CYLINDRICAL COMBINATORICS AND REPRESENTATIONS OF
CHEREDNIK ALGEBRAS OF TYPE A

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ABSTRACT. We investigate the representation theory of the rational and trigonometric Cherednik algebra of type \( GL_n \) by means of combinatorics on periodic (or cylindrical) skew diagrams.

We introduce and study standard tableaux and plane partitions on periodic diagrams, and in particular, compute some generating functions concerning plane partitions, where Kostka polynomials and their level restricted generalization appear.

On representation side, we study representations of Cherednik algebras which admit weight decomposition with respect to a certain commutative subalgebra. All the irreducible representations of this class are constructed combinatorially using standard tableaux on periodic diagrams, and this realization as ”tableaux representations” provides a new combinatorial approach to the investigation of these representations.

As consequences, we describe the decomposition of a tableau representation as a representation of the degenerate affine Hecke algebra, which is a subalgebra of the Cherednik algebra, and also describe the spectral decomposition of the spherical subspace (the invariant subspace under the action of the Weyl group) of a tableau representation with respect to the center of the degenerate affine Hecke algebra. In particular, the computation of the character of the spherical subspace is reduced to the computation of the generating function for the set of column strict plane partitions, and we obtain an expression of the characters in terms of Kostka polynomials as announced in [Su2].

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1. PLANE PARTITIONS AND STANDARD TABLEAUX ON PERIODIC DIAGRAMS

We study cylindrical combinatorics, that is, combinatorics on periodic (or cylindrical) skew diagrams, which are introduced by Gessel and Krattenthaler [GK] as a cylindrical analogue of skew Young diagrams, and have appeared in the representation theory of the double affine Hecke algebras [Ch4, SV].
1.1. **Root system and Weyl group.** Let \( \mathbb{F} \) denote a field of characteristic 0, which includes the field \( \mathbb{Q} \) of rational numbers and the ring \( \mathbb{Z} \) of integers.

Throughout this article, we use the following notation:

\[
[a, b] = \begin{cases} 
\{a, a + 1, \ldots, b\} & \text{for } a, b \in \mathbb{F} \text{ with } b - a \in \mathbb{Z}_{\geq 0}, \\
\emptyset & \text{otherwise}.
\end{cases}
\]

Fix \( n \in \mathbb{Z}_{\geq 2} \). Let \( \mathfrak{h} \) be an \( n \)-dimensional vector space over \( \mathbb{F} \) with the basis \( \{\epsilon_i^\vee, \epsilon_2^\vee, \ldots, \epsilon_n^\vee\} \):

\[
\mathfrak{h} = \bigoplus_{i \in [1, n]} \mathbb{F} \epsilon_i^\vee.
\]

Introduce the non-degenerate symmetric bilinear form \( ( \langle | \rangle ) \) on \( \mathfrak{h} \) by \( (\epsilon_i^\vee | \epsilon_j^\vee) = \delta_{ij} \). Let \( \mathfrak{h}^* = \bigoplus_{i=1}^n \mathbb{F} \epsilon_i \) be the dual space of \( \mathfrak{h} \), where \( \epsilon_i \) are the dual vectors of \( \epsilon_i^\vee \). The natural pairing is denoted by \( (\langle | \rangle ) : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{F} \).

Put \( \alpha_{ij} = \epsilon_i - \epsilon_j, \quad \alpha_{ij}^\vee = \epsilon_i^\vee - \epsilon_j^\vee (1 \leq i \neq j \leq n) \) and \( \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad \alpha_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee (i \in [1, n-1]) \). Then

\[
R = \{\alpha_{ij} | i, j \in [1, n], \ i \neq j\}, \quad R^+ = \{\alpha_{ij} | i, j \in [1, n], \ i < j\}
\]
give the system of roots and positive roots of type \( A_{n-1} \) respectively.

Let \( \mathcal{W} \) denote the Weyl group associated with the root system \( R \). The group \( \mathcal{W} \) acts on \( \mathfrak{h} \) and \( \mathfrak{h}^* \), and it is isomorphic to the symmetric group of degree \( n \).

Denote by \( s_\alpha \) the reflection in \( \mathcal{W} \) corresponding to \( \alpha \in R \). We write \( s_{ij} = s_{\alpha_{ij}} (i, j \in [1, n]) \) and \( s_i = s_{\alpha_i} (i \in [1, n-1]) \). We have \( \mathcal{W} = \langle s_1, s_2, \ldots, s_{n-1} \rangle \).

Put

\[
P = \bigoplus_{i \in [1, n]} \mathbb{Z} \epsilon_i,
\]

which is a subset of \( \mathfrak{h}^* \) and is preserved by the action of \( \mathcal{W} \). Define the **extended affine Weyl group** \( \hat{\mathcal{W}} \) of \( \mathfrak{g}_n \) as the semidirect product \( P \rtimes \mathcal{W} \).

For \( \eta \in P \), let \( t_\eta \) denote the corresponding element in \( \hat{\mathcal{W}} \). Put \( s_0 = t_{\epsilon_1 - \epsilon_{n-1}} \) and \( \pi = t_{\epsilon_1} s_1 t_{\epsilon_2} s_2 \ldots s_{n-1} \). Then \( \{s_0, s_1, \ldots, s_{n-1}, \pi^\pm 1\} \) gives a set of generators of \( \hat{\mathcal{W}} \), and the subgroup generated by \( s_0, s_1, \ldots, s_{n-1} \) is the affine Weyl group associated with the root system of type \( A_n^{(1)} \).

Define an action of the extended affine Weyl group \( \hat{\mathcal{W}} \) on the set \( \mathbb{Z} \) of integers by

\[
(1.1.1) \quad s_i(j) = \begin{cases} 
 j + 1 & \text{for } j \equiv i \mod n, \\
 j - 1 & \text{for } j \equiv i + 1 \mod n, \\
 j & \text{for } j \not\equiv i, i + 1 \mod n,
\end{cases} \quad (i \in [1, n-1])
\]

\[
(1.1.2) \quad t_{\epsilon_i}(j) = \begin{cases} 
 j + n & \text{for } j \equiv i \mod n, \\
 j & \text{for } j \not\equiv i \mod n.
\end{cases} \quad (i \in [1, n])
\]

Note in particular that \( \pi(j) = j + 1 \) for all \( j \in \mathbb{Z} \).

1.2. **Periodic skew diagrams.** We need a slightly generalized definition of skew diagrams.
Definition 1.1. A subset $\theta$ of $\mathbb{Z} \times \mathbb{F}$ is called a skew diagram if the following conditions are satisfied:

- (D1) The set $\theta$ consists of finitely many elements.
- (D2) For any $i \in \mathbb{Z}$, there exist $\lambda_i, \mu_i \in \mathbb{F}$ such that $\{(a, b) \in \theta \mid a = i\} = [\mu_i + 1, \lambda_i]$.
- (D3) (Skew property) If $(a, b), (a', b') \in \theta$ with $a' - a \in \mathbb{Z}_{\geq 1}$ and $(b' - a') - (b - a) \in \mathbb{Z}_{\geq 0}$ then $(a, b + 1), (a', b' - 1) \in \theta$.

In the sequel, we regard $\mathbb{Z} \times \mathbb{F}$ as a $\mathbb{Z}$-module.

Let $\gamma \in \mathbb{Z} \times \mathbb{F}$. Denote by $\mathbb{Z}_\gamma$ the subgroup of $\mathbb{Z} \times \mathbb{F}$ generated by $\gamma$.

Definition 1.2. A subset $\Theta$ of $\mathbb{Z} \times \mathbb{F}$ is called a periodic skew diagram (or a cylindrical skew diagram) of period $\gamma$ if the following conditions are satisfied:

- (D’1) The group $\mathbb{Z}_\gamma$ acts on $\Theta$ by parallel translation, i.e., $\Theta + \gamma = \Theta$, and a fundamental domain of this action on $\Theta$ consists of finitely many elements.
- (D’2) For any $i \in \mathbb{Z}$, there exist $\lambda_i, \mu_i \in \mathbb{F}$ such that $\{(a, b) \in \Theta \mid a = i\} = [\mu_i + 1, \lambda_i]$.
- (D’3) (Skew property) If $(a, b), (a', b') \in \Theta$ with $a' - a \in \mathbb{Z}_{\geq 1}$ and $(b' - a') - (b - a) \in \mathbb{Z}_{\geq 0}$ then $(a, b + 1), (a', b' - 1) \in \Theta$.

Let $Y^n$ denote the set of skew diagrams consisting of $n$-elements, and let $\hat{Y}^n_\gamma$ denote the set of periodic diagrams of period $\gamma$ consisting of $n$ numbers of $\mathbb{Z}_\gamma$-orbits.

For a skew diagram $\theta$, define a subset of $\mathbb{Z} \times \mathbb{F}$ by

$$\hat{\theta}_\gamma = \theta + \mathbb{Z}_\gamma.$$ 

We often write $\hat{\theta} = \hat{\theta}_\gamma$ when $\gamma$ is fixed. For $\gamma = (\pm m, l) \in \mathbb{Z} \times \mathbb{F}$ with $m \in \mathbb{Z}_{\geq 1}$, set

$$Y^n_\gamma = \{\theta \in Y^n \mid \theta \subset [1, m] \times \mathbb{F} \text{ and } \hat{\theta}_\gamma \in \hat{Y}^n_\gamma\}.$$ 

Note that $\hat{Y}^n_\gamma = Y^n_\gamma$ and $Y^n_\gamma = Y^n_\gamma$.

For a periodic skew diagram $\Theta$ of period $\gamma = (\pm m, l)$, the subset $\Theta \cap ([1, m] \times \mathbb{F})$ is a skew diagram and it gives a fundamental domain of $\mathbb{Z}_\gamma$ on $\Theta$. Hence any periodic diagram of period $\gamma$ is of the form $\Theta$ for some $\theta \in Y^n_\gamma$, and the map $\Theta \mapsto \hat{\theta}$, $Y^n_\gamma \mapsto \hat{Y}^n_\gamma$ is bijective.

By the condition (D2), any skew diagram $\theta \subset [1, m] \times \mathbb{F}$ is expressed as $\lambda/\mu$ for some $\lambda, \mu \in \mathbb{F}^m$, where

$$\lambda/\mu = \{(a, b) \in \mathbb{Z} \times \mathbb{F} \mid a \in [1, m], b \in [\lambda_a + \mu_a]\}.$$ 

Moreover, it follows from the condition (D3) that $\lambda$ and $\mu$ can be chosen from the set of dominant elements

$$\mathcal{D}_m = \{\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{F}^m \mid (\nu_i - i) - (\nu_j - j) \notin \mathbb{Z}_{\leq 0} \text{ for any } i < j\}.$$ 

For $\kappa \in \mathbb{F}$, define $\mathcal{D}_{m, \kappa}$ as the subset of $\mathcal{D}_m$ consisting of those elements $\nu$ satisfying the following conditions:

- (1.2.1) $p\kappa + (\nu_i - i) - (\nu_j - j) \notin \mathbb{Z}_{\leq 0}$ for any $i < j$ and $p \in \mathbb{Z}_{\geq 0}$,
- (1.2.2) $p\kappa - (\nu_i - i) + (\nu_j - j) \notin \mathbb{Z}_{\leq 0}$ for any $i < j$ and $p \in \mathbb{Z}_{> 0}$.

Note that $\mathcal{D}_{m, \kappa} = \emptyset$ unless $\kappa \in \mathbb{F} \setminus \mathbb{Q}_{\leq 0}$. For $\nu \in \mathbb{Z}^m$, we write $\nu \models n$ if $\nu$ is a composition of $n$; namely, $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_m$ and $\sum_{i \in [1, m]} \nu_i = n$. 
Lemma 1.3. Let \( m \in \mathbb{Z}_{\geq 1} \) and \( \kappa \in \mathbb{F} \setminus \mathbb{Q}_{\leq 0} \). If \( \lambda, \mu \in \mathcal{D}_{m, \kappa} \) and \( \lambda - \mu \nmid n \), then \( \lambda/\mu \in Y_{\mathcal{N}(\mathcal{D}_{m, \kappa}; m)} \). Conversely, for any \( \theta \in Y_{\mathcal{N}(\mathcal{D}_{m, \kappa}; m)} \), there exist \( \lambda, \mu \in \mathcal{D}_{m, \kappa} \) such that \( \theta = \lambda/\mu \).

Define \( Y_{\gamma, \mu}^{\mathcal{N}} \) as the subset of \( Y_{\gamma, \mu} \) consisting of diagrams without empty rows:
\[
Y_{\gamma, \mu}^{\mathcal{N}} = \left\{ \theta \in Y_{\gamma, \mu} \mid \forall a \in \mathbb{Z}, \ \exists b \in \mathbb{F} \text{ such that } (a, b) \in \hat{\theta} \right\}.
\]

Corollary 1.4. Let \( m \in [1, n] \) and \( \kappa \in \mathbb{F} \setminus \mathbb{Q}_{\leq 0} \). Put \( \gamma = (-m, \kappa - m) \).

(i) Let \( \lambda, \mu \in \mathbb{F}^{m} \) such that \( \lambda - \mu \nmid n \) and \( \lambda_i - \mu_i \geq 1 \) for all \( i \in [1, m] \). Then \( \lambda/\mu \) is in \( Y_{\gamma, \mu}^{\mathcal{N}} \) if and only if \( \lambda, \mu \in \mathcal{D}_{m, \kappa} \).

(ii) For any \( \theta \in Y_{\gamma, \mu}^{\mathcal{N}} \), there exist unique \( \lambda, \mu \in \mathcal{D}_{m, \kappa} \) such that \( \theta = \lambda/\mu \).

1.3. Plane partitions on periodic diagrams. Let \( \theta \) be a skew diagram.

Definition 1.5. (i) A map \( p : \theta \to \mathbb{Z} \) is called a plane partition on \( \theta \) if it is weakly row-column increasing; namely, if it satisfies the following two conditions (P2) and (P3):

- (P2) \( p(a, b) \leq p(a, b + 1) \) for any \( (a, b) \in \theta \) and \( (a, b + 1) \in \theta \).
- (P3) \( p(a, b) \leq p(a + k + 1, b + k) \) for any \( (a, b) \in \theta \) and \( (a + k + 1, b + k) \in \theta \) with \( k \in \mathbb{Z}_{\geq 0} \).

(ii) A plane partition on \( \theta \) is said to be row strict (resp., column strict) if the strict inequality always holds in (P2) (resp., (P3)).

Let \( \mathcal{P}(\theta) \) denote the set of the plane partitions on \( \theta \), and let \( \mathcal{P}^{R}(\theta) \) and \( \mathcal{P}^{C}(\theta) \) denote the set of the row strict and column strict plane partitions on \( \theta \) respectively. Define
\[
(1.3.1) \quad \mathcal{P}(\theta) = \{ p \in \mathcal{P}(\theta) \mid p(u) \geq 0 \ \forall u \in \theta \},
\]
and define \( \mathcal{P}^{R}(\theta) \) and \( \mathcal{P}^{C}(\theta) \) similarly.

Remark 1.6. In literature (e.g. [ST3]), a plane partition on \( \theta \subset \mathbb{Z} \times \mathbb{F} \) is defined as a map \( p : \theta \to \mathbb{Z}_{\geq 1} \) which is weakly decreasing in both row and column directions. In [ST3], an element of \( \mathcal{P}^{R}(\theta) \) is called a weak reverse plane partition.

Let \( \gamma \in \mathbb{Z} \times \mathbb{F} \). Let \( \theta \in Y_{\gamma, \mu}^{\mathcal{N}} \).

Definition 1.7. (i) A map \( p : \hat{\theta} \to \mathbb{Z} \) is called a plane partition on \( \hat{\theta} \) if it satisfies the following conditions:

- (P1) \( p(u + \gamma) = p(u) - 1 \) for all \( u \in \hat{\theta} \).
- (P2) \( p(a, b) \leq p(a, b + 1) \) for any \( (a, b) \in \hat{\theta} \) and \( (a, b + 1) \in \hat{\theta} \).
- (P3) \( p(a, b) \leq p(a + k + 1, b + k) \) for any \( (a, b) \in \hat{\theta} \) and \( (a + k + 1, b + k) \in \hat{\theta} \) with \( k \in \mathbb{Z}_{\geq 0} \).

(ii) A plane partition on \( \hat{\theta} \) is said to be row strict (resp., column strict) if the strict inequality always holds in the condition (P2) (resp., (P3)).
Let \( \tilde{PP}_\gamma(\theta) \) denote the set of the plane partitions on \( \hat{\theta} \), and let \( \tilde{PP}_R^\gamma(\theta) \) (resp., \( \tilde{PP}_C^\gamma(\theta) \)) denote the set of the row strict (resp., column strict) plane partitions on \( \hat{\theta} \). Define

\[
PP_\gamma(\theta) = \{ p \in \tilde{PP}_\gamma(\theta) \mid p(u) \geq 0 \ \forall u \in \theta \},
\]

and define \( PP_R^\gamma(\theta) \) and \( PP_C^\gamma(\theta) \) similarly.

**Example 1.8.** Let \( n = 7 \), \( \gamma = (-2,3) \), and let \( \theta = \lambda/\mu \) with \( \lambda = (5,3) \) and \( \mu = (1,0) \). Then \( \theta \in \mathcal{Y}_n^\theta \). The following figure represents the associated periodic diagram \( \hat{\theta} \) and a column strict plane partition on \( \hat{\theta} \).

![Diagram of a column strict plane partition](image)

**Figure 1.**

A plane partition \( p \in \tilde{PP}(\theta) \) on \( \theta \) can be uniquely extended to a function \( \hat{p} : \hat{\theta} \to \mathbb{Z} \) by setting \( \hat{p}(u + k\gamma) = p(u) - k \) for \( u \in \theta \) and \( k \in \mathbb{Z} \). The correspondence \( p \mapsto \hat{p} \) gives an embedding from \( \tilde{PP}(\theta) \) into the set \( \text{Map}(\hat{\theta}, \mathbb{Z}) \) of maps \( \hat{\theta} \to \mathbb{Z} \).

