A ROBUST LOWER ORDER MIXED FINITE ELEMENT METHOD FOR A STRAIN GRADIENT ELASTICITY MODEL

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Abstract. A robust nonconforming mixed finite element method is developed for a strain gradient elasticity (SGE) model. In two and three dimensional cases, a lower order $\mathbb{C}^0$-continuous $H^2$-nonconforming finite element is constructed for the displacement field through enriching the quadratic Lagrange element with bubble functions. This together with the linear Lagrange element is exploited to discretize a mixed formulation of the SGE model. The robust discrete inf-sup condition is established. The sharp and uniform error estimates with respect to both the small size parameter and the Lamé coefficient are achieved, which is also verified by numerical results. In addition, the uniform regularity of the SGE model is derived under two reasonable assumptions.

1. Introduction

Size effect of microstructures at the nanoscale has been observed in many experiments. In order to account for such phenomenon, higher-order continuum theory has begun to emerge even since the early twentieth century. Unlike the classic continuum theory, the corresponding constitutive relations contain additional parameters to characterize the size of materials, giving rise to various elastic mathematical models (cf. [17, 54, 39, 28, 38, 2, 5, 46]). Historically, Cosserat brothers propose in their celebrated work [17] a continuum model concerning the effect of couple-stresses. Later on, Mindlin [38] develops a more general linear elasticity theory, but the associated model contains quite a number of size parameters. To make the balance between effective prediction and computation cost, Aifantis and his collaborators introduce in [5, 46] only one size parameter to produce a strain gradient elasticity (SGE) model for material deformation at the micro/nano scale. This SGE model is well accepted in engineering areas to deal with elastic-plastic problems. We refer the reader to the survey papers [7, 44] for details along this line. It is worth noting that for a microarchitecture or microstructure, if its separation of scales is not fully valid, such as pantographic [45, 4], lattice [26, 55] and cellular [27] metamaterials, one requires to invoke higher-order theory with many size parameters for mechanical computation. Moreover, homogenization methods are actively used to determine generalized material parameters (cf. [56, 22]).

2020 Mathematics Subject Classification. 65N12; 65N15; 65N22; 65N30;
Key words and phrases. Strain gradient elasticity model, nonconforming mixed finite element method, regularity, robustness.

The work of J. Huang was partially supported by NSFC (Grant No. 12071289) and the Fundamental Research Funds for the Central Universities. The work of X. Huang was partially supported by NSFC (Grant Nos. 12171300, 12071289), and the Natural Science Foundation of Shanghai (Grant No. 21ZR1480500).
Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) be a bounded domain with Lipschitz boundary. The SGE model given in \([5, 46]\) can be described as follows.

\[
\begin{align*}
- \text{div} \left( (I - \iota^2 \Delta) \sigma(u) \right) &= f & \text{in } \Omega, \\
u &= \partial_n u = 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( \n \) is the unit outward normal to \( \partial \Omega \), \( u = (u_1, \ldots, u_d)^T \) is the displacement field, \( f \) is the applied force, \( \sigma(u) = 2\mu \varepsilon(u) + \lambda (\text{div} u)I \), and \( \varepsilon(u) = (\varepsilon_{ij}(u))_{d \times d} \) is the strain tensor field with \( \varepsilon_{ij}(u) = 1/2(\partial_i u_j + \partial_j u_i) \). Here \( \lambda \) and \( \mu \) are the Lamé constants, and \( I \) is the identity tensor field. The resulting stress field is given by \( \tilde{\sigma}(u) = (I - \iota^2 \Delta) \sigma(u) \), where \( \iota \in (0, 1) \) denotes the size parameter of the material under discussion. The SGE model (1.1) is a fourth order elliptic singular perturbation problem with small parameter \( \iota \), which will reduce to a second-order linear elasticity model (3.15) when \( \iota = 0 \). That means, the solution of the reduced problem would not meet one of the boundary conditions of problem (1.1), namely the value of the normal derivative on the boundary. This would lead to the phenomenon of boundary layer. Moreover, as \( \lambda \) goes to infinity, one can see from (3.21) that \( \| \text{div} u \|_2 \to 0 \). In other words, with \( \lambda \) becoming very large, the elastic material becomes nearly incompressible, and this would deteriorate the approximation accuracy of the lower-order conforming finite element methods and lead to the locking phenomenon. Hence, it is a very challenging issue to develop a robust numerical method with respect to both size parameter \( \iota \) and Lamé constant \( \lambda \).

Finite element method (FEM) is a well-known numerical method for solving elastic problems. For example, many \( C^1 \)-conforming FEMs are used for solving strain gradient elasticity problems in two or three dimensions (cf. \([3, 57, 58, 36, 42, 50, 52]\)). For avoiding higher order shape functions so as to alleviate the computational cost, many non-conforming FEMs are developed accordingly (cf. \([49, 53, 59, 30, 31, 33, 32]\)). Another way is to use the mixed method, and in this case the displacement and stress fields can be approximated simultaneously (cf. \([6, 11, 48, 43]\)).

More recently, the isogeometric analysis method has also been used for solving higher-order strain gradient models (cf. \([8, 22, 25, 35, 40]\)). As far as we know, the method is first proposed by Hughes and his collaborators for solving various engineering problems (see \([24, 19]\)), and the systematic error analysis is developed by Beirão da Veiga et al. in \([9]\). We mention that the optimal error estimates are obtained in some of the previous papers \([8, 22, 40]\), but the parameter dependence is not considered theoretically there.

Now, let us review some existing results for error analysis of the SGE model (1.1). In \([30, 31, 33, 32]\), Ming and his collaborators analyze several robust non-conforming FEMs with respect to the size parameter \( \iota \) under some specific assumptions, but not robust with respect to the Lamé constant \( \lambda \). To fix this problem, as the Lamé system for the linear elasticity, they introduce a “pressure field” \( p = \lambda \text{div} u \) to reformulate (1.1) in the form \([34]\)

\[
\begin{align*}
- \text{div} \left( (I - \iota^2 \Delta) (2\mu \varepsilon(u)) \right) - \nabla ((I - \iota^2 \Delta)p) &= f & \text{in } \Omega, \\
\text{div} \ u - \frac{1}{\lambda} p &= 0 & \text{in } \Omega, \\
u &= \partial_n u = 0 & \text{on } \partial \Omega.
\end{align*}
\]
And they propose lower order nonconforming finite element methods based on (1.2), and derived the robust error estimate of $\bm{u} - \bm{u}_h$ and $p_0 - p_h$ in energy norm, rather than $\bm{u} - \bm{u}_h$ and $p - p_h$, with respect to both Lamé constant $\lambda$ and size parameter $\iota$ under the assumption $\iota$ is much smaller than the mesh size $h$. Here $(\bm{u}_0, p_0)$ is the solution of problem (1.2) with $\iota = 0$, i.e. the Lamé system. It is rather involved to acquire robust error estimate of $\bm{u} - \bm{u}_h$ and $p - p_h$ with respect to both Lamé constant $\lambda$ and size parameter $\iota$, which motivates this paper. Besides, based on some assumptions of weak continuity, in his PhD thesis [51] supervised by Prof. Jun Hu, Tian constructed a family of robust nonconforming finite element methods with the reduced integration technique for the primal formulation of SGE model (1.1) in two dimensions.

In this paper we shall devise a robust nonconforming mixed finite element method based on (1.2) with respect to both the size parameter $\iota$ and the Lamé constant $\lambda$. The well-posedness of problem (1.2) is related to the surjectivity of operator $\text{div}: H^1_0(\Omega; \mathbb{R}^d) \cap \iota H^2_0(\Omega; \mathbb{R}^d) \rightarrow L^2_0(\Omega) \cap \iota H^1_0(\Omega)$, which is equivalent to the inf-sup condition

$$
(1.3) \quad \|q\|_0 + \iota |q|_1 \lesssim \sup_{\bm{v} \in H^1_0(\Omega; \mathbb{R}^d)} \frac{(\text{div} \bm{v}, q) + \iota^2 (\nabla \text{div} \bm{v}, \nabla q)}{|\bm{v}|_1 + \iota |\bm{v}|_2} \quad \forall q \in H^1_0(\Omega) \cap L^2_0(\Omega).
$$

Here, the symbol $H^1_0(\Omega; \mathbb{R}^d) \cap \iota H^2_0(\Omega; \mathbb{R}^d)$ denotes the space $H^2_0(\Omega; \mathbb{R}^d)$ equipped with norm $|\bm{v}|_1 + \iota |\bm{v}|_2$, while $L^2_0(\Omega) \cap \iota H^1_0(\Omega)$ denotes the space $H^1_0(\Omega) \cap L^2_0(\Omega)$ equipped with norm $\|q\|_0 + \iota |q|_1$. To develop robust finite element methods for problem (1.2), we need to find a pair of finite element spaces $V_h$ and $Q_h$ satisfying the discrete analogue of the inf-sup condition (1.3), where $V_h$ and $Q_h$ are used to discretize $H^2_0(\Omega; \mathbb{R}^d)$ and $H^1_0(\Omega) \cap L^2_0(\Omega)$, respectively. The smooth finite element de Rham complexes in [23, 14, 16] will ensure the discrete inf-sup condition, but suffer from large number of degrees of freedom (DoFs) and supersmooth DoFs. To this end, we construct a lower order $C^0$-continuous $H^2$-nonconforming finite element in two and three dimensions for the displacement field $\bm{u}$, and use the continuous linear element to discretize the pressure field $p$. The DoFs of the $H^2$-nonconforming finite element involve

$$(\text{div} \bm{v}, q)_F \quad \forall q \in P_0(F), F \in \mathcal{F}(K),$$

which is vital to prove the robust discrete inf-sup condition with respect to the size parameter $\iota$. With finite element spaces $V_h$ and $Q_h$, we propose a robust nonconforming finite element method for problem (1.2). After establishing the discrete inf-sup condition and some interpolation error estimates, we achieve the robust error estimate $O(h^{1/2})$ of $\bm{u} - \bm{u}_h$ and $p - p_h$ with respect to both the size parameter $\iota$ and the Lamé constant $\lambda$.

Another contribution of this work is building up the regularity of problem (1.1) $|\bm{u} - \bm{u}_0|_1 + \iota \|\bm{u}\|_2 + \iota^2 \|\bm{u}\|_3 + \lambda \|\text{div} (\bm{u} - \bm{u}_0)\|_0 + \lambda \|\text{div} \bm{u}\|_1 + \lambda \iota^2 \|\text{div} \bm{u}\|_2 \lesssim \iota^{1/2} \|f\|_0$, under two reasonable assumptions. These assumptions are interesting problems in the field of partial differential equations. We prove the first assumption in two dimensions with the aid of the regularity of the triharmonic equation.

