Bounds and semiparametric inference in $L^\infty$- and $L^2$-sensitivity analysis for observational studies

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Abstract

Sensitivity analysis for the unconfoundedness assumption is a crucial component of observational studies. The marginal sensitivity model has become increasingly popular for this purpose due to its interpretability and mathematical properties. After reviewing the original marginal sensitivity model that imposes a $L^\infty$-constraint on the maximum logit difference between the observed and full data propensity scores, we introduce a more flexible $L^2$-analysis framework; sensitivity value is interpreted as the “average” amount of unmeasured confounding in the analysis. We derive analytic solutions to the stochastic optimization problems under the $L^2$-model, which can be used to bound the average treatment effect (ATE). We obtain the efficient influence functions for the optimal values and use them to develop efficient one-step estimators. We show that multiplier bootstrap can be applied to construct a simultaneous confidence band of the ATE. Our proposed methods are illustrated by simulation and real-data studies.

1 Introduction

A critical assumption for causal inference from observational studies is unconfoundedness, also known as strong ignorability [Rosenbaum and Rubin, 1983b, 1984]. The assumption is satisfied if a study has measured a sufficiently rich set of covariates ($X$) so that the control and treated outcomes, $Y(0)$ and $Y(1)$, are conditionally independent of the treatment assignment ($Z$), i.e., $Y(0), Y(1) \perp \perp Z \mid X$. Historically, this assumption has been central to the development of causal inference on observational data, by allowing researchers to estimate treatment effects through modelling the outcomes, the treatment assignment mechanism [Rosenbaum and Rubin, 1983b] or both together [Robins et al., 1994]. Recently, this assumption has been invoked and extended with the growth of applying machine learning models to handle datasets with high-dimensional covariates [Chernozhukov et al., 2018]. Despite these existing developments, there is a reluctance to establish causality through observational data partly because unconfoundedness is an untestable assumption.

Sensitivity analysis in causal inference is a class of methods that can partially identify treatment effects under certain violations from the unconfoundedness assumption. Even before unconfoundedness

* A working draft done partly when Y.Z. was a PhD student at the University of Cambridge.
is formalized, concerns of unmeasured confounding existed and have motivated the invention of sensitivity analysis. For example, Fisher [1958] opposed the conclusion that the strong association between smoking and lung cancer implies causality by suggesting that it may be due to their shared causal genes. This inspired the first formal sensitivity analysis in the literature conducted by Cornfield et al. [1959], who postulated an unmeasured confounder $U$ of smoking (a binary indicator $Z$) and lung cancer (a binary indicator $Y$). They found that $U$ must have an unrealistically strong effect on smoking to completely explain away the association. This result was instrumental in the historical debate on the relationship between smoking and lung cancer.

Modern sensitivity analysis methods can be broadly categorized into three types depending on their model assumption and inferential result: parametric, point-identified and partially identified. First, like Cornfield et al. [1959], many methods postulate a random variable $U$ and use several parameters in the distribution of the full data to measure the strength of confounding, e.g., [Rosenbaum and Rubin, 1983a; Imbens, 2003; VanderWeele and Arah, 2011; VanderWeele and Ding, 2017]. It is well known that these sensitivity parameters, if not carefully specified, may be identifiable from the observed data [Scharfstein et al., 1999; Gusta\v{f}son and McCandless, 2018; Zhang and Tchetgen Tchetgen, 2019]. Consequently, the model with some chosen sensitivity parameter may fit the observed data poorly, rendering the sensitivity analysis non-interpretable. More generally, sensitivity analysis methods have sought to relax the unconfoundedness assumption by a difference between the distributions of $Y(z) | Z = z, X$ and $Y(z) | X, Z = 1 - z$ [Robins et al., 2000; Birmingham et al., 2003; Vansteelandt et al., 2006; Scharfstein et al., 2021] or between the distributions of $Z | X$ and $Z | X, Y(z)$ [Scharfstein et al., 1999; Gilbert et al., 2003, 2013; Franks et al., 2019]. The former distribution is observed from the data and used in lieu of the latter unobserved distribution to re-establish the identification formulas of causal effects without unconfoundedness. In these methods, the effect is point-identified so reporting the results for a range of differences is needed to form an objective analysis. Alternatively, Rosenbaum’s sensitivity analysis [Rosenbaum, 1987, 2002] focuses on randomization tests for sharp null hypotheses. The analysis can directly consider a range of differences and return a worst-case p-value or a bound to partially identify the effect.

The inverse probability weighting (IPW) estimator [Horvitz and Thompson, 1952] and its extension [Robins et al., 1994; Van der Laan et al., 2011; Chernozhukov et al., 2018] play a crucial role in treatment effect estimation from observational data. Tan [2006] was the first to consider sensitivity analysis for the estimator and proposed the marginal sensitivity model (MSM). This is popularized after Aronow and Lee [2013]; Miratrix et al. [2018]; Zhao et al. [2019] who showed that maximizing/minimizing the IPW estimator under this model corresponds to a linear fractional program. For statistical inference of the bound derived from the optimization problem, Zhao et al. [2019] proposed a computationally efficient percentile bootstrap method based on a generalized minimax/maximin inequality. Soriano et al. [2021] extended this idea to a broader class of balancing weights estimators. Interestingly, Dorn and Guo [2022] showed that the bound obtained by Zhao et al. [2019] are conservative even asymptotically. They sharpened the bound by adding a moment constraint to the optimization step. The refined program takes a closed-form “quantile-balancing” solution. Dorn et al. [2021] considered semiparametric inference for the sharp bound and proposed a “doubly-valid/doubly-sharp” (DVDS) estimator under a distributional-shift formulation of MSM.

The strength of unmeasured confounding may vary across units and is unbounded for some units in a study. In this case, the $L^\infty$-constraint of MSM on the maximum logit difference between the observed and full data propensity scores is violated. Weakening the model assumption and moving towards a more average-case analysis [Hasegawa and Small, 2017; Fogarty and Hasegawa, 2019] are two important directions for methods to close the gap with practice. Some recent works in the literature are motivated by these challenges. Jin et al. [2022] generalized the distributional-shift formulation of MSM to a wider class of models subject to $f$-divergence constraints. Bertsimas et al. [2021] extended this idea to a broader class of balancing weights estimators. Interestingly, Dorn and Guo [2022] showed that the bound obtained by Zhao et al. [2019] are conservative even asymptotically. They sharpened the bound by adding a moment constraint to the optimization step. The refined program takes a closed-form “quantile-balancing” solution. Dorn et al. [2021] considered semiparametric inference for the sharp bound and proposed a “doubly-valid/doubly-sharp” (DVDS) estimator under a distributional-shift formulation of MSM.
[2022] also made a generalization through the lens of distributionally robust optimization. Huang and Pimentel [2022] considered weighting estimators of treatment effects and introduced a new set of sensitivity models which bound the variance change of the weights by a standardized $R^2$ measure.

In Section 2, we first review the popular $L^\infty$-sensitivity analysis under the original MSM proposed by Tan [2006]. Then we introduce our new $L^2$-analysis framework which makes a great departure from the previous works in two aspects: (1) quantify the deviation from unconfoundedness based on the second moment of the observed and full data propensity score ratio, but (2) do not rely on any boundedness assumption on the second moment. Alternatively, we define the “sensitivity value” as the minimum second moment to quantify the minimum deviation needed to attain any upper or lower bounds on the conditional average outcomes. Computing the population average sensitivity value leads to a more flexible analysis for the average outcomes or ATE. The flexibility comes from allowing the strength of unmeasured confounding to vary across units. Taking into account how significantly the unmeasured confounding on each unit can influence the result, the average sensitivity value truly represents the minimum deviation needed to change the conclusion of the study, i.e., explain away the effect of interest. Finally, simple methods are provided to interpret $L^2$-analysis and connect it with the inferential result from $L^\infty$-analysis.

In Section 3, we study statistical inference for the target parameters introduced in $L^\infty$- and $L^2$-analysis. First, we derive the efficient influence function (EIF) of the sharp bound in $L^\infty$-analysis and follow it to construct a one-step efficient estimator for the bound. This result is different from the DVDS estimator discussed above because of the distinct formulations of MSM. In $L^2$-analysis, we also propose one-step estimators for the population average sensitivity value and the corresponding bounds on the average outcomes, based on their EIFs. The nuisance parameters such as conditional quantile and truncated outcomes in our estimation problems are interpretable and have existing implementations. In all cases, we show that our estimators are root-$n$ consistent and asymptotically normal if the nuisance estimators are estimated consistently with errors converging faster than $n^{-1/4}$. Furthermore, we consider the common situation that researchers want to report multiple bounds of the causal effect under various levels of unconfounding. Point-wise inference for a particular bound or value is inadequate for this purpose. To fill this gap, we show how to construct a simultaneously valid confidence band for a range of bounds and values based on the proposed estimators. This development is an extension of the method in Kennedy [2019]; Bonvini and Kennedy [2021] for different sensitivity models with different target parameters based on multiplier bootstrap [Giné and Zinn, 1984; Vaart and Wellner, 1996; Belloni et al., 2018].

Section 4 provides a simulation study to examine the finite-sample performance of our estimators, confidence intervals and bands. Section 5 demonstrate the analysis methods on a real-data example for estimating ATE of fish consumption on the blood mercury level. We show that $L^2$-analysis can also quantify the strength of unconfounding through a high-confidence bound on the propensity score ratio and provide a more reliable calibration process using observed confounders.

1.1 Preliminary: assumption & notation

Consider an observational study of $n$ units. Assume that $m = n/K$ is a positive integer for some $K \geq 2$. For each $i \in [n] := \{1, \ldots, n\}$, unit $i$ has two potential outcomes, $Y_i(0)$ under control and $Y_i(1)$ under treatment, and an observed triple $O_i = (X_i, Z_i, Y_i)$ including a set confounders $X_i$, a treatment variable $Z_i$ and an outcome variable $Y_i$.

**Assumption 1.** The following is a set of assumptions we make regarding the observed data distribution and the underlying potential outcomes throughout this article:
(i) The observations $O_{1:n} := (O_1, \ldots, O_n)$ are i.i.d. random copies of $O = (X, Z, Y) \sim \mathbb{P}$ on $\mathcal{O} := \mathcal{X} \times \mathcal{Z} \times \mathcal{Y}$ where $\mathcal{X}$ is a compact subset of $\mathbb{R}^d$, $\mathcal{Z} = \{0, 1\}$, $\mathcal{Y} = \mathbb{R}$ and $Y \mid X, Z$ is square-integrable and continuous with a positive density function on $\mathcal{Y}$.

(ii) Overlap assumption: $\rho \leq e(X) \leq 1 - \rho$ for some constant $\rho \in (0, 1/2)$.

(iii) Stable unit treatment value assumption: $Y = Y(Z) = Y(z)$ if $Z = z \in \{0, 1\}$.

For $z = 0, 1$, we denote the conditional density $p_{Y\mid X,z} \equiv p_{Y\mid X,Z}(\cdot \mid X, z)$, the conditional expectation $E_{Y\mid X,z} [\cdot] \equiv E[\cdot \mid X, Z = z]$ and the conditional probability $\mathbb{P}_{Y\mid X,z} \{\cdot\} \equiv \mathbb{P}\{\cdot \mid X, Z = z\}$. Denote propensity scores $e(X) := \mathbb{P}\{Z = 1 \mid X\} \equiv p_{Z\mid X}(1 \mid X)$, $e(X, Y) := \mathbb{P}\{Z = 1 \mid X, Y(1) = Y\} \equiv p_{Z\mid X,Y(1)}(1 \mid X, Y)$ and their ratio $h(X, Y) := e(X)/e(X, Y)$. The introduction in this article is mostly based on $Y(1)$ and $Z = 1$ as an example so we would omit $Z = 1$ to ease the notation. Let $a \preceq b$ denote $a \leq Cb$ for some positive constant $C$, $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$. Finally, we say $\hat{\eta}$ is $L^2$- or $L^\infty$- consistent estimator of $\eta$ when $\|\hat{\eta} - \eta\| := (\mathbb{E}[\|\hat{\eta}(O) - \eta(O)\|^2 \mid \eta])^{1/2} = o_p(1)$ or $\|\hat{\eta} - \eta\|_{\infty} := \sup_{o \in \mathcal{O}} |\hat{\eta}(o) - \eta(o)| = o_p(1)$, respectively.

2 IPW sensitivity analysis

2.1 $L^\infty$-analysis

The idea of IPW sensitivity analysis is based on the following identification formula,

$$
E[Y(1) \mid X] = E\left[\frac{YZ(1)}{e(X, Y(1))} \mid X\right] = E_{Y(1)\mid X, 1} \left[\frac{e(X)Y(1)}{e(X, Y(1))}\right] = E_{Y\mid X, 1} [h(X, Y)Y],
$$

where the first equality is attained by $E[ZY(1) \mid X, Y(1)] = Y(1)e(X, Y(1))$ and the last equality is obtained by Assumption 1 (SUTVA). The formula implies that if the propensity score ratio $h(X, Y)$ is assumed to be bounded like in the marginal sensitivity model below, we can bound the (conditional) average treated outcomes $E[Y(1) \mid X]$ and $E[Y(1)]$.

**Definition 1** (Marginal Sensitivity Model (MSM) [Tan, 2006]). Given a sensitivity parameter $\Gamma \in (1, \infty)$, the class of MSMs is defined as

$$
\mathcal{E}(\Gamma) = \left\{e(X, Y) \in [0, 1] : \Gamma^{-1} \leq \frac{e(X)/[1 - e(X)]}{e(X, Y)/[1 - e(X, Y)]} \leq \Gamma \right\},
$$

i.e., the logit difference between $e(X)$ and $e(X, Y)$ is $L^\infty$-bounded by $\log \Gamma$.

Equivalently, we can define the class of MSMs as

$$
\mathcal{H}(\Gamma) = \{h(X, Y) : W_- (X) \leq h(X, Y) = e(X)/e(X, Y) \leq W_+ (X)\},
$$

where $W_- (X) := (1 - \Gamma^{-1})e(X) + \Gamma^{-1}$ and $W_+ (X) := (1 - \Gamma)e(X) + \Gamma$.

Zhao et al. [2019] showed that an upper bound and a lower bound of $E[Y(1)]$ can be derived by solving the programs $\sup_{h \in \mathcal{H}(\Gamma)} E_{Y\mid X, 1} [h(X, Y)Y]$ and $\inf_{h \in \mathcal{H}(\Gamma)} E_{Y\mid X, 1} [h(X, Y)Y]$, respectively. Subtracting the upper bound of $E[Y(1)]$ by the lower bound of $E[Y(0)]$ gives an upper bound of the
average treatment effect, $\text{ATE} := \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)]$. Dorn and Guo [2022] observed that Zhao et al. [2019] did not consider the following constraint on the ratio $h(X, Y)$,

$$
\mathbb{E}_{Y|X, 1}[h(X, Y)] = \int \frac{p_{Y(1)|X}(y | X)}{p_{Y(1)|X,Z}(y | X, 1)} p_{Y|X,Z}(y | X, 1) dy = \int p_{Y(1)|X}(y | X) dy = 1,
$$

where the first equality is attained by $e(X, y) = p_{Z,Y(1)|X}(1, y | X)/p_{Y(1)|X}(y | X)$ and the second equality is attained by $Y = Y(1)$ if $Z = 1$ under Assumption 1 (SUTVA). Taking the constraint into account, a sharp upper bound\footnote{For a sharp upper bound $c'$, $\mathbb{E}[Y(1) | X] \leq c'$ holds for any valid distribution of $O$ and $U$ ($U$ is an arbitrary unmeasured confounder). For any non-sharp bound $c''$ (e.g. the one obtained by solving (4) without the added moment constraint), there exists $c \in (c', c'')$ s.t. $\mathbb{E}[Y(1) | X] = c$ is unattainable for any distribution of $O$ and $U$.} for $\mathbb{E}[Y(1)] = \mathbb{E}[\mathbb{E}[Y(1) | X]]$ can be obtained by solving

$$
\sup_{h \in \mathcal{H}(\Gamma)} \mathbb{E}_{Y|X, 1}[h(X, Y)Y] \text{ subject to } \mathbb{E}_{Y|X, 1}[h(X, Y)] = 1. \quad (4)
$$

**Proposition 1 ([Dorn and Guo, 2022]).** The program (4) has the “quantile balancing” solution

$$
h^*_X(Y) = \begin{cases}
W_-(X), & \text{if } Y < Q(X), \\
W_+(X), & \text{if } Y > Q(X),
\end{cases} \quad (5)
$$

where $Q(\cdot) = Q(\cdot; \Gamma) := \inf \left\{ y \in \mathcal{Y} : F_{Y|X,Z}(y | 1, 1) \geq \alpha_* \right\}$ with $\alpha_* := \Gamma/(1 + \Gamma)$. The sharp upper bound $\psi \equiv \psi(\Gamma) := \mathbb{E}[h^*_X(X, Y)]$ takes the form $\psi \equiv \psi(\Gamma) = \psi_+ + \psi_-$ with

$$
\psi_+ = \psi_+(\Gamma) := \mathbb{E}[W_+(X)\mu_+(X)] \quad \text{and} \quad \psi_- = \psi_-(\Gamma) := \mathbb{E}[W_-(X)\mu_-(X)], \quad (6)
$$

where $\mu_+(X) = \mathbb{E}_{Y|X, 1}[Y 1_{Y < Q(X)}]$ and $\mu_-(X) = \mathbb{E}_{Y|X, 1}[Y 1_{Y > Q(X)}]^2$. The sharp lower bound is given by the solution of (4) with minimization, which takes the same form as (5) but with $W_-(X)$ and $W_+(X)$ swapped and $\alpha_* = 1/(1 + \Gamma)$.

Dorn and Guo [2022] proved the analytical solution (5) using the Neyman-Pearson lemma. To see this connection, rewrite the constraint in $\mathcal{H}(\Gamma)$ as $h(X, Y) = [h(X, Y) - W_-(X)]/[W_+(X) - W_-(X)] \in [0, 1]$ and substitute $h(X, Y) = [W_+(X) - W_-(X)]h(X, Y) + W_-(X)$ into (4). The program is now an analogue of finding the most powerful testing function $h(X, Y)$ subject to a Type I error constraint. Like Neyman-Pearson, the solution (5) can be derived from the KKT conditions [Kuhn, 1951; Kuhn and Tucker, 2014]; a proof for this is given in Appendix D.

### 2.2 $L^2$-analysis

Recall that $\mathbb{E}[Y(1) | X] = \mathbb{E}_{Y|X, 1}[h(X, Y)Y]$ from the identification formula (1) and the first half of the unconfoundedness assumption, $Y(1) \perp Z | X$, holds if and only if $h(X, Y) = e(X)/e(X, Y) = 1$ almost surely. The moment constraint in (4) implies that $h(X, Y)$ is a mean 1 random variable. Any deviation from $Y(1) \perp Z | X$ can be captured by the deviation of $h(X, Y)$ from its mean 1, i.e., the second moment of $h(X, Y)$ larger than 1 because

$$
\text{Var}_{Y|X, 1}[h(X, Y)] > 0 \iff \mathbb{E}_{Y|X, 1}[h^2(X, Y)] > 1.
$$

Building upon this observation, perhaps the most obvious transition from $L^\infty$- to $L^2$-analysis is to optimize $\mathbb{E}_{Y|X, 1}[h(X, Y)Y]$ subject to $\mathbb{E}_{Y|X, 1}[h^2(X, Y)] \leq M, \mathbb{E}_{Y|X, 1}[h(X, Y)] = 1$ and $h(X, Y) \geq 0$. But because of the non-negativity constraint, $\mathbb{E}_{Y|X, 1}[h^2(X, Y)] = M$ and $\mathbb{E}_{Y|X, 1}[h(X, Y)] = 1$.

\footnote{\[\mu_-(X)\] is often called the expected shortfall or the conditional value at risk in the literature.}
does not always hold simultaneously. When the first constraint holds with strict inequality, its Lagrangian multiplier is 0. The Lagrangian function of the problem is linear in terms of \( h(X,Y) \), so differentiating it leads to a stationarity condition without \( h(X,Y) \) involved. Also, because the problem is \( X \)-dependent, any approximate solution may not generalize to the unseen values of \( X \).

We next propose two reformulations of \( L^2 \)-analysis that have analytical solutions. We aim to not only bound the (conditional) average outcome \( \mathbb{E}[Y(1) \mid X] \) and \( \mathbb{E}[Y(1)] \) but also estimate the minimum second moment of \( h(X,Y) \) to achieve the bound. In other words, we no longer need to specify an unknown upper bound \( M \) ourselves. The first formulation is not immediately useful for bounding \( \mathbb{E}[Y(1)] \), but it gives a more general understanding of the minimization problem of the second moment of \( h(X,Y) \). From the optimization point of view, the second formulation is a simplification of the first one and will become practical for bounding \( \mathbb{E}[Y(1)] \).

### 2.2.1 Sensitivity value formulation

The first formulation considers finding the “sensitivity value”, which is defined as the minimum second moment of \( h(X,Y) \) subject to the following constraints:

\[
\inf_h \frac{1}{2} \mathbb{E}_{Y \mid X, 1} [h^2(X,Y)]
\]

subject to

\[
\begin{align*}
\mathbb{E}_{Y \mid X, 1} [h(X,Y)] &\leq \mathbb{E}_{Y \mid X, 1} [Y] - \theta, \\
\mathbb{E}_{Y \mid X, 1} [h(X,Y)] &= 1, \\
h(X,Y) &\geq 0.
\end{align*}
\]

With \( \theta > 0 \), (7) considers the case that the conditional average treated outcome \( \mathbb{E}[Y(1) \mid X] = \mathbb{E}_{Y \mid X, 1} [h(X,Y)Y] \) is smaller than the conditional average outcome on the treated \( \mathbb{E}_{Y \mid X, 1} [Y] \) due to unmeasured confounding. For instance, after observing a positive \( \mathbb{E}_{Y \mid X, 1} [Y] \), we let \( \theta = \mathbb{E}_{Y \mid X, 1} [Y] \), which hypothesizes that the positiveness can be completely explained away by some deviation from unconfoundedness. The sensitivity value quantifies the minimum deviation needed for \( \mathbb{E}[Y(1) \mid X] \leq 0 \) to hold.\(^3\) Like the quantile-balancing solution above, (7) has an interpretable closed-form solution.

