A regulator for smooth manifolds and an index theorem

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Abstract

For a smooth manifold $X$ and an integer $d \geq \dim(X)$ we construct and investigate a natural map

$$\sigma_d : K_d(C^\infty(X)) \to \text{ku} \mathbb{C}/\mathbb{Z}^{-d-1}(X).$$

Here $K_d(C^\infty(X))$ is the algebraic $K$-theory group of the algebra of complex valued smooth functions on $X$, and $\text{ku} \mathbb{C}/\mathbb{Z}$ is the generalized cohomology theory called connective complex $K$-theory with coefficients in $\mathbb{C}/\mathbb{Z}$.

If the manifold $X$ is closed of odd dimension $d-1$ and equipped with a Dirac operator $\mathcal{D}$, then we state and partially prove the conjecture stating that the following two maps $K_d(C^\infty(X)) \to \mathbb{C}/\mathbb{Z}$ coincide:

1. Pair the result of $\sigma_d$ with the $K$-homology class of $\mathcal{D}$.
2. Compose the Connes-Karoubi multiplicative character with the classifying map of the $d$-summable Fredholm module of $\mathcal{D}$.

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1 Introduction

The torsion subgroup of the algebraic $K$-theory of the field $\mathbb{C}$ of complex numbers was calculated by Suslin \cite{Sus84}. An important tool for this calculation was a collection of homomorphisms

$$r_{2n+1}: K_{2n+1}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}$$

for $n \in \mathbb{N}$ which turned out to induce isomorphisms of torsion subgroups $K_{2n+1}(\mathbb{C})_{\text{tor}} \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$.

One may interpret $\mathbb{C}$ as the algebra of complex-valued smooth functions $C^\infty(X)$ on the one-point manifold $X = \ast$. The first goal of the present paper is to generalize the construction of the homomorphism (1) to higher-dimensional smooth manifolds $X$. In order to state the result we need the following notation:

1. We write $\text{ku} := K^{\text{top}}(\mathbb{C})$ for the connective topological complex $K$-theory spectrum. Its homotopy groups are given by

$$\pi_n(\text{ku}) \cong \begin{cases} 0 & n < 0 \\ \mathbb{Z} & n \geq 0 \text{ even} \\ 0 & n \geq 0 \text{ odd} \end{cases}$$

(2)

2. In general, we write $E^\ast(X)$ for the cohomology groups of the manifold $X$ with coefficients in the spectrum $E$.

3. If $A$ is an abelian group, then we let $MA$ denote the Moore spectrum of $A$ and write

$$EA := E \wedge MA.$$ (3)

For example, we can form the spectrum $\text{ku} \mathbb{C}/\mathbb{Z}$. In view of (2) and \cite{Bou79} Eq. (2.1)) its homotopy groups are given by

$$\pi_n(\text{ku} \mathbb{C}/\mathbb{Z}) \cong \begin{cases} 0 & n < 0 \\ \mathbb{C}/\mathbb{Z} & n \geq 0 \text{ even} \\ 0 & n \geq 0 \text{ odd} \end{cases}$$ (4)
Theorem 1.1. Let $X$ be a smooth manifold and $d \in \mathbb{N}$. If $d > \dim(X)$, then we have a construction of a homomorphism

$$\sigma_d : K_d(C^\infty(X)) \to \text{ku}C/\mathbb{Z}^{-d-1}(X)$$

which is natural in $X$ and induces the map (1) in homotopy groups for $X = *$

Remark 1.2. The main point of the theorem is the assertion that there is some interesting generalization of the homomorphism (1) to higher-dimensional manifolds. In this paper we just give a construction of such a natural homomorphism. We do not address the problem of characterizing it by a collection of natural properties. Example 1.4 below shows that the map $\sigma_d$ contains certain “higher information”. As opposed to the case $X = *$, in the higher-dimensional case we do not understand its kernel or cokernel.

Remark 1.3. The classical construction of the homomorphism (1) relies on the observation by Quillen that the natural map $K_*(\mathbb{C}) \to K_{\text{top}}(\mathbb{C})$ from the algebraic to the topological $K$-theory of $\mathbb{C}$ vanishes rationally in positive degrees. In order to employ this fact for the construction of $r_{2n+1}$ we work in the stable $\infty$-category $\text{Sp}$ of spectra. We consider the diagram

$$
\begin{array}{ccc}
K(\mathbb{C})[1..\infty] & \to & \Sigma^{-1}K(\mathbb{C})_{\text{top}}C/\mathbb{Z} \\
\downarrow & & \downarrow \\
K(\mathbb{C}) & \to & K_{\text{top}}(\mathbb{C}) \\
\downarrow & & \downarrow \\
K_{\text{top}}(\mathbb{C})C & \to & K_{\text{top}}(\mathbb{C})C
\end{array}
$$

Here $K(\mathbb{C})[1..\infty] \to K(\mathbb{C})$ denotes the connective covering of the algebraic $K$-theory spectrum $K(\mathbb{C})$. The right column is a segment of a Bockstein fibre sequence for the topological $K$-theory spectrum $K_{\text{top}}(\mathbb{C}) = \text{ku}$ of $\mathbb{C}$. Now, by Quillen’s observation, the dashed arrow induces the trivial map in homotopy groups. A choice of a zero homotopy of this arrow thus induces the dotted arrow which in turn induces the map $r_{2n+1}$ after applying $\pi_{2n+1}(-)$ and identifying $\pi_{2n+1}(\Sigma^{-1}K(\mathbb{C})_{\text{top}}C/\mathbb{Z})$ with $\mathbb{C}/\mathbb{Z}$ using (4). Note that the restriction of $r_{2n+1}$ to the torsion subgroup does not depend on the choices of the zero homotopy. A reference for this construction of (1) formulated in a slightly different language is [Wei84]. In the present paper we will generalize the alternative construction [BNV13, Ex. 6.9].

Example 1.4. For a unital algebra $A$ let

$$\iota : A^\times \to K_1(A)$$

be the natural homomorphism from the group $A^\times$ of units of $A$ to the first algebraic $K$-theory group.
A complex-valued smooth function $f \in C^\infty(S^1)$ gives rise to an invertible function $\exp(f) \in C^\infty(S^1)$. If $u \in C^\infty(S^1)$ is a second invertible function, then we have two algebraic $K$-theory classes $\iota(\exp(f)), \iota(u) \in K_1(C^\infty(S^1))$. Using the multiplicative structure of the algebraic $K$-theory for commutative algebras we define the class

$$\iota(\exp(f)) \cup \iota(u) \in K_2(C^\infty(S^1)).$$

We use the identification

$$\text{ku} \mathbb{C}/\mathbb{Z}^{-3}(S^1) \cong \text{ku} \mathbb{C}/\mathbb{Z}^{-4}(\ast) \cong \mathbb{C}/\mathbb{Z}$$

given by suspension and (4). With this identification we have

$$\sigma_2(\iota(\exp(f)) \cup \iota(u)) = \left[ \frac{1}{(2\pi i)^2} \int_{S^1} f \frac{du}{u} \right]_{\mathbb{C}/\mathbb{Z}} .$$

The formula (7) is a special case of (73). \hfill \square

In [BNV13, Example 6.9] we explained how one can construct the map (1) using techniques of differential cohomology. The differential cohomology approach in particular provides a canonical choice for the dotted arrow in (5). The main idea for the construction of $\sigma_d$ is to apply the framework of differential cohomology to the algebraic $K$-theory of the Fréchet algebra $C^\infty(X)$. The details will be worked out in Section 2. The final construction of $\sigma_d$ will be given in Definition 2.33.

We now come to the second theme of this paper. Let us assume that $X$ is a closed manifold of odd dimension $d$ which carries a generalized Dirac operator $\mathcal{D}$. This Dirac operator gives rise to a $K$-homology class $[\mathcal{D}] \in KU_d(X)$ and a $d+1$-summable Fredholm module which we will describe in Subsection 3.2 in greater detail. This Fredholm module is classified by a homomorphism

$$b_\mathcal{D} : C^\infty(X) \to \mathcal{M}_d ,$$

which is unique up to unitary equivalence, and where $\mathcal{M}_d$ denotes the classifying algebra for $d+1$-summable Fredholm modules introduced in [CK88], see Remark 3.10 for an explicit description. In [CK88] Connes and Karoubi further introduced the multiplicative character

$$\delta : K_{d+1}(\mathcal{M}_d) \to \mathbb{C}/\mathbb{Z} .$$

Since $\text{ku}$ is a $KU$-module spectrum we can define a map

$$r_\mathcal{D} : \text{ku} \mathbb{C}/\mathbb{Z}^{-d-2}(X) \xrightarrow{(-, [\mathcal{D}])} \text{ku} \mathbb{C}/\mathbb{Z}^{-2d-2}(\ast) \cong \mathbb{C}/\mathbb{Z}$$

given by the pairing with the $K$-homology class $[\mathcal{D}]$. An explicit construction of this map using elements of local index theory will be given in (97). We now make the following conjecture:
**Conjecture 1.5.** Assume that $X$ is a closed odd-dimensional manifold of dimension $d$ with a Dirac operator $\mathcal{D}$. Then the following diagram commutes:

This conjecture is supported by our second main result which asserts that it holds true if one replaces $K_d(C^\infty(X))$ by its subgroup of classes which are topologically trivial. Note that we do not know any example of a topologically non-trivial class in $K_n(C^\infty(X))$ for $n > \dim(X)$, see Remark 1.10. In order to explain what topologically trivial means we consider the fibre sequence in spectra (see (33) for details)

$$K^{rel}(C^\infty(X)) \xrightarrow{\partial} K(C^\infty(X)) \rightarrow K^{top}(C^\infty(X)) \rightarrow \Sigma K^{rel}(C^\infty(X))$$

relating the algebraic $K$-theory spectrum of $C^\infty(X)$ with its topological and relative $K$-theory spectra. A class in $K_{d+1}(C^\infty(X))$ is called topologically trivial if its image in $K^{top}_{d+1}(C^\infty(X))$ vanishes, or equivalently, if it belongs to the image of $\partial : K^{rel}_{d+1}(C^\infty(X)) \rightarrow K_{d+1}(C^\infty(X))$. We have the following theorem:

**Theorem 1.6.** Assume that $X$ is a closed odd-dimensional manifold of dimension $d$ with a Dirac operator $\mathcal{D}$. Then the following diagram commutes:

In the remainder of this introduction we describe how our constructions are related with other results in the literature relating index and spectral theory of operators with algebraic $K$-theory of smooth functions.

Since $C^\infty(X)$ is a commutative algebra, the algebraic $K$-theory $K_*(C^\infty(X))$ is a graded commutative ring. As in Example 1.4 we can use the map

$$\iota : C^\infty(X)^\times \rightarrow K_1(C^\infty(X))$$

and the $\cup$-product in algebraic $K$-theory in order to construct higher algebraic $K$-theory classes.
**Example 1.7.** Assume that $X$ is a closed odd-dimensional manifold with a Dirac operator $D$. If $f \in C^\infty(X)$, then we can form the unit $e^f \in C^\infty(X)^\times$, and we can consider the class $\iota(e^f) \in K_1(C^\infty(X))$. Note that this element is topologically trivial. Given a collection $f_1, \ldots, f_d$ of such smooth functions we can form the topologically trivial algebraic $K$-theory class 
\[ \{e^{f_1}, \ldots, e^{f_d}\} \in \iota(e^{f_1}) \cup \cdots \cup \iota(e^{f_d}) \in K_d(C^\infty(X)) \, . \]

The main result of [Kaa11b] is an explicit formula [Kaa11b, (1.2)] for the number 
\[ (\delta \circ b_D)(\{e^{f_1}, \ldots, e^{f_d}\}) \in \mathbb{C}/\mathbb{Z} \, . \]

It involves the traces of algebraic expressions build from the $f_i$ and the positive spectral projection of $D$. Kaad’s formula can be considered as the analytical side of an index formula. One can interpret our Conjecture 1.5 as providing the topological counterpart. Indeed, since $\{e^{f_1}, \ldots, e^{f_d}\}$ is topologically trivial, by Theorem 1.6 we have the equality 
\[ (\delta \circ b_D)(\{e^{f_1}, \ldots, e^{f_d}\}) = (\rho_{D} \circ \sigma_d)(\{e^{f_1}, \ldots, e^{f_d}\}) \]

where $d - 1 = \dim(X)$.

**Example 1.8.** We consider the case $X = S^1$ with the Dirac operator $D := i\partial_t$ acting as an unbounded essentially selfadjoint operator with domain $C^\infty(S^1)$ on the Hilbert space $L^2(S^1)$. Let $u_1, u_2 \in C^\infty(S^1)^\times$ be two invertible complex-valued functions. Then we form the algebraic $K$-theory class 
\[ \{u_1, u_2\} \in K_2(C^\infty(S^1)) \, . \]

Let $P \in B(L^2(S^1))$ be the projection onto the subspace of positive Fourier modes, i.e. the positive spectral projection of $D$. For $f \in C^\infty(S^1)$ we consider the Toeplitz operator 
\[ T_f := PfP \in B(L^2(S^1)) \, , \]

where $f$ acts as multiplication operator. For two functions $f_1, f_2 \in C^\infty(S^1)$ the commutator $[T_{f_1}, T_{f_2}]$ is a trace class operator.

We let $\mathcal{A} \subset B(L^2(S^1))$ be the algebra generated by all Toeplitz operators $T_f$ for $f \in C^\infty(S^1)$ and the algebra of trace class operators $\mathcal{L}^1 := \mathcal{L}^1(L^2(S^1))$. We then get the Toeplitz extension 
\[ 0 \to \mathcal{L}^1 \to \mathcal{A} \to C^\infty(S^1) \to 0 \, . \quad (8) \]

Associated to an extension of the trace class operators one has the determinant invariant (see e.g. [Bro73]) 
\[ d := \det \circ \partial : K_2(C^\infty(S^1)) \to \mathbb{C}^* \, , \]

where $\partial : K_2(C^\infty(S^1)) \to K_1(\mathcal{A}, \mathcal{L}^1)$ is the boundary operator in algebraic $K$-theory associated to the sequence (8) and $\det : K_1(\mathcal{A}, \mathcal{L}^1) \to \mathbb{C}^*$ is induced by the Fredholm determinant. The diagram [Kaa11a, (3)] states that 
\[ d(\{u_1, u_2\}) = \exp(2\pi i \delta(b_D(\{u_1, u_2\}))) \, . \quad (9) \]
The determinant invariant was identified by Carey-Pincus with the joint torsion \( \tau(A, B) \in \mathbb{C}^* \) which is defined for the pair \( A, B \) of Fredholm operators which commute up to trace class operators. In the special case of the pair \( T_{u_1}, T_{u_2} \) on \( \text{im}(P) \) we thus have

\[
d(\{u_1, u_2\}) = \tau(T_{u_1}, T_{u_2}) .
\]

We refer to [Mig14] and [Kaa12] for a gentle introduction to joint torsion. The joint torsion \( \tau(T_{u_1}, T_{u_2}) \) in turn has been calculated explicitly. In the special case where \( u_1 = e^{f_1} \) we have

\[
\tau(T_{u_1}, T_{u_2}) = \exp\left(\frac{1}{2\pi i} \int f_1 \, d\log u_2 \right) .
\]

In order to state the result of the calculation in the general case in a comprehensive way we will use the cup product in Deligne cohomology. Using the isomorphism (11) (to be explained below) the invertible functions \( u_i \) can be interpreted as classes in Deligne cohomology \( H^1_{Del}(S^1, \mathbb{Z}) \). Their cup product is the class

\[
u_1 \cup u_2 \in H^2_{Del}(S^1, \mathbb{Z}) .
\]

We have an isomorphism

\[
\langle -, [S^1] \rangle : H^3_{Del}(S^1, \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}
\]

given by evaluation. In [CP99] (1.2),(1.3) Carey-Pincus calculate the determinant invariant and joint torsion:

\[
d(\{u_1, u_2\}) = \tau(T_{u_1}, T_{u_2}) = \exp(2\pi i \langle u_1 \cup u_2, [S^1] \rangle) .
\]

Combining this equality with (9) we get the equality

\[
\delta(b_{\hat{\text{reg}}}(\{u_1, u_2\})) = \langle u_1 \cup u_2, [S^1] \rangle .
\]

(10)

Note that we do not know whether the class \( \{u_1, u_2\} \in K_2(C^\infty(S^1)) \) is topologically non-trivial.

Using the multiplicative features of the differential regulator map \( \hat{\text{reg}}_X \) (see Remark 2.27) one can also calculate \( r_{\hat{\text{reg}}}(\sigma_2(\{u_1, u_2\})) \) explicitly. The result is again the right-hand side of (10) as expected by Conjecture 1.5. We will not give the details of the multiplicative theory since it requires a set-up which is similar to [BT13] but differs from the one used in the present paper.

**Remark 1.9.** In this remark we recall the basic features of Deligne cohomology used above. For \( p \in \mathbb{N} \) the Deligne cohomology group \( H^p_{Del}(X, \mathbb{Z}) \) is defined as the \( p \)'th hypercohomology of the complex of sheaves

\[
0 \to \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{p-1} \to 0 .
\]
We refer to [CS1v] for a first definition and to [Bry93], [Bun, Sec. 3], or [BNV13, 4.3]) for introductions to Deligne cohomology. In the original paper [CS1v] Deligne cohomology classes are called differential characters and a different grading convention was used. We have a cup product

\[ \cup : H^p_{Del}(X, \mathbb{Z}) \otimes H^q_{Del}(X, \mathbb{Z}) \to H^{p+q}_{Del}(X, \mathbb{Z}) \]

which turns Deligne cohomology into a graded commutative ring. Moreover, we have a natural isomorphism of groups

\[ H^1_{Del}(X, \mathbb{Z}) \cong C^\infty(X)^\times. \]  

(11)

Note that get invertible complex-valued functions since \( \Omega^* \) is the de Rham complex of complex-valued forms. Finally, for a closed connected and oriented manifold \( M \) of dimension \( n - 1 \) we have an evaluation isomorphism

\[ \langle -, [M] \rangle : H^n_{Del}(M; \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}. \]

Remark 1.10. We refer to [Kar87, Appendix 4] for some information about the algebraic \( K \)-theory of the algebra of smooth functions on a manifold.

