DISTRIBUTIONS OF THE DIFFUSION COEFFICIENT FOR THE
QUANTUM AND CLASSICAL DIFFUSION IN DISORDERED MEDIA

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Abstract. It is shown that the distribution functions of the diffusion coefficient are
very similar in the standard model of quantum diffusion in a disordered metal and in
a model of classical diffusion in a disordered medium: in both cases the distribution
functions have lognormal tails, their part increasing with the increase of the disor-
der. The similarity is based on a similar behaviour of the high-gradient operators
determining the high-order cumulants. The one-loop renormalization-group correc-
tions make the anomalous dimension of the operator that governs the $s$-th cumulant
proportional to $s(s - 1)$ thus overtaking for large $s$ the negative normal dimension.
As behaviour of the ensemble-averaged diffusion coefficient is quite different in these
models, it suggests that a possible universality in the distribution functions is inde-
pendent of the behaviour of average quantities.

1 Introduction

The absence of self-averaging in mesoscopic systems (see recent collections of review pa-
pers [1, 2]) put forward a question about probability distributions of different physical
quantities. The conventional approach to systems with quenched disorder was based on
describing measurements in a single sample by the ensemble-averaged quantities, i.e. the
quantities averaged over all realizations of the quenched disorder. Discovery of the meso-
scopic fluctuations (first of all, of the “universal” conductance fluctuations) manifested that
this approach failed for the mesoscopic systems, at a length scale quite large as compared
to a microscopic one. Instead, a statistical approach based on evaluating the distribution
function of any physical quantity should be implemented. It could be reduced to evaluating
the average value and the variance only, if a distribution is normal. However, a question
about the shape of the distribution could not be solved a priori.

For the problem of quantum diffusion, the mesoscopic scale is defined by either the ther-
mal length or the phase-breaking length due to inelastic scatterings, both diverging when
temperature goes to zero. Then, at $T = 0$ the statistical approach is required for describ-
ing any physical quantity at any scale. Such an approach, based on the renormalization-
group (RG) analysis in the framework of the field-theoretical ($\sigma$ model) description of the

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quantum-diffusion problem [3, 4], has been constructed for different physical quantities at the metallic side of the Anderson metal-insulator transition, and extended by means of the $2 + \epsilon$ expansion (near the critical dimensionality $d = 2$) for describing these quantities in the vicinity of the transition (see ref. [3] for reviews). A complete description of the transition itself should be in terms of changing the distribution of conductance rather than of a critical change of the average conductance only, as all irreducible moments (i.e. cumulants) of the conductance are of the same order in the region of the transition so that the conductance distribution is completely non-Gaussian. However, in this region the $2 + \epsilon$ expansion provides quite a poor (at $d = 3$) description even for the average conductance, as the four-loop calculations by Wegner [6] proved. In view of this, is it possible that the statistical approach based on the $2 + \epsilon$ expansion could provide at least a qualitative description of changing the distribution function in approaching the transition?

I will argue here that such a possibility does exist. A main qualitative prediction of the statistical approach is the appearance of the lognormal tails in the distribution functions of different physical quantities, their part increasing with the increase of the disorder. For some local quantities the distributions become completely lognormal still on the metal side of the transition [3]. It provides a basis for the conjecture that statistically the metal-insulator transition reveals itself as a crossover from the normal to the lognormal distribution. Such a conjecture seems to be confirmed by recent numerical simulations [7, 8] and by the analytical continuation of the one-loop RG results for different distribution functions from $d = 2 + \epsilon$ to $d = 1$ that reproduces, after substituting a 1-\(d\) value for the mean conductance, exact 1-\(d\) results for these functions. It appears, therefore, that although the $2 + \epsilon$ expansion could not provide the reliable values for critical exponents, it could describe relations between the cumulants of the same physical quantities. The reason is that these relations could be independent of those particular properties of the model which determine behaviour of the ensemble-averaged quantities (e.g. the conductance). An important property of the nonlinear $\sigma$ model is the asymptotic freedom. The coupling constant (inversely proportional to the conductance) increases making the perturbative approach inapplicable in the region of the transition. The behaviour of the cumulants, however, is determined essentially by some additional operators. For each cumulant, the whole set of the additional operators is generated under the RG transformations. The eigenvalues of the RG equations that determine the critical exponents for the cumulants are governed by the operations of the group of permutations [9] acting in this set. These operations seem independent of the renormalization of the coupling constant.
To provide arguments in favour of this viewpoint, I will consider a model of classical diffusion in a medium with the quenched disorder. The coupling constant of this model is not renormalized to all orders of the loop expansion. It does not make the model to be trivial, though, as the long-time diffusion is anomalous and characterized by the mean-square displacement that increases with time $t$ slower than $t$. Although the behaviour of the coupling constant has nothing to do with the asymptotic freedom characteristic of the quantum-diffusion problem, the set of the eigenvalues of the RG equations governing the cumulants of the diffusion coefficients turns out to coincide with that for the quantum-diffusion problem. In both cases, the additional operators that govern the cumulants are irrelevant within the naive scaling but become relevant allowing for the one-loop RG corrections which give rise to nontrivial deviations of the distribution functions from Gaussian. It suggests that such a behaviour could be “universal” for very different models which by no means belong to the same universality class. As the classical-diffusion model is much simpler than the quantum one, a further analysis would be possible which is quite cumbersome in the quantum case. Due to this similarity, it could shed light on some properties of the Anderson transition.

Before considering the classical-diffusion model, I will describe the principal results on the conductance distribution in a disordered metal.

2 Conductance distribution in the quantum diffusion problem

For the sake of comparison with the classical diffusion problem, I will outline a way of calculating the moments of conductance distribution using the field-theoretical technique. A detailed description can be found in ref. [5].

