcal{I}^z$.

3. For $I \subseteq K$, let $y := \bigvee_{e \in I} e^1_x$. If $I \in \mathcal{I}^x$, $K \in B^y$, and $e^1_z \not\in y$ for $e \in K \setminus I$, then $K \in B^x$.

Proof. (1) We show the contrapositive; suppose $|K| > r_x(\bigvee_{e \in K} e^1_z)$. By $z \not\in e^1_z$, it holds $e^1_z = z \vee e^1_z$ for $e \in K$. Then $r_x(z) + |K| > r_x(z) + r_x(\bigvee_{e \in K} e^1_z) \geq r_x(\bigvee_{e \in K} e^1_z) + r_x(z)$. Thus $|K| > r_x(\bigvee_{e \in K} e^1_z)$, and $K \not\in \mathcal{I}^z$.

(2) Let $y := \bigvee_{e \in K} e^1_x$. We can choose an $x$-independent subset $K' \subseteq K$ such that $y = \bigvee_{e \in K'} e^1_x$. Also we can choose an $x$-independent subset $J \subseteq E \setminus K$ such that $y \vee (\bigvee_{e \in J} e^1_z) = z$. Then $K' \cup J$ is $x$-independent. Now $z$ belongs to the sublattice generated by independent $x$-rays $e_x \ (e \in K' \cup J)$. From Proposition 4.7, we conclude that $K'$ is independent at $z$. Hence $r_z[x, \bigvee_{e \in K} e^1_z] = |K'| = r_z[z, \bigvee_{e \in K} e^1_z] \leq r_z[z, \bigvee_{e \in K} e^1_z]$. Thus $r[z, \bigvee_{e \in K} e^1_z] = |K| - |I| = n - |I|$. Thus $r[z, \bigvee_{e \in K} e^1_z] = r[z, \bigvee_{e \in K} e^1_z] = n$. This implies that $K$ is a base at $x$.

Lemma 4.18. For $I \subseteq E$ and $x \in \mathcal{L}$, define $x = x^0, x^1, \ldots$ by

$$x^k := \bigvee_{e \in I} e^k_x \ (k = 0, 1, 2, \ldots)$$

If $I \in \mathcal{I}^\infty$, then there is $m \geq 0$ such that $I$ is independent at $x^m$. 

\[ \tag{4.4} \]
Proof. By the definition of $\mathcal{I}^\infty$, there is $y \in \mathcal{L}$ such that $I$ is independent at $y$. We can assume that $y \succeq x$ (Lemma 4.4). Consider the $x$-coordinate $y_x \in \mathbb{Z}^E$ of $y$, and let $z := \bigvee_{e \in I} y_{x}^{e} (\preceq y)$. By Lemma 4.12 (1), it holds $y_{x}^{e} (e) = 0$ for all $e \in I$. This means that $e^1_{x} \not\preceq y$ for all $e \in I$. Therefore, by Lemma 4.17 (1) and $I \in \mathcal{I}_y$, $I$ is independent at $z$. By $z \not\preceq e^y_{x} + 1$ and Lemma 4.8, it holds $e^1_{x} = z \lor e^y_{x} + 1$ for $e \in I$ and $I \geq 0$.

Let $m := \max_{e \in I} y_{x}^{e} (e)$. Then $x^{m} = \bigvee_{e \in I} y_{x}^{e} (e) \lor e^{m}_{x} = \bigvee_{e \in I} z \lor e^{m}_{x} = \bigvee_{e \in I} e^{m}_{x} - y_{x}^{e} (e)$. Thus $x^{m}$ belongs to the sublattice generated by independent $z$ rays, which implies that $I$ is independent at $x^{m}$.

□

Lemma 4.19. For $I \subseteq E$ and $x \in \mathcal{L}$, define $x = x^{0}, x^{1}, \ldots$ by (4.4). Then we have

$$x^{k} = \bigvee_{e \in I} e^{1}_{x^{k-1}} \quad (k = 1, 2, \ldots).$$

(4.5)

Proof. We show by induction on $k$ that $e^{k}_{x} \not\preceq x^{k-1}$ for $e \in I$. This implies $e^{1}_{x^{k-1}} = x^{k-1} \lor e^{k}_{x}$ by Lemma 4.8, and implies (4.5): $x^{k} = \bigvee_{e \in I} e^{1}_{x^{k-1}} \lor e^{k}_{x} = \bigvee_{e \in I} e^{1}_{x^{k-1}} \lor e^{k}_{x}$. For $e \in I$, by induction, $e^{k-1}_{x} \not\preceq x^{k-2}$. Then $e^{k}_{x} \lor x^{k-2} = e^{2}_{x^{k-2}}$ (by Lemma 4.8). If $e^{k}_{x} \preceq x^{k-1}$, then $e^{2}_{x^{k-2}} = e^{k}_{x} \lor x^{k-2} \preceq x^{k-1}$, and $x^{k-2} = e^{0}_{x^{k-2}} \preceq e^{1}_{x^{k-2}} \preceq e^{2}_{x^{k-2}} \preceq x^{k-1} = \bigvee_{e \in I} e^{1}_{x^{k-1}} \succeq (x^{k-2})^{+}$, contradicting $e^{2}_{x^{k-2}} \not\succeq [x^{k-2}, (x^{k-2})^{+}]$. Thus $e^{k}_{x} \not\preceq x^{k-1}$, as required. □

Proof of Proposition 4.16. We verify the axiom of independent sets. Choose $I, J \in \mathcal{I}^\infty$ with $|I| < |J|$. By the definition of $\mathcal{I}^\infty$, there is $x \in \mathcal{L}$ with $I \subseteq \mathcal{I}^x$. Consider $x^{1} := \bigvee_{e \in I} e^{1}_{x}$ and $y^{1} := \bigvee_{e \in J} e^{1}_{x}$. If $y^{1} \not\preceq x^{1}$, then we can choose $e^{*} \in J \setminus I$ with $(e^{*})^{1}_{x} \not\preceq x^{1}$, and $I + e^{*}$ is independent at $x^{1}$; $I + e^{*} \in \mathcal{L} \subseteq \mathcal{I}_x$, as required.

So suppose $y^{1} \preceq x^{1}$. For $k = 1, 2, \ldots$, let $x^{k} := \bigvee_{e \in I} e^{k}_{x}$, and let $y^{k} := \bigvee_{e \in J} e^{k}_{x}$. By Lemma 4.17 (2) and Lemma 4.19, $I$ is independent at all $x^{k}$. By Lemma 4.18 and $J \in \mathcal{I}^\infty$, there is $\ell$ such that $J$ is independent at all $y^{k}$ for $k \geq \ell$. With Lemma 4.19, it holds $r[x^{k}, x^{k+1}] = r[y^{k}, \bigvee_{e \in J} e^{1}_{x}] = |I| < |J| = r[y^{k}, \bigvee_{e \in J} e^{1}_{x}] = r[y^{k}, y^{k+1}]$ for $k \geq \ell$.

