Surfaces obtained from $\mathbb{C}P^{N-1}$ sigma models

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Abstract

In this paper, the Weierstrass technique for harmonic maps $S^2 \to \mathbb{C}P^{N-1}$ is employed in order to obtain surfaces immersed in multidimensional Euclidean spaces. It is shown that if the $\mathbb{C}P^{N-1}$ model equations are defined on the sphere $S^2$ and the associated action functional of this model is finite, then the generalized Weierstrass formula for immersion describes conformally parametrized surfaces in the $su(N)$ algebra. In particular, for any holomorphic or antiholomorphic solution of this model the associated surface can be expressed in terms of an orthogonal projector of rank $(N-1)$. The implementation of this method is presented for two-dimensional conformally parametrized surfaces immersed in the $su(3)$ algebra. The usefulness of the proposed approach is illustrated with examples, including the dilation-invariant meron-type solutions and the Veronese solutions for the $\mathbb{C}P^2$ model. Depending on the location of the critical points (zeros and poles) of the first fundamental form associated with the meron solution, it is shown that the associated surfaces are semi-infinite cylinders. It is also demonstrated that surfaces related to holomorphic and mixed Veronese solutions are immersed in $\mathbb{R}^8$ and $\mathbb{R}^3$, respectively.

Key words: Sigma models, Weierstrass formula for immersion, surfaces immersed in low-dimensional $su(N)$ algebras.

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1 Introduction

The expression describing surfaces with zero mean curvature (i.e. minimal surfaces) which are immersed in three-dimensional Euclidean space was first formulated by A. Enneper [1] and K. Weierstrass [2] one and a half centuries ago. Since

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then this idea has been thoroughly generalized and developed (e.g. [3, 4, 5, 6]). The subject was implemented by several authors (e.g. [7, 8, 9]) who produced several variants of the Weierstrass representation. For a comprehensive review of this topic see e.g. [10, 11, 12, 13, 14, 15] and references therein.

More recently, this subject was substantially elaborated by B. Konopelchenko and I. Taimanov [16], who first established the Weierstrass formulae for any generic surface immersed in \( \mathbb{R}^3 \). These formulae have been used extensively to study the global properties of surfaces in \( \mathbb{R}^3 \), as well as their integrable deformations [17]. By simple analogy with surfaces in the \( \mathbb{R}^3 \) case an extension of the Weierstrass procedure to multi-dimensional Euclidean and Riemannian spaces was proposed by B. Konopelchenko and G. Landolfi [18]. Their approach was successful for certain classes of conformally parametrized surfaces immersed in these spaces. However, this procedure has some limitations due to the assumption of a specific form of the Weierstrass system of \( 2N \) complex-valued functions which satisfy Dirac-type equations.

It was only in the past few years that the approach to the same problem was reformulated by exploiting the connection between generalized Weierstrass representations and the \( CP^1 \) sigma models, first established in \( \mathbb{R}^3 \) [19]. This idea allows one to generalize this connection for the \( CP^{N-1} \) case and derive in the adjoint \( SU(N) \) representation the corresponding moving frame of conformally parametrized surfaces in \( \mathbb{R}^{N^2-1} \) space. This modified Weierstrass representation [20, 21, 22, 23] has proven to be more general than the one proposed in [24] and to generate more diverse classes of surfaces (e.g. the Veronese surfaces). This algebraic description of surfaces on Lie groups and homogeneous spaces allows us to calculate some new expressions in closed form which determine the fundamental characteristics of these surfaces. For this purpose, using Cartan’s language of moving frames we derive the structural equations for immersion (e.g. the fundamental forms, the Gaussian curvature and the mean curvature vector) for the \( CP^{N-1} \) model. The \( CP^{N-1} \) models have found many applications in physics, to such areas as two-dimensional gravity [25], string theory [26], quantum field theory [27], statistical physics [28] and fluid mechanics [29].

This paper is concerned with smooth, orientable two-dimensional surfaces immersed in multi-dimensional Euclidean spaces. The crux of the matter is that the equations determining the formula for immersion are formulated directly in terms of matrices which take their values in the Lie algebra \( su(N) \). The main advantage of this procedure is that, in using an orthogonal projector satisfying the Euler-Lagrange equations of the given sigma model, it leads to simpler formulae and allows us to write the explicit form of some expressions which previously were too involved to be presented.

The objective of this paper is to study certain geometrical aspects of surfaces associated with the \( CP^{N-1} \) sigma models. In particular, we discuss in detail the necessary conditions for the existence of the radius vectors of surfaces associated with the \( CP^{N-1} \) sigma model, which are expressed in terms of an orthogonal projector of rank \( N - 1 \). Furthermore, we have shown that the Weierstrass formula for immersion of surfaces associated with mixed solutions of the \( CP^{N-1} \) model is no longer proportional to a rank-one projector (unlike the case for holomorphic and antiholomorphic solutions). Next, it is demonstrated that a parametrized surface, related to a Veronese mixed solution (i.e. an extension of the holomorphic case) is immersed in three-dimensional Euclidean space.
Finally, we construct meron-like solutions of the \(\mathbb{C}P^2\) model and determine their geometric characteristics.

The plan of this paper is as follows. Section 2 contains a brief account of basic definitions and properties concerning the \(\mathbb{C}P^{N-1}\) models and fixes the notation. We give a geometric formulation for the generalized Weierstrass formula for immersion of a surface \(\mathcal{F}\) in \(\mathbb{R}^{N^2-1}\). Next, we show that if the \(\mathbb{C}P^{N-1}\) model is defined on the sphere \(S^2\) and the corresponding action functional of this model is finite, then a specific holomorphic function (corresponding to a component of the energy-momentum tensor of the \(\mathbb{C}P^{N-1}\) model) vanishes. In Section 3, we investigate in great detail the Veronese surfaces related to the \(\mathbb{C}P^2\) model and construct their geometric characteristics. We show that the holomorphic and mixed solutions are associated with surfaces immersed in \(\mathbb{R}^8\) and \(\mathbb{R}^3\), respectively. In Section 4, we discuss certain aspects of the projector formalism in the context of surfaces. In Section 5, we present examples of the application of our approach to the dilation-invariant solutions of meron type. We perform the analysis using quadratic differentials and calculate the geometric implications. Section 6 contains final remarks, identifies some open questions on the subject and proposes some possible future developments.

2 Harmonic maps from \(S^2\) to \(\mathbb{C}P^{N-1}\) and the Weierstrass representation

This paper is devoted to the exploration of relations between the \(\mathbb{C}P^{N-1}\) sigma models and the generalized Weierstrass formula for the immersion of two-dimensional surfaces in multi-dimensional Euclidean spaces. To this end we briefly review some basic notions and properties of the \(\mathbb{C}P^{N-1}\) sigma models. For further details on this subject we refer the reader to e.g. [10, 11, 12, 13, 30, 31] and references therein.

