Locally Anisotropic Kinetic Processes and Thermodynamics in Curved Spaces

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Abstract

The kinetic theory is formulated with respect to anholonomic frames of reference on curved spacetimes. By using the concept of nonlinear connection we develop an approach to modeling locally anisotropic kinetic processes and, in corresponding limits, the relativistic non–equilibrium thermodynamics with local anisotropy. This lead to a unified formulation of the kinetic equations on (pseudo) Riemannian spaces and in various higher dimensional models of Kaluza–Klein type and/or generalized Lagrange and Finsler spaces. The transition rate considered for the locally anisotropic transport equations is related to the differential cross section and spacetime parameters of anisotropy. The equations of states for pressure and energy in locally anisotropic thermodynamics are derived. The obtained general expressions for heat conductivity, shear and volume viscosity coefficients are applied to determine the transport coefficients of cosmic fluids in spacetimes with generic local anisotropy. We emphasize that such locally anisotropic structures are induced also in general relativity if we are modelling physical processes with respect to frames with mixed sets of holonomic and anholonomic basis vectors which naturally admits an associated nonlinear connection structure.

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1 Introduction

The experimental data on anisotropies in the microwave background radiation (see, for instance, Ref. [27] and [13]) and modern physical theories support the idea that in the very beginning the Universe has to be described as an anisotropic and higher dimension spacetime. It is of interest the investigation of higher dimension generalized Kaluza–Klein and string anisotropic cosmologies. Therefore, in order to understand the initial dynamical behavior of an anisotropic universe, in particular, to study possible mechanisms of anisotropic inflation connected with higher dimensions [32] we have to know how we can compute the parameters (transport coefficients, damping terms and viscosity coefficients) that characterize the cosmological fluids in spacetimes with generic anisotropy.

The first relativistic macroscopic thermodynamic theories have been proposed in Refs. [9] and [22]. The further developments and applications in gravitational physics, astrophysics and cosmology are connected with papers [28, 37, 16, 24]. The Israel’s approach to a microscopic Boltzmann like kinetic theory for relativistic gases [18] makes possible to express the transport coefficients via the differential cross sections of the fluid’s particles. Here we also note the Chernikov’s results on Boltzmann equations with collision integral on (pseudo) Riemannian spaces [7]. A complete formulation of relativistic kinetics is contained in the monograph [15] by de Groot, van Leeuwen and van Weert. We also emphasize the Vlasov’s monograph [36] were an attempt to statistical motivation of kinetic and thermodynamic theory on phase spaces enabled with Finsler like metrics and connection structures was proposed. Recent developments and applications of kinetics and nonequilibrium thermodynamics could be found in [38].

The extension of the four dimensional considerations to higher dimensions is due to [29] and [3]. The generalization of kinetic and thermodynamic equations and formulas to curved spaces is not a trivial task. In order to consider flows of particles with noninteger spins, interactions with gauge fields, and various anisotropic processes of microscopic or macroscopic nature it is necessary a reformulation of the kinetic theory in curved spacetimes by using the Cartan’s moving frame method [6]. A general approach with respect to anholonomic frames contains the possibility to take into account generic spacetime anisotropies which play an important role in the vicinity of cosmological or astrophysical singularities and for non–trivial reductions of higher dimension theories to lower dimensional ones. In our works [30, 31, 32] we proposed to describe such locally anisotropic spacetimes and interactions by applying the concept of nonlinear connection (in brief, N–connection) [4] field which models the local splitting of spacetime into horizontal (isotropic) and vertical (anisotropic) subspaces. For some values of components the N–connection could parametrize, for instance, toroidal compactifications of higher dimensions but, in general, its dynamics is to be determined, in different approaches, by field equations or constraints in some generalized Kaluza–Klein, gauge gravity or (super)string theories [30, 31, 32, 35].

The concept of N–connection was firstly applied in the framework of Finsler and Lagrange geometry and gravity and their higher order extensions (see Refs. [23, 1].
Here it is to be emphasized that every generalized Finsler like geometry could be modelled equivalently on higher dimensions (pseudo)Riemannian spacetimes (for some models by introducing additional torsion and/or nonmetricity structures) with correspondingly adapted anholonomic frame structures. If we restrict our considerations only to the higher dimensional Einstein gravity, the induced N–connection structure becomes a "pure" anholonomic frame effect which points to the fact that the set of dynamical gravitational field variables given by metric’s components was redefined by introducing local frame variables. The components of a fixed basis define a system (equivalently, frame) of reference (in four dimensions one uses, for instance, the terms of vierbien, or tetradic field) with respect to which, in its turn, one states the coefficients of curved spacetime’s metric and of fundamental physical values and field equations. It should be noted that the procedure of choosing (establishing) of a system of reference must be also physically motivated and that in general relativity this task is not considered as a dynamical one following from the field equations. For trivial models of physical interactions on curved spacetimes one can restrict our considerations only with holonomic frames which locally are linearly equivalent to some coordinate basis. Extending the class of physical fields and interactions (even in the framework of Einstein’s gravity), for example, by introducing spinor fields, statistical and fluid models with spinning particles we have to apply frame bundle methods and deal with general anholonomic frames and phase spaces provided with Cartan’s Finsler like connections (induced by the Levi Civita connection on (pseudo)Riemannian spacetime). The modelling of kinetic processes with respect to anholonomic frames (which induces corresponding N–connection structures) is very useful with the aim to elucidate flows of fluids of particles not being in local equilibrium. More exactly, such generic anisotropic fluids are considered to be like in a thermodynamic equilibrium with respect to some frames which are locally adapted both to the spacetime metric and N–connection structures but in general (for instance, with respect to local coordinate frames) the conditions of local equilibrium are not satisfied.

We developed a proper concept of locally anisotropic spacetime (in brief, la–spacetime) in order to provide a unified (super) geometric background for generalized Kaluza–Klein (super) gravities and low energy (super) string models when the higher dimension spacetime is characterized by generic local anisotropies and compatible metric, N–connection and correspondingly adapted linear connection structures. It should be noted that the term la–spacetime can be used even for Einstein spaces if they are provided with anholonomic frame structures. Our treatment of local anisotropy is more general than that from Refs. which is used for a subclass of Finsler like metrics being conformally equivalent (with conformal factors depending both on spacetime coordinates and tangent vectors) to the flat (pseudo) Euclidean, equivalently, Minkovski metric. If in the Bogoslovsky, Goenner and Asanov works on Finsler like spacetimes there are investigated possible effects of violation of local Lorentz and/or Poincaré symmetries, our approach to locally anisotropic strings, field interactions and stochastics is back-grounded on the fact that such models could be constructed as to be locally Lorentz invariant with respect to frames locally adapted to N–connection structures.
The purpose of the present article is to formulate a Boltzmann type kinetic theory and non-equilibrium thermodynamics on spacetimes of higher dimension with respect to anholonomic frames which models local anisotropies in both type Einstein or generalized Finsler–Kaluza–Klein theories. We shall also discuss possible applications in modern cosmology and astrophysics.

On kinetic and thermodynamic part our notations and derivations are often inspired by the Refs. [3] and [15]. A number of formulas will be similar for both locally isotropic and anistropic spacetimes if in the last case we shall consider the equations and vector, tensor, spinor and connection objects with respect to anholonomic rest frames locally adapted to the N–connection structures. In consequence, some tedious proofs and intermediary formulas will be omitted by referring the reader to the corresponding works.

To keep the article self-consistent we start up in Section 2 with an overview of the so-called locally anisotropic spacetime geometry and gravity. In Sections 3 and 4 we present the basic definitions for locally anisotropic distribution functions, particle flow and energy momentum tensors and generalize the kinetic equations for la–spacetimes. The equilibrium state and derivation of expressions for the particle density and the entropy density for systems that are closed to a locally anisotropic state of chemical equilibrium are considered in Section 5. Section 6 is devoted to the linearized locally anisotropic transport theory. There we discuss the problem of solution of kinetic equations, prove the linear lows for locally anisotropic non–equilibrium thermodynamics and obtain the explicit formulas for transport coefficients. As an example, in Section 7, we examine the transport theory in curved spaces with rotation ellipsoidal horizons. Concluding remarks are contained in Section 8.

2 Spacetimes with Local Anisotropy

We outline the necessary background on anholonomic frames and nonlinear connections (N–connection) modelling local anisotropies (la) in curved spaces and on locally anisotropic gravity [30, 31] (see Refs. [17] and [23] for details on space-time differential geometry and N–connections structures). We shall prove that the Cartan’s moving frame method [6] allows a geometric treatment of both type of locally isotropic (for simplicity, we shall consider (pseudo) Riemannian spaces) and anisotropic (the so-called generalized Finsler–Kaluza–Klein spaces).

2.1 Anholonomic frames and Einstein equations

In this paper spacetimes are modelled as smooth (i.e class $C^\infty$) manifolds $V(d)$ of finite integer dimension $d \geq 3, 4, ..., $ being Hausdorff, paracompact and connected. We denote the local coordinates on $V(d)$ by variables $u^\alpha$, where Greek indices $\alpha, \beta, ... = 3, 4, ...$ could be both type coordinate or abstract (Penrose’s) ones. A spacetime is provided with corresponding geometric structures of symmetric metric $g_{\alpha\beta}$ and of linear, in general nonsymmetric, connection $\Gamma^\alpha_{\beta\gamma}$ defining the covariant derivation $\nabla_\alpha$ satisfying the metricity conditions $\nabla_\alpha g_{\beta\gamma} = 0$. We shall underline indices, $\underline{\alpha}, \underline{\beta}, ..., $
if one would like to emphasize them as abstract ones.

Let a set of basis (frame) vectors \( e^\alpha = \{e^\alpha_\alpha = g^{\alpha\beta}e_\alpha^\beta\} \) on \( V(d) \) be numbered by an underlined index. We shall consider only frames associated to symmetric metric structures via relations of type

\[
e^\alpha_\alpha e^\beta_\alpha = \eta^{\alpha\beta} \quad \text{and} \quad e^\alpha_\alpha \eta^{\alpha\beta} = g_{\alpha\beta},
\]

where the Einstein summation rule is accepted and \( \eta^{\alpha\beta} \) is a given constant symmetric matrix, for simplicity a pseudo–Euclidean metric of signature \((-,...,+)\) (the sign minus is used in this work for the time like coordinate of spacetime). Operations with underlined and non–underlined indices are correspondingly performed by using the matrix \( \eta^{\alpha\beta} \), its inverse \( \eta_{\alpha\beta} \), and the metric \( g_{\alpha\beta} \) and its inverse \( g^{\alpha\beta} \). A frame (local basis) structure \( e_\alpha \) on \( V(d) \) is characterized by its anholonomy coefficients \( w^{\gamma}_{\alpha\beta} \) defined from relations

\[
e_\alpha e_\beta - e_\beta e_\alpha = w^\gamma_{\alpha\beta} e_\gamma. \quad (1)
\]

With respect to a fixed basis \( e_\alpha \) and its dual \( e^\beta \) we can decompose tensors and write down their components, for instance,

\[
T = T^\gamma_{\alpha\beta} e_\gamma \otimes e^\alpha \otimes e^\beta
\]

where by \( \otimes \) it is denoted the tensor product.

A spacetime \( V(d) \) is \textbf{holonomic} (locally integrable) if it admits a frame structure for which the anholonomy coefficients from \( (1) \) vanishes, i.e. \( w^\gamma_{\alpha\beta} = 0 \). In this case we can introduce local coordinate bases,

\[
\partial_\alpha = \partial / \partial u^\alpha \quad (2)
\]

and their duals

\[
d^\alpha = du^\alpha \quad (3)
\]

and consider components of geometrical objects with respect to such frames.

We note that the general relativity theory was formally defined on holonomic pseudo-Riemannian manifolds. Even on holonomic spacetimes, for various (geometrical, computational and physically motivated) purposes, it is convenient to use anholonomic frames \( e_\alpha \), but we emphasize that for such spacetimes one can always define some linear transforms of frames to a coordinate basis, \( e_\alpha = a^\alpha_{\alpha'} \partial_{\alpha'} \). By applying both holonomic and anholonomic frames and theirs mutual transforms on holonomic pseudo-Riemannian spaces there were developed different variants of tetradic and spinor gravity and extensions to linear, affine and de Sitter gauge group gravity models \([1, 23, 12]\).

A spacetime is generically \textbf{anholonomic} (locally non-integrable) if it does not admit a frame structure for which the anholonomy coefficients from \( (1) \) vanishes, i.e. \( w^\gamma_{\alpha\beta} \neq 0 \). In this case the anholonomy becomes a proper spacetime characteristics. For instance, a generic anholonomy could be obtained if we consider nontrivial reductions from higher dimension spaces to lower dimension ones. It induces non-vanishing additional terms into the torsion,

\[
T(\delta_\gamma, \delta_\beta) = T^\alpha_{\beta\gamma} \delta_\alpha,
\]
and curvature,

\[ R(\delta_\tau, \delta_\gamma) \delta_\beta = R^\alpha_{\beta \gamma \tau} \delta_\alpha, \]

tensors of a linear connection \( \Gamma^\alpha_{\beta \gamma} \), with coefficients defined respectively as

\[ T^\alpha_{\beta \gamma} = \Gamma^\alpha_{\beta \gamma} - \Gamma^\alpha_{\gamma \beta} + u^\alpha_{\beta \gamma}, \quad (4) \]

and

\[ R^\alpha_{\beta \gamma \tau} = \delta_\tau \Gamma^\alpha_{\beta \gamma} - \delta_\gamma \Gamma^\alpha_{\beta \delta} + \Gamma^\varphi_{\beta \gamma} \Gamma^\alpha_{\varphi \tau} - \Gamma^\varphi_{\beta \tau} \Gamma^\alpha_{\varphi \gamma} + \Gamma^\alpha_{\beta \varphi} u^\varphi_{\gamma \tau}. \quad (5) \]

The Ricci tensor is defined

\[ R_{\beta \gamma} = R^\alpha_{\beta \gamma \alpha}, \quad (6) \]

and the scalar curvature is

\[ R = g^\beta_\gamma R_{\beta \gamma}. \quad (7) \]

The Einstein equations on an anholonomic spacetime are introduced in a standard manner,

\[ R_{\beta \gamma} - \frac{1}{2} g_{\beta \gamma} R = k \Upsilon_{\beta \gamma}, \quad (8) \]

where the energy–momentum d–tensor \( \Upsilon_{\beta \gamma} \) includes the cosmological constant terms and possible contributions of torsion \( (\ref{eq:4}) \) and matter and \( k \) is the coupling constant.

For a symmetric linear connection the torsion field can be considered as induced by some anholonomy (or equivalently, by some imposed constraints) conditions. For dynamical torsions there are necessary additional field equations, see, for instance, the case of locally anisotropic gauge like theories \( \ref{eq:5} \).

The usual locally isotropic Einstein gravity is obtained on the supposition that for every anholonomic frame could be defined corresponding linear transforms to a coordinate frame basis.

It is a topic of further theoretical and experimental investigations to establish if the present day experimental data on anisotropic structure of Universe is a consequence of matter and quantum fluctuation induced anisotropies and for some scales the anisotropy is a consequence of anholonomy of observer’s frame. The spacetime anisotropy could be also a generic property following, for instance, from string theory, and from a more general self–consistent gravitational theory when both the left (geometric) and right (matter energy–momentum tensor) parts of Einstein equations depend on anisotropic parameters.

### 2.2 The local anisotropy and nonlinear connection

A subclass of anholonomic spacetimes consists from those with local anisotropy modelled by a nonlinear connection structure. In this subsection we briefly outline the geometry of anholonomic frames with induced nonlinear connection structure.

The la–spacetime dimension is split locally into two components, \( n \) for isotropic coordinates and \( m \) for anisotropic coordinates, when \( n_{(a)} = n + m \) with \( n \geq 2 \) and \( m \geq 1 \). We shall use local coordinates \( u^a = (x^i, y^a) \), where Greek indices \( \alpha, \beta, ... \) take values \( 1, 2, ..., n + m \) and Latin indices \( i \) and \( a \) are correspondingly \( n \) and \( m \) dimensional, i.e. \( i, j, k, ... = 1, 2, ..., n \) and \( a, b, c, ... = 1, 2, ..., m \).
Now, we consider an invariant geometric definition of spacetime’s splitting into isotropic and anisotropic components. For modelling la–spacetimes one uses a vector bundle $E = (E_{n+m}, p, M(n), F(m), Gr)$ provided with nonlinear connection (in brief, N–connection) structure $N = \{N^a_j(u^a)\}$, where $N^a_j(u^a)$ are its coefficients [23]. We use denotations: $E_{m+n}$ is a $(n + m)$–dimensional total space of a vector bundle, $M(n)$ is the $n$–dimensional base manifold, $F(m)$ is the typical fiber being a $m$–dimensional real vector space, $Gr$ is the group of automorphisms of $F(m)$ and $p$ is a surjective map. For simplicity, we shall consider only local constructions on vector bundles.

