EXHAUSTING DOMAINS OF THE SYMMETRIZED BIDISC

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Abstract. We show that the symmetrized bidisc may be exhausted by strongly linearly convex domains. It shows in particular the existence of a strongly linearly convex domain that cannot be exhausted by domains biholomorphic to convex ones.

In our paper we show that the symmetrized bidisc can be exhausted by strongly linearly convex domains. Since the symmetrized bidisc is a $\mathbb{C}$-convex domain that cannot be exhausted by domains biholomorphic to convex ones, this fact has many interesting consequences. It gives a solution to open problems and implies alternate proofs of known results for the symmetrized bidisc.

Recall that a domain $D \subset \mathbb{C}^n$ is $\mathbb{C}$-convex if for any complex line $\ell$ intersecting $D$ the intersection $\ell \cap D$ is connected and simply connected. A bounded domain $D \subset \mathbb{C}^n$ with $C^2$-boundary is called strongly linearly convex if the defining function $r$ of $D$ satisfies the inequality

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k} (z_0) X_j \bar{X}_k > \left| \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k} (z_0) X_j X_k \right|$$

for any boundary point $z_0$ and any non-zero vector $X$ from the complex tangent space to $\partial D$ at $z_0$.

Basic facts on $\mathbb{C}$-convex domains and strongly linearly convex ones that we use in the paper can be found in [2] and [6]. Let us recall only that strong linear convexity implies $\mathbb{C}$-convexity.

For $\epsilon \in [0, 1)$ let us define

$$D_\epsilon := \{(s, p) \in \mathbb{C}^2 : \sqrt{|s - \bar{s}p|^2 + \epsilon + |p|^2} < 1\}.$$

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Note that $D_0$ is the symmetrized bidisc $\mathbb{G}_2$ (see [1] for the above description of the symmetrized bidisc) and $D_\epsilon \nearrow \mathbb{G}_2$ as $\epsilon \to 0^+$. Moreover, $D_\epsilon \subset \mathbb{C} \times \mathbb{D}$, $\epsilon \in (0, 1)$.

Note that the mapping

$$\mathbb{C} \times \mathbb{D} \ni (s, p) \mapsto (s - \bar{s}p, p) \in \mathbb{C}^2$$

is an $\mathbb{R}$-diffeomorphism onto the image. It shows in particular that $D_\epsilon$ is $\mathbb{R}$-diffeomorphic to the convex domain $G_\epsilon = \{(w, z) \in \mathbb{C}^2 : \sqrt{|w|^2 + \epsilon + |z|^2} < 1\}$. Moreover, it is elementary to see that the strongly convex domains $G_\epsilon$ ($\epsilon \in (0, 1)$) exhaust the (non-strongly) convex domain $G_0$.

We show that a similar result holds for the domains $D_\epsilon$.

**Theorem 1.** The domain $D_\epsilon$ is strongly linearly convex, $\epsilon \in (0, 1)$. Consequently, the symmetrized bidisc can be exhausted by an increasing sequence of strongly linearly convex domains.

Combining Theorem [1] with the fact that the symmetrized bidisc cannot be exhausted by domains biholomorphic to convex ones (see [4]) we get the following corollary which gives a negative answer to a long-standing open problem on the existence of a strongly linearly convex domain not biholomorphic to a convex domain. Note that examples of strongly linearly convex domains which are not convex are well known (see [11] and also [2]).

**Corollary 2.** The domains $D_\epsilon$ for $\epsilon > 0$ small enough are examples of strongly linearly convex domains that are not biholomorphic to convex ones (and even cannot be exhausted by such domains).

**Remark.** Recall that the equality between the Lempert function and the Carathéodory distance (i.e. the Lempert Theorem) holds for strongly linearly convex domains (see [9]). Therefore, Theorem [1] implies that the equality between the two functions on the symmetrized bidisc follows directly from the Lempert Theorem. It gives an alternate proof of that fact (to that in [1] and [3]). Moreover, it also implies that the tetra-block is the only known non-trivial example of a domain (i.e. bounded and pseudoconvex) for which the fact that Lempert Theorem holds does not follow directly from the papers [8] and [9] (see [5]).

**Remark.** Theorem [1] shows that the two papers of Lempert (see [8] and [9]) verify the equality of the Lempert function and the Carathéodory distance for different classes of domains (convex ones and strongly linearly convex). This fact seemed to be unknown.

