POLYNOMIALS ASSOCIATED TO NON-CONVEX BODIES

N. LEVENBERG\textsuperscript{1,*} and F. WIELONSKY\textsuperscript{2}

\textsuperscript{1}Indiana University, Bloomington, IN 47405, USA
e-mail: nlevenbe@indiana.edu

\textsuperscript{2}Laboratoire I2M–UMR CNRS 7373, Université Aix-Marseille, CMI 39 Rue Joliot Curie,
F-13453 Marseille Cedex 20, France
e-mail: franck.wielonsky@univ-amu.fr

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Abstract. Polynomial spaces associated to a convex body $C$ in $(\mathbb{R}^+)^d$ have been the object of recent studies. In this work, we consider polynomial spaces associated to non-convex $C$. We develop some basic pluripotential theory including notions of $C$–extremal plurisubharmonic functions $V_{C,K}$ for $K \subset \mathbb{C}^d$ compact. Using this, we discuss Bernstein–Walsh type polynomial approximation results and asymptotics of random polynomials in this non-convex setting.

1. Introduction

Pluripotential theory is the study of plurisubharmonic (psh) functions. A fundamental result, known as the Siciak–Zaharjuta theorem (see [10] for references and history), is that the extremal function (or pluricomplex Green function)

$$V_K(z) := \sup \{ u(z) : u \in L(\mathbb{C}^d), u \leq 0 \text{ on } K \}$$

associated to a compact set $K \subset \mathbb{C}^d$, where $L(\mathbb{C}^d)$ is the Lelong class of all plurisubharmonic functions $u$ on $\mathbb{C}^d$ with the property that

$$u(z) \leq \log^+ |z| + c_u := \max[0, \log |z|] + c_u,$$

\textsuperscript{*}Corresponding author.
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where \( c_u \) is a constant depending on \( u \), may be obtained from the subclass of \( L(\mathbb{C}^d) \) arising from polynomials:

\[
V_K(z) = \sup \left\{ \frac{1}{\deg(p)} \log |p(z)| : p \in \mathcal{P}_d, \deg(p) \neq 0, ||p||_K := \max_{\zeta \in K} |p(\zeta)| \leq 1 \right\},
\]

where \( \mathcal{P}_d = \mathbb{C}[z_1, \ldots, z_d] \) denotes the family of all holomorphic polynomials of \( d \) complex variables. This gives a connection with polynomial approximation; see Theorem 1.1 below. It is known that \( V^*_K(z) := \limsup_{\zeta \to z} V_K(\zeta) \) is either a psh function in \( L(\mathbb{C}^d) \) or else \( V^*_K \equiv +\infty \) (this latter case occurs precisely when \( K \) is pluripolar). We say \( K \) is regular if \( V_K = V^*_K \); i.e., \( V_K \) is continuous.

In recent years, pluripotential theory associated to a convex body \( C \) in \( (\mathbb{R}^+)^d \) has been developed. Let \( \mathbb{R}^+ = [0, \infty) \) and fix a convex body \( C \subset (\mathbb{R}^+)^d \) (\( C \) is compact, convex and \( C^o \neq \emptyset \)). Associated with \( C \) we consider the finite-dimensional polynomial spaces

\[
\text{Poly}(nC) := \left\{ p(z) = \sum_{J \in nC \cap (\mathbb{Z}^+)^d} c_J z^J : c_J \in \mathbb{C} \right\}
\]

for \( n = 1, 2, \ldots \) where \( z^J = z_1^{j_1} \cdots z_d^{j_d} \) for \( J = (j_1, \ldots, j_d) \). For \( C = \Sigma \) where

\[
\Sigma := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_i \leq 1, \sum_{i=1}^d x_i \leq 1 \right\}
\]

is a simplex in \( \mathbb{R}^d \), we have Poly\((n\Sigma)\) is the usual space of holomorphic polynomials of degree at most \( n \) in \( \mathbb{C}^d \). For a nonconstant polynomial \( p \) we define

\[
\deg_C(p) = \min \{ n \in \mathbb{N} : p \in \text{Poly}(nC) \}.
\]

As in [3], [4], [9], except for the case of \( C = C_0 \) defined in (1.12), we make the assumption throughout the entire paper on \( C \) that

\[
\Sigma \subset kC \quad \text{for some } k \in \mathbb{Z}^+.
\]

Note that under hypothesis (1.4), we have \( \bigcup_n \text{Poly}(nC) = \mathcal{P}_d \).

Recall the indicator function of a convex body \( C \) is

\[
\phi_C(x_1, \ldots, x_d) := \sup_{(y_1, \ldots, y_d) \in C} (x_1y_1 + \cdots x_dy_d).
\]

We define the logarithmic indicator function of \( C \) on \( \mathbb{C}^d \)

\[
H_C(z) := \sup_{J \in C} \log |z^J| := \sup_{J \in C} \log(\max |z_1|^{j_1} \cdots |z_d|^{j_d}) = \phi_C(\log |z_1|, \ldots, \log |z_d|)
\]

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(the exponents $j_k$ need not be integers) and we use $H_C$ to define a generalization of $L(C^d)$:

$$L_C = L_C(C^d) := \{ u \in \text{PSH}(C^d) : u(z) \leq H_C(z) + c_u \}.$$  

Since $\phi_\Sigma(x_1, \ldots, x_d) = \max(x_1, \ldots, x_d, 0)$, we have $L(C^d) = L_\Sigma$. Moreover, it was stated in [2] and shown in [5] that the $C$-extremal function

$$V_{C,K}(z) := \sup\{ u(z) : u \in L_C(C^d), u \leq 0 \text{ on } K \}$$

of a compact set $K$ can be given as

$$V_{C,K} = \lim_{n \to \infty} \frac{1}{n} \log \Phi_n = \lim_{n \to \infty} \frac{1}{n} \log \Phi_{n,C,K}$$

pointwise on $C^d$ where

$$\Phi_n(z) := \Phi_{n,C,K}(z) := \sup\{ |p(z)| : p \in \text{Poly}(nC), \|p\|_K \leq 1 \}.$$  

For $p \in \text{Poly}(nC), n \geq 1$ we have $\frac{1}{n} \log |p| \in L_C$; hence we have a Bernstein–Walsh inequality

$$|p(z)| \leq \|p\|_K e^{\deg_C(p)V_{C,K}(z)}.$$  

We add that for $C$ satisfying (1.4), $K$ regular is equivalent to $V_{C,K} = V_{C,K}^*$ ($V_{C,K}$ is continuous); cf. [9].

The inequality (1.8) leads to a connection with polynomial approximation. For $K$ a compact subset of $C^d$, and $F \in C(K)$, a complex-valued, continuous function on $K$, we define

$$d_{n}^{C}(F, K) = \inf_{p \in \text{Poly}(nC)} \|F - p\|_K.$$  

The following Bernstein–Walsh type theorem was proved in [9] to explain the use of various notions of degree for multivariate polynomials introduced by Trefethen in [11].

**Theorem 1.1.** Let $K$ be a compact subset of $C^d$ with $V_{C,K}$ continuous and $F \in C(K)$. The following assertions are equivalent.

1) $\limsup_{n \to \infty} d_{n}^{C}(F, K)^{1/n} = 1/R < 1$;

2) the function $F$ is the restriction to $K$ of a function $\tilde{F}$ holomorphic on a domain $\Omega$ containing $K$, and $R$ is the largest real number such that

$$\Omega_{C,R} = \{ z \in C^d : V_{C,K}(z) < \log R \} \subset \Omega.$$
This is a quantitative version of the Oka–Weil theorem: any \( F \)-holomorphic in a neighborhood of the polynomial hull

\[
\hat{K} := \{ z \in \mathbb{C}^d : |p(z)| \leq \|p\|_K, \ p \in \mathcal{P}_d \}
\]

of \( K \) can be uniformly approximated on \( \hat{K} \) by polynomials in \( \mathcal{P}_d \). Note that the “smaller” the convex body \( C \), the sparser the collection \( \text{Poly}(nC) \) will be. For purposes of numerical analysis, sparseness is desirable. Indeed, provided \( C \in (\mathbb{R}^+)^d \) is the closure of an open, connected set satisfying

\[
\varepsilon \Sigma \subset C \subset \delta \Sigma \quad \text{for some } \delta > \varepsilon > 0,
\]

regardless of whether \( C \) is convex, the finite-dimensional spaces \( \text{Poly}(nC) \) defined as in (1.1) make sense (as does the notion of \( C \)-degree in (1.3)) and \( \bigcup_n \text{Poly}(nC) = \mathcal{P}_d \). Thus, appealing to Oka–Weil, there is at least a possibility of a version of Theorem 1.1 in this setting. Examples of such \( C \) are the \( l^p \) balls

\[
C = C_p := \{(x_1, \ldots, x_d) \in (\mathbb{R}^+)^d : x_1^p + \cdots + x_d^p \leq 1\}
\]

with \( 0 < p < 1 \). Numerical analysts even consider the limiting case of \( p = 0 \),

\[
C_0 := \bigcup_{j=1}^d \{(0, \ldots, x_j, \ldots, 0), \ 0 \leq x_j \leq 1\}
\]

(but see Example 3.11).

In this note, we begin a study of polynomial classes associated to non-convex \( C \) satisfying (1.10). The next section discusses general results on \( C \)-extremal functions \( V_{C,K} \) defined in a fashion similar to (1.7). In Section 3 we show that while one direction of Theorem 1.1 trivially generalizes, the other allows for interesting contrasts.

These \( C \)-extremal functions \( V_{C,K} \) are difficult to compute explicitly. However, in a probabilistic sense, one can “generically” recover them. Let \( \tau \) be a probability measure on \( K \) which is nondegenerate in the sense that \( \|p\|_{\tau} := \|p\|_{L^2(\tau)} = 0 \) for a polynomial \( p \) implies \( p \equiv 0 \). Letting \( \{p_j\} \) be an orthonormal basis in \( L^2(\tau) \) for \( \text{Poly}(nC) \) constructed via Gram–Schmidt applied to a monomial basis \( \{z^n\} \) of \( \text{Poly}(nC) \), we consider random polynomials of \( C \)-degree at most \( n \) of the form

\[
H_n(z) := \sum_{j=1}^{m_n} a_j^{(n)} p_j(z)
\]
where the \( a_j^{(n)} \) are i.i.d. complex random variables and \( m_n := \text{dim}(\text{Poly}(nC)) \). This places a probability measure \( \mathcal{H}_n \) on \( \text{Poly}(nC) \). We form the product probability space of sequences of polynomials:

\[
\mathcal{H} := \bigotimes_{n=1}^{\infty} (\text{Poly}(nC), \mathcal{H}_n).
\]

The following was proved for \( C = \Sigma \) in [6] and for general convex \( C \) in [2].

**Theorem 1.2.** Let \( \tau \) be a probability measure on \( K \) such that \( (K, \tau) \) satisfies a Bernstein–Markov property and let \( a_j^{(n)} \) be i.i.d. complex random variables having distribution \( \phi(z)dm_2(z) \) where \( dm_2 \) denotes Lebesgue measure on \( \mathbb{R}^2 = \mathbb{C} \). Assume for some \( T > 0 \),

\[
|\phi(z)| \leq T \quad \text{for all } z \in \mathbb{C}; \quad \text{and}
\]

\[
\left| \int_{|z| \geq R} \phi(z)dm_2(z) \right| \leq T/R^2 \quad \text{for all } R \text{ sufficiently large}.
\]

Then almost surely in \( \mathcal{H} \) we have

\[
\left( \limsup_{n \to \infty} \frac{1}{n} \log |H_n(z)| \right)^* = V_{C,K}^*(z), \quad z \in \mathbb{C}^d.
\]

The *Bernstein–Markov property* will be defined in the next section.

In Section 4 we give a version of Theorem 1.2 in the nonconvex setting. We conclude in Section 5 with some open questions.

