VIRASORO GROUPS AND HURWITZ SCHEMES

J. M. MUÑOZ PORRAS
F. J. PLAZA MARTÍN

Abstract. In this paper we study the Hurwitz scheme in terms of the Sato Grassmannian and the algebro-geometric theory of solitons. We will give a characterization, its equations and a show that there is a group of Virasoro type which uniformizes it.

1. Introduction

Nowadays, the theory of Hurwitz spaces, which parametrize covers of curves, has shown to be relevant. It has been applied to the study of the moduli space of curves ([F]) but it is also important in enumerative geometry ([OP]).

In those problems one focusses on covers of curves $Y \to X$ with prescribed ramification data. In this paper, since we use the Sato Grassmannian, these data will be decorated with formal trivializations. The corresponding functor will be called the Hurwitz functor and it will be studied in §3. The Krichever map embeds it into the Sato Grassmannian and its image will be characterized. In particular, the Hurwitz functor is shown to be representable and its equations will be given.

Let us be more precise. Let us fix a natural number $n$ and a set of $r$ partitions of it $E = \{\overline{e}_1, \ldots, \overline{e}_r\}$, where $\overline{e}_i = \{e^{(i)}_1, \ldots, e^{(i)}_{k_i}\}$. Let us denote by $V$ the $\mathbb{C}$-algebra $\mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_r))$ and by $W$ the $V$-algebra

$$W = \mathbb{C}((z^{1/e^{(1)}_1})) \times \cdots \times \mathbb{C}((z^{1/e^{(1)}_{k_1}})) \times \cdots \times \mathbb{C}((z^{1/e^{(r)}_1})) \times \cdots \times \mathbb{C}((z^{1/e^{(r)}_{k_r}}))$$

Then, the Hurwitz functor parametrizes data $(Y, X, \pi, \overline{x}, \overline{\gamma}, t_{\overline{x}}, t_{\overline{y}})$ where $\pi : Y \to X$ is a cover of curves, $\overline{x} = x_1 + \cdots + x_r \subset X$ and $\sum_{i,j} y^{(i)}_j \subset Y$.
are divisors such that \( \pi^{-1}(x_i) = \sum_j e_j^{(i)} y_j^{(i)} \) and \( t_{\bar{x}} \) and \( t_{\bar{y}} \) are formal trivializations along \( \bar{x} \) and \( \bar{y} \) respectively (in particular, they induce isomorphisms \( t_{\bar{x}} : (\mathcal{O}_{X,\bar{x}})_{(0)} \sim \to V \) and \( t_{\bar{y}} : (\mathcal{O}_{Y,\bar{y}})_{(0)} \sim \to W \)). Let \( \mathcal{H}_E^\infty[g, g] \) denote the subscheme of the Grassmannian of \( W \) representing the Hurwitz functor.

In our previous paper [MP2] we proved that the Virasoro group scheme, which was defined as the group representing the functor of automorphisms of \( \mathbb{C}((z)) \), uniformizes the moduli space of curves (see Theorem 4.11 of [MP2] for the precise statement).

In §5, we introduce the group \( G^W_V \) as a certain subgroup of the group of automorphisms of \( W \) that induce an automorphism on \( V \). We show that this formal group scheme acts canonically on \( \text{Gr}(W) \) leaving \( \mathcal{H}_E^\infty[g, g] \) stable. Further, we prove that this group “uniformizes” \( \mathcal{H}_E^\infty[g, g] \), more precisely, we prove that the previous action is locally transitive (Theorem 5.13). The proof of these fact requires the explicit computation of the tangent space to the Hurwitz space.

In the last section, we add to the previous data a pair \((L, \phi_{\bar{y}})\) consisting of a line bundle on \( Y \) together with a formal trivialization of \( L \) along \( \bar{y} \). This functor is representable by a subscheme of the Grassmannian of \( W \) which will be denoted by \( \text{Pic}^\infty_E[g, g] \) (see Definition 6.1). Let \( \Gamma_W \) denote the connected component of 1 in the scheme representing the functor of invertible elements of \( W \) (see §2). Since \( G^W_V \) acts canonically on \( \Gamma_W \) we consider the semidirect product \( G^W_V \ltimes \Gamma_W \). Then we show that the group \( G^W_V \ltimes \Gamma_W \) acts on \( \text{Pic}^\infty_E[g, g] \) and that this actions is locally transitive (Theorem 6.3).

2. Vector Grassmannians and generalized \( \tau \)-functions

We assume the base field to be the field of complex numbers. However, all our results hold for an algebraically closed field of characteristic zero.

This section recalls and generalizes some results proved in §§2–3 of [MP1].

Let \( V \) be the trivial \( \mathbb{C} \)-algebra \( \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_r)) \). Let us fix an integer \( n > 0 \) and a set \( r \) partitions of it:

\[
E = \{ \bar{e}_1, \ldots, \bar{e}_r \}
\]

that is, \( \bar{e}_i = \{ e_1^{(i)}, \ldots, e_r^{(i)} \} \) and \( n = e_1^{(i)} + \cdots + e_r^{(i)} \). Finally, let us denote \( \bar{r} = k_1 + \cdots + k_r \).

Associated to this data we consider the following \( V \)-algebra

\[
W^i := \mathbb{C}((z_1^{1/e_i^{(i)}})) \times \cdots \times \mathbb{C}((z_1^{1/e_i^{(i)}}))
\]
and

\[ W := W^1 \times \cdots \times W^r \]

Let \( V_+ \), \( W_+^i \), and \( W_+ \) denote the subalgebras corresponding to the power series, that is

\[
V_+ := \mathbb{C}[z_1] \times \cdots \times \mathbb{C}[z_r] \\
W_+^i := \mathbb{C}[z_1^{1/\epsilon_{i1}^{(i)}}] \times \cdots \times \mathbb{C}[z_i^{1/\epsilon_{ki}^{(i)}}] \\
W_+ := W_+^1 \times \cdots \times W_+^r
\]

Analogously to the section 2 of [MP1], one introduces formal group schemes whose rational points are the groups of invertible elements of \( V \) and \( W \). Let \( \Gamma_V \) and \( \Gamma_W \) denote the connected components of the origin of these groups.

Let \( R \) be a \( \mathbb{C} \)-algebra. Recall that the set of \( R \)-valued points of \( \Gamma_V \) is the set of \( r \)-tuples \((\gamma_1, \ldots, \gamma_r) \in V \otimes_{\mathbb{C}} R\) with \( \gamma_i = \sum_j a_{ij}^i z_i \) where \( a_{ij}^i \in \text{Rad}(R) \) for \( j < 0 \) and \( a_{ij}^i \) are invertible. The set of \( R \)-valued points of \( \Gamma_W \) are described similarly. Indeed, it is the set of \( r \)-tuples \((\bar{\gamma}_1, \ldots, \bar{\gamma}_r) \in W \otimes_{\mathbb{C}} R\) with

\[
\bar{\gamma}_i = \left( \sum_j a_{ij}^{(i1)} z_i^{j/\epsilon_{i1}^{(i)}}, \ldots, \sum_j a_{ij}^{(ik_i)} z_i^{j/\epsilon_{ki}^{(i)}} \right) \in W ^i \otimes_{\mathbb{C}} R
\]

where \( a_{ij}^{(i,k)} \in \text{Rad}(R) \) for \( j < 0 \) and \( a_{ij}^{(i,k)} \) are invertible.

Let us recall briefly some facts on the theory of infinite Grassmannian. For the sake of simplicity, we will restrict now the pair \((W, W_+)\) although all facts hold for \((V, V_+)\) as well. It is well known that there is a \( \mathbb{C} \)-scheme \( \text{Gr}(W) \), which is called infinite Grassmannian, whose set of rational points is

\[
\left\{ \text{subspaces } U \subset W \text{ such that } U \to W/W_+ \text{ has finite dimensional kernel and cokernel} \right\}.
\]

This scheme is equipped with the determinant bundle, \( \text{Det}_W \), which is the determinant of the complex of \( \mathcal{O}_{\text{Gr}(W)} \)-modules

\[ \mathcal{L} \to W/W_+ \otimes_{\mathbb{C}} \mathcal{O}_{\text{Gr}(W)} , \]

where \( \mathcal{L} \) is the universal submodule of \( \text{Gr}(W) \) and the morphism is the natural projection. The connected components of the Grassmannian are indexed by the Euler–Poincaré characteristic of the complex. The connected component of index \( m \) will be denoted by \( \text{Gr}^m(W) \).

