ON $\tau$-TILTING FINITENESS OF BLOCK ALGEBRAS OF DIRECT PRODUCTS OF FINITE GROUPS

YUTA KOZAKAI

Abstract. We discuss finiteness/infiniteness of $\tau$-tilting modules over tensor products of two symmetric algebras. As an application, we consider block algebras of direct products of finite groups.

1. Introduction

In this paper, by an algebra we mean a finite-dimensional algebra over a fixed algebraically closed field $k$. In 2014, the notion of support $\tau$-tilting modules was introduced by Adachi, Iyama and Reiten ([1]), and has been studied since by many researchers. For example, the support $\tau$-tilting modules are in bijection with many representation theoretical objects such as two-term silting complexes ([1]), functorially finite torsion classes ([1]), left finite semibricks ([6]), and $t$-structures ([12]) and more. Since the notion of $\tau$-tilting finiteness/infiniteness was introduced by Demonet, Iyama and Jasso [9], several cases of $\tau$-tilting finite/infinite algebras have been verified. For example the $\tau$-tilting finiteness/infiniteness of the following classes of algebras are verified: preprojective algebras of Dynkin type ([19]), radical square zero algebras ([2]), cycle finite algebras ([17]), Brauer graph algebras ([3]), tame blocks of group algebras ([10]) biserial algebras ([20]), simply connected algebras ([17]), tilted and cluster tilted algebras ([22]), certain block algebras of finite groups covering cyclic blocks ([13, 14, 15]), triangular matrix algebras ([4]), certain self-injective algebras of tubular type ([3]), tensor product algebras of simply connected algebras ([18]), classical Schur algebras ([24]) and other kinds of $\tau$-tilting finite algebras can be seen in the list in Section 10 of [21]. Our aim is to give such examples among block algebras of finite groups. In fact, there are few studies associated to both $\tau$-tilting theory and modular representation theory nevertheless the former has been developing rapidly since it was introduced. Therefore we consider the $\tau$-tilting finiteness/infiniteness of block algebras of direct products of finite groups. In order to do so, we investigate more generally the $\tau$-tilting finiteness/infiniteness of the tensor products of two symmetric algebras.

Theorem 1.1 (See Theorem 3.4). Let $A$ and $B$ be indecomposable symmetric algebras over an algebraically closed field $k$ which are not local algebras. Then the tensor product algebra $A \otimes_k B$ is a $\tau$-tilting infinite algebra.

Finally, combining this result and previous work on $\tau$-tilting theory, we get the following result on the $\tau$-tilting finiteness/infiniteness of block algebras of direct products of finite groups.

Theorem 1.2 (See Theorem 3.7). Let $G$ and $H$ be finite groups, $k$ an algebraically closed field and $B$ a block of the group algebra $k[G \times H]$ of the direct product of $G$ and $H$. If the block algebra

2020 Mathematics Subject Classification. 16G20, 20C20.
Key words and phrases. $\tau$-tilting finiteness, block algebras of finite groups, tensor products of symmetric algebras.
A is τ-tilting finite, then there is a block A of kG or of kH such that the set of support τ-tilting modules over B is isomorphic to that of A as posets.

Throughout this paper, the symbol k denotes an algebraically closed field. Algebras are always finite-dimensional over k and tensor products are always taken over k. For the notation of quivers and the basic results on k-linear representations of quivers, we refer to the book [7]. In particular, for a finite dimensional algebra A, we denote by \( Q_A \) the Ext-quiver of A and by \( \mathcal{I}_A \) the ideal of \( kQ_A \) with \( A \cong kQ_A/\mathcal{I}_A \).

2. Tensor products of algebras

In this section we recall basic definitions and properties of tensor products of algebras and their structures of \( k \)-algebras. Let \( A \cong kQ_A/\mathcal{I}_A \) and \( B \cong kQ_B/\mathcal{I}_B \) be algebras. Then their tensor product \( A \otimes B := A \otimes_k B \) is again a \( k \)-algebra with the multiplication induced by the rule \((a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2\). To determine the structure of this \( k \)-algebra, the following theorem is essential.

