Equilibration of Small and Large Subsystems in Field Theories and Matrix Models

Nima Lashkari

Department of Physics and Astronomy, University of British Columbia
6224 Agricultural Road, Vancouver, B.C., V6T 1W9, Canada

nima@phas.ubc.ca

Abstract

It has been recently shown that small subsystems of finite quantum systems generically equilibrate. We extend these results to infinite-dimensional Hilbert spaces of field theories and matrix models. We consider a quench setup, where initial states are chosen from a microcanonical ensemble of finite energy in free theory, and then evolve with an arbitrary non-perturbative Hamiltonian. Given a dynamical assumption on the expectation value of particle number density, we prove that small subsystems reach equilibrium at the level of quantum wavefunction, and with respect to all observables. The picture that emerges is that at higher energies, larger subsystems can reach equilibrium. For bosonic fields on a lattice, in the limit of large number of bosons per site, all subsystem smaller than half equilibrate. In the Hermitian matrix model, by contrast, this occurs in the limit of large energy per matrix element, emphasizing the importance of the $O(N^2)$ energy scale for the fast scrambling conjecture. Applying our techniques to continuum field theories on compact spaces, we show that the density matrix of small momentum-space observables equilibrate. Finally, we discuss the connection with scrambling, and provide a sufficient condition for a time-independent Hamiltonian to be a scrambler in terms of the entanglement entropy of its energy eigenstates.
1 Introduction

It is generally accepted that most systems equilibrate even if they are perturbed far away from equilibrium. This is to say that the information about the initial state
spreads out such that after some time all observables restricted to small subsystems become almost independent of time and the initial state. Our current understanding of equilibration in field theories remains at the level of perturbation theory, either at small coupling or large $N$ expansion. Perturbation theory enables us to keep track of the time dependence of the expectation value of observables with a small number of field operators, what we refer to as small observables. However, the formalism quickly becomes cumbersome as we look at large observables with many field insertions. Knowledge of all observables is required to specify the quantum state of field theory. It is not clear whether the effective state one infers from the expectation value of small observables is a good description of the system.

In this work, we address equilibration at the level of subsystems’ density matrices non-perturbatively. We argue, quite generally, that the reduced density matrix of small subsystems becomes indistinguishable from the equilibrium state with respect to all observables living in the subsystem Hilbert space. The techniques we employ in this work apply to Hamiltonians with discrete spectrum as in lattice field theory, continuum field theories on compact manifolds and matrix models. Our results suggest that at higher energies larger subsystems can equilibrate and in the limit of infinite energy, on a lattice and in matrix models, any subsystem smaller than half the size of the system equilibrates.

Our work is a generalization of quantum information techniques \cite{1, 2, 3} that have been recently used to discuss equilibration of finite systems to the case of infinite dimensional Hilbert spaces\footnote{For a discussion of equilibration in integrable systems see \cite{4}.}. Motivated by the fast scrambling conjecture \cite{5}, this generalization enables us to address the question of the equilibration of large subsystems in field theories and matrix models. We consider a quench setting where the initial state is chosen from a free theory, and evolved with an arbitrary non-perturbative interacting Hamiltonian. With some reasonable physical assumptions we prove that small subsystems always equilibrate. Furthermore, on a lattice of $N$ sites, initial states of energy larger than $O(N)$ equilibrate on all subsystems larger than half. In the Hermitian matrix model the large-scale equilibration occurs for energies larger than $O(N^2)$. In continuum field theories, our results imply that there are scales $\mu_1$ and $\mu_2$, where the infra-red and ultra-violet density matrices constructed by tracing out all momentum modes, respectively, higher than $\mu_1$ and lower than $\mu_2$, are almost non-dynamical and time-independent. In a quench setting from an eigenspace of large energy $\Delta$ in a $d$-dimensional free theory, we find that both $\mu_1$ and $\mu_2$ are $O(\Delta^{1/d})$.

In spite of the technical details of proofs, the equilibration results are based on a rather simple picture. The initial state of the system lives in a large Hilbert space and contains a significant amount of information. If the Hilbert space that the subsystem
explores is small, then it can not contain all the information about the initial state and quickly loses track of it. An interesting question is what happens if one tries to encode a message in the initial state. We say that the message is scrambled if it cannot be retrieved from the state of any subsystem smaller than half the size of the system [5, 6]. With this definition, scrambling is a strong form of equilibration in which the state of large subsystems become independent of initial states including atypical ones such as a tensor product states. Hamiltonians, which are very powerful in generating entanglement, leave almost no room in the Hilbert space of subsystems, even the large ones. We will discuss the implication of this for scrambling of time-independent Hamiltonians.

In section 2, we start by defining what we mean by equilibration. Our method closely follows the arguments and approach to equilibration developed in the quantum information community over the past few years. Therefore, we devote section 3 to a quick review of results in finite systems. Our main results appear in section 4 which generalizes equilibration theorems to infinite dimensional Hilbert spaces in quench settings and includes as an example the equilibration in the following three systems: bosons on a lattice, the Hermitian matrix model and field theory in momentum space. All results up to section 5 hold for generic interactions. The dynamical input from the Hamiltonian appears in section 5 where we expand on the connection with scrambling.

2 Basic setup

The many-particle quantum systems we are interested in are described by a set of physical states living in a tensor product of Fock spaces: \( \psi \in \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \ldots \otimes \mathcal{F}_N \). The tower of states in Fock spaces \( \mathcal{F}_i \) are generated by the creation operators \( a_i^\dagger \). In particular we consider three systems described in this manner:

1. Bosons on a lattice where \( a_i^\dagger \) creates a boson at site \( i \).
2. An \( N \times N \) Hermitian matrix model which is equivalent \( N \) interacting fermions. The creation operator \( a_i^\dagger \) increases the energy of \( i^{th} \) fermion by one unit.
3. Quantum fields in momentum space on a compact space where \( a_k^\dagger \) creates a particle with momentum \( k \).

Note that for bosons on a lattice and quantum fields in momentum space, any \( \psi \in \mathcal{F} \) is physical, whereas in the matrix model the physical states are only the ones invariant under gauge transformations. In the Hermitian matrix model we consider here, due to
the fermionic nature of degrees of freedom, the physical states are anti-symmetrized over \( N \) sites \[7\].

In physical problems of interest, there are typically global constraints on the Hilbert space of initial states. They are chosen from a subspace \( \mathcal{H}_R \) of \( \mathcal{F} \). Note that the restriction is imposed only on initial states and the wave-function can leave \( \mathcal{H}_R \) as it evolves. We denote the subspace the time-evolved initial states explore by \( \mathcal{H}_T \). It is the subspace spanned by energy eigenstates with nonzero projection into \( \mathcal{H}_R \).

Consider an initial physical state \( \psi \in \mathcal{H}_R \subset \mathcal{F} \). As \( \psi \) evolves in time, if the system equilibrates with respect to subsystem \( S \), the reduced density matrix on \( S \) becomes close to an equilibrium state \( \rho_{\text{equil}}^S \). Any comparison of distances between physical states requires a choice of metric in the Hilbert space. Trace norm provides a metric with a natural operational meaning. If two states \( \rho_S \) and \( \rho_{\text{equil}}^S \) are close in trace norm, i.e. \( ||\rho_S - \rho_{\text{equil}}^S||_1 < \epsilon \), then any measurement to distinguish them succeeds with probability at most \( \frac{1}{2} + \epsilon \). For some choice of Hamiltonians and initial states it is possible that that \( \rho_S(t) \) approaches its equilibrium state monotonically in time \[9, 10\]. We say \( \rho_S \) equilibrates in a strong sense, if for any \( \epsilon > 0 \) there exists a \( \tau \) such that for all \( t > \tau \), \( ||\rho_S(t) - \rho_{\text{equil}}^S||_1 < \epsilon \). More generically, systems tend to equilibrate in a weak sense: \( \rho_S(t) \) becomes close to its equilibrium value, and spends most of its time in a small neighbourhood around it. We say \( \rho_S \) equilibrates in a weak sense, if there exists an \( \epsilon \ll 1 \) such that:

\[
\langle ||\rho_S(t) - \rho_{\text{equil}}^S||_1 \rangle_t \leq \epsilon,
\]

(2.1)

where \( \langle \sigma(t) \rangle_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T \sigma(t) dt \); see figure 1.

Conceptually, the weak equilibration matches our intuition of equilibration better, since it leaves the possibility for \( \rho_S \) to fluctuate away from equilibrium in a time-independent way. The bound in (2.1) is the statement that large fluctuations are rare.

We would like to stay as generic as possible; therefore we focus on weak equilibration. Following \[1\], our strategy is to show first that the distance between a subsystem’s density matrix and its time-averages \( \omega \equiv \lim_{T \to \infty} \langle \rho_S(t) \rangle_t \) is bounded by a constant:

\[
\langle ||\rho_S(t) - \omega||_1 \rangle_t \leq \eta_S.
\]

(2.2)

Then, we show that for most initial states in the initial ensemble the constant \( \eta_S \) is small. This establishes the time-independence of the subsystem state. In principle \( \omega_S \) is a function of the initial state \( \psi \). Next, we need to show that \( \omega_S(\psi) \) is well-approximated by its average over all \( \psi \in \mathcal{H}_R \):

\[
\langle ||\omega_S(\psi) - \langle \omega_S \rangle_\psi||_1 \rangle_\psi \leq \epsilon'.
\]

(2.3)

\[2\] For a basic reference on trace norm, and other commonly used measures of distance in Hilbert space see \[8\].
Figure 1: This figure schematically shows the difference between weak and strong equilibrations. As the subsystem density matrix evolves it comes close to its equilibrium value with respect to the trace norm. In weak equilibration (left-hand side) $\rho_S(t)$ spends most of its time in an $\epsilon$ neighborhood of the equilibrium state which is independent of the initial state $\rho_0$, $\phi_0$ or $\chi_0$. This is in contrast with strong equilibration (right-hand side) where equilibrium is approached monotonically in time.

