On classification of finite dimensional algebras
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Abstract
Classification and invariants, with respect to basis changes, of finite dimensional algebras are considered. An invariant open, dense (in the Zariski topology) subset of the space of structural constants is defined. The algebras with structural constants from this set are classified and a basis to the field of invariant rational functions of structural constants is provided.
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1 Introduction
The classification of finite dimensional algebras is an important problem in Algebra. For example, the classification of finite dimensional simple and semi-simple associative algebras by Wedderburn, the classification of finite dimensional simple and semi-simple Lie algebras by Cartan are considered as key results in the theory of corresponding algebras. In these both and many other cases the used approach to the classification problem is structural (basis free, invariant). Unlike the structural approach in this paper we are going to deal with all algebras by the use of their structural constants. For the given dimension we show an invariant, open, dense subset of the space of structural constants and classify all algebras who’s system of structural constants are in this set. We provide a basis for the field of invariant rational functions of structural constants as well.

The paper is organized in the following way. The key results which are used to obtain the classification and invariants of algebras are presented in Section 2. Section 3 is a realization of Section 2 results in the case of representation of general linear group in the space structural constants.

2 Preliminaries
In this section we consider a linear representation of the general linear group and under an assumption prove some general results about the equivalence and invariance problems with respect to this representation.

Let $n$, $m$ be any natural numbers, $\tau : (G, V) \to V$ be a fixed linear algebraic representation of an algebraic subgroup $G$ of $GL(m, F)$ on $V$, where $F$ is any field and $V$ is $n$-dimensional vector space over $F$. Further in this section it is assumed that the following assumption holds true.

Assumption. There exists a nonempty open(in Zariski topology) $G$-invariant subset $V_0$ of $V$ and an algebraic map $P : V_0 \to G$ such that
\[ P(\tau(g, v)) = P(v)g^{-1} \]  \hspace{1cm} (1)
whenever \( v \in V_0 \) and \( g \in G \).

**Theorem 2.1.** Elements \( u, v \in V_0 \) are \( G \)-equivalent, that is \( u = \tau(g, v) \) for some \( g \in G \), if and only if \( \tau(P(u), u) = \tau(P(v), v) \).

**Proof.** If \( u = \tau(g, v) \) then \( \tau(P(u), u) = \tau(P(\tau(g, v)), \tau(g, v)) = \tau(P(v)g^{-1}, \tau(g, v)) = \tau(P(v), \tau(g^{-1}, \tau(g, v))) = \tau(P(v), v) \).

Visa versa, if \( \tau(P(u), u) = \tau(P(v), v) \) then

\[ \tau(P(u)^{-1}P(v), v) = \tau((P(u))^{-1}, \tau(P(v), v)) = \tau((P(u))^{-1}, \tau(P(u), u)) = u \]

that is \( u = \tau(g, u) \), where \( g = P(u)^{-1}P(v) \).

This proposition shows that the system of components of \( \tau(P(x), x) \) is a separating system of invariants for the \( G \)-orbits in \( V_0 \), where \( x = (x_1, x_2, ..., x_n) \) is an algebraic independent system of variables over \( F \).

Let \( V_{00} \) stand for \( \{ v \in V_0 : P(v) = I_m \} \), where \( I_m \) stands for the \( m \)-order identity matrix. If \( w = \tau(g, v) \) and \( w, v \in V_{00} \) then \( P(w) = P(\tau(g, v)) = P(v)g^{-1} \) that is \( g = I_m \).

It implies that each \( G \)-orbit from \( V_0 \) has only one common element with \( V_{00} \). Each \( G \)-invariant function on \( V_0 \) is uniquely defined by its values on \( V_{00} \).

Further in this paper it is assumed that \( F \) is an infinite field.

**Theorem 2.2.** The field extension \( F \subset F(x)^G \) is a pure transcendental extension. Moreover if \( G = GL(m, F) \) then \( F(x)^G \) is generated over \( F \) by the system of components of \( \tau(P(x), x) \), that is the equality

\[ F(x)^G = F(\tau(P(x), x)) \]

holds true.

