Dynamical heredity from f(R)-bulk to braneworld: curvature dynamical constraint and emergent unimodular gravity

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A few years ago, Borzou et al (BSSY) provided a $F(R)$ generalization of Shiromizu-Maeda-Sazaki (SMS) formulation. The main result of them is an effective tensor that provides a correction in the Einstein equations right side besides SMS correction. Instead of this perspective, we require it in the left side acting as a generator of $f(R)$ theories on the brane. Thus we have additionally a $f(R)$ theory in the left side and the SMS stress-tensor in the right side. As the BSSY tensor carries the $F(R)$ functions, we will introduce a procedure in which is possible relate a $F(R)$-bulk with an effective $f(R)$-brane using the concept of curvature dynamical constraint (CDC). With a dynamical equation involving the extrinsic and 5D/4D intrinsic curvatures, the CDC relates the bulk-brane scalaron theories, i.e., the 5D/4D Ricci curvature dynamics while the Gauss equations trace (GDC) gives us a geometrical relation among the objects. We will show also that inside of our formulation, there is hidden a generalized $f(R)$-unimodular gravity in which it becomes the usual case when $f(R) \to R$. The connection between the $f(R)$-theory and the unimodular theory is given by an eigenvalue-like equation. Finally we should present some algebrical/cosmological manifestations connected with our formulation.

I. INTRODUCTION

Presenting a great variety of alternative models, the braneworld paradigm had as precursors the following studies [1–5]. Considering a 5D bulk, SMS obtained the projected Einstein equations on a 3-thin braneworld [6][7]. The celebrated Randall-Sundrum model (RS) with infinite extra dimension [3] can be obtained of SMS formulation when the bulk is an anti-de Sitter space-time: $AdS_5$. A lot of these ideas were inspired on string theory advances and same the RS scenario can be obtained of Horava-Witten theory [8]. Furthermore, the SMS formulation provides us two corrections the usual gravity theory, namely the projected Weyl tensor on the brane $\mathcal{E}_{\mu\nu}$ and a high-energy correction $\pi_{\mu\nu}$. The term $\pi_{\mu\nu}$ should be considered in the early universe when the quadratic matter-energy density could overcome the brane tension of our universe [6]. Despite $\pi_{\mu\nu}$ change the 4D Einstein
theory in the matter-energy presence, the object $E_{\mu\nu}$ can modify the vacuum theory, e.g., the black holes theories \cite{9,12} as also to induce a LTB space-time on the brane \cite{12}. For a cosmological scenario, both $\pi_{\mu\nu}$ and $E_{\mu\nu}$ lead the a correction in the Friedmann equation \cite{13,14} as also in the black hole metric with electromagnetic radiation \cite{15}. Actually $E_{\mu\nu}$ generates an effective radiation on the brane that is called of dark radiation, whose its source can be for example a Schwarzschild-AdS$_5$ bulk \cite{16}. A possible connection of $E_{\mu\nu}$ and $\pi_{\mu\nu}$ with the AdS/CFT correspondence can be seen at \cite{17,18}.

Instead of extra dimensional generalization of General Relativity, it is viable also a dynamical extension of the mentioned theory. A possible modification is provided by the $f(R)$ theories \cite{19,22} (see also \cite{23} and \cite{24}). Basically it substitutes the curvature term of Einstein-Hilbert functional by a generalized form: $f(R)$. Thus, the least action principle leads us to an alternative field equations in which they keep the fundamental Einstein idea. The trace of the field equations provides us a dynamical scalar equation with an additional freedom degree that is sometimes called of scalaron field by several authors. The scalaron can be mapped in the inflaton field when our gravity theory presents by itself as the Starobinsky theory: $f(R) = R + aR^2$ \cite{23}. Joining a $\mathcal{F}(\mathcal{R})$ bulk with the SMS procedure, BSSY \cite{26} found a new effective tensor on the brane: $Q_{\mu\nu}$. This correction carries the $\mathcal{F}(\mathcal{R})$ functions and increases with $E_{\mu\nu}$ and $\pi_{\mu\nu}$ the full modification of the Einstein equations. In the ref. \cite{27}, topological brane-world black hole have been obtained for constant scalar curvature. Since $Q_{\mu\nu}$ appears in the metric of topologically charged black holes as an effective cosmological constant on the brane, in the ref. \cite{28}, it is studied it by the classical tests of General Relativity. For a cosmological application, to see \cite{26}.

Thinking at a mechanism in which the $\mathcal{F}(\mathcal{R})$-bulk projects an effective $f(R)$-brane, we will use the BSSY equations and we will state an additional requirement:

"when $Q_{\mu\nu}$ is in the left side, it leads the an effective $f(R)$-theory on the brane."

With the mentioned requirement, we can obtain a general bulk-brane scalaron equation where $Q = Q_{\mu}^{\mu}$ plays a fundamental role. The philosophical idea is that $Q$ inherits the information of $\mathcal{F}(\mathcal{R})$ such that the $f(R)$ solution of our differential equation is directly given by $\mathcal{F}(\mathcal{R})$-bulk dynamics. There are two stages in this process. First, the $f(R)$-brane can be obtained by express only a particular equation for $Q$. In this stage, we do not need the previous knowledge of the $\mathcal{F}(\mathcal{R})$-bulk. Already in the second stage, we should choose an equation that reports dynamically the extrinsic and intrinsic curvatures. It is a dynamical version of the Gauss equations trace. Still, it is necessary when we want to provide a specific scenario where a $\mathcal{F}(\mathcal{R})$-bulk projects a $f(R)$-brane.
Thus a $\mathcal{F}(\mathcal{R})$-bulk and a $f(\mathcal{R})$-brane imply at a CDC as follows in the pictorial scheme:

A crucial point that will be observed here is that the simultaneous existence of BSSY model with our formulation implies in the unimodular gravity at a specific case. Actually, it implies a generalized $f(\mathcal{R})$-unimodular gravity formally identical with the obtained by Nojiri, Odintsov & Oikonomou (NOO) [29, 30]. Nowadays, the traceless Einstein tensor theory or equivalently unimodular gravity [31, 32] gives us hope for the cosmological constant problem [33, 34]. The problem is obtained when calculated its value by General Relativity or by Quantum field theory. The values obtained for each one of them are shamelessly without agreement. The point is that the unimodular gravity provides an effective cosmological constant unrelated directly with the usual cosmological constant. Strictly speaking, we will show that the unimodular gravity is obtained by an application of traceless differential operator $\Delta_{\mu\nu}$ at the $f(\mathcal{R}) \neq \mathcal{R}$ function. We will show that when $Q$ is constant it can be identified as the cosmological constant of unimodular gravity. Still, we will verificade that our formulation agrees with the obtained at [29] and provides a correction for it. We will find also cosmological expressions such as the Friedmann equation for example.

