Ethical Dilemmas of Strategic Coalitions

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ABSTRACT

A coalition of agents, or a single agent, has an ethical dilemma between several statements if each joint action of the coalition forces at least one specific statement among them to be true. For example, any action in the trolley dilemma forces one specific group of people to die. In many cases, agents face ethical dilemmas because they are restricted in the amount of the resources they are ready to sacrifice to overcome the dilemma. The paper presents a sound and complete modal logical system that describes properties of dilemmas for a given limit on a sacrifice.

1 INTRODUCTION

In this paper we study ethical dilemmas faced by agents and coalitions of agents in multiagent systems. Consider, for example, a setting in which the entire populations of four remote villages: $v_1$, $v_2$, $v_3$, and $v_4$ are affected by an epidemic caused by diseases $a$ and $b$. Villages $v_1$, $v_2$, and $v_3$ are affected by disease $a$ and villages $v_2$, $v_3$, and $v_4$ are affected by disease $b$, see Figure 1. The medications against diseases $a$ and $b$ are produced by pharmaceutical companies $m_a$ and $m_b$, respectively. The entire population of a village will be wiped out unless each resident of the village is given medicine against all diseases her village is affected by. As a part of a philanthropic initiative, each of the manufacturers $m_a$ and $m_b$ is ready to donate enough medicine to eradicate diseases $a$ and $b$, respectively, from any two villages.

Let us assume that manufacturer $m_a$ is distributing the medications directly. Thus, it is facing a moral dilemma: $m_a$ must decide which two out of the three villages $v_1$, $v_2$, and $v_3$ it should send the medicine to. No matter what decision it makes, this decision will guarantee the death of an entire population of one of the villages. We write this dilemma as

$$[m_a : d_1, d_2, d_3]_{m_a, m_b \rightarrow 2.2},$$

where statement $d_i$ stands for “the entire population of village $v_i$ is dead”. In general, statement $[C: \varphi_1, \ldots, \varphi_n]$ means that coalition $C$ is facing a moral dilemma because each joint action of coalition $C$ that satisfies the constraints imposed by a sacrifice function $s$ predetermines at least one specific statement among $\varphi_1, \ldots, \varphi_n$ to be true. This type of moral dilemma is referred to in the literature as the “trolley dilemma” [28, 51, 52]. Consider now statement $r = "There is going to be rain tomorrow."$ Note that each action of $m_a$, satisfying the constraints $m_a, m_b \rightarrow 2.2$ also predetermines that one of the four outcomes $d_1, d_2, d_3, r$ will be true. However, the actual dilemma $m_a$ faces is only between the first three alternatives. To capture this intuition, we will assume that, in order for statement $[C: \varphi_1, \ldots, \varphi_n]$ to be true, two conditions must hold: (i) each joint action of coalition $C$ that satisfies the constraints imposed by a sacrifice function $s$ predetermines at least one specific statement among $\varphi_1, \ldots, \varphi_n$ to be true and (ii) set $\varphi_1, \ldots, \varphi_n$ must be a minimal set for which condition (i) holds.

Suppose that manufacturer $m_a$ delegates the distribution of its medication to two helper organizations. Organization $h_1$ distributes medications between villages $v_1$ and $v_2$ and organization $h_2$ between $v_2$ and $v_4$, see Figure 1. Manufacturer $m_b$ has no control over how these organizations distribute the medications given to them or whether they decide to distribute them at all. Then, the action of $m_b$ does not always predetermine which one of $v_2, v_3$, or $v_4$ dies:

$$\neg [m_a : d_2, d_3, d_4]_{m_a, m_b \rightarrow 2.2}.$$

However each possible action of manufacturer $m_b$ guarantees either the death of $v_2$ or the death of $v_3$ or $v_4$:

$$[m_b : d_2, d_3 \lor d_4]_{m_a, m_b \rightarrow 2.2}.$$

Next, suppose that manufacturer $m_b$ is ready to donate enough medicine to eradicate disease $b$ in only one village. Then each possible action of manufacturer $m_b$ will guarantee either the death of $v_2$ or the death of both $v_3$ and $v_4$:

$$[m_b : d_2, d_3 \land d_4]_{m_a, m_b \rightarrow 2.1}.$$

Figure 1: Arrows represent distribution channels.

Figure 2: $m_a$ and $m'_a$ produce medication against disease $a$.

As the last example, consider the situation depicted in Figure 2. Here, all four villages are affected by the same disease $a$. The medication against this disease is produced by manufacturers $m_a$ and $m'_a$. Suppose that manufacturer $m_a$ is ready to donate enough medicine to eradicate disease $a$ from any two villages while manufacturer $m'_a$ is only ready to donate enough medicine to eradicate the disease.
from a single village. If manufacturer \( m_a \) sends the medicine to, say, villages \( v_1 \) and \( v_2 \), then no specific village among \( v_3 \) and \( v_4 \) is guaranteed to die because manufacturer \( m_a' \) might save it:

\[ \neg [m_a : d_1, d_2, d_3, d_4]_{m_a, m_a' \rightarrow 2.1}. \]

Similarly, \( \neg [m_a' : d_1, d_2, d_3, d_4]_{m_a, m_a' \rightarrow 2.1} \). However, no matter how manufacturers \( m_a \) and \( m_a' \) coordinate their actions, one specific village among \( v_1 \), \( v_2 \), \( v_3 \), and \( v_4 \) is guaranteed to die:

\[ [m_a, m_a' : d_1, d_2, d_3, d_4]_{m_a, m_a' \rightarrow 2.1}. \]

Ethical dilemmas without explicit sacrifice are faced by self-driving cars choosing between outcomes where different people are harmed [39, 44] and by policy makers balancing public safety with individual rights to privacy [9]. Dilemmas with sacrifice (constraints on time and resources) are faced by doctors in emergency hurricane evacuations [27] and by designers of organ transplantation policies [18]. Financial constraints also create dilemmas for policy makers funding different directions of medical research [49]. As AI technology matures, more of these moral choices will be made by AI agents. As a result, society is tasked with developing laws that regulate choices made by such agents. Some of this work has already begun. For example, Germany introduced ethics rules for autonomous vehicles [25].

Our work provides a framework for specifying and reasoning about ethical dilemmas, rather than suggesting how AI agents should resolve them. Such frameworks could be used as the language for writing new laws and regulations as well as to interpret the existing laws as they apply to AI systems. The main technical result of our work is a complete logical system that describes the properties of the ethical dilemma modality \([C : \varphi_1, \ldots, \varphi_n]\).

2 OVERVIEW

The rest of the paper is organized as follows. In the next section we review the related literature. In Section 4, we define the syntax and the semantics of our logical system. In Section 5, we list and discuss its axioms and inference rules, as well as state its soundness. Section 6 contains the proof of the completeness. Section 7 concludes the paper.