**Theorem 2.1** ([18, Lemma 1.3]). Let \( Q_{A \otimes B} \) be the following quiver

\[
(Q_{A \otimes B})_0 = (Q_A)_0 \times (Q_B)_0, (Q_{A \otimes B})_1 = ((Q_A)_1 \times (Q_B)_0) \cup ((Q_A)_0 \times (Q_B)_1).
\]

Let \( \mathcal{I}_{A \otimes B} \) be an ideal of \( kQ_{A \otimes B} \) generated by all elements in \( (\mathcal{I}_A \times (Q_B)_0) \cup ((Q_A)_0 \times \mathcal{I}_B) \) and by all elements in \( \{(e, e') - (e, e')(\alpha, \beta) \mid e \in (Q_A)_0, e' \in (Q_B)_0, \alpha \in (Q_A)_1, \beta \in (Q_B)_1\} \). Then there is a \( k \)-algebra isomorphism

\[
A \otimes B \cong kQ_{A \otimes B}/\mathcal{I}_{A \otimes B}.
\]

**Example 2.2.** Let \( A \) be given by the quiver

\[
e_1 \overset{\alpha_1}{\underset{\alpha_2}{\longrightarrow}} e_2
\]

bound by \( \alpha^3 = 0 \), and \( B \) be given by the quiver

\[
\begin{array}{ccc}
& e_1' & \\
\beta_1 & & \beta_3 \\
\beta_2 & & \\
e_2' & & e_3'
\end{array}
\]

bound by \( \beta^4 = 0 \). Then the quiver of the tensor product \( A \otimes B \) is given by
bound by \((\alpha, e')^3 = 0, (e, \beta)^4 = 0\) and \((\alpha, e')(e, \beta) = (e, \beta)(\alpha, e')\).

3. Main Theorem

In this section we state the main theorems stated in Section \[\text{I}\] and give their proofs. Let \(A\) be an algebra. We recall that an \(A\)-module \(M\) is a brick if \(\text{End}_A(M, M) \cong k\). Moreover we recall that the algebra \(A\) is \(\tau\)-tilting finite if and only if there are only finitely many isomorphism classes of bricks over \(A\) (see [9, Theorem 1.4]). We prepare the following lemmas.

Lemma 3.1. Let \(A\) be an algebra and \(U\) an indecomposable \(A\)-module. For the indecomposable decomposition \(\text{soc} \, U = \bigoplus_i S_i\) of \(\text{soc} \, U\), if all \(S_i\) are non-isomorphic to each other and each \(S_i\) does not appear as a composition factor of \(U/\text{soc} \, U\), then the \(A\)-module \(U\) is a brick.

Proof. To prove that the \(A\)-module \(U\) is a brick, assume that there is a nonzero endomorphism \(f : U \to U\) which is not isomorphism. Since \(\text{End}_A(U)\) is a local algebra by the indecomposability of \(U\), the endomorphism \(f\) is nilpotent. Also since the endomorphism \(f\) is nonzero, we have that \(0 \neq f(U) \leq U\). In particular, we have that \(0 \neq \text{soc} \, f(U) \leq \text{soc} \, U\). Take an indecomposable summand \(S\) of \(\text{soc} \, f(U)\), then it is also an indecomposable summand of \(\text{soc} \, U\) too. Now, if we take a minimal submodule \(V\) of \(U\) with \(f(V) = S\), it holds that \(\text{top} \, V \cong S\). Since the simple \(A\)-module \(S\) appears as a composition factor of \(U\) only one time and it is a direct summand of \(\text{soc} \, U\), the \(A\)-submodule \(V\) of \(U\) must be \(S\). Hence we have that \(f(S) = S\), which implies that \(f^n(S) = S\) for all positive integer \(n\), but this contradicts the fact that \(f\) is nilpotent. Therefore all nonzero endomorphisms of \(U\) are isomorphisms. \(\square\)

