ON MONOTONE FOURIER COEFFICIENTS OF A FUNCTION
BELONGING TO NIKOL’SKIĬ–BESOV CLASSES

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Abstract. In this paper, necessary and sufficient conditions on terms of
monotone Fourier coefficients for a function to belong to a Nikol’skii–Besov
type class are given.

1. Let \( f \in L^p[0, 2\pi] \), \( 1 < p < \infty \), be a \( 2\pi \)-periodic function having a cosine Fourier
series with monotone coefficients, i.e.

\[
f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx, \quad a_n \downarrow 0.
\]

and \( \omega_k(f, t)_p \) the modulus of smoothness of order \( k \) in \( L^p[0, 2\pi] \) metrics of the
function \( f \), i.e.

\[
\omega_k(f, t)_p = \sup_{|h| \leq t} \| \Delta_k^h f \|_p,
\]

where is

\[
\Delta_k^h f(x) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x + \nu h).
\]

We say that a \( 2\pi \)-periodic function \( f \) belongs to the Nikol’skii–Besov class
\( N(p, \theta, r, \lambda, \varphi) \), \( 1 < p < \infty \), if the following conditions are satisfied

1. \( f \in L^p[0, 2\pi] \);
2. Numbers \( \theta, r, \lambda \) belong to the interval \( (0, \infty) \), and \( k \) is an integer satisfying
   \( k > r + \lambda \);
3. The following inequality holds true
   \[
   \left( \int_{\delta}^{1} t^{-r \theta - 1} \omega_k(f, t)^{\theta} dt + \delta^{\lambda \theta} \int_{\delta}^{1} t^{-(r + \lambda) \theta - 1} \omega_k(f, t)^{\theta} dt \right)^{1/\theta} \leq C \varphi(\delta),
   \]
   while the function \( \varphi \) satisfies the conditions
   
   (4) \( \varphi \) is a non-negative continuous function on \( (0, 1) \) and \( \varphi \neq 0 \);
   (5) For every \( \delta_1, \delta_2 \) such that \( 0 \leq \delta_1 \leq \delta_2 \leq 1 \) holds \( \varphi(\delta_1) \leq C_1 \varphi(\delta_2) \);
   (6) For every \( \delta \) such that \( 0 \leq \delta \leq \frac{1}{2} \) holds \( \varphi(2\delta) \leq C_2 \varphi(\delta) \),

where constant \( C, C_1 \) and \( C_2 \) do not depend on \( \delta_1, \delta_2 \) and \( \delta \).

A more detailed approach to the classes \( N(p, \theta, r, \lambda, \varphi) \) is given in [8] (see also [5]
p. 298)). In our paper we give the necessary and sufficient condition in terms of
monotone Fourier coefficients for a function \( f \in L^p[0, 2\pi] \) to belong to the class
\( N(p, \theta, r, \lambda, \varphi) \).

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1Without mentioning it explicitly, we will consider all the constants positive.
2. Now we formulate our results.

**Theorem 2.1.** A function $f$ belongs to the class $N(p, \theta, r, \lambda, \varphi)$ if and only if

$$
\left( \sum_{\nu=n+1}^{\infty} \omega_k \left( \frac{1}{\nu} \right)^{\theta} \nu^{\theta-1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} \omega_k \left( \frac{1}{\nu} \right)^{\theta} \nu^{(r+\lambda)(\theta-1)} \right)^{1/\theta} \leq C \varphi \left( \frac{1}{n} \right),
$$

(2.1)

where constant $C$ does not depend on $n$.

**Theorem 2.2.** For a function $f \in L_p[0, 2\pi]$, $1 < p < \infty$, such that

$$
f(x) \sim \sum_{\nu=1}^{\infty} a_{\nu} \cos \nu x, \quad a_{\nu} \downarrow 0,
$$

(2.2)

to belong to the class $N(p, \theta, r, \lambda, \varphi)$ it is necessary and sufficient that its Fourier coefficients satisfy the condition

$$
\left( \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{\theta+\theta-\theta/p-1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^{\theta} \nu^{\theta+\theta-\theta/p-1} \right)^{1/\theta} \leq C \varphi \left( \frac{1}{n} \right),
$$

where constant $C$ does not depend on $n$.

