CYLINDER ABSOLUTE GAMES ON SOLENOIDS

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Abstract. Let $A$ be any affine surjective endomorphism of a solenoid $\Sigma_P$ over the circle $S^1$ which is not an infinite-order translation of $\Sigma_P$. We prove the existence of a cylinder absolute winning (CAW) subset $F \subseteq \Sigma_P$ with the property that for any $x \in F$, the orbit closure $\{A^\ell x | \ell \in \mathbb{N}\}$ does not contain any periodic orbits. A measure $\mu$ on a metric space is said to be Federer if for all small enough balls around any generic point with respect to $\mu$, the measure grows by at most some constant multiple on doubling the radius of the ball. The class of infinite solenoids considered in this paper provides, to the best of our knowledge, some of the early natural examples of non-Federer spaces where absolute games can be played and won. Dimension maximality and incompressibility of CAW sets is also discussed for a number of possibilities in addition to their winning nature for the games known from before.

1. Introduction. Let $\mathcal{P}$ be an (finite or infinite) ordered set of prime numbers with $p_1 < p_2 < \cdots$ and the restricted product space

$$X_\mathcal{P} := \mathbb{R} \times \prod_{p \in \mathcal{P}} \mathbb{Q}_p,$$

where $\prod'$ denotes that for each element $x \in X_\mathcal{P}$, the entries $x_p$ are in the compact ring $\mathbb{Z}_p$ for all but finitely many $p$’s. A $\mathcal{P}$-solenoid over the unit circle $S^1$ is the quotient space $\Sigma_\mathcal{P} := X_\mathcal{P}/\Delta(R)$, where

$$R = \mathbb{Z} \left[ \left\{ \frac{1}{p} \mid p \in \mathcal{P} \right\} \right]$$

is a subring of $\mathbb{Q}$ embedded diagonally as the uniform lattice

$$\Delta(R) = \{(r)_{p \in \{\infty\} \cup \mathcal{P}} \mid r \in R \} \subset X_\mathcal{P}. $$

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We denote the quotient map $X_{\mathcal{P}} \to \Sigma_{\mathcal{P}}$ to be $\Pi$. Solenoids are compact, connected metrizable abelian groups. Their higher topological-dimensional cousins have sometimes been called “fractal versions of tori” [21, Abstract]. When $\mathcal{P}$ is a finite set of cardinality $l - 1$, the Hausdorff dimension of $X_{\mathcal{P}}$ (and therefore of $\Sigma_{\mathcal{P}}$ too) under the natural metric given by (4) is $l$. This also implies that the dimension is infinite when $\mathcal{P}$ is so, as the increasing sequence of finite product spaces associated with the finite truncations of $\mathcal{P}$ are isometrically embedded inside $X_{\mathcal{P}}$.

The set of endomorphisms of $\Sigma_{\mathcal{P}}$ is precisely the ring $R$ whose elements act multiplicatively componentwise. An affine transformation $A : \Sigma_{\mathcal{P}} \to \Sigma_{\mathcal{P}}$ is meant to denote the map $x \mapsto \left(\frac{m}{n}\right)x + a$ (1) where $\frac{m}{n} \in R \setminus \{0\}$ and $a \in \Sigma_{\mathcal{P}}$. It is well known that when $A = \frac{m}{n}$ is a surjective endomorphism of the solenoid, it acts ergodically on $\Sigma_{\mathcal{P}}$ iff $m/n \notin \{0, \pm 1\}$ [23, Proposition 1.4]. We also learn from [4, Theorem 3.2] that for every compact group $G$, any semigroup of its affine transformations lying above an ergodic semigroup of surjective endomorphisms is ergodic as well. This gives us a sufficient condition for the transformation $Ax = \left(\frac{m}{n}\right)x + a$ to be ergodic, namely that $m/n \neq \pm 1$. Ergodicity of the action guarantees that almost all orbits of $A$ are dense in $\Sigma_{\mathcal{P}}$. However, just like [8], this work is concerned with understanding the complementary set. Given an affine transformation $A$, we would like to know how large is the set of points of $\Sigma_{\mathcal{P}}$ whose $A$-orbits remain away from periodic $B$-orbits for all $B \in R \setminus \{\pm 1\}$.

When $\mathcal{P}$ is the set consisting of all the primes in $\mathbb{N}$, the space $\Sigma_{\mathcal{P}}$ is called the full solenoid over $S^1$ with the field $\mathbb{Q}$ being its ring of endomorphisms. Let $B$ be a non-zero rational number. The growth of the number of $B$-periodic orbits as a function of the period is determined by the entropy of the action on $\Sigma_{\mathcal{P}}$. The latter has been computed first in [12] and recovered in [17], where it was explained to be the sum of the Euclidean and the $p$-adic contributions. In fact, Lind and Ward [17] achieve it for all automorphisms of solenoids over higher-dimensional tori as well. We remark that each such epimorphism lifts uniquely to a homomorphism from $X_{\mathcal{P}}$ to itself, which we shall continue to denote by the same rational number. For an affine transformation, we however have a choice involved in terms of a representative for the translation part $a$.

Let $y \in \Sigma_{\mathcal{P}}$ be arbitrary and $A$ be an affine surjective transformation of $\Sigma_{\mathcal{P}}$ as in (1) with either $m/n \neq 1$ or $a \in \Pi(\Delta(\mathbb{Q}))$. We intend to show that the set of points $x \in X_{\mathcal{P}}$ whose forward orbit under the map $x \mapsto Ax$ maintains some positive distance from the 1-uniformly discrete subset $\Pi^{-1}(\{y\}) \subset X_{\mathcal{P}}$ is cylinder absolute winning (CAW) in a similar sense as [11]. To begin with, one observes that the affine transformation $A$ is a finite-order translation of the solenoid when $m/n = 1$ and $a \in \Pi(\Delta(\mathbb{Q}))$. The set of points of $X_{\mathcal{P}}$ whose (forward) orbit avoids some neighbourhood of $\Pi^{-1}(y)$ is then the complement of some uniformly discrete subset of $X_{\mathcal{P}}$ and is certainly CAW. This is also the case when $m/n = -1$ and $A$ becomes an involution of the space $\Sigma_{\mathcal{P}}$.

Once we have a statement as advertised early in the paragraph above, we can take intersection of countably many of these sets to conclude about $A$-orbits which avoid neighbourhoods of all periodic orbits of surjective endomorphisms. This strategy is in similar taste and builds upon the work of Dani [8] on orbits of semisimple toral automorphisms.
Our setup has two players in which one of them (Alice) will be blocking open cylinder subsets of \(X_P\) at every stage of a two-player game. To elaborate, one such cylinder is given by

\[
C(x, \varepsilon, i) := \begin{cases} 
\mathbb{R} \times \prod_{j<i} \mathbb{Q}_{p_j} \times B(x_i, p_i \varepsilon) \times \prod_{j>i} \mathbb{Q}_{p_j} & \text{if } i > 0 \\
B(x_0, \varepsilon) \times \prod_{j>0} \mathbb{Q}_{p_j} & \text{otherwise,}
\end{cases}
\]

where \(x = (x_0, \ldots, x_i, \ldots) \in X_P\). For us, \(B(x_i, r)\) will always be the set of points in \(\mathbb{Q}_{p_i}\) whose distance from \(x_i\) is strictly less than \(r\) while \(\overline{B}(x_i, r)\) will also include those whose distance from \(x_i\) is exactly \(r\). We explain this game in §3 after a brief tour of some of its older and related versions. The aim is to prove the following statement in this paper:

**Theorem 1.1.** Let \(\Sigma_P\) be a solenoid over the circle \(S^1\) and

\[
\{A_j : x \mapsto (m_j/n_j) x + a_j \mid j \in \mathbb{N}\}
\]

be any countable family of affine surjective endomorphisms of \(\Sigma_P\) such that

1. none of the \(A_j\)'s is a translation of \(\Sigma_P\) of infinite order, and
2. the collection of rational numbers \(\{m_j/n_j\}_{j \in \mathbb{N}}\) lying below the family \(\{A_j\}_{j \in \mathbb{N}}\) belong to some finite ring extension of \(\mathbb{Z}\).

Then, there exists a cylinder absolute winning subset \(F \subseteq \Sigma_P\) such that for any \(x \in F\) and \(j \in \mathbb{N}\), the orbit closure \(\overline{\{A_j^k x \mid k \in \mathbb{N}\}}\) contains no periodic \(B\)-orbit for all \(B \in R \setminus \{\pm 1\}\).

This is done in §4. Note that the first condition in our hypothesis can possibly be relaxed in terms of some independence criterion on the co-ordinate entries of the translation vector \(a_j\). However, Haar-almost all infinite-order translations will lead to all orbits being dense. The second condition is a technicality of the proof and will be interesting to explore further.

In the last section, we illustrate as to how information about the existence of winning strategies for Alice can be used to infer something about the \(f\)-dimensional Hausdorff measure of \(F\). We also discuss the strong winning and incompressible nature of CAW subsets when \(P\) is finite.

### 1.1. Comparison with the work of Weil.

It is plausible that some of our results given here may also be obtained from the more general framework discussed in [22]. We take some time to expound the import of his main result.