Lemma 3.2. Let \(A \cong kQ_A/I_A\) be a non-local indecomposable symmetric algebra, where \(Q_A\) is an Ext-quiver of \(A\) and \(I_A\) is a relation ideal. Then there exists a path

\[
(1 \overset{\beta_2^{(1)}}{\longrightarrow} \cdots \overset{\beta_2^{(t_1-1)}}{\longrightarrow} 1 \overset{\beta_2^{(t_1)}}{\longrightarrow} 2 \overset{\beta_2^{(t_2-1)}}{\longrightarrow} \cdots \overset{\beta_2^{(t_2)}}{\longrightarrow} 2 \overset{\beta_2^{(t_3-1)}}{\longrightarrow} 3 \overset{\beta_2^{(t_3)}}{\longrightarrow} \cdots \overset{\beta_2^{(t_{m-1}-1)}}{\longrightarrow} (m) \overset{\beta_2^{(t_{m-1})}}{\longrightarrow} (m) \overset{\beta_2^{(t_{m-1})}}{\longrightarrow} 1)
\]

which is not a loop in \(Q_A\) with \(\beta_2^{(1)}(1) \beta_2^{(2)} \cdots \beta_2^{(t_1-1)} \beta_2^{(t_1)} \beta_2^{(t_2)} \beta_2^{(2)} \cdots \beta_2^{(t_2)} \beta_2^{(t_3)} \beta_2^{(2)} \cdots \beta_2^{(t_3)} \beta_2^{(t_4)} \beta_2^{(2)} \cdots \beta_2^{(t_4)} \beta_2^{(t_5)} \cdots \beta_2^{(t_{m-1})} \beta_2^{(t_{m-1})} \beta_2^{(t_m)} \notin \mathcal{I}_A\).

Proof. Since \(A\) is a non-local symmetric algebra, for any indecomposable projective \(A\)-module \(P\) we have that \(P/\text{rad} \, P \cong \text{soc} \, P\) and that \(\text{rad} \, P/\text{soc} \, P\) contains a simple \(A\)-module distinct to \(\text{soc} \, P\). Hence we can take a path satisfying the following conditions:

(1) it is not a loop.
(2) the source coincides with the target.
(3) the path is not in \(\mathcal{I}_A\).

Here, if there is a path with satisfying the three properties within the path, we retake the shorter path. By repeating this, we will get the desired path. \(\square\)

Remark 3.3. Without the assumption that the algebra \(A\) is symmetric, Lemma 3.2 does not hold in general. For example, for the self-injective algebra \(A := 1 \overset{\alpha}{\longrightarrow} 2 \overset{\beta}{\longrightarrow} /\langle \alpha \beta, \beta \alpha \rangle\) which is a not symmetric algebra, there cannot be a path as the one stated in Lemma 3.2.
We are now ready to state and prove our result on tensor products of symmetric algebras. For an algebra $A \cong kQ_A/\mathcal{I}_A$, we denote by $\text{rep}_k(Q, \mathcal{I})$ the category of finite dimensional $k$-linear representation of $Q_A$ bound by $\mathcal{I}_A$ (see [7, Section III]).

**Theorem 3.4.** Let $A$ and $B$ be indecomposable symmetric algebras over an algebraically closed field $k$ which are not local algebras. Then the tensor product algebra $A \otimes B$ is a $\tau$-tilting infinite algebra.

**Proof.** If there is a multiple arrow which is not a loop in $Q_A$ or $Q_B$, then we can easily show that $A \otimes B$ is a $\tau$-tilting infinite algebra (for example, see [8, Corollary 1.9] or [11, Main Theorem 2]) so we may assume that there are no multiple arrows which are not loops in $Q_A$ and $Q_B$. Since the algebras $A$ and $B$ are symmetric algebras which are not local algebras, by Lemma [3, 2] we can take a path $C_A := (1 \xrightarrow{\alpha_0} i_1 \xrightarrow{\alpha_1} i_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{i-1}} i_i = n \xrightarrow{\alpha_i} 1)$ in $Q_A$ satisfying the following conditions:

1. $1 \leq i_1 \leq i_2 \leq \cdots \leq i_i = n$ and $i_{c+1} - i_c$ is equal to 0 or 1 for $0 \leq c \leq l - 1$.
2. the path is not a loop (hence $1 \neq n$).
3. $\alpha_{i_0} \alpha_{i_1} \cdots \alpha_{i_i} \not\in \mathcal{I}_A$.

