NON-LINEAR SIGMA MODEL ON THE FUZZY SUPERSPHERE

Seçkin Kürkçüoğlu

Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA

Abstract

In this note we develop fuzzy versions of the supersymmetric non-linear sigma model on the supersphere $S^{(2,2)}$. In [1] Bott projectors have been used to obtain the fuzzy $\mathbb{C}P^1$ model. Our approach utilizes the use of supersymmetric extensions of these projectors. Here we obtain these (super)-projectors and quantize them in a fashion similar to the one given in [1]. We discuss the interpretation of the resulting model as a finite dimensional matrix model.

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1E-mail: skurkcuo@phy.syr.edu
1 Introduction

In past few years studies of field theories on non-commutative manifolds have been very fruitful. To construct such theories one usually starts with a continuum theory on a manifold $M$ and replaces the commutative algebra $A$ of functions on $M$ by a non-commutative algebra $\mathcal{A}$ which preserves most of the symmetries of the continuum theory and which approximates the commutative algebra $A$ and hence the continuum theory in the commutative limit. It is possible to realize a large class of such non-commutative field theories as finite dimensional matrix models. Field theories on the non-commutative (fuzzy) sphere $S^2_F$ and the fuzzy supersphere $S^{(2,2)}_F$ are two such examples. As non-commutative manifolds the former is based on the irreducible representations of the $su(2)$ Lie algebra, whereas the latter is described by the irreducible representations of the Lie superalgebra $osp(2,1)$. To date many studies on different and novel aspects of field theories on $S^2_F$ have been carried out \[1, 2, 3, 4, 5, 6\].

Recently, $\mathbb{C}P^1$ model on the fuzzy sphere $S^2_F$ have been studied from several different points of view \[1, 7, 8\]. In \[1\] the commutative theory have been reformulated by replacing the non-linear fields with a certain class of projectors called ”Bott Projectors”. A discrete (fuzzy) version of these projectors are easily obtained and they have permitted the construction of a fuzzy $\mathbb{C}P^1$ model in a rather straightforward way.

In this paper we address the question of constructing a fuzzy supersymmetric non-linear sigma model on $S^{(2,2)}$. For this purpose we obtain the supersymmetric extensions of the Bott projectors and quantize them in a similar manner as discussed in \[1\]. Using the quantized (super)-projectors and the already known description of $S^{(2,2)}$ in terms of the Lie superalgebras $osp(2,1)$ and $osp(2,2)$ and their associated Lie supergroups we construct the fuzzy supersymmetric non-linear sigma model on $S^{(2,2)}$. We interpret the resulting theory as a finite dimensional matrix model and comment on its various physical properties.

2 $\mathbb{C}P^1$ Sigma Model and Bott Projectors

Non-linear sigma models are customarily defined in terms of a field that maps the world-sheet to the target manifold. In the case of the $\mathbb{C}P^1$ models both world-sheet and the target manifolds are 2-spheres ($S^2$) and the field $n$ maps the point $x$ of the world-sheet

$$S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i x_i = 1\}$$

(1)

to a point on the target manifold

$$S^2 = \{\vec{n}(x) = (n_1(x), n_2(x), n_3(x)) \in \mathbb{R}^3 \mid n_a(x)n_a(x) = 1\}.$$ (2)

As is well known these maps are classified in terms of an integer $\kappa$ called the winding number since the second homotopy class $\pi_2(S^2) = \mathbb{Z}$.

An alternative formulation of $\mathbb{C}P^1$ model which happens to be more convenient for passage to fuzzy $\mathbb{C}P^1$ model have been considered in \[1\]. This formulation uses certain class of projectors, known as Bott projectors instead of the non-linear fields. At the topological sector $\kappa = 1$ the Bott projector can be expressed in terms of $\vec{n}$ as

$$P(x) = \frac{1 + \vec{n} \cdot \vec{n}(x)}{2}$$ (3)
where $\vec{\tau}$ are the Pauli matrices. $P(x)$ is a projector since $P^2(x) = P(x)$ and $P^\dagger(x) = P(x)$. At the topological sector $\kappa$, Bott projector can be expressed by introducing the partial isometries\(^1\) $\vartheta_\kappa^\dagger$ (for $\kappa > 0$)

$$\vartheta_\kappa^\dagger(z) = \left( \begin{array}{cc} z_1^\kappa & z_2^\kappa \\ \bar{z}_1^\kappa & \bar{z}_2^\kappa \end{array} \right) \frac{1}{\sqrt{Z_\kappa}}, \quad \vartheta_\kappa(z) = \left( \begin{array}{c} z_1^\kappa \\ z_2^\kappa \end{array} \right) \frac{1}{\sqrt{Z_\kappa}}, \quad Z_\kappa = |z_1|^{2\kappa} + |z_2|^{2\kappa}$$