### 2. Non-convex preliminaries

Let \( C \) be the closure of an open, connected set satisfying (1.10). For simplicity, we take \( \delta = 1 \) in (1.10). As noted in the introduction, the definitions of the vector spaces

\[
\text{Poly}(nC) := \left\{ p(z) = \sum_{J \in nC \cap \mathbb{N}^d} c_J z^J = \sum_{J \in nC \cap \mathbb{N}^d} c_J z_1^{j_1} \cdots z_d^{j_d}, \quad c_J \in \mathbb{C} \right\},
\]

\( n = 1, 2, \ldots \) and, for a nonconstant polynomial \( p \), the \( C \)-degree

\[
\text{deg}_C(p) := \min\{ n \in \mathbb{N} : p \in \text{Poly}(nC) \},
\]

can be defined as in the convex case. However, if \( C \) is not convex, two vital ingredients are lacking:

1. \( \text{Poly}(nC) \cdot \text{Poly}(mC) \) may *not* be contained in \( \text{Poly}(n + m)C \); and
2. there is no good analogue/replacement for the logarithmic indicator function \( H_C \) in (1.6) and hence the \( L_C \) Lelong class.
Item 1. is crucial in proving 2) implies 1) in Theorem 1.1. To explain 2., for $C$ the closure of an open, connected set satisfying (1.10), using (1.5) and (1.6) yields $H_C = H_{\text{co}(C)}$ where $\text{co}(C)$ denotes the convex hull of $C$. If, e.g., $C = C_p$ in (1.11) with $0 < p < 1$, then $H_{C_p} = H_{\Sigma}$.

2.1. $C$-extremal function for non-convex $C$. Given a compact set $K \subset \mathbb{C}^d$, we will define a $C$-extremal function using the $\text{Poly}(nC)$ classes: for $n = 1, 2, \ldots$, let

$$\Phi_n(z) = \Phi_{n,C,K}(z) := \sup \{ |p(z)| : p \in \text{Poly}(nC), \|p\|_K \leq 1 \}$$

(note taking $p \equiv 1$ shows $\Phi_n(z) \geq 1$) and

$$V_{C,K}(z) := \limsup_{n \to \infty} \frac{1}{n} \log \Phi_n(z).$$

For a polynomial $p \in \text{Poly}(nC)$ we have $\deg_C(p) \leq n$. Thus

$$\Phi_n(z)^{1/n} \leq \Phi_n(z)^{1/\deg_C(p)}.$$ 

This shows that

$$V_{C,K}(z) \leq \sup \left\{ \frac{1}{\deg_C(p)} \log |p(z)| : p \in \mathcal{P}_d, \|p\|_K \leq 1 \right\}.$$ 

We do not know if equality holds (in general) in (2.3). From (1.10),

$$\varepsilon V_K \leq V_{C,K} \leq V_K$$

so that for $K$ nonpluripolar, $V_{C,K}^*$ is a plurisubharmonic function (indeed, $V_{C,K}^* \in L(\mathbb{C}^d)$). Furthermore,

$$C \subset C' \implies V_{C,K} \leq V_{C',K}$$

which follows from the facts that $\text{Poly}(nC) \subset \text{Poly}(nC')$ and $p \in \text{Poly}(nC)$ implies $\deg_C(p) \geq \deg_{C'}(p)$.

**Remark 2.1.** From the definition (2.2), for $K \subset \mathbb{C}^d$ compact we have $V_{C,K} = V_{C,\hat{K}}$ where $\hat{K}$ is the polynomial hull of $K$ (recall (1.9)). Let $C$ be the closure of an open, connected set satisfying (1.10). We remark that

$$Z(K) := \{ z \in \mathbb{C}^d : V_{C,K}(z) = 0 \} = \hat{K}.$$ 

To see this, since $V_{C,K} \geq 0$ we clearly have $\hat{K} \subset Z(K)$. For the reverse inclusion, if $z_0 \notin \hat{K}$ there exists a polynomial $p$ with

$$\frac{|p(z_0)|}{\|p\|_K} =: 1 + \lambda > 1.$$
By (1.10), for each positive integer $k$, $p^k \in \text{Poly}(n_k C)$ for some positive integer $n_k = \deg_C(p^k)$ with $n_k \uparrow \infty$ and

$$n_k \leq \deg_{\Sigma}(p^k) \leq k \deg_{\Sigma}(p).$$

Thus

$$\limsup_{k \to \infty} \frac{1}{n_k} \log \frac{|p^k(z_0)|}{\|p^k\|^K} \geq \limsup_{k \to \infty} \frac{1}{k \cdot \deg_{\Sigma}(p)} \log(1 + \lambda)^k$$

$$= \frac{1}{\deg_{\Sigma}(p)} \log(1 + \lambda) > 0$$

so that $V_{C,K}(z_0) > 0$.

One class of compact sets $K$ for which $C$-extremal functions can be computed are products of planar compacta. For $E \subset \mathbb{C}$ a planar compacta, let $g_E$ be the classical Green function of (the unbounded component of the complement of) $E$.

**Proposition 2.2.** Let $C \subset \Sigma$ be a connected set containing $C_0$ in (1.12). For $K = E_1 \times \cdots \times E_d$, a product of regular, planar compacta $E_j$, we have

$$V_{C,K}(z) = V_{\Sigma,K}(z) = \max_{j=1,\ldots,d} g_{E_j}(z_j).$$

In particular, this holds for $C$ the closure of an open, connected set satisfying (1.10).

**Proof.** It is classical that $V_{\Sigma,K}(z) = \max_{j=1,\ldots,d} g_{E_j}(z_j)$ (cf. [10, Theorem 5.1.8]). Since $C \subset \Sigma$, clearly $V_{C,K}(z) \leq V_{\Sigma,K}(z)$. For the reverse inequality, for $n = 1,2,\ldots$ define

$$\Psi_n(z) := \sup \{ |p(z)| : p(z) = p_1(z_1) + \cdots + p_d(z_d), \ \deg p_i \leq n, \ \|p\|_K \leq 1 \}$$

(here $\deg = \deg_{\Sigma}$) and

$$V(z) := \limsup_{n \to \infty} \frac{1}{n} \log \Psi_n(z).$$

Since $C_0 \subset C$, $\Psi_n(z) \leq \Phi_n(z)$ in (2.1) and hence $V(z) \leq V_{C,K}(z)$. Fixing $j \in \{1,\ldots,d\}$ and taking $p_k \equiv 0$ for $k \neq j$, we have

$$\Psi_n(z) \geq \sup \{ |p_j(z_j)| : \deg p_j \leq n, \ \|p_j\|_{E_j} \leq 1 \}.$$

Thus

$$V(z) \geq \limsup_{n \to \infty} \left( \frac{1}{n} \log \sup \{ |p_j(z_j)| : \deg p_j \leq n, \ \|p_j\|_{E_j} \leq 1 \} \right) = g_{E_j}(z_j).$$

This holds for $j = 1,\ldots,d$ and the proposition is proved. $\square$
Remark 2.3. Since, in this setting,
\[ \sup \{ |p_j(z_j)| : \deg p_j \leq n, \|p_j\|_{E_j} \} \leq \Psi_n(z) \leq \Phi_n(z), \]
and
\[ \lim_{n \to \infty} \frac{1}{n} \log \sup \{ |p_j(z_j)| : \deg p_j \leq n, \|p_j\|_{E_j} \} = g_{E_j}(z_j) \]
we have a true limit in (2.2). Moreover, the proof of Proposition 2.2 shows that for \( K \) compact in \( \mathbb{C}^d \), if we let \( E_j = \pi_j(K) \) where \( \pi_j \) is the projection from \( z = (z_1, \ldots, z_d) \) to the \( j \)-th coordinate \( z_j \), then
\[ V_{\Sigma, K}^*(z) \geq V_{C, K}^*(z) \geq \max_{j=1,\ldots,d} g_{E_j}(z_j). \]
We mention that for \( C \) convex and \( K = E_1 \times \cdots \times E_d \), a product of regular, planar compacta \( E_j \), we have (cf., [9, Prop. 2.4])
\[ V_{C, K}(z_1, \ldots, z_d) = \phi_C\left(g_{E_1}(z_1), \ldots, g_{E_d}(z_d)\right) \]
where \( \phi_C \) is defined in (1.5).

Examples of nonconvex \( C \) satisfying the hypotheses of Proposition 2.2 are the \( l^p \) balls
\[ C_p = \{ (x_1, \ldots, x_d) \in (\mathbb{R}^+)^d : x_1^p + \cdots + x_d^p \leq 1 \} \]
from (1.11) with \( 0 < p < 1 \). Note Proposition 2.2 is also valid in the limiting case \( p = 0 \).

2.2. \( L^2 \)-approach to \( V_{C, K} \). We next discuss several ways of recovering \( V_{C, K} \) in this non-convex setting, motivated by the standard (and convex) settings. Often \( L^2 \)-norms are more convenient to work with than \( L^\infty \)-norms. To this end, for \( K \) a compact set in \( \mathbb{C}^d \) and \( \tau \) a positive Borel measure on \( K \), we say that \( (K, \tau) \) satisfies a Bernstein–Markov property if for any polynomial \( p_n \) of degree \( n \) and any \( n \)
\[ \|p_n\|_K \leq M_n \|p_n\|_\tau \quad \text{where } \limsup_{n \to \infty} M_n^{1/n} = 1. \]
From our hypothesis (1.10), this is equivalent to (2.6) for \( p_n \in \text{Poly}(nC) \). For simplicity, we assume \( \tau(K) = 1 \).

In the standard pluripotential setting of \( C = \Sigma \), let
\[ m_n = m_n(\Sigma) = \dim(\text{Poly}(n \Sigma)) = \binom{d + n}{n} = \mathcal{O}(n^d). \]
We take a lexicographical ordering of the monomials \( \{z^\nu\}_{|\nu|\leq n} \) in Poly\((n\Sigma)\) and write these as \( \{e_j(z)\}_{j=1}^{m_n} \). Let \( \{p_j\}_{j=1,...,m_n} \) be a set of orthonormal polynomials of degree at most \( n \) in \( L^2(\tau) \) gotten by applying the Gram–Schmidt process to these monomials in Poly\((n\Sigma)\). For each \( n = 1, 2, \ldots \) consider the corresponding Bergman kernel

\[
S_n(z, \zeta) := \sum_{j=1}^{m_n} p_j(z)p_j(\zeta)
\]

and the restriction to the diagonal

\[
S_n(z, z) = \sum_{j=1}^{m_n} |p_j(z)|^2. \tag{2.7}
\]

By the reasoning in [8], we have the following.

**Proposition 2.4.** Let \( K \subset \mathbb{C}^d \) be compact and nonpluripolar and let \( \tau \) be a probability measure on \( K \) such that \((K, \tau)\) satisfies (2.6). Then with \( S_n(z, z) \) defined in (2.7),

\[
\lim_{n \to \infty} \frac{1}{2n} \log S_n(z, z) = V_{\Sigma, K}(z), \quad z \in \mathbb{C}^d.
\]

If \( V_{\Sigma, K} \) is continuous, the convergence is uniform on compact subsets of \( \mathbb{C}^d \).

