Proof of some congruence conjectures of Z.-H. Sun involving Apéry-like numbers

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ABSTRACT
In this paper, we mainly prove the following conjecture of Sun [Congruences involving binomial coefficients and Apery-like numbers, Publ. Math. Debrecen 96 (2020), pp. 315–346]: Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{3k + 1}{(-16)^k} f_k \equiv (-1)^{(p-1)/2} p + p^3 E_{p-3} \pmod{p^4},$$

where $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ and $E_n$ stand for the $n$th Franel number and $n$th Euler number, respectively.

1. Introduction

In 1894, Franel [3] found that the numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots)$$

satisfy the recurrence relation (cf. [16, A000172]):

$$(n + 1)^2 f_{n+1} = (7n^2 + 7n + 2) f_n + 8n^2 f_{n-1} \quad (n = 1, 2, 3, \ldots).$$

These numbers are now called Franel numbers. Callan [2] found a combinatorial interpretation of the Franel numbers. The Franel numbers play important roles in combinatorics and number theory. The sequence $(f_n)_{n \geq 0}$ is one of the five sporadic sequences (cf. [28, Section 4]) which are integral solutions of certain Apéry-like recurrence equations and closely related to the theory of modular forms. In 2013, Sun [24] revealed some unexpected connections between numbers $f_n$ and representations of primes $p \equiv 1 \pmod{3}$ in
the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, for example, Sun [24, (1.2)] showed that
\[
\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.
\]

For more studies on Franel numbers, we refer the readers to Refs [6, 7, 10, 13, 23, 24].

For $n \in \mathbb{N}$, define
\[
H_n := \sum_{0 < k \leq n} \frac{1}{k}, \quad \text{and} \quad H_n^{(2)} := \sum_{0 < k \leq n} \frac{1}{k^2} \quad \text{with} \quad H_0 = H_0^{(2)} = 0,
\]
where $H_n$ with $n \in \mathbb{N}$ are often called the classical harmonic numbers. Let $p > 3$ be a prime. Wolstenholme [27] proved that
\[
H_{p-1} \equiv 0 \pmod{p^2} \quad \text{and} \quad H_{p-1}^{(2)} \equiv 0 \pmod{p},
\]
which implies that
\[
\binom{2p - 1}{p - 1} \equiv 1 \pmod{p^3}.
\]

In 1990, Glaisher [4, 5] showed further that
\[
\binom{2p - 1}{p - 1} \equiv 1 - \frac{2}{3}p^3B_{p-3} \pmod{p^4},
\]
where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. (See Ref. [8] for an introduction to Bernoulli numbers).

Recall that the Euler numbers $\{E_n\}$ are defined by
\[
\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}).
\]

In Ref. [19], Z.-H. Sun studied some Apéry-like numbers, he got many beautiful congruences involving these numbers and proposed a lot of conjectures at the end of the paper, one of the conjectures is,

**Conjecture 1.1** ([19, Conjecture 4.29]): Let $p > 3$ be a prime. Then
\[
\sum_{k=0}^{p-1} \binom{2k}{k} \frac{3k + 1}{(-16)^k} f_k \equiv (-1)^{(p-1)/2}p + p^3E_{p-3} \pmod{p^4}.
\]

**Remark 1.1:** In Ref. [20, Conjecture 4.23], Sun conjectured the congruence in Conjecture 1.1 modulo $p^2$ and Guo [7] proved it modulo $p^3$.

In this paper, our first goal is to prove the above conjecture.
Theorem 1.1: Conjecture 1.1 is true.

Recall that another Apéry-like numbers

\[ T_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2 \quad (n = 0, 1, 2, \ldots) \]

which satisfy the recurrence relation:

\[ (n + 1)^3 T_{n+1} = (2n + 1)(12n(n + 1) + 4)T_n - 16n^3 T_{n-1} \quad (n \geq 1). \]

Sun [18, Theorem 2.1] obtained that for any nonnegative integer \( n \),

\[ T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n + 2k}{4k} 4^{n-2k}. \quad (4) \]

