Abstract

A gapped ground state of a quantum spin system has a natural length scale set by the gap. This length scale governs the decay of correlations. A common intuition is that this length scale also controls the spatial relaxation towards the ground state away from impurities or boundaries. The aim of this article is to take a step towards a proof of this intuition. To make the problem more tractable, we assume that there is a unique ground state that is frustration-free and invertible (i.e. no long-range entanglement). Moreover, we assume the property that we are aiming to prove for one specific kind of boundary condition; namely open boundary conditions. With these assumptions we can prove stretched exponential decay away from boundaries for any boundary conditions or (large) perturbations and for all ground states of the perturbed system. In particular, the perturbed system itself can certainly have long-range entanglement.

1 Informal statement of the result

Since a full ab initio statement of our assumptions and theorem requires quite some setup and definitions, we first state some slightly simplified assumptions and our result, freely using terminology that is probably known to most of our readers. We consider a spin system on a finite discrete set \( \Gamma \), say, a subset of \( \mathbb{Z}^d \). We define a Hamiltonian \( H + J \) where both terms \( H \) and \( J \) are local Hamiltonians, i.e. sums of local terms \( H = \sum_{X \subset \Gamma} h_X, J = \sum_{X \subset \Gamma} j_X \) with \( ||h_X||, ||j_X|| \) decaying rapidly in \( \text{diam}(X) \). We assume moreover the following properties:

1. The spatial support of \( J \) is confined to a region \( \Gamma_j \) with arbitrary size. Crucially, we do not assume that the terms \( j_X \) are small.

2. The Hamiltonian \( H \) has a unique (up to phase) ground state \( \Omega \) that is invertible. This means there is an auxiliary state \( \Omega' \) such that \( \Omega \otimes \Omega' \) is connected to a product state by a locally generated unitary, i.e. it is automorphically equivalent to a product state.

3. \( H \) is frustration-free, i.e. the local terms \( h_X \) are all minimized by the state \( \Omega \).
4. The open boundary restrictions (OBC) $H_Z = \sum_{X \subset Z} h_Z$ for balls $Z$ with radius $r$ have a local gap $\gamma(r)$ that decays no faster than an inverse polynomial in $r$.

5. The ground states of the open-boundary restrictions $H_Z$ for balls $Z$ satisfy the so-called 'local topological quantum order' condition: Their local restrictions to a set $X$ approach the local restriction of the state $\Omega$, quasi-exponentially fast in the distance $\text{dist}(X, Z^c)$.

The result is that local restrictions of any ground state of $H + J$ approach the local restriction of $\Omega$ fast, as a function of the distance to the impurity or boundary region $\Gamma_j$:

**Informal claim** There is a stretching exponent $\beta > 0$ such that for any ground state $\Phi$ of $H + J$ and for any local observable $O_X$ supported in $X \subset \Gamma$ with $\text{diam}(X) \leq C$,

$$|\langle \Phi, O_X \Phi \rangle - \langle \Omega, O_X \Omega \rangle| \leq C ||O_X|| e^{-\left(\text{dist}(X, \Gamma_j)\right)\beta}$$

As far as we can see, there is no natural sense in which $\Phi$ resembles $\Omega$, other than the one stated above. In particular, $\Phi$ is not necessarily the unique ground state, it has in general no gap, no short range entanglement properties and it is not frustration-free. To see this, consider $\Gamma = \{1, 2, \ldots, L\}$ and imagine that at each site there is a qubit with base states $|\downarrow\rangle$, $|\uparrow\rangle$. The Hamiltonian is $H = \sum_{i \in \Gamma} h_i$ where each $h_i = (|\uparrow\rangle \langle\uparrow|)_i$ penalizes the up-state $|\uparrow\rangle$ at site $i$. This Hamiltonian has a unique ground state that is a product. Then the perturbation $J$ is chosen as $J = -h_1 - h_L + e^{-cL} j_{1,L}$ with some $c > 0$ and $j_{1,L}$ a two-qubit operator acting on sites 1 and $L$. Because of the very small prefactor $e^{-cL}$, this perturbation $J$ satisfies any reasonable locality requirement. By taking $j_{1,L} = 0$, the perturbed system $H + J$ has 4-fold ground state degeneracy. By taking $j_{1,L}$ to be minus the rank-1 projector on the entangled Bell state $\frac{1}{\sqrt{2}} (|\uparrow\rangle_1 \otimes |\uparrow\rangle_L + |\downarrow\rangle_1 \otimes |\downarrow\rangle_L)$, the perturbed system $H + J$ has a unique ground state that has maximal entanglement between qubits 1 and $L$.

### 1.1 Discussion

The above result should be contrasted with recent work on the stability of the spectral gap towards locally small perturbations. That recent work can be split in two classes. The first class concerns systems where the unperturbed ground state is a product, see [6, 7, 9, 23, 5, 2, 17]. The second class is based on the so-called Bravyi-Hastings-Michalakis (BHM) argument which applies to frustration-free ground states, see [3, 19, 21, 22]. Both classes are eventually based on some form of perturbation theory. Our result is different in spirit, as our perturbations are not assumed to be small, but we obtain only information about the perturbed ground states in regions far away from the region $\Gamma_j$ where the perturbation acts. On the other hand, and in contrast to the quoted works, our result relies crucially on the variational principle. One should realize that it is not only the proof of the results in [3, 19, 21] that does not apply in our case, but also the results should not be expected, as already stressed above.

Despite the previous contrast, the BHM argument is important for interpretation of our result. As stated, our assumptions are rather restrictive but we hope that the BHM argument would allow to establish the following: If we were to tighten our assumptions so that the local gap $\gamma(r) \geq \gamma > 0$, i.e. the gap is uniformly bounded from below, then that set of tightened assumptions might be

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1. Actually, our assumptions do not explicitly require a global gap either, but such a gap is there in spirit because of the invertibility assumption, and we have left it in the abstract to fix thoughts.
stable with respect to locally small perturbations, by a variant of the BHM argument. Once that would be established, our results would hence also become relevant for small perturbations of frustration-free, locally gapped models.

Results similar to ours have recently been obtained independently in [13], as a corollary to [23]. The result of [13, 23] yields full exponential decay and requires no explicit frustration freeness, but it is restricted to weakly interacting spins, i.e. perturbations of products. Instead, our result relies on the automorphic equivalence of the unperturbed ground state to a product state, possibly upon adjoining an auxiliary state. Both results need hence an underlying product structure. Another line of research that is loosely connected to ours, concerns stability at nonzero temperature in spatial dimension 1, see [11, 15].

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2 Setup

2.1 Preliminaries

2.1.1 Spatial structure
We consider a finite graph $\Gamma$, equipped with the graph distance $\text{dist}(\cdot, \cdot)$. Let $B_r(x) = \{ y : \text{dist}(x, y) \leq r \}$ be the ball of radius $r$ centered at $x \in \Gamma$. The graph is assumed to have a finite dimension $d$, i.e. for some $C_{\Gamma}$

$$\sup_{x \in \Gamma} |B_r(x)| \leq 1 + C_{\Gamma} r^d.$$  

For reasons of recognizability, we refer to vertices as 'sites'. To every site $x$ is associated a finite-dimensional Hilbert space $H_x$ and we define the total Hilbert space $H = H_\Gamma = \bigotimes_{x \in \Gamma} H_x$ and the algebra $A = A_\Gamma = B(H)$ of bounded operators on $H$. For any $X \subset \Gamma$, we have the Hilbert space $H_X = \bigotimes_{x \in X} H_x$ and the algebra $A_X = B(H_X)$. We use the usual embedding $A_X \rightarrow A$ given by $O_X \rightarrow O_X \otimes 1_{X^c}$. As is customary, we will identify $O_X \otimes 1_{X^c}$ with $O_X$ and say that $O_X$ is supported in $X$.

2.1.2 Locality
We write $\mathbb{N}^+$ for the strictly positive naturals and we let $\mathcal{M}$ be a class of functions $m : \mathbb{N}^+ \rightarrow \mathbb{R}^+$ of quasi-exponential decay. The class $\mathcal{M}$ is defined by the following two conditions.

1. $m$ is non-increasing.
2. For every $0 < \alpha < 1$, there exists $C_\alpha, c_\alpha$ such that $m(r) \leq C_\alpha e^{-c_\alpha r^\alpha}$.

