HIGH ORDER PARAMETER-ROBUST NUMERICAL
METHOD FOR A SYSTEM OF \((M \geq 2)\) COUPLED
SINGULARLY PERTURBED PARABOLIC
REACTION-DIFFUSION PROBLEMS

MUKESH KUMAR AND S. CHANDRA SEKHARA RAO

Abstract. We present a high order parameter-robust numerical method for a system of \((M \geq 2)\) coupled singularly perturbed parabolic reaction-diffusion problems. A small perturbation parameter \(\varepsilon\) is multiplied with the second order spatial derivatives in all the equations. The parabolic boundary layer appears in the solution of the problem when the perturbation parameter \(\varepsilon\) tends to zero. To obtain a high order approximation to the solution of this problem, we propose a numerical method that employs the Crank-Nicolson method on an uniform mesh in time direction, together with a hybrid finite difference scheme on a generalized Shishkin mesh in spatial direction. We prove that the resulting method is parameter-robust or \(\varepsilon\)-uniform of second order in time and almost fourth order in spatial variable, if the discretization parameters satisfy a non-restrictive relation. Numerical experiments are presented to validate the theoretical results and also indicate that the relation between the discretization parameters is not necessary in practice.

Key words. Singular perturbation, Parabolic reaction-diffusion problems, Coupled systems, High order compact scheme, Crank-Nicolson method, Parameter robust method, Generalized Shishkin mesh.

1. Introduction

We consider the following system of \((M \geq 2)\) coupled singularly perturbed parabolic reaction-diffusion problems

\[
\begin{align*}
L_\varepsilon u &= \frac{\partial u}{\partial t} + L_{x,\varepsilon} u = f, \quad (x, t) \in D := \Omega \times (0, T] = (0, 1) \times (0, T], \\
(1) &\quad u(0, t) = 0, \quad u(1, t) = 0, \quad \forall t \in [0, T], \\
(2) &\quad u(x, 0) = 0, \quad \forall x \in \Omega.
\end{align*}
\]

The spatial differential operator \(L_{x,\varepsilon}\) is defined by

\[
L_{x,\varepsilon} = \begin{pmatrix}
-\varepsilon \frac{\partial^2}{\partial x^2} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -\varepsilon \frac{\partial^2}{\partial x^2}
\end{pmatrix} + A,
\]

where \(\varepsilon\) is a small parameter that satisfies \(0 < \varepsilon \ll 1\). Denote the boundaries of the domain \(D\) by \(\Gamma := \Gamma_0 \cup \Gamma_1\), with \(\Gamma_0 = \{(x, 0)|x \in \Omega\}\)

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\[ \Gamma_1 = \{(x, t) | x = 0, 1, t \in [0, T]\} \]. We assume that the coupling matrix \( A = (a_{ij}(x))_{M \times M} \) satisfies the following positivity conditions at each \( x \in \Omega \)

\begin{align*}
(4) & \quad a_{ij} \leq 0, \ i \neq j, \\
(5) & \quad a_{ii} > 0, \sum_{j=1}^{M} a_{ij} \geq \beta^* > 0, \ i = 1, \ldots, M.
\end{align*}

If (5) is not satisfied directly, we consider the transformation \( \tilde{u}(x, t) = u(x, t) \exp(-\beta_0 t) \) with \( \beta_0 > 0 \) (sufficiently large) in order to transform the diagonal entries such that (5) holds. Also, we assume that sufficient regularity and compatibility conditions hold among the data of the problem (1)-(3) such that the exact solution \( u \in C^{6,3}(D)^M \). In the analysis we assume the following compatibility conditions (see [8])

\[ \frac{\partial^{s+q} f}{\partial x^s \partial t^q}(0,0) = \frac{\partial^{s+q} f}{\partial x^s \partial t^q}(1,0) = 0, \ \text{for} \ 0 \leq s + 2q \leq 4. \]

The numerical analysis of singular perturbation problems has always suffered from serious difficulties due to the boundary layer behavior of the solution when the perturbation parameter becomes small. Recent years have witnessed substantial progress in the development of layer adapted meshes to design a special class of numerical methods, so called parameter-robust numerical methods, that converge uniformly with respect to the perturbation parameter (see [15]). Parameter-robust numerical methods based on fitted meshes, particularly the Shishkin meshes gained popularity because of their simplicity and applicability to more complicated problems in higher dimensions, see [6] for more details. Several numerical studies for coupled system of singularly perturbed reaction-diffusion problems are considered in [10], [11], [12], [13], [16] and the references therein.

To solve the system of two coupled singularly perturbed parabolic reaction-diffusion problems with the distinct small perturbation parameters in each equations, Gracia and Lisbona [5] proposed a uniformly convergent numerical method by using the classical backward Euler scheme in time and the central difference scheme in spatial direction, and proved that the error bound is \( O(\Delta t + N^{-2+q} \ln^2 N) \) with the assumption \( N^{-q} \leq C \Delta t, \ 0 < q < 1 \). High order numerical methods have always been an interest for the numerical community as they provide good numerical approximations with low computational cost. Recently, Clavero et al. [4] gave an attempt to design a high order uniformly convergent numerical method for solving the system of two coupled singularly perturbed parabolic reaction-diffusion problems with
the distinct small perturbation parameters in each equations. To increase the order of uniform convergence, the authors in [4] considered the Crank-Nicolson method on an uniform mesh in time direction and central difference scheme on a standard Shishkin mesh in spatial direction, and proved that the error bound is $O((\Delta t)^2 + N^{-2+q} \ln^2 N)$ with the assumption $N^{-q} \leq C\Delta t$, $0 < q < 1$. To our knowledge this is the only high order parameter-robust numerical method is available in the literature for solving parabolic reaction-diffusion system (1)-(3). In the present paper, our objective is to integrate the available techniques for high order approximations (eg. [4] and [7]), to design a high order parameter-robust numerical method for solving parabolic reaction-diffusion system (1)-(3). For a high order approximation, we consider the Crank-Nicolson method on an uniform mesh in time, together with a hybrid scheme which is a suitable combination of the fourth order compact difference scheme and the standard central difference scheme on a generalized Shishkin mesh in spatial direction. It can be seen that the combination of Crank-Nicolson method in time direction with hybrid scheme in spatial direction does not satisfy the discrete maximum principle except if the restrictive condition $\Delta t \leq C(L/N)^2$ is imposed. In this article, we follow the approach of Clavero et al. [2] to overcome this difficulty. First, some auxiliary problems are considered which permits to prove appropriate bounds for local error of the Crank-Nicolson method. Then the uniform convergence analysis of the scheme used to discretize these auxiliary problems is discussed. Finally, using the recursive arguments and the uniform stability of the totally discrete scheme, we claim that the present method is uniformly convergent of second order in time and almost fourth order in spatial variable. It should be noted here that in the theoretical proof we assume the totally discrete scheme operator satisfy the uniform stability as a conjecture in Section 5. As so far it is an open problem to prove the uniform stability of totally discrete scheme theoretically (see also [4]). While in the support we presented the numerical tables (Tables 2, 4 and 6) that shows the spectral radius of the totally discrete operator is strictly less than one, independent of $\varepsilon$ and discretization parameters, in Section 6.

This paper is arranged as follows. In Section 2, a priori bounds on the solution of (1)-(3) and its derivative are constructed. The time semidiscretization using the Crank-Nicolson method and its local consistency error is given in Section 3. In this section we also discuss the asymptotic behavior of the solution of semidiscretized problems and their spatial derivatives. In Section 4, the generalized Shishkin mesh is given and the spatial semidiscretization with a hybrid scheme which is
a suitable combination of the fourth order compact difference scheme and the central difference scheme is described on generalized Shishkin mesh for the set of stationary singularly perturbed problems studied in Section 3. It is also proved that the spatial semidiscretization is almost fourth order uniformly convergent on generalized Shishkin mesh. In Section 5, semidiscretization steps are combined to give the total discretization and its uniform convergence is proved. The numerical experiments are conducted to demonstrate the efficiency of the proposed method in Section 6. Finally, conclusions are included in Section 7.

**Notations:** In the remaining parts of the paper, $C$ and $\mathbf{C} = C(1, \ldots, 1)^T$ are the generic positive constant independent of $\varepsilon$ and discretization parameters. Define $v \leq w$ if $v_i \leq w_i$, $1 \leq i \leq M$ and $|v| = (|v_1|, \ldots, |v_M|)^T$. We consider the maximum norm and it is denoted by $||.||_H$, where $H$ is a closed and bounded set. For a real valued function $v \in C(H)$ and for a vector valued function $\mathbf{v} = (v_1, \ldots, v_M)^T \in C(H)^M$, we define

$$||v||_H = \max_{x \in H} |v(x)| \text{ and } ||\mathbf{v}||_H = \max \{||v_1||_H, \ldots, ||v_M||_H\}.$$  

If $H = \overline{\Omega}$, we drop $H$ from the notation. The analogous discrete maximum norm on the mesh $\mathcal{N}$ is denoted by $||.||_{\mathcal{N}}$. For any function $g \in C(\overline{\Omega})$, $g_i$ is used for $g(x_i)$; if $\mathbf{g} \in C(\overline{\Omega})^M$ then $\mathbf{g}_i = \mathbf{g}(x_i) = (g_{1,i}, \ldots, g_{M,i})^T$. For simplicity, we use $L_{N_0}$ for $L(N_0)$. If $N_0 = N$, we drop $N$ as subscript from the notation $L_N$.

2. Properties of the exact solution

Following the technique of Theorem 1 in [5], we can show that the operator $L_\varepsilon$ in (1) satisfies the following maximum principle.

