SHEAVES ON $\mathbb{P}^2$ AND GENERALIZED APPELL FUNCTIONS

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ABSTRACT. A closed expression is given for the generating functions of (virtual) Poincaré polynomials of moduli spaces of semi-stable sheaves on the projective plane $\mathbb{P}^2$ with arbitrary rank and Chern classes. To classify and study these generating functions, the notion of Appell functions with signature $(n_+, n_-)$ is introduced. For $n_- = 1$, these novel functions reduce to the known class of Appell functions with multiple variables or higher level.

CONTENTS

1. Introduction
2. Semi-stable sheaves on rational surfaces and change of polarization
3. Generating functions for $\mathcal{I}(\gamma, w; J)$
4. Appell functions
   4.1. The classical Appell function
   4.2. Appell functions with signature $(n_+, n_-)$
   4.3. An example

References

1. Introduction

Moduli spaces and their topological invariants are of fundamental interest for mathematics and physics. The Donaldson-Uhlenbeck-Yau theorem [8, 39] rigorously establishes the close relation between moduli spaces of instanton solutions in Yang-Mills theory and moduli spaces of semi-stable vector bundles and sheaves. A lot of progress is made in recent years on the properties and computation of topological invariants of these moduli spaces for rational and
ruled algebraic surfaces. Among the important used techniques are wall-crossing [18, 44, 45, 29, 31, 21], toric localization in moduli spaces [24, 11], and the Hall algebra [33].

Generating functions of topological invariants of moduli spaces exhibit often interesting arithmetic and modular properties. On the arithmetic side, the topological invariants are known to equal counts of (colored) partitions for rank 1 sheaves [13], dimensions of representations of the Mathieu group for the K3 surface [9, 20], and class numbers for rank 2 sheaves on the projective plane $\mathbb{P}^2$ [22, 40, 46]. The appearance of modularity is understood physically by the relation to gauge theory, and the $SL(2,\mathbb{Z})$ electric-magnetic duality group of this theory. The path integral of topologically twisted $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group $U(r)$ (also known as Vafa-Witten partition function) can be shown to enumerate basic topological invariants as the Euler number and Poincaré polynomial of the moduli spaces of vector bundles of rank $r$ [40]. The electric-magnetic duality group then implies modular transformation properties for the generating functions of these invariants.

The class of rational and ruled surfaces allows to study the generating functions very explicitly, and in particular their dependence on the polarization $J$. In the limit of vanishing volume of the fibre of the ruled surface, the generating functions of the topological invariants take the form of a beautiful infinite product formula [31, 33] which transforms as a Jacobi form [11] under modular transformations. Application of wall-crossing formulas [23, 17, 18] allows to determine the invariants for other choices of the polarization $J$. The change of polarization is taken into account by so-called indefinite theta functions [14, 29, 47]. These are convergent and holomorphic sums over a subset of an indefinite lattice of signature $(r−1, r−1)$. They typically destroy the nice modular transformation properties. However using the theory of mock modular forms [18], one can add a specific non-holomorphic completion such that the modular properties are restored for rank 2 [40, 30]. The non-holomorphic completion is however not very well understood from gauge theory, and the holomorphic anomaly equation is only conjectured for $r > 2$ [34].

In this brief note we derive a closed expression for the generating functions for arbitrary rank $r$ and Chern classes. This closed form is given by Equations (3.10) to (3.13). For rank 3, the function simplifies considerably the expression given in [29], and also allows to relatively quickly determine invariants for $r > 3$. The key to this simplification is the fact that the wall-crossing formula of Joyce [17] for virtual Poincaré polynomials is very
suitable for application in generating functions. For \( r = 2 \), we find immediately the familiar Appell functions \([13, 30]\). To describe the functions for \( r \geq 3 \), we introduce the notion of Appell functions with signature \((n_+, n_-)\). Appell functions with signature \((n_+, 1)\) reduce to the multi-variable and higher level Appell functions previously described in the literature \([26, 48]\). The novel form of the generating functions is much more suitable for the study of their arithmetic and modular properties than the form in \([29, 30]\). The properties are currently being determined \([42]\).

Appell functions have by now a wide variety of applications in number theory, algebraic geometry and mathematical physics \([10, 40, 35, 19, 47, 36, 27, 28, 38, 9, 6]\). The functions found here for \( \mathcal{N} = 4 \) Yang-Mills theory have even more subtle transformation properties, which are also likely to appear in other contexts. We mention only a few here:

- Yang-Mills theory on \( \mathbb{P}^2 \) and \( \Sigma_1 \) is a very useful model for the more difficult problem in string theory of D4-D2-D0 branes supported on divisors in Calabi-Yau 3-folds \([26]\). This problem is relevant for describing black holes in \( \mathbb{R}^{3,1} \) in \( \mathcal{N} = 2 \) supergravity. From the string theory perspective, the modular properties of the generating functions are also important to understand S-duality of IIB string theory \([1]\). The period integrals which appear in the modular completion of the generating function and which render the partition function continuous as function of the stability parameters are expected to be related to twistor integrals occuring in the Darboux coordinates. Recently it was proposed that these integrals also occur in the multi-particle Witten index for \( \mathcal{N} = 2 \) supersymmetric theories in \( \mathbb{R}^{3,1} \) \([2]\).
- Another interesting aspect of the generating functions is that they are expected to appear as partition functions of two dimensional theories. The 6-dimensional M5-brane of M-theory relates \( \mathcal{N} = 4 \) Yang-Mills to a 2-dimensional field theory on an elliptic curve \([26, 40, 16, 32, 12]\). It would be interesting if the generalized Appell functions with signature \((n_+, n_-)\) could be derived from this point of view.
- Application of the wall-crossing for virtual Poincaré polynomials \([2, 2]\) will also simplify the analysis for other complex surfaces and give in this way more examples of the generalized Appell functions. A particularly interesting surface is \( \frac{1}{2} \) K3 (the rational elliptic surface), for which the Vafa-Witten partition function equals (for suitable \( J \)) the partition function of topological strings \([34]\).