**Sketch of proof.** We briefly indicate the two main steps in the proof since these will be generalized. First, for each \( n = 1, 2, \ldots \) define

\[
\phi_n(z) := \sup \{|p(z)| : p \in \text{Poly}(n\Sigma), \|p\|_K \leq 1\}.
\]

Then, from [10, Theorem 5.1.7],

\[
\lim_{n \to \infty} \frac{1}{n} \log \phi_n(z) = V_{\Sigma, K}(z) \tag{2.8}
\]

pointwise on \( \mathbb{C}^d \); and the convergence is uniform on compact subsets of \( \mathbb{C}^d \) if \( V_{\Sigma, K} \) is continuous. This is [8, Lemma 3.4]. The next step is a comparison between \( \phi_n(z) \) and \( S_n(z, z) \):

\[
1 \leq \frac{S_n(z, z)}{\phi_n(z)^2} \leq M_n^2 m_n, \quad z \in \mathbb{C}^d, \tag{2.9}
\]

where \( M_n \) is as in (2.6). The left-hand inequality follows from the reproducing property of the Bergman kernel \( S_n(z, \zeta) \) and the Cauchy–Schwarz inequality and is valid for any \( \tau \) for which one has an orthonormal basis.
in $L^2(\tau)$ for $\text{Poly}(n\Sigma)$. Let $p$ be a polynomial of degree at most $n$ with $\|p\|_K \leq 1$. Writing $p(z) = \sum_{j=1}^{m_n} a_j p_j(z)$,

$$\|p(z)\|^2 \leq \sum_{j=1}^{m_n} |a_j|^2 \cdot \sum_{j=1}^{m_n} |p_j(z)|^2 = \|p\|^2_\tau \cdot \sum_{j=1}^{m_n} |p_j(z)|^2 \leq \|p\|^2_K \cdot \sum_{j=1}^{m_n} |p_j(z)|^2 \leq S_n(z, z).$$

Since $\phi_n(z) = \sup \{|p(z)| : p \in \text{Poly}(n\Sigma), \|p\|_K \leq 1\}$, taking the supremum over all such $p$ gives the left-hand inequality. The right-hand inequality uses the Bernstein–Markov property of $(K, \tau)$. We have $\|p_j\|_K \leq M_n$ so that $|p_j(z)|/M_n \leq \phi_n(z)$ and

$$S_n(z, z) = \sum_{j=1}^{m_n} |p_j(z)|^2 \leq m_n \cdot M_n^2 \cdot [\phi_n(z)]^2. \quad \square$$

The exact same proof is valid for $C$ convex satisfying (1.4) (cf. [2, Proposition 2.11]). We note that the analogue of (2.8) in this setting uses the fact that

$$\text{Poly}(nC) \cdot \text{Poly}(mC) \subset \text{Poly}(n + mC).$$

Given $C$, the closure of an open, connected set satisfying (1.10) which contains $C_0$ in (1.12), if we know for a given compact set $K$ that we have the pointwise limit

$$\lim_{n \to \infty} \frac{1}{n} \log \Phi_n(z) = V_{C,K}(z), \quad (2.10)$$

i.e., the limit exists and equals $V_{C,K}(z)$, then the analogue of Proposition 2.4 holds in this setting, except perhaps for the local uniform convergence. We state this as Proposition 2.6. Moreover, we get convergence in $L^1_{loc}(\mathbb{C}^d)$ as well. Here we use an ordering $\prec_C$ on $\mathbb{N}^d$ which respects $\text{deg}_C(p)$ in the sense that $\alpha \prec_C \beta$ whenever $\text{deg}_C(z^\alpha) < \text{deg}_C(z^\beta)$, and

$$S^C_n(z, z) = \sum_{j=1}^{m_n} |p_j(z)|^2$$

where $\{p_j\}_{j=1}^{m_n}$ is an orthonormal basis in $L^2(\tau)$ for $\text{Poly}(nC)$ constructed via Gram–Schmidt applied to an ordered monomial basis $\{z^\nu\}$ of $\text{Poly}(nC)$. Here

$$m_n = m_n(C) = \dim(\text{Poly}(nC)).$$
For the $L^1_{\text{loc}}(\mathbb{C}^d)$ convergence, we will use the following standard result; this will also be needed in section 4. The proof is identical to that of [6, Proposition 4.4].

**Proposition 2.5.** Let $\{\psi_n\}$ be a locally uniformly bounded above family of plurisubharmonic functions on $\mathbb{C}^d$. Suppose for any subsequence $J$ of positive integers we have

$$\left(\limsup_{n \in J} \psi_n(z)\right)^* = V(z)$$

for all $z \in \mathbb{C}^d$ where $V \in \text{PSH}(\mathbb{C}^d) \cap L^\infty_{\text{loc}}(\mathbb{C}^d)$. Then $\psi_n \to V$ in $L^1_{\text{loc}}(\mathbb{C}^d)$.

**Proposition 2.6.** Let $C$ be the closure of an open, connected set satisfying (1.10). Let $K \subset \mathbb{C}^d$ be compact, nonpluripolar and satisfying (2.10). Finally, let $\tau$ be a positive Borel measure on $K$ such that $(K, \tau)$ satisfies the Bernstein–Markov property (2.6). Then the sequences $\{\frac{1}{2n} \log S_n^C\}$ and $\{\frac{1}{n} \log \Phi_n\}$ are locally uniformly bounded above and

$$\lim_{n \to \infty} \frac{1}{2n} \log S_n^C(z, z) = V_{C,K}(z)$$

pointwise on $\mathbb{C}^d$. Furthermore, both sequences $\{\frac{1}{2n} \log S_n^C\}$ and $\{\frac{1}{n} \log \Phi_n\}$ converge to $V_{C,K}^*$ in $L^1_{\text{loc}}(\mathbb{C}^d)$.

**Proof.** The analogue of (2.9) holds with $S_n, \phi_n, m_n = m_n(\Sigma)$ replaced by $S_n^C, \Phi_n, m_n = m_n(C)$ with the exact same proof:

$$1 \leq \frac{S_n^C(z, z)}{\Phi_n(z)^2} \leq M_n^2 m_n.$$

Under the hypothesis (2.10), the pointwise convergence of $\frac{1}{2n} \log S_n^C(z, z)$ to $V_{C,K}(z)$ follows. Moreover, from (2.11), $C \subset \Sigma$ (so that $\Phi_n \leq \phi_n$), and $\frac{1}{n} \log \phi_n \leq V_K$,

$$\frac{1}{2n} \log S_n^C(z, z) \leq \frac{1}{n} \log \Phi_n(z) + \frac{1}{2n} \log(M_n^2 m_n)$$

$$\leq V_{K}^*(z) + \limsup_{n \to \infty} \frac{1}{2n} \log(M_n^2 m_n) = V_{K}^*(z)$$

which shows the sequences $\{\frac{1}{2n} \log S_n^C\}$ and $\{\frac{1}{n} \log \Phi_n\}$ are locally uniformly bounded above. Proposition 2.5 immediately shows

$$\frac{1}{2n} \log S_n^C \to V_{C,K}^*$$

in $L^1_{\text{loc}}(\mathbb{C}^d)$.
Finally, for each $n$, the function $\frac{1}{n} \log \Phi_n^*$ is psh and is equal to $\frac{1}{n} \log \Phi_n$ except perhaps for a pluripolar set. Since a countable union of pluripolar sets is pluripolar,

$$\lim_{n \to \infty} \frac{1}{n} \log \Phi_n^*(z) = \lim_{n \to \infty} \frac{1}{n} \log \Phi_n(z) = V_{C,K}(z)$$

outside of a pluripolar set. Hence $[\lim_{n \to \infty} \frac{1}{n} \log \Phi_n^*(z)]^* = V_{C,K}^*(z)$ for all $z \in \mathbb{C}^d$. By Proposition 2.5, $\frac{1}{n} \log \Phi_n^* \to V_{C,K}^*$ in $L_{\text{loc}}^1(\mathbb{C}^d)$ and hence the same is true for $\{ \frac{1}{n} \log \Phi_n \}$. □

From Remark 2.3, the full conclusion of Proposition 2.6 holds if $C \subset \Sigma$ contains $C_0$ in (1.12) and $K = E_1 \times \cdots \times E_d$, a product of regular, planar compacta $E_j$. In fact, we get slightly more, namely local uniform convergence, for certain Bernstein–Markov measures on $\mu$. Suppose $\mu_j$ is a Bernstein–Markov measure on $E_j$ for $j = 1, \ldots, d$. Then $\mu := \bigotimes_{j=1}^d \mu_j$ is a Bernstein–Markov measure on $K$. If $\{e_{k}^{(j)}\}$ is an orthonormal basis of polynomials for $L^2(\mu_j)$ with $\deg(e_{k}^{(j)}) = k$, then we know that

$$\lim_{n \to \infty} \frac{1}{2n} \log \sum_{k=0}^n |e_{k}^{(j)}(z_j)|^2 = g_{E_j}(z_j)$$

locally uniformly on $\mathbb{C} = \mathbb{C}_{z_j}$. Then, we have

**Proposition 2.7.** With $K = E_1 \times \cdots \times E_d$ and $\mu := \bigotimes_{j=1}^d \mu_j$ as in the preceding paragraph, the $n$-th Bergman function

$$B_{n}^{\mu,C}(z) := S_{n}^{C}(z,z)$$

for $\mu$ associated to the vector space Poly$(nC)$ satisfies

(2.13) $$\lim_{n \to \infty} \frac{1}{2n} \log B_{n}^{\mu,C}(z_1, \ldots, z_d) = \max_{j=1, \ldots, d} g_{E_j}(z_j)$$

locally uniformly on $\mathbb{C}^d$ (and hence in $L_{\text{loc}}^1(\mathbb{C}^d)$).

**Proof.** We first consider the $n$-th Bergman function $B_{n}^{\mu,C_0}(z) := S_{n}^{C_0}(z,z)$ for

$$\text{Poly}(nC_0) = \text{span}\{1, z_1, \ldots, z_d, z_1^2, \ldots, z_d^2, z_1^n, \ldots, z_d^n\}$$

where $C_0$ is the $p = 0$ case of the $l^p$ ball $C_0$ in (1.12). Then, assuming $\mu$ is a probability measure, we have

$$d - 1 + B_{n}^{\mu,C_0}(z_1, \ldots, z_d) = d + \sum_{j=1}^d \sum_{k=1}^n |e_{k}^{(j)}(z_j)|^2 = d + \sum_{j=1}^d \sum_{k=0}^n |e_{k}^{(j)}(z_j)|^2.$$
Now recall that for real $a_1, \ldots, a_d$, 
\begin{equation}
\lim_{n \to \infty} \frac{1}{2n} \log (e^{2na_1} + \cdots + e^{2na_d}) = \max_{j=1, \ldots, d} a_j.
\end{equation}

Taking sequences \( \{a_n^{(j)}(z_j) := \sum_{k=0}^{n}|e_k^{(j)}(z_j)|^2\} \) so that 
\[ \lim_{n \to \infty} \frac{1}{2n} \log a_n^{(j)}(z_j) = a_j(z_j) := g_E(z_j), \quad j = 1, \ldots, d \]
where the convergence is locally uniform in each $C = \mathbb{C}z_j$, using (2.14) we have 
\[ \lim_{n \to \infty} \frac{1}{2n} \log \left(\sum_{j=1}^{d} a_n^{(j)}(z_j)\right) = \max_{j=1, \ldots, d} a_j(z_j) \]
locally uniformly in $\mathbb{C}^d$, and thus 
\[ \lim_{n \to \infty} \frac{1}{2n} \log B_{\mu,C_0}^\mu(z_1, \ldots, z_d) = \lim_{n \to \infty} \frac{1}{2n} \log (d - 1 + B_{\mu,C_0}^\mu(z_1, \ldots, z_d)) = \max_{j=1, \ldots, d} g_E(z_j) \]
locally uniformly in $\mathbb{C}^d$, which is (2.13) for $C = C_0$. The case $C = C_1 = \Sigma$ follows from the more general Proposition 2.4. For other $C$ as in Proposition 2.2, $C_0 \subset C \subset C_1$ which implies the inequality 
\[ B_{\mu,C_0}^\mu \leq B_{\mu,C}^\mu \leq B_{\mu,C_1}^\mu \]
and hence the general case of (2.13). \( \square \)

**Remark 2.8.** Unlike the case where $C$ is convex, in the non-convex setting, it is unclear whether one has 
\[ \lim_{n \to \infty} \frac{1}{n} \log \Phi_n(z) = V_{C,K}(z) \]
pointwise; and, even if this holds and $V_{C,K}$ is continuous, it is unclear whether the limit is locally uniform. From Proposition 2.7 and (2.11), for $C$ the closure of an open, connected set satisfying (1.10) which contains $C_0$ in (1.12), all of these properties hold for $K = E_1 \times \cdots \times E_d$ a product of regular, planar compacta $E_j$.

More generally, let $\mu$ be any positive measure on $K$ such that one can form orthonormal polynomials $\{p_\alpha\}$ using Gram–Schmidt on the monomials $\{z^\alpha\}$. As before we use an ordering $\prec_C$ on $\mathbb{N}^d$ which respects $\deg_C(p)$. The following argument of Zeriahi [12] is valid in this setting.
Proposition 2.9. Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar and let $C$ be the closure of an open, connected set satisfying (1.10). Then

$$
\limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_\alpha)} \log |p_\alpha(z)| \geq V_{C,K}(z), \quad z \not\in \hat{K}.
$$

Proof. Let $Q_n \in \text{Poly}(nC)$ and $\|Q_n\|_K \leq 1$. From the property of the ordering $\prec_C$, we can write $Q_n = \sum_{\alpha \in nC} c_\alpha p_\alpha$. Then

$$
|c_\alpha| = \left| \int_K Q_n \overline{p_\alpha} \, d\mu \right| \leq \int_K |\overline{p_\alpha}| \, d\mu \leq \sqrt{\mu(K)}
$$

by Cauchy–Schwarz. Hence

$$
|Q_n(z)| \leq m_n \sqrt{\mu(K)} \max_{\alpha \in nC} |p_\alpha(z)|
$$

where $m_n = \dim(\text{Poly}(nC))$.