In studying the \(\mathbb{C}P^{N-1}\) models one is interested in maps of the form \([z]: \Omega \rightarrow \mathbb{C}P^{N-1}\) (where \(\Omega\) is an open, connected subset of a complex plane \(\mathbb{C}\)) which are stationary points of the action functional \([30]\)

\[
S = \frac{1}{4} \int_{\Omega} (D_\mu z) \dagger (D_\mu z) d\xi d\bar{\xi}, \quad z \dagger \cdot z = 1,
\]

\(\mathbb{C} \ni \xi = \xi^1 + i\xi^2 \rightarrow z = (z_0, z_1, \ldots, z_{N-1}) \in \mathbb{C}^N, (1)\)

and thus are determined as solutions of the corresponding Euler-Lagrange equations. Here, \(D_\mu\) denote covariant derivatives acting on \(z: \Omega \rightarrow \mathbb{C}^N\), defined by

\[
D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z) z \in T_z S^{2N-1}, \quad \partial_\mu = \partial_\xi^\mu, \quad \mu = 1, 2, (2)
\]

where \(\xi\) and \(\bar{\xi}\) are local coordinates in \(\Omega\) and the symbol \(\dagger\) denotes Hermitian conjugation. The covariant derivatives \(D_\mu\) are orthogonal to the inhomogeneous coordinates \(z\), since \(z^\dagger D_\mu z = 0\) holds. They can be expressed in terms of a composite gauge field

\[
A_\mu = z^\dagger \partial_\mu z, \quad A_\mu^\dagger = -A_\mu. (3)
\]

Here, \(A_\mu\) is a pure imaginary function of \(\xi^1\) and \(\xi^2\). The action functional \([1]\) is invariant under global \(U(N)\) transformations and also under the local \(U(1)\)
gauge transformation $z \rightarrow z' = ze^{i\phi}$, where $\phi$ is a real-valued function. Note that the covariant derivatives $D_\mu z$ transform under the gauge transformation $D_\mu z' = (D_\mu z)e^{i\phi}$, so that the dependence on the phase $\phi$ drops out of the action functional \((1)\) and so the model is really based on $\mathbb{C}P^{N-1}$. In the homogeneous coordinates
\[ z = f(f^\dagger \cdot f)^{-\frac{i}{2}} \]
the equations of motion can be written in the form of a conservation law
\[ \partial K - \bar{\partial} K^\dagger = 0, \quad -i\partial K \in su(N), \]
where $K$ and $K^\dagger$ are $N \times N$ matrices of the form
\[ K = \frac{1}{f^\dagger \cdot f} \left( \bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger \right) + \frac{f \otimes f^\dagger}{(f^\dagger \cdot f)^2} (\bar{\partial} f^\dagger \cdot f - f^\dagger \cdot \bar{\partial} f), \]
\[ K^\dagger = \frac{1}{f^\dagger \cdot f} (f \otimes \bar{\partial} f^\dagger - \bar{\partial} f \otimes f^\dagger) + \frac{f \otimes f^\dagger}{(f^\dagger \cdot f)^2} (\bar{\partial} f^\dagger \cdot f - f^\dagger \cdot \bar{\partial} f). \] (6)
The symbols $\partial$ and $\bar{\partial}$ denote the standard derivatives with respect to $\xi$ and $\bar{\xi}$ respectively, i.e.
\[ \partial = \frac{1}{2} (\partial_{\xi^1} - i\partial_{\xi^2}) , \quad \bar{\partial} = \frac{1}{2} (\partial_{\bar{\xi}^1} + i\partial_{\bar{\xi}^2}) . \] (7)

Since the action \((1)\) is invariant under a global $U(N)$ transformation, without loss of generality we can set one of the components of the vector field $f$ equal to 1. Thus, in terms of these variables $f = (1, \bar{w}_1, \ldots, \bar{w}_N)^T$ the equations of motion for the $\mathbb{C}P^{N-1}$ sigma model take the following form
\[ \partial \bar{\partial} w_i - \frac{2\bar{w}_i}{A_{N-1}} \partial w_i \bar{\partial} w_i - \frac{1}{A_{N-1}} \sum_{j \neq i}^{N-1} \bar{w}_j (\partial w_i \bar{\partial} w_j + \bar{\partial} w_i \partial w_j) = 0, \]
\[ \bar{\partial} \partial \bar{w}_i - \frac{2w_i}{A_{N-1}} \bar{\partial} w_i \partial \bar{w}_i - \frac{1}{A_{N-1}} \sum_{j \neq i}^{N-1} w_j (\bar{\partial} w_i \partial \bar{w}_j + \partial \bar{w}_i \bar{\partial} w_j) = 0, \] (8)
where $i = 1, 2, \ldots, N - 1$ and $A_{N-1} = 1 + \sum_i^{N-1} w_i \bar{w}_i$. In what follows we refer to [8] as the equations of the $\mathbb{C}P^{N-1}$ sigma model.

It is instructive to express the Euler-Lagrange equations using the $N \times N$ orthogonal projector $P$ of rank $(N - 1)$ defined on the orthogonal complement to the complex line in $\mathbb{C}^N$,
\[ P = I_N - \frac{f \otimes f^\dagger}{f^\dagger \cdot f}, \quad P^\dagger = P, \quad P^2 = P, \] (9)
where $I_N$ is the $N \times N$ identity matrix. Hence, the Euler-Lagrange equation \((10)\) takes the simpler form
\[ \partial [\bar{\partial} P, P] + \bar{\partial} [\partial P, P] = 0 . \] (10)

After expressing the Euler-Lagrange equations \((10)\) as a conservation law, we are able to formulate the Weierstrass formula for the immersion of two-dimensional surfaces in multi-dimensional Euclidean space. Based on Poincaré's lemma, there exists a closed matrix-valued 1-form,
\[ dX = i(-[\partial P, P]d\xi + [\bar{\partial} P, P]d\bar{\xi}). \] (11)
From the closure of the 1-form \(dX\) (i.e. \(d(dX) = 0\)) it follows that the integral

\[
X(\xi, \bar{\xi}) = i \int_\gamma \left( -[\partial P, P]d\xi + [\bar{\partial} P, P]d\bar{\xi} \right),
\]

depends only on the end points of the curve \(\gamma\) (i.e. it is locally independent of the trajectory in \(\mathbb{C}\)). Note that (10) is invariant under the conformal transformation (i.e. the change of independent variables \(\xi \rightarrow \alpha(\xi)\) and \(\bar{\xi} \rightarrow \bar{\alpha}(\bar{\xi})\)). Such a transformation establishes a reparametrization of the surface \(F\) written in terms of an integral of a 1-form (12) which remains the same geometrical object.

For the analytical description of a two-dimensional surface \(F\) it is convenient to use the Lie algebra isomorphism and identify the \((N^2 - 1)\)-dimensional Euclidean space with the \(su(N)\) algebra

\[
\mathbb{R}^{N^2 - 1} \simeq su(N).
\]

For uniformity we use the scalar product on \(su(N)\) in the form

\[
< A, B > = -\frac{1}{2} \text{tr}(AB), \quad A, B \in su(N),
\]

rather than the Killing form of \(su(N)\) given by the formula

\[
B(A, B) = 2N\text{tr}(AB),
\]

which is negative definite \([32]\). Consequently the first fundamental form \(I\) is given by \([33]\)

\[
I = -Jd\xi^2 + \frac{2}{f^\dagger f} \bar{\partial} f^\dagger P \partial f d\xi d\bar{\xi} - \bar{J}d\bar{\xi}^2,
\]

where the complex-valued functions \(J\) and \(\bar{J}\) are given by

\[
J = \frac{1}{f^\dagger f} \partial f^\dagger P \partial f, \quad \bar{J} = \frac{1}{f^\dagger f} \bar{\partial} f^\dagger P \bar{\partial} f.
\]

They satisfy

\[
\bar{\partial} J = 0, \quad \partial \bar{J} = 0,
\]

whenever \(f\) is a solution of the equations of motion \([4]\). The quantities \(J\) and \(\bar{J}\) are invariant under the global \(U(N)\) transformation, i.e., \(f \rightarrow af, a \in U(N)\). From the physical point of view, \(J = (Dz)^\dagger \cdot Dz\) is related to the energy-momentum tensor \([30]\).

