3-POST-LIE ALGEBRAS AND RELATIVE ROTA-BAXTER OPERATORS OF NONZERO WEIGHT ON 3-LIE ALGEBRAS

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ABSTRACT. In this paper, first we introduce the notions of relative Rota-Baxter operators of nonzero weight on 3-Lie algebras and 3-post-Lie algebras. A 3-post-Lie algebra consists of a 3-Lie algebra structure and a ternary operation such that some compatibility conditions are satisfied. We show that a relative Rota-Baxter operator of nonzero weight induces a 3-post-Lie algebra naturally. Conversely, a 3-post-Lie algebra gives rise to a new 3-Lie algebra, which is called the subadjacent 3-Lie algebra, and an action on the original 3-Lie algebra. Then we construct an $L_\infty$-algebra whose Maurer-Cartan elements are relative Rota-Baxter operators of nonzero weight. Consequently, we obtain the twisted $L_\infty$-algebra that controls deformations of a given relative Rota-Baxter operator of nonzero weight on 3-Lie algebras. Finally, we introduce a cohomology theory for a relative Rota-Baxter operator of nonzero weight on 3-Lie algebras and use the second cohomology group to classify infinitesimal deformations.

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1. INTRODUCTION

Rota-Baxter associative algebras originated from the probability study of G. Baxter [7]. In the Lie algebra context, Kupershmidt introduced the notion of an $O$-operator (also called a relative Rota-Baxter operator) in [26] to better understand the classical Yang-Baxter equation. (Relative) Rota-Baxter operators on Lie algebras and associative algebras have important applications in various fields, such as the classical Yang-Baxter equation and integrable systems [2, 27, 33], splitting of operads [3, 31], double Lie algebras [21], Connes-Kreimer’s algebraic approach to renormalization of quantum field theory [11], etc. See the book [22] for more details. Recently, the deformation and cohomology theories of relative Rota-Baxter operators on both Lie and associative algebras were studied in [14, 35, 39]. Post-Lie algebras, as natural generalizations of
pre-Lie algebras [8], were introduced by Vallette in [17], and have important applications in geometric numerical integration and mathematical physics [5, 3, 12, 13, 8, 29]. In particular, a relative Rota-Baxter operator of nonzero weight on Lie algebras induces a post-Lie algebra.

The notion of 3-Lie algebras and more generally, $n$-Lie algebras was introduced by Filippov in [19], which appeared naturally in various areas of theoretical and mathematical physics [1, 16, 18]. See the review article [13, 28] for more details. Indeed, the $n$-Lie algebra is the algebraic structure corresponding to the Nambu mechanics [30]. In [8], R. Bai et al. introduced the notion of Rota-Baxter operators of weight $\lambda$ on 3-Lie algebras and give various constructions from Rota-Baxter operators on Lie algebras and pre-Lie algebras. Then the notion of an $O$-operator on a 3-Lie algebra with respect to a representation was introduced in [11] to study solutions of the 3-Lie classical Yang-Baxter equation. In particular, when the representation is the adjoint representation, an $O$-operator is exactly a Rota-Baxter operator of weight 0. $O$-operators on 3-Lie algebras were also used to study matched pairs of 3-Lie algebras and 3-Lie bialgebras in [24]. Cohomologies and deformations of $O$-operators on 3-Lie algebras were studied in [36], where the terminology of relative Rota-Baxter operators was used instead of $O$-operators.

The first purpose of this paper is to study relative Rota-Baxter operators of nonzero weight on 3-Lie algebras and associated structures. For this purpose, first we study actions of a 3-Lie algebra $\mathfrak{g}$ on a 3-Lie algebra $\mathfrak{h}$, which is totally different from the case of Lie algebra actions. We introduce the notion of a relative Rota-Baxter operator of weight $\lambda$ from a 3-Lie algebra $\mathfrak{h}$ to a 3-Lie algebra $\mathfrak{g}$ with respect to an action $\rho$, and characterize it using graphs of the semidirect product 3-Lie algebra. We further establish the deformation and cohomology theories for relative Rota-Baxter operators of weight $\lambda$ on 3-Lie algebras. Note that relative Rota-Baxter operators and Rota-Baxter operators of nonzero weight introduced in [6] are not consistent, see Remark 2.7 for details of the explanation. The cohomology theory for Rota-Baxter operators of weight $\lambda$ on 3-Lie algebras is considered in [23] separately.

The second purpose is to investigate the 3-ary generalizations of post-Lie algebras and analyze the relation with the aforementioned relative Rota-Baxter operators of nonzero weight on 3-Lie algebras. We introduce a new algebraic structure, which is called a 3-post-Lie algebra. A 3-post-Lie algebra naturally gives rise to a new 3-Lie algebra and an action on the original 3-Lie algebra such that the identity map is a relative Rota-Baxter operator of weight 1. We show that a relative Rota-Baxter operator of weight $\lambda$ induces a 3-post-Lie algebra structure naturally. We will explore further applications of 3-post-Lie algebras along the line of applications of post-Lie algebras in future works.

The paper is organized as follows. In Section 2, we introduce the notion of a relative Rota-Baxter operator of weight $\lambda$ from a 3-Lie algebra $\mathfrak{h}$ to a 3-Lie algebra $\mathfrak{g}$ with respect to an action $\rho$. In Section 3, we introduce the notion of a 3-post-Lie algebra and show that a relative Rota-Baxter operator of weight $\lambda$ on 3-Lie algebras naturally induces a 3-post-Lie algebra. Moreover, a 3-post-Lie algebra also gives rise to a new 3-Lie algebra together with an action. In Section 4, we construct an $L_\infty$-algebra whose Maurer-Cartan elements are precisely relative Rota-Baxter operators of weight $\lambda$ on 3-Lie algebras. In Section 5, we establish a cohomology theory for a relative Rota-Baxter operator of weight $\lambda$ on 3-Lie algebras, and classify infinitesimal deformations using the second cohomology group.

In this paper, we work over an algebraically closed filed $\mathbb{K}$ of characteristic 0.

**Acknowledgements.** This research is supported by NSFC (11922110). We give warmest thanks to the referee for helpful suggestions.
2. Relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras

In this section, we first introduce the notion of action of 3-Lie algebras, which give rise to the semidirect product 3-Lie algebras. Then we introduce the notion of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras, which can be characterized by the graphs of the semidirect product 3-Lie algebras. Finally we establish the relation between relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras and Nijenhuis operators on 3-Lie algebras. A class of examples are given via certain projections.

**Definition 2.1.** ([10]) A 3-Lie algebra is a vector space \( \mathfrak{g} \) together with a skew-symmetric linear map \( \{\cdot, \cdot, \cdot\}_3 : \wedge^3 \mathfrak{g} \to \mathfrak{g} \), such that for \( x_i \in \mathfrak{g}, 1 \leq i \leq 5 \), the following Fundamental Identity holds:

\[
\{x_1, x_2, [x_3, x_4, x_5]\}_3 = \{[[x_1, x_2, x_3]_3, x_4], x_5\}_3 + \{[x_3, [x_1, x_2, x_4]]_3, x_5\}_3 + \{[x_3, x_4, [x_1, x_2, x_5]]_3, \}
\]

For \( x_1, x_2 \in \mathfrak{g} \), define \( \text{ad}_{x_1, x_2} \in \mathfrak{gl}(\mathfrak{g}) \) by

\[
\text{ad}_{x_1, x_2} x := \{x_1, x_2, x\}_3, \quad \forall x \in \mathfrak{g}.
\]

Then \( \text{ad}_{x_1, x_2} \) is a derivation, i.e.

\[
\text{ad}_{x_1, x_2} [x_3, x_4, x_5]_3 = [\text{ad}_{x_1, x_2} x_3, x_4, x_5]_3 + [x_3, \text{ad}_{x_1, x_2} x_4, x_5]_3 + [x_3, x_4, \text{ad}_{x_1, x_2} x_5]_3.
\]

**Definition 2.2.** ([5]) A representation of a 3-Lie algebra \( (\mathfrak{g}, \{\cdot, \cdot, \cdot\}_3) \) on a vector space \( V \) is a linear map: \( \rho : \wedge^3 \mathfrak{g} \to \mathfrak{gl}(V) \), such that for all \( x_1, x_2, x_3, x_4 \in \mathfrak{g} \), the following equalities hold:

\[
(2) \quad \rho(x_1, x_2) \rho(x_3, x_4) = \rho(x_1, x_3, x_4, x_5) + \rho(x_1, x_2, x_3) \rho(x_3, x_4) + \rho(x_3, x_4) \rho(x_1, x_2);
\]

\[
(3) \quad \rho(x_1, x_2, x_3, x_4, x_5) = \rho(x_1, x_2, x_3, x_4) \rho(x_1, x_2, x_3) + \rho(x_1, x_2, x_3, x_4) \rho(x_1, x_2, x_3).
\]

Let \( (\mathfrak{g}, \{\cdot, \cdot, \cdot\}_3) \) be a 3-Lie algebra. The linear map \( \text{ad} : \wedge^3 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \) defines a representation of the 3-Lie algebra \( \mathfrak{g} \) on itself, which is called the adjoint representation of \( \mathfrak{g} \).

**Definition 2.3.** ([1]) Let \( (\mathfrak{g}, \{\cdot, \cdot, \cdot\}_3) \) be a 3-Lie algebra. Then the subalgebra \( [\mathfrak{g}, \mathfrak{g}, \mathfrak{g}]_3 \) is called the derived algebra of \( \mathfrak{g} \), and denoted by \( \mathfrak{g}^1 \). The subspace

\[
C(\mathfrak{g}) = \{x \in \mathfrak{g} : \{x, y, z\}_3 = 0, \forall y, z \in \mathfrak{g}\}
\]

is called the center of \( \mathfrak{g} \).

**Definition 2.4.** Let \( (\mathfrak{g}, \{\cdot, \cdot, \cdot\}_3) \) and \( (\mathfrak{h}, \{\cdot, \cdot, \cdot\}_3) \) be two 3-Lie algebras. Let \( \rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h}) \) be a representation of the 3-Lie algebra \( \mathfrak{g} \) on the vector space \( \mathfrak{h} \). If for all \( x, y \in \mathfrak{g}, u, v, w \in \mathfrak{h} \),

\[
(4) \quad \rho(x, y) u \in C(\mathfrak{h}),
\]

\[
(5) \quad \rho(x, y)[u, v, w] = 0,
\]

then \( \rho \) is called an action of \( \mathfrak{g} \) on \( \mathfrak{h} \).

