Abstract: Based on the solution of Paulsen Problem by Kwok, Lau, Lee, and Ramachandran [STOC’18-Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, 2018] and independently by Hamilton, and Moitra [Isr. J. Math., 2021] we study Paulsen Problem and Projection Problem in the context of Hilbert C*-modules. We show that for commutative C*-algebras, if Modular Paulsen Problem has a solution, then Modular Projection Problem also has a solution. We formulate the problem of operator scaling for matrices over C*-algebras.

Keywords: Frame, C*-algebra, Hilbert C*-module, invariant basis number, Paulsen Problem, Projection Problem, Operator Scaling Problem.

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1. Introduction

We begin from the definition of frames and various special sub classes of it for finite dimensional Hilbert spaces which originated from the work of Duffin and Schaeffer [34, 40].

Definition 1.1. [34, 51] A collection \( \{\tau_j\}_{j=1}^n \) in a finite dimensional Hilbert space \( H \) over \( \mathbb{K} (\mathbb{R} \text{ or } \mathbb{C}) \) is said to be a frame for \( H \) if there exist \( a, b > 0 \) such that

\[
a \|h\|^2 \leq \sum_{j=1}^n |\langle h, \tau_j \rangle|^2 \leq b \|h\|^2, \quad \forall h \in H.
\]

Definition 1.2. [34, 51] A frame \( \{\tau_j\}_{j=1}^n \) for a finite dimensional Hilbert space \( H \) is said to be a

(i) Parseval frame if

\[
h = \sum_{j=1}^n \langle h, \tau_j \rangle \tau_j, \quad \forall h \in H.
\]

(ii) tight frame if there exists \( a > 0 \) such that

\[
a \|h\|^2 = \sum_{j=1}^n |\langle h, \tau_j \rangle|^2, \quad \forall h \in H.
\]

(iii) unit norm frame if \( \|\tau_j\| = 1 \), for all \( 1 \leq j \leq n \).

From a frame \( \{\tau_j\}_{j=1}^n \) for \( H \) we get the following three operators. We use \( H \) to denote a finite dimensional Hilbert space. We denote the identity operator on \( H \) by \( I_H \).

(i) Analysis operator \( \theta_\tau : H \ni h \mapsto \theta_\tau h := (\langle h, \tau_j \rangle)_{j=1}^n \in \mathbb{K}^n \).

(ii) Synthesis operator \( \theta_\tau^* : \mathbb{K}^n \ni (a_j)_{j=1}^n \mapsto \theta_\tau^* (a_j)_{j=1}^n := \sum_{j=1}^n a_j \tau_j \in H \).

(iii) Frame operator \( S_\tau : H \ni h \mapsto S_\tau h := \sum_{j=1}^n \langle h, \tau_j \rangle \tau_j \in H \).
It then follows that $S_r = \theta_r^* \theta_r$ and the frame operator is positive and invertible. Therefore $\{S_r^{-1/2} \tau_j\}_{j=1}^n$ is a Parseval frame for $\mathcal{H}$. It is known that in this way the construction of Parseval frame is difficult due to inversion and square root process of frame operator. On the other side, given a collection, it is easy (even not always) to verify that whether it is a frame than to verify that it is a Parseval frame. This leads to the following notions.

**Definition 1.3.** [20] A Parseval frame $\{\tau_j\}_{j=1}^n$ for a d-dimensional Hilbert space $\mathcal{H}$ is called an **equal norm Parseval frame** if

$$\|\tau_j\|^2 = \frac{d}{n}, \quad \forall 1 \leq j \leq n.$$ 

**Definition 1.4.** [20] A frame $\{\tau_j\}_{j=1}^n$ for a d-dimensional Hilbert space $\mathcal{H}$ is called an **$\varepsilon$-nearly Parseval frame** ($\varepsilon < 1$) if

$$(1 - \varepsilon)I_\mathcal{H} \leq S_r \leq (1 + \varepsilon)I_\mathcal{H}.$$ 

**Definition 1.5.** [20] A frame $\{\tau_j\}_{j=1}^n$ for a d-dimensional Hilbert space $\mathcal{H}$ is called an **$\varepsilon$-nearly equal norm frame** ($\varepsilon < 1$) if

$$(1 - \varepsilon)\frac{d}{n} \leq \|\tau_j\|^2 \leq (1 + \varepsilon)\frac{d}{n}, \quad \forall 1 \leq j \leq n.$$ 

**Definition 1.6.** [20] A frame for a d-dimensional Hilbert space $\mathcal{H}$ which is both $\varepsilon$-nearly equal norm and $\varepsilon$-nearly Parseval is called as **$\varepsilon$-nearly equal norm Parseval frame**.

Following notion is often used to measure the distance between two collections in Hilbert space.

**Definition 1.7.** [20] **Distance** between two collections $\{\tau_j\}_{j=1}^n$ and $\{\omega_j\}_{j=1}^n$ in a Hilbert space $\mathcal{H}$ is defined as

$$\text{dist}(\{\tau_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n) := \left( \sum_{j=1}^n \|\tau_j - \omega_j\|^2 \right)^{\frac{1}{2}}.$$ 

Fundamental Paulsen problem now arises from the work of Holmes and Paulsen [53]. It is grounded on following three results.

**Theorem 1.8.** [53] There is an algorithm for turning every frame into an equal norm frame with same frame operator.

**Theorem 1.9.** [23] Let $\{\tau_j\}_{j=1}^n$ be an $\varepsilon$-nearly equal norm frame for a d-dimensional Hilbert space $\mathcal{H}$. Then the closest equal norm frame to $\{\tau_j\}_{j=1}^n$ is given by $\{\omega_j\}_{j=1}^n$, where

$$\omega_j := \left( \frac{\sum_{k=1}^n \|\tau_k\|}{n} \right) \frac{\tau_j}{\|\tau_j\|}, \quad \forall 1 \leq j \leq n.$$ 

