Weak convergence of the number of zero increments in the random walk with barrier

Alexander Marynych* Glib Verovkin†

Abstract
We continue the line of research of random walks with barrier initiated by Iksanov and Möhle (2008). Assuming that the tail of the step of the underlying random walk has a power-like behavior at infinity with exponent $-\alpha$, $\alpha \in (0,1)$, we prove that the number $V_n$ of zero increments in the random walk with barrier, properly centered and normalized, converges weakly to the standard normal law. This refines previously known weak law of large numbers for $V_n$ proved in Iksanov and Negadailov (2008).

Keywords: random walk with barrier, recursion with random indices, renewal process, undershot

1 Introduction
Let $(\xi_k)_{k \in \mathbb{N}}$ be independent copies of a random variable $\xi$ with distribution $p_k = \mathbb{P}\{\xi = k\}$, $k \in \mathbb{N}$. The random walk with barrier $n \in \mathbb{N}$ is a sequence $(R_k^{(n)})_{k \in \mathbb{N}_0}$ (where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) defined as follows:

$$R_0^{(n)} := 0 \quad \text{and} \quad R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)} + \xi_k < n\}}, \quad k \in \mathbb{N}.$$ 

Plainly, $(R_k^{(n)})_{k \in \mathbb{N}_0}$ is a non-decreasing Markov chain which cannot reach the state $n$. In what follows we always assume that $p_1 > 0$ which implies that the random walk with barrier $n$ will eventually get absorbed in the state $n - 1$.

The equalities

$$M_n := \#\{k \in \mathbb{N} : R_{k-1}^{(n)} \neq R_k^{(n)}\} = \sum_{l=0}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}};$$

$$T_n := \inf\{k \in \mathbb{N}_0 : R_k^{(n)} = n - 1\} = \sum_{l \geq 0} 1_{\{R_l^{(n)} < n-1\}};$$

*Faculty of Cybernetics, Taras Shevchenko National University of Kyiv, Ukraine. E-mail: marynych@unicyb.kiev.ua
†Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Ukraine. E-mail: glebverov@gmail.com
\[ V_n := T_n - M_n = \#\{ i \leq T_n : \xi_i = R_{i-1}^{(n)} \} = \sum_{l=0}^{T_n-1} 1_{\{ R_{i+1}^{(n)} \geq n \}} \]
define, respectively, the number of jumps, the absorption time and the number of zero increments before the absorption in the random walk with barrier \( n \).

There is a large number of real life situations where the random walk with barrier appears naturally. Let PTC be a transport company, offering a tour to the national park. The PTC uses buses with total amount of seats \( n \). Various groups of people book seats in order to visit the park. If the size of the group is less than remaining number of vacant seats, the request satisfied, otherwise it is turned down. The quantities of interest are the total number of groups applied \( T_{n+1} \), the number of accepted groups \( M_{n+1} \) and the number of rejections \( V_{n+1} \).

Another example is the work of a server. Imagine that a client has bought an internet-package \( n \) Mb in size. Consider the downloading of files with the size being a multiple of \( 1 \) Mb: the server receives requests on download, if the size of file is lower than remaining size, then it starts downloading it, else blocks the request. Similarly to the example above, the quantities of interest in this case are the total number of requests \( T_{n+1} \), the number of downloaded files \( M_{n+1} \) and the number of blocked requests \( V_{n+1} \).

In [10] (see also [8] for a particular case) it was shown that, if the law of \( \xi \) belongs to the domain of attraction of a stable law, \( M_n \), properly normalized and centered, weakly converges. Furthermore, the set of limiting laws is comprised of stable laws and the law of exponential subordinator. In [12] it was checked that the same group of results hold on replacing \( M_n \) by \( T_n \). Finally, in [11] it was proved that: (a) if \( \mathbb{E}\xi < \infty \) then \( V_n \) weakly converges (without normalization); (b) if the law of \( \xi \) belongs to the domain of attraction of an \( \alpha \)-stable law with \( \alpha \in (0,1] \), equivalently if

\[ \mathbb{P}\{ \xi \geq n \} \sim n^{-\alpha} \ell(n), \quad n \to \infty, \quad (1) \]

for some \( \ell \) slowly varying at infinity, then \( V_n/\mathbb{E}V_n \stackrel{P}{\to} 1 \) as \( n \to \infty \).

To complete the picture, in this paper we give results about the weak convergence of \( V_n \). The treatment of \( V_n \) calls for more delicate argument than that for \( M_n \) and/or \( T_n \). Crudely speaking, while the asymptotics of \( M_n \) and \( T_n \) is based on the "first order" arguments, the asymptotics of \( V_n \) needs the "second order" reasoning. As a result, the approach exploited in [10, 11] does not help in the present situation. Moreover, regular variation [11] alone seems not to be enough to ensure the weak convergence of properly scaled and normalized \( V_n \) and one has to impose more restrictive "second order" condition on the tail \( \mathbb{P}\{ \xi \geq n \} \). In this work we prove a central limit theorem-type result for \( V_n \) assuming

\[ \mathbb{P}\{ \xi \geq n \} = cn^{-\alpha} + O(n^{-(\alpha+\varepsilon)}), \quad n \to \infty, \quad (2) \]

for some \( c > 0, \alpha \in (0,1) \) and \( \varepsilon > 0 \).

