Modular proximal optimization for multidimensional total-variation regularization

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Abstract

One of the most frequently used notions of “structured sparsity” is that of sparse (discrete) gradients, a structure typically elicited through Total-Variation (TV) regularizers. This paper focuses on anisotropic TV-regularizers, in particular on \( \ell_p \)-norm weighted TV regularizers for which it develops efficient algorithms to compute the corresponding proximity operators. Our algorithms enable one to scalably incorporate TV regularization of vector, matrix, or tensor data into a proximal convex optimization solvers. For the special case of vectors, we derive and implement a highly efficient weighted 1D-TV solver. This solver provides a backbone for subsequently handling the more complex task of higher-dimensional (two or more) TV by means of a modular proximal optimization approach. We present numerical experiments that demonstrate how our 1D-TV solver matches or exceeds the best known 1D-TV solvers. Thereafter, we illustrate the benefits of our modular design through extensive experiments on: (i) image denoising; (ii) image deconvolution; and (iii) four variants of fused-lasso. Our results show the flexibility and speed our TV solvers offer over competing approaches. To underscore our claims, we provide our TV solvers in an easy to use multi-threaded C++ library (which also aids reproducibility of our results).

1 Introduction

Sparsity impacts the entire data analysis pipeline, touching algorithmic, mathematical modeling, as well as practical aspects. Sparsity is frequently elicited through \( \ell_1 \)-norm regularization [Tibshirani, 1996, Candès and Tao, 2004]. But more refined, “structured” notions of sparsity are also greatly valuable: e.g., groupwise-sparsity [Meier et al., 2008, Liu and Zhang, 2009, Yuan and Lin, 2006, Bach et al., 2011], hierarchical sparsity [Bach, 2010, Mairal et al., 2010], gradient sparsity [Rudin et al., 1992, Vogel and Oman, 1996, Tibshirani et al., 2005], or sparsity over structured ‘atoms’ [Chandrasekaran et al., 2012].

Typical sparse models in machine learning involve problems of the form

\[
\min_{x \in \mathbb{R}^n} \quad \Phi(x) := \ell(x) + r(x),
\]

where \( \ell : \mathbb{R}^n \to \mathbb{R} \) is (usually) a convex and smooth loss function, while \( r : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is (usually) a lower semicontinuous, convex, and nonsmooth regularizer that induces sparsity. For example, if \( \ell(x) = \frac{1}{2} \| Ax - b \|_2^2 \) and \( r(x) = \| x \|_1 \), we obtain the Lasso problem [Tibshirani, 1996]: more complex choices of \( r(x) \) help model richer structure—for a longer exposition of several such choices of \( r(x) \) see e.g., Bach et al. [2011].

We focus on instances of (1.1) where \( r \) is a weighted anisotropic Total-Variation (TV) regularizer\(^1\), which, for a vector \( x \in \mathbb{R}^n \) and a fixed set of weights \( w \geq 0 \) is defined as

\[
r(x) \overset{\text{def}}{=} \text{Tv}_p^w(w; x) \overset{\text{def}}{=} \left( \sum_{j=1}^{n-1} w_j |x_{j+1} - x_j|^p \right)^{1/p} \quad p \geq 1.
\]

\(^1\)We use the term “anisotropic” to refer to the specific TV penalties considered in this paper.
More generally, if $X$ is an order-$m$ tensor in $\mathbb{R}^{n_{i_1} \times \cdots \times n_{i_m}}$ with entries $X_{i_1, i_2, \ldots, i_m}$ ($1 \leq i_j \leq n_j$ for $1 \leq j \leq m$); we define the weighted $m$-dimensional anisotropic TV regularizer

$$
Tv^m_p(W; X) \equiv \sum_{k=1}^{m} \sum_{I_k = \{i_1, \ldots, i_m\} \setminus \{i_k\}} \left( \sum_{j=1}^{n_k-1} w_{I_k,j} |X_{j+1}^{[k]} - X_j^{[k]}|^{p_k} \right)^{1/p_k}, 
$$

(1.3)

where $X_{j}^{[k]} = X_{i_1, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_m}$, $w_{I_k,j} \geq 0$ are weights, and $p \equiv [p_k \geq 1]$ for $1 \leq k \leq m$. If $X$ is a matrix, expression (1.3) reduces to (note, $p, q \geq 1$)

$$
Tv^2_{p,q}(W; X) = \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2-1} w_{1,j} |x_{i,j+1} - x_{i,j}|^p \right)^{1/p} + \sum_{j=1}^{n_2} \left( \sum_{i=1}^{n_1-1} w_{2,i} |x_{i+1,j} - x_{i,j}|^q \right)^{1/q}, 
$$

(1.4)

These definitions look formidable, and are indeed nontrivial; already 2D-TV (1.4) or even the simplest 1D-TV (1.2) are fairly complex, which can complicate the overall optimization problem (1.1). Fortunately, their complexity can be “localized” by wrapping into using the notion of prox-operators [Moreau, 1962], which are now widely used across machine learning [Sra et al., 2011]—see also the classic work on the proximal-point method [Martinet, 1970, Rockafellar, 1976].

The main idea of using prox-operators while solving (1.1) is as follows. Suppose $\Phi$ is a convex lsc function on a set $\mathcal{X} \subset \mathbb{R}^n$. The prox-operator of $\Phi$ is defined as the map

$$
\text{prox}_\Phi : \mathbb{R}^n \mapsto \arg\min_{x \in \mathcal{X}} \left\{ \frac{1}{2} \|x - y\|_2^2 + \Phi(x) \right\} \quad \text{for} \quad y \in \mathbb{R}^n.
$$

(1.5)

This map is used in the proximal-point method to minimize $\Phi(x)$ over $\mathcal{X}$ by iterating

$$
x_{k+1} = \text{prox}_\Phi(x_k), \quad k = 0, 1, \ldots.
$$

(1.6)

Applying (1.6) directly can be impractical: computing $\text{prox}_\Phi$ may be even harder than minimizing $\Phi$! But for problems that have a composite objective $\Phi(x) \equiv \ell(x) + r(x)$, instead of iteration (1.6), closely related proximal splitting methods may be more effective.

Here, one ‘splits’ the optimization into two parts: a part that depends only on $\nabla \ell$ and another that computes $\text{prox}_r$, a step typically much easier than computing $\text{prox}_\Phi$. This idea is realized by the proximal gradient method (also known as ‘forward backward splitting’), which performs a gradient (forward) step followed by a proximal (backward) step to iterate

$$
x_{k+1} = \text{prox}_{\eta_k r}(x_k - \eta_k \nabla \ell(x_k)), \quad k = 0, 1, \ldots.
$$

(1.7)

Numerous other proximal splitting methods exist—please see [Beck and Teboulle, 2009, Nesterov, 2007, Combettes and Pesquet, 2009, Kim et al., 2010, Schmidt et al., 2011] and the references therein for additional examples.

Iteration (1.7) suggests that to profit from proximal splitting we must implement the prox-operator $\text{prox}_r$ efficiently. An additional concern is whether the overall proximal splitting algorithm requires exact computation of $\text{prox}_r$, or moderately inexact computations are sufficient. This concern is very practical: rarely does $r$ admit an exact algorithm for computing $\text{prox}_r$. Fortunately, proximal methods easily admit inexactness, e.g., [Schmidt et al., 2011, Salzo and Villa, 2012, Sra, 2012], which allows approximate prox-operators (as long as the approximation is sufficiently accurate).

### 1.1 Contributions

In light of the above background and motivation, we focus on computing prox-operators for TV regularizers, for which we develop, analyze, and implement a variety of fast algorithms. The ensuing contributions of this paper are summarized below.
• Efficient prox-operators for $\ell_p$-norm ($p \geq 1$) weighted 1D-TV. In particular,
  – For $p = 1$ we derive a new algorithm for weighted TV. This algorithm not only matches Condat’s remarkably fast method [Condat, 2012] for unweighted TV, but provides the fastest (to our knowledge) method for weighted TV; moreover, it has an intuitive derivation.
  – For $p = 1$ we also derive a projected-Newton method, which, though slower than our new algorithm, is of instructive value while also providing a practical framework for some subroutines needed for handling the difficult $p > 1$ case. Moreover, our derivations may be useful to readers seeking efficient prox-operators for other problems with structures similar to TV (e.g., $\ell_1$-trend filtering [Kim et al., 2009, Tibshirani, 2014]); in fact our projected-Newton ideas provided a basis for the recent fast “group fused-lasso” algorithms of [Wytock et al., 2014].
  – For $p = 2$, we present a specialized Newton method based on root-finding, while for the general $p > 1$ case we describe both “projection-free” and projection based first-order methods.

• Scalable algorithms based on proximal-splitting techniques for computing 2D (1.4) and higher-D TV (1.3) prox-operators: our algorithms are designed to reuse our fast 1D-TV routines and to exploit the massive parallelization inherent in matrix and tensor TV problems.

• The practically most important contribution of our paper is a well-tuned, multi-threaded open-source C++ implementation of our algorithms.\(^2\)

To complement our algorithmic contribution, we illustrate several applications of TV prox-operators: (i) to image and video denoising; (ii) as subroutines in some image deconvolution solvers; and (iii) as regularizers in four variants of fused-lasso.

**Note:** Our paper strives to support reproducible research. Given the vast attention that TV problems have received in the literature, we find it is valuable to both users of TV and other researchers to have access to our code, datasets, and scripts, to independently verify our claims, if desired.\(^4\)

### 1.2 Related work

The literature on TV is dauntingly large, so we will not attempt any pretense at comprehensiveness. Instead, we mention some of the most directly related work for placing our contributions in perspective.

We focus on anisotropic-TV (in the sense of [Bioucas-Dias and Figueiredo, 2007]), in contrast to isotropic-TV [Rudin et al., 1992]. Like isotropic TV, anisotropic-TV is also amenable to efficient optimization, and proves quite useful in applications seeking parameters with sparse (discrete) gradients.

The anisotropic TV regularizers $T_{1D}$ and $T_{2D}$ arise in image denoising and deconvolution [Dahl et al., 2010], in the fused-lasso [Tibshirani et al., 2005], in logistic fused-lasso [Kolar et al., 2010], in change-point detection [Harchaoui and Lévy-Leduc, 2010], in graph-cut based image segmentation [Chambolle and Darbon, 2009], in submodular optimization [Jegelka et al., 2013]; see also the related work in [Vert and Bleakley, 2010]. This broad applicability and importance of anisotropic TV is the key motivation towards developing carefully tuned proximity operators.

Isotropic TV regularization arises frequently in image denoising and signal processing, and quite a few TV-based denoising algorithms exist [Zhu and Chan, 2008, see e.g.]. Most TV-based methods, however, use the standard ROF model [Rudin et al., 1992]. There are also several methods tailored to anisotropic TV, e.g., those developed in the context of fused-lasso [Friedman et al., 2007, Liu et al., 2010], graph-cuts [Chambolle and Darbon, 2009], ADMM-style approaches [Combettes and Pesquet, 2009, Wahlberg et al., 2012], or fast methods based on dynamic programming [Condat, 2012, Johnson, 2013]. However, it seems

\(^2\)See [http://arantxa.ii.uam.es/~gaa/software.html](http://arantxa.ii.uam.es/~gaa/software.html)

\(^3\)For instance, the recent paper of Jegelka et al. [2013] relies on our toolbox for achieving several of its state-of-the-art results on fast submodular function minimization.

\(^4\)This material shall be made available at: [http://suvrit.de/work/soft/tv.html](http://suvrit.de/work/soft/tv.html)
that anisotropic TV norms other than $\ell_1$ has not been studied much in the literature, although recognized as a form of Sobolev semi-norms [Pontow and Scherzer, 2009].

Previously, Vogel and Oman [1996] suggested a Newton approach for TV; they smoothed the objective, but noted that it leads to numerical difficulties. In contrast, we directly solve the nonsmooth problem. Recently, Liu et al. [2010] presented tuned algorithms for $\text{TV}_1$-proximity based on a careful “restart” heuristic; their methods show strong empirical performance but do not extend easily to higher-D TV. Our Newton-type methods outperform the tuned methods of [Liu et al., 2010], and fit nicely in a general algorithmic framework that allows tackling the harder two- and higher-D TV problems. The gains obtained are especially prominent for 2D fused-lasso, for which previously Friedman et al. [2007] presented a coordinate descent approach that turns out to be slower.

Recently, Goldstein T. [2009] presented a so-called “Split-Bregman” (SB) method that applies directly to 2D-TV. It turns out that this method is essentially a variant of the well-known ADMM method. In contrast to our 2D approach, the SB strategy followed by Goldstein T. [2009] is to rely on $\ell_1$-soft thresholding substeps instead of 1D-TV substeps. From an implementation viewpoint, the SB approach is somewhat simpler, but not necessarily more accurate. Incidentally, sometimes such direct ADMM approaches turn out to be less effective than ADMM methods that rely on more complex 1D-TV prox-operators [Ramdas and Tibshirani, 2014].

For 1D-TV, there exist several direct methods that are exceptionally fast. We treat the 1D-TV problem in detail in Section 2, hence refer the reader to that section for discussion of closely related work on fast 1D-TV solvers. We note here, however, that in contrast to all previous fast solvers, our 1D-TV solver allows weights, a capability that can be very important in applications [Jegelka et al., 2013].

It is tempting to assume that existing isotropic algorithms, such as the state-of-the-art PDHG [Zhu and Chan, 2008] method, can be easily adapted to our problems. But this is not true. PDHG requires fine-tuning of its parameters, and to obtain fast performance its authors apply non-trivial adaptive rules that fail on our anisotropic model. In stark contrast, our solvers do not require any parameter tuning; the default values work across a wide spectrum of inputs.

It is worth highlighting that it is not just proximal solvers such as FISTA [Beck and Teboulle, 2009], SpaRSA [Wright et al., 2009], SALSA [Afonso et al., 2010], TwiIST [Bioucas-Dias and Figueiredo, 2007], TRIP [Kim et al., 2010], that can benefit from our fast prox-operators. All other 2D and higher-D TV solvers, e.g., [Yang et al., 2013], as well as the recent ADMM based trend-filtering solvers of Tibshirani [2014] immediately benefit (and in fact, even gain the ability to solve weighted versions of their problems, a feature that they currently lack).

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2 Prox Operators for 1D-TV

We begin with the 1D-TV problem (1.2), for which we develop below carefully tuned algorithms. This tuning pays off: our weighted-TV algorithm offers a fast, robust, and low-memory (in fact, in place) algorithm, which is not only of independent value, but is also an ideal building block for scalably solving 2D- and higher-D TV problems.