Fix $z_0 \in \mathbb{C}^d \setminus \hat{K}$ and let $\alpha_n \in nC$ be a multiindex with $\deg_C(p_{\alpha_n})$ largest such that

$$
|p_{\alpha_n}(z_0)| = \max_{\alpha \in nC} |p_\alpha(z_0)|.
$$

We claim that taking any such sequence $\{\alpha_n = \alpha_n(z_0)\}_{n=1,2,\ldots}$,

$$
\lim_{n \to \infty} \deg_C(p_{\alpha_n}) = +\infty.
$$

For if not, then by the above argument, there exists $A < \infty$ such that for any $n$ and any $Q_n \in \text{Poly}(nC)$ with $\|Q_n\|_K \leq 1$,

$$
|Q_n(z_0)| \leq m_n \sqrt{\mu(K)} \max_{\deg_C(p_\alpha) \leq A} |p_\alpha(z_0)| = m_n M(z_0)
$$

where $M(z_0)$ is independent of $n$. But then $\Phi_n(z_0) \leq m_n M(z_0)$ so that, from definition (2.2),

$$
V_{C,K}(z_0) \leq \limsup_{n \to \infty} \left[ \frac{1}{n} \log m_n + \frac{1}{n} \log M(z_0) \right] = 0
$$

which contradicts $z_0 \in \mathbb{C}^d \setminus \hat{K}$ from (2.4). We conclude that for any $z \in \mathbb{C}^d \setminus \hat{K}$, for any $n$ and any $Q_n \in \text{Poly}(nC)$ with $\|Q_n\|_K \leq 1$,

$$
\frac{1}{n} \log |Q_n(z)| \leq \frac{1}{n} \log m_n + \frac{1}{n} \log |p_{\alpha_n}(z)| + \frac{1}{n} \log \sqrt{\mu(K)}
$$
where we can assume \( \deg_C(p_{\alpha_n}) \uparrow \infty \). Thus, for such \( z \), again from definition (2.2),
\[
V_{C,K}(z) \leq \limsup_{n \to \infty} \frac{1}{n} \log |p_{\alpha_n}(z)| \leq \limsup_{n \to \infty} \frac{1}{\deg_C(p_{\alpha_n})} \log |p_{\alpha_n}(z)|
\]
\[
\leq \limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_{\alpha})} \log |p_{\alpha}(z)|
\]
where we have used \( \deg_C(p_{\alpha_n}) \leq n \). □

**Corollary 2.10.** Let \( K \subset \mathbb{C}^d \) be compact and nonpluripolar and let \( C \) be the closure of an open, connected set satisfying (1.10). Then for any Bernstein–Markov measure \( \mu \) for \( K \),
\[
V_{C,K}(z) = \limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_{\alpha})} \log |p_{\alpha}(z)|, \quad z \notin \hat{K}.
\]

**Proof.** In view of Proposition 2.9, it is sufficient to prove that \( V_{C,K}(z), \ z \notin \hat{K} \), is larger than the left-hand side of (2.15). Assuming \( \mu \) to be a probability measure, for the orthonormal polynomials \( \{p_{\alpha}\} \), we have
\[
1 \leq \|p_{\alpha}\|_K \leq M_{\deg_C(p_{\alpha})} \quad \text{and} \quad M_{\deg_C(p_{\alpha})}^{1/\deg_C(p_{\alpha})} \to 1
\]
using (1.10) and the Bernstein–Markov property (2.6). Thus
\[
\lim_{|\alpha| \to \infty} \|p_{\alpha}\|_K^{1/\deg_C(p_{\alpha})} = 1.
\]

Consequently, from definition (2.2),
\[
V_{C,K}(z) \geq \limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_{\alpha})} \log \|p_{\alpha}(z)\|_K = \limsup_{|\alpha| \to \infty} \frac{1}{\deg_C(p_{\alpha})} \log |p_{\alpha}(z)|. \quad \Box
\]

By [7, Proposition 3.1], for any compact set \( K \subset \mathbb{C}^d \) there exists a Bernstein–Markov measure \( \mu \) for \( K \). We will use Corollary 2.10 in the next subsection.

**2.3. \( K = B \), the complex Euclidean ball.** We first remark that some version of assumption (1.10) seems natural in order that we have \( \bigcup_n \text{Poly}(nC) = \mathcal{P}_d \). Moreover, if \( \bigcup_n \text{Poly}(nC) \subsetneq \mathcal{P}_d \), equality in (2.4) may fail. Indeed, let \( C = C_0 \) and consider \( K = B := \{z \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 \leq 1\} \), the complex Euclidean ball in \( \mathbb{C}^d \). Let \( \mu \) be normalized surface area measure on \( \partial B \) and let \( \nu \) be normalized Haar measure on the torus.

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\( T := \{ z \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1 \} \). The monomials \( z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d} \) are orthonormal with respect to \( \nu \) while the monomials \( z_j^{\alpha_j} \) are orthogonal with respect to \( \mu \) with

\[
 a_k := \| z_j^k \|_\mu^2 = \frac{(d-1)! k!}{(d-1+k)!}, \quad j = 1, \ldots, d.
\]

We consider the \( n \)-th Bergman functions \( B_{n,C_0}^\mu(z) \) and \( B_{n,C_0}^\nu(z) \) for

\[
 \text{Poly}(nC_0) = \text{span}\{ 1, z_1, \ldots, z_d, z_1^2, \ldots, z_d^2, \ldots, z_1^n, \ldots, z_d^n \}.
\]

We have

\[
 B_{n,C_0}^\nu(z) = 1 + (|z_1|^2 + \cdots + |z_d|^2) + (|z_1|^4 + \cdots + |z_d|^4)
 + \cdots + (|z_1|^{2n} + \cdots + |z_d|^{2n}).
\]

and

\[
 B_{n,C_0}^\mu(z) = 1 + a_1^{-1}(|z_1|^2 + \cdots + |z_d|^2) + a_2^{-1}(|z_1|^4 + \cdots + |z_d|^4)
 + \cdots + a_n^{-1}(|z_1|^{2n} + \cdots + |z_d|^{2n})
\]

Thus

\[
 (2.16) \quad B_{n,C_0}^\nu(z) \leq B_{n,C_0}^\mu(z) \leq 1 + a_n^{-1}[B_{n,C_0}^\nu(z) - 1].
\]

Similar to the proof of Proposition 2.7, we have

\[
 \lim_{n \to \infty} \frac{1}{2n} \log B_{n,C_0}^\nu(z) = \max[0, \log |z_1|, \ldots, \log |z_d|]
\]

locally uniformly for all \( z \in \mathbb{C}^d \). From (2.16) we also have

\[
 \lim_{n \to \infty} \frac{1}{2n} \log B_{n,C_0}^\mu(z) = \max[0, \log |z_1|, \ldots, \log |z_d|]
\]

locally uniformly for all \( z \in \mathbb{C}^d \). However, the inequality (2.11) is valid for \( C_0, B \) and \( \mu \); in the above notation, since \( \dim(\text{Poly}(nC_0)) = dn + 1 \),

\[
 1 \leq \frac{B_{n,C_0}^\mu(z)}{\Phi_n(z)^2} \leq M_n^2(dn + 1)
\]

where

\[
 \Phi_n(z) = \Phi_{n,C_0,B}(z) := \sup\{ |p(z)| : p \in \text{Poly}(nC_0), \|p\|_B \leq 1 \}.
\]
Hence \( \lim_{n \to \infty} \frac{1}{n} \log \Phi_n(z) \) exists and equals \( \max[0, \log |z_1|, \ldots, \log |z_d|] \) as well.

This shows

\[
V_{C_0,B}(z) = \max (0, \log |z_1|, \ldots, \log |z_d|).
\]

In particular,

\[
Z(B) = \{ z : V_{C_0,B}(z) = 0 \} = \{ z : \max_{j=1,\ldots,d} |z_j| \leq 1 \}
\]

so that \( B = \hat{B} \subset Z(B) \) when \( d > 1 \).

Moreover, given \( z_0 = (z_{0,1}, \ldots, z_{0,d}) \not\in B \), writing \( z_{0,j} = |z_{0,j}| e^{i\phi_j} \) and defining \( p(z) := \sum_{j=1}^d e^{-2i\phi_j} z_j^2 \), we have

\[
\|p\|_B \leq \max_{z \in B} \left( |z_1|^2 + \cdots + |z_d|^2 \right) = 1
\]

while

\[
|p(z_0)| = |z_{0,1}|^2 + \cdots + |z_{0,d}|^2 > 1.
\]

Since \( p \in \text{Poly}(2C_0) \), this shows that if one defines

\[
\tilde{V}_{C_0,B}(z) := \sup \left\{ \frac{1}{\deg_{C_0}(p)} \log |p(z)| : p \in \bigcup_n \text{Poly}(nC_0), \|p\|_B \leq 1 \right\},
\]

then \( \tilde{V}_{C_0,B}(z) > 0 \) for all \( z \not\in B \) so that \( \tilde{V}_{C_0,B} \neq V_{C_0,B} \). Thus equality fails to hold in (2.3) for \( C = C_0 \) and \( K = B \).

Next we show that, unlike the case of product sets \( K = E_1 \times \cdots \times E_d \) in Proposition 2.2, for \( K = B \), the \( C_p \)-extremal functions for \( 0 < p < 1 \) do not coincide with the \( C_0 \)- and \( C_1 = \Sigma \)-extremal functions.

**Proposition 2.11.** For \( 0 < p < 1 \), we have

\[
V_{C_0,B}(z) < V_{C_p,B}(z) < V_{C_1,B}(z)
\]

at certain points \( z \in \mathbb{C}^d \).

**Proof.** For simplicity, we let \( d = 2 \) and use variables \((z, w)\). As shown above, together with (2.4)

\[
V_{C_0,B}(z, w) = \max[0, \log |z|, \log |w|] = 0 < V_{C_p,B}(z, w)
\]

for \((z, w) \in Z(B) \setminus B \neq \emptyset\) and \( 0 < p < 1 \). We next verify for \( 0 < p < 1 \) that

\[
V_{C_p,B}(z, w) \neq V_{C_1,B}(z, w)
\]
for certain points. We utilize Corollary 2.10: taking $K = B \subset \mathbb{C}^2$, $\mu$ normalized surface area measure on $\partial B$, and $C = C_p$ for $0 < p \leq 1$,

$$ V_{C_p, B}(z, w) = \limsup_{|\alpha| \to \infty} \frac{1}{\deg_{C_p}(p_\alpha)} \log |p_\alpha(z, w)|, \quad (z, w) \notin B. $$

We look at points on the diagonal $w = z$ for $|z|$ large; indeed, we may consider any points $(z, w)$ with $|z| = |w|$ large. For $C_1 = \Sigma$ we have $V_{C_1, B}(z, w) = \frac{1}{2} \log^+(|z|^2 + |w|^2)$ so for $|z| \geq 1/\sqrt{2}$,

$$ V_{C_1, B}(z, z) = \frac{1}{2} \log(|z|^2 + |z|^2) = \frac{1}{2} \log 2 + \log |z|. $$

Since $\|z^a w^b\|_{L^2(\mu)}^2 = ab!/(a + b + 1)!$, at points $(z, z)$ with $|z| > 1$ for $C_1$ we need the orthonormal monomials $\{z^a w^b/\|z^a w^b\|_{L^2(\mu)}\}$ with $a, b$ near equal to achieve this value $\frac{1}{2} \log 2 + \log |z|$ using (2.18). Precisely, for $n$ large we need

$$ \frac{1}{n} \log \|z^a w^b\|_{L^2(\mu)}^{-1} \to \frac{1}{2} \log 2. $$

For $a = b = n/2$ (we assume $n$ even for simplicity), using Stirling’s formula,

$$ \|z^{n/2} w^{n/2}\|_{L^2(\mu)}^{-1} = \frac{\sqrt{(n+1)!}}{(n/2)!} \approx \frac{\sqrt{(n/e)^n}}{(n/2e)^{n/2}} = 2^{n/2} $$

so that

$$ \frac{1}{n} \log \|z^{n/2} w^{n/2}\|_{L^2(\mu)}^{-1} \approx \frac{1}{n} \log 2^{n/2} = \frac{1}{2} \log 2, $$

as desired.