Let \( X \) be a compact manifold and \( n \in \mathbb{N} \) be odd. By [Kar87, Thm A.4.6] the rank of

\[ \text{im} (K_n(C^\infty(X)) \to K^\text{top}_n(C^\infty(X))) \]

is at least \( \dim H^n(X; \mathbb{R}) \). By [Kar87, Thm A.4.6] this is true also if \( X \) is oriented and \( n = \dim(X) \) (not necessarily odd).

We have a decomposition

\[ K_*(C^\infty(X)) \cong K_*(\mathbb{C}) \oplus \tilde{K}_*(C^\infty(X)), \]

where the first summand is induced by the inclusion \( \mathbb{C} \to C^\infty(X) \) as constant functions, and the second summand is the kernel of the restriction to some point in \( X \). There are non-trivial classes in \( K_d(C^\infty(X)) \) for arbitrary large odd \( d \in \mathbb{N} \). Note that classes coming from the summand \( K_d(\mathbb{C}) \) are topologically trivial.

Theorem [Kar87, Thm A.4.3] shows that the group \( \tilde{K}_n(C^\infty(X)) \) itself is huge for \( 1 \leq n \leq \dim(X) \).

We do not know whether there exists topologically non-trivial classes in degrees strictly larger than \( \dim(X) \).

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2 The construction of $\sigma_d$

2.1 Elements of sheaf theory on manifolds

Our main idea is to analyse the algebraic $K$-theory spectrum $K(C\infty(X))$ of the algebra of complex-valued smooth functions on a manifold $X$ using the techniques of differential cohomology theory as developed in [BNV13]. In the following we recall some of the basic notions.

Let $\text{Mf}$ denote the site of smooth manifolds with corners with the open covering topology.

**Remark 2.1.** An $n$-dimensional manifold with corners is modeled on the quadrant $[0, \infty)^n \subset \mathbb{R}^n$. The category of manifolds with corners contains the unit interval $[0, 1]$, manifolds with boundary, simplices $\Delta^n$. Furthermore, the category of manifolds with corners is closed under taking products.

For a presentable $\infty$-category $\mathcal{C}$ (see [Lur09, Ch. 5]) we will consider the $\infty$-category of presheaves $\text{PSh}_\mathcal{C}(\text{Mf})$ and its full subcategory of sheaves $\text{Sh}_\mathcal{C}(\text{Mf})$ with values in $\mathcal{C}$ on the site $\text{Mf}$.

**Definition 2.2.** A presheaf $G \in \text{PSh}_\mathcal{C}(\text{Mf})$ is a sheaf if for every manifold $M$ and every open covering $U \to M$ the natural map

$$G(M) \to \lim_{\Delta} G(U^\bullet)$$

is an equivalence.

In this definition the simplicial manifold $U^\bullet \in \text{Mf}^{\Delta^{op}}$ is the Čech nerve of the open covering and the map (12) is induced from the natural map $U^\bullet \to M$, were $M$ is considered as a constant simplicial manifold. By a application of the general theory [Lur09, 6.2.2.7] we get that $\text{PSh}_\mathcal{C}(\text{Mf})$ and $\text{Sh}_\mathcal{C}(\text{Mf})$ are again presentable $\infty$-categories and that there is an adjunction

$$L : \text{PSh}_\mathcal{C}(\text{Mf}) \rightleftarrows \text{Sh}_\mathcal{C}(\text{Mf}) : \text{incl}$$

between the inclusion of sheaves into presheaves and the sheafification functor $L$.

We use the unit interval $I := [0, 1]$ in order to define the notion of homotopy invariance.

**Definition 2.3.** A sheaf or presheaf $G$ on $\text{Mf}$ is called homotopy invariant, if the map

$$G(M) \to G(I \times M)$$

induced by the projection $I \times M \to M$ is an equivalence for every smooth manifold $M$.  

As in [BNV13, Sec. 2] one argues that the full subcategories of homotopy invariant sheaves $\mathbf{Sh}_C^h(Mf)$ or homotopy invariant presheaves $\mathbf{PSh}_C^h(Mf)$ are presentable. Their inclusions into all sheaves or presheaves fit into adjunctions

$$\mathcal{H}^{pre} : \mathbf{PSh}_C(Mf) \dashv \mathbf{PSh}_C^h(Mf) : \text{incl}, \quad \mathcal{H} : \mathbf{Sh}_C(Mf) \dashv \mathbf{Sh}_C^h(Mf) : \text{incl}.$$ (13)

The left adjoints are called homotopification functors. The homotopification functors for sheaves and presheaves are related by the equivalence

$$\mathcal{H} \simeq L \circ \mathcal{H}^{pre} \circ \text{incl},$$ (14)

see [BNV13, Prop. 2.6].

Let $X$ be a smooth manifold. By $i_X : Mf \to Mf$ we denote the map of sites given by $M \mapsto X \times M$. It induces a pull-back

$$i_X^* : \mathbf{PSh}_C(Mf) \to \mathbf{PSh}_C^h(Mf).$$ (15)

The functor $i_X^*$ has the following properties:

**Lemma 2.4.** 1. The functor $i_X^*$ preserves sheaves.

2. The map $i_X^*$ preserves homotopy invariant presheaves and sheaves.

3. For presheaves $i_X^*$ commutes with homotopification, i.e. the natural transformation $\mathcal{H}^{pre} \circ i_X^* \to i_X^* \circ \mathcal{H}^{pre}$ is an equivalence.

4. If $X$ is compact, then the analogous statement holds true for sheaves, i.e. the natural map $\mathcal{H} \circ i_X^* \to i_X^* \circ \mathcal{H}$ is an equivalence.

**Proof.** If $U \to M$ is an open covering covering of $M$, then $i_X(U) \to i_X(M)$ is an open covering of $X \times M$. If $G$ is a presheaf, then the descent map of $i_X^* G$ with respect to $U \to M$ is the same as the descent map of $G$ with respect to $i_X(U) \to i_X(M)$. This implies the first assertion.

A sheaf or presheaf $G$ is homotopy invariant by definition if the natural transformation $G \to i_X^* G$ (induced by the map $I \to *$) is an equivalence. We have equivalences of functors $i_X^* i_X^* \simeq i_X^* i_X \simeq i_X^* i_X^*$. This implies that $i_X^*$ preserves homotopy invariant sheaves or presheaves.

In order to see that $i_X^*$ commutes with homotopification of presheaves we use the explicit formula for the homotopification given in [BNV13, Section 7]. We define the functor

$$s : \mathbf{PSh}_C(Mf) \to \mathbf{PSh}_C(Mf), \quad s(G) := \lim_{\Delta^\text{op}} i_X^* G,$$ (16)

where $\Delta^*$ is the cosimplicial manifold of standard simplices. Then the homotopification on presheaves $\mathcal{H}^{pre}$ is given by

$$\mathcal{H}^{pre} \simeq s.$$ (17)
Since the colimit for presheaves is taken object wise and \( i_X^* \Delta \cdot i_X^* \simeq i_X^* i_X^* \), we see that the homotopification for presheaves commutes with \( i_X^* \).

We now assume that \( X \) is compact and \( G \) is a sheaf. For \( n \in \mathbb{N} \) we let \( S^n \in \text{Mf} \) denote the \( n \)-dimensional sphere. For every \( n \in \mathbb{N} \), using [BNV13, Prop. 7.6] at the marked places, we get the following chain of equivalences:

\[
(i_X^* \mathcal{H}G)(S^n) \simeq (\mathcal{H}G)(X \times S^n) \simeq (\mathcal{H}^{pre}G)(X \times S^n) \simeq (i_X^* \mathcal{H}^{pre}G)(S^n) \simeq (\mathcal{H}^{pre}i_X^*G)(S^n).
\]

The Assertion 4. now follows from [BNV13, Lemma 7.3] which states that an equivalence between objects of \( \text{Sh}_C(\text{Mf}) \) can be detected on the collection of spheres \( S^n, n \in \mathbb{N} \).

**Remark 2.5.** Since \( i_X^* \) preserves sheaves we have a natural transformation

\[
L \circ i_X^* \to i_X^* \circ L.
\]

In general it is not an equivalence. For example, let \( \mathcal{C} := \text{Ab} \) and \( \mathbb{Z}^{pre} \) be the constant presheaf with value \( \mathbb{Z} \) and \( X \) consist of two points. Then we have \( i_X^* (L(\mathbb{Z}^{pre}))(\ast) \simeq \mathbb{Z} \oplus \mathbb{Z} \), but \( L(i_X^* \mathbb{Z}^{pre})(\ast) \simeq \mathbb{Z} \) and the map (18) is the diagonal inclusion.

In the present paper we will use the language of diffeological algebras.

**Definition 2.6.** A diffeological structure on an algebra \( A \) over \( \mathbb{C} \) is a subsheaf of algebras \( A^\infty \) of the sheaf of algebras \( M \mapsto \text{Hom}_{\text{Set}}(M, A) \) such that \( A^\infty(\ast) = A \). A diffeological algebra is an algebra equipped with a diffeological structure.

**Remark 2.7.** A sheaf \( F \) of sets on \( \text{Mf} \) which is a subsheaf of the sheaf of set-valued functions to \( F(\ast) \) is also called a concrete sheaf. We refer to [Sta11] for a discussion of various variants of the definition of a diffeology. Our version is most similar to the notion of a Chen space, but not equal. A Chen space is a concrete sheaf of sets on the site of convex subsets with non-empty interior of euclidean spaces. In contrast, our sheaves are defined on all manifolds with corners.

**Example 2.8.** In the following we list some examples of diffeological algebras.

1. The constant sheaf \( A \) generated by \( A \) is the minimal diffeological structure, while the sheaf \( M \mapsto \text{Hom}_{\text{Set}}(M, A) \) is the maximal diffeological structure on \( A \).

2. If \( A \) is a diffeological algebra and \( X \) is a smooth manifold, then we define the algebra \( C^\infty(X, A) := A^\infty(X) \). It has again a diffeological structure given by the sheaf \( i_X^* A^\infty \).

3. The algebra \( \mathbb{C} \) has a diffeological structure such that \( \mathbb{C}^\infty \) is the sheaf \( M \mapsto C^\infty(M) \) of smooth \( \mathbb{C} \)-valued functions on \( \text{Mf} \).

4. For a manifold \( X \) we equip \( C^\infty(X) \) with the diffeological structure defined in 2.
5. If $A$ is a locally convex algebra, then we have a natural notion of a smooth function $X \to A$. The diffeological structure is given by $A^\infty(X) := C^\infty(X, A)$, where $C^\infty(X, A)$ denotes the algebra of smooth functions on $X$ with values in $A$. See Remark 2.9 for more details.

Remark 2.9. In this remark we fix our conventions about smooth functions on manifolds with values in a locally convex algebra. A locally convex vector space is a complex vector space whose topology is defined by a collection of seminorms. A locally convex algebra is a locally convex vector space such that the product induces a continuous bilinear map $A \times A \to A$.

A locally convex vector space has a natural uniform structure. Therefore the notions of completeness and completion are defined.

We now consider smooth functions with values in a locally convex vector space $A$ (see e.g. [Tre67, Sec. 40]). Let $U \subseteq \mathbb{R}^n$ open and consider a continuous function $f : U \to A$.

Definition 2.10. The function $f$ continuously differentiable if there exists a continuous function $f' : U \to \text{Hom}(\mathbb{R}^n, A)$ such that for every seminorm $p$ on $A$ and every compact subset $K \subset U$ we have

$$\lim_{D \to 0} \sup_{x \in K} p \left( \frac{f(x + D) - f(x) - f'(x)(D)}{t} \right) = 0.$$ 

We call $\partial_i f := f'(\cdot)(e_i)$ the partial derivatives of $f$ in the $i$’th direction. We call $f$ smooth if it has all iterated continuous partial derivatives.

We denote the iterated partial derivatives by $f_{i_1, \ldots, i_k}^{(n)}$. We equip the complex vector space $C^\infty(U, A)$ with the locally convex structure determined by the seminorms

$$f \mapsto \sup_{x \in K} p(f_{i_1, \ldots, i_k}^{(n)}(x)).$$

The set of seminorms which generates the topology of $C^\infty(U, A)$ is thus indexed by compact subsets $K \subset U$, tuples $(i_1, \ldots, i_k)$ of elements of $\{1, \ldots, n\}$, and seminorms $p$ of $A$. If $A$ is complete, then so is $C^\infty(U, A)$.

This definition of smooth $A$-valued functions extends to manifolds in a straightforward manner.

Let $X$ be a smooth manifold. Then the algebra $C^\infty(X, A)$ has two diffeological structures:

1. The first comes from the construction 2. in Example 2.8 above.

2. The second is induced from its locally convex structure.
These two structures coincide in view of the exponential law:

\[ C^\infty(X, C^\infty(Y, A)) \cong C^\infty(X \times Y, A) . \]

Since \( C^\infty(M, A) \) is a subset of the set of all functions from \( M \) to \( A \) it is clear that these spaces of smooth functions for varying \( M \) define a concrete sheaf, i.e. a diffeological structure on \( A \).

In the non-commutative geometry literature instead of \( C^\infty(X, A) \) one often uses the projective tensor product \( C^\infty(X) \otimes_\pi A \). If \( A \) is complete, then this gives an equivalent structure as we will explain below. We have a natural map

\[ C^\infty(X) \otimes A \to C^\infty(X, A) \]

which is continuous with respect to projective topology on the algebraic tensor product.

In general, for locally convex vector spaces \( V, W \) we let \( V \otimes_\pi W \) denote the completion of the algebraic tensor product \( V \otimes W \) with respect to the projective topology. If \( A \) is a complete locally convex vector space, then we get an isomorphism

\[ C^\infty(X) \otimes_\pi A \cong C^\infty(X, A) . \]

Here is a reference for this classical fact:

1. It follows from \( \text{[Tre67, Thm 44.1]} \) that for a complete \( A \) we have an isomorphism \( C^\infty(X) \otimes_\epsilon A \cong C^\infty(X, A) \), where \( \otimes_\epsilon \) denotes the completion of the algebraic tensor product in the \( \epsilon \)-topology.

2. It follows from \( \text{[Tre67, Thm 50.1]} \) that the locally convex vector space \( C^\infty(X) \) is nuclear.

3. If one of the tensor factors is nuclear, then the natural map from the \( \pi \) to the \( \epsilon \)-tensor product is an isomorphism by \( \text{[Tre67, Thm 50.1]} \).

Remark 2.11. In this remark we explain the relationship between the notions of homotopy invariance according to Definition 2.3 and diffeotopy invariance of functors defined on locally convex algebras as considered e.g. in \( \text{[CT08, Sec. 4.1]} \).

Consider the category \( \mathcal{L}oc\mathcal{A}lg_1 \) of unital complete locally convex algebras. We have a functor

\[ \mathcal{L}oc\mathcal{A}lg_1 \to \mathbf{Sh}_{\mathcal{L}oc\mathcal{A}lg_1}(\mathbf{Mf}) , \quad A \mapsto A^\infty := (M \mapsto C^\infty(M) \otimes_\pi A) . \]

Let \( C \) be a presentable \( \infty \)-category.

Lemma 2.12. A functor \( F : \mathcal{L}oc\mathcal{A}lg_1 \to C \) is a diffeotopy invariant functor in the sense of \( \text{[CT08, Sec. 4.1]} \) if and only if the presheaf \( F(A^\infty) \in \mathbf{PSh}_C(\mathbf{Mf}) \) is homotopy invariant in the sense of Definition 2.3.
Proof. This is immediate from the definitions if one uses the associativity of $\otimes_\pi$ and $C^\infty(I \times X) \cong C^\infty(I) \otimes_\pi C^\infty(X)$.

\[ \square \]

2.2 Algebraic $K$-theory and cyclic homology of smooth functions

We start with fixing our notation concerning chain complexes.

1. Let $\text{Ch}$ be the category of chain complexes and chain morphisms. We identify chain complexes with cochain complexes such that the chain complex

\[ \cdots \to C_{n+1} \to C_n \to C_{n-1} \to \cdots \]

corresponds to the cochain complex

\[ \cdots \to C^{-n-1} \to C^{-n} \to C^{-n+1} \to \cdots \].

2. For an integer $p \in \mathbb{N}$ and a chain complex $(C,d) \in \text{Ch}$ we define its shift by $p$ by $C[p] := C^{n+p}$. The differential of the shifted complex is given by $(-1)^p d$.

3. If we invert the quasi-isomorphisms in $\text{Ch}$, then we get the stable $\infty$-category $\text{Ch}[W^{-1}]$. We have a natural functor $\iota : \text{Ch} \to \text{Ch}[W^{-1}]$. Our usual notation convention is that if the italic letter $C$ denotes an object of $\text{Ch}$, then we will use the roman letter $C$ in order to denote the object $\iota(C) \in \text{Ch}[W^{-1}]$.

4. For an integer $p \in \mathbb{Z}$ and a chain complex $C \in \text{Ch}$

\[ \cdots \to C^{p-1} \to C^p \to C^{p+1} \to \cdots \]

we form its naive truncation $\sigma^{\geq p}C$ at $p$ by

\[ \cdots \to 0 \to C^p \to C^{p+1} \to \cdots \].

We have a natural inclusion morphism $\sigma^{\geq p}C \to C$.