We denote an action by \( (\mathfrak{h}; \rho) \). Note that (4) and (5) imply that \( \rho(x, y) \) is a derivation.

**Example 2.5.** Let \( (\mathfrak{g}, \{\cdot, \cdot, \cdot\}_3) \) be a 3-Lie algebra. If \( \mathfrak{g} \) satisfies \( \mathfrak{g}^1 \subset C(\mathfrak{g}) \), then the adjoint representation \( \text{ad} : \wedge^3 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \) is an action of \( \mathfrak{g} \) on itself.

**Definition 2.6.** Let \( \rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h}) \) be an action of a 3-Lie algebra \( (\mathfrak{g}, \{\cdot, \cdot, \cdot\}_3) \) on a 3-Lie algebra \( (\mathfrak{h}, \{\cdot, \cdot, \cdot\}_3) \). A linear map \( T : \mathfrak{h} \to \mathfrak{g} \) is called a relative Rota-Baxter operator of weight \( \lambda \in \mathbb{K} \) from a 3-Lie algebra \( \mathfrak{h} \) to a 3-Lie algebra \( \mathfrak{g} \) with respect to an action \( \rho \) if

\[
(6) \quad [Tu, Tv, Tw] = T(\rho(Tu, Tv) w + \rho(Tv, Tw) u + \rho(Tw, Tu) v + \lambda[u, v, w]_3), \quad \forall u, v, w \in \mathfrak{h}.
\]
Remark 2.7. The notion of a relative Rota-Baxter operator of weight \( \lambda \) on a 3-Lie algebra given above is a natural generalization of the \( O \)-operator introduced in [3]. While it is not consistent with the Rota-Baxter operator of weight \( \lambda \) introduced in [3]. Therefore, there are two different theories for relative Rota-Baxter operators of weight \( \lambda \) introduced above and Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras introduced in [3]. We will see that relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras can be characterized by Maurer-Cartan elements of the controlling \( L_\infty \)-algebras, and naturally related to 3-post-Lie algebras and Nijenhuis structures. These facts can be viewed as justifications of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras being interesting structures.

Definition 2.8. Let \( T \) and \( T' \) be two relative Rota-Baxter operators of weight \( \lambda \) from a 3-Lie algebra \( (\mathfrak{g}, [\cdot, \cdot, \cdot])_\mathfrak{g} \) to a 3-Lie algebra \( (\mathfrak{g}, [\cdot, \cdot, \cdot])_\mathfrak{g} \) with respect to an action \( \rho \). A homomorphism from \( T \) to \( T' \) consists of 3-Lie algebra homomorphisms \( \psi_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g} \) and \( \psi_\mathfrak{g} : \mathfrak{h} \to \mathfrak{h} \) such that

\[
\begin{align*}
\psi_\mathfrak{g} \circ T &= T' \circ \psi_\mathfrak{g}, \\
\psi_\mathfrak{g}(\rho(x,y)u) &= \rho(\psi_\mathfrak{g}(x), \psi_\mathfrak{g}(y))(\psi_\mathfrak{g}(u)), \quad \forall x, y \in \mathfrak{g}, u \in \mathfrak{h}.
\end{align*}
\]

In particular, if both \( \psi_\mathfrak{g} \) and \( \psi_\mathfrak{g} \) are invertible, \( (\psi_\mathfrak{g}, \psi_\mathfrak{g}) \) is called an isomorphism from \( T \) to \( T' \).

Proposition 2.9. With above notations, \( (\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot, \cdot])_\rho \) is a 3-Lie algebra, which is called the semidirect product of the 3-Lie algebra \( \mathfrak{g} \) and the 3-Lie algebra \( \mathfrak{h} \) with respect to the action \( \rho \), and denoted by \( \mathfrak{g} \rtimes_\rho \mathfrak{h} \).

Proof. It follows from straightforward computations, and we omit details. \( \square \)

Theorem 2.10. Let \( \rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h}) \) be an action of a 3-Lie algebra \( (\mathfrak{g}, [\cdot, \cdot, \cdot])_\mathfrak{g} \) on a 3-Lie algebra \( (\mathfrak{h}, [\cdot, \cdot, \cdot])_\mathfrak{h} \). Then a linear map \( T : \mathfrak{h} \to \mathfrak{g} \) is a relative Rota-Baxter operator of weight \( \lambda \) if and only if the graph

\[
\text{Gr}(T) = \{ Tu + u | u \in \mathfrak{h} \}
\]

is a subalgebra of the 3-Lie algebra \( \mathfrak{g} \rtimes_\rho \mathfrak{h} \).

Proof. Let \( T : \mathfrak{h} \to \mathfrak{g} \) be a linear map. For all \( u, v, w \in \mathfrak{h} \), we have

\[
[Tu + u, Tv + v, Tw + w]_\rho = [Tu, Tv, Tw]_\mathfrak{g} + \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \lambda[u, v, w]_\mathfrak{h},
\]

which implies that the graph \( \text{Gr}(T) = \{ Tu + u | u \in \mathfrak{h} \} \) is a subalgebra of the 3-Lie algebra \( \mathfrak{g} \rtimes_\rho \mathfrak{h} \) if and only if \( T \) satisfies

\[
[Tu, Tv, Tw]_\mathfrak{g} = T\left( \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \lambda[u, v, w]_\mathfrak{h} \right),
\]

which means that \( T \) is a relative Rota-Baxter operator of weight \( \lambda \). \( \square \)

Since the graph \( \text{Gr}(T) \) is isomorphic to \( \mathfrak{h} \) as a vector space, there is an induced 3-Lie algebra structure on \( \mathfrak{h} \).
Corollary 2.11. Let $T : \mathfrak{h} \to \mathfrak{g}$ be a relative Rota-Baxter operator of weight $\lambda$ from a 3-Lie algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot])$ to a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ with respect to an action $\rho$. Then $(\mathfrak{h}, [\cdot, \cdot, \cdot])$ is a 3-Lie algebra, called the descendent 3-Lie algebra of $T$, where

$$[u, v, w]_T = \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \lambda[u, v, w]_\mathfrak{h}, \quad \forall u, v, w \in \mathfrak{h}.$$ 

Moreover, $T$ is a 3-Lie algebra homomorphism from $(\mathfrak{h}, [\cdot, \cdot, \cdot])$ to $(\mathfrak{g}, [\cdot, \cdot, \cdot])$.

In the sequel, we give the relationship between relative Rota-Baxter operators of weight $\lambda$ and Nijenhuis operators. Recall from [27] that a Nijenhuis operator on a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ is a linear map $N : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[Nx, Ny, Nz]_\mathfrak{g} = N([Nx, Ny, Nz]_\mathfrak{g}) + [x, Ny, Nz]_\mathfrak{g} + [Nx, y, Nz]_\mathfrak{g} - N[x, y, Nz]_\mathfrak{g} + N^2[x, y, z]_\mathfrak{g}, \quad \forall x, y, z \in \mathfrak{g}. \tag{11}$$

Proposition 2.12. Let $\rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ be an action of a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ on a 3-Lie algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot])$. Then a linear map $T : \mathfrak{h} \to \mathfrak{g}$ is a relative Rota-Baxter operator of weight $\lambda$ if and only if

$$\overline{T} = \begin{pmatrix} \text{Id} & T \\ 0 & 0 \end{pmatrix} : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g} \oplus \mathfrak{h},$$

is a Nijenhuis operator acting on the semidirect product 3-Lie algebra $\mathfrak{g} \ltimes_\rho \mathfrak{h}$.

Proof. For all $x, y, z, u, v, w \in \mathfrak{h}$, on the one hand, we have

$$[\overline{T}(x + u), \overline{T}(y + v), \overline{T}(z + w)]_{\rho} = [x, y, z]_{\mathfrak{g}} + [Tu, y, z]_{\mathfrak{g}} + [x, Tv, z]_{\mathfrak{g}} + [x, y, Tw]_{\mathfrak{g}} + [x, Tv, Tw]_{\mathfrak{g}} + [Tu, y, Tw]_{\mathfrak{g}} + [Tu, Tv, z]_{\mathfrak{g}} + [Tu, Tv, Tw]_{\mathfrak{g}}.$$ 

On the other hand, since $\overline{T}^2 = \overline{T}$, we have

$$\overline{T}([\overline{T}(x + u), \overline{T}(y + v), z + w]_{\rho} + [x + u, \overline{T}(y + v), z + w]_{\rho} + [x + u, y + v, \overline{T}(z + w)]_{\rho})$$

$$\overline{T}^2([x + u, y + v, z + w]_{\rho})$$

$$\overline{T}^3([x + u, y + v, z + w]_{\rho})$$

$$= [x, y, z]_{\mathfrak{g}} + [Tu, y, z]_{\mathfrak{g}} + [x, Tv, z]_{\mathfrak{g}} + [x, y, Tw]_{\mathfrak{g}} + [x, Tv, Tw]_{\mathfrak{g}} + [Tu, y, Tw]_{\mathfrak{g}} + [Tu, Tv, z]_{\mathfrak{g}} + [Tu, Tv, Tw]_{\mathfrak{g}} + \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \lambda[u, v, w]_{\mathfrak{h}},$$

which implies that $\overline{T}$ is a Nijenhuis operator on the semidirect product 3-Lie algebra $\mathfrak{g} \ltimes_\rho \mathfrak{h}$ if and only if (3) is satisfied. \hfill \Box

At the end of this section, we show that certain projections provide a class of examples of relative Rota-Baxter operators on 3-Lie algebras.

