DETERMINATION OF THE ORDER OF FRACTIONAL DERIVATIVE FOR SUBDIFFUSION EQUATIONS

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Dedicated to Professor Shavkat Alimov
on the occasion of his 75th birthday

Abstract

The identification of the right order of the equation in applied fractional modeling plays an important role. In this paper we consider an inverse problem for determining the order of time fractional derivative in a subdiffusion equation with an arbitrary second order elliptic differential operator. We prove that the additional information about the solution at a fixed time instant at a monitoring location, as “the observation data”, identifies uniquely the order of the fractional derivative.

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Key Words and Phrases: subdiffusion equation; Riemann-Liouville derivatives; inverse and initial-boundary value problem; determination of order of derivative; Fourier method

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1. Introduction

It is well known (see, for example, [1] - [4]) that Brownian motion, discovered in the first half of the 19th century, models motions of molecules in gases, electrons in semiconductors, neutrons in nuclear reactors, and much more. The main difference between Brownian motion and processes...
obeying Newton’s laws is that the diffusion packet spreads according to the law $t^{1/2}$ (and not like $t$). The subdiffusion process is characterized by a fractional exponent $\rho \in (0, 1)$, which is included in the diffusion equation as the order of the fractional time derivative.

The theory of differential equations with fractional derivatives has gained considerable popularity and importance in the past few decades, mainly due to its applications in numerous seemingly distant fields of science and technology (see, for example, [1] - [8]). In turn, the mathematical aspects of fractional differential equations and methods for solving them have been studied by many authors (see, for example, [9] - [33]).

By inverse problems in the theory of partial differential equations are commonly called problems in which, together with solving a differential equation, it is also necessary to determine a coefficient(s) of the equation or/and the right side (source function). Naturally, in this case additional information should be given to find a new unknown function. Note that the interest in studying inverse problems for equations of mathematical physics is due to the importance of their applications in many branches of modern science, including mechanics, seismology, medical tomography, epidemics, and geophysics, just to mention a few. A significant number of studies are devoted to inverse problems of determining the right-hand side of subdiffusion equations (see, for example, [1], [14] - [22] and references therein).

The present paper is devoted to the other important type of inverse problems, namely to determining of the order of fractional derivative in a subdiffusion equation, which is considered to govern the anomaly of diffusion. More precisely, this inverse problems is the determination of the unknown order of time-derivative in order to match available data such as $u(x_0, t), 0 < t < T$, at a monitoring point $x_0 \in \Omega$. One of the practical example is a modeling of COVID-19 outbreak. The data [23] presented by Johns Hopkins University about the outbreak from different countries seem to show fractional order dynamical processes, in which the identification of fractional order rate of change is a key issue [24, 25].

The problem of identification of fractional order of the model was considered by some researchers. Note that all the publications assumed the fractional derivative of order $0 < \rho < 1$ in the sense of Caputo and studied mainly the uniqueness problem. In paper [26] by J. Cheng et al. an inverse problem for determining the order of the Caputo fractional derivative and a coefficient of one-dimensional time-fractional diffusion equation is studied. The authors attached the homogeneous Neumann boundary condition and the initial value given by the Dirac delta function. They proved that the order of derivative and the unknown coefficient are uniquely determined by
the known datum $u(0, t), 0 < t < T$. The uniqueness of a solution of the two parameter inverse problem is considered in paper [27] by Tatar and Ulusoy for the differential equation

$$\partial_t^\rho u(t, x) = -(-\Delta)^\gamma u(t, x), \quad t > 0, \ x \in \mathbb{R}^N.$$  

The multi-term time-fractional diffusion equation and distributed order fractional diffusion equations considered in papers Li et al. [28] and [29], correspondingly.

In paper [30] by X. Zheng et al. the authors tried to solve the most difficult problem of determining the variable order of the Caputo fractional differentiation. In this work, as in many other papers, only the question of uniqueness is considered. But in our opinion, Lemma 4.1 of this paper is questionable, since there exist functions (see, for example, [35]) whose Fourier series converge to zero in a certain region, but not all Fourier coefficients are zero.

The following two papers [31] and [32] deals with the existence problem. J. Janno [31] considered a one-dimensional time-fractional diffusion equation with Caputo derivatives. Giving an extra boundary condition $Bu(\cdot, t) = h(t), 0 < t < T$ the author succeeded to prove the existence theorem for determining the order of the derivative and the kernel of the integral operator in the equation. The complexity of the proof of the existence can be seen from the statement of corresponding theorem (Theorem 7.2 is formulated on more than one journal page). In the paper of Hatano et al. [32] the equation $\partial_t^\rho u = \Delta u$, where $\Delta$ is the Laplace operator, is considered with the Dirichlet boundary condition and the initial function $\varphi(x)$. They proved that if $\varphi \in C_0^\infty(\Omega)$ and $\Delta \varphi(x_0) \neq 0$, then

$$\rho = \lim_{t \to 0} \left[ t^{\rho} u(x_0, t)[u(x_0, t) - \varphi(x_0)]^{-1} \right].$$

In the recent survey paper [33] by Z. Li et al. in the section of Open Problems they noted: “The studies on inverse problems of the recovery of the fractional orders are far from satisfactory since all the publications either assumed the homogeneous boundary condition or studied this inverse problem by the measurement in $t \in (0, \infty)$. It would be interesting to investigate inverse problem by the value of the solution at a fixed time as the observation data”.

