Vector logic and counterfactuals

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Abstract
In this work we investigate the representation of counterfactual conditionals using the vector logic, a matrix-vectors formalism for logical functions and truth values. With this formalism, we can describe the counterfactuals as complex matrix operators that appear preprocessing the implication matrix with one of the square roots of the negation, a complex matrix. This mathematical approach puts in evidence the virtual character of the counterfactuals. The reason of this fact, is that this representation of a counterfactual proposition produces a valuation that is the superposition the two opposite truth values weighted, respectively, by two complex conjugated coefficients. This result shows that this procedure produces a uncertain evaluation projected on the complex domain. After this basic representation, the judgment of the plausibility of a given counterfactual allows us to shift the decision towards an acceptance or a refusal represented by the real vectors “true” or “false”, and we can represent symbolically this shift applying for a second time the two square roots of the negation.

Keywords: Vector logic; Conditionals; Counterfactuals; Square Root of NOT.

1. Introduction
The comprehension and further formalization of the conditionals have a long and complex history, full of controversies, famously described in an article of Lukasiewicz (1934). Plausibly, material implication finally was relatively accepted due to its advantages to formalize deductive processes and demonstration rules. In fact, Lukasiewicz built the first version of his three-valued logic with postulates based on the implication (Lukasiewicz 1920). A well known alternative to material implication was the strict implication proposed by Lewis (Lewis and Langford 1932), a modal representation for the conditionals.

A topic in which the theory of the conditionals becomes important is the search of a logical formalization of counterfactuals, as was shown in the works of Lewis (1973), Ginsberg (1986) and Rescher (2007) among many others. In this case, all the problems associated with the formalization of the conditionals become acute because counterfactuals add the no minor trouble of referring to facts or propositions placed out of the reality. This “virtuality” of counterfactual propositions produced an interesting connection of this problem with the logical theory of “possible worlds” (Lewis 1973).

In the present work, we explore the use of an operator emerged from the theory of quantum computing, the square root of NOT (symbolically, $\sqrt{\text{NOT}}$ or $\sqrt{\text{\neg}}$ ) to represent the virtual structure of conditional counterfactuals.

In our approach we translate the logic operations into the formalism of vector logic, with truth values represented by vectors and the logical functions by
matrices (Mizraji 1992, 1994, 2008). Consequently, linear algebra will be the
basic language used along the present work. Vector logic was inspired by
neurocomputational models of reasoning (Mizraji 1996, 2008, 2019; Mizraji
and Lin 2011). This formalism produces a very natural fuzzy logic when the
logical truth values are linear combinations of the basic "true" and "false"
vectors (Mizraji (1992). In addition, this linear algebra approach was relevant
in the development of fuzzy logics based on complex vectors (Dick 2005,
Yazdanbakhsh and Dick 2018).

We organize of the present work as follows. First, we present a brief review of
some of the matrix operators involved in the representation of monadic
connective, including the square root of NOT, and the dyadic connectives
disjunction and implication. Then, we show a way to describe, inside this
operator formalism, some subtle aspects of counterfactuals. Finally, we discuss
the potential extensions of this approach.

2. A brief review of vector logic

Let \( \tau = \{s, n : s, n \in \mathbb{R}^Q \} \) be a binary set of truth values composed by the two
orthonormal \( Q \)-dimensional column vectors “true”, \( s \), and “false”, \( n \). These truth
vector values are arbitrary; we only ask for orthonormality. These truth values
result from the mapping of a proposition \( \text{Prop}(u) \) on a vector truth value \( u \in \tau \).
the structure of \( s \) and \( n \), as well as their dimension \( Q \), are designed according to
the structure of the problem. In this format, and adopting the conventional
matrix expressions, the identity logical operator is a matrix \( I \in \mathbb{R}^{Q\times Q} \) given by
\[
I = nn^T + ss^T
\]
and the negation \( N \in \mathbb{R}^{Q\times Q} \) is given by
\[
N = sn^T + ns^T.
\]

The superindex \( T \) means tranposition. Remark that \( N^2 = I \).

In Mizraji (2008) is presented a general expression of the square root of \( N \), a
logical gate initially discovered in the context of quantum computing (Hayes
1995, Deutsch et al 2000). Here we adopt the following representation of the
two roots \( \sqrt{N} \):
\[
A = \sqrt{N}_1 = \frac{1}{2}(1 + i)I + \frac{1}{2}(1 - i)N, \tag{3}
\]
\[
B = \sqrt{N}_2 = \frac{1}{2}(1 - i)I + \frac{1}{2}(1 + i)N, \tag{4}
\]
with \( i = \sqrt{-1} \).

