A singular series average 
and the zeros of the Riemann zeta-function

by

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1. Introduction. This paper is concerned with averages of the arithmetic function

\[ S(k) = \begin{cases} 
2C_2 \prod_{p|k} \frac{p-1}{p-2} & \text{if } k \text{ is even, } k \neq 0, \\
0 & \text{if } k \text{ is odd}, 
\end{cases} \]  

(1.1)

where

\[ C_2 = \prod_{p>2} \left( 1 - \frac{1}{(p-1)^2} \right) = 0.66016 \ldots. \]  

(1.2)

The function \( S(k) \) occurs as the arithmetic factor in conjectured asymptotic formulas for the Goldbach Conjecture and also the Hardy–Littlewood Prime Pair Conjecture and is referred to as the “singular series” for these problems (see [HL19] and [HL22]). Averages of \( S(k) \) arise frequently in a variety of problems in analytic number theory. For \( x \geq 2 \), the Cesàro mean of \( S(k) \),

\[ S_1(x) := \sum_{k \leq x} (x-k)S(k), \]  

(1.3)

appears naturally in evaluating the variance for the number of primes in a short interval, and originated in Hardy and Littlewood’s unpublished paper Partitio Numerorum VII (see [Ran40] Lemma 8]). This variance for the number of primes in intervals is equivalent to a pair correlation type conjecture for the zeros of the Riemann zeta-function [GM87], and this connection partly motivated work on obtaining precise asymptotic formulas for \( S_1(x) \) (see [FG95] and [MS02]). The best results currently known are due to

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Vaughan [Vau01], who proved that, for \( x \) sufficiently large,  
\[
S_1(x) = \frac{1}{2} x^2 - \frac{1}{2} x \log x + \frac{1}{2} \left( 1 - \gamma - \log 2\pi \right) x + E(x)
\]
with  
\[
E(x) \ll x^{1/2} \exp \left( -c \frac{\left( \log x \right)^{3/5}}{\left( \log \log x \right)^{1/5}} \right),
\]
for some positive constant \( c \), and assuming the Riemann Hypothesis that  
\[
E(x) \ll x^{5/12 + \epsilon}.
\]
In a recent paper [GS21] we proved that  
\[
E(x) = \Omega_{\pm}(x^{1/4})
\]
so that \( E(x) \) oscillates and cannot be too small. Vaughan’s method shows that the size of the error term \( E(x) \) depends on a zero-free region of the Riemann zeta-function, but does not determine this dependence explicitly.

Our goal in this paper is to make this dependence between the complex zeros \( \rho = \beta + i\gamma \) of the Riemann zeta-function and the error in the singular series average more explicit. The problem that is encountered in doing this for \( S_1(x) \) is that the Cesàro mean is not smooth enough for absolute convergence in the contour integrals one encounters. Inspired by recent work on a related problem [GZHN17], [GL17] concerning the square of the singular series, we can sidestep this problem by using smoother weights. We found while writing this paper that this idea has already been applied to other arithmetic functions, and some results of the same type as ours have been obtained by similar methods for these functions. For \( 1/\phi(k) \) and \( k/\phi(k) \), see [GV97], [SS13], [SS14], and [IK19]; for \( k/\psi(k) \), where \( \psi \) is the Dedekind totient function, see [II17] and [IK17]; and for the Möbius function see [Ino19] and [SS19].

We define, for \( m \geq 1 \) and \( x \geq 2 \),  
\[
S_m(x) := \sum_{k \leq x} (x - k)^m \mathcal{S}(k).
\]
This is the \( m \)th Riesz mean of the singular series, although we chose not to normalize it; when \( m = 1 \) this is the Cesàro mean. We generalize (1.4) by defining \( E_m(x) \) for \( m \geq 1 \) by  
\[
S_m(x) = \frac{1}{m+1} x^{m+1} - \frac{1}{2} x^m \log x - H_m + c_E + \log 2\pi + E_m(x),
\]
where \( H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \) is the \( m \)th harmonic number, \( c_E \) denotes Euler’s constant, and we will write \( E_1(x) = E(x) \) in agreement with (1.4).

To describe our main result, we note that if \( \rho = \beta + i\gamma \) is a complex zero of \( \zeta(s) \) then \( \overline{\rho} = \beta - i\gamma \) is also a zero, so that the zeros above and below the real axis are reflections of each other. The Riemann Hypothesis
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is that all these complex zeros lie on the vertical line with $\beta = 1/2$, so that $\rho = 1/2 + i\gamma$. We now define

$$ a(\rho) = a(\rho, m) := \frac{2C_2m!\zeta\left(\frac{\rho}{2} - 1\right)\zeta\left(\frac{\rho}{2}\right)G\left(\frac{\rho}{2} - 1\right)}{(2^{\rho/2} + 1)\zeta'(\rho)\left(\frac{\rho}{2} - 1\right)\left(\frac{\rho}{2} + 1\right)\cdots \left(\frac{\rho}{2} + m - 1\right)} $$

if $\rho$ is a simple zero of the Riemann zeta-function. Here

$$ G(s) = \prod_{p > 2} \left(1 + 2\left(p^{s+1} + 1\right)\right). $$

If $\rho$ is a multiple zero of multiplicity $m_\rho = \ell$, then $a(\rho)$ is a polynomial of degree $\ell - 1$ in powers of $\log x$,

$$ a(\rho) = a(\rho, m, \ell, x) = \sum_{j=0}^{\ell-1} A_j(\rho, m)(\log x)^j, $$

defined in the next section. When $m \geq 2$ we prove the following unconditional result.

**Theorem 1.** For fixed $m \geq 2$, and any $\epsilon > 0$ and $x$ sufficiently large, there exists a number $U$, $x^5 \leq U \leq 2x^5$, for which

$$ E_m(x) = x^{m-1} \sum_{|\gamma| \leq U} a(\rho)x^{\rho/2} + O(x^{m-1+\epsilon}). $$

The sum over zeros in Theorem 1 is difficult to estimate because of our lack of knowledge of the size of $a(\rho)$, and even the Riemann Hypothesis does not directly help estimate this type of sum. However, assuming the Riemann Hypothesis, the method used to prove Theorem 1 comes within an extra factor of $x^\epsilon$ of proving the correct upper bound for $E_m(x)$ when $m \geq 2$.

