There are no topologically transitive operators in the noncommutative Schwartz space

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Abstract. The aim of this note is to prove that there are no topologically transitive operators in the noncommutative Schwartz space.

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1. Introduction. Let $s$ and $s'$ be the spaces of rapidly decreasing and slowly increasing sequences, respectively, equipped with their natural locally convex topologies. The so-called noncommutative Schwartz space is the space $S := L(s', s)$ of all bounded linear operators acting from $s'$ into $s$. This space becomes a Fréchet algebra in a natural way: if $T_1$ and $T_2$ are in $S$, then the product $T_1T_2$ is defined by the formula $T_1T_2 = T_1 \circ \iota \circ T_2$, where $\iota$ is the natural embedding of $s$ into $s'$. In fact, $S$ embeds algebraically into the $C^*$-algebra $B(\ell_2)$ of all bounded and linear operators on the Hilbert space $\ell_2$.

The noncommutative Schwartz space is isomorphic (as a Fréchet $*$-algebra) to a number of other natural objects of analysis, e.g., $S \simeq S(\mathbb{R}^2)$—the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$ equipped with the Volterra multiplication $(f \cdot g)(x, y) := \int_\mathbb{R} f(x, z)g(z, y)dz$ and involution $f^*(x, y) := \overline{f(y, x)}$. It plays an important role, e.g., in K-theory—see [4,12], cyclic cohomology for crossed products—see [9,14], noncommutative geometry—see [3], operator spaces—see [7,8]. Another motivation comes from quantum mechanics where $S$ is called the space of physical states and its dual is the so-called space of observables—see [6] for details.

Since $S$ is a Fréchet algebra of operators, it is natural to ask about the dynamical properties of the elements of $S$. Recall that an operator $T : s' \to s$ is topologically transitive if for every two non-empty and open sets $U \subset s'$,
V ⊂ s, there exists $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$ and hypercyclic if there exists $x \in s'$ such that the set $\{T^n x : n \in \mathbb{N}\}$ is dense in $s$. It is clear that hypercyclicity of $T$ implies that it is topologically transitive. Formally it could happen that the latter is a weaker property since $s'$ is not a metric space.

The main goal of this note is to show that there are no topologically transitive operators in $S$. The main difficulty of the paper is to understand the spectral properties of operators from $S$, those are investigated in Sect. 2.

We refer the reader to [1,10,11] for unexplained details from linear dynamics and functional analysis, respectively.

2. Notation and terminology. Recall that

$$s = \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \subset \mathbb{C}^\mathbb{N}: |\xi|^2 := \sum_{j=1}^{+\infty} |\xi_j|^2 j^{2t} < +\infty \text{ for all } t \geq 0 \right\}$$

and its topological dual

$$s' = \left\{ \eta = (\eta_j)_{j \in \mathbb{N}} \subset \mathbb{C}^\mathbb{N}: (|\eta|^2)_{t} := \sum_{j=1}^{+\infty} |\eta_j|^2 j^{-2t} < +\infty \text{ for some } t \geq 0 \right\}$$

are the so-called spaces of rapidly decreasing and slowly increasing sequences, respectively.

Furthermore we consider the space $S := L(s',s)$ of all linear and continuous operators from $s'$ into $s$, equipped with the topology of uniform convergence on bounded sets. Consequently, the topology of $S$ is given by the scale $(\|\cdot\|_t)_{t \geq 0}$ of norms, defined as

$$\|T\|_t := \sup\{|T\eta|_t: |\eta|_t \leq 1\} \quad (T \in S, t \geq 0).$$

If we denote $H_t := \ell_2((j^t)_{j \in \mathbb{N}})$, $t \in \mathbb{R}$, then $H_t' \cong H_{-t}$ and every $T \in S$ is a Hilbert space operator in the sense that $T: H_t' \to H_t$ and

$$\|T\|_t = \|T\|_{H_t' \to H_t} \quad (t \geq 0).$$

In other words, if we denote by $D_t := \text{diag}(j^t)$, $t \in \mathbb{R}$, an infinite diagonal matrix, then $D_t$ becomes simultaneously an isometry $D_t: H_t \to \ell_2$ and $D_t: \ell_2 \to H_t'$ and

$$\|T\|_t = \|D_t T D_t\|_{B(\ell_2)} \quad (T \in S, t \geq 0). \quad (1)$$

In particular, $S = \bigcap_{t \geq 0} B(H_t', H_t) = \bigcup_{k \in \mathbb{N}} B(H_k', H_k)$. We will be using these properties interchangeably.