\(^1\)Toda \([37]\) pointed out recently that application of the wall-crossing formula of Joyce for numerical invariants (Euler numbers) \([13]\) is compatible with the theory of indefinite theta functions.
• The usual Appell functions are known to appear as global sections of rank 2 bundles on elliptic curves [35]. Properties of the Appell functions can be understood as $A_\infty$ constraints of the Fukaya category of the elliptic curve. It would be interesting to explore whether generalized Appell functions have similarly an interpretation as global sections of rank $r > 2$ bundles on elliptic curves.

The outline of this note is as follows. We start in Section 2 with a brief review of the wall-crossing formula for virtual Poincaré polynomials. In Section 3 we apply this to the rational surfaces $\mathbb{P}^2$ and $\Sigma_1$ and derive the closed form for arbitrary rank. In Section 4 we discuss the classical Appell function and introduce a larger class of Appell functions with signature $(n_+, n_-)$. The generating functions of Section 3 have are specializations in terms of the generalized Appell functions.

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2. Semi-stable sheaves on rational surfaces and change of polarization

We start in this section with briefly recalling necessary ingredients of semi-stable sheaves on rational surfaces and changes of the polarization to make the discussion self-contained. We denote by $\gamma$ the Chern character of a coherent sheaf: $\gamma = (r, c_1, ch_2)$. The polarization $J$ of an algebraic surface $S$ is an element of the closure of the ample cone $C(S)$ of $S$. The polarization enters in the definition of a stability condition $\varphi_J(\gamma)$ for coherent sheaves on $S$. The two relevant examples of stability conditions for this paper are $\mu$-stability, $\varphi_J^\mu(\gamma) = \mu(\gamma) \cdot J = c_1 \cdot J/r$, and Gieseker stability with $\varphi_J^{G}(\gamma) = p_J(\gamma)$ where $p_J(\gamma)$ is the Hilbert polynomial of the sheaf:

$$p_J(\gamma) = J^2/2 + \left(\frac{c_1(F) \cdot J}{r(F)} - \frac{K_S \cdot J}{2}\right) + \frac{1}{r(F)} \left(\frac{c_1(F)^2 - K_S \cdot c_1(F)}{2} - c_2(F)\right) + \chi(\mathcal{O}_S).$$

Most of the discussion in this article considers generating functions of so-called virtual Poincaré polynomials $\mathcal{I}_J(\gamma, w; J)$ of the moduli stack $\mathcal{M}_J(\gamma)$ of $\mu$-semi-stable sheaves with respect to $J$ [17]. Their formal definition is rather abstract and involved [17], however their
change under variations of $J$ (wall-crossing formula) as well as their generating functions take a rather simple form. Conjecturally, they uniquely determine the integer BPS invariants $\Omega(\gamma, w; J)$ appearing physics. See Section 3 for the conjectured, but explicit, relation between the $\mathcal{I}(\gamma, w; J)$ and $\Omega(\gamma, w; J)$.

To state the change of $\mathcal{I}(\gamma, w; J)$ under wall-crossing, we define the function $S(\{\gamma_i\}, \varphi, J, J')$ as in [17, Definition 4.2].

**Definition 2.1.** Let $\{\gamma_1, \gamma_2, \ldots, \gamma_\ell\}$ be a set of Chern characters with $r_i \in \mathbb{N}^*$, $i = 1, \ldots, \ell$. If for all $i = 1, \ldots, \ell - 1$ we have either

(a) $\varphi_J(\gamma_i) \leq \varphi_J(\gamma_{i+1})$ and $\varphi_J(\sum_{j=1}^{i} \gamma_j) > \varphi_J(\sum_{j=i+1}^{\ell} \gamma_j)$, or

(b) $\varphi_J(\gamma_i) > \varphi_J(\gamma_{i+1})$ and $\varphi_J(\sum_{j=1}^{i} \gamma_j) \leq \varphi_J(\sum_{j=i+1}^{\ell} \gamma_j)$,

then define $S(\{\gamma_i\}, \varphi, J, J') = (-1)^k$ where $k$ is the number of $i = 1, \ldots, \ell - 1$ satisfying (a). Otherwise, $S(\{\gamma_i\}, \varphi, J, J') = 0$.

Ref. [17] shows that for surfaces whose anti-canonical class $-K_S$ is numerically effective, the change of the invariants $\mathcal{I}(\gamma, w; J)$ under wall-crossing is expressed in terms of $S(\{\gamma_i\}, \varphi, J, J')$ as [17, Theorem 6.21]:

**Theorem 2.2.** Under a change of polarization $J \to J'$, the invariants $\mathcal{I}(\gamma, w; J')$ are expressed in terms of $\mathcal{I}(\gamma, w; J)$ by:

$$
\mathcal{I}(\gamma, w; J') = \sum_{\sum_{r_i \geq 1, i=1}^{\ell} \gamma_i = \gamma} S(\{\gamma_i\}, \varphi_J, \varphi_{J'}) w^{-\sum_{i<j} r_i r_j (\mu_j - \mu_i)} K_S \prod_{i=1}^{\ell} \mathcal{I}(\gamma_i, w; J).
$$

The orderings $\leq$ and $>$ in Definition 2.1 are to be replaced by the lexicographic ordering, $\preceq$ and $\succeq$, respectively. In the following, we will consider mostly $\mu$-stability and we therefore define $S(\{\gamma_i\}, J, J') := S(\{\gamma_i\}, \varphi_{J'}, \varphi_{J'})$.

We will restrict the computations in this article to only two rational surfaces, namely the projective plane $\mathbb{P}^2$ and its blow-up, the Hirzebruch surface $\Sigma_1$. Let $C \cong \mathbb{P}^1$ be the base curve and $f \cong \mathbb{P}^1$ be the fibre of $\Sigma_1$, then $H_2(\Sigma_1, \mathbb{Z}) = \mathbb{Z}C \oplus \mathbb{Z}f$, with intersection numbers $C^2 = -1$, $f^2 = 0$ and $C \cdot f = 1$. The anti-canonical class $K_{\Sigma_1}$ is numerically effective and given by $-K_{\Sigma_1} = 2C + 3f$. We parametrize the closure $\overline{C(S)}$ by:

$$
J_{m,n} = m(C + f) + nf, \quad m, n \geq 0.
$$

The blow-up of $\mathbb{P}^2$ be given by $\phi : \Sigma_1 \to \mathbb{P}^2$. The exceptional divisor of $\phi$ is $C$, and the hyperplane class $H$ of $\mathbb{P}^2$ is the pullback $\phi^*(C + f)$. 