For large $k$, the increase of the height of $y^{k}$ is greater than that of $x^{k}$. Therefore there is $k^{*}$ such that $y^{k^{*}} \preceq x^{k}$ and $y^{k^{*}+1} \not\preceq x^{k^{*}+1}$. This implies that $x^{k^{*}+1} \not\preceq x^{k^{*}} \lor y^{k^{*}+1} = x^{k^{*}} \lor \bigvee_{e \in J} e^{1}_{x^{k^{*}}} \succeq \bigvee_{e \in J} e^{1}_{x^{k^{*}}}$, and there is $e^{*} \in J \setminus I$ with $I + e^{*} \in \mathcal{I}_{x^{k^{*}}}$, as above.

For distinct $e, f \in E$ and $x \in \mathcal{L}$, let $y := e^{\delta x_{e} (e, f)} f^{\delta x_{e} (e, f)}$. Then $e^{1}_{y} \neq f^{1}_{y}$; see Section 4.1.3. This means that $(e, f)$ is independent on $M^{\infty}$. Thus $M^{\infty}$ is a simple matroid. □

Lemma 4.20. Let $x \in \mathcal{L}$. For a bounded vector $c \in \mathbb{Z}^E_{+}$, let $y := \bigvee_{e \in E} c^{e}_{x}$. Then there is $B \in \mathcal{B}^{x}$ such that $y_{x}^{e} (e) = c^{e}_{x}$ for $e \in B$, and

$$y = \bigvee_{e \in B} c^{e}_{x}.$$  (4.6)

Notice that $\bigvee_{e \in E} c^{e}_{x}$ exists by $\bigvee_{e \in E} c^{e}_{x} \preceq (x)^{+} \max_{e \in E} c^{e}_{x}$.

Proof. We use the induction on $\max_{e \in E} c^{e}_{x}$. Define $c^{e} \in \mathbb{Z}^{E}_{+}$ by $c^{e}_{x} := \max \{ c^{e}_{x} - 1, 0 \}$. Let $z := \bigvee_{e \in E} c^{e}_{x}$. By induction, there is $B' \in \mathcal{B}^{x}$ such that $z_{x}^{e} (e) = c^{e}_{x}$ for $e \in B'$ and $z = \bigvee_{e \in B'} c^{e}_{x}$. Define $I := \{ e \in B' \mid c^{e}_{x} > 0 \}$. Then $I \in \mathcal{I}^{x}$ (by $B' \in \mathcal{B}^{x}$).
Let $Z := \{e \in E \mid e^{(e)}_x \not\in z\}$. Then $I \subseteq Z$ (by $c(e) - 1 = z_x(e)$ for $e \in I$). Now $y = \bigvee_{e \in E} z \vee e^{(e)}_x = \bigvee_{e \in Z} z \vee e^{(e)}_x + 1 = \bigvee_{e \in Z} e^{(e)}_x$. There is $J \in \mathcal{I}$ such that $I \subseteq J \subseteq Z$ and $y = \bigvee_{e \in J} e^{(e)}_x = \bigvee_{e \in J} e^{(e)}_x$. If $J \in B^y$, then $J \in B^y$ (by Proposition 4.7), and $J$ is a desired subset. Suppose not. By the independence axiom for $B^y$, $J \in \mathcal{I}$ with $|B^y| > |J|$ we can choose a subset $K \subseteq B^y \setminus J$ with $J \cup K \in B^y$. Then $B := J \cup K$ is a desired base in $B^y$, since $c(e) = c'(e) = z_x(e) = 0$ and $y_x(e) = y_{z}(e) + z_x(e) = 0$ for $e \in K$.

4.1.6 $\mathbb{Z}^n$-skeleton

Let $x \in \mathcal{L}$, and $B \in B^x$. By Proposition 4.7, the sublattice $S^x(B)$ generated by elements in $x$-rays $e_x \in B$ is isomorphic to $\mathbb{Z}^n$, where $n$ is the uniform rank of $\mathcal{L}$. This sublattice is closed under the ascending operation. Define sublattice $S(B)$ by

$$S(B) := \bigcup_{k \in \mathbb{Z}} (S^x(B))^k.$$ 

Then $S(B)$ is isomorphic to $\mathbb{Z}^n$ with $(y)^+ = y + 1$ for $y \in S(B)$ (identified with $\mathbb{Z}^n$). We call $S(B)$ the $\mathbb{Z}^n$-skeleton generated by $B$. The next lemma shows that $S(B)$ is independent of the choice of $x$, and is well-defined for $B \in B^\infty$.

Lemma 4.21. For $B \in B^x$, it holds $S(B) = \{y \in \mathcal{L} \mid B \in B^y\}$.

Proof. From Proposition 4.7, the inclusion ($\subseteq$) is obvious. We show the converse. Let $y \in \mathcal{L}$ with $B \in B^y$. We may assume that $y \geq x$ by considering $(y)^{+k}$ and by $(S(B))^{+k} = S(B)$. Let $y' := \bigvee_{e \in B} e^{(e)}_x$. Then $y' \leq y$. We show $y' = y$. Suppose not: $y' < y$. There is an atom $a$ of $[y', (y')^+]$ with $a \leq y$; necessarily $a \not\in e^1_y$, for $e \in B$. By Lemma 4.17 (1), $B$ is also a maximal independent set at $y'$. Hence $r_{y'}(a \vee \bigvee_{e \in B} e^1_y) = r_{y'}((y')^+) = n$ and $n - 1 = r_a(\bigvee_{e \in B} (a \vee e^1_y)) = r_a(\bigvee_{e \in B} e^1_a)$. Namely $B$ is dependent at $a$ with $a \leq y \not\in e^1_a$ for $e \in B$. By Lemma 4.17 (1), $B$ is dependent at $y$, contradicting $B \in B^y$.

