SESHADRI CONSTANTS AND OKOUNKOV BODIES REVISITED

JINHYUNG PARK AND JAESUN SHIN

Abstract. In this paper, we give simple uniform proofs of the results of Choi–Hyun–Park–Won [CHPW] and Küronya–Lozovanu [KL1, KL2, KL3], which describe the local positivity of divisors in terms of convex geometry of Okounkov bodies. We then introduce the integrated volume function to investigate the relation between Seshadri constants and filtered Okounkov bodies introduced by Boucksom–Chen [BC].

1. Introduction

Throughout the paper, we work over an algebraically closed field $k$ of arbitrary characteristic unless otherwise stated. Let $X$ be a smooth projective variety of dimension $n$, and fix an admissible flag on $X$, that is a sequence of irreducible subvarieties $Y^•: X = Y^0 \supseteq Y^1 \supseteq \cdots \supseteq Y^{n-1} \supseteq Y^n = \{x\}$ where each $Y^i$ is of codimension $i$ in $X$ and is smooth at the point $x$. The Okounkov body $\Delta_{Y^i}(D)$ of an $\mathbb{R}$-divisor $D$ with respect to $Y^i$ is a compact convex subset of the Euclidean space $\mathbb{R}^{n}_{\geq 0}$. It was independently introduced by Lazarsfeld–Mustaţă [LM] and Kaveh–Khovanskii [KK] based on the pioneering works of Okounkov [O1, O2].

Now, consider the blow-up $\pi: \tilde{X} \to X$ at a point $x \in X$ with the exceptional divisor $E$. An infinitesimal admissible flag over $x$ is an admissible flag on $\tilde{X}$ defined as $\tilde{Y}^•: \tilde{X} = \tilde{Y}^0 \supseteq \tilde{E} = \tilde{Y}^1 \supseteq \cdots \supseteq \tilde{Y}^{n-1} \supseteq \tilde{Y}^n = \{x'\}$ where each $\tilde{Y}^i$ is a linear subspace of $\tilde{E} \simeq \mathbb{P}^{n-1}$ for $2 \leq i \leq n$. The infinitesimal Okounkov body is $\tilde{\Delta}_{\tilde{Y}^i}(D) := \Delta_{\tilde{Y}^i}(f^* D) \subseteq \mathbb{R}^n_{\geq 0}$. See Section 2 for more details.

In recent years, a considerable amount of research has been devoted to the study of the connection between local positivity of divisors and (infinitesimal) Okounkov bodies (see e.g., [CHPW, CPW1, CPW2, CPW3], [DKMS], [I], [KL1, KL2, KL3], [R]). It was explained in [CHPW], [KL1, KL2, KL3] that (inverted) standard simplices arise naturally in (infinitesimal) Okounkov bodies. Let $e_1, \ldots, e_n$ be the standard basis vectors for $\mathbb{R}^n$, and $0$ be the origin of $\mathbb{R}^n$. The standard simplex $\Delta^n_\xi \subseteq \mathbb{R}^n_{\geq 0}$ of size $\xi \geq 0$ (resp. the
inverted standard simplex $\Delta^{\#}_{\xi} \subseteq \mathbb{R}^{n}_{\geq 0}$ of size $\xi \geq 0$ is the convex hull of $\{0, \xi e_1, \ldots, \xi e_n\}$ (resp. $\{0, \xi e_1, \xi(e_1+e_2), \ldots, \xi(e_1+e_n)\}$). In this paper, we prove the following ampleness criterion in terms of Okounkov bodies, which is an analogue result of Seshadri’s ampleness criterion (cf. [La, Theorem 1.4.13]).

**Theorem 1.1.** Let $X$ be a smooth projective variety of dimension $n$, and $D$ be a big $\mathbb{R}$-divisor on $X$. Then the following are equivalent:

1. $D$ is ample.
2. For every admissible flag $Y_\bullet$ on $X$, the Okounkov body $\Delta_{Y_\bullet}(D)$ contains a nontrivial standard simplex.
3. For every point $x \in X$, there is an admissible flag $Y_\bullet$ centered at $x$ such that the Okounkov body $\Delta_{Y_\bullet}(D)$ contains a nontrivial standard simplex.
4. For every infinitesimal admissible flag $\tilde{Y}_\bullet$ over $X$, the infinitesimal Okounkov body $\tilde{\Delta}_{\tilde{Y}_\bullet}(D)$ contains a nontrivial inverted standard simplex.
5. For every point $x \in X$, there is an infinitesimal admissible flag $\tilde{Y}_\bullet$ over $x$ such that the infinitesimal Okounkov body $\tilde{\Delta}_{\tilde{Y}_\bullet}(D)$ contains a nontrivial inverted standard simplex.

In characteristic zero, the equivalences (1) $\iff$ (2) $\iff$ (3) and (1) $\iff$ (4) $\iff$ (5) were proved in [CHPW, Corollary D] and [KL1, Theorem B], respectively.

Theorem 1.1 follows from the relation between the Seshadri constants and (infinitesimal) Okounkov bodies. The Seshadri constant is a measure of local positivity of a nef and big divisor $D$ at $x$. It was first introduced by Demailly [D], and there has been a great deal of effort over the decades to study the Seshadri constants. See Section 2 for the definition. For an admissible flag $Y_\bullet$ centered at $x$ on $X$ and an infinitesimal admissible flag $\tilde{Y}_\bullet$ over $x$, we define nonnegative numbers

$$
\begin{align*}
\xi_{Y_\bullet}(D; x) &:= \max\{\xi \mid \Delta^n_{\xi} \subseteq \Delta_{Y_\bullet}(D)\} \quad \text{and} \quad \tilde{\xi}_{\tilde{Y}_\bullet}(D; x) := \max\{\tilde{\xi} \mid \tilde{\Delta}^{\#}_{\xi} \subseteq \tilde{\Delta}_{\tilde{Y}_\bullet}(D)\}, \\
\xi(D; x) &:= \sup_{Y_\bullet}\{\xi_{Y_\bullet}(D; x)\} \quad \text{and} \quad \tilde{\xi}(D; x) := \sup_{\tilde{Y}_\bullet}\{\tilde{\xi}_{\tilde{Y}_\bullet}(D; x)\},
\end{align*}
$$

where the supremums are taken over all admissible flags $Y_\bullet$ centered at $x$ and all infinitesimal admissible flags over $x$, respectively. If no (inverted) standard simplex is contained in the (infinitesimal) Okounkov body, then we let $\xi_{Y_\bullet}(D; x) = 0$ (or $\tilde{\xi}_{\tilde{Y}_\bullet}(D; x) = 0$). We provide a description of Seshadri constants in terms of Okounkov bodies.

**Theorem 1.2.** Let $X$ be a smooth projective variety of dimension $n$, and $D$ be a nef and big $\mathbb{R}$-divisor on $X$. For any point $x \in X$, we have

$$
\varepsilon(D; x) = \tilde{\xi}(D; x) \geq \xi(D; x).
$$

In characteristic zero, the equality $\varepsilon(D; x) = \tilde{\xi}(D; x)$ was shown in [KL1, Theorem C], and the inequality $\varepsilon(D; x) \geq \xi(D; x)$ was shown in [CHPW, Theorem E]. Note that Theorem 1.2 holds for moving Seshadri constants of arbitrary big $\mathbb{R}$-divisors by [KL1, Theorem C] and [CHPW, Theorem E]. These works can be regarded as an attempt to find a satisfactory theory of positivity of divisors in terms of convex geometry of Okounkov
bodies. Another important results in this direction are [KL1, Theorem 4.1] and [CHPW, Theorem C], which describe the local ampleness via (infinitesimal) Okounkov bodies. All the results are higher dimensional generalizations of [KL3] (see also [KL2]). We note that these four theorems were shown separately in lengthy. The interaction between infinitesimal Okounkov bodies of $D$ and jet separation of the adjoint divisor $K_X + D$ (see [KL1, Proposition 4.10]) plays a crucial role in [KL1]. The main ingredients of [CHPW] are the slice theorem [CPW1, Theorem 1.1] and a version of Fujita approximation [Le, Proposition 3.7]. Those results are based on Nadel vanishing theorem for multiplier ideal sheaves, so the characteristic zero assumption is necessary.

In Section 3, we give a simple new outlook on this theory by proving the main results of [KL1] and [CHPW] in a uniform way. We first give a quick direct proof of Theorem 1.2. Note that the inequality $\varepsilon(D; x) \geq \xi(D; x)$ follows from the equality $\varepsilon(D; x) = \xi(D; x)$ by Lemma 3.3, which is one of the main contributions of this paper. Our approach is elementary and to avoid using vanishing theorems so that Theorems 1.1 and 1.2 hold in arbitrary characteristic. Next, in Theorem 3.5, we recover [KL1, Theorem C] from Theorem 1.2 using [Le, Proposition 3.7]. By applying Lemma 3.3 as before, we immediately obtain [CHPW, Theorem E]. Then, in Theorem 3.7, we show that Theorem 3.5 immediately implies [KL1, Theorem 4.1] and [CHPW, Theorem C]. We refer to [CPW3] and [R] for another direction of the story.