The group \( \Gamma_W \) acts by homotheties on \( W \), and this action gives rise to a natural action on \( \text{Gr}(W) \)

\[ \Gamma_W \times \text{Gr}(W) \to \text{Gr}(W) . \]
Furthermore, this action preserves the determinant bundle.

**Remark 1.** The linear group \( \text{Gl}(W) \) acts on \( \text{Gr}(W) \) preserving its determinant bundle (see [MP2] §2.2). This fact implies that there is a natural central extension

\[
0 \to \mathbb{G}_m \to \tilde{\text{Gl}}(W) \to \text{Gl}(W) \to 0
\]

In fact, one has such an extension for every subgroup of these linear groups.

The loop group of the pair \((V, W)\) is defined as the group scheme \( \text{LGl}(W/V) \) representing the subfunctor of groups \( \text{LGl}(W/V) \subset \text{Gl}(W) \) defined by

\[
\text{LGl}(W/V)(S) := \left\{ \text{automorphisms of } W \hat{\otimes}_C H^0(S, \mathcal{O}_S) \text{ as } V \hat{\otimes}_C H^0(S, \mathcal{O}_S)-\text{module} \right\}
\]

Since the functors of points of \( \text{LGl}(W/V) \) and \( \Gamma_W \) are subfunctors of \( \text{Gl}(W) \), one obtains central extensions of group schemes

\[
0 \to \mathbb{G}_m \to \tilde{\text{LGl}}(W/V) \to \text{LGl}(W/V) \to 0
\]

\[
0 \to \mathbb{G}_m \to \tilde{\Gamma}_W \to \Gamma_W \to 0
\]

These facts allow us to introduce \( \tau \)-functions and Baker-Akhiezer functions of points of \( \text{Gr}(W) \). Let us recall the definition and some properties of these functions ([MP1], §3).

The determinant of the morphism \( \mathcal{L} \to W/W \hat{\otimes}_C \mathcal{O}_{\text{Gr}(W)} \) gives rise to a canonical global section

\[
\Omega_+ \in H^0(\text{Gr}^0(W), \text{Det}_W^*).
\]

In order to extend this section to \( \text{Gr}(W) \) (in a non-trivial way), we fix elements \( \{v_m|m \in \mathbb{Z}\} \) such that: i) the multiplication by \( v_m \) shifts the index by \( m \); and, ii) \( v_m \cdot v_{r-n-m} = 1 \). We define \( \Omega_+(U) := \Omega_+(v_m^{-1}U) \) for \( U \in \text{Gr}^m(W) \).

Now, the \( \tau \)-function and BA functions will be introduced following [MP1]. Recall that

\[
W = \mathbb{C}(\{z_{1}^{1/e_{1}}\}) \times \cdots \times \mathbb{C}(\{z_{1}^{1/e_{1}}\}) \times \cdots \times \mathbb{C}(\{z_{r}^{1/e_{r}}\}) \times \cdots \times \mathbb{C}(\{z_{r}^{1/e_{r}}\})
\]

and that \( \Gamma_W \) parametrices a certain subgroup of invertible elements of \( W \). Let \( t \) be the set of variables \( \{t^{(1,1)}, \ldots, t^{(1,k_1)}, \ldots, t^{(r,1)}, \ldots, t^{(r,k_r)}\} \) where \( t^{(a,b)} = (t_1^{(a,b)}, t_2^{(a,b)}, \ldots) \). Consider the element of \( \Gamma_W \) given by

\[
g = \left(1 + \sum_{j<0} t_j^{(1,1)} z_1^{j/e_{1}}, \ldots, 1 + \sum_{j<0} t_j^{(r,k_r)} z_r^{j/e_{r}}\right) \in \Gamma_W
\]
Then, the $\tau$-function of $U$, $\tau_U(t)$, is defined by

$$\tau_U(t) := \frac{\Omega_+(gU)}{g\delta_U}$$

where $\delta_U$ being a non-zero element in the fibre of $\text{Det}_W^*$ over $U$.

Let $z$, denote $(z_{1/\varepsilon_1}^{1}, \ldots, z_{1/\varepsilon_{k_1}}^{1}, \ldots, z_{r/\varepsilon_{r,k_r}}^{r}) \in W$.

Let $(a, b)$ be a pair of natural numbers such that $a \in \{1, \ldots, r\}$ and $b \in \{1, \ldots, k_a\}$. Let $(c, d)$ be another pair satisfying the corresponding constrains.

The $(a, b)$-th Baker-Akhiezer function of a point $U \in \text{Gr}(W)$ is the $W$-valued function whose $(c, d)$-th entry is given by

$$\psi_{a,b,U}^{(c,d)}(z, t) := \exp \left(-\sum_{i \geq 1} \frac{t_{i/\varepsilon_d}^{(c,d)}}{i! \varepsilon_d} \tau_{U_{a,b}}^{(c,d)}(t + [z_{c/\varepsilon_d}]_{a,b}) \tau_U(t) \right)$$

where

- $[z_i] := (z_{1/\varepsilon_i}^{2}, \ldots, z_{1/\varepsilon_i}^{k_i}, \ldots, z_{r/\varepsilon_{r,k_r}}^{r})$,
- $t + [z_{c/\varepsilon_d}] := (t(1), \ldots, t^{(1,k_1)}, \ldots, t^{(c,d)} + [z_{c/\varepsilon_d}], \ldots, t^{(r,1)}, \ldots, t^{(r,k_r)})$,
- and $U_{a,b}^{c,d} := (1, \ldots, z_{a/\varepsilon_b}^{1}, \ldots, (z_{c/\varepsilon_d})^{-1} - 1) \cdot U$.

The main property of these Baker-Akhiezer functions is that they can be understood as generating functions for $U$ as a subspace of $W$, as we recall next.

**Theorem 2.1 ([MP1]).** Let $U \in \text{Gr}^m(W)$. Then

$$\psi_{a,b,U}(z, t) = v_m^{-1}(1, \ldots, z_{a/\varepsilon_b}^{1}, \ldots, 1) \cdot \sum_{i > 0} \left( \psi_{a,b,U}^{(1,1)}(z_{1/\varepsilon_1}^{1}), \ldots, \psi_{a,b,U}^{(r,k_r)}(z_{r/\varepsilon_{r,k_r}}^{r}) \right) p_{a,b,i,U}(t)$$

where

$$\left\{ \left( \psi_{a,b,U}^{(1,1)}(z_{1/\varepsilon_1}^{1}), \ldots, \psi_{a,b,U}^{(r,k_r)}(z_{r/\varepsilon_{r,k_r}}^{r}) \right) \middle| a \in \{1, \ldots, r\}, \ b \in \{1, \ldots, k_a\} \right\}$$

is a basis of $U$ and $p_{a,b,i,U}(t)$ are functions in $t$.

Now, we will follow [MP1] to prove generalized Bilinear identities as well as to generalize the associated hierarchy, which will be called $E$-KP hierarchy.

Recall that $W = W^1 \times \cdots \times W^r$. Being $W^i$ a $\mathbb{C}(\langle z_i \rangle)$-algebra, one considers pairing given by

$$W^i \times W^i \rightarrow \mathbb{C}(\langle z_i \rangle)$$

$$(w_1^{(i)}, w_2^{(i)}) \rightarrow \text{Tr}^i(w_1^{(i)} w_2^{(i)})$$
where $\text{Tr}^i : W^i \to \mathbb{C}(z_i)$ is the trace map.

These give rise to a pairing

$$T_2 : W \times W \to \mathbb{C}$$

$$(w_1, w_2) \mapsto \sum_{i=1}^r \text{Res}_{z_i=0} \text{Tr}(w_1^{(i)} w_2^{(i)}) dz_i$$

where $w_j = (w_j^{(1)}, \ldots, w_j^{(r)})$ with respect to the decomposition $W = W^1 \times \cdots \times W^r$.

From the separability of $W^i$ as $\mathbb{C}(z_i)$-algebra, it follows that $T_2$ is a non-degenerate bilinear pairing. Furthermore, it induces an involution of the Grassmannian

$$\text{Gr}(W) \to \text{Gr}(W)$$

$$U \mapsto U^\perp$$

where $U^\perp$ is the orthogonal of $U$ w.r.t. $T_2$. This involution sends the connected component of index $m$ to that of index $\bar{r} - r \cdot n - m$.

Finally, the $(a, b)$-th adjoint Baker-Akhiezer functions of $U$ are defined by

$$\psi_{a,b,U}(z, t) := \psi_{a,b,U^\perp}(z, -t)$$

Now, we state the corresponding generalizations whose proofs are similar to those given in [MP1].

