Differential game of alternating pursuit of three targets by two pursuers with a Time-type criterion

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Abstract. On the plane, we consider the differential game of two identical pursuers against three consistently evading players (targets) forming a coalition. Players’ movements are simple. The speed of the pursuers is limited to one, the speed of the targets is limited by a given constant of less than one. At the beginning of the game, the pursuers and targets are localized at two different points on the plane: the pursuers are at a certain point $P^0$, and the targets are at a point $E^0$. A game is conducted with complete information. The task of the pursuers is to pinpoint all the targets in the shortest possible time. The task of evading players is the opposite.

1. Introduction
The statement under consideration belongs to the problems of pursuit-evasion on the plane, in which there are fewer pursuers than targets. In these problems, as a rule, the criteria traditional for the theory of differential games such as ”time” or ”miss” are used. For simple player movements, problems with one pursuer versus a coalition of two targets, one of which could be false, were investigated in [1, 2] for the criteria of total pursuit time of all targets and average pursuit time of the true target [3]. In the latter case, the target classification probabilities (true or false) were considered to be set. In [4], a similar problem is solved with the true target miss criterion. It should be emphasized that the solutions to the above-mentioned problems of alternate pursuit have a common specificity, namely: for certain initial positions of the players, the stage of the actual alternate pursuit is always preceded by the stage of joint pursuit of a group of targets by one pursuer. The physical meaning of having this stage is that during the joint pursuit, the pursuer holds the targets in conditions of additional uncertainty about the upcoming sequence of meetings, which is chosen by the pursuer at random at the end of the joint pursuit stage. Thus, the pursuer’s strategy is mixed.

In the proposed paper, two pursuers starting from one point should catch (point-by-point) three identical evaders starting from another point and having opposite interests in the shortest possible time. It is assumed that this game is the game with complete information, in which the players’ movements are simple. In this game the optimal trajectories of the players are found and it is shown that in contrast to the similar game ”one against two” setting mentioned above [2, 3], there is no stage of joint pursuit in this game.
2. Problem statement

On the plane, we consider the differential game of pursuit-evasion of two pursuers $P_1$ and $P_2$ against three consistently evading identical targets $E_1$, $E_2$ and $E_3$, having the same maximum speeds, limited by a given constant $\beta < 1$. The speed of the pursuers is limited to one. The players’ movements are assumed to be simple [5]. The performance index (criterion) is the time of point capture of all targets by the pursuers.

It is assumed that at the initial instant $t_0 = 0$, all three targets are located at the same point $E^0$ of the axis $OX$ at a distance $\varrho_0 = 1$ from the pursuers located at the point $P^0$ that coincides with the origin $O$ of some Cartesian coordinate system $XOY$.

Using $Z_{ji}(t)$, we denote a two-dimensional vector directed at instant $t$ from $P_i$ to the target $E_j$, $(i = 1, 2; j = 1, 2, 3)$. We assign index 3 to the last target in the order of pursuit, and index 1 to the first. Let $T_i$ be the instant when players $P_i$ and $E_i$ meet, and $T$ be the time moment when the game ends. Under the assumption $T_1 \leq T_2$, the mathematical model of the problem has the following form.

**Players motion equations** (the argument $t$ is omitted):

\[
\begin{align*}
\dot{Z}_{11} & = v_1 - u_1, & t & \in [0, T_1] \\
\dot{Z}_{31} & = v_3 - u_1, & t & \in [T_1, T_2] \\
\dot{Z}_{22} & = v_2 - u_2, & t & \in [T_2, T] \\
\dot{Z}_{32} & = v_3 - u_2, & t & \in [T_2, T] \\
\dot{Z}_0 & = 1, & Z_0(0) & = 0.
\end{align*}
\]

(1)

**Initial conditions:**

\[Z_{11}(0) = Z_{22}(0) = Z_{31}(0) = Z_{32}(0) = z^0 = \{1, 0\}, \quad Z_0(0) = 0.\]  

(2)