**Theorem 2.** Assume the Riemann Hypothesis. For fixed $m \geq 2$, and any $\epsilon > 0$ and $x$ sufficiently large, we have

$$ S_m(x) = \frac{1}{m+1}x^{m+1} - \frac{1}{2}x^m(\log x - H_m + c_E + \log 2\pi) + O(x^{m-3/4+\epsilon}), $$

and therefore

$$ E_m(x) \ll x^{m-3/4+\epsilon}. $$

We can conjecturally deal with the sum in Theorem 1 using conjectures related to the Mertens Hypothesis. Let

$$ M(x) = \sum_{n \leq x} \mu(n). $$

The Mertens Hypothesis is the inequality $|M(n)| < \sqrt{n}$ for integers $n > 1$, but this was disproved in 1985 by Odlyzko and te Riele [OtR85]. More recent work on violations of the Mertens Hypothesis may be found in [BT15] and [Hur18]. The conjecture $M(x) \ll \sqrt{x}$ is probably false as well because
Ingham [Ing42] proved this is false assuming the reasonable conjecture that the imaginary parts of the zeros of the Riemann zeta-function are linearly independent over the integers. On the other hand, it is generally believed that the following conjecture is true.

**Weak Mertens Conjecture.** We have

\[
\int_1^T \left( \frac{M(x)}{x} \right)^2 \, dx \ll \log T.
\]

(1.14)

It is not hard to prove that the Weak Mertens Conjecture implies the Riemann Hypothesis and that all the zeros are simple zeros, and that

\[
\sum \frac{1}{|\rho \zeta'(\rho)|^2} \ll 1
\]

(see [Tit86, §14.29]). Ng [Ng04] found evidence that a stronger form of the Weak Mertens Conjecture is probably true. For this he made use of the following conjecture of Gonek [Gon89] and also Hejhal [Hej89].

**Gonek–Hejhal Conjecture.** We have

\[
\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^2 \ll T.
\]

(1.16)

Ng proved that assuming the Riemann Hypothesis, the simplicity of the zeros, and the Gonek–Hejhal Conjecture, which implicitly implies all zeros are simple, the Weak Mertens Conjecture can be strengthened to the asymptotic formula

\[
\int_1^T \left( \frac{M(x)}{x} \right)^2 \, dx \sim \sum \frac{2}{|\rho \zeta'(\rho)|^2} \log T.
\]

(1.17)

For more recent results on this topic, see [Ino19] and [SS19].

**Theorem 3.** Assume the Riemann Hypothesis and the simplicity of zeros, so that \( \rho = \frac{1}{2} + i\gamma \) is simple. If in addition the Gonek–Hejhal Conjecture is true, then for \( m \geq 2 \) and any \( \epsilon > 0 \) and \( x \) sufficiently large, we have

\[
E_m(x) = x^{m-\frac{3}{4}} \sum_{\gamma} \frac{2C_2 m! \zeta(\frac{\rho}{2} - 1) \zeta(\frac{\rho}{2}) G(\frac{\rho}{2} - 1)}{(2\frac{\rho}{2} + 1) \zeta'(\rho)(\frac{\rho}{2} - 1)(\frac{\rho}{2})(\frac{\rho}{2} + 1) \cdots (\frac{\rho}{2} + m - 1)} x^{\frac{\rho}{2}} + O(x^{m-1+\epsilon}),
\]

where the sum over zeros is absolutely convergent. Therefore

\[
E_m(x) \ll x^{m-3/4},
\]

(1.18)

and more precisely, letting \( c_m := \sum_{\gamma} |a(\rho)| \), where \( a(\rho) \) is as defined in (1.8), we have

\[
|E_m(x)| \leq (1 + o(1)) c_m x^{m-3/4}.
\]

(1.19)
For $m \geq 3$ the same results hold when we assume the Weak Mertens Conjecture.

Unconditionally we prove that the bounds in Theorem 3 are sharp up to constants.

**Theorem 4.** For $m \geq 1$, we have

\begin{equation}
E_m(x) = \Omega_\pm(x^{m-3/4}).
\end{equation}

Conditionally we can strengthen Theorem 4 and show that the bounds in (1.20) are sharp.

**Theorem 5.** If the Riemann Hypothesis is false, or if the zeros are not all simple, then for $m \geq 1$,

\begin{equation}
\limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} = \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} = -\infty.
\end{equation}

For the alternative situation, we assume the Riemann Hypothesis is true and all the zeros are simple. Now let $S$ be the set of distinct positive imaginary parts of the zeros of the Riemann zeta-function, and assume the hypothesis that every finite subset of $S$ is linearly independent over the integers. If $\sum_{\gamma} |a(\rho)|$ diverges then (1.22) holds. However if $m \geq 2$ and $c_m = \sum_{\gamma} |a(\rho)|$ converges, then

\begin{equation}
\limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} \geq c_m \quad \text{and} \quad \liminf_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} \leq -c_m.
\end{equation}

When $m = 1$, $\sum_{\gamma} |a(\rho)|$ diverges, and therefore writing $E_1(x) = E(x)$, we have, assuming the Linear Independence Hypothesis,

\begin{equation}
\limsup_{x \to \infty} \frac{E(x)}{x^{1/4}} = \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{E(x)}{x^{1/4}} = -\infty.
\end{equation}

By Theorem 3 the upper bound in (1.20) together with (1.23) implies that for $m \geq 2$,

\begin{equation}
\limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} = c_m \quad \text{and} \quad \liminf_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} = -c_m.
\end{equation}

This is under the assumption of the Riemann Hypothesis, simplicity of zeros of the Riemann zeta-function, the Gonek–Hejhal Conjecture, and the Linear Independence Hypothesis. This result can be proved more easily by directly applying Kronecker’s theorem in the explicit formula (1.18) of Theorem 3.

Theorem 4 was proved in [GS21] in the case $m = 1$ and the same proof with minor modifications works for all $m \geq 1$. Here we will prove Theorems 4 and 5 by making use of a theorem of Ingham [Ing42]. In proving Theorem 5 this allows us to avoid the Gonek–Hejhal Conjecture or the Weak Mertens Conjecture by providing a replacement for (1.18). The methods of
Mossinghoff and Trudgian [MT20a, MT20b, MT20c] may be used to obtain bounds on the constants in the $\Omega$-results.

The proof of (1.23) requires that the imaginary parts of the zeros are linearly independent over all the integers, but following [BB+71] and later authors, we can prove that the linear independence needed to obtain (1.24) can be limited to coefficients $-2, -1, 0, 1, 2$, with at most one coefficient of $2$ or $-2$. The proof uses Ingham’s theorem and a Riesz product that first occurred in a proof of Kronecker’s theorem by Bohr and Jessen [BJ32]; see also [Kat76, 9.3].

2. The inverse Mellin transform of $S_m(x)$. Define the singular series generating function

$$F(s) = \sum_{k=1}^{\infty} \frac{\mathcal{G}(k)}{k^s},$$

where as usual $s = \sigma + it$. This series converges for $\sigma > 1$. The first lemma from [GS21] provides the analytic continuation of $F(s)$ to $\sigma > -1$.

**Lemma 2.1.** We have, for $\sigma > -1$,

$$F(s) = \frac{4C_2}{2^{s+1} + 1} \frac{\zeta(s)\zeta(s+1)}{\zeta(2s+2)} G(s),$$

where $C_2$ and $G(s)$ are given in (1.2) and (1.9), respectively.