In the same paper, he showed that for any prime \( p > 3 \), we have

\[ \sum_{k=0}^{p-1} (2k + 1) \frac{T_k}{4^k} \equiv p \pmod{p^4} \]

and

\[ \sum_{k=0}^{p-1} (2k + 1) \frac{T_k}{(-4)^k} \equiv (-1)^{(p-1)/2}p \pmod{p^3}. \]

Our second goal is to prove the following two congruences which were conjectured by Z.-H. Sun [18, Conjecture 2.4]:

Theorem 1.2: Let \( p > 3 \) be a prime. Then

\[ \sum_{k=0}^{p-1} (2k + 1) \frac{T_k}{4^k} \equiv p + \frac{7}{6} p^4 B_{p-3} \pmod{p^5} \quad (5) \]

and

\[ \sum_{k=0}^{p-1} (2k + 1) \frac{T_k}{(-4)^k} \equiv (-1)^{(p-1)/2}p + p^3 E_{p-3} \pmod{p^4}. \quad (6) \]

Remark 1.2: A few days after I submitted this paper to arXiv, Z.-H. Sun told me that the two congruences in Theorem 1.2 have been proved by Liu [11], and then I found that our proof differs from his because we could obtain the results directly by using one congruence involving harmonic numbers which was obtained by the author, Wang and Wang [12] and by the results in Refs [18] and [20, (1.7)]. And one of the referees suggested me to delete this theorem and related paragraphs and references, while another referee thought that I keep this theorem in this paper is a reasonable point, so I think I may keep this theorem and its proof in this paper.

We are going to prove Theorem 1.1 in Section 2. And the last Section is devoted to proving Theorem 1.2. Our proofs make use of some new combinatorial identities which were found by the package Sigma [15] via the software Mathematica.
2. Proof of Theorem 1.1

We need the following identity due to Sun [24]:

$$f_n = \sum_{k=0}^{n} \binom{n + 2k}{3k} \binom{2k}{k} \binom{3k}{k} (-4)^{n-k}. \tag{7}$$

We also rely on the following identity which can be proved by induction on $n$:

$$\sum_{k=j}^{n-1} \frac{3k + 1}{4^k} \binom{2k}{k} \binom{k + 2j}{3j} = \frac{2n(3j + 1) \binom{n + 2j}{3j+1} \binom{2n}{2j}}{(2j + 1)4^n}. \tag{8}$$

By loading the package Sigma in the software Mathematica, we find an interesting and useful identity:

**Lemma 2.1:** For any positive integer $n$, we have

$$\sum_{j=0}^{n-1} \binom{n}{j} (-1)^j \frac{H_{2j} - H_j}{2j + 1}$$

$$= \frac{4^n}{2(2n + 1)\binom{2n}{n}} \sum_{k=1}^{n} \binom{2k}{k} 4^k + \frac{(-1)^n H_n}{2n + 1} - \frac{2nH_n}{2(2n + 1)\binom{2n}{n}} - \frac{(-1)^n H_{2n}}{2n + 1}.$$

**Proof:** We can prove the above identity by using the package Sigma [15] in Mathematica as follows, other similar identities in the whole paper also can be proved in the same way. We first insert

\begin{verbatim}
In[1] := mysum = SigmaSum[SigmaBinomial[n, k] * (-1)^k
* (SigmaHNumber[2 * k] - SigmaHNumber[k])/(2 * k + 1), {k, 0, n - 1}]
Out[1] = \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k \frac{(H_{2k} - H_k)}{2k + 1}
\end{verbatim}

Then we can obtain the recurrence for the above sum:

\begin{verbatim}
In[2] := rec = GenerateRecurrence[mysum]
Out[2] = \left\{-4(1 + n)^2 \text{SUM}[n] + 2(3 + 2n)^2 \text{SUM}[1 + n] \right. \\
- (3 + 2n)(5 + 2n)\text{SUM}[2 + n] = \frac{1}{2(2 + n)} \\
+ \frac{(19 + 24n + 8n^2)(-1)^n}{2(1 + n)(2 + n)(1 + 2n)} + \frac{(-13 - 32n - 16n^2)(-1)^n H_n}{1 + 2n} \\
+ \frac{(13 + 32n + 16n^2)(-1)^n H_{2n}}{1 + 2n} \right\}
\end{verbatim}
Now we solve the above recurrence:

