A HERBRAND-RIBET THEOREM FOR FUNCTION FIELDS

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Abstract. We prove a function field analogue of the Herbrand-Ribet theorem on cyclotomic number fields. The Herbrand-Ribet theorem can be interpreted as a result about cohomology with \( \mu_p \)-coefficients over the splitting field of \( \mu_p \), and in our analogue both occurrences of \( \mu_p \) are replaced with the \( p \)-torsion scheme of the Carlitz module for a prime \( p \) in \( F_q[t] \).

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1. Introduction and statement of the theorem

Let \( p \) be a prime number, \( F = \mathbb{Q}(\zeta_p) \) the \( p \)-th cyclotomic number field and \( \text{Pic} \, \mathcal{O}_F \) its class group. Then \( F_p \otimes \mathbb{Z} \text{Pic} \, \mathcal{O}_F \) decomposes in eigenspaces under the action of the Galois group \( \text{Gal}(F/\mathbb{Q}) \) as

\[
F_p \otimes \mathbb{Z} \text{Pic} \, \mathcal{O}_F = \bigoplus_{n=1}^{p-1} (F_p \otimes \mathbb{Z} \text{Pic} \, \mathcal{O}_F) (\omega^n)
\]

where \( \omega : \text{Gal}(F/\mathbb{Q}) \to F_p^\times \) is the cyclotomic character.

If \( n \) is a nonnegative integer we denote by \( B_n \) the \( n \)-th Bernoulli number, defined by the identity

\[
\frac{\exp z - 1}{z} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.
\]

If \( n \) is smaller than \( p \) then \( B_n \) is \( p \)-integral. The Herbrand-Ribet theorem \([9]\) \([14]\) states that if \( n \) is even and \( 1 < n < p \) then

\[
(F_p \otimes \mathbb{Z} \text{Pic} \, \mathcal{O}_F) (\omega^{1-n}) \neq 0 \text{ if and only if } p \mid B_n.
\]
The Kummer-Vandiver conjecture asserts that for all odd $n$ we have
\[(\mathbb{F}_p \otimes \mathbb{Z} \text{Pic} \mathcal{O}_F) (\omega^{1-n}) = 0.\]

In this paper we will state and prove a function field analogue of the Herbrand-Ribet theorem and state an analogue of the Kummer-Vandiver conjecture.

Let $k$ be a finite field of $q$ elements and $A = k[t]$ the polynomial ring in one variable $t$ over $k$. Let $K$ be the fraction field of $A$.

**Definition 1.** The Carlitz module is the $A$-module scheme $C$ over $\text{Spec} \, A$ whose underlying $k$-vector space scheme is the additive group $G_a$ and whose $k[t]$-module structure is given by the $k$-algebra homomorphism
\[\varphi: A \to \text{End}(G_a), t \mapsto t + F,\]
where $F$ is the $q$-th power Frobenius endomorphism of $G_a$.

The Carlitz module is in many ways an $A$-module analogue of the $\mathbb{Z}$-module scheme $G_n$. For example, the $\text{Gal}(K^{\text{sep}}/K)$-action on torsion points is formally similar to the $\text{Gal}(\mathbb{Q}/\mathbb{Q})$-action on roots of unity:

**Proposition 1** ([2] §7.5]). Let $p \subset A$ be a nonzero prime ideal, then $C[p](K^{\text{sep}}) \cong A/p$ and the resulting Galois representation
\[\rho: \text{Gal}(K^{\text{sep}}/K) \longrightarrow (A/p)^\times,\]
satisfies

1. if a prime $q \subset A$ is coprime with $p$ then $\rho$ is unramified at $q$ and maps a Frobenius element to the class in $(A/p)^\times$ of the monic generator of $q$;
2. $\rho(D_\infty) = \rho(I_\infty) = k^\times$;
3. $\rho(D_p) = \rho(I_p) = (A/p)^\times$,

where the $D$’s and $I$’s denote decomposition and inertia subgroups. \[\square\]

Now fix a nonzero prime ideal $p \subset A$ of degree $d$. Let $L$ be the splitting field of $\rho$. Then $L/K$ is unramified outside $p$ and $\infty$, and $\rho$ induces an isomorphism $\chi: G = \text{Gal}(L/K) \xrightarrow{\sim} (A/p)^\times$.

Let $R$ be the normalization of $A$ in $L$ and $Y = \text{Spec} \, R$. Let $D_{\text{fl}}$ be the flat site on $Y$: the category of schemes locally of finite type over $Y$, with covering families being the jointly surjective families of flat morphisms.

The $p$-torsion $C[p]$ of $C$ is a finite flat group scheme of rank $q^d$ over $\text{Spec} \, A$. Let $C[p]^D$ be the Cartier dual of $C[p]$ and consider the decomposition
\[H^1(Y_{\text{fl}}, C[p]^D) = \bigoplus_{n=1}^{q^d-1} H^1(Y_{\text{fl}}, C[p]^D)(\chi^n)\]
of the $A/p$-vector space $H^1(Y_{\text{fl}}, C[p]^D)$ under the natural action of $G$.

Our analogue of the Herbrand-Ribet theorem will give a criterion for the vanishing of some of these eigenspaces in terms of divisibility by $p$ of the so-called Bernoulli-Carlitz numbers, which we now define.

The **Carlitz exponential** is the unique power series $e(z) \in K[[z]]$ which satisfies

1. $e(z) = z + e_1z^q + e_2z^{q^2} + \cdots$ with $e_i \in K$;
2. $e(tz) = e(z)^q + te(z)$.
The Carlitz exponential converges on any finite extension of $K_\infty$ and on an algebraic closure $\bar{K}_\infty$. We define $BC_n \in K$ by the power series identity

$$
\frac{z}{e(z)} = \sum_{n=0}^{\infty} BC_n z^n.
$$

If $n$ is not divisible by $q-1$ then $BC_n$ is zero. If $n$ is less than $q^d$ then $BC_n$ is $p$-integral.

**Theorem 1.** Let $0 < n < q^d - 1$ be divisible by $q-1$. Then $p$ divides $BC_n$ if and only if $H^1(Y_{fl}, C[p]^D)(\chi^{n-1})$ is nonzero.

This is the analogue of the Herbrand-Ribet theorem. The proof is given in section 4, modulo auxiliary results which are proven in sections 6–9.

In this context a natural analogue of the Kummer-Vandiver conjecture is the following:

**Question 1.** Does $H^1(Y_{fl}, C[p]^D)(\chi^{n-1})$ vanish if $n$ is not divisible by $q-1$?

By computer calculation we have verified that these groups indeed vanish for small $q$ and primes $p$ of small degree, see §2. However, if one believes in a function field version of Washington’s heuristics [18, §9.3] then one should expect that counterexamples do exist, but are very sparse, making it difficult to obtain convincing numerical evidence towards Question 1.

**Remark 1.** Our $BC_n$ differ from the commonly used Bernoulli-Carlitz numbers by a Carlitz factorial factor (see for example [7, §9.2]). This factor is innocent for our purposes since it is a unit at $p$ for $n < q^d$.