**Proof.** To prove the pure transcendence we show that \( V_{00} \) is dense in \( Span_F(V_{00}) \).

If a polynomial map \( p : Span_F(V_{00}) \to F \) is identically zero on \( V_{00} \) then consider the polynomial map \( p' : V_0 \to F \) defined by \( p'(v) = p(v_0) \), whenever \( v = \tau(g, v_0) \) for some \( g \in G \) and \( v_0 \in V_{00} \), which is well defined. It implies that \( p' \) has to be zero polynomial on \( V \) as far as \( V_0 \) is dense in \( V \). In particular \( p' \) is zero on \( Span_F(V_{00}) \) as well, that is \( p \) has to be zero on \( Span_F(V_{00}) \).

It is evident that all components of \( \tau(P(x), x) \) are in \( F(x)^G \). If \( f(x) = f(\tau(g, x)) \) for all \( g \in G = GL(m, F) \) then \( f(x) = f(\tau(g, x)) = f(\tau(P(\tau(g, x))^{-1}P(x), x)) = f(\tau(P(\tau(g, x))^{-1}, \tau(P(x), x))) \).

In particular it is true for \( g = P(x) \). But in infinite field case the equality

\[ P(\tau(P(x), x)) = I_m \]

is valid as far as \( P(\tau(g, v)) = P(v)g^{-1} \) whenever \( v \in V_0 \) and \( g \in G \). Therefore due to it one has \( f(x) = f(\tau(P(x), x)) \).

Further it is assumed that \( G = GL(m, F) \).

**Corollary.** The field \( F(x) \) is generated over \( F(x)^G \) by the system of components of \( P(x) \) that is the following equality is valid

\[ F(x)^G(P(x)) = F(x). \]
Proof. Indeed
\[ F(x)^G(P(x)) = F(\tau(P(x), x))(P(x)) = F(\tau(P(x), x), P(x)) \]
and \( \tau(P(x)^{-1}, \tau(P(x), x)) = x \) and therefore \( F(x)^G(P(x)) = F(x) \).

**Theorem 2.3.** The transcendence degree of \( F(x)^G \) over \( F \) equals to \( n - m^2 \) and the field extension \( F(x)^G \subset F(x) \) is a pure transcendental extension.

**Proof.** We show that the system of components of \( P(x) \) is algebraic independent over \( F(x)^G \). Let \( f((y_{ij})_{i,j=1,2,...,m}) \) be any polynomial over \( F(x)^G \) and \( f(P(x)) = 0 \) which means that \( f_v(P(v)) = 0 \) for all \( v \in V_1 \), where \( V_1 \) is a \( G \)-invariant nonempty open subset of \( V_0 \), \( f_v((y_{ij})_{i,j=1,2,...,m}) \) stands for the polynomial obtained from \( f((y_{ij})_{i,j=1,2,...,m}) \) by substitution \( v \) for \( x \). Therefore for any \( v \in V_1 \) and \( g \in G \) one has \( 0 = f_v(P(\tau(g, v))) = f_v(P(v)g^{-1}) \). In particular \( 0 = f_v(g) \) for any \( g \in G \) that is \( 0 = f_v((y_{ij})_{i,j=1,2,...,m}) \) whenever \( v \in V_1 \). It implies that \( f((y_{ij})_{i,j=1,2,...,m}) = 0 \) is the zero polynomial. Now due to \( F \subset F(x)^G \subset F(x) \), \( \text{tr.deg.} F(x)/F = n \) and the Corollary one has the required result.

**Question 1.** Due to Theorem 2.2 the above assumption is a sufficient condition to state that \( F \subset F(x)^G \) is a pure transcendental extension. Is it (somehow) a necessary condition to state that as well?

**Question 2.** Is it a typical situation for any representation of \( G = GL(m, F) \) that if one of the extensions \( F \subset F(x)^G \), \( F(x)^G \subset F(x) \) is a pure transcendental extension then the second one also has the same property?

3 Classification of algebras

3.1 General case

In this paper we use the standard notation (the Einstein notation) for tensors as well as the matrix representation for tensors which is more convenient in dealing with equivalence and invariance problems of tensors with respect to basis changes. The use of matrix representation for tensors makes the descriptions more transparent as well.