In the present work we address this problem. Namely, as follows from our main result, in the case of the initial-boundary (Neumann) value problem for the equation with the Riemann-Liouville fractional derivative

$$\partial_t^\rho u(x, t) = \Delta u(x, t), \quad x \in \Omega \subset \mathbb{R}^N, \ t > 0,$$

the only condition
\[
\int_{\Omega} u(x, t_0) dx = d_0 \neq 0,
\]
where \( t_0 \geq 1 \) is an observation time, recovers the order \( \rho \in (0, 1) \), and if we have two pairs of solutions \( \{u_1(x, t), \rho_1\} \) and \( \{u_2(x, t), \rho_2\} \), then \( u_1(x, t) \equiv u_2(x, t) \) and \( \rho_1 = \rho_2 \).

The paper is organized as follows. In the next section we formulate the main result. In Section 3 we prove the existence of a unique solution to the forward problem. This result will be used to prove the main result in Section 4. Throughout the paper we assume that the fractional order \( \rho \) of the main equation is constant. The solution of the forward problem is obtained under this assumption. The Cauchy problem in the case of piece-wise constant \( \rho \) was studied in [34] when \( x \in \mathbb{R}^N \). Modifying the result of this paper to the case of bounded domain \( \Omega \), one can extend the main result presented in this paper to the case of piece-wise constant \( \rho \), as well.

2. Main result

Let \( \Omega \) be an arbitrary \( N \)-dimensional domain with twice differentiable boundary \( \partial \Omega \). Namely, the functions, defining the boundary equation in local coordinates, are continuously twice differentiable. Let the second order differential operator

\[
A(x, D)u = \sum_{i,j=1}^{N} D_i [a_{i,j}(x) D_j u] - c(x) u
\]

be a symmetric elliptic operator in \( \Omega \), i.e.

\[
a_{i,j}(x) = a_{j,i}(x) \quad \text{and} \quad \sum_{i,j=1}^{N} a_{i,j}(x) \xi_i \xi_j \geq a \sum_{i,j=1}^{N} \xi_i^2,
\]

for all \( x \in \Omega \) and \( \xi_i \), where \( a = \text{const} > 0 \) and \( D_j u = \frac{\partial u}{\partial x_j}, \quad j = 1, \ldots, N \).

Consider the spectral problem

\[
-A(x, D)v(x) = \lambda v(x), \quad x \in \Omega; \quad \text{(2.1)}
\]

\[
Bv(x) \equiv \frac{\partial}{\partial n} v(x) + h(x) v(x) = 0, \quad x \in \partial \Omega, \quad \text{(2.2)}
\]

where \( n \) is an external normal to the surface \( \partial \Omega \). It is known (see, for example, [36], p. 100, [37], p. 111), that if \( c(x) \geq 0 \) and

\[
a_{i,j}(x) \in C^{1,\frac{N}{2}+1}(\Omega), \quad i,j = 1, \ldots, N,
\]

\[
c(x) \in C^{1,\frac{N}{2}}(\Omega), \quad h(x) \in C^{1,\frac{N}{2}+2}(\partial \Omega),
\]

(2.3)
where \( \lfloor a \rfloor \) stands for the integer part of \( a \), then the corresponding inverse operator is compact, i.e. spectral problem (2.1) - (2.2) has a complete in \( L_2(\Omega) \) set of orthonormal eigenfunctions \( \{ v_k(x) \} \) and a countable set of nonnegative eigenvalues \( \{ \lambda_k \} \).

The fractional part of our equation will be defined through the Riemann-Liouville fractional derivative \( \partial^\rho_t \) of order \( 0 < \rho < 1 \). To define the Riemann-Liouville fractional derivative, one can define the fractional integration of order \( \rho < 0 \) of a function \( f \) defined on \( [0, \infty) \) by the formula
\[
\partial^\rho_t f(t) = \frac{1}{\Gamma(-\rho)} \int_0^t f(\xi) (t-\xi)^{\rho+1} d\xi, \quad t > 0,
\]
provided the right-hand side exists. Here \( \Gamma(z) \) is Euler’s gamma function. Using this definition one can define the Riemann-Liouville fractional derivative of order \( \rho, k-1 < \rho \leq k, k \in \mathbb{N} \), as (see, for example, [9, 10])
\[
\partial^\rho_t f(t) = \frac{d^k}{dt^k} \partial^{\rho-k}_t f(t).
\]
Note that if \( \rho = k \), then fractional derivative coincides with the ordinary classical derivative of order \( k \):
\[
\partial^k_t f(t) = \frac{d^k}{dt^k} f(t).
\]