The integral representation (12) defines a mapping \(X : \Omega \ni (\xi, \bar{\xi}) \rightarrow X(\xi, \bar{\xi}) \in su(N)\). We treat each element of the real-valued \(su(N)\) matrix function \(X\) as coordinates of a two-dimensional surface \(F\) immersed in \(\mathbb{R}^{N^2 - 1}\). This map \(X\) is called the generalized Weierstrass formula for immersion. The projector \(P\) is invariant under the transformation \(P \rightarrow UP\), where \(U \in U(N)\) and thus the geometry of the surface \(F\) associated with a solution of (10) admits the symmetry equivalence class of solutions of (10). In this setting, our generalization lies in the realization that most of the properties of the associated surfaces with the \(\mathbb{C}P^{N-1}\) sigma models can be described using an orthogonal projector. The complex tangent vectors of this immersion are

\[
\partial X = iK^\dagger, \quad \bar{\partial} X = iK,
\]

\[5\]
where we use \( K = [\bar{\partial}P, P] \), \( K^\dagger = -[\partial P, P] \).

From the conservation law (5), it is convenient to decompose the matrix \( K \) as follows
\[
K = M + L, \tag{21}
\]
where
\[
M = (I_N - P)\bar{\partial}P, \quad L = -\bar{\partial}P(I_N - P). \tag{22}
\]
It was shown in [24] that the matrices \( M \) and \( L \) satisfy the same conservation law as the matrix \( K \)
\[
\partial M = \bar{\partial}M^\dagger, \quad \partial L = \bar{\partial}L^\dagger, \tag{23}
\]
and the matrices \( M \) and \( L \) differ by a total divergence
\[
M = L + \bar{\partial}P. \tag{24}
\]

Let us now discuss the existence of certain classes of surfaces immersed in the \( su(N) \) algebra under the hypotheses that the \( \C P^{N-1} \) model is defined on the sphere \( S^2 \) and its corresponding action functional (1) is finite. In this case, the procedure for constructing the general class of solutions of the Euclidean two-dimensional \( \C P^{N-1} \) model was derived by A. Din and W. Zakrzewski [24] and R. Sasaki [35]. As a result, one gets three classes of solutions, namely (i) holomorphic (i.e. \( \bar{\partial}f = 0 \)), (ii) antiholomorphic (i.e. \( \partial f = 0 \)) and (iii) mixed. The mixed solutions can be determined from either the holomorphic or the antiholomorphic nonconstant functions by the following procedure. The successive application, say \( k \) times with \( k \leq N - 1 \), of the operator \( P_+ \) defined by its action on vector-valued functions on \( \C N \)
\[
P_+: f \in \C N \rightarrow P_+ f = \partial f - f \frac{f^\dagger \partial f}{f^\dagger f}, \quad \bar{\partial} f = 0, \tag{25}
\]
starting from any nonconstant holomorphic function \( f \in \C N \), allows one to find mixed solutions
\[
f^k = P^k_+ f, \quad k = 0, 1, \ldots, N - 1, \tag{26}
\]
which represent harmonic maps from \( S^2 \) to the \( \C P^{N-1} \) sigma model. Here, \( P^0_+ = id \).

Note that the holomorphic function \( f \in \C N \), used in (25), could be replaced by any nonconstant antiholomorphic function. The mixed solutions \( f^k \) are constructed in the same way, except that the derivative \( \partial \) is replaced by \( \bar{\partial} \) in the definition of the operator \( P_- \). Thus, we have
\[
P_- f = \bar{\partial} f - f \frac{f^\dagger \bar{\partial} f}{f^\dagger f}, \quad \bar{\partial} f = 0, \tag{27}
\]
which yields complementary results.

Under the above hypotheses, the considered surfaces are conformally parametrized and the first fundamental form (16) becomes
\[
I = \frac{2}{f^\dagger f} \bar{\partial} f^\dagger P \partial f d\xi d\bar{\xi}. \tag{28}
\]
In order to demonstrate that the complex-valued functions $J$ and $\bar{J}$ vanish it is sufficient to consider the orthogonality relation
\[(P^i_+ f)^\dagger \cdot P^i_+ f = 0, \quad i \neq j. \tag{29}\]
for $i = k$ and $j = k + 2$ with arbitrary $k = 0, 1, \ldots, N - 1$. Denoting $\tilde{f} = P^k_+ f$, we get
\[0 = \tilde{f}^\dagger \cdot (P^2_+ \tilde{f}) = \tilde{f}^\dagger \cdot \left( \partial (P_+ \tilde{f}) - (P_+ \tilde{f}) \frac{(P_+ \tilde{f})^\dagger \partial (P_+ \tilde{f})}{(P_+ \tilde{f})^\dagger (P_+ \tilde{f})} \right)
= \tilde{f}^\dagger \cdot \partial (P_+ \tilde{f})
= -\partial \tilde{f}^\dagger \cdot (P_+ \tilde{f}), \tag{30}\]
where for the last two equalities we used the orthogonality condition $\tilde{f}^\dagger \cdot P_+ \tilde{f} = 0$. The right hand side of the last equality in (30) can also be written in terms of the complex-valued functions $J$ and $\bar{J}$ given in (17)
\[0 = -\partial \tilde{f}^\dagger \cdot \left( \partial \tilde{f} - \tilde{f} \otimes \tilde{f} \right) = -(\tilde{f}^\dagger \cdot \tilde{f}) \bar{J}. \tag{31}\]
Since $\tilde{f}^\dagger \cdot \tilde{f} \neq 0$, we have $\bar{J} = 0$. Note that for the holomorphic and antiholomorphic solutions $f$ of the $\mathbb{CP}^{N-1}$ model equations (5) the corresponding complex-valued functions $J$ and $\bar{J}$, given in (17), vanish identically.

In particular one can present an analogue of the Bonnet theorem. If we consider the holomorphic or antiholomorphic solutions of the $\mathbb{CP}^{N-1}$ model under the above hypotheses, the Weierstrass formula for immersion $X$ of a surface $\mathcal{F}$ can be expressed in terms of the orthogonal projector of rank $(N - 1)$ by the formula
\[X(\xi, \bar{\xi}) = \epsilon i \left( \frac{1 - N}{N} I_N + P \right), \quad \epsilon = \pm 1. \tag{32}\]
The surface $\mathcal{F}$ is determined uniquely up to Euclidean motions by its first and second fundamental forms
\[I = \text{tr}(\partial P \tilde{\partial} P) d\xi d\bar{\xi}, \tag{33}\]
and
\[II = \epsilon i \left( \partial^2 P - \Gamma^1_{11} \partial P - \Gamma^1_{12} \partial \bar{P} \right) d\xi^2 + 2 \partial \bar{P} P d\xi d\bar{\xi} + (\partial^2 P - \Gamma^1_{22} \partial P - \Gamma^2_{22} \partial \bar{P}) d\xi^2, \tag{34}\]
respectively, where the Christoffel symbols of the second kind are given by
\[\Gamma_{11}^1 = \frac{\text{tr}(\partial^2 P \tilde{\partial} P)}{\text{tr}(\partial P \tilde{\partial} P)}, \quad \Gamma_{11}^2 = \frac{\text{tr}(\partial^2 P \bar{\partial} P)}{\text{tr}(\partial P \bar{\partial} P)}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = 0, \tag{35}\]
\[\Gamma_{12}^2 = \frac{\text{tr}(\partial^2 P \bar{\partial} P)}{\text{tr}(\partial P \bar{\partial} P)}, \quad \Gamma_{22}^2 = \frac{\text{tr}(\bar{\partial}^2 P \partial P)}{\text{tr}(\bar{\partial} P \partial P)}, \quad \Gamma_{22}^1 = \Gamma_{21}^2 = 0. \]
In the case of holomorphic or antiholomorphic solutions $f$ of the $\mathbb{CP}^{N-1}$ model, according to [33], the matrix $K$ can be expressed in the simple form
\[K = \epsilon \bar{\partial} P, \quad K^\dagger = \epsilon \partial P, \quad \epsilon = \pm 1. \tag{36}\]
3 Veronese surfaces for the $\mathbb{C}P^2$ model