To express (1.2) more compactly we introduce the usual differencing matrix

\[ D = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ \\ \vdots \\ -1 & 1 \end{pmatrix} \in \mathbb{R}^{(n-1) \times n}, \]

using which, (1.2) becomes \( T_{\ell_p}^{1D}(x) = \| Dx \|_p \). The associated prox-operator solves

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad \frac{1}{2} \| x - y \|_2^2 + \lambda \| Dx \|_p \\
\text{s.t.} & \quad \| u \|_q \leq \lambda,
\end{align*}
\]

(2.1)

The presence of \( D \) and well as the \( \ell_p \)-norm makes (2.1) fairly challenging to solve; even the simpler variants for \( p \in \{ 1, 2, \infty \} \) are nontrivial. When solving (2.1) we may equivalently consider its dual form (obtained using Proposition A.4 and formula (A.3), for instance):

\[
\begin{align*}
\min_{u \in \mathbb{R}^{n-1}} & \quad \frac{1}{2} \| D^T u \|_2^2 - u^T Dy, \\
\text{s.t.} & \quad \| u \|_q \leq \lambda,
\end{align*}
\]

(2.2)

where \( p^{-1} + q^{-1} = 1 \) and \( \| \cdot \|_q \) is the dual-norm of \( \| \cdot \|_p \). If \( u^* \) is the dual solution, then using KKT conditions we can recover the primal solution \( x^* \): indeed, using \( \nabla_x L = 0 \), we obtain \( \nabla_x \left( \frac{1}{2} \| x - y \|_2^2 + \lambda \| z \|_p + u^T (Dx - z) \right) = 0 \), from which we obtain the primal solution

\[ x^* = y - D^T u^*. \]

If we have a feasible dual variable \( u \), we can compute a candidate primal solution \( x = y - D^T u \); the associated duality gap is given by (noting that we wrote (2.2) as a minimization):

\[
\begin{align*}
\text{gap}(x, u) & \overset{\text{def}}{=} \frac{1}{2} \| x - y \|_2^2 + \lambda \| Dx \|_p - \left( -\frac{1}{2} \| D^T u \|_2^2 - u^T Dy \right) \\
& = \lambda \| Dx \|_p - u^T Dx.
\end{align*}
\]

(2.3)

The duality gap provides a handy stopping criterion for many of our algorithms.

With these basic ingredients we are now ready to present specialized 1D-TV algorithms for different choices of \( p \). The choices \( p \in \{ 1, 2 \} \) are the most common, so we treat them in greater detail—especially the case \( p = 1 \), which is the most important of all the TV problems studied below.
2.1 TV-L1: Proximity for Tv\textsuperscript{1D}

In this section we present two efficient methods for computing the Tv\textsuperscript{1D} prox-operator. The first method employs a carefully tuned “taut-string” approach, while the second method adapts the classical projected-Newton (PN) method [Bertsekas, 1982] to TV. We first derive our methods for unweighted TV, before presenting details for the elementwise weighted TV problem (2.8). The previous fastest methods handle only unweighted-TV. It is is nontrivial to extend them to handle weighted-TV, a problem that is crucial to several applications, e.g., segmentation [Chambolle and Darbon, 2009] and certain submodular optimization problems Jegelka et al. [2013].

Taut-string approaches are already known to provide extremely efficient solutions to unweighted TV-L1, as seen in [Condat, 2012]. In addition to our optimized taut-string approach for weighted-TV, we also present details of a PN approach for its instructive value, and also because it provides key subroutines for solving \( p > 1 \) problems. Moreover, our derivation may be helpful to readers seeking to implement efficient prox-operators for other problems that have structure similar to TV, for instance \( \ell_1 \)-trend filtering [Kim et al., 2009, Tibshirani, 2014]. Indeed, the PN approach proved foundational for the recent fast “group fused-lasso” algorithms of [Wytock et al., 2014].

Before proceeding we note that other than [Condat, 2012], other efficient methods to address unweighted Tv\textsuperscript{1D} proximity have been proposed. Johnson [2013] shows how solving Tv\textsuperscript{1D} proximity is equivalent to computing the data likelihood of a specific Hidden Markov Model (HMM), which suggests a dynamic programming approach based on the well-known Viterbi algorithm for HMMs. The resulting algorithm is very competitive, and guarantees an overall \( O(n) \) performance while requiring approximately \( 8n \) storage. We will also consider this algorithm in our experimental comparison in §4. Another, roughly similar approach to the proposed projected-Newton method was already presented in Ito and Kunisch [1999], which by means of an augmented Langrangian method a dual formulation is obtained and solved following a Newton–like strategy, though with the additional burden of requiring manual setting of a number of optimization parameters to ensure convergence.

2.1.1 Optimized taut-string method for Tv\textsuperscript{1D}

While taut-string methods seem to be largely unknown in machine learning, they have been widely applied in the field of statistics—see e.g. [Grasmair, 2007, Davies and Kovac, 2001]. Even the recent efficient method of Condat [2012]—though derived using KKT conditions of the dual problem (2.2) (for \( p = 1, q = \infty \))—can be given a taut-string interpretation.

Below we introduce the general idea behind the taut-string approach and present our optimized version of it. Surprisingly, the resulting algorithm is equivalent to the fast algorithm of [Condat, 2012], though now with a clearer interpretation based on taut-strings, which proves key to obtaining a similarly fast method for weighted-TV.

For TV-L1 the dual problem (2.2) becomes

\[
\min_u \frac{1}{2} \| D^T u \|_2^2 - u^T D y, \text{ s.t. } \| u \|_\infty \leq \lambda, \quad (2.4)
\]

whose objective can be (i.e., without changing the argmin) replaced by \( \| D^T u - y \|_2^2 \), which upon using the structure of matrix \( D \) unfolds into

\[
\left( (u_1 - y_1)^2 + \sum_{i=2}^{n-1} (-u_{i-1} + u_i - y_i)^2 + (-u_{n-1} - y_n)^2 \right)^2.
\]

Introducing the fixed extreme points \( u_0 = u_n = 0 \), we can thus replace the dual (2.4) by

\[
\min_u \sum_{i=1}^n (y_i - u_i + u_{i-1})^2, \text{ s.t. } \| u \|_\infty \leq \lambda, \quad u_0 = u_n = 0. \quad (2.5)
\]
Now we perform a change of variables by defining the new set of variables \( s = r - u \), where \( r_i := \sum_{k=1}^{i} y_k \) is the cumulative sum of input signal values. Thus, (2.5) becomes

\[
\min_{s} \sum_{i=1}^{n} (y_i - r_i + s_i + r_{i-1} - s_{i-1})^2, \quad \text{s.t.} \quad \|s - r\|_{\infty} \leq \lambda, \quad r_0 - s_0 = r_n - s_n = 0,
\]

which upon simplification becomes

\[
\min_{s} \sum_{i=1}^{n} (s_i - s_{i-1})^2, \quad \text{s.t.} \quad \|s - r\|_{\infty} \leq \lambda, \quad s_0 = 0, \quad s_n = r_n. \tag{2.6}
\]

Now the key trick: Problem (2.6) can be shown to share the same optimum as

\[
\min_{s} \sum_{i=1}^{n} \sqrt{1 + (s_i - s_{i-1})^2}, \quad \text{s.t.} \quad \|s - r\|_{\infty} \leq \lambda, \quad s_0 = 0, \quad s_n = r_n. \tag{2.7}
\]

A proof of this relationship may be found in [Steidl et al., 2005]; for completeness and also because this proof will serve us for the weighted TV\(_{1D}\) variant, we include an alternative proof in Appendix C.

The name “taut-string” is explained as follows. The objective in (2.7) can be interpreted as the euclidean length of a polyline through the points \((i, s_i)\). Thus, (2.7) seeks the minimum length polyline (the taut string) crossing a tube of height \(\lambda\) with center the cumulative sum \(r\), and with the fixed endpoints \((s_0, s_n)\). An example illustrating this is shown in Figure 1.

Once the taut string is found, the solution for the original proximity problem can be recovered by observing that

\[
s_i - s_{i-1} = r_i - u_i - (r_{i-1} - u_{i-1}) = y_i - u_i + u_{i-1} = x_i,
\]

where we used the primal-dual relation \(x = y - D^T u\). Intuitively, the above argument shows that the solution to the TV-L1 proximity problem is obtained as the discrete gradient of the taut string, or as the slope of its segments.

**Figure 1:** Example of the taut string method. The cumulative sum \(r\) of the input signal values \(y\) is shown as the dashed line; the black dots mark the points \((i, r_i)\). The bottom and top of the \(\lambda\)-width tube are shown in red. The taut string solution \(s\) is shown as a blue line.
can be identified and added to the solution. The procedure is then restarted at the end of the identified segment, and iterated until all taut string segments have been obtained.

In the worst-case, the identification of a single segment can require analyzing on the order of \( n \) points in the signal, and the taut string may contain up to \( n \) segments. Thus, the worst-case complexity of the taut-string approach is \( O(n^2) \). However, as also noted in [Condat, 2012], in practice the performance is close to the best case \( O(n) \). But in [Condat, 2012], it is noted that maintaining the minorant and majorant functions in memory is inefficient, whereby a taut-string approach is viewed as potentially inferior to their proposed method. Surprisingly, the geometry-aware construction of taut-strings that we propose below, yields in a method analogous to [Condat, 2012], both in space and time complexity. Owing to its intuitive derivation, our geometry-aware ideas extend naturally to weighted-TV—in our opinion, vastly more easily than the KKT-based derivations in [Condat, 2012].

Details. Our geometry-aware method requires only the following bookkeeping variables.

1. \( i_0 \): index of the current segment start
2. \( \delta \): slope of the line joining segment start with majorant at the current point
3. \( \hat{\delta} \): slope of the line joining segment start with minorant at the current point
4. \( \bar{h} \): height of majorant w.r.t. the \( \lambda \)-tube center
5. \( \bar{h} \): height of minorant w.r.t. \( \lambda \)-tube center
6. \( \bar{i} \): index of last point where \( \delta \) was updated—potential majorant break point
7. \( \bar{i} \): index of last point where \( \hat{\delta} \) was updated—potential minorant break point.

Figure 2 gives a geometric interpretation of these variables; we use these variables to detect minorant-majorant intersections, without the need to compute or store them explicitly.

Algorithm 1 presents full pseudocode of our optimized taut-string method. Starting from the initial point, the method tries to build a single segment traversing the \( \lambda \)-tube up to the end point \( r_n \). In this way, at each iteration the method steps one point further through the tube, updating the maximum/minimum slope of each such hypothetical segment (\( \delta \)) as well as the minimum/maximum height that each segment could have at the current point (\( \bar{h} \)). If at some point the tube has limits such that the maximum height \( \bar{h} \) falls below the tube bottom, or such that the minimum height \( \bar{h} \) grows above the tube ceiling, then it
Algorithm 1 Optimized taut string algorithm for TV-L1-proximity

1: Initialize $i = \bar{i} = \bar{j} = h = \bar{h} = 0$, $\delta = \infty$, $\bar{\delta} = -\infty$.
2: while $i < n$ do
3:    Find tube height: $\tilde{\lambda} = \lambda$ if $i < n - 1$, else $\tilde{\lambda} = 0$
4:    Update minorant height following current slope: $\bar{h} = \bar{h} + \bar{\delta} - y_i$.
5:    /* Check for ceiling violation: minorant is above tube ceiling */
6:      if $\bar{h} > \tilde{\lambda}$ then
7:         Build valid segment up to last minorant breaking point: $x_{i_0+1:t_0} = \delta$.
8:         Start new segment after break: $(i_0, i) = \bar{i}, i = i_0 + 1, \bar{h} = \bar{h} = \lambda, \bar{\delta} = \infty, \bar{\delta} = -\infty$
9:         continue
10: end if
11: Update majorant height following current slope: $\bar{h} = \bar{h} + \bar{\delta} - y_i$.
12: /* Check for bottom violation: majorant is below tube bottom */
13:      if $\bar{h} < -\tilde{\lambda}$ then
14:         Build valid segment up to last majorant breaking point: $x_{i_0+1:t_0} = \delta$.
15:         Start new segment after break: $(i_0, i) = \bar{i}, i = i_0 + 1, \bar{h} = \bar{h} = \lambda, \bar{\delta} = \infty, \bar{\delta} = -\infty$
16:         continue
17: end if
18: /* Check if majorant height is above the ceiling */
19:      if $\bar{h} \geq \tilde{\lambda}$ then
20:         Correct slope: $\delta = \delta + \frac{\tilde{\lambda} - \bar{h}}{i - i_0}$
21:         We are touching the majorant: $\bar{h} = \tilde{\lambda}$
22:         This is a possible majorant breaking point: $\bar{i} = i$
23: end if
24: /* Check if minorant height is below the bottom */
25:      if $\bar{h} \leq -\tilde{\lambda}$ then
26:         Correct slope: $\delta = \delta + \frac{-\tilde{\lambda} - \bar{h}}{i - i_0}$
27:         We are touching the minorant: $\bar{h} = -\tilde{\lambda}$
28:         This is a possible minorant breaking point: $\bar{i} = i$
29: end if
30: end while
31: Build last valid segment: $x_{i_0+1:n} = \delta$.

is not possible to continue building the segment. In this event, the segment under construction must be broken at some point, and a new segment must be started after it. It can be shown that the segment must be broken at the point where it last touched the tube limits: either at the bottom limit if the segment has grown above the tube, or at the top limit if the segment has grown under the tube. With this criterion the segment is fixed from the starting point to such a breaking point and the procedure is restarted to build the next segment. Figure 3 visually shows an example of the algorithm’s workings.

Height variables. It should be noted that in order to implement the method described above, the height variables $h$ are not strictly necessary as they can be obtained from the slopes $\delta$. However, explicitly including them leads to efficient updating rules at each iteration, as we show below.

Suppose we are updating the heights and slopes from their estimates at step $i - 1$ to step $i$. Updating the heights is immediate given the slopes, as we have that

$$h_i = h_{i-1} + \delta - y_i,$$

which is to say, since we are following a line with slope $\delta$, the change in height from one step to the next is given by precisely such slope. Note, however, that in our algorithm we do not compute absolute heights but instead relative heights with respect to the $\lambda$–tube center. Therefore we need to account for
the change in the tube center between steps \(i - 1\) and \(i\), which is given by \(r_i - r_{i-1} = y_i\). This completes the update, which is shown in Algorithm 1 as lines 4 and 11.

