Surfaces that are covered by two pencils of circles

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Abstract

We list up to Möbius equivalence all possible degrees and embedding dimensions of real surfaces that are covered by at least two pencils of circles, together with the number of such pencils. In addition, we classify incidences between the contained circles, complex lines and isolated singularities. Such geometric characteristics are encoded in the Néron-Severi lattices of such surfaces and is of potential interest to geometric modelers and architects. As an application we confirm Blum’s conjecture in higher dimensional space and we address the Blaschke-Bol problem by classifying surfaces that are covered by hexagonal webs of circles. In particular, we find new examples of such webs that cannot be embedded in 3-dimensional space.

Keywords: families of curves, circles, Möbius geometry, real surfaces, del Pezzo surfaces, Néron-Severi lattices, root subsystems, hexagonal webs, Blum’s conjecture, Blaschke-Bol problem

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1 Introduction

Sir Christopher Wren discovered that a one-sheeted hyperboloid contains two lines through each point [42, 1669] and he used his discovery for an “engine designed for grinding hyperbolic lenses” [8, page 92]. We now know that a surface that is covered by two pencils of lines is either the plane, a hyperboloid of one sheet, or a hyperbolic paraboloid [13, Lecture 16]. In this article we consider surfaces that are covered by two analytic pencils of circles instead of lines. Such surfaces must be algebraic by [33] (see also [36, Theorem 4.1]). With surface we shall therefore mean an irreducible algebraic surface. We call a surface \(\lambda\)-circled if it contains no more and no less than \(\lambda\) circles through a general point and if the real points in the surface are not contained in a reducible curve. For example, the leftmost ring cyclide in Figure 1 is a 4-circled quartic surface and appears in physical models such as Twistor theory [28, Fig. 33.14]. A celestial surface is defined as a \(\lambda\)-circled surface such that \(\lambda \geq 2\). The name “celestial” is inspired by a model where planetary orbits are described by circles in 4-dimensional space [1]. Celestial surfaces are of interest to geometric modelers and architects [2, 19, 30–32]. In this section we introduce and motivate our main results, namely Theorem 1, Theorem 2, Theorem 3, Theorem 4 and its corollaries (see §6 for the proofs).

Figure 1: celestial surfaces in \(\mathbb{R}^3\) and their types.

Let \(\pi: S^n \to \mathbb{R}^n\) denote a stereographic projection from the \(n\)-dimensional unit-sphere \(S^n \subset \mathbb{R}^{n+1}\). Hipparchus of Nicaea (190–120 BCE) discovered that \(\pi\) sends circles to either circles or lines. We say that a surface \(Z \subset \mathbb{R}^n\) has type \((\lambda, d, n)\) if \(\pi^{-1}(Z) \subset S^n\) is a \(\lambda\)-circled surface of degree \(d\) that is not contained in a hyperplane section (see Figure 1). Such types are invariant under Möbius transformations of \(\mathbb{R}^n\). We assume without loss of generality that a celestial surface in \(\mathbb{R}^n\) is not contained in a hyperplane or hypersphere.
We know from [33] that if $\lambda \geq 2$, then $n < d \leq 8$ and either $\lambda \leq 10$ or $\lambda = \infty$.

**Theorem 1.** The type $(\lambda, d, n)$ of a celestial surface, is one of the following and each listed type is realized by some surface: $(2, 8, n)$ for $3 \leq n \leq 7$, $(3, 6, 5)$, $(3, 6, 4)$, $(2, 6, 5)$ $(2, 6, 4)$, $(\lambda, 4, 3)$ for $2 \leq \lambda \leq 6$, $(\infty, 4, 4)$ and $(\infty, 2, 2)$.

In Figure 2 we see projections into $\mathbb{R}^3$ of celestial surfaces in $\mathbb{R}^n$ for $n > 3$.

![Projected Celestial Surfaces](image)

**Figure 2:** Projected celestial surfaces are covered by ellipses.

We recovered a result from [18], namely that an $\infty$-circled surface is of type either $(\infty, 4, 4)$ or $(\infty, 2, 2)$. It follows from [36, Main Theorem 1.1] that a celestial surface of type $(2, 8, 3)$ is Möbius equivalent to either $\{a + b \in \mathbb{R}^3 \mid a \in A, \ b \in B\}$ or a stereographic projection $\pi(\{c \star d \in S^3 \mid a \in C, \ b \in D\})$, where $A, B \subset \mathbb{R}^3$ and $C, D \subset S^3$ are circles and $\star$ denotes the Hamiltonian product for the unit quaternions.

A surface of type $(\lambda, 4, 3)$ for some $\lambda > 0$ is called a *Darboux cyclide*. It follows from Remark 11 that this definition coincides with the definition in [32, Section 2]. A systematic overview of equations for Darboux cyclides can be found in [39]. The following corollary answers a question in [32, Section 5] and will be proven in the almost self-contained §4.

**Corollary 1.** A $\lambda$-circled surface in $\mathbb{R}^3$ such that $\lambda \geq 3$ is of degree at most four and Möbius equivalent to either a sphere or a Darboux cyclide.

The following corollary confirms the *Blum’s conjecture*, which was known for manifolds in $\mathbb{R}^3$ that are homeomorphic to a torus $S^1 \times S^1$ [38] and for Darboux cyclides [32, Remark 8].

**Corollary 2.** If a surface in $\mathbb{R}^n$ is $\lambda$-circled with $n \geq 2$, then either $\lambda = \infty$ or $\lambda \leq 6$. 

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In order to refine and prove Theorem 1 we propose to consider geometric aspects that happen at complex infinity. To uncover this hidden structure we define a real variety $X$ to be a complex variety together with an antiholomorphic involution $\sigma: X \to X$ (see [35, Section I.1] and [3, Introduction]) and we denote its real points by $X(\mathbb{R}) := \{ p \in X \mid \sigma(p) = p \}$. Such varieties can always be defined by polynomials with real coefficients [34, Section 6.1].

Points, curves, surfaces and projective spaces $\mathbb{P}^n$ are real algebraic varieties and maps between such varieties are compatible with the real structure $\sigma$ unless explicitly stated otherwise. Moreover, a variety is irreducible and a hypersurface or hyperplane section of a variety inherits the real structure unless explicitly stated otherwise. By default we assume that $X(\mathbb{R})$ for a surface $X$ is not contained in a reducible curve and that the real structure $\sigma: \mathbb{P}^n \to \mathbb{P}^n$ sends $x$ to $(x_0: \ldots: x_n)$.

As circles play a central role, it is natural to consider the Möbius quadric for our space: $S^n := \{ x \in \mathbb{P}^{n+1} \mid - x_0^2 + x_1^2 + \ldots + x_{n+1}^2 = 0 \}$. The Möbius transformations of $S^n$ are defined as biregular automorphisms $\text{Aut}(S^n)$ and are linear so that $\text{Aut}(S^n) \subset \text{Aut}(\mathbb{P}^{n+1})$. If $V \subset S^n$ is a variety, then we define $V(\mathbb{R}) := \gamma(V(\mathbb{R}))$ where $\gamma: S^n(\mathbb{R}) \to S^n$ is the isomorphism that sends $x$ to \((x_1/x_0, \ldots, x_{n+1}/x_0)\). Notice that $\pi^{-1}(\mathbb{R}^n)$ defines an isomorphic copy of $\mathbb{R}^n$ inside $S^n(\mathbb{R})$ such that the Möbius transformations of $S^n$ restrict via $\gamma$ and/or $\pi$ to Möbius transformations of $S^n$ and $\mathbb{R}^n$.

**Definition 1.** We call $C \subset S^n$ a circle if $C(\mathbb{R}) \subset S^n$ is a circle and a surface $X \subset S^n$ is $\lambda$-circled or celestial if $X(\mathbb{R}) \subset S^n$ as such. We say that $X \subset S^n$ is of type $(\lambda, d, n)$ if $X$ is a $\lambda$-circled surface of degree $d$ that is not contained in a hyperplane section. Notice that if $X \subset S^n$ is a celestial surface, then its stereographic projection $\pi(X(\mathbb{R}))$ is either a celestial surface in $\mathbb{R}^n$ or covered by lines. We call $X$ in this case the Möbius model of $\pi(X(\mathbb{R}))$. We call $X \subset S^3$ a Darboux cyclide if it is the Möbius model of a Darboux cyclide in $\mathbb{R}^3$. A complex circle is an irreducible complex conic $C \subset S^n$.

An irreducible hypersurface $F \subset X \times \mathbb{P}^1$ is called a pencil on a surface $X$ if $\pi_1(F) = X$ and $\pi_2(F) = \mathbb{P}^1$ for the projections of $F$ to its two factors. We require that a member $F_i := \pi_1(F \cap X \times \{i\})$ is a curve for almost all points $i \in$
and two pencils are equal if their members are the same. The common complex points in the intersection $\cap_{i \in \mathbb{P}^1} F_i$ are called base points. Notice that a point is real by convention, but that a “base point” is complex. A pencil of circles is defined as a pencil whose general member is a circle.

The circle graph of a $\lambda$-circled surface $Z \subset \mathbb{R}^n$ such that $\lambda < \infty$ is defined as a labeled graph whose vertices correspond to the pencils of circles that cover its Möbius model $X \subset \mathbb{S}^n$. Each vertex is labeled with either $-$, $+$, $\times$ or no label, if the pencil has two complex conjugate base points, one real base point, two real base points and no base points, respectively. Two vertices are connected by a solid or dashed labeled edge if general circles in the respective pencils intersect in two complex or real points. We dash the edge if and only if at least one of the two intersection points coincides with a common base point of the two pencils.

**Example 1.** Below we colored the vertices of the circle graph of the ring cyclide to match the corresponding pencils of circles:

![Diagram of a ring cyclide with colored vertices and edges]

The two pencils of cospherical circles that cover the ring cyclide, are called Villarceau circles [40, 1848]. These circles can be found in a sculpture of a staircase in the Strasbourg cathedral, which was built from 1176 until 1439 [4, Fig. II.7.7]. The pencil of horizontal brown circles are parallel and thus all the circles in this pencil meet at complex conjugate base points at infinity. The blue circles in the remaining pencil have complex conjugate base points at the axis of revolution.

**Definition 2.** Suppose that $\mathcal{W}$ is a set of curves in a surface $Z$. We define $\mathcal{W}_p := \{C \in \mathcal{W} \mid p \in C\}$ for all $p \in Z$ and let

$$V(\mathcal{W}) := \{v \in Z \mid v \text{ is a point such that } |\mathcal{W}_v| = 3\}.$$ 

We call $\mathcal{W}$ a 3-web if $V(\mathcal{W})$ is not contained in some reducible curve. We define $G(\mathcal{W})$ to be the graph with vertex set $V(\mathcal{W})$ and labeled edge set

$$\{\{(v, w), C\} \mid C \in \mathcal{W} \text{ and } v, w \in C \text{ are pairwise distinct}\}.$$
We call a 3-web $\mathcal{W}$ a hexagonal web if a general edge $\{p, q\}$ of the graph $G(\mathcal{W})$ is contained in a subgraph as defined in Figure 3, where the edge-labels $A, B, C, D, E, F, G, H, I \in \mathcal{W}$ are pairwise distinct.

![Figure 3](image1.png)

**Figure 3:** See Definition 2.

**Remark 1.** In order to understand Definition 2 we suppose that $\mathcal{W}$ is a hexagonal web on a surface and that $\{p, q\}$ in Figure 4a is a general edge of $G(\mathcal{W})$. In Figure 4b we draw all the curves in $\mathcal{W}_p \cup \mathcal{W}_q$ and we obtain at least two new intersection points $r$ and $s$. In Figure 4c we draw all the curves in $\mathcal{W}_r \cup \mathcal{W}_s$ and we obtain again at least two new intersection points. We repeat the last step one more time so that we obtain a closed hexagon as in Figure 4d. Figure 4e is an example for the case that $\mathcal{W}$ is not a hexagonal web. We refer to [26] for more information.

![Figure 4](image2.png)

**Figure 4:** See Remark 1.

**Remark 2.** Discrete realizations of hexagonal webs lead to nice triangularizations of the underlying surface. We translate [32, Theorem 18] using the concept of circle graphs: If $Z \subset \mathbb{R}^3$ is a Darboux cyclide, then three vertices of its circle graph form a hexagonal web if and only if the vertices are not contained in the following labeled subgraph of the circle graph: $\circlearrowleft \circlearrowright$. Notice that each vertex in the latter subgraph corresponds to a base point free pencil.
Example 2. Up to symmetries of the circle graph, there are two hexagonal webs of circles on a ring cyclide:

We colored the three vertices corresponding to the web.

The following result addresses the Blaschke-Bol problem, which refers to the classification problem for hexagonal webs of circles [5, §3, Aufgabe 1, page 31].

Theorem 2. If $Z \subset \mathbb{R}^n$ is a $\lambda$-circled surface such that $\lambda \geq 3$, then $Z$ is covered by a hexagonal web of circles.

See Figure 2 for linear projections of hexagonal webs of circles on celestial surfaces of types $(\infty, 4, 4)$ and $(3, 6, 5)$.

We will now introduce some concepts from algebraic geometry such as “smooth model” and “Néron-Severi lattice”. Although these concepts are well-known to algebraic geometers, we would like to convince also non-experts of its usefulness. For example, we will see that the Néron-Severi lattice of a celestial surface encodes its circle graph.

The smooth model of a surface $X \subset \mathbb{P}^n$ is a birational morphism $\varphi: Y \to X$ from a nonsingular surface $Y$, that does not contract complex $(-1)$-curves. See [17, Theorem 2.16] for the existence and uniqueness of the smooth model.

Suppose that $X \subset \mathbb{S}^n$ is a surface with smooth model $Y \to X$ such that $Y$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in either zero, two or four sufficiently general complex points that are left invariant as a set by the real structure $\sigma$. With sufficiently general is meant that at most two centers of blowup are contained in a complex fiber of a projection of $\mathbb{P}^1 \times \mathbb{P}^1$ to its first or second factor. Under this assumption, $Y$ is an example of a weak del Pezzo surface (see Remark 6). Let $V$ denote the vector space of all forms of bidegree $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. An anticanonical model of $Y$ is the image of a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X_N \subset \mathbb{P}^m$, whose components form a basis for the $(m + 1)$-dimensional subspace of $V$ defined by the forms that vanish at the centers of
blowup. If $X$ is a degree preserving linear projection of $X_N$, then we show in §3 how this point of view can be translated into an algorithm for constructing parametrizations and thus visualizations of celestial surfaces.

