Asymptotically-flat Black Hole in EiBI-Born-Infeld Theory

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Abstract. We study static spherically-symmetric solutions of charged Eddington-inspired Born-Infeld (EiBI) theory of gravity in asymptotically-flat D-dimensional spacetime. We consider nonlinear electrodynamics for the matter fields. In this particular work, the nonlinear theory we specifically consider is the Born-Infeld electrodynamics. We found that metrics can not be integrated in general dimensions. In order to solve it, we focus on studying on the case $\alpha \equiv 4\kappa b^2 = 1$. We study the electric fields. We find that this theory produces finite electric fields for any dimension. In Ricci scalar calculations, we find only point singularity.

1. Introduction

Black hole is an interesting object in the field of gravity and cosmology. The concept of this black hole appears in Einstein’s field equation solution. When we consider electromagnetic field as a source of matter sector to construct black hole solution in Einstein field equations, one usually use Maxwell electrodynamics. However, if we look at a very strong field, Maxwell’s theory does not apply. In 1930, Born-Infeld [1] in his proposal on electrodynamics found a non-linear form with the aim of obtaining a finite value on the electron’s self-energy. This makes scientists interested in researching more about this topic. Many proposals on Born-Infeld is widely applied to black hole object. Demianski finds a solution from Einstein-Born-infeld in 4 dimensions in asymptotically-flat space-time and finds that when the internal mass is zero, the solution is regular and describes static electromagnetic geon [2]. This led many researchers to further analyze with the addition of cosmological constant [3] or the addition of dimensions [4, 5].

In 2010, Eddington-inspired Born-Infeld (EiBI) is an alternative theory of gravity proposed by Banados and Ferreira [6]. This alternative theory successfully covered the singularity in the early universe. The characteristic of EiBI theory is in vacuum ($\kappa \to 0$), the action of EiBI theory will be reduced to Einstein-Hilbert action. When discussing a non-vacuum case ($T_{\mu\nu} \neq 0$), there will be a deviation from the general theory of relativity. Although still fairly new, this EiBI theory has been widely studied in black hole and other compact stars both 4d or higher dimension [7, 8, 9, 10, 11].

In this paper, we will study higher dimensional EiBI theory with Born-infeld as a matter sector. The 4d has been studied by Jana-Kar [12]. In this work, we restrict ourselves to finding the exact solution in the electrostatic ansatz with zero cosmological constant $\Lambda$. This paper is structured as follows. Section 2 discusses EiBI Formalism. Section 3 discusses the structure of
higher dimensional EiBI-Born-Infeld. In section 4 we will discuss metric solution, electric fields and the curvature in each dimension. Finally, some concluding remarks are given in Section 5.

2. EiBI Action

We start with EiBI action [6] where in vacuum limit, this action reduces to Einstein-Hilbert action,

\[
S(g, \Gamma, \Phi) = \frac{1}{8\pi\kappa} \int d^Dx \left( \sqrt{-|g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma)|} - \lambda \sqrt{-|g_{\mu\nu}|} \right) + S_M(g, \Phi). \tag{1}
\]

Where \( \lambda = 1 + \kappa \Lambda \), \( q_{\mu\nu} \) is auxiliary metric and \( \kappa \) is so called EiBI parameter. Here, we set \( G = 1 \).

From equation (1), varying the action with respect to pure metric \( g \) can appear a ghost term. In order to avoid it, Vollick [13] use Palatini formalism and he treat \( \Gamma \) and \( g \) as two independent fields and Ricci tensor is depent on \( \Gamma \),

\[
q_{\mu\nu} = g_{\mu\nu} + \kappa R_{\mu\nu}(\Gamma). \tag{2}
\]

The fields equation from EiBI theory can be obtained from varying EiBI action with respect \( \Gamma \) and leads us to \( \nabla_\beta q_{\mu\nu} = 0 \). After some algebra, we get

\[
\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} q^{\rho\sigma} (\partial_\alpha q_{\beta\rho} + \partial_\beta q_{\rho\alpha} - \partial_\rho q_{\alpha\beta}). \tag{3}
\]

If we varying the action with respect to metric tensor \( g_{\mu\nu} \), we get EiBI equation

\[
\sqrt{-q} q^{\mu\nu} = \lambda \sqrt{-g} g^{\mu\nu} - 8\pi\kappa \sqrt{-g} T^{\mu\nu}. \tag{4}
\]

with \( q \) and \( g \) is determinant of \( q_{\mu\nu} \) and \( g_{\mu\nu} \), \( T^{\mu\nu} \) is energy-momentum tensor and \( \kappa \) is EiBI parameter.