3 LITERATURE REVIEW

The ethical dilemmas that we study in this paper are usually referred to in the literature as variations of the "trolley dilemma". The original trolley dilemma was proposed by Foot [28] as a dilemma faced by an agent who must choose between allowing five people to die and killing one person to prevent the death of those five. The distinction between letting one die and killing someone is also emphasised by Thomson [51, 52] as well as by Bruers and Braeckman [19]. Navarrete, McDonald, Mott, and Asher study the same distinction in a virtual reality environment [43].

At the same time, others shift the focus of the trolley problem away from the distinction between letting things happen and making things happen. Marczyk and Marks empirically studied whether perceived moral permissibility changed when the person making a decision in the trolley dilemma stands to benefit from or be harmed by one of the outcomes [40]. Pan and Slater analysed participants’ ethical reasoning when they were confronted with the trolley dilemma through an online survey versus through immersive virtual realities [46]. Chen, Qiu, Li, and Zhang examined the differences in brain activity of Chinese undergraduates who experienced the great Sichuan earthquake when confronted with trolley dilemmic situations where they must choose to rescue one of two relatives and one of two strangers [24]. Indick, Kim, Oelberger, and Semino investigated how the gender of a person affects the decision that she makes in the trolley dilemma-like settings [37]. Bleske-Rechek, Nelson, Baker, Remiker, and Brandt observed that people are less likely to sacrifice the life of one person for the lives of five if the one person is young, a genetic relative, or a current romantic partner [15]. In a related work, Kawai, Kubo, and Kubo-Kawai showed that most people are more inclined to sacrifice an older person rather than a younger one [38]. In this paper, we also consider trolley-like ethical dilemmas in this broader sense.

Although we are not aware of any works treating ethical dilemma as a modality, there are papers that use existing logical formalism to capture ethical dilemmas. Berreby, Bourgne, and Ganascia used simplified event calculus to model dilemmas within answer set programming [12]. Horty suggested using nonmonotonic logic for reasoning about moral dilemmas [34]. Marc Pauly proposed a logic of coalition power that captures properties of modality "coalition \( C \) has a strategy to achieve \( \varphi \)" [47, 48]. His approach has been widely studied in the literature [1, 2, 10, 16, 29, 31, 42, 50, 53]. An alternative logical system for coalition power was proposed in [41]. Alur, Henzinger, and Kupferman introduced Alternating-Time Temporal Logic (ATL) that combines temporal and coalition modalities [7]. Goranko and van Drimmelen [32] gave a complete axiomatization of ATL. Additionally, decidability and model checking problems for ATL-like systems has also been studied in recent works [8, 13, 14]. Another approach to express "power to achieve" in a temporal setting is STIT logic [11, 33, 35, 36, 45]. Broersen, Herzig, and Troquard have shown that coalition logic can be embedded into a variation of STIT logic with the temporal modality "next-step" [17].

A logical system for constraints by a renewable resource, such as money, was proposed by Cao and Naumov [23]. Their system labels modalities with three parameters: agent, profit, and budget. Unlike this paper, they deal only with power of a single agent, not a coalition of agents. Alechina, Logan, Nguyen, and Rakib introduced resource bounded coalitional logic (RBCL) [6]. Many recent works on resource bounded coalitions have been focused on complexity of model-checking for different versions of RBCL and ATL with bounded resources [3–5, 20–22, 26].

The ethical dilemma modality \([C : X]\), even without the sacrifice subscript \( s \), cannot be expressed in Marc Pauly logic or any of its modifications mentioned above. However, this modality, without the sacrifice subscript, can be expressed via socially friendly coalition power modality introduced by Goranko and Enqvist [30]. They proposed several versions of social friendly modality. The basic one, \([C :(\varphi_1, \ldots, \varphi_n)]\) stands for "coalition \( C \) has an action profile that guarantees \( \varphi \) and enables the complementary coalition \( C \) to realise any one of \( \varphi_1, \ldots, \varphi_n \) by a suitable action profile". Our modality without the sacrifice function is expressible through socially friendly modality as

\[ [C : \varphi_1, \ldots, \varphi_n] \equiv [C : (\varphi_1, \ldots, \varphi_n)] \land \bigwedge_{D \subseteq C} \neg [D : (\varphi_1, \ldots, \varphi_n)]. \]
The logical system proposed by Goranko and Enqvist [30], unlike ours, does not consider resource bounded actions. Thus, our modality \([C : X]\), with the sacrifice function \(s\) is not expressible in their system. They sketch the proof that their axiomatization of socially friendly modality is complete, but, unlike us, do not claim strong completeness. The completeness proofs here and in [30] use different constructions — see our discussion in Section 6. Additionally, none of the axioms in [30] is similar to our main axiom, the Combination axiom.

4 SYNTAX AND SEMANTICS

In this paper we assume a fixed set \(\mathcal{A}\) of agents. By a coalition we mean any nonempty subset of \(\mathcal{A}\). By \(X^Y\) we denote the set of all functions from set \(Y\) to set \(X\). Although in the introduction the value of the sacrifice function was a positive integer (number of villages the company is ready to help), in general we assume that sacrifice is an arbitrary real-valued function representing the maximal cost of the sacrifice that each individual agent is ready to bear. We allow the sacrifice to be negative. Negative sacrifice is the minimal profit that an agent expects from an action. In other words, sacrifice function is an arbitrary function from set \(\mathbb{R}\).

The language \(\Phi\) of our logical system is defined by the grammar

\[
\phi ::= p \mid \neg \phi \mid \phi \rightarrow \phi \mid [C : X]
\]

where \(C\) is a coalition, \(X\) is a nonempty set of formulae, and \(s\) is a sacrifice function. For the sake of simplicity, we abbreviate \([C : \{\phi_1, \ldots, \phi_n\}]\) as \([C : \phi_1, \ldots, \phi_n]\). We assume that Boolean connectives \(\land\) and \(\lor\) as well as constants truth \(\top\) and false \(\bot\) are defined as usual. By \(\land\) and \(\lor\) we denote the conjunction and the disjunction of all formulae in \(X\) respectively. As usual, \(\land\) and \(\lor\) are defined to be \(\top\) and \(\bot\), respectively.

**Definition 4.1.** A game is a tuple \((W, \Delta, \cdot, |\cdot, M, \pi)\), where

1. \(W\) is a set of states,
2. \(\Delta\) is a non-empty set of actions,
3. \(|d|_w \in \mathbb{R}\) is the “cost” of action \(d \in \Delta\) for agent \(a \in \mathcal{A}\) in state \(w \in W\),
4. \(M \subseteq W \times \Delta^\mathcal{A} \times W\) is a relation, called “mechanism”,
5. \(\pi(p)\) is a subset of \(W\) for each propositional variable \(p\).