Let \( 0 < \rho < 1 \) be an unknown number to be determined. Consider the initial-boundary value problem
\[
\partial^\rho_t u(x,t) - A(x,D)u(x,t) = 0, \quad x \in \Omega, \quad 0 < t \leq T, \quad (2.4)
\]
\[
Bu(x,t) \equiv \frac{\partial}{\partial n} u(x,t) + h(x)u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \quad (2.5)
\]
\[
\lim_{t \to 0} \partial^{\rho-1}_t u(x,t) = \varphi(x), \quad x \in \overline{\Omega}. \quad (2.6)
\]
Under some conditions on initial function \( \varphi \) the solution of this problem exists and is unique. This solution obviously depends on \( \rho \). The purpose of this paper is not only to find the solution \( u(x,t) \), but also to determine the order \( \rho \in (0, 1) \) of the time derivative. To do this one needs an extra condition. As was mentioned above, different types of such conditions were considered by a number of authors. We formulate our inverse problem in the following way. Let \( \omega \in L_2(\Omega) \) be a weight function with a property \( ||\omega||_{L_2(\Omega)} = 1 \). Is it possible to determine the order of time fractional derivative \( 0 < \rho < 1 \) with the additional information
\[
\int_{\Omega} u(x,t_0)\omega(x)dx = d_0,
\]
at a fixed time instant \( t_0 \geq 1 \)? This integral can be considered as the average distribution of the quantity \( u(t,x) \) over the region \( \Omega \) at the time instant \( t = t_0 \) with the weight function \( \omega \).
Below, using the classical Fourier method, we give a positive answer to this question in the case when a weight function is equal to the first eigenfunction of spectral problem (2.1), (2.2): \( \omega(x) = v_1(x) \), i.e. when the latter integral has the form
\[
f(\rho; t_0) \equiv \int_\Omega u(x, t_0)v_1(x)dx = d_0 \neq 0.
\] (2.7)

The quantity \( f(\rho; t_0) \) is, in fact, the projection of the solution \( u(x, t_0) \) onto the first eigenfunction, and as is shown below, using the Fourier method, under certain conditions, \( f(\rho; t_0) \) has a simple form \( f(\rho; t_0) = d\rho^{-1} \), with a constant \( d \) depending on \( \rho \).

We call the initial-boundary value problem (2.4) - (2.6) (for an arbitrary, but fixed \( 0 < \rho < 1 \)) the forward problem. The initial-boundary value problem (2.4) - (2.6) for an unknown \( \rho \), together with extra condition (2.7), is called an inverse problem.

**Definition 2.1.** A pair \( \{u(x, t), \rho\} \) of the function \( u(x, t) \) and the parameter \( \rho \) with the properties:

1. \( \rho \in (0, 1) \),
2. \( \partial^\alpha u(x, t), A(x, D)u(x, t) \in C(\bar{\Omega} \times (0, \infty)) \),
3. \( \partial^{\alpha-1} u(x, t) \in C(\bar{\Omega} \times [0, \infty)) \),

and satisfying all the conditions of problem (2.4) - (2.7) in the classical sense is called a classical solution of inverse problem (2.4) - (2.7).

We will also call the classical solution simply the solution to the inverse problem. We draw attention to the fact, that in the definition of the classical solution the requirement of continuity in the closed domain of all derivatives included in equation (2.4) is not significant. However, on the one hand, the uniqueness of such a solution is proved quite simply, and on the other hand, the solution found by the Fourier method satisfies the above conditions.

Further, let us denote by \( \varphi_j \) the Fourier coefficients of the function \( \varphi(x) \) with respect to the system of eigenfunctions \( \{v_k(x)\} \), defined as a scalar product on \( L_2(\Omega) \), i.e. \( \varphi_j = (\varphi, v_j) \). Let \( D^\alpha \) stands for \( D_1^\alpha \cdots D_N^\alpha \). For the seek of simplicity, we describe the proposed method, which is based on the classical Fourier method, for finding the order of fractional differentiation in the case of \( \lambda_1 = 0 \) and \( \varphi_1 \neq 0 \). Otherwise, the method becomes technically cumbersome.

