A review of Lie 2-algebras

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Abstract

We first recall two equivalent definitions of Lie 2-algebras, categorification of Lie algebras and 2-term $L_{\infty}$-algebras. Then we present four different kinds of Lie 2-algebras from 2-plectic manifolds, Courant algebroids, homotopy Poisson manifolds and affine multivector fields on a Lie groupoid respectively. Moreover, we recall the cohomology theory of Lie 2-algebras and analyze its lower degree cases. The integration of strict Lie 2-algebras to strict Lie 2-groups is also discussed.

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1 Introduction

As categorification of Lie algebras, the notion of Lie 2-algebras was introduced by Baez and Crans in 2004 [2]. It is one of the fundamental objects in higher Lie theory and has close connection with strongly homotopy Lie algebras (also called $L_\infty$-algebras) introduced by Schlessinger and Stasheff in [53]. See also [30, 31]. A Lie 2-algebra is given by replacing the Jacobi identity in a Lie algebra with an isomorphism, called the Jacobiator, which satisfies a new law of its own. Likewise, if the associative law in a Lie group is substituted with an isomorphism, called the associator, one gets a Lie 2-group; see [1]. Since its appearance, the Lie 2-algebra structure has enjoyed significant applications in both geometry and mathematical physics. Its own algebraic properties have also drawn people’s attention.

The purpose of this paper is to give a brief overview of Lie 2-algebras on certain aspects. We first present the definition from two equivalent points of view, the categorification and the 2-term $L_\infty$-algebra. Explicitly, a Lie 2-algebra is given by a bilinear skew-symmetric bracket on a linear category, which is a functor and obeys the Jacobi identity only up to a trilinear coherent natural transformation (the Jacobiator). It is equivalent to a 2-term $L_\infty$-algebra, which is a 2-term chain complex of vector spaces with a 2-bracket and a 3-bracket. The Jacobi identity of the 2-bracket fails but is controlled by the 3-bracket which satisfies its own law. In particular, when the Jacobiator is an identity, i.e., the 3-bracket is trivial, one gets a strict Lie 2-algebra. Strict Lie 2-algebras are equivalent to Lie algebra crossed modules, which are classified by the third cohomology of a Lie algebra [21]. When the source and target of any morphism in a Lie 2-algebra are equal, i.e., the differential of the 2-term complex is zero, we get a skeletal Lie 2-algebra. Skeletal Lie 2-algebras one-to-one correspond to triples that consist of a Lie algebra, a Lie algebra module and a 3-cocycle. In fact, every Lie 2-algebra is equivalent to a skeletal one. Thus Lie 2-algebras are classified by using the third cohomology of Lie algebras. See [2].

Then we review several Lie 2-algebras that come from geometric structures, namely, 2-plectic manifolds, Courant algebroids, homotopy Poisson manifolds and affine multivector fields on Lie groupoids. A 2-plectic structure on a manifold is a nondegenerate closed 3-form. Similar to the Poisson bracket on the functions of a symplectic manifold, there is a Lie 2-algebra structure on functions and Hamiltonian 1-forms of a 2-plectic manifold [3, 47, 48]. This Lie 2-algebra is further used to define homotopy moment maps [11, 20, 41]. A Courant algebroid [38] is a vector bundle together with a bilinear form, a skew-symmetric bracket and an anchor map. The bracket satisfies the Jacobi identity up to a coboundary, which generates a Lie 2-algebra on the spaces of sections of the bundle and functions on the base manifold [52]. Parallel to the fact that there is a one-to-one correspondence between Lie algebra structures on a vector space and linear Poisson structures on the dual space, there is a one-to-one correspondence between Lie 2-algebra structures on a pair of vector spaces and linear homotopy Poisson structures on the dual [35]. A homotopy Poisson structure on a graded manifold is an $L_\infty$-algebra structure on the functions which is compatible with the graded commutative algebra structure by the Leibniz rule; see [12, 27, 35, 59]. At last, on a Lie groupoid, multivector fields that are compatible with the groupoid multiplication are called multiplicative. Multiplicative multivector fields with the Schouten bracket form a graded Lie algebra, which is not invariant under the Morita equivalence of Lie groupoids. Thus, to define multivector fields on a differentiable stack, one needs to extend the Lie algebra to a Lie 2-algebra formed by affine multivector fields on a Lie groupoid, which is Morita invariant; see [8, 9, 34, 45].

The cohomology theory of a Lie 2-algebra with a representation on a 2-vector space is built up and lower degree cases are studied. The first cocycles with respect to the adjoint representation on itself is used to define derivations of a Lie 2-algebra [33]. A derivation Lie 2-algebra and a derivation Lie 3-algebra are further constructed in order to classify nonabelian extensions of a Lie 2-algebra [13]. The second cohomology is to study deformations of a Lie 2-algebra and it classifies the abelian extension of a Lie 2-algebra [37]. The third cohomology is to classify strong crossed modules for Lie 2-algebras [32]. The corresponding results for Lie algebras are
well-known.

At last, we review the integration of strict Lie 2-algebras to strict Lie 2-groups, which is the same to integrating a Lie algebra crossed module to a Lie group crossed module. In particular, the integration of the derivation Lie 2-algebra is the automorphism Lie 2-group. The review is far from complete. Due to the limitations of our knowledge and the space of the paper, many important applications of Lie 2-algebras are missed here. For example, (1): as Lie algebras and Lie groups describe the symmetries of particles, Lie 2-algebras and Lie 2-groups are used to illustrate symmetries of symmetries. So Lie 2-algebras play essential roles in higher gauge theory. In fact, connections on principal 2-bundles are defined as Lie 2-algebra valued 1-forms; (2): the integration of a nonstrict Lie 2-algebra is not discussed in detail here. We refer to [2, 6, 22, 24, 57, 58]; (3): Lie 2-bialgebras ([7, 14, 29]) and the integration to Poisson Lie 2-groups are missed here; see [15].

This paper is organized as follows. Section 2 is about the two equivalent definitions of Lie 2-algebras and some basic facts about strict and skeletal Lie 2-algebras. A Lie 2-algebra from the skew-symmetrization of a Leibniz algebra is also given. In Section 3, we review four Lie 2-algebra structures from 2-plectic manifolds, Courant algebroids, homotopy Poisson manifolds and affine multivector fields on a Lie groupoid, respectively. In section 4, we recall the cohomology theory of Lie 2-algebras and analyze the significance of lower degree cohomologies. The last section is about the integration of strict Lie 2-algebras to strict Lie 2-groups. We discuss the integration of the derivation Lie 2-algebra to the automorphism Lie 2-group. Appendix A and B include a brief introduction on strict 2-categories and Lie groupoids and algebroids with some examples.

2 Definitions and examples

In this section, we first introduce Lie 2-algebras from the categorification point of view and by explicit formulas, under the names of semistrict Lie 2-algebras and 2-term $L_\infty$-algebras respectively. See [30, 31] for more about $L_\infty$-algebras. Then two special cases, strict Lie 2-algebras and skeletal Lie 2-algebras, are discussed. Strict Lie 2-algebras are equivalent to Lie algebra crossed modules, which can be classified by the 3rd cohomology of Lie algebras. Skeletal Lie 2-algebras are used to classify Lie 2-algebras in terms of the 3rd cohomology of Lie algebras. At last, a class of Lie 2-algebras is constructed by the skew-symmetrization of Leibniz algebras.

2.1 Lie 2-algebras and 2-term $L_\infty$-algebras

A Lie 2-algebra is the categorification of a Lie algebra. It is a linear category with a functorial skew-symmetric bilinear operation satisfying the Jacobi identity up to natural transformations which obey coherence laws of their own. To get a Lie 2-algebra, we first categorify a vector space to a 2-vector space. Let Vect denote the category of vector spaces. A 2-vector space is a category in Vect. Explicitly, a 2-vector space is a category whose objects and morphisms form vector spaces and all the structure maps are linear.

**Definition 2.1.** (2) A (semistrict) Lie 2-algebra is a 2-vector space $L$ equipped with

- a skew-symmetric bilinear functor, the bracket, $[\cdot, \cdot] : L \times L \to L$;

- a completely skew-symmetric trilinear natural isomorphism, the Jacobiator,

$$J_{x,y,z} : [[x, y], z] \to [x, [y, z]] + [[x, z], y],$$

that is required to satisfy the **Jacobiator identity**:

$$J_{w,x,y}([J_{w,x,z}, y] + 1)(J_{w,[x,z], y} + J_{w,[z,x], y} + J_{w,x,[y,z]}) =$$

$$[J_{w,x,y}, z](J_{w,[y,z], x} + J_{w,[x,y], z})([J_{w,y,z}, x] + 1)([w, J_{x,y,z}] + 1),$$
for all $w,x,y,z \in L_0$, where $L_0$ is the object space of $L$. It is called a strict Lie 2-algebra if the Jacobiator is the identity and a skeletal Lie 2-algebra if the source and target of any morphism are equal.

**Remark 2.2.** The Jacobiator identity can be interpreted by a commutative octagon; see [2] for the diagram.

The bracket in a semistrict Lie 2-algebra is strictly skew-symmetric but obeys the Jacobi identity only up to a coherent natural transformation. If both the skew-symmetry and the Jacobi identity are allowed to hold up to natural transformations, we arrive at a notion called weak Lie 2-algebras introduced by Rotenberg in [51]. Weak Lie 2-algebras are the complete categorification of Lie algebras. In particular, weak Lie 2-algebras with strict skew-symmetrizator are called semistrict and with trivial Jacobiator are called hemistrict. Semistrict and hemistrict Lie 2-algebras form full sub-2-categories of weak Lie 2-algebras. Throughout this paper, we only study semistrict Lie 2-algebras and call them simply as Lie 2-algebras.

Lie 2-algebras are equivalent to an earlier notion called 2-term $L_\infty$-algebras. $L_\infty$-algebras, also called strongly homotopy Lie algebras, were introduced by Stasheff and Schlessinger [53] in the study of deformation theory.

**Definition 2.3.** A 2-term $L_\infty$-algebra $\mathfrak{g}$ is a 2-term chain complex of vector spaces $\mathfrak{g}_{-1} \overset{d}{\rightarrow} \mathfrak{g}_0$ with

- two skew-symmetric bilinear maps $l^0_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ and $l^1_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$, which are always denoted as $l_2$ or $[\cdot , \cdot]$;
- a totally skew-symmetric trilinear map $l_3 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \wedge \mathfrak{g}_0 \rightarrow \mathfrak{g}_{-1}$,

satisfying that

1. $d[x,a] = [x, da], \quad [da,b] = [a, db]$;
2. $[[x,y],z] + [[y,z],x] + [[z,x],y] = -dl_3(x,y,z)$;
3. $[[x,y],a] + [[y,a],x] + [[a,x],y] = -l_3(x,y,da)$;
4. $l_3([w,x,y],z) - l_3([w,y],x,z) + l_3([w,z],y,x) + l_3([x,y],w,z) + l_3([y,z],w,x) - l_3([x,z],w,y) - l_3([w,y,z],x) + l_3([w,x,z],y) + l_3([x,y,z],w) = 0$,

for all $w,x,y,z \in \mathfrak{g}_0$ and $a,b \in \mathfrak{g}_{-1}$.

To build the equivalence of Lie 2-algebras and 2-term $L_\infty$-algebras, we first relate 2-vector spaces with 2-term chain complexes of vector spaces. A 2-term chain complex of vector spaces is a pair of vector spaces with a linear map between them, i.e., $C_{-1} \overset{d}{\rightarrow} C_0$. Given a 2-vector space $\mathbb{V}$, denote by $V_{-1}$ and $V_0$ the spaces of morphisms and objects, and $s,t : V_{-1} \rightarrow V_0$ the source and target maps. Define $d := t|_{\ker s}$, then

$$\ker s \overset{d}{\rightarrow} V_0$$

is a 2-term chain complex of vector spaces. Conversely, for a 2-term chain complex $C_{-1} \overset{d}{\rightarrow} C_0$, define a 2-vector space $\mathbb{V}$ by

$$V_{-1} = C_{-1} \oplus C_0, \quad V_0 = C_0.$$

The source, target maps and the composition of $\mathbb{V}$ are

$$s(\mathcal{f},x) = x, \quad t(\mathcal{f},x) = x + d\mathcal{f}, \quad (\mathcal{f},x) \cdot (\mathcal{g},y) = (\mathcal{f} + \mathcal{g},y),$$
Lemma 2.6. Let $g$ is a Lie 2-algebra isomorphism from $\phi = \psi$ morphisms. Their composition (the identity homomorphism $\phi$ invertible, then $\phi$ is an isomorphism, and $\phi^{-1}$ is given by
\[\phi^{-1} = (\phi_0^{-1}, \phi_1^{-1}, -\phi_1^{-1} \circ \phi_2 \circ (\phi_0^{-1} \times \phi_0^{-1})).\]

2.2 Strict and skeletal Lie 2-algebras

A Lie 2-algebra $(g, d, [\cdot, \cdot], l_3)$ is called strict if $l_3 = 0$. Then the 2-bracket $[\cdot, \cdot]$ satisfies the Jacobi identity. For a 2-vector space (or, a 2-term chain complex), its functors and natural transformations of a 2-vector space constitute a strict Lie 2-algebra.
Example 2.7. Let $\mathcal{V} : V_{-1} \xrightarrow{\partial} V_0$ be a 2-term chain complex of vector spaces. Its truncated morphisms constitute a strict Lie 2-algebra

$$\text{gl}(\mathcal{V}) : \text{gl}_{-1}(\mathcal{V}) := \text{Hom}(V_0, V_{-1}) \xrightarrow{\delta} \text{gl}_0(\mathcal{V}), \quad \delta(D) = (\partial \circ D, D \circ \partial),$$

where

$$\text{gl}_0(\mathcal{V}) := \{(A_0, A_1) \in \text{gl}(V_0) \oplus \text{gl}(V_{-1}); A_0 \circ \partial = \partial \circ A_1\},$$

and the 2-bracket is defined by the commutator:

$$[(A_0, A_1), (B_0, B_1)]_C = (A_0 \circ B_0 - B_0 \circ A_0, A_1 \circ B_1 - B_1 \circ A_1),$$

$$[(A_0, A_1), D]_C = A_1 \circ D - D \circ A_0,$$

for $(A_0, A_1), (B_0, B_1) \in \text{gl}_0(\mathcal{V})$ and $D \in \text{Hom}(V_0, V_{-1})$. We see that $\text{gl}_0(\mathcal{V})$ is the space of functors from $\mathcal{V}$ to $\mathcal{V}$ and $\text{gl}_{-1}(\mathcal{V})$ is the space of natural transformations.

This strict Lie 2-algebra will be used to define representations of Lie 2-algebras; see Subsection 4.1.

An equivalent description of strict Lie 2-algebras is the concept of Lie algebra crossed modules, which first appeared in the work of Gerstenhaber [21].