Proposition 2.13. Let $(\mathfrak{g}, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra such that the adjoint representation $\text{ad} : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an action of the 3-Lie algebra $\mathfrak{g}$ on itself. Let $\mathfrak{h}$ be an abelian 3-Lie subalgebra of $\mathfrak{g}$ satisfying $\mathfrak{g}^1 \cap \mathfrak{h} = 0$. Let $\mathfrak{t}$ be a complement of $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}$ as a vector space. Then the projection $P : \mathfrak{g} \to \mathfrak{h}$ onto the subspace $\mathfrak{h}$, i.e., $P(k, u) = u$ for all $k \in \mathfrak{t}$, $u \in \mathfrak{h}$, is a relative Rota-Baxter operator of weight $\lambda$ from $\mathfrak{g}$ to $\mathfrak{g}$ with respect to the adjoint action $\text{ad}$.
Proof. For all \( x, y, z \in \mathfrak{g} \), denote by \( \bar{x}, \bar{y}, \bar{z} \) their images under the projection \( P \). Since \( \mathfrak{h} \) is abelian and \( \mathfrak{g}^1 \cap \mathfrak{h} = 0 \), we have

\[
[P(x), P(y), P(z)]_\mathfrak{h} = P([P(x), P(y), P(z)]_\mathfrak{h}) = \lambda \cdot \mathfrak{h} + \lambda[\mathfrak{h}, \mathfrak{h}]_\mathfrak{h} = 0
\]

which implies that the projection \( P \) is a relative Rota-Baxter operator of weight \( \lambda \) from \( \mathfrak{g} \) to \( \mathfrak{g} \) with respect to the adjoint action \( \text{ad} \).

\[\square\]

Example 2.14. Let \( (\mathfrak{g}, [\cdot, \cdot, \cdot])_\mathfrak{h} \) be a 4-dimensional 3-Lie algebra with a basis \( \{e_1, e_2, e_3, e_4\} \) and the nonzero multiplication is given by

\[\{e_2, e_3, e_4\} = e_1.\]

The center of \( \mathfrak{g} \) is the subspace generated by \( \{e_1\} \). It is obvious that the adjoint representation \( \text{ad} : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \) is an action of \( \mathfrak{g} \) on itself. Let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \) generated by \( \{e_3, e_4\} \).

By Proposition 2.13, the projection \( P : \mathfrak{g} \to \mathfrak{g} \) given by \( \begin{cases} P(e_1) = 0, \\
 P(e_2) = 0, \\
 P(e_3) = e_3, \\
 P(e_4) = e_4,
\end{cases} \)

is a relative Rota-Baxter operator of weight \( \lambda \) from \( \mathfrak{g} \) to \( \mathfrak{g} \) with respect to the adjoint action \( \text{ad} \).

3. 3-POST-LIE ALGEBRAS

In this section, we introduce the notion of 3-post-Lie algebras. A relative Rota-Baxter operator of nonzero weight induces a 3-post-Lie algebra naturally. Therefore, 3-post-Lie algebras can be viewed as the underlying algebraic structures of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras. A 3-post-Lie algebra also gives rise to a new 3-Lie algebra and an action on the original 3-Lie algebra, which leads to the fact that the identity map is a relative Rota-Baxter operator of weight 1.

Definition 3.1. A 3-post-Lie algebra \( (A, [\cdot, \cdot, \cdot], [\cdot, \cdot, \cdot]) \) consists of a 3-Lie algebra \( (A, [\cdot, \cdot, \cdot]) \) and a ternary product \( [\cdot, \cdot, \cdot] : A \otimes A \otimes A \to A \) such that

\begin{align}
(12) \quad \{x_1, x_2, x_3\} &= \{-x_2, x_1, x_3\}, \\
(13) \quad \{x_1, x_2, [x_3, x_4, x_5]\} &= \{x_3, x_4, [x_1, x_2, x_5]\} + \{\{x_1, x_2, x_3\}, x_4, x_5\} + \{x_1, \{x_1, x_2, x_4\}, x_5\}, \\
(14) \quad \{\{x_1, x_2, x_3\}, x_4, x_5\} &= \{x_1, x_2, [x_3, x_4, x_5]\} + \{x_2, x_3, [x_1, x_4, x_5]\} + \{x_3, x_1, [x_2, x_4, x_5]\}, \\
(15) \quad \{x_1, x_2, [x_3, x_4, x_5]\} &= 0, \\
(16) \quad \{[x_1, x_2, x_3], x_4, x_5\} &= 0, \\
(17) \quad [x, y, z] &= \{x, y, \bar{z}\} + \{y, z, \bar{x}\} + \{z, x, \bar{y}\} + \{x, y, z\}, \quad \forall x, y, z \in A.
\end{align}

Remark 3.2. Let \( (A, [\cdot, \cdot, \cdot], [\cdot, \cdot, \cdot]) \) be a 3-post-Lie algebra. If the 3-Lie bracket \([\cdot, \cdot, \cdot] = 0\), then \( (A, [\cdot, \cdot, \cdot]) \) becomes a 3-pre-Lie algebra which was introduced in [3] in the study of the 3-Lie Yang-Baxter equation.
Definition 3.3. A homomorphism from a 3-post-Lie algebra \((A, \cdot, \cdot, \cdot)\) to \((A', \cdot, \cdot, \cdot')\) is a linear map \(\psi : A \to A'\) satisfying
\[
\begin{align*}
\psi([x, y, z]) &= \{\psi(x), \psi(y), \psi(z)\}', \\
\psi([x, y, z]) &= [\psi(x), \psi(y), \psi(z)]',
\end{align*}
\]
for all \(x, y, z \in A\).

Theorem 3.4. Let \((A, \cdot, \cdot, \cdot)\) be a 3-post-Lie algebra. Then

(i) \((A, \cdot, \cdot, \cdot)\) is a 3-Lie algebra, which is called the sub-adjacent 3-Lie algebra of the 3-post Lie algebra \((A, \cdot, \cdot, \cdot, \cdot)\), where \(\cdot, \cdot, \cdot\) is defined by \((7)\).

(ii) \((A; L)\) is an action of the 3-Lie algebra \((A, \cdot, \cdot, \cdot)\) on the 3-Lie algebra \((A, [\cdot, \cdot, \cdot])\), where the action \(L : \otimes^2 A \to \mathfrak{gl}(A)\) is defined by
\[
L(x, y)z = [x, y, z], \quad \forall x, y, z \in A.
\]

(iii) The identity map \(\text{Id} : A \to A\) is a relative Rota-Baxter operator of weight 1 from the 3-Lie algebra \((A, [\cdot, \cdot, \cdot])\) to the 3-Lie algebra \((A, \cdot, \cdot, \cdot)\) with respect to the action \(L\).

Proof. (i) By \((12)\), it is obvious that the operation \(\cdot, \cdot, \cdot\) given by \((7)\) is skew-symmetric. For all \(x_1, x_2, x_3, x_4, x_5 \in A\), by \((13)\)–\((16)\), we have
\[
\begin{align*}
&\llbracket x_1, x_2, \llbracket x_3, x_4, x_5 \rrbracket \rrbracket - \llbracket \llbracket x_1, x_2, x_3 \rrbracket, x_4, x_5 \rrbracket \\
&= \{x_1, x_2, \{x_3, x_4, x_5\}\} + \{x_1, x_2, \{x_4, x_5, x_1\}\} + \{x_1, x_2, \{x_5, x_1, x_4\}\} \\
&+ \{x_1, x_2, \{x_3, x_4, x_5\}\} + \{\llbracket x_3, x_4, x_5 \rrbracket, x_1, x_2\} + \{x_2, \llbracket x_3, x_4, x_5 \rrbracket, x_1\} \\
&+ \{x_4, x_5, \{x_1, x_2, x_3\}\} - \llbracket x_1, x_5, \{x_1, x_2, x_3\}\rrbracket - \{x_4, x_5, \{x_2, x_3, x_1\}\} \\
&- \llbracket x_5, \llbracket x_1, x_2, x_3 \rrbracket, x_4 \rrbracket - \llbracket \llbracket x_1, x_2, x_3 \rrbracket, x_4, x_5 \rrbracket - \{x_5, x_3, \{x_1, x_2, x_4\}\} \\
&- \{x_5, x_3, \{x_2, x_4, x_1\}\} - \{x_5, x_3, \{x_4, x_1, x_2\}\} - \{x_5, x_3, \{x_1, x_2, x_4\}\} \\
&- \{x_3, \llbracket x_1, x_2, x_4 \rrbracket, x_5\} - \{\llbracket x_1, x_2, x_4 \rrbracket, x_5, x_3\} - \{x_3, \llbracket x_1, x_2, x_4 \rrbracket, x_5\} \\
&- \{x_3, \llbracket x_3, x_4, x_1 \rrbracket, x_5\} - \{x_3, \llbracket x_4, x_1, x_2 \rrbracket, x_5\} - \{x_3, \llbracket x_3, x_4, x_1 \rrbracket, x_5\} \\
&- \{x_3, x_4, \{x_1, x_2, x_5\}\} - \{x_3, \llbracket x_1, x_2, x_5 \rrbracket, x_3, x_4\} - \{x_4, \llbracket x_1, x_2, x_5 \rrbracket, x_3\} \\
&- \{x_3, x_4, \{x_1, x_2, x_5\}\} = 0,
\end{align*}
\]
which implies that \((A, \cdot, \cdot, \cdot)\) is a 3-Lie algebra.

(ii) By \((13)\) and \((14)\), we have
\[
\begin{align*}
[L(x_1, x_2), L(x_3, x_4)](x_5) &= L(\llbracket x_1, x_2, x_3 \rrbracket, x_4)(x_5) + L(x_3, \llbracket x_1, x_2, x_4 \rrbracket)(x_5), \\
L(\llbracket x_1, x_2, x_3 \rrbracket, x_4)(x_5) &= (L(x_1, x_2)L(x_3, x_4) + L(x_2, x_3)L(x_1, x_4) \\
&+ L(x_3, x_1)L(x_2, x_4))(x_5),
\end{align*}
\]
which implies that \(L\) is a representation of the 3-Lie algebra \((A, \cdot, \cdot, \cdot)\) on the vector space \(A\). By \((16)\), \(L(x_1, x_2)x_3\) is in the center of the 3-Lie algebra \((A, [\cdot, \cdot, \cdot])\). Then by \((15)\), we have \(L(x_1, x_2)[x_3, x_4, x_5] = 0\). Therefore, \((A; L)\) is an action of the 3-Lie algebra \((A, \cdot, \cdot, \cdot)\) on the 3-Lie algebra \((A, [\cdot, \cdot, \cdot])\).