Remark 2.13. Note that $\iota(\sigma^{\geq p}C)$ is well-defined, but $\sigma^{\geq p}\iota(C)$ does not make sense.

\[ \square \]

5. Let $\Omega \in \text{ShCh}(\text{Mf})$ denote the sheaf of de Rham complexes on $\text{Mf}$ of complex-valued differential forms. For every $p \in \mathbb{Z}$ its truncation $\iota(\sigma^{\geq p}\Omega)$ is a sheaf, i.e

\[ \iota(\sigma^{\geq p}\Omega) \in \text{ShCh}_{[W^{-1}]}(\text{Mf}), \]

by [BNV13, Lemma 7.12].
6. Let $\mathbf{Sp}$ denotes the stable $\infty$-category of spectra. We will frequently use the Eilenberg-MacLane correspondence $H : \mathbf{Ch}[W^{-1}] \to \mathbf{Sp}$ (see [Lur11, 8.1.2.13]) which maps a chain complex to its associated Eilenberg-MacLane spectrum. For $C \in \mathbf{Ch}[W^{-1}]$ we have the relations

$$\pi_n(H(C)) \cong H^{-n}(C), \quad H(C[p]) \simeq \Sigma^p H(C)$$

between the homotopy groups of $H(C)$ and the cohomology of $C$ on the one hand, and the shifts by $p \in \mathbb{Z}$ in spectra and chain complexes, on the other.

7. The Eilenberg-MacLane equivalence preserves limits. Hence it induces a map $H : \mathbf{Sh}_{\mathbf{Ch}[W^{-1}]}(\text{Mf}) \to \mathbf{Sh}_{\mathbf{Sp}}(\text{Mf})$ by objectwise application. For example, by (20) we have the sheaf

$$H(\iota(\omega^{2p}\Omega)) \in \mathbf{Sh}_{\mathbf{Sp}}(\text{Mf}). \quad (21)$$

We will use the following ingredients from algebraic $K$-theory:

1. We let $\mathbf{Alg}$ denote the category of associative unital algebras. We have a functor

$$K : \mathbf{Alg} \to \mathbf{Sp}$$

which maps an associative unital algebra to its connective algebraic $K$-theory spectrum $K(A)$.

**Remark 2.14.** One way to construct this functor is as the following composition:

$$K(A) := \text{sp}(\text{GrCompl}(\mathbb{N}(\text{Iso}(\text{Proj}(A))))).$$

Here $\text{Proj}(A)$ is the symmetric monoidal category of finitely generated projective $A$-modules with respect to the direct sum and $\text{Iso}$ takes the underlying groupoid. The functor $\mathbb{N}$ maps a symmetric monoidal category to a commutative monoid in spaces. The group completion functor $\text{GrCompl}$ turns this monoid into a commutative group or equivalently into an infinite loop space. Finally, the functor $\text{sp}$ maps the infinite loop space to the corresponding spectrum. We refer to [BNV13, Sec 6] and [BG13, Sec. 6] for more details. In the present paper will not need any explicit construction of the algebraic $K$-theory functor. \hfill \Box

2. Let $\mathbf{Alg}_C$ denote the category of unital algebras over $\mathbb{C}$. We have a functor

$$CC^- : \mathbf{Alg}_C \to \mathbf{Ch}$$

which maps an associative unital algebra $A$ over $\mathbb{C}$ to its negative cyclic cohomology complex $CC^-(A)$. We define the negative cyclic homology of $A$ by

$$HC_n^-(A) := H_n(CC^-(A)). \quad (22)$$
Remark 2.15. For concreteness we will choose for $CC^{-}(A)$ the standard negative cyclic homology complex denoted by $\text{ToT}BC^{-}$ in [Lod98, 5.1.7]. Some constructions in the present paper will use this model explicitly. 

3. We define the functor $CC^{-}: \text{Alg}_C \to \text{Sp}$ as the composition 

$$\text{Alg}_C \xrightarrow{CC^{-}} \text{Ch} \xrightarrow{i} \text{Ch}[W^{-1}] \xrightarrow{H} \text{Sp}.$$ 

4. We have a natural transformation of functors 

$$\text{ch}^{GJ}: K \to CC^{-},$$

given by the Goodwillie-Jones Chern character. For the construction of the Goodwillie-Jones Chern character we refer to [McC94] and [LR06, Sec. 5].

We consider a diffeological algebra $A$ (see Definition 2.6) and form the presheaf of spectra 

$$\check{K}_A \in P\text{Sh}_{\text{Sp}}(\text{Mf}), \quad M \mapsto K(A^\infty(M)).$$

Its sheafification is a sheaf of spectra and will be denoted by 

$$K_A := L(\check{K}_A) \in \text{Sh}_{\text{Sp}}(\text{Mf}).$$

We apply these constructions to the diffeological algebra $C$. We then have the equivalences 

$$\check{K}_{C^\infty(X)} \simeq i_X^* \check{K}_C, \quad K_{C^\infty(X)} \simeq Li_X^* K_C.$$ 

The first follows from the definition of the diffeological structure on $C^\infty(X)$, and the second is then a reformulation of the definition above.

Applying the negative cyclic cohomology complex to the sheaf $\mathbb{C}^\infty$ we obtain the presheaf of chain complexes 

$$CC^{-}(\mathbb{C}^\infty) \in P\text{Sh}_{\text{Ch}}(\text{Mf}), \quad M \mapsto CC^{-}(C^\infty(M)).$$

Remark 2.16. Note that we do not complete the sheafify the tensor products involved in the definition of the negative cyclic cohomology complex, but see Remark 2.18.

In the following we define a differential geometric analog $DD^{-}$ of $CC^{-}(\mathbb{C}^\infty)$ and a comparison map 

$$\pi^{-}: CC^{-}(\mathbb{C}^\infty) \to DD^{-}.$$
Definition 2.17. We define the sheaf of chain complexes \( DD^- \in \text{Sh}_{\text{Ch}}(\text{Mf}) \)

\[
DD^- := \prod_{p \in \mathbb{Z}} DD^-(p) , \quad DD^-(p) := (\sigma^{\geq p} \Omega)[2p] .
\]

We further define a map of presheaves of chain complexes

\[
\pi^- : CC^- (\mathbb{C}^\infty) \to DD^-
\]

by

\[
CC^- (\mathbb{C}^\infty(X))_{q,p} \ni f_0 \otimes \cdots \otimes f_{p-q} \mapsto \frac{1}{(p-q)!} f_0 df_1 \wedge \cdots df_{p-q} \\
\in F^{p-q}Q^{p-q}(X) \subset DD^-(p)(X)^{p-q} .
\]

Here the index \((\ldots)_{q,p}\) refers to the component of the bicomplex \( BC^-\) in [Lod98, 5.1.7].

Using the formulas [Lod98, Sec.2.3.2] we conclude that \( \pi^- \) is compatible with the differentials.

Remark 2.18. For a manifold \( X \) let \( CC^{cont,-}(\mathbb{C}^\infty(X)) \) be the analog of \( CC^- (\mathbb{C}^\infty(X)) \) defined using completed (but not sheafified) tensor products. Then we have a factorization of \( \pi^-\):

\[
CC^- (\mathbb{C}^\infty(X)) \to CC^{cont,-}(\mathbb{C}^\infty(X)) \xrightarrow{\pi^{cont,-}} DD^-(X) .
\]

The second map is quasi-isomorphism by the well-known calculation of the continuous negative cyclic homology of the algebra of smooth functions on a smooth manifold. We will use the continuous version of cyclic homology and \( \pi^{cont,-} \) in Subsection 3.5 below. □

We further define the presheaves

\[
CC_C^- := H \circ \iota \circ CC^- (\mathbb{C}^\infty) , \quad DD^- := \iota (DD^-) , \quad DD^- := H \circ DD^- .
\]

The \( \text{Ch}[W^{-1}]-\)valued presheaf \( DD^- \) is a sheaf by (21). Since \( H \) preserves limits, \( DD^- \) is a \( \text{Sp} \)-valued sheaf, too.

By its naturality the Goodwillie-Jones Chern character [23] provides a map

\[
\text{ch}^{GJ} : K_C \to CC^- 
\]

between presheaves of spectra.

Definition 2.19. We define the regulator morphism \( \text{rég} \) of presheaves of spectra as the composition

\[
\text{rég} : K_C \xrightarrow{\text{ch}^{GJ}} CC^- \xrightarrow{\pi^-} DD^- .
\]

Furthermore, for a smooth manifold \( X \), we define the morphism of sheaves of spectra

\[
\text{reg}_X : K_{C^\infty(X)} \xrightarrow{\text{ch}^{GJ}} Li_X K_C \xrightarrow{L_i^{\text{rég}}} Li_X DD^- \simeq i_X DD^-.
\]
The last equivalence in (31) follows from the fact that $DD^-$ and therefore $i_\ast^X DD^-$ are sheaves.

\[ \square \]

**Remark 2.20.** In this paper we usually call transformations from $K$-theory to cyclic cohomology Chern characters, and transformations from $K$-theory to differential forms regulators. There is one exception, namely the usual Chern character from topological $K$-theory to cohomology with complex coefficients calculated by the de Rham cohomology.

### 2.3 Homotopification and regulator maps

For a sheaf $G$ with values in a stable $\infty$-category (e.g. $\text{Ch}[W^{-1}]$ or $\text{Sp}$) we have a functorial homotopification fibre sequence of sheaves

\[ \mathcal{A}(G) \to G \to \mathcal{H}(G) \to \Sigma \mathcal{A}(G), \] (32)

see [BNV13, Def. 3.1]. The functor $\mathcal{A}$ by definition takes the fibre of the homotopification map $G \to \mathcal{H}(G)$ given by the unit of the homotopification functor $\mathcal{H}$ introduced in (13). The sheaf $G$ is homotopy invariant if and only if $\mathcal{A}(G) \cong 0$.

Let $A$ be a diffeological algebra (Definition 2.6).

**Definition 2.21.** We define the sheaves of spectra

\[ K^\text{top}_A := \mathcal{H}(K_A), \quad K^\text{rel}_A := \mathcal{A}(K_A). \]

We call the evaluations $K^\text{top}(A) := K^\text{top}_A(\ast)$ and $K^\text{rel}(A) := K^\text{rel}_A(\ast)$ the topological and relative $K$-theory spectra of $A$.

Note that the topological and relative $K$-theory spectra depend on the diffeological structure on $A$. They fit into the fibre sequence of spectra

\[ K^\text{rel}(A) \to K(A) \to K^\text{top}(A) \to \Sigma K^\text{rel}(A) \] (33)

derived from (32) by evaluation at $\ast$.

**Remark 2.22.** A Fréchet algebra has a natural diffeological structure such that $A^\infty(M)$ is the algebra of smooth maps $M \to A$. In this case our definition of $K^\text{top}(A)$ coincides with that given in [CK88, Sec. 3.1]. Indeed, in this reference the authors apply Quillen’s $+$-construction to the classifying space of the simplicial group $GL(C^\infty(\Delta^\ast, A))$. Using [16] we can identify the resulting space with $\Omega^\infty(s(K_A)(\ast))$. We now use the equivalence $s(K_A)(\ast) \cong K^\text{top}_A(\ast)$ which follows from the combination of (17) and (14). As a consequence the relative $K$-theory of a Fréchet algebra as defined in [CK88, Sec. 3.1] is isomorphic to our version.

More generally, if $A$ is a complete locally convex algebra, then in [CT08, Def. 4.1.3] the notion of the diffeotopy $K$-theory spectrum was defined. This definition is just $s(K_A)(\ast)$ written down in different symbols. Therefore our topological or relative $K$-theory of a
complete locally convex algebra also coincides with the diffeotopy or relative $K$-theory of $\mathcal{C}^{108}$. 

We let

$$DD^\text{per} := \prod_{p\in\mathbb{Z}} \Omega[2p] \in \text{Sh}_{\text{Ch}}(\text{Mf})$$

(34)

be the two-periodic de Rham complex. Its cohomology

$$HP^*(X) := H^*(DD^\text{per}(X))$$

(35)

is called the periodic cohomology of the manifold $X$. The periodicity is implemented by the shift isomorphism, which for $k \in \mathbb{Z}$ is given by

$$\iota_k : HP^*(X) \xrightarrow{\cong} HP^{*-2k}(X), \quad \iota_k([\omega(p)])_{p\in\mathbb{Z}} = ([\omega(p+2k)])_{p\in\mathbb{Z}}.$$  

(36)

We further define the sheaf

$$DD^\text{per} := \iota(DD^\text{per}) \in \text{Sh}^h_{\text{Ch}[\mathcal{W}^{-1}]}(\text{Mf}).$$  

(37)

Since $\Omega$ resolves the constant sheaf $\mathbb{C}$, the sheaf $\iota(\Omega)$ is homotopy invariant. Consequently, the sheaf $DD^\text{per}$ is homotopy invariant, too. In view of the definition (29) of $DD^-$ we have a natural inclusion of sheaves of chain complexes $DD^- \rightarrow DD^\text{per}$.

**Lemma 2.23.** The induced map $DD^- \rightarrow DD^\text{per}$ is equivalent to the homotopification morphisms of $DD^-$. 

**Proof.** By [BNV13, Lemma 7.15] we know that the inclusion $\iota(\sigma^p\Omega) \rightarrow \iota(\Omega)$ is equivalent to the homotopification map. This implies that the natural inclusion $DD^- \rightarrow DD^\text{per}$ is equivalent to the homotopification map $DD^- \rightarrow \mathcal{H}(DD^-)$. 

If we set

$$DD^\text{per} := H(DD^\text{per}) \in \text{Sh}^h_{\text{Sp}}(\text{Mf}),$$

(38)

then by Lemma 2.23 we know that $DD^- \rightarrow DD^\text{per}$ is equivalent to the homotopification $DD^- \rightarrow \mathcal{H}(DD^-)$. 

Since $DD^-$ is a sheaf we can factorize the regulator (30) uniquely as

$$\mathcal{K}_C \rightarrow \mathcal{K}_C \xrightarrow{\text{reg}} DD^-.$$ 

We now apply homotopification to $\text{reg}$. We use the equivalence $\mathcal{H}(DD^-) \simeq DD^\text{per}$ and the Definition 2.21 of $K^{\text{top}}_C := \mathcal{H}(K_C)$ in order to get the second line of the following
The two upper vertical maps are the units of the homotopification. By definition we have

\[ \text{reg}^{\text{top}} := \mathcal{H}(\text{reg}) . \] (40)

Finally, the identification of \( K^{\text{top}}_C \simeq \text{ku} \) follows from [BNV13, Lemma 6.3], where \( \text{ku} \) is the constant sheaf of spectra generated by the connective topological \( K \)-theory spectrum \( \text{ku} \simeq K^{\text{top}}_C(\ast) \) of \( \mathbb{C} \).

Let \( X \) be a smooth manifold. We apply \( i^*_X \) (see (15)) to the diagram (39). We use that \( Li^*_X \tilde{K}_C \simeq K_{\infty}(X) \) and \( \mathcal{H}Li^*_X \tilde{K}_C \simeq K^{\text{top}}_{C\infty}(X) \). The left horizontal arrows in the second and third line of the following diagram are natural factorizations of the left upper horizontal arrow using the facts that \( i_X^* \tilde{K}_C \) is a sheaf, and that \( i_X^* K^{\text{top}}_C \) is homotopy invariant.

\[ (41) \]

Note that \( \text{reg}_{X}^{\text{top}} \simeq \mathcal{H}(\text{reg}_X) \).

If we evaluate \( i_X^* \text{reg}^{\text{top}} \) at a point, identify \((i^*_X K^{\text{top}}_C)(\ast) \simeq \text{ku}(X)\), and take homotopy groups, then we get a homomorphism (natural in \( X \))

\[ \text{ch}^{gj} : \text{ku}^*(X) \to \pi_{-*}(\text{DD}^{\text{per}}(X)) \cong HP^*(X) . \]
The origin of this map is the Goodwillie-Jones Chern character $\text{ch}^{GJ}$, see [23]. In this situation we also have the usual Chern character defined by Chern-Weil theory

$$\text{ch}^{cw} : \text{ku}^*(X) \to H P^*(X).$$

We refer to [BNV13, Sec. 6.1] for a construction of the Chern character $\text{ch}^{cw}$ using differential cohomology methods. The following lemma is probably well-known. For completeness we sketch an argument.

Lemma 2.24. We have the equality of Chern character maps $\text{ch}^{GJ} = \text{ch}^{cw} : \text{ku}^*(X) \to H P^*(X)$.

Proof. Since both Chern characters arise from maps between homotopy invariant sheaves of spectra they are characterized by their evaluation at the point. Since the target is rational they are determined by their actions on homotopy groups. Hence it suffices to check that

$$\text{ch}^{GJ} = \text{ch}^{cw} : \text{ku}^*(*) \to H P^*(*) .$$

One can now use the fact that both Chern characters are multiplicative (see [McC94] for the multiplicity of $\text{ch}^{GJ}$) in order to reduce to the cases $* = 0$ and 2. For $* = 0$ one checks the equality directly going through the definitions. For $* = 2$ we argue as follows. We know by an explicit calculation that $\text{ch}^{GJ} : K_1(C^\infty(S^1)) \to H_1(C^\text{cont.}, - (C^\infty(S^1))) \cong \Omega^1(S^1)$ maps the class $\iota(u) \in K_1(C^\infty(S^1))$ of a unit $u \in C^\infty(S^1)^\times$ to the form $\frac{1}{2\pi i} d \log u \in \Omega^1(S^1)$.