4.2 Valuated matroid from uniform semimodular lattice

Let $\mathcal{L}$ be a uniform semimodular lattice with uniform-rank $n$. For $x \in \mathcal{L}$ and $B \in B^\infty$, define $x_B \in \mathcal{L}$ as the maximum element $y \in S(B)$ with $y \preceq x$:

$$x_B := \bigvee \{y \in S(B) \mid y \preceq x\}.$$ 

The maximum element $x_B$ indeed exists by (F) and the fact that $S(B)$ is a sublattice. Now define $\omega = \omega^{x}: B^\infty \rightarrow \mathbb{Z}$ by

$$\omega(B) := -r[x_B, x] \quad (B \in B^\infty).$$ 

This quantity $\omega(B)$ is the negative of a “distance” between $x$ and $S(B)$; see Figure 1 for intuition. One of the main theorems is as follows:

Theorem 4.22. Let $\mathcal{L}$ be a uniform semimodular lattice with uniform-rank $n$, and let $x \in \mathcal{L}$. Then $\omega = \omega^{x}$ is a complete valuated matroid with rank $n$, where

1. $\mathcal{T}(\omega) \cap \mathbb{Z}^E$ is isomorphic to $\mathcal{L}$, and
Figure 1: A distance between $x$ and $S(B)$

(2) $T(\omega)$ is a geometric realization of simplicial complex $C(\mathcal{L})$ consisting of all chains $x^0 \prec x^1 \prec \cdots \prec x^m$ with $x^m \succeq (x^0)^+$. To prove this theorem, we show several properties of $x_B$.

Lemma 4.23. Let $x, y \in \mathcal{L}$ with $y \preceq x$, and $B \in \mathcal{B}^\infty$.

(1) $y = x_B$ if and only if $B \in \mathcal{B}^y$ and $x_y(e) = 0$ for all $e \in B$.

(2) $x_B \preceq y$ if and only if $x_y(e) = 0$ for all $e \in B$.

Proof. (1). If $y \in S(B)$, $y \preceq x$, and $x_y(e) > 0$ for some $e \in B$, then $\bigvee_{e \in B} e_y(e)$ belongs to $S(B)$, is greater than $y$, and is not greater than $x$. The claim follows from this fact. In particular, $x_{x_B}(e) = 0$ for $e \in B$.

(2). The only-if part follows from $x_{x_B} = x_y + x_{x_B}$ and $x_{x_B}(e) = 0$ of all $e \in B$. We show the if part. Suppose that $B$ is dependent at $y$ (otherwise $y = x_B$ by (1)). Define the sequence $y = y^0, y^1, y^2, \ldots$ by

$$y^k := \left( \bigvee_{e \in B} e_y^k \right)^{-k} \quad (k = 0, 1, 2, \ldots). \quad (4.7)$$

Then it holds that

$$y^k = \left( \bigvee_{e \in B} e_{y^k-1}^1 \right)^{-1} \quad (k = 1, 2, \ldots). \quad (4.8)$$

Indeed, let $z^k := \bigvee_{e \in B} e_{y^k}^k$. Then $y^k = (z^k)^{-k} = (\bigvee e_{z^k-1}^1)^{-k} = (\bigvee e_{y^k-1}^1)^{-1}$, where the second equality follows from Lemma 4.19 and the forth one follows from the observation $(e^1_k)^{-1} = e^{1}_{k-1}$. In particular, $x \succeq y \succeq y^1 \succeq y^2 \succeq \cdots$ holds (by Lemmas 2.2 and 4.3). Also $x_{y^k}(e) = 0$ for all $e \in B$ (by Lemma 4.12 (1). By Lemma 4.18, there is $\ell$ such that $B \in \mathcal{B}^{y^\ell}$. By (1), we have $y^\ell = x_B$, and $x_B \preceq y$, as required.

Lemma 4.24. For $x, y \in \mathcal{L}$ with $y \preceq x$, we have the following:

$$r(x_B) + \sum_{e \in B} y_x(e) \begin{cases} = r(y) & \text{if } y \in S(B) \iff B \in \mathcal{B}^y, \\ < r(y) & \text{otherwise.} \end{cases} \quad (B \in \mathcal{B}^\infty).$$
Proof. Suppose that \( y \in S(B) \). By Lemmas \[4.12\] (1) and \[4.23\] (1), \( \bigvee_{e \in B} c_y(e) \) is equal to \( x_B \). By Proposition \[1.7\] \( r[y, x_B] = \sum_{e \in B} x_y(e) \). Therefore \( r(y) + \sum_{e \in B} x_y(e) = r(x_B) \) holds, which implies \( r(y) = r(x_B) + \sum_{e \in B} y_x(e) \) by \( y_x = -x_y \); see Lemma \[4.13\].

Suppose that \( y \notin S(B) \). Let \( y' := \bigvee_{e \in B} c_y(e) \). By Lemma \[4.23\] (2), we have \( x_B \leq y' \leq x \), and

\[
\begin{align*}
r[y, y'] &\leq \sum_{e \in B} x_y(e), \\
r(x_B) &\leq r(y').
\end{align*}
\]

It suffices to show that one of the inequalities is strict. If \( y' \succ x_B \), then \( (\prec) \) holds in the second inequality. Suppose that \( y' = x_B \), and suppose to the contrary that equality holds in the first inequality. Let \( I := \{ e \in B \mid x_y(e) > 0 \} \neq \emptyset \), and let \( y'' := \bigvee_{e \in I} c_y(e) \). Then \( y'' = x_B = \bigvee_{e \in I} c_{y''} \). By the equality in the first inequality and Lemma \[4.12\] (1), \( I \) must be independent at \( y'' \), and \( c_{y''} \not\prec y' \) for \( e \in B \setminus I \) (otherwise \( x_y(e) > 0 \) for \( e \in B \setminus I \)). By Lemma \[4.17\] (3), \( B \) is independent at \( y'' \). Also \( r[y, y''] = \sum_{e \in B} \max\{x_y(e) - 1, 0\} \) holds. By repeating this argument (to \( y'' \)), we eventually obtain a contradiction that \( B \) is independent at \( y \not\in S(B) \).

\( \square \)

Proof of Theorem \[4.22\] Observe that \( \omega \) is upper-bounded. By Lemma \[3.3\] we show that for any bounded vector \( c \in \mathbb{Z}^E \), the maximizer family \( B_{\omega+c} \) is a matroid base family.

Suppose that \( c = y_x \) for some \( y \leq x \). By Lemma \[4.24\] the maximizer family \( B_{\omega+c} \) is nothing but \( B^y \).