These matrices \( A \) and \( B \) have the following remarkable properties:
\[
A^2 = N; \quad B^2 = N; \tag{5}
\]
\[
AB = BA = I. \tag{6}
\]
Another interesting point is the analogy with the two squares roots of -1. The positive root \( \sqrt[3]{1} \) corresponds to
\[
\left( \sqrt[3]{1} \right)_1 = IA,
\]
and the negative root \( -\sqrt[3]{1} \) corresponds to
\[
\left( \sqrt[3]{1} \right)_2 = NA;
\]
as a consequence, \( NA = B \).

The basic table that defines the material implication \( p \rightarrow q \) and the disjunction \( p \lor q \) defined for the symbolic truth values “true”, \( t \), and “false”, \( f \), is the following:

|   |   | \( p \rightarrow q \) | \( p \lor q \) |
|---|---|-----------------|----------------|
| t | t | t               | t              |
| t | f | f               | t              |
| f | t | t               | t              |
| f | f | t               | f              |

We remind a basic equivalence that defines the material implication in terms of the disjunction: \( p \rightarrow q \equiv \neg p \lor q \).

Now, we present the matrix operators that implement the material implication \( p \rightarrow q \) and the disjunction \( p \lor q \) over the vector truth values \( s \) and \( n \). First, consider the following matrix:

\[
H = \begin{bmatrix}
(s \otimes s) & (s \otimes n) & (n \otimes s) & (n \otimes n)
\end{bmatrix}
\]

with \( u \otimes v \ (u, v \in \tau) \) being the Kronecker product. We can represent the matrices implication \( L \) and disjunction \( D \), respectively, by the following compact equations (Mizraji 1996):

\[
L = [s \ s \ s \ s]H^T,
\]
\[
D = [s \ s \ s \ n]H^T,
\]

being \( [s \ s \ s \ s] \) and \( [s \ s \ s \ n] \) partitioned matrices in \( \mathbb{R}^{Q \times 4} \). Remark that
\[
H^T \in \mathbb{R}^{4 \times Q^2}, \text{ hence, } L, D \in \mathbb{R}^{Q \times Q^2}. \]
It is interesting to note the similarity of these matrix representation with the definitions of implication and disjunction displayed in Table 5.101 of the Wittgenstein’s Tractatus (1921). In an extended version, these matrices can be written as

\[
L = s(s \otimes s)^T + n(s \otimes n)^T + s(n \otimes s)^T + s(n \otimes n)^T,
\]
\[ D = s(s \otimes s)^T + s(s \otimes n)^T + s(n \otimes s)^T + n(n \otimes n)^T. \]  

(11)

These matrices are connected by the equation \( L = D(N \otimes I) \), an operator version of the equivalence \( p \rightarrow q \equiv \neg p \lor q \).

The computing capacity of these operators is based on the properties of the Kronecker product. We only show here two of these properties (for details, see Graham 1981):

\begin{align*}
(a) & \quad (U \otimes V)^T = U^T \otimes V^T \\
(b) & \quad (U_1 \otimes V_1)(U_2 \otimes V_2) = (U_1U_2) \otimes (V_1V_2).
\end{align*}

In (a) the matrices have no dimensional restrictions; in (b) the products \( U_1U_2 \) and \( V_1V_2 \) must be possible. In the case of four column vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^Q \), we have

\[ (a \otimes b)^T(c \otimes d) = (a^Tc) \otimes (b^Td) = \langle a, b \rangle \otimes \langle c, d \rangle, \]  

(12)

with the symbol \( \langle \ , \ \rangle \) indicating the scalar product. These two scalar products are the mathematical filters that allow us to represent the dyadic logical operators in terms of linear algebra (Mizraji 1992, 1996). As a consequence, we have for the matrix implication and disjunction, the following results:

\[
\begin{align*}
L(s \otimes s) & = L(n \otimes s) = L(n \otimes n) = s \quad \text{and} \quad L(s \otimes n) = n; \\
D(s \otimes s) & = D(s \otimes n) = D(n \otimes s) = s \quad \text{and} \quad D(n \otimes n) = n.
\end{align*}
\]

### 3. Counterfactuals

The central aspect of counterfactuals is that they refer to non-existent situations and establish implicative rapports between events contradicted by the reality. This interaction with the reality put in front the issue of counterfactuals belong both to the field of logic and that of physics, including in “physics” all the events that happen in our real world (to be as clear as possible, let us do without using the word “nomological” here). This double face of counterfactual, formal and physical, explains the fact that theorizing about counterfactuals becomes mixed with theorizing about causality or the sense of time (Lewis 1973, Horwich 1989, Pearl 2000). By the moment we are going to omit this important points and to concentrate in trying to state a formalism capable to deal with the fictional character of counterfactuals. In fact, our intention is to introduce this new formalism to the wide and diverse community of researchers that approaches the subtleties of counterfactuals from so many angles.