**Remark 3.** If the real structure $\sigma: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ maps $(x_0 : x_1 ; x_2 : x_3)$ to either $(\overline{x_0} : \overline{x_1} ; \overline{x_2} : \overline{x_3})$ or $(\overline{x_2} : \overline{x_3} ; \overline{x_0} : \overline{x_1})$, then $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^2$, respectively.◆

We call $X \subset \mathbb{S}^n$ a Veronese surface if it is isomorphic to the image of the biregular isomorphism $\mathbb{P}^2 \to X_N \subset \mathbb{P}^5$, whose components form a basis for the vector space of quadratic forms on $\mathbb{P}^2$.

**Theorem 3.** If $X \subset \mathbb{S}^n$ is celestial surface of type $(\lambda, d, n)$ with $n \geq 3$ and smooth model $Y \to X$, then $d$, $\lambda$ and $Y$ are characterized by one of the rows in Table 1, where the centers of blowup are in sufficiently general position. If $\lambda < \infty$, then $X$ is either an anticanonical model of $Y$ or a degree preserving linear projection of this anticanonical model. If $\lambda = \infty$, then $X$ is a Veronese surface.

**Table 1:** See Theorem 3.

| $d$ | $\lambda$ | smooth model |
|-----|-----------|--------------|
| 8   | 2         | $\mathbb{S}^1 \times \mathbb{S}^1$ |
| 6   | $\leq 3$  | $\mathbb{S}^1 \times \mathbb{S}^1$ blown up in a pair of complex conjugate points |
| 4   | $\infty$  | $\mathbb{P}^2$ |
| 4   | $\leq 6$  | $\mathbb{S}^1 \times \mathbb{S}^1$ blown up in two pairs of complex conjugate points |
| 4   | 2         | $\mathbb{S}^2$ blown up in two pairs of complex conjugate points |
| 4   | 2         | $X$ itself and $X(\mathbb{R})$ consist of two disjoint spheres |

We remark that if $X(\mathbb{R})$ is not connected, then $X$ is not real birational to the plane $\mathbb{P}^2$ [35, VI.6.5].

**Definition 3** (names). If $X \subset \mathbb{S}^3$ is of type $(4, 4, 3)$, $(5, 4, 3)$ or $(6, 4, 3)$ and $X(\mathbb{R})$ is smooth, then $X$ is called a ring cyclide, Perseus cyclide and Blum cyclide, respectively (see Figure 1). The latter two names have been introduced in [6] and [11], respectively. We call $X \subset \mathbb{S}^3$ a $S1$ cyclide or $S2$ cyclide if $X(\mathbb{R})$ is homeomorphic to $S^2$ and the disjoint union $S^2 \cup S^2$, respectively (see [32,
We use the following mnemonics for names of quadric surfaces whose equations are up to Euclidean similarity as in Table 11:

- E = elliptic/ellipsoid
- P = parabolic/paraboloid
- O = cone
- C = circular
- H = hyperbolic/hyperboloid
- Y = cylinder

For example, we call $X \subset S^3$ a CH1 cyclide if it is the Möbius model of a Circular Hyperboloid of 1 sheet. The CO cyclide and CY cyclide are also known as spindle cyclide and horn cyclide, respectively (see Figure 5). We call a celestial of type $(2,8,n)$, $(3,6,n)$ or $(2,6,n)$ for some $n > 0$, $dS$, $dP6$ and $wdP6$, respectively (see Figure 1 and Figure 2). If the Möbius model of a surface $Z \subset \mathbb{R}^3$ is a CH1 cyclide, then we call $Z$ also a CH1 cyclide. Similarly, for the other names. We remark that “wdP” stands for “weak del Pezzo surface” (see Remark 6) and “dS” stands for “double Segre surface” [10, 8.4.1].

![Figure 5: Blum cyclide, CO cyclide, CY cyclide and CH1 cyclide.](image)

Suppose that $\varphi : Y \to X$ is the smooth model of a surface $X \subset \mathbb{P}^n$.

The *Néron-Severi lattice* $N(X)$ is an additive group defined by the divisor classes on $Y$ up to numerical equivalence. This group comes with an unimodular intersection product $\cdot$ and a unimodular involution $\sigma_* : N(X) \to N(X)$ induced by the real structure $\sigma : X \to X$. We denote by Aut $N(X)$ the group automorphisms that are compatible with both $\cdot$ and $\sigma_*$. The *class* $[C] \in N(X)$ of a complex curve $C \subset X$ is defined as the divisor class of the one-dimensional part of its complex preimage $\varphi^{-1}(C)$ minus the components that are contracted by the smooth model $\varphi : Y \to X$. We consider the following subsets of $N(X)$.

- $B(X)$ denotes the set of classes of irreducible complex curves $C \subset Y$ such that $C$ is contracted by $\varphi$ to a complex point in $X$, and
- $G(X)$ denotes the set of classes of complex irreducible conics in $X$ that are not components of the singular locus of $X$. 


We call \( W \subset B(X) \) a component if it defines a maximal connected subgraph of the graph with vertex set \( B(X) \) and edge set \( \{(a,b) \mid a \cdot b > 0\} \). We write \( c \cdot W \succ 0 \) for \( c \in N(X) \) and \( W \subset N(X) \), if there exists \( w \in W \) such that \( c \cdot w > 0 \). Similarly, we define \( c \cdot W \not\succ 0 \), if there does not exists \( w \in W \) such that \( c \cdot w < 0 \).

**Proposition 1.** Suppose that \( X \subset \mathbb{S}^n \) is a celestial surface that is not \( \infty \)-circled.

a) General circles \( C, C' \subset X \) are members of the same pencil of circles if and only if \( [C] = [C'] \), \( [C] \in G(X) \) and \( \sigma_*[C] = [C] \).

b) General circles \( C, C' \subset X \) intersect in two complex points if and only if there exists \( 2 - [C] \cdot [C'] \) components \( W \subset B(X) \) such that \( [C] \cdot W \succ 0 \) and \( [C'] \cdot W \succ 0 \).

c) The base points of a pencil of circles with member \( C \subset X \) are in one-to-one correspondence to the set of components \( W \subset B(X) \) such that \( [C] \cdot W \succ 0 \). The base point is real if and only if \( \sigma_* (W) = W \).

We will consider Néron-Severi lattices that are generated by \( \langle \ell_0, \ell_1, \varepsilon_1, \ldots, \varepsilon_r \rangle \) for some \( r \geq 0 \) such that the nonzero intersections between its generators are \( \ell_0 \cdot \ell_1 = 1 \) and \( \varepsilon_1^2 = \ldots = \varepsilon_r^2 = -1 \). We define explicit coordinates for five different unimodular involutions \( \sigma_* \) that act on such lattices (see below for \( g_3 \)):

\[
A_0 : \quad r = 0, \quad \sigma_*(\ell_0) = \ell_0, \quad \sigma_*(\ell_1) = \ell_1,
\]

\[
A_1 : \quad r = 2, \quad \sigma_*(\ell_0) = \ell_0, \quad \sigma_*(\ell_1) = \ell_1, \quad \sigma_*(\varepsilon_1) = \varepsilon_2,
\]

\[
2A_1 : \quad r = 4, \quad \sigma_*(\ell_0) = \ell_0, \quad \sigma_*(\ell_1) = \ell_1, \quad \sigma_*(\varepsilon_1) = \varepsilon_2, \quad \sigma_*(\varepsilon_3) = \varepsilon_4,
\]

\[
3A_1 : \quad r = 4, \quad \sigma_*(\ell_0) = \ell_1, \quad \sigma_*(\varepsilon_1) = \varepsilon_2, \quad \sigma_*(\varepsilon_3) = \varepsilon_4,
\]

\[
D_4 : \quad r = 4, \quad \sigma_*(\ell_0) = g_3, \quad \sigma_*(\ell_1) = \ell_1, \quad \sigma_*(\varepsilon_i) = \ell_1 - \varepsilon_i \text{ for } 1 \leq i \leq 4.
\]

**Remark 4.** The names \( A_1, 2A_1, 3A_1 \) and \( D_4 \) correspond to the Dynkin types of root subsystems associated to \( \sigma_* \) and these types are invariant under \( \text{Aut } N(X) \). See [41] or the proof of Lemma 9 for details.

We use the following shorthand notation for elements in \( B(X) \) and \( G(X) \):

\[
b_1 := \varepsilon_1 - \varepsilon_3, \quad b_{ij} := \ell_0 - \varepsilon_i - \varepsilon_j \quad b_0 := \ell_0 + \ell_1 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4,
\]

\[
b_2 := \varepsilon_2 - \varepsilon_4, \quad b'_{ij} := \ell_1 - \varepsilon_i - \varepsilon_j.
\]
Example 3. If \( X \subset S^3 \) is a ring cyclide with smooth model \( Y \to X \), then \( Y \) is isomorphic to \( S^1 \times S^1 \) blown up in two pairs of complex conjugate points and \( \sigma_* \) is up to \( \text{Aut} \, N(X) \) equal to \( 2A_1 \). Moreover, \( B(X) = \{b_{13}, b_{24}, b_{14}'b_{23}'\} \) and \( G(X) = \{g_0, g_1, g_{12}, g_{34}\} \). Notice that the vertices of the circle graph in Example 1 correspond to the elements of \( G(X) \). It follows from Proposition 1 that \( g_0 \) is the class of a circle in a pencil that has to two complex base points corresponding to the components \( \{b_{14}'\} \) and \( \{b_{23}'\} \) of \( B(X) \). The classes of the Villarceau circles are \( g_{12} \) and \( g_{34} \).

If \( c \in N(X) \) and \( \Psi \subset N(X) \), then we write \( c \sim \Psi \) if \( c \) is up to permutation of the generators \( \varepsilon_1, \ldots, \varepsilon_4 \) and up to switching generators \( \ell_0 \) and \( \ell_1 \), equal to an element in \( \Psi \). For example, \( g_{34} \sim \{g_{12}\}, g_0 \sim \{g_1\} \) and \( g_2 \sim \{g_3\} \).

Theorem 4. If \( X \subset S^n \) is celestial surface of type \( (\lambda, d, n) \) such that \( \lambda < \infty \), then \( G(X) = \{c \in N(X) \mid c \sim \{g_0, g_2, g_{12}\}, c \cdot B(X) \neq 0\} \). Moreover, \( d, \lambda, \sigma_* \), \( B(X) \) and the name of \( X \) correspond up to \( \text{Aut} \, N(X) \) to exactly one row in Table 2.

| \( d \) | \( \lambda \) | \( \sigma_* \) | \( B(X) \) | name | \( d \) | \( \lambda \) | \( \sigma_* \) | \( B(X) \) | name |
|--------|--------|--------|--------|------|--------|--------|--------|--------|------|
| 8      | 2      | 0      | \( A_0 \) | dS    | 4      | 3      | 2A_1 | \( \{b_1, b_2, b_{12}\} \) | EY   |
| 6      | 3      | 0      | \( A_1 \) | dP6   | 4      | 2      | 2A_1 | \( \{b_1, b_2, b_{12}, b_{13}', b_{24}'\} \) | CY   |
| 6      | 2      | \( \{b_{12}\} \) | wdP6 | 4      | 3      | 2A_1 | \( \{b_{12}, b_{34}\} \) | EO   |
| 4      | 6      | \( 2A_1 \) | \( \emptyset \) | Blum | 4      | 2      | 2A_1 | \( \{b_{12}, b_{34}, b_{13}', b_{24}'\} \) | CO   |
| 4      | 5      | \( 2A_1 \) | \( \{b_1, b_2\} \) | Perseus | 4      | 2      | 3A_1 | \( \{b_0\} \) | EE/EH2 |
| 4      | 4      | \( 2A_1 \) | \( \{b_{13}, b_{24}, b_{14}', b_{23}'\} \) | ring | 4      | 2      | 3A_1 | \( \{b_{13}, b_{24}'\} \) | EP   |
| 4      | 4      | \( 2A_1 \) | \( \{b_{12}\} \) | EH1   | 4      | 2      | 3A_1 | \( \emptyset \) | S1   |
| 4      | 3      | \( 2A_1 \) | \( \{b_{13}, b_{24}, b_{12}\} \) | CH1   | 4      | 2      | \( D_4 \) | \( \emptyset \) | S2   |
| 4      | 2      | \( 2A_1 \) | \( \{b_{12}, b_{34}'\} \) | HP    | 4      | 2      | \( \emptyset \) |  |

Corollary 3 below generalizes [9, Theorem 20, page 296] to \( n > 3 \): if the circle graph of \( X \subset S^n \) contains an edge, then \( X \) must be a Darboux cyclide. Recall from Proposition 1 that the circle graph of \( X \) is encoded by \( B(X) \) and \( G(X) \).
Corollary 3. If a celestial surface is not $\infty$-circled, then its circle graph is in Table 3.

Table 3: See Corollary 3.

| dS | dP6 | wdP6 | Blum | Perseus | ring | S1/S2 |
|----|-----|------|------|---------|------|-------|
| +  | −   | +    | +    | ×       | ×    |       |
| −  |     | +    | +    | −       | −    |       |
|     |     |      |      |         |      |       |
| EH1 | CH1 | HP   | EY   | CY      | EO   | CO    | EE/EH2/EP |

Notice that if $X \subset \mathbb{S}^n$ is covered by a pencil of circles with a real base point, then the stereographic projection $\pi(X(\mathbb{R}))$ from this base point is a surface in $\mathbb{R}^n$ that is covered by lines. Corollary 4 below together with Theorem 3 and §3 addresses [36, Problem 5.6]. See also [27, Theorem 1.1] for $n \leq 3$.