Definition 2.8. A Lie algebra crossed module is a quadruple $(m, g, \varphi, \triangleright)$ consisting of Lie algebras $m$ and $g$, a Lie algebra homomorphism $\varphi : m \to g$, and an action $\triangleright$ of $g$ as derivations of $m$ (that is, a Lie algebra homomorphism $\triangleright : g \to \text{Der}(m)$), satisfying

$$\varphi(x \triangleright a) = [x, \varphi(a)];$$

$$\varphi(a) \triangleright b = [a, b],$$

for $x \in g$ and $a, b \in m$.

A Lie algebra crossed module becomes a strict Lie 2-algebra if taking the action of $g$ on $m$ as the bracket. Conversely, given a strict Lie 2-algebra $g_{-1} \xrightarrow{\delta} g_0$, it is obvious that $g_{-1}$ with the bracket $[a, b] = [da, b]$ is a Lie algebra and the Lie algebra $g_0$ acts on $g_{-1}$ by $x \triangleright a = [x, a]$. Then $g$ is a Lie algebra crossed module. By this analysis, we have

Proposition 2.9. There is a one-to-one correspondence between strict Lie 2-algebras and Lie algebra crossed modules.

Example 2.10. Let $g$ be a Lie algebra and $\mathfrak{h} \subset g$ an ideal. Then $\mathfrak{h} \hookrightarrow g$ is a Lie algebra crossed module, where the action of $g$ on $\mathfrak{h}$ is given by the Lie bracket on $g$.

Example 2.11. Let $g$ be a Lie algebra and $\text{Der}(g)$ its derivation Lie algebra. Then $\text{ad} : g \to \text{Der}(g)$ defined by $\text{ad}(x) = [x, \cdot]$ is a Lie algebra crossed module, where $\text{Der}(g)$ acts on $g$ naturally.

It was first shown by Gerstenhaber [21] that Lie algebra crossed modules are classified by the third cohomology of a Lie algebra. A direct proof can be found in [60]. Given a Lie algebra crossed module $m \xrightarrow{\varphi} g$, there exists a 4-term exact sequence

$$0 \xrightarrow{} V \xrightarrow{i} m \xrightarrow{\varphi} g \xrightarrow{\pi} \mathfrak{h} \xrightarrow{} 0,$$

where the cokernel $\mathfrak{h}$ is a Lie algebra and the kernel $V$ is an $\mathfrak{h}$-module induced by the action of $g$ on $m$. 

Definition 2.12. Two crossed modules \( \varphi : m \to g \) and \( \varphi' : m' \to g' \) such that \( \ker \varphi = \ker \varphi' = V \) and \( \coker \varphi = \coker \varphi' = h \) are called elementary equivalent if there are Lie algebra homomorphisms \( \phi_1 : m \to m' \) and \( \phi_0 : g \to g' \) such that \( \phi_1(x \triangleright a) = \phi_0(x) \triangleright' \phi_1(a) \) and the following diagram is commutative:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & V & \rightarrow & m & \xrightarrow{\varphi} & g & \xrightarrow{\phi_0} & h & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\phi_1} & & \downarrow{\phi_0} & & \downarrow{id} & & \\
0 & \rightarrow & V & \rightarrow & m' & \xrightarrow{\varphi'} & g' & \xrightarrow{\phi_0} & h & \rightarrow & 0.
\end{array}
\]

Define the equivalence relation on Lie algebra crossed modules by the equivalence relation generated by elementary equivalences. Denote by \( \text{crmod}(h, V) \) the set of equivalence classes of crossed modules with fixed kernel \( V \), fixed cokernel \( h \) and fixed action of \( h \) on \( V \).

Theorem 2.13. (Gerstenhaber) With notations above, there is a bijection

\[
\text{crmod}(h, V) \cong H^3(h, V).
\]

Another special case is the skeletal Lie 2-algebra. A Lie 2-algebra \( (g, d, [\cdot, \cdot], l_3) \) is called skeletal if \( d = 0 \). Then the 2-bracket \( [\cdot, \cdot] \) satisfies the Jacobi identity. Thus \( (g_0, [\cdot, \cdot]) \) is a Lie algebra, \( g_{-1} \) becomes a \( g_0 \)-module, and \( l_3 : \wedge^3 g_0 \to g_{-1} \) is a 3-cocycle.

Example 2.14. Given a semisimple Lie algebra \( g \) with the Killing form \( (\cdot, \cdot) \), we have a skeletal Lie 2-algebra

\[
\mathbb{R} \xrightarrow{0} g,
\]

where the 2-bracket \( [\cdot, \cdot] \) is the Lie bracket on \( g \) and trivial otherwise, and \( l_3(x, y, z) = (x, [y, z]) \). Note that \( l_3 \) is a 3-cocycle of \( g \) with coefficients in \( \mathbb{R} \), i.e., \( l_3 \in Z^3(g, \mathbb{R}) \). This Lie 2-algebra is called the string Lie 2-algebra and denoted by \( \text{String}(g) \).

Proposition 2.15. ([2]) There is a one-to-one correspondence between isomorphism classes of skeletal Lie 2-algebras and isomorphism classes of quadruples consisting of a Lie algebra \( g \), a vector space \( V \), a representation of \( g \) on \( V \), and an element in the third cohomology \( H^3(g, V) \).

For a Lie 2-algebra \( g_{-1} \xrightarrow{d} g_0 \), write \( g_0 \) and \( g_{-1} \) as: \( g_0 = \text{img} \ d \oplus g'_0 \) and \( g_{-1} = \ker \ d \oplus g'_{-1} \). This allows us to define a 2-term chain complex with trivial differential:

\[
\ker d \xrightarrow{0} g'_0.
\]

Actually, as shown in [2], we have

Lemma 2.16. Every Lie 2-algebra is equivalent, as an object of \( \text{Lie2Alg} \), to a skeletal one.

This result together with Proposition 2.15 leads to a classification of all Lie 2-algebras up to equivalence.

Theorem 2.17. ([2]) There is a one-to-one correspondence between equivalence classes of Lie 2-algebras and isomorphism classes of quadruples as in Proposition 2.15.

For the string Lie 2-algebra in Example 2.14 since \( g \) is semi-simple, the Killing form is non-degenerate. So the 3-cocycle \( l_3 \) described above represents a nontrivial cohomology class. By Theorem 2.17 the Lie 2-algebra \( \text{String}(g) \) is not equivalent to a skeletal strict Lie 2-algebra.
2.3 Leibniz algebras and Lie 2-algebras

The skew-symmetrization of a Leibniz algebra gives a Lie 2-algebra \[55\]. In fact, a Leibniz algebra gives rise to a hemistrict Lie 2-algebra and the skew-symmetrization of any hemistrict Lie 2-algebra is a semistrict Lie 2-algebra; see \[51\]. Besides this, the skew-symmetrization of a Leibniz algebra can be extended in a natural way to a Lie-Yamaguti algebra structure; see \[28\].

A Leibniz algebra is a vector space \(\mathfrak{g}\) endowed with a bilinear bracket \(\{\cdot,\cdot\}\) satisfying the Leibniz rule:
\[
\{x,\{y,z\}\} = \{\{x,y\},z\} + \{y,\{x,z\}\}, \quad \forall x, y, z \in \mathfrak{g}.
\]

The left center of a Leibniz algebra \(\mathfrak{g}\) is defined by
\[
Z(\mathfrak{g}) = \{x \in \mathfrak{g}; \{x,y\} = 0, \forall y \in \mathfrak{g}\}.
\]

By \(J_{x,y,z}\), we mean the corresponding Jacobiator, i.e.,
\[
J_{x,y,z} = [x, [y, z]] + [y, [z, x]] + [z, [x, y]], \quad \forall x, y, z \in \mathfrak{g}.
\]

It is shown that \(J_{x,y,z} \in Z(\mathfrak{g})\).

**Theorem 2.18.** (\[55\]) Let \((\mathfrak{g},\{\cdot,\cdot\}\) be a Leibniz algebra and \(Z(\mathfrak{g})\) its left center. Then we have a Lie 2-algebra

\[
Z(\mathfrak{g}) \hookrightarrow \mathfrak{g},
\]

where
\[
\begin{align*}
\ell_2(x, y) &= [x, y] = \frac{1}{2}(\{x,y\} - \{y,x\}); \\
\ell_2(x, c) &= [x, c] = \frac{1}{2}[x, c]; \\
\ell_3(x, y, z) &= J_{x,y,z},
\end{align*}
\]

for all \(x, y, z \in \mathfrak{g}\) and \(c \in Z(\mathfrak{g})\).

An omni-Lie algebra is a typical example of Leibniz algebras, which was introduced in \[63\] by linearizing Courant algebroids. Its Dirac structures characterize all the Lie algebra structures on a vector space. An omni-Lie algebra is a triple \((\text{gl}(V) \oplus V, \langle \cdot, \cdot \rangle, \{\cdot,\cdot\})\), where \(V\) is a vector space, \(\langle \cdot, \cdot \rangle\) is a nondegenerate pairing
\[
\langle A + u, B + v \rangle = Av + Bu, \quad \forall A + u, B + v \in \text{gl}(V) \oplus V,
\]

and the bracket is given by
\[
\{A + u, B + v\} = [A, B]_C + Av,
\]

where \([\cdot,\cdot]_C\) is the commutator bracket. Omni-Lie algebras are Leibniz algebras, i.e.,
\[
\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + \{e_2, \{e_1, e_3\}\}, \quad \forall e_1, e_2, e_3 \in \text{gl}(V) \oplus V.
\]

By Theorem 2.18, we have the following example of Lie 2-algebras from an omni-Lie algebra.

**Example 2.19.** For a vector space \(V\), there is a natural Lie 2-algebra

\[
V \hookrightarrow \text{gl}(V) \oplus V,
\]

where
\[
\begin{align*}
\ell_2^0(A + u, B + v) &= [A, B] + \frac{1}{2}(Av - Bu); \\
\ell_2^1(A + u, v) &= \frac{1}{2}Av; \\
\ell_3(A + u, B + v, C + w) &= -\frac{1}{4}([A, B]w + [B, C]u + [C, A]v),
\end{align*}
\]

for all \(A + u, B + v, C + w \in \text{gl}(V) \oplus V\). Here \([\cdot,\cdot]\) is the commutator bracket \([\cdot,\cdot]_C\).
3 Lie 2-algebras from geometric structures

Lie 2-algebras appear in various geometric structures. In this part, we shall review four geometric structures, 2-plectic manifolds, Courant algebroids, homotopy Poisson manifolds and affine multivector fields on Lie groupoids that give rise to Lie 2-algebras. Some applications are also given. We refer to Appendix B for a brief introduction of Lie groupoids and algebroids.

3.1 Lie 2-algebras from 2-plectic manifolds

In the classical mechanics of point particles, the phase space is often a symplectic manifold and the poisson bracket of its functions makes the space of observables a Lie algebra. Passing from point particles to strings, the phase space becomes a 2-plectic manifold and a Lie 2-algebra of observables is constructed. This Lie 2-algebra is used to describe the dynamics of a classical bosonic string. See [3].

Definition 3.1. A 3-form $\omega \in \Omega^3(M)$ on a smooth manifold $M$ is called a 2-plectic structure if it is both closed:

$$dR\omega = 0,$$

and nondegenerate, i.e., for $X \in T_mM$,

$$\omega(X, \cdot, \cdot) = 0 \Rightarrow X = 0.$$

We call the pair $(M, \omega)$ a 2-plectic manifold.

In classical mechanics, the position and momentum of a particle are given by a point in $T^*M$, which is a symplectic manifold with the symplectic structure given by the differential of a canonical 1-form. To describe the position and momentum of a string, we need $\wedge^2T^*M$, which is a 2-plectic manifold with the 3-form given by the differential of a canonical 2-form.

Example 3.2. Let $M$ be a manifold. On $\wedge^2T^*M$, there is a canonical 2-form $\alpha \in \Omega^2(\wedge^2T^*M)$ given by:

$$\alpha_\xi(v_1, v_2) = \xi(\pi_*(v_1), \pi_*(v_2)), \quad \forall \xi \in \wedge^2T^*M, v_1, v_2 \in T_\xi(\wedge^2T^*M),$$

where $\pi : \wedge^2T^*M \to M$ is the projection. Then $\wedge^2T^*M$ with the 3-form

$$\omega = dR\alpha \in \Omega^3(\wedge^2T^*M)$$

is 2-plectic manifold.

Let $(M, \omega)$ be a 2-plectic manifold. From the nondegeneracy of $\omega$, we have an injective map

$$T_mM \to \wedge^2T_m^*M, \quad X \mapsto \omega(X, \cdot, \cdot),$$

which is not an isomorphism in general.

Definition 3.3. Let $(M, \omega)$ be a 2-plectic manifold. A 1-form $\alpha$ on $M$ is called Hamiltonian if there exists a vector field $X_\alpha$ on $M$ such that

$$dR\alpha = \omega(X_\alpha, \cdot, \cdot).$$

$X_\alpha$ is called the Hamiltonian vector field corresponding to $\alpha$.

Denote by $\Omega^1_{\text{Ham}}(M)$ the vector space of Hamiltonian 1-forms. Similar to the Poisson bracket on functions in the symplectic case, there is a bracket on Hamiltonian 1-forms:

$$\{\alpha, \beta\} = \omega(X_\alpha, X_\beta, \cdot), \quad \forall \alpha, \beta \in \Omega^1_{\text{Ham}}(M).$$
**Theorem 3.4.** ([41]) If \((M, \omega)\) is a 2-plectic manifold, there is a Lie 2-algebra

\[
L_\infty(M, \omega) : C^\infty(M) \xrightarrow{d_{dR}} \Omega^1_{\text{Ham}}(M),
\]

where \(d_{dR}\) is the de Rham differential, and the brackets are

\[
\begin{align*}
\ell_2(\alpha, \beta) &= \{\alpha, \beta\}; \\
\ell_2(\alpha, f) &= 0; \\
\ell_3(\alpha, \beta, \gamma) &= \omega(X_\alpha, X_\beta, X_\gamma),
\end{align*}
\]

for \(\alpha, \beta, \gamma \in \Omega^1_{\text{Ham}}(M)\) and \(f \in C^\infty(M)\).

This result can be generalized to an arbitrary \(n\). An \(n\)-plectic manifold is a manifold with a nondegenerate closed \((n+1)\)-form, which is canonically equipped with a Lie \(n\)-algebra structure [47].

The following example is related to the Wess-Zumino-Witten model and loop groups.

**Example 3.5.** Let \(G\) be a compact simple Lie group with Lie algebra \(\mathfrak{g}\) and let \((\cdot, \cdot)\) be the Killing form on \(\mathfrak{g}\). There is a 2-plectic structure \(\omega \in \Omega^3(G)\) on \(G\) such that

\[
\omega(x, y, z) = (x, [y, z]), \quad \forall x, y, z \in T_eG = \mathfrak{g},
\]

and \(\omega\) is both left and right-invariant.

Let \(\Omega^1_{\text{Ham}}(G)^L\) be the set of left-invariant Hamiltonian 1-forms. Actually, \(\Omega^1_{\text{Ham}}(G)^L = \mathfrak{g}^*\).

