A-renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources

J. A. Shifflett
Department of Physics, Washington University, St. Louis, Missouri 63130

Abstract. The Einstein-Schrödinger theory is extended to include spin-0 and spin-1/2 sources, and the theory is derived from a Lagrangian density which allows other fields to be easily added. The original theory is also modified by including a cosmological constant caused by zero-point fluctuations. This cosmological constant which multiplies the symmetric metric is assumed to be nearly cancelled by Schrödinger’s “bare” cosmological constant which multiplies the nonsymmetric fundamental tensor, such that the total “physical” cosmological matches measurement. We show that the resulting Λ-renormalized Einstein-Schrödinger theory closely approximates ordinary Einstein-Maxwell theory and one-particle quantum mechanics. In particular, the field equations match the ordinary Einstein and Maxwell equations except for additional terms which are < 10^{-16} of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. We also show that the theory predicts the exact Lorentz force equation and the exact Klein-Gordon and Dirac equations. And the theory becomes exactly Einstein-Maxwell theory and one-particle quantum mechanics in the limit as the cosmological constant from zero-point fluctuations goes to infinity. Lastly, we discuss the merits of our Lagrangian density compared to the Einstein-Maxwell Lagrangian density.

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E-mail: shifflet@hbar.wustl.edu

1. Introduction

The Einstein-Schrödinger theory has had a long and distinguished history. Two full length books and over 100 research articles have been published on it (see the bibliographies of [1] [2]). Both Einstein and Schrödinger devoted a section of one of their books to it. All of this was done without anyone being able to definitely connect the theory to reality, and interest in the theory has faded as a result.

In this paper and in two previous papers[2, 3] it is shown that the Einstein-Schrödinger theory can be made to closely approximate ordinary Einstein-Maxwell theory by simply including a large cosmological term Λ_z g_{μν} in the field equations, where g_{μν} is the symmetric metric. It is well known that zero-point fluctuations should be expected to cause just such a cosmological term[4, 5, 6], so this can be viewed as a kind of zeroth order quantization effect. We assume the cosmological term Λ_z g_{μν} is nearly cancelled by Schrödinger’s “bare” cosmological term Λ_b N_{μν}, where N_{μν} is the nonsymmetric fundamental tensor and g_{μν} \neq N_{(μν)}. The resulting “physical” cosmological constant Λ = Λ_b + Λ_z can then be consistent with measurement, hence the name Λ-renormalized Einstein-Schrödinger theory (LRES theory).
In [2], the electro-vac version of this theory was described in detail, and close approximations of Maxwell’s equations and the Einstein equations were derived. An exact electric monopole solution was derived which closely approximates the Reissner-Nordström solution, and the equations-of-motion for such solutions found using the Einstein-Infeld-Hoffman method[7, 26] were shown to definitely exhibit the Lorentz force. In [3], the theory was expressed in Newman-Penrose tetrad form, some critical calculations were confirmed using tetrad methods, and the electric monopole solution was shown to be of Petrov-type D. A classical hydrodynamics extension of the theory was also developed, and the exact Lorentz force equation was derived. In the present paper, the theory is further extended to include spin-0 and spin-1/2 sources. It is also derived from a Palatini type of Lagrangian density which allows other fields to be easily added, including the additional fields of the Standard Model.

Einstein’s work[8, 9, 10, 11, 12] on this theory began in 1946, and he held to the same theory until his death in 1955. In his papers he never calls the theory a unified field theory, being careful not to claim anything he could not prove. He derived the theory using arguments similar to those he used to derive ordinary general relativity[13, 10], and he used a Palatini type of Lagrangian density[10].

Schrödinger’s contributions[14, 15, 16] to the theory began shortly after Einstein’s. However, he came upon the theory independently, by considering the simplest Lagrangian density he could think of, namely the square-root of the determinant of the Ricci tensor composed of only an affinity. He found that the resulting field equations were identical to Einstein’s theory, but with a cosmological constant. In fact, Schrödinger’s simple derivation only works if this cosmological constant is not zero. This is somewhat important for the present theory in the sense that one can claim that this “bare” cosmological constant is not simply being appended onto the theory, but is instead an inherent part of it.

Many others have made important contributions to the theory, most of which are listed in the bibliographies of [1, 2]. Of particular significance to our modified theory are contributions related to the choice of metric[17, 18, 19, 20], the generalized contracted Bianchi identity[17, 19, 20], the inclusion of charge currents[20, 21], the inclusion of an energy-momentum tensor[18], and exact solutions with a cosmological constant[22, 23].

In our modified theory, Schrödinger’s bare cosmological constant nearly cancels a large cosmological constant caused by zero-point fluctuations[4]. This is not a radical proposal! In fact, the apparent absence of a cosmological constant from zero-point fluctuations has been a longstanding problem in conventional Standard-Model physics[5]. Zero-point fluctuations are essential to quantum electrodynamics, and are the cause of the Casimir force[5] and other effects. To quote from the quantum field theory text Peskin and Schroeder[6] “We have no understanding of why $\lambda$ is so much smaller than the vacuum energy shifts generated in the known phase transitions of particle physics, and so much again smaller than the underlying field zero-point energies.” In the present theory, the fine-tuning of cosmological constants is not so objectionable when one considers that it is similar to mass/charge/field-strength renormalization in quantum electrodynamics. For example, to cancel electron self-energy in quantum electrodynamics, the “bare” electron mass becomes large for a cutoff frequency $\omega_c \sim 1/(\text{Planck length})$, and infinite if $\omega_c \to \infty$, but the total “physical” mass remains small. In a similar manner, to cancel zero-point energy in the present theory, the “bare” cosmological constant $\Lambda_b \sim \omega_c^2 (\text{Planck length})^2$ becomes large if $\omega_c \sim 1/(\text{Planck length})$, and infinite if $\omega_c \to \infty$, but the total “physical” $\Lambda$
remains small. This can be viewed as a kind of vacuum energy renormalization of the original Einstein-Schrödinger theory to account for zero-point fluctuations. With this quantum mechanical effect included, the theory closely approximates Einstein-Maxwell theory when $\omega_c \sim 1/(\text{Planck length})$, and it becomes exactly Einstein-Maxwell theory in the limit as $\omega_c \to \infty$.

There are other reasons for pursuing this theory besides the unification of gravitation and electromagnetism. Firstly, the theory suggests untried approaches to a complete unified field theory. For example, one could consider the theory with higher dimensions as in the Kaluza-Klein approach. Another possibility is to consider the theory with non-Abelian fields as in the Standard Model. This theory also offers an untried approach to the quantization of gravity. While the theory is almost identical to Einstein-Maxwell theory at ordinary frequencies, it becomes different near the Planck frequency, and it is possible that this could fix some of the infinities that occur when attempting to quantize ordinary general relativity. Since quantization of gravity is such an important topic, and previous approaches have been worked over exhaustively, any new and well motivated approach should be welcome. In the present paper we are considering a quantization effect, namely a cosmological constant $\Lambda_c$ caused by zero-point fluctuations, but the second quantization or multiparticle quantization of the theory is otherwise outside the scope of the paper. The hypothesis behind this paper is that an unquantized unified field theory might come in the form of a purely classical theory where fermions are represented not by fields, but by singular solutions of the field equations. We assume that the quantization process must also include a first quantization step in which wave-functions are substituted for particles to create a one-particle quantum mechanical theory. With this assumption, a charged spin-1/2 field must be added to the Lagrangian density, but this could perhaps be viewed as the first quantization of an elementary Kerr-Newman-like solution. While this idea is speculative, the theory presented here is rather conventional. We show that spin-0 and spin-1/2 fields can be added to the theory in the same way they are added to ordinary Einstein-Maxwell theory.

This paper is organized as follows. In §2 we discuss the Lagrangian density, and its extensions for the electro-vac, classical hydrodynamics, spin-0 and spin-1/2 cases. In §3-§5 we show that all of the extra terms in our Einstein and Maxwell equations have a $\Lambda_b$ in the denominator so that they go to zero as $\Lambda_b \to \infty$. And with a large $\Lambda_b$ caused by zero-point fluctuations, we show that these extra terms are all $< 10^{-16}$ of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. In §6-§7 we show that the theory predicts the ordinary Lorentz force equation and the ordinary Klein-Gordon and Dirac equations. In §8 we discuss the merits of our Lagrangian density compared to the Einstein-Maxwell Lagrangian density.

2. The Lagrangian Density

Ordinary vacuum general relativity can be derived from a Palatini Lagrangian density,

$$L(\Gamma^\alpha_{\rho\tau}, g_{\rho\tau}) = -\frac{1}{16\pi} \sqrt{-g} \left[ g^{\mu\nu} R_{\nu\mu}(\Gamma) + (n-2)\Lambda_b \right].$$

(1)

Here and throughout this paper we are assuming that $n = 4$, but the dimension “n” will be included in the equations to show how easily the results can be generalized to arbitrary dimension. The original unmodified Einstein-Schrödinger theory can be derived from a generalization of (1) formed from a connection $\Gamma^\alpha_{\nu\mu}$ and a fundamental
tensor $N_{\nu\mu}$ with no symmetry properties,\)
\[
\mathcal{L}(\tilde{\Gamma}_\nu^\alpha, N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{-\nu\mu}\mathcal{R}_{\nu\mu}(\tilde{\Gamma}) + (n-2)\Lambda_b\right].
\]  \hfill (2)

Our $\Lambda$-renormalized Einstein-Schrödinger (LRES) theory includes a cosmological constant $\Lambda_z$ caused by zero-point fluctuations, and allows other fields,\)
\[
\mathcal{L}(\tilde{\Gamma}_\nu^\alpha, N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{-\nu\mu}\mathcal{R}_{\nu\mu}(\tilde{\Gamma}) + (n-2)\Lambda_b\right]
- \frac{1}{16\pi}\sqrt{-g}(n-2)\Lambda_z + \mathcal{L}_m(\psi_e, g_{\mu\nu}, A_\sigma \ldots),
\]  \hfill (3)

where the “bare” $\Lambda_b$ obeys $\Lambda_b \approx -\Lambda_z$ so that the “physical” $\Lambda$ matches measurement,
\[
\Lambda = \Lambda_b + \Lambda_z,
\]  \hfill (4)

and the metric and electromagnetic potential are defined as\)
\[
\sqrt{-g}g^{\mu\nu} = \sqrt{-N}N^{-\nu\mu},
\]  \hfill (5)

\[
A_\nu = \frac{1}{2(n-1)}\tilde{\Gamma}^\sigma_{[\nu\mu]}\sqrt{2}i\Lambda_b^{-1/2}.
\]  \hfill (6)

Here and throughout this paper we use geometrized units with $c = G = 1$, the symbols ( ) and [ ] around indices indicate symmetrization and antisymmetrization, “$n$” is the dimension, $N = \det(N_{\nu\mu})$, and $N^{-\nu\mu}$ is the inverse of $N_{\nu\mu}$ so that $N_{[\nu\sigma]}N_{\nu\mu} = \delta_{\mu}^\sigma$. The $\mathcal{L}_m$ term is not to include a $\sqrt{-g}F^{\mu\nu}F_{\mu\nu}$ part but may contain the rest of the Standard Model. In (3), $\mathcal{R}_{\nu\mu}(\Gamma)$ is a form of the so-called Hermitianized Ricci tensor\[8\].

\[
\mathcal{R}_{\nu\mu}(\tilde{\Gamma}) = \tilde{\Gamma}_\nu^\alpha\alpha_{(\nu)} - \tilde{\Gamma}_{\nu}^{(\alpha)}\mu_{(\nu)} + \tilde{\Gamma}_\nu^\sigma\tilde{\Gamma}_\sigma\alpha_{(\nu)} - \tilde{\Gamma}_\nu^\sigma\tilde{\Gamma}_{\sigma\alpha_{(\nu)}} - \tilde{\Gamma}_{[\nu\sigma]}\tilde{\Gamma}^{\alpha}_{[\alpha_{(\nu)]}}/(n-1).
\]  \hfill (7)

This tensor reduces to the ordinary Ricci tensor for symmetric fields, where we have $\tilde{\Gamma}_\nu^\alpha = 0$ and $\Gamma_{\nu[\mu]} = R_{\alpha\nu\mu\rho}/2 = 0$.

It is helpful to decompose $\tilde{\Gamma}_\nu^\alpha$ into another connection $\tilde{\Gamma}_\nu^\alpha_{\mu}$, and $A_\sigma$ from (5),

\[
\tilde{\Gamma}_\nu^\alpha_{\mu} = \tilde{\Gamma}_\nu^\alpha + (\delta^\alpha_{\mu}A_\nu - \delta_\alpha^\nu A_\mu)\sqrt{2}i\Lambda_b^{1/2},
\]  \hfill (8)

where

\[
\tilde{\Gamma}_\nu^\alpha = \tilde{\Gamma}_\nu^\alpha + (\delta_\mu^\nu\tilde{\Gamma}_{[\nu\sigma]} - \delta_\nu^\sigma\tilde{\Gamma}_{[\nu\sigma]})/(n-1).
\]  \hfill (9)

By contracting (9) on the right and left we see that $\tilde{\Gamma}_\nu^\alpha$ has the symmetry

\[
\tilde{\Gamma}_{\nu}^\alpha = \tilde{\Gamma}_\nu^\alpha_{(\alpha)} = \tilde{\Gamma}_\nu^\alpha_{\alpha},
\]  \hfill (10)

so it has only $n^3-n$ independent components. Using $\mathcal{R}_{\nu\mu}(\tilde{\Gamma}) = \mathcal{R}_{\nu\mu}(\tilde{\Gamma}) + 2A_{\nu[\mu]}\sqrt{2}i\Lambda_b^{1/2}$ from (3), the Lagrangian density (3) can be rewritten in terms of $\tilde{\Gamma}_\nu^\alpha_{\mu}$ and $A_\sigma$,\)
\[
\mathcal{L}(\tilde{\Gamma}_\nu^\lambda, N_{\rho\tau}) = -\frac{1}{16\pi}\sqrt{-N}\left[N^{-\nu\mu}\tilde{\mathcal{R}}_{\nu\mu}(\tilde{\Gamma}) + 2A_{[\nu\mu]}\sqrt{2}i\Lambda_b^{1/2} + (n-2)\Lambda_b\right]
- \frac{1}{16\pi}\sqrt{-g}(n-2)\Lambda_z + \mathcal{L}_m(\psi_e, g_{\mu\nu}, A_\sigma \ldots).
\]  \hfill (11)

Here $\tilde{\mathcal{R}}_{\nu\mu} = \mathcal{R}_{\nu\mu}(\tilde{\Gamma})$, and from (10) the Hermitianized Ricci tensor (7) simplifies to

\[
\tilde{\mathcal{R}}_{\nu\mu} = \tilde{\mathcal{R}}_{\nu\mu} - \tilde{\Gamma}_{\nu(\alpha)}\tilde{\Gamma}_{\alpha(\mu)} + \tilde{\Gamma}_\nu^\sigma\tilde{\Gamma}_{\sigma\alpha_{(\mu)}} - \tilde{\Gamma}_{\nu\sigma}\tilde{\Gamma}_\sigma_{\alpha_{(\mu)}}.
\]  \hfill (12)

From (12) $\tilde{\Gamma}_\nu^\alpha_{\mu}$ and $A_\sigma$ fully parameterize $\tilde{\Gamma}_\nu^\alpha_{\mu}$ and can be treated as independent variables. So when we set $\delta \mathcal{L}/\delta \tilde{\Gamma}_\nu^\alpha_{\mu} = 0$ and $\delta \mathcal{L}/\delta A_\nu = 0$, the same field equations must result as with $\delta \mathcal{L}/\delta \tilde{\Gamma}_\nu^\alpha_{\mu} = 0$. It is simpler to calculate the field equations using $\tilde{\Gamma}_\nu^\alpha_{\mu}$ and $A_\nu$ instead of $\tilde{\Gamma}_\nu^\alpha$, so we will follow this method.
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We will usually assume that \( \Lambda_z \) is limited by a cutoff frequency\(^{[20, 23, 27, 30]} \)
\[
\omega_c \sim 1/l_P,
\]
where \( l_P = (\text{Planck length}) = \sqrt{\hbar G/c^3} = 1.6 \times 10^{-33} \text{cm} \). Then from \(^{[4, 13]} \) and assuming all of the known fundamental particles we have\(^{[5]} \),
\[
\Lambda_b \approx - \Lambda_z \sim C_z \omega_c^4 l_P^2 \sim 10^{66} \text{ cm}^{-2},
\]
\[
C_z = \frac{1}{2\pi} \left( \text{fermion } \frac{\text{spin states}}{\text{spin states}} \right) \sim \frac{60}{2\pi}
\]
and from astronomical measurements
\[
\Lambda \approx 1.4 \times 10^{-56} \text{ cm}^{-2}, \quad \Lambda/\Lambda_b \sim 10^{-122}.
\]
However, it might be more correct to fully renormalize with \( \omega_c \to \infty, |\Lambda_z| \to \infty, \Lambda_b \to \infty \) as in quantum electrodynamics. To account for this possibility we will prove that
\[
\lim_{\Lambda_b \to \infty} \left( \text{LRES} \right) = \left( \text{Einstein-Maxwell} \right).
\]
The Hermitianized Ricci tensor \(^{[7]} \) has the following invariance properties
\[
R_{\mu\nu}(\Gamma^T) = R_{\mu\nu}(\hat{\Gamma}), \quad (T = \text{transpose})\quad (18)
\]
\[
R_{\mu\nu}(\hat{\Gamma}^\alpha + \delta^\alpha_{[\rho\varphi,\tau]}) = R_{\nu\mu}(\hat{\Gamma}^\alpha) \quad \text{for an arbitrary } \varphi(x^\sigma).\quad (19)
\]
From \(^{[18, 19]} \), the Lagrangian densities \(^{[11]} \) are invariant under charge conjugation,
\[
Q \to -Q, \quad A_\sigma \to -A_\sigma, \quad \hat{\Gamma}^\alpha_{\mu\nu} \to \hat{\Gamma}^\alpha_{\nu\mu}, \quad \hat{\Gamma}^\alpha_{\rho\tau} \to \hat{\Gamma}^\alpha_{\tau\rho}, \quad N_{\nu\mu} \to N_{\mu\nu}, \quad N^{-\nu\mu} \to N^{-\mu\nu},
\]
and also under an electromagnetic gauge transformation
\[
\psi \to \psi e^{i\phi}, \quad A_\alpha \to A_\alpha - \frac{\hbar}{Q} \phi_\alpha, \quad \hat{\Gamma}^\alpha_{\rho\tau} \to \hat{\Gamma}^\alpha_{\rho\tau}, \quad \hat{\Gamma}^\alpha_{\mu\nu} \to \hat{\Gamma}^\alpha_{\mu\nu} + \frac{2\hbar}{Q} \delta^\alpha_{[\rho\varphi,\tau]} \sqrt{2} i \Lambda_b^{1/2},
\]
assuming that \( L_m \) is invariant. If \( \Lambda_b > 0, \Lambda_z < 0 \) as in \(^{[14]} \) then \( \hat{\Gamma}^\alpha_{\mu\nu} \), \( \hat{\Gamma}^\alpha_{\nu\mu} \), \( N_{\nu\mu} \) and \( N^{-\nu\mu} \) are all Hermitian, \( \hat{R}_{\nu\mu} \) and \( \hat{R}_{\nu\mu}(\hat{\Gamma}) \) are Hermitian from \(^{[18]} \), and \( g_{\nu\mu}, A_\sigma \) and \( \mathcal{L} \) are real from \(^{[6, 8, 11]} \). If instead \( \Lambda_b < 0, \Lambda_z > 0 \), then all of the fields are real.