Before defining and determining the generating functions, we briefly recall the relations between the virtual Poincaré polynomials and integer and rational BPS invariants. The rational BPS invariant $\overline{\Omega}(\gamma, w; J)$ is defined in terms of $\mathcal{I}(\gamma, w; J)$ by the relation \[17\]:

\[ \Omega(\gamma, w; J) := \sum_{\gamma_1 + \cdots + \gamma_\ell = \gamma} \prod_{i=1}^{\ell} \mathcal{I}(\gamma_i, w; J) \]

with inverse:

\[ \mathcal{I}(\gamma, w; J) = \sum_{\gamma_1 + \cdots + \gamma_\ell = \gamma} \frac{1}{\ell!} \prod_{i=1}^{\ell} \Omega(\gamma_i, w; J). \]

At generic points of the polarization, away from walls of marginal stability and boundary points, the $\overline{\Omega}(\gamma, w; J)$ can be further related to Laurent polynomials $P(\gamma, w; J) \in \mathbb{Z}[w, w^{-1}]$, which are symmetric under $w \leftrightarrow w^{-1}$. To this end, define $\Omega(\gamma, w; J)$ by:

\[ \Omega(\gamma, w; J) := \sum_{m \mid \gamma} \frac{\mu(m)}{m} \overline{\Omega}(\gamma/m, -(w)^m; J), \]

where $\mu(m)$ is the arithmetic Möbius function. Eq. \[3.2\] has inverse:

\[ \Omega(\gamma, w; J) = \sum_{m \mid \gamma} \frac{\Omega(\gamma, -(w)^m); J}{m} \]

Then away from walls of marginal stability $\Omega(\gamma, w; J)$ takes the form:

\[ \Omega(\gamma, w; J) = \frac{P(\gamma, w; J)}{w - w^{-1}}, \]

where $P(\gamma, w; J)$ is a Laurent polynomial symmetric under $w \leftrightarrow w^{-1}$. The integer BPS invariants $\Omega(\gamma; J)$ are obtained from these by

\[ \Omega(\gamma; J) = \lim_{w \to -1} (w - w^{-1}) \Omega(\gamma, w; J) \]

and similarly for the rational numerical invariants $\overline{\Omega}(\gamma; J)$.

If $\gamma$ is primitive and semi-stability implies stability than the moduli space $\mathcal{M}_J(\gamma)$ is smooth and compact and $w^{\dimc \mathcal{M}_J(\gamma)} P(\gamma, w; J)$ equals the Poincaré polynomial $\sum_{\ell=0}^{2 \dimc \mathcal{M}_J(\gamma)} b_\ell(\mathcal{M}_J(\gamma)) w^\ell$ of $\mathcal{M}_J(\gamma))$. If semi-stable does not imply stable, the precise cohomological meaning of $P(\gamma, w; J)$ is not completely clear. But following \[14\] one expects that the Laurent polynomial gives dimensions of intersection cohomology groups.
We now define the two generating functions $H_{r,c_1}(\tau, z; J)$ and $h_{r,c_1}(\tau, z; J)$ with $\text{Im}(\tau) > 0$ and $z \in \mathbb{C} \setminus \{\text{poles}\}$. As usual, we let $q := e^{2\pi i \tau}$ and $w := e^{2\pi i z}$. The generating series are then defined as:

$$H_{r,c_1}(\tau, z; J) := \sum_{c_2} T(\gamma, w; J) q^{r\Delta(\gamma) - \frac{r^2}{24}}$$

where $\Delta(\gamma)$ is the discriminant defined by:

$$\Delta(\gamma) = \frac{1}{r} \left( c_2 - \frac{r - 1}{r} c_1^2 \right) \in \mathbb{Q},$$

and $\chi(S)$ the Euler number of the surface $S$. The second class of generating functions $h_{r,c_1}(\tau, z; J)$ is defined by:

$$h_{r,c_1}(\tau, z; J) := \sum_{c_2} \bar{\Omega}(\gamma, w; J) q^{r\Delta(\gamma) - \frac{r^2}{24}},$$

and is argued to equal the path integral of $\mathcal{N} = 4$ Yang-Mills for $w \to -1$ \cite{40}. Using (3.2), $h_{r,c_1}(\tau, z; J)$ can be expressed in terms of $H_{r,c_1}(\tau, z; J)$ and vice versa. In the following we will consider only two surfaces, the Hirzebruch surface $\Sigma_1$ and the projective plane $\mathbb{P}^2$. We let $H_{r,c_1}(\tau, z; J)$ be the generating function for invariants of $\Sigma_1$ with respect to the polarization $J$, and $H_{r,c_1}(\tau, z; \mathbb{P}^2)$ the generating function for $\mathbb{P}^2$ which has no explicit dependence on its polarization.

Mozgovoy \cite{33} proved using the Hall algebra of $\mathbb{P}^1$ the conjecture in \cite{31} that the generating functions $H_{r,c_1}(z, \tau; J)$ take a particularly simple form for $J = J_{0,1} = f$. One has:

$$H_{r,c_1}(z, \tau; J_{0,1}) = \begin{cases} H_r(z, \tau), & \text{if } c_1 \cdot f = 0 \mod r, \quad r \geq 1, \\ 0, & \text{if } c_1 \cdot f \neq 0 \mod r, \quad r > 1. \end{cases}$$

with

$$H_r(z, \tau) := \frac{i (-1)^{r-1} \eta(\tau)^{2r-3}}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)^2 \cdots \theta_1((2r - 2)z, \tau)^2 \theta_1(2rz, \tau)},$$

with

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^\infty (1 - q^n),$$

$$\theta_1(z, \tau) := i \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^{r-\frac{1}{2}} q^{\frac{r^2}{2}} w^r$$

$$= i(w^{\frac{1}{2}} - w^{-\frac{1}{2}}) \prod_{n \geq 1} (1 - q^n)(1 - q^nw)(1 - w^{-1}q^n).$$