**Theorem 1.10.** [10,27] Let $\{\tau_j\}_{j=1}^n$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$. Then $\{S_r^{-1/2} \tau_j\}_{j=1}^n$ is the closest Parseval frame to $\{\tau_j\}_{j=1}^n$, i.e.,

$$\sum_{j=1}^n \left\| S_r^{-1/2} \tau_j - \tau_j \right\|^2 = \inf \left\{ \sum_{j=1}^n \|\tau_j - \omega_j\|^2 : \{\omega_j\}_{j=1}^n \text{ is a Parseval frame for } \mathcal{H} \right\}.$$
Further, \( \{S^{-1/2}_\tau \tau_j\}_{j=1}^n \) is the unique minimizer. Moreover, if \( \{\tau_j\}_{j=1}^n \) is any \( \varepsilon \)-nearly Parseval frame for \( \mathcal{H} \), then
\[
\sum_{j=1}^n \left\| S^{-1/2}_\tau \tau_j - \tau_j \right\|^2 \leq d(2 - \varepsilon - 2\sqrt{1 - \varepsilon}) \leq \frac{d\varepsilon^2}{4}.
\]

**Problem 1.11.** [20–22] (Paulsen problem) Find the function \( f : (0, 1) \times \mathbb{N} \times \mathbb{N} \rightarrow [0, \infty) \) so that for any \( \varepsilon \)-nearly equal norm Parseval frame \( \{\tau_j\}_{j=1}^n \) for \( d \)-dimensional Hilbert space \( \mathcal{H} \), there is an equal norm Parseval frame \( \{\omega_j\}_{j=1}^n \) for \( \mathcal{H} \) satisfying
\[
\text{dist}^2(\{\tau_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n) = \sum_{j=1}^n \|\tau_j - \omega_j\|^2 \leq f(\varepsilon, n, d).
\]
Moreover, whether \( f \) depends on \( n \)?

It is clear that Problem 1.11 can also be stated as follows.

**Problem 1.12.** [20] (Paulsen problem) Find the function \( f : (0, 1) \times \mathbb{N} \times \mathbb{N} \rightarrow [0, \infty) \) so that for any \( \varepsilon \)-nearly equal norm Parseval frame \( \{\tau_j\}_{j=1}^n \) for \( d \)-dimensional Hilbert space \( \mathcal{H} \),
\[
\inf\left\{\text{dist}^2(\{\tau_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n) : \{\omega_j\}_{j=1}^n \text{ is an equal norm Parseval frame for } \mathcal{H}\right\} \leq f(\varepsilon, n, d).
\]
Moreover, whether \( f \) depends on \( n \)?

The function \( f \) in Definition 1.11 and in Definition 1.12 is known as Paulsen function. First question is whether such a function exists. Using compactness of the unit sphere in finite dimensional Hilbert space, Hadwin proved the following.

**Theorem 1.13.** [10] Solution \( f \) to Paulsen problem exists.

Casazza [24] observed that Paulsen function is bounded below and it is independent of the number of elements in the frame.

**Proposition 1.14.** [24] Paulsen function \( f \) satisfies
\[
f(\varepsilon, n, d) \geq \varepsilon^2 d, \quad \forall \varepsilon > 0, \forall d \in \mathbb{N}.
\]

Using the system of ordinary differential equations and frame energy, Bodmann and Casazza were able to obtain the first striking result for Paulsen problem in 2010 [10].

**Theorem 1.15.** [10] Let \( n \) and \( d \) be relatively prime and let \( \varepsilon < 1/2 \). If \( \{\tau_j\}_{j=1}^n \) is any \( \varepsilon \)-nearly equal norm frame for a \( d \)-dimensional Hilbert space \( \mathcal{H} \), then there is an equal norm Parseval frame \( \{\omega_j\}_{j=1}^n \) for \( \mathcal{H} \) such that
\[
\text{dist}^2(\{\tau_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n) \leq \frac{29d}{8}d^2n(n-1)^8\varepsilon.
\]

Casazza, Fickus, and Mixon [25] gave the following version of the Paulsen problem in 2012.

**Problem 1.16.** [25] (Paulsen problem) Given \( n, d \in \mathbb{N} \), find positive \( \delta(n, d), c(n, d) \) and \( \alpha(n, d) \) such that given any unit norm frame \( \{\tau_j\}_{j=1}^n \) for \( d \)-dimensional Hilbert space \( \mathcal{H} \) satisfying
\[
\left\| S - \frac{n}{d} I_\mathcal{H} \right\|_{HS} \leq \delta(n, d),
\]
there exists a unit norm tight frame $\{\omega_j\}_{j=1}^n$ for $H$ such that
\[ \|\theta^*_w - \theta^*_r\|_{\text{HS}} \leq c(n, d) \left\| S_r - \frac{n}{d} I_H \right\|_{\text{HS}}^{\alpha(n, d)}, \]
where $\text{HS}$ denotes the Hilbert-Schmidt norm of the operator.

In the same paper [25] Casazza, Fickus, and Mixon proved the following particular cases of Problem[1,16]

**Theorem 1.17.** [25] Let $n$ and $d$ be relatively prime and let $0 < t < 1/2n$. Let $\{\tau_j^{(0)}\}_{j=1}^n$ be a unit norm tight frame for a $d$-dimensional Hilbert space $H$ such that
\[ \left\| S_r^{(0)} - \frac{n}{d} I_H \right\|_{\text{HS}}^2 \leq \frac{2}{d^3}. \]
Define $\{\omega_j^{(k)}\}_{j=1}^n$ as follows.
\[ \omega_j^{(k)} := S_r^{(k)} \tau_j^{(k)} - \langle S_r^{(k)} \tau_j^{(k)}, \tau_j^{(k)} \rangle \tau_j^{(k)}, \quad \forall 1 \leq j \leq n, \forall k \geq 0. \]
Now define $\{\tau_j^{(k)}\}_{j=1}^n$ as follows.
\[ \tau_j^{(k+1)} := \begin{cases} \cos(\langle \omega_j^{(k)}, t\rangle \tau_j^{(k)} - \sin(\langle \omega_j^{(k)}, t\rangle \tau_j^{(k)} \rangle \frac{\omega_j^{(k)}}{\| \omega_j^{(k)} \|}) & \text{if } \omega_j^{(k)} \neq 0, \\ \tau_j^{(k)} & \text{if } \omega_j^{(k)} = 0. \end{cases} \]
Then the limit of $\{\tau_j^{(k)}\}_{j=1}^n$ as $k \to \infty$ exists, denoted by $\{\tau_j^{(\infty)}\}_{j=1}^n$ is a unit norm tight frame for $H$ and satisfies
\[ \left\| S_r^{(\infty)} - S_r^{(0)} \right\|_{\text{HS}} \leq \frac{4d^{2n+5.5}}{1 - 2nt} \left\| S_r^{(0)} - \frac{n}{d} I_H \right\|_{\text{HS}}. \]

