The kinetic equation solutions and Kolmogorov spectra

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Abstract. The Hasselmann kinetic equation (KE) for stochastic nonlinear surface waves is studied numerically with the aim of searching for features of the Kolmogorov turbulence (KT). To this aim, solutions of the KE for the long-term wave-spectrum evolution are executed. As far as the total wave action \( N \) and wave energy \( E \) are not preserved simultaneously in the course of the KE solution, two versions of the numerical algorithm are used, preserving values of \( N \) or \( E \) in separate. In every case, the KE solutions result in formation of the self-similar spectrum shape, \( S_{ssf}(\omega) \), with the frequency tail \( S_{ssf}(\omega) \sim \omega^{-4} \), independently of the \( N \)- or \( E \)-fluxes generated by the nonlinear interactions. This urges us to state that the used KE does not obey to regulations of the KT. The reason of this fact resides in the mathematical feature of the kinetic integral, which, in any case of solving the KE, results in formation of the nonlinear energy-transfer tail of kind \( N_{1234}(\omega) \sim -\omega^{-4} \), what stabilizes the spectral tail in form \( S_{ssf}(\omega) \sim \omega^{-4} \).

1. Introduction

The four-wave kinetic equation (KE), describing time-evolution of 2D wave-action spectrum \( N(k) \) in the wave-vector \( k \)-space, can be written in the form

\[
\frac{\partial N(k)}{\partial t} = N_{nl}(k) = I_{nl}(N) = 4\pi \int M^{2}(k_{1}, k_{2}, k_{3}, k_{4}) F_{X}(N_{1}, N_{2}, N_{3}, N_{4}) \delta_{1234} dk_{1} dk_{2} dk_{3}, \tag{1}
\]

Here, \( N_{nl}(k) \) is the wave-action nonlinear transfer though the \( k \)-space, \( I_{nl}(N) \) is the kinetic integral (KI), \( M^{2}(k_{1}, k_{2}, k_{3}, k_{4}) \) (or \( M_{1234} \)) are the matrix elements for the four-wave nonlinear interactions, \( F_{X}(N_{1}, N_{2}, N_{3}, N_{4}) \) is the cubic function in \( N(k) \), \( \delta_{1234} \equiv \delta(\omega_{1} + \omega_{2} - \omega_{3} - \omega_{4}) \cdot \delta(k_{1} + k_{2} - k_{3} - k_{4}) \) is the Dirac delta-function describing the resonant feature of the four-wave interactions, and \( \omega(k) \) is the radian frequency of the wave component, related with wave vector \( k \) by the dispersion relation. For gravity waves in deep water, considered here, it is \( \omega(k) = (gk)^{1/2} \), where \( g \) is the gravity acceleration. Equation (1) was derived for the first time by Hasselmann [1] with the perturbation theory methods applied directly to the Euler equations in the potential approximation. Later, KE (1) was rederived by Zakharov [2, 3, 4] with using the Hamiltonian formalism. The kinematics of stochastic wave fields in this presentation was called as the “weak wave turbulence” (WWT) [3, 4]. In our work we use KE (1) in the presentation for the wave-energy spectrum, \( S(\omega, \theta) \), given in the frequency-angular space, \( (\omega, \theta) \). Energy spectrum is related linearly with \( N(k) \) by the ratio [2-4]:

\[
S(\omega, \theta) = \omega N(\omega, \theta) = 2(\omega^{4}/g^{2}) N(k). \tag{2}
\]
The expressions for matrix elements \( M_{1,2,3,4} \) are well known [1-5], we use the elements from [5]. In any representation, formulas for elements \( M_{1,2,3,4} \) are very cumbersome to be given here. It is important to note only that the KI formally preserves the total wave energy, \( E = \int S(\omega,\theta)d\omega d\theta \), total wave action, \( N = \int \int (S(\omega,\theta)/\omega)d\omega d\theta \), and total wave momentum, \( M = \int \int (\text{K}S(\omega,\theta)/\omega)d\omega d\theta \), under the condition of uniform convergence of the KI [3, 6], what is very important for the below. Since late 60s, the KE became a separate subject of studying (e.g., [4, 7-16] and references herein). Among them, analytical results found by Zakharov and co-authors (see below) play the crucial role.