To determine the generating functions $H_{r,c_1}(z, \tau; \mathbb{P}^2)$ we use the techniques originally put forward by Yoshioka \cite{43}: change of polarization from $J_{0,1}$ to $J_{1,0}$ using Theorem 2.2 followed
by the blow-up formula. Let $\phi : \Sigma_1 \to \mathbb{P}^2$ be the blow-up map of a point of $\mathbb{P}^2$, such that $\phi^*c_1 \in H^2(\Sigma_1, \mathbb{Z})$ is the pull back of the first Chern class of sheaves on $\mathbb{P}^2$. Then one has [15, 13, 25]:

$$
H_{r,c_1}(z, \tau; \mathbb{P}^2) = \frac{H_{r,\phi^*c_1-kC}(z, \tau; J_{1,0})}{B_{r,k}(z, \tau)},
$$

with

$$
B_{r,k}(z, \tau) = \frac{1}{\eta(\tau)^r} \sum_{\substack{a_i=0 \\ a_j \in \mathbb{Z}+\frac{r}{2}}} q^{-\sum_{i<j} a_i a_j} w^{\sum_{i<j} a_i - a_j}.
$$

The crucial step to determine $H_{r,c_1}(z, \tau; \mathbb{P}^2)$ is to obtain a closed form for $H_{r,c_1}(z, \tau; J_{0,1})$. This is given by the following proposition.

**Proposition 3.1.** The function $H_{r,bC-a\phi}(z, \tau; J_{0,1})$ is given for any choice of $r \in \mathbb{N}^+$ and $a, b \in \mathbb{Z}$ by:

$$
H_{r,b\rho C-a\phi}(z, \tau; J_{0,1}) = \sum_{r_1+\cdots+r_\ell=r, r_i \in \mathbb{N}^+} (-1)^{\ell-1} \Psi(r_1, \ldots, r_\ell, (a,b))(z, \tau) \prod_{j=1}^\ell H_{r_j}(z, \tau),
$$

with

$$
\Psi(r_1, \ldots, r_\ell, (a,b))(z, \tau) := \sum_{r_1 b_1 + \cdots + r_\ell b_\ell = b} w^{\sum_{i<j} r_i r_j (b_i-b_j)+2(r_i+r_{i-1})\left(\sum_{k=i}^{\ell} r_k\right)-1} \prod_{i=2}^{\ell} \left(1 - w^{-2(r_i+r_{i-1})q^{b_i-b_{i-1}}}\right)
\times q^{\sum_{i=1}^{\ell} \frac{r_i(r_i-1)}{2}} \sum_{r_1}^{r_\ell} r_1 r_\ell b_\ell - \frac{1}{2} \sum_{j<i} r_i r_j b_j + \sum_{i=2}^{\ell} (b_i-b_{i-1}) \left(1-\left\{ \frac{1}{2} \sum_{k=i}^{\ell} r_k \right\} \right),
$$

where $\{ \lambda \} = \lambda - \lfloor \lambda \rfloor$ is the rational part of $\lambda$.

**Proof.** Substitution of Eq. (2.2) gives for the generating function:

$$
H_{r,c_1}(z, \tau; J_{1,0}) = \sum_{\sum_{i=1}^\ell \gamma_i = (r_1, c_1, c_2)} S(\{\gamma_i\}, J_{0,1}, J_{1,0}) w^{-\sum_{j<i} r_i r_j (\mu_i-\mu_j) - K_{\Sigma_1} q^{\Delta(\{\gamma_i\}) - \frac{i}{2}}},
$$

where $\Delta(\{\gamma_i\})$ is the discriminant of a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = F$ of the sheaf $F$, whose quotients $E_i = F_i/F_{i-1}$ have Chern character $\gamma_i$. The discriminant $\Delta(\{\gamma_i\})$ is expressed in terms of $\gamma_i = (r_i, c_1, c_2)$ by:

$$
\Delta(\{\gamma_i\}) = \sum_{i=1}^{\ell} r_i \Delta(E_i) - \sum_{i=2}^{\ell} \frac{1}{2r_i} \sum_{j=1}^{r_i} \frac{1}{r_j} \sum_{k=1}^{r_i-1} r_k \left( \sum_{j=1}^{i-1} r_j c_{1,j} - r_j c_{1,i} \right)^2.
$$
Since $S(\{\gamma_i\}, J, J')$ is independent of the $\text{ch}_2, i = 1, \ldots, \ell$, the sum over $\text{ch}_2$ can be replaced by $H_{r,c_1}(z, \tau; J_{0,1})$:

$$H_{r,c_1}(z, \tau; J_{1,0}) = \sum_{\sum_{i=1}^\ell (r_{c_1,i})^{r_{c_1}}} S(\{\gamma_i\}, J_{0,1}, J_{1,0}) w^{-\sum_{j=1}^\ell r_j (\mu_j - \mu_j) - K_S}$$

(3.16)

Furthermore, from Equation (3.10) we know that $H_{r,c_1}(z, \tau; J_{0,1})$ vanishes for $c_1 \cdot f \neq 0 \mod r$. Therefore the only contributing terms in the sum in Eq. (3.10) are of the form $(r_i, c_{1,i}) = (r_i, r_i b_i C - a_i f)$, with $r_i \in \mathbb{N}^*$ and $a_i, b_i \in \mathbb{Z}$. We parametrize the total first Chern class as $c_1 = b i C - a_i f$, such that the sets $\{a_i\}$ and $\{b_i\}$ have to satisfy $\sum_{i=1}^\ell a_i = a$ and $\sum_{i=1}^\ell r_i b_i = b$. We continue with bringing $H_{r,i C - a f}(z, \tau; J_{1,0})$ to a form which allows to carry out the sums over $a_i, i = 2, \ldots, \ell$ of the different contributions.