This is the statement that the equilibrium state keeps almost no memory of the initial perturbation. However, this does not necessarily imply that the equilibrium state is well approximated by the Gibbs state. In cases where the asymptotic state is the Gibbs state we say that the system has thermalized. It is known that thermalization is not as ubiquitous in nature as equilibration. There are many known examples of systems and initial states that equilibrate but never thermalize \cite{10,11}. It is a trivial extension of the above definitions to introduce notions of weak or strong scrambling or thermalization.

Another noteworthy comment is the difference between the equilibration of subsystems as defined above and equilibration with respect to coarse-grained observables \cite{2,3,12}. Coarse-grained observables need not be the result of only local measurements. In appendix B we briefly summarize the current understanding of equilibration for coarse-grained observables.

3 Intuition from finite systems

Recent developments in quantum information theory have made it possible to reformulate the foundations of thermodynamics in a more precise way. The authors in \cite{1,2,3}, among many others, revisited the question of equilibration in the quantum world and proved the following three statements about weak equilibration in finite-dimensional systems that we refer to as fluctuation, typicality and universality theorems:
Theorem 3.1 Fluctuations: Consider an arbitrary initial state $\psi_0$ in a finite dimensional Hilbert space $\mathcal{H}_R \subset \mathcal{H}$ evolving with a Hamiltonian with non-degenerate energy gaps. Let $\omega = \langle \psi(t) \rangle_t$ be the time-averaged density matrix of the whole system, and $d_{\text{eff}}(\omega) = \frac{1}{\text{tr}(\omega^2)}$ be a measure of the effective dimension of the Hilbert space it explores. For any subsystem $S$ we define:

$$\eta_S(\psi_0) = \sqrt{\frac{d_S^2}{d_{\text{eff}}(\omega)}},$$

where $d_S$ is the dimension of $\rho_S$. Then,

$$\langle \|\rho_S(t, \psi_0) - \omega_S(\psi_0)\| \rangle_t \leq \eta_S(\psi_0). \quad (3.1)$$

Therefore, for all initial states $\psi_0$, $\rho_S(t, \psi_0)$ approaches its time average in a weak sense if $\eta_S(\psi_0) \ll 1$.

Hereafter, we refer to $\eta_S$ as the equilibration parameter. Roughly speaking, $\eta_S$ bounds the size of fluctuations around equilibrium. We expect $\eta_S$ to be small whenever the subsystem $S$ is much smaller than the system. One way to make this intuition more precise is to ask: what is the probability of finding a large $d_{\text{eff}}(\omega)$ if we pick a random $\psi_0$ from $\mathcal{H}_R$? The answer to this question is given by the typicality theorem which makes use of Levy’s lemma.

Theorem 3.2 Typicality: For a state $\psi_0$ chosen at random with uniform measure in $\mathcal{H}_R$:

$$\text{Prob} \left( d_{\text{eff}}(\omega) < \frac{d_R}{4} \right) \leq 2e^{-c\sqrt{d_R}}, \quad (3.2)$$

where $c = \frac{(\ln 2)^2}{2\pi^3} \approx 2 \times 10^{-4}$.

Putting the first two theorems together we conclude that as long as the dimension of the initial ensemble’s Hilbert space $\mathcal{H}_R$ is much larger than the dimension of a subsystem’s Hilbert space, any interacting Hamiltonian evolves the subsystem towards equilibration. Naturally, we expect the equilibrium state to be almost independent of the initial state $\psi_0$. The universality theorem quantifies this by showing that the equilibrium states corresponding to all states in $\mathcal{H}_R$ are almost indistinguishable:

\[\text{For any equilibration to happen in Hilbert space it is essential that the degrees of freedom interact. One way to assure this is to restrict the problem to Hamiltonians with no energy gap $E_i - E_j$ that is hugely degenerate. In a free theory $E = E_S + E_{\bar{S}}$, so for any arbitrary $E_S$ and energy gap $\delta E_{\bar{S}}$ in $S$, there is a gap $\delta E$ in $E$. Note that even an arbitrarily small generic interaction lifts all degeneracies. The generalization of this result to the case of degenerate interacting Hamiltonians is discussed in [13].}

\[\text{For completeness, we have included Levy's lemma in appendix A.}\]
\[ \rho_S(t, \psi_0) \rightarrow \omega_S(\psi_0) = \langle \rho_S(t, \psi_0) \rangle_t \rightarrow \omega_S = \langle \omega_S(\psi_0) \rangle_{\psi_0} \rightarrow e^{-\beta H} \]

| time-independence | initial-state independence | Equilibration | thermalization |
|--------------------|---------------------------|---------------|---------------|

Figure 2: Information-theoretic approach to equilibration: First, the fluctuation and typicality theorems establish the time-independence of the subsystem density matrix by showing that \( \rho_S(t) \) is close to its time-average for a typical state. Then, the time-averaged state is shown to be close to its initial ensemble average.

**Theorem 3.3 Universality:** For a random state \( \psi_0 \) chosen with uniform measure in \( \mathcal{H}_R \):

\[
Prob\left( \| (\omega(\psi_0), \langle \omega \rangle_{\psi_0}) \| > \frac{1}{2} \sqrt{\frac{d_S}{d_R}} + \epsilon \right) \leq 2e^{-c'\epsilon^2 d_R},
\]

where \( c' = \frac{2}{9\pi^4} \). Here \( \langle \omega \rangle_{\psi_0} \) is the state \( \omega(\psi_0) \) averaged over \( \mathcal{H}_R \).

Figure 2 summarizes the approach discussed above.

### 4 Equilibration in infinite-dimensional Hilbert spaces

In systems with bosonic degrees of freedom, \( \rho_S \) lives in an infinite dimensional Hilbert space. Equilibration in trace norm is the condition that \( \rho_S \) at any time is indistinguishable from its equilibrium value with respect to an infinite number of independent measurements. It is not apriori clear that such a strong condition holds in equilibrated field theories. This problem is manifest in the appearance of \( d_S \) in the equilibration parameter. Subsystems can have infinite dimensions, and since the denominator in \( \eta_S \) only depends on the initial state and is finite when \( d_R \) is finite, the bound seems useless.

What saves us is the intuition that in any physical equilibration process the support of the wave function on arbitrarily large energy eigenstates is small. Therefore, one should be able to truncate the Hilbert space by reasonable dynamical assumptions without missing out the relevant physics. As long as the errors caused by truncation

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5The tighter version of this bound introduced in [1] is crucial for our discussion of scrambling in section 5. However, this form suffices for our purposes here.
are small, the finite dimensional calculations are reliable. When the equilibration parameter is small but truncation errors are not, it is only the set of observables which act on the truncated subspace that equilibrate. Significant deviations from equilibrium can be detected using operators that act beyond the truncated space.

Cutting off the full Hilbert space in the interacting Hamiltonian basis at energy $\Lambda$ does not guarantee that the reduced density matrix on $S$ is finite-dimensional. However, if we instead truncate in the free Hamiltonian basis, not only does it regulate $d_S$ but also it allows us to estimate the equilibration parameter by counting partitions.

In order to make the discussion more concrete, we choose to study the following “quench” problem. Consider a free theory on $\mathcal{F} = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_N$ with Hamiltonian $H_{\text{free}} = \sum_{i=1}^N \mu_i H_i$. The Hilbert space is a direct sum $\mathcal{H} = \bigoplus E \mathcal{H}_E$, where $\mathcal{H}_E$ is the eigenspace corresponding to energy $E$ spanned by $| n_1, \ldots, n_N \rangle$:

$$E = \sum_i \mu_i n_i.$$  \hspace{1cm} (4.1)

In a similar fashion, $P^\Delta$ denotes the projector to the initial ensemble $\mathcal{H}_{\Delta}$. We expect that there exists a truncation such that $\rho^\Lambda_S(t) = \text{tr}_S(P^\Lambda \psi(t) P^\Lambda)$ approximates $\rho_S(t)$ well. Here $\bar{S}$ denotes the complement of $S$. The triangle inequality tells us:

$$\| \rho_S(t) - \omega_S \| \leq \| \rho^\Lambda_S(t) - \omega^\Lambda_S \| + \| (\rho_S(t) - \omega_S) - (\rho^\Lambda_S(t) - \omega^\Lambda_S) \| \hspace{1cm} (4.2)$$

We call the second term on the right hand side the truncation error, and denote it by $E^\Lambda$. The trick is to choose a cut-off $\Lambda$ such that both the equilibration parameter in the truncated subspace $\eta^\Lambda_S$, and the errors caused by truncation are small. For this

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6Typically the term quench is used when the initial state is the ground state of an initial Hamiltonian. Here, the initial state is chosen from some large energy sector of the free theory.