Since the left-invariant smooth functions on \(G\) are constants, we have a Lie 2-algebra \(L_\infty(G, \omega)\) on the complex:

\[
0 : \mathbb{R} \to \mathfrak{g}^*.
\]

This Lie 2-algebra is in fact the string Lie 2-algebra \(\text{String}(\mathfrak{g})\) in Example 2.14.

The Lie 2-algebra \(L_\infty(M, \omega)\) associated with a 2-plectic manifold \((M, \omega)\) is used to define Lie algebra and Lie 2-algebra moment maps [11, 20, 41].

**Definition 3.6.** ([11]) Let \(\mathfrak{g} \to \mathfrak{X}(M), u \mapsto \hat{u}\) be a Lie algebra action on a 2-plectic manifold \((M, \omega)\) such that \(\hat{u}\) for all \(u \in \mathfrak{g}\) are Hamiltonian vector fields. A homotopy moment map for this action (or a \(\mathfrak{g}\)-moment map) is a Lie 2-algebra homomorphism

\[
(\phi_0, \phi_1, \phi_2) : \mathfrak{g} \to L_\infty(M, \omega),
\]

where \(\phi_0 : \mathfrak{g} \to \Omega^1_{\text{Ham}}(M)\) and \(\phi_2 : \wedge^3 \mathfrak{g} \to C^\infty(M)\), such that

\[
-\iota_u \omega = d_{dR}(\phi_0(u)), \quad \forall u \in \mathfrak{g}.
\]

Fixing a point \(p \in M\), define \(\omega_{3p} : \wedge^3 \mathfrak{g} \to \mathbb{R}\) by

\[
\omega_{3p}(u_1, u_2, u_3) := \omega(u_1, \hat{u}_2, \hat{u}_3)|_p, \quad \forall u_1, u_2, u_3 \in \mathfrak{g}.
\]

It can be checked that \(\omega_{3p}\) is a 3-cocycle of the Lie algebra \(\mathfrak{g}\) with coefficients in \(\mathbb{R}\), i.e., \(\omega_{3p} \in Z^3(\mathfrak{g}, \mathbb{R})\). Moreover, the cohomology class \([\omega_{3p}] \in H^3(\mathfrak{g}, \mathbb{R})\) is independent of the choice of the point \(p\).

**Theorem 3.7.** ([11]) The existence of a \(\mathfrak{g}\)-moment map implies that \([\omega_{3p}] = 0\). The converse is true if \(H^1(M) = 0\).

In [41], the authors went one step further by replacing the Lie algebra \(\mathfrak{g}\) by a skeletal Lie 2-algebra. We would not go into the details here.
3.2 Lie 2-algebras from Courant algebroids

Courant algebroids were originally introduced in [38] for studying the double of Lie bialgebroids. They include as examples the double of Lie bialgebras and the bundle $TM \oplus T^*M$ with the bracket introduced by Courant [16] for the study of Dirac structures. Courant algebroids play important roles in generalized complex geometry [23]. See also [18, 49, 54] for other applications. Courant algebroids give rise to Lie 2-algebras naturally [52].

**Definition 3.8.** ([38]) A Courant algebroid is a vector bundle $E \to M$ equipped with

- a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle;
- a skew-symmetric bracket $\lbrack, \rbrack$, called the Courant bracket, on $\Gamma(E)$;
- a bundle map $\rho : E \to TM$, called the anchor,

such that the following properties hold:

1. $\lbrack\lbrack e_1, e_2 \rbrack, e_3 \rbrack + \lbrack\lbrack e_2, e_3 \rbrack, e_1 \rbrack + \lbrack\lbrack e_3, e_1 \rbrack, e_2 \rbrack = DT(e_1, e_2, e_3)$;
2. $\rho(e_1, e_2) = [\rho(e_1), \rho(e_2)];$
3. $\lbrack e_1, f e_2 \rbrack = f \lbrack e_1, e_2 \rbrack + \rho(e_1)(f)e_2 - \frac{1}{2}\langle e_1, e_2 \rangle Df$;
4. $\rho \circ D = 0$, i.e., $\langle Df, Dg \rangle = 0$;
5. $\rho(e_1)(e_2, e_3) = \langle [e_1, e_2], e_3 \rangle + \frac{1}{2}D\langle e_1, e_2 \rangle, e_3 \rangle + \langle e_2, [e_1, e_3] \rangle + \langle e_3, [e_1, e_2] \rangle,$

for $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$. Here $D : C^\infty(M) \to \Gamma(E)$ is defined by

$$\langle Df, e \rangle = \rho(e)(f), \quad \forall e \in \Gamma(E), f \in C^\infty(M),$$

and $T(e_1, e_2, e_3) \in C^\infty(M)$ is defined by

$$T(e_1, e_2, e_3) = \frac{1}{6}(\lbrack\lbrack e_1, e_2 \rbrack, e_3 \rbrack + \lbrack\lbrack e_2, e_3 \rbrack, e_1 \rbrack + \lbrack\lbrack e_3, e_1 \rbrack, e_2 \rbrack).$$

In [38], the authors also introduced a bracket:

$$\{e_1, e_2\} := \lbrack\lbrack e_1, e_2 \rbrack, \rho(e_1, e_2) \rangle, \quad \forall e_1, e_2 \in \Gamma(E).$$

It is not skew-symmetric. Later, this bracket is found to satisfy the Jacobi identity

$$\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + \{e_2, \{e_1, e_3\}\},$$

and is called the **Dorfman bracket**. Rotenberg [49] gives a simplified definition of Courant algebroids by using the Dorfman bracket.

**Theorem 3.9.** ([52]) Let $(E, \langle \cdot, \cdot \rangle, \lbrack\lbrack \cdot, \cdot \rbrack, \rho)$ be a Courant algebroid. Then we obtain a Lie 2-algebra

$$L_\infty(E) : C^\infty(M) \xrightarrow{\mathcal{D}} \Gamma(E),$$

where $l_2$ and $l_3$ are given by

$$ \left\{ \begin{array} {lcl} l_2(e_1, e_2) & = & \lbrack\lbrack e_1, e_2 \rbrack, \rho(e_1, e_2) \rbrack; \\
 l_2(e_1, f) & = & \frac{1}{2}\langle e_1, Df \rangle; \\
 l_3(e_1, e_2, e_3) & = & -T(e_1, e_2, e_3), \end{array} \right.$$
More generally, from a pre-Courant algebroid, one is also able to get a Lie 2-algebra; see [30].

**Example 3.10.** On the bundle $TM \oplus T^*M$, define a bilinear form by the pairing

$$\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X), \quad \forall X, Y \in \mathfrak{X}(M), \alpha, \beta \in \Omega^1(M),$$

the Courant bracket by

$$\llbracket X + \alpha, Y + \beta \rrbracket = [X, Y] + L_X\beta - L_Y\alpha + \frac{1}{2}d_{dR}(\alpha(Y) - \beta(X)),$$

and the anchor $\rho$ by the projection to $TM$. Then $(TM \oplus T^*M, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ is a Courant algebroid. It was used in [16] to study Dirac structures.

In particular, let $V$ be a vector space and $M = V^*$. Choosing linear vector fields and constant forms on $V^*$, one gets $gl(V) \oplus V$, which inherits the pairing, Courant bracket and the anchor from the Courant algebroid $TV^* \oplus T^*V^*$. It turns out to be the omni-Lie algebra we recalled in Subsection 2.3. So omni-Lie algebras are the linearization of Courant algebroids. In fact, the Lie 2-algebra in Example 2.19 is a particular case of Theorem 3.9.

It was then developed in [54] that the Courant bracket (3) on $TM \oplus T^*M$ can be twisted by a 3-form $\omega \in \Omega^3(M)$ in the way:

$$\llbracket X + \alpha, Y + \beta \rrbracket_\omega = \llbracket X + \alpha, Y + \beta \rrbracket - \iota_Y\iota_X\omega.$$

**Example 3.11.** Let $M$ be a manifold with a 3-form $\omega \in \Omega^3(M)$. Then $TM \oplus T^*M$ with the bilinear form and anchor the same as in Example 3.10 and the bracket $\llbracket \cdot, \cdot \rrbracket_\omega$ is a Courant algebroid if and only if $d_{dR}\omega = 0$.

By Theorem 3.9 from a manifold $M$ with a closed 3-form $\omega \in \Omega^3(M)$, we get a Lie 2-algebra $C^\infty(M) \xrightarrow{d_{dR}} \mathfrak{X}(M) \oplus \Omega^1(M)$. Denote it by $L_\infty(TM \oplus T^*M)$. Recall from Theorem 3.4 that we have another Lie 2-algebra $L_\infty(M, \omega)$ if $\omega \in \Omega^3(M)$ is closed and nondegenerate. The relation of these two Lie 2-algebras is clarified in [48] as follows.

**Theorem 3.12.** With the notations above, there exists a Lie 2-algebra homomorphism $(\phi_0, \phi_1, \phi_2)$ embedding $L_\infty(M, \omega)$ into $L_\infty(TM \oplus T^*M)$:

$$
\begin{array}{c}
C^\infty(M) \xrightarrow{\phi_0 = \text{id}} C^\infty(M) \\
d_{dR} \downarrow \quad \quad \downarrow d_{dR} \\
\Omega^1_{\text{Ham}}(M) \xrightarrow{\phi_1} \mathfrak{X}(M) \oplus \Omega^1(M),
\end{array}
$$

where $\phi_0(\alpha) = (X_\alpha, \alpha)$ with $X_\alpha$ being the Hamiltonian vector field of $\alpha$, i.e., $d_{dR}\alpha = \iota_{X_\alpha}\omega$, and $\phi_2 : \wedge^2 \Omega^1_{\text{Ham}}(M) \to C^\infty(M)$ is given by

$$\phi_2(\alpha, \beta) = \frac{1}{2}(\alpha(X_\beta) - \beta(X_\alpha)), \quad \forall \alpha, \beta \in \Omega^1_{\text{Ham}}(M).$$

We refer to [48] for the discussion of this result for exact Courant algebroids.

### 3.3 Lie 2-algebras from homotopy Poisson manifolds

For a vector space $V$, there is a one-to-one correspondence between Lie algebra structures on $V$ and linear Poisson structures on $V^*$. Such a result for Lie 2-algebras was explored in [35]. In fact, on the dual of a Lie 2-algebra, there is a linear homotopy Poisson structure and also a linear quasi-Poisson groupoid structure. Moreover, from a Lie 2-algebra, there associates a natural Courant algebroid structure.

A homotopy Poisson algebra is a graded commutative algebra with an $L_\infty$-structure whose brackets satisfy the Leibniz rule. It has appeared as higher Poisson structures in [27, 59] and $P_{\infty}$-structures in [12].
Definition 3.13. A homotopy Poisson algebra of degree $n$ is a graded commutative algebra $\mathfrak{a}$ with an $L_\infty$-algebra structure $\{l_m\}_{m \geq 1}$ on $\mathfrak{a}[n]$, such that the map
\[ x \mapsto l_m(x_1, \cdots, x_{m-1}, x), \quad \forall x_1, \cdots, x_{m-1}, x \in \mathfrak{a} \]
is a derivation of degree $2 - m - n(m - 1) + \sum_{i=1}^{m-1} |x_i|$. Here, $|x|$ denotes the degree of $x \in \mathfrak{a}$.

A homotopy Poisson algebra of degree $n$ is of finite type if there exists a $q$ such that $l_m = 0$ for all $m > q$. A homotopy Poisson manifold of degree $n$ is a graded manifold $\mathcal{M}$ whose algebra of functions $C^\infty(\mathcal{M})$ is equipped with a degree $n$ homotopy Poisson algebra structure of finite type.

Given a Lie 2-algebra $(\mathfrak{g}, d, [\cdot, \cdot], l_3)$, its graded dual space $\mathfrak{g}^*[1] = \mathfrak{g}_0^*[1] \oplus \mathfrak{g}_{-1}^*[1]$ is an $N$-manifold of degree 1 with the base manifold $\mathfrak{g}_{-1}^*$. Its algebra of functions is
\[ \cdots \oplus (C^\infty(\mathfrak{g}_{-1}^*) \otimes \Lambda^2 \mathfrak{g}_0) \oplus (C^\infty(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0) \oplus C^\infty(\mathfrak{g}_{-1}^*). \]
There is a degree 1 homotopy Poisson algebra structure on it obtained by extending the original Lie 2-algebra structure using the Leibniz rule. Thus, the dual of a Lie 2-algebra is a linear homotopy Poisson manifold of degree 1. Here linear means that the brackets of linear functions are still linear functions. The converse is easy to see.

Proposition 3.14. Let $\mathbb{V} = V_{-1} \oplus V_0$ be a graded vector space. Then there is a one-to-one correspondence between Lie 2-algebra structures on $\mathbb{V}$ and linear homotopy Poisson structures on $\mathbb{V}^*[1]$.

Let $(\mathfrak{g}, d, [\cdot, \cdot], l_3)$ be a Lie 2-algebra. The cotangent bundle $T^*[2] \mathfrak{g}^*[1]$ is a symplectic NQ-manifold of degree 2. Symplectic NQ-manifolds of degree 2 are in one-to-one correspondence with Courant algebroids [50]. Thus we obtain a Courant algebroid $E$ from a Lie 2-algebra $\mathfrak{g}$:
\[ E = \mathfrak{g}_{-1}^* \times (\mathfrak{g}_0^* \oplus \mathfrak{g}_0) \longrightarrow \mathfrak{g}_{-1}^*, \]
in which the anchor and the Dorfman bracket are defined by the derived bracket using the degree 3 function $l = d - l_2 - l_3$. Here elements in $\mathfrak{g}_0, \mathfrak{g}_0^*, \mathfrak{g}_{-1}, \mathfrak{g}_{-1}^*$ as functions are of degree 1, 1, 0, 2 respectively.

Proposition 3.15. (35) The Courant algebroid structure on $E$ is as follows: the bilinear form is the pairing between $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$; the anchor and Dorfman bracket are determined by
\begin{enumerate}
\item $\rho(x)(a) = [x, a]$, the anchor of $x$ is a linear vector field on $\mathfrak{g}_{-1}^*$;
\item $\rho(\xi) = d^*(\xi)$, the anchor of $\xi$ is a constant vector field on $\mathfrak{g}_{-1}^*$;
\item $\{x, y\} = [x, y] + l_3(x, y, \cdot) \in \mathfrak{g}_0 + \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$;
\item $\{\xi, \eta\} = 0$ and $\{x, \xi\} = -ad^*_{\xi} \xi$, where $ad^*_{\xi} : \mathfrak{g}_0 \to \mathfrak{g}_0^*$ is given by $\langle ad^*_{\xi} \xi, y \rangle = -\langle \xi, [x, y] \rangle$, for constant sections $x, y \in \mathfrak{g}_0, \xi, \eta \in \mathfrak{g}_0^*$ and linear function $a \in \mathfrak{g}_{-1}$.
\end{enumerate}
So a Courant algebroid is derived from a Lie 2-algebra, which further gives rise to a new Lie 2-algebra as in Theorem 3.9. See 35 for the relation of these two Lie 2-algebras. This Courant algebroid $E$ is actually the double of a Lie quasi-bialgebroid; see [35]. By integration, we shall further get a quasi-Poisson groupoid from a Lie 2-algebra.