First, after substitution of $c_{1,i} = r_i b_i C - a_i f$ in $\Delta(\{\gamma_i\})$ and $a_1 = a - \sum_{i=2}^\ell a_i$ in $\Delta(\{\gamma_i\})$, one obtains:

$$r \Delta(\{\gamma_i\}) = \sum_{i=1}^\ell r_i \Delta(\gamma_i) + \sum_{i=2}^\ell \frac{1}{2 \sum_{j=1}^{i-1} r_j \sum_{k=1}^{i-1} r_k} \left( \sum_{j=1}^{i-1} r_j (b_j - b_i) \right)^2$$

(3.17)

$$+ \sum_{i=2}^\ell \sum_{j=1}^{i-1} r_j \sum_{k=1}^{i-1} r_k \left( \sum_{j=1}^{i-1} r_j (b_i - b_j) \right) \left( \sum_{k=i+1}^\ell r_{i+k} + \sum_{k=1}^{i-1} r_{i+k} - r_i a_{i+k} \right).$$

We now make the following change of variables:

$$a_i = s_i - s_{i+1}, \quad i = 2, \ldots, \ell - 1, \quad a_\ell = s_\ell.$$

This transforms the second line of (3.17) to:

$$\sum_{i=2}^\ell \left( \sum_{j=1}^{i-1} r_j (b_i - b_j) \right) \left( \frac{s_i}{\sum_{j=1}^{i-1} r_j} - \frac{s_{i+1}}{\sum_{j=1}^{i} r_j} - \frac{r_i a_i}{\sum_{j=1}^{i-1} r_j \sum_{k=1}^{i-1} r_k} \right),$$

(3.19)

which can be further simplified to:

$$\sum_{i=2}^\ell (b_i - b_{i-1}) s_i - a \sum_{i=2}^\ell \sum_{j=1}^{i-1} r_j r_{j} (b_i - b_j) - \sum_{j=1}^\ell r_j r_{j} (b_i - b_j).$$

The exponent of $w$ in Eq. (3.14) is easily evaluated in terms of $a_i$ and $b_i$:

$$- \sum_{j<i} r_i r_j (\mu_i - \mu_j) \cdot K_{\Sigma_1} = \sum_{j<i} r_i r_j (b_i - b_j) - 2(r_j a_i - r_i a_j).$$
Replacing $a_1$ as before this becomes:

\[(3.20) \quad \sum_{j<i} r_ir_j(b_i - b_j) - 2 \left( 2 \sum_{1<j<i} r_ja_i + (r_1 + r_j)a_j - (r - r_1)a \right).\]

The substitution (3.18) then gives:

\[2(r - r_1)a + \sum_{j<i} r_ir_j(b_i - b_j) - 2 \sum_{i=2}^\ell (r_{i-1} + r_i)s_i.\]

Now we come to the third term of the summand in (3.16): $S(\{\gamma_i\}, J, J')$. Interestingly, this can be written as a product of differences of signs, which are familiar from the literature on indefinite theta functions \[14, 47\]. To this end, define:

\[(3.21) \quad \text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}\]

Then:

\[(3.22) \quad S(\{\gamma_i\}, J, J') = \frac{1}{2^\ell} \prod_{i=2}^\ell \left( \text{sgn}(\varphi_J(\gamma_{i-1}) - \varphi_J(\gamma_i) - v_1) - \text{sgn}\left( \varphi_J'\left( \sum_{j=1}^{i-1} \gamma_j \right) - \varphi_J'\left( \sum_{j=i}^{\ell} \gamma_j \right) - v_1 \right) \right),\]

where $v_1 > 0$ is a sufficiently small positive constant such that $0 < v_1 < |\varphi_J(\gamma) - \varphi_J(\gamma')|$ for each $\gamma, \gamma'$ such that $\varphi_J(\gamma) - \varphi_J(\gamma') \neq 0$. Specializing to $\varphi_J = \varphi_J'$ and substitution of $c_{1,i} = b_ir_iC - a_if$ gives:

\[(3.23) \quad S(\{\gamma_i\}; J_{0,1}, J_{1,0}) = \frac{(-1)^{\ell-1}}{2^{\ell-1}} \prod_{i=2}^\ell \left( \text{sgn}(b_i - b_{i-1} + v_2) + \text{sgn}\left( \sum_{j=1}^{i-1} \sum_{k=i+1}^\ell a_kr_j - a_jr_k - v_2 \right) \right).\]

where $0 < v_2 < 1$. Making again the substitution for $a_1$ brings the argument in the second sign to the form:

\[\sum_{j=1}^i \sum_{k=i+1}^\ell a_kr_j - a_jr_k = r \sum_{k=i+1}^\ell a_k - a \sum_{k=i+1}^\ell r_k.\]

With the substitution (3.18), this simplifies to:

\[(3.24) \quad S(\{\gamma_i\}; J_{0,1}, J_{1,0}) = \frac{(-1)^{\ell-1}}{2^{\ell-1}} \prod_{i=2}^\ell \left( \text{sgn}(b_i - b_{i-1} + v_2) + \text{sgn}\left( rs_i - a \sum_{k=i}^\ell r_k - v_2 \right) \right),\]

We now observe that the sum over the $s_i$’s are simply geometric sums and can be carried out if $z$ is such that $|w^{-4}| < 1 \text{ and } |w^4q| < 1$. This gives brings $\Psi_{(\tau_1, ..., \tau_l), (a, b)}(z, \tau)$ to the
following form:

\[
\Psi_{(r_1, \ldots, r_\ell), (a,b)}(z, \tau) := \sum_{r_1 b_1 + \cdots + r_\ell b_\ell = b} \frac{w^{2(r-r_1)a + \sum_i r_i r_j (b_i-b_j) - 2(r_i + r_j - 1)}}{\prod_{i=2}^\ell (1 - w^{-2(r_i+r_j-1)}q^{b_i-b_j})} \prod_{i=2}^\ell \frac{r_i}{2(\sum_{j=1}^{i-1} r_j (b_j-b_i))]^2} \times q^{\sum_{i=2}^\ell (b_i-b_{i-1})(1 + [\sum_{k=1}^{i-1} r_k])} \frac{1}{\sum_{k=1}^{i-1} r_k \sum_{m=1}^{i-1} r_m},
\]