Let \((\overline{X}, d)\) be a proper metric space (i.e. all closed balls are compact) and \(X\) a closed subset of \(\overline{X}\). A contraction for us would be a map \(\psi\) from the half-open interval \((0, 1]\) to the class of non-empty compact subsets of \(\overline{X}\) satisfying \(\psi(t_1) \subseteq \psi(t_2)\) for all \(0 < t_1 < t_2\). Consider a family of subsets \(\{R_\lambda \subseteq \overline{X} \mid \lambda \in \Lambda\}\) which are called resonant sets and a family of contractions \(\{\psi_\lambda \mid \lambda \in \Lambda\}\), indexed by some (same) countable set \(\Lambda\). It is further required that \(R_\lambda \subseteq \psi_\lambda(t)\) for all \(\lambda \in \Lambda\) and \(t > 0\). This data is written in a concise form as \(F = (\Lambda, R_\lambda, \psi_\lambda)\). The set of badly approximable points in \(S\) with respect to the family \(F\) is defined as

\[
BA_X(F) := \left\{ x \in X \mid \exists c = c(x) > 0 \text{ such that } x \notin \bigcup_{\lambda \in \Lambda} \psi_\lambda(c) \right\}.
\]

Next, each \(R_\lambda\) is assigned a height \(h_\lambda\) with \(\inf_\lambda h_\lambda > 0\). The standard contraction \(\psi_\lambda\) is then determined as \(\psi_\lambda(c) := \overline{B}_{c/h_\lambda}(R_\lambda)\), where \(\overline{B}_\varepsilon(S)\) now denotes the set of all points of \(\overline{X}\) whose distance is at most \(\varepsilon\) from some element of \(S\). We further
assume that our resonant sets \( R_\lambda \) are nested with respect to the height function \( h \), i.e., \( R_\lambda \subseteq R_\beta \) for every \( \lambda, \beta \in \Lambda \) such that \( h_\lambda \leq h_\beta \) and that the values taken by \( h \) form a discrete subset of \((0, \infty)\). For any collection \( \mathcal{S} \) of subsets of \( X \), the set \( X \) is said to be \( b_* \)-diffuse with respect to \( \mathcal{S} \) for some \( 0 < b_* < 1 \) if there exists some \( r_0 > 0 \) such that for all balls \( \overline{B}(x, r) \), \( x \in X \), \( 0 < r < r_0 \) and \( S \in \mathcal{S} \), there exists a sub-ball

\[
\overline{B}(y, b_*r) \subseteq \overline{B}(x, r) \setminus B_{b_*r}(S) \text{ with } y \in X.
\]

The family \( \mathcal{F} \) is locally contained in \( \mathcal{S} \) if for any \( B = \overline{B}(x, r) \) for some \( x \in X \) with \( r < r_0 \) and \( \lambda \in \Lambda \) such that \( h_\lambda \leq 1/r \), there is an \( S \in \mathcal{S} \) such that \( B \cap R_\lambda \subseteq S \). Concrete realizations of this abstract formalism include the case when \( X \) is the Euclidean space \( \mathbb{R}^n \) and \( \mathcal{S} \) consists of all affine hyperplanes in \( X \). Some examples of hyperplane diffuse sets are Cartesian products of the middle-third Cantor set and the Sierpiński triangle.

**Theorem 1.2** (Weil [22]). Let \( X \subseteq \overline{X} \) be closed and \( b_* \)-diffuse with respect to a collection \( \mathcal{S} \) of subsets of \( \overline{X} \). Also, \( \mathcal{F} \) is a family with nested resonant sets \( R_\lambda \) and discrete heights, locally contained in \( \mathcal{S} \). Then, the set \( \text{BA}_X(\mathcal{F}) \) defined in (3) is a Schmidt winning subset of \( X \).

Many of the terms used above will be explained in § 3. Actually, it is shown in [22] that \( \text{BA}_X(\mathcal{F}) \) is absolute winning with respect to \( \mathcal{S} \). This covers many important examples like the set of \((s_1, \ldots, s_n)\)-badly approximable vectors in \( \mathbb{R}^n \), \( \mathbb{C}^2 \) and \( \mathbb{Z}_p^2 \), the set of sequences in the Bernoulli-shift which avoid all periodic sequences and the set of orbits of toral endomorphisms which stay away from periodic orbits. Curiously for us, we do not find any discussion on solenoidal endomorphisms in his work.

In order to be able to use Weil’s results to prove Theorem 1.1, we would have to show that (a) the space \( X_\mathcal{P} \) is \( b_* \)-diffuse with respect to the collection \( \mathcal{C} \) of open cylinders in \( X_\mathcal{P} \) for an appropriate value of \( b_* \), and (b) the family of pre-images

\[
\{ A^{-k}B(y + z, t) \mid k \in \mathbb{N}, z \in \Delta(R) \}
\]

is locally contained in \( \mathcal{C} \) for any fixed \( y \in X_\mathcal{P} \) and some \( t = t(A, y) > 0 \). We have endeavoured to provide a more direct proof here. In this sense, our work may be considered as an addition to the list of examples given in [22]. While it simplifies the proofs to an appreciable extent and cuts down on many metric space notations, the approach taken in the present paper also helps to establish the strong winning nature (Proposition 3) of the set of non-dense solenoidal orbits which is missing in the abstract regime of [22]. In addition, we delve into issues of incompressibility of CAW subsets.

2. **Metric and measure structure on solenoids.** Before we discuss the game, it is imperative that we say a few words about how balls in our space \( X_{\mathcal{P}, n} \) “look like.” A metric on \( X_\mathcal{P} \) is given by

\[
d(x, z) := \max \left\{ |x_0 - z_0|, \sup_{p \in \mathcal{P}} \left\{ p^{-1} |x_p - z_p|_p \right\} \right\}
\]

where \( \cdot, \cdot \) is the usual Euclidean metric on \( \mathbb{R} \) and \( | \cdot |_p \) refers to the \( p \)-adic ultrametric on \( \mathbb{Q}_p \) such that the diameter of the clopen ball \( p^{-1}\mathbb{Z}_p \) is \( p \). Clearly, the distance between any two distinct points of \( \Delta(R) \) is at least 1 or in other words, the injectivity radius for the quotient map \( \Pi : X_\mathcal{P} \to \Sigma_\mathcal{P} \) equals 1. More generally, as per [5], a subset \( Z \) of any metric space \( X \) is said to be \( \delta \)-uniformly discrete if the distance
Taking negative logarithms on both sides, we have

\[ \ell \to \text{determining} \quad P \]

Then, the 'closed' ball \( \mathcal{B}(x, r) \) is a compact neighbourhood of \( x \). It is easy to see that the diameter of \( B(x, r) \) is \( 2r \) but in most (all but one) of the places \( p \in \mathcal{P} \), the \( p \)-adic diameters of the projections will be strictly less than \( pr \) as distances in the non-archimedean fields are a discrete set. For a given \( r > 0 \), there might not be any integer \( j \) such that \( p_j^r = pr \). This seemingly minor issue is very crucial in our next set of calculations. If \( j_i \in \mathbb{Z} \) is such that \( p_i^{j_i} \leq r < p_i^{j_i+1} \), then we call \([r]_i := p_i^{j_i}\). It should be noted that \( \lfloor p_i^{mj} \rfloor_i = p_i^{mj} \lfloor r \rfloor_i \) for all \( m \in \mathbb{Z} \) and more generally, \( \lfloor tr \rfloor_i \leq \lfloor p_i \lfloor t \lfloor r \rfloor_i \rfloor_i = p_i \lfloor t \rfloor_i \lfloor r \rfloor_i \) for all \( r, t > 0 \). The lemma below may be of independent interest to the reader:

**Lemma 2.1.** Let \( x > 1 \) and \( P_{\mathcal{P}}(x) \) denote the product

\[ \prod_{i \in \mathbb{N}} \left[ p_i x \right]_i = \prod_{i \in \mathbb{N}, p_i < x} \left[ p_i x \right]_i. \]

Then,

\[ \ln P_{\mathcal{P}}(x) \leq -\theta_{\mathcal{P}}(x) + O((\ln x)^2), \text{ where } \theta_{\mathcal{P}}(x) := \sum_{p \leq x} \ln p. \]

**Proof.** Let \( i \in \mathbb{N} \) be such that \( p_i < x \) and \( m_i \in \mathbb{N} \) be the unique integer for which

\[ \frac{1}{p_i^{m_i}} \leq \frac{1}{x} < \frac{1}{p_i^{m_i-1}} \quad \Leftrightarrow \quad p_i^{m_i} < x \leq p_i^{m_i+1}. \]

We know that \( m_i = k \) if and only if \( x^{1/(k+1)} \leq p_i < x^{1/k} \). This observation leads to decomposing \( P_{\mathcal{P}}(x) \) as a double product

\[ P_{\mathcal{P}}(x) = \prod_{k=1}^{\ell} \prod_{p \in \mathcal{P}, x^{1/(k+1)} \leq p < x^{1/k}} p^{-k} \]

where \( \ell \in \mathbb{N} \) is such that \( x^{1/(\ell+1)} \leq 2 < x^{1/\ell} \) (i.e., \( \log x / \log 2 - 1 \leq \ell < \log x / \log 2 \)). Taking negative logarithms on both sides, we have

\[ -\ln P_{\mathcal{P}}(x) = \sum_{k=1}^{\ell} k \sum_{p \in \mathcal{P}, x^{1/(k+1)} \leq p < x^{1/k}} \ln p \]

\[ \geq \sum_{k=1}^{\ell} k \left( \theta_{\mathcal{P}} \left( x^{1/k} \right) - \theta_{\mathcal{P}} \left( x^{1/(k+1)} \right) - \frac{1}{k} \ln x \right) \]

\[ = \sum_{k=1}^{\ell} \theta_{\mathcal{P}} \left( x^{1/k} \right) - \ell \left( \theta_{\mathcal{P}} \left( x^{1/(\ell+1)} \right) + \ln x \right) \]

\[ \geq \theta_{\mathcal{P}}(x) - \ell \left( \theta_{\mathcal{P}} \left( x^{1/(\ell+1)} \right) + \ln x \right). \]
As $\theta_P\left(x^{1/(\ell+1)}\right) \leq \theta_P\left(2\right) \leq \ln 2$ and $\ell \ll \ln x$, we are done. 

When $\mathcal{P}$ is the full subset of primes, $\theta_P\left(x\right) \sim x$ as $x \to \infty$. If $\mathcal{P}$ consists of all primes in an arithmetic progression with common difference $k$ and $(a, k) = 1$ where $a$ is one of the terms in the progression, then $\theta_P\left(x\right)$ goes roughly as $x/\varphi(k)$ as $x \to \infty$. Here, $\varphi$ stands for the Euler’s totient function. For more on this, the reader is redirected to [19]. We conclude that for all subsets $\mathcal{P}$ of primes which come from arithmetic progressions (and in particular the full subset), there exist some $0 < c_1 = c_1(k) < 1$ such that

$$P_{\mathcal{P}}\left(\frac{1}{r}\right) \leq r^{c_1 \ln(1/r)}$$

for all $0 < r < 1$. 