In the section 2 we will review quickly the $f(\mathcal{R})$ theories, unimodular gravity, NOO $f(\mathcal{R})$-unimodular gravity and SMS/BSSY brane formulations. Already in the section 3 we will promote our main ideas about the effective $f(\mathcal{R})$-branes. A general approach will be elaborate in the section 4, where we will see to emerge the unimodular gravity. In the section 5 we will conclude our study. In order to fix the notation, hereupon, $\{\theta_\mu\}$, with $\mu = 0, 1, 2, 3$ [and $\{\theta_a\}$, with $a = 0, 1, 2, 3, 5$] denotes a basis for the cotangent bundle on a braneworld, embedded in the 5D bulk. Furthermore, $\{e_a\}$ is its dual basis and $\theta^a = dx^a$, when a coordinate chart is chosen. Let $n = n_a \theta^a$ be a timelike covector field normal to the brane and $y$ the associated Gaussian coordinate. In particular, $n_a dx^a = dy$ on the hypersurface defined by $y = 0$. The brane metric $q_{\mu\nu}$ and the corresponding components of the bulk metric $g_{ab}$ are in general related by $g_{ab} = q_{ab} + n_a n_b$. With these choices it follows that $g_{55} = 1$ and $g_{\mu 5} = 0$, the 5D bulk metric

$$g_{ab} dx^a dx^b = q_{\mu\nu}(x^\alpha, y) dx^\mu dx^\nu + dy^2.$$ (1)
II. SHORT REVIEWS: $f(R)$ THEORIES, UNIMODULAR GRAVITY, $f(R)$-UNIMODULAR GRAVITY AND SMS/BSSY BRANE FORMULATIONS

A. $f(R)$ Theories of Gravitation

We can substitute the curvature scalar $R$ of Einstein-Hilbert action by a generic function of curvature $f(R)$ as follows

$$S = \frac{1}{8\pi G} \int d^4x \sqrt{-g} f(R) + S_{\text{matter}},$$

where $S_{\text{matter}}$ the action for the fields living in the space-time, $g$ the metric determinant and $G$ the usual Newton gravitation constant. Varying it with respect to the metric, this substitution leads the called $f(R)$ theories of gravitation \[19–24\]. Their respective field equations are given by

$$R_{\mu\nu} d_R f(R) - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) d_R f(R) = \kappa_4^2 T_{\mu\nu},$$

with $d_R \equiv d/dR$ and $\kappa_4^2 \equiv 8\pi G$. In general, the $f(R)$-theories are scalar-tensor theories, in the sense that the trace of (3) provides us a scalar dynamical equation

$$[(R + 3\Box) d_R - 2] f(R) = \kappa_4^2 T.$$

This expression has as source the trace of stress-energy tensor: $T$. For instance, if we have the theory $f(R) = R + aR^2$, then the equation (4) becomes

$$[\Box - m^2] R = m^2 \kappa_4^2 T, \quad \text{where } m \equiv \pm 1/6a^2.$$

The scalar $R$ satisfies a Klein-Gordon equation with associated mass $m \equiv 1/6a^2$ and source $\kappa_4^2 T$. Therewith, several authors sometimes have called $R$ (or same $d_R f(R)$) of scalaron field. Considering the Friedmann-Robertson-Walker (FRW) metric and $T = 0$, the expression (3) yields us (for details \[23\])

$$F \equiv 6H \partial_t \partial_t H + 18H^2 \partial_t H - 3(\partial_t H)^2 = -3m^2H^2,$$

with $H$ being the Hubble function. The result (6) composes the Starobinsky theory proposed in 1980 \[25\] and it is the first Friedmann equation in this case. This is the first model of inflation. During the inflation $F \simeq 18H^2 \partial_t H \simeq -3m^2H^2$ so that $H \simeq H_0 - (m^2/6)(t - t_0)$ leads us to an inflationary scale factor $a \simeq a_0 \exp[H_0(t - t_0) - (m^2/12)(t - t_0)^2]$. Here, $H_0$ and $a_0$ are defined in the start of the inflation $t_0$ where $H \equiv a^{-1} \partial_t a$. 


B. Unimodular Gravity

The traceless Einstein tensor theory, or the oftentime called unimodular gravity [31, 32], is based on the substitution of the Einstein tensor $G_{\mu\nu}$ in the left side by a traceless, i.e.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \rightarrow R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}. \quad (7)$$

The full field equations are here given by

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = \kappa^2 \left( T_{\mu\nu} - \frac{1}{4}Tg_{\mu\nu} \right), \quad (8)$$

because the trace should be simultaneously null in left/right side. Taking the divergence of (8) we will have

$$\nabla^\mu G_{\mu\nu} = 0 = \nabla^\mu T_{\mu\nu} : \partial_\mu (R + \frac{\kappa^2}{4}T) = 0, \quad (9)$$

such that $\Lambda^{(U)} \equiv R + \frac{\kappa^2}{4}T$ is constant. Substituting $T$ in the expression (8) we should obtain the effective General Relativity theory

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda^{(U)} g_{\mu\nu} + \kappa^2 T_{\mu\nu}. \quad (10)$$

The obtained constant is does not related with the cosmological constant found in the Einstein theory and therefore it is unrelated to the vacuum energy regarded in the Quantum field theory. Thus the unimodular gravity can “soften” the cosmological constant problem [33, 34].

C. $f(R)$-Unimodular Gravity

The combination among $f(R)$ theory with unimodular gravity have been done in the references [29, 30]. Taking the 4D action

$$S = \frac{1}{\kappa^2_4} \int d^4x \left[ \sqrt{-g} (f(R) - \mathcal{L}) + \mathcal{L} \right] + S_{\text{matter}}, \quad (11)$$

and then varying it with respect to the metric, it yields the field equations

$$R_{\mu\nu}d_R f(R) - \frac{1}{2} \left[ f(R) - \mathcal{L} \right] g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) d_R f(R) = \kappa^2_4 T_{\mu\nu}. \quad (12)$$

Here, $\mathcal{L}$ is the Lagrange multiplier function (for details [29]). When $S$ is varied by $\mathcal{L}$ is obtained the relation $\sqrt{-g} = 1$, that is called of unimodular constraint. As showed in the refs [29, 30], several interesting physical applications have been obtained. In [29] have been studied inflationary scenarios. Still, at [29] is obtained the behavior of Newton law in the $f(R)$-unimodular gravity as well as a more deepened study about is given in the ref. [30].


D. Shiromizu-Maeda-Sazaki Formulation and Its $f(R)$-Generalization

Improving the Randall-Sundrum model [2, 3], SMS extracted the effective Einstein equations for thin branes models [6, 7] (for a review see [35]). Taking the Gauss equations given by

$$R_{abcf} = q^f e_q a^d q^g b^e R_{dgh} e + 2K_{c[a} K_{b]} f,$$

as also the Israel junction condition [6, 37]

$$K_{\mu\nu} = -\frac{1}{2} k^2 \left[ \tau_{\mu\nu} + \frac{1}{3} (\lambda - \tau) q_{\mu\nu} \right],$$

and still the 5D Einstein theory

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa_5^2 T_{ab}, \quad T_{ab} = -\Lambda_5 g_{ab} + (-\lambda g_{ab} + \tau_{ab}) \delta(y),$$

they obtained the 4D effective Einstein theory

$$R_{\mu\nu} - \frac{1}{2} R q_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + \kappa_4^2 \pi_{\mu\nu} + \frac{6\kappa_4^2}{\lambda} \pi_{\mu\nu} - \mathcal{E}_{\mu\nu} \equiv J_{\mu\nu},$$

where

$$\pi_{\mu\nu} = \frac{1}{12} \tau_{\mu\nu} - \frac{1}{4} \tau_{\rho\sigma} \tau_{\mu\nu} + \frac{1}{24} (3 \tau_{\sigma\delta\rho\sigma\delta} - \tau^2) q_{\mu\nu}, \quad \tau = \tau_{\mu}^\mu.$$

Here $R_{dgh} e$ is the five dimensional Riemann tensor and $R_{abcf}$ its four dimensional version while $E_{ab} = C^d_{\;abcd} n^a n^b$ is the contraction of the bulk Weyl tensor $C^a_{\;bcd}$ with respect to the vectors $n^a$ and $q_{\mu\nu}$. The object $E_{ab}$ is projected on the brane by the action of induced metric: $E_{\mu\nu} = q_{\mu}^a q_{\nu}^b E_{ab}$. The brane extrinsic curvature at $y = 0$ is given by $K_{\mu\nu} \sim \partial_\mu q_{\nu\mu}$ at Gaussian coordinates and the $\mathbb{Z}_2$-symmetry is assumed. The brane tension is represented by $\lambda$ and $\tau_{\mu\nu}$ is the stress-energy tensor of fields on the brane. The function $\delta(y)$ provides us the thin brane localization: $y = 0$. The object $\Lambda_5$ is our bulk cosmological constant where the effective cosmological constant is given by $\Lambda_4$. The relation among $\Lambda_4$ and $\Lambda_5$ is provided by

$$\Lambda_4 = \frac{1}{2} k_5^2 \left[ \Lambda_5 + \frac{1}{6} k_5^2 \lambda^2 \right], \quad \text{with} \quad \kappa_4^2 = \frac{\lambda}{6} \kappa_5^4,$$

where $\kappa_4$ ($\kappa_5$) the four (five) dimensional gravitation constant.