(3.25)

which can immediately be analytically continued to \(z \in \mathbb{C}\setminus\{\text{poles}\}\). To bring it to the simpler form (3.13), one proves easily with induction on \(\ell\) that:

\[
\sum_{i=2}^\ell (r_i + r_{i-1}) \sum_{k=1}^\ell r_k = (r - r_1)r,
\]

\[
r \sum_{i=2}^\ell \sum_{j=1}^{i-1} \frac{r_i r_j (b_i-b_j)}{r_k \sum_{m=1}^{i-1} r_m} = b - b_1 r = \sum_{i=2}^\ell (b_i - b_{i-1}) \sum_{k=1}^\ell r_k,
\]

\[
\sum_{i=2}^\ell \frac{r_i}{2 \sum_{j=1}^{i-1} r_j \sum_{k=1}^{i-1} r_k} \left(\sum_{j=1}^{i-1} r_j (b_j-b_i)\right)^2 = \sum_{i=1}^\ell \sum_{j \neq \ell} \frac{r_j r_i}{2r} (b_i^2 - 2b_i b_j)
\]

Substitution of these expressions in Equation (3.25) gives the proposition. Note that it is manifestly invariants under shifts of \((a, b) \rightarrow (a, b) + (k_1, k_2)\) with \(k_1, k_2 \in \mathbb{Z}\).

We note that Proposition 3.1 gives already for \(r = 3\) much simpler expressions than those in [24, 41, 29, 31]. This allows to rather quickly determine the invariants. We have verified that the first coefficients of \(H_{r,c_1}(z, \tau; \mathbb{P}^2)\) reproduce those in all known cases.

We finish this section with an example. We compute the integer invariants \(\Omega(\gamma, w; \mathbb{P}^2)\) of sheaves on \(\mathbb{P}^2\) with \((r, c_1) = (4, 2H)\). Eq. (3.11) shows that we need to determine both \(H_{2,H}(z, \tau; \mathbb{P}^2)\) and \(H_{4,2H}(z, \tau; \mathbb{P}^2)\). For \(H_{2,H}(z, \tau; \mathbb{P}^2)\) we determine first \(H_{2,C+f}(z, \tau; J_{1,0})\). The only contributing term of the sum \(\Sigma_{r_1, \ldots, r_\ell = 2}\) with solutions to \(r_1 b_1 + r_2 b_2 = 1\) in Eq. (3.12) is \((r_1, r_2) = (1, 1)\). This gives immediately the result of Yoshioka [33]:

\[
H_{2,H}(z, \tau; \mathbb{P}^2) = -\frac{H_1(z, \tau)^2}{B_{2,0}(z, \tau)} \sum_{k \in \mathbb{Z}} \frac{w^{2k-3}q^{k^2-k}}{1 - w^{-4}q^{2k-1}}.
\]

(3.26)

For \(H_{4,2H}(z, \tau; \mathbb{P}^2)\) the contributing terms in the sum \(\Sigma_{r_1, \ldots, r_\ell = r}\) are \((r_1, \ldots, r_\ell) = (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2)\) and \((1, 1, 1, 1)\). The different \(\Psi_{(r_1, \ldots, r_\ell), (a,b)}(z, \tau)\) with
\[(a, b) = (-2, 2)\] are given by:

\[
\Psi_{(3,1),(a,b)}(z, \tau) = \Psi_{(1,3),(a,b)} = \sum_{k \in \mathbb{Z}} \frac{w^{12k+2} q^{6k^2+8k+\frac{7}{2}}}{1 - w^{-8} q^{4k+2}},
\]

\[
\Psi_{(2,2),(a,b)}(z, \tau) = \sum_{k \in \mathbb{Z}} \frac{w^{8k-12} q^{2k^2-2k-\frac{1}{8}}}{1 - w^{-8} q^{2k-1}},
\]

\[
\Psi_{(2,1,1),(a,b)}(z, \tau) = \Psi_{(1,1,2),(a,b)}(z, \tau) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{10k_1+2k_2-2q^{3k^2_1+2k_1+k^2_2+6k_1+2k_2+\frac{7}{2}}}{(1 - w^{-6} q^{k_1-k_2})(1 - w^{-4} q^{2k_1+2k_2+2})},
\]

\[
\Psi_{(1,2,1),(a,b)}(z, \tau) = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{6k_1+6k_2} q^{k^2_1+2k_1+k^2_2+3k_1+5k_2+1}}{(1 - w^{-6} q^{k_1-k_2})(1 - w^{-4} q^{2k_1+2k_2+2})},
\]

\[
\Psi_{(1,1,1),(a,b)}(z, \tau) = \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \frac{w^{6k_1+4k_2+2k_3-2q^{k^2_1+k^2_2+k^2_3+k_1+k_2+k_3+3k_1+3k_2+2k_3+2k_3+\frac{7}{2}}}{(1 - w^{-4} q^{k_1-k_2})(1 - w^{-4} q^{k_2-k_3})(1 - w^{-4} q^{k_1+k_2+k_3+2})}.
\]

Summing up these functions as prescribed by Proposition 3.1 one determines \(H_{4,2c+2f}(z, \tau; J_{1,0})\).

After application of the blow-up formula and using the formulas in Section 3, one obtains the integer invariants for Gieseker semi-stable sheaves on \(\mathbb{P}^2\) with \(\gamma = (4, 2H, c_2)\):

\[
\frac{H_{4,2c+2f}(z, \tau; J_{1,0})}{B_{4,0}(z, \tau)} \sim - \frac{1}{2} H_{2,H}(z, \tau; \mathbb{P}^2)^2 + \frac{1}{2} H_{2,H}(2z, 2\tau; \mathbb{P}^2).
\]

Taking the limit \(z \to \frac{1}{2}\), one finds that the first few non-vanishing Euler numbers are: 6, 162, 1846, 14766, \ldots.

## 4. Appell Functions

In this section, we first review the definition and main properties of the classical Appell function. In Subsection 4.2 we introduce the generalized Appell functions with signature \((n_+, n_-)\). Subsection 4.3 shows with an example that the functions \(\Psi_{(r_1, \ldots, r_\ell),(a,b)}(z, \tau)\) of Section 3 are specializations of Appell functions with signature \((\ell - 1, \ell - 1)\).