In what follows we reserve notation \( \eta \) for a random variable with the beta \((1 - \alpha, \alpha)\) law, \( \alpha \in (0,1) \), i.e.,

\[ \mathbb{P}\{ \eta \in dx \} = \frac{\sin \pi \alpha}{\pi} x^{-\alpha}(1-x)^{\alpha-1}1_{(0,1)}(x)dx; \quad (3) \]
\[ \mu_\alpha := \mathbb{E} |\log \eta| = \psi(1) - \psi(1 - \alpha) \]

and

\[ \sigma^2_\alpha := \text{Var} (\log \eta) = \psi'(1 - \alpha) - \psi'(1), \]

where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the logarithmic derivative of the gamma function.

The main result of this paper is given by the next theorem

**Theorem 1.** Assume that (2) holds with \( \alpha \in (0, 1) \), \( \varepsilon > 0 \) and \( c > 0 \). If \( \alpha \in (0, 1/2] \) assume additionally

\[ \sup_{n \geq 1} \frac{np_n}{\mathbb{P}\left\{ \xi > n \right\}} < \infty. \]  

(4) Then

\[ \frac{V_n - \mu_\alpha^{-1} \log n}{\sqrt{\sigma^2_\alpha \mu_\alpha^{-3} \log n}} \xrightarrow{d} \mathcal{N}(0, 1), \quad n \to \infty, \]

where \( \mathcal{N}(0, 1) \) is a random variable with the standard normal law. Moreover, there is a convergence of the first absolute moments.

Our approach is based on the analysis of random recursive equation for \( (V_n) \). It is shown that the sequence \( (V_n) \) can be approximated by a suitable renewal counting process and the error of such an approximation is estimated in terms of an appropriate probability distance. A similar method has already been used in [7] to derive the weak convergence result for the number of collisions in beta coalescents.

The rest of the paper is organized as follows. In Section 2 we define the approximating renewal process and give random recursive equations for related quantities. The proofs are presented in Section 3. An auxiliary lemma is formulated and proved in Appendix.

## 2 Renewal process and recursion with random indices

Given the sequence \( (\xi_n)_{n \in \mathbb{N}} \), define a zero-delayed random walk

\[ S_0 = 0, \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N}, \]

and the first passage process

\[ N_n := \inf\{k \in \mathbb{N}_0 : S_k \geq n\}, \quad n \in \mathbb{N}. \]

The random variable \( Y_n := n - S_{N_n-1} \) is called undershot. It was shown\(^1\) in [11] that the sequence \( (V_n)_{n \in \mathbb{N}} \) satisfies the following recursion with random index

\[ V_1 = 0, \quad V_n \overset{d}{=} 1_{\{Y_n > 1\}} + V'_{Y_n}, \quad n \geq 2, \]  

(5)

where \( V'_k \overset{d}{=} V_k \) for all \( k \in \mathbb{N} \) and \( (V'_k)_{k \in \mathbb{N}} \) and \( Y_n \) are independent.

---

\(^1\)Note that in [11] the definition of \( T_n \) is slightly different from our which results in different recursion for \( (V_n) \).
The recursion (5) can be slightly simplified by setting $X_n := V_n + 1_{\{n > 1\}}$, then
\[ X_1 = 0, \quad X_n \overset{d}{=} 1 + X_n', \quad n \geq 2, \quad \text{(6)} \]
where likewise $X_k' \overset{d}{=} X_k$ for all $k \in \mathbb{N}$ and $(X_k')_{k \in \mathbb{N}}$ and $Y_n$ are independent. Clearly, the asymptotic behavior of $X_n$ is the same as of $V_n$.

It is a classical observation due to Dynkin [3] that under the assumption (1) with $\alpha \in (0,1)$ we have
\[ Y_n/n \overset{d}{\to} \eta, \quad n \to \infty, \quad \text{(7)} \]
where $\eta$ has density [3].

Let $(\eta_k)_{k \in \mathbb{N}}$ be iid copies of $\eta$. Define a zero-delayed random walk
\[ T_0 = 0; \quad T_k = |\log \eta_1| + \ldots + |\log \eta_k|, \quad k \in \mathbb{N}; \]
the corresponding renewal counting process
\[ \nu_t := \#\{k \in \mathbb{N} : T_k \leq t\} = \sum_{k=1}^{\infty} 1_{\{T_k \leq t\}}, \quad t \in \mathbb{R}, \]
and set $W_t := \nu_{\log t} + 1_{\{t > 1\}}$ for $t > 0$. Since $\nu_t = 0$ a.s. for $t \leq 0$ we have $W_t = 0$ for $t \in (0,1]$, while for $t > 1$ the strong Markov property implies
\[ W_t \overset{d}{=} 1 + W'_{tn}, \quad \text{(8)} \]
where $W_t \overset{d}{=} W'_t$ for every $t > 0$ and $(W'_t)_{t \geq 0}$ and $\eta$ are independent.