As previously mentioned, the result obtained by SMS provides us two corrections to the 4D theory: $\pi_{\mu\nu}$ and $E_{\mu\nu}$. The object $\pi_{\mu\nu}$ represents a high-energy correction while $E_{\mu\nu}$ has origin in the 5D Weyl tensor. Being $E_{\mu\nu}$ traceless, we can see it as an effective radiation tensor on the brane in which it carries informations about the bulk geometry (for contextualize [6, 35, 36]). Its several
physical implications were analysed in the references \cite{9-18}. The Randall-Sundrum model \cite{3} is obtained when $\Lambda_4 = 0$, that is when the bulk is $AdS_5$. The null divergence requeriment implies

$$\nabla^\mu E_{\mu\nu} = -\frac{6\kappa_4^2}{\lambda} \nabla^\mu \pi_{\mu\nu},$$

(19)

because the divergenceless given by $\nabla^\mu G_{\mu\nu} = 0$ implies $\nabla^\mu J_{\mu\nu} = 0$.

If we want to incorporate a $\mathcal{F}(\mathcal{R})$-bulk, we should take a 5D theory described by

$$R_{ab}d_{\mathcal{R}}\mathcal{F}(\mathcal{R}) - \frac{1}{2} \mathcal{F}(\mathcal{R}) g_{ab} + (g_{ab} \Box - D_a D_b) d_{\mathcal{R}}\mathcal{F}(\mathcal{R}) = \kappa_5^2 T_{ab},$$

(20)

where $\Box \equiv D^a D_a$ is the 5D box operator while $D_a$ is the covariant derivative with respect the bulk metric $g_{ab}$. Here we have used the notation: $d_{\mathcal{R}} \equiv d/d\mathcal{R}$.

Using the SMS procedure, BSSY obtained the effective equations as follows \cite{26}

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = J_{\mu\nu} + Q_{\mu\nu},$$

(21)

where we have the new correction besides SMS formulation given by

$$Q_{\mu\nu} = \left[ F(\mathcal{R}) q_{\mu\nu} + \frac{2}{3} D_a D_b (d_{\mathcal{R}}\mathcal{F}) \left( \delta_{\mu}^a \delta_{\nu}^b + n^a n^b q_{\mu\nu} \right) \right]_{y=0},$$

(22)

with

$$F(\mathcal{R}) = -\frac{4}{15} \Box (d_{\mathcal{R}}\mathcal{F}) - \frac{\mathcal{R}}{10} \left( \frac{3}{2} + d_{\mathcal{R}}\mathcal{F} \right) + \frac{1}{4} \mathcal{F} - \frac{2}{5} \Box (d_{\mathcal{R}}\mathcal{F}).$$

(23)

Consequently, we have here three corrections in the Einstein equations. We should note that $Q_{\mu\nu}$ encopasses the $\mathcal{F}(\mathcal{R})$ effects on the brane and effectively it acts “merely” as a new stress-energy tensor on the brane.

The trace of (21) gives us

$$R = -Q - J,$$

(24)

with $J = J_{\mu}^\mu$ and $Q = Q_{\mu}^\mu$ given respectively by

$$J = -4 \Lambda_4 + \kappa_4^2 \tau + \frac{6\kappa_4^2}{\lambda} \pi_{\mu}^\mu \quad \text{and} \quad Q = \left[ 4 F(\mathcal{R}) + \frac{2}{3} D_a D_b (d_{\mathcal{R}}\mathcal{F}) \left( \delta_{ab} + 4 n^a n^b \right) \right]_{y=0},$$

(25)

where this result will be largely used here. The null divergence requirement becomes here

$$\nabla^\mu Q_{\mu\nu} = \nabla^\mu \left( E_{\mu\nu} - \frac{6\kappa_4^2}{\lambda} \pi_{\mu\nu} \right),$$

(26)

once $\nabla^\mu (J_{\mu\nu} + Q_{\mu\nu}) = 0$. In the vacuum and for a conformally flat bulk $\nabla^\mu Q_{\mu\nu} = 0$, so that we can identify it as a kind of matter. A set of approaches projecting $f(R)$-bulk at thin braneworlds can be seen in the refs. \cite{38-40}.
III. \( f(R) \)-BRANES AND CURVATURE DYNAMICAL CONSTRAINT

A. \( f(R) \)-branes

In the BSSY formulation, the \( \mathcal{F}(\mathcal{R}) \)-bulk leads for an Einstein brane with an extra stress-tensor besides of the SMS correction for 4D General Relativity. Here, we want to formulate a model whose the \( \mathcal{F}(\mathcal{R}) \)-bulk generates an effective \( f(R) \)-brane. We can make this by a trivial way. Instead of original idea \([21]\), we will require that the object \( Q_{\mu\nu} \) can operate in the left side of Einstein equations as follows

\[
G^{[f(R)]}_{\mu\nu} \equiv G^{[R]}_{\mu\nu} - Q_{\mu\nu} = J_{\mu\nu}. \tag{27}
\]

The notation \( G^{[f(R)]}_{\mu\nu} \) denotes the Einstein tensor component of a \( f(R) \) theory at four dimensions.

Therefore we can rewrite \([21]\) and \([27]\) as

\[
R_{\mu\nu} - \frac{1}{2} R q_{\mu\nu} - Q_{\mu\nu} = R_{\mu\nu} d_R f - \frac{1}{2} f q_{\mu\nu} + (q_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) d_R f, \tag{28}
\]

and

\[
R_{\mu\nu} d_R f - \frac{1}{2} f q_{\mu\nu} + (q_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) d_R f = J_{\mu\nu}, \tag{29}
\]

with their respective traces yielding us respectively

\[
[(R + 3 \Box) d_R - 2] f(R) + R = -Q, \tag{30}
\]

and

\[
[(R + 3 \Box) d_R - 2] f(R) = J. \tag{31}
\]

The expression \([31]\) is the SMS version of the scalaron dynamical equation \([4]\). Basically, in the vacuum, we do not have explicitly any difference with the scalaron theory without extra dimension. This result provides the information that the scalaron field does not explicitly affected by \( \mathcal{E}_{\mu\nu} \), namely it does not influenced by the tilde radiation. Obviously when \( \tau \neq 0 \) we will have that consider the quadratic term: \( \pi_\alpha^{\alpha} \). We can see also that the expression \([30]\) is obtained directly of \([31]\) by the application of \([24]\).