7Choosing $\psi_0 \in \bigoplus_{E=\Delta-\delta} \mathcal{H}_E$ with some small $\delta/\Delta$ will not change any of our general conclusions.
procedure to work, we need to make a dynamical assumption that limits the support of wave-function on \( \mathcal{H}_E \) for \( E > \Lambda \), and quantifies the truncation errors. Here, we employ a generalization of the method in [14] that uses the mean photon number in quantum optics to quantify errors caused by truncation to a finite-dimensional Hilbert space. The following theorem provides upper bounds on both terms on the right hand side of (4.4):

**Theorem 4.1 Equilibration in infinite-dimensional \( \mathcal{H} \):** Consider an arbitrary state \( \psi_0 \), chosen at random from a subspace corresponding to energy \( \Delta \) in the Hilbert space of a free theory. For any subsystem \( S \), and projectors \( P^\Lambda \) and \( P^\Delta \) defined as in (4.1), we define:

\[
\eta^\Lambda_S(\psi_0) = \sqrt{\frac{(d^\Lambda_S)^2}{d_{eff}(\omega)}}
\]

where \( d^\Lambda_S \) is the dimension of the projected Hilbert space available to \( S \). After a quench by an arbitrary interaction Hamiltonian:

(i) **Fluctuations:**

\[
\langle \|\rho_S(t)^\Lambda - \omega^\Lambda_S\|\rangle_t \leq \eta^\Lambda_S.
\]  

(ii) **Truncation error:**

\[
\mathcal{E}^\Lambda \equiv \langle \|\rho_S(t) - \omega_S\| - (\rho_S^\Lambda(t) - \omega^\Lambda_S)\|\rangle_t \leq 6\sqrt{\frac{\text{tr}(\omega_{H_{\text{free}}})}{\Lambda}}
\]  

(iii) **Typicality:**

\[
\text{Prob} \left( d_{eff}(\omega) < \frac{d^\Delta}{4} \right) \leq 2e^{-c\sqrt{d^\Delta}},
\]  

where \( c = \frac{(\ln 2)^2}{72\pi^3} \approx 2 \times 10^{-4} \).

(iv) **Universality:**

\[
\text{Prob} \left( \|\omega^\Lambda_S(\psi_0) - \langle \omega^\Lambda_S \rangle_{\psi_0}\| > \frac{1}{2}\sqrt{\frac{d^{\Lambda\delta^\Lambda}}{d^\Delta}} + \epsilon \right) \leq 2e^{-c'\epsilon^2d^\Delta},
\]  

\[
\delta^\Lambda = \sum_k \langle k | \frac{P^\Delta}{d^\Delta} | k \rangle \text{ tr} [\text{tr}_S(P^\Lambda | k \rangle \langle k | P^\Lambda)^2],
\]

where \( c' = \frac{2}{9\pi^3} \). Here \( \langle \omega \rangle_{\psi_0} \) is the state \( \omega^\Lambda(\psi_0) \) averaged over \( \mathcal{H}_\Delta \), and \( | k \rangle \) are energy eigenstate of the interacting Hamiltonian.
The above theorem implies that the state of subsystem $S$ equilibrates for almost all initial states $\psi_0 \in \mathcal{H}_\Delta$, if there exist a cut-off $\Lambda$ such that $\eta_\Lambda S \ll 1$ and $\text{tr}(\omega H_{\text{free}}) \ll \Lambda$. The proof appears in appendix C.

The dynamical condition to meet to justify a truncation with $P^\Lambda$ is that the time-averaged expectation value of the free part of the Hamiltonian, which was initially $\Delta$, is much smaller than the cut-off scale $\Lambda$. For systems with $N$ degrees of freedom, it is more convenient to define new variables $p$, $m$, $\bar{m}$ and $\lambda$:

$$m = \Delta/N^p$$
$$\bar{m} = \text{tr}(\omega H_{\text{free}})/N^p$$
$$\lambda = \Lambda/\Delta,$$

that remain finite in the thermodynamic limit $N \to \infty$. In terms of these new variables, the truncation error $E^\lambda \leq 6\sqrt{\bar{m}/m\lambda}$. We devote the rest of this section to discussing the implications of theorem 4.1 in three different physical systems.

### 4.1 Bosons on a lattice

As our first example of equilibration in multi-particle quantum systems we consider bosons on a lattice with $N$ sites. The Hamiltonian of the free theory, in appropriate units, is simply the total boson number:

$$H = \mu N = \mu \sum_i a_i^\dagger a_i,$$

where $a_i$ and $a_i^\dagger$ are the creation and annihilation operators defined at each site on the lattice. Setting $\mu = 1$, the energy spectrum is given by all non-negative integers $\Delta$, each of which appear with degeneracy $d_\Delta = (\Delta+N-1)\cdot N^{-1}$, which is the number of weak compositions of $\Delta$ in $N$ parts. We restrict the ensemble of initial states to $\mathcal{H}_\Delta$. The effective dimension of $\psi_0$ only depends on the initial ensemble, and for almost all $\psi_0$ is larger than $d_\Delta/4$. Whereas, the dimension of the subspace available to subsystem $S$ after introducing the cut-off $\Lambda$ is the number of distinct compositions one finds by restricting to a sequence of length $s = |S|$ in weak compositions of $\Lambda$ in $N$ parts:

$$d_\Lambda^S \leq \sum_{i=1}^{\Lambda-1} \binom{i+s-1}{s-1} = \frac{\Lambda}{s} \left( \frac{\Lambda+s-1}{s-1} \right) - 1$$

We are interested in investigating how subsystems of different sizes equilibrate in the thermodynamic limit $N \to \infty$:

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8Roughly speaking, this is a constraint on the number density of particles.

9A weak composition of $\Delta$ in $N$ parts is a sequence of $N$ non-negative integers which sum to $\Delta$. 
Finite-site subsystems: The density matrix of subsystems of size \( s = O(1) \) contains information about small observables of the sort \( \langle \phi^{m_1}(x_1) \ldots \phi^{m_s}(x_s) \rangle \). Let us assume for the moment that the number of bosons per site is initially finite and remains so at all times, i.e. \( p \leq 1 \). From (4.4) we find that the error caused by truncation by \( \lambda = N^\epsilon \) is negligible for any \( \epsilon > 0 \). Plugging these into the expression for the equilibration parameter gives

\[
\eta^\Lambda_S < \frac{N^\epsilon \Delta}{s} \sqrt{\frac{(N^\epsilon \Delta + s - 1)!^2 (N - 1)! \Delta!}{(N^\epsilon \Delta)!^2 (\Delta + N - 1)!}} \tag{4.11}
\]

Using the Stirling approximation we find that:

\[
\eta^\Lambda_S \leq \begin{cases} 
    f(s) N^{s(p+\epsilon)+p/4-(1-p)mN^\epsilon/2} & \text{if } 0 < p < 1 \\
    g(s) N^{(1+\epsilon)s+1/4} \left( \frac{m}{1+m} \right)^{mN/2} & \text{if } p = 1,
\end{cases}
\tag{4.12}
\]

for some \( O(1) \) functions \( f(s) \) and \( g(s) \). Hence, in the thermodynamic limit \( N \gg 1 \) the density matrix of all finite subsystems equilibrate.

A small but finite fraction of all sites: Consider subsystems of size \( s/N \) order one but small, with a finite initial boson number density per site: \( m = O(1) \). Applying the Stirling approximation one finds that the equilibration parameter is bounded by

\[
\eta^\Lambda_S \leq c \left( \frac{mN\lambda \sqrt{\epsilon}}{s} \right)^{2s/N} \frac{1}{(1+m)} \left( \frac{m}{1+m} \right)^{mN/2} \left( \frac{N}{s^2} \right)^{1/4} \leq c \left( \frac{mN\lambda \sqrt{\epsilon}}{s} \right)^{2s/N} \frac{1}{(1+m)} \left( \frac{m}{1+m} \right)^{mN/2} \left( \frac{N}{s^2} \right)^{1/4}, \tag{4.13}
\]

for some \( O(1) \) constant \( c \). For any initial energy per site \( m \) and small error \( \mathcal{E}^\lambda \), there exists an \( s_0 \) found from

\[
\left( \frac{mN\lambda \sqrt{\epsilon}}{s_0} \right)^{2s_0/N} = (1+m) \tag{4.14}
\]

such that any subsystem smaller than \( s_0 \) equilibrates.

Large subsystems: A careful look at equation (4.14) shows that for any fixed \( \lambda \), \( s_0(m) \) is an increasing function of \( m \) with the asymptotic value \( \lim_{m \to \infty} s_0 = N/2 \); see figure 4.11. It is a curious fact that for large enough initial boson number density, not only small subsystems but also all subsystems smaller than half the size of the system equilibrate. Intuitively, this is due to the exponential growth in density of

\[101 \leq \frac{n!}{\sqrt{2\pi n(n/e)^n}} \leq \frac{4^n}{2\pi}.\]
states as a function of energy which makes the fraction of the Hilbert space available to $S$ vanishingly small for $N \gg 1$. In fact, for any $\Delta > O(N)$, in the limit of $N \to \infty$, both the error and the equilibration parameter vanish for any subsystem smaller than half.