**Lemma 2.1.** Let $\mathbf{y} \in (C^{2,1}(D) \cap C^{0,0}(\overline{D}))^M$. Let $y(x, 0) \geq 0$ on $\overline{\Omega}$ and $y(0, t) \geq 0$, $y(1, t) \geq 0$ on $[0, T]$. Then $L_\varepsilon \mathbf{y} \geq 0$ in $D$ implies $\mathbf{y} \geq 0$ on $\overline{D}$.

An immediate consequence of Lemma 2.1 is the following stability result.

**Lemma 2.2.** Let $\mathbf{u}$ be the solution of (1)-(3). Then

$$||\mathbf{u}||_{\overline{D}} \leq \frac{1}{\beta^*} ||\mathbf{f}||_{\overline{D}}.$$  

To obtain the bounds on the solution $\mathbf{u}$ of (1)-(3), the variable $x$ is transformed to the stretched variable $\tilde{x}$ defined by $\tilde{x} = x/\sqrt{\varepsilon}$, this results that (1)-(3) transformed as
\begin{equation}
\widetilde{L}_\varepsilon \tilde{u} := \frac{\partial \tilde{u}}{\partial t} + \tilde{L}_{x,\varepsilon} \tilde{u} = \tilde{f}, \quad (x, t) \in \widetilde{D}_\varepsilon,
\end{equation}

\begin{equation}
\tilde{u}(\bar{x}, t) = 0, \quad (\bar{x}, t) \in \tilde{\Gamma}_\varepsilon,
\end{equation}

where

\[
\tilde{L}_{x,\varepsilon} = \begin{pmatrix}
-\frac{\partial^2}{\partial x^2} & 0 & \cdots & 0 \\
0 & -\frac{\partial^2}{\partial x^2} & \ddots & \vdots \\
0 & 0 & \ddots & -\frac{\partial^2}{\partial x^2} \\
\end{pmatrix} + \tilde{A}, \quad \text{with} \quad \tilde{A} = \begin{pmatrix}
\tilde{a}_{11}(\bar{x}) & \cdots & \tilde{a}_{1M}(\bar{x}) \\
\vdots & \ddots & \vdots \\
\tilde{a}_{M1}(\bar{x}) & \cdots & \tilde{a}_{MM}(\bar{x}) \\
\end{pmatrix},
\]

\(\tilde{D}_\varepsilon = \bar{\Omega}_\varepsilon \times (0, T] = (0, 1/\sqrt{\varepsilon}) \times (0, T]\) and \(\tilde{\Gamma}_\varepsilon\) is its boundary analogous to \(\Gamma\). Here the differential equation (6) is independent of \(\varepsilon\). Using the standard local estimate for the solution of system of time dependent partial differential equations (see [8]), we obtain the bounds on the solution of (6)-(7) and its derivative. On returning in term to the original variable \(x\) and using \(||u||_{\overline{\Omega}} \leq C\), obtained from Lemma 2.2 with \(\varepsilon\)-uniform boundedness of \(f\), yields the following result.

**Lemma 2.3.** Let \(u\) be the solution of (1)-(3). Then it satisfies

\[
\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\|_{\overline{\Omega}} \leq C \varepsilon^{-i/2}, \quad \text{for} \quad 0 \leq i + 2j \leq 6.
\]

In the following result, we derive sharper bounds on the derivatives of \(u\) to show that the large values seen in Lemma 2.3 do in fact decay rapidly as one moves away from the boundary \(\Gamma\).

**Lemma 2.4.** Let \(u\) be the solution of (1)-(3). Let \(\beta \in (0, \beta^*)\) be arbitrary but has a fixed value. Then there exists a constant \(C\), independent of \(\varepsilon\), such that

\[
\left| \frac{\partial^m u(x, t)}{\partial x^m} \right| \leq C \left(1 + \varepsilon^{-m/2}(\exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon}))\right)
\]

for \((x, t) \in \overline{D}\) and \(m = 0, \ldots, 6\).

**Proof.** Fix \(\beta \in (0, \beta^*)\) and set \(P_m(x) = 1 + \varepsilon^{-m/2}(\exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon}))\). The proof is by mathematical induction. The bound (8) for \(m = 0\) follows from Lemma 2.2. Assume that (8) holds for \(m = 0, \ldots, \nu - 1, 1 \leq \nu \leq 6\). We now prove (8) for \(m = \nu\). Letting

\[
y = \frac{\partial^\nu u}{\partial x^\nu},
\]
note that
\[
\begin{aligned}
\partial_t y - E \frac{\partial^2 y}{\partial x^2} + Ay &= \frac{\partial y}{\partial \nu} \bigg|_\nu - \sum_{l=0}^{\nu-1} \left( \begin{array}{c} \nu \\ l \end{array} \right) A^{(\nu-l)} \frac{\partial u}{\partial x} := \Psi_\nu \quad \text{in } D, \\
y(x, 0) &= 0 \quad \text{in } \bar{\Omega}, \\
||y(0, t)|| &\leq C \varepsilon^{-\nu/2}, ||y(1, t)|| \leq C \varepsilon^{-\nu/2} \quad \text{in } (0, T],
\end{aligned}
\]
where boundary conditions follow from Lemma 2.3. From the inductive hypothesis, it is clear that
\[
||\Psi_\nu(x, t)|| \leq C P_{\nu-1} \left( x \right).
\]
Applying the maximum principle with the barrier function \(C P_{\nu} \left( x \right)\), we obtain the required result, i.e., for \((x, t) \in D\)
\[
\left| \frac{\partial^s u(x, t)}{\partial x^s} \right| \leq C \left( 1 + \varepsilon^{-\nu/2}(\exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon})) \right).
\]
This proves the lemma.

Now a special decomposition of the exact solution \(u\) into a regular part \(v\) and a layer part \(w\) can be obtained as follows. Set \(x^* = 4 \sqrt{\varepsilon/\beta} \ln(1/\sqrt{\varepsilon})\). Define for each \(j \in \{1, \ldots, n\}\) and \((x, t) \in \bar{D}\)
\[
v_j(x, t) = \begin{cases} 
\sum_{\nu=0}^{4} \frac{(x - x^*)^\nu}{\nu!} \partial_x^\nu u_j(x^*, t) & \text{for } 0 \leq x \leq x^*, t \in [0, T]; \\
u_j(x, t) & \text{for } x^* \leq x \leq 1 - x^*, t \in [0, T]; \\
\sum_{\nu=0}^{4} \frac{(x - x^*)^\nu}{\nu!} \partial_x^\nu u_j(1 - x^*, t) & \text{for } 1 - x^* \leq x \leq 1, t \in [0, T],
\end{cases}
\]
and \(w_j(x, t) = u_j(x, t) - v_j(x, t)\). Then Lemma 2.4 and the choice of \(x^*\) yields, for \(s = 0, \ldots, 6\), (cf. Linss [9])
\[
\begin{align}
\left| \partial^s v_j(x, t) \right| &\leq C(1 + \varepsilon^{2-s/2}) \tag{10a} \\
\left| \partial^s w_j(x, t) \right| &\leq C \varepsilon^{-s/2} \left( \exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon}) \right). \tag{10b}
\end{align}
\]
It should be noted here that this decomposition does not, in general, satisfy \(L_\varepsilon v = f\) and \(L_\varepsilon w = 0\).

3. The time semidiscretization

We introduce the time semidiscretization of (1)-(3) by using the classical Crank-Nicolson method, with constant time step \(\Delta t\) on uniform mesh \(\varpi = \{n \Delta t, 0 \leq n \leq T/\Delta t\}\). The time semidiscretization is given
by

\begin{align}
\begin{cases}
    u^0 = u(x, 0) = 0, \\
    (I + \frac{\Delta t}{2} L_{x, \varepsilon}) u^{n+1} = (I - \frac{\Delta t}{2} L_{x, \varepsilon}) u^n + \frac{\Delta t}{2} (f^n + f^{n+1}), \\
    u^{n+1}(0) = u^{n+1}(1) = 0,
\end{cases}
\end{align}

where \( u^n \) is the approximation of the exact solution \( u \) of (1)-(3) at the time level \( t_n = n \Delta t, \ n = 0, 1, \ldots, T/\Delta t - 1 \).

To study the consistency of (11), we define the following auxiliary problem

\begin{align}
\begin{cases}
    \hat{L}_{x, \varepsilon} \hat{u}^{n+1} := (I + \frac{\Delta t}{2} L_{x, \varepsilon}) \hat{u}^{n+1} = (I - \frac{\Delta t}{2} L_{x, \varepsilon}) u(x, t_n) + \frac{\Delta t}{2} (f^n + f^{n+1}), \\
    \hat{u}^{n+1}(0) = \hat{u}^{n+1}(1) = 0,
\end{cases}
\end{align}

where \( \hat{u}^{n+1} \) is the approximation to \( u(x, t_{n+1}) \). Let \( e_{n+1}(x) = u(x, t_{n+1}) - \hat{u}^{n+1}(x) \) be the local truncation error of (11) and it satisfies the following lemma.

**Lemma 3.1.** If

\[
\left| \frac{\partial^i u(x, t)}{\partial t^i} \right| \leq C, \quad (x, t) \in \overline{D}, \quad 0 \leq i \leq 3,
\]

then the local error associated to the scheme (11) satisfies

\[
|e_{n+1}(x)| \leq C(\Delta t)^3, \quad x \in \overline{\Omega}.
\]

**Proof.** The results follows from the arguments given in [2]. \(\square\)

Now we prove that the asymptotic behavior of the solution of the semidiscretize problem (12) and its spatial derivative have essentially the same asymptotic behavior that the solution of a stationary system of coupled singularly perturbed reaction-diffusion problems. Using the approach Clavero et al. [1], such feature is given by the following lemma.

**Lemma 3.2.** Let \( \hat{u}^{n+1} \) be the solution of (12). Then it satisfies

\[
\left| \frac{d^k \hat{u}^{n+1}}{dx^k} \right| \leq C \left( 1 + \varepsilon^{-k/2}(\exp(-x\sqrt{\beta}/\varepsilon) + \exp(-(1-x)\sqrt{\beta}/\varepsilon)) \right),
\]

where \( 0 \leq k \leq 6 \) and \( C \) is a constant independent of \( \varepsilon \) and \( \Delta t \).