In the sequel, we often identify \( \tilde{PP}(\theta) \) with a subset of \( \text{Map}(\hat{\theta}, \mathbb{Z}) \). Under this identification, we have

\[
\tilde{PP}_\gamma(\theta) \subseteq PP(\theta), \quad \tilde{PP}_R^\gamma(\theta) \subseteq PP_R(\theta), \quad \tilde{PP}_C^\gamma(\theta) \subseteq PP_C(\theta)
\]

\[
PP_\gamma(\theta) \subseteq PP(\theta), \quad PP_R^\gamma(\theta) \subseteq PP_R(\theta), \quad PP_C^\gamma(\theta) \subseteq PP_C(\theta).
\]

The following statement can be shown easily using the skew property of \( \hat{\theta} \):

**Lemma 1.9.** A plane partition \( p \in \tilde{PP}(\theta) \) is in \( \tilde{PP}_\gamma(\theta) \) if and only if

\[
p(a,b) \leq p(a + k + 1, b + k)
\]

for any \( (a,b) \in \theta \) with \( k = \min\{j \in \mathbb{Z} \cup 0 \mid (a + j + 1, b + j) \in \hat{\theta}\} \).

**Remark 1.10.** Another important generalization of plane partitions is given by imposing the periodicity \( p(u + \gamma) = p(u) \) instead of (P1). Such plane partitions (called cylindric partitions) are introduced and studied by Gessel and Krattenthaler \textbf{GK}, and they are related to the theory of quantum Schubert calculus (see \textbf{Po}).
1.4. **Tableaux on periodic diagrams.** Let \( \gamma \in \mathbb{Z} \times \mathbb{F} \) and \( \theta \in Y^n_\gamma \). Following [SV], we introduce standard tableaux on \( \hat{\theta} \), which connect the representation theory of Cherednik algebras and the combinatorics on periodic diagrams.

**Definition 1.11.** Let \( \theta \in Y^n_\gamma \).

(i) A map \( T : \hat{\theta} \to \mathbb{Z} \) is called a *tableau* on \( \hat{\theta} \) if it is a bijection and satisfies the following condition:

\[
(T1) \quad T(u + \gamma) = T(u) - n \quad \text{for all } u \in \hat{\theta}.
\]

(ii) A tableau \( T \) on \( \hat{\theta} \) is called a *standard tableaux* if it satisfies the following conditions:

\[
(T2) \quad T(a,b) < T(a,b + 1) \text{ for any } (a,b) \in \hat{\theta}.
\]

\[
(T3) \quad T(a,b) < T(a + k + 1,b + k) \text{ for any } (a,b) \in \hat{\theta} \text{ and } (a + k + 1,b + k) \in \hat{\theta} \text{ with } k \in \mathbb{Z}_{\geq 0}.
\]

Denote by \( \text{Tab}_\gamma(\hat{\theta}) \) and \( \text{St}_\gamma(\hat{\theta}) \) the set of tableaux and the set of standard tableaux on \( \hat{\theta} \) respectively. Similarly to Lemma 1.9, we have

**Lemma 1.12.** A tableau \( T \) on \( \hat{\theta} \) is a standard tableaux if and only if it satisfies the following conditions:

\[
(T'2) \quad T(a,b) < T(a,b + 1) \text{ for any } (a,b) \in \theta \text{ and } (a,b + 1) \in \theta.
\]

\[
(T'3) \quad T(a,b) < T(a + k + 1,b + k) \text{ for any } (a,b) \in \theta \text{ and } (a + k + 1,b + k) \in \theta \text{ with } k = \min\{j \in \mathbb{Z}_{\geq 0} \mid (a + j + 1,b + j) \in \hat{\theta}\}.
\]

For \( T \in \text{Tab}_\gamma(\hat{\theta}) \) and \( w \in \hat{W} \), the map \( wT : \hat{\theta} \to \mathbb{Z} \) given by \( (wT)(u) = w(T(u)) \) \( u \in \hat{\theta} \) is also a tableau on \( \theta \). The assignment \( T \mapsto wT \) gives an action of \( \hat{W} \) on \( \text{Tab}_\gamma(\hat{\theta}) \).

**Proposition 1.13 ([SV]).** For any fixed \( T \in \text{Tab}_\gamma(\hat{\theta}) \), the assignment \( w \mapsto wT \) gives the bijection \( \psi_T : W \to \text{Tab}_\gamma(\hat{\theta}) \) between sets.

**Remark 1.14.** It is possible to give the inverse image of \( \text{St}_\gamma(\hat{\theta}) \) by \( \psi_T \) explicitly. See [SV, Theorem 3.19].

We define a tableau on a (classical) skew diagram \( \theta \) as a bijection \( T : \theta \to [1,n] \), and define standard tableaux analogously. Denote by \( \text{tab}(\theta) \) and \( \text{st}(\theta) \) the set of tableaux and the set of standard tableaux on \( \theta \) respectively.

A tableau \( T \) on \( \theta \in Y^n_\gamma \) can be extended to a tableau on \( \hat{\theta} \) by the (quasi) periodicity \( T(u + \gamma) = T(u) - n \).

This gives an embedding

\[
\text{tab}(\theta) \to \text{Tab}_\gamma(\hat{\theta}),
\]

through which we often regard \( \text{tab}(\theta) \) as a subset of \( \text{Tab}_\gamma(\hat{\theta}) \).

In the sequel, we treat the case \( \gamma \in \mathbb{Z}_{\leq -1} \times \mathbb{F} \).

Then, under the identification above, \( \text{st}(\theta) \) is thought as a subset of \( \text{St}_\gamma(\hat{\theta}) \).
Define a special tableau, which we call the row reading tableau on $\theta$, by

$$(1.4.1) \quad t_\theta(a, \mu + j) = \sum_{k=1}^{a-1} (\lambda_k - \mu_k) + j \quad \text{for} \quad a \in [1, m], \quad j \in [1, \lambda_i - \mu_i],$$

and extend it to the tableau on $\hat{\theta}$. Here $\lambda_i$ and $\mu_i$ ($i \in [1, m]$) are components of $\lambda, \mu \in F_m$ such that $\theta = \lambda/\mu$. Observe that $t_\theta \in \mathbf{st}(\theta) \subseteq \text{St}_\gamma(\hat{\theta})$.

**Example 1.15.** Let $n = 7$, $\gamma = (-2, 3)$, and let $\theta = \lambda/\mu$ with $\lambda = (5, 3)$ and $\mu = (1, 0)$. Then the row reading tableau $t_\theta$ on $\hat{\theta}$ is expressed as follows:

\[
\begin{array}{cccccccc}
-6 & -5 & -4 & -3 \\
-2 & -1 & 0 \\
1 & 2 & 3 & 4 \\
5 & 6 & 7 \\
8 & 9 & 10 & 11 \\
12 & 13 & 14 \\
\end{array}
\]

Figure 2

1.5. **Connection between standard tableaux and plane partitions.** Let $\gamma \in \mathbb{Z}_{\leq -1} \times \mathbb{F}$ and $\theta \in Y^n$.

For $T \in \text{Tab}_\gamma(\hat{\theta})$, define the map $\varrho(T) : \hat{\theta} \to \mathbb{Z}$ by

$$(1.5.1) \quad \varrho(T)(u) = \left\lceil \frac{T(u) - 1}{n} \right\rceil \quad \text{for} \quad u \in \hat{\theta},$$

where $\left\lceil t \right\rceil$ ($t \in \mathbb{Q}$) denotes the maximum integer which is not greater than $t$.

In other words, $\varrho(T)$ is defined as the map $\varrho(T) : \hat{\theta} \to \mathbb{Z}$ such that $\bar{T} = T - n\varrho(T)$ is a map $\hat{\theta} \to [1, n]$ (see Figure 3).

\[
\begin{array}{cccc}
0 & 6 & 17 \\
8 & 9 \\
\end{array}
= \quad
\begin{array}{cccc}
-1 & 1 & 3 \\
1 & 1 \\
\end{array}
+ \quad
\begin{array}{cccc}
5 & 1 & 2 \\
3 & 4 \\
\end{array}
\]

Figure 3. $T = 5\varrho(T) + \bar{T}$

It is easy to see that if $T \in \text{St}_\gamma(\hat{\theta})$ then $\varrho(T)$ is a plane partition on $\hat{\theta}$. Namely, the assignment $T \mapsto \varrho(T)$ gives a map $\varrho : \text{St}_\gamma(\hat{\theta}) \to \tilde{\text{PP}}_\gamma(\theta)$.

For $T \in \text{Tab}_\gamma(\hat{\theta})$, we put

$$\hat{\theta}^T = T^{-1}([1, n]) \subset \hat{\theta}.$$ 

The following is easy.
Lemma 1.16. (i) For any $T \in \text{St}_\gamma(\hat{\theta})$, the set $\hat{\theta} T$ is a skew diagram consisting of $n$ elements.
(ii) For any $p \in \mathcal{PP}_\gamma(\theta)$ and $k \in \mathbb{Z}$, the set $p^{-1}(k)$ is a skew diagram consisting of $n$ elements.
(iii) For any $T \in \text{St}_\gamma(\hat{\theta})$ and $k \in \mathbb{Z}$, it holds that $\varrho(T)^{-1}(k) = \hat{\theta} T - k \gamma$.

Consider the set $\mathcal{W} \setminus \text{Tab}_\gamma(\hat{\theta})$ of the $\mathcal{W}$-orbits on $\text{Tab}_\gamma(\hat{\theta})$. Let $\gamma \text{St}_\gamma(\hat{\theta})$ denote the image of $\text{St}_\gamma(\hat{\theta})$ under the projection $\text{Tab}_\gamma(\hat{\theta}) \to \mathcal{W} \setminus \text{Tab}_\gamma(\hat{\theta})$.

Proposition 1.17. Let $\gamma \in \mathbb{Z}_{\leq -1} \times \mathbb{F}$ and $\theta \in Y^n_\gamma$. The map $\varrho : \text{St}_\gamma(\hat{\theta}) \to \mathcal{PP}_\gamma(\theta)$ is surjective, and moreover it factors the bijection

$$\varrho : \gamma \text{St}_\gamma(\hat{\theta}) \xrightarrow{\sim} \mathcal{PP}_\gamma(\theta).$$

Proof. Take any $p \in \mathcal{PP}_\gamma(\theta)$. Then $\theta = \bigcup_{k \in \mathbb{Z}} \theta(k)$, where we put $\theta(k) = p^{-1}(k) \cap \theta$. Observe that $p^{-1}(k)$ is a skew diagram, and so is $\theta(k)$. One can find a tableau $S$ on $\theta$ such that the restriction $S|_{\theta(k)}$ is a standard tableau on $\theta(k)$, and moreover $S(u) < S(v)$ for any $u, v \in \theta$ with $p(u) > p(v)$.

Extend $S$ to a map $\hat{\theta} \to \mathbb{Z}$ periodically by $S(u + \gamma) = S(u)$. Then, it follows that $T_p := S + np$ is a standard tableau on $\hat{\theta}$. It is obvious that $\varrho(T_p) = p$. Therefore $\varrho$ is surjective.

Let $T \in \text{St}_\gamma(\hat{\theta})$. Then there exists $\hat{T} \in \text{tab}(\theta)$ such that $T(u) = \hat{T}(u) + n\varrho(T)(u)$ for any $u \in \theta$. Observe that $w T(u) = w(\hat{T}(u)) + n\varrho(T)(u)$ for any $w \in \mathcal{W}$ and $u \in \theta$. This implies that $\varrho(T_1) = \varrho(T_2)$ if and only if $T_1 = wT_2$ for some $w \in \mathcal{W}$, and hence the map $\varrho$ factors the bijection $\gamma \text{St}_\gamma(\hat{\theta}) \xrightarrow{\sim} \mathcal{PP}_\gamma(\theta)$. \hfill \Box

1.6. **Content.** Let $\gamma \in \mathbb{Z}_{\leq -1} \times \mathbb{F}$ and $\theta \in Y^n_\gamma$. Let $c$ denote the map $\mathbb{Z} \times \mathbb{F} \to \mathbb{F}$ given by $c(a, b) = b - a$. For a tableau $T$ on $\hat{\theta}$, define a function $c_T : \mathbb{Z} \to \mathbb{F}$ by

$$c_T(i) = c(T^{-1}(i)) \quad (i \in \mathbb{Z}).$$

The function $c_T$ is called the **content** of $T$. For later use, we give several lemmas below. The first one is easy:

**Lemma 1.18.** Let $T \in \text{Tab}_\gamma(\hat{\theta})$. Then
(i) $c_T(i + n) = c_T(i) - c(\gamma)$ for any $i \in \mathbb{Z}$.
(ii) $c_T(i) = c_T(w^{-1}(i))$ for any $i \in \mathbb{Z}$ and $w \in \overline{\mathcal{W}}$.

**Lemma 1.19.** Let $S, T \in \text{St}_\gamma(\hat{\theta})$. Then $c_S = c_T$ if and only if $S = T$.

Proof. Let $T \in \text{St}_\gamma(\hat{\theta})$. Then it follows from the definition of the standard tableaux that $T(a, b) < T(a + k, b + k)$ for any $(a, b), (a + k, b + k) \in \hat{\theta}$ with $k \in \mathbb{Z}_{\geq 1}$. The statement follows easily from this property. \hfill \Box

**Lemma 1.20.** Let $T \in \text{St}_\gamma(\hat{\theta})$ and $i \in [0, n - 1]$.
(i) $c_T(i) - c_T(i + 1) \neq 0$.
(ii) $s_i T \in \text{St}_\gamma(\hat{\theta})$ if and only if $c_T(i) - c_T(i + 1) \notin \{-1, 1\}$. 
Proof. Follows easily from the skew property and the definition of the standard tableaux. \(\square\)

**Lemma 1.21.** Let \(T \in \text{St}_\gamma(\hat{\theta})\). Let \(w \in \hat{\mathcal{W}}\) and \(i \in [0, n-1]\) such that \(l(s_i w) < l(w)\). If \(wT \in \text{St}_\gamma(\hat{\theta})\) then \(s_i wT \in \text{St}_\gamma(\hat{\theta})\).

Proof. It follows from \(l(s_i w) < l(w)\) that \(w^{-1}(i) > w^{-1}(i+1)\). Put \(S = wT\) and suppose that \(S \in \text{St}_\gamma(\hat{\theta})\) and \(s_i S \notin \text{St}_\gamma(\hat{\theta})\). Then \(c_S(i) - c_S(i+1) = \pm 1\). Suppose that \(c_S(i) - c_S(i+1) = 1\). Then it follows from \(S \in \text{St}_\gamma(\hat{\theta})\) that \(S^{-1}(i) + (j + 1, j)\) for some \(j \in \mathbb{Z}_{\geq 0}\). But then we have \(T(S^{-1}(i)) = w^{-1}(i) > w^{-1}(i+1) = T(S^{-1}(i+1))\). This contradicts \(T \in \text{St}_\gamma(\hat{\theta})\). Similarly we have a contradiction in the case \(c_S(i) - c_S(i+1) = -1\). Hence \(s_i S = s_i w T \in \text{St}_\gamma(\hat{\theta})\). \(\square\)

### 2. Generating functions

Let \(\theta\) be a skew diagram. For \(p \in \text{PP}(\theta)\), define

\[
|p| = \sum_{u \in \theta} p(u).
\]

For a subset \(A\) of \(\text{PP}(\theta)\), we define the generating function for \(S\) by

\[
\Psi(A; q) = \sum_{p \in A} q^{|p|}.
\]

The purpose of this section is to compute the generating functions \(\Psi(\text{PP}_R^\gamma(\theta); q)\) and \(\Psi(\text{PP}_C^\gamma(\theta); q)\).

#### 2.1. Count on single columns.

As a first example, we compute the generating functions for a special skew diagram \(\delta_n = \{(a, 1) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, n]\}\) by a naive enumerative method. We will see in Section 5.4 that they are related with character formulas for finite-dimensional representations of the rational Cherednik algebra of type \(A\) obtained by Berest-Etingof-Ginzburg [BEG2].

It turns out that the computation is easy for most \(\kappa \in \mathbb{P} \setminus \mathbb{Q}_{\leq 0}\) (see Lemma 5.14), and the only interesting case is

\[
(2.1.1) \quad \kappa = n/r \text{ with } r \in \mathbb{Z}_{\geq 1}, \ (n, r) = 1,
\]

which we treat in the sequel. For \(k \in \mathbb{Z}_{\geq 1}\), put

\[
(2.1.2) \quad [k]_q! = (1 - q)(1 - q^2) \ldots (1 - q^k),
\]

and put \([0]_q! = 1\).