The rest of this paper is organized as follows. In Sect. 2, we recall some notations and existing results. Then the regularity result for SGE model is shown in Sect. 3. In Sect. 4, we construct a lower order $H^2$-nonconforming finite element, and propose a mixed finite element method for SGE model. In Sect. 5, we derive the robust
error estimates with respect to Lamé parameter \( \lambda \) and size parameter \( \iota \). Finally, numerical examples are provided in Sect. 6.

2. Preliminaries

2.1. Notation and some basic inequalities. Throughout this paper, let \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) be a convex polytope, which is partitioned into a family of shape regular simplices \( \mathcal{T}_h = \{ K \}; h_K = \text{diam}(K) \) and \( h = \max_{K \in \mathcal{T}_h} h_K \). Let \( V(\mathcal{T}_h) \) and \( V^j(\mathcal{T}_h) \) be the sets of all and interior vertices of \( \mathcal{T}_h \), respectively; denote by \( E(\mathcal{T}_h) \) and \( E^i(\mathcal{T}_h) \) (resp. \( \mathcal{F}(\mathcal{T}_h) \) and \( \mathcal{F}^i(\mathcal{T}_h) \)) the sets of all and interior one-dimensional edges (resp. \((d-1)\)-dimensional faces) of \( \mathcal{T}_h \), respectively. When \( d = 2 \), it is evident that \( \mathcal{F}(\mathcal{T}_h) = E(\mathcal{T}_h) \) and \( \mathcal{F}^3(\mathcal{T}_h) = E^3(\mathcal{T}_h) \). We next introduce the following macro elements for later requirement. For vertex \( \delta \in V(\mathcal{T}_h) \), edge \( e \in E(\mathcal{T}_h) \) and face \( F \in \mathcal{F}(\mathcal{T}_h) \), we need the first Korn’s inequality \([12, \text{Corollary 11.2.25}]\).

As usual, we use \( \| \cdot \|_m \) and \( | \cdot |_m \) for \( m \)-seminorm and \( m \)-norm, respectively for \( D = \Omega \). The notation \( (\cdot, \cdot)_D \) symbolizes the standard \( L^2 \) inner product on \( D \), and the subscript will be omitted when \( D = \Omega \). Let \( \mathbb{P}_k(D) \) be the set of all polynomials on \( D \) with the total degree up to \( k \). For a natural number \( s \), set \( \mathcal{H}^m(D; \mathbb{R}^s) := \mathcal{H}^m(D) \otimes \mathbb{R}^s \), \( \mathcal{H}^m_0(D; \mathbb{R}^s) := \mathcal{H}^m_0(D) \otimes \mathbb{R}^s \) and \( \mathbb{P}_k(D; \mathbb{R}^s) := \mathbb{P}_k(D) \otimes \mathbb{R}^s \). Let \( L^2_0(D) \) be the space of functions in \( L^2(D) \) with vanishing integral average values.

For a scalar function \( w \) and a vector valued function \( \mathbf{v} = (v_1, v_2) \) in two dimensions, introduce two usual differential operations curl \( w = (\partial_2 w, -\partial_1 w) \) and \( \text{rot} \mathbf{v} = \partial_1 v_2 - \partial_2 v_1 \).

As usual, we use \( \lesssim \) to represent \( \leq \), where \( C \) may be a generic positive constant independent of the mesh size \( h \), the material parameter \( \iota \) and the Lamé coefficient \( \lambda \). And \( a \approx b \) indicates \( a \lesssim b \lesssim a \). Assume \( \Omega \) is a star-shaped domain \([12]\). To deal with strain tensors, we need the first Korn’s inequality \([12, \text{Corollary 11.2.25}]\)

\[
\| \nabla \mathbf{v} \|_0 \lesssim \| \varepsilon(\mathbf{v}) \|_0 \quad \forall \mathbf{v} \in H^1_0(\Omega; \mathbb{R}^d),
\]

and the \( H^2 \)-Korn’s inequality \([32, \text{Theorem 1}]\)

\[
(1 - 1/\sqrt{2}) \| \nabla \mathbf{v} \|_0^2 \leq \| \nabla \varepsilon(\mathbf{v}) \|_0^2 \quad \forall \mathbf{v} \in H^1_0(\Omega; \mathbb{R}^d) \cap H^2(\Omega; \mathbb{R}^d).
\]

Recall the continuity of the right inverse of the divergence operator in \([18]\).

**Lemma 2.1.** For \( p \in H^{m-1}_0(\Omega) \cap L^2_0(\Omega) \) with non-negative integer \( m \), there exists \( \mathbf{v} \in H^{m+1}_0(\Omega; \mathbb{R}^d) \) such that \( \text{div} \mathbf{v} = p \) and

\[
\| \nabla \mathbf{v} \|_j \lesssim \| p \|_j \quad \text{for} \ j = 0, 1, \ldots, m.
\]
2.2. Mixed formulation. In order to solve the problem (1.1) with a mixed finite element method, we recall its mixed formulation in [34]. The variational formulation of (1.2) is to find $u \in V := H^2_0(\Omega; \mathbb{R}^d)$ and $p \in Q := H_0^1(\Omega) \cap L^2_0(\Omega)$ such that

$$
\begin{align*}
\begin{cases}
a(u, v) + b(v, p) = (f, v) & \forall v \in V, \\
b(u, q) - c(p, q) = 0 & \forall q \in Q,
\end{cases}
\end{align*}
$$

(2.3)

where

$$
a(u, v) := 2\mu(\varepsilon(u), \varepsilon(v)), = 2\mu(\varepsilon(u), \varepsilon(v)) + \nu^2(\nabla \varepsilon(u), \nabla \varepsilon(v)),
b(v, p) := (\text{div } v, p) = (\text{div } v, p) + \nu^2(\nabla \text{div } v, \nabla p),
c(p, q) := (\langle p, q \rangle_\lambda = \langle (p, q) + \nu^2(\nabla p, \nabla q) \rangle_\lambda
$$

with a weighted $H^1$ inner product $(p, q) := \langle (p, q) + \nu^2(\nabla p, \nabla q) \rangle$. To make the forthcoming discussion in a compact way, we further introduce the following weighted Sobolev norms over $V$ and $Q$:

$$
\|v\|_V := (\|v\|^2 + \nu^2\|\nabla v\|^2)^{1/2} \quad \forall v \in V; \quad \|q\|_Q := (\|q\|^2 + \nu^2\|\nabla q\|^2)^{1/2} \quad \forall q \in Q.
$$

Now, it is easy to check that the linear forms $a(\cdot, \cdot), b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are bounded on $V \times V, V \times Q$ and $Q \times Q$, respectively. The coercivity of the linear form $a(\cdot, \cdot)$ on $V \times V$ results from Korn’s inequalities (2.1)-(2.2). And the inf-sup condition

$$
\|q\|_Q \lesssim \sup_{v \in V} \frac{b(v, q)}{\|v\|_V} \quad \forall q \in Q
$$

holds from Lemma 2.1. Applying the Babuška-Brezzi theory [10], we have the stability

$$
\|u\|_V + \|p\|_Q \lesssim \sup_{v \in V, q \in Q} \frac{a(u, v) + b(v, p) - b(u, q) + c(p, q)}{\|v\|_V + \|q\|_Q} \quad \forall u \in V, p \in Q.
$$

Therefore, the mixed formulation (2.3) is well-posed.

3. Some regularity estimates

In order to derive the robust error estimates for the proposed mixed finite element method in the next section, we require to develop a series of regularity estimates for the underlying (system of) partial differential equations. In fact, these results are interesting themselves.

Lemma 3.1. If $f \in H^{-2}(\Omega; \mathbb{R}^d)$, then the following problem

$$
\begin{align*}
\begin{cases}
\Delta^2 \phi + \nabla \Delta p = f & \text{in } \Omega, \\
\Delta \text{div } \phi = 0 & \text{in } \Omega, \\
\phi = \partial_n \phi = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}
$$

(3.1)

has a unique solution $(\phi, p) \in H^2_0(\Omega; \mathbb{R}^d) \times (H_0^1(\Omega) \cap L^2_0(\Omega))$ such that

$$
\|\phi\|_2 + \|p\|_1 \lesssim \|f\|_{-2}.
$$

(3.2)

Proof. It is routine that the weak formulation of problem (3.1) is to find $(\phi, p) \in H^2_0(\Omega; \mathbb{R}^d) \times (H_0^1(\Omega) \cap L^2_0(\Omega))$ such that

$$
\begin{align*}
\begin{cases}
\langle \nabla^2 \phi, \nabla^2 \psi \rangle + \langle \nabla \text{div } \psi, \nabla p \rangle = \langle f, \psi \rangle & \forall \psi \in H^2_0(\Omega; \mathbb{R}^d), \\
\langle \nabla \text{div } \phi, \nabla q \rangle = 0 & \forall q \in H^1_0(\Omega) \cap L^2_0(\Omega).
\end{cases}
\end{align*}
$$

(3.3)
Clearly, the bilinear form \((\nabla^2, \nabla^2)\) is uniformly coercive over \(H^2_0(\Omega; \mathbb{R}^d)\). On the other hand, thanks to Lemma 2.1, for every \(q \in H^2_0(\Omega) \cap L^2_0(\Omega)\), there exists a \(\chi \in H^2_0(\Omega; \mathbb{R}^d)\) such that \(q = \text{div}\chi\) which admits the estimate \(\|\nabla\chi\|_1 \lesssim |q|_1\). Hence,

\[
\|q\|_1 \lesssim (\nabla q, \nabla q)/\|\nabla q\|_0 \lesssim (\nabla \text{div}\chi, \nabla q)/\|\nabla\chi\|_1 \lesssim \sup_{\psi \in H^2_0(\Omega; \mathbb{R}^d)} \frac{(\nabla \text{div}\psi, \nabla q)}{\|\psi\|_2}.
\]

Thus, we conclude the desired result from the Babuška-Brezzi theory [10]. \(\square\)

Next, we need to introduce two assumptions on the two auxiliary problems, which are the basis to develop subtle estimates for problem (1.1).

