A HEURISTIC DERIVATION OF LINEAR RECURRANCE RELATIONS FOR $\zeta'(-2k)$ AND $\zeta(2k+1)$

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Abstract. We have gone back to old methods found in the historical part of Hardy’s Divergent Series well before the invention of the modern analytic continuation to use formal manipulation of harmonic sums which produce some interesting formulae. These are linear recurrence relations for $\sum_{n=1}^{\infty} H_n n^k$ which in turn yield linear recurrence relations for $\zeta'(-k)$ and hence using the functional equation to a linear recurrence relation for $\zeta'(2k)$ and $\zeta(2k+1)$. Questions of rigor have been postponed to a subsequent preprint.

1. Introduction

For several years the authors have been studying the Euler sum

$$h(s) = \sum_{n=1}^{\infty} \frac{H_n}{n^s}.$$  \hspace{1cm} (1)

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$, the $n^{th}$ harmonic number. This has led to published works with K.N. Boyadzhiev [1], [2] and B. Candelpergher [4] which was presented in the ICMZeta 2010 in Chennai. Also see [3]. We now turn to a curious formula in the paper with Boyadzhiev [1].

Lemma 1. (Fundamental Lemma) For $\Re(s) > 1$

$$\sum_{n=1}^{\infty} \frac{H_n}{(n + 1)^s} = \sum_{n=1}^{\infty} \frac{H_n}{n^s} - \zeta(s + 1)$$  \hspace{1cm} (2)

where $\zeta(s)$ is the Riemann zeta function.

Proof. From the definition of $H_n$

$$H_{n+1} = H_n + \frac{1}{n + 1}.$$  \hspace{1cm} (3)
Using this,

\[
\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^s} = \sum_{n=1}^{\infty} \frac{H_{n+1} - \frac{1}{n+1}}{(n+1)^s}
\]

(4)

\[
= \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^s} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^{s+1}}
\]

\[
= \left( \sum_{n=1}^{\infty} \frac{H_n}{n^s} - 1 \right) - (\zeta(s + 1) - 1)
\]

\[
= \sum_{n=1}^{\infty} \frac{H_n}{n^s} - \zeta(s + 1)
\]

\[
\Box
\]

Now we naively analytically continue the formula (2) and to our surprise we get new results. Replacing \(s\) by \(-s\) in (2) we get

\[
\sum_{n=1}^{\infty} H_n(n + 1)^s = \sum_{n=1}^{\infty} H_n n^s - \zeta(1 - s)
\]

(5)

Using the functional equation of the Riemann zeta-function

\[
\zeta(1 - s) = 2(2\pi)^{-s}\Gamma(s) \cos \frac{\pi s}{2} \zeta(s),
\]

(6)

(5) becomes

\[
\sum_{n=1}^{\infty} H_n(n + 1)^s = \sum_{n=1}^{\infty} H_n n^s - 2(2\pi)^{-s}\Gamma(s) \cos \frac{\pi s}{2} \zeta(s)
\]

(7)

For \(s = 2\) this becomes

\[
\sum_{n=1}^{\infty} H_n(n + 1)^2 = \sum_{n=1}^{\infty} H_n n^2 = \frac{1}{12}
\]

(8)

where we have used the formula \(\zeta(2) = \frac{\pi^2}{6}\). Expanding the left hand side and simplifying

\[
2 \sum_{n=1}^{\infty} n H_n + \sum_{n=1}^{\infty} H_n = \frac{1}{12}
\]

(9)

When \(s = 3\) we get

\[
3 \sum_{n=1}^{\infty} n^2 H_n + 3 \sum_{n=1}^{\infty} n H_n + \sum_{n=1}^{\infty} H_n = 0
\]

(10)

as \(\cos \frac{3\pi}{2} = 0\). Thus we get a recurrence relation for \(\sum_{n=1}^{\infty} H_n n^k\).
2. Connection to $\zeta'(-k)$

In this section we would like to give the relation between $\sum_{n=1}^{\infty} H_n n^k$ and $\zeta'(1-k)$.

To do this we need an interpolation of the Bernoulli numbers. The Bernoulli numbers are defined as

$$
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \frac{z^n}{n!} \tag{11}
$$

With the definition given above $B_1(= \frac{1}{2})$. Note that it differs by the conventional value of $B_1(= -\frac{1}{2})$ by a sign. The other values of Bernoulli numbers are not affected. These numbers can be got from the contour integral as follows.

**Lemma 2.** For $n \geq 1$,

$$
\frac{1}{2\pi i} \int_C \frac{1}{z^n} \frac{-z}{e^{-z} - 1} \, dz = \frac{B_{n-1}}{(n-1)!} \tag{12}
$$

where $C$ is the Hankel contour.

Using the above integral we get an interpolation $B_s$ of the Bernoulli numbers. It is given by

$$
-\frac{B_{s+1}}{\Gamma(s+2)} = \frac{1}{2\pi i} \int_C \frac{1}{z^{s+1}} \frac{e^z}{1-e^z} \, dz \tag{13}
$$

The formulae connecting $B_s$ and its derivative $B'_s$ with the Riemann zeta function are given below.