Of course it might well happen that the new height \(h\) runs over or under the tube. This would mean that we cannot continue using the current slope in the majorant or minorant, and a recalculation is needed, which again can be done efficiently by now using the height information. Let us assume, without loss of generality, that the starting index of the current segment is 0 and the absolute height of the starting point of the segment is given by \(\alpha\). Then, for adjusting the majorant slope \(\bar{\delta}_i\) so that it touches the tube ceiling at the current point we note that

\[
\bar{\delta}_i = \frac{\lambda + r_i - \alpha}{i} = \frac{\lambda + (\bar{h}_i - \bar{h}_i) + r_i - \alpha}{i},
\]

where we have also added and subtracted the current value of \(\bar{h}_i\). Observe that this value was computed using the estimate of the slope so far \(\bar{\delta}_{i-1}\), so we can rewrite it as the projection of the initial point in the segment following such slope, that is, as \(\bar{h}_i = i\bar{\delta}_i - r_i + \alpha\). Doing so for one of the added heights \(\bar{h}_i\) produces

\[
\bar{\delta}_i = \frac{\lambda + (i\bar{\delta}_{i-1} - r_i + \alpha) - \bar{h}_i + r_i - \alpha}{i} = \bar{\delta}_{i-1} + \frac{\lambda - \bar{h}_i}{i},
\]

which generates a simple updating rule. A similar derivation holds for the minorant. The resultant updates are included in the algorithm in lines 20 and 26. After recomputing this slope we need to adjust the corresponding height back to the tube: since the heights are relative to the tube center we can just set \(\bar{h} = \lambda\), \(\bar{h} = -\lambda\); this is done in lines 21 and 27.

Notice also that the special case of the last point in the tube where the taut-string must meet \(s_n = r_n\) is handled by line 3, where \(\lambda\) is set to 0 at such a point to enforce this constraint. In practice, it is more efficient to add a separate step to the algorithm’s main loop for handling this special case, as was already observed in Condat [2012]. Overall, one iteration of the method is very efficient, as mostly just additions are performed at every iteration. Moreover, no additional memory is required beyond the constant number of bookkeeping variables, and in-place updates are also possible due to the fact that \(y_i\) values for already fixed sections of the taut-string are not required again, so the output \(x\) and the input \(y\) can both refer to the same memory locations.

### 2.1.2 Optimized taut–string method for weighted TV\(_1\)\(^{\text{LD}}\)

Several applications TV require penalizing the discrete gradients individually, which can be done by solving the weighted TV-L1 problem

\[
\min_x \frac{1}{2} \|x - y\|_2^2 + \sum_{i=1}^{n-1} w_i |x_{i+1} - x_i|,
\]

where the weights \(w_i\) are all positive. To solve (2.8) using our optimized taut-string approach, we again begin with its dual

\[
\min_u \frac{1}{2} \|D^T u\|_2^2 - u^T D y \quad \text{s.t.} \quad |u_i| \leq w_i, \quad 1 \leq i < n.
\]

Then, we repeat the derivation of the unweighted taut-string method with a few key modifications. More precisely, we transform (2.9) by introducing \(u_0 = u_n = 0\) to obtain

\[
\min_u \sum_{i=1}^{n} (y_i - u_i + u_{i-1})^2 \quad \text{s.t.} \quad |u_i| \leq w_i, \quad 1 \leq i < n.
\]

Next, we perform the change of variable \(s = r - u\), where \(r_i := \sum_{k=1}^{i} y_k\), and consider

\[
\min_s \sum_{i=1}^{n} (s_i - s_{i-1})^2 \quad \text{s.t.} \quad |s_i - r_i| \leq w_i, \quad 1 \leq i < n, \quad s_0 = 0, \quad s_n = r_n.
\]
Figure 3: Example of the evolution of the proposed optimized taut string method. The majorant/minorant slopes are shown in green/blue colors, respectively. At step (1) the algorithm is initialized. Steps (2) to (4) successfully manage to update majorant/minorant slopes respecting the tube limits. At step (5), however, the minorant grows over the tube ceiling, and so it is necessary to break the segment. Since the one under violation is the minorant, it is broken at the point where it last touched the tube bottom. A segment piece is thus fixed at step (6), and the algorithm is restarted at its end. The slopes are updated at step (7), though at step (8) once again the minorant grows over the tube ceiling. Hence, at step (9) a breaking point is introduced again and the algorithm is restarted. Following this, step (10) manages to update majorant/minorant slopes up to the end of the tube, and so at step (11) the final segment is built using the (now equal) slopes.

Finally, applying Theorem C.1 we obtain the equivalent weighted taut-string problem

$$\min_{s} \sum_{i=1}^{n} \sqrt{1 + (s_i - s_{i-1})^2} \quad \text{s.t.} \ |s_i - r_i| \leq w_i, \ 1 \leq i < n, \ s_0 = 0, \ s_n = r_n. \quad (2.10)$$

Problem (2.10) differs from its unweighted counterpart (2.7) in the constraints $|s_i - r_i| \leq w_i$ ($1 \leq i < n$), which allow different weights for each component instead of using the same value $\lambda$. Our geometric intuition also carries over to the weighted problem, albeit with a slight modification: the tube we are
trying to traverse now has varying widths at each step (instead of fixed \( \lambda \) width)—Figure 4 illustrate this idea.

Figure 4: Example of the weighted taut string method with \( \mathbf{w} = (1.35, 3.03, 0.73, 0.06, 0.71, 0.20, 0.12, 1.49, 1.41) \). The cumulative sum \( \mathbf{r} \) of the input signal values \( \mathbf{y} \) is shown as the dashed line, with the black dots marking the points \((i, r_i)\). The bottom and ceiling of the tube are shown in red, which vary in width at each step following the weights \( w_i \). The weighted taut string solution \( s \) is shown as a blue line.

As a consequence of the above derivation and intuition, we ultimately obtain a weighted taut-string algorithm that can handle varying tube width by suitably modifying Algorithm 1. In particular, when checking ceiling/floor violations as well as when checking slope recomputations and restarts, we must account for varying tube heights. Algorithm 2 presents the precise modifications that we must make to Algorithm 1 to handle weights.

Algorithm 2 Modified lines for weighted version of Algorithm 1

3: Find tube height: \( \bar{\lambda} = \lambda_{i+1} \) if \( i < n - 1 \), else \( \bar{\lambda} = 0 \)

8: Start new segment after break: \( (i_0, i) = i, i = i_0 + 1, \bar{h} = h = -w_i, \bar{\delta} = \infty, \delta = -\infty \)

15: Start new segment after break: \( (i_0, i) = i, i = i + 1, \bar{h} = h = w_i, \bar{\delta} = \infty, \delta = -\infty \)

2.1.3 Projected-newton for weighted TV\(_1\)D

In this section we present details of a projected-Newton (PN) approach to solving the weighted-TV problem (2.8). Although, our optimized taut-string approach is empirically superior to our PN approach, as mentioned before we still present its details. These details prove useful when developing subroutines for handling \( \ell_p \)-norm TV prox-operators, but perhaps their greatest use lies in presenting a general method that could be applied to other problems that have structures similar to TV, e.g., group total-variation [Alaiz et al., 2013, Wytock et al., 2014] and \( \ell_1 \)-trend filtering [Kim et al., 2009, Tibshirani, 2014].

The weighted-TV dual problem (2.9) is a bound-constrained QP, so it could be solved using a variety of methods such as TRON [Lin and Moré, 1999], L-BFGS-B [Byrd et al., 1994], or projected-Newton (PN) [Bertsekas, 1982]. Obviously, these methods will be inefficient if invoked off-the-shelf; exploitation of problem structure is a must for solving (2.9) efficiently. PN lends itself well to such structure exploitation; we describe the details below.
PN runs iteratively in three key steps: first it identifies a special subset of active variables and uses these to compute a reduced Hessian. Then, it uses this Hessian to scale the gradient and move in the direction opposite to it, damping with a stepsize, if needed. Finally, the next iterate is obtained by projecting onto the constraints, and the cycle repeats. PN can be regarded as an extension of the gradient-projection method (GP, Bertsekas [1999]), where the components of the gradient that make the updating direction infeasible are removed; in PN both the gradient and the Hessian are reduced to guarantee this feasibility.

At each iteration PN selects the active variables

\[ I := \{ i \mid (u_i = -w_i \text{ and } [\nabla \phi(u)]_i > \epsilon) \text{ or } (u_i = w_i \text{ and } [\nabla \phi(u)]_i < -\epsilon) \} \]  (2.11)

where \( \epsilon \geq 0 \) is small scalar. This corresponds to the set of variables at a bound, and for which the gradient points inside the feasible region; that is, for these variables to further improve the objective function we would have to step out of bounds. It is thus clear that these variables are of no use for this iteration, so we define the complementary set \( \bar{I} := \{ 1, \ldots, n \} \setminus I \) of indices not in \( I \), which are the variables we are interested in updating. From the Hessian \( H = \nabla^2 \phi(u) \) we extract the reduced Hessian \( H_{\bar{I}} \) by selecting rows and columns indexed by \( I \), and in a similar way the reduced gradient \( [\nabla \phi(u)]_{\bar{I}} \). Using these we perform a Newton–like “reduced” update in the form

\[ u_{\bar{I}} \leftarrow P(u_{\bar{I}} - \alpha H_{\bar{I}}^{-1} [\nabla \phi(u)]_{\bar{I}}), \]  (2.12)

where \( \alpha \) is a stepsize, and \( P \) denotes projection onto the constraints, which for box–constraints reduces to simple element–wise projection. Note that only the variables in the set \( \bar{I} \) are updated in this iterate, leaving the rest unchanged. While such update requires computing the inverse of the reduced Hessian \( H_{\bar{I}} \), which in the general case can amount to computational costs in the \( O(n^3) \) order, we will see now how exploiting the structure of the problem allows us to perform all the steps above efficiently.

First, observe that for (2.9) the Hessian is

\[ H = DD^T = \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& -1 & 2 & \\
& & & & \\
& & \ddots & -1 & \\
& & & & & 2
\end{pmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}. \]

Next, observe that whatever the active set \( I \), the corresponding reduced Hessian \( H_{\bar{I}} \) remains symmetric tridiagonal. This observation is crucial because then we can quickly compute the updating direction \( d_{\bar{I}} = H_{\bar{I}}^{-1} [\nabla \phi(u)]_{\bar{I}} \), which can be done by solving the linear system \( H_{\bar{I}} d_{\bar{I}} = [\nabla \phi(u^i)]_{\bar{I}} \) as follows:

1. Compute the Cholesky decomposition \( H_{\bar{I}} = R^T R \).
2. Solve the linear system \( R^T v = [\nabla \phi(u)]_{\bar{I}} \) to obtain \( v \).
3. Solve the linear system \( R d_{\bar{I}} = v \) to obtain \( d_{\bar{I}} \).

Because the reduced Hessian is also tridiagonal, its Cholesky decomposition can be computed in linear time to yield a bidiagonal matrix \( R \), which in turn allows to solve the subsequent linear systems also in linear time. Extremely efficient routines to perform all these tasks are available in the LAPACK libraries [Anderson et al., 1999].

The next crucial ingredient is efficient selection of the stepsize \( \alpha \). The original PN algorithm Bertsekas [1982] recommends Armijo-search along projection arc. However, for our problem this search is inordinately expensive. So we resort to a backtracking strategy using quadratic interpolation [Nocedal and Wright, 2000], which works admirably well. This strategy is as follows: start with an initial stepsize \( \alpha_0 = 1 \). If
In the event that at some point of the procedure the new stepsize \( \alpha \) is found. The goodness of a stepsize is measured using the following Armijo-like sufficient descent rule

\[
\alpha_{k+1} = \min\left\{ \alpha_k, \frac{\phi(u_k) - \phi(P[u_k - \alpha_k d])}{\nabla \phi(u_k) \cdot d} : \beta \geq 0 \right\},
\]

where \( \beta \) is a tolerance parameter. If \( \alpha_{k+1} > \alpha_k \) or \( \alpha_{k+1} \approx \alpha_k \), then \( \alpha_{k+1} = \frac{1}{2} \alpha_k \).

The whole stepsize selection procedure is shown in Algorithm 3. The costliest operation in this procedure is the evaluation of \( \phi(u) \), which, nevertheless can be done in linear time. Furthermore, in practice a few iterations more than suffice to obtain a good stepsize.

Overall, a full PN iteration as described above runs at \( O(n) \) cost. Thus, by exploiting the structure of the problem, we manage to reduce the \( O(n^3) \) cost per iteration of a general PN algorithm to a linear-cost method. The pseudocode of the resulting method is shown as Algorithm 4. Note that in the special case when the weights \( W := \text{Diag}(w_i) \) are so large that the unconstrained optimum coincides with the constrained one, we can obtain \( u^* \) directly via solving \( DD^T W u^* = Dy \) (which can also be done at \( O(n) \) cost). The duality gap of the current solution (see formula 2.3) is used as a stopping criterion, where we use a tolerance of \( \epsilon = 10^{-5} \) in practice.

---

**Algorithm 3** Stepsize selection for Projected Newton

- **Initialize:** \( \alpha_0 = 1, \quad k = 0, \quad d \), tolerance parameter \( \sigma \)

- **while** \( \phi(u) - \phi(P[u - \alpha_k d]) < \sigma \cdot \alpha_k \cdot (\nabla \phi(u) \cdot d) \)
  - Minimize quadratic model: \( \alpha_{k+1} = \min\left\{ \alpha_k, \frac{\alpha^2 \nabla \phi(u)^2}{2(1 - \alpha \nabla \phi(u)^2)} \right\} \).
  - if \( \alpha_{k+1} > \alpha_k \) or \( \alpha_{k+1} \approx \alpha_k \), then \( \alpha_{k+1} = \frac{1}{2} \alpha_k \).
  - \( k \leftarrow k + 1 \)

- **end while**

- **return** \( \alpha_k \)

---

**Algorithm 4** PN algorithm for TV-L1-proximity

Let \( W = \text{Diag}(w_i) \); solve \( DD^T W u^* = Dy \).

- if \( \|W^{-1} u^*\|_\infty \leq 1 \), **return** \( u^* \).

- \( u_0 = P[u^*], \quad t = 0 \).

- **while** \( \text{gap}(u) > \epsilon \)
  - Identify set of active constraints \( I \); let \( \bar{I} = \{1 \ldots n\} \setminus I \).
  - Construct reduced Hessian \( H_{\bar{I}} \).
  - Solve \( H_{\bar{I}} d_{\bar{I}} = [\nabla \phi(u^*)]_{\bar{I}} \).
  - Compute stepsize \( \alpha \) using backtracking + interpolation (Alg. 3).
  - Update \( u_{\bar{I}}^{t+1} = P[u_{\bar{I}}^t - \alpha d_{\bar{I}}] \).
  - \( t \leftarrow t + 1 \).

- **end while**

- **return** \( u^t \).
2.2 TV-L2: Proximity for TV$_2$ID

For TV-L2 proximity ($p = 2$) the dual (2.2) reduces to

$$\min_u \phi(u) := \frac{1}{2} \|D^T u\|^2_2 - u^T Dy, \quad \text{s.t. } \|u\|_2 \leq \lambda.$$  

(2.13)

Problem (2.13) is nothing but a version of the well-known trust-region subproblem (TRS), for which a variety of numerical approaches are known [Conn et al., 2000].

We derive a specialized algorithm based on the classic Moré-Sorensen Newton (MSN) method [Moré and Sorensen, 1983]. This method in general can be quite expensive, but again we can exploit the tridiagonal Hessian to make it efficient. Curiously, experiments show that for a limited range of $\lambda$ values, even ordinary gradient-projection (GP) can be competitive. Thus for overall best performance, one may prefer a hybrid MSN-GP approach.