Note that \(^{[5]} \) defines \( g^{\mu\nu} \) unambiguously because \( \sqrt{-g} = [-\text{det}(\sqrt{-g} g^{\mu\nu})]^{1/(n-2)} \).

In this theory the metric \( g_{\mu\nu} \) is used for measuring space-time intervals, and for calculating geodesics, and for raising and lowering of indices. The covariant derivative \( \gamma^\alpha_{\nu} \) is always done using the Christoffel connection formed from \( g_{\mu\nu} \),
\[
\Gamma^\alpha_{\nu\mu} = \frac{1}{2} g^{\alpha\sigma} (g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}).
\]
With the metric \(^{[5]} \), the divergence of the Einstein equations vanishes if \( L_m = 0 \), and it gives the exact Lorentz force equation if \( L_m \neq 0 \). And when \( N_{\mu\nu} \) and \( \hat{\Gamma}^\alpha_{\mu\nu} \) are symmetric, the definition \(^{[5]} \) requires \( g_{\mu\nu} = N_{\mu\nu} \), the definition \(^{[6]} \) requires \( A_\sigma = 0 \), and the theory reduces to ordinary general relativity without electromagnetism.

The electromagnetic field is defined in terms of the potential \(^{[6]} \) as usual
\[
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.
\]
However, we will also define a lowercase \( f_{\mu\nu} \)
\[
\sqrt{-g} f^{\mu\nu} = \sqrt{-N} N^{-[\nu\mu]} \Lambda_b^{1/2} / \sqrt{2} i.
\]
Then from \(^{[5]} \), \( g^{\mu\nu} \) and \( f^{\mu\nu} \sqrt{2} i \Lambda_b^{-1/2} \) are the symmetric and antisymmetric parts of a total field,
\[
W^{\mu\nu} = (\sqrt{-N} / \sqrt{-g}) N^{-\nu\mu} = g^{\mu\nu} + f^{\mu\nu} \sqrt{2} i \Lambda_b^{-1/2}.
\]

We will see that the field equations require \( f_{\mu \nu} \approx F_{\mu \nu} \) to a very high precision, so it is mainly just a matter of terminology which one is called the electromagnetic field.

Note that there are many possible nonsymmetric generalizations of the Ricci tensor besides the Hermitianized Ricci tensor \( R_{\alpha \beta \gamma \delta} \) from (12), and the ordinary Ricci tensor \( R_{\alpha \beta}(\hat{\Gamma}) \). For example, we could form any weighted average of \( R_{\alpha \beta}(\hat{\Gamma}), R_{\mu \nu}(\hat{\Gamma}), \) \( R_{\alpha \beta}(\hat{\Gamma}^T), \) and then add any linear combination of the tensors \( R_{\alpha \beta}(\hat{\Gamma}), R_{\alpha \beta}(\hat{\Gamma}^T), \) \( \hat{\Gamma}_{\alpha \beta \gamma \delta} [\alpha \beta \gamma \delta] \) and \( \hat{\Gamma}_{\alpha \beta \gamma \delta} [\alpha \beta \gamma \delta] \). All of these generalized Ricci tensors would be linear in \( \hat{\Gamma}_{\alpha \beta \gamma \delta} ; \) quadratic in \( \hat{\Gamma}_{\alpha \beta \gamma \delta} \), and would reduce to the ordinary Ricci tensor for symmetric fields. Even if we limit the tensor to only four terms, there are still eight possibilities. We assert that invariance properties like (18,19) are the most sensible way to choose among the different alternatives, not criteria such as the number of terms in the expression.

For the electro-vac case as in (2), matter is represented by solutions to the field equations (which are allowed to have singularities) and we have

\[
\mathcal{L}_m = 0.
\]

For the classical hydrodynamics case as in (3), we can form a rather artificial \( \mathcal{L}_m \) which depends on a mass scalar density \( \mu \) and a velocity vector \( u^\nu \), neither of which is constrained (that is we will not require \( \delta \mathcal{L}/\delta \mu = 0 \) or \( \delta \mathcal{L}/\delta u^\sigma = 0 \)),

\[
\mathcal{L}_m = -\frac{\mu Q}{m} u^\nu A_\nu - \frac{\mu}{2} g_{\alpha \sigma} u^\sigma.
\]

For the spin-0 case as in (24), matter is represented with a scalar wave-function \( \psi \),

\[
\mathcal{L}_m = \sqrt{-g} \left( \frac{h^2}{2m} \bar{\psi} D_\mu \psi - m \bar{\psi} \psi \right),
\]

\[
D_\mu = \partial_\mu + \frac{iQ}{\hbar} A_\mu,
\]

For the spin-1/2 case as in (24), matter is represented by a four-component wave-function \( \psi \), and things are defined using tetrads \( e_{(a)}^\sigma \),

\[
\mathcal{L}_m = \sqrt{-g} \left( \frac{i\hbar}{2} (\bar{\psi} e_{(a)}^\sigma D_\sigma \psi - \bar{\psi} \delta_{\sigma \sigma} \gamma^{(a)} \gamma^{(a)} \psi) - \bar{\psi} \psi \right),
\]

\[
\gamma^{(a)} = \gamma(e_{(a)}) \sigma \quad \gamma(0) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma(i) = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},
\]

\[
g^{(a)(b)} = e_{(a)}^\tau e_{(b)}^\gamma g^{\tau \gamma} = \frac{1}{2} (\gamma^{(a)} \gamma^{(b)} + \gamma^{(b)} \gamma^{(a)}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
e^{(a)}_{\tau} e^{(b)}_{\gamma} = \delta^{(a)}_{(b)} \sigma , \quad e^{(a)}_{\tau} e^{(b)}_{\gamma} = \delta^{(a)}_{(b)} \sigma ,
\]

\[
D_\mu = \partial_\mu + \hat{\Gamma}_\mu + \frac{iQ}{\hbar} A_\mu, \quad \hat{D}_\mu = \partial_\mu + \hat{\Gamma}_\mu - \frac{iQ}{\hbar} A_\mu,
\]

In the equations above, \( m \) is mass, \( Q \) is charge and the \( \sigma_i \) are the Pauli spin matrices.

In (29,35) the conjugate derivative operator \( \hat{D}_\mu \) is made to operate from right to left to simplify subsequent calculations. The spin-0 and spin-1/2 \( \mathcal{L}_m/s \) are the ordinary expressions for quantum fields in curved space (24).
3. The Einstein Equations

The Einstein equations are obtained by setting \( \delta L / \delta N_{\nu\mu} = 0 \). However, the field equations must be the same if we instead use \( g^{\nu\mu} \) and \( f^{\nu\mu} \) as the independent variables. The field equations must also be the same if we use \( \sqrt{-N} N^{\nu\mu} \) as the independent variable, and since this is simplest, we will follow this method. Before calculating the field equations, we need some preliminary results. From (3) we get,

\[
\frac{\partial}{\partial(\sqrt{-g} g^{\sigma\tau})} = \delta_\mu^\sigma \delta_\nu^\tau, \tag{37}
\]

\[
\frac{\partial (g_{\sigma\tau}/\sqrt{-g})}{\partial(\sqrt{-N} N^{\nu\mu})} = -\frac{g_{\tau(\nu \mu)\sigma}}{\sqrt{-g} \sqrt{-g}} \quad \text{(because \( \frac{\partial}{\partial(\sqrt{-N} N^{\nu\mu})} = 0 \)).} \tag{38}
\]

Using (5) and the identities \( \det(sM) = s^n \det(M) \), \( \det(M^{-1}) = 1/\det(M) \) gives

\[
\sqrt{-N} = (-\det(\sqrt{-N} N^{\nu\mu}))^{1/(n-2)}, \tag{39}
\]

\[
\sqrt{-g} = (-\det(\sqrt{-g} g^{\nu\mu}))^{1/(n-2)} = (-\det(\sqrt{-N} N^{\nu\mu}))^{1/(n-2)}. \tag{40}
\]

Using (39,40) and the identity \( \partial(\det(M^-)) / \partial M^{\nu\mu} = N_{\nu\mu}^{-1} \det(M^-) \) gives

\[
\frac{\partial \sqrt{-N}}{\partial(\sqrt{-N} N^{\nu\mu})} = \frac{(-\det(\sqrt{-N} N^{\nu\mu}))^{1/(n-2)-1+1}}{(n-2)} N_{\nu\mu} = \frac{N_{\nu\mu}}{(n-2)}, \tag{41}
\]

\[
\frac{\partial \sqrt{-g}}{\partial(\sqrt{-N} N^{\nu\mu})} = \frac{(-\det(\sqrt{-g} g^{\nu\mu}))^{1/(n-2)-1+1}}{(n-2)} g_{\nu\mu} = \frac{g_{\nu\mu}}{(n-2)}. \tag{42}
\]

Note that from (37,38,42), if \( L_m \) depends only on \( g_{\nu\mu} \) and \( \sqrt{-g} \), and not on \( N_{\nu\mu} \) and \( \sqrt{-N} \), then \( \partial L_m / \partial(\sqrt{-N} N^{\nu\mu}) \) gives the same result as \( \partial L_m / \partial(\sqrt{-g} g^{\nu\mu}) \).

For the spin-1/2 case we need the derivative \( \partial(\sqrt{-g} e_{(a)\tau}) / \partial(\sqrt{-N} N^{\nu\mu}) \). Multiplying (33) by \( \sqrt{-g} g^{\sigma\tau} e_{(b)\sigma} \) and taking its derivative with respect to \( \sqrt{-g} g^{\nu\mu} \) gives

\[
\sqrt{-g} g^{\sigma\tau} e_{(b)\sigma} = \sqrt{-g} e_{(a)\tau} g_{(a)(b)}, \tag{43}
\]

\[
\delta_{(a)}^{\sigma} \delta_{(b)}^{\tau} e_{(b)\sigma} + \sqrt{-g} g^{\sigma\tau} \frac{\partial e_{(b)\sigma}}{\partial(\sqrt{-g} g^{\nu\mu})} = \frac{\partial(\sqrt{-g} e_{(a)\tau})}{\partial(\sqrt{-g} g^{\nu\mu})} g_{(a)(b)}. \tag{44}
\]

Taking the derivative of (44) with respect to \( \sqrt{-g} g^{\nu\mu} \) and using (42) gives

\[
0 = e_{(a)\tau} \frac{\partial(\sqrt{-g} e_{(a)\tau})}{\partial(\sqrt{-g} g^{\nu\mu})} - \frac{g_{\nu\mu}}{n-2} g_{(a)\tau} + \sqrt{-g} \frac{\partial e_{(a)\tau}}{\partial(\sqrt{-g} g^{\nu\mu})} e_{(c)\tau}. \tag{45}
\]

Substituting (45) into (44) we finally get

\[
\frac{\partial(\sqrt{-g} e_{(a)\tau})}{\partial(\sqrt{-g} g^{\nu\mu})} g_{(a)(b)} - e_{(b)(c)\tau} = \sqrt{-g} g^{\tau\sigma} \frac{\partial e_{(b)\sigma}}{\partial(\sqrt{-g} g^{\nu\mu})} \tag{46}
\]

\[
= g^{\tau\sigma} \left( \frac{g_{\nu\mu}}{n-2} e_{(b)\sigma} - e_{(b)\tau} \frac{\partial(\sqrt{-g} e_{(a)\tau})}{\partial(\sqrt{-g} g^{\nu\mu})} e_{(a)\sigma} \right) \tag{47}
\]

\[
= \frac{g_{\nu\mu}}{n-2} e_{(b)\tau} - g_{(a)(b)} \frac{\partial(\sqrt{-g} e_{(a)\tau})}{\partial(\sqrt{-g} g^{\nu\mu})}, \tag{48}
\]

\[
\frac{\partial(\sqrt{-g} e_{(a)\tau})}{\partial(\sqrt{-N} N^{\nu\mu})} = \frac{\partial(\sqrt{-g} e_{(a)\tau})}{\partial(\sqrt{-g} g^{\nu\mu})} = \frac{1}{2} e_{(a)(b)\mu} + \frac{g_{\nu\mu}}{2(n-2)} e_{(a)\tau}. \tag{49}
\]
Now we are ready to calculate the field equations. Setting $\delta \mathcal{L}/\delta (\sqrt{-N}N^{-\mu\nu}) = 0$ and using (51,12) gives,

$$0 = -16\pi \frac{\partial \mathcal{L}}{\partial (\sqrt{-N}N^{-\mu\nu})} = \left( \frac{\partial \mathcal{L}}{\partial (\sqrt{-N}N^{-\mu\nu})} \right)_{\omega}$$  

$$= \tilde{R}_{\nu\mu} + 2A_{[\nu,\mu]}\sqrt{2}i\Lambda_b^{1/2} + \Lambda_bN_{\nu\mu} + \Lambda_2g_{\nu\mu} - 8\pi S_{\nu\mu},$$  

(50)

where $S_{\nu\mu}$ and the energy-momentum tensor $T_{\nu\mu}$ are defined by

$$S_{\nu\mu} = 2\frac{\delta \mathcal{L}_m}{\delta (\sqrt{-N}N^{-\mu\nu})}$$  

$$= T_{\nu\mu} - \frac{1}{(n-2)}g_{\nu\mu}T^\alpha_\alpha,$$  

(52)

$$T_{\nu\mu} = S_{\nu\mu} - \frac{1}{2}g_{\nu\mu}S^\alpha_\alpha.$$  

(53)

Taking the symmetric and antisymmetric parts of (51) and using (23) gives

$$\tilde{R}_{(\nu\mu)} + \Lambda_bN_{(\nu\mu)} + \Lambda_2g_{\nu\mu} = 8\pi \left( T_{\nu\mu} - \frac{1}{(n-2)}g_{\nu\mu}T^\alpha_\alpha \right),$$  

(55)

$$N_{(\nu\mu)} = F_{\nu\mu}\sqrt{2}i\Lambda_b^{1/2} - \tilde{R}_{(\nu\mu)}\Lambda_b^{-1}.$$  

(56)

Also from the curl of (56) we get

$$\tilde{R}_{(\nu\mu,\sigma)} + \Lambda_bN_{(\nu\mu,\sigma)} = 0.$$  

(57)

To put (55) into a form which looks more like the ordinary Einstein equations, we need some preliminary results. The definitions (54,24) of $g_{\nu\mu}$ and $f_{\nu\mu}$ can be inverted to give $N_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$. An expansion in powers of $\Lambda_b^{-1}$ is derived in Appendix F and confirmed by tetrad methods in [3],

$$N_{(\nu\mu)} = g_{\nu\mu} - 2 \left( f^\alpha_{\nu}f_{\mu\alpha} - \frac{1}{2(n-2)}g_{\nu\mu}f^{\mu\sigma}f_{\sigma\rho} \right)\Lambda_b^{-1} + (f^3)\Lambda_b^{-2} \ldots$$  

(58)

$$N_{(\nu\mu)} = f_{\nu\mu}\sqrt{2}i\Lambda_b^{-1/2} + (f^3)\Lambda_b^{-3/2} \ldots.$$  

(59)