In [I], Ito studied the relation between Seshadri constants and Okounkov bodies of a birational graded linear series $V_\bullet$. In Theorem 3.8, we show that

$$\varepsilon(V_\bullet; x) = \tilde{\xi}(V_\bullet; x) \geq \xi(V_\bullet; x)$$

for a very general point $x \in X$. The inequality $\varepsilon(V_\bullet; x) \geq \xi(V_\bullet; x)$ follows from [I, Theorem 1.2], but we give an alternative proof. The equality $\varepsilon(V_\bullet; x) = \tilde{\xi}(V_\bullet; x)$ a new result. The main ingredient is the “differentiation” result [EKL, Proposition 2.3], [N, Lemma 1.3]. Theorem 3.8 does not hold if $x$ is not a general point (see Remark 3.9).

As was observed in [KL3, Remark 4.9] and [CHPW, Example 7.4], the inequality $\varepsilon(D; x) \geq \xi(D; x)$ in Theorem 1.2 can be strict in general. Moreover, one can conclude from [CPW3, Remark 3.12] that it is impossible to extract the exact value of $\varepsilon(D; x)$ from the set of Okounkov bodies. Thus it is necessary to consider finer structures on Okounkov bodies in order to read off the exact value of the Seshadri constant. In this paper, as in [DKMS], [KMS], we consider the multiplicative filtration determined by the geometric valuation $\ord_x$ for a point $x \in X$. Let $V_\bullet := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, O(X([mD])))$ be the complete graded linear series of a big $\mathbb{R}$-divisor $D$ on $X$, and $\mathcal{F}_x$ be a multiplicative filtration on $V_\bullet$ defined by

$$\mathcal{F}_x V_m := \{ s \in V_m \mid \ord_x(s) \geq t \}.$$  

Fix an admissible flag $Y_\bullet$ on $X$ centered at $x$. The filtered Okounkov body $\tilde{\Delta}_{Y_\bullet}(V_\bullet, \mathcal{F}_x)$ of $D, \mathcal{F}_x$ with respect to $Y_\bullet$ is a compact convex subset in $\mathbb{R}^{n+1}_{\geq 0}$, and it was introduced in [BC]. See Section 2 for the definition. In Section 4, we introduce and study the integrated
volume function
\[ \widehat{\varphi}_x : \text{Big}(X) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad (D, t) \mapsto \int_{u=0}^{t} \text{vol}_{\mathbb{R}^n}(\widehat{\Delta}_{Y_x}(V_\bullet, F_x)_{x_{n+1}=u})du. \]

Note that \( \widehat{\varphi}_x \) is continuous on the whole domain \( \text{Big}(X) \times \mathbb{R}_{\geq 0} \). By fixing a big \( \mathbb{R} \)-divisor \( D \), the derivative \( \widehat{\varphi}_x'(D, t) := \frac{d\widehat{\varphi}_x(D, t)}{dt} \) always exists. The Euclidean volume of the filtered Okounkov body \( \widehat{\Delta}_{Y_x}(V_\bullet, F_x) \) is given by \( \widehat{\varphi}_x(D, \infty) \), which was used to study diophantine approximation on algebraic varieties in [MR]. This number also plays an important role in theory of K-stability (cf. [BJ], [F], [Li]). In this paper, we give a new characterization of the Seshadri constant in terms of the integrated volume function.

**Theorem 1.3.** Let \( X \) be a smooth projective variety of dimension \( n \), and \( D \) be a nef and big \( \mathbb{R} \)-divisor on \( X \). For any point \( x \in X \), we have
\[ \varepsilon(D; x) = \inf \left\{ t \geq 0 \mid \widehat{\varphi}_x'(D, 0) - \widehat{\varphi}_x'(D, t) < \frac{t^n}{n!} \right\}. \]

In Section 4, we define the bounded mass function \( \text{mass}_+(V_m, F_x, t) \) for \( t \geq 0 \) as an “appropriate” sum of jumping numbers of \( (V_m, F_x) \). We will see in Theorem 4.8 that
\[ \widehat{\varphi}_x(D, t) = \lim_{m \to \infty} \frac{\text{mass}_+(V_m, F_x, mt)}{m^{n+1}}. \]

Thus the integrated volume function is independent of the choice of the admissible flags to define the filtered Okounkov body.

The rest of the paper is organized as follows. We begin in Section 2 with recalling basic definitions. In Section 3, we show Theorems 1.1 and 1.2, and give simple uniform proofs of the main results in [CHPW], [KL1, KL2, KL3]. Section 4 is devoted to the study of integrated volume functions; in particular, we prove Theorem 1.3.

## 2. Preliminaries

Throughout the paper, we use the following notations: \( X \) is a smooth projective variety of dimension \( n \) defined over an algebraically closed field \( k \) of arbitrary characteristic, and \( D \) is an \( \mathbb{R} \)-divisor on \( X \). We fix a point \( x \in X \), and let \( \pi : \tilde{X} \to X \) be the blow-up of \( X \) at \( x \) with the exceptional divisor \( E \). Let \( V_\bullet \) be a graded linear series associated to \( D \) so that \( V_m \) is a linear subspace of \( H^0(X, \mathcal{O}_X([mD])) \) for every \( m \geq 0 \).

### 2.1. Okounkov bodies

Let \( Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{x\} \) be an admissible flag on \( X \). We define a valuation-like function
\[ \nu_\cdot : |D|_\mathbb{R} := \{ D' \mid D \sim D' \geq 0 \} \to \mathbb{R}^n_{\geq 0} \]
as follows. For any \( D' \in |D|_\mathbb{R} \), let \( \nu_1 = \nu_1(D') := \text{ord}_{Y_1}(D') \). Then \( D' - \nu_1 Y_1 \) is effective and does not contain \( Y_2 \) in the support. Let \( \nu_2 = \nu_2(D') := \text{ord}_{Y_2}((D' - \nu_1 Y_1)|_{Y_1}). \) Similarly let \( \nu_{i+1} = \nu_{i+1}(D') := \text{ord}_{Y_{i+1}}((D' - \nu_1 Y_1|_{Y_1} - \nu_2 Y_2|_{Y_2} - \cdots - \nu_i Y_i)|_{Y_i}) \) for
infinitesimal admissible flags are studied in \[ \text{linear series} \]

One can also define the infinitesimal Okounkov body \( \Delta_0 \) such that \( t > m \) with \( Y \) birational onto its image for any \( t \). Following [KL1, Definition 2.1] (see also [KL2, Remark 2.2]), we define the \textit{infinitesimal Okounkov body} of \( D \) with respect to \( Y \) as

\[
\tilde{\Delta}_Y(D) := \Delta_Y(\pi^* D) \subseteq \mathbb{R}^n_0.
\]

One can also define the infinitesimal Okounkov body \( \tilde{\Delta}_Y(V) \) associated to a graded linear series \( V \). We remark that the Okounkov bodies with respect to more general infinitesimal admissible flags are studied in [R] and [CPW3].

For a graded linear series \( V \) on \( X \), we define nonnegative numbers

\[
\xi_Y(V; x) := \max\{\xi \mid \Delta_\xi \subseteq \Delta_Y(V)\} \quad \text{and} \quad \tilde{\xi}_Y(V; x) := \max\{\tilde{\xi} \mid \tilde{\Delta}_\tilde{\xi} \subseteq \tilde{\Delta}_Y(V)\},
\]

\[
\tilde{\xi}_Y(V; x) := \sup_{Y_0}(\xi_Y(V; x)) \quad \text{and} \quad \tilde{\xi}(V; x) := \sup_{\tilde{Y}_n}(\tilde{\xi}_Y(V; x)),
\]

where the supremums are taken over all admissible flags \( Y \) centered at \( x \) and all infinitesimal admissible flags over \( x \), respectively. If no (inverted) standard simplex is contained in the (infinitesimal) Okounkov body, then we let \( \xi_Y(V; x) = 0 \) (or \( \tilde{\xi}_Y(V; x) = 0 \)).

We say that a graded linear series \( V \) is \textit{birational} if the rational map given by \( |V_m| \) is birational onto its image for any \( m \gg 0 \). It is exactly Condition (B) in [LM, Definition 2.5]. Now, let \( F \) be a multiplicative filtration on \( V \). For any \( t \in \mathbb{R} \), we have a new graded linear series \( V^{(t)} \), which is defined as \( V^{(t)}_m := F^t m V_m \) for all \( m \in \mathbb{Z}_{\geq 0} \). Notice that the Okounkov bodies \( \Delta_Y(V^{(t)}) \) with respect to an admissible flag \( Y \) on \( X \) form a nonincreasing family of convex subsets \( \Delta_Y(V^{(t)}) \subseteq \mathbb{R}^n_0 \). See [BC] for more details.