**Discontinuity surfaces of the right-hand sides of the motion equations:**

\[Z_{11}(T_1) = 0, \quad Z_{22}(T_2) = 0.\]  

(3)

**Phase restrictions:**

\[|Z_{31}| \geq |Z_{11}|, \quad |Z_{32}| \geq |Z_{22}|.\]  

(4)

**Terminal condition:**

\[|Z_{31}(T)| \cdot |Z_{32}(T)| = 0.\]  

(5)

**Restrictions on control:**

\[P_i : \quad |u_i| \leq 1, \quad E_j : \quad |v_j| \leq \beta < 1.\]  

(6)

**Performance index (criterion):**

\[G = Z_0(T) \to \min_{u_1, u_2, v_1, v_2, v_3} \max_{u_1, u_2, v_1, v_2, v_3}.\]  

(7)

Here the scalar variable $Z_0 = Z_0(t)$ plays the role of time.
3. Problem solution
The problem is solved by applying Pontryagin maximum principle for problems with a game situation [5]. The following proposition is established.

**Proposition 1**
(i) At intervals \([0, T_1], [T_1, T_2], [T_2, T_3]\) the players move along straight lines with maximum modulo speeds.
(ii) Players’ movements are symmetrical with respect to the \(OX\) axis.
(iii) The movement of the target \(E_3\) is along the \(OX\) axis.
(iv) There is an equality of meetings time moments: \(T_1 = T_2\).
(v) There is no joint pursuit stage.

Statements (i) – (v) define the structure of optimal players’ trajectories (Figure 1).

In Figure 1: \(P^0\) and \(E^0\) are the starting points of the pursuers and targets, respectively, \(P^T_1 = E^T_1\) and \(P^T_2 = E^T_2\) are the meeting points with the first and second targets, respectively, and \(P^T_3 = E^T_3\) is the point where the pursuers catch the third target. Point \(C = \{\beta^2/(1-\beta^2), 0\}\) is the center of the Apollonius circle (the geometric location of the players’ meeting points when approaching in parallel) of radius \(R = \beta/(1-\beta^2)\). From Figure 1 it can be seen that from instant

\(t_0 = 0\), the pursuers immediately switch to alternate pursuit of the targets \(E_1\) and \(E_2\) along the trajectories of parallel convergence. After instant \(T_1 = T_2\), the pursuers simultaneously turn around to capture the target \(E_3\), performing a kind of ”roundabout” maneuver that does not allow the target \(E_3\) to leave the \(OX\) axis. The following theorem is true.

**Theorem 1**
The angles \(\alpha\) of evasion of the targets \(E_1\) and \(E_2\) are constant and equal to \(60^\circ\) for any values of \(\beta\).

**Proof.** Knowing the structure of the optimal trajectories of the players (piecewise straightness and symmetry with respect to the \(OX\) axis), it is easy to explicitly derive the values of the criterion \(G\) (up to a constant multiplier), as a function of the evasion angle’ \(\alpha\) cosine \(c\) of the targets \(E_1\) and \(E_2\):

\[
G(c) = \frac{1 - \beta^2 c + \beta \sqrt{2(1-c)} - \beta^2(1-c^2)}{\sqrt{1-\beta^2(1-c^2)} - \beta c},
\]

...
where \( c = \cos \alpha \). To do this it is enough to write the cosine theorem for triangles \( \triangle O\!E^0P_1^{T_1} \) and \( \triangle E^0P_1^{T_1}P^T \). The solution of the equation \( dG(c)/dc = 0 \) gives the root \( c^o = 1/2 \). Whence it follows that \( \alpha^o = \pm 60^o \). Theorem 1 is proved.

4. Conclusion

The results obtained can be extended to the case when only one of the three targets is true, and the other two are false with the specified probabilities of classification of true and false targets. In this case, the pursuers need to minimize the mathematical expectation of the time of catching the true target. However, in this formulation, the problem can only be solved numerically.