(4)

where $z = (z_1, z_2)$ is a point on $S^3 = \langle z = (z_1, z_2) \in \mathbb{C}^2 \mid |z|^2 := |z_1|^2 + |z_2|^2 = 1 \rangle$ and “bar” stands for complex conjugation. Using the Hopf fibration $U(1) \rightarrow S^3 \rightarrow S^2$, points $x$ on the world-sheet $S^2$ is expressed in terms of $z$ as

$$x_i = z^\dagger \tau_i z.$$ 

(5)

By definition $\vartheta_\kappa^\dagger$ is a partial isometry if and only if $\vartheta_\kappa(z) \vartheta_\kappa^\dagger(z)$ is a projection. It is straightforward to check that $P_\kappa(x)$ in the topological sector $\kappa$ given as

$$P_\kappa(x) = \vartheta_\kappa(z) \vartheta_\kappa^\dagger(z) = \frac{1}{Z_\kappa} \left( \begin{array}{cc} |z_1|^{2\kappa} & z_1^\kappa \bar{z}_2^\kappa \\ z_2^\kappa \bar{z}_1^\kappa & |z_2|^{2\kappa} \end{array} \right)$$

is a projector: $P_\kappa(x)^2 = P_\kappa(x), P_\kappa(x)^\dagger = P_\kappa(x)$.

The field $n^\kappa_a(x)$ is associated to $P_\kappa(x)$ by the formulas

$$n^\kappa_a(x) = Tr(x) \tau_a P_\kappa(x) = \vartheta_\kappa^\dagger(z) \tau_a \vartheta_\kappa(z), \quad P_\kappa(x) = \frac{1 + \vec{\tau} \cdot \vec{n}(x)}{2}.$$ 

(7)

A phase change $z \rightarrow ze^{i\theta}$ induces the change $\vartheta_\kappa(z) \rightarrow \vartheta_\kappa(z)e^{i\kappa\theta}$. Nevertheless, this phase cancels in $\vartheta_\kappa(z) \vartheta_\kappa^\dagger(z)$ and $P_\kappa(x)$ is a function of $x$ only.

In [1] an intuitive argument as well as an explicit calculation is given to show that $\kappa$ appearing in equations [1] through [7] is indeed the winding number. Here we recollect the former. For $\kappa > 0$ consider the $\kappa$ points (up to an overall phase of $z$ which cancels out on $x$) of $S^2$ labeled by $\ell$:

$$z_\ell = (z_1 e^{i\frac{2\pi \ell}{\kappa}}, z_2) \quad \ell \in (0, \kappa - 1).$$

(8)

All $z_\ell$ map to the same point on the target manifold $S^2$ or equivalently, they all have the same projection via $P_\kappa(x)$, giving winding number $\kappa$.

It must be noticed that the form of $P_\kappa(x)$ is very particular. Nevertheless, the most general projector $P_\kappa(x)$ can be obtained from

$$P_\kappa(x) = U(x) P_\kappa(x) U(x)^\dagger$$

(9)

where $U(x) \in U(2)$ is a $2 \times 2$ unitary matrix. The field associated to $P_\kappa(x)$ is nothing but

$$n^\kappa_a(x) = Tr(\tau_a P_\kappa(x)$$

(10)

where $n^\kappa_a(x) = R_{ab} n^\kappa_b(x), U^\dagger \tau_a U = R_{ab} \tau_b$ and $R \in O(3)$. The unitary transformation do not affect the the winding number since $\pi_2(U(2)) = \{e\}$.