The ensemble-averaged conductance $g$ can be represented directly in terms of the generalized nonlinear $\sigma$ model as follows:

$$\langle g_{\alpha,\beta} \rangle = \frac{\pi}{8N^2L^2} \text{Tr} \left\{ \frac{\partial^2 Z[A]}{\partial A_\alpha \partial A_\beta} \right\}_{A=0, N=0}$$

(1)

where $L$ is size of the system, $\alpha$ and $\beta$ are the vector indices in the $d$-dimensional space, the brackets $\langle \ldots \rangle$ stand for the averaging over all realizations of the disorder, and $g$ is chosen to be dimensionless quantity measured in the units of $e^2/2\pi^2\hbar$. The generating functional...
is given by

\[ Z[A] = \frac{\int \exp\{-F[Q; A]\} \mathcal{D}Q}{\int \exp\{-F[Q]\} \mathcal{D}Q} \quad (2) \]

with \( F[Q] \equiv F[Q; A=0] \) being the functional of the standard nonlinear \( \sigma \) model\[3, 4\]

\[ F[Q] \equiv \frac{1}{\tilde{t}} \int d^d r \text{Tr} \left( \frac{\partial Q}{\partial Q} \right)^2 . \quad (3) \]

Here \( \partial \) stands for a usual gradient, and the coupling constant \( \tilde{t} \equiv (\pi \nu D/8)^{-1} \) is related to \( \tilde{g} \equiv \langle g_{xx} \rangle \) as \( \tilde{t} = 16\pi L^{d-2}/\tilde{g} \) where \( D = v_e l/d \) is the diffusion coefficient, \( \nu \) is the one-electron density of states, and \( l \) is the length of the mean free path. The \( 2N \times 2N \) matrix \( Q \) obeys the constraints

\[ Q^2 = I, \quad \text{Tr} \ Q = 0 \quad (4) \]

and the integration in Eq.(2) is carried out over independent elements of \( Q \). In deriving the above field-theoretical representation from the initial model of the non-interacting electrons in the random potential, the replica trick has been used for the ensemble averaging so that the limit \( N = O \) should be taken on calculating the conductance (1). Alternatively, the supersymmetric formulation \[10, 11\] could be used for calculating both the average value of conductance and its higher moments.

The functional \( F[Q; A] \) in Eq.(2) is obtained from \( F[Q] \) by substituting the gradient \( \partial Q \) by the covariant derivative \( \nabla Q \):

\[ \partial Q \rightarrow \nabla Q \equiv \partial_\alpha Q - [A_\alpha, Q] , \quad (5) \]

where the source field \( A \) is introduced to provide the direct representation of the conductance in Eq.(1).

A generalization of Eq.(2) for calculating the \( s \)-th cumulant of the conductance is straightforward \[4\]:

\[ \left\langle \prod_{i=1}^s g_{\alpha_i \beta_i} \right\rangle = \left( \frac{\pi}{8N^2 L^2} \right)^s \left\{ \prod_{i=1}^s \text{Tr} \left( \frac{\partial^2}{\partial A_{\alpha_i} \partial A_{\beta_i}} \right) \right\} Z[A] \bigg|_{A=0 \atop N=0} . \quad (6) \]

However, a further extension of the effective field-theoretical model was required in Eq.(6) to obtain all relevant contribution to the cumulants. So the generating functional \( Z[A] \) of Eq.(2) has been replaced by \( Z[A] \) obtained by substituting \( F[Q, A] \) for \( F[Q, A] \) into Eq.(2) where the functional \( F[Q, A] \) of the extended nonlinear \( \sigma \) model reads

\[ F[Q, A] = \frac{\partial_\alpha Q}{\partial_\alpha Q} + \sum_{s=2}^\infty F_s[Q, A] \]
\[ F_s[Q, A] = z_s \delta_{\alpha_1, \ldots, \alpha_{2s}} \int d^d r \operatorname{Tr} \left( \prod_{i=1}^{2n} \nabla_{\alpha_i} Q \right). \] (7)

The bare values of the charges \( z_s \) are inversely proportional to the bare value of the coupling constant \( t \). The tensor \( \delta_{\alpha_1, \ldots, \alpha_{2s}} \) in this expression arises from angular integration of the product \( n_{\alpha_1} \ldots n_{\alpha_{2s}} (n \equiv p/p_F) \) so that it is proportional to the sum of all the possible products of the Kronecker symbols.

The functional (7) alongside with the functional of the standard \( \sigma \) model (3) has been derived microscopically from the initial model of free electrons in the random potential \( V(r) \) chosen to be Gaussian with zero average and the short-range correlator

\[ \langle V(r)V(r') \rangle = \frac{1}{2\pi
\nu \tau^2} \delta(r - r') \]

where \( \tau = l/v_F \) is the mean time between elastic scatterings.

The conductance cumulants (5) have been calculated with the renormalized functional Eqs.(3), (7). It is important that the latter functional is not closed under the RG procedure, and may be renormalizable only in a larger space which includes for any integer \( s \geq 2 \) (where \( 2s \) is the total number of \( \nabla Q \) all possible vertices of the following type:

\[ F^s[Q; A] = z^s \int d^d r \left[ \operatorname{Tr}(\nabla_{\alpha_1} Q \nabla_{\alpha_{2s}} Q) \right]^{s_1} \ldots \left[ \operatorname{Tr}(\nabla_{\beta_1} Q \ldots \nabla_{\beta_{2m}} Q) \right]^{s_m} \ldots \] (8)

Here a set of integers \( s \equiv (s_1, \ldots, s_m, \ldots) \) obeys the constraint

\[ \sum_{m \geq 1} ms_m = s, \] (9)

and vertices with all possible partitions into the product of traces and all possible transpositions of the vector indices \( \alpha_i \) and \( \beta_i \) (each of them being repeated twice) are included.

The notation (8) is somewhat symbolic as it means that there are \( s_m \) traces of the length \( 2m \) while their vector-indices structures are different.

The bare values of charges attached to all the additional vertices are equal zero. However, they are not less “physical” then the original ones derived microscopically, Eq.(6). Thus, the vertices which made direct contribution to the cumulants (5) are those with the set \( s_1 = s \) (and \( s_m = 0 \) for \( m \neq 1 \)) rather than the original ones (with the set \( s_s = 1 \) and \( s_m = 0 \) for \( m \neq s \)).