First, Zakharov and Filonenko [7] have found the analytic solution of KE (1) of form \( S_1(\omega) \propto \omega^4 \), which puts the KI identically to zero \( (I_{\infty}[S_1(\omega)] = 0) \), in the case of an angular isotropic spectrum spread throughout the infinite frequency band, \( 0 < \omega < \infty \). Later, Zakharov and Zaslavskii [8] found that KE (1) has the second analogous solution of form \( S_2(\omega) \propto \omega^{-4/3} \). They have proposed to interpret these solutions as the Kolmogorov-type spectra of the constant energy flux, \( P_E \), directed upward in frequencies, and constant wave-action flux, \( P_N \), downward in frequencies, respectively. In this interpretation, the mentioned spectra acquire the proper Kolmogorov-type representation [3, 4, 8]

\[
S_1(\omega) = c_1 P_E^{1/3} \omega^{4/3}, \quad S_2(\omega) = c_2 P_N^{1/3} \omega^{-4/3},
\]

(3a, b)

where \( c_1 \) and \( c_2 \) are the dimensionless Kolmogorov constants. The proper sources and sinks of these fluxes, obligatory in the Kolmogorov’s theory [17], were assumed to be located at the zero and infinity frequency-points, respectively, in dependence on the flux [8]. It should be noted, however, that this interpretation was a hypothesis, at that time, because it was based only on dimensional considerations and correspondence of forms \( (3a,b) \) to the Kolmogorov’s theory. But this does not follow from the Euler equations without external forces. Later, the mentioned analytical results were sophisticated in different directions by Zakharov and co-authors (e.g., [2, 4, 18-20] and numerous references therein).

In parallel, the study of the KE is carried out numerically, beginning with the works by Webb [9] Masuda [10], Hasselmann and Hasselmann [11], devoted to the developing methods for calculating the KI. Two methods are widely used at now [9, 10]. The first of them was elaborated in [14, 15] and used by these authors and the Zakharov’s group. The second one was elaborated in works [12, 21-23], authors of which use it. Comparative evaluation of these methods showed the same efficiency [24].

Here it should be noted that the most important part of the KI-calculation algorithm is the method of estimating the contribution of integrable singularities in the integrand (‘singular points’), resulting out from integration of the frequency delta-function [10, 12]. The different approaches to this issue represent the main differences in the algorithms mentioned. Herewith, in [12] it was numerically shown that the contribution of the singular points determines crucially the conservation balance for the integral values \( E, N, M \), and the accuracy of the KI-estimate as a whole.

For the first time, a detailed numerical studying the KI-properties was carried out by Masuda [10]. It was expanded by Polnikov [12]. After that, Polnikov [13] has performed a numerical solution of KE (1), where it was shown the fact of establishing the self-similar shape of 2D wave spectrum, \( S_d(\omega,\theta) \), in the course of the long-term evolution of nonlinear waves. This result was later confirmed in a series of papers by the Zakharov’s group (see references in [4, 19]) and others authors (e.g., [21, 23]).

In addition to the Polnikov’s result [13], authors of [4, 19] have established that the tail of the self-similar spectrum \( S_d(\omega) \) falls according to the law \( S_d(\omega) \sim \omega^4 \), and the shape of spectral peak for \( S_d(\omega,\theta) \) is close to one for the JONSWAP spectrum with \( \gamma = 3.3 \) [4]. They first noted that the preservation for total wave action \( N \) takes only place, during wave spectrum evolution due to KE (1), and this evolution is accompanied by a leak of the total wave energy \( E \) and appearance of a certain energy flux \( P_E \) directed to the higher frequencies. This result was treated as the formation of the Kolmogorov spectrum of form \( (3a) \) in the frame of solving KE (1).

The mentioned effects represent the most interesting points for further investigations, taking in mind that all three integral values: \( E, N, \) and \( M \), should be formally preserved simultaneously, as far as
KE (1) is derived from the Euler equations without external forces [1-3]. Moreover, the conservative feature of these equations was proved analytically by Zakharov [25]. It needs to clarify this point in more details.

In addition to the said, other essential features of the process of forming the spectral-peak shape and the entire shape of 2D self-similar spectrum $S_{sf}(\omega, \theta)$ are not sufficiently described yet, including: the high-frequency asymptotes of nonlinear energy transfer $Nl_{s}(\omega)$ on short- and long-time scales of evolution; quantitative integral parameters of the spectral peak and angular distribution for $S_{sf}(\omega, \theta)$; and accompanying details. There is also a need to analyze the impact of the preservation condition for total wave energy $E$ on the shape of self-similar spectrum and Kolmogorov spectra formation, in addition to the case $N = \text{const}$ considered already in [4, 19]. This task could be especially realized as a theoretical scenario for the numerical solution of KE (1). Comparison of these two scenarios for numerical solution of KE (1) is very important, in view of results of work [8], for understanding the nature of establishing self-similar spectrum $S_{sf}(\omega, \theta)$ and the Kolmogorov spectra formation.

The present paper is devoted to elaboration all the mentioned points.