In the introductory example depicted in Figure 1, the set \(W\) consists of an initial state \(init\) and 16 outcome states. Each outcome state specifies villages among \(v_1, v_2, v_3, v_4\) the entire population of which survives after the medicine is distributed. In other words, \(W = \{init\} \cup \{X \mid X \subseteq \{v_1, v_2, v_3, v_4\}\}\). The set of agents \(\mathcal{A}\) in this example is set \(\{m_a, m_b\}\). For the sake of simplicity, we do not consider helper organizations \(h_1\) and \(h_2\) as agents facing ethical dilemmas in this game. Note that, in the introductory example, different agents have different sets of available actions. Agent \(m_a\) decides to which villages it will send the medicine. Thus, the set of actions \(\Delta_a\) of agent \(m_a\) could be represented by sets \(\{X \mid X \subseteq \{v_1, v_2, v_3, v_4\}\}\). Agent \(m_b\) decides how much medicine each of the helpers \(h_1\) and \(h_2\) gets. Then, we can assume that the set of actions \(\Delta_b\) of agent \(m_b\) is \(\mathbb{N} \times \mathbb{N}\). For example, action \((1, 2) \in \Delta_b\) means that manufacturer \(m_b\) gives helper \(h_1\) just enough medicine to save one village and helper \(h_2\) just enough medicine to save two villages. Having agent-specific domains of actions is not significant. We can always consider a combined domain of actions and interpret any additional actions as one fixed “default” action in the original domain. For the sake of simplicity, Definition 4.1 assumes that the set of actions is the same for all agents.

In our example, the cost function \(|d|_w\) measures the amount of medicine donated by pharmaceutical company \(m\). In our example, the cost does not depend on the current state. In general, the cost may vary from state to state. We measure the amount of medicine by how many villages the medicine can eradicate the disease in. For example, \(|\{(v_1, v_2)\}|_{init}^m = 2\).

The mechanism \(M\) of a game specifies the rules by which the game transitions from one state to another. For example,

\[
\{(init, \{(m_a, \{v_1, v_2\}), (m_b, (1, 1))\}, \{v_1, v_2, v_4\}) \in M\}
\]

because, if in the initial state \(init\), manufacturer \(m_a\) sends medicine to villages \(v_1\) and \(v_2\) and manufacturer \(m_b\) gives a supply of medicine sufficient to save one village to helper \(h_1\) and the same amount to helper \(h_2\), then the game might transition into the state where only population of villages \(v_1, v_2,\) and \(v_4\) survives.

Note that our semantics is more general than those in Marc Pauly’s original works on coalitional logic [47, 48] and Goranko and Enqvist paper on socially friendly coalition power [30]. Namely, we assume that mechanism is a relation and not necessarily a function. In other words, we allow a complete action profile to transition the game into one of several different states. This means that our statement \([C : \phi_1, \ldots, \phi_2]\) does not imply that the complement of coalition \(C\) has a strategy to force each of the statements \(\phi_1, \ldots, \phi_2\). Goranko and Enqvist’s statement \([C \mid (\top, \phi_1, \ldots, \phi_n)]\) does imply this. Our approach also allows some complete action profiles to not result in any next state at all. We interpret this as a termination of the game.

Throughout this paper, we write \(f =_X g\) if \(f(x) = g(x)\) for each \(x \in X\). We also use shorthand notation captured in the following definition.

**Definition 4.2.** \(|\delta|_w \leq s\) if \(|\delta(a)|_w \leq s(a)\) for each agent \(a \in \mathcal{A}\).

The next definition is the key definition of this paper. Part (4) of it specifies the formal meaning of the ethical dilemma statement \([C : X]\). Item 4(a) states that any strategy of coalition \(C\) forces a specific statement \(\varphi\) in \(X\) to be true. Item 4(b) states that \(X\) is a minimal set with such property.

**Definition 4.3.** For each game \((W, \Delta, \cdot, |\cdot, M, \pi)\), each state \(w \in W\), and each formula \(\varphi \in \Phi\), the satisfiability relation \(w \models \varphi\) is defined recursively:

1. \(w \models p\) if \(p \in \pi(p)\), where \(p\) is a propositional variable,
2. \(w \models \neg \varphi\) if \(w \not\models \varphi\),
3. \(w \models \varphi \rightarrow \psi\) if \(w \models \varphi\) or \(w \not\models \psi\),
4. \(w \models [C : X]\) if
   a. for any strategy \(t \in \Delta^C\) of coalition \(C\) there is a formula \(\varphi \in X\) such that for any action profile \(\delta \in \Delta^\mathcal{A}\) and any state \(u \in W\) if \(|\delta|_w \leq s\), \(t =_C \delta\), and \((w, \delta, u) \in M\), then \(u \models \varphi\).
   b. for any nonempty subset \(Y \subseteq X\) there is a strategy \(t \in \Delta^C\) of coalition \(C\) such that for any formula \(\varphi \in Y\) there is an action profile \(\delta \in \Delta^\mathcal{A}\) and a state \(u \in W\) where \(|\delta|_w \leq s\), \(t =_C \delta\), \((w, \delta, u) \in M\), and \(u \not\models \varphi\).
Recall that we allow a game to terminate as a result of agents’ actions. For example, suppose that in a state \( w \) an agent \( a \) has three actions \( d_1, d_2, d_3 \) all of which have a “cost” of 1. Let action \( d_1 \) transition the system into a state in which statement \( \psi_1 \) is true, action \( d_2 \) transition the system into a state in which statement \( \psi_2 \) is true, and action \( d_3 \) be an action that terminates the system. Then, \( w \models \{ a: \phi_1, \phi_2 \} \) is true, because each action of agent \( a \) predetermines a specific \( \psi_1 \) to be true in each outcome state. In other words, being able to terminate the system does not provide a way for an agent to "escape" the dilemma.

In the next section we state the axioms of our logical system that capture the properties of modality \([C:X]_s\). When stating these axioms, it will be convenient to define \([C:X]_s\) as an abbreviation for formula \( \forall x \in X \left( [C:X]_s \right) \). In other words, \([C:X]_s\) means that each action profile of coalition \( C \) forces a specific formula in set \( X \) to be true, but set \( X \) is not necessarily a minimal such set. We call expression \([C:X]_s\) a weak dilemma.

### 5 AXIOMS

In this section we list and discuss the axioms and inference rules of our logical system. The first of these axioms uses the notation \( X \otimes Y \). For any two sets of formulae \( X \) and \( Y \), let \( X \otimes Y \) be the set of formulae \( \{ \phi \land \psi \mid \phi \in X, \psi \in Y \} \).

In addition to propositional tautologies in language \( \Phi \), our logical system contains the following axioms:

1. **Combination**: \( [C:X]_s \rightarrow ([C:Y]_s \rightarrow [C:X \otimes Y]_s) \).
2. **Monotonicity**: \( [C:X]_s \rightarrow [D:X]_s \), where \( C \subseteq D \) and \( s \leq s' \).
3. **Minimality**: \( [C:X]_s \rightarrow [C:Y]_s \), where \( Y \subseteq X \).
4. **No Alternatives**: \( [C:X]_s \rightarrow [D:X]_s \), where \( |X| = 1 \).