**Remark 1.** Put $\varphi(\delta) = \delta^\alpha$, $0 < \alpha < \lambda$, in the definition of the class $N(p, \theta, r, \lambda, \varphi)$, we obtain the Nikol’skiĭ class $H_p^{r+\alpha}$. Thus Theorems 2.1 and 2.2 give the single coefficient condition

$$
a_{\nu} \leq \frac{C}{\nu^{r+\alpha+1-\frac{1}{p}}},
$$

for $f \in H_p^{r+\alpha}$, given in [7] (see also [3]), where the function $f$ is given by (2.2).

**Remark 2.** If $\varphi(\delta) \geq C$, then we obtain the Besov class $B_p^{\theta r}$. Thus Theorems 2.1 and 2.2 give the necessary and sufficient condition

$$
\sum_{\nu=1}^{\infty} a_{\nu}^{\theta} \nu^{\theta-\theta/p-1} < \infty
$$

for $f \in B_p^{\theta r}$, given in [8] (see also [4]), where the function $f$ is given by (2.2).

3. In order to establish our results, we use the following lemmas.

**Lemma 3.1.** Let $0 < \alpha < \beta < \infty$ and $a_{\nu} \geq 0$. The following inequality holds true

$$
\left( \sum_{\nu=1}^{n} a_{\nu}^{\beta} \right)^{1/\beta} \leq \left( \sum_{\nu=1}^{n} a_{\nu}^{\alpha} \right)^{1/\alpha}.
$$

Proof of the lemma is due to Jensen [4, p. 43].

**Lemma 3.2.** Let $\{a_{\nu}\}_{\nu=1}^{\infty}$ be a sequence of non-negative numbers, $\alpha > 0$, $\lambda$ a real number, $m$ and $n$ positive integers such that $m < n$. Then

1. for $1 \leq p < \infty$ the following equalities hold

$$
\sum_{\mu=m}^{n} \mu^{\alpha-1} \left( \sum_{\nu=\mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \leq C_1 \sum_{\mu=m}^{n} \mu^{\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
$$

$$
\sum_{\mu=m}^{n} \mu^{-\alpha-1} \left( \sum_{\nu=\mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \leq C_2 \sum_{\mu=m}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p};
$$

Here and below we assume that the parameters $\theta$, $r$, $\lambda$ and $k$ satisfy the condition [2] and the function $\varphi$ satisfies the conditions [3] of the definition of the class $N(p, \theta, r, \lambda, \varphi).$
(2) for $0 < p \leq 1$ the following equalities hold
\[
\sum_{\mu = m}^{n} \mu^{-\alpha} \left( \sum_{\nu = \mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \geq C_{3} \sum_{\mu = m}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]
\[
\sum_{\mu = m}^{n} \mu^{-\alpha-1} \left( \sum_{\nu = \mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \geq C_{4} \sum_{\mu = m}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]
where constants $C_{1}$, $C_{2}$, $C_{3}$ and $C_{4}$ depend only on numbers $\alpha$, $\lambda$ and $p$, and do not depend on $m$, $n$ as well as on the sequence $\{a_{\nu}\}_{\nu=1}^{\infty}$.

Proof of the lemma is given in [2].

We write $a_{\nu} \downarrow$ if $\{a_{\nu}\}_{\nu=1}^{\infty}$ is a monotone–decreasing sequence of non-negative numbers, i.e. if $a_{\nu} \geq a_{\nu+1} \geq 0$ ($\nu = 1, 2, \ldots$).