This short exercise serves dual purpose. On one hand, it is a roundabout way of establishing the infiniteness of the Hausdorff dimension of $X_{\mathcal{P}}$ when $\mathcal{P}$ is as above using the mass distribution principle. A finite non-zero measure $\mu$ whose support is a bounded subset of a metric space $M$ is called a mass distribution on $M$. We require our dimension functions (also known as gauge functions) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ to be increasing in some neighbourhood $[0, r)$, continuous on $(0, r)$, right continuous at 0 and $f(r) = 0$ if $r$ equals zero [10, pg. 33].

**Proposition 1** (cf. [10]). Let $\mu$ be a mass distribution on a second countable metric space $M$ such that for some dimension function $f$ and $\delta_0 > 0$,

$$\mu(U) \leq c_2 f(|U|)$$

for some fixed $c_2 > 0$ and all subsets $U$ with $|U| \leq \delta_0$. Then, $\mathcal{H}^f(M) \geq \mu(M)/c_2$. If $f$ is the power rule $r \mapsto r^s$, we can say that $\dim M \geq s$.

Now, let $\mu$ be the restriction to $[0, 1] \times \prod_{p \in \mathcal{P}} \mathbb{Z}_p$ of the Haar measure $\nu$ on $X_{\mathcal{P}}$ which is the product of the Haar measures on $\mathbb{R}$ and on each $\mathbb{Q}_p$, $p \in \mathcal{P}$. We normalize it so that $\mu$ is a probability measure on $X_{\mathcal{P}}$. Any ball $B(x, r)$ with radius $0 < r \ll 1$ will then have

$$\mu(B(x, r)) \leq 2r \times \prod_{i \in \mathbb{N}} |p_i r|_i = 2r \cdot P_{\mathcal{P}}(1/r) \leq 2r^{c_1 \ln(1/r) + 1}$$

by (5) when $\mathcal{P}$ is the set of all primes in any infinite arithmetic progression with $(a, k) = 1$. After some more work, the above proposition will give us that the Hausdorff dimension of the space $X_{\mathcal{P}}$ is infinite in such cases.

On the other hand, Lemma 2.1 will help us again in §5 when we examine the dimension-theoretic largeness of various winning subsets of cylinder absolute games. In our version, Alice shall be dealing with the family $C$ of closed subsets exactly one of whose co-ordinates $x_i$ is a fixed constant where $i \in \mathbb{N}$. The resulting $\varepsilon$-neighbourhood $C(x, \varepsilon, i)$ of such a set $P \in C$ as also defined in (2) will be called an (open) cylinder. Given a cylinder $C = C(x, \varepsilon, i)$, we say that the radius of $C$ is $\varepsilon$ if $i = 0$ and the minimum such $\varepsilon'$ for which $C(x, \varepsilon', i) = C(x, \varepsilon, i)$ otherwise. The index $i$ is called the constraining coordinate of $C$. We emphasize that both Alice and Bob are fully aware of the radii of the balls chosen by the latter at any stage of our game by reading the real coordinate.

3. **Infinite games on complete metric spaces.** Let $M$ be a complete metric space and $F$ be a fixed subset of $M$. In the original game introduced by Schmidt [20], Alice and Bob are two players who each take turns to pick closed balls in $M$ in
the following manner: We have $\alpha, \beta \in (0, 1)$ to be two real numbers such that $1 - 2\alpha + \alpha \beta > 0$. The game begins with Bob choosing any closed ball $B_0 = \overline{B}(b_0, r) \subseteq M$ subsequent to which Alice has to make a choice of some $A_1 = \overline{B}(a_1, \alpha r)$ such that $A_1 \subseteq B_0$. After this, Bob picks $B_1 = \overline{B}(b_1, \beta r) \subseteq A_1$ and the game goes on till infinity. We thus get a decreasing sequence of closed, non-empty subsets of a complete metric space $M \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq A_2 \supseteq \ldots$

Alice is declared the winner if $\bigcap_j B_j = \bigcap_j A_j = \{a_\infty\} \subseteq F$. A set $F$ is called $(\alpha, \beta)$-winning if Alice has a strategy to win the above game regardless of Bob’s moves. Further, it is $\alpha$-winning if it is $(\alpha, \beta)$-winning for all $\beta \in (0, 1)$ and Schmidt winning if it is $\alpha$-winning for some $\alpha \in (0, 1)$.

For various applications of practical interest, one finds out that Alice need not bother herself too much about choosing the balls $A_j$’s as long as she is able to block out neighbourhoods of certain undesirable points. This is true for example when $M$ is the real line and $F$ is the set $\mathcal{B}A$ of badly approximable numbers as discussed in [20] where Alice needs to be far from rational numbers with small denominators. Moreover if she is careful enough about her strategy, she has to worry about very few of such rationals – at times just one of them and hence, she need only shift the game outside of a ball $B$ centered at some $b$ which retains its strong winning property under quasisymmetric mappings [18, Theorem 1.2].

We again have two parameters $\alpha, \beta$ in the form of balls $A_{i+1} \subseteq B_i$ such that $|A_{i+1}| \geq \alpha |B_i|$ for all $i \in \mathbb{N}$ while for Bob, $|B_i| \geq \beta |A_i|$ for $i > 1$. It continues to be mandatory that $A_i \subseteq B_{i-1}$ and $B_i \subseteq A_i$ for all $i$. A subset $F$ is said to be Schmidt winning if Alice has a winning strategy in this game is called an $(\alpha, \beta)$-strong winning set. The subset $F$ is said to be $\alpha$-strong winning if it is $(\alpha, \beta)$-strong winning for all $0 < \beta < 1$ and strong winning if it is $\alpha$-strong winning for some $\alpha > 0$. For Euclidean spaces, a strong winning subset is Schmidt winning too and retains its strong winning property under quasisymmetric mappings [18, Theorem 1.2].

The absolute game has an obvious drawback that if $F$ is the set of badly approximable vectors in $\mathbb{R}^n$ for any $n > 1$, then Bob can force the game to be always centered on the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ and Alice is not able to win trivially. Therefore, it was proposed in [6] that she be allowed to block out a neighbourhood of some $k$-dimensional affine subspace of $\mathbb{R}^n$ at each stage of the game. Taking this into consideration, they gave a family of games played on the Euclidean space $\mathbb{R}^n$ called $k$-dimensional $\beta$-absolute games ($0 < \beta < 1/3, 0 \leq k < n$) where Bob
having chosen $B_0 = B(b_0, r_0) \subset \mathbb{R}^n$, Alice picks some affine subspace $V_1$ of dimension $k$ and for some $0 < \varepsilon_1 \leq \beta r_1$ removes the $\varepsilon_1$-neighbourhood of $V_1$, namely $A_1 = V_1^{(\varepsilon_1)}$ from $B_0$. This is followed by Bob picking a closed ball $B_1 \subseteq B_0 \setminus A_1$ with radius $(B_1) \geq \beta r_0$ and the game proceeds in a similar fashion. In general, the parameter $\varepsilon_j$ is allowed to depend on $j$ subject only to $0 < \varepsilon_j \leq \beta \cdot \text{radius}(B_j)$. Alice wins if $\cap_j B_j \cap F \neq \emptyset$. As before, $F \subseteq \mathbb{R}^n$ is $k$-dimensional $\beta$-absolute winning if Alice can win the $k$-dimensional $\beta$-absolute game over $F$ irrespective of Bob’s strategy. It is called $k$-dimensional absolute winning if it is $k$-dimensional $\beta$-absolute winning for all $\beta \in (0, 1/3)$. It is clear from the definitions that for $0 \leq k_1 < k_2 < n$, if a set $F \subseteq \mathbb{R}^n$ is $k_1$-dimensional $\beta$-absolute winning, then it is $k_2$-dimensional $\beta$-absolute winning too. Also, 0-dimensional $\beta$-absolute winning is the same as $\beta$-absolute winning.

All of this culminated in the axiomatization in [11] by Fishman, Simmons and Urbanski such that $M$ is a complete metric space, $\mathcal{H}$ is a non-empty collection of closed subsets of $M$ and $F \subseteq M$ is fixed before the start of play. For $0 < \beta < 1$, the set $F$ is called $(\mathcal{H}, \beta)$-absolute winning if Alice can ensure the intersection $\cap_j B_j \cap F \neq \emptyset$ by removing neighbourhoods $A_j = H_j^{(\varepsilon_j)}$ for some $H_j \in \mathcal{H}$ and $0 < \varepsilon_j \leq \beta \cdot \text{radius}(B_{j-1})$ at every $j$-th stage of the game. We follow [14] to declare Bob the winner by default if at any (finite) stage of the game, he is left with no legal choice of the ball $B_j$ to make. In the course of the game, Alice will have to make sure that such an event does not ever occur. This is keeping in mind the example of a Schmidt game illustrated in [15, Proposition 5.2] where Bob is not able to win because he has no option of choosing the ball $B_j$.

Ever since [20] came out, Schmidt games have been played and won over subsets of various metric spaces. We were unable to find any reasonable survey article covering the developments in the area. It will also be impossible to give here a comprehensive account of all the progress that has been made by different people and groups. We will have to contend ourselves by pointing to only a few representative works. Dani [7] formulated and proved results about the winning nature of the set of points in a homogeneous space $G/\Gamma$ of a semisimple Lie group $G$ whose orbits under a one-parameter subgroup action are bounded. Aravinda [2] showed that the set of points on any non-constant $C^1$ curve $\sigma$ on the unit tangent sphere $S_p$ of any point $p$ on a complete, non-compact Riemannian manifold $M$ with constant negative curvature and finite Riemannian volume which lead to bounded geodesic orbits is Schmidt winning.