Let us consider any \( m \) dimensional algebra \( W \) with multiplication \( \cdot \) given by a bilinear map \( (u, v) \mapsto u \cdot v \). If \( e = (e^1, e^2, ..., e^m) \) is a basis for \( W \) then one can represent the bilinear map by a matrix \( A \in \text{Mat}(m \times m^2; F) \) such that
\[
u \cdot v = eA(u \otimes v)\]
for any \( u = eu, v = ev \), where \( u = (u_1, u_2, ..., u_m), v = (v_1, v_2, ..., v_m) \) are column vectors. So the binary operation (bilinear map, tensor) is presented by the matrix \( A \in \text{Mat}(m \times m^2; F) \) with respect to the basis \( e \). Further we deal only with such matrices of rank \( m \).

If \( e' = (e'^1, e'^2, ..., e'^m) \) is also a basis for \( W, g \in G = GL(m, F) \), \( e'g = e \) and \( u \cdot v = e'B(u' \otimes v') \), where \( u = e'u, v = e'v' \), then \( u \cdot v = eA(u \otimes v) = e'B(u' \otimes v') = eg^{-1}B(gu \otimes gv) = eg^{-1}B(g \otimes g)(u \otimes v) \) as far as \( u = eu = e'u = eg^{-1}u', v = ev = e'v' = eg^{-1}v' \). Therefore the equality
\[
B = gA(g^{-1}) \otimes g^{-2}
\]
is valid.

Now let \( \tau \) stand for the representation of \( G = GL(m, F) \) on the \( n = m^3 \) dimensional vector space \( V = \text{Mat}(m \times m^2; F) \) defined by
\[
\tau : (g, A) \mapsto B = gA(g^{-1} \otimes g^{-1}).
\]

3
To have Theorems 2.1-2.3 for this case we will construct a map \( P : V_0 \rightarrow GL(m, F) \) with property (1) in the following way. For any natural number \( k \) due to (2) one has
\[
B^\otimes k = g^\otimes k A^\otimes k (g^{-1})^\otimes 2k \tag{3}
\]
Let us consider all its possible contractions with respect to \( k \) upper and \( k \) lower indices. It is clear that the result of each of such contraction will be \( f(B) = f(A)(g^{-1})^\otimes k \) type equality, where \( f(A) \) is a row vector with \( m^k \) entries.

In \( k = 1 \) case one gets the following \( 2^1! = 2 \) different row equalities: \( Tr_1(B) = Tr_1(A)g^{-1} \), \( Tr_2(B) = Tr_2(A)g^{-1} \), where \( Tr_1(A) \) stands for the row vector with entries
\[
A_{j,i} = \sum_{j=1}^{n} A_{j,i} \text{ the contraction on the first upper and lower indices}
\]
and \( Tr_2(A) \) stands for the row vector with entries
\[
A_{i,j} = \sum_{j=1}^{n} A_{i,j} \text{ the contraction on the first upper and second lower indices.}
\]

In \( k = 2 \) case one gets the following \( 2^22! + 2^11! = 10 \) different row equalities:
\[
Tr_1(B) \otimes Tr_1(B) = (Tr_1(A) \otimes Tr_1(A))(g^{-1})^\otimes 2, \quad Tr_1(B)B = Tr_1(A)A(g^{-1})^\otimes 2,
\]
where \( i, j = 1, 2, \) and
\[
(B^i_j B^j_{i,q}) = (A^i_{j,p} A^j_{i,q})(g^{-1})^\otimes 2, \quad (B^i_j B^j_{q,i}) = (A^i_{j,p} A^j_{i,q})(g^{-1})^\otimes 2,
\]
\[
(B^i_{p,j} B^j_{q,i}) = (A^i_{p,j} A^j_{q,i})(g^{-1})^\otimes 2, \quad (B^i_{p,j} B^j_{i,q}) = (A^i_{p,j} A^j_{i,q})(g^{-1})^\otimes 2.
\]

In any \( k \) case only the number of contractions of \( A^\otimes k \) when all \( k \) different upper indices are contracted with lower indices of different \( A \) is
\[
(2k) \times (2(k - 1)) \times (2(k - 2)) \times ... \times 2 = 2^k k!.
\]
In general it is nearly clear that the corresponding resulting system of \( 2^k k! \) rows depending on the variable matrix \( A := X = (X^i_{j,k})_{i,j,k=1,2,...,m} \) is linear independent over \( F \). But for big enough \( k \) the inequality \( 2^k k! \geq m^k \) holds true as well. Therefore in general for big enough \( k \) it is possible to choose \( m^k \) contractions (rows) among the all contractions of \( X^\otimes k \) for which the matrix \( Q(X) \) consisting of these \( m^k \) rows is a nonsingular matrix. For the matrix \( Q(X) \) one has equality \( Q(Y) = Q(X)(g^{-1})^\otimes k \) whenever \( g \in G, Y = gX(g^{-1})^\otimes 2 \).

