Non-perturbative improvement and renormalization of lattice operators

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The Alpha Collaboration has proposed an optimal value for \(c_{SW}\) in the Sheikholeslami-Wohlert action, chosen to remove \(O(a)\) effects. To measure hadronic matrix elements to the same accuracy we need a method of finding \(O(a)\) improved operators, and their renormalization constants. We determine the \(Z\) factors by a non-perturbative method, measuring the matrix elements for single quark states propagating through gauge fields in the Landau gauge. The data show large effects coming from chiral symmetry breaking. This allows us to find the improvement coefficients too, by requiring that the amount of chiral symmetry breaking agrees with that predicted by the chiral Ward identities.

1. INTRODUCTION

There is current interest in improving the Wilson fermion action to remove all \(O(a)\) discretization errors. An improved action is already known \cite{1}, however to measure hadronic matrix elements to the same accuracy we need a method of finding \(O(a)\) improved operators, and their renormalization constants. We determine the \(Z\) factors by a non-perturbative method, measuring the matrix elements for single quark states propagating through gauge fields in the Landau gauge. The data show large effects coming from chiral symmetry breaking. This allows us to find the improvement coefficients too, by requiring that the amount of chiral symmetry breaking agrees with that predicted by the chiral Ward identities.

2. CHIRAL WARD IDENTITIES

We need a way to identify the \(O(a)\) effects in three-point functions, so that we can tune the improvement parameters to remove them. There is a way to do this, since the Wilson mass term and the Sheikholeslami-Wohlert clover term both violate chiral symmetry.

If we were simulating a theory which respected chiral symmetry, it would be easy to use this fact to pick out \(O(a)\) effects, and we could recognise the correct improvement parameters by the restoration of chiral symmetry. However for lattice QCD things are not quite this simple. As well as the chiral symmetry violation coming from lattice artifacts, which we want to remove, there is also genuine chiral symmetry violation, some coming from the fact that we work with finite quark mass, and some from the well-known fact that QCD spontaneously breaks chiral symmetry. We need to know how big the real violation of chiral symmetry is.

Fortunately there is a way to identify this genuine component of chiral symmetry breaking \cite{4}. Chiral Ward identities can be constructed for any...
Green function, which tell us exactly how much violation of chiral symmetry should be present.

The simplest case to consider is the fermion propagator. In theories, such as the naive fermion action, where the only explicit breaking of chiral symmetry is a mass term $m_0\bar{\psi}\psi$, the quark propagator satisfies \[ \gamma_5 S(p) + S(p)\gamma_5 = 2m_0 G_{\gamma_5}(p) \]

where $\Lambda_{\gamma_5}$ is the amputated three-point function. Because $\gamma_5$ and $\gamma_5$ anticommute, this identity relates the running mass (the scalar part of $S^{-1}$) to the pseudoscalar density. With naive fermions $Z_m Z_{\gamma_5} = 1$, so the Ward identity Eq. (1) holds in the same form both for bare and renormalized Green functions. If chiral symmetry is spontaneously broken the right hand side doesn’t vanish as $m_0 \to 0$, and the fermion propagator looks massive even in the chiral limit [3].

In our action this chiral Ward identity is not automatically true. As well as chiral symmetry breaking from the mass term there is additional breaking from the Wilson and clover terms. We can now determine improvement coefficients, and find some information about the $Z$s, if we impose the chiral Ward identity, by insisting that the improved Green functions obey the identity

\[
\gamma_5 S_{\text{imp}}^{-1}(p) + S_{\text{imp}}^{-1}(p)\gamma_5 = Z_m Z_{\gamma_5} 2m_q \Lambda_{\gamma_5}^{\text{imp}}(p), \tag{2}
\]

where $m_q \equiv 1/(2\kappa) - 1/(2\kappa_c)$.

We are fortunate that improving $G_{\gamma_5}$ is easier than improving most operators. There is no single-derivative operator to add, all we have to do is multiply $G_{\gamma_5}$ by a factor of the type $(1 + b \bar{m}_q)$.

The improvement of the quark propagator is slightly more complicated. As well as multiplying by a $m_q$-dependent factor, we also have to subtract a contact term from the propagator [3]

\[
S(p) = \frac{Z_2}{(1 + b_2 \bar{m}_q) (i \not{p} + m_R(p))} + a\lambda + O(a^2)
\]

where $\lambda$ is a constant in momentum space, or a $\delta$ function in position space. This makes the chiral symmetry breaking term as short-range as possible. In perturbation theory we can see that the $O(a)$ corrections to the propagator only take this simple form if $c_{SW}$ is correctly chosen [3].

In Fig. 3. we show how well the Ward identities can be satisfied when the improvement coefficients are correctly chosen. The agreement is good while $a^2 p^2 \ll 2$.

3. THE LOCAL VECTOR CURRENT

There are similar Ward identities for the flavour non-singlet three-point functions, but they require the measurement of four-point functions $H$.

\[
H_{\Gamma, \gamma_5}(p) \equiv \sum_{ijkl} \langle M_{ij}^{-1} \Gamma M_{kl}^{-1} \gamma_5 M_{jk}^{-1} \rangle e^{i(p|x_i - x_l)}
\]

\[
H_{\gamma_5, \Gamma}(p) \equiv \sum_{ijkl} \langle M_{ij}^{-1} \gamma_5 M_{kl}^{-1} \Gamma M_{jk}^{-1} \rangle e^{i(p|x_i - x_l)} \tag{4}
\]

where $M$ is the fermion matrix. In naive fermions Ward identities analogous to Eq. (3) relate the
chiral violation in the three-point functions $G$ to $m_0H$. For example, for the local vector current, 
\[ \gamma_5 G \gamma_\mu + G \gamma_\mu \gamma_5 = m_0 (H \gamma_\mu \gamma_5 + H \gamma_5 \gamma_\mu). \] (5)
The $H$s also appear in Ward identities for the parity-splitting between the operators $\Gamma$ and $\Gamma \gamma_5$.

The amputated Green function for the improved local vector current, with the value it should have, according to the Ward identity Eq.(5). As in amputations are performed by dividing by the improved fermion propagator. We see that the comparison is very sensitive to the value of the improvement coefficient $c_1v$, defined by

\[ (\bar{\psi} \gamma_\mu \psi)^{\text{imp}} = (1 + abm_0) \bar{\psi} \gamma_\mu \psi - ac_1v \bar{\psi} D_\mu \psi. \] (6)

When we use a poor value (as in the upper graph) the violation of Eq.(6) is clear. Our preliminary result is that we find the best agreement with the Ward identity when $c_{1v} \approx 1.16$, $b \approx 0.1$ and $Z_v \approx 0.87$. Since we are working with off-shell quantities we can not use the equations of motion to eliminate either $b$ or $c_{1v}$.

We can test these coefficients, derived from off-shell quarks, by using them to renormalize our on-shell nucleon measurements. The results are in good agreement with the known value of the conserved vector current \[8\]. So we see that using chiral Ward identities to tell us when $O(a)$ effects have been removed seems a promising method.

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