### 4.2 Hermitian matrix model

For bosons on a lattice we found that states with energies $\Delta > O(N)$ equilibrate on all subsystems smaller than half. However, we do not expect such a sector to survive in continuum field theories. An example where one is interested in studying such high energy states is the matrix model. Consider the Hermitian one-matrix model with a free Hamiltonian

$$H = \text{tr} \left( \dot{X}^2 + m^2 X^2 \right), \quad (4.15)$$

where $X$ is an $N \times N$ matrix. The Hamiltonian is $U(N)$ invariant, a freedom which can be partially fixed by diagonalizing the matrix. In the diagonal form, each eigenvalue $\lambda_i$ becomes a simple harmonic oscillator. The change of variables in the quantum partition function gives a Van der Mond determinant which can be absorbed by a redefinition of eigenvalues \[7, 15\]. In terms of redefined eigenvalues, the wave-function $\psi(\lambda_1, \ldots, \lambda_N)$ is a tensor product of $N$ harmonic oscillators states, anti-symmetrized.
\[ \psi(n_1, \ldots, n_N) = \frac{1}{\sqrt{N!}} \det \begin{pmatrix} |n_1(\lambda_1)\rangle & |n_1(\lambda_2)\rangle & \cdots & |n_1(\lambda_N)\rangle \\ |n_2(\lambda_1)\rangle & |n_2(\lambda_2)\rangle & \cdots & |n_2(\lambda_N)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |n_N(\lambda_1)\rangle & |n_N(\lambda_2)\rangle & \cdots & |n_N(\lambda_N)\rangle \end{pmatrix}, \tag{4.16} \]

where \(|n_i(\lambda_j)\rangle\) is the \(n_i\)th excited state of the \(j\)th eigenvalue and multiplication inside the determinant is tensor product of states. The fermionic nature of the degrees of freedom implies that \(n_i\) is ordered as \(n_N > n_{N-1} > \ldots > n_1\). The ground state has energy

\[ E^{(0)} = \sum_{k=0}^{N-1} \left( k + \frac{1}{2} \right) \omega = \frac{N^2}{2} \omega. \tag{4.17} \]

We will set \(\omega = 1\) and measure energy in units of \(\omega\). The excitations are described by an ordered set of positive numbers \(r_N \geq r_{N-1} \geq \ldots \geq r_1 \geq 0\) where each \(r_i\) corresponds to the number of times the \(i\)th energy level is excited over its ground state, i.e.

\[ r_i = E_i - E_i^{(0)}. \tag{4.18} \]

If \(\Delta = E - E^{(0)}\), then to each total energy \(E\) corresponds a sub-Hilbert space \(H_\Delta\) spanned by degenerate eigenstates of that energy level. The dimension of \(H_\Delta\) is \(P(\Delta, N)\), the number of partitions of \(\Delta\) into at most \(N\) parts. To each of these states one can associate a Young diagrams with \(r_{N-i+1}\) boxes in the \(i\)th row from the top \[15, 16\]. Then, \(P(\Delta, N)\) counts the number of Young diagrams with \(\Delta\) boxes and maximum \(N\) rows.

Consider a subsystem of \(s\) eigenvalues. The dimension of \(d_\Lambda^s\) is bounded above by the number of distinct subdiagrams one finds by picking \(s\) rows from Young diagrams corresponding to \(P(\Lambda, N)\), and discarding the rest of the rows. Sub-diagrams have any number of boxes from one to \(\Lambda\), but have at most \(s\) rows. Therefore,

\[ d_\Lambda^s \leq \sum_{E=1}^{\Lambda} P(E, s) < \Lambda P(\Lambda, s). \tag{4.19} \]

Then, the equilibration parameter satisfies

\[ \eta_\Lambda^s \leq \Lambda \sqrt{\frac{P(\Lambda, s)^2}{P(\Delta, N)}}. \tag{4.20} \]

By studying the large \(N\) asymptotics of restricted partitions functions \(P(N, s)\) we find\[1,11\]

---

\[11\text{See appendix D.} \]
A finite set of eigenvalues: For \( s = O(1) \) and \( p \geq 0 \), the equilibration parameter is small:

\[
\eta_s^A \leq c \Lambda^s e^{-1/2(N^{1/2 \min[p, 1]})}
\] (4.21)

Choosing \( \Lambda \) to be any polynomial of finite order larger than one in \( \Delta \) keeps both the truncation error and the equilibration parameter small. Hence, in the thermodynamic limit, small subsystems equilibrate.

Large subsystems: One particularly interesting regime is \( \Delta = O(N^2) \). In this regime, we find that for large \( N \) and \( s/N = O(1) \), the equilibration parameter is bounded by

\[
\eta_s^A \leq c \sqrt{N} e^{f(s, m, \lambda)N/2},
\] (4.22)

for an order one constant \( c \). The exponent on the right hand side is given by:

\[
f(s, m, \lambda) = 2 \left( \beta_c \frac{s}{N} - \frac{mN\lambda}{s} \log(1 - e^{-\beta_c}) \right) - (\beta - 2m \log(1 - e^{-\beta}))
\] (4.23)

where \( \beta \) and \( \beta_c \) are found by solving

\[
m = \frac{\text{Li}_2(e^{-\beta})}{\log(1 - e^{-\beta})^2},
\]

\[
\frac{mN^2\lambda}{s^2} = \frac{\text{Li}_2(e^{-\beta_c})}{\log(1 - e^{-\beta_c})^2}.
\] (4.24)

For fixed \( m \) and arbitrary small error \( \mathcal{E}^A \leq 6 \sqrt{m/(m\lambda)} \) all subsystems of size smaller than \( s_0N \) equilibrate, where \( s_0 \) is the solution to \( f(s_0, m, \lambda) = 0 \). In the limit of a large initial boson number \( m \gg 1 \), and small errors \( \mathcal{E}^A \ll 1 \), it is possible to simplify the expressions in (4.23) to obtain:

\[
m \simeq \left( \frac{N^2\lambda}{s_0^2} \right)^{2s_0/(N-2s_0)}
\] (4.25)

The interesting fact is that by increasing \( m \) it becomes possible for larger and larger subsystems to equilibrate. In the energy range \( \Delta > O(N^2) \) we find that all subsystems smaller than half the size of the system equilibrate with a vanishing error. Note that in the Hermitian matrix model, the regime relevant for equilibration of large subsystems has an initial energy per degree of freedom that scales with \( N \), as opposed to bosons on a lattice where it was finite.
Figure 4: The size of the largest subsystem that provably equilibrates plotted as a function of the average initial energy per matrix element; $\Delta/N^2$. The error parameter is suppressed by $\mathcal{E}^\lambda \leq 6 \times 10^{-6} \sqrt{m/m}$. In the limit of infinite initial energy any subsystem smaller than half the size of the whole system equilibrates.

4.3 Field theories in momentum space

Our discussion so far has been restricted to discrete models. In continuum field theories the Hilbert space of free theory has a natural decomposition in terms of momentum modes. On a compact space and in the absence of any internal symmetries, the Hilbert space is a tensor product of infinite Fock spaces $\mathcal{F}_k$ corresponding to momentum modes $k$. There is no analogue of parameter $N$ in momentum space, so we only consider high energy ensembles, i.e. $\Delta \gg 1$.

1+1-dimensional field theories

For simplicity, let us start with a scalar field in a one-dimensional box and normalize such that $k$’s are positive integers. The Hamiltonian of a free massless theory has the form

$$H = \sum_k \infty k a_k^\dagger a_k.$$  

The dimension of the energy eigenspace $\Delta$ is given by the number of partitions of $\Delta$, which according to the Hardy-Ramanujan expansion scales as

$$d_\Delta = \mathcal{P}(\Delta) \simeq e^{\pi \sqrt{2\Delta/3}}. \quad (4.27)$$

12The generalization to fields with non-zero spin is straightforward.
On the other hand, the dimension of the Hilbert space available for a set of momentum modes \( S = \{ k_1, \ldots, k_s \} \) with a momentum cut-off \( \Lambda \) satisfies

\[
d_S^\Lambda \leq \sum_{E=1}^{\Lambda} D(E; S),
\]

and \( D(E; S) \) is the denumerant; the number of ways one can partition the number \( E \) as \( a_1 k_1 + \ldots + a_s k_s \).

**Finite momentum modes:** The large \( E \) asymptotic of denumerant for finite \( s \) is given by \( ^{13} \)

\[
D(E; S) \simeq \frac{E^{s-1}}{(s-1)! k_1 k_2 \ldots k_s}.
\]

It is clear from \( ^{13} \) that for large \( \Lambda \), \( d_S^\Lambda \) is bounded by a polynomial of degree \( s \) in \( \Lambda \). As a result, for any \( \Lambda \) polynomial in \( \Delta \) of a finite order larger than one, both the equilibration parameter and the truncation error are small. The theorem \( ^{14} \) implies that \( \rho_S \) equilibrates for any finite set of momentum modes. In other words, the results of all field theory measurements and observables that involve a finite number of \( k \) modes, almost at all times after the quench, are well approximated by those of a stationary state, conditioned that whatever the non-perturbative dynamics is, the expectation value of kinetic energy does not grow indefinitely in time.

**Large subsystems:** An interesting observation is that for any subsystem size \( s \), \( d_S^\Lambda \) is at its maximum value when \( S \) is the “infra-red” density matrix; i.e. \( S = \{ k : |k| \leq s \} \) \( ^{14} \). The reason is that for all \( S \) and \( E \) the denumerant satisfies \( D(E; S) \leq P(E, s) \), where \( P(E, s) \) is the number of partitions of \( E \) into at most \( s \) parts with the equality holding when \( S = \{ k : |k| \leq s \} \). \( ^{15} \)

Let \( \rho^{IR}_\mu \) and \( \rho^{UV}_\mu \) denote, respectively, the “infra-red” and “ultra-violet” density matrices corresponding to the set of modes \( S = \{ k : k \leq \mu \} \) and \( S = \{ k : k > \mu \} \). For a finite energy state, the expectation is that we do not lose details of the dynamics by tracing out large momentum modes. In this context of equilibration, we ask what are the largest and smallest \( \mu \) such that, respectively, \( \rho^{IR}_\mu \) and \( \rho^{UV}_\mu \) are time-independent, and contains no information about the dynamics or the initial state.

