Approximate functional equation for the derivatives of functions in Selberg class

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Abstract

Let $F(s)$ be a function belonging to Selberg class. Chandrasekharan and Narasimhan proved the approximate functional equation for $F(s)$. In this paper, we shall generalize this formula for the derivatives of $F(s)$.

1 Introduction

Selberg [8] introduced a class of zeta functions satisfying the following properties:

(a) The function $F(s)$ is written as an absolutely convergent Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_F(n)n^{-s}$ for $\text{Re } s > 1$.

(b) There exists $m \in \mathbb{Z} \geq 0$ such that $(s - 1)^m F(s)$ is an entire function of finite order.

(c) The function $\Phi(s) = Q^s \prod_{j=1}^{q} \Gamma(\lambda_j s + \mu_j) F(s)$ satisfies $\Phi(s) = \omega \overline{\Phi(1 - s)}$ where $q \in \mathbb{Z}_{\geq 1}$, $Q \in \mathbb{R}_{> 0}$, $\lambda_j \in \mathbb{R}_{> 0}$, $\mu_j \in \mathbb{C}$ : $\text{Re } \mu_j \geq 0$, $\omega \in \mathbb{C}$ : $|\omega| = 1$, and $X$ denotes $X(s) = X(\overline{s})$.

(d) The Dirichlet coefficients $a_F(n)$ satisfy $a_F(n) = O(n^\varepsilon)$ for any $\varepsilon \in \mathbb{R}_{> 0}$.

(e) The function $\log F(s)$ is written as $\log F(s) = \sum_{n=1}^{\infty} b_F(n)n^{-s}$ where $b_F(n)$ satisfy $b_F(n) = 0$ for $n \neq p^r$ ($r \in \mathbb{Z}_{\geq 1}$) and $b_F(n) = O(n^{\theta})$ for some $\theta \in \mathbb{R}_{< 1/2}$, which is called the Selberg class $S$. For example, we see that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and the $L$-function attached to cusp forms $L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}$ belong to $S$, where $f$ is a cusp form of weight $k$ given by $f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{k-1/2} e^{2\pi i nz}$. By Landau’s [5] result (see (15) of p.214), the conditions (a)–(d) of $S$ imply that the average of $a_F(s)$ is approximated as

$$\sum_{n \leq x} a_F(n) = \begin{cases} x \sum_{p_F=1}^{p_F-1} c_r \left( \log x \right)^r + O(x^{\frac{d_F}{d_F+1} + \varepsilon}), & p_F \in \mathbb{Z}_{\geq 1}, \\ O(x^{\frac{d_F}{d_F+1} + \varepsilon}), & p_F \in \mathbb{Z}_{\leq 0}, \end{cases}$$

(1.1)

where $c_r \in \mathbb{C}$, $p_F$ is the order of pole at $s = 1$ for $F(s)$, and $d_F$ is given by $d_F = 2 \sum_{j=1}^{q} \lambda_j$ called the degree of $F$. The condition (c) implies that the $m$-th derivatives of $F(s)$ holds

$$F^{(m)}(s) = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda_F^{(m-r)}(s) \overline{F^{(r)}(1 - s)}$$

(1.2)
for $s \in \mathbb{C}$ where $\chi_F(s)$ is given by

$$
\chi_F(s) = \omega Q^{1-2s} \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(1-s) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)}.
$$

(1.3)

Chandrasekharan and Narasimhan [2] proved the approximate functional equation for a class of zeta functions (see Theorem 2 of p.53). In the Selberg class, this equation is written as

$$
F(s) = \sum_{n \leq y} \frac{a_F(n)}{n^s} + \chi_F(s) \sum_{n \leq y} \frac{a_F(n)}{n^{1-s}} + O(|t|^{\frac{d_F}{2}(1-\sigma)-\frac{1}{2F} \log y})^{1\over 2}
$$

under the condition $a_n \geq 0$ where $F \in \mathcal{S}$, $s = \sigma + it$; $0 \leq \sigma \leq 1$, $|t| \geq 1$ and $y = (Q^{\lambda_1^4} \cdots \lambda_\sigma^\lambda)|t|^{d_F/2}$. On the other hand, the author [9] showed the approximate functional equation for the derivatives of $L$-function attached for cusp form as

$$
L_f^{(m)}(s) = \sum_{n \leq |n|} \frac{\lambda_f(n)(-\log n)^m}{n^s} + \chi_{L_f}(s) \sum_{n \leq |n|} \frac{\lambda_f(n)(-\log n)^m}{n^{1-s}} + O(|t|^{1/2-\sigma+\varepsilon}),
$$

and introduced the mean value formula for $L_f^{(m)}(s)$ as

$$
\int_0^T |L_f^{(m)}(\sigma + it)|^2 dt
$$

$$
= \begin{cases}
A_f T(\log T)^{2m+1} + O(T(\log T)^{2m}), & \sigma = 1/2, \\
T \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} + O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in (1/2, 1), \\
T \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} + O((\log T)^{2m+2}), & \sigma = 1.
\end{cases}
$$

where $\chi_{L_f}(s) = (-1)^k/2(2\pi)^{1-s} \Gamma(1-s+k^{-1}1/2) \Gamma(s+k^{-1}1/2)$ and $A_f$ is a positive constant depending on $f$. These results are generalized for those results for $L_f(s)$ proved by Good [4]. For $F \in \mathcal{S}$, $\sigma > 1/2$, and $T > 0$, let $N_{F^{(m)}}(\sigma, T)$ be a number of zeros for $F^{(m)}(s)$ in the region $\text{Re } s \geq \sigma$, $0 < \text{Im } s \leq T$. As the application of those results, the author [10] obtained a zero-density estimate for $L_f^{(m)}(s)$, which is

$$
N_{L_f^{(m)}}(\sigma, T) = O \left( \frac{T}{\sigma - 1/2} \log \frac{1}{\sigma - 1/2} \right) \quad (T \to \infty)
$$

for $\sigma > 1/2$. This result corresponds the estimate of zero-density estimate for $\zeta^{(m)}(s)$

$$
N_{\zeta^{(m)}}(\sigma, T) = O \left( \frac{T}{\sigma - 1/2} \log \frac{1}{\sigma - 1/2} \right) \quad (T \to \infty)
$$

for $\sigma > 1/2$ which is shown by Aoki–Minamide [1]. However, zero-density estimates for derivatives for zeta-function belonging to Selberg class is not known.

In this paper we study the approximate functional equation for $F^{(m)}(s)$ in order to establish tools for estimating $N_{F^{(m)}}(\sigma, T)$ in next paper. To attain the above object we shall use Good’s [4] method and the result (1.1). Let $\mathcal{R}$ be a class of $C^\infty$-class functions $\varphi : [0, \infty) \to \mathbb{R}$ satisfying $\varphi(\rho) = 1$ for $\rho \in [0, 1/2]$ and $\varphi(\rho) = 0$ for
\( \rho \in [2, \infty) \), which is called characteristic functions. Put \( \varphi_0(\rho) := 1 - \varphi(1/\rho) \) and \( \|\varphi^{(j)}\|_1 = \int_0^\infty |\varphi^{(j)}(\rho)|d\rho \) where \( \varphi^{(j)} \) is the \( j \)-th derivative function of \( \varphi \in \mathcal{R} \). Then we see that \( \varphi_0 \in \mathcal{R} \) and \( \|\varphi^{(j)}\|_1 < \infty \). For \( j \in \mathbb{Z}_{\geq 0}, \ r \in \{0, \ldots, m\}, \ \rho \in \mathbb{R}_{> 0} \) and \( s \in \mathbb{C} : |t| \gg 1 \), we define

\[ \gamma_j^{(r)}(s; \rho) := \frac{1}{2\pi i} \int_{\mathcal{F}} \frac{g_F(s + w)}{g_F(s)} \frac{1}{w \cdots (w + j)} \chi_F(1 - (s + w)) \times \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} \left( \rho e^{-\frac{\pi}{2} \text{sgn}(t)} \right) \frac{dw}{w} \]  

(1.5)

\[ \delta_j^{(r)}(s; \rho) := \frac{1}{2\pi i} \int_{\mathcal{F}} \frac{g_F(s + w)}{g_F(s)} \frac{1}{w \cdots (w + j)} \chi_F(1 - (s + w)) \times \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} \left( \rho e^{-\frac{\pi}{2} \text{sgn}(t)} \right) \frac{dw}{w} \]  

(1.6)

where \( \text{sgn}(t) = \pm 1 \) if \( t \gtrless 0 \), \( \mathcal{F} = \{(1/2 \pm 1) - \sigma + \sqrt{t} e^{i\pi(1/2 \pm \theta)} | \theta \in [0, 1]\} \cup \{u - \sigma \pm i\sqrt{t} | u \in [-1/2, 3/2]\} \) (double-sign corresponds), and \( g_F(s) \) is given by

\[ g_F(s) = (s(1 - s))^{p_F + m} \prod_{j=1}^{q} (A_j(s) \chi_j(1 - s))^{m+1}. \]  

(1.7)

Here \( A_j(s) \) is given by

\[ A_j(s) = \begin{cases} 1, & \text{Re } \mu_j > \lambda_j/2, \\ \prod_{n=0}^{\nu_j} (\lambda_j s + \mu_j - n), & \text{Re } \mu_j \leq \lambda_j/2 \end{cases} \]  

(1.8)

where \( \nu_j = 0 \) when \( \text{Re } \mu_j > \lambda_j/2 \) and \( \nu_j = [\lambda_j/2 - \text{Re } \mu_j] \) when \( \text{Re } \mu_j \leq \lambda_j/2 \).