\(^1\)To be more precise the partial isometry $\vartheta_\kappa^\dagger$ in the algebra $A = C^\infty(S^3) \otimes Mat_{2 \times 2}\mathbb{C}$ is the matrix $\left( \begin{array}{cc} z_1^\kappa & z_2^\kappa \\ 0 & 0 \end{array} \right)$. But for all practical calculations it is perfectly safe to call $\vartheta_\kappa$ as the partial isometry, thus we do so from now on.
3 On the Actions

A Euclidean action in the $\kappa$-th topological sector is given in terms of the fields $n^\alpha_\kappa(x)$ by

$$S_\kappa = -\frac{1}{8\pi} \int_{S^2} d\Omega(\mathcal{L}_i n^\alpha_\kappa)(\mathcal{L}_i n^\alpha_\kappa)$$

(11)

where $\mathcal{L}_i = -i(x \wedge \nabla)_i$ is the angular momentum operator and $d\Omega = d\cos \theta d\psi$. In terms of the projectors, $S_\kappa$ can be expressed as

$$S_\kappa = -\frac{1}{4\pi} \int_{S^2} d\Omega Tr(\mathcal{L}_i \mathcal{P}_\kappa)(\mathcal{L}_i \mathcal{P}_\kappa).$$

(12)

The well known formulae for the winding number and BPS bound of this model can also be rewritten in terms of the projectors $\mathcal{P}_\kappa$. The actions given in (11) and (12) both do have discrete versions when the $\mathbb{C}P^1$ model is formulated on the fuzzy sphere $S^2_F$. However, it seems that the latter is better adapted for formulation of fuzzy $\mathbb{C}P^1$ sigma models; as will be discussed in section 6 it is possible to quantize the projectors in a straightforward manner. For a detailed discussion on the fuzzy $\mathbb{C}P^1$ model the reader is referred to [1].

In section 5 we develop the supersymmetric extension of the projectors $\mathcal{P}_\kappa(x)$ and apply this result to the description of non-linear sigma model first on the supersphere and then on the fuzzy supersphere. The latter will require the supersymmetric extension of quantized projectors.

4 The Commutative and Non-Commutative (Fuzzy) Superspheres

4.1 The Supersphere $S^{(2,2)}$

In this section we would like to collect some preliminary differential geometric and group theoretical formulae that is used to characterize the supersphere $S^{2,2}$ and its non-commutative (fuzzy) version $S^{(2,2)}_F$. The details of the very brief discussion below can be found in [10, 11].

The structure underlying the supersphere $S^{(2,2)}$ comes from the Lie superalgebras $osp(2,1)$ and $osp(2,2)$ and their associated Lie supergroups $OSP(2,1)$ and $OSP(2,2)$. $osp(2,1)$ is build up of the Lie algebra $su(2)$ (even part) with generators $L_i$, $(i = 1, 2, 3)$ and $su(2)$ spinors $V_\alpha(\alpha = +, -)$ (odd part). $osp(2,2)$ Lie superalgebra is constructed by augmenting $osp(2,1)$ generators with an additional pair of spinors $D_\alpha(\alpha = +, -)$ and an additional even generator $\Gamma$. The graded commutation relations of $osp(2,2)$ generators read

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad [L_i, V_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}V_\beta, \quad \{V_\alpha, V_\beta\} = \frac{1}{2}(C\sigma_i)_{\alpha\beta}L_i,$$

$$[L_i, \Gamma] = 0, \quad [\Gamma, V_\alpha] = D_\alpha, \quad [\Gamma, D_\alpha] = V_\alpha, \quad [L_i, D_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}D_\beta,$$

$$\{D_\alpha, D_\beta\} = -\frac{1}{2}(C\sigma_i)_{\alpha\beta}L_i, \quad \{D_\alpha, V_\beta\} = \frac{1}{4}C_{\alpha\beta}\Gamma.$$  

(13)

where $i, j = 1, 2, 3$, $\alpha, \beta = \pm$ and $C = i\sigma_3$. The graded commutation relations for the $osp(2,1)$ generators is given by the first line of [13].

For brevity we drop the “prime” on the fields $n_\alpha(x)$. 

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In the corresponding enveloping algebras there are central polynomials - the Casimir operators in representations given by the formulas:

\[ K_{2}^{osp(2,1)} = L_i L_i + C_{\alpha\beta} V_{\alpha} V_{\beta}, \]
\[ K_{2}^{osp(2,2)} = L_i L_i + C_{\alpha\beta} V_{\alpha} V_{\beta} - \left( C_{\alpha\beta} D_{\alpha} D_{\beta} + \frac{1}{4} \Gamma^2 \right). \]  

(14)

These Lie superalgebras are endowed with a grade dagger operation \( \dagger \) replacing the usual adjoint operation on the Lie algebras. Generators of \( osp(2,2) \) fulfill the following reality conditions implemented by \( \dagger \):

\[ L_i^\dagger = L_i, \quad V_{\alpha}^\dagger = C_{\alpha\beta} V_{\beta}, \quad D_{\alpha}^\dagger = -C_{\alpha\beta} D_{\beta}, \quad \Gamma^\dagger = \Gamma^\dagger = \Gamma. \]  

(15)

The reality conditions fulfilled by \( osp(2,1) \) is obtained by restricting to the relations fulfilled by \( L_i \) and \( V_{\alpha} \). The graded conjugation is extended to homogeneous elements \( A \) and \( B \) in enveloping algebras by

\[ (AB)^\dagger = (-1)^{|A||B|} B^\dagger A^\dagger. \]  

(16)

Here \(|A|\) and \(|B|\) denote the degrees of \( A \) and \( B \), respectively. By linearity the conjugation is extended to the whole enveloping algebra. The Casimir elements, given above, are real.