**Proposition 2.** The program (7) with \( \theta \leq 0 \) is solved by \( h_s(X, Y) = 1 \). Under Assumption 1, for \( \theta > 0 \), the solution is given by

\[
h_s(X,Y) = \begin{cases} 
\lambda_X (\xi_X - Y), & \text{if } Y \leq \xi_X, \\
0, & \text{if } Y > \xi_X,
\end{cases}
\]

where \( \xi_X \) is the unique root of the strictly increasing function,

\[
f_{\theta, X}(\xi) := \frac{\mathbb{E}_{Y \mid X, 1} [(\xi - Y)Y1_{\{Y \leq \xi\}}]}{\mathbb{E}_{Y \mid X, 1} [(\xi - Y)1_{\{Y \leq \xi\}}]} - (\mathbb{E}_{Y \mid X, 1} [Y] - \theta),
\]

and \( \lambda_X = 1/\mathbb{E}_{Y \mid X, 1} [(\xi_X - Y)1_{\{Y \leq \xi_X\}}] \).

If \( \theta \leq 0 \), \( h_s(X,Y) = 1 \) is the solution of (7) simply because it satisfies all the constraints and achieves the lowest second moment 1 for a mean 1 random variable. Figure 1b illustrates the solution in (8). Figure 1c is the solution for a generalization of (7) subject to the bound (3) in \( L^\infty \)-analysis.

\(^3\) Under Assumption 1 (Overlap), \( h(X,Y) = \epsilon(X)/\epsilon(X,Y) \geq \rho > 0 \). However, the constant \( \rho \) is unknown. A growing literature [Crump et al., 2009; Khan and Tamer, 2010; Yang and Ding, 2017; D’Amour et al., 2021] reveals that the assumption can be violated easily in practice. By using \( h(X,Y) \geq 0 \), the program (7) gets relaxed and will return a conservative result that works for any \( \rho \in (0,1/2) \).
We prove all the solutions together in Appendix E. Consider flipping Figure 1c horizontally, it is a reminiscence of the winsorizing technique [Dixon, 1960] and the derivative of Huber’s loss [Huber, 2011] in robust statistics. This is not a surprise since Huber’s minimax idea includes finding the least favourable contaminating distribution to maximize the asymptotic variance of an estimator. Suppose we change \( \leq \) to \( \geq \) in the first constraint of (7), i.e.,

\[
E_{Y \mid X,1}[h(X,Y)Y] \geq E_{Y \mid X,1}[Y] + \theta. \tag{10}
\]

The program solution is essentially a horizontal flip of Figure 1b,

\[
h^*(X,Y) = \lambda_X(Y - \xi_X)1_{(Y \geq \xi_X)} \text{ if } \theta > 0,
\]

and \( h^*(X,Y) = 1 \) if \( \theta \leq 0 \). Proposition 2 still holds for this modified program after replacing \( Y \) by \( -Y \) and \( \xi_X \) by \( -\xi_X \). There is no added difficulty in estimation because of the change.

As the constraint gets stricter with an increasing \( \theta > 0 \), the sensitivity value increases by definition, as illustrated by the red and blue curves in Figure 2. We call the bending parts of the curves together as the “sensitivity curve” of \( E_{Y \mid X,1}[Y] \). Consider the points \((0, 2)\) and \((4, 2)\) on the curve. The left and right grey regions show that the sensitivity value of 2 is the minimum second moment for the hypothesis \( E[Y(1) \mid X] < 0 \) or \( > 4 \) to hold. Conversely, if assuming the true moment is bounded between 1 and 2, we can partially identify \( E[Y(1) \mid X] \in [0, 4] \) (i.e. the middle grey region).

\[\text{Figure 2: The curves of sensitivity values } E_{Y \mid X,1}[h^2(X,Y)] \text{ from the program (7) (in red) and with a change of constraint to (10) (in blue).}\]

\[\text{Figure 1: Illustration of program solutions: figures (b) and (c) are for } \theta > 0 \text{ in the prgm. (7), and (b) is a special case of (c) with } W_-(X) = 0 \text{ and } W_+(X) = \infty \text{ in } \mathcal{H}(\Gamma).}\]
For (7), the average sensitivity value over $X$ is defined as $\psi_0(\theta) := \mathbb{E}_{Y|Z=1} [h^2(X,Y)]$. It is the minimum second moment when $\mathbb{E}[Y(1) | X] \leq \mathbb{E}_{Y|X,1}[Y] - \theta$ holds over all values of $X$. In practice, we may be interested in knowing the minimum strength of unconfounding under which more than 10% of the units have a negative outcome. In this case, we can estimate $\mathbb{E}_{Y|X,1}[Y]$ for all the treated units and let $\theta$ be the 10% sample quantile. Some key estimation results for this application are provided in Appendix A and we leave this for future work. In this article, we focus on sensitivity analysis for the average outcomes and ATE. The averaged outcome hypothesis, $\mathbb{E}[Y(1)] \leq \mathbb{E}_{Y|Z=1}[Y] - \theta$, is weaker so the average sensitivity value from (7) is not the minimum second moment for it to hold. Even though we can average the first constraint in (7) over $X$, $\mathbb{E}_{Y|Z=1}[Y] - \theta$ is not a suitable lower bound for $\mathbb{E}[Y(1)]$. The second formulation below offers a more appropriate $L^2$-analysis for $\mathbb{E}[Y(1)]$.

2.2.2 Lagrangian formulation

The red curve below $\mathbb{E}_{Y|X,1}[Y]$ in Figure 2 can be viewed as a curve of Pareto frontiers in multi-objective optimization [Miettinen, 2012]. Consider $\mathbb{E}_{Y|X,1}[h^2(X,Y)]$ and $\mathbb{E}_{Y|X,1}[h(X,Y)Y]$ as two competing objectives to minimize, like the cost of an asset and the risk of devaluation. The frontier points on the curve dominate all feasible points above the curve on at least one of the objectives. In other words, they give the best trade-off between the two objectives under different weights. Following this idea, we reformulate the program (7) through a trade-off parameter $\lambda > 0$:

$$\inf_h \frac{1}{2} \mathbb{E}_{Y|X,1} [h^2(X,Y)] + \lambda \mathbb{E}_{Y|X,1} [h(X,Y)Y]$$

subject to $\mathbb{E}_{Y|X,1} [h(X,Y)] = 1,$

$h(X,Y) \geq 0.$

(12)

Proposition 3. Under Assumption 1, the program (12) with $\lambda > 0$ is solved by

$$h_*(X,Y) = \begin{cases} \lambda (\xi_X - Y), & \text{if } Y \leq \xi_X, \\ 0, & \text{if } Y > \xi_X, \end{cases}$$

(13)

where $\xi_X$ is the unique root of the strictly increasing function,

$$f_{\lambda,X}(\xi) := \mathbb{E}_{Y|X,1} [[\xi - Y]1_{Y \leq \xi}] - 1/\lambda.$$

(14)

For $\lambda = 0$, the solution of (12) is $h_*(X,Y) = 1$. For $\lambda < 0$, it is easy to show that the solution takes the form $h_*(X,Y) = -\lambda Y - \xi_X 1_{Y \geq \xi_X}$ which corresponds to the solution (11) discussed earlier. Compared with the solution in the first formulation, (12) fixes $\lambda_X = \lambda$ while leaving $\mathbb{E}[Y(1) | X] = \mathbb{E}_{Y|X,1} [h(X,Y)Y]$ as an unknown parameter to estimate. We average the sensitivity values over $X$ to give an average sensitivity value $\psi_1 := \mathbb{E}_{Y|Z=1} [h^2(X,Y)]$ for $\mathbb{E}[Y(1)]$. Correspondingly, we have a lower bound $\psi_2 := \mathbb{E}_{Y|Z=1} [h^*(X,Y)Y]$ for $\mathbb{E}[Y(1)]$ from the solution. As $\lambda$ increases, $\psi_1$ increases (i.e., a larger deviation from unconfoundedness) and $\psi_2$ decreases (i.e. we are more uncertain of $\mathbb{E}[Y(1)]$). Gathering ($\psi_1, \psi_2$) for a range of $\lambda$ together form a lower sensitivity curve for $\mathbb{E}[Y(1)]$.

Sensitivity curve for ATE. We can define the program (12) with $Z = 0$ and solve it for a range of negative $\lambda$ to obtain an upper sensitivity curve for $\mathbb{E}[Y(0)]$. In the implementation, we can simply change $Z = 1$ to $Z = 0$ and $Y$ to $-Y$ in the dataset, then apply our proposed estimator (in the next section) for $\mathbb{E}[Y(1)]$ and multiply it by $-1$ at the end. Subtracting the lower curve for $\mathbb{E}[Y(1)]$ by the upper curve for $\mathbb{E}[Y(0)]$ gives the lower curve for the ATE $= \mathbb{E}[Y(1)] - \mathbb{E}[Y(0)]$; the upper curve for the ATE can be obtained similarly. Finally, summing up the average sensitivity values weighted by $P(Z = 1)$ (i.e. the proportion of treated units in the study) and $P(Z = 0)$ (i.e. the proportion of control units in the study) gives the average sensitivity value for the ATE, which measures the deviation from the full unconfoundedness assumption $Y(1), Y(0) \perp Z | X$. 8
2.3 Comparison: $L^\infty$-analysis vs. $L^2$-analysis

**Definitions.** The definitions of $L^\infty$- and $L^2$-analysis can be made consistent on $h(X,Y)$. Even though (3) reformulates the constraint in $L^\infty$- analysis as $W_-(X) \leq h(X,Y) \leq W_+(X)$, it is not strictly assuming that $h(X,Y)$ is $L^\infty$-bounded. If $W_-(X) = 0$, $h(X,Y)$ is $L^\infty$-bounded but it violates the bound in (3). Although Proposition 1 still holds with $\psi_- = 0$ and $\psi = \psi_+$, it is known from our proof that in this case, $\alpha_s = 1 - 1/W_+(X)$ becomes $X$-dependent in the quantile function $Q(X)$. In practice, we expect the strength of an unmeasured confounder, if bounded in the $L^\infty$ sense, to be bounded in both directions. Therefore, we keep the bound in $L^\infty$-analysis as it is.

**Assumptions.** The Lagrangian formulation (12) is a flexible approach to quantifying the minimum deviation from unconfoundedness. To see this, consider two units with $X = x_1$ and $X = x_2$ respectively. Suppose that the distribution of $Y \mid X = x_1, Z = 1$ concentrates somewhere far away from 0 while the distribution of $Y \mid X = x_2, Z = 1$ concentrates at 0. Then for $X = x_1$, (12) will focus on minimizing the outcome in the objective, while for $X = x_2$, it will mainly minimize the second moment. More generally, for every value of $X$, the sensitivity value $E_Y[X,1] \left[ h_2(X,Y) \right]$ is larger if the distribution of $Y \mid X, Z = 1$ has a larger influence. At the other extreme, if the distribution has no influence, then the sensitivity value is 1 representing the least sensitivity. Existing sensitivity analysis methods often assume the same bound on the deviation for all the units. The consequence of this is not truly estimating the minimum deviation needed to explain away the effect. The average sensitivity value $\psi_1$ derived from (12) overcomes this issue. But it needs to be estimated because the average minimum deviation is never a known quantity by nature.

**Interpretability.** $L^2$-analysis can be interpreted similarly as $L^\infty$-analysis. There are two approaches to obtaining a high-confidence bound on $h(X,Y)$ given the first and second moments of $h(X,Y)$ are known from the analysis. First, we can assume that $h(X,Y)$ follows a right-skewed distribution (e.g. Gamma) and take an upper and lower quantile to form a high-confidence bound. Second, more generally, for the non-negative random variable $h(X,Y)$, we can apply Theorem 4 in Ghosh [2002] to derive the sharpest bound $-h_{\min} < h(X,Y) - 1 < h_{\max}$ with probability at least $1 - \delta$, where $h_{\min}, h_{\max} > 0$ and $\delta \in (0, 1)$ are user-specified. In sensitivity analysis, researchers also like to leave out covariates to compare the strengths of confounding and unconfounding. This approach is called calibration in the literature, e.g., see Hsu and Small [2013]. The average sensitivity value $\psi_1$ from (12) fits seamlessly into the calibration process. We will demonstrate the discussed advantages of $L^2$-analysis through a real data example in Section 5.

3 Statistical inference

The challenge in sensitivity analysis beyond optimization is to estimate the optimal value under some nuisance parameters introduced by the optimizer. In what follows, we first describe how to apply the general one-step debiased estimation strategy in our context. Then building upon our proposed estimators, we will show how to construct simultaneous confidence bands for the sensitivity curves.

3.1 One-step estimation: $L^\infty$-analysis

Following (6), a researcher can directly estimate $\psi$ based on some estimators for the nuisance parameters $\eta = (e, Q, \mu_+, \mu_-)$. Suppose that we let $\hat{e}$ estimate $e$, and $\hat{Q}$ estimate $Q$, and $(\hat{\mu}_+, \hat{\mu}_-)$
Assumption 1. With probability 1,

\[ \text{Theorem 1.} \quad \text{The uncentered EIF of } \hat{\phi} \text{ where } \hat{\phi} = \mathbb{P}_{n}(K) \{ \hat{W}_{\mu+} + \hat{W}_{\mu-} \}, \]

where \( \hat{W}_{\mu}(\cdot) := (1 - \Gamma)\hat{e}(\cdot) + \Gamma, \hat{W}_{\mu-}(\cdot) := (1 - \Gamma^{-1})\hat{e}(\cdot) + \Gamma^{-1} \) and \( \mathbb{P}_{n}(K) = m^{-1} \sum_{i=n-m+1}^{n} \delta_{0_{i}} \) is the empirical measure based on the \( K \)-th fold of the \( n \) observations, while the other folds of observations, \( O_{[n-m]} \), is fitted to the nuisance estimator \( \hat{\eta} = (\hat{e}, \hat{Q}, \hat{\mu}_{+}, \hat{\mu}_{-}) \).

Suppose that we make no parametric assumption on \( \eta \) and use a nonparametric estimator for \( \hat{\eta} \). It follows from the theory of nonparametric regression [Wasserman, 2006] that \( \hat{\eta} \) converges to \( \eta \) at a rate slower than \( n^{-1/2} \), then the bias of \( \hat{\psi}_{\text{initial}} \) converges to 0 slower than \( n^{-1/2} \). The consequence of this is that \( \hat{\psi}_{\text{initial}} \) fails to be asymptotically normal and rules out the possibility of asymptotic inference based on the central limit theorem (CLT). Fortunately, it is known from semiparametric statistics [Bickel et al., 1993] that we can lower the rate requirement of \( \hat{\eta} \) by applying a one-step correction to \( \hat{\psi}_{\text{initial}} \) based on the efficient influence function (EIF) of \( \psi \). The correction is essentially a nonparametric analogue of one-step MLE for parametric models [Shao, 2003, Section 4.5.3] when \( \eta \) is nonparametric (i.e. infinite-dimensional) and does not have a well-defined score function.

More background of influence functions can be found in Appendix C and the review article [Kennedy, 2022]. The EIF of \( \psi \) is essentially the first-order functional derivative in the Von-mises expansion of \( \psi \) [Mises, 1947] (i.e. the functional version of Taylor expansion). It is a zero-mean function, taking the form ofEIF (ψ) = ϕ(0) − ψ, where ϕ(O) = 1. The EIF is given below.

**Theorem 1.** The uncentered EIF of \( \psi \) is given by \( \phi(O) = \phi_{\mu+}(O) + \phi_{\mu-}(O) \) with

\[
\phi_{\mu+}(O) = \frac{ZW_{\mu+}(X)}{e(X)} \left[ (1 - \alpha_{s} - \mathbb{I}_{Y<Q(X)}) Q(X) + \mathbb{I}_{Y>Q(X)} + \mathbb{I}_{Y>Q(X)} - \mu_{+}(X) + (1 - \Gamma)Z + \Gamma \right] \mu_{+}(X),
\]

\[
\phi_{\mu-}(O) = \frac{ZW_{\mu-}(X)}{e(X)} \left[ (\alpha_{s} - \mathbb{I}_{Y<Q(X)}) Q(X) + \mathbb{I}_{Y>Q(X)} - \mu_{-}(X) + (1 - \Gamma^{-1})Z + \Gamma^{-1} \right] \mu_{-}(X).
\]

Making the dependence of \( \psi \) and \( \phi \) on \( \mathbb{P} \) explicit, the first-order expansion\(^5\) is given by

\[
\psi(\mathbb{P}) = \psi(\tilde{\mathbb{P}}) + \int \left[ \phi(\mathbb{P}) - \psi(\tilde{\mathbb{P}}) \right] d(\mathbb{P} - \tilde{\mathbb{P}}) + R(\mathbb{P}, \tilde{\mathbb{P}})
\]

\[
= \psi(\tilde{\mathbb{P}}) + \mathbb{P} \phi(\mathbb{P}) - \psi(\tilde{\mathbb{P}}) + R(\mathbb{P}, \tilde{\mathbb{P}})
\]

\[
= \mathbb{P} \phi(\mathbb{P}) + R(\mathbb{P}, \tilde{\mathbb{P}}),
\]

where \( \mathbb{P} \{ \cdot \} = \mathbb{E}_{\mathbb{P} \sim \tilde{\mathbb{P}}}[\cdot] \). The second equality is attained by the zero-mean property of the EIF, i.e., \( \mathbb{P} \phi - \psi(\tilde{\mathbb{P}}) \) characterizes the first-order bias of \( \psi(\mathbb{P}) \) in estimating \( \psi(\tilde{\mathbb{P}}) \). This motivates us to approximate and correct this bias for the initial estimator. Following the equation, the correction is to use the one-step estimator,

\[
\hat{\psi} = \hat{\psi}_{\text{initial}} + \mathbb{P}_{n}(K) \hat{\phi} - \hat{\psi}_{\text{initial}} = \mathbb{P}_{n}(K) \hat{\phi},
\]

where \( \hat{\phi}(O) = \phi(O; \hat{\eta}) \). Unlike the initial estimator, \( \hat{\psi} \) only has second and higher order biases \( R(\mathbb{P}, \tilde{\mathbb{P}}) \), so asymptotic inference is feasible even when \( \hat{\eta} \) is estimated at a nonparametric rate lower than \( n^{-1/2} \). This result is formalized below under some standard regularity assumptions.

**Assumption 2.** With probability 1, \( (Q, \mu+, \mu-, \hat{Q}, \hat{\mu}_{+}, \hat{\mu}_{-}) \) are bounded and \( \hat{e} \in (0, 1) \).

**Proposition 4.** Under Assumptions 1 and 2, \( \text{Bias}(\hat{\phi} | \hat{\psi}) = o_{\mathbb{P}}(1/\sqrt{n}) \) if \( \| \hat{\eta} - \eta \| = o_{\mathbb{P}}(n^{-1/4}) \).

\(^5\)See Fisher and Kennedy [2021, equ. (2) and (8)] for a simple derivation.
To remedy the asymptotic efficiency loss in sample splitting, we can use the popular K-fold cross-fitting strategy [Schick, 1986; Zheng and Van Der Laan, 2010; Chernozhukov et al., 2018]. The cross-fitted estimator of \( \psi \) is given by averaging \( K \) cross-fitted one-step estimators,

\[
\hat{\psi}_{\text{cf}} := \frac{1}{K} \sum_{k=1}^{K} \hat{\psi}^{(k)} = \frac{1}{K} \sum_{k=1}^{K} \mathbb{P}_n^{(k)} \hat{\psi}^{(k)} = \mathbb{P}_n \hat{\psi}_{\text{cf}},
\]

where \( \mathbb{P}_n^{(k)} \) is the empirical measure based on the \( k \)-th fold of the \( n \) observations, \( \hat{\psi}^{(k)}(O) = \hat{\phi}(O; \hat{\eta}_{-k}) \) and \( \hat{\eta}_{-k} \) is the nuisance estimator fitted to the other folds. After obtaining \( \hat{\eta}_{-1}, \ldots, \hat{\eta}_{-K} \), the estimator is simply given by an average of \( \hat{\phi}(O_i; \hat{\eta}_{-i}), i \in [n] \), where \( k_i \) is the fold that contains \( O_i \), i.e., the fold that \( \hat{\eta}_{-k_i} \) is not fitted to. In the equation, we denote this adaptive average by \( \mathbb{P}_n \hat{\psi}_{\text{cf}} \).

By our construction, sample splitting and cross-fitting overlap in the \( K \)-th fold. It holds that \( \hat{\psi} = \hat{\psi}^{(K)} \), \( \hat{\phi}^{(K)} = \hat{\phi} \) and \( \hat{\eta} = \hat{\eta} \). By Proposition 4, \( \text{Bias}(\hat{\phi}_{\text{cf}} | \hat{\eta}) = o_P(1/\sqrt{n}) \) for all \( k \in [K] \) and

\[
\hat{\psi}_{\text{cf}} - \psi = (\mathbb{P}_n - \mathbb{P}) \phi_{\text{cf}} + \text{Bias}(\hat{\phi}_{\text{cf}} | \hat{\eta}_{\text{cf}}) = o_P(1/\sqrt{n}),
\]

which implies that \( \hat{\psi}_{\text{cf}} \) is a root-\( n \) consistent and asymptotically normal (CAN) estimator of \( \psi \), i.e.,

\[
\sqrt{n}(\hat{\psi}_{\text{cf}} - \psi) \overset{d}{\to} \mathcal{N}(0, \sigma^2 := \text{Var}[\phi(O)])
\]

It is worth noting that the uncentered EIF \( \phi(O) \) achieves the lowest variance in the CLT, so \( \hat{\psi}_{\text{cf}} \) is also an efficient estimator of \( \psi \) [Bickel et al., 1993; Tsiatis, 2006]. Building upon this result, the \( (1 - \alpha) \)-confidence interval (CI) for \( \psi \) is given by

\[
\hat{C}_\alpha := \left[ \hat{\psi}_{\text{cf}} \pm z_{2\alpha} \hat{\sigma}_{\text{cf}} / \sqrt{n} \right]
\]

where \( z_{2\alpha} \) is the upper \( \frac{\alpha}{2} \)-quantile of the standard normal.

### 3.2 One-step estimation: \( L^2 \)-analysis

For the program (12) from \( L^2 \)-analysis, we have two parameters to estimate,

\[
\psi_1 = \psi_1(\lambda) := \mathbb{E}_{Y|Z=1} [h_\lambda^2(X, Y)] \quad \text{and} \quad \psi_2 = \psi_2(\lambda) := \mathbb{E}_{Y|Z=1} [h_\lambda(X, Y)Y]
\]

for the solution \( h_\lambda(X, Y) = \lambda(\xi_X - Y) 1_{\{Y \leq \xi_X \}} \) in Proposition 3. In Appendix A, we provide similar results on the average sensitivity value from the program (7) and discuss some potential applications on heterogeneous causal effects. The technical difference there is that \( \lambda_X \) is no longer fixed to \( \lambda \).