**Theorem 2.2 (Bilinear Identity).** Let $U, U' \in \text{Gr}^m(W)$ be two rational points lying on the same connected component. Then, $U = U'$ if and only if the following condition holds

$$T_2 \left( \frac{1}{z} \psi_U(z, t), \frac{1}{z} \psi_{U'}^*(z, t') \right) = 0$$

### 3. Hurwitz Schemes

**The Krichever morphism.** Let $\pi : Y \to X$ be a finite morphism between proper curves over $\mathbb{C}$. Let us suppose $Y$ and $X$ to be reduced. Fix a set of pairwise distinct smooth points in $X$, $x = \{x_1, \ldots, x_r\}$ and let $y := \pi^{-1}(x_1) + \cdots + \pi^{-1}(x_r)$.

Define $A := H^0(X - x, \mathcal{O}_X)$, $B := H^0(Y - y, \mathcal{O}_Y)$, $\Sigma_X$ (resp. $\Sigma_Y$) to be the total quotient ring of $A$ (resp. $B$). Let $\text{Tr}_{\Sigma_X}^\Sigma_Y$ denote the trace of $\Sigma_Y$ as a finite $\Sigma_X$-algebra.

The triple $(Y, X, x)$ is said to have the property $(\ast)$ if $\text{Tr}_{\Sigma_X}^\Sigma_Y(B) \subseteq A$.

It worth pointing out that every covering $\pi : Y \to X$ has the property $(\ast)$ whenever $X$ is smooth or $\pi$ is flat.

Let us fix a set of numerical data as in the previous section

$$E = \{\bar{e}_1, \ldots, \bar{e}_r\}$$
with \( \bar{e}_i = \{ e_1^{(i)}, \ldots, e_{k_i}^{(i)} \} \) with \( n = e_1^{(i)} + \cdots + e_{k_i}^{(i)} > 0 \). Let \( V \) and \( W \) be the \( \mathbb{C}((z)) \)-algebras defined by the data \( E \) as in \( \S 2 \). For a \( \mathbb{C} \)-scheme \( S \), we write \( \hat{V}_S := V \otimes \mathbb{C}O_S \) and \( \hat{W}_S := W \otimes \mathbb{C}O_S \).

**Definition 3.1.** The Hurwitz functor \( \mathcal{H}_E^\infty \) of pointed coverings of curves of degree \( n \) with fibres of type \( E \) and formal parameters along the fibers is the contravariant functor on the category of \( \mathbb{C} \)-schemes

\[
\mathcal{H}_E^\infty : \left\{ \text{category of } \mathbb{C} \text{-schemes} \right\} \longrightarrow \left\{ \text{category of sets} \right\}
\]

\[
S \mapsto \mathcal{H}_E^\infty (S) := \{(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})\}
\]

where

1. \( p_Y : Y \to S \) and \( p_X : X \to S \) are proper and flat morphisms whose fibres are geometrically reduced curves.
2. \( \pi : Y \to X \) is a finite morphism of \( S \)-schemes of degree \( n \) such that its fibres over closed points \( s \in S \) have the property (\( * \)).
3. \( \bar{x} = \{ x_1, \ldots, x_r \} \) is a set of disjoint smooth sections of \( p_X \) such that the Cartier divisors \( x_i(s) \) for \( i = 1, \ldots, r \) are smooth points of \( X_s := p_X^{-1}(s) \) for all closed points \( s \in S \).
4. \( \bar{y} = \{ y_1, \ldots, y_r \} \) and, for each \( i \), \( y_i = \{ y_1^{(i)}, \ldots, y_{k_i}^{(i)} \} \) is a set of disjoint smooth sections of \( p_Y \) such that the Cartier divisor \( \pi^{-1}(x_i(S)) \) is \( e_1^{(i)} y_1^{(i)}(S) + \cdots + e_{k_i}^{(i)} y_{k_i}^{(i)}(S) \).
5. For all closed point \( s \in S \) and each irreducible component of the fibre \( X_s \), there is at least one point \( x_i(s) \) lying on that component.
6. For all closed point \( s \in S \) and each irreducible component of the fibre \( Y_s \), there is at least one point \( y_j^{(i)}(s) \) lying on that component.
7. \( t_{\bar{x}} \) is a formal parameter along \( \bar{x}(S) \), \( t_{\bar{x}} : \hat{O}_{X, \bar{x}(S)} \cong \hat{V}_S \), such that it induces

\[
t_{x_i} := (t_{\bar{x}})_{x_i} : \hat{O}_{X, x_i(S)} \cong \hat{O}_S[z_i]
\]

for all \( i \).
8. \( t_{\bar{y}} = \{ t_{\bar{y}_1}, \ldots, t_{\bar{y}_r} \} \) are formal parameters along \( \bar{y}_1(S), \ldots, \bar{y}_r(S) \) such that

\[
\pi^*(t_{x_i})_{y_j^{(i)}(S)} = t_{y_j^{(i)}}^{(i)}
\]
(9) \((Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})\) and \((Y', X', \pi', \bar{x}', \bar{y}', t_{\bar{x}'}, t_{\bar{y}'})\) are said to be equivalent when there is a commutative diagram of \(S\)-schemes

\[
\begin{array}{ccc}
Y & \xrightarrow{\sim} & Y' \\
\downarrow \pi & & \downarrow \pi' \\
X & \xrightarrow{\sim} & X'
\end{array}
\]

compatible with all the data.

The Krichever morphism for the Hurwitz functor is the morphism of functors

\[
\text{Kr}: \overline{\mathcal{H}_E}^\infty \to \text{Gr}(W)
\]

which sends \((Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}) \in \overline{\mathcal{H}_E}^\infty(S)\) to the following submodule of \(W \hat{\otimes}_C O_S\)

\[
t_{\bar{y}} \left( \lim_{m} (p_{Y})_* O_{Y} (m \cdot \pi^{-1}(\bar{x})) \right) \subset W \hat{\otimes}_C O_S
\]

Let \(\mathcal{M}^\infty(\bar{r})\) be the moduli functor parametrizing the classes of sets of data \((Y; y_1, \ldots, y_{\bar{r}}; t_1, \ldots, t_{\bar{r}})\) of geometrically reduced curves with \(\bar{r}\) pairwise distinct marked smooth points \(\{y_1, \ldots, y_{\bar{r}}\}\) and formal parameters \(\{t_1, \ldots, t_{\bar{r}}\}\) at these points such that each irreducible component of \(Y\) contains at least one of the marked points. Applying Theorem 4.3 of [MP1], one gets the following

**Theorem 3.3.** The Krichever morphism (3.2) identifies \(\mathcal{M}^\infty(\bar{r})(S)\) with the set of submodules \(U \in \text{Gr}(W)(S)\) such that \(U \cdot U \subseteq U\) and \(O_S \subseteq U\). In particular, this functor is representable by a closed subscheme of the infinite Grassmannian which will be also denoted by \(\mathcal{M}^\infty(\bar{r})\).

Let \(\text{Tr}: W \to V\) denotes the trace map of \(W\) as a \(V\)-algebra, then it is clear that \(\text{Tr} = \bigoplus_{i=1}^{\bar{r}} \text{Tr}_i\) where \(\text{Tr}_i: W^i \to \mathbb{C}(z_i)\) is the trace map of the \(\mathbb{C}(z_i)\)-algebra \(W^i\). Furthermore, one has a commutative diagram

\[
\begin{array}{ccc}
H^0(Y - y, O_Y)^\circ & \xrightarrow{t_y} & \widehat{W}_K := W \hat{\otimes}_C K = (W^1 \times \cdots \times W^r) \hat{\otimes}_C K \\
\downarrow \text{Tr}_Y & & \downarrow \text{Tr} \\
H^0(X - x, O_X)^\circ & \xrightarrow{t_x} & \widehat{V}_K := V \hat{\otimes}_C K = K((z_1)) \times \cdots \times K((z_r))
\end{array}
\]

for every geometric point \((Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})\) in \(\overline{\mathcal{H}_E}^\infty(K)\) (\(K\) being an extension of \(\mathbb{C}\)).

The study of the functor \(\overline{\mathcal{H}_E}^\infty\) for the case \(r = 1\) has been carried out exhaustively in [MP1]. Its main results (given in its fourth section) are
easily generalized for our case. Let us simply state these generalizations.

