Odd-Spin Yrast States as Multiple Quadrupole-Phonon Excitations

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The wavefunctions of the lowest odd spin positive parity yrast states in the IBA are shown to be nearly pure multiple quadrupole-phonon excitations even outside the three dynamical symmetries. The empirical data for collective nuclei with $30 \leq Z \leq 80$ confirm these predictions. The quadrupole-phonon purity of the $2^+_1$ state can be measured from E2-branching ratios of the $3^+_1$ state. These data show a high correlation to the $2^+_1$ Q-phonon purity deduced from the E2-decay of $2^+$ states.

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Single phonon excitations of atomic nuclei have been well studied [1]. In the last years multi-phonon modes have also attracted wider interest, e.g. [2–10]. Recently the introduction of the quadrupole-phonon (Q-phonon) scheme [11–13] for describing the wavefunctions of low lying collective excitations for the O(6)-symmetry of the Interacting Boson Approximation model (IBA) [14,15] has led to an intuitive understanding of the E2-decay of $\gamma$-soft nuclei [16,17]. In nature there are no nuclei, however, that exactly exhibit the properties of the dynamical symmetries of the IBA. Better descriptions of real nuclei are achieved by adding a symmetry breaking part to the Hamiltonians of the dynamical symmetries. Then, the wavefunctions of the eigenstates have no longer a simple structure rendering the understanding of their properties more difficult. Therefore it is desirable to find simple expressions for the wavefunctions of at least some nuclear states also outside the analytical limits. It will be shown in this Letter that a generalization of the Q-phonon scheme can fulfil this for the restricted set of the low lying positive parity yrast states that we will discuss in the following.

A pure n-Q-phonon configuration with angular momentum $L$ is obtained by an n-fold application of the quadrupole operator to the ground state $|gs\rangle$

$$| L, n \rangle = N^{(L,n)} \frac{(QQ \cdots Q)^{(L)}}{n=L/2} | gs \rangle$$

for even $L$ and

$$| L, n \rangle = N^{(L,n)} \frac{(Q^n QQ \cdots Q)^{(L+1)}/(2)}{n-1=(L+1)/2} | gs \rangle$$

for odd $L$ where $Q = Q^x = s^+ \tilde{d} + d^+ s + \chi (d^+ \tilde{d})^{(2)}$ is the general quadrupole operator of the IBA which includes a $(d^+ \tilde{d})$ term and $N$ is a normalization factor.

For an investigation of the structure of states in the full symmetry triangle of the IBA, the properly scaled consistent Q (CQF) Hamiltonian

$$H = Q^x \cdot Q^x - \frac{\epsilon}{\kappa} n_d = H(\epsilon/\kappa, \chi)$$

and the E2-transition operator

$$T(E2) = q \cdot Q^x$$

are useful [18,20]. The limiting cases of the three dynamical symmetries correspond to the following parameter combinations: U(5): $(\epsilon/\kappa = \infty, \chi = 0)$, SU(3): $(\epsilon/\kappa = 0, \chi = -\sqrt{7}/2)$ and for O(6): $(\epsilon/\kappa = 0, \chi = 0)$. The parameter $q$ in the E2-transition operator is the effective quadrupole charge. For arbitrary values of the Hamiltonian parameters the multiple Q-phonon configurations are not eigenstates of the Hamiltonian. However, they can be expressed as linear combinations of the true eigenstates $|L_i\rangle$ of the CQF-Hamiltonian $H$. 
Here \(| L_i, n \rangle = \sum_i \alpha_i^{(L,n)} \langle L_i \rangle\).