Comparing recursions (6) and (8) and in view of (7) we may expect that the weak asymptotic behavior of $X_n$ is the same as of $W_n$. We will show, assuming (2), that this heuristic can be made rigorous and leads to the desired result on the asymptotic of $V_n$.

## 3 Proofs

We start with a refinement of (7) by estimating the speed of convergence of $Y_n/n$ to $\eta$ in terms of so-called minimal $L_1$-distance. Let us recall its definition. Let $\mathcal{D}_1$ be the set of probability laws on $\mathbb{R}$ with finite first absolute moment. The $L_1$-minimal (or Wasserstein) distance on $\mathcal{D}_1$ is defined by
\[ d_1(X,Y) = \inf \mathbb{E}|\hat{X} - \hat{Y}|, \quad \text{(9)} \]
where the infimum is taken over all couplings $(\hat{X}, \hat{Y})$ such that $X \overset{d}{=} \hat{X}$ and $Y \overset{d}{=} \hat{Y}$.

For ease of reference we summarize the properties of $d_1$ to be used in this work in the following proposition.

**Proposition 3.1.** Let $X,Y$ be random variables with finite first absolute moments. The distance $d_1$ has the following properties:
The first equality follows from (Lin) property of Proof.

From the distributional identity metrics, in particular for the proofs of the aforementioned properties of $Y$ where

\begin{align}
\underbrace{\text{Under the assumptions of Theorem 1.1 there exists Proposition 3.2.}}_\text{(Conv) For $X, X_n \in D_1$ convergence $d_1(X_n, X) \to 0$, $n \to \infty$, is equivalent to $X_n \xrightarrow{d} X$ and $E|X_n| \to E|X|$, $n \to \infty$.}
\end{align}

We refer the reader to Chapter 1 in [13] for an introduction to the theory of probability metrics, in particular for the proofs of the aforementioned properties of $d_1$.

In view of (Conv) characterization of $d_1$ the next lemma is indeed a refinement of (7).

**Proposition 3.2.** Under the assumptions of Theorem 1.1 there exists $\delta > 0$ such that

\begin{align}
d_1\left(\log \frac{Y_n}{n}, \log \eta\right) = d_1\left(\log Y_n, \log (n\eta)\right) = O(n^{-\delta}), \ n \to \infty.
\end{align}

**Proof.** The first equality follows from (Lin) property of $d_1$. Using (Rep) we have

\begin{align}
d_1\left(\log Y_n, \log (n\eta)\right) = \sup_{f \in F_1} \left| E_f(\log Y_n) - E_f(\log (n\eta)) \right|.
\end{align}

From the distributional identity

\begin{align}
Y_1 = 1, \ Y_n = n_1\{\xi \geq n\} + Y_n'1\{\xi < n\}, \ n \geq 2,
\end{align}

where $Y_k' \xrightarrow{d} Y_k$ for all $k \in \mathbb{N}$ and $(Y_k')_{k \in \mathbb{N}}$ is independent from $\xi$, we infer

\begin{align}
E_f(\log Y_n) = P\{\xi \geq n\} f(\log n) + \sum_{j=1}^{n-1} p_j E_f(\log Y_{n-j}), \ n \geq 2.
\end{align}

Substituting this into (10) and using the triangle inequality gives

\begin{align}
d_1\left(\log Y_n, \log (n\eta)\right)
&\leq \sup_{f \in F_1} \left| P\{\xi \geq n\} f(\log n) + \sum_{j=1}^{n-1} p_j E_f(\log (n-j)\eta) - E_f(\log (n\eta)) \right|
\quad + \sum_{j=1}^{n-1} p_j \sup_{f \in F_1} \left| E_f(\log Y_{n-j}) - E_f(\log (n-j)\eta) \right|
\quad + \sum_{j=1}^{n-1} p_j d_1\left(\log Y_{n-j}, \log (n-j)\eta\right).
\end{align}
Let $\tilde{\xi}$ be independent of $\tilde{\eta}$ and $\tilde{\xi} \overset{d}{=} \xi$, $\tilde{\eta} \overset{d}{=} \eta$. The first term can be written as

$$
\sup_{j \in \mathcal{F}_1} \left| \mathbb{P}\{\xi \geq n\} f(\log n) + \sum_{j=1}^{n-1} p_j \mathbb{E} f(\log(n-j)\eta) - \mathbb{E} f(\log(n\eta)) \right|
$$

$$
= d_1 \left( \log(n_1\{\tilde{\xi} \geq n\} + (n - \tilde{\xi})\tilde{\eta}1_{\{\tilde{\xi} < n\}}), \log(n\tilde{\eta}) \right)
= d_1 \left( \log((1 - \tilde{\xi}n^{-1})\tilde{\eta})1_{\{\tilde{\xi} < n\}}, \log \tilde{\eta} \right),
$$

where we have utilized (Lin) property of $d_1$ in the second equality.

