Bures distance as a measure of entanglement for symmetric two-mode Gaussian states

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We evaluate a Gaussian entanglement measure for a symmetric two-mode Gaussian state of the quantum electromagnetic field in terms of its Bures distance to the set of all separable Gaussian states. The required minimization procedure was considerably simplified by using the remarkable properties of the Uhlmann fidelity as well as the standard form II of the covariance matrix of a symmetric state. Our result for the Gaussian degree of entanglement measured by the Bures distance depends only on the smallest symplectic eigenvalue of the covariance matrix of the partially transposed density operator. It is thus consistent to the exact expression of the entanglement of formation for symmetric two-mode Gaussian states. This non-trivial agreement is specific to the Bures metric.

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I. INTRODUCTION

In recent years many attempts to quantify the entanglement of a Gaussian state have been made due to the experimental interest in using such states in quantum information processing [1, 2]. Work on the inseparability properties of two-mode Gaussian states (TMGSs) was also stimulated by the formulation of their separability criterion [3, 4]. Simon proved that a TMGS is separable if and only if the non-negativity of its density matrix is preserved under partial transposition (PT) [3]. Simon has written the PT-criterion in a $\text{Sp}(2,\mathbb{R}) \times \text{Sp}(2,\mathbb{R})$ invariant form which allows one to easily check whether a two-mode Gaussian state is separable or not [3]. Denoting by $\tilde{k}_-$ the smallest symplectic eigenvalue of the covariance matrix (CM) of the PT-operator $\rho^{\text{PT}}$, the separability criterion reads $\tilde{k}_- \geq 1/2$ for a separable TMGS and $\tilde{k}_- < 1/2$ for an entangled one.

It is interesting to recall that a computable inseparability measure for an arbitrary bipartite state was proposed in Refs. [5, 6] in terms of negativity defined as the absolute value of the sum of the negative eigenvalues of the PT-density operator. As proved by Vidal and Werner, the negativity is an entanglement monotone. For TMGSs the negativity turned out to be an expression depending only on $\tilde{k}_-$ [5].

With regard to other accepted measures of entanglement for a TMGS, the only exact evaluation at present appears to be the entanglement of formation (EF) for a symmetric TMGS [4]. The optimal pure-state decomposition required to define the EF was found in this case to be in terms of Gaussian states. For an arbitrary TMGS, a Gaussian entanglement of formation was further defined using its optimal decomposition in pure Gaussian states [4]. Following the prescription of Ref. [4], an evaluation of the Gaussian EF for a two-mode squeezed thermal state (STS) was given in Ref. [4]. In the general case an insightful formula for the Gaussian EF was not yet written. One can notice that, for a symmetric TMGS, the amount of entanglement is described by monotonous functions (negativity and EF) depending on only $\tilde{k}_-$. The situation is different for other special TMGSs. A disagreement between the Gaussian EF and the negativity of the Gaussian states having extremal negativity at fixed global and local purities [12] was recently noticed in Ref. [13].

Following the earlier distance-type proposal for quantifying entanglement due to Vedral and co-workers [11], a class of distance-type Gaussian measures of entanglement could be defined with respect to only the set of separable Gaussian states identified by the separability criterion [4]. To our knowledge, the first authors who used and evaluated numerically a Gaussian measure of entanglement were Scheel and Welsch in Ref. [15]. In our paper [16] co-authored with Scutaru, an explicit analytic Gaussian amount of entanglement was calculated for a STS by using the Bures distance. Note that the STSs are important non-symmetric TMGSs that can be produced experimentally and are used in the protocols for quantum teleportation. Interestingly, in the STS-case the Gaussian entanglement measured by Bures distance [15] and the Gaussian EF [11] were found to be in agreement. They are monotonous functions of the same parameter [17] which cannot be expressed only in terms of $\tilde{k}_-$. Therefore, the negativity of a STS is not equivalent to the two Gaussian measures of entanglement evaluated at present [11] [16].