(iii) It follows from the definition of the subadjacent 3-Lie bracket \(\cdot, \cdot, \cdot\) directly. \(\Box\)
Corollary 3.5. Let \( \psi : A \rightarrow A' \) be a homomorphism from a 3-post-Lie algebra \((A, \{\cdot, \cdot, \cdot\}, \{\cdot, \cdot\})\) to \((A', \{\cdot, \cdot, \cdot\}', \{\cdot, \cdot\}')\). Then \( \psi \) is also a homomorphism from the sub-adjacent 3-Lie algebra \((A, [\cdot, \cdot, \cdot])\) to \((A', [\cdot, \cdot, \cdot]')\).

The following results illustrate that 3-post-Lie algebras can be viewed as the underlying algebraic structures of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras.

Theorem 3.6. Let \( T : \mathfrak{h} \rightarrow \mathfrak{g} \) be a relative Rota-Baxter operator of weight \( \lambda \) from a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) to a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) with respect to an action \( \rho \).

(i) \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) is a 3-post-Lie algebra, where \([\cdot, \cdot, \cdot] \) and \([\cdot, \cdot] \) are given by
\[
\{u_1, u_2, u_3\} = \rho(Tu_1, Tu_2)u_3, \quad [u_1, u_2, u_3] = \lambda[u_1, u_2, u_3]_{\mathfrak{h}}, \quad \forall u_1, u_2, u_3 \in \mathfrak{h}.
\]

(ii) \( T \) is a 3-Lie algebra homomorphism from the sub-adjacent 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) to \((\mathfrak{g}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\), where the 3-Lie bracket \([\cdot, \cdot, \cdot]\) is defined by
\[
\{u_1, u_2, u_3\} = \rho(Tu_1, Tu_2)u_3 + \rho(Tu_3, Tu_1)u_2 + \rho(Tu_2, Tu_3)u_1 + \lambda[u_1, u_2, u_3]_{\mathfrak{h}}.
\]

Proof. (i) For any \( u_1, u_2, u_3 \in \mathfrak{h} \), it is obvious that
\[
\{u_1, u_2, u_3\} = \rho(Tu_1, Tu_2)u_3 = -\rho(Tu_2, Tu_1)u_3 = -\{u_2, u_1, u_3\}.
\]
Furthermore, for \( u_1, u_2, u_3, u_4, u_5 \in \mathfrak{h} \), by (1), (3) and (17), we have
\[
\{u_1, u_2, [u_3, u_4, u_5]\} - \{u_3, u_4, [u_1, u_2, u_5]\}
= \{u_1, u_2, \rho(Tu_3, Tu_4)u_5\} - \{u_3, u_4, \rho(Tu_1, Tu_2)u_5\}
- \rho(Tu_1, Tu_2)u_3 + \rho(Tu_3, Tu_1)u_2 + \rho(Tu_2, Tu_3)u_1 + \lambda[u_1, u_2, u_3]_{\mathfrak{h}}, u_4, u_5
- \{u_3, \rho(Tu_1, Tu_2)u_4 + \rho(Tu_2, Tu_4)u_1 + \rho(Tu_4, Tu_1)u_2 + \lambda[u_1, u_2, u_3]_{\mathfrak{h}}, u_4, u_5
\]
\[
= \rho(Tu_1, Tu_2)\rho(Tu_3, Tu_4)u_5 - \rho(Tu_3, Tu_4)\rho(Tu_1, Tu_2)u_5
- \rho(Tu_1, Tu_2)u_3 + \rho(Tu_3, Tu_1)u_2 + \rho(Tu_2, Tu_3)u_1 + \lambda[u_1, u_2, u_3]_{\mathfrak{h}}, Tu_4)u_5
- \rho(Tu_3, Tu_1)u_2 + \rho(Tu_2, Tu_3)u_1 + T\lambda[u_1, u_2, u_3]_{\mathfrak{h}} + T\lambda[u_1, u_2, u_3]_{\mathfrak{h}} + Tu_4)u_5
= 0.
\]
This implies that (13) in Definition 3.1 holds. Similarly, by (3), we can verify that (14) holds. Moreover, by (3) and (5), (15) and (16) hold, too. Hence \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) is a 3-post-Lie algebra.

(ii) Note that the sub-adjacent 3-Lie algebra of the above 3-post-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) is exactly the 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]' [\cdot, \cdot])\) given in Corollary 2.1. Then the result follows. \(\square\)

Proposition 3.7. Let \( T \) and \( T' \) be two relative Rota-Baxter operators of weight \( \lambda \) from a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) to a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) with respect to an action \( \rho \). Let \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) and \((\mathfrak{h}', [\cdot, \cdot, \cdot]', [\cdot, \cdot]')\) be the induced 3-post-Lie algebras and \((\psi_0, \psi_0)\) be a homomorphism from \( T' \) to \( T \). Then \( \psi_0 \) is a homomorphism from the 3-post-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot], [\cdot, \cdot])\) to the 3-post-Lie algebra \((\mathfrak{h}', [\cdot, \cdot, \cdot]', [\cdot, \cdot]'\))

Proof. For all \( u, v, w \in \mathfrak{h} \), by (7)-(8) and (20), we have
\[
\psi_0([u, v, w]) = \psi_0(\rho(Tu, Tv)w)
= \rho(\psi_0(Tu), \psi_0(Tv))\psi_0(w)
= \rho(T'\psi_0(u), T'\psi_0(v))\psi_0(w)
= \{\psi_0(u), \psi_0(v), \psi_0(w)'\}.
\[ \psi_b([u, v, w]) = \psi_b\lambda[u, v, w] \]
\[ = \lambda[\psi_b(u), \psi_b(v), \psi_b(w)]_b \]
\[ = [\psi(u), \psi(v), \psi(w)]', \]
which implies that \( \psi_b \) is a homomorphism between the induced 3-post-Lie algebras. \( \square \)

Thus, Theorem 3.6 can be enhanced to a functor from the category of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras to the category of 3-post-Lie algebras.

4. Maurer-Cartan characterization of relative Rota-Baxter operators of weight \( \lambda \)

In this section, given an action of 3-Lie algebras, we construct an \( L_\infty \)-algebra whose Maurer-Cartan elements are relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras. Then we obtain the \( L_\infty \)-algebra that controls deformations of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras. This fact can be viewed as certain justification of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras being interesting structures.

Definition 4.1. An \( L_\infty \)-algebra is a \( \mathbb{Z} \)-graded vector space \( g = \oplus_{k \in \mathbb{Z}} g_k \) equipped with a collection (\( k \geq 1 \)) of linear maps \( l_k : \otimes^k g \to g \) of degree 1 with the property that, for any homogeneous elements \( x_1, \ldots, x_n \in g \), we have

(i) (graded symmetry) for every \( \sigma \in S_n \),

\[ l_n(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}, x_{\sigma(n)}) = \varepsilon(\sigma)l_n(x_1, \ldots, x_{n-1}, x_n), \]

(ii) (generalized Jacobi Identity) for all \( n \geq 1 \),

\[ \sum_{i=1}^{n} \sum_{\sigma \in S_{i-n-i}} \varepsilon(\sigma)l_{n-i+1}(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0. \]

Definition 4.2. A Maurer-Cartan element of an \( L_\infty \)-algebra \( (g, \{l_i\}_{i=1}^{\infty}) \) is an element \( \alpha \in g^0 \) satisfying the Maurer-Cartan equation

\[ \sum_{n=1}^{\infty} \frac{1}{n!} l_n(\alpha, \ldots, \alpha) = 0. \]

Let \( \alpha \) be a Maurer-Cartan element of an \( L_\infty \)-algebra \( (g, \{l_i\}_{i=1}^{\infty}) \). For all \( k \geq 1 \) and \( x_1, \ldots, x_n \in g \), define a series of linear maps \( l_k : \otimes^k g \to g \) of degree 1 by

\[ l_k(x_1, \ldots, x_k) = \sum_{n=0}^{\infty} \frac{1}{n!} l_{n+k} \{\underbrace{\alpha, \ldots, \alpha}_{n}, x_1, \ldots, x_k\}. \]

Theorem 4.3. ([29]) With the above notations, \( (g, \{l_i\}_{i=1}^{\infty}) \) is an \( L_\infty \)-algebra, obtained from the \( L_\infty \)-algebra \( (g, \{l_i\}_{i=1}^{\infty}) \) by twisting with the Maurer-Cartan element \( \alpha \). Moreover, \( \alpha + \alpha' \) is a Maurer-Cartan element of \( (g, \{l_i\}_{i=1}^{\infty}) \) if and only if \( \alpha' \) is a Maurer-Cartan element of the twisted \( L_\infty \)-algebra \( (g, \{l_i\}_{i=1}^{\infty}) \).

In [38], Th. Voronov developed the theory of higher derived brackets, which is a useful tool to construct explicit \( L_\infty \)-algebras.

Definition 4.4. ([38]) A V-data consists of a quadruple \( (L, F, \mathcal{P}, \Delta) \), where

- \( (L, [\cdot, \cdot]) \) is a graded Lie algebra,
- \( F \) is an abelian graded Lie subalgebra of \( (L, [\cdot, \cdot]) \),
• \( \mathcal{P} : L \rightarrow L \) is a projection, that is \( \mathcal{P} \circ \mathcal{P} = \mathcal{P} \), whose image is \( F \) and kernel is a graded Lie subalgebra of (\( L, [\cdot, \cdot] \)).

• \( \Delta \) is an element in \( \ker(\mathcal{P}) \) such that \( [\Delta, \Delta] = 0 \).

**Theorem 4.5.** ([38]) Let \( (L, F, \mathcal{P}, \Delta) \) be a V-data. Then \( (F, \{l_k\}_{k=1}^{\infty}) \) is an \( L_\infty \)-algebra, where
\[
l_k(a_1, \cdots, a_k) = \mathcal{P}[[ \cdots [[ \Delta, a_1], a_2], \cdots, a_k] \quad \text{for homogeneous } a_1, \cdots, a_k \in F.
\]

We call \( \{l_k\}_{k=1}^{\infty} \) the **higher derived brackets** of the V-data \( (L, F, \mathcal{P}, \Delta) \).