To express the locality properties of Hamiltonians, we consider collections $g$ of local terms $(g_X)_{X \subset \Lambda}$, $g_X \in \mathcal{A}_X$, sometimes called 'interactions', and we endow them with a family of norms, parametrized by $m \in \mathcal{M}$:

$$\|g\|_m := \sup_{x \in \Gamma} \sum_{X \ni x} \frac{\|g_X\|}{m(1 + \text{diam}(X))}$$
The locality property is then expressed by the finiteness of $\|q\|_m$ for some $m \in M$.

### 2.1.3 Trace norms

We recall the trace norm $\| \cdot \|_{1,X}$ on $A_X = B(H_X)$, defined by

$$\|O\|_{1,X} = \text{tr}(X \sqrt{OO^*})$$

where it is understood that $\text{tr}(X)$ is the trace on the Hilbert space $H_X$, not on the whole of $H$. We then denote by $\text{tr}_X$ the partial trace $\text{tr}_X : A \mapsto A_X$ defined by $\text{tr}(X) \text{tr}_X O = \text{tr} O$ for any $O \in A$. Since we will often use trace norms of operators resulting from the partial trace we introduce a short-hand notation:

$$|O|_X := \| \text{tr}_X O \|_{1,X}.$$ 

Nonnegative operators $\rho$ that have unit trace $\text{tr} \rho = 1$, are called density matrices. For $\Psi \in H$, we write $\rho_{\Psi}$ for the pure density matrix equal to the orthogonal projector on $\mathbb{C}\Psi$.

### 2.2 Spaces and Hamiltonian

The Hamiltonian $H$ and the perturbation $J$ are written as

$$H = \sum_{X \subset \Gamma} h_X, \quad J = \sum_{X \subset \Gamma} j_X \quad h_X, j_X \in A_X$$

(1)

The perturbation $J$ is spatially restricted to a region $\Gamma_j \subset \Gamma$ in a mild sense

$$j_X = 0 \quad \text{unless} \quad X \cap \Gamma_j \neq \emptyset$$

Then, we assume, for the collections $h, j,$

$$\|h\|_{m_h}, \|j\|_{m_j} < \infty$$

for some $m_h, m_j \in M$.

### 2.3 Assumptions

Our first assumption states that the Hamiltonian $H$ has a frustration free ground state $\Omega$:

**Assumption 1** (Frustration-free ground state). *All the terms $h_X$ are nonnegative: $h_X \geq 0$. There is a $\Omega \in H, \|\Omega\| = 1$ such that $\Omega \in \text{Ker}(h_X)$ for any $X$. We denote by $\mu = \rho_{\Omega}$ the corresponding density matrix.*

We note that the frustration free property depends on the way the Hamiltonian is written as a sum of local terms, i.e. on the interaction. Next, we introduce 'open boundary condition' (OBC) restrictions of $H$,

$$H_Z = \sum_{X \subset Z} h_X,$$

Just as the frustration free property, the notion of OBC restriction depends on the interaction. We let $P_Z$ be the orthogonal projector on $\text{Ker}(H_Z)$. Since $h_X \geq 0$, it holds that

$$\text{Ker}(H_Z) = \cap_{X \subset Z} \text{Ker}(h_X)$$
and hence also
\[ P_Z P_{Z'} = P_{Z'}, \quad Z \subset Z' \]
The following assumption has come to be known as 'Local Topological Quantum Order' (LTQO) but, in our case, it is better described as the fact that the density matrix of any ground state of the OBC Hamiltonian \( H_Z \) looks similar to the global ground state \( \mu \) in the deep interior of \( Z \). Recall that \( B_r(x) \subset \Gamma \) is a ball.

**Assumption 2** (OBC-regularity). There is \( m_O \in \mathcal{M} \) and \( d_O < \infty \) such that, for any \( x \in \Gamma \) and \( \Psi \in \text{Ker}(H_{B_r(x)}) \),
\[
|\rho_{\Psi} - \mu|_{B_{r-k}(x)} \leq r^{d_O} m_O(k), \quad k < r
\]
Let us briefly comment on the precise form of the above bound. We have in mind, roughly, that, any \( \Psi \in \text{Ker}(H_{B_r(x)}) \) differs from \( \Omega \) only through the presence of boundary modes. One realization of this would be that any such \( \Psi \) is of the form \( e^{iF}\Omega \) with \( F \) a sum of local or quasilocal terms supported near the boundary of \( B_r(x) \). This would indeed lead to the bound above with \( r^{d_O} \) an upper bound for \( C|\partial B_r(x)| \).

The next assumption concerns the local gap of OBC Hamiltonians. Let \( \gamma(Z) = \min(\text{spec}(H_Z) \setminus \{0\}) \) be the spectral gap of \( H_Z \) (\( \text{spec}(\cdot) \) is the spectrum).

**Assumption 3** (Local Gap). There is a \( C_\gamma, d_\gamma < \infty \) such that
\[
\sup_{x \in \Gamma} \gamma(B_r(x))^{-1} \leq C_\gamma r^{d_\gamma}
\]
This assumption might be slightly misleading. In most examples that we are aware of, OBC restrictions \( H_Z = \sum_{X \subset Z} h_Z \) of gapped frustration-free states have a gap that is actually bounded below in \( Z \), and the physical edge modes are found as ground states of \( H_Z \), i.e. the eigenvalue at 0 will typically be degenerate. In that case, the genuine restriction is not so much assumption 3 but rather assumption 2.

### 2.3.1 Invertibility

Let \( \mathcal{H}' \) be \( \mathcal{H}' = \mathcal{H}'_\Gamma = \otimes_{x \in \Gamma} \mathcal{H}_x' \) with \( \mathcal{H}_x' \) a finite-dimensional Hilbert space. We will now consider the tensor product \( \mathcal{H} \otimes \mathcal{H}' \) which we denote by
\[
\hat{\mathcal{H}} = \mathcal{H} \otimes^{\text{aux}} \mathcal{H}'.
\]
The superscript \( \otimes^{\text{aux}} \) reminds us of the fact that this is not a tensor product between disjoint spatial regions, but between the 'original' Hilbert space and an 'auxiliary' Hilbert space. This is helpful because we also view \( \hat{\mathcal{H}} \) again as a tensor product over sites
\[
\hat{\mathcal{H}} = \otimes_{x \in \Gamma} \hat{\mathcal{H}}_x, \quad \hat{\mathcal{H}}_x = \mathcal{H}_x \otimes^{\text{aux}} \mathcal{H}_x'.
\]
The local structure of the space \( \hat{\mathcal{H}} \) is completely analogous to the one of \( \mathcal{H} \), except that now the dimension of the on-site spaces is larger. We consistently put a prime on algebras derived from \( \mathcal{H}_x', A'_X = B(\mathcal{H}') \) and we put a tilde on algebras derived from \( \hat{\mathcal{H}} \), i.e. \( \tilde{A} = B(\hat{\mathcal{H}}) \) and \( \tilde{A}_X' \subset A', \tilde{A}_X \subset \tilde{A} \) are subalgebras of operators supported in \( X \). We also copy the definition of collections \( q \) of local operators \((q_X)_X\) and their norms \(|| \cdot ||_m\). The next assumption expresses that the state \( \Omega \) has no long-range entanglement. If we were working in an infinite-volume setting, one would describe this assumption as 'upon adjoining an auxiliary state \( \Omega' \), \( \Omega \otimes \Omega' \) is automorphically equivalent to a product state'.
Assumption 4 (Invertibility). There are collections \(q(s)\) of local operators \((q_X(s))_{X \subseteq \Gamma}, q_X(s) \in \tilde{A}_X\), indexed by \(s \in [0,1]\) such that

1. \(q_X(s) = q_X^*(s)\) and \(s \mapsto q_X(s)\) is measurable, for any \(X\).
2. \(\sup_{s \in [0,1]} ||q(s)||_{m_I} \leq C_I < \infty\) for some \(m_I \in \mathcal{M}\).
3. There is a product state \(\Phi = \otimes_{x \in \Gamma} \Phi_x\) and a state \(\Omega' \in \mathcal{H}'\) such that

\[
\Omega \otimes_{\text{aux}} \Omega' = U(1)\Phi, \quad U(s) = 1 + i \int_0^s du H_q(u)U(u)
\]

where \(H_q(s) = \sum_{X \subseteq \Gamma} q_X(s)\).