Our aim in the present work is to apply the Bures distance as a measure of entanglement for symmetric two-mode Gaussian states in the framework of the Gaussian approach and compare the result to the exact EF for a symmetric TMGS. The body of the paper is structured as follows. We recall in Sec. II several properties of two-mode Gaussian states. Here we show that the CM of a symmetric TMGS can be diagonalized by the beam-splitter transformation which is orthogonal and symplectic. In Sec. III we define a Gaussian amount of entanglement for a symmetric TMGS in terms of its Bures distance to the set of all separable TMGSs. By using the properties of the Uhlmann fidelity between
we get the $Sp(2, \mathbb{R})_x$ with $k$, where we have used the $\sigma$-dependent amount of entanglement is expressed in terms of only $k_\pm$ being thus consistent to the exact evaluation of the EF [4]. Our final conclusions are presented in Sec. V.

II. TWO-MODE GAUSSIAN STATES

An undisplaced TMGS is entirely specified by its CM denoted by $\mathcal{V}$ which determines the characteristic function of the state

$$\chi_G(x) = \exp \left( -\frac{1}{2} x^T \mathcal{V} x \right), \quad (2.1)$$

with $x^T$ denoting a real row vector $(x_1, x_2, x_3, x_4)$. $\mathcal{V}$ is a symmetric and positive $4 \times 4$ matrix which has the following block structure:

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{C} \\ \mathcal{C}^T & \mathcal{V}_2 \end{pmatrix}. \quad (2.2)$$

Here $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{C}$ are $2 \times 2$ matrices. Their entries are second-order correlations of the canonical operators $q_j = (a_j + a_j^\dagger)/\sqrt{2}$, $p_j = (a_j - a_j^\dagger)/\sqrt{2}$, where $a_j$ and $a_j^\dagger$, $(j = 1, 2)$, are the amplitude operators of the modes. $\mathcal{V}_1$ and $\mathcal{V}_2$ denote the symmetric covariance matrices for the individual reduced one-mode STSs [18], while the matrix $\mathcal{C}$ contains the cross-correlations between modes. The Robertson-Schrödinger form of the uncertainty relations for the canonical variables can be cast as

$$\mathcal{V} + \frac{i}{2} \Omega \geq 0, \quad \Omega := i(\sigma_2 \odot \sigma_2), \quad (2.3)$$

where we have used the $\sigma_2$-Pauli matrix. From Eq. 2.3 we get the $Sp(2, \mathbb{R}) \times Sp(2, \mathbb{R})$ invariant inequality [4, 8]

$$\det(\mathcal{V} + \frac{i}{2} \Omega) = \det(\mathcal{V}) - \frac{1}{4} (\det(\mathcal{V}_1) + \det(\mathcal{V}_2) + 2\det(\mathcal{C}) + \frac{1}{16} \geq 0. \quad (2.4)$$

A factorized form of the condition (2.4) in terms of the symplectic eigenvalues $k_+$ and $k_-$ of the CM,

$$\det(\mathcal{V} + \frac{i}{2} \Omega) = \left( k_+^2 - \frac{1}{4} \right) \left( k_-^2 - \frac{1}{4} \right) \geq 0, \quad (2.5)$$

shows that $k_+ \geq k_- \geq 1/2$.

According to the separability criterion derived by Simon [3], a TMGS is separable if the PT-density operator $\rho^{PT}$ describes a Gaussian state. This means that its CM, hereafter denoted by $\mathcal{V}$, obeys the uncertainty relation

$$\det(\mathcal{V} + \frac{i}{2} \Omega) = \left( k_+^2 - \frac{1}{4} \right) \left( k_-^2 - \frac{1}{4} \right) \geq 0, \quad (2.6)$$

which is equivalent to the condition $k_- \geq 1/2$.