On a Lie groupoid $\mathcal{G}$, a bivector field $\Pi$ is called multiplicative if the graph of the groupoid multiplication $\Lambda = \{(g, h, gh): s(g) = t(h)\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ is coisotropic relative to $\Pi \oplus \Pi \oplus -\Pi$. Namely,
\[ (\Pi \times \Pi \times -\Pi)(\xi_1, \xi_2) = 0, \quad \forall \xi_1, \xi_2 \in \Lambda^\perp, \]
where
\[ \Lambda^\perp = \{\xi \in T^*_x(\mathcal{G} \times \mathcal{G} \times \mathcal{G}) \cap \Lambda : \lambda \in \Lambda, \langle \xi, v \rangle = 0, \forall v \in T_\lambda \Lambda\}. \]
Definition 3.16. (25) A quasi-Poisson groupoid is a Lie groupoid \( G \) with a multiplicative bivector field \( \Pi \in \mathfrak{X}^2(\mathcal{G}) \) and a 3-section \( \phi \in \Gamma(\wedge^3\mathfrak{A}) \), where \( \mathfrak{A} \) is the Lie algebroid of \( G \), such that

\[
\frac{1}{2}[\Pi, \Pi] = \overrightarrow{\phi} - \overleftarrow{\phi}, \quad [\Pi, \overrightarrow{\phi}] = 0.
\]

If \( \phi = 0 \), it is called a Poisson groupoid.

For the 2-term chain complex \( \mathfrak{g}_0 \xrightarrow{d} \mathfrak{g}^-_1 \), the abelian Lie group \( \mathfrak{g}_0^* \) acts on \( \mathfrak{g}^-_1 \) by

\[
\xi \triangleright \alpha = \alpha + d^* (\xi), \quad \forall \xi \in \mathfrak{g}_0^*, \alpha \in \mathfrak{g}^-_1.
\]

This gives rise to an action Lie groupoid \( \mathcal{G} : \mathfrak{g}_0^* \times \mathfrak{g}^-_1 \rightrightarrows \mathfrak{g}^-_1 \), where the source, target and multiplication are

\[
s(\xi, \alpha) = \alpha, \quad t(\xi, \alpha) = \alpha + d^* (\xi), \quad (\xi_1, \alpha_1)(\xi_2, \alpha_2) = (\xi_1 + \xi_2, \alpha_2),
\]

where \( \alpha_1 = \alpha_2 + d^*(\xi_2) \).

The Lie algebroid of this Lie groupoid is the action Lie algebroid \( \mathcal{A} : \mathfrak{g}_0^* \times \mathfrak{g}^-_1 \rightrightarrows \mathfrak{g}^-_1 \). See Example 3.16 in Appendix B for action Lie groupoids and algebroids. Observe that an element \( l_3 \in \wedge^3 \mathfrak{g}_0^* \otimes \mathfrak{g}^-_1 \) can be viewed as a 3-section of \( \mathcal{A} \).

Theorem 3.17. (33) Let \( \mathfrak{g} \) be a Lie 2-algebra. Then we get a linear quasi-Poisson groupoid \( (\mathcal{G}, \Pi, \phi) \), where \( \mathcal{G} : \mathfrak{g}_0^* \times \mathfrak{g}^-_1 \rightrightarrows \mathfrak{g}^-_1 \) is the action Lie groupoid described above, \( \phi = -l_3 \in \Gamma(\wedge^3\mathfrak{A}) \) and \( \Pi \) is characterized by

\[
\Pi(d_{\mathcal{A}R}x, d_{\mathcal{A}R}y) = -[x, y], \quad \Pi(d_{\mathcal{A}R}x, d_{\mathcal{A}R}a) = -[x, a], \quad \Pi(d_{\mathcal{A}R}a, d_{\mathcal{A}R}b) = -[da, b],
\]

where \( x, y \in \mathfrak{g}_0^* \) and \( a, b \in \mathfrak{g}^-_1 \) are linear functions on \( \mathfrak{g}_0^* \times \mathfrak{g}^-_1 \).

In summary, we have

\[
\text{Lie 2-algebra} \quad \overset{\text{equivalent}}{\longleftrightarrow} \quad \text{2-term } L_\infty\text{-algebra } \mathfrak{g} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \text{HPM } \mathfrak{g}^*[1]
\]

\[
\text{Lie quasi-bialgebroid } (A, \delta, \phi) \quad \overset{\text{integration}}{\longleftrightarrow} \quad \text{Courant algebroid } E \quad \overset{T^*[2]\mathfrak{g}^*[1]}{\longleftrightarrow} \quad \text{quasi-Poisson groupoid } (\mathcal{G}, \Pi, \phi)
\]

where HPM stands for homotopy Poisson manifolds.

3.4 Lie 2-algebras from affine structures

Geometric structures on a Lie groupoid that are compatible with the groupoid multiplication are called multiplicative structures. For example, a \( k \)-vector field \( \Pi \) on a Lie groupoid \( \mathcal{G} \) is called multiplicative (25) if the graph of the groupoid multiplication \( \{(g, h, gh) ; s(g) = t(h)\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G} \) is coisotropic with respect to \( \Pi \oplus \Pi \oplus (-1)^{k+1}\Pi \). This generalizes the multiplicative bivector field for a quasi-Poisson groupoid in Definition 3.16.

In [8, 11], the authors constructed a strict Lie 2-algebra on the multiplicative 1-vector fields on a Lie groupoid and their natural transformations. This construction is Morita invariant and gives rise to a strict Lie 2-algebra structure on the differentiable stack. Later this idea is generalized to multiplicative multivector fields in [9]. A strict graded Lie 2-algebra is obtained which is used to define multivector fields on a differentiable stack. A geometric explanation of this Lie 2-algebra is given in [34] by using affine multivector fields. See [39, 62] for affine structures on a Lie group. In this subsection, we always suppose that \( \mathcal{G} \) is a Lie groupoid with Lie algebroid \( \mathfrak{A} \).
Definition 3.18. A $k$-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is called affine if the submanifold

$$S := \{(g,h,l, hg^{-1}l); s(g) = s(h), t(g) = t(l)\} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$$

is coisotropic with respect to $\Pi \oplus (-1)^{k+1}\Pi \oplus (-1)^{k+1}\Pi \oplus \Pi$.

It is shown in [25] that a $k$-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ is multiplicative if and only if it is affine and the base manifold $M$ is coisotropic with respect to $\Pi$.

Lemma 3.19. (15) A $k$-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is affine if and only if the following two conditions hold:

(i) for any $(g, h) \in \mathcal{G}^{(2)}$,

$$\Pi(gh) = L_{\mathcal{X}} \Pi(h) + R_{\mathcal{Y}} \Pi(g) - L_{\mathcal{X}} \circ R_{\mathcal{Y}} (\Pi(s(g))),$$

where $\mathcal{X}$ and $\mathcal{Y}$ are any two local bisections passing through $g$ and $h$ respectively.

(ii) for any $\xi \in \Omega^1(M)$, $t_{\xi} \Pi$ is right-invariant.

Lemma 3.20. Let $\Pi$ be a $k$-vector field on the Lie groupoid $\mathcal{G}$ with $\pi = \text{pr}_{\mathcal{X}^k A} \Pi|_M \in \Gamma(\wedge^k A)$. Define

$$\Pi_r = \Pi - \frac{\partial}{\partial \pi}, \quad \Pi_t = \Pi - \frac{\partial}{\partial \pi}.$$ (4)

Then $\Pi$ is affine if and only if $\Pi_t$ or $\Pi_r$ is a multiplicative $k$-vector field on $\mathcal{G}$.

Denote by $\mathfrak{X}^k_{\text{aff}}(\mathcal{G})$ and $\mathfrak{X}^k_{\text{mult}}(\mathcal{G})$ the spaces of affine and multiplicative $k$-vector fields respectively. We have $\mathfrak{X}^k_{\text{mult}}(\mathcal{G}) \subset \mathfrak{X}^k_{\text{aff}}(\mathcal{G})$. The following result is from [33].

Theorem 3.21. (i) We have a 2-vector space $\mathfrak{X}^k_{\text{aff}}(\mathcal{G})$, whose spaces of morphisms and objects are $\mathfrak{X}^k_{\text{aff}}(\mathcal{G})$ and $\mathfrak{X}^k_{\text{mult}}(\mathcal{G})$ respectively, and the source and target maps are given by $s(\Pi) = \Pi_r$ and $t(\Pi) = \Pi_t$ as defined in (4).

(ii) The graded 2-vector space $\oplus_k \mathfrak{X}^k_{\text{aff}}(\mathcal{G})$ with the Schouten bracket is a strict graded Lie 2-algebra.

Here by a strict graded Lie 2-algebra, we say that the spaces of objects and morphisms are graded vector spaces and the Lie bracket is a graded Lie bracket. See [9] for the explicit definition. The 2-term $L_{\infty}$-algebra associated to this Lie 2-algebra is the one in [22] stated as follows. Define $\Gamma(\wedge^\bullet A) := \oplus_k \Gamma(\wedge^k A)$ and $\mathfrak{X}^\bullet_{\text{mult}}(\mathcal{G}) := \oplus_k \mathfrak{X}^k_{\text{mult}}(\mathcal{G})$.

Proposition 3.22. There is a strict graded Lie 2-algebra structure on

$$\Gamma(\wedge^\bullet A) \xrightarrow{d} \mathfrak{X}^\bullet_{\text{mult}}(\mathcal{G}),$$

where $d(u) = \overleftarrow{u} - \overrightarrow{u}$, the bracket on $\mathfrak{X}^\bullet_{\text{mult}}(\mathcal{G})$ is the Schouten bracket and the bracket $[\Pi, u]$ for $\Pi \in \mathfrak{X}^\bullet_{\text{mult}}(\mathcal{G})$ and $u \in \Gamma(\wedge^l A)$ is determined by the relation $[\Pi, u] = \Pi \circ [\overleftarrow{u}, \overrightarrow{u}]$.

The homotopy equivalence class of this strict graded Lie 2-algebra is invariant under the Morita equivalence of Lie groupoids, thus is considered as the space of multivector fields on the corresponding differentiable stack. We refer to Appendix A for the definition of homotopy equivalences of Lie 2-algebras.

Let $\mathcal{G} \to M$ be a Lie groupoid and $J : X \to M$ a surjective smooth map. A left action of $\mathcal{G}$ on $X$ along $J$, which is called the moment map, is a smooth map

$$\mathcal{G} \times_M X = \{(g, x) \in \mathcal{G} \times X; s(g) = J(x)\} \to X, \quad (g, x) \mapsto g \cdot x = gx$$
such that
\[ J(gx) = t(g), \quad (gh)x = g(hx), \quad 1_J(x)x = x. \]

We then say that \( X \) is a **left \( G \)-space**. A **left \( G \)-bundle** is a left \( G \)-space \( X \) together with a \( G \)-invariant surjective submersion \( \beta : X \to N \). A left \( G \)-bundle is called **principal** if the map
\[ G \times_M X \to X \times_N X, \quad (g, x) \mapsto (gx, x) \]
is a diffeomorphism.

**Definition 3.23.** A *Morita equivalence* between two Lie groupoids \( G_1 \rightrightarrows M_1 \) and \( G_2 \rightrightarrows M_2 \) is given by a principal \( G_1 \)-\( G_2 \)-bibundle, i.e., a manifold \( X \) with moment maps \( \alpha : X \to M_1 \) and \( \beta : X \to M_2 \), such that \( \beta : X \to M_2 \) is a left principal \( G_1 \)-bundle, \( \alpha : X \to M_1 \) is a right principal \( G_2 \)-bundle and the two actions commute: \( g_1 \cdot (x \cdot g_2) = (g_1 \cdot x) \cdot g_2 \) for any \( g_1 \in G_1, x \in X \) and \( g_2 \in G_2 \):

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\alpha} & X \\
\downarrow \quad \swarrow & & \searrow \downarrow \quad \nearrow \\
M_1 & & M_2
\end{array}
\]

**Theorem 3.24.** ([9]) Let \( G_1 \rightrightarrows M_1 \) and \( G_2 \rightrightarrows M_2 \) be Morita equivalent Lie groupoids. Then any \( G_1 \)-\( G_2 \)-bibundle \( M_1 \leftarrow X \to M_2 \) induces a homotopy equivalence between the strict graded Lie 2-algebras \( \Gamma(\wedge^\bullet A_1) \to \mathcal{X}_{\text{mult}}(G_1) \) and \( \Gamma(\wedge^\bullet A_2) \to \mathcal{X}_{\text{mult}}(G_2) \).

**Definition 3.25.** Let \( \mathfrak{X} \) be a differentiable stack. The *space of multivector fields* on \( \mathfrak{X} \) is defined to be the homotopy equivalence class of strict graded Lie 2-algebras \( \Gamma(\wedge^\bullet A) \to \mathcal{X}_{\text{mult}}(G) \), where \( G \rightrightarrows M \) is any Lie groupoid representing \( \mathfrak{X} \).

In other words, a multivector field on a differentiable stack \( \mathfrak{X} \) is defined as a homotopy equivalence class of affine multivector fields on any groupoid representing \( \mathfrak{X} \). Following [9], a shifted (+1) Poisson structure on a differentiable stack \( \mathfrak{X} \) is simply an element of the Maurer-Cartan moduli set of the dgla decided by the homotopy equivalence class of strict graded Lie 2-algebras corresponding to \( \mathfrak{X} \).

## 4 Modules and cohomology

The cohomology theories for \( L_\infty \)-algebras and \( A_\infty \)-algebras were developed in [30, 42, 46] and also in [43] for a bigger framework. In particular, the Lie 2-algebra cohomology was formulated in [9] for the strict case to characterize strict Lie 2-bialgebras and in [37] for the general to study deformations of Lie 2-algebras. See also [10] for the cohomology of hemistrict Lie 2-algebras and [1] for a cohomological theory of both strict Lie 2-algebras and Lie 2-groups. The 1-cocycles of a Lie 2-algebra with respect to the adjoint representation are used to define derivations for a Lie 2-algebra. We shall also see that the second cohomology is used to classify the deformations and abelian extensions of Lie 2-algebras [37] and the third cohomology is for classifying crossed modules of Lie 2-algebras [32]. See [18, 21, 30] for the corresponding results for Lie algebras.

### 4.1 Definition

**Definition 4.1.** Let \( \mathfrak{g} \) be a Lie 2-algebra and \( \mathbb{V} \) a 2-vector space. A **representation** of \( \mathfrak{g} \) on \( \mathbb{V} \) is a Lie 2-algebra homomorphism \( \phi : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V}) \). Such a 2-vector space \( \mathbb{V} \) is called a **\( \mathfrak{g} \)-module**.