We now evaluate $S_m(x)$ as a contour integral. By the formula in Theorem B of Ingham [Ing32], if $m$ is a positive integer and $c, x > 0$, then

$$m! \frac{c+i\infty}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+m}}{s(s+1)(s+2)\cdots(s+m)} ds = \begin{cases} 0 & \text{if } 0 < x \leq 1, \\ (x-1)^m & \text{if } x \geq 1. \end{cases}$$

Hence, for $c > 1$,

$$S_m(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{F}(s) ds,$$

where

$$\mathcal{F}(s) := \frac{m! F(s)x^{s+m}}{s(s+1)(s+2)\cdots(s+m)} = \frac{4C_2 m! \zeta(s)\zeta(s+1)G(s)x^{s+m}}{(2^{s+1} + 1)\zeta(2s+2)s(s+1)(s+2)\cdots(s+m)}.$$

We see by Lemma 2.1 that for $\sigma > -1$ the singularities of $\mathcal{F}(s)$ consist of a simple pole at $s = 1$, a double pole at $s = 0$, and poles of order $m(\rho)$ at $s = \rho/2 - 1$, where $\rho$ denotes any complex zero of $\zeta(s)$. We need to compute the residues at these poles. Letting $R_\mathcal{F}(a) = \text{Res}(\mathcal{F}(s); s = a)$, we collect
our results in the following lemma. The case when \( m = 1 \) was already done in [GS21].

**Lemma 2.2.** The residues of the singularities of \( F(s) \) for \( \sigma > -1 \) are

\[
R_F(1) = \frac{x^{m+1}}{m+1},
\]

\[
R_F(0) = -\frac{1}{2} (\log x - H_m + c_E + \log 2\pi)x^m,
\]

where \( H_m = \sum_{n=1}^{m} \frac{1}{n} \) is the \( m \)th harmonic number,

\[
(2.6)
\]

\[
(2.7)
\]

and in general

\[
R_F\left(\frac{\rho}{2} - 1\right) = \frac{2C_2m!\zeta(\frac{\rho}{2} - 1)\zeta(\frac{\rho}{2})G(\frac{\rho}{2} - 1)}{(2^{\rho/2} + 1)\zeta'(\rho)(\frac{\rho}{2} - 1)(\frac{\rho}{2})(\frac{\rho}{2} + 1)\cdots(\frac{\rho}{2} + m - 1)}x^{\rho/2+m-1}, \quad \text{if} \ m_\rho = 1,
\]

\[
(2.8)
\]

for constants \( A_j(\rho, m) \).

**Proof.** For the simple pole of \( \zeta(s) \) at \( s = 1 \) with residue 1, we have

\[
R_F(1) = \frac{4C_2m!\zeta(2)G(1)x^{m+1}}{5\zeta(4)(m+1)!} = \frac{x^{m+1}}{m+1}
\]

since

\[
\frac{4\zeta(2)G(1)}{5\zeta(4)} = \prod_{p > 2}\left(1 - \frac{1}{p^2}\right)^{-1}\left(1 - \frac{1}{p^4}\right)\left(1 + \frac{2}{(p - 2)(p^2 + 1)}\right) = \prod_{p > 2}\frac{(p - 1)^2}{p(p - 2)} = \frac{1}{C_2}.
\]

Next, for the double pole at \( s = 0 \) of \( \zeta(s+1)/s \), we let

\[
U(s) := \frac{s}{\zeta(s+1)}F(s) = \frac{4C_2m!\zeta(s)G(s)x^{s+m}}{(2s+1+1)\zeta(2s+2)(s+1)(s+2)\cdots(s+m)}.
\]

Since \( U(s) \) is analytic at \( s = 0 \) and \( \zeta(s) = \frac{1}{s-1} + c_E + O(|s-1|) \) for \( s \) near 1 (see [Tit86], eq. (2.1.16)), we obtain the Laurent expansion around \( s = 0 \):

\[
F(s) = \frac{\zeta(s+1)}{s}U(s) = \left(\frac{1}{s^2} + \frac{c_E}{s} + O(1)\right)(U(0) + sU'(0) + O(|s|^2))
\]

\[
= \frac{U(0)}{s^2} + \frac{c_EU(0)}{s} + U'(0) + O(1),
\]
and therefore

\[ R_F(0) = \left( \frac{U'(0)}{U(0)} + c_E \right) U(0). \]

Here \( c_E \approx 0.577216 \) is the Euler–Mascheroni constant. We will make use below of the special values

\[ \frac{\zeta'(0)}{\zeta(0)} = \log 2\pi \quad \text{and} \quad \zeta(0) = -\frac{1}{2} \]

(see [Tit86 §2.4]), and also the harmonic numbers \( H_m \). First,

\[ U(0) = \frac{4C_2\zeta(0)G(0)x^m}{3\zeta(2)} = -\frac{1}{2}x^m \]

since

\[ \frac{4G(0)}{3\zeta(2)} = \prod_{p>2} \left( \frac{1}{1 - \frac{1}{p^2}} \right) \left( 1 + \frac{2}{(p-2)(p+1)} \right) = \prod_{p>2} \frac{(p-1)^2}{p(p-2)} = \frac{1}{C_2}. \]

Next, logarithmically differentiating \( U(s) \) and evaluating at \( s = 0 \), we obtain

\[ \frac{U'(0)}{U(0)} = \frac{\zeta'(0)}{\zeta(0)} + \frac{G'}{G}(0) + \log x - \frac{2}{3} \log 2 - 2\frac{\zeta'(2)}{\zeta(2)} - H_m \]

\[ = \log x + \log 2\pi - H_m, \]

since

\[ \frac{G'}{G}(0) - \frac{2}{3} \log 2 - 2\frac{\zeta'(2)}{\zeta(2)} \]

\[ = \sum_{p>2} \frac{-2p \log p}{(p-2)(p+1)(1 + \frac{2}{(p-2)(p+1)})} + \sum_{p>2} \frac{2 \log p}{p^2 - 1} = 0. \]

Therefore we conclude

\[ R_F(0) = -\frac{1}{2}(\log x - H_m + c_E + \log 2\pi)x^m. \]

Finally, we evaluate \( R_F(\rho/2 - 1) \) where the \( \rho \)'s are the complex zeros of \( \zeta(s) \). If \( \rho \) is a simple zero of \( \zeta(s) \), then \( \frac{1}{\zeta(2s+2)} \) will have a corresponding simple pole at \( s = \rho/2 - 1 \) with residue \( \frac{1}{2\zeta'(\rho)} \). Thus

\[ R_F\left( \frac{\rho}{2} - 1 \right) = \frac{2C_2m!\zeta\left( \frac{\rho}{2} - 1 \right)\zeta\left( \frac{\rho}{2} \right)G\left( \frac{\rho}{2} - 1 \right)}{(2^{\rho/2} + 1)\zeta'(\rho)\left( \frac{\rho}{2} - 1 \right)\left( \frac{\rho}{2} \right)\left( \frac{\rho}{2} + 1 \right) \cdots \left( \frac{\rho}{2} + m - 1 \right)}x^{\rho/2 + m - 1}. \]