A. Scaled standard states

Following Refs. [3, 4] we define an equivalence class of locally similar TMGSs. The states belonging to this class have the same amount of entanglement and a scaled standard form of their CMs. These are four-parameter and two-variable matrices [3]:

$$\mathcal{V}(u_1, u_2) = \begin{pmatrix} b_1 u_1 & 0 & c \sqrt{u_1 u_2} & 0 \\ 0 & b_1 u_1 & 0 & d \sqrt{u_1 u_2} \\ c \sqrt{u_1 u_2} & 0 & b_2 u_2 & 0 \\ 0 & d \sqrt{u_1 u_2} & 0 & b_2 u_2 \end{pmatrix} \quad (b_1 \geq 1/2, \ b_2 \geq 1/2). \quad (2.7)$$

We denote by $\rho^{(I)}$ the Gaussian density operator whose CM is

$$\mathcal{V}_I := \mathcal{V}(1, 1). \quad (2.8)$$

In fact the CMs (2.7) are obtained by applying to $\mathcal{V}_I$ two independent one-mode squeeze transformations

$$S = S_1 \oplus S_2, \ S \in Sp(2, \mathbb{R}) \times Sp(2, \mathbb{R}), \quad (2.9)$$

with the squeeze factors $u_1, u_2$. $\mathcal{V}_I$ was called the standard form I of the CM for this equivalence class [3]. There is an obvious one-to-one correspondence between the set of the four-standard-form parameters $b_1, b_2, c, d$ appearing as entries in $\mathcal{V}_I$ and the set of the $Sp(2, \mathbb{R}) \times Sp(2, \mathbb{R})$ invariants $(\det(\mathcal{V}_1), \det(\mathcal{V}_2), \det(\mathcal{C}), \det(\mathcal{V}))$. According to Simon [3], entangled TMGSs should have a negative $d$ parameter.

Among the scaled standard states, there is another important one introduced and discussed by Duan et al. [3]: the TMGS for which the separability and classicality conditions coincide [10]. Let us denote its CM by $\mathcal{V}_{II} = \mathcal{V}(v_1, v_2)$ and term it the standard form II. The scaling factors $v_1, v_2$ satisfy the algebraic system

$$\frac{b_1 (v_1^2 - 1)}{2b_1 - v_1} = \frac{b_2 (v_2^2 - 1)}{2b_2 - v_2} \quad (2.10)$$

$$b_1 b_2 (v_1^2 - 1)(v_2^2 - 1) = (c v_1 v_2 - |d|^2)^2. \quad (2.11)$$

Although the solution of the system (2.10) - (2.11) for an arbitrary TMGS arises finally from a still unsolved eighth-order one-variable algebraic equation, it is possible to find it for some particular useful TMGSs.

B. Symmetric TMGSs

When having det $\mathcal{V}_1 = \det \mathcal{V}_2$ we are dealing with symmetric TMGSs. The standard parameters of the CMs for entangled symmetric TMGSs we are considering in the
following are denoted as \( b_1 = b_2 =: b, c \geq |d|, d = -|d| \). We shall review below several useful properties of these states.

The symplectic eigenvalues of the CM are

\[
    k_+ = \sqrt{(b - |d|)(b + c)}, \quad k_- = \sqrt{(b + |d|)(b - c)}. \tag{2.12}
\]

Similarly, for the PT-density operator we find

\[
    \tilde{k}_+ = \sqrt{(b + |d|)(b + c)}, \quad \tilde{k}_- = \sqrt{(b - |d|)(b - c)}. \tag{2.13}
\]

leading to the separability condition \[3\]

\[
    (b - |d|)(b - c) - \frac{1}{4} \geq 0. \tag{2.14}
\]

Further, Eqs. (2.10) and (2.11) can be solved for a symmetric TMGS. We readily get the squeezed factors in the standard form II:

\[
    v_1 = v_2 = \sqrt{\frac{b - |d|}{b - c}}. \tag{2.15}
\]