Let \( \mathfrak{g} \) be a vector space. We consider the graded vector space
\[
C^*(\mathfrak{g}, \mathfrak{g}) = \oplus_{n \geq 0} C^n(\mathfrak{g}, \mathfrak{g}) = \oplus_{n \geq 0} \text{Hom}(\wedge^n \mathfrak{g} \otimes \cdots \otimes \wedge^2 \mathfrak{g} \wedge \mathfrak{g}, \mathfrak{g}).
\]

**Theorem 4.6.** ([32]) The graded vector space \( C^*(\mathfrak{g}, \mathfrak{g}) \) equipped with the graded commutator bracket
\[
[P, Q]_R = P \circ Q - (-1)^{pq} Q \circ P, \quad \forall P \in C^p(\mathfrak{g}, \mathfrak{g}), Q \in C^q(\mathfrak{g}, \mathfrak{g}),
\]
is a graded Lie algebra, where \( P \circ Q \in C^{p+q}(\mathfrak{g}, \mathfrak{g}) \) is defined by
\[
(P \circ Q)(x_1, \cdots, x_{p+q}, x)
\]
\[
= \sum_{\sigma \in S(p+q)} \sum_{k=1}^{p} (-1)^{(k-1)q} \left( P(x_{\sigma(1)}, \cdots, x_{\sigma(k-1)}, Q(x_{\sigma(k)}, \cdots, x_{\sigma(k+q-1)}, x_{k+q}) \wedge y_{k+q}, x_{k+q+1}, \cdots, x_{p+q}, x) \right)
\]
\[
+ \sum_{\sigma \in S(p+q)} \sum_{k=1}^{p} (-1)^{(k-1)q} \left( Q(x_{\sigma(1)}, \cdots, x_{\sigma(k-1)}, x_{k+q} \wedge Q(x_{\sigma(k)}, \cdots, x_{\sigma(k+q-1)}, y_{k+q}), x_{k+q+1}, \cdots, x_{p+q}, x) \right)
\]
\[
+ \sum_{\sigma \in S(p+q)} (-1)^{pq} (-1)^{\sigma} P(x_{\sigma(1)}, \cdots, x_{\sigma(p)}, Q(x_{\sigma(p+1)}, \cdots, x_{\sigma(p+q-1)}, x_{\sigma(p+q)}, x))
\]
for all \( x_i, y_i \in \wedge^2 \mathfrak{g}, \ i = 1, 2, \cdots, p + q \) and \( x \in \mathfrak{g} \).

Moreover, \( \mu : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g} \) is a 3-Lie bracket if and only if \( [\mu, \mu]_R = 0 \), i.e. \( \mu \) is a Maurer-Cartan element of the graded Lie algebra \( (C^*(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot, \cdot], R) \).

Let \( \rho : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{gl}(h) \) be an action of a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot]_b)\) on a 3-Lie algebra \((h, [\cdot, \cdot, \cdot]_b)\).

For convenience, we use \( \pi : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g} \) to indicate the 3-Lie bracket \([\cdot, \cdot, \cdot]_b\) and \( \mu : \wedge^2 \mathfrak{h} \rightarrow \mathfrak{h} \) to indicate the 3-Lie bracket \([\cdot, \cdot, \cdot]_b\). In the sequel, we use \( \pi + \rho + \lambda \mu \) to denote the element in \( \text{Hom}(\wedge^2 \mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h}) \) given by
\[
(\pi + \rho + \lambda \mu)(x + u, y + v, z + w) = [x, y, z]_h + \rho(x, y)w + \rho(y, z)u + \rho(z, x)v + \lambda[u, v, w]_b,
\]
for all \( x, y, z \in \mathfrak{g}, u, v, w \in \mathfrak{h}, \lambda \in \mathbb{K} \). Note that the right hand side is exactly the semidirect product 3-Lie algebra structure given in Proposition 2.9. Therefore by Theorem 4.6, we have
\[
[\pi + \rho + \lambda \mu, \pi + \rho + \lambda \mu]_R = 0.
\]

**Proposition 4.7.** Let \( \rho : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{gl}(h) \) be an action of a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot]_b)\) on a 3-Lie algebra \((h, [\cdot, \cdot, \cdot]_b)\). Then we have a V-data \((L, F, \mathcal{P}, \Delta)\) as follows:

• the graded Lie algebra \((L, [\cdot, \cdot])\) is given by \((C^*(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h}), [\cdot, \cdot], R)\);

• the abelian graded Lie subalgebra \( F \) is given by
\[
F = C^*(h, \mathfrak{g}) = \oplus_{n \geq 0} C^n(h, \mathfrak{g}) = \oplus_{n \geq 0} \text{Hom}(\wedge^n \mathfrak{h} \otimes \cdots \otimes \wedge^2 \mathfrak{h} \wedge \mathfrak{h}, \mathfrak{g});
\]
\( \mathcal{P} : L \to L \) is the projection onto the subspace \( F \);
\( \Delta = \pi + \rho + \lambda \mu. \)

Consequently, we obtain an \( L_\infty \)-algebra \( (C^*(\mathfrak{h}, \mathfrak{g}), l_1, l_3) \), where
\[
\begin{align*}
  l_1(P) & = \mathcal{P}[\pi + \rho + \lambda \mu, P]_R, \\
  l_3(P, Q, R) & = \mathcal{P}[[[\pi + \rho + \lambda \mu, P]_R, Q]_R, R]_R,
\end{align*}
\]
for all \( P \in C^m(\mathfrak{h}, \mathfrak{g}), Q \in C^n(\mathfrak{h}, \mathfrak{g}) \) and \( R \in C^k(\mathfrak{h}, \mathfrak{g}) \).

**Proof.** By Theorem 4.5, \((F, \{l_k\}_{k=1}^{\infty})\) is an \( L_\infty \)-algebra, where \( l_k \) is given by (23). It is obvious that \( \Delta = \pi + \rho + \lambda \mu \in \ker(\mathcal{P})^1 \). For all \( P \in C^m(\mathfrak{h}, \mathfrak{g}), Q \in C^n(\mathfrak{h}, \mathfrak{g}) \) and \( R \in C^k(\mathfrak{h}, \mathfrak{g}) \), we have
\[
[[\pi + \rho + \lambda \mu, P]_R, Q]_R \in \ker(\mathcal{P}),
\]
which implies that \( l_2 = 0 \). Similarly, we have \( l_k = 0 \), when \( k \geq 4 \). Therefore, the graded vector space \( C^*(\mathfrak{h}, \mathfrak{g}) \) is an \( L_\infty \)-algebra with nontrivial \( l_1, l_3 \), and other maps are trivial. \( \square \)

**Theorem 4.8.** Let \( \rho : \wedge^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h}) \) be an action of a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g})\) on a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_\mathfrak{h})\). Then Maurer-Cartan elements of the \( L_\infty \)-algebra \((C^*(\mathfrak{h}, \mathfrak{g}), l_1, l_3)\) are precisely relative Rota-Baxter operators of weight \( \lambda \) from the 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_\mathfrak{h})\) to the 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g})\) with respect to the action \( \rho \).

**Proof.** It is straightforward to deduce that
\[
\begin{align*}
  \mathcal{P}[\pi + \rho + \lambda \mu, T]_R(u, v, w) & = -T \lambda \mu(u, v, w), \\
  \mathcal{P}[[[\pi + \rho + \lambda \mu, T]_R, T]_R, T]_R(u, v, w) & = 6(\pi(Tu, Tv, Tw) - T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v)).
\end{align*}
\]
Let \( T \) be a Maurer-Cartan element of the \( L_\infty \)-algebra \((C^*(\mathfrak{h}, \mathfrak{g}), l_1, l_3)\). We have
\[
\sum_{n=1}^{+\infty} \frac{1}{n!} l_n(T, \ldots, T)(u, v, w) = \mathcal{P}[\pi + \rho + \lambda \mu, T]_R(u, v, w) + \frac{1}{3!} \mathcal{P}[[[\pi + \rho + \lambda \mu, T]_R, T]_R, T]_R(u, v, w)
\]
\[
= \pi(Tu, Tv, Tw) - T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v + \lambda \mu(u, v, w))
\]
\[
= 0,
\]
which implies that \( T \) is a relative Rota-Baxter operator of weight \( \lambda \) from the 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_\mathfrak{h})\) to the 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g})\) with respect to the action \( \rho \). \( \square \)

**Proposition 4.9.** Let \( T \) be a relative Rota-Baxter operator of weight \( \lambda \) from a 3-Lie algebra \( \mathfrak{h} \) to a 3-Lie algebra \( \mathfrak{g} \) with respect to an action \( \rho \). Then \( C^*(\mathfrak{h}, \mathfrak{g}) \) carries a twisted \( L_\infty \)-algebra structure as following:
\[
\text{(26)} \quad I^T_1(P) = l_1(P) + \frac{1}{2} l_3(T, T, P),
\]
\[
\text{(27)} \quad I^T_3(P, Q) = l_3(T, P, Q),
\]
\[
\text{(28)} \quad I^T_3(P, Q, R) = l_3(P, Q, R),
\]
\[
\text{(29)} \quad I^T_k = 0, \quad k \geq 4,
\]
where \( P \in C^m(\mathfrak{h}, \mathfrak{g}), Q \in C^n(\mathfrak{h}, \mathfrak{g}) \) and \( R \in C^k(\mathfrak{h}, \mathfrak{g}) \).
Proof. Since $T$ is a Maurer-Cartan element of the $L_\infty$-algebra $(C^*(b, g), l_1, l_3)$, by Theorem 4.3, we have the conclusions. □

The above $L_\infty$-algebra controls deformations of relative Rota-Baxter operators of weight $\lambda$ on 3-Lie algebras.