**Theorem 1.18.** [25] Let $\varepsilon \leq 1/2n$. If $\{\tau_j\}_{j=1}^n$ is any $\varepsilon$-orthogonally partitionable unit norm frame for a $d$-dimensional Hilbert space $H$, then there is an orthogonally partitionable unit norm frame $\{\omega_j\}_{j=1}^n$ for $H$ such that
\[ \|\theta^*_w - \theta^*_r\|_{\text{HS}} \leq \sqrt{2n(\varepsilon d)^{\frac{3}{2}}}. \]

**Theorem 1.19.** [25] Let $n$ and $d$ be not relatively prime. If $\{\tau_j\}_{j=1}^n$ is any unit norm frame for a $d$-dimensional Hilbert space $H$, then there is a unit norm frame $\{\omega_j\}_{j=1}^n$ for $H$ which is either tight or is orthogonally partitionable with equal redundancies in each of the two partitioned subsets such that
\[ \|\theta^*_w - \theta^*_r\|_{\text{HS}} \leq 3d^{\frac{3}{2}} \sqrt{n} \left\| S_r - \frac{n}{d} I_H \right\|_{\text{HS}}^\frac{1}{2}. \]

Projection problem is another prime problem in finite dimensional Hilbert space theory.

**Problem 1.20.** [20] (Projection problem) Let $H$ be a $d$-dimensional Hilbert space with orthonormal basis $\{u_k\}_{k=1}^d$. Find the function $g : (0, 1) \times \mathbb{N} \times \mathbb{N} \to [0, \infty)$ satisfying the following: If $P : H \to H$ is an orthogonal projection of rank $n$ satisfying
\[ (1 - \varepsilon) \frac{n}{d} \leq \|Pu_k\|^2 \leq (1 + \varepsilon) \frac{n}{d}, \quad \forall 1 \leq k \leq d, \]
then there exists an orthogonal projection $Q : H \to H$ with
\[ \|Qu_k\|^2 = \frac{n}{d}, \quad \forall 1 \leq k \leq d, \]
satisfying
\[ \sum_{k=1}^{d} \|Pu_k - Qu_k\|^2 \leq g(\varepsilon, n, d). \]

Moreover, whether \( g \) depends on \( n \)?

By employing the chordal distance between subspaces [35], Cahill and Casazza [20] showed that Paulsen problem can be solved if and only if projection problem can be solved.

**Theorem 1.21.** [20] If \( f \) is the function for the Paulsen problem and \( g \) is the function for the projection problem, then
\[ g(\varepsilon, n, d) \leq 4f(\varepsilon, n, d) \leq 8g(\varepsilon, n, d), \quad \forall \varepsilon, n, d. \]

Again, in the same paper, Cahill and Casazza [20] were able to derive the following using Naimark complement of frames (see [26, 28, 29, 36]).

**Theorem 1.22.** [20] Let \( n > d \). If \( f \) is the function for the Paulsen problem, then
\[ f(\varepsilon, n, d) \leq 8f \left( \frac{d}{n-d}, n-n \right). \]

A big benefit of Theorem 1.22 is the following result.

**Theorem 1.23.** [20] To solve the Paulsen problem for a \( d \)-dimensional Hilbert space \( \mathcal{H} \), it suffices to solve it for Parseval frames \( \{\tau_j\}_{j=1}^{n} \) for \( \mathcal{H} \) with \( d \leq n \leq 2d \).

In 2017, using operator scaling algorithm and smoothed analysis, Kwok, Lau, Lee, and Ramachandran [70, 71] resolved Paulsen problem by deriving the following result.

**Theorem 1.24.** [70, 71, 86, 87] (Kwok-Lau-Lee-Ramachandran Theorem) For any \( \varepsilon \)-nearly equal norm Parseval frame \( \{\tau_j\}_{j=1}^{n} \) for \( \mathbb{R}^{d} \), there is an equal norm Parseval frame \( \{\omega_j\}_{j=1}^{n} \) for \( \mathbb{R}^{d} \) satisfying
\[ \text{dist}^2(\{\tau_j\}_{j=1}^{n}, \{\omega_j\}_{j=1}^{n}) \leq O(\varepsilon d^{\frac{13}{2}}). \]

In other words, \( f \) does not depend upon \( n \) and \( f(\varepsilon, n, d) = c\varepsilon d^{\frac{13}{2}} \), for some constant \( c > 0 \).

In 2018, using radial isotropic position, Hamilton and Moitra [49, 50] gave another proof of Paulsen problem which improved Theorem 1.24.

**Theorem 1.25.** [49, 50] (Hamilton-Moitra Theorem) For any \( \varepsilon \)-nearly equal norm Parseval frame \( \{\tau_j\}_{j=1}^{n} \) for \( \mathbb{R}^{d} \), there is an equal norm Parseval frame \( \{\omega_j\}_{j=1}^{n} \) for \( \mathbb{R}^{d} \) satisfying
\[ \text{dist}^2(\{\tau_j\}_{j=1}^{n}, \{\omega_j\}_{j=1}^{n}) \leq 20\varepsilon d^{2}. \]

In other words, \( f(\varepsilon, n, d) = 20\varepsilon d^{2} \).

After the solution of Paulsen problem (and hence projection problem), both Paulsen and projection problems are stated for Banach spaces which is open till today [69]. In this paper, we formulate Paulsen and projection problems for Hilbert C*-modules based on frame theory for Hilbert C*-modules. We show that solution of modular Paulsen problem gives a solution to modular projection problem.
2. Modular Paulsen Problem and Modular Projection Problem

Originated from the work of Kaplansky [63] for commutative C*-algebras and developed from the work of Paschke [84] and Rieffel [88] for noncommutative C*-algebras, through a development of half a century, it became evident that Hilbert C*-modules play a prominent role in the noncommutative geometry [73, 77, 95]. This demanded a necessity of developing frame theory for Hilbert C*-modules. In their fundamental paper Frank and Larson introduced the notion of frames for Hilbert C*-modules [42]. In this paper we only consider the following particular Hilbert C*-module. Let \( A \) be a unital C*-algebra, \( d \in \mathbb{N} \) and \( A^d \) be the left module over \( A \) w.r.t. natural operations. Modular \( A \)-inner product on \( A^d \) is defined as
\[
\langle (x_j)_{j=1}^d, (y_j)_{j=1}^d \rangle := \sum_{j=1}^d x_j y_j^*, \quad \forall (x_j)_{j=1}^d, (y_j)_{j=1}^d \in A^d.
\]
Hence the norm on \( A^d \) becomes
\[
\| (x_j)_{j=1}^d \| := \left\| \sum_{j=1}^d x_j x_j^* \right\|^{\frac{1}{2}}, \quad \forall (x_j)_{j=1}^d \in A^d.
\]

Standard orthonormal basis for \( A^d \) is denoted by \( \{ e_j \}_{j=1}^d \).