\textbf{Example 2.1.} As in Introduction, we define the multiplicative filtration \( F_t V \) on \( V \) as

\[
F_t V := \{ s \in V_m \mid \text{ord}_x(s) \geq t \}.
\]

Then \( F_t \) is pointwise bounded below and linearly bounded above in the sense of [BC, Definition 1.3] (see [KMS, Proposition 3.5]). If \( V \) is birational, then so is \( V^{(t)} \) for any \( t > 0 \) such that \( V^{(t+\epsilon)}_m \neq \emptyset \) for any integer \( m \gg 0 \) and a sufficiently small number \( \epsilon > 0 \). To see why, fix an integer \( m' \gg 0 \) such that \( V^{(t)}_m \) defines a birational map and a sufficiently small number \( \epsilon > 0 \) with \( V^{(t+\epsilon)}_m \neq \emptyset \) for \( m \gg 0 \). For any integer \( m \gg 0 \) with \( m(t + \epsilon) > (m + m')t \), we have \( s \cdot V^{(t)}_m \subseteq V^{(t)}_{m + m'} \) for any nontrivial \( s \in V^{(t+\epsilon)}_m \) so that \( V^{(t)}_{m+m'} \) defines a birational map.
Following [BC], we define the \textit{concave transform} of a multiplicative filtration $\mathcal{F}$ on $V_\bullet$ to be a real-valued function on $\Delta_{V_\bullet}(V_\bullet)$ given by
\[
\varphi_{\mathcal{F},V_\bullet}(x) := \sup \{t \in \mathbb{R} \mid x \in \Delta_{V_\bullet}(V_\bullet(t))\},
\]
and the \textit{filtered Okounkov body} associated to $V_\bullet$, $\mathcal{F}$ with respect to $Y_\bullet$ to be a compact convex subset of $\mathbb{R}_\geq 0 \times \mathbb{R}_\geq 0 = \mathbb{R}_{\geq 0}^{n+1}$ given by
\[
\widehat{\Delta}_{V_\bullet}(V_\bullet, \mathcal{F}) := \{ (x, t) \in \Delta_{V_\bullet}(V_\bullet) \times \mathbb{R} \mid t \in [0, \varphi_{\mathcal{F},V_\bullet}(x)]\} \subseteq \mathbb{R}_{\geq 0}^{n+1}.
\]

2.2. \textbf{Seshadri constants.} Suppose that $D$ is nef. The \textit{Seshadri constant} of $D$ at a point $x \in X$ is the nonnegative real number
\[
\varepsilon(D; x) := \sup \{k \mid \pi^*D - kE \text{ is nef}\} = \inf_{x \in C} \left\{ \frac{D.C}{\mult_x C} \right\},
\]
where the infimum in the middle is taken over all irreducible curves $C$ on $X$ passing through $x$. When $D$ is a $\mathbb{Z}$-divisor, for an integer $m \geq 0$, we let $s(mD; x)$ be the supremum of integers $s \geq -1$ such that the natural map $H^0(X, \mathcal{O}_X(mD)) \to \mathcal{O}_X(mD) \otimes \mathcal{O}_X/m_x^{s+1}$ is surjective. Then $\varepsilon(D; x) = \sup_{m \geq 0} \frac{s(mD; x)}{m}$. The Seshadri constant was first introduced by Demailly [D]. See [La, Chapter 5] for more details.

The \textit{Nakayama constant} of a graded linear series $V_\bullet$ at $x$ is similarly defined by
\[
\mu(V_\bullet; x) = \sup \left\{ \frac{\ord_x(s)}{m} \mid s \in V_m \right\}. \quad \text{When } V_\bullet \text{ is the complete graded linear series of a pseudoeffective divisor } D, \text{ we have } \mu(V_\bullet; x) = \mu(D; x) = \sup \{k \mid \pi^*D - kE \text{ is pseudoeffective}\}.
\]

For an arbitrary $\mathbb{R}$-divisor $D$, the \textit{stable base locus} of $D$ is $\text{SB}(D) := \bigcap_{m \geq 0} \text{Supp}(mD)$. The \textit{restricted base locus} of $D$ is $\text{B}_+(D) := \bigcup D$, and the \textit{augmented base locus} of $D$ is $\text{B}_+(D) := \bigcup_{A} \text{SB}(D - A)$, where the intersection is taken over all ample divisors $A$ on $X$. We refer to [ELMNP1], [M] for further properties.

Now, we assume that $\text{char}(k) = 0$. If $x \notin \text{B}_+(D)$, then the \textit{moving Seshadri constant} of $D$ at $x$ is defined as
\[
\varepsilon(\|D\|; x) := \sup_{f \circ D = A + E} \varepsilon(A; x), \quad \text{where the supremum runs over all birational morphisms } f : \hat{X} \to X \text{ with } \hat{X} \text{ smooth, that are isomorphic over a neighborhood of } x, \text{ and decompositions } f^*D = A + E \text{ with an ample divisor } A \text{ and an effective divisor } E \text{ such that } f^{-1}(x) \text{ is not in the support of } E. \text{ If } x \in \text{B}_+(D), \text{ then we put } \varepsilon(\|D\|; x) := 0. \text{ If } D \text{ is nef, then we have } \varepsilon(\|D\|; x) = \varepsilon(D; x). \text{ For further details, see [ELMNP2].}
\]

Let $V_\bullet$ be a graded linear series associated to a $\mathbb{Z}$-divisor $D$. As before, for an integer $m \geq 0$, let $s(V_m; x)$ be the supremum of integers $s \geq -1$ such that the natural map $V_m \to \mathcal{O}_X(mD) \otimes \mathcal{O}_X/m_x^{s+1}$ is surjective. Then we define the \textit{Seshadri constant} of $V_\bullet$ at $x$ to be $\varepsilon(V_\bullet; x) := \sup_{m \geq 0} \frac{s(V_m; x)}{m}$. We refer to [I, Section 3] for more details.

3. Local positivity via Okounkov bodies

In this section, we show Theorems 1.1 and 1.2, and then present alternative approach to the main results in [CHPW], [KL1, KL2, KL3]. We continue to use the notations in Section 2.
3.1. Basic Lemmas. We start by proving some useful lemmas.

Lemma 3.1. Suppose that $D$ is big. For $0 < k < \mu(D; x)$, we have the following:

1. $E \not\subseteq B_+(\pi^*D - kE)$.
2. $\tilde{\Delta}_{\tilde{Y}_\bullet}(D)_{x_1=k} = \Delta_{\tilde{Y}_\bullet}(V^k)$, where $V^k$ is the graded linear series on $Y_1$ such that $V^k_m = \text{Image}[H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}([m(\pi^*D - kE)])) \to H^0(Y_1, \mathcal{O}_{Y_1}([m(\pi^*D - kE)]))]$ for $m \geq 0$.
3. $\tilde{\Delta}_{\tilde{Y}_\bullet}(D) \subseteq \tilde{\Delta}_{\tilde{Y}_\bullet}^n(D)$ for any infinitesimal admissible flag $\tilde{Y}_\bullet$ over $x$.
4. If $\tilde{\Delta}_n(x) \subseteq \tilde{\Delta}_{\tilde{Y}_\bullet}(D)$ for some infinitesimal admissible flag $\tilde{Y}_\bullet$ over $x$, then the same is true for every infinitesimal admissible flag over $x$. In particular, $\tilde{\xi}(D; x) = \tilde{\xi}_{\tilde{Y}_\bullet}(D; x)$ for any infinitesimal admissible flag $\tilde{Y}_\bullet$ over $x$.

Proof. This lemma was shown in [KL1, Propositions 2.5 and 4.7], but we give an alternative proof here. The assertion (1) directly follows from [FKL, Theorem B]. Then [LM, Theorem 4.26] implies (2). For (3), let $\tilde{Y}_\bullet$ be an infinitesimal flag over $x$. Note that $E \simeq \mathbb{P}^{n-1}$ and $V^k_m \subseteq H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}([mkH]))$, where $H$ is a hyperplane section of $\mathbb{P}^{n-1}$. Thus we have

$$\Delta_{\tilde{Y}_\bullet}(V^k) \subseteq \Delta_{\tilde{Y}_\bullet}(kH) = \Delta_{\tilde{Y}_\bullet}^{n-1}.$$  

This proves (3). For (4), we now assume that $\tilde{\Delta}_n(x) \subseteq \tilde{\Delta}_{\tilde{Y}_\bullet}(D)$ for an infinitesimal admissible flag $\tilde{Y}_\bullet$ over $x$. Let $F_x$ be the multiplicative filtration in Example 2.1 for the complete graded linear series $V_x$ associated to $D$. For $0 < k < \tilde{\xi}$, we consider the induced birational graded linear series $V^{(k)}_x$. Note that $\tilde{\Delta}_{\tilde{Y}_\bullet}(V^{(k)}_x) = \tilde{\Delta}_{\tilde{Y}_\bullet}(D)_{x_1=k}$. Thus we have

$$\tilde{\Delta}_{\tilde{Y}_\bullet}(D) = \tilde{\Delta}_n(x) \cup \tilde{\Delta}_{\tilde{Y}_\bullet}(V^{(k)}),$$

so we obtain $\text{vol}_x(D) = k^n + \text{vol}_x(V^{(k)}_x)$ by [LM, Theorem 2.13]. For any infinitesimal admissible flag $\tilde{Y}_\bullet$ over $x$, we also have $\tilde{\Delta}_{\tilde{Y}_\bullet}(V^{(k)}_x) = \tilde{\Delta}_{\tilde{Y}_\bullet}(D)_{x_1=k}$. By considering (3) and [LM, Theorem 2.13], we see that $\tilde{\Delta}_{\tilde{Y}_\bullet}(D)_{x_1=k} = \tilde{\Delta}_n(x)$. This proves (4). \qed

We note that Lemma 3.1 (3) and (4) hold for a birational graded linear series $V_x$.

Lemma 3.2. Suppose that $D$ is big. If $0 \in \Delta_{\tilde{Y}_\bullet}(D)$ for some admissible flag $Y_\bullet$ on $X$ centered at $x$, then $x \not\in B_-(D)$.