We now consider specifically the embedding $u : S^1 \to \mathbb{C}^\times$. The class $\iota(u)$ is then mapped to the generator $x \in \text{ku}^{-1}(S^1) \cong \mathbb{Z}$ under the composition

$$K_1(C^\infty(S^1)) \to K_1^\text{top}(C^\infty(S^1)) \xrightarrow{[\mathbb{I}]} \pi_1(K_1^\text{top}(S^1)) \cong \text{ku}^{-1}(S^1) .$$

Therefore $\text{ch}^{GJ}(x) \in HP^{-1}(S^1)$ is given by the class $(c(p))_{p \in \mathbb{Z}} \in H^{-1}(DD^\text{per}(S^1))$ with

$$c(p) := \begin{cases} [\text{vol}_{S^1}] \in H^{-1}(DD^\text{per}(p)(S^1)) & p = 1 \\ 0 & \text{else} \end{cases} ,$$

where $\text{vol}_{S^1} = \frac{1}{2\pi i} d \log u$ is the normalized volume form of $S^1$. It follows by suspension that $\text{ch}^{GJ} : \text{ku}^{-2}(*) \to H P^{-2}(*)$ maps the generator of $\text{ku}^{-2}(*) \cong \mathbb{Z}$ to the class $(b(p))_{p \in \mathbb{Z}} \in H^{-2}(DD^\text{per}(*))$ given by

$$b(p) := \begin{cases} [1] \in H^{-2}(DD^\text{per}(p)(*)) & p = 1 \\ 0 & \text{else} \end{cases} .$$

The same holds true for $\text{ch}^{cw}$. \qed
2.4 The differential regulator map

In this subsection we introduce the differential algebraic $K$-theory $\hat{K}^k(\mathcal{C}_\infty(X))$ refining $K_*(\mathcal{C}_\infty(X))$ and define a differential regulator map $\text{reg}_X$ from this differential algebraic $K$-theory to the Hopkins-Singer differential cohomology associated with $\text{ku}$.

Modifying the Definition 2.17 of the sheaf of chain complexes $DD^-$, for any integer $k \in \mathbb{Z}$ we define its naively truncated version (see (19) for notation)

$$\geq k DD^- := \prod_{p \in \mathbb{Z}} \geq k DD^-(p) , \quad \geq k DD^-(p) := \sigma^{\geq k}((\sigma^{\geq p} \Omega)[2p]) = (\sigma^{\geq \max(2p+k,p)} \Omega)[2p] . \quad (42)$$

We further define the sheaf of chain complexes

$$\geq k DD^{per} := \prod_{p \in \mathbb{Z}} \geq k DD^{per}(p) , \quad \geq k DD^{per}(p) := \sigma^{\geq k}(\Omega[2p]) = (\sigma^{\geq 2p+k} \Omega)[2p] .$$

These sheaves of chain complexes are related by natural inclusion morphisms

$$\geq k DD^- \to \geq k DD^{per} , \quad \geq k DD^- \to DD^- , \quad \geq k DD^{per} \to DD^{per} . \quad (43)$$

We now consider the induced $\infty$-categorical objects

$$\geq k DD^- := \iota(\geq k DD^-) , \quad \geq k DD^{per} := \iota(\geq k DD^{per}) ,$$

which are a priori presheaves on $\text{Mf}$ with values in $\text{Ch}[W^{-1}]$. By [BNV13, Lemma 7.12] they are indeed sheaves. We finally define the sheaves of spectra

$$\geq k DD^- := H(\geq k DD^-) , \quad \geq k DD^{per} := H(\geq k DD^{per}) .$$

These sheaves of spectra are related by natural morphisms so that the following square in $\text{Sh}_{\text{sp}}(\text{Mf})$

$$\begin{array}{ccc}
\geq k DD^- & \to & DD^- \\
\downarrow e & & \downarrow \\
\geq k DD^{per} & \to & DD^{per}
\end{array} \quad (44)$$

commutes. The map denoted by $e$ will be used later.

We fix a smooth manifold $X$ and recall the definitions (31) of $\text{reg}_X$ and the (Chern-Weyl version) of the Chern character $\text{ch}^{cw}$ which is equivalent to $\text{ch}^{gj}$ by Lemma 2.24.

**Definition 2.25.** For $k \in \mathbb{Z}$ we define the $k$'th differential algebraic $K$-theory sheaf

$$\hat{K}_C^k(X) \in \text{Sh}_{\text{sp}}(\text{Mf})$$
by the pull-back in sheaves of spectra

\[
\begin{array}{c}
\hat K^k_{\mathcal{C}^{\infty}(X)} \xrightarrow{R} i_X^* \geq^k \text{DD}^- \\
\downarrow I \\
K_{\mathcal{C}^{\infty}(X)} \xrightarrow{\text{reg}_X} i_X^* \text{DD}^- \\
\end{array}
\]  

(45)

We define the Hopkins-Singer differential connective complex K-theory $\hat{\text{ku}}^k \in \text{Sh}_{\text{Sp}}(\text{Mf})$ by

\[
\begin{array}{c}
\hat{\text{ku}}^k \xrightarrow{R} \geq^k \text{DD}^{\text{per}} \\
\downarrow I \\
\hat{\text{ku}} \xrightarrow{\text{ch}^{\text{cw}}} \text{DD}^{\text{per}} \\
\end{array}
\]  

(46)

For a general discussion of Hopkins-Singer differential cohomology theories we refer to [BNV13, Sec. 4.4]. The maps denoted by $I$ take the underlying (non-differential) classes, and the maps denoted by $R$ are called the curvature morphisms.

We get an obvious induced map of pull-back diagrams (45) → (46)

\[
\begin{array}{c}
K_{\mathcal{C}^{\infty}(X)} \xrightarrow{\text{reg}_X} i_X^* \text{DD}^- \xrightarrow{\text{e}} i_X^* \geq^k \text{DD}^- \\
\downarrow \\
i_X^* \hat{\text{ku}} \xrightarrow{i_X^* \text{ch}^{\text{cw}}} i_X^* \text{DD}^{\text{per}} \xleftarrow{i_X^* \geq^k \text{DD}^{\text{per}}} \\
\end{array}
\]  

(47)

The left square is supplied by (41) in combination with an identification $\text{ch}^{\text{cw}} \cong \text{ch}^{g_j}$ by Lemma 2.24, while the right square is (44).

**Definition 2.26.** For $k \in \mathbb{Z}$ we define the $k$’th differential algebraic $K$-theory spectrum of $\mathcal{C}^{\infty}(X)$ by

\[
\hat{K}^k(\mathcal{C}^{\infty}(X)) := \hat{K}^k_{\mathcal{C}^{\infty}(X)}(*)
\]

and the differential algebraic $K$-theory groups by

\[
\hat{K}^k_*(\mathcal{C}^{\infty}(X)) := \pi_*(\hat{K}^k(\mathcal{C}^{\infty}(X))).
\]

Further we define the differential connective topological $K$-theory of $X$ by

\[
\hat{\text{ku}}^k_*(X) := \pi_*(\hat{\text{ku}}^k(X)).
\]

The evaluation at the manifold $*$ of the map $\hat{K}^k_{\mathcal{C}^{\infty}(X)} \rightarrow i_X^* \hat{\text{ku}}^k$ induced by (47) is called the differential regulator map

\[
\hat{\text{reg}}_X : \hat{K}^k(\mathcal{C}^{\infty}(X)) \rightarrow \hat{\text{ku}}^k(X).
\]
Remark 2.27. For commutative algebras it is possible to refine the Goodwillie-Jones Chern character $\text{ch}^{GJ}$ to a morphism between commutative ring spectra. The spectra occurring on the right lower corners of the diagrams (45) and (46) are obtained by an application of $H \circ \iota$ to sheaves of commutative differential graded algebras and therefore are commutative ring spectra, as well. The morphisms $\text{reg}_X$ and $\text{ch}^{cw}$ have refinements to morphisms between sheaves of commutative ring spectra. Finally we have products

$$\geq k \mathbb{D}^+ \land \geq l \mathbb{D}^+ \rightarrow \geq k+l \mathbb{D}^+, \geq k \mathbb{D}^+ \land \geq l \mathbb{D}^+ \rightarrow \geq k+l \mathbb{D}^+. \quad (48)$$

Using these multiplicative data we can refine the sums

$$\bigvee_{k \in \mathbb{Z}} \hat{K}^k(C^\infty(X)) \quad \text{and} \quad \bigvee_{k \in \mathbb{Z}} \hat{ku}^k(X)$$

to commutative ring spectra, see e.g. [Bun Sec. 4.6]. The differential regulator then becomes a morphism between commutative ring spectra.

The multiplicative structure is helpful if one wants to calculate $\sigma_d$ of products like

$$\iota(u_1) \cup \cdots \cup \iota(u_d) \in K_d(C^\infty(X))$$

for a collection of invertible functions $(u_i)_{i=1, \ldots, d}$ in $C^\infty(X)$. Since our main results do not use the multiplicative structure, in the present paper we will not discuss its details further. \hfill \Box

2.5 The construction of $\sigma_d$.

We consider a smooth manifold $X$. We will analyse the map induced in homotopy by the morphism $I$ in (45). Recall that $I$ maps a differential algebraic $K$-theory class to its underlying algebraic $K$-theory class.

We get the $K$-theory spectrum of the algebra of smooth functions on $X$ by evaluating the sheaf of spectra $K_{C^\infty(X)}$ (see (26)) at a point. We have the equivalence $K(C^\infty(X)) \simeq K_{C^\infty(X)}(*)$ and the isomorphism $K_*(C^\infty(X)) \cong \pi_*(K(C^\infty(X)))$.

Let $d \in \mathbb{Z}$.

Lemma 2.28. If $d \geq \dim(X)$, then the underlying class map

$$I : \hat{K}_d^{-d}(C^\infty(X))) \rightarrow K_d(C^\infty(X))$$

is an isomorphism.

Proof. We define the sheaf of chain complexes

$$\leq -d-1 \mathbb{D}^- := \prod_{p \in \mathbb{Z}} \leq -d-1 \mathbb{D}^-(p), \quad \leq -d-1 \mathbb{D}^-(p) := \sigma^{\leq -d-1}((\sigma^{\geq p} \Omega)[2p]) \quad (48)$$

$$= (\sigma^{\geq 2p-d-1} \sigma^{\geq p} \Omega)[2p].$$

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For every $k \in \mathbb{Z}$ we have an exact sequence of sheaves of chain complexes
\[ 0 \to \sigma^{\geq k} \Omega \to \Omega \xrightarrow{1} \sigma^{\leq k-1} \Omega \to 0 , \] (49)
where the first map is the inclusion and the second marked by $!$ is the projection. Forming an appropriate product of sequences of this type we obtain an exact sequence of sheaves of chain complexes
\[ 0 \to \geq -d \dd^+ \to \dd^+ \to \leq -d-1 \dd^+ \to 0 . \]
Applying $H \circ \iota$ we get a fibre sequence in $\text{Sh}_{\text{Sp}}(\text{Mf})$
\[ \Sigma^{-1}(\leq -d-1 \dd^+) \to \geq -d \dd^+ \xrightarrow{!} \dd^+ \to \leq -d-1 \dd^+ . \] (50)
In view of the pull-back (45) we can identify the fibre of $I$ with the fibre of the map marked by $!$ in (50). We thus have a fibre sequence of spectra
\[ \Sigma^{-1}(\leq d-1 \dd^-)(X) \xrightarrow{a} \hat{K}^{-d}(C^\infty(X)) \xrightarrow{I} K(C^\infty(X)) \to (\leq d-1 \dd^-)(X) . \] (51)
The pull-back square (45) provides a long exact sequence
\[ \ldots \to \hat{K}^{-d}(C^\infty(X)) \xrightarrow{1 \oplus R} K_d(C^\infty(X)) \oplus H^{-d}(\geq -d \dd^-(X)) \to H^{-d}(\dd^-(X)) \to \ldots \]
The map
\[ H^{-d}(\geq -d \dd^-(X)) \to H^{-d}(\dd^-(X)) \]
is obviously surjective since it just maps a cycle to its cohomology class. Therefore
\[ I : \hat{K}^{-d}(C^\infty(X)) \to K_d(C^\infty(X)) \]
is surjective.

The fibre sequence (51) now gives the exact sequence
\[ H^{-d-1}(\leq -d-1 \dd^-)(X) \xrightarrow{a} \hat{K}^{-d}(C^\infty(X)) \xrightarrow{I} K_d(C^\infty(X)) \to 0 . \]
In order to show the assertion of the Lemma it suffices to show that
\[ H^{-d-1}(\leq -d-1 \dd^-)(p)(X)) = 0 \]
for all $p \in \mathbb{Z}$, see (48) for notation. A class $[\omega] \in H^{-d-1}(\leq -d-1 \dd^-)(p)(X))$ is represented by a form
\[ \omega \in \sigma^{2p-d-1} \sigma^{\geq p} \Omega^{2p-d-1}(X) . \]
If $\omega \neq 0$, then we have the inequalities
\[ 2p - d - 1 \geq p , \quad 2p - d - 1 \leq \dim(X) \leq d , \]
which contradict each other. \qed

We fix integers $k, d \in \mathbb{Z}$. In the following definition the symbol $R$ denotes the curvature map of the respective differential cohomology theory, see (45) and (46).
Definition 2.29. A class \( x \in \hat{K}^k_d(C^\infty(X)) \) is called flat if \( R(x) = 0 \). We let
\[
\hat{K}^k_{d,\text{flat}}(C^\infty(X)) \subseteq \hat{K}^k_d(C^\infty(X))
\]
denote the subgroup of flat classes. Similarly we define the subgroup of flat classes
\[
\hat{ku}_{d,\text{flat}}(X) := \ker \left( R : \hat{ku}^{k-d}(X) \rightarrow \pi_d(\geq^k DD_{\text{per}}(X)) \right)
\]
The flat subgroup of a Hopkins-Singer differential cohomology theory can be calculated explicitly.

Lemma 2.30. We have a natural isomorphism
\[
\hat{ku}_{d,\text{flat}}(X) \cong ku\mathbb{C}/\mathbb{Z}^{d-1}(X) \oplus \bigoplus_{p=1}^\infty H^{-2p-d-1}(X; \mathbb{C})
\]
(52)

Proof. The complex of sheaves of abelian groups \( \Omega \) is a resolution of the constant sheaf \( \mathbb{C} \). Consequently \( DD_{\text{per}} \) resolves the constant sheaf \( \prod_{p \in \mathbb{Z}} \mathbb{C}[2p] \). In view of (37), (38) we thus obtain an equivalence
\[
DD_{\text{per}} \simeq H(\iota(\prod_{p \in \mathbb{Z}} \mathbb{C}[2p]))
\]
(53)
It follows from the general properties of Hopkins-Singer differential cohomology theories (see (25) in [BNV13]) that the flat subgroup is the evaluation on \( X \) of a cohomology theory. The latter is represented by the fibre of the evaluation at \( * \) of lower horizontal map in the defining pull-back square. In our present case this square is (46), and the respective map is
\[
\text{ch}^{cw} : ku \rightarrow H(\iota(\prod_{p \in \mathbb{Z}} \mathbb{C}[2p]))
\]
(54)
Using the decomposition
\[
\prod_{p \in \mathbb{Z}} \mathbb{C}[2p] \cong \prod_{p \in \mathbb{N}} \mathbb{C}[2p] \oplus \bigoplus_{p=1}^\infty \mathbb{C}[-2p]
\]
and the fact that \( ku \) is connective we get the decomposition of the fibre of (54) into the sum of the fibres of the morphisms of spectra
\[
ku \rightarrow H(\iota(\prod_{p \in \mathbb{N}} \mathbb{C}[2p])) , \quad 0 \rightarrow H(\iota(\prod_{p=1}^\infty \mathbb{C}[-2p]))
\]
The fibre of the first morphism is equivalent to \( \Sigma^{-1} ku\mathbb{C}/\mathbb{Z} \), and the fibre of the second is \( \Sigma^{-1} H(\iota(\prod_{p=1}^\infty \mathbb{C}[-2p])) \). The assertion of the Lemma now follows. \( \square \)

The curvature morphism \( R \) in (45) decomposes into a product of components \( R(p) \), \( p \in \mathbb{Z} \), according to the product decomposition of \( \geq^k DD^- \), see (42). Similarly we decompose \( \text{reg}_X \) into a product of components \( \text{reg}_X(p) \) for \( p \in \mathbb{Z} \) according to the decomposition of \( DD^- \), see Definition 2.17. Note that \( \pi_d(DD^- (d)(X)) \cong \Omega^d_{\text{cl}}(X) \).
Lemma 2.31.

1. If \( \dim(X) = d \) and \( x \in \hat{K}_d^{-d}(\mathcal{C}^\infty(X)) \), then we have
   \[
   R(p)(x) = \begin{cases} 
   0 & p \neq d \\
   \text{reg}_X(d)(I(x)) \in \Omega^d_d(X) & p = d 
   \end{cases} .
   \]

2. If \( \dim(X) \leq d - 1 \), then we have an isomorphism
   \[
   \hat{K}_d^{-d,\text{flat}}(\mathcal{C}^\infty(X)) \sim \hat{K}_d^{-d}(\mathcal{C}^\infty(X)) .
   \]

Proof. Let \( x \in \hat{K}_d^{-d}(\mathcal{C}^\infty(X)) \). For \( p \in \mathbb{Z} \) the component \( R(p)(x) \) of the curvature of \( x \) is represented by a cycle in \( \sigma_{\geq \max(p,2p-d)} \Omega^{2p-d}(X) \). If it is non-zero, then we get the inequalities
   \[
   2p - d \leq d , \quad p \leq 2p - d
   \]
   which contradict each other except if \( d = p \). This shows the first assertion of the Lemma. The second assertion follows from the first since \( \Omega^d(X) = 0 \) if \( \dim(X) \leq d - 1 \).

In view of the Definition 2.26 of the differential regulator \( \text{reg}_X \) as the map associated to the map of pull-back squares \( (17) \) \( \text{reg}_X \) is compatible with the curvature maps, i.e. the following diagram of spectra commutes:

\[
\begin{array}{ccc}
\hat{K}^{k}(\mathcal{C}^\infty(X)) & \xrightarrow{R} & \geq k\text{DD}^{-}(X) \\
\downarrow \text{reg}_X & & \downarrow e \\
\hat{ku}^{k}(X) & \xrightarrow{R} & \geq k\text{DD}^{\text{per}}(X)
\end{array}
\]

In particular, \( \text{reg}_X \) maps flat classes to flat classes.