Now we formulate the main result of this paper.
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THEOREM 2.1. Let the coefficients of operator $A(x, D)$ and function $h(x)$ satisfy conditions (2.3) and let the first eigenvalue of spectral problem (2.1)-(2.2) is equal to 0: $\lambda_1 = 0$ and $\varphi_1 \neq 0$. Moreover, let the initial function $\varphi(x)$ satisfy the conditions:

$$\varphi(x) \in C^{\lceil \frac{N}{2} \rceil}(\Omega), \quad (2.8)$$

$$D^\alpha \varphi(x) \in L_2(\Omega), \quad |\alpha| = \left\lfloor \frac{N}{2} \right\rfloor + 1, \quad (2.9)$$

$$B\varphi(x) = BA(x, D)\varphi(x) = \cdots = BA^{\lceil \frac{N}{2} \rceil}(x, D)\varphi(x) = 0, \quad x \in \partial \Omega. \quad (2.10)$$

Then inverse problem (2.4) - (2.7) has a unique solution $\{u(x, t), \rho\}$ if and only if

$$0 < \frac{d_0}{\varphi_1} < 1. \quad (2.11)$$

REMARK 2.1. (1) Conditions (2.8)-(2.10) are standard for the existence of a solution to the forward problem (see for example, [21]). Condition $\lambda_1 = 0$ of the theorem is satisfied, for example, in the case of the Neumann condition on the boundary for the Laplace operator.

(2) Theorem defines the unique $\rho$ from (2.7). Hence, if we define the integral (2.7) at another time instant $t_1$ and get a new $\rho_1$, i.e. $f(\rho_1; t_1) = d_1$, then from the equality $f(\rho_1; t_0) = d_0$, by virtue of the theorem, we obtain $\rho_1 = \rho$.

3. Forward problem

DEFINITION 3.1. A function $u(x, t)$ with the properties

(1) $\partial_t^\alpha u(x, t), A(x, D)u(x, t) \in C(\overline{\Omega} \times (0, \infty))$,
(2) $\partial_t^{\alpha - 1} u(x, t) \in C(\overline{\Omega} \times [0, \infty))$

and satisfying all the conditions of problem (2.4) - (2.6) in the classical sense is called a classical solution of forward problem (2.4) - (2.6).

In this section we prove existence and uniqueness of the solution of the forward problem by the Fourier method. In accordance with the Fourier method, we will look for a solution to problem (2.4) - (2.6) in the form of a series:

$$u(x, t) = \sum_{j=1}^{\infty} T_j(t)v_j(x), \quad t > 0, \ x \in \Omega, \quad (3.1)$$
where functions $T_j(t)$ are solutions to the Cauchy type problem

$$\partial_t^\rho T_j + \lambda_j T_j = 0, \quad \lim_{t \to 0} \partial_t^{\rho-1} T_j(t) = \varphi_j, \quad \forall j = 1, \ldots, \infty. \quad (3.2)$$

The unique solution of problem (3.2) has the form (see, for example, [10], p. 16)

$$T_j(t) = \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho), \quad (3.3)$$

where $E_{\rho,\mu}$ is the Mittag-Leffler function

$$E_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(pk + \mu)}. \quad (3.4)$$

**Theorem 3.1.** Let conditions (2.3) and (2.8) - (2.10) be satisfied. Then there exists a unique solution of the forward problem (2.4) - (2.6) and it has the representation

$$u(x, t) = \sum_{j=1}^{\infty} \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho)v_j(x), \quad (3.4)$$

which absolutely and uniformly converges on $x \in \bar{\Omega}$ for each $t \in (0, T]$.

**Proof.** The uniqueness of the solution can be proved by the standard technique based on completeness of the set of eigenfunctions $\{v_k(x)\}$ in $L_2(\Omega)$ (see, for example, [21]).

To prove the existence we need to introduce for any real number $\tau$ an operator $\hat{A}^\tau$, acting in $L_2(\Omega)$ in the following way

$$\hat{A}^\tau g(x) = \sum_{k=1}^{\infty} \lambda_k^\tau g_k v_k(x), \quad g_k = (g, v_k).$$

Obviously, the operator $\hat{A}^\tau$ with the domain of definition

$$D(\hat{A}^\tau) = \{g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |g_k|^2 < \infty\}$$

is selfadjoint. If we denote by $A$ the operator in $L_2(\Omega)$, acting as

$$Av(x) = A(x, D)v(x)$$

and with the domain of definition

$$D(A) = \{v \in C^2(\bar{\Omega}) : Bv(x) = 0, \quad x \in \partial \Omega\},$$

then it is not hard to show, that the operator $\hat{A} \equiv \hat{A}^1$ is the selfadjoint extension of the operator $A$ in $L_2(\Omega)$. In the same way one can define the operator $(\hat{A} + I)^\tau$, where $I$ is the identity operator in $L_2(\Omega)$.
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Further, we use the following lemma (see [39], p. 453):

**Lemma 3.1.** Let $\sigma > 1 + \frac{N}{4}$. Then for any multi-index $\alpha$ satisfying $|\alpha| \leq 2$ the operator $D^{\alpha}(\hat{A} + I)^{-\sigma}$ (completely) continuously maps the space $L_2(\Omega)$ into $C(\Omega)$, and moreover, the following estimate holds

$$\|D^{\alpha}(\hat{A} + I)^{-\sigma}g\|_{C(\Omega)} \leq C\|g\|_{L_2(\Omega)}. \quad (3.5)$$