**Lemma 3.3.** Let $a_{\nu} \downarrow$, $\alpha > 0$, $\lambda$ a real number, $m$ and $n$ positive integers. Then

1. for $1 \leq p < \infty$, $n \geq 16m$ the following equalities hold
\[
\sum_{\mu = m}^{n} \mu^{-\alpha} \left( \sum_{\nu = \mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \geq C_{1} \sum_{\mu = m}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]
\[
\sum_{\mu = m}^{n} \mu^{-\alpha-1} \left( \sum_{\nu = \mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \geq C_{2} \sum_{\mu = m}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]

2. for $0 < p \leq 1$, $n \geq 4m$ the following equalities hold
\[
\sum_{\mu = 4m}^{n} \mu^{-\alpha} \left( \sum_{\nu = \mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \leq C_{3} \sum_{\mu = m}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]
\[
\sum_{\mu = 4m}^{n} \mu^{-\alpha-1} \left( \sum_{\nu = \mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \leq C_{4} \sum_{\mu = m}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]

where constants $C_{1}$, $C_{2}$, $C_{3}$ and $C_{4}$ depend only on numbers $\alpha$, $\lambda$ and $p$, and do not depend on $m$, $n$ as well as on the sequence $\{a_{\nu}\}_{\nu=1}^{\infty}$.

Proof of the lemma is given in [2].

**Lemma 3.4.** Let $a_{\nu} \downarrow$, $\alpha > 0$, $\lambda$ a real number, $m$ and $n$ positive integers. For $0 < p < \infty$ the following inequalities hold
\[
C_{1} \sum_{\mu = 1}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p} \leq \sum_{\mu = 1}^{n} \mu^{-\alpha} \left( \sum_{\nu = \mu}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \leq C_{2} \sum_{\mu = 1}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]
\[
C_{3} \sum_{\mu = 1}^{n} \mu^{-\alpha} (a_{\mu} \mu^{\lambda+1})^{p} \leq \sum_{\mu = 1}^{n} \mu^{-\alpha-1} \left( \sum_{\nu = 1}^{n} a_{\nu} \nu^{\lambda} \right)^{p} \leq C_{4} \sum_{\mu = 1}^{n} \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^{p},
\]
where constants $C_{1}$, $C_{2}$, $C_{3}$ and $C_{4}$ depend only on numbers $\alpha$, $\lambda$ and $p$, and do not depend on $m$, $n$ as well as on the sequence $\{a_{\nu}\}_{\nu=1}^{\infty}$.

The lemma is also proved in [2].

**Lemma 3.5.** Let $f \in L_{p}[0, 2\pi]$ for a fixed $p$ from the interval $1 < p < \infty$ and let
\[
f(x) \sim \sum_{\nu = 1}^{\infty} a_{\nu} \cos \nu x, \quad a_{\nu} \downarrow 0.
\]
The following inequalities hold
\[
C_1 \frac{1}{n^2} \left( \sum_{\nu=1}^{n} a_{\nu \nu}^p \nu^{(k+1)p-2} \right)^{1/p} + \left( \sum_{\nu=n+1}^{\infty} a_{\nu \nu}^p \nu^{p-2} \right)^{1/p} \leq \omega_k \left( f, \frac{1}{n} \right)_p
\]
\[
\leq C_2 \frac{1}{n^2} \left( \sum_{\nu=1}^{n} a_{\nu \nu}^p \nu^{(k+1)p-2} \right)^{1/p} + \left( \sum_{\nu=n+1}^{\infty} a_{\nu \nu}^p \nu^{p-2} \right)^{1/p},
\]
where constants $C_1$ and $C_2$ do not depend on $n$ and $f$.

The lemma is proved in [9].