Proof. The proof of [CHPW, Theorem A] or [KL2, Theorem A] works for positive characteristic, but we include the proof here. Suppose that $x \in B_-(D)$. Then $\text{ord}_x(||D||) > 0$ by [ELMNP1, Proposition 2.8] and [M, Theorem C]. Let $Y_\bullet$ be any admissible flag on $X$ centered at $x$. For any $x = (\nu_1, \ldots, \nu_n) \in \Delta_{\tilde{Y}_\bullet}(D)$, we have $\nu_1 + \cdots + \nu_n \geq \text{ord}_x(||D||) > 0$, which implies that $0 \not\in \Delta_{\tilde{Y}_\bullet}(D)$.

Lemma 3.3. Suppose that $V_x$ is birational. Then we have

$$\tilde{\xi}(V_x; x) \geq \xi(V_x; x).$$
Proof. Fix an admissible flag $\Y_\bullet$ on $X$ centered at $x$ and an infinitesimal admissible flag $\widetilde{\Y}_\bullet$ over $x$. Put $\xi := \xi_{\widetilde{\Y}_\bullet}(\Y_\bullet; x)$ and $\xi := \xi_{\Y_\bullet}(\Y_\bullet; x)$. By Lemma 3.1 (4), we know that $\xi = \xi(\Y_\bullet; x)$. Thus it suffices to show that $\xi \geq \xi$. Fix a sufficiently small number $\varepsilon > 0$. Let $\mathcal{F}_x$ be the multiplicative filtration in Example 2.1, and $V^{(\xi-\varepsilon)}_\bullet$ be the induced birational graded linear series. Note that $\Delta_{\Y_\bullet}(V^{(\xi-\varepsilon)}_\bullet) \subseteq \Delta_{\Y_\bullet}(V^\bullet) \setminus \Delta_{\xi-\varepsilon}$. By applying [LM, Theorem 2.13], we see that

$$\text{vol}_{\xi}(V^{(\xi-\varepsilon)}_\bullet) \leq \text{vol}_{\xi}(V^\bullet) - (\xi - \varepsilon)^n.$$ 

On the other hand, note that $\Delta_{\Y_\bullet}(V^{(\xi-\varepsilon)}_\bullet) = \Delta_{\Y_\bullet}(V^\bullet)_{x_1 \geq \xi - \varepsilon}$. Then, by Lemma 3.1 (3), we have $\Delta_{\Y_\bullet}(V^\bullet) \subseteq \Delta_{\xi-\varepsilon} \cup \Delta_{\Y_\bullet}(V^{(\xi-\varepsilon)}_\bullet)$. By [LM, Theorem 2.13], we obtain

$$\text{vol}_{\xi}(V^\bullet) \leq (\xi - \varepsilon)^n + \text{vol}_{\xi}(V^{(\xi-\varepsilon)}_\bullet).$$

We can now conclude that the equality holds, and hence, $\Delta_{\Y_\bullet}(V^\bullet) = \Delta_{\xi-\varepsilon} \cup \Delta_{\Y_\bullet}(V^{(\xi-\varepsilon)}_\bullet)$. Since $\varepsilon > 0$ can be arbitrarily small, it follows that $\Delta^n_{\xi} \subseteq \Delta_{\Y_\bullet}(V^\bullet)$. This implies the desired inequality $\xi \geq \xi$. \hfill $\Box$

3.2. Nef and big divisor case. In this subsection, we assume that $D$ is nef and big. We are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.2. By Lemma 3.3, we have $\xi(D; x) \geq \xi(D; x)$. Thus it is sufficient to prove that $\varepsilon := \varepsilon(D; x) = \xi(D; x) =: \xi$. First, we show that $\varepsilon \leq \xi$. We only have to consider the case that $\varepsilon > 0$. For $0 < k \leq \varepsilon$, by Lemma 3.1 (3), we have $\Delta_{\Y_\bullet}(D)_{x_1 = k} \subseteq \Delta_{\varepsilon}^{n-1}$. Lemma 3.1 (1) says that $E \notin B_+(\pi^* D - kE)$. Then [LM, (2.7) in p.804] implies that $\text{vol}_{\xi}(E)_{\xi} = \text{vol}_{\xi}(D - kE) \leq k^n$. However, since $\pi^* D - kE$ is nef, it follows that $\text{vol}_{\xi}(E)_{\xi} = \text{vol}_{\xi}(D - kE) = k^{n-1}$. By [LM, Theorem 2.13], $\text{vol}_{\xi}(\Delta_{\Y_\bullet}(D)_{x_1 = k}) = \text{vol}_{\xi}(\Delta_{\varepsilon}^{n-1}(D)_{x_1 = k})$. This means that $\Delta_{\Y_\bullet}(D)_{x_1 = k} = \Delta_{\varepsilon}^{n-1}$, which shows that $\varepsilon \leq \xi$.

For the converse, suppose that $\varepsilon < \xi$. Take any number $k$ with $\varepsilon < k < \xi$. Let $A$ be an ample divisor on $X$. We can choose a sufficiently small number $\delta > 0$ such that $\varepsilon(D + \delta A; x) < k$. Let $\delta' > 0$ be a number such that $\pi^* A - \delta' E$ is ample. We consider an irreducible curve $C$ on $X$ passing through $x$, and denote by $\overline{C}$ the strict transform of $C$ by $\pi$. If $\overline{C} \notin \SB(\pi^* D - kE + \delta(\pi^* A - \delta'E))$, then

$$D.C - k \text{mult}_x C + \delta(A.C - \delta' \text{mult}_x C) = (\pi^* D - kE + \delta(\pi^* A - \delta'E))\overline{C} \geq 0.$$ 

This implies that $(D + \delta A)_C \geq k + \delta'$. Since $\varepsilon(D + \delta A; x) < k$, there is an irreducible curve $C$ on $X$ passing through $x$ such that its strict transform $\overline{C}$ by $\pi$ is contained in $\SB(\pi^* D - kE + \delta(\pi^* A - \delta'E))$. Thus $\overline{C} \subseteq \overline{B}_-(\pi^* D - kE)$. Take a point $x' \in \overline{C} \cap E$, and let $\widetilde{\Y}_\bullet$ be an infinitesimal admissible flag over $x$ centered at $x'$. By Lemma 3.2, we have $0 \notin \Delta_{\Y_\bullet}(\pi^* D - kE)$. However, by Lemma 3.1 (4), $\widetilde{\Y}_\bullet(D; x) = \xi$ so that $0 \in \Delta_{\Y_\bullet}(\pi^* D - kE)$, which is a contradiction. Hence $\varepsilon = \xi$, so we complete the proof. \hfill $\Box$

Remark 3.4. Theorem 1.2 holds for the moving Seshadri constant of an $\mathbb{R}$-divisor $D$ at any point $x$ provided that $D$ admits the Zariski decomposition $D = P + N$. In
characteristic zero, we have \( \varepsilon(||D||; x) = \varepsilon(P; x) \). When \( \text{char}(k) > 0 \), we define the moving Seshadri constant to be \( \varepsilon(||D||; x) := \varepsilon(P; x) \). If \( x \in \text{Supp}(N) \), then \( \varepsilon(||D||; x) = 0 = \xi(D; x) \). If \( x \not\in \text{Supp}(N) \), then \( \Delta_{V*}(D) = \Delta_{V*}(P) \) and \( \tilde{\Delta}_{V*}(D) = \tilde{\Delta}_{V*}(P) \) for any admissible flag \( Y \) centered at \( x \) and any infinitesimal admissible flag \( \tilde{Y} \) over \( x \). Thus, by Theorem 1.2, we have \( \varepsilon(||D||; x) = \tilde{\xi}(D; x) \geq \xi(D; x) \).

**Proof of Theorem 1.1.** (1) \( \Rightarrow \) (2): It can be shown by a standard argument (see e.g., [CHPW, Lemma 6.1]), so we omit the proof.

(2) \( \Rightarrow \) (3): It is trivial.

(3) \( \Rightarrow \) (4): Assume that (3) holds. Let \( \tilde{Y} \) be an infinitesimal admissible flag over \( x \). By Lemma 3.3 and Lemma 3.1 (4), we have \( 0 < \xi(D; x) \leq \tilde{\xi}(D; x) = \tilde{\xi}_{Y*}(D; x) \), which implies that (4) holds.

(4) \( \Rightarrow \) (5): It is trivial.