**Proposition 3.4.** Let \( \mathcal{Y} = (Y, X, x, y, t_x, t_y) \) be an \( S \)-valued point of \( \mathcal{H}_E^\infty \). Then, it holds that

\[
\Kr(X, x, t_x) = \Kr(\mathcal{Y}) \cap \hat{V}_S = \Tr(\Kr(\mathcal{Y})) \in \Gr(V)(S)
\]

In particular, the Krichever map (3.2) is injective.

**Theorem 3.5.** Let \( U \in \mathcal{M}^\infty(\bar{r}) \subset \Gr(W) \) be an \( S \)-valued point. Then, the following conditions are equivalent

1. \( U \in \mathcal{H}_E^\infty(\bar{r}) \)
2. \( \Tr(U) \subseteq U \)

In particular, the functor \( \mathcal{H}_E^\infty \) is representable by a closed subscheme \( \mathcal{H}_E^\infty \) of \( \Gr(W) \).

**Theorem 3.6** (**E-KP hierarchy**). Let \( B \in \mathcal{M}^\infty(\bar{r}) \subset \Gr(W) \) be a closed point. Let \( \{u_{1,1}, \ldots, u_{r,k_r}\} \) be integer numbers defined by

\[
u_m = \frac{u_{1,1}^{1/e_1}}{\cdots} \frac{u_{r,k_r}^{1/e_r}}{z}.
\]

Then, \( B \in \mathcal{H}_E^\infty \) if and only if the following “bilinear identities” are satisfied; that is, the form

\[
\left( \sum_{l=1}^r \sum_{j=1}^{e_j} \sum_{i=1}^{e_i} \psi_{a,b,B}^{(l,j)}(\xi_i^{z_l} z_1^{1/e_i}, t) \right) \left( \sum_{l=1}^r \sum_{j=1}^{e_j} \sum_{i=1}^{e_i} \psi_{c,d,B}^{(l,j)}(\xi_i^{z_l} z_1^{1/e_i}, t) \right) dz
\]

has residue zero at \( z = 0 \) for all \( a, b, c, d \) and where \( \xi_e \) is a primitive \( e \)-th root of unity and \( \delta \) is the Kronecker symbol.

**Proof.** The proof is modeled in the proof of Theorem 4.10 or [MP1]. The idea consists of translating the second condition of the previous statement in terms of Baker-Akhiezer functions by means of Theorem 2.1. \( \square \)

**Definition 3.7.** The Hurwitz functor \( \mathcal{H}_E^\infty \) of pointed coverings of smooth curves of degree \( n \) with fibres of type \( E \) and formal parameters along the fibres is the subfunctor of \( \mathcal{H}_E^\infty \) consisting of data \((Y, X, \pi, \bar{x}, \bar{y}, t_x, t_y)\) where the fibres of \( Y \to S \) are nonsingular curves.

**Theorem 3.8.** The functor \( \mathcal{H}_E^\infty \) is representable by a subscheme of \( \Gr(W) \) which will be denoted by \( \mathcal{H}_E^\infty \).

**Proof.** Observe that if \( Y \to X \) is a family has the property (\( \ast \)) and the fibres of \( Y \) are nonsingular curves, then the fibres of \( X \) are also nonsingular curves. Now let \( \mathcal{Y} \to \mathcal{H}_E^\infty \) be the universal family given by
the representability of the functor $\overline{\mathcal{H}}^\infty_{E}$. Then the desired subscheme, $\mathcal{H}^\infty_{E}$, consists precisely of the points $s \in \overline{\mathcal{H}}^\infty_{E}$ such that $Y_s$ is smooth. □

**Remark 2.** Standard procedures can be used to rewrite the identities of Theorems 2.2 and 3.6 as hierarchies of partial differential equations for the $\tau$-function.

**Coverings with prescribed ramification.** The above results, which generalize the results proved in [MP1], allow us to describe the structure of the Hurwitz scheme parametrizing coverings with prescribed ramification points and formal parameters at the given points.

Let $\pi : Y \to X$ be a finite covering of degree $n$ of smooth integral curves. Let $\bar{g}$ and $g$ be the genus of $Y$ and $X$ respectively. Then, the Hurwitz formula reads

$$1 - g = n(1 - g) - \frac{1}{2} \sum_{y \in Y} (e_y - 1)$$

where $e_y$ is the ramification index of the point $y \in Y$.

Let $\{x_1, \ldots, x_r\} \subset X$ be the branch locus and denote

$$\bar{y} = \pi^{-1}(x_1) + \cdots + \pi^{-1}(x_r)$$

$$\pi^{-1}(x_i) = e^{(i)}_1 y^{(i)}_1 + \cdots + e^{(i)}_{k_i} y^{(i)}_{k_i}$$

Considering $E = \{\bar{e}_1, \ldots, \bar{e}_r\}$, $\bar{e}_i = \{e^{(i)}_1, \ldots, e^{(i)}_{k_i}\}$ and $\bar{r} = \sum_{i=1}^{r} k_i$, one has that

$$\sum_{y \in Y} (e_y - 1) = \sum_{i=1}^{r} \sum_{j=1}^{k_i} (e^{(i)}_j - 1) = rn - \bar{r}$$

then the Hurwitz formula can be rewritten as

$$1 - g = n(1 - g) - \frac{1}{2}(rn - \bar{r})$$

**Definition 3.9.** For integers $i, j$, we define the following subschemes of $\mathcal{H}^\infty_{E}$

$$\mathcal{H}^\infty_{E}[j] := \{U \in \mathcal{H}^\infty_{E} \cap \text{Gr}^{1-j}(W) \text{ such that } U \text{ is an integral domain}\}$$

$$\mathcal{H}^\infty_{E}[j, i] := \{U \in \mathcal{H}^\infty_{E}[j] \text{ such that } \text{Tr}(U) \in \text{Gr}^{1-i}(V)\}$$

Note that, since $\text{Tr}(U) \subseteq U$ for any $U \in \mathcal{H}^\infty_{E}$, the condition that $U$ is an integral domain implies that $\text{Tr}(U)$ is integral too.

From the representability of $\mathcal{H}^\infty_{E}$ and the Hurwitz formula for coverings, one has the following

**Theorem 3.10.** Let $(E, n, \bar{r})$ be a set of numerical data as in §2 and $g, \bar{g}$ be two non-negative integer numbers satisfying

$$\bar{g} - 1 = n(g - 1) + \frac{1}{2}(rn - \bar{r})$$
Then, the subscheme $\mathcal{H}_E^\infty[\bar{g}, g] \subset \text{Gr}^{1-\bar{g}}(W)$ is the moduli scheme parametrizing geometrical data $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}) \in \mathcal{H}_E^\infty$ where $Y$ has genus $\bar{g}$, $X$ has genus $g$, the covering $\pi : Y \to X$ is non-ramified outside $\bar{y}$ and the ramification index at the point $y_j \in Y$ is $e(j)^{i}$.

Remark 3. Let us observe that there exists a natural forgetful morphism
\[ \Phi : \mathcal{H}_E^\infty[\bar{g}, g] \to \mathcal{M}_g^\infty(r) \]
\[ (Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}) \mapsto (X, \bar{x}, t_{\bar{x}}) \]
Given $(X, \bar{x}, t_{\bar{x}}) \in \mathcal{M}_g^\infty(r)$ with $X$ smooth and integral, the fiber of this point will be denoted by $\mathcal{H}_E^\infty(X, \bar{x}, t_{\bar{x}})$. Recall that there is a finite number of coverings $\pi : Y \to X$ with $Y$ and $X$ smooth, $X$ integral, $\pi$ is dominant on each component of $Y$, non-ramified outside $\bar{x}$, and ramification indexes $\bar{e}_i$ at $x_i$ (see [ELSV, OP]). Then, we conclude that $\mathcal{H}_E^\infty(X, \bar{x}, t_{\bar{x}})$ is a finite set.

4. Tangent space to the Hurwitz scheme

This section is devoted to an explicit computation of the tangent space of the Hurwitz schemes constructed in the previous section. To this goal, we begin by recalling the computation of the tangent spaces to the infinite Grassmannian and to the moduli space of pointed curves.

Proposition 4.1. Let $U$ be a rational point of $\text{Gr}(W)$. There is a canonical isomorphism
\[ T_U \text{Gr}(W) \cong \text{Hom}(U, W/U) \]

Proof. Let $A \sim W_+$ and $U \in \text{Gr}(W)$ such that $U \oplus A \simeq W$. Let $F_A$ be the open subscheme of $\text{Gr}(W)$ parametrizing those subspaces $U' \in \text{Gr}(W)$ such that $U' \oplus A \simeq W$. Then, it is well known that $F_A$ is isomorphic to the affine space $\text{Hom}(U, A)$ and that the embedding
\[ \text{Hom}(U, A) \xrightarrow{\sim} F_A \hookrightarrow \text{Gr}(W) \]
maps $f : U \to A$ to its graph $\Gamma_f := \{ u + f(u) | u \in U \}$.