Before this, we should mention that, besides of studying self-similar spectrum $S_{sf}(\omega, \theta)$ formation, a lot of papers (e.g., [4, 19, 24, 26] and numerous references therein) are devoted to study establishing Kolmogorov spectra (3a,b) as the result of numerical solution for the extended KE of the form

$$\frac{\partial S(\omega, \theta)}{\partial t} = I_{nl}[S(\omega, \theta)] + In(\omega, \theta) - \text{Dis}(\omega, \theta).$$

(4)

In (4), $In(\omega, \theta)$ and $\text{Dis}(\omega, \theta)$ are the functions of the external source and sink, respectively, which are separated in the frequency space, to ensure the presence of the inertial interval, as the main condition for applicability of the Kolmogorov’s theory [17]. All studies have shown that Kolmogorov spectra (3a,b) are realised in the case of solving KE (4). But here we restrict ourselves by KE (1), with no external forcing, taking in mind that KEs (1) and (4) describe absolutely different physical systems.

2. Methods of research

2.1. Scenarios of the KE solution

Long ago, Pushkarev and Zakharov[27] have pointed out that the wave energy should leak to the higher frequencies due to the four-wave nonlinear interactions, basing on numerical solutions of KE (1). Since that time, this idea is applied in any case of considering KE (1) by the Zakharov’s group (e.g. [4, 16, 19, 20]). Usually, these authors say about conservation of $N$ and leakage of $E$ and $M$ in the course of KE (1) solution, explaining these numerical facts by an appearing the $E$- and $M$-fluxes, as the manifestation the Kolmogorov turbulence. Though, in our mind, there is no convincing proof of it.

Indeed, establishing self-similar spectrum $S_{sf}(\omega, \theta)$ contradicts to the preserving three integral values: values: $E$, $N$, and $M$. It was many times noted and can be easily proved (e.g., [4, 19]), taking in mind the two-lobe shape of the self-similar nonlinear transfer $Nl_{sf}(\omega)$ (see figures below).

On the other hand, the conservative feature of the Euler equations was proved by Zakharov long ago [25]. It means that the energy of the system does not change in time. Thus, the KE derived from these equations, should also preserve wave energy, at least. Moreover, it was proved that KE (1) formally preserve three integral wave values: energy $E$, action $N$, and momentum $M$, as was already mentioned. Though, a proper special testing the known and actively used algorithms for the KI-calculation is not presented in literature in all details (e.g. [10, 12, 14, 15, 19]). Thus, the numerical solution of KE (1) makes a mathematical paradox which has its explanation [16], though not so full.

As we have no convincing (say, categorical) prohibition for saving $E$ during solution of KE (1), in our mind, two scenarios of preservation for both integral quantities $N$ and $E$ may be chosen, at least, as an attracting theoretical alternative. It occurs that this way gives another paradox (see below).

2.2. Numerical details

In our study, to examine the asymptotes of the non-linear transfer of energy (NLT), $Nl(\omega, \theta) = I_{nl}[S(\omega, \theta)]$, the extended frequency grid and enhanced angular resolution are chosen. They are given by the ratios:
with \( \omega_1 = \omega_0 q^{-1} \), \( \omega_i = 0.64 \text{ rad/s}, q = 1.05, 1 \leq i \leq I = 90 \), and \( \Delta \theta = 5^\circ \).

The initial high-frequency asymptote of NLT is determined from the results of calculating KI at the first time-step of the KE solution. To study the asymptote of NLT and spectral shape resulting from numerical solutions of KE on long-term scales of evolution, the rarer grid is used:

\[
0.64 \leq \omega \leq 7 \text{ rad/s with } I = 50, \Delta \theta = 10^\circ,\]

with previous values for \( \omega_1 \) and \( q \).

The initial wave spectrum is given in the modified form for JONSWAP spectrum (J-spectrum)

\[
S_j(\omega, \theta, n, \gamma) = S_{PM}(\omega, n)^{\gamma/(\omega_0 q^{-1})^2/2 \Delta^2} \Psi(\omega, \theta) \tag{7}
\]

where \( \Delta = 0.07 \) to 0.09 is the non-dimensional (n.d.) peak-width parameter of the J-spectrum, \( \gamma = 1 \) to 7 is the n.d. ‘peakness’ parameter, \( \Psi(\omega, \theta) \) is the n.d. frequency-angular form, and

\[
S_{PM}(\omega, n) = 0.01 \gamma^2 (\omega^n / \omega_p^{5-n}) \exp\left[ -\left( n / 4 \right)(\omega_p / \omega)^4 \right] \tag{8}
\]

is the Pearson-Moskowitz spectrum, modified to the case of an arbitrary degree of the spectrum-tail decay, \( n \) [12]. Every time, the initial peak frequency is: \( \omega_p(0) = 2 \text{ rad/s} \); the initial form of function \( \Psi(\omega, \theta) \) is assumed to be independent of the frequency and given in the form

\[
\Psi(\omega, \theta) \equiv \Psi(\theta) = \cos^a(\theta / m) \tag{9}
\]

Variation of parameters \( a \) and \( m \) ensure the assignment of a wide degree of initial spectrum anisotropy.