**Proposition 2.1.** Let \(r \in \mathbb{Z}_{\geq 1}\) such that \((n, r) = 1\). Put \(\gamma = (-n, n/r - n)\). Then

\[
\Psi(\text{PP}_\gamma(\delta_n); q) = \Psi(\text{PP}_R^\gamma(\delta_n); q) = \frac{[n + r - 1]_q!}{[n]_q! [r]_q!},
\]
Proof. Let $p \in \text{PP}_\gamma(\delta_n)$. It follows from Lemma \[1.3\] that $p \in \text{PP}_\gamma(\delta_n)$ if and only if $p(n, 1) \leq p(1 + r n, 1) = p(1, 1) + r$. Identifying the diagram $\delta_n$ with the set $[1, n]$ via the correspondence $(a, 1) \mapsto a$, we have

$$
\text{PP}_\gamma(\delta_n) = \text{PP}_\gamma^R(\delta_n) = \{p : [1, n] \to \mathbb{Z}_{\geq 0} | p(1) \leq p(2) \leq \cdots \leq p(n) \leq p(1) + r\},
$$

and

$$
\text{PP}_\gamma^C(\delta_n) = \{p : [1, n] \to \mathbb{Z}_{\geq 0} | p(1) < p(2) \leq \cdots < p(n) < p(1) + r\}.
$$

Note that $\text{PP}_\gamma^R(\delta_n) = \sqcup_{k \in \mathbb{Z}_{\geq 0}} \text{PP}_\gamma^R(\delta_n)_k$ and $\text{PP}_\gamma^C(\delta_n) = \sqcup_{k \in \mathbb{Z}_{\geq 0}} \text{PP}_\gamma^C(\delta_n)_k$, where

$$
\text{PP}_\gamma^R(\delta_n)_k = \{p \in \text{PP}_\gamma^R(\delta_n) | p(1) = k\}
$$

and

$$
\text{PP}_\gamma^C(\delta_n)_k = \{p \in \text{PP}_\gamma^C(\delta_n) | p(1) = k\}.
$$

The generating functions for these sets have been calculated classically (see e.g., \[1.3\] Section 1.3). In particular for $k = 0$, we have

$$
\Psi(\text{PP}_\gamma^R(\delta_n)_0; q) = \frac{[n + r - 1]_q!}{[n - 1]_q! [r]_q!},
$$

and

$$
\Psi(\text{PP}_\gamma^C(\delta_n)_0; q) = \begin{cases} 
\frac{[r-1]_q!}{[n-1]_q! [r-n]_q!} & \text{if } r \geq n, \\
0 & \text{if } r < n.
\end{cases}
$$

Noting the equalities $\Psi(\text{PP}_\gamma^R(\delta_n)_k; q) = q^k \Psi(\text{PP}_\gamma(\delta_n)_0; q)$ and $\Psi(\text{PP}_\gamma^C(\delta_n)_k; q) = q^k \Psi(\text{PP}_\gamma^C(\delta_n)_0; q)$, the statement follows. \hfill $\square$

2.2. Kostka polynomial. The main purpose in the rest of Section 2 is to compute the generating function

$$
\Psi(\text{PP}_\gamma^C(\theta); q) = \sum_{p \in \text{PP}_\gamma^C(\theta)} q^{\|p\}.
$$

when $\theta \subset \mathbb{Z} \times \mathbb{Z}$. It turns out that $\Psi(\text{PP}_\gamma^C(\theta); q)$ is expressed by level restricted Kostka polynomials, and it is proved using Lascoux-Schützenberger type expression \[LS\], which we will see below.

Put

$$
Z^Y_n = Y_n \cap (\mathbb{Z} \times \mathbb{Z}), \quad Z^\gamma_n = Y^\gamma_n \cap (\mathbb{Z} \times \mathbb{Z}).
$$

Let $\theta \subset Z^Y_n$. For $T \in \text{tab}(\theta)$ and $i \in [1, n]$, define

$$
d_i(T) = \begin{cases} 
1 & \text{if } T^{-1}(i + 1) - T^{-1}(i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}, \\
0 & \text{otherwise},
\end{cases}
$$

(2.2.1)
and put

\[ h_T = \sum_{i \in [1,n]} \left( \sum_{k \in [1,i-1]} d_k(T) \right) \epsilon_i \in P \] where we put $P = \{ \zeta \in P \mid \langle \zeta \mid \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+ \}$. 

Then $\langle h_T \mid \alpha^\vee \rangle = -d_i(T)$ for $i \in [1,n]$ and hence $h_T \in P^-$.

For $\zeta \in P$, we write $|\zeta| = \sum_{i \in [1,n]} \langle \zeta \mid \epsilon_i^\vee \rangle$.

Define

\[ \hat{K}_\theta(1^n)(q) = \sum_{T \in \text{st}(\theta)} q^{|h_T|}, \]
\[ K_\theta(1^n)(q) = q^{\frac{1}{2}(n-1)} \hat{K}_\theta(1^n)(q^{-1}), \]

where $(1^n)$ denote the partition $(1,1,\ldots,1)$ of $n$. The polynomial $K_\theta(1^n)(q)$ is called the Kostka polynomial associated with the skew diagram $\theta$ and the partition $(1^n)$. It is known that

\[ K_\theta(1^n)(q) = \hat{K}_{\theta'}(1^n)(q), \]

where $\theta'$ is the conjugate of $\theta$: $\theta' = \{(b,a) \mid (b,a) \in \theta\}$.

Remark 2.2. For partitions $\lambda$ and $\mu$, the Kostka polynomials $K_{\lambda \mu}(q)$ can be defined as the coefficients in the expansion

\[ s_\lambda(z_1,\ldots,z_n) = \sum_{\mu \vdash n} K_{\lambda \mu}(q)p_\mu(z_1,\ldots,z_n;q), \]

where $s_\lambda$ is the Schur polynomial and $p_\mu$ is the Hall-Littlewood polynomial (see e.g., [Mac]). The expression of the Kostka polynomials of the form (2.2.3) (2.2.4) is given in [LS].

2.3. Generating functions for classical plane partitions. Concerning the computation of the generating function $\Psi(PP^C(\theta);q)$ for (classical) column strict plane partitions, several approaches have been known. (See [St1] [St3, Section 7.21] [Mac, Example 1.5.12].) We show the formula for $\Psi(PP^C(\theta);q)$ using the following statement, and will generalize it to the periodic case later.

Recall that we identify $\text{st}(\theta)$ as a subset of $\text{St}_1(\hat{\theta}) \subset \text{Map}(\hat{\theta},Z)$.

Proposition 2.3. Let $\theta \in \mathcal{Y}^n$. The assignment $\zeta \mapsto \varrho(t_\zeta+h_TT)$, where $\varrho$ is given by (1.5.1), gives a bijection

\[ P^- \times \text{st}(\theta) \cong \widehat{PP}^C(\theta). \]

Proof. It follows from the definition of $h_T$ that $\varrho(t_{h_T}T) \in \widehat{PP}^C(\theta)$ for any $T \in \text{st}(\theta)$. Hence $\varrho(t_\zeta+h_TT) \in \widehat{PP}^C(\theta)$ for any $\zeta \in P^-$ and $T \in \text{st}(\theta)$. 

To show that this map is a bijection, we define a map \( \beta : \widehat{PP}^C(\theta) \to P^- \times st(\theta) \) as follows. Take \( p \in PP^C(\theta) \). Introduce an order in the set \( \theta \) by

\[
(a, b) \prec (a', b') \iff \begin{cases} \quad p(a, b) < p(a', b'), \text{ or} \\ \quad p(a, b) = p(a', b') \text{ and } b < b'. \end{cases}
\]

Observe that this gives a total order because \( p \) is column strict.

Let \( u_1, \ldots, u_n \) be the elements in \( \theta \) such that \( u_1 \prec u_2 \prec \cdots \prec u_n \). Define \( T_p \in \text{tab}(\theta) \) by \( T_p(u_i) = i \), and define \( \zeta_p \in P \) by \( \langle \zeta_p | \epsilon_i^\gamma \rangle = p(u_i) - \langle h_{T_p} | \epsilon_i^\gamma \rangle \). Then it is easy to check that \( T_p \in \text{st}(\theta) \).

Let us check \( \zeta_p \in P^- \). We have

\[
\langle \zeta_p | \alpha_i^\gamma \rangle = p(u_i) - p(u_{i+1}) - \langle h_{T_p} | \alpha_i^\gamma \rangle.
\]

Observe that \( \langle h_{T_p} | \alpha_i^\gamma \rangle = -d_i(T_p) \geq -1 \).

Suppose that \( \langle \zeta_p | \alpha_i^\gamma \rangle > 0 \). This occurs only if \( p(u_i) - p(u_{i+1}) = 0 \) and \( \langle h_{T_p} | \alpha_i^\gamma \rangle = -1 \). The latter equality implies \( u_{i+1} - u_i \in Z_{\geq 1} \times Z \) by (2.3.1). By \( p(u_i) = p(u_{i+1}) \), it follows from the definition (2.3.1) of the order \( \prec \) that \( u_{i+1} - u_i \in Z \times Z_{\geq 1} \). Hence \( u_{i+1} - u_i \in Z_{\geq 1} \times Z_{\geq 1} \), and this implies \( p(u_{i+1}) = p(u_i) \) as \( p \) is column strict. This is a contradiction. Therefore \( \langle \zeta_p | \alpha_i^\gamma \rangle \geq 0 \) for any \( i \in [1, n] \), namely, \( \zeta_p \in P^- \).

We set \( \beta(p) = (\zeta_p, T_p) \). It is easy to see that \( \beta \) gives an inverse map of the map \( (\zeta, T) \mapsto \varrho(t_{\zeta+h_T}T) \).

By restricting the map in Proposition 2.3 we have a bijection

\[
P_o^- \times \text{st}(\theta) \sim \widehat{PP}^C(\theta),
\]

where we put

\[
P_o^- = \{ \zeta \in P^- | \langle \zeta | \epsilon_i^\gamma \rangle \geq 0 \}.
\]

As a consequence, we have the following well-known result.

**Proposition 2.4 (SI).** Let \( \theta \in \mathbb{Z}^n \). Then

\[
\Psi(PP^R(\theta); q) = \frac{1}{|n|!}K_{\theta(1^n)}(q),
\]

\[
\Psi(PP^C(\theta); q) = \frac{1}{|n|!}K_{\theta(1^n)}(q).
\]

**Proof.** Note that \( |\varrho(t_{\zeta+h_T}T)| = \sum_{i \in [1, n]} \langle \zeta + h_T | \epsilon_i^\gamma \rangle = |\zeta| + |h_T| \). The bijection (2.3.2) derived from Proposition 2.3 implies

\[
\Psi(PP^C(\theta); q) = \sum_{\zeta \in P_o^-} \sum_{T \in \text{st}(\theta)} q^{|\zeta| + |h_T|} = \frac{1}{|n|!}K_{\theta(1^n)}(q).
\]

To show the formula for \( \Psi(PP^R(\theta); q) \), we consider the conjugate operation. For \( p \in PP^R(\theta) \), the function \( p'(a,b) = p(b,a) \) for \( (b, a) \in \theta \) gives a column strict plane partition on \( \theta' \). We have \( |p| = |p'| \) and the correspondence \( p \mapsto p' \) gives a bijection \( PP^R(\theta) \sim PP^C(\theta') \). Hence \( \Psi(PP^R(\theta); q) = \Psi(PP^C(\theta'); q) = K_{\theta'(1^n)}(q) = K_{\theta(1^n)}(q) \).
2.4. **Restricted tableaux.** In the rest of this section we restrict ourselves to the case where \( \theta \in \mathbb{Z}^n_\gamma \) with \( \epsilon(\gamma) \in \mathbb{Z} \geq 1 \).

Let \( \kappa \in \mathbb{Z} \geq 1 \) and \( m \in \mathbb{Z} \geq 1 \). Put \( \gamma = (-m, \kappa - m) \) and let \( \theta \in \mathbb{Z}^n_\gamma \). Then \( \theta \) is expressed as \( \theta = \lambda/\mu \) for some \( \lambda, \mu \in \mathcal{D}_{m, \kappa} \). Here, recall that \( \mathcal{D}_{m, \kappa} \) is defined as the subset of \( \mathbb{F}^m \) consisting of the elements satisfying the dominance condition (1.2.1) (1.2.2).

For \( T \in \text{st}(\theta) \), we have a sequence \( T^{-1}(\{1\}), T^{-1}(\{1, 2\}), \ldots, T^{-1}(\{1, n\}) \) of skew diagrams. Define \( \lambda^{(k)}_T \in \mathbb{F}^m \) as the unique element such that \( T^{-1}(\{1, k\}) = \lambda^{(k)}_T/\mu \).

**Definition 2.5.** A standard tableaux \( T \) on \( \theta \) is said to be \( \gamma \)-restricted if \( \lambda^{(k)}_T \in \mathcal{D}_{m, \kappa} \) for all \( k \in \{1, n\} \).

Define
\[
(2.4.1) \quad \text{st}_\gamma(\theta) = \{ T \in \text{st}(\theta) \mid T \text{ is } \gamma \text{-restricted} \}.
\]

**Lemma 2.6.** (i) If \( T \in \text{st}_\gamma(\theta) \), then for any \( i, j \in \{1, n\} \) with \( i > j \), it holds that \( c_T(j) - c_T(i) \leq \kappa - 2 \).

(ii) If \( T \in \text{st}(\theta) \) and \( T \notin \text{st}_\gamma(\theta) \), then there exist \( i, j \in \{1, n\} \) such that \( i > j \) and \( c_T(j) - c_T(i) = \kappa - 1 \).

**Proof.** (i) Suppose that \( T \in \text{st}_\gamma(\theta) \). Take any \( i \in \{1, n\} \). It follows from \( \lambda^{(i)}_T \in \mathcal{D}_{m, \kappa} \) that \( c_T(j) - c_T(i) \leq \kappa - 1 \) for any \( j \in \{1, i - 1\} \). Suppose that the equality holds. Then, it is easy to see that \( T^{-1}(j) - T^{-1}(i) \in \mathbb{Z} \) (namely, \( T^{-1}(i) \) is located below \( T^{-1}(j) \)). But this implies \( \lambda^{(i-1)}_T \notin \mathcal{D}_{m, \kappa} \), and this is a contradiction.

(ii) Suppose that \( T \notin \text{st}_\gamma(\theta) \). Since \( \lambda^{(n)}_T = \lambda \in \mathcal{D}_{m, \kappa} \), there exists \( i \in \{1, n\} \) such that \( \lambda^{(i-1)}_T \notin \mathcal{D}_{m, \kappa} \) and \( \lambda^{(i)}_T \in \mathcal{D}_{m, \kappa} \). It is easy to see that \( c_T(j) - c_T(i) = \kappa - 1 \), where \( j \) is a number such that \( c_T(j) = \max\{c_T(k) \mid k \in \{1, i - 1\}\} \).

Recall that we regard \( \text{st}(\theta) \) as a subset of \( \text{St}_\gamma(\hat{\theta}) \).

**Proposition 2.7.** Let \( m \in \mathbb{Z} \geq 1 \), \( \kappa \in \mathbb{Z} \geq 1 \) and \( \theta \in \mathbb{Z}^n_\gamma \) with \( \gamma = (-m, \kappa - m) \).

(i) The assignment \( (\zeta, T) \mapsto t_{\zeta + h_T} T \) gives an embedding
\[
P^- \times \text{st}_\gamma(\theta) \rightarrow \text{St}_\gamma(\hat{\theta}).
\]

(ii) The assignment \( (\zeta, T) \mapsto \phi(t_{\zeta + h_T} T) \) gives bijections
\[
P^- \times \text{st}_\gamma(\theta) \sim \widetilde{\text{PP}}^C_{\gamma}(\theta), \quad P^- \times \text{st}_\gamma(\theta) \sim \text{PP}^C_{\gamma}(\theta).
\]

**Proof.** In this proof, we put \( \xi_i = \langle \zeta + h_T \mid \xi_i \rangle \) for \( i \in \{1, n\} \).

(i) First we shall prove that the image of \( P^- \times \text{st}_\gamma(\theta) \) is included in \( \text{St}_\gamma(\hat{\theta}) \).

Let \( \zeta \in P^- \) and \( T \in \text{st}(\theta) \). Put \( S = t_{\zeta + h_T} T \).

Let us check the row increasing condition (T2) in Definition 1.11. Take any \( (a, b), (a, b + 1) \in \theta \). Put \( i = T(a, b) \) and \( j = T(a, b + 1) \). Then we have \( i < j \) as \( T \in \text{st}(\theta) \), and we have \( \xi_i < \xi_j \) as \( \zeta + h_T \in P^- \). Hence
\[
S(a, b) = t_{\zeta + h_T} T(a, b) = i + n\xi_i < j + n\xi_j = S(a, b + 1).
\]

Therefore \( S \) satisfies the condition (T2).
Let us check the condition (T3). Since \( \hat{\theta} \subset Z \times Z \), it is enough to check \( S(u) < S(v) \) for any \( u = (a, b) \in \theta \) and \( v = (a + k + 1, b + k) \in \hat{\theta} \) with \( k \in \mathbb{Z}_{\geq 0} \).

Put \( i = T(u) \in [1, n] \). Let \( p \in \mathbb{Z}_{\geq 0} \) be the number such that \( v + pr \in \theta \).

Suppose that \( p = 0 \), namely, \( v \in \theta \). Then \( T(v) = j \) for some \( j \in [1, n] \), and, by similar argument as above, we have \( i < j \) and \( \xi_i \leq \xi_j \), and hence \( S(u) < S(v) \).

Suppose that \( v + pr \in \theta \) with \( p > 0 \). Then \( T(v) = j + pn \) for some \( j \in [1, n] \).

We have \( c_T(j) - c_T(i) = c_T(j + pn) + pk - c_T(i) = p(k - 1) > \kappa - 2 \). By \( T \in \text{st}_\gamma(\hat{\theta}) \), Lemma 2.6 implies \( i < j \), and hence we have \( \xi_i \leq \xi_j \) as \( \zeta + h_T = P^- \). We have

\[
S(u) = t_{\zeta + h_T}T(u) = i + n\xi_i < j + n(\xi_j + p) = t_{\zeta + h_T}T(v) = S(v).
\]

Hence \( S \) satisfies the condition (T3). We have proved \( S \in \text{St}_\gamma(\hat{\theta}) \).

Now, we shall prove the statement (ii). Note that this implies the injectivity of the map \( P^- \times \text{st}_\gamma(\hat{\theta}) \to \text{St}_\gamma(\hat{\theta}) \).

(ii) (Step 1) We take \( \zeta \in P^- \) and \( T \in \text{st}_\gamma(\theta) \), and shall prove that \( \varrho(t_{\zeta + h_T}T) \in \overline{\text{PP} \gamma}(\theta) \).

Put \( S = t_{\zeta + h_T}T \). We have shown \( S \in \text{St}_\gamma(\hat{\theta}) \), and hence \( \varrho(S) \in \overline{\text{PP} \gamma}(\theta) \) by Proposition 1.17. Assume that \( \varrho(S) \not\in \overline{\text{PP} \gamma}(\theta) \). Then there exist \( u = (a, b) \in \hat{\theta} \) and \( v = (a + k + 1, b + k) \in \hat{\theta} \) with \( k \in \mathbb{Z}_{\geq 0} \) such that \( \varrho(S)(u) = \varrho(S)(v) \).