One says that \(\Omega'\) is an 'inverse' to \(\Omega\).

The 'invertibility' assumption should roughly correspond to states that do not have anyonic excitations. Symmetry protected topological states (SPT’s) are invertible \[8,10\], states characterized by an integer quantum Hall effect are invertible \[14\], etc.. We refer to the extensive literature for a more thorough discussion. For us, invertible states simply form a natural class for which our proof works. The motivation for considering the type of spatial decay expressed by the class \(\mathcal{M}\) is because this corresponds to the decay one can prove for the (quasi-)adiabatic dynamics connecting ground states of a uniformly gapped Hamiltonian, by the technique of (quasi-)adiabatic continuation, see \[12,1\].

2.4 Result

We say that \(\Phi \in \mathcal{H}_\Gamma\) is a ground state of the perturbed Hamiltonian \(H + J\) if, for the associated density matrix \(\rho_\Phi\),

\[
\langle (H + J) \rangle_{\rho_\Phi} = \min(\text{spec}(H + J)).
\]

Recall that \(\mu\) is the global ground state density matrix of the unperturbed Hamiltonian \(H\). By a ‘constant’, we mean a quantity that can depend only on \(m_h, m_j, m_I \in \mathcal{M}\), the numbers \(C_\gamma, C_\Gamma\) and \(d, d_0, d_\gamma\). In particular, constants can not depend on the size \(|\Gamma|\).

**Theorem 1.** For any \(w > 0\), there is a constant \(C(w)\) such that, for any ground state \(\Phi\) of \(H + J\), and any \(x \in \Gamma\) with \(R := \text{dist}(x, \Gamma_j)\),

\[
|\rho_\Phi - \mu|_{B_{R^p-w}(x)} \leq C(w)e^{-Rp-w}, \quad p = \frac{1}{d + d_\gamma + 2}
\]

To connect this theorem to the informal claim in Section 1 note that the left hand side is equal to \(\sup_{O \in \mathcal{A}_X, ||O|| = 1} |\langle O \rangle_{\rho_\Phi} - \langle O \rangle_{\mu}|\) with \(X = B_{R^p-w}(x)\).

3 Proof of Theorem 1

We define some additional notation to be used in this section. For regions \(Z \subset \Gamma\), we define the \(r\) fattening

\[
(Z)_r := \{x \in \Gamma, \text{dist}(x, Z) \leq r\}
\]

and the boundary

\[
\partial Z = (Z)_1 \cap (Z^c)_1
\]
We use constants $C, c$ as introduced above Theorem 1, i.e. depending on a number of fixed parameters. We also use the generic notation $m$ for a function in $\mathcal{M}$ that depends possibly on those same fixed parameters. Just as the constants $C, c$, the precise function $m$ can change from line to line. In the same vein, we also use $p$ to denote a polynomial that depends only on the fixed parameters.

### 3.1 Stitching maps

Recall that $\mu$ is the global ground state density matrix of the unperturbed Hamiltonian $H$. We say that a family of maps $\Sigma_Z$, indexed by $Z \subset \Gamma$, are stitching maps if they satisfy the following properties.

**Definition 1.** Stitching maps $\Sigma_Z$ with $Z \subset \Gamma$ are trace-preserving completely positive maps $\mathcal{A} \to \mathcal{A}$ iff. there is an $m \in \mathcal{M}$ such that, for any $Z, X \subset \Gamma$ and density matrices $\rho, \omega$,

1. $|\Sigma_Z(\rho) - \text{tr}_Z(\mu \otimes \Sigma_Z \rho)|_X \leq |X|m(\text{dist}(\partial Z, X))$
2. $|\Sigma_Z(\rho) - \Sigma_Z(\omega)|_X \leq |\rho - \omega|_X + |X|m(r)$
3. $\Sigma_Z(\mu) = \mu$

To state this in a rough way, stitching maps are such that $\Sigma_Z(\mu) = \mu$ and further

$$\Sigma_Z(\rho) = \begin{cases} \mu & \text{deep inside } Z \\ \rho & \text{far outside } Z \end{cases}$$

The stitching maps will be used for regions $Z$ that are far away from the perturbation region $\Gamma_j$.

The purpose of the invertibility assumption 4 is precisely to ensure the existence of stitching maps, as we show now. Recall that $\tilde{\mathcal{H}}$ is the enlarged Hilbert space.

**Proposition 1.** For any region $Z \subset \Gamma$, there are unitaries $V = V_Z$ acting on $\tilde{\mathcal{H}}$ such that the following family of CP maps $\Sigma = \Sigma_Z$ are stitching maps in the sense of Definition 1:

$$\Sigma(\rho) = \text{tr}_{\mathcal{H}'_s} \left[ V^* \left( \text{tr}_{\tilde{\mathcal{H}}_s}(V \mu \otimes \text{aux} \mu^* V^*) \otimes \text{tr}_{\mathcal{H}_s}(V \rho \otimes \text{aux} \mu^* V^*) \right) V \right] (2)$$

where $\mu = |\Omega\rangle\langle\Omega|$ and $\mu' = |\Omega'\rangle\langle\Omega'|$ with $\Omega' \in \mathcal{H}'$ is the 'inverse state' to $\Omega$, see assumption 4.

To study the map $\Sigma = \Sigma_Z$ proposed above, we introduce also the adjoint map $\Sigma^*$, defined by $\text{tr}(\Sigma^*O\rho) = \text{tr}(OS(\rho))$ for all $O \in \mathcal{A}$. We decompose $\Sigma^* = \Upsilon_1 \circ \ldots \circ \Upsilon_6$ with $\Upsilon_i$ the following norm-contracting CP maps

$\Upsilon_6$: $A \to \tilde{A}: O \mapsto O \otimes \text{aux} 1_{\mathcal{A}}$

$\Upsilon_5$: $\tilde{A} \to \tilde{A}: O \mapsto VOV^*$

$\Upsilon_4$: $\tilde{A} \to \tilde{A}_{Z^c}: O \mapsto \text{tr}_Z \kappa O$ for a density matrix $\kappa$ on $\mathcal{H}_Z$.

$\Upsilon_3$: $\tilde{A}_{Z^c} \to \tilde{A}: O \mapsto 1_Z \otimes O$

$\Upsilon_2$: $\tilde{A} \to \tilde{A}: O \mapsto V^*OV$

$\Upsilon_1$: $\tilde{A} \to \mathcal{A}: O \mapsto \text{tr}_{\mathcal{H}'} \kappa O$ for a density matrix $\kappa$ on $\mathcal{H}'$.

**Proof.** We first construct the appropriate unitaries $V$. To that end, we introduce the interactions $\hat{q}(s)$ by

$$\hat{q}_X(s) = \begin{cases} q_X(s) & X \subset Z \quad \text{or} \quad X \subset Z^c \\ 0 & \text{otherwise} \end{cases}$$
where the family $q_X$ is given by assumption 3. We let $\hat{U}(s)$ be the unitary defined analogously to $U(s)$ in the invertibility assumption 4, but with $H_q(s)$ replaced by $H_q(s) = \sum_X q_X(s)$. We then set

$$V(s) = \hat{U}(s)U^*(s), \quad V = V(1)$$

and we observe that

$$V\Omega \otimes^{aux} \Omega' = \Omega_Z \otimes \Omega_{Z^c}$$

(3)

for some $\Omega_Z \in \tilde{H}_Z, \Omega_{Z^c} \in \tilde{H}_{Z^c}$. This shows that $\Sigma_Z$ satisfies Property 3) of Definition 1 and we now proceed to checking Property 2). First we state a locality property of the adjoint $\Sigma^* = \Sigma_Z^*$:

$$||\Sigma^*(O) - \mathbb{E}_{(X_r),\xi}(\Sigma^*(O))|| \leq ||O||X|m(r), \quad \forall O \in A_X$$

(4)

where $\mathbb{E}_{(X_r),\xi}(\cdot) = \frac{tr_{(X_r),\xi}(\cdot)}{tr_{(X_r),\xi}(1)}$ is the normalized partial trace. To establish this locality bound, it suffices to prove it, i.e. (4), separately for each map $\Upsilon_i$. This is because we have the freedom to adjust the functions $m$, in particular the function $m'(r) = \sum_{r_1=1}^{r-1} m(r_1)m(r-r_1)$ is dominated by a function $\mathcal{M}$ (see appendix). The locality bound (4) holds trivially for $\Upsilon_i$ with $i = 1, 3, 4, 6$, whereas for $i = 2, 5$ it is a consequence of Lemma 1 below. This yields Property 2 of Definition 1.