**Remark.** Theorem [1] also implies that $\mathbb{G}_2$ is a $\mathbb{C}$-convex domain - it gives an alternate proof to that in [10].
Below we choose one of possible (global) defining $C^\infty$ functions for the domain $D_\epsilon$ ($\epsilon \in (0, 1)$):

\begin{equation}
(4) \quad r_\epsilon(s, p) := r(s, p) := |s - \bar{s}p|^2 + \epsilon - (1 - |p|^2)^2, \quad (s, p) \in \mathbb{C} \times \mathbb{D}.
\end{equation}

Note that the defining function is even real analytic.

**Proof of Theorem** Let us fix $\epsilon \in (0, 1)$.

First we note that the gradient of $r$ does not vanish on $\partial D_\epsilon$ (we shall calculate the complex tangent below).

Now for a point $(s_0, p_0) \in \partial D_\epsilon$ and $(s, p)$ being a non-zero tangent (in the complex sense) vector to $\partial D_\epsilon$, we shall show that $\rho_{\lambda\lambda}(0) > |\rho_{\lambda}(0)|$, where $\rho(\lambda) := r(s_0 + \lambda s, p_0 + \lambda p)$, $\lambda \in \mathbb{C}$. Note that for $\rho(s_0, p_0) = 0$ and arbitrary $(s, p)$ we have

\begin{equation}
(5) \quad \rho(\lambda) = 2 \Re \left( (\bar{s_0}s_0 - \bar{s_0}\bar{p_0})(s - \bar{s_0}p) - (s_0 - \bar{s_0}p_0)s\bar{p_0} - 2\bar{p_0}p - 2|p_0|^2\bar{p_0}p)\lambda \right) + |\lambda|^2 \left( |s - \bar{s_0}p|^2 + |s|^2|p_0|^2 - 2\Re((\bar{s_0}s_0 - \bar{s_0}\bar{p_0})\bar{s}p) + 2|p|^2 - 2|p_0|^2|p|^2 \right)
\end{equation}

\[- \Re \left( 2(s - \bar{s_0}p)s\bar{p_0}\lambda^2 \right) - \left( \Re(2\bar{p_0}p\lambda) \right)^2 + o(\lambda^2). \]

The above formula shows in particular that tangent vectors $(s, p)$ to $\partial D_\epsilon$ are given by the formula

\begin{equation}
(6) \quad s(s_0 - s_0\bar{p_0} - \bar{p_0}(s_0 - \bar{s_0}p_0)) = p(s_0(s_0 - s_0\bar{p_0}) + 2\bar{p_0} + 2|p_0|^2\bar{p_0}).
\end{equation}

It is also elementary to see that for a $C^2$-function $v(\lambda) = \Re(A\lambda) + a|\lambda|^2 + \Re(b\lambda^2) - (\Re(c\lambda))^2 + o(\lambda^2)$, where $a \in \mathbb{R}$, $A, b, c \in \mathbb{C}$, the condition for $v_{\lambda\lambda}(0) > |v_{\lambda}(0)|$ is

\begin{equation}
(7) \quad a - \frac{|c|^2}{2} > \left| b - \frac{c^2}{2} \right|.
\end{equation}

Applying this information to the function $\rho$ we get the following inequality

\begin{equation}
(8) \quad |s - \bar{s_0}p|^2 + |s|^2|p_0|^2 - 2\Re((\bar{s_0}s_0 - \bar{s_0}\bar{p_0})\bar{s}p) + 2|p|^2 - 2|p_0|^2|p|^2 - \frac{2|\bar{p_0}p|^2}{2} > \left| 2(s - \bar{s_0}p)s\bar{p_0} + \frac{(2\bar{p_0}p)^2}{2} \right|
\end{equation}

that when proven for boundary points $(s_0, p_0)$ and non-zero tangent $(s, p)$ will finish the proof of the theorem.

Substitute the condition on the tangency of the vector $(s, p)$. Since the inequality is trivial when $s_0 = 0$ we may neglect this case.
Then we divide both sides by \( |p|^2 \) and after reductions we get the inequality

\[
(9) \quad \left| 2|p_0|^2\bar{p}_0 - 2\bar{p}_0 + \bar{s}_0\bar{p}_0(s_0 - \bar{s}_0p_0) \right|^2 + |p_0|^2|\bar{s}_0 - s_0\bar{p}_0| - 2\bar{p}_0 + 2|p_0|^2|\bar{p}_0|^2 - \\
2 \Re \left( ((s_0-s_0\bar{p}_0)(s_0(s_0-s_0p_0) - 2p_0|p_0|^2p_0)(\bar{s}_0 - s_0\bar{p}_0 - \bar{p}_0(s_0 - \bar{s}_0p_0)) \right) + \\
2|\bar{s}_0 - s_0\bar{p}_0 - \bar{p}_0(s_0 - \bar{s}_0p_0)|^2 - 4|p_0|^2|\bar{s}_0 - s_0\bar{p}_0 - \bar{p}_0(s_0 - \bar{s}_0p_0)|^2 > \\
\left| 2(2|p_0|^2\bar{p}_0 - 2\bar{p}_0 + \bar{s}_0\bar{p}_0(s_0 - \bar{s}_0p_0)) (\bar{s}_0 - s_0\bar{p}_0) - 2\bar{p}_0 + 2|p_0|^2\bar{p}_0)\bar{p}_0 + \\
2\bar{p}_0^2(\bar{s}_0 - s_0\bar{p}_0 - \bar{p}_0(s_0 - \bar{s}_0p_0))^2 \right|.
\]