**Remark 2.** Let $p$ be an odd prime number, $F = \mathbb{Q}(\zeta_p)$ and $D = \text{Spec} \mathcal{O}_F$. Global duality [10] provides a perfect pairing between

$$
F_p \otimes \mathbb{Z} \text{Pic } D = \text{Ext}^2_{D_{\text{et}}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m, D
$$

and

$$
H^1(D_{\text{et}}, \mathbb{Z}/p\mathbb{Z}) = H^1(D_{\text{fl}}, \mathbb{Z}/p\mathbb{Z}).
$$

The Herbrand-Ribet theorem thus says that (for $1 < n < p-1$ even)

$$
p \mid B_n \text{ if and only if } H^1(D_{\text{fl}}, \mu_p^D)(\chi^{n-1}) \neq 0,
$$

in perfect analogy with the statement of Theorem 1.

**Remark 3.** The analogy goes even further. In [16] and [15] we have defined a finite $A$-module $H(C/R)$, analogue of the class group $\text{Pic} \mathcal{O}_F$, and although we will not use this in the proof of Theorem 1 we show in Section 10 of this paper that there are canonical isomorphisms

$$
A/p \otimes_A H(C/R) \xrightarrow{\sim} \text{Hom}(H^1(Y_{fl}, C[p]^D), F_p).
$$

**Remark 4.** A more naive attempt to obtain a function field analogue of the Herbrand-Ribet theorem would be to compare the $p$-divisibility of the Bernoulli-Carlitz numbers with the $p$-torsion of the divisor class groups of $\bar{Y}$ and $\bar{L}$ (where $p$ is the characteristic of $k$). In other words, to consider cohomology with $\mu_p$-coefficients
on the curves defined by the splitting of $C[p]$. Several results of this kind have in fact been obtained by Goss [6], Gekeler [5], Okada [12], and Anglès [2], but there appears to be no complete analogue of the Herbrand-Ribet theorem in this context.

In the proof of Theorem 1 we will see that the $A$-module $H^1(Y_{\text{fl}}, C[p]D)$ and the group $(\text{Pic} Y)[p]$ are related, and this relationship might shed some new light on these older results.

**Remark 5.** I do not know if there is a relation between Question 1 and Anderson’s analogue of the Kummer-Vandiver conjecture [1].

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2. **Tables of small irregular primes**

The results of section 1 indicate a method for computing the modules $H^1(Y_{\text{fl}}, C[p]D)$ with their $G$-action in terms of finite-dimensional vector spaces of differential forms on the compactification $X$ of $Y$.

Assisted by the computer algebra package MAGMA we were able to compute them in the following ranges:

- (1) $q = 2$ and $\deg p \leq 5$;
- (2) $q = 3$ and $\deg p \leq 4$;
- (3) $q = 4$ and $\deg p \leq 3$;
- (4) $q = 5$ and $\deg p \leq 3$.

In all these cases $H^1(Y_{\text{fl}}, C[p]D)$ turns out to be at most one-dimensional, and to fall in the $\chi^{q-1}$-component with $n$ divisible by $q - 1$ (and hence with $p$ dividing $BC_n$.) In particular we have not found any counterexamples to Question 1.

In tables 1–3 we list all cases where the cohomology group is nontrivial. For $q = 5$ and $\deg p \leq 3$ the group turns out to vanish. In the middle columns, only $n$ in the range $1 \leq n \leq q^{\deg p}$ are printed.

| $p$ | $\{n : p | BC_n\}$ | $\dim H^1(Y_{\text{fl}}, C[p]D)$ |
|-----|---------------------|-------------------------------|
| $(t^4 + t + 1)$ | $\{9\}$ | 1 |

Table 1. All irregular primes in $F_2[t]$ of degree at most 5

3. **Notation and conventions**

**Basic setup.** $k$ is a finite field of $q$ elements, $p$ its characteristic. $A = k[t]$ and $p \subset A$ a nonzero prime. These data are fixed throughout the text. We denote by $d$ the degree of $p$, so that $A/p$ is a field of $q^d$ elements.

**The Carlitz module.** The Carlitz module is the $A$-module scheme $C$ over $\text{Spec} A$ defined in Definition 1.

**Cyclotomic curves and fields.** $K$ is the fraction field of $A$, and $L/K$ the splitting field of $C[p][K]$. The integral closure of $A$ in $L$ is denoted by $R$, and $Y = \text{Spec} R$. We denote by $\mathfrak{P} \subset R$ the unique prime lying above $p \subset A$.

**Sites.** For any scheme $S$ we denote by $S_{\text{et}}$ the small étale site on $S$ and by $S_{\text{fl}}$ the flat site in the sense of [11]: the category of schemes locally of finite type over $S$. 

where covering families are jointly surjective families of flat morphisms. For every $S$ there is a canonical morphism of sites $f : S_{fl} \to S_{et}$. Any commutative group scheme over $S$ defines a sheaf of abelian groups on $S_{fl}$ and on $S_{et}$.

**Cartier dual.** If $G$ is a finite flat commutative group scheme, then $G^D$ denotes the Cartier dual of $G$.

**Frobenius and Cartier operators.** For any $k$-scheme $S$ we denote by

$$F : G_{a,S} \to G_{a,S}, \ x \mapsto x^q$$

the $q$-power Frobenius endomorphism of sheaves on $S_{fl}$ or $S_{et}$, and by

$$c : \Omega_S \to \Omega_S$$

the $q$-Cartier operator of sheaves on $S_{et}$. If $q = p^r$ with $p$ prime this is the $r$-th power of the usual Cartier operator. The endomorphism $c$ satisfies $c(f^q \omega) = f c(\omega)$ for all local sections $f$ of $\mathcal{O}_S$ and $\omega$ of $\Omega_S$. In particular it is $k$-linear.

### 4. Overview of the proof

Choose a generator $\lambda$ of $C[p](L)$. It defines a map of finite flat group schemes

$$\lambda : (A/p)_Y \to C[p]_Y$$

which is an isomorphism over $Y - \mathfrak{P}$. It induces a map of Cartier duals

$$C[p]_Y^D \to (A/p)_Y^D$$

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
$p$ & $\{ n : p \mid BC_n \}$ & $\dim H^1(Y_{fl}, C[p]^D)$ \\
\hline
$(t^3 - t + 1)$ & $\{10\}$ & 1 \\
$(t^3 - t - 1)$ & $\{10\}$ & 1 \\
$(t^4 - t^3 + t^2 + 1)$ & $\{40\}$ & 1 \\
$(t^4 - t^2 - 1)$ & $\{32\}$ & 1 \\
$(t^4 - t^3 - t^2 + t - 1)$ & $\{32\}$ & 1 \\
$(t^4 + t^3 + t^2 + 1)$ & $\{40\}$ & 1 \\
$(t^4 + t^3 - t^2 - t - 1)$ & $\{32\}$ & 1 \\
$(t^4 + t^2 - 1)$ & $\{40\}$ & 1 \\
\hline
\end{tabular}
\caption{All irregular primes in $F_3[t]$ of degree at most 4}
\end{table}