**Lemma 3.**

$$
B_s = -s\zeta(1-s) \tag{14}
$$

$$
B'_s = -\zeta(1-s) + s\zeta'(1-s) \tag{15}
$$

In particular,

$$
B'_k = -\zeta(1-k) + k\zeta'(1-k), \; k = 1, 2, \cdots \tag{16}
$$

where $\zeta'(-k) = \sum_{n=1}^{\infty} n^k \log n$ in an appropriate sense.

The functional equation of $\zeta(s)$ gives the following relation between $\zeta(2k+1)$ and $\zeta'(-2k)$

$$
\zeta(2k+1) = (-1)^k 2^{2k} \frac{(2\pi)^{2k}}{(2k)!} \zeta'(-2k) \tag{17}
$$

Using this we get

**Lemma 4.**

$$
\zeta(2k+1) = (-1)^k \frac{2\pi^{2k+1}}{(2k+1)!} \frac{B'(2k+1)}{\pi} \tag{18}
$$

In [5] we proved the following identity.
Lemma 5. For $k \geq 1$,

$$(-1)^{k-1} k \sum_{n=1}^{\infty} H_n n^{k-1} = B'_k + kB_{k-1} + \gamma B_k - \sum_{l+m=k} \binom{k}{l} B_l B_m - B_k H_k$$

Substituting for $B'_k$ from (16) in terms of $\zeta'$, the above formula takes the form

$$(-1)^{k-1} k \sum_{n=1}^{\infty} H_n n^{k-1} = -\zeta(1-k) + k\zeta'(1-k) + kB_{k-1} + \gamma B_k - \sum_{l+m=k} \binom{k}{l} B_l B_m - B_k H_k$$

(20)

3. Comments

1. As it can be seen that the formulae will lead to a reduction in the number of unknown constants in the theory of the Riemann zeta function. If $\zeta'(0)$ is fed into the recurrence relations, all the $\zeta'(k)$ for $k \geq 1$ will be obtained.

2. We are trying to develop the theory of $\sum_{n=1}^{\infty} H_n n^k$ in strict analogy with Bernoulli numbers. In particular we want to get the analogue of

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$$ for $n \geq 2$ (21)

This is got from the identity

$$\frac{xe^x}{e^x - 1} - \frac{x}{e^x - 1} = x$$

(22)

3. An analogue of this for $\sum_{n=1}^{\infty} H_n n^k$ can be got by using the generating functions. Let us start with the identity

$$\frac{\log(1 - e^{-x})}{1 - e^{-x}} - \frac{e^{-x}\log(1 - e^{-x})}{1 - e^{-x}} = \log(1 - e^{-x})$$

(23)

Taking the Mellin transform of the above equation (23)

$$\int_{0}^{\infty} x^{s-1} \log(1 - e^{-x}) dx - \int_{0}^{\infty} x^{s-1} e^{-x} \log(1 - e^{-x}) dx = \int_{0}^{\infty} x^{s-1} \log(1 - e^{-x}) dx$$

(24)

and using the generating functions

$$\log(1 - e^{-x}) = -\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}$$

(25)

and

$$\frac{\log(1 - e^{-x})}{1 - e^{-x}} = -\sum_{n=1}^{\infty} H_n e^{-nx}$$

(26)

we get

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^s} = \sum_{n=1}^{\infty} \frac{H_n}{n^s} - \zeta(s+1)$$

(27)

which is our Fundamental Lemma (2).
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4. Conclusion

We paraphrase G. H. Hardy, see Section 1.3, [6] (because we cannot improve on his language):

The results of the formal calculations of · · · are correct wherever they can be checked: thus all of the formulae · · · are correct. It is natural to suppose that the other formulae will prove to be correct, and our transformations justifiable, if they are interpreted appropriately. We should then be able to regard the transformations as shorthand representations of more complex processes justifiable by the ordinary canons of analysis. It is plain that the first step towards such an interpretation must be some definition, or definitions, of the ‘sum’ of an infinite series, more widely applicable than the classical definition of Cauchy.

This remark is trivial now: it does not occur to a modern mathematician that a collection of mathematical symbols should have a ‘meaning’ until one has been assigned to it by definition. It was not a trivality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, ‘by X we mean Y’. There are reservations to be made, to which we shall return in · · ·; but it is broadly true to say that mathematicians before Cauchy asked not ‘How shall we define 1 − 1 + 1 − · · · ?’ but ‘What is 1 − 1 + 1 − · · · ?’ and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.

It is well known to experts in summability theory that different methods of summation assign slightly different values as the sum of a divergent series. All these points will have to be taken into account to get the value of ζ(2k + 1).

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6. Dedication

We would like to dedicate this preprint to the memory of Prof. K. Ramachandra who encouraged study of simple ideas. We would also remember with gratitude our parents Hemalatha Gadiyar, Dr. H. G. Madhav Gadiyar, R. Gowri and Prof. S. Ramanathan.

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