### 4.1. The classical Appell function

The classical Appell function is defined as [3]:

\[
A(u, v, \tau) := e^{\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2} e^{2\pi i n u}}{1 - e^{2\pi i u} q^n}
\]

which is a meromorphic function of \(u \in \mathbb{C}\), and holomorphic in \(\tau \in \mathcal{H}\) and \(v \in \mathbb{C}\). It is well-known that the transformation properties of \(A(u, v, \tau)\) are not exactly those of a modular or Jacobi form [36, 47]. However, define the “completed” Appell function as:

\[
\hat{A}(u, v, \tau) := A(u, v, \tau) + \frac{i}{2} \theta_1(v, \tau) R(u - v, \tau)
\]
with
\begin{equation}
R(u, \tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \left( \text{sgn}(n) - E\left(\frac{n + \text{Im}(u)/y}{\sqrt{2y}}\right) (-1)^{n-\frac{1}{2}} e^{-2\pi i n u} q^{-n^2/2} \right)
\end{equation}
and \( E(x) = 2 \int_{0}^{x} e^{-\pi u^2} du \). Then \( \hat{A}(u, v, \tau) \) satisfies the following properties [48]:

1. Modular transformations: for \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}) \)

\[
\hat{A}\left( \frac{u}{ct + d}, \frac{v}{ct + d} \right) = (ct + d) e^{\pi ic(-u^2+2uv)/\pi(c+d)} \hat{A}(u, v, \tau).
\]

2. Elliptic transformations: for \( k, l, m, n \in \mathbb{Z} \)

\[
\hat{A}(u + k\tau + l, v + m\tau + n; \tau) = (-1)^{k+m} e^{2\pi i(k-m)u-2\pi iuv} q^{k^2/2-km} \hat{A}(u, v, \tau).
\]

3. Periodicity relation:

\[
\theta_1(v, \tau) A(u + z, v + z, \tau) - \theta_1(v, z, \tau) A(u, v, \tau) = \frac{\eta(\tau)^3 \theta_1(u + v + z, \tau) \theta_1(z, \tau)}{\theta_1(u, \tau) \theta_1(v, \tau) \theta_1(u + z, \tau) \theta_1(v + z, \tau)}
\]

4. They can be seen as coefficients of a meromorphic Jacobi form:

\[
\sum_{n \in \mathbb{Z}} A(u + m\tau, v, \tau) e^{2\pi im(z-\frac{1}{2}\tau) - \pi i u} = \frac{\eta(\tau)^3 \theta_1(u + z, \tau) \theta_1(v - z)}{i \theta_1(u, \tau) \theta_1(z, \tau)}
\]

4.2. Appell functions with signature \((n_+, n_-)\). Based on characters of conformal field theory [19, 36], various generalisations of the classical Appell functions are proposed. In particular the higher level Appell functions [36] and the multivariable Appell functions [19, 48]. Appell functions appear also in the context of \( \mathcal{N} = 4 \) black holes [6]. And of course in this paper we see that Equation (3.26) contains \( A(u, v, \tau) \) with \( u \) and \( v \) appropriately specialized.

The functions \( \Psi((r_1, \ldots, r_\ell), (a, b)) (z, \tau) \) for \( \ell > 2 \) form an interesting extension to the known class of Appell functions. To describe and study these functions in more detail, we introduce Appell functions with signature \((n_+, n_-)\). These functions depend on a \( n_+ \)-dimensional lattice \( \Lambda \cong \mathbb{Z}^{n_+} \) with positive definite quadratic form \( Q(k) = k^T Q k \). The scalar product \( k \cdot m \) denotes as usual \( \sum_{i=1}^{n_+} k_i m_i \). We have furthermore an \( n_+ \) by \( n_- \) matrix \( M \) such that the following determinant does not vanish:

\begin{equation}
\left| \begin{array}{cc} Q & M^T \\ M & 0 \end{array} \right| \neq 0.
\end{equation}

The “signature” of the Appell function can thus be seen as the signature \((n_+, n_-)\) of the above matrix. The column vectors of \( M \) are denoted by \( m_i \in \Lambda^*, \ i = 1, \ldots, n_- \). We furthermore
have a vector \( m_0 \in \Lambda \times \mathbb{Q} \), two complex vectors \( \mathbf{u} = (u_1, \ldots, u_{n_-}) \), \( \mathbf{v} = (v_1, \ldots, v_{n_+}) \in \Lambda \otimes \mathbb{C} \) of length \( n_- \) and \( n_+ \) respectively, and a constant \( R \in \mathbb{Q} \). In terms of this data we define an Appell function of signature \((n_+, n_-)\) as a function of the form:

\[
A_{\mathbf{m}, \{m_i\}}(\mathbf{u}, \mathbf{v}, \tau) = e^{2\pi i m_0 \cdot \mathbf{u}} \sum_{k \in \Lambda} q^{\frac{1}{2}Q(k) + R e^{2\pi i \mathbf{v} \cdot \mathbf{k}}} \prod_{j=1}^{n_-} (1 - q^{m_j} e^{2\pi i u_j}).
\]

Note that expanding the denominators as a geometric sum will bring \( A_{\mathbf{m}, \{m_i\}}(\mathbf{u}, \mathbf{v}, \tau) \) to the form of an indefinite theta function of a lattice with the quadratic form given by the matrix in (4.4). Appell functions with signature \((1, 1)\) are the classical Appell functions, possibly of higher level. Appell functions of signature \((n_+, 1)\) with \( n_+ \geq 2 \) are the multi-variable Appell functions studied in [48]. To my knowledge, the functions for \( n_- > 1 \) have not appeared earlier in the literature.