The N–connection is a new geometric object which generalizes that of linear connection. This concept came from Finsler geometry (see the Cartan’s monograph [3]), the global formulation of it is due to W. Barthel [2], and it is studied in details in Miron and Anastasiei works [23]. We have extended the geometric constructions for spinor bundles and superbundles with further applications in locally anisotropic field theory and strings and modern cosmology and astrophysics [30, 31, 32]. We have also illustrated [34] that the N–connection could be introduced on (pseudo) Riemannian spacetimes and in Einstein gravity if we consider anholonomic frames consisting from sets of basic vectors some of them being holonomic and the rest anholonomic. In this case the N–connection coefficients are associated to the frame structure and transforms into some metric components if the considerations are transferred with respect to a coordinate basis.

The rigorous mathematical definition of N–connection is based on the formalism of horizontal and vertical subbundles and on exact sequences of vector bundles. Here, for simplicity, we define a N–connection as a distribution which for every point $u = (x, y) \in E$ defines a local decomposition of the tangent space of our vector bundle, $T_uE$, into horizontal, $H_uE$, and vertical (anisotropy), $V_uE$, subspaces, i.e.

$$T_uE = H_uE \oplus V_uE.$$  

If a N–connection with coefficients $N^a_j(u^a)$ is introduced on the vector bundle $E$ the modelled spacetime posses a generic local anisotropy and in this case we can not apply in a usual manner the operators of partial derivatives and their duals, differentials. Instead of coordinate bases (2) and (3) we must consider some bases adapted to the N–connection structure:

$$\delta_\alpha = (\delta_i, \partial_a) = \frac{\delta}{\partial u^\alpha} \equiv \left(\delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N^b_i \left(x^j, y^b\right) \frac{\partial}{\partial y^b}, \partial_a = \frac{\partial}{\partial y^a}\right),$$  

and

$$\delta^\beta = \left(d^i, \delta^a\right) = \delta u^\beta = \left(d^i = dx^i, \delta^a = dy^a + N^a_a \left(x^j, y^b\right) dx^k\right).$$  

A nonlinear connection (N–connection) is characterized by its curvature

$$\Omega^a_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}.$$  


Here we note that the class of usual linear connections can be considered as a particular case when
\[ N^a_j(x, y) = \Gamma^a_{bj}(x)y^b. \]

The elongation (by N–connection) of partial derivatives in the adapted to the N–connection partial derivatives \( (\mathcal{I}) \), or the locally adapted basis (la–basis) \( \delta_\beta \), reflects the fact that the spacetime \( \mathcal{E} \) is locally anisotropic and generically anholonomic because there are satisfied anholonomy relations \( (\mathcal{I}) \),
\[ \delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w^\gamma_\alpha_\beta \delta_\gamma, \]
where anholonomy coefficients are as follows
\[ w^k_{ij} = 0, w^k_{aj} = 0, w^k_{ia} = 0, w^c_{ab} = 0, w^{\alpha}_{ab} = 0, \]
\[ w^a_{ij} = -\Omega^a_{ij}, w^b_{aj} = -\partial^a N^b_i, w^b_{ia} = \partial_a N^b_i. \]

On a la–spacetime the geometrical objects have a distinguished (by N–connection), into horizontal and vertical components, character. They are briefly called d–tensors, d–metrics and/or d–connections. Their components are defined with respect to a la–basis of type \( (\mathcal{I}) \), its dual \( (\mathcal{I}) \), or their tensor products (d–linear or d–affine transforms of such frames could also be considered). For instance a covariant and contravariant d–tensor \( Z \), is expressed as
\[ Z = Z^\alpha_\delta_\alpha \otimes \delta^\beta = Z^i_j \delta_i \otimes dx^i + Z^i_\delta_\delta \otimes \delta^\alpha + Z^b_j \partial_b \otimes dx^j + Z^b_\delta_\delta \otimes \delta^\alpha. \] (12)

A symmetric d–metric on la–space \( \mathcal{E} \) is written as
\[ \delta s^2 = g_{\alpha\beta}(u) \delta^\alpha \otimes \delta^\beta = g_{ij}(x, y)dx^i dx^j + h_{ab}(x, y)\delta y^a \delta y^b. \] (13)

A linear d–connection \( D \) on la–space \( \mathcal{E} \),
\[ D_{\delta \alpha \beta} = \Gamma^\alpha_{\beta \gamma}(x^k, y) \delta_\alpha, \]
is parametrized by non–trivial h–v–components,
\[ \Gamma^\alpha_{\beta \gamma} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}). \] (14)

Some d–connection and d–metric structures are compatible if there are satisfied the conditions
\[ D_{\alpha \beta \gamma} = 0. \]

For instance, a canonical compatible d–connection
\[ c\Gamma^\alpha_{\beta \gamma} = (c L^i_{jk}, c L^a_{bk}, c C^i_{jc}, c C^a_{bc}) \]
is defined by the coefficients of d–metric \( (\mathcal{I}) \), \( g_{ij}(x, y) \) and \( h_{ab}(x, y) \), and by the coefficients of N–connection,
\[ c L^i_{jk} = \frac{1}{2} g^{in} (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}), \] (15)
\[ c L^a_{bk} = \partial_b N^a_k + \frac{1}{2} h^{ac} \left( \delta_k h_{bc} - h_{dc} \partial_b N^d_i - h_{db} \partial_c N^d_i \right), \]
\[ c C^i_{jc} = \frac{1}{2} g^{jk} \partial_c g_{jk}, \]
\[ c C^a_{bc} = \frac{1}{2} h^{ad} (\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}). \]
The coefficients of the canonical d–connection generalize for la–spacetimes the well known Cristoffel symbols.

For a d–connection \((14)\) we can compute the components of, in our case d–torsion, \((16)\)
\[
T_{jk}^i = T_{jk}^i = L_{jk}^i - L_{kj}^i, \quad T_{ia}^j = C_{ija}, \quad T_{ai}^j = -C_{ja}, \quad T_{aj}^i = 0, \quad T_{bc}^a = S_{bc}^a = C_{bc}^a - C_{cb}^a, \quad T_{ij}^a = -\Omega_{ij}^a, \quad T_{bi}^a = \partial_b N_i^a - L_{bi}^a, \quad T_{ib}^a = -T_{bi}^a.
\]

In a similar manner, putting non–vanishing coefficients \((14)\) into the formula for curvature \((17)\), we can compute the non–trivial components of a d–curvature
\[
R_{h,jk}^i = \delta_k L_{hj}^i - \delta_j L_{hk}^i + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{h,ia}^i \Omega_{jk}^a, \quad R_{b,jk}^i = \delta_k L_{bj}^i - \delta_j L_{bk}^i + L_{bj}^c L_{ck}^i - L_{bk}^c L_{cj}^i - C_{b,ia}^i \Omega_{jk}^a, \quad P_{j,ka}^i = \partial_j L_{ik}^a + C_{i,j}^a T_{ka}^b - (\partial_k C_{ij}^a + L_{ik}^a C_{ja}^j + L_{jk}^a C_{ia}^j - L_{jk}^a C_{ia}^j - L_{ja}^a C_{ik}^j, \quad P_{b,ka}^c = \partial_b L_{ck}^c + C_{c,ia}^c T_{ka}^d - (\partial_k C_{ca}^c + L_{ck}^c C_{ba}^a - L_{ck}^d C_{da}^a - L_{ck}^d C_{da}^a - L_{ck}^d C_{da}^a) \quad S_{j,ba}^c = \partial_b C_{ja}^c - \partial_j C_{ba}^c + C_{ja}^c C_{hc}^c - C_{ja}^c C_{hc}^c \quad S_{b,cd}^a = \partial_b C_{ba}^a - \partial_a C_{bd}^a + C_{ba}^a C_{cd}^a - C_{bd}^a C_{cd}^a.
\]

The components of the Ricci tensor \((16)\) with respect to locally adapted frames \((1)\) and \((14)\) (in this case, d–tensor) are as follows:
\[
R_{ij} = R_{ij,k}, \quad R_{ia} = -2P_{ia} = -P_{i,ka}, \quad R_{ai} = 1P_{ai} = P_{a,ib}, \quad R_{ab} = S_{ab}^c.
\]
We point out that because, in general, \(1P_{ai} \neq 2P_{ia}\) the Ricci d-tensor is non symmetric.

Having defined a d-metric of type \((13)\) in \(E\) we can compute the scalar curvature \((17)\) of a d-connection \(D\),
\[
\hat{R} = G^{a\beta} R_{a\beta} = \hat{R} + S,
\]
where \(\hat{R} = g^{ij} R_{ij}\) and \(S = h_{ab} S_{ab}\).

Now, by introducing the values \((16)\) and \((17)\) into anholonomic gravity field equations \((8)\) we can write down the system of Einstein equations for la–gravity with prescribed N–connection structure \((23)\):
\[
R_{ij} - \frac{1}{2} \left( \hat{R} + S \right) g_{ij} = k \Upsilon_{ij}, \quad S_{ab} - \frac{1}{2} \left( \hat{R} + S \right) h_{ab} = k \Upsilon_{ab}, \quad 1P_{ai} = k \Upsilon_{ai}, \quad 2P_{ia} = -k \Upsilon_{ia},
\]
where \(\Upsilon_{ij}, \Upsilon_{ab}, \Upsilon_{ai}\) and \(\Upsilon_{ia}\) are the components of the energy–momentum d–tensor field. We note that such decompositions into h– and v–components of gravitational
field equations have to be considered even in general relativity if physical interactions are examined with respect to an anholonomic frame of reference with associated N–connection structure.

There are variants of la–gravitational field equations derived in the low–energy limits of the theory of locally anisotropic (super)strings [31] or in the framework of gauge like la–gravity [35, 32] when the N–connection and torsions are dynamical fields and satisfy some additional field equations.

2.3 Modelling of generalized Finsler geometries in (pseudo) Riemannian spaces

The present day trend is to consider the Finsler like geometries and their generalizations as to be quite sophisticated for straightforward applications in quantum and classical field theory. The aim of this subsection is to prove that, a matter of principle, such geometries could be equivalently modelled on corresponding (pseudo) Riemannian manifolds (tangent or vector bundles) by using the Cartan’s moving frame method and from this viewpoint a wide class of Finsler like metrics could be treated as some solutions of usual Einstein field equations.

2.3.1 Almost Hermitian Models of Lagrange and Finsler Spaces

This topic was originally investigated by Miron and Anastasiei [23]; here we outline some basic results.

Let us model a la–spacetime not on a vector bundle \( \mathcal{E} \) but on a manifold \( \overline{T M} = TM \setminus \{0\} \) associated to a tangent bundle \( TM \) of a \( n \)–dimensional base space \( M \) (when the dimensions of the typical fiber and base are equal, \( n = m \) and \( \{0\} \) means that there is eliminated the null cross–section of the bundle projection \( \tau : TM \rightarrow M \)) and consider d–metrics of type

\[
\delta s^2 = g_{\alpha\beta}(u) \delta^\alpha \otimes \delta^\beta = g_{ij}(x, y)dx^i dx^j + g_{ij}(x, y)dy^i dy^j.
\]  (19)

On \( TM \) we can define a natural almost complex structure \( C_{(a)} \) as follows

\[
C_{(a)} (\delta_i) = -\partial/\partial y^i \quad \text{and} \quad C_{(a)} \left( \partial/\partial y^i \right) = \delta_i
\]

where the la–derivative \( \delta_i = \partial/\partial x^i - N^k_i \partial/\partial y^k \) (9) and la–differential \( \delta^i = dy^i + N^k_i dx^k \) (10) act on \( \overline{T M} \) being adapted to a nontrivial N–connection structure \( N = \left\{ N^k_j(x, y) \right\} \) in \( TM \). It is obvious that \( C^2_{(a)} = -I \). The pair \( (\delta s^2, C_{(a)}) \) defines an almost Hermitian structure on \( \overline{T M} \) with an associate 2–form

\[
\theta = g_{ij}(x, y) \delta^i \Lambda dx^j
\]

and the triad \( K^{2n} = (\overline{T M}, \delta s^2, C_{(a)}) \) is an almost Kählerian space. By straightforward calculations we can verify that the canonical d–connection (14) satisfies the conditions

\[
^cD_X \left( \delta s^2 \right) = 0, \quad ^cD_X \left( C_{(a)} \right) = 0
\]

10
for any d–vector $X$ on $TM$ and has zero $hhh$– and $vvv$–torsions.

The notion of **Lagrange space** \[20, 23\] was introduced as a generalization of Finsler geometry in order to geometrize the fundamental concepts in mechanics. A regular Lagrangian $L(x^i, y^i)$ on $\tilde{T}M$ is introduced as a continuity class $C^\infty$ function $L : TM \to \mathbb{R}$ for which the matrix

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}$$

has rank $n$ and is of constant signature on $\tilde{T}M$. A d–metric \[20\] with coefficients of form \[20\], a corresponding canonical d–connection \[14\] and almost complex structure $C(a)$ defines an almost Hermitian model of Lagrange geometry.

For arbitrary metrics $g_{ij}(x, y)$ of rank $n$ and constant signature on $\tilde{T}M$, which can not be determined as a second derivative of a Lagrangian, one defines the so–called generalized Lagrange geometry on $TM$ (see details in \[23\]).

A particular subclass of metrics of type \[20\] consists from those instead of a regular Lagrangian one considers a Finsler metric function $F$ on $M$ defined as $F : TM \to \mathbb{R}$ having the properties that it is of class $C^\infty$ on $\tilde{T}M$ and only continuous on the image of the null cross–section in $TM$, the restriction of $F$ on $\tilde{T}M$ is a positive function homogeneous of degree 1 with respect to the variables $y^i$, i. e.

$$F(x, \lambda y) = \lambda F(x, y), \lambda \in \mathbb{R}^n,$$

and the quadratic form on $\mathbb{R}^n$ with coefficients

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

defined on $\tilde{T}M$, is positive definite. Different approaches to Finsler geometry, its generalizations and applications are examined in a number of monographs \[11, 5, 23\] and as a rule they are based on the assertion that in this type of geometries the usual (pseudo)Riemannian metric interval

$$ds = \sqrt{g_{ij}(x) \, dx^i \, dx^j}$$

on a manifold $M$ is changed into a nonlinear one defined by the Finsler metric $F$ (fundamental function) on $\tilde{T}M$ (we note an ambiguity in terminology used in monographs on Finsler geometry and on gravity theories with respect to such terms as Minkowschi space, metric function and so on)

$$ds = F(x^i, dx^j).$$

Geometric spaces with a 'combersome' variational calculus and a number of curvatures, torsions and invariants connected with nonlinear metric intervals of type \[20\] are considered as less suitable for purposes of modern field and particle physics.

In our investigations of generalized Finsler geometries in (super) string, gravity and gauge theories \[30, 31\] we advocated the idea that instead of usual geometric
constructions based on straightforward applications of derivatives of (21) following from a nonlinear interval (22) one should consider d–metrics (13) and/or (19) with the coefficients of necessity determined via an almost Hermitian model of a Lagrange (20), Finsler geometry (21) and/or their extended variants. This way, by a synthesis of the moving frame method with the geometry of N–connection, we can investigate a various class of higher and lower dimension gravitational models with generic or induced anisotropies in a unified manner on some anholonomic and/or Kaluza–Klein spacetimes.

