On the local density problem for graphs of given odd-girth

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\textbf{Abstract}
Erdős conjectured that every $n$-vertex triangle-free graph contains a subset of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. Extending a recent result of Norin and Yepremyan, we confirm this for graphs homomorphic to so-called Andrásfai graphs. As a consequence, Erdős’ conjecture holds for every triangle-free graph $G$ with minimum degree $\delta(G) > 10n/29$ and if $\chi(G) \leq 3$ the degree condition can be relaxed to $\delta(G) > n/3$. In fact, we obtain a more general result for graphs of higher odd-girth.

\textit{Keywords:} Andrásfai graphs, Erdős (1/2, 1/50) - conjecture, sparse halves
1 Introduction

We say an $n$-vertex graph $G$ is $(\alpha, \beta)$-dense if every subset of $\lfloor \alpha n \rfloor$ vertices spans more than $\beta n^2$ edges. Given $\alpha \in (0, 1]$ Erdős, Faudree, Rousseau, and Schelp [5] asked for the minimum $\beta = \beta(\alpha)$ such that every $(\alpha, \beta)$-dense graph contains a triangle. For example, Mantel’s theorem asserts that $\beta(1) = 1/4$. More generally, Erdős et al. conjectured that for $\alpha \geq 17/30$ the balanced complete bipartite graph gives the best lower bound for the function $\beta(\alpha)$, which leads to

$$\beta(\alpha) = \frac{1}{4}(2\alpha - 1).$$  \hfill (1)

The same authors verified this conjecture for $\alpha \geq 0.648$ and the best result in this direction is due to Krivelevich [9], who verified it for every $\alpha \geq 3/5$. We say a graph $G$ is a blow-up of some graph $F$, if there exists a partition $V(G) = \bigcup_{x \in V(F)} V_x$ such that

$$\forall x, y \in V(F) \forall a \in V_x \forall b \in V_y : ab \in E(G) \iff xy \in E(F).$$

For $\alpha < 17/30$ balanced blow-ups of the 5-cycle yield a better lower bound for $\beta(\alpha)$ and Erdős et al. conjectured

$$\beta(\alpha) = \frac{1}{25}(5\alpha - 2)$$  \hfill (2)

for $\alpha \in [53/120, 17/30]$. For $\alpha < 53/120$ it is known that balanced blow-ups of the Andrásfai graph $F_3$ (see Figure 1) lead to a better bound. The special case $\beta(1/2) = 1/50$ was considered before by Erdős (see, e.g., [4] for a monetary bounty for this problem).

Conjecture 1.1 (Erdős) Every $(1/2, 1/50)$-dense graph contains a triangle.

Besides the balanced blow-up of the 5-cycle Simonovits (see, e.g., [4]) noted that balanced blow-ups of the Petersen graph yield the same lower bound for Conjecture 1.1 and, more generally, for (2) in the corresponding range.

Conjecture 1.1 asserts that every triangle-free $n$-vertex graph $G$ contains a subset of $\lfloor n/2 \rfloor$ vertices that spans at most $n^2/50$ edges. Our first result

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(see Theorem 1.2 below) verifies this for graphs $G$ that are homomorphic to a triangle-free graph from a special class.

1.1 Andrásfai graphs

A well studied family of triangle-free graphs, which appear in the lower bound constructions for the function $\beta(\alpha)$ above, are the so-called Andrásfai graphs (see also Woodall [12]). For an integer $d \geq 1$ the Andrásfai graph $F_d$ is the $d$-regular graph with vertex set

$$V(F_d) = \{v_1, \ldots, v_{3d-1}\},$$

where $\{v_i, v_j\}$ forms an edge if

$$d \leq |i - j| \leq 2d - 1. \quad (3)$$

Note that $F_1 = K_2$ and $F_2 = C_5$ (see Figure 1). It is easy to check that Andrásfai graphs are triangle-free and balanced blow-ups of these graphs play a prominent role in connection with extremal problems for triangle-free graphs (see, e.g., [1],[6],[7],[3]).

![Fig. 1. Andrásfai graphs $F_2$, $F_3$, and $F_4$.](image)

Our first result validates Conjecture 1.1 (stated in the contrapositive) for graphs homomorphic to some Andrásfai graph.

**Theorem 1.2** If a graph $G$ is homomorphic to an Andrásfai graph $F_d$ for some integer $d \geq 1$, then $G$ is not $(1/2, 1/50)$-dense.

Since $F_d$ is homomorphic to $F_{d'}$ if and only if $d' \geq d$, Theorem 1.2 extends recent work of Norin and Yepremyan [11], who obtained such a result for $n$-vertex graphs $G$ homomorphic to $F_5$ with the additional minimum degree assumption $\delta(G) \geq 5n/14$. 