Corollary 4. A surface of degree $\delta$ in $\mathbb{R}^n$ for $n \geq 3$ that is not contained in a hyperplane section, and that contains $c \geq 1$ circles and $\ell \geq 1$ lines through a general point, is characterized by a row in Table 4.

Table 4: See Corollary 4.

| $n$ | $\delta$ | $c$ | $\ell$ | type | name |
|-----|----------|-----|--------|------|------|
| 3   | 2        | 2   | 2      | (4, 4, 3) | EH1 |
| 3   | 2        | 1   | 2      | (3, 4, 3) | CH1 |
| 3   | 2        | 2   | 1      | (3, 4, 3) | EO or EY |
| 3   | 2        | 1   | 1      | (2, 4, 3) | CO or CY |
| 4   | 3        | $\infty$ | 1    | ($\infty$, 4, 4) |      |
| 4   | 4        | 1   | 1      | (2, 6, 4) |      |
| 5   | 4        | 1   | 1      | (2, 6, 5) |      |

If $X \subset \mathbb{S}^n$ is a celestial surface, then, by Lemma 1, $N(X)$ encodes beside the circle graph also partial information about complex lines in $X$ and the singular locus $\text{sng} X$. In addition to $B(X)$ and $G(X)$ we let $E(X)$ denote the classes of complex lines in $X$ and we use the following notation for its elements:

$$e_i = \varepsilon_i, \quad e_{ij} = \ell_i - \varepsilon_j, \quad e'_i = b_0 + \varepsilon_i.$$
A component $W \subset B(X)$ in Theorem 4 defines a Dynkin graph of type either $A_1$, $A_2$ or $A_3$ and corresponds by Lemma 3 to an isolated double point that is a node, cusp or tacnode, respectively. We underline if the isolated singularity is real (for example $\underline{A_1}$) and take formal sums to denote the disjoint union of singularities (for example $2\underline{A_1} + \underline{A_1}$).

**Corollary 5.** If $X \subset \mathbb{S}^n$ is celestial surface of type $(\lambda, d, n)$ such that $n = d - 1$, then $E(X) = \{ c \in N(X) \mid c \sim \{e_1, e_{01}, e'_1\}, \; c \cdot B(X) \neq 0 \}$ and $\lambda$, $\sigma_*$, $\text{sng} \, X$, $|E(X)|$ and the name of $X$ correspond up to $\text{Aut} \, N(X)$ to a row in Table 5.

**Table 5:** See Corollary 5.

| $\lambda$ | $\sigma_*$ | sng $X$ | $|E(X)|$ | name   | $\lambda$ | $\sigma_*$ | sng $X$ | $|E(X)|$ | name   |
|-----------|------------|--------|----------|--------|-----------|------------|--------|----------|--------|
| 2         | $A_0$      |        | 0        | dS     | 3         | $2\underline{A_1}$ | $A_3$ | 4        | EY     |
| 3         | $A_1$      |        | 6        | dP6    | 2         | $2\underline{A_1}$ + $2A_1$ | 2    | CY      |
| 2         | $A_1$      | $\underline{A_1}$ | 3        | wdP6   | 3         | $2\underline{A_1}$ + $2A_1$ | 8    | EO      |
| 6         | $2\underline{A_1}$ |        | 16       | Blum   | 2         | $2\underline{A_1}$ + $2A_1$ | 4    | CO      |
| 5         | $2A_1$     | $2A_1$ | 8        | Perseus | 2         | $3\underline{A_1}$ | $A_1$ | 12       | EE/EH2 |
| 4         | $2A_1$     | $4A_1$ | 4        | ring   | 2         | $3\underline{A_1}$ | $A_2$ | 8        | EP     |
| 4         | $2A_1$     | $\underline{A_1}$ | 12       | EH1    | 2         | $3\underline{A_1}$ | $\emptyset$ | 16       | S1     |
| 3         | $2A_1$     | $A_1 + 2A_1$ | 6        | CH1    | 2         | $\underline{D_4}$ | $\emptyset$ | 16       | S2     |
| 2         | $2A_1$     | $\underline{A_2}$ | 8        | HP     |           |            |        |          |        |

**Example 4.** We continue with Example 3, where $X \subset \mathbb{S}^3$ is a ring cyclide and $B(X) = \{b_{13}, b_{24}, b'_{14}, b'_{23}\}$ so that $E(X) = \{e_1, e_2, e_3, e_4\}$. The complex line with class $e_1$ intersects the complex line with class $e_3$ at the complex isolated singularity corresponding to the component $\{b_{13}\}$, since $e_1 \cdot b_{13}, e_3 \cdot b_{12} > 0$.

We find that $X$ contains two pairs of complex conjugate lines that intersect in two pairs of complex conjugate nodes.

For convenience of the reader we computed $E(X)$ and $G(X)$ from $B(X)$ in Theorem 4 for all celestial Darboux cyclides.

**Corollary 6.** If $X \subset \mathbb{S}^3$ is a celestial Darboux cyclide, then its name together with $B(X)$, $E(X)$ and $G(X)$ are up to $\text{Aut} \, N(X)$ defined by a row in Table 6.
The dashed row dividers indicate that $\sigma_*$ is $2A_1$, $3A_1$ and $D_4$, respectively.

Table 6: See Corollary 6. A class is send by the unimodular involution $\sigma_*$ to itself if underlined and to its left or right neighbor in the listing otherwise.

| name         | $B(X)$, $E(X), G(X)$                                                                 |
|--------------|--------------------------------------------------------------------------------------|
| Blum         | $\{\}, \{e_1, e_2, e_3, e_4, e_0, e_2, e_0, e_3, e_4, e_1, e_2, e_3, e_4, e_1\}$,    |
|              | $\{g_0, g_1, g_{12}, g_{34}, g_2, g_3, g_{13}, g_{24}, g_{14}, g_{23}\}$              |
| Perseus      | $\{b_1, b_2\}, \{e_1, e_2, e_3, e_4, e_0, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{12}, g_2, g_3, g_{13}, g_{24}\}$ |
| ring         | $\{b_{13}, b_{24}, b_{14}, b_{23}\}, \{e_1, e_2, e_3, e_4, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{34}\}$ |
| EH1          | $\{b_{12}\}, \{e_1, e_2, e_3, e_4, e_0, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{34}, g_2, g_3, g_{13}, g_{24}\}$ |
| CH1          | $\{b_{13}, b_{24}, b_{12}\}, \{e_1, e_2, e_3, e_4, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{34}, g_{14}, g_{23}\}$ |
| HP           | $\{b_{12}, b_{34}\}, \{e_1, e_2, e_3, e_4, e_0, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{13}, g_{24}, g_{14}, g_{23}\}$ |
| EY           | $\{b_{23}, b_{1}, b_{2}\}, \{e_3, e_4, e_1, e_2, e_3, e_4\}, \{g_0, g_1, g_{34}, g_2, g_3, g_{13}, g_{24}\}$ |
| CY           | $\{b_{12}, b_{1}, b_2, b_{13}, b_{24}\}, \{e_3, e_4\}, \{g_0, g_1\}$                |
| EO           | $\{b_{12}, b_{34}\}, \{e_1, e_2, e_3, e_4, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{34}, g_2, g_3, g_{13}, g_{24}, g_{14}, g_{23}\}$ |
| CO           | $\{b_{23}, b_{13}, b_{24}\}, \{e_1, e_2, e_3, e_4\}, \{g_0, g_1, g_{14}, g_{23}\}$ |
| EE/EH2       | $\{b_0\}, \{e_1, e_2, e_3, e_4, e_0, e_1, e_2, e_3, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{12}, g_{34}, g_2, g_3, g_{24}, g_{14}, g_{23}\}$ |
| EP           | $\{b_{13}, b_{24}\}, \{e_1, e_2, e_3, e_4, e_0, e_1, e_2, e_3, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{12}, g_{34}, g_{14}, g_{23}\}$ |
| S1           | $\{\}, \{e_1, e_2, e_3, e_4, e_0, e_1, e_2, e_3, e_1, e_2, e_3, e_1\}, \{g_0, g_1, g_{12}, g_{34}, g_{13}, g_{24}, g_{23}, g_{24}\}$ |
| S2           | $\{\}, \{e_1, e_2, e_3, e_4, e_0, e_1, e_2, e_3, e_1, e_2, e_3, e_1\}, \{g_0, g_{3}, g_{12}, g_{34}, g_{13}, g_{24}, g_{14}, g_{23}, g_{24}, g_{11}, g_{2}\}$ |

2 Smooth models

In Proposition 2 we will characterize the possible types $(\lambda, d, n)$ and smooth models of a celestial surface $X \subset \mathbb{S}^n$. For this purpose, we start by collecting known results from the literature and we prove Proposition 1.

Let $X \subset \mathbb{P}^n$ be a surface with smooth model $\varphi: Y \to X$. Its linear normalization $X_N \subset \mathbb{P}^m$ is defined as the image of $Y$ via the map $\varphi_h$ associated to the linear equivalence class $h$ of the pullback to $Y$ of a hyperplane section of $X$.
Remark 5. The associated map \( \varphi_h : Y \to X_N \subset \mathbb{P}^m \) is compatible with the real structures and \( X_N \) is unique up to \( \text{Aut}(\mathbb{P}^m) \) as a direct consequence of the definitions (see [16, Remark II.7.8.1] and [20, Section 1.1.B]). We have that \( m \geq n \) and there exists a degree-preserving linear projection \( \eta : \mathbb{P}^m \to \mathbb{P}^n \) such that \( \eta(X_N) = X \) (see [33, Theorem 6]).

Definition 4. A surface \( Y_{r+1} \) is the **blowup of \( Y_1 \) in \( r \) complex points** if there exists a birational morphism \( \tau : Y_{r+1} \to Y_1 \) and complex blowups \( \tau_i : Y_{i+1} \to Y_i \) of complex points \( p_i \in Y_i \) such that \( \tau = \tau_1 \circ \cdots \circ \tau_r \) (see [16, Example I.4.9.1]). We refer to \( p_i \) as a **center of blowup**. If \( p_{i+1} \) lies on the complex \((-1)\)-curve that is contracted by \( \tau_i \), then we say that \( p_{i+1} \) is **infinitely near** to \( p_i \). If \( C \subset Y_1 \) is a complex curve, then its **strict transform** \( C_i \subset Y_i \) for some \( i \) is defined as the Zariski closure of the preimage \( (\tau_1 \circ \cdots \circ \tau_{i-1})^{-1}(C) \) minus the complex curve components that are contracted by \( \tau_1 \circ \cdots \circ \tau_{i-1} \).

Remark 6. Suppose that \( X \subset \mathbb{P}^n \) is a surface with smooth model \( Y \to X \) such that \( Y \) is complex isomorphic to the blown up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) in \( 0 \leq r \leq 7 \) sufficiently general points. Notice that if \( r > 0 \), then \( Y \) is complex isomorphic to a blowup of \( \mathbb{P}^2 \) in \( r+1 \) complex points. Let \( k \in N(X) \) denote the canonical class of \( Y \) (see [16, Example V.1.4.4]). If \(-k\) is the class of a hyperplane section of \( X \), then \( Y \) is a **weak del Pezzo surface** and we call its linear normalization \( X_N \) an **anticanonical model** of \( Y \) (see [10, Definition 8.1.18]).

Theorem A (Schicho, 2001). **Suppose that \( X \subset \mathbb{P}^n \) is a surface of degree \( d \) that contains \( \lambda \geq 2 \) complex conics and no complex line through a general complex point.** Let \( Y \to X \) be its smooth model and let \( k \) denote its canonical class.

- *If \( \lambda < \infty \), then \( 3 \leq d \leq 8 \), \( Y \) is complex isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up in \( 8-d \) sufficiently general points, \( X_N \subset \mathbb{P}^d \) and the class of a hyperplane section is equal to \(-k\).*

- *If \( \lambda = \infty \), then \( d = 4 \), \( Y \) is isomorphic to \( \mathbb{P}^2 \), \( X_N \subset \mathbb{P}^5 \) and the class of a hyperplane section is equal to \(-\frac{2}{3}k\).*

**Proof.** Direct consequence of [33, Theorems 5–8 and Proposition 1] and Remark 6. Notice that \(-\frac{1}{3}k\) is the class of a line in \( \mathbb{P}^2 \) [16, Example II.8.20.3].
Lemma 1. Suppose that $X \subset \mathbb{P}^n$ is a surface with smooth model $Y \to X$ such that $Y$ is complex isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in $r \in \{0, 2, 4\}$ sufficiently general complex points. Let $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the first and second projection.

a) $N(X)$ is generated by $\langle \ell_0, \ell_1, \varepsilon_1, \ldots, \varepsilon_r \rangle_\mathbb{Z}$ such that
- $\ell_0$ and $\ell_1$ are the pullbacks of the classes of the general fibers of $\pi_1$ and $\pi_2$, respectively,
- $\varepsilon_1, \ldots, \varepsilon_r$ are the pullbacks of the classes of complex $(−1)$-curves that contract to the centers $p_1, \ldots, p_r$ of blowup, and
- the canonical class $k$ of $X$ is equal to $−2\ell_0 − 2\ell_1 + \varepsilon_1 + \ldots + \varepsilon_r$.

b) $b \in B(X)$ if and only if there exists $1 \leq i < j \leq 4$ such that either
- $b = b_{ij}$ so that $p_i$ and $p_j$ lie on strict transforms of a complex fiber of $\pi_1$,
- $b = b'_{ij}$ so that $p_i$ and $p_j$ lie on strict transforms of a complex fiber of $\pi_2$,
- $b = b_0$ so that $p_1, \ldots, p_4$ lie on strict transforms of a complex curve of bidegree $(1, 1)$, or
- $b = \varepsilon_i − \varepsilon_j$ so that $p_j$ is infinitely near to $p_i$.

c) $G(X) = \{ c \in N(X) \mid c \sim \{g_0, g_2, g_{12}\}, c \cdot B(X) \neq 0 \}$.

d) $E(X_N) = \{ c \in N(X) \mid c \sim \{e_1, e'_1, e_{01}\}, c \cdot B(X) \neq 0 \} \subset E(X)$.