Here the notation $(f^3)$ and $(f^4)$ refers to terms like $f_{\nu\sigma}f^{\alpha\sigma}f_{\mu\rho}$ and $f_{\nu\sigma}f^{\alpha\sigma}f^{\beta\rho}f^{\gamma\mu}$. Let us consider the size of these higher order terms relative to the leading order term for worst-case fields accessible to measurement. In geometrized units an elementary charge has

$$Q_e = e\sqrt{\frac{G}{\hbar c^3}} = \sqrt{\frac{e^2G\hbar c^3}{\alpha \hbar c^3}} = \sqrt{\alpha} \ell_P = 1.38 \times 10^{-34} \text{cm}$$  

(60)

where $\alpha = e^2/\hbar c$ is the fine structure constant and $\ell_P = \sqrt{G\hbar c^3}$ is Planck’s constant. If we assume that charged particles retain $f^1_{\nu\mu} \sim Q/e^2$ down to the smallest radii probed by high energy particle physics experiments ($10^{-17}\text{cm}$) we have from (60,44),

$$|f^1_{\nu\mu}|^2/\Lambda_b \sim (Q_e/(10^{-17})^2)/\Lambda_b \sim 10^{-66}.$$  

(61)

Here $|f^1_{\nu\mu}|$ is assumed to be in some standard spherical or cartesian coordinate system. If an equation has a tensor term which can be neglected in one coordinate system, it can be neglected in any coordinate system, so it is only necessary to prove it in one coordinate system. The fields at $10^{-17}\text{cm}$ from an elementary charge would be larger than near any macroscopic charged object, and would also be larger than the strongest plane-wave fields. Therefore the higher order terms in (58,59) must be $< 10^{-66}$ of the leading order terms, so they will be completely negligible for most purposes.
In \(\Lambda\)-renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources

In \(\Lambda\)-renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources, we will calculate the connection equations resulting from \(\delta \mathcal{L}/\delta \tilde{\Gamma}^\alpha_{\nu\mu} = 0\). Solving these equations gives \(\delta \mathcal{L}/\delta \tilde{\Gamma}^\alpha_{\nu\mu} = 0\), which can be abbreviated as

\[
\tilde{\Gamma}^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} + \mathcal{O}(\Lambda_b^{-1}), \quad \tilde{\Gamma}^\alpha_{\nu\mu} = \mathcal{O}(\Lambda_b^{-1/2}),
\]

(62)

\[
\tilde{G}_{\nu\mu} = G_{\nu\mu} + \mathcal{O}(\Lambda_b^{-1}), \quad \tilde{R}_{\nu\mu} = \mathcal{O}(\Lambda_b^{-1/2}),
\]

(63)

where \(\Gamma^\alpha_{\nu\mu}\) is the Christoffel connection (22), \(\tilde{\mathcal{R}}_{\nu\mu} = \mathcal{R}_{\nu\mu}(\tilde{\Gamma})\), \(R_{\nu\mu} = \mathcal{R}_{\nu\mu}(\Gamma)\) and

\[
\tilde{G}_{\nu\mu} = \tilde{\mathcal{R}}_{\nu\mu} - \frac{1}{2} g_{\nu\mu} \tilde{\mathcal{R}}^\rho\rho, \quad G_{\nu\mu} = R_{\nu\mu} - \frac{1}{2} g_{\nu\mu} R.
\]

(64)

In (63), \(\mathcal{O}(\Lambda_b^{-1})\) and \(\mathcal{O}(\Lambda_b^{-1/2})\) indicate terms like \(f^\alpha_{\nu\sigma\mu} f^\alpha_{\mu\sigma} \Lambda_b^{-1}\) and \(f^\alpha_{\nu\mu\sigma} \Lambda_b^{-1/2}\).

From the antisymmetric part of the field equations (56) and (59, 63) we get

\[
f_{\nu\mu} = f_{\nu\mu} + \mathcal{O}(\Lambda_b^{-1}).
\]

(65)

So \(f_{\nu\mu}\) and \(F_{\nu\mu}\) only differ by terms with \(\Lambda_b\) in the denominator, and the two become identical in the limit as \(\Lambda_b \to \infty\). Combining the symmetric part of the field equations (56) with its contraction, and substituting (51, 53)

\[
N_{(\nu\mu)} - \frac{1}{2} g_{\nu\mu} N^\rho_{\rho} = g_{\nu\mu} - 2 \left( f^\sigma_{\nu\sigma\mu} f_{\sigma\mu} - \frac{1}{2(n-2)} g_{\nu\mu} f^{\rho\sigma} f_{\rho\sigma} \right) \Lambda_b^{-1}
\]

\[
- \frac{1}{2} g_{\nu\mu} n + g_{\nu\mu} \left( f^{\rho\sigma} f_{\rho\sigma} - \frac{1}{2(n-2)} n f^{\rho\sigma} f_{\rho\sigma} \right) \Lambda_b^{-1} + (f^4) \Lambda_b^{-2} \ldots
\]

\[
= g_{\nu\mu} \left( 1 - \frac{n}{2} \right) - 2 f^\sigma_{\nu\sigma\mu} \Lambda_b^{-1}
\]

\[
+ g_{\nu\mu} \left( \frac{1}{n-2} + 1 - \frac{n}{2(n-2)} \right) f^{\rho\sigma} f_{\rho\sigma} \Lambda_b^{-1} + (f^4) \Lambda_b^{-2} \ldots
\]

\[
= -2 \left( f^\sigma_{\nu\sigma\mu} - \frac{1}{4} g_{\nu\mu} f^{\rho\sigma} f_{\rho\sigma} \right) \Lambda_b^{-1} - \left( \frac{n}{2} - 1 \right) g_{\nu\mu} + (f^4) \Lambda_b^{-2} \ldots
\]

gives the Einstein equations

\[
\tilde{G}_{\nu\mu} = 8\pi T_{\nu\mu} - \Lambda_b \left( N_{(\nu\mu)} - \frac{1}{2} g_{\nu\mu} N^\rho_{\rho} \right) + \Lambda \left( \frac{n}{2} - 1 \right) g_{\nu\mu},
\]

(66)

\[
= 8\pi T_{\nu\mu} + 2 \left( f^\sigma_{\nu\sigma\mu} - \frac{1}{4} g_{\nu\mu} f^{\rho\sigma} f_{\rho\sigma} \right) + \Lambda \left( \frac{n}{2} - 1 \right) g_{\nu\mu} + (f^4) \Lambda_b^{-1} \ldots.
\]

(67)

So from (63), equation (67) differs from the ordinary Einstein equations only by terms with \(\Lambda_b\) in the denominator, and it becomes identical to the ordinary Einstein equations in the limit as \(\Lambda_b \to \infty\). In (65) we will examine how close the approximation is for the very large value \(\Lambda_b \sim 10^{26} \text{ cm}^{-2}\) from (14).

From (52) we see that \(S_{\nu\mu}\) and \(T_{\nu\mu}\) are different for each \(\mathcal{L}_m\) case. For the electro-vac case

\[
S_{\nu\mu} = 0,
\]

(68)

\[
T_{\nu\mu} = 0.
\]

(69)

For the classical hydrodynamics case (27),

\[
S_{\nu\mu} = \frac{\mu}{\sqrt{-g}} \left( u_{\nu\mu} u_{\mu} \frac{1}{n-2} g_{\nu\mu} u^\alpha u_\alpha \right),
\]

(70)

\[
T_{\nu\mu} = \frac{\mu}{\sqrt{-g}} u_{\nu\mu} u_{\mu}.
\]

(71)
A renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources

For the spin-0 case (28) as in [24],

\[
S_{\nu\mu} = \frac{1}{m} \left( \hbar^2 \bar{\psi} \mathring{D}_{\nu} D_{\mu} \psi - \frac{1}{(n-2)} g_{\nu\mu} m^2 \bar{\psi} \psi \right),
\]

(72)

\[
T_{\nu\mu} = \frac{1}{m} \left( \hbar^2 \bar{\psi} \mathring{D}_{\nu} D_{\mu} \psi - \frac{1}{2} g_{\nu\mu} (\hbar^2 \bar{\psi} \mathring{D}_\sigma \psi - m^2 \bar{\psi} \psi) \right).
\]

(73)

For the spin-1/2 case (30) as in [24],

\[
S_{\nu\mu} = \frac{i \hbar}{2} \left( \bar{\psi} \gamma_{(\nu} D_{\mu)} \psi - \bar{\psi} \mathring{D}_{(\nu} \gamma_{\mu)} \psi - \frac{1}{(n-2)} g_{\nu\mu} (\bar{\psi} \gamma_{\sigma} D_{\sigma} \psi - \bar{\psi} \mathring{D}_{\sigma} \gamma_{\sigma} \psi) \right),
\]

(74)

\[
T_{\nu\mu} = \frac{i \hbar}{2} \left( \bar{\psi} \gamma_{(\nu} D_{\mu)} \psi - \bar{\psi} \mathring{D}_{(\nu} \gamma_{\mu)} \psi \right).
\]

(75)

Note that in the purely classical limit as \(i \hbar D_{\sigma} \psi \rightarrow p_{\sigma} \psi\), \(-i \hbar \bar{\psi} \mathring{D}_{\sigma} \rightarrow \bar{\psi} p_{\sigma}\), the energy-momentum tensors (73) for spin-0 and (75) for spin-1/2 both go to the classical hydrodynamics case (71).

4. Maxwell’s Equations

Setting \(\delta L/\delta A_\tau = 0\) and using the definition (24) gives

\[
0 = 4\pi \sqrt{-g} \left[ \frac{\partial L}{\partial A_\tau} - \left( \frac{\partial L}{\partial A_{\tau,\omega}} \right)_\omega \right],
\]

(76)

\[
= \frac{\sqrt{2} i \Lambda^{1/2}_b}{2\sqrt{-g}} (\sqrt{-N} N^{-1}[\omega])_\omega - 4\pi j^\tau = \left( \frac{\sqrt{-g} f^{\omega\tau}}{\sqrt{-g}} \right)_\omega - 4\pi j^\tau.
\]

(77)

where

\[
j^\tau = -\frac{1}{\sqrt{-g}} \left[ \frac{\partial L_m}{\partial A_\tau} - \left( \frac{\partial L_m}{\partial A_{\tau,\omega}} \right)_\omega \right].
\]

(78)

From (77, 23) we get Maxwell’s equations,

\[
f^{\omega\tau}_\omega = 4\pi j^\tau,
\]

(79)

\[
F_{[\mu,\nu]} = 0.
\]

(80)

Using \(f_{\nu\mu} = F_{\nu\mu} + \mathcal{O}(\Lambda^{-1}_b)\) from (64), we see that equations (79, 80) differ from the ordinary Maxwell equations only by terms with \(\Lambda_b\) in the denominator, and these equations become identical to the ordinary Maxwell equations in the limit as \(\Lambda_b \rightarrow \infty\). In § 5 we will examine how close the approximation is for the very large value \(\Lambda_b \sim 10^{66} \text{cm}^{-2}\) from (14).

From (78) we see that \(j^\tau\) is different for each \(L_m\) case. For the electro-vac case, \(j^\alpha\) may have a singularity at the position of a particle, but otherwise we have,

\[
j^\alpha = 0.
\]

(81)

For the classical hydrodynamics case (27),

\[
j^\alpha = \frac{\mu Q}{m \sqrt{-g}} u^\alpha.
\]

(82)

For the spin-0 case (28) as in [24],

\[
j^\alpha = \frac{i \hbar Q}{2m} (\bar{\psi} D^\alpha \psi - \mathring{D}^\alpha \psi).
\]

(83)
A-renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources

For the spin-1/2 case \( \text{[20]} \) as in \( \text{[24]} \),

\[
j^\alpha = Q\tilde{\psi}^\alpha \psi.
\]  
(84)

A continuity equation follows from \( \text{[20]} \) regardless of the type of source,

\[
j_{\mu \rho} = \frac{1}{4\pi} f^{\tau \rho}_{\mu \nu \rho} = 0.
\]  
(85)

Note that the covariant derivative in \( \text{[79, 85]} \) is done using the Christoffel connection \( \text{[11, 12]} \) we can calculate,

\[
\delta L/\delta \Gamma_\alpha^{\mu \rho} = 0 \text{ requires some preliminary calculations. With the definition}
\]
\[
\frac{\Delta L}{\Delta \Gamma_\beta^\alpha_{\tau \rho}} = \frac{\partial L}{\partial \Gamma_\beta^\alpha_{\tau \rho}} - \left( \frac{\partial L}{\partial \Gamma_\beta^\alpha_{\tau \rho, \omega}} \right) \omega \ldots
\]  
(86)

and \( \text{[111, 12]} \) we can calculate,

\[
-16\pi \frac{\Delta L}{\Delta \Gamma_\alpha^{\mu \rho}} = 2\sqrt{-N}N^{-\mu \nu \rho} (\delta_\beta^\gamma \delta_\nu^\rho \delta_\mu^\alpha \Gamma_\gamma^\alpha_{\sigma [\alpha]} + \Gamma_\gamma^\alpha_{\nu \rho \beta} \delta_\beta^\sigma \delta_\sigma^\alpha) \omega - (\sqrt{-N} \delta_\alpha^\gamma \delta_\rho^\gamma \delta_\nu^\mu \omega) \omega
\]
\[
= - (\sqrt{-N} \delta_\beta^\gamma \delta_\nu^\mu \omega) \omega + \Gamma_\gamma^\alpha_{\nu \rho \beta} \sqrt{-N} \delta_\beta^\gamma \delta_\rho^\gamma \delta_\nu^\mu \omega
\]
\[
+ \delta_\alpha^\gamma \delta_\beta^\gamma \delta_\nu^\mu \omega) \omega + \Gamma_\gamma^\alpha_{\nu \rho \beta} \sqrt{-N} \delta_\beta^\gamma \delta_\rho^\gamma \delta_\nu^\mu \omega
\]
\[
= (n - 1) (\sqrt{-N} \delta_\beta^\gamma \delta_\nu^\mu \omega) \omega + \Gamma_\gamma^\alpha_{\nu \rho \beta} \sqrt{-N} \delta_\beta^\gamma \delta_\rho^\gamma \delta_\nu^\mu \omega
\]  
(87)

In these last two equations, the index contractions occur after the derivatives. At this point we must be careful. Because \( \Gamma_\alpha^{\mu \rho} \) has the symmetry \( \text{[10]} \), it has only \( n^3 - n \) independent components, so there can only be \( n^3 - n \) independent field equations associated with it. It is shown in \( \text{Appendix D} \) that instead of just setting \( \text{[87]} \) to zero, the field equations associated with such a field are given by the expression,

\[
0 = 16\pi \left[ \frac{\Delta L}{\Delta \Gamma_\beta^\alpha_{\tau \rho}} - \frac{\delta_\beta^\gamma \Delta L}{(n - 1) \Delta \Gamma_\gamma^\alpha_{\tau \rho}} - \frac{\delta_\beta^\gamma \Delta L}{(n - 1) \Delta \Gamma^\alpha_{\tau \beta}} \right]
\]  
(90)

\[
= (\sqrt{-N} \delta_\beta^\gamma \delta_\nu^\mu \omega) \omega + \Gamma_\gamma^\alpha_{\nu \rho \beta} \sqrt{-N} \delta_\beta^\gamma \delta_\rho^\gamma \delta_\nu^\mu \omega
\]
\[
- \delta_\alpha^\gamma \delta_\beta^\gamma \delta_\nu^\mu \omega) \omega + \Gamma_\gamma^\alpha_{\nu \rho \beta} \sqrt{-N} \delta_\beta^\gamma \delta_\rho^\gamma \delta_\nu^\mu \omega
\]  
(91)

These are the connection equations, like \( (\sqrt{-g} g^{\rho \tau})_{;\beta} = 0 \) in the symmetric case.
From the definition of matrix inverse $N^{-\rho\sigma} = (1/N)\partial N/\partial N_{\rho\sigma}$, $N^{-\rho\sigma}N_{\rho\mu} = \delta^\alpha_\mu$ we get the identity

$$(\sqrt{-N})_{,\sigma} = \partial\sqrt{-N}/\partial N_{\rho\sigma} = \frac{\sqrt{-N}}{2}N^{-\rho\sigma}N_{\rho\sigma} = -\frac{\sqrt{-N}}{2}N^{-\mu,\sigma}N_{\rho\sigma}. \quad (92)$$

Contracting (91) with $N_{\rho\sigma}$ using (10) we get

$$0 = (n-2)((\sqrt{-N}),_\beta - \tilde{\Gamma}^\alpha_{\alpha\beta}\sqrt{-N}) + \frac{8\pi}{(n-1)}\sqrt{-g}j^\rho N_{[\rho\beta]}\sqrt{2}i\Lambda_\beta^{1/2},$$

and dividing this by $(n-2)$ gives,

$$(\sqrt{-N}),_\beta - \tilde{\Gamma}^\alpha_{\alpha\beta}\sqrt{-N} = -\frac{8\pi\sqrt{2}i}{(n-1)(n-2)\Lambda_\beta^{1/2}}\sqrt{-g}j^\rho N_{[\rho\beta]}.$$ \hspace{1cm} (93)

From (93) we get

$$\tilde{\Gamma}^\alpha_{\alpha[\nu\rho]} - \frac{8\pi\sqrt{2}i}{(n-1)(n-2)\Lambda_\beta^{1/2}}\left(\sqrt{-g}j^\rho N_{[\rho\nu]}\right)_{,\mu} = (ln\sqrt{-N})_{,[\nu\rho\mu]} = 0. \quad (94)$$

From (91,93) we get the contravariant connection equations,

$$N^{-\rho\sigma},_\beta + \tilde{\Gamma}^\tau_{\alpha\beta\tau}N^{-\rho\sigma} + \tilde{\Gamma}^\rho_{\alpha\beta}N^{-\rho\tau} = \frac{8\pi\sqrt{2}i}{(n-1)\Lambda_\beta^{1/2}}\sqrt{-g}\left(j^\rho j^\tau + \frac{1}{(n-2)}j^\alpha N_{[\alpha\beta]}N^{-\rho\tau}\right). \quad (95)$$

Multiplying this by $-N_{\nu\rho}N_{\tau\mu}$ gives the covariant connection equations,

$$N_{\nu\rho,\beta} - \tilde{\Gamma}^\alpha_{\alpha\beta\nu}N_{\alpha\rho\mu} - \tilde{\Gamma}^\rho_{\beta\rho\nu}N_{\nu\rho\mu} = -\frac{8\pi\sqrt{2}i}{(n-1)\Lambda_\beta^{1/2}}\sqrt{-g}\left(N_{[\alpha\beta\nu]}N_{[\beta\rho\mu]} + \frac{1}{(n-2)}N_{[\alpha\beta]}N_{[\nu\rho\mu]}\right)j^\alpha. \quad (96)$$

Equation (96) together with (55,57,10) are often used to define the Einstein-Schrödinger theory, particularly when $T_{\nu\mu} = 0$, $j^\alpha = 0$.