**Theorem 4.10.** Let $T : b \to g$ be a relative Rota-Baxter operator of weight $\lambda$ from a 3-Lie algebra $(b, [\cdot, \cdot, \cdot]_b)$ to a 3-Lie algebra $(g, [\cdot, \cdot, \cdot]_g)$ with respect to an action $\rho$. Then for a linear map $T' : b \to g$, $T + T'$ is a relative Rota-Baxter operator if and only if $T'$ is a Maurer-Cartan element of the twisted $L_\infty$-algebra $(C^*(b, g), l_1', l_2', l_3')$, that is $T'$ satisfies the Maurer-Cartan equation:

$$l_1'(T') + \frac{1}{2}l_2'(T', T') + \frac{1}{3!}l_3'(T', T', T') = 0.$$

**Proof.** By Theorem 4.8, $T + T'$ is a relative Rota-Baxter operator if and only if

$$l_1(T + T') + \frac{1}{3!}(T + T') = 0.$$

Applying $l_1(T) + \frac{1}{3!}l_3(T, T, T) = 0$, the above condition is equivalent to

$$l_1(T') + \frac{1}{2}l_2(T, T', T') + \frac{1}{2}l_3(T, T', T') + \frac{1}{6}l_3(T', T', T') = 0.$$

That is, $l_1'(T') + \frac{1}{2}l_2'(T', T') + \frac{1}{3!}l_3'(T', T', T') = 0$, which implies that $T'$ is a Maurer-Cartan element of the twisted $L_\infty$-algebra $(C^*(b, g), l_1', l_2', l_3')$. □

5. Cohomologies of Relative Rota-Baxter Operators and Infinitesimal Deformations

Let $(V; \rho)$ be a representation of a 3-Lie algebra $(g, [\cdot, \cdot, \cdot]_g)$. Denote by

$$\mathcal{C}_{3\text{Lie}}^n(g; V) := \text{Hom}((\wedge^2 g \otimes \cdots \otimes \wedge^2 g) \otimes g, V), \quad (n \geq 1),$$

which is the space of $n$-cochains. The coboundary operator $d : \mathcal{C}_{3\text{Lie}}^n(g; V) \to \mathcal{C}_{3\text{Lie}}^{n+1}(g; V)$ is defined by

$$(df)(\bar{x}_1, \cdots, \bar{x}_n, x_{n+1}) = \sum_{1 \leq j \leq k \leq n} (-1)^j f(\bar{x}_1, \cdots, \bar{x}_j, \cdots, \bar{x}_{k-1}, [x_j, y_j, x_k]_g \wedge y_k$$

$$+ x_k \wedge [x_j, y_j, y_k]_g, \bar{x}_{k+1}, \cdots, \bar{x}_n, x_{n+1})$$

$$+ \sum_{j=1}^n (-1)^j f(\bar{x}_1, \cdots, \bar{x}_j, x_n, [x_j, y_j, x_n]_g$$

$$+ \sum_{j=1}^n (-1)^{j+1} \rho(x_j, y_j) f(\bar{x}_1, \cdots, \bar{x}_j, \cdots, \bar{x}_n, x_{n+1})$$

$$+ (-1)^{n+1} \left( \rho(y_n, x_{n+1}) f(\bar{x}_1, \cdots, \bar{x}_{n-1}, x_n) + \rho(x_{n+1}, y_n) f(\bar{x}_1, \cdots, \bar{x}_{n-1}, y_n) \right),$$

for all $\bar{x}_i = x_i \wedge y_i \in \wedge^2 g$, $i = 1, 2, \cdots, n$ and $x_{n+1} \in g$. It was proved in [10, 52] that $d \circ d = 0$. Thus, $(\oplus_{n=1}^{\infty} \mathcal{C}_{3\text{Lie}}^n(g; V), d)$ is a cochain complex.
Definition 5.1. The cohomology of the 3-Lie algebra \( \mathfrak{g} \) with coefficients in \( V \) is the cohomology of the cochain complex \( (\mathfrak{g}^\infty, \mathbb{Z}^\infty_3, \mathbb{B}^\infty_3, g; V) \). Denote by \( \mathbb{Z}^n_3(\mathfrak{g}; V) \) and \( \mathbb{B}^n_3(\mathfrak{g}; V) \) the set of \( n \)-cocycles and the set of \( n \)-coboundaries, respectively. The \( n \)-th cohomology group is defined by

\[
\mathcal{H}^n_3(\mathfrak{g}; V) = \mathbb{Z}^n_3(\mathfrak{g}; V) / \mathbb{B}^n_3(\mathfrak{g}; V).
\]

5.1. Cohomologies of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras. In this subsection, we construct a representation of the 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\) on the vector space \( \mathfrak{g} \) from a relative Rota-Baxter operator \( T : \mathfrak{h} \to \mathfrak{g} \) of weight \( \lambda \) and define the cohomologies of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras.

Lemma 5.2. Let \( T : \mathfrak{h} \to \mathfrak{g} \) be a relative Rota-Baxter operator of weight \( \lambda \) from a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\) to a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot]_\rho)\) with respect to an action \( \rho \). Define \( \varphi : \Lambda^2 \mathfrak{h} \to \mathfrak{g} \) \( L(\mathfrak{g}) \) by

\[
\varphi(u, v)(x) = [Tu, Tv, x]_\rho - T(\rho(x, Tu)v + \rho(Tv, xu)),
\]

for all \( x \in \mathfrak{g}, u, v \in \mathfrak{h} \). Then \((\mathfrak{g}, \varphi)\) is a representation of the descendent 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\).

Proof. By a direct calculation using \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\) and \((\mathfrak{g}, [\cdot, \cdot, \cdot]_\rho)\), for all \( u_i \in \mathfrak{h}, 1 \leq i \leq 4, x \in \mathfrak{g} \), we have

\[
\begin{align*}
\varphi(u_1, u_2)\varphi(u_3, u_4) - \varphi(u_3, u_4)\varphi(u_1, u_2) &- \varphi([u_1, u_2, u_3]_T, u_4) + \varphi([u_1, u_2, u_4]_T, u_3)) \in \mathfrak{g}
\end{align*}
\]

and

\[
\begin{align*}
\varphi([u_1, u_2, u_3]_T, u_4) - \varphi(u_1, u_2)\varphi(u_3, u_4) - \varphi(u_2, u_3)\varphi(u_1, u_4) - \varphi(u_3, u_1)\varphi(u_2, u_4) \in \mathfrak{g}
\end{align*}
\]
\[
= \langle [T u_1, T u_2, T u_3]_3, T u_4, x \rangle - T\left(\rho(x, [T u_1, T u_2, T u_3])u_4 + \rho(T u_4, x)\right)u_4 + \rho(T u_4, x)u_4 - T p(T u_4, x)u_4
\]
\[
- \psi(u_1, u_2)([T u_3, T u_4, x]_3 - T p(x, T u_3)u_4 - T p(T u_4, x)u_3)
\]
\[
- \psi(u_2, u_3)([T u_1, T u_4, x]_3 - T p(x, T u_1)u_4 - T p(T u_4, x)u_1)
\]
\[
- \psi(u_3, u_1)([T u_2, T u_4, x]_3 - T p(x, T u_2)u_4 - T p(T u_4, x)u_2)
\]
\[
= -T\left(\rho(x, [T u_1, T u_2, T u_3])u_4 + \lambda\rho(T u_4, x)\right)u_4 + T u_3, T u_2, T p(x, T u_3)u_4 + T p(T u_4, x)u_3
\]
\[
+ T\left(\rho([T u_3, T u_4, x]_3, T u_1)u_2 + \rho(T u_2, [T u_3, T u_4, x]_3)u_1\right)
\]
\[
- T\left(\rho(T p(T u_4, x)u_3, T u_1)u_2 + \rho(T u_2, T p(T u_4, x)u_3)u_1\right)
\]
\[
+ T\left(\rho([T u_1, T u_3, T u_4, x]_3, T u_1, T u_2)u_3 + \rho(T u_3, [T u_1, T u_4, x]_3)u_2\right)
\]
\[
- T\left(\rho(T p(T u_4, x)u_3, T u_1, T u_2)u_3 + \rho(T u_3, T p(T u_4, x)u_3)u_1\right)
\]
\[
+ T\left(\rho([T u_2, T u_4, x]_3, T u_3)u_1 + \rho(T u_1, [T u_2, T u_4, x]_3)u_3\right)
\]
\[
- T\left(\rho(T p(T u_4, x)u_2, T u_3)u_1 + \rho(T u_1, T p(T u_4, x)u_2)u_3\right)
\]
\[
= 0.
\]

Thus, we deduce that \((\mathfrak{g}; \varrho)\) is a representation of the descendent 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\). \(\square\)

**Proposition 5.3.** Let \(T\) and \(T'\) be relative Rota-Baxter operators of weight \(\lambda\) from a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_\mathfrak{h})\) to a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g})\) with respect to an action \(\rho\). Let \((\psi_\mathfrak{h}, \psi_\mathfrak{g})\) be a homomorphism from \(T\) to \(T'\).

(i) \(\psi_\mathfrak{g}\) is also a 3-Lie algebra homomorphism from the descendent 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\) of \(T\) to the descendent 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T')\) of \(T'\);

(ii) The induced representation \((\mathfrak{g}; \varrho)\) of the 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\) and the induced representation \((\mathfrak{g}; \varrho')\) of the 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot]_T)\) satisfy the following relation:

\[
\psi_\mathfrak{g} \circ \varrho(u, v) = \varrho'(\psi_\mathfrak{h}(u), \psi_\mathfrak{h}(v)) \circ \psi_\mathfrak{g}, \quad \forall u, v \in \mathfrak{h}.
\]

That is, the following diagram commutes:

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\psi_\mathfrak{g}} & \mathfrak{g} \\
\varrho(u, v) \downarrow & & \downarrow \varrho'(\psi_\mathfrak{h}(u), \psi_\mathfrak{h}(v)) \\
\mathfrak{g} & \xrightarrow{\psi_\mathfrak{g}} & \mathfrak{g}
\end{array}
\]
Proof. (i) It follows from Corollary 5.5 and Proposition 5.7 directly.