One of the simplest applications of a result concerning solutions of the $\mathbb{C}P^{N-1}$ sigma model \cite{3} is the Veronese sequence \cite{36}

$$f = \left(1, \sqrt{\binom{N-1}{1}}, \xi, \ldots, \sqrt{\binom{N-1}{r}}, \xi^r, \ldots, \xi^{N-1}\right).$$ \hspace{1cm} (37)

For all of the above Veronese solutions the first fundamental form is conformal and given by

$$I = (N - 1)(1 + |\xi|^2)^{-2}d\xi d\bar{\xi}. \hspace{1cm} (38)$$

Since $g_{11} = g_{22} = 0$, the Gaussian curvature is computed from the following formula \cite{33}

$$K = -\frac{1}{g_{12}}\partial\partial\ln g_{12}, \hspace{1cm} (39)$$

and for the Veronese solutions it is found to be

$$K = \frac{4}{N-1}. \hspace{1cm} (40)$$

From now on we will only be concerned with the $\mathbb{C}P^2$ model ($N = 3$) for which \cite{3} becomes

$$\partial\bar{\partial}w_1 - \frac{2\bar{w}_1}{A_2}\partial w_1\bar{\partial}w_1 - \frac{\bar{w}_2}{A_2}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0,$$

$$\partial\bar{\partial}w_2 - \frac{2\bar{w}_2}{A_2}\partial w_2\bar{\partial}w_2 - \frac{\bar{w}_1}{A_2}(\partial w_1\bar{\partial}w_2 + \bar{\partial}w_1\partial w_2) = 0,$$

$$A_2 = 1 + w_1\bar{w}_1 + w_2\bar{w}_2, \hspace{1cm} (41)$$

together with their complex conjugate equations. The Veronese vector $f$ for this model is given by

$$f = (1, \sqrt{2}\xi, \xi^2). \hspace{1cm} (42)$$

The method for finding the radius vector $\vec{X}$ through the use of the generalized Weierstrass formula for immersion of 2D surfaces in $\mathbb{R}^8$ was proposed in \cite{20} and \cite{33}. According to \cite{33}, the real components of the corresponding 1-forms for any solution of the $\mathbb{C}P^2$ model are

\begin{align*}
    dX_1 &= \frac{1}{2A_2}\left(\left|(w_1^2 - w_1^1)(\bar{w}_1\partial\bar{w}_2 - \bar{w}_2\partial\bar{w}_1) - (\bar{w}_2^2 - \bar{w}_2^1)(w_1\partial w_2 - w_2\partial w_1)\right|d\xi + \text{c.c.}\right), \\d X_2 &= \frac{i}{2A_2}\left(\left|(w_1^2 + w_1^1)(\bar{w}_1\partial\bar{w}_2 - \bar{w}_2\partial\bar{w}_1) + (\bar{w}_2^2 + \bar{w}_2^1)(w_2\partial w_1 - w_1\partial w_2)\right|d\xi - \text{c.c.}\right), \\d X_3 &= \frac{1}{2A_2}\left(\left|w_2\partial w_2 - w_1\partial w_1 - \bar{w}_2\partial w_2 + \bar{w}_1\partial w_1
\quad + 2|w_1|^2(w_2\partial w_2 - \bar{w}_2\partial w_2) - 2|w_2|^2(w_1\partial w_1 - \bar{w}_1\partial w_1)\right|d\xi + \text{c.c.}\right), \\d X_4 &= \frac{\sqrt{3}}{2A_2}\left(\left|w_1\partial w_1 + w_2\partial w_2 - \bar{w}_1\partial w_1 - \bar{w}_2\partial w_2\right|d\xi + \text{c.c.}\right),
\end{align*}
\[ dX_5 = -\frac{i}{2A^2} \left( (1 + \bar{w}_1^2 + |w_2|^2) \partial w_1 + (1 + w_1^2 + |w_2|^2) \partial \bar{w}_1 + (w_2 \partial \bar{w}_1 - \bar{w}_2 \partial w_1)(w_1 - \bar{w}_1) \right) d\xi - \text{c.c.}, \]
\[ dX_6 = -\frac{i}{2A^2} \left( (1 + \bar{w}_2^2 + |w_1|^2) \partial w_2 + (1 + w_2^2 + |w_1|^2) \partial \bar{w}_2 + (w_1 \partial \bar{w}_2 - \bar{w}_1 \partial w_2)(w_1 + \bar{w}_1) \right) d\xi - \text{c.c.}, \]
\[ dX_7 = \frac{1}{2A^2} \left( (1 - w_1^2 + |w_2|^2) \partial \bar{w}_1 - (1 - \bar{w}_1^2 + |w_2|^2) \partial w_1 + (\bar{w}_2 \partial w_1 - w_2 \partial \bar{w}_1)(w_1 + \bar{w}_1) \right) d\xi + \text{c.c.}, \]
\[ dX_8 = \frac{1}{2A^2} \left( (1 - w_2^2 + |w_1|^2) \partial w_2 - (1 - \bar{w}_2^2 + |w_1|^2) \partial \bar{w}_2 + (\bar{w}_1 \partial w_1 - w_1 \partial \bar{w}_1)(w_2 + \bar{w}_2) \right) d\xi + \text{c.c.}. \] (43)

For any holomorphic solution \((w_1, w_2)\) of the \(\mathbb{C}P^2\) model the above 8 real-valued 1-forms can easily be integrated to give the components of the radius vector
\[ \vec{X}(\xi, \bar{\xi}) = \left( X_1(\xi, \bar{\xi}), \ldots, X_8(\xi, \bar{\xi}) \right), \] (44)

of a two-dimensional surface in \(\mathbb{R}^8\)
\[ X_1 = \frac{w_1 \bar{w}_2 + \bar{w}_1 w_2}{2A^2}, \quad X_2 = \frac{i w_1 \bar{w}_2 - \bar{w}_1 w_2}{2A^2}, \quad X_3 = \frac{|w_1|^2 - |w_2|^2}{2A^2}, \]
\[ X_4 = -\sqrt{3} \frac{|w_1|^2 + |w_2|^2}{2A^2}, \quad X_5 = -i \frac{w_1 - \bar{w}_1}{2A^2}, \quad X_6 = -i \frac{w_2 - \bar{w}_2}{2A^2}, \]
\[ X_7 = -\frac{w_1 + \bar{w}_1}{2A^2}, \quad X_8 = -\frac{w_2 + \bar{w}_2}{2A^2}, \] (45)

where we choose the integration constants to be zero.