Definition 2.1. \([42]\) A collection \( \{ \tau_j \}_{j=1}^n \) in \( A^d \) is said to be a (modular) frame for \( A^d \) if there exist real \( a, b > 0 \) such that
\[
a \langle x, x \rangle \leq \sum_{j=1}^n \langle x, \tau_j \rangle \langle \tau_j, x \rangle \leq b \langle x, x \rangle, \quad \forall x \in A^d.
\]
A frame \( \{ \tau_j \}_{j=1}^n \) for \( A^d \) is said to be Parseval if
\[
\langle x, x \rangle = \sum_{j=1}^n \langle x, \tau_j \rangle \langle \tau_j, x \rangle, \quad \forall x \in A^d.
\]

It has to be noted that the theory of frames for Hilbert C*-modules behaves mysteriously compared to the theory of frames for Hilbert spaces [8, 10, 11, 59, 76, 85]. Thus every result of Hilbert space frame theory has to be rechecked for Hilbert C*-modules. Through detailed working, Frank and Larson obtained the following result.

Theorem 2.2. \([42]\) Let \( \{ \tau_j \}_{j=1}^n \) be a frame for \( A^d \). Then

(i) Analysis homomorphism \( \theta_\tau : A^d \ni x \mapsto \theta_\tau x := \langle x, \tau_j \rangle_{j=1}^n \in A^n \) is adjointable and bounded below.
(ii) Synthesis homomorphism \( \theta_\tau^* : A^n \ni (a_j)_{j=1}^n \mapsto \theta_\tau^*(a_j)_{j=1}^n := \sum_{j=1}^n a_j \tau_j \in A^d \) is surjective
(iii) Frame homomorphism \( S_\tau : A^d \ni x \mapsto S_\tau x := \sum_{j=1}^n \langle x, \tau_j \rangle \tau_j \in A^d \) is self-adjoint positive and invertible.
(iv) \( S_\tau = \theta_\tau^* \theta_\tau \).
(v) \( P_\tau := \theta_\tau S_\tau^{-1} \theta_\tau^* \) is a projection onto \( \theta_\tau(A^d) \).

To state modular Paulsen problem we need modular versions of Definitions 1.3, 1.4, 1.5, 1.6 and 1.7 (see [68]).
Definition 2.3. A Parseval frame \( \{ \tau_j \}_{j=1}^n \) for \( A^d \) is called an equal inner product Parseval frame if
\[
\langle \tau_j, \tau_j \rangle = \frac{d}{n}, \quad \forall 1 \leq j \leq n.
\]

Definition 2.4. A frame \( \{ \tau_j \}_{j=1}^n \) for \( A^d \) is called an \( \varepsilon \)-nearly Parseval frame if
\[
(1-\varepsilon) I_{A^d} \leq S_{\tau} \leq (1+\varepsilon) I_{A^d}.
\]

Definition 2.5. A frame \( \{ \tau_j \}_{j=1}^n \) for \( A^d \) is called an \( \varepsilon \)-nearly equal inner product frame if
\[
(1-\varepsilon) \frac{d}{n} \leq \langle \tau_j, \tau_j \rangle \leq (1+\varepsilon) \frac{d}{n}, \quad \forall 1 \leq j \leq n.
\]

Definition 2.6. A frame for \( A^d \) which is both \( \varepsilon \)-nearly equal inner product and \( \varepsilon \)-nearly Parseval is called as \( \varepsilon \)-nearly equal inner product Parseval frame.

Definition 2.7. Modular distance between two collections \( \{ \tau_j \}_{j=1}^n, \{ \omega_j \}_{j=1}^n \) in \( A^d \) is defined as
\[
\text{dist}(\{ \tau_j \}_{j=1}^n, \{ \omega_j \}_{j=1}^n) := \left\| \sum_{j=1}^n (\tau_j - \omega_j, \tau_j - \omega_j) \right\|^{\frac{1}{2}}.
\]

In view of Theorems 1.8, 1.9 and 1.10 we ask following three problems.

Problem 2.8. Whether there is an algorithm for turning every modular frame into an equal inner product modular frame with same modular frame homomorphism?

Problem 2.9. What is the closest (in terms of distance given in Definition 2.7) modular Parseval frame to a given modular frame?

Problem 2.10. What is the closest equal inner product modular frame (in terms of distance given in Definition 2.7) to a given \( \varepsilon \)-nearly inner product modular frame?

We can now formulate the most important problem of this paper.

Problem 2.11. (Modular Paulsen problem) Find the function \( f : (0, \infty) \times \mathbb{N} \times \mathbb{N} \to [0, \infty) \) so that for any \( \varepsilon \)-nearly equal inner product Parseval frame \( \{ \tau_j \}_{j=1}^n \) for \( A^d \), there is an equal inner product Parseval frame \( \{ \omega_j \}_{j=1}^n \) for \( A^d \) satisfying
\[
\text{dist}^2(\{ \tau_j \}_{j=1}^n, \{ \omega_j \}_{j=1}^n) = \left\| \sum_{j=1}^n (\tau_j - \omega_j, \tau_j - \omega_j) \right\| \leq f(\varepsilon, n, d).
\]

Moreover, whether \( f \) depends on \( n \)?

It is clear that, just like Hilbert spaces, we can reformulate Problem 2.11 as follows.

Problem 2.12. (Modular Paulsen problem) Find the function \( f : (0, \infty) \times \mathbb{N} \times \mathbb{N} \to [0, \infty) \) so that for any \( \varepsilon \)-nearly equal inner product Parseval frame \( \{ \tau_j \}_{j=1}^n \) for \( A^d \),
\[
\inf \{ \text{dist}^2(\{ \tau_j \}_{j=1}^n, \{ \omega_j \}_{j=1}^n) : \{ \omega_j \}_{j=1}^n \text{ is an equal inner product Parseval frame for } A^d \} \leq f(\varepsilon, n, d).
\]

Moreover, whether \( f \) depends on \( n \)?