(ii) By (30), (1)-(3) and the fact that \( \psi_\varrho \) is a 3-Lie algebra homomorphism, for all \( x \in \mathfrak{g}, u, v \in \mathfrak{h} \), we have

\[
\psi_\varrho(g(u, v)x) = \psi_\varrho([Tu, Tv, x]_\mathfrak{g}) - T(\rho(x, Tu)v) - T(\rho(Tv, x)u)
\]

\[
= [\psi_\varrho(Tu), \psi_\varrho(Tv), \psi_\varrho(x)]_\mathfrak{g} - T'\psi_\varrho(\rho(x, Tu)v) - T'\psi_\varrho(\rho(Tv, x)u)
\]

\[
= [T'\varrho_\vartheta(u), T'\varrho_\vartheta(v), \varrho_\vartheta(x)]_\mathfrak{g} - T'(\varrho(\varrho_\vartheta(x), T'\varrho_\vartheta(u))\varrho_\vartheta(v)) - T'(\varrho(\varrho_\vartheta(x), T'\varrho_\vartheta(u))\varrho_\vartheta(v))
\]

\[
= \varrho'(\varrho_\vartheta(u), \varrho_\vartheta(v))\varrho_\vartheta(x).
\]

We finish the proof. \( \square \)

Let \( d_T : \mathcal{C}_{3\text{Lie}}^n(\mathfrak{h}; \mathfrak{g}) \to \mathcal{C}_{3\text{Lie}}^{n+1}(\mathfrak{h}; \mathfrak{g}), (n \geq 1) \) be the corresponding coboundary operator of the descendant 3-Lie algebra \( (\mathfrak{h}, [\cdot, \cdot, \cdot]_T) \) with coefficients in the representation \( (\mathfrak{g}; \varrho) \). More precisely, for all \( f \in \text{Hom}(\wedge^3 \mathfrak{h} \otimes \cdots \otimes \wedge^3 \mathfrak{h}; \mathfrak{g}) \), \( U_i = u_i \wedge v_i \in \wedge^3 \mathfrak{h}, i = 1, 2, \cdots, n \) and \( u_{n+1} \in \mathfrak{h} \), we have

\[
(d_T f)(U_1, \cdots, U_n, u_{n+1}) = \sum_{1 \leq j < k \leq n} (-1)^j f(U_1, \cdots, \hat{U}_j, \cdots, U_k, [u_j, v_j, u_k]_T) \wedge v_k
\]

\[
+ u_k \wedge [u_j, v_j, v_k]_T U_{k+1}, \cdots, U_n, u_{n+1})
\]

\[
+ \sum_{j=1}^n (-1)^j f(U_1, \cdots, \hat{U}_j, \cdots, U_n, [u_j, v_j, u_{n+1}]_T)
\]

\[
+ \sum_{j=1}^n (-1)^{j+1} \varrho(g(u_j, v_j)) f(U_1, \cdots, \hat{U}_j, \cdots, U_n, u_{n+1})
\]

\[
+ (-1)^{n+1}(\varrho(g(u_j, v_j)) f(U_1, \cdots, \hat{U}_j, \cdots, U_n, u_{n+1})) + \varrho(g(u_{n+1}, u_n)) f(U_1, \cdots, U_{n-1}, u_n) + \varrho(g(u_{n+1}, u_n)) f(U_1, \cdots, U_{n-1}, v_n).
\]

It is obvious that \( f \in \mathcal{C}_{3\text{Lie}}^1(\mathfrak{h}; \mathfrak{g}) \) is closed if and only if

\[
[f(u_1), Tu_2, Tu_3]_\mathfrak{g} + [Tu_1, f(u_2), Tu_3]_\mathfrak{g} + [Tu_1, Tu_2, f(u_3)]_\mathfrak{g}
\]

\[
= T(\rho(Tu_2, f(u_3))u_1 + \rho(f(u_3), Tu_1)u_2 + \rho(f(u_1), Tu_2)u_3)
\]

\[
+ T(\rho(f(u_2), Tu_3)u_1 + \rho(Tu_3, f(u_1))u_2 + \rho(Tu_1, f(u_2))u_3)
\]

\[
+f(\rho(Tu_2, Tu_3)u_1 + \rho(Tu_3, Tu_1)u_2 + \rho(Tu_1, Tu_2)u_3 + \lambda[u_1, u_2, u_3]_\mathfrak{g}).
\]

Define \( \delta : \wedge^2 \mathfrak{g} \to \text{Hom}(\mathfrak{h}, \mathfrak{g}) \) by

\[
\delta(\mathfrak{x})u = T\rho(\mathfrak{x}u - [\mathfrak{x}, Tu])_\mathfrak{g}, \quad \forall \mathfrak{x} \in \wedge^2 \mathfrak{g}, u \in \mathfrak{h}.
\]

**Proposition 5.4.** Let \( T : \mathfrak{h} \to \mathfrak{g} \) be a relative Rota-Baxter operator of weight \( \lambda \) from \( \mathfrak{h} \) to \( \mathfrak{g} \) with respect to an action \( \rho \). Then \( \delta(\mathfrak{x}) \) is a 1-cocycle of the 3-Lie algebra \( (\mathfrak{h}, [\cdot, \cdot, \cdot]_T) \) with coefficients in \( (\mathfrak{g}; \varrho) \).

**Proof.** It follows from straightforward computations, and we omit details. \( \square \)

We now introduce the cohomology theory of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras.
Let \( T \) be a relative Rota-Baxter operator of weight \( \lambda \) from a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot])\) to a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot])\) with respect to an action \( \rho \). Define the space of \( n \)-cochains by

\[
\mathcal{C}_T^n(\mathfrak{h}; \mathfrak{g}) = \begin{cases} \mathcal{C}_{\text{3Lie}}^{n-1}(\mathfrak{h}; \mathfrak{g}), & n \geq 2, \\ \mathfrak{g} \land \mathfrak{g}, & n = 1. \end{cases}
\]

Define \( \partial : \mathcal{C}_T^n(\mathfrak{h}; \mathfrak{g}) \to \mathcal{C}_T^{n+1}(\mathfrak{h}; \mathfrak{g}) \) by

\[
\partial = \begin{cases} d_T, & n \geq 2, \\ \delta, & n = 1. \end{cases}
\]

**Theorem 5.5.** \((\bigoplus_{n=1}^\infty \mathcal{C}_T^n(\mathfrak{h}; \mathfrak{g}), \partial)\) is a cochain complex.

**Proof.** It follows from Proposition 5.4 and the fact that \( d_T \) is the corresponding coboundary operator of the descendant 3-Lie algebra \( (\mathfrak{h}, [\cdot, \cdot, \cdot], \mathfrak{T}) \) with coefficients in the representation \( \mathfrak{g} \) directly. \( \square \)

**Definition 5.6.** The cohomology of the cochain complex \((\bigoplus_{n=1}^\infty \mathcal{C}_T^n(\mathfrak{h}; \mathfrak{g}), \partial)\) is taken to be the cohomology for the relative Rota-Baxter operator \( T \) of weight \( \lambda \). Denote the set of \( n \)-cocycles by \( \mathcal{Z}_T^n(\mathfrak{h}; \mathfrak{g}) \), the set of \( n \)-coboundaries by \( \mathcal{B}_T^n(\mathfrak{h}; \mathfrak{g}) \) and \( n \)-th cohomology group by

\[
\mathcal{H}_T^n(\mathfrak{h}; \mathfrak{g}) = \mathcal{Z}_T^n(\mathfrak{h}; \mathfrak{g})/\mathcal{B}_T^n(\mathfrak{h}; \mathfrak{g}), \quad n \geq 1.
\]

**Remark 5.7.** The cohomology theory for relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras enjoys certain functorial properties. Let \( T \) and \( T' \) be relative Rota-Baxter operators of weight \( \lambda \) from a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot], \mathfrak{b})\) to a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot], \mathfrak{b})\) with respect to an action \( \rho \). Let \((\psi_\mathfrak{h}, \psi_\mathfrak{g})\) be a homomorphism from \( T \) to \( T' \) in which \( \psi_\mathfrak{h} \) is invertible. Define a map \( p : \mathcal{C}_T^n(\mathfrak{h}; \mathfrak{g}) \to \mathcal{C}_{T'}^n(\mathfrak{h}; \mathfrak{g}) \) by

\[
p(\omega)(U_1, \cdots, U_{n-2}, u_{n-1}) = \psi_\mathfrak{g}\left(\omega(\psi_\mathfrak{h}^{-1}(u_1) \land \psi_\mathfrak{h}^{-1}(v_1), \cdots, \psi_\mathfrak{h}^{-1}(u_{n-2}) \land \psi_\mathfrak{h}^{-1}(v_{n-2}), \psi_\mathfrak{h}^{-1}(u_{n-1}))\right),
\]

for all \( \omega \in \mathcal{C}_T^n(\mathfrak{h}; \mathfrak{g}) \), \( U_i = u_i \land v_i \in \land^2 \mathfrak{h}, \ i = 1, 2, \cdots, n-2 \) and \( u_{n-1} \in \mathfrak{h} \). Then it is straightforward to deduce that \( p \) is a cochain map from the cochain complex \((\bigoplus_{n=2}^\infty \mathcal{C}_T^n(\mathfrak{h}; \mathfrak{g}), d_T)\) to the cochain complex \((\bigoplus_{n=2}^\infty \mathcal{C}_{T'}^n(\mathfrak{h}; \mathfrak{g}), d_{T'})\). Consequently, it induces a homomorphism \( p_* \) from the cohomology group \( \mathcal{H}_T^n(\mathfrak{h}; \mathfrak{g}) \) to \( \mathcal{H}_{T'}^n(\mathfrak{h}; \mathfrak{g}) \).

At the end of this subsection, we give the relationship between the coboundary operator \( d_T \) and the differential \( l_T^1 \) defined by (2.3) using the Maurer-Cartan element \( T \) of the \( L_\infty \)-algebra \((C^*(\mathfrak{h}, \mathfrak{g}), l_1, l_3)\).