Proof. Assertion a) follows from [16, Proposition V.2.3 and Proposition V.3.3]. The classes in $B(X)$ and $E(X)$ are classes of complex $(−2)$-curves and complex $(−1)$-curves, respectively (see [10, 8.2.7 and 8.2.6]). Assertions b), c) and d) follow from [23, Proposition 1, Lemma 1 and Proposition 2] after converting to a different set of generators for $N(X)$. See alternatively [10, Proposition 8.2.7 and Lemma 8.2.22] for assertions b) and d). For d), notice that a complex non-linear curve in the linear normalization $X_N$ may be linearly projected to a complex line in the singular locus of $X$.

Proof of Proposition 1. Let $\varphi : Y \to X$ be the smooth model of a celestial surface $X \subset S^n$ that is not $\infty$-circled and suppose that $C, C' \subset X$ are general circles. Notice that $[C] \in G(X)$ and that $\sigma_*([C]) = [C]$ as a direct consequence of the definitions. It follows from Theorem A, Lemma 1 and the Riemann-Roch and Kawamata-Viehweg vanishing theorems that a pencil of
complex circles that cover $X$ are the images of complex curves in $Y$ that form a one-dimensional complete linear series without base points. In particular, $[C] = [C']$ if and only if $C$ and $C'$ are members of the same pencil, and thus assertion a) holds. Now let $D, D' \subset Y$ denote the preimages of $C$ and $C'$ in $Y$, respectively. We find that $C$ and $C'$ intersect in at least $[C] \cdot [C'] = |D \cap D'|$ complex “moving intersection” points. Suppose that there exists complex points $p \in D$ and $q \in D'$ such that $p, q \notin D \cap D'$ and $\varphi(p) = \varphi(q)$. Thus $C$ and $C'$ will intersect in addition to the moving intersection points at the complex point $\varphi(p)$. As $C$ and $C'$ are chosen generally, we find that $p$ and $q$ must lie on a possibly reducible complex curve $R \subset Y$ such that each of its irreducible complex components is contracted to $\varphi(p)$. Therefore, $\varphi(p)$ must be a base point for each pencil that has the circle $C$ or $C'$ as member. We have proven assertion b) as $|C \cap C'|$ minus the moving intersection points is equal to the number of base points that are met by both $C$ and $C'$. Notice that $R$ corresponds to a component $W \subset B(X)$ and thus $\sigma(R) = R$ if and only if $\sigma_*(W) = W$ if and only if the base point $\varphi(p)$ is real. This concludes the proof for assertion c).

Lemma 2. If $X \subset S^n$ is a celestial surface, then its degree is even and $X(\mathbb{R})$ does not contain lines.

Proof. Suppose by contradiction that $X$ is of odd degree. The intersection of $X(\mathbb{R}) \subset \mathbb{P}^{n+1}$ with the hyperplane at infinity is an odd degree curve and thus non-empty. We arrived at a contradiction as $X(\mathbb{R}) \subset S^n$. In particular, $X(\mathbb{R})$ is compact and thus does not contain lines.

An $(2/3)$-anticanonical model of $\mathbb{P}^2$ is defined as the image of a map associated to minus two-thirds times the canonical class of $\mathbb{P}^2$.

Proposition 2. If $X \subset S^n$ is a celestial surface of type $(\lambda, d, n)$ for $n > 2$ with linear normalization $X_\lambda$, then up to $\text{Aut} \, N(X)$ either

- $d = 8$, $\lambda = 2$, $n \leq 7$, $X_\lambda \subset \mathbb{P}^8$ is an anticanonical model of $S^1 \times S^1$, $\sigma_* = A_0$, $B(X) = \emptyset$ and $G(X) = \{g_0, g_1\}$,
- $d = 6$, $\lambda \leq 3$, $n \leq 5$, $X_\lambda \subset \mathbb{P}^6$ is an anticanonical model of $S^1 \times S^1$ blown up in a pair of complex conjugate points, $\sigma_* = A_1$, $B(X) \subseteq \{b_{12}\}$ and $G(X) \subseteq \{g_0, g_1, g_{12}\}$,
• $d = 4$, $\lambda = \infty$, $n \leq 4$, the Veronese surface $X_N \subset \mathbb{P}^5$ is a $(2/3)$-anticanonical model of $\mathbb{P}^2$ and $\sigma_\ast$ is the identity, or

• $d = 4$, $\lambda \leq 10$, $n = 3$ and $X \subset \mathbb{P}^4$ is complex isomorphic to an anticanonical model of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in four sufficiently general points.

Proof. If $\lambda = \infty$, then the assertion follows from Theorem A. We henceforth assume that $\lambda < \infty$ and let $\varphi : Y \to X$ be the smooth model of $X$. It follows from Theorem A and Lemma 2 that $d \in \{4, 6, 8\}$ so that the assumption of Lemma 1 holds. Moreover, $X_N \subset \mathbb{P}^d$ is an anticanonical model of $Y$. Notice that if $g \in G(X)$ is the class of a circle then $\sigma_\ast(g) = g$. For the following three if-statements we apply Lemma 1 and Proposition 1. If $d = 4$, then $\lambda \leq 10$ and thus the proof is concluded by Theorem A. If $d = 8$, then $\{g \in G(X) \mid \sigma_\ast(g) = g\} = \{g_0, g_1\}$ and thus $Y \cong S^1 \times S^1$ by Remark 3 as asserted. If $d = 6$, then $2 \leq \lambda \leq 3$ and we may assume without loss of generality that $g_0$ and $g_1$ are the classes of circles. Since $g_0 \cdot g_1 = 1$, the fiber product of the associated maps $\varphi_{g_0}, \varphi_{g_1} : Y \to \mathbb{P}^1$ defines a birational morphism $\varphi_{g_0} \times \varphi_{g_1} : Y \to S^1 \times S^1$. Hence, $Y$ must be isomorphic to $S^1 \times S^1$ blown up in $8 - d$ centers. If $E \subset Y$ is a resulting complex $(-1)$-curve, then $-k \cdot [E] = 1$ and thus $\varphi(E)$ is a complex line in $X$, which must be non-real by Lemma 2 so that the centers are complex conjugate as is asserted.

Lemma 3. Suppose that $\varphi : Y \to X_N$ defines the smooth model of a linear normal surface $X_N$, such that $Y$ is complex isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in $0 \leq r \leq 4$ sufficiently general points. If $W \subset B(X)$ is a component, then the union of curves $C \subset Y$ such that $[C] \in W$ is contracted by $\varphi$ to an isolated double point of $X_N$. If the graph with vertex set $W$ and edge set $\{(a, b) \mid a \cdot b > 0\}$ has Dynkin type $A_1$, $A_2$ or $A_3$, then this double point is a node, cusp or tacnode, respectively.

Proof. See [10, Proposition 8.1.10] and [10, Theorem 8.2.28].

3 Constructing celestial surfaces

In this section we propose a method for constructing examples of celestial surfaces of given type. Each celestial surface can be constructed with this method.
and we show that if \((d, n) \notin \{(6, 3), (4, 3)\}\), then each type in Theorem 1 is realized by a celestial surface. However, our method leaves room for improvement and this will be formulated as an open problem.

Each entry in Table 7 is marked with a type \((\lambda, d, n)\) and consists of three items \((u, R, Q)\) which encode a rational map \(\mu: S^1 \times S^1 \rightarrow \mathbb{P}^d\), a linear projection \(\rho: \mathbb{P}^d \rightarrow \mathbb{P}^{n+1}\) and a hyperquadric \(Q \subset \mathbb{P}^{n+1}\). We will show that if \(\nu: \mathbb{P}^{n+1} \rightarrow \mathbb{P}^{n+1}\) is a projective transformation such that \(\nu(Q) = S^n\), then \(\nu \circ \rho \circ \mu\) is a birational maps of bidegree \((2, 2)\) that parametrizes a celestial surface of type \((\lambda, d, n)\). We start by explaining the encoding with an example (recall Remark 3).

**Table 7:** See Lemma 5. The list \(u\), matrix \(R\) and symmetric matrix \(Q\), defines the map \(\mu\), projection \(\rho\) and hyperquadric \(Q\), respectively.

| Type | Example |
|------|---------|
| \((2, 6, 5)\) | \([x^2 v^2 + y^2 w^2, x^2 v w, x^2 w^2, x v y w, x y w^2, y^2 v w, y^2 v^2, y^2 w^2]\) |
| \((2, 8, 4)\) | \([x^2 v^2, x^2 v w, x^2 w^2, x y v^2, x y v w, x y w^2, y^2 v^2, y^2 v w, y^2 w^2]\) |
| \((2, 8, 5)\) | \([x^2 v^2, x^2 v w, x^2 w^2, x y v^2, x y v w, x y w^2, y^2 v^2, y^2 v w, y^2 w^2]\) |
| \((2, 8, 6)\) | \([x^2 v^2, x^2 v w, x^2 w^2, x y v^2, x y v w, x y w^2, y^2 v^2, y^2 v w, y^2 w^2]\) |
| \((2, 8, 7)\) | \([x^2 v^2, x^2 v w, x^2 w^2, x y v^2, x y v w, x y w^2, y^2 v^2, y^2 v w, y^2 w^2]\) |
| \((2, 8, 8)\) | \([x^2 v^2, x^2 v w, x^2 w^2, x y v^2, x y v w, x y w^2, y^2 v^2, y^2 v w, y^2 w^2]\) |
| \((2, 6, 4)\) | \([x^2 v^2 - y^2 v w, x^2 v w + y^2 v^2, x^2 w^2 + y^2 v^2, x y v^2 - y^2 v w, x y v w - y^2 v^2, y^2 v w + x y w^2, y^2 v^2 + y^2 w^2]\) |
| \((2, 6, 5)\) | \([x^2 v^2 - y^2 v w, x^2 v w + y^2 v^2, x^2 w^2 + y^2 v^2, x y v^2 - y^2 v w, x y v w - y^2 v^2, y^2 v w + x y w^2, y^2 v^2 + y^2 w^2]\) |
| \((2, 6, 6)\) | \([x^2 v^2 - y^2 v w, x^2 v w + y^2 v^2, x^2 w^2 + y^2 v^2, x y v^2 - y^2 v w, x y v w - y^2 v^2, y^2 v w + x y w^2, y^2 v^2 + y^2 w^2]\) |
| \((2, 6, 7)\) | \([x^2 v^2 - y^2 v w, x^2 v w + y^2 v^2, x^2 w^2 + y^2 v^2, x y v^2 - y^2 v w, x y v w - y^2 v^2, y^2 v w + x y w^2, y^2 v^2 + y^2 w^2]\) |
| \((2, 6, 8)\) | \([x^2 v^2 - y^2 v w, x^2 v w + y^2 v^2, x^2 w^2 + y^2 v^2, x y v^2 - y^2 v w, x y v w - y^2 v^2, y^2 v w + x y w^2, y^2 v^2 + y^2 w^2]\) |
linear projection $\rho: \mathbb{P}^6 \to \mathbb{P}^5$ that sends $x$ to $(x_0 + x_3 + x_4 + x_5 : x_0 + x_3 + x_5 + x_6 : x_1 + x_2 + x_3 + x_5 : x_4 : x_1 + x_3 + x_4 + x_5 : x_3 + x_4 + x_5)$. The quadratic form associated to the symmetric matrix $Q$ defines a hyperquadric $Q \subset \mathbb{P}^5$ so that $Q := \{ x \in \mathbb{P}^5 \mid -6 x_0^2 + 6 x_0 x_1 - 7 x_0 x_2 + 19 x_1 x_2 + 11 x_0 x_3 + 19 x_1 x_3 - 2 x_2 x_3 - 27 x_3^2 + 28 x_0 x_4 - 40 x_1 x_4 - 28 x_3 x_4 - 12 x_2^2 - 15 x_0 x_5 + 15 x_1 x_5 + 9 x_2 x_5 + 33 x_3 x_5 + 3 x_4 x_5 - 6 x_5^2 = 0 \}$. Since $Q$ has signature $(4, 1, 1)$, there exists projective transformation $\nu: \mathbb{P}^5 \to \mathbb{P}^5$ such that $\nu(Q) = \mathbb{S}^4$. The Zariski closed image of $\nu \circ \rho \circ \mu$ is a celestial surface of type $(2, 6, 4)$ (see forward Lemma 5).

Lemma 4. Let $(\mu, \rho, Q)$ be encoded by the entry $(u, R, Q)$ in Table 7 that is marked with type $(\lambda, d, n)$.

- The matrix $Q$ is symmetric and has signature $(n + 1, 1)$.
- The rank of the $(n + 2) \times (d + 1)$ matrix $R$ is $n + 2$.
- If $q$ is the quadratic form associated to the matrix $Q$ then $q \circ \rho \circ \mu = 0$.
- The components of $\rho \circ \mu$ are $n + 1$ linearly independent forms of bidegree $(2, 2)$.
- If $d = 8$, then the linear series of $\rho \circ \mu$ is base point free.
- If $(\lambda, d) = (3, 6)$, then the linear series of $\rho \circ \mu$ has base points $(1 : -i; 1 : i)$ and $(1 : i; 1 : -i)$.
- If $(\lambda, d) = (2, 6)$, then the linear series of $\rho \circ \mu$ has base points $(1 : i; 1 : 0)$ and $(1 : -i; 1 : 0)$.

Proof. Straightforward verification. See [21, Algorithm 1] for computing base points of a linear series. \hfill \square

Lemma 5. Let $(\mu, \rho, Q)$ be encoded by an entry $(u, R, Q)$ in Table 7 that is marked with type $(\lambda, d, n)$. Suppose that $Z_N \subset \mathbb{P}^d$ and $Z \subset \mathbb{P}^{n+1}$ are the Zariski closed images of the maps $\mu$ and $\rho \circ \mu$, respectively.

a) The map $\mu: \mathbb{S}^1 \times \mathbb{S}^1 \to Z_N$ is birational, $\deg Z_N = d$ and $Z_N$ is covered by $\lambda$ pencils of conics.

b) If $\nu: \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ is a projective transformation such that $\nu(Q) = \mathbb{S}^n$, then the Zariski closed image of the bidegree $(2, 2)$ birational map $\nu \circ \rho \circ \mu$ is a celestial surface of type $(\lambda, d, n)$. There exists a projective transformation $\nu$ such that $\nu(Q) = \mathbb{S}^n$.  