When $\Gamma \subset G$ is an irreducible lattice of a connected, semisimple $G$ with no compact factors, Kleinbock and Margulis [13] established that the subset of points in $G/\Gamma$ with bounded $H$-orbits is of full Hausdorff dimension whenever $H$ is a nonquasi-nilpotent one-parameter subgroup of $G$. Later, Kleinbock and Weiss [15] allowed for Alice’s and Bob’s choices of subsets to be more flexible than just metric balls and used this to settle that the set of $s$-badly approximable vectors in $\mathbb{R}^n$ is winning for the modified game given any fixed $s \in \mathbb{R}_+^n$. This was part of an effort to understand Schmidt’s conjecture on the intersection of the sets of weighted badly approximable vectors for different weights which was finally resolved by Badziahin, Pollington and Velani [3]. More generally, one can define a hyperplane absolute game on any $C^1$
manifold. It was recently proved in [1] that for any one parameter Ad-semisimple
subsemigroup \( \{g_t\}_{t \geq 0} \) of the product \( G \) of finitely many copies of \( \text{SL}_2(\mathbb{R}) \)'s, the set
of points \( x \) belonging to any lattice quotient \( G/\Gamma \) of \( G \) and with bounded \( \{g_t\}\)-orbit
in \( G/\Gamma \) is hyperplane absolute winning.

In our setting, \( M \) shall be \( X_\mathcal{P} \) (or \( \Sigma_\mathcal{P} \) if you prefer), \( \mathcal{H} \) is the family \( C \) of subsets
described in §2 and an example of the target set \( F \) is given below. A less contrived
one will be available in the next section. A \textit{cylinder} \( \beta \)-absolute game begins with
Bob choosing a closed ball \( B_0 = \overline{B}(x_0, r_0) \). Subsequent to this, Alice blocks an
open cylinder \( C_1 \) whose radius has to be less than or equal to \( \beta r_0 \). The ball chosen
by Bob on his \( j \)-th move must satisfy radius \( (B_j) \geq \beta \cdot \text{radius} \ (B_{j-1}) \) for all \( j \in \mathbb{N} \).
And, the game of our interest goes as

\[
B_0 \supseteq B_0 \setminus C_1 \supseteq B_1 \supseteq B_1 \setminus C_2 \supseteq \cdots
\]

and \( F \) is said to be \textit{cylinder} \( \beta \)-absolute winning if Alice can devise a method to
win this game, i.e., \( \bigcap_j B_j \cap F \neq \emptyset \). It is \textit{cylinder absolute winning} (abbr. CAW) if
there exists a \( 0 < \beta = \beta_\mathcal{P}(F) \leq 1/3 \) such that \( F \) is cylinder \( \beta \)-absolute winning
for all \( \beta \in [0, \beta_\mathcal{P}[ \). The supremum of such \( \beta_\mathcal{P}'s \) is christened the \textit{CAW dimension}
of \( F \) or more briefly, \textit{CAW dim}(\( F \)). The cylinders seem to us to be the appropriate
replacement for the hyperplane neighbourhoods of [6] in metric spaces like solenoids.
Recall that the exact value of the radius can be read off from the real coordinate.

**Proposition 2.** A countable intersection of CAW subsets of \( X_\mathcal{P} \) with winning di-
mension \( \geq \beta_0 \) each is a CAW set with winning dimension \( \geq \beta_0 \).

The basic idea of the proof remains the same as in [20, Theorem 2] and is being
skipped here. We instead provide an auxiliary lemma which helps to simplify the
proof of the theorem following it.

**Lemma 3.1.** If \( F \subseteq X_\mathcal{P} \) is a CAW set, then Alice also has a winning strategy for
\( F \) which ensures that the radius of Bob’s choices goes to zero.

**Proof.** Let \( 0 < \beta < \text{CAW dim}(F) \). On odd-numbered moves, Alice chooses cylinders
whose constraining coordinate is archimedean, center same as that of the latest ball
\( B_{j-1} \) chosen by Bob and radius is \( \beta \cdot \text{radius} \ (B_{j-1}) \). Bob’s next choice \( B_j \) can have
radius at most \( (1 - \beta) \cdot \text{radius} \ (B_{j-1})/2 \). On even-numbered moves, she tries to win
with \( F \) as her target set for the cylinder \( \beta^2 \)-absolute game on \( X_\mathcal{P} \). \( \square \)

The theorem below is largely inspired by [7, Theorem 3.2].

**Theorem 3.2.** Let \( N \) be a countable indexing set and

\[
\{ A_{n,t} \subseteq C_{n,t} \subset X_{\mathcal{P}} \mid n \in N, \ t \in (0,1) \}
\]

be a family of set pairs where \( C_{n,t} \) are restricted to be open cylinders in \( X_\mathcal{P} \) with
the same fixed constraining coordinate \( i \). If for any compact \( K \subset X_\mathcal{P} \) and \( \mu \in (0,1) \),
there exist \( R \geq 1, \ v \in (0,1) \) and a sequence \( \langle R_n \rangle \) of positive reals with the following properties:

1. if \( n \in N \) and \( t \in (0, \varepsilon) \) are such that \( A_{n,t} \cap K \neq \emptyset \), then \( R_n \leq R \) and the
   radius \( r(C_{n,t}) \) of the cylinder \( C_{n,t} \) is at most \( tR_n \),
2. if \( n_1, n_2 \in N \) and \( t \in (0, \varepsilon) \) are such that both \( A_{n,t} \) intersect \( K \) non-trivially
   and the radius bounds of the associated cylinders are comparable, i.e., \( \mu R_{n_1} \leq
   R_{n_2} \leq \mu^{-1} R_{n_1} \), then either \( n_1 = n_2 \) or \( d(A_{n_1,t}, A_{n_2,t}) \geq \varepsilon(R_{n_1} + R_{n_2}) \).
Then, \( F = \bigcup_{j > 0} \{ x \in \mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_{(n, \delta)} \} \) is a cylinder absolute winning set with CAW dimension at least \( \beta_0 \) where \( \beta_0 := 1/p_i \) if \( i > 0 \) and \( 1/3 \) otherwise.

Proof. Given any \( 0 < \beta < \beta_0 \), let \( B_0 = \overline{B}(x, r_0) \) be the initial closed ball of radius \( r_0 \) chosen by Bob to kick start the cylinder \( \beta \)-absolute game. In light of Lemma 3.1, we may assume that \( r_0 < 1/2 \) as well as that the balls \( B_i \) chosen by Bob have radii \( r_1 \to 0 \). We let \( \varepsilon \) and \( (R_n) \) take the values dictated by our hypothesis for \( K = B_0 \) and \( \mu = \beta^2/2 \). Then, let \( k_0 \in \mathbb{N} \) be the smallest such that \( \mu^{k_0} < \varepsilon \) and \( \delta := \mu^{k_0+1}r_0 < \varepsilon \). For \( k \geq 1 \), mark \( h_k \to \infty \) to be any strictly increasing subsequence such that

\[
\beta \mu^kr_0 < \text{radius} (B_{h_k}) =: r_k \leq \mu^k r_0.
\]

This is well-defined as \( r_{k+1} \geq \beta r_k \) for all \( k \in \mathbb{N} \) and \( \mu^{k+1} < \beta \mu^k \). We claim that Alice is able to play in such a manner that the closed ball \( B_{h_{k+i}} \) does not intersect any \( A_{(n, \delta)} \) with \( R_n \geq \mu^{k-k_0} \). The limit point \( \{ b_\infty \} = \cap_{k=0}^{\infty} B_k = \cap_{k=0}^{\infty} B_{h_k} \) shall then be in \( F \) and the proof of the theorem will be done (as \( \beta < \beta_0 \) is arbitrary).

Our claim is vacuously true for \( k = 0 \) as \( R_n \leq R < \mu^{-k_0} \) for all \( A_{(n, \delta)} \) intersecting \( B_0 \) non-trivially by our assumption. Thereafter, supposing that the claim holds for \( k \), we show it to be true for \( k+1 \). Since the sets \( A_{(n, \delta)} \) with the corresponding cylinder radii bounds \( R_n \geq \mu^{k-k_0} \) have already been taken care of, we only need to show that Alice can now ensure \( B_{h_{k+1}} \) does not intersect \( A_{(n, \delta)} \) for any \( n \in N \) such that \( \mu^{k+1-k_0} \leq R_n < \mu^{k-k_0} \). As hinted before, she has to worry about exactly one such subset. For, if both \( A_{(n_1, \delta)} \cap B_{h_k}, A_{(n_2, \delta)} \cap B_{h_k} \neq \emptyset \) and \( R_{n_1}, R_{n_2} \in (\mu^{k+1-k_0}, \mu^{k-k_0}) \), then the second condition of the theorem says that

\[
d(A_{(n_1, \delta)}, A_{(n_2, \delta)}) \geq \varepsilon (r_{n_1} + r_{n_2}) \geq 2\varepsilon \mu^{k+1-k_0} \text{ while } |B_{h_k}| \leq 2r_k \leq 2\mu^kr_0 \text{ and we have a contradiction.}
\]

If \( n \in N \) is the unique index for which \( A_{(n, \delta)} \cap B_{h_k} \neq \emptyset \) and \( R_n \in (\mu^{k+1-k_0}, \mu^{k-k_0}) \) where \( B_{h_k} = \overline{B}(x_k, r_k) \), Alice chooses \( C_{h_{k+1}} \) to be the open cylinder \( C_{(n, \delta)} \) and since \( \text{radius} (C_{h_{k+1}}) \leq \delta R_n < \mu^{k+1}r_0 \cdot \mu^{k-k_0} < \beta \cdot r_k \), this constitutes a legal move. It only remains to be argued that Bob has some choice of \( B_{h_{k+1}} \subseteq B_{h_k} \setminus C_{h_{k+1}} \) left (in fact, plenty of them). If the constraining coordinate \( i \) of \( C_{h_{k+1}} \) equals zero, we only need to find a point in the closed ball \( \overline{B}(x_{k,0}, (1-\beta)r_k) \subset \mathbb{R} \) which is at a Euclidean distance \( \beta r_k \) from some open ball \( B(y, \beta r_k) \) containing the projection \( \pi_0(C_{h_{k+1}}) \) of \( C_{(n, \delta)} \) in the archimedean coordinate. This is clearly possible as long as \( \beta < 1/3 \).