**Theorem 2.** The uncentered EIFs of \( \psi_1 \) and \( \psi_2 \) are given by

\[
\phi_1(O) = \frac{Z}{e(X)} \left\{ 2\Pi(X, Y) + h_\lambda^2(X, Y) - \mathbb{E}_{Y|X,1} [h_\lambda^2(X, Y)] \right\} + \mathbb{E}_{Y|X,1} [h_\lambda^2(X, Y)]
\]

and

\[
\phi_2(O) = \frac{Z}{e(X)} \left\{ \Pi(X, Y) \mathbb{E}_{Y|X,1} [1_{\{Y \leq \xi_X \}}] + h_\lambda(X, Y)Y - \mathbb{E}_{Y|X,1} [h_\lambda(X, Y)Y] \right\} + \mathbb{E}_{Y|X,1} [h_\lambda(X, Y)Y],
\]

where \( \Pi(X, Y) := [1 - h_\lambda(X, Y)] / E_{Y|X,1} [1_{\{Y \leq \xi_X \}}] \) has a mean of 0 by the first constraint in (12).
Like in Section 3.1, the one-step estimator $\hat{\psi}_j = \mathbb{P}_n^{(K)} e_j$ for $\psi_j$ is constructed by an empirical average of the uncentered EIF over the $K$-th fold while using the other folds to fit the nuisance estimator. In the EIFs above, the nuisance parameters based on $\xi_X$ are

$$\mathbb{E}_Y|X,1[\mathbb{I}_{Y\leq \xi_X}], \mathbb{E}_Y|X,1[Y\mathbb{I}_{Y\leq \xi_X}] \text{ and } \mathbb{E}_Y|X,1[Y^2\mathbb{I}_{Y\leq \xi_X}].$$

(17)

We consider estimating $\mathbb{E}_Y|X,1[\cdot]$ by $\hat{\mathbb{E}}_Y|X,1[\cdot] = \int_{\mathcal{X}} \hat{p}_Y|X,1(y)dy$, where $\hat{p}_Y|X,1$ is a Nadaraya-Watson kernel estimator [Nadaraya, 1964; Watson, 1964],

$$\hat{p}_Y|X,1(y) = \sum_{i=1}^{n-m} Z_i K_1(X-X_i)K_2(y-Y_i) / \sum_{j=1}^{n-m} Z_j K_1(X-X_j),$$

(18)

with two continuous and non-negative kernel functions $K_1 : \mathbb{R}^d \rightarrow [0, \infty]$ and $K_2 : \mathbb{R} \rightarrow [0, \infty]$. Similarly, we can estimate the function $f_{\lambda,X}(\xi)$ in (14) by $\hat{f}_{\lambda,X}(\xi) = \mathbb{E}_Y|X,1[\langle \xi - Y \rangle \mathbb{I}_{Y\leq \xi}] - 1/\lambda$, which is a strictly increasing function of $\xi$ for $\lambda > 0$ as long as $\hat{p}_Y|X,1 > 0$. Then we can use the bisection method to search for the unique root $\hat{\xi}_X$ for $\hat{f}_{\lambda,X}(\xi)$. The complexity to obtain every root $\xi_X$ is $O(nt)$ from summing up the $n - m$ observations in $\hat{p}_Y|X,1$ for $t$ bisections, where $t$ is small because the root-finding error decays as fast as $O(2^{-t})$.

A machine learning model $\hat{\mu}$ can still be applied to estimate the conditional mean of $Y$ in the kernel $K_2$, i.e., replacing $Y_i$ by $\hat{\mu}(X) + Y_i - \hat{\mu}(X_i)$; see Hansen [2004, Section 4] for more details of this two-step modelling approach. The following result shows that one-step estimation makes asymptotic inference feasible in $L^2$-analysis even when the nuisance estimator converges at a slower nonparametric rate. The $L^\infty$-consistency assumption on the estimator $\hat{\psi}_Y|X,1$ is needed to show that the root $\hat{\xi}_X$ derived using $\hat{p}_Y|X,1$ has the same order of error $o_P(n^{-1/4})$.

**Assumption 3.** With probability 1, $\hat{\epsilon} \in (0, 1)$ and $\hat{p}_Y|X,1 : \mathcal{X} \times \mathcal{Y} \rightarrow (0, \infty)$ is continuous function. Furthermore, the nuisance estimator $\hat{\eta} = (\hat{\epsilon}, \hat{p}_Y|X,1)$ satisfies that

$$\|\hat{\epsilon} - \epsilon\| = o_P(n^{-1/4}) \text{ and } \|\hat{p}_Y|X,1 - p_Y|X,1\|_\infty = o_P(n^{-1/4}).$$

**Proposition 5.** Under Assumptions 1 and 3, Bias($\hat{\phi}_1 | \hat{\eta}$) = $o(1/\sqrt{n})$ and Bias($\hat{\phi}_2 | \hat{\eta}$) = $o(1/\sqrt{n})$.

### 3.3 Simultaneous confidence bands

In causal inference, researchers often anticipate different sources or strengths of unmeasured confounding that can change the conclusion of the study. Reporting multiple results is increasingly needed in practice. This encourages us to perform uniform inference for all the parameters we estimate. For example in $L^\infty$-analysis, we should consider a user-specified range of $\Gamma$, and construct a simultaneous valid confidence band for the upper sensitivity curve,

$$\Psi(D) := \{(\Gamma, \psi(\Gamma)) : \Gamma \in D := [\Gamma_{\min}, \Gamma_{\max}]\}.$$

We now keep $\Gamma$ explicit in the notation to distinguish parameters and estimators for different $\Gamma$'s.

Taking a union of the CIs for different $\Gamma \in D$ does not lead to a valid confidence band for $\Psi(D)$. But intuitively, if both $\psi(\Gamma)$ and $\hat{\psi}(\Gamma)$ change smoothly with respect to $\Gamma$, the events $\psi(\Gamma) \in \tilde{C}_\alpha(\Gamma)$ for different $\Gamma$ are correlated, i.e., if one of them happens, the others are expected to happen with a higher probability. The following theorem implies the point-wise CLT above and establishes the joint limiting distribution of $\{\psi(\Gamma) : \Gamma \in D\}$ under some assumptions.
Theorem 3. Under Assumptions 1 and 2 and the conditions

\begin{enumerate}[(i)]
  \item \( \sup_{\Gamma \in \mathcal{D}} |\hat{\sigma}_{\text{cf}}(\Gamma)/\sigma^2(\Gamma) - 1| = o_P(1) \),
  \item \( \mathbb{E} \left\{ \left[ \sup_{\Gamma \in \mathcal{D}} |\hat{\phi}(O; \Gamma) - \phi(O; \Gamma)| \right]^2 \mid \hat{\eta} \right\} = o_P(1) \),
  \item \( \sup_{\Gamma \in \mathcal{D}} \|\hat{Q} - Q\|, \sup_{\Gamma \in \mathcal{D}} \|\hat{\mu}_+ - \mu_+\|, \sup_{\Gamma \in \mathcal{D}} \|\hat{\mu}_- - \mu_-\| \text{ and } \|\hat{\ell} - \ell\| \text{ are } o_P(n^{-1/4}) \),
  \item \( (\hat{Q}, \hat{\mu}_+, \hat{\mu}_-) \) are Lipschitz continuous functions of \( \Gamma \in \mathcal{D} \) with probability 1.\(^6\)
\end{enumerate}

it holds that \( \sqrt{n} \left[ \hat{\psi}_{\text{cf}}(\cdot) - \psi(\cdot) \right]/\hat{\sigma}_{\text{cf}}^2(\cdot) \overset{d}{\rightarrow} \mathcal{G}(\cdot) \in \mathcal{L}^\infty(\mathcal{D}) \), where \( \mathcal{G}(\cdot) = |\phi(O; \cdot) - \psi(\cdot)|/\sigma(\cdot) \) is a Gaussian process with mean 0 and covariance smoothness.

The first two conditions in the theorem are mild because they only require the estimators to be consistent. The third condition essentially requires \( \sup_{\Gamma \in \mathcal{D}} \text{Bias}(\hat{\phi}(O; \Gamma) \mid \hat{\eta}) = o_P(1/\sqrt{n}) \). It is slightly stronger than Proposition 4 that gives \( \sup_{\Gamma \in \mathcal{D}} \text{Bias}(\hat{\phi}(O; \Gamma) \mid \hat{\eta}) = o_P(1/\sqrt{n}) \) for all \( \Gamma \in \mathcal{D} \). The fourth condition is introduced to make sure \( \hat{\psi}_{\text{cf}}(\Gamma) \) behaves smoothly with respect to \( \Gamma \). We can let \( Q(\cdot) = \inf \{ y \in \mathcal{Y} : \hat{F}_{Y|X,Z}(y; \cdot, 1) \geq \Gamma/(1 + \Gamma) \} \) for a conditional c.d.f. estimator \( \hat{F}_{Y|X,Z}(y; \cdot, 1) \) obtained by integrating the kernel estimator \( \hat{p}_{Y|X,1} \) in (18). Then let \( \hat{\mu}_+(X) = \int Q(X) y \hat{p}_{Y|X,1}(y) dy \) and \( \hat{\mu}_-(X) = \int \hat{Q}(X) y \hat{p}_{Y|X,1}(y) dy \). The proof of Theorem 3 follows the idea by Kennedy [2019, Section 8.4], but is different in verifying the smoothness of the EIFs and the convergence of the bias.

Building upon Theorem 3, we can apply multiplier bootstrap (MB) [Giné and Zinn, 1984; Vaart and Wellner, 1996; Belloni et al., 2018] to construct the desired confidence band for \( \Psi(\mathcal{D}) \). Because of the Gaussian process, at every \( \Gamma \in \mathcal{D} \), the band still takes the form \( \hat{\psi}_{\text{cf}}(\Gamma) \pm q_\alpha \hat{\sigma}_{\text{cf}}(\Gamma)/\sqrt{n} \). MB approximates the supremum of the process,

\[ \sup_{\Gamma \in \mathcal{D}} |\mathcal{G}(\Gamma)| = \sup_{\Gamma \in \mathcal{D}} |\phi(O; \Gamma) - \psi(\Gamma)|/\sigma(\Gamma), \]

then \( q_\alpha \) is chosen to upper bound the supremum with probability \( 1 - \alpha \). The validity of MB is proven in Kennedy [2019, Theorem 4] using Chernozhukov et al. [2014]’s theory on the suprema of empirical processes. The following result is a restatement of the theorem there.

Theorem 4. In the setup of Theorem 3, let \( \hat{q}_\alpha \) denote the critical value of the supremum of the multiplier bootstrap process such that

\[ \mathbb{P} \left\{ \sup_{\Gamma \in \mathcal{D}} \left| \sqrt{n} P_n \left[ A \left( \phi_{\text{cf}}(O; \Gamma) - \hat{\psi}_{\text{cf}}(\Gamma) \right)/\hat{\sigma}_{\text{cf}}(\Gamma) \right] \right| \geq \hat{q}_\alpha \mid O_{1:n} \right\} = \alpha, \]

where \( A_{1:n} \) are i.i.d. Rademacher variables drawn independently of \( O_{1:n} \), then

\[ \mathbb{P} \left\{ \psi(\Gamma) \not\in \hat{\mathcal{C}}_\alpha(\Gamma) := [\hat{\psi}_{\text{cf}}(\Gamma) \pm \hat{q}_\alpha \hat{\sigma}_{\text{cf}}(\Gamma)/\sqrt{n}], \forall \Gamma \in \mathcal{D} \right\} = \alpha + o(1), \]

i.e., \( \hat{\mathcal{C}}_\alpha(\mathcal{D}) := \left\{ (\Gamma, \hat{\mathcal{C}}_\alpha(\Gamma)) : \Gamma \in \mathcal{D} \right\} \) is a \((1 - \alpha)\)-confidence band for \( \Psi(\mathcal{D}) \).

\(^6\)For example, \( |Q(X; \Gamma) - \hat{Q}(X; \Gamma)| \leq C(X)|\Gamma - \Gamma| \) for some \( \mathbb{P}\)-integrable function \( C(X) \).

\(^7\)\( L^\infty(\mathcal{D}) \) is the space of bounded functions on \( \mathcal{D} \).
In $L^2$-analysis, we can also apply MB to construct a confidence band for the lower sensitivity curve

$$\Psi_1(D_1) := \{(\psi_1(\lambda), \psi_2(\lambda)) : \lambda \in D_1 := [\lambda_{\min}, \lambda_{\max}]\}.$$ 

The formal result is similar to the ones above so deferred to Appendix B and summarized below. As in (15), for $j = 1, 2$, we define $\hat{\psi}_{j,cf}(\lambda) = \mathbb{P}_A(\phi_{j,cf}$ as the $K$-fold cross-fitted estimator for $\psi_j(\lambda)$, and $\hat{\sigma}_{j,cf}^2(\lambda)$ the estimator of the asymptotic variance $\sigma_j^2(\lambda) := \text{Var}[\phi_j(O; \lambda)]$. Like Theorem 3, we prove that the estimators $\hat{\psi}_{j,cf}(\lambda)$ for $\lambda \in D$ converge to a Gaussian process. The assumption we make in addition to Assumptions 1 and 3 is

$$\sup_{\lambda \in D_1} \|\xi_X - \xi_{\lambda}\| = o_p(n^{-1/4}),$$

which leads to

$$\text{sup}_{\lambda \in D_1} \text{Bias}(\hat{\phi}_j | \hat{q}) = o(1/\sqrt{n}),$$

a stronger version of Proposition 5. Compared with $L^\infty$-analysis, we additionally estimate the average sensitivity value $\psi_1(\lambda)$ to measure the minimum deviation from unconfoundedness. Based on a critical value $\hat{q}_{j,\alpha}$ from MB, it holds that for any $\lambda \in D_1$,

$$\psi_j(\lambda) \notin \hat{C}_{j,\alpha}(\lambda) := [\hat{\psi}_{j,cf}(\lambda) \pm \hat{q}_{j,\alpha}\hat{\sigma}_{j,cf}(\lambda)/\sqrt{n}]$$

with probability $\alpha + o(1)$.

Then by a step of Bonferroni correction, it is straightforward to verify that $\cup_{\lambda \in D_1} [\hat{C}_{1,\alpha}(\lambda) \times \hat{C}_{2,\alpha}(\lambda)]$ is a $(1 - 2\alpha)$-confidence band for the sensitivity curve $\Psi_1(D_1)$.

### 4 Simulation study

In this section, we examine the finite-sample performance of the proposed estimators, confidence intervals and bands through a simple simulation study.

The data-generating process of the study is given by

$$X_i \sim \mathcal{N}_{[-1,1]}(0, 0.5), \quad U_i \sim \mathcal{N}(0, 0.2),$$

$$e(X_i) = 1/[1 + \exp(1 - 2X_i)], \quad Z_i \sim \text{Bern}(e(X_i)),$$

$$Y_i(0) = 0.5 + X_i + 0.5U_i + 0.5\{X_i > 0\}U_i, \quad Y_i(0) = Y_i(1) - 0.5, \quad Y_i = Z_iY_i(1) + (1 - Z_i)Y_i(0).$$

where $\mathcal{N}_{[-1,1]}$ is the normal distribution truncated at -1 and 1.

In Section 2.2.2, we emphasized that the average sensitivity value $\psi_1(\lambda)$ is flexible in measuring the minimum deviation from unconfoundedness. To illustrate this point, we compare the values and $\Gamma$ in Figure 3a when they give the same lower bound for $\mathbb{E}[Y(1)] = 1/2$ in this study. The lower bound of $\mathbb{E}[Y(1)]$ in Proposition 1 is exactly $1 - \psi(\Gamma)$ in this study. For the bound $1 - \psi(\Gamma) \in [0, 1]$, the corresponding $\Gamma$ is shown as the grey curve. To make a fair comparison, in Figure 3b, we turn the minimum second moments $\psi_0(\theta)$ from (7) and $\psi_1(\lambda)$ from (12) into standard deviation, and compute the width of the bound of $h(X, Y)$ in (3), which is $(1 - e(X))(\Gamma - \Gamma^{-1}) = (\Gamma - \Gamma^{-1})/2$ assuming that $e(X) = 1/2$. In either figure, there are clear gaps between the sensitivity curves.

The flexibility of $\psi_1(\lambda)$ leads to the gap between the red and blue curves. Fix $\psi_1(\lambda)$ and $\psi_0(\theta)$ at the same value on the vertical axis in Figure 3a, the red curve implies that $\mathbb{E}[Y(1)]$ can be below the blue curve, which aligns with our conclusion earlier that the blue curve is not a valid lower bound for $\mathbb{E}[Y(1)]$. We can not fix $\psi_1(\lambda)$ and $\Gamma$ at the same value because they are not the same quantity. But the gap between the grey and red curves highlights that without the bound (3), it is easy for an unmeasured confounder (with a relatively small average sensitivity value $\psi_1(\lambda)$) to explain away $\mathbb{E}[Y(1)]$. Therefore, in practice, even when an effect can only be explained away by a very large $\Gamma$ in $L^\infty$-analysis, $L^2$-analysis is still worthy to conduct for drawing a more robust conclusion.

We next compare one-step estimation with the initial direct estimation on the target parameters $\psi(\Gamma), \psi_1(\lambda)$ and $\psi_2(\lambda)$. For example, $\psi_1(\lambda) = \mathbb{E}_{Y \mid Z=1}[h^2(X, Y)]$ is estimated directly by an average
Figure 3: Lower sensitivity curves of $E[Y(1)]$ in the simulation study.

We use 10-fold cross-fitting in both cases. We compare the estimators in terms of the root-mean-squared errors over 500 runs, e.g., the RMSE of $\hat{\psi}_{1,\text{cf}}(\lambda)$ is given by

$$\text{RMSE}\left\{\hat{\psi}_{1,\text{cf}}(\lambda)\right\} = \left\{\frac{1}{500} \sum_{j=1}^{500} \left(\hat{\psi}_{1,\text{cf}}^{(j)}(\lambda) - \psi(\lambda)\right)\right\}^{1/2}.$$ 

We estimate the propensity score $e$ by a logistic regression model, and the other nuisance parameters $Q, \mu_+, \mu_-$ and $p_{Y|X,1}$ based on a Nadaraya-Watson estimator (18) with two Gaussian kernels $K_1$ and $K_2$ with bandwidths $\propto n^{-1/6}$. In the data-generating process, we add a quadratic term $-X^2_i$ to the equations of $e(X_i)$ and $Y_i(1)$ so that the nuisance estimators are partially misspecified. The simulation results are reported in Table 1. Overall, the RMSEs drop as the sample size increases and/or more true nuisance parameters are used in one-step estimation. In the third and fourth columns, plugging the true propensity score $e$ does not necessarily lead to an improvement when the other nuisance parameters are estimated with errors in finite samples. After plugging in the other true nuisance parameters in the last two columns, we see the general benefits of using $e$.

Finally, we check the coverage rates of the point-wise confidence intervals based on the cross-fitted estimators and the simultaneous confidence bands given by multiplier bootstrap (MB) in Theorems 4 and 7. In the simulation, we consider two discretized parameter ranges, $D_1 = \{2, 3, \ldots, 9, 10\}$ and $D_1 = \{0.5, 1, 1.5, \ldots, 4.5, 5\}$. For $\Gamma = 2, 4, 6$ and $\Lambda = 1, 2, 3$, Table 2 first shows the coverage rates of the point-wise 95%-CIs for $\psi(\Gamma), \psi_1(\lambda)$ and $\psi_2(\lambda)$ over 500 runs; the average CIs over the runs are also provided in the table. The CIs for $\psi(\Gamma)$ and $\psi_2(\lambda)$ achieve around 95% coverage at all sample sizes, while the CI for $\psi_1(\lambda)$ does so when $n = 300$. The CIs shrinks as the sample size increases.

Table 3 shows that by MB (resampling the multipliers $A_{1,n}$ for 2500 times), the confidence bands with adjusted critical values around 2.70 (i.e. roughly 30-40% wider than 1.96) achieve (more than) 95%-uniform coverage for all the sensitivity parameters in $D$ and $D_1$. This is not an easy price to pay but the point-wise CIs with a critical value of 1.96 often fail to achieve uniform coverage, e.g., especially for $\{\psi_1(\lambda) : \lambda \in D_1\}$. In practice, the critical value inflation can be much smaller than the one we observe here. The real data experiment in the next section is an example.
Table 1: RMSEs of direct and one-step estimation over 500 simulations.

| Methods | Direct | One-step |
|---------|--------|----------|
| Nuisance | $\hat{e}, \hat{Q}, \hat{\mu}_\pm$ | $\hat{e}, \hat{Q}, \hat{\mu}_\pm$ | $\hat{e}, \hat{Q}, \hat{\mu}_\pm$ | $\hat{e}, \hat{Q}, \hat{\mu}_\pm$ |
| $n$ | $\Gamma = 4$, $\psi(\Gamma) = 0.324$ |  |  |  |
| 100 | 0.0885 | 0.0694 | 0.0713 | 0.0697 | 0.0689 |
| 200 | 0.0678 | 0.0498 | 0.0514 | 0.0501 | 0.0497 |
| 300 | 0.0597 | 0.0425 | 0.0438 | 0.0424 | 0.0422 |

| Methods | Direct | One-step |
|---------|--------|----------|
| Nuisance | $\hat{e}, \hat{\mu}_Y | X,1$ | $\hat{e}, \hat{\mu}_Y | X,1$ | $\hat{e}, \hat{\mu}_Y | X,1$ | $\hat{e}, \hat{\mu}_Y | X,1$ |
| $n$ | $\lambda = 2$, $\psi_1(\lambda) = 1.404$ |  |  |  |
| 100 | 0.2007 | 0.0857 | 0.0834 | 0.0645 | 0.0633 |
| 200 | 0.1740 | 0.0549 | 0.0557 | 0.0464 | 0.0463 |
| 300 | 0.1551 | 0.0426 | 0.0432 | 0.0373 | 0.0369 |

| $\lambda$ | 1 | 2 | 3 |
| $\psi_1(\lambda)$ |  |  |  |
| 2 |  |  |  |
| 4 |  |  |  |
| 6 |  |  |  |

| $\psi(\Gamma)$ | 0.546 | 0.589 | 0.612 |
| $n$ | CI | Coverage | CI | Coverage | CI | Coverage |
| 100 | [0.419, 0.675] | 0.952 | [0.463, 0.724] | 0.962 | [0.487, 0.752] | 0.950 |
| 200 | [0.459, 0.639] | 0.946 | [0.503, 0.685] | 0.932 | [0.522, 0.706] | 0.944 |
| 300 | [0.474, 0.620] | 0.942 | [0.519, 0.668] | 0.956 | [0.538, 0.687] | 0.948 |

| $\lambda$ | 1 | 2 | 3 |
| $\psi_2(\lambda)$ | 0.376 | 0.280 | 0.216 |
| $n$ | CI | Coverage | CI | Coverage | CI | Coverage |
| 100 | [1.081, 1.174] | 0.914 | [1.287, 1.516] | 0.936 | [1.530, 2.011] | 0.920 |
| 200 | [1.093, 1.158] | 0.962 | [1.322, 1.509] | 0.936 | [1.585, 1.913] | 0.936 |
| 300 | [1.098, 1.150] | 0.944 | [1.335, 1.486] | 0.940 | [1.611, 1.874] | 0.950 |

Table 2: Point-wise coverage rates over 500 simulations.