Now note that for any \( A \in \{ X : \det(Q(X)) \neq 0 \} \) and \( g \in G \) one has, for example,
\[
(B \otimes (Tr_1(B))^\otimes (k-2))Q(B)^{-1} = g(A \otimes (Tr_1(A))^\otimes (k-2))(g^{-1})^\otimes k Q(A)(g^{-1})^\otimes k)^{-1} = g(A \otimes (Tr_1(A))^\otimes (k-2))Q(A)^{-1}.
\]
Therefore if \( P(A)^{-1} \) stands for arbitrary nonsingular \( m \times m \) size sub-matrix of \( (A \otimes (Tr_1(A))^\otimes (k-2))Q(A)^{-1} \) then one has the equality \( P(B)^{-1} = gP(A)^{-1} \), where \( g \in G, B = gA(g^{-1})^\otimes 2 \). It implies that whenever \( A \in V_0 = \{ A : \det(P(A)) \det(Q(A)) \neq 0 \} \) the equality \( P(B) = P(A)g^{-1} \) holds true for any \( g \in G \) and \( B = gA(g^{-1})^\otimes 2 \). Note that
\[
V_0 = \{ A : \det(P(A)) \det(Q(A)) \neq 0 \}
\]
is a \( G \)-invariant, open and dense subset of \( V \).

Therefore we have the following results.

**Theorem 3.1.** Two algebras with matrices of structural constants \( A, B \in V_0 \) are same (isomorph) algebras if and only if
\[
P(A)A(P(A)^{-1} \otimes P(A)^{-1}) = P(B)B(P(B)^{-1} \otimes P(B)^{-1}).
\]
Theorem 3.2. The field of $G$-invariant rational functions $F(X)^G$ of structural constants defined by variable matrix $X = (X^{i,j,k})_{i,j,k=1,2,...,m}$ is generated by the system of entries of $P(X)X(P(X)^{-1} \otimes P(X)^{-1})$ over $F$, that is the equality

$$F(X)^G = F(P(X)X(P(X)^{-1} \otimes P(X)^{-1}))$$

holds true. Moreover the extension $F \subset F(X)^G$ is a pure transcendental extension.

Theorem 3.3. The transcendence degree of $F(X)^G$ over $F$ equals to $m^3 - m^2$ and the field extension $F(X)^G \subset F(X)$ is a pure transcendental extension.

Now let us consider two and three dimensional algebra cases.

Example 1. Two dimensional $(m = 2)$ case. Let

$$A = \begin{pmatrix}
A_{1,1}^1 & A_{1,2}^1 & A_{2,1}^1 & A_{2,2}^1 \\
A_{1,1}^2 & A_{1,2}^2 & A_{2,1}^2 & A_{2,2}^2
\end{pmatrix}$$

be the matrix of structural constants with respect to a basis. In this case at $k = 1$ already $2^41! = m^4$ and therefore for the rows of $P(A)$ on can take

$$\text{Tr}_1(A) = (A_{1,1}^1 + A_{2,1}^2, A_{1,2}^1 + A_{2,2}^2)$$

and $V_0$ consists of all $A$ for which

$$\det P(A) = (A_{1,1}^1 + A_{2,1}^2)(A_{2,1}^1 + A_{2,2}^2) - (A_{1,1}^1 + A_{2,2}^2)(A_{1,1}^1 + A_{1,2}^2) \neq 0.$$ 

To see the corresponding system of generators one can evaluate

$$P(X)X(P(X)^{-1} \otimes P(X)^{-1}), \text{ where } X = \begin{pmatrix}
X_{1,1}^1 & X_{1,2}^1 & X_{1,2}^1 & X_{1,2}^1 \\
X_{2,1}^2 & X_{2,1}^2 & X_{2,1}^2 & X_{2,1}^2
\end{pmatrix}. $$

For other approaches to the classification problem of two dimensional algebras one can see [1,2].