As a matter of principle, having a physical model with a d–metric and geometrical objects associated to la–frames, we can redefine the physical values with respect to a local coordinate base on a (pseudo) Riemannian space. The coefficients \( g_{ij} (x, y) \) and \( h_{ab} (x, y) \) of d–metric written for a la–basis \( \delta u^\alpha = (dx^i, \delta y^a) \) transforms into a usual (pseudo) Riemannian metric if we rearrange the components with respect to a local coordinate basis \( du^{\hat{\alpha}} = (dx^{\hat{i}}, dy^{\hat{a}}) \),

\[
g^{\hat{\alpha}\hat{\beta}} = \left( \begin{array}{ccc} g^{\hat{i}\hat{j}} + N^k_j h^{\hat{a}\hat{b}} N^e_j h^{\hat{a}\hat{e}} & N^e_j h^{\hat{a}\hat{e}} h^{\hat{b}\hat{e}} & N^e_j h^{\hat{a}\hat{e}} h^{\hat{b}\hat{e}} h^{\hat{a}\hat{b}} \\ N^k_i h^{\hat{a}\hat{e}} & h^{\hat{a}\hat{b}} & h^{\hat{a}\hat{b}} h^{\hat{a}\hat{b}} \end{array} \right),
\]

where 'hats' on indices emphasize that coefficients of the metric are given with respect to a coordinate (holonomic) basis on a spacetime \( V^{n+m} \). Parametrizations (ansatzs) of metrics of type (23) are largely applied in Kaluza–Klein gravity and its generalizations [25]. In our works [30, 31, 32, 34], following the geometric constructions from [23], we proved that the physical model of interactions is substantially simplified, as well we can correctly elucidate anisotropic effects, if we work with diagonal blocks of d–metrics (13) with respect to anholonomic frames determined by a N–connection structure.

### 2.3.2 Finsler like metrics in Einstein’s gravity

There are obtained [32, 34] (see Appendix) some classes of locally anisotropic cosmological and black hole like solutions (in three, four and higher dimensions) which can be treated as generalized Finsler metrics being of nonspheric symmetry (with rotation ellipsoid, torus and cylindrical event horizons, or with elliptical oscillations of horizons). Under corresponding conditions such metrics could be solutions of field equations in general relativity or its lower or higher dimension variants. Here we shall formulate the general criteria when a Finsler like metric could be a solution of gravitational field equations in Einstein gravity.

Let consider on \( T\tilde{M} \) an ansatz of type (23) when \( g_{ij} = h_{ij} = \frac{1}{2} \partial^2 F^2 / \partial y^i \partial y^j \) (for simplicity, we omit 'hats' on indices) i.e.

\[
g^{\hat{\alpha}\hat{\beta}} = \frac{1}{2} \left( \begin{array}{ccc} \partial^2 F^2 / \partial y^i \partial y^j + N^k_j \partial^2 F^2 / \partial y^i \partial y^k \partial y^j \\ N^k_j \partial^2 F^2 / \partial y^i \partial y^k \partial y^j \partial y^l \\ N^l_j \partial^2 F^2 / \partial y^i \partial y^k \partial y^j \partial y^l \end{array} \right),
\]

A metric (24), induced by a Finsler quadratic form (21) could be treated in a framework of a Kaluza–Klein model if for some values of Finsler metric \( F (x, y) \) and N–connection coefficients \( N^k_j (x, y) \) this metric is a solution of the Einstein equations
written with respect to a holonomic frame. For the dimension \( n = 2 \), when the values \( F \) and \( N \) are chosen to induce locally a (pseudo) Riemannian metric \( \hat{g}^{\alpha\beta} \) of signature \((-+, +, +, +)\) and with coefficients satisfying the four dimensional Einstein equations we define a subclass of Finsler metrics in the framework of general relativity. Here we note that, in general, a \( N \)-connection, on a Finsler space, subjected to the condition that the induced (pseudo) Riemannian metric is a solution of usual Einstein equations does not coincide with the well known Cartan’s \( N \)-connection [5, 11]. We have to examine possible compatible deformations of \( N \)-connection structures [23].

Instead of Finsler like quadratic forms we can consider ansatzs of type (23) with \( g_{ij} \) and \( h_{ij} \) induced by a Lagrange quadratic form (20). A general approach to the geometry of spacetimes with generic local anisotropy can be developed on embeddings into corresponding Kaluza–Klein theories and adequate modelling of la–interactions with respect to anholonomic or holonomic frames and associated \( N \)-connection structures.

### 3 Collisionless relativistic kinetic equation

As argued in Section 2, the spacetimes could be of generic local anisotropy after non-trivial reductions from some higher dimension theories or posses a local anisotropy induced by anholonomic frame structures even we restrict our considerations to the general relativity theory. In this line of particular interest is the formulation of relativistic kinetic theory with respect to general anholonomic frames and elucidation of locally anisotropic kinetic and thermodynamic processes.

#### 3.1 The distribution function and its moments

We use the relativistic approach to kinetic theory (we refer readers to monographs [13, 34] for history and complete treatment). Let us consider a simple system consisting from \( \varpi \) point particles of mass \( \tilde{m} \) in a la–spacetime with a \( \tilde{d} \)-metric \( g_{\alpha\beta} \). Every particle is characterized by its coordinates \( u^\alpha_{(l)} = \left(x^i_{(l)}, y^a_{(l)}\right) \), \( (u^T = x^1 = ct \) is considered the time like coordinate, for simplicity we put hereafter the light velocity \( c = 1 \); \( x^2_{(l)}, x^3_{(l)}, \ldots, x^n_{(l)} \) and \( y^1_{(l)}, y^2_{(l)}, \ldots, y^m_{(l)} \) are respectively space like and anisotropy coordinates, where the index \( (l) \) enumerates the particles in the system. The particles momenta are denoted by \( p_{\alpha(l)} = g_{\alpha\beta} p^\beta_{(l)} \). We shall use the distribution function \( \Phi(u^\alpha, p_\beta) \), given on the space of supporting elements \( (u^\alpha, p_\beta) \), as a general characteristic of particle system.

The system (of particles) is to be defined by using the random function

\[
\phi(u^\alpha, p_\beta) = \sum_{l=1}^\infty \int \delta s \delta^{(\nu(a))} \left(u^\alpha - u^\alpha_{(l)}(s)\right) \delta^{(\nu(a))} \left(p^\beta - p^\beta_{(l)}(s)\right),
\]

where the sum is taken on all system’s particles,

\[
\delta s = \sqrt{g_{\alpha\beta} \delta u^\alpha \delta u^\beta}
\]
is the interval element along the particle trajectory $u_{(l)}^\alpha(s)$ parametrized by a natural parameter $s$ and $\delta^n(\alpha)(\alpha)$ is the $n(\alpha)$–dimensional delta function. The functions $u_{(l)}^\alpha(s)$ and $p_{(l)}^\beta(s)$ describing the propagation of the $l$–particle are found from the motion equations on la–spacetime
\begin{equation}
\tilde{m} \frac{\delta u_{(l)}^\alpha}{ds} = p_{(l)}^\alpha \tag{26}
\end{equation}
\begin{equation}
\tilde{m} \frac{D \delta p_{(l)}^\alpha}{ds} = \tilde{m} \frac{\delta p_{(l)}^\alpha}{ds} - \Gamma^\tau_{\alpha \beta} \left( u_{(l)}^\gamma \right) p_{\tau(l)}^\beta = F_{\alpha(l)},
\end{equation}
where $\Gamma^\alpha_{\beta \gamma}$ is the canonical d–connection with coefficients [13] and by $F_{\alpha(l)}$ is denoted an exterior force (electromagnetic or another type) acting on the $l$th particle.

The distribution function $\Phi(u, p)$ is defined by averaging on paths of random function
\begin{equation}
\Phi(u, p) = \langle \langle \phi(u, p) \rangle \rangle
\end{equation}
where brackets $\langle \langle \ldots \rangle \rangle$ denote path averaging.

Let consider a space like hypersurface $\mathcal{F}(u^\alpha) = \text{const}$ with elements $\delta \Sigma_\alpha = n_\alpha \delta \Sigma$, where $n_\alpha = \frac{d\mathcal{F}}{dT}$ and $d\Sigma = \sqrt{|g|} \delta^n(\alpha) u$, $|D\mathcal{T}| = \sqrt{g_{\alpha \beta} D_\alpha D_\beta \mathcal{T}}$ and $|g| \delta^n(\alpha) u$ is the invariant space–time volume. A local system of reference in a point $u_{(0)}^\alpha$, with the metric $g_{ij}^{(0)} = \eta_{ij} = \text{diag}(-1, 1, 1, \ldots, 1)$, is obtained if $\mathcal{F}(u_{(0)}^\alpha) = x_{(0)}^1$. In this case $\delta \Sigma_\alpha = \delta^1 \delta^n(\alpha)-1 u$, $\delta^n(\alpha)-1 u = dx^2 dx^3 \ldots dx^n \delta y^1 \delta y^2 \ldots \delta y^n$ and $\delta^\alpha_\beta$ is the Kronecker symbol. The value
\begin{equation}
\Phi(u, p) v^\alpha d\Sigma_\alpha \frac{\delta^n(\alpha)p}{\sqrt{|g|}},
\end{equation}
where $v^\alpha = p^\alpha/\tilde{m}$ and $\delta P = \delta^n(\alpha)p/\sqrt{|g|} = \delta p_1 \delta p_2 \ldots \delta p_{n(\alpha)}/\sqrt{|g|}$ is the invariant volume in the momentum space and defines the quantity of particles intersecting the hypersurface element $\delta \Sigma_\alpha$ with momenta $p_\alpha$ included into the element $\delta P$ in the vicinity of the point $u^\alpha$. The first $< p^\alpha >$ and second moments $< p^\alpha p^\beta >$ of distribution function $\Phi(x, p)$ give respectively the flux of particles $n^\alpha$ and the energy–momentum $T^{\alpha \beta}$,
\begin{equation}
< p^\alpha > = \int \Phi(u, p) p^\alpha \delta P = \tilde{m} n^\alpha \tag{27}
\end{equation}
and
\begin{equation}
< p^\alpha p^\beta > = \int \Phi(u, p) p^\alpha p^\beta \delta P = \tilde{m} T^{\alpha \beta}. \tag{28}
\end{equation}

We emphasize that the motion equations (26) have the first integral, $g_{\alpha \beta} p_{\alpha}^{(l)} p_{\beta}^{(l)} = \tilde{m}^2 = \text{const}$ for every $l$–particle, so the functions (29) and, in consequence, $\Phi(u, p)$ are non–zero only on the mass hypersurface
\begin{equation}
g_{\alpha \beta} p_\alpha p_\beta = \tilde{m}^2, \tag{29}
\end{equation}
which in distinguished by a N–connection form (see the dual to d–metric [13]) is written
\begin{equation}
g^{ij} p_i p_j + g^{ab} p_a p_b = \tilde{m}^2.
\end{equation}
For computations it is convenient to use a new distribution function \( f(u, p) \) on a \( n(a) - 1 \) dimensional la–hyperspace (29)

\[
\Phi(u^\alpha, p_\beta) = f(u^\alpha, p_\beta) \delta \left( \sqrt{g^{\alpha\beta} p_\alpha p_\beta - \tilde{m}} \right) \theta(p_T)
\]

where

\[
\theta(p_T) = \begin{cases} 
1, & p_T \geq 0 \\
0, & p_T < 0 
\end{cases}
\]

and the index \( \hat{\beta} \) runs values in \( n(a) - 1 \) dimensional la–space. The flux of particles (27) and of energy–momentum (28) are computed by using \( f(u, p) \) as

\[
n^\alpha = \int \delta \varsigma \ p^\alpha f(u^\gamma, p_\beta),
\]

with respective base \( n^i \) and fiber \( n^a \) components,

\[
n^i = \int \delta \varsigma \ p^i f(u^\gamma, p_\beta) \quad \text{and} \quad n^a = \int \delta \varsigma \ p^a f(u^\gamma, p_\beta),
\]

when

\[
\Upsilon^{\alpha\beta} = \int \delta \varsigma \ p^\alpha p^\beta f(u^\gamma, p_\beta)
\]

where \( p_T \) is expressed via \( p^2, p^3, \ldots, p^{n(a)} \) by using the equation (29) and it is used the abbreviation \( \delta \varsigma = \delta^{n(a)} - 1 p / (\sqrt{|g|} p_T) \). The energy–momentum can be also split into base–fiber (horizontal–vertical, in brief, h– and v–, or hv–components) by a corresponding distinguishing of momenta, \( \Upsilon^{\alpha\beta} = \{ \Upsilon^{ij}, \Upsilon^{aj}, \Upsilon^{ib}, \Upsilon^{ab} \} \).

The one–particle distribution function \( f(u, p) \) characterizes the number of particles (with mass \( \tilde{m} \), or massless if \( \tilde{m} = 0 \)) at a point \( u^\alpha \) of a la–spacetime of dimension \( n(a) \) having distinguished by N–connection momentum vector (in brief, momentum d–vector) \( p^\alpha = (p^i, p^a) = (p_T, \vec{p}) \). One states that

\[
f(u, p) p^\beta \ \delta \sigma_\beta \ \delta \varsigma = f(u, p) p^i \ \delta \sigma_i \ \delta \varsigma + f(u, p) p^a \ \delta \sigma_a \ \delta \varsigma,
\]

gives the number of world–lines of particles with momentum d–vector \( p^\alpha \) in an interval \( \delta p^\alpha \) around \( p \), crossing a space like hypersurface \( \delta \sigma_\beta \) at a point \( u \). When \( \delta \sigma_\beta \) is taken to be time like one has \( \delta \sigma_\beta = (\delta^{n(a)} u, 0, \ldots, 0) \).

The velocity d–field of a fluid \( U^\mu \) in la–spacetime can be defined in some way like in relativistic kinetic theory [13]. To obtain a particularly simple form for the energy–momentum d–tensor one should follow the Landau–Lifshitz approach [22] when the locally anisotropic fluid velocity \( U^\mu(u) = (U^i(u), U^a(u)) \) and the energy density \( \varepsilon(u) \) are defined respectively as the eigenvector and eigenvalue of the eigenvalue equation

\[
\Upsilon^{\alpha\beta} U_\beta = \varepsilon U^\alpha. \tag{30}
\]

A unique value of \( U^\alpha \) is to be found from the conditions to be a time like and normalized to the unity d–vector, \( U^\alpha U_\alpha = 1 \).
Having so fixed the fluid d–velocity we can define correspondingly the particle density
\[ \tilde{n}(u) = n^\alpha U_\alpha, \]  
(31)
the energy density
\[ \tilde{\varepsilon}(u) = U_\alpha \Upsilon^{\alpha\beta} U_\beta, \]  
(32)
and the average energy per particle
\[ e = \frac{\tilde{\varepsilon}}{\tilde{n}}. \]  
(33)

We can introduce also the pressure d–tensor
\[ P^{\alpha\beta}(u) = \Delta^{\alpha\tau} \Upsilon_{\tau\nu} \Delta^{\nu\beta}, \]  
(34)
by applying the projector
\[ \Delta^{\alpha\beta} = g^{\alpha\beta} - U^\alpha U^\beta \]  
(35)
with properties
\[ \Delta^{\alpha\beta} \Delta_{\beta\mu} = \delta^{\alpha}_\mu, \quad \Delta^{\alpha\beta} U_\beta = 0 \quad \text{and} \quad \Delta^\alpha_\mu = n^{(a)} - 1. \]

With respect to a la–frame \( \mathcal{F} \) we can introduce a particular Lorentz system, called the locally anisotropic rest frame of the fluid when \( U^\mu = (1, 0, 0, ..., 0) \) and \( \Delta^\alpha_\mu = diag (0, 1, 1, ..., 1) \). So the pressure d–tensor was defined as to coincide with the la–space part of the energy–momentum d–tensor with respect to a locally anisotropic rest frame.

### 3.2 Collisionless kinetic equation

We start our proof by applying the identity
\[ \int ds \frac{d}{ds} \delta^{(a)} \left( u^\alpha - u^\alpha_{(s)} \right) \delta^{(a)} \left( p^\beta - p^\beta_{(s)} \right) = 0. \]
Differentiating under this integral and taking into account the equation (29) we get the relation
\[ \frac{\delta (p^\alpha \phi)}{\partial u^\alpha} + \frac{\partial}{\partial p^\beta} \left( c \Gamma^\mu_{\beta\xi} p_\mu p^\xi \phi \right) = 0. \]  
(36)
In our further considerations we neglect the interactions between the particles and suppose that the metric of the background gravitational field \( g_{\alpha\beta} \) does not depend on motion of particles. After averaging (36) on paths we define the equation for the one–particle distribution function
\[ \frac{\delta (p^\alpha \Phi)}{\partial u^\alpha} + \frac{\partial}{\partial p^\beta} \left( c \Gamma^\mu_{\beta\xi} p_\mu p^\xi \Phi \right) = 0. \]  
(37)
As a matter of principle we can generalize the problem \( \mathcal{F} \) when particles are considered to be also sources of gravitational field which is self–consistently defined.
from the Einstein equations with the energy–momentum tensor defined by the formula (28).