Towards solving (2.13), consider its KKT conditions:

$$\begin{align*}
(DD^T + \alpha I)u &= Dy, \\
\alpha(\|u\|_2 - \lambda) &= 0, \quad \alpha \geq 0,
\end{align*}$$  

(2.14)

where $\alpha$ is a Lagrange multiplier. There are two possible cases regarding the $\|u\|_2$ value: either $\|u\|_2 < \lambda$, or $\|u\|_2 = \lambda$.

If $\|u\|_2 < \lambda$, then the KKT condition $\alpha(\|u\|_2 - \lambda) = 0$, implies that $\alpha = 0$ must hold and $u$ can be obtained immediately by solving the linear system $DD^T u = Dy$. This can be done in $O(n)$ time owing to the bidiagonal structure of $D$. Conversely, if the solution to $DD^T u = Dy$ lies in the interior of the $\|u\|_2 = 1$, then it solves (2.14). Therefore, this case is trivial to solve, and we need to consider only the harder case $\|u\|_2 = \lambda$.

For any given $\alpha$ one can obtain the corresponding vector $u$ as $u(\alpha) = (DD^T + \alpha I)^{-1} Dy$. Therefore, optimizing for $u$ reduces to the problem of finding the “true” value of $\alpha$.

An obvious approach is to solve $\|u(\alpha)\|^2_2 = \lambda^2$. Less obvious is the **MSN equation**

$$h(\alpha) := \lambda^{-1} - \|u(\alpha)\|^2_2 = 0,$$  

(2.15)

which has the benefit of being almost linear in the search interval, which results in fast convergence [Moré and Sorensen, 1983]. Thus, the task is to find the root of the function $h(\alpha)$, for which we use Newton’s method, which in this case leads to the iteration

$$\alpha \leftarrow \alpha - h(\alpha)/h'(\alpha).$$  

(2.16)

Some calculation shows that the derivative $h'$ can be computed as

$$\frac{1}{h'(\alpha)} = \frac{\|u(\alpha)\|^2_2}{u(\alpha)^T (DD^T + \alpha I)^{-1} u(\alpha)}.$$  

(2.17)

The key idea in MSN is to eliminate the matrix inverse in (2.17) by using the Cholesky decomposition $DD^T + \alpha I = R^T R$ and defining a vector $q = (R^T)^{-1} u$, so that $\|q\|^2_2 = u(\alpha)^T (DD^T + \alpha I)^{-1} u(\alpha)$. As a result, the Newton iteration (2.16) becomes

$$\begin{align*}
\alpha - \frac{h(\alpha)}{h'(\alpha)} &= \alpha - (\|u(\alpha)\|_2^{-1} - \lambda^{-1}) \cdot \frac{\|u(\alpha)\|^3_2}{u(\alpha)^T (DD^T + \alpha I)^{-1} u(\alpha)}, \\
&= \alpha - \frac{\|u(\alpha)\|^3_2 - \lambda^{-1} \|u(\alpha)\|_2^3}{\|q\|^3_2}, \\
&= \alpha - \frac{\|u(\alpha)\|^3_2}{\|q\|^3_2} \left(1 - \frac{\|u(\alpha)\|_2}{\lambda}\right),
\end{align*}$$
Algorithm 5 MSN based TV-L2 proximity

Initialize: $\alpha^0 = 0$, $t = 0$, $u = 0$.

while $|\|u\|^2_2 - \lambda| > \epsilon_\lambda$ or gap($u$) > $\epsilon_{\text{gap}}$ do

  Compute Cholesky decomp. $DD^T + \alpha t I = R^T R$.

  Obtain $u$ by solving $R^T Ru = Dy$.

  Obtain $q$ by solving $R^T q = u$.

  $\alpha^{t+1} = \alpha^t + \alpha - \frac{\|u(\alpha)\|^2_2}{\|q\|^2_2} \left(1 - \frac{\|u(\alpha)\|_2}{\lambda}\right)$.

  $t \leftarrow t + 1$.

end while

return $u^t$.

Algorithm 6 GP algorithm for TV-L2 proximity

Initialize $u_0 \in \mathbb{R}^N$, $t = 0$.

while (~converged) do

  Gradient update: $v^t = u^t - \frac{1}{2} \nabla f(u^t)$.

  Projection: $u^{t+1} = \max(1 - \lambda/\|v^t\|_2, 0) \cdot v^t$.

  $t \leftarrow t + 1$.

end while

return $u^t$.

and therefore

$$\alpha \leftarrow \alpha - \frac{\|u(\alpha)\|^2_2}{\|q\|^2_2} \left(1 - \frac{\|u(\alpha)\|_2}{\lambda}\right).$$ (2.18)

As in TV-L1, the tridiagonal structure of $(DD^T + \alpha I)$ allows to compute both $R$ and $q$ in linear time, so the overall iteration runs in $O(n)$ time.

The above ideas are presented as pseudocode in Algorithm 5. As a stopping criterion two conditions are checked: whether the duality gap is small enough, and whether $u$ is close enough to the boundary. This latter check is useful because intermediate solutions could be dual-infeasible, thus making the duality gap an inadequate optimality measure on its own. In practice we use tolerance values $\epsilon_\lambda = 10^{-6}$ and $\epsilon_{\text{gap}} = 10^{-5}$.

Even though Algorithm 5 requires only linear time per iteration, it is fairly sophisticated, and in fact a much simpler method can be devised. This is illustrated here by a gradient-projection method with a fixed stepsize $\alpha_0$, whose iteration is

$$u^{t+1}_i = P_{\|\cdot\|_2 \leq \lambda}(u^t_i - \alpha_0 \nabla \phi(u^t_i)).$$ (2.19)

The theoretically ideal choice for the stepsize $\alpha_0$ is given by the inverse of the Lipschitz constant $L$ of the gradient $\nabla \phi(u)$ [Nesterov, 2007, Beck and Teboulle, 2009]. Since $\phi(u)$ is a convex quadratic, $L$ is simply the largest eigenvalue of the Hessian $DD^T$. Owing to its special structure, the eigenvalues of the Hessian have closed-form expressions, namely $\lambda_i = 2 - 2 \cos \left(\frac{i\pi}{n+1}\right)$ (for $1 \leq i \leq n$). The largest one is $\lambda_n = 2 - 2 \cos \left(\frac{(n-1)\pi}{n}\right)$, which tends to 4 as $n \to \infty$; thus the choice $\alpha_0 = 1/4$ is a good approximation.

Pseudocode showing the whole procedure is presented in Algorithm 6. Combining this with the fact that the projection $P_{\|\cdot\|_2 \leq \lambda}$ is also trivial to compute, the GP iteration (2.19) turns out to be very attractive. Indeed, sometimes it can even outperform the more sophisticated MSN method, though only for a very limited range of $\lambda$ values. Therefore, in practice we recommend a hybrid of GP and MSN, as suggested by our experiments.
2.3 TV-Lp: Proximity for $TV^1_p$

For TV-$L_p$ proximity (for $1 < p < \infty$) the dual (2.2) problem becomes

$$\min_{\mathbf{u}} \phi(\mathbf{u}) := \frac{1}{2} \|D^T \mathbf{u}\|_2^2 - \mathbf{u}^T D \mathbf{y}, \quad \text{s.t.} \quad \|\mathbf{u}\|_q \leq \lambda, \quad (2.20)$$

where $q = 1/(1 - 1/p)$. Problem (2.20) is not particularly amenable to the Newton-type approaches taken above, as neither PN, nor MSN-type methods can be applied easily. It is somewhat amenable to a gradient-projection (GP) strategy, for which the same update rule as in (2.19) applies, but unlike the $q = 2$ case, the projection step here is much more involved. Thus, to complement GP, we also propose an alternative strategy using the projection-free Frank-Wolfe (FW) method. As expected, the overall best performing approach is actually a hybrid of GP and FW. Let us thus present details of both below.

2.3.1 Efficient projection onto the $\ell_q$-ball

The problem of projecting onto the $\ell_q$-norm ball is

$$\min_{\mathbf{w}} \ d(\mathbf{w}) := \frac{1}{2} \|\mathbf{w} - \mathbf{u}\|_2^2, \quad \text{s.t.} \quad \|\mathbf{w}\|_q \leq \lambda. \quad (2.21)$$

For this problem, it turns out to be more convenient to address its Fenchel dual

$$\min_{\mathbf{w}} \ d^*(\mathbf{w}) := \frac{1}{2} \|\mathbf{w} - \mathbf{u}\|_2^2 + \lambda \|\mathbf{w}\|_p, \quad (2.22)$$

which is actually nothing but $\text{prox}_{\lambda \|\cdot\|_p}(\mathbf{u})$. The optimal solution, say $\mathbf{w}^*$, to (2.21) can be obtained by solving (2.22), by using the Moreau-decomposition (A.6) which yields

$$\mathbf{w}^* = \mathbf{u} - \text{prox}_{\lambda \|\cdot\|_p}(\mathbf{u}).$$

Projection (2.21) is computed many times within GP, so it is crucial to solve it rapidly and accurately. To this end, we first turn (2.22) into a differentiable problem and then derive a projected-Newton method following our approach of §2.1.

Assume therefore, without loss of generality that $\mathbf{u} \geq 0$, so that $\mathbf{w} \geq 0$ also holds (the signs can be restored after solving this problem). Thus, instead of (2.22), we solve

$$\min_{\mathbf{w}} \ d^*(\mathbf{w}) := \frac{1}{2} \|\mathbf{w} - \mathbf{u}\|_2^2 + \lambda \left(\sum_i w_i^p\right)^{1/p} \quad \text{s.t.} \quad \mathbf{w} \geq 0. \quad (2.23)$$

The gradient of $d^*$ may be compactly written as

$$\nabla d^*(\mathbf{w}) = \mathbf{w} - \mathbf{u} + \lambda \|\mathbf{w}\|_p^{1-p} \mathbf{w}^{p-1}, \quad (2.24)$$

where $\mathbf{w}^{p-1}$ denotes elementwise exponentiation of $\mathbf{w}$. Elementary calculation yields

$$\frac{\partial^2}{\partial w_i \partial w_j} d^*(\mathbf{w}) = \delta_{ij} \left(1 + \lambda(p - 1) \left(\frac{w_i}{\|\mathbf{w}\|_p}\right)^{p-2}\|\mathbf{w}\|_p^{-1}\right) + \lambda(1 - p) \left(\frac{w_i}{\|\mathbf{w}\|_p}\right)^{p-1} \left(\frac{w_j}{\|\mathbf{w}\|_p}\right)^{p-1}\|\mathbf{w}\|_p^{-1}$$

$$= \delta_{ij} \left(1 - c\hat{w}_i^{p-2}\right) + c\hat{w}_i \hat{w}_j,$$

where $c := \lambda(1 - p)\|\mathbf{w}\|_p^{-1}$, $\hat{w} := \mathbf{w}/\|\mathbf{w}\|_p$, $\hat{\mathbf{w}} := (\mathbf{w}/\|\mathbf{w}\|_p)^{p-1}$, and $\delta_{ij}$ is the Dirac delta. In matrix notation, this Hessian’s diagonal plus rank-1 structure becomes apparent

$$H(\mathbf{w}) = \text{Diag}(1 - c\hat{w}_i^{p-2}) + c\hat{w} \cdot \hat{\mathbf{w}}^T \quad (2.25)$$

To develop an efficient Newton method it is imperative to exploit this structure. It is not hard to see that for a set of non-active variables $\tilde{I}$ the reduced Hessian takes the form

$$H_{\tilde{I}}(\mathbf{w}) = \text{Diag}(1 - c\hat{w}_{\tilde{I}}^{p-2}) + c\hat{w}_{\tilde{I}} \hat{w}_{\tilde{I}}^T. \quad (2.26)$$
With the shorthand $\Delta = \text{Diag}(1 - c \bar{w}_f^{p-2})$, the matrix-inversion lemma yields

$$H_f^{-1}(w) = (\Delta + c \bar{w}_f \bar{w}_f^T)^{-1} = \Delta^{-1} - \Delta^{-1}c \bar{w}_f \bar{w}_f^T \Delta^{-1} \frac{1}{1 + c \bar{w}_f \Delta^{-1} \bar{w}_f}, \quad (2.27)$$

Furthermore, since in PN the inverse of the reduced Hessian always operates on the reduced gradient, we can rearrange the terms in this operation for further efficiency; that is,

$$H_i(w)^{-1} \nabla_i f(w) = v \odot \nabla_i f(w) - \frac{(v \odot \bar{w}_f)(v \odot \bar{w}_f)^T \nabla_i f(w)}{1/c + \bar{w}_f^T(v \odot \bar{w}_f)}, \quad (2.28)$$

where $v := (1 - c \bar{w}_f^{p-2})^{-1}$, and $\odot$ denotes componentwise product.

To import of the above derivations is that the Newton direction, and thus the overall PN iteration can be computed in $O(n)$ time, which results in a highly effective solver.

2.3.2 Frank-Wolf algorithm for TV $\ell_p$ proximity

The Frank-Wolfe (FW) algorithm (see e.g., [Jaggi, 2013] for a recent overview), also known as the conditional gradient method [Bertsekas, 1999] solves differentiable optimization problems over compact convex sets, and can be quite effective if we have access to a subroutine to solve linear problems over the constraint set.

The generic FW iteration is illustrated in Algorithm 7. FW offers an attractive strategy for TV $\ell_p$ proximity because both the descent-direction as well as stepsizes can be computed very easily. Specifically, to find the descent direction we need to solve

$$\min_s \ s^T (DD^T u - Dy), \quad \text{s.t. } \|s\|_q \leq \lambda. \quad (2.29)$$

This problem can be solved by observing that $\max_{\|s\|_q \leq 1} s^T z$ is attained by some vector $s$ proportional to $z$, of the form $|s^*| \propto |z|^{p-1}$. Therefore, $s^*$ in (2.29) is found by taking $z = DD^T u - Dy$, computing $s = -\text{sgn}(z) \odot |z|^{p-1}$ and then rescaling $s$ to meet $\|s\|_q = \lambda$.

**Algorithm 7** Frank-Wolfe (FW)

**Inputs:** $f$, compact convex set $\mathcal{D}$.
Initialize $x_0 \in \mathcal{D}$, $t = 0$.

**while** stopping criteria not met **do**

Find descent direction: $\min_s \ s \cdot \nabla f(x_t)$ s.t. $s \in \mathcal{D}$.
Determine stepsize: $\min_{\gamma} f(x_t + \gamma(s - x_t))$ s.t. $\gamma \in [0,1]$.
Update: $x_{t+1} = x_t + \gamma(s - x_t)$
$t \leftarrow t + 1$.