**Lemma 5.8.** Let \( T : \mathfrak{h} \to \mathfrak{g} \) be a relative Rota-Baxter operator of weight \( \lambda \) from a 3-Lie algebra \((\mathfrak{h}, [\cdot, \cdot, \cdot], \mathfrak{b})\) to a 3-Lie algebra \((\mathfrak{g}, [\cdot, \cdot, \cdot], \mathfrak{b})\) with respect to an action \( \rho \). For all \( x, y, z \in \mathfrak{g}, u, v, w \in V \), we have

\[
[l_T^1(x, y, z), T](x + u, y + v, z + w) = \begin{cases} 2(Tu, Tv, z_\mathfrak{b}) + \rho(Tu, Tv)w + [Tv, Tw, x_\mathfrak{b}] + \rho(Tw, T)u + [Tw, Tu, y_\mathfrak{b}] + \rho(Tw, Tu)v \\ -2T(\rho(Tu, v)w + \rho(z, Tu)v + \rho(x, Tv)w + \rho(Tv, z)u + \rho(y, Tw)u + \rho(Tw, x)v) \end{cases}
\]

**Proof.** It follows from straightforward computations. \( \square \)
Theorem 5.9. Let $T$ be a relative Rota-Baxter operator of weight $\lambda$ from a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_\mathfrak{g})$ to a 3-Lie algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_\mathfrak{h})$ with respect to an action $\rho$. Then we have
\[
d_T f = (-1)^{n-1} l_1^T f, \quad \forall f \in \text{Hom}(\bigwedge^2 \mathfrak{h} \otimes \cdots \otimes \bigwedge^2 \mathfrak{h} \wedge \mathfrak{g}), \quad n = 1, 2, \cdots.
\]

Proof. For all $\mathfrak{U}_i = u_i \wedge v_i \in \bigwedge^2 \mathfrak{h}$, $i = 1, 2, \cdots, n$ and $u_{n+1} \in \mathfrak{h}$, we have
\[
l_1(f)(\mathfrak{U}_1, \cdots, \mathfrak{U}_n, u_{n+1})
= [\pi + \rho + \lambda \mu, f]_\mathfrak{R}(\mathfrak{U}_1, \cdots, \mathfrak{U}_n, u_{n+1})
= ((\pi + \rho + \lambda \mu) \circ f - (-1)^{n-1} f \circ (\pi + \rho + \lambda \mu))(\mathfrak{U}_1, \cdots, \mathfrak{U}_n, u_{n+1})
= (\pi + \rho + \lambda \mu)(f(\mathfrak{U}_1, \cdots, \mathfrak{U}_{n-1}, u_n) \wedge v_n, u_{n+1})
= + (\pi + \rho + \lambda \mu)(u_n \wedge f(\mathfrak{U}_1, \cdots, \mathfrak{U}_{n-1}, u_n), u_{n+1})
+ \sum_{i=1}^n (-1)^{n-1} (-1)^{i-1} (\pi + \rho + \lambda \mu)(\mathfrak{U}_i, f(\mathfrak{U}_1, \cdots, \mathfrak{U}_i, \cdots, \mathfrak{U}_n, u_{n+1})).
\]

By Lemma 5.8, we have
\[
\frac{1}{2} l_3(T, T, f)(\mathfrak{U}_1, \cdots, \mathfrak{U}_n, u_{n+1})
= [[(\pi + \rho + \lambda \mu, T]_\mathfrak{R}, T]_\mathfrak{R}, f]_\mathfrak{R}(\mathfrak{U}_1, \cdots, \mathfrak{U}_n, u_{n+1})
= [[(\pi + \rho + \lambda \mu, T]_\mathfrak{R}, T]_\mathfrak{R}(f(\mathfrak{U}_1, \cdots, \mathfrak{U}_{n-1}, u_n) \wedge v_n, u_{n+1})
= + [[(\pi + \rho + \lambda \mu, T]_\mathfrak{R}, T]_\mathfrak{R}(u_n \wedge f(\mathfrak{U}_1, \cdots, \mathfrak{U}_{n-1}, v_n), u_{n+1})
+ \sum_{i=1}^n (-1)^{n-1} (-1)^{i-1} [[(\pi + \rho + \lambda \mu, T]_\mathfrak{R}, T]_\mathfrak{R}(\mathfrak{U}_i, f(\mathfrak{U}_1, \cdots, \mathfrak{U}_i, \cdots, \mathfrak{U}_n, u_{n+1})).
\]
Thus, we deduce that \( d_T f = (-1)^{n+1} (t_1(f) + \frac{1}{2} t_3(T, T, f)) \), that is \( d_T f = (-1)^{n+1} t_1 f \). \( \square \)

5.2. Infinitesimal deformations of relative Rota-Baxter operators. In this subsection, we use the established cohomology theory to characterize infinitesimal deformations of relative Rota-Baxter operators of weight \( \lambda \) on 3-Lie algebras.

Let \( (\mathfrak{g}, [\cdot, \cdot, \cdot]_h) \) be a 3-Lie algebra over \( \mathbb{K} \) and \( \mathbb{K}[t] \) be the polynomial ring in one variable \( t \). Then \( \mathbb{K}[t]/(t^2) \otimes_{\mathbb{K}} \mathfrak{g} \) is an \( \mathbb{K}[t]/(t^2) \)-module. Moreover, \( \mathbb{K}[t]/(t^2) \otimes_{\mathbb{K}} \mathfrak{g} \) is a 3-Lie algebra over \( \mathbb{K}[t]/(t^2) \), where the 3-Lie algebra structure is defined by

\[
[f_i(t) \otimes_{\mathbb{K}} x_1, f_2(t) \otimes_{\mathbb{K}} x_2, f_3(t) \otimes_{\mathbb{K}} x_3] = f_1(t) f_2(t) f_3(t) \otimes_{\mathbb{K}} [x_1, x_2, x_3],
\]

for \( f_i(t) \in \mathbb{K}[t]/(t^2) \), \( 1 \leq i \leq 3 \), \( x_1, x_2, x_3 \in \mathfrak{g} \).

In the sequel, all the vector spaces are finite dimensional vector spaces over \( \mathbb{K} \) and we denote \( f(t) \otimes_{\mathbb{K}} x \) by \( f(t)x \), where \( f(t) \in \mathbb{K}[t]/(t^2) \).

**Definition 5.10.** Let \( T : \mathfrak{h} \to \mathfrak{g} \) be a relative Rota-Baxter operator of weight \( \lambda \) from a 3-Lie algebra \( (\mathfrak{h}, [\cdot, \cdot, \cdot]_h) \) to a 3-Lie algebra \( (\mathfrak{g}, [\cdot, \cdot, \cdot]_h) \) with respect to an action \( \rho \). Let \( \mathcal{I} : \mathfrak{h} \to \mathfrak{g} \) be a linear map. If \( T_i = T + i \mathcal{I} \) is a relative Rota-Baxter operator of weight \( \lambda \) modulo \( t^2 \), we say that \( \mathcal{I} \) generates an infinitesimal deformation of \( T \).

Since \( T_i = T + i \mathcal{I} \) is a relative Rota-Baxter operator of weight \( \lambda \) modulo \( t^2 \), by consider the coefficients of \( t \), for any \( u, v, w \in \mathfrak{h} \), we have

\[
[T \mathcal{I} u, T v, T w]_\mathfrak{g} + [T u, T \mathcal{I} v, T w]_\mathfrak{g} + [T u, T v, T \mathcal{I} w]_\mathfrak{g} = T (\rho(T \mathcal{I} u, T v) + \rho(T v, T u) + \rho(T u, T \mathcal{I} v)) + \rho(T \mathcal{I} v, T u) + \rho(T u, T \mathcal{I} v) + \rho(T u, T v) + \rho(T v, T u) + \rho(T \mathcal{I} u, T w) + \rho(T w, T \mathcal{I} u) + \rho(T u, T \mathcal{I} w) + \rho(T \mathcal{I} w, T u) + \rho(T w, T \mathcal{I} u) + \rho(T u, T \mathcal{I} w) + \rho(T \mathcal{I} w, T u) + \rho(T w, T \mathcal{I} u) + \rho(T u, T \mathcal{I} w) + \rho(T \mathcal{I} w, T u).
\]
Note that (35) means that $\mathcal{I}$ is a 2-cocycle of the relative Rota-Baxter operator $T$. Hence, $\mathcal{I}$ defines a cohomology class in $\mathcal{H}_T^2(\mathfrak{h}; \mathfrak{g})$.

**Definition 5.11.** Let $T$ be a relative Rota-Baxter operator of weight $\lambda$ from a 3-Lie algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_h)$ to a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_g)$ with respect to an action $\rho$. Two one-parameter infinitesimal deformations $T_1^t = T + t\mathcal{I}_1$ and $T_2^t = T + t\mathcal{I}_2$ are said to be equivalent if there exists $X \in \mathfrak{g} \land \mathfrak{g}$ such that $(\text{Id}_h + t\text{ad}_X, \text{Id}_h + t\rho(X))$ is a homomorphism from $T_1^t$ to $T_2^t$ modulo $t^2$. In particular, an infinitesimal deformation $T_1 = T + t\mathcal{I}_1$ of a relative Rota-Baxter operator $T$ is said to be trivial if there exists $X \in \mathfrak{g} \land \mathfrak{g}$ such that $(\text{Id}_h + t\text{ad}_X, \text{Id}_h + t\rho(X))$ is a homomorphism from $T_1$ to $T$ modulo $t^2$.

Let $(\text{Id}_h + t\text{ad}_X, \text{Id}_h + t\rho(X))$ be a homomorphism from $T_1^t$ to $T_2^t$ modulo $t^2$. By (35) we get,

$$\begin{align*}
(\text{Id}_h + t\text{ad}_X)(T + t\mathcal{I}_1)(u) = (T + t\mathcal{I}_2)(\text{Id}_h + t\rho(X))(u),
\end{align*}$$

which implies

$$\begin{align*}
\mathcal{I}_1(u) - \mathcal{I}_2(u) = T\rho(X)u - [X, Tu].
\end{align*}$$

Now we are ready to give the main result in this section.

**Theorem 5.12.** Let $T$ be a relative Rota-Baxter operator of weight $\lambda$ from a 3-Lie algebra $\mathfrak{h}$ to a 3-Lie algebra $\mathfrak{g}$ with respect to an action $\rho$. If two one-parameter infinitesimal deformations $T_1^t = T + t\mathcal{I}_1$ and $T_2^t = T + t\mathcal{I}_2$ are equivalent, then $\mathcal{I}_1$ and $\mathcal{I}_2$ are in the same cohomology class in $\mathcal{H}_T^2(\mathfrak{h}; \mathfrak{g})$.

**Proof.** It is easy to see from (35) that

$$\mathcal{I}_1(u) = \mathcal{I}_2(u) + (\partial X)(u), \quad \forall u \in \mathfrak{h},$$

which implies that $\mathcal{I}_1$ and $\mathcal{I}_2$ are in the same cohomology class. $\square$

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