**Proof.** Let us first start by studying the behaviour of \( \hat{u}^{n+1} \), that means the result (13) for \( k = 0 \). As the data \( f \) is \( \varepsilon \)-uniformly bounded, \( |u(x, t_n)| \leq C \) and \( |L_{x, \varepsilon} u(x, t_n)| \leq C \); similar to [12], the operator \((I + \frac{\Delta t}{2} L_{x, \varepsilon})\) satisfies a maximum principle and using this it follows that

\[
|\hat{u}^{n+1}| \leq C.
\]
To prove the result (13) for the derivatives of \( \hat{u}^{n+1} \), we introduce the following auxiliary function

\[
\phi^{n+1} = \frac{2}{\Delta t}(\hat{u}^{n+1}(x) - u(x, t_n)),
\]

which is the solution of the following boundary value problem

\[
\begin{cases}
(I + \frac{\Delta t}{2}L_{x, \varepsilon})\phi^{n+1} = -2L_{x, \varepsilon}u(x, t_n) + f^n + f^{n+1}, \\
\phi^{n+1}(0) = 0, \quad \phi^{n+1}(1) = 0.
\end{cases}
\]

Using \( |L_{x, \varepsilon}u(x, t_n)| = |f(x, t_n) - \frac{\partial u}{\partial t}(x, t_n)| \leq C \) and \( |\phi^{n+1}(0)| \leq C, \ |\phi^{n+1}(1)| \leq C \) with the maximum principle for \((I + \frac{\Delta t}{2}L_{x, \varepsilon})\) we get

\[
|\phi^{n+1}| \leq C.
\]

Next we write the problem (12) as

\[
\begin{cases}
L_{x, \varepsilon}\hat{u}^{n+1} = -\phi^n + L_{x, \varepsilon}u(x, t_n) + f^n + f^{n+1}, \\
\hat{u}^{n+1}(0) = 0, \quad \hat{u}^{n+1}(1) = 0.
\end{cases}
\]

From \( |\phi^n| \leq C \), it can be seen that the right side of (15) is \( \varepsilon \)-uniformly bounded. Using this with \( |\hat{u}^{n+1}| \leq C \) we get

\[
\left\| \frac{d^2\hat{u}^{n+1}}{dx^2} \right\|_{\Omega} \leq C\varepsilon^{-1},
\]

From (16) and using the mean value theorem argument as used in [12], we obtain

\[
\left\| \frac{d\hat{u}^{n+1}}{dx} \right\|_{\Omega} \leq C\varepsilon^{-1/2}.
\]

On differentiating (12) with respect to \( x \), we define \( \zeta^{n+1}_i = \frac{d\hat{u}^{n+1}}{dx_i}, \ i = 1, 2 \), are the solutions of boundary value problems

\[
\begin{cases}
(I + \frac{\Delta t}{2}L_{x, \varepsilon})\zeta^{n+1}_i = g_i(x), \\
\zeta^{n+1}_i(0) = s^0_i, \quad \zeta^{n+1}_i(1) = s^1_i,
\end{cases}
\]

where \( |s^0_i| \leq C\varepsilon^{-i/2}, \ i = 1, 2, \ j = 0, 1 \) and using Lemma 2.4 \( |g_i(x)| \leq C(1 + \varepsilon^{-i/2})(\exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon})) \), \( i = 1, 2 \).

Now taking the barrier function as

\[
\tilde{\zeta}(x) = C_1(1 + x) + C_2\varepsilon^{-i/2}(\exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon}))
\]

and for sufficiently large value of \( C_1 \) and \( C_2 \) using the maximum principle for \((I + \frac{\Delta t}{2}L_{x, \varepsilon})\), we deduce that (13) is true for \( k = 1, 2 \).
Now to prove the bound (13) for higher value of \( k \), we follow similar arguments given in [1]. This proves the lemma. \( \square \)

Next we define the Shishkin-type decomposition for the solution of semidiscretize problem (12). This type of decomposition has been discussed earlier in Linss [9], for scalar singularly perturbed boundary value problem. To define this, let \( x^* = (4\sqrt{\varepsilon/\beta}) \ln(1/\sqrt{\varepsilon}) \). Similar to (9), for each \( x \in \overline{\Omega} \) and \( k = 1, \ldots, M \), we set \( \hat{u}^{n+1}_{k} = \hat{u}^{n+1}_{k} \) for \( x \in [x^*, 1 - x^*] \) and \( \hat{v}^{n+1} \) extends to a smooth function defined on \( \Omega \) and define \( \hat{w}^{n+1}_{k} = \hat{u}^{n+1}_{k} - \hat{v}^{n+1}_{k} \) for \( x \in \overline{\Omega} \). Then the results of Lemma 3.2 and the choice of \( x^* \) implies the following decomposition of \( \hat{u}^{n+1} \) (cf. Linss [9])

**Lemma 3.3.** Let \( \hat{u}^{n+1} \) be the solution of (12). Then it can be represented as \( \hat{u}^{n+1} = \hat{v}^{n+1} + \hat{w}^{n+1} \), where the regular part \( \hat{v}^{n+1} \) satisfies

\[
\left| \frac{dm_{ik}^{n+1} \hat{v}^{n+1}_{k}}{dx^{m}} \right| \leq C(1 + \varepsilon^{2-m/2}),
\]

and the layer part \( \hat{w}^{n+1} \) satisfies

\[
\left| \frac{dm_{ik}^{n+1} \hat{w}^{n+1}_{k}}{dx^{m}} \right| \leq C \varepsilon^{-m/2} \left( \exp(-x\sqrt{\beta/\varepsilon}) + \exp(-(1-x)\sqrt{\beta/\varepsilon}) \right),
\]

for \( 0 \leq m \leq 6 \), \( k = 1, \ldots, M \), and \( C \) is a constant independent of \( \varepsilon \) and \( \Delta t \).

The above lemma shows that the solution \( \hat{u}^{n+1} \) of (12) is decomposed into a sum of regular part \( \hat{v}^{n+1} = (\hat{v}^{n+1}_{1}, \ldots, \hat{v}^{n+1}_{M})^{T} \) and layer part \( \hat{w}^{n+1} = (\hat{w}^{n+1}_{1}, \ldots, \hat{w}^{n+1}_{M})^{T} \), that is, it can be written as \( \hat{u}^{n+1} = \hat{v}^{n+1} + \hat{w}^{n+1} \). This decomposition is said to be a Shishkin-type decomposition (not a standard Shishkin decomposition) as it does not in general satisfy (I + \( \frac{\Delta t}{2} L_{x,\varepsilon} \))\( \hat{v}^{n+1} = (I - \frac{\Delta t}{2} L_{x,\varepsilon})\hat{v}(x, t_n) + \frac{\Delta t}{2} (f^{n} + f^{n+1}) \) and (I + \( \frac{\Delta t}{2} L_{x,\varepsilon} \))\( \hat{w}^{n+1} = (I - \frac{\Delta t}{2} L_{x,\varepsilon})\hat{w}(x, t_n) \), as these additional properties are not needed in the error analysis of present method.

4. The spatial semidiscretization

In this section, first, we construct a generalized Shishkin mesh \( S(L) \) to discretized the domain \( \overline{\Omega} := [0, 1] \) by using a suitable mesh generating function \( \mathcal{K} \) as described in [17]. Define the transition parameter

\[
\sigma = \min \left\{ \frac{1}{4}, \sigma_0 \sqrt{\varepsilon} L \right\},
\]

where \( \sigma_0 (\geq 4/\sqrt{\beta}) \) is a positive constant and \( L = L(N) \) the value of \( L \) with \( N \) elements that satisfy \( \ln(\ln N) < L \leq \ln N \) and

\[
e^{-L} \leq \frac{L}{N}.
\]
The mesh points of generalized-Shishkin discretized domain \( \Omega^S_N \) are given by \( x_j = \mathcal{K}(j/N), \ j = 0, 1, \ldots, N/2, \) and by symmetry \( x_{N-j} = 1 - x_j, \ j = 0, 1, \ldots, N/2, \) where \( \mathcal{K} \in C^2[0, 1/2] \) and defined as

\[
\mathcal{K}(t) = \begin{cases} 
4\sigma t, & \text{for } t \in [0, 1/4]; \\
p(t - 1/4)^3 + 4\sigma(t - 1/4) + \sigma, & \text{for } t \in [1/4, 1/2]. 
\end{cases}
\]

Here the coefficient \( p \) is determined by \( \mathcal{K}(1/2) = 1/2. \)

Note that the mesh \( \Omega^S_N \) is uniform in \([0, \sigma]\) and \([1 - \sigma, 1]\), and it changes smoothly in the transition points \( \{\sigma, 1 - \sigma\} \). However, the mesh width \( h_j = x_{j+1} - x_j \), for \( j = N/4, \ldots, 3N/4, \) satisfies (see \([17]\))

\[
h_{j+1} \leq N^{-1} \max_{[\{i-1\}/N, \{i+1\}/N]} \mathcal{K}'(t) \leq CN^{-1}
\]

\[
h_{j+1} - h_j \leq N^{-2} \max_{[\{i-1\}/N, \{i+1\}/N]} \mathcal{K}''(t) \leq CN^{-2}.
\]

We shall let \( h_{\text{max}} = \max_{v_j} h_j, j = 1, 2, \ldots, N, \) and by symmetry it is easy to verify that \( h_{\text{max}} = h_{N/2} = h_{N/2+1}. \)