All the high-gradient operators are naïvely irrelevant as the couplings \( z^s \rightarrow \lambda^{d-2s} z^s \) after the scaling \( \lambda L \rightarrow L \). However, the one-loop RG analysis which reduced, after deriving the
RG equations for the whole set of the couplings \( z^s \), to the analysis in terms of the group of permutations\(^3\),\(^4\) gives the dimension \( \alpha_s \) of the couplings as follows

\[
\alpha_s = d - 2s + \epsilon s(s - 1)
\]  

where only the largest eigenvalue \( s(s - 1) \) of the RG transformations has been taken into account. At \( d = 2 \) one substitutes \( \epsilon \) by \( \mathbf{g}^{-1} \) in this equation.

There are two different types of contributions to the renormalized values of the conductance cumulants. The first one is obtained by the repeated differentiation in Eq.(8) of the generating functional \( Z \), which includes only the standard functional of the nonlinear \( \sigma \) model (3) with the substitution (5). The contributions of the second type are obtained with allowance for the high gradient vertices Eqs.(7) and (8). On keeping the most important contribution of the second type (which is made for the \( s \)-th cumulant by the vertices with \( 2s \) gradient operators), one finds \(^5\) omitting the vector indices that

\[
\langle g^s \rangle \sim \begin{cases} 
\mathbf{g}^{1-s}, & s \ll n_0 \\
g_0 (l/L)^{2s-d} \exp[u(s^2 - s)], & s \gg n_0 
\end{cases} 
\]  

Here \( g_0 \gg 1 \) is the bare value of the conductance,

\[
u = \ln \frac{\sigma_0}{\sigma},
\]

\( \sigma_0 \) and \( \sigma \) are the bare and renormalized value of the conductivity (not conductance). The number \( n_0 \) that separates “high” and “low” moments turns out to be quite large in the region of validity of the RG approach, \( n_0 \sim u^{-1} \ln(L/l) \), i.e. \( n_0 \sim g_0 \) in the region of the weak localization where \( \mathbf{g} = g_0 - \ln(L/l) \approx g_0 \gg 1 \), and \( n_0 \sim g_0 / \ln(g_0) \) on approaching the strong localization region where \( \mathbf{g} \) drops down to 1.

The \( \exp(s^2) \)-dependence of the moments is a generic feature of the lognormal distribution as one can see from the following identity

\[
u^s \exp(\nu s^2) = \int_0^\infty w^s \exp \left( -\frac{1}{4u} \ln^2 \frac{w}{v} \right) \frac{dw}{w}.
\]  

As only the high moments of the distribution obey this dependence, only its tails are lognormal. In the weak-localization region, the low cumulants (11a) are small as compared to variance (which is of order 1), and the bulk of the distribution is normal. On approaching the strong disorder region, the distribution becomes wide and non-Gaussian and the part of the lognormal tails increases. However, the small parameter \( (l/L)^{2s} \) in Eq.(11b) which is
due to the normal part of the dimension \(|10|\) of the couplings \(z^s\), prevents the tails becoming more essential at the metallic side of the transition. In calculating the cumulants of some other quantities, like a local density of states (DoS), the negative normal dimension of the high-gradient operators is exactly compensated by the positive normal dimension of the operators coupled to the cumulants of DoS so that the behaviour of the cumulants is governed completely by the anomalous dimension. In this case, the distribution function becomes completely lognormal when \(u \sim 1\) which is still in the metal regime. [5]

The distribution of the global DoS is similar to that of the conductance.[5] Thus, due to the Einstein relation, the distribution of the diffusion coefficient (which, for the quantum diffusion problem, has not been directly calculated) has similar features. I will show below that the same behaviour of the distribution of the diffusion coefficient is characteristic of some model for classical diffusion in disordered media.

### 3 Classical diffusion in media with the quenched disorder

Diffusion of classical particles in random media has attracted considerable attention (see ref.[12] for a recent review). A set of models [13]–[17] is given by the Langevin equation that describes random walks of a particle under the influence of a thermal noise \(\eta(t)\) in the medium with the quenched disorder governed by a random-drift field \(v(r)\):

\[
\dot{r} = v(r) + \eta(t)
\]

The thermal noise \(\eta\) is chosen to be Gaussian white noise with a zero average and

\[
\overline{\eta_\alpha(t) \eta_\beta(t')} = 2D \delta_{\alpha\beta} \delta(t - t'),
\]

where \(D\) is the diffusion coefficient.

Equivalently, averaging over the random noise one describes these random walks in terms of the Fokker-Planck (FP) equation

\[
\left[ \frac{\partial}{\partial t} + \partial_\alpha (v_\alpha - D \partial_\alpha) \right] \rho(r, t) = 0
\]

with \(\rho(r, t)\) being the local density of the diffusing particles.

The exact solution obtained by Sinai [13] to the model Eqs. (14)–(16) in one dimension (with the Gaussian short-range correlator for the random-drift fields \(v\)) has revealed the
anomalous diffusion manifested by the strongly sub-diffusive long-time behaviour of the random walks. Later such a diffusion has been considered as a possible universal source for the excess \((1/f)\) current noise \([18]\).

However, it appeared that anomalous diffusion occurs only for \(d \leq 2\), at least in the weak-disorder case, as \(d = 2\) turned out to be the upper critical dimensionality, and the long-time behaviour of the random walks could be super-diffusive as well as sub-diffusive, depending on some features of the model. The point is that a vector character of the quenched disorder in \(d > 1\) gives more freedom in choosing correlator of the fields \(v\). In general, assuming the Gaussian nature of the random-drift fields, \(i.e.\) irrelevancy of the higher-order correlations (the assumption that could be proved in a subsequent RG analysis), and choosing \(\langle v \rangle = 0\), one may write has

\[
\langle v_\alpha(r)v_\beta(r') \rangle = \gamma_0 F_{\alpha\beta}(r - r') \tag{17}
\]

where \(\gamma_0\) characterizes the strength of the disorder.