(5) \( \Rightarrow \) (1): Assume that (5) holds. For any point \( x \in X \), there exists an infinitesimal admissible flag \( \tilde{Y} \) over \( x \) centered at \( x' \) such that \( 0 \in \tilde{\Delta}_{Y*}(D) \). By Lemma 3.2, \( x' \not\in B_-(\pi^*D) \). Thus [ELMNP1, Proposition 2.8] and [M, Theorem C] imply that \( \text{mult}_x(||\pi^*D||) = 0 \). However, we have \( \text{mult}_x(||D||) = \text{mult}_E(||\pi^*D||) \leq \text{mult}_x(||\pi^*D||) = 0 \), so we obtain \( \text{mult}_x(||D||) = 0 \). By [ELMNP1, Proposition 2.8] and [M, Theorem C], \( x \not\in B_-(D) \). Since \( x \) is an arbitrary point, it follows that \( B_-(D) = \emptyset \), i.e., \( D \) is nef. By applying Theorem 1.2, we see that \( \varepsilon(D; x) > 0 \) under the condition (5). Suppose that (1) does not hold, i.e., \( D \) is not ample. By Nakai-Moishezon criterion, there is a positive dimensional subvariety \( V \subseteq X \) such that \( D|_V \) is not big. Then \( \varepsilon(D; x) \leq \varepsilon(D|_V; x) = 0 \) for any point \( x \in V \). We get a contradiction to that \( \varepsilon(D; x) > 0 \). Thus (1) holds. \( \square \)

### 3.3. Big divisor case.

From now on, we assume that \( \text{char}(k) = 0 \). In this subsection, we assume that \( D \) is an arbitrary big \( \mathbb{R} \)-divisor on \( X \) and \( V \) is a graded linear series associated to \( D \). We fix some notations. Let \( f_m: X_m \rightarrow X \) be a resolution of the base ideal \( \mathfrak{b}(V_m) \) for any integer \( m \geq 1 \). Then we have a decomposition

\[
f_m^*|V_m| = |M_m| + F_m
\]

into the base point free part \( |M_m| \) and the fixed part \( F_m \), i.e., \( \mathfrak{b}(V_m)O_{X_m} = O_{X_m}(-F_m) \).

Let \( M'_m := \frac{1}{m}M_m \) and \( F'_m := \frac{1}{m}F_m \). Suppose that \( f_m \) is isomorphic over a neighborhood of \( x \) and \( f_m^{-1}(x) \not\subseteq \text{Supp}(F_m) \). Let \( \pi_m: \tilde{X}_m \rightarrow X_m \) be the blow-up at \( f_m^{-1}(x) \) with the exceptional divisor \( E_m \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
\tilde{X}_m & \xrightarrow{\pi_m} & X_m \\
\downarrow{\tilde{f}_m} & & \downarrow{f_m} \\
\tilde{X} & \xrightarrow{\pi} & X.
\end{array}
\]

Let \( \tilde{Y} \) be an infinitesimal admissible flag over \( x \) on \( \tilde{X} \). We can identify and \( E_m \) with \( E \), so we may regard \( \tilde{Y} \) as an infinitesimal admissible flag over \( f_m^{-1}(x) \) on \( \tilde{X}_m \).
Now, we give alternative proofs of [KL1, Theorem C] and [CHPW, Theorem E] using Theorem 1.2. The proof uses the strategy in [CHPW, Proof of Theorem 7.2].

**Theorem 3.5** (char(\(k\)) = 0). For any point \(x \in X\), we have
\[
\epsilon(||D||; x) = \xi(D; x) \geq \xi(D; x).
\]

**Proof.** By Lemma 3.3, we have \(\tilde{\xi}(D; x) \geq \xi(D; x)\). Thus we only have to show that \(\varepsilon := \varepsilon(||D||; x) = \tilde{\xi}(D; x) =: \tilde{\xi}\). Fix an infinitesimal admissible flag \(\tilde{V}_*\) over \(x\). By Lemma 3.1 (4), we know that \(\tilde{\xi} = \tilde{\xi}_{\tilde{V}_*}(D; x)\). Suppose that \(x \in B_-(D)\). Then \(\varepsilon = 0\). On the other hand, let \(\tilde{V}_*\) be centered at \(x'\). Since \(x' \in B_-(\pi^*D)\), it follows from Lemma 3.2 that \(\tilde{\xi}(D; x) = 0\). Thus we may assume that \(x \notin B_-(D)\). Take an ample divisor \(A\) on \(X\). Then \(x \notin B_+(D + \varepsilon A)\) for any number \(\varepsilon > 0\). Since
\[
\tilde{\Delta}_{\tilde{V}_*}(D) \subseteq \tilde{\Delta}_{\tilde{V}_*}(D + \varepsilon A) \quad \text{for any } \varepsilon > 0 \quad \text{and} \quad \tilde{\Delta}_{\tilde{V}_*}(D) = \bigcap_{\varepsilon > 0} \tilde{\Delta}_{\tilde{V}_*}(D + \varepsilon A),
\]
we obtain \(\tilde{\xi} = \lim_{\varepsilon \to 0} \tilde{\xi}(D + \varepsilon A; x)\). We have \(\varepsilon(||D||; x) = \lim_{\varepsilon \to 0} \varepsilon(||D + \varepsilon A||; x)\) by [ELMNLP2, Theorem 6.2]. By replacing \(D\) by \(D + \varepsilon A\), we may assume that \(x \notin B_+(D)\).

We use the notations in the beginning of this subsection. In our case, \(V_*\) is the complete graded linear series of \(D\). We may assume that \(f_m\) is also a resolution of \(\mathcal{F}(||mD||)\) and it is isomorphic over a neighborhood of \(x\). Take a general member \(G \in ||bD| - (K_X + (n + 1)H)|\) for a sufficiently large integer \(b > 0\), where \(H\) is a sufficiently positive very ample divisor on \(X\). By adding a sufficiently positive ample divisor to \(G\), we may assume that \(G\) is also ample. Let \(f_m^*D = P_m + N_m\) be the divisorial Zariski decomposition. Since \(f_m^{-1}(x) \notin \text{Supp}(N_m)\), it follows from [CHPW, Lemma 3.9] that
\[
\bar{\Delta}_{\tilde{V}_*}(D) = \bar{\Delta}_{\tilde{V}_*}(P_m).
\]
By [Le, Proposition 3.7], for a sufficiently large integer \(m \gg 0\), we have
\[
M_m \leq P_m \leq M_m + \frac{1}{m} f_m^*G.
\]
Since \(f_m^{-1}(x) \notin \text{Supp}(F_m)\), it follows that \(f_m^{-1}(x) \notin \text{SB}(P_m - M_m)\). We can also assume that \(f_m^{-1}(x) \notin \text{SB}(M_m + \frac{1}{m} f_m^*G - P_m)\). Thus we obtain
\[
\bar{\Delta}_{\tilde{V}_*}(M_m) \subseteq \bar{\Delta}_{\tilde{V}_*}(D) \subseteq \bar{\Delta}_{\tilde{V}_*}
\left(M_m + \frac{1}{m} f_m^*G\right),
\]
and hence, we get \(\tilde{\xi}(M_m; f_m^{-1}(x)) \leq \tilde{\xi} \leq \tilde{\xi}(M_m + \frac{1}{m} f_m^*G; f_m^{-1}(x))\). Note that we have
\[
\varepsilon = \lim_{m \to \infty} \varepsilon(M_m; f_m^{-1}(x)) = \lim_{m \to \infty} \varepsilon
\left(M_m + \frac{1}{m} f_m^*G; f_m^{-1}(x)\right).
\]
It then follows from Theorem 1.2 that
\[
\varepsilon = \lim_{m \to \infty} \tilde{\xi}(M_m; f_m^{-1}(x)) = \lim_{m \to \infty} \tilde{\xi}
\left(M_m + \frac{1}{m} f_m^*G; f_m^{-1}(x)\right).
\]
Thus we obtain \(\varepsilon = \tilde{\xi}\), and complete the proof. \(\square\)
Remark 3.6. In [KL1], jet separation of $K_X + D$ ([KL1, Proposition 4.10]) plays a crucial role in the proof of [KL1, Theorem C]. In contrasts, our proof does not depend on [KL1, Proposition 4.10]. In fact, it is a consequence of Theorem 3.5 together with [ELMNP2, Proposition 6.8].

Next, we recover [CHPW, Theorem C] and [KL1, Theorem 4.1].

Theorem 3.7 (char($k) = 0$). The following are equivalent:

1. $x \notin B_+(D)$.
2. The Okounkov body $\Delta_{Y_0}(D)$ contains a nontrivial standard simplex for every admissible flag $Y_0$ on $X$ centered at $x$.
3. The Okounkov body $\Delta_{Y_0}(D)$ contains a nontrivial standard simplex for some admissible flag $Y_0$ on $X$ centered at $x$.
4. The infinitesimal Okounkov body $\tilde{\Delta}_{Y_0}(D)$ contains a nontrivial inverted standard simplex for every infinitesimal admissible flag $\tilde{Y}_0$ over $x$.
5. The infinitesimal Okounkov body $\tilde{\Delta}_{Y_0}(D)$ contains a nontrivial inverted standard simplex for some infinitesimal admissible flag $\tilde{Y}_0$ over $x$.

Proof. (1) $\Rightarrow$ (2): It can be shown by a standard argument (see e.g., [CHPW, Theorem C]), so we omit the proof.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): The proofs are identical to those of Theorem 1.1.

(5) $\Rightarrow$ (1): Assume that (5) holds. By Theorem 3.5, we have $\varepsilon(||D||; x) > 0$. Then $x \notin B_+(D)$ (see [ELMNP2, p. 644]), i.e., (1) holds.