4. Now we prove our results.

Proof of Theorem 2.1 Put
\[
I_1 = \int_{0}^{\pi/\nu} t^{-(r+\theta-1)} \omega_k(f, t)_p^p dt, \quad I_2 = \int_{\pi/\nu}^{1} t^{-(r+\lambda+\theta-1)} \omega_k(f, t)_p^p dt.
\]

We have [9] p. 55]
\[
I_1 = \int_{0}^{\pi/\nu} t^{-(r+\theta-1)} \omega_k(f, t)_p^p dt = \sum_{\nu=1+1}^{\infty} \int_{\pi/\nu}^{1} t^{-(r+\theta-1)} \omega_k(f, t)_p^p dt
\]
\[
\leq \sum_{\nu=1+1}^{\infty} \omega_k \left( f, \frac{1}{\nu+1} \right)_p^p \int_{\pi/\nu}^{1} t^{-(r+\theta-1)} dt \leq C_1 \sum_{\nu=1+1}^{\infty} \omega_k \left( f, \frac{1}{\nu+1} \right)_p^p \nu^{r+\theta-1}
\]
and, taking into account properties of modulus of smoothness [10] p. 116],
\[
I_1 \geq \sum_{\nu=n+1}^{\infty} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \int_{\pi/\nu}^{1} t^{-(r+\theta-1)} dt \geq C_2 \sum_{\nu=n+1}^{\infty} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \nu^{r+\theta-1}.
\]

In an analogous way we estimate
\[
I_2 \leq \sum_{\nu=1}^{n} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \int_{\pi/\nu}^{1} t^{-(r+\lambda+\theta-1)} dt \leq C_3 \sum_{\nu=1}^{n} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \nu^{r+\lambda+\theta-1}
\]
and
\[
I_2 \geq \sum_{\nu=1}^{n} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \int_{\pi/\nu}^{1} t^{-(r+\lambda+\theta-1)} dt \geq C_3 \sum_{\nu=1}^{n} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \nu^{r+\lambda+\theta-1}.
\]

Let $f \in N(p, \theta, r, \lambda, \phi)$. For a positive integer $n$ we put $\delta = \frac{1}{n+1}$. Then we have
\[
l^\theta = I_1 + \delta^\theta I_2.
\]

Hence we obtain
\[
J = \left( \sum_{\nu=n+1}^{\infty} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \nu^{r+\theta-1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} \omega_k \left( f, \frac{1}{\nu} \right)_p^p \nu^{r+\lambda+\theta-1} \right)^{1/\theta}
\]
\[
\leq C_6 I \leq C_7 \varphi(\delta) = C_7 \varphi \left( \frac{1}{n+1} \right) \leq C_8 \varphi \left( \frac{1}{n} \right),
\]
which proves inequality (2.1).
Now we suppose that inequality (2.1) holds. For \( \delta \in (0, 1) \) we choose the positive integer \( n \) satisfying \( \frac{1}{n+1} < \delta \leq \frac{1}{n} \). Then, taking into consideration the estimates from above for \( I_1 \) and \( I_2 \) we have

\[
I^\theta = \int_0^{\frac{1}{n+\varepsilon}} t^{-\theta-1} \omega_k(f, t)^{\theta}_p dt + \int_{\frac{1}{n+\varepsilon}}^{\delta} t^{-\theta-1} \omega_k(f, t)^{\theta}_p dt + \delta^{\lambda \theta} \int_{\delta}^{1} t^{-(r+\lambda)\theta-1} \omega_k(f, t)^{\theta}_p dt \leq I_1 + \delta^{\lambda \theta} I_2 \leq C_9 \left( \sum_{\nu=n+1}^{\infty} \omega_k \left( f, \frac{1}{\nu} \right)^{\theta}_p + \nu^{-\lambda \theta} \sum_{\nu=1}^{n} \omega_k \left( f, \frac{1}{\nu} \right)^{\theta}_p \right) \nu^{(r+\lambda)\theta-1}.
\]

Whence

\[
I \leq C_{10} J \leq C_{11} \phi \left( \frac{1}{n} \right) \leq C_{12} \phi \left( \frac{1}{2n} \right) \leq C_{13} \phi(\delta),
\]

implying \( f \in N(p, \theta, r, \lambda, \varphi) \).