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c) If \( X \) is a celestial surface of type \((\lambda, d, n)\), then there exists a linear projection \(\hat{\rho}: Z_N \to X\) such that \(\hat{\rho} \circ \mu: S^1 \times S^1 \dashrightarrow X\) is birational and of bidegree \((2, 2)\).

Proof. a) First suppose that \(d = 6\). It follows from Lemma 4 that the linear series of \(\mu\) consists of linear independent bidegree \((2, 2)\) forms that vanish at the complex conjugate base points \(p_1\) and \(p_2\). These base points lie either in general position or are contained in a fiber of \(\pi_1: S^1 \times S^1 \to S^1\) (recall Remark 3). We illustrated the configurations in Figure 6, where the vertical and horizontal line segments correspond to the fibers of \(\pi_1\) and \(\pi_2\), respectively.

Let \(\tau: Y \to S^1 \times S^1\) be the birational morphism that blows up \(p_1\) and \(p_2\) and recall from Lemma 1a that \(-k = 2\ell_0 + 2\ell_1 - \varepsilon_1 - \varepsilon_2\) is the anticanonical class of \(Y\). Notice that \(\mu \circ \tau: Y \to Z_N\) is the map associated to \(-k\) and thus \(Z_N\) is the anticanonical model of \(Y\). It follows from \(-k\) being nef and big and from Reider’s lemma that \(\mu \circ \tau\) and thus \(\mu\) itself is birational. Since \(-k\) is not orthogonal to the class of a \((-1)\)-curve, we find that \(\mu \circ \tau\) is a smooth model for \(Z_N\). It follows from Lemma 4 and Lemma 1b that \(B(Z_N) = \{b_{12}\}\) or \(B(Z_N) = \emptyset\) if \((\lambda, d) = (2, 6)\) and \((\lambda, d) = (3, 6)\), respectively. Therefore, by Lemma 1c, we have that \(G(Z_N)\) is equal to either \(\{g_0, g_1, g_{12}\}\) or \(\{g_0, g_1\}\) and we verify that \(\lambda = |G(Z_N)|\). Suppose that \(C \subset Z_N\) is a conic so that \([C] \in G(Z_N)\). It follows from the Riemann-Roch and Kawamata vanishing theorems that \(h^0([C]) = 2\) and thus \(C\) is a member of a pencil of conics. Hence, we conclude that \(Z_N\) is covered by \(\lambda\) pencils of conics as asserted. If \(d = 8\), then the proof is similar, except in this case the linear series of \(\mu\) is base point free, \(B(Z_N) = \emptyset\) and \(G(Z_N) = \{g_0, g_1\}\).

b) It follows from Lemma 4 that \(Z_N\) and \(Z\) are not contained in a hyperplane section. Moreover, we verified that the base points of the linear series associated to \(\rho \circ \mu\) and \(\mu\) are the same. Therefore the linear series of \(\mu\) is a completion of the linear series of \(\rho \circ \mu\), or equivalently, \(Z_N\) is a linear normalization of \(Z\).
We also know from Lemma 4 that $Z \subset Q$ and that $Q$ has the same signature as $S^\infty$. Hence, there exists a projective transformation $\nu: \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ such that $\nu(Q) = S^\infty$ and $\nu(Z) \subset S^\infty$. We now conclude from a) that $\nu(Z)$ is a celestial surface of type $(\lambda, d, n)$.

c) Let $X_N \subset \mathbb{P}^d$ be linear normalization of $X$ and let $\varphi: Y \to X_N$ be its smooth model. Recall from Remark 5 that there exists a linear projection $\eta: X_N \to X$ and notice that $\eta \circ \varphi: Y \to X$ is a smooth model for $X$. First suppose that $d = 6$. We know from Proposition 2 that $B(X) \subseteq \{b_{12}\}$, $G(X) \subseteq \{g_0, g_1, g_{12}\}$ and there exists a birational morphism $\tau: Y \to S^1 \times S^1$ that contracts two complex conjugate $(-1)$-curves to the complex conjugate blowup centers $p_1$ and $p_2$. It follows from Lemma 1[b,c] that $p_1$ and $p_2$ either lie in a general position or are contained in a fiber of $\pi_1$. Up to $\text{Aut}(S^1 \times S^1)$ we may assume that one of the following two cases holds:

- $\lambda = 3$, $p_1 = (1 : -i ; 1 : i)$ and $p_2 = (1 : i ; 1 : -i)$ (see $C_1$ in Figure 6), or
- $\lambda = 2$, $p_1 = (1 : i ; 1 : 0)$ and $p_2 = (1 : -i ; 1 : 0)$ (see $C_2$ in Figure 6).

We know from Lemma 1a that $-k = 2\ell_0 + 2\ell_1 - \varepsilon_1 - \varepsilon_2$ is the anticanonical class and thus the class of a hyperplane section of $X_N$. Therefore, the linear series of $\varphi \circ \tau^{-1}$ consist of all the bidegree $(2, 2)$ forms that vanish at $p_1$ and $p_2$ with multiplicity one. It follows from Lemma 4 that the components of $\mu$ define a basis for such a linear series. Therefore, there exists a projective isomorphism $\alpha: Z_N \to X_N$ so that $\breve{\rho} := \eta \circ \alpha$ exists as is asserted. If $d = 8$, then the proof is similar, except in this $Y \cong S^1 \times S^1$ and $-k = 2\ell_0 + 2\ell_1$. □

Lemma 5 suggests the following algorithm for computing examples of celestial surfaces. Each celestial surface is reached by this algorithm, except for Veronese surfaces and 2-circled Darboux cyclides whose circle graphs are connected.

**Algorithm 1** (constructing examples of celestial surfaces).

- **Input:** The set $B(X)$ and type $(\lambda, d, n)$ of some celestial surface $X$ with smooth model $Y \to X$ such that $Y$ is isomorphic to a blowup of $S^1 \times S^1$.
- **Output:** A birational map $S^1 \times S^1 \dashrightarrow X$ of bidegree $(2, 2)$ such that $X \subset S^a$ is a celestial surface of type $(\lambda, d, n)$. 

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• Method:

1. Use Lemma 1 to translate \( B(X) \) into an explicit configuration of base points of a linear series (see Figure 6 and Lemma 11 for examples).

2. Apply [21, Algorithm 2] to construct a linear series of bidegree \((2, 2)\) forms that pass through the base points with multiplicity one and let \( \mu : S^1 \times S^1 \to \mathbb{P}^d \) be the map associated to this linear series.

3. Let \( \rho : \mathbb{P}^d \to \mathbb{P}^{n+1} \) be a random linear projection. Compute the graded ideal \( I(Z) \subset \mathbb{Q}[y_0, \ldots, y_d] \) of the image \( Z \subset \mathbb{P}^{n+1} \) of \( \rho \circ \mu \) and let \( (q_i) \), be the generators of the vector space of quadratic forms in \( I(Z) \). Let \( q := \sum_i c_i q_i \), where \( c_i \in \mathbb{Z} \) are random coefficients. We compute \( \deg Z \) and \( \dim Z \) via the Hilbert polynomial of \( I(Z) \).

4. If either \((\deg Z, \dim Z) \neq (d, 2)\) or if the signature of \( q \) is not equal to \((n+1, 1)\), then go back to and repeat step 3. Otherwise, we diagonalize the symmetric matrix \( Q \) associated to \( q \) and obtain a projective isomorphism \( \nu : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1} \) such that \( \nu(Q) = S^n \), where \( Q \subset \mathbb{P}^{n+1} \) is defined as the zero-set of \( q \).

5. We output the composition \( \nu \circ \rho \circ \mu : S^1 \times S^1 \to X \).

Algorithm 1 has been implemented in [22, orbital] and was used for computing the examples in Table 7. An open problem is compute \( \rho \) and \( q \) in step 3 such that each component of the projective transformation \( \nu \) has small coefficients in say \( \{-1, 0, 1\} \).

A Veronese surface of type \((\infty, 4, 4)\) can be constructed with the same method as Algorithm 1, but instead we set \( \mu : \mathbb{P}^2 \to \mathbb{P}^5 \) to be a map whose components generate the vector space of quadratic forms on \( \mathbb{P}^2 \).

**Proposition 3.** Each of the following types is realized by some celestial surface: \((2,8,n)\) for \(3 \leq n \leq 7\), \((3,6,5), (3,6,4), (2,6,5), (2,6,4)\) and \((\infty,4,4)\).

**Proof.** See [18, (23.6), (23.7)] for implicit equations of a celestial surface of type \((\infty,4,4)\). The remaining cases are now a direct consequence of Lemma 5b.

**Remark 7.** Suppose that \( \nu \circ \rho \circ \mu : S^1 \times S^1 \to X \) is the output of Algorithm 1 with input \( B(X) = \{b_{12}\} \) and \((2,6,4)\). Let \( q \in S^1 \times S^1 \) be a general point that lies in the same fiber as the complex conjugate base points (see C2 at
Figure 6). It follows from Lemma 3 that this fiber is contracted by \( \nu \circ \rho \circ \mu \) so that \((\nu \circ \rho \circ \mu)(q) \) is a nodal singularity of \( X \subset S^4 \). If we choose the center of stereographic projection \( \pi \) on this singular point in \( X(\mathbb{R}) \), then \( \pi(X(\mathbb{R})) \subset \mathbb{R}^4 \) is covered by a pencil of lines and a pencil of circles. A particularly nice example of such a surface was provided by Mikhail Skopenkov during private communication:

\[
\mathbb{R}^2 \to \mathbb{R}^4, \quad (u, v) \mapsto (u v^2 - v, \ u v + v^2, \ u v + 1, u - v)/(v^2 + 1).
\]

If \( d = 8 \) then an alternative parametrization for the real points \( X(\mathbb{R}) \subset S^7 \) of a surface of type \((2,8,7)\) is as follows with \( 0 \leq \alpha, \beta \leq 2\pi \):

\[
(\cos \alpha, \sin \alpha, \cos \beta, \sin \beta, \cos(\alpha + \beta), \sin(\alpha + \beta), \cos(\alpha - \beta), \sin(\alpha - \beta))/2.
\]

The surface of type \((2,8,3)\) in Figure 1 was constructed as a stereographic projection \( \pi(\{c \star d \in S^3 \mid a \in A, \ b \in B\}) \), for some circles \( A, B \subset S^3 \), where \( \star \) denotes the Hamiltonian product for unit quaternions.

4 Impossible types

In this section we show that a celestial surface cannot have type \((3,6,3), (2,6,3)\) or \((\infty,4,3)\) so that a surface that contains \( \lambda \geq 3 \) circles through a general point must be of type \((\infty,2,2), (\infty,4,4), (3,6,5), (3,6,4)\) or \((\lambda,4,3)\) by Proposition 2. It will follow from \S5 that \( \lambda \leq 6 \) so that Theorem 1 holds.

Definition 5. Suppose that \( Z \subset \mathbb{R}^3 \) is a surface. We denote by \( \mathbb{P}(Z) \subset \mathbb{P}^3 \) the Zariski closure of \( \iota(Z) \subset \mathbb{P}^3 \), where \( \iota: \mathbb{R}^3 \hookrightarrow \mathbb{P}^3 \) sends \((x_1, x_2, x_3)\) to the projective point \((1 : x_1 : x_2 : x_3)\).

The delta invariant \( \delta_p(C) \) of a complex point \( p \) in a planar curve \( C \subset \mathbb{P}^2 \) is defined as \( \Sigma_{q \in \mathcal{I}_p} m_q (m_q - 1)/2 \), where \( \mathcal{I}_p \) is the set that contains \( p \) and complex points that are infinitely near to \( p \), and \( m_q \) denotes the multiplicity of a strict transform of \( C \) at \( q \) (see Definition 4 and [16, Example V.3.9.2]). Informally, we may think of \( \delta_p(C) \) as the number of double points that are concentrated at \( p \) (see [25, page 85]). Notice that \( \delta_p(C) > 0 \) if and only if \( p \in \text{sgn} \ C \), and that \( \delta_p(C) = \delta_{\sigma(p)}(C) \).

Lemma 6. If \( Z \subset \mathbb{R}^3 \) is a \( \lambda \)-circled celestial surface of degree \( d \) and if \( H \) is a general hyperplane section of \( \mathbb{P}(Z) \), then \( \sum_{p \in H} \delta_p(H) = \frac{1}{2}(d - 1)(d - 2) - s \), where \( s = 0 \) or \( s = 1 \) if \( \lambda = \infty \) and \( \lambda < \infty \), respectively.
Proof. Since $H$ is a planar curve of degree $d$ it follows from [12, Section 2.4.6] that the geometric genus of $H$ is equal to $p_g(H) = \frac{1}{2}(d-1)(d-1) - \sum_{p \in H} \delta_p(H)$.

It follows from [33, Theorem 5 and Theorem 8] that the sectional genus $p_g(H)$ is as asserted. Alternatively, notice that $p_g(H) = \frac{1}{2}([H]^2 + [H] \cdot k) + 1$ by [16, Proposition IV.1.1 and Exercise V.1.3] and it follows from Theorem A that $[H] = -\frac{2}{3}k$ or $[H] = -k$ if $\lambda = \infty$ and $\lambda < \infty$, respectively.

Lemma 7. If $X \subset \mathbb{P}^n$ is a surface and $H \subset X$ a general hyperplane section, then $p \in sng H$ if and only if $p \in H \cap sng X$.

Proof. First suppose that $p \in sng H$ and let $\varphi: Y \to X$ denote the smooth model of $X$. By Bertini’s theorem [15, page 137] the preimage $\varphi^{-1}(H)$ for the general hyperplane section $H$ is smooth. Since $\varphi$ is an isomorphism outside the preimage of singular locus of $X$, it follows that $H$ is smooth outside singular locus of $X$ and thus $p \in sng X$ as asserted. Now suppose that $p \in H \cap sng X$. Recall from Remark 5 that there exists a linear projection $\eta: X_N \to X$ from the linear normalization $X_N$. Let $H_N := \eta^{-1}(H)$, $V_N := \eta^{-1}(sng X)$ and $p_N := \eta^{-1}(p)$ so that $p_N \subseteq V_N \cap H_N$. Since $H$ is general, it follows that either $H_N$ is already singular at $p_N$ or $|p_N| > 1$. If $|p_N| > 1$, then the local complex analytic branches of $H_N$ with respective centers in $p_N$ are mapped to branches of $H$ each centered at $p$. Thus $H$ must be singular at $p$ as asserted.