The simplest possible choice of the correlator \([14]\)

\[
F_{\alpha\beta} = \delta_{\alpha\beta}\delta(r - r') \tag{18}
\]

leads to the model where corrections to the effective diffusion coefficient \(D(t)\) vanish in the long-time limit. Two other specific models for \(F_{\alpha\beta}\) arise naturally \([15] - [17]\) on imposing constraints on the random field \(v\) choosing it either potential (\(v_\alpha = \partial_\alpha \Phi\)) or solenoidal (\(\partial_\alpha v_\alpha(r) = 0\)). For these models the Fourier transform of the correlator \([17]\) reads

\[
F_{\alpha\beta} = \begin{cases} 
\frac{q_\alpha q_\beta}{q^2}, & \text{potential field} \\
\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}, & \text{solenoidal field}
\end{cases} \tag{19}
\]

The RG analysis has shown \([13] - [15]\) that at \(d = 2\) the potential disorder leads to the sub-diffusive behaviour in the long-time limit while the solenoidal disorder leads to the super-diffusive one:

\[
D(t) \equiv \frac{\partial}{\partial t} \langle r^2(t) \rangle \propto \begin{cases} 
D_0(t/\tau)^{-g_0/2}, & \text{potential field} \\
D_0 \ln^{1/2}(t/\tau), & \text{solenoidal field}
\end{cases} \tag{20}
\]

Here

\[
g_0 = \frac{\gamma_0}{4\pi D_0^2} \tag{21}
\]
is the bare value of the coupling constant $g$ (dimensionless at $d = 2$) which plays a role of the effective weak-disorder parameter, $D_0$ is the bare value of the diffusion coefficient (describing diffusion at $t \sim \tau$), and $\tau$ is some small time which provides the ultraviolet cutoff necessary to regularize the model.

Note that the results of Eq.(20) are asymptotically exact in the weak-disorder limit. In the case of the solenoidal disorder, the coupling $g$ is decreased by the RG transformation from its bare value $g_0 \ll 1$ (the “zero-charge” situation) which makes the strong-disorder region unattainable. In the case of the potential disorder, there is no renormalization of the coupling constant $g$ at all. It has been firstly demonstrated up to the two-loop order of the RG analysis [17] but has later been proved [19] to be perturbatively exact: the renormalization is absent in all the orders of the loop expansion. It does not make the model trivial as both the strength of the disorder $\gamma$ and the diffusion coefficient $D$ (as well as other physical quantities, such as mobility) are renormalized which leads, in particular, to the sub-diffusion, Eq.(20a).

The quenched disorder may enter the FP equation (16) also via the spatial variation of the diffusion coefficient $D(r) \equiv \mathcal{D} + 2\delta D(r)$ where $\mathcal{D}$ is the uniform part of $D(r)$. One can take the limit $v = 0$ and define the disorder in terms of the distribution $P(D)$ of the spatially uncorrelated diffusion coefficients. The distribution is governed by its cumulants

\[
\begin{align*}
\langle \delta D(r_1) \delta D(r_2) \rangle &= 2! g^{(2)}(r_1 - r_2), \\
\cdots \\
\langle \delta D(r_1) \cdots \delta D(r_s) \rangle &= s! g^{(s)}(r_1 - r_2) \cdots (r_1 - r_s),
\end{align*}
\]

where the additional couplings $g^{(s)}$ could have changed under the renormalization.

Such a model was believed to be trivial in the weak-disorder limit as the operators described the quenched fluctuations of the diffusion coefficient are irrelevant under the RG transformations [17] at any dimensionality $d > 0$ so that the long-time diffusive behaviour remains normal and the limiting distribution of the diffusion coefficients turns out to be Gaussian whatever is the initial choice of the constants $g^{(s)}$. However, as it has been recently suggested by Tsai and Shapir [20], this model could be analyzed in terms of the $d = 0 + \epsilon$ expansion which could shed some light at the diffusive behaviour in the strong-disorder limit. In the case of the annealed disorder, the analysis of ref.[20] has shown that in the “strong-disorder” limit (i.e. for some nonzero $\epsilon$) a nontrivial fixed distribution of the diffusion coefficient existed which turned out to be the inverse Gaussian one. Unfortunately, it appeared to be impossible [20] to have extended the results of this analysis to the more
interesting case of the quenched disorder that corresponds to the $N = 0$ replica limit. (The annealed limit corresponds to the $N = 1$ case, i.e. to the averaging of the non-replicated functional).

Notwithstanding a viability of $0 + \epsilon$ expansion for describing the strong disorder in any physical dimensionality and the lack of analysis in the case of the quenched disorder, the appearance of the nontrivial fixed point [21] seems to be quite important. However, the “zero-dimensional” model defined by the FP equation (16) with $\nu = 0$ and by the distribution (22) has some quite specific features. The most important is that all the operators coupled to the diffusion cumulants become marginal at the critical dimensionality $d = 0$ (i.e. the naïve scaling dimension for each operator becomes equal zero). In any model with $d \neq 0$, the higher the order of cumulant, the more irrelevant becomes the appropriate operator. Thus, in the model of the quantum diffusion described in the Section (2) the naïve dimension of the $2s$-th high-gradient operator (which defines the $s$-th cumulant of the conductance and, to some extent, of the diffusion coefficient) decreases proportionally to $-s$ as in Eq.(10). It makes impossible an impact of the operators with the higher number of gradients on those with the lower one that, in turn, makes implausible the appearance of a new fixed point for the distribution (at least, starting from the weak-disorder limit where the dimensional analysis has some sense). It does not mean, however, that the distribution remains trivial (i.e. normal). As was shown in the Section (4), to the one-loop order the dimension of the high-gradient operators becomes proportional to $s^2$ so that they could become relevant. It leads to the increasing (11) of the higher cumulants which define nontrivial tails of the distribution. As these tails become more essential on approaching the strong disorder regime, the shape of the distribution could have changed completely in the strong-disorder limit [4]. As all the cumulants have different critical exponents (which are not proportional to the order of the cumulant), the change in the shape of the distribution has a character of a smooth crossover that is not associated with a new fixed point.

