Accelerated Variance Reduced Stochastic ADMM

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Abstract

Recently, many variance reduced stochastic alternating direction method of multipliers (ADMM) methods (e.g. SAG-ADMM, SDCA-ADMM and SVRG-ADMM) have made exciting progress such as linear convergence rates for strongly convex problems. However, the best known convergence rate for general convex problems is $O(1/T)$ as opposed to $O(1/T^2)$ of accelerated batch algorithms, where $T$ is the number of iterations. Thus, there still remains a gap in convergence rates between existing stochastic ADMM and batch algorithms. To bridge this gap, we introduce the momentum acceleration trick for batch optimization into the stochastic variance reduced gradient based ADMM (SVRG-ADMM), which leads to an accelerated (ASVRG-ADMM) method. Then we design two different momentum term update rules for strongly convex and general convex cases. We prove that ASVRG-ADMM converges linearly for strongly convex problems, respectively, as compared to sub-linear rates of SGD. More recently, the Nesterov’s acceleration technique (Nesterov 2004) was introduced in (Allen-Zhu 2016; Hien et al. 2016) to further speed up the stochastic variance-reduced algorithms, which results in the best known convergence rates for both strongly convex and general convex problems. This motivates us to integrate the momentum acceleration trick into the stochastic alternating direction method of multipliers (ADMM) below.

When $A$ is a more general matrix, i.e. $A \neq I_{d_1}$, the formulation (1) becomes many more complicated problems arising from machine learning, e.g. graph-guided fused Lasso (Kim, Sohn, and Xing 2009) and generalized Lasso (Tibshirani and Taylor 2011). To solve this class of composite optimization problems with an auxiliary variable $y = Ax$, which are the special case of the general ADMM form,

$$\min_{x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^{d_2}} f(x) + h(y), \text{ s.t. } Ax + By = c,$$

the ADMM is an effective optimization tool (Boyd et al. 2011), and has shown attractive performance in a wide range of real-world problems, such as big data classification (Nie et al. 2014). To tackle the issue of high per-iteration complexity of batch (deterministic) ADMM (as a popular first-order optimization method), Wang and Banerjee (2012), Suzuki (2013) and Ouyang et al. (2013) proposed some online or stochastic ADMM algorithms. However, all these variants only achieve the convergence rate of $O(1/\sqrt{T})$ for general convex problems and $O(\log T/T)$ for strongly convex problems, respectively, as compared with the $O(1/T^2)$ and linear convergence rates of accelerated batch algorithms (Nesterov 1983), e.g. FISTA, where $T$ is the number of iterations. By now several accelerated and faster converging versions of stochastic ADMM, which are all based on variance reduction techniques, have been proposed, e.g. SAG-ADMM (Zhong and Kwok 2014b), SDCA-ADMM (Suzuki 2014) and SVRG-ADMM (Zheng and Kwok 2016). With regard to strongly convex problems,
Table 1: Comparison of convergence rates and memory requirements of some stochastic ADMM algorithms.

| Algorithm     | General convex | Strongly-convex | Space requirement |
|---------------|----------------|-----------------|-------------------|
| SAG-ADMM      | $O(1/T)$       | unknown         | $O(d_dz + nd_d)$   |
| SDCA-ADMM     | unknown        | linear rate     | $O(d_dz + n)$     |
| SCAS-ADMM     | $O(1/T)$       | $O(1/T)$        | $O(d_dz)$         |
| SVRG-ADMM     | $O(1/T)$       | linear rate     | $O(d_dz)$         |
| ASVRG-ADMM    | $O(1/T^2)$     | linear rate     | $O(d_dz)$         |

Suzuki (2014) and Zheng and Kwok (2016) proved that linear convergence can be obtained for the special ADMM form (i.e. $B = -I_d$ and $c = 0$) and the general ADMM form, respectively. In SAG-ADMM and SVRG-ADMM, an $O(1/T)$ convergence rate can be guaranteed for general convex problems, which implies that there still remains a gap in convergence rates between the stochastic ADMM and accelerated batch algorithms.

We also prove that ASVRG-ADMM has a convergence rate of $O(1/T^2)$ for non-strongly convex problems, which is a factor of $T$ faster than SAG-ADMM and SVRG-ADMM, whose convergence rates are $O(1/T)$.

Our experimental results further verified that our ASVRG-ADMM method has much better performance than the state-of-the-art stochastic ADMM methods.

Related Work

Introducing $y = Ax \in \mathbb{R}^{d_l}$, problem (1) becomes

$$\min_{x \in \mathbb{R}^{d_l}, y \in \mathbb{R}^{d_2}} f(x) + h(y), \text{ s.t. } Ax - y = 0. \tag{3}$$

Although (3) is only a special case of the general ADMM form (2), when $B = -I_d$ and $c = 0$, the stochastic (or online) ADMM algorithms and theoretical results in (Wang and Banerjee 2012; Ouyang et al. 2013; Zhong and Kwok 2014b; Zhong and Kwok 2016) and this paper are all for the more general problem (2). To minimize (2), together with the dual variable $\lambda$, the update steps of batch ADMM are

$$y_k = \arg \min_y h(y) + \frac{\beta}{2} \|Ax_{k-1} + By - c - \lambda_{k-1}\|^2, \tag{4}$$

$$x_k = \arg \min_x f(x) + \frac{\beta}{2} \|Ax + By - c - \lambda_{k-1}\|^2, \tag{5}$$

$$\lambda_k = \lambda_{k-1} + Ax_k + By_k - c, \tag{6}$$

where $\beta > 0$ is a penalty parameter.

To extend the batch ADMM to the online and stochastic settings, the update steps for $y_k$ and $\lambda_k$ remain unchanged. In (Wang and Banerjee 2012; Ouyang et al. 2013), the update step of $x_k$ is approximated as follows:

$$x_k = \arg \min_x x^T \nabla f_{\tilde{x}}(x_{k-1}) + \frac{1}{2\eta_k} \|x - x_{k-1}\|_G^2 + \frac{\beta}{2} \|Ax + By_k - c - \lambda_{k-1}\|^2, \tag{7}$$

where we draw $i_k$ uniformly at random from $[n] := \{1, \ldots, n\}$, $\eta_k \propto 1/\sqrt{k}$ is the step-size, and $\|z\|_G^2 = z^T G z$ with given positive semi-definite matrix $G$, e.g. $G = I_{d_k}$ in (Ouyang et al. 2013). Analogous to SGD, the stochastic ADMM variants use an unbiased estimate of the gradient at each iteration. However, all those algorithms have much slower convergence rates than their batch counterpart, as mentioned above. This barrier is mainly due to the variance introduced by the stochasticity of the gradients. Besides, to guarantee convergence, they employ a decaying sequence of step sizes $\eta_k$, which in turn impacts the rates.

More recently, a number of variance reduced stochastic ADMM methods (e.g. SAG-ADMM, SDCA-ADMM and SVRG-ADMM) have been proposed and made exciting progress such as linear convergence rates. SVRG-ADMM in (Zheng and Kwok 2016) is particularly attractive here because of its low storage requirement compared with the algorithms in (Zheng and Kwok 2014a, 2014b; Suzuki 2014). Within each epoch of SVRG-ADMM, the full gradient $p = \nabla f(\bar{x})$ is first computed, where $\bar{x}$ is the average point of the previous epoch. Then $\nabla f_{i_k}(x_{k-1})$ and $\eta_k$ in (7) are replaced by

$$\nabla f_{i_k}(x_{k-1}) = \frac{1}{|I_k|} \sum_{i_k \in I_k} (\nabla f_{i_k}(x_{k-1}) - \nabla f_{i_k}(\bar{x})) + \bar{p} \tag{8}$$

and a constant step-size $\eta$, respectively, where $I_k \subset [n]$ is a mini-batch of size $b$ (which is a useful technique to reduce the variance). In fact, $\nabla f_{I_k}(x_{k-1})$ is an unbiased estimator of the gradient $\nabla f(x_{k-1})$, i.e. $\mathbb{E}[\nabla f_{i_k}(x_{k-1})] = \nabla f(x_{k-1})$.

Accelerated Variance Reduced Stochastic ADMM

In this section, we design an accelerated variance reduced stochastic ADMM method for both strongly convex and general convex problems. We first make the following assumptions: Each convex $f_i(\cdot)$ is $L_i$-smooth, i.e. there exists a constant $L_i > 0$ such that $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L_i \|x - y\|$, $\forall x, y \in \mathbb{R}^{d_i}$, and $L = \max_i L_i$; $f(\cdot)$ is $\mu$-strongly convex, i.e. there is $\mu > 0$ such that $f(x) \geq f(y) + \nabla f(y)^T(x - y) + \frac{\mu}{2} \|x - y\|^2$ for all $x, y \in \mathbb{R}^{d_i}$; The matrix $A$ has full row rank. The first two assumptions are common in the analysis of first-order optimization methods, while the last one has
Algorithm 1 ASVRG-ADMM for strongly-convex case

Input: $m, \eta, \beta > 0, 0 \leq b \leq n$.
Initialize: $\tilde{x}_0 = \tilde{y}_0 = \tilde{s}_0 = 0, \theta_0 = -\frac{1}{2}(A^T)^T\nabla f(\tilde{x}_0)$. 
1: for $s = 1, 2, \ldots, T$ do
2: \[ x^s_0 = \tilde{x}^s_0, \quad y^s_0 = \tilde{y}^s_1, \quad z^s_0 = \tilde{s}^s_1; \]
3: \[ p = \tilde{V}(\tilde{x}^s_0) \]
4: for $k = 1, 2, \ldots, m$ do
5: Choose $I_k \subseteq [n]$ of size $b$, uniformly at random;
6: \[ y^s_k = \arg\min_{y_k} h(y_k) + \frac{\beta}{2}\|A z^s_k + By_k - c + \lambda^s_{k-1}\|^2 \]
7: \[ z^s_k = z^s_{k-1} - \eta(\tilde{V}_I(x^s_{k-1}) + \beta A^T(A z^s_{k-1} + By^s_k - c + \lambda^s_{k-1})) \]
8: \[ x^s_k = (1 - \theta)\tilde{x}^s_{k-1} + \theta z^s_k \]
9: \[ \lambda^s_k = \lambda^s_{k-1} + \beta A z^s_k + B y^s_k - c \]
10: end for
11: \[ \tilde{x}^s = \frac{1}{m}\sum_{k=1}^{m} x^s_k, \quad \tilde{y}^s = (1 - \theta)\tilde{y}^s_{1-1} + \frac{2}{m}\sum_{k=1}^{m} y^s_k \]
12: \[ \lambda^s = \frac{1}{2}(A^T)^T\nabla f(\tilde{x}^s) \]
13: end for
Output: $\tilde{x}^T, \tilde{y}^T$.

been used in the convergence analysis of batch ADMM (Nishihara et al. 2015, Deng and Yin 2016) and stochastic ADMM (Zheng and Kwok 2016).