We write \( \models \phi \) if formula \( \phi \in \Phi \) is derivable in our logical system using the Modus Ponens, the Necessitation, and the Substitution inference rules

\[
\frac{\phi, \phi \rightarrow \psi}{\psi}[C: \phi]_s \quad \frac{\phi \rightarrow \tau(\phi)}{[C: \phi]_s \rightarrow [C: \tau(\phi)]_s}
\]

for each function \( \tau \) that maps set \( \Phi \) into set \( \Phi \). If \( \models \phi \), then we say that formula \( \phi \) is a theorem of our system. We write \( \models \phi \) if formula \( \phi \) is provable from all theorems of our logical system and an additional set of formulae \( X \) using the Modus Ponens inference rule only.

The Combination axiom states that if each action profile of coalition \( C \) forces a specific formula in set \( X \) to be true and a specific formula in set \( Y \) to be true, then each action profile of coalition \( C \) forces a specific formula in set \( X \otimes Y \) to be true. Indeed, if a particular action profile forces \( \phi \in X \) to be true and \( \psi \in Y \) to be true, then this profile also forces \( \phi \land \psi \) to be true.

A hypothetical Combination axiom with the single bracket modality in the conclusion \([C:X]_s \rightarrow ([C:Y]_s \rightarrow [C:X \otimes Y]_s)\) is not sound. Indeed, suppose that \( X = Y = \{ \phi \land \psi, \phi \land \psi \lor \chi \land \psi \land \psi \} \). Let \( w \models [C: \phi, \psi]_s \) for some state \( w \) of a game. It follows that \( w \not\models [C: \phi \land \psi, \psi \land \psi]_s \). Thus, we have \( w \models [C:X]_s \) and \( w \not\models [C:Y]_s \), but \( w \not\models [C:X \otimes Y]_s \). Since \( \phi \land \psi \land \psi \leq X \otimes Y \), statement \( w \models [C:X \otimes Y]_s \) would violate the minimality condition 4(b) of Definition 4.3.

The Monotonicity axiom states that if each action profile of coalition \( C \) forces a specific formula in set \( X \) to be true under a more relaxed constraint \( s' \) on sacrifice, then each action profile of a stronger coalition \( D \) forces a specific formula in set \( X \) to be true under a stronger constraint \( s \). A hypothetical Monotonicity axiom with single bracket modality in the conclusion is also not sound. For example, \( [m_a: d_1, d_2, d_3]_{m_a,m_b,m_\geq 2,1} \) in the setting of Figure 1. However, \( [m_a, m_b: d_1, d_2, d_3]_{m_a,m_b,m_\geq 2,1} \). Once an action of agent \( m_b \) is fixed, this action will guarantee that the entire population of village \( v_2 \) will die or it will guarantee that the entire population of village \( v_3 \) will die, see Figure 1. Hence, \( [m_a, m_b: d_1, d_2, d_3]_{m_a,m_b,m_\geq 2,1} \) and \( [m_a, m_b: d_1, d_2, d_3]_{m_a,m_b,m_\geq 2,1} \) due to the minimality condition.

The Minimality axiom captures the minimality requirement of item 4(b) in Definition 4.3.

The No Alternatives axiom deals with extreme case of a singleton set \( X = \{ \phi \} \). Note that statement \([C: \phi]_s \) means that statement \( \phi \) is predetermined to be true under any action profile of coalition \( C \) as long as actions of all agents are constrained by \( s \). In other words, \( \phi \) is true as long as actions of all agents are constrained by \( s \). Since the last statement does not depend on the coalition \( C \), we may conclude that validity of statement \([C: \phi]_s \) does not depend on the choice of coalition \( C \). This observation is captured in the No Alternatives axiom.

The Necessitation rule states that if formula \( \phi \) is true in all states of all games, then statement \( \phi \) is predetermined to be true under any action profile of coalition \( C \) as long as actions of all agents are constrained by \( s \). The Substitution rule says that if \([C:X]_s \) and statement \( \phi \) in set \( X \) is replaced with a logically weaker statement \( \psi(\phi) \), then each action profile of coalition \( C \) still forces a specific formula in the set \( X(\tau(\phi)) \) to be true, but \( X(\tau(\phi)) \) is not necessarily the smallest such set. An example of an instance of this rule is

\[
\models \neg \neg \phi \rightarrow \phi, \quad \models \psi \rightarrow (\chi \rightarrow \psi)
\]

Note that \( X \) and \( \tau(\phi) \) are sets, not lists. Thus, set \( \tau(\phi) \) might have fewer elements than set \( X \):

\[
\models \phi \rightarrow (\phi \land \psi), \quad \models (\phi \lor \psi) \rightarrow (\phi \lor \psi)
\]

Our last example illustrates that even if \( X \) satisfies the minimality condition, set \( \tau(\phi) \) might not:

\[
\models (\phi \lor \psi), \quad \models (\psi \lor \phi)
\]

Theorem 5.1 (Strong soundness). If \( w \models \chi \) for each formula \( \chi \in X \), then \( w \models \phi \). □

### 6 COMPLETENESS

In this section we prove a strong completeness theorem for our logical system. This proof is split into three subsections. First, we establish auxiliary properties of modalities \([C:X]_s \) and \([C:Y]_s \). Then, we define the canonical game and prove several lemmas about this game. Finally, we state and prove completeness theorem.

#### 6.1 Preliminaries

**Lemma 6.1.** Inference rule \( \frac{\phi}{[C: \phi]_s} \) is derivable.
Proof. Suppose that \( \vdash \phi \). Thus, \( \vdash [C : \phi]_s \) by the Necessitation inference rule. Hence, \( \vdash \bigvee_{\phi \subseteq Y \subseteq \mathcal{A}} [C : \phi]_s \), because singleton set \( \{ \phi \} \) has only one nonempty subset. Therefore, \( \vdash [C : \phi]_s \) by the definition of the modality \( [\ ] \).

**Lemma 6.2.** \( \vdash [C : X]_s \rightarrow ([C : Y]_s \rightarrow [C : X \otimes Y]_s) \).

**Proof.** By the Combination axiom,
\[
\vdash [C : X]_s \rightarrow ([C : Y]_s \rightarrow [C : X' \otimes Y']_s),
\]
for any nonempty sets \( X', Y' \subseteq \mathcal{A} \). Thus, by the definition of the modality \( [\ ] \),

\[
\vdash [C : X]_s \rightarrow \left( [C : Y]_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq X' \otimes Y'} [C : Z]_s \right).
\]
Suppose that \( X' \subseteq X \) and \( Y' \subseteq Y \). Then \( X' \otimes Y' \subseteq X \otimes Y \). Hence, by the laws of propositional reasoning,

\[
\vdash [C : X']_s \rightarrow \left( [C : Y']_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq X \otimes Y} [C : Z]_s \right).
\]
Since set \( X' \) and set \( Y' \) are arbitrary nonempty subsets of \( X \) and \( Y \) respectively, by propositional reasoning,

\[
\vdash \bigvee_{\phi \subseteq X' \subseteq X} [C : X']_s \rightarrow \left( \bigvee_{\phi \subseteq Y' \subseteq Y} [C : Y']_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq X \otimes Y} [C : Z]_s \right).
\]
Therefore, \( \vdash [C : X]_s \rightarrow ([C : Y]_s \rightarrow [C : X \otimes Y]_s) \) by the definition of the modality \( [\ ] \).