For every $x \geq 1$,

$$
\mathbb{P}\{\xi \geq x\} = \mathbb{P}\{\xi \geq [x]\} = c([x])^{-\alpha} + O(([x])^{-(\alpha + \varepsilon)}) = cx^{-\alpha} + O(x^{-(\alpha + \varepsilon)\wedge 1}),
$$

hence, by Lemma 4.1 with $\beta = 1$ and $x = n$, there exist $K > 0$ and $\delta \in (0, 1 - \alpha)$ such that

$$
d_1 \left( \log Y_n, \log(n\eta) \right) \leq Kn^{-(\alpha + \delta)} + \sum_{j=1}^{n-1} p_j d_1 \left( \log Y_{n-j}, \log(n-j)\eta \right).
$$

Using 1-arithmetic variant of Theorem 1 in [1] and also Theorem B in [2] if $\alpha \in (0, 1/2]$ (see also Theorem 1 in [5]), we obtain

$$
d_1 \left( \log Y_n, \log(n\eta) \right) = O(n^{-\delta}), \ n \to \infty.
$$

The proof is complete.

\[\square\]

3.1 Proof of Theorem 1.1

It is enough to prove Theorem 1.1 for $V_n$ replaced by $X_n$. In view of (Conv) property of $d_1$, in order to prove Theorem 1.1 we need to check

$$
d_1 \left( \frac{X_n - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}}, \mathcal{N}(0, 1) \right) \to 0, \ n \to \infty.
$$

Using the triangle inequality yields for $n \geq 2$,

$$
\begin{align*}
\begin{aligned}
d_1 \left( \frac{X_n - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}}, \mathcal{N}(0, 1) \right) &\leq d_1 \left( \frac{X_n - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}}, \frac{W_n - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}} \right) + d_1 \left( \frac{W_n - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}}, \mathcal{N}(0, 1) \right) \\
&= d_1 \left( \frac{X_n - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}}, \frac{W_n - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}} \right) + d_1 \left( \frac{\mu_{\log n} + 1 - \mu_a^{-1} \log n}{\sqrt{\sigma_a^2 \mu_a^{-3} \log n}}, \mathcal{N}(0, 1) \right)
\end{aligned}
\end{align*}
$$

The second term converges to zero in view of the CLT for the renewal process with finite variance (see Chapter XI.5 in [4]) as well as the convergence of first absolute moments (see Proposition A.1 in [9]). From (Lin) property of $d_1$ we see that it is enough to prove

$$
d_1(X_n, W_n) = O(1), \ n \to \infty.
$$

(11)
Using the recursions for $X_n$ and $W_n$ we have, in view of (Lin) property of $d_1$,

\[ t_n := d_1(X_n, W_n) = d_1(X'_n, W'_{nn}) \leq d_1(W'_{nn}, W'_{Y_n}) + d_1(W'_{Y_n}, X'_n) + \mathbb{E}|\hat{W}_{Y_n} - \hat{X}_{Y_n}| \]

\[ =: c_n + \sum_{k=2}^{n} \mathbb{P}\{Y_n = k\} \mathbb{E}|\hat{X}_k - \hat{W}_k|, \]

for arbitrary pairs $\{(\hat{X}_k, \hat{W}_k) : 2 \leq k \leq n\}$ independent of $Y_n$ such that $\hat{X}_k \overset{d}{=} X_k$, $\hat{W}_k \overset{d}{=} W_k$. Passing to infimum over all such pairs in both sides of inequality leads to

\[ t_n \leq c_n + \sum_{k=2}^{n} \mathbb{P}\{Y_n = k\} t_k. \quad (12) \]