The connection equations (91) can be solved similar to the way that $g_{\rho\tau;\beta} = 0$ is solved to get the Christoffel connection [25,11]. An expansion in powers of $\Lambda_\beta^{-1}$ is derived in [Appendix E] confirmed by tetrad methods in [3], and is also stated without derivation in [20].

$$\tilde{\Gamma}^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} + \Upsilon^\alpha_{\nu\mu}, \quad (97)$$

$$\Upsilon^\alpha_{\nu\mu} = -2\left[f_{(\nu\mu)}^\alpha + f^\alpha f_{(\nu\mu)} + \frac{1}{4(n-2)}(f^\rho f_{\rho\sigma}f^\alpha f_{\sigma\mu} + 2f^\rho f_{\rho\sigma}f^\alpha f_{\sigma\mu}) + \frac{4\pi}{(n-2)}j^\rho f^\alpha f_{\rho\mu} + \frac{2}{(n-1)}f^\rho f_{\rho\mu}\right] - \frac{1}{2}f_{\nu\rho\mu}, \quad (98)$$

$$\Upsilon^\alpha_{[\nu\mu]} = \left[\frac{1}{2}(f_{\nu\rho\mu} + f^\alpha f_{\rho\mu} + f^\alpha f_{\nu\mu}) + 8\pi j^\alpha (f^\rho f_{\rho\mu})\right] + \frac{8\pi}{(n-1)}j^\alpha f_{\nu\mu} + \frac{8\pi}{(n-1)(n-2)}j^\alpha f_{\nu\mu} + \frac{8\pi}{(n-1)(n-2)}j^\alpha f_{\nu\mu} + \frac{8\pi}{(n-1)(n-2)}j^\alpha f_{\nu\mu} + \frac{8\pi}{(n-1)(n-2)}j^\alpha f_{\nu\mu} \quad (99)$$

$$\Upsilon^\alpha_{\nu\mu} \quad (100)$$

In (97), $\Gamma^\alpha_{\nu\mu}$ is the Christoffel connection (22). The notation $(f^\rho f^\sigma)$ and $(f^\nu f^\rho)$ refers to terms like or $f^\rho f^\sigma f^\nu_{\nu\mu}$ and $f^\rho f^\sigma f^\nu_{\nu\mu}$. As in (58,99), we see from (98) that the higher order terms in (98,100) must be $< 10^{-66}$ of the leading order terms, so they will be completely negligible for most purposes.
Substituting (97-100, 79) into (C.5), with $\ell = f^{\alpha\rho}g_{\sigma\rho}$,

$$\tilde{\mathcal{R}}_{(\nu\mu)} = R_{(\nu\mu)} + \nabla_{(\nu\mu)}^{\alpha} - \nabla_{\nu(\nu\mu)}^{\alpha} - \nabla_{\nu\mu}^{\alpha} [\nabla_{\nu\mu}] \cdots$$

$$= R_{\nu\mu} - 2 \left[ f^{\alpha\nu} f_{\nu\mu}^{\alpha\tau} + 2 f^{\alpha\nu} f_{\nu\mu}^{\alpha} + \frac{1}{4(n-2)} (f^{\alpha\nu} g_{\nu\mu} - 2\ell \delta^{\alpha}_{\nu\mu})^{1}\alpha + \frac{4\pi}{(n-2)} g^{\alpha\nu} g_{\nu\mu} + \frac{2}{(n-1)} \ell \delta^{\alpha}_{\nu\mu} \right] + \frac{1}{2(n-2)} \ell_{(\nu\mu)} - \frac{8\pi}{n(n-1)(n-2)} (f^{\alpha\nu} f_{(\nu\mu)})$$

$$= \tilde{\mathcal{R}}^{\beta}_{\rho} = R - \left[ 2 f^{\alpha\beta} f^{\alpha\beta} + f^{\alpha\nu} f_{\nu\mu}^{\alpha\beta} + \frac{3}{2} f^{\alpha\nu} f^{\alpha\beta} \ell \delta^{\alpha}_{\nu\mu} \right] \Lambda_{b}^{-1} \cdots$$

and using (63) gives

$$\tilde{G}_{\nu\mu} = G_{\nu\mu} - \left( R^{\alpha\beta} f^{\alpha\beta} + 2 f^{\alpha\nu} f_{\nu\mu}^{\alpha} + \frac{3}{2} f^{\alpha\nu} f_{\nu\mu}^{\alpha} \right) \Lambda_{b}^{-1} \cdots$$

Substituting (99, 79) into (C.5) gives

$$\tilde{\mathcal{R}}_{(\nu\mu)} = \nabla^{\alpha}_{(\nu\mu)} + \mathcal{O}(\Lambda_{b}^{-3/2}) \cdots$$

$$= \left( \frac{1}{2} (f_{\nu\mu}^{\alpha} + f_{\mu\nu}^{\alpha} - f_{\nu\mu}^{\alpha}) + \frac{8\pi}{n(n-1)} f_{\nu\mu} j_{\nu\mu}^{\alpha} \right) \sqrt{2} t \Lambda_{b}^{-1/2} \cdots$$
Λ-renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources

and Maxwell equations in the limit as \( \omega \to \infty \), \( |A_z| \to \infty \), \( \Lambda_b \to \infty \), and it also causes the relation \( f_{\nu \mu} \approx F_{\nu \mu} \) from (65) to become exact in this limit. Let us examine how close these approximations are when \( \Lambda_b \approx 10^{16} \text{ cm}^{-2} \) as in (114).

We will start with the Einstein equations (67). Let us consider worst-case values of \( G_{\nu \mu} - G_{\nu \mu} \) accessible to measurement, and compare these to the ordinary electromagnetic term in the Einstein equations (67). If we assume that charged particles retain \( f_{\nu \mu} \sim Q/e^2 \) down to the smallest radii probed by high energy particle physics experiments \( (10^{-17} \text{ cm}) \) we have,

\[
|f_{\nu \mu}|^2/\Lambda_b \sim 4/\Lambda_b (10^{-17})^2 \sim 10^{-32},
\]

\[
|f_{\nu \mu}|^2/\Lambda_b \sim 6/\Lambda_b (10^{-17})^2 \sim 10^{-32}.
\]

So for electric monopole fields, terms like \( f_{\nu \mu} f_{\nu \mu}^\alpha \Lambda_b^{-1} \) and \( \Lambda_b \) in (101) must be \( < 10^{-32} \) of the ordinary electromagnetic term in (67). And regarding \( j^\tau \) as a substitute for \( (1/4\pi)f_{\nu \mu} \) from (72), the same is true for the \( j^\tau \) terms. For an electromagnetic plane-wave in a flat background space we have \( j^\tau = 0 \) and

\[ A_\mu = A [\alpha, x^\alpha] , \quad \epsilon^\alpha \epsilon_\alpha = -1 , \quad k^\alpha k_\alpha = k^\alpha \epsilon_\alpha = 0, \]

\[ f_{\nu \mu} = 2 \Lambda [\mu, \nu] \]

(105)

(106)

Here \( A \) is the magnitude, \( k^\alpha \) is the frequency, and \( \epsilon^\alpha \) is the polarization. Substituting (105) into (101), all of the terms vanish for a flat background space. Also, for the highest energy gamma rays known in nature \( (10^{20} \text{ eV}, 10^{34} \text{ Hz}) \) we have from (114),

\[
|f_{\nu \mu}|^2/\Lambda_b \sim (E/\hbar c)^2/\Lambda_b \sim 10^{-16},
\]

\[
|f_{\nu \mu}|^2/\Lambda_b \sim (E/\hbar c)^2/\Lambda_b \sim 10^{-16}.
\]

So for electromagnetic plane-wave fields, even if some of the extra terms in (67) were non-zero because of spatial curvatures, they must still be \( < 10^{-16} \) of the ordinary electromagnetic term. Therefore even for the most extreme worst-case fields accessible to measurement, the extra terms in the Einstein equations (67) must all be \( < 10^{-16} \) of the ordinary electromagnetic term.

Now let us look at the approximation \( f_{\nu \mu} \approx F_{\nu \mu} \) from (65). From the covariant derivative commutation rule, the definition of the Weyl tensor \( C_{\nu \mu \alpha \tau} \), and the Einstein equations \( R_{\nu \mu} = -\Lambda g_{\nu \mu} + (f^2) \ldots \) from (67) we get

\[ 2f_{\nu \mu} = R_{\nu \mu \alpha \tau} - R_{\nu \mu} + f_{\nu \mu \alpha \tau} \]

\[ = \frac{1}{2} \left( C_{\nu \mu \alpha \tau} + \frac{4}{(n-2)} \delta_{[\nu}^{[\alpha} R_{\mu]^{\tau]} - \frac{2}{(n-1)(n-2)} \delta_{[\nu}^{[\alpha} \delta_{\mu]^{\tau]} R \right) f_{\alpha \tau} - R_{\nu \mu} f_{\nu \tau}
\]

\[ = \frac{1}{2} f_{\nu \mu} C_{\nu \mu \alpha \tau} + \frac{n-2}{(n-1)} \Lambda f_{\nu \mu} + (f^3) \ldots
\]

(109)

Substituting (108) into the antisymmetric field equations (58) gives

\[ f_{\nu \mu} = F_{\nu \mu} + \mathcal{R}_{[\nu |\mu]} \sqrt{2} i \Lambda_b^{-1/2} / 2 + (f^3) \Lambda_b^{-1} \ldots,
\]

(110)
and using (102) we get
\[ f_{\nu\mu} = F_{\nu\mu} + \left( \theta_{\nu\alpha} \varepsilon_{\nu\mu\alpha} + f^{\alpha\tau} C_{\alpha\tau\nu\mu} + \frac{2(n-2)\Lambda}{(n-1)} f_{\nu\mu} + \frac{8\pi(n-2)}{(n-1)} j_{[\nu,\mu]} + (f^3) \right) \Lambda^{-1}_b. \] (111)

where
\[ \theta_{\nu\alpha} = \frac{1}{4} f_{[\nu,\alpha]} \varepsilon_{\nu\mu\alpha}, \quad f_{[\nu,\alpha]} = -\frac{2}{3} \theta_{\nu\mu\alpha}, \] (112)
\[ \varepsilon_{\nu\mu\alpha} = \text{(Levi–Civita tensor)}, \] (113)
\[ C_{\alpha\tau\nu\mu} = \text{(Weyl tensor)}. \] (114)

The \( \theta_{\nu\alpha} \varepsilon_{\nu\mu\alpha} \Lambda^{-1}_b \) term in (111) is divergenceless so that it has no effect on Ampere’s law (79). The \( f_{\nu\mu} \Lambda / \Lambda_b \) term is approximately \( 10^{-122} \) of \( f_{\nu\mu} \) from (10). The \( (f^3) \Lambda^{-1}_b \) term is approximately \( 10^{-66} \) of \( f_{\nu\mu} \) from (61). The largest observable values of the Weyl tensor might be expected to occur near the Schwarzschild radius, \( r_s = 2GM/c^2 \), of black holes, where it takes on values around \( r_s/r_s \). However, since the lightest black holes have the smallest Schwarzschild radius, they will create the largest value of \( r_s/r_s = 1/r_s^2 \). The lightest black hole that we might observe would be of about one solar mass, where from (14),
\[ \frac{C_{0101}}{\Lambda_b} \sim \frac{1}{\Lambda_b r_s^2} = \frac{1}{\Lambda_b} \left( \frac{c^2}{2GM} \right)^2 \sim 10^{-77}. \] (115)

And regarding \( j^\tau \) as a substitute for \( (1/4\pi)f^{\nu\tau} \) from (59), the \( j_{[\nu,\alpha]} \Lambda^{-1}_b \) term is approximately \( 10^{-32} \) of \( f_{\nu\mu} \) from (104). Therefore even for the most extreme worst-case fields accessible to measurement, the last four terms in (111) must all be \( 10^{-32} \) of \( f_{\nu\mu} \).

From (111) the extra terms in Maxwell’s equations (79) must be \( 10^{-32} \) of the ordinary terms. In most so-called “exact” equations in physics, there are really many known corrections due to quantum electrodynamics and other effects which are routinely ignored.

The divergenceless term \( \theta_{\nu\alpha} \varepsilon_{\nu\mu\alpha} \Lambda^{-1}_b \) of (111) should also be expected to be \( 10^{-32} \) of \( f_{\nu\mu} \) from (104). However, we need to consider the possibility where \( \theta_{\nu\mu\alpha} \) changes extremely rapidly, so let us consider the “dual” part of (111). Taking the curl of (111), the \( F_{\nu\mu} \) and \( j_{[\nu,\alpha]} \) terms drop out,
\[ f_{[\nu,\sigma]} = \left( \theta_{\nu\alpha;\sigma} \varepsilon_{\nu\mu\alpha} + f^{\alpha\tau} C_{\alpha\tau[\nu\mu],\sigma} + \frac{2(n-2)\Lambda}{(n-1)} f_{[\nu,\sigma]} + (f^3) \right) \Lambda^{-1}_b. \]

Contracting this with \( \Lambda_b \varepsilon^{\rho\sigma\tau\mu} / 2 \) and using (112) gives
\[ 2\Lambda_b \theta^\rho = -2 \theta_{[\rho;\sigma]} + \frac{1}{2} \varepsilon^{\rho\sigma\mu\nu} (f^{\alpha\tau} C_{\alpha\tau[\nu\mu],\sigma}) + \frac{4(n-2)\Lambda}{(n-1)} \theta^\rho + (f^3) \ldots \]

Using \( \theta_{\nu\mu} = 0 \) from (112), the covariant derivative commutation rule, and the Einstein equations \( R_{\nu\mu} = -\Lambda g_{\nu\mu} + (f^2) \ldots \) from (67), gives \( \theta_{\nu\mu;\sigma} = R_{\sigma\rho} \theta^\sigma = -\theta_{\rho\sigma} + (f^3) \ldots \), and we get something similar to the Proca equation,
\[ 2\Lambda_b \theta_{\rho} = -\theta_{\rho;\sigma} + \frac{1}{2} \varepsilon^{\rho\sigma\mu\nu} (f^{\alpha\tau} C_{\alpha\tau[\nu\mu],\sigma}) + \frac{3(n-7)\Lambda}{(n-1)} \theta_{\rho} + (f^3) \ldots. \] (116)

This equation suggests the possibility of \( \theta_{\rho} \) waves, and this is discussed in detail in [2]. There it is shown that if \( \theta_{\rho} \) waves do result from (116), they would appear to have a negative energy. However, this theory avoids ghosts in an unusual way. Recall that this theory is the original Einstein-Schrödinger theory, but with a \( \Lambda_b g_{\mu\nu} \) in the field.
equations to account for zero-point fluctuations, and $\Lambda_z = -C_z \omega_c^2 l_P^2$, from (115) is finite only because of a cutoff frequency $\omega_c \sim 1/l_P$ from (13). From these equations and (116), Proca waves would be cut off because they would have a minimum frequency

$$\omega_{\text{Proca}} = \sqrt{2\Lambda_b} = \sqrt{-2\Lambda_z} = \sqrt{2C_z \omega_c^2 l_P} > \omega_c.$$  

(117)

Whether the cutoff of zero-point fluctuations is caused by a discreteness, uncertainty or foaminess of spacetime near the Planck length, or some other effect, the same $\omega_c$ which cuts off $\Lambda_z$ should also cut off Proca waves in this theory. So we should expect to observe only the trivial solution $\vartheta_{\rho} \approx 0$ to (116) and no ghosts. Comparing $\omega_{\text{Proca}}$ and $\omega_c$ from above, we see that this argument only applies if

$$\omega_c > \frac{1}{l_P \sqrt{2C_z}}.$$  

(118)

Here $C_z$ is defined by (15), and the inequality is satisfied for this theory when $\omega_c$ and $C_z$ are chosen as in (13,14) to be consistent with a cosmological constant caused by zero-point fluctuations. Since the prediction of negative energy waves would probably be inconsistent with reality, this theory should be approached cautiously when considering it with values of $\omega_c$ and $C_z$ which do not satisfy (118).