**Assumption 3.1.** Assume that \(f \in H^{-1}(\Omega; \mathbb{R}^d)\). Let \((\phi, p) \in H^2_0(\Omega; \mathbb{R}^2) \times (H^1_0(\Omega) \cap L^2_0(\Omega))\) be the solution of problem (3.1). Then \(\phi \in H^3(\Omega; \mathbb{R}^2)\) and \(p \in H^2(\Omega)\) which admit the estimate

\[
(3.4) \quad \|\phi\|_3 + \|p\|_2 \lesssim \|f\|_{-1}.
\]

We will prove Assumption 3.1 holds for a convex polygon in Section 3.1. Since the solution \(\phi\) of (3.3) is zero when \(f \in \nabla L^2(\Omega)\), by replacing \(f\) in problem (3.1) with \(f + \nabla q\), we have from (3.4) that

\[
(3.5) \quad \|\phi\|_3 \lesssim \inf_{q \in L^2(\Omega)} \|f + \nabla q\|_{-1}.
\]

**Assumption 3.2.** Assume that \(f \in H^{-1}(\Omega; \mathbb{R}^d)\). Let \(u \in H^2_0(\Omega; \mathbb{R}^d)\) be the solution of the following problem:

\[
\begin{align*}
\Delta (Lu) &= f & \text{in } \Omega, \\
u = \partial_n u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \(Lu = \mu \Delta u + (\lambda + \mu)\nabla (\text{div} u)\) is the usual Lamé operator, with \(\lambda \in [0, \Lambda]\) and \(\mu \in [\mu_0, \mu_1]\) being the Lamé constants. Then \(u \in H^3(\Omega; \mathbb{R}^d)\) and it admits the estimate

\[
\|u\|_3 \lesssim \|f\|_{-1},
\]

where the hidden constant may depend on \(\Lambda, \mu_0\) and \(\mu_1\).

**Remark 3.2.** If \(\Omega \in C^\infty\), Assumption 3.2 holds in terms of the mathematical theory in [1]. However, if \(\Omega\) is a polytope domain, as shown in [29], one should first figure out the operator pencil of the underlying problem and then discover the spectrum distribution in order to determine under which conditions Assumption 3.2 holds. Such study is rather involved, and we cannot ensure this assumption holds even for convex polygons until now.

### 3.1. Regularity results for triharmonic equations in two dimensions and applications.

In this subsection, in view of the mathematical theory in [20, 29], we are going to develop the regularity theory for triharmonic equations with homogeneous boundary conditions over convex polygons. As far as we know, this result is new. It has its interest itself and will also be used to show Assumption 3.1 holds for a convex polygon in combination with some results in [18].

Before introducing and proving our regularity estimates, let us recall some existing results in [20]. As shown in Fig. 1, let \(\Omega\) be a convex polygon with \(X\) as a generic vertex which has an interior angle \(\alpha \in (0, \pi)\). Let \(L\) be a strongly elliptic \(2m\)-order differential operator defined on \(\overline{\Omega}\) with constant coefficients, where \(m\) is
any given positive integer. Consider the following homogeneous Dirichlet boundary problem

\[
Lu = f \quad \text{in } \Omega, \\
u = \partial_n u = \cdots = \partial_{n}^{m-1} u = 0 \quad \text{on } \partial\Omega,
\]

which induces the following operators for each positive real number \( s \):

\[
L^{(s)} : H^m_0(\Omega) \cap H^{s+m}(\Omega) \to H^{s-m}(\Omega), \\
u \to Lu.
\]

A key problem is to determine the regularity property of \( L^{(s)} \) for \( s > 0 \).

As a matter of fact, the problem can be answered locally. As shown in Fig. 1, for any given vertex \( X \), we might assume the unit circle \( S_X \) centered at \( X \) has intersection with each of two edges of \( \Omega \) which share the vertex \( X \), and then write \( \Omega_X = \Omega \cap S_X \). Let \( L_X \) be the principal part of \( L \). We introduce a polar coordinate system \((r, \theta)\) with \( X \) as the origin. Then the differential operator \( L_X \) can be expressed as

\[
L_X(D_x) = r^{-2m} L_X(\theta; r \partial_r, D_\theta),
\]

where \( D_x = \frac{1}{r} \partial_x, \quad D_\theta = \frac{1}{r} \partial_\theta \). Next, we replace \( r \partial_r \) by a complex variable \( \lambda \in \mathbb{C} \), so as to get a holomorphic operator \( L_X(\lambda) = L_X(\theta; \lambda, D_\theta) \) which acts from \( H^m_0(\Omega_X) \cap H^{s+m}(\Omega_X) \) into \( H^{s-m}(\Omega_X) \). \( L_X(\lambda) \) is simply written as \( L(\lambda) \) when there is no confusion caused.

As shown in [20], the angle singularities of solution of elliptic equations have very close relationship with spectrum properties of \( L(\lambda) \). Recall that a point \( \lambda_0 \in \mathbb{C} \) is said to be regular if the operator \( L(\lambda_0) \) is invertible. The set of all non-regular points is called the spectrum of the operator. Then, by the well-known Peetre’s theorem and the main theorem in [20, p. 10] we can have a simplified result from [20], described as follows.

**Theorem 3.3** (Dauge). Let \( \Omega \) be a convex polygon and let \( L \) be a strongly elliptic 2m-order differential operator defined on \( \Omega \) with constant coefficients. Assume that the boundary value problem (3.6) has a unique solution \( u \in H^m_0(\Omega) \). Moreover, \( s \geq 0 \) and \( s \neq \left\{ \frac{1}{2}, \frac{3}{2}, \cdots, m - \frac{1}{2} \right\} \). If any \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \in [m - 1, s + m - 1] \) is not spectrum of \( L(\lambda) \) and \( f \in H^{s-m}(\Omega) \), then \( u \in H^{s+m}(\Omega) \), and it holds the estimate

\[
\|u\|_{s+m} \ll \|f\|_{s-m}.
\]

On the other hand, it is rather involved to figure out the closed-form of \( L(\lambda) \) and analyze its spectrum distribution. However, it is very lucky that for the differential
operator $L = (-\Delta)^m$, we have from \cite[Chapter 8]{29} that
\[ L(\lambda) = (-1)^m \prod_{j=0}^{m-1} (\partial^2_\theta + (\lambda - 2j)^2), \]
and the following result holds.

**Lemma 3.4.** \cite[Chapter 8]{29} If $\Omega$ is a convex polygon, and $L = (-\Delta)^m$ for a positive integer $m$, then we have

1. the spectrum of $L(\lambda)$ does not contain the points $m, m+1, \cdots, 2m-1$;
2. the strip $m-2 \leq \text{Re}\lambda \leq m$ does not contain any eigenvalues of $L(\lambda)$.

Making use of the above two results, we can derive the regularity theory for the triharmonic equation given below.

**Theorem 3.5.** Assume $\Omega$ is a convex polygon. Let $w \in H^3_0(\Omega)$ satisfy
\[ -\Delta^3 w = f \in H^{-2}(\Omega). \] (3.7)
Then $w \in H^4(\Omega)$ and it admits the estimate $\|w\|_4 \lesssim \|f\|_{-2}$.

**Proof.** The weak formulation of problem (3.7) is to find $w \in H^3_0(\Omega)$ such that
\[ \int_{\Omega} \nabla^3 w : \nabla^3 v \, dx = \langle f, v \rangle \quad \forall \, v \in H^3_0(\Omega). \]
Here, for any two third-order tensors $A$ and $B$, $A : B = \sum_{ijk} A_{ijk} B_{ijk}$, and $\langle \cdot, \cdot \rangle$ denotes the duality pair between $H^{-3}(\Omega)$ and $H^3_0(\Omega)$. Hence, by the Poincaré inequality and the Lax-Milgram theorem, the last problem has a unique solution $w \in H^3_0(\Omega)$. In other words, the boundary value problem (3.7) has a unique solution $w \in H^3_0(\Omega)$ for all $f \in H^{-3}(\Omega)$. According to Lemma 3.4 with $m = 3$, $L(\lambda)$ does not have a spectrum point in the strip $1 \leq \text{Re}\lambda \leq 3$. Hence, we can conclude the desired result from Theorem 3.3 with $s = 1$. \hfill \square

With the help of the above theorem, we can obtain the following result.

**Theorem 3.6.** Assumption 3.1 holds whenever $\Omega$ is a convex polygon and $f \in H^{-1}(\Omega; \mathbb{R}^2)$.

**Proof.** Step 1: According to the second and third equations in (3.1), $\text{div} \, \phi \in H^1_0(\Omega)$ satisfies the Laplace equation, and hence $\text{div} \, \phi = 0$. Since $\phi \in H^3_0(\Omega; \mathbb{R}^2)$, there exists a scalar function $w \in H^3_0(\Omega)$ \cite{18} such that $\phi = \text{curl} \, w$. Then apply the rot operator to the first equality of (3.1) to get
\[ -\Delta^3 w = \text{rot}(\Delta^2 \text{curl} \, w) = \text{rot}(\Delta^2 \phi) = \text{rot}(\Delta^2 \phi + \nabla^3 p) = \text{rot} \, f. \]
Due to Theorem 3.5, we know that $w \in H^4(\Omega)$ admits the estimate $\|w\|_4 \lesssim \|\text{rot} \, f\|_{-2} \lesssim \|f\|_{-1}$.

By the relation $\phi = \text{curl} \, w$, we immediately have $\phi \in H^3(\Omega; \mathbb{R}^2)$ and
\[ \|\phi\|_3 \lesssim \|f\|_{-1}. \] (3.8)

Step 2: Rewrite the first equation in (3.1) as $\nabla \Delta p = f - \Delta^2 \phi$. Since $f - \Delta^2 \phi \in H^{-1}(\Omega; \mathbb{R}^2)$ and $\Delta p \in H^{-1}(\Omega)$, apply Theorem 2.2 in \cite{21} to get $\Delta p \in L^2(\Omega)$ and
\[ \|\Delta p\|_0 \lesssim \|\Delta p\|_{-1} + \|f - \Delta^2 \phi\|_{-1} \lesssim \|p\|_1 + \|f - \Delta^2 \phi\|_{-1}, \]
which combined with (3.2) and (3.8) yields
\[ \| \Delta p \|_0 \lesssim \| f \|_{-1}. \]
Noting that \( \Omega \) is convex, by the regularity of Poisson’s equation,
\[ \| p \|_2 \lesssim \| \Delta p \|_0 \lesssim \| f \|_{-1}. \]
Now combining (3.8) and the last inequality gives (3.4).

3.2. Regularity of the SGE model. Now we are ready to show the regularity results of the SGE model. From now on we always suppose Assumptions 3.1-3.2 hold.