Theorem 1.13 says that solution of Paulsen problem exists for Hilbert spaces. However, the proof of Theorem 1.13 as given by Hadwin can not be executed for \( C^* \)-algebras. Therefore, we first ask the following question.
Problem 2.13. Whether there exists a solution to the modular Paulsen problem?

Since Hilbert C*-modules are generalizations of Hilbert spaces, we must have the following result (see Proposition 1.14).

Proposition 2.14. Modular Paulsen function \( f \), if it exists, satisfies
\[
f(\varepsilon, n, d) \geq \varepsilon^2 d, \quad \forall \varepsilon > 0, \forall d \in \mathbb{N}.
\]

We formulate Problem 1.20 to Hilbert C*-modules as follows.

Problem 2.15. (Modular projection problem) Let \( \mathcal{A} \) be a C*-algebra with invariant basis number property. Let \( \{e_k\}_{k=1}^d \) be the standard orthonormal basis for \( \mathcal{A}^d \). Find the function \( g : (0, \infty) \times \mathbb{N} \times \mathbb{N} \to [0, \infty) \) satisfying the following: If \( P : \mathcal{A}^d \to \mathcal{A}^d \) is an orthogonal projection of rank \( n \) satisfying
\[
(1 - \varepsilon) \frac{n}{d} \leq \langle P e_k, P e_k \rangle \leq (1 + \varepsilon) \frac{n}{d}, \quad \forall 1 \leq k \leq d,
\]
then there exists an orthogonal projection \( Q : \mathcal{A}^d \to \mathcal{A}^d \) with
\[
\langle Q e_k, Q e_k \rangle = \frac{n}{d}, \quad \forall 1 \leq k \leq d,
\]
satisfying
\[
\left\| \sum_{k=1}^d \langle P e_k - Q e_k, P e_k - Q e_k \rangle \right\| \leq g(\varepsilon, n, d).
\]

Moreover, whether \( g \) depends on \( n \)?

First natural question is the following.

Problem 2.16. Whether there is a relation between modular Paulsen problem and modular projection problem?

We answer Problem 2.16 partially. More precisely, we show that if we can solve modular Paulsen problem, then we can solve modular projection problem. Our developments are motivated from the arguments given in [20]. First we derive a lemma.

Theorem 2.17. Assume that the C*-algebra \( \mathcal{A} \) is commutative. Let \( \{\tau_j\}_{j=1}^n \) and \( \{\omega_j\}_{j=1}^n \) be two Parseval frames for \( \mathcal{A}^d \). If
\[
\text{dist}^2(\{\tau_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n) = \left\| \sum_{j=1}^n (\tau_j - \omega_j, \tau_j - \omega_j) \right\| < \varepsilon,
\]
then
\[
\text{dist}^2(\{\theta_\tau \tau_j\}_{j=1}^n, \{\theta_\omega \omega_j\}_{j=1}^n) = \left\| \sum_{j=1}^n (\theta_\tau \tau_j - \theta_\omega \omega_j, \theta_\tau \tau_j - \theta_\omega \omega_j) \right\| < 4\varepsilon.
\]

Proof. Let \( 1 \leq j \leq n \). Then
\[
\theta_\tau \tau_j = \sum_{k=1}^n \langle \tau_j, \tau_k \rangle e_k, \quad \theta_\omega \omega_j = \sum_{k=1}^n \langle \omega_j, \omega_k \rangle e_k.
\]
We now note the following. If $a, b$ are any two elements of a C*-algebra, then $(a+b)(a+b)^*\leq 2(aa^*+bb^*)$. In fact,

\[ 2(aa^*+bb^*)-(a+b)(a+b)^* = 2aa^*+2bb^*-aa^*-ab^*-ba^*-bb^* = (a-b)(a-b)^* \geq 0. \]

Therefore

\[
\langle \theta_2 \tau_j - \theta_1 \omega_j, \theta_2 \tau_j - \theta_1 \omega_j \rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} 2(\langle \tau_j, \tau_k \rangle - \langle \omega_j, \omega_k \rangle)(\langle \tau_j, \tau_k \rangle - \langle \omega_j, \omega_k \rangle)^* \\
\leq 2 \sum_{k=1}^{n} \langle \tau_j, \tau_k - \omega_k \rangle \langle \tau_j, \tau_k - \omega_k \rangle^* + 2 \sum_{j=1}^{n} \langle \tau_j - \omega_j, \tau_j - \omega_j \rangle \langle \tau_j - \omega_j, \tau_j - \omega_j \rangle^* \\
= 2 \sum_{k=1}^{n} \langle \tau_j, \tau_k - \omega_k \rangle \langle \tau_j, \tau_k - \omega_k \rangle^* + 2 \langle \tau_j - \omega_j, \tau_j - \omega_j \rangle^*.
\]

We now sum by varying $j$ and use commutativity of the C*-algebra to get

\[
\sum_{j=1}^{n} (\theta_2 \tau_j - \theta_1 \omega_j, \theta_2 \tau_j - \theta_1 \omega_j) \leq 2 \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \tau_j, \tau_k - \omega_k \rangle \langle \tau_j, \tau_k - \omega_k \rangle^* + 2 \sum_{j=1}^{n} \langle \tau_j - \omega_j, \tau_j - \omega_j \rangle^* \\
= 2 \sum_{k=1}^{n} \sum_{j=1}^{n} \langle \tau_j, \tau_k - \omega_k \rangle \langle \tau_j, \tau_k - \omega_k \rangle^* + 2 \sum_{j=1}^{n} \langle \tau_j - \omega_j, \tau_j - \omega_j \rangle^* \\
= 2 \sum_{k=1}^{n} \langle \tau_k - \omega_k, \tau_k - \omega_k \rangle + 2 \sum_{j=1}^{n} \langle \tau_j - \omega_j, \tau_j - \omega_j \rangle \\
= 4 \sum_{j=1}^{n} \langle \tau_j - \omega_j, \tau_j - \omega_j \rangle.
\]

By taking norm we get the conclusion. \hfill \square

**Theorem 2.18.** Assume that the C*-algebra is commutative. If Modular Paulsen Problem has a solution, then Modular Projection Problem also has a solution.