Then we obtain the approximate functional equation for \( F^{(m)}(s) \) containing characteristic functions:

**Theorem 1.1.** For any \( \varphi \in \mathcal{S}, \ \varphi \in \mathcal{R}, \ m \in \mathbb{Z}_{\geq 0}, \ l \in \mathbb{Z}_{> M_F}, \ s \in \mathbb{C} : \sigma \in [0, 1], \ |t| \gg 1 \) and \( y_1, y_2 \in \mathbb{R}_{> 0} : y_1 y_2 = (Q\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q})^2 |t|^{d_F} \), we have

\[ F^{(m)}(s) = \sum_{n=1}^{\infty} a_F(n) \frac{(- \log n)^m}{n^s} \varphi \left( \frac{n}{y_1} \right) + \sum_{r=0}^{m} (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n=1}^{\infty} \frac{a_F(n) (- \log n)^m}{n^{1-s}} \varphi_0 \left( \frac{n}{y_2} \right) + R_{\varphi}(s) \]

where \( M_F \) is some positive constant and \( R_{\varphi}(s) \) is given by

\[ R_{\varphi}(s) := \sum_{\frac{m}{2} \leq n \leq 2y_1} a_F(n) (- \log n)^m \sum_{j=1}^{l} \varphi^{(j)} \left( \frac{n}{y_1} \right) \left( \frac{n}{y_1} \right)^r \gamma_j^{(m)} \left( s; \frac{1}{(\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q})^{d_F}} |t| \right) \times \]

plus \( \chi_F(s) \sum_{r=0}^{m} (-1)^r \binom{m}{r} \sum_{\frac{m}{2} \leq n \leq 2y_2} \frac{a_F(n) (- \log n)^r}{n^{1-s}} \times \)
Using these lemmas we shall give Theorems 1.1 and 1.2 in Section 3 and 4 respectively.

By results of Rankin [6], Selberg [7] and Deligne [3], we see that
\[
\text{L}_{k}\text{ of weight for derivatives of Rankin-Selberg} L_{\infty}\sum
\]
above theorem, and choosing \(\alpha\) for \(\text{any}\) \(\text{the approximate functional equation for}\ \Phi\)
\[
\text{As an example of Theorem 1.2, we shall give the approximate functional}\ \text{equation}
\]
\[
\text{f}(\cdot) \in S\text{SL}_2(\mathbb{Z})\text{given by } f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \text{ and } g(z) = \sum_{n=1}^{\infty} \lambda_g(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \text{the Rankin-Selberg } L\function\text{ is defined by}
\]
\[
\text{L}_{f \times g}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n) \lambda_g(n)}{n^s}.
\]
By results of Rankin [6], Selberg [7] and Deligne [3], we see that \(L_{f \times g}(s)\) belongs to \(S\). Hence we obtain
\[
\text{L}_{f \times g}^{(m)}(s) = \sum_{n \leq \frac{|t|^{\frac{2}{d}}}{4\pi^2}} \lambda_{f \times g}(n)(-\log n)^m \frac{1}{n^s} + \sum_{r=0}^{m} (-1)^r \binom{m}{r} \chi_{L_{f \times g}}^{(m-r)}(s) \sum_{n \leq \frac{|t|^{\frac{2}{d}}}{4\pi^2}} \frac{\lambda_{f \times g}(n)(-\log n)^r}{n^{1-s}} + O(|t|^{\frac{3}{2} - 2\sigma + \epsilon})
\]
where \(m \in \mathbb{Z}_{\geq 0}, s = \sigma + it : \sigma \in [0, 1], |t| \gg 1\), and \(\chi_{L_{f \times g}}(s)\) is given by
\[
\chi_{L_{f \times g}}(s) = (2\pi)^{4s-2} \frac{\Gamma(1-s)\Gamma(1-s+k-1)}{\Gamma(s)\Gamma(s+k-1)}.
\]
In next section we shall show preliminary lemmas to prove Theorems 1.1 and 1.2. Using these lemmas we shall give Theorems 1.1 and 1.2 in Section 3 and 4 respectively.
2 Some Lemmas

First for $\varphi \in \mathcal{R}$ we define

$$K_\varphi(w) := w \int_0^\infty \varphi(\rho) \rho^{w-1} d\rho \quad (\text{Re } w > 0).$$

Then $K_\varphi(w)$ has the following properties:

**Lemma 2.1** ([4, LEMMA 3]). The function $K_\varphi(w)$ is analytically continued to the whole $w$-plane. Furthermore $K_\varphi(w)$ has the functional equation

$$K_\varphi(w) = K_{\varphi_0}(-w).$$

and the integral representation

$$K_\varphi(w) = \frac{(-1)^{l+1}}{(w+1) \cdots (w+l)} \int_0^\infty \varphi^{(l+1)}(\rho) \rho^{w+l} d\rho$$

for any $l \in \mathbb{Z}_{\geq 0}$. Especially $K_\varphi(0) = 1$.

Next to approximate $(\chi_{F}^{(r)}/\chi_{F})(s)$, we shall use the following lemma:

**Lemma 2.2** ([9, Lemma 2.3]). Let $F$ and $G$ be holomorphic functions in the region in $D$ satisfying $\log F(s) = G(s)$ and $F(s) \neq 0$ for $s \in D$. Then for any $r \in \mathbb{Z}_{\geq 1}$ there exist $\ell_1, \ldots, \ell_r \in \mathbb{Z}_{\geq 0}$ and $C_{\ell_1,\ldots,\ell_r} \in \mathbb{Z}_{\geq 0}$ such that

$$F^{(r)}(s) = \sum_{\ell_1+\cdots+\ell_r=r} C_{\ell_1,\ldots,\ell_r} (G^{(1)}(s))^{\ell_1} \cdots (G^{(r)}(s))^{\ell_r}. (2.3)$$

for $s \in D$. Especially $C_{r,0,\ldots,0} = 1$.

Before approximating $(\chi_{F}^{(r)}/\chi_{F})(s)$ we shall show the following formulas:

**Lemma 2.3.** Let $D = \{ z \in \mathbb{C} \mid \text{Re } z < \delta, |\text{Im } z| < 1 \}$ where $\delta \in \mathbb{R}_{\geq 0}$. For any $s \in \mathbb{C} \setminus D$ we satisfy the following formulas:

(i) \[ \sum_{n=1}^{\infty} \frac{1}{(s+n)^l} = \begin{cases} O(|t|^{-(l-1)}), & |t| \gg 1, \\ O(1), & |t| \ll 1, \end{cases} \quad \text{where } l \in \mathbb{Z}_{\geq 2}. \]

(ii) \[ \sum_{n=1}^{\infty} \left( \frac{1}{s+n} - \frac{1}{n} \right) = \begin{cases} -\log |t| + i\pi \text{sgn}(t)/2 - \gamma + O(|t|^{-1}), & |t| \gg 1, \\ O(1), & |t| \ll 1, \end{cases} \]

where $\gamma$ is the Euler’s constant given by $\gamma = 1 - \int_1^{\infty} \{u\} u^{-2} du$.

**Proof.** Using partial summation we have

$$\sum_{n=1}^{\infty} \frac{1}{(s+n)^l} = l \int_1^{\infty} \frac{u - \{u\}}{(s+u)^{l+1}} du \ll \int_1^{\infty} \left( \frac{1}{|s+u|^l} + \frac{|s|}{|s+u|^{l+1}} \right) du. (2.4)$$

The estimates

\[ |u+s| \geq \begin{cases} |\text{Im}(u+s)| \geq |s \sin \varepsilon|, & u \leq |s| \text{ and } \text{arg } s \in [-\pi + \varepsilon, \pi - \varepsilon], \\ |\text{Re}(u+s)| \geq |s| \cos \varepsilon, & u \leq |s| \text{ and } \text{arg } s \in [-\varepsilon, \varepsilon], \\ |\text{Re}(u+s)| \geq u, & u \geq |s|, \end{cases} \quad (2.5) \]
Lemma 2.4. For any \( r \in \mathbb{Z} \geq 2 \) we obtain the approximate formula for \( (\chi_F^{(r)})^{-1} \) as follows:

\[
\chi_F^{(r)}(s) = \left\{ \begin{array}{ll}
-\log |t| + i\pi \text{sgn}(t)/2 + O(|t|^{-1}), & |t| \gg 1, \\
O(1), & |t| \ll 1
\end{array} \right.
\]

and

\[
\chi_F^{(r)}(s) = \left\{ \begin{array}{ll}
-1 + O(|t|^{-1}), & |t| \gg 1, \\
O(1), & |t| \ll 1.
\end{array} \right.
\]

for any \( s \in \mathbb{C} \setminus D \) and \( l \in \mathbb{Z} \geq 2 \). Combining (2.4) and (2.6), and estimating \(|s|^{-(l-1)}\) we obtain the formula (i). Similarly to (i), we calculate

\[
\sum_{n=1}^{\infty} \left( \frac{1}{s+n} - \frac{1}{n} \right)
= \int_{1}^{\infty} \left( \frac{1}{(s+u)^2} - \frac{1}{u^2} \right) (u - \{u\}) du
= \int_{1}^{\infty} \left( \frac{1}{s+u} - \frac{1}{u} \right) du + \int_{1}^{\infty} \frac{(-s)}{(s+u)^2} du - \int_{1}^{\infty} \frac{u}{(s+u)^2} du + 1 - \gamma.
\]