Table 3: Uniform coverage rates over 500 simulations.
5 Real data study

We now apply our analysis methods to an observational study for estimating the causal effect of fish consumption on the blood mercury level. This study is publicly available in the R package CrossScreening [Zhao et al., 2018] on CRAN. The outcome variable “blood mercury” is obtained from the individuals who answered questionnaires about seafood consumption in the National Health and Nutrition Examination Survey (NHANES) 2013-2014. The binary “treatment” variable indicates if an individual has consumed more than 12 servings of fish or shellfish in the previous month. The study has 234 treated individuals (high consumption) and 873 controls (low consumption), and 8 covariates: gender, age, income, whether income is missing, race, education, ever smoked, and the number of cigarettes smoked last month. The outcome variable is scaled by $\log_2$.

In $L^\infty$-analysis, we consider the sensitivity parameter $\Gamma$ ranging from 1 to 7, while in the $L^2$-analysis, we use $\lambda$ ranging from 0 to 0.7. As in the last section, the nuisance parameters are estimated by the Nadaraya-Waston estimator and logistic regression. The sensitivity curves are estimated by our proposed estimators with 9-fold cross-fitting. The result is shown in Figure 4; a confidence interval version of the figure is provided in Appendix M. We can see that both parameter ranges lead to identifying the ATE of fish consumption to be between 0 and 3.5. The adjusted critical values obtained using multiplier bootstrap are around 1.9 and 2.1 for a significance level of 0.10 and 0.05 in the analyses respectively, so the confidence intervals based on the central limit theorem are expected to be roughly 15% narrower than the ones shown in the figure.

![Figure 4: Sensitivity curves of the ATE of fish consumption on the blood mercury level.](image)

For the bound in (2), $\Gamma = 7$ implies that the causal effect of fish consumption is insensitive to unmeasured confounding. In Figure 4b, the average sensitivity value of around 1.5 will explain away the ATE. We can apply the first interpretability method discussed in Section 2.3 by assuming a Gamma variable with mean 1 and variance $1.5 - 1^2 = 0.5$ (i.e. both the shape and rate parameters $= 2$). The variable is bounded between $1/6$ and 6 with a probability larger than 0.95. $L^2$-analysis draws the same conclusion as $L^\infty$-analysis without using any boundedness assumption.

We next consider calibration by leaving out some measured covariates. In $L^\infty$- and $L^2$-analysis, we assess the strength of confounding via the odd ratio or the second moment of the estimated...
propensity scores respectively. Leaving out the covariate “education” gives an odd ratio larger than 6 for a few individuals. Because the odd ratio bound (2) is uniform over all values of $X$, “education” shall be treated as an important covariate that can potentially explain away the effect if unobserved. But it is easy to imagine an unmeasured covariate that is more predictive of fish consumption and blood mercury level than “education”. This result contradicts our previous conclusion that $\Gamma = 6$ or 7 is large enough to confirm the effect in $L^\infty$-analysis. In contrast, no single covariate after leaving out can lead to a second moment larger than 1.1. Only leaving out three covariates “income”, “race” and “education” together can achieve a second moment right below 1.5, which explains away the effect. These covariates together can be viewed as a strong confounder, because they are highly predictive of an individual’s diet. Conditioning on them, it is hard to come up with another unmeasured but equally impactful confounder. Therefore, with the average sensitivity value of 1.5, we can confirm the effect of fish consumption. Because the sensitivity value is averaged over all the individuals, calibration is more stable to use in $L^2$-analysis than in $L^\infty$-analysis.

6 Discussion

In this article, we extend the $L^\infty$-sensitivity analysis based on Tan [2006]’s marginal sensitivity model (MSM) to a new $L^2$-sensitivity analysis framework and provide solutions to the semiparametric estimation problem and uniform inference problem in both analyses. Our extension has analytical solutions, keeps the interpretability of MSM and improves the stability of calibration.

The $L^2$-analysis is based on the mean 1 constraint of the propensity score ratio $h(X, Y)$ from the $L^\infty$-analysis, which implies that the second moment (i.e. variance) of $h(X, Y)$ can be applied to measure the deviation from unconfoundedness. But different from the $L^\infty$-analysis and other existing methods in the literature, our formulations of $L^2$-analysis allow the deviation to be unbounded and vary across units. We estimate the average sensitivity value as the minimum deviation to partially identify a causal effect. The term “sensitivity value” was first proposed by Zhao [2018] under Rosenbaum [2002, Chapter 4]’s sensitivity model for pair-matched observational studies. The application of this original proposal is limited by the constant effect hypothesis and the model assumption. Our sensitivity value is free of these limitations, but its estimation and inference are harder because of the nuisance parameters arising from optimization. In this article, we partially addressed this challenge by one-step estimation using efficient influence functions. The nuisance estimators are required to converge at a rate faster than $n^{-1/4}$. This is still a stringent condition if the dataset is high-dimensional. In future works of $L^2$-analysis, it will be useful to further simplify the nuisance estimation problem as well as develop methods for heterogeneous causal effects based on the sensitivity value formulation and its estimation results in the appendix.

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22
Appendices

A $L^2$-sensitivity analysis: heterogeneous causal effects

Let $\psi_0 \equiv \psi_0(\theta) \equiv \mathbb{E}_{Y \mid Z=1} [h^2_*(X,Y)]$ denote the average sensitivity value from the program (7) for the solution $h_*(X,Y)$ in Proposition 2 when $\theta > 0$.

Theorem 5. The uncentered EIF of $\psi_0$ is

$$
\phi_0(O) = \frac{Z}{e(X)} \left\{ -2\lambda_X (Y - \mathbb{E}_{Y \mid X,1}[Y]) + \mathbb{E}_{Y \mid X,1} [h^2_*(X,Y)] - h^2_*(X,Y) \right\} + \mathbb{E}_{Y \mid X,1} [h^2_*(X,Y)] - \psi_0.
$$

The one-step estimator $\hat{\psi}_0 = \mathbb{P}_n^{(K)} \hat{\psi}_0$ is given by

$$
\hat{\psi}_0 = \mathbb{P}_n^{(K)} \left\{ \frac{Z}{e(X)} \left\{ -2\lambda_X (Y - \hat{\mathbb{E}}_{Y \mid X,1}[Y]) + \hat{\mathbb{E}}_{Y \mid X,1} [\hat{h}^2_*(X,Y)] - \hat{h}^2_*(X,Y) \right\} + \hat{\mathbb{E}}_{Y \mid X,1} [\hat{h}^2_*(X,Y)] \right\}.
$$

where $\hat{h}_*(X,Y) = \lambda_X (\hat{\xi}_X - Y) \mathbb{1}_{\{Y \leq \xi_X\}}$ and $\lambda_X = 1/\hat{\mathbb{E}}_{Y \mid X,1}[(\hat{\xi}_X - Y) \mathbb{1}_{\{Y \leq \xi_X\}}]$ and $\hat{\xi}_X$ is the root of the estimator of $f_{\theta,X}(\xi)$ in (9),

$$
\hat{f}_{\theta,X}(\xi) = \frac{\xi \hat{\mathbb{E}}_{Y \mid X,1} [Y \mathbb{1}_{\{Y \leq \xi\}}]}{\xi \hat{\mathbb{E}}_{Y \mid X,1} [Y \mathbb{1}_{\{Y \leq \xi\}}]} - \frac{\hat{\mathbb{E}}_{Y \mid X,1} [Y^2 \mathbb{1}_{\{Y \leq \xi\}}]}{\hat{\mathbb{E}}_{Y \mid X,1} [Y \mathbb{1}_{\{Y \leq \xi\}}]} - \frac{\hat{\mathbb{E}}_{Y \mid X,1} [Y - \theta]}{\xi}.
$$

The estimator $\hat{f}_{\theta,X}(\xi)$ has a positive derivative if $\hat{p}_{Y \mid X,1} > 0$ and it is negative when $\xi < \hat{\mathbb{E}}_{Y \mid X,1}[Y] - \theta$; see the derivative calculation for $f_{\theta,X}(\xi)$ at the end of Appendix E with $\mathbb{E}_{Y \mid X,1}[\cdot]$ replaced by $\hat{\mathbb{E}}_{Y \mid X,1}[\cdot] = \int \cdot \hat{p}_{Y \mid X,1}(y) dy$. Compared with the result in Section 3.2, the root $\hat{\xi}_X$ has a new definition while the nuisance parameters and estimators remain the same.

Proposition 6. Under Assumptions 1 and 3, Bias($\hat{\psi}_0 \mid \hat{\eta}$) = $o(1/\sqrt{n})$.

The theorem and proposition are proven in Appendices I and K respectively. The average sensitivity value $\psi_0$ is derived for the stronger hypothesis $\mathbb{E}[Y(1) \mid X] \leq \mathbb{E}[Y(X) \mid X] - \theta$ for all values of $X$, we can consider using it for partially identifying the conditional average treatment effect, CATE($X$) := $\mathbb{E}[Y(1) \mid X] - \mathbb{E}[Y(0) \mid X]$. The sensitivity curve for $\mathbb{E}[Y(0) \mid X]$ can be generated in the same way as the one for $\mathbb{E}[Y(1) \mid X]$ in Figure 2. For $\theta'' = \theta + \theta'$, the CATE hypothesis

$$
\mathbb{E}[Y(1) \mid X] - \mathbb{E}[Y(0) \mid X] \leq \mathbb{E}[Y(X) \mid Y] - \mathbb{E}[Y(X,0) \mid Y] - \theta''
$$

can be decomposed into two sub-hypotheses

$$
\mathbb{E}[Y(1) \mid X] \leq \mathbb{E}[Y(X) \mid Y] - \theta \quad \text{and} \quad \mathbb{E}[Y(0) \mid X] \geq \mathbb{E}[Y(X,0) \mid Y] + \theta'.
$$

In other words, the lower sensitivity curve of CATE($X$) is given by the difference between the lower sensitivity curve of $\mathbb{E}[Y(1) \mid X]$ and the upper sensitivity curve of $\mathbb{E}[Y(0) \mid X]$. Fix the average sensitivity value, we can lower bound CATE($X$) by $\mathbb{E}[Y(X) \mid Y] - \mathbb{E}[Y(X,0) \mid Y] - \theta$. The difference $\mathbb{E}[Y(X) \mid Y] - \mathbb{E}[Y(X,0) \mid Y]$ can be estimated efficiently with a CATE learner [Künzel et al., 2019; Kennedy, 2020; Nie and Wager, 2021] that leverages the smoothness of the CATE function. Based on the one-step estimator and the bias analysis above, we can conduct $L^2$-analysis for CATE even when nuisance parameters are estimated at slow nonparametric rates. This is an interesting extension of the current work that we leave for future research.
The following theorem is similar to Theorem 3. The first two consistency conditions are mild. On top of Assumption 3, the third condition is introduced to achieve \( \sup_{\lambda \in \mathcal{D}_1} \text{Bias}(\hat{\phi}_j \mid \hat{\eta}) = o(1/\sqrt{n}) \). The continuity assumption of \( \hat{p}_{Y|X,1} \) makes sure the root \( \hat{\xi}_X \) behaves smoothly with respect to \( \lambda \) so that cross-fitted estimators \( \hat{\psi}_{j,cf}(\lambda) \) for all \( \lambda \in \mathcal{D}_1 \) converge to a Gaussian process.

**Theorem 6.** For \( j = 1, 2 \), under Assumption 1, 3 and the conditions

1. \( \sup_{\lambda \in \mathcal{D}_1} |\hat{\sigma}_{j,cf}(\lambda)/\sigma_{j}(\lambda) - 1| = o_P(1) \),
2. \( \mathbb{E}\left\{ \left[ \sup_{\lambda \in \mathcal{D}_1} |\hat{\phi}_j(O; \lambda) - \phi_j(O; \lambda)| \right]^2 \right\} = o_P(1) \),
3. \( \sup_{\lambda \in \mathcal{D}_1} \|\hat{\xi}_X - \xi_X\| = o_P(n^{-1/4}) \),

it holds that \( \sqrt{n}\left[ \hat{\psi}_{j,cf}(\cdot) - \psi_j(\cdot) \right]/\hat{\sigma}_{j,cf}(\cdot) \xrightarrow{d} \mathbb{G}_j(\cdot) \in L^\infty(\mathcal{D}_1) \), where \( \mathbb{G}_j(\cdot) = [\phi_j(O; \cdot) - \psi_j(\cdot)]/\sigma_j(\cdot) \) is a Gaussian process with mean 0 and covariance \( \mathbb{E}\left\{ \mathbb{G}_j(\lambda)\mathbb{G}_j(\hat{\lambda}) \right\} \).

**Theorem 7.** In the setup of Theorem 6, for \( j = 1, 2 \), let \( \hat{q}_{j,\alpha} \) denote the critical value of the supremum of the multiplier bootstrap process such that

\[
P\left\{ \sup_{\lambda \in \mathcal{D}_1} \sqrt{n}\mathbb{P}_n \left[ A \left( \hat{\phi}_{j,cf}(O; \lambda) - \hat{\psi}_{j,cf}(\lambda) \right)/\hat{\sigma}_{j,cf}(\lambda) \right] \geq \hat{q}_{j,\alpha} \middle| O_{1:n} \right\} = \alpha,
\]

where \( A_{1:n} \) are i.i.d Rademacher variables drawn independently of \( O_{1:n} \), then

\[
P\left\{ \psi_j(\lambda) \notin \hat{C}_{j,\alpha}(\lambda) \right\} = \left[ \hat{\psi}_{j,cf}(\lambda) \pm \hat{q}_{j,\alpha}\hat{\sigma}_{j,cf}(\lambda)/\sqrt{n} \right], \forall \lambda \in \mathcal{D}_1 \} = \alpha + o(1).
\]

The proof of Theorem 6 is provided in Appendix L.2. Theorem 7 is a restatement of Kennedy [2019, Theorem 4] in our context. For every \( \lambda \in \mathcal{D}_1 \) we estimate two parameters \( \psi_1(\lambda) \) and \( \psi_2(\lambda) \). By a step of Bonferroni correction and Theorem 7, \( \hat{C}_{1,\alpha}(\mathcal{D}_1) = \bigcup_{\lambda \in \mathcal{D}_1} [\hat{C}_{1,\alpha}(\lambda) \times \hat{C}_{2,\alpha}(\lambda)] \) is a \((1-2\alpha)\)-confidence band for \( \Psi_1(\mathcal{D}_1) \), i.e.,

\[
P\{\Psi_1(\mathcal{D}_1) \notin \hat{C}_{1,\alpha}(\mathcal{D}_1)\} = P\left\{ \bigcup_{\lambda \in \mathcal{D}_1} \left\{ (\psi_1(\lambda), \psi_2(\lambda)) \notin [\hat{C}_{1,\alpha}(\lambda) \times \hat{C}_{2,\alpha}(\lambda)] \right\} \right\} = P\left\{ \bigcup_{j=1,2} \bigcup_{\lambda \in \mathcal{D}_1} \left\{ \psi_j(\lambda) \notin \hat{C}_{j,\alpha}(\lambda) \right\} \right\} \leq 2\alpha + o(1).
\]

### C Background: influence functions

The theory of influence functions [van der Vaart and Bernard, 2002] is crucial to removing the first-order bias in semiparametric estimation. It starts by treating a one-dimensional parameter \( \tau \) as a mapping \( \tau(\mathbb{P}) \) from \( \mathcal{P} \) to \( \mathbb{R} \) where \( \mathcal{P} \) is the set of all possible observed data distributions. For any \( \mathbb{P} \in \mathcal{P} \), we define a path through \( \mathbb{P} \) as a one-dimensional submodel that passes through \( \mathbb{P} \) at \( \epsilon = 0 \) in the direction of a zero-mean function \( s \) satisfying that \( \|s\|_2 \leq C' \) and \( \epsilon \leq 1/C' \) for some constant \( C' > 0 \). The submodel \( \mathbb{P}_s \) takes a density \( p_{O,\epsilon}(\cdot) := p_{O}(\cdot)[1 + \epsilon s(\cdot)] \) for \( \epsilon \in O \). The tangent space \( \mathcal{S} \)
is defined as the set of zero-mean functions $s$ for any paths through $\mathbb{P}$. And $S$ is known to be the Hilbert space of zero-mean functions when we use a nonparametric model.

Suppose that the mapping $\tau(\mathbb{P})$ is differentiable at $\mathbb{P}$ relative to $S$, meaning that there is a linear mapping $\hat{\tau}(\cdot; \mathbb{P}) : S \to \mathbb{R}$ such that for any $s \in S$ and submodel $\mathbb{P}_\epsilon$,

$$
\hat{\tau}(s; \mathbb{P}_\epsilon) = \frac{d\psi}{d\epsilon}|_{\epsilon=0} = \mathbb{E}\left[EIF(\tau)(O)s(O)\right],
$$

where $\tau_\epsilon := \tau(\mathbb{P}_\epsilon)$ and $s(o) := \frac{d}{d\epsilon} \log p_{O,\epsilon}(o)|_{\epsilon=0}$. In this article, we let the subscript “$\epsilon$” denote the distribution shift from $\mathbb{P}$ to $\mathbb{P}_\epsilon$, “$s$” denote the score function for a random variable. The second equality above is established by the Riesz representation theorem for the Hilbert space $S$ that expresses $\hat{\tau}(\cdot; \mathbb{P})$ as an inner product with the unique efficient influence function $EIF(\tau) : O \to \mathbb{R}$ that lies in the closed linear span of $S$. The efficiency of $EIF(\tau)$ implies that it has lower variance than any other influence functions, i.e., any measurable function $IF(\tau) : O \to \mathbb{R}$ whose projection onto the closed linear span of $S$ is $EIF(\tau)$. Some basic EIFs of expectation, conditional expectation and truncated expectations are given below, and we will use them in the following sections.

**Lemma 1.** Consider a random variable $A$, it holds that $EIF(\mathbb{E}[A]) = A - \mathbb{E}[A]$.

**Proof.** By the zero-mean property of influence functions, it is straightforward to verify the EIF definition in (20) that $\frac{d}{d\epsilon} \mathbb{E}[A]_{\epsilon=0} = \mathbb{E}\left[\left(A - \mathbb{E}[A]\right)S(A)\right]$. \qed

**Lemma 2.** Given two random variables $A$ and $B$, it holds that

$$
{EIF}\left\{\mathbb{E}[A \mid B = b]\right\} = \frac{\mathbb{I}\{b=b\}}{p_B(b)} \left(A - \mathbb{E}[A \mid B = b]\right).
$$

**Proof.** We can directly verify the EIF definition in (20):

$$
\frac{d}{d\epsilon} \mathbb{E}[A \mid B = b]_{\epsilon=0} = \int a \frac{d}{d\epsilon} \left[ \log p_{A,B,\epsilon}(a,b) - \log p_{B,\epsilon}(b) \right]|_{\epsilon=0} p_{A\mid B}(a\mid b) da
$$

$$
= \int a s(a,b) p_{A\mid B}(a \mid b) da - \int a p_{A\mid B}(a\mid b) da \times \frac{d}{d\epsilon} \log p_{B,\epsilon}(b)|_{\epsilon=0}
$$

$$
= \int \int \frac{\mathbb{I}\{b=b\}}{p_B(b')} as(a, b') p_{A,B}(a, b') db' - \mathbb{E}[A \mid B = b] \int \frac{\mathbb{I}\{b=b\}}{p_B(b')} \left(\frac{d}{d\epsilon} \log p_{B,\epsilon}(b')\right)|_{\epsilon=0}
$$

$$
+ \left(\frac{d}{d\epsilon} \log p_{A\mid B,\epsilon}(a \mid b')|_{\epsilon=0} p_{A\mid B}(a \mid b') da\right) p_B(b') db'
$$

$$
= \int \int \frac{\mathbb{I}\{b=b\}}{p_B(b')} as(a, b') p_{A,B}(a, b') db' - \mathbb{E}[A \mid B = b] \int \frac{\mathbb{I}\{b'=b\}}{p_B(b')} s(a, b') p_{A,B}(a, b') db'
$$

$$
= \int \int \frac{\mathbb{I}\{b'=b\}}{p_B(b')} \left(a - \mathbb{E}[A \mid B = b']\right) s(a, b') p_{A,B}(a, b') db'
$$

$$
= \mathbb{E}\left[EIF\left\{\mathbb{E}[A \mid B = b]\right\} S(A, B)\right],
$$

as required. \qed
Lemma 3. In the setup of $L^\infty$-analysis, it holds that
\[
\operatorname{EIF}\left\{\mathbb{E}_{Y|X=x,z=1} [Y \mathbb{1}_{\{Y \leq Q(X)\}}]\right\} = \frac{\mathbb{1}_{\{X=x,z=1\}}}{p_{X,Z}(x,1)} \left\{ (\alpha_* - \mathbb{1}_{\{Y \leq Q(X)\}}) Q(X) + Y \mathbb{1}_{\{Y \leq Q(X)\}} - \mathbb{E}_{Y|X=x,z=1} [Y \mathbb{1}_{\{Y \leq Q(X)\}}] \right\},
\]

Proof. First,
\[
\alpha_* = \int^{Q(x)} p_{Y|X,Z}(y|x,1) dy
\]