The Strongly Convex Case

In this part, we consider the case of (2) when each $f_i(\cdot)$ is convex, $L$-smooth, and $h(\cdot)$ is a strongly-convex function. Recall that this class of problems include gradient descent and SVM as well as many other important examples of strongly convex problems. To efficiently solve this class of problems, we incorporate both the momentum acceleration and variance reduction techniques into stochastic ADMM. Our algorithm is divided into $T$ epochs, and each epoch consists of $m$ stochastic updates, where $m$ is usually chosen to be $O(n)$ as in (Johnson and Zhang 2013).

Let $z$ be an important auxiliary variable, its update rule is given as follows. Similar to (Zheng and Kwok 2014, Deng and Yin 2016), we also use the inexact Uzawa method (Zhang, Burger, and Osher 2011) to approximate the sub-problem (7), which can be used to stop the momentum weight $0 < \theta_s < 1$ (the update rule for $\theta_s$ is provided below) is introduced into the proximal term $\frac{1}{2\eta}\|x - x_{k-1}\|^2_G$ similar to that of (7), and then the sub-problem with respect to $z$ is formulated as follows:

\[
\min_z \left( z - z^s_{k-1} \right)^T \tilde{V} f_I(x^s_{k-1}) + \frac{\theta_s}{2\eta} \|z - z^s_{k-1}\|^2_G + \frac{\beta}{2} \|A z + B y^s_k - c + \lambda^s_{k-1}\|^2, \tag{9}
\]

where $\tilde{V} f_I(x^s_{k-1})$ is defined in (8), $\eta < \frac{1}{2\gamma}$, and $G = \gamma I_d - \frac{\eta^2}{\theta_s} A^T A$ with $\gamma \geq \gamma_{\min} = \frac{\|A^T A\|_{\infty}}{\theta_s} + 1$ to ensure that $G \succeq I$ similar to (Zheng and Kwok 2016), where $\|.\|_2$ is the spectral norm, i.e. the largest singular value of the matrix. Furthermore, the update rule for $\theta_s$ is given by:

\[
x^s_k = \tilde{x}^s_{k-1} + \theta_s(z^s_k - \tilde{x}^s_{k-1}) = (1 - \theta_s) \tilde{x}^s_{k-1} + \theta_s z^s_k, \quad \theta_s = 1 - \frac{1 - \theta_s}{\theta_s}, \tag{10}
\]

where $\theta_s(z^s_k - \tilde{x}^s_{k-1})$ is the key momentum term (similar to those in accelerated batch methods (Nesterov 2004)), which helps accelerate our algorithm by using the iterate of the previous epoch, i.e. $\tilde{x}^s_{k-1}$. Similar to $x^s_k$, $\tilde{y}^s = (1 - \theta_{s-1})\tilde{y}^s_{1-1} + \frac{\theta_{s-1}}{m}\sum_{k=1}^{m} y^s_k$. Moreover, $\theta_s$ can be set to be a constant $\theta$ in all epochs of our algorithm, which must satisfy $0 < \theta \leq 1 - \frac{\delta(b)}{\alpha - 1}$, and $\delta(b)$ is defined below. The optimal value of $\theta$ is provided in Proposition 1 below. The detailed procedure is shown in Algorithm 1 where we adopt the same initialization technique for $\lambda^s$ as in (Zheng and Kwok 2016), and $\cdot^s$ is the pseudo-inverse. Note that, when $\theta = 1$, ASVRG-ADMM degenerates to SVRG-ADMM in (Zheng and Kwok 2016).

The Non-Strongly Convex Case

In this part, we consider general convex problems of the form (2) when each $f_i(\cdot)$ is convex, L-smooth, and $h(\cdot)$ is not strongly convex (but possibly non-smooth). Different from the strongly convex case, the momentum weight $\theta_s$ is required to satisfy the following inequalities:

\[
\frac{1 - \theta_s}{\theta_s} \leq \frac{1}{\theta_{s-1}} \quad \text{and} \quad 0 < \theta_s \leq \frac{1 - \delta(b)}{\alpha - 1}, \tag{11}
\]

where $\delta(b) := \frac{n \cdot b}{b(n - 1)}$ is a decreasing function with respect to the mini-batch size $b$. The condition (20) allows the momentum weight to decrease, but not too fast, similar to the requirement on the step-size $\eta_b$ in classical SGD and stochastic ADMM (?). Unlike batch acceleration methods, the weight must satisfy both inequalities in (20).

Motivated by the momentum acceleration techniques in (Seng 2010, Nesterov 2004) for batch optimization, we give the update rule of the weight $\theta_s$ for the mini-batch case:

\[
\theta_s = \sqrt{\theta_{s-1}^4 + 4\theta_{s-1}^2 - 2\theta_{s-1}} - \frac{\theta_{s-1}}{2} = 1 - \frac{1 - \delta(b)}{\alpha - 1}. \tag{12}
\]

For the special case of $b = 1$, we have $\delta(1) = 1$ and $\theta_0 = 1 - \frac{1}{\alpha - 1}$, while $b = n$ (i.e. batch version), $\delta(n) = 0$ and $\theta_0 = 1$. Since $\{\theta_s\}$ is decreasing, then $\theta_s \leq 1 - \frac{\delta(b)}{\alpha - 1}$ is satisfied. The detailed procedure is shown in Algorithm 2, which has many slight differences in the initialization and output of each epoch from Algorithm 1. In addition, the key difference between them is the update rule for the momentum weight $\theta_s$. That is, $\theta_s$ in Algorithm 1 can be set to a constant, while that in Algorithm 2 is adaptively adjusted as in (12).

Convergence Analysis

This section provides the convergence analysis of our ASVRG-ADMM algorithms (i.e. Algorithms 1 and 2) for strongly convex and general convex problems, respectively. Following (Zheng and Kwok 2016), we first introduce the following function $P(x, y) := f(x) - f(x^\star) - \tilde{V}(x - x^\star) + h(y) - h(y^\star) - h'(y^\star)(y - y^\star)$ as a convergence criterion, where $h'(y)$ denotes the (sub)gradient of $h(\cdot)$ at $y$. Indeed, $P(x, y) \geq 0$ for all $x, y \in \mathbb{R}^d$. In the following, we give the intermediate key results for our analysis.
Algorithm 2 ASVRG-ADMM for general convex case

Input: $m, \eta, \beta > 0, 1 \leq b \leq n$.
Initialize: $\bar{x}_0 = \tilde{z}_0^0, \bar{y}_0 = 0, \bar{\xi}_0 = 1 - \frac{L_0 \delta(b)}{L \eta}$.

1: for $s = 1, 2, \ldots, T$ do
2: \[ x_s^k = (1 - \theta_s - 1)\bar{x}^s - 1 + \theta_s - 1 \bar{x}^s - 1, \quad y_s^k = y_0^s - 1 + \lambda_s^{- 1} \bar{z}^s; \]
3: \[ \tilde{p} = \nabla f(\bar{x}^s), \quad \bar{z}_0^s = \bar{z}^s - 1; \]
4: for $k = 1, 2, \ldots, m$ do
5: Choose $I_k \subseteq \{n\}$ of size $b$, uniformly at random;
6: \[ y_s^k = \arg\min_y h(y) + \frac{\beta}{2} \| A_s y_s^k + B y - c + A_s^* z_s^k \|_2^2; \]
7: \[ z_s^k = z_s^k - 1 \frac{\nabla h(\bar{y}^s \bar{z}_0^s)}{\lVert \nabla h(\bar{y}^s \bar{z}_0^s) \rVert_2}; \]
8: \[ x_s^k = (1 - \theta_s - 1)\bar{x}_0^s + \theta_s - 1 \bar{z}_0^s; \]
9: \[ \lambda_s^k = \lambda_s^k - 1 + A_s^* z_s^k + B y^k - c; \]
10: end for
11: $\bar{z}^s = 1 \sum_{k=1}^m x_k^s, \quad \bar{y}^s = (1 - \theta_s - 1)\bar{y}^s + \frac{1}{m} \sum_{k=1}^m y^k$,
12: $\bar{\xi}_s = \lambda_s^k - 1 + A_s^* z_s^k, \quad \theta_s = \sqrt{\theta_s^2 + 4 \theta_s^2 - \theta_s^2};$
13: end for

Output: $\bar{x}^T, \bar{y}^T$.

Lemma 1.
\[ E[\| \nabla f(x_k^s) - \nabla f(x_k^s) \|_2^2] \leq 2L_0 \delta \left[ (\bar{x}^s - f(x_k^s) + (x_k^s - 1)^T \nabla f(x_k^s) \right], \]
where $\delta(b) = \frac{a(n-b)}{b(n-b)} \leq 1$ and $1 \leq b \leq n$.