### 4.1. The hybrid scheme

We introduce a hybrid scheme to discretize the set of stationary coupled system of singularly perturbed reaction-diffusion problem \([12]\) on the generalized Shishkin mesh \( \Omega^S_N. \) The hybrid scheme is a combination of the fourth order compact difference scheme (where the coefficients \( q_i^k \)'s and \( r_i^k \)'s of the scheme are determined so that the scheme is exact for the polynomials up to degree four and satisfy the normalization conditions \( q_i^{k,-} + q_i^{k,c} + q_i^{k,+} = 1, i = 1, 2, \ldots, N - 1, k = 1, 2, \ldots, M \)) and the central difference scheme, and is given by

\[
\left[ \hat{L}^N_{x,\varepsilon} \hat{U}^{n+1}\right]_i = [Q^n]_i, \text{ for } i = 1, 2, \ldots, N - 1,
\]

\[
\hat{U}^{n+1}_0 = 0, \quad \hat{U}^{n+1}_N = 0,
\]

where

\[
\left[ \hat{L}^N_{x,\varepsilon} \hat{U}^{n+1}\right]_i := \begin{pmatrix} R(\hat{U}^{n+1}_1) + \frac{\Delta t}{2} Q(a_{12} \hat{U}^{n+1}_2) + \ldots + \frac{\Delta t}{2} Q(a_{1M} \hat{U}^{n+1}_M) \\
R(\hat{U}^{n+1}_2) + \frac{\Delta t}{2} Q(a_{21} \hat{U}^{n+1}_1) + \ldots + \frac{\Delta t}{2} Q(a_{2M} \hat{U}^{n+1}_M) \\
\vdots \\
R(\hat{U}^{n+1}_M) + \frac{\Delta t}{2} Q(a_{M1} \hat{U}^{n+1}_1) + \ldots + \frac{\Delta t}{2} Q(a_{MM-1} \hat{U}^{n+1}_{M-1}) \end{pmatrix}_i
\]
\[ [\hat{Q}^n]_i := \begin{pmatrix} Q(\hat{f}_1^n) \\ Q(f_2^n) \\ \vdots \\ Q(\hat{f}_M^n) \end{pmatrix}, \]

(27)

\[
\begin{cases}
R(V_{k,i}) = r_{i}^{k,-}V_{k,i-1} + r_{i}^{k,c}V_{k,i} + r_{i}^{k,+}V_{k,i+1}, \\
Q(V_{k,i}) = q_{i}^{k,-}V_{k,i-1} + q_{i}^{k,c}V_{k,i} + q_{i}^{k,+}V_{k,i+1}.
\end{cases}
\]

(28)

with

\[
\hat{f}^n(x_i) = u(x_i, t_n) + \frac{\Delta t}{2}(-[L_{x,c}u](x_i, t_n) + f(x_i, t_n) + f(x_i, t_{n+1})).
\]

The coefficients \(r_i^{k,*}, i = 1, \ldots, N - 1, k = 1, 2, \ldots, M, * = -, c, +\) are given by

(29)

\[
\begin{align*}
& r_i^{k,-} = \frac{\Delta t}{2} (h_{\text{max}}^{-2 \varepsilon} - (a_{k,k;i-1} + \frac{2 \varepsilon}{\Delta t})) \\
& r_i^{k,c} = \frac{\Delta t}{2} (q_{i}^{k,-} - a_{k,k;i-1} + q_{i}^{k,c} + q_{i}^{k,+}) - r_i^{k,-} + r_i^{k,+}
\end{align*}
\]

The coefficients \(q_i^{k,*}, i = 1, \ldots, N - 1, k = 1, 2 \ldots, M, * = -, c, +\) are defined in two different ways.

(i) For the mesh points located in \((0, \tau) \cup (1 - \tau, 1)\), the coefficients \(q_i^{k,*}, i = \{1, \ldots, N/4 - 1\} \cup \{3N/4 + 1, \ldots, N - 1\}, k = 1, 2, \ldots, M, * = -, c, +\) are given by

(30)

\[ q_i^{k,-} = \frac{1}{12}, \quad q_i^{k,c} = \frac{5}{6}, \quad q_i^{k,+} = \frac{1}{12}. \]

(ii) For the mesh points located in \([\tau, 1 - \tau]\), depending on the relation between \(h_{\text{max}}\) and \(\varepsilon\), the coefficients \(q_i^{k,*}, i = 1, \ldots, N - 1, k = 1, 2, \ldots, M, * = -, c, +\) are defined in two different cases. Define \(\hat{a}_{kk} = a_{kk} + 2/\Delta t\) for \(k = 1, 2, \ldots, M\).

In the first case, when \(\gamma h_{\text{max}}^2 ||\hat{a}_{kk}||_{\infty} \leq \varepsilon\), where \(\gamma\) is a positive constant independent of \(\varepsilon\) and \(\Delta t\), the coefficients \(q_i^{k,*}, i = N/4, \ldots, 3N/4, k = 1, 2, \ldots, M, * = -, c, +\) are given by

(31)

\[ q_j^{k,-} = \frac{2h_j - h_{j+1}}{6(h_j + h_{j+1})}, \quad q_j^{k,c} = \frac{5}{6}, \quad q_j^{k,+} = \frac{2h_{j+1} - h_j}{6(h_j + h_{j+1})}. \]

While in the second case, when \(\gamma h_{\text{max}}^2 ||\hat{a}_{kk}||_{\infty} > \varepsilon\), where \(\gamma\) is a positive constant independent of \(\varepsilon\) and \(\Delta t\), the coefficients \(q_i^{k,*}, i = \ldots, 3N/4, k = \ldots, 1, 2, \ldots, M, * = -, c, +\) are given by

(31)
\[N/4, \ldots, 3N/4, \ k = 1, 2, \ldots, M, \ * = -, c, +, \] are given by
\[(32) \quad q_i^{k,-} = 0, \quad q_i^{k,c} = 1, \quad q_i^{k,+} = 0.\]

The above definition of coefficients \(q_i^{k}\)'s and \(r_i^{k}\)'s show that the scheme (24)-(25) is defined by the fourth order compact difference scheme within the boundary layer region \((0, \tau) \cup (1 - \tau, 1)\). While in the regular region \(\tau, 1 - \tau\), the scheme (24)-(25) is defined by a modified high order non-equidistant difference scheme when \(\gamma h_{\text{max}}^2 \| \tilde{a}_{kk} \|_\infty \leq \varepsilon\) and is defined by the central difference scheme when \(\gamma h_{\text{max}}^2 \| \tilde{a}_{kk} \|_\infty > \varepsilon\). This means that the scheme (24)-(25) considers the high-order approximation only when the local mesh width is small enough to give non-positive off-diagonal entries while at all other mesh points the central difference scheme is used. This combination leads to the following lemma.

**Lemma 4.1.** Let \(\gamma = 1/6\) and \(N_0\) be the smallest positive integer such that
\[\max_k \{4\sigma_0^2(\|a_{kk}\|_\infty + 2/\Delta t)/3\} < (N_0/L_{N_0})^2,\]
where \(L_{N_0} = L(N_0)\) defined in (21). Then, for any \(N \geq N_0\), the discrete operator defined by (24)-(25) is of positive type.

**Proof.** Firstly, for \(x_i \in (0, \tau) \cup (1 - \tau, 1)\), the fourth order compact difference scheme is considered in this region. The condition \[\max_k \{4\sigma_0^2(\|a_{kk}\|_\infty + 2/\Delta t)/3\} < (N_0/L_{N_0})^2\]
for any \(N \geq N_0\), where \(L_{N_0} = L(N_0)\) as defined in (21), with the the coefficients \(q_i^{k,*}\), \(r_i^{k,*}\), \(* = -, c, +, \ k = 1, 2, \ldots, M\), defined by (29)-(30) and the assumption (4)-(5), concludes the lemma.

Secondly, for \(x_i \in [\tau, 1 - \tau]\) when \(\gamma h_{\text{max}}^2 \| \tilde{a}_{kk} \|_\infty > \varepsilon\), the central difference scheme is considered. Hence the proof is trivial.

While in the opposite case, for \(x_i \in [\tau, 1 - \tau]\) when \(\gamma h_{\text{max}}^2 \| \tilde{a}_{kk} \|_\infty \leq \varepsilon\), we decompose \([\tau, 1 - \tau] := [\tau, x_{N/2}] \cup [x_{N/2}, 1 - \tau]\) and study the sign of coefficients \(q_i\)'s in these two cases, separately. First, when \(x_i \in [\tau, x_{N/2}]\) the coefficients \(q_i^{k,+}\) is clearly non-negative on generalized Shishkin mesh \(\Omega_N^S\) while the coefficient \(q_i^{k,-}\) will be non-negative when \(2h_i - h_{i+1} \geq 0\) for \(N/4 \leq i \leq N/2\). The assertion is trivially true for \(i = N/2\) because of uniform mesh (at symmetry). For \(N/4 \leq i \leq N/2 - 1\), we can write
\[h_i \geq h_{i+1} - h_i,\]
follows if (cf. (23))
\[N\mathcal{K}'((i - 1)/N) \geq \mathcal{K}''((i + 1)/N),\]
that is, if
\[ \tilde{w}(z) = 3pz^2 - 6pz - 12p + 4\sigma N^2 \geq 0, \]
where \( z = i - 1 - N/4 \geq -1 \). It is not hard to verify that the discriminant of the quadratic function \( \tilde{w} \) is non-positive if \( 4\sigma N^2 \geq 15p \). Since \( \gamma h_{\text{max}}^2 \| \hat{a}_{kk} \|_{\infty} \leq \varepsilon \) definitely implies \( 4\sigma N^2 \geq 15p \), it follows that \( \tilde{w}(z) \geq 0 \) for all \( z \). This completes the proof.