The supersphere \( S^{(2,2)} \) is the adjoint orbit of the Lie supergroup \( OSP(2,1) \). It can be obtained through a super generalization of the Hopf fibration for the 2-sphere. In the supersymmetric case this becomes \( U(1) \rightarrow S^{(3,2)} \rightarrow S^{(2,2)} \) where \( S^{(3,2)} \equiv OSP(2,1) \) and

\[ S^{(2,2)} = S^{(3,2)} / U(1). \]  

(17)

The superspace \( \mathbb{R}^{(3,2)} \) is defined as the algebra of polynomials in generators \( x_i \) and \( \theta_{\alpha} \) satisfying reality conditions

\[ x_i^\dagger = x_i, \quad \theta_{\alpha}^\dagger = C_{\alpha\beta} \theta_{\beta}. \]  

(18)

These conditions are extended as in \[16\] to all polynomials. The equation characterizing the adjoint orbit \( S^{(2,2)} \) of \( osp(2,1) \) is

\[ S^{(2,2)} = \left\{ (x_i, \theta_{\alpha}) \in \mathbb{R}^{(3,2)} \mid x_i^2 + C_{\alpha\beta} \theta_{\alpha} \theta_{\beta} = \frac{1}{4} \right\}. \]  

(19)

The action of \( osp(2,1) \) on \( S^{(2,2)} \) is the adjoint action and is given in terms of the differential operators

\[ \ell_i = -i \varepsilon_{ijk} x_j \partial_k - \frac{1}{2} (\sigma_i)_{\beta\alpha} \theta_{\beta} \partial_{\alpha}, \]
\[ v_{\alpha} = -\frac{1}{2} (\sigma_i)_{\beta\alpha} \theta_{\beta} \partial_i + \frac{1}{2} (C_{\sigma_i})_{\alpha\beta} x_i \partial_{\beta}. \]  

(20)

corresponding to the \( osp(2,1) \) generators \( L_i \) and \( V_{\alpha} \), respectively. It can be extended to an \( osp(2,2) \) action which is not an adjoint action but it is closely related to it. (for details see \[10\ \[11\]). The additional differential operators have the form

\[ d_{\alpha} = -r \left( 1 + \frac{2}{r^2} \right) C_{\alpha\beta} \partial_{\beta} + \frac{1}{2r} (\sigma_i)_{\beta\alpha} \theta_{\beta} \ell_i - \frac{\theta_{\alpha}}{2r} x_i \partial_i, \]
\[ \gamma = \left( \frac{\theta_+ x_3}{r} + \frac{\theta_- x_3}{r} \right) \partial_+ + \left( \frac{\theta_+ x_-}{r} - \frac{\theta_- x_-}{r} \right) \equiv 2(\theta_+ v_+ - \theta_- v_-). \]  

(21)

corresponding to the generators \( D_{\alpha} \) and \( \Gamma \) of \( osp(2,2) \).
4.2 The Fuzzy Supersphere $S^{(2,2)}$

The fuzzy supersphere $S^{(2,2)}_F$ is obtained replacing $(x_i, \theta_\alpha) \in \mathbb{R}^{(3,2)}$ by suitable rescaled $osp(2,1)$ generators $X_i = \lambda L_i$ and $\Theta = \lambda V_\alpha$ with $\lambda$ determined by the value of $osp(2,1)$ Casimir operator:

$$\frac{1}{4\lambda^2} = K_{osp(2,1)}^2. \tag{22}$$

The fuzzy parameters then satisfy the supersphere’s defining relation

$$X_i X_i + C_{\alpha\beta} \Theta_\alpha \Theta_\beta = \frac{1}{4}. \tag{23}$$

The non-commutativity of the supersphere follows from the graded commutation relations of $X_i$ and $\Theta_\alpha$. For details we refer the reader to [10], [11].