Then for \( s \in \mathbb{C} \setminus D \) the first and second term of right-hand side of (2.7) are

\[
-\log(s+1)
= -\log s + O(|s|^{-1}) = \left\{ \begin{array}{ll}
-\log |t| + i\pi \text{sgn}(t)/2 + O(|t|^{-1}), & |t| \gg 1, \\
O(1), & |t| \ll 1
\end{array} \right.
\]

and

\[
-1 + \frac{1}{1+s} = \left\{ \begin{array}{ll}
-1 + O(|t|^{-1}), & |t| \gg 1, \\
O(1), & |t| \ll 1.
\end{array} \right.
\]

respectively, where (2.5) and \( \log s = \log |t| + i\pi \text{sgn}(t)/2 + O(|t|^{-1}) \) (see [4, p.335]) were used. By combining (2.7)–(2.9) the formula (ii) is obtained.

Using the infinite product of \( \Gamma(s) \) and applying the above lemma with \( F = \chi_F \), we obtain the approximate formula for \( (\chi_F^{(r)}/\chi_F)(s) \) as follows:

**Lemma 2.4.** For any \( F \in \mathcal{S} \) and \( r \in \mathbb{Z} \geq 2 \) the function \( (\chi_F^{(r)}/\chi_F)(s) \) has pole of order \( r \) at \( s = -(\mu_j + n)/\lambda_j, \ 1 + (\mu_j + n)/\lambda_j \) where \( n \in \mathbb{Z} \geq 0 \) and \( j \in \{1, \ldots, q\} \). Put

\[
D = \{ z \in \mathbb{C} \mid a \leq \text{Re} \ z \leq b \},
\]

\[
E_1 = \left\{ z \in \mathbb{C} \mid \text{Re} \ z < \max_j \frac{-\text{Re} \mu_j + \delta}{\lambda_j}, \ \min_j \frac{-\text{Im} \mu_j - 1}{\lambda_j} < \text{Im} \ z < \max_j \frac{-\text{Im} \mu_j + 1}{\lambda_j} \right\},
\]

\[
E_2 = \left\{ z \in \mathbb{C} \mid \text{Re} \ z > \max_j \frac{\text{Re} \mu_j - \delta}{\lambda_j} - 1, \ \min_j \frac{-\text{Im} \mu_j - 1}{\lambda_j} + 1 < \text{Im} \ z < \max_j \frac{\text{Im} \mu_j + 1}{\lambda_j} + 1 \right\}
\]

where \( a, b \in \mathbb{R} \) and \( \delta \in \mathbb{R}_{>0} \). Then for any \( s \in \mathbb{C} \setminus (E_1 \cup E_2) \) we have

\[
\frac{\chi_F^{(r)}}{\chi_F}(s) = \left\{ \begin{array}{ll}
-\log(C_F|t|^{d_F})^m + O\left( \frac{(\log |t|)^{m-1}}{|t|} \right), & |t| \gg 1, \\
O(1), & |t| \ll 1
\end{array} \right.
\]

where \( C_F = (Q\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q})^2 \).

**Proof.** First we check the location and order of pole for \( (\chi_F^{(m)}/\chi_F)(s) \). Taking loga-
arithmic differentiation in the both-hand side of \((1/\Gamma)(s) = se^{\gamma s} \prod_{n=1}^{\infty} (1 + s/n)e^{-s/n}\) and \((1.3)\) we have
\[
-\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{s + n} - \frac{1}{n} \right), \quad (2.10)
\]
\[
\frac{\chi'_F}{\chi_F}(s) = -2 \log Q - \sum_{j=1}^{q} \lambda_j \left( \frac{\Gamma'(\lambda_j s + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} + \frac{\Gamma'(\lambda_j(1-s) + \mu_j)}{\Gamma(\lambda_j(1-s) + \mu_j)} \right), \quad (2.11)
\]
respectively. Put \(G^{(l)}(s) = (d^{l-1}/ds^{l-1})G^{(1)}(s)\) and let \(G^{(1)}(s)\) be the right-hand side of \((2.11)\). Applying \((2.10)\) to \((2.11)\), we get
\[
G^{(l)}(s) = \begin{cases} 
-2 \log Q + d_F \gamma + \sum_{j=1}^{q} \lambda_j \left( \frac{1}{\lambda_j s + \mu_j} + \frac{1}{\lambda_j(1-s) + \mu_j} \right) + \\
\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_j s + \mu_j + n} - \frac{1}{\lambda_j(1-s) + \mu_j + n} \right), & l = 1,
\end{cases}
\]
\[
\sum_{j=1}^{q} \sum_{n=0}^{\infty} \left( \frac{(-1)^{l-1}(l-1)!}{(\lambda_j s + \mu_j + n)^l} + \frac{(l-1)!}{(\lambda_j(1-s) + \mu_j + n)^l} \right), & l \in \mathbb{Z}_{\geq 2}.
\]
\[(2.12)\]
Hence \(G^{(l)}(s)\) has pole of order \(l\) at \(s = -(\mu_j + n)/\lambda_j, 1+(\mu_j + n)/\lambda_j (n \in \mathbb{Z}_{\geq 0})\). By using Lemma 2.2 the first statement of Lemma 2.4 is showed.

Lastly we shall approximate \(G^{(l)}(s)\). Since \(\text{Re}(\lambda_j s + \mu_j) < \delta, |\text{Im}(\lambda_j s + \mu_j)| < 1\) for \(s \in E_1\) and \(\text{Re}(\lambda_j(1-s) + \mu_j) < \delta, |\text{Im}(\lambda_j(1-s) + \mu_j)| < 1\) for \(s \in E_2\), Lemma 2.3 gives that
\[
G^{(j)}(s) = \begin{cases} 
O(|t|^{-(j-1)}), & j \in \mathbb{Z}_{\geq 2}, |t| \gg 1, \\
O(1), & j \in \mathbb{Z}_{\geq 1}, |t| \ll 1.
\end{cases}
\]
for \(s \in \mathbb{C} \setminus (E_1 \cup E_2)\). Especially in the case of \(j = 1\) and \(|t| \gg 1\), since \(\text{sgn}(\text{Im}(\lambda_j s + \mu_j)) + \text{sgn}(\text{Im}(\lambda_j(1-s) + \mu_j)) = 0\), \(G^{(1)}(s)\) is approximated as
\[
G^{(1)}(s) = -2 \log Q + d_F \gamma + \sum_{j=1}^{q} \lambda_j(-2 \log |\lambda_j t + \text{Im} \mu_j| - 2\gamma) + O(|t|^{-1})
\]
\[
= - \log ((Q\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q})^2|t|^{d_F}) + O(|t|^{-1})
\]
for \(s \in \mathbb{C} \setminus (E_1 \cup E_2)\) and \(|t| \gg 1\). By Lemma 2.2 a desired approximation is obtained:
\[
\frac{\chi_F^{(m)}}{\chi_F}(s) = (G^{(1)}(s))^m + O(|G^{(1)}(s)|^{m-2}|G^{(2)}(s)|)
\]
\[
= \begin{cases} 
(-d_F \log(C_F|t|))^m + O(|t|^{-1}(\log t)^{m-1}), & |t| \gg 1, \\
O(1), & |t| \ll 1.
\end{cases}
\]
\[
\square
\]
Next in order to estimate the gamma-factors of \((1.5)\) and \((1.6)\), we shall use the following estimates:
Lemma 2.5 ([4, Lemma 2 of p.334]). For \(a, b \in \mathbb{R}\) put \(D := \{z \in \mathbb{C} \mid a \leq \text{Re} z \leq b\}\) and \(E_{-} := \{z \in \mathbb{C} \mid \text{Re} z < 1/2, |\text{Im} z| < 1\}\). Then for any fixed \(s \in D\) and \(C_0 \in \mathbb{R}_{>0}\) we have
\[
\left| \frac{\Gamma(s + w)}{\Gamma(s)} (e^{-i \frac{\pi}{2} \text{sgn}(t) w}) \right| \leq \begin{cases} C_1 \frac{(1 + |t + v|)^{\sigma + u - 1/2}}{|t|^{|\sigma| - 1/2}}, & \text{if } s + w \in D \setminus E_{-}, \\ C_2 |t|^u, & \text{if } |w| \leq C_0 \sqrt{|t|}. \end{cases}
\]
where \(C_1\) and \(C_2\) are constants.