In the general case of a zero \( \rho \) with multiplicity \( m_\rho = \ell \), we expand around \( s = \rho/2 - 1 \) and have

\[ x^{s+m} = x^{\rho/2 + m - 1}x^{s-\rho/2+1} = x^{\rho/2 + m - 1} \sum_{j=0}^{\infty} \frac{(s - \rho/2 + 1) \log x)^j}{j!}, \]
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while the remaining part of $\mathcal{F}(s)$ has Laurent expansion

$$
\frac{4C_2m!\zeta(s)\zeta(s+1)G(s)}{(2^{s+1}+1)\zeta(2s+2)s(s+1)(s+2)\cdots(s+m)} = \sum_{k=-\ell}^{\infty} a_k(\rho, m) \left(s - \frac{\rho}{2} + 1\right)^k
$$

which does not depend on $x$, and one obtains (2.9) on multiplying out and collecting the coefficients of $(s - \rho/2 + 1)^{-1}$, that is, in (2.9),

$$
A_j(\rho, m) := a_{-j-1}(\rho, m) \frac{(\log x)^j}{j!}.
$$

3. Proof of Theorem 1. We start with the inverse Mellin transform formula (2.4) for $S_m(x)$ and move the contour to the left. To justify this we need various bounds for $F(s)$.

For the Riemann zeta-function, we have the classical bound

$$
\zeta(s) - \frac{1}{s-1} \ll |t|^\mu(\sigma) + \epsilon, \quad s = \sigma + it,
$$

which holds for any $\epsilon > 0$, with

$$
\mu(\sigma) \leq \begin{cases} 
0 & \text{if } \sigma > 1, \\
(1 - \sigma)/2 & \text{if } 0 \leq \sigma \leq 1, \\
1/2 - \sigma & \text{if } \sigma < 0
\end{cases}
$$

(see [Tit86, §5.1]). Next, we need the zero-free region of $\zeta(s)$ (see [Tit86, Theorem 3.8]) given by

$$
\sigma \geq 1 - \frac{A}{\log(|t| + 3)}
$$

for some $A > 0$. In this region, the reciprocal of $\zeta(s)$ is analytic and satisfies the bound

$$
\frac{1}{\zeta(\sigma + it)} \ll \log(|t| + 3)
$$

(see [Tit86, eq. (3.11.8)]), and for values of $A$ and the constant in (3.4), see [Tru15]).

By the functional equation, for any fixed $B > 0$ we have

$$
|\zeta(s)| \asymp (|t| + 3)^{1/2 - \sigma} |\zeta(1 - s)|
$$

uniformly for $|\sigma| \leq B$ and $|t| \geq 1$ (see [MV07, Corollary 10.5]). We therefore conclude that $\zeta(s)$ has no zeros in the region

$$
-B \leq \sigma \leq \frac{A}{\log(|t| + 3)}, \quad |t| \geq 1,
$$

and in this region

$$
\frac{1}{\zeta(\sigma + it)} \ll (|t| + 3)^{\epsilon + \sigma - 1/2}.
$$
Finally, we will make use of a nice result of Ramachandra and Sankaranarayanan [RS91, Theorem 2] which allows us to avoid the need to assume the Riemann Hypothesis: for \(T\) sufficiently large and \(C > 0\) a constant, we have

\[
\min_{T \leq t \leq T + T^{1/3}} \max_{1/2 \leq \sigma \leq 2} |\zeta(\sigma + it)|^{-1} \leq \exp(C(\log \log T)^2).
\]

From this result and (3.5) we conclude that for any \(\epsilon > 0\) and \(T\) sufficiently large there exists a \(T' \), \(T \leq T' \leq T + T^{1/3}\), such that

\[
|\zeta(\sigma + iT')|^{-1} \ll (T')^{-\max(1/2-\sigma,0)+\epsilon} \ll (T')^{\epsilon}
\]

for all \(-B \leq \sigma \leq B\).

**Proof of Theorem 1.** We now assume \(m \geq 2\) is a fixed integer, and do not include the dependence on \(m\) in error estimates. Taking \(c = 2\) in (2.4), we have

\[
S_m(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \mathcal{F}(s) \, ds,
\]

where

\[
\mathcal{F}(s) = \frac{4C2m!\zeta(s)\zeta(s+1)G(s)x^{s+m}}{(2s+1+1)\zeta(2s+2)s(s+1)(s+2)\cdots(s+m)}.
\]

Since all the zeta-functions in \(\mathcal{F}(s)\) are bounded and have no zeros when \(s = 2 + it\), we have

\[
\mathcal{F}(2 + it) \ll \frac{x^{2+m}}{(|t| + 3)m+1},
\]

and therefore we can truncate the integral in (3.8) to the range \(|t| \leq T\) with an error \(\ll x^{2+m}/T^m\). We conclude that for \(m \geq 2\) and \(T \geq x^2\),

\[
S_m(x) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \mathcal{F}(s) \, ds + O(1).
\]

We now consider the rectangle with corners \(2 + iT^*\) and \(b \pm iT^*\), where \(b = -1 + a/\log T\) and \(T^*\) is chosen as in (3.7) so that \(T \leq T^* \leq 2T\), and \(a\) is chosen small enough so that all the poles of \(1/\zeta(2s+2)\) at \(s = \rho/2 - 1\) with \(|\gamma|/2 \leq T^*\) have \(\beta/2 > a/\log T^*\). Thus all the singularities of \(\mathcal{F}(s)\) for \(-1 < \sigma \leq 2\) and \(|t| \leq T^*\) are inside the rectangle and not on the boundary. On the horizontal contours \(s = \sigma \pm iT^*\) and for \(-1 \leq \sigma \leq 2\) we have

\[
\frac{1}{\zeta(2s+2)} \ll (T^*)^{\epsilon}.
\]

On the vertical contour \(s = b + it, \ |t| \leq T^*\), we have

\[
\frac{1}{\zeta(2s+2)} \ll (|t| + 3)^{2b+3/2+\epsilon}.
\]
We integrate around this contour counterclockwise and find by the residue theorem and Lemma 2.2 that

\[
S_m(x) = R_F(1) + R_F(0) + \sum_{|\gamma| \leq 2T^*} R_F(\rho/2 - 1) + \frac{1}{2\pi i} \left( \int_{2-iT^*}^{b-iT^*} + \int_{b+iT^*}^{b-iT^*} + \int_{2+iT^*}^{b+iT^*} \right) F(s) \, ds + O(1).
\]

We now bound the contribution from the contour integrals along those three sides of the rectangle. To do this we need a bound on \( G(s) \) provided by the following lemma.