Corollary 2.32. If \( \dim(X) \leq d \), then by composing \( \text{reg}_X \) with the inverse of \( I \) from Lemma 2.28 we get a map

\[
\text{reg}_X : K_d(\mathcal{C}^\infty(X)) \to \hat{ku}^{-d,-d}(X) . \tag{55}
\]

If \( \dim(X) \leq d - 1 \), then \( \text{reg}_X \) maps to the subgroup of flat classes.

We can now finish the definition of the transformation \( \sigma_d \).

Definition 2.33. If \( \dim(X) \leq d - 1 \), then we define \( \sigma_d \) as the composition

\[
\sigma_d : K_d(\mathcal{C}^\infty(X)) \xrightarrow{\text{reg}_X} \hat{ku}^{-d,-d}_{\text{flat}}(X) \xrightarrow{\text{pr}} \text{ku}\mathbb{C}/\mathbb{Z}^{-d-1}(X) , \tag{56}
\]

where the second map \( \text{pr} \) is the projection onto the first summand in the r.h.s. of \( (52) \).

This finishes the proof of Theorem 1.1. □
2.6 Restriction to relative $K$-theory

In this subsection we derive an explicit formula for the restriction of $\sigma_d$ defined in 2.33 to topologically trivial classes. The result will be formulated as Proposition 2.34. It will be used in the proof of Theorem 1.6.

The statement of Proposition 2.34 involves some notation, in particular a relative version of the regulator, which we introduce in the following. We start with defining the sheaf of chain complexes

$$ DD := \prod_{p \in \mathbb{Z}} DD(p) \in \text{Sh}_{\text{Ch}}(\text{Mf}) , \quad DD(p) := (\sigma \leq_p \Omega)[2p] . $$

(57)

It fits into exact sequence of sheaves of chain complexes

$$ 0 \to DD_- \to DD^{\text{per}} \to DD[2] \to 0 . $$

(58)

The second map in this sequence is induced by the family of maps of chain complexes

$$ DD^{\text{per}}(p) \cong \Omega[2p] \xrightarrow{1} (\sigma < p \Omega)[2p] \cong ((\sigma \leq_{p-1} \Omega)[2(p-1)][2] \cong DD(p-1)[2], $$

(59)

where the arrow marked by ! is an in (49). By [BNV13, Lemma 7.12] the object $DD := \iota(DD)$ is a sheaf with values in $\text{Ch}[W^{-1}]$. We define $DD := H(DD) \in \text{Sh}_{\text{Sp}}(\text{Mf})$. If we apply $H \circ \iota$ to the sequence (58), then we get the fibre sequence of spectra

$$ \Sigma DD \to DD_- \xrightarrow{1} DD^{\text{per}} \to \Sigma^2 DD . $$

(60)

Since by Lemma 2.23 the marked map is equivalent to the homotopification map we get an equivalence

$$ \Sigma DD \simeq A(DD^\sim) , $$

(61)

where $A$ is defined in (32), and an equivalence of (60) with the homotopification sequence of $DD^\sim$.

If $X$ is a compact manifold, then by Lemma 2.4. 4. we have an equivalence

$$ i^*_X \circ A \simeq A \circ i^*_X . $$

(62)

This implies the equivalence of spectra

$$ A(i^*_X DD^\sim)(\ast) \simeq i^*_X A(DD^\sim)(\ast) \simeq A(DD^\sim)(X) \simeq \Sigma DD(X) $$

(63)

which will frequently use below.

For a compact manifold $X$ we consider the map of homotopification fibre sequences (32) induced by the regulator map $\text{reg}_X : K_{C_\infty(X)} \to i^*_X DD^\sim$ (see (31)) and the evaluation at $\ast$:

$$
\begin{array}{cccccc}
K^{\text{rel}}(C^\infty(X)) & \xrightarrow{\partial} & K(C^\infty(X)) & \xrightarrow{\text{reg}_X^{\text{rel}}} & K^{\text{top}}(C^\infty(X)) & \xrightarrow{\text{reg}_X^{\text{rel}}} & \Sigma K^{\text{rel}}(C^\infty(X)) \\
\downarrow \text{reg}_X^{\text{rel}} & & \downarrow \text{reg}_X & & \downarrow \text{reg}_X^{\text{top}} & & \downarrow \text{reg}_X^{\text{rel}} \\
\Sigma DD(X) & \to & DD^\sim(X) & \to & DD^{\text{per}}(X) & \to & \Sigma^2 DD(X)
\end{array}
$$

(64)
In order to write the first and the last spectrum in the lower sequence in this way we have used (63). The upper sequence in (64) contains the map \( \partial : K^{rel}(C^\infty(X)) \to K(C^\infty(X)) \). In this special case we use \( \partial \) and not the generic notation \( \alpha \) as in (32). Furthermore, the left vertical morphism is by definition the relative version of the regulator

\[
\text{reg}^{rel}_X := A(\text{reg}_X)(*)
\]  

We now assume that \( \dim(X) \leq d \). In order to state Proposition 2.34 we use the following notation:

1. \( \partial : K^{rel}(C^\infty(X)) \to K(C^\infty(X)) \), see (64).
2. \( \text{reg}_X : K_{d+1}(C^\infty(X)) \to \hat{k}u^{-d-1,d-1}(X) \), see (55).
3. \( \text{reg}^{rel}_X : K^{rel}(C^\infty(X)) \to A(DD^-)(X) \), see (65).
4. \( c := A(e) : A(i_X^* \geq -d-1 DD^-) \to A(i_X^* \geq -d-1 DD^\text{per}) \), where \( e \) is as in (47).
5. If \( X \) is compact and \( \dim(X) \leq d \), then we have an isomorphism

\[
\kappa : \pi_{d+1}(A(\geq -d-1 DD^-)(X)) \to \pi_{d+1}(A(DD^-)(X))
\]

(see Lemma 2.35).
6. \( a_{\hat{k}u} : A(i_X^* \geq -d-1 DD^\text{per}) \to i_X^* \hat{k}u^{-d-1} \) is the structure map of the Hopkins-Singer differential cohomology theory associated to \( \hat{k}u \), see (68) below.

Note that we consider classes in degree \( d+1 \) in order to be compatible with what is used in Subsection 3.6.

**Proposition 2.34.** If \( X \) is compact and \( \dim(X) \leq d \), then we have the equality

\[
\text{reg}_X \circ \partial = a_{\hat{k}u} \circ c \circ \kappa^{-1} \circ \text{reg}^{rel}_X : K_{d+1}^{rel}(C^\infty(X)) \to \hat{k}u^{-d-1,d-1}(X) .
\]

**Proof.**

**Lemma 2.35.** If \( X \) is compact and \( \dim(X) \leq d \), then we have an isomorphism

\[
\kappa : \pi_{d+1}(A(\geq -d-1 DD^-)(X)) \to \pi_{d+1}(A(DD^-)(X)) .
\]

**Proof.** We use the decompositions of \( DD^- \) and \( \geq -d-1 DD^- \) into products of components indexed by \( p \in \mathbb{Z} \). Thus we can fix \( p \in \mathbb{Z} \) and must show that

\[
\kappa(p) : \pi_{d+1}(A(\geq -d-1 DD^-)(p))(X)) \to \pi_{d+1}(A(DD^-)(p))(X))
\]

is an isomorphism. The map \( \kappa(p) \) is induced by the inclusion map

\[
\sigma^{\geq -d-1}(\sigma^{\geq 0} \Omega)[2p] \to (\sigma^{\geq 0} \Omega)[2p] ,
\]

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application of $\mathcal{A} \circ H \circ \iota$, evaluation at $X$, and $\pi_{d+1}$. We now use that for $k, \ell \in \mathbb{Z}$
\[ \mathcal{A}(\iota(\sigma^{\geq k}\Omega)[\ell]) \simeq \iota((\sigma^{< k}\Omega)[\ell - 1]) , \]
see [BNV13 Lemma 4.2, 2]. So the map $\kappa(p)$ is induced by the projection
\[ H^{2p-d-2}(\sigma^{< \max(2p-d-1,p)}\Omega(X)) \rightarrow H^{2p-d-2}(\sigma^{< p}\Omega(X)) . \]
Both sides are zero if $2p - d - 2 > \dim(X)$. So it remains to consider the case where
$2p - d - 2 \leq \dim(X) \leq d$, i.e. $p - 1 \leq d$. This condition implies $2p - d - 1 \leq p$ and hence $\max(2p - d - 1, p) = p$. In this case (66) is induced by an isomorphism of chain complexes.

Since we assume that $X$ is compact we can freely employ the equivalence (62). The functor $\mathcal{A}$ preserves pull-back diagrams and annihilates homotopy invariant sheaves. We apply the functor $\mathcal{A}$ to the map of pull-back diagrams (47) and get the map of pull-back diagrams
\[ K_{C^\infty(X)}^{rel} \xrightarrow{\mathcal{A}(\hat{k})} \Sigma i_X^* \mathcal{D}D \leftarrow i_X^* \mathcal{A}(\geq k \mathcal{D}D^-) . \]
Its evaluation at $*$ defines the morphism
\[ \mathcal{A}(\hat{\mathcal{K}}^k_X) : \mathcal{A}(\hat{K}^k_{C^\infty(X)})(*) \rightarrow \mathcal{A}(\hat{k})_X(X) . \]
The following diagram commutes by the naturality of the map $\alpha$ in (32) and the definition of $\partial$ as $\alpha$ in the case of $K_{C^\infty(X)}$:
\[ K_{C^\infty(X)}^{rel} \xrightarrow{\mathcal{A}(I)} \mathcal{A}(K^k_{C^\infty(X)}) \xrightarrow{\mathcal{A}(\hat{\mathcal{K}}^k_X)} \mathcal{A}(i_X^* \mathcal{K}^k_{C^\infty(X)}) \xrightarrow{\iota_X^* \mathcal{A}(\geq k \mathcal{D}D^\perp)} . \]

**Lemma 2.36.** We assume that $X$ is compact and $\dim(X) \leq d$.

1. If $d + 1 + k \leq 0$, then the map
\[ \mathcal{A}(I) : \pi_{d+1}(\mathcal{A}(K^k_{C^\infty(X)})(*)) \rightarrow K_{d+1}^{rel}(C^\infty(X)) \]
is surjective.

2. If $\dim(X) \leq d$ and $k + d + 1 = 0$, then (69) is an isomorphism.
Proof. We have a pull-back diagram

\[
\begin{array}{ccc}
\mathcal{A}(\hat{K}^{k}_{C_{\infty}(X)}) & \xrightarrow{A(R)} & i_{X}^{*}\mathcal{A}(\geq k DD^-) \\
\downarrow A(I) & & \downarrow \\
K_{rel}^{C_{\infty}(X)} & \xrightarrow{A(\text{reg}_{X})} & i_{X}^{*}\mathcal{A}(DD^-)
\end{array}
\]  

(70)

As in the proof of Lemma 2.35 a model of the map

\[ A(\geq k DD^-) \rightarrow A(DD^-) \]

can be obtained by applying \( H \circ \iota \) to the product of projection maps

\[ (\sigma^{< \max(2p+k,p)} \Omega)[2p-1] \rightarrow (\sigma^{<p} \Omega)[2p-1]. \]

(71)

It follows that

\[ H^{2p-d-2}(\sigma^{< \max(2p+k,p)} \Omega)(X)) \rightarrow H^{2p-d-2}(\sigma^{<p} \Omega)(X)) \]

is surjective as long as

1. \( 2p + k \leq p \) or
2. \( 2p + k > p \) and \( 2p - d - 2 \neq p - 1. \)

If \( k + d + 1 \leq 0 \), then this condition on \( p \) is true for all \( p \in \mathbb{Z} \). We conclude that \( A(I) \) is surjective using the exact sequences associated to the pull-back (70) similarly as in the proof of Lemma 2.28.

The second assertion follows from the fact, that \( \kappa \) in Lemma 2.35 is an isomorphism. \( \square \)

It follows from the commutativity of the diagram (68) and the definition (55) of \( \text{reg}_{X} \) that the following two compositions coincide:

\[ \text{reg}_{X} \circ \partial \circ A(I) = a_{ku} \circ A(\text{reg}_{X}) : \pi_{d+1}(\mathcal{A}(\hat{K}^{-d-1}_{C_{\infty}(X)})(\ast)) \rightarrow \hat{k u}^{-d-1,-d-1}(X). \]

From (67) we conclude that

\[ A(\text{reg}_{X}) = c \circ A(R) : \pi_{d+1}(\mathcal{A}(\hat{K}^{-d-1}_{C_{\infty}(X)})(\ast)) \rightarrow \pi_{d+1}(\mathcal{A}(\geq -d-1 DD^{	ext{per}})(X)) , \]

where \( R \) is as in (45) and \( c \) is defined in (67). For \( d \leq \dim(X) \)

\[ \kappa : \pi_{d+1}(i_{X}^{*}\mathcal{A}(\geq -d-1 DD^-)(\ast)) \rightarrow \pi_{d+1}(\mathcal{A}(DD^-)(X)) \]

is an isomorphism. It follows from (70) and Lemma 2.36 1. that

\[ A(R) = \kappa^{-1} \circ \text{reg}_{X}^{rel} \circ A(I) : \pi_{d+1}(\mathcal{A}(\hat{K}^{-d-1}_{C_{\infty}(X)})(\ast)) \rightarrow \pi_{d+1}(\mathcal{A}(DD^-)(X)) . \]

The assertion of the proposition now follows from a combination of these equalities and the surjectivity of \( A(I) \). \( \square \)
2.7 Explicit calculations of $\sigma_d$

In this subsection we use the homotopy formula for $\hat{\text{ku}}^{-d,-d}$ in order to provide a formula for $\hat{\text{reg}}_X(x)$ for certain topologically trivial classes $x \in K_d(C^\infty(X))$. We assume that $X$ is a manifold of dimension $\dim(X) = d - 1$. We consider a class $\tilde{x} \in K_d(C^\infty(I \times X))$ which has the property that $\tilde{x}_{|\{0\} \times X} = 0$. We define

$$x := \tilde{x}_{|\{1\} \times X} \in K_d(C^\infty(X)) .$$

By construction the class $x$ is topologically trivial, i.e. it belongs to the kernel of $K_d(C^\infty(X)) \to K_{d \text{top}}(C^\infty(X))$.

Since the degree of $\tilde{x}$ and the dimension of $I \times X$ match, by Lemma 2.31 the only non-trivial component of the regulator of $\tilde{x}$ is

$$\text{reg}_{I \times X}(d)(\tilde{x}) \in \Omega^d_{cl}(I \times X) .$$

Furthermore we have a map

$$i_d : \Omega^{d-1}_{cl}(X) \to H^{d-1}(\sigma^{< d} \Omega(X)) \to \pi_d(\mathcal{A}(\tilde{\sigma}^{-d} \text{DD}_{\text{per}})(X)) .$$

Proposition 2.37. We have

$$\hat{\text{reg}}_X(x) = a_{\text{ku}}(i_d \int_I \text{reg}_{I \times X}(d)(\tilde{x})) .$$

Proof. By the homotopy formula [BNV13, (27)] we have

$$\hat{\text{reg}}_X(x) = a_{\text{ku}}(\int_I R_{\text{ku}}(\hat{\text{reg}}_{I \times X}(\tilde{x}))) ,$$

where $R_{\text{ku}}$ is the curvature morphism of the differential cohomology theory $\hat{\text{ku}}^{-d}$ as in (46). We use the additional subscript in order to distinguish it from the curvature morphism $R$ for $\hat{K}_{C^\infty(X)}^{-d}$ in [45] which appears in the formula below, too. We have

$$R_{\text{ku}}(\hat{\text{reg}}_{I \times X}(\tilde{x})) = e(R(\tilde{x})) ,$$

where $\tilde{x} \in \hat{K}_{\text{d}}^{-d}(C^\infty(I \times X))$ is a lift of $\tilde{x}$ (under isomorphism given in Lemma 2.28), and where $e$ is as in (47). This map $e$ in particular preserves the decomposition into the $p$-components. By Lemma 2.31 the only nontrivial component of $R(\tilde{x})$ is the component

$$R(d)(\hat{\text{reg}}_{I \times X}(\tilde{x})) = \text{reg}_{I \times X}(d)(\tilde{x}) .$$

So our final formula is

$$\hat{\text{reg}}_X(x) = a_{\text{ku}}(i_d \int_I \text{reg}_{I \times X}(d)(\tilde{x})) .$$

$\square$
Example 2.38. We now perform some explicit calculations:

1. We consider $X = S^1$. Let $u : S^1 \to \mathbb{C}$ be the inclusion. We consider $u$ as a unit in $C^\infty(S^1)$ and $\iota(u) \in K_1(C^\infty(S^1))$. Note that

$$\pi_1(\text{DD}^-(S^1)) \cong \pi_1(\text{DD}^-(1)(S^1)) \cong \Omega^1(S^1).$$

Under this identification have

$$\text{reg}_{S^1}(\iota(u)) = \frac{1}{2\pi i} \frac{du}{u}.$$  

2. Next we consider the inclusion $t : \mathbb{R}_+ \to \mathbb{C}$ as a unit in $C^\infty(\mathbb{R}_+)$. We obtain

$$\text{reg}_{\mathbb{R}_+}(1)(\iota(t)) = \frac{1}{2\pi i} \frac{dt}{t}.$$  

3. We now consider the inclusion $z : \mathbb{C}^* \to \mathbb{C}$. We write $z = u(z)t(z)$ with $u := z/|z|^{-1}$ and $t := |z|$. Then we get by naturality and additivity of the regulator

$$\text{reg}_{\mathbb{C}^*}(1)(\iota(z)) = \frac{1}{2\pi i} \left( \frac{d(z/|z|^{-1})}{z/|z|^{-1}} + \frac{d|z|}{|z|} \right) = \frac{1}{2\pi i} \frac{dz}{z}.$$  