5 Non-Linear Sigma Model on $S^{(2,2)}$

5.1 Preliminaries

The superfield $\Phi$ on $S^{(2,2)}$ is a function of the variables $(x_i, \theta_\alpha)$; it is real provided that $\Phi^\dagger = \Phi$. For a free real superfield multiplet the action is related to the $osp(2,1)$ invariant given as the difference of the quadratic Casimir operators:

$$K_{osp(2,1)}^2 - K_{osp(2,2)}^2 = C_{\alpha\beta} D_\alpha D_\beta + \frac{1}{4}\Gamma^2. \tag{24}$$

The action takes the form

$$S_{SUSY}^{SU3} = \frac{1}{4\pi} \int d\mu \left( d_\alpha \Phi d_\alpha \Phi + \frac{1}{2} \gamma \Phi \frac{1}{2} \gamma \Phi \right) \tag{25}$$

where $d_\mu = d^3 x^i d\theta^+ d\theta^- \delta(x_i^2 + C_{\alpha\beta} \theta_\alpha \theta_\beta - \frac{1}{4})$, and $d_\alpha$ and $\gamma$ are the differential operators given in (24).

For a free triplet real superfield $\Phi^a = \Phi^a(x_i, \theta_\alpha), (a = 1, 2, 3)$, we just replace in $\Phi$ by $\Phi^a$ (with the summation over repeated index $a$ understood). Now we define the $O(3)$ sigma model [12] by putting on $\Phi^a$ the constraint

$$\Phi^a \Phi^a = 1 \quad (a = 1, 2, 3). \tag{26}$$

Then (25) and (26) defines the non-linear sigma model on the supersphere $S^{(2,2)}$ with the target manifold being $S^2$.

The superfield $\Phi^a(x_i, \theta_\alpha)$ can be expanded in powers of $\theta_\alpha$ as

$$\Phi^a(x_i, \theta_\alpha) = n^a(x_i) + C_{\alpha\beta} \theta_\beta \psi^a_\alpha(x_i) + \frac{1}{2} F^a(x_i) C_{\alpha\beta} \theta_\alpha \theta_\beta \tag{27}$$

where $\psi^a(x_i)$ are two component Majorana spinors : $\psi^a_\alpha = C_{\alpha\beta} \psi^a_\beta$, and $F^a(x_i)$ are auxiliary scalar fields. In terms of the component fields the constraint equation (26) splits to

$$n^a n^a = 1, \quad n^a F^a = \frac{1}{2} \psi^a n^a \psi^a, \quad n^a \psi^a_\alpha = 0. \tag{28}$$
5.2 Supersymmetric Extensions of Bott Projectors

A possible supersymmetric extension of the projector $P_\kappa(x)$ can be obtained in the following way. Let $U(x_i, \theta_\alpha)$ be a graded unitary operator with $U^\dagger U = 1$. $U(x_i, \theta_\alpha)$ in general can be thought as a $2 \times 2$ supermatrix whose entries are functions on $S^{(2,2)}$. $U(x_i, \theta_\alpha)$ acts on $P_\kappa$ by conjugation and generates a set of supersymmetric extensions $Q_\kappa(x_i, \theta_\alpha)$:

$$ Q_\kappa(x_i, \theta_\alpha) = U^\dagger P_\kappa(x) U. $$

(29)

It is easy to see that $Q_\kappa(x_i, \theta_\alpha)$ satisfies $Q_\kappa^2(x_i, \theta_\alpha) = Q_\kappa(x_i, \theta_\alpha)$ and $Q_\kappa^\dagger(x_i, \theta_\alpha) = Q_\kappa(x_i, \theta_\alpha)$. Thus $Q_\kappa(x_i, \theta_\alpha)$ is a (super)-projector. The real superfield on $S^{(2,2)}$ associated to $Q_\kappa(x_i, \theta_\alpha)$ is given by

$$ \Phi'_a(x_i, \theta_\alpha) = Tr \tau_a Q_\kappa. $$

(30)

In order to perform a check that establishes that $Q_\kappa(x_i, \theta_\alpha)$ are indeed the supersymmetric projectors that reproduces the superfields on $S^{(2,2)}$ subject to

$$ \Phi'_a \Phi'_a = 1, $$

we proceed as follows. First we expand $U(x_i, \theta_\alpha)$ in powers of the Grassmann variables as

$$ U(x_i, \theta_\alpha) = U_0(x_i) + C_{\alpha\beta} \theta_\alpha U_\alpha(x_i) + \frac{1}{2} U_2(x_i) C_{\alpha\beta} \theta_\alpha \theta_\beta $$