Note also that the CM of any symmetric TMGS can be diagonalized with a beam-splitter transformation. The optical effect of a lossless beam splitter is described by the wave mixing operator \[20, 21\]

\[
    B(\theta, \phi) = \exp \left[ -\frac{\theta}{2} (e^{i\phi} a_1^\dagger a_2 - e^{-i\phi} a_1 a_2^\dagger) \right], \tag{2.16}
\]

with \( \theta \in [0, \pi], \phi \in (-\pi, \pi] \). Transformation of an arbitrary CM is governed by a \( 4 \times 4 \) symplectic and orthogonal matrix \( M(\theta, \phi) \in SO(4) \cap Sp(4, \mathbb{R}) \)

\[
    \mathcal{V}^B = M^T \mathcal{V} M, \tag{2.17}
\]

where the superscript \( T \) stands for transpose. Explicitly we get

\[
    M(\theta, \phi) = \begin{pmatrix} \cos \frac{\theta}{2} I_2 & -\sin \frac{\theta}{2} R(\phi) \\ \sin \frac{\theta}{2} R(-\phi) & \cos \frac{\theta}{2} I_2 \end{pmatrix}, \tag{2.18}
\]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix and \( R(\phi) \) is the \( 2 \times 2 \) rotation matrix

\[
    R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}. \tag{2.19}
\]

Now, the CM of a symmetric equally scaled standard state \((u_1 = u_2 = u)\) has the nice property of being diagonalized by a beam-splitter transformation having the angles \( \phi = 0 \) and \( \theta = \pi/2 \). By applying Eq. (2.17) via Eqs. (2.18) and (2.19) we get

\[
    \mathcal{V}^B(u, u) = \text{diag} \left[ (b + c)u, \frac{b - |d|}{u}, (b - c)u, \frac{b + |d|}{u} \right]. \tag{2.20}
\]

In the particular case of a symmetric TMGS having its CM in the standard form II the congruent matrix \[22, 20\]
reads

\[
    \mathcal{V}^B_{II} = \text{diag} \left[ (b + c) \sqrt{\frac{b - |d|}{b - c}}, \tilde{k}_-, \tilde{k}_-, (b + |d|) \sqrt{\frac{b - c}{b - |d|}} \right]. \tag{2.21}
\]

III. DEFINING GAUSSIAN ENTANGLEMENT

We follow now the idea of Vedral and co-workers \[14\] and characterize the degree of inseparability of a TMGS by its Bures distance to the set of all separable TMGSs of the given system. The original and rigorous proposal in Ref. \[14\] is thus modified by restricting the set of all separable states to a relevant one identified by a separability criterion. As for the continuous-variable two-mode systems a separability criterion was proved for only TMGSs \[3, 4\], we find natural to use the separable TMGSs as reference set when defining an entanglement measure for a symmetric TMGS.

A. Properties of Uhlmann fidelity

The virtues of the Bures distance \[22\] as a measure of entanglement were first revealed in Ref. \[14\]. In our paper \[16\], we took advantage of having derived an explicit formula for the Uhlmann fidelity \[23, 24\] between two-mode squeezed thermal states and gave the first Gaussian amount of entanglement measured by Bures distance. Notice that the Uhlmann fidelity is tightly related to the Bures metric:

\[
    d_B(\rho, \sigma) := \left[ 2 - 2\sqrt{\mathcal{F}(\rho, \sigma)} \right]^{1/2}. \tag{3.1}
\]

In Eq. (3.1) \( \rho \) and \( \sigma \) are density operators acting on a Hilbert space \( \mathcal{H} \) and the function \( \mathcal{F}(\rho, \sigma) \) is the Uhlmann fidelity of the two states \[23, 24\]:

\[
    \mathcal{F}(\rho, \sigma) = \left\{ \text{Tr}[(\sqrt{\rho} \sqrt{\sigma})^{1/2}] \right\}^2. \tag{3.2}
\]

Some of the remarkable general properties of the fidelity \[23, 24, 25\] listed below will be explicitly used in the rest of the paper.