4. We consider the unit $\exp(z) = \exp^* z \in C^\infty(\mathbb{C})$. We get

$$\text{reg}_{\mathbb{C}}(1)(\iota(\exp(z))) = \exp^*(\text{reg}_{\mathbb{C}^*}(1)(\iota(z))) = \frac{1}{2\pi i} \frac{dz}{z}.$$  

5. Let now $X$ be a smooth manifold and $f \in C^\infty(X)$. Then we have

$$\text{reg}_X(1)(\iota(\exp(f))) = f^*(\text{reg}_{\mathbb{C}}(1)(\iota(\exp(z)))) = \frac{1}{2\pi i} df.$$  

6. The regulator $\text{reg}_X : K_*(C^\infty(X)) \to \pi_*(\text{DD}^-(X))$ is given by the composition of the Goodwillie-Jones Chern character and the map $[27]$. Since $C^\infty(X)$ is commutative both maps are in fact multiplicative, where the ring structure on $\text{DD}^-$ is induced by the obvious bigraded differential algebra structure on $\prod_{p \in \mathbb{Z}}(\sigma_{\geq p}\Omega)[2p]$. Now assume that $\dim(X) = d - 1$ and consider a collection $u_2, \ldots, u_d$ of units. Let $t \in I$ be the coordinate. On $I \times X$ we consider the collection of $d$ units

$$\exp(tf), u_2, \ldots, u_d$$

and define

$$\tilde{x} := \iota(\exp(tf)) \cup \iota(u_2) \cup \cdots \cup \iota(u_d) \in K_d(C^\infty(I \times X)).$$
By multiplicativity of the regulator and \((72)\) we have
\[
\text{reg}_{I \times \chi}(d(\exp(tf)) \cup \iota(u_2) \cup \cdots \cup \iota(u_d)) = \frac{1}{(2\pi i)^d} (fdt + tdf) \wedge \frac{du_2}{u_2} \wedge \cdots \wedge \frac{du_d}{u_d}
\]
(and this is the only non-trivial component). From Proposition 2.37 we conclude that
\[
\text{reg}_{X}(\iota(\exp(f)) \cup \iota(u_2) \cup \cdots \cup \iota(u_d)) = a(i_d(\frac{1}{(2\pi i)^d} f \frac{du_2}{u_2} \wedge \cdots \wedge \frac{du_d}{u_d})) . \tag{73}
\]

\[\square\]

3 An index theorem

3.1 The index pairing

In this subsection we introduce the pairing between Dirac operators and the Hopkins-Singer version of differential periodic complex \(K\)-theory. This pairing refines the pairing between periodic complex \(K\)-theory and the \(K\)-homology. If the Dirac operator comes from a \(\text{Spin}^c\)-structure, then our pairing is a special case of the integration in differential cohomology. In any case, the pairing has a simple description in terms of standard constructions of local index theory.

First we introduce the Hopkins-Singer version \(\hat{K}U_{k,*}(-)\) of differential periodic complex \(K\)-theory \(KU\) associated to an integer \(k \in \mathbb{Z}\). By
\[
\text{ch}_{\text{cw}}^{\text{per}} : KU \to H(\iota(\prod_{p \in \mathbb{Z}} \mathbb{C}[2p]))
\]
we denote the usual Chern character. We will use the same symbol also in order to denote the composition of morphisms of sheaves of spectra
\[
\text{ch}_{\text{cw}}^{\text{per}} : KU \xrightarrow{\text{ch}_{\text{cw}}^{\text{per}}} H(\iota(\prod_{p \in \mathbb{Z}} \mathbb{C}[2p])) \overset{\text{(53)}}{=} \text{DD}_{\text{per}}. \tag{53}
\]

Remark 3.1. The Chern character \(\text{ch}_{\text{cw}}\) discussed in Lemma 2.24 is equivalent to the composition
\[
\text{ku} \to KU \xrightarrow{\text{ch}_{\text{cw}}^{\text{per}}} H(\iota(\prod_{p \in \mathbb{Z}} \mathbb{C}[2p])) ,
\]
where the first map
\[
\text{ku} \to KU \tag{74}
\]
is the connective covering map of \(KU\).
Similarly as in the connective case (Definition 2.25, (46)) we define the differential extension of $\text{KU}$ as follows:

**Definition 3.2.** For $k \in \mathbb{Z}$ we define the Hopkins-Singer differential cohomology theory associated to $\text{KU}$ as the sheaf of spectra $\hat{\text{KU}}^k \in \text{ShSp}(\text{Mf})$ given by the following pull-back

\[
\begin{array}{ccc}
\hat{\text{KU}}^k & \xrightarrow{R} & \geq k \text{DD}^{\text{per}} \\
\downarrow I \downarrow & & \downarrow \downarrow \\
\text{KU} & \xrightarrow{\text{ch}_{\text{per}}} & \text{DD}^{\text{per}}
\end{array}
\]

We further define the differential periodic complex $K$-theory groups of a manifold $X$ by

\[
\hat{\text{KU}}^{k,*}(X) := \pi_-(\hat{\text{KU}}^k(X))
\]

(compare with Definition 2.26).

Since $H^0(\geq 0 \text{DD}^{\text{per}}(X)) \to H^0(\text{DD}^{\text{per}}(X))$ is surjective, the differential periodic complex $K$-theory in degree zero fits into the exact sequence

\[
\text{KU}^{-1}(X) \xrightarrow{\text{ch}_{\text{per}}} \text{DD}^{\text{per}}(X)^{-1}/\text{im}(d) \xrightarrow{a_{\text{KU}}} \hat{\text{KU}}^{0,0}(X) \xrightarrow{I} \text{KU}^0(X) \to 0.
\]

(75)

This can be shown by similar arguments as in the proof of Lemma 2.28. This exact sequence is one of the basic features of differential periodic complex $K$-theory, see e.g. [BS09, Prop. 2.20].

By [BS10] for a compact manifold $X$ the group $\hat{\text{KU}}^{0,0}(X)$ is canonically isomorphic to the differential $K$-theory groups defined using geometric models [BS09], [SS10]. In these models geometric vector bundles are cycles for differential $K$-theory classes. Recall that a geometric $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V := (V, h, \nabla)$ is a triple consisting of a $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle $V \to X$, a hermitean metric $h$ on $V$ such that the even and odd summands are orthogonal, and a connection $\nabla$ which preserves $h$ and the grading. In the geometric models for $\hat{\text{KU}}^{0,0}(X)$ a geometric $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V := (V, h, \nabla)$ tautologically represents a class

\[
[V] \in \hat{\text{KU}}^{0,0}(X).
\]

We refer to [BNV13, Sec. 6.1] for an alternative construction of this class in terms of a cycle map.

We now assume that $X$ is a closed Riemannian manifold of odd dimension $d$. We further assume that we are given a generalized Dirac operator $\mathbf{D}$ on $X$. By definition, $\mathbf{D}$ is the Dirac operator associated to a Dirac bundle, see e.g. [Bun09, Sec. 3.1].
Remark 3.3. A generalized Dirac operator provides a $K$-homology class which can be paired with $K$-theory classes on $X$. The basic idea of the following Lemma is that the Dirac operator as a geometric object provides a sort of differential refinement of its $K$-homology class which can be paired with differential $K$-theory classes. The map $\rho_D$ defined in Proposition 3.4 below only captures the secondary information contained in this pairing. Its value on a differential $K$-theory class can be considered as the reduced $\eta$-invariant of the Dirac operator twisted with this class. A very similar construction has been used in order to define the intrinsic universal $\eta$ invariant in [Bun11].

Proposition 3.4. We have a canonical evaluation map

$$\rho_D : \widehat{KU}^{0,0}(X) \to \mathbb{C}/\mathbb{Z}.$$  

Proof. Let $x \in \widehat{KU}^{0,0}(X)$. In view of the sequence \ref{eq:sevenfive} we can choose a geometric $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V := (V, h, \nabla)$ and a form $\gamma \in DD^{per}(X)^{-1}/\text{im}(d)$ such that the following identity holds true in $\widehat{KU}^{0,0}(X)$:

$$x = [V] + a_{KU}(\gamma).$$

We need the following standard constructions from local index theory:

1. We form the twist $\mathcal{D} \otimes V$ of the Dirac operator by $V$ (see [Bun09, Sec. 3.1] for details if necessary).

2. We let $\xi(\mathcal{D} \otimes V) \in \mathbb{R}/\mathbb{Z}$ denote the reduced $\eta$-invariant of $\mathcal{D} \otimes V$ given by

$$\xi(\mathcal{D} \otimes V) := \left[ \frac{\eta(\mathcal{D} \otimes V) + \dim(\ker(\mathcal{D} \otimes V))}{2} \right], \quad (76)$$

where $\eta(\mathcal{D} \otimes V)$ is the Atiyah-Patodi-Singer $\eta$-invariant introduced in [APS75].

3. We let $\hat{A}(\mathcal{D}) \in \prod_{p \in \mathbb{Z}}(\Omega(X, \Lambda)[2p])^0_{cl}$ denote the local index form associated to $\mathcal{D}$ where $\Lambda$ denotes the orientation twist of $X$.

Remark 3.5. The local index form has the following explicit description. Locally on $X$ we can write $\mathcal{D} = \mathcal{D}_{spin} \otimes E$ for the spin Dirac operator $\mathcal{D}_{spin}$ and a geometric $\mathbb{Z}/2\mathbb{Z}$-graded twisting bundle $E = (E, h^E, \nabla^E)$. If we can write $\mathcal{D}$ in this way, then

$$\hat{A}(\mathcal{D}) = ([\hat{A}(\nabla^{LC}) \wedge \text{ch}(\nabla^E)]_{2p})_{p \in \mathbb{Z}} \in \prod_{p \in \mathbb{Z}}(\Omega(X, \Lambda)[2p])^0_{cl},$$

where $\nabla^{LC}$ is the Levi-Civita connection of $X$, $[\omega]_{2p}$ denotes the degree-2$p$-component of the inhomogeneous even form $\omega$, and $\hat{A}(\nabla)$ and $\text{ch}(\nabla)$ are the usual characteristic forms defined in terms of the curvature of the connections (including the $2\pi i$-factors), see [Bun09, Sec. 4.3] for explicit formulas.
The following observation will make it unnecessary to use the explicit formula for the index density. This fact will be particularly helpful in the proof of Lemma 3.19 below. We define the integral

$$\int_X : \prod_{p \in \mathbb{Z}} \Omega(X, \Lambda)[2p] \rightarrow \mathbb{C}, \quad \int_X (\omega(p))_{p \in \mathbb{Z}} := \sum_{p \in \mathbb{Z}} \int_X [\omega(p)]_{\dim(X)} . \quad (77)$$

It induces an evaluation of cohomology classes which we will denote by the same symbol. By the Atiyah-Singer index theorem we can calculate for every class $u \in KU^{-1}(X)$ the index pairing by

$$\langle [u], [\mathcal{D}] \rangle = \int_X [\hat{A}(\mathcal{D})] \cup \iota_{d+1} \text{ch}_{\text{per}}(u) \in \mathbb{Z} , \quad (78)$$

where $\iota_{d+1}$ is the shift isomorphism defined in (36).

Using the integral (77) we now define

$$\rho_{\mathcal{D}}(x) := \xi(\mathcal{D} \otimes V) + \left[ \int_X \hat{A}(\mathcal{D}) \wedge \iota_{d+1} \gamma \right] \in \mathbb{C}/\mathbb{Z} . \quad (79)$$

We must show that $\rho_{\mathcal{D}}$ is a well-defined homomorphism.

1. First observe that the right-hand side of (79) does not depend on the choice of $\gamma$. Indeed, by (75) two choices differ by a closed form representing an element in the image of $\text{ch}_{\text{per}} : KU^{-1}(X) \rightarrow DD_{\text{per}}(X)^{-1}/\text{im}(d)$, and the integral of the product of those elements with $\hat{A}(\mathcal{D})$ is an integer by the Atiyah-Singer index theorem, see Remark 3.5.

2. We observe that the right-hand side of (79) is invariant under stabilization by bundles which admit an odd $\mathbb{Z}/2\mathbb{Z}$ symmetry.

3. Next we observe, using the variation formula for the classes $[V]$ (the homotopy formula for $\widehat{KU}^{0,0}$) and $\xi(\mathcal{D} \otimes V)$, that the right-hand side of (79) does not depend on the choice of the geometry of $V$.

4. If we choose a different bundle $V'$ and form $\gamma'$, then after stabilization we can assume that $V \cong V'$ as graded bundles. Therefore the right-hand side of (79) does not depend on the choice of $V$.

5. Finally we observe that $\rho_{\mathcal{D}}$ is a homomorphism.
Using the periodicity $\widehat{KU}^{0,0}(X) \cong \widehat{KU}^{-n,-n}(X)$ for all $n \in 2\mathbb{Z}$ we also have evaluation maps

$$\rho_\mathcal{P} : \widehat{KU}^{-n,-n}(X) \to \mathbb{C}/\mathbb{Z} \, .$$

(80)

**Remark 3.6.** Assume that $X$ is a Riemannian spin manifold of odd dimension $d$. Then the map $p : X \to \ast$ is differentially $K$-oriented and we have an integration

$$\hat{p}_! : \widehat{KU}^{0,0}(X) \to \widehat{KU}^{-d,-d}(\ast) \cong \mathbb{C}/\mathbb{Z} \, .$$

We refer to [BS09, Sec. 3] and [Bun, Sec. 4.10 and 4.11] for details on the integration in differential cohomology.

Let us now assume that $\mathcal{P} = \mathcal{P}_{\text{spin}} \otimes \mathcal{E}$ for some twist $\mathcal{E} = (E, h^E, \nabla^E)$. From [BS09, Cor. 5.5] we conclude that $\rho_\mathcal{P}$ can be expressed in terms of the integration $\hat{p}_!$ in differential $K$-theory as follows

$$\rho_\mathcal{P}(x) = \hat{p}_!(\mathcal{E} \cup x) \, , \quad x \in \widehat{KU}^{0,0}(X) \, .$$

The spin structure on $X$ provides the underlying topological $K$-orientation of $X$ given by the fundamental class $[\mathcal{P}_{\text{spin}}] \in \text{KU}_d(X)$. The restriction of $\rho_\mathcal{P}$ to the flat subgroup corresponds under the identification

$$\widehat{KU}^{0,0}_{\text{flat}}(X) \cong \text{KU}/\mathbb{Z}^{-1}(X)$$

(81)

(this is the analog of Lemma 2.30) to the evaluation pairing

$$\langle - \cup [E], [\mathcal{P}_{\text{spin}}] \rangle : \text{KU}/\mathbb{Z}^{-1}(X) \to \text{KU}/\mathbb{Z}^{-1}(X) \to \text{KU}/\mathbb{Z}^{-d-1}(\ast) \cong \mathbb{C}/\mathbb{Z}$$

where the first map uses the KU-module structure of KU$/\mathbb{Z}$.

**Remark 3.7.** In this remark we explain the relation between the evaluation map $\rho_\mathcal{P}$ and the index theorem for flat vector bundles by Atiyah-Patodi-Singer [APS76, Thm. 5.3]. Let $\mathcal{V}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded flat geometric bundle of virtual dimension zero. It represents a class $[\mathcal{V}] \in \widehat{KU}^{0,0}_{\text{flat}}(X) \cong \text{KU}/\mathbb{Z}^{-1}(X)$. In this case

$$\rho_\mathcal{P}([\mathcal{V}]) = \xi(\mathcal{P} \otimes \mathcal{V})$$

is exactly the analytic index of the flat bundle introduced by Atiyah-Patodi-Singer. Their index theorem for flat bundles states that this analytic index is equal to the pairing of the class $[\mathcal{V}]$ with the $K$-homology class of $\mathcal{P}$. This also follows from the last assertion in Remark 3.6. 

$\square$
Example 3.8. We consider $S^1$ as a spin manifold with the standard metric of length 1 and with the non-bounding spin structure. The spinor bundle is one-dimensional and can be trivialized such that $\mathcal{D}_{\text{spin}} = i \partial_t$. Assume now that $L$ is a geometric line bundle with holonomy $v \in U(1)$. Then we can trivialize $L$ such that its connection is given by $\nabla^L = d - \log(v) dt$. We get

$$\mathcal{D}_{\text{spin}} \otimes L = i(\partial_t - \log(v)) .$$

Its spectrum is $\{2\pi n - \log v \mid n \in \mathbb{Z}\}$ with multiplicity 1. For $v \neq 1$ we get by an explicit calculation

$$\eta(\mathcal{D}_{\text{spin}} \otimes L) = 1 - \frac{\log(v)}{\pi i} ,$$

where the sheet of the logarithm is chosen such that $\frac{\log(v)}{\pi i} \in (0, 2)$. Using (76) and (79) we get

$$\rho_{\mathcal{D}_{\text{spin}}}([L]) = \left[ \frac{1}{2} - \frac{\log(v)}{2\pi i} \right]_{\mathbb{C}/\mathbb{Z}}$$

because in this case we can take $\gamma = 0$. This formula holds true also for $v = 1$.

If $L$ is trivial, then $[L] = 1$ and we have $\rho_{\mathcal{D}_{\text{spin}}}(1) = \left[ \frac{1}{2} \right]$. We have an isomorphism $K\tilde{U}_{\text{flat}}^0(S^1) \cong \mathbb{C}/\mathbb{Z}$ which maps $[L] - 1$ to $\frac{\log v}{2\pi i}$. With this identification the restriction of the evaluation map to the flat subgroup is given by the homomorphism

$$\rho_{\mathcal{D}_{\text{spin}}} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z} , \quad [z] \mapsto [-z] .$$

3.2 Fredholm modules

We consider a closed Riemannian manifold $X$ of odd dimension $d$ with a generalized Dirac operator $\mathcal{D}$ associated to a Dirac bundle $E$. The Dirac operator $\mathcal{D}$ gives rise to a $d + 1$-summable Fredholm module $(H, P)$ over $C^\infty(X)$ as follows (see [Con85]):

1. The Hilbert space of the Fredholm module is $H := L^2(X, E)$. The algebra $C^\infty(X)$ acts on $H$ in the usual way by multiplication operators.