The key for we obtain an effective \( f(R) \)-brane lives in the equation \([30]\). We should note here that \( Q = Q(\mathcal{F}(\mathcal{R}(R))) \), where the relation \( \mathcal{R}(R) \) dictated by the trace of Gauss equations \([13]\), i.e.

\[
\mathcal{R} = R + \mathcal{K}, \quad \text{with} \quad \mathcal{K} = K_{ab} K^{ab} - K^2. \tag{32}
\]
We can rewrite (30) as follows

\[ \Pi[f(R)] f(R) + R = \left[\frac{2}{5} \Pi[F(R)] F(R) + \frac{3R}{5} - (d_R F)^{-1} O (d_R F) \right]_{y=0}, \]  

(33)

where we have used (25) with \( O \), \( \Pi[f(R)] \) and \( \Pi[F(R)] \) defined respectively by

\[ O \equiv \frac{2}{3} \left( \delta^{ab} + 4n^a n^b \right) D_a D_b - \frac{16}{15} \Box, \]  

(34)

\[ \Pi[f(R)] \equiv (R + 3 \Box) d_R - 2 \quad \text{and} \quad \Pi[F(R)] \equiv (R + 4 \Box) d_R - \frac{5}{2}. \]  

(35)

Taking the trace of (20), we have the 5D scalaron theory

\[ \Pi[F(R)] F(R) = \kappa^2 T, \]  

(36)

so that \( \Pi[F(R)] \) (by (31)) and \( \Pi[F(R)] \) are the brane (bulk) scalaron operators respectively. Thus, the expression (33) provides us a closed relation among the 5D scalaron theory with its 4D version. It is easy to see that when \( F(R) = R \) (\( Q = 0 \)) then \( f(R) = R \). This result gives us the original SMS theory.

Now, we will provide some non-trivial examples. For example, when the equation \( 3 \Box d_R f(R) = -Q \) is satisfied at (30), we must to solve \((-Rd_R + 2) f = 1 \). Thus, for a “Starobinsky-Shiromizu-Maeda-Sazaki” brane (SSMS brane) we will have then

\[ 3 \Box d_R f(R) = -Q: \quad Rd_R f(R) - 2 f(R) + 1 = 0 \Rightarrow f(R) = R + aR^2, \]  

(37)

with \( a \) being an arbitrary constant. With the before result, we can write the SSMS theory, whose is represented by the pair of equations

\[ G[^{R+aR^2}_{\mu\nu}] = J_{\mu\nu} = -\Lambda 4q_{\mu\nu} + \kappa^4_j \pi_{\mu\nu} + \frac{6\kappa^2}{\lambda} \pi_{\mu\nu} - E_{\mu\nu}, \]  

(38)

and

\[ a\Box R = \frac{1}{6} \left[\frac{2}{5} \Pi[F(R)] F(R) + \frac{3R}{5} - (d_R F)^{-1} O (d_R F) \right]_{y=0}. \]  

(39)

The relations (38) and (39) give us a hybrid theory in which it combines the inflationary Starobinsky model with the SMS theory. When \( a = 0 \) we recover the original SMS formulation. Using the expressions (38) and (39) we can obtain alternative forms to SSMS scalaron theory, i.e.,

\[ \Delta_{KG}^{(-)} R = m^2 J \quad \text{or} \quad \Box R = -m^2 Q \quad \text{or} \quad \Delta_{KG}^{(-)} Q = -\Box J, \]  

(40)
where $m^2 \equiv 1/6a$ and $\Delta_{K_G}^{(\pm)} \equiv \Box \pm m^2$ is the positive (negative) Klein Gordon operator. Others two examples that we can see easily are given by

\[ 3\Box d_R f + R = -Q : \quad Rd_R f = 2f \Rightarrow f(R) = \frac{fR^2}{2} \Rightarrow \Delta_{K_G}^{(+)} R = -m^2 Q, \quad (41) \]

and

\[ 3\Box d_R f - 2f = -Q : \quad d_R f = -1 \Rightarrow f(R) = -R + \epsilon \Rightarrow 2R + Q = 2\epsilon, \quad (42) \]

where $f = f_0/R_0^2$ and $\epsilon$ are constants. The expressions (42) relate us an “exotic” SMS theory given by the changes $G_{\mu\nu}^{[R]} \mapsto G_{\mu\nu}^{[R]}$ and $J_{\mu\nu}^{b=0} \mapsto -J_{\mu\nu}$. Resuming the before results, we will provide the following table:

| $R\cdot Q$ relation | $R\cdot j$ relation | $Q\cdot j$ relation |brane | bulk |
|---------------------|---------------------|---------------------|------|------|
| $\Box f(R) = 0 = Q$ | $R = -J$           | there is not        | $R$  | $R$  |
| $2R + Q = 2\epsilon$| $R - J = 2\epsilon$ | $Q + 2J = -2\epsilon$| $-R + \epsilon$ ? |
| $\Delta_{K_G}^{(+)} R = -m^2 Q$ | $\Box R = m^2 J$ | $\Box Q = -\Delta_{K_G}^{(+) } J$ | $\frac{fR^2}{2}$ ? |
| $\Box R = -m^2 Q$ | $\Delta_{K_G}^{(-)} R = m^2 J$ | $\Delta_{K_G}^{(-)} Q = -\Box J$ | $R + aR^2$ ? |

In the table above, we must to note that we do not have obtained the bulk theory. In the follows, we will provide a mecanism for this.

B. Projecting $F(R) \rightarrow f(R)$: Curvature Dynamical Constraint

Here, we will develop a mechanism that provides an effective $f(R)$-brane from $F(R)$-bulk. Thereunto, we should introduce the concept of curvature dynamical constraint (CDC). Going ahead, the equation (32) will be called of curvature geometrical constraint (CGC). The CDC concept does necessary when we want to know the $F(R)$-bulk nature explicitly. The philosophical idea is that when we require any gravitation theory with a specific dynamics on the brane, we must to have a CDC besides CGC. For example, here we have a bulk with a $F(R)$ dynamical, so that when we postulate that this bulk projects on the brane a $f(R)$ dynamical, we should provide a extra relation among the curvature objects.

Will be convenient define the objects $\Pi_s^{[f]}$, $\Pi_s^{[F]}$ and $\Theta$ as follows

\[ \Pi_s^{[f(R)]} \equiv \Pi_s^{[f(R)]} + \frac{R}{f}, \quad \Pi_s^{[F(R)]} \equiv \frac{2}{5} \Pi_s^{[F(R)]} + \frac{3R}{5F} \quad \text{and} \quad \Theta \equiv \frac{\mathcal{O}d_RF}{d_RF}. \quad (43) \]

With the mentioned definition, we will consider the equation (33) rewritten in the following form

\[ \Pi_s^{[f(R)]} f(R) = \left[ \Pi_s^{[F(R)]} F(R) - \Theta(R) \right]_{y=0}. \quad (44) \]
Still, we can consider now all the triad of equations (30), (31) and (36) given respectively in the compact form

\[
\Pi^{[f(R)]} f(R) = J, \quad \Pi^{[f(R)]} f(R) = -Q \quad \text{and} \quad \Pi^{[F(R)]} F(R) = \kappa^2 T. \tag{45}
\]

We are fit for introduce the CDC concept.

Formally, it is obtained when the “extended” bulk scalaron term \( \Pi^{[F]} F \) taked at \( y = 0 \) follows the prescription

\[
\left[ \Pi^{[F(R)]} F(R) \right]_{y=0} = \Pi^{[f(R)]} f(R) + \left[ D (R, \mathcal{K}) \right]_{y=0}, \tag{46}
\]

so that putting (46) in (44), it yields us

\[
D (R, \mathcal{K}) = \Theta (R). \tag{47}
\]

Here, \( D (K, R) \) is a dynamical term that involves the curvatute objects. The expression (47) is our generic CDC. Using the CGC, we have of (47) the following formal prescriptions

\[
\hat{D}_1 (R) = \hat{\Theta}_1 (R) \quad \text{or} \quad \hat{D}_2 (\mathcal{K}) = \hat{\Theta}_2 (R) \quad \text{or} \quad \hat{D}_3 (R, \mathcal{K}) = \hat{\Theta}_3 (R + \mathcal{K}), \tag{48}
\]

so that each (48) relates us only two dynamical variables once the CGC eliminates one of them. Here, the \( \leftrightarrow \) symbol denotes that \( D \) and \( \Theta \) already are merged.