\[
\text{In[3]} := \text{resol} = \text{SolveRecurrence}[\text{rec}[1], \text{SUM}[n]]
\]

\[
\text{Out[3]} = \left\{ \left\{ 0, \frac{\prod_{l=1}^{n} \frac{2l}{1+2l}}{1+2n} \right\}, \left\{ 0, \frac{H_n \prod_{l=1}^{n} \frac{2l}{1+2l}}{-1-2n} \right\}, \right. \\
\left. 1, \frac{\left( \prod_{l=1}^{n} \frac{2l}{1+2l} \right) \sum_{l=1}^{n} \prod_{l=1}^{n} \frac{-1+2l}{l}}{2(1+2n)} + (-1)^n \left( \frac{H_n}{1+2n} - \frac{H_{2n}}{1+2n} \right) \right\}
\]

At last, we obtain another form of mysum by finding the linear combination of the solutions:

\[
\text{In[4]} := \text{FindLinearCombination}[\text{resol}, \text{mysum}, 2]
\]

\[
\text{Out[4]} = \frac{\left( \prod_{l=1}^{n} \frac{2l}{1+2l} \right) \sum_{l=1}^{n} \prod_{l=1}^{n} \frac{-1+2l}{l}}{2(1+2n)} + \frac{(-1)^n H_n}{1+2n}
\]

\[
- \frac{H_n \prod_{l=1}^{n} \frac{2l}{1+2l}}{2(1+2n)} - \frac{(-1)^n H_{2n}}{1+2n}
\]

Thus we get the desired identity by simple computation. Therefore, the proof of Lemma 2.1 is finished.

Lemma 2.2: Let \( p > 2 \) be a prime. If \( 0 \leq j \leq (p - 3)/2 \), then we have

\[
(3j + 1) \binom{3j}{j} \left( \begin{array}{c} p + 2j \\ 3j + 1 \end{array} \right) \equiv p(-1)^j(1 + pH_{2j} - pH_j) \pmod{p^3}.
\]

If \( (p + 1)/2 \leq j \leq p - 1 \), then

\[
(3j + 1) \binom{2j}{j} \binom{3j}{j} \left( \begin{array}{c} p + 2j \\ 3j + 1 \end{array} \right) \equiv 2p(-1)^j \binom{2j}{j} \pmod{p^3}.
\]

Proof: If \( 0 \leq j \leq (p - 3)/2 \), then we have

\[
(3j + 1) \binom{3j}{j} \left( \begin{array}{c} p + 2j \\ 3j + 1 \end{array} \right) = \frac{(p + 2j) \cdots (p + 1)p(p - 1) \cdots (p - j)}{j!(2j)!}
\]

\[
\equiv \frac{p(2j)!(1 + pH_{2j})(-1)^j(j)!(1 - pH_j)}{j!(2j)!}
\]

\[
\equiv p(-1)^j(1 + pH_{2j} - pH_j) \pmod{p^3}.
\]
If \((p + 1)/2 \leq j \leq p - 1\), then we have \(\binom{2j}{j} \equiv 0 \pmod{p}\), and hence

\[
(3j + 1) \binom{2j}{j} \binom{3j}{j} \binom{p + 2j}{3j + 1} \equiv \binom{2j}{j} (p + 2j) \cdots (2p + 1)(2p)(2p - 1) \cdots (p + 1)p(p - 1) \cdots (p - j) \mod{2j(2j)!} \\
= \binom{2j}{j} 2^p (p + 1)(p - 1)!(-1)^j j! \mod{2p(-1)^j \binom{2j}{j} (\mod{p^3})}.
\]

Now the proof of Lemma 2.2 is complete.