Analogues of all four properties of the classical Appell function listed above are expected to exist for the Appell functions with general signature. After addition of a suitable completion, the generalized Appell functions are expected to transform as a multivariable Jacobi form with weight \((n_+ + n_-)/2\) modular form. The modular properties will also fix as usual the values \( R \) and \( m_0 \). The analogue of the fourth property is most easily established. We have

**Proposition 4.1.** Let \( \mathbf{z} = (z_1, \ldots, z_{n_-}) \) be a complex vector of length \( n_- \)

\[
\sum_{l \in \mathbb{Z}^{n_-}} A_{\mathbf{m}_i}(\mathbf{u} + l \tau, \mathbf{v}, \tau) e^{2\pi i l \cdot (\mathbf{z} - m_0 \tau) - 2\pi i m_0 \mathbf{u}} = \Theta_{\mathbf{Q}}(\mathbf{v} - \mathbf{Mz}, \tau) \prod_{j=1}^{n_-} \left( \frac{\eta(\tau)^3 \theta_1(u_j + z_j, \tau)}{\theta_1(u_j, \tau) \theta_1(z_j, \tau)} \right),
\]

where \( \Theta_{\mathbf{Q}}(\mathbf{v}, \tau) \) is a theta function for the lattice with quadratic form \( \mathbf{Q} \):

\[
\Theta_{\mathbf{Q}}(\mathbf{v}, \tau) = \sum_{k \in \Lambda} q^{\frac{1}{2}Q(k) + R e^{2\pi i \mathbf{v} \cdot \mathbf{k}}}. 
\]

**Proof.** The proof follows almost immediately from the change of the summation variables \( \mathbf{l} \rightarrow \mathbf{l} - \mathbf{M} \cdot \mathbf{k} \), and application of the identity:

\[
\sum_{m \in \mathbb{Z}} e^{2\pi i m z} = \eta(\tau)^3 \frac{\theta_1(u + z, \tau)}{\theta_1(u, \tau) \theta_1(z, \tau)}.
\]

\[\square\]

### 4.3. An example

The functions in Section 3 are clearly specializations of the generalized Appell functions as defined above. For \( \ell = 2 \) the \( \Psi_{(r_1, r_2), (a, b)} \) are specializations of higher level Appell functions. And for general \( \ell \), they have signature \((\ell - 1, \ell - 1)\). For explicitness
we consider $\Psi_{(1,1,1),(-2,2)}(z, \tau)$, whose associated quadratic form is the one of $A_3$:

$$Q_{A_3} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

(4.6)

The vectors $m_i$, $i = 0, \ldots, 3$ are given by:

$$m_0 = \frac{1}{2}(1,0,0), \quad m_1 = (1,-1,0), \quad m_2 = (0,1,-1), \quad m_3 = (1,1,2),$$

$u = -4z(1,1,1) + 2\tau(0,0,1)$, $v = (6,4,2)z + (3,3,2)\tau$, and $R = \frac{5}{2}$.

We note that the elliptic periodicity transformations of $u$ and $v$ in $A_{Q,(m_i)}(u, v, \tau)$ should imply the identities implied by the blow-up formulas among $H_{r,c_1}(z, \tau; J_{1,0})$. They also provide non-trivial information about the zeros and poles of the Appell functions. We illustrate this with the example of $(r, c_1) = (3, H)$. The blow-up formula gives for this case:

$$\frac{H_{3,C+f}(z, \tau; J_{1,0})}{B_{3,0}(z, \tau)} = \frac{H_{3-C+f}(z, \tau; J_{1,0})}{B_{3,1}(z, \tau)} = \frac{H_{3,f}(z, \tau; J_{1,0})}{B_{3,1}(z, \tau)},$$

(4.7)

where

$$H_{3,C+f}(z, \tau; J_{1,0}) = -H_1(z, \tau)H_2(z, \tau) \left( \sum_{k \in \mathbb{Z}} \frac{w^{6k}q^{\frac{1}{2}(3k+1)^2+\frac{1}{2}(3k+1)}}{1-w^{-6}q^{3k+1}} + \sum_{k \in \mathbb{Z}} \frac{w^{6k-6}q^{\frac{1}{2}(3k-1)^2+\frac{1}{2}(3k-1)}}{1-w^{-6}q^{3k-1}} \right)$$

$$+ H_{1,0}(z, \tau)^3 \sum_{k_1,k_2 \in \mathbb{Z}} \frac{w^{2(k_1+2k_2-3)}q^{k_1^2+k_2^2+k_1k_2-\frac{1}{2}}}{(1-w^{-4}q^{2k_1+k_2-1})(1-w^{-4}q^{k_1+k_2})},$$

(4.8)

and

$$H_{3,f}(z, \tau; J_{1,0}) = -H_1(z, \tau)H_2(z, \tau) \left( \sum_{k \in \mathbb{Z}} \frac{w^{6k-2}q^{3k^2+k}}{1-w^{-6}q^{3k}} + \sum_{k \in \mathbb{Z}} \frac{w^{6k-4}q^{3k^2+2k}}{1-w^{-6}q^{3k}} \right)$$

$$+ H_1(z, \tau)^3 \sum_{k_1,k_2 \in \mathbb{Z}} \frac{w^{2(k_1+2k_2-2)}q^{k_1^2+k_2^2+k_1k_2+k_1+k_2}}{(1-w^{-4}q^{2k_1+k_2})(1-w^{-4}q^{k_1+k_2})} + H_3(z, \tau).$$

(4.9)

One can show using techniques from [III] that the function $B_{3,1}(z, \tau)$ has zeroes at torsion points: $z_0 = \frac{2}{3}, n = 1, 2 \mod 3$, whereas $B_{3,0}(z, \tau)$ has zeros for $z_0 = \pm(\frac{1}{2} + \frac{1}{4} + \nu(\tau))$ where

$$\nu(\tau) = -2\pi \int_{\tau}^{i\infty} (t-\tau)F(t)dt$$

$$= \frac{q^{\frac{1}{2}}}{2\pi} \left( 1 - \frac{11}{6}q + \frac{243}{40}q^2 + \ldots \right),$$

(4.10)

with $F(\tau) = \frac{\eta(\tau)^{12}\eta(3\tau)^6}{(q^{1/2} \prod_{n=0}^{\infty} \eta(\tau)^5 B_{3,0}(\tau, n4+\mu/4))^{3/2}}$. It is not hard to verify that the two functions on the right hand side of the first $= \text{sign in (4.7)}$ do not have poles at points where $B_{3,0}(z, \tau)$
vanishes. Indeed one can verify order by order that $H_{3,C^+}(z, \tau; J_{1,0})$ also vanishes at those points.

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