Since $U \in F_A$ (it corresponds to the zero map) and $F_A$ is open, we obtain an isomorphism of vector spaces
\[ \text{Hom}(U, A) \xrightarrow{\sim} T_0 \text{Hom}(U, A) \xrightarrow{\sim} T_U \text{Gr}(W) \xrightarrow{\sim} \text{Gr}(W)(k[\epsilon]/\epsilon^2) \times_{\text{Gr}(W)(k)} \{ U \} \]
which maps $f \in \text{Hom}(U, A)$ to the $(k[\epsilon]/\epsilon^2)$-valued point of $\text{Gr}(W)$ given by $\{ u + \epsilon f(u) | u \in U \}$.

Composing the inverse of this map with the isomorphism
\[ \text{Hom}(U, A) \xrightarrow{\sim} \text{Hom}(U, W/U) \]
\[ f \mapsto U \xrightarrow{\text{Id} + f} U \oplus A \xrightarrow{\sim} W \to W/U \]
one gets the desired isomorphism (observe that it does not depend on the choice of $A$).

**Proposition 4.2.** Let $U$ be a rational point of $\mathcal{M}^\infty(\bar{r})$. The isomorphism of Proposition 4.1 induces a canonical identification

$$T_U \mathcal{M}^\infty(\bar{r}) \simeq \text{Der}(U, W/U)$$

(where Der means derivations trivial over $\mathbb{C}$).

Furthermore, if $U$ is associated to the geometrical data $(C, \bar{p}, \bar{z})$ under the Krichever map, then $W/U \simeq H^0(C - \bar{p}, \omega_C)^*$.  

**Proof.** For $U \in \mathcal{M}^\infty(\bar{r})$, one has that

$$T_U \mathcal{M}^\infty(\bar{r}) = \{ \bar{U} \in T_U \text{Gr}(W) \text{ such that } \bar{U} \cdot \bar{U} \subseteq \bar{U} \}$$

From Proposition 4.1, there is a map $f \in \text{Hom}(U, A)$ such that $\bar{U} = \{ u + \epsilon f(u) | u \in U \}$.

The condition $\bar{U} \cdot \bar{U} \subseteq \bar{U}$ means that for $u, u' \in U$ there exists $u'' \in U$ satisfying $(u + \epsilon f(u)) \cdot (u' + \epsilon f(u')) = u'' + \epsilon f(u'')$; that is

$$f(u \cdot u') = uf(u') + f(u'u') \quad (4.3)$$

The second condition, $k[\epsilon] \subseteq \bar{U}$, implies that there exists $u_0 \in U$ such that $u_0 + \epsilon f(u_0) = 1$; or, in other words

$$f(1) = 0 \quad (4.4)$$

It is now easy to check that the image of $f \in \text{Hom}(U, A)$ in $\text{Hom}(U, W/U)$ gives rise to a derivation $Df \in \text{Der}(U, W/U)$ (note that $W/U$ is an $U$-module). The claim follows from a straightforward check.

The second part of the statement follows easily from the exact sequence

$$0 \rightarrow H^0(C - \bar{p}, \mathcal{O}_C) \rightarrow W \simeq (\mathcal{O}_\bar{p})_{(0)} \rightarrow \lim_{m \rightarrow \infty} H^1(C, \mathcal{O}_C(-m\bar{p})) \rightarrow 0$$

Note that a similar result holds for points of $\mathcal{M}^\infty(r) \subset \text{Gr}(V)$.

Let us denote

$$\text{Der}(U, W/U)^{\text{Tr}} := \left\{ D \in \text{Der}(U, W/U) \text{ such that } \text{Tr}^1 U \rightarrow W/U \right\}$$

where $\text{Tr}^1$ is the map induced by the trace (since $\text{Tr} U \subseteq U$).
Theorem 4.6. Let $U$ be a rational point of $\overline{\mathcal{H}}^\infty_E$. The embedding $\overline{\mathcal{H}}^\infty_E \hookrightarrow \mathcal{M}^\infty(\bar{r})$ yields an identification

$$T_U \overline{\mathcal{H}}^\infty_E \simeq \text{Der}(U, W/U)^{Tr}$$

Moreover, if $U$ corresponds to the geometrical data $(Y, X, \pi, \bar{x}, \bar{y}, t_\bar{x}, t_\bar{y})$ in $\mathcal{H}^\infty_E$, then $T^1$ is the map

$$\text{Tr}^1: H^0(Y - y, \omega_Y)^* \longrightarrow H^0(X - x, \omega_X)^*$$

canonically induced by the trace $\pi_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$.

Proof. Let us keep the notations of the proofs of the two previous propositions. Then, let $\bar{U} \in \text{Gr}(\mathcal{W})$ be a $k[\epsilon]/\epsilon^2$-valued point lying in $T_U \mathcal{M}^\infty(\bar{r})$. Let $f \in \text{Hom}(U, A)$ correspond to $\bar{U}$.

Then, the condition $\text{Tr}(U) \subseteq U$ (Theorem 3.5) is equivalent to say that for each $u \in U$ there exists $u' \in U$ satisfying $\text{Tr}(u + \epsilon f(u)) = u' + \epsilon f(u')$. Since $\text{Tr}(u) \in U$, this condition is

$$\text{Tr}^1(f(u)) = f(\text{Tr}(u)) \quad \forall u \in U$$

And the first part of the claim follows easily.

Let us prove the second part. Note that the trace is a sheaf homomorphism

$$\text{Tr}: \pi_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$$

This map induces

$$H^1(X, (\pi_* \mathcal{O}_Y)(-n\bar{x})) \longrightarrow H^1(X, \mathcal{O}_X(-n\bar{x}))$$

Bearing in mind the adjunction formula, that $\pi$ is affine, Serre duality and taking limits, one obtains the desired map

$$\text{Tr}^1: H^0(Y - \bar{y}, \omega_Y)^* \longrightarrow H^0(X - \bar{x}, \omega_X)^*$$

Using arguments similar to those of the proof of Proposition 4.2, it is not difficult to check that this map is compatible with the isomorphisms $H^0(Y - \bar{y}, \omega_Y)^* \simeq W/U$ and $H^0(X - \bar{x}, \omega_X)^* \simeq V/\text{Tr}(U)$.

Theorem 4.8. Let $(E, n, r, \bar{g}, g)$ as in Theorem 3.10. Let $U \in \mathcal{H}^\infty_E[\bar{g}, g]$ be a rational point. Then, there is a canonical injection

$$T_U \mathcal{H}^\infty_E[\bar{g}, g] \hookrightarrow T_{\text{Tr}} \mathcal{M}^\infty(r)$$

Proof. By Proposition 4.2 and Theorem 4.6, the claim is equivalent to the injectivity of the restriction map

$$\text{Der}(U, W/U)^{\text{Tr}} \longrightarrow \text{Der}(\text{Tr} U, V/\text{Tr} U)$$

Let $D \in \text{Der}(U, W/U)^{\text{Tr}}$ be in the kernel of the above map; that is, $D|_{\text{Tr}(U)} = 0$. Let $u$ be any element in $U$ and let us see that $Du = 0$. Since $A := \text{Tr}(U) \rightarrow U$ is an integral morphism, there is a monic
minimal polynomial \( p(x) = \sum a_i x^i \in A[x] \) such that \( p(u) = 0 \). Then, the following identity holds true
\[
0 = Dp(u) = \sum (Da_i)u^i + p'(u)Du = p'(u)Du
\]
\((p'(x)\) denoting the derivative w.r.t. \( x \)).

Since \( p(x) \) is separable and \( \pi \) is unramified over \( X - \bar{x} = \text{Spec} A \), then \( \frac{1}{p'(u)} \in A[u] \subseteq U \). Therefore, one has \( Du = 0 \).

\[\square\]

5. The Multicomponent Virasoro Group

The algebraic Virasoro group, defined as \( G := \text{Aut}_C C((z)) \), was introduced and studied in [MP2]. In this section it will be generalized for certain \( C((z)) \)-algebras. Note that \( V \) carries the linear topology given by \( \{z^nV_+\}_{n \in \mathbb{Z}} \).