Appendix A

Proof of Proposition 1. Following the methodology of the maximum principle [6], we write Hamiltonians for the system (1) (the argument \( t \) is omitted):

\[
\mathcal{H} = \lambda_{11} \cdot (v_1 - u_1) + \lambda_{31} \cdot (v_3 - u_1) + \lambda_{22} \cdot (v_2 - u_2) + \lambda_{32} \cdot (v_3 - u_2) + \lambda_0, \quad t \in [0, T_1] \quad (A.1)
\]

\[
\tilde{\mathcal{H}} = \lambda_{31} \cdot (\dot{v}_3 - \dot{u}_1) + \lambda_{22} \cdot (\dot{v}_2 - \dot{u}_2) + \lambda_{32} \cdot (\dot{v}_3 - \dot{u}_2) + \lambda_0, \quad t \in [T_1, T_2] \quad (A.2)
\]

\[
\tilde{\mathcal{H}} = \lambda_{31} \cdot (\dot{v}_3 - \dot{u}_1) + \lambda_{32} \cdot (\dot{v}_3 - \dot{u}_2) + \lambda_0, \quad t \in [T_2, T] \quad (A.3)
\]

Here the symbols \( \wedge \) and \( \sim \) indicate variables corresponding to the intervals \([T_1, T_2]\) and \([T_2, T]\). A point between vectors means a two-dimensional scalar product.

In (A.1) – (A.3), the functions \( \lambda_k \) (similarly for \( \tilde{\lambda}_k \) and \( \lambda_k \)), where \( k \) is any of the subscripts: 0, 11, \ldots , 32, play the role of conjugate variables and satisfy the equations \( \lambda_k = -\partial\mathcal{H}/\partial Z_k \). In (A.1) – (A.3), due to the independence of the right-hand sides from the phase variables \( Z_k \), these partial derivatives are zero. Therefore, all the functions \( \lambda_k \) (similarly \( \tilde{\lambda}_k \) and \( \lambda_k \)) are two-dimensional vector-constants for \( k \neq 0 \) and for \( k = 0 \): \( \lambda_0 = \text{const} \) (similarly \( \tilde{\lambda}_0 \) and \( \lambda_0 \)).

Due to the linearity of the controls, the saddle points of the Hamiltonians are realized by the controls (see (6), (7)):

\[
\begin{align*}
    & t \in [0, T_1] & t \in [T_1, T_2] & t \in [T_2, T] \\
    v_1^* = -\beta \frac{\lambda_{11}}{|\lambda_{11}|} & \quad \dot{v}_1^* = 0 & \quad \ddot{v}_1^* = 0 \\
    v_2^* = -\beta \frac{\lambda_{22}}{|\lambda_{22}|} & \quad \dot{v}_2^* = -\beta \frac{\tilde{\lambda}_{22}}{|\lambda_{22}|} & \quad \ddot{v}_2^* = 0 \\
    v_3^* = -\beta \frac{\lambda_{31} + \lambda_{32}}{|\lambda_{31} + \lambda_{32}|} & \quad \dot{v}_3^* = -\beta \frac{\tilde{\lambda}_{31} + \tilde{\lambda}_{32}}{|\lambda_{31} + \lambda_{32}|} & \quad \ddot{v}_3^* = -\beta \frac{\lambda_{31} + \lambda_{32}}{|\lambda_{31} + \lambda_{32}|} \\
    u_1^* = -\frac{\lambda_{11} + \lambda_{31}}{|\lambda_{11} + \lambda_{31}|} & \quad \dot{u}_1^* = \frac{\tilde{\lambda}_{31}}{|\lambda_{31}|} & \quad \ddot{u}_1^* = -\frac{\lambda_{31}}{|\lambda_{31}|} \\
    u_2^* = -\frac{\lambda_{22} + \lambda_{32}}{|\lambda_{22} + \lambda_{32}|} & \quad \dot{u}_2^* = -\frac{\tilde{\lambda}_{22} + \tilde{\lambda}_{32}}{|\lambda_{22} + \lambda_{32}|} & \quad \ddot{u}_2^* = -\frac{\lambda_{32}}{|\lambda_{32}|}
\end{align*}
\]

Here an asterisk marks the optimal player’ controls.