2. The operator $P \in B(H)$ is the orthogonal projection $P^+$ onto the positive eigenspace of $\mathcal{D}$

3. The condition that $(H, P)$ is $d + 1$-summable means that

$$[P, f] \in \mathcal{L}^{d+1}(H) ,$$

for all $f \in C^\infty(X)$, where $\mathcal{L}^{d+1}(H)$ denotes the $d + 1$'th Schatten class.
Remark 3.9. In some references odd Fredholm modules are denoted by \((H,F)\), where \(F \in B(H)\) is a selfadjoint involution such that \([F,f] \in \mathcal{L}^{d+1}(H)\). The relation with our notation is given by the equation \(F = P - (1 - P)\).

We let \(\mathcal{M}^d\) be the universal algebra for \(d + 1\)-summable Fredholm modules introduced by Connes-Karoubi \cite{CK88}. Then we get a homomorphism

\[
b : \mathcal{C}^\infty(X) \to \mathcal{M}^d
\]  

classifying the Fredholm module \((H,P)\). Note that \(b\) is uniquely determined up to unitary equivalence.

Remark 3.10. In this remark we give an explicit description of \(b\). The algebra \(\mathcal{M}^d\) for odd \(d \in \mathbb{N}\) is a subalgebra of the algebra of \(2 \times 2\) matrices of bounded operators on the standard separable Hilbert space \(\ell^2\) consisting of the matrices

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}, \quad a_{12}, a_{21} \in \mathcal{L}^{d+1}(\ell^2), \quad a_{11}, a_{22} \in B(\ell^2).
\]  

Let \(P^+, P^-\) be the positive and non-positive spectral projections of \(\mathcal{D}\). Then we choose identifications \(\ell^2 \cong \text{im}(P^+) \cong \text{im}(P^-)\). The homomorphism \(b : \mathcal{C}^\infty(X) \to \mathcal{M}^d\) is then given by

\[
f \mapsto \begin{pmatrix} P^+ fp^+ & P^+ f p^- \\
P^- f p^+ & P^- f p^-
\end{pmatrix}.
\]  

\[\tag{84}
\]

\[\square\]

3.3 The multiplicative character

In \cite{CK88} 4.10 Connes and Karoubi constructed the ”multiplicative” character

\[
\delta : K_{d+1}(\mathcal{M}^d) \to \mathbb{C}/\mathbb{Z}.
\]  

\[\tag{85}
\]

In this subsection we explain how the construction of the multiplicative character \(\delta\) fits into the framework of differential cohomology. The details of the construction will be needed later in Subsection 3.6.

We consider a unital locally convex algebra \(A\). It has a natural diffeological structure \(A^\infty\), see Example 2.8 5. From the sheaf of algebras \(A^\infty\) we derive the sheaves of spectra

\[
\mathcal{C}^\infty_A := L(H(\iota(\mathcal{C}^\infty(A^\infty)))) , \quad K_A \overset{23}{=} L(K(A^\infty)).
\]

The Goodwillie-Jones Chern character \(23\) gives a morphism

\[
\text{ch}^{gj} : K_A \to \mathcal{C}^\infty_A.
\]  

\[\tag{86}
\]
Remark 3.11. In principle we want to apply the homotopification sequence (32) to \( \text{ch}^{ij} \). This leads to the problem of understanding the homotopification of \( \mathbb{C}C_A^{-} \). The known facts are contained e.g. in [CT08, Cor. 4.1.2]. In particular the homotopification of \( \mathbb{C}C_A^{-} \) is equivalent to the homotopification of its periodic version \( \mathbb{C}C_A^{\text{per}}^{-} \), and the homotopification of the cyclic homology \( \mathbb{C}C_A \) vanishes. The problem is that \( \mathbb{C}C_A^{\text{per}} \) is not known to be homotopy invariant. We will get a better theory if we use the continuous versions of cyclic homology. The main advantage is that the continuous periodic cyclic homology for complete locally convex algebras is known to be diffeotopy invariant, see Theorem 3.12.

If we define the cyclic bicomplex \( \mathcal{B}C^{\text{cont}}(A) \) of a locally convex algebra \( A \) similarly as in [Lod98, 5.1.7] but using projective tensor products, then we get the continuous versions of cyclic, negative cyclic and periodic cyclic homology complexes

\[
CC^{\text{cont}}(A), \quad CC^{\text{cont},-}(A), \quad CC^{\text{cont},\text{per}}(A). \tag{87}
\]

In the natural extension of the notation of [Lod98, 5.1.7] to the continuous case these complexes would have been denoted by \( \text{Tot} \mathcal{B}C^{\text{cont}}, \text{Tot} \mathcal{B}C^{\text{cont},-}, \) and \( \text{Tot} \mathcal{B}C^{\text{cont},\text{per}}. \) We have an exact sequence of chain complexes

\[
0 \rightarrow CC^{\text{cont},-}(A) \rightarrow CC^{\text{cont},\text{per}}(A) \xrightarrow{q} CC^{\text{cont}}(A)[2] \rightarrow 0. \tag{88}
\]

Note that \( A^{\infty} \) is a presheaf of locally convex algebras by Remark 2.9. In (88) we can thus replace \( A \) by \( A^{\infty} \) in order to get an exact sequence of presheaves with values in \( \text{Ch}. \) Then we apply \( L \circ H \circ \iota \) in order to get the fibre sequence of sheaves of spectra

\[
\Sigma CC_A^{\text{cont}} \rightarrow CC_A^{\text{cont},-} \rightarrow CC_A^{\text{cont},\text{per}} \rightarrow \Sigma^2 CC_A^{\text{cont}} \tag{89}
\]

which is very similar to (60).

We now use the well-known fact that the continuous periodic cyclic homology is diffeotopy invariant [Kar97, Theoreme 2.7] (for Fréchet algebras) and [Val00] (for complete locally convex algebras):

**Theorem 3.12.** Assume that \( A \) is a complete locally convex algebra. Then the projection \( I \times M \rightarrow M \) induces a quasi-isomorphism

\[
CC^{\text{cont},\text{per}}(C^{\infty}(M, A)) \rightarrow CC^{\text{cont},\text{per}}(C^{\infty}(I \times M, A)) .
\]

As a consequence, the sheaf \( CC_A^{\text{cont},\text{per}} \) is homotopy invariant in the sense of Definition 2.3.

From now one we assume that \( A \) is complete. We apply the homotopification sequence (32) to the morphism (86). Using the Definition 2.21 we obtain the upper two columns of the following diagram:
The lower sequence is \((89)\). The map \(t : CC^{-}_A \to CC_{A}^{cont,-}\) is induced by the canonical map from algebraic to projectively completed tensor products. The composition \(p \circ t\) maps to a homotopy invariant target. The dotted arrow marked by \(i\) and the filler of the lower middle square is obtained from the universal property of the homotopification as a left adjoint in \((13)\). The dashed arrows marked by \(ii\) are now induced naturally.

We now drop out the middle row and evaluate the diagram at \(*\). We then get the map of fibre sequences

\[
\begin{align*}
K^{rel}_A & \longrightarrow K_A \longrightarrow K^{top}_A \longrightarrow \Sigma K^{rel}_A \\
A(ch^{gj}) & \downarrow \quad \downarrow \quad \downarrow H(ch^{gj}) \\
A(CC^{-}_A) & \longrightarrow CC^{-}_A \longrightarrow \mathcal{H}(CC^{-}_A) \longrightarrow A(CC^{-}_A) \\
ii & \downarrow \quad \downarrow \quad \downarrow i \\
\mathcal{H}(CC^{-}_A) & \longrightarrow CC^{cont,-}_A \longrightarrow p \longrightarrow CC^{cont,per}_A \longrightarrow \Sigma^2 CC^{cont}_A
\end{align*}
\]

which defines various versions of the continuous Chern character.

**Remark 3.13.** By \([CT08, \text{Lemma 4.2.2}]\) the lower sequence in \((91)\) can be identified with the homotopification sequence of \(CC^{cont,-}_A\). The whole diagram is thus the result of applying the homotopification sequence to the map \(t \circ ch^{gj} : K(A) \to CC^{cont,-}_A\).

The construction of the various versions of the continuous Chern characters above is thus completely parallel to what is done in \([CT08, \text{Sec. 4.2}]\). The diagram \((91)\) is exactly the last diagram in \([CT08, \text{Sec. 4.2}]\). \(\square\)

We finally define the Chern character \(ch^{cont}_{top}\) by the following diagram which involves the factorization of \(q\) over the \(S\)-operator:

\[
\begin{align*}
K^{top}(A) & \longrightarrow K(A) \longrightarrow K^{top}(A) \longrightarrow \Sigma K^{rel}(A) \\
\downarrow ch^{cont}_{per} & \quad \downarrow ch^{cont} & \quad \downarrow ch^{cont} & \quad \downarrow ch^{cont}_{rel} \\
CC^{cont,per}_A & \longrightarrow CC^{cont}_A \longrightarrow p \longrightarrow CC^{cont,per}_A \longrightarrow \Sigma^2 CC^{cont}_A
\end{align*}
\]

For \(\sharp \in \{\emptyset, -, per\}\) we let \(HC^{cont,\sharp}_*(A) := H_*(CC^{cont,\sharp}_*(A))\) denote the respective versions of continuous cyclic homology groups of \(A\).
We can now explain the construction of Connes-Karoubi character [CK88], see also [CT08, Sec. 7.3]. The algebra \( \mathcal{M}^d \) has a natural Fréchet structure so that the notions of topological and relative \( K \)-theory used in [CK88] or [CT08] coincide with our versions, see Remark 2.22. We start with the diagram derived from the right part of (92) and the upper sequence in (90) (see [CK88, 4.10])

\[
\begin{array}{cccc}
K_{d+2}^{\text{top}}(\mathcal{M}^d) & \longrightarrow & K_{d+1}^{\text{rel}}(\mathcal{M}^d) & \longrightarrow & K_{d+1}^{\text{top}}(\mathcal{M}^d) \\
\downarrow \text{ch}_{\text{top}}^{\text{cont}} & & \alpha & \downarrow \text{ch}_{\text{rel}}^{\text{cont}} & \downarrow \delta \\
HC_{d+2}^{\text{cont}}(\mathcal{M}^d) & S & HC_d^{\text{cont}}(\mathcal{M}^d) & \phi & \downarrow \\
\downarrow \phi_d & & \downarrow & & \\
\mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}/\mathbb{Z}.
\end{array}
\] (93)

It is a theorem of Karoubi [Kar86] that the right upper map (marked by 0) vanishes. The map \( \phi_d \) is given by the pairing with an explicit continuous cocycle \( \phi_d \in HC_d^{\text{cont}}(\mathcal{M}^d) \) which we will describe in (94) below. It has been verified in [CK88] that elements coming from \( \mathbb{Z} \cong K_{d+2}^{\text{top}}(\mathcal{M}^d) \) are mapped to integers under the obvious composition indicated by the left dotted arrow. The right dotted arrow is the multiplicative character. It is defined by the obvious diagram chase.

**Remark 3.14.** In this remark we describe the cocycle \( \phi_d \) explicitly. The formula will be used in the Remarks 3.15 and 3.20 below. Our description of \( \phi_d \) employs the chain complex \( C^\lambda_{\text{cont}}(A) \) given in [Lod98, 2.1.4] in order to calculate \( HC_n^{\text{cont}}(A) \) for a unital locally convex algebra over \( \mathbb{C} \). In particular \( C^\lambda_{\text{cont}}(A) \) is the space of coinvariants for the action of the cyclic permutation group on \( A^{\otimes_{n+1}} \). We use the notation \([a^0 \otimes \cdots \otimes a^n]\) in order to denote elements in \( C^\lambda_{\text{cont}}(A) \).

Furthermore, for a locally convex algebra \( A \) we calculate the cyclic cohomology \( HC_d^{\text{cont}}(A) \) using the complex \( C_*^{\lambda,\text{cont}}(A) \), where \( C_*^{\lambda,\text{cont}}(A) \) is the \( \mathbb{C} \)-vector space of continuous multilinear and cyclically invariant maps \( A^{\otimes_{n+1}} \rightarrow \mathbb{C} \). We have a natural pairing

\[
C_n^{\text{cont}}(A) \times C_n^{\lambda,\text{cont}}(A) \rightarrow \mathbb{C}
\]

given by

\[
(\phi, [a^0 \otimes \cdots \otimes a^n]) \rightarrow \phi(a^0, \ldots, a^n).
\]

Using these conventions the map \( \phi_d : HC_d^{\text{cont}}(\mathcal{M}^d) \rightarrow \mathbb{C} \) in (93) is given for odd \( d \) by the pairing with the cocycle (using the notation introduced in Remark 3.10)

\[
\phi_d(a^0, \ldots, a^d) := (-1)^{d+1} \frac{d!}{(2\pi i)^{d+1}(d+1)!} \text{Tr} \left[ \begin{pmatrix}
0 & a^0_{12} \\
0 & 0
\end{pmatrix} \ldots \begin{pmatrix}
0 & a^0_{21} \\
0 & 0
\end{pmatrix}
\right],
\] (94)

where

\[
z := \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]
Remark 3.15. In this remark we approach an explicit formula for the composition

\[ \delta \circ K(b_D) \circ \partial : K_{d+1}^r(C^\infty(X)) \to \mathbb{C}/\mathbb{Z}. \]

Here for a homomorphism \( b \) between algebras we denote by \( K(b) \) or \( HC(b) \) the induced maps in \( K \)-theory or cyclic homology. In view of (93) and the naturality of \( \text{ch}_{\text{rel}}^{cont} \) we have the equality

\[ \delta \circ K(b_D) \circ \partial = [-]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ \text{ch}_{\text{rel}}^{cont} \circ K(b_D) = [-]_{\mathbb{C}/\mathbb{Z}} \circ \phi_d \circ HC(b_D) \circ \text{ch}_{\text{rel}}^{cont}. \]

It is clearly complicated to write down an explicit formula for the relative Chern character \( \text{ch}_{\text{rel}}^{cont} \). But we can give an explicit formula for the composition

\[ \phi_d \circ HC(b_D) : HC_d(C^\infty(X)) \to HC_d(M^d) \to \mathbb{C}. \]

We continue with the notation introduced in Remark 3.10. For \( f \in C^\infty(X) \) we have

\[ [(P^+ - P^-), f] = 2(P^+fP^- - P^-fP^+) = 2 \begin{pmatrix} 0 & P^+fP^- \\ -P^-fP^+ & 0 \end{pmatrix}. \]

Combining (94) with (84) we see that \( \phi_d \circ HC(b_D) \) is represented by the cochain

\[ (f_0, \ldots, f_d) \mapsto \frac{-2^{d+1}d!}{(2\pi i)^{d-1}(d-1)!} \text{Tr} \left( (P^+ - P^-)[(P^+ - P^-), f_0] \ldots [(P^+ - P^-), f_d] \right) \] \quad (95)

This formula is a first step in the direction of the main result of [Kaa11b]. On the other hand it is still a complicated non-local formula. By standard methods of local index theory using e.g. the heat kernel and Getzler rescaling one can produce local cocycles representing the same cohomology class, see e.g. [BF90, CM95]. In Lemma 3.19 we avoid complicated analysis and the struggle with normalizations by using the Atiyah-Singer index theorem. The explicit local formula will be stated in Remark 3.20.

\[ \square \]

### 3.4 The conjecture

The connective covering morphism of spectra (74) induces a morphism of spectra

\[ ku\mathbb{C}/\mathbb{Z} \to KU\mathbb{C}/\mathbb{Z}. \] \quad (96)

Let \( X \) be a closed Riemannian manifold of odd dimension \( d \) equipped with a generalized Dirac operator \( \mathcal{D} \). Then we define the map \( r_\mathcal{D} \) as the following composition:

\[ r_\mathcal{D} : ku\mathbb{C}/\mathbb{Z}^{-d-2}(X) \xrightarrow{(94)} KU\mathbb{C}/\mathbb{Z}^{-d-2}(X) \xrightarrow{(81)} KU_{\text{flat}}^{-d-1,-d-1}(X) \xrightarrow{(80)} \mathbb{C}/\mathbb{Z}. \] \quad (97)
We now consider the diagram:

\[
\begin{array}{ccc}
K_{d+1}(C^\infty(X)) & \xrightarrow{\delta} & K_{d+1}(\mathcal{M}^d) \\
\xrightarrow{b_{D}} & & \xrightarrow{\sigma_{d+1}} \\
\xrightarrow{r_{\Phi}} & \xrightarrow{\sigma_{d+1}} & K_{d+1}(C/\mathbb{Z}) \\
\xrightarrow{uC/\mathbb{Z}^{-d-2}(X)} & & ,
\end{array}
\]

where \(b_{D}\) is defined in (82) and classifies the Fredholm module of \(\mathcal{D}\), \(\delta\) is the multiplicative character of Connes-Karoubi (85), and \(\sigma_{d+1}\) is defined in Definition 2.33.

**Conjecture 3.16.** Let \(X\) be a closed Riemannian manifold of odd dimension \(d\) equipped with a generalized Dirac operator \(\mathcal{D}\). Then the diagram (98) commutes.

In the present paper we show this conjecture for topologically trivial classes in \(K_{d+1}(C^\infty(X))\). The precise formulation of this result is Theorem 1.6.

### 3.5 Comparison of certain cocycles

In this subsection we prepare the proof of Theorem 1.6 by providing a differential geometric formula for the composition \(\delta \circ HC(b_{D})\). The main result of this subsection is Lemma 3.19.