Then,
\[
\frac{d}{de} \mathbb{E}_{Y|X=x,z=1,e} [Y \mathbb{1}_{\{Y \leq Q(X)\}}]_{e=0}
= \int^{Q(x)} y p_{Y|X,Z}(y|x,1) dy_{e=0}
= Q(x) p_{Y|X,Z}(Q(x) | x,1) \frac{d}{de} Q_e(x) \bigg|_{e=0} + \frac{d}{de} \int \mathbb{1}_{\{y \leq Q(x)\}} p_{Y|X,Z}(y|x,1) dy_{e=0}
= \frac{d}{de} \int \left[ \int \mathbb{1}_{\{y \leq Q(x)\}} Q(x') + y \mathbb{1}_{\{y \leq Q(x)\}} \right] p_{Y|X,Z}(y|x,1) dy_{e=0}
= \int \int \mathbb{1}_{\{x'=x,z'=1\}} Q(x') + y \mathbb{1}_{\{y \leq Q(x')\}} - \alpha_* Q(x')
+ \mathbb{E}_{Y|X=x',Z=1} [Y \mathbb{1}_{\{Y \leq Q(X)\}}] S(y, z', x') dydz'dx'
= \mathbb{E} \left[ \operatorname{EIF}\left\{\mathbb{E}_{Y|X=x,z=1} [Y \mathbb{1}_{\{Y \leq Q(X)\}}]\right\} S(Y, Z, X) \right],
\]

where the penultimate equality is obtained by the proof of Lemma 2 above. \hfill \Box

D Proof of Proposition 1

Proof. The Lagrangian function of (4) is
\[
\mathcal{L} = \mathbb{E}_{Y|X=x} \left\{ -h(X,Y)Y + \lambda_{Y,1} [W_-(X) - h(X,Y)] + h(X,Y) - W_+(X) \right\}
+ \lambda_{X,3} \left[ 1 - \mathbb{E}_{Y|X=x} [h(X,Y)] \right].
\]

Setting the functional derivative of $\mathcal{L}$ w.r.t. $h$ to 0, we obtain the Euler-Lagrangian equation (i.e. the stationarity condition in the KKT conditions),
\[
-Y - \lambda_{Y,1} + \lambda_{Y,2} - \lambda_{X,3} = 0 \iff \lambda_{Y,1} - \lambda_{Y,2} = -\lambda_{X,3} - Y.
\]

By complementary slackness,
\[
\lambda_{Y,1} [W_-(X) - h(X,Y)] = 0 \quad \text{and} \quad \lambda_{Y,2} [h(X,Y) - W_+(X)] = 0.
\]
Combined the above with the dual feasibility \((\lambda_{Y,1}, \lambda_{Y,2} \geq 0)\), we have

\[
[\lambda_{Y,1}, \lambda_{Y,2}, h(X, Y)] = \begin{cases} 
[-\lambda_{X,3} - Y, 0, W_-(X)], & \text{if } Y < -\lambda_{X,3}; \\
[0, \lambda_{X,3} + Y, W_+(X)], & \text{if } Y > -\lambda_{X,3}.
\end{cases}
\] (21)

Let \(\alpha_* := \mathbb{P}\{Y < -\lambda_{X,3} \mid X = x, Z = 1\}\). By the primal feasibility,

\[
\mathbb{E}_{Y \mid X, 1} \{h(X, Y)\} = \alpha_* W_-(X) + (1 - \alpha_*) W_+(X) = 1,
\]

which implies that

\[
\alpha_* = \frac{1 - W_+(X)}{W_-(X) - W_+(X)} = \frac{1 - (1 - \Gamma)e(X) - \Gamma}{(1 - \Gamma^{-1})e(X) + \Gamma^{-1} - (1 - \Gamma)e(X) - \Gamma} = \frac{\Gamma}{1 + \Gamma},
\]

then \(-\lambda_{X,3}\) is the \(\frac{\Gamma}{1 + \Gamma}\)-quantile \(Q(X)\). The solution of (4) with minimization can be derived in the same way with a change from \(-h(X, Y)\) to \(h(X, Y)\) in the definition of \(L\).

\[
\square
\]

**E Proof of Proposition 2**

**Proof.** More generally, when \(\theta > 0\), if subject to \(h \in \mathcal{H}(\Gamma)\) in (3), the solution of (7) is given by

\[
h_*(X, Y) = \begin{cases} 
W_+(X), & \text{if } Y < \xi_X - W_+(X)/\lambda_X, \\
\lambda_X (\xi_X - Y), & \text{if } \xi_X - W_+(X)/\lambda_X \leq Y \leq \xi_X - W_-(X)/\lambda_X, \\
W_-(X), & \text{if } Y > \xi_X - W_-(X)/\lambda_X.
\end{cases}
\] (22)

If \(W_-(X) = 0\) and \(W_+(X) = \infty\), we have (8), \(h_*(X, Y) = \lambda_X (\xi_X - Y)\) if \(Y \leq \xi_X - 0/\lambda_X = \xi_X\), otherwise 0. To obtain the result for (7) subject to the constraint (10), the proof below can be applied with \(W_-(X) = -\infty\) and \(W_+(X) = 0\) in (22).

The Lagrangian function of (7) further subject to \(h \in \mathcal{H}(\Gamma)\) is given by

\[
L = \frac{1}{2} \mathbb{E}_{Y \mid X, 1} \left\{ h^2(X, Y) \right\} + \lambda_X \left[ \mathbb{E}_{Y \mid X, 1} \{h(X, Y)\} - \theta \right] + \lambda_{Y,2} \left[ 1 - \mathbb{E}_{Y \mid X, 1} \{h(X, Y)\} \right]
+ \mathbb{E}_{Y \mid X, 1} \left\{ \lambda_{Y,3} [W_-(X) - h(X, Y)] + \lambda_{Y,4} [h(X, Y) - W_+(X)] \right\}.
\]

Setting the functional derivative of \(L\) w.r.t. \(h\) to 0, we obtain the Euler-Lagrangian equation (i.e. the stationarity condition in the KKT conditions),

\[
h(X, Y) + \lambda_X Y - \lambda_{Y,2} - \lambda_{Y,3} + \lambda_{Y,4} = 0.
\]

By complementary slackness, \(\lambda_X \left[ \mathbb{E}_{Y \mid X, 1} \{h(X, Y)\} - \theta \right] = 0\),

\[
\lambda_{Y,3} [W_-(X) - h(X, Y)] = 0 \quad \text{and} \quad \lambda_{Y,4} [h(X, Y) - W_+(X)] = 0.
\]

By dual feasibility, \(\lambda_X, \lambda_{Y,3}, \lambda_{Y,4} \geq 0\). Because \(W_+(X) > W_-(X)\), we cannot have \(\lambda_{Y,3}, \lambda_{Y,4} > 0\). If \(\lambda_{Y,3} > 0\) and \(\lambda_{Y,4} = 0\), \(h(X, Y) = W_-(X)\). Further if \(\lambda_X = 0\), by the stationarity condition,

\[
\lambda_{Y,3} = W_-(X) - \lambda_{Y,3} + \lambda_{Y,4} < W_-(X).
\]

If \(\lambda_{Y,4} > 0\) and \(\lambda_{Y,3} = 0\), \(h(X, Y) = W_+(X)\). Further if \(\lambda_X = 0\),

\[
\lambda_{Y,3} = W_+(X) - \lambda_{Y,3} + \lambda_{Y,4} > W_+(X).
\]
Because \( W_+(X) > W_-(X) \), \( \lambda_{X,2} < W_-(X) \) contradicts with \( \lambda_{X,2} > W_+(X) \) in the last two equations, so we cannot have \( h(X, Y) = W_-(X) \) and \( W_+(X) \) for two different values of \( Y \) if \( \lambda_X = 0 \). If \( \lambda_{Y,3} \) or \( \lambda_{Y,4} \) is always positive, i.e., \( h(X, Y) \) is always equal to \( W_-(X) \) or \( W_+(X) \), \( h(X, Y) \) does not satisfy the equality constraint \( \mathbb{E}_{Y|X,1}[h(X, Y)] = 1 \). So overall, \( \lambda_X \neq 0 \) unless \( \lambda_{Y,3} = \lambda_{Y,4} = 0 \).

If \( \lambda_X = \lambda_{Y,3} = \lambda_{Y,4} = 0 \), \( h(X, Y) = \lambda_{X,2} \) by the stationarity condition. By primal feasibility, \( \mathbb{E}_{Y|X,1}[h(X, Y)] = \lambda_{X,2} = 1 \) \( \Rightarrow h(X, Y) = 1 \), then \( \mathbb{E}_{Y|X,1}[Y] \leq \mathbb{E}_{Y|X,1}[Y] - \theta \Rightarrow \theta \leq 0 \) and \( W_-(X) \leq 1 \leq W_+(X) \). This completes the proof for \( h_+(X, Y) = 1 \) if \( \theta \leq 0 \).

We now turn to the case that \( \lambda_X > 0 \). If \( \lambda_X > 0 \) and \( \lambda_{Y,3} = \lambda_{Y,4} = 0 \),

\[
h(X, Y) = \lambda_{X,2} - \lambda_X Y = \lambda_X (\xi_X - Y)
\]

with \( \xi_X := \lambda_{X,2}/\lambda_X \),

By primal feasibility,

\[
W_-(X) \leq \lambda_X (\xi_X - Y) \leq W_+(X) \Leftrightarrow \xi_X - W_+(X)/\lambda_X \leq Y \leq \xi_X - W_-(X)/\lambda_X,
\]

If \( \lambda_X, \lambda_{Y,3} > 0 \) and \( \lambda_{Y,4} = 0 \), \( h(X, Y) = W_-(X) \), by the stationarity condition,

\[
W_-(X) + \lambda_X Y - \lambda_{X,2} > 0 \Leftrightarrow Y > [\lambda_{X,2} - W_-(X)]/\lambda_X = \xi_X - W_-(X)/\lambda_X,
\]

If \( \lambda_X, \lambda_{Y,4} > 0 \) and \( \lambda_{Y,3} = 0 \), \( h(X, Y) = W_+(X) \), then

\[
W_+(X) + \lambda_X Y - \lambda_{X,2} < 0 \Leftrightarrow Y < [\lambda_{X,2} - W_+(X)]/\lambda_X = \xi_X - W_+(X)/\lambda_X.
\]

The last three equations complete the proof for (22).

We now verify the existence and uniqueness of \( \xi_X \) in the solution (8) for \( \theta > 0 \). In this case, we have proved that \( \lambda_X > 0 \). Then by complementary slackness, \( \mathbb{E}_{Y|X,1}[h_+(X, Y)Y] = \mathbb{E}_{Y|X,1}[Y] - \theta \).

Dividing it by \( \mathbb{E}_{Y|X,1}[h_+(X, Y)] = 1 \) removes \( \lambda_X \), then we can find \( \xi_X \) by solving the equation

\[
\mathbb{E}\left[ (\xi_X - Y)Y1_{\{Y \leq \xi_X\}} \right] / \mathbb{E}\left[ (\xi_X - Y)1_{\{Y \leq \xi_X\}} \right] = \mathbb{E}_{Y|X,1}[Y] - \theta.
\]

This leads to the definition of \( f_{\theta, X}(\xi) \) in (9). We first show that \( f_{\theta, X}(\xi) \) has a positive derivative,

\[
\frac{df_{\theta, X}(\xi)}{d\xi} = \frac{(0 + \mathbb{E}_{Y|X,1}[Y1_{\{Y \leq \xi\}}]) \mathbb{E}_{Y|X,1,Y \leq \xi}[\xi - Y] - (0 + \mathbb{P}\{Y \leq \xi\}) \mathbb{E}_{Y|X,1,Y \leq \xi}[\xi - Y]}{\mathbb{E}_{Y|X,1,Y \leq \xi}[\xi - Y]^2 - \mathbb{E}_{Y|X,1,Y \leq \xi}[\xi - Y]^2}.
\]

The function \( f_{\theta, X}(\xi) \) can be written as

\[
f_{\theta, X}(\xi) = \mathbb{E}_{Y|X,1}[\xi - Y(Y - \mathbb{E}_{Y|X,1}[Y] + \theta)1_{\{Y \leq \xi\}}]/\mathbb{E}_{Y|X,1}[(\xi - Y)1_{\{Y \leq \xi\}}].
\]

If \( \xi < \mathbb{E}_{Y|X,1}[Y] - \theta \),

\[
(\xi - Y)(Y - \mathbb{E}_{Y|X,1}[Y] + \theta)1_{\{Y \leq \xi\}} = -\xi Y^21_{\{Y \leq \xi\}} + \xi - Y\xi1_{\{Y \leq \xi\}}(\xi - \mathbb{E}_{Y|X,1}[Y] + \theta) < 0.
\]

Because \( \mathbb{E}_{Y|X,1}[(\xi - Y)1_{\{Y \leq \xi\}}] > 0 \), \( f_{\theta, X}(\xi) < 0 \) for \( \xi < \mathbb{E}_{Y|X,1}[Y] - \theta \). Together with the positive derivative, we know that \( f_{\theta, X}(\xi) \) has a unique root \( \xi_X \). Finally, the expression of \( \lambda_X \) is derived from the equality constraint \( \mathbb{E}[h_+(X, Y)] = 1 \).
F Proof of Proposition 3

Proof. The Lagrangian function of (12) is given by
\[ \mathcal{L} = \mathbb{E}_{Y|X,1} \left\{ h^2(X, Y)/2 + \lambda h(X, Y)Y \right\} + \lambda_{X,2} \left[ 1 - \mathbb{E}_{Y|X,1} \{ h(X, Y) \} \right] - \mathbb{E}_{Y|X,1} \{ \lambda_{Y,3}h(X, Y) \}. \]
Setting the functional derivative of \( \mathcal{L} \) w.r.t. \( h \) to 0, we obtain the Euler-Lagrangian equation (i.e. the stationarity condition in the KKT conditions),
\[ h(X, Y) + \lambda Y - \lambda_{X,2} - \lambda_{Y,3} = 0. \]
By complementary slackness, \( \lambda_{Y,3}h(X, Y) = 0 \). By dual feasibility, \( \lambda_{Y,3} \geq 0 \). If \( \lambda_{Y,3} = 0 \),
\[ h(X, Y) = \lambda_{X,2} - \lambda Y = \lambda (\xi_X - Y), \]
where \( \xi_X := \lambda_{X,2}/\lambda \). By primal feasibility and \( \lambda > 0 \), \( h(X, Y) \geq 0 \iff Y \leq \xi_X \). If \( \lambda_{Y,3} > 0 \), \( h(X, Y) = 0 \). By the stationarity condition and \( \lambda > 0 \), \( \lambda Y - \lambda_{X,2} > 0 \iff Y > \xi_X \). Finally, the function \( f_{\lambda, X}(\xi) \) in (14) is attained by substituting (13) into the first constraint in (12). Under Assumption 1, \( \mathbb{E}_{Y|X,1} [ (\xi - Y)1_{Y \leq \xi} ] \) in \( f_{\lambda, X}(\xi) \) is strictly increasing function \( \xi \) and it is positive for any \( \xi \). Then for \( \lambda > 0 \), \( f_{\lambda, X}(\xi) \) has a unique root \( \xi_X \). \hfill \( \Box \)

G Proof of Theorem 1

Proof. Following the EIF definition in (20), we can derive the EIF\( (\psi_+) = \phi_+(O) - \psi_+ \) by taking the derivative of \( \psi_+(P_s) \) w.r.t. \( \epsilon \) and rewriting it as the inner product with \( s(o) \) [Levy, 2019]. More conveniently, we can first derive the expression of EIF via the calculus of IFs (e.g. chain rule and product rule) [Kennedy, 2022, Section 3.4.3], then verify that the expression satisfies (20).

\[ IF \{ \psi_+ \} = IF \left\{ \sum_{x \in \mathcal{X}} p_X(x)W_+(x)\mu_+(x) \right\} \]
\[ = \sum_{x \in \mathcal{X}} IF \{ p_X(x) \} W_+(x)\mu_+(x) + \sum_{x \in \mathcal{X}} p_X(x)IF \{ W_+(x) \} \mu_+(x) + \sum_{x \in \mathcal{X}} p_X(x)W_+(x)IF \{ \mu_+(x) \} \]
\[ = \sum_{x \in \mathcal{X}} [1_{(X=x)} - p_X(x)] W_+(x)\mu_+(x) + \sum_{x \in \mathcal{X}} p_X(x) (1 - \Gamma) IF \{ e(x) \} \mu_+(x) \]
\[ + \sum_{x \in \mathcal{X}} p_X(x)W_+(x) \left( IF \{ \mathbb{E}_{Y|X,1}[\epsilon] \} - IF \{ \mathbb{E}_{Y|X,1}\{ Y \mathbb{1}_{Y \leq Q(X)} \} \} \right) \]
\[ = W_+(X)\mu_+(X) - \psi_+ + (1 - \Gamma) \sum_{x \in \mathcal{X}} p_X(x)\mu_+(x) \frac{1_{(X=x)}}{p_X(x)} [Z - e(X)] \]
\[ + \sum_{x \in \mathcal{X}} p_X(x)W_+(x) \frac{1_{(X=x,z=1)}}{p_{X,Z}(x,z)} \left[ Y - \mathbb{E}_{Y|X,1}[Y] - Y\mathbb{1}_{Y \leq Q(X)} + \mathbb{E}_{Y|X,1}[Y\mathbb{1}_{Y \leq Q(X)}] \right] \]
\[ - (\alpha_* - \mathbb{1}_{(Y \leq Q(X))}) Q(X) \]
\[ = [(1 - \Gamma)Z + \Gamma] \mu_+(X) - \psi_+ + \frac{W_+(X)Z}{e(X)} \left[ (1 - \alpha_* - \mathbb{1}_{(Y > Q(X))}) Q(X) + Y\mathbb{1}_{(Y > Q(X))} - \mu_+(X) \right]. \]
The first term in the third equality is attained by Lemma 1. In the fourth equality, the second term is obtained by Lemma 2, and the third term is obtained by Lemmas 2 and 3. Next, we define
\[ g_+(X,Y) = (1 - \alpha_* - \mathbb{1}_{(Y > Q(X))}) Q(X) + Y\mathbb{1}_{(Y > Q(X))}. \]
Denote $\sum_{z=0}^{1}$ by $\int dz$. We next verify the expression of $EIF(\psi_+)$:

$$
\mathbb{E}[EIF(\psi_+)(O)s(O)] = \int \int \int EIF(\psi_+)(y, z, x) \frac{d}{dt} \log p_{Y, Z, X}(y, z, x) \bigg|_{\epsilon=0} p_{Y, Z, X}(y, z, x) dydzdx
$$

$$
= \int \int \int EIF(\psi_+)(y, z, x) \frac{d}{dt} p_{Y|X, z, \epsilon}(y \mid x, z) \bigg|_{\epsilon=0} p_{Z|X}(z \mid x) p_{X}(x) dydzdx
$$

$$
+ \int \int \int EIF(\psi_+)(y, z, x) \frac{d}{dt} p_{Z|X, \epsilon}(z \mid x) \bigg|_{\epsilon=0} p_{Y|X, Z}(y \mid x, z) p_{X}(x) dydzdx
$$

$$
+ \int \int \int EIF(\psi_+)(y, z, x) \frac{d}{dt} p_{X, \epsilon}(x) \bigg|_{\epsilon=0} p_{Y|X, Z}(y \mid x, z) p_{Z|X}(z \mid x) dydzdx
$$

$$
= \int \int [W_+(x)[g_+(x, y) - \mu_+(x)] + e(x)\mu_+(x)] \frac{d}{dt} p_{Y|X, Z, \epsilon}(y \mid x, 1) \bigg|_{\epsilon=0} p_{X}(x) dydzdx
$$

$$
+ \int \int \Gamma \mu_+(x) \frac{d}{dt} p_{Y|X, Z, \epsilon}(y \mid x, 0) \bigg|_{\epsilon=0} [1 - e(x)] p_{X}(x) dydzdx
$$

$$
+ \int \mu_+(x) \frac{d}{dt} e(x) \bigg|_{\epsilon=0} p_{X}(x) dydzdx + \int \Gamma \mu_+(x) \frac{d}{dt} [1 - e(x)] \bigg|_{\epsilon=0} p_{X}(x) dydzdx
$$

$$
+ \int \mu_+(x) e(x) \frac{d}{dt} p_{X, \epsilon}(x) \bigg|_{\epsilon=0} p_{X}(x) dydzdx + \int \Gamma \mu_+(x) [1 - e(x)] \frac{d}{dt} p_{X, \epsilon}(x) \bigg|_{\epsilon=0} dydzdx
$$

$$
= W_+(x) \frac{d}{dt} \mu_+(x) \bigg|_{\epsilon=0} p_{X}(x) dydzdx + \frac{d}{dt} \int W_+(x) \mu_+(x) p_{X}(x) dx \bigg|_{\epsilon=0}
$$

$$
+ \int \mu_+(x) \frac{d}{dt} W_+(x) \bigg|_{\epsilon=0} p_{X}(x) dydzdx + \int W_+(x) \mu_+(x) \frac{d}{dt} p_{X, \epsilon}(x) \bigg|_{\epsilon=0} dydzdx
$$

$$
= \frac{d\psi_+}{dt} \bigg|_{\epsilon=0},
$$

as required by (20). The first term in the penultimate equality attained by

$$
\int [g_+(x, y) - \mu_+(x)] \frac{d}{dt} p_{Y|X, Z, \epsilon}(y \mid x, 1) \bigg|_{\epsilon=0} dy
$$

$$
= \int \int \frac{1}{p_{X, Z}(x', z)} [g_+(x', y) - \mu_+(x')] \log p_{Y|X, Z, \epsilon}(y \mid x', z) \bigg|_{\epsilon=0} p_{Y, Z, X}(y, z, x') dydzdx'
$$

$$
= EIF\{\mu_+(x)\} \text{ by Lemma 2}
$$

$$
= \int \int EIF\{\mu_+(x)\} \frac{d}{dt} \log p_{Y, Z, X}(y, z, x') \bigg|_{\epsilon=0} p_{Y, Z, X}(y, z, x') dydzdx'
$$

$$
- \int \int EIF\{\mu_+(x)\} p_{Y|X, Z}(y \mid x', z) \frac{d}{dt} \log p_{X, Z, \epsilon}(x, z) \bigg|_{\epsilon=0} p_{X,Z}(x', z) dzdx'
$$

$$
= \int \int EIF\{\mu_+(x)\} s(y, z, x') p_{Y, Z, X}(y, z, x') dydzdx'
$$

$$
= \frac{d}{dt} \mu_+(x) \bigg|_{\epsilon=0}.
$$

The proof for $EIF(\psi_-) = \phi_- (O) - \psi_-$ follows the same steps so omitted here. The only differences are replacing $\Gamma$ by $\Gamma^{-1}$ and truncating $Y$ below the quantile rather than above. \qed

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Proof of Proposition 4

Proof Sketch. Let $\hat{\mu}_+(X) = E_{Y \mid X} [Y \mathbb{1}_{\{Y > \hat{Q}(X)\}}]$ and $\hat{\mu}_-(X) = E_{Y \mid X} [Y \mathbb{1}_{\{Y < \hat{Q}(X)\}}]$. In the subsection below, we prove that Bias($\hat{\phi} \mid \hat{\eta}$) = Bias$_1(\hat{\phi} \mid \hat{\eta}) +$ Bias$_2(\hat{\phi} \mid \hat{\eta})$, where

$$\text{Bias}_1(\hat{\phi} \mid \hat{\eta}) = \Gamma \mathbb{E} \left\{ \frac{\hat{e}(X) - e(X)}{\hat{e}(X)} \left[ \left( \mathbb{P}_{Y \mid X} [Y > \hat{Q}(X)] - [1 - \alpha_+] \right) \hat{Q}(X) + \hat{\mu}_+(X) - \mu_+(X) \right] \mid \hat{\eta} \right\}$$

$$+ \Gamma^{-1} \mathbb{E} \left\{ \frac{\hat{e}(X) - \hat{e}(X)}{\hat{e}(X)} \left[ \left( \mathbb{P}_{Y \mid X} [Y < \hat{Q}(X)] - \alpha_+ \right) \hat{Q}(X) + \hat{\mu}_-(X) - \mu_-(X) \right] \mid \hat{\eta} \right\}$$

and

$$\text{Bias}_2(\hat{\phi} \mid \hat{\eta}) = \mathbb{E} \left\{ \left( W_+(X) - W_-(X) \right) \left[ \left( \hat{Q}(X) - Q(X) \right) \left( \mathbb{P}_{Y \mid X} [Y < \hat{Q}(X)] - \alpha_+ \right) \right] \right\}.$$

In Bias$_1(\hat{\phi} \mid \hat{\eta})$, we can rewrite $\hat{\mu}_+(X) - \mu_+(X) = \hat{\mu}_+(X) - \mu_+(X) + \mu_+(X) - \mu_+(X)$. By definition, $\hat{\mu}_+$ converges to $\mu_+$ as fast as $\hat{Q}$ converges to $Q$,

$$\hat{\mu}_+(X) - \mu_+(X) \leq E_{Y \mid X} [Y \mathbb{1}_{\{Q(X) < Y < Q(X) \land Q(X) \leq \hat{Q}(X)\}}] \leq |\hat{Q}(X) - Q(X)|.$$

The same argument applies to $\hat{\mu}_-(X) - \mu_-(X)$. Thus, Bias$_1(\hat{\phi} \mid \hat{\eta}) = o_P(1/\sqrt{m})$ by Cauchy-Schwarz and Assumption 2. Next, Bias$_2(\hat{\phi} \mid \hat{\eta}) \leq E\{(|\hat{Q}(X) - Q(X)|^2 \mid \hat{\eta}\} = o_P(1/\sqrt{m})$ by

$$W_+(X) - W_-(X) < \Gamma - \Gamma^{-1} \mathbb{P}_{Y \mid X} [Y < \hat{Q}(X)] - \alpha^* = O_P(|\hat{Q}(X) - Q(X)|),$$

and $|Y - Q(X)| \leq |\hat{Q}(X) - Q(X)|$ under the event $Q(X) \land \hat{Q}(X) < Y < Q(X) \lor \hat{Q}(X)$ which has probability decaying as fast as $|\hat{Q}(X) - Q(X)|$.