**Lemma 6.3.** For each integer \( n \geq 1 \),
\[
\vdash [C : \phi]_s \rightarrow ([C : \psi_1, \ldots, \psi_n]_s \rightarrow [C : \phi \land \psi_1, \ldots, \phi \land \psi_n]_s).
\]
**Proof.** The statement of the lemma follows from Lemma 6.2 for \( X = \{ \phi \} \) and \( Y = \{ \psi_1, \ldots, \psi_n \} \).

Earlier we defined \( \otimes \) as a binary operation on the sets of formulæ. In the next lemma we use notation \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \). If \( n > 2 \), then \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \) stands for \((\cdots((X_1 \otimes X_2) \otimes X_3) \cdots) \otimes X_n \). If \( n = 1 \), then \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \) is set \( X_1 \). If \( n = 0 \), then \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \), is the singleton set \( \{ T \} \).

**Lemma 6.4.** For each integer \( n \geq 0 \),
\[
\vdash \bigvee_{i \leq n} [C : X_i]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_n]_s.
\]
**Proof.** We prove the lemma by induction on \( n \).

If \( n = 0 \), then \( X_1 \otimes X_2 \otimes \cdots \otimes X_n = \{ T \} \). Note that \( \{ T \} \) is a tautology. Thus, \( \vdash [C : X_1]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_n]_s \) by the Lemma 6.1. Then, \( \vdash \bigvee_{i \leq n} [C : X_i]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_n]_s \), by the laws of propositional reasoning.

If \( n = 1 \), then the following formula is a propositional tautology:
\[
\vdash [C : X_1]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_n]_s.
\]
If \( n > 1 \), then, by the induction hypothesis,
\[
\vdash \bigvee_{i \leq n-1} [C : X_i]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_{n-1}]_s.
\]
At the same time, by Lemma 6.2,
\[
\vdash [C : X_1]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_{n-1}]_s \rightarrow \left( [C : X_n]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_n]_s \right).
\]
Therefore, \( \vdash \bigwedge_{i \leq n} [C : X_i]_s \rightarrow [C : X_1 \otimes X_2 \otimes \cdots \otimes X_n]_s \), by propositional reasoning from the last two statements.

**Lemma 6.5.** For each \( n \geq 0 \),
\[
\vdash [C : \varphi_1]_s \land \cdots \land [C : \varphi_n]_s \rightarrow [C : \varphi_1 \land \cdots \land \varphi_n]_s.
\]
**Proof.** The statement of the lemma follows from Lemma 6.4 for sets \( X_1 = \{ \varphi_1 \}, \ldots, X_n = \{ \varphi_n \} \).

**Lemma 6.6.** \( \vdash [C : X]_s \rightarrow [D : X]_s \), where \( |X| = 1 \).

**Proof.** Note that formula \( [C : X]_s \rightarrow [D : X]_s \) is an instance of the No Alternatives axiom. Thus, because singleton set \( X \) has only one nonempty subset, \( \vdash \bigvee_{\phi \subseteq Y \subseteq X} [C : \phi]_s \rightarrow [D : Y]_s \). Hence, \( \vdash [C : X]_s \rightarrow [D : X]_s \), by the definition of modality \( [\ ] \).

**Lemma 6.7.** \( \vdash [C : X']_s \rightarrow [D : X']_s \), where \( C \subseteq D \) and \( s \leq s' \).

**Proof.** By the Monotonicity axiom, \( \vdash [C : Y']_s \rightarrow [D : Y]_s \), for each nonempty set of formula \( Y \subseteq X \). Thus, by the definition of modality \( [\ ] \),
\[
\vdash [C : Y']_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq X} [D : Z]_s.
\]
Suppose now that \( Y \subseteq X \). Then, by propositional reasoning,
\[
\vdash [C : Y']_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq X} [D : Z]_s.
\]
for any nonempty set \( Y \subseteq X \). Thus, by propositional reasoning,
\[
\vdash \bigvee_{\phi \subseteq Y \subseteq X} [C : \phi]_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq X} [D : Z]_s.
\]
Therefore, \( \vdash [C : X']_s \rightarrow [D : X']_s \) by the definition of modality \( [\ ] \).

**Lemma 6.8.** Inference rule
\[
\{ \varphi \rightarrow \tau(\varphi) \mid \varphi \in X \} \rightarrow [C : \tau(X)]_s,
\]
is derivable for any function \( \tau \) from set \( \varphi \) to set \( \Phi \).

**Proof.** Suppose \( \vdash \varphi (\varphi) \) for each \( \varphi \in X \). Consider an arbitrary nonempty subset \( Y \subseteq X \). Then, \( \vdash \varphi (\varphi) \) for each \( \varphi \in Y \). Thus, by the Substitution inference rule, \( \vdash [C : Y]_s \rightarrow [C : \tau(Y)]_s \).

Hence,
\[
\vdash [C : Y]_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq \tau(X)} [D : Z]_s.
\]
by the definition of modality \( [\ ] \). Then, because \( [C : Y]_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq \tau(X)} [D : Z]_s \), thus, since set \( Y \) is an arbitrary nonempty subset of \( X \),
\[
\vdash \bigvee_{\phi \subseteq Y \subseteq X} [C : \phi]_s \rightarrow \bigvee_{\phi \subseteq Z \subseteq \tau(X)} [D : Z]_s.
\]
Hence, \( \vdash [C : X]_s \rightarrow [C : \tau(X)]_s \) by the definition of modality \( [\ ] \).

**Lemma 6.9.** For each \( n \geq 1 \),
\[
\vdash [C : \phi_1, \ldots, \phi_n]_s \rightarrow [C : \phi_1 \lor \cdots \lor \phi_n]_s.
\]
**Proof.** Note that \( \phi_1 \lor \phi_1 \lor \cdots \lor \phi_n \) is a tautology for each \( i \leq n \). Therefore, \( \vdash [C : \phi_1, \ldots, \phi_n]_s \rightarrow [C : \phi_1 \lor \cdots \lor \phi_n]_s \) by Lemma 6.8.
Lemma 6.10. $\vdash [c : x]_s \rightarrow \lbrack c : x \rbrack_s$.

Proof. Formula $[c : x]_s \rightarrow \bigvee_{\phi \neq Y \subseteq X} [c : y]_s$ is a propositional tautology. Therefore, $\vdash [c : x]_s \rightarrow \lbrack c : x \rbrack_s$ by the definition of modality $\lbrack \rbrack$.

Lemma 6.11. $\vdash [c : x]_s \rightarrow \neg \bigvee_{\phi \neq Y \subseteq X} [c : y]_s$.