3.4. Graded linear series case. We show an analogue result of Theorem 1.2 and Theorem 3.5 for the graded linear series.

Theorem 3.8 (char($k) = 0$). Assume that $D$ is a $\mathbb{Z}$-divisor and $V_0$ is a birational graded linear series associated to $D$. For a very general point $x \in X$, we have

$$\varepsilon(V_0; x) = \tilde{\xi}(V_0; x) \geq \xi(V_0; x).$$

Proof. By Lemma 3.3, we have $\tilde{\xi}(V_0; x) \geq \xi(V_0; x)$. Thus we only have to prove that $\varepsilon := \varepsilon(V_0; x) = \tilde{\xi}(V_0; x) =: \tilde{\xi}$. We use the notations in the beginning of Subsection 3.3. Since $x$ is a very general point, we may assume that $f_m$ is isomorphic over a neighborhood of $x$ and $f_m^{-1}(x) \not\subseteq \text{Supp}(F_m)$. It then follows that

$$\tilde{\Delta}_{Y_0}(M'_m) \subseteq \tilde{\Delta}_{Y_0}(V_0)$$

for any infinitesimal admissible flag $\tilde{Y}_0$ over $x$. By [I, Lemma 3.10], we have

$$\varepsilon(V_0; x) = \sup_{m>0} \varepsilon(M'_m; x) = \lim_{m \to \infty} \varepsilon(M'_m; x).$$

Thus we obtain $\varepsilon \leq \tilde{\xi}$.

Now, suppose that $\varepsilon < \tilde{\xi}$. By [LM, Theorem D and Theorem 2.13], we can fix a sufficiently large integer $m \gg 0$ such that $\text{vol}_X(V_0) - \text{vol}_{X_m}(M'_m)$ is arbitrarily small.
Take a real number $k$ with $\varepsilon(M'_m; x) < k < \xi$. Consider an irreducible curve $C$ on $X$ passing through $x$, and its strict transform $\overline{C}$ by $\pi$. We identify $f_m^{-1}(x)$ with $x$, so by abuse of notation, we use the same notation for the strict transforms of $C$ by $f_m$ and $\overline{C}$ by $\overline{f}_m$. If $\overline{C} \not\subseteq SB(\pi^*_m M'_m - kE)$, then

$$M'_m C - k \operatorname{mult}_x C = (\pi^*_m M'_m - kE) \overline{C} \geq 0$$

so that we obtain

$$\frac{M'_m C}{\operatorname{mult}_x C} \geq k > \varepsilon(M'_m; x).$$

Thus there is an irreducible curve $C$ on $X_m$ passing through $x$ such that its strict transform $\overline{C}$ by $\pi_m$ is contained in $SB(\pi^*_m M'_m - kE)$. By taking $k \to \varepsilon(M'_m; x)$ and suitably choosing a curve $C$ accordingly, we may assume that

$$\alpha(C) := \inf \{ k \in \mathbb{Q} \mid \overline{C} \subseteq SB(\pi^*_m M'_m - kE) \} = \varepsilon(M'_m; x) \leq \varepsilon.$$

Take a point $x' \in \overline{C} \cap E$, and let $\tilde{Y}_{\bullet}$ be an infinitesimal admissible flag over $x$ centered at $x'$. By [N, Lemma 1.3] (see also [EKL, Proposition 2.3]), for any $\beta > \alpha(C)$, we have

$$\operatorname{ord}_{x'}(||\pi^*_m M'_m - \beta E||) \geq \operatorname{ord}_C(||\pi^*_m M'_m - \beta E||) \geq \beta - \alpha(C).$$

For any point $x = (\nu_1, \ldots, \nu_{n-1}) \in \Delta_{\tilde{Y}_{\bullet}}(M'_m)_{x_1 = \beta} = \Delta_{\tilde{Y}_{\bullet}}(\pi^*_m M'_m - \beta E)_{x_1 = 0}$, by considering Lemma 3.1 (2), we have

$$\nu_1 + \cdots + \nu_{n-1} \geq \beta - \alpha(C).$$

Thus $\Delta_{\tilde{Y}_{\bullet}}(\pi^*_m M'_m - \beta E)_{x_1 = 0} \cap \Delta^{n-1}_{\beta - \alpha(C)}$ in $\mathbb{R}^{n-1}$ has smaller dimension than $n - 1$ for any $\beta > \alpha(C)$. Hence $\Delta_{\tilde{Y}_{\bullet}}(\pi^*_m M'_m - \alpha(C)E) \cap \Delta^n_{\xi - \alpha(C)}$ in $\mathbb{R}^n$ has smaller dimension than $n$, i.e., $\Delta_{\tilde{Y}_{\bullet}}(M'_m)_{x_1 \geq \alpha(C)} \cap \left((\alpha(C), 0, \ldots, 0) + \Delta^n_{\xi - \alpha(C)} \right)$ in $\mathbb{R}^n$ has smaller dimension than $n$. Recall that $\Delta_{\tilde{Y}_{\bullet}}(V_{\bullet})_{x_1 \geq \alpha(C)}$ contains $((\alpha(C), 0, \ldots, 0) + \Delta^n_{\xi - \alpha(C)})$ and $\varepsilon \geq \alpha(C)$. By applying [LM, Theorem 2.13], we see that

$$\operatorname{vol}(V_{\bullet}) \geq \operatorname{vol}_{x_m}(M'_m) + (\xi - \alpha(C))^n \geq \operatorname{vol}_{x_m}(M'_m) + (\xi - \varepsilon)^n.$$}

Thus $\operatorname{vol}(V_{\bullet}) - \operatorname{vol}_{x_m}(M'_m) \geq (\xi - \varepsilon)^n$, which is contradiction to that $\operatorname{vol}(V_{\bullet}) - \operatorname{vol}_{x_m}(M'_m)$ is arbitrarily small. Therefore, $\varepsilon = \xi$.

**Remark 3.9.** One can easily check that if $V_{\bullet}$ is finitely generated, then Theorem 3.8 holds for every point $x \in X$. In general, it does not hold when the point $x$ is not general. For example, we fix a point $x \in \mathbb{P}^2$, and consider a graded linear series $V_{\bullet}$ associated to $\mathcal{O}_{\mathbb{P}^2}(1)$ given by

$$V_m := \{ s \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \mid \operatorname{ord}_x(s) \geq 1 \}$$

for any $m \geq 1$. Evidently, $V_{\bullet}$ is birational. For any infinitesimal admissible flag $\tilde{Y}_{\bullet}$ over $x$ or any admissible flag $Y_{\bullet}$ centered at $x$, we have

$$\Delta_{\tilde{Y}_{\bullet}}(V_{\bullet}) = \Delta_{\tilde{Y}_{\bullet}}(\mathcal{O}_{\mathbb{P}^2}(1)) \quad \text{and} \quad \Delta_{Y_{\bullet}}(V_{\bullet}) = \Delta_{Y_{\bullet}}(\mathcal{O}_{\mathbb{P}^2}(1)),$$

so we obtain $\xi(V_{\bullet}; x) = \xi(V_{\bullet}; x) = 1$. However, we have $\varepsilon(V_{\bullet}; x) = 0$. 


4. INTEGRATED VOLUME FUNCTIONS

This section is devoted to the study of integrated volume functions. In particular, we prove Theorem 1.3. We continue to use the notations in Section 2. The base field \( \mathbb{k} \) may have arbitrary characteristic. Assume that a graded linear series \( V_* \) associated to a big \( \mathbb{R} \)-divisor \( D \) is birational, and consider the multiplicative filtration \( \mathcal{F}_x \) induced by \( \text{ord}_x \) in Example 2.1. Fix an admissible flag \( Y_* \) on \( X \) centered at \( x \). We defined the filtered Okounkov body \( \hat{\Delta}_{Y_*}(V_*, \mathcal{F}_x) \subseteq \mathbb{R}^n_{\geq 0} \). Note that \( \hat{\Delta}_{Y_*}(V_*, \mathcal{F}_x)_{x_{n+1}=t} = \Delta_{Y_*}(V_*^{(t)}) \).

**Definition 4.1.** The integrated volume function of \((V_*, \mathcal{F}_x)\) at \( x \) is defined by

\[
\hat{\varphi}_x(V_*, \mathcal{F}_x, t) := \int_{u=0}^{t} \text{vol}_{\mathbb{R}^n}(\hat{\Delta}_{Y_*}(V_*, \mathcal{F}_x)_{x_{n+1}=u})du.
\]

**Remark 4.2.** By fixing \( x \) and \((V_*, \mathcal{F}_x)\), we can easily check that the integrated volume function \( \hat{\varphi}_x(V_*, \mathcal{F}_x, -) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is nondecreasing continuous. When \( V_* \) is a complete graded linear series of \( D \), we simply let \( \hat{\varphi}_x(D, t) := \hat{\varphi}_x(V_*, \mathcal{F}_x, t) \). Clearly, if \( D \equiv D' \), then \( \hat{\varphi}_x(D, t) = \hat{\varphi}_x(D', t) \). We then have a function

\[
\hat{\varphi}_x : \text{Big}(X) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad (D, t) \mapsto \int_{u=0}^{t} \text{vol}_{\mathbb{R}^n}(\hat{\Delta}_{Y_*}(V_*, \mathcal{F}_x)_{x_{n+1}=u})du,
\]

which is continuous on the whole domain.