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
$p$ & $\{ n : p \mid BC_n \}$ & $\dim H^1(Y_{fl}, C[p]^D)$ \\
\hline
$(t^3 + t^2 + t + \alpha)$ & $\{33\}$ & 1 \\
$(t^3 + t^2 + t + \alpha^2)$ & $\{33\}$ & 1 \\
$(t^3 + \alpha)$ & $\{33\}$ & 1 \\
$(t^3 + \alpha^2)$ & $\{33\}$ & 1 \\
$(t^3 + \alpha^2t^2 + \alpha t + \alpha^2)$ & $\{33\}$ & 1 \\
$(t^3 + \alpha^2 + \alpha^2t + \alpha)$ & $\{33\}$ & 1 \\
$(t^3 + \alpha^2t^2 + \alpha^2 t + \alpha)$ & $\{33\}$ & 1 \\
$(t^3 + \alpha^2t^2 + \alpha t + \alpha)$ & $\{33\}$ & 1 \\
\hline
\end{tabular}
\caption{All irregular primes in $F_4[t]$ of degree at most 3 (with $F_4 = F_2(\alpha)$).}
and a map on cohomology
\[ H^1(Y_{\text{fl}}, C[p]^D) \to H^1(Y_{\text{fl}}, (A/p)^D). \]

This map is not \(G\)-equivariant (since \(\lambda\) is not \(G\)-invariant), but rather restricts for every \(n\) to a map
\[ H^1(Y_{\text{fl}}, C[p]^D)_{(\chi^{n-1})} \to H^1(Y_{\text{fl}}, (A/p)^D)_{(\chi^n)}. \]

We will see in section \(\S\) that there is a natural \(G\)-equivariant isomorphism
\[ H^1(Y_{\text{fl}}, (A/p)^D) \simto A/p \otimes_k \Omega_{R}^{\omega=1} \]
where \(\Omega_{R}^{\omega=1}\) is the \(k\)-vector space of \(q\)-Cartier invariant Kähler differentials. Also, we will see that the Kummer sequence induces a short exact sequence
\[
0 \to A/p \otimes_{\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y^\times) \xrightarrow{\text{dlog}} A/p \otimes_k \Omega_{R}^{\omega=1} \to A/p \otimes_{\mathcal{F}_p} (\text{Pic} Y)[p] \to 0.
\]

Note that the residue field of the completion \(R_p\) is \(A/p\), so \(R_p\) is naturally an \(A/p\)-algebra. In particular, for all \(m\) the \(R\)-module \(\Omega_{R}/\mathfrak{m}^m\Omega_{R}\) is naturally an \(A/p\)-module. Using this the quotient map \(\Omega_{R} \to \Omega_{R}/\mathfrak{m}^m\Omega_{R}\) extends to an \(A/p\)-linear map
\[ A/p \otimes_k \Omega_{R} \to \Omega_{R}/\mathfrak{m}^m\Omega_{R}. \]

In section \(\S\) we will use the results on flat duality of Artin and Milne \(\cite{3}\) to show the following.

**Theorem 2.** For all \(n\) the sequence of \(A/p\)-vector spaces
\[ 0 \to H^1(Y_{\text{fl}}, C[p]^D)_{(\chi^{n-1})} \xrightarrow{\lambda} A/p \otimes_k \Omega_{R}^{\omega=1} \to \Omega_{R}/\mathfrak{m}^d\Omega_{R} \]
is exact.

The function \(\lambda\) is invertible on \(Y - \mathfrak{m}\). Consider the decomposition of \(1 \otimes \lambda \in A/p \otimes_{\mathbb{Z}} \Gamma(Y - \mathfrak{m}, \mathcal{O}_Y^\times)\) in isotypical components:
\[
1 \otimes \lambda = \sum_{n=1}^{q^d-1} \lambda_n \quad \text{with} \quad \lambda_n \in A/p \otimes_{\mathbb{Z}} \Gamma(Y - \mathfrak{m}, \mathcal{O}_Y^\times)_{(\chi^n)}.
\]

The homomorphism \(\text{dlog}: R^\times \to \Omega_{R}\) extends to an \(A/p\)-linear map
\[ A/p \otimes_{\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y^\times) \to \Omega_{R}. \]

Inspired by Okada’s construction \(\cite{12}\) of a Kummer homomorphism for function fields we prove in section \(\S\) the following result.

**Theorem 3.** If \(1 \leq n < q^d - 1\) then \(\lambda_n \in A/p \otimes_{\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y^\times)\) and the following are equivalent:

1. \(p\) divides \(BC_n\);
2. \(\text{dlog}\lambda_n\) lies in the kernel of \(A/p \otimes_k \Omega_{R} \to \Omega_{R}/\mathfrak{m}^d\Omega_{R}\).

It may (and does) happen that \(\lambda_n\) vanishes for some \(n\) divisible by \(q - 1\). However, the following theorem provides us with sufficient control over the vanishing of \(\lambda_n\).

**Theorem 4.** If \(n\) is divisible by \(q - 1\) but not by \(q^d - 1\) then the following are equivalent:

1. \(\lambda_n = 0\);
2. \(A/p \otimes_{\mathcal{F}_p} (\text{Pic} Y)[p](\chi^n) \neq 0\).
The proof is an adaptation of work of Galovich and Rosen [4], and uses $L$-functions in characteristic 0. It is given in section 9.

Assuming the three theorems above, we can now prove the main result.

**Proof of Theorem 4.** Assume $q - 1$ divides $n$ and $p$ divides $BC_n$. We need to show that $H^1(Y_{fl}, C[p]^D)(\chi^{1-n})$ is nonzero. Being a (component of) a differential logarithm $d\log \lambda_n$ is Cartier-invariant and Theorem 3 tells us that $d\log \lambda_n \in A/p \otimes_k \Omega_{R}^{c=1}(\chi^n)$ maps to 0 in $\Omega_R/\mathcal{P}^d\Omega_R$. If $\lambda_n \neq 0$ then by Theorem 2 we conclude that $H^1(Y_{fl}, C[p]^D)(\chi^{n-1})$ is nonzero and we are done. So assume that $\lambda_n = 0$. Consider the short exact sequence (2). By Theorem 4 we have that

$$\dim_{A/p} A/p \otimes_k \Gamma(Y, C^\wedge_1)^{\chi^n} \geq 1,$$

and since $A/p \otimes k \Gamma(Y, C^\wedge_1)\chi^n$ is one-dimensional, we find that

$$\dim_{A/p} A/p \otimes_k \Omega_{R}^{c=1}(\chi^n) \geq 2.$$

But $\Omega_R/\mathcal{P}^d\Omega_R(\chi^n)$ is one-dimensional, so it follows from Theorem 2 that $H^1(Y_{fl}, C[p]^D)(\chi^{n-1}) \neq 0$.

Conversely, assume that $q - 1$ divides $n$ and $p$ does not divide $BC_n$. Then Theorem 3 guarantees that $d\log \lambda_n$ is nonzero and it follows from Theorem 4 and the short exact sequence (2) that

$$\dim A/p \otimes_k \Omega_{R}^{c=1}(\chi^n) = 1.$$

Therefore $A/p \otimes_k \Omega_{R}^{c=1}(\chi^n)$ is generated by $d\log \lambda_n$ and since the image of $d\log \lambda_n$ in $\Omega_R/\mathcal{P}^d\Omega_R$ is nonzero we conclude from Theorem 2 that $H^1(Y_{fl}, C[p]^D)(\chi^{n-1})$ vanishes. □

5. Flat duality

In this section we summarize some of the results of Artin and Milne [3] on duality for flat cohomology in characteristic $p$.