Suppose that \( c \) is general. From \( B_{\omega+c} = B_{\omega+c+k1} \), we can assume that \( c \geq 0 \). Let \( y := \bigvee_{e \in E} c(e) \). By Lemma \[4.20\] there is \( B \in B^y \) such that \( y = \bigvee_{e \in B} c(e) \). Let \( \tilde{c} := y_x \). Then \( \tilde{c} \geq c \). Thus \( -r[x_B, x] + \sum_{e \in B} c(e) \leq -r[x_B, x] + \sum_{e \in B} \tilde{c}(e) \) for arbitrary \( B' \in B^\omega \), and the equality holds for \( B \) by \( c(e) = y_x(e) = \tilde{c}(e) \) (\( e \in B \)). Since \( B \in B^y = B_{\omega+c} \) (by above), the maximum of \( \omega + c \) is the same as that of \( \omega + \tilde{c} \). This implies that \( B_{\omega+c} \subseteq B_{\omega+c} \). Now \( B_{\omega+c} \) is viewed as the maximizer family of a linear function \( B \mapsto \sum_{e \in B} (c - \tilde{c})(e) \) over the matroid base family \( B_{\omega+c} \), and is a matroid base family, as required.

(1) follows from Proposition \[4.14\] and the next claim.

Claim. \( T(\omega) \cap \mathbb{Z}^E = \mathbb{Z}(\mathcal{L}, x) \).

Proof. For \( c = y_x \in \mathbb{Z}(\mathcal{L}, x) \), the maximizer family \( B_{\omega+c} \) is equal to \( B^y \), which is loop-free. Hence \((\supseteq)\).

Let \( c \in \mathbb{Z}^E \) with \( c \not\in \mathbb{Z}(\mathcal{L}, x) \). Consider \( \tilde{c} \) as above. Then \( \tilde{c} \geq c \), and \( \tilde{c} \neq c \). As seen above, \( \max_{B} -r[x_B, x] + \sum_{e \in B} c(e) = \max_{B} -r[x_B, x] + \sum_{e \in B} \tilde{c}(e) \). This means that an element \( e \in E \) with \( \tilde{c}(e) > c(e) \) cannot belong to any maximizer in \( B_{\omega+c} \). Namely \( e \) is a loop in \( B_{\omega+c} \). Thus \( c \not\in T(\omega) \cap \mathbb{Z}^E \), implying \((\subseteq)\).

(2) is a corollary of this claim and Lemma \[3.7\] (2). By Proposition \[4.16\] \((E, \omega)\) is a simple valued matroid. Lemma \[4.30\] in the next section shows that topologies on \( E \) induced by \( d_x \) and by \( D_p \) from \( \omega \) coincide. By Proposition \[4.11\] (2), \( \omega \) is complete. \( \square \)
4.3 Uniform semimodular lattice from valuated matroid

The main statement for the uniform semimodular lattice of a valuated matroid is as follows.

**Theorem 4.25.** Let \((E, \omega)\) be an integer-valued valuated matroid with rank \(n\). Then \(\mathcal{L}(\omega) := T(\omega) \cap \mathbb{Z}^{E}\) is a uniform semimodular lattice with uniform-rank \(n\), in which the following hold:

1. The ascending operator is equal to \(x \mapsto x + 1\).
2. A height function \(r\) is given by \[ x \mapsto \max_{B \in \mathcal{B}}(\omega + x)(B). \]
3. The meet \(\land\) and the join \(\lor\) are given by
   \[
   x \land y = \min(x, y), \\
   x \lor y = \bigwedge \{ z \in \mathcal{L}(\omega) | x \leq z \leq y \} \quad (x, y \in \mathcal{L}(\omega)).
   \]
4. For \(x \in \mathcal{L}(\omega)\), the valuated matroid \(\mathcal{M}(\omega, \omega^{\mathcal{L}(\omega), x})\) is a completion of a valuated matroid projectively equivalent to \((E, \omega)\).

The rest of this section is devoted to the proof. Let \(\mathcal{M} = (E, \mathcal{B})\) be the underlying matroid of \(\omega\). By \((\text{TC}_+^1)\), if \(x \in \mathcal{L}(\omega)\) then \(x + 1 \in \mathcal{L}(\omega)\). We first show that the interval \([x, x + 1]\) in \(\mathcal{L}(\omega)\) is a geometric lattice corresponding to \(\mathcal{M}_{\omega + x}\).

**Lemma 4.26.** Let \(x \in \mathcal{L}(\omega)\).

1. \([x, x + 1]\) is isomorphic to the lattice of flats of \(\mathcal{M}_{\omega + x}\), where the isomorphism is given by the map \(x + 1_F \mapsto F\).
2. \(y \in \mathcal{L}(\omega)\) covers \(x\) if and only if \(y = x + 1_F\) for a parallel class \(F\) in \(\mathcal{M}_{\omega + x}\).

**Proof.** (1). By replacing \(\omega\) by \(\omega + x\), we can assume \(x = 0\). By Lemma 3.9, for a flat \(F\) of \(\mathcal{B}_{\omega}\), and any \(e \in F\) and \(f \not\in F\) we can choose \(B \in \mathcal{B}_{\omega} \cap \mathcal{B}_{\omega + 1_F}\) containing \(e, f\). This implies \(x + 1_F \in \mathcal{L}(\omega)\). Suppose that \(F\) is not a flat of \(\mathcal{B}_{\omega}\). Consider \(e \in \text{cl}(F) \setminus F\). Then \(\max\{ |B \cap (F + e)| | B \in \mathcal{B}_{\omega}\} = \max\{ |B \cap F| | B \in \mathcal{B}_{\omega}\}\). This implies that \(\max_B(\omega + 1_F)(B) = \max_B(\omega + 1_{F + e})(B)\). Thus no base in \(\mathcal{B}_{\omega + 1_F}\) contains \(e\), implying \(x + 1_F \not\in \mathcal{L}(\omega)\).

2. By (1), it suffices to the only-if part. We first show that for \(F \subseteq E\) and \(e \in E \setminus F\), if \(e\) is a loop in \(\mathcal{M}_{\omega}\), then so is \(\mathcal{M}_{\omega + 1_F}\). Choose \(B \in \mathcal{B}_{\omega}\) with maximal \(B \cap F\). By Lemma 3.9, it holds \(B \in \mathcal{B}_{\omega + 1_F}\). If there is a base in \(\mathcal{B}_{\omega + 1_F}\) containing \(e\), then by exchange axiom there is \(f \in B\) such that \(B + e - f \in \mathcal{B}_{\omega + 1_F}\). Then \(B + e - f \not\in \mathcal{B}_{\omega}\), and \(\omega(B + e - f) \leq \omega(B) - 1\). By \(e \not\in F\), it holds \(|(B + e - f) \cap F| \leq |B \cap F|\). Therefore \((\omega + 1_F)(B + e - f) < (\omega + 1_F)(B)\), contradicting \(B + e - f \in \mathcal{B}_{\omega + 1_F}\). Thus no base in \(\mathcal{B}_{\omega + 1_F}\) contains \(e\).