We can take a similar path $C_B := (1 \xrightarrow{\beta_{j_0}} j_1 \xrightarrow{\beta_{j_1}} j_2 \xrightarrow{\beta_{j_2}} \cdots \xrightarrow{\beta_{j_{m-1}}} j_m \xrightarrow{\beta_{j_m}} 1)$ in $Q_B$. We denote a unique arrow $i \rightarrow i + 1$ in $Q_A$ by $\alpha_i$ and a unique arrow $j \rightarrow j + 1$ in $Q_B$ by $\beta_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. In these setting, we will construct a brick $M^{(\lambda)}$ over $A \otimes B$ parameterized by the nonzero element $\lambda$ in $k$, and show that $M^{(\lambda)}$ is not isomorphic to $M^{(\mu)}$ if $\lambda$ is different from $\mu$. This shows that the tensor product algebra $A \otimes B$ is a $\tau$-tilting infinite algebra by [9, Theorem 1.4].

First, for an arbitrary nonzero element $\lambda$ in $k$ we define $M^{(\lambda)}$ by constructing a $k$-linear representation $M^{(\lambda)} = (M^{(\lambda)}_{(i,j)}, \phi^{(\lambda)}_{(\alpha,\beta)})_{(i,j) \in (Q_{A \otimes B})_0, (\alpha,\beta) \in (Q_{A \otimes B})_1}$ corresponding to the module $M^{(\lambda)}$ in the category $\text{rep}_k(Q_{A \otimes B}, \mathcal{I}_{A \otimes B})$ of finite dimensional $k$-linear representations of $Q_{A \otimes B}$ bound by $\mathcal{I}_{A \otimes B}$. Without loss of generality, we may suppose that $n \leq m$. For the projective $B$-module $P(T_1)$ corresponding to the simple $B$-module $T_1$, let $U$ be a minimal quotient of $P(T_1)$ such that the simple $B$-module $T_{m-n+2}$ appears as a composition factor just one time (we remark that we can choose such $U$ because $\beta_{j_0}\beta_{j_1}\cdots\beta_{j_m} \not\in \mathcal{I}_B$). Then we remark that the socle of $U$ is isomorphic to $T_{m-n+2}$, so the socle of $S_1 \otimes U$ is isomorphic to $S_1 \otimes T_{m-n+2}$ for the simple $A$-module $S_1$ and that $S_1 \otimes T_{m-n+2}$ appears as a composition factor of $S_1 \otimes U$ just one time. Hence in the $k$-linear representation corresponding to $S_1 \otimes U$, the vertex $(1, m-n+2)$ in $(Q_{A \otimes B})_0$ is a sink and is associated to 1-dimensional vector space $k$. We will construct the $k$-linear representation $M^{(\lambda)} = (M^{(\lambda)}_{(i,j)}, \phi^{(\lambda)}_{(\alpha,\beta)})_{(i,j) \in (Q_{A \otimes B})_0, (\alpha,\beta) \in (Q_{A \otimes B})_1}$ by adding some extra nonzero spaces and nonzero maps to the $k$-linear representation corresponding to $S_1 \otimes U$ in $\text{rep}_k(Q_{A \otimes B}, \mathcal{I}_{A \otimes B})$. At first, in addition to the $k$-linear representation of $S_1 \otimes U$, we associate all vertices $(n-i, m-n+2+i)$ and $(n-i, m-n+3+i)$ in $(Q_{A \otimes B})_0$ for $0 \leq i \leq n-2$ to the 1-dimensional vector space $k$, where the second component is taken modulo $m$. Moreover we associate all arrows $(\alpha_{n-i}, \epsilon_{m-n+2+i})$ and $(\epsilon_{n-i}, \beta_{m-n+2+i})$ for $0 \leq i \leq n-2$ to the identity maps, and we associate the arrow $(\alpha_i, \epsilon'_i)$ from the vertex $(1, 1)$ to $(2, 1)$ to the multiplication by the $1 \times l$ matrix $[\lambda, 0, \cdots, 0]$, where $l$ means the dimension of the vector space corresponding to the vertex $(1, 1)$ and where the first component of this vector space $k^l$ corresponds to top $(S_1 \otimes U)$ isomorphic to $S_1 \otimes T_1$. This finishes the construction of the $k$-linear representation corresponding to $M^{(\lambda)}$. We will show that the $A \otimes B$-module
$M^{(\lambda)}$ is a brick, and that it is not isomorphic to $M^{(\mu)}$ for the nonzero element in $\mu \in k$ different from $\lambda$.