H.1 Bias expression

Proof. By definition, $\phi(O; \hat{\eta}) = \phi_+(O; \hat{\eta}) + \phi_-(O; \hat{\eta})$. Following (23), we define

$$\hat{g}_+(X, Y) = \left( 1 - \alpha_+ - \mathbb{1}_{\{Y > \hat{Q}(X)\}} \right) \hat{Q}(X) + Y \mathbb{1}_{\{Y > \hat{Q}(X)\}},$$

We rewrite the uncentered EIF $\phi_+(O; \hat{\eta})$ as

$$\phi_+(O; \hat{\epsilon}, \hat{Q}, \hat{\mu}_+) := \frac{Z \hat{W}_+(X)}{\hat{e}(X)} \left[ \hat{g}_+(X, Y) - \hat{\mu}_+(X) \right] + [(1 - \Gamma)Z + \Gamma] \hat{\mu}_+(X)$$

$$= \frac{Z \hat{W}_+(X)}{\hat{e}(X)} \left[ \hat{g}_+(X, Y) - \hat{\mu}_+(X) \right] + [(1 - \Gamma)Z + \Gamma] \left[ \hat{\mu}_+(X) - \hat{g}_+(X, Y) \right]$$

$$+ [(1 - \Gamma)Z + \Gamma] \hat{g}_+(X, Y).$$

The difference $\phi_+(O; \hat{\epsilon}, \hat{Q}, \hat{\mu}_+) - \phi_+(O; e, Q, \mu_+)$ can be written as

$$\phi_+(O; \hat{\epsilon}, \hat{Q}, \hat{\mu}_+) - \phi_+(O; e, Q, \mu_+) + \phi_+(O; e, \hat{Q}, \hat{\mu}_+) - \phi_+(O; e, Q, \mu_+)$$

$$\overset{(a)}{=} \Gamma Z [1/\hat{e}(X) - 1/e(X)] \left[ \hat{g}_+(X, Y) - \hat{\mu}_+(X) \right] + \Gamma [Z/e(X) - 1] \left[ \hat{g}_+(X, Y) - \hat{\mu}_+(X) - g_+(X, Y) + \mu_+(X) \right].$$

(a) by the first expression of $\phi_+$ above
Because the expectation of \[\frac{Z}{e(X)} - 1\] conditional on \(X\) is 0, we have

\[
\text{Bias}(\hat{\phi}_+ | \hat{\eta}) = E \left\{ \phi_+(O; \hat{e}, \hat{Q}, \hat{\mu}_+) - \phi_+(O; e, \hat{Q}, \hat{\mu}_+) + \phi_+(O; e, Q, \hat{\mu}_+) - \phi_+(O; e, Q, \mu_+) | \hat{\eta} \right\} + \Gamma E \left\{ \frac{\hat{\mu}_+(X) - E_{Y|X}[\hat{g}_+(X,Y)]}{\hat{e}(X)} \right\} + \Gamma E \left\{ \frac{Z \hat{W}_+(X)}{\hat{e}(X)} [\hat{g}_+(X,Y) - g_+(X,Y)] | \hat{\eta} \right\}.
\]

Define the functions

\[
\hat{g}_-(X,Y) = \left( \alpha_s - 1_{\{Y<Q(X)\}} \right) \hat{Q}(X) + Y 1_{\{Y<Q(X)\}} \\
\hat{g}_-(X,Y) = \left( \alpha_s - 1_{\{Y<\hat{Q}(X)\}} \right) \hat{Q}(X) + Y 1_{\{Y<\hat{Q}(X)\}}.
\]

We can rewrite \(\phi_-(O; \hat{e}, \hat{Q}, \hat{\mu}_-)\) as

\[
\phi_-(O; \hat{e}, \hat{Q}, \hat{\mu}_-) = \frac{Z \hat{W}_-(X)}{\hat{e}(X)} \left[ \hat{g}_-(X,Y) - \hat{\mu}_-(X) \right] + [(1 - \Gamma^{-1})Z + \Gamma^{-1}] \hat{\mu}_-(X)
\]

and show that

\[
\text{Bias}(\hat{\phi}_- | \hat{\eta}) = \Gamma^{-1} E \left\{ \frac{\hat{\mu}_-(X) - \hat{\mu}_-(X)}{\hat{\mu}_-(X)} \right\} + \Gamma E \left\{ \frac{Z \hat{W}_-(X)}{\hat{e}(X)} [\hat{g}_-(X,Y) - g_-(X,Y)] | \hat{\eta} \right\}.
\]

Summing up \(\text{Bias}(\hat{\phi}_+ | \hat{\eta})\) and \(\text{Bias}(\hat{\phi}_- | \hat{\eta})\) above proves the expression of \(\text{Bias}_1(\phi | \hat{\eta})\) and

\[
\text{Bias}_2(\hat{\phi} | \hat{\eta}) = E \left\{ W_+(X) [\hat{g}_+(X,Y) - g_+(X,Y)] + W_- (X) [\hat{g}_-(X,Y) - g_-(X,Y)] | Z = 1, \hat{\eta} \right\}. \ (24)
\]

First, consider the terms of trimmed outcomes in \(\text{Bias}_2(\hat{\phi} | \hat{\eta})\),

\[
W_+(X) \left[ Y 1_{\{Y>\hat{Q}(X)\}} - Y 1_{\{Y>Q(X)\}} \right] + W_- (X) \left[ Y 1_{\{Y<\hat{Q}(X)\}} - Y 1_{\{Y<Q(X)\}} \right]
\]

Second, consider the other terms in \(\text{Bias}_2(\hat{\psi})\),

\[
W_+(X) \left[ 1 - \alpha_s - 1_{\{Y>\hat{Q}(X)\}} \right] \hat{Q}(X) + W_- (X) \left[ \alpha_s - 1_{\{Y<\hat{Q}(X)\}} \right] \hat{Q}(X)
\]

\[
- W_+(X) \left[ 1 - \alpha_s - 1_{\{Y>Q(X)\}} \right] Q(X) - W_- (X) \left[ \alpha_s - 1_{\{Y<Q(X)\}} \right] Q(X)
\]

\[
W_+(X) \left[ 1 - \alpha_s - 1_{\{Y>\hat{Q}(X)\}} \right] Q(X) + W_- (X) \left[ \alpha_s - 1_{\{Y<\hat{Q}(X)\}} \right] Q(X)
\]

\[
- W_+(X) \left[ 1 - \alpha_s - 1_{\{Y>Q(X)\}} \right] Q(X) - W_- (X) \left[ \alpha_s - 1_{\{Y<Q(X)\}} \right] Q(X)
\]

\[
= W_+(X) \left[ 1 - \alpha_s - 1_{\{Y>\hat{Q}(X)\}} \right] [\hat{Q}(X) - Q(X)] + W_- (X) \left[ \alpha_s - 1_{\{Y<\hat{Q}(X)\}} \right] [\hat{Q}(X) - Q(X)]
\]
\begin{equation}
+ W_+(X) \left[ -\mathbb{1}_{\{Y > Q(X)\}} + \mathbb{1}_{\{Y > Q(X)\}} \right] Q(X) + W_-(X) \left[ -\mathbb{1}_{\{Y < Q(X)\}} + \mathbb{1}_{\{Y < Q(X)\}} \right] Q(X)
\end{equation}

\((b) - (c)\)

\begin{align*}
&= [W_+(X) - W_-(X)] \left[ \mathbb{1}_{\{Y < Q(X)\}} - \alpha_* \right] [\hat{Q}(X) - Q(X)] + W_+(X) \mathbb{1}_{\{Y = Q(X)\}} [\hat{Q}(X) - Q(X)] \\
&\quad + \begin{cases} 
0, & \text{if } Y > \hat{Q}(X), Y > Q(X); \\
W_+(X)Q(X) - W_-(X)Q(X), & \text{if } Y < \hat{Q}(X), Y > Q(X); \\
W_-(X)Q(X) - W_+(X)Q(X), & \text{if } Y > \hat{Q}(X), Y < Q(X); \\
0, & \text{if } Y < \hat{Q}(X), Y < Q(X). 
\end{cases}
\end{align*}

Summing up the end of the last two equations, we know that the right-hand side of (24) is the expectation of the product of \(W_+(X) - W_-(X)\) and

\(\hat{Q}(X) - Q(X)\) \(\mathbb{1}_{\{Y < Q(X)\}} - \alpha_*\) + \(\begin{cases} 
0, & \text{if } Y > \hat{Q}(X), Y > Q(X); \\
Q(X) - Y, & \text{if } Y < \hat{Q}(X), Y > Q(X); \\
Y - Q(X), & \text{if } Y > \hat{Q}(X), Y < Q(X); \\
0, & \text{if } Y < \hat{Q}(X), Y < Q(X), 
\end{cases}\) conditioning on \(Z = 1\) and \(\hat{\eta}\). Further conditioning on \(X\) gives the expression of \(\text{Bias}_2(\hat{\phi} | \hat{\eta})\).

\section{Proof of Theorems 2 and 5}

\subsection{Two derivatives}

We start by deriving two derivatives. To simplify the exposition, we denote

\[
\kappa_0(X, Y) := (\xi_X - Y)\mathbb{1}_{\{Y \leq \xi_X\}} \quad \text{and} \quad \kappa_0(X) := \mathbb{E}_{Y | X, 1}[\{\xi_X - Y\} \mathbb{1}_{\{Y \leq \xi_X\}}],
\]

\[
\kappa_1(X, Y) := (\xi_X - Y)Y \mathbb{1}_{\{Y \leq \xi_X\}} \quad \text{and} \quad \kappa_1(X) := \mathbb{E}_{Y | X, 1}[\{\xi_X - Y\} \mathbb{1}_{\{Y \leq \xi_X\}}].
\]

It holds that \(\kappa_1(X, Y) = \kappa_0(X, Y)Y\). First,

\[
\frac{d\tilde{\kappa}_{0, \epsilon}(X)}{d\epsilon} \bigg|_{\epsilon = 0} = \frac{d}{d\epsilon} \left\{ \int_{\epsilon_X, \epsilon} (\xi_X, \epsilon - y) p_Y | X, 1, \epsilon(y)dy \right\} \bigg|_{\epsilon = 0}
\]

\[
= 0 \times \frac{d\xi_{X, \epsilon}}{d\epsilon} \bigg|_{\epsilon = 0} + \frac{d\xi_{X, \epsilon}}{d\epsilon} \bigg|_{\epsilon = 0} \mathbb{E}_{Y | X, 1}[\mathbb{1}_{\{Y \leq \xi_X\}}]
\]

\[
+ \frac{d}{d\epsilon} \left\{ \int (\xi_X - y) \mathbb{1}_{\{Y \leq \xi_X\}} p_Y | X, 1, \epsilon(y)dy \right\} \bigg|_{\epsilon = 0}
\]

\[
= \frac{d\xi_{X, \epsilon}}{d\epsilon} \bigg|_{\epsilon = 0} \mathbb{E}_{Y | X, 1}[\mathbb{1}_{\{Y \leq \xi_X\}}] + \frac{d}{d\epsilon} \mathbb{E}_{Y | X, 1}[\kappa_0(X, Y)] \bigg|_{\epsilon = 0}
\]

\[
= \mathbb{E}_{Y | X, 1}[\mathbb{1}_{\{Y \leq \xi_X\}}] + \mathbb{E} \left\{ \frac{Z}{e(X)} [\kappa_0(X, Y) - \tilde{\kappa}_0(X)] S(O) \right\}. \tag{25}
\]

The last line follows from the proof of Lemma 2, which implies that

\[
\frac{d}{d\epsilon} \mathbb{E}_{Y | X = x, Z = 1, \epsilon}[\kappa_0(X, Y)] \bigg|_{\epsilon = 0} = \mathbb{E} \left\{ \frac{\mathbb{1}_{\{X = x, Z = z\}}}{e(X)p_X(X)} [\kappa_0(X, Y) - \tilde{\kappa}_0(X)] S(O) \right\}
\]

\[
= \mathbb{E} \left\{ \frac{Z}{e(X)} [\kappa_0(X, Y) - \tilde{\kappa}_0(X)] S(O) \right\}. \tag{26}
\]
We write the solution (13) as
\[ \Pi(\kappa) \]
where \( \kappa \) and \( \bar{\kappa} \).

Similarly, we can show that
\[ \psi \]
as required by the EIF definition in (20). We now turn to
Proof.

I.2 Proof of Theorem 2

Proof. We first prove the expression of \( EIF(\psi_2) = \phi_2(O) - \psi_2 \). We will use \( \kappa_0(X,Y), \bar{\kappa}_0(X,Y) \) and \( \bar{\kappa}_1(X) \) defined in Appendix I.1. It follows from (25) that
\[ \frac{d\bar{\kappa}_0(X)}{de} \bigg|_{e=0} = 0 \iff \frac{d\xi_{X,e}}{de} \bigg|_{e=0} = -E \left[ \frac{Z}{e(X)} \left( \kappa_0(X,Y) - \bar{\kappa}_0(X) \right) S(O) \mid X \right] / E_{Y|X,1} \left[ I_{\{Y \leq \xi_X\}} \right]. \]

We write the solution (13) as \( h_\ast(X,Y) = \lambda \kappa_0(X,Y) \) and \( \lambda \bar{\kappa}_0(X) = E_{Y|X,1} \left[ h_\ast(X,Y) \right] = 1. \) Plugging \( \kappa_0(X,Y) = h_\ast(X,Y) / \lambda \) and \( \bar{\kappa}_0(X) = 1 / \lambda \) into above leads to
\[ \lambda \frac{d\xi_{X,e}}{de} \bigg|_{e=0} = E \left[ \frac{Z\Pi(X,Y)}{e(X)} S(O) \mid X \right]. \]

where \( \Pi(X,Y) = \left[ 1 - h_\ast(X,Y) \right] / E_{Y|X,1} \left[ I_{\{Y \leq \xi_X\}} \right] \). Then,
\[ \frac{d}{de} E_{Y|X,1,e} \left[ h_\ast(e(X),Y) \right] \bigg|_{e=0} = \]
\[ = E \left[ \frac{Z}{e(X)} \left( \Pi(X,Y) E_{Y|X,1} \left[ I_{\{Y \leq \xi_X\}} \right] + h_\ast(X,Y) Y - E_{Y|X,1} \left[ h_\ast(X,Y) Y \right] \right) S(O) \mid X \right]. \]

By \( \psi_2 = E_{Y|Z=1} \left[ h_\ast(X,Y) Y \right] = E_{X} \left[ E_{Y|X,1} \left[ h_\ast(X,Y) Y \right] \right] \) and Lemma 1,
\[ \frac{d\psi_2}{de} \bigg|_{e=0} = E_{X} \left[ \frac{d}{de} E_{Y|X,1,e} \left[ h_\ast(e(X),Y) \right] \bigg|_{e=0} \right] + \frac{d}{de} E_{X,e} \left[ E_{Y|X,1} \left[ h_\ast(X,Y) Y \right] \right] \bigg|_{e=0} \]
\[ = E \left[ \left( \frac{Z}{e(X)} \left[ \Pi(X,Y) E_{Y|X,1} \left[ I_{\{Y \leq \xi_X\}} \right] + h_\ast(X,Y) Y - E_{Y|X,1} \left[ h_\ast(X,Y) Y \right] \right) + E_{Y|X,1} \left[ h_\ast(X,Y) Y - \psi_2 \right] S(O) \mid X \right] \right], \]

as required by the EIF definition in (20). We now turn to \( EIF(\psi_1) = \phi_1(O) - \psi_1 \). Like (25),
\[ \frac{d}{de} E_{Y|X,1} \left[ h_\ast^2(e(X),Y) \right] \bigg|_{e=0} = \]
\[ = 2 \lambda \frac{d\xi_{X,e}}{de} \bigg|_{e=0} E_{Y|Z=1} \left[ h_\ast(X,Y) \right] + \]
By \( \psi_1 = \mathbb{E}_{Y|Z=1} [h_\epsilon^2(X,Y)] = \mathbb{E} [\mathbb{E}_{Y|X,1}[h_\epsilon^2(X,Y)]] \) and Lemma 1,

\[
\frac{d\psi_1}{d\epsilon} \bigg|_{\epsilon=0} = \mathbb{E}_{X} \left[ \frac{d}{d\epsilon} \mathbb{E}_{Y|X,1}[h_\epsilon^2(X,Y)] \bigg|_{\epsilon=0} \right] + \frac{d}{d\epsilon} \mathbb{E}_{X,\epsilon} [\mathbb{E}_{Y|X,1}[h_\epsilon^2(X,Y)]] \bigg|_{\epsilon=0} \\
= \mathbb{E} \left[ \left( \frac{Z}{e(X)} \{ 2\Pi(X,Y) + h_\epsilon^2(X,Y) \} - \mathbb{E}_{Y|X,1}[h_\epsilon^2(X,Y)] \right) + \mathbb{E}_{Y|X,1}[h_\epsilon^2(X,Y)] - \psi_1 \right) S(O),
\]

which shows that our expression of \( EIF(\psi_1) = \phi_1(O) - \psi_1 \) satisfies the definition in (20). \( \square \)

### I.3 Proof of Theorem 5

**Proof.** By the definition of \( \xi_X \) in (9), \( \bar{\kappa}_1(X) = \Delta_X \bar{\kappa}_0(X) \) where \( \Delta_X := \mathbb{E}_{Y|X,1}[Y] - \theta \). Applying this equality to connect the derivative expressions for \( \bar{\kappa}_1(X) \) and \( \bar{\kappa}_0(X) \) above,

\[
\frac{d\bar{\kappa}_1}{d\epsilon} \bigg|_{\epsilon=0} = \frac{d\Delta_X}{d\epsilon} \bigg|_{\epsilon=0} \bar{\kappa}_0(X) + \Delta_X \frac{d\bar{\kappa}_0}{d\epsilon} \bigg|_{\epsilon=0}.
\]

Plugging in the expressions leads to

\[
\frac{d\xi_X}{d\epsilon} \bigg|_{\epsilon=0} = \mathbb{E}_{Y|X,1} \left[ \left( (Y - \mathbb{E}_{Y|Y}|Y) \bar{\kappa}_0(X) - \Delta_X \mathbb{E}_{Y|X,1}[Y|Y] \right) \bar{\kappa}_0(X) + \bar{\kappa}_1(X) \right] S(O) | X
\]

\[
= \mathbb{E}_{Y|X,1} \left[ (Y - \mathbb{E}_{Y|Y}|Y) \bar{\kappa}_0(X) - \Delta_X \mathbb{E}_{Y|X,1}[Y|Y] \right] S(O) | X
\]

\[
= -\mathbb{E}_{Y|X,1} \left[ (Y - \mathbb{E}_{Y|Y}|Y) \bar{\kappa}_0(X) - \Delta_X \mathbb{E}_{Y|X,1}[Y|Y] \right] S(O) | X
\]

By the equality that

\[
\mathbb{E}_{Y|X,1} \left[ (Y - \Delta_X) \xi_X | Y \leq \xi_X \right] = \bar{\kappa}_1(X) - \Delta_X \bar{\kappa}_0(X) = 0,
\]

we have \( (\xi_X - \Delta_X) \mathbb{E}_{Y|X,1} \left[ [(\xi_X - Y) \bar{\kappa}_0(X) + \Delta_X \mathbb{E}_{Y|X,1}[Y|Y] \right] S(O) | X \]

\[
= \lambda_X^2 \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y) \bar{\kappa}_0(X) + \Delta_X \mathbb{E}_{Y|X,1}[Y|Y] \right] S(O) | X
\]

\[
= \lambda_X (\xi_X - \Delta_X).
\]