The goal of further considerations is to show that a similar scenario can take place in a classical-diffusion model. I will consider the FP equation (16) that includes both the random drifts $\nu(r)$ defined with the correlator Eqs.(17)–(19), and the random diffusion coefficients defined with the cumulants (22).

Before proceeding with the renormalization, I will describe a possible realization of the random-drifts models.
3.1 RELATION TO LATTICE HOPPING MODELS

The lattice hopping model which goes over to the FP equation (16) in the continuum limit is defined by the master equation for the probability $\rho_r(t)$ of finding the particle at the $r$-th site at the moment $t$:

$$\frac{\partial \rho_r}{\partial t} = \sum_{r'} (W_{rr'} \rho_{r'} - W_{r'r} \rho_r).$$

(23)

Here $W_{rr'}$ is the probability of hopping from site $r'$ to site $r$ which slightly fluctuates around the hopping probability $W^0$ in a regular lattice:

$$W_{rr'} = W^0_{r-r'} + \delta W_{rr'},$$

(24)

with $\delta W_{rr'}$ describing the quenched weak disorder on the lattice.

The parameters of the continuum model described by the FP equation (16) are readily expressed in terms of the hopping probabilities:

$$D(r) = \frac{1}{2} \sum_{r'} (r - r')^2 W_{rr'},$$

(25)

$$v(r) = \sum_{r'} (r - r') W_{rr'},$$

(26)

The random drift velocity given by Eq.(26) vanishes in all realizations of the disorder until the asymmetry in the hopping ($\delta W_{rr'} \neq \delta W_{r'r}$) is included. Therefore, the symmetric hopping model goes over to the FP equations with $v = 0$ where the disorder in $D$ described by the cumulants (22) reflects the symmetric disorder in $W$. In this case, the analysis of the continuum limit showed the weak disorder to be irrelevant for any $d$.

The asymmetry could be due to the presence of magnetic or charged impurities which leads to the continuum model with either the solenoidal or the potential disorder. Following ref. [17], I will show that in the presence of a random electric field $E(r)$ of charged impurities the continuum-limit model includes the potential random drifts (19a). If $E$ is the only source of randomness, the probability of thermally activated hops at a distance $b$ is

$$W_{r,r+b} = W^0_b \exp \left[ -\frac{eE(r)b}{2kT} \right].$$

(27)

Assuming then a random distribution of the charged impurities on the lattice and a global electro-neutrality, and representing the Fourier transform of $E(r)$ as

$$E(q) = \frac{2\pi i q}{\varepsilon q} \sum_j e_j \exp(iq \cdot r_j),$$

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with $e_j = \pm e$ being the charges of impurities with random coordinates $r_j$ and $\varepsilon$ being the dielectric constant, one finds

$$\langle E_{\alpha}(q)E_{\beta}(q') \rangle = \frac{(2\pi e)^2 C q_{\alpha}q_{\beta}}{q^2} \delta_{qq'}$$  \hspace{1cm} (28)$$

where $C$ is the concentration of the charged impurities. In the continuum limit, using the linear in $E$ approximation ($\delta W \propto E$) and assuming for simplicity the nearest-neighbor hopping only, one arrives at the potential-random-drifts model with the correlator Eqs.(17), (19). The strength of disorder $\gamma_0$ in Eq.(17) is related to the parameters of the hopping model as

$$\gamma_0 = \frac{\pi^2 e^4 C}{\varepsilon^2 (kT)^2} \left( \frac{W_0 z a^2}{2} \right)^2 \equiv \frac{\pi^2 e^4 C}{\varepsilon^2 (kT)^2} D_0^2$$  \hspace{1cm} (29)$$

where $z$ is the coordination number of the lattice, $a$ is the lattice constant, $W_0 \equiv W_0^0$ is a regular part of the nearest-neighbor hopping probability, and $D_0$ is the bare value of the diffusion coefficient. The randomness in $W$ results also in some distribution $P(D)$ of the diffusion coefficient whose cumulants (22) could be directly expressed in terms of the cumulants of $\delta W$.

In real lattices the correlator (28) becomes non-singular ($\propto q_{\alpha}q_{\beta}$) for $q \gg r_0$ where $r_0$ is a screening radius. The continuum model with the correlator (19a) could then describe only the random walks at a distance not exceeding $r_0$. Some models where $r_0$ is macroscopically large are described in ref.[17]. In general, $r_0$ takes the part of the infrared cutoff similar to that of the phase-breaking length in the weak-localization theory and defines the mesoscopic scale for the classical-diffusion problem. Below I will discuss the continuum model with the unscreened correlator (19).

### 3.2 THE EFFECTIVE FUNCTIONAL

Random walks of the particle started at the initial moment $t = 0$ at the point $r$ is characterized by the probability to find it at the point $r'$ in time $t$

$$G(r',r;t) = \langle \rho(r',t) \rho(r,0) \rangle$$ \hspace{1cm} (30)$$

which is the Green’s function of the FP equation (13). The asymptotic properties are governed by the long-time behaviour of this function, or by low-frequency behaviour of its Fourier transform, $G(r,r';\omega)$. To perform the ensemble averaging, it is convenient to represent the latter as a functional integral over the conjugate complex fields $\varphi(r)$ and
\[ G(r, r'; \omega) = \frac{i \int \overline{\varphi}(r) \varphi(r') e^{iS[\overline{\varphi}, \varphi]} D\overline{\varphi} D\varphi}{\int e^{iS[\overline{\varphi}, \varphi]} D\overline{\varphi} D\varphi}, \]

where the effective action functional is given by

\[ S[\overline{\varphi}, \varphi] = \int d^dr [i\overline{\varphi}\omega \varphi + v_\alpha (\partial_\alpha \varphi) \varphi - D\partial_\alpha \overline{\varphi} \partial_\alpha \varphi] \]