For simplicity, we always denote an action \( \phi = (\phi_0, \phi_1, \phi_2) : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V}) \) by
\[
x \triangleright (u + m) := \phi_0(x)(u + m), \quad a \triangleright u := \phi_1(a)(u), \quad (x, y) \triangleright u := \phi_2(x, y)(u),
\]
for all $x, y \in g_0, a \in g_{-1}, u \in V_0$ and $m \in V_{-1}$.

A Lie algebra $g$ is a Lie $2$-algebra $0 \to g$. A $2$-term representation up to homotopy of a Lie algebra $g$ on a $2$-vector space $\mathcal{V}$ is defined to be a Lie $2$-algebra homomorphism $\phi : g \to \mathfrak{gl}(\mathcal{V})$ \cite{56}.

**Example 4.2.** Given a Lie algebra $(g, [\cdot, \cdot]_g)$ with a $2$-term representation up to homotopy on $\mathcal{V} : V_{-1} \xrightarrow{\partial} V_0$, we get a Lie $2$-algebra

$$g \ltimes \mathcal{V} : V_{-1} \xrightarrow{0+\partial} g \oplus V_0,$$

where the brackets are given by

$$
\begin{aligned}
l_2(x + u, y + v) &= [x, y]_g + x \triangleright v - y \triangleright u; \\
l_2(x + u, m) &= x \triangleright m; \\
l_3(x + u, y + v, z + w) &= -(x, y) \triangleright w - (y, z) \triangleright u - (z, x) \triangleright v,
\end{aligned}
$$

for $x, y, z \in g, u, v, w \in V_0$ and $m \in V_{-1}$. This Lie $2$-algebra is called the semidirect product Lie $2$-algebra. We refer to \cite{56} for more explicit examples.

**Example 4.3.** Every Lie $2$-algebra $(g, d, [\cdot, \cdot], l_3)$ has a natural representation on itself, called the adjoint representation, which is given by $\text{ad} = (\text{ad}_0, \text{ad}_1, \text{ad}_2) : g \to \mathfrak{gl}(g)$, where

$$
\text{ad}_0(x) = [x, \cdot], \quad \text{ad}_1(a) = [a, \cdot], \quad \text{ad}_2(x, y) = -l_3(x, y, \cdot),
$$

for $x, y, z \in g_0$ and $a \in g_{-1}$.

Let $g$ be a Lie $2$-algebra and $\mathcal{V} : V_{-1} \xrightarrow{\partial} V_0$ a $g$-module given by $\phi : g \to \mathfrak{gl}(\mathcal{V})$. The cohomology of $g$ comes from the generalized Chevalley-Eilenberg complex as follows:

| Degree | Description |
|--------|-------------|
| $-1$   | $V_{-1} \xrightarrow{D}$ |
| $0$    | $V_0 \oplus \text{Hom}(g_0, V_{-1}) \xrightarrow{D}$ |
| $1$    | $\text{Hom}(g_0, V_0) \oplus \text{Hom}(g_{-1}, V_{-1}) \oplus \text{Hom}(\wedge^2 g_0, V_{-1}) \xrightarrow{D}$ |
| $2$    | $\text{Hom}(g_{-1}, V_0) \oplus \text{Hom}(\wedge^2 g_0, V_0) \oplus \text{Hom}(g_0 \wedge g_{-1}, V_{-1}) \oplus \text{Hom}(\wedge^3 g_0, V_{-1}) \xrightarrow{D}$ |

where the degrees of $g_0, g_{-1}, V_0, V_{-1}$ are $-1, -2, 0, -1$ respectively. Denote by $C^k(g, \mathcal{V})$ the set of $k$-cochains. The coboundary operator $D$ can be decomposed as:

$$D = \hat{d} + \hat{\partial} + d^{(1,0)} + d^{(0,1)} + d_{\phi_2} + d_{l_3}, \quad (5)$$

where, for $s = 0, -1,$

$$
\begin{aligned}
\hat{d} : \text{Hom}(\wedge^p g_0 \wedge \bigodot^q g_{-1}, V_s) &\longrightarrow \text{Hom}(\wedge^{p-1} g_0 \wedge \bigodot^{q+1} g_{-1}, V_s), \\
\hat{\partial} : \text{Hom}(\wedge^p g_0 \wedge \bigodot^q g_{-1}, V_{-1}) &\longrightarrow \text{Hom}(\wedge^p g_0 \wedge \bigodot^q g_{-1}, V_0), \\
d^{(1,0)} : \text{Hom}(\wedge^p g_0 \wedge \bigodot^q g_{-1}, V_s) &\longrightarrow \text{Hom}(\wedge^{p+1} g_0 \wedge \bigodot^q g_{-1}, V_s), \\
d^{(0,1)} : \text{Hom}(\wedge^p g_0 \wedge \bigodot^q g_{-1}, V_0) &\longrightarrow \text{Hom}(\wedge^p g_0 \wedge \bigodot^{q+1} g_{-1}, V_{-1}), \\
d_{\phi_2} : \text{Hom}(\wedge^p g_0 \wedge \bigodot^q g_{-1}, V_0) &\longrightarrow \text{Hom}(\wedge^{p+2} g_0 \wedge \bigodot^q g_{-1}, V_{-1}), \\
d_{l_3} : \text{Hom}(\wedge^p g_0 \wedge \bigodot^q g_{-1}, V_s) &\longrightarrow \text{Hom}(\wedge^{p+2} g_0 \wedge \bigodot^{q-1} g_{-1}, V_s).
\end{aligned}
$$
More concretely, for all \(x_i \in g_0, a_i \in g_{-1}, i \in \mathbb{N},\)
\[
d f(x_1, \ldots, x_{p+1}, a_1, \ldots, a_{q+1}) = (-1)^p f(x_1, \ldots, x_{p+1}, da_1, \ldots, a_{q+1}) + c.p. (a_1, \ldots, a_{q+1}),
\]
\[
\partial f = (-1)^{p+2q} \partial \circ f,
\]
\[
d^{(1,0)}_p f(x_1, \ldots, x_{p+1}, a_1, \ldots, a_q) = \sum_{i=1}^{p+1} (-1)^{i+1} x_i \triangleright f(x_1, \ldots, x_{p+1}, a_1, \ldots, a_q)
\]
\[
+ \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1}, a_1, \ldots, a_q)
\]
\[
+ \sum_{i,j} (-1)^i f(x_1, \ldots, \hat{x}_i, \ldots, x_{p+1}, a_1, \ldots, [x_i, a_j], \ldots, a_q),
\]
\[
d^{(0,1)}_p f(x_1, \ldots, x_p, a_1, \ldots, a_{q-1}) = \sum_{i=1}^{q-1} (-1)^p a_i \triangleright f(x_1, \ldots, x_p, a_1, \ldots, \hat{a}_i, \ldots, a_{q-1}),
\]
\[
d_{b_{2}} f(x_1, \ldots, x_{p+2}, a_1, \ldots, a_q) = \sum_\sigma (-1)^{p+2q} (-1)^\sigma (\sigma(1), \sigma(2)) \triangleright f(x_{\sigma(3)}, \ldots, x_{\sigma(p+2)}, a_1, \ldots, a_q),
\]
\[
d_{i_{2}} f(x_1, \ldots, x_p, a_1, \ldots, a_{q-1}) = \sum_{i=1}^{p+1} (-1)^{q} f(x_{i+1}, \ldots, x_{p+1}, a_1, \ldots, a_{q-1}, l_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})),
\]
where \(\sigma\) and \(\tau\) are taken over all \((2,p)\)-unshuffles and \((3,p)\)-unshuffles respectively. Define

\[
Z^k(g, \mathbb{V}) := \{ f \in C^k(g, \mathbb{V}); Df = 0 \}, \quad B^k(g, \mathbb{V}) := \{ Dg \in C^k(g, \mathbb{V}); g \in C^{k-1}(g, \mathbb{V}) \}.
\]

Elements in \(Z^k(g, \mathbb{V})\) and \(B^k(g, \mathbb{V})\) are called \(k\)-**cocycles** and \(k\)-**coboundaries** respectively. The space \(H^k(g, \mathbb{V}) := Z^k(g, \mathbb{V})/B^k(g, \mathbb{V})\) is called the \(k\)th **cohomology** of the Lie 2-algebra \(g\) with respect to the \(g\)-module \(\mathbb{V}\).

### 4.2 The first cohomology and derivation Lie 2-algebras

Let us write down the expressions of 1-cocycles and 1-coboundaries for a Lie 2-algebra \(g\) with a \(g\)-module \(\mathbb{V}\).

**Lemma 4.4.** Let \((X, l_X) \in C^1(g, \mathbb{V})\), where \(X = (X_0, X_1) \in \text{Hom}(g_0, V_0) \oplus \text{Hom}(g_{-1}, V_{-1})\) and \(l_X \in \text{Hom}(\wedge^2 g_0, V_{-1})\). Then

(i) \((X, l_X) \in Z^1(g, \mathbb{V})\) if and only if

\[
\partial \circ X_1 = X_0 \circ d,
\]
\[
\partial l_X(x, y) = X_0 [x, y] + y \triangleright X_0 x - x \triangleright X_0 y,
\]
\[
l_X(x, da) = X_1 [x, a] + a \triangleright X_0 x - x \triangleright X_1 a,
\]
\[
l_X l_3(x, y, z) = (l_X(x, [y, z]) + x \triangleright l_X(y, z) - (y, z) \triangleright X_0 x) + c.p.,
\]

where \(x, y, z \in g_0\) and \(a \in g_{-1}\).

(ii) \((X, l_X) \in B^1(g, \mathbb{V})\) if and only if \(\exists (u, \Theta) \in V_0 \oplus \text{Hom}(g_0, V_{-1}) = C^0(g, \mathbb{V})\), s.t.,

\[
X(x + a) = x \triangleright u + a \triangleright u - \partial \Theta(x) - \Theta(da),
\]
\[
l_X(x, y) = \Theta(x + y) \triangleright u + x \triangleright \Theta(y) - y \triangleright \Theta(x) - \Theta([x, y]).
\]

Recall the adjoint representation of a Lie 2-algebra \(g\) on itself as defined in Definition 4.3.

**Definition 4.5.** Let \(g\) be a Lie 2-algebra. A **derivation** of degree 0 of \(g\) is a 1-cocycle of \(g\) with respect to the adjoint representation on itself. An **inner derivation** of degree 0 of \(g\) is a 1-coboundary of \(g\) with respect to the adjoint representation on itself.
A notion of a homotopy derivation was introduced in [19] using the theory of operads. For Lie 2-algebras, the similarities and differences of these two definitions were clarified in [33].

Denote by \( \text{Der}_0(\mathfrak{g}) \) the set of derivations of degree 0 of \( \mathfrak{g} \). Define

\[
\text{Der}_{-1}(\mathfrak{g}) = \text{gl}_{-1}(\mathfrak{g}) := \text{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-1}).
\]

One obtains a 2-vector space:

\[
\text{Der}(\mathfrak{g}) : \text{Der}_{-1}(\mathfrak{g}) \xrightarrow{\partial} \text{Der}_0(\mathfrak{g}),
\]

where \( \partial \) is given by \( \partial (\Theta) = (\delta(\Theta), l_{\delta(\Theta)}) \), in which \( \delta(\Theta) = (d \circ \Theta, \Theta \circ d) \in \text{gl}(\mathfrak{g}_0) \oplus \text{gl}(\mathfrak{g}_{-1}) \) and \( l_{\delta(\Theta)} \in \text{Hom}(\wedge^2 \mathfrak{g}_0, \mathfrak{g}_{-1}) \) is

\[
l_{\delta(\Theta)}(x, y) = \Theta[x, y] - [x, \Theta y] - [\Theta x, y], \quad \forall x, y \in \mathfrak{g}_0.
\]

In addition, define

\[
\begin{align*}
\{(X, l_X), \Theta\} &= [X, \Theta]_C, \quad \text{(6)} \\
\{(X, l_X), (Y, l_Y)\} &= (\{X, Y\}_C, L_X(l_Y) - L_Y(l_X)), \quad \text{(7)}
\end{align*}
\]

where \( \{\cdot, \cdot\}_C \) is the commutator bracket and for all \( X = (X_0, X_1) \in \text{gl}(\mathfrak{g}_0) \oplus \text{gl}(\mathfrak{g}_{-1}), L_X : \text{Hom}(\wedge^2 \mathfrak{g}_0, \mathfrak{g}_{-1}) \longrightarrow \text{Hom}(\wedge^2 \mathfrak{g}_0, \mathfrak{g}_{-1}) \) is given by

\[
L_X(l_Y)(x, y) = X_1 l_Y(x, y) - l_Y(X_0 x, y) - l_Y(x, X_0 y).
\]

**Theorem 4.6.** ([33]) With the notations above, \( (\text{Der}(\mathfrak{g}), \partial, \{\cdot, \cdot\}) \) is a strict Lie 2-algebra. It is called the **derivation Lie 2-algebra** of \( \mathfrak{g} \).

Associated with the 2-term complex \( \text{Der}_{-1}(\mathfrak{g}) \xrightarrow{\partial} \text{Der}_0(\mathfrak{g}) \), there is a 3-term complex of vector spaces

\[
\begin{array}{c}
\text{DER}(\mathfrak{g}) : \mathfrak{g}_{-1} \xrightarrow{d_0} \text{Der}_{-1}(\mathfrak{g}) \oplus \mathfrak{g}_0 \xrightarrow{d_0} \text{Der}_0(\mathfrak{g}),
\end{array}
\]

where the differential \( d_0 \) is given by

\[
\begin{align*}
d_0(a) &= ([a, \cdot], -da), \quad \forall a \in \mathfrak{g}_{-1}; \\
d_0(\Theta, x) &= \partial(\Theta) + ([x, \cdot], l_3(x, \cdot, \cdot)), \quad \forall (\Theta, x) \in \text{Der}_{-1}(\mathfrak{g}) \oplus \mathfrak{g}_0.
\end{align*}
\]

Moreover, there is also a degree 0 bracket \( [\cdot, \cdot]_D \) on \( \text{DER}(\mathfrak{g}) \) given by

\[
\begin{array}{l}
[(X, l_X), (Y, l_Y)]_D = \{(X, l_X), (Y, l_Y)\}; \\
[(X, l_X), (\Theta, x)]_D = \{(X, l_X), \Theta\} + l_X(x, \cdot, X_0 x); \\
[(\Theta, x), (\Theta', x')]_D = -\Theta x' - \Theta' x; \\
[(X, l_X), a]_D = X_1 a,
\end{array}
\]

for any \( (X, l_X), (Y, l_Y) \in \text{Der}_0(\mathfrak{g}), \Theta, \Theta' \in \text{Der}_{-1}(\mathfrak{g}), x, x' \in \mathfrak{g}_0 \) and \( a \in \mathfrak{g}_{-1} \), where the bracket \( \{\cdot, \cdot\} \) is defined in (6) and (7).

**Lemma 4.7.** ([33]) With the above notations, \( (\text{DER}(\mathfrak{g}), d_0, [\cdot, \cdot]_D) \) is a strict Lie 3-algebra. It is called the **derivation Lie 3-algebra** of \( \mathfrak{g} \).