Similar to this, we can prove \( q_i^{k,+} \geq 0 \) for \( N/2 \leq i \leq 3N/4 \). Thus the condition \( \gamma h_{\text{max}}^2 \| \hat{a}_{kk} \|_{\infty} \leq \varepsilon \) with the coefficients \( q_i^{k,*}, r_i^{k,*}, * = -, c, +, k = 1, 2, \ldots, M \), defined by (29),(31) and the assumption (4)-(5), concludes the lemma.

**Remark 4.2.** It can be seen that the scheme (24)-(25) on standard Shishkin mesh does not satisfy the above Lemma 4.1 because the coefficients \( q_i^{k,*} \) are not always non-negative at the transition points, due to the fact that the standard Shishkin mesh is very anisotropic in nature. While if we consider the scheme (24)-(25) on the generalized Shishkin mesh \( \Omega_S^N \), then the coefficients \( q_i^{k,*} \) are always non-negative. This is used in the proof of Lemma 4.1.

Using the Lemma 4.1, the discretization operator defined by (24)-(25) is of positive type and it satisfies the following discrete comparison principle.

**Lemma 4.3.** (Discrete Comparison Principle) Let \( \hat{V} \) and \( \hat{W} \) be two mesh functions and satisfy
\[ [\hat{L}_{x,\varepsilon}^N \hat{V}]_i \geq [\hat{L}_{x,\varepsilon}^N \hat{W}]_i, \quad i = 1, 2, \ldots, N-1, \quad \hat{V}_0 \geq \hat{W}_0 \quad \text{and} \quad \hat{V}_N \geq \hat{W}_N, \]
then \( \hat{V}_i \geq \hat{W}_i \; i = 0, 1, \ldots, N \).

Using the above discrete comparison principle we obtain the following discrete stability estimate.

**Lemma 4.4.** (Discrete Stability Estimate) Let \( \hat{V} \) be the mesh function with \( \hat{V}_0 = \hat{V}_N = 0 \). Then
\[ \| \hat{V} \| \leq C \| \hat{L}_{x,\varepsilon}^N \hat{V} \| \]
where \( C \) is independent of \( N, \Delta t \) and \( \varepsilon \).

Let \( \Gamma_{\tilde{u}^{n+1}}(x_i) \) be the truncation error associated to the scheme (24)-(25) and is defined by
\[ \Gamma_{\tilde{u}^{n+1}}(x_i) = [\hat{L}_{x,\varepsilon}^N (\tilde{u}^{n+1} - \hat{U}^{n+1})]_i. \]
Lemma 4.5. Let \( \hat{u}^{n+1} \) be the solution of \( (12) \) and \( \hat{U}^{n+1} \) be the approximate solution of the spatial discretized scheme \((24)-(25)\). Let the hypothesis of Lemma 4.1 be satisfied. Then the global error satisfies

\[
|\hat{u}^{n+1}(x_i) - \hat{U}^{n+1}_i| \leq C\Delta t(L/N)^4,
\]

with the assumption that \( L^{-4} \leq C\Delta t \), where \( C \) and \( \mathcal{C} \) are positive constants independent of \( N, \Delta t \) and \( \varepsilon \).

Proof. If \( \tau = 1/4 \), then the mesh \( D_N \) is uniform, that is, \( N^{-1} \) is very small respect to \( \varepsilon \) and therefore a classical analysis can be used to prove the convergence of the scheme. So, in the analysis we only consider the case \( \tau = \sigma_0 \sqrt{\varepsilon} L \).

The truncation error estimate \( \Gamma_{\hat{u}^{n+1}}(x_i) \) of the scheme \((24)-(25)\) on the generalized Shishkin mesh \( D_N \) is discussed in the following cases.

(I) When \( x_i \in (0, \tau) \cup (1 - \tau, 1) \), we have \( h_i = h_{i+1} = 4\sigma_0 \sqrt{\varepsilon} N^{-1} L \). By Taylor expansions we obtain

\[
|\Gamma_{\hat{u}^{n+1}}(x_i)| \leq C\varepsilon \Delta t h_i \left\| \frac{d^6 \hat{u}^{n+1}}{dx^6} \right\|_{[x_{i-1}, x_{i+1}]},
\]

Using \( h_i = 4\sigma_0 \sqrt{\varepsilon} N^{-1} L \) and \( \left\| \frac{d^6 \hat{u}^{n+1}}{dx^6} \right\| \leq C\varepsilon^{-3} \), it follows that

\[
|\Gamma(u)|_{i} \leq C\Delta t(L/N)^4,
\]

(II) When \( x_i \in [\tau, 1 - \tau] \), according to the decomposition \( \hat{u}^{n+1} = \hat{v}^{n+1} + \hat{w}^{n+1} \), split the truncation error into two parts to obtain

\[
|\Gamma_{\hat{u}^{n+1}}(x_i)| \leq |\Gamma_{\hat{v}^{n+1}}(x_i)| + |\Gamma_{\hat{w}^{n+1}}(x_i)|.
\]

For the mesh points located in \([\tau, 1 - \tau]\), depending on the relation between \( h_{\text{max}} \) and \( \varepsilon \), the scheme \((24)-(25)\) is defined by the combination modified high order non-equidistant difference scheme and the central difference scheme. The error analysis for both cases are given as follows.

(i) For the case \( \gamma h_{\text{max}}^2 \leq \varepsilon \), suppose \( g \in C^6[0, 1]^M \), then by Taylor expansions we obtain

\[
|\Gamma_{g,k}(x_i)| \leq C\varepsilon \Delta t(P_{k,i} + Q_{k,i} + R_{k,i}), \quad \text{for } k = 1, \ldots, M,
\]

where

\[
\begin{align*}
P_{k,i} &= (h_{i+1} - h_i)^2 \left\| g_k^{(4)} \right\|_{[x_{i-1}, x_{i+1}]}, & Q_{k,i} &= |h_{i+1} - h_i| (h_{i+1} + h_i)^2 \left\| g_k^{(5)} \right\|_{[x_{i-1}, x_{i+1}]}, \quad R_{k,i} &= (h_i^4 + h_{i+1}^4) \left\| g_k^{(6)} \right\|_{[x_{i-1}, x_{i+1}]}.\end{align*}
\]

Using \((23)\) and \((19)\), we obtain the bound of the truncation error with respect to the regular part \( \hat{v}^{n+1} \).
by Taylor expansions we obtain

\[ |\Gamma_{\tilde{v}^{n+1}}(x_i)| \leq C \Delta t N^{-4}. \tag{37} \]

Again using (23) and (20), we obtain the bound of the truncation error with respect to the layer part \( \tilde{w}^{n+1} \)

\[ |\Gamma_{\tilde{w}^{n+1}}(x_i)| \leq C \varepsilon^{-2} \Delta t N^{-4} \|B_{\varepsilon}\|_{[x_{i-1},x_{i+1}].} \tag{38} \]

For \( x_i \in [\tau, 1 - \tau] \),

\[
|B_{\varepsilon}|_{[x_{i-1},x_{i+1}]} \leq e^{(-xN/4 - 1)\sqrt{\beta/\varepsilon})} + e^{(1-x3N/4+1)\sqrt{\beta/\varepsilon}} = 2e^{(-xN/4 - 1)\sqrt{\beta/\varepsilon})} \]

\[
= 2e^{(-\tau\sqrt{\beta/\varepsilon})} e^{(hN/4\sqrt{\beta/\varepsilon})} \leq Ce^{-4L}, \text{ where } \tau = \sigma_0 \sqrt{\varepsilon}L, \sigma_0 \geq 4/\sqrt{\beta}. \]

Then \( e^{-L} \leq L/N \) leads to

\[ ||B_{\varepsilon}||_{[x_{i-1},x_{i+1}]} \leq C(L/N)^4. \tag{39} \]

Using (39) in (38) with \( \gamma h_{\text{max}}^2 \|\hat{a}_{kk}\|_{\infty} \leq \varepsilon \), we get

\[ |\Gamma_{\tilde{w}^{n+1}}(x_i)| \leq C \Delta t (L/N)^4. \tag{40} \]

On combining (37) and (40) with (35), we obtain

\[ |\Gamma_{\tilde{v}^{n+1}}(x_i)| \leq C \varepsilon \Delta t (L/N)^4, \text{ for } \gamma h_{\text{max}}^2 \|\hat{a}_{kk}\|_{\infty} \leq \varepsilon. \tag{41} \]

(ii) Now, for the case \( \gamma h_{\text{max}}^2 \|\hat{a}_{kk}\|_{\infty} > \varepsilon \), suppose \( g \in C^4[0,1]^M \), then by Taylor expansions we obtain

\[ |\Gamma_{g,k}(x_i)| \leq C \varepsilon \Delta t (Y_{k,i} + Z_{k,i}), \text{ for } k = 1, \ldots, M, \tag{42} \]

where

\[
Y_{k,i} = |h_{i+1} - h_i ||g_k^{(3)}||_{[x_{i-1},x_{i+1}]}, \quad Z_{k,i} = h_{i+1}^2 ||g_k^{(4)}||_{[x_{i-1},x_{i+1}]}. \]

Using (23) and (19), we obtain the bound of the truncation error with respect to the regular part \( \tilde{v}^{n+1} \)

\[ |\Gamma_{\tilde{v}^{n+1}}(x_i)| \leq C \varepsilon \Delta t N^{-2}. \tag{43} \]

Now using the condition \( \gamma h_{\text{max}}^2 \|\hat{a}_{kk}\|_{\infty} > \varepsilon \), we obtain

\[ |\Gamma_{\tilde{v}^{n+1}}(x_i)| \leq C N^{-4}. \tag{44} \]

Note that in (44) the term \( \Delta t \) disappears from the bound for the error associated with the regular part; this fact is important in order to impose the relation between the discretization parameters \( \Delta t \) and \( N \).