The algorithm for calculating the KI is used according to work [13]. Comparison of the KI-estimates on grids (5a,b) and (6) has showed the accuracy for one-step KI-calculations within 3% to 5% for peak values of the NLT, what gives the upper limits for the errors of calculations.

To check the conservation balance \( Q \), it is used the ratio

\[
Q_{E,N} = \left( \int_{N=0}^{N_I} N_{E,N}(\omega, \theta)d\omega d\theta + \int_{N=0}^{N_I} N_{E,N}(\omega, \theta)d\omega d\theta \right) / \int \int_{N=0} N_{E,N}(\omega, \theta)d\omega d\theta, \tag{10}
\]

where the sub-indexes \( E \) and \( N \) mean the cases of NLT calculated for energy spectrum \( S(\omega, \theta) \) or wave action one \( N(\omega, \theta) \), respectively. Our results have shown that for the fast falling spectra (faster \( \omega^2 \)), balance \( Q \) has the value about of 1-2% for both values: \( E \) and \( N \) (\( M \) was out of our considerations), For the slower falling spectra, the balance for \( N \) retains the same, whilst for \( E \) it becomes negative and has the order of 5-10% (depending on the shape of spectrum and stage of evolution).

In the KE-solution, the exact balance, \( Q_E = 0 \) or \( Q_N = 0 \), is made by correcting the whole negative part of the relevant NLT, i.e., \( N_{E,N}(\omega, \theta) \), with a proper coefficient, at each time-step. It is done for the “purity” of the scenarios mentioned. This correction does not influence on the high frequency asymptotes of NLT, absolutely. Hereafter, it saves the physics of spectrum evolution, because the mentioned corrections are weaker than one-step changes of the spectra in the course of solving KE (1).

The behaviour of the high-frequency asymptote for the one-dimensional energy NLT, \( NL(\omega) \), is determined by the least-squares estimation of the decay parameter \( p \) in the power-law dependence

\[
\int J_{NL} \{ S(\omega, \theta, t) \} d\theta = NL(\omega, t) = const(t) \cdot \omega^{-p}. \tag{11}
\]

The estimation of \( p \) is executed in frequency band \( S_{0.2} < \omega < (15-20)\omega_p \) for numerical grid (5a,b). The quantitative characteristics of the 2D spectrum-shape are determined in the integral quantitative representation by introducing the spectral frequency-width parameter, \( B \), and the angular-narrowness function of spectrum, \( A(\omega) \), given by ratios [12, 13]:

\[
B = E / \omega_p S_p \quad \text{and} \quad A(\omega) = S(\omega, \theta_p) / S(\omega). \tag{12}
\]

Here, \( S_p = S(\omega_p) \) is the value of the 1D spectrum at the peak frequency, and \( \theta_p \) is the direction of propagating the peak-component of the 2D wave spectrum. Hereafter, \( \theta_p = 0 \).
The numerical solution of KE (1) is performed according to the algorithm of paper [13], which includes an explicit numerical scheme of the first order of accuracy, linear interpolation of the spectrum from the nodes of the computational grid to the resonance points of four-wave quadruplets, and the choice of the time-step, \( \Delta t \), according to the condition
\[
\Delta t = 0.03 / \min \left( \text{abs} \left[ I_{NL} S(\omega, \theta, t) / S(\omega, \theta, t) \right] \right).
\] (13)

In this paper, all calculations were performed in the dimensional units. First, the scenario of preservation for \( E \) is realized, as the less studied, assuming that the wave system is the conservative one. Then, some KE-solutions were done with the scenario of preservation for \( N \), to demonstrate the difference between them. Below, these scenarios are named as the algorithms versions, or the cases.

3. Calculation results and analysis

3.1. Asymptotes of the NLT

The one-step calculations of KI on grid (5) were performed for several spectral forms some of which are given in table 1 with the estimates of parameter \( p \) for NLT of form (11). It was found that the high-frequency tail of the NLT, for all kinds of J-spectrum (6-9), is positive only when \( n \geq 4 \); when \( n < 4 \), it is negative and poorly defined due to the weak convergence of the KI for such spectra [4, 12, 16].