**Lemma 3.1.** For any fixed \( \delta > 0 \) and \( -1 + \delta \leq \sigma \leq 2 \), we have

\[
G(s) \approx_\delta 1.
\]

Further, with \( b = -1 + \frac{a}{\log T} \), uniformly for \( b \leq \sigma \leq 2 \) and \( |t| \ll T \) we have

\[
G(s) \ll \exp \left( B \frac{\log T}{\log \log T} \right),
\]

where \( B \) is a constant. In particular \( G(s) \ll T^\epsilon \) for any \( \epsilon > 0 \) and for \( T \) sufficiently large.

We prove this lemma after completing the proof of Theorem 1.

Consider first the horizontal line segments \( \sigma \pm iT^* \) with \( b \leq \sigma \leq 2 \). Applying the estimates above we have

\[
F(\sigma \pm iT^*) \ll \frac{T^{\mu(\sigma)+\mu(\sigma+1)+4\epsilon}}{T_m+1} x^{\sigma+m} \ll \frac{T^{\mu(-1)+\mu(0)+4\epsilon}}{T_m+1} x^{m+2} \ll T^{1-m+4\epsilon} x^{m+2} \ll 1
\]

when \( T \geq x^5 \) since \( m \geq 2 \). Hence the contribution from these horizontal contours is \( \ll 1 \).

Next, on the vertical contour with \( \sigma = b + it, |t| \leq T^* \), we have

\[
F(b + it) \ll (|t| + 3)^{1/2-m} T^\epsilon x^{m+b},
\]

since for the range \( 1 \leq |t| \leq T^* \) we have

\[
F(b + it) \ll \frac{(|t| + 3)^{\mu(b)+\mu(b+1)+2b+3/2+3\epsilon}}{|t| + 3} T^{\epsilon} x^{m+b} \ll (|t| + 3)^{1/2-m+3\epsilon} T^\epsilon x^{m+b},
\]

while for \( |t| \leq 1 \) we have

\[
F(b + it) \ll \frac{x^{m+b}}{|b + 1 + it|} |G(b + it)| \ll \frac{x^{m+b}}{|b + 1 + it|} T^\epsilon \ll x^{m+b} T^\epsilon \log T \ll x^{m+b} T^{2\epsilon},
\]
where we applied Lemma 3.1 in the second inequality. Taking \( T = x^5 \) and integrating (3.14) over \(|t| \leq T^* \) shows that the contribution from this vertical contour is \( \ll x^{m-1+\epsilon} \).

**Proof of Lemma 3.1.** For \(-1 + \delta \leq \sigma \leq 2\) we have
\[
\prod_{p > 2} \left(1 - \frac{2}{p^{1+\delta}}\right) \ll |G(s)| \ll \prod_{p > 2} \left(1 + \frac{2}{p^{1+\delta}}\right),
\]
and (3.12) follows because the products here converge for any fixed \( \delta > 0 \). For \( b \leq \sigma \leq 2 \) we have
\[
|G(s)| \leq \prod_{p > 2} \left(1 + \frac{2 \log T}{a(p-2) \log p}\right).
\]
The terms in the product with \( p \leq Y := \frac{4}{a} \frac{\log T}{\log \log T} \) make a contribution
\[
\ll \prod_{p \leq Y} \frac{2}{a} \log T = \left(\frac{2}{a} \log T\right)^{\pi(Y)} \ll \exp\left(C_1 \frac{\log T}{\log \log T}\right).
\]
Using the inequality \( \log(1+x) \leq x \) for \( 0 \leq x < 1 \), we find that the terms in the product with \( p > Y \) make a contribution
\[
= \exp\left(\sum_{p > Y} \log \left(1 + \frac{2 \log T}{a(p-2) \log p}\right)\right) \leq \exp\left(\sum_{p > Y} \frac{2 \log T}{a(p-2) \log p}\right)
\ll \exp\left(C_2 \log T \left(\int_{Y}^{\infty} \frac{du}{u \log^2 u} + \frac{1}{\log^2 Y}\right)\right)
\ll \exp\left(C_2 \log T \frac{\log Y}{\log \log T}\right) \ll \exp\left(2C_2 \log T \frac{\log T}{\log \log T}\right).
\]
Multiplying these contributions proves (3.13) with \( B = C_1 + 2C_2 \).

4. **Proof of Theorem 2.** Starting with (3.10), we use the same rectangle as before but now take \( b = b_1 \), a fixed number with \(-3/4 < b_1 < -1/2\). By the Riemann Hypothesis, \( \rho/2 - 1 = -3/4 + i\gamma/2 \), and therefore none of the singularities of \( 1/\zeta(2s+2) \) are in this rectangle. Hence (3.11) becomes
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\[ S_m(x) = R_F(1) + R_F(0) + \frac{1}{2\pi i} \left( \int_{2-iT^*}^{b_1-iT^*} + \int_{b_1-iT^*}^{b_1+iT^*} + \int_{b_1+iT^*}^{2+iT^*} \right) F(s) \, ds + O(1). \]

As before, when \( T \geq x^5 \) and \( m \geq 2 \) the integrals along the horizontal contours are \( O(1) \). On the Riemann Hypothesis, we have

\[ \frac{1}{\zeta(\sigma + it)} \ll (|t| + 3)^\epsilon \quad \text{for every } \sigma > 1/2 \]

(see \[Tit86\] eq. (14.2.6))) and we may replace the bound for \( \mu(\sigma) \) in (3.2) by

\[ \mu(\sigma) \leq \max(1/2 - \sigma, 0) \]

(4.2)

Therefore on the vertical contour with \( \sigma = b_1 + it, |t| \leq T^* \), (3.14) becomes

\[ \mathcal{F}(b_1 + it) \ll (|t| + 3)^{-2b_1 - m - 1} T^\epsilon x^{m+b_1}. \]

On integrating and taking \( T = x^5 \) we find that the contribution from this vertical contour is \( \ll x^{m+b_1+\epsilon} \). Since \( b_1 \) can be taken as close to \(-3/4\) as we wish, this proves Theorem 2.

5. Proof of Theorem 3

We first prove the following lemma, partially contained in \[Ng04\] Lemma 1.

**Lemma 5.1 (Ng, 2004).** Under the Riemann Hypothesis, the simplicity of all zeros, and the Gonek–Hejhal Conjecture, the sums

\[ T_1(b) := \sum_\gamma \frac{1}{|\gamma| b |\zeta'(\rho)|} \quad \text{and} \quad T_2(b) := \sum_\gamma \frac{1}{|\gamma| b |\zeta'(\rho)|^2} \]

both converge for any fixed number \( b > 1 \). If we assume instead the Weak Mertens Conjecture, these sums converge for any \( b \geq 2 \).