Let \(y = x + \sum_{i} 1_{F_i}\) for \(F_1 \supseteq F_2 \supseteq \cdots \supseteq F_m\). By repeated use of the above property, one can see that \(F_1\) must be a flat in \(\mathcal{B}_{\omega + x}\); otherwise \(e \in \text{cl}(F_1) \setminus F_1\) is a loop in \(\mathcal{B}_{\omega + y}\). Consider the parallel class \(F\) of \(e \in F_1\) in \(\mathcal{M}_{\omega + x}\). By (1), \(x + 1_F\) belongs to \(\mathcal{L}(\omega)\). Therefore \(x \leq x + 1_F \leq y\), implying \(y = x + 1_F\). \(\square\)
Proof of Theorem 4.25 (1-3). First we show (2) that a height function \( r \) of \( L(\omega) \) is given by \( x \mapsto \max_{B \in B} (\omega + x)(B) \). Consider \( x, y \in L(\omega) \) such that \( y \) covers \( x \). By Lemma 4.26 (2), \( y = x + 1_F \) for a parallel class \( F \). Then \( B_{\omega+y} \supseteq \{ B \in B_{\omega+x} \mid |B \cap F| = 1 \} (\neq \emptyset) \) by Lemma 3.9. Therefore \( r(y) = r(x) + 1 \).

Next we show that \( L(\omega) \) is a lattice with property (3). Let \( x, y \in L(\omega) \), and let \( z := \min(x, y) \). By the tropical convexity (Lemma 3.10), \( z \) belongs to \( L(\omega) \), and necessarily \( x \land y = z \). By Lemma 4.26 (2) and (2) shown above, \( x - z \) and \( y - z \) are upper-bounded. This implies that \( \max(x, y) - x \) and \( \max(x, y) - y \) are upper-bounded. Thus \( \{ z \in L(\omega) \mid z \geq \max(x, y) \} \) is nonempty; for example, consider \( x + \alpha 1 \) for large \( \alpha \). By this fact and the existence of a height function, \( \bigwedge \{ z \in L(\omega) \mid z \geq \max(x, y) \} \) exists, and is the join of \( x, y \).

By Lemma 4.26, if \( a, b \) cover \( a \land b \), then \( a \lor b \) covers \( a, b \). Hence \( L(\omega) \) is semimodular (Lemma 2.1). The property (1) is also an immediate corollary of the same lemma. The map \( x \mapsto x + 1 \) is obviously an automorphism. Thus \( L(\omega) \) is a uniform semimodular lattice. The uniform-rank is equal to the rank of \([x, x + 1] \) that is equal to the rank of \( \mathbf{M} \).

To show the property (4), we have to study the relationship between \( E \) and the space \( E^{L(\omega)} \) of ends in \( L(\omega) \).

**Lemma 4.27.** Let \((a^\ell)\) be a ray in \( L(\omega) \).

1. There is a decreasing sequence \( F_0 \supseteq F_1 \supseteq \cdots \) of nonempty subsets in \( E \) such that

   \[ a^{\ell+1} = a^\ell + 1_{F_\ell} \quad (\ell = 0, 1, \ldots), \]

   where \( F_\ell \) is a parallel class of \( M_{\omega+a^\ell} \).

2. If \( \bigcap_\ell F_\ell \) is nonempty, then \( \bigcap_\ell F_\ell \) is a parallel class of \( M \).

**Proof.** (1). By Lemma 4.26 (2), \( F_\ell \) is a parallel class of \( M_{\omega+a^\ell} \). It suffices to show \( F_0 \supseteq F_1 \). Here \( F_0 \cap F_1 = \emptyset \) is impossible, since otherwise \( a^2 \in [a^0, a^0 + 1] \) contradicting the fact that \((a^\ell)\) is a ray. Suppose \( F_1 \setminus F_0 \neq \emptyset \). Choose \( e \in F_0 \cap F_1 \) and \( f \in F_1 \setminus F_0 \). Then there is a base \( B \in B_{\omega+a^0} \) containing \( e, f \). By Lemma 3.9, \( B \) is also a base in \( B_{\omega+a^1} \). Namely, \( e, f \) are independent in \( M_{\omega+a^1} \). However this is a contradiction to the fact that \( F_1 \) is a parallel class of \( M_{\omega+a^1} \).

(2). Suppose that there are distinct non-parallel elements \( e, f \in \bigcap_\ell F_\ell \). There is \( B \in \mathcal{B} \) containing \( e, f \). Then \((\omega + a^{\ell+1})(B) - (\omega + a^\ell)(B) \geq 2 \). On the other hand, \( r(a^{\ell+1}) - r(a^\ell) = 1 \). Therefore \( B \) must be in \( B_{\omega+a^\ell} \) for some \( \ell \); this is a contradiction to the fact that \( e, f \) are parallel in \( M_{\omega+a^\ell} \) for all \( \ell \).

In the case of (2), ray \((a^\ell)\) is said to be normal and have \( \infty \)-direction \( F = \bigcap_\ell F_\ell \).

**Lemma 4.28.**

1. Two normal rays are parallel if and only if they have the same \( \infty \)-direction.

2. For \( x \in L(\omega) \) and a parallel class \( F \) of \( M \), there is a normal \( x \)-ray having \( \infty \)-direction \( F \).
Proof. (1). Let \((a^\ell)\) be a normal ray having \(\infty\)-direction \(F\), and let \(y \in L(\omega)\) with \(a^{\ell+1} \not\geq y \geq a^\ell\). We show that ray \((a^\ell) \vee y = (y = a^\ell \vee y < a^{\ell+1} \vee y < \cdots)\) is a normal ray having \(\infty\)-direction \(F\). By Lemma 4.27 (1), we can suppose that \(y \lor a^{\ell+k+1} = y \lor a^{\ell+k+1} + I_{G_k}\) for \(G_k \subseteq E\). By \(\min(y, a^{\ell+1}) = y \land a^{\ell+1} = a^\ell\) and \(a^{\ell+1} = a^\ell + I_{F_k}\), it holds \(y \lor a^{\ell+1} - y \geq \max(y, a^{\ell+1}) - y = 1_{F_k}\). Necessarily \(G_0 \supseteq F_{\ell+1}\). Consequently \(G_k \supseteq F_{\ell+k+1}\) for all \(k\). Therefore \(\bigcap_k G_k\) contains \(F\), and must be equal to \(F\), since \(\bigcap_k G_k\) is also a parallel class of \(M\) (Lemma 4.27 (2)). Thus \((a^\ell) \lor y\) has \(\infty\)-direction \(F\). The only-if part is immediate from this property. The if-part also follows from this property and the observation that if two normal rays at the same starting point have the same \(\infty\)-direction, then the two rays must be equal.