Since $s \hookrightarrow s'$, we can define multiplication in $S$ as

$$T_1 T_2 := T_1 \circ \iota \circ T_2 \quad (T_1, T_2 \in S),$$

where $\iota: s \hookrightarrow s'$, $\iota(\xi) := \xi$ is the formal embedding. Altogether it turns $S$ into an $m$-convex Fréchet algebra. It comes endowed also with the involution (or the adjoint map) given as

$$\langle T^* \xi, \eta \rangle := \langle \xi, T \eta \rangle \quad (\xi, \eta \in s', T \in S).$$

It is worth noting that $s \hookrightarrow \ell_2$ and, by dualization, also $\ell_2 \hookrightarrow s'$ therefore $S$ is algebraically contained in the $C^*$-algebra $B(\ell_2)$ of all bounded and linear
operators on the Hilbert space $\ell_2$. Therefore multiplication in $\mathcal{S}$ is essentially the multiplication in $\mathcal{B}(\ell_2)$ with the additional property that the resulting operator belongs to $\mathcal{S}$. The same applies to involution.

The unitization of $\mathcal{S}$ will be denoted by $\mathcal{S}_1$. Clearly, the unit in $\mathcal{S}_1$ is the identity operator on $\ell_2$ denoted by $1$. The algebra $\mathcal{S}$ is called the noncommutative Schwartz space and the elements of $\mathcal{S}$ are called smooth operators. We refer the reader to [2,13] for more information on the properties of this algebra.

3. Spectral properties of operators in $\mathcal{S}$. We start by showing some spectral properties of smooth operators.

**Proposition 3.1** ([5, Proposition 3.1 and Theorem 3.3]). Every smooth operator is compact, i.e., $\mathcal{S} \hookrightarrow \mathcal{K}(\ell_2)$ and

$$\sigma_{\mathcal{S}_1}(T) = \sigma_{\mathcal{B}(\ell_2)}(T) \quad (T \in \mathcal{S}).$$

In particular, the spectrum of every smooth operator consists of zero and a (possibly) null sequence of eigenvalues.

**Lemma 3.2.** For any $t \in \mathbb{R}$ and every smooth operator $T \in \mathcal{S}$, we have

$$\sigma_{\mathcal{B}(\ell_2)}(D_tD_{-t}) \subset \sigma_{\mathcal{S}_1}(T).$$

**Proof.** Let $t \in \mathbb{R}$ and $T \in \mathcal{S}$ be fixed. Suppose that $\lambda \in \rho_{\mathcal{S}_1}(T)$, i.e., there is $S \in \mathcal{S}$ such that

$$(S - \frac{1}{\lambda}1)(T - \lambda1) = (T - \lambda1)(S - \frac{1}{\lambda}1) = 1,$$

where $1$ is the identity operator on $\ell_2$. Then

$$\ell_2 \xrightarrow{D_{-t}} H_t \hookrightarrow s' \xrightarrow{S} s \hookrightarrow H_t \xrightarrow{D_t} \ell_2$$

and

$$\left(D_tD_{-t} - \frac{1}{\lambda}1\right)(D_tD_{-t} - \lambda1) = (D_tD_{-t} - \lambda1)\left(D_tD_{-t} - \frac{1}{\lambda}1\right) = 1.$$ Consequently, $\lambda \in \rho_{\mathcal{B}(\ell_2)}(D_tD_{-t})$ and the proof is thereby complete. \hfill $\Box$

**Corollary 3.3.** If a smooth operator $T \in \mathcal{S}$ satisfies

$$\sigma_{\mathcal{S}_1}(T) \subset \mathbb{D},$$

then the sequence $(T^n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{S}$.