Taking into account the identity
\[ \delta p^\alpha \partial u^\alpha + \partial \partial p^\beta (\Gamma^\mu_{\beta \varepsilon} p^\mu p^\varepsilon) = 0 \]
we get from (37) the collisionless kinetic equation for the distribution function \( \Phi(u^\alpha, p^\beta) \)
\[ p^\beta \hat{D}_\beta \Phi = 0, \tag{38} \]
where
\[ \hat{D}_\beta = \frac{\delta}{\partial u^\beta} - \Gamma^\varepsilon_{\beta \alpha} p^\alpha \partial \partial p^\varepsilon \tag{39} \]
generalizes the Cartan’s covariant derivation [3] for a space with higher order anisotropy (locally parametrized by supporting elements \( (u^\alpha, p^\beta) \)), provided with a (extended to momenta coordinates) higher order nonlinear connection
\[ \hat{\nabla}_\beta = \delta^\varepsilon_{\alpha} \delta^i_{\beta} N^\alpha_i - \Gamma^\varepsilon_{\beta \alpha} p^\alpha \tag{40} \]
(on the geometry of higher order anisotropic spaces and superspaces and possible applications in physics see [23, 30, 31]).

The equation (38) can be written in equivalent form for the distribution function \( f(u^\alpha, p^\beta) \),
\[ p^\lambda \hat{D}_\lambda f = 0, \tag{41} \]
with the Cartan’s operator defined on \( n_{(a)} - 1 \) dimensional momentum space.

The kinetic equations (38) and, equivalently, (41) reflect the conservation law of the quantity of particles in every volume of the space of supporting elements which holds in absence of collisions.

Finally, in this subsection, we note that the kinetic equation (38) is formulated on the space of supporting elements \( (u^\alpha, p^\beta) \), which is characterized by coordinate transforms
\[ x'^i = x'^i (x^k), y'^a = y^a K_a^i (x^i) \tag{42} \]
and
\[ p'^j = p_j \frac{\partial x^i}{\partial x'^j}, p^a = p_a K_a^i (x^i), \]
where \( K_a^i (x^i) \) and \( K_a^i (x^i) \) take values correspondingly in the set of matrices parametrizing the group of linear transforms \( GL(m, \mathbb{R}^m) \), where \( \mathbb{R} \) denotes the set of real numbers. In a particular case, for dimensions \( n = m \) we can parametrize
\[ K_a^i (x^i) = \frac{\partial x^i}{\partial x'^i} \text{ and } K_a^i (x^i) = \frac{\partial x^i}{\partial x'^i} \mid_{x^i=x'^i(x^i)}. \]

A distinguished by higher order nonlinear connection [10] tensor \( Q_{\beta_1 \beta_2 ... \beta_q}^\alpha (u^\varepsilon, p_\tau) \) being contravariant of rang \( r \) and covariant of rang \( q \) satisfy the next transformation laws under chainings of coordinates of type (41):
\[ Q_{\beta_1 \beta_2 ... \beta_q}^\alpha \left( u^\varepsilon, p_\tau \right) = \frac{\partial u^\alpha_{\varepsilon_i}}{\partial u^{\alpha_{\varepsilon_i}}} \cdots \frac{\partial u^\alpha_{\varepsilon_r}}{\partial u^{\alpha_{\varepsilon_r}}} \frac{\partial u^\beta_{\tau_i}}{\partial u^{\beta_{\tau_i}}} \cdots \frac{\partial u^\beta_{\tau_q}}{\partial u^{\beta_{\tau_q}}} Q_{\beta_1 \beta_2 ... \beta_q}^\alpha \left( u^\varepsilon, p_\tau \right). \tag{43} \]
Operators of type (40) and transformations (42) and (43), on first order of anisotropy, on tangent bundle $TM$ on a $n$ dimensional manifold $M$ were considered by E. Cartan in his approach to Finsler geometry [5] and by A. A. Vlasov [36] in order to formulate the statistical kinetic theory on a Finsler geometry background. It should be emphasized that the Cartan–Vlasov approach is to be applied even in general relativity because the kinetic processes are to be examined in a phase space provided with local coordinates $(u^\epsilon, p_\tau)$. Our recent generalizations [30, 31] to higher order anisotropy (including spinor and supersymmetric spaces) are to applied in the case of models with nontrivial reductions (modelled by $N$–connections) from higher dimensions to lower dimensional ones.

4 Kinetic equation with pair collisions

In this section we shall prove the relativistic kinetic equations for one–particle distribution function $f(u^\alpha, p_\beta)$ with pair collisions [7]. We summarize the related results and generalize the constructions for Minkowski spaces [15].

4.1 Integral of collisions, differential cross–section and velocity of transitions

If collisions of particles are taken into account (for simplicity we shall consider only pair collisions), the quantity of particles from a volume in the space of supporting elements is not constant. We have to introduce a source of particles $C(x, p)$, for instance, in the kinetic equations (41),

$$p^\alpha \, \delta_\alpha f = C(f) = C(u, p).$$ (44)

The scalar function $C(u, p)$ is called the integral of collisions (in the space of supporting elements). The value

$$\Delta^{n(\alpha)} u \, \Delta^{n(\alpha)-1} p / p^i C(u, p)$$

is the changing of the quantity of particles under pair collisions in a region $\Delta^{n(\alpha)} u \, \Delta^{n(\alpha)-1} p$.

Let us denote by

$$W(p, p^{[1]}|p', p'^{[1]}) / p^i p'^{[1]} p'^{[1]}$$ (45)

the probability of transition for two particles which before scattering have the momenta $p_i$ and $p_{i[1]}$ and after scattering the momenta $p'_i$ and $p'_{i[1]}$ with respective inaccuracies $\Delta^{n(\alpha)-1} p_i$ and $\Delta^{n(\alpha)-1} p'_{i[1]}$. The function $W(p, p^{[1]}|p', p'^{[1]})$ (the so–called collision rate) is symmetric on arguments $p, p^{[1]}$ and $p', p'^{[1]}$ and describes the velocity of transitions with conservation of momenta of type

$$p^\alpha + p^{\alpha [1]} = p'^\alpha + p'^{\alpha [1]}$$
when the conditions (29) hold.

The number of binary collisions within a la–spacetime interval \( \delta u \) around a point \( u \) between particles with initial momenta in the ranges

\[
(\vec{p}, \vec{p} + \delta \vec{p}) \quad \text{and} \quad (\vec{p}_{[1]}, \vec{p}_{[1]} + \delta \vec{p}_{[1]})
\]

and final momenta in the ranges

\[
(\vec{p}', \vec{p}' + \delta \vec{p}') \quad \text{and} \quad (\vec{p}'_{[1]}, \vec{p}'_{[1]} + \delta \vec{p}'_{[1]})
\]

is given by

\[
f(u, p) f(u, p_{[1]}) W(p, p_{[1]}; p', p'_{[1]}) \delta \varsigma_p \delta \varsigma_{p_{[1]}} \delta \varsigma_{p'} \delta \varsigma_{p'_{[1]}} \delta u
\]

(46)

were, for instance, we abbreviated \( \delta \varsigma_p = \delta n(a)p/\sqrt{|g|p^T} \).

The collision integral of the Boltzmann equation in la–spacetime (44) is expressed in terms of the collision rate (45)

\[
C(f) = \int f(u, p) f(u, p_{[1]}) W(p, p_{[1]}; p', p'_{[1]}) \delta \varsigma_{p_{[1]}} \delta \varsigma_{p'} \delta \varsigma_{p'_{[1]}}.
\]

(47)

The collision integral is related to the differential cross section (see below).

### 4.2 The Cross Section in La–spacetime

The number \( n_{(\text{bin})} \) of binary collisions per unit time and unit volume when the initial momenta of the colliding particles lie in the ranges

\[
(\vec{p}, \vec{p} + \delta \vec{p}) \quad \text{and} \quad (\vec{p}_{[1]}, \vec{p}_{[1]} + \delta \vec{p}_{[1]})
\]

and the final momenta are in some interval \( \varsigma \) in the space of variables \( \vec{p}' \) and \( \vec{p}'_{[1]} \) is to be obtained from (46) by dividing on \( \delta u = \delta t \delta x^2 \ldots \delta x^n dy^1 \ldots dy^m \) and integrating with respect to the primed variables,

\[
n_{(\text{bin})} = f(u, p) f(u, p_{[1]}) \times \delta \varsigma_p \delta \varsigma_{p_{[1]}} \int_{\varsigma} W(p, p_{[1]}; p', p'_{[1]}) \delta \varsigma_{p'} \delta \varsigma_{p'_{[1]}}.
\]

(48)

Let us introduce an auxiliary velocity

\[
v = F/(p^T p_{[1]})
\]

(49)

were the so–called Möller flux factor is defined by

\[
F = \left[ (p^\alpha p_{[1]})^2 - \tilde{m}^4 \right]^{1/2}.
\]

(50)

It may be verified that the speed \( v \) reduces to the relative speed of particles in a frame in which one of the particles initially is at rest. We also consider the product of the number density of target particles with momenta in the range \( (\vec{p}_{[1]}, \vec{p}_{[1]} + \delta \vec{p}_{[1]}) \) (given by \( f(u, p_{[1]}) \delta^{n(a)-1} p_{[1]} \)) and the flux of incoming particles with momenta in
the range \((\vec{p}, \vec{p} + \delta\vec{p})\) (given by \(f(u, p)\delta^{n(a)-1}p\nu\), where \(\nu\) is the speed \((19)\). This product can be written

\[
Ff(u, p)f(u, p_{[1]})\delta\varsigma_{\nu}\delta\varsigma_{p_{[1]}}.
\]

By definition (like in usual the relativistic Boltzmann theory \([7, 15, 3]\)) the cross section \(\tilde{\sigma}\) is the division of the number \((48)\) to the number \((51)\)

\[
\tilde{\sigma} = \frac{1}{F} \int W(p, p_{[1]}; p', p'_{[1]})\delta\varsigma_{\nu}\delta\varsigma_{p_{[1]}}.
\]

The condition that collisions are local implies that the collision rate must contain \(n(a)\) delta–functions

\[
W(p, p_{[1]}; p', p'_{[1]}) = w(p, p_{[1]}; p', p'_{[1]}) \times \delta^{n(a)}(p + p_{[1]} - p' - p'_{[1]}).
\]

Let \(\mathbb{R}^{n(a)-1}\) be a \((n(a) - 1)\)–dimensional Euclidean space enabled with an orthonormal basis \((e_1, e_2, ..., e_{n(a)-1})\) and denote by \((u^1, u^2, ..., u^{n(a)-1})\) the Cartesian coordinates with respect to this basis. For calculations on scattering of particles it is useful to apply the system of spherical coordinates \((r_{n(a)} = r, \theta_{n(a)-1}, \theta_{n(a)-2}, ..., \theta_1)\) associated with the Cartesian coordinates \((u_1, u_2, ..., u_n)\). The volume element can be expressed as

\[
d^{n(a)-1}u = (r)^{n(a)-2}dr\, d^{n(a)-2}\Omega_\theta,
\]

where the element of solid angle, the \((n(a) - 1)\)–spherical element, is given by

\[
d^{n+m-2}\Omega_\theta = \sin^{n+m-3}\theta_{n+m-2} \cdot \sin^{n+m-4}\theta_{n+m-3} \cdot ...
\]

\[
\sin^2\theta_3 \cdot \sin\theta_2 \times d\theta_{n+m-2} \cdot ... \cdot d\theta_2 \cdot d\theta_1.
\]

In our further considerations we shall consider that the Cartesian and spherical \((n(a) - 1)\)–dimensional coordinates are given with respect to a la–frame of type \((18)\).

In order to eliminate the delta functions from \((53)\) put into \((52)\) we fix as a reference frame the center of mass frame for the collision between two particles with initial momenta \(\vec{p}\) and \(\vec{p}_{[1]}\). The quantities defined with respect to the center of mass will be enabled with the subindex \(CM\). One denotes by \(\vec{p}_{CM}\) the polar axis and characterizes the directions \(\vec{p}'_{CM}\) of the outgoing particles with respect to polar axis by means of generalized spherical coordinates. In this case

\[
\delta^{n(a)-1}p'_{CM} = \left|\vec{p}'_{CM}\right|^{n(a)-2}d\left|\vec{p}'_{CM}\right|d\Omega_{CM}
\]

where \(d\Omega_{CM}\) is given by the formula \((54)\).

The total \((n(a) + 1)\)–momenta before and after collision in la–spacetime are given by d–vectors

\[
P^\alpha = p^\alpha + p^\alpha_{[1]}, \quad P'^\alpha = p'^\alpha + p'^\alpha_{[1]}.
\]

Following from

\[
P'^\alpha_{CM} = 2p'^\alpha_{CM} = 2\sqrt{\left|\vec{p}'_{CM}\right|^2 + m^2}
\]

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we have
\[ \left| \mathbf{p}'_{CM} \right|^2 d \left| \mathbf{p}'_{CM} \right| = \frac{1}{4} P_{CM}' d P_{CM}' . \]

In result the formula for the volume element (55) transforms into
\[ \delta^{n(a)-1} p'_{CM} = \frac{1}{4} \left| \mathbf{p}'_{CM} \right|^{n(a)-2} P_{CM}' d P_{CM}' d \Omega_{CM} . \]

Inserting this collision rate (53) into (52) we obtain an integral which can be rewritten (see the Appendices 1 and 7 to the paper [3] for details on transition to the center of mass variables; in la–spacetimes one holds good similar considerations with that difference that me must work with respect to la–frames (9) and (10) and d–metric (13)
\[ \sigma = \frac{F_{n(a)-4}}{E_{n(a)-2}} \int w d \Omega_{CM} \] (57)

where \( E = \sqrt{P \cdot P} , \) is the total energy (devided by \( c = 1 \)).

The differential section in the center of mass system is found from (57)
\[ \frac{\delta \sigma}{d \Omega} \bigg|_{CM} = \frac{F_{n(a)-4}}{E_{n(a)-2}} w \]

which allow to express the collision rate (53) as
\[ W \left( p, p_{[1]} ; p', p'_{[1]} \right) = \frac{E_{n(a)-2}}{E_{n(a)-4}} \delta \sigma \bigg|_{CM} \cdot \delta^{n(a)} \left( p + p_{[1]} - p' - p'_{[1]} \right) . \] (58)

We shall use the formula (58) in order to compute the transport coefficients.