(2). Note that \(F\) is a rank-1 subset in \(M_{\omega+x}\) for every \(y \in L(\omega)\). Let \(a^0 := x\). For \(\ell = 0, 1, 2, \ldots\), define \(F^\ell\) as \(cl(F)\) in \(M_{\omega+x}\), and \(a^{\ell+1} := a^\ell + 1_{F^\ell}\). Then \((a^\ell)\) is a ray, since \(a^{\ell+1} \geq a^{\ell-1} + 2I_F\) and \(a^{\ell+1} \not\in [a^{\ell-1}, a^{\ell-1} + 1]\). Also \((a^\ell)\) is normal with \(\infty\)-direction \(F\) (since parallel class \(\bigcap_\ell F^\ell\) contains \(F\) and equals \(F\)).

In the case where \(\omega\) is simple, by associating \(e \in E\) with the end having \(\infty\)-direction \(\{e\}\), we can regard \(E\) as a subset of \(E^{\omega}(\omega)\). Then each local matroid \(M_{\omega+x}\) is the restriction of \(M^{\omega, x}\) to \(E\):

\[\text{Lemma 4.29.}\quad\text{For } x \in L(\omega), \text{ it holds } B_{\omega+x} = \{B \in B^{\omega, x} \mid B \subseteq E\}.\]

Proof. By Lemma 4.26, \(B \in B_{\omega+x}\) if and only if \(x + 1_{F_e}\) (\(e \in B\)) are independent atoms in geometric lattice \([x, x+1]\), where \(F_e\) is the parallel class of \(e\) in \(M_{\omega+x}\). If \(e \in E\) is regarded as a normal ray, then \(e_1 = x + 1_{F_e}\). From this, we see the equality to hold.

We verify that \(d_x\) and \(D_p\) induce the same topology on the set \(E\) of normal rays.

\[\text{Lemma 4.30.}\quad\text{Suppose that } \omega \text{ is simple. For } x \in L(\omega), \text{ if } r(x) = 0, \text{ then } -x \in TS(\omega), \text{ and } D_{-x}(e, f) = d_x(e, f) \text{ for } e, f \in E.\]

Proof. The fact \(-x \in TS(\omega)\) follows from \((3.2)\) and \(r(x) = \max_B(\omega + x)(B)\). It suffices to show that for two normal rays \(e, f \in E\), it holds

\[\delta_x(e, f) = -\max\{(\omega + x)(B) \mid B \in B : \{e, f\} \subseteq B\} \geq 0.\] (4.9)

Consider the sequence \(x = x^0, x^1, \ldots\) defined by \(x^{i+1} := e_1^x \lor f_1^x = e_1^{x+1} \lor f_1^{x+1}\); recall Lemmas 4.18 and 4.19. Then \(\delta_x(e, f)\) is the minimum index \(i^*\) such that \(r(x^{i^*}) = r(x^{i^* - 1} + 2)\) or equivalently that there is \(B \in B_{\omega+x}^*\) with \(e, f \in B\). Now \(r(x^i) = r(x^{i-1}) + 1,\) and \((\omega + x^i)(B) = (\omega + x^{i-1})(B) + 2\) for base \(B \in B\) with \(e, f \in B\). Therefore the index \(i^*\) must be the right hand side of \((4.9)\).

\[\text{Lemma 4.31.}\quad\text{The set } E \text{ of normal rays is dense in } E^{\omega}(\omega).\]

Proof. Consider a ray \(e \in E^{\omega}(\omega)\). Let \(x \in L\). Then \(x\)-ray \(e_x\) is represented as in Lemma 4.27 for some decreasing sequence \(F_1 \supseteq F_2 \supseteq \cdots\) of nonempty subsets in \(E\). For each \(i\), choose \(e_i \in F_i\). Then the sequence \((e_i)\) of normal rays satisfies \(\lim_{i \to \infty} d_x(e, e_i) = 0\).

\[\text{Proof of Theorem 4.25(4).}\quad\text{We can assume that } \omega \text{ is simple. Let } x \in L(\omega). \text{ By Lemmas 4.30 and 4.31, } E^{\omega}(\omega) \text{ coincides with the Dress-Terhalle completion of } E. \text{ Finally we verify the linear equivalence between } \omega \text{ and } \omega^{\omega, x} \text{ (restricted to } E).\]
Claim. For $B \in \mathcal{B}$, it holds $(\omega + x)(B) = r(x_B) = \omega^L_x(B) + r(x)$.

Proof. Consider the sequence $x = x^0 \succeq x^1 \succeq \cdots$ defined by $x^i := (\bigvee_{e \in B} e_{x^i-1})^{-1} = \bigvee_{e \in B} e_{x^i} - 1$. As seen in the proof of Lemma 4.23 (see (4.7) and (4.8)), for some $k$ it holds $x^k = x_B$. We prove the statement by induction on $k$. In the case of $k = 0$, $x = x_B$, $B \in \mathcal{B}_{\omega}^{\pm}$, and $B \in \mathcal{B}_{\omega+1}$ by Lemma 4.29 Then $r(x_B) = r(x) = (\omega + x)(B)$, implying the base case.

Suppose $k > 0$. Notice $(x^i)_B = x_B$. By induction, $(\omega + x^i)(B) = r(x_B)$. By definition of $x^k$, it holds $x^1(e) = x(e)$ for $e \in B$. Therefore, $(\omega + x)(B) = (\omega + x^1)(B) + (x - x^1)(B) = r(x_B)$, as required.

Note the constant term $r(x)$ is represented as linear term $(r(x)/n)\mathbf{1}$. Thus $\omega$ is projectively equivalent to the restriction of $\omega^L_{\omega,x}$ to $E$. This completes the proof of Theorem 4.25.

5 Example

Tree metric. Tree metrics may be viewed as valuated matroids of rank 2; see e.g., [7]. We here study tree metrics from our framework of uniform semimodular lattice. Let $T = (V, E)$ be a tree, and let $X$ be a subset of vertices of $T$. Let $\mathcal{B} := \{\{u, v\} \subseteq X \mid u \neq v\}$. Then $\mathcal{M} = (X, \mathcal{B})$ is a uniform matroid of rank 2. Define $d : \mathcal{B} \to \mathbb{Z}$ by

$$d(u, v) := \text{the number of edges in the unique path in } T \text{ connecting } u \text{ and } v,$$

where $d(\{u, v\})$ is written as $d(u, v)$. Then the classical four-point condition of tree-metrics says

$$d(u, v) + d(u', v') \leq \max\{d(u, u') + d(u', v), d(u', v) + d(u, v')\}$$

for distinct $u, v, u', v' \in X$. This is nothing but the exchange axiom (EXC). Thus $d$ is a valuated matroid on $\mathcal{M}$.