Else if \( i > 0 \), let \( \overline{B}(x_{k,i}, |p_i r_k|_i) \), \( B(y, p_i r'_k+1) \subset \mathbb{Q}_{p_i} \) be the respective images of \( B_{h_k} \) and \( C_{h_{k+1}} \) under the projection \( \pi_i \) onto the \( i \)-th coordinate. As \( B_{h_k} \cap C_{h_{k+1}} \neq \emptyset \) by our assumption, we get that \( \overline{B}(x_{k,i}, |p_i r_k|_i) \cap B(y, r'_k+1) \neq \emptyset \) too. Being balls in an ultrametric space, one of them then has to be contained in the other and because \( \beta < 1/p_i \), we have

\[
r'_k+1 \leq \frac{1}{p_i} |r_k|_i,
\]

and thereby \( B(y, p_i r'_k+1) \subseteq \overline{B}(x_{k,i}, |p_i r_k|_i) = \overline{B}(y, p_i |r_k|_i) \). Bob picks a point \( z_i \in \overline{B}(y, p_i |r_k|_i) \) whose distance from \( y \) is equal to \( p_i |r_k|_i \) and if

\[ (x_{k,0}, \ldots, x_{k,i-1}, x_{k,i+1}, \ldots) = \pi_i^+(x_k), \]
Lemma 3.3. Let \( \pi^i_0 : \Sigma_{\mathcal{P}} \to \mathbb{R} \times \prod_{j \neq i} \mathbb{Q}_{p_j} \) be the complementary projection of \( \pi_i \), he defines
\[
B_{h_k+1} = \overline{B}(\langle x_{k,0}, \ldots x_{k,i-1}, z_i, x_{k,i+1}, \ldots \rangle, \beta r_k).
\]
This has diameter \( 2\beta r_k \), is contained in \( B_{h_k} \) and avoids the cylinder \( C_{h_k+1} \).

Given any positive lower bound on the CAW dimension, we can boost it to an absolute quantity (depending on \( \mathcal{P} \) alone) for finite solenoids.

Lemma 3.3. Let \( \mathcal{P} = \{ p_1 < \cdots < p_{l-1} \} \) be finite. Any CAW subset \( S \) of \( \Sigma_{\mathcal{P}} \) has winning dimension greater than or equal to \( \beta_\mathcal{P} := \min \{ 1/3, 1/p_{l-1} \} \).

Proof. Assume \( 0 < \beta < \beta_\mathcal{P} \). We are guaranteed that there exists some \( 0 < \beta' < \beta \) such that \( S \) is cylinder \( \beta' \)-absolute winning. Changing the game parameter from \( \beta' \) to \( \beta \) only enlarges the set of choices available to Alice while Bob continues to have some legal choice left as long as \( \beta < 1/3 \) and \( \beta < 1/p_{l-1} \leq 1/p_i \), if \( i > 0 \) is the constraining coordinate of the cylinder blocked by Alice in the previous move. Also, all of his valid moves in the cylinder \( \beta \)-absolute game remain so in the \( \beta' \)-game. Alice just needs to pretend that the game parameter is \( \beta' \) and follow her winning strategy for the same.

4. Non-dense orbits of solenoidal maps. As already mentioned in \( \S \) 1, an affine endomorphism \( A : \Sigma_{\mathcal{P}} \to \Sigma_{\mathcal{P}} \) is of the form \( x \mapsto (m/n)x + a \) for some \( m/n \in R \) and \( a \in \Sigma_{\mathcal{P}} \). Here, \( R \) is the set of endomorphisms of the solenoid \( \Sigma_{\mathcal{P}} = \Sigma_{\mathcal{P}/\Delta(R)} \) given by the ring \( R = \mathbb{Z} \{ \{1/p_i \mid p_i \in \mathcal{P} \} \} \). The affine transformation \( A \) is invertible iff \( n/m \in R \) too.

Next, the cylinder \( \beta \)-absolute game on \( \Sigma_{\mathcal{P}} \) can be shifted to a game played on \( \Sigma_{\mathcal{P}} \) once the radii of the balls \( B_j \subset \Sigma_{\mathcal{P}} \) become small enough (say < 1/2) for all large \( j \in \mathbb{N} \). This can be forced on Bob in finitely many steps after the beginning of the game.

Pick some \( y \in \Sigma_{\mathcal{P}} \) and let \( A \) be any fixed affine transformation of \( \Sigma_{\mathcal{P}} \) with its linear part \( m/n \in R \setminus \{ 0, \pm 1 \} \). We abuse notation and call any of its lifts from \( \Sigma_{\mathcal{P}} \to \Sigma_{\mathcal{P}} \) to be \( A \) too. Note that any such lift is an invertible self map of \( \Sigma_{\mathcal{P}} \) as long as \( A : \Sigma_{\mathcal{P}} \to \Sigma_{\mathcal{P}} \) is surjective. Further, let \( F_A(y) \) denote the set of points \( x \in \Sigma_{\mathcal{P}} \) whose image \( x := A(y) \) has its \( A \)-orbit not entering some \( \delta(x, y, A) \)-neighbourhood of \( y \). The goal for Alice is to avoid \( \varepsilon \)-neighbourhoods of the grid points \( \Pi^{-1}(\{y\}) = y + \Delta(R) \) (for some \( y \in \Pi^{-1}(\{y\}) \)) which are all at least a unit distance away from each other. Otherwise said, the set \( F_A(y) \) that Alice should aim for is
\[
\bigcup_{t \geq 0} \left( \bigcup_{j=0}^{\infty} A^{-j}(\Delta(R)) + B(y, t) \right)
\]
and \( F_A(y) \) shall be the image set for the game played on \( \Sigma_{\mathcal{P}} \). We let \( \lambda_i := |m/n|_{p_i} \) for every \( i \geq 0 \) and \( \lambda_A \) to be sup, \( \lambda_i \). By our assumption about \( A \), the linear part \( m/n \) belongs to \( R \setminus \{ 0, \pm 1 \} \) and thus, there exists some \( i_0 \geq 0 \) such that
\[
\lambda_A = \lambda_{i_0} > 1.
\]
As \( \lambda_i = 1 \) for all but finitely many \( i \), the quantity \( \lambda_A \) is finite. In particular for any \( x_1, x_2 \in \Sigma_{\mathcal{P}} \), we have that
\[
d(A^{-j}x_1, A^{-j}x_2) \geq \lambda_A^{-j}d(x_1, x_2) \quad \text{for all } j \geq 0.
\]
as translation by any \( a \) is an isometry of \( \Sigma_P \) (and \( X_P \)). Equivalently for any two subsets \( F_1, F_2 \subseteq X_P \),

\[
d( A^{-j} F_1, A^{-j} F_2 ) \geq \lambda^{-j}_A d( F_1, F_2 ) \text{ for all } j \geq 0.
\]

The constraining coordinate of all the cylinders removed by Alice will be some fixed \( i_0 \) for which \( \lambda_{i_0} = \lambda_A \). Let \( 0 < \mu < 1 \) and \( \ell \in \mathbb{N} \) be the smallest for which \( \lambda^{-\ell}_A < \mu \). If \( a = m^{-\ell} \), then \( (m/n)^{-j} R \subseteq aR \) for all \( j \in \{0, \ldots, \ell \} \). As is the case for \( R \), the points of \( aR \) too constitute a \( \delta \)-uniformly discrete set for some \( 0 < \delta = \delta(a, P) \leq 1 \).

We also choose \( b \geq 1 \) given by

\[
b = \max_{0 \leq j \leq \ell} \sup_{x \in \overline{B(0,1)}} d(A^{-j}x, 0)
\]

and let

\[
t_0 = \min \left\{ \frac{1}{3b} \left\{ d(y, A^{-j}y, a \nu) > 0 \mid 0 \leq j \leq \ell, \ \nu \in \Delta(R) \right\} \right\}.
\]

This belongs to \( ]0, 1/3[ \) and thereby for any \( z_1, z_2 \in \Delta(R) \) such that \( y + z_1 \neq A^{-j}(y + z_2) \) for some \( 0 \leq j \leq \ell \), we have

\[
d \left( B(y + z_1, t_0), A^{-j} \left( B(y + z_2, t_0) \right) \right) \geq d \left( B(y + z_1, bt_0), B \left( A^{-j}(y + z_2), bt_0 \right) \right) \geq 3bt_0 - 2bt_0 = bt_0 \geq t_0.
\]

If \( j_1 \leq j_2 \) are any two exponents such that \( \mu^{j_1 + 1} \leq \lambda^{-j_2}_A \leq \lambda^{-j_1}_A \leq \mu^k \) for some \( k \in \mathbb{N} \), then \( j_2 - j_1 \leq \ell \) by the very definition of \( \ell \). Hence, for \( 0 < t < \mu t_0/2 \) and any \( z_1, z_2 \in \Delta(R) \) for which \( y + z_1 \neq A^{-j}(y + z_2) \), we get that

\[
d \left( A^{-j_1} B(y + z_1, t), A^{-j_2} \left( B(y + z_2, t) \right) \right) > \mu^{j_1 + 1} t_0 = \frac{\mu t_0}{2} (\mu^k + \mu^k)
\]

while \( A^{-j} B(y + z, t) \subset C(A^{-j}(y + z), \lambda^{-j}_A t, i_0) \subset C(A^{-j}(y + z), \mu^k t, i_0) \) for \( \lambda^{-j}_A \leq \mu^k \). We let

\[
N = \{ (j, y + z) \mid (j, y + z) \in A(0) \}
\]

be the countable indexing set in our Theorem 3.2. The first hypothesis therein is satisfied by taking \( R = 1 \) and for \( n = (j, y + z) \), letting

\[
A(n, t) = A^{-j} B(y + z, t) \subset C(n, t) := C(A^{-j}(y + z), \lambda^{-j}_A t, i_0)
\]

which suggests that we should take \( R_n = \lambda^{-j}_A \). Clearly, the second hypothesis has been shown to hold here in (6). Hence, we infer that \( F_A(y) = \cup_{t>0} \left( X_P \setminus \cup_{j=0}^\infty A^{-j} \left( R + B(y, t) \right) \right) \) is a CAW subset of \( X_P \) with winning dimension as in the statement of Theorem 3.2 and so its image in \( \Sigma_P \). Because Proposition 2, we can extend this result to the set of points whose \( A \)-orbits avoid some neighbourhoods of countably many points \( \{y_k\}_{k \in \mathbb{N}} \subset \Sigma_P \).