Hence, using the Weierstrass formula for immersion \([\text{45}]\) we obtain that the radius vector \(\vec{X}\) of a two-dimensional parametrized surface (for the Veronese solution \([\text{42}]\)) is immersed in \(\mathbb{R}^8\). Its components are
\[ X_1 = \frac{|\xi|^2(\xi + \bar{\xi})}{\sqrt{2}(1 + |\xi|^2)^2} = \frac{\sqrt{2} x(x^2 + y^2)}{(1 + x^2 + y^2)^2}, \]
\[ X_2 = \frac{-i |\xi|^2(\xi - \bar{\xi})}{\sqrt{2}(1 + |\xi|^2)^2} = \frac{\sqrt{2} y(x^2 + y^2)}{(1 + x^2 + y^2)^2}, \]
\[ X_3 = \frac{-|\xi|^2(|\xi|^2 - 2)}{2(1 + |\xi|^2)^2} = \frac{(x^2 + y^2)(x^2 + y^2 - 2)}{2(1 + x^2 + y^2)^2}, \]
\[ X_4 = \frac{-\sqrt{3} |\xi|^2(|\xi|^2 + 2)}{2(1 + |\xi|^2)^2} = \frac{-\sqrt{3}(x^2 + y^2)(x^2 + y^2 + 2)}{2(1 + x^2 + y^2)^2}, \]
\[ X_5 = \frac{-i \xi - \bar{\xi}}{\sqrt{2}(1 + |\xi|^2)^2} = \frac{\sqrt{2} y}{(1 + x^2 + y^2)^2}, \]
\[ X_6 = \frac{-i \xi^2 - \bar{\xi}^2}{2(1 + |\xi|^2)^2} = \frac{2 xy}{(1 + x^2 + y^2)^2}. \]
\[
X_7 = -\frac{\xi + \bar{\xi}}{\sqrt{2}(1 + |\xi|^2)^2} = -\frac{\sqrt{2}x}{(1+x^2+y^2)^2}, \\
X_8 = -\frac{\xi^2 + \bar{\xi}^2}{2(1 + |\xi|^2)^2} = -\frac{x^2+y^2}{(1+x^2+y^2)^2},
\]
where we used \(\xi = x + iy\). The components \(X_i\) \((i = 1, \ldots, 8)\) given in (46) satisfy the equation of an affine sphere
\[
4X_1^2 + 4X_2^2 + 4X_3^2 + \frac{2}{\sqrt{3}}X_4 + X_5^2 + X_6^2 + X_7^2 + X_8^2 = 0. \tag{47}
\]

We can now proceed to construct a mixed solution which, as is well-known \([30]\), can be obtained directly from the holomorphic one. Applying the operator \(P_+\), given by (25), to the vector field (42), we obtain the mixed solution in the form
\[
P_+f = \frac{\sqrt{2}}{1+|\xi|^2}(-\sqrt{2}\bar{\xi}, 1-|\xi|^2, \sqrt{2}\xi). \tag{48}
\]
Let us note that for the \(\mathbb{CP}^2\) model, the repeated applications of the operator \(P_+\) to a holomorphic solution \(f\) only lead to the mixed solution (48) and an antiholomorphic one \(P_+^3f\), since \(P_+^3f = 0\). Thus, the holomorphic and mixed solutions considered here indeed constitute a complete set of solutions for the \(\mathbb{CP}^2\) model. Using \(U(1)\) invariance of the \(\mathbb{CP}^2\) model we can normalize (48) to the following vector
\[
f_1 = (1, \bar{w}_1, \bar{w}_2), \tag{49}
\]
where we denote
\[
\bar{w}_1 = \frac{|\xi|^2 - 1}{\sqrt{2}|\xi|}, \quad \bar{w}_2 = -\frac{\xi}{\bar{\xi}}. \tag{50}
\]
Then, substituting (50) into (43) and integrating, we obtain a two-dimensional parametrized surface immersed in \(\mathbb{R}^3\)
\[
X_1 = -X_7 = -\frac{\xi + \bar{\xi}}{\sqrt{2}(1 + |\xi|^2)^2} = \frac{\sqrt{2}x}{1+x^2+y^2}, \\
X_3 = X_4 = \frac{1}{\sqrt{2}(1 + |\xi|^2)^2} = \frac{1}{1+x^2+y^2}, \\
X_2 = X_5 = -i\frac{\xi - \bar{\xi}}{\sqrt{2}(1 + |\xi|^2)^2} = \frac{\sqrt{2}y}{1+x^2+y^2}, \\
X_6 = X_8 = 0. \tag{51}
\]
Note that the components of the radius vector \(\vec{X}\) in (51) satisfy the following relation
\[
X_1^2 + X_2^2 + (\sqrt{2}X_3 - \frac{1}{\sqrt{2}})^2 = \frac{1}{2}. \tag{52}
\]
Equation (52) represents an ellipsoid, centered at the point \((0, 0, \frac{1}{2})\) in \(\mathbb{R}^3\). So, this case corresponds to the immersion of the \(\mathbb{CP}^2\) model into the \(\mathbb{CP}^1\) model.

Let us now explore some geometrical characteristics of surfaces corresponding to two different solutions of the \(\mathbb{CP}^2\) model. In the holomorphic case (42) the orthogonal projector has the following form
\[
P = \frac{1}{(1 + |\xi|^2)^2} \begin{pmatrix}
|\xi|^2(2 + |\xi|^2) & -\sqrt{2}\xi & -\xi^2 \\
-\sqrt{2}\bar{\xi} & 1 + |\xi|^4 & -\sqrt{2}|\xi|^2\xi \\
-\xi^2 & -\sqrt{2}|\xi|^2\xi & 1 + 2|\xi|^2
\end{pmatrix}, \tag{53}
\]
where \( \text{rank} P = 2 \) and \( \text{tr} P = 2 \). The surface is determined by \( \xi \) and its induced metric is conformal

\[
g_{11} = g_{22} = 0, \quad g_{12} = \frac{1}{(1 + |\xi|^2)^2}. \tag{54}
\]

The nonzero Christoffel symbols of the second kind are

\[
\Gamma^1_{11} = -\frac{2\xi}{1 + |\xi|^2}, \quad \Gamma^2_{22} = -\frac{2\xi}{1 + |\xi|^2}. \tag{55}
\]

The first fundamental form and the Gaussian curvature are given by

\[
I = \frac{2}{(1 + |\xi|^2)^2} d\xi d\bar{\xi}, \quad K = 2, \tag{56}
\]

respectively. Making use of the expression \( \xi \) for the radius vector \( \vec{X} \) we can explicitly write the second fundamental form \( II \) of the surface in the equivalent matrix form. The components of the matrix \( II \) are

\[
II_{11} = \frac{2i}{(1 + |\xi|^2)^4} (\xi^2 d\xi^2 + (4|\xi|^2 - 2)d\xi d\bar{\xi} + \xi^2 d\bar{\xi}^2),
\]

\[
II_{12} = \frac{2\sqrt{2}i}{(1 + |\xi|^2)^3} (-\bar{\xi} d\bar{\xi}^2 + 2\bar{\xi}(|\xi|^2 - 2)d\xi d\bar{\xi} + 3\xi^2 d\bar{\xi}^2),
\]

\[
II_{13} = \frac{2i}{(1 + |\xi|^2)^4} (d\bar{\xi}^2 - 6\xi^2 d\xi d\bar{\xi} + \xi^4 d\bar{\xi}^2),
\]

\[
II_{21} = \frac{2\sqrt{2}i}{(1 + |\xi|^2)^3} (\bar{\xi} d\bar{\xi}^2 + 2\bar{\xi}(|\xi|^2 - 2)d\xi d\bar{\xi} - \xi d\bar{\xi}^2),
\]