In order to estimate $c_n$ we proceed as follows. Let $(\hat{Y}_n, \hat{\eta})$ be a coupling of $Y_n$ and $\eta$ such that $d_1(\log Y_n, \log(n\eta)) = \mathbb{E}|\log \hat{Y}_n - \log(n\hat{\eta})|$. Let $(\hat{\nu}_t)_{t \in \mathbb{R}}$ be a copy of $(\nu_t)_{t \in \mathbb{R}}$ independent of $(\hat{Y}_n, \hat{\eta})$. We have

\[ c_n = d_1(W'_{Y_n}, W'_{nn(a)}) = d_1(\hat{\nu}_{\log \hat{Y}_n} + 1_{\{\hat{y}_n > 1\}}, \hat{\nu}_{\log(n\hat{\eta})} + 1_{\{n\hat{\eta} > 1\}}) \leq \mathbb{E}|\hat{\nu}_{\log \hat{Y}_n} + 1_{\{\hat{y}_n > 1\}} - \hat{\nu}_{\log(n\hat{\eta})} - 1_{\{n\hat{\eta} > 1\}}| \leq \mathbb{E}|\hat{\nu}_{\log \hat{Y}_n} - \hat{\nu}_{\log(n\hat{\eta})}| + \mathbb{P}\{Y_n = 1\} + \mathbb{P}\{n\eta \leq 1\} \]

where the penultimate inequality follows from the definition of $d_1$, since $(\hat{Y}_n, \hat{\eta}, (\hat{\nu}(t)))$ is a particular coupling. There exists $\rho > 0$ such that the last two summands are $O(n^{-\rho})$.

To bound the first term, we apply the distributional subadditivity of $(\nu_t)$:

\[ \nu_{x+y} - \nu_x \overset{d}{=} \nu_y, \quad x, y \in \mathbb{R}, \]

which yields

\[ c_n \leq \mathbb{E}\hat{\nu}_{|\log \hat{Y}_n - \log(n\hat{\eta})|} + O(n^{-\rho}). \quad (13) \]

Note that for every $x \geq 0$,

\[ \mathbb{P}\{T_1 \leq x\} \leq \mathbb{E}\nu_x = \sum_{k=1}^{\infty} \mathbb{P}\{T_k \leq x\} \leq \sum_{k=1}^{\infty} (\mathbb{P}\{T_1 \leq x\})^k = \frac{\mathbb{P}\{T_1 \leq x\}}{\mathbb{P}\{T_1 > x\}}, \]

hence, by the standard sandwich argument,

\[ \lim_{x \downarrow 0} \frac{\mathbb{E}\nu_x}{x^\alpha} = \frac{\sin \pi \alpha}{\pi \alpha}. \]

On the other hand, from the elementary renewal theorem we have

\[ \lim_{x \to \infty} \frac{\mathbb{E}\nu_x}{x} = \frac{1}{\mathbb{E}T_1}, \]

therefore there exist constants $c_1, c_2 > 0$ such that for all $x \geq 0$,

\[ \mathbb{E}\nu_x \leq c_1 x^\alpha + c_2 x. \quad (14) \]
Using (14) and (13) we obtain
\begin{align*}
c_n & \leq c_1 E|\log \hat{Y}_n - \log (n\eta)| + c_2 E|\log \hat{Y}_n - \log (n\eta)| + O(n^{-\rho}) \\
& \leq c_1 d_1^\alpha (\log Y_n, \log (n\eta)) + c_2 d_1 (\log Y_n, \log (n\eta)) + O(n^{-\rho}).
\end{align*}

By Lemma 3.2 we conclude $c_n = O(n^{-\rho'})$ for some $\rho' > 0$ as $n \to \infty$.

It remains to apply Lemma A.1 from [6] with $\phi_n \equiv 1$ to (12) to conclude that
\[ t_n = O\left( \sum_{k=1}^{n} \frac{k^{-\rho'}}{k} \right) = O(1), \quad n \to \infty. \]

The proof of Theorem 1.1 is complete.

4 Appendix

The next lemma is the main ingredient in the proof of Proposition 3.2.

Lemma 4.1. Assume that $\theta$ is a random variable on $[1, +\infty)$ such that for some $c > 0$, $\alpha \in (0, 1)$ and $\varepsilon > 0$
\[ 1 - F_\theta(x) := P\{\theta \geq x\} = cx^{-\alpha} + O(x^{-(\alpha+\varepsilon)}), \quad x \to \infty. \quad (15) \]

Let $\eta$ be a random variable with density $(3)$ independent of $\theta$. Then for every $\beta > 0$ there exists $\delta > 0$ such that
\[ d_1 \left( \log((1 - \theta x^{-1})\eta)1_{\{\theta < x^{-1}\}}, \log \eta \right) = O(x^{-(\alpha+\delta)}), \quad x \to \infty. \quad (16) \]

Proof. Denote the left-hand side of (16) by $s_\theta(x, \beta)$. In view of relations
\[ s_\theta(x, \beta) = s_{c^{-1/\alpha}(c^{-1/\alpha} x, c^{-1/\alpha} \beta)}, \quad x \geq 1, \]
and
\[ P\{c^{-1/\alpha} \theta \geq x\} = x^{-\alpha} + O(x^{-(\alpha+\varepsilon)}), \quad x \to \infty, \]
it is enough to prove the result for $c = 1$. Fix $\beta$ for the rest of the proof. Using representation (Int) from Proposition 3.1 we have
\begin{align*}
s_\theta(x, \beta) &= \int_{-\infty}^{0} \left| P\{\log(1_{\theta \geq x^{-1}}) + (1 - \theta x^{-1})\eta 1_{\{\theta < x^{-1}\}} \leq z\} - P\{\log \eta \leq z\} \right| dz \\
&= \int_{0}^{1} \left| P\{1_{\theta \geq x^{-1}} + (1 - \theta x^{-1})\eta 1_{\{\theta < x^{-1}\}} \leq z\} - P\{\eta \leq z\} \right| z^{-1} dz.
\end{align*}