The averaging is performed by the standard replica trick: the fields \( \overline{\varphi}(r) \) and \( \varphi(r) \) and the functional integration in Eq. (31) are \( N \)-replicated and the independent averaging over the numerator and denominator in Eq. (31) is justified in the replica limit \( N = 0 \) (that should be taken in the final results). Taking into account the disorder both in the random drifts \( v \) and in the diffusion coefficients \( D \) which is defined by Eqs. (17) and (22), one deduces the effective action

\[ S[\overline{\varphi}, \varphi] = \{ S_0 + S_{int} + S_{cum} \}[\overline{\varphi}, \varphi] \]

which should be substituted for that given by Eq. (32) into Eq. (31) where the functional integration should be performed over all components of the fields \( \varphi \equiv \varphi_1, \ldots, \varphi_N \) and \( \overline{\varphi} \equiv \overline{\varphi}_1, \ldots, \overline{\varphi}_N \). Here

\[ S_0[\overline{\varphi}, \varphi] = \int d^dr \overline{\varphi}(i\omega + \overline{D}\partial^2)\varphi \]

\[ S_{int}[\overline{\varphi}, \varphi] = \frac{i\gamma}{2} \int d^dr d^dr' (\partial_\alpha \overline{\varphi} \varphi)_r F_{\alpha\beta}(r - r') (\partial_\beta \overline{\varphi} \varphi)_{r'} \]

\[ S_{cum}[\overline{\varphi}, \varphi] = i \sum_{s=2}^{\infty} g^{(s)} S_{(s)}; \quad S_{(s)} = \int d^dr \prod_{i=1}^{s} (\partial_\alpha \overline{\varphi} \partial_\alpha \varphi)_r \]

The RG analysis of the functional (34), (35) in the upper critical dimensionality \( d = 2 \) has been used for describing the anomalous long-time behaviour of the average diffusion coefficient, Eq. (20). The higher-order gradient operators, Eq. (36), describe the renormalization of the cumulants (22) of the distribution \( P(D) \). Under rescaling \( L \to \lambda L, g^{(s)} \to \lambda^{(s-1)d} g^{(s)} \) so that the naive dimension of \( g^{(s)} \) behaves like that for the high-gradient operators in the quantum diffusion problem, and \( g^{(s)} \) are irrelevant. I will show now that in the case of the potential random drifts, Eq. (19a), the one-loop RG corrections overturn this conclusion and make the scaling dimensions of the high-order gradient operators positive.
3.3 ONE-LOOP RENORMALIZATION

The renormalization is performed by expanding $\exp(iS_{\text{int}} + iS_{\text{cum}})$ in a power series, integrating with the weight $\exp(iS_0[\varphi_0, \varphi_0])$ over the “fast” components of the fields (this integration will be denoted as $\langle \ldots \rangle_0$), and exponentiating the results of the integration. Here $\varphi(r)$ (and $\overline{\varphi}(r)$) is decomposed into the sum of “slow”, $\tilde{\varphi}(r)$, and “fast”, $\varphi_0(r)$ components where

$$\tilde{\varphi}(r) = \sum_{q<\lambda q_0} \varphi(q)e^{iq\cdot r}, \quad \varphi_0(r) = \sum_{\lambda q_0<q<q_0} \varphi(q)e^{iq\cdot r},$$

$0 < \lambda < 1$ is the scaling parameter, $q_0$ is the ultraviolet cutoff, e.g. the inverse lattice constant in the lattice realization of the model. In the one-loop order (i.e. in the first order in powers of the coupling constant $g$, Eq.(21)), the logarithmic (at $d = 2$) contribution to $g^{(s)}$ is made only by the term

$$-\langle S_{\text{int}}S^{(s)} \rangle_0 \equiv -g\Lambda \hat{R}\{S^{(s)}\}.$$  \hspace{1cm} (37)

Here the one-loop RG operator $\hat{R}$ is introduced, $\Lambda \equiv \ln \lambda^{-1}$. The relevant contractions (i.e. those which give the logarithmic term) have the structure

$$\langle \varphi_0^a \partial^b_\alpha \overline{\varphi}_0^b \rangle_0 \langle \varphi_0^c \partial^d_\beta \overline{\varphi}_0^d \rangle_0,$$

where fields $\varphi_0$ are taken from $S_{\text{int}}$, and fields $\overline{\varphi}_0$ from $S^{(s)}$. As $\langle \overline{\varphi}_0 \varphi_0 \rangle_0$ is just a diffusion propagator, the contractions containing more gradients are irrelevant. That is why there is no contributions from the terms like $\langle S^{(s)}_n \rangle_0$, and no feedback from the terms with the higher number of gradients to those with the lower one.

It is evident from Eq.(38) that the index structure of the action (36) is not conserved under the renormalization. A set of additional operators is generated having the structure

$$\partial^a_\alpha \overline{\varphi}^a \partial^b_\beta \varphi^b \partial^c_\gamma \overline{\varphi}^c \partial^d_\delta \varphi^d \ldots$$

with all possible permutations of the vector indices $\alpha, \beta, \ldots$ and the replica indices $a, b, \ldots$, each index being repeated twice, where the summation over repeated replica indices from 1 to $N$ and over repeated vector indices from 1 to $d$ is implied. These operators are “unphysical” in a sense that the distribution $P(D)$ is defined by the renormalization of the initial operators (36) only. Then, only those operators are of interest which have the RG feedback to the initial ones.
To classify all the additional operators one introduces matrix notations
\[ Q^{ab} = \partial_\alpha \varphi^a \partial_\alpha \varphi^b, \quad P^{ab} = \partial_\alpha \varphi^a \partial_\alpha \varphi^b, \quad \overline{P}^{ab} = \partial_\alpha \overline{\varphi}^a \partial_\alpha \overline{\varphi}^b. \] (40)

In these notations, the contractions (38) could involve only the matrices \( Q \) and \( P \). On calculating the action of the RG operator (37) on the product \( QQ \) one finds
\[ \hat{R}\{Q^{ab}Q^{cd}\} = A(Q^{ab}Q^{cd} + Q^{ad}Q^{cb}) + (2 - |A|)\overline{P}^{ac}P^{bd}. \] (41)

Here \( A \) discriminates the models described above: \( A = 1 \) for the potential random drifts, Eq. (19a), \( A = -1 \) for the solenoidal ones, Eq. (19b), and \( A = 0 \) for the spatially uncorrelated random drifts, Eq. (18).