Proof. Consider any nonempty set $Z \subseteq X$. Then, by the Minimality axiom, $\vdash [c : x]_s \rightarrow \neg [c : z]_s$. Hence, by propositional reasoning, $\vdash [c : x]_s \rightarrow \neg \bigvee_{\phi \neq Z \subseteq Y} [c : z]_s$. Thus, by propositional reasoning,

$$\vdash [c : x]_s \rightarrow \neg \bigvee_{\phi \neq Y \subseteq X} [c : y]_s.$$ 

Therefore, $\vdash [c : x]_s \rightarrow \neg \bigvee_{\phi \neq Y \subseteq X} [c : y]_s$ by the definition of modality $\lbrack \rbrack$.

Lemma 6.12. $\vdash [c : x]_s \land \neg \bigvee_{\phi \neq Y \subseteq X} [c : y]_s \rightarrow [c : x]_s$.

Proof. The following formula is a propositional tautology:

$$\bigvee_{\phi \neq Y \subseteq X} [c : y]_s \land \neg \bigvee_{\phi \neq Y \subseteq X} [c : z]_s \rightarrow [c : x]_s.$$ 

Thus, by propositional reasoning,

$$\vdash \bigvee_{\phi \neq Y \subseteq X} [c : y]_s \land \neg \bigvee_{\phi \neq Y \subseteq X} [c : z]_s \rightarrow [c : x]_s.$$ 

Therefore, $\vdash [c : x]_s \land \neg \bigvee_{\phi \neq Y \subseteq X} [c : y]_s \rightarrow [c : x]_s$ by the definition of modality $\lbrack \rbrack$.

6.2 Canonical Game

In this section we define the canonical game $(W, \Delta, \lbrack \rbrack, M, \pi)$.

Definition 6.13. $W$ is the set of all maximal consistent sets of formulae.

An action of an agent consists in specifying a set of formula $\Phi$ and a real number $c$. Informally, set $\Psi$ is a set of "demands" by the agent and $c$ is the price that she offers to pay for her demands. If $c$ is negative, then $|c|$ is the profit that the agent requests in addition to the fulfillment of her demands.

Definition 6.14. $\Delta$ is the set of all pairs $(\Psi, c)$, where $\Psi \subseteq \Phi$ is a set of formulae and $c \in \mathbb{R}$ is a real number.

When an agent offers to pay or when she requests a profit, the amount she specifies is exactly the amount she pays or gets.

Definition 6.15. $\lbrack (\Psi, c) \rbrack_s^w = c$.

In the next definition we specify the mechanism of the canonical game. Informally, this mechanism enforces two conditions: (1) it either grants all "demands" of all agents or terminates the game and (2) it makes sure that each time a coalition faces a weak dilemma $\lbrack c : x \rbrack_s$, at least one agent in coalition $C$ demands at least one formula in set $X$; if not, the game is terminated.

For any pair $u = (\bar{x}, \bar{y})$ by $pr_1(u)$ and $pr_2(u)$ we mean elements $x$ and $y$ respectively.

Definition 6.16. Mechanism $M$ is the set of all triples $(w, \delta, u) \in W \times \Delta^A \times W$ such that:

1. $pr_1(\delta(a)) \leq u$ for each agent $a \in A$,
2. for each formula $\lbrack c : x \rbrack_s \in w$, if $|\delta|_w \leq s$, then there is an agent $a \in C$ such that set $pr_1(\delta(a)) \land X \not= \emptyset$.

Definition 6.17. $\pi(p) = \{ w \in W \mid p \in w \}$.

This concludes the definition of the canonical game. Both our construction and the one sketched by Goranko and Enqvist in their proof of completeness of socially friendly coalition logic [30] modify Marc Pauly’s construction for the completeness of coalition power logic in a significant way. Marc Pauly’s construction could be interpreted in terms of each agent voting for a single formula in a single-round voting [48]. If the right group of agents votes for the right formula, then the group’s wish is guaranteed to be granted. Goranko and Enqvist propose a two-round voting mechanism in which agents that fail to achieve their goals in the first round get a chance to vote for a second formula. Technically, both rounds are carried out on the same ballot. In our construction, each single agent votes for a set of formulae and specifies a price that she is ready to pay. The mechanism either satisfies the demands of all agents or terminates the game.

As usual, the key step in a proof of completeness is an induction, or “truth”, lemma. In our case this is Lemma 6.20. The induction step in the proof of this lemma for a formula of the form $\lbrack c : x \rbrack_s$ is based on the next two auxiliary lemmas.

Lemma 6.18. For any state $w \in W$, any formula $\lbrack c : x \rbrack_s \in w$, and any strategy $t \in \Delta^A$ of coalition $C$, there is a formula $\phi \in X$ such that for each action profile $\delta \in \Delta^A$ and each state $u \in W$, if $|\delta|_w \leq s$, $t =_C \delta$, and $(w, \delta, u) \in M$, then $\phi \in u$.

Proof. Consider any state $w \in W$, any formula $\lbrack c : x \rbrack_s \in w$, and any strategy $t \in \Delta^A$ of coalition $C$.

Case I: There is an agent $a_0 \in C$ such that $pr_1(t(a_0)) \land X \not= \emptyset$. Let $\phi$ be any such formula such that $\phi \in pr_1(t(a_0)) \land X$. Then $\phi \in X$. Consider any action profile $\delta \in \Delta^A$ and any state $u \in W$ such that $|\delta|_w \leq s$, $t =_C \delta$, and $(w, \delta, u) \in M$. To prove the lemma, it suffices to show that $\phi \in u$. Indeed, assumption $t =_C \delta$ implies that $pr_1(t(a_0)) = pr_1(\delta(a_0))$ because $a_0 \in C$. Thus, $\phi \in pr_1(t(a_0)) = pr_1(\delta(a_0))$ by the choice of formula $\phi$. Note that $pr_1(\delta(a_0)) \subseteq u$ by the assumption $(w, \delta, u) \in M$ and item 2 of Definition 6.16. Therefore, $\phi \in u$.

Case II: $pr_1(t(a)) \land X = \emptyset$ for each agent $a \in C$. Let $\phi$ be an arbitrary formula in language $\Phi$. To finish the proof of the lemma, it suffices to show that there is no action profile $\delta \in \Delta^A$ and state $u \in W$, such that $|\delta|_w \leq s$, $t =_C \delta$, and $(w, \delta, u) \in M$. Suppose, for the sake of contradiction, that such an action profile and a state exist. Then statement $t =_C \delta$ implies that $pr_1(\delta(a)) \land X = \emptyset$ for each agent $a \in C$ by the assumption of the case. Hence, $(w, \delta, u) \not\in M$ by part 2 of Definition 6.16 and because $|\delta|_w \leq s$, which is a contradiction.

Lemma 6.19. For any state $w \in W$ and any formula $\lbrack c : x \rbrack_s \in w$, there is a strategy $t \in \Delta^A$ of coalition $C$ such that for any formula $\phi \in X$ there is an action profile $\delta \in \Delta^A$ and a state $u \in W$ where $|\delta|_w \leq s$, $t =_C \delta$, $(w, \delta, u) \in M$, and $\phi \notin u$.