Integrating by parts the first probability in the integrand, we obtain for $z \in [0, 1)$ and $x > 1 + \beta$,
\begin{align*}
P\{1_{\theta \geq x^{-1}} + (1 - \theta x^{-1})\eta 1_{\{\theta < x^{-1}\}} \leq z\}
&= -\int_{1^{-}}^{1_{x^{-1}}} P\{(1 - yx^{-1})\eta \leq z\} d(1 - F_\theta(y)) \\
&= -P\{\eta \leq \beta^{-1} x z\} \left( 1 - F_\theta((x - \beta) -) \right) + P\{\eta \leq z x(x - 1)^{-1}\} \\
&+ \int_{1^{-}}^{1_{x^{-1}}} (1 - F_\theta(y)) dy P\{(1 - yx^{-1})\eta \leq z\}.
\end{align*}
Let $\theta_\alpha$ be a random variable independent of $\eta$ and with distribution

$$1 - F_{\theta_\alpha}(x) := \mathbb{P}\{\theta_\alpha \geq x\} = x^{-\alpha}, \ x \geq 1.$$  

By the same reasoning as above,

$$\mathbb{P}\{1_{\{\theta_\alpha \geq x - \beta\}} + (1 - \theta_\alpha^{-1})\eta 1_{\{\theta_\alpha < x - \beta\}} \leq z\} = -\int_{[1,x-\beta]} \mathbb{P}\{1 - yx^{-1}\eta \leq z\}d(1 - F_{\theta_\alpha}(y)) = -\mathbb{P}\{\eta \leq \beta^{-1}xz\}(1 - F_{\theta_\alpha}((x - \beta))) + \mathbb{P}\{\eta \leq zx(x - 1)^{-1}\} + \int_{[1,x-\beta]} (1 - F_{\theta_\alpha}(y))dy\mathbb{P}\{(1 - yx^{-1})\eta \leq z\}.$$

Subtracting the corresponding equations and using (15) we have for $z \in [0, 1)$ and $x > 1 + \beta$,

$$\left|\mathbb{P}\{1_{\{\theta_\alpha \geq x - \beta\}} + (1 - \theta_\alpha^{-1})\eta 1_{\{\theta_\alpha < x - \beta\}} \leq z\} - \mathbb{P}\{1_{\{\theta_\alpha \geq x - \beta\}} + (1 - \theta_\alpha^{-1})\eta 1_{\{\theta_\alpha < x - \beta\}} \leq z\}\right|$$

$$\leq K\left(\mathbb{P}\{\eta \leq \beta^{-1}xz\}(x - \beta)^{-(\alpha + \varepsilon)} + \int_{[1,x-\beta]} y^{-(\alpha + \varepsilon)}dy\mathbb{P}\{(1 - yx^{-1})\eta \leq z\}\right),$$

for some $K > 0$ which does not depend on $x$ and $z$. Therefore,

$$s_\theta(x, \beta)$$

$$\leq \int_0^1 \left|\mathbb{P}\{1_{\{\theta_\alpha \geq x - \beta\}} + (1 - \theta_\alpha^{-1})\eta 1_{\{\theta_\alpha < x - \beta\}} \leq z\} - \mathbb{P}\{\eta \leq z\}\right|z^{-1}dz$$

$$+ K \int_0^1 z^{-1}\mathbb{P}\{\eta \leq \beta^{-1}xz\}(x - \beta)^{-(\alpha + \varepsilon)}dz$$

$$+ K \int_0^1 z^{-1}\int_{[1,x-\beta]} y^{-(\alpha + \varepsilon)}dy\mathbb{P}\{(1 - yx^{-1})\eta \leq z\}dz =: I_1(x) + I_2(x) + I_3(x).$$

Firstly we calculate $I_2(x)$ explicitly as follows:

$$I_2(x) = K(x - \beta)^{-(\alpha + \varepsilon)} \int_0^1 \mathbb{P}\{\eta \leq \beta^{-1}xz\}z^{-1}dz$$

$$= K(x - \beta)^{-(\alpha + \varepsilon)} \int_0^{\beta x^{-1}} \mathbb{P}\{\eta \leq \beta^{-1}xz\}z^{-1}dz + K(x - \beta)^{-(\alpha + \varepsilon)}(\log x - \log \beta)$$

$$= K(x - \beta)^{-(\alpha + \varepsilon)} \int_0^1 \mathbb{P}\{\eta \leq z\}z^{-1}dz + K(x - \beta)^{-(\alpha + \varepsilon)}(\log x - \log \beta)$$

$$= K(x - \beta)^{-(\alpha + \varepsilon)}(\mathbb{E}\log \eta + \log x - \log \beta) = O(x^{-(\alpha + \varepsilon)\log x}).$$