Lemma 8. If $Z \subset \mathbb{R}^n$ is a celestial surface of type $(\lambda, d, n)$, then $(\lambda, d, n) \notin \{ (\infty, 4, 3), (2, 6, 3), (3, 6, 3) \}$.

Proof. Suppose by contradiction that $(d, n) = (6, 3)$ with $\lambda < \infty$. We may assume after a Möbius transformation that $\deg Z = d$ so that $Z$ is compact in $\mathbb{R}^3$.

Thus there exists a general hyperplane section $H$ of $\mathbb{P}(Z)$ such that $H(\mathbb{R}) = \emptyset$. We apply Lemma 6 and Lemma 7 and find that $\sum_{p \in H \cap sng \mathbb{P}(Z)} \delta_p(H) = 9$. If $p, q \in H$ are complex conjugate points, then $\delta_p(H) = \delta_q(H)$ as a direct consequence of the definitions. We arrived at a contradiction as $\sum_{p \in H \cap sng \mathbb{P}(Z)} \delta_p(H)$ must be even. To show that $(\lambda, d, n) \neq (\infty, 4, 3)$ we use the same argument as before, but instead $\sum_{p \in H \cap sng \mathbb{P}(Z)} \delta_p(H) = 3$.

We are now ready to answer the question in [32, Section 5].
Proof of Corollary 1. Suppose that \( Z \subset \mathbb{R}^3 \) contains \( \lambda \geq 3 \) circles through a general point. We know from Proposition 2 (or alternatively from Lemma 2 and [33, Theorems 5–8 and Proposition 1]) that \( Z \) has type either \((\infty, 2, 2), (\lambda, 4, 3), (2, 6, 3), (3, 6, 3)\) or \((\infty, 4, 3)\). The proof is now concluded by Lemma 8 as the latter three types are impossible.

\[ \square \]

5 Darboux cyclides

In this section we classify celestial Darboux cyclides in order to complete the proof of Theorem 4 and its corollaries. The geometry of a celestial Darboux cyclide \( X \subset S^3 \) is to a large extent determined by its Néron-Severi lattice \( N(X) \). We classify \( N(X) \) by classifying unimodular involutions \( \sigma_* \) and subsets \( B(X) \) up to \( \text{Aut} \, N(X) \).

Remark 8. Corollary 3 can be approached using the methods from [9, 32, 39] and it would be interesting to see such an alternative proof.

\[ \langle \right \]

Lemma 9. If \( X \subset S^3 \) is a celestial Darboux cyclide with smooth model \( Y \to X \), then up to \( \text{Aut} \, N(X) \) the unimodular involution \( \sigma_* \) and \( Y \) are characterized by a row of Table 8, where the centers of blow up are in sufficiently general position. Moreover, if \( \sigma_* \in \{2A_1, 3A_1, D_4\} \), then \( \{g \in G(X) \mid \sigma_*(g) = g\} \) is contained in \( \{g_0, g_1, g_2, g_3, g_{12}, g_{34}\}, \{g_{12}, g_{34}\} \) and \( \{g_1, g_2\} \), respectively. If \( X \) is in addition smooth and \( \sigma_* \) is \( 3A_1 \) or \( D_4 \), then \( X \) is a \( S1 \) cyclide and \( S2 \) cyclide, respectively.

\begin{center}
\begin{tabular}{cl}
\hline
\( \sigma_* \) & smooth model \\
\hline
\( 2A_1 \) & \( S^1 \times S^1 \) blown up in two pairs of complex conjugate points \\
\( 3A_1 \) & \( S^2 \) blown up in two pairs of complex conjugate points \\
\( D_4 \) & \( X \) \\
\hline
\end{tabular}
\end{center}

Proof. Recall from Remark 6 and Proposition 2 that \( Y \) is a weak del Pezzo surface. As \( \sigma_*: N(X) \to N(X) \) is induced by the real structure it leaves the canonical class \( k \) of \( X \) invariant. The matrix defining \( \sigma_* \) is an involution and thus has eigenvalues \( \pm 1 \). Notice that \( \sigma_*(v) \cdot \sigma_*(k) = -v \cdot k \) for all \( v \) in the
eigenspace of $-1$. Hence this eigenspace is contained in the following inner product space $V_k(X) := \{\{c \in N(X) \mid k \cdot c = 0\} \otimes \mathbb{R}, \cdot\}$. By [10, 8.6.3] the set $R(X) := \{c \in V_k(X) \mid c^2 = -2\}$ forms a root system of Dynkin type $D_5$ in $V_k(X)$. The intersection of a root system with a subspace is a root subsystem and thus $S_\sigma := \{c \in R(X) \mid \sigma(c) = -c\}$ forms a root subsystem of $R(X)$. We know from [41, Corollary 2.1] that $S_\sigma(X)$ has Dynkin type either

$$A_0 := \emptyset, \quad A_1, \quad 2A_1, \quad 2A'_1, \quad 3A_1 \quad \text{or} \quad D_4.$$  

Root subsystems with the same Dynkin type $2A_1$ may not be the same up to $\text{Aut} N(X)$ and therefore one is marked with $'$.

A linearly independent subset $\{s_1, \ldots, s_r\} \subset S_\sigma(X)$ is called a root base if $s_i \cdot s_j \geq 0$ for all $1 \leq i, j \leq r$. The Dynkin type of $S_\sigma(X)$ is defined by the type of the incidence diagram of a root base and does not depend on the choice. We may assume without loss of generality that $s_i \sim \{b_0, b_1, b_{12}\}$ (see [10, 8.2.3]).

Let $G'(X) := \{c \in N(X) \mid \sigma(c) = c, \ c \sim \{g_0, g_2, g_{12}\}\}$ and $E'(X) := \{c \in N(X) \mid \sigma(c) = c, \ c \sim \{e_1, e'_1, e_{01}\}\}$. If $E'(X) > 0$, then it is a straightforward consequence of [10, Lemma 8.2.22] that $\{c \in E(X) \mid \sigma(c) = c\} > 0$. Thus by Lemma 1 and Lemma 2, we require that $|G'(X)| \geq 2$ and $|E'(X)| = 0$.

For each of the six Dynkin types of $S_\sigma(X)$ we find a basis of given type and from this basis we construct explicit coordinates for $\sigma_*$. We require that $|G'(X)| \geq 2$ and $|E'(X)| = 0$ for the constructed $\sigma_*$. For example, suppose that $S_\sigma(X)$ has Dynkin type $D_4$. Up to $\text{Aut} N(X)$, $S_\sigma(X)$ has root base $\{\ell_1 - \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4\}$. We consider the following three matrices:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}.$$  

The four columns of $B$ correspond to the generators of this root base and are eigenvectors for the eigenvalue $-1$ of $\sigma_*$. The matrix $J$ defines the intersection product for $N(X)$. The matrix $V$ is obtained by augmenting $B$ with two column vectors that generate the kernel of the matrix $B^\top \cdot J$. The appended two columns are eigenvectors for the eigenvalue 1 of $\sigma_*$. Let $D$ be the diagonal matrix with eigenvalues $(-1, -1, -1, -1, 1, 1)$ on the diagonal. We denote the matrix corresponding to $\sigma_*$ by $M$. Since $M \cdot V = V \cdot D$, it follows that $V \cdot D \cdot V^{-1}$
is the matrix corresponding to involution $\sigma_*$ such that $\sigma_*(\ell_0) = g_3$, $\sigma_*(\ell_1) = \ell_1$ and $\sigma_*(\varepsilon_i) = \ell_1 - \varepsilon_i$ for $1 \leq i \leq 4$. We verify that $|E'(X)| = 0$ and find that $G'(X) = \{g_1, g_2\}$ so that $G'(X) = G(X)$. Notice that $\sigma_*$ is unique up to inner automorphisms in Aut $N(X)$ and that we used the same notation $D_4$ in $\S 1$. This concludes the proof of this lemma for the case that $S_\sigma(X)$ has Dynkin type $D_4$.

The remaining cases are similar and we verify that $|E'(X)| = 0$ and $|G'(X)| \geq 2$ only if $S_\sigma(X)$ has Dynkin type either $2A_1$, $3A_1$ or $D_4$. The characterizations of $Y$ is now a direct consequence of Remark 3 and Proposition 2. The topological characterizations as $S_1$ cyclide or $S_2$ cyclide follows from [41, Corollary 3.2].

**Lemma 10.** If $X \subset S^3$ is a celestial Darboux cyclide and either $B(X) = \emptyset$ or $\sigma_* \neq 2A_1$, then $\sigma_*$, $B(X)$, $\text{sng} X$ and $G(X)$ are up to Aut $N(X)$ characterized by a row in Table 9. We overlined a class $g$ in $G(X)$ if $\sigma_*(g) = g$.

**Table 9:** See Lemma 10.

| $\sigma_*$ | $B(X)$ | $\text{sng} X$ | $G(X)$ |
|------------|--------|---------------|--------|
| $2A_1$     | $\emptyset$ | $\emptyset$ | $\{\overline{g}_0, \overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_{12}, \overline{g}_{34}, g_{13}, g_{24}, g_{14}, g_{23}\}$ |
| $3A_1$     | $\{b_0\}$ | $A_1$ | $\{g_0, g_1, \overline{g}_{12}, \overline{g}_{34}, g_{13}, g_{24}, g_{14}, g_{23}\}$ |
| $3A_1$     | $\{b_{13}, b_{24}\}$ | $A_2$ | $\{g_0, g_1, \overline{g}_{12}, \overline{g}_{34}, g_{14}, g_{23}\}$ |
| $3A_1$     | $\emptyset$ | $\emptyset$ | $\{g_0, g_1, g_2, g_3, \overline{g}_{12}, \overline{g}_{34}, g_{13}, g_{24}, g_{14}, g_{23}\}$ |
| $D_4$      | $\emptyset$ | $\emptyset$ | $\{\overline{g}_1, \overline{g}_2, g_0, g_3, \overline{g}_{12}, \overline{g}_{34}, g_{13}, g_{24}, g_{14}, g_{23}\}$ |

**Proof.** By Lemma 9 we have $\sigma_* \in \{2A_1, 3A_1, D_4\}$ and we verify for each case the assertion using Lemma 1. For example, if $\sigma_*$ is $D_4$, then $G(X) = \{g_1, g_2\}$ by Lemma 9, $\sigma_*(\varepsilon_i - \varepsilon_j) = \varepsilon_j - \varepsilon_i$, $\sigma_*(b_{ij}) = -b_{ij}'$, $g_2 \cdot b_{ij}, g_2 \cdot b_0 < 0$ for all $1 \leq i < j \leq 4$ so that $B(X) = \emptyset$ by Lemma 1. The remaining cases are similar and thus straightforward.

**Remark 9.** Suppose that $X \subset S^3$ is a celestial Darboux cyclide with smooth model $Y \to X$ such that $\sigma_*$ is $2A_1$. Thus $Y$ is by Lemma 9 isomorphic to $S^1 \times S^1$ blown up in two pairs of complex conjugate points $p_1$, $p_2$, $p_3$ and $p_4$. We visualize $S^1 \times S^1$ as a square such that horizontal and vertical line segments in the square correspond to the fibers of $\pi_1$ and $\pi_2$, respectively. The real
structure acts on this square by sending horizontal line segments to horizontal line segments and vertical line segments to vertical line segments. A pair of complex conjugate blowup centers are depicted by either two squares or two circles. We consider in Lemma 11 below the following configurations, where the four points do not lie on a curve of bidegree (1,1).

We denote an infinitely near blowup center as an overlapping square. If $p_2$ lies additionally on the pullback of a fiber containing $p_1$, then the overlapping square lies in the horizontal or vertical direction of the blowup center, like in $E_3$ and $E_4$.

It follows from Lemma 1 and Lemma 3 that a fiber in $Y \cong S^1 \times S^1$ that contains two centers of blowup is contracted via the smooth model $Y \to X$ to a complex isolated singularity.

**Lemma 11.** If $X \subset S^3$ is a celestial Darboux cyclide such that $B(X) \neq \emptyset$ and $\sigma_* = 2A_1$, then $B(X)$, sng $X$ and $G(X)$ are up to Aut $N(X)$ characterized by a row in Table 10. Moreover, we included configurations for the centers of blowup from Remark 9 and overlined classes in $G(X)$ such that $\sigma_*(g) = g$.

**Proof.** We follow the notation of Remark 9. Since $B(X) \neq \emptyset$ it follows from Lemma 1 that the four centers of blowup do not lie in general position, but only in sufficiently general position. We first consider the case that the four base points do not lie on a curve of bidegree (1, 1).

For each value for $\alpha := |\pi_1(P)|$ and $\beta := |\pi_2(P)|$ with $P := \{p_1, p_2, p_3, p_4\}$, we list all possible configurations up to symmetry by using the following three restrictions: base points must come in complex conjugate pairs, fibers that contain two non-conjugate base points come in pairs, and at most two non-infinitely near base points are allowed to lie in the same fiber. Observe that
For infinitely near base points we find that sends \( \ell \) \( B \) their associated Néron-Severi lattices. For the first configuration we have that two cases are equivalent, in the sense that there exists an isomorphism between

\[ X \sng \] thus

\[ \{ \} \]

It follows from Proposition 1 that we need to verify using Lemma 1 that rations, as infinity near base points must come in complex conjugate pairs.