**end while**

**return** $x_t$.

The stepsize can also be computed in closed form owing as the objective function is quadratic. Note the update in FW takes the form $u + \gamma(s - u)$, which can be rewritten as $u + \gamma d$ with $d = s - u$. Using this notation the optimal stepsize is obtained by solving

$$\min_{\gamma \in [0,1]} \frac{1}{2} \|D^T (u + \gamma d)\|_2^2 - (u + \gamma d)^T Dy.$$

A brief calculation on the above problem yields

$$\gamma^* = \min \{ \max \{\hat{\gamma}, 1\}, 0\},$$
where $\hat{\gamma} = -\left(d^T D D^T u + d^T D y\right) / \left(d^T D D^T d\right)$ is the unconstrained optimal stepsize. We note that following [Jaggi, 2013] we also check a “surrogate duality-gap”

$$ g(x) = x^T \nabla f(x) - \min_{s \in D} s^T \nabla f(x) = (x - s^*)^T \nabla f(x), $$

at the end of each iteration. If this gap is smaller than the desired tolerance, the real duality gap is computed and checked; if it also meets the tolerance, the algorithm stops.

### 2.4 Prox operator for TV-L_∞

The final case is TV_{1D}^{L_∞}. We mention this case only for completeness, but do not spend much time in developing tuned methods. The dual (2.2) here becomes

$$ \min_u \frac{1}{2} \| D^T u \|_2^2 - u^T D y, \quad \text{s.t. \ } \| u \|_1 \leq \lambda. \tag{2.30} $$

This problem can be again easily solved by invoking GP, where the only non-trivial step is projection onto the \( \ell_1 \)-ball. But the latter is an extremely well-studied operation (see e.g., [Liu and Ye, 2009, Kiwiel, 2008]), and so \( O(n) \) time routines for this purpose are readily available. By integrating them in our GP framework an efficient prox solver is obtained.

### 3 Prox operators for multidimensional TV

This section shows how to build on our efficient 1D-TV prox operators to handle proximity for the much harder multidimensional TV (1.3). To that end, we follow classical proximal optimization techniques [Combettes and Pesquet, 2009, Bauschke and Combettes, 2011].

#### 3.1 Proximity stacking

The basic composite objective (1.1) is a special case of the more general class of models where one may have several regularizers, so that we now solve

$$ \min_x f(x) + \sum_{i=1}^m r_i(x), \tag{3.1} $$

where each \( r_i \) (for \( 1 \leq i \leq m \)) is lsc and convex.

Just like the basic problem (1.1), the more complex problem (3.1) can also be tackled via proximal methods. The key to doing so is to use inexact proximal methods along with a technique that we call proximity stacking. Inexact proximal methods allow one to use approximately computed prox operators without impeding overall convergence, while proximity stacking allows one to compute the prox operator for the entire sum \( r(x) = \sum_{i=1}^m r_i(x) \) by “stacking” the individual \( r_i \) prox operators. This stacking leads to a highly modular design; see Figure 5 for a visualization. In other words, proximity stacking involves computing the prox operator

$$ \prox_r(y) := \argmin_x \frac{1}{2} \| x - y \|_2^2 + \sum_{i=1}^m r_i(x), \tag{3.2} $$

by iteratively invoking the individual prox operators \( \prox_{r_i} \) and then combining their outputs. This mixing is done by means of a combiner method, which guarantees convergence to the solution of the overall \( \prox_r(y) \).
Figure 5: Design schema in proximal optimization for minimizing the function $f(x) + \sum_{i=1}^{m} r_i(x)$. Proximal stacking makes the sum of regularizers appear as a single one to the proximal method, while retaining modularity in the design of each proximity step through the use of a combiner method. For non-smooth $f$ the same schema applies by just replacing the $f$ gradient operator by its corresponding proximity operator.

Different proximal combiners can be used for computing $\text{prox}_r$ (3.2). For instance, for $m = 2$, the Proximal Dykstra (PD) [Combettes and Pesquet, 2009] method is a reasonable choice; alternatively, as shown recently in [Jegelka et al., 2013], the Douglas-Rachford (DR) scheme proves to be a more effective choice. The crux of both PD and DR, however lies in that they require only subroutines to compute the individual prox operators $\text{prox}_{r_1}$ and $\text{prox}_{r_2}$, which allows us to reuse our 1D-TV subroutines.

More generally, if more than two regularizers are present (i.e., $m > 2$), then a it is easier to use Parallel-Proximal Dykstra (PPD) [Combettes, 2009] or Parallel Douglas-Rachford (PDR) as the proximal combiners—both methods being generalizations obtained via the “product-space trick” of Pierra [1984]. Empirically, PDR seems to run faster than PPD, so we prefer it in our implementation and following [Jegelka et al., 2013], we also prefer to implement PDR for solving the corresponding best-approximation problem.

These parallel proximal methods are attractive because they not only combine an arbitrary number of regularizers, but also allows parallelizing the calls to the individual prox operators. This feature allows us to develop a highly parallel implementation for multidimensional TV proximity (§3.3).

Thus, using proximal stacking and combination, any convex machine learning problem with multiple regularizers can be solved in a highly modular proximal framework. Below we exemplify these ideas by applying them to two- and higher-dimensional TV proximity, which we then use within proximal solvers for addressing a wide array of applications.

### 3.2 Two-dimensional TV

Recall that for a matrix $X \in \mathbb{R}^{n_1 \times n_2}$, our anisotropic 2D-TV regularizer takes the form

$$
Tv_{p,q}^2(X) := \sum_{i=1}^{n_1} \left( \sum_{j=1}^{n_2-1} |x_{i,j+1} - x_{i,j}|^p \right)^{1/p} + \sum_{j=1}^{n_2} \left( \sum_{i=1}^{n_1-1} |x_{i+1,j} - x_{i,j}|^q \right)^{1/q}.
$$

This regularizer applies a $Tv_{p}^1$ regularization over each row of $X$, and a $Tv_{q}^1$ regularization over each column. Introducing differencing matrices $D_n$ and $D_m$ for the row and column dimensions, the regularizer (3.3) can be rewritten as

$$
Tv_{p,q}^{2D}(X) = \sum_{i=1}^{n} \|D_n x_i : \|_p + \sum_{j=1}^{m} \|D_m x : j \|_q,
$$

where $x_i :$ denotes the $i$-th row of $X$, and $x : j$ its $j$-th column. The corresponding $Tv_{p,q}^{2D}$ proximity problem is

$$\min_X \; \frac{1}{2} \|X - Y\|_F^2 + \lambda Tv_{p,q}^{2D}(X),$$

where $X$ is the solution to the TV minimization problem and $Y$ is the data matrix.
where we use the Frobenius norm \( \|X\|_F = \sqrt{\sum_{ij} x_{ij}^2} = \|\text{vec}(X)\|_2 \), where \( \text{vec}(X) \) is the vectorization of \( X \). Using (3.4), problem (3.5) becomes

\[
\min_X \quad \frac{1}{2} \|X - Y\|_F^2 + \lambda \left( \sum_i \|D_{n_i} x_{i,\cdot}\|_p \right) + \lambda \left( \sum_j \|D_{m_j} x_{\cdot,j}\|_q \right),
\]

where the parentheses make explicit that \( T_{2D} \) is a combination of two regularizers: one acting over the rows and the other over the columns. Formulation (3.6) fits the model solvable by PD or DR, though with an important difference: each of the two regularizers that make up \( T_{2D} \) is itself composed of a sum of several (\( n \) or \( m \)) 1D-TV regularizers. Moreover, each of the 1D row (column) regularizers operates on a different row (column), and can thus be solved independently.

### 3.3 Higher-dimensional TV

Going even beyond \( T_{2D} \) is the general multidimensional TV \((1.3)\), which we recall below.

Let \( X \) be an order-\( m \) tensor in \( \mathbb{R}^{n_1 \times \cdots \times n_m} \), whose components are indexed as \( X_{i_1,i_2,\ldots,i_m} \) \((1 \leq i_j \leq n_j \) for \( 1 \leq j \leq m)\); we define TV for \( X \) as

\[
T_{p,q}^m(X) \overset{\text{def}}{=} \sum_{k=1}^m \sum_{\{i_2,\ldots,i_m\}\setminus i_k} \left( \sum_{j=1}^{n_k-1} \left| X_{i_1,i_k-1,j+1,i_k+1,\ldots,i_m} - X_{i_1,i_k-1,j,i_k+1,\ldots,i_m} \right|^{p_k} \right)^{1/p_k},
\]

where \( p = [p_1,\ldots,p_m] \) is a vector of scalars \( p_k \geq 1 \). This corresponds to applying a 1D-TV to each of the 1D fibers of \( X \) along each of the dimensions.

Introducing the multi-index \( i(k) = (i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m) \), which iterates over every 1-dimensional fiber of \( X \) along the \( k \)-th dimension, the regularizer (3.7) can be written more compactly as

\[
T_{p,q}^m(X) = \sum_{k=1}^m \sum_{i(k)} \|D_{n_k} x_{i(k)}\|_{p_k},
\]

where \( x_{i(k)} \) denotes a row of \( X \) along the \( k \)-th dimension, and \( D_{n_k} \) is a differencing matrix of appropriate size for the 1D-fibers along dimension \( k \) (of size \( n_k \)). The corresponding \( m \)-dimensional-TV proximity problem is

\[
\min_X \quad \frac{1}{2} \|X - Y\|_F^2 + \lambda T_{p,q}^m(X),
\]

where \( \lambda > 0 \) is a penalty parameter, and the Frobenius norm for a tensor just denotes the ordinary sum-of-squares norm over the vectorization of such tensor.

Problem (3.9) looks very challenging, but it enjoys decomposability as suggested by (3.8) and made more explicit by writing it as a sum of \( \text{TV}^{1D} \) terms

\[
\min_X \quad \frac{1}{2} \|X - Y\|_F^2 + \sum_{k=1}^m \sum_{i(k)} \text{TV}^{1D}_{p,q}(x_{i(k)}).
\]

The proximity task (3.10) can be regarded as the sum of \( m \) proximity terms, each of which further decomposes into a number of inner \( \text{TV}^{1D} \) terms. These inner terms are trivial to address since, as in the 2D-TV case, each of the \( \text{TV}^{1D} \) terms operates on different entries of \( X \). Regarding the \( m \) major terms, we can handle them by applying the PDR combiner algorithm (\S 3), which ultimately yields the prox operator for \( T_{p,q}^m \) by just repeatedly calling \( \text{TV}^{1D} \) prox operators. Most importantly, both PDR and the natural decomposition of the problem provide a vast potential for parallel multithreaded computing, which is valuable when dealing with such complex and high-dimensional data.
4 1D-TV: Experiments and Applications

Since the most important components of our methods are the efficient 1D-TV prox operators, let us begin by highlighting their empirical performance. In particular, we compare our methods against state-of-the-art algorithms, showing the advantages of our approach.

Our solvers are implemented in C++, with calls to the LAPACK (FORTRAN) library [Anderson et al., 1999]. To permit reproducibility and to facilitate use of our TV prox-operators, we have made our complete toolbox available at: http://arantxa.ii.uam.es/~gaa/software.html

We test solvers for 1D-TV regularization under two scenarios:

I) Increasing input size ranging from $n = 10^1$ to $n = 10^7$. A penalty $\lambda \in [0, 50]$ chosen at random for each run, and the data vector $y$ with uniformly random entries $y_i \in [-2\lambda, 2\lambda]$ (proportionally scaled to $\lambda$).

II) Varying penalty parameter $\lambda$ ranging from $10^{-3}$ (negligible regularization) to $10^3$ (the TV term dominates); here $n$ is set to 1000 and $y_i$ is randomly generated in the range $[-2, 2]$ (uniformly).

4.1 Running time results for TV-L1

![Figure 6: Running times (in secs) for SLEP, Projected Newton, Taut String, Condat and Johnson solvers for $T_{1D}$-proximity with increasing a) input sizes, b) penalties. Both axes are on a log-scale. The proposed methods are marked as bold lines.](image)

Starting with the proposed methods for $T_{1D}$-proximity (Projected Newton and Optimized Taut String), running times are compared against the following competing solvers:

- The FLSA function (C implementation) of the SLEP library of Liu et al. [2009] for $T_{1D}$-proximity [Liu et al., 2010].
- The state-of-the-art method of Condat [2012], which by a different reasoning arrives at an algorithm essentially equivalent to the (unweighted) taut-string method.
- The dynamic programming method of Johnson [2013], which guarantees linear running time.

For Projected Newton and SLEP a duality gap of $10^{-5}$ is used as the stopping criterion. The rest of algorithms are direct methods, and thus they are run until completion. Timing results are presented in Figure 6 for both experimental scenarios. The following interesting facts are drawn from these results
• Direct methods (Taut string, Condat, Johnson) prove to be much faster than iterative methods (Projected Newton, SLEP). This is true even for the case of the taut string and Condat’s methods, which have a theoretical worst-case performance of $O(n^2)$. This possibility was already noted at Condat [2012].

• Even more, though Johnson’s method has a guaranteed running time of $O(n)$, it turns out to be slower than the taut string and Condat’s methods. This is explained by the need of Johnson’s method of an extra $\sim 8n$ memory storage, which becomes a burden in contrast to the in-place strategies of taut string and Condat’s.

• The best methods are taut string and Condat’s solvers; both show equivalent performance for this unweighted TV problem.

4.2 Running time results for weighted TV-L1

![Figure 7: Running times (in secs) for Projected Newton and Taut String solvers for weighted and uniform TV1D-proximity with increasing a) input sizes, b) penalties. Both axes are on a log-scale.](image)

An advantage of the solvers proposed in this paper is their flexibility to easily deal with the more difficult, weighted version of the TV-L1 proximity problem. To illustrate this, Figure 7 shows the running times of the Projected Newton and Optimized Taut String methods when solving both the standard and weighted TV-L1 prox operators.

Since for this set of experiments a whole vector of weights $\mathbf{w}$ is needed, we have adjusted the experimental scenarios as follows:

I) $n$ is generated as in the general setting, penalties $\mathbf{w} \in [0, 100]$ are chosen at random for each run, and the data vector $\mathbf{y}$ with uniformly random entries $y_i \in [-2\lambda, 2\lambda]$, with $\lambda$ the mean of $\mathbf{w}$, using also this $\lambda$ choice for the uniform (unweighted) case.

II) $\lambda$ and $n$ are generated as in the general setting, and the weights vector $\mathbf{w}$ is drawn randomly from the uniform distribution $\mathbf{w}_i \in [0.5\lambda, 1.5\lambda]$.