Lemma 2. Using the same notation as in Lemma 1, let $(x^s, y^s, \lambda^s)$ denote an optimal solution of problem 2 and \{(z_k^s, x_k^s, y_k^s, \lambda_k^s, \bar{y}^s)\} be the sequence generated by Algorithm 2 with $\theta_s \leq 1 - \theta_s \delta(b)$, and $\alpha = \frac{1}{L \eta}$. Then the following holds for all $k, \eta, \theta, \lambda, \mu, \delta$.
\[ E[P(\bar{x}^s, \bar{y}^s) - \frac{1}{m} \sum_{k=1}^m \| (x^s - z^s) A^T \varphi_k + (y^s - y_k^s) B^T \varphi_k \|_2^2] \leq \frac{2 \theta_s \| \varphi_k \|_2^2 + \| y^s - y_k^s \|_2^2 + \| x^s - z^s \|_2^2}{2m \eta}, \]
where $\varphi_k = \beta(\lambda_k^s - \lambda^s)$.

The detailed proofs of Lemmas 2 and 3 are provided in the Supplementary Material.

Linear Convergence

Our first main result is the following theorem which gives the convergence rate of Algorithm 1.

Theorem 1. Using the same notation as in Lemma 2, with $\theta_s \leq 1 - \theta_s \delta(b)$, and suppose $f(\cdot)$ is $\mu$-strongly convex and $L_f$-smooth, and $m$ is sufficiently large so that
\[ \rho = \theta \beta \| A^T A \|_2^2 \frac{1}{\eta \mu} + 1 - \theta \beta \frac{L_f \theta}{\beta \mu \sigma_{\min}(A^T A)} < 1, \quad (13) \]
where $\sigma_{\min}(A^T A)$ is the smallest eigenvalue of the positive semi-definite matrix $A^T A$, and $G$ is defined in (9). Then
\[ E[P(\bar{x}^T, \bar{y}^T)] \leq \rho^T P(\bar{x}^T, \bar{y}^T). \]

The proof of Theorem 1 is provided in the Supplementary Material. From Theorem 1, we can see that ASVRG-ADMM achieves linear convergence, which is consistent with that of SVRG-ADMM, while SCAS-ADMM has only an $O(1/T)$ convergence rate.

Remark 1. Theorem 1 shows that our result improves slightly upon the rate of $\rho$ in [Zeng and Kwok 2016] with the same $\eta$ and $\beta$. Specifically, as shown in (13), $\rho$ consists of three components, corresponding to those of Theorem 1 in [Zeng and Kwok 2016]. In Algorithm 1, recall that $\theta_s \leq 1$ and $G$ is defined in (9). Thus, both the first and third terms in (13) are slightly smaller than those of Theorem 1 in [Zeng and Kwok 2016]. In addition, one can set $\eta = 1/8L (i.e. \alpha = 8)$ and $\theta_s = 1 - \delta(b)/(\alpha - 1) = 1 - \delta(b)/7$. Thus, the second term in (13) equals to $\delta(b)/7$, while of that of SVRG-ADMM is approximately equal to $4L_0 \delta(b)/(1 - 4L_0 \delta(b)) \geq \delta(b)/2$. In summary, the convergence bound of SVRG-ADMM can be slightly improved by ASVRG-ADMM.

Selecting Scheme of $\theta$

The rate $\rho$ in (13) of Theorem 1 can be expressed as the function with respect to the parameters $\theta$ and $\beta$ with given $m, \eta, L_f, L, A, \mu$. Similar to [Nishihara et al. 2013, Zeng and Kwok 2016], one can obtain the optimal parameter $\beta^* = \sqrt{L_f \mu}/(\min_{\theta} (A^T A))$, which produces a smaller rate $\rho$. In addition, as shown in (13), all the three terms are with respect to the weight $\theta$. Therefore, we give the following selecting scheme for $\theta$.

Proposition 1. Given $\alpha = L_f \mu, \beta^*, \kappa = L_f \mu, b, A$, and let $\omega = \| A^T A \|_2/\min_{\theta} (A^T A)$, we set $m > 2\kappa + 2\sqrt{\omega}$ and $\eta = 1/(L \alpha)$, where $\alpha = \frac{m - 2\sqrt{\omega}}{\kappa} + \delta(b) + 1$. Then the optimal $\theta^*$ of Algorithm 1 is given by
\[ \theta^* = \frac{m - 2\sqrt{\omega}}{m - 2\sqrt{\omega} + 2\kappa} + \delta(b) + 1. \]

The proof of Proposition 1 is provided in the Supplementary Material.

Convergence Rate of $O(1/T^2)$

We first assume that $z \in Z$, where $Z$ is a convex compact set with diameter $D_Z = \sup_{z_1, z_2 \in Z} \| z_1 - z_2 \|$, and the dual variable $\lambda$ is also bounded with $D_\lambda = \sup_{\lambda \in \lambda} \| \lambda_1 - \lambda_2 \|$. For Algorithm 2, we give the following result.

Theorem 2. Using the same notation as in Lemma 2, with $\theta_0 = 1 - \delta(b)/(a-1)$, then we have
\[ E[P(\bar{x}^T, \bar{y}^T) + \gamma \| A^T \bar{x}^T + B \bar{y}^T - c \|] \leq \frac{4(\alpha - 1) \delta(b)}{(\alpha - 1 - \delta(b))^2 (T + 1)^2} \left( \| A^T \|_2^2 + 2 \| B^T \|_2^2 \right) + \frac{2L_0 \| A^T \|_2^2 D_Z^2 + 4D_\lambda^2}{m(T + 1)^2}. \]

(14)
Figure 1: Comparison of different stochastic ADMM methods for graph-guided fused Lasso problems on the four data sets. The $x$-axis represents the objective value minus the minimum (top) or testing loss (bottom), and the $y$-axis corresponds to the running time (seconds).

The proof of Theorem 2 is provided in the Supplementary Material. Theorem 2 shows that the convergence bound consists of the three components, which converge as $O(1/T^2)$, $O(1/mT^2)$ and $O(1/mT)$, respectively, while the three components of SVRG-ADMM converge as $O(1/T)$, $O(1/mT)$ and $O(1/mT)$. Clearly, ASVRG-ADMM achieves the convergence rate of $O(1/T^2)$ as opposed to $O(1/T)$ of SVRG-ADMM and SAG-ADMM ($m \gg T$). All the components in the convergence bound of SCAS-ADMM converge as $O(1/T)$. Thus, it is clear from this comparison that ASVRG-ADMM is a factor of $T$ faster than SAG-ADMM, SVRG-ADMM and SCAS-ADMM.

Connections to Related Work

Our algorithms and convergence results can be extended to the following settings. When the mini-batch size $b = n$ and $m = 1$, then $d(n) = 0$, that is, the first term of (14) vanishes, and ASVRG-ADMM degenerates to the batch version. Its convergence rate becomes $O(D_x^2/(T+1)^2 + D_z^2/(T+1) + D_\lambda^2/(T+1))$ (which is consistent with the optimal result for accelerated deterministic ADMM methods (Goldstein et al. 2014; Lu et al. 2016), where $D_x = \|x^* - x^0\|_G$. Many empirical risk minimization problems can be viewed as the special case of (1) when $A = I$. Thus, our method can be extended to solve them, and has an $O(1/T^2 + 1/(mT^2))$ rate, which is consistent with the best known result as in (Allen-Zhu 2016; Hien et al. 2016).

Experiments

In this section, we use our ASVRG-ADMM method to solve the general convex graph-guided fused Lasso, strongly convex graph-guided logistic regression and graph-guided SVM problems. We compare ASVRG-ADMM with the following state-of-the-art methods: STOC-ADMM (Ouyang et al. 2013), OPG-ADMM (Suzuki 2013), SAG-ADMM (Zhong and Kwok 2014b), and SCAS-ADMM (Zhao, Li, and Zhou 2015) and SVRG-ADMM (Zheng and Kwok 2016). All methods were implemented in MATLAB, and the experiments were performed on a PC with an Intel i5-2400 CPU and 16GB RAM.

Graph-Guided Fused Lasso

We first evaluate the empirical performance of the proposed method for solving the graph-guided fused Lasso problem:

$$\min_{x} \frac{1}{n} \sum_{i=1}^{n} \ell_i(x) + \lambda_1 \|Ax\|_1,$$

where $\ell_i$ is the logistic loss function on the feature-label pair $(a_i, b_i)$, i.e., $\log(1+\exp(-b_i a_i^T x))$, and $\lambda_1 \geq 0$ is the regularization parameter. Here, we set $A = [G; I]$ as in (Ouyang et al. 2013; Zhong and Kwok 2014b; Azadi and Sra 2014; Zheng and Kwok 2016), where $G$ is the sparsity pattern of the graph obtained by sparse inverse covariance selection (Banerjee, Ghaoui, and d’Aspremont 2008). We used four publicly available data sets in our experiments, as listed in Table 2. Note that except STOC-ADMM, all the other algorithms adopted the linearization of the penalty term $\frac{1}{2}\|Ax + z\|^2$ to avoid the inversion of $\frac{1}{2}I_{d_1} + \beta A^T A$ at each iteration, which can be computationally expensive for large matrices. The parameters of ASVRG-ADMM are set as follows: $m = 2n/b$ and $\gamma = 1$ as in (Zhong and Kwok 2014b; Zheng and Kwok 2016), as well as $\eta$ and $\beta$.

Figure 1 shows the training error (i.e. the training objective value minus the minimum) and testing loss of all the algorithms for the general convex problem on the four data

\[\text{http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/}\]
Table 2: Summary of data sets and regularization parameters used in our experiments.

| Data sets | # training | # test  | # mini-batch | λ1    | λ2    |
|-----------|------------|---------|--------------|-------|-------|
| a9a       | 16,281     | 16,280  | 20           | 1e-5  | 1e-2  |
| w8a       | 32,350     | 32,350  | 20           | 1e-5  | 1e-2  |
| SUSY      | 3,500,000  | 1,500,000 | 100       | 1e-5  | 1e-2  |
| HIGGS     | 7,700,000  | 3,300,000 | 150       | 1e-5  | 1e-2  |

Figure 2: Comparison of different stochastic ADMM methods for graph-guided logistic regression problems on the two data sets: a9a (top) and w8a (bottom).

sets. SAG-ADMM could not generate experimental results on the HIGGS data set because it ran out of memory. These figures clearly indicate that the variance reduced stochastic ADMM algorithms (including SAG-ADMM, SCAS-ADMM, SVRG-ADMM and ASVRG-ADMM) converge much faster than those without variance reduction techniques, e.g. STOC-ADMM and OPG-ADMM. Notably, ASVRG-ADMM consistently outperforms all other algorithms in terms of the convergence rate under all settings, which empirically verifies our theoretical result that ASVRG-ADMM has a faster convergence rate of $O(1/T^2)$, as opposed to the best known rate of $O(1/T)$.