\(^{13}\)See appendix \( ^{12} \) for details.

\(^{14}\)Note that in a theory with non-perturbative interactions, low-momentum modes can have overlap with large energy eigenstates. Therefore, the reduced density matrix \( \rho_S \) is not necessarily capturing the low-energy dynamics. However, the constraint on the time-average of the free Hamiltonian expectation value implies that this overlap is small for eigenstates of energy larger than \( \Lambda \).

\(^{15}\)We are using the following well-known result in number theory: the number of partitions of \( E \) into at most \( m \) parts is equal to the number of partitions of \( E \) with each part at most \( m \).
The dimension of the Hilbert space available to $\rho^\text{IR}_\mu$ is

$$d^\text{IR}_\mu \leq \sum_{E=1}^\Lambda P(E, \mu).$$  \hspace{1cm} (4.30)

The asymptotic behaviour of $P(E, \mu)$ for all large $\mu$ and $\Delta$ is such that the equilibration parameter is never small unless $\mu = a\sqrt{\Delta}$, for some $a$ at least order one. The analysis of asymptotics of restricted partitions in appendix D shows that for $\mu = a\sqrt{\Delta}$:

$$\eta^\Lambda_S \leq c \sqrt{\Delta} e^{\frac{1}{2}g(a)\sqrt{\Delta}},$$  \hspace{1cm} (4.31)

where

$$g(a) = a \left(\frac{2}{\beta} \int_0^\beta \frac{dt}{e^t - 1} - \log(1 - e^{-\beta})\right) - \sqrt{\frac{2}{3}}\pi,$$

$$\beta^2 = \frac{a^2}{\lambda} \int_0^\beta \frac{dt}{e^t - 1},$$  \hspace{1cm} (4.32)

and $c$ is some order one constant. For $\lambda$ large enough so that $\mathcal{E}^\lambda \ll 1$, the equilibration parameter is small for all $a \leq a_{IR}$, where $a_{IR}$ is the solution to $g(a_{IR}) = 0$.

Similarly, the dimension of the Hilbert space available to $\rho^\text{UV}_\mu$ is

$$d^\text{UV}_\mu = \sum_{E=1}^\Lambda P(E, \geq \mu),$$  \hspace{1cm} (4.33)

where $P(E, \geq \mu)$ is the number of partitions of $E$ into parts at least $\mu$. Since these partitions have at most $E/\mu$ parts, then $P(E, \geq \mu) \leq P(E, E/\mu)$. It follows that $\rho^\text{UV}_\mu$ equilibrates if $a > a_{UV}$ where $a_{UV}$ is found by replacing $a$ by $\lambda/a$ in (4.32) and solving for $g(a_{UV}) = 0$.

**Higher-dimensional field theories**

In dimensions higher than one, it is easier to work with generating partition functions:

$$Z(q) = \sum_\Delta P(\Delta)q^\Delta = \prod_{k_{x_1}, \ldots, k_{x_d}} \frac{1}{1 - q^{\Delta(k)}}.$$  \hspace{1cm} (4.34)
For simplicity we assume that the field theory lives in a cubic box. Then, the free energy at large $\Delta$ is approximated by an integral:

$$
F(e^{-\beta}) = -\frac{1}{\beta} \log Z(e^{-\beta}) = \Omega(d-1)\beta^{-1} \int_0^\infty dk \, k^{d-1} \log(1 - e^{-\beta k})
$$

$$
= -\Omega(d-1)\beta^{-1} \sum_{n=1}^\infty \int_0^\infty dk \, k^{d-1} \frac{e^{-\beta nk}}{n}
$$

$$
= -\beta^{-(d+1)} f(d),
$$

(4.35)

where $f(d) = \Omega(d-1)\Gamma(d)/\zeta(d+1)$. The expectation value of energy and entropy are found in terms of $\beta$:

$$
\bar{E} = F + \beta \partial_\beta F = \beta^{-(d+1)} \int f(d)
$$

$$
S(\beta) = \beta^2 \partial_\beta F = \beta^{-d} (d+1) f(d).
$$

(4.36)

The number of partitions at large $E$ is well approximated by $e^{S(\bar{E})}$:

$$
P(E) \simeq e^{AE/(d+1)}
$$

(4.37)

where $A = (d+1) (d^{-d} f(d))^{1/(d+1)}$. This is a well-known result for density of states in free $d$-dimensional theories.

The partition function that generates $P(E, s)$ is also easy to write down:

$$
Z_S(\beta) = \sum_E P(E, s) e^{-\beta E} = \prod_{k=1, \ldots, k_d} \frac{1}{(1 - e^{-\beta E(k)})}
$$

(4.38)

The free energy associated with this partition function is

$$
F = \Omega(d-1)\beta^{-1} \sum_{n=1}^\infty \int_0^s dk \, k^{d-1} \log(1 - e^{-\beta k}).
$$

(4.39)

The expectation value of energy and the entropy are

$$
\bar{E} = \Omega(d-1) \int_0^s \frac{dk \, k^d}{e^{k\beta} - 1} = \Omega(d-1)\beta^{-(d+1)} \int \frac{\beta}{e^t - 1} dt \frac{t^d}{e^t - 1}
$$

(4.40)

$$
S(\beta) = \Omega(d-1) \left( \beta \int_0^s \frac{dk \, k^d}{e^{k\beta} - 1} - \int_0^s dk \, k^{d-1} \log(1 - e^{-\beta k}) \right)
$$

$$
= \Omega(d-1) s^d \left( 2\beta^{d-2} \int \frac{\beta}{e^t - 1} \frac{dt \, t^d}{e^t - 1} - \frac{\log(1 - e^{-\beta})}{d} \right),
$$

(4.41)

Knowledge of the spectrum of Laplacian is sufficient to generalize this method to arbitrary compact spaces.
where we have defined \( t = \beta k \) and \( \tilde{\beta} = \beta s \). When \( \bar{E} \gg 1 \) and \( \frac{s}{E^{1/(d+1)}} \gg 1 \), (4.40) implies that \( \tilde{\beta} \) is small. The entropy at small \( \tilde{\beta} \) is dominated by the term \( (s/\tilde{\beta})^d \) in (4.41). Solving for entropy as a function of energy in this limit reproduces the familiar expression in (4.37) for entropy of free theories. In this regime, since \( d_S^\Delta \simeq d_\Delta \), the equilibration parameter is large. However, if \( \frac{s}{E^{1/(d+1)}} = O(1) \) then \( \tilde{\beta} \) is also order one, and \( P(E, s) \) grows as \( E^{d/(d+1)} \), but with a smaller coefficient. Similar to the situation in one spatial dimension, any subsystem of size \( s = aE^{1/(d+1)} \) with \( a < a_{IR} \) equilibrates, where \( a_{IR} \) is the solution to

\[
\Omega(d-1) a^d \left( \frac{A}{2} - \frac{\log(1 - e^{-\tilde{\beta}})}{d} \right) = \frac{\bar{E} \Delta}{\Omega(d-1)}
\]

(4.42)

## 5 Atypical initial states and scrambling

The typicality theorem implies that almost all initial states (typical states) in \( \mathcal{H}_R \) equilibrate; however, in scrambling we are interested in far from equilibrium initial states which are highly atypical. For instance, consider an initial product state \( \psi = \phi_M \otimes \psi_M \) for a small subsystem \( M \). Then, from the fluctuation theorem we know that for a fixed \( \phi_M \in \mathcal{H}_M \), and typical states \( \psi_M \)

\[
\langle \parallel \rho_S(t) - \omega_S(\psi_M, \phi_M) \parallel \rangle < \sqrt{\frac{d_S^2}{d_{eff}(\omega_M)}}.
\]

(5.1)

As a result, if \( d_M \gg d_S^2 \) the density matrix of \( S \) is independent of \( \psi_M \). As we saw previously, this condition is easily satisfied for small messages. However, the equilibrium density matrix can still depend on the message \( \phi_M \). We say a Hamiltonian scrambles an ensemble of initial states, if all subsystems \( S \) smaller than half the size of the system \( \omega_S \) become independent of \( \phi_M \). It is clear that this does not always occur. As was argued in [1], in systems where the eigenstates of the interacting Hamiltonian remain close to tensor products on \( M \) and \( \bar{M} \) (as in some weakly coupled systems), \( \omega_S \) retains information about \( \phi_M \).

One might think that the type of arguments we have been using so far fails to provide insight about scrambling of small messages. The typicality arguments we have been using are based on the idea that a small subsystem \( S \) does not have enough

---

\(^{17}\) For \( d = 1 \), after some algebra, (4.40) reproduces the asymptotic behaviour we quote in appendix [D].