\[
II_{22} = \frac{4i}{(1 + |\xi|^2)^4} (-\xi^2 d\xi^2 + (1 + |\xi|^4 - 4|\xi|^2)d\xi d\bar{\xi} - \xi^2 d\bar{\xi}^2),
\]

\[
II_{23} = \frac{2\sqrt{2}i}{(1 + |\xi|^2)^4} (\xi d\xi^2 + 2\xi(1 - 2|\xi|^2)d\xi d\bar{\xi} - \xi^4 d\bar{\xi}^2),
\]

\[
II_{31} = \frac{2i}{(1 + |\xi|^2)^4} (\xi^4 d\xi^2 - 6\bar{\xi}^2 d\xi d\bar{\xi} + d\bar{\xi}^2),
\]

\[
II_{32} = \frac{2\sqrt{2}i}{(1 + |\xi|^2)^4} (-\bar{\xi}^3 d\xi^2 + 2\bar{\xi}(1 - 2|\xi|^2)d\xi d\bar{\xi} + \xi^4 d\bar{\xi}^2),
\]

\[
II_{33} = \frac{2i}{(1 + |\xi|^2)^4} (\xi^2 d\xi^2 + 2|\xi|^2(2 - |\xi|^2)d\xi d\bar{\xi} + \xi^2 d\bar{\xi}^2). \tag{57}
\]

The mean curvature \( \mathcal{H} = \partial \bar{\partial} X / g_{12} \), written as a matrix, takes the form

\[
\mathcal{H} = \frac{4i}{(1 + |\xi|^2)^2} \begin{pmatrix}
2|\xi|^2 - 1 & \sqrt{2} \xi (|\xi|^2 - 2) \\
\sqrt{2} \bar{\xi} (|\xi|^2 - 2) & 1 + |\xi|^2(|\xi|^2 - 4) - \sqrt{2} \bar{\xi} (2|\xi|^2 - 1) \\
-3\xi^2 & -\sqrt{2} \xi (2|\xi|^2 - 1) - |\xi|^2(|\xi|^2 - 2)
\end{pmatrix}, \tag{58}
\]

where \( \text{rank} \mathcal{H} = 2 \) and \( \text{tr} \mathcal{H} = 0 \). The total energy \( \mathcal{E} \) for the holomorphic solution \( \xi \) is finite over all space

\[
u = \ln \left( \frac{|\partial w_1|^2 + |\partial w_2|^2 + |w_2 \partial w_1 - w_1 \partial w_2|^2}{A_2} \right) = \ln \left( \frac{2}{(1 + |\xi|^2)^2} \right). \tag{59}
\]
A particularly significant quantity for the solution (42) satisfying the $\mathbb{CP}^2$ model equations (41) is the topological charge

$$Q = -\frac{1}{\pi} \int_{S^2} g_{12} d\xi d\bar{\xi},$$

(60)

defined on the whole Riemann unit sphere $S^2$. The integral (60) exists and is an invariant of the surface (40). It characterizes globally the surface and is an integer

$$Q = 1.$$  

(61)

In the second case, for mixed solutions (50) the corresponding orthogonal projector takes the form

$$P_1 = \frac{1}{(1 + |\xi|^2)^2} \begin{pmatrix}
(1 + |\xi|^4) & -\sqrt{2} \xi (|\xi|^2 - 1) & 2\xi^2 \\
-\sqrt{2} \bar{\xi} (|\xi|^2 - 1) & 4|\xi|^2 & \sqrt{2} \bar{\xi} (|\xi|^2 - 1) \\
2\xi^2 & \sqrt{2} \bar{\xi} (|\xi|^2 - 1) & 1 + |\xi|^4
\end{pmatrix},$$

(62)

where $\text{rank}P_1 = 2$ and $\text{tr}P_1 = 2$. The surface is determined by (51) and its induced metric associated with the projector (62) is also conformal

$$g_{11} = g_{22} = 0, \quad g_{12} = \frac{2}{(1 + |\xi|^2)^2}.$$  

(63)

The first fundamental form and the Gaussian curvature are

$$I = \frac{4}{(1 + |\xi|^2)^2} d\xi d\bar{\xi}, \quad K = 1,$$

(64)

respectively.

It is worth mentioning that from the Veronese sequences we can obtain associated surfaces with constant Gaussian curvature as stated in [37]. However, the converse statement does not apply in general. In this paper and in [33] we give examples of surfaces with constant Gaussian curvature which are not associated with Veronese sequences. Such surfaces correspond to the dilation-invariant solutions or mixed soliton solutions of the $\mathbb{CP}^2$ model.

### 4 Comments on surfaces obtained via projector formalism

Certain geometrical aspects of surfaces have been studied recently in [37] using the generalized Weierstrass representation associated with the $\mathbb{CP}^{N-1}$ sigma models. In the context of [37] a sequence of rank-one projectors of the form

$$\mathcal{P}_k := \mathcal{P}(V_k) = \frac{V_k \otimes V_k^\dagger}{V_k^\dagger \cdot V_k}, \quad \text{where} \quad V_k = P_k^f, \quad k \in \mathbb{Z}^+,$$

(65)

were used to construct a family of surfaces associated with a given solution of the $\mathbb{CP}^{N-1}$ model. More specifically, starting with any nonconstant holomorphic solution of the $\mathbb{CP}^{N-1}$ model, one can successively (say $k$ times) apply the operator $P_+$ (given in (25)) in order to find a new solution $P_k = P_k^f$, which
represents a harmonic map \( S^2 \to \mathbb{C}P^{N-1} \). So, for every \( k \leq N - 1 \) the quantity \( P_k^+ \) constitutes a building element for the construction of a set of rank-one projectors \( \{ P_0, P_1, \ldots, P_k \} \) as described in [33]. This set of projectors determines new conservation laws of the form \([33]\), which can be considered new in the sense that only the first one is related to the holomorphic (or antiholomorphic) solutions and the rest are related to the mixed solutions obtained from the nonconstant holomorphic ones by applying the operator \( P_+ \) successively. Consequently, according to this procedure it is claimed that one can obtain new surfaces for each projector. However, there are some questions concerning this procedure. Using the following properties (given in [30])

\[
\bar{\partial}(P_k^+ f) = -P_k^+ f \frac{\overline{P_k^+ f}^2}{|P_k^+ f|^2},
\]

\[
\partial \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) = \frac{P_k^+ f}{|P_k^+ f|^2},
\]

(66)

together with the orthogonality relation \([29]\), it is straightforward to compute

\[
\partial P_k = \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) \otimes \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) - \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) \otimes \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right)
\]

(67)

and

\[
[\partial P_k, P_k] = \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) \otimes \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) + \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) \otimes \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right)
\]

(68)

which can also be written as

\[
[\partial P_k, P_k] = \partial P_k + 2\left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) \otimes \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right).
\]

(69)

Similarly, we can write \([\partial P_k, P_k]\) as

\[
[\partial P_k, P_k] = -\partial P_k - 2\left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) \otimes \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right).
\]

(70)

As a consequence of the commutators given in \([69]\) and \([70]\), the Weierstrass data \([11]\) becomes

\[
dX = -i \left[ \partial P_k + 2 \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right) \right] d\xi + (\partial P_k + 2 \left( \frac{P_k^+ f}{|P_k^+ f|^2} \right)) d\xi. \tag{71}
\]