We obtain the following table for non-infinitely near base points:

| \( B(X) \) | \( \sng X \) | \( G(X) \) | \( D3/E1 \) |
|---|---|---|---|
| \{1, 2\} | \( 2A1 \) | \( \{ \emptyset, \emptyset, \emptyset, \emptyset_2, \emptyset_3, \emptyset_1, \emptyset_2, \emptyset_3, \emptyset_{1, 2}, g_{13}, g_{24} \} \) |
| \{1, 3, 2, 4, \emptyset_1, \emptyset_2, \emptyset_{1, 2}, \emptyset_3, \emptyset_4, \emptyset_{1, 2}, \emptyset_1, \emptyset_4 \} | \( 4A1 \) | \( \{ \emptyset, \emptyset, \emptyset, \emptyset_2, \emptyset_3, \emptyset_1, \emptyset_2, \emptyset_3, \emptyset_{1, 2}, g_{13}, g_{24} \} \) |
| \{1, 2\} | \( \overline{A}_1 \) | \( \{ \emptyset, \emptyset, \emptyset, \emptyset_2, g_{13}, g_{24}, g_{14}, g_{23} \} \) |
| \{1, 2\} | \( \overline{A}_1 + 2A_1 \) | \( \{ \emptyset, \emptyset, \emptyset_1, g_{13}, g_{24}, g_{14}, g_{23} \} \) |
| \{1, 2\} | \( \overline{A}_2 \) | \( \{ \emptyset, \emptyset, \emptyset_1, g_{13}, g_{24}, g_{14}, g_{23} \} \) |
| \{1, 2\} | \( \overline{A}_3 \) | \( \{ \emptyset, \emptyset, \emptyset_1, g_{13}, g_{24}, g_{14}, g_{23} \} \) |
| \{1, 2\} | \( \overline{A}_4 \) | \( \{ \emptyset, \emptyset, \emptyset_1, g_{13}, g_{24}, g_{14}, g_{23} \} \) |
| \{1, 2\} | \( \overline{A}_5 \) | \( \{ \emptyset, \emptyset, \emptyset_1, g_{13}, g_{24}, g_{14}, g_{23} \} \) |

\[ 2 \leq \alpha, \beta \leq 4, (\alpha, \beta) \neq (4, 4) \) and \( (\alpha, \beta) = (\beta, \alpha) \) up to symmetry. Thus we obtain the following table for non-infinitely near base points:

| \( (\alpha, \beta) \) | \( (3, 4) \) | \( (2, 4) \) | \( (3, 3) \) | \( (2, 3) \) | \( (2, 2) \) |
|---|---|---|---|---|---|
| configuration | \( D1 \) | \( D2, D3 \) | \( D5 \) | \( D4 \) | \( D6, D7 \) |

For infinitely near base points we find that \( E1–E4 \) are all possible configurations, as infinity near base points must come in complex conjugate pairs. It follows from Proposition 1 that we need to verify using Lemma 1 that

\[ |\{g \in G(X) \mid \sigma_\alpha(g) = g\}| \geq 2 \) for each configuration. For example, for configuration \( D4 \) we have \( B(X) = \{b_{13}, b_{24}, b_{12}\}, G(X) = \{\emptyset, \emptyset_1, \emptyset_3, g_{14}, g_{23}\} \) and thus \( \sng X \) is \( \overline{A}_1 + 2A_1 \) by Lemma 3. The other cases are similar.

Notice that both \( D7 \) and \( E4 \) are configurations such that \( \sng X \) is \( 4A_1 \). These two cases are equivalent, in the sense that there exists an isomorphism between their associated Néron-Severi lattices. For the first configuration we have that \( B(X) = \{b_{12}, b_{24}, b_{14}, b_{23}\} \). We now apply the isomorphism \( \mu: N(X) \cong N(X) \) that sends \( \ell_0, \ell_1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \) to \( g_3, g_{34}, \ell_0 - \varepsilon_4, \ell_0 - \varepsilon_3, \ell_1 - \varepsilon_2, \ell_1 - \varepsilon_1 \), respectively. The image of \( B(X) \) via this isomorphism corresponds to the configuration of \( E4 \) as \( \mu(B(X)) = \{b_{13}, b_{24}, b_1, b_2\} \). The configurations \( D3 \) and \( E1 \) are equivalent for similar reasons.

Finally, we include the configuration where all four base points lie on a curve of bidegree \( (1,1) \). We go through each of the configurations \( D1–D7 \) and \( E1–E4 \), but now include \( b_0 \) as an additional element for \( B(X) \). We find that \( |G(X)| < \)
2, unless $B(X) = \{b_0\}$. However, this case is the same as $B(X) = \{b_{12}\}$ up to $\text{Aut } N(X)$ and thus we concluded the proof of this lemma.

\begin{definition}

The hyperplane at infinity $H_\infty \subset \mathbb{P}^3$ is the hyperplane that contains the absolute conic $U := \{x \in \mathbb{P}^3 \mid x_0 = x_1^2 + x_2^2 + x_3^2 = 0\}$. A complex circle in $\mathbb{P}^3$ is defined as an irreducible complex conic that intersects $U$ either tangentially or in two complex points.

\end{definition}

We included a proof of the following classically known proposition due to the lack of suitable reference.

\begin{proposition}

If $Z \subset \mathbb{R}^3$ is irreducible quadric surface, then, up to Euclidean similarity, the following data is defined by exactly one row of Table 11:

\begin{itemize}
    \item the quadratic form $q(x)$ such that $\mathbb{P}(Z) = \{x \in \mathbb{P}^3 \mid q(x) = 0\}$,
    \item the number of circles through a general point that are not lines,
    \item the number of lines through a general point, and
    \item $|G(X)|$, where $X \subset \mathbb{S}^3$ is the Möbius model of $Z$.
\end{itemize}

\begin{proof}

Let $Q$ denote the projective quadric $\mathbb{P}(Z)$. The Euclidean similarities act via the embedding $\iota: \mathbb{R}^3 \hookrightarrow \mathbb{P}^3$ on the hyperplane at infinity $H_\infty$ as the orthogonal group while leaving the absolute conic $U$ invariant. It follows that we can diagonalize the quadratic form associated to $Q \cap H_\infty$ and thus we may assume up to rotations and translations that $q(x)$ is of the form

\begin{equation}
    q(x) = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + x_0 (b_0 x_0 + b_1 x_1 + b_2 x_2 + b_3 x_3),
\end{equation}

for $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. We make a case distinction on the curve $Q \cap H_\infty$ and $U \cap Q$ which are represented below by a solid red curve and a dashed blue circle, respectively.

Thus $|Q \cap U|$ is equal to $\infty$, 4, 2, 4, 2 and 2 at $H_1$–$H_6$, respectively.

\end{proof}
Table 11: See Proposition 4 and let $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$ with $\alpha \neq \beta$ and $\gamma \neq 1$.

| $q(x)$ | singular | #circles | #lines | $|G(X)|$ | name     |
|--------|----------|-----------|--------|--------|----------|
| $\alpha x_1^2 + \beta x_2^2 + x_3^2 - x_0^2$ | no       | 2         | 0      | 8      | EE       |
| $\gamma x_1^2 + \gamma x_2^2 + x_3^2 - x_0^2$ | no       | 1         | 0      | 5      | CE       |
| $x_1^2 + x_2^2 + x_3^2 - x_0^2$ | no       | $\infty$ | 0      | $\infty$ | S$^2$ |
| $\alpha x_1^2 + \beta x_2^2 + x_3^2 + x_0^2$ | no       | 0         | 0      | 8      | empty-1 |
| $\gamma x_1^2 + \gamma x_2^2 + x_3^2 + x_0^2$ | no       | 0         | 0      | 5      | empty-2 |
| $x_1^2 + x_2^2 + x_3^2 + x_0^2$ | no       | 0         | 0      | $\infty$ | empty-3 |
| $\alpha x_1^2 + \beta x_2^2 + x_3^2$ | yes      | 0         | 0      | 7      | point-1 |
| $\gamma x_1^2 + \gamma x_2^2 + x_3^2$ | yes      | 0         | 0      | 4      | point-2 |
| $x_1^2 + x_2^2 + x_3^2$ | yes      | 0         | 0      | $\infty$ | point-3 |
| $\alpha x_1^2 + \beta x_2^2 - x_3^2 + x_0^2$ | no       | 2         | 0      | 8      | EH2      |
| $\alpha x_1^2 + \alpha x_2^2 - x_3^2 + x_0^2$ | no       | 1         | 0      | 5      | CH2      |
| $\alpha x_1^2 + \beta x_2^2 - x_3^2 - x_0^2$ | no       | 2         | 2      | 8      | EH1      |
| $\alpha x_1^2 + \alpha x_2^2 - x_3^2 - x_0^2$ | no       | 1         | 2      | 5      | CH1      |
| $\alpha x_1^2 + \beta x_2^2 - x_3^2$ | yes      | 2         | 1      | 7      | EO       |
| $\alpha x_1^2 + \alpha x_2^2 - x_3^2$ | yes      | 1         | 1      | 4      | CO       |
| $\gamma x_1^2 + x_3 x_0$ | no       | 2         | 0      | 6      | EP       |
| $\alpha x_1^2 + \alpha x_2^2 + x_3 x_0$ | no       | 1         | 0      | 3      | CP       |
| $\gamma x_1^2 + x_2^2 - x_0^2$ | yes      | 2         | 1      | 5      | EY       |
| $\alpha x_1^2 + \alpha x_2^2 - x_0^2$ | yes      | 1         | 1      | 2      | CY       |
| $\gamma x_1^2 + x_2^2 + x_0^2$ | yes      | 0         | 0      | 5      | point-4 |
| $x_1^2 + x_2^2 + x_0^2$ | yes      | 0         | 0      | 2      | point-5 |
| $\alpha x_1^2 - x_2^2 + x_3 x_0$ | no       | 0         | 2      | 6      | HP       |
| $\alpha x_1^2 - x_3^2 + x_0^2$ | yes      | 0         | 1      | 5      | HY       |
| $\alpha x_1^2 + x_0 x_2$ | yes      | 0         | 1      | 1      | PY       |
We make an additional case distinction on the singular locus and real structure of a quadric surface $Q$:

**S1.** $Q$ contains two real lines through each point.

**S2.** $Q$ contains two complex conjugate lines through each point.

**S3.** $Q$ is singular.

It follows from (1) that up to Euclidean similarity $q(0 : x_1 : x_2 : x_3)$ is either

\[
a_1 x_1^2 + a_2 x_2^2 + x_3^2, \quad a_1 x_1^2 + a_2 x_2^2 - x_3^2, \quad a_1 x_1^2 + a_2 x_3^2, \quad a_1 x_1^2 - a_2 x_2^2 \quad \text{or} \quad x_1^2,
\]

for $a_1, a_2 \in \mathbb{R}_{>0}$. We separated the rows of Table 11 according to these 5 cases.

If $a_i \neq 0$ in (1), then $b_i = 0$ after the translation $x_i \mapsto x_i - \frac{b_i}{2a_i} x_0$ for $1 \leq i \leq 3$.

After an Euclidean similarity we may assume that $b_0 \in \{-1, 0, 1\}$. If $a_i = 0$ and $b_i \neq 0$ for $i \in \{2, 3\}$, then $b_0 = 0$ after the translation $x_i \mapsto x_i - \frac{1}{b_0} x_0$.

Notice that $a_1 \neq a_2$, $a_1 = a_2 \neq 1$ and $a_1 = a_2 = 1$ are, a priori, treated as distinct cases with respect to H1–H5. It follows that the first column of Table 11 lists, up to Euclidean similarity, all possible $q(x)$.

Complex circles in $\mathbb{S}^3$ are via a stereographic projection in one to one correspondence with complex lines and complex circles in $\mathbb{P}^3$. It follows that $|G(X)|$ corresponds to the number of pencils of complex lines or complex circles that cover $Q$.

Let us start with the row in Table 11 corresponding to EH1. We see from the equation that we are in case H2 with S1 so that $\#\text{lines}=2$. A pencil of planes, that contain a line spanned by two complex points in $Q \cap U$, defines a pencil of complex circles on $Q$. Thus $Q$ is covered by \( \binom{4}{2} = 6 \) pencils of complex circles so that $|G(X)| = 6 + 2 = 8$. There are two pairs of complex conjugate points in $Q \cap U$ and thus $\#\text{circles}=2$.

Let us now consider the row in Table 11 corresponding to CH2. We see from the equation that we are in case H3 with S2 so that $\#\text{lines}=0$. The pencil of planes, that contain the real line spanned by the two complex conjugate points in $Q \cap U$, or the two complex conjugate lines that are tangent to both $Q$ and $U$, define a pencil of complex circles that covers $Q$. It follows that $\#\text{circles}=1$ and $|G(X)| = 2 + 3 = 5$. 

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We treat the remaining rows analogously so that we obtain the following table, where \( M_i := \text{empty-}i \) and \( N_i := \text{point-}i \):

|   | H1 | H2 | H3 | H4 | H5 | H6 |
|---|----|----|----|----|----|----|
| S1 | EH1 | CH1 | HP |    |    |    |
| S2 | S2,M3 | EE,M1,EH2 | CE,M2,CH2 | EP | CP |    |
| S3 | N3 | N1,EO | N2,CO | EY,N4,BY | CY,N5 | PY |

The details of the remaining cases are now straightforward. \( \square \)

**Lemma 12.** If \( X \subset \mathbb{S}^3 \) is a celestial Darboux cyclide with a real isolated singularity, then \( \sigma_*, B(X) \) and its name are up to \( \text{Aut} \ N(X) \) characterized by a row in Table 12.