In view of (7) and (8), we have

\[
\sum_{k=0}^{p-1} \frac{3k + 1}{(-16)^k} \binom{2k}{k} f_k = \frac{\sum_{k=0}^{p-1} 3k + 1}{(-16)^k} \frac{2k}{k} \sum_{j=0}^{k} \binom{k + 2j}{3j} \binom{2j}{j} (-4)^{k-j} \\
= \sum_{j=0}^{p-1} \frac{\binom{2j}{j}}{(-4)^j} \sum_{k=j}^{p-1} \frac{3k + 1}{4^k} \frac{2k}{k} \frac{(k + 2j)}{3j} \\
= \sum_{j=0}^{p-1} \frac{\binom{2j}{j}}{(-4)^j} 2p(3j + 1)(p + 2j) \frac{\binom{2p}{3j + 1}}{(2j + 1)4^j} \\
= \frac{p}{4p - 1} \frac{(2p - 1)}{p - 1} \sum_{j=0}^{p-1} \frac{\binom{2j}{j} (3j + 1)(\binom{3j}{j})(p + 2j)(\binom{3j + 1}{j})}{(2j + 1)(-4)^{j}}.
\]

This, with Lemma 2.2 yields that

\[
\sum_{k=0}^{p-1} \frac{3k + 1}{(-16)^k} \binom{2k}{k} f_k = \frac{p}{4p - 1} (S_1 + S_2 + S_3) \pmod{p^4},
\tag{9}
\]

where

\[
S_1 = p \sum_{j=0}^{(p - 3)/2} \frac{\binom{2j}{j}}{4^j} \frac{1 + pH_j}{2j + 1},
\]

\[
S_2 = \frac{\binom{2p - 1}{p - 1} (\binom{p - 1}{2})^2}{(-4)^{p - 1/2} / 4^{p}} \quad \text{and} \quad S_3 = 2p \sum_{j=(p + 1)/2}^{p - 1} \frac{\binom{2j}{j}}{(2j + 1)4^j}.
\]

It is well known the Morley’s congruence [14]:

\[
\binom{p - 1}{(p - 1)/2} \equiv (-1)\binom{p - 1}{2} 4^{p - 1} \pmod{p^3} \quad \text{for } p > 3.
\tag{10}
\]
And Calitz \cite{1} further showed that
\[
\left( \frac{p - 1}{(p - 1)/2} \right) \equiv (-1)^{(p - 1)/2} \left( 4^{p-1} + \frac{p^3}{12} B_{p-3} \right) \pmod{p^4}. \tag{11}
\]
It is easy to see that
\[
S_1 = p \sum_{j=0}^{p-3} \frac{(2j)}{(2j + 1)4^j} + p^2 \sum_{j=0}^{p-3} \frac{(2j)}{4^j} \frac{H_{2j} - H_j}{2j + 1}
\]
\[= p \sum_{j=0}^{p-3} \frac{(2j)}{(2j + 1)4^j} + p^2 \sum_{j=0}^{p-3} \binom{n}{j} (-1)^j \frac{H_{2j} - H_j}{2j + 1} \pmod{p^3}. \tag{12}
\]
We know that Tauraso \cite{26} and Sun \cite{22}, \eqref{1.5}], respectively, proved that
\[
\sum_{k=1}^{p-1} \frac{(2k)}{k4^k} \equiv - H_{(p-1)/2} \pmod{p^3}, \quad \sum_{k=\frac{p+1}{2}}^{p-1} \frac{(2k)}{k4^k} \equiv (-1)^{\frac{p-1}{2}} 2p E_{p-3} \pmod{p^2}.
\]
And it is easy to check that by \eqref{10}, we have
\[
(-1)^{\frac{p-1}{2}} \left( \frac{p - 1}{\frac{p - 1}{2}} \right) - 2^{p-1} \equiv 4^{p-1} - 2^{p-1} = 2^{p-1} p q_p(2) \equiv p q_p(2) \pmod{p^2},
\]
where \(q_p(a) = (a^{p-1} - 1)/p\) stands for the Fermat quotient.
These, with Lemma 2.1, \eqref{1}, \eqref{10} and \(H_{(p-1)/2} \equiv -2d_{p}(2) \pmod{p} \) \cite{9} yield that
\[
\sum_{j=0}^{p-3} \binom{n}{j} (-1)^j \frac{H_{2j} - H_j}{2j + 1}
\]
\[= \frac{2^{p-1} \sum_{k=1}^{p-1} \frac{(2k)}{k4^k}}{2p \left( \frac{p-1}{2} \right)} + \frac{(-1)^{\frac{p-1}{2}} H_{\frac{p-1}{2}}}{p} - \frac{2^{p-1} H_{\frac{p-1}{2}}}{2p \left( \frac{p-1}{2} \right)} - \frac{(-1)^{\frac{p-1}{2}} H_{p-1}}{p}
\]
\[\equiv \frac{((-1)^{\frac{p-1}{2}} \left( \frac{p-1}{2} \right) - 2^{p-1}) H_{p-1}}{2^{p-1} \left( \frac{p-1}{2} \right)} - E_{p-3}
\]
\[\equiv (-1)^{(p-1)/2} q_p(2) H_{(p-1)/2} - E_{p-3} \equiv (-1)^{\frac{p+1}{2}} 2q_p^2(2) - E_{p-3} \pmod{p}.
\]
Substituting this into \eqref{12} and by Sun \cite{21}, \eqref{1.1}], we have
\[
S_1 \equiv (-1)^{\frac{p+1}{2}} p q_p(2) + (-1)^{\frac{p+1}{2}} 2^2 q_p^2(2) - p^2 E_{p-3} \pmod{p^3}.
\]
This, with \(2^{p-1} = 1 + p q_p(2)\) and Fermat’s little theorem yields that
\[
\frac{p}{4^{p-1}} S_1 \equiv \frac{p}{4^{p-1}}((-1)^{\frac{p+1}{2}} p q_p(2) + (-1)^{\frac{p+1}{2}} 2^2 q_p^2(2) - p^2 E_{p-3})
\]
\[\equiv (-1)^{\frac{p+1}{2}} p^2 q_p(2) - p^3 E_{p-3} \pmod{p^4}. \tag{13}
\]
It is easy to check that by (2), (10) and \(2^{p-1} = 1 + pq_p(2)\), we have
\[
\frac{p}{4^{p-1}}S_2 \equiv (-1)^{\frac{p-1}{2}}p2^{p-1} = (-1)^{\frac{p-1}{2}}p + (-1)^{\frac{p-1}{2}}p^2q_p(2) \pmod{p^4}. \tag{14}
\]
In view of \([21, (1.2)]\), we have
\[
\frac{p}{4^{p-1}}S_3 \equiv 2p^3E_{p-3} \pmod{p^4}.
\]
Therefore, combining this with (9), (13) and (14), we immediately obtain the desired result
\[
\sum_{k=0}^{p-1} \binom{2k}{k} \frac{3k+1}{(-16)^k} f_k \equiv (-1)^{\frac{p-1}{2}}p + p^3E_{p-3} \pmod{p^4}.
\]