By definition, \( \lambda_X = 1/\bar{\kappa}_0(X) \). By (28) and the derivative expressions for \( \bar{\kappa}_0(X) \) and \( \xi_X \) above,

\[
\frac{d}{d\epsilon} \mathbb{E}_{Y|X,1}[h_\epsilon^2(X,Y)] \bigg|_{\epsilon=0}
\]

\[
= \frac{d}{d\epsilon} \left( (\xi_X - \Delta_X) \bar{\kappa}_0(X) \right) \bigg|_{\epsilon=0}
\]

\[
= \left[ \left( \frac{d\xi_X}{d\epsilon} \bigg|_{\epsilon=0} - \left( \frac{d\Delta_X}{d\epsilon} \bigg|_{\epsilon=0} \right) \bar{\kappa}_0(X) \right) + \mathbb{E}_{Y|X,1} \left[ (\Delta_X - Y) \bar{\kappa}_0(X) \right] \right] S(O) | X
\]

\[
= \lambda_X^2 \left\{ \left( \frac{d\xi_X}{d\epsilon} \bigg|_{\epsilon=0} - \left( \frac{d\Delta_X}{d\epsilon} \bigg|_{\epsilon=0} \right) \bar{\kappa}_0(X) \right) + \mathbb{E}_{Y|X,1} \left[ (\Delta_X - Y) \bar{\kappa}_0(X) \right] \right\} S(O) | X
\]

\[
= \lambda_X (\xi_X - \Delta_X).
\]

35
\[ \psi(X) = \frac{Z}{e(X)}\left[2(Y - E_{Y|X,Y}[Y])\tilde{\rho}(X) + (\Delta_X - Y)\kappa_0(X, Y) + (\xi_X - \Delta_X)(\kappa_0(X, Y) - \tilde{\rho}(X))\right] S(O) \mid X \]

\[ = -\lambda_X^2 \psi(X) \left[2(Y - E_{Y|X,Y}[Y])\tilde{\rho}(X) + (\xi_X - Y)\kappa_0(X, Y) - (\xi_X - \Delta_X)\tilde{\rho}(X)\right] S(O) \mid X \]

\[ = \kappa_0(X, Y) - (\xi_X - \Delta_X)\tilde{\rho}(X) \right] S(O) \mid X \]

\[ = E\left[ \left\{ -2\lambda_X(Y - E_{Y|X,Y}[Y]) - \lambda_X^2(Y - \xi_X)^2 \mathbb{1}_{\{Y \leq \xi_X\}} + \lambda_X(\xi_X - \Delta_X) \right\} S(O) \mid X \right] \]

By \( \psi_0 = E_{Y|Z=1}[h_2^2(X, Y)] = E[E_{Y|X,Y}[h_2^2(X, Y)]] \) and Lemma 1,

\[ \frac{d\psi_0}{d\varepsilon} \bigg|_{\varepsilon=0} = E\left[ \left( \frac{Z}{e(X)} \left\{ -2\lambda_X(Y - E_{Y|X,Y}[Y]) + E_{Y|X,Y}[h_2^2(X, Y)] - h_2^2(X, Y) \right\} \right) S(O) \right], \]

which shows that our expression of \( EIF(\psi_0) = \phi_0(O) - \psi_0 \) satisfies the definition in (20). \( \square \)

### J Proof of Proposition 5

**Proof.** Bias of \( \hat{\xi}_X \). First, by the roots \( \xi_X \) and \( \hat{\xi}_X \), we have \( \hat{f}_{\lambda,X}(\hat{\xi}_X) = f_{\lambda,X}(\xi_X) \), which implies that

\[ E_{Y|X,Y}[\hat{\xi}_X - \xi_X] = 0 \]

\[ (\hat{\xi}_X - \xi_X)E_{Y|X,Y}[\mathbb{1}_{\{Y \leq \xi_X\}} \mid \hat{\eta}] - E_{Y|X,Y}[\mathbb{1}_{\{Y \leq \xi_X\}} \mid \hat{\eta}] = \frac{op(n^{-1/4})}{error \ of \ \hat{\rho}_{Y|X,Y}} \] (29)

The conditional bias \( Bias_{Y|X}(\hat{\xi}_X \mid \hat{\eta}) \) can be written as

\[ \hat{e}(X) - e(X) \left( E_{Y|X,Y}[\hat{h}_2^2(X, Y)] - E_{Y|X,Y}[\hat{h}_2^2(X, Y) \mid \hat{\eta}] \right) + Bias_{Y|X,Y}[\hat{h}_2^2(X, Y) + 2\hat{P}(X, Y) \mid \hat{\eta}] \].

Under Assumption 3,

\[ E_{Y|X,Y}[\hat{h}_2^2(X, Y)] - E_{Y|X,Y}[\hat{h}_2^2(X, Y) \mid \hat{\eta}] = Op\left( \int |\hat{p}_{Y|X,Y}(y) - p_{Y|X}(y)| dy \right) = op(n^{-1/4}), \]

then the expectation of the first multiplicative term in (30) is \( o(1/\sqrt{n}) \) by the Cauchy-Schwarz inequality. Using the constraint \( E_{Y|X,Y}[h_2(X, Y)] = 1 \), the second term in (30) can be written as

\[ E_{Y|X,Y}[\hat{h}_2^2(X, Y) \mid \hat{\eta}] = E_{Y|X,Y}[\hat{h}_2^2(X, Y) \mid \hat{\eta}] + E_{Y|X,Y}[h_2(X, Y) \mid \hat{\eta}] / E_{Y|X,Y}[\mathbb{1}_{\{Y \leq \xi_X\}}] \]

\[ = \lambda E_{Y|X,Y}\left\{ \hat{h}_2^2(X, Y) + h_2(X, Y) - 2\hat{E}_{Y|X,Y}[\mathbb{1}_{\{Y \leq \xi_X\}}] \right\} (\hat{h}_2(X, Y) - h_2(X, Y) \mid \hat{\eta}) \]

\[ \leq \lambda E_{Y|X,Y}\left\{ |\hat{h}_2(X, Y) - h_2(X, Y)| - 2\hat{E}_{Y|X,Y}[\mathbb{1}_{\{Y \leq \xi_X\}}] \right\} \times |\hat{\xi}_X - \xi_X| \times \mathbb{1}_{\{\xi_X \land \xi_X \leq Y \leq \xi_X \lor \xi_X\}} \mid \hat{\eta} \]
\[
+ \lambda E_{Y|X,1} \left\{ \left( \hat{h}_*(X,Y) + h_*(X,Y) - 2 \hat{E}_{Y|X,1}[1_{Y \leq \xi_X}] \right) \times \left( \hat{\xi}_X - \xi_X \right) \times 1_{Y \leq \xi_X \wedge \hat{\xi}_X} \mid \hat{\eta} \right\} \\
+ \lambda E_{Y|X,1} \left\{ \left( \hat{h}_*(X,Y) + h_*(X,Y) - 2 \hat{E}_{Y|X,1}[1_{Y \leq \xi_X}] \right) \times 0 \times 1_{Y > \xi_X \vee \hat{\xi}_X} \mid \hat{\eta} \right\} \\
\leq O_p(\|\hat{\xi}_X - \xi_X\|^2) + \lambda |\hat{\xi}_X - \xi_X| \times |\lambda E_{Y|X,1,Y \leq \xi_X \wedge \xi_X}[\xi_X - Y \mid \hat{\eta}] + \lambda E_{Y|X,1,Y \leq \xi_X \wedge \xi_X}[\xi_X - Y \mid \hat{\eta}] \\
- 2\lambda \hat{E}_{Y|X,1}[\hat{\xi}_X - Y\beta_{Y \leq \xi_X}] / \hat{E}_{Y|X,1}[\beta_{Y \leq \xi_X}] \\
= O_p(\|\hat{\xi}_X - \xi_X\|^2) + \lambda^2 |\hat{\xi}_X - \xi_X| \times |E_{Y|X,1,Y \leq \xi_X \wedge \xi_X}[\hat{\xi}_X - Y + \xi_X - Y \mid \hat{\eta}] - 2\hat{E}_{Y|X,1,Y \leq \xi_X}[\hat{\xi}_X - Y] | \\
\leq O_p(\|\hat{\xi}_X - \xi_X\|^2) + \lambda^2 |\hat{\xi}_X - \xi_X| \times \left( 2|E_{Y|X,1,Y \leq \xi_X \wedge \xi_X}[Y \mid \hat{\eta}] - \hat{E}_{Y|X,1,Y \leq \xi_X}[Y]| + |\hat{\xi}_X - \xi_X| \right) \\
\leq O_p(\|\hat{\xi}_X - \xi_X\|^2) + O_p(\|\hat{\xi}_X - \xi_X| \times \int |\hat{p}_{Y|X,1}(y) - p_{Y|X}(y)|dy) \\
= o_p(1/\sqrt{n}). 
\]

In the second inequality, the first term uses \( P_{Y|X,1}\{\xi_X \land \hat{\xi}_X < Y \leq \xi_X \lor \hat{\xi}_X \mid \hat{\eta} \} = O_p(\|\hat{\xi}_X - \xi_X\|), \)
and the second term is obtained by the root \( \hat{\xi}_X \) of \( f_\lambda(X, \xi), \) i.e., \( \hat{E}_{Y|X,1}[\hat{\xi}_X - Y \beta_{Y \leq \xi_X}] = 1/\lambda. \)
Under Assumption 3, the last equality is intuitive and proven in (32) below. The last equality is attained by (29) and Assumption 3. Using (30), we have the following result,

\[
\text{Bias}(\hat{\phi}_1 \mid \hat{\eta}) \leq E_X^{1/2} \left\{ \left| \hat{e}(X) - c(X) \right| \left| \hat{\eta} \right| \times E_X^{1/2} \left\{ \left| \hat{p}_{Y|X,1}(y) - p_{Y|X}(y) \right| dy \right\}^2 \mid \hat{\eta} \right\} + E_X \left\{ \left| \hat{\xi}_X - \xi_X \right| \mid \hat{\eta} \right\} + E_X \left\{ \left| \hat{\xi}_X - \xi_X \right| \times \int |\hat{p}_{Y|X,1}(y) - p_{Y|X}(y)|dy \mid \hat{\eta} \right\}. 
\]

(31)

The same bound for \( \text{Bias}(\hat{\phi}_2 \mid \hat{\eta}) \) can be derived in the same way following the proof below.

**Bias of \( \hat{\phi}_2 \).** First, consider the bias of \( \hat{\phi}_2 \) conditioning on \( X \) and \( \hat{\eta} \),

\[
\text{Bias}_{Y,Z|X}(\hat{\phi}_2 \mid \hat{\eta}) = \frac{\hat{e}(X) - c(X)}{\hat{e}(X)} \left( \hat{E}_{Y|X,1} \left[ \hat{h}_*(X,Y)Y \right] - E_{Y|X,1} \left[ h_*(X,Y)Y \right] \mid \hat{\eta} \right) + \text{Bias}_{Y|X,1} \left( \hat{I}(X,Y)\hat{E}_{Y|X,1}[Y\beta_{Y \leq \xi_X}] + \hat{h}_*(X,Y)Y \mid \hat{\eta} \right).
\]

Under Assumption 3, \( \hat{E}_{Y|X,1} \left[ h_*(X,Y)Y \right] - E_{Y|X,1} \left[ h_*(X,Y)Y \mid \hat{\eta} \right] = o_p(n^{-1/4}), \) then the expectation of the first term in \( \text{Bias}(\hat{\phi}_2 \mid \hat{\eta}) \) over \( X \) is \( o_p(1/\sqrt{n}) \) by the Cauchy-Schwarz inequality. Using the equality constraint \( E_{Y|X,1}[h_*(X,Y)] = 1, \) we can rewrite the second term as

\[
E_{Y|X,1} \left[ h_*(X,Y)Y \mid \hat{\eta} \right] - E_{Y|X,1} \left[ h_*(X,Y)Y \mid \hat{\eta} \right] + \frac{E_{Y|X,1}[Y\beta_{Y \leq \xi_X}]}{E_{Y|X,1}[\beta_{Y \leq \xi_X}]} \left( E_{Y|X,1}[h_*(X,Y)] - E_{Y|X,1}[h_*(X,Y) \mid \hat{\eta}] \right)
\]

\[
= \lambda E_{Y|X,1} \left\{ \left( \hat{\xi}_X - Y \right)1_{Y \leq \xi_X} - \left( \xi_X - Y \right)1_{Y \leq \xi_X} \right\} \times Y - \hat{E}_{Y|X,1,Y \leq \xi_X}[Y] \mid \hat{\eta} \right\} \leq \lambda E_{Y|X,1} \left\{ \left| \hat{\xi}_X - \xi_X \right| \times \left| Y - \hat{E}_{Y|X,1,Y \leq \xi_X}[Y] \right| \times 1_{\xi_X \leq Y \leq \xi_X \vee \xi_X} \mid \hat{\eta} \right\} + \lambda E_{Y|X,1} \left\{ \left| \hat{\xi}_X - \xi_X \right| \times \left| Y - \hat{E}_{Y|X,1,Y \leq \xi_X}[Y] \right| \times 1_{\xi_X \leq \xi_X \land Y} \mid \hat{\eta} \right\} + \lambda E_{Y|X,1} \left\{ 0 \times \left| Y - \hat{E}_{Y|X,1,Y \leq \xi_X}[Y] \right| \times 1_{Y > \xi_X \vee \xi_X} \mid \hat{\eta} \right\} \leq O_p(\|\hat{\xi}_X - \xi_X\|^2) + \lambda |\hat{\xi}_X - \xi_X| \times \left| E_{Y|X,1,Y \leq \xi_X \wedge \xi_X}[Y \mid \hat{\eta}] - \hat{E}_{Y|X,1,Y \leq \xi_X}[Y] \right| = o_p(1/\sqrt{n}).
\]

The last two lines use \( P_{Y|X,1}\{\xi_X \land \hat{\xi}_X < Y \leq \xi_X \lor \hat{\xi}_X \mid \hat{\eta} \} = O_p(\|\hat{\xi}_X - \xi_X\|). \) (29) and (32).
We now prove the claim

\[ |E_{\hat{Y}}|_{X,1,Y \leq \xi_x} \leq |E_{\hat{Y}}|_{X,1,Y \leq \xi_x} - \hat{E}_{\hat{Y}}|_{X,1,Y \leq \xi_x} | \]

\[ \leq O_p\left( (\hat{\xi}_x - \xi_x) \right) + O_p\left( \int |\hat{p}_{\hat{Y}}|_{X,1} - p_{\hat{Y}}|_{X} |dy \right)\]

\[ = E_{X,1} + E_{X,2}. \]

Proof. The second equality is obtained as follows:

\[ E_{\hat{Y}}|_{X,1,Y \leq \xi_x} - \hat{E}_{\hat{Y}}|_{X,1,Y \leq \xi_x} = \]

\[ = \hat{E}_{\hat{Y}}|_{X,1}[\hat{E}_{\hat{Y}}|_{X,1}Y_{1}(Y \leq \xi_x)] - \hat{E}_{\hat{Y}}|_{X,1}[\hat{E}_{\hat{Y}}|_{X,1}Y_{1}(Y \leq \xi_x)] \]

\[ = E_{\hat{Y}}|_{X,1}[\hat{E}_{\hat{Y}}|_{X,1}Y_{1}(Y \leq \xi_x)] - E_{\hat{Y}}|_{X,1}[\hat{E}_{\hat{Y}}|_{X,1}Y_{1}(Y \leq \xi_x)] + E_{X,2} \]

\[ = O_p\left( E_{\hat{Y}}|_{X,1}[\hat{E}_{\hat{Y}}|_{X,1}Y_{1}(Y \leq \xi_x)] - E_{\hat{Y}}|_{X,1}[\hat{E}_{\hat{Y}}|_{X,1}Y_{1}(Y \leq \xi_x)] \right) + E_{X,2} \]

\[ = E_{X,1} + E_{X,2}; \]

similarly, \( E_{\hat{Y}}|_{X,1,Y \leq \xi_x} = \hat{E}_{\hat{Y}}|_{X,1,Y \leq \xi_x} = E_{X,1} + E_{X,2} \). 

\[ \square \]

K Proof of Proposition 6

Proof. We start by showing that

\[ \hat{\xi}_x - \xi_x = o_p(n^{-1/4}). \] (33)

First, we have

\[ E_{Y}|_{X,1}[(\hat{\xi}_x - Y)Y_{1}(Y \leq \xi_x)] - (\xi_x - Y)Y_{1}(Y \leq \xi_x) | \hat{\eta} \]

\[ = E_{Y}|_{X,1}[(\hat{\xi}_x - Y)Y_{1}(Y \leq \xi_x)] - (\xi_x - Y)Y_{1}(Y \leq \xi_x) + (\xi_x - Y)Y_{1}(Y \leq \xi_x) - (\xi_x - Y)Y_{1}(Y \leq \xi_x) | \hat{\eta} \]

\[ = O_p(\hat{\xi}_x - \xi_x) + O_p\left( E_{Y}|_{X,1}[\xi_x \land \hat{\xi}_x \leq Y \leq \xi_x \lor \hat{\xi}_x | \hat{\eta}] \right) \]

\[ = O_p(\hat{\xi}_x - \xi_x). \]

Similarly, we can show that

\[ E_{Y}|_{X,1}[(\hat{\xi}_x - Y)Y_{1}(Y \leq \xi_x)] - (\xi_x - Y)Y_{1}(Y \leq \xi_x) | \hat{\eta} = O_p(\hat{\xi}_x - \xi_x). \]

By the roots \( \xi_x \) and \( \hat{\xi}_x \), we have \( \hat{f}_{\theta,X}(\hat{\xi}_x) + \hat{E}_{Y}|_{X,1}[Y] = f_{\theta,X}(\xi_x) + E_{Y}|_{X,1}[Y] \), which leads to

\[ \hat{f}_{\theta,X}(\hat{\xi}_x) = f_{\theta,X}(\xi_x) = o_p(n^{-1/4}) \]

\[ E_{Y}|_{X,1}[(\xi_x - Y)Y_{1}(Y \leq \xi_x)] = \hat{E}_{Y}|_{X,1}[(\xi_x - Y)Y_{1}(Y \leq \xi_x)] - \hat{E}_{Y}|_{X,1}[(\xi_x - Y)Y_{1}(Y \leq \xi_x)] \]

\[ = E_{Y}|_{X,1}[(\xi_x - Y)Y_{1}(Y \leq \xi_x)] - \hat{E}_{Y}|_{X,1}[(\xi_x - Y)Y_{1}(Y \leq \xi_x)] = o_p(n^{-1/4}) \]

\[ E_{Y}|_{X,1}[(\xi_x - Y)Y_{1}(Y \leq \xi_x)] = E_{Y}|_{X,1}[(\xi_x - Y)Y_{1}(Y \leq \xi_x) | \hat{\eta}] \]
We can upper bound the last expectation:

\[ \mathbb{E}_{Y|X,1} \left[ (\hat{\xi}_X - Y) \mathbb{I}_{\{Y \leq \xi_X\}} | \hat{\eta} \right] \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)Y \mathbb{I}_{\{Y \leq \xi_X\}} \right] = o_p(n^{-1/4}) + o_p(n^{-1/4}) \]

\[ \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y) \mathbb{I}_{\{Y \leq \xi_X\}} \right] \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)Y \mathbb{I}_{\{Y \leq \xi_X\}} \right] - (\hat{\xi}_X - Y)Y \mathbb{I}_{\{Y \leq \xi_X\}} | \hat{\eta} \right]

+ \mathbb{E}_{Y|X,1} \left[ (\hat{\xi}_X - Y) \mathbb{I}_{\{Y \leq \xi_X\}} \right] \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)Y \mathbb{I}_{\{Y \leq \xi_X\}} \right] = o_p(n^{-1/4})

\[ \hat{\xi}_X - \xi_X = o_p(n^{-1/4}). \]

The second equality is obtained by Assumption 3. It follows from the last two equations that

\[ \hat{\lambda}_X - \lambda_X = 1/\mathbb{E}_{Y|X,1} \left[ (\hat{\xi}_X - Y) \mathbb{I}_{\{Y \leq \xi_X\}} \right] - 1/\mathbb{E}_{Y|X,1} \left[ (\xi_X - Y) \mathbb{I}_{\{Y \leq \xi_X\}} \right] \]

\[ = O_p \left( \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y) \mathbb{I}_{\{Y \leq \xi_X\}} \right] - \mathbb{E}_{Y|X,1} \left[ (\hat{\xi}_X - Y) \mathbb{I}_{\{Y \leq \xi_X\}} \right] \right) \]

\[ = O_p(\hat{\xi}_X - \xi_X) = o_p(n^{-1/4}). \]

The conditional bias \( \text{Bias}_{Y|X} (\hat{\phi}_0 | \hat{\eta}) \) can be written as

\[ \frac{e(X)}{\sigma(X)} \left\{ -2\hat{\lambda}_X (\mathbb{E}_{Y|X,1}[Y] - \hat{\mathbb{E}}_{Y|X,1}[Y]) + \mathbb{E}_{Y|X,1} \left[ \hat{h}_2^2(X,Y) \right] - \mathbb{E}_{Y|X,1} \left[ h_2^2(X,Y) \right] \right\} + \mathbb{E}_{Y|X,1} \left[ \hat{h}_2^2(X,Y) \right] - \mathbb{E}_{Y|X,1} \left[ h_2^2(X,Y) \right] \]

\[ = \left\{ \frac{e(X)}{\sigma(X)} - 1 \right\} \left\{ -2\hat{\lambda}_X (\mathbb{E}_{Y|X,1}[Y] - \hat{\mathbb{E}}_{Y|X,1}[Y]) + \mathbb{E}_{Y|X,1} \left[ \hat{h}_2^2(X,Y) \right] - \mathbb{E}_{Y|X,1} \left[ h_2^2(X,Y) \right] \right\} 

- 2\hat{\lambda}_X (\mathbb{E}_{Y|X,1}[Y] - \hat{\mathbb{E}}_{Y|X,1}[Y]) + 2\mathbb{E}_{Y|X,1} \left[ \hat{h}_2^2(X,Y) \right] - \mathbb{E}_{Y|X,1} \left[ h_2^2(X,Y) \right] \right\} \]

\[ = 2\lambda_X (\hat{\xi}_X - \xi_X) + 2(\hat{\lambda}_X - \lambda_X)(\xi_X - \Delta_X) + 2(\hat{\lambda}_X - \lambda_X)(\hat{\xi}_X - \Delta_X) / \lambda_X + \frac{\Delta_X - \hat{\xi}_X}{\lambda_X} (\hat{\lambda}_X - \lambda_X)^2 \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} \right] \]

\[ = \frac{\Delta_X - \hat{\xi}_X}{\lambda_X} (\hat{\lambda}_X - \lambda_X)^2 + \frac{\lambda_X}{\lambda_X} \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} \right] - (\hat{\xi}_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} | \hat{\eta} \right] \]

Under Assumption 3, \( \mathbb{E}_{Y|X,1}[\cdot] - \mathbb{E}_{Y|X,1}[\cdot] = o_p(n^{-1/4}) \), then the expectation of the first line over \( X \) in the second equality of (35) is \( o_p(1/\sqrt{n}) \) by the Cauchy-Schwarz inequality. Like (28),

\[ \hat{\mathbb{E}}_{Y|X,1} \left[ \hat{h}_2^2(X,Y) \right] = \lambda_X (\hat{\xi}_X - \hat{\Delta}_X) \]

where \( \hat{\Delta}_X := \hat{\mathbb{E}}_{Y|X,1}[Y] - \theta \).