Graph-Guided Logistic Regression

We further discuss the performance of ASVRG-ADMM for solving the strongly convex graph-guided logistic regression problem (Ouyang et al. 2013; Zhong and Kwok 2014a):

$$\min_x \frac{1}{n} \sum_{i=1}^{n} \left( \ell_i(x) + \frac{\lambda_2}{2} \|x\|_2^2 \right) + \lambda_1 \|Ax\|_1. \quad (16)$$

Due to limited space and similar experimental phenomena on the four data sets, we only report the experimental results on the a9a and w8a data sets in Figure 2, from which we observe that SVRG-ADMM and ASVRG-ADMM achieve comparable performance, and they significantly outperform the other methods in terms of the convergence rate, which is consistent with their linear (geometric) convergence guarantees. Moreover, ASVRG-ADMM converges slightly faster than SVRG-ADMM, which shows the effectiveness of the momentum trick to accelerate variance reduced stochastic ADMM, as we expected.

Graph-Guided SVM

Finally, we evaluate the performance of ASVRG-ADMM for solving the graph-guided SVM problem,

$$\min_x \frac{1}{n} \sum_{i=1}^{n} \left( 1 - b_i a_i^T x \right)_+ + \frac{\lambda_2}{2} \|x\|_2^2 + \lambda_1 \|Ax\|_1, \quad (17)$$

where $[x]_+ = \max(0, x)$ is the non-smooth hinge loss. To effectively solve problem (17), we used the smooth Huberized hinge loss in (Rosset and Zhu 2007) to approximate the hinge loss. For the 20news dataset we randomly divide it into 80% training set and 20% test set. Following (Ouyang et al. 2013), we set $\lambda_1 = \lambda_2 = 10^{-5}$, and use the one-vs-rest scheme for the multi-class classification.

Figure 3 shows the average prediction accuracies and standard deviations of testing accuracies over 10 different runs. Since STOC-ADMM, OPG-ADMM, SAG-ADMM and SCAS-ADMM consistently perform worse than SVRG-ADMM and ASVRG-ADMM in all settings, we only report the results of STOC-ADMM. We observe that SVRG-ADMM and ASVRG-ADMM consistently outperform the classical SVM and STOC-ADMM. Moreover, ASVRG-ADMM performs much better than the other methods in all settings, which again verifies the effectiveness of our ASVRG-ADMM method.

Conclusions

In this paper, we proposed an accelerated stochastic variance reduced ADMM (ASVRG-ADMM) method, in which we combined both the momentum acceleration trick for batch optimization and the variance reduction technique. We designed two different momentum term update rules for strongly convex and general convex cases, respectively. Moreover, we also theoretically analyzed the convergence properties of ASVRG-ADMM, from which it is clear that ASVRG-ADMM achieves linear convergence and $O(1/T^2)$ rates for both cases. Especially, ASVRG-ADMM is at least a factor of $T$ faster than existing stochastic ADMM methods for general convex problems.

http://www.cs.nyu.edu/~roweis/data.html
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Supplementary Materials for “Accelerated Variance Reduced Stochastic ADMM”

In this supplementary material, we give the detailed proofs for two important lemmas (i.e., Lemmas 1 and 2), two key theorems (i.e., Theorems 1 and 2) and a proposition (i.e., Proposition 1).

Proof of Lemma 1:
Our convergence analysis will use a bound on the variance term \( \mathbb{E}[\|\mathbf{W}_{f_i}(x_k^s) - \nabla f(x_k^s)\|^2] \), as shown in Lemma 1. Before giving the proof of Lemma 1, we first give the following lemma.

Lemma 3. Since each \( f_j(x) \) is convex, \( L_j \)-smooth \((j = 1, \ldots, n)\), then the following holds
\[
\|\nabla f_i(x_{k-1}^s) - \nabla f_i(x_{k-1}^s)\|^2 \\
\leq 2L \left[ f_i(x_{k-1}^s) - f_i(x_{k-1}^s) + \langle \nabla f_i(x_{k-1}^s), x_{k-1}^s - x_{k-1}^s \rangle \right],
\]
where \( i_k \in [n] \), and \( L := \max_j L_j \).

Proof. This result follows immediately from Theorem 2.1.5 in [Nesterov 2004]. □

Proof of Lemma 1:
Proof. \( \overline{\nabla f_i}(x_{k-1}^s) = \nabla f_i(x_{k-1}^s) - \nabla f_i(x_{k-1}^s) + \nabla f(x_k^s) \). Taking expectation over the random choice of \( i_k \), we have
\[
\mathbb{E}\left[ \|\overline{\nabla f_i}(x_{k-1}^s) - \nabla f(x_k^s)\|^2 \right] \\
= \mathbb{E}\left[ \|\nabla f(x_{k-1}^s) - \nabla f(x_{k-1}^s)\|^2 \right] - 2 \mathbb{E}[\|\nabla f_i(x_{k-1}^s) - \nabla f_i(x_{k-1}^s)\|^2] \\
\leq 2L \mathbb{E}\left[ f_i(x_{k-1}^s) - f_i(x_{k-1}^s) + \langle \nabla f_i(x_{k-1}^s), x_{k-1}^s - x_{k-1}^s \rangle \right] \\
= 2L \mathbb{E}\left[ f_i(x_{k-1}^s) - f(x_{k-1}^s) + \langle \nabla f_i(x_{k-1}^s), x_{k-1}^s - x_{k-1}^s \rangle \right],
\]
where the first inequality follows from the fact that \( \mathbb{E}[\|\mathbb{E}[x] - x\|^2] = \mathbb{E}[\|x\|^2] - \|\mathbb{E}[x]\|^2 \), and the second inequality is due to Lemma 3 given above. Note that the similar result for (19) was also proved in [Allen-Zhu 2016] (see Lemma 3.4 in [Allen-Zhu 2016]). Next, we extend the result to the mini-batch setting.

Let \( b \) be the size of mini-batch \( I_k \). We prove the result of Lemma 1 for the mini-batch case, i.e. \( b \geq 2 \).
\[
\mathbb{E}\left[ \|\overline{\nabla f_i}(x_{k-1}^s) - \nabla f(x_k^s)\|^2 \right] \\
= \mathbb{E}\left[ \left\| \frac{1}{b} \sum_{i \in I_k} (\nabla f_i(x_{k-1}^s) - \nabla f_i(x_{k-1}^s)) + \nabla f(x_{k-1}^s) - \nabla f(x_{k-1}^s) \right\|^2 \right] \\
= \frac{n-b}{b(n-1)} \mathbb{E}\left[ \|\overline{\nabla f_i}(x_{k-1}^s) - \nabla f(x_{k-1}^s) + \nabla f(x_{k-1}^s)\|^2 \right] \\
\leq \frac{2L(n-b)}{b(n-1)} \left( f(x_{k-1}^s) - f(x_{k-1}^s) + \langle \nabla f(x_{k-1}^s), x_{k-1}^s - x_{k-1}^s \rangle \right),
\]
where the second equality follows from Lemma 4 in [Koneeney et al. 2016], and the inequality holds due to the result in (19). □

Proof of Lemma 2:
Before proving the key Lemma 2, we first give the following a property [Baldassarre and Pontil 2013] [Lan 2012], which is useful for the convergence analysis of ASVRG-ADMM.

Property 1. Given any \( w_1, w_2, w_3, w_4 \in \mathbb{R}^d \), then we have
\[
\langle w_1 - w_2, w_1 - w_3 \rangle = \frac{1}{2} (\|w_1 - w_2\|^2 + \|w_1 - w_3\|^2 - \|w_2 - w_3\|^2),
\]
\[
\langle w_1 - w_2, w_3 - w_4 \rangle = \frac{1}{2} (\|w_1 - w_4\|^2 - \|w_1 - w_3\|^2 + \|w_2 - w_3\|^2 - \|w_2 - w_4\|^2).
\]
In order to prove Lemma 2, we first give and prove the following two key lemmas.