Finally, if we fully renormalize with $\omega_c \to \infty$ as in quantum electrodynamics, then $\Lambda_b \to \infty$ and $\omega_{\text{Proca}} \to \infty$, so the potential ghost goes away completely. In the limit $\omega_c \to \infty$ our theory becomes exactly Einstein-Maxwell theory. Assuming such a full renormalization does not diminish the value of the theory in any way. It still unifies gravitation with electromagnetism, it still suggests untried approaches to a complete unified field theory, and it still offers an untried approach to the quantization of gravity. In any attempt to quantize this theory, the cutoff frequency $\omega_c$ would need to be the same cutoff which is taken to infinity during renormalization. For example, Pauli-Villars masses would probably go as $M = \hbar \omega_c$ if Pauli-Villars renormalization was used. Since $\Lambda_b$ and $\Lambda_z$ in the Lagrangian density go as $\omega_c^4$, quantization and renormalization would certainly need to be done a bit different than usual. Also, because $\omega_{\text{Proca}}$ goes as $\omega_c^2$, Proca waves would not represent a ghost from the standpoint of quantization.

6. The Lorentz Force Equation

A generalized contracted Bianchi identity is derived in Appendix B using only the connection equations (91) and the symmetry (10) of $\Gamma_{\nu\mu}$. The identity can also be written in a manifestly covariant form

$$\sqrt{-\text{NN}} N^{-\nu\sigma} \tilde{R}_{\sigma\lambda} + \sqrt{-\text{NN}} N^{-\nu\sigma} \tilde{R}_{\lambda\nu} = 0.$$  

(119)

or in terms of $g^{\rho\nu}, f^{\nu\tau}$ and $\tilde{G}_{\nu\mu}$ from (119 121),

$$\tilde{G}_{\nu\cdot} = \left(\frac{3}{2} f^{\sigma\nu} \tilde{R}_{[\sigma\rho]\nu} + f^{\alpha\sigma} \tilde{R}_{[\sigma\nu]}\right) \sqrt{2} i \Lambda_b^{-1/2}.$$  

(121)

Clearly (119 121) are simple generalizations of the ordinary contracted Bianchi identity $2(\sqrt{-\text{NN}} R^\nu\lambda)_{\nu\cdot} - \sqrt{-\text{NN}} g^{\nu\tau} R_{\sigma\nu\cdot} = 0$, or $G_{\nu\cdot} = 0$. This identity was first derived in (10 15) without assuming charge currents, and later expressed in terms of the metric (9) by (17 19 18). The derivation with charge currents was first done in [20] by applying an infinitesimal coordinate transformation to an invariant integral.
Another useful identity is derived in Appendix A using only the definitions of $g_{\mu\nu}$ and $f_{\mu\nu}$,
\begin{equation}
\left( N^{(\mu\nu)} - \frac{1}{2} \delta^{\mu\nu} N_{\rho\sigma} \right)_{\rho\sigma} = \left( \frac{3}{2} f_{\tau\rho} N_{\rho_\sigma} + f_{\sigma\rho} N_{\rho_\tau} \right) \sqrt{2} i \Lambda_b^{1/2}.
\end{equation}

The ordinary Lorentz force equation of Einstein-Maxwell theory results from taking the divergence of the Einstein equations (66) using (121,79,56,122,23)
\begin{equation}
8\pi T^\sigma_{\nu\sigma} = (3/2) f_{\tau\rho} \tilde{R}_{\rho_\sigma} + 4\pi j^\sigma \tilde{R}_{\rho_\sigma} \sqrt{2} i \Lambda_b^{1/2} + \Lambda_b \left( N^{(\mu\nu)} - \frac{1}{2} \delta^{\mu\nu} N_{\rho\sigma} \right)_{\rho\sigma}.
\end{equation}

Another useful identity is derived in Appendix A using only the definitions (5,24)
\begin{equation}
\delta_{\mu\nu} = \left( \frac{1}{2} \delta^{\mu\nu} N_{\rho\sigma} \right)_{\rho\sigma} = \left( \frac{3}{2} f_{\tau\rho} N_{\rho_\sigma} + f_{\sigma\rho} N_{\rho_\tau} \right) \sqrt{2} i \Lambda_b^{1/2}.
\end{equation}
Both the Klein Gordon and Dirac equations match those of ordinary one-particle quantum mechanics in curved space. Note that for the spin-0 case, instead of deriving the continuity equation from the divergence of Ampere’s law, it can be derived from the Klein-Gordon equation,

\[
0 = \frac{iQ}{2\hbar} \left[ \psi \left( \text{one side of Klein-Gordon equation} \right) - \left( \text{one side of conjugate Klein-Gordon equation} \right) \psi \right]
\]

Similarly, instead of deriving the Lorentz force equation from the divergence of the Einstein equations, it can be derived from the Klein-Gordon equation,

\[
0 = \frac{\bar{\psi} D_\mu}{2} \left( \text{one side of Klein-Gordon equation} \right) + \left( \text{one side of conjugate Klein-Gordon equation} \right) \frac{D_\mu \psi}{2}
\]

The calculations can be found in a commented-out appendix of this paper’s .tex file. Presumably, similar results occur for the spin-1/2 case, but this was not verified.

8. Discussion

The original Einstein-Schrödinger theory results from many different Lagrangian densities. In fact it results from any Lagrangian density of the form,

\[
\mathcal{L}(\tilde{\Gamma}_\rho^\lambda, N_\rho^\alpha) = -\frac{1}{16\pi} \sqrt{-N} \left[ N^{\alpha\mu\nu}(R_{\mu\nu}(\tilde{\Gamma}) + c_1 \tilde{\Gamma}_\alpha^\lambda_{\mu\nu} + 2A_{\rho\nu\mu}) \sqrt{2} i\Lambda_b^{1/2} \right] + (n-2)\Lambda_b.
\]

where \(c_1, c_2, c_3\) are arbitrary constants and

\[
\tilde{\Gamma}_{\rho\mu} = \tilde{\Gamma}_{\rho\mu} - \tilde{\Gamma}_{\rho\alpha\mu} + \tilde{\Gamma}_{\rho\alpha\mu} - \tilde{\Gamma}_{\rho\mu\alpha} \tilde{\Gamma}_{\rho\mu\alpha} - \tilde{\Gamma}_{\rho\mu\alpha} \tilde{\Gamma}_{\rho\mu\alpha},
\]

\[
\tilde{\Gamma}_{\rho\mu} = \tilde{\Gamma}_{\rho\mu} + \frac{2}{(n-1)} \left[ c_2 \delta_{\rho\nu} \tilde{\Gamma}_{\sigma\nu} + (c_2 - 1) \delta_{\rho\nu} \tilde{\Gamma}_{\sigma\nu} \right],
\]

\[
A_\nu = \tilde{\Gamma}_{\sigma\nu} / c_3.
\]

Contracting \([145]\) on the right and left gives

\[
\tilde{\Gamma}_{\rho\alpha} = \frac{1}{(n-1)} \left[ (c_2 n + c_2 - 1)\tilde{\Gamma}_{\rho\alpha} - (c_2 n + c_2 - n)\tilde{\Gamma}_{\alpha\rho} \right] = \tilde{\Gamma}_{\rho\alpha},
\]

so \(\tilde{\Gamma}_{\rho\mu}\) has only \(n^3 - n\) independent components. Also, from \([145][146]\) we have

\[
\tilde{\Gamma}_{\rho\mu} = \tilde{\Gamma}_{\rho\mu} -\frac{2c_1}{(n-1)} \left[ c_2 \delta_{\rho\mu} A_\nu + (c_2 - 1) \delta_{\rho\nu} A_\mu \right],
\]

so \(\tilde{\Gamma}_{\rho\mu}\) and \(A_\nu\) fully parameterize \(\tilde{\Gamma}_{\rho\mu}\) and can be treated as independent variables. Therefore setting \(\delta \mathcal{L} / \delta \tilde{\Gamma}_{\rho\mu} = 0\) and \(\delta \mathcal{L} / \delta A_\nu = 0\) must give the same field equations as \(\delta \mathcal{L} / \delta \tilde{\Gamma}_{\rho\mu} = 0\). Because the field equations can be derived in this way, the constants \(c_2\) and \(c_3\) are clearly arbitrary, and because of \([24]\) with \(j^\sigma = 0\), the constant \(c_1\) is also arbitrary.

For \(c_1 = 1, c_2 = 1/2, c_3 = -(n-1)\sqrt{2} i\Lambda_b^{1/2}\), \([143]\) reduces to \([2]\) from the Hermitianized Ricci tensor \([7]\), where we have the invariance properties from \([20][21]\),

\[
A_{\nu} \rightarrow A_{\nu}, \quad \tilde{\Gamma}_{\rho\mu} \rightarrow \tilde{\Gamma}_{\rho\mu}, \quad N_{\rho\nu} \rightarrow N_{\rho\nu}, \quad N^{\rho\mu\nu} \rightarrow N^{\rho\mu\nu} \Rightarrow \mathcal{L} \rightarrow \mathcal{L}, \quad (149)
\]

\[
A_\alpha \rightarrow A_\alpha - \frac{\hbar}{Q} \phi_{\alpha}, \quad \tilde{\Gamma}_{\rho\tau} \rightarrow \tilde{\Gamma}_{\rho\tau}, \quad \tilde{\Gamma}_{\rho\tau} \rightarrow \tilde{\Gamma}_{\rho\tau} + \frac{2\hbar}{Q} \delta_{\rho\mu} \phi_{\tau} \sqrt{2} i\Lambda_b^{1/2} \Rightarrow \mathcal{L} \rightarrow \mathcal{L}. \quad (150)
\]
A renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources

For this case we have $\hat{\Gamma}_a^{\alpha} = \hat{\Gamma}_a^{\alpha} = \hat{\Gamma}_a^{\alpha}$, and from (88), (51), (143) the field equations require a generalization of the result $\mathcal{L}_\sigma - \Gamma_{\alpha\sigma} \mathcal{L} = 0$ that occurs with the Lagrangian density (1) of ordinary vacuum general relativity, that is

$$\mathcal{L}_\sigma - \hat{\Gamma}_{(\alpha\sigma)} \mathcal{L} = 0 \quad \text{or} \quad \mathcal{L}_\sigma - \Re(\hat{\Gamma}_{\alpha\sigma}) \mathcal{L} = 0. \quad (151)$$

For the alternative choice, $c_1 = 0, c_2 = n/(n+1), c_3 = -(n-1)\sqrt{2} \Lambda_b^{1/2}/2$, we have $\hat{\Gamma}_a^{\alpha} = \hat{\Gamma}_a^{\alpha} = \hat{\Gamma}_a^{\alpha}$, and from (88), (51), (143) the field equations require

$$\mathcal{L}_\sigma - \hat{\Gamma}_{a\sigma} \mathcal{L} = 0. \quad (152)$$

For the alternative choice $c_1 = 1, c_2 = 0, c_3 = -(n-1)\sqrt{2} \Lambda_b^{1/2}/2$, (143) reduces to

$$\mathcal{L}(\hat{\Gamma}_\rho\lambda, N_{\rho\tau}) = -\frac{1}{16\pi} \sqrt{-N} \left[ N^{-\mu\nu} \Re_{\nu\mu}(\hat{\Gamma}) + (n-2)\Lambda_b \right], \quad (153)$$

where $\Re_{\nu\mu}(\hat{\Gamma})$ is a fairly simple generalization of the ordinary Ricci tensor

$$\Re_{\nu\mu}(\hat{\Gamma}) = \hat{\Gamma}_\mu^{\alpha}_{\nu\alpha} - \hat{\Gamma}_\nu^{\alpha}_{\mu\alpha} + \hat{\Gamma}_\nu^{\sigma}_{\mu\alpha} \hat{\Gamma}_\sigma^{\alpha}_{\nu\mu} - \hat{\Gamma}_\sigma^{\rho}_{\nu\alpha} \hat{\Gamma}_\nu^{\alpha}_{\sigma\mu}. \quad (154)$$

For another alternative choice $c_1 = 0, c_2 = 0, c_3 = -(n-1)\sqrt{2} \Lambda_b^{1/2}/2$, (143) reduces to

$$\mathcal{L}(\hat{\Gamma}_\rho\lambda, N_{\rho\tau}) = -\frac{1}{16\pi} \sqrt{-N} \left[ N^{-\mu\nu} R_{\nu\mu}(\hat{\Gamma}) + (n-2)\Lambda_b \right], \quad (155)$$

where $R_{\nu\mu}(\hat{\Gamma})$ is the ordinary Ricci tensor

$$R_{\nu\mu}(\hat{\Gamma}) = \hat{\Gamma}_\mu^{\alpha}_{\nu\alpha} - \hat{\Gamma}_\nu^{\alpha}_{\mu\alpha} + \hat{\Gamma}_\nu^{\sigma}_{\mu\alpha} \hat{\Gamma}_\sigma^{\alpha}_{\nu\mu} - \hat{\Gamma}_\sigma^{\rho}_{\nu\alpha} \hat{\Gamma}_\nu^{\alpha}_{\sigma\mu}. \quad (156)$$

The original Einstein-Schrödinger theory (including the cosmological constant) can even be derived from purely affine versions of the Lagrangian densities described above, such as the Lagrangian density used by Schrödinger [13],

$$\mathcal{L}(\hat{\Gamma}) = \sqrt{-\det(R_{\nu\mu}(\hat{\Gamma}))}. \quad (157)$$

Whether one prefers the Lagrangian density (2) with the properties (149), (150), (151) or one of the alternatives, it is clear that the original Einstein-Schrödinger theory can be derived from rather simple principles. The theory proposed in this paper is a natural extension of the original Einstein-Schrödinger theory to account for zero-point fluctuations and first quantization. The search for simple principles has led to many advances in physics, and is what led Einstein to general relativity and also to the Einstein-Schrödinger theory [13] [10]. Einstein disliked the term $\sqrt{-g} F^{\nu\mu} F_{\mu\nu}/16\pi$ in the Einstein-Maxwell Lagrangian density. Referring to the equation $G_{\nu\mu} = 8\pi T_{\nu\mu}$ he states [13] “The right side is a formal condensation of all things whose comprehension in the sense of a field-theory is still problematic. Not for a moment, of course, did I doubt that this formulation was merely a makeshift in order to give the general principle of relativity a preliminary closed expression. For it was essentially not anything more than a theory of the gravitational field, which was somewhat artificially isolated from a total field of as yet unknown structure.” In modern times the term $\sqrt{-g} F^{\nu\mu} F_{\mu\nu}/16\pi$ has become standard and is rarely questioned. The theory presented here suggests that this term should be questioned, and offers an alternative which is based on simple principles and which genuinely unifies gravitation and electromagnetism.
9. Conclusions

The Einstein-Schrödinger theory is extended to include spin-0 and spin-1/2 sources. The theory is also modified by including a cosmological constant caused by zero-point fluctuations. This cosmological constant which multiplies the symmetric metric is assumed to be nearly cancelled by Schrödinger’s “bare” cosmological constant which multiplies the nonsymmetric fundamental tensor, such that the total “physical” cosmological constant matches measurement. The resulting Λ-renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources. And the theory becomes exactly Einstein-Maxwell theory and one-particle quantum mechanics. In particular, the field equations match the ordinary Einstein and Maxwell equations except for additional terms which are < 10^{-16} of the usual terms for worst-case field strengths and rates-of-change accessible to measurement. The theory also predicts the ordinary Lorentz force equation and the ordinary Klein-Gordon and Dirac equations. And the theory becomes exactly Einstein-Maxwell theory and one-particle quantum mechanics in the limit as the cosmological constant from zero-point fluctuations goes to infinity.