We may assume without loss of generality that \( \varrho(S)(u) = 0 \). Then, putting \( i = S(u) \) and \( j = S(v) \), we have \( i, j \in [1, n] \). We have \( T(u) = t_{-\zeta - h_T}S(u) = i - n\xi_i \) and, similarly, \( T(v) = j - n\xi_j \).

Since \( T \in \text{st}(\theta) \subset \text{St}_\gamma(\hat{\theta}) \), it must hold that \( \xi_i \geq \xi_j \). On the other hand, since we have shown that \( S \in \text{St}_\gamma(\hat{\theta}) \), it must hold that \( i < j \), and hence \( \xi_i \leq \xi_j \) as \( \zeta + h_T = P^- \). Therefore we have \( \xi_i = \xi_j \).

Look at two elements \( T^{-1}(i) = u - \xi_j \gamma \) and \( T^{-1}(j) = v - \xi_j \gamma \) in \( \theta \). Since \( i < j \) and \( a + \xi_i m < a + k + 1 + \xi_j m = a + k + 1 + \xi_j m \), it follows from the definition of \( h_T \) that \( \langle h_T \mid \epsilon_i \rangle < \langle h_T \mid \epsilon_j \rangle \). Combining with \( \zeta \in P^- \), this implies \( \xi_i < \xi_j \). This is a contradiction. Therefore \( \varrho(S) \in \overline{\text{PP} \gamma}(\theta) \).

(Step 2) Recall the bijection \( P^- \times \text{st}(\theta) \xrightarrow{\sim} \overline{\text{PP} \gamma}(\theta) \) in Proposition 2.8. We have to show is the surjectivity of the map \( P^- \times \text{st}_\gamma(\hat{\theta}) \to \overline{\text{PP} \gamma}(\theta) \).

Let \( T \in \text{st}(\theta) \) and \( \zeta \in P^- \), and suppose that \( \varrho(t_{\zeta + h_T}T) \in \overline{\text{PP} \gamma}(\theta) \). We shall show that \( T \in \text{st}_\gamma(\hat{\theta}) \). Note that this will complete the proof of the statement.

Suppose that \( T \notin \text{st}_\gamma(\hat{\theta}) \). Then, by Lemma 2.6 there exist \( i, j \in [1, n] \) such that \( i > j \) and \( c_T(j) - c_T(i) = \kappa - 1 \). Combining with \( T \in \text{st}(\theta) \), we have \( a > a' \), where we put \( T^{-1}(i) = (a, b) \) and \( T^{-1}(j) = (a', b') \).

Observe that \( (a', b') - \gamma = (a, b) + (k + 1, k) \), where \( k = m - 1 - (a - a') \in \mathbb{Z}_{\geq 0} \).

By \( i > j \) and \( a > a' \), it follows from the definition of \( h_T \) that \( \langle h_T \mid \epsilon_i \rangle > \langle h_T \mid \epsilon_j \rangle \), and hence \( \xi_i > \xi_j \).

We have \( t_{\zeta + h_T}(a, b) = i + n\xi_i \) and

\[
t_{\zeta + h_T}(a + k + 1, b + k) = t_{\zeta + h_T}((a', b') - \gamma) = t_{\zeta + h_T}(j + n) = j + n(\xi_j + 1),
\]
and hence $g(t_{\xi+\mathbf{h}_T}T)(a,b) = \xi_i$ and $g(t_{\xi+\mathbf{h}_T}T)(a+k+1,b+k) = \xi_j + 1$. The assumption $g(t_{\xi+\mathbf{h}_T}T) \in \mathcal{PP}_C^C(\theta)$ implies $\xi_i < \xi_j + 1$. This is a contradiction and hence $T \in \mathbf{st}_\gamma(\theta)$. The statement has been proved. 

2.5. Generating functions and level restricted Kostka polynomials. Define 

\[ \tilde{K}^{(\kappa-m)}_{\theta}(\mathcal{1}^n)(q) = \sum_{T \in \mathbf{st}_\gamma(\theta)} q^{|\mathbf{h}_T|}. \] 

(2.5.1)

The polynomials given by $K^{(\kappa-m)}_{\theta}(\mathcal{1}^n)(q) = q^{\frac{1}{2}n(n-1)} \tilde{K}^{(\kappa-m)}_{\theta}(\mathcal{1}^n)(q^{-1})$ is called the level restricted Kostka polynomial of level $\kappa-m$ associated with the skew diagram $\theta$ and the partition $(1^n)$ (see e.g. [SS]). We obtain a periodic analogue (or level restricted analogue) of the classical formulas in Proposition 2.4.

**Theorem 2.8.** Let $\kappa, m \in \mathbb{Z}_{\geq 1}$ and put $\gamma = (-m, \kappa - m)$. Let $\theta \in \bar{\mathcal{Y}}^n_\gamma$. Then 

\[ \Psi(\mathcal{PP}_C^C(\theta); q) = \mathbf{1}_{[n]} q! \tilde{K}^{(\kappa-m)}_{\theta}(\mathcal{1}^n)(q^{-1}). \]

*Proof.* Follows from Proposition 2.4(ii) using $|g(t_{\xi+\mathbf{h}_T}T)| = |\xi + \mathbf{h}_T| = |\xi| + |\mathbf{h}_T|$. 

**Remark 2.9.** The formula for $\Psi(\mathcal{PP}_R^R(\theta); q)$ (for $\kappa \in \mathbb{Z}_{\geq 1}$) has not been obtained by the argument above. Classically, to compute the generating function $\Psi(\mathcal{PP}_R^R(\theta); q)$ for the set of row strict plane partitions is an equivalent problem with to compute $\Psi(\mathcal{PP}_C^C(\theta); q)$ for the set of column strict plane partitions, since they are transferred to each other through the conjugate (transpose) operation.

On the other hand, for plane partitions on a periodic diagram, we do not have $\mathcal{PP}_R^R(\theta) \cong \mathcal{PP}_C^C(\theta)$ in general any more. Actually, plane partitions (and standard tableaux) on a periodic diagram of period $\gamma$ are transformed by the conjugation into those on the conjugated periodic diagram of period $\gamma'$, for which we have $c(\gamma') = -c(\gamma) = -\kappa$. For a period with negative content, our method can be applied to compute $\Psi(\mathcal{PP}_R^R(\theta'); q)$, but fails to compute $\Psi(\mathcal{PP}_C^C(\theta'); q)$.*

3. Tableaux representations of the trigonometric Cherednik algebra

We apply combinatorics on periodic diagrams to the study of the representation theory of Cherednik algebras.

First, we give a combinatorial construction of a class of representations of the trigonometric Cherednik algebra by modifying the construction in [SV] for the double affine Hecke algebra.*

Another approach to compute the generating functions is to use an interpretation of plane partitions in terms of non-intersecting lattice paths. By this method, Gessel and Krattenthaler obtained a determinant expression of the generating functions for cylindrical partitions ([GR, Theorem 2, Theorem 3]). Connection between their results and our formula in Theorem 2.8 will be discussed in another place.

*Another approach to compute the generating functions is to use an interpretation of plane partitions in terms of non-intersecting lattice paths. By this method, Gessel and Krattenthaler obtained a determinant expression of the generating functions for cylindrical partitions ([GR, Theorem 2, Theorem 3]). Connection between their results and our formula in Theorem 2.8 will be discussed in another place.
3.1. Trigonometric Cherednik algebra. We introduce several more notations.

For a set $G$, we let $\mathbb{F}G$ denote the vector space of $\mathbb{F}$-valued functions on $G$. In particular, if $G$ is a (semi)group, then $\mathbb{F}G$ denotes the group algebra.

We let $\mathbb{F}[x]$ denote the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$, and let $\mathbb{F}[x^\pm 1]$ denote the Laurent polynomial ring $\mathbb{F}[x_1^\pm 1, \ldots, x_n^\pm 1]$.

For a vector space $V$, we let $S(V)$ denote the symmetric algebra of $V$.

For an algebra $A$, we denote by $A$-mod the category of finitely generated $A$-modules.

For an element $\eta = \sum_{i \in [1, n]} \eta_i e_i$ of $P$, we denote by $x^\eta = x_1^{\eta_1} x_2^{\eta_2} \cdots x_n^{\eta_n}$ the corresponding element in the group algebra $\mathbb{F}P$. Via the correspondence $\eta \mapsto x^\eta$, we often identify $\mathbb{F}P$ with the Laurent polynomial ring $\mathbb{F}[x^\pm 1]$. The action of $\mathcal{W}$ on $\mathbb{F}P = \mathbb{F}[x^\pm 1]$ induced from the action on $P$ is given by the permutation of variables $x_1, \ldots, x_n$.

**Definition 3.1.** For $\kappa \in \mathbb{F}$, the trigonometric Cherednik algebra (or the degenerate double affine Hecke algebra) $\tilde{\mathcal{H}}_\kappa$ of type $GL_n$ is defined as the unital associative $\mathbb{F}$-algebra generated by the algebras $\mathbb{F}P$, $\mathbb{F}\mathcal{W}$ and $S(\mathfrak{h})$ with the following relations:

\[
s_i \hbar = s_i(h) s_i - \langle \alpha_i | h \rangle \quad (i \in [1, n-1], \ h \in \mathfrak{h}),
\]

\[
s_i x^\eta s_i^{-1} = x^{s_i(\eta)} \quad (i \in [1, n-1], \ \eta \in P),
\]

\[
[h, x^\eta] = \kappa \langle \eta | h \rangle x^\eta + \sum_{\alpha \in \mathbb{R}^+} \langle \alpha | h \rangle (x^\eta - x^{s_\alpha(\eta)}) \frac{1}{1 - x^{-\alpha}} s_\alpha \quad (h \in \mathfrak{h}, \ \eta \in P).
\]

Define the degenerate affine Hecke algebra of type $GL_n$ as the subalgebra of $\tilde{\mathcal{H}}_\kappa$ generated by $\mathbb{F}\mathcal{W}$ and $S(\mathfrak{h})$. Observe that $\mathcal{H}^{\text{aff}}$ is a subalgebra of $\tilde{\mathcal{H}}_\kappa$.

It is known due to Cherednik that the natural multiplication map in $\tilde{\mathcal{H}}_\kappa$ induces a linear isomorphism

\[\mathbb{F}P \otimes \mathbb{F}\mathcal{W} \otimes S(\mathfrak{h}) \xrightarrow{\sim} \tilde{\mathcal{H}}_\kappa.\]

Define the category $\mathcal{O}(\tilde{\mathcal{H}}_\kappa)$ as the full subcategory of $\tilde{\mathcal{H}}_\kappa$-mod consisting of the modules which are locally finite for $S(\mathfrak{h})$, i.e., a finitely generated $\tilde{\mathcal{H}}_\kappa$-module $M$ is in $\mathcal{O}(\tilde{\mathcal{H}}_\kappa)$ if and only if $\dim_{\mathbb{F}} S(\mathfrak{h})v < \infty$ for any $v \in M$.

Recall that the elements $e_i^\vee \in \tilde{\mathcal{H}}_\kappa$ ($i \in [1, n]$) are realized as Cherednik-Dunkl operators (of trigonometric Dunkl operators)

\[
\kappa x_i \frac{\partial}{\partial x_i} + \sum_{\alpha \in \mathbb{R}^+} \langle \alpha | e_i^\vee \rangle \frac{1}{1 - x^{-\alpha}} (1 - s_\alpha) + i - 1
\]

on the polynomial representation $\mathbb{F}[x^\pm 1]$ of $\tilde{\mathcal{H}}_\kappa$ \((\text{[11]})\). We call the subalgebra $S(\mathfrak{h}) = \mathbb{F}[e_1^\vee, \ldots, e_n^\vee]$ the Cherednik-Dunkl subalgebra.

For $\zeta \in \mathfrak{h}^+$ and an $S(\mathfrak{h})$-module $M$, define

\[M_\zeta = \{ v \in M | e_i^\vee v = \langle \zeta | e_i^\vee \rangle v \ \forall i \in [1, n] \}\]

An element $\zeta$ of $\mathfrak{h}^+$ is said to be a weight of $M$ if $M_\zeta \neq 0$, and an element of $M_\zeta$ a weight vector of weight $\zeta$. 

Define $O^h(\widehat{H}_\kappa)$ as the full subcategory of $O(\widehat{H}_\kappa)$ consisting of those modules $M$ such that $M = \bigoplus_{\zeta \in \mathfrak{h}^*} M_\zeta$.

3.2. **Tableaux representations.** Now, we introduce representations of $\widehat{H}_\kappa$ associated with periodic diagrams.

Fix $\kappa \in \mathbb{F}$. Let $m \in \mathbb{Z}_{\geq 1}$ and put $\gamma = (-m, \kappa - m)$.

For $\theta \in \mathbf{Y}_n^\gamma$, we denote by $\tilde{\mathcal{V}}_\gamma(\hat{\theta})$ the space $\mathbb{F}\text{St}_\gamma(\hat{\theta})$ of functions on the set $\text{St}_\gamma(\hat{\theta})$, and denote by $v_T$ the element of $\tilde{\mathcal{V}}_\gamma(\hat{\theta})$ corresponding to $T \in \text{St}_\gamma(\hat{\theta})$. Namely, we set

$$
\tilde{\mathcal{V}}_\gamma(\hat{\theta}) = \mathbb{F}\text{St}_\gamma(\hat{\theta}) = \bigoplus_{\hat{T} \in \text{St}_\gamma(\hat{\theta})} \mathbb{F}v_T.
$$

For $T \in \text{tab}(\theta)$, define $c_T \in \mathfrak{h}^*$ by

$$(3.2.1) \quad c_T = \sum_{i \in [1,n]} c_T(i)\epsilon_i,$$

where $c_T$ is the content of $T$.

**Lemma 3.2.** (i) Let $\theta \in \mathbf{Y}_n^\gamma$ and $T \in \text{st}(\theta)$. Let $M$ be an $\mathcal{H}^\text{aff}$-module and suppose that $M_{c_T} \neq 0$. Then $\mathcal{H}^\text{aff}v \cap M_{c_T} \neq 0$ for any $v \in M_{c_T}$ and $S \in \text{st}(\theta)$.

(ii) Let $\theta \in \mathbf{Y}_n^\gamma$ and $T \in \text{St}_\gamma(\hat{\theta})$. Let $M$ be an $\widehat{H}_\kappa$-module and suppose that $M_{c_T} \neq 0$. Then $\widehat{H}_\kappa v \cap M_{c_T} \neq 0$ for any $v \in M_{c_T}$ and $S \in \text{St}_\gamma(\hat{\theta})$.

**Proof.** We will prove the statement (ii). The statement (i) follows similarly.

Let $v \in M_{c_T}$. Let $S \in \text{St}_\gamma(\hat{\theta})$ and take $w \in \widehat{W}$ such that $S = wT$. We will prove that $\widehat{H}_\kappa v \cap M_{c_T} = \widehat{H}_\kappa v \cap M_{c_T} 
eq 0$ by induction on the length $l(w)$ of $w$. If $l(w) = 0$ then $w = \pi^k$ and we have $\pi^k v \in M_{c_T} \setminus \{0\}$, and the statement holds.

Suppose that $l(w) = k$ and the claim holds for all $z \in \widehat{W}$ with $l(z) < k$. Take $i \in [0,n-1]$ and $z \in \widehat{W}$ such that $w = s_i z$ and $l(w) = l(z) + 1$. Then by Lemma [1.21] we have $zT \in \text{St}_\gamma(\hat{\theta})$, and by the induction hypothesis, we can find a non-zero element $v'$ in $\mathcal{H}_\kappa v \cap M_{c_T}$. Put $c = c_{zT}(i) - c_{zT}(i + 1)$. Then it is easy to check that $(1 + cs_i)v' \in M_{c_{zT}}$. Note that $c \neq \pm 1$ by Lemma [1.20] and hence $(1 - cs_i)(1 + cs_i)v' = (1 - c^2)v' \neq 0$. Therefore $(1 + cs_i)v'$ is a non-zero element in $\mathcal{H}_\kappa v \cap M_{c_{zT}}$.

**Theorem 3.3.** (cf. Theorem 3.16, Theorem 3.17 in [SV]) Let $\kappa \in \mathbb{F}$, $m \in \mathbb{Z}_{\geq 1}$ and put $\gamma = (-m, \kappa - m)$. Let $\theta \in \mathbf{Y}_n^\gamma$.

(i) There exists a unique $\widehat{H}_\kappa$-module structure on $\tilde{\mathcal{V}}_\gamma(\hat{\theta})$ such that $\text{St}_\gamma(\hat{\theta})$:

$$
\begin{align*}
\epsilon_i^\gamma v_T &= c_T(i)v_T \quad (i \in [1,n]), \\
\pi v_T &= v_{\pi T}, \\
s_i v_T &= \begin{cases} 
\frac{1 + c_T(i) - c_T(i + 1)}{c_T(i) - c_T(i + 1)} v_{s_i T} - \frac{1}{c_T(i) - c_T(i + 1)} v_T & \text{if } s_i T \in \text{St}_\gamma(\hat{\theta}) \\
-\frac{1}{c_T(i) - c_T(i + 1)} v_T & \text{if } s_i T \notin \text{St}_\gamma(\hat{\theta}) \quad (i \in [0,n-1]).
\end{cases}
\end{align*}
$$

(Note that $c_T(i) - c_T(i + 1) \neq 0$ by Lemma [1.20].)
(ii) The \( \tilde{\mathcal{V}}_\gamma(\hat{\theta}) \) admits a weight space decomposition with respect to the subalgebra \( S(\mathfrak{h}) \):

\[
\tilde{\mathcal{V}}_\gamma(\hat{\theta}) = \bigoplus_{T \in \text{St}_\gamma(\hat{\theta})} \tilde{\mathcal{V}}_\gamma(\hat{\theta})_{c_T},
\]

and, moreover, \( \tilde{\mathcal{V}}_\gamma(\hat{\theta})_{c_T} = Fv_T \) for all \( T \in \text{St}_\gamma(\hat{\theta}) \).