Proof of Theorem 2.2. Theorem 2.1 implies that the condition \( f \in N(p, \theta, r, \lambda, \varphi) \) is equivalent to the condition

\[
\sum_{\nu=n+1}^{\infty} \omega_k \left( f, \frac{1}{\nu} \right)^{\theta}_p + \nu^{-\lambda \theta} \sum_{\nu=1}^{n} \omega_k \left( f, \frac{1}{\nu} \right)^{\theta}_p \leq C_1 \phi \left( \frac{1}{n} \right)^{\theta},
\]

where constant \( C_1 \) does not depend on \( n \). Lemma 3.5 yields that the last estimate is equivalent to the estimate \[ p. 31\]

\[
\sum_{\nu=n+1}^{\infty} \nu^{(r-k)\theta-1} \left( \sum_{\mu=1}^{\nu} a^{(k+1)p-2}_\mu \right)^{\theta/p} + \nu^{-\lambda \theta} \sum_{\nu=1}^{n} \nu^{(r-k)\theta-1} \left( \sum_{\mu=1}^{\nu} a^{(k+1)p-2}_\mu \right)^{\theta/p} + \nu^{-\lambda \theta} \sum_{\nu=1}^{n} \nu^{(r-k)\theta-1} \left( \sum_{\mu=1}^{\nu} a^{(k+2)p-2}_\mu \right)^{\theta/p} \leq C_2 \phi \left( \frac{1}{n} \right)^{\theta},
\]

where constant \( C_2 \) does not depend on \( n \). Hence, if we denote the terms on the left-hand side of the inequality by \( J_1, J_2, J_3 \) and \( J_4 \) respectively, then condition \( f \in N(p, \theta, r, \lambda, \varphi) \) is equivalent to the condition

\[
J_1 + J_2 + J_3 + J_4 \leq C_2 \phi \left( \frac{1}{n} \right)^{\theta}. \tag{4.1}
\]

Now we estimate the terms \( J_1, J_2, J_3 \) and \( J_4 \) from below and above by means of expression taking part in the condition of the theorem.

First we estimate \( J_1 \) and \( J_2 \) from below. We have

\[
J_1 = \sum_{\nu=n+1}^{\infty} \nu^{(r-k)\theta-1} \left( \sum_{\mu=1}^{\nu} a^{(k+1)p-2}_\mu \right)^{\theta/p} \geq \sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \left( \sum_{\mu=1}^{\nu} a^{(k+1)p-2}_\mu \right)^{\theta/p}.
\]
For $k - r > 0$, making use of Lemmas 3.2 and 3.3 we obtain
\[ J_1 \geq C_3 \sum_{\nu=4(n+1)}^{\infty} \nu^{-(k-r)}(a_p \nu^{(k+1)p-2} \nu)^{\theta/p} = C_3 \sum_{\nu=4(n+1)}^{\infty} a_p \nu^{\theta+k-\theta/p} - 1. \quad (4.2) \]

In an analogous way, for $r \theta > 0$ we get
\[ J_2 = \sum_{\nu=n+1}^{\infty} \nu^{\theta-1} \left( \sum_{\mu=\nu}^{\infty} a_p \mu^{(k+1)p-2} \right)^{\theta/p} \geq C_4 \sum_{\nu=8(n+1)}^{\infty} a_p \nu^{\theta+k-\theta/p} - 1. \quad (4.3) \]

We estimate the term $J_2$ from above:
\[ J_2 \leq C_5 \sum_{\nu=\lceil \frac{n}{4} \rceil}^{\infty} \nu^{\theta-1} (a_p \nu)^{\theta+k-\theta/p} = C_5 \sum_{\nu=\lceil \frac{n}{4} \rceil}^{\infty} a_p \nu^{\theta+k-\theta/p} - 1. \quad (4.4) \]