Table 1. Asymptotes of the NLT at the first time-step of solving KE (1).

| Run | Initial spectrum shape | Nl-asymptote |
|-----|------------------------|--------------|
| No  | \( n \) | \( \gamma \) | \( \Psi(\theta) \) | \( p \) |
| 1   | 6   | 3.3 | const | 4.9 |
| 2   | 6   | 3.3 | \( \cos^2(\theta) \) | 5.3 |
| 3   | 6   | 1.0 | const | 5.2 |
| 4   | 5   | 3.3 | const | 3.3 |
| 5   | 5   | 3.3 | \( \cos^2(\theta) \) | 3.8 |
| 6   | 5   | 1.0 | const | 3.8 |
| 7   | 4   | 3.3 | const | 0.85 |

The features of \( N_l(\omega) \)-asymptotes at the first-step of solving KE (1) can be formulated as follows:
1) The high-frequency tail of the NLT for J-spectra is positive when \( n \geq 4 \) and negative when \( n < 4 \);
2) Representation of \( N_l(\omega) \) in form (11) has a varying power-law of decay along the whole tail;
3) For values \( n \geq 5 \), parameter \( p \) is close to value \( n-1 \), in the intermediate range, \( 5 < \omega / \omega_p < 15 \); in the entire frequency band, \( 5 < \omega / \omega_p < 25 \), the decay of \( N_l(\omega) \) is somewhat weaker than \( \omega S(\omega) \);
4) For fixed \( n > 5 \), parameter \( p \) increases with a decrease of parameter \( \gamma \) and increase of angular anisotropy parameter \( a \) (see table 1);
5) When parameter \( n \) approaches to 4, the relative intensity of the NLT-tail decreases radically, and when \( n \leq 4 \), the decay-features of NLT begin to depend significantly on the spectrum parameters: \( \gamma, a \), and on the limits of the computing grid in units of \( \omega / \omega_p \), due to the slow convergence of the KI.

Thus, the nonlinear interactions at high frequencies are sensitive to both the frequency-shape of the spectral peak (variations of \( \gamma \)) and the angular shape of the spectrum (variations of \( a \)) for any fixed \( n \). It means: the nonlinear interactions have an explicit non-local feature (see also [11, 12, 15]), what is very important for explaining the reasons of the self-similar spectral shape formation (see below).

3.2. Kinetic equation solutions on long-time scales

To achieve the goals posed, we performed a large series of numerical solutions for KE of type (1) on grid (6) with the algorithm versions both preserving total wave action \( N \) and total wave energy \( E \). For the first time, our calculations have shown that on long scales of evolution time, this difference impacts slightly on the features of the self-similar shapes for \( N_l(\omega) \) and \( S(\omega) \), in the peak-domain.
(0.5\omega_p < \omega < 2\omega_p), and does not impact on the asymptotes for their high-frequency tails (\omega > 2\omega_p). To discuss these results, we dwell, mainly, on the features for one-dimensional functions.

Figure 1. Normalized self-similar shapes on long-time scales: (a) \text{S}_{sf}(\omega); (b) \text{N}_{l_{sf}}(\omega).

For each version of the long-time solutions of KE (1), the following results were established:

1) The self-similar spectra, \text{S}_{sf}(\omega), in any algorithm case, have nearly the same shapes (figure 1a), having a slightly different frequency width \(B\) and decaying for \(\omega > 2\omega_p\) with the law
\[
\text{S}_{sf}(\omega) \sim \omega^{-4.0 \pm 0.02};
\]

2) The self-similar shapes of \text{N}_{l_{sf}}(\omega) have a certain, visually seen discrepancy, in different cases (figure 1b), though they have the same tail-asymptotes (\omega > 2\omega_p)
\[
\text{N}_{l}(\omega) \sim -\omega^{-4.15 \pm 0.05};
\]

3) The shape parameters for 2D spectrum \text{S}_{sf}(\omega, \theta): A_p = A(\omega_p) and \(B\), vary slightly in time within \(\pm 5\%\), and their average values depend on the degree of anisotropy for the initial spectrum \text{S}(\omega, \theta), as shown in table 2 for case \(E=\text{const}\). If \(N=\text{const}\), values of \(B\) are on 1-2\% smaller; \(A(\omega)\) is the same.