As can be readily observed, performance for both versions of the problem is almost identical, even if the weighted problem is conceptually harder. Conversely, adapting the other reviewed algorithms to address this problem while keeping up with performance is not a straightforward task.
4.3 Running time results for TV-L2

Next we show results for TV$_{L2}^{2D}$ proximity. To our knowledge, this version of TV has not been explicitly treated before, so there do not exist highly-tuned solvers for it. Thus, we show running time results only for the MSN and GP methods. We use a duality gap of $10^{-5}$ as the stopping criterion; we also add an extra boundary check for MSN with tolerance $10^{-6}$ to avoid early stopping due to potentially infeasible intermediate iterates. Figure 8 shows results for the two experimental scenarios under test.

The results indicate that the performance of MSN and GP differs noticeably in the two experimental scenarios. While the results for the first scenario (Figure 8(a)) might suggest that GP converges faster than MSN for large inputs, it actually does so depending on the size of $\lambda$ relative to $\|y\|_2$. Indeed, the second scenario (Figure 8(b)) shows that although for small values of $\lambda$, GP runs faster than MSN, as $\lambda$ increases, GP’s performance worsens dramatically, so much that for moderately large $\lambda$, it is unable to find an acceptable solution even after 10,000 iterations (an upper limit imposed in our implementation). Conversely, MSN finds a solution satisfying the stopping criterion under every situation, thus showing a more robust behavior.

These results suggest that it is preferable to employ a hybrid approach that combines the strengths of MSN and GP. Such a hybrid approach is guided using the following (empirically determined) rule of thumb: if $\lambda < \|y\|_2$ use GP, otherwise use MSN. Further, as a safeguard, if GP is invoked but fails to find a solution within 50 iterations, the hybrid should switch to MSN. This combination guarantees rapid convergence in practice. Results for this hybrid approach are also included in the plots in Figure 8, and show how it successfully mimics the behavior of the better algorithm amongst MSN and GP.

4.4 Running time results for TV-Lp

Now we show results for TV$_{Lp}^{2D}$ proximity. Again, to our knowledge efficient solvers for this version of TV are not available; still proposals for solving the $\ell_q$-ball projection problem do exist, such as the epp function in SLEP library [Liu et al., 2009], based on a zero finding approach. Consequently, we present here a comparison between this reference projection subroutine and our PN–based projection when embedded in our proposed Gradient Projection solver of §2.3. The alternative proposal given by the Frank–Wolfe algorithm of §2.3.2 is also present in the comparison. We use a duality gap of $10^{-5}$ as
stopping criterion both for GP and FW. Figure 9 shows results for the two experimental scenarios under test, for \( p \) values of 1.5, 1.9 and 3.

Figure 9: Running times (in secs) for GP with PN projection, GP with SLEP’s \( epp \) projection, FW and a hybrid GP+FW algorithm, for \( T_{1D} \) -proximity with increasing input sizes and three different choices of \( p \). Both axes are on a log-scale.

Figure 10: Attained duality gaps (a-c) and running times (d-f, in secs) for GP with PN projection, GP with SLEP’s \( epp \) projection, FW and a hybrid GP+FW algorithm, for \( T_{1D} \) -proximity with increasing penalties and three different choices of \( p \). Both axes are on a log-scale.

A number of interesting conclusions can be drawn from the results. First, our Projected Newton \( \ell_q \)-ball subroutine is far more efficient than \( epp \) when in the context of the GP solver. Two factors seem...
to be the cause of this: in the first place our Projected Newton approach proves to be faster than the zero finding method used by epp. Secondly, in order for the GP solver to find a solution within the desired duality gap, the projection subroutine must provide very accurate results (about $10^{-12}$ in terms of duality gap). Given its Newton nature, our $\ell_q$-ball subroutine scales better in term of running times as a factor of the desired accuracy, which explains he observed differences in performance.

It is also of relevance noting that Frank–Wolfe is significantly faster than Projected Newton. This should discourage the use of Projected Newton, but we find it to be extremely useful in the range of $\lambda$ penalties where $\lambda$ is large, but not enough to render the problem trivial ($w = 0$ solution). In this range the two variants of PN and also FW are unable to find a solution within the desired duality gap ($10^{-5}$), getting stuck at suboptimal solutions. We solve this issue by means of a hybrid GP+FW algorithm, in which updates from both methods are interleaved at a ratio of 10 FW updates per 1 GP update, as FW updates are faster. As both algorithms guarantee improvement in each iteration but follow different procedures for doing so, they complement each other nicely, resulting a superior method attaining the objective duality gap and performing faster than GP.

4.5 Running time results for TV-L∞

For completeness we also include results for our $\TV^1_\infty$ solver based on GP + a standard $\ell_1$-projection subroutine. Figure 11 presents running times for the two experimental scenarios under test. Since $\ell_1$-projection is an easier problem than the general $\ell_q$-projection the resultant algorithm converges faster to the solution than the general GP $\TV^1_p$ prox solver, as expected.

![Figure 11: Running times (in secs) for GP for $\TV^1_\infty$-proximity with increasing a) input sizes, b) penalties. Both axes are on a log-scale.](image)

4.6 Application: Proximal optimization for Fused-Lasso

We now present a key application that benefits from our TV prox operators: **Fused-Lasso** (FL) [Tibshirani et al., 2005], a model that takes the form

$$\min_x \frac{1}{2} \| Ax - y \|^2_2 + \lambda_1 \| x \|_1 + \lambda_2 \TV^1_1(x).$$

The $\ell_1$-norm in (4.1) forces many $x_i$ to be zero, while $\TV^1_1$ favors nonzero components to appear in blocks of equal values $x_{i-1} = x_i = x_{i+1} = \ldots$. The FL model has been successfully applied in several
bioinformatics applications [Tibshirani and Wang, 2008, Rapaport and Vert, 2008, Friedman et al., 2007],
as it encodes prior knowledge about consecutive elements in microarrays becoming active at once.

Following the ideas presented in Sec. 3, since the FL model uses two regularizers, we can use Proximal
Dykstra as the combiner to handle the prox operator. To illustrate the benefits of this framework in
terms of reusability, we apply it to several variants of FL.

- **Fused-Lasso (FL):** Least-squares loss + \(\ell_1 + TV_1^D\) as in (4.1)
- **\(\ell_p\)-Variable Fusion (VF):** Least-squares loss + \(\ell_1 + TV_p^D\). Though Variable Fusion was already
  studied by Land and Friedman [1997], their approach proposed an \(\ell_p\)-like regularizer in the sense
  that \(r(x) = \sum_{i=1}^{n-1} |x_{i+1} - x_i|^p\) is used instead of the TV regularizer \(TV_1^D(x) = \left(\sum_{i=1}^{n-1} |x_{i+1} - x_i|^p\right)^{1/p}\).
  Using \(TV_p\) leads to a more conservative penalty that does not oversmooth the estimates. This FL
  variant seems to be new.
- **Logistic-fused lasso (LFL):** Logistic-loss + \(\ell_1 + TV_1^D\), where the loss takes the form
  \(\ell(x, c) = \sum_i \log \left(1 + e^{-y_i(a_i^T x + c)}\right)\), and can be used in a FL formulation to obtain models more appropriate
  for classification on a dataset \(\{(a_i, y_i)\}\) [Kolar et al., 2010].
- **Logistic + \(\ell_p\)-fusion (LVF):** Logistic loss + \(\ell_1 + TV_1^D\).

To solve these variants of FL, all that remains is to compute the gradients of the loss functions, but
this task is trivial. Each of these four models can be then solved easily by invoking any proximal splitting
method by appropriately plugging in gradient and prox operators. We use TRIP [Kim et al., 2010] as the
splitting method of choice. Figure 12 shows a schematic of the algorithmic modules for solving each FL
model.

**Remark:** A further algorithmic improvement can be obtained by realizing that for \(r(x) = \lambda_1 \|x\|_1 + \lambda_2 TV_1^D(x)\) the prox operator \(\text{prox}_r \equiv \text{prox}_{\lambda_1 \|\cdot\|_1} \circ \text{prox}_{\lambda_2 TV_1^D}(\cdot)\). Such a decomposition does not usually
hold, but it can be shown to hold for this particular case [Yu, 2013, Rinaldo, 2009, Tibshirani et al.,
2005]. Therefore, for FL and LFL we can compute the proximal operator for the combined regularizer \(r\)
directly, thus removing the need for a combiner algorithm. This is also shown in Figure 12.
Table 1: Running times (secs) for SLEP and Trip for optimizing different versions of fused-lasso with increasing input sizes. Both methods were run to satisfy the same convergence criterion.

| Model | SLEP $n=10^3$ | SLEP $n=10^4$ | SLEP $n=10^5$ | Trip $n=10^3$ | Trip $n=10^4$ | Trip $n=10^5$ |
|-------|---------------|---------------|---------------|---------------|---------------|---------------|
| FL    | 0.089         | 1.43          | 41.80         | 0.02          | 0.10          | 0.86          |
| VF    | 0.16          | 1.26          | 35.77         | 0.02          | 0.10          | 0.90          |
| LFL   | 0.21          | 15.01         | 144.81        | 0.78          | 5.35          | 53.88         |
| LVF   | 0.86          | 0.02          | 132.13        | 0.81          | 0.15          | 11.24         |

Table 2: Classification accuracies for the presented Fused–Lasso models on microarray data.

| Dataset  | FL   | VF   | LFL  | LVF  |
|----------|------|------|------|------|
| ArrayCGH | 73.6%| 78.9%| 73.6%| 73.6%|
| Leukemias| 92.0%| 92.0%| 96.0%| 96.0%|
| Colon    | 77.2%| 77.2%| 77.2%| 77.2%|
| Ovarian  | 88.8%| 83.3%| 77.7%| 77.7%|
| Rat      | 67.2%| 65.5%| 70.4%| 70.4%|

4.6.1 Fused-Lasso experiments: simulation

The standard FL model has been well-studied in the literature, so a number of practical algorithms addressing it have already been proposed. One outstanding example is the recent Fused-Lasso algorithm in the SLEP library [Liu et al., 2010], which implements a carefully tuned variant of FISTA [Beck and Teboulle, 2009] based on an efficient proximity step (FLSA). Our experiments on $T_1^{1D}$-proximity have already shown superiority of our prox solvers over FLSA; what remains to be checked is whether our choice of using Trip also proves superior to SLEP. To do so, we begin with a test with synthetic data.

We generate random matrices $A \in \mathbb{R}^{n \times m}$ with i.i.d. entries drawn from a zero mean, unit variance gaussian. We fix $m = 100$ and set the penalties to $\lambda_1 = \lambda_2 = 0.01$. Then, we sample matrices with number of columns (patterns) $n$ being $10^3$, $10^4$, and $10^5$. We select the vector of responses $y$ using the formula $y = \text{sgn}(Ax + v)$, where $x$, and $v$ are random vectors whose entries have variances 1 and 0.01, respectively. The numerical results are summarized in Table 1, which compares SLEP (version 4.0) [Liu et al., 2009] against our Trip-based approach. Since SLEP only supports the FL and LFL models, we endow it with our $T_2^{1D}$ prox operator to permit a comparison for the VF and LVF models. While for smaller matrices with $n = 10^3$ both methods run similarly fast, for larger matrices the Trip-based fused-lasso solvers run much faster.

4.6.2 Fused-Lasso Experiments: Microarray classification

Now we report results of applying the four FL model on a problem from bioinformatics. We test the FL models on binary classification tasks for the following real microarray datasets: ArrayCGH [Stransky et al., 2006], Leukemias [Golub et al., 1999], Colon [U. Alon et al., 1999], Ovarian [Rogers et al., 2005] and Rat [Hua et al., 2009]. Each dataset was split into three equal parts (ensuring both classes were present in every split) for training, validation and test. The penalty parameters were found by exhaustive grid search in the range $\lambda_1, \lambda_2 \in [10^{-3}, 10^1]$ to maximize classification accuracy on the validation splits.

Table 2 shows test accuracies. In general, as expected the logistic-loss based FL models yield better
classification accuracies than those based on least-squares. Regarding the TV-regularizer, three out of five datasets seem to be insensitive to this choice, though the $\text{Tv}_{1}^{1D}$-penalty performs better for Ovarian, while $\text{Tv}_{2}^{1D}$ works best for ArrayCGH. It can be concluded, thus, that new proposed variants of Fused-Lasso using $\text{Tv}_{2}^{1D}$ regularization can be of use in some situations.

5 2D-TV: Experiments and Applications

We address now several practical applications that benefit from two-dimensional TV regularization; our results show again how our $\text{Tv}_{p,q}^{2D}$ prox operator fit in seamlessly into our modular framework to produce efficient proximal splitting solvers.

5.1 Image denoising through anisotropic filtering

Our first example is to the classic problem of image denoising, but with the twist that we deal with noise of an anisotropic character. More specifically, suppose that the true image $\mu \in \mathbb{R}^{n \times m}$ is contaminated by additive noise $N$, so that only $\mu_0 = \mu + N$ is observed. The denoising problem estimates $\mu$ given just the noisy version $\mu_0$. This problem is highly ill-posed and as such not approachable unless additional assumptions on the noise (or on the underlying image) are made.

5.1.1 Isotropic and anisotropic models

An extremely common choice is to simply assume the noise to be gaussian, or some other zero-mean distribution. Under these conditions, a classic method to perform such denoising task is the Rudin-Osher-Fatemi (ROF) model [Rudin et al., 1992], which finds an approximation $X$ to the original image by solving

$$\min_{X} \| X - \mu_0 \|^2_F + \lambda \sum_{i=2}^{n} \sum_{j=2}^{m} \| \partial x_{i,j} \|_2,$$

(5.1)

where $\partial x_{i,j}$ is the discrete gradient

$$\partial x_{i,j} = \begin{bmatrix} x_{i,j} - x_{i-1,j} \\ x_{i,j} - x_{i,j-1} \end{bmatrix}.$$

That is, it is the vector of differences of $X_{i,j}$ and its neighbors along both axes.

The objective of the first term in the ROF model is to penalize any deviation of $X$ from the observed image $\mu_0$, while the second term can be readily recognized as a mixed $(2,1)$-norm over the discrete gradient of $X$. This regularizer models cater to some prior knowledge: in natural images sharp discontinuities in intensity between neighboring points only appear in borders of objects, while the rest of the pixels usually show smooth variations in intensity. It makes sense, therefore, to penalize large values of the gradient, as sharp changes have a higher probability of having been produced by noise. Conversely, as the mean of the noise is zero, it is also sensible to maintain the denoised image $X$ close to the observed $\mu_0$. Merging these two goals produces the ROF model (5.1).