Let us provide now a solid example. We will consider a \((\mathcal{R} + b\mathcal{R}^2)\)-bulk projecting a \((R + a\mathcal{R}^2)\)-brane. The \( \Pi_* \)-operators acting at their respective theories, give us

\[
\Pi_*^{[R+a\mathcal{R}^2]} f = 6a\Box R \quad \text{and} \quad \Pi_*^{[R+b\mathcal{R}^2]} F = \frac{1}{5} \left[ 16b - b\mathcal{R} \right] \mathcal{R}. \tag{49}
\]

Taking the expressions (49), so as the CGC, and then putting them in (44), we obtain the following equation

\[
6a\Box R = \frac{16b}{5} \Box R + \left[ \frac{b}{5} \left( 16 \Box \mathcal{K} + 16D^\nu D_y R - \mathcal{R}^2 \right) - \frac{2b}{(1 + 2b\mathcal{R})} \mathcal{O} \mathcal{R} \right]_{y=0}, \tag{50}
\]

where we have used \( \Box = \Box + D^\nu D_y \), as also

\[
\Theta (\mathcal{R}) = \frac{2b}{(1 + 2b\mathcal{R})} \mathcal{O} \mathcal{R}. \tag{51}
\]

Comparing the left/right sides of (50), we see clearly the relation among the constants \( a \) and \( b \): \( a = 8b/15 \). Thus, taking \([\ldots]_{y=0} \) term equal to zero, we should find our first CDC, that is

\[
\Box \mathcal{K} + D^\nu D_y R = \left[ \frac{5}{8 (1 + 2b\mathcal{R})} \mathcal{O} + \frac{1}{16} \mathcal{R} \right] \mathcal{R}. \tag{52}
\]
We can then reformulate our result as follows: the projection represented by
\[ F(R) = R + bR^2 \implies f(R) = R + \frac{8b}{15}R^2, \] (53)
whose it has the following 5D/4D scalaron theories
\[ \left( \Box - m_5^2 - \frac{1}{16}R \right) R = \frac{2}{3} \kappa_5^2 m_5^2 T \quad \text{and} \quad \Delta^{(-)}_{KG} R = m_4^2 J, \] (54)
it should contain also the additional relation (52). It is viable to note here that the 5D/4D masses associated with the 5D/4D scalaron fields are related by
\[ m_5^2 = \left( \frac{3}{5} \right) m_4^2 = \frac{3}{16b}. \]

We must also want to find the relations (48) for this case. Defining \( \Gamma_1 \) as
\[ \Gamma_1 \equiv \frac{1}{16} \left( R + \frac{10O}{1 + 2bR} \right) = \frac{1}{16} \left( R + \mathcal{K} + \frac{10O}{1 + 2b(R + \mathcal{K})} \right), \] (55)
and so using the CGC in (52), it implies in the \( \not\! D/\Theta \)-objects as follows
\[ \not\! D_1 = \Box R, \quad \not\! D_2 = \Box \mathcal{K}, \quad \not\! D_3 = \Box \mathcal{K} + D^yD_y R, \] (56)
\[ \not\! \Theta_1 = (\Box - \Gamma_1) R, \quad \not\! \Theta_2 = (-D^yD_y + \Gamma_1) R \quad \text{and} \quad \not\! \Theta_3 = \Gamma_1 (R + \mathcal{K}). \]

Before continuing, will be useful define also
\[ \Gamma_2 \equiv \frac{1}{16} \left( R + \frac{5O}{5R} \right) = \frac{1}{16} \left( R + \mathcal{K} + \frac{5O}{5R + \mathcal{K}} \right). \] (57)

Now, we will consider others possible projections. Thus, we must to write then the \( \Pi \)-operators acting at the \( fR^2/bR^2 \)-theories, i.e.,
\[ \Pi^{[fR^2]} \! f = (6\Box + 1) R \quad \text{and} \quad \Pi^{[bR^2]} \! F = \frac{1}{5} [16b \Box + 2 - b\mathcal{R}] R. \] (58)
Taking the expressions (58) and (49), considering their full combinations and putting them in the equation (44), we should obtain separately each projection with its respective CDC. Proceeding as mentioned, we have
\[ bR^2 \implies \frac{8b}{15} R^2 : \left( \Box - \frac{1}{2b} \right) \mathcal{K} + \left( D^yD_y - \frac{3}{16b} \right) R = \frac{1}{16} \left( R + \frac{5O}{5R} \right) R, \] (59)
\[ bR^2 \implies R + \frac{8b}{15} R^2 : \left( \Box + \frac{1}{8b} \right) \mathcal{K} + \left( D^yD_y + \frac{1}{8b} \right) R = \frac{1}{16} \left( R + \frac{5O}{5R} \right) R, \] (60)
and
\[ \mathcal{R} + bR^2 \implies \frac{8b}{15} R^2 : \left( \Box - \frac{5}{8b} \right) \mathcal{K} + \left( D^yD_y - \frac{5}{16b} \right) R = \frac{1}{16} \left( R + \frac{10O}{1 + 2bR} \right) R. \] (61)
Let us rewrite each CDC found here at a general form. Looking the expressions (52) and (59)-(61), it is easy see that all of them contain the general structure:

\[ \mathcal{D}_R \mathcal{R} = \mathcal{D}_R \mathcal{R} + \mathcal{D}_K \mathcal{K}. \]  
(62)

Evidently, the \( \mathcal{D} \)-operators for each projection are given separately as follows in the below table

| \( \mathcal{F}(\mathcal{R}) \Rightarrow f(\mathcal{R}) \) | \( \mathcal{D}_R \mathcal{R} \) | \( \mathcal{D}_R \) | \( \mathcal{D}_K \) |
|---|---|---|---|
| \( \mathcal{R} + b \mathcal{R}^2 \Rightarrow R + \frac{8b}{15} \mathcal{R}^2 \) \( \Gamma_1 \) | \( D^y D_y \) | \( \square \) | \( \square \) |
| \( h \mathcal{R}^2 \Rightarrow \frac{8h}{15} \mathcal{R}^2 \) \( \Gamma_2 \) | \( D^y D_y - \frac{3}{100} \) | \( \square \) | \( \square \) |
| \( h \mathcal{R}^2 \Rightarrow R + \frac{8h}{15} \mathcal{R}^2 \) \( \Gamma_2 \) | \( D^y D_y + \frac{1}{60} \) | \( \square \) | \( \square \) |
| \( \mathcal{R} + b \mathcal{R}^2 \Rightarrow \frac{8b}{15} \mathcal{R}^2 \) \( \Gamma_1 \) | \( D^y D_y - \frac{5}{100} \) | \( \square \) | \( \square \) |

Similarly as previously done for obtain the expressions (56)-(52), we should use the CGC at (62) for find

\[ \hat{\mathcal{D}}_R \mathcal{R} = \hat{\mathcal{D}}_R \mathcal{R}, \quad \hat{\mathcal{D}}_K \mathcal{R} = \hat{\mathcal{D}}_R \mathcal{K} \quad \text{and} \quad \hat{\mathcal{D}}_K R = \hat{\mathcal{D}}_R K, \]  
(63)

where

\[ \hat{\mathcal{D}}_R \equiv \mathcal{D}_K - \mathcal{D}_R, \quad \hat{\mathcal{D}}_K \equiv \mathcal{D}_K - \mathcal{D}_R \quad \text{and} \quad \hat{\mathcal{D}}_R \equiv \mathcal{D}_R - \mathcal{D}_R. \]  
(64)