| Run No | Initial spectrum shape | Evolution time, s | Asymp. of \text{N}(\omega) | Parameters of \text{S}(\omega, \theta) |
|--------|------------------------|-------------------|-----------------------------|-------------------------------------|
| \(n\) | \(\gamma\) | \(\Psi(\theta)\) | \(p\) | \(B*100\) | \(A_p*100\) |
| 1     | 6         | 3.3       | \text{const}     | 1.3 \cdot 10^6 | 4.1 | 22 | 16 |
| 2     | 6         | 1.0       | \text{const}     | 4.2 \cdot 10^6 | 4.2 | 25 | 16 |
| 3     | 5         | 3.3       | \text{const}     | 1.3 \cdot 10^5 | 4.1 | 25 | 16 |
| 4     | 5         | 1.0       | \text{const}     | 7.9 \cdot 10^4 | 4.2 | 25 | 16 |
| 5     | 4         | 3.3       | \text{const}     | 6.9 \cdot 10^4 | 4.1 | 23 | 16 |
| 6     | 4         | 1.0       | \text{const}     | 5.4 \cdot 10^4 | 4.2 | 26 | 16 |
| 7     | 5         | 1.0       | \cos^2(\theta/2) | 4.1 \cdot 10^6 | 4.2 | 32 | 46 |
| 8     | 5         | 1.0       | \cos^2(\theta/2) | 3.6 \cdot 10^6 | 4.1 | 34 | 63 |
| 9     | 5         | 1.0       | \cos(\theta)     | 4.1 \cdot 10^5 | 4.2 | 33 | 64 |
| 10    | 5         | 1.0       | \cos(\theta)     | 8.7 \cdot 10^6 | 4.2 | 33 | 66 |
| 11    | 5         | 3.3       | \cos^2(\theta)   | 5.6 \cdot 10^6 | 4.2 | 31 | 61 |

Note. Different degrees of shadowing show differences in angular forms for the self-similar spectra.

The normalized shapes of self-similar 1D functions of spectrum \text{S}(\omega) and NLT \text{N}(\omega) are shown in figures 1(a,b). They demonstrate the proper differences for cases \(N=\text{const}\) and \(E=\text{const}\). Note that in
each separate case, transfers $N_{sf}(\omega)$, shown in figure 1(b), provides the proper balances for $N$ and $E$. In terms of value $Q$ (Eq. 10), the relevant balances are practically zero (less 0.05%).

For anisotropic initial spectra (runs 7-11), function of angular directivity $A(\omega)$ has the sharp peak, slightly below peak frequency $\omega_p$, with the frequency increasing, $A(\omega)$ quickly becomes constant: $A(\omega) \approx 0.16$ (for $\omega \geq 2 \omega_p$), corresponding to the isotropic distribution of the spectrum (figure 2a).

![Figure 2](image)

Figure 2. (a) Shapes of angular directivity $A(\omega)$: 1- mean for runs 7-11; 2 - run 10; 3 - run 11. (b) Shapes $S_{sf}(\omega/\omega_p)$ for run 3: 1 - J-spectrum; 2 - time $t \approx 1 \cdot 10^4$ s; 3 - $t = 1 \cdot 10^5$ s.

In turn, the frequency shape of one-dimensional spectrum $S_{sf}(\omega)$ has the sharp peak too. For isotropic initial spectra, its shape is very similar to one for the J-spectrum with parameters $\gamma = 3.3$ and $n = 5$ (figure 2b). For the anisotropic initial spectra, the shape of peak for $S_{sf}(\omega)$ is similar to that above, although the peak is 1.5 times wider (in the value of $B$). Such a detailed description of the integral parameters for 2D spectrum $S_{sf}(\omega, \theta)$ supplements essentially the earlier results [4, 13, 19].

It needs to note results for very instructive run 6 from table 2 (figures 3a, b). In this case, for the initial J-spectrum decaying with the law $S(\omega) \sim \omega^4$, at the first time-step nonlinear transfer $NI(\omega)$ has a rather intensive tail not corresponding to analytical expectations [7, 8]. But on the relatively long evolution scales, $t > 10^4 \cdot 1/\omega_p(\theta)$, the self-similar spectrum has formed, having the same decay-law as the initial spectrum: $S_{sf}(\omega) \sim \omega^{-4}$. Though, on these scales, transfer $NI(\omega)$ gets the rapidly decaying tail of form (15), corresponding to the theory [7,8]. Similar results we have got for initial spectra with $n < 4$ (run 11). The said is very interesting and important result shown here for the first time. It demonstrates the crucial role of nonlocality for NLT in formation $S_{sf}(\omega)$, already mentioned above.