Let $S$ be a scheme over $k$ and $\mathcal{V}$ a quasi-coherent $O_S$-module. Then the pull-back $F^*\mathcal{V}$ of $\mathcal{V}$ under $F$: $S \to S$ is a quasi-coherent $O_S$-module and there is a $k$-linear (typically not $O_S$-linear) isomorphism

$$F: \mathcal{V} \longrightarrow F^*\mathcal{V}$$

of sheaves on $S_{fl}$.

If $S$ is smooth of relative dimension 1 over $k$ then the $q$-Cartier operator induces a canonical map

$$c: \mathcal{H}om(F^*\mathcal{V}, \Omega_{S/k}) \longrightarrow \mathcal{H}om(\mathcal{V}, \Omega_{S/k})$$

of sheaves on $S_{ct}$.

Recall that we denote the canonical map $S_{fl} \to S_{ct}$ by $f$.

**Theorem 5** (Artin & Milne). Let $S$ be smooth of relative dimension 1 over $\text{Spec } k$. Let

$$0 \longrightarrow G \longrightarrow \mathcal{V} \xrightarrow{a} F^*\mathcal{V} \longrightarrow 0$$

be a short exact sequence of sheaves on $S_{fl}$ with

1. $\mathcal{V}$ a locally free coherent $O_S$-module;
(2) $\alpha : \mathcal{V} \to F^*\mathcal{V}$ a morphism of $\mathcal{O}_S$-modules. Then $G$ is a finite flat group scheme and there is a short exact sequence

$$(4) \quad 0 \to R^1f_*G^D \to \mathcal{H}_{\text{om}}(F^*\mathcal{V}, \Omega_{S/k}) \xrightarrow{\alpha} \mathcal{H}_{\text{om}}(\mathcal{V}, \Omega_{S/k}) \to 0$$

of sheaves on $S_{\text{et}}$, functorial in $\mathcal{V}$. Moreover, for all $i \neq 1$ one has $R^i f_* G^D = 0$.

**Proof.** Locally on $S$, we have that $G$ is given as a closed subgroup scheme of $\mathcal{G}_a$ defined by equations of the form $FX - \alpha X = 0$. In particular $G$ is flat of degree $q \text{rk} \mathcal{V}$. The Cartier dual $G^D$ of $G$ is a finite flat group scheme of height 1.

If $q$ is prime then the existence of (4) is shown in [3, §2]. One can deduce the general case from this as follows. Assume $n$ is a positive integer, and assume given a short exact sequence

$$0 \to G \to V \xrightarrow{\alpha F} (F^n)^*V \to 0$$

of sheaves on $S_{\text{fl}}$, with $\alpha : V \to (F^n)^*V$ an $\mathcal{O}_S$-linear map. Define $V' := V \oplus F^*V \oplus \cdots \oplus (F^{n-1})^*V$.

The map $\alpha$ induces an $\mathcal{O}_S$-linear map

$$\alpha' : V' \to F^*V'$$

defined by mapping the component $V$ to the component $(F^n)^*V$ using $\alpha$, and mapping all other components to zero. We thus have a short exact sequence

$$0 \to G \to V' \xrightarrow{\alpha' - F} F^*V' \to 0$$

and one deduces the theorem for $F^n$ from the theorem for $F$. \qed

**Example 5.1.** If $k = \mathbb{F}_p$ then the Artin-Schreier exact sequence

$$0 \to \mathbb{Z}/p \mathbb{Z} \to G_a \xrightarrow{1-F} G_a \to 0$$

on $S_{\text{fl}}$ induces a dual exact sequence

$$0 \to R^1 f_* \mu_p \to \Omega_{S/k} \xrightarrow{1-\xi} \Omega_{S/k} \to 0$$

on $S_{\text{et}}$, and the exact sequence

$$0 \to \alpha_p \to G_a \xrightarrow{1-\xi} G_a \to 0$$

on $S_{\text{fl}}$ induces a dual exact sequence

$$0 \to R^1 f_* \alpha_p \to \Omega_{S/k} \xrightarrow{1-\xi} \Omega_{S/k} \to 0$$

on $S_{\text{et}}$.

6. **Flat cohomology with $(A/p)^D$ coefficients**

The constant sheaf $A/p$ on $Y_{\text{fl}}$ has a resolution

$$0 \to A/p \to A/p \otimes_k G_{a,Y} \xrightarrow{1-\xi} A/p \otimes_k \Omega_{Y} \xrightarrow{1-\xi} A/p \otimes_k \Omega_{Y} \to 0$$

so by Theorem 5 we have $R^i f_*(A/p)^D = 0$ for $i \neq 1$, and $R^1 f_*(A/p)^D$ sits in a short exact sequence

$$1 \to R^1 f_*(A/p)^D \to A/p \otimes_k \Omega_{Y} \xrightarrow{1-\xi} A/p \otimes_k \Omega_{Y} \to 0$$

of sheaves on $Y_{\text{et}}$. Taking global sections now yields an isomorphism

$$H^1(Y_{\text{fl}}, (A/p)^D) \xrightarrow{\sim} A/p \otimes_k \Omega_{Y}^{e-1}_{k}.$$
where \( \Omega^{c=1}_{R/k} \) denotes the \( k \)-vector space of Cartier-invariant Kähler differentials.

On the other hand, we have a natural isomorphism

\[(A/p)^{D} \overset{\sim}{\longrightarrow} A/p \otimes_{F_{p}} \mu_{p},\]

of sheaves on \( Y_{\bar{R}} \) and the Kummer sequence

\[1 \longrightarrow \mu_{p} \longrightarrow \mathbb{G}_{m} \overset{p}{\longrightarrow} \mathbb{G}_{m} \longrightarrow 1\]

gives rise to a short exact sequence

\[(5) \quad 0 \longrightarrow A/p \otimes_{Z} \Gamma(Y, \mathcal{O}) \longrightarrow H^{1}(Y_{\bar{R}}, (A/p)^{D}) \longrightarrow A/p \otimes_{k} \Omega^{c=1}_{R} \rightarrow 0.\]

The proof of Theorem 5 shows that the resulting composed morphism

\[A/p \otimes_{k} \Gamma(Y, \mathcal{O}) \longrightarrow H^{2}(Y_{\bar{R}}, (A/p)^{D}) \overset{\sim}{\longrightarrow} A/p \otimes_{k} \Omega^{c=1}_{R}\]

is the map induced from

\[\text{dlog}: \Gamma(Y, \mathcal{O}) \longrightarrow \Omega^{c=1}_{R}; \quad u \mapsto \frac{du}{u},\]

so that (5) becomes the short exact sequence (2).