The derivation Lie 2-algebra and derivation Lie 3-algebra were introduced in [33] to classify the nonabelian extensions of Lie 2-algebras.
Definition 4.8. (i) Let \((\mathfrak{g}, d_\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \mathfrak{h}, d_\mathfrak{h}, [\cdot, \cdot]_\mathfrak{h}, l^3_3)\), \((\hat{\mathfrak{g}}, d, [\cdot, \cdot]_\hat{\mathfrak{g}}, \hat{l}_3)\) be Lie 2-algebras and \(i = (i_0, i_1) : \mathfrak{h} \rightarrow \hat{\mathfrak{g}}\), \(p = (p_0, p_1) : \mathfrak{h} \rightarrow \mathfrak{g}\) be strict Lie 2-algebra homomorphisms. The following sequence of Lie 2-algebras is a short exact sequence if \(\ker i = \ker p\), \(\ker i = 0\) and \(\im p = \mathfrak{g}\):

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathfrak{h}_{-1} & \longrightarrow & \hat{\mathfrak{g}}_{-1} & \longrightarrow & \mathfrak{g}_{-1} & \longrightarrow & 0 \\
& & \downarrow d_\mathfrak{h} & & \downarrow d & & \downarrow d_\mathfrak{g} & & \\
0 & \longrightarrow & \mathfrak{h}_0 & \longrightarrow & \hat{\mathfrak{g}}_0 & \longrightarrow & \mathfrak{g}_0 & \longrightarrow & 0.
\end{array}
\]

(8)

We call \(\hat{\mathfrak{g}}\) an extension of \(\mathfrak{g}\) by \(\mathfrak{h}\), and denote it by \(E_{\hat{\mathfrak{g}}}\). It is called an abelian extension if \(\mathfrak{h}\) is abelian, i.e., \([\cdot, \cdot]_\mathfrak{h} = 0\) and \(l^3_3(\cdot, \cdot, \cdot) = 0\).

(ii) Two extensions of Lie 2-algebras \(E_{\hat{\mathfrak{g}}} : 0 \rightarrow \mathfrak{h} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p\circ F} \mathfrak{g} \rightarrow 0\) and \(E_{\bar{\mathfrak{g}}} : 0 \rightarrow \mathfrak{h} \xrightarrow{i} \bar{\mathfrak{g}} \xrightarrow{p\circ F} \mathfrak{g} \rightarrow 0\) are equivalent, if there exists a Lie 2-algebra homomorphism \(F : \hat{\mathfrak{g}} \rightarrow \bar{\mathfrak{g}}\) such that \(F \circ i = j\), \(q \circ F = p\) and \(F_2(i(u), \alpha) = 0\), for all \(u \in \mathfrak{h}_0\), \(\alpha \in \mathfrak{g}_0\).

Theorem 4.9. ([3]) There is a one-to-one correspondence between isomorphism classes of extensions of Lie 2-algebras given by (8) for \(\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{h}\) and equivalence classes of morphisms from the Lie 2-algebra \(\mathfrak{g}\) to the derivation Lie 3-algebra \(\text{DER}(\mathfrak{h})\).

4.3 The second cohomology and deformations of Lie 2-algebras

The infinitesimal deformation of a Lie 2-algebra is characterized by the second cohomology of this Lie 2-algebra with respect to the adjoint representation on itself defined in Example [13]. Moreover, the abelian extension of a Lie 2-algebra, as a special case of Theorem [43], has a simpler description by means of the second cohomology.

Let \((\mathfrak{g}, d, [\cdot, \cdot], l_3)\) be a Lie 2-algebra. A Lie 2-algebra infinitesimal deformation of \(\mathfrak{g}\) is a sequence of linear maps

\[
d_1 : \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_0, \quad [\cdot, \cdot]_1 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \longrightarrow \mathfrak{g}_1, j = 0, -1, \quad l_3, i : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_{-1}, \quad i = 0, 1,
\]

with \(d_0 = d, [\cdot, \cdot]_0 = [\cdot, \cdot]\) and \(l_{3,0} = l_3\), such that the \(\mathbb{R}[[\lambda]]/(\lambda^2)\)-linear operations \((d_\lambda, [\cdot, \cdot]_\lambda, l^3_3)\) on \(\mathfrak{g}[[\lambda]]/(\lambda^2)\) determined by

\[
d_\lambda a := da + \lambda d_1 a, \quad a \in \mathfrak{g}_{-1},
\]

\[
[x, y]_\lambda := [x, y] + \lambda [x, y]_1, \quad (9)
\]

\[
[x, a]_\lambda := [x, a] + \lambda [x, a]_1, \quad (10)
\]

\[
l^3_3(x, y, z) := l_3(x, y, z) + \lambda l_{3,1}(x, y, z), \quad x, y, z \in \mathfrak{g}_0, \quad (11)
\]

\[
l^3_3(x, y, z) := l_3(x, y, z) + \lambda l_{3,1}(x, y, z), \quad x, y, z \in \mathfrak{g}_0 \quad (12)
\]

give a Lie 2-algebra structure on \(\mathfrak{g}\) for each \(\lambda\). Denote an infinitesimal deformation by \((\mathfrak{g}, d_1, [\cdot, \cdot]_1, l_{3,1})\). Two infinitesimal deformations \(\mathfrak{g}_\lambda = (\mathfrak{g}, d_1, [\cdot, \cdot]_1, l_{3,1})\) and \(\mathfrak{g}_\lambda' = (\mathfrak{g}, d'_1, [\cdot, \cdot]_1', l'_{3,1})\) of a Lie 2-algebra \(\mathfrak{g}\) are said to be equivalent if there exists a triple \(\phi = (\phi_0, \phi_1, \phi_2)\) with

\[
\phi_0 \in \text{gl}(\mathfrak{g}_0), \quad \phi_1 \in \text{gl}(\mathfrak{g}_1), \quad \phi_2 \in \text{Hom}(\mathfrak{g}_0 \otimes \mathfrak{g}_0, \mathfrak{g}_{-1})
\]

such that Id + \(\lambda \phi : \mathfrak{g}_\lambda \rightarrow \mathfrak{g}_\lambda'\) modulo \(\lambda^2\) is a Lie 2-algebra homomorphism for each \(\lambda\).

Theorem 4.10. ([37]) Let \(\mathfrak{g}\) be a Lie 2-algebra. There is a one-to-one correspondence between equivalence classes of Lie 2-algebra infinitesimal deformations of \(\mathfrak{g}\) and elements in the second cohomology group \(H^2(\mathfrak{g}, \mathfrak{g})\) of \(\mathfrak{g}\) with respect to the adjoint representation on itself.

Let us write down the infinitesimal deformation of \((\mathfrak{g}, d, [\cdot, \cdot], l_3)\) given by a 2-coboundary. Let \((X, l_X) \in C^1(\mathfrak{g}, \mathfrak{g})\), with \(X = (X_0, X_1) \in \text{gl}(\mathfrak{g}_0) \oplus \text{gl}(\mathfrak{g}_{-1})\) and \(l_X \in \text{Hom}(\mathfrak{g}_0 \otimes \mathfrak{g}_0, \mathfrak{g}_{-1})\). Note that

\[
C^2(\mathfrak{g}, \mathfrak{g}) = \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0) \oplus \text{Hom}(\mathfrak{g}_0 \otimes \mathfrak{g}_0, \mathfrak{g}_{-1}) \oplus \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \oplus \text{Hom}(\mathfrak{g}_0 \wedge \mathfrak{g}_{-1}, \mathfrak{g}_{-1})
\]

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The 2-coboundary \( D(X, l_X) \in C^2(\mathfrak{g}, \mathfrak{g}) \) has four components, denoted by \( d_1, [\cdot, \cdot]_1 \) (two components) and \( l_{3,1} \) respectively. By Lemma 4.4 we have

\[
d_1 a = dX_1 a - X_0 da, \tag{13}
\]
\[
[x, y]_1 = [x, X_0 y] + [X_0 x, y] - X_0 [x, y] + dl_X(x, y), \tag{14}
\]
\[
[x, a]_1 = [x, X_1 a] + [X_0 x, a] - X_1 [x, a] + l_X(x, da), \tag{15}
\]
\[
l_{3,1}(x, y, z) = l_X(x, [y, z]) + [x, l_X(y, z)] + l_3(X_0 x, y, z) + c.p. - X_1 l_3(x, y, z). \tag{16}
\]

Then the infinitesimal deformation of \( \mathfrak{g} \) given by the 2-coboundary \( D(X, l_X) \) is given by formulas (9)-(12) with \( d_1, [\cdot, \cdot]_1 \), and \( l_{3,1} \) given by the formulas (13)-(16).

Besides describing the deformations of a Lie 2-algebra, the second cohomology can also be used to classify abelian extensions of a Lie 2-algebra, which simplifies and strengthens the result in Theorem 4.9 for general extensions.

**Theorem 4.11.** (37) Given a representation \((\phi_0, \phi_1, \phi_2) : \mathfrak{g} \to gl(\mathfrak{h})\) of a Lie 2-algebra \( \mathfrak{g} \) on a 2-vector space \( \mathfrak{h} \), there is a one-to-one correspondence between equivalence classes of abelian extensions of the Lie algebra \( \mathfrak{g} \) by \( \mathfrak{h} \) and elements in the second cohomology group \( H^2(\mathfrak{g}, \mathfrak{h}) \).

### 4.4 The third cohomology and crossed modules of Lie 2-algebras

The third cohomology of a Lie algebra classifies Lie algebra crossed modules due to Gerstenhaber; see Theorem 2.13. This motivated the authors in [32] to find a notion of crossed modules for Lie 2-algebras, which is classified by the third cohomology of Lie 2-algebras. We shall recall this result in this subsection.

Let \((\mathfrak{m}, d_\mathfrak{m}, [\cdot, \cdot]_\mathfrak{m}, l^\mathfrak{m})\) and \((\mathfrak{g}, d_\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, l^\mathfrak{g})\) be two Lie 2-algebras. We say that \( \mathfrak{g} \) acts on \( \mathfrak{m} \) by derivations if there exists a Lie 2-algebra homomorphism

\[
\phi = (\phi_0, \phi_1, \phi_2) : \mathfrak{g} \to gl(\mathfrak{m})
\]

and a linear map \( l_{\phi_0}(x) : \wedge^2 \mathfrak{m}_0 \to \mathfrak{m}_{-1} \) such that \((\phi_0(x), l_{\phi_0}(x)) \in \text{Der}_0(\mathfrak{m})\) and the map

\[
((\phi_0, l_{\phi_0}), \phi_1, \phi_2) : \mathfrak{g} \to \text{Der}(\mathfrak{m})
\]

is a Lie 2-algebra homomorphism. See Theorem 4.6 for the derivation Lie 2-algebra \( \text{Der}(\mathfrak{m}) \). Then we shall define a crossed product of \( \mathfrak{g} \) and \( \mathfrak{m} \) as follows: on \( \mathfrak{g} \oplus \mathfrak{m} \), define \( L_1 = d_{\mathfrak{g}} + d_{\mathfrak{m}} \) and

\[
\begin{aligned}
L_2(x + \alpha, y + \beta) &= [x, y]_\mathfrak{g} + [\alpha, \beta]_\mathfrak{m} + x \triangleright y - y \triangleright x, \quad \forall x, y \in \mathfrak{g}, \forall \alpha, \beta \in \mathfrak{m}, \\
L_3(x + \alpha, y + \beta, z + \gamma) &= l^\mathfrak{g}_3(x, y, z) + l^\mathfrak{g}_0_1(\alpha, \beta, \gamma, \gamma) - (x, y) \triangleright \gamma - (y, z) \triangleright \alpha \\
&\quad - (z, x) \triangleright \beta + l_{\phi_0}(x)(\beta, \gamma) + l_{\phi_0}(y)(\gamma, \alpha) + l_{\phi_0}(z)(\alpha, \beta), \\
&\quad \forall x, y, z \in \mathfrak{g}_0, \forall \alpha, \beta, \gamma \in \mathfrak{m}_0.
\end{aligned}
\]

It can be checked that \((\mathfrak{g} \oplus \mathfrak{m}, L_1, L_2, L_3)\) is still a Lie 2-algebra with \( \mathfrak{g} \) as a Lie 2-subalgebra and \( \mathfrak{m} \) as an ideal. Denote it by \( \mathfrak{g} \triangleright_\phi \mathfrak{m} \).

**Definition 4.12.** (32) A crossed module of Lie 2-algebras is a quadruple \((\mathfrak{m}, \phi, \Pi)\), where \( \mathfrak{m} \) and \( \mathfrak{g} \) are two Lie 2-algebras, \( \phi \) is an action of \( \mathfrak{g} \) on \( \mathfrak{m} \) by derivations, and \( \Pi : \mathfrak{g} \triangleright_\phi \mathfrak{m} \to \mathfrak{g} \) is a Lie 2-algebra homomorphism such that \( \Pi|_\mathfrak{g} = \text{Id}_\mathfrak{g} = (\text{id}_{\mathfrak{g}_0}, \text{id}_{\mathfrak{g}_{-1}}, 0) \) and

\[
\begin{align*}
(i) &\quad [\alpha, \beta]_\mathfrak{m} = \Pi(\alpha) \triangleright \beta, \quad \forall \alpha, \beta \in \mathfrak{m}, \\
(ii) &\quad l^\mathfrak{m}_3(\alpha, \beta, \gamma) = - (\Pi_0 \alpha, \Pi_0 \beta) \triangleright \gamma - \Pi_2(\Pi_0 \alpha, \beta) \triangleright \gamma, \quad \forall \alpha, \beta, \gamma \in \mathfrak{m}_0, \\
(iii) &\quad l_{\phi_0}(x)(\beta, \gamma) = - (x, \Pi_0 \beta) \triangleright \gamma - \Pi_2(x, \beta) \triangleright \gamma, \quad \forall \beta, \gamma \in \mathfrak{m}_0, x \in \mathfrak{g}_0, \\
(iv) &\quad \Pi_2(\alpha, \beta) = \Pi_2(\Pi_0 \alpha, \beta) = \Pi_2(\alpha, \Pi_0 \beta), \quad \forall \alpha, \beta \in \mathfrak{m}_0.
\end{align*}
\]
In particular, it is called a strong crossed module of Lie 2-algebras if \( \Pi_2 = 0 \).

For a crossed module \((\mathfrak{m}, \mathfrak{g}, \phi, \Pi)\), decompose \(\Pi\) into
\[
\Pi = (\Pi_0, \Pi_1, \Pi_2) = \text{Id}_\mathfrak{g} + \sigma + \varphi = (\text{id}_\mathfrak{g}, \varphi_0), (\text{id}_\mathfrak{g}, \varphi_1), (0, \sigma, \varphi_2)),
\]
where \(\varphi = \Pi|_{\mathfrak{m}}\) and \(\sigma = \Pi_2|_{\mathfrak{g} \otimes \mathfrak{m}}\). We see that \(\varphi = (\varphi_0, \varphi_1, \varphi_2) : \mathfrak{m} \to \mathfrak{g}\) is a Lie 2-algebra homomorphism and \(\varphi_2\) is determined by \(\sigma\). A crossed module is also denoted by \((\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)\).