Replace \(s \mapsto \lambda_j s + \mu_j\) and \(w \mapsto \lambda_j w\) in the above lemma. Using the trivial estimate \((1 + |\lambda_j(t + v) + \text{Im} \mu_j|)_{\lambda_j(\sigma+u)+\text{Re} \mu_j-1/2} \approx \lambda_j^{\lambda_j u}(1 + |t + v|)_{\lambda_j(\sigma+u)+\text{Re} \mu_j-1/2}\) for \(s \in D\) and \(s + w \in D \setminus E_1\), and multiplying the above formula for \(j \in \{1, \ldots, q\}\), a desired estimate is obtained:

Lemma 2.6. Let \(D\) and \(E_1\) be those of Lemma 2.4 respectively. Then for any fixed \(s \in D\) and \(c_0 \in \mathbb{R}_{>0}\) we have
\[
\left| \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} (e^{-i \frac{\pi}{2} \text{sgn}(t)} \frac{dF}{\sigma})^w \right| \leq \begin{cases} c_1(\lambda_1^1 \cdots \lambda_q^1)^u(1 + |t + v|)_{\frac{dF}{\sigma} + \frac{\text{Re} \mu_j}{q} - \frac{1}{2}} |t|_{\frac{dF}{\sigma} + \frac{\text{Re} \mu_j}{q} - \frac{1}{2}}^u, & \text{if } s + w \in D \setminus E_1, \\ c_2(\lambda_1^1 \cdots \lambda_q^1)^u |t|_{\frac{dF}{\sigma} + \frac{\text{Re} \mu_j}{q} - \frac{1}{2}}^u, & \text{if } |w| \leq c_0 \sqrt{|t|}, \end{cases}
\]
where \(e_F := 2 \sum_{j=1}^{q} \text{Re} \mu_j\) and \(c_1, c_2\) are constants depending on \(\lambda_1, \mu_1, \ldots, \lambda_q, \mu_q\).

By using Stirling’s formula and residue theorem the functions \(\gamma_j^{(r)}(s; \rho)\) and \(\delta_j^{(r)}(s; \rho)\) are approximate as follows:

Lemma 2.7. For any \(j, r \in \mathbb{Z}_{\geq 0}\) and \(s \in \mathbb{C}: |t| \gg 1\) we have
\[
\gamma_j^{(r)}(s; \frac{1}{(\lambda_1^1 \cdots \lambda_q^1)^{\frac{2}{\sigma_F}} |t|}) = \begin{cases} (\chi_F^{(r)}/\chi_F)(1 - s), & j = 0, \\ O(|t|^{-1}(\log |t|)^r), & j = 1, \\ O(|t|^{-\frac{3}{2}}(\log |t|)^r), & j \in \mathbb{Z}_{\geq 2}. \end{cases}
\] (2.13)

The function \(\delta_j^{(r)}(s; (\lambda_1^1 \cdots \lambda_q^1)^{-\frac{2}{\sigma_F}} |t|^{-1})\) equals the right hand side of (2.13).

Proof. Since \(|w| \ll \sqrt{|t|}\) for \(w \in \mathcal{F}\), from Lemma 2.6 we can obtain a desired formula in the case of \(j \in \mathbb{Z}_{\geq 2}\):
\[
\gamma_j^{(r)}(s; \frac{1}{(\lambda_1^1 \cdots \lambda_q^1)^{\frac{2}{\sigma_F}} |t|}) \ll \int_{\mathcal{F}} \frac{(\log |t|)^r}{|t|^{\frac{11}{2}}} |dw| \ll \frac{(\log |t|)^r}{|t|^{\frac{3}{2}}}.
\]

Using Cauchy’s residue theorem we have \(\gamma_0^{(r)}(s, \rho) = (\chi_F^{(r)}/\chi_F)(1 - s)\) and
\[
\gamma_1^{(r)}(s; \frac{1}{(\lambda_1^1 \cdots \lambda_q^1)^{\frac{2}{\sigma_F}} |t|})
\]
\[ \frac{\Gamma(s-1)\chi_F(r)}{\Gamma(s)} = \frac{g_F(s-1)}{g_F(s)} \frac{\chi_F(r)}{\chi_F(r)} (2-s) \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s-1)+\mu_j)}{\Gamma(\lambda_j s+\mu_j)} (\lambda_j it)^{\lambda_j} \] (2.14)

where we used \(|t| e^{\frac{\pi}{2} \text{sgn}(t)} = it\). It is clear that
\[ \frac{g_F(s-1)}{g_F(s)} = \frac{1 + O(|s|^{-1})}{1 + O(|s|^{-1})} = 1 + O\left(\frac{1}{|t|}\right). \] (2.15)

Stirling’s formula \(\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} (1 + O(|s|^{-1}))\) and the trivial approximation \(\lambda_j s + \mu_j = i\lambda_j t (1 + O(|t|^{-1}))\) give that
\[ \frac{\Gamma(\lambda_j(s-1)+\mu_j)}{\Gamma(\lambda_j s+\mu_j)} = \frac{(i\lambda_j t)^{\lambda_j(s-1)+\mu_j} e^{-i\lambda_j t} (1 + O(|t|^{-1}))}{(i\lambda_j t)^{\lambda_j s+\mu_j} e^{-i\lambda_j t} (1 + O(|t|^{-1}))} = (\lambda_j it)^{-\lambda_j} (1 + O(|t|^{-1})), \] (2.16)

Combining (2.14)–(2.16) and using Lemma 2.4 we obtain
\[ \gamma_1^{(r)}(s; \frac{1}{(\lambda_1^{\alpha_1} \ldots \lambda_q^{\alpha_q})^{\frac{2}{d_F}}} |t|) = \frac{\chi_F(r)}{\chi_F(r)} (1-s) - \frac{\chi_F(r)}{\chi_F(r)} (2-s) + O\left(\frac{1}{|t|} \left| \frac{\chi_F(r)}{\chi_F(r)} (2-s) \right| \right) = O(|t|^{-1}(\log |t|)^r). \]

By the same discussion, the approximate formula of \(\delta_j^{(r)}(s; \frac{1}{(\lambda_1^{\alpha_1} \ldots \lambda_q^{\alpha_q})^{\frac{2}{d_F}}} |t|)\) is also obtained.

Finally, in order to prove Theorem 1.2 from Theorem 1.1, we introduce new functions. For \(\varphi \in \mathcal{R}\), \(\alpha \in \mathbb{R}_{\geq 0}\) and \(|t| \gg 1\) we set
\[ \xi(\rho) := \begin{cases} 1, & \rho \in [0,1], \\ 0, & \rho \in [1,\infty). \end{cases} \]
\[ \varphi(\rho) := \begin{cases} 1, & \rho \in [0,1-(2|t|^\alpha)^{-1}], \\ \varphi(1+(\rho-1)|t|^\alpha), & \rho \in [1-(2|t|^\alpha)^{-1}, 1+|t|^{-\alpha}], \\ 0, & \rho \in [1+|t|^{-\alpha}, \infty), \end{cases} \]
\[ \varphi(\rho) := 1 - \varphi(1/\rho). \]

Then these function have the following properties:

**Lemma 2.8 ([4, (12)–(15)])**. For any \(\alpha \in \mathbb{R}_{\geq 0}\) and \(\varphi \in \mathcal{R}\) we satisfy the following statements (i)–(iv):

(i) \(\varphi, \varphi_0 \in \mathcal{R}\).

(ii) \((\varphi - \xi)(\rho) = 0, (\varphi_0 - \xi)(\rho) = 0\) for \(\rho \in [0,1-(2|t|^\alpha)^{-1}] \cup [1+|t|^{-\alpha}, \infty]\).

(iii) \(\varphi_j^{(j)}(\rho) = 0, \varphi_0^{(j)}(\rho) = 0\) for \(j \in \mathbb{Z}_{\geq 1}\) and \(\rho \in [0,1-(2|t|^\alpha)^{-1}] \cup [1+|t|^{-\alpha}, \infty]\).

(iv) \(\varphi^{(j)}(\rho) \ll |t|^{\alpha j}, \varphi_0^{(j)}(\rho) \ll |t|^{\alpha j}, \|\varphi^{(j)}\|_1 \ll |t|^{\alpha(j-1)}, \|\varphi_0^{(j)}\|_1 \ll |t|^{\alpha(j-1)}\) for \(\rho \in [0,\infty)\) and \(j \in \mathbb{Z}_{\geq 0}\).
3 Proof of Theorem 1.1

First for $s = \sigma + it : \sigma \in [0, 1], |t| \gg 1$, we shall use Cauchy’s integral theorem in the region

$$D_\sigma = \{ w \in \mathbb{C} \mid w = u + iv, -1/2 - \sigma \leq u \leq 3/2 - \sigma, v \in \mathbb{R} \}.$$ 

Then from (1.2) and Lemma 2.2 the following lemma is obtained:

**Proposition 3.1.** For any $m \in \mathbb{Z}_{\geq 0}$, $F \in S$, $s = \sigma + it : \sigma \in [0, 1], |t| \gg 1$, $\varphi \in \mathcal{R}$ and $x \in \mathbb{R}_{>0}$ we have

$$F^{(m)}(s) = G_m(s; x, \varphi) + \chi_F(s) \sum_{r=0}^{m} (-1)^r \binom{m}{r} H_r(1 - s; 1/x, \varphi_0)$$

where $G_r(s; x, \varphi)$ and $H_r(s; x, \varphi)$ are given by

$$G_r(s; x, \varphi) = \frac{1}{2\pi i} \int_{(\frac{3}{2}-\sigma)} g_F(s + w) K_\varphi(w) \frac{\chi_F^{(m-r)}(1 - (s + w)) F^{(r)}(s + w) \times}
\quad \times \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} (Q \frac{2}{F}xe^{-i\frac{\pi}{2}sgn(t)}) \frac{dF}{\chi_F} dw,$$

$$H_r(s; x, \varphi) = \frac{1}{2\pi i} \int_{(\frac{3}{2}-\sigma)} g_F(s + w) K_\varphi(w) \frac{\chi_F^{(m-r)}(1 - (s + w)) F^{(r)}(s + w) \times}
\quad \times \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} (Q \frac{2}{F}xe^{-i\frac{\pi}{2}sgn(t)}) \frac{dF}{\chi_F} dw.$$

respectively. Here $(3/2 - \sigma)$ denotes $\{3/2 - \sigma + iv \mid v \in \mathbb{R} \}$.