7. Comparing \((A/p)^{D}\) and \(C[p]^{D}\)-coefficients

Choose a nonzero torsion point \( \lambda \in C[p](L) \). Then \( \lambda \) defines a morphism \((A/p)_{Y} \to C[p]_{Y}\) and hence a morphism of Cartier duals

\[C[p]^{D}_{Y} \overset{\lambda}{\longrightarrow} (A/p)^{D}_{Y}.\]

Let \( \mathfrak{p} \in Y \) be the unique prime above \( p \subset A \). We have \( \mathfrak{p} = \mathfrak{R} \lambda \).

**Proposition 2.** The sequence

\[(6) \quad 0 \longrightarrow R^{1}f_{*}C[p]^{D}_{Y} \overset{\lambda}{\longrightarrow} R^{1}f_{*}(A/p)^{D} \longrightarrow \Omega_{Y}/\mathfrak{p}^{N}\Omega_{Y} \longrightarrow 0,\]

of sheaves on \( Y_{\bar{R}} \) is exact and if \( i \neq 1 \) then \( R^{i}f_{*}C[p]^{D} = 0 \).

Note that for all \( N \) the sheaf \( \Omega_{Y}/\mathfrak{p}^{N}\Omega_{Y} \) on \( Y_{\text{et}} \) is naturally a sheaf of \( A/p \)-modules. The middle map in the proposition is the composition

\[R^{1}f_{*}(A/p)^{D} \longrightarrow A/p \otimes_{k} \Omega_{Y} \longrightarrow \Omega_{Y}/\mathfrak{p}^{N}\Omega_{Y}.\]

Taking global sections in (6) we obtain an exact sequence of \( A/p \)-vector spaces

\[0 \longrightarrow H^{1}(Y_{\bar{R}}, C[p]^{D}_{Y}) \overset{\lambda}{\longrightarrow} A/p \otimes_{k} \Omega^{c=1}_{R} \longrightarrow \Omega_{R}/\mathfrak{p}^{N}\Omega_{R}\]

and considering the \( G \)-action on \( \lambda \) we see that Proposition 2 implies Theorem 2.

As one may expect, the proof of Proposition 2 relies on a careful analysis of the group scheme \( C[p]_{Y} \) near the prime \( \mathfrak{p} \).

Let \( \bar{s} \to Y \) be a geometric point lying above \( \mathfrak{p} \in Y \),

**Lemma 1.** There is an étale neighborhood \( V \to Y \) of \( \bar{s} \) and a short exact sequence

\[0 \longrightarrow C[p]_{V} \longrightarrow G_{n}^{d-1} \longrightarrow G_{n}^{d} \longrightarrow 0\]

of sheaves of \( A/p \)-vector spaces on \( V_{\bar{R}} \).
Proof. Let \( O_{Y,s} \) be the étale stalk of \( O_Y \) at \( s \) (a strict henselization of \( O_{Y,s} \)) and let \( S = \text{Spec} O_{Y,s} \). We have that \( C[p]_S \) is a finite flat \( A/p \)-vector space scheme of rank \( q^d \) over \( S \), étale over the generic fibre. Such vector space schemes have been classified by Raynaud [13] \S 1.5) (generalizing the results of Oort and Tate [17]). Let \( q = p^r \), with \( p = \text{char} \ k \), then the classification says that \( C[p]_S \) is a subgroupscheme of \( G_a^{q^d} \) given by equations
\[
X_i = a_i X_{i+1}
\]
for some \( a_i \in O_{Y,s} \), and where the index \( i \) runs over \( \mathbb{Z}/rd\mathbb{Z} \). Since the special fibre of \( C[p]_S \) is the kernel of \( F^d \) on \( G_a \), we find that all but one \( a_i \) are units. In particular, we can eliminate all but one variable and find that \( C[p]_S \) sits in a short exact sequence
\[
0 \to C[p]_S \to G_{a,S} \xrightarrow{a-F^d} G_{a,S} \to 0
\]
for some \( a \in O_{Y,s} \), well-defined up to a unit. We claim that \( a = \lambda^{q^d-1} \) (up to a unit). To see this, we compute the discriminant of the finite flat \( S \)-scheme \( C[p]_S \) in two ways. On the one hand \( C[p]_S \) is defined by the equation \( X^{q^d} - aX \), with discriminant \( a^{q^d} \) (modulo squares of units). On the other hand, \( C[p] \) is the \( p \)-torsion scheme of the Carlitz module and hence it is given by an equation
\[
X^{q^d} + b_{d-1} X^{q^d-1} + \ldots + b_0 X
\]
with \( b_i \in A \), and with \( b_0 \) a generator of \( p \). In this way we find that the discriminant equals \( b_0^{q^d} \) (modulo squares of units). Comparing the two expressions we conclude that we can take \( a = \lambda^{q^d-1} \), which proves the claim.

To finish the proof it suffices to observe that this short exact sequence is already defined over some étale neighbourhood \( V \to Y \) of \( s \). \( \square \)

Using this lemma we can now prove Proposition 2.

Proof of Proposition 2. Let \( V \) be as in the lemma and \( U := Y - \mathcal{Q} \). Then \( \{U, V\} \) is an étale cover of \( Y \) and it suffices to prove that the pull-backs of (6) to \( U_{\text{et}} \) and \( V_{\text{et}} \) are exact.

The pull-back to \( U_{\text{et}} \) is the sequence
\[
0 \to R^1 f_* C[p]_U^D \xrightarrow{\lambda} R^1 f_* (A/p)_U^D \to 0
\]
which is exact because \( \lambda : (A/p)_U \to C[p]_U \) is an isomorphism of sheaves on \( U_{\text{et}} \).

For the exactness over \( V_{\text{et}} \), consider the commutative square
\[
\begin{array}{ccc}
G_{a,V} & \xrightarrow{1-F^d} & G_{a,V} \\
\downarrow & & \downarrow \\
G_{a,V} & \xrightarrow{\lambda^{q^d-1-F^d}} & G_{a,V}
\end{array}
\]
It extends to a map of short exact sequences
\[
\begin{array}{ccccccccc}
0 & \to & (A/p)_V & \to & G_{a,V} & \xrightarrow{1-F^d} & G_{a,V} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C[p]_V & \to & G_{a,V} & \xrightarrow{\lambda^{q^d-1-F^d}} & G_{a,V} & \to & 0
\end{array}
\]
and without loss of generality we may assume that the leftmost vertical map is the
one induced by $\lambda$. Now Theorem 5 (with $k, F, S$ replaced by $A/p, F^d$ and $V$)
yields a commutative diagram of sheaves of $A/p$-vector spaces on $V_{\text{et}}$ with exact
rows:

$$
\begin{array}{c}
0 & \rightarrow & R^1f_*C[p]^D_V & \rightarrow & \Omega_V & \rightarrow & 0 \\
& & \downarrow & \lambda & & & \\
0 & \rightarrow & R^1f_*(A/p)^D_V & \rightarrow & \Omega_V & \rightarrow & 0
\end{array}
$$

(where by abuse of notation, we denote the canonical maps of sites $V_{\text{fl}} \rightarrow V_{\text{et}}$ and
$Y_{\text{fl}} \rightarrow Y_{\text{et}}$ by the same symbol $f$.) This shows that on $V_{\text{et}}$ we have an exact sequence

$$
0 \rightarrow R^1f_*C[p]^D_V \xrightarrow{\lambda} R^1f_*(A/p)^D_V \rightarrow \Omega_V/\lambda \Omega_V \rightarrow 0.
$$

so the pullback of (6) to $V_{\text{et}}$ is exact. □

8. A CANDIDATE COHOMOLOGY CLASS

Let $\lambda \in R$ be a primitive $p$-torsion point of the Carlitz module. Consider the
decomposition

$$
1 \otimes \lambda = \sum_{n=1}^{q^d-1} \lambda_n
$$

in $A/p \otimes \mathbb{Z} \Gamma(Y - \mathfrak{P}, \mathcal{O}_Y^\times)$. In this section we will prove Theorem 3 which states
that for $1 \leq n < q^d - 1$ we have

$$
\lambda_n \in A/p \otimes \Gamma(Y, \mathcal{O}_Y^\times)
$$

and that the following are equivalent

1. $p$ divides $BC_n$;
2. $d\log \lambda_n$ lies in the kernel of $A/p \otimes_k \Omega_R \rightarrow \Omega_R/\mathfrak{P}^{q^d} \Omega_R$.