Proof. Let the set of pairs $R$ and the set of formulae $\Sigma$ be defined as follows:

$$R = \{(D, Y) \mid \llbracket D : Y \rbrack_s \in w, D \subseteq C\}, \tag{1}$$

$$\Sigma = \{\forall Z \mid \llbracket E : Z \rbrack_s \in w, E \not\subseteq C\}. \tag{2}$$
By a choice function $\lambda$ we mean any function on set $R$ such that for each pair $(D,Y) \in R$,
\[ \lambda(D,Y) \in Y. \] (3)
The set of all choice functions is denoted by $\Lambda$.

**Claim 1.** $w \vdash [C : \land \Sigma']$, for each finite set $\Sigma' \subseteq \Sigma$.

**Proof of Claim.** Consider any formula $\sigma \in \Sigma'$. Then, because $\Sigma' \subseteq \Sigma$, by equation (2), there must exist a formula $[E : Z] \in w$ such that $\sigma$ is equal to $\lor Z$. Then, $w \vdash [E : \sigma]_s$, by Lemma 6.9. Hence, $w \vdash [C : \sigma]_s$, by Lemma 6.6 and the Modus Ponens inference rule. Therefore, $w \vdash [C : \land \Sigma']_s$, by Lemma 6.5.

**Claim 2.** For each finite set $R' \subseteq R$,
\[ w \vdash \left[ C : \bigwedge_{(D,Y) \in R'} \lambda(D,Y) \mid \lambda \in \Lambda \right]_s \] (4)

**Proof of Claim.** Consider any $(D,Y) \in R' \subseteq R$. Then, $[D : Y]_s \in w$ and $D \subseteq C$ due to equation (1). Thus, by Lemma 6.7 and the Modus Ponens rule, $w \vdash [C : Y]_s$. Hence, $w \vdash [C : Y]_s$ for each $(D,Y) \in R'$. Thus,
\[ w \vdash \left[ C : \bigwedge_{(D,Y) \in R'} \lambda(D,Y) \mid \lambda \in \Lambda \right]_s \] (5)

Next, note that the following formula is provable in our logical system by Lemma 6.4:
\[ \bigwedge_{(D,Y) \in R'} [C : Y]_s \implies \left[ C : \left\{ \bigwedge_{(D,Y) \in R'} \lambda(D,Y) \mid \lambda \in \Lambda \right\} \right]_s. \]
Therefore,
\[ w \vdash \left[ C : \left\{ \bigwedge_{(D,Y) \in R'} \lambda(D,Y) \mid \lambda \in \Lambda \right\} \right]_s \]
by the Modus Ponens inference rule.

**Claim 3.** For all finite sets $\Sigma' \subseteq \Sigma$ and $R' \subseteq R$,
\[ w \vdash \left[ C : \bigwedge \Sigma' \land \bigwedge_{(D,Y) \in R'} \lambda(D,Y) \mid \lambda \in \Lambda \right]_s \] (6)

**Proof of Claim.** The statement of the claim follows from Claim 1 and Claim 2 by Lemma 6.3.

**Claim 4.** There is a choice function $\lambda \in \Lambda$ such that for any formula $\varphi \in X$ the following set is consistent:
\[ \{ \neg \varphi \} \cup \Sigma \cup \{ \lambda(D,Y) \mid (D,Y) \in R \}. \]

**Proof of Claim.** Suppose the opposite. Thus, for any choice function $\lambda \in \Lambda$ there is a formula $\varphi_\lambda \in X$ such that
\[ \Sigma, \{ \lambda(D,Y) \mid (D,Y) \in R \} \vdash \varphi_\lambda. \]
Hence, since any derivation uses only finitely many assumptions, for any choice function $\lambda \in \Lambda$ there are finite sets $\Sigma_\lambda \subseteq \Sigma$ and $R_\lambda \subseteq R$ such that
\[ \Sigma_\lambda, \{ \lambda(D,Y) \mid (D,Y) \in R_\lambda \} \vdash \varphi_\lambda. \]
Recall that set $X$ is finite by definition of language $\Phi$. Then, set $\{ \varphi_\lambda \mid \lambda \in \Lambda \} \subseteq X$ is also finite (although set $\Lambda$ might be infinite). Hence, there are finite sets $R' \subseteq R$ and $\Sigma' \subseteq \Sigma$ such that, for each choice function $\lambda \in \Lambda$,
\[ \Sigma', \{ \lambda(D,Y) \mid (D,Y) \in R' \} \vdash \varphi_\lambda. \]

Thus, by the deduction theorem and the laws of propositional reasoning, for each choice function $\lambda \in \Lambda$,
\[ \vdash \land \Sigma' \land \bigwedge_{(D,Y) \in R'} \lambda(D,Y) \rightarrow \varphi_\lambda. \]
Hence, $w \vdash [\{ \varphi_\lambda \mid \lambda \in \Lambda \}]_s$, by Claim 3 and Lemma 6.8. Then, $w \vdash [C : X]_s$, by Lemma 6.7 and because $\{ \varphi_\lambda \mid \lambda \in \Lambda \} \subseteq X$. Therefore, $\neg [C : X]_s \notin w$ because set $w$ is consistent, which contradicts the assumption of the lemma.

We now return to the proof of the lemma. Let $\lambda_0$ be the choice function that exists by Claim 4 and strategy $t \in \Delta^C$ be such that for each agent $a \in C$,
\[ t(a) = (\{ \lambda_0(D,Y) \mid (D,Y) \in R \}, s(a)). \] (7)
Consider an arbitrary formula $\varphi \in X$. By the selection of the choice function $\lambda_0$, the set
\[ U = \{ \neg \varphi \} \cup \Sigma \cup \{ \lambda_0(D,Y) \mid (D,Y) \in R \} \]
(8)
is consistent. Let $u$ be any maximal consistent extension of set $U$. Thus, $\neg \varphi \notin u$. Hence, $\varphi \notin u$ because set $u$ is consistent. Define action profile $\delta \in \Delta^A$ as follows:
\[ \delta(a) = \begin{cases} t(a), & \text{if } a \in C, \\ (u, s(a)), & \text{otherwise}. \end{cases} \] (9)
Then, $t = \delta$.

**Claim 5.** $\triangledown u^a \in s(a)$ for each agent $a \in \cal A$.

**Proof of Claim.** We consider the following two cases separately:

**Case I:** $a \in C$. Thus, by equation (6), equation (4), and Definition 6.15,
\[ \triangledown u^a = |t(a)|^a_w = |(\{ \lambda_0(D,Y) \mid (D,Y) \in R \}, s(a))|^a_w = s(a). \]

**Case II:** $a \notin C$. Hence, $\triangledown u^a = |u, s(a)|^a_w = s(a)$ by equation (6) and Definition 6.15.

Then, $\triangledown u^a \leq s$ by Definition 4.2.