Let $|\alpha| \leq 2$. First we prove that one can validly apply the operators $D^{\alpha}$ and $\partial_t^\rho$ to the series in (3.4) term-by-term. Suppose that the function $\varphi(x)$ satisfies the following condition for some $\tau > \frac{N}{4}$:

$$\sum_{j=1}^{\infty} (\lambda_j + 1)^{2\tau} |\varphi_j|^2 \leq C \varphi < \infty. \quad (3.6)$$

Consider the sum

$$S_k(x,t) = \sum_{j=1}^{k} v_j(x) \varphi_j t^{\rho - 1} E_{\rho,\rho}(-\lambda_j t^\rho). \quad (3.7)$$

Since $(\hat{A} + I)^{-\tau - 1} v_j(x) = (\lambda_j + 1)^{-\tau - 1} v_j(x)$, we can rewrite the latter in the form

$$S_k(x,t) = (\hat{A} + I)^{-\tau - 1} \sum_{j=1}^{k} v_j(x)(\lambda_j + 1)^{\tau + 1} \varphi_j t^{\rho - 1} E_{\rho,\rho}(-\lambda_j t^\rho).$$

Therefore, by virtue of Lemma 3.1, one has

$$\|D^{\alpha} S_k\|_{C(\Omega)} = \left\| D^{\alpha}(\hat{A} + I)^{-\tau - 1} \sum_{j=1}^{k} v_j(x)(\lambda_j + 1)^{\tau + 1} \varphi_j t^{\rho - 1} E_{\rho,\rho}(-\lambda_j t^\rho) \right\|_{C(\Omega)}$$

$$\leq C \left\| \sum_{j=1}^{k} v_j(x)(\lambda_j + 1)^{\tau + 1} \varphi_j t^{\rho - 1} E_{\rho,\rho}(-\lambda_j t^\rho) \right\|_{L_2(\Omega)}. \quad (3.8)$$

Using the orthonormality of the system $\{v_j\}$, we have

$$\|D^{\alpha} S_k\|_{C(\Omega)}^2 \leq C \sum_{j=1}^{k} \left| (\lambda_j + 1)^{\tau + 1} \varphi_j t^{\rho - 1} E_{\rho,\rho}(-\lambda_j t^\rho) \right|^2. \quad (3.9)$$

For the Mittag-Leffler function with a negative argument we have an estimate (see, for example, [10], p.13)

$$|E_{\rho,\rho}(-t)| \leq \frac{C}{1 + t}, \quad t > 0.$$
Applying this inequality, we have

\[
\sum_{j=1}^{k} |(\lambda_j + 1)^{\tau+1} \varphi_j t^{\rho-1} E_{\rho, \rho}(-\lambda_j t^\rho)|^2 = \sum_{\lambda_j < t^{-\rho}} |(\lambda_j + 1)^{\tau+1} \varphi_j t^{\rho-1} E_{\rho, \rho}(-\lambda_j t^\rho)|^2 \\
+ \sum_{\lambda_j > t^{-\rho}} |(\lambda_j + 1)^{\tau+1} \varphi_j t^{\rho-1} E_{\rho, \rho}(-\lambda_j t^\rho)|^2 \\
\leq Ct^{-2}(1 + t^\rho)^2 \sum_{j=1}^{k} (\lambda_j + 1)^{2\tau} |\varphi_j|^2 \leq Ct^{-2}(1 + T^\rho)^2 C\varphi. \tag{3.10}
\]

Here we used the inequality \( E_{\rho, \rho}(-\lambda_j t^\rho) < C \) in the case \( \lambda_j < t^{-\rho} \), and the inequality \( E_{\rho, \rho}(-\lambda_j t^\rho) < C \frac{1}{\lambda_j t^\rho} \) in the case \( \lambda_j > t^{-\rho} \). Taking into account (3.10), one can rewrite the estimate (3.9) as

\[
\|D^\alpha S_k^1\|_{C(\Omega)}^2 \leq Ct^{-2}(1 + T^\rho)^2 C\varphi.
\]

This implies uniform convergence on \( x \in \bar{\Omega} \) of the differentiated sum (3.7) with respect to variables \( x_j, j = 1, \ldots, N \), for each \( t \in (0, T] \). On the other hand, the sum (3.8) converges for any permutation of its members, as well, since these terms are mutually orthogonal. This implies the absolute convergence of the differentiated sum (3.7) on the same interval \( t \in (0, T] \).

Further, it is not hard to see that

\[
\partial_t^\rho \sum_{j=1}^{k} T_j(t) v_j(x) = -\sum_{j=1}^{k} \lambda_j T_j(t) v_j(x) \\
= -A(x, D)(\hat{A} + 1)^{-\tau-1} \sum_{j=1}^{k} (1 + \lambda_j)^{\tau+1} T_j(t) v_j(x).
\]

Absolute and uniform convergence of the latter series can be proved as above.