We start with the first assertion.

**Proposition 3.** If $1 \leq n < q^d - 1$ then $\lambda_n \in A/p \otimes \mathbb{Z} \Gamma(Y, \mathcal{O}_Y^\times)$.

**Proof.** For all integers $n$ we have

$$
\lambda_n = -\sum_{g \in G} \chi(g)^{-n} \otimes g\lambda.
$$

If moreover $n$ is not divisible by $q^d - 1$ then $\sum_{g \in G} \chi(g)^{-n} = 0$ so that we can rewrite the above identity as

$$
\lambda_n = -\sum_{g \in G} \chi(g)^{-n} \otimes \frac{g\lambda}{\lambda}.
$$

Since the point $\mathfrak{P}$ is fixed under $G$ it follows that for all $g \in G$ one has that $g\lambda/\lambda$
has valuation 0 at $\mathfrak{P}$ and therefore for all $1 \leq n < q^d - 1$ we have

$$
\lambda_n \in A/p \otimes \Gamma(Y, \mathcal{O}_Y^\times),
$$

as was claimed. □

Now let $L_{\mathfrak{P}}$ be the completion of $L$ at $\mathfrak{P}$ and $\mathfrak{m}$ the maximal ideal of its valuation
ring $\mathcal{O}_{Y, \mathfrak{P}}$. Note that $\mathfrak{m} = (\lambda)$.

Consider the quotient $\mathfrak{m}/\mathfrak{m}^{q^d}$. It carries two $A$-module structures:
the linear action coming from the $A$-algebra structure of $O_{Y,p}^\circ$;
(2) the Carlitz action defined using $\varphi$.

Also, the Galois group $G$ acts on $m/m^q$ and the action commutes with both $A$-module structures.

**Lemma 2.** Both actions on $m/m^q$ factor over $A/p$.

**Proof.** Note that $pO_{Y,p}^\circ = m^{q^d-1}$. In particular the assertion is immediate for the linear action. For the Carlitz action, consider a generator $f$ of $p$. Then
\[ \varphi(f) = a_0 + a_1 F + \cdots + a_{d-1} F^{d-1} + F^d \]
with $a_i \in p$ for all $i$. From this it follows that $\varphi(f)$ maps $m \subset O_{Y,p}^\circ$ into $m^{q^d}$, as desired.

The Carlitz exponential series
\[ e(z) = \sum_{n=1}^{\infty} e_n z^n \in K[[z]] \]
has the property that for all $n < q^d$ the coefficient $e_n$ is $p$-integral, so the truncated and reduced exponential power series
\[ \bar{e}(z) = \sum_{n=1}^{q^d-1} e_n z^n \in (A/p)[[z]]/(z^{q^d}) \]
defines a $k$-linear map
\[ \bar{e} : m/m^{q^d} \to m/m^{q^d} \]
which is an isomorphism because it induces the identity map on the intermediate quotients $m^i/m^{i+1}$. Note that $\bar{e}$ is $G$-equivariant, as the coefficients $e_i$ of the Carlitz exponential lie in $K$.

**Lemma 3.** For all $x \in m/m^{q^d}$ and $a \in A$ we have $\bar{e}(ax) = \varphi(a)\bar{e}(x)$.

**Proof.** In $K[[z]]$ we have the identity
\[ e(tz) = te(z) + e(z)^q \]
of formal power series. Identifying coefficients on both sides we find that in $(A/p)[[z]]/(z^{q^d})$ we have
\[ \bar{e}(tz) = t\bar{e}(z) + \bar{e}(z)^q, \]
and we deduce that for all $a \in A$ and $x \in m/m^{q^d}$ we have $\bar{e}(ax) = \varphi(a)\bar{e}(x)$. □

Put $\bar{\pi} := \bar{e}^{-1}(\bar{\lambda})$, where $\bar{\lambda}$ is the image of $\lambda \in m$ in $m/m^{q^d}$.

**Lemma 4.** For all $g \in G$ we have $g\bar{\pi} = \chi(g)\bar{\pi}$.

In other words $\bar{\pi} \in m/m^{q^d}(\chi)$.

**Proof of Lemma 4** Let $g \in G$ and $a \in A$ be so that $a$ reduces to $g$ in $G = (A/p)^\times$. Since $\lambda$ is a $p$-torsion point of the Carlitz module we have that
\[ g\bar{\lambda} = \varphi(a)\bar{\lambda}. \]
Applying $\bar{e}^{-1}$ to both sides we find with Lemma 3 that
\[ g\bar{\pi} = a\bar{\pi} \]
and by definition $a\bar{\pi}$ equals $\chi(g)\bar{\pi}$.

□
Choose a lift \( \pi \in m \) of \( \bar{\pi} \) such that \( g\pi = \chi(g)\pi \) for all \( g \). Then \( \pi \) is a uniformizing element of \( Lp \).

**Proposition 4.** Let \( 1 \leq n < q^d - 1 \). Then
\[
d\log \lambda_n = (BC_n\pi^n + \delta) d\log \pi
\]
for some \( \delta \in m^{n+q^d-1} \).

**Proof.** Since \( \bar{\lambda} = \bar{\chi}(\bar{\pi}) \) we have in \( \mathcal{O}_{Y,p}^\chi \) the identity
\[
\lambda = \sum_{n=1}^{q^d-1} e_n\pi^n + \delta_1
\]
for some \( \delta_1 \in m^{q^d} \). Since \( d\pi^n = 0 \) for any \( n \) divisible by \( q \) we find
\[
d\lambda = (1 + \delta_2)d\pi
\]
for some \( \delta_2 \in m^{q^d} \). Dividing both expressions we find
\[
d\log \lambda = \left( \sum_{n=0}^{q^d-2} BC_n\pi^n + \delta_3 \right) d\log \pi
\]
for some \( \delta_3 \in m^{q^d-1} \). Now the proposition follows from decomposing this identity in isotypical components, since \( d\log \pi \) is \( G \)-invariant and \( g\pi = \chi(g)\pi \) for all \( g \in G \). \( \Box \)

We can now finish the proof of Theorem 3.