5 Equilibrium States in La–Spacetimes

When the system is in equilibrium we can derive an expression for the particle distribution function \( f = f_{eq} = f_{eq} (u, p) \) in a similar way as for locally isotropic spaces and write
\[ f_{eq} (u, p) = \frac{1}{(2 \pi \hbar)^{n(a)-1}} \exp \left( \frac{\mu - p_i U_i - p^a U_a}{k_B T} \right) , \] (59)

where \( k_B \) is Boltzmann’s constant and \( \hbar \) is Planck’s constant divided by \( 2 \pi \) and \( \mu = \mu (u) \) and \( T = T (u) \) are respectively the thermodynamic potential and temperature. For simplicity, in this section we shall omit explicit dependencies of \( \mu \) and \( T \) on la–spacetime coordinates \( u \). Our thermodynamic systems will be considered in local equilibrium in a vicinity of a point \( u_{[0]} \) with respect to a rest frame locally adapted to N–connection structure, like (9) and (10). In the simplest case the \( n + m \) splitting is trivially given by a N–connection with vanishing curvature (11). For such conditions of trivial la–spacetime \( \mu \) and \( T \) can be considered as constant values.
5.1 Particle Density

The formula relating the particle density $\tilde{n}$, temperature $T$, and thermodynamic potential $\mu$ is obtained by inserting (59) into (31), with (27). Choosing the locally adapted to the $\bar{N}$–connection (9) to be the rest frame, when $U_\mu = (1, 0, \ldots, 0)$ the calculus is to be performed as for isotropic $n(a)$ dimensional spaces [3]. The integral for the number of particles per unit volume in the rest la–frame is

$$\tilde{n}(\mu, T) = \frac{1}{(2\pi\bar{h})^{n(a)-1}} \exp \left(\frac{\mu}{k_BT}\right) \times \int p^\uparrow \exp \left(-\frac{p^\uparrow}{k_BT}\right) \frac{d^{n(a)-1}p}{p^\uparrow},$$

(60)

with

$$p^\uparrow = \left(|\vec{p}|^2 + \tilde{m}^2\right)^{1/2}$$

(61)

and depends only on length of $n(a)-1$ dimensional d–vector $\vec{p}$. By applying spherical coordinates, when

$$d^{n(a)-1}p = |\vec{p}|^{n(a)-2} d|\vec{p}| \ d\Omega$$

and $d\Omega$ is given by the expression (54), and differentiating on radius $\rho$ the well known formula for the volume $V_{n(a)-1}$ in a $n(a) - 1$ dimensional Euclidean space

$$V_{n(a)-1} = \frac{(n(a) - 1) \pi^{(n(a)-1)/2}}{\Gamma[(n(a) + 1)/2]} \rho,$$

(62)

where $\Gamma$ is the Euler gamma function, then putting $\rho = 1$ we get

$$\int d\Omega = \frac{(n(a) - 1) \pi^{(n(a)-1)/2}}{\Gamma[(n(a) + 1)/2]}.$$

(63)

We note that from (61) one follows

$$|\vec{p}| \ d|\vec{p}| = p^\uparrow \ dp^\uparrow$$

and we can transform (54) into an integral with respect to the angular variables through the replacement

$$\frac{d^{n(a)-1}p}{p^\uparrow} \rightarrow \frac{(n(a) - 1) \pi^{(n(a)-1)/2}}{\Gamma[(n(a) + 1)/2]} |\vec{p}|^{n(a)-3} dp^\uparrow.$$

Introducing the dimensionless quantities

$$\xi = p^\uparrow / k_BT, \text{ and } \vartheta = \tilde{m} / k_BT,$$

(64)

for which, respectively,

$$dp^\uparrow = k_BT \ d\xi, \text{ and } |\vec{p}| = k_BT \sqrt{\xi^2 - \vartheta^2},$$

(65)
the integral (60) is computed

\[ \tilde{n}(\mu, T) = 2^{n(a)/2} \left(\frac{\pi \tilde{m}}{2\pi \hbar}\right)^{n(a)-1} \left(\frac{k_B T}{\tilde{m}}\right)^{(n(a)-2)/2} \]

(66)

\[ \times \exp\left(\frac{\mu}{k_B T}\right) K_{(n(a)+1)/2}\left(\frac{\tilde{m}}{k_B T}\right), \]

where the modified Bessel function of the second kind of order \( n(a)/2 \) has the integral representation

\[ K_s(\eta) = \frac{\sqrt{\pi}}{(2\eta)^s} \Gamma\left(s + \frac{1}{2}\right) \int_\infty^{\infty} e^{-q^2 - \eta^2} dq. \]

(67)

We note the dependence of (66) (but not of (65)) on \( m \) anisotropic parameters (coordinates). The formulas proved in this subsection transforms into locally anisotropic ones \([3]\) if the N–connection is fixed to be trivial (with vanishing N–connection curvature \([1]\)) and the d–metric \([3]\) transforms into a usual (pseudo) Riemannian one.

### 5.2 Average energy and pressure

The average energy per particle, for a system in equilibrium, can be calculated by introducing the distribution function (59) into (33) and applying the formulas (28), (32) and (66). In terms of dimensionless variables (64) we have

\[ \varepsilon(\mu, T) = \frac{(k_B T)^{n(a)}}{(2\pi \hbar)^{n(a)-1}} \frac{\left(n(a) - 1\right) \pi^{(n(a)-1)/2}}{\Gamma\left((n(a) + 1)/2\right)} \]

\[ \times \exp\left(\frac{\mu}{k_B T}\right) \int_\vartheta^{\infty} \left(\xi^2 - \vartheta^2\right)^{(n(a)-3)/2} \xi^2 d\xi. \]

After carrying out partial integrations together with applications of the formula

\[ \left(q^2 - \zeta^2\right)^{(s-2)/2} = \frac{1}{s} \frac{d}{dq} \left[\left(q^2 - \zeta^2\right)^{s/2}\right] \]

we obtain

\[ \varepsilon(\vartheta) = \pi^{(n(a)-2)/2} \frac{\tilde{m}^{n(a)}}{(2\pi \hbar)^{n(a)-1}} \left(\frac{2}{\vartheta}\right)^{n(a)/2} \]

\[ \times \exp\left(\frac{\mu}{k_B T}\right) \left[\vartheta K_{(n(a)+2)/2}(\vartheta) - K_{n(a)/2}(\vartheta)\right]. \]

Dividing the energy density (68) to the particle density (60) and substituting the inverse dimensionless temperature \( \vartheta = \vartheta(T) \) \([3]\) we get the energy per particle

\[ e(T) = k_B T \left(\frac{\tilde{m}}{k_B T} \frac{K_{(n(a)+2)/2}}{K_{n(a)/2}} \left(\frac{\tilde{m}}{k_B T}\right) - 1\right). \]

(69)
This formula is the thermal equation of state of an equilibrium system having m anisotropy parameters (we remember that \(n_{(a)} = n + m\) with respect to a la–frame \(\mathcal{I}\)).

The pressure \(\delta\)-tensor is to be computed by substituting (59) into (34), applying also the integral (28). We get

\[
P^{\alpha\beta} = -k_B T \bar{n}(\mu, T) \Delta^{\alpha\beta},
\]

were, by definition, the coefficients before \(\Delta^{\alpha\beta}\) determine the pressure

\[
\bar{P} = k_B T \bar{n}(\mu, T).
\]

So, for a system of \(n_{(a)} - 1 = n + m - 1\) dimensions the expression (71) defines the equation of state of ideal gas of particles with respect to a la–frame, having \(m\) anisotropic parameters.

5.3 Enthalpy, specific heats and entropy

The average enthalpy per particle

\[
h(T) = e(T) + \bar{P}(\mu, T) / \bar{n}(\mu, T)
\]

is computed directly by substituting in this formula the values (69),(71) and (66). The result is

\[
h(T) = \bar{m} K^{(n_{(a)}+2)/2} \left( \frac{\bar{m}}{k_B T} \right)^{n_{(a)}/2}.
\]

Using (59) and (73) we can compute respectively the specific heats at constant pressure and at constant volume

\[
c_p = \left( \frac{\partial h}{\partial T} \right)_p \quad \text{and} \quad c_v = \left( \frac{\partial e}{\partial T} \right)_v.
\]

Subtracting of \(h\) and \(e\), after carrying out the differentiation with respect to \(T\), one finds Mayer’s relation,

\[
c_p - c_v = k_B
\]

and, introducing the adiabatic constant \(\gamma = c_p/c_v\),

\[
\frac{\gamma}{\gamma - 1} = \vartheta^2 + (n_{(a)} + 1) \frac{h}{k_B T} - \left( \frac{h}{k_B T} \right)^2.
\]

The relation (74) can be proven by straightforward calculations by using the properties of Bessel’s function \(J_\nu\) (see a similar proof for isotropic spacetimes in [2]).

The entropy per particle is introduced as in the isotropic case

\[
s = \frac{h - \mu}{T}.
\]
We have to insert the values of $\mu$ (found from (66))

$$
\mu (\bar{n}, T) = k_B T \times \ln \frac{(2\pi\hbar)^{n(a)-1} \bar{n}}{(2\bar{m})^{n(a)/2}(\pi k_B T)^{(n(a)-2)/2}K_{n(a)/2}^{n(a)}(\frac{m}{k_B T})} \tag{76}
$$

and of $h$ (see (73) in order to obtain the dependence $s = s(T)$ (for simplicity, we omit this cumbersome formula).

## 5.4 Low and high energy limits

Let us investigate the non–relativistic locally anisotropic limit by applying the asymptotic formula [1]

$$
K_s (\vartheta) \sim \sqrt{\frac{\pi}{2\vartheta}} e^{-\vartheta}
$$

valid for large $\vartheta$. The formula (69) gives

$$
e \sim \bar{m} + \frac{n + m - 1}{2} k_B T.
$$

Thus, each isotropic and anisotropic dimension contributes with $k_B T/2$ to the average energy per particle.

Now, we consider high temperatures. For $\vartheta \to 0$ one holds the asymptotic formula [1]

$$
K_s (\vartheta) \sim 2^{s-1} \Gamma (s) \vartheta^{-s}
$$

and the particle number density (66), the energy (69), enthalpy (73) and entropy (75) can be approximated respectively (we state explicitly the dimensions $n + m$) by

$$
\bar{n} (\mu, T) = 2^{n+m-1} \pi^{n+m-2} \Gamma \left( \frac{n + m}{2} \right) \times \left( \frac{k_B T}{2\pi\hbar} \right)^{n+m-1} \exp \left( \frac{\mu}{k_B T} \right), \tag{77}
\quad e = (n + m - 1) k_B T, \quad h = (n + m) k_B T, \\
\quad s = (n + m) k_B.
$$

In consequence, the corresponding specific heats and adiabatic constant are

$$
c_p = (n + m) k_B, c_v = (n + m - 1) k_B \quad \text{and} \quad \gamma = \frac{n + m}{n + m - 1}.
$$

These formulas imply that for large $T$ the spacetime anisotropy (very possible at the beginning of our Universe) could modify substantially the thermodynamic parameters.

Finally we emphasize that in the ultrarelativistic limit

$$
s = 2^{n+m-1} \pi^{n+m-2} (n + m) \times \Gamma \left( \frac{n + m}{2} \right) \left( \frac{k_B T}{2\pi\hbar} \right)^{n+m-1} \frac{k_B}{\bar{n}}. \tag{78}
$$

We also note that our treatment was based on Maxwell–Boltzmann instead of Bose–Einstein statistics.
6 Linearized Locally Anisotropic Transport Theory

For systems with local anisotropy outside equilibrium the distribution function \( f_{(eq)}(u, p) \) (\[59\]) must be generalized to another one, \( f(u, p) \), solving the kinetic la–equation (\[44\]). We follow a standard procedure of linearization and construction of solutions of kinetic equations by generalizing to la–spacetimes the Chapman–Enskog approach (we shall extend to locally anisotropic backgrounds the results presented in \[15\]).

We write

\[ f(u, p) = f_{[0]}(u, p) [1 + \varphi(u, p)] \] (79)

with the lowest order of approximation to \( f \) taken similarly to (59)

\[ f_{[0]}(u, p) = \frac{1}{(2\pi\hbar)^{n(a)-1}} \times \exp \left( \frac{\mu(u) - p^i U_i(u) - p^a U_a(u)}{k_B T(u)} \right) \] (80)

where the constant variables of a trivial equilibrium \( \mu, T \) and \( U_\alpha \) are changed by some local counterparts \( \mu(u), T(u) \) and \( U_\alpha(u) \). Outside equilibrium the first two dependencies \( \mu(u) \) and \( T(u) \) are defined respectively from relations (76) and (69).

6.1 Linearized Transport Equations

The so–called deviation function \( \varphi(u, p) \) from (79) describes the deformation by nonequilibrium flows of \( f_{[0]}(u, p) \) into \( f(u, p) \). It is considered that in equilibrium \( \varphi \) vanishes and not too far from equilibrium states it must be small. The Chapman–Enskog method states that after substituting \( f_{[0]} \) into the left side and \( f_{[0]}[1 + \varphi] \) into the right hand of (44) we shall neglect the quadratic and higher terms in \( \varphi \). In result we obtain a linearized equation for the deviation function (for simplicity, hereafter we shall not point to the explicit dependence of functions, kinetic and thermodynamic values on spacetime coordinates)

\[-p^\alpha \widetilde{D}_\alpha f_{[0]} = f_{[0]} L[\varphi] \]

where, having introduced (80) into (44), we write for the right side

\[ L[\varphi] = -p^\alpha \widetilde{D}_\alpha \left( \frac{\mu - p^i U_i - p^a U_a}{k_B T} \right) \].

The left side is to be computed by applying the generalized Cartan derivative (39).

We introduce a locally anisotropic generalization of the gradient operator by considering the operator (in brief, la–gradient)

\[ \nabla_\alpha = \Delta_\alpha^\beta D_\beta \] (81)

with \( \Delta_\alpha^\beta \) given by (33) (we use the a d–covariant derivative defined by a d–connection (14) instead of partial and/or isotropic derivatives for isotropic spaces (13, 3)). In a
local rest frame of the isotropic fluid $\nabla_{\alpha} \to (0, \partial_1, \partial_2, ...)$, i.e. in the flat isotropic space the la–gradient reduces to the ordinary space gradient. By applying $\nabla_{\alpha}$ we can eliminate the time la–derivatives

$$k_B T L[\phi] = p^\alpha p^\beta \nabla_{\alpha} U_{\beta} - T p^\alpha \nabla_{\alpha} \left( \frac{\mu}{T} \right) - p^\alpha p^\beta U_{\alpha} \left( \frac{\nabla_{\beta} T}{T} - \frac{\nabla_{\beta} p}{h \tilde{n}} \right)$$

$$+ [ (\gamma - 1) p^\alpha U_{\alpha} + T^2 (\gamma - 1) \frac{\partial}{\partial T} \left( \frac{\mu}{T} \right) + \tilde{n} \left( \frac{\partial \mu}{\partial \tilde{n}} \right) ] p^\epsilon U_{\epsilon} \nabla^\mu U_{\nu}. \tag{82}$$

We can verify, using the expression $\mu = \mu(\tilde{n}, T)$ from (76) that one holds the equalities

$$\frac{\partial}{\partial T} \left( \frac{\mu}{T} \right) = - \frac{e}{T^2}, \quad \frac{\partial \mu}{\partial \tilde{n}} = \frac{k_B T}{\tilde{n}}$$

and

$$\frac{1}{\tilde{n}} \nabla_{\alpha} p = h \frac{\nabla_{\alpha} T}{T} + T \nabla_{\alpha} \left( \frac{\mu}{T} \right).$$

Introducing these expressions into (82) we get

$$k_B T L[\phi] = \Xi X + (h - p^\alpha U_{\alpha}) \Delta_{\epsilon}^\alpha p_{\beta} X_{\beta}$$

$$+ \left( \Delta_{\epsilon}^\beta \Delta_{\gamma}^\alpha - \frac{1}{n + m - 1} \Delta_{\epsilon\gamma} \Delta_{\beta\alpha} \right) p_{\beta} p_{\alpha} (0) X^{\epsilon\gamma},$$

where

$$\Xi = \left( \frac{n + m}{n + m - 1} - \gamma \right) (p^\alpha U_{\alpha})^2 + \left[ (\gamma - 1) h - \gamma k_B T \right] p^\alpha U_{\alpha} - \frac{\tilde{m}^2}{n + m - 1} \tag{83}$$

and there were considered forces deriving the system towards equilibrium

$$X = - \nabla^\mu U_{\mu}, \quad X_{\alpha} = \frac{\nabla_{\alpha} T}{T} - \frac{\nabla_{\alpha} p}{\tilde{n} h}$$

and

$$(0) X^{\epsilon\gamma} = \frac{1}{2} \left( \nabla^\epsilon U^\gamma + \nabla^\gamma U^\epsilon \right) - \frac{1}{n + m - 1} \Delta_{\epsilon\gamma} \nabla^\mu U_{\mu}.$$

We note that the values $k_B T L[\phi], \Xi$ and $(0) X^{\epsilon\gamma}$ depend explicitly on the dimensions of the base subspace, $n$, and of the fiber subspace, $m$. The anisotropy also modify both the thermodynamic and kinetic values via operators $\nabla^\mu$ and $\Delta_{\epsilon}^\beta$ which depend explicitly on d–metric and d–connection coefficients.