**Lemma 4.** Since $\eta = \frac{1}{L_f}$, $1 - \theta_{s-1} \geq \frac{\delta(b)}{\alpha - 1}$, and $\delta(b) = \frac{n-b}{\theta(n-1)}$, then we have

$$
\mathbb{E} \left[ f(\bar{x}^s) - f(x^*) - \langle \nabla f(x^*), \bar{x}^s - x^* \rangle - \frac{\theta_{s-1}}{m} \sum_{k=1}^{m} \langle A^T \varphi_k, x^* - z_k^s \rangle \right]
$$

$$
\leq \mathbb{E} \left[ (1 - \theta_{s-1}) (f(\bar{x}^{s-1}) - f(x^*) - \langle \nabla f(x^*), \bar{x}^{s-1} - x^* \rangle) + \frac{L \theta_{s-1}^2}{2m} (\|x^* - z_0^s\|^2 - \|x^* - z_m^s\|^2) \right].
$$

**Proof.** Let $g_k = \frac{1}{b} \sum_{i \in I_k} \langle \nabla f_{i,s}(x_{k-1}^s) - \nabla f_{i,s}(\bar{x}^{s-1}) \rangle$ + $\nabla f(\bar{x}^{s-1})$. Since the function $f$ is convex, differentiable with an $L_f$-Lipschitz-continuous gradient, where $L_f \leq L = \max_{j=1,\ldots,n} L_j$, then

$$
f(x_k^s) \leq f(x_{k-1}^s) + \langle \nabla f(x_{k-1}^s), x_k^s - x_{k-1}^s \rangle + \frac{L \alpha}{2} \|x_k^s - x_{k-1}^s\|^2 - \frac{L(\alpha - 1)}{2} \|x_k^s - x_{k-1}^s\|^2
$$

$$
= f(x_{k-1}^s) + \langle g_k, x_k^s - x_{k-1}^s \rangle + \frac{L \alpha}{2} \|x_k^s - x_{k-1}^s\|^2
$$

$$
+ \langle \nabla f(x_{k-1}^s) - g_k, x_k^s - x_{k-1}^s \rangle - \frac{L(\alpha - 1)}{2} \|x_k^s - x_{k-1}^s\|^2. \quad (20)
$$

Using Lemma 1, then we get

$$
\mathbb{E} \left[ \langle \nabla f(x_{k-1}^s) - g_k, x_k^s - x_{k-1}^s \rangle - \frac{L(\alpha - 1)}{2} \|x_k^s - x_{k-1}^s\|^2 \right]
$$

$$
\leq \mathbb{E} \left[ \frac{1}{2L(\alpha - 1)} \|\nabla f(x_{k-1}^s) - g_k\|^2 + \frac{L(\alpha - 1)}{2} \|x_k^s - x_{k-1}^s\|^2 - \frac{L(\alpha - 1)}{2} \|x_k^s - x_{k-1}^s\|^2 \right]
$$

$$
\leq \frac{\delta(b)}{\alpha - 1} (f(x_{k-1}^s) - f(x_{k-1}^s) + \langle \nabla f(x_{k-1}^s), x_k^s - x_{k-1} - \bar{x}^{s-1} \rangle), \quad (21)
$$

where the first inequality holds due to the Young’s inequality, and the second inequality follows from Lemma 1. Taking the expectation over the random choice of $I_k$ and substituting the inequality (21) into the inequality (20), we have

$$
\mathbb{E}[f(x_k^s)] \leq f(x_{k-1}^s) + \mathbb{E} \left[ \langle g_k, x_k^s - x_{k-1}^s \rangle + \frac{L \alpha}{2} \|x_k^s - x_{k-1}^s\|^2 \right]
$$

$$
+ \frac{\delta(b)}{\alpha - 1} \left[ f(\bar{x}^{s-1}) - f(x_{k-1}^s) + \langle \nabla f(x_{k-1}^s), x_k^s - x_{k-1} - \bar{x}^{s-1} \rangle \right]
$$

$$
= f(x_{k-1}^s) + \mathbb{E} \left[ \langle g_k, x_k^s - v^s + v^s - x_{k-1}^s \rangle + \frac{L \alpha}{2} \|x_k^s - x_{k-1}^s\|^2 \right]
$$

$$
+ \frac{\delta(b)}{\alpha - 1} \left[ f(\bar{x}^{s-1}) - f(x_{k-1}^s) + \langle \nabla f(x_{k-1}^s), x_k^s - x_{k-1} - \bar{x}^{s-1} \rangle \right]
$$

$$
= f(x_{k-1}^s) + \mathbb{E} \left[ \langle g_k, x_k^s - v^s \rangle + \frac{L \alpha}{2} \|x_k^s - x_{k-1}^s\|^2 \right] + \frac{\delta(b)}{\alpha - 1} (f(\bar{x}^{s-1}) - f(x_{k-1}^s))
$$

$$
+ \langle \nabla f(x_{k-1}^s), v^s - x_{k-1}^s + \frac{\delta(b)}{\alpha - 1} (x_k^s - x_{k-1} - \bar{x}^{s-1}) \rangle + \mathbb{E} \left[ \langle \frac{1}{b} \sum_{i \in I_k} \nabla f_{i,s}(\bar{x}^{s-1}) + \nabla f(\bar{x}^{s-1}), v^s - x_{k-1}^s \rangle \right],
$$

$$
= f(x_{k-1}^s) + \mathbb{E} \left[ \langle g_k, x_k^s - v^s \rangle + \frac{L \alpha}{2} \|x_k^s - x_{k-1}^s\|^2 \right] + \frac{\delta(b)}{\alpha - 1} (f(\bar{x}^{s-1}) - f(x_{k-1}^s))
$$

$$
+ \langle \nabla f(x_{k-1}^s), v^s - x_{k-1}^s + \frac{\delta(b)}{\alpha - 1} (x_k^s - x_{k-1} - \bar{x}^{s-1}) \rangle,
$$

where $v^s = (1 - \theta_{s-1}) \bar{x}^{s-1} + \theta_{s-1} x_k^s$, the second equality holds due to that $\langle g_k, v^s - x_{k-1}^s \rangle = \langle \frac{1}{b} \sum_{i \in I_k} \nabla f_{i,s}(x_{k-1}^s), v^s - x_{k-1}^s \rangle + \langle \frac{1}{b} \sum_{i \in I_k} \nabla f_{i,s}(\bar{x}^{s-1}) + \nabla f(\bar{x}^{s-1}), v^s - x_{k-1}^s \rangle$ and $\mathbb{E} \left[ \frac{1}{b} \sum_{i \in I_k} \nabla f_{i,s}(x_{k-1}^s) \right] = \nabla f(x_{k-1}^s)$, and the last equality follows from the fact that $\mathbb{E} \left[ \langle \frac{1}{b} \sum_{i \in I_k} \nabla f_{i,s}(\bar{x}^{s-1}) + \nabla f(\bar{x}^{s-1}), v^s - x_{k-1}^s \rangle \right] = 0.$
Furthermore,
\[
\langle \nabla f(x_k^{s-1}), v^* - x_k^{s-1} + \frac{\delta(b)}{\alpha - 1}(x_k^{s-1} - \bar{x}^{s-1}) \rangle \\
= \langle \nabla f(x_k^{s-1}), (1 - \theta_{s-1})\bar{x}^{s-1} + \theta_{s-1}x^* - x_k^{s-1} + \frac{\delta(b)}{\alpha - 1}(x_k^{s-1} - \bar{x}^{s-1}) \rangle \\
= \langle \nabla f(x_k^{s-1}), \theta_{s-1}x^* + (1 - \theta_{s-1}) - \frac{\delta(b)}{\alpha - 1}(x_k^{s-1} - \bar{x}^{s-1}) - x_k^{s-1} - x_k^{s-1} \rangle \\
\leq f\left(\theta_{s-1}x^* + (1 - \theta_{s-1}) - \frac{\delta(b)}{\alpha - 1}(x_k^{s-1} - \bar{x}^{s-1}) + \frac{\delta(b)}{\alpha - 1}f(x_k^{s-1}) - f(x_k^{s-1})\right) \\
\leq \theta_{s-1}f(x^*) + (1 - \theta_{s-1}) - \frac{\delta(b)}{\alpha - 1}f(\bar{x}^{s-1}) + \frac{\delta(b)}{\alpha - 1}f(x_k^{s-1}) - f(x_k^{s-1})
\]
where the first inequality holds due to the fact that \(\langle \nabla f(x), y - x \rangle \leq f(y) - f(x)\), and the last inequality follows from the convexity of the function \(f\) and the assumption that \(1 - \theta_{s-1} - \frac{\delta(b)}{\alpha - 1} \geq 0\).

Substituting the inequality (23) into the inequality (22), we have
\[
\mathbb{E}[f(x_k^s)] \leq f(x_{k-1}^s) + \mathbb{E}\left[\langle g_k, x_k^s - v^* \rangle + \frac{L_0}{2}\|x_k^s - x_{k-1}^s\|^2 + \frac{\delta(b)}{\alpha - 1}(f(\bar{x}^{s-1}) - f(x_k^{s-1}))\right] \\
+ \theta_{s-1}f(x^*) + (1 - \theta_{s-1}) - \frac{\delta(b)}{\alpha - 1}f(\bar{x}^{s-1}) + \frac{\delta(b)}{\alpha - 1}f(x_k^{s-1}) - f(x_k^{s-1}) \\
= \theta_{s-1}f(x^*) + (1 - \theta_{s-1})f(\bar{x}^{s-1}) + \mathbb{E}\left[\langle g_k, x_k^s - v^* \rangle + \frac{L_0}{2}\|x_k^s - x_{k-1}^s\|^2\right].
\]