Ginsberg (1986) proposed the following definition: “A counterfactual is a statement such as, ‘if \( p \), then \( q \),’ where the premise \( p \) is either know or expected to be false”, then he adds that “falsehood implies anything”. This author gives us two examples of counterfactuals with a different qualitative level of plausibility. On the one hand, we can state “If the electricity hadn’t failed, dinner would have been ready on time”; on the other, we can state “If the electricity hadn’t failed, pigs would fly” (Ginsberg 1986). Based on his notation,
we can write the “truth table” (assuming that this expression is acceptable in this context) of a counterfactual conditional \( p^* < q^* \) as follows:

| \( p^* \) | \( q^* \) | \( p^* < q^* \) |
|---|---|---|
| t | t | t |
| t | f | f |
| f | t | ? |
| f | f | ? |

Here, \( p^* \) and \( q^* \) are the modified counterfactual propositions derived from a true conditional \( p \rightarrow q \).

Rescher (2007) analyzes the logical possibilities of the “what .... if” situations and organizes these possibilities of creating counterfactuals in two classes:

I. “If p (which is true) were false, then what?”
II “If p (which is false) were true, then what?”

In this work, we are going to explore a minimalist approach that focuses a restricted category of counterfactuals. First, we define the "factual implication" FI as an implication necessarily true. This assumption is based on the nature of our real world that shows factual conditionals, like "if we don't eat then we die", empirically true. The reality of this conditional can, in fact, be considered a definition of "true". Then, we are going to assume that the counterfactual implication associated to a particular FI, CF, is a priori true. Hence, the paradigm of this minimalist approach is that \( p \rightarrow q^* \) "true" becomes \( p^* < q^* \) "hypothetically true". The heuristic behind the last supposition is the following.

In our cognitive life, we construct alternatives looking for plausible conjectural conditionals. These counterfactuals are created usually to investigate interesting organizational, economical, sociological, or technological strategic possibilities not existent yet. The counterfactual builders usually try to explore future possible real situations. For this reason we can tentatively suppose an a priori hypothetical true for any counterfactual. In the following subsection a refined way to judge the value of this tentative valuation is described.

Using the matrix formalism, the link between factual and counterfactual conditionals looks as follows:

\[
L(p \otimes q) = s \xrightarrow{\text{counterfactual}} L^c(p^* \otimes q^*) \triangleq s
\]

with \( L^c \) being the counterfactual conditional, \( p^*, q^* \in \tau \) and the symbol \( \triangleq \) is used here to represent the conjectural a priori assumed truth value. This format excludes counterfactuals with \( p^* = s \) and \( q^* = n \).
3.1. Virtualization

Taking into account that counterfactuals don’t belong to the real world, in order to transform these statements into virtual propositions, that we symbolize by CF(v), we can exploit the properties of the $\sqrt{NOT}$ and premultiply CF by the matrix A or the matrix B (in this stage, it is not relevant the selected root). If we select A, the results of this “virtualization” are the following:

(a) $\text{CF}(v): \text{AL}^c(s \otimes s) = \text{As}$
(b) $\text{CF}(v): \text{AL}^c(n \otimes s) = \text{As}$
(c) $\text{CF}(v): \text{AL}^c(n \otimes n) = \text{As}$

Note that $L \in \mathbb{R}^{Q \times Q^2}$ but $\text{AL}^c, \text{BL}^c \in \mathbb{C}^{Q \times Q^2}$. Particularly important is what happen with the truth value of these virtual conditionals. Remark that

$$\text{As} = \frac{1}{2}(1+i)s + \frac{1}{2}(1-i)n,$$
$$\text{Bs} = \frac{1}{2}(1-i)s + \frac{1}{2}(1+i)n.$$  \hspace{1cm} (16)  

This is an suggesting result because in this formalism the uncertainty of a counterfactual valuation is represented by the splitting, by the matrices A and B, of the truth value associated to the counterfactual conditional into the two superimposed complex truth values, $\frac{1}{2}(1 \pm i)s$, a complex version of “true”, and $\frac{1}{2}(1 \mp i)n$, the complex version of “false”. In the virtual world, both complex truth values are equally valid. Hence, under this representation, the counterfactual plausibility is uncertain.