Let us construct the corresponding uniform semimodular lattice in a combinatorial way. First delete all redundant vertices not belonging to the (shortest) path between any pair of $X$. Fix a vertex $z \in V$ (as a root). Next, for each $u \in X$, consider an infinite path $P_u$ (with $V(P_u) \cap V(T) = \emptyset$) having a vertex $u'$ of degree one. Glue $T$ and $P_u$ by identifying $u$ and $u'$. Let $\mathcal{L}$ denote the union of $V \times 2\mathbb{Z}$ and $E \times (2\mathbb{Z} + 1)$. For each $(uv, k) \in E \times (2\mathbb{Z} + 1)$, consider binary relations (directed edges) $(uv, k) \leftarrow (u, k + 1)$, $(uv, k) \leftarrow (v, k + 1)$, $(u, k - 1) \leftarrow (uv, k)$, and $(v, k - 1) \leftarrow (uv, k)$. The partial order $\preceq$ on $\mathcal{L}$ is induced by the transitive closure of $\leftarrow$. Then $\mathcal{L}$ is a uniform (semi)modular lattice of uniform-rank 2, where the ascending operator is given by $(x, k) \mapsto (x, k + 2)$; see [13]. Ends are naturally identified with $P_u$ ($u \in X$). In particular $\mathcal{B} = \mathcal{B}_{\infty}$. For two ends $P_u, P_v$, there is a simple path $P$ of $T$ containing $P_u, P_v$. The $\mathbb{Z}^2$-skeleton $\mathcal{S}\{\{u, v\}\}$ is the sublattice of $\mathcal{L}$ induced by the union of $V(P) \times \mathbb{Z}$ and $E(P) \times (2\mathbb{Z} + 1)$. Let $x := (z, 0)$. For the lowest common ancestor $z_{u,v}$ of $u, v$ in $T$, $x_{\{u,v\}}$ is given by $(z_{u,v}, -2d(z, z_{u,v}))$. Thus the valuated matroid $\omega = \omega^L_{\omega,x}$ is given by

$$\omega(u, v) = -2d(z, z_{u,v}) \quad (\{u, v\} \in \mathcal{B}).$$

From the relation $-2d(z, z_{u,v}) = d(u, v) - d(z, u) - d(z, v)$, we see the projective-equivalence between $\omega$ and $d$. 

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Representable valued matroid. Let $K$ be a field, and $K(t)$ the field of rational functions with indeterminate $t$. The degree $\deg(p/q)$ of $p/q \in K(t)$ with polynomials $p, q$ is defined by $\deg(p) - \deg(q)$. Consider the vector space $K(t)^n$ over $K(t)$. Let $E$ be a subset of $K(t)^n$, and let $\mathcal{B}$ be the family of $K(t)$-bases $B \subseteq E$ of $K(t)^n$. Then $\mathbf{M} = (E, \mathcal{B})$ is a matroid. Define $\omega = \omega^E : \mathcal{B} \to \mathbb{Z}$ by

$$\omega^E(B) := \deg \det(B) \quad (B \in \mathcal{B}),$$

where $B \in \mathcal{B}$ is regarded as a nonsingular $n \times n$ matrix consisting of vectors in $B$. Then $\omega^E$ is a valued matroid. Such a valued matroid is called representable (over $K(t)$).

A tropical interpretation [19 20 21] of $L(\omega) = T(\omega) \cap \mathbb{Z}^E$ is the set of degree vectors $(\deg(q^Te) : e \in E)$ for all $q \in K(t)^n$, where we need to add $-\infty$ to $\mathbb{Z}$ for $\deg(0) := -\infty$. We here provide a different algebraic interpretation, which is viewed as an analogue of: The lattice of flats of the matroid represented by a matrix $M$ is the lattice of vector spaces spanned by columns of $M$.

Let $K(t)^-$ denote the ring of elements $p/q$ in $K(t)$ with $\deg(p/q) \leq 0$. Then $K(t)^n$ is also viewed as a $K(t)^-$-module. For a subset $F \subseteq K(t)^n$, let $(F)$ denote the $K(t)^-$-module generated by $F$, i.e., $(F) = \{\sum_{u \in F} \lambda_u u \mid \lambda_u \in K^-(t), F' \subseteq F : |F'| < \infty\}$. Also, for $z \in \mathbb{Z}^F$, let $F^z := \{t^{z(u)} u \mid u \in F\}$.

Suppose that $E \subseteq K(t)^n$ contains a $K(t)$-basis of $K(t)^n$. Let $\mathcal{B} \subseteq 2^E$ be the family of $K(t)$-bases, which is the underlying matroid of $(E, \omega)$. Define the family $\mathcal{L}(E)$ of $K^-(t)$-submodules of $K(t)^n$:

$$\mathcal{L}(E) := \{\langle E^z \rangle \mid z \in \mathbb{Z}^E\}.$$ 

The partial order on $\mathcal{L}(E)$ is defined as the inclusion relation. For $L \in \mathcal{L}(E)$, define $z^L \in \mathbb{Z}^E$ by

$$z^L(p) := \max\{\alpha \in \mathbb{Z} \mid t^\alpha p \in L\} \quad (p \in E).$$

**Proposition 5.1.** $\mathcal{L}(E)$ is a uniform semimodular lattice that is isomorphic to $L(\omega^E)$ by the maps $L \mapsto z^L$ and $z \mapsto \langle E^z \rangle$, where the following hold:

1. The ascending operator is given by $L \mapsto tL$.

2. The $\mathbb{Z}^n$-skeleton $\mathcal{S}(B)$ of $B \in \mathcal{B}$ is equal to $\mathcal{L}(B)$.

3. A height function $r$ of $\mathcal{L}(E)$ is given by

$$r(L) = \deg \det(Q) \quad (L \in \mathcal{L}(E)),$$

where $Q$ is a $K^-(t)$-basis of $L$.

4. For $x \in \mathcal{L}(\omega)$, it holds

$$\langle E^x \rangle_B = \langle B^x \rangle, \quad \omega^E(x)(B) = (\omega^E + x)(B) - r(\langle E^x \rangle) \quad (B \in \mathcal{B}).$$

Here, for $F \subseteq E$ and $x \in \mathbb{Z}^E$, we denote $F^x|_F$ by $F^x$. The proof uses the following basic lemma.