3. Proof of Theorem 1.2

In Ref. [25, (1.7)], Sun showed that
\[
\frac{p-3}{2} \sum_{k=0}^{\frac{p-1}{2}} \left(\frac{2k}{k}\right)^2 (2k+1)16^k = -2q_p(2) - pq_p^2(2) + \frac{5}{12} p^2B_{p-3} \pmod{p^3} \tag{15}
\]
and in Ref. [20, (1.7)], he proved that
\[
\sum_{k=0}^{\frac{p-1}{2}} \left(\frac{2k}{k}\right)^2 16^k \equiv (-1)^{\frac{p-1}{2}} + p^2E_{p-3} \pmod{p^3}. \tag{16}
\]
In view of the following congruence which was confirmed by the author, Wang and Wang [12]
\[
\sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)^2}{k16^k} H_{2k}^{(2)} \equiv -\frac{5}{2} B_{p-3} \pmod{p},
\]
we can immediately obtain the following important lemma.

**Lemma 3.1:** For any prime \(p > 3\), we have
\[
\sum_{k=0}^{\frac{p-3}{2}} \frac{(2k)^2}{(2k+1)16^k} H_{2k}^{(2)} \equiv -\frac{5}{4} B_{p-3} \pmod{p}.
\]
**Proof:** In view of (1) and \( \binom{2k}{k}/(-4)^k \equiv \binom{(p-1)/2}{k} \pmod{p} \) for each \( 0 \leq k \leq (p-1)/2 \), we have

\[
\sum_{k=0}^{p-3} \frac{\binom{2k}{k}^2}{(2k+1)16^k} H_{2k}^{(2)} = \sum_{k=0}^{p-3} \frac{\binom{p-1}{k}^2}{2k+1} H_{2k}^{(2)} = \sum_{j=1}^{p-1} \frac{\binom{(p-1)/2}{k}^2}{p-2k} H_{p-1-2k}^{(2)}
\]