From the optimality condition of (9) with respect to \(z_k^s\) and \(\eta = \frac{1}{L_0}\), we have
\[
\langle g_k + \beta A^T(Az_k^s + Bg_k - c) + \beta A^T\lambda_k^s - 1 + L_0\theta_{s-1}(G(z_k^s - z_{k-1}^s), z_k^s - z_{k-1}^s) \geq 0, \forall z \in Z,\]
where \(Z\) is a convex compact set. Since \(x_k^s = \theta_{s-1}z_k^s + (1 - \theta_{s-1})\bar{x}^{s-1}\) and \(v^* = \theta_{s-1}x^* + (1 - \theta_{s-1})\bar{x}^{s-1}\), and the above inequality with \(z = x^*\), we obtain
\[
\langle g_k, x_k^s - v^* \rangle = \theta_{s-1}\langle g_k, z_k^s - x^* \rangle \\
\leq \beta\theta_{s-1}\langle A^T(Az_k^s + Bg_k - b) + A^T\lambda_k^s - 1, x^* - z_k^s \rangle + L_0\theta_{s-1}^2\langle G(z_k^s - z_{k-1}^s), x^* - z_k^s \rangle \\
\leq \beta\theta_{s-1}\langle A^T(Az_k^s + Bg_k - b) + A^T\lambda_k^s - 1, x^* - z_k^s \rangle + \frac{L_0\theta_{s-1}^2}{2}\left(||x^* - z_{k-1}^s||_G^2 - ||x^* - z_k^s||_G^2 - ||z_k^s - z_{k-1}^s||_G^2\right) \\
\leq \beta\theta_{s-1}\langle A^T\lambda_k^s, x^* - z_k^s \rangle + \frac{L_0\theta_{s-1}^2}{2}\left(||x^* - z_{k-1}^s||_G^2 - ||x^* - z_k^s||_G^2 - ||z_k^s - z_{k-1}^s||_G^2\right),
\]
where the second inequality follows from Property 1. Using the optimality condition \(\nabla f(x^*) + \beta A^T\lambda^* = 0\) of problem (2) and let \(\varphi_k^s = \beta(\lambda_k^s - \lambda^*)\), then
\[
\theta_{s-1}\langle A^T\lambda_k^s, x^* - z_k^s \rangle = \theta_{s-1}\langle \nabla f(x^*), z_k^s - x^* \rangle + \theta_{s-1}\langle A^T\varphi_k^s, x^* - z_k^s \rangle + \theta_{s-1}\langle \beta A^T\lambda_k^s, x^* - z_k^s \rangle \\
= \theta_{s-1}\langle \nabla f(x^*), z_k^s - x^* \rangle + \theta_{s-1}\langle \varphi_k^s, x^* - z_k^s \rangle.
\]
Taking the expectation of both sides of (25) over the random choice of \(I_k\), we have
\[
\mathbb{E}[\langle g_k, x_k^s - v^* \rangle] \\
\leq \mathbb{E}\left[\theta_{s-1}\langle \nabla f(x^*), z_k^s - x^* \rangle + \theta_{s-1}\langle A^T\varphi_k^s, x^* - z_k^s \rangle + \frac{L_0\theta_{s-1}^2}{2}\left(||x^* - z_{k-1}^s||_G^2 - ||x^* - z_k^s||_G^2 - ||z_k^s - z_{k-1}^s||_G^2\right)\right].
\]
Substituting the inequality (26) into the inequality (24), and \(x_k^s - x_{k-1}^s = (1 - \theta_{s-1})\bar{x}^{s-1} + \theta_{s-1}z_k^s - (1 - \theta_{s-1})\bar{x}^{s-1} = \theta_{s-1}z_k^s - z_{k-1}^s\), we obtain
\[
\mathbb{E}[f(x_k^s) - f(x^*) - \theta_{s-1}\langle \nabla f(x^*), z_k^s - x^* \rangle - \theta_{s-1}\langle A^T\varphi_k^s, x^* - z_k^s \rangle] \\
\leq (1 - \theta_{s-1})(f(\bar{x}^{s-1}) - f(x^*)) + \frac{L_0\theta_{s-1}^2}{2}\mathbb{E}[||x^* - z_{k-1}^s||_G^2 - ||x^* - z_k^s||_G^2 - ||z_k^s - z_{k-1}^s||_G^2].
\]
where the last inequality holds due to $G \geq 1$ in Algorithms 1 and 2, that is, $\|z^k - z^k_G\|_{L^2} \geq 0$. Using the update rule $x^*_k = (1 - \theta_{s-1})\bar{x}^{s-1} + \theta_{s-1}z^*_k$ and subtracting $(1 - \theta_{s-1})(\nabla f(x^*), \bar{x}^{s-1} - x^*)$ from both sides, we have

$$
\mathbb{E}[f(x^*_k) - f(x^*) - \langle \nabla f(x^*), x^*_k - x^* \rangle - \theta_{s-1}\langle AT\psi^*_k, x^* - z^*_k \rangle] \\
\leq \mathbb{E}
\left[
(1 - \theta_{s-1})
\left(f(\bar{x}^{s-1}) - f(x^*) - \langle \nabla f(x^*), \bar{x}^{s-1} - x^* \rangle + \frac{L\theta^2_{s-1}}{2}
\left(\|x^* - z^*_k\|_G^2 - \|z^*_k\|_G^2\right)
\right)
\right].
$$

(27)

Since $\bar{x}^* = \frac{1}{m}\sum_{k=1}^m x^*_k$, and taking the expectation over the random choice of the history of random variables $I_1, \ldots, I_m$ on the inequality (27), summing it over $k = 1, \ldots, m$ at the $s$-th stage and $f\left(\frac{1}{m}\sum_{k=1}^m x^*_k\right) \leq \frac{1}{m}\sum_{k=1}^m f(x^*_k)$, we have

$$
\mathbb{E}
\left[f(\bar{x}^*) - f(x^*) - \langle \nabla f(x^*), \bar{x}^* - x^* \rangle - \frac{\theta_{s-1}}{m}\sum_{k=1}^m \langle AT\psi^*_k, x^* - z^*_k \rangle
\right] \\
\leq \mathbb{E}
\left[
(1 - \theta_{s-1})
\left(f(\bar{x}^{s-1}) - f(x^*) - \langle \nabla f(x^*), \bar{x}^{s-1} - x^* \rangle + \frac{L\theta^2_{s-1}}{2m}
\left(\|x^* - z^*_0\|_G^2 - \|x^* - z^*_0\|_G^2\right)
\right)
\right].
$$

This completes the proof.

\[\square\]

Lemma 5. Let $\bar{y}^* = (1 - \theta_{s-1})\bar{y}^{s-1} + \frac{\theta_{s-1}}{m}\sum_{k=1}^m y^*_k$, then

$$
\mathbb{E}
\left[
\|h(\bar{y}^*) - h(y^*) - h'(y^*)^T(\bar{y}^* - y^*) - \frac{\theta_{s-1}}{m}\sum_{k=1}^m \langle B^T\psi^*_k, y^* - y^*_k \rangle \right]
\leq (1 - \theta_{s-1})\mathbb{E}
\left[
\|h(\bar{y}^{s-1}) - h(y^*) - h'(y^*)^T(\bar{y}^{s-1} - y^*)\right]
+ \frac{\beta\theta_{s-1}}{2m}\mathbb{E}
\left[
\|A\bar{z}^*_0 + By^* - c\|^2 - \|A\bar{z}^*_m + By^* - c\|^2 + \sum_{k=1}^m \|\lambda^*_k - \lambda^*_k\|^2
\right].
$$

Proof. Since $\lambda^*_k = \lambda^*_{k-1} + A\bar{z}^*_k + By^*_k - c$, and using Lemma 3 in (Zheng and Kwok 2016), we obtain

$$
\mathbb{E}
\left[
\|h(\bar{y}^*) - h(y^*) - h'(y^*)^T(\bar{y}^* - y^*) - \frac{\theta_{s-1}}{m}\sum_{k=1}^m \langle B^T\psi^*_k, y^* - y^*_k \rangle \right]
\leq (1 - \theta_{s-1})\mathbb{E}
\left[
\|h(\bar{y}^{s-1}) - h(y^*) - h'(y^*)^T(\bar{y}^{s-1} - y^*)\right]
+ \frac{\beta\theta_{s-1}}{2m}\mathbb{E}
\left[
\|A\bar{z}^*_0 + By^* - c\|^2 - \|A\bar{z}^*_m + By^* - c\|^2 + \sum_{k=1}^m \|\lambda^*_k - \lambda^*_k\|^2
\right].
$$

Using the update rule $\bar{y}^* = (1 - \theta_{s-1})\bar{y}^{s-1} + \frac{\theta_{s-1}}{m}\sum_{k=1}^m y^*_k$, $h(\bar{y}) \leq (1 - \theta_{s-1})h(\bar{y}^{s-1}) + \frac{\theta_{s-1}}{m}\sum_{k=1}^m h(y^*_k)$, and taking expectation over whole history and summing the above inequality over $k = 1, \ldots, m$, we have

$$
\mathbb{E}
\left[
\|h(\bar{y}^*) - h(y^*) - h'(y^*)^T(\bar{y}^* - y^*) - \frac{\theta_{s-1}}{m}\sum_{k=1}^m \langle B^T\psi^*_k, y^* - y^*_k \rangle \right]
\leq (1 - \theta_{s-1})\mathbb{E}
\left[
\|h(\bar{y}^{s-1}) - h(y^*) - h'(y^*)^T(\bar{y}^{s-1} - y^*)\right]
+ \frac{\beta\theta_{s-1}}{2m}\mathbb{E}
\left[
\|A\bar{z}^*_0 + By^* - c\|^2 - \|A\bar{z}^*_m + By^* - c\|^2 + \sum_{k=1}^m \|\lambda^*_k - \lambda^*_k\|^2
\right].
$$

This completes the proof.

\[\square\]

Proof of Lemma 2:
Proof. Using Lemmas 4 and 5 and the definition of $P(x, y)$, we have
\[
E\left[ P(\tilde{x}^s, \tilde{y}^s) - \frac{\theta_{s-1}}{m} \sum_{k=1}^{m} (\langle A^T \varphi_k^s, x^* - z_k^s \rangle + \langle B^T \varphi_k^s, y^* - y_k^s \rangle) \right]
\leq (1 - \theta_{s-1})E\left[ P(\tilde{x}^{s-1}, \tilde{y}^{s-1}) \right] + \frac{L_0 \theta_{s-1}}{2m} E\left[ \|x^* - z_0^s\|_G^2 - \|y^* - z_m^s\|_G^2 \right]
+ \frac{\beta \theta_{s-1}}{2m} E\left[ \|Ax_0^s + By^* - c\|^2 - \|Ax_m^s + By^* - c\|^2 + \sum_{k=1}^{m} \|\lambda_k^s - \lambda_{k-1}^s\|^2 \right]
\leq E\left[ (1 - \theta_{s-1})P(\tilde{x}^{s-1}, \tilde{y}^{s-1}) \right] + \frac{\theta_{s-1}^2 (\|x^* - z_0^s\|_G^2 - \|x^* - z_m^s\|_G^2)}{2m \eta}
+ \frac{\beta \theta_{s-1}}{2m} E\left[ \|Ax_0^s - Ax^*\|^2 - \|Ax_m^s - Ax^*\|^2 + \sum_{k=1}^{m} \|\lambda_k^s - \lambda_{k-1}^s\|^2 \right].
\]
This completes the proof.