3. Structure of Higher Dimensional Charged Black Hole in EiBI-Born-Infeld Theory

We start with Born-Infeld lagrangian for matter fields,

\[
\mathcal{L}_{BI} = \frac{b^2}{4\pi} \left[ 1 - \sqrt{1 + \frac{F}{b^2}} \right], \tag{5}
\]

with \( F = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \) and \( b \) is Born-Infeld parameter. When \( b \to \infty \), Maxwell’s theory was recovered with higher-order corrections,

\[
\mathcal{L}_M = -F_{\mu\nu} F^{\mu\nu} + O(F^4). \tag{6}
\]

Hence, the resulting of tensor energy-momentum is

\[
T_{\mu\nu} = -\frac{b^2}{4\pi} \left[ g_{\mu\nu} \left( \sqrt{1 + \frac{F_{\alpha\beta} F^{\alpha\beta}}{2b^2}} - 1 \right) - \frac{F_{\mu\sigma} F_{\nu}^{\sigma}}{b^2 \sqrt{1 + \frac{F_{\alpha\beta} F^{\alpha\beta}}{2b^2}}} \right]. \tag{7}
\]

In this paper, we limit our work in electrostatic \( A_\mu = \{ \phi, 0, 0, 0 \} \). Varying equation (1) with respect to \( A_\mu \) yields

\[
\nabla_\mu \left( \frac{F^{\mu\nu}}{\sqrt{1 + \frac{F_{\alpha\beta} F^{\alpha\beta}}{2b^2}}} \right) = 0. \tag{8}
\]
For spherically symmetric work, we use ansatz

\[ g_{\mu\nu} dx^\mu dx^\nu = -U(\bar{r}) e^{2\psi(\bar{r})} dt^2 + U(\bar{r}) e^{2\nu(\bar{r})} dr^2 + V(\bar{r}) \bar{r}^2 d\Omega_{D-2}^2, \]

\[ q_{\mu\nu} dx^\mu dx^\nu = -e^{2\psi(\bar{r})} dt^2 + e^{2\nu(\bar{r})} dr^2 + \bar{r}^2 d\Omega_{D-2}^2, \]

where \( d\Omega_{D-2}^2 \) denotes the metric of an unit \((D - 2)\) sphere. From ansatz above, \( \bar{r} \) is radial coordinate, but the physical radial distance is \( r^2 = V(\bar{r})\bar{r}^2 \). This physical radial distance is interpreted as area of \((D - 2)\) sphere. Here, this solution can’t solved easily, so we follow Jana-Kar [12] ansatz. The advantage of this metric is absence of differential expression in field equation (4). Hence, as we shall see, \( U(\bar{r}) \) and \( V(\bar{r}) \) can be solved in polynomial equation. Before evaluate these field equation, its important to solving the electromagnetic equation first.

In electrostatics, the non-zero component of Maxwell stress tensor \( F_{\mu\nu} \) is \( F_{10} = \partial_\nu \phi \). Solving the electromagnetic equation, we obtain

\[ E(\bar{r}) = -\frac{d\phi}{d\bar{r}} = \frac{q U e^{\nu+\psi}}{\sqrt{V D-2}} \]

with \( q \) an integration constant which can usually be identified as the charge. Hence, the non-zero of tensor energy-momentum is

\[ T^{00} = \frac{e^{-2\psi} b^2}{4\pi} \left( \frac{\sqrt{V D-2} \bar{r}^2 (D-2) + q^2}{b^2 - \bar{r}^2} - 1 \right), \]

\[ T^{11} = -\frac{e^{-2\nu} b^2}{4\pi} \left( \frac{\sqrt{V D-2} \bar{r}^2 (D-2) + q^2}{b^2 - \bar{r}^2} - 1 \right), \]

\[ T^{22} = \frac{b^2}{4\pi \bar{r}^2} \left( 1 - \frac{V^{D-2} \bar{r}^2 (D-2)}{\sqrt{V D-2} \bar{r}^2 (D-2) + q^2} \right), \]

\[ T^{33} = \frac{T^{22}}{\sin^2 \theta}, \quad T^{ab} = \frac{T^{22}}{\prod_{j=1}^{D-2} \sin^2 \theta_j}. \]