\[\begin{align*}
    \textbf{P1} & \quad 0 \leq \mathcal{F}(\rho, \sigma) \leq 1, \quad \text{and} \quad \mathcal{F}(\rho, \sigma) = 1 \quad \text{if and only if} \quad \rho = \sigma. \\
    \textbf{P2} & \quad \mathcal{F}(\sigma, \rho) = \mathcal{F}(\rho, \sigma), \quad (\text{symmetry}). \\
    \textbf{P3} & \quad \mathcal{F}(\rho, \sigma) \geq \text{Tr}(\rho \sigma); \\
    & \quad \text{if at least one of the states is pure, Eq. (3.2) reduces to the usual transition probability \text{Tr}(\rho \sigma), i.e., if \rho} \quad \text{or/and \sigma is pure, then} \quad \mathcal{F}(\rho, \sigma) = \text{Tr}(\rho \sigma). \\
    \textbf{P4} & \quad \mathcal{F}(U \rho U^\dagger, U \sigma U^\dagger) = \mathcal{F}(\rho, \sigma), \quad (\text{invariance under unitary transformations}). \\
    \textbf{P5} & \quad \mathcal{F}(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = \mathcal{F}(\rho_1, \sigma_1)\mathcal{F}(\rho_2, \sigma_2), \quad (\text{multiplicativity}). 
\end{align*}\]

It was also proved that the fidelity cannot decrease under local general measurements and classical communications. This property of fidelity is important in properly defining a measure of entanglement.
B. Defining Gaussian entanglement by Bures metric

Let us denote in the following by $\rho_s$ the density operator of an inseparable symmetric TMGS whose CM, Eq. (2.2), has the following structure:

$$\mathcal{V}_1 = \mathcal{V}_2 = \begin{pmatrix} bu & 0 \\ 0 & b/u \end{pmatrix}, \mathcal{C} = \begin{pmatrix} cu & 0 \\ 0 & -d/u \end{pmatrix},$$

$$b \geq \frac{1}{2}. \quad (3.3)$$

Equation (3.3) describes the CM of a symmetric scaled standard state with equal local squeeze factors. For later convenience, we define the amount of Gaussian entanglement of the entangled state $\rho_s$

$$E_0(\rho_s) := \min_{\rho' \in D_0} \frac{1}{2} d_B^2(\rho_s, \rho') = 1 - \max_{\rho' \in D_0} \sqrt{F(\rho_s, \rho')}. \quad (3.4)$$

In Eq. (3.4) we have introduced the set $D_0^{sep}$ of all separable scaled standard TMGSs which is included in the set of all separable TMGSs displaying the property (separability threshold) [10]

$$k' = 1/2. \quad (3.5)$$

Our task is to maximize the fidelity between the entangled symmetric TMGS $\rho_s$ and a state $\rho' \in D_0^{sep}$. Obviously, as the inseparability does not depend on local operations we have

$$E_0(\rho_s) = E_0(\rho_1). \quad (3.6)$$

IV. EVALUATING GAUSSIAN ENTANGLEMENT

A. Results on the fidelity between two Gaussian states

Evaluation of the fidelity between one-mode Gaussian states was possible by taking advantage of the exponential form of their density operators. According to Ref. [26], the fidelity between the undisplaced one-mode Gaussian states $\sigma_1$ and $\sigma_2$ is

$$F(\sigma_1, \sigma_2) = [(\Delta + \Lambda)^{1/2} - \Lambda^{1/2}]^{-1} \quad (4.1)$$

with

$$\Delta = \text{det}(\mathcal{V}_{\sigma_1} + \mathcal{V}_{\sigma_2}),$$

$$\Lambda = 4[\text{det}(\mathcal{V}_{\sigma_1}) - \frac{1}{4}][\text{det}(\mathcal{V}_{\sigma_2}) - \frac{1}{4}]. \quad (4.2)$$

A main consequence of having an analytic formula for the fidelity was to define and calculate a degree of nonclassicality for one-mode Gaussian states [27]. The explicit formula of the fidelity was then used to quantify the accuracy of teleportation of mixed one-mode Gaussian states through a Gaussian channel in Refs. [28, 29].