The amplitudes \(\alpha_i^{(L,n)}\) in Eq. (7) can be calculated in the sd-IBA-1 with standard programs like PHINT by the use of the reduced E2 matrix elements \(r(L_i \rightarrow L_f)\) defined by:

\[ r(L_i \rightarrow L_f) = \frac{\langle L_f \parallel Q \parallel L_i \rangle}{\sqrt{2L_i + 1}}. \]

The expression is obtained from Eq. (2) by multiplying with the bra \(\langle L_i \parallel \) from the left, by decoupling the quadrupole operators from Eq. (2), and by introducing the unity operator \((1 = \sum_i \sum_{r_i} \langle L_i \parallel I_{rr} \rangle \parallel I_{rr} \rangle\) between every two following quadrupole operators. The intermediate spin is \(I_{n-1} = 2(n-1) = L-2\) for even \(L\) and \(I_{n-1} = 2(n-1) = L+1\) for odd \(L\). These requirements allow unique decoupling of the quadrupole operators. The positive constant \(N^{(L,n)}\) is fixed by the normalization condition \(\Sigma_i \alpha_i^{(L,n)} = 1\). It has been shown recently that the lowest members of the ground state band with even spin, e.g., the \(2^+_1\) and the \(4^+_1\) states, have a nearly pure Q-phonon configuration in the \(\langle \epsilon/\kappa, \chi \rangle\) space.

It is the main purpose of this Letter to show that this is true also for the positive parity yrast states with odd spin in the full \(\langle \epsilon/\kappa, \chi \rangle\) parameter space with the exception of the SU(3) symmetry. A comparison to the exact expressions found in the O(6)-limit suggests that the lowest odd spin states with odd spin can be written as Q-phonon configurations consisting of \(n = (L+3)/2\) Q-phonons, where \(L = 3, 5,...\). In particular one finds:

\[ | 3^+_1 \rangle = \alpha_3 | 3, 3 \rangle + | r_3 \rangle = \alpha_3 N^{(3,3)}(QQQ^{(4)}(3) | 0^+_1 \rangle + | r_3 \rangle, \]
\[ | 5^+_1 \rangle = \alpha_5 | 5, 4 \rangle + | r_5 \rangle = \alpha_5 N^{(5,4)}(QQQ^{(6)}(5) | 0^+_1 \rangle + | r_5 \rangle \]

where the labels \(| L_i, n \rangle = | 3, 3 \rangle\) and \(| 5, 4 \rangle\) give the total angular momentum and the Q-phonon number respectively, as defined in Eq. (2). The "rest" wavefunction \(| r_L \rangle\) is defined by Eqs. (8) and (9) and the condition that it is orthogonal to the corresponding yrast state \(| L^+_1 \rangle\). These yrast states have a pure Q-phonon configuration if the rest wavefunction vanishes, i.e., if \(\langle r_L | r_L \rangle = 0\) which implies \(\alpha_3^2 = 1\) and \(\alpha_5^2 = 1\) (see Eqs. (8) and (9)) as is the case in the O(6)-symmetry. Note that we use "state" for the eigenstates of the Hamiltonian as opposed to "Q-phonon configuration" defined in Eqs. (1) and (2). Therefore the amount of the Q-phonon impurity of an eigenstate of \(H\) is expressed by the norm of the rest wavefunction: \(R^{(3,3)} = | r_3 \rangle | r_3 \rangle = 1 - \alpha_3^2\) and \(R^{(5,4)} = | r_5 \rangle | r_5 \rangle = 1 - \alpha_5^2\). For the boson number \(N = 10\) the Q-phonon impurities \(R^{(3,3)}\) and \(R^{(5,4)}\) are shown in Fig. 6 for the CQF Hamiltonian as a function of the parameters \(\chi\) and \(\epsilon/\kappa\). It is concluded that at least the lowest odd spin positive parity yrast states have a rather pure Q-phonon configuration. One finds that the squared amplitudes of the rest wavefunctions \(\langle r_3 | r_3 \rangle\) and \(\langle r_5 | r_5 \rangle\) are less than 10%.