By a similar argument, Property 1) of Definition 1 follows from the bound (5) of Lemma 1.

**Lemma 1.** For any $X, Z$ and $O \in A_X$, with $V = V_Z$ as defined above,

$$||V^*OV - \mathbb{E}_{(X_r),\xi}(V^*OV)|| \leq ||O||X|m(r)$$

(5)

and

$$||V^*OV - O|| \leq ||O||X|m(\text{dist}(X, \partial Z))$$

(6)

The same estimate holds as well if we exchange $V$ and $V^*$.

The proof of Lemma 1 follows from well-known considerations based on Lieb-Robinson bounds, and therefore we postpone it to the appendix.

We will now establish a property of stitching maps, which we will henceforth denote by $\Sigma_Z$, that relies crucially on the OBC-regularity assumption 2. It makes explicit that stitching maps are ‘seamless’, i.e. they do not introduce errors at the cut $\partial Z$. We fix a set $Z$ and a ball $B = B_r(x)$ for some $x, r$. We will drop $Z$ and write $\Sigma = \Sigma_Z$. Recall that $P_X$ is the ground state projector corresponding to the OBC restriction $H_X$ and let $\bar{P}_X = 1 - P_X$.

**Proposition 2.** Let $\sigma = \rho_B$ be the pure density matrix associated to some $\Psi \in \mathcal{H}_r$. Consider a set $Z$ and balls $B_r = B_r(x)$ for some site $x$. Then (recall that $p(\cdot)$ is a polynomial),

$$\langle \bar{P}_{B_{r-k}} \Sigma_Z(\sigma) \rangle \leq 3\langle \bar{P}_{B(r)} \rangle_\sigma + p(r)m(k).$$

**Proof.** Since $Z$ will be fixed, we drop it and write $\Sigma = \Sigma_Z$. Let us denote $\epsilon := \langle \bar{P}_{B(r)} \rangle_\sigma$ and remark that $||P_B \Psi - \Psi|| \leq \sqrt{\epsilon}$. If $\epsilon = 1$, then the required bound is trivial, hence we assume $\epsilon < 1$. We split

$$\sigma = \sum_{i=1}^{4} \zeta_i = P_B \sigma P_B + P_B \sigma \bar{P}_B + \bar{P}_B \sigma P_B + \bar{P}_B \sigma \bar{P}_B$$

where we abbreviated $P_B = P_{B_r}$, as we will also do below. We will treat these terms separately, i.e. find bounds on

$$\text{tr} \bar{P}_{B_{r-k}} \Sigma(\zeta_i), \quad i = 1, 2, 3, 4.$$
The term $\zeta_1 = P_B \sigma P_B$

Since $\epsilon < 1$, $\text{tr} \zeta_1 > 0$ and we can define the density matrix $\zeta = \zeta_1 / \text{tr} \zeta_1$, satisfying $\langle \bar{P}_B \rangle_{\zeta} = 0$. By the properties of Definition[1] we get the equality and first inequality in

$$|\Sigma(\zeta) - \mu|_{B(r-k)} = |\Sigma(\zeta) - \Sigma|_{B(r-k)} \leq |\zeta - \mu|_{B(r-k/2)} + p(r)m(k) \leq p(r)m(k)$$

whereas the second inequality follows from the OBC regularity assumption[2] and the fact that $\zeta$ is the density matrix of a ground state of $H_B$. Since, by the frustration-free property, $\langle \bar{P}_{B(r-k)} \rangle_{\mu} = 0$, we get $\text{tr} \bar{P}_{B_{r-k}} \Sigma(\zeta) \leq p(r)m(k)$ and hence also

$$\text{tr} \bar{P}_{B_{r-k}} \Sigma(\zeta_1) \leq p(r)m(k)$$

The term $\zeta_4 = \bar{P}_B \sigma \bar{P}_B$

Here we have $|\zeta_4|_{\Gamma} = \text{tr} \bar{P}_B \sigma \bar{P}_B = \epsilon$ and since $\Sigma$ is trace-preserving,

$$\text{tr} \bar{P}_{B_{r-k}} \Sigma(\zeta_4) \leq \epsilon.$$ 

The term $\zeta_2 = P_B \sigma \bar{P}_B$

Since $\sigma$ is pure, we can write

$$\zeta_2 = P_B \sigma \bar{P}_B = \langle \bar{P}_B \rangle_{\sigma}^{1/2} |a\rangle\langle b|, \quad ||a|| \leq ||b|| = 1, \quad P_B a = a$$

and we note that $|a\rangle\langle a| = \zeta_1$. To estimate $\text{tr} \bar{P}_{B_{r-k}} \Sigma(|a\rangle\langle b|)$, we use Lemma[2] below to get

$$\text{tr} \bar{P}_{B_{r-k}} \Sigma(\zeta_2) \leq \sqrt{\langle \bar{P}_B \rangle_{\sigma}} \sqrt{\text{tr} \bar{P}_{B_{r-k}} \Sigma(\zeta_1)} \leq \sqrt{\epsilon \sqrt{p(r)m(k)}} \leq \epsilon + p(r)m(k)$$

The second inequality follows from the bound for the term $\zeta_1$ above.

**Lemma 2.** For an orthogonal projector $P$, a trace-conserving CP map $\Gamma$ and $||a||, ||b|| \leq 1$, we have

$$|\text{tr}(P\Gamma(|a\rangle\langle b|))| \leq \sqrt{\text{tr}(P\Gamma(|a\rangle\langle a|)) \text{tr}(P\Gamma(|b\rangle\langle b|))} \leq \sqrt{\text{tr}(P\Gamma(|a\rangle\langle a|))}$$

**Proof.** Introducing again the adjoint map $\Gamma^*$, the first inequality reads

$$||b, \Gamma^*(P)a|| \leq \sqrt{\langle a, \Gamma^*(P)a \rangle / \langle b, \Gamma^*(P)b \rangle}.$$ 

It follows from positivity of $\Gamma^*(P)$ and the Cauchy-Schwarz inequality. $\square$

The term $\zeta_3 = \bar{P}_B \sigma \bar{P}_B$

The bound and its proof are analogous to the case of $\zeta_2$. The claim of the lemma follows by adding the bounds on $\text{tr} \bar{P}_{B_{r-k}} \Sigma(\zeta_i)$ for $i = 1, 2, 3, 4$. $\square$

### 3.2 The basic inequality

If the energy of a pure density matrix $\sigma$ at the boundary $\partial Z$ of $Z$ is small, then application of the stitching map $\Sigma_Z$ does not significantly alter $\sigma$ near $\partial Z$. This statement is the content of the following lemma, and it is a consequence of Proposition[2]

**Lemma 3.** Let $\sigma$ be a pure density matrix, i.e. $\sigma = \rho_\Psi$ for some $\Psi \in \mathcal{H}_\Gamma$, and let $\Sigma = \Sigma_Z$ for some region $Z \subset \Gamma$. Then

$$\sum_{X \in Z \setminus \partial Z} (\langle h_X \rangle_{\Sigma(\sigma)} - \langle h_X \rangle_{\sigma}) \leq Cr^{d+d_f} \langle H_{\partial Z} \rangle_{\sigma} + m(r)|\partial Z|$$
Proof. We split the class of $X$ contributing the left hand side in disjoint classes $a, b, c, d, e$, defined by the condition $X \not\subset Z \setminus (\partial Z)_r$ and the additional condition(s)

a) $X \subset Z \setminus (\partial Z)_{r/2}$
b) $X \subset Z^c \setminus (\partial Z)_{r/2}$
c) $X \cap (\partial Z)_{r/2} \neq \emptyset$ and $\text{diam}(X) < r/4$.
d) $X \cap (\partial Z)_{r/2} \neq \emptyset$ and $\text{diam}(X) \geq r/4$.
e) $X \subset ((\partial Z)_{r/2})^c$ and $X \cap Z \neq \emptyset$ and $X \cap Z^c \neq \emptyset$.