Obviously, the function in (3.4) satisfies boundary conditions (2.5). Considering the initial condition as (see, for example, [10] p. 104)

\[
\lim_{t \to 0} t^{1-\rho} u(x, t) = \frac{\varphi(x)}{\Gamma(\rho)}, \tag{3.11}
\]

it is not hard to verify, that this condition is also satisfied.

Hence, if the function \( \varphi(x) \) satisfies condition (3.6), then all the statements of Theorem 3.1 hold. As is shown in work [36] by V.A. Il'in (see also [37] p. 111) the fulfillment of conditions (2.3)-(2.9) guarantee the convergence of the series in (3.6). Thus, Theorem 3.1 is completely proved.
4. The inverse problem. Proof of Theorem 2.1

In this section we prove the main result. First we prove the following auxiliary lemma:

**Lemma 4.1.** Let the first eigenvalue of the operator \( A(x, D) \) be zero and the Fourier coefficient \( \varphi_1 \) of \( \varphi(x) \) is not zero. Then \( f(\rho; t_0) \) as a function of \( \rho \in (0, 1) \) is strictly monotone: if \( \varphi_1 > 0 \), then \( f(\rho; t_0) \) increases, and if \( \varphi_1 < 0 \), then \( f(\rho; t_0) \) decreases. Moreover,

\[
\lim_{\rho \to 0^+} f(\rho; t_0) = 0, \quad f(1; t_0) = \varphi_1.
\] (4.1)

**Proof.** Since the system of eigenfunctions \( \{v_j(x)\} \) are orthonormal and \( \lambda_1 = 0 \), then from (3.4) one has

\[
f(\rho; t_0) = \varphi_1 t_0^{\rho-1} E_{\rho, \rho} (0) = \frac{\varphi_1 t_0^{\rho-1}}{\Gamma(\rho)}.
\] (4.2)

Let \( \Psi(\rho) \) be the logarithmic derivative of the gamma function \( \Gamma(\rho) \) (see, for example, [38]). Then \( \Gamma'(\rho) = \Gamma(\rho) \Psi(\rho) \), and for \( \rho \in (0, 1) \) we have \( \Gamma(\rho) > 0 \) and \( \Psi(\rho) < 0 \). Therefore,

\[
\frac{d}{d\rho} \left( \frac{t_0^{\rho-1}}{\Gamma(\rho)} \right) = \frac{t_0^{\rho-1}}{\Gamma(\rho)} [\ln t_0 - \Psi(\rho)] > 0.
\]

for \( t_0 > 1 \) (the case \( t_0 = 1 \) is obvious). Thus function \( f(\rho; t_0) \) increases or decreases depending on sign of \( \varphi_1 \). It is easy to verify equalities (4.1).

\[ \square \]

**Proof of Theorem 2.1.** First we show existence of the order of the fractional derivative \( \rho \), which satisfies condition (2.7). We have

\[
f(\rho; t_0) = \int_{\Omega} u(x, t_0)v_1(x)dx = d_0.
\]

It follows from Lemma 4.1 and representation (4.2) immediately, that if

\[
0 < \frac{d_0}{\varphi_1} < 1,
\]

then there exists a unique \( \rho \), which satisfies condition (2.7).

To prove the uniqueness of a solution of inverse problem (2.4)-(2.7) we suppose that there exist two pairs of solutions \( \{u_1, \rho_1\} \) and \( \{u_2, \rho_2\} \) such, that \( 0 < \rho_k < 1, \ k = 1, 2 \), and

\[
\partial_t^{\rho_k} u_k(x, t) - A(x, D) u_k(x, t) = 0, \quad k = 1, 2, \quad x \in \Omega, \ 0 < t \leq T; \quad (4.3)
\]

\[
\lim_{t \to 0} \partial_t^{\rho_k-1} u_k(x, t) = \varphi(x), \quad k = 1, 2, \quad x \in \overline{\Omega}; \quad (4.4)
\]