The maximum of the Q-phonon impurity of the \(3^+_1\) state is 8% for \(N = 10\) and 5% for \(N = 6\) and for the \(5^+_1\) state it is 6% for \(N = 10\) and 3% for \(N = 6\). The maxima of the phonon impurities are reached for parameter combinations \((\chi = -\sqrt{7}/2, \epsilon/\kappa \neq 0)\) that do not correspond to real nuclei. Even well deformed nuclei such as \(^{168}\text{Er}\) or \(^{178}\text{Hf}\) are described best with values of the parameter \(| \chi | < 0.7\). For real nuclei the Q-phonon impurity of these states is therefore predicted to be considerably less than 10%. Furthermore, the Q-phonon impurity decreases with increasing spin and decreasing boson number. Here it should be noted that the Q-phonon picture of odd spin states is not defined in the SU(3)-symmetry because, there, no E2-transitions can lead out of the ground state band which consists only of states with even spin because \(Q^x = -\sqrt{7}/2\) is a generator of the subgroup SU(3) and cannot connect different irreducible representations of SU(3). Infinitesimally breaking of the SU(3)-symmetry allows one to calculate the Q-phonon purity. The \(3^+_1\) and \(5^+_1\) states then have practically pure Q-phonon configurations as in the two other symmetries. But the normalization factor \(N\) goes to infinity when the SU(3) singularity is approached. However, as mentioned above, for describing real nuclei a \(\chi\)-value is needed which considerably deviates from its value in the SU(3)-symmetry. So, problems with the definition of the Q-phonon purity for the odd spin states in the SU(3)-symmetry do not affect the conclusions that are drawn in this Letter concerning atomic nuclei.
Of course one could also consider the $3^+_1$ state as a 2-Q-phonon configuration $\Lambda^{(3,2)}(QQ)\,(3)\,|\,0^+_1\rangle$. This suggestion fails, however, in all three dynamical symmetries of the IBA whereas the configuration $\Lambda^{(3,3)}(QQQ)\,(3)\,|\,0^+_1\rangle$ fails only in the SU(3)-limit. The antisymmetric coupling of the two quadrupole operators $(Q^\chi Q^\chi)\,(3) = (1-4\chi^2/7)(d_+^\dagger d_+^\dagger)\,(3)$ vanishes identically in the SU(3) dynamical symmetry ($\chi = -\sqrt{7}/2$). In addition it is a generator of the O(5) Lie-subgroup and must conserve the d-boson seniority $\tau$. Therefore, it cannot connect the ground state ($\tau = 0$) to any $3^+_1$ state if the seniority is a good quantum number, as it is on the whole transition between the U(5) and O(6) dynamical symmetries. We note that this problem and problems of uniqueness of configurations can be completely solved for the O(6)-symmetry by ref. [13] by application of a proper symmetrizer to the Q-phonon operators. The configurations created by Eq. (1) are identical to those symmetrized. Also Eq. (2) produces Q-phonon configurations which in practice coincide with the symmetric ones.

Combining the present results for the odd spin states with the previous results for the even spin yrast states [24] one finds that the low spin yrast states can be considered approximately pure Q-phonon configurations. This applies even for deformed nuclei outside the SU(3)-symmetry. One additional advantage of this description consists in the occurrence of approximate selection rules for electromagnetic E2-decay between pure Q-phonon configurations with different numbers of Q-phonons. In order to demonstrate the selection rules we have investigated the E2-decay properties of the pure Q-phonon configurations of Q-phonon number of $n = 3, 4$ and total angular momenta $L = 3, 5$ respectively as defined in Eqs. (2), (3), and (4). In Fig. 3 the ratios of the E2-transition strengths $B(E2;\, L, n \rightarrow L', n')$ between pure Q-phonon configurations are shown:

$$\frac{B(E2;\, |\, 3, 3\rangle \rightarrow |\, 2, 1\rangle)}{B(E2;\, |\, 3, 3\rangle \rightarrow |\, 2, 2\rangle)}$$

and

$$\frac{B(E2;\, |\, 5, 4\rangle \rightarrow |\, 4, 2\rangle)}{B(E2;\, |\, 5, 4\rangle \rightarrow |\, 4, 3\rangle)}$$