**Proof.** By the well-known unconditional estimate

\[ U(T) := \sum_{0 < \gamma \leq T} \frac{1}{\gamma} \ll \log^2 T \]

(see \[Ing32\] Theorem 25b]), \( \sum_\gamma \frac{1}{|\gamma|^b} \) converges for \( b > 1 \) since

\[ \sum_{0 < \gamma \leq T} \frac{1}{\gamma^b} = \int_1^T \frac{1}{t^{b-1}} \, dU(t) = \frac{U(T)}{T^{b-1}} + (b - 1) \int_1^T \frac{U(t)}{t^b} \, dt \ll 1. \]

Hence by Cauchy’s inequality

\[ T_1(b) \leq \sqrt{T_2(b) \sum_\gamma \frac{1}{|\gamma|^b}}, \]
and therefore when $b > 1$ the convergence of $T_1(b)$ follows from the convergence of $T_2(b)$. By (1.15) the Weak Mertens Conjecture immediately shows $T_2(b)$ converges for $b \geq 2$. Denote

$$V(T) := \sum_{0<\gamma \leq T} \frac{1}{|\zeta'(\rho)|^2};$$

then for $b > 1$,

$$T_2(b) = \int_1^\infty \frac{1}{t^b} dV(t) = b \int_1^\infty \frac{V(t)}{t^{b+1}} dt \ll 1,$$

since by the Gonek–Hejhal Conjecture $V(t) \ll t$.

**Proof of Theorem 3.** By (3.12) and the Riemann Hypothesis bounds used to obtain (4.3) we have

$$a(\rho) \ll \frac{1}{|\gamma|^{m-1/2-\epsilon}|\zeta'(\rho)|}.$$

Here, assuming the simplicity of zeros, $a(\rho)$ is as defined in (1.8). Hence by Lemma 5.1 we see that $\sum \gamma a(\rho)$ is absolutely convergent for $m \geq 2$ on the Gonek–Hejhal Conjecture and for $m \geq 3$ on the Weak Mertens Conjecture. To prove (1.18), by (1.10) we find that there exists a number $U$ satisfying $x^5 \leq U \leq 2x^5$ for which, assuming the Riemann Hypothesis,

$$E_m(x) = x^{m-3/4} \sum_{|\gamma| \leq U} a(\rho)x^{i\gamma/2} + O(x^{m-1+\epsilon})$$

$$= x^{m-3/4} \sum_{\gamma} a(\rho)x^{i\gamma/2} - x^{m-3/4} \sum_{|\gamma| > U} a(\rho)x^{i\gamma/2} + O(x^{m-1+\epsilon}).$$

By Lemma 5.1

$$\sum_{|\gamma| > U} a(\rho)x^{i\gamma/2} \ll \sum_{|\gamma| > U} \frac{1}{|\gamma|^{m-1/2-\epsilon}|\zeta'(\rho)|} \leq \frac{1}{U^{1/5}} \sum_{|\gamma| > U} \frac{1}{|\gamma|^{m-3/4}|\zeta'(\rho)|}$$

$$\leq \frac{1}{U^{1/5}} \sum_{|\gamma|} \frac{1}{|\gamma|^{m-3/4}|\zeta'(\rho)|} \ll \frac{1}{U^{1/5}} \ll \frac{1}{x},$$

for $m \geq 2$ on the Gonek–Hejhal Conjecture and for $m \geq 3$ on the Weak Mertens Conjecture. Thus we obtain (1.18), and equations (1.19) and (1.20) are immediate consequences of (1.18).

**6. Proof of Theorems 4 and 5.** It is well-known for the Mertens problem that the existence of a zero off the half-line or of a zero with multiplicity $m_\rho \geq 2$ creates large oscillations in the error term. Thus if $\rho = \Theta + i\gamma$ is a zero of multiplicity $m_\rho \geq 1$, then

$$M(x) = \Omega_\pm(x^{\Theta} \log^{m_\rho-1} x)$$
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The same method implies for $E_m(x)$ that for $m \geq 1$,

$$E_m(x) = \Omega_\pm(x^{m+\Theta/2-1} \log^{m-1} m),$$

and therefore in particular if the Riemann Hypothesis is false or if there is a zero that is not simple then (1.22) follows. Thus we are left to consider the case where the Riemann Hypothesis is true and all the zeros are simple, which we henceforth assume.

Our proof is a simple application of a theorem of Ingham [Ing42]. Ingham stated his result in terms of Laplace transforms by making the change of variables $x = e^u$ in the Mellin transform; the form we give below is for the Mellin transform in the form given in [AS81] (see also [BD04, Theorem 11.12]). Note we are considering here Mellin transforms that have a sequence of simple poles going to infinity on the line $s = \sigma_0$ symmetrically above and below the real axis.

**Proposition (Ingham).** Let $A(x)$ be a bounded Riemann integrable function in any finite interval $1 \leq x \leq X$, and suppose that

$$\hat{A}(s) := \int_1^\infty \frac{A(x)}{x^{s+1}} \, dx$$

converges for $\sigma > \sigma_1$ and $\hat{A}(s)$ has an analytic continuation to $\sigma > \sigma_0$. Further, suppose we have a sequence of real numbers $0 < \gamma_1 < \gamma_2 < \cdots$ with $\gamma_n \to \infty$, and define $\gamma_0 = 0$ and $\gamma_{-n} = -\gamma_n$. Also define a corresponding sequence of complex numbers $r_n \neq 0$, $r_{-n} = \overline{r}_n$, and $r_0$ real and allowed to be zero. By (6.2) we obtain the Mellin transform pair, for any $T > 0$,

$$S_T(x) = x^{\sigma_0} \sum_{|\gamma_n| \leq T} r_n x^{i\gamma_n}, \quad \hat{S}_T(s) = \sum_{|\gamma_n| \leq T} \frac{r_n}{s - (\sigma_0 + i\gamma_n)}.$$

If $\hat{A}(s) - \hat{S}_T(s)$ can be analytically continued to the region $\sigma \geq \sigma_0$, $-T \leq t \leq T$, for some $T > 0$, then

$$\liminf_{x \to \infty} \frac{A(x)}{x^{\sigma_0}} \leq S^*_T(x_0) \leq \limsup_{x \to \infty} \frac{A(x)}{x^{\sigma_0}}$$

for any real number $x_0$, where

$$S^*_T(x) = \sum_{|\gamma_n| \leq T} \left(1 - \frac{|\gamma_n|}{T}\right) r_n x^{i\gamma_n} = r_0 + 2 \text{ Re} \sum_{0 < |\gamma_n| \leq T} \left(1 - \frac{|\gamma_n|}{T}\right) r_n x^{i\gamma_n}.$$

Notice that

$$x^{\sigma_0} S^*_T(x) = \frac{1}{T} \int_0^T S_t(x) \, dt$$

has smoothed out some of the variation in $S_T(x)$, but in applications where we take $T \to \infty$ there is no loss in using $S^*_T(x)$ in place of $S_T(x)$. In numerical
work however where $T$ is fixed, there is a better choice than $S_T^*(x)$; see [OTR85] and [BT15].