As \( \hat{R} \) does not act on the matrices \( P \) (they do not contain the field \( \overline{\varphi} \) to be contracted with the field \( \varphi \) from the \( S_{int} \)), their number in any operator could not be reduced under the renormalization. Therefore, the RG equations have a triangular structure: the operators containing only the matrices \( Q \) are not influenced on by those containing \( \overline{P}P \) (the number of \( \overline{P} \) is evidently equal to the number of \( P \) in any operator).

The initial operator (36) is proportional to \( [\text{Tr} Q]^s \), so that one can restrict the RG analysis to the subset of the operators containing only the matrices \( Q \). There is no one-loop corrections to \( g^{(s)} \) in the case of the uncorrelated random drifts, Eq. (18), as only the \( \overline{P}P \) term arises on renormalizing the initial action (36). I consider the case of the potential random drifts, Eq. (19a). The action (41) of the RG operator \( \hat{R} \) on the chosen subset reduces to simple permutations of the matrix indices of \( Q \). It can be analyzed in terms of the group of permutations, similar to the case of quantum diffusion [5, 9]. Any operator containing \( s \) matrices \( Q \) (i.e. \( 2s \) gradients of the fields \( \overline{\varphi} \) and \( \varphi \)) may be written down in a form similar to that for \( 2s \)-gradient operators in the quantum-diffusion problem (compare to Eq. (8)):
\[ S_{[\overline{\varphi}]Q} = g^{(s)} \int d^dr \left\{ \left[ \text{Tr}(Q) \right]^{s_1} \ldots \left[ \text{Tr}(Q) \right]^{s_m} \ldots \right\} \] (42)

Here the set of integers \( \overline{s} \) obeys the constraint (1), the initial operator (36) corresponding to \( \overline{s} = (0, \ldots, 0, s) \). The renormalization of the \( s \)-th cumulants results from solving the RG equations for the whole set of \( g^{(\overline{s})} \), the bare values of all the additional charges being equal to zero.

The RG equations may be written down as follows
\[ \frac{d|\overline{s}\rangle}{d\Lambda} = g\hat{R}|\overline{s}\rangle \] (43)
where $g^{(\vec{s})}$ is represented as some ket-vector defined by the “occupation numbers” $s_m$

$$g^{(\vec{s})} \equiv |\vec{s}\rangle \equiv |s_1 2^{s_2} \ldots m^{s_m} \ldots\rangle,$$

and $\hat{R}$ is expressed in terms of the “creation” and “annihilation” Bose operators $a^\dagger$, $a$

$$\hat{R} = \frac{1}{2} \left\{ \left( \sum_{m \geq 1} m a^\dagger_m a_m \right)^2 - \sum_{m \geq 1} m a^\dagger_m a_m \right. \right.
\left. + \sum_{m,l \geq 1} \left[ (m + l) a^\dagger_m a^\dagger_l a_{m+l} + ml a^\dagger_{m+l} a_m a_l \right] \right\},$$

these operators acting on the states $|\vec{s}\rangle$ as follows:

$$a_m |1^{s_1} \ldots m^{s_m} \ldots\rangle = s_m |1^{s_1} \ldots m^{s_m-1} \ldots\rangle,$$

$$a^\dagger_m |1^{s_1} \ldots m^{s_m} \ldots\rangle = |1^{s_1} \ldots m^{s_m+1} \ldots\rangle.$$

So, the RG transformation is expressed in the form of some one-dimensional field theory with the “hamiltonian” \(\hat{H}\). The first line in Eq.(45) represents the trivial diagonal part in the operation \(\hat{Q}\) while the second one represents the transposition between indices of $Q$ in Eq.(11). The hamiltonian (45) differs only by the trivial diagonal part from that derived for representing the RG transformations for the composite operators of the unitary nonlinear $\sigma$ model that describes the quantum diffusion problem in the presence of a magnetic field.

The operator (45) which is just a representation of the simple symmetric group operation (all possible pair transpositions \(\hat{Q}\) in a set of $2s$ pairs of indices) can be diagonalized exactly \[5, 9\]. The eigenvectors are given by

$$|\vec{\rho}\rangle = \sum_{\{\vec{s}\}} g(\vec{s}) \chi_\rho(\vec{s}) |\vec{s}\rangle$$

where the summation is performed over all the partitions $\vec{s} \equiv s_1, \ldots, s_m \ldots$ of $s$ obeying the constraint \[2\]. $g(\vec{s}) = s! / \prod_m m^{s_m} s_m!$ is the number of elements in the class defined by the partition $\vec{s}$, and $\chi_\rho(\vec{s})$ are the characters of irreducible representation of the symmetric group characterized by the Young frame $\vec{\rho}$ having boxes of length $\rho_1, \ldots, \rho_m$ where $\sum \rho_m = s$. The appropriate eigenvalues are given by

$$\alpha_s(\vec{\rho}) = \frac{s(s - 1)}{2} + \sum m \rho_m (\rho_m - 2m + 1)$$

To verify Eqs.(17) and (48), it is necessary to remember that the operator (13) is non-Hermitian, and $a^\dagger$ and $a$ defined by Eq.(16) are not mutually conjugate. In view of this, the set of bra-vectors orthogonal to the ket-vectors (47) is given by

$$\langle \vec{\rho} | = (1/s!) \sum_{\{\vec{s}\}} \chi_\rho(\vec{s}) \langle \vec{s} |.$$
The maximum eigenvalue corresponds to the eigenvector characterized by the one-line Young frame with $\rho_1 = s$, $\rho_m = 0$ for $m > 1$ for which $\chi_\rho(\vec{s}) = 1$ for all $\vec{s}$, so that it is equal to $s(s-1)$, as in the case of the quantum diffusion. Note that it could be verified without any reference to the representations of the permutation group, acting by the transverse operator $\hat{R}$ on the bra-vector (49) which is proportional in this case just to $\sum(\vec{s})$. Thus, with the one-loop accuracy, the dimension of the operators coupled to the moments of the diffusion coefficient is given by

$$\alpha_s = -(s-1)d + g_0s(s-1), \quad (50)$$

very similar to the case of the quantum diffusion, Eq.(44).