**Example 4.3.** Let \( X = \mathbb{P}^1 \times \mathbb{P}^1 \), and \( V_* \) be the complete graded linear series associated to \( D = \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \). Note that \( \mu(V_*; x) = 2 \). For any admissible flag \( Y_* \) on \( X \) centered at \( x \), we have

\[
\text{vol}_{\mathbb{R}^2}(\hat{\Delta}_{Y_*}(V_*, \mathcal{F}_x)_{x_{n+1}=t}) = \text{vol}_{\mathbb{R}^2}(\Delta_{Y_*}(V_*^{(t)})) = \left\{ \begin{array}{ll} -\frac{1}{2}t^2 + 1 & \text{if } 0 \leq t \leq 1, \\ \frac{1}{2}t^2 - 2t + 2 & \text{if } 1 \leq t \leq 2. \end{array} \right.
\]

It then follows that

\[
\hat{\varphi}_x(V_*, \mathcal{F}_x, t) = \left\{ \begin{array}{ll} -\frac{1}{6}t^3 + t & \text{if } 0 \leq t \leq 1, \\ \frac{1}{6}t^3 - t^2 + 2t - \frac{1}{3} & \text{if } 1 \leq t \leq 2. \end{array} \right.
\]

See [MR, Example in Section 4] for the case that \( D = \mathcal{O}_{\mathbb{P}^1}(d_1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d_2) \) with \( d_1, d_2 \geq 1 \).

We are now in a position to extract several important invariants of graded linear series from the integrated volume function.

**Proposition 4.4.** We have the following:

1. \( \hat{\varphi}_x(V_*, \mathcal{F}_x, t) = \hat{\varphi}_x(V_*, \mathcal{F}_x, \mu(V_*; x)) = \text{vol}_{\mathbb{R}^{n+1}}(\hat{\Delta}_{Y_*}(V_*, \mathcal{F}_x)) \) for any \( t \geq \mu(V_*; x) \).
2. \( \hat{\varphi}'_x(V_*, \mathcal{F}_x, t) = \frac{d\hat{\varphi}_x(V_*, \mathcal{F}_x, t)}{dt} = \text{vol}_{\mathbb{R}^n}(\Delta_{Y_*}(V_*^{(t)})) \) for all \( t \geq 0 \).
3. \( \text{vol}_X(V_*^{(t)}) = n! \cdot \hat{\varphi}_x(V_*, \mathcal{F}_x, t) \) for all \( t \geq 0 \).
4. \( \hat{\varphi}'_x(V_*, \mathcal{F}_x, 0) - \hat{\varphi}'_x(V_*, \mathcal{F}_x, t) \leq \frac{n}{n!} \) for all \( t \geq 0 \).
5. \( \mu(V_*; x) = \text{inf}\{ t \geq 0 \mid \hat{\varphi}_x(V_*, \mathcal{F}_x, t) = 0 \} \).
6. \( \xi(V_*; x) = \text{inf}\{ t \geq 0 \mid \hat{\varphi}_x(V_*, \mathcal{F}_x, 0) - \hat{\varphi}_x(V_*, \mathcal{F}_x, t) < \frac{n}{n!} \} \).
Proof. (1) and (2) are clear by the definition. Then (3) follows from (2) and [LM, Theorem 2.13]. Now, fix an infinitesimal admissible flag \( \bar{\mathcal{Y}} \) over \( x \). Then we have
\[
\overline{\phi}_x(V_\bullet, F_x, t) = \text{vol}_{\mathbb{R}^n}(\Delta_{V_\bullet}^{(0)}(V^*_\bullet)) = \text{vol}_{\mathbb{R}^n}(\tilde{\Delta}_{V_\bullet}^{(0)}(V^*_\bullet)_{x_{1}\geq t}).
\]
By Lemma 3.1 (3) and [LM, Theorem 2.13], we obtain (4) (see [MR, Lemma 4.1] for an alternative proof of (4) when \( V_\bullet \) is a complete graded linear series). Observe that \( \mu(V_\bullet, x) = \sup\{\nu_n \mid x = (\nu_1, \ldots, \nu_n) \in \tilde{\Delta}_{V_\bullet}^{(0)}(V^*_\bullet)\} \). This implies (5). Note that
\[
\overline{\phi}_x^e(V_\bullet, F_x, 0) - \overline{\phi}_x^e(V_\bullet, F_x, t) = \text{vol}_{\mathbb{R}^n}(\tilde{\Delta}_{V_\bullet}^{(0)}(V^*_\bullet)_{0 \leq x_{1} \leq t}).
\]
Then (6) follows from Lemma 3.1 (3) and (4). \( \square \)

Remark 4.5 \((\text{char}(k) = 0)\). Theorem 3.5 and Proposition 4.4 (5) show a new characterization of moving Seshadri constant of a big \( \mathbb{R} \)-divisor \( D \) at any point \( x \in X \) as
\[
\varepsilon(||D||; x) = \inf \left\{ t \geq 0 \left| \overline{\phi}_x^e(D, 0) - \overline{\phi}_x^e(D, t) < \frac{t^n}{n!} \right. \right\}.
\]
Let \( V_\bullet \) be a birational graded linear series associated to a \( \mathbb{Z} \)-divisor \( D \). Similarly, for a very general point \( x \in X \), Theorem 3.8 and Proposition 4.4 (5) show that
\[
\varepsilon(V_\bullet; x) = \inf \left\{ t \geq 0 \left| \overline{\phi}_x(V_\bullet, F_x, 0) - \overline{\phi}_x(V_\bullet, F_x, t) < \frac{t^n}{n!} \right. \right\}.
\]
Proof of Theorem 1.3. It follows from Theorem 1.2 and Proposition 4.4 (5). \( \square \)

Recall from [BC, Definition 1.2] that the jumping numbers of \((V_m, F_x)\) are defined as
\[
e_{j}(V_m) = e_{j}(V_m, F_x) := \sup\{t \in \mathbb{R} \mid \dim F_x^{t} V_m \geq j\} \quad \text{for} \quad j = 1, \ldots, \dim V_m = v_m.
\]
We have \( 0 \leq e_{v_m}(V_m) \leq \cdots \leq e_{1}(V_m) \). The positive mass of \((V_m, F_x)\) is defined as
\[
\text{mass}_+(V_m) = \text{mass}_+(V_m, F_x) := \sum_{e_j(V_m) > 0} e_j(V_m) = \sum_{1 \leq j \leq v_m} e_j(V_m).
\]

Definition 4.6. (1) Let \( S(V_m) = S(V_m, F_x) := \{e_{v_m}(V_m), \ldots, e_{1}(V_m)\} \), and \( N(V_m) = N(V_m, F_x) := |S(V_m)| \).
(2) We define the effective jumping numbers of \((V_m, F_x)\) as
\[
\alpha_j(V_m) = \alpha_j(V_m, F_x) := \text{the } j\text{-th largest element in } S(V_m) \text{ for } j = 1, \ldots, N(V_m).
\]
For convention, we put \( \alpha_{N(V_m)+1} := 0 \). Then we have
\[
0 = \alpha_{N(V_m)+1}(V_m) \leq \alpha_{N(V_m)}(V_m) < \alpha_{N(V_m)-1}(V_m) < \cdots < \alpha_{1}(V_m).
\]
(3) Let \( \beta_j(V_m) = \beta_j(V_m, F_x) := \max\{\ell \in [1, v_m] \mid e_{\ell}(V_m) = \alpha_{j}(V_m)\} \) for \( j = 1, \ldots, N(V_m) \), and \( j_t(V_m) = j_t(V_m, F_x) := \inf\{j \in [1, N(V_m) + 1] \mid \alpha_j(V_m) \leq t\} \) for \( t \geq 0 \).
(4) For \( t \geq 0 \), the bounded mass function of \((V_m, F_x)\) is defined as
\[
\text{mass}_+(V_m, t) = \text{mass}_+(V_m, F_x, t) := \beta_{j_{t}-1}(t - \alpha_{j_{t}}) + \sum_{j = j_{t}}^{N(V_m)} \beta_{j}(\alpha_{j} - \alpha_{j+1}),
\]
where \( \alpha_j = \alpha_j(V_m) \), \( \beta_j = \beta_j(V_m) \), and \( j_t = j_t(V_m) \). When \( j_t(V_m) = N(V_m) + 1 \), we put 
\[
\sum_{j=j_t(V_m)}^{N(V_m)} \beta_j(V_m)(\alpha_j(V_m) - \alpha_{j+1}(V_m)) := 0.
\]

**Remark 4.7.** One can check that
\[
\text{mass}_+(V_m) = \sum_{e_j(V_m) > 0} e_j(V_m) = \sum_{j=1}^{N(V_m)} \beta_j(V_m)(\alpha_j(V_m) - \alpha_{j+1}(V_m)) = \text{mass}_+(V_m, \infty).
\]
However, \( \text{mass}_+(V_m, t) \neq \sum_{0 < e_j(V_m) < t} e_j(V_m) \) in general.

We now show that the integrated volume function can be expressed in terms of the bounded mass functions. In particular, we see that the integrated volume function \( \hat{\varphi}_x(V_\bullet, \mathcal{F}_x, t) \) only depends on the multiplicative filtration \( \mathcal{F}_x \) on \( V_\bullet \).

**Theorem 4.8.** For all \( t \geq 0 \), we have
\[
\hat{\varphi}_x(V_\bullet, \mathcal{F}_x, t) = \lim_{m \to \infty} \frac{\text{mass}_+(V_m, \mathcal{F}_x, mt)}{m^{n+1}}.
\]

**Proof.** For any \( t \geq 0 \), we have
\[
\hat{\varphi}_x(V_\bullet, \mathcal{F}_x, t) = \int_0^t \text{vol}_{\mathbb{R}^n} (\Delta V_\bullet(V^{(u)}) \text{du}) = \int_0^t \lim_{m \to \infty} \frac{\dim \mathcal{F}_{x}^{mu} V_m}{m^n} \text{du}.
\]
By [LM, Theorem 2.13], a sequence of functions \( \{f_m(u) := \frac{\dim \mathcal{F}_{x}^{mu} V_m}{m^n}\} \) converges pointwise to a function \( f(u) := \text{vol}_{\mathbb{R}^n} (\Delta V_\bullet(V^{(u)}) \) as \( m \to \infty \). We have \( |f_m(u)| \leq \frac{\dim V_\bullet}{m^n} \).