**Proof.** Let $\mathcal{A}$ be a commutative C*-algebra, $d \in \mathbb{N}$. Let $\{e_k\}_{k=1}^{d}$ be the standard orthonormal basis for $\mathcal{A}^d$. Assume that Modular Paulsen Problem has a solution. We wish to show that Modular Projection Problem also has a solution. Let $P: \mathcal{A}^d \to \mathcal{A}^d$ be an orthogonal projection of rank $n$ satisfying

\[
(1) \quad (1-\varepsilon)\frac{n}{d} \leq \langle Pe_k, Pe_k \rangle \leq (1+\varepsilon)\frac{n}{d}, \quad \forall 1 \leq k \leq d.
\]

Since $P$ is a projection, it is then clear that $\{Pe_k\}_{k=1}^{d}$ is a modular Parseval frame for $P(\mathcal{A}^d)$. Note that Inequality (1) tells that $\{Pe_k\}_{k=1}^{d}$ is an $\varepsilon$-nearly equal inner product Parseval frame for $P(\mathcal{A}^d)$ (note that rank of this module is $n$). Since Modular Paulsen Problem has a solution, there is an equal inner product Parseval frame $\{\omega_k\}_{k=1}^{d}$ for $P(\mathcal{A}^d)$ satisfying

\[
\text{dist}^2(\{Pe_k\}_{k=1}^{d}, \{\omega_k\}_{k=1}^{d}) = \left\| \sum_{k=1}^{d} (Pe_k - \omega_k, Pe_k - \omega_k) \right\| \leq f(\varepsilon, d, n).
\]
Define \( Q := P_{\omega} = \theta_{\omega} \theta_{\omega}^* : A^d \to A^d \). Then \( Q \) is an orthogonal projection and \( Q e_k = \theta_{\omega} \theta_{\omega}^* e_k = \theta_{\omega} \omega_k \) for all \( 1 \leq k \leq d \). Since \( \{\omega_k\}_{k=1}^d \) is a Parseval Parseval frame for \( P(A^d) \), we have

\[
\langle Q e_k, Q e_k \rangle = \langle \theta_{\omega} \omega_k, \theta_{\omega} \omega_k \rangle = \langle \omega_k, \omega_k \rangle = \frac{n}{d}, \quad \forall 1 \leq k \leq d.
\]

Define \( \tau_k := P e_k \) for all \( 1 \leq k \leq d \). Then we find that

\[
\theta_{\tau} P e_k = \sum_{j=1}^d \langle P e_k, \tau_j \rangle e_j = \sum_{j=1}^d \langle P e_k, e_j \rangle e_j = \sum_{j=1}^d \langle P e_k, e_j \rangle e_j = P e_k, \quad \forall 1 \leq k \leq d.
\]

Now using Theorem 2.17 we get

\[
\left\| \sum_{k=1}^d (P e_k - Q e_k, P e_k - Q e_k) \right\| = \left\| \sum_{k=1}^d (P e_k - \omega_k, P e_k - \omega_k) \right\|
\]

\[
= \left\| \sum_{k=1}^d \langle \theta_{\tau} P e_k - \theta_{\omega} e_k, \theta_{\tau} P e_k - \theta_{\omega} e_k \rangle \right\|
\]

\[
\leq 4f(\varepsilon, d, n).
\]

Therefore Modular Projection Problem holds. \( \square \)

**Corollary 2.19.** Assume that the C*-algebra is commutative. If Modular Paulsen function \( f \) exists, then modular projection function \( g \) also exists and

\[
g(\varepsilon, n, d) \leq 4f(\varepsilon, n, d), \quad \forall (\varepsilon, n, d) \in (0, \infty) \times \mathbb{N} \times \mathbb{N}.
\]

3. **Appendix**

As mentioned in the introduction, Paulsen problem is solved in [70][71] using operator scaling methods [45][46][48][70][72]. Therefore we wish to set following problems which are important. For this, we need some notions. Let \( A \) be a unital C*-algebra and \( m, n \) be a natural numbers. Define \( M_{m \times n}(A) \) as the set of all \( m \) by \( n \) matrices over \( A \) with natural matrix operations. The adjoint of an element \( A := [a_{j,k}]_{1 \leq j \leq m, 1 \leq k \leq n} \in M_{m \times n}(A) \) is defined as \( A^* := [a_{k,j}^*]_{1 \leq k \leq n, 1 \leq j \leq m} \in M_{n \times m}(A) \). We define the **modular Hilbert-Schmidt inner product** of \( A := [a_{j,k}]_{1 \leq j \leq m, 1 \leq k \leq n} \), \( B := [b_{j,k}]_{1 \leq j \leq m, 1 \leq k \leq n} \in M_{m \times n}(A) \) as

\[
\langle A, B \rangle_{\text{MHS}} := \sum_{j=1}^m \sum_{k=1}^n a_{j,k} b_{j,k}^*.
\]

We define **modular Hilbert-Schmidt norm** of \( A := [a_{j,k}]_{1 \leq j \leq m, 1 \leq k \leq n} \in M_{m \times n}(A) \) as

\[
\|A\|_{\text{MHS}} := \|\langle A, A \rangle_{\text{MHS}}\|^{\frac{1}{2}} = \left\| \sum_{j=1}^m \sum_{k=1}^n a_{j,k} a_{j,k}^* \right\|^{\frac{1}{2}}.
\]

Whenever \( n = m \), we denote \( M_{m \times n}(A) \) by \( M_m(A) \). Identity matrix in \( M_m(A) \) is denoted by \( I_m \).

**Problem 3.1.** (**Modular Morphism Scaling Problem**) Let \( m, n \in \mathbb{N} \) and \( A \) be a unital C*-algebra. Given \( U_1, \ldots, U_k \in M_{m \times n}(A) \), find matrices \( L \in M_m(A) \) and \( R \in M_n(A) \) such that matrices

\[
V_j := LV_j R, \quad \forall 1 \leq j \leq k
\]
satisfy
\[
\sum_{j=1}^{k} V_j V_j^* = I_m \quad \text{and} \quad \sum_{j=1}^{k} V_j^* V_j = \frac{m}{n} I_n.
\]
Moreover, whether there is a polynomial time algorithm for modular morphism scaling problem?