To estimate the error with respect to the layer part \( \tilde{w}^{n+1} \), suppose \( g \in C^2[0,1]^M \), then using

\[ |\Gamma_{g}(x_i)| \leq C \varepsilon \Delta t ||g''||_{[x_{i-1},x_{i+1}]}, \]
we have
\[ |\Gamma_{\hat{u}}^{n+1}(x_i)| \leq C \Delta t |B_\varepsilon|_{[x_{i-1},x_{i+1}]} \]
Using (39), we have
\[ |\Gamma_{\hat{u}}^{n+1}(x_i)| \leq C \Delta t (L/N)^4 \]
Combining (44) and (45) in (35) with the assumption such that \( L^{-4} \leq C \Delta t \), we obtain
\[ |\Gamma_{\hat{u}}^{n+1}(x_i)| \leq C \Delta t (L/N)^4, \quad \text{for } \gamma h^2 \max_i |\hat{a}_{kk}|_\infty > \varepsilon. \]

On combining the case (I) and case (II), we obtain the truncation error estimate for the scheme (24)-(25) on the generalized Shishkin mesh \( \mathcal{D}_N \) and it is given by
\[ |\Gamma_{\hat{u}}^{n+1}(x_i)| \leq C \Delta t (L/N)^4. \]
Therefore, from the truncation error estimate (47) and the uniform stability result given in Lemma 4.4, we conclude the lemma. \( \square \)

5. Total discretization

In this section, we write the total discretization by combining the time semidiscretization and spatial semidiscretization to compute the approximate solution of (1)-(3) and after that we prove that the resultant scheme is uniformly convergent of second order in time and almost fourth order in spatial variable. Concretely, the numerical approximate \( U_i^n \) of \( u(x_i,n\Delta t) \) for \( i = 1, \ldots, N \) and \( n = 0,1,\ldots,T/\Delta t \), are obtained by the following totally discrete scheme

\[
\begin{align*}
U_i^0 &= 0, \quad L^N_{x,\varepsilon} U_i^0 = L_{x,\varepsilon} u(x_i,0), \quad i = 0(1)N, \\
[\hat{L}^N_{x,\varepsilon} U^{n+1}]_i &= [QF^n]_i, \quad \text{for } i = 1(1)N - 1, \\
U_{0+1}^n &= 0, \quad U_{N+1}^n = 0, \quad \text{for } n = 0,1,\ldots,T/\Delta t - 1,
\end{align*}
\]
where
\[
[\hat{L}^N_{x,\varepsilon} U^{n+1}]_i := \left( \begin{array}{c}
R(U_{1+1}^{n+1}) + \frac{\Delta t}{2} Q(a_{12} U_{2}^{n+1}) + \cdots + \frac{\Delta t}{2} Q(a_{1M} U_{M}^{n+1}) \\
R(U_{2+1}^{n+1}) + \frac{\Delta t}{2} Q(a_{21} U_{1}^{n+1}) + \cdots + \frac{\Delta t}{2} Q(a_{2M} U_{M}^{n+1}) \\
\vdots \\
R(U_{M+1}^{n+1}) + \frac{\Delta t}{2} Q(a_{M1} U_{1}^{n+1}) + \cdots + \frac{\Delta t}{2} Q(a_{MM-1} U_{M-1}^{n+1}) \\
\end{array} \right)_i,
\]
\[
[QF^n]_i := \left( \begin{array}{c}
Q(F^n_1) \\
Q(F^n_2) \\
\vdots \\
Q(F^n_M) 
\end{array} \right)_i,
\]
\[
F^n(x_i) = U^n_i + \frac{\Delta t}{2} (-L^N_{x,\varepsilon} U^n_i + f(x_i,t_n) + f(x_i,t_{n+1})),
\]
for $i = 1(1)N - 1$, $n = 0, 1, \ldots, T/\Delta t - 1$,

$$L^N_{x,i} U_{i}^{n+1} = -L^N_{x,i} U_{i}^{n} - 2 \frac{U_{i}^{n+1} - U_{i}^{n}}{\Delta t} + f(x_i, t_n) + f(x_i, t_{n+1}),$$

and

$$
\begin{align*}
R(V_{k,i}) &= r_{k}^{k} V_{k,i-1} + r_{k}^{*} V_{k,i} + r_{k}^{k+1} V_{k,i+1}, \\
Q(V_{k,i}) &= q_{k}^{k} - V_{k,i-1} + q_{k}^{*} V_{k,i} + q_{k}^{k+1} V_{k,i+1}.
\end{align*}
$$

The coefficients $q_{k}^{k}$, $*$ = $-, c, +$ and $r_{k}^{k}$, $*$ = $-, c, +$, $k = 1, 2, \ldots, M$
are defined as in Section 4.

**Theorem 5.1.** Let $u$ be the exact solution of (1)-(3) and let $\{ U_{i}^{n+1} \}$
be the numerical solution of the scheme (48). Under the hypothesis of
Lemma 4.1, the global error $u(x_i, t_{n+1}) - U_{i}^{n+1}$ at the time $t_{n+1}$ satisfies

$$||u(x_i, t_{n+1}) - U_{i}^{n+1}||_{\mathcal{D}_N} \leq C((\Delta t)^2 + (L/N)^4),$$

with the assumption that $L^{-4} \leq C \Delta t$, where $C$ is a positive constant
independent of $N$, $\Delta t$ and $\varepsilon$.

**Proof.** The global error $u(x_i, t_{n+1}) - U_{i}^{n+1}$ of the totally discrete
scheme at the time $t_{n+1}$ can be split in the form

$$u(x_i, t_{n+1}) - U_{i}^{n+1} \leq (u(x_i, t_{n+1}) - \hat{u}_{i}^{n+1}(x_i)) + (\hat{u}_{i}^{n+1}(x_i) - \hat{U}_{i}^{n+1})$$

$$+ (\hat{U}_{i}^{n+1} - U_{i}^{n+1}).$$

On combining the result from the Lemma 3.1 and Lemma 4.5 with
(52), we obtain

$$||u(x_i, t_{n+1}) - U_{i}^{n+1}||_{\mathcal{D}_N} \leq C((\Delta t)^2 + \Delta t(L/N)^4) + ||\hat{U}_{i}^{n+1} - U_{i}^{n+1}||_{\mathcal{D}_N}.$$  

To bound the term $||\hat{U}_{i}^{n+1} - U_{i}^{n+1}||_{\mathcal{D}_N}$, we consider that $\hat{U}_{i}^{n+1} - U_{i}^{n+1}$
can be written as the solution of one step of (48) with starting value
$u(x_i, t_{n}) - U_{i}^{n}$, taking the source term $f$ equal to zero together with
zero boundary conditions. Then it follows that

$$\hat{U}_{i}^{n+1} - U_{i}^{n+1} = R_N(u(x_i, t_{n}) - U_{i}^{n}),$$

where $R_N$ is a linear operator, called the transition operator associated
to the totally discrete scheme (48). Using this with (53) we obtain a
recursive argument as

$$||u(x_i, t_{n+1}) - U_{i}^{n+1}||_{\mathcal{D}_N} \leq C \sum_{k=1}^{n} ||R_{N}^{n-k}||_{\mathcal{D}_N} ((\Delta t)^3 + \Delta t(L/N)^4).$$
To get the required result for the uniform convergence of totally discrete scheme a sufficient condition is that
\[ ||R^j_N||_{\mathcal{D}_N} \leq C, \ j = 1, \ldots, n. \]
By assuming the uniform boundedness condition on power of discrete transition operator \( R_N \) with (53) (see the Remark 5.2 below) and the hypothesis of Lemma 4.5 that \( L^{-4} \leq C \Delta t \), we conclude the main results of this section.

**Remark 5.2.** We assume here the uniform boundedness condition as a conjuncture holds for the transition operator \( R_N \), as the theoretical proof of this is an open problem so far in the literature. Some partial results in this direction can be obtained by using a result by Palencia [14], but for the present problem this would require an \( \varepsilon \)-uniform estimate of the resolvent of the spatial operator \( L_{x,\varepsilon} \). Here due to lack of available theoretical result in this direction we assume the uniform boundedness of the power of discrete transition operator \( R_N \) as a conjuncture. For the support of this conjuncture we show some numerical evidence for the spectral radius of \( R_N \). From the numerical results of the Tables 2, 4, and 6 we observe that the spectral radius of \( R_N \) is strictly less than one and it stabilize as the singular perturbation parameter \( \varepsilon \) becomes small.

**Remark 5.3.** Theorem 5.1 proves almost fourth order uniform convergence of the method in spatial variable under the relation \( L^{-4} \leq C \Delta t \). Nevertheless, from the numerical point of view in Section 6, this condition is an artificial relation that we never needed in the experiments. Note that this relation appeared when we prove the convergence of the regular components in regular region, see eq. (44) in Section 4.

**6. Numerical experiments**

The proposed method is implemented on three test examples. In all the cases we begin with total number of nodal points \( N = 64 \) and the time step \( \Delta t = 0.5 \). The maximum error at the nodal points is calculated for the different values of \( \varepsilon \) and \( N \).

**Example 1:** Consider the following system of two coupled singularly perturbed parabolic problem
\[ \begin{align*}
\frac{\partial u_1}{\partial t} - \varepsilon \frac{\partial^2 u_1}{\partial x^2} + (2 + x)u_1 - (1 + x)u_2 &= x^2(1 - x)^2, \\
\frac{\partial u_2}{\partial t} - \varepsilon \frac{\partial^2 u_2}{\partial x^2} + (\varepsilon^2 + 1)u_2 - (1 + x)u_1 &= x^2(1 - x)^2,
\end{align*} \]
for \((x,t) \in (0,1) \times (0,1]\), with the initial-boundary conditions
\[ u(x,0) = 0, \ x \in (0,1), \]
\( u(0, t) = u(1, t) = 0, \ t \in [0, 1], \)
and the exact solution is not known.