It appears that our paper [14] co-authored with Scutaru was the only one to give and exploit an explicit formula for the fidelity of two TMGSs. After evaluating the fidelity between two two-mode STSs, we obtained an explicit expression for a properly defined Gaussian amount of entanglement of a STS. Recently, we have shown that the fidelity between two TMGSs having the density operators $\rho_1$ and $\rho_2$ is determined by the properties of the non-Hermitian Gaussian operator $\rho_1 \rho_2^\dagger$ [31]. Here we want to deal with the Bures entanglement (3.4) of a symmetric TMGS. Fortunately, the details of the explicit general formula of the fidelity between two TMGSs are not necessary here. Instead, we have to use the following property which we have proved by using the general formula for the fidelity between scaled standard states [30]: The closest separable scaled standard state to a given symmetric scaled standard state having equal local squeeze factors $u_1 = u_2 = u$ is a symmetric scaled standard state observing the threshold condition [35]. Therefore, the amount of Gaussian entanglement for a symmetric TMGS can be calculated in a simpler way, because the separable reference set $D_0^{sep}$ used in Eq. (3.4) is in fact restricted to its subset of equally squeezed symmetric states.

By using property P1 of the fidelity for the beamsplitter operator $B(\pi/2, 0)$, Eq. (2.16), we first write the fidelity between $\rho_s$ and any symmetric scaled standard state $\rho'$ having equal local squeeze factors:

$$F(\rho_s, \rho') = F[B(\pi/2, 0) \rho_s B^\dagger(\pi/2, 0), B(\pi/2, 0) \rho' B^\dagger(\pi/2, 0)]. \quad (4.3)$$

According to Eq. (2.24), the transformed density operators describe two-mode product states:

$$B(\pi/2, 0) \rho_s B^\dagger(\pi/2, 0) = \sigma_1 \otimes \sigma_2 \quad (4.4)$$

and respectively

$$B(\pi/2, 0) \rho' B^\dagger(\pi/2, 0) = \sigma'_1 \otimes \sigma'_2. \quad (4.5)$$

Here the one-mode states $\sigma_1$ and respectively $\sigma_2$ have the CMs

$$\mathcal{V}_{\sigma_1}(u, u) = \text{diag}[(b + c)u, (b - |d|)/u],$$

$$\mathcal{V}_{\sigma_2}(u, u) = \text{diag}[(b - c)u, (b + |d|)/u]. \quad (4.6)$$

Similarly, the transformed separable density operator (4.5) has the CMs of its one-mode reductions

$$\mathcal{V}_{\sigma'_1}(u', u') = \text{diag}[(b' + c')u', (b' - |d'|)/u'],$$

$$\mathcal{V}_{\sigma'_2}(u', u') = \text{diag}[(b' - c')u', (b' + |d'|)/u']. \quad (4.7)$$

However, the parameters appearing in the above equation are related by the threshold separability condition [35]. Now we can apply the multiplicativity property P5 of the fidelity and reduce the evaluation of fidelity to a single-mode problem [31]:

$$F(\rho_s, \rho') = F(\sigma_1, \sigma'_1) F(\sigma_2, \sigma'_2). \quad (4.8)$$
B. Explicit evaluation

Maximization of the product-fidelity \((4.8)\) with respect to the parameters of the CM \((4.7)\) is still a complicated problem. At this point we choose to use the standard form II of the covariance matrix, Eq. \((2.21)\), for the given entangled state \(\rho_s\). The separability threshold condition \((3.5)\) is manifestly satisfied when the closest separable state \(\rho'\) is in the standard form II, too. We are now left to maximize the product of the one-mode fidelities between states described by the CMs

\[
\mathcal{V}_{\sigma_1} = \text{diag} \left[ (b + c) \sqrt{\frac{b - |d|}{b - c}} \right], \\
\mathcal{V}_{\sigma_1'} = \text{diag} \left[ 2(b' + c')(b' - |d'|) \right], \quad \text{(4.9)}
\]

and

\[
\mathcal{V}_{\sigma_2} = \text{diag} \left[ \tilde{k}_-, \frac{b + |d|}{\sqrt{b - |d|}} \right], \\
\mathcal{V}_{\sigma_2'} = \text{diag} \left[ \frac{1}{2} 2(b' + |d'|)(b' - c') \right]. \quad \text{(4.10)}
\]