Then a strong crossed module means that \(\sigma = 0\).

The following example is another motivation for the definition of crossed modules of Lie 2-algebras.

**Example 4.13.** For a Lie 2-algebra \((\mathfrak{g}, d, [\cdot, \cdot], l_3)\), the derivation Lie 2-algebra \(\text{Der}(\mathfrak{g})\) acts on \(\mathfrak{g}\) by derivations with the identity \(\text{Id} : \text{Der}(\mathfrak{g}) \to \text{Der}(\mathfrak{g})\). Consider the extended adjoint representation \(\text{ad} : \mathfrak{g} \to \text{Der}(\mathfrak{g})\) given by
\[
\text{ad}_0(x) = ([x, \cdot], l_3(x, \cdot)), \quad \text{ad}_1(a) = [a, \cdot], \quad \text{ad}_2(y,z) = -l_3(y, z, \cdot)
\]
for \(x, y, z \in \mathfrak{g}_0, a \in \mathfrak{g}_{-1}\) and a linear map \(\sigma : \text{Der}_0(\mathfrak{g}) \wedge \mathfrak{g}_0 \to \text{Der}_{-1}(\mathfrak{g})\) defined by
\[
\sigma((X, l_X), x) = -l_X(x, \cdot), \quad \forall (X, l_X) \in \text{Der}_0(\mathfrak{g}), x \in \mathfrak{g}_0.
\]
It can be verified that \((\mathfrak{g}, \text{Der}(\mathfrak{g}), \text{Id}, \text{ad}, \sigma)\) is a crossed module of Lie 2-algebras. This crossed module is not strong even if \(\mathfrak{g}\) is a strict Lie 2-algebra.

Associated with a crossed module \((\mathfrak{m}, \mathfrak{g}, \phi, \Pi)\) of Lie 2-algebras, there is a strict Lie 3-algebra on the complex
\[
\begin{array}{c}
\mathfrak{m}_{-1} \xrightarrow{d_0} \mathfrak{g}_{-1} \oplus \mathfrak{m}_0 \xrightarrow{d_0} \mathfrak{g}_0;
\end{array}
\]
see [32] for details. In particular, for Example 4.13, it recovers the derivation Lie 3-algebra \(\text{DER}(\mathfrak{g})\) in Lemma 4.7.

A crossed module \((\mathfrak{m}, \mathfrak{g}, \phi, \varphi, \sigma)\) yields a 4-term exact sequence of 2-vector spaces
\[
0 \to \mathcal{V} \xrightarrow{i} \mathfrak{m} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \to 0,
\]
where \(\mathcal{V} := \ker \varphi, \mathfrak{h} := \coker \varphi,\) and \(i, \pi\) are the canonical inclusion and projection. By definition, \(\mathcal{V}\) is in the center of \(\mathfrak{m}\). If this crossed module is strong, we further get that \(\mathfrak{h}\) is a Lie 2-algebra and there exists an induced action of \(\mathfrak{h}\) on \(\mathcal{V}\).

Denote by \(\mathcal{C}(\mathfrak{h}, \mathcal{V})\) the set of strong crossed modules with respect to fixed kernel \(\mathcal{V}\), fixed cokernel \(\mathfrak{h}\) and fixed action of \(\mathfrak{h}\) on \(\mathcal{V}\).

**Theorem 4.14.** ([32]) For a Lie 2-algebra \(\mathfrak{h}\) and an \(\mathfrak{h}\)-module \(\mathcal{V}\), there exists a canonical bijection
\[
\mathcal{C}(\mathfrak{h}, \mathcal{V}) / \sim \xrightarrow{\cong} H^3(\mathfrak{h}, \mathcal{V}).
\]
We refer the reader to [32, Definition 5.8] for the definition of the equivalence relation \(\sim\).

## 5 Integration of Lie 2-algebras

Like Lie 2-algebras, Lie 2-groups are the categorification of Lie groups [4]. In this section, we shall focus on strict Lie 2-groups and the corresponding strict Lie 2-algebras. In particular, the automorphisms of a Lie 2-algebra form a strict Lie 2-group, whose infinitesimal is the derivation Lie 2-algebra in Theorem 4.6.
5.1 Strict Lie 2-groups and strict Lie 2-algebras

Lie 2-groups, as the categorification of Lie groups, were introduced and studied in [4], where the associative law is replaced by an isomorphism, called the associator, which satisfies the pentagon equation. We will focus on the strict case. A strict 2-group is a group object in Cat. It is a strict 2-category with one object and all the 1-morphisms and 2-morphisms are invertible. See Appendix A. Explicitly, a strict Lie 2-group is a Lie groupoid \( \Gamma \in \text{Cat} \), where \( \Gamma_1 \) and \( \Gamma_0 \) are Lie groups and all the structure maps are group homomorphisms.

We have a Lie group counterpart of the Lie algebra crossed module defined in Definition 2.8.

**Definition 5.1.** A Lie group crossed module consists of a Lie group homomorphism \( \Phi : H \to G \) and an action of \( G \) on \( H \) by automorphisms (i.e., \( \triangleright : G \to \text{Aut}(H) \) is a Lie group homomorphism) satisfying

\[
\Phi(g \triangleright h) = g\Phi(h)g^{-1}; \\
\Phi(h) \triangleright h' = hh'h^{-1},
\]

for all \( g \in G \) and \( h, h' \in H \).

**Proposition 5.2.** There is a bijection between strict Lie 2-groups and crossed modules of Lie groups.

Let us sketch the correspondence. If \( \Gamma_1 \rightrightarrows \Gamma_0 \) is a strict Lie 2-group, taking the restriction of the target map \( t \) on \( \ker s \subset \Gamma_1 \), we obtain a Lie group crossed module \( t : \ker s \to \Gamma_0 \), where the action of \( \Gamma_0 \) on \( \ker s \) is \( g \triangleright h = \iota(g)h\iota(g)^{-1} \) for \( g \in \Gamma_0 \) and \( h \in \ker s \). Here \( \iota : \Gamma_0 \to \Gamma_1 \) is the inclusion map and the multiplication in \( \iota(g)\iota(h)^{-1} \) is the group multiplication in \( \Gamma_1 \).

Conversely, given a Lie group crossed module \( \Phi : H \to G \), we build a strict Lie 2-group structure on \( H \times G \rightrightarrows G \), denoted by \( H \ltimes G \), as follows: The group structure on \( H \ltimes G \) is given by

\[
(h_1, g_1) \ast (h_2, g_2) = (h_1(g_1 \triangleright h_2), g_1g_2), \quad \forall g_1, g_2 \in G, h_1, h_2 \in H,
\]

and the groupoid structure on \( H \ltimes G \) is:

- source and target maps: \( s(h, g) = g, t(h, g) = \Phi(h)g \);
- inclusion map: \( \iota(g) = (e, g) \);
- groupoid multiplication \( (h_1, g_1)(h_2, g_2) = (h_1h_2, g_1g_2) \) if \( g_1 = \Phi(h_2)g_2 \).

The notation \( H \ltimes G \) is to emphasize the action of \( G \) on \( H \) in the crossed module. The Lie group \( H \) also acts on \( G \) through \( \Phi \), namely, \( h \triangleright g = \Phi(h)g \). The Lie groupoid \( H \ltimes G \rightrightarrows G \) is actually an action Lie groupoid given by the action of \( H \) on \( G \).

Various examples of Lie group crossed modules are listed in Section 8.4 of [4]. We recall two examples here.

**Example 5.3.** Let \( H \) be a Lie group and \( \text{Aut}(H) \) the automorphism group. Then we have a Lie group crossed module

\[
\Phi : H \to \text{Aut}(H), \quad \Phi(h)(h') = hh'h^{-1}, \quad \forall h, h' \in H,
\]

where \( \text{Aut}(H) \) acts on \( H \) naturally. The corresponding strict Lie 2-group is called the strict automorphism 2-group of \( H \). Particularly, take \( H = \text{SU}(2) \) or the multiplicative group of nonzero quaternions. Then \( \text{Aut}(H) = \text{SO}(3) \) and \( \Phi : \text{SU}(2) \to \text{SO}(3) \) is the universal covering map. These examples of strict 2-groups play explicit roles in physics.
Example 5.4. Suppose that $\Phi : H \rightarrow G$ is a surjective homomorphism of Lie groups. Then it is equipped with a Lie group crossed module structure if and only if $\Phi$ is a central extension, meaning that $\ker \Phi$ is contained in the center of $H$. Moreover, when this crossed module exists, it is unique.

Suppose $V$ is a finite dimensional real vector space with a skew-symmetric bilinear form $\omega : V \times V \rightarrow \mathbb{R}$. Let $H = V \oplus \mathbb{R}$ be the Lie group with the product

$$(v, \alpha)(w, \beta) = (v + w, \alpha + \beta + \omega(v, w)).$$

Then we obtain a Lie group crossed module

$$\Phi : V \oplus \mathbb{R} \rightarrow V, \quad \Phi(v, \alpha) = v.$$

The corresponding strict 2-group is called the **Heisenberg 2-group** of $(V, \omega)$.

Lie groups have Lie algebras. Taking differentials of the structure maps in a Lie group crossed module, we obtain a Lie algebra crossed module.

**Proposition 5.5.** There is a unique functor from the category of Lie group crossed modules to the category of Lie algebra crossed modules.

So by integrating a strict Lie 2-algebra, we mean to find a Lie group crossed module whose infinitesimal Lie algebra crossed module is the given strict Lie 2-algebra. Actually, there exists a unique 2-functor from the 2-category of Lie group crossed modules to that of Lie algebra crossed modules; see [2, Proposition 45].

The integration of a general Lie 2-algebra is rather complicated. For the string Lie 2-algebra in Example 2.14 we refer to [2, 6]. In [6], the authors constructed an infinite-dimensional Lie 2-group $P_1G$ whose Lie 2-algebra is equivalent to the string Lie 2-algebra. The objects of $P_1G$ are based paths in $G$, while the automorphisms of any object form the level-1 Kac-Moody central extension of the loop group $\Omega G$.

More recently, for a Lie $n$-algebra, the author in [24] provided an explicit construction of its integrating Lie $n$-group. This extends the work in [22] for nilpotent $L_\infty$-algebras. In particular, for Lie algebras, this construction gives the simplicial classifying space of the corresponding simply-connected Lie group. For string Lie 2-algebras, it yields the model of the simplicial nerve of the string groups [6].

In the case of strict Lie 2-algebras, the integration by [24] is proved in [58] to be Morita equivalent to the integration of Lie algebra crossed modules to Lie group crossed modules. As an application, the authors in [58] further integrated a non-strict morphism between Lie algebra crossed modules to a generalized morphism between their corresponding Lie group crossed modules, which includes examples of 2-term representations up to homotopy, nonabelian extensions and up to homotopy Poisson actions of Lie algebras. An integration of such morphisms was also achieved by the technique of butterflies [44].

Another quite interesting case is to integrate semidirect product Lie 2-algebras in Example 4.2. In [57], the authors integrated such a class of nonstrict Lie 2-algebras and they obtained strict Lie 2-groups in the finite dimensional case by using the butterfly method.

### 5.2 Integration of derivations for Lie 2-algebras

In this subsection, for a Lie 2-algebra $(\mathfrak{g}, d, [\cdot, \cdot], l_3)$, we give the integration of the derivation Lie 2-algebra $\text{Der}(\mathfrak{g})$ in Theorem 4.6 which is the automorphism Lie 2-group $\text{Aut}(\mathfrak{g})$.

To define $\text{Aut}(\mathfrak{g})$, first denote by $\text{Aut}_0(\mathfrak{g})$ the set of Lie 2-algebra automorphisms of $\mathfrak{g}$. See Subsection 2.11 for the definition of automorphisms. It is evident that $\text{Aut}_0(\mathfrak{g})$ with the composition $\circ$ is a Lie group. Next, define a multiplication on $\text{gl}_{-1}(\mathfrak{g}) := \text{Hom}(\mathfrak{g}_0, \mathfrak{g}_{-1})$ by

$$\tau \ast \tau' = \tau + \tau' + \tau \circ d \circ \tau', \quad \forall \tau, \tau' \in \text{gl}_{-1}(\mathfrak{g}).$$
Note that \( \ast \) satisfies the associative law. So \((\mathfrak{gl}_1(\mathfrak{g}), \ast)\) is a monoid, in which the zero map is the identity element. \(\text{Aut}_1(\mathfrak{g})\) is defined to be the group of units of \(\mathfrak{gl}_1(\mathfrak{g})\), which is a Lie group. Define \( \partial : \text{Aut}_1(\mathfrak{g}) \to \text{Aut}_0(\mathfrak{g}) \) by

\[
\partial(\tau) = (\text{id} + d \circ \tau, \text{id} + \tau \circ d, l^0_{\tau}) , \quad \forall \tau \in \text{Aut}_1(\mathfrak{g}),
\]

where \( l^0_{\tau} : \wedge^2 \mathfrak{g}_0 \to \mathfrak{g}_{-1} \) is defined by

\[
l^0_{\tau}(x,y) = \tau[x,y] - [x,\tau y] - [\tau x,y] - [\tau x,d\tau y] , \quad \forall x,y \in \mathfrak{g}_0.
\]

Based on the fact that \( \tau \in \text{Aut}_1(\mathfrak{g}) \) if and only if \( \text{id} + d \circ \tau \in \text{GL}(\mathfrak{g}_0) \), we see that \( \partial(\tau) \in \text{Aut}_0(\mathfrak{g}) \). Furthermore, \( \text{Aut}_0(\mathfrak{g}) \) acts on \( \text{Aut}_1(\mathfrak{g}) \) naturally:

\[
\mathfrak{g} > \tau = A_1 \circ \tau \circ A_0^{-1} \quad \forall A = (A_0, A_1, A_2) \in \text{Aut}_0(\mathfrak{g}), \tau \in \text{Aut}_1(\mathfrak{g}).
\]

**Theorem 5.6.** With the notations above, we have that

(1) the quadruple \((\text{Aut}_1(\mathfrak{g}), \text{Aut}_0(\mathfrak{g}), \partial, >)\) is a Lie group crossed module, which is called the **automorphism 2-group** of \(\mathfrak{g}\), and denoted by \(\text{Aut}(\mathfrak{g})\).

(2) the differentiation of the Lie group crossed module \((\text{Aut}_1(\mathfrak{g}), \text{Aut}_0(\mathfrak{g}), \partial, >)\) is the Lie algebra crossed module \((\text{Der}_1(\mathfrak{g}), \text{Der}_0(\mathfrak{g}), d, \phi)\).