Lemma 3.7. Let \( w \in H^2_0(\Omega; \mathbb{R}^d) \) be the solution of
\[
\begin{align*}
\text{(3.9)} & \quad \begin{cases} 
\text{div} \Delta (2\mu \varepsilon(w) + \lambda(\text{div} w) I) = f & \text{in } \Omega, \\
\text{div} w = \partial_n w = 0 & \text{on } \partial \Omega
\end{cases} \\
\text{with } f \in H^{-1}(\Omega; \mathbb{R}^d). \text{ Suppose Assumptions 3.1-3.2 hold. We have}
\end{align*}
\]
\[
(3.10) \quad \| w \|_{2+j} + \lambda \| \text{div} w \|_{1+j} \lesssim \| f \|_{j-2} \quad \text{for } j = 0, 1,
\]
\[
(3.11) \quad \| w \|_3 \lesssim \inf_{q \in L^2(\Omega)} \| f + \nabla q \|_{-1} + \frac{1}{\lambda} \| f \|_{-1}.
\]

Proof. We follow the argument of Lemma 2.2 in [13] to prove (3.10). The weak formulation of (3.9) is
\[
\begin{align*}
\text{(3.12)} & \quad 2\mu(\nabla \varepsilon(w), \nabla \varepsilon(v)) + \lambda(\nabla \text{div} w, \nabla \text{div} v) = (f, v) \quad \forall \ v \in H^2_0(\Omega; \mathbb{R}^d),
\end{align*}
\]
which is well-posed by the \( H^2 \)-Korn’s inequality (2.2), and it holds \( \| w \|_2 \lesssim \| f \|_{-2} \). By Lemma 2.1, there exists \( \tilde{w} \in H^2_0(\Omega; \mathbb{R}^d) \) such that \( \text{div} \tilde{w} = \text{div} w \) and \( \| w \|_2 \lesssim \| \text{div} w \|_1 \). Taking \( v = \tilde{w} \) in (3.12), it follows that
\[
\lambda |\text{div} w|^2 = (f, \tilde{w}) - 2\mu(\nabla \varepsilon(w), \nabla \varepsilon(\tilde{w})) \lesssim \| f \|_{-2} \| w \|_2 + \| w \|_2 \| \tilde{w} \|_2,
\]
which gives (3.10) for \( j = 0 \).

Next we prove (3.10) for \( j = 1 \). The first equation in (3.9) can be rewritten as
\[
\Delta^2 w + \frac{\mu + \lambda}{\mu} \nabla \Delta \text{div} w = \frac{1}{\mu} f \quad \text{in } \Omega.
\]
Clearly \( w \in H^3(\Omega; \mathbb{R}^d) \) under Assumption 3.2. Employing lemma 2.1 again, there exists \( \tilde{w} \in H^3(\Omega; \mathbb{R}^d) \cap H^2_0(\Omega; \mathbb{R}^d) \) such that \( \text{div} \tilde{w} = \text{div} w \) and \( \| \tilde{w} \|_3 \lesssim \| \text{div} w \|_2 \).

With the help of \( \tilde{w} \), we have
\[
\begin{align*}
\text{(3.13)} & \quad \| w - \tilde{w} \|_3 + \frac{\mu + \lambda}{\mu} \| \text{div} w \|_2 \lesssim \| f \|_{-1} + \| \tilde{w} \|_3, \\
\text{(3.14)} & \quad \| w - \tilde{w} \|_3 \lesssim \inf_{q \in L^2(\Omega)} \| f + \nabla q \|_{-1} + \| \tilde{w} \|_3.
\end{align*}
\]
By (3.13), there exists a constant $C > 0$ such that
\[ \|w\|_3 + \frac{\mu + \lambda}{\mu} \|\text{div } w\|_2 \leq C(\|f\|_{-1} + \|\text{div } w\|_2). \]
When $\lambda > 2\mu C$, i.e. $C < \frac{\lambda}{2\mu}$, we get
\[ \|w\|_3 + \frac{2\mu + \lambda}{2\mu} \|\text{div } w\|_2 \leq C\|f\|_{-1}. \]
Thus, (3.10) holds for $j = 1$. When $0 < \lambda \leq 2\mu C$, (3.10) for $j = 1$ is guaranteed by Assumption 3.2.

It follows from (3.14) that
\[ \|w\|_3 \leq \inf_{q \in L^2(\Omega)} \|f + \nabla q\|_{-1} + \|\tilde{w}\|_3 \leq \inf_{q \in L^2(\Omega)} \|f + \nabla q\|_{-1} + \|\text{div } w\|_2, \]
which together with (3.10) yields (3.11). \hfill \Box

Taking $\varepsilon = 0$, problem (1.1) becomes the linear elasticity model. Let $u_0 \in H^2_0(\Omega; \mathbb{R}^d)$ be the solution of the linear elasticity problem
\begin{equation}
\label{eq:lemma3.8}
\begin{cases}
-\mu \Delta u_0 - (\lambda + \mu) \nabla \div u_0 = f & \text{in } \Omega, \\
u_0 = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}
According to [13, 37], it holds
\begin{equation}
\label{eq:lemma3.8-1}
\|u_0\|_2 + \lambda \|\div u_0\|_1 \lesssim \|f\|_0.
\end{equation}

**Lemma 3.8.** With the same assumptions as Lemma 3.7, let $u \in H^2(\Omega; \mathbb{R}^d)$ be the solution of problem (1.1), and $u_0 \in H^1(\Omega; \mathbb{R}^d)$ satisfy the linear elasticity problem (3.15). It holds
\begin{equation}
\label{eq:lemma3.8-2}
|u - u_0|_1 + \varepsilon \|u\|_2 + \varepsilon^2 \|u\|_3 \lesssim \varepsilon^{1/2} \|f\|_0.
\end{equation}

**Proof.** By (1.1) and (3.15), we have
\begin{equation}
\label{eq:lemma3.8-3}
\div \Delta \sigma(u) = \varepsilon^{-2} \div \sigma(u - u_0).
\end{equation}
Then it follows from (3.11) that
\begin{equation}
\label{eq:lemma3.8-4}
\varepsilon^2 \|u\|_3 \lesssim \inf_{q \in L^2(\Omega)} \|\div \sigma(u - u_0) + \nabla q\|_{-1} + \frac{1}{\lambda} \|\div \sigma(u - u_0)\|_{-1}
\end{equation}
\[ \lesssim |u - u_0|_1 + \frac{1}{\lambda} |u - u_0|_1 + \|\div (u - u_0)\|_0 \lesssim |u - u_0|_1. \]
Multiply (3.18) by $u - u_0 \in H^2(\Omega; \mathbb{R}^d) \cap H^1(\Omega; \mathbb{R}^d)$ and apply the integration by parts to get
\begin{equation}
\label{eq:lemma3.8-5}
\varepsilon^2 (\nabla \sigma(u), \nabla \varepsilon(u)) + (\sigma(u - u_0), \varepsilon(u - u_0))
\end{equation}
\[ = \varepsilon^2 (\nabla \sigma(u), \nabla \varepsilon(u_0)) - \varepsilon^2 (\partial_n \sigma(u), \varepsilon(u_0))_{\partial \Omega}. \]
By (3.16),
\begin{equation}
\label{eq:lemma3.8-6}
(\nabla \sigma(u), \nabla \varepsilon(u_0)) = 2\mu (\nabla \varepsilon(u), \nabla \varepsilon(u_0)) + \lambda (\nabla \div u, \nabla \div u_0)
\lesssim |u|_2 (|u_0|_2 + \lambda) \div u_0 |\lesssim |u|_2 \|f\|_0.
\end{equation}
Applying the multiplicative trace inequality, (3.16) and (3.19),

\[ - (\partial_n \sigma(u), \epsilon(u_0))_{\partial \Omega} = - 2 \mu (\partial_n \epsilon(u), \epsilon(u_0))_{\partial \Omega} - \lambda (\partial_n \text{div} u, \text{div} u_0)_{\partial \Omega} \]

\[ \lesssim \| \partial_n \epsilon(u) \|_{0, \partial \Omega} \| \epsilon(u_0) \|_{0, \partial \Omega} + \lambda \| \partial_n \text{div} u \|_{0, \partial \Omega} \| \text{div} u_0 \|_{0, \partial \Omega} \]

\[ \lesssim \| u \|_{2/2}^2 \| u_0 \|_{3/2}^2 (\| u_0 \|_2^2 + \lambda \| \text{div} u_0 \|_1) \lesssim \| u \|_{2/2}^2 \| u_0 \|_{3/2}^2 \| f \|_0 \]

\[ \lesssim \epsilon^{-1} \| u \|_{2/2}^2 \| f \|_0 \| u - u_0 \|_{1/2}. \]

Combining the last two inequalities and (3.20) yields

\[ \epsilon^2 (\nabla \sigma(u), \nabla \epsilon(u)) + (\sigma(u - u_0), \epsilon(u - u_0)) \]

\[ \lesssim \left( \epsilon^{3/2} |u|_2 + \epsilon^{1/2} |u|_{2/2} |u - u_0|_{1/2} \right) \epsilon^{1/2} |f|_0 \lesssim (\epsilon |u|_2 + |u - u_0|_1) \epsilon^{1/2} |f|_0, \]

which implies

\[ \epsilon |u|_2 + |u - u_0|_1 + \lambda^{1/2} \| \text{div} (u - u_0) \|_0 + \lambda^{1/2} \epsilon | \text{div} u |_1 \lesssim \epsilon^{1/2} |f|_0. \]

Finally, we get (3.17) from (3.19).

**Theorem 3.9.** Let \( u \in H^2_0(\Omega; \mathbb{R}^d) \) be the solution of problem (1.1), and \( u_0 \in H^1_0(\Omega; \mathbb{R}^d) \) satisfy the linear elasticity problem (3.15). Under the same assumptions as Lemma 3.7, it holds

\[ \lambda \| \text{div} (u - u_0) \|_0 + \lambda \epsilon | \text{div} u |_1 \lesssim \epsilon^{1/2} |f|_0. \]

**Proof.** Applying Lemma 3.7 to (3.18), we obtain from (3.10) and (3.17) that

\[ \lambda^{1/2} \| u \|_2 \lesssim \| \text{div} \sigma(u - u_0) \|_{-1} \lesssim \| \sigma(u - u_0) \|_0 \lesssim |u - u_0|_1 + \lambda | \text{div} (u - u_0) |_0 \]

\[ \lesssim \epsilon^{1/2} |f|_0 + \lambda \| \text{div} (u - u_0) \|_0. \]

Hence it suffices to prove

\[ \lambda \| \text{div} (u - u_0) \|_0 + \lambda \epsilon | \text{div} u |_1 \lesssim \epsilon^{1/2} |f|_0. \]

Thanks to Lemma 2.1, there exists \( v \in H^2(\Omega; \mathbb{R}^d) \cap H^1_0(\Omega; \mathbb{R}^d) \) such that

\[ \| v \|_1 \lesssim \| \text{div} (u - u_0) \|_0. \]

Multiply (3.18) by \( v \) and apply the integration by parts to get

\[ \lambda \| \text{div} (u - u_0) \|_0^2 + \lambda \epsilon^{2} | \text{div} u |_1^2 \]

\[ = -2\epsilon (\epsilon (u - u_0), (v) + 2 \mu (\Delta \epsilon(u), (v)) - \epsilon^{2} (\Delta \text{div} u, (u - u_0)). \]