**Proof of Theorem 3.** If \( n > 1 \) then the Theorem follows from the above proposition. If \( n = 1 \) we consider two cases. Either \( q > 2 \) and then \( BC_1 = 0 \) and \( d\log \lambda_1 = 0 \), or else \( q = 2 \) and then \( p \) does not divide \( BC_1 \) and from the above \( \pi \)-adic expansion we see that \( d\log \lambda_1 \) does not map to zero in \( \Omega_R/\mathfrak{p}^{q^d}\Omega_R \). In both cases the theorem holds. \( \Box \)

### 9. Vanishing of \( \lambda_n \)

Let \( W \) be the ring of Witt vectors of \( A/p \). For \( a \in (A/p)^x \) we denote by \( \hat{a} \in W^x \) the Teichmüller lift of \( a \). Also, we denote by \( \hat{\chi}: G \to W^x \) the Teichmüller lift of the character \( \chi: G \to (A/p)^x \). If \( M \) is a \( W[G] \)-module then it decomposes into isotypical components
\[
M = \bigoplus_{n=1}^{q^d-1} M(\hat{\chi}^n)
\]
with \( G \) acting via \( \hat{\chi}^n \) on \( M(\hat{\chi}^n) \).

Put \( U := W \otimes_{\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y^\chi) \) and let \( D \) be the \( W \)-module of degree zero \( W \)-divisors on \( X - Y \). Then we have a natural inclusion \( U \hookrightarrow D \) with finite quotient. Consider the decomposition of \( 1 \otimes \lambda \in W \otimes \Gamma(Y - \mathcal{P}, \mathcal{O}_Y^\chi) \) in isotypical components:
\[
1 \otimes \lambda = \sum_{n=1}^{q^d-1} \hat{\lambda}_n \quad \text{with} \quad \hat{\lambda}_n \in W \otimes_{\mathbb{Z}} \Gamma(Y - \mathcal{P}, \mathcal{O}_Y^\chi)(\hat{\chi}^n).
\]
We have
\[
\hat{\lambda}_n = \sum_{g \in G} \chi(g)^{-n} \otimes g\lambda
\]
and for $1 < n < q^d - 1$ we have that $\tilde{\lambda}_n$ lies in $U(\tilde{\chi}^n)$ and it maps to $\lambda_n$ under the reduction map 

$$U \longrightarrow A/p \otimes \mathbb{Z} \Gamma(Y, O_Y^\times).$$

If $n$ is divisible by $q - 1$ but not by $q^d - 1$, the $W$-modules $D(\tilde{\chi}^n)$ and $U(\tilde{\chi}^n)$ are free of rank one. In particular

$$\lambda_n = 0 \text{ if and only if } \frac{U(\tilde{\chi}^n)}{W\lambda_n} \neq 0,$$

and Theorem 4 follows from the following.

**Proposition 5.** Let $n$ be divisible by $q - 1$ but not by $q^d - 1$. Then the finite $W$-modules

$$\frac{U(\tilde{\chi}^n)}{W\lambda_n}$$

and

$$W \otimes \mathbb{Z} \text{Pic}_Y(\tilde{\chi}^n)$$

have the same length.

**Proof.** Let $X$ be the canonical compactification of $Y$. Since we have a short exact sequence of $W$-modules

$$0 \longrightarrow \frac{D(\tilde{\chi}^n)}{U(\tilde{\chi}^n)} \longrightarrow W \otimes \mathbb{Z} (\text{Pic}^0 X)(\tilde{\chi}^n) \longrightarrow W \otimes \mathbb{Z} (\text{Pic} Y)(\tilde{\chi}^n) \longrightarrow 0,$$

it suffices to show that

$$\frac{D(\tilde{\chi}^n)}{W\lambda_n} \text{ and } W \otimes \mathbb{Z} (\text{Pic}^0 X)(\tilde{\chi}^n)$$

have the same length. By Goss and Sinnott [8] the length of $W \otimes \mathbb{Z} (\text{Pic}^0 X)(\tilde{\chi}^n)$ is the $p$-adic valuation of $L(1, \tilde{\chi}^{-n}) \in W$. We will show that also the length of $D(\tilde{\chi}^n)/W\lambda_n$ equals the $p$-adic valuation of $L(1, \tilde{\chi}^{-n})$.

Since $n$ is divisible by $q - 1$, the representation $\tilde{\chi}^{-n}$ is unramified at $\infty$. Since all the points of $X$ lying above $\infty$ are $k$-rational, the local $L$-factor at $\infty$ of $L(T, \tilde{\chi}^{-n})$ is $(1 - T)^{-1}$. Since $n$ is not divisible by $q^d - 1$, the representation is ramified at $p$ and hence the local $L$-factor at $p$ is $1$. Recall that for a prime $q \subset A$ coprime with $p$ we have that $\chi(\text{Frob}_q)$ is the image of the monic generator of $q$ in $(A/p)^\times$. Together with unique factorization in $A$ we obtain

$$L(T, \tilde{\chi}^{-n}) = (1 - T)^{-1} \sum_{a \in A_+, a \not\in p} \tilde{a}^{-n} T^{\deg a},$$

where $A_+$ is the set of monic elements of $A$. In fact it is easy to see that for $m \geq d$ the coefficient of $T^m$ in the sum vanishes, so we have

$$L(T, \chi^{-n}) = (1 - T)^{-1} \sum_{a \in A_+^{<d}} \tilde{a}^{-n} T^{\deg a},$$

where $A_+^{<d}$ is the set of monic elements of degree smaller than $d$.

Since $n$ is divisible by $q - 1$ we have

$$\sum_{a \in A_+^{<d}} \tilde{a}^{-n} T^{\deg a} = \frac{1}{q - 1} \sum_{a \in A^{<d}} \tilde{a}^{-n} T^{\deg a}.$$
We conclude from (7) that
\[
L(1, \chi^{-n}) = \frac{1}{q-1} \sum_{a \in A_{<d}} (\deg a) \tilde{a}^{-n}.
\]

Consider the function
\[
\deg : G \to \{0, 1, \ldots, d-1\}
\]
which maps \(g \in G\) to the degree of its unique representative in \(A_{<d}\). Then the above identity can be rewritten as
\[
L(1, \chi^{-n}) = \frac{1}{q-1} \sum_{g \in G} (\deg g) \tilde{g}^{-n}.
\]

By [4, p. 372] there is a point in \(X - Y\) with associated valuation \(v\) and integers \(u, w\) with \((u, p) = 1\) such that
\[
v(g\lambda) = u \deg g + w
\]
for all \(g \in G\). The valuation \(v\) extends to an isomorphism of \(W\)-modules
\[
v : D(\chi^n) \to W,
\]
and we have
\[
v(\lambda_n) = \sum_{g \in G} \tilde{g}^{-n} v(g\lambda) = u(q-1)L(1, \chi^{-n}) + w \sum_{g \in G} \tilde{g}^{-n} = u(q-1)L(1, \chi^{-n}).
\]

In particular, the length of \(D(\chi^n)/\lambda_n\) is the \(p\)-adic valuation of \(L(1, \chi^{-n})\) and the proposition follows. \(\square\)

10. Complement: the class module of \(Y\)

Let \(L\) be an arbitrary finite extension of \(K\) and \(R\) the integral closure of \(A\) in \(L\). Put \(Y = \text{Spec } R\). In [10] and [15] we have given several equivalent definitions of a finite \(A\)-module \(H(C/Y)\) depending on \(Y\), that is analogous to the class group of a number field. One of these definitions is the following.