If \( A(x) = (m/n)x + a \) is such that \( m/n = -1 \), then \( A^2 \) is the identity endomorphism. In this case, Alice only needs to move the game away from the countable set \( \{y_k\} \cup \{a - y_k\} \). This is trivial. The situation is even simpler when \( \lambda_A \) is just the identity map. Now, let \( Y \) be the set consisting of all those points of \( \Sigma_P \) which have a periodic orbit for some \( B \in R \setminus \{\pm 1\} \). This is countable and leads us to conclude:

**Theorem 4.1.** Let \( \{A_j : x \mapsto (m_j/n_j)x + a_j\}_{j \in \mathbb{N}} \) be any subset of affine surjective endomorphisms of the solenoid \( \Sigma_P \) such that

1. none of the \( A_j \)'s is a non-trivial translation of \( \Sigma_P \), and

...
2. the collection of rational numbers \(\{m_j/n_j\}_{j \in \mathbb{N}}\) belong to some finite extension of \(\mathbb{Z}\).

Then, the set of points whose orbit closure under the action of any of the \(A_j\)’s does not contain any periodic \(B\)-orbit for all \(B \in R \setminus \{\pm 1\}\) is cylinder absolute winning.

**Proof.** If \(\{A_j\}_{j \in \mathbb{N}} \subseteq \mathbb{Z}[\{1/p_1, \ldots, 1/p_n \mid p_i \in \mathcal{P}\}\right]\), then the winning dimension of each of the subsets \(F_{A_j}\) is at least \(\min\{1/3, \min\{1/p_i \mid 1 \leq i \leq n\}\} > 0\). This is also a lower bound on \(\text{CAW dim} \left( \cap_j F_{A_j} \right)\) invoking Proposition 2 once again. \(\square\)

Note that even though \(R\) is a countable set, we cannot further this argument to take intersections over any arbitrarily chosen sequences of affine surjective endomorphisms of \(\Sigma\). This is because the lower bound on the winning dimension of the CAW subsets of \(\Sigma\) corresponding to each \(A\) is dependent on \(A\) itself in terms of \(i_0\) for which \(\lambda_{i_0} = \lambda_A\). However, for finite \(\mathcal{P}\), each such \(i_0\) is at least \(\min\{1/3, \min\{1/p_i \mid p_i \in \mathcal{P}\}\}\). We can then remove the second condition in Theorem 4.1 to get

**Theorem 4.2.** Let \(\mathcal{P}\) be a finite set of rational primes and \(\{A_j\}\) be any sequence of affine surjective endomorphisms of \(\Sigma\) such that none of the \(A_j\)’s is a translation. The set of points whose orbit closure under the action of any \(A_j\) does not contain any periodic \(B\)-orbit for all \(B \in R \setminus \{\pm 1\}\) is CAW with winning dimension at least \(\min\{1/3, 1/p_{i-1}\}\). Here, \(p_{i-1}\) is the largest prime in \(\mathcal{P}\).

In particular, this is true of the collection of all surjective endomorphisms of \(\Sigma\).

5. **Sizes of CAW subsets.** Let \(\mathcal{P}\) be finite. We start by discussing the implications of the cylinder absolute winning property of a subset \(F\) for a strong game played on \(X_{\mathcal{P}}\) with \(F\) as its target.

**Proposition 3.** A CAW subset of \(X_{\mathcal{P}}\) is \(\alpha\)-strong winning for all \(\alpha < \beta_{\mathcal{P}}\).

**Proof.** Without loss of generality, we may take \(\text{CAW dim}(F)\) to be \(\beta_{\mathcal{P}}\) due to Lemma 3.3. This means our target set \(F\) is \(\beta\)-CAW for all \(0 < \beta < \beta_{\mathcal{P}}\). Now, suppose that \(\alpha \in [0, \beta_{\mathcal{P}}\} \) and \(\gamma \in [0, 1\} \) are any fixed (strong) game parameters for Alice and Bob, respectively.

Given a ball \(B = \overline{B}(x, r)\) chosen by Bob at any stage of the strong game, Alice checks the cylinder \(C\) with radius \((C) \leq \alpha \gamma r\) to be removed by her in accordance with her winning strategy for \(F\) when playing the cylinder \((\alpha \gamma)\)-absolute game. If \(B \cap C = \emptyset\), she chooses any \(A \subseteq B\) allowed by the rules of the strong game. Assume this to not be the case for the rest of this proof.

If the constraining coordinate \(i\) of \(C\) is archimedean, Alice has no problem in choosing a Euclidean ball \(A \subseteq \pi_0(B) \setminus \pi_0(C)\) with radius \((A) \geq \alpha r\) as \(\alpha \leq \beta_{\mathcal{P}} \leq 1/3\).

Else again when \(i > 0\), we have \(\pi_i(C) \subseteq \pi_i(B)\) because

\[
\text{radius}(\pi_i(C)) \leq p_i \lfloor \beta_{\mathcal{P}} r \rfloor_i \leq \lfloor r \rfloor_i \quad \text{while} \quad \text{radius}(\pi_i(B)) \geq p_i \lfloor r \rfloor_i.
\]

We can moreover take the center \(x_i\) of \(\pi_i(B)\) to be the same as that of \(\pi_i(C)\). Let \(z_i \in \pi_i(B) \setminus \pi_i(C)\). Then, \(\{z_i - x_i\}_{p_i} = p_i \lfloor r \rfloor_i\) and the ultrametric also gives us that \(B_{\mathcal{Q}}(z_i, \lfloor r \rfloor_i) \subseteq \pi_i(B) \setminus \pi_i(C)\). In either case, the pre-images \(\pi_{i-1}^{-1}(A)\) or \(\pi_{i-1}^{-1}(B_{\mathcal{Q}}(z_i, \lfloor r \rfloor_i))\) contain a ball of \(X_{\mathcal{P}}\) of radius at least \(\beta_{\mathcal{P}} r \geq \alpha r\) which lies inside \(B \setminus C\). Alice chooses one such \(A_1\) to be her next move. Bob’s choice of any \(B_1 \subseteq A_1\) with radius \((B_1) \geq \gamma \cdot \text{radius}(A_1) \geq \alpha \gamma r\) immediately after is also a valid move in cylinder \((\alpha \gamma)\)-absolute game. \(\square\)
It should be mentioned here that the relationship between winning sets for strong games and quasisymmetric homeomorphisms of \( X_\mathcal{P} \) is not clear to us. Nor do we have the analogous statement of Proposition 3 for the full solenoid.

5.1. Incompressibility. The next result is about the incompressible behaviour of cylinder absolute winning subsets of \( X_\mathcal{P} \). A set \( S \subseteq X_\mathcal{P} \) is strongly affinely incompressible if for any non-empty open subset \( U \) and any sequence of invertible affine homeomorphisms \( (\Psi_i)_{i \in \mathbb{N}} \), the set \( \cap_{i \in \mathbb{N}} \Psi_i^{-1} S \cap U \) has the same Hausdorff dimension as \( U \) [6, 9]. It is our claim that CAW subsets of \( X_\mathcal{P} \) are strongly affinely incompressible for finite \( \mathcal{P} \). We show this by proving a lower bound on the CAW dimension of \( \Psi^{-1} S \cap U \) in terms of \( \text{CAW dim}(S) \) for any affine map \( \Psi \) of \( X_\mathcal{P} \).

Together with Proposition 2, this will give us that the intersection of any countably many pre-images of a CAW subset under invertible affine homeomorphisms is CAW too.

**Theorem 5.1.** Let \( \mathcal{P} \) be finite, \( U \subseteq X_\mathcal{P} \) open and \( S \) be any CAW subset. Also, let \( \Psi : X_\mathcal{P} \to X_\mathcal{P} \) be an invertible affine homeomorphism. Then, the set \( \Psi^{-1} S \cup (X_\mathcal{P} \setminus U) \) is also cylinder absolute winning with \( \text{CAW dim}(\Psi^{-1} S \cup (X_\mathcal{P} \setminus U)) \geq \beta_\mathcal{P} \).

**Proof.** As in Proposition 3, we may take the CAW dimension of \( S \) to be \( \beta_\mathcal{P} \) without loss of generality. Let us first make some reductions to simpler situations. If the diameters \( |B_k| \) of balls chosen by Bob don’t go to zero as \( k \to \infty \), then \( \cap_k B_k \) contains an open ball inside it. As \( S \) is a winning subset, it has to be dense and in turn its pre-image \( \Psi^{-1} S \) is also dense in \( X_\mathcal{P} \). Second, if \( B_k \cap (X_\mathcal{P} \setminus U) \neq \emptyset \) for infinitely many \( k \), then they form a decreasing sequence of closed subsets of the compact ball \( B_1 \). Their intersection cannot be empty and hence \( \cap_k B_k \) contains a point of \( X_\mathcal{P} \setminus U \) resulting in Alice’s victory. It is safe to exclude both of these events from the rest of the proof. We can moreover take that \( B_0 \subseteq U \).