**6.2 On the solution of locally anisotropic transport equations**

Let us suppose that is known the solution of these three equations:

$$k_B T L[A(p)] = \Xi,$$

$$k_B T L[B(p) \Delta_{\alpha}^\beta p_{\beta}] = (h - p^\alpha U_{\alpha}) \Delta_{\alpha}^\beta p_{\beta} \tag{84}$$
and
\[ k_B T L \left[ C(p) \left( \triangle^\beta_\alpha \triangle^\sigma_\epsilon - \frac{1}{n+m-1} \triangle^\alpha_\epsilon \triangle^\beta_\sigma \right) p_\beta p_\sigma \right] = \left( \triangle^\beta_\alpha \triangle^\sigma_\epsilon - \frac{1}{n+m-1} \triangle^\alpha_\epsilon \triangle^\beta_\sigma \right) p_\beta p_\sigma \]

for some functions \( A(p), B(p) \) and \( C(p) \). Because the integral operator \( L \) is linear
the linear combination of thermodynamic forces
\[ \varphi = AX + B_\alpha X^\alpha + C_{\alpha\beta}^{(0)} X^{\alpha\beta}, \]
where
\[ B_\alpha = B(p) \triangle^\beta_\alpha p_\beta \]

and
\[ C_{\alpha\beta} = B(p) \left( \triangle^\epsilon_\alpha \triangle^\sigma_\epsilon - \frac{1}{n+m-1} \triangle^\alpha_\epsilon \triangle^\beta_\sigma \right) p_\epsilon p_\sigma, \]
is a solution for the deviation function \( \varphi \).

It can also be verified that every function of the form \( \varphi = a + b_\alpha p^\alpha \) with some parameters \( a \) and \( b_\alpha \) not depending on \( p^\alpha \) is a solution of the homogeneous equation \( L[\varphi] = 0 \). In consequence, it was proved (see \cite{8} and \cite{3}) that the scalar parts of \( A(p) \) and \( B(p) \) are determined respectively up to functions of the forms \( \varphi = a + b_\alpha p^\alpha \) and \( b^\alpha p_\alpha \). We emphasize that solutions of the transport equations (84) (see next subsections) will be expressed in terms of \( A, B, \) and \( C \).

### 6.3 Linear laws for locally anisotropic non-equilibrium thermodynamics

Let \( U_\beta(u) \) be a velocity field of Landau–Lifshitz type characterizing a locally anisotropic fluid flow. The heat flow and the viscous pressure \( \sigma \)-tensor with respect to a such \( \sigma \)-field are correspondingly defined similarly to \cite{15} but in terms of objects on \( \sigma \)-spacetime (see formulas (73), (70), (85), (34), (28), (27), (14), (9), (13)
\[ I^\alpha_{(heat)} = \left( U_\alpha \right)^{\alpha\beta} - nh^\beta \right) \triangle^\epsilon_\beta, \]
\[ \Pi^{\alpha\beta} = P^{\alpha\beta} + p \triangle^\alpha_\beta. \]

It should be noted that if the Landau–Lifshitz condition (30) is satisfied the first term in \( I^\nu_{(heat)} \) vanishes and the heat flow is the enthalpy carried away by the particles. The pressure \( P(u) \) is defined by \( \tilde{n}(u) k_B T(u) \), see \cite{17}. Inserting \( \tilde{\nu} \) and \( \tilde{\eta} \) into (83) we prove (see locally isotropic cases in \cite{15, 3}) the linear laws of locally anisotropic non-equilibrium thermodynamics
\[ I^\nu_{(heat)} = \lambda T X^\nu \]
and
\[ \Pi^{\alpha\beta} = 2\eta^{(0)} X^{\alpha\beta} - \eta_{(0)} X \triangle^\epsilon_\sigma \]

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with the transport coefficients (the heat conductivity $\lambda$, the shear viscosity $\eta$, and the volume viscosity coefficient $\eta(\nu)$; in order to compare with formulas from [3] we shall introduce explicitly the light velocity constant $c$) defined as

$$\lambda = -\frac{c}{(n + m - 1)T} \int p^\sigma B_\sigma \triangle_\epsilon^\varepsilon (h - p^\nu U_\nu) f_{[0]} d\kappa_p,$$

$$\eta(\nu) = -\frac{c}{n + m - 1} \int p_\epsilon p_\sigma \triangle_\varepsilon^\sigma A f_{[0]} d\kappa_p,$$

$$\eta = \frac{c}{(n + m)(n + m - 4)} \int p^\sigma p^\rho C_{\nu\mu} \triangle_\nu^\rho \triangle_\mu^\sigma f_{[0]} d\kappa_p,$$

where one uses the d–tensor

$$\Delta_\nu^\mu = \frac{1}{2} (\Delta_\nu^\rho \Delta_\rho^\mu - \Delta_\nu^\mu \Delta_\rho^\rho) - \frac{1}{n + m - 1} \Delta_\nu^\mu \Delta_\rho^\rho$$

with the properties

$$\Delta_\nu^\rho \Delta_\rho^\phi = \Delta_\nu^\phi$$

and

$$\Delta_\nu^\rho = -1 + \frac{(n + m)(n + m - 1)}{2}.$$

The main purpose of non–equilibrium thermodynamics is the calculation of transport coefficients. With respect to la–frames the formulas are quite similar with those for isotropic spaces with that difference that we have to consider the values as d–tensors and take into account the number $m$ of anisotropic variables.

On both type of locally isotropic and anisotropic spacetimes one holds the so–called conditions of fit (see, for instance, [3])

$$\int p_\epsilon U^\epsilon A f_{[0]} d\kappa_p = 0 \text{ and } \int (p_\epsilon U^\epsilon)^2 A f_{[0]} d\kappa_p = 0$$

which allow us to write the volume viscosity (see (83))

$$\eta(\nu) = c \int \Xi A f_{[0]} d\kappa_p.$$

For some d–tensors $H^{\alpha_1...\alpha_q} (p)$ and $S_{\alpha_1...\alpha_q} (p)$ we define the symmetric bracket

$$\{H, S\} = \frac{1}{\tilde{n}^2} \int H^{\alpha_1...\alpha_q} (p) L [S_{\alpha_1...\alpha_q} (p)] f_{[0]} d\kappa_p$$

where $\tilde{n}$ is the particle density and $L$ is a linearized operator. In terms of such brackets the coefficients (86) can be rewritten in an equivalent form

$$\lambda = -\frac{ck_B \tilde{n}^2}{n + m - 1} \{B^\alpha, B_\alpha\},$$

$$\eta = \frac{ck_B T \tilde{n}^2}{(n + m - 1)^2 - 3} \{C^{\alpha\beta}, C_{\alpha\beta}\},$$

$$\eta(\nu) = ck_B T \tilde{n}^2 \{A, A\}.$$

The addition of invariants of type $\varphi = a + b_\alpha p^\alpha$ to some solutions for $A (p), B_\alpha (p)$, or $C_{\alpha\beta} (p)$ does not change the values of the transport coefficients.
6.4 Integral and algebraic equations

The solutions of transport equations (84) are approximated [3] by considering power series on \( \zeta = p^\alpha U_\alpha / k_B T \) for functions

\[
A(\zeta) = \sum_{\alpha=2}^{\infty} A_\alpha \zeta^\alpha, \quad B(\zeta) = \sum_{\beta=1}^{\infty} B_\beta \zeta^\beta, \quad C(\zeta) = \sum_{\gamma=0}^{\infty} C_\gamma \zeta^\gamma.
\]

The starting values \( \hat{a} = 2 \) and \( \hat{b} = 1 \) have been introduced with the aim to determine the scalar functions \( A(\zeta) \) and \( B(\zeta) \), respectively, up to contributions of the forms \( a + b \alpha U_\alpha \) and \( b \alpha p_\alpha \). Inserting these power series into the integral equations (84), multiply respectively on \( \zeta^\alpha f[0], \zeta^\alpha p^\alpha f[0], \) and \( \zeta^\alpha p^\alpha p^\beta f[0] \), after integrating on \( d\kappa \), we get

\[
\sum_{\alpha=2}^{\infty} a_{\hat{a} \alpha} A_\alpha = \alpha_{\hat{a}}, \quad \sum_{\beta=1}^{\infty} b_{\hat{b} \beta} B_\beta = \beta_{\hat{b}}, \quad \sum_{\gamma=0}^{\infty} c_{\hat{c} \gamma} C_\gamma = \gamma_{\hat{c}},
\]

where \( \hat{a} = 2, 3, \ldots, \hat{b} = 1, 2, \ldots, \hat{c} = 0, 1, \ldots, \) and there are symmetric brackets

\[
a_{\hat{a} \alpha} = \{ \zeta^\alpha, \zeta^\alpha \}, \quad b_{\hat{b} \beta} = \{ \zeta^\beta p^\alpha, \zeta^\beta \triangle^\mu p^\mu \}, \quad c_{\hat{c} \gamma} = \{ \zeta^\gamma p^\alpha p^\beta, \zeta^\gamma \triangle^{\mu \nu} p_\mu p_\nu \},
\]

and integrals

\[
\alpha_{\hat{a}} = \int \zeta^\hat{a} \Xi f[0] / k_B T \hat{n}^2 d\kappa, \quad (88)
\]

\[
\beta_{\hat{b}} = \int \zeta^\hat{b} (h - p^\alpha U_\alpha) \triangle^\mu p^\mu p^\nu f[0] / k_B T \hat{n}^2 d\kappa, \quad (89)
\]

\[
\gamma_{\hat{c}} = \int \zeta^\hat{c} \triangle^{\mu \nu} p_\mu p_\nu p^\alpha p^\beta f[0] / k_B T \hat{n}^2 d\kappa.
\]

The lowest approximation is given by the coefficients

\[
A_2 = \alpha_2 / a_{22}, \quad B_1 = \beta_1 / b_{11}, \quad C_0 = \gamma_0 / c_{00}.
\]

Introducing these values into (87) we obtain the first–order approximations to the transport coefficients

\[
\lambda = -\frac{ck_B \hat{n}^2 \beta_1^2}{n + m - 1} a_{22}^{-1}, \quad (90)
\]

\[
\eta = \frac{c k_B T \hat{n}^2 \gamma_0^2}{(n + m) (n + m - 1) - 2 c_{00}},
\]

\[
\eta(v) = c k_B T \hat{n}^2 \frac{\alpha_2^2}{a_{22}^2}.
\]

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The values $\alpha_2$, $\beta_1$ and $\gamma_0$ are $(n+m-1)$–fold integrals which can be expressed in terms of enthalpy $H$ and temperature $T$,

$$\alpha_2 = \frac{n+m+1-(n+m-1)}{ck_BT\tilde{n}} - \frac{(n+m-1)}{cn} \to 0, \quad \text{as } T \to \infty$$

$$\beta_1 = \frac{k_BT}{cn} \frac{(n+m-1)}{\gamma - 1} \to 0, \quad \text{as } T \to \infty$$

$$\gamma_0 = \frac{k_BT}{c^2n} (n+m-2)(n+m+1) \to \frac{1}{n} \left( \frac{k_BT}{c} \right)^2 \left[ (n+m)^2 - 1 \right] (n+m).$$

In order to complete the calculus of the transport coefficients it is necessary to compute the brackets from the denominators of (91).

### 6.5 Brackets from flat to la–spacetimes

The calculation of brackets is a quite tedious task (see [3]) which should involve the curved spacetime metric and connection. For simplicity, we consider a background flat spacetime with trivial local anisotropy. In this case we can apply directly the formulas proved for the Boltzmann theory in $n_a$ dimension but with respect to la–frames and by introducing the locally anisotropic d–connections and d–metrics instead of their isotropic analogous.

Let us consider a locally adapted to N–connection rest frame, $\hat{\eta}$ being a unit vector and denote by $(\Theta^{n_a-1}, \Theta^{n_a-2}, ..., \Theta_1)$ the angles of spherical coordinates with respect to a Cartesian one $(X^1, X^2, ..., X^{n_a})$ chosen as the $X^{n_a}$–axis to be along $\hat{\eta}_{CM}$ (where $|\hat{\eta}_{CM}| = F/P, P = |P^a|$, see [50] (56), and $\hat{\eta}_{CM}$ $\hat{\eta}$ in the plane of $X^{n_a}$ and $X^{n_a-1}$ axes. We suppose that the scattering angle is $\Theta^{n_a-1}$ and the differential cross section $(d\sigma/d\Omega)_{CM}$ is a function only on the module of total energy and on scattering angle $\Theta = \Theta^{n_a-1}$,

$$(d\sigma/d\Omega)_{CM} = \left( d\sigma/d\Omega \right)_{CM} (\hat{P}, \Theta).$$

The results of calculations of symmetric brackets will be expressed into terms of twofold integrals, which for arbitrary integers

$$J^{(i,j,k)}_{rs} \left( \zeta, \hat{\tau}, \hat{\eta}, \hat{\nu}, \hat{\omega} \right) = \nu(\zeta) \left( \hat{\tau} \hat{\nu} \right) \left( \hat{\eta} \hat{\omega} \right) \left( \hat{\tau}/\hat{\nu} \right) \left( \hat{\eta}/\hat{\omega} \right) \frac{2^{b+\nu-d}}{\sqrt{\pi}} \times \Gamma \left[ b + \nu + \frac{1}{2} \right] F \left( \hat{\tau}, \hat{\eta}, \hat{\nu}, \hat{\omega} \right) \int d\chi \chi^{(n+m-1+b-\hat{\nu})}$$

$$\times \left( \frac{\chi^2}{4} - \zeta \right)^{(n+m-2+b+\hat{\nu})/2} \left( (n+m-1+b-\hat{\nu}) \right) \left( (n+m-2+b+\hat{\nu}) \right) \left( n+m-1+b-\hat{\nu} \right) \left( n+m-2+b+\hat{\nu} \right)$$

$$\times K_{b+\nu-\hat{\nu}}(\zeta) \left[ \delta_{0 \hat{\omega}} - \left( \hat{\eta} \hat{\nu} \right) \sin \hat{\nu} \Theta \cos \hat{\nu} \Theta \right]$$

$$\times \int d\Theta \sin^{n+m-1} \Theta \left( d\sigma/d\Omega \right)_{CM} (\hat{P}, \Theta),$$

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where there are introduced the integers

\[ \hat{d} = \hat{r} + \hat{s} - \hat{\ell} - \hat{u}, \hat{b} = (\hat{\ell} + \hat{u} + n + m - 2)/2, \hat{h} = [1 - (-1)^{\hat{d}}] \]

and variables \( \chi = c/(k_B T) \) and \( \zeta = \tilde{m} c^2/(k_B T) \). The factors in (92) are defined

\[
\nu(\zeta) = \frac{\pi^{(n+m-3)/2}}{2^{(n+m-1)/2} \Gamma(n + m - 3) \zeta^2 \left[ K_{n+m}(\zeta) \right]^2},
\]

and

\[
F(\hat{\ell}, \hat{u}, \hat{w}) = \frac{1}{4} \left[1 + (-1)^{\hat{\ell} - \hat{u} + \hat{w}}\right] \frac{1}{\hat{\ell} + \hat{u} + \hat{w}}
\times B\left(\frac{n + m - 3}{2}, \frac{\hat{w} + 1}{2}\right) \times B\left(\frac{n + m - 2 + \hat{w}}{2}, \frac{\hat{\ell} + \hat{u} - \hat{w} + 1}{2}\right),
\]

where the beta function is given by gamma functions,

\[
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)},
\]

and by \( \binom{\hat{r}}{\hat{\ell}} \) it is denoted the Newton’s binomium.