3.2. Back to real

Given a virtual conditional expressed by a counterfactual, $p^* < q^*$ derived from a real conditional $p \rightarrow q$, we need to explore its level of plausibility. The evaluation of the sources of plausibility have been soundly analyzed by many authors using different approaches and methods (e.g.: Goodman 1946, Lewis 1973, Horwich 1989, Pearl 2000, Bochman 2018). Here, our criteria of plausibility is double. On the one hand, we require that the hypothetical relation between the antecedent $p^*$ and consequent $q^*$ can happen in the real world. On the other hand, we need this hypothetical implication to be logically consistent. Let us assume that there are a set including the plausible factual evidence F and a set of logical laws C. Now, given the counterfactual propositions $p^*$ and $q^*$, we are going to use the logical formula (18) to evaluate the plausibility of the counterfactual:

$$\text{PL}(p^*, q^*) = \left[ (p^* \in F) \land (q^* \in F) \right] \land \left[ (p^* \in C) \land (q^* \in C) \right].$$  \hspace{1cm} (18)
Based on this formula we assume that a counterfactual is plausible if

\[ PL(p^*,q^*) = \text{true} \quad \text{and implausible if} \quad PL(p^*,q^*) = \text{false}. \]

If we consider the counterfactual “If the electricity hadn’t failed, then dinner would have been ready on time” under the form \( p^* \wedge q^* \), we see that the antecedent and the consequent are both consistent with possible empirical facts \( F \), and that they are not inconsistent with any formal logic law. Hence, the three conjunctions of eq. (18) are true and the final evaluation is true. Consequently, we can consider plausible this counterfactual. In the case of “If the electricity hadn’t failed, then pigs would fly”, it is clear that the second part of this counterfactual, \( q^* \), is not real. Consequently we can assume that \( q^* \not\in F \), and being the first conjunction false, the whole expression (18) is false. And we can assume implausible this second counterfactual. Remark that the factual implication associated with the previous counterfactuals can be expressed as follows (1) “If the electricity fails, then dinner cannot be ready on time”, and (2) “If the electricity fails, then pigs cannot fly”. In both cases, the associated factual conditionals \( FI \) have the same structure: \( L(p \odot q) = L(s \odot s) \). This is because in both cases the propositions \( p \) and \( q \) are factually plausible. And the associated counterfactuals result from the negation of both \( p \) and \( q \) given \( L^c(p \odot q^*) = L^c(n \odot n) \).

A nice aspect of the formal virtualization generated by the square roots of NOT, is that we have a simple way to turn the virtual (and neutral) evaluations given in equations (16-17) into real evaluations established according some plausibility criteria. The key is to premultiply the virtual expressions by one of the square roots of NOT, \( A \) or \( B \), according to the structure of the virtual counterfactual and the decision about plausibility or implausibility.

To define a way to project the level of plausibility into the matrix formalism, we proceed as follows. First we notice that matrices \( A \) and \( B \) are complex conjugate: \( A^* = B \) and conversely \( B^* = A \). Now we can define a plausibility function for a given counterfactual \( CF \) by:

\[
\text{pl}(CF(v)) = \begin{cases} 
X^* [XL(p^* \odot q^*)] & \text{if } PL(p^*,q^*) = \text{true} \\
X [XL(p^* \odot q^*)] & \text{if } PL(p^*,q^*) = \text{false}
\end{cases}
\]  

(19)

with \( X \in \{A,B\} \).

We illustrate these evaluations presenting an example of counterfactuals having the same syntactic structure but different plausibility. We begin quoting the insightful comment of Emerson (1850) about the nature of genius, included in his essay “Shakespeare, or the Poet”. Emerson wrote “There is no choice to genius. (...) he finds himself in the river of the thoughts and events, forced onward by the ideas and necessities of his contemporaries. He stands where all
the eyes of men look one way, and their hands all point in the direction in which he should go.” Let us assume that Emerson is right and that any genius fulfill necessities of its own epoch. However, there is an important difference between artistic genius and scientific genius: the creation of the genial artist would be impossible to recreate if he had not existed, but the creation of a genial scientist would be very likely to be recreated if he had not existed. With this in mind, we present the following two counterfactuals referred to persons considered genius of their times:

(a) If Jorge Luis Borges had not been born, the story “Death and the Compass” would not have been written.
(b) If Thomas Willis had not been born, the cerebral arterial circle would not have been discovered.