(iii) The \( \tilde{H}_\kappa \)-module \( \tilde{\mathcal{V}}_\gamma(\hat{\theta}) \) is irreducible.

Proof. (i) The statement is proved by verifying the defining relation of \( \tilde{H}_\kappa \) by direct calculations.

(ii) Follows directly from Lemma \[\text{4.9}\].

(iii) Take any \( S, T \in \text{St}_\gamma(\hat{\theta}) \) and take \( w \in \hat{W} \) such that \( S = wT \). By Lemma \[\text{3.2}\] we have \( v_S \in \tilde{H}_\kappa v_T \). This implies the irreducibility. \( \square \)

We call \( \tilde{\mathcal{V}}_\gamma(\hat{\theta}) \) the tableaux representation of \( \tilde{H}_\kappa \) associated to \( \hat{\theta} \).

**Remark 3.4.** For \( \kappa \in F \setminus \mathbb{Q}_{\leq 0} \), it can be shown that any irreducible module in \( \mathcal{O}_k(\tilde{H}_\kappa) \) is isomorphic to a tableaux representation \( \tilde{\mathcal{V}}_\gamma(\hat{\theta}) \) for some \( m \in [1, n] \) and \( \theta \in Y_{m, \kappa-n}^n \). A proof of this statement is given by modifying the proof of [SV] Theorem 4.20, where the corresponding theorem for the double affine Hecke algebra is proved with the restriction \( \kappa \in \mathbb{Z}_{\geq 1} \) and \( \theta \subset \mathbb{Z} \times \mathbb{Z} \).

The same statement has been given by Cherednik in [Ch4, Ch5], where the condition for two diagrams to give isomorphic representations is also given.

Similarly, putting \( \mathcal{V}^{\text{aff}}(\theta) = \bigoplus_{T \in \text{st}(\theta)} Fv_T \), we have the following:

**Proposition 3.5 ([Ch3, Ra]).** There exists a unique \( \mathcal{H}^{\text{aff}} \)-module structure on \( \mathcal{V}^{\text{aff}}(\theta) \) such that

\[
epsilon_i^\gamma v_T = c_T(i)v_T \quad (i \in [1, n]),
\]

\[
s_i v_T = \begin{cases} 
1 + c_T(i) - c_T(i+1) & \text{if } s_i T \in \text{st}(\theta) \in \mathcal{V}^{\text{aff}}(\theta) \\
-1 & \text{if } s_i T \notin \text{st}(\theta) \quad (i \in [1, n-1]).
\end{cases}
\]

Moreover, \( \mathcal{V}^{\text{aff}}(\theta) \) is an irreducible \( \mathcal{H}^{\text{aff}} \)-module.

3.3. **Restriction rule and plane partitions.** Let \( \kappa \in F, m \in \mathbb{Z}_{\geq 1} \) and put \( \gamma = (-m, \kappa - m) \). Let \( \theta \in Y_\kappa^m \). We put \( \hat{\theta}^T = T^{-1}([1, n]) \subset \hat{\theta} \) for \( T \in \text{Tab}_\gamma(\theta) \) as before. Recall the surjection \( \varrho : \text{St}_\gamma(\hat{\theta}) \rightarrow \widehat{\text{PP}}_\gamma(\theta) \) in Proposition \[\text{4.17}\] for which we have \( \hat{\theta}^T = \varrho(T)^{-1}(0) \).

**Lemma 3.6.** Let \( T \in \text{St}_\gamma(\hat{\theta}) \).

(i) Any \( S \in \text{st}(\hat{\theta}^T) \) can be extended to a standard tableau \( \hat{S} \) on \( \hat{\theta} \) by setting \( \hat{S}(u + j \gamma) = S(u) - jn \) \( (u \in \hat{\theta}^T, j \in \mathbb{Z}) \).

(ii) The assignment \( \tilde{v}_S \mapsto v_S \) \( (S \in \text{st}(\hat{\theta}^T)) \) gives an embedding \( \mathcal{V}^{\text{aff}}(\hat{\theta}^T) \rightarrow \tilde{\mathcal{V}}_\gamma(\hat{\theta}) \) as an \( \mathcal{H}^{\text{aff}} \)-module.
3.4. \[ W \text{-invariant subspace.} \] Define the elements \( e_{\pm} \) in \( \mathbb{F}W \) by

\[
e_{\pm} = \frac{1}{n!} \sum_{w \in W} w,
\]

\[
e_{-} = \frac{1}{n!} \sum_{w \in W} (-1)^{l(w)} w.
\]

The space \( e_{\pm}M \) (resp., \( e_{-}M \)) is the \( \mathcal{W} \)-invariant (resp., \( \mathcal{W} \)-anti-invariant) subspace of \( M \): \( e_{\pm}M = \{ v \in M \mid vw = (\pm 1)^{l(w)}v \ \forall w \in W \} \). The algebra \( S(h)^{\mathcal{W}} \) acts on \( e_{\pm}M \). We will give a decomposition of \( e_{\pm} \mathcal{V}_{\gamma}(\hat{\theta}) \) as an \( S(h)^{\mathcal{W}} \)-module.

Definition 3.8. A skew diagram \( \theta \) is said to be \textit{linked} if there exist \( (a, b) \in \theta \) and \( (a + k + 1, b + k) \in \theta \) with \( k \in \mathbb{Z}_{\geq 0} \).

A skew diagram \( \theta \) is said to be \textit{unlinked} if it is not linked.

Remark 3.9. When \( \theta \subset \mathbb{Z} \times \mathbb{Z} \), an unlinked diagram \( \theta \) is also called a horizontal strip.

A proof of the following proposition is given in the next section.
Proposition 3.10. Let \( \theta \) be a skew diagram. Then
\[
\dim_F e_+ \nu^{\text{aff}}(\theta) = \begin{cases} 
1 & \text{if } \theta \text{ is unlinked,} \\
0 & \text{if } \theta \text{ is linked,}
\end{cases}
\]
\[
\dim_F e_- \nu^{\text{aff}}(\theta) = \begin{cases} 
1 & \text{if } \theta' \text{ is unlinked,} \\
0 & \text{if } \theta' \text{ is linked,}
\end{cases}
\]
where \( \theta' \) is the conjugate of \( \theta \) : \( \theta' = \{(a, b) \in \mathbb{Z} \times \mathbb{F} \mid (b, a) \in \theta\} \).

For \( \zeta \in \mathfrak{h}^* \), let \( \chi_\zeta : S(\mathfrak{h})^W \to \mathbb{F} \) denote the character corresponding to the image \( \mathcal{W}_\zeta \) of \( \zeta \) under the natural projection \( \mathfrak{h}^* \to \mathcal{W} \setminus \mathfrak{h}^* \).

Let \( \gamma = (-m, \kappa - m) \) with \( m \in \mathbb{Z}_{\geq 1} \). Recall the isomorphism \( g : \nu_0 \text{St}_\gamma(\hat{\theta}) \to \overline{\text{PP}}_\gamma(\theta) \). For \( p \in \overline{\text{PP}}_\gamma(\theta) \), define \( \chi_p = \chi_{\hat{\theta}^T} \), where \( T \) is any standard tableau such that \( g(T) = p \).

Lemma 3.11. Let \( p, q \in \overline{\text{PP}}_\gamma(\theta) \). Then \( \chi_p = \chi_q \) if and only if \( p = q \).

Proof. We have the map \( \text{St}_\gamma(\hat{\theta}) \to \mathfrak{h}^* \) given by \( T \mapsto \hat{\iota}_T \). This map is injective by Lemma 1.19. Moreover, it follows from Lemma 1.18 that \( w(\hat{\iota}_T) = \hat{\iota}_{wT} \). Hence it factors the injection \( \nu_0 \text{St}_\gamma(\hat{\theta}) \to \mathcal{W} \setminus \mathfrak{h}^* \). Therefore we have an injection \( \overline{\text{PP}}_\gamma(\theta) \to \mathcal{W} \setminus \mathfrak{h}^* \). This implies the statement.

For a character \( \chi \) of \( S(\mathfrak{h})^W \), put
\[
ev_{\pm \nu}_\gamma(\hat{\theta}) = \{v \in e_{\pm \nu}_\gamma(\hat{\theta}) \mid hv = \chi(h)v \ \forall h \in S(\mathfrak{h})^W\}.
\]
For \( T \in \text{St}_\gamma(\hat{\theta}) \), observe that the vectors \( e_+ v_T \in \overline{\nu}_\gamma(\hat{\theta}) \) and \( e_- v_T \in \overline{\nu}_\gamma(\hat{\theta}) \) have the character \( \chi_{\theta(T)} = \chi_{\hat{\iota}^T} \).

Theorem 3.12. Let \( \kappa \in \mathbb{F} \) and \( m \in \mathbb{Z}_{\geq 1} \), and let \( \theta \in Y^\kappa_\gamma \) with \( \gamma = (-m, \kappa - m) \). Then
\[
e_+ \nu_\gamma(\hat{\theta}) = \bigoplus_{p \in \overline{\text{PP}}_\gamma^C(\theta)} e_+ \nu_\gamma(\hat{\theta})^{x_p},
\]
\[
e_- \nu_\gamma(\hat{\theta}) = \bigoplus_{p \in \overline{\text{PP}}_\gamma^R(\theta)} e_- \nu_\gamma(\hat{\theta})^{x_p}.
\]
Moreover, \( \dim_F e_+ \nu_\gamma(\hat{\theta})^{x_p} = 1 \) for all \( p \in \overline{\text{PP}}_\gamma^C(\theta) \), and \( \dim_F e_- \nu_\gamma(\hat{\theta})^{x_p} = 1 \) for all \( p \in \overline{\text{PP}}_\gamma^R(\theta) \).

Proof. By Theorem 3.7 and Proposition 3.10, we have
\[
e_+ \nu_\gamma(\hat{\theta}) = \bigoplus_{p \in \overline{\text{PP}}_\gamma(\theta), \ p^{-1}(0) : \text{unlinked}} e_+ \nu^{\text{aff}}(p^{-1}(0))
\]
To prove the formula for \( e_+ \nu_\gamma(\hat{\theta}) \), it is enough to show that \( p^{-1}(0) \) is unlinked if and only if \( p \in \overline{\text{PP}}_\gamma^C(\theta) \). The “if” part is obvious. Let us prove the opposite implication. Suppose that \( p \) is not column strict. Then there exists \( u_1 = (a, b) \in \hat{\theta} \)
and \( u_2 = (a + k + 1, b + k) \in \hat{\theta} \) with \( k \in \mathbb{Z}_{\geq 0} \) such that \( p(u_1) = p(u_2) \). Then
\[
p(u_1 + p(u_1)\gamma) = p(u_2 + p(u_1)\gamma) = 0.
\]
Hence \( u_1 + p(u_1)\gamma \) and \( u_2 + p(u_1)\gamma \) belongs to \( p^{-1}(0) \), and this implies that \( p^{-1}(0) \) is linked.

Therefore we have proved the formula for \( e_+ \tilde{\mathcal{V}}_\gamma(\hat{\theta}) \). The formula for \( e_+ \tilde{\mathcal{V}}_\gamma(\hat{\theta}) \) follows similarly.

The following decomposition formula was conjectured in [AS] Conjecture 6.2.6.

**Theorem 3.13.** Let \( \kappa \in \mathbb{Z}_{\geq 1} \), \( m \in \mathbb{Z}_{\geq 1} \) and \( \theta \in \mathbb{Z}^n_\gamma \) with \( \gamma = (-m, \kappa - m) \). Then
\[
e_+ \tilde{\mathcal{V}}_\gamma(\hat{\theta}) = \bigoplus_{\zeta \in P^-, T \in \text{st}_\gamma(\theta)} e_+ \tilde{\mathcal{V}}_\gamma(\hat{\theta})^{\chi_{\zeta + h_T}},
\]
Moreover \( e_+ \tilde{\mathcal{V}}_\gamma(\hat{\theta})^{\chi_{\zeta + h_T}} = F e_+ \mathcal{V}_{\zeta + h_T} \) and it is non-zero for all \( \zeta \in P^- \) and \( T \in \text{st}_\gamma(\theta) \).

**Proof.** Follows from Proposition 2.7 and Theorem 3.12 noting that \( \chi_{\theta'(\zeta + h_T)} = \chi_{\zeta + h_T} \) for \( T \in \text{st}_\gamma(\theta) \).

**3.5. Proof of Proposition 3.10.** First, we prepare notations concerning cosets of the Weyl group.

For \( w \in \mathcal{W} \), set \( R(w) = R^+ \cap w^{-1}R^- \), where \( R^- = R \setminus R^+ \).

Let \( \nu = (\nu_1, \ldots, \nu_m) \models n \) (a composition of \( n \)). Put \( I(\nu) = [1, n] \setminus \{\nu_1, \nu_1 + \nu_2, \ldots, \sum_{k \in [1, m]} \nu_k\} \) and define
\[
\mathcal{W}_\nu = \langle s_i \mid i \in I(\nu) \rangle \subseteq \mathcal{W}.
\]
The subgroup \( \mathcal{W}_\nu \) is called the parabolic subgroup (or Young subgroup) associated with \( \nu \). Define
\[
(3.5.1) \quad \mathcal{W}^\nu = \{w \in \mathcal{W} \mid R(w) \cap R^+_w = \emptyset\},
\]
where \( R^+_w = \{\alpha \in R^+ \mid s_\alpha \in \mathcal{W}_\nu\} \). As is well-known, \( \mathcal{W}^\nu \) complete set of representatives of the coset \( \mathcal{W}/\mathcal{W}_\nu \).

**Proof of Proposition 3.10.** Suppose that \( \theta \) is linked. Then there exist \( (a, b) \in \theta \) and \( (a + k + 1, b + k) \in \theta \) with \( k \in \mathbb{Z}_{\geq 0} \). We take \( k \) as small as possible. Then it is easy to see that there exists \( T \in \text{st}(\theta) \) such that \( T(a + k + 1, b + k) = T(a, b) + 1 \).

Put \( i = T(a, b) \). Then \( s_i T \notin \text{st}(\theta) \), and it follows from the definition of the action of \( s_i \) (Theorem 3.3) that \( s_i \nu_T = -\nu_T \), and hence
\[
e_+ \nu_T = \frac{1}{n!} \sum_{w \in \mathcal{W}} w\nu_T = \frac{1}{n!} \sum_{w \in \mathcal{W}_\nu(i)} w(1 + s_i)\nu_T = 0.
\]
where \( \nu(i) \) denotes the composition \( (i, n - i) \), and \( \mathcal{W}_\nu(i) \) denotes the set of coset representatives of \( \mathcal{W}/\mathcal{W}_\nu(i) \).

Since \( \mathcal{V}^{\text{aff}}(\theta) = \mathcal{H}^{\text{aff}} \cdot \nu_T = F \mathcal{W} \cdot \nu_T \), we have
\[
e_+ \mathcal{V}^{\text{aff}}(\theta) = e_+ F \mathcal{W} \cdot \nu_T = F e_+ \nu_T = 0.
\]
Suppose that \( \theta \) is unlinked and take \( \lambda, \mu \in \mathbb{F}^m \) such that \( \theta = \lambda/\mu \). Let \( M \) denote the \( F \mathcal{W} \)-module \( F \mathcal{W} \otimes F \mathcal{W}_{\lambda-\mu} \mathbf{1}_{\lambda-\mu} \), where \( \mathbf{1}_{\lambda-\mu} \) denotes the trivial module of the parabolic subgroup \( \mathcal{W}_{\lambda-\mu} \).
Observe that \( s_i \bar{v}_\theta = \bar{v}_\theta \) for any \( s_i \in \mathcal{W}_{\lambda - \mu} \). Here, \( \bar{v}_\theta \) is the row reading tableau on \( \theta \) in \([14.1]\). Hence there exists an \( \mathbb{F} \mathcal{W} \)-homomorphism
\[
M \to \mathcal{V}^{\text{aff}}(\theta)
\]
such that \( 1_{\lambda - \mu} \mapsto \bar{v}_\theta \). Note that we have \( \mathcal{V}^{\text{aff}}(\theta) = \mathcal{H}^{\text{aff}} \cdot \bar{v}_\theta = \mathbb{F} \mathcal{W} \cdot \bar{v}_\theta \) as \( \bar{v}_\theta \) is a weight vector with respect to \( S(\mathfrak{h}) \). Hence the map \( M \to \mathcal{V}^{\text{aff}}(\theta) \) is surjective.

It is easy to see that
\[
\sharp \text{st}(\theta) = \frac{n!}{(\lambda_1 - \mu_1)! \ldots (\lambda_m - \mu_m)!} = \sharp \mathcal{W} / \mathcal{W}_{\lambda - \mu}
\]
for an unlinked diagram \( \theta \). Therefore \( \dim \mathcal{V}^{\text{aff}}(\theta) = \dim M \), which implies that the map above is an \( \mathbb{F} \mathcal{W} \)-isomorphism. Therefore \( e_+ \mathcal{V}^{\text{aff}}(\theta) = e_+ M = \mathbb{F} e_+ 1_{\lambda - \mu} \) and it is one-dimensional. This completes the proof of the statement for \( e_+ \mathcal{V}^{\text{aff}}(\theta) \). The statement for \( e_- \mathcal{V}^{\text{aff}}(\theta) \) is proved similarly. \( \square \)

4. Application to the rational Cherednik algebra

We will apply the combinatorial method developed in the previous section to the study of representations of the rational Cherednik algebra \( \mathcal{H}_\kappa \).