To apply the Proposition to $E_m(x)$, we note by integration by parts that for $\sigma > 1$,
\[
\int \frac{S_m(x)}{x^{s+m+1}} \, dx = \sum_{k=1}^{\infty} \mathcal{G}(k) \int \frac{(x-k)^m}{x^{s+m+1}} \, dx = \frac{m!}{s(s+1)(s+2)\cdots(s+m)} F(s),
\]
which may also be obtained from (2.4) and (2.5) with the Mellin inversion formula. On substituting (1.7) into this equation we obtain
\[
\int \frac{E_m(x)}{x^{s+m+1}} \, dx = \frac{m!}{s(s+1)(s+2)\cdots(s+m)} F(s) - \frac{1}{s+1} \frac{1}{s} \frac{1}{s^2} - \frac{1}{s} (H_m - \gamma - \log 2\pi).
\]
The case $m = 1$ was used in [GS21]. In the Proposition, we take
\[
A(x) = \frac{E_m(x)}{x^m},
\]
\[
\hat{A}(s) = \frac{m!}{s(s+1)(s+2)\cdots(s+m)} F(s) - \frac{1}{s+1} \frac{1}{s} \frac{1}{s^2} - \frac{1}{s} (H_m - \gamma - \log 2\pi),
\]
and see from Lemmas 2.1 and 2.2 that $\hat{A}(s)$ is analytic for $\sigma > -3/4$ and meromorphic for $\sigma > -1$ with simple poles at $s = \rho/2 - 1 = -3/4 + i\gamma/2$, where $\rho$ are the complex zeros of $\zeta(s)$, and the residues at these simple poles are $a(\rho)$. Thus in the Proposition we have $\sigma_0 = -3/4$, $\hat{\gamma}_n = \gamma_n/2$, $r_0 = 0$, and $r_n = a(\rho_n)$. Therefore we conclude that for any number $x_0$,
\[
\liminf_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} \leq 2 \Re \sum_{0 < \gamma_n \leq 2T} \left(1 - \frac{\gamma_n}{2T}\right) a(\rho_n)(x_0)^{i\gamma_n/2} \leq \limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}}.
\]

Proof of Theorem 4. Since $\gamma_1 = 14.134725 \ldots$ and $\gamma_2 = 21.022039 \ldots$, we choose $T = 10$ in (6.6), and obtain
\[
\liminf_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} \leq 2 \left(1 - \frac{\gamma_1}{20}\right) \Re (a(\rho_1)(x_0)^{i\gamma_1/2}) \leq \limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}}.
\]
We can clearly find values for $x_0$ so that
\[
a(\rho_1)(x_0)^{i\gamma_1/2} = |a(\rho_1)|e^{i(\arg a(\rho_1)+\gamma_1/2)\log x_0} = \pm |a(\rho_1)|,
\]
and Theorem 4 follows. ■
Proof of Theorem 5. We will only prove the lim sup parts of the theorem since the lim inf is handled the same way. Pick an $\epsilon > 0$. If the imaginary parts of the zeros are linearly independent over the integers, by Kronecker’s theorem we can find values of $x_0 = x_0(T)$ such that

\begin{equation}
(6.7) \quad \limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} \geq 2 \text{Re} \sum_{0 < \gamma_n \leq 2T} \left(1 - \frac{\gamma_n}{2T}\right) a(\rho_n)(x_0)^i\gamma_n/2
\end{equation}

> \begin{align*}
2(1 - \epsilon) \sum_{0 < \gamma_n \leq 2T} \left(1 - \frac{\gamma_n}{2T}\right) |a(\rho_n)|.
\end{align*}

Case 1. Suppose $\sum_{\gamma} |a(\rho)|$ diverges. Then (1.22) holds since

\begin{align*}
\limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} & \gg \sum_{0 < \gamma_n \leq T} |a(\rho_n)| \to \infty \quad \text{as } T \to \infty.
\end{align*}

Case 2. Suppose $c_m = \sum_{\gamma} |a(\rho)|$ converges. With the $\epsilon$ in (6.7), on taking $T$ sufficiently large we have

\begin{align*}
\limsup_{x \to \infty} \frac{E_m(x)}{x^{m-3/4}} & > 2(1 - \epsilon) \sum_{0 < \gamma_n \leq 2\epsilon T} \left(1 - \frac{\gamma_n}{2T}\right) |a(\rho_n)| \bigg| \\
& > 2(1 - \epsilon)^2 \sum_{0 < \gamma_n \leq 2\epsilon T} |a(\rho_n)| \\
& > (1 - \epsilon)^3 c_m;
\end{align*}

we conclude that $\limsup_{x \to \infty} E_m(x)/x^{m-3/4} \geq c_m$, and (1.23) follows.

It remains to prove that $\sum_{\gamma} |a(\rho)|$ diverges when $m = 1$. By (3.5) and (3.12) we have

\begin{align*}
|a(\rho)| \asymp \left|\frac{\zeta(\rho/2 - 1)\zeta(\rho/2)}{|\rho^2\zeta'(\rho)|}\right| \asymp \left|\frac{\zeta(2 - \rho/2)\zeta(1 - \rho/2)}{|\rho^{1/2}\zeta'(\rho)|}\right| \\
& \gg \frac{1}{|\rho^{1/2+\epsilon}\zeta'(\rho)|},
\end{align*}

since assuming the Riemann Hypothesis we have $|1/\zeta(s)| = O((|t| + 3)^\epsilon)$ for $\sigma > 1/2$ (see [Tit86 eq. (14.2.6)]). By [MV07 Theorem 15.6], the Riemann Hypothesis implies that

\begin{align*}
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|} \gg T,
\end{align*}

and the divergence follows from

\begin{align*}
\sum_{0 < \gamma \leq T} |a(\rho)| \gg \sum_{0 < \gamma \leq T} \frac{1}{|\gamma^{1/2+\epsilon}\zeta'(\rho)|} \gg \frac{1}{T^{1/2+\epsilon}} \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|} \gg T^{1/2-\epsilon}.
\end{align*}
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References

[AS81] R. J. Anderson and H. M. Stark, Oscillation theorems, in: Analytic Number Theory (Philadelphia, PA, 1980), Lecture Notes in Math. 899, Springer, Berlin, 1981, 79–106.

[BB+71] P. T. Bateman, J. W. Brown, R. S. Hall, K. E. Kloss, and R. M. Stemmler, Linear relations connecting the imaginary parts of the zeros of the zeta function, in: Computers in Number Theory (Oxford, 1969), Academic Press, London, 1971, 11–19.