The final configurations in the denominators are coupled as follows: $|\, 2, 2\rangle = \mathcal{N}(QQ)\,(2)\,|\,0^+_1\rangle$ and $|\, 4, 3\rangle = \mathcal{N}(QQQ)\,(4)\,|\,0^+_1\rangle$ respectively. From these calculations one finds that these branching ratios are very small in the full symmetry triangle. Therefore rather good selection rules exist for E2-transitions between pure Q-phonon configurations.

This observation leads to simple predictions for experimental E2-branching ratios. From recent work [24] it is known that, empirically, the one Q-phonon configuration

$$|\, 2, 1\rangle = \mathcal{N}^{(2,1)}Q\,|\,0^+_1\rangle \approx \alpha\, |\, 2^+_1\rangle + \gamma\, |\, 2^+_1\rangle; \quad \gamma^2 \leq 7\%$$

consists mainly of the first excited state with a small admixture of the $2^+_1$ state. We use the notation $2^+_1$ for either the $2^+_1$ state or the $2^+_1$ state, whichever is connected to the $0^+_1$ state by only a small E2 matrix element. The first excited state has therefore the form

$$|\, 2^+_1\rangle \approx \frac{1}{\alpha}\, |\, 2, 1\rangle + \epsilon\, |\, 2^+_1\rangle; \quad \epsilon^2 = \left(\frac{\gamma}{\alpha}\right)^2 \leq 8\% .$$

The branching ratio for E2-transitions of the $3^+_1$ state to the $2^+_1$ and the $2^+_1$ states is given by

$$\frac{B(E2;\, 3^+_1 \rightarrow 2^+_1)}{B(E2;\, 3^+_1 \rightarrow 2^+_1)} \approx \left(\frac{\langle 3, 3\, |\, Q\, |\, 2, 1\rangle + \epsilon\, |\, 2^+_1\rangle}{\langle 3, 3\, |\, Q\, |\, 2^+_1\rangle}\right)^2 \approx \epsilon^2 \approx 1 - \alpha^2 .$$

This E2-branching ratio from the $3^+_1$ to the $2^+_1$ and the $2^+_1$ states is thus a measure of the Q-phonon impurity $1 - \alpha^2$ of the $2^+_1$ state. An alternative measure of this Q-phonon impurity of the $2^+_1$ state is the ratio

$$R^{(2,1)} = \frac{\sum_{i>1} B(E2;\, 0^+_1 \rightarrow 2^+_1)}{\sum_{i>1} B(E2;\, 0^+_1 \rightarrow 2^+_1)} = 1 - \alpha^2$$

which has been studied in ref. [24]. Fig. 3 shows according to Eq. (14) the experimentally known branching ratios for the $3^+_1$ state for collective nuclei ($E_{4^+_1}/E_{2^+_1} > 1.9$) with $A < 200$ [24]. This histogram confirms the parameter-free prediction of the sd-CQF-IBA-1 [23] that the Q-phonon purity of the $2^+_1$ state exceeds 93% for all nuclei in the symmetry triangle. The largest deviations from a pure Q-phonon configuration are found for the $2^+_1$ states of the Os.
isotopes which are transitional nuclei and lie between the O(6) and SU(3) symmetry. Interestingly, it is also for the \( \text{O(6) } \rightarrow \text{SU(3)} \) transition that the IBA predicts the largest Q-phonon impurities of the \( 2^+ \) state \[24\].

The two E2-branching ratios of Eqs. (14) and (15) involve different observables and thus are independent measures of the purity of the Q-phonon configuration of the \( 2^+ \) state. Therefore, the Q-phonon purity of the \( 2^+ \) state obtained with the two methods can be compared. The empirical correlation of these two "purity" observables is plotted in Fig. 4 for those nuclei for which both observables are known experimentally with a relative error smaller than 20%. The good correlation is clear; in fact, the correlation coefficient of \( r \approx 0.84 \) is rather large. These results confirm the predictive strength of the Q-phonon picture even outside the three dynamical symmetries.