Representations of \( \mathcal{H}_\kappa \) with weight decomposition with respect to the Cherednik-Dunkl subalgebra are constructed as tableau representations by giving the action of generators of \( \mathcal{H}_\kappa \) on a basis vector labeled by standard tableaux. On the other hand, such construction has not been known for \( \mathcal{H}_\kappa \). But it will be shown that any irreducible \( \mathcal{H}_\kappa \)-modules of the corresponding class can be realized as a subspace of tableau representations of \( \mathcal{H}_\kappa \) using the functor given in \([Su1]\), which relates the representation theory of \( \mathcal{H}_\kappa \) and \( \mathcal{H}_\kappa \).

4.1. Rational Cherednik algebra. We put \( P_\circ = \bigoplus_{i \in [1, n]} \mathbb{Z}_{\geq 0} \epsilon_i \) as before, which is the semi subgroup of \( P \) generated by \( \{ \epsilon_1, \ldots, \epsilon_n \} \). Recall also that \( \mathbb{F} P_\circ = \mathbb{F}[x] \) under our identification. We define
\[
\mathcal{W}_\circ = P_\circ \rtimes \mathcal{W} \subset \mathcal{W},
\]
which is a semigroup (with unit) with generators \( \{ \pi, s_1, s_2, \ldots, s_{n-1} \} \).

**Definition 4.1.** For \( \kappa \in \mathbb{F} \), the *the rational Cherednik algebra* \( \mathcal{H}_\kappa \) of type \( GL_n \) is the unital associative \( \mathbb{F} \)-algebra generated by \( \mathbb{F} \mathcal{W}_\circ \) and \( \mathbb{F}[y] = \mathbb{F}[y_1, \ldots, y_n] \) with the following relations:

\[
s_i y_j = y_{s_i(j)} s_i \quad (i \in [1, n - 1], \ j \in [1, n]),
\]

\[
[y_i, x_j] = \begin{cases} 
\kappa + \sum_{k \neq i} s_{ik} & (i = j) \\
-s_{ij} & (i \neq j)
\end{cases} \quad (i, j \in [1, n]).
\]

The natural map gives a linear isomorphism
\[
\mathbb{F} P_\circ \otimes \mathbb{F} \mathcal{W} \otimes \mathbb{F}[y] \sim \mathcal{H}_\kappa.
\]
Put $\mathbb{F}[y]_+ = \sum_{i \in [1, n]} y_i \mathbb{F}[y] \subset H_\kappa$. An element $v$ of an $H_\kappa$-module is said to be $\mathbb{F}[y]_+$-nilpotent if there exists $k \in \mathbb{Z}_{>0}$ such that $(y_i)^k v = 0$ for all $i \in [1, n]$. An $H_\kappa$-module $M$ is said to be locally nilpotent for $\mathbb{F}[y]_+$, if any element of $M$ is $\mathbb{F}[y]_+$-nilpotent.

Define $\mathcal{O}(H_\kappa)$ as the full subcategory of $H_\kappa$-mod consisting of those modules which are locally nilpotent for $\mathbb{F}[y]_+$. The following statement is easy to show but will play an important role.

**Proposition 4.2 (Sn1).** There exists an algebra embedding $\iota : H_\kappa \to \tilde{H}_\kappa$ such that

\[
\iota(w) = w \quad (w \in \mathcal{W}), \quad \iota(x_i) = \epsilon_i \quad (i \in [1, n]),
\]

\[
\iota(y_i) = x_i^{-1}\left(\epsilon_i^\vee - \sum_{1 \leq j < i} s_{ji}\right) \quad (i \in [1, n]).
\]

Moreover, the embedding $\iota : H_\kappa \to \tilde{H}_\kappa$ is extended to the algebra isomorphism $\mathbb{F}[x^{\pm 1}] \otimes_{\mathbb{F}[y]} H_\kappa \sim \tilde{H}_\kappa$.

In the sequel, we often identify $H_\kappa$ with the subalgebra of $\tilde{H}_\kappa$ generated by $\mathbb{F}[\mathcal{W}]$ and $y_i = x_i^{-1}\left(\epsilon_i^\vee - \sum_{1 \leq j < i} s_{ji}\right) \quad (i \in [1, n])$ via the embedding $\iota$.

Under this identification, $\epsilon_i^\vee \in \tilde{H}_\kappa$ is also contained in $H_\kappa$, and it is expressed as

\[
\epsilon_i^\vee = x_i y_i + \sum_{j < i} s_{ji} \quad \text{in terms of the generators of } H_\kappa.
\]

Define $\mathcal{O}^h(H_\kappa)$ as a full subcategory of $\mathcal{O}(H_\kappa)$ consisting of those modules $M$ such that $M = \bigoplus_{\zeta \in \mathbb{N}} M_\zeta$, where $M_\zeta = \{v \in M \mid \epsilon_i^\vee v = \langle \zeta \mid \epsilon_i^\vee \rangle v \quad \forall i \in [1, n]\}$ as before.

The following lemma is easily checked.

**Lemma 4.3.** (i) There exists an algebra isomorphism $\sigma : \tilde{H}_\kappa \to \tilde{H}_{-\kappa}$ such that

\[
\sigma(x_i) = (-1)^{n-1} x_i, \quad \sigma(\epsilon_i^\vee) = -\epsilon_i^\vee \quad (i \in [1, n]),
\]

\[
\sigma(s_i) = -s_i \quad (i \in [0, n-1]),
\]

and its restriction to $H_\kappa$ gives an algebra isomorphism $\sigma : H_\kappa \sim \tilde{H}_{-\kappa}$.

(ii) The algebra isomorphism $\sigma$ induces categorical equivalences

\[
\mathcal{O}(\tilde{H}_\kappa) \sim \mathcal{O}(\tilde{H}_{-\kappa}), \quad \mathcal{O}^h(\tilde{H}_\kappa) \sim \mathcal{O}^h(\tilde{H}_{-\kappa}),
\]

\[
\mathcal{O}(H_\kappa) \sim \mathcal{O}(H_{-\kappa}), \quad \mathcal{O}^h(H_\kappa) \sim \mathcal{O}^h(H_{-\kappa}).
\]

Consider the induction functor $H_\kappa$-mod $\to \tilde{H}_\kappa$-mod given by

\[
M \to \tilde{H}_\kappa \otimes_{H_\kappa} M.
\]

We denote its restriction to $\mathcal{O}(H_\kappa)$ by $\text{ind}$. Then it turns out that $\text{ind}$ gives an exact functor into $\mathcal{O}(H_\kappa)$ (Sn1 Corollary 3.4, Proposition 4.2)).

For an $H_\kappa$-module $N$, denote by $\text{nil}(N)$ the subspace of $N$ consisting of the $\mathbb{F}[y]_+$-nilpotent elements:

\[
(4.1.1) \quad \text{nil}(N) = \{v \in N \mid v \text{ is } \mathbb{F}[y]_+\text{-nilpotent}\}.
\]

It follows that $\text{nil}(N)$ is an $H_\kappa$-submodule of $N_{\downarrow H_\kappa}$ and it is finitely generated over $H_\kappa$ (Sn1 Lemma 4.4)). Therefore we have the functor $\text{nil} : \mathcal{O}(H_\kappa) \to \mathcal{O}(H_\kappa)$. 

Theorem 4.4. ([Sni Section 6]) (i) The functor nil is the right adjoint functor of functor ind. Moreover, nil ∘ ind(M) ≅ M for any M ∈ O(H_κ).

(ii) The functor ind : O(H_κ) → O(\widetilde{H}_κ) is exact and fully-faithful.

4.2. The category \( O^b \) for the rational algebra. Let \( m \in \mathbb{Z}_{≥1} \) and \( \kappa \in \mathbb{F} \setminus \mathbb{Q}_{≤0} \), and put \( \gamma = (-m, \kappa - m) \).

Let \( \Lambda^+(m, n) \) denote the set of partitions of \( n \) consisting \( m \) nonzero components:

\[
\Lambda^+(m, n) = \{ \lambda = (\lambda_1, \ldots, \lambda_m) \in (\mathbb{Z}_{≥1})^m \mid \lambda \vdash n \}
\]

Set \( \Lambda_κ^+(m, n) = \Lambda^+(m, n) \) for \( \kappa \in \mathbb{F} \setminus \mathbb{Q} \), and set

\[
\Lambda_κ^+(m, n) = \{ \lambda \in \Lambda^+(m, n) \mid s - m - \lambda_1 + \lambda_m \in \mathbb{Z}_{≥0} \}
\]

for \( \kappa = s/r \in \mathbb{Q}_{>0} \) with \( s, r \in \mathbb{Z}_{>0} \), \((s, r) = 1\).

We identify a partition \( \lambda \in \Lambda^+(m, n) \) with the associated diagram \( \{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \in [1, m], b \in [1, \lambda_a] \} \). By this identification, \( \Lambda^+(m, n) \) and \( \Lambda_κ^+(m, n) \) are thought as a subset of \( Y^m \) and \( Y_γ^n \) respectively.

Recall the functor nil : \( O(\mathcal{H}_κ) \rightarrow O(\mathcal{H}_κ) \), and consider its image of the tableau representation \( \mathcal{V}_γ(\lambda) \) for \( \lambda \in \Lambda_κ^+(m, n) \). By Theorem 4.4 nil(\( \mathcal{V}_γ(\lambda) \)) is irreducible or zero. Moreover, it admits a weight decomposition with respect to the Cherednik-Dunkl subalgebra \( S(\mathfrak{h}) \), since it is an \( S(\mathfrak{h}) \)-submodule of \( \mathcal{V}_γ(\lambda) \). Hence nil(\( \mathcal{V}_γ(\lambda) \)) belongs to \( O^b(\mathcal{H}_κ) \).

Remark 4.5. (See [Sni Proposition 8.1].) More generally, it holds for any \( M \in O(\mathcal{H}_κ) \) that

\[
M \in O^b(\mathcal{H}_κ) \Leftrightarrow \text{ind}(M) \in O^b(\mathcal{H}_κ).
\]

For a partition \( \lambda \vdash n \), let \( S_\lambda \) denote the corresponding irreducible \( \mathcal{W} \)-module. We define the standard module of \( \mathcal{H}_κ \) associated with \( S_\lambda \) by

\[
\Delta_κ(\lambda) = \mathcal{H}_κ \otimes_{\mathcal{F}W, \mathcal{F}[y]} S_\lambda,
\]

where we let \( \mathcal{F}[y] \) act on \( S_\lambda \) through the augmentation map \( \mathcal{F}[y] \rightarrow \mathbb{F} \) given by \( y_i \mapsto 0 \) \((i \in [1, n])\).

Let \( \text{Irr}O(\mathcal{H}_κ) \) (resp., \( \text{Irr}O^b(\mathcal{H}_κ) \)) denote the set of isomorphism classes of the irreducible modules in \( O(\mathcal{H}_κ) \) (resp., \( O^b(\mathcal{H}_κ) \)).

Proposition 4.6. ([DO] [GGOR]) Let \( \kappa \in \mathbb{F} \setminus \{0\} \).

For \( \lambda \in \bigsqcup_{m \in [1, n]} \Lambda^+(m, n) \), the standard module \( \Delta_κ(\lambda) \) has a unique simple quotient module, which is denoted by \( \mathcal{L}_κ(\lambda) \). Moreover, the assignment \( \lambda \mapsto \mathcal{L}_κ(\lambda) \) gives a one-to-one correspondence

\[
\bigsqcup_{m \in [1, n]} \Lambda^+(m, n) \xrightarrow{\sim} \text{Irr}O(\mathcal{H}_κ).
\]

Proposition 4.7. Let \( \kappa \in \mathbb{F} \setminus \mathbb{Q}_{≤0} \) and \( m \in \mathbb{Z}_{≥1} \). For \( \lambda \in \Lambda_κ^+(m, n) \), it holds that \( \mathcal{L}_κ(\lambda) \cong \text{nil}(\mathcal{V}_γ(\lambda)) \), where \( \gamma = (-m, \kappa - m) \).
Proof. Recall that \( \mathcal{V}^{\text{aff}}(\lambda) \) is embedded into \( \tilde{\mathcal{V}}_\gamma(\bar{\lambda}) \) via \( \bar{v}_T \mapsto v_T \) as an \( \mathcal{H}^{\text{aff}} \)-submodule. Obviously, (the image of) \( \mathcal{V}^{\text{aff}}(\lambda) \) is locally \( \mathbb{F}[y]_+ \)-nilpotent, and hence \( \mathcal{V}^{\text{aff}}(\lambda) \subset \text{nil}(\tilde{\mathcal{V}}_\gamma(\bar{\lambda})) \). Remark that for \( \lambda \in \Lambda^+(m,n) \), the restriction \( \mathcal{V}^{\text{aff}}(\lambda)\mid_{\mathbb{F}W} \) is irreducible as an \( \mathbb{F}W \)-module, and it is isomorphic to \( S_\lambda \). Therefore we have a surjective homomorphism \( \Delta_\kappa(\lambda) \to \text{nil}(\tilde{\mathcal{V}}_\gamma(\bar{\lambda})) \). This implies that \( \text{nil}(\tilde{\mathcal{V}}_\gamma(\bar{\lambda})) \) is the simple quotient of \( \Delta_\kappa(\lambda) \). \( \square \)

We have the following result for \( \mathcal{O}^h(\mathcal{H}_\kappa) \), whose proof will be given in Appendix.

**Theorem 4.8.** (cf. [Su1 Theorem 8.2]) Let \( \kappa \in \mathbb{F}\backslash \mathbb{Q}_{\leq 0} \). The assignment \( \lambda \mapsto \mathcal{L}_\kappa(\lambda) \) gives a one-to-one correspondence

\[
\bigcup_{m \in [1,n]} \Lambda^+_\kappa(m,n) \xrightarrow{\sim} \text{Irr}^h(\mathcal{H}_\kappa).
\]

**Remark 4.9.** (i) The classification result in Theorem 4.8 for \( \kappa \in \mathbb{Q}_{< 0} \) can be obtained via the isomorphism \( \mathcal{H}_\kappa \cong \mathcal{H}_{-\kappa} \).

(ii) In the case where \( \kappa \in \mathbb{Z} \setminus \{0\} \), the classification is given in [Su1] as a consequence of the classification theorem for \( \mathcal{O}^h(\mathcal{H}_\kappa) \) ([Ch4 Theorem 6.5] [SV Corollary 4.23]). The statement for general \( \kappa \) is also stated in [Su1 Remark 8.3] without a precise proof. It should be also mentioned that, in the proof given in Appendix, the classification result for \( \mathcal{H}_\kappa \), which is more complicated, is not used.

### 4.3. Tableaux description for irreducible \( \mathcal{H}_\kappa \)-modules.

We will give a weight decomposition of \( \mathcal{L}_\kappa(\lambda) = \text{nil}(\tilde{\mathcal{V}}_\gamma(\bar{\lambda})) \) with respect to the subalgebra \( S(\mathfrak{h}) \) by giving a description of the subspace \( \text{nil}(\tilde{\mathcal{V}}_\gamma(\bar{\lambda})) \) of \( \tilde{\mathcal{V}}_\gamma(\bar{\lambda}) \) in terms of the basis \( \{v_T\}_{T \in \text{St}_\gamma(\bar{\lambda})} \).

Define \( \mathcal{H}^\circ_\kappa \) as the subalgebra of \( \tilde{\mathcal{H}}_\kappa \) generated by \( \mathbb{F}P_0, \mathbb{F}W \) and \( S(\mathfrak{h}) \):

\[ \mathcal{H}^\circ_\kappa = \mathcal{H}^\circ_\kappa \cdot \mathbb{F}P \cdot \mathbb{F}W \cdot S(\mathfrak{h}), \]

which is also a subalgebra of \( \mathcal{H}_\kappa \).

We treat a general skew diagram \( \theta \in \mathcal{Y}_\gamma^n \) for a while. Identify the \( \mathcal{H}^{\text{aff}} \)-module \( \mathcal{V}^{\text{aff}}(\theta) \) with the subspace \( \bigoplus_{T \in \text{St}(\theta)} \mathbb{F}v_T \) of \( \tilde{\mathcal{V}}_\gamma(\bar{\theta}) \). Let \( \tilde{\mathcal{V}}_\gamma(\bar{\theta})_+ \) be the \( \mathcal{H}^\circ_\kappa \)-submodule of \( \tilde{\mathcal{V}}_\gamma(\bar{\theta}) \) generated by \( \mathcal{V}^{\text{aff}}(\theta) \):

\[ \tilde{\mathcal{V}}_\gamma(\bar{\theta})_+ = \mathcal{H}^\circ_\kappa \cdot \mathcal{V}^{\text{aff}}(\theta). \]

Define

\[ (4.3.1) \quad \text{St}_\gamma(\bar{\theta})_+ = \{ T \in \text{St}_\gamma(\bar{\theta}) \mid T(u) \in \mathbb{Z}_{\geq 1} \ \forall u \in \theta \}. \]

**Lemma 4.10.** As a subspace of \( \tilde{\mathcal{V}}_\gamma(\bar{\theta}) = \mathcal{F}\text{St}_\gamma(\bar{\theta}) \), it holds that

\[ \tilde{\mathcal{V}}_\gamma(\bar{\theta})_+ = \mathcal{F}\text{St}_\gamma(\bar{\theta})_+. \]

**Proof.** Let \( T \in \text{St}_\gamma(\bar{\theta})_+ \). If \( s_1 T, \ldots, s_{n-1} T \) and \( \pi T \) are in \( \text{St}_\gamma(\bar{\theta})_+ \), then they are actually in \( \text{St}_\gamma(\bar{\theta})_+ \). By the formulas in Theorem 4.8(i), it follows that \( v_{s_1 T}, \ldots, v_{s_{n-1} T}, \ldots, v_{s_{n-1} T}, \ldots, v_{s_{n-1} T} \),
and $\pi \nu_T$ are in $\bigoplus_{T \in \text{St}_\gamma(\hat{\theta})_+} \mathbb{F}v_T = \mathbb{F}\text{St}_\gamma(\hat{\theta})_+$. Therefore $\mathbb{F}\text{St}_\gamma(\hat{\theta})_+$ is closed under the action of $\hat{\mathbb{W}}_\gamma$. Since $\mathbb{F}\text{St}_\gamma(\hat{\theta})_+$ includes $\mathcal{V}\text{aff}(\theta) = \bigoplus_{T \in \text{st}(\theta)} \mathbb{F}v_T$, we have $\tilde{\mathcal{V}}_\gamma(\hat{\theta})_+ = \mathbb{F}\hat{\mathbb{W}}_\gamma \cdot \mathcal{V}\text{aff}(\theta) \subseteq \mathbb{F}\text{St}_\gamma(\hat{\theta})_+$.