If $0 < p < 1$, the only monomials $z^a w^b \in \text{Poly}(nC_p)$ with $a + b$ “near” $n$ are “near” $z^n$ and $w^n$ (i.e., corresponding to integer lattice points near the coordinate axes); while those with $a, b$ near equal have $a + b$ “well away” from $n$ by concavity of the curve $x^p + y^p = n$. Fix $0 < p < 1$ and fix any $0 < \lambda < 1$. For any monomial $z^a w^b$ with $(a, b) \in n(1 - \lambda)C_1$ we have $a + b \leq n(1 - \lambda)$ so that at a point $(z, z)$ the function $\frac{1}{n} \log |z^a w^b|$ takes the value

$$ \frac{1}{n} \log |z|^{a + b} \leq \frac{1}{n} \log |z|^{n(1 - \lambda)} = (1 - \lambda) \log |z|. $$

Since $(1 - \lambda) < 1$, for points $(z, z)$ with $|z|$ sufficiently large, these monomials cannot approach $\log |z| + C$ regardless of $n$.

For $\lambda$ sufficiently close to 1, any remaining monomials $z^a w^b \in \text{Poly}(nC_p)$ with $(a, b) \in nC_p \setminus n(1 - \lambda)C_1$ must have exponents close to $(n, 0)$ or $(0, n)$.
In the sequel we only consider the first case, the second case being identical. Then, there exists some $0 < \lambda_0 < 1/4$, say, such that

$$a > n(1-\lambda_0) \quad \text{and} \quad b < n\lambda_0 \quad \text{with} \quad a + b \leq n.$$  

(2.19)

At a point $(z, z)$ with $|z| \geq 1$ the function $\frac{1}{n} \log |z^a w^b|$ has an upper bound of

$$\frac{1}{n} \log |z^n| = \log |z|,$$

so to complete our argument it suffices to show that

$$\limsup_{n \to \infty} \frac{1}{n} \log \|z^a w^b\|_{L^2(\mu)}^{-1} < \frac{1}{2} \log 2$$

where the lim sup is taken over monomials $z^a w^b$ satisfying (2.19). Estimating

$$\|z^a w^b\|_{L^2(\mu)}^{-2} \simeq \frac{(a + b)!}{a! b!} \leq \frac{(n\lambda_0 + 1) \cdots n}{a!} \leq \frac{n!}{(n(1 - \lambda_0))! (n\lambda_0)!},$$

and using Stirling’s formula,

$$\frac{n!}{(n(1 - \lambda_0))! (n\lambda_0)!} \simeq \frac{(n/e)^n}{(n(1 - \lambda_0)/e)^{n(1-\lambda_0)} (n\lambda_0/e)^{n\lambda_0}} = (1 - \lambda_0)^{-n(1-\lambda_0)} \lambda_0^{-n\lambda_0}.$$

Setting $L_n$ for this last expression, we have

$$\frac{1}{n} \log L_n = - \left( (1 - \lambda_0) \log(1 - \lambda_0) + \lambda_0 \log \lambda_0 \right).$$

We want to show that, for $\lambda_0 < 1/4$, this last quantity is smaller than $\log 2$. The function $f(x) := -[(1-x) \log(1-x) + x \log x]$ is increasing on $(0, 1/2)$, decreasing on $(1/2, 1)$, and has a maximum at $x = 1/2$ with $f(1/2) = \log 2$; this gives the result. □

3. Non-convex Bernstein–Walsh

We continue to let $C$ be the closure of an open, connected set satisfying (1.10). Given $K \subset \mathbb{C}^d$ compact, as in the convex setting for $f \in C(K)$ we define

$$d_n^C(f, K) := \inf_{p \in \text{Poly}(nC)} \|f - p\|_K$$
where as before

\[ \text{Poly}(nC) := \{ p(z) = \sum_{J \in nC \cap \mathbb{N}^d} c_J z^J = \sum_{J \in nC \cap \mathbb{N}^d} c_J z_1^{j_1} \cdots z_d^{j_d}, \; c_J \in \mathbb{C} \}, \]

\( n = 1, 2, \ldots \). Note that the dimension of Poly\((nC)\) is proportional to \( \text{vol}(C) \cdot n^d \) where \( \text{vol}(C) \) is the \( d \)-dimensional volume of \( C \). In this section, we consider generalizations of Theorem 1.1.

**Proposition 3.1.** Let \( K \subset C^d \) be compact with \( V_{C,K} \) continuous and satisfying (2.10) locally uniformly in \( C^d \). Suppose for some \( R > 1 \) we have \( f \in C(K) \) which satisfies

\[ \limsup_{n \to \infty} \left[ d_n^C(f, K) \right]^{1/n} \leq 1/R. \]

Then \( f \) extends holomorphically to the open set

\[ \Omega_{R,C} := \{ z \in C^d : V_{C,K}(z) < \log R \}. \]

**Proof.** Under the assumption that

\[ \lim_{n \to \infty} \frac{1}{n} \log \Phi_n(z) = V_{C,K}(z) \quad \text{locally uniformly in } C^d, \]

we have an asymptotic Bernstein–Walsh inequality: for any \( E \subset C^d \) compact and \( \varepsilon > 0 \), we have \( n_0 = n_0(\varepsilon, K) \) so that

\[ |p_n(z)| \leq \| p_n \|_K e^{n(V_{C,K}(z)+\varepsilon)}, \; z \in E \]

for any \( p \in \text{Poly}(nC) \) and \( n > n_0 \). This follows since

\[ \frac{1}{n} \log \frac{|p_n(z)|}{\| p_n \|_K} \leq \frac{1}{n} \log \Phi_n(z) \leq V_{C,K}(z) + \varepsilon \]

for \( z \in E \) and \( n > n_0 \) by (3.1). We use this to show that if \( p_n \in \text{Poly}(nC) \) satisfies \( d_n^C(f, K) = \| f - p_n \|_K \), then the series \( p_0 + \sum_1^\infty (p_n - p_{n-1}) \) converges uniformly on compact subsets of \( \Omega_{R,C} \) to a holomorphic function \( F \) which agrees with \( f \) on \( K \). To this end, choose \( R' \) with \( 1 < R' < R \); by hypothesis the polynomials \( p_n \) satisfy

\[ \| f - p_n \|_K \leq \frac{M}{R^n}, \quad n = 0, 1, 2, \ldots, \]

for some \( M > 0 \). Let \( \rho \) satisfy \( 1 < \rho < R' < R \). Fix \( \varepsilon > 0 \) sufficiently small so that \( 1 < \rho < \rho e^\varepsilon < R' \), and apply the definition of \( V_{C,K} \) and \( \Omega_{\rho,C} \) with
the asymptotic Bernstein–Walsh estimate on $E = \overline{\Omega}_{\rho,C}$ and the polynomial $p_n - p_{n-1} \in \text{Poly}(nC)$ to obtain

$$\sup_{\Omega_{\rho,C}} |p_n(z) - p_{n-1}(z)| \leq \rho^n e^{n\varepsilon} \|p_n - p_{n-1}\|_K$$

$$\leq \rho^n e^{n\varepsilon} (\|p_n - f\|_K + \|f - p_{n-1}\|_K) \leq \rho^n e^{n\varepsilon} \frac{M(1 + R')}{R'^n}.$$ 

Since $\rho$ and $R'$ were arbitrary numbers satisfying $1 < \rho < R' < R$, we conclude that $p_0 + \sum_{n=1}^{\infty} (p_n - p_{n-1})$ converges locally uniformly on $\Omega_{\rho,C}$ to a holomorphic function $F$. From (3.2), $F = f$ on $K$. □

The direct converse of Proposition 3.1 is false, in general; simple examples can be constructed using product sets (use Propositions 2.2 and 3.8 (below)). Our goal is to determine what one can say in the opposite direction. If $F$ is holomorphic in a neighborhood of $K \subset \mathbb{C}^d$ compact, then $\limsup_{n \to \infty} d_n^C(F, K)^{1/n} < 1$ and from (1.10) we then have $\limsup_{n \to \infty} d_n^C(F, K)^{1/n} < 1$. We will use the following lemma, which says that the asymptotic behavior of the rates of polynomial approximation of $F \in C(K)$ by $\text{Poly}(nC)$ in the sup norm on $K$ and in the $L^2$ norm with respect to a Bernstein–Markov measure (2.6) on $K$ are the same, in this $n$-th root sense.

**Lemma 3.2.** Let $K \subset \mathbb{C}^d$ be compact and nonpluripolar. Let $\mu$ be a probability measure on $K$ which satisfies the Bernstein–Markov property (2.6). Let $F \in C(K)$. Let $C$ be a compact, connected subset of $\mathbb{R}^d_+$ with nonempty interior, such that $a\Sigma \subset C$ for some $a > 0$. Assume that

$$\limsup_{n \to \infty} (d_n^C(F, K))^{1/n} =: \rho_\infty < 1.$$ 

If $\{p_n\}$ is a sequence of best $L^2_\mu$ approximants to $F$ with $p_n \in \text{Poly}(nC)$ then

$$(3.3) \quad \limsup_{n \to \infty} \|F - p_n\|_{\mu}^{1/n} = \limsup_{n \to \infty} \|F - p_n\|_K^{1/n} = \rho_\infty.$$ 

**Proof.** Let $r$ such that $\rho_\infty < r < 1$. For $n$ large enough, there exists $q_n \in \text{Poly}(nC)$ such that

$$\|F - q_n\|_K \leq r^n.$$ 

Hence, for the sequence $p_n$, we have

$$\|F - p_n\|_\mu \leq \|F - q_n\|_\mu \leq \|F - q_n\|_K \leq r^n,$$

and, in particular, $p_n$ converges to $F$ in $L^2_\mu$. Moreover, for $k$ large,

$$\|p_{k+1} - p_k\|_\mu \leq \|F - p_{k+1}\|_\mu + \|F - p_k\|_\mu \leq 2r^k.$$
By the Bernstein–Markov property (2.6), for a given $\varepsilon > 0$ such that $\tilde{r} = r(1 + \varepsilon) < 1$, we have, for $k$ large,

$$\|p_{k+1} - p_k\|_K \leq (1 + \varepsilon)^k\|p_{k+1} - p_k\|_\mu \leq 2\tilde{r}^k,$$

and thus $p_n = p_1 + \sum_{k=1}^{n-1} (p_{k+1} - p_k)$ converges uniformly to $F$ on $K$. Moreover,

$$\|F - p_n\|_K = \left\| \sum_{k=n}^{\infty} (p_{k+1} - p_k) \right\|_K \leq 2\frac{\tilde{r}^n}{1 - \tilde{r}}.$$

Letting $r$ tend to $\rho_\infty$, $\varepsilon$ tend to 0, and taking $n$-th roots, proves the second equality in (3.3). For the first equality, set

$$\rho_\mu := \limsup_{n \to \infty} \|F - p_n\|_\mu^{1/n} \leq \rho_\infty.$$

We have, for $\rho_\mu < r < 1$, $\tilde{r} = r(1 + \varepsilon)$, and $n$ large,

$$\|F - p_n\|_K \leq \sum_{k=n}^{\infty} \|p_{k+1} - p_k\|_K \leq \sum_{k=n}^{\infty} (1 + \varepsilon)^k\|p_{k+1} - p_k\|_\mu \leq 2\sum_{k=n}^{\infty} \tilde{r}^k = 2\frac{\tilde{r}^n}{1 - \tilde{r}}.$$

Letting $r$ tend to $\rho_\mu$, $\varepsilon$ tend to 0, and taking $n$-th roots finishes the proof. □

For simplicity, in the rest of this section we work in $\mathbb{C}^2$. The "standard" version of Theorem 1.1 is the case where $C$ is the simplex $\Sigma$ defined in (1.2) with $d = 2$. As a first attempt at a converse to Proposition 3.1, we let $C_p$ be the set in $\mathbb{R}^2_+$ defined by

$$C_p = \{ (x, y) \in \mathbb{R}^2, x, y \geq 0, x^p + y^p \leq 1 \}, \quad 0 < p \leq 1,$$

i.e., the $d = 2$ case of (1.11). For $0 < p < 1$, $C_p$ is a non-convex body satisfying (1.10), and for $p = 1$, $C_1 = \Sigma$. For $0 < \alpha < 1$, let $T_\alpha$ be the triangle with vertices $(0, 0), (0, \alpha), (\beta, 0)$ where $\beta > 0$ is such that the side from $(0, \alpha)$ to $(\beta, 0)$ is tangent to the curve $x^p + y^p = 1$. Note $\beta$ is a function of $\alpha$. Let $A_p$ and $A_\alpha$ denote the square roots of the areas of $C_p$ and $T_\alpha$, i.e.,

$$(3.4)\quad A_p^2 := \frac{\Gamma(1/p)\Gamma(1 + 1/p)}{p\Gamma(1 + 2/p)}, \quad A_\alpha^2 := \frac{\alpha \beta}{2},$$

where $\Gamma(x)$ denotes the Gamma function.