Since all variables \( \lambda_k \) are constants, the statement (i) of Proposition 1 follows from (A.4).

Now we need to write the transversality conditions at instants \( T_1, T_2 \) and \( T \), taking into account that at the terminal instant \( T \), generally speaking, three outcomes are possible: \( Z_{31}(T) = 0, Z_{32}(T) = 0 \), and simultaneously \( Z_{31}(T) = Z_{32}(T) = 0 \) (see (5)).
The transversality conditions at instant $T_1$ have the form:
\[
\lambda_{11} \cdot \delta Z_{11} + \lambda_{31} \cdot \delta Z_{31} + \lambda_{22} \cdot \delta Z_{22} + \lambda_{32} \cdot \delta Z_{32} + \lambda_0 \delta Z_0 - H\delta T_1 = 0
\]
where $\delta Z_{11} = 0$ due to (3). Variations of other variables are arbitrary. Hence (A.5) gives
\[
\lambda_{31} = \hat{\lambda}_{31}, \lambda_{22} = \hat{\lambda}_{22}, \lambda_{32} = \hat{\lambda}_{32}, \lambda_0 = \hat{\lambda}_0, H = \tilde{H},
\] (A.6)
which means the continuity of these variables at instant $T_1$. Similarly, the transversality conditions at instant $T_2$, taking into account $\delta Z_{11} = 0$ and $\delta Z_{22} = 0$ by virtue of (3), implies continuity of functions
\[
\lambda_{31} = \hat{\lambda}_{31}, \lambda_{32} = \hat{\lambda}_{32}, \lambda_0 = \hat{\lambda}_0, \tilde{H} = \tilde{H}.
\] (A.7)
Finally from the transversality conditions at terminal instant $T$ in the case $Z_{31}(T) = Z_{32}(T) = 0$ we find
\[
\hat{\lambda}_0 = -1, \tilde{H}(T) = 0.
\] (A.8)
The continuity of variables from (A.6) – (A.8) means that the symbols $\wedge$ and $\sim$ can be removed in (A.1) – (A.4) for all $\lambda_k$. Further, due to the stationarity of the system (1) and the conditions (A.6) – (A.8), we have
\[
H = \tilde{H} = \tilde{H} \equiv 0, \quad t \in [0, T].
\] (A.9)
Explicitly, (A.9) has the form:
\[
1 + \beta |\lambda_{31} + \lambda_{32}| = |\lambda_{31}| + |\lambda_{32}|,
1 + \beta |\lambda_{31} + \lambda_{32}| + \beta |\lambda_{22}| = |\lambda_{31}| + |\lambda_{22} + \lambda_{32}|,
1 + \beta |\lambda_{31} + \lambda_{32}| + \beta |\lambda_{22}| + \beta |\lambda_{11}| = |\lambda_{11} + \lambda_{31}| + |\lambda_{22} + \lambda_{32}|.
\] (A.10)
So, we got eleven scalar variables: $T_1$, $T_2$, $T$ and eight components of four two-dimensional vectors-constants $\lambda_{11}$, $\lambda_{22}$, $\lambda_{31}$, $\lambda_{32}$, subject to three scalar equations (A.10) and four vector equations, which we obtain by integrating the system (1) under the controls (A.4), taking into account the conditions (2), (3) and (5).

From these relations, it is easy to get that $E_3$ always moves in a straight line with the speed $\beta$ in the continuation of the vector $z^0$ (see (2)). This implies that the problem is symmetric with respect to indices 1 and 2, which implies that the instants of meetings are equal $T_1 = T_2$, and therefore the terminal condition $Z_{31}(T) = Z_{31}(T) = 0$ is met (see (5)). Note that in this case the condition (4) is executed automatically. Proposition 1 is proved.

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