By the multiplicative trace inequality and (3.22), it holds

\[ - \lambda^{2} (\Delta \text{div} u, (u - u_0)) = \lambda^{2} (\nabla \text{div} u, \nabla (u - u_0)) - \lambda^{2} (\partial_n \text{div} u, \text{div} u_0)_{\partial \Omega} \]

\[ \lesssim \lambda^{2} \| \text{div} u |_1 | \text{div} u_0 |_1 + \lambda \epsilon^{2} \| \text{div} u |_1^{1/2} | \text{div} u_0 |_1^{1/2} \]

\[ \lesssim \lambda^{2} \| \text{div} u |_1 | \text{div} u_0 |_1 + \lambda^{1/2} | \epsilon^{1/4} \| \text{div} u |_1^{1/2} | f |_0^{1/2} \| \text{div} u_0 |_1 \]

\[ + \lambda \epsilon | \text{div} u |_1^{1/2} | \text{div} (u - u_0) |_1^{1/2} \| \text{div} u_0 |_1. \]
Then we acquire from (3.24)-(3.25) that
\[ \lambda \| \text{div}(u - u_0) \|^2_0 + \lambda^2 \| \text{div} u_0 \|^2_0 \]
\[ \lesssim (|u - u_0|_1 + \epsilon^2 |u_0|_1 |v_1|_1 + \lambda |\text{div} u_0|_1) \| \text{div}(u - u_0) \|_1^{1/2} \| \text{div} u_0 \|_1 \]
\[ + \lambda^{1/2} \epsilon^{5/4} \| f \|_0^{1/2} \| \text{div} u_0 \|_1 + \lambda \| \text{div} u_0 \|_1 \| \text{div}(u - u_0) \|_1 \]
\[ \lesssim (|u - u_0|_1 + \epsilon^2 |u_0|_1 |v_1|_1 + \lambda |\text{div} u_0|_1) \| \text{div}(u - u_0) \|_1 \]
\[ + (\lambda \| \text{div} u_1 \| \text{div}(u - u_0)\|_1^{1/2} (\lambda \| \text{div} u_0\|_1^{2})^{1/2} \]
\[ + (\lambda \| \text{div} u_1 \| \text{div}(u - u_0)\|_1^{1/2} (\lambda \| \text{div} u_0\|_1^{2})^{1/2} \]
\[ + (\lambda \| \text{div} u_1 \| \text{div}(u - u_0)\|_1^{1/2} (\lambda \| \text{div} u_0\|_1^{2})^{1/2} \]
\[ i.e., \]
\[ \lambda \| \text{div}(u - u_0) \|_0 + \lambda |\text{div} u_1| \lesssim |u - u_0|_1 + \epsilon^2 |u_0|_1 + \lambda^{1/2} |u_0|_1 + \epsilon^{1/2} |f|_0. \]

Therefore, we conclude (3.23) from (3.17) and (3.16).

\[ \square \]

4. Mixed finite element method

In this section we will construct a lower order $C^0$-continuous $H^2$-nonconforming finite element, and apply to discretize the variational formulation (2.3) of SGE model (1.2).

4.1. $H^2$-nonconforming finite element. For a $d$-dimensional simplex $K$ ($d = 2, 3$) with vertices $a_1, \ldots, a_{d+1}$, let $\mathcal{V}(K)$, $\mathcal{E}(K)$ and $\mathcal{F}(K)$ be the sets of all vertices, one-dimensional edges and $(d-1)$-dimensional faces of $K$ respectively. Set $\mathcal{V}^s(K) := \mathcal{V}(K) \cap \mathcal{V}^s(\mathcal{T}_h)$. For $s = 1, \ldots, d+1$, denote by $\lambda_s$ the barycentric coordinate corresponding to $a_s$ and there exists $F_s \in \mathcal{F}(K)$ such that $\lambda_s|_{F_s} = 0$. Let $b_K = \prod_{s=1}^{d+1} \lambda_s = \lambda_j b_{F_j}$ be the bubble function of $K$, where $b_{F_j}$ is the bubble function of $F_j$ with $j = 1, \ldots, d+1$.

Given a face $F \in \mathcal{F}(K)$ with unit normal vector $n$, for a vector function $v$, define the tangential component $\Pi_F v = v - (v \cdot n)n$. Then
\[ (4.1) \quad \text{div} v = \text{div}((v \cdot n)n + \Pi_F v) = \partial_n (v \cdot n) + \text{div}_F v, \]

where face divergence $\text{div}_F v := \text{div}(\Pi_F v)$.

With previous preparation, take
\[ \mathit{V}(K) := F_2(K; \mathbb{R}^d) + b_K P_1(K; \mathbb{R}^d) + b_K^2 P_0(K; \mathbb{R}^d) \]
as the space of shape functions. Then
\[ \dim \mathit{V}(K) = d(d + 2)/2 + d(d + 1) + d = 1/2 d(d + 2)(d + 3). \]
The degrees of freedom (DoFs) are chosen as
\begin{align}
(4.2) & \quad \mathbf{v}(\delta) \quad \forall \delta \in \mathcal{V}(K), \\
(4.3) & \quad \frac{1}{|e|} \langle \mathbf{v}, \mathbf{q} \rangle_e \quad \forall \mathbf{q} \in \mathbb{P}_0(e; \mathbb{R}^d), e \in \mathcal{E}(K), \\
(4.4) & \quad \frac{1}{|F|} \langle \partial_n(\mathbf{v} \cdot \mathbf{t}_i), q \rangle_F \quad \forall q \in \mathbb{P}_0(F), F \in \mathcal{F}(K), i = 1, \ldots, d - 1, \\
(4.5) & \quad \frac{1}{|F|} \langle \text{div} \mathbf{v}, q \rangle_F \quad \forall q \in \mathbb{P}_0(F), F \in \mathcal{F}(K), \\
(4.6) & \quad \frac{1}{|K|} \langle \mathbf{v}, \mathbf{q} \rangle_K \quad \forall \mathbf{q} \in \mathbb{P}_0(K; \mathbb{R}^d),
\end{align}
where \( \mathbf{t}_1, \ldots, \mathbf{t}_{d - 1} \) are \((d - 1)\) unit orthogonal tangential vectors of \( F \). DoF (4.5) is vital to prove the robust discrete inf-sup condition with respect to the size parameter \( \iota \).

**Lemma 4.1.** The degrees of freedom (4.2)-(4.6) are uni-solvent for \( V(K) \).

**Proof.** The number of the degrees of freedom (4.2)-(4.6) is
\[
d(d + 1) + d \left( \frac{d + 1}{2} \right) + d(d + 1) + d = \frac{1}{2} d(d + 2)(d + 3) = \dim V(K).
\]

Take \( \mathbf{v} \in V(K) \) and assume all the degrees of freedom (4.2)-(4.6) vanish, then we prove \( \mathbf{v} = 0 \). Noting that \( \mathbf{v}|_F \in \mathbb{P}_2(F; \mathbb{R}^d) \) for each \( F \in \mathcal{F}(K) \), we get from the vanishing DoFs (4.2)-(4.3) that \( \mathbf{v}|_{\partial K} = 0 \). As a result there exist \( \mathbf{v}_1 \in \mathbb{P}_1(K; \mathbb{R}^d) \) and \( \mathbf{v}_2 \in \mathbb{P}_0(K; \mathbb{R}^d) \) such that \( \mathbf{v} = b_K \mathbf{v}_1 + b^*_K \mathbf{v}_2 \). Thanks to (4.1), the vanishing DoF (4.5) implies \( \langle \partial_n(\mathbf{v} \cdot \mathbf{n}), q \rangle_F = 0 \) for all \( q \in \mathbb{P}_0(F) \), which combined with the vanishing DoF (4.4) yields \( \langle \partial_n \mathbf{v}, \mathbf{q} \rangle_F = 0 \) for all \( \mathbf{q} \in \mathbb{P}_0(F; \mathbb{R}^d) \), and thus
\[
\langle b_F \mathbf{v}_1, \mathbf{q} \rangle_F = 0 \quad \forall \mathbf{q} \in \mathbb{P}_0(F; \mathbb{R}^d), F \in \mathcal{F}(K).
\]
As a result \( \mathbf{v}_1 = 0 \). Finally, we conclude \( \mathbf{v} = \mathbf{v}_2 = 0 \) from the vanishing DoF (4.6). \( \square \)

Define a global \( H^2 \)-nonconforming finite element space:
\[
V_h = \{ \mathbf{v}_h \in L^2(\Omega; \mathbb{R}^d) \mid \mathbf{v}_h|_K \in V(K) \text{ for each } K \in \mathcal{T}_h, \text{ all the DoFs (4.2)-(4.6)} \}
\]
are single-valued, and DoFs (4.2)-(4.5) on boundary vanish}.

Clearly \( V_h \subset H^2_0(\Omega; \mathbb{R}^d) \), but \( V_h \not\subset H^2(\Omega; \mathbb{R}^d) \). The finite element space \( V_h \) has the weak continuity
\begin{align}
(4.7) & \quad \int_F \| \nabla_h \mathbf{v}_h \| dS = 0 \quad \forall \mathbf{v}_h \in V_h, F \in \mathcal{F}(\mathcal{T}_h).
\end{align}

Here \( \nabla_h \) is the elementwise version of \( \nabla \) with respect to \( \mathcal{T}_h \).

Introduce the Lagrange element space \( Q_h = Q_h^1 \cap Q \), where
\[
Q_h^1 := \{ q \in H^1(\Omega) \mid q|_K \in \mathbb{P}_1(K) \text{ for each } K \in \mathcal{T}_h \}.
\]

Then the dicretization formulation of (2.3) is to find \((\mathbf{u}_h, q_h) \in V_h \times Q_h \) such that
\begin{align}
(4.8) & \quad \begin{cases}
\begin{align}
a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, q_h) &= (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h, \\
b_h(\mathbf{u}_h, q_h) - c(p_h, q_h) &= 0 \quad \forall q_h \in Q_h,
\end{align}
\end{cases}
\end{align}
where
\[
\begin{align*}
    a_h(u_h, v_h) &:= 2\mu((\varepsilon(u_h), \varepsilon(v_h)) + \nu^2(\nabla_h \varepsilon(u_h), \nabla_h \varepsilon(v_h))), \\
    b_h(v_h, p_h) &:= (\mathop{\text{div}} v_h, p_h) + \nu^2(\nabla_h \mathop{\text{div}} v_h, \nabla p_h).
\end{align*}
\]
Define squared norm \(\|v_h\|_{V,h}^2 := |v_h|_h^2 + \nu^2|v_h|_{2,h}^2\) with \(|v_h|_{2,h}^2 = \sum_{K \in T_h} |v_h|_{2,K}^2\).

Clearly the discrete linear forms \(a_h(\cdot, \cdot)\) and \(b_h(\cdot, \cdot)\) are continuous on \(V_h \times V_h\) and \(V_h \times Q_h\), respectively.