Our aim is to compare the rates of approximation with respect to a non-convex $C_p$ ($0 < p < 1$), and with respect to the family of inscribed convex

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triangles $T_\alpha$, $0 < \alpha < 1$. Of course, since $T_\alpha \subset C_p$, for a compact set $K \subset \mathbb{C}^2$ and a function $F$ on $K$, we have

\[(3.5)\quad d_{n^p}(F, K) \leq \inf_{0 < \alpha < 1} d_{n^\alpha}(F, K).\]

The comparison is less clear, and more interesting, if we take into account that the number of monomials in $nC_p$ is larger than the number of monomials in $nT_\alpha$, and thus normalize accordingly, by comparing $d_{n^p}(F, K)^1/A_p$ and $d_{n^\alpha}(F, K)^1/A_\alpha$; i.e.,

\[(3.6)\quad d_{n^p/A_p}(F, K) \quad \text{and} \quad d_{n^\alpha/A_\alpha}(F, K)\]

where by abuse of notation, we write $d_{n^p/A_p}(F, K)$ to denote $d_{n^p/A_p}(F, K)$. In the sequel, we estimate these two quantities explicitly in two extreme cases, namely when the function $F$ is of the form

\[F(z, w) = f(z) + g(w)\]

and when \[F(z, w) = f(zw)\].

Let us start with the first case, and consider a subset $K = A \times B \subset \mathbb{C}^2$ where $A$ and $B$ are regular compact subsets of $\mathbb{C}$. We will denote by $\mu := \mu_A \otimes \mu_B$ the measure on $K$ arising from Bernstein–Markov measures $\mu_A, \mu_B$ on $A, B$. Let $f$ and $g$ be holomorphic functions in neighborhoods of $A$ and $B$. We denote by $\rho_A := \rho_A(f)$ and $\rho_B := \rho_B(g)$ the asymptotic rate (in the $n$-th root sense) of best uniform univariate polynomial approximation to $f$ on $A$ and to $g$ on $B$; i.e., letting $P_n(\mathbb{C})$ denote the univariate (holomorphic) polynomials of degree at most $n$,

\[\rho_A(f) := \limsup_{n \to \infty} \sqrt[n]{\inf_{p_n \in P_n(\mathbb{C})} \|f - p_n\|_A} \quad \text{and} \quad \rho_B(g) \leq \rho_B(g) \leq 1.\]

Define the function of two variables

\[F(z, w) = f(z) + g(w)\]

We begin with a lemma which is applicable to the sets $C_p$ when $0 \leq p \leq 1$.

**Lemma 3.3.** Let $C$ be a subset of $\mathbb{R}^2_+$ and assume that

\[C \subset [0, 1]^2, \quad C \cap (\mathbb{R}_+ \times \{0\}) = [0, 1] \times \{0\}, \quad C \cap (\{0\} \times \mathbb{R}_+) = \{0\} \times [0, 1].\]

Let $P_n(z, w)$ be the best $L_2^p$ approximant to $F(z, w)$ in $\text{Poly}(nC)$. Then

\[P_n(z, w) = t_n^f(z) + t_n^g(w),\]

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where $t^f_n$ and $t^g_n$ are the best $L^2_{\mu_A}$ and best $L^2_{\mu_B}$ approximants to $f$ and $g$ in $P_n(\mathbb{C})$, the space of polynomials in one variable of degree less than or equal to $n$.

**Proof.** Assume that

$$p_0(z), p_1(z), \ldots, p_n(z), \quad \deg p_k = k, \quad k = 0, \ldots, n,$$

$$q_0(z), q_1(z), \ldots, q_n(z), \quad \deg q_k = k, \quad q = 0, \ldots, n,$$

are orthonormal bases in $L^2_{\mu_A}$ and $L^2_{\mu_B}$. Then, the family of polynomials $p_k(z)q_l(w), \ (k, l) \in n\mathbb{C}$, is an orthonormal basis of $\text{Poly}(n\mathbb{C})$. Moreover,

$$\int f(z)p_k(z)q_l(w) \, d\mu_A(z) \, d\mu_B(w) = \delta_{l,0} \int f(z)p_k(z) \, d\mu_A(z), \quad l \geq 0,$$

and similarly,

$$\int g(w)p_l(z)q_k(w) \, d\mu_A(z) \, d\mu_B(w) = \delta_{l,0} \int g(w)q_k(w) \, d\mu_B(w), \quad l \geq 0.$$

The statement of the lemma follows. □

**Remark 3.4.** If for some $\alpha, \beta \leq 1$, $C \subset [0, 1]^2$ satisfies

$$C \cap (\mathbb{R}_+ \times \{0\}) = [0, \alpha] \times \{0\}, \quad C \cap (\{0\} \times \mathbb{R}_+) = \{0\} \times [0, \beta],$$

a similar proof shows that $P_n(z, w) = t^f_{\lfloor \alpha n \rfloor}(z) + t^g_{\lfloor \beta n \rfloor}(w)$.

We now compute the asymptotic rates of approximation with respect to the sets $C_p$ and the family $T_\alpha, 0 < \alpha < 1$.

**Proposition 3.5.** We have, for $0 < p \leq 1$,

$$(3.7) \quad \limsup_n d_n^{C_p}(F, K)^{1/n} = \max(\rho_A, \rho_B).$$

**Proof.** In view of Lemma 3.2, it is equivalent to estimate the rate of best $L^2$ approximation to $F$. Let $P_n \in \text{Poly}(nC_p)$ be the best $L^2$ approximants to $F$ with respect to the measure $\mu$. From Lemma 3.3, we get

$$\|F - P_n\|_\mu^2 = \|f(z) - t^f_n(z) + g(w) - t^g_n(w)\|_\mu^2 = \|f - t^f_n\|_{\mu_A}^2 + \|g - t^g_n\|_{\mu_B}^2,$$

where we use the fact that

$$\int (f - t^f_n)(z) \, d\mu_A(z) = 0, \quad \int (g - t^g_n)(w) \, d\mu_B(w) = 0.$$
Next, making use of the one-variable version of Lemma 3.2, we have

$$\limsup_{n \to \infty} \| f - t_n^f \|_{\mu_A}^{1/n} = \rho_A, \quad \limsup_{n \to \infty} \| g - t_n^g \|_{\mu_B}^{1/n} = \rho_B.$$ 

The proposition follows. □

**Proposition 3.6.** We have, for $0 < p \leq 1$,

$$\limsup_n d_{n}^{T_\alpha}(F, K)^{1/n} \leq \max(\rho_A^\alpha, \rho_B^\beta).$$

**Proof.** From Remark 3.4, we now get

$$\| F - P_n \|_{\mu}^2 = \| f(z) - t_{[\alpha n]}^f(z) + g(w) - t_{[\beta n]}^g(w) \|_{\mu}^2 = \| f - t_{[\alpha n]}^f \|_{\mu_A}^2 + \| g - t_{[\beta n]}^g \|_{\mu_B}^2.$$ 

Since

$$\limsup_{n \to \infty} \| f - t_{[\alpha n]}^f \|_{\mu_A}^{1/n} = \rho_A^\alpha, \quad \limsup_{n \to \infty} \| g - t_{[\beta n]}^g \|_{\mu_B}^{1/n} = \rho_B^\beta,$$

the proposition follows. □

Now recall that comparing the limits of the $n$-th roots of the two rates in (3.6) is equivalent to comparing

$$\limsup_n d_{n}^{C_p}(F, K)^{1/(A_p n)} = \max(\rho_A, \rho_B)^{1/A_p}$$

and

$$\limsup_n d_{n}^{T_\alpha}(F, K)^{1/(A_\alpha n)} = (\max(\rho_A^\alpha, \rho_B^\beta))^{1/A_\alpha}.$$

**Theorem 3.7.** When $\alpha$ and $\beta$ are close to each other, that is close to $(1/2)^{1/p-1}$, and the triangle $T_\alpha$ is close to an isosceles triangle, one has

$$\limsup_n (d_{n/A_\alpha}^{T_\alpha}(F, K))^{1/n} \leq \limsup_n (d_{n/A_p}^{C_p}(F, K))^{1/n},$$

while, when $\alpha$ is close to 0 and $\beta$ close to 1 (or the reverse), and the triangle $T_\alpha$ becomes very small, one has the opposite inequality

$$\limsup_n (d_{n/A_p}^{C_p}(F, K))^{1/n} \leq \limsup_n (d_{n/A_\alpha}^{T_\alpha}(F, K))^{1/n}.$$ 

**Proof.** By symmetry, we may assume without loss of generality that $\rho_B \leq \rho_A$.

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Assume first that $\alpha = \beta$. Then, we have to show that $\rho_A^{\sqrt{2}} \leq \rho_A^{1/A_p}$, that is $1/A_p \leq \sqrt{2}$. In view of (3.4), this is equivalent to

$$p\Gamma(1 + 2/p) \leq 2\Gamma(1/p)\Gamma(1 + 1/p) \iff \Gamma(1 + 2/p) \leq 2\Gamma(1 + 1/p)^2,$$

which is easily seen to be true by computing derivatives. Moreover, equality holds only if $p = 1$. Thus, by continuity, if $p \neq 1$, inequality (3.8) still holds when $\alpha$ and $\beta$ are close to each other.

Now, consider the case when $\alpha$ is close to 0 and $\beta$ is close to 1. Then $\rho_A^\alpha$ is close to 1 while $\rho_B^\beta$ is close to $\rho_B$, so $\max(\rho_A^\alpha, \rho_B^\beta) = \rho_A^\alpha$. Moreover $1/A_\alpha \simeq \sqrt{2/\alpha}$, so the limit of the rates corresponding to $T_\alpha$ is close to 1, while the limit of the rates corresponding to $C_p$ remains a number less than 1. □

We now consider the case of a function $F(z, w)$ of the form

$$F(z, w) = f(zw),$$

where we assume that the largest disk centered at the origin in $\mathbb{C}$ contained in the domain of analyticity of $f$ is the disk $D_R$ of radius $R > 1$. Also let $K = \mathbb{D} \times \mathbb{D} = \{(z, w), |z| \leq 1, |w| \leq 1\}$ be the unit polydisk in $\mathbb{C}^2$.

**Proposition 3.8.** We have, for $0 < p \leq 1$, and with $r = 1/R$,

$$(3.9) \quad \limsup_n d_n^{C_p}(F, K)^{1/n} = r^{(1/2)^{1/p}}.$$

**Proof.** Let $\mu_K = \mu \otimes \mu$, $\mu = d\theta/2\pi$, be the normalized measure supported on $\mathbb{T} \times \mathbb{T} = \{(z, w), |z| = 1, |w| = 1\}$. In view of Lemma 3.2, it is sufficient to consider a sequence of best $L^2(\mu_K)$ approximants to $F$. Since the family $\{z^j w^k\}$, $(j, k) \in \mathbb{N}^2$, is orthogonal with respect to $\mu_K$, we have

$$\limsup_n d_n^{C_p}(F, K)^{1/n} = \limsup_n \|f - p_{na_p}\|_{\mu}^{1/n},$$

where $a_p = (1/2)^{1/p}$ is such that point $(a_p, a_p)$ is the intersection of the curve $x^p + y^p = 1$ and the line $x = y$, and $p_{na_p}$ denotes the best $L^2(\mu)$ polynomial approximant to $f$ of degree at most $na_p$. Moreover, by using the one variable versions of Lemma 3.2 and Theorem 1.1, we get

$$\limsup_n \|f - p_{na_p}\|_{\mu}^{1/n} = \limsup_n d_{na_p}(f, \mathbb{D})^{1/n} = r^a_p,$$

which proves (3.9). □

**Proposition 3.9.** We have, for $0 < p \leq 1$,

$$(3.10) \quad \limsup_n d_n^{T_\alpha}(F, K)^{1/n} = r^{\alpha\beta/\alpha + \beta}.$$
Proof. The proof is identical to the proof of Proposition 3.8. The only change is that we now need to consider the point which is at the intersection of the line $\beta y = \alpha (\beta - x)$, the side of the triangle tangent to $C_p$, and the line $x = y$. This point has both coordinates equal to $\alpha \beta / (\alpha + \beta)$, which implies the result. □

From the two previous propositions, we may make more precise inequality (3.5), and also compare the normalized rates of approximation.