\[
\equiv -\frac{1}{2} \sum_{j=1}^{p-1} \frac{\binom{2k}{k}^2 (H_{p-1}^{(2)} - H_{2k}^{(2)})}{k16^k} \equiv \frac{1}{2} \sum_{j=1}^{p-1} \frac{\binom{2k}{k}^2 H_{2k}^{(2)}}{k16^k}
\]

\[
\equiv -\frac{5}{4} B_{p-3} \pmod{p},
\]

where we used the following congruence

\[
H_{p-1-2k}^{(2)} = \sum_{j=1}^{p-1-2k} \frac{1}{j^2} = \sum_{j=2k+1}^{p-1} \frac{1}{(p-j)^2} \equiv H_{p-1}^{(2)} - H_{2k}^{(2)} \pmod{p}. \tag{17}
\]

So the proof of Lemma 3.1 is finished. 

**Proof:** In view of [18, Theorem 2.6], Sun already proved that

\[
\sum_{k=0}^{p-1} (2k+1) \frac{T_k}{4^k} \equiv p^{(p-1)/2} \sum_{k=0}^{p-3} \frac{\binom{2k}{k}^2}{(2k+1)16^k} (1 - p^2 H_{2k}^{(2)}) \pmod{p^5}.
\]

This, with (15), Lemma 3.1, (11), [17, Corollary 5.1] and Fermat’s little theorem yields that

\[
\sum_{k=0}^{p-1} (2k+1) \frac{T_k}{4^k}
\]

\[
\equiv p^2 \sum_{k=0}^{p-3} \frac{\binom{2k}{k}^2}{(2k+1)16^k} - p^4 \sum_{k=0}^{p-3} \frac{\binom{2k}{k}^2}{(2k+1)16^k} H_{2k}^{(2)} + p \frac{(p-1)^{p-1}}{2} \frac{(1 - p^2 H_{p-1}^{(2)})}{4p-1}
\]

\[
\equiv -2p^2 q_p(2) - p^3 q_p^2(2) + \frac{5}{12} p^4 B_{p-3} + \frac{5}{4} p^4 B_{p-3} + p 4^{p-1} - \frac{p^4}{2} B_{p-3}
\]

\[
\equiv p + \frac{7}{6} p^4 B_{p-3} \pmod{p^5},
\]

where we also used a fact that \( 4^{p-1} = 1 + 2p q_p(2) + p^2 q_p^2(2) \). So we finish the proof of (5).
Proof of (6): Also, Sun [18, Theorem 2.7] already showed that
\[ \sum_{k=0}^{p-1} (2k+1) \frac{T_k}{(-4)^k} \equiv p \sum_{k=0}^{(p-1)/2} \frac{(2k)^2}{16^k} (1 - p^2 H_{2k}^{(2)}) \pmod{p^5}. \] (18)

As seen in (17) and by using (1), we have
\[ H_{p-1-2k}^{(2)} \equiv -H_{2k}^{(2)} \pmod{p}, \]
and hence
\[ \sum_{k=0}^{(p-1)/2} \frac{(2k)^2}{16^k} H_{2k}^{(2)} \equiv \sum_{k=0}^{(p-1)/2} \left( \frac{p-1}{k} \right)^2 H_{2k}^{(2)} = \sum_{j=0}^{(p-1)/2} \left( \frac{p-1}{j} \right)^2 H_{p-1-2j}^{(2)} \]
\[ \equiv -\sum_{k=0}^{(p-1)/2} \left( \frac{p-1}{k} \right)^2 H_{2k}^{(2)} \pmod{p}. \]

Thus,
\[ \sum_{k=0}^{(p-1)/2} \frac{(2k)^2}{16^k} H_{2k}^{(2)} \equiv 0 \pmod{p}. \]

This, with (16) and (18) yields the desired result
\[ \sum_{k=0}^{p-1} (2k+1) \frac{T_k}{(-4)^k} \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \pmod{p^4}. \]

Now the proof of Theorem 1.1 is complete.

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