![Figure 3](image)

Figure 3. (a) Time-history of $S(\omega, t)$ for run 6. Lines: 0 - $t = 0$ s; 1 - $t = 5.7 \cdot 10^2$ s; 2 - $t = 3.9 \cdot 10^3$ s; 3 - $t = 5.4 \cdot 10^3$ s; 3' - tail-part of 3 (weighted by 0.3); 3'' - trend of 3' (equation $y = 0.0248x^{-4}$). (b) The same for $NI(\omega, t)$. Lines 1, 2, 3 are weighted by 3, 10, 100, respectively.
Here, it is important to emphasise that the results presented above were repeated for versions of solutions for KE (1) with both preservation of total wave energy $E$ and total wave action $N$. Moreover, the same results were found for the algorithms without exact preservation of $N$ or $E$, when the proper control was not carried out. These facts are very important for understanding features of the considered WWT in the frame of KE (1) and treating its difference from the Kolmogorov’s one.

At the same time, some asymptotes are different for the various versions of the algorithms. It is evidently true for the time-dependence of total wave action $N(t)$ and total wave energy $E(t)$. The rate of downshifting the peak frequency, $\omega_p(t)$ is also different (see below). But the most interesting is the frequency dependence of the fluxes for wave energy $P_E(\omega)$ and wave action $P_N(\omega)$, taking place in two different versions for the KE-solution algorithms.

Functions $P_E(\omega)$ and $P_N(\omega)$ can be estimated with formulas similar to ones from [4]:

$$P_E(\omega) = -\int_{\omega_1}^{\omega} N_l(\omega, \theta)d\theta d\omega \quad \text{and} \quad P_N(\omega) = -\int_{\omega_1}^{\omega} (N_l(\omega, \theta)/\omega)d\theta d\omega,$$

where $\omega_1$ is the lower limit of the numerical frequency band. According to [4], the positive value of $P_E$ (or $P_N$) indicates the flux upward in frequency and the negative value does downward.

![Graph](image)

Figure 4. (a) Time-history of flux $P_E(\omega,t)$ for run 2, lines: $0 - t = 0$ s; $1 - t = 700$ s; $2 - t = 2500$ s.
(b) The same for $P_N(\omega,t)$. Lines correspond to the same time moments.

For run 2 from table 2, the time-histories of fluxes $P_E(\omega,t)$ and $P_N(\omega,t)$ for different cases are presented in figures 4(a,b). It is seen that, if wave action $N$ is constant, a quasi-constant tail for upward energy flux $P_E(\omega)$ is established in range $\omega > 2\omega_p(t)$, on long evolution scales: $t > 10^3\omega_p(0)$. This fact means the loss of wave energy in the system (herewith, $P_N(\omega_{max},t) = 0$, ensuring $N = \text{const}$). If wave energy $E$ is constant, the same is true for flux $P_N(\omega)$. Due to very week $\omega$-dependences of the tails for $P_E(\omega)$ and $P_N(\omega)$, we call them as constant ones, following to [4,19], for simplicity of analysis.

According to the main conclusions of theories for WWT (e.g., [2, 4, 7, 8, 19]) and Kolmogorov’s one [17], different fluxed should provide two types of Kolmogorov spectra: (3a) and (3b). Though, our results show that this is not the case. For any algorithm, we have got the same self-similar spectral shapes with the tail $S(\omega) \sim \omega^{-4}$. This is another paradox which will be treated below. But previously we dwell on point of reliability of our numerical results.

### 3.3 Correspondence between numerical and analytical results

Despite of the shape of $S_d(\omega)$, obtained in both versions of the KE-solution is nearly the same, the difference between these cases is well expressed in the following temporal asymptotes: different peak frequency downshifting $\omega_p(t)$, peak values evolution, $S_d(t) = S_d(\omega_p(t))$, and, evidently, via temporal
asymptotes for integral characteristics: $E(t)$ or $N(t)$ (in proper case). All these asymptotes could be derived analytically, among them the basic one is the temporal downshifting of peak frequency, $\omega_p(t)$.

Basing on the self-similarity of $S_p(\omega)$ and $N_p(\omega)$, one can analytically derive the asymptotes [4, 28]

$$\omega_p(t) \propto t^{-1/11} \quad \text{and} \quad \omega_p(t) \propto t^{-1/9},$$  \hspace{1cm} (17a, b)

for constant $N$ and constant $E$, respectively. Herewith, theoretical asymptotes for $S_p(t)$, $E(t)$ and $N(t)$ should be as follows (see Appendix in [28]): in the case $N=\text{const}$,

$$S_p(t) = \text{const} ; \quad E(t) \sim \omega_p(t);$$  \hspace{1cm} (18a, b)

and, in the case $E=\text{const}$,

$$S_p(t) \sim 1/\omega_p(t); \quad N(t) \sim 1/\omega_p(t).$$  \hspace{1cm} (19a, b)

In our calculations, the basic ratios are confirmed rather well (estimations are weaker on 5-10% due to small range of downshifting). Though, ratios (18a,b) and (19a,b) are confirmed very well, what allow us to state a high reliability of the results obtained, permitting to draw reasonable conclusions.