**Lemma 5.2.** $\langle E \rangle$ is a free $K^-(t)$-module having any $B \in \mathcal{B}_\omega$ as a basis.
Proof. Choose any $B \in \mathcal{B}_\omega$. Since $B$ is a $K(t)$-basis of $K(t)^n$, every element $u \in E$ is represented as $u = B\lambda$ for $\lambda \in K(t)^n$, where $B$ is regarded as a matrix. By Cramer’s rule, the $i$-th component $\lambda_i$ of $\lambda$ is equal to $\det(B^i)/\det(B)$, where $B^i$ is obtained from $B$ by replacing the $i$-th column with $u$. Then $\deg(\lambda_i) = \deg(\det(B^i)) - \deg(\det(B)) = \omega(B^i) - \omega(B) \leq 0$ by $B \in \mathcal{B}_\omega$. This means that $\lambda \in K^-(t)$. Consequently $\langle E \rangle$ is a free $K^-(t)$-module of basis $B$.

Proof of Proposition 5.1. Obviously we have $L = \langle E^{z^k} \rangle$. We show that $z^k \in \mathcal{L}(\omega)$. Suppose indirectly that $p \in E$ is a loop in $\mathcal{B}_{\omega+z^k}$. By the above lemma, for any $B \in \mathcal{B}_{\omega+z^k}$, $B^{z^k}$ is a basis of $L$. Consider equation $B^{z^k} = t^{z^k}(p)$. By using Cramer’s rule as above, we have $\deg(\lambda_i) = (\omega + z^k)(B - e_i + p) - (\omega + z^k)(B) \leq -1$ for each $i = 1, 2, \ldots, n$, where $e_i$ is the $i$-th column of $B$. The inequality follows from the fact that $p$ is a loop in $\mathcal{B}_{\omega+z^k}$. This means that $t^{z^k}(p+1) \in \mathcal{E}(E^k)$ also belongs to $\langle B^{z^k} \rangle = L$. This is a contradiction to the definition of $z^k$. Thus $z^k \in \mathcal{L}(\omega)$. Also $L \mapsto z^k$ is the inverse of $z \mapsto \langle E^k \rangle$. Indeed, $z^{E^k} \geq z$. If $z^{E^k}(e) > z(e)$, then one can see as above that $e$ does not belong to any base of $\mathcal{B}_{\omega+z^k}$, contradicting $z \in \mathcal{L}(\omega)$.

(1). This follows from $z^{E^k} = z^k + 1$.

(2). Observe that the sublattice $\mathcal{L}(B) = \{ \langle B^r \rangle \mid z \in \mathbb{Z}^B \}$ of $\mathcal{L}(E)$ is isomorphic to $\mathbb{Z}^n$. By Lemma 5.2, we have $B \in \mathcal{B}_{\omega+z^k}$ for $L = \langle B^r \rangle \in \mathcal{L}(B)$. By Lemma 4.21, we have $B \in \mathcal{S}(B)$. Thus $\mathcal{L}(B) \subseteq \mathcal{S}(B)$. Both $\mathcal{L}(B)$ and $\mathcal{S}(B)$ are isomorphic to $\mathbb{Z}^n$ with the same ascending operator. Consequently, it must hold $\mathcal{L}(B) = \mathcal{S}(B)$.

(3). Suppose that $L'$ covers $L$. We can choose $B \in B$ with $L', L' \in \mathcal{S}(B)$. Necessarily $L' = \langle B^{z^k} \rangle$ and $L = \langle B^r \rangle$ for $z - z' = 1_e$ for some $e \in B$, where $1_e := 1_{\{e\}}$. Then $\det(B^r) = \deg \det(B^r) + 1$.

(4). It obviously holds that $\langle B^r \rangle \in \mathcal{L}(B) = \mathcal{S}(B)$, and $\langle B^r \rangle \subseteq \langle E^k \rangle_B$. Suppose indirectly that the inclusion is strict. Then, for some $e \in B$, it holds $\langle B^{r+1}e \rangle \subseteq \langle E^k \rangle_B \subseteq \langle E^k \rangle_B$. This means that $\langle E^{k+1} \rangle = \langle E^k \rangle$. However this is a contradiction to $x = z^{(E^k)}$.

From the definition, we have $\omega^{\mathcal{L}(\omega),x}(B) = -r(\langle B^r \rangle, \langle E^k \rangle) = \deg(\det(B^r) - r(\langle E^k \rangle)) = \omega^{E^k}(B) - r(\langle E^k \rangle) = (\omega^E + x)(B) - r(\langle E^k \rangle)$.

Modular valued matroid and Euclidean building. Analogous to a modular matroid — a matroid whose lattice of flats is a modular lattice, a modular valued matroid is defined as an integer-valued valued matroid $(E, \omega)$ such that the corresponding $\mathcal{L}(\omega)$ is a uniform modular lattice. The companion work [13] showed that uniform modular lattices and Euclidean buildings of type $\Lambda$ are cryptomorphically equivalent in the following sense. For a uniform modular lattice $\mathcal{L}$, define equivalence relation $\simeq$ on $\mathcal{L}$ by $x \simeq y$ if $x = (y)^{+k}$ for some $k$. Then the simplicial complex $\mathcal{C}(\mathcal{L})$ modulo $\simeq$ is a Euclidean building of type $\Lambda$; recall Theorem 4.22 for simplicial complex $\mathcal{C}(\mathcal{L})$. Conversely, every Euclidean building of type $\Lambda$ is obtained in this way. Thus we have the following:

**Theorem 5.3.** For a modular valued matroid $(E, \omega)$, the tropical linear space $T(\omega)/\mathbb{R}1$ is a geometric realization of the Euclidean building associated with uniform modular lattice $\mathcal{L}(\omega)$.

Dress and Terhalle [7] claimed this result on the Euclidean building for $\text{SL}(F^n)$, where $F$ is a field with a discrete valuation. In the previous example, take the whole set $K(t)^n$ as $E$. In this case, $\mathcal{L}(E)$ is the lattice of all full-rank free $K^-(t)$-submodules of $K(t)^n$, and is a uniform modular lattice of uniform-rank $n$; see [13] Example 3.3.
In particular, valuated matroid \((E, \omega^E)\) is a modular valuated matroid. The simplicial complex \(C(L(E))\) is nothing but the Euclidean building for \(\text{SL}(K(t)^n)\); see [11, Section 19].

**Acknowledgments**

The author thanks Kazuo Murota, Yuni Iwamasa, and Koyo Hayashi for careful reading and helpful comments. This work was partially supported by JSPS KAKENHI Grant Numbers JP25280004, JP26330023, JP26280004, JP17K00029.

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