Inserting this tensor energy-momentum to equation (4) and after some algebra, we get the expression of \( V(\bar{r}) \) and \( U(\bar{r}) \) from polynomial equation

\[ (1 - \alpha) V^{D-2} - (2 - \alpha) \frac{V^{D-2}}{b^2} + 1 - \frac{\kappa q^2 \alpha}{\bar{r}^2 (D-2)} = 0. \]

Where \( \alpha \equiv 4\kappa b^2 \). Of course, there are many possible solution for \( V(\bar{r}) \) and also \( U(\bar{r}) \). We can manipulate this solution into two possible solution. Therefore, we have

\[ V(\bar{r}) = \left[ 1 - \frac{\alpha}{2} \left( 1 + \frac{q^2}{\bar{r}^2 (D-2)} - \frac{4\kappa b^2}{\bar{r}^2 (D-2)} \right) \right] \frac{V^{D-2}}{b^2}, \]

\[ U(\bar{r}) = \left( 1 - \frac{\alpha}{2} \right) \frac{V^{D-2} \bar{r}^2 (D-2) + q^2}{(1 - \alpha) \left( 1 + \frac{4\kappa q^2}{\alpha \bar{r}^2 (D-2)} - \frac{4\kappa b^2}{\bar{r}^2 (D-2)} \right)} V(\bar{r}) \frac{(D-2)}{2}. \]
On the other hand, from (2) one obtains

\[
\left(1 - \frac{U}{\kappa}\right) e^{2\psi} = e^{-2\nu + 2\psi} \left(-\frac{(D-2)}{\bar{r}} \psi' + \nu' \psi' - \psi'^2 - \psi''\right),
\]

(19)

\[
\left(1 - \frac{V}{\kappa}\right) e^{2\nu} = \left(\frac{(D-2)}{\bar{r}} \nu' + \nu' \psi' - \psi'^2 - \psi''\right),
\]

(20)

\[
\left(1 - \frac{\bar{r}}{\kappa}\right) \bar{r}^2 = (D-3) - \frac{1}{(D-2)xD^{-3}} \left(\bar{r}^{D-2}\right)' e^{2\psi}'.
\]

(21)

The first two equations are solved by \(\psi = -\nu\). Inserting it into (21) yields

\[
e^{2\psi(\bar{r})} = 1 - \frac{2M}{\bar{r}^{(D-3)}} - \frac{\bar{r}^2}{(D-1)\kappa} + \frac{1}{\kappa \bar{r}^{(D-3)}} \int V(\bar{r}) \bar{r}^{(D-2)} d\bar{r}.
\]

(22)

Finally, equation (22) is the last metric solution that we want to solve. Unfortunately, this equation cannot be solved analytically for arbitrary D. We therefore specify the value of \(\alpha\). In this work, we specify \(\alpha = 1\) from polynomial equation (19). Hence, the metric functions \(V, U\) and \(e^{2\psi}\) becomes

\[
V(\bar{r}) = \left(1 - \frac{\kappa q^2}{\bar{r}^{2(D-2)}}\right) \frac{\bar{r}^2}{(D-1)\kappa},
\]

(23)

\[
U(\bar{r}) = \left(1 + \frac{\kappa q^2}{\bar{r}^{2(D-2)}}\right) V(\bar{r}) \frac{4-D}{2},
\]

(24)

and

\[
e^{2\psi(\bar{r})} = 1 - \frac{2M}{\bar{r}^{(D-3)}} - \frac{\bar{r}^2}{(D-1)\kappa} + \frac{\bar{r}^2}{(D-1)\kappa} \frac{2F_1\left(-\frac{2}{D-2}, -\frac{D-1}{2(D-2)}; \frac{D-3}{2(D-2)}; q^2\frac{4-D}{2}\kappa\right)}{\bar{r}^{(D-2)}}.
\]

(25)

Where \(2F_1\) is hypergeometric function. When \(q \rightarrow 0\), the solution reduces to the standard Tangherlini form. Equations (23)-(25) are the exact solution of higher dimensional Born-Infeld gravity coupled to Born-Infeld electrodynamics in radial coordinate \(\bar{r}\) at specific value of \(\alpha\). This exact solution comes from gauge (9) that has \(\bar{r}\) on its metric. Fortunately, it is possible to bring it into another gauge that has \(r\) on metric functions, called Tangherlini gauge.