Then by (28) and (36), the second line in the second equality of (35) can be rewritten as

\[ -2\hat{\lambda}_X (\Delta_X - \hat{\Delta}_X) + 2\mathbb{E}_{Y|X,1} \left[ \hat{h}_2^2(X,Y) \right] - 2\mathbb{E}_{Y|X,1} \left[ h_2^2(X,Y) \right] - \mathbb{E}_{Y|X,1} \left[ h_2^2(X,Y) \right] \]

\[ = 2\lambda_X (\hat{\xi}_X - \xi_X) + 2(\hat{\lambda}_X - \lambda_X)(\xi_X - \Delta_X) + 2(\hat{\lambda}_X - \lambda_X)(\hat{\xi}_X - \Delta_X) / \lambda_X + \frac{\Delta_X - \hat{\xi}_X}{\lambda_X} (\hat{\lambda}_X - \lambda_X)^2 \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} \right] \]

\[ = 2\lambda_X (\hat{\xi}_X - \xi_X) + 2(\hat{\lambda}_X - \lambda_X)(\xi_X - \Delta_X) - (\Lambda_X + \hat{\lambda}_X)(\hat{\xi}_X - \lambda_X)(\xi_X - \Delta_X) / \lambda_X + \frac{\Delta_X - \hat{\xi}_X}{\lambda_X} (\hat{\lambda}_X - \lambda_X)^2 \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} \right] - (\hat{\xi}_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} | \hat{\eta} \right] \]

\[ = 2\lambda_X (\hat{\xi}_X - \xi_X) + (2 - 1 - \hat{\lambda}_X / \lambda_X)(\hat{\lambda}_X - \lambda_X)(\xi_X - \Delta_X) \]

\[ + \frac{\Delta_X - \hat{\xi}_X}{\lambda_X} (\hat{\lambda}_X - \lambda_X)^2 \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} \right] - (\hat{\xi}_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} | \hat{\eta} \right] \]

\[ = 2\lambda_X (\hat{\xi}_X - \xi_X) + \frac{\Delta_X - \hat{\xi}_X}{\lambda_X} (\hat{\lambda}_X - \lambda_X)^2 + \frac{\lambda_X}{\lambda_X} \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} \right] - (\hat{\xi}_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} | \hat{\eta} \right]. \]

We can upper bound the last expectation:

\[ \mathbb{E}_{Y|X,1} \left[ (\xi_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} \right] - (\hat{\xi}_X - Y)^2 \mathbb{I}_{\{Y \leq \xi_X\}} | \hat{\eta} \right] \]

\[ \leq \mathbb{E}_{Y|X,1} \left[ \mathbb{I}_{\{Y > \hat{\xi}_X \cap \check{\xi}_X \leq Y \leq \xi_X \cap \check{\xi}_X \}} | \hat{\eta} \right] + \mathbb{E}_{Y|X,1} \left[ (\hat{\xi}_X - \xi_X)^2 \times \mathbb{I}_{\{\hat{\xi}_X \leq \xi_X \cap \check{\xi}_X \leq Y \leq \xi_X \}} | \hat{\eta} \right] \]
Then the penultimate equation can be upper bounded by

\[
\lambda_X^2 (\xi - \xi) \left( \frac{2}{\lambda_X} - E_{Y|X} \left[ (\xi - Y) \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] - E_{Y|X} \left[ (\xi - Y) \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] \right)
\]

\[+ O_P \left[ (\lambda_X - \lambda_X)^2 \right] + O_P \left[ (\xi - \xi)^2 \right] \]

\[
= \lambda_X^2 (\xi - \xi) \left( 2E_{Y|X} \left[ (\xi - Y) \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] - E_{Y|X} \left[ (\xi - Y) \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] \right)
\]

\[+ o_P (1/\sqrt{n}) \]

\[
= o_P (n^{-1/4}) \times o_P (n^{-1/4}) + o_P (1/\sqrt{n}) = o_P (1/\sqrt{n}).
\]

In the first equality, the first term is obtained by \( \lambda_X = 1 - \frac{\hat{E}_{Y|X}}{E_{Y|X}} \left[ (\xi - Y) \mathbb{1}_{\{Y \leq \xi \}} \right] \), and the second term is attained by (33) and (34). The penultimate equality is obtained by Assumption 3 and (33):

\[
E_{Y|X} \left[ Y \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] \leq \left| \hat{E}_{Y|X} \left[ Y \mathbb{1}_{\{Y \leq \xi \}} \right] - E_{Y|X} \left[ Y \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] \right| + o_P (n^{-1/4})
\]

\[
E_{Y|X} \left[ Y \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] - E_{Y|X} \left[ Y \mathbb{1}_{\{Y \leq \xi \}} | \hat{\eta} \right] = o_P (n^{-1/4}).
\]

\[\square\]

L Proofs of Theorems 3 and 6

L.1 Proof of Theorem 3

Proof. We follow the steps of the proof by Kennedy [2019, Section 8.4] while skipping the details of the generic steps there. Denote the full-sample empirical process by \( \mathbb{G}_n = \sqrt{n}(\hat{\mathcal{P}} - \mathcal{P}) \). Define

\[\Omega_n (\Gamma) = \sqrt{n} \left[ \tilde{\psi}_{\text{cf}} (\Gamma) - \psi (\Gamma) \right] / \hat{\sigma}_{\text{cf}} (\Gamma)\text{ and } \Omega_n (\Gamma) = \mathbb{G}_n \left\{ [\phi (O; \Gamma) - \psi (\Gamma)] / \sigma (\Gamma) \right\},\]

The proof is completed by verifying two statements:

\[\Omega_n (\cdot) \overset{d}{\to} \mathcal{G} (\cdot) \in L^\infty (\mathcal{D}) \text{ and } \sup_{\Gamma \in \mathcal{D}} |\Omega_n (\Gamma) - \Omega_n (\Gamma)| = o_P (1).\]

Lemma 4 below shows that \( \mathcal{F} = \{ \phi (O; \Gamma) : \Gamma \in \mathcal{D} \} \) is Lipschitz under Assumptions 1 and 2, which proves the first statement. Because in this case, \( \mathcal{F} \) has a finite bracketing integral so Donsker; see Vaart and Wellner [1996, Chapter 2.5.6] and Kennedy [2016, Section 4.3] for more details. For the second statement, it follows from Kennedy [2019, Section 8.4] that in our context, for any \( \Gamma \in \mathcal{D}, \)

\[
\sup_{\Gamma \in \mathcal{D}} |\Omega_n (\Gamma) - \Omega_n (\Gamma)| \leq \sup_{\Gamma \in \mathcal{D}} |\Omega_n (\Gamma) - \Omega_n (\Gamma)| + \sup_{\Gamma \in \mathcal{D}} |\sigma (\Gamma) / \hat{\sigma}_{\text{cf}} (\Gamma) - 1|
\]

\[= \sup_{\Gamma \in \mathcal{D}} |\Omega_n (\Gamma) - \Omega_n (\Gamma)| + o_P (1),\]

(38)
where \( \hat{\Omega}_n(\Gamma) = \sqrt{n} \left[ \hat{\psi}_e(\Gamma) - \psi(\Gamma) \right] / \sigma(\Gamma) \), and the equality is given by the first condition in the theorem. Let \( \mathcal{G}_n^{(k)} = \sqrt{m(P_m^{(k)} - \mathbb{P})} \). It is proven in Kennedy [2019, Section 8.4] that

\[
\hat{\Omega}_n(\Gamma) - \Omega_n(\Gamma) = \frac{\sqrt{n}}{K \sigma(\Gamma)} \sum_{k=1}^{K} \frac{1}{\sqrt{m}} \mathcal{G}_n^{(k)} \left\{ \hat{\phi}^{(k)}(O; \Gamma) - \phi(O; \Gamma) \right\} + \text{Bias} \left\{ \hat{\phi}^{(k)}(O; \Gamma) \mid \hat{\eta} - k \right\}
\equiv B_{n,1}(\Gamma) + B_{n,2}(\Gamma),
\]

and that if \( \mathcal{F}_n^{(k)} = \left\{ \hat{\phi}^{(k)}(\cdot; \Gamma) - \phi(\cdot; \Gamma) : \Gamma \in \mathcal{D} \right\} \) is Lipschitz, then

\[
\sup_{\Gamma \in \mathcal{D}} |B_{n,1}(\Gamma)| \lesssim \max_{k \in [K]} \sup_{k \in \mathcal{F}(k)} |\mathcal{G}_n(f)| = o_p(1),
\]

under the second condition in the theorem. We already proved that \( \phi(\cdot; \Gamma) \) is Lipschitz function of \( \Gamma \) in Lemma 4. Under the fourth condition, the same proof also shows that \( \hat{\phi}^{(k)}(\cdot; \Gamma) \) is a Lipschitz function of \( \Gamma \). Then, the function class \( \mathcal{F}_n^{(k)} \) is Lipschitz. Finally, as explained in the main manuscript, under the third condition, we have

\[
\sup_{\Gamma \in \mathcal{D}} \text{Bias} \left\{ \hat{\phi}(O; \Gamma) \mid \hat{\eta} \right\} = o_p(1/\sqrt{n}) \Leftrightarrow \sup_{\Gamma \in \mathcal{D}} \text{Bias} \left\{ \hat{\phi}^{(k)}(O; \Gamma) \mid \hat{\eta} - k \right\} = o_p(1/\sqrt{n}),
\]

which implies that \( \sup_{\Gamma \in \mathcal{D}} B_{n,2}(\Gamma) = o_p(1) \). The condition is derived by applying \( \sup_{\Gamma \in \mathcal{D}} \) to the expression of \( \text{Bias}\{ \hat{\phi}(O; \Gamma) \mid \hat{\eta} \} \) in the proof sketch of Proposition 4 in Appendix H, then keeping \( \sup_{\Gamma \in \mathcal{D}} \) in front of the terms that depend on \( \Gamma \). Finally, together with (39), \( \sup_{\Gamma \in \mathcal{D}} |\hat{\Omega}_n(\Gamma) - \Omega_n(\Gamma)| = o_p(1) \) and then the second statement in (37) holds through (38).

\[\square\]

L.2 Proof of Theorem 6

Proof. The proof is similar to the one in Appendix L.1 with minor changes which we detail below. Like (37), the goal is to prove the two statements: for \( j = 1, 2 \),

\[
\Omega_{j,n}(\cdot) \overset{d}{=} \mathcal{G}_j(\cdot) \in L^\infty(\mathcal{D}_1) \quad \text{and} \quad \sup_{\lambda \in \mathcal{D}_1} |\hat{\Omega}_{j,n}(\lambda) - \Omega_{j,n}(\lambda)| = o_p(1),
\]

where \( \hat{\Omega}_{j,n}(\cdot) = \sqrt{n} \left[ \hat{\psi}_{e,j}(\cdot) - \psi_j(\cdot) \right] / \hat{\sigma}_{e,j}(\cdot) \) and \( \Omega_{j,n}(\cdot) = \mathcal{G}_n \left\{ [\phi_j(O; \cdot) - \psi_j(\cdot)] / \sigma_j(\cdot) \right\} \).

Lemma 5 below shows that the function classes \( \mathcal{F}_j = \{ \phi_j(O; \lambda) : \lambda \in \mathcal{D}_1 \} \), \( j = 1, 2 \), are Lipschitz, then the first statement in (40) holds. Like (38), under the first condition,

\[
\sup_{\lambda \in \mathcal{D}_1} |\hat{\Omega}_{j,n}(\lambda) - \Omega_{j,n}(\lambda)| \lesssim \sup_{\lambda \in \mathcal{D}_1} |\hat{\Omega}_{j,n}(\lambda) - \Omega_{j,n}(\lambda)| + o_p(1),
\]

where \( \hat{\Omega}_{j,n}(\lambda) - \Omega_{j,n}(\lambda) \) can be rewritten as

\[
\frac{\sqrt{n}}{K \sigma_j(\lambda)} \sum_{k=1}^{K} \left( \frac{1}{\sqrt{m}} \mathcal{G}_n^{(k)} \left\{ \hat{\phi}_j^{(k)}(O; \lambda) - \phi_j(O; \lambda) \right\} + \text{Bias} \left\{ \hat{\phi}_j^{(k)}(O; \lambda) \mid \hat{\eta} - k \right\} \right) \equiv B_{j,n,1}(\lambda) + B_{j,n,2}(\lambda),
\]

Under Assumption 3, \( \hat{\xi}_X \) is the root of the strictly increasing function \( \hat{f}_{X,1}(\xi) \). The proof for Lemma 5 still holds with \( \xi_X \) replaced by \( \hat{\xi}_X \) and \( \mathbb{E}_{Y|X,1}[\cdot] \) replaced by \( \mathbb{E}_{Y|X,1}[\cdot] \), so \( \hat{\phi}_j^{(k)}(\cdot; \lambda) \) is a Lipschitz function of \( \lambda \in \mathcal{D}_1 \). The function class \( \mathcal{F}_{j,n}^{(k)} = \left\{ \hat{\phi}_j^{(k)}(\cdot; \lambda) - \phi(\cdot; \lambda) : \lambda \in \mathcal{D}_j \right\} \) is Lipschitz, which
implies that \( \sup_{\lambda \in \mathcal{D}_1} B_{j,n,1}(\lambda) = o_{p}(1) \) as in (39). In bias bound (31), \( ||\hat{\xi}_X - \xi_X|| \) is the only term depending on \( \lambda \). Then by Assumption 3 and the third condition, 
\[
\sup_{\lambda \in \mathcal{D}_1} \text{Bias} \left\{ \hat{\phi}_j^{(k)}(O; \lambda) \mid \hat{\eta}_k \right\} = o_{p}(1/\sqrt{n}), \forall k \in [K] \Rightarrow \sup_{\lambda \in \mathcal{D}_1} B_{j,n,2}(\lambda) = o_{p}(1),
\]
which proves that \( \sup_{\lambda \in \mathcal{D}_1} |\hat{\Omega}_{j,n}(\lambda) - \Omega_{j,n}(\lambda)| = o_{p}(1) \) in (41) and the second statement in (40). \( \square \)

### L.3 Lipschitz continuity

Let \( \lesssim_p \) denote smaller than up to some \( \mathbb{P} \)-integrable function which can depend on \( O \) but not the sensitivity parameter \( \Gamma \) or \( \lambda \). The following definition of Lipschitz continuity follows from Vaart and Wellner [1996, Chapter 2.7.4] and Kennedy [2016, Section 4.3].

**Lemma 4.** Under Assumptions 1 and 2, \( \mathcal{F} = \{ \phi(O; \Gamma) : \Gamma \in \mathcal{D} = [\Gamma_{\min}, \Gamma_{\max}] \} \) is Lipschitz such that \( |\phi(O; \Gamma) - \phi(O; \tilde{\Gamma})| \lesssim_p |\Gamma - \tilde{\Gamma}| \) for any \( \Gamma, \tilde{\Gamma} \in \mathcal{D} \).

**Proof.** From Theorem 1, the uncentered EIF \( \phi_+ \) can be rewritten as
\[
\phi_+(O; \Gamma) = \frac{ZW(X)}{e(X)} \left\{ [Y - Q(X)] 1_{Y > Q(X)} - \mathbb{E}_{Y|X,1} [Y - Q(X)] 1_{Y > Q(X)} \right\} + [(1 - \Gamma)Z + \Gamma] (\mathbb{E}_{Y|X,1} [Y - Q(X)] 1_{Y > Q(X)}) + Q(X)\mathbb{P}_{Y|X,1} [Y > Q(X)].
\]
Recall from Proposition 1 that \( Q(X) \) is the \( \Gamma/(1+\Gamma) \)-quantile. Let \( \hat{Q}(X) \) denote the \( \hat{\Gamma}/(1+\hat{\Gamma}) \)-quantile. Under Assumption 1, we have \( p_{Y|X,1}(Q(X)) \neq 0 \) for any \( \Gamma \in \mathcal{D} \), then
\[
\frac{dQ(X)}{d\Gamma} = (1 + \Gamma)^{-2}/p_{Y|X,1}(Q(X)) < \infty,
\]
which shows that \( Q(X) \) is a Lipschitz function of \( \Gamma \in \mathcal{D} \). So is \( \mathbb{P}_{Y|X,1} [Y > Q(X)] \) because
\[
|\mathbb{P}_{Y|X,1} [Y > \hat{Q}(X)] - \mathbb{P}_{Y|X,1} [Y > Q(X)]| \lesssim |\hat{Q}(X) - Q(X)|.
\]
Next, it is easy to see that
\[
|[Y - \hat{Q}(X)] 1_{Y > \hat{Q}(X)} - [Y - Q(X)] 1_{Y > Q(X)}| \leq |[Y - \hat{Q}(X)] - [Y - Q(X)]| = |\hat{Q}(X) - Q(X)|,
\]
which shows that \( [Y - Q(X)] 1_{Y > Q(X)} \) and \( \mathbb{E}_{Y|X,1} [Y - Q(X)] 1_{Y > Q(X)} \) are Lipschitz functions of \( \Gamma \). So are \( (1 - \Gamma)Z + \Gamma \) and \( W_+(X) = (1 - \Gamma)e(X) + \Gamma \) because their derivatives with respect to \( \Gamma \) are upper bounded by 1. We have proven that all the components in \( \phi_+(O; \Gamma) \) are Lipschitz. Under Assumption 3, fix \( O = o \), all these components are also bounded functions of \( \Gamma \in \mathcal{D} \). As the products and sums of these bounded Lipschitz functions, \( \phi_+(O; \Gamma) \) is a Lipschitz function of \( \Gamma \).

The uncentered EIF \( \phi_- \) can be rewritten as
\[
\phi_-(O; \Gamma) = \frac{ZW(X)}{e(X)} \left\{ [Y - Q(X)] 1_{Y < Q(X)} - \mathbb{E}_{Y|X,1} [Y - Q(X)] 1_{Y < Q(X)} \right\} + [(1 - \Gamma)Z + \Gamma] (\mathbb{E}_{Y|X,1} [Y - Q(X)] 1_{Y < Q(X)}) + Q(X)\mathbb{P}_{Y|X,1} [Y < Q(X)].
\]
By the proof for \( \phi_+(O; \Gamma) \), we can show that \( \phi_-(O; \Gamma) \) is a Lipschitz function of \( \Gamma \). Then, \( \phi(O; \Gamma) = \phi_+(O; \Gamma) + \phi_-(O; \Gamma) \) is also a Lipschitz function of \( \Gamma \) as required. \( \square \)
Lemma 5. For $j = 1, 2$, $\mathcal{F}_j = \{\phi_j(O; \lambda), \forall \lambda \in \mathcal{D}_1 = [\lambda_{\text{min}}, \lambda_{\text{max}}]\}$ is Lipschitz such that $|\phi_j(O; \lambda) - \phi_j(O; \hat{\lambda})| \leq \rho |\lambda - \hat{\lambda}|$ for any $\lambda, \hat{\lambda} \in \mathcal{D}_1$, under Assumptions 1 and 3.

Proof. The uncentered EIFs $\phi_1(O; \lambda)$ and $\phi_2(O; \lambda)$ are given in Theorem 2. By the definition of $\xi_X$ in Proposition 3, we can view $\lambda = \zeta_1(\xi_X) := 1/E \left[ (\xi_X - Y) \mathbb{1}_{\{Y \leq \xi_X\}} \right]$ as a strictly decreasing function of $\xi_X$, and its inverse $\xi_X = \zeta_1^{-1}(\lambda)$ is a strictly decreasing function of $\xi_X$. Then, $d\zeta_1^{-1}(\lambda)/d\lambda = E \left[ (\zeta_1^{-1}(\lambda) - Y) \mathbb{1}_{\{Y \leq \zeta_1^{-1}(\lambda)\}} \right] / P \{ Y \leq \zeta_1^{-1}(\lambda) \}$, which is negative and bounded for any $\lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}]$. This implies that $\xi_X = \zeta_1^{-1}(\lambda)$ is a Lipschitz function of $\lambda$, and that 

$$|\mathbb{1}_{\{\xi_X - Y \geq \xi_X\}} - \mathbb{1}_{\{\xi_X - Y \geq \xi_X\}}| \leq |\xi_X - Y - (\xi_X - Y)| = |\xi_X - \xi_X| \leq \rho |\lambda - \lambda|.$$ 

Because the product of bounded Lipschitz functions is Lipschitz, $h_*(X, Y) = \lambda(\xi_X - Y) \mathbb{1}_{\{Y \leq \xi_X\}}$, $h_2^2(X, Y)$, $h_*(X, Y)Y$ and their expectations are also Lipschitz functions of $\lambda \in \mathcal{D}_1$. The derivatives of $1/P_{Y|X, 1} \{ Y \leq \xi_X \}$ and $E_{Y|X, 1} \left[ Y \mathbb{1}_{\{Y \leq \xi_X\}} \right]$ with respect to $\xi_X$ are bounded for any $\xi_X \in [\zeta_1^{-1}(\lambda_{\text{max}}), \zeta_1^{-1}(\lambda_{\text{min}})]$. Then their derivatives with respect to $\lambda$ are also bounded by the chain rule and the bounded derivative in (43). Since all the components of $\phi_1(O; \lambda)$ and $\phi_2(O; \lambda)$ are bounded Lipschitz functions of $\lambda \in \mathcal{D}_1$, they are also Lipschitz functions of $\lambda \in \mathcal{D}_1$.

M Additional figure

Figure 5: Sensitivity curves of the ATE of fish consumption on the blood mercury level. This is the confidence interval version of Figure 4 also obtained by multiplier bootstrap.