Since \((1 - 1/\sqrt{2})|v_h|_{2,h}^2 \lesssim \|\nabla_h \varepsilon(v_h)\|_0^2\) and the Korn’s inequality (2.1), we achieve the discrete coercivity
\[
\|v_h\|_{V,h}^2 \lesssim a_h(v_h, v_h) \quad \forall \, v_h \in V_h.
\]

4.2. Discrete inf-sup condition and uni-solvence. To derive the unisolvence of the mixed finite element method (4.8), we first introduce the second order Brezzi-Douglas-Marini (BDM) element [10, 15].

The second order BDM element takes \(P_2(K; \mathbb{R}^d)\) as the shape function space, and the degrees of freedom are chosen as [15]
\[
(4.10) \quad (v, n, q)_F \quad \forall \, q \in P_2(F), \, F \in F(K),
\]
\[
(4.11) \quad (v, q)_K \quad \forall \, q \in P_1(K; \mathbb{R}^d) \text{ satisfying } x \cdot q \in P_1(K).
\]

Let \(I_h^{BDM} : H^1(K; \mathbb{R}^d) \rightarrow P_2(K; \mathbb{R}^d)\) be the nodal interpolation operator based on DoFs (4.10)-(4.11). It holds
\[
(4.12) \quad \mathop{\text{div}}(I_h^{BDM} v) = Q_K(\mathop{\text{div}} v) \quad \forall \, v \in H^1(K; \mathbb{R}^d),
\]
where \(Q_K\) is the standard \(L^2\) projection operator from \(L^2(K)\) to \(P_1(K)\). Let \(I_h^{BDM}\) be the global version of \(I_h^{BDM}\).

Now we define interpolation operator \(I_h : H^2_0(\Omega; \mathbb{R}^d) \rightarrow V_h\) as follows:
\[
(4.13) \quad (I_h v)(\delta) = \frac{1}{\#T_\delta} \sum_{K \in T_\delta} (I_h^{BDM} v)(\delta),
\]
\[
(I_h v, q)_e = \frac{1}{\#T_e} \sum_{K \in T_e} (I_h^{BDM} v, q)_e \quad \forall \, q \in P_0(e; \mathbb{R}^d),
\]
\[
(\partial_n(\Pi_F(I_h v), q)_F = \frac{1}{\#F} \sum_{K \in F} (\partial_n(\Pi_F(I_h^{BDM} v)), q)_F \quad \forall \, q \in P_0(F; \mathbb{R}^{d-1}),
\]
\[
(4.14) \quad (I_h v, q)_K = (v, q)_K \quad \forall \, q \in P_0(K; \mathbb{R}^d),
\]
for \(\delta \in V^i(T_h), \, e \in E^i(T_h), \, F \in F^i(T_h)\) and \(K \in T_h\).

Lemma 4.2. We have
\[
(4.15) \quad |I_h v|_{j,h} \lesssim |v|_j \quad \forall \, v \in H^2_0(\Omega; \mathbb{R}^d), \, j = 1, 2,
\]
\[
(4.16) \quad \sum_{i=1}^{3} h_K^i |v - I_h v|_{i,K} \lesssim h_K^3 \sum_{\delta \in V(K)} |v|_{3,\delta} \quad \forall \, v \in H^2_0(\Omega; \mathbb{R}^d) \cap H^3(\Omega; \mathbb{R}^d).
\]

Proof. Thanks to (4.11) and (4.14), it follows that
\[
(I_h v - I_h^{BDM} v, q)_K = 0 \quad \forall \, q \in P_0(K; \mathbb{R}^d).
\]
Then by the inverse inequality, scaling argument and (4.12),
\[ h_K^2 | \mathbf{v} - I_{K}^{BDM} \mathbf{v} |_{1,K}^2 \]
\[ \lesssim h_K^d \sum_{\delta \in \mathcal{V}(K)} (I_{K} \mathbf{v} - I_{K}^{BDM} \mathbf{v})^2(\delta) + h_K^{d-1} \sum_{e \in \mathcal{E}(K)} \| I_{K} \mathbf{v} - I_{K}^{BDM} \mathbf{v} \|_{0,e}^2 \]
\[ + h_K^{3} \sum_{F \in \mathcal{F}(K)} \| \partial_n(I_{K} \mathbf{v} - I_{K}^{BDM} \mathbf{v}) \|_{0,F}^2 + h_K^{3} \sum_{F \in \mathcal{F}(K)} \| \nabla(I_{K} \mathbf{v} - I_{K}^{BDM} \mathbf{v}) \|_{F}^2 \]
\[ \lesssim h_K \sum_{\delta \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}(T_h)} \| I_{K}^{BDM} \mathbf{v} \|_{0,F}^2 + h_K^{3} \sum_{F \in \mathcal{F}(K)} \| \partial_n(I_{K}^{BDM} \mathbf{v}) \|_{0,F}^2 \]
\[ + h_K^{3} \sum_{F \in \mathcal{F}(K)} \| \nabla(I_{K}^{BDM} \mathbf{v}) \|_{F}^2. \]

This implies
\[ h_K^2 | \mathbf{v} - I_{K} \mathbf{v} |_{1,K}^2 \lesssim h_K^d | \mathbf{v} - I_{K}^{BDM} \mathbf{v} |_{1,K}^2 + h_K \sum_{\delta \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}(T_h)} \| I_{K}^{BDM} \mathbf{v} \|_{0,F}^2 \]
\[ + h_K^{3} \sum_{F \in \mathcal{F}(K)} \| \partial_n(I_{K}^{BDM} \mathbf{v}) \|_{0,F}^2 + h_K^{3} \sum_{F \in \mathcal{F}(K)} \| \nabla(I_{K}^{BDM} \mathbf{v}) \|_{0,F}^2. \]

Finally, we end the proof by employing the inverse inequality, the trace inequality, and the error estimates of \( I_{K}^{BDM} \) and \( Q_K \).

Next we present the discrete inf-sup condition.

**Lemma 4.3.** It holds the discrete inf-sup condition

\[ (4.17) \quad \|
\]
\[ b_h(v_h, q_h) \|_Q = \| v_h \|_0 + i \| q_h \|_1 \lesssim \sup_{\| v_h \|_{V,h}} \frac{b_h(v_h, q_h)}{\| v_h \|_{V,h}} \quad \forall q_h \in Q_h. \]

**Proof.** By Lemma 2.1, there exists \( \mathbf{v} \in H_0^1(\Omega; \mathbb{R}^d) \) satisfying \( \text{div} \ \mathbf{v} = q_h \) and \( \| \mathbf{v} \|_1 + i \| \mathbf{v} \|_2 \lesssim \| q_h \|_0 + i \| q_h \|_1 \). Take \( \mathbf{v}_h = I_h \mathbf{v} \in H_0^1(\Omega; \mathbb{R}^d) \). By (4.15),
\[ (4.18) \quad \| \mathbf{v}_h \|_1 + i \| \mathbf{v}_h \|_2 \lesssim \| q_h \|_0 + i \| q_h \|_1. \]

For \( K \in \mathcal{T}_h \), by (4.12) and \( \text{div} \ \mathbf{v} \in Q_h \), we have \( \text{div}(I_{K}^{BDM} \mathbf{v}) = Q_K(\text{div} \ \mathbf{v}) = \text{div} \ \mathbf{v} \).

Then we get from (4.13) and (4.14) that
\[ (\text{div}(v_h - v), q_h) = -(v_h - v, \nabla q_h) = 0, \]
\[ (\nabla_h \text{div}(v_h - v), \nabla q_h) = \sum_{K \in \mathcal{T}_h} (\text{div}(v_h - v), \partial_n q_h)_{\partial K} = 0. \]

Combining the last two equations gives
\[ b_h(v_h, q_h) = (\text{div} \ \mathbf{v}, q_h) + i^2 (\nabla \text{div} \ \mathbf{v}, \nabla q_h) = \| q_h \|_0^2 + i^2 \| q_h \|_1^2, \]
which together with (4.18) ends the proof.

**Theorem 4.4.** It holds the discrete stability

\[ (4.19) \quad \| \mathbf{u}_h \|_{V,h} + \| \mathbf{p}_h \|_Q \lesssim \sup_{\mathbf{u}_h, \mathbf{p}_h} \frac{a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(v_h, p_h) - b_h(\mathbf{u}_h, q_h) + c(\mathbf{p}_h, q_h)}{\| \mathbf{v}_h \|_{V,h} + \| q_h \|_Q} \]

for \( \mathbf{u}_h \in V_h \) and \( \mathbf{p}_h \in Q_h \). Then the mixed finite element method (4.8) is well-posed.
Proof. Applying the Babuška-Brezzi theory [10], we get from the discrete coercivity (4.9) and the discrete inf-sup condition (4.17) that
\[
\|\tilde{u}_h\|_{V_h} + \|\tilde{p}_h\|_{Q_h} \lesssim \sup_{v_h \in V_h, q_h \in Q_h} \frac{a_h(\tilde{u}_h, v_h) + b_h(v_h, \tilde{p}_h) - b_h(\tilde{u}_h, q_h)}{\|v_h\|_{V_h} + \|q_h\|_{Q_h}}
\]
for \(\tilde{u}_h \in V_h\) and \(\tilde{p}_h \in Q_h\), which indicates (4.19). □

5. Error analysis

We will analyze the mixed finite element method (4.8) in this section, and derive robust error estimates of \(\|u - u_h\|_{V_h} + \|p - p_h\|_{Q_h}\) with respect to parameters \(\iota\) and \(\lambda\).

5.1. Interpolation estimates. Denote by \(I_h^{SZ} : H^1_0(\Omega) \to Q_h^1 \cap H^1_0(\Omega)\) the Scott-Zhang interpolation operator with the homogeneous boundary condition [47]. Since \(I_h^{SZ} v \notin Q_h\) for \(I_h^{SZ} v \notin L^2_0(\Omega)\), we will modify \(I_h^{SZ} v\).

For \(K \in T_h\), let \(N_K := \#V^i(K)\) be the number of interior vertices of \(K\). We assume the triangulation \(T_h\) satisfies \(\min_{K \in T_h} N_K \geq 1\). Define operator \(Q_h : L^2(\Omega) \to Q_h^1 \cap H^1_0(\Omega)\) by
\[
(Q_h v)(\delta) = \sum_{K \in T_h} \frac{d+1}{N_K |\omega_\delta|} \int_K v \, dx \quad \forall \, \delta \in V^i(T_h),
\]
where \(|\omega_\delta|\) is the geometrical measure of \(\omega_\delta\).