Let us get rid of subscripts. After elementary calculations we get the inequality

\[
(10) \quad |p|^2|2p|^2 - 2 + \bar{s}(s - \bar{s}p) |^2 + |p|^2|\bar{s}(s - \bar{s}p) + 2|p|^2\bar{p} - 2\bar{p} |^2 - \\
2 \Re \left( (\bar{s} - \bar{s}\bar{p})(s(s - \bar{s}p) - 2p + 2|p|^2p)(\bar{s} - s\bar{p} - \bar{p}(s - \bar{s}p)) \right) + \\
2|\bar{s} - s\bar{p} - \bar{p}(s - \bar{s}p)|^2 - 4|p|^2|\bar{s} - s\bar{p} - \bar{p}(s - \bar{s}p)|^2 > \\
2|p|^2|2|p|^2 - 2 + \bar{s}(s - \bar{s}p))(\bar{s}(s - \bar{s}p) - 2\bar{p} + 2|p|^2\bar{p}) + \\
(\bar{s} - s\bar{p} - \bar{p}(s - \bar{s}p))^2 |.
\]

Note that the above function is invariant with respect to the mapping \((s, p) \mapsto (e^{it}s, e^{-2it}p)\) which means that we may assume that \( s \geq 0 \). Since \( \rho(s, p) = 0 \) we get that \( s^2 = \frac{(1-|p|^2)^2 - \epsilon}{1-|p|^2} \) (and \( p \) may be arbitrary complex number satisfying the inequality \( \epsilon \leq (1-|p|^2)^2 \)). Substituting the above in the inequality we get that
(11) \[ |p|^2 \left| 2(|p|^2 - 1)(1 - \bar{p}) + (1 - |p|^2)^2 - \epsilon \right|^2 + |p|^2 \left| (1 - |p|^2)^2 - \epsilon - 2\bar{p}(1 - |p|^2)(1 - p) \right|^2 - 2((1 - |p|^2)^2 - \epsilon) \cdot \text{Re} \left( (1 - \bar{p}) (\frac{(1 - |p|^2)^2 - \epsilon}{1 - \bar{p}} - 2p(1 - |p|^2))(1 - 2\bar{p} + |p|^2) \right) \]
\[ + 2((1 - |p|^2)^2 - \epsilon)(1 - 2\bar{p} + |p|^2)^2 > 0 \]
\[ > 2\epsilon((1 - |p|^2)^2 - \epsilon) \text{Re}(1 - 2p + |p|^2) > 2\epsilon|p|^2 \epsilon^2 - 2\epsilon(1 - 2p + |p|^2)^2. \]

which is equivalent to the inequality

(12) \[ |1 - 2p + |p|^2|^2 |p|^2 \epsilon + 2|p|^2 \epsilon^2 + 2\epsilon((1 - |p|^2)^2 - \epsilon) \text{Re}(1 - 2p + |p|^2) > 2\epsilon|p|^2 \epsilon^2 - \epsilon(1 - 2p + |p|^2)^2 |p| |p - \epsilon - 4|p|^2((1 - |p|^2)^2 - \epsilon)(1 - 2\bar{p} + |p|^2)^2 \]
\[ > 0. \]

Note that \( \text{Re}(1 - 2p + |p|^2) = |1 - p|^2 > 0 \) which easily implies that the above inequality holds for all possible \( p \) (i.e. satisfying the inequality \( (1 - |p|^2)^2 \geq \epsilon \)).

\[ \square \]

**Remark** Let us recall some of the open questions concerning the strongly linearly convex and \( \mathbb{C} \)-convex domains that still remain open and that can be found in [2] and [12]:

(a) Does the Lempert theorem hold for any bounded \( \mathbb{C} \)-convex domain?

(b) Can any bounded \( \mathbb{C} \)-convex domain be exhausted by strongly linearly convex ones? The answer is positive under an additional assumption of smoothness of \( D \); see [7].

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