We will now estimate

$$\sum_{X \in \text{class } x} (\langle h_X \rangle_{\Sigma(\sigma)} - \langle h_X \rangle_\sigma)$$

(class $a$

$$\sum_{X \in \text{class } a} (\langle h_X \rangle_{\Sigma(\sigma)} - \langle h_X \rangle_\sigma) \leq \sum_{X \subset Z \setminus (\partial Z)_{r/2}} \langle h_X \rangle_{\Sigma(\sigma)}$$

$$\leq \sum_{X \subset Z \setminus (\partial Z)_{r/2}} (\langle h_X \rangle_\mu + ||h_X|| ||X|| m(\text{dist}(X, \partial Z)))$$

$$\leq \sum_{x \in Z \setminus (\partial Z)_{r/2}} m(\text{dist}(x, \partial Z)) \sum_{X \ni x} ||h_X|| ||X|| \leq |\partial Z| m(r)$$

The first inequality follows from non-negativity of $h_X$. The second from Property 1) of Definition 1 and the third from $\langle h_X \rangle_\mu = 0$. Then we use $\sup_x \sum_{X \ni x} ||h_X|| ||X|| \leq C$ and the rapid decay of $m$ to absorb a geometrical polynomial factor in $r$.

(class $b$

$$\sum_{X \in \text{class } b} (\langle h_X \rangle_{\Sigma(\sigma)} - \langle h_X \rangle_\sigma) \leq \sum_{X : \text{dist}(X, Z) \geq r/2} ||h_X|| ||X|| m(\text{dist}(X, Z))$$

$$\leq \sum_{x : \text{dist}(x, Z) \geq r/2} m(\text{dist}(x, Z)) \sum_{X \ni x} ||h_X|| ||X|| \leq |\partial Z| m(r)$$

The first inequality is Property 1) of Definition 1 the rest is analogous to case $a$ above.

(class $c$}
Lemma 4. Let $X$ classes of sets

\[ \sum_{X \in \text{class } e} (\langle h_X \rangle_{\Sigma(\sigma)} - \langle h_X \rangle_\sigma) \leq \sum_{x \in (\partial Z)_{r/2}} \sum_{X \exists x, \text{diam}(X) \leq r/4} \langle h_X \rangle_{\Sigma(\sigma)} \]

\[ \leq \sum_{x \in (\partial Z)_{r/2}} \sum_{X \exists x, \text{diam}(X) \leq r/4} \|h_X\| \langle \tilde{P}_X \rangle_{\Sigma(\sigma)} \]

\[ \leq \sum_{x \in (\partial Z)_{r/2}} C \langle \tilde{P}_{B_{r/4}(x)} \rangle_{\Sigma(\sigma)} \]

\[ \leq \sum_{x \in (\partial Z)_{r/2}} C \langle \tilde{P}_{B_{r/2}(x)} \rangle_{\sigma} + p(r)m(r) \]

\[ \leq \sum_{x \in (\partial Z)_{r/2}} \gamma(B_{r/2}(x))^{-1}\langle H_{B_{r/2}(x)} \rangle_{\sigma} + m(r) \]

\[ \leq \frac{\sup_x |B_{r/2}(x)|}{\inf_x \gamma(B_{r/2}(x))} \langle H(\partial Z) \rangle_{\sigma} + |(\partial Z)_{r/2}|m(r) \]

\[ \leq C_1 r^{d+d'} \langle H(\partial Z) \rangle_{\sigma} + |\partial Z|m(r) \]

The first inequality is by nonnegativity of $h_X$. The third inequality uses $\tilde{P}_X \subset \tilde{P}_X'$ for $X \subset X'$, i.e. the frustration freeness of assumption $[1]$ and $\sum_{X \exists x} \|h_X\| < C$. The fourth inequality uses Proposition $2$. The sixth inequality follows because any $h_X$ contributing to $H(\partial Z)_r$ appears in at most a number $\sup_x \|B_r(x)\|$ of balls with radius $r/2$. The last inequality uses the finite dimension of $\Gamma$ and assumption $[3]$.

class $d$

\[ \sum_{X \in \text{class } d} (\langle h_X \rangle_{\Sigma(\sigma)} - \langle h_X \rangle_\sigma) \leq \sum_{x \in (\partial Z)_{r/2}} \sum_{X \exists x, \text{diam}(X) \geq r/4} 2||h_X|| \leq |\partial Z|m(r) \]

We just use $\langle h_X \rangle_{\Sigma(\sigma)}, \langle h_X \rangle_\sigma \leq ||h_X||$ and then analogous reasoning as used for class $a$.

class $e$

\[ \sum_{X \in \text{class } e} (\langle h_X \rangle_{\Sigma(\sigma)} - \langle h_X \rangle_\sigma) \leq \sum_{x \in (\partial Z)_{r/2}} \sum_{X \exists x, r/2 \leq \text{dist}(X, \partial Z) \leq \text{diam}(X)} 2||h_X|| \leq |\partial Z|m(r) \]

Reasoning is analogous to that for class $d$ and class $a.$

The following lemma shows why the interaction $j$ does not play any role in controlling the energy along the stitch $\partial Z$. Its proof is similar, but simpler than that of Lemma $[3]$.

Lemma 4. Let $\sigma$ be a density matrix and let $Z$ be a set such that $\text{dist}(Z, \Gamma_j) \geq r$. Then

\[ |\langle J \rangle_{\Sigma(\sigma)} - \langle J \rangle_\sigma| \leq m(r)|\partial Z| \quad (8) \]

Proof. We split $|\langle J \rangle_{\Sigma(\sigma)} - \langle J \rangle_\sigma| \leq \sum_X |\langle jX \rangle_{\Sigma(\sigma)} - \langle jX \rangle_\sigma|$ and we consider three (non-disjoint) classes of sets $X$. 

a) $\text{dist}(X, Z) \geq r/2$

b) $\text{dist}(X, \partial Z) \leq r/2$
c) \( \text{dist}(X, Z^c) \geq r/2 \)

We estimate the contribution of each class and we get in each case the desired bound.

\textbf{class a}

\[
\sum_{X \in \text{class a}} |\langle j_X \rangle_{\Sigma_Z(\sigma)} - \langle j_X \rangle_{\sigma}| \leq \sum_{x: \text{dist}(x, Z) > r/2} m(\text{dist}(x, Z)) \sum_{X \ni x} ||j_X||
\]

\[
\leq C \sum_{x: \text{dist}(x, Z) > r/2} m(\text{dist}(x, Z)) \leq m(r)|\partial Z|
\]

where we used property 1 of Definition 1 to get the first inequality and the rapid decay of \( m \) and finite dimension of \( \Gamma \) to get the third inequality.

\textbf{class b}

\[
\sum_{X \in \text{class b}} |\langle j_X \rangle_{\Sigma_Z(\sigma)} - \langle j_X \rangle_{\sigma}| \leq 2 \sum_{x: (\partial Z)_{r/2}} \sum_{X \ni x: \text{diam}(X) \geq r/2} ||j_X|| \leq m(r)|\partial Z|
\]

The first inequality follows because \( j_X = 0 \) unless \( X \cap \Gamma_j \neq \emptyset \). The second inequality follows because \((\partial Z)_{r/2} \leq Cr^d\) and this polynomial factor can be absorbed in \( m \).