It is convenient to recall the following result: let \( R \) be a \( C \)-algebra and let \( f(z) \in R((z)) \). Then \( f(z) \) is invertible if and only if there exists \( n \in \mathbb{Z} \) such that \( a_i \in \text{Rad}(R) \) for \( i < n \) and \( a_n \) is invertible.

**Definition 5.1.** The functor of automorphisms of \( V \) is the functor defined from the category of \( C \)-schemes to the category of groups defined as follows
\[
S \rightsquigarrow \text{Aut}_C(V)(S) := \text{Aut}_{H^0(S,O_S)} V \hat{\otimes}_C H^0(S,O_S)
\]
where \( \text{Aut}_R \) means continuous automorphisms of \( R \)-algebras.

**Lemma 5.2.** Let \( S \) be a \( C \)-scheme and let \( \phi \in \text{Aut}_C(V)(S) \). For any point \( s \in S \), let \( p_{\phi(s)} \) be the permutation defined by \( \phi(s) \) on the set \( \text{Spec}(V_s) \) (note that \( V_s := V \hat{\otimes}_C k(s) \) consists of \( r \) points).

Then, the map
\[
S \rightarrow S_r,
\]
\[
s \mapsto p_{\phi(s)}
\]
(\( S_r \) being the symmetric group of \( r \) letters) is locally constant.

**Proof.** Let us assume that \( S = \text{Spec}(R) \) is an irreducible \( C \)-scheme. Let \( \phi \in \text{Aut}_C(V)(S) \). Since \( V_s := V \hat{\otimes}_C k(s) \simeq \prod k((z)) \) is a product of fields, it follows that \( \phi(s) \in \text{Aut}_C(V(k(s))) \) acts by permutation on \( \text{Spec}(V_R) \); that is, on the set of ideals \( I_1 := ((z_1,0,\ldots,0)), \ldots, I_r := ((0,\ldots,0,z_r)) \). That is, there exists \( p_{\phi(s)} \in S_r \) such that
\[
\phi(s)(I_i) = I_{p_{\phi(s)}(i)}
\]

Let us write \( \phi(0,\ldots,z_i,\ldots,0) = (\phi_1^{(i)},\ldots,\phi_r^{(i)}) \in R((z_1)) \times \cdots \times R((z_r)) \). Let \( s_0 \) denote the point associated to the minimal prime ideal.
The very definition of the permutation \( p_{\phi(s_0)} \) says that

\[
\phi_j^{(i)}(s_0) \in k(s_0)((z_j)) \text{ is } \begin{cases} 
\text{invertible if } j = p_{\phi(s_0)}(i) \\
\text{zero if } j \neq p_{\phi(s_0)}(i)
\end{cases}
\]

and, therefore, \( s_0 \in (\phi_j^{(i)})_0 \) for \( j \neq p_{\phi(s_0)}(i) \). Since \( s_0 \) is minimal, \( S \) is irreducible and \((\phi_j^{(i)})_0\) is closed, it follows that the closure of \( s_0 \), \( S \), is contained in \((\phi_j^{(i)})_0\) for \( j \neq p_{\phi(s_0)}(i) \).

Then, it turns out that \( \phi_j^{(i)}(s) \neq 0 \) for \( j = p_{\phi(s_0)}(i) \) and for all \( s \in S \); equivalently, \( \phi_j^{(i)} \in R((z_j)) \) is invertible for \( j = p_{\phi(s_0)}(i) \) and, therefore, \( p_{\phi(s_0)} = p_{\phi(s)} \) for all \( s \in S \). The statement follows.

As a consequence of the previous lemma we have a well defined map of functors on groups

\[
\text{Aut}_C(V) \xrightarrow{p} S_r
\]

given by \( p(\phi) := p_{\phi} \). Here, the scheme structure of the finite group \( S_r \) is considered to be isomorphic to the \( C \)-scheme \( \text{Spec}(\prod \mathbb{C}) \).

**Theorem 5.3.** Let \( V \) be the \( C \)-algebra \( \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_r)) \) and \( V_+ \) the subalgebra \( \mathbb{C}[z_1] \times \cdots \times \mathbb{C}[z_r] \).

The canonical exact sequence of functors on groups

\[
0 \to G \times \cdots \times G \xrightarrow{i} \text{Aut}_C(V) \xrightarrow{p} S_r \to 0
\]

splits.

In particular, \( \text{Aut}_C(V) \) is representable by a formal group \( C \)-scheme which will be denoted by \( G_V \).

**Proof.** The map \( i \) is the canonical inclusion

\[
\text{Aut}_C(\mathbb{C}((z_1))) \times \cdots \times \text{Aut}_C(\mathbb{C}((z_r))) \hookrightarrow \text{Aut}_C(V)
\]

Now, observe that one can associate to every permutation \( \sigma \in S_r \) the automorphism of \( V \) defined by \( z_i \mapsto z_{\sigma^{-1}(i)} \).

**Corollary 5.4.** If \( G_V^0 \) denotes the connected component of the origin of \( G_V \), then there is an identification of groups

\[
G_V^0 = G \times \cdots \times G
\]

There are scheme isomorphisms

\[
G_V \simeq (G \times \cdots \times G) \times S_r
\]

\[
G_V^0 \simeq \Gamma_V
\]

(which are not group homomorphisms).
The formal group scheme $G^0_V$ acts on Gr($V$) and this action yields an action on the subscheme $\mathcal{M}^\infty(r) \subset \text{Gr}(V)$.

The main results of [MP2] can be generalized to the present setting. Now, let us consider the groups $G_V$ and $G_W$ corresponding to the $\mathbb{C}$-algebras $V$ and $W$ respectively.

The $V$-algebra structure of $W$ allows us to define a subgroup $G^W_V \subseteq G^0_W$ as follows

\begin{equation}
G^W_V := \left\{ \begin{array}{c}
W \xrightarrow{\bar{g}} W \\
V \xrightarrow{g} V
\end{array} \right\} \quad \text{where} \quad \bar{g} \in G^0_W \text{ and } g \in G^0_V
\end{equation}

The element of $G^W_V$ given by such a diagram will be denoted by $\bar{g}$.

**Proposition 5.6.** There exists an exact sequence of formal group $\mathbb{C}$-schemes

$$0 \to \mu_E \to G^W_V \xrightarrow{\pi} G^0_V \to 0$$

where $\pi(\bar{g}) = g$ and

$$\mu_E := (\mu_{e_1^{(1)}} \times \cdots \times \mu_{e_{k_1}^{(1)}}) \times \cdots \times (\mu_{e_1^{(r)}} \times \cdots \times \mu_{e_{k_r}^{(r)}})$$

($\mu_{e_i^{(j)}} \subset \mathbb{C}^*$ being the group of $e_i^{(j)}$-th roots of unity).

**Proof.** By the previous Corollary it can be assumed that $r = 1$ and $k_1 =$; that is, $V = \mathbb{C}((z))$ and $W = \mathbb{C}((z^{1/e}))$. Then, one has that $G_V \simeq G$ and $G_W \simeq G$.

Observe that an element $\bar{g} \in G^0_W$ belongs to $G^W_V$ if and only if $g(z) = \bar{g}(z^{1/e})^e$. And the result follows. $\square$

**Corollary 5.7.** The canonical restriction map $G^W_V \to G^0_V$ yields an isomorphism of Lie algebras

$$\text{Lie } G^W_V \xrightarrow{\sim} \text{Lie } G^0_V$$

**Lemma 5.8.** Let $R$ be a $\mathbb{C}$-algebra and $f(z^{1/e}) \in R((z^{1/e}))$.

If $f(z^{1/e})^e \in R((z))$ and $f(z^{1/e})$ is invertible, then there exist $i$ such that $z^{i/e} f(z^{1/e}) \in R((z))$.

**Proof.** We may assume that $f(z^{1/e}) = \sum_i a_i z^{i/e}$ where $a_i \in \text{Rad}(R)$ for $i < 0$ and $a_0$ is invertible. Let $I \subseteq \text{Rad}(R)$ be the ideal generated by $\{a_i | i < 0\}$ (recall that this set is finite) and let $n \geq 0$ be the smallest integer such that $I^{n+1}$ vanishes. Let us proceed by induction on $n$. 


Proof. Note that it suffices to prove the claim for the following case

\[ f(z^{1/e}) = e \cdot a_0 \cdot z^{i_0/e} + \text{(higher order terms)} \in R((z)) \]

and, therefore, \( i_0 = \hat{e} \). To conclude is suffices to consider \( (a_0 + a_{i_0} z^{i_0/e})^{-1} \cdot f(z^{1/e}) \) and iterate this argument.