**Example 2:** Consider the following system of three coupled singularly perturbed parabolic problem

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - \varepsilon \frac{\partial^2 u_1}{\partial x^2} + 3u_1 - (1 - x)u_2 - (1 - x)u_3 &= 16x^2(1 - x)^2, \\
\frac{\partial u_2}{\partial t} - \varepsilon \frac{\partial^2 u_2}{\partial x^2} + (4 + x)u_2 - 2u_1 - u_3 &= t^3, \\
\frac{\partial u_3}{\partial t} - \varepsilon \frac{\partial^2 u_3}{\partial x^2} + (6 + x)u_3 - 2u_1 - 3u_2 &= 16x^2(1 - x)^2,
\end{align*}
\]

for \((x, t) \in D\), with the initial-boundary conditions

\[
\begin{align*}
u(x, 0) &= 0, \ x \in (0, 1), \\
u(0, t) &= u(1, t) = 0, \ t \in [0, 1],
\end{align*}
\]
and the exact solution is not known.

**Example 3:** Consider the following system of two coupled singularly perturbed parabolic problem

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - \varepsilon \frac{\partial^2 u_1}{\partial x^2} + 2u_1 - u_2 &= 1, \\
\frac{\partial u_2}{\partial t} - \varepsilon \frac{\partial^2 u_2}{\partial x^2} + 2u_2 - u_1 &= 1,
\end{align*}
\]

for \((x, t) \in (0, 1) \times (0, 1]\), with the initial-boundary conditions

\[
\begin{align*}
u(x, 0) &= 0, \ x \in (0, 1), \\
u(0, t) &= u(1, t) = 0, \ t \in [0, 1],
\end{align*}
\]
and the exact solution is not known.

As the exact solution is not known for these examples, we estimate the maximum nodal error, \( E_{\varepsilon,N,\Delta t} = \max_{i,n} \tilde{e}_{\varepsilon,N,\Delta t}(i,n;\Delta t) \), for different values of \( \varepsilon \) and \( N \) where \( \tilde{e}_{\varepsilon,N,\Delta t}(i,n;\Delta t) = \| u_N(x_i,t_n) - \tilde{u}_N(x_i,t_n) \| \). We use a variant of the double mesh principle, assume \( u_N(x_i,t_n) \) denotes the numerical solution at the nodal point \((x_i,t_n)\) on the tensor product mesh of the generalized Shishkin mesh \( \mathcal{M}_N \) with \( N + 1 \) nodal points in spatial direction and a uniform mesh of step size \( \Delta t \) in time direction, and \( \tilde{u}_N(x_i,t_n) \) denotes the numerical solution at the nodal point \((x_i,t_n)\) on the tensor product mesh \( \{ (\tilde{x}_{i},\tilde{t}_{n}) \} \) that contains the mesh points of the original mesh and their midpoints.
Table 1. Maximum error and numerical rate of convergence of the present method with uniform step size $\Delta t$ in time direction and generalized Shishkin mesh $S(L)$ with $L = L^*$ in spatial direction for the Example 1.

| $\varepsilon = 2^{-k}$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|------------------------|----------|----------|----------|-----------|-----------|
| $\Delta t = 0.5$ | $\Delta t = 0.5/4$ | $\Delta t = 0.5/4^2$ | $\Delta t = 0.5/4^3$ | $\Delta t = 0.5/4^4$ |
| $k=4$ | $8.82E-04$ | $4.42E-05$ | $2.71E-06$ | $1.69E-07$ | $1.06E-08$ |
| & | 4.32 | 4.03 | 4.00 | 4.00 |
| 8 | $4.19E-04$ | $2.57E-05$ | $1.60E-06$ | $1.00E-07$ | $6.26E-09$ |
| & | 4.03 | 4.00 | 4.00 | 4.00 |
| 12 | $3.91E-04$ | $2.43E-05$ | $1.52E-06$ | $9.52E-08$ | $5.95E-09$ |
| & | 4.01 | 3.99 | 4.00 | 4.00 |
| 16 | $3.88E-04$ | $2.41E-05$ | $1.52E-06$ | $9.51E-08$ | $6.00E-09$ |
| & | 4.01 | 3.99 | 4.00 | 3.99 |
| 20 | $3.86E-04$ | $2.41E-05$ | $1.52E-06$ | $9.48E-08$ | $5.93E-09$ |
| & | 4.00 | 3.99 | 4.00 | 4.00 |
| 24 | $3.86E-04$ | $2.41E-05$ | $1.52E-06$ | $9.48E-08$ | $5.93E-09$ |
| & | 4.00 | 3.99 | 4.00 | 4.00 |
| 28 | $3.86E-04$ | $2.41E-05$ | $1.52E-06$ | $9.48E-08$ | $5.93E-09$ |
| & | 4.00 | 3.99 | 4.00 | 4.00 |
| 32 | $3.86E-04$ | $2.41E-05$ | $1.52E-06$ | $9.48E-08$ | $5.93E-09$ |
| & | 4.00 | 3.99 | 4.00 | 4.00 |
| $E_{N,\Delta t}$ | $8.82E-04$ | $4.42E-05$ | $2.71E-06$ | $1.69E-07$ | $1.06E-08$ |
| $p^N$ | 4.32 | 4.03 | 4.00 | 4.00 |

In the standard way, we estimate the classical convergence rate, for each fixed $\varepsilon$, by

$$p^N = \frac{\ln(\tilde{E}_{\varepsilon,N,\Delta t}) - \ln(\tilde{E}_{\varepsilon,2N,\Delta t/4})}{\ln 2},$$

and the parameter-robust convergence rate $p^N$ by

$$p^N = \frac{\ln(E_{N,\Delta t}) - \ln(E_{2N,\Delta t/4})}{\ln 2},$$

where $E_{N,\Delta t} = \max_{\varepsilon} \tilde{E}_{\varepsilon,N,\Delta t}$. 
Table 2. Spectral radius of the transition operator $R_N$ for the Example 1.

| $\varepsilon = 2^{-k}$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|------------------------|---------|-----------|-----------|-----------|-----------|
| $\Delta t = 0.5$      | 0.40350 | 0.80792   | 0.94825   | 0.98681   | 0.99670   |
| $\Delta t = 0.5/4$    |         |           |           |           |           |
| $\Delta t = 0.5/4^2$  |         |           |           |           |           |
| $\Delta t = 0.5/4^3$  |         |           |           |           |           |

Using $L < \ln N$ instead of $\ln N$; this means we are trying to bring the point $x_1$ closer to $x = 0$ and this provides the higher density of the mesh points in the layers. The motivation for this is the fact that the better performance of the mesh $S(L)$ can be governed by the high density of mesh points in the layers. The smallest value of $L$ is chosen to be $L^* = L^*(N)$ which satisfies

$$e^{-L^*} = L^*/N.$$  

For the different values of $N$ and $\varepsilon$, Table 1, Table 2, and Table 3 represent the maximum error $\tilde{E}_{\varepsilon,N,\Delta t}$ and the classical rate of convergence $p_N^\varepsilon$ of the present method for the Example 1, Example 2, and Example 3, respectively. The last two rows in each of the tables (Table 1, Table 3, and Table 5) represent the maximum error with respect to each nodal point for all value of $\varepsilon$, that is $E_{N,\Delta t}$; and the parameter-robust numerical rate of convergence $p^N$.

To show the numerical evidence for the uniform stability of the transition operator $R_N$, we calculate the spectral radius of $R_N$ for different value of $N$, $\Delta t$ and $\varepsilon$. Table 2, Table 4, and Table 6 display the spectral radius of this operator for the Example 1, Example 2, and Example 3, respectively. We clearly observe that the spectral radius for all value of $N$, $\Delta t$ and $\varepsilon$ is always strictly less than one. Moreover, we observe that the spectral radius stabilized for the small value of singular perturbation parameter $\varepsilon$. This stabilization of spectral radius for small value of $\varepsilon$ indicates the uniform stability of the operator $R_N$. 
Table 3. Maximum error and numerical rate of convergence of the present method with uniform step size $\Delta t$ in time direction and generalized Shishkin mesh $S(L)$ with $L = L^*$ in spatial direction for the Example 2.

| $\varepsilon = 2^{-k}$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|------------------------|---------|---------|---------|---------|---------|
| $\Delta t = 0.5$       | $\Delta t = 0.5/4$ | $\Delta t = 0.5/4^2$ | $\Delta t = 0.5/4^3$ | $\Delta t = 0.5/4^4$ |
| $k=4$                  | $4.48E-02$ | $4.09E-03$ | $2.17E-04$ | $1.35E-05$ | $8.44E-07$ |
|                        | $3.45$   | $4.23$   | $4.01$   | $4.00$   |          |
| 8                      | $4.44E-02$ | $3.58E-03$ | $1.97E-04$ | $1.22E-05$ | $7.63E-07$ |
|                        | $3.63$   | $4.18$   | $4.01$   | $4.00$   |          |
| 12                     | $4.33E-02$ | $3.54E-03$ | $1.95E-04$ | $1.21E-05$ | $7.58E-07$ |
|                        | $3.65$   | $4.18$   | $4.01$   | $4.00$   |          |
| 16                     | $4.43E-02$ | $3.54E-03$ | $1.95E-04$ | $1.21E-05$ | $7.58E-07$ |
|                        | $3.65$   | $4.18$   | $4.01$   | $4.00$   |          |
| 20                     | $4.43E-02$ | $3.54E-03$ | $1.95E-04$ | $1.21E-05$ | $7.58E-07$ |
|                        | $3.65$   | $4.18$   | $4.01$   | $4.00$   |          |
| 24                     | $4.43E-02$ | $3.54E-03$ | $1.95E-04$ | $1.21E-05$ | $7.58E-07$ |
|                        | $3.65$   | $4.18$   | $4.01$   | $4.00$   |          |
| 28                     | $4.43E-02$ | $3.54E-03$ | $1.95E-04$ | $1.21E-05$ | $7.58E-07$ |
|                        | $3.65$   | $4.18$   | $4.01$   | $4.00$   |          |
| 32                     | $4.43E-02$ | $3.54E-03$ | $1.95E-04$ | $1.21E-05$ | $7.58E-07$ |
|                        | $3.65$   | $4.18$   | $4.01$   | $4.00$   |          |