\textbf{class c}

\[
\sum_{X \in \text{class c}} |\langle j_X \rangle_{\Sigma_Z(\sigma)} - \langle j_X \rangle_{\sigma}| \leq 2 \sum_{k \geq r/2} \sum_{x: \text{dist}(x, Z^c) = k} \sum_{X \ni x: \text{diam}(X) \geq k} ||j_X||
\]

\[
\leq \sum_{k \geq r/2} \sum_{x: \text{dist}(x, Z^c) = k} m(k) \leq \sum_{k \geq r/2} Ck^d|\partial Z|m(k) \leq m(r)|\partial Z|
\]

The first inequality follows because \( j_X = 0 \) unless \( X \cap \Gamma_j \neq \emptyset \). The second inequality follows because the function \( m(t) \) is decreasing and so \( \sum_{X \ni x: \text{diam}(X) \geq k} ||j_X|| \leq ||j||m_j m_j(k) \). In the last inequality, we again absorbed a polynomial factor in \( m \).

\[\square\]

### 3.3 Isoperimetry

We will use Lemma’s 3 and 4 to relate the energy in a region \( Z \) to the energy around \( \partial Z \), hence the title of this subsection. We again choose the region \( Z \) to be far away from \( \Gamma_j \): \( \text{dist}(Z, \Gamma_j) \geq r \) with \( r \gg 1 \). We take a density matrix \( \sigma \) that is a minimizer of \( H + J \), i.e.

\[
\langle H + J \rangle_{\Sigma_Z(\sigma)} - \langle H + J \rangle_{\sigma} \geq 0.
\]

(9)

By Lemma 4 we can discard \( J \) in the above inequality at small cost, and we obtain

\[
\langle H \rangle_{\Sigma_Z(\sigma)} - \langle H \rangle_{\sigma} + m(r)|\partial Z| \geq 0
\]

(10)

We split this inequality as

\[
\langle H_{Z \setminus (\partial Z)} \rangle_{\Sigma_Z(\sigma)} - \langle H_{Z \setminus (\partial Z)} \rangle_{\sigma} + \sum_{X \not\subseteq Z \setminus (\partial Z)} \langle (h_X)_{\Sigma_Z(\sigma)} - (h_X)_{\sigma} \rangle + m(r)|\partial Z| \geq 0
\]

(11)

from which we get
Lemma 5.

\[ \langle H_{Z \setminus \partial Z} \rangle_\sigma \leq C r^{d + d_\gamma} \langle H_{\partial Z} \rangle_\sigma + m(r) |\partial Z| \]

Proof. We start from (11). By the same easy reasoning as used in the proof of lemma 3 for case a, we bound the term \( \langle H_{Z \setminus \partial Z} \rangle_\Sigma(\sigma) \) by \( m(r) |\partial Z| \). Then the upper bound of Lemma 3 gives us immediately the desired claim.

Let us now see how Lemma 5 can help us. We will consider a sequence of regions \( Z_i \) chosen as concentric balls \( Z_i = B_{R_i + r} = B_{R_i + r}(x) \), with \( R_i \) to be specified. With this choice, we will get information on

\[ B_{R_i} = Z_i \setminus (\partial Z_i)_r \]

and so we abbreviate

\[ E_i = \langle H_{B_{R_i}} \rangle_\sigma, \quad \delta_i = \langle H_{(\partial B_{R_i})_r} \rangle_\sigma \]

Then, the inequality in Lemma 5 reads

\[ E_i \leq C r^{d + d_\gamma} \delta_i + m(r) |\partial B_{R_i + r}| \]

(12)

Note also that \( E_{i+1} \geq E_i + \delta_i \) provided that \( R_{i+1} \geq R_i + 2r \). Our strategy will be to establish a lower bound on \( E_i \), depending on \( E_1 \), and then eventually use the a priori upper bound

\[ E_i = \langle H_{B_{R_i}} \rangle_\sigma < C R_i^d \]

to get an upper bound on \( E_1 \). Proceeding in this way, we obtain

Lemma 6. Let \( R \) be the distance from \( x \) (the center of the balls) to \( \Gamma_j \). Then, for any \( w > 0 \)

\[ \langle H_{B_{Rw}} \rangle_\sigma \leq C(w) e^{-Rw-w}, \quad p = \frac{1}{d + d_\gamma + 2}. \]

Proof. We choose \( R_i = (2i - 2)r \) for \( i = 1, \ldots, i_* \) with \( i_* \) the largest integer that is smaller than \( R/(2r) \). Using inequality (12) and \( E_{i+1} \geq E_i + \delta_i \) we obtain

\[ E_{i+1} \geq a E_i - b_i, \quad a = (1 + cr^{-d-\gamma}), \quad b_i = m(r) |\partial B_{R_i + r}| \]

(13)

We will now use a (discrete) Grönwall inequality. Multiplying the inequality (13) by \( a^{-i} \) and summing over \( i \) yields

\[ \sum_{i=1}^k E_{i+1} a^{-i} \geq \sum_{i=1}^k a^{-i+1} E_i + \sum_{i=1}^k b_i a^{-i}, \]

which implies

\[ E_i \geq a^{-i} E_1 - \sum_{j=1}^{i-1} b_j a^{-j} a^{i-1}. \]

and hence, choosing \( i = i_* \),

\[ E_1 \leq E_{i_*} a^{-i_*+1} + \sum_{j=1}^{i_*-1} b_j a^{-j}. \]

To estimate the sum on the right hand side, we bound \( |\partial B_{R_j+r}| \leq |B_{R_j+r}| \leq C(2jr)^d \) and estimate

\[ \sum_{j=1}^{i_*-1} b_j a^{-j} \leq m(r) r^d \sum_{j=1}^\infty (3j)^d a^{-j} \leq m(r). \]
Using in addition the a priori bound \( \langle H_{B_R} \rangle_\sigma \leq CR^d \) we get

\[
E_1 \leq CR^d a^{-\frac{R}{2\pi}} + m(r) \leq CR^d e^{-(d+\gamma+1)} + m(r).
\]

To optimize the inequality we let \( r \) grow with \( R \). Inspecting the definition of the class of functions \( M \), we can upper bound \( m(r) \leq C(b)e^{-r^{1-b}} \) for any \( 0 < b < 1 \). Therefore, we can choose

\[
cR^{-(d+\gamma+1)} = r^{1-b/2}
\]

and obtain

\[
E_1 \leq C(b)e^{-(cR)^{\frac{2-b/2}{2+d+\gamma}}}.
\]

The bound then yields the claim of the lemma upon relating parameters \( w \) and \( b \).

\( \square \)

**Conclusion of the proof.** We start from Lemma 3 which gives a bound of the form

\[
\langle H_{B_r} \rangle_\sigma \leq \epsilon
\]

By using the local gap assumption 3 we get

\[
\langle \bar{P}_{B_r} \rangle_\sigma \leq Cr^{d\gamma}\epsilon
\]

and by using the OBC regularity assumption 2 we get

\[
|\sigma - \mu|_{B_{R-k}} \leq Cr^{d\gamma}\epsilon + p(r)m(k)
\]

In our case, we choose \( r = R^p \) and \( k = R^{w_1} \) with a small \( w_1 > 0 \). Then we get

\[
|\sigma - \mu|_{B_{R^p-w_1}} \leq C(w, w_1)e^{-R^{b-2/2}}
\]

Since \( w, w_1 \) are arbitrary, we can just as well replace \( w_1 \to w, w/2 \to w \). This yields the statement of Theorem 1

\( \square \)

## 4 Appendix: Locality estimates

We review the standard propagation bounds that are necessary for the proofs of Lemma 3. These bounds go back to [18] but we use the recent formulation in [20].

### 4.1 Functions \( \mathbb{N}^+ \to \mathbb{R}^+ \)

We define transformations \( M, S, P \) on bounded functions \( \mathbb{N}^+ \to \mathbb{R}^+ \);

- \( M(f)(r) = \max_{r' \geq r} f(r') \)
- \( P(f)(r) = (1 + C \Gamma r^d)f(r) \)
- \( S(f)(r) = \sup_{t \in \mathbb{N}^+} \sup_{(r_1, \ldots, r_t) \in (\mathbb{N}^+)^t} \prod_i f(r_i) \)
The use of $P$ is to bound and abbreviate expressions like the following, for $S \subset \Gamma$,

$$|S|f(\text{diam}(S) + 1) \leq (1 + C_\Gamma r^d)f(r) = Pf(r), \quad r = \text{diam}(S) + 1,$$

where $C_\Gamma$ is the geometric constant introduced in Section 2.1.1. The definition of $S(f)$ is taken from [4] and its use lies in the fact that $f' = S(f)$, for $f < 1$, is logarithmically superadditive, i.e,

$$f'(r_1)f'(r_2) \leq f'(r_1 + r_2) \quad r_1, r_2 \in \mathbb{N}^+.$$  \hspace{1cm} (14)

We now consider the following transformations $f \mapsto f'$. Let $a \in \mathbb{N}^+, b \in \mathbb{N}$.

1. $f' = M(pf)$ for some polynomial $p$ with positive coefficients;
2. $f'(r) = M\left\{f\left(\frac{r-b}{a}\right) \quad \frac{r-b}{a} \in \mathbb{N}^+ \right. \left. 0 \quad \text{otherwise}\right\}$
3. $f' = S(f)$ for $f < 1$.