Let us consider the example for the string Lie 2-algebra \(\text{String}(\mathfrak{g}) : \mathbb{R} \to \mathfrak{g}\) defined in Definition 2.14, where \(\mathfrak{g}\) is a semisimple Lie algebra. A triple \((X_0, t, l_X)\), with \(X_0 \in \mathfrak{gl}(\mathfrak{g}), t \in \mathbb{R}\) and \(l_X \in \wedge^2 \mathfrak{g}^*\), is a derivation of degree 0 if and only if

\[
X_0 \in \text{Der}(\mathfrak{g}), \quad (\mathfrak{D}l_X)(x, y, z) = tl_3(x, y, z) - l_3(X_0x, y, z) - l_3(x, X_0y, z) - l_3(x, y, X_0z),
\]

where \(\mathfrak{D} : \wedge^* \mathfrak{g}^* \to \wedge^{*+1} \mathfrak{g}^*\) is the differential of \(\mathfrak{g}\) with coefficients in \(\mathbb{R}\). Since \(\mathfrak{g}\) is semisimple, we have \(X_0 = \text{ad}_x = [x, \cdot]_\mathfrak{g}\) for a unique \(x \in \mathfrak{g}\). By the invariance of the Killing form, we have

\[
l_3(X_0x, y, z) + l_3(X_0y, z, x) + l_3(X_0z, x, y) = 0,
\]

and hence

\[
(\mathfrak{D}l_X)(x, y, z) = tl_3(x, y, z) = 0,
\]

i.e., \(t = 0\), as the Cartan 3-form \(l_3\) is not exact. By the fact that \(H^1(\mathfrak{g}) = 0, i = 1, 2\) as \(\mathfrak{g}\) is semisimple, there exists a unique \(\xi \in \mathfrak{g}^*\) such that \(l_X = \mathfrak{D}(\xi)\). In summary, we have

\[
\text{Der}_0(\text{String}(\mathfrak{g})) = \{(\text{ad}_x, \mathfrak{D}(\xi)), x \in \mathfrak{g}, \xi \in \mathfrak{g}^*\}.
\]

Moreover, the bracket is

\[
\{(\text{ad}_x, 0, \mathfrak{D}(\xi)), (\text{ad}_y, 0, \mathfrak{D}(\eta))\} = (\text{ad}_{[x,y]_\mathfrak{g}}, 0, \mathfrak{D}(\text{ad}_x^\ast \eta - \text{ad}_y^\ast \xi)).
\]

**Example 5.7.** Proposition 3.4 For the string Lie 2-algebra \(\text{String}(\mathfrak{g})\), the derivation Lie 2-algebra \(\text{Der}(\text{String}(\mathfrak{g}))\) is as follows:

- \(\text{Der}_0(\text{String}(\mathfrak{g}))\) is isomorphic to the semidirect product Lie algebra \(\mathfrak{g} \ltimes \mathfrak{g}^*\);
- \(\text{Der}_1(\text{String}(\mathfrak{g}))\) is \(\mathfrak{g}^*\), which is abelian;
- the differential \(\tilde{d} : \text{Der}_1(\text{String}(\mathfrak{g})) \to \text{Der}_0(\text{String}(\mathfrak{g}))\) is given by \(\tilde{d}(\Theta) = (0, -\Theta)\).

Following a similar discussion as above, we have the automorphism 2-group of the string Lie 2-algebra.

**Example 5.8.** The automorphism 2-group \(\text{Aut}(\text{String}(\mathfrak{g}))\) of the string Lie 2-algebra is
• \(\text{Aut}_0(\text{String}(g))\) is isomorphic to the semidirect product \(G \rtimes g^*\), where \(G\) is the connected and simply-connected Lie group that integrating \(g\);

• \(\text{Aut}_{-1}(\text{String}(g)) = g^*\), the abelian Lie group;

• The map \(\partial : \text{Aut}_{-1}(\text{String}(g)) \to \text{Aut}_0(\text{String}(g))\) is given by \(\partial(\tau) = (e, -\tau)\), where \(e\) is the identity in \(G\).

### Appendix

#### A 2-Categories

Roughly speaking, a 2-category consists of objects, 1-morphisms (also called morphisms) between objects, and 2-morphisms between morphisms. The morphisms can be composed along the objects, while the 2-morphisms can be composed in two different directions: along objects-called horizontal composition, and along morphisms-called vertical composition. The composition of morphisms is allowed to be associative only up to coherent associator 2-morphisms. When the associativity holds, it is called a strict 2-category. See [26].

**Definition A.1.** A strict 2-category \(C\) consists of the following data

1. a set of objects \(\text{Ob}(C)\);

2. for each pair \(x, y \in \text{Ob}(C)\), a category \(\text{Mor}_C(x, y)\);
   - The objects of \(\text{Mor}_C(x, y)\) are called 1-morphisms and denoted by \(F : x \to y\).
   - the morphisms between these 1-morphisms are called 2-morphisms and denoted by \(\theta : F' \Rightarrow F\).
   - The composition of 2-morphisms in \(\text{Mor}_C(x, y)\) is called vertical composition and denoted as \(\theta' \circ \theta\) for \(\theta' : F'' \Rightarrow F'\) and \(\theta : F'' \Rightarrow F'\):

     \[
     \begin{array}{c}
     F'' \\
     \theta' \Downarrow \\
     \theta \Downarrow \\
     F'
     \end{array}
     =
     \begin{array}{c}
     F'' \\
     \theta' \circ \theta \Downarrow \\
     F
     \end{array}
     \]

3. For each triple \(x, y, z \in \text{Ob}(C)\), a functor

\[
(\circ, *) : \text{Mor}_C(y, z) \times \text{Mor}_C(x, y) \to \text{Mor}_C(x, z),
\]

the image of the pair of 1-morphisms \((F, G)\) is called the composition of \(F\) and \(G\), denoted by \(F \circ G\). The image of the pair of 2-morphisms \((\theta, \tau)\) is called the horizontal composition and denoted by \(\theta \circ \tau\):

\[
\begin{array}{c}
F' \Theta \\
\theta \Downarrow \\
F
\end{array}
\begin{array}{c}
G' \\
\tau \Downarrow \\
G
\end{array}
= \begin{array}{c}
F' \circ G' \\
\theta \circ \tau \Downarrow \\
F \circ G
\end{array}
\]

These data satisfy the following rules:

1. The sets of objects and 1-morphisms with composition of 1-morphisms form a category;

2. The horizontal composition of 2-morphisms is associative;
(3) The identity 2-morphism \( \text{id}_{\text{id}_x} \) of the identity 1-morphism \( \text{id}_x \) is a unit for the horizontal composition.

Since \((\circ, \ast) : \text{Mor}_C(y, z) \times \text{Mor}_C(x, y) \to \text{Mor}_C(x, z)\) is a functor, we have the interchange law:

\[
(\theta' \circ \theta) \ast (\tau' \circ \tau) = (\theta' \ast \tau') \circ (\theta \ast \tau),
\]

for \( \theta : F \Rightarrow F', \theta' : F' \Rightarrow F'' \Rightarrow G \Rightarrow G' \) and \( \tau : G \Rightarrow G' \).

A strict Lie 2-group is a strict 2-category with one object and all the 1-morphisms and 2-morphisms are invertible.

**Definition A.2.** Let \( g \) and \( h \) be two 2-term \( L_\infty \)-algebras and \( \phi, \psi : g \to h \) be two 2-term \( L_\infty \)-algebra homomorphisms. A 2-homomorphism \( \tau : \phi \Rightarrow \psi \) is a linear map \( \tau : g_0 \to h_{-1} \) such that

1. \( \tau \) is a homotopy between the chain maps \( (\phi_0, \phi_1) \) and \( (\psi_0, \psi_1) \), i.e.,
   \[
   \psi_0 - \phi_0 = d_h \circ \tau, \quad \psi_1 - \phi_1 = \tau \circ d_g.
   \]
2. \( \psi_2(x, y) - \phi_2(x, y) = \tau([x, y]_g) - [\phi_0(x), \tau(y)]_h - [\tau(x), \psi_0(y)]_h \) for \( x, y \in g_0 \).

**Definition A.3.** A homotopy equivalence between two 2-term \( L_\infty \)-algebras \( g \) and \( h \) is a pair of \( L_\infty \)-algebra homomorphisms \( \phi : g \to h \) and \( \psi : h \to g \) such that the compositions \( \phi \circ \psi \) and \( \psi \circ \phi \) are homotopic to the identity map, i.e., there are 2-homomorphisms \( \theta : \phi \circ \psi \Rightarrow \text{Id}_h \) and \( \eta : \psi \circ \phi \Rightarrow \text{Id}_g \).

Homotopy equivalences for strict graded Lie 2-algebras we used in Theorem 3.24 are similar to be defined. We refer to the Appendix A in [9] for details.

**Example A.4.** There is a strict 2-category \( 2\text{Term}_{L_\infty} \) with 2-term \( L_\infty \)-algebras as objects, homomorphisms as 1-morphisms and 2-homomorphisms as 2-morphisms. See Definitions 2.3 and 3.2 for homomorphisms and 2-homomorphisms respectively.

Generally, let \( K \) be a category. There is a strict 2-category, denoted by \( K\text{Cat} \), with categories in \( K \) as objects, functors in \( K \) as 1-morphisms and natural transformations in \( K \) as 2-morphisms; see [2] Proposition 4. For instance, when \( K \) is the category of vector spaces, \( K\text{Cat} \) is the strict 2-category \( 2\text{Vect} \) of 2-vector spaces we mentioned in Subsection 2.1.

### B Lie algebroids and Lie groupoids

We refer to Mackenzie's book [10] for a thorough description of the theories of Lie algebroids and groupoids.

**Definition B.1.** A **Lie algebroid** is a vector bundle \( A \to M \) with a Lie algebra structure \( [\cdot, \cdot]_A \) on the section space \( \Gamma(A) \) and a vector bundle morphism \( \rho_A : A \to TM \) from \( A \) to the tangent bundle \( TM \), called the anchor, such that

\[
[u, fv]_A = f[u, v]_A + \rho_A(u)(f)v, \quad \forall u, v \in \Gamma(A), \quad f \in C^\infty(M).
\]

A groupoid is a small category such that every arrow is invertible. Explicitly,

**Definition B.2.** A **groupoid** is a pair \((\mathcal{G}, M)\), where \( M \) is the set of objects and \( \mathcal{G} \) is the set of arrows, with the structure maps

- two surjection maps \( s, t : \mathcal{G} \to M \), called the source map and target map, respectively;
- the multiplication \( \cdot : \mathcal{G}^{(2)} \to \mathcal{G} \), where \( \mathcal{G}^{(2)} = \{(g_1, g_2) \in \mathcal{G} \times \mathcal{G} | s(g_1) = t(g_2)\} \);
• the inverse map \((\cdot)^{-1} : G \to G\);
• the inclusion map \(\iota : M \to G\), called the identity map,
satisfying the following properties:

1. (associativity) \((g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)\), whenever the multiplications are well-defined;
2. (unitality) \(\iota(t(g)) \cdot g = g = g \cdot \iota(s(g))\);
3. (invertibility) \(g \cdot g^{-1} = \iota(t(g)), \quad g^{-1} \cdot g = \iota(s(g))\).

Denote a groupoid by \((G \rightrightarrows M, s, t)\) or simply by \(G\).

A Lie groupoid is a groupoid such that both the set of objects and the set of arrows are smooth manifolds, all structure maps are smooth, and source and target maps are subjective submersions.

As the tangent space of a Lie group at the identity has a Lie algebra structure, for a Lie groupoid \((G \rightrightarrows M, s, t)\), on the vector bundle \(A := \ker(s_*)|_M \to M\), there is a Lie algebroid structure defined as follows: the anchor map \(A \to TM\) is simply \(t_*\), and the Lie bracket \([u, v]_A\) is determined by

\[
[u, v]_A = [\overrightarrow{u}, \overrightarrow{v}], \quad \forall u, v \in \Gamma(A),
\]

where \(\overrightarrow{u}\) denotes the right-invariant vector field on \(G\) given by \(\overrightarrow{u}_g = R_{g*}u_{t(g)}\) and \([\cdot, \cdot]\) is the Schouten bracket on \(\mathfrak{X}(G)\).

Unlike the Lie algebra case, not every Lie algebroid can be integrated to a Lie groupoid. We refer to [17] for the integrability condition.

**Example B.3.** A Lie algebra is a Lie algebroid with the base manifold being a point; A Lie group is a Lie groupoid with the base manifold being a point.

**Example B.4.** For a manifold \(M\), the tangent bundle \(TM\) with the Schouten bracket is a Lie algebroid whose anchor is the identity map. The Lie groupoid of this Lie algebroid is the pair groupoid \(M \times M \rightrightarrows M\), where the source and target maps are the two projections, the inclusion map is \(\iota(x) = (x, x)\), and the multiplication is

\[(x, y) \cdot (y, z) = (x, z), \quad \forall x, y, z \in M.\]

Another Lie groupoid integrating \(TM\) is the fundamental groupoid, or homotopy groupoid:

\[\pi_1(M) := \{\text{paths in } M\}/\text{homotopies} \rightrightarrows M,\]

whose source and target are end points of a path, and multiplication is the concatenation of paths. This is the unique source simply-connected groupoid that integrating \(TM\); see [17, 40] for details.

**Example B.5.** For a Poisson manifold \((M, \pi)\), there is a Lie algebroid structure on the cotangent bundle \(T^*M\), with the anchor

\[\pi^\sharp : T^*M \to TM, \quad \pi^\sharp(\xi)(\eta) := \pi(\xi, \eta), \quad \forall \xi, \eta \in \Omega^1(M),\]

and the Lie bracket

\[[\xi, \eta]_{T^*M} = L_{\pi^\sharp(\xi)}\eta - L_{\pi^\sharp(\eta)}\xi - d_{\pi^\sharp}s(\xi, \eta).\]

The Lie groupoid of this Lie algebroid is the symplectic groupoid of the Poisson manifold \((M, \pi)\).
Example B.6. Let $g$ be a Lie algebra which acts on a manifold $M$ by the Lie algebra homomorphism $\phi : g \to \mathfrak{X}(M)$. Then there is a Lie algebroid structure on the trivial bundle $A := g \times M \to M$, where the anchor $\rho_A$ is
\[
\rho_A(fu) = f\phi(u),
\]
and the Lie bracket $[\cdot,\cdot]_A$ is given by
\[
[fu,gv]_A = fg[u,v]_g + f\phi(u)(g)v - g\phi(v)(f)u, \quad \forall f, g \in C^\infty(M), u, v \in g.
\]
This Lie algebroid is called the action Lie algebroid. The corresponding Lie groupoid is the action Lie groupoid given as follows. Let $G$ be a Lie group which acts a manifold $M$ by $\Phi : G \to \text{Diff}(M)$. Then we have a Lie groupoid structure on $G \times M \rightrightarrows M$, whose source and target maps are
\[
s(g, m) = m, \quad t(g, m) = \Phi(g)(m),
\]
the inclusion map is $\iota(m) = (e, m)$, and the multiplication is
\[
(g, m) \cdot (h, n) = (gh, n), \quad \forall m, n \in M, g, h \in G,
\]
whenever $m = \Phi(h)(n)$. 

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