It is easily seen that for \( k = 0 \) (e.g. for the holomorphic solutions, or equivalently the antiholomorphic ones) equation \([71]\) reduces to

\[
dX = -i \left[ \partial P_0 d\xi + \bar{\partial} P_0 d\bar{\xi} \right], \tag{72}
\]

since the other terms in \([71]\) do not appear for \( k = 0 \). Hence, it is concluded that \( X \) is proportional to the projector \( P_0 \). This point has been fully discussed both in this paper and in \([33]\) and \([38]\). However, for \( k \neq 0 \) (i.e. for the mixed solutions) the integral of the Weierstrass representation \([71]\) cannot be proportional to \( P_k \).
This point can be further discussed for the example of the mixed solutions given in Section 3. This example is also analyzed in [37] by a different approach. In [37] it is stated that the mixed solution, obtained from the Veronese vector \( f = (1, \sqrt{2} \xi, \xi^2) \), for the \( CP^2 \) model associated with the projector

\[
P_1 = \frac{1}{(1 + |\xi|^2)^2} \begin{pmatrix}
2|\xi|^2 & \sqrt{2} \xi((|\xi|^2 - 1) & -2\xi^2 \\
\sqrt{2} \xi((|\xi|^2 - 1) & (1 - |\xi|^2)^2 & -\sqrt{2} \xi((|\xi|^2 - 1) \\
-2\xi^2 & -\sqrt{2} \xi((|\xi|^2 - 1) & 2|\xi|^2
\end{pmatrix},
\]

leads to a radius vector \( \vec{Y} \) which lies in a 5-dimensional subspace of \( \mathbb{R}^8 \). Moreover, the components of the radius vector \( \vec{Y} \) are given as

\[
Y_1 = \frac{2x(1 - x^2 - y^2)}{(1 + x^2 + y^2)^2}, \quad Y_2 = \frac{2y(1 - x^2 - y^2)}{(1 + x^2 + y^2)^2}, \quad Y_3 = \frac{2(x^2 - y^2)}{(1 + x^2 + y^2)^2},
\]

\[
Y_4 = \frac{4xy}{(1 + x^2 + y^2)^2}, \quad Y_5 = \frac{\sqrt{3}(1 - x^2 - y^2)^2}{(1 + x^2 + y^2)^2},
\]

which satisfy the following surface

\[
Y_1^2 + Y_2^2 + 4Y_3^2 + 4Y_4^2 + \frac{1}{\sqrt{3}}Y_5 = 1.
\]

However, using the same solution together with the projector (62) and the procedure summarized in Section 2, we obtain an associated surface with the radius vector \( \vec{X} \) immersed in a 3-dimensional subspace of \( \mathbb{R}^8 \). The components of the radius vector \( \vec{X} \) are given in (51) and they satisfy equation (52). Since the two surfaces are obtained from the same mixed solutions of the \( CP^2 \) model, constructed by the same procedure from the Veronese vector \( f = (1, \sqrt{2} \xi, \xi^2) \), we expect them to be the same geometrical object in accordance with the Bonnet theorem. However, it can easily be verified that the two surfaces cannot be transformed into each other by rotations and translations.

5 Dilation-invariant solutions

The objective of this section is to construct dilation-invariant solutions of the \( CP^2 \) model and then to calculate some geometric properties of the surface associated with this model by using the Weierstrass formula for immersion in \( \mathbb{R}^8 \).

Let us discuss the solutions of (41) which are invariant under the scaling symmetries

\[
S = w_i \partial w_i - \bar{w}_i \partial \bar{w}_i, \quad i = 1, 2.
\]

For this purpose we determine the invariants of the vector fields (76), which imply the algebraic constraints \( w_i \bar{w}_i = D_i \in \mathbb{R}, \ i = 1, 2 \). Without loss of generality we may choose \( D_i = 1 \). Then the invariant solution is given by

\[
w_i = \frac{F_i(\xi)}{\bar{F}_i(\xi)}, \quad i = 1, 2,
\]

where \( F_i \) and \( \bar{F}_i \) are arbitrary complex-valued functions of one complex variable \( \xi \) and \( \bar{\xi} \), respectively. After substituting (77) into the \( CP^2 \) model equations (41)
it is immediately seen that the unknown functions $F_i$ and $\bar{F}_i$ must satisfy the following differential relation

$$|F_2'|^2 |F_1|^2 = |F_1'|^2 |F_2|^2,$$  (78)

where prime means differentiation with respect to the argument (i.e. with respect to either $\xi$ or $\bar{\xi}$). Equation (78) implies

$$F_2' (\xi) = \frac{F_2 (\xi) F_1'(\xi)}{F_1(\xi)} e^{i\psi},$$  (79)

which has the following solution

$$F_2 (\xi) = c F_1(\xi) e^{i\psi}, \quad c \in \mathbb{C},$$  (80)

where $\psi$ is an arbitrary constant. By substituting (77) and (79) into (41) it is seen that $\psi$ must satisfy

$$\psi = \pm \frac{\pi}{3} + 2\pi m, \quad m \in \mathbb{Z}.$$  (81)

Thus we obtain a class of scaling invariant solutions of the $\mathbb{C}P^2$ model equations (41) which depend on one arbitrary complex-valued function of one variable $\xi$ and its conjugate

$$w_1 = \frac{F_1(\xi)}{F_2(\xi)}, \quad w_2 = \frac{c F_1(\xi) e^{i\psi}}{c \bar{F}_1(\xi) e^{-i\psi}}.$$  (82)

We now perform a detailed investigation of the geometric implications of the induced metric associated with a quadratic differential. In [33] it was shown that the induced metric for the $\mathbb{C}P^2$ model equations subjected to the DCs (i.e. $w_i \bar{w}_i = 1, \ i = 1, 2$) is conformal and the Gaussian curvature for the associated surfaces vanishes. It was also shown that the coordinates of the radius vector $\vec{X}$ for the nonsplitting solutions of the $\mathbb{C}P^2$ model equations are given by

$$X_1 = \frac{i}{6 \sqrt{3} |c|^2} |F|^{-2e^{i\psi}} (c^2 F - c^2 \bar{F}|F|^{2i\sqrt{3}}),$$

$$X_2 = -\frac{1}{6 \sqrt{3} |c|^2} |F|^{-2e^{i\psi}} (c^2 F + c^2 \bar{F}|F|^{2i\sqrt{3}}),$$

$$X_3 = \frac{1}{6} ((1 - i \sqrt{3}) \ln F + (1 + i \sqrt{3}) \ln F),$$

$$X_4 = -\frac{1}{6} ((i + \sqrt{3}) \ln F + (-i + \sqrt{3}) \ln F),$$

$$X_5 = \frac{F^2 + \bar{F}^2}{6 \sqrt{3} |F|^2},$$

$$X_6 = \frac{1}{6 \sqrt{3} |c|^2} |F|^{-2e^{i\psi}} (c^2 \bar{F} + c^2 F|F|^{2i\sqrt{3}}),$$

$$X_7 = \frac{i(F^2 - \bar{F}^2)}{6 \sqrt{3} |F|^2},$$

$$X_8 = \frac{i}{6 \sqrt{3} |c|^2} |F|^{-2e^{i\psi}} (c^2 \bar{F} - c^2 F|F|^{2i\sqrt{3}}).$$  (83)
The corresponding first fundamental form is immediately given as

\[ I = \frac{2}{3} \frac{|F'|^2}{|F|^2} d\xi d\breve{\xi}. \]  

(84)

Note that the components of the radius vector \( \vec{X} \) in (83) satisfy the following relations

\[ X_1^2 + X_2^2 = X_5^2 + X_7^2 = X_6^2 + X_8^2 = \frac{1}{27}. \]  

(85)