[BD04] P. T. Bateman and H. G. Diamond, Analytic Number Theory. An Introductory Course, World Sci., Singapore, 2004.

[BT15] D. G. Best and T. S. Trudgian, Linear relations of zeroes of the zeta-function, Math. Comp. 84 (2015), no. 294, 2047–2058.

[BJ32] H. Bohr and B. Jessen, One more proof of Kronecker’s theorem, J. London Math. Soc. 7 (1932), 274–275.

[FG95] J. B. Friedlander and D. A. Goldston, Some singular series averages and the distribution of Goldbach numbers in short intervals, Illinois J. Math. 39 (1995), 158–180.

[GL17] W. Ge and H. Liu, On the singular series for primes in arithmetic progressions, Lith. Math. J. 57 (2017), 294–318.

[GM87] D. A. Goldston and H. L. Montgomery, Pair correlation of zeros and primes in short intervals, in: Analytic Number Theory and Diophantine Problems (Stillwater, OK, 1984), A. C. Adolphson et al. (eds.), Birkhäuser, Boston, 1987, 183–203.

[GS21] D. A. Goldston and A. I. Suriajaya, The error term in the Cesàro mean of the prime pair singular series, J. Number Theory 227 (2021), 144–157.

[GV97] D. A. Goldston and R. C. Vaughan, On the Montgomery–Hooley asymptotic formula, in: Sieve Methods, Exponential Sums, and Their Application in Number Theory (Cardiff, 1995), G. R. H. Greaves et al. (eds.), Cambridge Univ. Press, Cambridge, 1997, 117–142.

[GZHN17] D. A. Goldston, J. Ziegler Hunts, and T. Ngotiaoco, The tail of the singular series for the prime pair and Goldbach problems, Funct. Approx. Comment. Math. 56 (2017), 117–141.

[Gon89] S. M. Gonek, On negative moments of the Riemann zeta-function, Mathematika 36 (1989), 71–88.

[HL19] G. H. Hardy and J. E. Littlewood, Note on Messrs. Shah and Wilson’s paper entitled: On an empirical formula connected with Goldbach’s Theorem, Proc. Cambridge Philos. Soc. 19 (1919), 245–254; reprinted in: Collected Papers of G. H. Hardy, Vol. I, Clarendon Press, Oxford, 1966, 535–544.

[HL22] G. H. Hardy and J. E. Littlewood, Some problems of ‘Partitio numerorum’: III: On the expression of a number as a sum of primes, Acta Math. 44 (1922), 1–70; reprinted in: Collected Papers of G. H. Hardy, Vol. I, Clarendon Press, Oxford, 1966, 561–630.
A singular series average

[Hej89] D. A. Hejhal, *On the distribution of* $\log |\zeta'(1/2 + it)|$, in: Number Theory, Trace Formulas and Discrete Groups, Sympos. in Honor of Atle Selberg (Oslo, 1987), K. E. Aubert et al. (eds.), Academic Press, Boston, 1989, 343–370.

[Hur18] G. Hurst, *Computations of the Mertens function and improved bounds on the Mertens conjecture*, Math. Comp. 87 (2018), no. 310, 1013–1028.

[II17] T. Inaba and S. Inoue, *Riesz means of the Dedekind function II*, arXiv:1705.05594 (2017).

[Ing32] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge Tracts in Math. Math. Phys. 30, Cambridge Univ. Press, Cambridge, 1932.

[Ing42] A. E. Ingham, *On two conjectures in the theory of numbers*, Amer. J. Math. 64 (1942), 313–319.

[Ino19] S. Inoue, *Relations among some conjectures on the Möbius function and the Riemann zeta-function*, Acta Arith. 191 (2019), 1–32.

[IK17] S. Inoue and I. Kiuchi, *Riesz means of the Dedekind function*, Kyushu J. Math. 71 (2017), 105–114.

[IK19] S. Inoue and I. Kiuchi, *Riesz means of the Euler totient function*, Funct. Approx. Comment. Math. 60 (2019), 31–40.

[Kat76] Y. Katznelson, *An Introduction to Harmonic Analysis*, 2nd corrected ed., Dover Publ., New York, 1976.

[MS02] H. L. Montgomery and K. Soundararajan, *Beyond pair correlation*, in: Paul Erdős and His Mathematics, I (Budapest, 1999), Bolyai Soc. Math. Stud. 11, János Bolyai Math. Soc., Budapest, 2002, 507–514.

[MV07] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory*, Cambridge Stud. Adv. Math. 97, Cambridge Univ. Press, Cambridge, 2007.

[MT20a] M. J. Mossinghoff and T. S. Trudgian, *A tale of two omegas*, in: 75 Years of Mathematics of Computation, Contemp. Math. 754, Amer. Math. Soc., Providence, RI, 2020, 343–364.

[MT20b] M. J. Mossinghoff and T. S. Trudgian, *The size of oscillations in the Goldbach conjecture*, arXiv:2006.14742 (2020).

[MT20c] M. J. Mossinghoff and T. S. Trudgian, *Oscillations in weighted arithmetic sums*, arXiv:2007.14537 (2020).

[Ng04] N. Ng, *The distribution of the summatory function of the Möbius function*, Proc. London Math. Soc. (3) 89 (2004), 361–389.

[OtR85] A. M. Odlyzko and H. J. J. te Riele, *Disproof of the Mertens conjecture*, J. Reine Angew. Math. 357 (1985), 138–160.

[RS91] K. Ramachandra and A. Sankaranarayanan, *Notes on the Riemann zeta-function*, J. Indian Math. Soc. 57 (1991), 67–77.

[Ran40] R. A. Rankin, *The difference between consecutive prime numbers, II*, Proc. Cambridge Philos. Soc. 36 (1940), 255–266.

[SS19] B. Saha and A. Sankaranarayanan, *On estimates of the Mertens function*, Int. J. Number Theory 15 (2019), 327–337.

[SS13] A. Sankaranarayanan and S. K. Singh, *On the Riesz means of* $\frac{n}{\varphi(n)}$, Hardy–Ramanujan J. 36 (2013), 8–20.

[SS14] A. Sankaranarayanan and S. K. Singh, *On the Riesz means of* $\frac{n}{\varphi(n)}$ — II, Arch. Math. (Basel) 103 (2014), 329–343.

[Tit86] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Clarendon Press, Oxford, 1986.

[Tru15] T. Trudgian, *Explicit bounds on the logarithmic derivative and the reciprocal of the Riemann zeta-function*, Funct. Approx. Comment. Math. 52 (2015), 253–261.
R. C. Vaughan, *On a variance associated with the distribution of primes in arithmetic progressions*, Proc. London Math. Soc. (3) 82 (2001), 533–553.

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