Let us see the opposite inclusion. Fix any $S \in \text{st}(\theta) \subseteq \text{St}_\gamma(\hat{\theta})$ and put

$$Z_S = \{ w \in \hat{\mathbb{W}} \mid w S \in \text{St}_\gamma(\hat{\theta}) \}, \quad Z^o_S = \{ w \in \hat{\mathbb{W}} \mid w S \in \text{St}_\gamma(\hat{\theta})_+ \}.$$  

Observe that $Z_S \cdot S = \text{St}_\gamma(\hat{\theta})$, $Z^o_S \cdot S = \text{St}_\gamma(\hat{\theta})_+$ and $Z_S^o = \{ w \in Z_S \mid w(i) \geq 1 \forall i \in [1,n] \}$.

We will prove that $v_{wS} \in \tilde{\mathcal{V}}_\gamma(\hat{\theta})_+$ for every $w \in Z^o_S$ by induction on the length of $w$.

Let $w \in Z^o_S$ with $l(w) = 0$. Then $w = \pi^p$ for some $p \in \mathbb{Z}$ and it is easy to see that $p \in \mathbb{Z}_{\geq 0}$. Assume that the statement is true for all $w \in Z^o_S$ with $l(w) < k$.

Let $w \in Z^o_S$ with $l(w) = k$. Take $i \in [0,n-1]$ and $x \in \hat{\mathbb{W}}$ such that $w = s_i x$ and $l(w) = l(x) + 1$. Then $x \in Z_S$ by Lemma 1.21.

Suppose that $i \in [1,n-1]$. Then it follows that $x = s_i w \in Z^o_S$ and hence $v_{xS} \in \tilde{\mathcal{V}}_\gamma(\hat{\theta})_+$ by induction hypothesis. We have

$$v_{wS} = v_{s_i xS} = (1+a)^{-1}(v_{xS} + a s_i v_{xS}) \in \mathbb{F}\hat{\mathbb{W}}_\gamma v_{xS},$$

where $a := c_{xS}(i) - c_{xS}(i+1)$, and $1+a \neq 0$ by Lemma 1.20. Therefore $v_{wS} \in \tilde{\mathcal{V}}_\gamma(\hat{\theta})_+$.

Suppose that $i = 0$. As a step, we will show that $\pi^{-1} x \in Z^o_S$. If $\pi^{-1} x \notin Z^o_S$, then there exists $j \in [1,n]$ such that $\pi^{-1} x(j) \in \mathbb{Z}_{<0}$. This means $x(j) = 1$ since $x(j) \in Z^o_S$ and $\pi^{-1} x(j) = x(j) - 1$. Therefore $w(j) = s_0 x(j) = 0 \notin [1,n]$. This contradicts $w \in Z^o_S$. Therefore we have $\pi^{-1} x \in Z^o_S$.

Put $y = \pi^{-1} x$. The induction hypothesis implies $v_{yS} \in \tilde{\mathcal{V}}_\gamma(\hat{\theta})_+$. Since $y \in Z_S$ and $s_{n-1} y = \pi^{-1} w \in Z_S$, we have $a := c_{yS}(n-1) - c_{yS}(n) \neq -1$. Therefore we have

$$v_{wS} = v_{s_{n-1} yS} = (1+a)^{-1}(v_{yS} + a s_{n-1} v_{yS}) \in \mathbb{F}\hat{\mathbb{W}}_\gamma v_{yS},$$

and hence $v_{wS} \in \tilde{\mathcal{V}}_\gamma(\hat{\theta})_+$. Therefore we have $\tilde{\mathcal{V}}_\gamma(\hat{\theta})_+ \subseteq \bigoplus_{T \in \text{St}_\gamma(\hat{\theta})_+} \mathbb{F}v_T = \mathbb{F}\text{St}_\gamma(\hat{\theta})_+$, and the statement is proved. \hfill $\Box$

**Proposition 4.11.** Let $\kappa \in \mathbb{F} \setminus \mathbb{Q}_{\leq 0}$ and let $\lambda \in \Lambda^\perp_\kappa(m,n)$. Then

$$\text{nil}(\tilde{\mathcal{V}}_\gamma(\hat{\lambda})) = \mathbb{F}\text{St}_\gamma(\hat{\theta})_+$$

as a subspace of $\tilde{\mathcal{V}}_\gamma(\hat{\lambda}) = \mathbb{F}\text{St}_\gamma(\hat{\theta})$.

**Proof.** It is enough to prove $\text{nil}(\tilde{\mathcal{V}}_\gamma(\hat{\lambda})) = \tilde{\mathcal{V}}_\gamma(\hat{\lambda})_+$. Since $\mathcal{V}\text{aff}(\lambda) \subset \text{nil}(\tilde{\mathcal{V}}_\gamma(\hat{\lambda}))$, it holds that $\tilde{\mathcal{V}}_\gamma(\hat{\lambda})_+ \subseteq \text{nil}(\tilde{\mathcal{V}}_\gamma(\hat{\lambda}))$. Since $\text{nil}(\tilde{\mathcal{V}}_\gamma(\hat{\lambda}))$ is an irreducible $\mathcal{H}_\kappa$-module, it is enough to show that $\tilde{\mathcal{V}}_\gamma(\hat{\lambda})_+$ is closed under the action of $\mathcal{H}_\kappa$. Put $\omega = \sum_{i \in [1,n]} y_i \in \mathcal{H}_\kappa \subset \hat{\mathcal{H}}_\kappa$. Then

$$[\omega, e_i^\gamma] = \kappa y_i, \quad [\omega, x_i] = \kappa \quad (i \in [1,n]),$$

$$[\omega, s_i] = 0 \quad (i \in [1,n-1]).$$
This implies that \( \omega \) and the elements in \( \mathcal{H}_\kappa^a \) generate the algebra \( \mathcal{H}_\kappa \). It is easy to check that \( \omega v = 0 \) for all \( v \in \mathcal{V}^{\text{aff}}(\lambda) \). Hence it follows that \( \omega \) preserves \( \mathcal{V}_\gamma(\hat{\lambda})_+ = F\hat{W}_0 \cdot \mathcal{V}^{\text{aff}}(\lambda) = FP_0 \otimes FW \cdot \mathcal{V}^{\text{aff}}(\lambda) \). Therefore \( \mathcal{H}_\kappa \cdot \mathcal{V}_\gamma(\hat{\lambda})_+ \subseteq \mathcal{V}_\gamma(\hat{\lambda})_+ \). \( \square \)

Combined with Proposition 4.7, we obtained

\[
\mathcal{L}_\kappa(\lambda) \cong F\mathcal{St}_\gamma(\hat{\theta})_+.
\]

4.4. Consequences. Let \( \kappa \in F \setminus Q_{\leq 0} \) and \( m \in \mathbb{Z}_{\geq 1} \), and put \( \gamma = (-m, \kappa - m) \). Let \( \lambda \in \Lambda_\kappa^+(m, n) \). As a consequence of the realization \((4.3.2)\) of \( \mathcal{L}_\kappa(\lambda) \), we obtain the following:

**Theorem 4.12.** The \( \mathcal{H}_\kappa \)-module \( \mathcal{L}_\kappa(\lambda) \) admits the following weight decomposition with respect to the subalgebra \( \mathcal{S}(\mathfrak{h}) = F[\mathcal{V}_\gamma] \):

\[
\mathcal{L}_\kappa(\lambda) = \bigoplus_{T \in \mathcal{St}_\gamma(\hat{\lambda})_+} \mathcal{L}_\kappa(\lambda)_T.
\]

Moreover, \( \dim_F \mathcal{L}_\kappa(\lambda)_T = 1 \) for all \( T \in \mathcal{St}_\gamma(\hat{\lambda})_+ \).

Observe that the degenerate affine Hecke algebra \( \mathcal{H}^{\text{aff}} = \mathcal{S}(\mathfrak{h})FW \subset \mathcal{H}_\kappa \) is a subalgebra of \( \mathcal{H}_\kappa \) under our identification \( \mathcal{H}_\kappa \subset \mathcal{H}_\kappa \). As a consequence of Theorem 4.11, we have

**Theorem 4.13.** The \( \mathcal{H}_\kappa \)-module \( \mathcal{L}_\kappa(\lambda) \) admits the following decomposition as an \( \mathcal{H}^{\text{aff}} \)-module:

\[
\mathcal{L}_\kappa(\lambda)_{\mathcal{H}^{\text{aff}}} = \bigoplus_{p \in \mathcal{PP}_\gamma(\lambda)} \mathcal{V}^{\text{aff}}(p^{-1}(0)).
\]

Recall that \( \chi_\zeta : \mathcal{S}(\mathfrak{h})^{\mathcal{V}} \to F \) denotes the character corresponding to \( \zeta \in \mathfrak{h}^* \). We denote \( \chi_p = \chi_\zeta_T \) for \( p \in \mathcal{PP}_\gamma(\theta) \) as before, where \( T \) is a standard tableau on \( \hat{\theta} \) such that \( \theta(T) = p \). By Theorem 4.12, we have

**Theorem 4.14.** The \( \mathcal{H}_\kappa \)-module \( \mathcal{L}_\kappa(\lambda) \) admits the following decomposition as an \( \mathcal{S}(\mathfrak{h})^{\mathcal{V}} \)-module:

\[
e_+ \mathcal{L}_\kappa(\lambda) = \bigoplus_{p \in \mathcal{PP}_\zeta^-(\lambda)} e_+ \mathcal{L}_\kappa(\lambda)^{\chi_p},
\]

\[
e_- \mathcal{L}_\kappa(\lambda) = \bigoplus_{p \in \mathcal{PP}_\zeta^+(\lambda)} e_- \mathcal{L}_\kappa(\lambda)^{\chi_p}.
\]

Moreover, \( \dim_F e_+ \mathcal{L}_\kappa(\lambda)^{\chi_p} = 1 \) for all \( p \in \mathcal{PP}_\zeta^-(\lambda) \), and \( \dim_F e_- \mathcal{L}_\kappa(\lambda)^{\chi_p} = 1 \) for all \( p \in \mathcal{PP}_\zeta^+(\lambda) \).

By Proposition 4.11 and Theorem 4.14, we obtain the following:

**Theorem 4.15.** The \( \mathcal{H}_\kappa \)-module \( \mathcal{L}_\kappa(\lambda) \) admits the following decomposition as an \( \mathcal{S}(\mathfrak{h})^{\mathcal{V}} \)-module:

\[
e_+ \mathcal{L}_\kappa(\lambda) = \bigoplus_{\zeta \in P_\gamma^-, T \in \mathcal{St}_\gamma(\lambda)} e_+ \mathcal{L}_\kappa(\lambda)^{\chi_{\zeta + h_T}}.
\]

Moreover, \( e_+ \mathcal{L}_\kappa(\lambda)^{\chi_{\zeta + h_T}} = F e_+ v_{\zeta + h_T} T \) for all \( \zeta \in P_\gamma^- \) and \( T \in \mathcal{St}_\gamma(\lambda) \).
Remark 4.16. Put \( C = \mathbb{Q}_{>0} \cup \{-2/(2k+1) \mid k \in \mathbb{Z}_{>0}\} \). Then it is shown by Gordon-Stafford [GS] (see also [BEG1]) that the assignment \( M \mapsto e_+ M \) gives a Morita equivalence \( \mathcal{H}_\kappa \text{-mod} \to e_+ \mathcal{H}_\kappa e_+ \text{-mod} \) if \( \kappa \not\in C \). Via the isomorphism \( \mathcal{H}_\kappa \cong \mathcal{H}_{-\kappa} \) in Lemma 4.3, it also holds that the correspondence \( M \mapsto e_- M \) gives a Morita equivalence \( \mathcal{H}_\kappa \text{-mod} \to e_- \mathcal{H}_\kappa e_- \text{-mod} \) if \(-\kappa \not\in C \).

The algebra \( e_+ \mathcal{H}_\kappa e_+ \) is called the spherical subalgebra of \( \mathcal{H}_\kappa \).

It should be mentioned that parallel statements can be shown for \( \tilde{\mathcal{H}}_\kappa \)-modules using the isomorphism \( \mathbb{F}[\mathbb{Z}^\pm] \otimes \mathbb{F}[\mathbb{Z}] \mathcal{H}_\kappa \cong \tilde{\mathcal{H}}_\kappa \).

5. Characters

We introduce characters for \( e_\pm \mathcal{L}_\kappa(\lambda) \), and compute them in several cases using the results in [2] and [4,14].

5.1. Characters and generating functions. The algebra \( \mathcal{H}_\kappa \) has the grading operator

\[
\partial = \kappa^{-1} \sum_{i=1}^n \epsilon_i^\vee = \kappa^{-1} \sum_{i=1}^n x_i y_i + \kappa^{-1} \sum_{1 \leq i < j \leq n} s_{ij}.
\]

The element \( \partial \) belongs to \( S(\mathfrak{h})^W \) and satisfies

\[
[\partial, x_i] = x_i, \quad [\partial, \epsilon_i^\vee] = -\epsilon_i^\vee \quad (i \in [1, n]),
\]

\[
[\partial, w] = 0 \quad (w \in W).
\]

In particular, \( \partial \) preserves the \( W \)-(anti-)invariant subspace of any \( \mathcal{H}_\kappa \)-module.

We put \( \partial' = \kappa^{-1} \sum_{i=1}^n x_i y_i \) and \( \partial_W = \kappa^{-1} \sum_{1 \leq i < j \leq n} s_{ij} \). Then \( \partial = \partial' + \partial_W \) and \( [\partial', \partial_W] = 0 \).

For an \( \mathbb{F}[\partial] \)-module \( M \) and \( d \in \mathbb{F} \), put \( M[d] = \{ v \in M \mid (\partial - d)^k v = 0 \text{ for } k \gg 1 \} \).

For \( M \in \mathcal{O}(\mathcal{H}_\kappa) \), it follows that \( M \) is finitely generated over \( \mathbb{F}[x] \), and hence we have \( \dim_{\mathbb{F}} M[d] < \infty \), and the trace

\[
\text{Tr}(M; q^\partial) = \sum_{d \in \mathbb{F}} (\dim_{\mathbb{F}} M[d]) q^d
\]

is defined. Similarly, we can define \( \text{Tr}(e_\pm M; q^\partial) \), which we call the spherical character of \( M \).

Let \( \lambda \in \Lambda^+_\kappa(m, n) \). Put \( |p| = \sum_{u \in \lambda} p(u) \) for \( p \in \mathbb{PP}(\lambda) \) as before. The following can be checked easily.

**Lemma 5.1.** Let \( T \in \mathbf{St}_\lambda(\hat{\lambda}) \). Then for \( \nu_T \in \mathcal{V}_\lambda(\hat{\lambda}) \),

\[
\partial \nu_T = (|\varphi(T)| + d_\lambda) \nu_T,
\]

where

\[
d_\lambda = \frac{1}{2\kappa} \sum_{i=1}^m \lambda_i (\lambda_i - 2i + 1).
\]

By Theorem 4.13, the computation of the characters are reduced to the computation of the generating functions for plane partitions:
Corollary 5.2. Let $\kappa \in \mathbb{F} \setminus \mathbb{Q}_{\leq 0}$ and $m \in \mathbb{Z}_{\geq 1}$, and put $\gamma = (-m, \kappa - m)$. Let $\lambda \in \Lambda_\kappa^+(m,n)$. Then

$$\text{Tr}(e_+L_\kappa^+(\lambda); q^\partial) = q^{d_\lambda} \sum_{p \in PP_\gamma^+(\lambda)} q^{|p|},$$

$$\text{Tr}(e_-L_\kappa^+(\lambda); q^\partial) = q^{d_\lambda} \sum_{p \in PP_\gamma^-(\lambda)} q^{|p|}.$$ 

5.2. Spherical characters of irreducible modules and Kostka polynomials.

Combining Corollary 5.2 and the results in Section 2, we obtain character formulas in the following cases:

(i) $\kappa \in \mathbb{Z}_{\geq 1}$.

(ii) $\kappa$ is "generic".

First, let $\kappa \in \mathbb{Z}_{\geq 1}$. By Theorem 4.15 and Theorem 2.8, we obtain the following formula, which is announced in the previous paper by the author [Su2].

Theorem 5.3. Let $m, \kappa \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \Lambda_\kappa^+(m,n)$. Then

$$\text{Tr}(e_+L_\kappa^+(\lambda); q^\partial) = q^{d_\lambda} \left[\frac{n}{q}! \tilde{K}_{\lambda}(1^{n})(q)\right],$$

where $[n]_q! = (1 - q)(1 - q^2) \ldots (1 - q^n)$.

Remark 5.4. The formula for $e_-(L_\kappa^+(\lambda))$ with $\kappa \in \mathbb{Z}_{-1}$ can be obtained through the isomorphism $\mathcal{H}_\kappa \cong \mathcal{H}_{-\kappa}$. But, the formula for $e_-(L_\kappa^+(\lambda))$ with $\kappa \in \mathbb{Z}_{1}$ has not been obtained (see also Remark 2.9). It seems remarkable that $e_+(L_\kappa^+(\lambda))$ has such a simple character formula.

Next, we consider the case where $\kappa$ is generic. Let $\kappa \in \mathbb{F} \setminus \{0\}$, $m \in \mathbb{Z}_{\geq 1}$, and let $\theta \in Y\gamma^n$ with $\gamma = (-m, \kappa - m)$.