Application of Eqs. \((4.1)\) and \((4.2)\) for one-mode fidelities shows that the product \((4.8)\) is a function of only two independent variables \[32\]:

\[
x := (b' + c')(b' - |d'|), \quad y := (b' + |d'|)(b' - c'). \quad \text{(4.11)}
\]

In this way a considerable simplification of maximization procedure is obtained. We easily find that the maximal fidelity

\[
\max_{\rho' \in \mathcal{D}_0^{\rho'}} \mathcal{F}(\rho_s, \rho') = \frac{2\sqrt{\tilde{k}_-}}{(\tilde{k}_- + 1/2)^2}, \quad \text{(4.12)}
\]

is reached when the following conditions are met:

\[
x_{\text{max}} - \frac{1}{4} = \frac{1}{2\tilde{k}_-} \left( \tilde{k}_+ - \frac{1}{4} \right), \\
y_{\text{max}} - \frac{1}{4} = \frac{1}{2\tilde{k}_-} \left( \tilde{k}_+ - \frac{1}{4} \right). \quad \text{(4.13)}
\]

Remark that the maximal fidelity is a function of only \(\tilde{k}_-\) while the parameters \(x_{\text{max}}\) and \(y_{\text{max}}\) depend on the symplectic eigenvalues \(2.12\) as well. We can write now the final expression for the Gaussian entanglement measured by the Bures metric, Eq. \((3.4)\),

\[
E_0(\rho_s) = \frac{(1 - \sqrt{2\tilde{k}_-})^2}{2\tilde{k}_- + 1}, \quad \tilde{k}_- < 1/2, \\
E_0(\rho_s) = 0, \quad \tilde{k}_- \geq 1/2. \quad \text{(4.14)}
\]

Recall that in Ref.\[9\] it was proved that the exact EF of a symmetric TMGS is a function of only \(\tilde{k}_-\). Our result \((4.14)\) obtained in the Gaussian approach of Bures metric is thus in agreement with the exact EF. Accordingly, one could assume that, in terms of the Bures metric, the closest separable state to an entangled symmetric TMGS is also a symmetric TMGS.

C. The closest separable state

It is instructive to determine the parameters \(b'', c'', d'' = -|d''|\) of the closest separable state. To this end we use Eqs. \((3.4)\) and \((4.13)\) and get

\[
(b'')^2 = \frac{1}{4\tilde{k}_-} \left[ \tilde{k}_+ + k_+^2 - \frac{1}{4} \right] \left[ \tilde{k}_+ + k_+^2 - \frac{1}{4} \right], \\
c'' = \frac{1}{2\tilde{k}_-} \left[ \tilde{k}_+ + k_+^2 - 1/4 \right] \left( k_+^2 - 1/4 \right), \\
|d''| = \frac{1}{2\tilde{k}_-} \left[ \tilde{k}_+ + k_+^2 - 1/4 \right] \left( k_+^2 - 1/4 \right). \quad \text{(4.15)}
\]

Equations \((4.15)\) tell us that the parameters of the closest separable state to a symmetric TMGS are determined by the symplectic eigenvalues \(k_-\) and \(k_+\) of its CM and by the smallest symplectic eigenvalue \(\tilde{k}_-\) of the CM \(\mathcal{V}\) of the PT-density operator.

V. CONCLUSIONS

The intricate expressions \((4.15)\) give one an idea about the considerable simplification we introduced in the maximization procedure of the fidelity in two ways: first by applying its property \(P4\) under the beam-splitter transformation, second by considering the given state with its CM in the standard form II. Notice that this form of the CM is involved in giving an inseparability criterion for a TMGS \[3\]. Our result for the Gaussian degree of entanglement measured by Bures distance depends only on the smallest symplectic eigenvalue of the covariance matrix of the PT-density operator. Thus, it is in agreement with the exact EF found in Ref.\[9\]. One could therefore conjecture that the closest separable state to an entangled symmetric TMGS in terms of Bures metric is a Gaussian. This is not the case with other distance-type measures of entanglement such as relative entropy \[14\] for which it is known that, even for pure states, the closest separable state to a Gaussian one is non-Gaussian.

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