By the construction of $M^{(\lambda)}$, it is clear that the socle of $M^{(\lambda)}$ is isomorphic to $(S_1 \oplus T_{m-n+2}) \oplus (\bigoplus_{0 \leq i \leq n-2} S_{n-i} \otimes T_{m-n+3+i}) = \bigoplus_{-1 \leq i \leq n-2} S_{n-i} \otimes T_{m-n+3+i}$, where the indices of $S_j$ are taken modulo $n$ and those of $T_j$ are taken modulo $m$, whose any indecomposable summand does not appear as composition factor of $M^{(\lambda)}/\text{soc } M^{(\lambda)}$. Therefore, by Lemma 3.1, it is enough to show that the $A \otimes B$-module $M^{(\lambda)}$ is indecomposable in order to prove that it is a brick.

Assume that the $A \otimes B$-module $M^{(\lambda)}$ is not indecomposable. Then there is an endomorphism $f : M^{(\lambda)} \rightarrow M^{(\lambda)}$ with $M^{(\lambda)} \cong \text{Im } f \oplus \text{Ker } f$. In particular, it holds that $\text{soc } M^{(\lambda)} \cong \text{soc } \text{Im } f \oplus \text{soc } \text{Ker } f$, where $\text{soc } \text{Im } f$ and $\text{soc } \text{Ker } f$ are nonzero. By the construction of $M^{(\lambda)}$, it holds either of the following:

1. For some $-1 \leq i \leq n - 3$ it holds that $S_{n-i} \otimes T_{m-n+3+i}$ is a direct summand of $\text{soc } \text{Im } f$ and that $S_{n-i} \otimes T_{m-n+3+i}$ is a direct summand of $\text{soc } \text{Ker } f$.
2. For some $-1 \leq i \leq n - 3$ it holds that $S_{n-i} \otimes T_{m-n+3+i}$ is a direct summand of $\text{soc } \text{Ker } f$ and that $S_{n-i} \otimes T_{m-n+3+i}$ is a direct summand of $\text{soc } \text{Im } f$.

If the case (1) holds, then the morphism $(f_{(i,j)})_{(i,j) \in (Q_{A \otimes B})_0}$ in $\text{rep}_k(Q_{A \otimes B}, \mathcal{T}_{A \otimes B})$ corresponding to the endomorphism $f$ satisfies the following two equations by the commutativity of $(\varphi^{(\lambda)}_{(\alpha,\beta)})_{(\alpha,\beta) \in (Q_{A \otimes B})_1}$ and $(f_{(i,j)})_{(i,j) \in (Q_{A \otimes B})_0}$ (see the diagram below):

\[
\begin{cases}
    f_{(n-i,m-n+3+i)} \circ \text{id} = \text{id} \circ f_{(n-(i+1),m-n+3+i)} \\
    \text{id} \circ f_{(n-(i+1),m-n+3+i)} = f_{(n-(i+1),m-n+3+(i+1))} \circ \text{id}.
\end{cases}
\]

This shows that the two linear maps $f_{(n-i,m-n+3+i)} : k \rightarrow k$ and $f_{(n-(i+1),m-n+3+(i+1))} : k \rightarrow k$ are equal, but this contradicts the assumption (1). Similarly we can show that the case (2) cannot arise. Therefore the $A \otimes B$-module $M^{(\lambda)}$ is indecomposable.