It is direct observe the relation \( \hat{\mathcal{D}}_R = \hat{\mathcal{D}}_R + \hat{\mathcal{D}}_K \), whose formally contain an analogous CGC structure where the analogy is given by \( \hat{\mathcal{D}}_{\mathcal{L}_i} \leftrightarrow \mathcal{L}_i \). The object \( \mathcal{L}_i \) is defined here only for convenience as \( (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) \equiv (\mathcal{R}, \mathcal{R}, \mathcal{K}) \). For the \( \hat{\mathcal{D}} \)-operators, we can create the following table

| \( \mathcal{F}(\mathcal{R}) \Rightarrow f(\mathcal{R}) \) | \( \hat{\mathcal{D}}_R \mathcal{R} \) | \( \hat{\mathcal{D}}_R \) | \( \hat{\mathcal{D}}_K \) |
|---|---|---|---|
| \( \mathcal{R} + b \mathcal{R}^2 \Rightarrow R + \frac{8b}{15} \mathcal{R}^2 \) \( \square \) | \( \square \) | \( \square \) | \( -D^y D_y + \Gamma_1 \) |
| \( h \mathcal{R}^2 \Rightarrow \frac{8h}{15} \mathcal{R}^2 \) \( \square \) | \( \square \) | \( \square \) | \( -D^y D_y + \frac{3}{100} + \Gamma_2 \) |
| \( h \mathcal{R}^2 \Rightarrow R + \frac{8h}{15} \mathcal{R}^2 \) \( \square \) | \( \square \) | \( \square \) | \( -D^y D_y - \frac{1}{60} + \Gamma_2 \) |
| \( \mathcal{R} + b \mathcal{R}^2 \Rightarrow \frac{8b}{15} \mathcal{R}^2 \) \( \square \) | \( \square \) | \( \square \) | \( -D^y D_y + \frac{5}{100} + \Gamma_1 \) |

We will define the concept of canonical CDC or CCDC as follows: defining the operator \( \mathcal{C}_{\mathcal{L}_i} \) as \( \mathcal{C}_{\mathcal{L}_i} \equiv \lambda^2 \mathcal{L}_i \), being \( \lambda \) a constant, if \( \mathcal{D}_{\mathcal{L}_i} = \mathcal{C}_{\mathcal{L}_i} \), then (62) is our CCDC for the projection \( \mathcal{F} \Rightarrow f \). In this case, we observe that (62) implies in the CGC canonically, i.e,

\[ \mathcal{C}_R \mathcal{R} = \mathcal{C}_R R + \mathcal{C}_K \mathcal{K} = \lambda^2 \mathcal{R} = \lambda^2 (R + K) \Rightarrow \mathcal{R} = R + K. \]  
(65)

We will call \( \{ \mathcal{C}_{\mathcal{L}_i} \} \) of canonical operator set of the projection \( \mathcal{F} \Rightarrow f \).
The canonical versions of the expressions (63) are
\[ \overset{\leftrightarrow}{C}K R = \overset{\leftrightarrow}{C}K R, \quad \overset{\leftrightarrow}{C}R R = \overset{\leftrightarrow}{C}R K \quad \text{and} \quad \overset{\leftrightarrow}{C}R R = -\overset{\leftrightarrow}{C}R K, \] or simply \( \overset{\leftrightarrow}{C} \mathcal{L}_i \mathcal{L}_i = 0 \). To finish, in the sense of definitions, we can also characterise the several possible projections. For example, a given projection is symmetric \( \overset{\star}{f} \) when we have
\[ \mathcal{F} = \mathcal{C}(\mathcal{R}) \quad \overset{\star}{\Rightarrow} \quad f = \mathcal{C}(\mathcal{R}). \] Thus, the \( \mathcal{C}(\mathcal{L}_i) \) function of the symmetric Starobinsky-Starobinsky projection is \( \mathcal{C}(\mathcal{L}_i) = \mathcal{L}_i + c_i \mathcal{L}_i^2 \), where \( c_i \) is a constant associated with \( \mathcal{L}_i \). Others kinds of the projection will be given in a future work considering a more formal rigorous study. In the follows, we will see to emerge a generalized unimodular gravity for a specific case.

IV. EMERGENT \( f(\mathcal{R}) \)-UNIMODULAR GRAVITY: \( \mathcal{Q}_{\mu\nu} = Q(\mathcal{R})q_{\mu\nu} \) CASE

In the present section, we must to explore the case where is possible decompose \( \mathcal{Q}_{\mu\nu} \) in the following fashion: \( \mathcal{Q}_{\mu\nu} = Q(\mathcal{R})q_{\mu\nu} \). Such decomposition turns possible when
\[
\left[ \frac{2}{3} \delta^a_b \delta^b_a D_a D_b + q_{\mu\nu} \left( \frac{2}{3} n^a n^b D_a D_b - \frac{4}{15} \Box \right) \right] (d_\mathcal{R} \mathcal{F}) = q_{\mu\nu} \mathcal{O}_s (d_\mathcal{R} \mathcal{F}). \] (68)

In this case, we can rewrite the expression (21) as also the second of (45) respectively as
\[ R_{\mu\nu} - \frac{1}{2} Rq_{\mu\nu} = J_{\mu\nu} + q_{\mu\nu} Q \quad \text{and} \quad \Pi^{[f(\mathcal{R})]}_s f(\mathcal{R}) = -4Q. \] (69)

Here, the scalar object \( Q \) is given by
\[ Q = \left[ \Theta_s (\mathcal{R}) - \frac{1}{4} \Pi^{[F(\mathcal{R})]}_s \mathcal{F}(\mathcal{R}) \right]_{y=0}, \quad \text{where} \quad \Theta_s \equiv (d_\mathcal{R} \mathcal{F})^{-1} \mathcal{O}_s (d_\mathcal{R} \mathcal{F}). \] (70)

Before we see to emerge the \( f(\mathcal{R}) \)-unimodular gravity, we must to understand the basic of the traceless Einstein tensor for the case (69). Let us also present the notation that we will use several times along this section. For any generic tensor \( A_{\mu\nu} \), we will use of notation \( ^o \) which it represents the composition:
\[ A_{\mu\nu}^o \equiv A_{\mu\nu} - \frac{1}{4} Aq_{\mu\nu}. \] (71)

We should stress that \( A_{\mu\nu}^o \) is traceless.