**Claim 6.** $(w, \delta, u) \in M$.

**Proof of Claim.** We will show the two statements from Definition 6.16 separately.

(1) To prove that $pr_1(\delta(a)) \subseteq u$ for each agent $a \in \cal A$, consider the following two cases:

**Case I:** $a \in C$. Thus, by equation (6), equation (4), equation (5), and the choice of set $u$,
\[ pr_1(\delta(a)) = pr_1(t(a)) = pr_1(\{ \lambda_0(D,Y) \mid (D,Y) \in R \}, s(a)) = \{ \lambda_0(D,Y) \mid (D,Y) \in R \} \subseteq U \subseteq u. \]

**Case II:** $a \notin C$. Hence, by equation (6),
\[ pr_1(\delta(a)) = pr_1(u, s(a)) = u. \]
ψ
of coalition each state ψ a formula t
Definition 4.3.
Definition 4.3. u and a state ψ such that for any formula
Case I
Definition 4.3 and the maximality and the consistency of set w.
Proof.
Lemma 6.20. w ⊩ φ iff φ ∈ w where w ∈ W and φ ∈ Φ.
Proof. We prove the statement of the lemma by induction on
structural complexity of formula φ. If formula φ is a propositional
variable, then the statement of the lemma follows from item 1 of
Definition 4.3 and Definition 6.17. The cases when formula φ is
a negation or an implication follow from item 2 and item 3 of
Definition 4.3 and the maximality and the consistency of set w in
the standard way.
Suppose now that formula φ has form [C : X]₁.
(⇒) : Suppose that [C : X]₁ ̸∈ w. Thus, w ⊬ [C : X]₁ because set w is maximal.
Hence, either w ⊬ [C : X]₁ or w ⊬ ¬ ∆ₓ ∈ X by Lemma 6.12. We consider these two cases separately.
Case I: w ⊬ [C : X]₁. Thus, ¬[C : X]₁ ̸∈ w because set w is maximal.
Hence, by Lemma 6.19, there is a strategy t ∈ ∆₅ of coalition C
such that for any formula ψ ∈ X there is an action profile δ ∈ ∆₅
and a state u ∈ W where |δ|u ≤ s, t =₆ δ, (w, δ, u) ∈ M, and ψ ̸∈ u.
Then, by the induction hypothesis, for any formula ψ ∈ X there is
an action profile δ ∈ ∆₅ and a state u ∈ W where |δ|u ≤ s, t =₆ δ, (w, δ, u) ∈ M, and u ⊬ ψ. Therefore, w ⊬ [C : X]₁ by item 4(a) of
Definition 4.3.
Case II: w ⊬ ¬ ∆ₓ ∈ X by Lemma 6.12. Thus, because set w is maximal,
∆ₓ ∈ X. Then, again by the maximality of set w, there exists a nonempty set Y₀ ⊆ X such that [C : Y₀]₁ ̸∈ w. Then,
by Lemma 6.18, for any strategy t ∈ ∆₅ of coalition C, there is
a formula ψ ∈ Y₀ such that for each action profile δ ∈ ∆₅
and each state u ∈ W, if |δ|u ≤ s, t =₆ δ, and (w, δ, u) ∈ M, then
ψ ̸∈ u. Thus, by the induction hypothesis, for any strategy t ∈ ∆₅
of coalition C, there is a formula ψ ∈ Y₀ such that for each profile
δ ∈ ∆₅ and each state u ∈ W, if |δ|u ≤ s, t =₆ δ, and (w, δ, u) ∈ M,
then u ⊬ ψ. Therefore, w ̸⊨ [C : X]₁ by item 4(b) of Definition 4.3
and the assumption Y₀ ⊆ X.
(⇐) : Suppose [C : X]₁ ̸∈ w. To prove that w ⊬ [C : X]₁, we will verify conditions (a) and (b) in item 4 of Definition 4.3.
(a) By Lemma 6.10 and the Modus Ponens inference rule, assumption
[C : X]₁ ̸∈ w implies w ⊬ [C : X]₁. Hence, [C : X]₁ ̸∈ w because set w is maximal.
Thus, by Lemma 6.18, for any strategy t ∈ ∆₅ of coalition C, there is a formula
ψ ∈ X such that for each action profile δ ∈ ∆₅
and each state u ∈ W, if |δ|u ≤ s, t =₆ δ, and (w, δ, u) ∈ M, then
ψ ̸∈ u. Therefore, by the induction hypothesis, for any strategy
t ∈ ∆₅ of coalition C, there is a formula ψ ∈ X such that for each action profile δ ∈ ∆₅
and each state u ∈ W, if |δ|u ≤ s, t =₆ δ, and (w, δ, u) ∈ M, then u ⊬ ψ.
(b) By Lemma 6.11 and the Modus Ponens inference rule, assumption
[C : X]₁ ̸∈ w implies w ⊬ [C : X]₁. Hence, [C : X]₁ ̸∈ w because set w is maximal.
Thus, by Lemma 6.19, for any nonempty subset Y ⊆ X there is a strategy t ∈ ∆₅
of coalition C such that for any formula ψ ∈ Y there is an action profile δ ∈ ∆₅
and a state u ∈ W where |δ|u ≤ s, t =₆ δ, (w, δ, u) ∈ M, and ψ ̸∈ u. Therefore, by the induction hypothesis,
for any nonempty subset Y ⊆ X there is a strategy t ∈ ∆₅ of coalition C such that for any formula
ψ ∈ Y there is an action profile δ ∈ ∆₅
and a state u ∈ W
where |δ|u ≤ s, t =₆ δ, (w, δ, u) ∈ M, and u ⊬ ψ.
This concludes the proof of the lemma.
□

6.3 Strong Completeness Theorem
Theorem 6.21. For any set of formulae X and any formula φ, if X ⊬ φ, then there is a game and a state w of this game such that
w ⊬ φ for each formula χ ∈ X and w ⊬ φ.
Proof. Suppose X ⊬ φ. Thus, set X ∪ {¬φ} is consistent. Let w be any maximal consistent extension of this set. Then, w ⊬ φ
for each formula χ ∈ X by Lemma 6.20 and the choice of set w.
In addition, w ⊬ φ also by Lemma 6.20 and the choice of set w.
Therefore, w ⊬ φ by item 2 of Definition 4.3.
□

7 CONCLUSION
The contribution of this paper is two-fold. Firstly, we introduce a formal semantics for trolley-like ethical dilemmas in a strategic
game settings expressed through the “trolley” modality [C : X]₁. Secondly, we give a sound and complete axiomatization of universally
true properties of this modality. The completeness result is the strong completeness theorem with respect to the proposed
semantics. We believe that the standard filtration technique could be used to prove weak completeness with respect to the class of finite games.
This would imply decidability of our logical system, assuming the sacrifice function is rational-valued functions. Another possible
extension of this work is to study ethical dilemmas in an imperfect information setting, where [C : X]₁ would mean that not only does
collection C force a specific formula in set X to be true, but coalition
C also knows which one.