Finally, we show that $A \otimes B$-module $M^{(\lambda)}$ is not isomorphic to $M^{(\mu)}$ for pairwise distinct nonzero elements $\lambda$ and $\mu$ in $k$. Assume that there is an isomorphism $f : M^{(\lambda)} \rightarrow M^{(\mu)}$, and we denote the corresponding morphism in $\text{rep}_k(Q_{A \otimes B}, \mathcal{T}_{A \otimes B})$ by $(f_a)_{a \in (Q_{A \otimes B})_0}$. By the constructions of $M^{(\lambda)}$ and $M^{(\mu)}$, via a similar argument as above, we have the following two equations.

\[
\begin{cases}
    f_{(n-i,m-n+3+i)} \circ \text{id} = \text{id} \circ f_{(n-(i+1),m-n+3+i)} \\
    \text{id} \circ f_{(n-(i+1),m-n+3+i)} = f_{(n-(i+1),m-n+3+(i+1))} \circ \text{id}.
\end{cases}
\]
Summarizing the above equations, we have that $f_{(n-i,m-n+3+i)} = f_{(n-(i+1),m-n+3+(i+1))}$ for $-1 \leq i \leq n - 3$. Hence we have that $f_{(1,m-n+2)} = f_{(n,m-n+3)} = f_{(n-1,m-n+4)} = \cdots = f_{(2,1)}$. In particular, we may suppose that $f_{(1,m-n+2)} = f_{(2,1)} = 1$. Moreover we regard $f_{(1,1)}$ as the multiplication by an $l \times l$ matrix $[a_{ij}]$ from the left. Also the direct summand $S_1 \otimes T_{m-n-2}$ of soc $M^{(\lambda)}$ comes from the socle of the tensor product of the simple $A$-module $S_1$ and the $B$-module $U$ which is a quotient module of the projective $B$-module $P(T_1)$, and $S_1 \otimes U$ has the simple top $S_1 \otimes T_1$. Hence there is a path

$$(1, 1) \xrightarrow{(e_1, \gamma_1)} \cdots \xrightarrow{(e_1, \beta_{m-n+1})} (1, m - n + 2)$$

from the vertex $(1, 1)$ to $(1, m - n + 2)$ such that the composition of the all maps on the arrows composing the path can be considered as the multiplication by $1 \times l$ matrix $[x, 0, \cdots, 0]$ from the left for some nonzero element $x$ in $k$ (we remark that the arrow $(e_1, \gamma_1)$ is not necessarily the arrow $(e_1, \beta_1)$). By the commutativity of $(\varphi^{(\lambda)}_{(i,j)})_{(i,j) \in (Q_{A \otimes B})}$, $(\varphi^{(\mu)}_{(i,j)})_{(i,j) \in (Q_{A \otimes B})}$ and $(f_{(i,j)})_{(i,j) \in (Q_{A \otimes B})}$ we have that

$$\begin{cases}
  f_{(1,m-n+2)} \circ \varphi_{(e_1, \beta_{m-n+1})} \circ \cdots \circ \varphi_{(e_1, \gamma_1)} = \varphi_{(e_1, \beta_{m-n+1})} \circ \cdots \circ \varphi_{(e_1, \gamma_1)} \circ f_{(1,1)}, \\
  f_{(2,1)} \circ \varphi_{(\alpha_1, \epsilon_1')} = \varphi_{(\alpha_1, \epsilon_1')} \circ f_{(1,1)},
\end{cases}$$

which implies that

$$\begin{cases}
  [x, 0, \cdots, 0] = [x, 0, \cdots, 0][a_{ij}], \\
  [\lambda, 0, \cdots, 0] = [\mu, 0, \cdots, 0][a_{ij}].
\end{cases}$$