A closer look at the ROF regularizer reveals that it follows the spirit of our 2D-TV regularizer which also penalizes sharp variations between neighboring pixels. Indeed, all such regularizers are broadly categorized as TV regularizers within the image processing community. It is clear, though, that the ROF regularizer (5.1) does not coincide with our $\text{Tv}_{p,q}^{2D}$ regularizer. Some authors [Bioucas-Dias and Figueiredo, 2007] differentiate between these regularizers by naming the ROF approach as isotropic TV and the $\text{Tv}_{p,q}^{2D}$-style approach as anisotropic TV. This naming comes from the fact that isotropic TV penalizes each component of the discrete gradient $\partial x_{i,j}$ following an $\ell_2$ norm, whereas the anisotropic $\text{Tv}_{p,q}^{2D}$-norm and in particular $\text{Tv}_{1,1}^{2D}$-norm, penalize rows and columns independently.
While image filtering using isotropic TV is generally preferred for natural images denoising [Bioucas-Dias et al., 2006], in some settings anisotropic filtering can produce better results, and in fact has been favored by some authors in the past [Choksi et al., 2010, Li and Santosa, 1996]. This is specially true on those images that present a “blocky” structure, and thus are better suited to the structure modeled by the $T_{p,q}^{2D}$-norm. Therefore, efficient methods to perform anisotropic filtering are also important.

5.1.2 Anisotropic denoising experiments

Denoising using the anisotropic $T_{p,q}^{2D}$-norm reduces to solving

$$\min_X \| X - \mu_0 \|_F^2 + \lambda T_{p,q}^{2D}(X).$$

(5.2)

But (5.2) is nothing but the $T_{p,q}^{2D}$-proximity problem, and hence can be directly solved by applying the 2D-TV prox operators developed above. We solve (5.2) below for the choice $p = q = 1$ (which is common in practice), and compare it against the following state of the art algorithms:

- The Split Bregman method of Goldstein T. [2009], which follows an ADMM–like approach to split the $\ell_1$ norm apart from the discrete gradient operator, thus not requiring the use of a 1D-TV prox operator.
- The splitting method by Chambolle and Pock [2011], which can be thought as a kind of preconditioned ADMM algorithm.
- The general splitting method by Condat [2014], which is a generalization of the method by Chambolle and Pock.
- The method by Yang et al. [2013] which also proposes an ADMM approach, though splitting the problem in a way that takes advantage of fast 1D-TV prox operator.
- The maximum flow approach by Goldfarb and Yin [2009], which shows the relationship between the 2D-TV proximity minimization and the maximum flow problem over a grid, and thus applies an efficient maximum flow method to solve a discrete-valued version of 2D-TV.

The method by Yang et al. is the closest in spirit to our stacking proposal, also relying on basic 1D-TV prox solvers. Both for this method and our proposed algorithms we use the taut–string strategy presented before as this basic piece. All algorithm parameters were set as recommended in their corresponding papers or public implementations, except our proposed methods, which are parameter free.

The images used in the experiments are displayed in Appendix D as Figure 18. To test the filters under a variety of scenarios, different kinds of noise were introduced for each image. Table 3 gives details on this, while the noisy images are shown in Figure 19. All QR barcode images used the same kind and parameters of noise. Noise was introduced using Matlab’s `imnoise` function.

Values for the regularization parameter $\lambda$ were found by maximizing the quality of the reconstruction, measured using Improved Signal-to-Noise Ratio (ISNR) [Afonso et al., 2010]. ISNR is defined as

$$\text{ISNR}(X, \mu, \mu_0) = 10 \log_{10} \frac{\| \mu - X \|_F^2}{\| \mu - \mu_0 \|_F^2},$$

where $\mu$ is the original image, $\mu_0$ its noisy variant, and $X$ the reconstruction.

To compare the algorithms we run all of them for each image and measured its ISNR and relative distance to the optimal objective value of the current solution at each iteration through their execution. The only exception to this procedure is the method of Goldfarb and Yin, which is non–iterative and thus always returns an exact solution, and so we just measure the time required to finish. The optimal objective value was estimated by running all methods for very large number of iterations and taking the minimum value of them all. This produced the plots shown in Figures 13-14. From them the following observations are of relevance.
Figure 13: Relative distance to optimum vs time of the denoising 2D-TV algorithms under comparison, for the different images considered in the experiments.
Figure 14: Increased Signal to Noise Ratio (ISNR) vs time of the denoising 2D-TV algorithms under comparison, for the different images considered in the experiments.
Table 3: Types of noise and parameters for each test image. A ∅ indicates that such noise was not applied for the image. Gaussian and Speckle correspond to gaussian additive and multiplicative (respectively) noises with zero mean and the indicated variance. Salt & Pepper noise turns into black or white the indicated fraction of image pixels. Poisson regenerates each pixel by drawing a random value from a Poisson distribution with mean equal to the original pixel value, thus producing a more realistic noise.

| Image    | Gaussian | Speckle | Poisson | Salt & Pepper |
|----------|----------|---------|---------|---------------|
| randomQR | 0.2      | 0.3     | ∅       | ∅             |
| shape    | 0.05     | ∅       | ∅       | ∅             |
| trollface| ∅        | 1       | ∅       | ∅             |
| diagram  | ∅        | ∅       | ✓       | ∅             |
| text     | ∅        | ∅       | ∅       | 0.1           |
| comic    | 0.05     | ∅       | ✓       | ∅             |
| contour  | ∅        | ∅       | ✓       | 0.4           |
| phantom  | ∅        | 2       | ✓       | ∅             |

- Condat’s method and Chambolle-Pock’s method are reduced to essentially the same algorithm when applied to the particular case of TV denoising. Furthermore, they seem to perform slowly when compared to other methods.

- Yang’s method exhibits slow performance at the beginning, but when run for sufficient time is able to achieve the best approximation to the optimum.

- The Split Bregman method, in spite of being an ADMM–like method much like Condat’s or Chambolle-Pock, performs significantly better than those. We attribute this to the very efficient implementation provided by its authors, and likely also to a good choice of the ADMM parameters for the problem at hand.

- The method by Goldfarb and Yin is slower than other approaches and seems to provide suboptimal solutions. We attribute this to the fact that this method solves a discrete (integer–rounded) approximation to the problem. We acknowledge that other methods exploiting the Total Variation - Minimum-cut relationship have been proposed with varying speed results, e.g. [Duan and Tai, 2012], however the suboptimality issues still apply.

- The two approaches proposed in this paper are the fastest to achieve a mid-quality solution, with Douglas-Rachford performing better than Proximal Dykstra.

Considering these facts, the method of choice among the ones considered depends on the desired accuracy. We argue, however, that for the purpose of image processing a mid-quality solution is sufficient. The ISNR plots of Figure 14 certainly seem to support this, as the perceived quality of the reconstruction, roughly approximated by the ISNR, saturates rapidly and no significant improvements are obtained through further optimization. Given this, the proposed methods seem to be the best suited for the considered task.

For quick reference, Table 4 presents a summary of key points of the compared methods, along with some recommendations about when to put them to use.

5.1.3 Parallelization experiments

In addition to the previous experiments and to illustrate the parallelization potential of the developed anisotropic filtering method, Figure 15 plots running times for the PD algorithm as the number of processor core ranges from 1 through 16. We see that for the smaller images, the gains due to more processors
Table 4: Summary of key points of the compared $T_{\mathcal{P},\mathcal{Q}}^{2D}$ proximity (denoising) methods.

| Method              | Key points                                                                 |
|---------------------|-----------------------------------------------------------------------------|
| **Douglas Rachford**| + Fast convergence to medium-quality                                        |
|                     | + Embarrassingly parallel                                                   |
|                     | - Slow for higher accuracies                                                |
|                     | ⇒ Ideal for standard denoising tasks                                        |
| **Proximal Dykstra**| + Similar to DR                                                              |
|                     | - Slower than DR                                                            |
|                     | ⇒ Use DR instead                                                            |
| **Split Bregman**   | + Eventually performs similarly to DR                                       |
|                     | - Slow convergence at first iterations                                      |
|                     | ⇒ Use DR instead                                                            |
| **Chambolle–Pock**  | - Slow                                                                      |
|                     | ⇒ Use other method instead                                                  |
| **Condat**          | + Solves objectives involving a sum of smooth/non–smooth functions with    |
|                     | linear operators                                                            |
|                     | - Reduces to Chambolle–Pock when solving basic image denoising              |
|                     | ⇒ Use only when dealing with more complex functionals                       |
| **Yang**            | + More accurate                                                             |
|                     | - Slow                                                                      |
|                     | ⇒ Useful when extremely accurate solutions are required                      |
| **Goldfarb-Yin**    | + Solves the discrete version of the problem                               |
|                     | - Slow                                                                      |
|                     | - Poor accuracy for the continuous version                                  |
|                     | ⇒ Apply only when solving the discrete problem                              |

essentially flatten out by 8 cores, where synchronization and memory contention offsets potential computational gains (first row). For the larger images, there is steadier speedup as the number of cores increase (in each plot there seems to be a “bump” at 14 processors; we attribute this to a quirk of the multicore machine that we used). From all the plots, however, the message is clear: our TV prox operators exploit parallelization well, and show substantial speedups as more processor cores become available.

5.2 Anisotropic image deconvolution

Taking a step forward we now confront the problem of image deconvolution (or image deblurring). This setting is more complex since the task of image recovery is made harder by the presence of a convolution kernel $K$ that distorts the image as

$$\mu_0 = K * \mu + N,$$

where $N$ is noise as before and $*$ denotes convolution. To recover the original image $\mu$ from the observed $\mu_0$, it is common to solve the following deconvolution problem

$$\min_X \frac{1}{2} \| K * X - \mu \|_F^2 + \lambda r(X).$$

(5.3)
As before the regularizer \( r(X) \) can be isotropic or anisotropic TV, among others. Here we focus again on the anisotropic TV case to show how our solvers can also be used for this image task.

Problem (5.3) also fits the proximal splitting framework, and so we employ the popular FISTA [Beck and Teboulle, 2009] method for image processing. The gradient of the loss can be dealt efficiently by exploiting \( K \) being a convolution operator, which through the well–known convolution theorem is equivalent to a dot product in the frequencies space, and so the computation is done by means of fast Fourier transforms and products. Several other solvers that explicitly deal with convolution operators are also available [Afonso et al., 2010, Bioucas-Dias and Figueiredo, 2007]. A notable solver specific for the isotropic case is given by the work of Krishnan and Fergus [2009], that handles even nonconvex isotropic TV-norms \((0 < p < 1)\). But this approach does not extend to the anisotropic case, so we focus on general proximal splitting.

We use the same test images as for our denoising experiments (Figure 18), with identical noise patterns (Table 3) for the QR images, and gaussian noise with variance 0.05 for the rest. In addition, we convolve each image with a different type of kernel to assess the behavior for a variety of convolutions; Table 5 shows the kernels applied. We constructed these kernels using Matlab’s \texttt{fspecial} function; the convolved images are shown in Figure 21.

The values for the regularizer \( \lambda \) were determined by maximizing the reconstruction quality measured in ISNR. Since deconvolution is much more expensive than denoising, instead of performing an exhaustive search for the best \( \lambda \), we used a Focused Grid Search strategy [Barbero et al., 2008, 2009] to find the best performing values.

Any denoising subroutine can be plugged into the aforementioned deconvolution methods, however for comparison purposes we run our experiments with our best proposed method, Douglas Rachford, and the best competing method, Split Bregman. A key parameter in deconvolution performance is for how long should these methods be run at each FISTA iteration. To select this, we first run FISTA with 100 iterations of Douglas Rachford per step, for a large number of FISTA steps, and take the final objective value as an estimate of the optimum. Then we find the minimum number of Douglas Rachford and Split Bregman iterations for which FISTA can achieve a relative distance to such optimum below

Figure 15: Multicore speedups on different images (see also Table 5)
Table 5: Convolution kernels used for each test image. *Average* substitutes each pixel with the average of its surrounding $n \times m$ neighbors. *Disk* performs the same operation within a disk-shaped neighborhood of the shown radius. *Gaussian* uses an $n \times n$ neighborhood and assigns different weights to each neighbor following the value of a gaussian distribution of the indicated deviation centered at the current pixel. *Motion* emulates the distortions produced when taking a picture in motion, defining a neighborhood following a vector of the indicated length and angle.

10$^{-3}$. The reason for doing this is that for larger distances the attained ISNR values are still far from convergence. This turned to be 5 iterations for Douglas Rachford and 20 for Split Bregman. We then run FISTA for such configurations of the inner solvers, and others with a larger number of inner iterations, for comparison purposes.

Figures 16-17 shows the evolution of objective values and ISNR for all the tested configurations. In general, Douglas Rachford seems to be slightly better at finding very accurate solutions, though in terms of ISNR both methods converge similarly. We explain this by the fact that the major advantage of Douglas Rachford is its aforementioned ability to find medium–quality solutions in a very small number of iterations. However, for image deconvolution we have seen that a minimum of 5 Douglas Rachford iterations are needed to achieve convergence in ISNR, and at such level the method tends to behave similarly to Split Bregman (see denoising experiments above), only becoming better at higher accuracies, as observed.

We conclude therefore that Douglas Rachford and Split Bregman perform similarly for image deconvolution, with Douglas Rachford being superior only when high–accuracy solutions are sought.

### 5.3 2D Fused-Lasso Signal Approximator

The **Fused–Lasso Signal Approximator** (FLSA) [Friedman et al., 2007] can be regarded as a particular case of Fused-Lasso where the input matrix $A$ is the identity matrix $I$, i.e.,

$$
\min_x \quad \frac{1}{2}\|x - y\|_2^2 + \lambda_1\|x\|_1 + \lambda_2 TV_{1D}(x).
$$

This problem can be solved immediately using the methods presented in §4.6. A slightly less trivial problem is the one posed by the 2D variant of FLSA:

$$
\min_X \quad \frac{1}{2}\|X - Y\|_F^2 + \lambda_1\|\text{vec}(X)\|_1 + \lambda_2 TV_{2D}(X).
$$

(F.4)

Friedman et al. [2007] used this model for denoising images where a large number of pixels are known to be completely black (intensity 0), which aligns well with the structure imposed by the $\ell_1$ regularizer.

Akin to the 1D-case, 2D-FLSA (F.4) can also be solved by decomposing its computation into two prox operators [Friedman et al., 2007]; formally,

$$
\text{prox}_{\lambda_1\|\cdot\|_1 + \lambda_2 TV_{2D}(\cdot)}(Y) = \text{prox}_{\lambda_1\|\cdot\|_1} \left( \text{prox}_{\lambda_2 TV_{2D}(\cdot)}(Y) \right).
$$
Figure 16: Relative distance to optimum vs time of the deconvolution 2D-TV algorithms under comparison, for the different images considered in the experiments.
Figure 17: Increased Signal to Noise Ratio (ISNR) vs time of the deconvolution 2D-TV algorithms under comparison, for the different images considered in the experiments.
Thus, to solve (5.4) we merely invoke our $T_{v,1,1}^2$ prox operator and then apply soft-thresholding to the results. Since soft-thresholding is done in closed form, the performance of a 2D-FLSA solver depends only on its ability to compute $T_{v,1,1}^2$-proximity efficiently. We can then safely claim that the results summarized in table 4 apply equivalently to 2D-FLSA, and so the proposed Douglas Rachford method performs best when reconstruction ISNR is the primary concern.