\(^{18}\) Replacing \( d_R \) with \( d_S^2 \) in theorem [L] implies typicality and universality for product states with typical \( \psi \in \mathcal{H}_R \).
Hilbert space (memory) available to keep track of all the information about the initial state. However, if we encode a small message (e.g. with one degree of freedom) in the initial state as in the case of a product state, it is likely that the message fits in subsystems with parametrically larger number of degrees of freedom. However, a careful look at theorem [4.1] shows that this is not necessarily true. Intuitively, if our Hamiltonian of interest is very efficient in generating entanglement, there might not be much room left in large subsystems to hold information. This is reflected in parameter $\delta^A$ in [4.6]. Consider the universality argument for initial state with $\psi_M$ fixed and a typical $\phi_M \in H_M$:

$$\langle \| \omega_S(M) - \langle \omega_S \rangle_M \| \rangle \leq \sqrt{\frac{d_S^A \delta^A}{d_M}},$$  \hspace{1cm} (5.2)

$$\delta^A = \sum_k \langle k \| \frac{P^M}{d_M} | k \rangle \text{ tr} \left[ \left( \text{tr}_{S} (P^A | k \rangle \langle k | P^A) \right)^2 \right],$$  \hspace{1cm} (5.3)

where $| k \rangle$ are the eigenstates of the full interacting Hamiltonian, $P^M$ is the projector to $H_M$. If the right hand side of (5.2) is small, the message $M$ is scrambled in subsystems of size less than or equal to $S$. The parameter $\delta^A$ is a weighted average of regularized subsystems’ purities in energy eigenstates. Note that if the subsystem purity (equivalently its second Renyi entropy) is bounded above (below) by an energy independent quantity $\alpha/d_S^A$, then $\delta^A d_S^A \leq \alpha$. The logarithm on the right hand side of (5.2) provides a sufficient condition for scrambling:

$$n_M - \log \alpha \gg 1,$$  \hspace{1cm} (5.4)

where $n_A = \log(d_A)$ denotes the number of degrees of freedom in subspace $H_A$.

The Renyi entropy of subsystems in vacuum is a topic that has attracted a lot of attention recently. The ground state of systems with local interactions seem to have an entanglement entropy proportional to the area of the subsystem; see [17] for a review. This goes with the name area law, and supports the simple picture that due to the locality of interactions, it is only the degrees of freedom near the boundary that matter in the calculation of entanglement entropy. The same is believed to be true for low-energy eigenstates and subsystems’ second Renyi entropy. The sufficient condition for scrambling a message $M$ in local systems with area law for all energy eigenstates in $d > 1$ dimension is:

$$n_M - n_S - 2c n_S^{(d-1)/d} \gg 1,$$  \hspace{1cm} (5.5)

and $c$ is the constant appearing in the area law formula. There is some evidence that in discrete systems there is a high energy sector of the theory that has an extensive
entanglement entropy [18, 19]. In such cases, the coefficient $c$ plays a crucial role in the size of the largest message that is scrambled. It is not clear whether such sectors survive in the continuum limit.

With regards to the scrambling conjecture, the Hamiltonian of the matrix model or other quantum models of black holes include highly non-local interactions. For systems with non-local interactions it is not hard to imagine sectors of the spectrum that have extensive entanglement entropy. It seems to us that finding bounds on the scaling of entanglement entropy of energy eigenstates in matrix model or other models of quantum black holes is a promising approach to proving that they are scramblers.

6 Conclusions and discussion

In this work, we have generalized the recent information-theoretic approach to equilibration to infinite-dimensional Hilbert spaces, including as examples field theories and matrix models. We generically find that small subsystems equilibrate, in the sense that their density matrices become almost independent of time and initial state. Our results suggest that at higher energies, larger subsystems can reach equilibrium. In the limit of infinite energy, we show that any subsystem smaller than half equilibrates. Our work emphasizes the importance of energy scales for equilibration and scrambling of large subsystems, both in field theory and matrix models. It is an interesting fact that even in Hermitian one-matrix model that has $N$ degrees of freedom, the energy-scale relevant for equilibration of large subsystems is order $N^2$. This is in contrast with lattice field theories, where the relevant scale is the same order as the number of degrees of freedom $N$.

For concrete estimation of the equilibration parameter we focused our attention on a quench problem from the high-energy sectors of free theories. However, the formalism is powerful enough to apply to more general ensembles of initial states. The challenge in each case is to find upper bounds on the dimensions of the Hilbert space available to subsystems. In our treatment of the problem, we find bounds on the support of the wave-function in the Hilbert space of free theory, and take advantage of our knowledge of the partition function at zero coupling. This is the reason why the dynamical constraint that appears is on the expectation value of the free Hamiltonian. Loosely speaking, the constraint we use, assumes an upper bound on the time average of particle number densities. We believe that this quantity is not a natural observable to consider in strongly coupled systems. It would be interesting to reformulate our approach in terms of energy expectation values in the interacting theory, and without any reference to the zero coupling Hilbert space, at least for some simple systems.
In spite of our powerful results in lattice field theories, we deliberately avoided addressing the equilibration of density matrices corresponding to spatial regions in continuum field theories. The missing step in applying typicality to these cases is a faithful truncation of the reduced density matrix to a finite-dimensional Hilbert space. Another subtlety appearing in the continuum limit is that one has to introduce a time scale the corresponding to turning on the interaction. However, this time-scale is clearly negligible compared to the time-scales required for the typicality argument to work.

An interesting question to ask is whether one can generalize our discussion of equilibration to multi-matrix models. In multi-matrix models, the notion of a subsystem, and how the true degrees of freedom interact is obscured by gauge transformation. Our philosophy is that for systems with gauge symmetry, the equilibration is more naturally formulated in terms of macro-observables discussed in appendix B.

Our focus in this work was on the equilibration of small and large subsystems by proving upper bounds on the time-averaged trace distance of states from their equilibrium value. However, in the context of information loss in black holes, one is also interested in deviations from equilibration. Finding a lower bound on the trace distance would be a natural way to address the question of how large is the smallest subsystem that does provably contain information about the full quantum state, but we leave this for future work.

Acknowledgements

We are grateful to Jens Eisert, Patrick Hayden, Alexander Maloney, Guy D. Moore, Robert Raussendorf, Joan Simon and Mark Van Raamsdonk for valuable conversations. This work is supported by the National Science and Engineering Research Council of Canada.

19 A truncation that introduces negligible errors.

20 Turning on the interaction instantaneously might introduce infinities in $\tilde{m}$. We thank Guy D. Moore for clarifying this issue.
Appendices

A Levy’s lemma

Lemma A.1 Given a function \( f : S^d \to \mathbb{R} \), with Lipschitz constant \( \lambda = \sup |\nabla f| \), at any random point \( x \in S^d \),

\[
\Pr[|f(x) - \langle f \rangle_{x \in S^d}| \geq \epsilon] \leq 2e^{-c(d+1)\epsilon^2/\lambda^2}, \tag{A.1}
\]

where \( c = \frac{1}{18\pi^3} \).

See \[21\] for a proof.

A pure \( d \)-dimensional quantum state \( |\psi\rangle = \sum_k c_k |k\rangle \) can be thought of as a point on \( S^{2d-1} \) with the real and imaginary parts of \( c_k \) as coordinates. We apply Levy’s lemma to functions \( f(\psi) \) to discuss the probability of finding \( f(\psi) \) far from its ensemble average \( \langle f(\psi) \rangle_\psi \).

B Coarse-grained observables

In a different approach to equilibration, Reimann \[2\] showed that, quite generically, the state of quantum systems becomes indistinguishable from its equilibrium with respect to coarse-grained observables. An observable is coarse-grained if it has finite precision, or in other words can have only a finite number of measurement outcomes.

Following \[3\], we define the distinguishability of two states \( \rho \) and \( \sigma \) with respect to a set of coarse-grained observables \( \mathcal{M} \) to be:

\[
D_\mathcal{M}(\rho, \sigma) = \frac{1}{2} \sum_r |\text{tr}(M_r \rho) - \text{tr}(M_r \sigma)|, \tag{B.1}
\]

where \( M_r \) are all POVMs that describe the set \( \mathcal{M} \). The operational interpretation of distinguishability is similar to the trace-norm: if \( D_\mathcal{M}(\rho, \sigma) \leq \epsilon \) the optimal probability of telling \( \rho \) and \( \sigma \) apart using observables \( \mathcal{M} \) is \( \frac{1}{2}(1 + \epsilon) \).

The following theorem was shown in \[3\].

**Theorem B.1** Consider a closed quantum system evolving with a Hamiltonian which has non-degenerate energy gaps. The time-averaged distinguishability of the state of the system from its time-average with respect to a set of observables \( \mathcal{M} \) satisfies:

\[
\langle D_\mathcal{M}(\rho(t), \omega) \rangle_t \leq \frac{N(\mathcal{M})}{4\sqrt{d_{\text{eff}}(\omega)}}. \tag{B.2}
\]
where $N(\mathcal{M})$ is the total number of outcomes in $\mathcal{M}$. The denominator $d_{eff}(\omega)$ is exponentially large in the number of degrees of freedom. Therefore, for all realistic measurements we expect the state to be indistinguishable from its time-average.

C Proof of theorem 4.1:

The proof we provide here for theorem 4.1 closely follows [1].