(32)

where $U_0, U_\alpha(\alpha = \pm)$ and $U_2$ are all $2 \times 2$ graded unitary matrices. The requirement that $U(x_i, \theta_\alpha)$ is graded unitary makes $U_0(x_i)$ unitary, whereas $U_\alpha(x_i)$ are uniquely determined by $U_\alpha(x_i) = H_\alpha(x_i) U_0(x_i)$ where $H_\alpha$ are $2 \times 2$ odd supermatrices with the reality condition $H_\alpha = -C_{\alpha\beta} H_\beta$. Moreover, with the ansatz that $U_2 = AU_0$ with $A$ being an arbitrary $2 \times 2$ even supermatrix, graded unitarity of $U(x_i, \theta_\alpha)$ further restricts the symmetric part of $A$ as:

$$ A + A^\dagger = -C_{\alpha\beta} H_\alpha H_\beta. $$

(33)

Using the expansion (32) in (29) and subsequently the resulting expression in (30) together with the properties listed above it is straightforward to extract the component fields of the superfield $\Phi'_a(x_i, \theta_\alpha)$. We find

$$ n^\alpha_{\dot{a}} := Tr \tau_\alpha U_0^\dagger P_\kappa U_0, $$

$$ \psi_{\dot{a}} := -2 i (\vec{\pi}^\alpha \times \vec{H}_\alpha)^\alpha = Tr \tau_\alpha U_0^\dagger [H_\alpha, P_\kappa] U_0, $$

(34)

(35)

and after using (33) that

$$ F^\alpha_{\dot{a}} := 4 (\vec{H}_\alpha^\dagger \cdot \vec{H}_\alpha^\dagger) n^\alpha_{\dot{a}} - 2 \vec{H}_\alpha^a (\vec{\pi}^\alpha \cdot \vec{H}_\alpha^\dagger) - (\vec{\pi}^\alpha \cdot \vec{H}_\alpha^\dagger) 2 \vec{H}_\alpha^a + i (\vec{\pi}^\alpha \times (\vec{A} - \vec{A})^\dagger)^\alpha $$

(36)

where $\vec{H}_\alpha^a = H_\alpha^a \tau^a$ and $\vec{A}^\dagger = A^a \tau^a$. By direct computation from above we find

$$ n_{\dot{a}} n_{\dot{a}} = 1, \quad n_{\dot{a}} F^\alpha_{\dot{a}} = \frac{1}{2} \psi_{\dot{a}} \psi_{\dot{a}}^\dagger, \quad n_{\dot{a}} \psi_{\dot{a}} \psi_{\pm} = 0. $$

(37)

Comparing (33) with (34) we observe that they are identical. Therefore we conclude that the superfield associated to the super-projector $Q_\kappa$ is the same as the superfield in supersymmetric non-linear sigma model of the previous subsection.
5.3 SUSY Action Revisited

We are now ready to give the formulation of non-linear sigma model on the supersphere using the (super)-projectors. In close analogy with the \( \mathbb{C}P^1 \) case the supersymmetric action in (25) with the constraint (26) translates to

\[
S_{\kappa}^{\text{SUSY}} = \frac{1}{2\pi} \int d\mu \text{Tr} \left[ (d_\alpha Q_\kappa)(d_\alpha Q_\kappa) + \frac{1}{4}(\gamma Q_\kappa)(\gamma Q_\kappa) \right].
\]  

(38)

The even part of this action, as well as the one given in (25) is nothing but the action \( S_\kappa \) of the \( \mathbb{C}P^1 \) theory given in (12) and (11), respectively. In other words, the action \( S_{\kappa}^{\text{SUSY}} \) is the supersymmetric extension of \( S_\kappa \) on \( S^2 \) to \( S^{(2,2)} \). Thus in the supersymmetric theory it is possible to interpret the index \( \kappa \) carried by the action as the winding number of the corresponding \( \mathbb{C}P^1 \) theory.