To conclude, we have shown that the lowest odd spin yrast states have nearly pure Q-phonon configurations in the IBA. Their Q-phonon purity is always larger than 90% and is calculated to increase with increasing spin and decreasing boson number. Between pure Q-phonon configurations there exist rather good selection rules for E2-transitions. Examples of these selection rules are \( Q\bar{Q} \neq 0 \), \( QQ \neq Q \), or \( Q\bar{Q} \neq QQ \). These approximate selection rules improve with decreasing spin and increasing boson number. The E2-branching ratio of the \( 3^+ \) to the \( 2^+ \) and the \( 2^+ \) states is predicted by the CQF-IBA to be less than 8% in the full \( (\epsilon/\kappa, \chi) \) parameter space (except at the SU(3) dynamical symmetry where the Q-phonon is not defined) and it correlates to the Q-phonon purity observable \( R^{(2)} \). These predictions are confirmed by the experimental data. The present Q-phonon picture is similar to the ordinary phonon concept for the vibrational nuclei but its range of application extends to the more deformed nuclei. Independent of the actual nuclear shape the Q-phonon scheme provides an intuitive understanding of low lying collective \textit{yrast spin} excitations of nuclei. The study of non-yrast states is beyond the scope of the present paper. Following ref. \[3\] the phonon picture has to be extended to include, e.g., symmetrization. This is considerably more complicated than the present approach, which works only for yrast states. Furthermore, literal extensions do not make the members of the \((\text{quasi-})\gamma\)-band or the \((\text{quasi-})\beta\)-band describable in terms of pure Q-phonon configurations, e.g., the \( 2^+ \) as an approximately pure \( (QQ)^{(2)} \mid 0^+_1 \) state. This is an interesting limitation of the present approach for non-yrast states, which we are presently investigating.

In the past, most IBA calculations have dealt with fits to a specific nucleus, to specific groups of nuclei (e.g., isotopic sequences), or to properties of specific classes of nuclei (e.g., axially deformed or \( \gamma \)-soft). With refs. \[24\] and the present work, studies of the IBA have entered a new phase focusing on its predictions of very general properties of collectivity in nearly all nuclei, essentially independent of parameters. The fact that, in every case studied, these properties are confirmed experimentally shows that the truncation embodied in the IBA, especially in the CQF, reflects nearly universal manifestations of nuclear collectivity. This merits microscopic study to better understand the underlying origins of collectivity in nuclei.

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FIG. 1. Q-phonon impurities $R^{(3,3)} = \langle r_3 | r_3 \rangle$ and $R^{(5,4)} = \langle r_5 | r_5 \rangle$ of the $3_1^+$ and the $5_1^+$ states for the whole IBA space calculated gridwise throughout the symmetry triangle for $N = 10$ bosons.
FIG. 2. E2-branching ratios for pure Q-phonon configurations as defined by Eqs. (10) and (11) for the whole IBA structure space calculated gridwise for $N = 10$ bosons. Transitions between configurations that differ by more than one Q-phonon are hindered by 2 orders of magnitude. The selection rule is better fulfilled for the lower spin. Note that the orientation of the plot differs from that in Fig. 1 for ease of visualization.
FIG. 3. Histogram of all experimentally known branching ratios [23] from the $3_1^+$ to the lower $2^+$ states (see text) for collective nuclei ($E_{4_1^+}/E_{2_1^+} > 1.9$) with $30 \leq Z \leq 80$. In all cases pure E2-transitions were assumed.
FIG. 4. Correlation of the two "purity" observables for those collective nuclei with $30 \leq Z \leq 80$ for which both observables are known with an experimental error of less than 20%. The dotted line shows a perfect correlation. All data have been taken from [23].