Since the physical radial distance is \(r^2 = \bar{r}^2 V(\bar{r})\), we therefore set \(r^2 = \left(1 - \frac{\kappa q^2}{\bar{r}^{2(D-2)}}\right) \frac{\bar{r}^2}{(D-1)\kappa} \bar{r}^2\). The resulting metric is

\[
ds^2 = -A(r)f(r) dt^2 + \frac{r^{D-4}(\frac{1}{2})^{\frac{D-2}{2}} A(r) \left(1 + \sqrt{B(r)}\right)}{B(r) f(r)} dr^2 + r^2 d\Omega^2,
\]

(26)

where

\[
A(r) = \left(1 + \frac{\kappa q^2}{y(r)^2(2D-2)}\right) \left(1 - \frac{\kappa q^2}{y(r)^2(2D-2)}\right)^{\frac{4-D}{2}},
\]

(27)

\[
f(r) = 1 - \frac{2M}{y(r)^{(D-3)}} - \frac{y(r)^2}{(D-1)\kappa} + \frac{y(r)^2}{(D-1)\kappa} \frac{2F_1\left(-\frac{2}{D-2}, -\frac{D-1}{2(D-2)}; \frac{D-3}{2(D-2)}; \frac{\kappa q^2}{y(r)^{(D-2)}}\right)}{(D-1)\kappa},
\]

(28)

\[
y(r) \equiv \bar{r}(r) = \left(\frac{1}{2}\right)^{\frac{1}{D-2}} \left(1 + \sqrt{B(r)}\right)^{\frac{1}{D-2}} r,
\]

(29)
and

\[ B(r) = 1 + \frac{4\kappa q^2}{r^{2(D-2)}}. \]  

(30)

Equation (26)-(30) are exact solutions in D-dimension which we want to analyze its behaviour. On the other hand, a curvature of this space-time is important to be analyzed. Also, electric fields that we have shown in (11) has interesting feature. All of these quantities will be discuss deeply in the next section.

4. Result and Discussion

From the previous section, we study static spherically-symmetric solutions of EiBI-Born-Infeld in higher dimension. Equation (26)-(30) constitute an exact solution in this paper. Since black holes are containing singularity that enclosed by horizon, it is interesting to get the information about the radius of its horizon by study the metric solution. For \( D \geq 4 \), metric solution in Einstein-Born-Infeld has been studied in [5] and found that these metric behaviour causing double horizons even in asymptotically-flat space-time. While in EiBI (Eddington inspired Born-Infeld) is absent with different values of parameter \( \kappa \). Next, we will present the metric behaviour of EiBI-Born-Infeld black hole in \( D \geq 4 \).

![Figure 1. Metric functions of \( g_{tt} \) as a function of \( r \) with \( M = 1 \) and \( q = 0.7 \) in asymptotically-flat space-time.](image)

In Figure 1, we show the metric of \( g_{tt} = U(r)e^{2\psi(r)} \) in specify dimension. Important to note that this plot as a function of \( r \) since a physical radius distance is \( r = \bar{r}\sqrt{V(\bar{r})} \), as we mention before. We varying the value of \( \kappa \) from weak to strong coupling. Based on this plot, 4\( d \) metric shape is the same as Jana-kar in \( \alpha = 1 \). It turns out that the behaviour of this metric in \( D > 4 \) is also the same, 1 horizon for different \( \kappa \). We note a generic feature that for \( \kappa \) positive the horizon radius \( r \) decreases with increasing \( \kappa \). In 5\( d \) and 6\( d \), also have the same amount of horizon and size on its radius (decreases with increasing \( \kappa \)). Note that the \( D = 4 \) solution we found
looks regular at the origin \[12\]. The regularity of this metric seems to indicate that the EM source is point-charge. In higher dimensions \((D \geq 5)\), all the metric solutions are singular at the origin. It is implied that the metric functions are limited to minimum radius where the charged is distributed. So there is no discontinuity between \(D = 4\) and \(D \geq 5\) as \(r \to 0\). To ensure this statement, as can we see later, the scalar Ricci is singular at the origin in all dimensions. Thus, we interpreted our solutions as a genuine black holes. In large \(r\), these metric can be seen as asymptotically-flat space-time.