6 Application of multidimensional TV

To our knowledge, TV regularization has not been previously applied to tensorial data (beyond matrices). We apply our multidimensional TV regularizer to anisotropic filtering for video denoising. The extension to videos from images is natural. Say a video contains $f$ frames of size $n \times m$ pixels; this video can be viewed as a 3D-tensor $X \in \mathbb{R}^{n \times m \times f}$, on which a 3D-TV based filter can be effected by

$$\min_X \frac{1}{2} \|X - U_0\|_F^2 + \lambda T_{v,1,1}^{3D}(X), \quad (6.1)$$

where $U_0$ is the observed noisy video, and $T_{v,1,1}^{3D} = T_{v,1}^3$ with $p = [p_1, p_2, p_3]$. Application of the filter (6.1) is nothing but computation of the prox operator, which can be done using the Parallel-Proximal Dykstra algorithm presented in Sec. 3.

We apply this idea to the salesman video sequence (frames from this video are shown in Appendix D). The video consists of 50 frames with a resolution of $288 \times 352$ grayscale pixels per frame; Figure 23 shows some of the frames of the sequence. Into each frame of this sequence, we inject independent additive gaussian noise with zero mean and variance 10 (Figure 24). Then we filter this noisy sequence following a two-step procedure:

1. Using only the first 25% frames of the video, look for the optimal value of the regularization parameter $\lambda$ in terms of ISNR value of the reconstruction. This was done testing for a range of values of $\lambda$, and selecting the best performing one.
2. Filter the whole sequence using the $\lambda$ value found.

The filtered sequence is shown in Figure 25. The anisotropic filter slightly blurs the result but successfully removes the noise, obtaining an ISNR value of 5.6. Once a good value for $\lambda$ has been chosen, filtering the entire video takes just 33 seconds. Given that the sequence contains in total $50 \times 288 \times 352 = 5068800$ pixels, the amount of data is comparable to filtering a 2D image of size $\approx 2251 \times 2251$ pixels. Looking at the previous results for image denoising, it seems like the Parallel-Proximal Dykstra algorithm performs well when compared against the filtering of the countour and phantom images. When the same filtering is performed in parallel using 2 processors, running time drops down to 19 seconds, confirming the suitability of the algorithm for multi-threaded computation.

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R. J. Tibshirani. Adaptive piecewise polynomial estimation via trend filtering. The Annals of Statistics, 42(1):285–323, 02 2014. doi: 10.1214/13-AOS1189.
A Mathematical background

We begin by recalling a few basic ideas from convex analysis; we recommend the recent book [Bauschke and Combettes, 2011] for more details.

Let $\mathcal{X} \subset \mathbb{R}^n$ be any set. A function $r : \mathcal{X} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called lower semicontinuous if for every $x \in \mathcal{X}$ and a sequence $(x_k)$ that converges to $x$, it holds that

$$x_k \to x \implies r(x) \leq \liminf_k r(x_k). \tag{A.1}$$

The set of proper lsc convex functions on $\mathcal{X}$ is denoted by $\Gamma_0(\mathcal{X})$ (such functions are also called closed convex functions). The indicator function of a set $C$ is defined as

$$\delta_C : \mathcal{X} \to [0, \infty] : x \mapsto \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{if } x \notin C, \end{cases} \tag{A.2}$$

which is lsc if and only if $C$ is closed.

The convex conjugate of $r$ is given by $r^*(z) := \sup_{x \in \text{dom } r} \langle x, z \rangle - r(x)$, and a particularly important example is the Fenchel conjugate of a norm $\|\cdot\|$

$$\text{if } r = \|\cdot\|, \text{ then } r^* = \delta_{\|\cdot\|_1}, \tag{A.3}$$

where the norm $\|\cdot\|_1$ is dual to $\|\cdot\|$. Let $r$ and $h$ be proper convex functions. The infimal convolution of $r$ with $h$ is the convex function given by $(r \boxplus h)(x) := \inf_{y \in \mathcal{X}} (r(y) + h(x - y))$. For our purposes, the most important special case is infimal convolution of a convex function with the squared euclidean norm, which yields the Moreau envelope [Moreau, 1962].
Proposition A.1. Let \( r \in \Gamma_0(X) \) and let \( \gamma > 0 \). The Moreau envelope of \( r \) indexed by \( \gamma \) is
\[
E^\gamma_r(\cdot) := r \square (\frac{1}{2\gamma} \| \cdot \|_2^2).
\] (A.4)

The Moreau envelope (A.4) is convex, real-valued, and continuous.

Proof. See e.g. [Bauschke and Combettes, 2011, Prop. 12.15]. \( \square \)

Using the Moreau envelope (A.4), we now formally introduce prox operators.

Definition A.2 (Prox operator). Let \( r \in \Gamma_0(X) \), and let \( y \in X \). Then \( \text{prox}_r y \) is the unique point in \( X \) that satisfies
\[
E^1_r(y) = \min_{x \in X} (r(x) + \frac{1}{2} \| x - y \|_2^2), \tag{A.5}
\]
and the nonlinear map \( \text{prox}_r : X \to X \) is called the prox operator of \( r \).

Sometimes the Fenchel conjugate \( r^* \) is easier to use than \( r \); similarly, sometimes the operator \( \text{prox}_{r^*} \) is easier to compute than \( \text{prox}_r \). The result below shows the connection.

Proposition A.3 (Moreau decomposition). Let \( r \in \Gamma_0(X) \), \( \gamma > 0 \), and \( y \in X \). Then,
\[
y = \text{prox}_{\gamma r} y + \gamma \text{prox}_{r^*/\gamma} (\gamma^{-1} y). \tag{A.6}
\]

Proof. A brief exercise; see e.g., [Bauschke and Combettes, 2011, Thm. 14.3]. \( \square \)

This decomposition provides the necessary tools to exploit useful primal–dual relations. For the sake of clarity we also present a last result regarding a particular primal-dual relation that plays a key role in our algorithms.

Proposition A.4. Let \( f \in \Gamma_0(X) \) and \( r \in \Gamma_0(Z) \). The problems below form a primal-dual pair.
\[
\begin{align*}
\inf_{x \in X} & \quad f(x) + r(Bx) \quad \text{s.t.} \quad Bx \in Z \tag{A.7} \\
\inf_{u \in Z} & \quad f^*(-B^Tu) + r^*(u). \tag{A.8}
\end{align*}
\]

Proof. Introduce an extra variable \( z = Bx \), so that the dual function is
\[
g(u) = \inf_{x \in X} f(x) + u^T Bx + \inf_{z \in Z} r(z) - u^T z,
\]
which upon rewriting using Fenchel conjugates yields (A.8). \( \square \)

B proxTV toolbox

All the Total–Variation proximity solvers in this paper have been implemented as the \texttt{proxTV} toolbox, available at \url{http://arantxa.ii.uam.es/~gaa/software.html}. The toolbox has been designed to be used out of the box in a user friendly way; the top–level Matlab function \texttt{TV} solves Total–Variation proximity for a given signal under a variety of settings. For instance
\[
>> \text{TV}(X, \text{lambda})
\]
solves \( T_{1} \) proximity for a signal \( X \) of any dimension and a regularization value \( \text{lambda} \). The weighted version of this problem is also seamlessly tackled by just providing a vector of weights of the appropriate length as the \texttt{lambda} parameter.

If a third parameter \( p \) is provided as

\[
\begin{align*}
\text{proxTV}(X, \text{lambda}, p)
\end{align*}
\]
the general $TV_p$ proximity problem is addressed, whereupon an adequate solver is chosen by the library. More advanced uses of the library are possible, allowing to specify which norm $p$ and regularizer $\lambda$ values to use for each dimension of the signal, and even applying combinations of several different $TV_p$ regularizers along the same dimension. Please refer to the documentation within the toolbox for further information.

C Proof on the equality of taut-string problems

**Theorem C.1 (Equality of taut-string problems).** Given the problems

\[
\min \sum_{i=1}^{n} (s_i - s_{i-1})^2, \quad \text{s.t. } |s_i - r_i| \leq w_i, \forall i = 1, \ldots, n-1, s_0 = 0, s_n = r_n,
\]

and

\[
\min \sum_{i=1}^{n} \sqrt{1 + (\hat{s}_i - \hat{s}_{i-1})^2}, \quad \text{s.t. } |\hat{s}_i - r_i| \leq w_i, \forall i = 1, \ldots, n-1, \hat{s}_0 = 0, \hat{s}_n = r_n,
\]

for a non-zero vectors $w$, both problems share the same minimum $s^\ast = \hat{s}^\ast$.

**Proof.** The Lagrangian of problem C.1 takes the form

\[
L(s, \alpha, \beta) = \sum_{i=1}^{n} (s_i - s_{i-1})^2 + \sum_{i=1}^{n-1} \alpha_i (s_i - r_i - w_i) + \sum_{i=1}^{n-1} \beta_i (-w_i - s_i + r_i),
\]

and its Karush-Kuhn-Tucker optimality conditions are given by

\[
(s_{i+1} - s_i) - (s_i - s_{i-1}) = \alpha_i - \beta_i,
\]

\[
|s_i - r_i| \leq w_i,
\]

\[
\alpha_i, \beta_i \geq 0,
\]

\[
\alpha_i (s_i - r_i - w_i) = 0,
\]

\[
\beta_i (-w_i - s_i + r_i) = 0,
\]

$\forall i = 1, \ldots, n-1$, and where the first equation comes from the fact that $\frac{\partial L(s, \alpha, \beta)}{\partial s_i} = 0$ at the minimum.

As the only difference between problems C.1 and C.2 is in the form of the objective, the KKT conditions for problem C.2 take the same form, but for the first one,

\[
\frac{(\hat{s}_{i+1} - \hat{s}_i)}{\sqrt{1 + (\hat{s}_{i+1} - \hat{s}_i)^2}} - \frac{(\hat{s}_i - \hat{s}_{i-1})}{\sqrt{1 + (\hat{s}_i - \hat{s}_{i-1})^2}} = \hat{\alpha}_i - \hat{\beta}_i,
\]

\[
|\hat{s}_i - r_i| \leq w_i,
\]

\[
\hat{\alpha}_i, \hat{\beta}_i \geq 0,
\]

\[
\hat{\alpha}_i (\hat{s}_i - r_i - w_i) = 0,
\]

\[
\hat{\beta}_i (-w_i - \hat{s}_i + \hat{r}_i) = 0,
\]

$\forall i = 1, \ldots, n-1$, and where we use hat notation for the dual coefficients to tell them apart from those of problem C.1.

Suppose $s^\ast$ minimizer to problem C.1, hence fulfilling the conditions C.3-C.7. In particular this means that it is feasible to assign values to the dual coefficients $\alpha, \beta$ in such a way that the conditions above are met. If we set $\hat{s} = s^\ast$ in the conditions C.8-C.12 the following observations are of relevance.
• Condition C.9 becomes the same as condition C.4, and so it is immediately met.

• The operator \( f(x) = \frac{x}{\sqrt{1+x^2}} \) is contractive and monotonous.

• The couple \((\alpha_i, \beta_i)\) cannot be both zero at the same time, since \(\alpha_i > 0\) enforces \(s_i = r_i + w_i\) and \(\beta_i > 0\) enforces \(s_i = r_i - w_i\).

• Hence and because \(\alpha_i, \beta_i \geq 0\) and condition C.3 holds, when \((s_{i+1} - s_i) - (s_i - s_{i-1}) > 0\) then \(\alpha_i > 0, \beta_i = 0\), and when \((s_{i+1} - s_i) - (s_i - s_{i-1}) < 0\) then \(\alpha_i = 0, \beta_i > 0\).

• \(f(s_{i+1} - s_i) - f(s_i - s_{i-1})\) has the same sign as \((s_{i+1} - s_i) - (s_i - s_{i-1})\), since \(f\) is monotonous and as such preserves ordering.

• Since \(f\) is contractive, condition C.8 can be met by setting \((\hat{\alpha}_i, \hat{\beta}_i) = (k\alpha_i, k\beta_i)\) for some \(0 \leq k < 1\).

• Condition C.10 is met for those choices of \(\hat{\alpha}_i, \hat{\beta}_i\), as C.5 was met for \(\alpha_i, \beta_i\) and \(0 \leq k < 1\).

• Conditions C.11 and C.12 are also met for those choices of \(\hat{\alpha}_i, \hat{\beta}_i\), as \(\hat{\alpha}_i(s_i - r_i - w_i) = k\alpha_i(s_i - r_i - w_i) = 0\) and \(\hat{\beta}_i(-w_i - s_i + r_i) = k\beta_i(-w_i - s_i + r_i) = 0\).

Therefore, all of the optimality conditions C.8-C.12 for problem C.2 are met for \(s^*\) solution of problem C.1, and so a minimum of problem C.1 is also a minimum for problem C.2.

The proof can be repeated the other way round by setting \(s = \hat{s}^*\) optimal for problem C.2, defining the operator \(f^{-1}(x) = \frac{x}{\sqrt{1-x^2}}\), and observing that this operator is monotonous and expansive, so we can establish \((\alpha_i, \beta_i) = (k\hat{\alpha}_i, k\hat{\beta}_i)\) for some \(k \geq 1\) and the optimality conditions C.3-C.7 for problem C.1 are met following a similar reasoning to the one presented above. Thus, a minimum for problem C.2 is also a minimum for problem C.1, which joined with the previous result completes the proof.

\[\square\]

D Testing images and videos, and experimental results

The images used in the experiments are displayed in what follows, along with their noisy/denoised and convoluted/deconvoluted versions for each algorithm tested. QR barcode images were generated by encoding random text using Google chart API. Images shape and phantom are publicly available and frequently used in image processing. trollface and comic are also publicly available. gaudi, used in the multicore experiments, is a high resolution 3197 \times 3361 photograph of Gaudi’s Casa Batlló. The rest of the images were originally created by the authors.

For the video experiments, the salesman sequence was used, which is publicly available at BM3D [2013]. Frames from the video are displayed in what follows, along with their noisy/denoised versions.

\[\footnote{http://code.google.com/intl/en-EN/apis/chart/}\]
\[\footnote{Extracted from \url{http://en.wikipedia.org/wiki/File:Shepp_logan.png}}\]
\[\footnote{Author: Francisco Molina. \url{http://www.afrikislife.net/english/}}\]
\[\footnote{Extracted from \url{http://www.flickr.com/photos/jeffschwartz/202423023/}}\]
Figure 18: Test images used in the experiments together with their sizes in pixels. Images displayed have been scaled down to fit in page.
Figure 19: Noisy versions of images used in the experiments.
Figure 20: Denoising results for the test images.
Figure 21: Noisy and convoluted versions of images used in the experiments.
Figure 22: Deconvolution results for the test images.
Figure 23: A selection of frames from the *salesman* video sequence.
Figure 24: Noisy frames from the salesman video sequence.
Figure 25: Denoised frames from the *salesman* video sequence using an anisotropic filter.