### 6.5.1 Scalar type brackets

The first type of brackets necessary for calculation of transport coefficients (87) (see the series approximation (88) are the so-called the scalar brackets, decomposed into a sum of two integrals

\[ a_{\hat{a}_1 \hat{a}} = a'_{\hat{a}_1 \hat{a}} + a''_{\hat{a}_1 \hat{a}}, \quad (93) \]

where \( \hat{a}_1, \hat{a}, \hat{b} = 2, 3, \ldots \),

\[
\begin{align*}
  a'_{\hat{a}_1 \hat{a}} &= \tilde{n}^{-2} \int f_{[0]}(p) \int f_{[0]}(p_{[1]}) (\zeta_p)^{\hat{a}_1} \times \left[ (\zeta_{p_{[1]}})^{\hat{\alpha}} - (\zeta_{p_{[1]}})^{\hat{\beta}} \right] W d\zeta_p d\zeta_{p_{[1]}} d\zeta_{p'} d\zeta_{p_{[1]}'}, \\
  a''_{\hat{a}_1 \hat{a}} &= \tilde{n}^{-2} \int f_{[0]}(p) \int f_{[0]}(p_{[1]}) (\zeta_p)^{\hat{a}_1} \times \left[ (\zeta_p)^{\hat{\alpha}} - (\zeta_{p'})^{\hat{\beta}} \right] W d\zeta_p d\zeta_{p_{[1]}} d\zeta_{p'} d\zeta_{p_{[1]}}',
\end{align*}
\]

with the integration variables

\[
\begin{align*}
  \zeta_p &= p^\alpha U_\alpha / k_B T, \quad \zeta_{p_{[1]}} = p_{[1]}^\alpha U_\alpha / k_B T, \\
  \zeta_{p'} &= p^\alpha U_\alpha / k_B T, \quad \zeta_{p_{[1]'}} = p_{[1]}^\alpha U_\alpha / k_B T.
\end{align*}
\]

The explicit calculations of integrals from (93) (see the method and basic intermediary formulas in (5); in this paper we deal with \( \hat{a} \)-values) give

\[
\begin{align*}
  a'_{\hat{a}_1 \hat{a}} &= \sum_{\hat{a}_1} \sum_{\hat{a}} \sum_{\hat{u}} \sum_{\hat{w}} \sum_{\hat{v}} (-1)^{\hat{v}} J_{\hat{a}_1 \hat{a}}^{(0,0,0)}(\hat{\ell}, \hat{u}, \hat{v}, \hat{w}) , \\
  a''_{\hat{a}_1 \hat{a}} &= \sum_{\hat{a}_1} \sum_{\hat{a}} \sum_{\hat{u}} \sum_{\hat{w}} \sum_{\hat{v}} J_{\hat{a}_1 \hat{a}}^{(0,0,0)}(\hat{\ell}, \hat{u}, \hat{v}, \hat{w}).
\end{align*}
\]
In the lowest approximation, for \( \hat{a}_1 = \hat{a} = 2 \), we obtain
\[
a_{22} = 2 \left[ J^{(0,0,0)}_{22} (2, 2, 0, 0) + J^{(0,0,0)}_{22} (2, 2, 0, 2) \right].
\]
So, we have reduced the scalar brackets to twofold integrals which are expressed in terms of spherical coordinates with respect to a locally anisotropic rest frame.

### 6.5.2 Vector type brackets

The vector type brackets from (87) and (88) are also split into two types of integrals
\[
b_{b_1 b} = b'_{b_1 b} + b''_{b_1 b} \tag{94}
\]
where \( \hat{b}_1, \hat{b} = 1, 2, \ldots \),
\[
b'_{b_1 b} = \frac{3}{4} \sum_{(i) = 1} b'_{(i) \hat{b}_1 \hat{b}} \quad \text{and} \quad b''_{b_1 b} = \frac{3}{4} \sum_{(i) = 1} b''_{(i) \hat{b}_1 \hat{b}},
\]
with corresponding sums
\[
b'_{(1) \hat{b}_1 \hat{b}} = \frac{1}{4} \left( \frac{k_{b T}}{c} \right)^2 \sum_{\hat{b}_1} \sum_{\hat{b}} \frac{\hat{d}}{2} \sum_{\hat{u}} (-1)^\hat{u} J^{(2,0,0)}_{a_1 \hat{a}} \left( \hat{t}, \hat{u}, \hat{\nu}, \hat{w} \right),
\]
\[
b'_{(2) \hat{b}_1 \hat{b}} = \left( \frac{k_{b T}}{c} \right)^2 \sum_{\hat{b}_1} \sum_{\hat{b}} \frac{\hat{d}}{2} \sum_{\hat{u}} (-1)^\hat{u} J^{(2,0,1)}_{a_1 \hat{a}} \left( \hat{t}, \hat{u}, \hat{\nu}, \hat{w} \right),
\]
\[
b'_{(2) \hat{b}_1 \hat{b}} = - \left( \frac{k_{b T}}{c} \right)^2 \sum_{\hat{b}_1} \sum_{\hat{b}} \frac{\hat{d}}{2} \sum_{\hat{u}} (-1)^\hat{u} J^{(2,0,1)}_{a_1 + 1 \hat{a} + 1} \left( \hat{t}, \hat{u}, \hat{\nu}, \hat{w} \right);
\]
with corresponding sum expressions for \( b''_{(i) \hat{b}_1 \hat{b}} \) by omitting the factor \((-1)^\hat{u}\).

In the lowest approximation (for \( \hat{b}_1 = \hat{b} = 1 \)) we have
\[
b^{(11)}_{11} = -2 \left( \frac{k_{b T}}{c} \right)^2 \sum_{a_1} \left[ J^{(2,0,1)}_{a_1 \hat{a}} (1, 1, 0, 0) + J^{(0,0,0)}_{22} (2, 2, 0, 2) \right].
\]
Here should be noted that in general \( T = T (u) \) is a function on la–spacetime coordinates.
6.5.3 Tensor type brackets

In a similar fashion as for scalar and vector type symmetric brackets from [5] and [8] one holds the decomposition

\[ c_{1\hat{c}} = c'_{1\hat{c}} + c''_{1\hat{c}} \]  

(95)

where \( \hat{c}_1, \hat{c} = 1, 2, \ldots, \)

\[
c'_{1\hat{c}} = \bar{n}^{-2} \int f_{[0]} \left( p \right) f_{[0]} \left( p_{[1]} \right) \Delta_{\epsilon \epsilon}^{\nu \mu} \left( \zeta_{\epsilon \epsilon} \right) \hat{c}_1 p_\nu p_\mu \\
\times \left[ p_{[1]} p'_{[1]} \left( \zeta_{p_{[1]} p'_{[1]}} \right) \right] - p_{[1]} p'_{[1]} \left( \zeta_{p_{[1]} p'_{[1]}} \right) \left[ W d\zeta_p d\zeta_{p'} d\zeta_{p'_{[1]}} \right] \\
\]

\[
c''_{1\hat{c}} = \bar{n}^{-2} \int f_{[0]} \left( p \right) f_{[0]} \left( p_{[1]} \right) \Delta_{\epsilon \epsilon}^{\nu \mu} \left( \zeta_{\epsilon \epsilon} \right) \hat{c}_1 p_\nu p_\mu \\
\times \left[ p_{[1]} p'_{[1]} \left( \zeta_{p_{[1]} p'_{[1]}} \right) \right] - p_{[1]} p'_{[1]} \left( \zeta_{p_{[1]} p'_{[1]}} \right) \left[ W d\zeta_p d\zeta_{p'} d\zeta_{p'_{[1]}} \right] \\
\]

For tensor like brackets we have to consider sums on nine terms

\[
c'_{(s)1\hat{c}} = \sum_{(s)=1}^9 c'_{(s)1\hat{c}} \quad \text{and} \quad c''_{(s)1\hat{c}} = \sum_{(s)=1}^9 c''_{(s)1\hat{c}} \\
\]

These terms are [3] (with that exception that we have dependencies on the number of anisotropic variables and \( T = T \left( u \right) \) is a function on \( \mathbb{R}^- \)–spacetime coordinates)

\[
c'_{(1)1\hat{c}} = \frac{1}{16} \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(0,0,0)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]

\[
c'_{(2)1\hat{c}} = \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(4,4,4)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]

\[
c'_{(3)1\hat{c}} = \frac{n + m - 2}{n + m - 1} \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(0,0,0)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]

\[
c'_{(4)1\hat{c}} = \frac{1}{2} \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(2,2,1)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]

\[
c'_{(5)1\hat{c}} = -2 \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(2,2,1)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]

\[
c'_{(6)1\hat{c}} = -\frac{1}{2} \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(0,0,0)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]

\[
c'_{(7)1\hat{c}} = -\frac{1}{n + m - 1} \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(0,0,0)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]

\[
c'_{(8)1\hat{c}} = \frac{1}{n + m - 1} \left( \frac{k_B T}{c} \right)^4 \sum_{l=0}^{n} \sum_{\hat{u} = 0}^{\hat{c}} \sum_{\hat{v} = 0}^{\hat{d}/2} \sum_{\hat{w} = 0}^{\hat{u} / 4} (-1)^{\hat{u}} J^{(0,0,0)}_{c_{1\hat{c}}} \left( \hat{t}, \hat{u}, \hat{v}, \hat{w} \right) \\
\]
The twice primed values $c''_{(s)c_1c}$ are given by similar sums by omitting the factor $(-1)^{\hat{u}}$ and by changing into $c''_{(4)c_1c}$ and $c''_{(5)c_1c}$ the overall signs.

In the lower approximation, for $c_1 = \hat{c} = 0$, one holds

$$c_{00} = 2 \left( \frac{k_B T}{c} \right)^4 \{ J_{22}^{(0,0,0)}(\hat{2},\hat{2},\hat{0},\hat{0}) + J_{22}^{(0,0,0)}(\hat{2},\hat{2},\hat{0},\hat{0}) \}. $$

### 6.6 Locally anisotropic transport coefficients for a class of cross sections

For astrophysical applications we can substitute

$$\left( \frac{d\sigma}{d\Omega} \right)_CM = \xi P_r, $$

with some scalar factor $\xi$ (with or without dimension) and $r$ being a positive or negative number into (92). In this subsection we put $\zeta = \tilde{m}c^2 / (k_B T)$ for high values of $T$. The chosen type of differential cross section (96) is used, for instance, for calculations of neutrino–neutrino scattering (when $r = 2$ and $\xi$ is connected with the weak coupling constant). In the first approximation the symmetric brackets (88) are

$$a_{22} = \xi \left( \frac{2k_B T}{c} \right)^r \frac{\pi^{(n+m-1)/2} \Gamma [n + m + 1 + r/2] \Gamma [(n + m + r) / 2]}{\Gamma [n + m + 1 + r/2] \Gamma [(n + m + r) / 2] \Gamma [(n + m + 1) / 2]}, $$

$$b_{11} = - \left( \frac{2k_B T}{c} \right)^2 \frac{n + m + 2 + r}{2} a_{22}, $$

$$c_{00} = \left( \frac{2k_B T}{c} \right)^2 \frac{(n + m + r)(n + m + 4 + r)}{2} + \frac{n + m - 2}{n + m - 1} a_{22}. $$

Putting these values into (91) we obtain the locally anisotropic variant of transport coefficients in the first approximation, when $\eta(\nu) \simeq 0$ but with nonzero

$$\lambda = \frac{2k_B c}{\xi} \left( \frac{c}{2k_B T} \right)^r \frac{(n + m)^2 (n + m - 1)}{n + m + 2 + r} \times \frac{\Gamma [n + m - 2] \Gamma [(n + m) / 2] \Gamma [(n + m + 1) / 2]}{\Gamma [n + m + 1 + r/2] \Gamma [(n + m + r) / 2]}, $$

and

$$\eta = \frac{1}{\xi} \left( \frac{c}{2k_B T} \right)^{r-1} \frac{1}{\pi^{(n+m-1)/2} \left[ (n + m)^2 - n - m - 2 \right]}.
\begin{align*}
\times \frac{[(n + m - 1)^2 - 1] (n + m + 1)^2}{(n + m + r) (n + m + 4 + r) + 2 \frac{(n + m - 2)}{(n + m - 1)}^{(n + m - 1)} / 2} \\
\times \frac{\Gamma [n + m - 2] \Gamma [(n + m) / 2] \Gamma [(n + m + 1) / 2]}{\Gamma [n + m + 1 + r / 2] \Gamma [(n + m + r) / 2]}.
\end{align*}

These formulas present a locally anisotropic generalization of the Boisseau and van Leeuwen results.

7 Transport Theory in Curved Spaces with Rotation Ellipsoidal Horizons

After having established in a general way the scheme for calculation of transport coefficients on la–spacetimes, we now specify an example for a four dimensional static metric (being a solution of Einstein equations in both general relativity and la–gravity) with the event horizon described by a hypersurface of rotation ellipsoid \([34]\) (see the Appendix). We note that in this case \(n = 3, m = 1\) and \(n(a) = 4\).

The main formulas for kinetic and thermodynamic variables from the previous Sections 3–6 were based on spherical symmetry of \(n(a) - 1\) dimensional volume \((62)\) and spherical integral \((63)\). For the rotation ellipsoid we have to modify the volume’s formula by introducing the ellipsoidal dependence

\[V_{n(a)-1} = \frac{(n(a) - 1) \pi^{(n(a)-1)/2}}{\Gamma [(n(a) - 1)/2]} \rho \left( \frac{\bar{\theta}}{n(a)} = 4 \right) \rho \left( \bar{\theta} \right) \]

where

\[\rho \left( \bar{\theta} \right) = \frac{\rho(0)}{1 - \varepsilon \cos \theta} \]

is the parametric formula of an ellipse with constant parameter \(\rho(0)\), angle variable \(\bar{\theta}\) and eccentricity \(\varepsilon = 1 / \sigma < 1\) is defined by the axes of rotation ellipsoid (see the formula \((119)\) from Appendix).

If in the case of spherical symmetry \(4 \int_0^{\pi/2} d\bar{\theta} = 2\pi\), the ellipse deformation gives the result

\[4 \int_0^{\pi/2} \frac{d\bar{\theta}}{1 - \varepsilon \cos \theta} = \frac{8}{\sqrt{1 - \varepsilon^2}} \arctan \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}.\]

So, performing integrations on solid angles in spaces with rotational ellipsoid symmetry we can use the same formulas as for spherical symmetry but multiplied on

\[q(\varepsilon) = \frac{4}{\pi \sqrt{1 - \varepsilon^2}} \arctan \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}}.\]

For instance, the integral \((63)\) transforms

\[\int d\Omega \to \int d\Omega (\varepsilon) = q(\varepsilon) \frac{(n(a) - 1) \pi^{(n(a)-1)/2}}{\Gamma [(n(a) - 1)/2]} \overset{n(a) = 4}{\rightarrow} q(\varepsilon) \frac{3 \pi^{3/2}}{\Gamma [3/2]}.\]
The formulas for the particle density \((60)\) and energy density \((68)\) of point particles must be multiplied on \(q(\varepsilon)\),
\[
\tilde{n} \rightarrow \tilde{n}(\varepsilon) = q(\varepsilon)\tilde{n} \quad \text{and} \quad \tilde{\varepsilon} \rightarrow \tilde{\varepsilon}(\varepsilon) = q(\varepsilon)\tilde{\varepsilon},
\]
but the energy per particle will remain constant. We also have to modify the formula for pressure \((71)\), been proportional to the particle density, but consider unchanged the averaged enthalpy \((72)\). The entropy per particle \((75)\) and chemical potential \((76)\) depends explicitly on \(q(\varepsilon)\)–factor because theirs formulas were derived by using the particle density \(\tilde{n}(\varepsilon)\). Here we note that all proved formulas depends on \(m (= 1, \text{in this Section})\) anisotropic parameters and on volume element determined by d–metric. In the first approximation of transport coefficients we could chose a locally isotropic background but introducing the factor \(q(\varepsilon)\) and taking into account the dependence on anisotropic dimension.