Lemma 5.1. For \(v \in L^2(\Omega)\), we have \(v - Q_h v \in L^2_0(\Omega)\), and
\[
\|Q_h v\|_{0,K} + h_K |Q_h v|_{1,K} \lesssim \sum_{\delta \in V^i(K)} \|v\|_{0,\omega_\delta}.
\]

Proof. By the fact \((Q_h v)|_K \in P_1(K)\) for \(K \in T_h\),
\[
\int_{\Omega} Q_h v \, dx = \sum_{K \in T_h} \int_K Q_h v \, dx = \frac{1}{d+1} \sum_{K} \sum_{\delta \in V^i(K)} |K|(Q_h v)(\delta) = \sum_{\delta \in V^i(T_h)} \sum_{K \in T_h} \sum_{K'} \frac{|K|}{N_K' |\omega_\delta|} \int_{K'} v \, dx
\]
\[
= \sum_{\delta \in V^i(T_h)} \sum_{K' \in T_h} \frac{1}{N_{K'}} \int_{K'} v \, dx = \int_{\Omega} v \, dx.
\]
On the other side, by the scaling argument,
\[
\|Q_h v\|_{0,K} \lesssim h^{d/2}_K \sum_{\delta \in V^i(K)} |(Q_h v)(\delta)| \lesssim \sum_{\delta \in V^i(K)} \|v\|_{0,\omega_\delta},
\]
which together with the inverse inequality produces (5.1). □

Define \(J_h : H^1_0(\Omega) \to Q_h^1 \cap H^1_0(\Omega)\) by \(J_h v := I_h^{SZ} v + Q_h(v - I_h^{SZ} v)\). Since \(\int_{\Omega} J_h v \, dx = \int_{\Omega} v \, dx\), it holds that \(J_h v \in Q_h\) when \(v \in Q\).

Lemma 5.2. We have
\[
h^i |v - J_h v|_i \lesssim h^j |v|_j \quad \forall \, v \in H^1_0(\Omega) \cap H^i(\Omega), \, i \leq j \leq 2, \, i = 0, 1.
\]
Proof. For each $K \in \mathcal{T}_h$, by (5.1) and $v - J_h v = v - I_h^{SZ} v - Q_h (v - I_h^{SZ} v)$, 

$$h_i^k |v - J_h v|_{i,K} \lesssim h_i^k |v - I_h^{SZ} v|_{i,K} + \sum_{\delta \in \mathcal{V}(K)} \|v - I_h^{SZ} v\|_{0,\omega_\delta}.$$ 

Therefore, we acquire (5.2) from the estimate of $I_h^{SZ}$. \hfill \Box

To derive robust error estimates of $\|u - u_h\|_{V,h} + \|p - p_h\|_Q$ with respect to parameters $\iota$ and $\lambda$, we introduce a finite element space 

$$\tilde{V}_h := \{ v_h \in L^2(\Omega; \mathbb{R}^d) | v_h |_K \in V(K) \text{ for each } K \in \mathcal{T}_h, \text{ all the DoFs (4.2)-(4.6) are single-valued, and DoFs (4.2)-(4.3) on boundary vanish}\}.$$

Define an interpolation operator $\tilde{I}_h : H^1_0(\Omega; \mathbb{R}^d) \rightarrow \tilde{V}_h$ as follows:

$$(\tilde{I}_h v)(\delta) = \frac{1}{\# \mathcal{T}_h} \sum_{K \in \mathcal{T}_h} (I_K^{BDM} v)(\delta),$$

$$(\tilde{I}_h v, q)_e = \frac{1}{\# \mathcal{T}_e} \sum_{K \in \mathcal{T}_e} (I_K^{BDM} v, q)_e \quad \forall q \in P_0(e; \mathbb{R}^d),$$

$$(\partial_n(\Pi_F(\tilde{I}_h v)), q)_F = \frac{1}{\# \mathcal{T}_F} \sum_{K \in \mathcal{T}_F} (\partial_n(\Pi_F(I_K^{BDM} v)), q)_F \quad \forall q \in P_0(F; \mathbb{R}^{d-1}),$$

$$(\text{div}(\tilde{I}_h v), q)_F = \frac{1}{\# \mathcal{T}_F} \sum_{K \in \mathcal{T}_F} (\text{div}(I_K^{BDM} v), q)_F \quad \forall q \in P_0(F),$$

$$(\tilde{I}_h v, q)_K = (v, q)_K \quad \forall q \in P_0(K; \mathbb{R}^d),$$

for $\delta \in \mathcal{V}(\mathcal{T}_h)$, $e \in \mathcal{E}(\mathcal{T}_h)$, $F \in \mathcal{F}(\mathcal{T}_h)$ and $K \in \mathcal{T}_h$.

Lemma 5.3. We have 

$$|\tilde{I}_h v|_1 \lesssim |v|_1 \quad \forall v \in H^1_0(\Omega; \mathbb{R}^d),$$

(5.3)

$$\sum_{i=1}^3 h_i^k |v - \tilde{I}_h v|_{i,K} \lesssim h_i^k \sum_{\delta \in \mathcal{V}(K)} |v|_{j,\omega_\delta} \quad \forall v \in H^1_0(\Omega; \mathbb{R}^d) \cap H^j(\Omega; \mathbb{R}^d), j = 2, 3.$$

(5.4)

Proof. Applying the similar argument as Lemma 4.2, we get for $i = 1, 2$ that 

$$h_i^k |\tilde{I}_h v - I_K^{BDM} v|_{i,K}^2 \lesssim h_i^k \sum_{\delta \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}(\mathcal{T}_h) \delta \in F} \|I_K^{BDM} v\|_{0,F}^2$$

$$+ h_i^3 \sum_{F \in \mathcal{F}^i(K)} \|\partial_n(I_K^{BDM} v)\|_{0,F}^2 + h_i^3 \sum_{F \in \mathcal{F}^i(K)} \|Q_K(\text{div} v)\|_{0,F}^2.$$

(5.5)

Employing the inverse inequality, the trace inequality, the error estimates of $I_K^{BDM}$ and $Q_K$,

$$|\tilde{I}_h v - I_K^{BDM} v|_{1,K} \lesssim \sum_{\delta \in \mathcal{V}(K)} |v|_{1,\omega_\delta},$$

which combined with the $H^1$ boundedness of $I_K^{BDM}$ yields (5.3).
On the other hand, by (5.5), we get
\[ h_K^2 |v - \tilde{I}_h u|^2_{1,K} \lesssim h_K^2 |v - I_h^{BDM} u|^2_{1,K} + h_K \sum_{\delta \in \mathcal{V}(K)} \sum_{F \in \mathcal{F}(\delta)} \left\| I_K^{BDM} v \right\|^2_{0,F} \]
\[ + h_K^3 \sum_{F \in \mathcal{F}(K)} \left\| \partial_n (I_K^{BDM} v) \right\|^2_{0,F} + h_K^3 \sum_{F \in \mathcal{F}(K)} \left\| Q_K (\text{div} \, v) \right\|^2_{0,F}. \]

Then (5.4) follows from the inverse inequality, the trace inequality, and the error estimates of $I_K^{BDM}$ and $Q_K$.

Lemma 5.4. We have
\[ |u - \tilde{I}_h u|_1 + \varepsilon |u - \tilde{I}_h u|_{2,h} \lesssim h^{1/2} \| f \|_0. \]  

Proof. By (5.3)-(5.4) and (3.16)-(3.17),
\[ |u - u_0 - \tilde{I}_h (u - u_0)|_1 \lesssim h^{1/2} |u - u_0|_{1/2} |u|_{1/2} \lesssim h^{3/2} \| f \|_0. \]

And it follows from (5.4) and (3.16) that
\[ |u_0 - \tilde{I}_h u_0|_1 \lesssim h |u_0 - u_0|_2 \lesssim h^{1/2} \| f \|_0. \]

Combining the last two inequalities gives
\[ |u - \tilde{I}_h u|_1 \lesssim h^{1/2} \| f \|_0. \]

According to (5.4) and (3.17), we have
\[ |u - \tilde{I}_h u|_{2,h} \lesssim h^{1/2} |u|_{1/2} |u|_{1/2} \lesssim \varepsilon^{-1} h^{1/2} \| f \|_0. \]

Therefore, (5.6) holds from the last two inequalities.

Lemma 5.5. We have
\[ |u - I_h u|_1 + \varepsilon |u - I_h u|_{2,h} \lesssim h^{1/2} \| f \|_0, \]  
\[ \| p - J_h p \|_0 + \varepsilon \| p - J_h p \|_1 \lesssim h^{1/2} \| f \|_0. \]  

Proof. We only prove (5.7), since the proof of (5.8) is similar. By the definitions of $I_h$ and $\tilde{I}_h$, we get from the multiplicative trace inequality, the estimate of $I_h^{BDM}$ and (3.16)-(3.17) that
\[ |\tilde{I}_h u - I_h u|_1^2 \lesssim \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F \| \nabla (u - I_h^{BDM} u) \|^2_{0,F} \]
\[ \lesssim \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F \| \nabla (u - u_0 - I_h^{BDM} (u - u_0)) \|^2_{0,F} + \| \nabla (u_0 - I_h^{BDM} u_0) \|^2_{0,F} \]
\[ \lesssim h |u - u_0|_1 |u - u_0|_2 + h^2 |u_0|_2^2 \lesssim h \| f \|_0^2. \]

Similarly, we have
\[ |\tilde{I}_h u - I_h u|_{2,h}^2 \lesssim \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F^{-1} \| \nabla (u - I_h^{BDM} u) \|^2_{0,F} \lesssim h |u|_2 |u|_3 \lesssim \varepsilon^{-2} h \| f \|_0^2. \]

Hence
\[ |\tilde{I}_h u - I_h u|_1 + \varepsilon |\tilde{I}_h u - I_h u|_{2,h} \lesssim h^{1/2} \| f \|_0. \]

Finally, (5.7) holds from (5.6) and the last inequality.
5.2. **Error estimates.** Similar to [41], we have the following abstract error estimates.

**Lemma 5.6.** Let \((u, p) \in V \times Q\) and \((u_h, p_h) \in V_h \times Q_h\) be the solution of problem (1.2) and problem (4.8), respectively. Then
\[
\|u - u_h\|_{V_h} + \|p - p_h\|_Q \lesssim \inf_{u_i \in V_h} \|u - u_i\|_{V_h} + \inf_{p_l \in Q_h} \|p - p_l\|_Q + E_h,
\]
where
\[
E_h = \sup_{v_h \in V_h} \frac{(f, v_h) - a_h(u, v_h) - b_h(v_h, p)}{\|v_h\|_{V_h}}.
\]

**Proof.** Let \(\tilde{v}_h = u_h - u\) and \(\tilde{p}_h = p_h - p\). It follows from the mixed finite element method (4.8) and the second equation of (2.3) that
\[
a_h(\tilde{u}_h, v_h) + b_h(v_h, \tilde{p}_h) - b_h(\tilde{u}_h, q_h) + c(\tilde{p}_h, q_h)
= (f, v_h) - a_h(u, v_h) + b_h(u, \tilde{v}_h, p) + b_h(u_1, v_h) - c(p_1, q_h)
\]
\[
= (f, v_h) - a_h(u, v_h) - b_h(v_h, p) + a_h(u - u_1, v_h) + b_h(v_h, p - p_1)
- b_h(u - u_1, q_h) + c(p - p_1, q_h),
\]
which together with the discrete stability (4.19) ends the proof.