General case. Let \( \tilde{f}_n(z^{1/e}) \) be the class of \( f(z^{1/e}) \) in \( R/I^n((z^{1/e})) \). From the induction hypothesis it follows that \( \tilde{f}_n(z^{1/e}) \in R/I^n((z)) \). Let \( f_n(z) \in R((z)) \) be a preimage of \( \tilde{f}_n \). Consider now the element \( f(z^{1/e})(f_n(z))^{-1} \in R((z)) \). From the \( n = 1 \) case, it follows that \( f(z^{1/e})(f_n(z))^{-1} \in R((z)) \), and the claim follows.

\[ \square \]

Lemma 5.9. Let \( R \) be a \( \mathbb{C} \) algebra, \( \text{Tr} : \hat{W}_R \to \hat{V}_R \) be the trace map and \( \bar{g} \) be an element of \( G_W^0 \).

Then, \( \bar{g} \in G_W^0 \) if and only if \( \text{Tr} \circ \bar{g} = \bar{g} \circ \text{Tr} \).

Proof. Note that it suffices to prove the claim for the following case

\[ V = R((z)) \hookrightarrow W = R((z^{1/e})) \]

Let us show that

\[ \text{Tr}(\bar{g}w) = \pi(\bar{g}) \text{Tr}(w) = g(\text{Tr}(w)) \quad \forall w \in \hat{W}_R \]

An element \( \bar{g} \in G_W^0(R) \) is of the form

\[ \bar{g}(z^{1/e}) = z^{1/e} \cdot (\sum_i a_i z^{i/e}) \in R((z^{1/e})) \]

where \( a_i \) is nilpotent for \( i < 0 \) and \( a_0 \) in invertible. The condition \( \bar{g} \in G_W^0 \) implies that

\[ \bar{g}(z^{1/e})^e = \bar{g}(z) \in R((z)) \]

and, by the previous lemma, it follows that \( \sum_i a_i z^{i/e} \in R((z)) \), that is, \( a_i = 0 \) if \( i \neq \hat{e} \). In particular, \( g = \pi(\bar{g}) \) is the automorphism given by

\[ g(z) = z (\sum_j a_{je} z^j) \in R((z)) \]

By linearity, it is enough to check the claim for \( z^{l/e} \) where \( l \in \mathbb{Z} \)

\[ \text{Tr}(\bar{g}(z^{l/e})) = \text{Tr}(z^{l/e} \cdot (\sum_j a_{je} z^j)) = (\sum_j a_{je} z^j) \cdot \text{Tr}(z^{l/e}) = g(\text{Tr}(z^{l/e})) \]

since \( \text{Tr}(z^{l/e}) = 0 \) for \( l \neq \hat{e} \) and \( \text{Tr}(z^{l/e}) = e \cdot z^{l/e} \) for \( l = \hat{e} \).

Conversely. For \( \bar{g} \in G_W^0 \) commuting with the trace, we have that

\[ \bar{g}(\text{Tr}(z)) = \bar{g}(e \cdot z) = e \cdot \bar{g}(z^{1/e})^e \]

\[ \text{Tr}(\bar{g}(z)) \in R((z)) \]
Therefore, one obtains that $\bar{g}(z^{1/e}) \in R((z))$. The previous lemma implies that $\bar{g}(z^{1/e})$ is of the form $z^{1/e}(\sum a_i z^i)$ which belongs to $G^W_V$.

**Theorem 5.11.** It holds that

1. $\text{Lie } G_V \simeq \bigoplus_{i=1}^r \text{Der}(\mathbb{C}((z_i)), \mathbb{C}((z_i))) \simeq \bigoplus_{i=1}^r \mathbb{C}((z_i)) \frac{\partial}{\partial z_i}$;
2. $\text{Lie } G_W \simeq \bigoplus_{i=1}^r \bigoplus_{j=1}^{k_i} \text{Der}(\mathbb{C}((z_i^{1/e_j})), \mathbb{C}((z_i^{1/e_j})))$;
3. $\text{Lie } G^W_V \simeq (\text{Lie } G^W_W)^{\text{Tr}}$ (those derivations commuting with the trace; as in (4.5)).

**Proof.** The first two follow from similar arguments as in the proof of Theorem 3.5 of [MP2]. The last claim is a consequence of the previous Lemma.

**Theorem 5.12.** The group $G^W_V$ acts on $\mathcal{H}_E^\infty$.

**Proof.** Recall that $\mathcal{H}_E^\infty$ consists of those points $U$ of $\mathcal{M}^\infty(\bar{r})$ such that $\text{Tr}(U) \subseteq U$ and that $G^W_V$ acts on $\mathcal{M}^\infty(\bar{r})$. Therefore, it suffices to show that the condition $\text{Tr}(U) \subseteq U$ implies that $\text{Tr}(\bar{g}U) \subseteq \bar{g}U$ for all $\bar{g} \in G^W_V$.

Let $U$ be a point of $\mathcal{H}_E^\infty$. From the defining property of $\bar{g}$ (see (5.5)) and from the inclusion $\text{Tr}(U) \subseteq U$, one has

$$\text{Tr}(\bar{g}U) = \bar{g}(\text{Tr}(U)) \subseteq \bar{g}U$$

and the statement follows.

**Theorem 5.13.** Let $(E, n, r, \bar{g}, g)$ as in Theorem 3.10. The group $G^W_V$ acts on $\mathcal{H}_E^{\infty}[\bar{g}, g]$ and this action is locally transitive.

**Proof.** It is straightforward that the action on $\mathcal{H}_E^{\infty}$ gives an action on $\mathcal{H}_E^{\infty}[\bar{g}, g]$. The rest of the proof is based on the ideas of [MP2] (Lemma 4.10, Theorem 4.11 and Lemma A.2). There it is shown that it is enough to check that the surjectivity of the map of tangent spaces

\begin{equation}
T_{id}G^W_V \longrightarrow T_U \mathcal{H}_E^{\infty}[\bar{g}, g]
\end{equation}

Let $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ be the data attached to $U \in \mathcal{H}_E^{\infty}[\bar{g}, g]$. Then, Theorems 4.6, 4.8 and 5.11 and Corollary 5.7 give the following commutative diagram

\begin{equation}
\begin{array}{cccccc}
0 & \longrightarrow & H^0(Y - \bar{y}, T_Y) & \longrightarrow & \text{Lie } G^0_W & \longrightarrow & \text{Der}(U, W/U) & \longrightarrow & 0 \\
& & & \downarrow & & & \downarrow & & \\
& & & \text{Lie } G^W_V & \psi & \longrightarrow & \text{Der}(\dot{U}, W/U)^{\text{Tr}} & & \\
& & & \downarrow & & & \downarrow & & \\
0 & \longrightarrow & H^0(X - \bar{x}, T_X) & \longrightarrow & \text{Lie } G^0_V & \longrightarrow & \text{Der}(\text{Tr }U, V/ \text{Tr }U) & \longrightarrow & 0
\end{array}
\end{equation}
(\mathcal{T} denoting the tangent sheaf). From the diagram, we deduce that \( \psi \) is surjective. Since \( \psi \) coincides with the map (5.14), we are done. □

**Theorem 5.15.** Let \((E, n, r, \bar{g}, g)\) as in Theorem 3.10. Let \( U \in \mathcal{H}_E^\infty[\bar{g}, g] \) be a rational point. Then, there is an isomorphism

\[
T_U \mathcal{H}_E^\infty[\bar{g}, g] \sim T_{U^V} \mathcal{M}^\infty(r)
\]

**Proof.** The injectivity follows from Theorem 4.8. The surjectivity is a consequence of the diagram of the previous proof. □

**Remark 4.** This Theorem is the analog of the fact that the map from the classical Hurwitz space to the moduli of curves is etale at those points corresponding to covers where both curves are smooth. Our approach also allows one to study the non-smooth case, however, because of our goals we have focus ourselves on the smooth case.