In this formulation, the traceless Einstein tensor is given as follows
\[ R_{\mu\nu}^o = J_{\mu\nu}^o. \] (72)
The expression (72) is obtained taking the trace of tensor equation given at (69), isoling $Q$ and putting it back in the tensor expression. It is easy verify that (72) does not contain the effective cosmological constant predicted in the SMS formulation because $q^\circ_{\mu\nu} = 0$. However, we will have two corrections in the traceless stress-tensor: $\pi^\circ_{\mu\nu}$ and $E^\circ_{\mu\nu}$. The traceless high energy term and $E^\circ_{\mu\nu}$ are respectively given by

$$\pi^\circ_{\mu\nu} = \frac{1}{4} \left[ \frac{1}{3} \tau^\tau_{\mu\nu} - \tau_{\mu\sigma} \tau^\sigma_{\nu} + \frac{1}{12} \left( 3 \tau_{\sigma\delta} \tau^\sigma_{\delta} - \tau^2 \right) q_{\mu\nu} \right]$$

and

$$E^\circ_{\mu\nu} = \pi^\circ_{\mu\nu} = \pi_{\mu\nu}.$$

When in the vacuum, the expression (72) is driven by the projected Weyl tensor, i.e.,

$$R^\circ_{\mu\nu} = -\pi_{\mu\nu}.$$

Let us obtain the $f(R)$-unimodular gravity on the brane. For this, we should combine the equations (69) and then isolate $J_{\mu\nu}$. Thereafter will be necessary only put $J_{\mu\nu}$ in the expression (29). Realizing the procedures cited here, we must to derive the generalized unimodular gravity, i.e.,

$$\Delta^\circ_{\mu\nu} (d_R f) = R^\circ_{\mu\nu} (d_R f - 1).$$

Here $\Delta^\circ_{\mu\nu}$ is our 4D traceless differential operator. The theory given at (75) should be supplemented by the equations (72). Mathematically we announce the $f(R)$-unimodular gravity by the symbolic pair:

$$\left[ R^\circ_{\mu\nu} - \Delta^\circ_{\mu\nu} \right] d_R f(R) = R^\circ_{\mu\nu} \quad \text{and} \quad R^\circ_{\mu\nu} = J^\circ_{\mu\nu}.$$

Therefore, we can see (76) as a natural $f(R)$-extension of unimodular gravity provided by the extra dimensional context.

We note that when $f(R) \rightarrow R$, the left equation tends to the right and then there is a redundancy. This aspect can be formally represented by

$$\left[ R^\circ_{\mu\nu} - \Delta^\circ_{\mu\nu} \right] d_R f(R) = J^\circ_{\mu\nu} \quad \text{and} \quad R^\circ_{\mu\nu} = J^\circ_{\mu\nu}.$$

When $f(R) = R$, we have $Q_{\mu\nu} = 0$ and thus the decomposition $Q q_{\mu\nu}$ does not make sense. It is not valid when $f(R) = R$. However, we argue that any $f(R)$ must to recover the $R$-theory.

We pointed here that there is an equivalence between the unimodular gravity and the SMS theory when $Q_{\mu\nu} \rightarrow 0$. If we consider (77) as also the limit

$$\left[ R^\circ_{\mu\nu} - \Delta^\circ_{\mu\nu} \right] d_R f(R) = J^\circ_{\mu\nu} \quad \Rightarrow \quad G^\circ_{\mu\nu} f[R] = J^\circ_{\mu\nu} \quad \text{and} \quad C^\circ_{\mu\nu} f[R] = J^\circ_{\mu\nu},$$

In the usual case would be $R^\circ_{\mu\nu} = 0$.\footnote{In the usual case would be $R^\circ_{\mu\nu} = 0$.}
it turns clear that when \( f(R) \to R \), the SMS formulation is equivalent the unimodular gravity in this approximation.

The theory obtained here is formally identical with the NOO theory given at (12). Taking the trace of (12), isoling \( \mathcal{L} \) and posteriorly putting it at (12), we will obtain the following expression

\[
(R^0_{\mu
u} - \Delta^0_{\mu
u}) \, d_{R} f(R) = \kappa_4^2 T^0_{\mu
u}.
\]  

Looking (79), it is direct to see that our theory represented by (76) is mathematically identical with the given at (12). The fact is that the Lagrange multiplier function \( \mathcal{L} \) is equivalent with our \( Q \) function \( (\mathcal{L} \sim Q) \) in the sense that them lead us for the \( f(R) \)-unimodular gravity\(^2\). However, our result provides a correction for NOO theory. Let us consider \( \Delta T^0_{\mu
u} \) as a possible deviation of (79). Still, we will consider also that \( \Delta T^0_{\mu
u} \) modifies (80) with the following prescription

\[
(R^0_{\mu
u} - \Delta^0_{\mu
u}) \, d_{R} f(R) = \kappa_4^2 T^0_{\mu
u} + \Delta T^0_{\mu
u}.
\]  

Comparing (80) with (76), we can identify \( \Delta T^0_{\mu
u} \) with the objects \( \pi^0_{\mu\nu} \) and \( \mathcal{E}_{\mu\nu} \), i.e.,

\[
\Delta T^0_{\mu\nu} \sim \frac{6\kappa_4^2}{\lambda} \pi^0_{\mu\nu} - \mathcal{E}_{\mu\nu},
\]  

where we have identified also the \( T^0_{\mu\nu} \sim \tau^0_{\mu\nu} \). Thus, \( \Delta T^0_{\mu\nu} \) can provide extra dimension signature when compared with results of NOO theory. In the vacuum, any deviation can be speculated as a possible manifestation of the dark radiation because \( \Delta T^0_{\mu\nu} \sim -\mathcal{E}_{\mu\nu} \).

Similarly as done in the subsection 2.2, we obtain by (72) the continuity relations

\[
\nabla^\mu J^\mu_{\nu} = \frac{1}{4} \partial_\nu (J + R) = -\partial_\nu Q,
\]  

at concordance with (24) and (26). When \( \partial_\mu Q = 0 \), we will have \( J + R = -4Q \) equal a constant \( c \equiv 4\Lambda^{(Q)} \). Isoling \( J \) and posteriorly substituting it at (72), we obtain the following equations

\[
G^{[R]}_{\mu\nu} = -\Lambda^{(Q)} q_{\mu\nu} + J_{\mu\nu}, \quad \text{where} \quad \Lambda^{(Q)} \sim \Lambda^{(U)}.
\]  

Thus, being \( Q \) a constant, we can identify it as the cosmological constant obtained in the traceless Einstein tensor theory: \( \Lambda^{(U)} \sim -Q \). The effective cosmological constant will be then: \( \Lambda_4^{eff} \equiv \Lambda_4 + \Lambda^{(Q)} \).

Let us define now a new function \( \mathfrak{A}(R) \) as follows

\[
\mathfrak{A}(R) \equiv d_{R} f(R) - 1.
\]

\(^2\) Here we will use the term equivalent (denoted by \( \sim \)) because in (79) the metric tensor must satisfies the unimodular constraint while in (76) it does not.
We note that $A$ excludes basically the $f(R) \neq R$ model on the brane. Using (84), we can rewrite the expression (76) in the following fashion

$$\Delta^\circ_{\mu\nu} A = R^\circ_{\mu\nu} A \quad \Rightarrow \quad R^\circ_{\mu\nu} = J^\circ_{\mu\nu} = \frac{1}{A} \Delta^\circ_{\mu\nu} A.$$  (85)

The right equation is obviously valid when $A \neq 0$.

Will be useful define also the objects $\Psi_{\mu\nu}$ and $\Phi$ as below:

$$\Psi_{\mu\nu} \equiv \frac{1}{A} \Delta_{\mu\nu} A \quad \text{and} \quad \Phi \equiv \frac{1}{3} \left[ \frac{2f}{A} - R + \frac{4Q}{A} \right].$$  (86)

Rewriting (30) using (84) and simultaneously taking $\Psi_{\mu}^\mu$, we will find the relations

$$\Box A = (\Psi_{\mu}^\mu) A = \Phi A \quad \Rightarrow \quad \Psi = \Phi.$$  (87)

Consequently, our brane theory has been here codificade at $\Psi_{\mu\nu}$.