Since \( \limsup_{m \to \infty} \frac{\dim V_\bullet}{m^n} = \lim_{m \to \infty} \frac{\dim V_\bullet}{m^n} = \frac{\text{vol}(V_\bullet)}{n!} \) by [LM, Theorem 2.13 and Remark 2.14], there exists a constant \( C > 0 \) such that \( \frac{\dim V_\bullet}{m^n} \leq C \text{ for all } m \gg 0 \). It is enough to consider only a sufficiently large \( m \gg 0 \) for the claim, so we may assume that \( |f_m(u)| \leq C \) for all \( m \). The integration is taken over an area \((0, t)\) with a finite measure, and hence, we see that \( |f_m(u)| \) is bounded by an integrable function. Now, by applying Lebesgue dominated convergence theorem and change of variable \( v = mu \), we obtain
\[
\int_0^t \lim_{m \to \infty} \frac{\dim \mathcal{F}_{x}^{mu} V_m}{m^n} \text{du} = \lim_{m \to \infty} \int_0^t \frac{\dim \mathcal{F}_{x}^{mu} V_m}{m^n} \text{du} = \lim_{m \to \infty} \frac{1}{m^{n+1}} \int_0^{mt} \dim \mathcal{F}_{x}^{v} V_m \text{dv}.
\]
By regarding \( \dim \mathcal{F}_{x}^{v} V_m \) as a function of \( v \), we can write
\[
\dim \mathcal{F}_{x}^{v} V_m = \begin{cases} 
\beta_{N(V_m)}(V_m) & \text{if } v = 0, \\
\beta_j(V_m) & \text{if } v \in (\alpha_{j+1}(V_m), \alpha_j(V_m)] \text{ for } j = N(V_m), \ldots, 1, \\
0 & \text{if } v \in (\alpha_1(V_m), \infty).
\end{cases}
\]
Now, it is immediate to see that
\[
\int_0^{mt} \dim \mathcal{F}_{x}^{v} V_m \text{dv} = \beta_{j_{mt}-1}(mt - \alpha_{j_{mt}}) + \sum_{j=j_{mt}}^{N(V_m)} \beta_j(\alpha_j - \alpha_{j+1}) = \text{mass}_+(V_m, \mathcal{F}_x, mt),
\]
where \( \alpha_j = \alpha_j(V_m) \), \( \beta_j = \beta_j(V_m) \), and \( j_{mt} = j_{mt}(V_m) \), which gives the desired result. \( \square \)

As a consequence of Theorem 4.8, we recover [BC, Corollary 1.13] in our situation.
Corollary 4.9. We have
\[ \text{vol}_{\mathbb{P}^{n+1}} \left( \hat{\Delta}_{Y*}(V_*, F_x) \right) = \varphi_x(V_*, F_x, \infty) = \lim_{m \to \infty} \frac{\text{mass}_+(V_m, F_x)}{m^{n+1}}. \]

Example 4.10. Let \( X = \mathbb{P}^2 \) be any point, and \( V_* \) be the complete graded linear series associated to \( O_{\mathbb{P}^2}(1) \). Note that \( \mu(V_*; x) = 1 \). Hence, \( \text{dim} \mathcal{F}_x V_m = \frac{(m+2)(m+1)}{2} - \frac{s(s+1)}{2} \). The jumping numbers \( e_j = e_j(V_m, F_x) \) are given by \( e_j = \frac{2}{m+1}, \ldots, e_j = \frac{2}{m+1} \frac{k(k+1)}{2} = k \) for all \( k \geq 0 \) with \( \frac{(m+2)(m+1)}{2} - \frac{k(k+3)}{2} \geq 1 \), so the effective jumping numbers are given by \( \alpha_j(V_m, F_x) = m + 1 - j \) for \( j = 1, \ldots, m + 1 = N(V_m, F_x) \). We then obtain \( \beta_j(V_m, F_x) = \frac{(m+2)(m+1)}{2} - \frac{(m+1-j)(m+2-j)}{2} \) and \( j_t(V_m, F_x) = m + 1 - \lfloor t \rfloor \). For \( 0 \leq t \leq 1 \), the bounded mass function is given by
\[ \text{mass}_+(V_m, F_x, mt) = \left( \frac{(m+1)(m+2)}{2} - \frac{\lfloor mt \rfloor + 1}{2} \right) (mt - \lfloor mt \rfloor) \]
\[ + \sum_{k=1}^{\lfloor mt \rfloor} \left( \frac{(m+1)(m+2)}{2} - \frac{k(k+1)}{2} \right). \]

By Theorem 4.8, we obtain
\[ \varphi_x(V_*, F_x, t) = \lim_{m \to \infty} \frac{\text{mass}_+(V_m, F_x, mt)}{m^3} = -\frac{1}{6} t^3 + \frac{1}{2} t \text{ for } 0 \leq t \leq 1. \]

References

[BJ] H. Blum and M. Jonsson, Thresholds, valuations, and K-stability, preprint: arXiv:1706.04548v1.

[BC] S. Boucksom and H. Chen, Okounkov bodies of filtered linear series, Compos. Math. 147 (2011), 1205–1229.

[CHPW] S. Choi, Y. Hyun, J. Park, and J. Won, Asymptotic base loci via Okounkov bodies, Adv. Math. 323 (2018), 784–810.

[CPW1] S. Choi, J. Park, and J. Won, Okounkov bodies associated to pseudoeffective divisors II, Taiwanese J. Math. 21 (2017), 601–620.

[CPW2] S. Choi, J. Park, and J. Won, Okounkov bodies associated to abundant divisors and Iitaka fibrations, preprint: arXiv:1711.07352v1.

[CPW3] S. Choi, J. Park, and J. Won, Local numerical equivalences and Okounkov bodies in higher dimensions, preprint: arXiv:1808.02226v1.

[D] J.-P. Demailly, Singular Hermitian metrics on positive line bundles in Complex Algebraic Varieties (Bayreuth, 1990), Lect. Notes Math. 1507 (1992), Springer-Verlag, 87–104.

[DKMS] M. Dumnicki, A. Küronya, C. Maclean, and T. Szemberg, Seshadri constants via functions on Newton-Okounkov bodies, Math. Nachr. 289 (2016), 2173–2177.

[EKL] L. Ein, O. Küchle, and R. Lazarsfeld, Local positivity of ample line bundles, J. Differential Geom. 42 (1995), 193–219.

[ELMNP1] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier (Grenoble) 56 (2006), 1701–1734.

[ELMNP2] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa, Restricted volumes and base loci of linear series, Amer. J. Math. 131 (2009), 607–651.

[F] K. Fujita, A valuative criterion for uniform K-stability of Q-Fano varieties, to appear in J. Reine Angew. Math.
M. Fulger, J. Kollár, and B. Lehmann, *Volume and Hilbert function of $\mathbb{R}$-divisors*, Michigan Math. J. **65** (2016), 371–387.

A. Ito, *Okounkov bodies and Seshadri constants*, Adv. Math. **241** (2013), 246–262.

K. Kaveh and A. G. Khovanskii, *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), 925–978.

A. Küronya and V. Lozovanu, *Infinitesimal Newton-Okounkov bodies and jet separation*, Duke Math. J. **166** (2017), 1349–1376.

A. Küronya and V. Lozovanu, *Positivity of line bundles and Newton-Okounkov bodies*, Doc. Math. **22** (2017), 1285–1302.

A. Küronya and V. Lozovanu, *Local positivity of linear series on surfaces*, Algebra Number Theory **12** (2018), 1–34.

A. Küronya, C. Maclean, and T. Szemberg, *Functions on Okounkov bodies coming from geometric valuations*, preprint, arXiv:1210.3523v2.

R. Lazarsfeld, *Positivity in Algebraic Geometry I and II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, A Series of Modern Surveys in Mathematics **48** and **49** (2004), Springer-Verlag, Berlin.

R. Lazarsfeld and M. Mustață, *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 783–835.

B. Lehmann, *Comparing numerical dimensions*, Algebra Number Theory **7** (2013), 1065–1100.

C. Li, *K-semistability is equivariant volume minimization*, Duke Math. J. **166**, (2017), 3147–3218.

D. McKinnon and M. Roth, *Seshadri constants, diophantine approximation, and Roth’s theorem for arbitrary varieties*, Invent. Math. **200** (2015), 513–583.

M. Mustață, *The non-nef locus in positive characteristic*, A celebration of algebraic geometry, 535–551, Clay Math. Proc. **18** (2013), Amer. Math. Soc., Providence, RI.

M. Nakamaye, *Seshadri constants at very general points*, Trans. Amer. Math. Soc. **357** (2004), 3285–3297.

A. Okounkov, *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996) 405–411.

A. Okounkov, *Why would multiplicities be log-concave?* in *The Orbit Method in Geometry and Physics*, Progr. Math. **213** (2003), Birkhäuser Boston, Boston, MA, 329–347.

J. Roé, *Local positivity in terms of Newton-Okounkov bodies*, Adv. Math. **301** (2016), 486–498.