Proof of Theorem 1:
Let $(x^*, y^*)$ be an optimal solution of the convex problem (2), and $\lambda^*$ the corresponding Lagrange multiplier that maximizes the dual. Then $x^*$, $y^*$ and $\lambda^*$ satisfy the following Karush-Kuhn-Tucker (KKT) conditions:
\[
\beta A^T \lambda^* + \nabla f(x^*) = 0, \quad \beta B^T \lambda^* + h'(y^*) = 0, \quad Ax^* + By^* = c.
\]
Before giving the proof of Theorem 1, we first present the following lemmas (Zheng and Kwok 2016).

Lemma 6. Let $\varphi_k = \beta (\lambda_k - \lambda^*)$, and $\lambda_k = \lambda_{k-1} + Ax_k + By_k - c$, then
\[
E\left[ -(Ax_k + By_k - c)^T \varphi_k \right] = \frac{\beta}{2} E\left[ \|\lambda_{k-1} - \lambda^*\|^2 - \|\lambda_k - \lambda^*\|^2 - \|\lambda_k - \lambda_{k-1}\|^2 \right].
\]

Lemma 7. Since the matrix $A$ has full row rank, then
\[
\lambda^* = -\frac{1}{\beta} (A^T)^T \nabla f(x^*).
\]

Lemma 8. Let $\tilde{\lambda}^{s-1} = \lambda_0^s = -\frac{1}{\beta} (A^T) \nabla f(\tilde{x}^{s-1})$, and $\lambda^* = -\frac{1}{\beta} (A^T) \nabla f(x^*)$, then
\[
\|\tilde{\lambda}^{s-1} - \lambda^*\|^2 \leq \frac{2Lf}{\beta^2 \sigma_{\min}(AA^T)} \left( f(\tilde{x}^{s-1}) - f(x^*) - \nabla f(x^*)^T (\tilde{x}^{s-1} - x^*) \right).
\]

Proof of Theorem 1:
Proof. Using the update rule $\lambda_k^s = \lambda_{k-1}^s + Az_k^s + By_k^s - c$, then
\[
\sum_{k=1}^{m} \langle A^T \varphi_k^s, x^* - z_k^s \rangle + \langle B^T \varphi_k^s, y^* - y_k^s \rangle + \langle Az_k^s + By_k^s - c, \varphi_k^s \rangle = 0,
\]
where $\varphi_k^s = \beta (\lambda_k^s - \lambda^*)$. Using Lemma 6 we have
\[
- \sum_{k=1}^{m} \langle Az_k^s + By_k^s - c, \varphi_k^s \rangle = \frac{\beta}{2} \sum_{k=1}^{m} (\|\lambda_{k-1}^s - \lambda^*\|^2 - \|\lambda_k^s - \lambda^*\|^2 - \|\lambda_{k-1}^s - \lambda_k^s\|^2)
= \frac{\beta}{2} \left( \|\lambda_0^s - \lambda^*\|^2 - \|\lambda_m^s - \lambda^*\|^2 - \sum_{k=1}^{m} \|\lambda_{k-1}^s - \lambda_k^s\|^2 \right).
\]
Combining Lemma 2 and the above results with $\theta_{s-1} = \theta$ at all stages, $z_0^* = \bar{x}^s$ and $\lambda_0^* = \bar{\lambda}^s$, we have
\[
\mathbb{E}[P(\bar{x}, \bar{y})] 
\leq (1 - \theta)\mathbb{E}[P(\bar{x}^s, \bar{y}^s)] + \frac{L_t \theta}{2m} \mathbb{E}[\|x^* - \bar{x}^s\|^2] + \frac{\beta \theta}{2m} \mathbb{E}[\|x^* - z_m^*\|^2] 
+ \frac{\beta \theta}{2m} \mathbb{E}[\|A\bar{x}^s - By^s - c\|^2] + \frac{\beta \theta}{2m} \mathbb{E}[\|A\bar{x}^s - A\bar{x}^s\|^2 + \|\bar{\lambda}^s - \lambda^s\|^2] 
\leq (1 - \theta)\mathbb{E}[P(\bar{x}^s, \bar{y}^s)] + \frac{L_t \theta}{2m} \mathbb{E}[\|x^* - \bar{x}^s\|^2] + \frac{\beta \theta}{2m} \mathbb{E}[\|A\bar{x}^s - A\bar{x}^s\|^2 + \|\bar{\lambda}^s - \lambda^s\|^2] 
= (1 - \theta)\mathbb{E}[P(\bar{x}^s, \bar{y}^s)] + \frac{1}{2m} \mathbb{E}[\|x^* - \bar{x}^s\|^2] + \frac{\beta \theta}{2m} \mathbb{E}[\|\bar{\lambda}^s - \lambda^s\|^2] 
\leq (1 - \theta)\mathbb{E}[P(\bar{x}^s, \bar{y}^s)] + \frac{L_t \theta^2 + \beta \theta \|AT\|_2}{\eta \mu} \mathbb{E}[\|x^* - \bar{x}^s\|^2] + \frac{\beta \theta}{2m} \mathbb{E}[\|\bar{\lambda}^s - \lambda^s\|^2] 
\leq (1 - \theta)\mathbb{E}[P(\bar{x}^s, \bar{y}^s)] + \frac{L_t \theta^2 + \beta \theta \|AT\|_2}{\eta \mu} \mathbb{E}[\|x^* - \bar{x}^s\|^2] + \frac{\beta \theta}{2m} \mathbb{E}[\|\bar{\lambda}^s - \lambda^s\|^2] ,
\]
where the second inequality holds due to the fact that $Ax^* + By^* - c = 0$, and the last inequality follows from the definition of the strongly convex function.

Using $\lambda^* = -\frac{1}{\rho}(A^t \nabla f(x^*))$, the update rule $\lambda_0^* = -\frac{1}{\rho}(A^t \nabla f(\bar{x}^s))$, and Lemma 8 we have
\[
\|\lambda^* - \lambda_0^*\|^2 \leq \frac{2L_t}{\beta^2 \sigma_{\min}(AA^T)} [f(\bar{x}^s) - f(x^*) - \nabla f(x^*)^T(\bar{x}^s - x^*)] .
\]
Combining the above results, $h(\bar{y}^s) - h(x^*) - h'(y^s)^T(\bar{y}^s - y^s) \geq 0$, $\eta = \frac{1}{L_t}$ and the definition of $P(x, y)$, then we have
\[
\mathbb{E}[P(\bar{x}, \bar{y})] 
\leq \left( 1 - \theta + \frac{L_t \theta}{\rho \|AT\|_2} \right) \mathbb{E}[P(\bar{x}^s, \bar{y}^s)] 
\leq \left( 1 - \theta + \frac{\theta^2 + \eta \beta \|AT\|_2}{\eta \mu} \right) \mathbb{E}[P(\bar{x}^s, \bar{y}^s)] .
\]
This completes the proof.

Proof of Proposition 1:

Proof. Since
\[
G = \gamma I_d - \frac{\eta \beta^* AT A}{\theta} , \quad \gamma = \frac{\eta \beta^* \|AT\|_2}{\theta} + 1 , \quad \text{and} \quad \beta^* = \sqrt{\frac{L_t \mu}{\|AT\|_2 \sigma_{\min}(AA^T)}} ,
\]
then the rate $\rho$ in Theorem 1 is rewritten as follows:
\[
\rho(\theta) = \frac{\theta \|\theta G + \eta \beta^* AT A\|_2}{\eta \mu} + 1 - \theta + \frac{L_t \theta}{\beta^* \sigma_{\min}(AA^T)} 
= \theta (\eta \beta^* \|AT\|_2 + \theta) / (\eta \mu) + 1 - \theta + \frac{L_t \theta \sqrt{\|AT\|_2 / \sigma_{\min}(AA^T)}}{\sqrt{\mu} L_f \mu} 
= \theta / (\eta + \beta^* \theta \|AT\|_2 / (\eta \mu) + 1 - \theta + \frac{\theta \sqrt{L_t / \mu \sqrt{\omega}}}{\sqrt{\mu} L_f \mu} 
= \frac{\kappa \theta^2}{m} + \frac{2 \theta \sqrt{\eta \omega}}{m} + 1 - \theta ,
\]
where $\alpha = \frac{1}{L_t}$. Therefore, the rate $\rho$ can be expressed as a simple convex function with respect to $\theta$ by fixing other variable. To minimize the quadratic function $\rho$ with respect to $\theta$, then we have
\[
\theta^* = \frac{m - \eta \sqrt{\eta \omega}}{2 \kappa \alpha} .
\]
Recall from the body of this paper that $\alpha = \frac{m-2\sqrt{\kappa \omega}}{2\kappa} + \delta(b) + 1$. Then it is not difficult to verify that

$$\theta^* = \frac{m-2\sqrt{\kappa \omega}}{2\kappa \alpha} = \frac{m-2\sqrt{\kappa \omega}}{\left(\left(m-2\sqrt{\kappa \omega}\right) + 2\kappa(\delta(b) + 1)\right)}$$

$$\leq \frac{1}{\left(m-2\sqrt{\kappa \omega}\right) + 2\kappa\delta(b)} = 1 - \frac{\delta(b)}{\alpha - 1}.$$  

By the above result, we have

$$\theta^* = \frac{m-2\sqrt{\kappa \omega}}{2\kappa \alpha} = \frac{m-2\sqrt{\kappa \omega}}{m-2\sqrt{\kappa \omega} + 2\kappa(\delta(b) + 1)}. \quad (29)$$

It is not difficult to verify that $0 < \rho(\theta^*) < 1$ is satisfied. Thus, $\theta^*$ is the optimal solution of (28) with $\alpha = \frac{m-2\sqrt{\kappa \omega}}{2\kappa} + \delta(b) + 1$. This proof is completed. \hfill \square

Proof of Theorem 2:

Before giving the proof of Theorem 2, we first present the following lemma (Zheng and Kwok 2016).