Definition 5.5. The $\kappa$ is said to be generic for $\theta$ if

$$\{(a + k + 1, b + k) \in \mathbb{F} \times \mathbb{Z} \mid k \in \mathbb{Z}_{\geq 0} \} \cap \hat{\theta}_\gamma \subseteq \theta \text{ for any } (a, b) \in \theta,$$

where $\gamma = (-m, \kappa - m)$.

It is obvious that for $\theta \subset \mathbb{Z} \times \mathbb{Z}$, irrational $\kappa$ is generic. The following can be shown easily.

Proposition 5.6. Let $\kappa \in \mathbb{F} \setminus \mathbb{Q}_{\leq 0}$ and $\lambda \in \Lambda_\kappa^+(m,n)$. Then $\kappa$ is generic for $\lambda$ if and only if one of the following holds:

(a) $\kappa \notin \mathbb{Q}$.

(b) $\kappa = s/r$ with $r, s \in \mathbb{Z}_{\geq 1}$, $(s, r) = 1$, and $\lambda_1 \geq s - m + 1$.

The following is obvious

Lemma 5.7. If $\kappa$ is generic for $\theta$, then $PP(\theta) = PP_\gamma(\theta)$, $PP_R(\theta) = PP_R_\gamma(\theta)$ and $PP_C(\theta) = PP_C_\gamma(\theta)$. 
As a direct consequence of Proposition 5.10 we have the following:

**Corollary 5.8.** Let $\kappa \in \mathbb{F}\setminus\{0\}$, $m \in \mathbb{Z}_{\geq 1}$ and $\lambda \in \Lambda^+(m,n)$. Suppose that $\kappa$ is generic for $\lambda$. Then

$$
\text{Tr}(e_+\mathcal{L}_\kappa(\lambda); q^\partial) = \frac{q^{d_\lambda}}{|n|q} K_\lambda(1^n)(q),
$$

$$
\text{Tr}(e_-\mathcal{L}_\kappa(\lambda); q^\partial) = \frac{q^{d_\lambda}}{|n|q} K_\lambda(1^n)(q).
$$

5.3. **Characters of standard modules and their irreducibility.** Let $\lambda \vdash n$. We have an isomorphism

$$
\Delta_\kappa(\lambda) \cong \mathbb{F}[x] \otimes S_\lambda
$$
as an $\mathbb{F}W$-module. Observe that it is also an isomorphism of $\mathbb{F}[\partial']$-modules, where $\partial'$ acts on $\mathbb{F}[x] \otimes S_\lambda$ as $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \otimes \text{id}_{S_\lambda}$. The following is straightforward:

**Proposition 5.9.** Let $\kappa \in \mathbb{F}\setminus\{0\}$ and let $\lambda \vdash n$. Then

$$
\text{Tr}(\Delta_\kappa(\lambda); q^\partial) = \frac{q^{d_\lambda}}{(1-q)^n} \dim_{\mathbb{F}} S_\lambda.
$$

We will see that the genericity of $\kappa$ is a sufficient condition for a standard module to be irreducible.

For $\zeta = \sum_{i \in [1,n]} \zeta_i \epsilon_i \in P$ and $T \in \text{st}(\theta)$, define $j_1 < j_2 < \cdots < j_n$ to be the increasing sequence in $\mathbb{Z}$ obtained as a rearrangement of $1+n\zeta_1, 2+n\zeta_2, \ldots, n+n\zeta_n$, and define the tableau $\Sigma = \Sigma(\zeta, T)$ on $\hat{\theta}$ by

$$
\Sigma(u + k\gamma) = j_{T(u)} - kn \quad (u \in \theta, \ k \in \mathbb{Z}).
$$

Then it follows from the definition that $\Sigma$ is row-column increasing on $\theta$. Suppose now that $\kappa$ is generic for $\theta$. Then it is easy to see that $\Sigma \in \text{St}_\gamma(\hat{\theta})$, and hence the assignment $(\zeta, T) \mapsto \Sigma(\zeta, T)$ gives a map $P \times \text{st}(\theta) \to \text{St}_\gamma(\hat{\theta})$.

**Lemma 5.10.** Let $\theta \in \text{Y}_n^\circ$ and suppose that $\kappa$ is generic for $\theta$. Then the correspondence $(\zeta, T) \mapsto \Sigma(\zeta, T)$ above gives bijections

$$
P \times \text{st}(\theta) \cong \text{St}_\gamma(\hat{\theta}), \quad P_0 \times \text{st}(\theta) \cong \text{St}_\gamma(\hat{\theta}).
$$

**Proof.** For $S \in \text{St}_\gamma(\hat{\theta})$, define $l_1 < \cdots < l_n$ as the increasing sequence in $\mathbb{Z}$ obtained as a rearrangement of $\{S(u) \mid u \in \theta\}$. Write $l_i = \tilde{l}_i + k_i n$ with $\tilde{l}_i \in [1,n]$ and $k_i \in \mathbb{Z}$, and define $\zeta_S = \sum_{i \in [1,n]} k_i \epsilon_i \in P$. Define $T_S \in \text{tab}(\theta)$ by

$$
T_S(u) = i \quad \text{for} \ u \in \theta \text{ such that } S(u) = l_i.
$$

Then $T_S \in \text{st}(\theta)$ as $S \in \text{St}_\gamma(\hat{\theta})$. It is easy to check that the map $\text{St}_\gamma(\hat{\theta}) \to P \times \text{st}(\theta)$ given by the correspondence $S \mapsto (\zeta_S, T_S)$ is the inverse map of $\Sigma$. \hfill \Box

**Proposition 5.11.** Let $\lambda \vdash n$. If $\kappa$ is generic for $\lambda$. Then $\Delta_\kappa(\lambda)$ is irreducible and isomorphic to $\mathcal{L}_\kappa(\lambda)$.\hfill \Box
Suppose $\delta$ responding to the single column $\delta$

Note that

Proof.

Lemma 5.14. $\delta$

Proposition 5.12.

2.1.

Proof.

Remark 5.13. Proposition 5.12 can be also derived from the result by Garsia-Procesi [GP]. As is well-known, $F[x]$ is a free module over $F[x]^W$ and furthermore $F[x] \cong F[x]^W \otimes \mathcal{R}$ as a graded $FW$-module, where $\mathcal{R}$ (called the coinvariant algebra) is by definition the quotient algebra of $F[x]$ by the ideal generated by the elementary symmetric functions of positive degree.

In [GP], it is proved that in the decomposition

(5.3.1)

as a graded $FW$-module, the graded dimension of $M_\lambda$ is given by $K_\lambda(1^n)(q)$.

This implies the formula in Proposition 5.12.

Conversely, our proof of Proposition 5.12 gives an alternative derivation of the formula $\text{Tr}(M_\lambda; q^{\partial'}) = K_\lambda(1^n)(q)$ in [GP].

5.4. Characters for single column representations. As a result for non-generic and non-integral $\kappa$, we give character formulas for the tableaux representation corresponding to the single column $\delta_n = \{(a, 1) \in Z \times Z | a \in [1, n]\}$ using Proposition 2.1

Lemma 5.14. Let $\kappa \in F \setminus Q_{\leq 0}$ and put $\gamma = (-n, \kappa - n)$.

(i) $\delta_n \in Y^n_\gamma$ if and only if $\kappa \notin Q$ or $\kappa = s/r$ with $s \in Z_{\geq n}$, $r \in Z_{\geq 1}$ and $(s, r) = 1$.

(ii) Suppose $\delta_n \in Y^n_\gamma$. Then $\Delta_\kappa(\delta_n) \cong \mathcal{L}_\kappa(\delta_n)$ unless $\kappa = n/r$ with $r \in Z_{\geq 1}$ and $(n, r) = 1$.

Proof. Note that $\delta_n \in Y^n_\gamma \iff \delta_n \in \Lambda^+(n, n)$, and the statement (i) is obvious.

By Proposition 5.11 $\kappa$ is generic unless $\kappa = n/r$. Hence the statement (ii) follows from Proposition 5.11.
In the rest, we suppose \( \kappa = n/r \) with \( r \in \mathbb{Z}_{\geq 1}, \ (n, r) = 1 \).

**Proposition 5.15.** The \( \mathcal{H}_\kappa \)-module \( \mathcal{L}_\kappa(\delta_n) \) is a free module over \( \mathbb{F}[\pi] \) of rank \( r^{n-1} \).

Moreover,

\[
\text{Tr}(\mathcal{L}_\kappa(\delta_n); q^\theta) = q^{-r(n-1)} \left( \frac{1 - q^r}{1 - q} \right)^{n-1},
\]

\[
\text{Tr}(\mathcal{L}_\kappa(\delta_n); q^\theta) = q^{-\frac{r(n-1)}{2}} \left( \frac{n + r - 1}{n} \right)_q \gamma_r \text{ if } r < n,
\]

\[
\text{Tr}(\mathcal{L}_\kappa(\delta_n); q^\theta) = \begin{cases} 
0 & \text{if } r < n, \\
q^{-\frac{r(n-1)}{2}} \left( \frac{r-1}{n} \right)_q & \text{if } r \geq n.
\end{cases}
\]

**Remark 5.16.** For the rational Cherednik algebra \( \mathcal{H}_\kappa(SL_n) \) of type \( SL_n \), formulas for \( \text{Tr}(\mathcal{L}_\kappa(\delta_n); q^\theta) \) and \( \text{Tr}(\mathcal{L}_\kappa(\delta_n); q^\theta) \) have been obtained by Berest-Etingof-Ginzburg [BEG2] by a different method. (They furthermore computed \( \text{Tr}(\mathcal{L}_\kappa(\delta_n); wat^\theta) \) for any \( w \in \mathcal{W} \).) In the same paper it is also proved that finite-dimensional representations for \( \mathcal{H}_\kappa(SL_n) \) exist only for \( \kappa = \pm n/r \) with \( r \in \mathbb{Z}_{\geq 1}, \ (r, n) = 1 \), and moreover any finite-dimensional irreducible module is isomorphic to \( \mathcal{L}_\kappa(\delta_n) \) in “+” case and is isomorphic to \( \mathcal{L}_\kappa(\delta_n) \) in “−” case (see also [Ch4, p.65]).

**Proof.** The formulas for \( \mathcal{L}_\kappa(\delta_n) \) are direct consequences of Proposition 2.1. We shall prove the statement for \( \mathcal{L}_\kappa(\delta_n) \).

By similar argument as in the proof of Proposition 2.1, it follows that

\[
\text{St}_\gamma(\delta_n)_+ = \left\{ T \in \text{Tab}_\gamma(\delta_n) \mid 0 < T(1, 1) < T(2, 1) < \cdots < T(n, 1) < T(1, 1) + rn \right\}.
\]

Observe that \( \pi \in \hat{\mathcal{W}} \) acts on the space \( \mathcal{L}_\kappa(\delta_n) = \mathbb{F}\text{St}_\gamma(\delta_n)_+ \) by \( \pi \nu_T = \nu_{\pi T} \).

Put \( \text{St}_\gamma(\delta_n)_1 = \left\{ T \in \text{St}_\gamma(\delta_n) \mid T(1, 1) = 1 \right\} \). Then it easily follows that \( \mathbb{F}\text{St}_\gamma(\delta_n)_+ \) is a free module over \( \mathbb{F}[\pi] \) with the basis \( \{ \nu_T \mid T \in \text{St}_\gamma(\delta_n)_1 \} \).

Let \( M(n, r) \) denote the set of all maps from \([2, n]\) to \([0, r - 1]\). For \( f \in M(n, r) \), there exists a unique \( T \in \text{St}_\gamma(\delta_n)_1 \) such that \( T(2, 1), \ldots, T(n, 1) = \{ i + f(i)n \mid i \in [1, n] \} \). From the expression of \( \text{St}_\gamma(\delta_n)_+ \), it follows that this correspondence gives a bijection \( M(n, r) \rightarrow \text{St}_\gamma(\delta_n)_1 \). In particular, we have \( \text{rank}_{\mathbb{F}[\pi]} \mathcal{L}_\kappa(\delta_n) = \# \text{St}_\gamma(\delta_n)_1 = r^{n-1} \).

Let \( T_f \) denote the element in \( \text{St}_\gamma(\delta_n)_1 \) corresponding to \( f \in M(n, r) \) through the bijection \( M(n, r) \rightarrow \text{St}_\gamma(\delta_n)_1 \) above. Then, we have \( \partial \nu_{T_f} = (|f| + d_{\delta_n}) \nu_{T_f} \), where \( |f| = \sum_{i \in [2, n]} f(i) \) and \( d_{\delta_n} = -\frac{r}{q} (n-1) \). It is easy to see that

\[
\sum_{f \in M(n, r)} q^{|f|} = \left( \frac{1 - q^r}{1 - q} \right)^{n-1},
\]

from which the statement follows. \( \square \)
APPENDIX A. CLASSIFICATION OF IRREDUCIBLE MODULES WITH WEIGHT DECOMPOSITION

We give a proof of Theorem 4.8. For $\zeta \in \mathfrak{h}^*$, we define a function $F_{\zeta}$ on $\mathbb{Z}$ by setting

$$F_{\zeta}(i) = \langle \zeta \mid \epsilon_i^\vee \rangle - k\kappa \quad \text{for } i = \bar{i} + kn \text{ with } \bar{i} \in [1,n], \ k \in \mathbb{Z}.$$ 

To prove Theorem 4.8, we use the following lemma:

**Lemma A.1.** (cf. [SV, Lemma 4.19]) Let $\kappa \in \mathbb{F} \setminus Q_{\leq 0}$. Let $L$ be an irreducible $\mathcal{H}_\kappa$-module which belongs to $\mathcal{O}^b(\mathcal{H}_\kappa)$, and let $\zeta \in \mathfrak{h}^*$ be a weight of $L$. For any $i, j \in \mathbb{Z}$ such that $i < j$ and $\langle \zeta \mid \alpha_{ij}^\vee \rangle = 0$,

there exist $k_+ \in [i+1, j-1]$ and $k_- \in [i+1, j-1]$ such that

$$F_{\zeta}(k_-) = F_{\zeta}(i) \pm 1.$$ 

**Proof.** The statement has been proved in [SV, Lemma 4.19] with the restriction $\zeta \in P$ and $\kappa \in \mathbb{Z}$. The proof can be generalized to our case with little modification. \hfill \square

**Proof of Theorem 4.8**

We have seen that $\mathcal{L}_\kappa(\lambda) \cong \text{nil}(\mathcal{H}_\gamma(\hat{\lambda}))$ when $\lambda \in \bigsqcup_{m \in [1,n]} \Lambda^+_{\kappa}(m,n)$, and hence $\mathcal{L}_\kappa(\lambda)$ is in $\mathcal{O}^b(\mathcal{H}_\kappa)$.

We suppose that $\lambda \in \Lambda^+(m,n) \setminus \Lambda^+_\kappa(m,n)$, and will prove that $\mathcal{L}_\kappa(\lambda)$ does not belong to $\mathcal{O}^b(\mathcal{H}_\kappa)$. The statement is easily checked when $\kappa \notin \mathbb{Q}$.

Assume that $\kappa \in \mathbb{Q}_{\geq 0}$ and write $\kappa = s/r$ with $r, s \in \mathbb{Z}_{\geq 1}$, $(r, s) = 1$.

Since $S_\lambda = \mathcal{V}^{\text{aff}}(\lambda) \subset \mathcal{L}_\kappa(\lambda)$, the content $c_{\hat{t}_\lambda}$ of the row reading tableau $t_\lambda$ on $\lambda$ gives a weight $\epsilon_{\hat{t}_\lambda}$ of $\mathcal{L}_\kappa(\lambda)$. By the assumption, we have

$$s - m - \lambda_1 + \lambda_m \in \mathbb{Z}_{<0}.$$ 

First, assume that $s < m$. Put $a = m - s$. Then $a \in [1, m - 1]$ and $(a, 1) \in \lambda$. We put $i = t_\lambda(m, 1)$ and $j = t_\lambda(a, 1) + rn$. We have

$$F_{\hat{t}_\lambda}(j) = F_{\hat{t}_\lambda}(t_\lambda(a, 1)) - r\kappa = 1 - a - s = 1 - m = F_{\hat{t}_\lambda}(i).$$

It is easy to see that

$$F_{\hat{t}_\lambda}(k) > F_{\hat{t}_\lambda}(i) \quad \text{for all } k \in [i+1, n],$$

$$F_{\hat{t}_\lambda}(k) > F_{\hat{t}_\lambda}(j) \quad \text{for all } k \in [1+rn, j-1],$$

$$F_{\hat{t}_\lambda}(k) \notin \mathbb{Z} \quad \text{for all } k \in [n+1, rn].$$

Therefore there are no $k \in [i+1, j-1]$ such that $F_{\hat{t}_\lambda}(k) = F_{\hat{t}_\lambda}(i)-1$. By Lemma A.1, $\mathcal{L}_\kappa(\lambda)$ is not in $\mathcal{O}^b(\mathcal{H}_\kappa)$.

Next, assume that $s \geq m$. Put $b = 1 + \lambda_m + s - m$. Then we have $b \in [1, \lambda_1]$ and hence $(1, b) \in \lambda$. Put $i = t_\lambda(m, \lambda_m) = n$ and $j = t_\lambda(1, b) + rn$. Then $F_{\hat{t}_\lambda}(j) = F_{\hat{t}_\lambda}(t_\lambda(1, b)) - r\kappa = (1 + \lambda_m + s - m) - 1 - s = \lambda_m - m = F_{\hat{t}_\lambda}(i)$. Similarly to the case $s < m$, it is shown that there are no $k \in [i+1, j-1]$ such that $F_{\hat{t}_\lambda}(k) = F_{\hat{t}_\lambda}(i) + 1$. By Lemma A.1, $\mathcal{L}_\kappa(\lambda)$ is not in $\mathcal{O}^b(\mathcal{H}_\kappa)$. \hfill \square
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