For $J_1$ we have
\[ J_1 \leq C_6 \left( \sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \right) \left( \sum_{\mu=n+1}^{\nu} a_p \mu^{(k+1)p-2} \right)^{\theta/p} + \sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \left( \sum_{\mu=1}^{n} a_p \mu^{(k+1)p-2} \right)^{\theta/p}, \]
and applying once more Lemmas 3.2 and 3.3 we obtain
\[ J_1 \leq C_7 \sum_{\nu=\lceil \frac{n}{4} \rceil}^{\infty} a_p \nu^{\theta+k-\theta/p} + n^{-(k-r)\theta} \left( \sum_{\mu=1}^{n} a_p \mu^{(k+1)p-2} \right)^{\theta/p}. \quad (4.5) \]

Put
\[ I_1 = n^{-(k-r)\theta} \sum_{\mu=1}^{n} a_p \mu^{(k+1)p-2}. \]

Then for
\[ I_2 = I_1 n^{(k-r)\theta}, \]

taking into account that $(k+1)p - 2 \geq 0$ and $a_\nu \downarrow 0$ we get
\[ I_2 = \sum_{\mu=1}^{n} a_p \mu^{(k+1)p-2} \leq \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_p \mu^{(k+1)p-2} + a_p \left[ \frac{n}{2} \right] + 1 \sum_{\mu=\left[ \frac{n}{2} \right] + 1}^{n} \mu^{(k+1)p-2} \leq \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_p \mu^{(k+1)p-2} + C_8 n^{(k+1)p-1} a_p \left[ \frac{n}{2} \right] + 1 \leq C_9 \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_p \mu^{(k+1)p-2}. \]
Since $k - r - \lambda > 0$, we have

\[
I_1^{\theta/p} \leq C_{10} n^{-(k-r)\theta} \left( \sum_{\mu=1}^{\left\lfloor \frac{n}{2} \right\rfloor} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}
\]

\[
\leq C_{11} n^{-\lambda \theta} \sum_{\nu=1}^{n} \nu^{-(k-r-\lambda)\theta-1} \left( \sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}
\]

\[
\leq C_{11} n^{-\lambda \theta} \sum_{\nu=1}^{n} \nu^{-(k-r-\lambda)\theta-1} \left( \sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}.
\]

Applying Lemma 3.4 we obtain

\[
I_1^{\theta/p} \leq C_{12} n^{-\lambda \theta} \sum_{\nu=1}^{n} \nu^{-(k-r-\lambda)\theta-1} (a_{\nu}^p \nu^{(k+1)p-2})^{\theta/p}
\]

\[
= C_{12} n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^p \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.
\]

From (4.5) it follows that

\[
J_1 \leq C_{13} \left( \sum_{\nu=1}^{\infty} a_{\nu}^p \nu^{r+\theta-\theta/p-1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^p \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right).
\]

This way, inequalities (4.2), (4.3), (4.4) and (4.6) yield

\[
C_{14} \sum_{\nu=8(n+1)}^{\infty} a_{\nu}^p \nu^{r+\theta-\theta/p-1} \leq J_1 + J_2
\]

\[
\leq C_{15} \left( \sum_{\nu=1}^{\infty} a_{\nu}^p \nu^{r+\theta-\theta/p-1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^p \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right).
\]

Now we estimate $J_3$ and $J_4$. Put

\[
A_1 = n^{\lambda \theta} J_3 = \sum_{\nu=1}^{n} \nu^{(r+\lambda-k)\theta-1} \left( \sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}
\]

and

\[
A_2 = n^{\lambda \theta} J_4 = \sum_{\nu=1}^{n} \nu^{(r+\lambda)\theta-1} \left( \sum_{\mu=\nu}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p},
\]

applying Lemma 3.4 for $r + \lambda - k < 0$ we get

\[
A_1 \leq C_{16} \sum_{\nu=1}^{n} a_{\nu}^p \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.
\]