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Appendix A. A divergence identity

Using only the definitions of $g_{\mu\nu}$ and $f_{\nu\mu}$, and the identity gives,

$$\left(N^{(\mu}_{\nu})-\frac{1}{2}\delta^{(\mu}_{\nu}N^{\rho}_{\rho}\right)_{;\mu}-\frac{3}{2}f^{\sigma\rho}N_{[\sigma\rho;\nu]}\sqrt{2}i\Lambda^{-1/2}_b \tag{A.1}$$

$$=\frac{1}{2}\delta^{\sigma\rho}(N_{(\mu\nu);\sigma}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})-\frac{1}{2}f^{\sigma\rho}(N_{[\sigma\rho;\nu]}+N_{[\nu\sigma;\rho]}+N_{[\nu\sigma;\rho]})\sqrt{2}i\Lambda^{-1/2}_b \tag{A.2}$$

$$=1\sqrt{-N}\left[N^{-\sigma\rho}(N_{(\rho\sigma);\nu}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})+N^{-\sigma\rho}(N_{[\rho\nu;\sigma]}+N_{[\nu\mu;\sigma]}+N_{[\nu\sigma;\rho]})\right] \tag{A.3}$$

$$=1\sqrt{-N}_{2g}\left[N^{-\sigma\rho}(N_{(\rho\sigma);\nu}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})+N^{-\sigma\rho}(N_{[\rho\nu;\sigma]}+N_{[\nu\mu;\sigma]}+N_{[\nu\sigma;\rho]})\right] \tag{A.4}$$

$$=1\sqrt{-N}_{2g}\left[N^{-\sigma\rho}(N_{(\rho\sigma);\nu}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})\right] \tag{A.5}$$

$$=1\sqrt{-N}_{2g}\left[N^{-\sigma\rho}(N_{(\rho\sigma);\nu}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})\right] \tag{A.6}$$

$$=1\sqrt{-N}_{2g}\left[N^{-\sigma\rho}(N_{(\rho\sigma);\nu}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})\right] \tag{A.7}$$

$$=1\sqrt{-N}_{2g}\left[N^{-\sigma\rho}(N_{(\rho\sigma);\nu}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})\right] \tag{A.8}$$

$$=1\sqrt{-N}_{2g}\left[N^{-\sigma\rho}(N_{(\rho\sigma);\nu}+N_{(\nu\sigma);\rho}-N_{(\rho\sigma);\nu})\right] \tag{A.9}$$

$$=f^{\sigma\rho}_{\sigma\rho}N_{[\rho\nu]}\sqrt{2}i\Lambda^{-1/2}_b \tag{A.10}$$
Appendix B. Derivation of the generalized contracted Bianchi identity

Here we derive the generalized contracted Bianchi identity \([121]\) from the connection equations \([91]\), and from the symmetry \([10]\) of \(\dot{\Gamma}_{\nu\mu}^{\sigma}\). Whereas \([20]\) derived the identity by performing an infinitesimal coordinate transformation on an invariant integral, we will instead use a direct method similar to \([10]\), but generalized to include charge currents. First we make the following definitions,

\[W^\tau_{\rho} = \sqrt{-g} W^{\tau}_{\rho} = \sqrt{-\Lambda N}逆\tau = \sqrt{-g}(g^{\tau\rho} + \dot{\Gamma}^{\tau\rho}),\]  
\[\dot{\Gamma}^{\nu\mu} = \Gamma^{\nu\mu}_{\alpha} - \Gamma^{\nu}_{\alpha\mu} + \Gamma^{\rho}_{\nu\mu} \Gamma^{\rho}_{\sigma\alpha} - \Gamma^{\rho}_{\nu\alpha} \Gamma^{\rho}_{\mu\sigma} + \delta^{\rho}_{\nu} \Gamma^{\sigma}_{\alpha\mu},\]  
\[R^{\tau}_{\nu\mu\alpha} = \dot{\Gamma}^{\tau}_{\nu\mu\alpha} - \Gamma^{\tau}_{\nu\mu\alpha} + \Gamma^{\rho}_{\nu\mu\alpha} \dot{\Gamma}^{\rho}_{\alpha\sigma} - \Gamma^{\rho}_{\nu\alpha\mu} \dot{\Gamma}^{\rho}_{\sigma\mu} + \delta^{\rho}_{\nu} \Gamma^{\sigma}_{\alpha\mu}.\]

Here \(R^{\alpha}_{\nu\mu}\) is the Hermitianized Ricci tensor \([12]\), which has the property from \([18]\),

\[\mathcal{R}^{\alpha}_{\nu\rho} = \mathcal{R}^{\alpha}_{\rho\nu}.\]  

The tensors \(\dot{\mathcal{R}}^{\mu}_{\nu\alpha}\) and \(\dot{\mathcal{R}}^{\nu\alpha}_{\mu}\) reduce to the ordinary Ricci and Riemann tensors for symmetric fields where \(\Gamma^{\sigma}_{\nu\mu} = \Gamma^{\nu\mu}_{\sigma}/2 = 0\).

Rewriting the connection equations \([91]\) in terms of the definitions above gives,

\[0 = W^{\tau\rho}_{\lambda\sigma} + \dot{\Gamma}^{\tau}_{\sigma\lambda} W^{\rho\sigma} + \dot{\Gamma}^{\sigma}_{\lambda\sigma} W^{\tau\rho} - \frac{4\pi}{(n-1)}(\dot{j}^{\delta}_{\lambda} - \dot{j}^{\delta}_{\sigma}).\]  

Differentiating \([B.0]\), antisymmetrizing, and substituting \([B.3]\) for \(W^{\tau\rho}_{\lambda\sigma}\) gives,

\[0 = \left(\mathcal{W}^{\tau\rho}_{\lambda\sigma} + \mathcal{\dot{\Gamma}}^{\tau}_{\sigma\lambda} \mathcal{W}^{\rho\sigma} + \mathcal{\dot{\Gamma}}^{\sigma}_{\lambda\sigma} \mathcal{W}^{\tau\rho} - \frac{4\pi}{(n-1)}(\dot{j}^{\delta}_{\lambda} - \dot{j}^{\delta}_{\sigma})\right).\]

Cancelling the terms 2B-3A, 2C-4A, 3C-4B and using \([B.2]\) gives,

\[0 = \frac{1}{2} \left[\mathcal{W}^{\tau\rho}_{\lambda\sigma} \mathcal{R}^{\tau\rho}_{\sigma\nu\lambda} + \mathcal{W}^{\tau\sigma} \mathcal{R}^{\rho}_{\sigma\nu\lambda}(\mathcal{T})\right] + \frac{4\pi}{(n-1)} \left[\dot{\Gamma}^{\tau}_{\lambda\nu} \dot{j}^{\nu} - \dot{\Gamma}^{\tau}_{\lambda\rho} \dot{j}^{\rho}\right] + \frac{4\pi}{(n-1)} \left[(\dot{j}^{\nu}_{\lambda} + \dot{\Gamma}^{\nu}_{\lambda\sigma} \dot{j}^{\sigma})\delta^{\rho}_{\lambda} - (\dot{j}^{\nu}_{\tau} + \dot{\Gamma}^{\nu}_{\tau\sigma} \dot{j}^{\sigma})\delta^{\rho}_{\tau}\right].\]

Multiplying by 2, contracting over \(\tau\), and using \([B.2]\) and \(\dot{j}^{\nu}_{\tau} = 0\) from \([88]\) gives,

\[0 = \mathcal{W}^{\nu\rho}_{\lambda\sigma} \mathcal{R}^{\tau\rho}_{\sigma\nu\lambda} + \mathcal{W}^{\tau\sigma} \mathcal{R}^{\rho}_{\sigma\nu\lambda}(\mathcal{T}) + \frac{8\pi}{(n-1)} \left[\dot{\Gamma}^{\tau}_{\lambda\nu} \dot{j}^{\nu} - \dot{\Gamma}^{\tau}_{\lambda\rho} \dot{j}^{\rho}\right].\]
\[ + \frac{8\pi}{(n-1)} \left[ (\tilde{\mathcal{J}}^\nu \,_{[\nu} \hat{\Gamma}^\nu_{\sigma]} \mathcal{J}^\sigma - \tilde{\Gamma}^\nu_{\sigma\lambda} \mathcal{J}^\sigma) \delta^\nu_{\lambda} \right] - (\mathcal{J}^\nu \,_{[\nu} \hat{\Gamma}^\nu_{\sigma]} \mathcal{J}^\sigma - \tilde{\Gamma}^\nu_{\sigma\lambda} \mathcal{J}^\sigma) \delta^\nu_{\lambda} \right] \] (B.11)
\[ \mathbf{W}^{\sigma\nu} \tilde{\mathcal{R}}^{\tau}_{\sigma\nu\lambda} + \mathbf{W}^{\tau\sigma} \tilde{\mathcal{R}}_{\lambda\sigma} - \frac{4\pi(n-2)}{(n-1)} (\tilde{\mathcal{J}}^{\tau}_{\lambda\lambda} + \hat{\Gamma}^{\tau}_{\sigma\lambda} \mathcal{J}^\sigma - \tilde{\Gamma}^{\tau}_{\sigma\lambda} \mathcal{J}^\sigma). \] (B.12)

This is a generalization of the symmetry \( R^\tau_{\lambda} = R^\lambda_{\tau} \) of the ordinary Ricci tensor.

Next we will use the generalized uncontracted Bianchi identity \[ \text{(10)}, \] which can be verified by direct computation,
\[ \tilde{\mathcal{R}}^{\tau}_{\sigma\nu\rho\lambda} + \tilde{\mathcal{R}}^{\tau}_{\sigma\alpha\lambda\nu} + \tilde{\mathcal{R}}^{\tau}_{\sigma\lambda\nu\alpha} = 0. \] (B.13)

The +/- notation is from \[ \text{(10)}, \] and indicates that covariant derivative is being done with \( \Gamma^\rho_{\nu\mu} \) instead of the usual \( \Gamma^\rho_{\nu\mu} \). A plus by an index means that the associated derivative index is to be placed on the right side of the connection, and a minus means that it is to be placed on the left side. Note that the identity \[ \text{(B.13)} \] is true for either the ordinary Riemann tensor or for our definition \[ \text{(B.13)}. \] This is because the two tensors differ by the term \( \delta^\nu_{\lambda} \Gamma^\sigma_{\sigma[\nu,\lambda]} \), so that the expression \[ \text{(B.13)} \] would differ by the term \( \delta^\nu_{\lambda} (\Gamma^\rho_{\nu[\nu,\lambda]} + \tilde{\Gamma}^\rho_{\nu[\alpha,\lambda]} + \tilde{\Gamma}^\rho_{\nu[\lambda,\nu]} \alpha, \nu) \). But this difference vanishes because for an arbitrary curl \( \mathbf{Y}_{[\alpha,\lambda]} \) we have
\[ Y_{[\nu,\alpha]} + Y_{[\alpha,\lambda]} + Y_{[\nu,\alpha]} = Y_{[\nu,\alpha]} - \tilde{\Gamma}^\nu_{\alpha,\nu} Y_{[\sigma,\alpha]} - \tilde{\Gamma}^\nu_{\alpha,\nu} Y_{[\nu,\sigma]} \]
\[ + Y_{[\alpha,\lambda]} - \tilde{\Gamma}^\sigma_{\alpha,\lambda} Y_{[\sigma,\alpha]} + Y_{[\nu,\sigma]} - \tilde{\Gamma}^\nu_{\alpha,\nu} Y_{[\lambda,\sigma]} = 0. \] (B.14)

A simple form of the generalized contracted Bianchi identity results if we contract \[ \text{(B.13)} \] over \( \mathbf{W}^{\sigma\nu} \) and \( \tilde{\mathcal{R}}^\tau_{\sigma\nu\lambda} \), then substitute \[ \text{(B.12)} \] for \( \mathbf{W}^{\sigma\nu} \tilde{\mathcal{R}}^{\tau}_{\sigma\nu\lambda} \) and \[ \text{(B.9)} \] for \( \mathbf{W}^{\tau\nu}_{\;\;\;\tau} \),
\[ 0 = \mathbf{W}^{\sigma\nu} (\tilde{\mathcal{R}}^{\tau}_{\sigma\nu\rho\lambda} + \mathbf{W}^{\tau\rho}_{\;\;\;\sigma} \tilde{\mathcal{R}}^{\rho}_{\sigma\nu\lambda} \mathcal{J}^\nu + \tilde{\mathcal{R}}^{\tau}_{\sigma\rho\nu\lambda}) \] (B.15)
\[ = - \mathbf{W}^{\tau\sigma} \tilde{\mathcal{R}}^{\rho}_{\sigma\nu\lambda} + \mathbf{W}^{\sigma\nu} \tilde{\mathcal{R}}^{\tau}_{\sigma\nu\lambda} \] (B.16)
\[ = - \mathbf{W}^{\tau\sigma} \tilde{\mathcal{R}}^{\rho}_{\sigma\nu\lambda} + \mathbf{W}^{\sigma\nu} \tilde{\mathcal{R}}^{\tau}_{\sigma\nu\lambda} \] (B.17)
\[ = - \mathbf{W}^{\tau\sigma} \tilde{\mathcal{R}}^{\rho}_{\sigma\nu\lambda} + \mathbf{W}^{\sigma\nu} \tilde{\mathcal{R}}^{\tau}_{\sigma\nu\lambda} \] (B.18)
\[ + \left( \frac{4\pi(n-2)}{(n-1)} (\tilde{\mathcal{J}}^{\nu}_{\lambda\lambda} + \hat{\mathcal{J}}^{\nu}_{\sigma\lambda} \mathcal{J}^\sigma - \tilde{\mathcal{J}}^{\nu}_{\sigma\lambda} \mathcal{J}^\sigma) \right) \] (B.19)
With the $\hat{R}$ has cancelled out of (B.21), the Christoffel connection $\Gamma_{\sigma\lambda}$. Rewriting the identity in terms of $2(\nu\mu)$, 4C-6A, 4D-8D, 5A-7A, 5B-7B, 5C-7C all cancel, 4A and 4E are zero because $j^\nu = 0$ from (B.20), and 4B, 6B, 6C, 8C cancel the Ricci tensor term 8A, 8B. With the $W^{\sigma}$ terms of (B.20), all those with a $\tilde{\Gamma}_\mu^{\alpha}$ factor cancel, which are the terms 1C-2C, 1B-3C, 2B-2D, 3B-3D. Doing the cancellations and using (B.11), we get

$$0 = (\sqrt{-N}N^{-\nu\sigma}_{\nu} \tilde{R}_{\sigma\lambda} + \sqrt{-N}N^{-\nu\sigma}_{\nu} \tilde{R}_{\sigma\lambda})_{\nu} - \sqrt{-N}N^{-\nu\sigma}_{\nu} \tilde{R}_{\sigma\lambda} \cdot (B.21)$$

Equation (B.21) is a simple generalization of the ordinary contracted Bianchi identity $2(\sqrt{-g} R^{\nu}_{\lambda})_{\nu} - \sqrt{-g} g^{\nu\sigma} R_{\sigma\lambda} = 0$, and it applies even when $j^\nu \neq 0$. Because $\tilde{\Gamma}_\mu^{\alpha}$ has cancelled out of (B.21), the Christoffel connection $\Gamma_\nu^{\alpha}$ would also cancel, so a manifestly tensor relation can be obtained by replacing the ordinary derivatives with covariant derivatives done with $\Gamma_\nu^{\alpha}$.

$$0 = (\sqrt{-N}N^{-\nu\sigma}_{\nu} \tilde{R}_{\sigma\lambda} + \sqrt{-N}N^{-\nu\sigma}_{\nu} \tilde{R}_{\sigma\lambda})_{\nu} - \sqrt{-N}N^{-\nu\sigma}_{\nu} \tilde{R}_{\sigma\lambda} \cdot (B.22)$$

Rewriting the identity in terms of $g^{\nu\sigma}$ and $\tilde{f}^{\nu\sigma}$ as defined by (B.11, B.2) gives,

$$0 = (\sqrt{-g} (g^{\nu\sigma} + \tilde{f}^{\nu\sigma}) \tilde{R}_{\sigma\lambda} + \sqrt{-g} (g^{\nu\sigma} + \tilde{f}^{\nu\sigma}) \tilde{R}_{\sigma\lambda})_{\nu} - \sqrt{-g} (g^{\nu\sigma} + \tilde{f}^{\nu\sigma}) \tilde{R}_{\sigma\lambda} \cdot (B.23)$$

$$= \sqrt{-g} [2\tilde{R}_\lambda^{(\nu\lambda)}]_{\nu} - \tilde{R}_{\sigma;\lambda} + \sqrt{-g} [2(\tilde{f}^{\nu\sigma} \tilde{R}_{\sigma\lambda})_{\nu} + \tilde{\tilde{f}}^{\nu\sigma} \tilde{R}_{\sigma\lambda}] \cdot (B.24)$$

$$= \sqrt{-g} [2\tilde{R}_\lambda^{(\nu\lambda)}]_{\nu} - \tilde{R}_{\sigma;\lambda} + \sqrt{-g} [3\tilde{f}^{\nu\sigma} \tilde{R}_{\sigma\lambda} + 2\tilde{\tilde{f}}^{\nu\sigma} \tilde{R}_{\sigma\lambda}] \cdot (B.25)$$

Dividing by $2\sqrt{-g}$ gives another form of the generalized contracted Bianchi identity

$$\left(\tilde{R}_\lambda^{(\nu\mu)} - \frac{1}{2}\delta^\chi_\nu \tilde{R}_\sigma^\chi\right)_{\nu} = \frac{3}{2} \tilde{f}^{\nu\sigma} \tilde{R}_{\nu\sigma\lambda} + \tilde{\tilde{f}}^{\nu\sigma} \tilde{R}_{\nu\sigma\lambda} \cdot (B.26)$$

From (B.26) we get the final result (121).