\[
Bu_k(x, t) = 0, \quad k = 1, 2, \quad x \in \partial \Omega, \ 0 \leq t \leq T. \quad (4.5)
\]
Consider the following functions
\[ w_j^k(t) = \int_{\Omega} u_k(x, t)v_j(x)dx, \quad k = 1, 2, \quad j = 1, 2, \cdots \]
Then, for each \( j = 1, 2, \ldots, \) we have
\[ \partial_t^{\rho_k} w_j^k(t) + \lambda_j w_j^k(t) = 0, \quad \lim_{t \to 0} \partial_t^{\rho_k-1} w_j^k(t) = \varphi_j, \quad k = 1, 2. \]
Therefore (see (3.3)), for each \( j = 1, 2, \ldots, \)
\[ w_j^1(t) = \varphi_j t^{\rho_k-1} E_{\rho_k, \rho_k}(-\lambda_j t^{\rho_k}), \quad k = 1, 2, \]
and condition (2.7) implies \( w_1^1(t_0) = w_2^2(t_0). \) Since \( \lambda_1 = 0 \) we obtain
\[ \varphi_1 t_0^{\rho_1-1} E_{\rho_1, \rho_1}(0) = \varphi_1 t_0^{\rho_2-1} E_{\rho_2, \rho_2}(0) = \rho_0. \]
As we have seen above (see Lemma 4.1), this equation yields \( \rho_1 = \rho_2. \) But in this case \( w_1^1(t) = w_2^2(t) \) for all \( t \) and \( j. \) Hence
\[ \int_{\Omega} [u_1(x, t) - u_2(x, t)]v_j(x)dx = 0 \]
for all \( j. \) Since the set of eigenfunctions \( \{v_j\} \) is complete in \( L_2(\Omega), \) then we finally have \( u_1(x, t) = u_2(x, t). \) Thus, the “if part” of the theorem is proved.

To prove the “only if” part of the theorem assume that condition (2.11) is not verified. In this case, as it follows evidently from representation (4.2), equation \( f(\rho; t_0) = d_0 \) has no solution on the interval \((0, 1). \) Hence, in this case the inverse problem does not have a solution. The proof of Theorem 2.1 is complete.

As an example of application of Theorem 2.1 consider the following initial-boundary value problem for one-dimensional diffusion equation
\[ \partial_t^\rho u(x, t) - u_{xx}(x, t) = 0, \quad x \in (0, \pi), \quad t > 0, \quad (4.6) \]
with the initial condition
\[ \lim_{t \to 0} \partial_t^{\rho-1} u(x, t) = \varphi(x), \quad x \in [0, \pi], \quad (4.7) \]
and the boundary condition
\[ u_x(0, t) = 0, \quad u_x(\pi, t) = 0, \quad t \geq 0, \quad (4.8) \]
where \( 0 < \rho < 1. \) In this case the corresponding spectral problem has the set of eigenfunctions \( \{\cos kx\} \) complete in \( L_2(0, \pi), \) and eigenvalues \( k^2, \quad k = 0, 1, \ldots. \) Note that the first eigenvalue in this case is \( \lambda_1 = 0 \) and the corresponding eigenfunction is \( v_1(x) = 1. \) Therefore, condition (2.7) takes the form
\[
\int_0^\pi u(x,t_0)dx = d_0, \quad t_0 \geq 1.
\] (4.9)

**Theorem 4.1.** Let \( \varphi \in C^1[0, \pi] \) and \( \varphi_x(0) = \varphi_x(0) = 0 \). If
\[
0 < \frac{d_0}{\varphi_1} < 1,
\]
where \( \varphi_1 = \int_0^\pi \varphi(x)dx \), then there exists a unique solution \( \{u(x,t), \rho\} \) to inverse problem (4.6) - (4.9). Moreover, for the solution the representation
\[
u(x,t) = \sum_{j=1}^{\infty} \varphi_j t^{\rho-1} E_{\rho,\rho} \left( -\left( j - 1 \right)^2 t^\rho \right) \cos(j-1)x \quad t > 0, \quad x \in [0, \pi],
\]
holds, where \( \rho \) is the unique root of the algebraic equation
\[
\frac{t_0^{\rho-1}}{\Gamma(\rho)} = \frac{d_0}{\varphi_1}.
\]

The proof of this theorem immediately follows from Theorem 2.1.

**References**

[1] J.A.T. Machado (Ed.), *Handbook of Fractional Calculus with Applications*, Vols. 1 - 8. DeGruyter (2019).

[2] S. Umarov, M. Hahn, K. Kobayashi, *Beyond the Triangle: Brownian Motion, Itô Calculus, and Fokker-Plank Equation - Fractional Generalizations*. World Scientific (2018).

[3] R. Metzler, J. Klafter, The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* 339, No 1 (2000), 1–77.

[4] Y. Zhang, D.A. Benson, M.M. Meerschaert, E.M. LaBolle, H.P. Scheffler, Random walk approximation of fractional-order multiscaling anomalous diffusion. *Phys. Rev. E.* 74 (2006), Art. ID 026706.

[5] R. Hilfer (Ed.), *Applications of Fractional Calculus in Physics*. Singapore, World Scientific (2000).

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, North-Holland, Mathematics Studies (2006).

[7] V.V. Uchaikin, *Fractional Derivatives for Physicists and Engineers, 1*, *Background and Theory Application, 2*. Springer (2013).

[8] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogozin, *Mittag-Leffler Functions, Related Topics and Applications*. Springer (2014).
[9] S. Umarov, *Introduction to Fractional and Pseudo-Differential Equations with Singular Symbols*. Springer (2015).

[10] A.V. Pskhu, *Fractional Partial Differential Equations* (in Russian). Nauka (2005).