4. Interpretation of the results

Here we try to answer the question: why self-similar spectrum $S_d(\omega)$ always has the tail $\sim \omega^{-4}$, both in cases $N=\text{const}$ and $E=\text{const}$, despite of arising constant upward energy flux $P_E(\omega)$ and downward action flux $P_N(\omega)$, respectively, what contradicts to the Kolmogorov turbulence paradigm [17] and theoretical expectations [7, 8].

First, we note that the possibility of using two numerical algorithms is stipulated by the fact that two integral parameters of the wave-system, $N$ and $E$, are not simultaneously preserved in the numerical solution of KE (1)[4, 16, 19]. This has several technical and mathematical reasons, the main of which is a weak convergence of the KI for slow-falling spectra [16]. These reasons are not due to the physics rather to shortages of used mathematical approximation [16], which violates the constancy of $N$ or $E$. Therefore, both cases could be considered, at least, as a compensating the mentioned errors.

Second, the reason of establishing self-similar spectral form $S_d(\omega) \sim \omega^{-4}$ resides in formation of $Nl(\omega)$-tails of form $\sim \omega^{-4}$, in any case of solving KE (1), what stabilizes a spectral shape. This effect is accompanied by formation a specific shape of the spectral peak-domain (figures 2a,b), improving convergence of KI due to the nonlocality of the four-wave interactions (Sect. 3.1 and figures 3a,b).

Third, arising constant fluxes $P_E(\omega)$ or $P_N(\omega)$ (in proper case) is simply the mathematical feature of KE (1), not affecting the formation of $S_d(\omega)$. It is proved by considering the case $E=\text{const}$, providing constant downwarp $P_E(\omega)$ without establishing expected Kolmogorov spectrum with the tail $\sim \omega^{-10/3}$. Instead, the same self-similar spectrum is established again, having the tail of form $S_d(\omega) \sim \omega^{-4}$.

5. Conclusions and final remarks

Establishing self-similar spectrum with tail $S_d(\omega) \sim \omega^{-4}$ is not due to the existence of constant fluxes $P_E(\omega)$ or $P_N(\omega)$, arising in a proper case of solving KE (1), rather due to an intrinsic mathematical property of the KI, analytically found still in [7]. We have checked that the tail $\sim \omega^{-4}$ is formed even in cases without simultaneous exact preservation of $N$ and $E$, and absence of proper constant fluxes.

In any case, the certain, self-similar shape of spectral peak (figures 2a,b) is formed during evolution in the frame of KE (1). Herewith, the shape of spectral peak governs the shape of $Nl(\omega)$ and entire spectrum (figures 3a,b), due to nonlocality of the nonlinear interactions.

Thus, one can state that, in the frame of KE (1) (i.e. in the frame of WWT), we do not deal with the Kolmogorov turbulence. There are at least three differences between WWW and KT, they are: a) the presence of a distinguished frequency scale, formed by the peak-domain of spectrum, governing the entire spectrum shape; b) the nonlocality of four-wave nonlinear interactions, providing the self-similar spectrum formation; and c) the absence of sources and sinks external to wave system.

In addition, it is pertinent to note here that, in this problem, a using condition of wave energy constancy is preferable, from the physical point of view. It is because of the fact that conservativity of the potential wave system is proved in [25], whilst the invariance of total action $N$ does not directly follow from the Euler equations. There are no physical reasons for the leakage of total energy in the considered system of homogeneous field of nonlinear potential waves without viscosity and external
forces. Moreover, some evidence of realization the case \( E = \text{const} \) can be found in [29], validated via empirical supporting the presence of asymptote (19a) in the wave-tank experiments.

On other hand, the fact of increasing \( N \), in the case \( E=\text{const} \), could be interpreted as the formal effect of ‘condensation’ of ‘virtual wave particles’ (according to the meaning of wave action \( N \) [3]) from the tail of the spectrum, during the nonlinear wave evolution in the frame of KE (1).

Principally another situation, relevant to the Kolmogorov turbulence, is realised in the case of presence of the source and sink external to the wave system and separated in the frequency band. Such cases were modelled in numerous works devoted to numerical solution of the KE of form (4) (e.g., [4, 19, 21, 26] among others). Discussion of the mentioned algorithms impact on the processes of Kolmogorov spectra formation in forms (3a) and (3b), in the frame KE (4), detailed analysis of these processes, and their reconciliation with the conclusions of this paper require a separate study in future.

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