Pick $\varepsilon' \in (0, \varepsilon]$ such that $\alpha + \varepsilon' < 1$. The third summand $I_3(x)$ is estimated using the
Fubini’s theorem:

\[
I_3(x) \leq K \int_0^1 z^{-1} \int_{[1,x-\beta]} y^{-(\alpha+\varepsilon')} d\eta \mathbb{P}\{(1- y x^{-1}) \eta \leq z\} dz
\]

\[
= K \int_0^1 z^{-1} \int_{[1,x-\beta]} y^{-(\alpha+\varepsilon')} \mathbb{P}\{(1- \eta^{-1} z) x \in dy\} dz
\]

\[
= \int_0^1 z^{-1} \mathbb{E}\left[(1- \eta^{-1} z) x\right]^{-(\alpha+\varepsilon')} 1_{\{1 \leq (1- \eta^{-1} z) x \leq x-\beta\}} dz
\]

\[
= K x^{-(\alpha+\varepsilon')} \int_0^{\eta^{-1}(1-x^{-1})} z^{-1} (1- \eta^{-1} z)^{-(\alpha+\varepsilon')} 1_{\{1 \leq (1- \eta^{-1} z) x \leq x-\beta\}} dz
\]

\[
= K x^{-(\alpha+\varepsilon')} \int_{\beta x^{-1}}^{1-x^{-1}} u^{-1} (1- u)^{-(\alpha+\varepsilon')} du = O(x^{-(\alpha+\varepsilon')} \log x).
\]

It remains to bound the first integral. To this end, note that for every \( z \in [0, 1) \) and \( x \geq 1 + \beta \),

\[
\{1_{\{\theta \geq x-\beta\}} + (1 - \theta x^{-1}) \eta 1_{\{\theta < x-\beta\}} \leq z\} = \{(1- \eta^{-1} z) x \leq \theta < x - \beta\},
\]

and therefore

\[
\mathbb{P}\{1_{\{\theta \geq x-\beta\}} + (1 - \theta x^{-1}) \eta 1_{\{\theta < x-\beta\}} \leq z\}
\]

\[
= \mathbb{P}\{(1- \eta^{-1} z) x \leq \theta < x - \beta\}
\]

\[
= \mathbb{P}\{(1- \eta^{-1} z) x \vee 1 \leq \theta < x - \beta\}
\]

\[
= \mathbb{P}\{\eta \leq \beta^{-1} x z, ((1- \eta^{-1} z) x) \vee 1 \leq \theta < x - \beta\}
\]

\[
= \mathbb{P}\{\eta \leq \beta^{-1} x z, ((1- \eta^{-1} z) x) \vee 1 \leq \theta < x\} - \mathbb{P}\{\eta \leq \beta^{-1} x z\}((1- \eta^{-1} z) x) - \mathbb{P}\{\eta \leq \beta^{-1} x z\}((x- \beta)^{-\alpha} - x^{-\alpha}).
\]

Putting this into \( I_1(x) \) yields

\[
I_1(x) \leq \int_0^1 \mathbb{P}\{\eta \leq \beta^{-1} x z, ((1- \eta^{-1} z) x) \vee 1 \leq \theta < x\} - \mathbb{P}\{\eta \leq z\}|z^{-1} dz
\]

\[
+ ((x- \beta)^{-\alpha} - x^{-\alpha}) \int_0^1 z^{-1} \mathbb{P}\{\eta \leq \beta^{-1} x z\} dz.
\]

The second term is \( O(x^{-\alpha-1} \log x) \) by the same argument as was used in the estimation of \( I_2(x) \). Using simple algebra we obtain that the first term is equal to

\[
\int_0^1 \mathbb{P}\{z < \eta \leq (x-1)^{-1} x z\}
\]

\[
+ x^{-\alpha}\left(\int_{((x-1)^{-1} x z)^{\wedge} 1} ((1- y^{-1} z)^{-\alpha}) \mathbb{P}\{\eta \in dy\} - \mathbb{P}\{\eta \leq \beta^{-1} x z\}\right)|z^{-1} dz =: J(x).
\]
By the triangle inequality, 
\[ J(x) \leq \int_0^1 z^{-1} \mathbb{P}\{z < \eta \leq (x - 1)^{-1}zx\}dz \]
\[ + x^{-\alpha} \int_0^1 |(\beta^{-1}xz)^{\lambda_1}(1 - y^{-1}z)^{-\alpha} \mathbb{P}\{\eta \in dy\} - \mathbb{P}\{\eta \leq \beta^{-1}xz\}|z^{-1}dz. \quad (17) \]