**Proof.** For $r \in \{0, 1, \ldots, m\}$ and $|v| \gg |t|$ let

$$I_r(v) = \frac{1}{2\pi i} \int_{(\frac{3}{2}-\sigma)} g_F(s + w) K_\varphi(w) \frac{\chi_F^{(m-r)}(1 - (s + w)) F^{(r)}(s + w) \times}
\quad \times \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} (Q \frac{2}{F}xe^{-i\frac{\pi}{2}sgn(t)}) \frac{dF}{\chi_F} dw.$$

First we shall show that the integrand of (3.3) is holomorphic in $D_\sigma \setminus \{0\}$. From Lemma 2.1, $K_\varphi(w)(Q^{2/d_F} xe^{-i\pi sgn(t)/2})^{d_F-w/2}$ is holomorphic in $D_\sigma$. Since $F^{(m)}(w + s)$ has pole of order $p_F + m$ at most at $w = 1 - s$, we see that

$$\quad (s + w - 1)^{p_F+m} F^{(m)}(s + w)$$

is holomorphic in $w \in D_\sigma$.

Here we shall consider the holomorphicity of gamma-factor of (3.3). In the case of $\text{Re} \, \mu_j > \lambda_j/2$, since $\text{Re}(\lambda_j(s + w) + \mu_j) \geq -\lambda_j/2 + \text{Re} \, \mu_j > 0$ for $w \in D_\sigma$ we see that $A_j(s + w) \Gamma(\lambda_j(s + w) + \mu_j) = \Gamma(\lambda_j(s + w) + \mu_j)$ is holomorphic in $D_\sigma$. On the other hand, in the case of $\text{Re} \, \mu_j \leq \lambda_j/2$, we have $\text{Re}(\lambda_j(s + w) + \mu_j + [\lambda_j/2 - \text{Re} \, \mu_j] + 1) =$
$1 - \{\lambda_j/2 - \text{Re } \mu_j\} > 0$ for $w \in D_\sigma$. Hence the functional equation for $\Gamma(s)$ implies that

$$A_j(s)\Gamma(\lambda_j(s + w) + \mu_j) = \Gamma(\lambda_j(s + w) + \mu_j + [\lambda_j/2 - \text{Re } \mu_j] + 1)$$

(3.5) is holomorphic in $D_\sigma$.

Next we consider the existence of pole for $(\chi_{F_1}^{(m-r)}/\chi_{F}(s + w))$ in $D_\sigma$. In the case of $\text{Re } \mu_j > \lambda_j/2$, since

$$\text{Re}(-s - (\mu_j + n)/\lambda_j) \leq -\sigma - (\text{Re } \mu_j)/\lambda_j < -1/2 - \sigma,$n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq \lfloor \lambda_j/2 - \text{Re } \mu_j \rfloor},$$

$$\text{Re}(1 - s + (\overline{\mu}_j + n)/\lambda_j) \geq 1 - \sigma + \text{Re } \mu_j/\lambda_j > 3/2 - \sigma$$

for $n \in \mathbb{Z}_{\geq 0}$, we see that $(\chi_{F_1}^{(m-r)}/\chi_{F}(s + w))$ does not have pole in $D_\sigma$. On the other hand, in the case of $\text{Re } \mu_j \leq \lambda_j/2$, since

$$\text{Re}(-s - (\mu_j + n)/\lambda_j) \in \begin{cases} 
[-\sigma - 1/2, -\sigma], & n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq \lfloor \lambda_j/2 - \text{Re } \mu_j \rfloor}, \\
(-\infty, -1/2 - \sigma], & n \in \mathbb{Z}_{> \lfloor \lambda_j/2 - \text{Re } \mu_j \rfloor},
\end{cases}$$

$$\text{Re}(1 - s + (\overline{\mu}_j + n)/\lambda_j) \in \begin{cases} 
[1 - \sigma, 3/2 - \sigma], & n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq \lfloor \lambda_j/2 - \text{Re } \mu_j \rfloor}, \\
(3/2 - \sigma, \infty), & n \in \mathbb{Z}_{> \lfloor \lambda_j/2 - \text{Re } \mu_j \rfloor},
\end{cases}$$

from Lemma 2.4 the points $w = -s - (\mu_j + n)/\lambda_j$, $1 - s + (\overline{\mu}_j + n)/\lambda_j$ for $n \in \mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq \lfloor \lambda_j/2 - \text{Re } \mu_j \rfloor}$ are pole of order $(m-r)$ for $(\chi_{F_1}^{(m-r)}/\chi_{F}(s + w))$ in $D_\sigma$. Therefore

$$\prod_{j=1}^{q}(A_j(s + w)A_j(1 - (s + w))^m - r \chi_{F_1}^{(m-r)}/\chi_{F})(s + w)$$

(3.6) is holomorphic in $D_\sigma$. Combining (3.4)–(3.6) we find that the integrand of (3.2) is holomorphic $w \in D_\sigma \setminus \{0\}$.

Next we shall show $I_\tau(v) \to 0$ when $v \to \infty$. Trivial estimate gives

$$\frac{g_{F}(s + w)}{g_{F}(s)} \ll |v|^{2(p_F+m)+(m+1)f_F}$$

(3.7) where $f_F = 2 \sum_{j=1}^{q} \max\{0, [\lambda_j/2 - \text{Re } \mu_j]\}$. To obtain an estimate of $F^{(r)}(s + w)$ in $w \in D_\sigma$, we shall consider estimates of $\chi_{F}(s)$ and $(\chi_{F}^{(r)})/(\chi_{F})(s)$. By the same method of (2.16) and $it = |t|e^{i\theta(t)}$, we have

$$\frac{\Gamma(\lambda_j(1-s) + \overline{\mu}_j)}{\Gamma(\lambda_j s + \mu_j)} = \frac{(i\lambda_j t)^{\lambda_j(1-s)+\mu_j-1/2}e^{i\lambda_j t}}{(i\lambda_j t)^{\lambda_j s+\mu_j-1/2}e^{-i\lambda_j t}}(1 + O(|t|^{-1}))$$

$$= (\lambda_j t)^{\lambda_j(1-2s)-2\text{Im } \mu_j}e^{(2\lambda_j + \frac{\pi}{2}(-\lambda_j - 2\text{Re } \mu_j + 1)\text{sgn}(t)))(1 + O(|t|^{-1}))$$

(3.8)

where $\theta_j(t) = 2\lambda_j + \text{sgn}(t) \cdot (-\lambda_j - 2\text{Re } \mu_j + 1)/2 - \log(\lambda_j t)^{2(\lambda_j + \text{Im } \mu_j)}$. Hence we obtain

$$\chi_{F}(s) = \omega C_F \frac{e^{i\theta_F(t)}|t|^{d_F}}{|d_F + \text{sgn}(t) \cdot (-d_F/2 - e_F + q)/2 - \log(C_F t)^{d_F + e_F}}$$