**Lemma 5.7.** We have
\[
E_h \lesssim ch(|u|_3 + \lambda |\text{div} u|_2),
\]
\[
E_h \lesssim h^{1/2}||f||_0.
\]

**Proof.** For \(v_h \in V_h \subset H^1_0(\Omega; \mathbb{R}^d)\), apply integration by parts to the first equation in problem (1.1) to acquire
\[
(f, v_h) - a_h(u, v_h) - b_h(v_h, p)
= (\sigma(u), \varepsilon(v_h)) - i^2(\Delta \sigma(u), \varepsilon(v_h)) - a_h(u_1, v_h) - b_h(v_h, p)
\]
\[
= -i^2 \sum_{K \in T_h} \int_{\partial K} \nabla \sigma(u) : \varepsilon(v_h) dS.
\]
Let \(Q^0_F\) be the standard \(L^2\)-projection from \(L^2(F)\) to \(\mathbb{P}_0(F)\), whose tensorial version is also denoted by \(Q^0_F\). By the weak continuity (4.7),
\[
(f, v_h) - a_h(u, v_h) - b_h(v_h, p)
= -i^2 \sum_{K \in T_h} \sum_{F \in \mathcal{F}(K)} \int_F (\nabla \sigma(u) - Q^0_F \nabla \varepsilon(v_h)) : (\varepsilon(v_h) - Q^0_F \varepsilon(v_h)) dS.
\]
From the error estimate of \(Q^0_F\) and the trace inequality, we acquire
\[
(f, v_h) - a_h(u, v_h) - b_h(v_h, p) \lesssim i^2 h_0(|u|_3 + \lambda |\text{div} u|_2) |v_h|_{2,h},
\]
\[
(f, v_h) - a_h(u, v_h) - b_h(v_h, p) \lesssim i^2 h^{1/2}(|u|_2 + \lambda |\text{div} u|_1) \sqrt{h}(|u|_3 + \lambda |\text{div} u|_2) |v_h|_{2,h}.
\]
Hence
\[
E_h \lesssim ch(|u|_3 + \lambda |\text{div} u|_2), \quad E_h \lesssim ch^{1/2}(|u|_2 + \lambda |\text{div} u|_1) \sqrt{h}(|u|_3 + \lambda |\text{div} u|_2).
\]
Therefore, we finish the proof by using (3.17) and (3.21).
Theorem 5.8. Let \((u, p) \in (V \cap H^3(\Omega; \mathbb{R}^d)) \times (Q \cap H^2(\Omega))\) and \((uh, ph) \in V_h \times Q_h\) be the solution of problem (1.2) and (4.8), respectively. We have
\[
\|u - uh\|_{V,h} + \|p - ph\|_Q \lesssim (h^2 + \theta h)|u|_3 + \lambda(h^2 + \theta h)|\text{div } u|_2.
\]
Proof. Let \(u_I = I_h u \in V_h\), and \(p_I = J_h p \in Q_h\) in Lemma 5.6. Then by (4.16) and (5.2),
\[
\|u - u_h\|_{V,h} + \|p - ph\|_Q \lesssim (h^2 + \theta h)|u|_3 + \lambda(h^2 + \theta h)|\text{div } u|_2 + E_h.
\]
Therefore (5.11) follows from (5.9).

Theorem 5.9. With the same assumption of Theorem 5.8, we have
\[
\|u - u_h\|_{V,h} + \|p - ph\|_Q \lesssim h^{1/2}\|f\|_0,
\]
\[
\|u_0 - u_h\|_{V,h} + \|p_0 - ph\|_Q \lesssim (\theta/2 + h^{1/2})\|f\|_0.
\]
Proof. The estimate (5.12) follows from Lemma 5.6, (5.7), (5.8) and (5.10). And estimate (5.13) holds from (5.12), (3.16)-(3.17) and (3.21).

6. NUMERICAL EXPERIMENTS

In this section, we present two numerical examples using the mixed FEM constructed in Sect 4.1 toward testifying its uniform convergence and robustness with respect to \(\lambda\) and \(\theta\) on a uniform triangulation.

To make the proposed method more accessible, we first present the mixed finite element spaces in two dimensions in detail. Let \(\mathcal{T}_h\) be a shape regular triangulation. The finite element spaces in two dimensions are
\[
V_h = \{v_h \in H^1_0(\Omega; \mathbb{R}^d)\mid v_h|_K \in V(K)\text{ for each } K \in \mathcal{T}_h, \text{ all the DoFs (6.1)-(6.4)}\}
\]
\[
Q_h = \{q_h \in H^1(\Omega) \cap L_0^2(\Omega)\mid q_h|_K \in P_1(K)\text{ for each } K \in \mathcal{T}_h\},
\]
where \(V(K) = P_2(K; \mathbb{R}^2) + b_K P_1(K; \mathbb{R}^2) + b_K \delta P_0(K; \mathbb{R}^2)\). The associated DoFs for \(V_h\) are given as
\[
v_h(\delta) \quad \forall \ \delta \in V'(\mathcal{T}_h),
\]
\[
v_h(e) \quad \forall \ e \in E'(\mathcal{T}_h),
\]
\[
1/|e| \int_e \partial_n v_h \text{ ds} \quad \forall \ e \in E'(\mathcal{T}_h),
\]
\[
1/|K| \int_K v_h \text{ dx} \quad \forall \ K \in \mathcal{T}_h,
\]
where \(m_e\) is the midpoint of \(e\). The DoFs for \(Q_h\) are all the function values at interior vertices. Note that the DoFs (6.1)-(6.3) on boundary vanish. The other point to be emphasized is that we here substitute DoFs (4.4)-(4.5) by the moments of the normal derivatives (6.3) owing to (4.1) in order to simplify computation. Also, we use the function values of midpoints on edges (6.2) instead of (4.3). With such substitution, the global finite element space remains the same.

Example 6.1. Let \(\Omega = (0,1)^2\). We choose the right side function \(f\) such that the exact solution of SGE model (1.1) is
\[
u = \begin{bmatrix}
3(c^2 - c) 2 \sin(2\pi x_1) \sin(\pi x_2) \\
8(c^2 - c) 2 \sin(2\pi x_1) \sin^3(\pi x_2)
\end{bmatrix}.
\]
Moreover, set $\mu = 1$. It is easy to check that $u$ is divergence-free without boundary layer, and hence $f$ is independent of the Lamé constant $\lambda$. The purpose of this example is to demonstrate the error estimate given in Theorem 5.8.

We measure the relative error by

$$E_u := \frac{\|u - u_h\|_{V,h}}{\|f\|_0}.$$ 

The numerical results of $E_u$ are displayed in Table 1 with different values of Lamé constant $\lambda$, size parameter $\iota$ and mesh size $h$. We may observe from Table 1 that when $\iota$ takes the value 1 and $1e - 01$, the relative error is linearly convergent and when $\iota = 1e - 08$, it is quadratically convergent. In particular, Fig. 2 performs the relative error curves for different $\iota$ with fixed $\lambda = 1e + 08$. In this situation, if $\iota = 1e - 08$, $E_u$ behaves like $O(h^2)$. If $\iota = 1e - 01$ or $1e - 00$, $E_u = O(h)$. Hence, we may conclude that the convergence order of the error is quadratic as $\iota \ll h$. All these results are in good agreement with the error bound given in Theorem 5.8.

Example 6.2. Let $\Omega = (0, 1)^2$. The exact solution of the reduced problem (3.15) is set to be a divergence-free function in the form

$$u_0 = \begin{bmatrix} -x_1^2(1-x_1)^2x_2(1-x_2)(1-2x_2) \\ x_1(1-x_1)(1-2x_1)x_2^2(1-x_2)^2 \end{bmatrix}.$$
Table 2. The performance of Example 6.2.

| λ   | ι      | 1/16   | 1/32   | 1/64   | 1/128  | 1/256  | rate |
|-----|--------|--------|--------|--------|--------|--------|------|
|     | 1e-04  |        |        |        |        |        |      |
| 1e+00| 1e-06  | 2.052e-02 | 1.445e-02 | 1.020e-02 | 7.206e-03 | 5.093e-03 | 0.50 |
|     | 1e-08  | 2.052e-02 | 1.445e-02 | 1.020e-02 | 7.206e-03 | 5.093e-03 | 0.50 |
|     | 1e+04  |        |        |        |        |        |      |
| 1e+04| 1e-06  | 2.053e-02 | 1.446e-02 | 1.020e-02 | 7.207e-03 | 5.094e-03 | 0.50 |
|     | 1e-08  | 2.053e-02 | 1.446e-02 | 1.020e-02 | 7.207e-03 | 5.094e-03 | 0.50 |

The right side term \( f \) computed from (3.15) is independent of both \( \lambda \) and \( \iota \). We use this \( f \) as the right side function of problem (1.1). Since \( \partial_n u_0|_{\partial \Omega} \neq 0 \), the function \( u \) has a strong boundary layer when \( \iota \) is sufficient small. We still set \( \mu = 1 \). We focus on investigating the robustness of our numerical method with respect to both \( \lambda \) and \( \iota \) in this example.

When \( \iota \ll h \), it follows from (3.16)-(3.17) and (3.21) that
\[
\|u - u_0\|_V \lesssim \iota^{1/2} \|f\|_0 \lesssim h^{1/2} \|f\|_0,
\]
which combined with (5.13) immediately implies (5.12). So we turn to verify the estimate (5.13), which depends on \( u_0 \) instead of the unknown function \( u \). Let \( E_{u_0} = \|u_h - u_0\|_{V,h} / \|f\|_0 \). We compute its values for different values of Lamé constant \( \lambda \) and the mesh size \( h \) in Table 2. We may observe that \( E_{u_0} = O(h^{0.43}) \) with \( \iota = 1e-04 \) and \( E_{u_0} = O(h^{1/2}) \) with \( \iota = 1e-06 \) or \( 1e-08 \) no matter what value \( \lambda \) takes. We can see that the best convergence order is really \( 1/2 \) when \( \iota \ll h \), which is consistent with the estimate (5.13) and shows the robustness of the mixed finite element method with respect to both the size parameter \( \iota \) and Lamé coefficient \( \lambda \).

Acknowledgments

The second author would like to thank Prof. M. Dauge from Université de Rennes 1 for helpful discussion about elliptic boundary value problems in domains with corners and bringing his attention to the monograph [29]. The authors are also indebted to the referees for valuable comments which improved an earlier version of the paper.

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