C.1 Fluctuations:

Proof of (i): Consider an initial state $\psi_0 = \sum_k c_k |k\rangle$, where $|k\rangle$ is an energy eigenstate of the interacting Hamiltonian after the quench. Then, in the absence of degenerate energy gaps $^{21}$,

$$\rho^\Lambda_S(t) - \omega^\Lambda_S = \sum_{k,l} c_k c_l^* (e^{-i(E_k - E_l)t} - \langle e^{-i(E_k - E_l)t} \rangle_t) \text{ tr } \rho \langle P^\Lambda | k \rangle \langle l | P^\Lambda \rangle$$

$$= \sum_{k \neq l} c_k c_l^* e^{-i(E_k - E_l)t} \text{ tr } \rho \langle P^\Lambda | k \rangle \langle l | P^\Lambda \rangle$$

(C.1)

where we have used $\langle e^{-i(E_k - E_l)t} \rangle_t = \delta_{kl}$. From the Cauchy-Schwarz inequality we know that the trace norm is bounded above by:

$$\langle \| \rho^\Lambda_S(t) - \omega^\Lambda_S \| \rangle_t \leq \sqrt{d^\Lambda_S \langle \text{ tr } [\rho^\Lambda_S(t) - \omega^\Lambda_S]^2 \rangle_t}$$

(C.2)

The right-hand side of (C.2) can be simplified further:

$$\langle \text{ tr } [\rho^\Lambda_S(t) - \omega^\Lambda_S]^2 \rangle_t = \sum_{k \neq l, m \neq n} c_k c_m c_n^* c_l^* (e^{-i(E_k - E_l + E_m - E_n)t})_t \text{ tr } \rho \langle P^\Lambda | k \rangle \langle l | P^\Lambda \rangle \text{ tr } \rho \langle P^\Lambda | m \rangle \langle n | P^\Lambda \rangle$$

$$= \sum_{k \neq l} |c_k|^2 |c_l|^2 \sum_{s,s',b,b'} \langle sb | P^\Lambda | k \rangle \langle l | P^\Lambda | s'b \rangle \langle s'b | P^\Lambda | k \rangle \langle l | P^\Lambda | s'b \rangle$$

$$= \sum_{k \neq l} |c_k|^2 |c_l|^2 \sum_{s,s',b,b'} \langle sb | P^\Lambda | k \rangle \langle l | P^\Lambda | s'b \rangle \langle s'b | P^\Lambda | l \rangle \langle l | P^\Lambda | s'b \rangle$$

$$= \sum_{k \neq l} \text{ tr } [\rho_S (|c_k|^2 P^\Lambda | k \rangle \langle l | P^\Lambda \rangle) \text{ tr } \rho_S (|c_l|^2 P^\Lambda | l \rangle \langle l | P^\Lambda \rangle)]$$

$$= \text{ tr } [(\omega^\Lambda_S)^2] - \sum_k |c_k|^4 \text{ tr } [(\rho \rho^\Lambda_S | k \rangle \langle k | P^\Lambda \rangle)^2]$$

$$\leq \text{ tr } [(\omega^\Lambda_S)^2]$$

(C.3)

$^{21}$For a treatment of cases with degenerate gaps see [13]
where \{\{s\}\} and \{\{b\}\} are orthonormal bases that span \(\mathcal{H}_S\) and \(\mathcal{H}_S\), and we have used the absence of degenerate energy gaps: \(\langle e^{-i(E_k - E_l + E_m - E_n)\tau} \rangle_t = \delta_{kn}\delta_{lm}\). The weak subadditivity of purity [22] implies \(\text{tr}[(\omega_S^A)^2] \leq \delta^A_S\text{tr}[(\omega^A)^2]\). Expanding in the orthonormal basis of free theory \{\{n\}\} it is clear that
\[
\text{tr}((\omega^A)^2) = \sum_{n,m,k,l} \omega_{nm}\omega_{kl} \langle l|P^A|n\rangle \langle m|P^A|k\rangle = \sum_{n,m,k,l \in \mathcal{H}_{E \leq \Lambda}} |\omega_{nm}|^2 \leq \sum_{n,m,k,l} |\omega_{nm}|^2 = \text{tr}(\omega^2).
\]
Putting this back into (C.2) we find:
\[
\langle \|\rho_S^A(t) - \omega_S^A\| \rangle_t \leq \sqrt{\frac{(\delta_S^A)^2}{d_{\text{eff}}(\omega)}} = \eta_S^A
\]
(C.5)

C.2 Truncation Error

Proof of (ii): Expand the state \(\psi(t)\) in the orthonormal basis of free theory:
\[
\psi(t) = \sum_n c_n(t)c_m(t)|n\rangle \langle m|
\]
with \(|n\rangle\) denoting the eigenstate \(|n_1, \ldots, n_N\rangle\). The expectation value of the free Hamiltonian at time \(t\) satisfies:
\[
\text{tr}(\psi(t)H_{\text{free}}) \geq \sum_{n: \sum_i \mu_i n_i = \Lambda} |c_n(t)|^2 \sum_i \mu_i n_i \geq \Lambda \sum_{n: \sum_i \mu_i n_i = \Lambda} |c_n(t)|^2 \geq \Lambda \text{tr}(\psi - \psi^A).
\]
(C.7)
The matrix \(\psi = \begin{pmatrix} \psi^A & A \overline{A}^\dagger \\ \overline{A}^\dagger \psi^{UV} \end{pmatrix}\) is positive semi-definite, and as a result \(\|A\|^2 \leq \|\psi^A\|\|\psi^{UV}\| \leq \|\psi^{UV}\|\Lambda\). From (C.7) and the triangle inequality we find
\[
\|\psi(t) - \psi^A(t)\| \leq 2\|A\| + \|\psi^{UV}\| \leq 3\sqrt{\|\psi^{UV}\|} \leq 3\sqrt{\frac{\text{tr}(\psi(t)H_{\text{free}})}{\Lambda}}
\]
Hence, the truncation error is bounded above by
\[
\langle \|\rho_S(t) - \omega_S\| - (\rho_S^A(t) - \omega_S^A)\| \rangle_t \leq \langle \|\psi(t) - \psi^A\|\rangle_t + \|\omega - \omega^A\| \leq 6\sqrt{\frac{\text{tr}(\omega H_{\text{free}})}{\Lambda}},
\]
(C.8)
where we have used the fact that partial trace as a quantum operation brings density operators closer, i.e. \( \| \rho_S - \sigma_S \| \leq \| \rho - \sigma \| \).

### C.3 Typicality

**Proof of (iii):** We are interested in finding the probability for \( d_{\text{eff}}(\omega) \) to be small. Let us define the following two maps:

\[
F[\rho] = \sum_k |k\rangle \langle k| \rho |k\rangle \langle k|
\]

\[
\tilde{F}[\rho] = \sum_k |\tilde{k}\rangle \langle k| \rho |k\rangle \langle \tilde{k}|
\]

where \(|k\rangle\) are eigenstates of the interacting Hamiltonian, and \(|\tilde{k}\rangle = \frac{P_{\Delta} |k\rangle}{\| P_{\Delta} |k\rangle \|}\) are their projections to the initial ensemble. The first map \( F \) dephases states by killing off-diagonal elements in the interacting Hamiltonian basis, e.g. \( \omega = F[\psi_0] \). The second map \( \tilde{F}[\psi] \) acts on states in \( \mathcal{H}_T \) to dephase and projects them back to the initial ensemble \( \mathcal{H}_R \).

An important observation is that the purity of \( \tilde{F}[\psi_0] \) is larger than that of \( \omega \), and therefore \( d_{\text{eff}}(\omega) \geq \frac{1}{\text{tr} (\tilde{F}[\psi_0]^2)} \). Then, to prove (iii) it suffices to show that

\[
\text{Prob} \left( \text{tr} (\tilde{F}[\psi_0]^2) > \frac{4}{d_{\Delta}} \right) \leq 2e^{-c\sqrt{d_{\Delta}}}
\]

by applying Levy’s lemma to the function

\[
f(\psi_0) = \ln \left( \text{tr} (\tilde{F}[\psi_0]^2) \right).
\]

We need to find upper bounds on the average and the Lipschitz constant of \( f \). We employ following lemmas:

**Lemma C.1** For \( \psi \in \mathcal{H}_R \):

\[
\langle |\psi_0\rangle \langle \psi_0| \otimes |\psi_0\rangle \langle \psi_0| \rangle_{\psi_0} = \frac{(P_{\Delta} \otimes P_{\Delta}) (I + \mathbb{S})}{d_{\Delta}(d_{\Delta} + 1)}.
\]

The swap operator \( \mathbb{S} \) is defined by \( \mathbb{S} |l, k\rangle = |k, l\rangle \), and we have used the trick \( \text{tr} (AB) = \text{tr} (\mathbb{S} (A \otimes B)) \).

**Lemma C.2** The Lipschitz constant of \( f(\psi) \equiv \ln \left( \text{tr} \left( (\tilde{F}[\psi])^2 \right) \right) \) is bounded above according to:

\[
\lambda = |\nabla f| \leq 4d_{\Delta}^{1/4}.
\]
The proofs appear in Appendix B of [1]. With above lemmas in mind, it is not hard to come by an upper bound on \( \langle f \rangle_{\psi_0} \):

\[
\langle f \rangle_{\psi_0} \leq \ln \left( \text{tr} (\overline{F}[\psi_0]^2) \right)_{\psi_0} \\
\leq \ln \text{tr} \left( S (\overline{F} \otimes \overline{F}) (|\psi_0\rangle \langle \psi_0|) \right) \\
= \ln \left( \sum_{k,l} \text{tr} \left( |\tilde{k}\rangle \langle kl| \left( \frac{(P^\Delta \otimes P^\Delta)(I + S)}{d_\Delta(d_\Delta + 1)} \right) |kl\rangle \langle \tilde{l}| \right) \right) \\
\leq \ln \left( \sum_{k,l} \langle \tilde{l}|k\tilde{l}\rangle \left( \frac{\langle kl|P^\Delta \otimes P^\Delta(|kl\rangle + |lk\rangle)}{d_\Delta(d_\Delta + 1)} \right) \right) \\
\leq \ln \left( \frac{2}{d_\Delta(d_\Delta + 1)} \sum_{kl} \langle kl|P^\Delta \otimes P^\Delta|kl\rangle \right) \\
\leq \ln \left( \frac{2}{d_\Delta} \right). \tag{C.14}
\]

Plugging (C.13) and (C.14) into Levy’s lemma for \( f[\psi_0] \) gives:

\[
\text{Prob} \left( \text{tr} (\overline{F}[\psi_0]^2) > \frac{4}{d_\Delta} \right) \leq 2e^{-c\sqrt{d_\Delta}}, \tag{C.15}
\]

which completes the proof of typicality.