These transformations have the following important property, referring to the class $\mathcal{M}$ defined in Section 2.1.2.

**Lemma 7.** If $f \in \mathcal{M}$, then $f'$ is in $\mathcal{M}$ as well.

**Proof.** We only comment on the proof of item 3. If $f \leq F$ with $F$ logarithmically superadditive, then $S(f) \leq S(F) = F$. Since $f < 1$ and $f \in \mathcal{M}$, we can find $c_\alpha$ small enough so that $f \leq e^{-c_\alpha r^\alpha}$. The map $r \mapsto e^{-c_\alpha r^\alpha}$ is logarithmically superadditive for $\alpha < 1$ and the claim follows. \( \square \)

Finally, we will often use without comment that $\max(f_1, f_2) \in \mathcal{M}$ if $f_1, f_2 \in \mathcal{M}$.

### 4.2 Lieb-Robinson bounds

#### 4.2.1 Local decompositions

We define a canonical decomposition of observables $A \in \mathcal{A}$ into finitely supported terms centered at some $x \in \Gamma$. We set

$$A_{x,r} = \begin{cases} 
\mathbb{E}_{B^c_{r}(x)}[A] - \mathbb{E}_{B^c_{r-1}(x)}[A] & r > 0, \\
\mathbb{E}_{B^c_{r}(x)}[A] & r = 0.
\end{cases} \hspace{1cm} (15)$$

Then $A = \sum_{r \geq 0} A_{x,r}$ and

$$||A_{x,r}|| \leq ||A - \mathbb{E}_{B^c_{r}(x)}[A]|| + ||A - \mathbb{E}_{B^c_{r-1}(x)}[A]||, \quad ||A_{x,0}|| \leq ||A|| \hspace{1cm} (16)$$

and the terms on the right-hand side will in practice be bounded by the following standard reasoning.

**Lemma 8.** For any $A \in \mathcal{A}$,

$$||A - \mathbb{E}_{Z}[A]|| \leq \sup_{O \in A_{Z^c}, ||O||=1} ||[O, A]||.$$

**Proof.** $\mathbb{E}_X[A] = \int dUU^*AU$ with $dU$ the Haar measure on unitaries in $\mathcal{A}_X$. \( \square \)
4.2.2 Evolution of observables

Let \( z(\cdot) \) be a time-dependent family of interactions, as in section Section 4.1, such that \( s \mapsto z_s(s) \) is measurable and such that, for a non-increasing \( f_z : \mathbb{N}^+ \to \mathbb{R}^+ \),

\[
||z||_{f_z} = \sup_s ||z_s||_{f_z} < \infty.
\]

We let \( \alpha_z(s)[\cdot] \) be the dynamics (a family of automorphisms \( A \to A \)) generated by the family \( z(\cdot) \), in the sense that

\[
\alpha_z(s)(A) = A + i \int_0^s du \alpha_z(s)[[H_z(u), A]], \quad H_z(u) = \sum_{S \subset \Gamma} z_s(u).
\]

The existence and uniqueness of this dynamics follows from elementary facts on matrix-valued ODE’s. We first state a version of the Lieb-Robinson bound, using the language introduced in section 4.1.

**Lemma 9.** Let

\[
h = S(MP f_z)^{v(f_z)}), \quad v(f) = 2 \sup_r (P f(r))
\]

Then, for any \( A \in A_X \) and \( B \in A_Y \),

\[
||[B, \alpha_z(1)(A)]|| \leq 2 e^{v(f_z)} ||z||_{f_z} ||A|| ||B|| (\text{dist}(X, Y)).
\]

Note that if \( f_z \in \mathcal{M} \) then \( h \in \mathcal{M} \) and \( v(f_z) < \infty \).

4.2.3 Evolution of interactions

Let \( g \) be an interaction and let \( z = (z_s) \) be as above. Let \( g \) be anchored in a region \( \Gamma_g \subset \Gamma \) in the sense that \( g_S = 0 \) unless \( S \cap \Gamma_g \neq \emptyset \). Note that we allow also \( \Gamma_g = \Gamma \). We then define the evolved interaction \( \alpha_s[g] \) by

\[
(\alpha_s[g])_{B_r(x)} = \sum_{X \ni x} \frac{1}{|X \cap \Gamma_g|} (\alpha_z(s)[g_X])_{x,r}
\]

whenever \( x \in \Gamma_g \) and \( (\alpha_s[g])_{S} = 0 \) for \( S \) that are not of the form \( B_r(x), x \in \Gamma_g \). We say that the interaction is centralized in \( \Gamma_g \) and we note that ‘centralized’ implies ‘anchored’. The motivation for definition (18) is that it gives the correct time-evolution on the level of the total Hamiltonian, i.e.,

\[
H_{\alpha_z(s)[g]} = \sum_{S \subset \Gamma} (\alpha_z(s)[g])_S = \alpha_z(s)[H_g], \quad H_g = \sum_{S \subset \Gamma} g_S
\]

but one should keep in mind that this requirement does of course not fix the definition \( \alpha_s(g) \) uniquely.

**Lemma 10.** Let

\[
h = S(MP f_g, MP f_z)^{v(f_g, v(f_z))}, \quad v(f) = 2 \sup_r (P f(r))
\]

\[
f'(r) = M \left\{ \begin{array}{ll} (\frac{r}{2})^2 P h & \frac{r}{2} \in \mathbb{N}^+ \\ 0 & \text{otherwise} \end{array} \right.
\]

Then

\[
||\alpha_s(g)||_{f'} \leq 12 e^{v(f_z)} ||z||_{f_z} ||v(f_g)|| ||g||_{f_g}
\]

Note that \( f' \in \mathcal{M} \) if \( f_z, f_g \in \mathcal{M} \).
4.3 Proof of Lemma 10

We will start from the bound (22) below. This bound follows from equations (3.41-3.42) of [20] once one proves that the remainder term \( R_N(t) \) appearing in (3.41) in [20] vanishes as \( N \to \infty \). The vanishing of this remainder term is proven analogously to the bounds that we derive below and so we do not comment on this further.

For \( A \in \mathcal{A}_{S_0} \) and \( O \in \mathcal{A}_{B_{r-1}(x)^c} \) with \( r \in \mathbb{N}^+ \),

\[
\frac{||\langle O, \alpha_s(A) \rangle||}{||O||} \leq 2 ||A|| \sum_{n \geq 0} \sum_{S_{j+1} \sim S_j, j=0, \ldots, n} \int_s^s du ||z_{S_n}(u_n)|| \ldots ||z_{S_1}(u_1)||,
\]  

(22)

where

1. we denote \( S \sim S' \) whenever \( S, S' \) intersect.
2. \( \int_s^s du \) is shorthand for \( \int_0^s du \int_{u_1}^s du_2 \ldots \int_{u_{n-1}}^s du_n \).
3. we introduced the dummy \( S_{n+1} = B_{r-1}^c \).

In particular, for \( n = 0 \), the sole condition is that \( S_0 \) intersects \( B_{r-1}^c \); If the condition holds we interpret the inside sum as being equal to 1. We now take \( A = \sum_{S_0 \ni x} \frac{1}{|S_0|} g_{S_0} \). Then the bound above, linearity of the commutator and the triangle inequality yield

\[
t_r := \sup_{O \in \mathcal{A}_{B_{r-1}^c}} \frac{||\langle O, \alpha_s(A) \rangle||}{||O||} \leq 2 \sum_{n \geq 0} \sum_{S_{j+1} \sim S_j, j=0, \ldots, n} \int_s^s du ||z_{S_n}(u_n)|| \ldots ||z_{S_1}(u_1)|| \frac{1}{|S_0|} ||g_{S_0}||.
\]

(23)

where we have introduced the dummy \( S_{-1} = \{x\} \), i.e. \( S_0 \) is from now on constrained to satisfy \( S_0 \sim \{x\} \). Recalling the definition of \( h \) in (20),

\[
\frac{f_g(\text{diam}(S_0) + 1)}{v(f_g)} \prod_{j=1}^n \frac{|S_j| f(\text{diam}(S_j) + 1)}{v(f_z)} \leq \prod_{j=1}^n h(\text{diam}(S_j) + 1) \leq h(\sum_{j=0}^n \text{diam}(S_j) + n + 1) \leq h(r)
\]