We estimate $A_2$ in an analogous way:

\[
A_2 \leq C_{17} \left( \sum_{\nu=1}^{n} \nu^{(r+\lambda)\theta-1} \left( \sum_{\mu=\nu}^{n} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right)
\]

\[
+ \sum_{\nu=1}^{n} \nu^{(r+\lambda)\theta-1} \left( \sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p}
\]

\[
\leq C_{18} \left( \sum_{\nu=1}^{n} a_{\nu}^p \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + n^{(r+\lambda)\theta} \left( \sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right).
\]
We estimate the series
\[ B = \left( \sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p}. \]

First let \( \frac{\theta}{p} > 1 \). Applying Hölder inequality we have
\[
\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \leq \left( \sum_{\mu=n+1}^{\infty} \left(a_{\mu}^p \mu^{p-1+rp-p/\theta}\right)^{p/\theta} \right)^{\theta/p} \times \left( \sum_{\mu=n+1}^{\infty} \left(\mu^{-\left(rp-p/\theta+1\right)}\right)^{(\theta-p)/\theta} \right). 
\]

Since \( (rp - \frac{\theta}{p} + 1) \frac{\theta}{p-\theta} = rp\frac{\theta}{p-\theta} + 1 > 1 \), we get
\[
\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \leq C_{19} n^{-rp} \left( \sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{\theta-\theta/p+1} \right)^{p/\theta}.
\]

So, for \( \frac{\theta}{p} > 1 \) we have proved that
\[
B \leq C_{20} n^{-r \theta} \sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{\theta-\theta/p-1}.
\]

Let \( \frac{\theta}{p} \leq 1 \). For given \( n \) we choose the positive integer \( N \) such that \( 2^N \leq n + 1 < 2^{N+1} \). Then we have
\[
B \leq \left( \sum_{\mu=2^N}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \leq \left( \sum_{\nu=N}^{\infty} a_{\nu}^p \sum_{\mu=2^\nu}^{2^{\nu+1}-1} \mu^{p-2} \right)^{\theta/p} \leq C_{21} \left( \sum_{\nu=N}^{\infty} a_{\nu}^p \nu^{p-1} \right)^{\theta/p}.
\]

Making use of Lemma \ref{lemma} we obtain
\[
B \leq C_{21} \sum_{\nu=N}^{\infty} a_{\nu}^p \nu^{\theta-\theta/p} \leq C_{22} \sum_{\nu=N}^{\infty} \sum_{\mu=2^\nu}^{2^{\nu+1}-1} a_{\mu}^p \mu^{\theta-\theta/p-1} \leq C_{22} \sum_{\nu=\left[\frac{n+1}{4}\right]}^{\infty} a_{\nu}^p \nu^{\theta-\theta/p-1} \leq C_{22} \left[ \frac{n+1}{4} \right]^{-r \theta} \sum_{\nu=\left[\frac{n+1}{4}\right]}^{\infty} a_{\nu}^p \nu^{\theta-\theta/p-1}.
\]

Since for \( n \geq 3 \) holds \( \left[\frac{n+1}{4}\right] \geq \frac{n}{7} \), we get
\[
B \leq C_{23} n^{-r \theta} \sum_{\nu=\left[\frac{n+1}{4}\right]}^{\infty} a_{\nu}^p \nu^{\theta-\theta/p-1}.
\]

This way, for \( 0 < \frac{\theta}{p} < \infty \) we proved that
\[
B \leq C_{24} n^{-r \theta} \sum_{\nu=\left[\frac{n+1}{4}\right]}^{\infty} a_{\nu}^p \nu^{\theta-\theta/p-1}.
\]
Hence \((4.10)\) yields
\[
A_2 \leq C_{25} \left( \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1} + n^{\lambda\theta} \sum_{\nu=|\nu|+1}^{\infty} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1} \right).
\]