C.4 Universality

**Proof of (iv):** We apply Levy’s lemma to the function \( g[\psi_0] = ||\omega^A_S - \langle \omega^A_S \rangle_{\psi_0}||. \) The Lipschitz constant of \( g \) satisfies \( |\nabla g| \leq 1. \) As for the upper bound on \( \langle g \rangle_{\psi_0} \) we use the Cauchy-Schwarz inequality:

\[
\langle g \rangle_{\psi_0} \equiv \langle ||\omega^A_S - \langle \omega^A_S \rangle_{\psi_0}|| \rangle_{\psi_0} \\
\leq \sqrt{d^A_S \langle \text{tr} [(\omega^A_S - \langle \omega^A_S \rangle_{\psi_0})^2] \rangle_{\psi_0}} \tag{C.16}
\]
The right hand side of (C.16) is bounded by:

\[
\langle \text{tr} \left[ (\omega_\Delta^e - \langle \omega_\Delta^e \rangle_{\psi_0})^2 \right] \rangle_{\psi_0} = \text{tr} \left[ S \left( \langle \omega_\Delta^e \otimes \omega_\Delta^e \rangle_{\psi_0} - \langle \omega_\Delta^e \rangle_{\psi_0} \otimes \langle \omega_\Delta^e \rangle_{\psi_0} \right) \right] \\
= \text{tr} \, \text{tr}_{SS} \left[ \left( \text{tr}_{SS} \left[ \mathcal{P}^\Delta(\mathcal{F}) \otimes \mathcal{P}^\Delta(\mathcal{F}) \left( \langle \psi_0 \otimes \psi_0 \rangle_{\psi_0} - \frac{\mathcal{P}^\Delta \otimes \mathcal{P}^\Delta}{d_\Delta^2} \right) \right] \right) \right] \\
= \text{tr} \, \text{tr}_{SS} \left[ \left( \text{tr}_{SS} \left[ \mathcal{P}^\Delta(\mathcal{F}) \otimes \mathcal{P}^\Delta(\mathcal{F}) \left( \frac{\mathcal{P}^\Delta \otimes \mathcal{P}^\Delta(\mathbb{I} + \mathcal{S})}{d_\Delta(d_\Delta + 1)} - \frac{\mathcal{P}^\Delta \otimes \mathcal{P}^\Delta}{d_\Delta^2} \right) \right] \right) \right] \\
\leq \text{tr} \, \text{tr}_{SS} \left[ \left( \text{tr}_{SS} \left[ \mathcal{P}^\Delta(\mathcal{F}) \otimes \mathcal{P}^\Delta(\mathcal{F}) \left( \frac{(\mathcal{P}^\Delta \otimes \mathcal{P}^\Delta)\mathcal{S}}{d_\Delta^2} \right) \right] \right) \right] \\
\leq \sum_{lk} \frac{\langle k|\mathcal{P}^\Delta \otimes \mathcal{P}^\Delta|lk \rangle}{d_\Delta^2} \text{tr} \, \text{tr}_{SS} \left[ \left( \text{tr}_{SS} \left[ \mathcal{P}^\Delta \otimes \mathcal{P}^\Delta|kl \rangle \langle kl|\mathcal{P}^\Delta \otimes \mathcal{P}^\Delta \right] \right) \right] \\
= \sum_{lk} \frac{\langle k|\mathcal{P}^\Delta \otimes \mathcal{P}^\Delta|lk \rangle}{d_\Delta^2} \text{tr} \, \text{tr}_{SS} \left[ \text{tr}_{S} \left[ \mathcal{P}^\Delta |k \rangle \langle k|\mathcal{P}^\Delta \right] \text{tr} \, \text{tr}_{SS} \left[ \mathcal{P}^\Delta |l \rangle \langle l|\mathcal{P}^\Delta \right] \right] \\
\leq \sum_{lk} \frac{\langle k|\mathcal{P}^\Delta |l \rangle \langle l|\mathcal{P}^\Delta|k \rangle}{2d_\Delta^2} \text{tr} \, \text{tr}_{SS} \left[ \left( \text{tr} \left[ \mathcal{P}^\Delta |k \rangle \langle k|\mathcal{P}^\Delta \right] \right)^2 + \left( \text{tr} \left[ \mathcal{P}^\Delta |l \rangle \langle l|\mathcal{P}^\Delta \right] \right)^2 \right] \\
= \frac{1}{d_\Delta^2} \sum_{k} \langle k|\mathcal{P}^\Delta|k \rangle \text{tr} \, \text{tr}_{SS} \left[ \left( \text{tr} \left[ \mathcal{P}^\Delta |k \rangle \langle k|\mathcal{P}^\Delta \right] \right)^2 \right] = \frac{\delta^\Lambda}{d_\Delta} \tag{C.17}
\]

where we have defined \( \mathcal{P}^\Delta(\mathcal{F})[\sigma] = \mathcal{P}^\Delta \mathcal{F}[\sigma] \mathcal{P}^\Delta \) and used (C.12). Therefore, \( \langle f \rangle_{\psi_0} \leq \sqrt{d_\Delta^3 \delta^\Lambda/d_\Delta} \). Plugging these into Levy’s lemma for \( g(\psi_0) \) completes the proof.

## D  Asymptotics of restricted partitions

The large \( E \) asymptotic formula for the number of partitions of integer \( E \) is given by

\[
P(E) \simeq \frac{1}{4\sqrt{3E}} e^{\pi \sqrt{2E/3}}, \tag{D.1}
\]

which is the first term that appears in the asymptotic expansion obtained by Hardy and Ramanujan in [23].

The asymptotics of the restricted partition function \( Q(E, s) \) defined by the number of partitions \( E \) into exactly \( s \) parts is studied [24], and has the following three regimes:

1. If \( s = O(1) \) and \( E \gg 1 \):

\[
Q(E, s) \simeq \frac{E^{s-1}}{s!(s-1)!}
\]

2. If \( s/\sqrt{E} = O(1) \) and \( E \gg 1 \):

\[
Q(E, s) \simeq \frac{a^2}{2\pi s^2} \exp \left( \frac{2a}{a} \int_0^a \frac{dt \, t^{-1}}{e^{t-1}} - s \log(1 - e^{-a}) - \frac{a}{2} \right) \left( (e^a - 1) \int_0^a \frac{dt \, e^{t-1}}{(e^{t-1})^2} \right)^{1/2},
\]

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where $a$ is found by solving
\[ \frac{E}{s^2} = \frac{1}{a^2} \int_0^a \frac{dt}{e^t - 1}. \]

3. If $s/\sqrt{E} \gg 1$, and $E \gg s \gg 1$:
\[ Q(E, s) \simeq \frac{1}{4\sqrt{3}E} \exp \left( \pi \sqrt{\frac{2E}{3}} - \frac{\pi s}{\sqrt{6E}} - \frac{\sqrt{6E}}{\pi} e^{-\frac{\pi s}{\sqrt{6E}}} \right) \]

The number of partitions of $E$ into exactly $s$ parts is equal to the number of partitions of $E - s$ into at most $s$ parts: $Q(E, s) = P(E - s, s)$. Since we are only interested in the limit $E \gg s$, the asymptotic behaviour of $Q(E, s)$ and $P(E, s)$ are the same. In section 4.3 we provide a physicist’s derivation of these formulae.

In number theory, the denumerant $D(E; a_1, \ldots, a_s)$ denotes the number of ways one can partition a positive integer $E$ into integer parts $a_1, \ldots, a_s$: $E = \sum_i n_i a_i$. The asymptotic behaviour of the denumerant for large $E$ and $s = O(1)$ is given by
\[ D(E; a_1, \ldots, a_s) = \frac{E^{s-1}}{(s-1)! a_1 \ldots a_s}. \quad (D.2) \]

For a reference on asymptotic properties of denumerant see [25].

## E Equilibration time

In this appendix, we would like to elaborate on time-scales relevant for weak equilibration. One way to approach this problem is to bound the trace distance $\|\rho_S(t) - \omega_S\|$ averaged over a finite time interval $[0, T]$. This approach was discussed in [13], and a generalization of the fluctuation theorem to finite-time averages was obtained. Applied to our regularized Hilbert spaces we find:
\[ \langle \|\rho^\Lambda_S(t) - \omega^\Lambda_S\| \rangle_T \leq \eta^\Lambda_S \sqrt{1 + \frac{8 \log_2 d_\Lambda}{\epsilon T}} \quad (E.1) \]

where $\langle f \rangle_T = \frac{1}{T} \int_0^T df(t)$, $d_\Lambda$ is the dimension of the total Hilbert space $\psi^\Lambda$ explores, and $\epsilon$ is defined by the minimum spaces between energy gaps:
\[ \epsilon = \min_{a,b} \{|G(a,b) - G(c,d)| : (a, b) \neq (c, d)\}. \quad (E.2) \]

The minimum is over all energy gaps $G(a,b) = E_a - E_b$ in the spectrum of the interacting Hamiltonian. We expect $\epsilon$ to be at least as small as $O(d^{-1}_\Lambda)$, which gives an equilibration
time-scale exponential in the number of degrees of freedom. This time scale is extremely long, and does not seem to provide any insights into the dynamics of equilibration. We could have anticipated this based on the fact that it takes a long time for $\psi(t)$ to explore the exponentially large Hilbert space needed for the typicality argument to work. A stronger bound would require restriction to a specific set of Hamiltonians or further dynamical assumptions.

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