(3.9) where $\theta_F(t) = d_F + \text{sgn}(t) \cdot (-d_F/2 - e_F + q)/2 - \log(C_F t)^{d_F + e_F}$ and $C_F =$
\((Q \prod_{j=1}^{q} \lambda_j^{\lambda_j+\text{Im} \mu_j})^2\). Combining (1.2), (3.9) and Lemma 2.4 we have
\[ F^{(r)}(s) \ll |t|^{d_F(\frac{3}{2}-\sigma)} \sum_{j=0}^{r} (\log |t|)^{-j} F^{(j)}(1 - \sigma + it) \ll |t|^{d_F(\frac{3}{2}-\sigma)}(\log |t|)^r \]
for \(s \in \mathbb{C} : \sigma \in \mathbb{R}_{<0}, |t| \gg 1\). Phragmén-Lindelöf theorem implies
\[ F^{(r)}(s + w) \ll |v|^{\frac{d_F(\frac{3}{2}-(\sigma+u))}{2}}(\log |v|)^r \]
uniformly for \(u \in [-1/2 - \sigma, 3/2 - \sigma]\). By (3.7), (3.10), Lemmas 2.4, 2.6 we get
\[ I_r(v) \ll \int_{-1/2-\sigma}^{3/2-\sigma} |v|^{2(p_F+m)+(m+1)f_F} \frac{\|\varphi(t+1)\|_1}{|v|^{l+1}}(\log |v|)^r x \]
\[ \times |v|^{\frac{d_F(\frac{3}{2}-(\sigma+u))}{2}}(\log |v|)^m |v|^{d_F(\sigma+u)+\text{Im} u} du \]
\[ \ll |v|^{3d_F+\text{Re} \frac{\pi}{2} + 2(p_F+m)+(m+1)f_F-l-1}(\log |v|)^m \|\varphi(t+1)\|_1 \]
when \(|v| \gg |t|\). Choosing \(l \in \mathbb{Z} > M_F\) we find that \(I_r(v) \to 0\) when \(v \to \infty\), where \(M_F = 3d_F/4 + (e_F - q)/2 + 2(p_F + m) + (m + 1)f_F\). Cauchy’s integral theorem gives
\[ F^{(m)}(s) = \frac{1}{2\pi i} \left( \int_{(3/2-\sigma)} - \int_{(-1/2-\sigma)} \right) \frac{g_F(s + w) K_{\varphi}(w)}{g_F(s)} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} (Q \prod_{j=1}^{q} \Gamma(1 - s - w) + \mu_j^{\lambda_j}) \frac{g_F(1 - s + w)}{g_F(1 - s)} \]
\[ \times \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(1 - s - w) + \mu_j)}{\Gamma(\lambda_j(1 - s) + \mu_j)} \frac{\chi_F^{(m-r)}(s + w)}{\chi_F^{(m-r)}}(1 - s - w) \]
\[ \times F^{(r)}(1 - s - w) \frac{\chi_F^{(m-r)}(1 - (1 - s + w))}{\chi_F^{(m-r)}} \prod_{j=1}^{m} \frac{\Gamma(\lambda_j(1 - s + w) + \mu_j)}{\Gamma(\lambda_j(1 - s) + \mu_j)} \]
(3.11).

Here the first term of right-hand side of (3.11) is
\[ = G_m(s; x, \varphi) \]
(3.12).

Since (1.3) implies
\[ \frac{\chi_F(s + w)}{\chi_F(s)} = \frac{1}{Q^{2w}} \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(1 - s - w) + \mu_j)}{\Gamma(\lambda_j(1 - s) + \mu_j)} \]
combining this formula and (1.2) we obtain
\[ F^{(m)}(s + w) \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_j s + \mu_j)} \]
\[ = \frac{\chi_F(s)}{Q^{2w}} \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(1 - s - w) + \mu_j)}{\Gamma(\lambda_j(1 - s) + \mu_j)} \sum_{r=0}^{m} (-1)^r \left( \begin{array}{c} m \\ r \end{array} \right) \frac{\chi_F^{(m-r)}(s + w)}{\chi_F^{(m-r)}}(s + w) F^{(r)}(1 - s - w) \]
By replacing \(w \mapsto -w\) and using this formula, (2.1) and \(\overline{g_F}(1 - s) = g_F(s)\), the second term of right-hand side of (3.11) is
\[ \frac{\chi_F(s)}{Q^{2w}} \sum_{r=0}^{m} (-1)^r \left( \begin{array}{c} m \\ r \end{array} \right) \frac{1}{2\pi i} \int_{(3/2-1-\sigma)}^{(3/2-\sigma)} \frac{g_F(1 - s + w) K_{\varphi}(w)}{g_F(1 - s)} w \]
\[ \times \frac{\Gamma(\lambda_j(1 - s + w) + \mu_j)}{\Gamma(\lambda_j(1 - s) + \mu_j)} \prod_{j=1}^{q} \frac{\Gamma(\lambda_j(1 - s + w) + \mu_j)}{\Gamma(\lambda_j(1 - s) + \mu_j)} \]
(3.11).
\[
\times \left( Q_{F_{r+1}}^2 x^{-1} e^{-i \frac{\pi}{4} \text{sgn}(t)} \right) \frac{dF}{d\rho} dw
\]
\[
= \chi_F(s) \sum_{r=0}^{m} (-1)^r \binom{m}{r} H_r(1 - s; 1/x, \varphi_0).
\]
(3.13)

Therefore, from (3.11) and (3.13) Proposition 3.1 is obtained.

Next applying residue theorem to the functions \( G_r(s; x, \varphi) \) and \( H_r(s; x, \varphi) \), then these functions are approximated as follows:

**Proposition 3.2.** For any \( F \in S \), \( m \in \mathbb{Z}_{\geq 0} \), \( r \in \{0, \ldots, m\} \), \( s = \sigma + it : \sigma \in [0, 1], |t| \gg 1 \), \( \varphi \in \mathcal{R} \), \( l \in \mathbb{Z}_{> M_F} \) and \( x, y \in \mathbb{R}_{>0} : \sqrt{C_F}|x|^{\frac{dF}{d\rho}} = y \), we have

\[
G_r(s; x, \varphi) = \sum_{n \leq 2y} \frac{a_F(n)(-\log n)^r}{n^s} \sum_{j=0}^{l} \varphi(j) \left( \frac{n}{y} \right) \left( -\frac{n}{y} \right)^j \gamma_j^{(m-r)} \left( s; \frac{1}{(\lambda_1^{\lambda_1} \cdots \lambda_r^{\lambda_r})^{\frac{dF}{d\rho}} |t|} \right) + O(y^{1-\sigma}(\log y)^{r+\max\{|F|, 0\}} |t|^{-\frac{1}{2}} (\log |t|)^{-m-r} \varphi^{(l+1)}(t))
\]
(3.14)

\[
H_r(s; x, \varphi) = \sum_{n \leq 2y} \frac{a_F(n)(-\log n)^r}{n^s} \sum_{j=0}^{l} \varphi(j) \left( \frac{n}{y} \right) \left( -\frac{n}{y} \right)^j \delta_j^{(m-r)} \left( s; \frac{1}{(\lambda_1^{\lambda_1} \cdots \lambda_r^{\lambda_r})^{\frac{dF}{d\rho}} |t|} \right) + O(y^{1-\sigma}(\log y)^{r+\max\{|F|, 0\}} |t|^{-\frac{1}{2}} (\log |t|)^{-m-r} \varphi^{(l+1)}(t)),
\]
(3.15)

where \( M_F \) is some positive constant, and \( \gamma_{j,r}(s; \rho), \delta_{j,r}(s; \rho) \) are given by (1.5), (1.6) respectively.

**Proof.** In order to show (3.14), we use (2.2) and write \( F^{(n)}(s) = (\sum_{n \leq \rho y} + \sum_{n > \rho y}) \times a_F(n)(-\log n)^r n^{-s} \) for \( Re s > 1 \) and \( \rho \in \mathbb{R}_{>0} \). Then

\[
G_r(s; x, \varphi) = I_1 + I_2
\]
(3.16)

where \( I_1, I_2 \) are given by

\[
I_1 = \frac{1}{2\pi i} \int_{\frac{3}{2} - \sigma}^{\frac{3}{2} + \sigma} \frac{g_F(s+w)}{g_F(s)} \left( \int_0^\infty \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n \leq \rho y} \frac{a_F(n)(-\log n)^r}{n^{s+w}} d\rho \right) \times \frac{(-1)^{t+1}}{w \cdots (w+l)} \chi_F^{(m-r)} \left( s + w \right) \prod_{j=1}^q \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_1 s + \mu_1)} (Q_{F_{r+1}}^2 x e^{-i \frac{\pi}{4} \text{sgn}(t)}) \frac{dF}{d\rho} dw,
\]
(3.17)

\[
I_2 = \frac{1}{2\pi i} \int_{\frac{3}{2} - \sigma}^{\frac{3}{2} + \sigma} \frac{g_F(s+w)}{g_F(s)} \left( \int_0^\infty \varphi^{(l+1)}(\rho) \rho^{w+l} \sum_{n > \rho y} \frac{a_F(n)(-\log n)^r}{n^{s+w}} d\rho \right) \times \frac{(-1)^{t+1}}{w \cdots (w+l)} \chi_F^{(m-r)} \left( s + w \right) \prod_{j=1}^q \frac{\Gamma(\lambda_j(s + w) + \mu_j)}{\Gamma(\lambda_1 s + \mu_1)} (Q_{F_{r+1}}^2 x e^{-i \frac{\pi}{4} \text{sgn}(t)}) \frac{dF}{d\rho} dw.
\]
(3.18)
To approximate $I_1$ and $I_2$, we define $L_{\pm j}$, $C_j$ ($j = 1, 2$) as

$$L_{\pm j} = \{\sigma_j \pm iv \mid v \in [\sqrt{t}, \infty)\}, \quad C_j = \{\sigma_j + \sqrt{|t|}e^{-i\pi(\pm 1/2 + \theta)} \mid \theta \in [0, 1]\},$$

respectively, where $\sigma_1 = -1/2 - \sigma$ and $\sigma_2 = 3/2 - \sigma$. The residue theorem gives