Example 2. Three dimensional $(m = 3)$ case. Let

$$A = \begin{pmatrix}
A_{1,1}^1 & A_{1,2}^1 & A_{1,3}^1 & A_{2,1}^1 & A_{2,2}^1 & A_{2,3}^1 & A_{3,1}^1 & A_{3,2}^1 & A_{3,3}^1 \\
A_{1,1}^2 & A_{1,2}^2 & A_{1,3}^2 & A_{2,1}^2 & A_{2,2}^2 & A_{2,3}^2 & A_{3,1}^2 & A_{3,2}^2 & A_{3,3}^2 \\
A_{1,1}^3 & A_{1,2}^3 & A_{1,3}^3 & A_{2,1}^3 & A_{2,2}^3 & A_{2,3}^3 & A_{3,1}^3 & A_{3,2}^3 & A_{3,3}^3
\end{pmatrix}$$

be the matrix of the structural constants with respect to a basis.

In this case at $k = 1$ one has $2^41! < 3!$. At $k = 2$ already $2^22! + 2^41! = 10 > 3^2$ and the following 10 equalities

$$\text{Tr}_i(B) \otimes \text{Tr}_j(B) = \text{Tr}_i(A) \otimes \text{Tr}_j(A)(g^{-1})^{\otimes 2}, \text{ Tr}_i(B)B = \text{Tr}_i(A)(g^{-1})^{\otimes 2},$$

where $i, j = 1, 2,$

$$(B_{j,i,p}^{i,j})(g^{-1})^{\otimes 2}, \text{ (B}_{j,i,p}^{i,j})(g^{-1})^{\otimes 2}, \text{ (B}_{j,i,p}^{i,j})(g^{-1})^{\otimes 2}, \text{ (B}_{j,i,p}^{i,j})(g^{-1})^{\otimes 2}$$

holds true.

Therefore, for example, for $Q(A)$ one can take the following matrix
\[
Q(A) = \begin{pmatrix}
A_{i,1}^1 A_{j,1}^2 & A_{i,1}^1 A_{j,2} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,2} \\
A_{i,1}^1 A_{j,1}^3 & A_{i,1}^1 A_{j,3} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,3} \\
A_{i,1}^1 A_{j,1}^2 & A_{i,1}^1 A_{j,2} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,2} \\
A_{i,1}^1 A_{j,1}^3 & A_{i,1}^1 A_{j,3} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,3} \\
A_{i,2}^1 A_{j,1}^i & A_{i,2}^1 A_{j,2} & A_{i,2}^1 A_{j,3} & A_{i,2}^1 A_{j,1}
\end{pmatrix}
\]

For \( P(A)^{-1} \) one can take any 3 \times 3 size nonsingular sub-matrix of 

\((A \otimes \text{Tr}_1(A))Q(A)^{-1} \), where 

\((A \otimes \text{Tr}_1(A)) = \begin{pmatrix}
A_{i,1}^1 A_{j,1}^1 & A_{i,1}^1 A_{j,2} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,2} \\
A_{i,1}^1 A_{j,1}^3 & A_{i,1}^1 A_{j,3} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,3} \\
A_{i,2}^1 A_{j,1}^2 & A_{i,2}^1 A_{j,2} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,2} \\
A_{i,2}^1 A_{j,1}^3 & A_{i,2}^1 A_{j,3} & A_{i,2}^1 A_{j,1} & A_{i,2}^1 A_{j,3} \\
A_{i,1}^2 A_{j,1}^i & A_{i,1}^2 A_{j,2} & A_{i,2}^2 A_{j,1} & A_{i,2}^2 A_{j,2} \\
A_{i,1}^2 A_{j,1}^3 & A_{i,1}^2 A_{j,3} & A_{i,2}^2 A_{j,1} & A_{i,2}^2 A_{j,3} \\
A_{i,2}^2 A_{j,1}^2 & A_{i,2}^2 A_{j,2} & A_{i,2}^2 A_{j,1} & A_{i,2}^2 A_{j,2} \\
A_{i,2}^2 A_{j,1}^3 & A_{i,2}^2 A_{j,3} & A_{i,2}^2 A_{j,1} & A_{i,2}^2 A_{j,3} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,1}^3 A_{j,1} & A_{i,1}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \\
A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} & A_{i,2}^3 A_{j,1} & A_{i,2}^3 A_{j,2} \
\end{pmatrix}
\]