Definition 3.2. \textit{(Doubly balanced modular matrix tuples)} Let \( A \) be a unital C*-algebra. A modular matrix tuple \((V_1, ..., V_k)\), \( V_j \in M_m(A) \), \( \forall 1 \leq j \leq k \) is called \textit{doubly balanced} if there exists a positive element \( c \in A \) such that
\[
\sum_{j=1}^{k} V_j V_j^* = cn I_m \quad \text{and} \quad \sum_{j=1}^{k} V_j^* V_j = cm I_n.
\]

Definition 3.3. \textit{(Doubly stochastic modular matrix tuples)} Let \( A \) be a unital C*-algebra. A modular matrix tuple \((V_1, ..., V_k)\), \( V_j \in M_m(A) \), \( \forall 1 \leq j \leq k \) is called \textit{doubly stochastic} if
\[
\sum_{j=1}^{k} V_j V_j^* = I_m \quad \text{and} \quad \sum_{j=1}^{k} V_j^* V_j = \frac{1}{n} m I_n.
\]

Definition 3.4. A modular matrix tuple \((V_1, ..., V_k)\), \( V_j \in M_m(A) \), \( \forall 1 \leq j \leq k \) is called \( \varepsilon \)-\textit{nearly doubly stochastic} if
\[
(1 - \varepsilon) I_m \leq \sum_{j=1}^{k} V_j V_j^* \leq (1 + \varepsilon) I_m
\]
and
\[
(1 - \varepsilon) \frac{m}{n} I_n \leq \sum_{j=1}^{k} V_j^* V_j \leq (1 + \varepsilon) \frac{m}{n} I_n.
\]

Definition 3.5. Let \( A \) be a unital C*-algebra. Given modular matrix tuples \((U_1, ..., U_k)\) and \((V_1, ..., V_k)\), \( U_j, V_j \in M_m(A) \), \( \forall 1 \leq j \leq k \), we define the \textit{modular distance} between \((U_1, ..., U_k)\) and \((V_1, ..., V_k)\) as
\[
\text{dist}((U_j)_{j=1}^{k}, (V_j)_{j=1}^{k}) := \left\| \sum_{j=1}^{k} (U_j - V_j, U_j - V_j)_{\text{MHS}} \right\|^\frac{1}{2}.
\]

Problem 3.6. \textit{(Matrix Modular Paulsen Problem)} Let \( A \) be a unital C*-algebra. Find the function \( h : (0, \infty) \times N \times N \times N \rightarrow [0, \infty) \) so that for any modular \( \varepsilon \)-nearly doubly stochastic matrix tuple \((U_1, ..., U_k)\), \( U_j \in M_m(A) \), \( \forall 1 \leq j \leq k \), there is a modular doubly stochastic matrix tuple \((V_1, ..., V_k)\), \( V_j \in M_m(A) \), \( \forall 1 \leq j \leq k \) satisfying
\[
\text{dist}^2((U_j)_{j=1}^{k}, (V_j)_{j=1}^{k}) := \left\| \sum_{j=1}^{k} (U_j - V_j, U_j - V_j)_{\text{MHS}} \right\| \leq h(\varepsilon, n, d, k).
\]
Moreover, whether \( f \) depends on \( n \)?

Next we wish to set up problem based on Barthe theorem [13] which played important role in the proof of Hamilton and Moitra [49,50] for Paulsen Problem. To formulate we set the definition of radial isotropic position vectors in modules as follows.
Definition 3.7. Let \( A \) be a unital C*-algebra and \( d \in \mathbb{N} \). We say that a set of vectors \( \{u_1, u_2, \ldots, u_n\} \) in \( A^d \) is in modular radial isotropic position with respect to a coefficient vector \( c := (c_1, c_1, \ldots, c_d)^T \in A^d \) if following conditions hold.

(i) \( \langle u_j, u_j \rangle \) is invertible for all \( 1 \leq j \leq n \).

(ii) 
\[
\sum_{j=1}^{n} c_j \langle u_j, u_j \rangle^2 u_j (\langle u_j, u_j \rangle^2 u_j)^* = I_d.
\]

Problem 3.8. Whether there is a characterization for a set of vectors in Hilbert C*-module which can be put into modular radial isotropic position with respect to a coefficient vectors?

Using chordal distance between subspaces it is proved in [20] that Projection Problem implies Paulsen Problem. However, even though we can define the notion of trigonometric functions in C*-algebras (see [67]) it seems that we can not define the notion of chordal distance as defined in [35]. However, motivated from characterization of chordal distance between subspaces, we propose the following definition of chordal distance.

Definition 3.9. (Modular Chordal Distance) Let \( A \) be a C*-algebra with invariant basis number property. Let \( P, Q : A^d \to A^d \) be rank \( m \) projections onto sub modules \( M \) and \( N \) of \( A^d \), respectively. Let \([P]\) and \([Q]\) be matrices of \( P \) and \( Q \), respectively, w.r.t. standard orthonormal basis for \( A^d \). We define the modular chordal distance between \( M \) and \( N \) as

\[
\text{dist}_{\text{ModChor}}(M, N) := \left\| m - \frac{1}{2} (\text{Trace}([P][Q]) + \text{Trace}([Q][P])) \right\|_F^2.
\]

Remark 3.10. We believe strongly that modular Paulsen problem and modular projection problem are solvable at least for W*-algebras (von Neumann algebras) or C*-algebras with invariant basis number (IBN) property (we refer [77] for IBN properties of C*-algebras).

We terminate by formulating following conjectures and a problem which we believe to have lot of importance in operator algebras. First two conjectures are based on Bourgain-Tzafriri Restricted Invertibility Theorem [17, 18, 20, 30, 31, 78, 82, 83, 91, 93, 96, 101] and the remaining is based on Johnson-Lindenstrauss Flattening Lemma [1, 7, 9, 12, 14, 19, 27, 39, 41, 43, 44, 52, 54, 58, 60, 62, 64, 66, 74, 75, 79, 81, 92, 94].