Appendix C. Extraction of a connection addition from the Hermitianized Ricci tensor

Substituting $\tilde{\Gamma}_\nu^{\mu} = \Gamma^{\alpha}_{\mu\nu} + \chi^{\alpha}_{\mu\nu}$ from (97, 22) into (12) gives

$$R_{\nu\mu}(\tilde{\Gamma}) = 2[(\Gamma^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu})_{\alpha} + (\Gamma^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu})_{(\alpha)} + (\Gamma^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu})_{(\alpha)}] + (\Gamma^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu}) \cdot (C.1)$$

$$= R_{\nu\mu}(\Gamma) + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} \cdot (C.2)$$

$$= R_{\nu\mu}(\Gamma) + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} \cdot (C.3)$$

$$= R_{\nu\mu}(\Gamma) + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} \cdot (C.4)$$

$$= R_{\nu\mu}(\Gamma) + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} + \chi^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\mu}\chi^{\alpha}_{\nu\mu} \cdot (C.5)$$
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Also, substituting $\tilde{\Gamma}_\alpha^{\mu} = \Gamma_\nu^{\mu} + [\delta^{\alpha}_\mu A_\nu - \delta^{\alpha}_\nu A_\mu] \sqrt{2} i \Lambda^1_b$ from (8) into (7) and using $\tilde{\Gamma}_\nu^{\alpha} = \hat{\Gamma}_\nu^{(\alpha)} = \hat{\Gamma}_\nu^{\alpha}$ gives

\[
\mathcal{R}_{\nu\mu}(\hat{\Gamma}) = \Gamma_\nu^{\mu,\alpha} + [\delta^{\alpha}_\mu A_\nu - \delta^{\alpha}_\nu A_\mu] \sqrt{2} i \Lambda^1_b - \hat{\Gamma}_\nu^{\alpha}
\]

\[
+ \left( \hat{\Gamma}_\nu^{\alpha} + [\delta^{\alpha}_\mu A_\nu - \delta^{\alpha}_\nu A_\mu] \sqrt{2} i \Lambda^1_b \right) \hat{\Gamma}_\sigma^{\alpha}
\]

\[
- \left( \hat{\Gamma}_\nu^{\alpha} + [\delta^{\alpha}_\mu A_\nu - \delta^{\alpha}_\nu A_\mu] \sqrt{2} i \Lambda^1_b \right) \left( \hat{\Gamma}_\sigma^{\alpha} + [\delta^{\alpha}_\mu A_\sigma - \delta^{\alpha}_\sigma A_\mu] \sqrt{2} i \Lambda^1_b \right)
\]

\[
+ 2(n - 1) A_\nu A_\mu A_\nu
\]

(C.6)

\[
= \Gamma_\nu^{\mu,\alpha} + 2A_{[\nu,\mu]} \sqrt{2} i \Lambda^1_b - \Gamma_\alpha^{\mu} + \hat{\Gamma}_\nu^{\alpha} + [A_\nu \hat{\Gamma}_\alpha^{\mu} - A_\mu \hat{\Gamma}_\nu^{\alpha}] \sqrt{2} i \Lambda^1_b
\]

\[
- \hat{\Gamma}_\nu^{\sigma} \hat{\Gamma}_\sigma^{\alpha} - \left[ \Gamma_\nu^{\sigma} \hat{\Gamma}_\sigma^{\alpha} - \hat{\Gamma}_\nu^{\sigma} \hat{\Gamma}_\sigma^{\alpha} \right] \sqrt{2} i \Lambda^1_b / 2 - \left[ A_\nu \hat{\Gamma}_\alpha^{\mu} - A_\mu \hat{\Gamma}_\nu^{\alpha} \right] \sqrt{2} i \Lambda^1_b
\]

\[
+ 2A_\nu A_\mu(1 - n - 1 + 1)A_\nu + 2(n - 1) A_\nu A_\mu A_\nu
\]

(C.7)

\[
= \Gamma_\nu^{\mu,\alpha} - \hat{\Gamma}_\nu^{\alpha} + \hat{\Gamma}_\nu^{\alpha} \hat{\Gamma}_\sigma^{\alpha} + A_{[\nu,\mu]} \sqrt{2} i \Lambda^1_b
\]

(C.8)

\[
= \mathcal{R}_{\nu\mu}(\hat{\Gamma}) + 2A_{[\nu,\mu]} \sqrt{2} i \Lambda^1_b / 2.
\]

(C.9)

Appendix D. Variational derivatives for fields with the symmetry $\hat{\Gamma}_\alpha^{\mu} = 0$

The field equations associated with a field with symmetry properties must have the same number of independent components as the field. For a field with the symmetry $\hat{\Gamma}_\alpha^{\mu} = 0$, the field equations can be found by introducing a Lagrange multiplier $\Omega^\mu$,

\[
0 = \delta \int (\mathcal{L} + \Omega^\mu \hat{\Gamma}_\alpha^{\mu}) d^3x.
\]

(D.1)

Minimizing the integral with respect to $\Omega^\mu$ shows that the symmetry is enforced. Using the definition,

\[
\frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\tau^\nu} = \frac{\partial \mathcal{L}}{\partial \hat{\Gamma}_\tau^\nu} - \left( \frac{\partial \mathcal{L}}{\partial \hat{\Gamma}_\tau^\nu} \right)_\omega \cdot \cdot \cdot,
\]

(D.2)

and minimizing the integral with respect to $\hat{\Gamma}_\tau^\nu$ gives

\[
0 = \frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\tau^\nu} + \Omega^\mu \delta_\tau^\nu \delta_\sigma^\rho = \frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\tau^\nu} + \frac{1}{2} \left( \Omega^\tau \delta_\tau^\rho - \delta_\rho^\tau \Omega^\rho \right).
\]

(D.3)

Contracting this on the left and right gives

\[
\Omega^\rho = \frac{2}{(n - 1)} \frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\nu^{\alpha}} = - \frac{2}{(n - 1)} \frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\nu^{\alpha}}.
\]

(D.4)

Substituting (D.4) back into (D.3) gives

\[
0 = \frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\tau^\nu} - \frac{\delta_\tau^\nu}{(n - 1)} \frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\nu^{\alpha}} - \frac{\delta_\rho^\tau}{(n - 1)} \frac{\Delta \mathcal{L}}{\Delta \hat{\Gamma}_\nu^{\alpha}}.
\]

(D.5)

In (D.4) (D.5) the index contractions occur after the derivatives. Contracting (D.5) on the right and left gives the same result, so it has the same number of independent components as $\hat{\Gamma}_\nu^{\alpha}$. This is a general expression for the field equations associated with a field having the symmetry $\hat{\Gamma}_\alpha^{\mu} = 0$. 
Appendix E. Solution for $\tilde{\Gamma}^\alpha_{\nu\mu}$ in terms of $g_{\nu\mu}$ and $f_{\nu\mu}$, with sources

Here we derive the approximate solution \[98\overset{99}{=}0\] to the connection equations \[91\]. First let us define the notation

$$
\tilde{\Gamma}^\sigma_{\nu\mu} = f^\nu\mu \sqrt{2} i \Lambda_b^{-1/2}, \quad \tilde{\sigma} = j^\sigma \sqrt{2} i \Lambda_b^{-1/2}, \quad \tilde{\gamma}^\nu_{\nu\mu} = \gamma^\nu_{(\nu\mu)}, \quad \tilde{\gamma}^\nu_{\nu\mu} = \gamma^\nu_{[\nu\mu]}.
$$  \tag{E.1}

We assume that $\vert \tilde{f}^\nu_{\nu\mu} \vert \ll 1$ for all components of the unit field $\tilde{f}^\nu_{\nu\mu}$, and find a solution in the form of a power series expansion in $\tilde{f}^\nu_{\nu\mu}$. Using \[93\] and

$$
\tilde{\Gamma}^\sigma_{\alpha\sigma} = \left(\sqrt{-N}\right)_{\alpha\sigma} + \frac{8\pi}{(n-2)(n-1)} \tilde{\sigma} N_{[\alpha\sigma]} \tag{E.2}
$$

from \[93\] and $\tilde{\Gamma}^\alpha_{\nu\mu} = \Gamma^\alpha_{\nu\mu} + \gamma^\alpha_{\nu\mu}$, \((\sqrt{-N}/\sqrt{-g}) N^{-\nu\mu} = g^{\nu\mu} + \tilde{f}^{\nu\mu}\) from \[91\overset{5\overset{24}{=}0}\] we get

$$
0 = \frac{\sqrt{-N}}{\sqrt{-g}} (N^{-\nu\mu})_{\alpha\sigma} + \tilde{\Gamma}^\nu_{\tau\alpha} N^{-\mu\tau} + \tilde{\Gamma}^\mu_{\nu\alpha} N^{-\tau\nu}
$$

\[\frac{8\pi}{(n-1)} \tilde{\sigma} N_{[\alpha\sigma]} N^{-\mu\nu} \]  \tag{E.3}

$$
= \left(\frac{\sqrt{-N}}{\sqrt{-g}} N^{-\mu\nu}\right)_{\alpha\sigma} + \frac{8\pi}{(n-1)} \tilde{\sigma} N_{[\alpha\sigma]} N^{-\mu\nu}
$$

\[\frac{8\pi}{(n-1)} \tilde{\sigma} N_{[\alpha\sigma]} N^{-\mu\nu} \]  \tag{E.4}

$$
= (g^{\mu\nu} + \tilde{f}^{\mu\nu})_{\alpha\sigma} + \gamma^\nu_{\tau\alpha} (g^{\tau\mu} + \tilde{f}^{\tau\mu}) + \gamma^\mu_{\alpha\tau} (g^{\nu\tau} + \tilde{f}^{\nu\tau}) - \gamma^\sigma_{\alpha\sigma} (g^{\nu\mu} + \tilde{f}^{\nu\mu})
$$

\[\frac{8\pi}{(n-1)} \tilde{\sigma} N_{[\alpha\sigma]} N^{-\mu\nu} \]  \tag{E.5}

$$
= \tilde{f}^{\nu\mu}_{\alpha\sigma} + \gamma^\nu_{\tau\alpha} g^{\tau\mu} + \gamma^\mu_{\alpha\tau} g^{\nu\tau} + \gamma^\mu_{\alpha\tau} g^{\nu\tau} - \gamma^\sigma_{\alpha\sigma} g^{\nu\mu} - \gamma^\sigma_{\alpha\sigma} \tilde{f}^{\nu\mu}
$$

\[\frac{4\pi}{(n-1)} \tilde{\sigma} N_{[\alpha\sigma]} N^{-\mu\nu} \]  \tag{E.6}

Contracting this with $g_{\nu\mu}$ gives

$$
0 = (2 - n) \gamma^\nu_{\alpha\sigma} - 2 \gamma^\nu_{\alpha\sigma} \tilde{f}_{\gamma\gamma} \Rightarrow \gamma^\sigma_{\alpha\sigma} = \frac{2}{(n-2)} \tilde{\gamma}_{\alpha\sigma} \tilde{\gamma}_{\gamma\gamma}.
$$  \tag{E.7}

Lowering the indices of \[E.6\] and making linear combinations of its permutations gives

$$
\gamma^\nu_{\alpha\nu} = \gamma^\nu_{\alpha\nu} + \left( \tilde{f}_{\nu\alpha} + \gamma_{\nu\alpha} + \gamma_{\nu\alpha} \tilde{f}_{\nu\alpha} + \gamma_{\nu\alpha} \tilde{f}_{\nu\alpha} - \gamma_{\alpha\nu} g_{\nu\mu} - \gamma_{\alpha\nu} \tilde{f}_{\nu\mu} \right.
$$

\[\frac{4\pi}{(n-1)} \gamma_{\nu\alpha} g_{\nu\alpha} - \gamma_{\nu\alpha} g_{\nu\alpha} \]  \tag{E.8}
Cancelling out the $\Upsilon_{\alpha\nu\mu}$ terms on the right-hand side, collecting terms, and separating out the symmetric and antisymmetric parts gives,

$$\mathcal{T}_{\alpha\nu\mu} = \Upsilon_{[\alpha\mu]} \hat{f}_{\nu}^\tau + \Upsilon_{[\alpha\nu]} \hat{f}_{\mu}^\tau + \Upsilon_{[\nu\mu]} \hat{f}_{\alpha}^\tau - \frac{1}{2} \Upsilon_{\alpha\sigma\mu} g_{\nu\mu} \alpha + \Upsilon_{\alpha\nu\mu} g_{\mu\sigma} \alpha \quad \text{(E.9)}$$

Substituting (E.9) into (E.10)

$$\mathcal{T}_{\alpha\nu\mu} = -\Upsilon_{[\alpha\mu]} \hat{f}_{\nu}^\tau + \Upsilon_{[\alpha\nu]} \hat{f}_{\mu}^\tau + \Upsilon_{[\nu\mu]} \hat{f}_{\alpha}^\tau - \frac{1}{2} \Upsilon_{\alpha\sigma\mu} g_{\nu\mu} \alpha + \Upsilon_{\alpha\nu\mu} g_{\mu\sigma} \alpha + \frac{1}{2} \left( f_{\nu\mu\alpha} + \hat{f}_{\alpha\mu\nu} - \hat{f}_{\alpha\nu\mu} \right) + \frac{8\pi}{(n-1)^2} j_{[\nu\mu\alpha]} \alpha.$$  \hspace{1cm} (E.10)

Substituting (E.9) into (E.10)

$$\mathcal{T}_{\alpha\nu\mu} = -\frac{1}{2} \left( \Upsilon_{[\alpha\tau\nu]} \hat{f}_{\mu}^\sigma + \Upsilon_{[\alpha\nu\mu]} \hat{f}_{\tau}^\sigma + \Upsilon_{[\nu\mu]} \hat{f}_{\tau}^\sigma \alpha - \frac{1}{2} \Upsilon_{\alpha\sigma\mu} g_{\nu\mu} \alpha + \Upsilon_{\alpha\nu\mu} g_{\mu\sigma} \alpha \right) \hat{f}_{\nu}^\tau$$

$$- \frac{1}{2} \left( \Upsilon_{[\nu\tau\sigma]} \hat{f}_{\rho}^\mu + \Upsilon_{[\nu\sigma\rho]} \hat{f}_{\tau}^\mu + \Upsilon_{[\nu\tau]} \hat{f}_{\rho}^\sigma \alpha - \frac{1}{2} \Upsilon_{\alpha\sigma\mu} g_{\nu\mu} \alpha + \Upsilon_{\alpha\nu\mu} g_{\mu\sigma} \alpha \right) \hat{f}_{\nu}^\tau$$

$$+ \frac{1}{2} \left( \Upsilon_{[\nu\tau]} \hat{f}_{\rho}^\mu + \Upsilon_{[\nu\rho]} \hat{f}_{\tau}^\mu \alpha - \frac{1}{2} \Upsilon_{\alpha\sigma\mu} g_{\nu\mu} \alpha + \Upsilon_{\alpha\nu\mu} g_{\mu\sigma} \alpha \right) \hat{f}_{\nu}^\tau$$

$$+ \frac{1}{2} \left( \Upsilon_{[\nu\rho]} \hat{f}_{\tau}^\mu \alpha - \Upsilon_{[\nu\tau]} \hat{f}_{\rho}^\mu \alpha - \frac{1}{2} \Upsilon_{\alpha\sigma\mu} g_{\nu\mu} \alpha + \Upsilon_{\alpha\nu\mu} g_{\mu\sigma} \alpha \right) \hat{f}_{\nu}^\tau$$

$$- \frac{1}{2} \Upsilon_{\alpha\sigma\mu} \hat{f}_{\nu\mu} + \Upsilon_{\alpha\nu\mu} \hat{f}_{\sigma\mu} \alpha$$

$$+ \frac{1}{2} \left( f_{\nu\mu\alpha} + \hat{f}_{\alpha\mu\nu} - \hat{f}_{\alpha\nu\mu} \right) + \frac{8\pi}{(n-1)^2} j_{[\nu\mu\alpha]} \alpha.$$ \hspace{1cm} (E.11)

and using (E.7) gives,

$$\mathcal{T}_{\alpha\nu\mu} = \Upsilon_{[\alpha\sigma\sigma]} \hat{f}_{\mu}^\tau + \Upsilon_{[\alpha\sigma]} \hat{f}_{\rho}^\sigma \alpha - \Upsilon_{[\alpha\nu]} \hat{f}_{\mu}^\sigma \alpha + \Upsilon_{[\nu\mu]} \hat{f}_{\alpha}^\sigma \alpha + \frac{1}{2} \left( f_{\nu\mu\alpha} + \hat{f}_{\alpha\mu\nu} - \hat{f}_{\alpha\nu\mu} \right) + \frac{8\pi}{(n-1)^2} j_{[\nu\mu\alpha]} \alpha.$$ \hspace{1cm} (E.11)

Equation (E.11) is useful for finding exact solutions to the connection equations because it consists of only $n^2(n-1)/2$ equations in the $n^2(n-1)/2$ unknowns $\Upsilon_{\alpha\nu\mu}$. 