Eliminating the functions \( F \) and \( \breve{F} \) in (83) we obtain

\[ X_1 = \frac{i}{6\sqrt{3}|c|^2} e^{-(v+\bar{v})e^{i\psi}} (c^2 e^v + c^2 e^{i\sqrt{3}(v+\bar{v})}), \]
\[ X_2 = -\frac{1}{6\sqrt{3}|c|^2} e^{-(v+\bar{v})e^{i\psi}} (ic^2 e^v + c^2 e^{i\sqrt{3}(v+\bar{v})}), \]
\[ X_5 = -\frac{1}{3\sqrt{3}} \cos \left( \frac{3}{2}(\sqrt{3}X_3 + X_4) \right), \]
\[ X_6 = \frac{1}{6\sqrt{3}|c|^2} e^{-(v+\bar{v})e^{i\psi}} (c^2 e^{i\sqrt{3}(v+\bar{v})}), \]
\[ X_7 = -\frac{1}{3\sqrt{3}} \sin \left( \frac{3}{2}(\sqrt{3}X_3 + X_4) \right), \]
\[ X_8 = \frac{i}{6\sqrt{3}|c|^2} e^{-(v+\bar{v})e^{i\psi}} (c^2 e^{i\sqrt{3}(v+\bar{v})}), \]  

(86)

where \( v = \frac{2}{3}(1 + i\sqrt{3})(X_3 + iX_4) \). The surface is parametrized in terms of \( X_3 \) and \( X_4 \). Now, the corresponding first fundamental form becomes

\[ I = \frac{3}{2} (dX_3^2 + dX_4^2). \]  

(87)

Note that this is just the real form of (84) where \( \xi^1 = X_3 \) and \( \xi^2 = X_4 \).

The induced metric (83) on the \((\xi, \breve{\xi})\) plane can be written as a quadratic differential

\[ I = \frac{2}{3} d(\ln F(\xi)) \wedge d(\ln \breve{F}(\breve{\xi})). \]  

(88)

Equation (88) defines a field of line elements on a surface \( \mathcal{F} \) with singularities at the critical points (i.e. the zeros and poles of the differential (88)). The geodesic trajectories \( \xi = \xi(t) \) of this metric are determined locally by the integral

\[ \text{Re} \left( e^{i\theta} \omega \right) = c, \quad \omega = \int_{\xi}^{\xi'} \frac{F'}{F} d\xi, \quad \theta \in \mathbb{R}, \]  

(89)

where we make use of the definitions and notations given in (89). The simplest local trajectory structure of the quadratic differential (88) can be found by assuming that \( F'/F \) has two simple zeros and one simple pole. Then

\[ F(\xi) = A\xi^n \left( 1 + \mathcal{O}(\xi) \right), \quad n \in \mathbb{Z}, \quad A \in \mathbb{C}, \]  

(90)

and

\[ \frac{F'}{F} = \frac{n}{\xi} \left( 1 + \mathcal{O}(\xi) \right) \text{ near a pole at } \xi = 0, \]
\[ \frac{F'}{F} = C(\xi - a) \left( 1 + \mathcal{O}(\xi - a) \right) \text{ near a simple zero of } F', \quad a \in \mathbb{C}. \]  

(91)
Locally, the flat coordinates of the metric $I$ are the real and imaginary parts of the function

$$\omega = \int \frac{F'}{F} d\xi = n \ln |\xi| + O(1), \quad (92)$$

where $O(1)$ denotes some analytic function near $\xi = 0$. The critical vertical trajectory is defined to be the maximal trajectory of the ODE

$$\text{Re} \left( \frac{F'}{F} d\xi \right) = 0 \quad \text{with} \quad \theta = 0. \quad (93)$$

As a consequence of this equation we obtain the following condition

$$\frac{d\xi}{dt} = i \frac{F'(\xi)}{F(\xi)}. \quad (94)$$

The monodromy of (92) is given by $2i\pi n$. Also, the function $q = e^{\omega/n}$ is analytic in a punctured neighborhood of $\xi = 0$, since $\text{Re}(\omega) = n \ln |\xi|$ and $|q| \sim |\xi|$ near $\xi = 0$. Thus, $q$ has a removable singularity at $\xi = 0$, and hence can be extended to an analytic function

$$q(\xi) = B\xi + O(\xi^2), \quad 0 \neq B \in \mathbb{C}. \quad (95)$$

Let us denote by $D$ the maximal connected domain foliated by closed trajectories homotopic to a small circle around $\xi = 0$. The function $q$ is a single valued conformal map of $D$ onto the disk of radius $|n|$, since the perimeter of the disk is $2\pi|n|$. Hence, we get one semi-infinite cylinder (homeomorphic to the disk $\{0 < |\xi| \leq 1\}$) for each simple pole of $F'/F$. For example, for $F = \xi(\xi - 1)$ we have

$$\frac{F'}{F} = \frac{2\xi - 1}{\xi(\xi - 1)} = \frac{1}{\xi} + \frac{1}{\xi - 1}, \quad \text{Res}_\infty \left( \frac{F'}{F} \right) = -2. \quad (96)$$

Hence we obtain three cylinders, two for the poles at 0 and 1, each of perimeter $2\pi$, and one of perimeter $4\pi$ for the pole at $\infty$. If we instead assume that $F'/F$ has two simple zeros and three simple poles, then we get four semi-infinite cylinders glued along the based perimeter with two points of conical singularities. It should be noted that for the $\mathbb{C}P^1$ model, the surfaces corresponding to the dilation invariant solutions are cylinders.[40]

Note that the singular solutions (82) are in fact the meron-like solutions (i.e. with logarithmically divergent action at isolated points) of the $\mathbb{C}P^2$ model equations (41). The meron solutions, obtained from the dilation invariance of the solutions of the $\mathbb{C}P^2$ model, are located at $\bar{F}(\bar{\xi}) = 0$. Mersons are more durable than instantons in the sense that they can exist in a constant (not necessarily zero) Higgs field. This property is shared by both the $\mathbb{C}P^{N-1}$ and Yang-Mills models [41, 42, 43].

6 Conclusions

In this paper we have shown that if the $\mathbb{C}P^{N-1}$ model equations are defined on the sphere $S^2$ and the associated action functional of this model is finite, then the specific holomorphic function $J$ (i.e. component of the energy-momentum tensor of the model) vanishes and consequently the surfaces are conformally
parametrized. We demonstrate that the holomorphic and the mixed Veronese solutions of the $\mathbb{C}P^2$ model are associated with a sphere and an ellipsoid immersed in $\mathbb{R}^8$ and $\mathbb{R}^3$, respectively. In this context we have shown that the Weierstrass formula for immersion of surfaces associated with the mixed solutions of the $\mathbb{C}P^{N-1}$ model cannot be proportional to the rank-one projectors $\mathcal{P}_k$ for $k \neq 0$, unlike the case $k = 0$. The analysis of the geometrical properties of the dilation-invariant meron type solutions shows that they represent semi-infinite cylinders.

This research could be expanded in several directions. The meaning of the new conservation laws could be investigated in the context of surfaces immersed in multi-dimensional spaces. Are they really independent? Can they differ from each other by a total divergence (as discussed in Section 2)? Other natural directions which could also be addressed involve families of solutions obtained recursively and whether they can be related through an auto-Bäcklund transformation (note that a Bäcklund parameter is not present in formula (25), but it can be introduced by a gauge transformation). In addition to these it is also important to ask if the symmetry operator (24) is expressible in terms of some combination of the known infinitesimal generators (as given in 33) of the Lie-point symmetry algebra of the $\mathbb{C}P^{N-1}$ model. These and other issues will be addressed in our future work.

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