Conjecture 3.11. [(Commutative) Modular Bourgain-Tzafriri Restricted Invertibility Conjecture] Let \( A \) be a unital commutative C*-algebra and \( I(A) \) be the set of all invertible elements of \( A \). For \( d \in \mathbb{N} \), let \( M_{d \times d}(A) \) be the set of all \( d \times d \) matrices over \( A \). For \( M \in M_{d \times d}(A) \), let \( \text{det}(M) \) be the determinant of \( M \). Let \( A^d \) be the standard (left) Hilbert C*-module over \( A \) and \( \{e_j\}_{j=1}^d \) be the canonical orthonormal basis for \( A^d \). There are universal real constants \( A > 0 \), \( c > 0 \) (\( A \) and \( c \) may depend upon C*-algebra \( A \)) satisfying the following property. If \( d \in \mathbb{N} \) and \( M \in M_{d \times d}(A) \) with \( \langle Me_j, Me_j \rangle = 1 \), \( \forall 1 \leq j \leq d \) and \( \text{det}(M) \in I(A) \cup \{0\} \), then there exists a subset \( \sigma \subseteq \{1, \ldots, d\} \) of cardinality

\[
\text{Card}(\sigma) \geq \frac{cd}{\|M\|^2}
\]

such that

\[
\sum_{j \in \sigma} \sum_{k \in \sigma} a_j \langle Me_j, Me_k \rangle a_k^* = \left( \sum_{j \in \sigma} a_j Me_j, \sum_{k \in \sigma} a_k Me_k \right) \geq A \sum_{j \in \sigma} a_j a_j^*, \quad \forall a_j \in A, \forall j \in \sigma,
\]

\[
12
\]
where \( \|M\| \) is the norm of the Hilbert C*-module homomorphism defined by \( M : A^d \ni x \mapsto Mx \in A^d \).

To formulate Conjecture 3.11 for noncommutative unital C*-algebras we use the notion of Manin matrices. We refer [32, 33] for the basics of Manin matrices.

Conjecture 3.12. [(Noncommutative) Modular Bourgain-Tzafriri Restricted Invertibility Conjecture] Let \( A \) be a unital C*-algebra and \( \mathcal{I}(A) \) be the set of all invertible elements of \( A \). For \( d \in \mathbb{N} \), let \( \mathbb{M}_d(A) \) be the set of all \( d \times d \) Manin matrices over \( A \). For \( M \in \mathbb{M}_d(A) \), let \( \det_{\text{column}}(M) \) be the Manin determinant of \( M \) by column expansion. Let \( A^d \) be the standard (left) Hilbert C*-module over \( A \) and \( \{e_j\}_{j=1}^d \) be the canonical orthonormal basis for \( A^d \).

There are universal real constants \( A > 0 \), \( c > 0 \) (\( A \) and \( c \) may depend upon C*-algebra \( A \)) satisfying the following property. If \( d \in \mathbb{N} \) and \( M \in \mathbb{M}_d(A) \) with \( \langle Me_j, Me_j \rangle = 1 \), \( \forall 1 \leq j \leq d \) and \( \det_{\text{column}}(M) \in \mathcal{I}(A) \cup \{0\} \), then there exists a subset \( \sigma \subseteq \{1, \ldots, d\} \) of cardinality

\[
\text{Card}(\sigma) \geq \frac{cd}{\|M\|^2}
\]

such that

\[
\sum_{j \in \sigma} \sum_{k \in \sigma} a_j \langle Me_j, Me_k \rangle a_k^* = \left( \sum_{j \in \sigma} a_j Me_j, \sum_{k \in \sigma} a_k Me_k \right) \geq A \sum_{j \in \sigma} a_j a_j^*, \quad \forall a_j \in A, \forall j \in \sigma,
\]

where \( \|M\| \) is the norm of the Hilbert C*-module homomorphism defined by \( M \) as \( M : A^d \ni x \mapsto Mx \in A^d \).

**Remark 3.13.** (i) We can surely formulate Conjecture 3.11 by removing the condition \( \det(M) \in \mathcal{I}(A) \cup \{0\} \) and Conjecture 3.12 by removing the condition Manin matrices and \( \det_{\text{column}}(M) \in \mathcal{I}(A) \cup \{0\} \). But we strongly believe that Conjectures 3.11 and 3.12 fail with this much of generality.

(ii) If Conjecture 3.11 holds but Conjecture 3.12 fails, then we can try Conjecture 3.12 for \( \mathcal{W}^* \)-algebras or C*-algebras with IBN property.

**Problem 3.14.** Let \( A \) be the set of all unital C*-algebras. What is the best function \( \phi : A \times (0, 1) \times \mathbb{N} \rightarrow (0, \infty) \) satisfying the following. Let \( A \) be a unital C*-algebra. There is a universal constant \( C > 0 \) (which may depend upon \( A \)) satisfying the following. Let \( 0 < \varepsilon < 1, M, N \in \mathbb{N} \) and \( x_1, x_2, \ldots, x_M \in A^N \). For each natural number

\[
m > C\phi(A, \varepsilon, M)
\]

there exists a matrix \( M \in \mathbb{M}_m \times N(A) \) such that

\[
(1 - \varepsilon)(x_j - x_k, x_j - x_k) \leq \langle M(x_j - x_k), M(x_j - x_k) \rangle \leq (1 - \varepsilon)(x_j - x_k, x_j - x_k), \quad \forall 1 \leq j, k \leq m.
\]

A particular case of Problem 3.14 is the following conjecture.

Conjecture 3.15. (Modular Johnson-Lindenstrauss Flattening Conjecture) Let \( A \) be a unital C*-algebra. There is a universal constant \( C > 0 \) (which may depend upon \( A \)) satisfying the following. Let \( 0 < \varepsilon < 1, M, N \in \mathbb{N} \) and \( x_1, x_2, \ldots, x_M \in A^N \). For each natural number

\[
m > C \varepsilon^2 \log M,
\]

\[
\sum_{j \in \sigma} \sum_{k \in \sigma} a_j \langle Me_j, Me_k \rangle a_k^* = \left( \sum_{j \in \sigma} a_j Me_j, \sum_{k \in \sigma} a_k Me_k \right) \geq A \sum_{j \in \sigma} a_j a_j^*, \quad \forall a_j \in A, \forall j \in \sigma,
\]
there exists a matrix $M \in M_{m \times N}(A)$ such that
\[
(1 - \varepsilon)\langle x_j - x_k, x_j - x_k \rangle \leq \langle M(x_j - x_k), M(x_j - x_k) \rangle \leq (1 - \varepsilon)\langle x_j - x_k, x_j - x_k \rangle, \quad \forall 1 \leq j, k \leq M.
\]

**Remark 3.16.** We believe that Conjecture 3.15 holds at least for $W^*$-algebras (von Neumann algebras) or $C^*$-algebras with IBN property.

**Remark 3.17.** Modular Welch bounds are derived in [68] and a collection of problems for Hilbert $C^*$-modules including Modular Zauner conjecture have been formulated there.

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