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Also, from (E.11) we can immediately see that
\[ \tilde{\Upsilon}_{\alpha\mu} = \frac{1}{2} (\hat{f}_{\nu\mu};\alpha + \hat{f}_{\alpha\mu;\nu} - \hat{f}_{\alpha\nu;\mu}) + \frac{4\pi}{(n-1)} (\hat{\jmath}_\nu g_{\mu\alpha} - \hat{\jmath}_\mu g_{\nu\alpha}) + (\hat{f}^3) \ldots \]  
(E.12)

Here the notation \((\hat{f}^3)\) refers to terms like \(\hat{f}_{\alpha\tau} \hat{f}_\sigma^\tau \hat{f}_\rho^\sigma \). With (E.12) as a starting point, one can calculate more accurate \(\tilde{\Upsilon}_{\alpha\mu}\) by recursively substituting the current \(\tilde{\Upsilon}_{\alpha\mu}\) into (E.11). Then this \(\tilde{\Upsilon}_{\alpha\mu}\) can be substituted into (E.7) to get \(\tilde{\Upsilon}_{\alpha\mu}\). For our purposes (E.12) will be accurate enough. Substituting (E.12) into (E.7) we get
\[ \tilde{\Upsilon}_{(\alpha\nu)\mu} = - \hat{f}_{\mu(\nu;\alpha)} + \frac{4\pi}{(n-1)} (\hat{\jmath}_{(\nu;\alpha)\mu} - \hat{\jmath}_{\mu(\nu;\alpha)}) + (\hat{f}^3) \ldots, \]  
(E.13)

\[ \tilde{\Upsilon}_{[\alpha\nu]\mu} = \frac{1}{2} \hat{f}_{\nu\alpha;\mu} + \frac{4\pi}{(n-1)} \hat{\jmath}_{[\nu\alpha]\mu} + (\hat{f}^3) \ldots, \]  
(E.14)

\[ \tilde{\Upsilon}_{\sigma\alpha} = \frac{2}{(n-2)} \left( \frac{1}{2} \hat{f}_{\sigma\alpha;\tau} + \frac{4\pi}{(n-1)} \hat{\jmath}_{\sigma\alpha;\tau} \right) f^\tau + (\hat{f}^4) \ldots \]  
(E.15)

Substituting these equations into (E.9) gives
\[ \tilde{\Upsilon}_{\alpha\mu} = - \left( \frac{1}{2} \hat{f}_{\alpha\mu;\tau} + \frac{2\pi}{(n-1)} (\hat{\jmath}_{\alpha g_{\mu\tau} - \hat{\jmath}_{\mu g_{\alpha\tau}}}) \right) f^\tau \]  
\[ + \left( \frac{1}{2} \hat{f}_{\alpha\tau;\nu} + \frac{2\pi}{(n-1)} (\hat{\jmath}_{\nu g_{\alpha\tau} - \hat{\jmath}_{\alpha g_{\nu\tau}}}) \right) f^\nu \]  
\[ + \left( - \hat{f}_{\tau(\mu;\nu)} + \frac{2\pi}{(n-1)} (\hat{\jmath}_{\mu g_{\nu\tau} + \hat{\jmath}_{\nu g_{\mu\tau}}}) \right) f^\tau + \left( \frac{8\pi}{(n-1)(n-2)} \hat{f}_{\tau(\nu)} g_{\mu\nu} \right) 
\[ + \frac{1}{4(n-2)} \left( \hat{f}_{\nu(\mu;\nu)} + \frac{2\pi}{(n-1)} \hat{\jmath}_{\nu g_{\mu\nu} - \hat{\jmath}_{\mu g_{\nu\nu}}} \right) f^\nu \]  
\[ = f^\tau \hat{f}_{\nu(\mu;\nu)} + \hat{f}_{\alpha;\tau} f^\tau (\nu;\mu) + \frac{1}{4(n-2)} \left( (\hat{f}_{\nu(\mu;\nu)};\alpha) g_{\nu\mu} - 2(\hat{f}_{\nu(\mu;\nu)};\nu) g_{\mu\nu} \right) 
\[ + \frac{4\pi}{(n-2)} \hat{f}_{\alpha;\tau} g_{\nu\mu} + \frac{2}{(n-1)} \hat{f}_{\tau(\nu)} g_{\mu\nu} \right) + (\hat{f}^4) \ldots. \]  
(E.16)

Here the notation \((\hat{f}^4)\) refers to terms like \(\hat{f}_{\alpha\tau} \hat{f}_\sigma^\tau \hat{f}_\rho^\sigma \hat{f}_\nu^\rho \). Raising the indices on (E.16)(E.12)(E.15) and using (E.11) gives the final result (98,99,100).

**Appendix F. Solution for \(N_{\nu\mu}\) in terms of \(g_{\nu\mu}\) and \(f_{\nu\mu}\)**

Here we invert the definitions (5.24) of \(g_{\nu\mu}\) and \(f_{\nu\mu}\) to obtain (5.59), the approximation of \(N_{\nu\mu}\) in terms of \(g_{\nu\mu}\) and \(f_{\nu\mu}\). First let us define the notation
\[ \hat{f}^\nu = f^\nu \sqrt{2} i \Lambda_b^{-1/2}. \]  
(F.1)

We assume that \(|\hat{f}^\nu| \ll 1\) for all components of the unitless field \(\hat{f}_{\nu}^\mu\), and find a solution in the form of a power series expansion in \(\hat{f}^\nu_{\nu}\). Lowering an index on the equation \((\sqrt{-N}/\sqrt{-g})N^{-\nu\mu} = g^\nu_{\mu} + \hat{f}^\nu_{\nu}\) from (5.24) gives
\[ \sqrt{-N}/\sqrt{-g} N^{-\nu\mu} = \delta^\nu_{\mu} - \hat{f}^\nu_{\mu}. \]  
(F.2)
Let us consider the tensor \( \hat{f}^{\mu \nu} \). Because \( g_{\nu \alpha} \) is symmetric and \( \hat{f}^{\mu \nu} \) is antisymmetric, it is clear that \( \hat{f}^{\mu \alpha} = 0 \). Also because \( f_{\nu \alpha} \hat{f}^{\sigma \mu} \) is symmetric it is clear that \( \hat{f}^{\nu \sigma} \hat{f}^{\rho \mu} \hat{f}^{\rho \nu} = 0 \). In matrix language therefore \( \text{tr}(\hat{f}) = 0 \), \( \text{tr}(\hat{f}^3) = 0 \), and in fact \( \text{tr}(\hat{f}^p) = 0 \) for any odd \( p \). Using the well known formula \( \text{det}(e^M) = \exp(\text{tr}(M)) \) and the power series \( \ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4 \ldots \) we then get \[ F.3 \]

\[
\ln(\text{det}(I - \hat{f})) = \text{tr}(\ln(I - \hat{f})) = -\frac{1}{2} \hat{f}^{\rho \sigma} \hat{f}_{\rho \sigma} + (\hat{f}^4) \ldots \]

Here the notation \( (\hat{f}^4) \) refers to terms like \( \hat{f}^{\nu \sigma} \hat{f}^{\rho \sigma} \hat{f}^{\rho \tau} \). Taking \( \ln(\text{det}) \) on both sides of \( F.2 \) using the result \( F.3 \) and the identities \( \text{det}(sM) = s^n \text{det}(M) \) and \( \text{det}(M^{-1}) = 1/\text{det}(M) \) gives

\[
\ln\left(\frac{\sqrt{-N}}{\sqrt{-g}}\right) = \frac{1}{(n-2)} \ln\left(\frac{N^{(n/2-1)}}{g^{(n/2-1)}}\right) = -\frac{1}{2(n-2)} \hat{f}^{\rho \sigma} \hat{f}_{\rho \sigma} + (\hat{f}^4) \ldots \]

Taking \( e^x \) on both sides of \( F.4 \) and using \( e^x = 1 + x + x^2/2 \ldots \) gives

\[
\frac{\sqrt{-N}}{\sqrt{-g}} = 1 - \frac{1}{2(n-2)} \hat{f}^{\rho \sigma} \hat{f}_{\rho \sigma} + (\hat{f}^4) \ldots \]

Using the power series \( (1-x)^{-1} = 1 + x + x^2 + x^3 \ldots \), or multiplying \( F.2 \) term by term, we can calculate the inverse of \( F.2 \) to get \( G.1 \)

\[
\frac{\sqrt{N}}{\sqrt{-g}} N^\nu_\mu = \delta^\nu_\mu + \hat{f}^{\nu \mu} + \hat{f}^{\nu \sigma} \hat{f}_{\sigma \mu} + \hat{f}^{\nu \rho} \hat{f}_{\rho \sigma} \hat{f}^{\sigma \mu} + (\hat{f}^4) \ldots \]

Here the notation \( (\hat{f}^4) \) refers to terms like \( \hat{f}_{\nu \alpha} \hat{f}^{\alpha \sigma} \hat{f}^{\rho \sigma} \hat{f}^{\rho \tau} \). Since \( \hat{f}_{\nu \alpha} \hat{f}^{\alpha \sigma} \hat{f}^{\rho \sigma} \hat{f}^{\rho \tau} \) is symmetric and \( \hat{f}_{\nu \alpha} \hat{f}^{\rho \sigma} \hat{f}^{\rho \tau} \) is antisymmetric, we obtain from \( F.7 \) the final result \( E.9 \).

**Appendix G. Derivation of the continuity equation and Lorentz force equation from the Klein-Gordon equation in curved space**

For the spin-0 case, instead of deriving the continuity equation \( E.9 \) from the divergence of Ampere’s law, it can also be derived from the Klein-Gordon equation. Using \( E.13 \) we get,

\[
0 = \frac{iQ}{2\hbar} \left[ \hat{\psi} \left( \text{one side of } \left( \text{Klein–Gordon equation} \right) \hat{\psi} \right) \right] \quad (G.1)
\]

\[
= \frac{iQ}{2m} \left[ \frac{\hbar^2}{\sqrt{-g}} D_{\mu} \sqrt{-g} D^{\mu} + m^2 - \hat{D}_{\mu} \sqrt{-g} D^{\mu} \right] \psi \quad (G.2)
\]

\[
= \frac{ihQ}{2m} \left[ \left( \hat{D}_{\mu} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \sqrt{-g} \right) D^{\mu} - \hat{D}_{\mu} \left( \sqrt{-g} \frac{\partial}{\partial x^{\mu}} \frac{1}{\sqrt{-g}} D^{\mu} \right) \right] \psi \quad (G.3)
\]

\[
= \frac{ihQ}{2m} \left[ \left( \frac{\partial}{\partial x^{\mu}} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \sqrt{-g} \right) D^{\mu} - \hat{D}_{\mu} \left( \sqrt{-g} \frac{\partial}{\partial x^{\mu}} \frac{1}{\sqrt{-g}} \right) \right] \psi \quad (G.4)
\]

\[
= \frac{ihQ}{2m} \left( \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\hat{\psi} \sqrt{-g} D^{\mu} \psi) - (\hat{\psi} \hat{D}_{\mu} \sqrt{-g} \psi) \frac{\partial}{\partial x^{\mu}} \frac{1}{\sqrt{-g}} \right) \quad (G.5)
\]

\[
= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{-g} \frac{ihQ}{2m} (\hat{\psi} D^{\mu} \psi - \hat{\psi} \hat{D}_{\mu} \psi)) \quad (G.6)
\]

\[
= j_{\mu}^{\text{\cdot}} \quad (G.7)
\]
Similarly, instead of deriving the Lorentz-force equation \[ \text{G.8} \] from the divergence of the Einstein equations, it can also be derived from the Klein-Gordon equation. Using \[ \text{G.9} \] we get,

\begin{equation}
0 = \left( \text{one side of conjugate Klein-Gordon equation} \right) \frac{D_\rho \bar{\psi}}{2} + \frac{\bar{\psi} D_\rho}{2} \left( \text{one side of Klein-Gordon equation} \right) \tag{G.8}
\end{equation}

\begin{equation}
= \bar{\psi} \left( D^\lambda \sqrt{-g} D_\lambda - \frac{h^2}{2m} \right) D_\rho \psi + \frac{\bar{\psi} D_\rho}{2m} \left( \frac{h^2}{\sqrt{-g}} D_\lambda \sqrt{-g} D^\lambda + m^2 \right) \psi \tag{G.9}
\end{equation}

\begin{equation}
= \frac{h^2}{2m} \left[ \frac{\partial( \bar{\psi} D^\lambda \sqrt{-g} )}{\partial x^\lambda} \frac{1}{\sqrt{-g}} D_\rho \psi + \frac{\bar{\psi} D_\rho}{\sqrt{-g}} \frac{1}{\partial x^\lambda} \partial( \sqrt{-g} D^\lambda \psi) \right. \\
\left. + \bar{\psi} D^\lambda \left( -iQ \hbar A_\lambda \right) D_\rho \psi + \bar{\psi} D_\rho \left( \frac{iQ}{\hbar} A_\lambda \right) D^\lambda \psi \right] \\
+ \frac{m}{2} \left( \bar{\psi} D_\rho \psi + \bar{\psi} D_\rho \psi \right) \tag{G.10}
\end{equation}

\begin{equation}
= \frac{h^2}{2m} \left[ \frac{\partial( \bar{\psi} D^\lambda \sqrt{-g} )}{\partial x^\lambda} \frac{1}{\sqrt{-g}} D_\rho \psi + \frac{\bar{\psi} D_\rho}{\sqrt{-g}} \frac{1}{\partial x^\lambda} \partial( \sqrt{-g} D^\lambda \psi) \right. \\
\left. - \bar{\psi} D^\lambda \left( iQ \hbar A_\lambda \right) \frac{\partial \psi}{\partial x^\rho} - \bar{\psi} D^\lambda \left( iQ \hbar A_\rho \right) \frac{\partial \psi}{\partial x^\lambda} + \bar{\psi} D^\lambda \left( -iQ \hbar A_\rho \right) D^\lambda \psi \right] \\
\left. - \frac{\partial \bar{\psi}}{\partial x^\rho} \left( \frac{iQ}{\hbar} A_\lambda \right) D^\lambda \psi + \left( \frac{iQ}{\hbar} A_\rho \right) \frac{\partial D^\lambda \psi}{\partial x^\lambda} - \bar{\psi} D^\lambda \left( -iQ \hbar A_\rho \right) D^\lambda \psi \right] \\
+ \frac{m}{2} \left( \frac{\partial( \bar{\psi} \psi) \right) \tag{G.11}
\end{equation}

\begin{equation}
= \frac{h^2}{2m} \left[ \frac{\partial( \bar{\psi} D^\lambda \sqrt{-g} )}{\partial x^\lambda} \frac{1}{\sqrt{-g}} D_\rho \psi + \frac{\bar{\psi} D_\rho}{\sqrt{-g}} \frac{1}{\partial x^\lambda} \partial( \sqrt{-g} D^\lambda \psi) \right. \\
\left. + \bar{\psi} D^\lambda \left( \frac{\partial}{\partial x^\lambda} \left( \frac{iQ}{\hbar} A_\rho \psi \right) - \frac{\partial}{\partial x^\rho} \left( \frac{iQ}{\hbar} A_\lambda \psi \right) - 2\frac{iQ}{\hbar} A_{(\rho,\lambda)} \psi \right) \right] \\
\left. + \left( \frac{\partial}{\partial x^\lambda} \left( -iQ \hbar A_\rho \psi \right) - \frac{\partial}{\partial x^\rho} \left( -iQ \hbar A_\lambda \psi \right) + 2\frac{iQ}{\hbar} A_{(\rho,\lambda)} \psi \right) D^\lambda \psi \right\} \\
+ \frac{m}{2} \left( \frac{\partial( \bar{\psi} \psi) \right) \tag{G.12}
\end{equation}

\begin{equation}
= \frac{h^2}{2m} \left[ \frac{\partial( \bar{\psi} D^\lambda \sqrt{-g} )}{\partial x^\lambda} \frac{1}{\sqrt{-g}} D_\rho \psi + \frac{\bar{\psi} D_\rho}{\sqrt{-g}} \frac{1}{\partial x^\lambda} \partial( \sqrt{-g} D^\lambda \psi) \right. \\
\left. - \frac{\partial \bar{\psi}}{\partial x^\rho} \left( \frac{iQ}{\hbar} A_\lambda \right) D^\lambda \psi + \left( \frac{iQ}{\hbar} A_\rho \right) \frac{\partial D^\lambda \psi}{\partial x^\lambda} - \bar{\psi} D^\lambda \left( -iQ \hbar A_\rho \right) D^\lambda \psi \right] \\
\left. + \frac{m}{2} \left( \frac{\partial( \bar{\psi} \psi) \right. \tag{G.13}
\end{equation}
A-renormalized Einstein-Schrödinger theory with spin-0 and spin-1/2 sources

\[ + \bar{\psi} \tilde{D}^\lambda \left( \frac{\partial (D_\rho \psi)}{\partial x^\lambda} - \frac{\partial (D_\lambda \psi)}{\partial x^\rho} \right) + \left( \frac{\partial (\bar{\psi} \tilde{D}_\rho)}{\partial x^\lambda} - \frac{\partial (\bar{\psi} \tilde{D}_\lambda)}{\partial x^\rho} \right) D^\lambda \psi \]

\[ + \frac{2iQ}{\hbar} \left( \bar{\psi} D^\lambda \psi - \bar{\psi} \tilde{D}^\lambda \psi \right) A_{[\rho, \lambda]} \]

\[ = \frac{\hbar^2}{2m} \left[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\lambda} \left( \sqrt{-g} (\bar{\psi} \tilde{D}^\lambda D_\rho \psi + \bar{\psi} \tilde{D}_\rho D^\lambda \psi) \right) + \bar{\psi} \tilde{D}_\rho \frac{\partial g^{\lambda\nu}}{\partial x^\rho} D_\nu \psi - \frac{\partial}{\partial x^\rho} \bar{\psi} \tilde{D}_\rho D^\lambda \psi \right] \]

\[ + \frac{m}{2} \frac{\partial (\bar{\psi} \psi)}{\partial x^\rho} + \frac{iQ}{m} \left( \bar{\psi} D^\lambda \psi - \bar{\psi} \tilde{D}^\lambda \psi \right) A_{[\rho, \lambda]} \]

\[ = \frac{1}{2m} \left[ \hbar^2 (\bar{\psi} \tilde{D}^\lambda D_\rho \psi + \bar{\psi} \tilde{D}_\rho D^\lambda \psi)_{\lambda} - (\hbar^2 \bar{\psi} \tilde{D}_\rho D^\lambda \psi - m^2 \bar{\psi} \psi)_\rho \right] \]

\[ + \frac{iQ}{m} \left( \bar{\psi} D^\lambda \psi - \bar{\psi} \tilde{D}^\lambda \psi \right) A_{[\rho, \lambda]} \]

\[ = T^\lambda_{\rho, \lambda} + 2j^\lambda A_{[\rho, \lambda]} \ldots \]

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