Let \( 0 < \beta < \beta_\mathcal{P} \) and \( \Psi(x) = Dx + a \) as explained before. Following [6], Alice will run a ‘hypothetical’ Game 2 (in her mind) where the target set is \( S \) and a different game parameter \( \beta' \) which is some positive power of \( \beta \). She carefully decides and projects some of Bob’s moves in the \( (\Psi^{-1}(S) \cup (X_\mathcal{P} \setminus U), \beta) \)-game to construct choices made by a hypothetical Bob II in Game 2. Since there is a winning strategy for the latter by hypothesis, she channels the winning moves in this second game via the inverse map \( \Psi^{-1} \) to win over \( \Psi^{-1}(S) \). We take

\[
\lambda_\Psi := \max \left\{ \max_{p \in \mathcal{P}} |D|_p, |D| \right\}
\]

which makes sense as \( D \) is a rational number. As \( D \neq 0 \), we have \( 1 \leq \lambda_\Psi < \infty \) for any \( \Psi \) and any \( \mathcal{P} \). It is clear that

\[
d(\Psi(x), \Psi(y)) \leq \lambda_\Psi d(x, y)
\]
as \( d \) is a translation-invariant metric on \( X_\mathcal{P} \). We re-label the choices made by Bob such that radius \( (B_0) < 1/\lambda_\Psi \). Let \( n \in \mathbb{N} \) be the smallest positive natural number for which

\[
\lambda_\Psi \lambda_{\Psi^{-1}}(\beta + 1)\beta^{n-2} < 1, \quad \beta' = \beta^n \quad \text{and} \quad \eta := (\beta + 1)\beta^{n-1}.
\]

Alice waits for the stages \( 0 = j_1 < j_2 < \cdots \) in the original \( (\Psi^{-1}(S) \cup (X_\mathcal{P} \setminus U), \beta) \)-cylinder absolute game when

\[
\beta^n \leq \text{radius} \left( B_{j_k} \right) / \text{radius} \left( B_{j_{k-1}} \right) < \beta^{n-1}
\]
for the first time. Notice that this is well-defined and exists because we assumed \( |B_k| \to 0 \) and the radius of Bob’s choice at \((k+1)\)-th step cannot shrink by a factor of more than \( \beta \) compared to that of his choice at the \( k \)-th step for all \( k \).

Imitating [6], denote
\[
B_{jk}=\overline{B}(x_k,r_k) \quad \text{and} \quad B'_k=\overline{B}(\Psi(x_k),r'_k) \quad \text{where} \quad r'_k=\lambda \Psi r_k
\]
so that \( \Psi(B_{jk}) \subseteq B'_k \) for all \( k \) by our definition of \( \lambda \Psi \). Then, \( \Psi(\cap_k B_k) = \Psi(\cap_k B_{jk}) \subseteq \cap_k B'_k \) while the intersections \( \cap_k B_k \) and \( \cap_k B'_k \) are both singleton sets (we are in the case when \( |B_k| \to 0 \Rightarrow |B'_k| \to 0 \) as \( k \to \infty \)). Thus, we see that the non-emptiness of \( \cap_k B_k \cap S \) will imply that of \( \cap_k B_k \cap \Psi^{-1}(S) \).

If \( C_{k+1} = C(y'_{k+1},\varepsilon'_{k+1},i_{k+1}) \) is the cylinder to be removed by Alice in Game 2 where \( \varepsilon'_{k+1} \leq \beta' r'_k \), she chooses \( C_{j_{k+1}} \) as
\[
C(\Psi^{-1}(y'_{k+1}),\lambda \Psi \cdot \eta r_k,i_{k+1}) \supseteq \Psi^{-1}(C(y'_{k+1},\varepsilon'_{k+1},i_{k+1}))
\]
to be blocked next in the \( \beta \)-cylinder absolute game with target set \( \Psi^{-1}(S) \). By design, \( \lambda \Psi \cdot \eta r_k < \beta \) and it only remains to show that Bob has some choice of \( B_{j_{k+1}} \subseteq B_{jk} \setminus C_{j_{k+1}} \) left with radius \( B_{j_{k+1}} \geq \beta \cdot \lambda r_k \). As we have seen in the proof of Theorem 3.2, this is clearly not a problem as \( \beta < \beta_P \leq \min \{1/3,1/p_{i_{k+1}}\} \) when \( i_{k+1} > 0 \) or otherwise. All of Bob’s subsequent choices in Game 1, including \( B_{j_{k+1}} = \overline{B}(x_{k+1},r_{k+1}) \), obey
\[
|x_{k,0} - x_{k+1,0}| \leq r_k - r_{k+1}
\]
in the archimedean coordinate and
\[
|x_{k,i} - x_{k+1,i}|_{p_i} \leq p_i r_k \quad \text{for all} \quad i > 0.
\]
Then,
\[
|\Psi(x_{k+1})_0 - \Psi(x_k)_0| = |Dx_{k+1,0} + a_0 - (Dx_{k,0} + a_0)| \\
\leq \lambda \Psi |x_{k+1,0} - x_{k,0}| \leq \lambda \Psi (r_k - r_{k+1}) = r'_k - r'_{k+1},
\]
and for all \( i > 0 \), \( |\Psi(x_{k+1})_i - \Psi(x_k)_i|_{p_i} \leq \lambda \Psi p_i r_k = p_i r'_{k+1} \) similarly. The conclusion cannot be escaped that the corresponding ball \( B'_{k+1} = \overline{B}(\Psi(x_{k+1}),r'_{k+1}) \subseteq B'_k \) in Game 2. It is also outside of \( C'_{k+1} \) when \( i_{k+1} > 0 \) as
\[
|x_{k+1,i_{k+1}} - \Psi^{-1}(y'_{k+1})|_{i_{k+1}} \geq p_{i_{k+1}} \lambda \Psi \eta r_k
\]
which gives that
\[
\Psi(x_{k+1})_{i_{k+1}} - y'_{k+1, i_{k+1}} \bigg|_{i_{k+1}} \geq p_{i_{k+1}} \lambda \Psi \eta r_k = p_{i_{k+1}} \eta r'_{k+1}
\]
\[
= p_{i_{k+1}} (\beta' \beta^{n-1}) r'_{k+1}
\]
\[
\geq p_{i_{k+1}} (\varepsilon'_{k+1} + r'_{k+1})
\]
by Alice’s choice of marking for \( B_{j_{k+1}} \). The computations are not very different when the constraining coordinate of \( C'_{k+1} \) is archimedean.

For general \( \mathcal{P} \), we are only able to show the largeness of countable intersections of pre-images under translations of \( X_P \).

**Proposition 4.** Let \( (a_k)_{k \in \mathbb{N}} \subseteq X_P \) be any arbitrary sequence and \( S \) be a CAW subset with winning dimension \( \beta_0 \). Then, so is \( S \cap \bigcap_{k \in \mathbb{N}} (S + a_k) \).
Proof. Each of the translations $\Psi_k(x) := x - a_k$ is an isometry and in particular, does not change the shape of the balls in $X_P$. Given any such single $\Psi$, we argue that $\Psi^{-1}(S)$ has the same CAW dimension as $S$. Alice simply translates back her choices for the Game 2 described above by $-a$ when $\Psi(x) = x + a$ and projects Bob’s succeeding choice forward by $\Psi$. Note that as $\lambda_{\Psi_k} = \lambda_{\Psi^{-1}} = 1$ for all $k$, she should take $\beta' = \beta$ for any $0 < \beta < \beta_0$. One should also replace $\eta = 1$ in the previous calculations. The countable intersection property then follows by Proposition 2.

5.2. Hausdorff dimension and measures. Lastly, we will try to understand the sizes of CAW sets in terms of Hausdorff dimensions and measures. Towards this goal, we will require an estimate on the number of legal choices that Bob has at any stage of the game.

Lemma 5.2. Let $0 < \beta < 1$. Then, the maximum number of pairwise disjoint balls of radius $\beta r$ contained in any closed ball $B(x, r) \subset X_P = \mathbb{R} \times \prod_{j>0} \mathbb{Q}_{p_j}$, which do not intersect an open cylinder $C(y, \beta r, i)$ and also maintain a distance at least $\beta r$ from each other is given by

$$N_C(\beta) \gg \beta^{-\frac{\theta_P(1/\beta)}{\ln(1/\beta)}}$$

where $\theta_P(t) = \sum_{p \in P, p \leq t} \ln p$.

Proof. Because of (4), every closed ball is the Cartesian product of its coordinatewise projections. In any non-archimedean coordinate $j$ such that $p_j \leq 1/\beta$, there are at least $(p_j \lceil \beta \rceil_j)^{-1}$ pairwise disjoint balls contained in the projection $\pi_j(B(x, r)) = B(x_j, \lceil p_j r \rceil_j)$ whose radius equals $\lfloor p_j \beta r \rfloor_j$. Each of them also maintain a distance of at least $p_j^2 \lfloor \beta r \rfloor_j$ from each other which means that the pre-images of any two such sub-balls in $X_P$ are $\geq p_j \lfloor \beta r \rfloor_j > \beta r$ away. The lower bound $(p_j \lfloor \beta \rfloor_j)^{-1} = (1/\beta)_j$ unless $\beta$ is an integral power of $p_j$ in which case it is $p_j^{-1} \lfloor 1/\beta \rfloor_j$. Note that this can happen for at most one prime for any given $\beta$ and we have already assumed $\beta < 1/p_j$. In the real coordinate, this number is $\gg \beta^{-1}$ even when we ask that the balls are at least $\beta r$ apart. The pre-images under the projection map $\pi_j : X_P \rightarrow \mathbb{R} \times \prod_{p_j \leq 1/\beta} \mathbb{Q}_{p_j}$ of any product of these sub-balls of $\pi_j(B(x, r))$ for $j = 0$ or $p_j \leq 1/\beta$ are pairwise disjoint, each contain at least one sub-ball of $B(x, r)$ of radius $\beta r$ and the minimum distance between any two of those pre-images is $\geq \beta r$.