### 3.2 Commutative and anti-commutative algebra cases

For the classification purpose instead of all \( m \) dimensional algebras one can consider only such commutative or anti-commutative algebras. The commutativity (anti-commutativity) of the binary operation in terms of the corresponding matrix \( A \) means \( A_{j,k}^i = A_{k,j}^i \) (respectively, \( A_{j,k}^i = -A_{k,j}^i \)) for all \( i, j, k = 1, 2, \ldots, m \). So in commutative (anti-commutative) algebra case for the \( V \) we consider

\[ V = \{ A \in \text{Mat}(m \times m^2; F) : A_{j,k}^i = A_{k,j}^i \text{ (resp. } A_{j,k}^i = -A_{k,j}^i \text{) for all } i, j, k = 1, 2, \ldots, m \} \]

Note that in commutative (anti-commutative) case the dimension of \( V \) is \( \frac{m(m+1)}{2} \) (respectively, \( \frac{m(m-1)}{2} \)).

To have Theorems 2.1-2.3 for these cases one can construct a map \( F : V_0 \to GL(m, F) \) with property (1) in a similar way as in the general algebra case. Consider once again equality (3) and all its possible contractions with respect to \( k \) upper and \( k \) lower indices.
In commutative (anti-commutative) case at \( k = 1 \) one gets the following \( 1! = 1 \) row equality: \( \text{Tr}_1(B) = \text{Tr}_1(A)g^{-1} = \text{Tr}_2(A)g^{-1} \) as far as \( A_{j,k}^i = A_{k,j}^i \) (respectively, \( \text{Tr}_1(B) = \text{Tr}_1(A)g^{-1} = -\text{Tr}_2(A)g^{-1} \) as far as \( A_{j,k}^i = -A_{k,j}^i \)) for all \( i, j, k = 1, 2, \ldots, m \).

In \( k = 2 \) case one gets the following \( 2! + 1! = 3 \) different row equalities:

\[
\text{Tr}_1(B) \otimes \text{Tr}_1(B) = (\text{Tr}_1(A) \otimes \text{Tr}_1(A))(g^{-1}) \otimes^2,
\]
\[
\text{Tr}_1(B)B = \text{Tr}_1(A)A(g^{-1}) \otimes^2, \quad (B_{j,p}^i B_{i,q}^j) = (A_{j,p}^i A_{i,q}^j)(g^{-1}) \otimes^2.
\]

In any \( k \) case only the number of contractions of \( A^\otimes_k \) when all \( k \) different upper indices are contracted with lower indices of different \( A \) is \( k! \). Once again in general it is nearly clear that the corresponding resulting system of \( k! \) rows depending on variable matrix \( A := X = (X_{j,k}^i)_{i,j,k=1,2,\ldots,m} \), where \( X_{j,k}^i = X_{k,j}^i \) (respectively, \( X_{j,k}^i = -X_{k,j}^i \)) for all \( i, j, k = 1, 2, \ldots, m \), is linear independent over \( F \). But for big enough \( k \) the inequality \( k! \geq m^k \) holds true as well. Therefore in general for big enough \( k \) it is possible to choose \( m^k \) contractions (rows) among the all contractions of \( X^\otimes_k \) for which the matrix \( Q(X) \) consisting of these \( m^k \) rows is nonsingular. For the matrix \( Q(X) \) one has equality \( Q(Y) = Q(X)(g^{-1}) \otimes^k \) whenever \( g \in G, Y = gX(g^{-1}) \otimes^2 \).

Therefore if \( P(A)^{-1} \) stands for arbitrary \( m \times m \)-size nonsingular sub-matrix of \( (A \otimes (\text{Tr}_1(A))^\otimes_k)^{-1} \) then one has the equality \( P(B)^{-1} = gP(A)^{-1} \), where \( g \in G, B = gA(g^{-1}) \otimes^2 \). It implies that whenever \( A \in V_0 = \{ A \in V : \det(P(A)) \det(Q(A)) \neq 0 \} \) the equality \( P(B) = P(A)g^{-1} \) holds true for any \( g \in G \), where \( B = gA(g^{-1}) \otimes^2 \).