The first summand, again by the Fubini’s theorem, is calculated easily:
\[ \int_0^1 z^{-1} \mathbb{P}\{z < \eta \leq (x - 1)^{-1}zx\}dz = E \int_0^1 z^{-1}1_{\{z<\eta\leq(x-1)^{-1}zx\}}dz \]
\[ = E \int_0^1 z^{-1}1_{\{x^{-1}(1-\eta)\leq z<\eta\}}dz \]
\[ = E \int_{x^{-1}(1-\eta)}^\eta z^{-1}dz = |\log(1-x^{-1})| = O(x^{-1}). \]

The inner integral in the second summand in rhs of (17) is equal
\[ \frac{\sin \pi \alpha}{\pi} \int_{(x-1)^{-1}xz \wedge 1}^{(z-xz) \wedge 1} u^{-\alpha}(1-u)^{-\alpha+1}du = \mathbb{P}\left\{ \frac{z}{(1-z)(x-1)} \wedge 1 \leq \eta \leq \frac{(\beta^{-1}x-1)z}{1-z} \wedge 1 \right\}. \]

Since for $z \in [0,1)$ and $x > 1 + \beta,$
\[ 0 \leq \frac{z}{(1-z)(x-1)} \wedge 1 \leq \frac{(\beta^{-1}x-1)z}{1-z} \wedge 1 \leq (\beta^{-1}xz) \wedge 1, \]
the integral in the second summand in (17) is
\[ \int_0^1 z^{-1} \mathbb{P}\left\{ \eta \leq \frac{z}{(1-z)(x-1)} \wedge 1 \right\}dz + \int_0^1 z^{-1} \mathbb{P}\left\{ \frac{(\beta^{-1}x-1)z}{1-z} \wedge 1 \leq \eta \leq (\beta^{-1}xz) \wedge 1 \right\}dz. \]

We will check that the second summand above is $O(x^{-1})$ as follows:
\[ \int_0^1 z^{-1} \mathbb{P}\left\{ \frac{(\beta^{-1}x-1)z}{1-z} \wedge 1 \leq \eta \leq (\beta^{-1}xz) \wedge 1 \right\}dz \]
\[ = E \int_0^1 z^{-1}1_{\{\eta \leq \beta^{-1}x - 1 \leq 1 + \eta\}}dz \]
\[ = E \left( \log(\beta^{-1}x) - \log(\beta^{-1}x - 1 + \eta) \right) = O(x^{-1}). \]

The first term can be treated analogously, hence $J(x) = O(x^{-1})$. Combining all the estimates we get $s_\theta(x, \beta) = O(x^{-\alpha+\delta})$ for sufficiently small $\delta > 0$. The proof is complete. \qed
References

[1] Anderson, K. K. and Athreya, K. B. (1987). A renewal theorem in the infinite mean case. *Ann. Probab.* 15, 388–393.

[2] Doney, R. A. (1997). One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* 107, 451–465.

[3] Dynkin, E. B. (1961). Some limit theorems for sums of independent random variables with infinite mathematical expectations. *Selected Transl. in Math. Statist. and Probability* 1, 171–189.

[4] Feller, W. (1970). An Introduction to Probability Theory and Its Applications. Vol 2. Secon Edition. John Wiley & Sons.

[5] Garsia, A. and Lamperti, J. (1963). A discrete renewal theorem with infinite mean. *Comment. Math. Helv.* 37, 221-234.

[6] Gnedin, A., Iksanov, A. and Marynych, A. (2011). Lambda-coalescents with dust component. *J. Appl. Prob.* 48(4), 1133–1151.

[7] Gnedin, A., Iksanov, A., Marynych, A. and Möhle, M. (2014). On asymptotics of beta-coalescents, *Adv. Appl. Prob.* 46(2), 496–515.

[8] Haas, B. and Miermont, G. (2011). Self-similar scaling limits of non-increasing Markov chains. *Bernoulli* 17, 1217–1247.

[9] Iksanov, A., Marynych, A. and Meiners, M. (2014). Limit theorems for renewal shot noise processes with eventually decreasing response functions. *Stoch. Proc. Appl.* 124, 2132–2170.

[10] Iksanov, A. and Möhle, M. (2008). On the number of jumps of random walks with a barrier. *Adv. Appl. Probab.* 40, 206–228.

[11] Iksanov, A. and Negadailov, P. (2008). On the number of zero increments of random walks with a barrier. *Discrete Math. Theor. Comput. Sci.*, Proceedings Series Volume AI, 247–254.

[12] Negadailov, P. A. (2009). Asymptotic results for the absorption times of random walks with a barrier. *Theor. Probab. Math. Statist.* 79, 127–138.

[13] Zolotarev, V. M. (1997). *Modern theory of summation of random variables*, VSP, Utrecht, The Netherlands.