Thus, knowing the information contained at $\Psi_{\mu\nu}$, the pair $[q_{\mu\nu}, \circ]$ generates the traceless and scalar equations of our $f(R)$-unimodular gravity, i.e.,

$$\Delta^\circ_{\mu\nu} A = \Psi^\circ_{\mu\nu} A \quad \text{and} \quad \Box A = \Psi A.$$  (88)

where $\Psi^\circ_{\mu\nu} = R^\circ_{\mu\nu}$. We stress that in general $\Psi_{\mu\nu} \neq R_{\mu\nu}$. The object $\Psi_{\mu\nu}$ was defined in the following way (86), in the sense that its traceless version coincides with $R^\circ_{\mu\nu}$ while its trace with $\Phi \neq R$. In general $R_{\mu\nu}$ can be written as

$$R_{\mu\nu} = \Psi_{\mu\nu} + q_{\mu\nu} \varphi, \quad \text{where} \quad 4\varphi \equiv R - A^{-1} \Box A.$$  (89)

When $\varphi = 0$, the inequalities cited anteriorly turn equalities. Therefore, our fundamental equations can be stated as

$$\Delta_{\mu\nu} A = \Psi_{\mu\nu} A.$$  (90)

The expression (90) is formally a generalized eigenvalue equation. Thus, $\Psi_{\mu\nu}$ and $A$ would be the “eigenvalue” and the “eigenstate” respectively while the “eigenoperator” would be $\Delta_{\mu\nu}$. Let us now constraint us for the vacuum case. When $\tau_{\mu\nu} = 0$, the crucial objects to put at (88) are

$$\Psi^\circ_{\mu\nu} = -\mathcal{E}_{\mu\nu} \quad \text{and} \quad \Psi = \frac{1}{3} \left[ \frac{2f}{A} - R - \frac{R}{A} \right].$$  (91)

The first equations are obtained comparing (74), (85) with the definition of $\Psi_{\mu\nu}$. The second expression is generated only taking (24) in the vacuum. In the follows, we will apply the formulation developed here.
Now, we will apply our model considering for metric background the FRW \textit{ansatz}, i.e.,

$$q_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \quad i, j = 1, 2, 3,$$ \hspace{1cm} (92)

with \(a(t)\) being the scale factor while \(\delta_{ij}\) is the Kronecker delta component. In general, here the traceless gravity can be written in the following form

$$R_{\mu\nu} - \frac{1}{2A}\Delta_{\mu\nu}A = \frac{1}{4}q_{\mu\nu}(R - \Psi A),$$ \hspace{1cm} (93)

where we have used the relations (85), (88) and the right expression of (91). Considering (92) and then taking the 00 and \(ii\) components, we will obtain the equality

$$3\frac{\partial_t\partial_t a}{a} + \frac{\partial_t\partial_t A}{A} = \frac{1}{4}(R - \Psi A) = \frac{2}{a^2} \left(\frac{\partial_t a}{a}\right)^2 + \frac{\partial_t\partial_t a}{a} + \frac{\partial_t A\partial_t a}{A},$$ \hspace{1cm} (94)

such that it yields us

$$[\partial_t\partial_t - H\partial_t + 2(\partial_t H)]A = 0.$$ \hspace{1cm} (95)

In another hand, the scalaron equation of movement (right expression of (88)), provides us

$$[\partial_t\partial_t + 3H\partial_t + \Psi]A = 0.$$ \hspace{1cm} (96)

The simultaneous existence of (95) and (96) implies in the following equation

$$2[\partial_t - 2(\partial_t \ln A)]H = \Psi.$$ \hspace{1cm} (97)

The expression (97) is our guide equation for study the universe expansion for some generic case. The Hubble function that satisfies (97) is given by

$$H(t) = 2A \left[ C + \frac{1}{2} \int \frac{\Psi}{A^2} dt \right],$$ \hspace{1cm} (98)

where \(C\) is a constant. Of course that \(H, A\) and \(\Psi\) of (98) must be also compatible with the relation \(R = 6(\partial_t H + 2H^2)\) in which is obtained of the definition of \(R\), remembering that \(A\) is given at terms of \(R\). Restricting (97) for \(f(R) = R + aR^n\) with \(n \geq 2\), taking vacuum brane and the RS fine-tuning, we have

$$\left[\partial_t - 2(n - 1)\left(\frac{\partial_t R}{R}\right)\right]H = \frac{1}{6naR^{n-1}} \left[R + a(2 - n)R^n\right].$$ \hspace{1cm} (99)

As an example, we will consider now the SSMS brane: \(n = 2\). Remembering that \(m_4^2 = 1/6a\) and considering also \(R = -4Q = 6(\partial_t H + 2H^2)\), we will obtain of (99) our modified first Starobinsky-Friedmann equation, i.e.,

$$F = m_4^2 Q.$$ \hspace{1cm} (100)
V. CONCLUSION

We have developed two different aspects associated with \( f(R) \) branewords. The first has been the establishment of rules to create it from \( \mathcal{F}(R) \)-bulk. Wanting generate a “copy” of the bulk dynamical on the brane, we have inserted the CDC concept. Heuristically, a copy is obtained when the geometrical reduction does not affect the dynamical structure of the left side, such that \( \mathcal{F}(R) \) at 5\( D \) is fully analog the \( f(R) \) at 4\( D \). For example, the idea of the effective \( f(R) \)-branes has been developed in the ref. [11], where the projection has been stipulates as \( \mathcal{F}(R) = f(R + \mathcal{K}) \), i.e., only applying the CGC in \( \mathcal{F} \). Therefore, we have obtained a new projection class: \( \mathcal{F}(R) \iff f(R) \).

As a future study we should formalise strictly the several kinds of the possible projections with the concept of copy. As we have seen, the CDC should specify \( \iff \) being it equivalent with the CGC, where canonical or not. Thus, we should want to study its “spectrum” with hope that it explains to us which projection is valid. Loosely speaking, maybe we can find a physical principle that provides us which projection is correct. Maybe, this principle can be founded observing the paper [42] where we have a 5\( D \) Einstein-Gauss-Bonnet bulk projecting \( f(R) \) brane. In order to solve the problem of predictability of brane cosmology, they argue that AdS/CFT correspondence can be implemented to solve the problem. Maybe we can generalize the CDC concept for generic cases: \( D \Rightarrow d \), where \( D/d \) any bulk/brane dynamical and \( \Rightarrow \) a general projection.

In the second step, we have taken \( Q_{\mu \nu} = Q_{\eta \nu} \) case. We have showed that the brane theory can be codificade in the object \( \Psi_{\mu \nu} \) such that \([ \circ , g^{\mu \nu} ]\) must generate respectively the \( f(R) \)-unimodular gravity and scalaron equation. We have showed also that in the appropriate approximation, the SMS formulation is equivalent with the usual unimodular gravity. We have seen yet that when \( \nabla^\mu Q_{\mu \nu} = 0 \), \( Q \) can be identified as the cosmological constant of the unimodular gravity. Moreover, when \( Q \) is not constant we have taked the FRW case with specific conditions where we have found a modified \( \mathcal{F} \) equation. When \( \iff \), this modified equation can be written as

\[
\mathcal{F} = \mathcal{F}_{\text{bulk}} \bigg|_{y=0}, \quad \text{where} \quad \mathcal{F}_{\text{bulk}} = \left[ \frac{5}{8 (1 + 2b \mathcal{R})} \mathcal{O}_4 - \frac{1}{4} \left( \Box - \frac{1}{16} \mathcal{R} \right) \right] \mathcal{R}.
\]  

(101)

In general \( \mathcal{F}_{\text{bulk}} \propto \Theta_4 - (1/4) \Pi^{[\mathcal{F}]} \mathcal{F} \), such that we can see directly \( \mathcal{F} \) as remnant of the 5\( D \) scalaron arising from \( \mathcal{F}_{\text{bulk}} \). Referring to NOO theory, we can study physical applications analyzing the exact equivalence of \( \Delta T^\infty_{\mu \nu} \) with \( \pi^o_{\mu \nu} \) and \( \mathcal{E}_{\mu \nu} \), providing then the truth correspondence with [29].

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