In the following we define the cyclic homology \(HC_*(A)\) of an associative unital algebra \(A\) over \(\mathbb{C}\) as the homology of the standard cyclic complex \(CC_*(A)\). For details we refer to [Lod98, 2.1.9] where this complex is denoted by \(\text{Tot} B\). Explicitly, we define

\[
CC_n(A) := \begin{cases} 
\bigoplus_{k=0}^{n/2} A^{\otimes 2k+1} & n \text{ even} \\
\bigoplus_{k=0}^{n-1} A^{\otimes 2k} & n \text{ odd}
\end{cases}.
\]

As in Subsection 3.3 for a unital locally convex algebra we define the continuous cyclic homology complex \(CC^\text{cont}_*(A)\) and its homology \(HC^\text{cont}_*(A)\) similarly but using projective tensor products \(\otimes_{\pi}\) instead of algebraic ones, see Remark 2.9.

**Remark 3.17.** As shown in [Lod98, 2.1.4] there is a natural quasi-isomorphism

\[
CC(A) \to C^\lambda(A)
\]

which we use in order to compare the present definition of cyclic homology with the one used in Remark 3.14. The quasi-isomorphism (99) is induced by a chain-complex level projection map, which in degree \(n\) is the homomorphism \(CC_n(A) \to C^\lambda_n(A)\) given by (we write the formula for odd \(n\))

\[
\bigoplus_{k=0}^{n-1} a_0^k \otimes \cdots \otimes a_{2k+1}^k \mapsto [a_0^{n-1} \otimes \cdots \otimes a_n^{n-1}] .
\]

There is a similar quasi-isomorphism in the continuous case. \(\square\)
We define a morphism of graded groups (see (57) for the definition of $DD(X)$)

$$\pi : CC^{\text{cont}}(C^\infty(X)) \to DD(X)$$

by the following prescription:

1. If $n$ is odd, then we define $CC^{\text{cont}}_n(C^\infty(X)) \to \prod_{p \in \mathbb{Z}} (\sigma^{\leq p}\Omega)[2p]^{-n}(X)$ by

$$\bigoplus_{k=0}^{\frac{n-1}{2}} f_0^k \otimes \cdots \otimes f_{2k+1}^k \mapsto \sum_{k=0}^{\frac{n-1}{2}} b^{\frac{n+1}{2}+k} \frac{f_0^k df_1^k \wedge \cdots \wedge df_{2k+1}^k}{(2k+1)!}.$$  \hspace{1cm} (102)

2. If $n$ is even, then we define $CC^{\text{cont}}_n(C^\infty(X)) \to \prod_{p \in \mathbb{Z}} (\sigma^{\leq p}\Omega)[2p]^{-n}(X)$ by

$$\bigoplus_{k=0}^{\frac{n}{2}} f_0^k \otimes \cdots \otimes f_{2k}^k \mapsto \sum_{k=0}^{\frac{n}{2}} b^{\frac{n}{2}+k} \frac{f_0^k df_1^k \wedge \cdots \wedge df_{2k}^k}{(2k)!}.$$  \hspace{1cm} (102)

In these formulas we use the variable $b$ of degree $-2$ and the identification

$$\prod_{p \in \mathbb{Z}} \Omega[2p](X) \cong \Omega[b, b^{-1}](X).$$

Under this identification the series $\sum_{p \in \mathbb{Z}} b^{p} \omega(p) \in \Omega[b, b^{-1}](X)$ corresponds to the family $(\omega(p))_{p \in \mathbb{Z}} \in \prod_{p \in \mathbb{Z}} \Omega[2p](X)$. By [Lod98, 2.3.6] the map $\pi$ is a morphism of chain complexes. By the calculation of the continuous cyclic homology of $C^\infty(X)$ by Connes $\pi$ is actually a quasi-isomorphism.

We have a projection

$$DD^{\text{per}}(X) \to DD(X)$$

induced by the projections in the components

$$DD^{\text{per}}(p) \to DD(p), \quad \Omega[2p] \to (\sigma^{\leq p}\Omega)[2p]$$

for all $p \in \mathbb{Z}$.

**Remark 3.18.** This projection (103) must not be confused with the projection (59). The latter is given by

$$DD^{\text{per}}(X) \xrightarrow{\text{103}} DD(X) \xrightarrow{S} DD(X)[2],$$

where $S((\omega(p))_{p \in \mathbb{Z}}) = (\omega(p+1))_{p \in \mathbb{Z}}$. \hspace{1cm} $\square$
Let \( d \in \mathbb{N} \) be odd and assume that \( \dim(X) \leq d \). One easily checks that \((103)\) induces an isomorphism
\[
H^{-d}(D\dd\text{per}(X)) \xrightarrow{\cong} H^{-d}(D\dd(X)).
\]
(104)

We consider the isomorphism \( \pi_d \) defined as the following composition
\[
\pi_d : HC_d^{\text{cont}}(C^\infty(X)) \xrightarrow{\cong} H^{-d}(D\dd(X)) \xrightarrow{\phi_d \circ HC(b_{\mathcal{D}})} \C.
\]
(104)

Let \( \mathcal{D} \) be a generalized Dirac operator on \( X \). Using the local index density \( \hat{A}(\mathcal{D}) \) (see Remark 3.5) we define the map \( \tilde{\rho}_{\mathcal{D}} \):
\[
\tilde{\rho}_{\mathcal{D}} : HP^{-1}(X) \rightarrow \C, \quad \tilde{\rho}_{\mathcal{D}}([\gamma]) := \int_X \hat{A}(\mathcal{D}) \wedge t_{d+1}\gamma.
\]
(106)

**Lemma 3.19.** Let \( X \) be a closed manifold of odd dimension \( d \) and \( \mathcal{D} \) be a generalized Dirac operator on \( X \). Then the triangle
\[
\begin{array}{ccc}
HC_d^{\text{cont}}(C^\infty(X)) & \xrightarrow{\pi_d} & \C \\
& \downarrow \phi_d \circ HC(b_{\mathcal{D}}) & \\
\pi_d & \downarrow \hat{A}(\mathcal{D}) \wedge t_{d+1}\gamma & \\
& \downarrow \tilde{\rho}_{\mathcal{D}} & \\
HP^{-1}(X) & \xrightarrow{\phi_d \circ HC(b_{\mathcal{D}})} & \C
\end{array}
\]
commutes.

**Proof.** Our task is to compare the composition of the quasi-isomorphism \((99)\) with the map \((95)\) on the one hand, and the map \( \tilde{\rho}_{\mathcal{D}} \) defined in \((106)\) on the other. It seems to be difficult to do this by an explicit calculation. Therefore we give an indirect argument based on the Atiyah-Singer index theorem. Our argument will not use explicit formulas. The convention for fixing normalizations described in Remark 3.5 automatically takes care of the correct normalizations of \( \hat{A}(\mathcal{D}) \) and \( \phi_d \) in Remark 3.14.

We consider the composition of the map marked by !!! in \((41)\) with the Chern character given by the third column in \((90)\) in the case \( A = \C \):
\[
K_{C^\infty(X)}^{\text{top}} \xrightarrow{!!!} i_X^! K_c^{\text{top}} \xrightarrow{\text{ch}_{\text{per}}} i_X^! \text{CC}_C^{\text{cont,per}}.
\]
By evaluation at \(*\) and taking the \((d+2)\)’th homotopy group we obtain the left triangle in the following diagram:
\[
\begin{array}{ccc}
K_{d+2}^{\text{top}}(C^\infty(X)) & \xrightarrow{\pi_d} & HC_d^{\text{cont}}(C^\infty(X)) \\
& \downarrow t_{d+1}\text{ch}_{\text{per}} & \downarrow \phi_d \circ HC(b_{\mathcal{D}}) \\
HC_{d+2}^{\text{cont,per}}(C^\infty(X)) & \xrightarrow{\pi_d} & \C
\end{array}
\]
(107)
The map marked by $q$ is induced by the map marked by this symbol in \[88\]. By construction the three triangles on the left commute.

Let $K^\ast(-)$ denote the usual $K$-theory for $C^\ast$-algebras [Bal98]. Since $X$ is $d$-dimensional, the connective covering [74] induces an isomorphism marked by $*$ in the following chain of isomorphisms:

$$
\pi_{d+2}(K^\text{top}_C(X)) \cong \text{ku}^{-d-2}(X) \cong \text{KU}^{-d-2}(X) \cong K^\ast_{d+2}(C(X)).
$$

Under this identification the map

$$
\text{KU}^{-d-2}(X) \to H^1(X)
$$

induced by the dotted arrow in \[107\] is the composition $\iota_{d+1} \circ \text{ch}^\text{cw}_{\text{per}}$ of the usual Chern character and the shift, use Lemma 2.24). In particular, its image is a lattice of full rank in $H^1(X)$. Since $\pi_d$ is an isomorphism, in order to show that the right triangle in \[107\] commutes is suffices to verify that the right pentagon (omit the left upper corner) commutes.

We consider the map

$$
K^\ast_{d+2}(C(X)) \cong \pi_{d+2}(K^\text{top}_C(X)) \to H^\ast_{\text{cont}}(X)
$$

in the upper line of \[107\]. The cocycle $\phi_d$ is normalized exactly such that the composition of \[109\] with $\phi_d \circ H^\ast_{\text{cont}}(b_{\text{D}})$ is the integer valued function obtained by the pairing of $K^\ast_{d+2}(C(X))$ with the Fredholm module of $\text{D}$.

The down right right-up composition in the pentagon maps the $K$-theory class $x \in K^\ast_{d+2}(C(X))$ to

$$
\int_X [\hat{A}(\text{D})] \cup \iota_{d+1} \text{ch}^\text{cw}_{\text{per}}(x)
$$

which is apriori a complex number. The Atiyah-Singer index theorem encoded in equation \[78\] shows that \[110\] is an integer and equal to the index pairing. So the down right right-up composition coincides with \[109\].

**Remark 3.20.** This is a continuation of Remark 3.15. The following two cocycles (a) and (b) on $CC^\ast_{d,\text{cont}}(C^\infty(X))$ represent the same map $H^\ast_{\text{cont}}(C^\infty(X)) \to \mathbb{C}$. We describe result of the application of the two cocyles to the chain

$$
\oplus_{k=0}^{d-1} f_0^k \otimes \cdots \otimes f_{2k+1} \in CC^\ast_{d}(C^\infty(X))
$$

\hspace{1cm}

(a) \hspace{1cm}

$$
\frac{-2^{d+1}d!}{(2\pi i)^{d+1} (d-1)!} \text{Tr} \left( [(P^+ - P^-), f_0^{d-1}] \ldots [(P^+ - P^-), f_d^{d-1}] \right)
$$

(b) \hspace{1cm}

$$
\sum_{k=0}^{d-1} \frac{1}{(2k+1)!} \int_X [\hat{A}(\text{D})]_{d-2k-1} \wedge f_0^k df_1^k \wedge \cdots \wedge df_{2k+1}^k
$$

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The formula (a) is one for $\phi_d \circ CC(b_d)$ obtained by combining (100) and (95). The formula (b) gives $\tilde{\rho}_d \circ \pi_d$ and is derived from a combination of (106) and (102). The equality of the cohomology classes of (a) and (b) is the assertion of Lemma 3.19.

3.6 Proof of the conjecture for topologically trivial classes

In this subsection we prove Theorem 1.6. We must show the equality of homomorphisms

$$r_B \circ \sigma_{d+1} \circ \partial = \delta \circ K(b_B) \circ \partial : K_{d+1}^{rel}(C^\infty(X)) \to \mathbb{C}/\mathbb{Z}.$$  

We first show a preliminary result. We consider the structure map

$$a_{\text{ku}} : \pi_{d+1}(A_{\geq d-1 \text{DD} \text{per}}(X)) \to \hat{\text{ku}}^{-d-1,-d-1}(X)$$

(see (68)) and the periodicity operator $\iota_{d-1} : HP^{-1}(X) \xrightarrow{\approx} HP^{-d-2}(X)$ given by (36). We have a natural isomorphism

$$\pi_{d+1}(A_{\geq d-1 \text{DD} \text{per}}(X)) \cong H^{-d-2}(\sigma_{\leq d-2 \text{DD} \text{per}}(X)) \cong \text{DD} \text{per}(X)^{-d-2}/\text{im}(d).$$

Hence there is a natural inclusion

$$HP^{-d-2}(X) \hookrightarrow \pi_{d+1}(A_{\geq d-1 \text{DD} \text{per}}(X)).$$

Lemma 3.21. For $[\gamma] \in HP^{-1}(X)$ we have the equality

$$r_B \circ \text{pr} \circ a_{\text{ku}} \circ \iota_{d-1}([\gamma]) = [\tilde{\rho}_B([\gamma])]_{\mathbb{C}/\mathbb{Z}}.$$  

Proof. Recall the structure map $a_{\text{KU}} : HP^{-1}(X) \to \hat{KU}^{0,0}(X)$ of the differential cohomology theory $\hat{KU}^{0,0}$ occurring in the exact sequence (75). By (79) we have

$$\rho_B(a_{\text{KU}}([\gamma])) = [\tilde{\rho}_B([\gamma])]_{\mathbb{C}/\mathbb{Z}}.$$  

In order to see (111) we now use (97) and that the following diagram commutes:

\[
\begin{array}{ccc}
HP^{-1}(X) & \xrightarrow{a_{\text{KU}}} & \hat{KU}^{0,0}_{\text{flat}}(X) \\
\downarrow^{\iota_{d-1}} & & \downarrow^{\approx} \\
HP^{-d-2}(X) & \xrightarrow{a_{\text{ku}}} & \hat{\text{ku}}^{-d-1,-d-1}(X) \\
\text{pr} & & \text{pr}
\end{array}
\]

We now come to the actual proof of Theorem 1.6. We first use Proposition 2.34 in order to replace

$$r_B \circ \sigma_{d+1} \circ \partial \quad \text{by} \quad r_B \circ \text{pr} \circ a_{\text{ku}} \circ \kappa^{-1} \circ \text{reg}_{X}^{\text{rel}},$$

In order to see (111) we now use (97) and that the following diagram commutes:

\[
\begin{array}{ccc}
HP^{-1}(X) & \xrightarrow{a_{\text{KU}}} & \hat{KU}^{0,0}_{\text{flat}}(X) \\
\downarrow^{\iota_{d-1}} & & \downarrow^{\approx} \\
HP^{-d-2}(X) & \xrightarrow{a_{\text{ku}}} & \hat{\text{ku}}^{-d-1,-d-1}(X) \\
\text{pr} & & \text{pr}
\end{array}
\]
where \( pr \) is the projection as in (56). We know from Corollary 2.32 (55) that the composition \( a_{\text{ku}} \circ c \circ \kappa^{-1} \circ \text{reg}_{X}^{\text{rel}} \) takes values in \( \kappa_{\text{flat}}^{-d-1,-d-1}(X) \). Explicitly, the target of \( c \) is given by

\[
\pi_{d+1}(\mathcal{A}(\Sigma DD^\perp))(X) \cong \prod_{p \in \mathbb{Z}} \Omega^{2p-d-2}(X)/\text{im}(d),
\]

and the image of \( c \circ \kappa^{-1} \circ \text{reg}_{X}^{\text{rel}} \) belongs to the subgroup

\[
\prod_{p \in \mathbb{Z}} \Omega^{2p-d-2}(X)/\text{im}(d) \cong HP^{-d-2}(X).
\]

Using the Lemmas 3.19 and 3.21 we can now replace

\[
r_{\mathcal{B}} \circ pr \circ a_{\text{ku}} \circ c \circ \kappa^{-1} \circ \text{reg}_{X}^{\text{rel}} \text{ by } [-]_{\mathcal{C}/\mathbb{Z}} \circ \phi_{d} \circ HC(b_{\mathcal{B}}) \circ \pi_{d}^{-1} \circ \imath_{d+1} \circ c \circ \kappa^{-1} \circ \text{reg}_{X}^{\text{rel}}.
\]

We now use the equality \( \text{reg}_{X}^{\text{rel}} = \pi \circ \text{ch}_{\text{rel}}^{\text{cont}} \), where \( \pi \) is as in (101). We observe by going through the explicit definitions of the maps that the composition

\[
HC_{d}^{\text{cont}}(C^{\infty}(X)) \xrightarrow{\pi} H^{-d}(DD(X)) \cong \pi_{d+1}(\Sigma DD(X)) \cong \pi_{d+1}(\mathcal{A}(DD^{-})(X)) \xrightarrow{\text{coK}^{-1}} \prod_{p \in \mathbb{Z}} \Omega^{2p-d-2}(X)/\text{im}(d) \cong HP^{-d-2}(X) \xrightarrow{\imath_{d+1}} HP^{-1}(X)
\]

is exactly \( \pi_{d} \) defined by (105). We can thus replace

\[
[-]_{\mathcal{C}/\mathbb{Z}} \circ \phi_{d} \circ HC(b_{\mathcal{B}}) \circ \pi_{d}^{-1} \circ \imath_{d+1} \circ c \circ \kappa^{-1} \circ \text{reg}_{X}^{\text{rel}} \text{ by } [-]_{\mathcal{C}/\mathbb{Z}} \circ \phi_{d} \circ HC(b_{\mathcal{B}}) \circ \text{ch}_{\text{rel}}^{\text{cont}}.
\]

We finally use the naturality of the Chern character in order to replace this by

\[
[-]_{\mathcal{C}/\mathbb{Z}} \circ \phi_{d} \circ \text{ch}_{\text{rel}}^{\text{cont}} \circ K(b_{\mathcal{B}}).
\]

In view of the definition (93) of the Connes-Karoubi multiplicative character and the naturality of \( \partial \) this is exactly \( \delta \circ K(b_{\mathcal{B}}) \circ \partial \) as asserted. \( \square \)

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