Let \(X\) be the canonical compactification of \(Y\) and let \(\infty\) be the divisor on \(X\) of zeroes of \(1/t \in L\). (This is also the inverse image of the divisor \(\infty\) on \(\mathbb{P}^1\).) Then \(H(C/Y)\) is defined by the exact sequence
\[
A \otimes_k H^1(X, \mathcal{O}_X) \xrightarrow{\partial} A \otimes_k H^1(X, \mathcal{O}_X(\infty)) \to H(C/Y) \to 0,
\]
where
\[
\partial = 1 \otimes (t + F) - t \otimes 1.
\]

**Theorem 6.** Let \(I \subset A\) be a nonzero ideal. Then there is a natural isomorphism
\[
H^1(Y_\text{fl}, C[I]^D) \cong H(C/Y) \otimes_A A/I
\]
where \((-)^\vee\) denotes the \(k\)-linear dual.
Proof. The starting point of the proof is the exact sequence of sheaves of $A$-modules
\[ 0 \rightarrow A \otimes_k \mathbb{G}_a \xrightarrow{\partial} A \otimes_k \mathbb{G}_a \xrightarrow{\alpha} C \rightarrow 0 \]
with $\partial(a \otimes f) = a \otimes (f^q + tf) - ta \otimes f$ and with $\alpha(a \otimes f) = \varphi(a)f$. From this we derive a short exact sequence
\[ 0 \rightarrow C[I]_Y \rightarrow A/I \otimes_k \mathbb{G}_a \xrightarrow{\partial} A/I \otimes_k \mathbb{G}_a \rightarrow 0. \]
Using Theorem 5 we obtain a dual resolution:
\[ 0 \rightarrow R^i f_* C[I]^D \rightarrow A/I \otimes_k \Omega_Y \xrightarrow{\partial^*} A/I \otimes_k \Omega_Y \rightarrow 0 \]
of sheaves of $A$-modules on $Y_{et}$, where $\partial^* = 1 \otimes (t+c) - t \otimes 1$. Since $R^i f_* C[I]^D = 0$ for $i \neq 1$, taking global sections we obtain an exact sequence of $A$-modules
\[ 0 \rightarrow H^1(Y_{fl}, C[I]^D) \rightarrow A/I \otimes_k \Gamma(Y, \Omega_Y) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(Y, \Omega_Y). \]

Now we claim that the natural inclusion of the complex
\[ A/I \otimes_k \Gamma(X, \Omega_X(-\infty)) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(X, \Omega_X) \]
in the complex
\[ A/I \otimes_k \Gamma(Y, \Omega_Y) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(Y, \Omega_Y) \]
is a quasi-isomorphism. Indeed, the quotient has a filtration with intermediate quotients of the form
\[ A/I \otimes_k \frac{\Gamma(X, \Omega_X(n\infty))}{\Gamma(X, \Omega_X((n-1)\infty))} \xrightarrow{\partial^*} A/I \otimes_k \frac{\Gamma(X, \Omega_X(n+1\infty))}{\Gamma(X, \Omega_X(n\infty))} \]
with $n \in \mathbb{Z}_{\geq 0}$. On these intermediate quotients we have that $1 \otimes c$ and $t \otimes 1$ are zero, so that $\partial^* = 1 \otimes t$, which is an isomorphism.

Hence we obtain from (9) a new exact sequence
\[ 0 \rightarrow H^1(Y_{fl}, C[I]^D) \rightarrow A/I \otimes_k \Gamma(X, \Omega_X(-\infty)) \xrightarrow{\partial^*} A/I \otimes_k \Gamma(X, \Omega_X). \]
Under Serre duality the $q$-Cartier operator $c$ on $\Omega_X$ is adjoint to the $q$-Frobenius $F$ on $\mathcal{O}_X$, so we obtain a dual exact sequence
\[ A/I \otimes_k H^1(X, \mathcal{O}_X) \xrightarrow{\partial} A/I \otimes_k H^1(X, \mathcal{O}_X(\infty)) \rightarrow H^1(Y_{fl}, C[I]^D)^{\vee} \rightarrow 0. \]
Theorem 5 now follows by comparing this sequence with the sequence obtained by reducing (8) modulo $I$. \qed

References

[1] Greg W. Anderson. Log-algebraicity of twisted $A$-harmonic series and special values of $L$-series in characteristic $p$. J. Number Theory, 60(1):165–209, 1996. MR1405732
[2] Bruno Anglès. On Gekeler’s conjecture for function fields. J. Number Theory, 87(2):242–252, 2001. MR1824146
[3] M. Artin and J. S. Milne. Duality in the flat cohomology of curves. Invent. Math., 35:111–129, 1976. MR0419350
[4] Steven Galovich and Michael Rosen. The class number of cyclotomic function fields. J. Number Theory, 13(3):363–375, 1981. MR642206
[5] Ernst-Ulrich Gekeler. On regularity of small primes in function fields. J. Number Theory, 34(1):114–127, 1990. MR1039771
[6] David Goss. Analogies between global fields. In Number theory (Montreal, Que., 1985), volume 7 of CMS Conf. Proc., pages 83–114. Amer. Math. Soc., Providence, RI, 1987. [MR894321]

[7] David Goss. Basic structures of function field arithmetic, volume 35 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, Berlin, 1996. [MR1423131]

[8] David Goss and Warren Sinnott. Class-groups of function fields. Duke Math. J., 52(2):507–516, 1985. [MR792185]

[9] J. Herbrand. Sur les classes des corps circulaires. J. Math. Pures Appl., IX. Sér., 11:417–441, 1932.

[10] Barry Mazur. Notes on étale cohomology of number fields. Ann. Sci. École Norm. Sup. (4), 6:521–552 (1974), 1973. [MR0344251]

[11] J. S. Milne. Arithmetic duality theorems, volume 1 of Perspectives in Mathematics. Academic Press Inc., Boston, MA, 1986. [MR881804]

[12] Shozo Okada. Kummer’s theory for function fields. J. Number Theory, 38(2):212–215, 1991. [MR1111373]

[13] Michel Raynaud. Schémas en groupes de type $(p,\ldots,p)$. Bull. Soc. Math. France, 102:241–280, 1974. [MR0419467]

[14] Kenneth A. Ribet. A modular construction of unramified $p$-extensions of $\mathbb{Q}(\mu_p)$. Invent. Math., 34(3):151–162, 1976. [MR0419403]

[15] Lenny Taelman. The Carlitz shtuka. J. Number Theory, 131(3):410–418, 2011. [MR2739613]

[16] Lenny Taelman. Special $L$-values of Drinfeld modules. To appear in Annals of Math., 2011.

[17] John Tate and Frans Oort. Group schemes of prime order. Ann. Sci. École Norm. Sup. (4), 3:1–21, 1970. [MR0255368]

[18] Lawrence C. Washington. Introduction to cyclotomic fields, volume 83 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1997. [MR1421575]

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