In our discussion about metric behaviour of asymptotically-flat EiBI-Born-Infeld black hole in \(D \geq 4\). To ensure this metric behaviour we use scalar quantities, Ricci scalar \(\mathcal{R} = g^{\mu\nu}R_{\mu\nu}\) in this paper. While there are several order that exist in scalar quantities such as \(R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\) and \(R_{\mu\nu}R^{\mu\nu}\), one can verify that the Ricci scalar is the lowest order. In Figure 2, we plot the Ricci scalar as a function of physical radial distance \(r\). According to metric functions, it has different feature in \(r \to 0\) in several dimensions. It motivate us to plot the Ricci scalar with varying dimension to test whether this curvature has the same characteristics with the metric functions or not. This typical plot, fortunately, still suffers singularity at the origin. This is not surprising since black hole has singularity that enclosed by horizon. However, this Ricci scalar calculation does not ensure the singularity of the electric field. In Reissner-Nordstrom, one can verify that the strength of the electric fields blow up near origin. This problem is covered by studying Born-Infeld electrodynamics in Einstein gravity, based on Born-infeld proposal. Surprisingly, EiBI come up with its remarkable feature when coupled to 4d Maxwell electrodynamics that has finite value at the origin \[7\] and 4d Born-Infeld electrodynamics has the same feature with its Einstein gravity, produces finite electric fields in a strong regime \[12\]. Hence, it is tempting to check whether EM field can still have regular on its self-energy in higher dimension \[11\]. In Figure 3, we present the solution of electric field in non-linear electrodynamics \(E(r)\) in equation \((11)\). Note that when \(b \to \infty\) this formalism reduces to electric field in Maxwell electrodynamics. In this plot, the electric fields is a function of physical radial distance \(r\) and this solution obtained with \(\alpha \equiv 4\kappa b^2 = 1\). Hence, we have a relation between \(\kappa\) and \(b\)

\[
b^2 = \frac{1}{4\kappa},
\]

where \(\kappa\) is EiBI parameter and \(b\) is non-linear parameter. Plots that we present based on the strength of parameter \(\kappa\) and \(b\). The strength is weak when \(\kappa \to 0\) and \(b > 1\) and strong when \(\kappa \geq 1\) and \(b < 1\). As we can see, we can’t find the electric field that has strong on \(\kappa\) and weak.
Figure 3. A plot of Born-Infeld electric fields $E(r)$ as a function of $r$ with $q = 0.7$ in asymptotically-flat space-time. This plot is the equation given in (11). For parameter values from left panel is $\kappa = 2$ and $b = 0.35$, right panel is $\kappa = 0.4$ and $b = 0.7$ and left panel at the bottom is $\kappa = 0.1$ and $b = 1.5$.

on $b$ because of its natural fraction on $b^2 = \frac{1}{4\kappa^2}$. On the other hand, the electric fields in this plot is always has finite value on its self-energy. In large $r$, $E(r)$ becomes zero.
5. Conclusion

In this paper, we study static spherically-symmetric solutions of charged Eddington-inspired Born-Infeld (EiBI) theory of gravity in D-dimensional spacetime. EiBI theory is a modified gravity that put born-infeld action on its gravity sector. This theory parametrized by $\kappa$. If $\kappa \to 0$, EiBI theory reduces to GR. Our solution in $D = 4$, reduces to Jana-Kar and reduces to Reissner-Nordstrom when $\kappa \to 0$. This solution have 1 horizon in all different value of $\kappa$. To ensure this metric, we use Ricci scalar $\mathcal{R}$. We found that the metric still suffers singularity near the origin. The electric fields of this solution is also finite in $D > 4$.

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