When asking for only those sub-balls that do not intersect the open cylinder $C(y, \beta r, i)$, it is necessary and sufficient that we restrict ourselves to only those from the above chosen collection whose images in $\pi(B(x, r))$ do not intersect $\pi_i(C(y, \beta r, i))$. Otherwise said, all coordinates but $i$ are not affected. If $i = 0$, the number of such balls in $\pi_0(B(x, r)) = B(x_0, r)$ that do not intersect $\pi_0(C(y, \beta r, 0)) = B(y_0, \beta r)$ is still $\gg \beta^{-1}$ albeit with a smaller constant. Else, it is at least $(p_i \beta)^{-1} - 1 > \beta^{-1}/2p_i$. The Cartesian product of these disjoint balls in $\mathbb{Q}_{p_j}$’s and $\mathbb{R}$ then gives us that

$$N_C(\beta) \gg \prod_{p_j \in P, p_j < 1/\beta} \lfloor \beta^{-1} \rfloor_j \geq \beta^{-\frac{\theta_P(1/\beta)}{\ln(1/\beta)}}$$

for all $0 < \beta < 1$, by a calculation similar to the one in Lemma 2.1. \qed
As discussed in §2, \( \theta_P(t) \sim t \) when \( t \to \infty \) and \( P \) is the full set of primes. More generally, it shows an asymptotic linear growth with \( t \) when \( P \) is any (infinite) set of all primes in an arithmetic progression. We record that the constant implied by the Vinogradov notation in (7) is independent of the constraining coordinate of the cylinder \( C \). For finite \( P \), we use a slightly different lower bound.

**Lemma 5.3.** Let \( |P| = l - 1 \) and \( p_{l-1} \) be the largest prime in \( P \). Then,
\[
N_C(\beta) \gg \beta^{-l} \text{ for all } 0 < \beta < 1/p_{l-1},
\]
where the implied constant may depend on the primes \( p_1, \ldots, p_{l-1} \) and \( l \).

**Proof.** We only need to replace the lower bound for the number of disjoint sub-balls in each non-archimedean coordinate by \( \beta^{-1}/p_j \) and the rest of the argument remains the same. \( \square \)

Suppose \( F \) is a cylinder absolute winning subset of \( X_P \) and that Alice always plays according to a winning strategy if it is available. We shall now construct a subset \( F^* \subseteq F \) which corresponds to the points obtained when Bob is only allowed to choose one of the \( N_C(\beta) \)-many sub-balls described in Lemmata 5.2 or 5.3 at each stage of the game. This resembles closely a device from Kristensen [16, Theorem 4.1] (see also [20, Theorem 6]).

**Proposition 5.** Let \( f \) be any dimension function such that
\[
\limsup_{\delta \to 0} \frac{\log f(\delta)}{\log \delta} < \liminf_{\beta \to 0} \frac{\log N_C(\beta)}{|\log \beta|}.
\]
Then, the \( f \)-dimensional Hausdorff measure of any CAW subset of \( X_P \) is greater than zero.

**Proof.** Let \( \beta_0, \delta_0 > 0 \) be small enough so that \( \beta_0 \) is less than the winning dimension of our CAW set \( F \), \( \delta_0 < 1 \) and
\[
\sup_{\delta < \delta_0} (\log f(\delta)/\log \delta) < \inf_{\beta \leq \beta_0} \left( \log N_C(\beta)/|\log \beta| \right).
\]
They exist by virtue of our hypothesis about \( f \). Now, let
\[
\Lambda := \{0, 1, \ldots, N_C(\beta_0) - 1\}^N
\]
be the sequence space each of whose element \( \lambda = (\lambda_k)_{k \in \mathbb{N}} \) corresponds to a sequence of choices made by Bob when he is only allowed to choose from one of the \( N_C(\beta_0) \)-many disjoint sub-balls inside \( B_{k-1} \). The choices made by him at the \( k \)-th stage are labelled \( B(\lambda_1, \lambda_2, \ldots, \lambda_k) \). If \( \lambda \neq \lambda' \), they differ in some entry \( k_0 \) and the corresponding balls \( B(\lambda_1, \ldots, \lambda_{k_0}) \) and \( B(\lambda'_1, \ldots, \lambda'_{k_0}) \) are disjoint. This implies that the points obtained at infinity, \( a_\infty(\lambda) = \cap_{k \to \infty} B(\lambda_1, \ldots, \lambda_k) \neq a_\infty(\lambda') = \cap_{k \to \infty} B(\lambda'_1, \ldots, \lambda'_k) \subseteq F \) under the belief that Alice is following a winning strategy for the target set \( F \) with game parameter \( \beta_0 \). Let
\[
F^* := \{a_\infty(\lambda) \mid \lambda \in \Lambda\} \subseteq F.
\]
We show that \( \mathcal{H}^f(F^*) > 0 \) and this shall in turn give us our desired statement. For this, the space \( F^* \) is mapped in a continuous fashion (via the bijection with \( \Lambda \)) onto \([0, 1]\) using the \( N_C(\beta_0) \)-adic expansion of real numbers, namely \( a_\infty(\lambda) \mapsto 0.\lambda_1\lambda_2 \cdots \).

Call this map \( \psi \) and let \((U_n)_{n \in \mathbb{N}}\) be any \( \delta \)-cover of \( F^* \) for some \( \delta < \delta_0 \). Without
loss of generality, let $U_n \subseteq F^*$ for all $n$. Plainly, $(\psi(U_n))_{n \in \mathbb{N}}$ is a cover for $[0, 1]$ and since diameter is an outer measure on $\mathbb{R}$, we get that
\[
1 \leq \sum_{n \in \mathbb{N}} |\psi(U_n)|.
\]
Define
\[
j_n = \left\lfloor \frac{\log(2 |U_n|)}{\log \beta_0} \right\rfloor
\]
so that $j_n > 0$ for all $|U_n|$ small enough and furthermore, $|U_n| < \beta_0^{j_n}$. Thus, $U_n$ intersects non-trivially with at most one of the balls $B(\lambda_1, \ldots, \lambda_{j_n})$ as any two such are at least $\beta_0^{j_n}$ apart. Further, being a subset of $F^*$ it is completely contained inside some $B(\lambda_1, \ldots, \lambda_{j_n})$. The latter itself is mapped by $\psi$ into the interval of length $N_c(\beta_0)^{-\lambda_j}$ of $I$ consisting of numbers whose $N_c(\beta_0)$-adic expansion begins with $0,\lambda_1 \cdots \lambda_{j_n}$. We conclude that $|\psi(U_n)| \leq N_c(\beta_0)^{-\lambda_j}$ and thereby,
\[
1 \leq \sum_{n \in \mathbb{N}} |\psi(U_n)| \leq \sum_{n \in \mathbb{N}} N_c(\beta_0)^{-\lambda_j}
\]
\[
= \sum_{n \in \mathbb{N}} N_c(\beta_0)^{-\lambda_j} \left\lfloor \frac{\log(2 |U_n|)}{\log \beta_0} \right\rfloor \leq N_c(\beta_0)^{-\lambda_j} \cdot 2^{-\lambda_j} \sum_{n \in \mathbb{N}} |U_n|^\frac{\log N_c(\beta_0)}{\log |U_n|}.
\]
As $|U_n| \leq \delta < \delta_0$ for all $n \in \mathbb{N}$ and
\[
\frac{\log N_c(\beta_0)}{\log \beta_0} > \sup_{\delta < \delta_0} \frac{\log f(\delta)}{\log \delta} \geq \frac{\log f(|U_n|)}{\log |U_n|}
\]
by our assumption, we have that
\[
\sum_{n \in \mathbb{N}} f(|U_n|) \geq (N_c(\beta_0))^{-\lambda_j} 2^{-\lambda_j} \frac{\log N_c(\beta_0)}{\log \beta_0}
\]
for any arbitrary $\delta$-cover $(U_n)$ of $F^*$. Thus, the infimum of the sums on the left side of (9) taken over all $\delta$-coverings of $F^*$ is a positive number independent of $\delta$. Letting $\delta \to 0$ from the right, we conclude that the $f$-dimensional Hausdorff measure of $F^*$ is strictly positive. In particular, this proves our claim.

Corollary 1. Let $\Sigma$ be the full solenoid over $S^1$. Then, the Hausdorff dimension of any CAW subset of $\Sigma$ is infinite.

Proof. We know $N_c(\beta)$ rises faster than $\beta^{c/(\beta \ln \beta)}$ as $\beta \to 0$ for some absolute constant $c > 0$, when $\mathcal{P}$ is the set of all rational primes. Take $f$ to be the power function $r \mapsto r^n$ for some $n \in \mathbb{N}$. The condition in Proposition 5 is satisfied then and we get that the $n$-dimensional Hausdorff measure of any CAW subset of $\Sigma$ is positive. Finally, we let $n \to \infty$.

The same is true for CAW subsets of $\Sigma_{\mathcal{P}}$, when $\mathcal{P}$ is an infinite set consisting of all primes in some arithmetic progression. A shortcoming of Proposition 5 is that it only tells us whether the $f$-dimensional Hausdorff measure of our winning sets is different from zero and no more. It will be interesting to study the class of exact dimension functions for the spaces $X_{\mathcal{P}}$ and their CAW subsets [10]. For finite $\mathcal{P}$, we are satisfied with a statement about maximality of Hausdorff dimension.

Proposition 6. Let $|\mathcal{P}| < \infty$ and $F \subseteq X_{\mathcal{P}}$ be a $\beta$-CAW set. Then,
\[
dim F \geq \frac{\log N_c(\beta)}{|\log \beta|}.
\]
Proof. The proof is the same as that for Proposition 5 till (8) with $\beta$ replacing $\beta_0$ everywhere.

Once again if we let $\beta \to 0$ for a CAW set, we get

**Corollary 2.** Any CAW subset of $X_P$ with $|P| = l - 1$ has Hausdorff dimension equal to $l$. In particular, the collection of points described in Theorem 4.2 has full dimension.

**Remark.** One can similarly consider orbits under surjective affine endomorphisms of solenoids defined over higher-dimensional tori by quotienting the restricted product space $\mathbb{R}^n \times \prod_{p \in P} \mathbb{Q}_p^\times$ by the diagonal embedding of the product ring $\mathbb{R}^n$. This will bring its own set of challenges and may be worth exploring.

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