Both can be represented by a conditional \( p \rightarrow q \) and a counterfactual resulting from the negation of the antecedent and the consequent, \( p^* < q^* \), with \( p^* = \neg p \) and \( q^* = \neg q \). We can associate to these counterfactuals the corresponding FI and the associated counterfactuals CF and virtual counterfactuals CF(v):

\[
\text{FI:} \quad L(s \otimes s) = s, \\
\text{CF:} \quad L^c(n \otimes n), \\
\text{CF(v):} \quad A L^c(n \otimes n) = A s = \frac{1}{2}(1 + i)s + \frac{1}{2}(1 - i)n.
\]

But these counterfactuals with the same syntactical structure, have different plausibility. The first, is almost sure. The factual probability that other person wrote the same story, in Spanish, with exactly the same words is almost zero, and consequently, the probability of \( q^* \) is almost one. Consequently our criteria of plausibility (18-19) give true and we can render true our counterfactual (a). To do this, we need to premultiply the virtual counterfactual by matrix B, resulting

\[
\text{pl(CF(a))} = B A L^c(n \otimes n) = B A s = s.
\]

In the second case the situation is different. In the time of Thomas Willis (1621-1675) the anatomical research was intense and it is almost sure that the beautiful arterial structure, named frequently “Willis polygon”, would have been equally discovered if Willis had not existed. In this case, the probability if a plausible reality of \( q^* = \neg q \) is near zero and according to our equations (18-19) the counterfactual is false. Consequently, we must negate our virtual counterfactual (b). In terms of the matrix format, we should have

\[
\text{pl(CF(b))} = A A L^c(n \otimes n) = A A s = n.
\]

It is clear that the decision on counterfactual plausibility depends on aspects that are outside its formal structure.
4. Discussion

The representation of material implications in terms of matrix operators, allows us to represent the virtual nature of counterfactuals using the matrix vector formalism. In this virtual representation, one of the square roots of NOT matrices pushes the eventual real vector truth values into the domain of complex vectors, generating a superposition of opposite truth values. This fact provides us with a nice representation of the two alternatives *a priori* compatible with the counterfactual.

Of course, this work does not pretend to describe the way cognitive neural systems try to solve the decisions concerning counterfactuals. It is clear that our minds can deal with counterfactuals, and that they are constantly present in our social life. The actual knowledge about brain function shows that the neural processing of complex information is carried out by large neural networks with the ability to process in parallel a large amount of complex information (Kohonen 1977, Rumelhart and McClelland 1986, Anderson 1995, Arbib 1995, Sporns 2010, Mizraji and Lin 2011). These large neural systems, that in fact model accurately a good deal of our cognitive functions, are capable to map into a memory a fictional counterfactual and to sustain the two contradictory options of the eventual counterfactual decision. For the modeling of these neural abilities, the complex quantum operators do not appear as a physical necessity. Hence, in the present work, the square roots of NOT are used as an interesting formal devices, not as a cognitive model. However, there is now a large amount of neural research that uses the statistical properties of quantum operators to analyze results emerged from experiments in the domain of cognitive sciences. This is territory of intensive research that exploits formalisms born in quantum physics (for a comprehensive introduction to these approaches, see Blutner and beim Graben 2016).

The research on counterfactuals shows nowadays an extraordinary expansion. As stated earlier, the purpose of this work is not to delve into the vast theoretical corpus about counterfactuals. The main objective of this article is to propose another instrument to be added to the large repertoire of theoretical methods used in this research area.

Finally, a comment about the introduction of the square roots of NOT in basic logic. It is well known that the introduction of complex numbers in many branches of science turned out to be a beneficial procedure to enlarge the technical power of a theory and to obtain unified concepts. In this sense, a possible extension of this work is to study the representation of the logical connectives and the associated tautologies using a matrix format with the square roots on NOT acting as logical variables. As an example, we comment that we can express the disjunction as $D(X,Y) = D(XB \oplus YN)$, being the truth values $s$ and $n$ parameters, and the variables being $X,Y \in \{A,B\}$. As a consequence, we can express all the other logical operations taking into account that $\text{NOR}(X,Y) = ND(X,Y)$. If we think in logical circuits, the fact that the gates are physically implementable, including the $\sqrt{\text{NOT}}$ (see Hayes 1995 and Deutsch et al. 2000), may add a further interest to the possibility to built logical circuits with the truth values acting as parameters and the gates as switching variables.
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