Now, from \((4.8)\) it follows that
\[
J_3 + J_4 = n^{-\lambda\theta} (A_1 + A_2)
\]
\[
\leq C_{26} \left( n^{-\lambda\theta} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1} \right).
\tag{4.10}
\]

Further, we estimate the series
\[
A_3 = \sum_{\nu=|\nu|+1}^{\infty} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1} = A_4 + \sum_{\nu=n+1}^{\infty} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1},
\]
where is
\[
A_4 = \sum_{\nu=|\nu|+1}^{n} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1} \leq C_{27} a_{\nu}^\theta \nu^{r\theta+\theta/p}
\]
\[
\leq C_{28} n^{-\lambda\theta} \sum_{\nu=1}^{n+1} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1} \leq C_{28} n^{-\lambda\theta} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1}.
\]

Whence
\[
A_3 \leq C_{29} \left( n^{-\lambda\theta} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1} \right).
\tag{4.11}
\]

Making use of \((4.11)\) and \((4.10)\) we have
\[
J_3 + J_4 \leq C_{30} \left( n^{-\lambda\theta} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1} \right).
\]

Hence, applying \((4.11)\) in \((4.7)\) we obtain
\[
J_1 + J_2 + J_3 + J_4
\]
\[
\leq C_{31} \left( n^{-\lambda\theta} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^\theta \nu^{r\theta+\theta-\theta/p-1} \right).
\tag{4.12}
\]

Now we estimate \(A_1\) and \(A_2\) from below. Making use of Lemma 3.3 we get
\[
A_1 \geq C_{32} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1},
\]
and in an analogous way
\[
A_2 \geq \sum_{\nu=1}^{n} \nu^{r(p+\lambda)\theta-1} \left( \sum_{\mu=0}^{n} a_{\nu}^\mu \nu^{\mu-\nu-2} \right)^{\theta/p} \geq C_{33} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1}.
\]

Hence
\[
A_1 + A_2 \geq C_{34} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1}.
\]

This way the following inequality holds
\[
J_3 + J_4 \geq C_{35} n^{-\lambda\theta} \sum_{\nu=1}^{n} a_{\nu}^\theta \nu^{(r+\lambda)\theta-\theta/p-1}.
\]
From (14.7) it follows that
\[ J_1 + J_2 + J_3 + J_4 \]
\[ \geq C_{36} \left( \sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{\theta - \theta/p - 1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^{\theta} (r + \lambda) \nu^{\theta - \theta/p - 1} \right). \] (4.13)

Since
\[ \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{\theta - \theta/p - 1} \leq C_{37} a_{n}^{\theta} n^{\theta - \theta/p} \]
\[ \leq C_{38} n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^{\theta} (r + \lambda) \nu^{\theta - \theta/p - 1} \]
holds, we have
\[ \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{\theta - \theta/p - 1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^{\theta} (r + \lambda) \nu^{\theta - \theta/p - 1} \]
\[ \leq C_{39} \left( \sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{\theta - \theta/p - 1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^{\theta} (r + \lambda) \nu^{\theta - \theta/p - 1} \right). \]

Now, estimates (14.13) and (14.12) imply
\[ C_{40} \left( \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{\theta - \theta/p - 1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^{\theta} (r + \lambda) \nu^{\theta - \theta/p - 1} \right) \]
\[ \leq J_1 + J_2 + J_3 + J_4 \]
\[ \leq C_{41} \left( \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{\theta - \theta/p - 1} + n^{-\lambda \theta} \sum_{\nu=1}^{n} a_{\nu}^{\theta} (r + \lambda) \nu^{\theta - \theta/p - 1} \right). \]

This way we proved that condition (2.1) is equivalent to the condition of the theorem. Since condition (2.1) is equivalent to the condition \( f \in N(p, \theta, r, \lambda, \varphi) \), proof of Theorem 2.2 is completed. \( \square \)

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