We recall that \( d_\alpha \) and \( \gamma \) are both derivations in the superalgebra \( Osp(2,2) \). Therefore they obey a graded Leibnitz rule and from \( Q_\kappa^2 = Q_\kappa \) we find

\[
Q_\kappa d_\alpha Q_\kappa = d_\alpha Q_\kappa (1 - Q_\kappa).
\]  

(39)

This enables us to write

\[
\text{Tr} d_\alpha Q_\kappa (1 - Q_\kappa) d_\alpha Q_\kappa = \text{Tr} (1 - Q_\kappa)(d_\alpha Q_\kappa)^2 = \frac{1}{2} \text{Tr}(d_\alpha Q_\kappa)^2.
\]  

(40)

Equations (39) and (40) continue to hold when \( d_\alpha \) is replaced by \( \gamma \) as well. The action then takes the form

\[
S_{\kappa}^{\text{SUSY}} = \frac{1}{\pi} \int d\mu \text{Tr} \left[ Q_\kappa (d_\alpha Q_\kappa)(d_\alpha Q_\kappa) + \frac{1}{4}Q_\kappa(\gamma Q_\kappa)(\gamma Q_\kappa) \right].
\]  

(41)

It is possible that this form of the action could play an important role for obtaining a supersymmetric generalization of the BPS equation since an analogous expression in the \( \mathbb{C}P^1 \) case \[1\] have been employed to achieve this result.

6 Fuzzy Projectors and Sigma Models

6.1 Fuzzy \( \mathbb{C}P^1 \) Model

In \[1\] the \( \mathbb{C}P^1 \) model has been quantized as follows. Let \( \xi = (\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{0\} \). In terms of \( \xi \) we define

\[
z = \frac{\xi}{|\xi|}, \quad |\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2}, \quad x_i = z^\dagger x_i z.
\]

(42)

\( \xi_\alpha \) and \( \bar{\xi}_\alpha \) are quantized by replacing them with a pair of annihilation \( a_\alpha \) and creation \( a_\alpha^\dagger \) operators respectively. With this substitution \( |\xi| \) becomes the square root of the number operator and we have

\[
\hat{N} = \hat{N}_1 + \hat{N}_2, \quad \hat{N}_1 = a_\alpha^\dagger a_\alpha, \quad \hat{N}_2 = a_\alpha^\dagger a_\alpha
\]

\[
\hat{z}_\alpha = \frac{1}{\sqrt{N}} a_\alpha = \frac{1}{\sqrt{N}} a_\alpha \frac{1}{\sqrt{N + 1}}, \quad \hat{x}_i = \frac{1}{N} a_\alpha \tau_i a.
\]

(43)
In the light of this conjecture it is easy to see that the quantized version of the partial isometry \( \hat{\psi}_\kappa^\dagger \) defined in (1) and its Hermitian conjugate reads

\[
\hat{\psi}_\kappa^\dagger = \frac{1}{\sqrt{Z_{\kappa}}} \left( a_1^{\dagger \kappa} \quad a_2^{\dagger \kappa} \right), \quad \hat{\psi}_\kappa = \left( a_1^\kappa \quad a_2^\kappa \right) \frac{1}{\sqrt{Z_{\kappa}}}, \quad \hat{\psi}_\kappa^\dagger \hat{\psi}_\kappa = 1,
\]

(44)

\[
\hat{Z}_{\kappa} = \hat{Z}_{\kappa}^{(1)} + \hat{Z}_{\kappa}^{(2)}, \quad \hat{Z}_{\kappa}^{(\alpha)} = \hat{N}_\alpha (\hat{N}_\alpha - 1) \ldots (\hat{N}_\alpha - \kappa + 1).
\]

The fuzzy analogue of (6) can now be written as

\[
\hat{P}_\kappa (x) = \hat{\psi}_\kappa \hat{\psi}_\kappa^\dagger = \left( \begin{array}{cc} a_1^{\dagger \kappa} \frac{1}{Z_{\kappa}} a_1^{\kappa} & a_1^{\dagger \kappa} \frac{1}{Z_{\kappa}} a_2^{\kappa} \\ a_2^{\dagger \kappa} \frac{1}{Z_{\kappa}} a_1^{\kappa} & a_2^{\dagger \kappa} \frac{1}{Z_{\kappa}} a_2^{\kappa}. \end{array} \right)
\]

where for example

\[
a_1^{\dagger \kappa} \frac{1}{Z_{\kappa}} a_1^{\kappa} = \frac{1}{(\hat{N}_1 + \kappa) \ldots (\hat{N}_1 + 1) + \hat{Z}_{\kappa}^{(2)}} a_1^{\kappa}, \quad a_2^{\dagger \kappa} a_1^{\kappa} = (\hat{N}_1 + \kappa) \ldots (\hat{N}_1 + 1).
\]

(46)