Observe that the data in Example 3 does not satisfy the zeroth order compatibility conditions at the nodal points $(0, 0)$ and $(1, 0)$. Moreover, Table 5 shows the low order of accuracy of the present method for the Example 3 in comparison with the numerical results presented in Table 1 and Table 3 for the Example 1 and Example 2, respectively; in which the sufficient compatibility conditions are satisfied. From this one can infer that, in practice some of the theoretical compatibility conditions seems to be very necessary for high order convergence of the present method. Clearly the numerical results presented in Table 1 and Table
Table 4. Spectral radius of the transition operator $R_N$ for the Example 2.

| $\varepsilon = 2^{-k}$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|-------------------------|---------|---------|---------|---------|---------|
| $\Delta t = 0.5$       | $\Delta t = 0.5/4$ | $\Delta t = 0.5/4^2$ | $\Delta t = 0.5/4^3$ | $\Delta t = 0.5/4^4$ |
| $k=4$                  |         |         |         |         |         |
| 0.01380                 | 0.68153 | 0.90960 | 0.97661 | 0.99410 |
| 0.25983                 | 0.72130 | 0.93118 | 0.98234 | 0.99556 |
| 0.32590                 | 0.77295 | 0.93796 | 0.98412 | 0.99601 |
| 0.34288                 | 0.78200 | 0.94065 | 0.98482 | 0.99618 |
| 0.35146                 | 0.78561 | 0.94172 | 0.98510 | 0.99625 |
| 0.35448                 | 0.78705 | 0.94214 | 0.98526 | 0.99628 |
| 0.35577                 | 0.78762 | 0.94231 | 0.98528 | 0.99628 |
| 0.35625                 | 0.78784 | 0.94231 | 0.98528 | 0.99628 |

3 verify our theoretical results.

Previously, the Crank-Nicolson method has been used in the framework of scalar singularly perturbed problem, for instance, in [2] to solve one dimensional parabolic problems of convection diffusion type. Recently, Clavero et al. [4] considered the Crank-Nicolson method on uniform mesh in time discretization and the central difference scheme on standard Shishkin mesh in spatial discretization for a system of two coupled time dependent singularly perturbed reaction-diffusion problems. In this article, to obtain a high order robust approximation we considered the Crank-Nicolson method in time direction and a hybrid scheme which is a suitable combination of fourth order compact difference scheme (or HODIE scheme ) and standard central difference scheme on a generalized Shishkin mesh in spatial direction. Here it is interesting to see how the HODIE technique permits to obtain a uniformly convergent method having order bigger than two in spatial direction. Earlier, the HODIE scheme for scalar singularly perturbed reaction-diffusion problems has been considered in Clavero and Gracia [3], and it is proved that the scheme is third order uniformly convergent on standard Shishkin mesh. But the extension of new HODIE scheme on standard Shishkin mesh is not possible in the case of system of coupled reaction-diffusion problems. It can be seen that the coefficients $q_k$'s in (24)-(25) is not always positive at the transition points, due to the fact that standard Shishkin mesh is very anisotropic in nature. This shows that the operator in (24)-(25) is not of positive type on standard Shishkin mesh. At the moment, when $N^{-1} < \sqrt{\varepsilon}$ we can not
Table 5. Maximum error and numerical rate of convergence of the present method with uniform step size $\Delta t$ in time direction and generalized Shishkin mesh $S(L)$ with $L = L^*$ in spatial direction for the Example 3.

| $\varepsilon = 2^{-k}$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|-------------------------|---------|---------|---------|---------|---------|
| $\Delta t = 0.5$ | $\Delta t = 0.5/4$ | $\Delta t = 0.5/4^2$ | $\Delta t = 0.5/4^3$ | $\Delta t = 0.5/4^4$ |
| $k=4$ | $3.32E-02$ | $7.75E-03$ | $1.91E-03$ | $4.76E-04$ | $9.52E-05$ |
| | $2.10$ | $2.02$ | $2.00$ | $2.32$ | |
| 8 | $3.29E-02$ | $7.75E-03$ | $1.91E-03$ | $4.76E-04$ | $9.53E-05$ |
| | $2.08$ | $2.02$ | $2.00$ | $2.32$ | |
| 12 | $1.67E-02$ | $2.41E-03$ | $5.17E-04$ | $1.24E-04$ | $2.49E-05$ |
| | $2.80$ | $2.22$ | $2.06$ | $2.32$ | |
| 16 | $1.67E-02$ | $2.16E-03$ | $4.41E-04$ | $1.24E-04$ | $2.47E-05$ |
| | $2.95$ | $2.29$ | $1.84$ | $2.32$ | |
| 20 | $1.67E-02$ | $2.16E-03$ | $4.41E-04$ | $1.24E-04$ | $2.47E-05$ |
| | $2.95$ | $2.29$ | $1.84$ | $2.32$ | |
| 24 | $1.67E-02$ | $2.16E-03$ | $4.41E-04$ | $1.24E-04$ | $2.47E-05$ |
| | $2.95$ | $2.29$ | $1.84$ | $2.32$ | |
| 28 | $1.67E-02$ | $2.16E-03$ | $4.41E-04$ | $1.24E-04$ | $2.47E-05$ |
| | $2.95$ | $2.29$ | $1.84$ | $2.32$ | |
| 32 | $1.67E-02$ | $2.16E-03$ | $4.41E-04$ | $1.24E-04$ | $2.47E-05$ |
| | $2.95$ | $2.29$ | $1.84$ | $2.32$ | |

Find a difference scheme of positive type which is high order uniformly convergent on standard Shishkin mesh for system of coupled reaction-diffusion problems. To avoid this, one can use the central difference scheme in the regular region $[\tau, 1 - \tau]$ and the fourth order compact difference scheme in $(0, \tau) \cup (1 - \tau, 1)$. But this combination gives only second order uniformly convergent result. In order to increase the order of convergence and to maintain the positivity of the present discrete operator in (24)-(25), we consider a generalized Shishkin mesh instead of standard Shishkin mesh. The Lemma 4.1 shows that the discrete operator in (24)-(25) on a generalized Shishkin mesh is of positive
Table 6. Spectral radius of the transition operator $R_N$ for the Example 3.

| $\varepsilon = 2^{-k}$ | $N = 64$ | $N = 128$ | $N = 256$ | $N = 512$ | $N = 1024$ |
|-------------------------|----------|-----------|-----------|-----------|-----------|
| $\Delta t = 0.5$       | $\Delta t = 0.5/4$ | $\Delta t = 0.5/4^2$ | $\Delta t = 0.5/4^3$ | $\Delta t = 0.5/4^4$ |
| $k=4$                  | 0.42429  | 0.88209   | 0.95077   | 0.98745   | 0.99685   |
| 8                      | 0.58776  | 0.88234   | 0.96806   | 0.99192   | 0.99792   |
| 12                     | 0.59923  | 0.88235   | 0.96916   | 0.99220   | 0.99793   |
| 16                     | 0.59995  | 0.88235   | 0.96923   | 0.99222   | 0.99794   |
| 20                     | 0.60000  | 0.88235   | 0.96923   | 0.99222   | 0.99794   |
| 24                     | 0.60000  | 0.88235   | 0.96923   | 0.99222   | 0.99794   |
| 28                     | 0.60000  | 0.88235   | 0.96923   | 0.99222   | 0.99794   |
| 32                     | 0.60000  | 0.88235   | 0.96923   | 0.99222   | 0.99794   |

Type and the analysis in Section 4 shows that the scheme (24)-(25) is almost fourth order uniformly convergent with respect to the perturbation parameter $\varepsilon$. Here we also want to point out one more benefit of generalized Shishkin mesh over standard Shishkin mesh in the numerical methods presented in [4] and [5] for parabolic reaction diffusion systems. It is proved that the numerical methods presented in [4] and [5] have almost second order uniform convergence under the theoretical relation $N^{-q} \leq C \Delta t$, where $0 < q < 1$. Note that the theoretical relation appeared in the analysis when the barrier function technique was used to prove the second order convergence of the regular component on standard Shishkin mesh. While if we use generalized Shishkin mesh instead of standard Shishkin mesh in [4] and [5] then we can claim almost second order uniform convergence in spatial variable without any theoretical relation by using the same analysis technique.

7. Conclusions

We presented a high order parameter-robust numerical method for a system of $(M \geq 2)$ coupled singularly perturbed parabolic reaction-diffusion problem (1)-(3). The problem is discretized using the Crank-Nicolson method on an uniform mesh in time direction and a suitable combination of the fourth order compact difference scheme and the central difference scheme on a generalized Shishkin mesh in spatial direction. The essential idea in this method is to use a generalized Shishkin mesh in order to attain a high order parameter-robust convergence in spatial variable. The fine parts of standard Shishkin mesh and generalized Shishkin mesh are identical, but the coarse part of
A generalized Shishkin mesh is a smooth continuation of the fine mesh and is no longer equidistant. Using this fact we proved that the present method is second order uniformly convergent in time and almost fourth order uniformly convergent in spatial variable, if the discretization parameters satisfy a non-restrictive relation. Numerical experiments are presented to validate the theoretical results and also the results of the experiments indicate that the relation between the discretization parameters is not necessary in practice.

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Differential Equations and Numerical Analysis Group, Department of Mathematical Sciences, Norwegian University of Sciences and Technology, NO-7491, Trondheim, Norway
E-mail: mukesh.kumar@math.ntnu.no

Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi-110016, India
E-mail: scsr@maths.iitd.ernet.in