$$I_1 = I'_1 + \text{Res} \left( I_1, F \right), \quad I_2 = I'_2$$

(3.19)

where $I'_1, I'_2$ are (3.17) replaced by $L_{-1} + C_1 + L_{+1}$, $L_{-2} + C_2 + L_{+2}$ respectively, and Res $(I_1, F)$ is the sum of residue for the integrand of (3.17) in $F$. Here by using the result

$$\int_{\mu}^{\infty} \varphi^{(l+1)}(\rho)\rho^{w+l}d\rho = \sum_{j=0}^{l} \varphi^{(l-j)}(\mu)\mu^{w+l-j}(1)(w-l+j+1)\cdots(w+l) +$$

$$+ (-1)^{l+1}w\cdots(w+l) \int_{\mu}^{\infty} \varphi(\rho)\rho^{w-1}d\rho$$

for $\mu \in \mathbb{R}_{\geq 0}$ (see p.337 of [4]) and the residue theorem, Res $(I_1, F)$ is written as

$$\text{Res} \left( I_1, F \right)$$

$$= \sum_{n \leq 2y} \frac{a_F(n)(-\log n)^r}{n^s} \times \frac{1}{2\pi i} \int_{F} \frac{g_F(s+w)}{g_F(s)} \frac{(-1)^l}{w\cdots(w+l)} \frac{\chi_F^{(m-r)}}{\chi_F} \left( s+w \right) \times$$

$$\times \left( \int_{\mu}^{\infty} \varphi^{(l+1)}(\rho)\rho^{w+l}d\rho \right) \prod_{j=1}^{q} \Gamma(\lambda_j(s+w)+\mu_j) \left( Q_{\sigma_j} \frac{x}{n^s} e^{-i\frac{\pi}{2} \text{sgn}(t)} \right) d\rho$$

$$= \sum_{n \leq 2y} \frac{a_F(n)(-\log n)^r}{n^s} \sum_{j=0}^{l} \varphi^{(j)} \left( \frac{n}{y} \right) \left( -\frac{n}{y} \right) \gamma_j^{(m-r)} \left( s; Q_{\sigma_j} \frac{x}{y^{s+j}} \right).$$

(3.20)

Next we shall estimate $I'_1$ and $I'_2$ as the error term of approximate functional equation. Partial summation and the assumption (1.1) give

$$\int_{0}^{\infty} \varphi^{(l+1)}(\rho)\rho^{w+l} \sum_{n \leq \rho y} \frac{a_F(n)(-\log n)^r}{n^s+w} d\rho$$

$$\ll \int_{1/2}^{\infty} |\varphi^{(l+1)}(\rho)|\rho^{w+l} \left( \frac{\log \rho y}{\rho y^{\sigma+u}} \right)^{r+\max \{p_F-1, 0\}} + \int_{1}^{\rho y} \frac{(\log u)^{r+\max \{p_F-1, 0\}}}{u^{\sigma+u}} du \right) d\rho$$

$$\ll y^{1-(\sigma+u)}(\log y)^{r+\max \{p_F-1, 0\}} \|\varphi^{(l+1)}\|_1$$

(3.21)

for $\text{Re}(s+w) \leq -1/2$ and

$$\int_{0}^{\infty} \varphi^{(l+1)}(\rho)\rho^{w+l} \sum_{n > \rho y} \frac{a_F(n)(-\log n)^r}{n^s+w} d\rho$$

$$\ll \int_{1/2}^{\infty} |\varphi^{(l+1)}(\rho)|\rho^{w+l} \left( \int_{\rho y}^{\infty} \frac{(\log u)^{r+\max \{p_F-1, 0\}}}{u^{\sigma+u}} du \right) d\rho$$

$$\ll y^{1-(\sigma+u)}(\log y)^{r+\max \{p_F-1, 0\}} \|\varphi^{(l+1)}\|_1.$$
for $\Re(s + w) \geq 3/2$. From Lemma 2.4 and 2.6 we get the following estimate:

$$
\frac{(-1)^{l}}{w \cdots (w + l)} \chi_F^{(m-r)}(1 - (s + w)) \prod_{j=1}^{q} \Gamma(\lambda_j(s + w) + \mu_j) \prod_{j=1}^{m} \Gamma(\lambda_j s + \mu_j) \left(\sqrt{\frac{2}{\pi}} \frac{x e^{-\frac{t}{2} \text{sgn}(t)}}{2} \right) \frac{d_F}{w}
$$

$$
\approx \begin{cases} 
\left(\frac{\log |t|}{|t|^{l+\frac{1}{2}}}ight)^{m-r} \left(\sqrt{C_F(x|t|)} \frac{d_x}{x}ight)^{u}, & w \in C_1 \cup C_2 \cup \mathcal{F}, \\
\left(\frac{\log |v|}{|v|^{l+\frac{1}{2}}}ight)^{m-r} \left(1 + |t + v|\right)^{\frac{d_F(x+v)e^{-q}}{2}} \left(\sqrt{C_F(x\frac{d_x}{x})} \frac{d_F}{x}ight)^{u}, & w \in L_{\pm 1} \cup L_{\pm 2}.
\end{cases} \tag{3.23}
$$

Trivial estimate gives

$$
g_F(s + w) g_F^*(s) \approx \begin{cases} 
1, & w \in C_1 \cup C_2 \cup \mathcal{F}, \\
\frac{1}{|t|^{2(m + p_F) + (m + 1)f_F}} & w \in L_{\pm 1} \cup L_{\pm 2}.
\end{cases} \tag{3.24}
$$

Hence by combining (3.21)–(3.24), $I_1'$ is estimated as

$$
I_1' \approx \int_{C_1} \left(\frac{\log |t|}{|t|^{l+\frac{1}{2}}}ight)^{m-r} \left(\sqrt{C_F(x|t|)} \frac{d_x}{x}ight)^{u} |y|^{1-(\sigma + u)} \left(\log y\right)^{r + \max\{p_F - 1, 0\}} \left\|\varphi^{(l+1)}\right\|_1 |dw| +
$$

$$
+ \int_{L_{\pm 1}} \left(\frac{\log |v|}{|v|^{l+\frac{1}{2}}}ight)^{m-r} \left(1 + |t + v|\right)^{\frac{d_F(x+v)e^{-q}}{2} + 2(m + p_F) + (m + 1)f_F} \times
$$

$$
\left(\sqrt{C_F(x|t|)} \frac{d_x}{x}ight)^{u} |y|^{1-(\sigma + u)} \left(\log y\right)^{r + \max\{p_F - 1, 0\}} \left\|\varphi^{(l+1)}\right\|_1 |dv|
$$

$$
\approx |y|^{-\sigma} \left(\log y\right)^{r + \max\{p_F - 1, 0\}} \left| t \right|^{-\frac{1}{2}} \left(\log |t|\right)^{m-r} \left\|\varphi^{(l+1)}\right\|_1 \tag{3.25}
$$

under the condition $\sqrt{C_F(x|t|)} \frac{d_x}{x} = y$, where the following estimate was used:

$$
\left(\int_{\pm \frac{2}{3}\sqrt{t}}^{\pm |t|} + \int_{\pm \frac{2}{3}\sqrt{t}}^{\pm 2|t|} + \int_{\pm \frac{2}{3}\sqrt{t}}^{\pm \infty} \right) \left(1 + |t + v|\right)^{M_F - d_F} \frac{d_F}{|v|^{l+\frac{1}{2}}} \left| v \right|^{M_F - d_F} \left(\log |v|\right)^{m-r} dv =: J_1 + J_2 + J_3.
$$

Since $1 \ll 1 + |t + v| \ll |t|$ when $v \in [-2|t|, -|t|/2] \cup [|t|/2, 2|t|]$ and

$$
1 + |t + v| \asymp \begin{cases} 
|t| & \text{when } v \in [-|t|/2, -\sqrt{|t|}] \cup [\sqrt{|t|}, |t|/2], \\
|v| & \text{when } v \in (-\infty, -2|t|] \cup [2|t|, \infty),
\end{cases}
$$

$J_j \ (j = 1, 2, 3)$ were estimated as

$$
J_1 \ll \left(\log |t|\right)^{m-r} \int_{\pm \frac{2}{3}\sqrt{t}}^{\pm |t|} \frac{dv}{|v|^{l+\frac{1}{2}}} \ll |t|^{-\frac{1}{2}} \left(\log |t|\right)^{m-r},
$$

$$
J_2 \ll |t|^{\max\{0, d_F - M_F\} - (l+1)} \left(\log |t|\right)^{m-r} \int_{\pm \frac{2}{3}\sqrt{t}}^{\pm 2|t|} \frac{dv}{(1 + |t + v|)^{\max\{0, d_F - M_F\}}} \ll |t|^{-l+\max\{0, d_F - M_F\}} \left(\log |t|\right)^{m-r} \ll |t|^{-\frac{1}{2}} \left(\log |t|\right)^{m-r},
$$

$$
J_3 \ll \frac{1}{|t|^{M_F - d_F}} \int_{\pm 2|t|}^{\pm \infty} \frac{d_F}{|v|^{l+1 - M_F + d_F}} \left(\log |v|\right)^{m-r} dv \ll |t|^{-l} \left(\log |t|\right)^{m-r},
$$

where $l$ was chosen as $l \in \mathbb{Z}_{\geq 2\max\{0, d_F - M_F\}}$. By the same discussion of estimate of $I_1'$,
the estimate of $I'_2$ is obtained:
\[ I'_2 \ll y^{-\sigma} (\log y)^{r+A} |t|^{-\frac{1}{2}} (\log |t|)^{m-r} \| \varphi^{(l+1)} \|_1. \] (3.26)