### 6. Picard schemes

**Definition 6.1.** Let \( \text{Pic}_E^\infty[\bar{g}, g] \) be the contravariant functor from the category of \( \mathcal{C} \)-schemes to the category of sets defined by

\[
S \rightsquigarrow \{(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}, L, \phi_{\bar{y}})\}
\]

where

1. \((Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}) \in \mathcal{H}_E^\infty[\bar{g}, g](S)\);
2. \(L\) is a line bundle on \( Y\) such that \((Y_s, \bar{y}(s), L|_{Y_s})\) is maximal for all closed point \(s \in S\);
3. \(\phi_y\) is a formal trivialization of \(L\) along \(\bar{y}\); that is, an isomorphism \(\phi_y : \tilde{L}_y \simeq \mathcal{O}_{Y, \bar{y}}\).
4. Two sets of data are said to be equivalent when there is an isomorphism of \(S\)-schemes \(Y \sim Y'\) compatible with all the data.

**Theorem 6.2.** The functor \( \text{Pic}_E^\infty[\bar{g}, g] \) is representable by a subscheme \( \text{Pic}_E^\infty[\bar{g}, g] \) of \( \text{Gr}(W) \).

**Proof.** Consider the morphism from \( \text{Pic}_E^\infty[\bar{g}, g] \) to \( \text{Gr}(V) \times \text{Gr}(V) \) which sends the \(S\)-valued point \((Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}, L, \phi_y)\) to the following pair of submodules as a point of \( \text{Gr}(W) \times \text{Gr}(W) \)

\[
(t_y \left( \lim_m \left( p_* \mathcal{O}_Y(m \cdot \pi^{-1}(\bar{x})) \right) \right), (t_y \circ \phi_y) \left( \lim_m \left( p_* L(m \cdot \pi^{-1}(\bar{x})) \right) \right))
\]

where \(p : Y \times S \to S\) is the projection.

It can be shown that this map is injective and that the image is contained in the subscheme \( Z \subset \text{Gr}(W) \times \text{Gr}(W) \) of those pairs \((\mathcal{A}, \mathcal{L})\) verifying

\[
\mathcal{O}_S \subset \mathcal{A} , \quad \mathcal{A} \cdot \mathcal{A} \subset \mathcal{A} , \quad \mathcal{A} \cdot \mathcal{L} \subset \mathcal{L}.
\]
Applying the converse construction of the Krichever correspondence to the algebra \( \mathcal{A} \) we obtain a curve \( \mathcal{X} \to Z \). Then, consider the subscheme \( Z' \subset Z \) defined by the points \( z \in Z \) such that \( \mathcal{X}_z \) is smooth.

Now, we claim that if \( (\mathcal{A}, \mathcal{L}) \in Z' \), then \( \mathcal{A} \) can be obtained from \( \mathcal{L} \). Indeed, it will be shown that \( \mathcal{A} \) is the stabilizer of \( \mathcal{L} \).

Consider \( (\mathcal{A}, \mathcal{L}) \in Z' \) and let \( \mathcal{A}_\mathcal{L} \) denote the stabilizer of \( \mathcal{L} \), that is, the \( \mathcal{O}_S \)-algebra

\[
\mathcal{A}_\mathcal{L} := \{ w \in \hat{W}_S \text{ such that } w \cdot \mathcal{L} \subseteq \mathcal{L} \},
\]

Since \( \mathcal{A}_z \) corresponds to a smooth curve for all \( z \in Z' \) and \( \mathcal{A} \subseteq \mathcal{A}_\mathcal{L} \) are points of \( \text{Gr}(W) \), then \( \mathcal{A}_\mathcal{L} \) is a finite \( \mathcal{A} \)-module such that \( \mathcal{A}_z = (\mathcal{A}_\mathcal{L})_z \) for all \( z \in Z' \). Therefore we have that \( \mathcal{A} = \mathcal{A}_\mathcal{L} \).

Now, one checks that the image of \( Z' \) by the projection onto the second factor, \( \text{Gr}(W) \times \text{Gr}(W) \to \text{Gr}(W) \), represents \( \text{Pic}_E^\infty[\bar{g},g] \). \( \square \)

**Remark 5.** For \( \chi \in \mathbb{Z} \), the subfunctor of \( \text{Pic}_E^\infty[\bar{g},g] \) consisting of those points such that \( L \) has Euler-Poincaré characteristic equal to \( \chi \) is representable by the subscheme \( \text{Pic}_E^\infty[\bar{g},g] \cap \text{Gr}^\chi(W) \).

Since \( \Gamma_W \) represents the group of invertible elements of \( W \) and \( G_V^W \) is a group of automorphisms of \( W \) as an algebra, one has a canonical actions of \( G_V^W \) on \( \Gamma_W \). Therefore, it makes sense to consider the semidirect product \( G_V^W \ltimes \Gamma_W \) as follows

\[
(g_2, \gamma_2)(g_1, \gamma_1) := (g_2g_1, g_1^{-1}(\gamma_2)\gamma_1)
\]

and the action of \( G_V^W \ltimes \Gamma_W \) on the Grassmannian induced by the action on \( W \)

\[
(g, \gamma)w := g(\gamma \cdot w)
\]

**Theorem 6.3.** There are canonical actions of the groups \( G_V^W, \Gamma_W \) and \( \Gamma_W \ltimes G_V^W \) on \( \text{Pic}_E^\infty[\bar{g},g] \).

Moreover, the action of \( G_V^W \ltimes \Gamma_W \) is locally transitive.

**Proof.** Let

\[
\Psi: \text{Pic}_E^\infty[\bar{g},g] \to \mathcal{H}_E^\infty[\bar{g},g]
\]

be the forgetful morphism. Let us consider a rational point \( p \in \text{Pic}_E^\infty[\bar{g},g] \) corresponding to the geometric data \( (Y, X, \pi, \bar{x}, \bar{y}, t_x, t_y, L, \phi_y) \). Let \( A := t_y(H^0(Y - \bar{y}, \mathcal{O}_Y)) \in \mathcal{H}_E^\infty[\bar{g},g] \) and let \( U := t_y(\phi_y(H^0(Y - \bar{y}, L))) \in \text{Pic}_E^\infty[\bar{g},g] \).
Considering the map induced by $\Psi$ at the level of tangent spaces and recalling Theorem 5.13, one easily gets the following diagram

\[
\begin{array}{c}
0 \longrightarrow \operatorname{Lie} \Gamma_W \longrightarrow \operatorname{Lie}(\Gamma_W \ltimes G^W_V) \longrightarrow \operatorname{Lie} G^W_V \longrightarrow 0 \\
\downarrow \psi \quad \downarrow \quad \downarrow \psi \quad \downarrow \psi \quad \downarrow \psi \\
0 \longrightarrow \operatorname{Hom}_{A-\text{mod}}(U, W/U) \longrightarrow T_p \operatorname{Pic}^\infty_E [\bar{g}, g] \longrightarrow \operatorname{Der}(A, W/A)^{\text{TV}} \longrightarrow 0
\end{array}
\]

The snake lemma implies that the middle vertical arrow is surjective. We conclude by similar ideas as in the proof of Theorem 5.13. □

Remark 6. In a future work and following ideas of [BF] we aim at studying how the deformations of a point $\operatorname{Pic}^\infty_E [\bar{g}, g]$ under the action of $G^W_V \ltimes \Gamma_W$ can be interpreted as isomonodromic deformations.

References

[BF] Ben-Zvi, D.; Frenkel, E., “Geometric Realization of the Segal–Sugawara Construction”, \url{http://arxiv.org/abs/math/0301206}.

[ELSV] Ekedahl, T.; Lando, S.; Shapiro, M.; Vainshtein, A., “Hurwitz numbers and intersections on moduli spaces of curves”, Invent. Math. 146 (2001), no. 2, pp. 297–327.

[F] Fulton, W.; “Hurwitz schemes and irreducibility of moduli of algebraic curves”, Ann. of Math. (2) 90 (1969), 542–575.

[LO] Levin, A. M.; Olshanetsky, M. A., “Hierarchies of isomonodromic deformations and Hitchin systems”, Moscow Seminar in Mathematical Physics, 223–262, Amer. Math. Soc. Transl. Ser. 2, 191, Amer. Math. Soc., Providence, RI, 1999.

[MP1] Muñoz Porras, J. M.; Plaza Martín, F. J., “Equations of Hurwitz schemes in the infinite Grassmannian”, preprint math.AG/0207091.

[MP2] Muñoz Porras, J.M.; Plaza Martín, F.J., “Automorphism group of $k((t))$: applications to the bosonic string”, Commun. Math. Phys. 216 (2001), pp. 609–634.

[OP] Okounkov, A.; Pandharipande, R., “Gromov-Witten theory, Hurwitz theory, and completed cycles”, math.AG/0204305

Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca. Spain., Tel: +34 923294460. Fax: +34 923294583

E-mail address: jmp@usal.es

E-mail address: fplaza@usal.es