Note that the minimum eigenvalue in Eq.(48) (which corresponds to the eigenvector characterized by the one-column Young frame) equals zero. As results from Eq.(41), the eigenvalues for the solenoidal random-drifts differ in sign from those for the potential case. Thus, there is no positive exponents in the solenoidal case so that the dimensions of the operators coupled to the diffusion cumulants remain negative, and the distribution scales to the normal one. Therefore, of the three random-drifts model considered, the distribution becomes nontrivial and similar to the quantum-diffusion case only for the potential drifts.

4 Comparison of the results for the quantum and classical diffusion

There are two types of contributions into the fluctuations of the diffusion coefficient in the random-walks model considered, similar to the quantum diffusion problem, Eq.(11). The “normal” one is given only by the functional (35). It diverges in the infrared limit thus making the fractional fluctuation $\langle \langle (\delta D)^2 \rangle \rangle / D^2 \propto g_0^2$ to be independent of the size of the system, analogous to the universal conductance fluctuation (the constant $g_0$ has nonuniversal dependence on the disorder parameter, Eq.(21), which restricts the analogy). This contribution has been considered in detail in ref.[17] and is of no interest for the present considerations. The additional contribution to the diffusion cumulants (22) is governed by the dimensions of the couplings $g^s$ in the “additional” functional (36). Keeping only the maximum eigenvalue, as in Eq.(50), one finds in the critical dimensionality $d = 2$

$$\langle \langle (\delta D)^s \rangle \rangle \propto \left( \frac{L}{L} \right)^{2(s-1)-g_0s(s-1)} \quad (51)$$
It is quite similar to the “additional” contribution to the conductance cumulants, Eq. (11b), in the quantum-diffusion problem (as the appropriate contribution to the cumulants of the density of states has the same form [5], one can write down the same expressions directly for the quantum-diffusion cumulants). To make the analogy more striking, one substitutes into Eq. (11b) the value of the parameter $u$, Eq. (12), in the weak-localization limit at $d = 2$, $u = g_0^{-1} \ln(L/l) \equiv \tilde{u}$, which gives

$$\langle g^s \rangle \propto \left( \frac{l}{L} \right)^{2(s-1) - \tilde{g}^{-1}_0 s(s-1)}.$$  \hspace{1cm} (52)

Note that both expressions (51) and (52) are applicable only for the high-order cumulants as all the eigenvalues of the RG equations but the maximum ones have been neglected in the derivation. Nevertheless, even for small $s \geq 2$ these expressions give a qualitatively valid result that the low-order cumulants are small as compared to the variance so that the bulk of the distribution is Gaussian. The high moments (with $s \geq g_0^{-1}$ for Eq. (51) and $s \geq \tilde{g}_0$ in Eq. (52)) govern, in accordance with the identity (13), the lognormal tails of the distributions which have the following shape

$$f(\delta X) \propto \frac{\alpha^2}{\delta X} \exp \left[ -\frac{1}{4\tilde{u}} \ln^2 \left( \delta X \alpha^2 e^{-\tilde{u}} \right) \right].$$  \hspace{1cm} (53)

An appropriate derivation is described in detail in ref. [5]. Here $\delta X$ stands for either $\delta D$ or for $g$ in the classical or quantum problems, respectively, $\alpha \equiv L/l$, and $\tilde{u}$ is given above for the quantum problem, and equals $g_0 \ln(L/l)$ for the classical one.

In the classical case, $\tilde{u}$ has only a trivial dependence on the scale given by the logarithm, as there is no renormalization of the coupling constant. In the quantum case, $\tilde{u}$ gives just the weak-disorder limit for the parameter $u = \ln(\sigma_0/\sigma)$, Eq. (12). So the whole dependence on the renormalization of the coupling constant is absorbed by the parameter $u$ which is substituted to the lognormal distribution, its appearance being governed by the renormalization of the high-gradient operators. That is why one can separate the renormalization of the average values from that of the higher order cumulants. In such a way, one reproduces [5] exact one-dimensional results for lognormal distributions by substituting the exact one-dimensional value of $u$ into the formulae similar to (53). It provides a basis for the conjecture that one can obtained a reasonable description of the distributions near the transition just by substituting $\exp(u) \sim |g - g_c|^\nu$ with a proper choice of the critical exponent $\nu$. 

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5 Discussion

It has been demonstrated that the shape of the distribution of the diffusion coefficient in the model of classical diffusion in a medium with quenched random drifts proved to be very similar for that of the conductance distribution in a weakly disordered metal. In both cases the distributions turn out to be almost Gaussian in the weak-disorder limit but have slowly decreasing lognormal tails, and the part of the tails increase with increasing the disorder. This similarity occurs although the coupling constants in the field-theoretical models describing the quantum and the classical diffusion behave very differently. The asymptotic freedom of the nonlinear $\sigma$ model, i.e. the increase of the coupling constant (inversely proportional to the conductance) with increasing a scale, is believed to govern the Anderson transition. No transition occurs in the classical diffusion problem described by the field theory with the $(\partial\phi\phi)^2$-type interaction where in the case of the potential disorder a perturbative renormalization of the coupling constant proves to be absent thus leading to the sub-diffusion, Eq.(20a). The reason for the similarity is that the RG equations governing the cumulants of the distributions in both cases are classified according to the same irreducible representations of the group of permutations. But deriving the RG equations for the high-gradient operators proved to be much easier in the classical-diffusion model. Then the higher-loop RG analysis of the high-gradient operators which could check a viability of the one-loop results seems hardly possible in the quantum model but could appear quite straightforward for the classical one. Thus studying the classical-diffusion problem described here gives a possibility to learn more about the quantum diffusion in disordered media.

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