Therefore combining (3.16)–(3.20), (3.25)–(3.26) we obtain (3.14). Since we can obtain (3.15) from the same discussion in the above, Proposition 3.2 is showed. \[ \square \]

Finally, for any $x \in \mathbb{R}_{>0}$ we choose parameters $y_1, y_2 \in \mathbb{R}_{>0}$ such that $\sqrt{C_F(x|t|)} \frac{d}{dt} = y_1$ and $\sqrt{C_F(x^{-1}|t|)} \frac{d}{dt} = y_2$, that is, $y_1 y_2 = C_F |t|^{d_F}$. By combining Propositions 3.1, 3.2 and using (3.9), $F^{(m)}(s)$ is approximated as

\[
F^{(m)}(s) \approx \sum_{n \leq y_1} a_F(n) (-\log n)^m n^s \sum_{j=0}^{l} \varphi^{(j)} \left( \frac{n}{y_1} \right) \left( -\frac{n}{y_1} \right)^j \gamma_j \left( s; \frac{1}{(\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q})^2} \right) + \\
+ \chi(s) \sum_{r=0}^{m} (-1)^r \binom{m}{r} \sum_{n \leq y_2} \frac{a_F(n) (-\log n)^r n^{1-s}}{n^r} \sum_{j=0}^{r} \varphi^{(j)} \left( \frac{n}{y_2} \right) \left( -\frac{n}{y_2} \right)^j \chi_j \left( s; \frac{1}{(\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q})^2} \right) + \\
\times \delta_j^{(m-r)} \left( 1 - s; \frac{1}{(\lambda_1^{\lambda_1} \cdots \lambda_q^{\lambda_q})^2} \right) + O(y_1^{1-\sigma} (\log y_1)^{m+A} |t|^{-\frac{1}{2}} \| \varphi^{(l+1)} \|_1) + \\
+ O(\frac{y_2^2}{2} |t|^{d_F(\frac{1}{2}-\sigma)-\frac{1}{2}} \| \varphi^{(l+1)} \|_1 \sum_{r=0}^{m} (\log y_2)^{r+A} (\log |t|)^{m-r})
\]

Using Lemma 2.7 and dividing sums of $j \in \mathbb{Z}_{\geq 0}$ to term of $j = 0$ and sum of $j \in \mathbb{Z}_{\geq 1}$, we complete proof of Theorem 1.1.

## 4 Proof of Theorem 1.2

In order to prove Theorem 1.2 from the approximate functional equation containing characteristic functions, we use Lemma 2.8. For any functions $X : [0, \infty) \to \mathbb{R}$, we define $M_X(s)$ to

\[
M_X(s) := \sum_{n=1}^{\infty} \frac{a_F(n) (-\log n)^m}{n^s} X \left( \frac{n}{y_1} \right) + \\
+ \sum_{r=0}^{m} (-1)^r \binom{m}{r} \chi(r)^{m-r} (s) \sum_{n=1}^{\infty} \frac{a_F(n) (-\log n)^r}{n^{1-s}} X_0 \left( \frac{n}{y_2} \right).
\]

Replacing $\varphi \mapsto \varphi^\alpha$ ($\alpha \in \mathbb{R}_{>0}$) in Theorem 1.1 we can write

\[
F^{(m)}(s) = M_{\varphi}^\alpha(s) + R_{\varphi}^\alpha(s) = M_{\xi}(s) + O(M_{\varphi, -\xi}(s) + R_{\varphi^\alpha}(s))
\] (4.1)

Here $M_{\xi}(s)$ and $M_{\varphi, -\xi}(s) + R_{\varphi^\alpha}(s)$ are written as

\[
M_{\xi}(s) = \sum_{n \leq y} \frac{a_F(n) (-\log n)^m}{n^s} + \sum_{r=0}^{m} (-1)^r \binom{m}{r} \chi(r)^{m-r} (s) \sum_{n \leq y} \frac{a_F(n) (-\log n)^m}{n^{1-s}}
\] (4.2)
Since $\rho$ and $\alpha$ for any $2.7$ and $2.8$, we have

\[(3.9), (4.3)–(4.5)\text{ and } (1 + |E_T| m), \]

respectively, where $E(s)$, $S_0(\rho)$ and $T_{m-r}(\rho)$ are given by

\[E(s) = O\left(y_1^{1-\sigma}(\log y_1)^{m+\max\{p_F-1,0\}|t|^{-\frac{1}{2}}||\varphi^{(l+1)}|\right) +
\]
\[O\left(y_2^2|t|^d_{F+1-\sigma}+\sum_{r=0}^m|E_T\rho|^{r+\max\{p_F-1,0\}}(\log |t|)^{m-r}\right).
\]

\[S_0(\rho) = (\varphi - \xi)(\rho) + \sum_{j=1}^{l} \varphi^{(j)}(\rho)(-\rho)^{j}\gamma_j^{(0)}\left(s; \frac{1}{(\lambda_1^{2j} \cdots \lambda_{\rho r})^{2F}|t|}\right),
\]

\[T_{m-r}(\rho) = (\varphi_0 - \xi)(\rho)\lambda_F^{(m-r)}(s) +
\]
\[\sum_{j=1}^{l} \varphi^{(j)}(\rho)(-\rho)^{j}\delta^{(m-r)}\left(1 - s; \frac{1}{(\lambda_1^{2j} \cdots \lambda_{\rho r})^{2F}|t|}\right),\]

Since $S_0(\rho) = 0$ and $T_{m-r}(\rho) = 0$ for $\rho \in [0, (1 + |t|^{-\alpha})^{-1}] \cup [1 + |t|^{-\alpha}, \infty)$ by Lemmas 2.7 and 2.8, we have

\[E(s) \ll y_1^{1-\sigma}(\log y_1)^{m+\max\{p_F-1,0\}|t|^{-\frac{1}{2}}|\varphi^{(l+1)}| +
\]
\[y_2^2|t|^d_{F+1-\sigma}+\sum_{r=0}^m|E_T\rho|^{r+\max\{p_F-1,0\}}(\log |t|)^{m-r} \quad (4.3)
\]

for any $\alpha \in \mathbb{R}_{\geq 0}$, and

\[S_0(\rho) \ll 1 + \sum_{j=1}^{l} |t|^\alpha|t|^{-\frac{1}{2}} \ll 1, \quad (4.4)
\]

\[T_{m-r}(\rho) \ll (\log |t|)^{m-r} + \sum_{j=1}^{l} |t|^\alpha|t|^{-\frac{1}{2}}(\log |t|)^{m-r} \ll (\log |t|)^{m-r} \quad (4.5)
\]

for $\rho \in [(1 + |t|^{-\alpha})^{-1}, 1 + |t|^{-\alpha}]$ under the condition $\alpha \in [0, 1/2]$. Now we choose $\alpha = 1/2 - \varepsilon$ and $l \in \mathbb{Z}_{\geq 1/(2\varepsilon)}$. By the condition (d) of Selberg class and the estimates (3.9), (4.3)–(4.5) and $(1 + |t|^{-\alpha})y_1 - (1 + |t|^{-\alpha})^{-1}y_1 \leq 2|t|^{-\alpha}y_1$, $M_{\varphi_0 - \xi}(s) + R_{\varphi_0}(s)$ is estimated as

\[M_{\varphi_0 - \xi}(s) + R_{\varphi_0}(s)
\]
\[\ll y_1^{1-\sigma}(\log y_1)^{m+\max\{p_F-1,0\}|t|^{-\frac{1}{2}}|\varphi^{(l+1)}| +
\]
\[y_2^2|t|^d_{F+1-\sigma}+\sum_{r=0}^m|E_T\rho|^{r+\max\{p_F-1,0\}}(\log |t|)^{m-r} +
\]
\[\sum_{(1 + |t|^{-\alpha})^{-1}y_1 \leq n \leq (1 + |t|^{-\alpha})y_1} \frac{|a_F(n)|(|\log n|)^{m}}{n^s} +
\]

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\[ + |t|^{d_F \left( \frac{1}{2} - \sigma \right)} \sum_{r=0}^{m} (\log |t|)^{m-r} \sum_{(1+|t|^{-\alpha})^{-1} y_2 \leq n \leq (1+|t|^{-\alpha}) y_2} \frac{|a_F(n)| (\log n)^r}{n^{1-\sigma}} + \]

\[ \ll y_1^{1-\sigma+c} |t|^{-\frac{1}{2}} + y_2^{\sigma+c} |t|^{d_F \left( \frac{1}{2} - \sigma \right) - \frac{1}{2}}. \]  

(4.6)

Hence combining (4.1), (4.2) and (4.6) we complete the proof of Theorem 1.2.

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