**Proof.** Let the smooth operator $T \in \mathcal{S}$ satisfy (2). Since $D_t^{-1} = D_{-t}$, we obtain that for every $n \in \mathbb{N}$ and every $t \geq 0$,

$$\|T^n\|_t = \|D_tD_t\|_{\mathcal{B}(\ell_2)} = \|D_tD_{-t}D_tD_{-t} \cdots D_tD_{-t}D_tD_t\|_{\mathcal{B}(\ell_2)} \leq \|(D_tD_{-t})^{n-1}\|_{\mathcal{B}(\ell_2)} \|T\|_t.$$
From Lemma 3.2, the spectral radius formula, and compactness of the spectrum, it now follows that there is \( \varepsilon > 0 \) such that
\[
\nu(D_tTD_{-t}) = \lim_{n \to \infty} \|(D_tTD_{-t})^n\|_{B(\ell_2)}^{1/n} \leq 1 - \varepsilon.
\]
Hence there is \( N \in \mathbb{N} \) such that for every \( n \geq N \), we have
\[
\|(D_tTD_{-t})^n\|_{B(\ell_2)} \leq 1.
\]
If we now define \( C_t := \max\{\|T\|_t, \|T^2\|_t, \ldots, \|T^N\|_t, 1\} \cdot \|T\|_t \), then
\[
\sup_{n \in \mathbb{N}} \|T^n\|_t \leq C_t < \infty.
\]
Consequently, \((T^n)_{n \in \mathbb{N}}\) is a bounded sequence in the noncommutative Schwartz space. \( \square \)

4. Main result.

Theorem 3.1. There are no topologically transitive operators in \( S \). In particular, the operators in \( S \) are not hypercyclic.

Proof. Let \( T \in S \) be arbitrary. There are two possible cases: either \( \sigma_{S_1}(T) = \{0\} \) or there exists \( 0 \neq \lambda \in \sigma_{S_1}(T) \).

If \( \sigma_{S_1}(T) = \{0\} \), then from Corollary 3.3, the sequence \((T^n)_{n \in \mathbb{N}}\) is bounded in \( S \) and therefore it is equicontinuous. In particular, for every zero neighbourhood \( U \subset s' \), there is a zero neighbourhood \( V \subset s \) such that
\[
T^n(U) \subset \frac{1}{2}V \quad (n \in \mathbb{N}).
\]
We choose now \( \xi \in s \setminus V \) and suppose that there is \( n \in \mathbb{N} \) and \( \eta \in s \) such that
\[
\eta \in T^n(U) \cap \left( \xi + \frac{1}{2}V \right).
\]
This implies that for some \( \zeta \in \frac{1}{2}V \), we have
\[
\xi = \eta - \zeta \in \frac{1}{2}V + \frac{1}{2}V = V.
\]
This contradicts the choice of \( \xi \in s \) and shows that in this case \( T \) is not topologically transitive.

If there exists \( 0 \neq \lambda \in \sigma_{S_1}(T) \), then from Proposition 3.1, it follows that \( \lambda \) is an eigenvalue and we let \( f \) be a holomorphic function on a neighbourhood of \( \sigma_{S_1}(T) \) such that \( f(\lambda) = 1 \) and \( f(z) = 0 \) for \( z \in \sigma_{S_1}(T) \setminus \{\lambda\} \). Using the holomorphic functional calculus (which is available in \( S \) by [12, Lemma 1.3]), we can now consider the operator \( f(T) \in S \). Let \( M = \text{Im}(f(T)) \). It is clear that \( M \) is a non-trivial and finite dimensional subspace of \( s \) (every non-zero element of \( M \) is an eigenvector of the compact operator \( f(T) \)). The properties of the functional calculus imply that the diagram
\[
\begin{array}{ccc}
s' & \overset{T}{\longrightarrow} & s \\
\downarrow f(T) & & \downarrow f(T) |_s \\
M & \overset{T_{|M}}{\longrightarrow} & M
\end{array}
\]
commutes and one can easily verify that topological transitivity of $T$ would imply topological transitivity of $T|_M$. Since $M$ is finite dimensional this implies that $T$ is not topologically transitive. □

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