Therefore we have that $\lambda = \mu$. \qed

**Example 3.5.** Let $A$ and $B$ be the algebras defined in Example 2.2. Since the algebras $A$ and $B$ are symmetric Nakayama algebras, they are finite representation type. Hence they are $\tau$-tilting finite algebras. We show that the tensor product $A \otimes B$ is a $\tau$-tilting infinite algebra. In fact, starting with the paths $C_A := (e_1 \xrightarrow{\alpha_1} e_2 \xrightarrow{\alpha_2} e_1)$ and $C_B := (e'_1 \xrightarrow{\beta_1} e'_2 \xrightarrow{\beta_2} e'_3 \xrightarrow{\beta_3} e'_1)$, for nonzero element $\lambda$ in $k$, let $M^{(\lambda)}$ be an $A \otimes B$-module whose $k$-linear representation is given as follows:

![Diagram](image)

Then the $A \otimes B$-module $M^{(\lambda)}$ parameterized by $\lambda$ is a brick. Also, for a nonzero element $\mu$ in $k$ different from $\lambda$, the $A \otimes B$-module $M^{(\lambda)}$ is not isomorphic to $M^{(\mu)}$, which means that $A \otimes B$ has infinitely many pairwise nonisomorphic brick. Therefore $A \otimes B$ is $\tau$-tilting infinite.

As an application of Theorem 3.4, we get the following result on the $\tau$-tilting finiteness of block algebras of direct products of finite groups.
Lemma 3.6 (see [1] Theorem 2.1). Let $A$ be a finite dimensional $\tau$-tilting finite algebra and $R$ a finite dimensional local algebra. Then we have a poset isomorphism $- \otimes_k R : s\tau$-tilt$A \rightarrow s\tau$-tilt$(A \otimes R)$.

Proof. It is enough to show that $- \otimes_k R : 2$-silt $A \rightarrow 2$-silt $(A \otimes R)$ is an isomorphism of partially ordered sets. By the similar argument as the proof of [1, Theorem 2.1], we have that the map $- \otimes_k R : 2$-silt $A \rightarrow 2$-silt $(A \otimes R)$ is an injection. However, under the assumptions that the algebra $A$ is a $\tau$-tilting finite algebra, the similar proof to the latter half of the proof of [1, Theorem 2.1] works, and we have that the injection is a poset isomorphism. □

Theorem 3.7. Let $G$ and $H$ be finite groups, $k$ an algebraically closed field and $B$ a block of the group algebra $k[G \times H]$ of the direct product of $G$ and $H$. If the block algebra $B$ is $\tau$-tilting finite, then there is a block $A$ of $kG$ or of $kH$ such that the set of support $\tau$-tilting modules over $B$ is isomorphic to that of $A$ as posets.

Proof. Let $kG = \bigoplus_i b_i$ and $kH = \bigoplus_j b_j'$ be block decompositions. Then the block decomposition of $k[G \times H]$ is given by $k[G \times H] = \bigoplus_{i,j} b_i \otimes b_j'$. Since the block algebras of finite groups are symmetric algebras, if both $b_i$ and $b_j'$ are not local algebras, then the block algebra $b_i \otimes b_j'$ is a $\tau$-tilting infinite algebra by Theorem 3.4. On the other hand, if the block algebra $b_i$ is a local algebra, then the set of the support $\tau$-tilting modules over $b_i \otimes b_j'$ is isomorphic to that of $b_j'$ by Lemma 3.6. The case the block algebra $b_j'$ is a local algebra follows similarly too. □

Acknowledgments The author would like to thank the referee for valuable comments to improve the paper.