Theorem 3.10. The following holds true,

$$\limsup_n \left( d_{C_p}^n(F, K) \right)^{1/n} = \inf_{\alpha} \limsup_n \left( d_{T_\alpha}^n(F, K) \right)^{1/n}. $$

The inf on the right-hand side is attained when $\alpha = \alpha_p = (1/2)^{1/p-1}$, which corresponds to the isosceles triangle $T_\alpha$ such that $\alpha = \beta$. For the rates normalized by the areas of $C_p$ and $T_\alpha$, we have the following. If $\alpha$ and $\beta$ are close to each other,

$$\limsup_n \left( d_{C_p/n/A_\alpha}^n(F, K) \right)^{1/n} \leq \limsup_n \left( d_{C_p/n/A_\beta}^n(F, K) \right)^{1/n},$$

while if one of $\alpha$ or $\beta$ is close to 0, we have the opposite inequality

$$\limsup_n \left( d_{C_p/n/A_\alpha}^n(F, K) \right)^{1/n} \leq \limsup_n \left( d_{C_p/n/A_\beta}^n(F, K) \right)^{1/n}.$$

Proof. In view of (3.10), the inf on the right of (3.11) is attained when $\alpha = \beta$. We obtain the value $r^{ap}$ which equals the lim sup on the left. For the normalized rate, and $\alpha$ and $\beta$ close to each other, the asserted inequality is just a consequence of the fact that $1/A_p < 1/A_\alpha$. When $\alpha$ or $\beta$ is close to 0, one may argue as in the proof of Theorem 3.7. □

Example 3.11. Related to the previous class of functions $F(z, w) = f(zw)$, simple examples show that in the limiting case $p = 0$, the corresponding polynomial classes are too sparse to uniformly approximate even simple bivariate polynomials. We offer a simple geometric argument to show $f(x, y) = xy$ is not uniformly approximable on $[0, 1] \times [0, 1]$ by a sum of univariate polynomials in $x$ and in $y$. Indeed, suppose, given $\epsilon > 0$, one could find $p(x)$ of degree $n$, say, and $q(y)$ of degree $n$, say, with

$$|p(x) + q(y) - xy| < \epsilon \quad \text{for} \quad 0 \leq x, y \leq 1.$$

Then for each fixed $y_0 \in [0, 1],$

$$|p(x) - [y_0 x - q(y_0)]| < \epsilon \quad \text{for} \quad 0 \leq x \leq 1.$$
This says that the function \( p(x) \) simultaneously uniformly approximates the whole family of linear functions \( l_{y_0}(x) := y_0x - q(y_0) \) for \( 0 \leq y_0 \leq 1 \) on the interval \([0, 1]\) (in the \( x \)-variable) which is impossible (note the slopes of the \( l_{y_0} \) vary from 0 to 1).

4. Non-convex random polynomials

Let \( C \) be the closure of an open, connected set satisfying (1.10) and which contains \( C_0 \) in (1.12). We let \( K \) be a nonpluripolar compact set in \( \mathbb{C}^d \) satisfying (2.10). We assume, moreover, that

\[
V_{C,K} \text{ is continuous; i.e., } V_{C,K} = V_{C,K}^*.
\]

Let \( \tau \) be a probability measure on \( K \) such that \((K, \tau)\) satisfies (2.6). Letting \( \{p_j\} \) be an orthonormal basis in \( L^2(\tau) \) for \( \text{Poly}(nC) \) constructed via Gram–Schmidt applied to an ordered monomial basis \( \{z^n\} \) of \( \text{Poly}(nC) \), as described in the introduction we consider random polynomials of \( C \)-degree at most \( n \) of the form

\[
H_n(z) := \sum_{j=1}^{m_n} a_j^{(n)} p_j(z)
\]

where the \( a_j^{(n)} \) are i.i.d. complex random variables with a distribution \( \phi \) satisfying (1.13) and (1.14). Here \( m_n = \dim(\text{Poly}(nC)) \). This gives a probability measure \( \mathcal{H}_n \) on \( \text{Poly}(nC) \) and we form the product probability space

\[
\mathcal{H} := \bigotimes_{n=1}^{\infty} (\text{Poly}(nC), \mathcal{H}_n)
\]

of sequences of random polynomials. We identify \( \mathcal{H} \) with

\[
\mathcal{C} := \bigotimes_{n=1}^{\infty} (\mathbb{C}^{m_n}, \text{Prob}_{m_n})
\]

where, for \( G \subset \mathbb{C}^{m_n} \),

\[
\text{Prob}_{m_n}(G) := \int_G \phi(z_1) \cdots \phi(z_{m_n}) \, dm_2(z_1) \cdots dm_2(z_{m_n}).
\]

In this setting, we recall a result from [6]. Here, we write

\[
a^{(m_n)} = (a_1^{(n)}, \ldots, a_{m_n}^{(n)}) \in \mathbb{C}^{m_n}
\]

and \( \langle \cdot, \cdot \rangle, \| \cdot \| \) denote the standard Hermitian inner product and associated norm on \( \mathbb{C}^{m_n} \).
**Corollary 4.1.** Let \( \{ w^{(m_n)} = (w_1^{(n)}, \ldots, w_{m_n}^{(n)}) \} \) be a sequence of vectors \( w^{(m_n)} \in \mathbb{C}^{m_n} \). For \( \phi \) satisfying (1.13) and (1.14), with probability one in \( \mathcal{C} \), if \( \{ m_n \} \) is a sequence of positive integers with \( m_n = O(n^M) \) for some \( M \), then

\[
\forall \{ w^{(m_n)} \}, \quad \limsup_{n \to \infty} \frac{1}{n} \log |\langle a^{(m_n)}, w^{(m_n)} \rangle| \leq \limsup_{n \to \infty} \frac{1}{n} \log \| w^{(m_n)} \|.
\]

Moreover, for each \( \{ w^{(m_n)} \} \),

\[
\liminf_{n \to \infty} \frac{1}{n} \log |\langle a^{(m_n)}, w^{(m_n)} \rangle| \geq \liminf_{n \to \infty} \frac{1}{n} \log \| w^{(m_n)} \|
\]

with probability one in \( \mathcal{C} \); i.e., for each \( \{ w^{(m_n)} \} \), the set

\[
\{ \{ a^{(m_n)} := (a_1^{(n)}, \ldots, a_{m_n}^{(n)}) \}_{n=1,2,\ldots} \in \mathcal{C} : (4.3) \text{ holds} \}
\]

depends on \( \{ w^{(m_n)} \} \) but is always of probability one.

Using Proposition 2.6 and Corollary 4.1, we can follow the proof of [6, Theorem 4.1].

**Theorem 4.2.** Let \( K \) satisfy (2.10) and (4.1) and let \( a_j^{(n)} \) be i.i.d. complex random variables with a distribution \( \phi \) satisfying (1.13) and (1.14). Then almost surely in \( \mathcal{H} \) we have

\[
\left( \limsup_{n \to \infty} \frac{1}{n} \log |H_n(z)| \right)^* = V_{C,K}(z), \quad z \in \mathbb{C}^d.
\]

**Proof.** Using the first part of Corollary 4.1, (4.2), with

\[
w^{(n)} = p^{(n)}(z) := (p_1(z), \ldots, p_{m_n}(z)) \in \mathbb{C}^{m_n},
\]

almost surely in \( \mathcal{H} \)

\[
\limsup_{n \to \infty} \frac{1}{n} \log |H_n(z)| \leq V_{C,K}(z), \quad z \in \mathbb{C}^d
\]

from Proposition 2.6. Fix a countable dense subset \( \{ z_t \}_{t \in S} \) of \( \mathbb{C}^d \). Using the second part of Corollary 4.1, (4.3), for each \( z_t \), almost surely in \( \mathcal{H} \) we have

\[
\liminf_{n \to \infty} \frac{1}{n} \log |H_n(z_t)| \geq V_{C,K}(z_t).
\]

A countable intersection of sets of probability one is a set of probability one; thus (4.5) holds almost surely in \( \mathcal{H} \) for each \( z_t, t \in S \).

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Define
\[ H(z) := \left( \limsup_{n \to \infty} \frac{1}{n} \log |H_n(z)| \right)^*. \]
From (4.4), since \( V_{C,K} = V_{C,K}^* \), almost surely in \( H \), \( H(z) \leq V_{C,K}(z) \) for all \( z \in \mathbb{C}^d \). Moreover, from (2.12), almost surely in \( H \) we have \( \{ \frac{1}{n} \log |H_n(z)| \} \) is locally bounded above and hence \( H \) is plurisubharmonic; indeed, \( H \in L^1(\mathbb{C}^d) \). By (4.5), \( H(z_t) \geq V_{C,K}(z_t) \) for all \( t \in S \). Now given \( z \in \mathbb{C}^d \), let \( S' \subset S \) with \( \{ z_t \}_{t \in S'} \) converging to \( z \). Then, since \( V_{C,K} \) is continuous at \( z \),
\[ V_{C,K}(z) = \lim_{t \in S' \atop z_t \to z} V_{C,K}(z_t) \leq \limsup_{t \in S' \atop z_t \to z} H(z_t) \leq H(z). \]

Thus \( H(z) = V_{C,K}(z) \) for all \( z \in \mathbb{C}^d \). \( \square \)

To obtain convergence of linear differential operators applied to
\[ \frac{1}{n} \log |H_n(z)|, \]
we verify convergence to \( V_{C,K}(z) \) in \( L^1_{loc}(\mathbb{C}^d) \).

**Theorem 4.3.** Let \( K \) satisfy (2.10) and (4.1) and let \( a_j^{(n)} \) be i.i.d. complex random variables with a distribution \( \phi \) satisfying (1.13) and (1.14). Then almost surely in \( H \) we have
\[ \lim_{n \to \infty} \frac{1}{n} \log |H_n(z)| = V_{C,K}(z) \]
in \( L^1_{loc}(\mathbb{C}^d) \) and hence
\[ \lim_{n \to \infty} dd^c \left( \frac{1}{n} \log |H_n(z)| \right) = dd^c V_{C,K}(z) \]
as positive currents, where \( dd^c = \frac{i}{\pi} \partial \overline{\partial} \).

As in [6], the proof of Theorem 4.3 will follow from Proposition 2.5 and a modification of the proof of Theorem 4.2.

**Proof of Theorem 4.3.** From Proposition 2.5, we need to show almost surely in \( H \) that for any subsequence \( J \) of positive integers, we have
\[ \left( \limsup_{n \in J} \frac{1}{n} \log |H_n(z)| \right)^*= V_{C,K}(z) \]
for all \( z \in \mathbb{C}^d \). Fix any subsequence \( J \). Following the proof of Theorem 4.2, almost surely in \( \mathcal{H} \)

\[
\limsup_{n \in J} \frac{1}{n} \log |H_n(z)| \leq \limsup_{n \to \infty} \frac{1}{n} \log |H_n(z)| \leq V_{C,K}(z)
\]

for all \( z \in \mathbb{C}^d \) from (4.4) and the fact that \( J \) is a subsequence of positive integers. Fix a countable dense subset \( \{z_t\}_{t \in S} \) of \( \mathbb{C}^d \). Then for each \( z_t \), almost surely in \( \mathcal{H} \) we have

\[
\liminf_{n \in J} \frac{1}{n} \log |H_n(z_t)| \geq \liminf_{n \to \infty} \frac{1}{n} \log |H_n(z_t)| \geq V_{C,K}(z_t)
\]

from (4.5) and the fact that \( J \) is a subsequence of positive integers. This relation holds almost surely in \( \mathcal{H} \) for each \( z_t, t \in S \).