The unitary matrix \( U \) introduced to generate all possible projectors \( P_\kappa \) from \( P_\kappa \) also have fuzzy analogue. It is a 2 \( \times \) 2 unitary matrix \( \hat{U} \), with matrix entries being polynomials in \( a_\alpha^\dagger a_\beta \). Thus the most general fuzzy projectors are

\[
\hat{P}_\kappa = \hat{U} \hat{P}_\kappa \hat{U}^\dagger.
\]

(47)

From (45) it is clear that \( \hat{P}_\kappa \) acts in general on \( F^2 := F \otimes \mathbb{C}^2 \) where \( F \) stands for the standard Fock space. It also follows from (45) that \( \hat{P}_\kappa \) commutes with the number operator \( \hat{N} \), as can be checked directly. Consequently, we can restrict ourselves to work on a finite dimensional subspace \( F_n \) of dimension \( n + 1 \) of \( F \). Then \( \hat{P}_\kappa \) act on the finite dimensional Hilbert space \( F_n^2 := F_n \otimes \mathbb{C}^2 \) and this allows one to formulate a finite dimensional matrix model for projectors \( \hat{P}_\kappa \).

In [1] this has been done to write down the fuzzy \( CP^1 \) model. They found that the fuzzy action corresponding to (12) is

\[
S_{F,\kappa} = \frac{1}{4\pi} \frac{1}{2(n + 1)} Tr_{n=0} (\mathcal{L}_i \hat{P}_\kappa)(\mathcal{L}_i \hat{P}_\kappa)
\]

(48)

where \( \mathcal{L}_i \hat{P}_\kappa = [L_i, \hat{P}_\kappa] \) and the trace is over \( F_n^{(2)} \).

6.2 Fuzzy Supersymmetric Model

In much the same way the supersymmetric projectors \( Q_\kappa \) have been constructed from \( P_\kappa \) in section 5, we can construct supersymmetric extensions of \( \hat{P}_\kappa \) by the graded unitary transformation

\[
\hat{Q}_\kappa = \hat{U} \hat{P}_\kappa \hat{U}^\dagger
\]

(49)

where \( \hat{U} \) is a 2 \( \times \) 2 supermatrix whose entries are polynomials in \( a_\alpha^\dagger a_\beta \) and \( b^\dagger b \) and where \( b \) and \( b^\dagger \) are fermionic annihilation and creation operators with the anti-commutation relation \( \{b, b^\dagger\} = 1 \).

\( \hat{Q}_\kappa \) defined (49) acts on the finite dimensional space \( \tilde{F}_n^2 = \tilde{F}_n \otimes \mathbb{C}^2 \). Here \( \tilde{F} \) is the \( n + 1 \) dimensional subspace of the Hilbert space \( \tilde{F} \) spanned by the kets \( |n_1, n_2, n_3 \rangle \) where \( n_3 \) labels the
fermionic part taking on the values 0 and 1 only. It is readily seen that $\hat{Q}_\kappa$ commutes with the supersymmetric number operator $\hat{N} = a_\alpha^\dagger a_\alpha + b_\beta^\dagger b_\beta$. In close analogy with the fuzzy $\mathbb{C}P^1$ model, it is now possible to write down a finite dimensional (super)matrix model for the (super)-projectors $\hat{Q}_\kappa$.

Making use of (24) once more the action for the fuzzy supersymmetric model is given as

$$S_{F,\kappa}^{SUSY} = \frac{1}{2\pi} \text{Str}_{\hat{N}=n} \left[ (D_\alpha \hat{Q}_\kappa)(D_\alpha \hat{Q}_\kappa) + \frac{1}{4}(\Xi \hat{Q}_\kappa)(\Xi \hat{Q}_\kappa) \right],$$

(50)

where $D_\alpha \hat{Q}_\kappa = \{D_\alpha, \hat{Q}_\kappa\}$ and $\Xi \hat{Q}_\kappa = [\Gamma, \hat{Q}_\kappa]$. “Str” in the above expression is the supertrace over the finite dimensional space $\tilde{F}_n^2$. Obviously, in the large $\hat{N} = n$ limit (50) approximates the action given in [38].

7 Conclusions

In this paper we have obtained the fuzzy version of supersymmetric non-linear sigma model on $S^{(2,2)}$. Our approach has utilized the use of supersymmetric extensions of the Bott projectors and generalized results of $\mathbb{C}P^1$ model to supersymmetric theories. A natural question to be addressed is the supersymmetric generalization of the BPS equation. We hope to report any development on this issue elsewhere.

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