[11] O.P. Agrawal, Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlin. Dynam.* 29 (2002), 145–155.

[12] R. Hilfer, Y. Luchko, Z. Tomovski, Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives. *Fract. Calc. Appl. Anal.* 12 (2009), 299–318.

[13] R. Ashurov, A. Cabada, B. Turmetov, Operator method for construction of solutions of linear fractional differential equations with constant coefficients. *Fract. Calc. Appl. Anal.* 19, No 1 (2016), 229–252; DOI: 10.1515/fca-2016-0013; https://www.degruyter.com/view/journals/fca/19/1/fca.19.issue-1.xml.

[14] Y. Zhang, X. Xu, Inverse source problem for a fractional differential equations. *Inverse Prob.* 27, No 3 (2011), 31–42.

[15] H.T. Nguyen, D.L. Le, V.T. Nguyen, Regularized solution of an inverse source problem for a time fractional diffusion equation. *Appl. Math. Modeling* 40 (2016), 8244–8264.

[16] N. Heymans, I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheol. Acta.* 45 (2006), 765–771.

[17] Z. Li, Y. Liu, M. Yamamoto, Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients. *Appl. Math. and Computation* 257 (2015), 381–397.

[18] W. Rundell, Z. Zhang, Recovering an unknown source in a fractional diffusion problem. *J. of Comput. Phys.* 368 (2018), 299–314.

[19] N.A. Asl, D. Rostamy, Identifying an unknown time-dependent boundary source ib time-fractional diffusion equation with a non-local boundary condition. *J. of Comput. and Appl. Math.* 335 (2019), 36–50.

[20] L. Sun, Y. Zhang, T. Wei, Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation. *Appl. Numer. Math.* 135 (2019), 228–245.

[21] M. Ruzhansky, N. Tokmagambetov, B.T. Torebek, Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. *J. Inverse Ill-Posed Probl.* 27 (2019), 891–911.

[22] S.A. Malik, S. Aziz, An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions. *Computers and Math. with Appl.* 3 (2017), 7–19.
Determination of the order of fractional ... 1661

[23] https://www.worldometers.info/coronavirus/.

[24] C. Xu, Y. Yu, Y.-Q. Chen, Z. Lu, Forecast analysis of the epidemic trend of COVID-19 in the United States by a generalized fractional-order SEIR model. ArXiV: 2004.12541v1 (2020).

[25] M.A. Khan, A. Atangana, Modeling the dynamics of novel coronavirus (2019-nCov) with fractional derivative. Alexandria Eng. J. (2020) (in press).

[26] J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. Inverse Prob. 4 (2009), 1–25.

[27] S. Tatar, S. Ulusoy, A uniqueness result for an inverse problem in a space-time fractional diffusion equation. Electron. J. Differ. Equ. 257 (2013), 1–9.

[28] Z. Li, M. Yamamoto, Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. Appl. Anal. 94 (2015), 570–579.

[29] Z. Li, Y. Luchko, M. Yamamoto, Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem. Comput. Math. Appl. 73 (2017), 1041–1052.

[30] X. Zheng, J. Cheng, H. Wang, Uniqueness of determining the variable fractional order in variable-order time-fractional diffusion equations. Inverse Problems 35 (2019), 1–11.

[31] J. Janno, Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time-fractional diffusion equation. Electr. J. Diff. Equations 2016 (2016), 1–28.

[32] Y. Hatano, J. Nakagawa, S. Wang, M. Yamamoto, Determination of order in fractional diffusion equation. J. Math.-for-Ind., 5A (2013), 51–57.

[33] Z. Li, Y. Liu, M. Yamamoto, Inverse problems of determining parameters of the fractional partial differential equations. Handbook of Fractional Calculus with Applications 2, DeGruyter (2019).

[34] S. Umarov, S. Steinberg, Variable order differential equations with piecewise constant order-function and diffusion with changing modes. Zeitschrift für Anal. und ihre Anwendungen 28 (2009), 431–450.

[35] Sh.A. Alimov, R.R. Ashurov, A.K. Pulatov, Multiple Fourier series and Fourier integrals. In: Ser. Commutative Harmonic Analysis IV, Encyclopaedia of Math. Sci. (Eds. V.P. Khavin, N.K. Nikolskii), Springer (1992), 95 pp.

[36] V.A. Il’in, On the solvability of mixed problems for hyperbolic and parabolic equations. Russian Math. Surveys 15 No 1 (1960), 85–142.
1662 R. Ashurov, S. Umarov

[37] O.A. Ladyjinskaya, *Mixed Problem for a Hyperbolic Equation*. Gostexizdat (1953).
[38] H. Bateman, *Higher Transcendental Functions*. McGraw-Hill (1953).
[39] M.A. Krasnoselski, P.P. Zabreyko, E.I. Pustilnik, P.S. Sobolevski, *Integral Operators in the Spaces of Integrable Functions* (in Russian). Nauka (1966).

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