**Lemma 9.** Let $\varphi_k = \beta(\lambda_k - \lambda^*)$ and any $\varphi = \beta \lambda$, and $\lambda_k = \lambda_{k-1} + Ax_k + By_k - c$, then

$$E[-(Ax_k + By_k - c)^T (\varphi_k - \varphi)] = \frac{\beta}{2} E[||\lambda_{k-1} - \lambda^* - \lambda||^2 - ||\lambda_k - \lambda^* - \lambda||^2 - ||\lambda_k - \lambda_{k-1}||^2].$$

**Proof of Theorem 2:**

For any $\varphi = \beta \lambda$, we have

$$\sum_{k=1}^m (\langle A^T \varphi_k, x^* - z_k^* \rangle + \langle B^T \varphi_k, y^* - y_k^* \rangle + \langle A z_k^* + By_k^* - c, \varphi_k - \varphi \rangle) = \sum_{k=1}^m (A z_k^* + By_k^* - c, \varphi) ,$$

where $\varphi_k = \beta (\lambda_k^* - \lambda^*)$. Using Lemma 9, we have

$$-\sum_{k=1}^m (A z_k^* + By_k^* - c, \varphi_k - \varphi) = \frac{\beta}{2} \sum_{k=1}^m (||\lambda_{k-1}^* - \lambda^* - \lambda||^2 - ||\lambda_k^* - \lambda^* - \lambda||^2 - ||\lambda_{k-1}^* - \lambda_k^*||^2) \]

$$= \frac{\beta}{2} (||\lambda_0^* - \lambda^* - \lambda||^2 - ||\lambda_m^* - \lambda^* - \lambda||^2 - \sum_{k=1}^m ||\lambda_{k-1}^* - \lambda_k^*||^2).$$

Combining the above results and Lemma 2, we have

$$E \left[ P(\bar{x}, \bar{y}) - \frac{\theta_{s-1}}{m} \sum_{k=1}^m (Az_k^* + By_k^* - c, \varphi) \right]$$

$$\leq (1-\theta_{s-1}) E \left[ P(\bar{x}^{s-1}, \bar{y}^{s-1}) \right] + \frac{L \alpha \theta_{s-1}^2}{2m} E \left[ ||x^* - z_0^*||^2 \right]$$

$$+ \frac{\beta \theta_{s-1}}{2m} E \left[ ||Az_0^* + By^* - c||^2 - ||Az_m^* + By^* - c||^2 + ||\lambda_0^* - \lambda^* - \lambda||^2 - ||\lambda_m^* - \lambda^* - \lambda||^2 \right]. \quad (30)$$

Using the update rule of $\bar{x}^s = \frac{1}{m} \sum_{k=1}^m x_k^s$, we have

$$\bar{x}^s = \frac{1}{m} \sum_{k=1}^m x_k^s = \frac{1}{m} \sum_{k=1}^m (\theta_{s-1} z_k^* + (1 - \theta_{s-1}) \bar{x}^{s-1}) = (1 - \theta_{s-1}) \bar{x}^{s-1} + \frac{\theta_{s-1}}{m} \sum_{k=1}^m z_k^s.$$  

Recall that

$$\bar{y}^s = (1 - \theta_{s-1}) \bar{y}^{s-1} + \frac{\theta_{s-1}}{m} \sum_{k=1}^m y_k^s.$$
Subtracting \((1 - \theta_{s-1})\langle A\tilde{x}^{s-1} + B\tilde{y}^{s-1} - c, \varphi \rangle\) and dividing both sides by \(\theta_{s-1}^2\), we have

\[
\frac{1}{\theta_{s-1}^2} \mathbb{E}[P(\tilde{x}, \tilde{y}) - \langle A\tilde{x} + B\tilde{y} - c, \varphi \rangle] \leq \frac{(1 - \theta_0)}{\theta_{s-1}^2} \mathbb{E}[P(\tilde{x}^{s-1}, \tilde{y}^{s-1}) - \langle A\tilde{x}^{s-1} + B\tilde{y}^{s-1} - c, \varphi \rangle] + \frac{L\alpha}{2m} \mathbb{E}[\|x^s - z_0^0\|^2_G - \|x^s - z_m^0\|^2_G] + \frac{\beta}{2m\theta_{s-1}} \mathbb{E}[\|Az_0^0 + By^0 - c\|^2 - \|Az_m^s + By^s - c\|^2 + \|\lambda_0^s - \lambda^*\|^2 - \|\lambda_m^s - \lambda^* - \lambda\|^2].
\]

By the update rule in (12), we have \((1 - \theta_s)/\theta_s^2 = 1/\theta_{s-1}^2\). Since \(z_0^0 = z_m^{s-1}\), \(\tilde{z}^0 = \tilde{x}^0\), and summing over all stages \((s = 1, \ldots, T)\), we have

\[
\frac{1}{\theta_{T-1}^2} \mathbb{E}[P(\tilde{x}^T, \tilde{y}^T) - \langle A\tilde{x}^T + B\tilde{y}^T - c, \varphi \rangle] \leq \mathbb{E}[P(\tilde{x}^0, \tilde{y}^0) - \langle A\tilde{x}^0 + B\tilde{y}^0 - c, \varphi \rangle] + \frac{L\alpha}{2m} \mathbb{E}[\|x^1 - \tilde{x}^0\|^2_G - \|x^1 - \tilde{z}_m^0\|^2_G] + \frac{\beta}{2m} \mathbb{E}[\|Az_0^0 - A\tilde{x}^*\|^2 - \|Az_m^s - A\tilde{x}^*\|^2 + \|\lambda_0^s - \lambda^*\|^2 - \|\lambda_m^s - \lambda^* - \lambda\|^2].
\]

Since \((1 - \theta_s)/\theta_s^2 = 1/\theta_{s-1}^2\), then

\[
0 \leq \frac{1}{\theta_s} - \frac{1}{\theta_{s-1}} \leq \frac{1}{\theta_s} - \frac{\sqrt{1 - \theta_s}}{1 + \sqrt{1 - \theta_s}} = \frac{1}{1 + \sqrt{1 - \theta_s}} < 1.
\]

Using \(z_0^0 = z_m^{s-1}\) and \(\lambda_0^s = \lambda_m^{s-1}\), we have

\[
\sum_{s=1}^T \frac{\beta}{2m\theta_{s-1}} \mathbb{E}[\|Az_0^0 - A\tilde{x}^s\|^2 - \|Az_m^s - A\tilde{x}^s\|^2 + \|\lambda_0^s - \lambda^* - \lambda\|^2 - \|\lambda_m^s - \lambda^* - \lambda\|^2] \leq \mathbb{E}\left[\frac{\beta(1 - \theta_0)}{2m\theta_0} \left(\|Az_0^1 - A\tilde{x}^*\|^2 + \|\lambda_0^1 - \lambda^* - \lambda\|^2\right) + \frac{\beta}{2m} \sum_{s=1}^T (\|Az_0^s - A\tilde{x}^s\|^2 + \|\lambda_0^s - \lambda^* - \lambda\|^2)\right]
\]

where the first inequality follows from that \(1/\theta_s - 1/\theta_{s-1} < 1\), \(z_1^s = \tilde{x}^0\) and \(\lambda_0^s = \tilde{\lambda}^0\), and the last inequality holds due to that \(\|Az_0^s - A\tilde{x}^s\|^2 \leq \|AT\|_2^2 \|x^s - \tilde{x}^0\|^2_G + 4D^2\), and \(\|\lambda_0^s - \lambda^* - \lambda\| \leq \|\lambda_0^s - \lambda^*\| + \|\lambda\| \leq 2D\).

By (31) and (32) with \(\theta_s \leq 2/(s + 2)\) for all \(s\), we have

\[
\mathbb{E}[P(\tilde{x}^T, \tilde{y}^T) - \langle A\tilde{x}^T + B\tilde{y}^T - c, \varphi \rangle] \leq \frac{4\tau}{(T + 1)^2} \left(\frac{P(\tilde{x}_0, \tilde{y}_0) - \langle A\tilde{x}_0 + B\tilde{y}_0 - c, \varphi \rangle}{m(T+1)^2} + \frac{2L\alpha}{m(T+1)^2} \|x^0 - \tilde{x}_0\|_G^2 + \frac{2\beta(T - 1 + 1/\theta_0)}{m(T+1)^2} \|AT\|_2^2 \|x^0 - \tilde{x}_0\|^2_G + 4D^2\right),
\]

where \(\theta_0 = 1 - \frac{\delta(b)}{\alpha - 1}\) and \(\tau = (1 - \theta_0)/\theta_0^2 = (\alpha - 1 - \delta(b))/\alpha\). Setting \(\varphi = \gamma \frac{A\tilde{x}^T + B\tilde{y}^T - c}{\|\varphi\|/\beta}\) with \(\gamma \leq \beta D\lambda\) such that \(\|\lambda\| = \|\varphi\|/\beta \leq D\lambda\), and \(-\langle A\tilde{x}^0 + B\tilde{y}^0 - c, \varphi \rangle \leq \|\varphi\| \|A\tilde{x}^0 + B\tilde{y}^0 - c\| \leq \gamma \|A\tilde{x}^0 + B\tilde{y}^0 - c\|\), we have

\[
\mathbb{E}[P(\tilde{x}^T, \tilde{y}^T) + \gamma \|A\tilde{x}^T + B\tilde{y}^T - c\|] \leq \frac{4(\alpha - 1 - \delta(b))^2}{(T + 1)^2} \left(\frac{P(\tilde{x}_0, \tilde{y}_0) + \gamma \|A\tilde{x}_0 + B\tilde{y}_0 - c\|}{m(T+1)^2} + \frac{2L\alpha}{m(T+1)^2} \|x^0 - \tilde{x}_0\|_G^2 + \frac{4\beta}{m(T+1)^2} \|AT\|_2^2 \|x^0 - \tilde{x}_0\|^2_G + 4D^2\right).
\]

This completes the proof. \qed