Owing to the work of Chen, Jin, and Koh [3], which asserts that every triangle-free 3-chromatic $n$-vertex graph $G$ with minimum degree $\delta(G) > n/3$ is homomorphic to some Andrásfai graph, we deduce from Theorem 1.2 that Conjecture 1.1 holds for all such graphs $G$.

Similarly, combining Theorem 1.2 with a result of Jin [7], which asserts that triangle-free graphs $G$ with $\delta(G) > 10n/29$ are homomorphic to $F_9$, implies Conjecture 1.1 for those graphs as well. We summarise these direct consequences of Theorem 1.2 in the following corollary.

**Corollary 1.3** Let $G$ be a triangle-free graph on $n$ vertices.

(i) If $\delta(G) > 10n/29$, then $G$ is not $(1/2, 1/50)$-dense.

(ii) If $\delta(G) > n/3$ and $\chi(G) \leq 3$, then $G$ is not $(1/2, 1/50)$-dense.

We remark that part (i) slightly improves earlier results of Krivelevich [9] and of Norin and Yepremyan [11] (see also [8] where an average degree condition was considered).

### 1.2 Generalised Andrásfai graphs of higher odd-girth

We consider the following straightforward variation of Andrásfai graphs of odd-girth at least $2k+1$, i.e., graphs without odd cycles of length at most $2k-1$. For integers $k \geq 2$ and $d \geq 1$ let $F_d^k$ be the $d$-regular graph with vertex set

$$V(F_d^k) = \{v_1, \ldots, v_{(2k-1)(d-1)+2}\},$$

where $\{v_i, v_j\}$ forms an edge if

$$(k-1)(d-1) + 1 \leq |i-j| \leq k(d-1) + 1. \tag{4}$$

In particular, for $k = 2$ we recover the definition of the Andrásfai graphs from (3) and for general $k \geq 2$ we have $F_1^k = K_2$, $F_2^k = C_{2k+1}$ and for every $d \geq 2$ the graph $F_d^k$ has odd-girth $2k+1$ (see Figure 2).

Our main result generalises Theorem 1.2 for graphs of odd-girth at least $2k+1$. In fact, the constant $\frac{1}{2(2k+1)^2}$ appearing in Theorem 1.4 is best possible as balanced blow-ups of $C_{2k+1}$ show.

**Theorem 1.4** If a graph $G$ is homomorphic to a generalised Andrásfai graph $F_d^k$ for some integers $k \geq 2$ and $d \geq 1$, then $G$ is not $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense.

Analogous to the relation between Conjecture 1.1 and Theorem 1.2 one may wonder if every $(\frac{1}{2}, \frac{1}{2(2k+1)^2})$-dense graph contains an odd cycle of length
Fig. 2. Generalised Andrásfai graphs $F_2^3$, $F_3^3$, and $F_4^3$ of odd-girth 7.

at most $2k - 1$. For $n$-vertex graphs $G$ with $\delta(G) > \frac{3n}{4k}$ such a result follows from Theorem 1.4 combined with the work from [10].

**Corollary 1.5** Let $G$ be a graph with odd-girth at least $2k + 1$ on $n$ vertices. If $\delta(G) > \frac{3n}{4k}$, then $G$ is not $(\frac{1}{2}, \frac{1}{2}(2k+1)^2)$-dense.

For $k = 2$ Theorem 1.4 reduces to Theorem 1.2. For the proof of Theorem 1.4 it will be convenient to work with a geometric representation of such graphs $G$. In that representation we will arrange the vertices of $G$ on the unit circle $\mathbb{R}/\mathbb{Z}$ and edges between two vertices $x$ and $y$ may only appear depending on their angle with respect to the centre of the circle.

Fig. 3. A copy of $F_2 = C_5$ and a representation of a blow-up on the unit circle.

For example, let $G$ be a blow-up of $F_2 = C_5$. One can distribute the vertices of $F_2$ equally spaced on the unit circle (see Figure 3). Then we place all vertices of $G$ that correspond to the blow-up class of $v_i$ into a small $\varepsilon$-ball around $v_i$ on the unit circle (cf. green arcs in Figure 3). For a sufficiently small $\varepsilon$, all vertices in an $\varepsilon$-ball around $v_i$ have the same neighbours and they can be characterised by having their smaller angle with respect to the centre bigger than $120^\circ$ (cf. red and blue lines in Figure 3).
For the proof of Theorem 1.2 we distinguish two cases depending on the independence number $\alpha(G)$ and refer to [2].

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