References

[1] T. Adachi, O. Iyama, I. Reiten, $\tau$-tilting theory. Compos. Math. 150 (2014), no. 3, 415–452.
[2] T. Adachi, Characterizing $\tau$-tilting finite algebras with radical square zero. Proc. Amer. Math. Soc. 144 (2016), no. 11, 4673–4685.
[3] T. Adachi, T. Aihara, A. Chan, Classification of two-term tilting complexes over Brauer graph algebras. Math. Z. 290 (2018), no. 1–2, 1–36.
[4] T. Aihara, T. Honma, $\tau$-tilting finite triangular matrix algebras. J. Pure Appl. Algebra 225 (2021), no. 12, Paper No. 106785, 10 pp.
[5] T. Aihara, T. Honma, K. Miyamoto, Q. Wang, Report on the finiteness of silting objects. Proc. Edinb. Math. Soc. (2) 64 (2021), no. 2, 217–233.
[6] S. Asai, Semibricks. Int. Math. Res. Not. IMRN 2020, no. 16, 4993–5054.
[7] I. Assem, D. Simson, A. Skowroński, Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.
[8] L. Demonet, O. Iyama, N. Reading, I. Reiten, H. Thomas, Lattice theory of torsion classes Preprint (2021), arXiv:1711.01785.
[9] L. Demonet, O. Iyama, G. Jasso, $\tau$-tilting finite algebras, bricks, and g-vectors. Int. Math. Res. Not. IMRN 2019, no. 3, 852–892.
[10] F. Eisele, G. Janssens, T. Raedschelders, A reduction theorem for $\tau$-rigid modules. Math. Z. 290 (2018), no. 3–4, 1377–1413.
[11] R. Kase, From support $\tau$-tilting posets to algebras, Preprint (2021), arXiv:1709.05049.
[12] S. Koenig, D. Yang, Silting objects, simple-minded collections, $t$-structures and co-$t$-structures for finite-dimensional algebras. Doc. Math. 19 (2014), 403–438.
[13] R. Koshio, Y. Kozakai, On support $\tau$-tilting modules over blocks covering cyclic blocks. J. Algebra 580 (2021), 84–103.
[14] R. Koshio, Y. Kozakai, *Induced modules of support $\tau$-tilting modules and extending modules of semibricks over blocks of finite groups*. Preprint (2021), [arXiv:2112.08897](https://arxiv.org/abs/2112.08897).

[15] Y. Kozakai, *On tilting complexes over blocks covering cyclic blocks*. Comm. Algebra, to appear. DOI 10.1080/00927872.2022.2162912.

[16] Z. Leszczyński, *On the representation type of tensor product algebras*. Fund. Math. **144** (1994), no. 2, 143–161.

[17] P. Malicki, A. Skowroński, *Cycle-finite algebras with finitely many $\tau$-rigid indecomposable modules*. Comm. Algebra **44** (2016), no. 5, 2048–2057.

[18] K. Miyamoto, Q. Wang, *On $\tau$-tilting finiteness of tensor products between simply connected algebras*. Preprint (2021), [arXiv:2106.06423](https://arxiv.org/abs/2106.06423).

[19] Y. Mizuno, *Classifying $\tau$-tilting modules over preprojective algebras of Dynkin type*. Math. Z. **277** (2014), no. 3–4, 665–690.

[20] K. Mousavand, *$\tau$-tilting finiteness of biserial algebras*. Algebr. Represent. Theory (2022), to appear.

[21] H. Treffinger, *$\tau$-tilting theory – An introduction*. Preprint (2021), arXiv: 2106.00426.

[22] S. Zito, *$\tau$-tilting finite cluster-tilted algebras*. Proc. Edinb. Math. Soc. (2) **63** (2020), no. 4, 950–955.

[23] Q. Wang, *On $\tau$-tilting finite simply connected algebras*. Tsukuba J. Math. **46** (2022), no. 1, 1–37.

[24] Q. Wang, *On $\tau$-tilting finiteness of the Schur algebra*. J. Pure Appl. Algebra **226** (2022), no. 1, Paper No. 106818, 29 pp.