Now define

\[
H_J(z) := \left( \limsup_{n \in J} \frac{1}{n} \log |H_n(z)| \right)^*.
\]

Then almost surely in \( \mathcal{H} \), \( H_J \) is plurisubharmonic and \( H_J(z) \leq V_{C,K}(z) \) for all \( z \in \mathbb{C}^d \); and \( H_J(z_t) \geq V_{C,K}(z_t) \) for all \( t \in S \). Given \( z \in \mathbb{C}^d \), let \( S' \subset S \) with \( \{z_t\}_{t \in S'} \) converging to \( z \). Then

\[
V_{C,K}(z) = \lim_{t \in S'} V_{C,K}(z_t) \leq \limsup_{t \in S'} H_J(z_t) \leq H_J(z).
\]

Thus \( H_J(z) = V_{C,K}(z) \) for all \( z \in \mathbb{C}^d \). \( \square \)

We write \( Z_{H_n} := dd^c \log |H_n| \) and \( \tilde{Z}_{H_n} := \frac{1}{n} dd^c \log |H_n| \), the normalized zero current of \( H_n \). The expectation \( \mathbb{E}(\tilde{Z}_{H_n}) \) of \( \tilde{Z}_{H_n} \) is a positive current of bidegree \((1,1)\) defined as follows: the action of \( \mathbb{E}(\tilde{Z}_{H_n}) \) on a \((d-1,d-1)\) form \( \alpha \) with \( C_0^\infty(\mathbb{C}^d) \) coefficients is given as the average of the action \((\tilde{Z}_{H_n}, \alpha)\) of the normalized zero current \( \tilde{Z}_{H_n} \) on \( \alpha \):

\[
(\mathbb{E}(\tilde{Z}_{H_n}), \alpha) := \int_{\mathbb{C}^{m_n}} (\tilde{Z}_{H_n}, \alpha) \, d\text{Prob}_{m_n}(\alpha^{(n)})
\]

\[
= \int_{\mathbb{C}^{m_n}} (\frac{1}{n} dd^c \log |H_n|, \alpha) \, d\text{Prob}_{m_n}(\alpha^{(n)}).
\]

Using Theorem 4.3, we verify the analogue of [6, Theorem 7.1].

**Theorem 4.4.** Let \( K \) satisfy (2.10) and (4.1) and let \( \alpha_j^{(n)} \) be i.i.d. complex random variables with a distribution \( \phi \) satisfying (1.13) and (1.14). Then

\[
\lim_{n \to \infty} \mathbb{E}(\tilde{Z}_{H_n}) = dd^c V_{C,K} \text{ as positive (1,1) currents.}
\]

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Proof. Theorem 4.3 gives
\[
\lim_{n \to \infty} dd^c \left( \frac{1}{n} \log |H_n| \right) = dd^c V_{C,K}
\]
as positive currents a.s. in $\mathcal{H}$. We want to show
\[
\lim_{n \to \infty} \left( \mathbb{E}(\tilde{Z}_{H_n}), \alpha \right) = (dd^c V_{C,K}, \alpha)
\]
for each $(d - 1, d - 1)$ form $\alpha$ with $C^\infty_0(\mathbb{C}^d)$ coefficients. As before, we write $a^{(n)}$ for the $m_n$-tuple $\{a_j^{(n)}\}$ of coefficients of $H_n$. Given $\alpha$, define
\[
f_n = f^{(\alpha)}_n : \mathbb{C}^{m_n} \to \mathbb{C}
\]
as $f_n^{(a^{(n)})} := (\tilde{Z}_{H_n}, \alpha)$.
Then $\{f_n\}$ are uniformly bounded by the norm of $\alpha$ on its support and extending $f_n$ to $F_n$ on $\mathcal{H}$ via
\[
F_n(\ldots, a^{(n)}, \ldots) := f_n(a^{(n)}),
\]
the $\{F_n\}$ are uniformly bounded on $\mathcal{H}$. Define
\[
\int_{\mathcal{H}} F_n(\ldots, a^{(n)}, \ldots) \otimes_{n=1}^\infty d \text{Prob}_{m_n}(a^{(n)}) := \int_{\mathbb{C}^{m_n}} f_n(a^{(n)}) d \text{Prob}_{m_n}(a^{(n)}).
\]
We apply dominated convergence to $\{F_n\}$ on $\mathcal{H}$ to conclude. □

Remark 4.5. Following [6], using different arguments one can eliminate the need for (1.13) in Theorem 4.4. Moreover, one can slightly weaken the hypothesis (1.14) as in [2].

For $2 \leq k \leq d$, we consider the common zeros of $k$ polynomials $H_n^{(1)}, \ldots, H_n^{(k)}$ where
\[
H_n^{(l)}(z) := \sum_{j=1}^{m_n} a_j^{(n,l)} p_j(z), \ l = 1, \ldots, k
\]
with the $a_j^{(n,l)}$ i.i.d. complex random variables with a distribution $\phi$ satisfying (1.13) and (1.14). For $k = 2, 3, \ldots, d$, we observe that the wedge product
\[
Z_k^{H_n} := dd^c \log |H_n^{(1)}| \wedge \cdots \wedge dd^c \log |H_n^{(k)}|
\]
is a.s. well-defined as a positive $(k, k)$ current; cf., [2] or [6]. We write $\tilde{Z}_n^{k} := (1/n^k)Z_k^{H_n}$ for the normalized zero current. The expectation $\mathbb{E}(\tilde{Z}_n^{k})$ of $\tilde{Z}_n^{k}$ is a positive $(k, k)$ current: for $k = 2, 3, \ldots, d$, the action of $\mathbb{E}(\tilde{Z}_n^{k})$ on a
(d – k, d – k) form α with $C_0^\infty(\mathbb{C}^d)$ coefficients is given as the average of the action $(\tilde{Z}_{H_n}, \alpha)$ of the normalized zero current $\tilde{Z}_{H_n}$ on α: writing $a^{(n,l)} = (a_1^{(n,l)}, \ldots, a_m^{(n,l)})$, $l = 1, \ldots, k$,

$$
(\mathbb{E}(\tilde{Z}_{H_n}), \alpha) := \int_{(\mathbb{C}^m)^k} (\tilde{Z}_{H_n}, \alpha) d\text{Prob}_{m_n}(a^{(n,1)}) \cdots d\text{Prob}_{m_n}(a^{(n,k)})
$$

$$
= \int_{(\mathbb{C}^m)^k} (dd^c \log |H_n^{(1)}| \land \cdots \land dd^c \log |H_n^{(k)}|, \alpha)
$$

$$
\times d\text{Prob}_{m_n}(a^{(n,1)}) \cdots d\text{Prob}_{m_n}(a^{(n,k)}).
$$

If, in addition, the distribution $\phi$ is smooth (e.g., $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\pi|z|^2}$, a standard complex Gaussian), by the independence of $H_n^{(1)}, \ldots, H_n^{(k)}$ we have

$$
\mathbb{E}(\tilde{Z}_{H_n}^k) = \mathbb{E}\left(\frac{1}{n^k} dd^c \log |H_n^{(1)}| \land \cdots \land dd^c \log |H_n^{(k)}|\right)
$$

$$
= \mathbb{E}(\tilde{Z}_{H_n^{(1)}}) \land \cdots \land \mathbb{E}(\tilde{Z}_{H_n^{(k)}}) = [\mathbb{E}(\tilde{Z}_{H_n^{(1)}})]^k
$$

(cf., the argument in Corollary 3.3 of [1]). Thus from Theorem 4.4 we obtain the asymptotics of these $(k, k)$ currents.

**Corollary 4.6.** Let $K$ satisfy (2.10) and (4.1) and let $a_j^{(n,l)}$ be i.i.d. complex random variables with a smooth distribution $\phi$ satisfying (1.13) and (1.14). Then for $k = 2, \ldots, d$,

$$
\lim_{n \to \infty} \mathbb{E}(\tilde{Z}_{H_n}^k) = \mathbb{E}\left(\frac{1}{n^k} dd^c \log |H_n^{(1)}| \land \cdots \land dd^c \log |H_n^{(k)}|\right) = (dd^c V_{C,K})^k.
$$

Taking $K = E_1 \times \cdots \times E_d$, a product of regular planar compacta $E_j$, we have

$$
V_{C,K}(z_1, \ldots, z_d) = \max_{j=1, \ldots, d} g_{E_j}(z_j).
$$

Letting $k = d$ in Corollary 4.6, we have the following result.

**Corollary 4.7.** For $K = E_1 \times \cdots \times E_d$, a product of regular planar compacta $E_j$, we have

$$
\lim_{n \to \infty} \mathbb{E}(\tilde{Z}_{H_n}^d) = (dd^c V_{C,K})^d = \bigotimes_{j=1}^d \mu_{E_j}
$$

where $\mu_{E_j} = \Delta g_{E_j}$.
Remark 4.8. This holds, e.g., for $C_p$ in (1.11) for all $0 \leq p \leq 1$ (for $p = 0$, see Remark 4.9 below). For $p = 0$ the classes $\text{Poly}(nC_0)$ are very sparse – they consist of sums $\sum_{j=1}^d p_j(z_j)$ of univariate polynomials $p_j$ of degree at most $n$ while for $p = 1$ the classes $\text{Poly}(nC_1) = \text{Poly}(n\Sigma)$ are the “standard” polynomials of degree at most $n$. On the other hand, for $p > 1$ the set $C_p$ is convex, and from (2.5) and (1.5), if we let $1/p + 1/q = 1$, the classes $\text{Poly}(nC_p)$ are the “standard” polynomials of degree at most $n$.

The results in [2] show that the expected normalized zero measures $E(\tilde{Z}_{H_n}^d)$ for the random polynomial mappings in this setting converge to

$$(dd^cV_{C_p,K})^d = dd^c([g_{E_1}(z_1)^q + \cdots + g_{E_d}(z_d)^q]^{1/q})^d$$

which clearly changes with $p$.

Remark 4.9. As in Subsection 2.3, if $\mu$ is a Bernstein–Markov measure on a nonpluripolar compact set $K \subset \mathbb{C}^d$ satisfying (2.10) and (4.1), the inequality (2.9) is valid for $C_0$, $K$ and $\mu$ and all the results of this section are valid. In particular, we can take $K = B := \{z \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 \leq 1\}$, the complex Euclidean ball in $\mathbb{C}^d$ and $\mu_B$ normalized surface area measure on $\partial B$, or $K = T := \{z \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$ the unit torus and $\mu_T$ normalized Haar measure on $T$. From Proposition 2.2 and (2.17), the $C_0$-extremal functions are the same: $V_{C_0,K}(z) = \max[0, \log |z_1|, \ldots, \log |z_d|]$. Thus in both cases the corresponding expected normalized zero measures $E(\tilde{Z}_{H_n}^d)$ converge to $(dd^cV_{C_0,K})^d = \mu_T$.

5. Questions and further directions

The reader will note that many basic issues in the non-convex theory are unresolved. We include a partial list. In 1. and 2. $C$ is the closure of an open, connected set satisfying (1.10) and $K$ is a compact set in $\mathbb{C}^d$.

1. Do we have equality in (2.3), i.e., does

$$V_{C,K}(z) = \sup \left\{ \frac{1}{\deg_C(p)} \log |p(z)| : p \in \mathcal{P}_d, \|p\|_K \leq 1 \right\}$$

exist for $z \in \mathbb{C}^d$?

2. Does the limit in (2.10)

$$\lim_{n \to \infty} \frac{1}{n} \log \Phi_n(z) = V_{C,K}(z)$$

exist for $z \in \mathbb{C}^d$?
3. For the complex Euclidean ball $B \subset \mathbb{C}^d$, are the $C_p$-extremal functions $V_{C_p,B}$ different for different $p \in (0,1)$? Proposition 2.11 simply asserts that $V_{C_p,B}(z) \neq V_{C_0,B}(z)$, $V_{C_1,B}(z)$ at certain points $z \in \mathbb{C}^d$.

4. For $A, B \subset \mathbb{C}$ and $0 < p \leq 1$ we saw that

$$V_{C_p,A \times B}(z,w) = \max[g_A(z),g_B(w)].$$

On the other hand, for the triangles $T_\alpha$ defined before (3.4), we have

$$V_{T_\alpha,A \times B}(z,w) = \max(\beta g_A(z),\alpha g_B(w))$$

(cf. [9, Proposition 2.4]) so that

$$V_{C_p,A \times B}(z,w) = \sup_{0<\alpha<1} V_{T_\alpha,A \times B}(z,w).$$

Is the equality

$$V_{C_p,K}(z,w) = \sup_{0<\alpha<1} V_{T_\alpha,K}(z,w)$$

true for more general $K \subset \mathbb{C}^2$, e.g., is this true for the complex Euclidean ball $B \subset \mathbb{C}^2$?

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