(24)

where the second inequality is because \( h \) is logarithmically superadditive and the last inequality is because \( \sum_{j=0}^n \text{diam}(S_j) \geq r + n \) and \( h \) is non-increasing. Hence we get, dropping the condition \( S_{n+1} \sim S_n \).

\[
\frac{t_r}{h(r)} \leq 2 \sum_{n \geq 0} \sum_{S_{j+1} \sim S_j, j=0, \ldots, n-1} \frac{v(f_g) ||g_{S_0}||}{|S_0| f_g(\text{diam}(S_0) + 1)} \int_s^s du \prod_{j=1}^n \frac{v(f_z) ||z_{S_j}(u_j)||}{|S_j| f_z(\text{diam}(S_j) + 1)}
\]

We can now perform iteratively the sum \( S_j \) iteratively, starting at \( j = n \) and using that

\[
\frac{1}{|S_{j-1}|} \sum_{S_{j-1} \sim S_{j-1}} \frac{||z_{S_j}(u)||}{f_z(1 + \text{diam}(S_j))} \leq ||z(u)||_{f_z},
\]

and for the last factor \( j = 0 \),

\[
\sum_{S_0 \ni x} \frac{||g_{S_0}||}{f_g(\text{diam}(S_0) + 1)} \leq ||g||_{f_g}.
\]
Taking sup over $u$ and performing the integrals, we get
\[
\frac{t_r}{h(r)} \leq 2 \sum_{n \geq 0} \frac{s^n}{n!} ||z|| f_z^n ||g|| f_g \leq 2e^{sv(f_z)} ||z|| f_z v(f_g) \leq 2e^{sv(f_z)} ||z|| f_z v(f_g) =: R. \tag{25}
\]

From the bound (10) and Lemma 8 we conclude that, with $K_{x,r} = (\alpha_s(A))_{x,r}$
\[
||K_{x,r}|| \leq t_r + t_{r+1} \leq 2t_r \leq 2h(r)R.
\]

To convert these bounds into a bound on $||\alpha_s(g)||$, we note that, for a non-increasing $f'$,
\[
\sum_{s \geq x} \frac{1}{f'(\text{diam}(S) + 1)} ||(\alpha_s(g))_s|| \leq \frac{1}{f'(1)} ||K_{x,0}|| + \sum_{r \in \mathbb{N}^+} \sum_{\text{dist}(y,x) = r} \frac{1}{f'(2r + 1)} ||K_{y,r}|| \tag{26}
\]
\[
\leq \frac{2h(1)}{f'(1)} R + 2 \sum_{r \in \mathbb{N}^+} \frac{1 + C_tr^d}{f'(2r + 1)} h(r)R. \tag{27}
\]

We choose $f'$ as in (21), so that in particular $f'(1) \geq (1 + C_t)h(1)$ and
\[
f'(2r + 1) \geq r^2(1 + C_tr^d)h(r).
\]

Then (27) is bounded by $2R + \frac{2\pi^2}{6} R \leq 6R$ from which the lemma follows.

### 4.4 Proof of Lemma 9

The proof starts proceeds analogously as the proof of Lemma 10 and we comment on the differences. Here, we do not replace $A$ by $\sum_{S_0 \geq x} \frac{1}{s_0} g_{S_0}$ and there is no sum over $S_0$, since $S_0$ is now the fixed set $X$. Similarly, $S_{n+1}$ is now a fixed set $Y$. Also, the sum over $n$ runs from 1 instead of 0. The definition of the function $h$ is different as it is now given by (17) and instead of the bound (23), we have
\[
\prod_{j=1}^{n} \frac{|S_j| f(\text{diam}(S_j) + 1)}{v(f_z)} \leq \prod_{j=1}^{n} h(\text{diam}(S_j) + 1) \leq h(\sum_{j=1}^{n} \text{diam}(S_j) + n) \leq h(\text{dist}(X,Y)). \tag{28}
\]

Next, to perform the sums in (24), we introduce $1 = |X|/|X|$. The factor $1/|X|$ is used to control the sum $\sum_{S_1 \sim S_0}$ and the factor $|X|$ appears then in our final result.

### 4.5 Proof of Lemma 1

We use the technical lemma’s 9 and 10 to prove Lemma 1 in the main text.

The first claim of the lemma does not depend on the specific form of $V$. It only requires that $V$ is generated by an interaction with a finite $f$-norm for some $f \in \mathcal{M}$. For an automorphism $\alpha_z \equiv \alpha_z(1)$ and $O \in \mathcal{A}_X$ we have using Lemma 8
\[
||\alpha_z(O) - E_{X^z} \alpha_z(O)|| \leq \sup_{U \in \mathcal{A}_X} ||[U, \alpha_z(O)]||.
\]

Using Lemma 9 to estimate the commutator on the RHS we then get
\[
||\alpha_z(O) - E_{X^z} \alpha_z(O)|| \leq C_z||O|||X|h(r).
\]
Using this for $\alpha_z$ such that $\alpha_z(O) = V^*OV$ resp. $VOV^*$ proves the first part of the lemma.

We proceed to prove the second claim. We will use $z = q$ and $z = \hat{q}$. By Duhamel formula,

$$\alpha_q(s)(A) = \alpha_q(s)(A) + i \int_0^s \alpha_q(u)[H_q(u) - H_q(u), \alpha_q(s, u)(A)] du.$$

This implies that

$$V^*OV - O = (\alpha_q(1))^{-1} \alpha_q(1)(O) - O = i(\alpha_q(1))^{-1} \int_0^1 \alpha_q(u)[H_q(u) - H_q(u), \alpha_q(1, u)(O)] du,$$

and hence

$$\|V^*OV - O\| \leq \sup_{0 \leq u \leq 1} \|[\alpha_q^{-1}(1, u)(H_q(u) - H_q(u)), O]\|.$$ 

Using the concept of time evolved interactions introduced above we write $\alpha_q^{-1}(1, u)(H_q(u) - H_q(u)) = H_{l(s)}$ with $l(s) = \alpha_q^{-1}(1, s)[z(s)]$ where $z(s) = \hat{q}(s) - q(s)$.

We now use Lemma 10 to find a $m_l \in M$ such that $||l||_{m_l} = \sup_{s \in [0, 1]} ||l(s)||_{m_l} < \infty$. Next, we argue that $l(s)$ can be chosen to be anchored in $\partial Z$ (without weakening of the above bounds). This follows from the following general observation, relying on the fact that the distance $\text{dist}(\cdot, \cdot)$ on $\Gamma$ is the graph distance. For any set $X \subset \Gamma$, we can find $X' \supset X$ such that $\text{diam}(X') = \text{diam}(X)$ and such that $X'$ is connected. We can then modify any given interaction $g$ into an interaction $g'$ such that $||g'||_m \leq ||g||_m$ (possibly both infinite) for any $m \in M$ and such that $g'_{X'} = 0$ unless $X$ is connected. Moreover, since $g'_{X'}$ is obtained from $g_X$ by a finite sum over $X$, measurability in some parameter is preserved. In particular, we do this modification for the family of interactions $q(s)$ featuring in Assumption 4. A consequence of this is that the interactions $\hat{q}(s) - q(s)$ are anchored in $\partial Z$. It then follows that both the families $z(s)$ and $l(s)$ are centralized in $\partial Z$. Now we can estimate, uniformly in $s$,

$$||[H_{l(s)}, O]\| \leq 2 \sum_{x \in \partial Z} m_l(\text{dist}(X, x)) \sum_{S \supset X} \frac{||l(S)||_m||O||}{m_l(\text{diam}(S) + 1)} \leq 2||O||_m ||l||_{m_l} \sum_{x \in \partial Z} m_l(\text{dist}(X, x))$$

where we used that $l$ is centralized in $\partial Z$ to bound $\text{diam}(S) + 1 \geq \text{dist}(x, X)$. By the finite-dimensionality of $\Gamma$, the sum $\sum_{x \in \partial Z} m_l(\text{dist}(X, x))$ is easily bounded by $|X|m(\text{dist}(X, \partial Z)$ for some $m \in M$. This proves the second claim of Lemma 10 the bound (10).

As a final remark, we note that the notion of time evolved interactions and Lemma 10 were crucial to prove (8) with the factor $|X|$ on the RHS. For example, proving the same bound with a factor $|X|^2$ instead of $|X|$ would only require Lemma 9.

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