Note that

\[
V_0 = \{ A \in V : \det(P(A)) \det(Q(A)) \neq 0 \}
\]

is a \( G \)-invariant, open and dense subset of \( V \).

Therefore we have the following results.

**Theorem 3.1’**. Two commutative (anti-commutative) algebras with the matrices of structural constants \( A, B \in V_0 \) are the same algebras if and only if

\[
P(A)A(P(A)^{-1} \otimes P(A)^{-1}) = P(B)B(P(B)^{-1} \otimes P(B)^{-1}).
\]

**Theorem 3.2’**. The field of \( G \)-invariant rational functions \( F(X)^G \) of the structural constants presented by the matrix \( X = ((X_{j,k}^i)_{i,j,k=1,2,\ldots,m} \) of the variable commutative (respectively, anti-commutative) algebras, where \( X_{j,k}^i = X_{k,j}^i \) (respectively, \( X_{j,k}^i = -X_{k,j}^i \)) for all \( i, j, k = 1, 2, \ldots, m \), is generated by the system of entries of \( P(X)X(P(X)^{-1} \otimes P(X)^{-1}) \) over \( F \), that is the equality

\[
F(X)^G = F(P(X)X(P(X)^{-1} \otimes P(X)^{-1}))
\]

holds true. Moreover the extension \( F \subset F(X)^G \) is a pure transcendental extension.

**Theorem 3.3’**. In commutative (anti-commutative) algebra case the transcendence degree of \( F(X)^G \) over \( F \) equals to \( \frac{m^2(m-1)}{2} \) (respectively, \( \frac{m^2(m-3)}{2} \), \( m \geq 3 \)) and the field extension \( F(X)^G \subset F(X) \) is a pure transcendental extension.

Now let us consider two dimensional commutative algebra case.

**Example 3.** Let

\[
A = \begin{pmatrix}
A_{1,1}^1 & A_{1,2}^1 & A_{2,1}^2 & A_{2,2}^2 \\
A_{1,1}^2 & A_{1,2}^2 & A_{2,1}^1 & A_{2,2}^1
\end{pmatrix}
\]
where $A_{1,2}^i = A_{2,1}^i$ at $i = 1, 2$, be the matrix of structural constants of a commutative algebra with respect to a basis. Consider $B^{\otimes 3} = g^{\otimes 3} A (g^{-1})^{\otimes 6}$ and its all contractions on 3 upper and 3 lower indices. Among them in particular one gets the following 6 equalities:

$$(B_{\sigma(i),p}^r B_{\sigma(j),q}^s B_{\sigma(k),r}^t) = (A_{\sigma(i),p}^r A_{\sigma(j),q}^s A_{\sigma(k),r}^t) (g^{-1})^{\otimes 3}$$

where $\sigma \in S_3$ - the symmetric group of permutations of symbols $i, j, k$, and

$$(B_{r,ij}^p B_{r,ij}^q B_{r,ij}^r) = (A_{r,ij}^p A_{r,ij}^q A_{r,ij}^r) (g^{-1})^{\otimes 3}, (B_{p,q}^r B_{r,ij}^q B_{r,ij}^r) = (A_{p,q}^r A_{r,ij}^q A_{r,ij}^r) (g^{-1})^{\otimes 3}.$$ 

So for $Q(A)$ one can take the matrix consisting of the following 8 rows

$$(A_{\sigma(i),p}^r A_{\sigma(j),q}^s A_{\sigma(k),r}^t)_{\sigma \in S_3}, (A_{p,q}^r A_{r,ij}^q A_{r,ij}^r), (A_{p,q}^r A_{r,ij}^q A_{r,ij}^r)$$

and for the $P(A)^{-1}$ any nonsingular 2 $\times$ 2-size sub-matrix of $(A \otimes Tr_1(A))Q(A)^{-1}$ provided that $\det(Q(A)) \neq 0$.

For an approach to classification of three dimensional anti-commutative algebras one can see [3].

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