Putting \(\varepsilon\)–corrections into \((90)\) we obtain the first–order approximations to the transport coefficients in a la–spacetime with the symmetry of rotation ellipsoid
\[
\lambda = -\frac{c k_B \tilde{n}^2(\varepsilon)}{3 a_{22}^2},
\]
\[
\eta = \frac{c k_B T \tilde{n}^2(\varepsilon)}{10 \sigma_{00}},
\]
\[
\eta(v) = \frac{c k_B T \tilde{n}^2(\varepsilon)}{a_{22}^2}.
\]
The values \(\alpha_2, \beta_1, \text{and} \gamma_0\) are 3–fold integrals expressed in terms of enthalpy \(h\) and temperature \(T\) and have the limits
\[
\alpha_2 = \frac{5\gamma}{c k_B T \tilde{n}(\varepsilon)} - \frac{3\gamma}{c \tilde{n}(\varepsilon)} \xrightarrow{T \rightarrow \infty} 0,
\]
\[
\beta_1 = \frac{k_B T}{c \tilde{n}(\varepsilon)} \frac{3\gamma}{\gamma - 1} \xrightarrow{T \rightarrow \infty} 12 \frac{k_B T}{c \tilde{n}(\varepsilon)},
\]
\[
\gamma_0 = \frac{k_B T}{c^2 \tilde{n}(\varepsilon)} 10h \xrightarrow{T \rightarrow \infty} 60 \frac{1}{\tilde{n}(\varepsilon)} \left(\frac{k_B T}{c}\right)^2.
\]
In the next step we compute the \(q(\varepsilon)\)–deformations of brackets from the denominators of \((100)\). Because the squares of \(\alpha, \beta\) and \(\gamma\) coefficients from \((89)\) are proportional to deformations of \(\tilde{n}^{-2}\) (this conclusion follows from the formulas \((97)\) in the \(T \rightarrow \infty\) limit, see \((101)\)) and the scalar \((93)\), vector \((94)\) and tensor \((95)\) type brackets do not change under \(q(\varepsilon)\)–deformations (see \((97)\)) we conclude that the transport coefficients \((100)\) do not contain the factor \(q(\varepsilon)\) but depends only on the number \(m\) of anisotropic dimensions. This conclusion is true only in the first approximation and for locally isotropic backgrounds. In consequence, the final formulas for the transport coefficient, see \((98)\) and \((99)\), in a \((3+1)\) locally anisotropic spacetime with rotation ellipsoid symmetry are
\[
\eta(v) \approx 0
\]
\[ \lambda = \frac{2k_Bc}{\xi} \left( \frac{c}{2k_BT} \right)^r \frac{48}{6+r}\Gamma[5+r/2] \Gamma[(4+r)/2] \]
\[ \eta = \frac{1}{\xi} \left( \frac{c}{2k_BT} \right)^{r-1} \frac{20}{\pi^{3/2}[4(r+8)+4/3]} \frac{\Gamma[2] \Gamma[5/2]}{\Gamma[5+r/2] \Gamma[(4+r)/2]} \]

This is consistent with the fact that we chose as an example a static metric with a local anisotropy that does not cause drastic changings in the structure of transport coefficients. Nevertheless, there are \( q(c) \)-deformations of such values as the density of particles, kinetic potential and entropy which reflects modifications of kinetic and thermodynamic processes even by static spacetime local anisotropies.

8 Concluding Remarks

The formulation of Einstein’s theory of relativity with respect to anholonomic frames raises a number of questions concerning locally anisotropic field interactions and kinetic and thermodynamic effects.

We argue that spacetime local anisotropy (la) can be modeled by applying the Cartan’s moving frame method \([6]\) with associated nonlinear connection (N–connection) structures. A remarkable fact is that this approach allows a unified treatment of various type of theories with generic local anisotropy like generalized Finsler like gravities, of standard Kaluza–Klein models with nontrivial compactifications (modelled by N–connection structures), of standard general relativity with anholonomic frames and even of low dimensional models with distinguished anisotropic parameters. We have shown a relationship between a subclass of Finsler like metrics with (pseudo) Riemannian ones being solutions of Einstein equations.

This paper has provided a generalization of relativistic kinetics and nonequilibrium thermodynamics in order to be included possible spacetime local anisotropies. It should not be considered as a work on definitions of some sophisticated theories on Finsler like spaces but developing an approach to kinetics and thermodynamics in (pseudo) Riemannian spacetimes of arbitrary dimension provided with anholonomic frame structures. The item of introducing into consideration of generalized Finsler and Kaluza–Klein spaces was imposed by the facts that one has recently constructed (see Appendix and \([34]\) solutions of the Einstein equations with generic anisotropy, of Finsler and another type, like ellipsoidal static black holes, black tora, anisotropic solitonic backgrounds and so on) and that in the low energy limits of string theories various classes of generalized Finsler–Kaluza–Klein metrics could be obtained alternatively to the well known (pseudo) Riemannian ones \([31]\). By applying the moving frame method it is possible to elaborate a general schema for defining of physical values, basic equations and approximated calculations of kinetic and thermodynamic values with respect to la–frames in all type of the mentioned theories.

The crucial ingredient in definition of collisionless relativistic locally anisotropic kinetic equation was the extension of the moving frame method to the space of supporting elements \((u^\alpha, p^\beta)\) provided with an induced higher order anisotropic
structure. The former Cartan–Vlasov approach \[5, 36\], proposing a variant of statistical kinetic theory on curved phase spaces provided with Finsler like and Cartan N–connection structures, was self–consistently modified for both type of locally isotropic (the Einstein theory) and anisotropic (generalized Finsler–Kaluza–Klein) spacetimes with N–connection structures induced by local anholonomic frame or via reductions from higher dimensions in (super) string or (super) gravity theories. The physics of pair collisions in la–spacetimes was examined by introducing on the space of supporting elements of (correspondingly adapted to the N–connection structure) integral of collisions, differential cross–sections and velocity of transitions.

Despite all the complexities of definition of equilibrium states with generic anisotropy it is possible a rigorous definition of local equilibrium particle distribution functions by fixing some anholonomic frames of reference adapted to the N–connection structure. The basic kinetic and thermodynamic values such as particle density, average energy and pressure, enthalpy, specific heats and entropy are derived via integrations on volume elements determined by metric components given with respect to la–bases. In the low and high energy limits the formulas reflect explicit dependencies on the number of anisotropic dimensions as well on anisotropic deformations of spacetime metric and linear connection.

One can linearize the transport equations and prove the linear laws for locally anisotropic non–equilibrium thermodynamics. It has also established a general scheme for calculation of transport coefficients (the heat conductivity, the shear viscosity and the volume viscosity) in la–spacetimes. An explicit computation of such values was performed for a metric with rotation ellipsoidal event horizon (an example of spacetime with static local anisotropy), recently found as a new solution of the Einstein equations.

Our overall conclusion is that in order to obtain a self–consistent formulation of the locally anisotropic kinetic and thermodynamic theory in curved spacetimes and calculation of basic physical values we must consider moving frames with correspondingly adapted nonlinear connection structures.

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Appendix: A Locally Anisotropic Solution of Einstein’s Equations

Before presenting an explicit construction \[34\] of a four dimensional solution with local anisotropy of the Einstein equations \[8\] we briefly review the properties of four dimensional metrics which transforms into (2+2) or (3+1) anisotropic d–metrics (we note that this is not a (space + time) but a (isotropic+anisotropic) decomposition of coordinates) by transitions to correspondingly defined anholonomic bases of tetrads (vierbeins).
Let us consider a four dimensional (in brief, 4D) spacetime $V^{(2+2)}$ (with two isotropic plus two anisotropic local coordinates) provided with a metric of signature $(+,+,−,+)$. Parametrized by a symmetric matrix of type

\[
\begin{pmatrix}
g_1 + q_1^2 h_3 + n_1^2 h_4 & 0 & q_1 h_3 & n_1 h_4 \\
0 & g_2 + q_2^2 h_3 + n_2^2 h_4 & q_2 h_3 & n_2 h_4 \\
n_1 h_3 & q_2 h_3 & h_3 & 0 \\
n_1 h_4 & n_2 h_4 & 0 & h_4
\end{pmatrix}
\]

(102)

with coefficients being some functions

\[g_i = g_i(x^j), q_i = q_i(x^j, t), n_i = n_i(x^j, t), h_a = h_a(x^j, t)\]

of necessary smooth class. With respect to a la–basis (10) the ansatz (102) results in diagonal $2 \times 2$ $h$– and $v$–metrics for a $d$–metric (13) (for simplicity, we shall consider only diagonal 2D nondegenerated metrics because for such dimensions every symmetric matrix can be diagonalized).

An equivalent diagonal $d$–metric of type (13) is obtained for the associated $N$–connection with the coefficients being functions on three coordinates $(x^i, z)$,

\[
N^3_1 = q_1(x^i, z), N^3_2 = q_2(x^i, z), N^4_1 = n_1(x^i, z), N^4_2 = n_2(x^i, z).
\]

For simplicity, we shall use brief denotations of partial derivatives, like $\dot{a} = \partial a/\partial x^1$, $\ddot{a} = \partial a/\partial x^2$, $a^* = \partial a/\partial z$, $\dot{a}' = \partial^2 a/\partial x^1 \partial x^2$, $a^{**} = \partial^2 a/\partial z \partial z$.

The non–trivial components of the Ricci $d$–tensor (14) (for the ansatz 102) when $R^1_1 = R^2_2$ and $S^3_3 = S^4_4$, are computed

\[
R^1_1 = R^2_2 = \frac{1}{2g_1g_2}[-(g''_1 + \dot{g}_2) + \frac{1}{2g_2}(\ddot{g}_2 + \dot{g}'_1g_2) + \frac{1}{2g_1}(g'_1^2 + \dot{g}_1\dot{g}_2)];
\]

(104)

\[
S^3_3 = S^4_4 = \frac{1}{h_3h_4}[-h^*_4 + \frac{1}{2h_4}(h^*_4)^2 + \frac{1}{2h_3}h^*_3h^*_4];
\]

(105)

\[
P_{31} = \frac{g_1}{2} \left[ \left(\frac{h^*_3}{h_3}\right)^2 - \frac{h^*_3}{h_3} + \frac{h^*_4}{2h_4} - \frac{h^*_3h^*_4}{2h_3h_4} \right] + \frac{1}{2h_4} \left[ \frac{\dot{h}_4}{h_4} h^*_4 - h^*_4 + \frac{\dot{h}_4}{2h_3} h^*_4 \right];
\]

(106)

\[
P_{32} = \frac{g_2}{2} \left[ \left(\frac{h^*_3}{h_3}\right)^2 - \frac{h^*_3}{h_3} + \frac{h^*_4}{2h_4} - \frac{h^*_3h^*_4}{2h_3h_4} \right] + \frac{1}{2h_4} \left[ \frac{\dot{h}_4}{h_4} h^*_4 - h^*_4 + \frac{\dot{h}_4}{2h_3} h^*_4 \right];
\]

\[
P_{41} = -\frac{h_4}{2h_3} n^*_1 + \frac{1}{4h_3} \left( \frac{h_4}{h_3} h^*_3 - 3h^*_4 \right)n^*_1;
\]

(107)

\[
P_{42} = -\frac{h_4}{2h_3} n^*_2 + \frac{1}{4h_3} \left( \frac{h_4}{h_3} h^*_3 - 3h^*_4 \right)n^*_2.
\]

The curvature scalar $\tilde{R}$ (17) is defined by the sum of two non–trivial components $\tilde{R} = 2R^1_1$ and $S = 2S^3_3$. 

40
The system of Einstein equations (18) transforms into

\[ R_1^1 = -\kappa \Upsilon_3^3 = -\kappa \Upsilon_4^4, \tag{108} \]
\[ S_3^3 = -\kappa \Upsilon_1^1 = -\kappa \Upsilon_2^2, \tag{109} \]
\[ P_{3i} = \kappa \Upsilon_{3i}, \tag{110} \]
\[ P_{4i} = \kappa \Upsilon_{4i}, \tag{111} \]

where the values of \( R_1^1, S_3^3, P_{ai} \), are taken respectively from (104), (105), (106), (107).

We note that we can define the N–coefficients (103), \( q_i(x^k, z) \) and \( n_i(x^k, z) \), by solving the equations (110) and (111) if the functions \( h_i(x^k, z) \) are known as solutions of the equations (109).

An elongated rotation ellipsoid hypersurface is given by the formula [21]

\[ \frac{x^2 + y^2}{\sigma^2 - 1} + \frac{z^2}{\sigma^2} = \tilde{\rho}^2, \tag{112} \]

where \( \sigma \geq 1 \) and \( \tilde{\rho} \) is similar to the radial coordinate in the spherical symmetric case.

The space 3D coordinate system is defined

\[ \tilde{x} = \tilde{\rho} \sinh u \sin v \cos \varphi, \quad \tilde{y} = \tilde{\rho} \sinh u \sin v \sin \varphi, \]
\[ \tilde{z} = \tilde{\rho} \cosh u \cos v, \]

where \( \sigma = \cosh u, (0 \leq u < \infty, 0 \leq v \leq \pi, 0 \leq \varphi < 2\pi) \). The hypersurface metric is

\[ g_{uu} = g_{vv} = \tilde{\rho}^2 \left( \sinh^2 u + \sin^2 v \right), \tag{113} \]
\[ g_{\varphi\varphi} = \tilde{\rho}^2 \sinh^2 u \sin^2 v. \]

Let us introduce a d–metric

\[ \delta s^2 = g_1(u, v) du^2 + dv^2 + h_3(u, v, \varphi) (\delta t)^2 + h_4(u, v, \varphi) (\delta \varphi)^2, \tag{114} \]

where \( \delta t \) and \( \delta \varphi \) are N–elongated differentials.

As a particular solution (108) for the h–metric, considering \( \Upsilon_3^3 = \Upsilon_4^4 \), we choose the coefficient

\[ g_1(u, v) = \cos^2 v. \tag{115} \]

The \( h_3(u, v, \varphi) = h_3(u, v, \tilde{\rho}(u, v, \varphi)) \) is considered as

\[ h_3(u, v, \tilde{\rho}) = \frac{1}{\sinh^2 u + \sin^2 v} \left[ 1 - \frac{r_0}{4\tilde{\rho}} \right]^2, \tag{116} \]

In order to define the \( h_4 \) coefficient solving the Einstein equations, for simplicity, with a diagonal energy–momentum d–tensor for vanishing pressure, we must solve
the equation (109) which transforms into a linear equation if \( \Upsilon_1 = 0 \). In our case \( s(u, v, \varphi) = \beta^{-1}(u, v, \varphi) \), where \( \beta = (\partial h_4 / \partial \varphi) / h_4 \), must be a solution of

\[
\frac{\partial s}{\partial \varphi} + \frac{\partial \ln \sqrt{|h_3|}}{\partial \varphi} s = \frac{1}{2}.
\]

After two integrations (see [19]) the general solution for \( h_4(u, v, \varphi) \), is

\[
h_4(u, v, \varphi) = a_4(u, v) \exp \left[ - \int_0^\varphi F(u, v, z) \, dz \right],
\]

where

\[
F(u, v, z) = \left( \sqrt{|h_3(u, v, z)|} \right) \left[ s_{1(0)}(u, v) + \frac{1}{2} \int_{z_0(u, v)}^z \sqrt{|h_3(u, v, z)|} \, dz \right]^{-1},
\]

\( s_{1(0)}(u, v) \) and \( z_0(u, v) \) are some functions of necessary smooth class. We note that if we put \( h_4 = a_4(u, v) \) the equations (109) are satisfied for every \( h_3 = h_3(u, v, \varphi) \).

Every d–metric (114) with coefficients of type (115), (116) and (117) solves the Einstein equations (108)–(111) with the diagonal momentum d–tensor

\[
\Upsilon_\alpha^\beta = \text{diag} [0, 0, -\epsilon = -m_0, 0],
\]

when \( r_g = 2\kappa m_0 \); we set the light constant \( c = 1 \). If we choose

\[
a_4(u, v) = \frac{\sinh^2 u \sin^2 v}{\sinh^2 u + \sin^2 v}
\]

our solution is conformally equivalent (if not considering the time–time component) to the hypersurface metric (113). The condition of vanishing of the coefficient (116) parametrizes the rotation ellipsoid for the horizon

\[
\frac{\tilde{x}^2 + \tilde{y}^2}{\sigma^2 - 1} + \frac{\tilde{z}^2}{\sigma^2} = \left( \frac{r_g}{4} \right)^2,
\]

where the radial coordinate is redefined via relation \( \tilde{r} = \tilde{\rho} \left( 1 + \frac{r_g}{4\tilde{\rho}} \right)^2 \). After multiplication on the conformal factor

\[
(\sinh^2 u + \sin^2 v) \left[ 1 + \frac{r_g}{4\tilde{\rho}} \right]^4,
\]

approximating \( g_1(u, v) = \sin^2 v \approx 0 \), in the limit of locally isotropic spherical symmetry,

\[
\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = r_g^2,
\]

the d–metric (114) reduces to

\[
ds^2 = \left[ 1 + \frac{r_g}{4\tilde{\rho}} \right]^4 \left( d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 \right) - \frac{\left[ 1 - \frac{r_g}{4\tilde{\rho}} \right]^2}{\left[ 1 + \frac{r_g}{4\tilde{\rho}} \right]^2} dt^2
\]
which is just the Schwazschild solution with the redefined radial coordinate when
the space component becomes conformally Euclidean.

So, the d–metric (114), the coefficients of N–connection being solutions of (110)
and (111), describe a static 4D solution of the Einstein equations when instead of a
spherical symmetric horizon one considers a locally anisotropic deformation to the
hypersurface of rotation elongated ellipsoid.

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