**Table 12:** See Lemma 12.

| \( \sigma_* \) | \( B(X) \)     | name    | \( \sigma_* \) | \( B(X) \)     | name    |
|---|------------|--------|---|------------|--------|
| 3A1 | \( \{b_0\} \) | EE/EH2 | 2A1 | \( \{b_1, b_2, b_{12}\} \) | EY |
| 3A1 | \( \{b_{13}, b_{24}\} \) | EP | 2A1 | \( \{b_1, b_2, b_{12}, b_{13}, b_{24}\} \) | CY |
| 2A1 | \( \{b_{12}\} \) | EH1 | 2A1 | \( \{b_{12}, b_{34}\} \) | EO |
| 2A1 | \( \{b_{13}, b_{24}, b'_{12}\} \) | CH1 | 2A1 | \( \{b_{12}, b_{34}, b'_{13}, b'_{24}\} \) | CO |
| 2A1 | \( \{b_{12}, b'_{34}\} \) | HP |   |   |   |

**Proof.** If the center of stereographic projection \( \pi \) lies on the real isolated singularity of \( X(\mathbb{R}) \), then \( Z := \pi(X(\mathbb{R})) \subset \mathbb{R}^3 \) is a quadric surface. We match \( |G(X)| \) in Lemma 10 and Lemma 11 with \( |G(X)| \) in Proposition 4. Moreover, if \( W \subset B(X) \) such that \( \sigma_*(W) = W \), then it follows from Proposition 1 that \( \# \text{circles} = \{ c \in G(X) \mid \sigma_*(c) = c, \ c \cdot W \neq 0 \} \) and \( \# \text{lines} = \{ c \in G(X) \mid \sigma_*(c) = c, \ c \cdot W > 0 \} \). For example, if \( \sigma_* \) is 3A1 and \( B(X) = \{b_0\} \), then \( G(X) = \{g_0, g_1, \overline{g}_{12}, \overline{g}_{34}, g_{13}, g_{24}, g_{14}, g_{23}\} \) so that \( (\# \text{circles}, \# \text{lines}, |G(X)|) = (2, 0, 8) \). Therefore, \( Z \) is by Proposition 4 either EE or EH2 so that \( X \) is either an EE cyclide or an EH2 cyclide. The remaining cases are similar so we conclude the proof of this lemma. \( \square \)

**Remark 10** (Darboux cyclides). Darboux cyclides such that \( \sigma_* \) is 2A1 can be constructed with Algorithm 1. See [39] for a systematic overview of implicit equations for Darboux cyclides up to Möbius transformations.

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If \((c_1, c_2, c_3, c_4)\) equals \((8, 2, 2, 2)\) or \((8, 2, 2, -2)\), then
\[
Z := \{ x \in \mathbb{P}^3 \mid (x_1^2 + x_2^2 + x_3^2)^2 - x_0^2(c_1x_1^2 - c_2x_2^2 - c_3x_3^2 - c_4x_0^2) = 0 \},
\]
is a celestial surface of type \((2, 4, 3)\) such that \(\sigma_s\) is \(3A_1\) and \(D_4\), respectively, up to \(\text{Aut} N(X)\). See [32, Figure 11] and [24] for Euclidean and kinematic constructions of Darboux cyclides, respectively.

6 Combining the results

We prove the assertions in §1 by applying the established results.

Proof of Theorem 1. If a Darboux cyclide is \(\lambda\)-circled, then \(\lambda \leq 6\) by Lemma 9. Thus the listed types now follow from Proposition 2 and Lemma 8. It follows from Proposition 3 and Remark 10 that all listed types are realized by celestial surfaces.

Corollary 1 is a direct consequence of Theorem 1, but has already been established in §4.

Proof of Corollary 2. Direct consequence of Theorem 1.

Proof of Theorem 3. Direct consequence of Proposition 2 and Lemma 9.

Proof of Theorem 4. Direct consequence of the following assertions: Proposition 2, Lemma 9, Lemma 10, Lemma 11 and Lemma 12.

Proof of Corollary 3. Direct consequence of Theorem 4 and Proposition 1.

Proof of Corollary 4. If \(X \subset S^n\) for \(n \geq 3\) is a celestial surface that contains infinitely many circles through some \(p \in S^n\), then either \(X\) is a Veronese surface in \(S^4\), or \(p\) is a real base point of a pencil of circles. In the latter case, the circle graph of \(X\) contains a vertex that is labeled with + or \(\times\). If we choose the center of stereographic projection \(\pi: S^n \rightarrow \mathbb{R}^n\) to be \(p\), then \(\pi(X(\mathbb{R}))\) is a surface that is covered by circles and lines. This corollary is thus a direct consequence of Theorem 4 and Corollary 3.

Proof of Corollary 5. We apply Theorem 4, Lemma 10, Lemma 11, Lemma 12. Moreover, \(|E(X)|\) follows from Lemma 1, since \(X \cong X_N\) by Proposition 2.
**Proof of Corollary 6.** Direct consequence of Theorem 4 and Lemma 1.  

**Remark 11.** The cyclicity of a surface $Z \subset \mathbb{R}^3$ is defined as the multiplicity of the absolute conic $U$ in $\mathbb{P}(Z)$ (recall Definition 5 and Definition 6). Notice that if $\mathbb{P}(Z) = \{x \in \mathbb{P}^4 \mid F(x) = 0\}$ is of cyclicity $c$ and degree $d$, then $F(0 : x_1 : x_2 : x_3) = (x_1^2 + x_2^2 + x_3^2)^c L(x)$ for some form $L(x)$ of degree $d - 2c$. It now follows from Proposition 5 below that our definition for “Darboux cyclide” is equivalent to the definition in [32, Section 2].  

**Proposition 5.** If $Z \subset \mathbb{R}^3$ is a surface of degree $d$ and cyclicity $c$, then $\deg \pi^{-1}(Z) = 2(d - c)$, for all stereographic projections $\pi : S^3 \to \mathbb{R}^3$.  

**Proof.** We may assume up to $\text{Aut}(S^3)$ that $\pi$ has center $(0,0,0,1)$. Let $\tilde{\pi} : S^3 \to \mathbb{P}^3$ be its projective closure so that $\tilde{\pi}(x) = (x_0 - x_4 : x_1 : x_2 : x_3)$ and $\tilde{\pi}^{-1}(y) = (\Delta + y_0^2 : 2y_0y_1 : 2y_0y_2 : 2y_0y_3 : \Delta - y_0^2)$ with $\Delta := y_1^2 + y_2^2 + y_3^2$. Let $X \subset S^3$ denote the Möbius model of $Z$ so that $\deg X = \deg \pi^{-1}(Z)$ and so that $\mathbb{P}(Z)$ coincides with the Zariski closure of $\tilde{\pi}(X)$. We consider the linear series associated to $\tilde{\pi}^{-1}$, namely the quadric surfaces in $\mathbb{P}^3$ that are preimages of hyperplanes in $\mathbb{P}^4$ parametrized by $\alpha \in \mathbb{P}^4$:  

$$Q_\alpha := \{y \in \mathbb{P}^3 \mid \alpha_0 (\Delta + y_0^2) + \alpha_1 y_0 y_1 + \alpha_2 y_0 y_2 + \alpha_3 y_0 y_3 + \alpha_4 (\Delta - y_0^2) = 0\}.$$  

The degree of $X \subset \mathbb{P}^4$ is by definition the number of common intersections between $X$ and two general hyperplanes [16, Section I.7]. Notice that $Q_\alpha$ contains the absolute conic $U$ for all $\alpha \in \mathbb{P}^4$ and thus $\tilde{\pi}^{-1}$ is not defined at $U$. Hence, the degree of $X$ is equal to the number of complex points in $Q_{\alpha_0} \cap Q_{\alpha_1} \cap X$ that do not lie in $U$, for some general $\alpha_0, \alpha_1 \in \mathbb{P}^4$. The complex conic $C$ defined by $Q_{\alpha_0} \cap Q_{\alpha_1}$ intersects $X$ at $U$ in $2c$ points when counted with multiplicity (see [12, Section 2.1]). Since $C$ intersects $X$ in $2d$ points by Bézout’s theorem it follows that $\deg X = 2d - 2c$ as was to be shown. We remark that if $|C(\mathbb{R})| = \infty$, then $\nu^{-1}(C(\mathbb{R})) \subset \mathbb{R}^3$ is a circle (see Definition 5) and thus $2(d - c)$ is equal to the maximal number of intersections of $Z \subset \mathbb{R}^3$ with a circle. \hfill $\square$  

**Definition 7.** A complex hexagonal web is defined by replacing the points, curves and surfaces in Definition 2 with complex points, complex curves and complex surfaces, respectively. \hfill $\square$
Example 6. Suppose that \( q_1, q_2, q_3 \in \mathbb{P}^2 \) are distinct complex points that do not lie on a complex line. It follows from [14] that the following set forms a complex hexagonal web of lines on \( \mathbb{P}^2 \) (see Definition 7):
\[
\mathcal{L} := \{ L \subset \mathbb{P}^2 \mid L \text{ is a complex line such that } |L \cap \{q_1, q_2, q_3\}| \geq 1 \}.
\]
See alternatively [5, Section 1.3, page 24].

Proof of Theorem 2. Suppose that \( X \subset \mathbb{S}^n \) is the Möbius model of \( Z \subset \mathbb{R}^n \) so that \( Z = \pi(X(\mathbb{R})) \) is \( \lambda \)-circled with \( \lambda \geq 3 \). It follows from Corollary 3 that \( Z \) is not covered by lines, because the graphs in Table 3 containing an \( \oplus \) vertex contain less than 3 vertices that are not labeled by \( \oplus \). Hence, \( Z \) is covered by hexagonal web of circles if and only if \( X \) is covered by a hexagonal web of circles.

Let \( \mathcal{L} \) be the complex hexagonal web of lines in \( \mathbb{P}^2 \) from Example 6 that pass through \( q_1, q_2 \) or \( q_3 \). Our plan is to construct a complex birational map \( \mu : X \to \mathbb{P}^2 \) and a 3-web \( W \) of circles on \( X \) such that \( \mu(W) \subset \mathcal{L} \) and \( |C \cap C'| \leq 1 \) for all \( C, C' \in W \). In particular, we require that \( \mu \) is almost everywhere a complex isomorphism that sends circles in \( W \) to complex lines in \( \mathcal{L} \). Notice that circles in \( W \) are either disjoint or intersect in a single point that must be real. We follow the procedure in Remark 1 and when we draw distinct circles \( C, C' \in W \) we also draw complex lines \( \mu(C), \mu(C') \in \mathcal{L} \). If \( \mu(C) \cap \mu(C') \cap \{q_1, q_2, q_3\} = \emptyset \), then \( C \) and \( C' \) must intersect in a point \( p \) and thus there exists a circle \( C'' \in W \) so that \( p \in C'' \) and \( C'' \notin \{C, C'\} \). Hence the procedure results in a closed hexagon so that \( W \) must by Definition 2 be a hexagonal web of circles.

We make a case distinction on the possible types of \( X \) as listed in Theorem 1.

First suppose that \( X \) is of type either \((3, 6, 4)\) or \((3, 6, 5)\). We let \( \varphi : Y \to X \) denote its smooth model. We know from Proposition 1a that \( |G(X)| \geq 3 \). Recall from Lemma 1[a,c] and Proposition 2 that \( N(X) \cong \langle \ell_0, \ell_1, \varepsilon_1, \varepsilon_2 \rangle_{\mathbb{Z}}, G(X) = \{g_0, g_1, g_{12}\} \) and \( B(X) = \emptyset \). It follows that \( Y \) is isomorphic to the blowup of \( \mathbb{P}^1 \times \mathbb{P}^1 \) in two complex conjugate centers \( p_1 \) and \( p_2 \) (see Remark 3). Thus \( Y \) is also complex isomorphic to the blowup of \( \mathbb{P}^2 \) in three complex centers that are not collinear. We may assume up to \( \text{Aut}_C \mathbb{P}^2 \) that these centers coincide with \( q_1, q_2 \) and \( q_3 \) in Example 6. Let \( \tau : Y \to \mathbb{P}^2 \) be the complex
birational morphism that contracts three complex \((-1)\)-curves to these centers. We know from [16, Example II.8.20.3, Proposition V.3.2, Proposition V.3.3] that \(N(X) \cong \langle \alpha, \beta_1, \beta_2, \beta_3 \rangle \mathbb{Z}\) and \(-k = 3 \alpha - \beta_1 - \beta_2 - \beta_3\) is the anticanonical class of \(Y\). Notice that \(\alpha\) is the pullback via \(\tau\) of the class of a general complex line in \(\mathbb{P}^2\) and that \(\beta_i\) is the class of the complex \((-1)\)-curve that is centered at \(q_i\) for \(1 \leq i \leq 3\). Without loss of generality we have \(\alpha = \ell_0 + \ell_1 - \varepsilon_1\), \(\beta_1 = \ell_1 - \varepsilon_1\), \(\beta_2 = \ell_0 - \varepsilon_1\) and \(\beta_3 = \varepsilon_2\). It follows that \(\alpha - \beta_i\) is the class of a curve in \(Y\) that is send by \(\tau\) to a complex line in \(\mathbb{P}^2\) that passes through \(q_i\) (alternatively see [16, Proposition V.3.6]). We remark that the image of the complex \((-1)\)-curve in \(Y\) with class \(\alpha - \beta_i - \beta_j\) is the complex line in \(\mathbb{P}^2\) that passes through \(q_i\) and \(q_j\) for all \(1 \leq i < j \leq 3\). Now let \(\mu\) be defined as the complex birational map \(\tau \circ \varphi^{-1}\): \(X \to \mathbb{P}^2\) and let \(\mathcal{W}\) be the set of circles in \(X\). Since \(G(X) = \{g_0, g_1, g_2\} = \{\alpha - \beta_1, \alpha - \beta_2, \alpha - \beta_3\}\), it follows that \(\mu(\mathcal{W}) \subset \{L \in \mathcal{L} \mid |L \cap \{q_1, q_2, q_3\}| = 1\}\) and \(|C \cap C'| = [C] \cdot [C'] \leq 1\) for all \(C, C' \in \mathcal{W}\). Hence, \(X\) is covered by a hexagonal web of circles so that \(Z\) is covered by a hexagonal web of circles as well.

Next suppose that \(X\) is of type \((\infty, 4, 4)\) so that \(X \subset \mathbb{P}^5\) is a Veronese surface by Theorem 3. By definition there exists a biregular isomorphism \(\mu: X \to \mathbb{P}^2\) such that circles in \(X\) are send to lines in \(\mathbb{P}^2\). We may assume up to \(\text{Aut}_\mathbb{C} \mathbb{P}^2\) that \(\mu^{-1}(\{q_1, q_2, q_3\})\) consists of three points which are real. We define \(\mathcal{W}\) to be the set of circles in \(X\) such that \(\mu(\mathcal{W}) \subset \mathcal{L}\) and thus \(|C \cap C'| = 1\) for all \(C, C' \in \mathcal{W}\). This concludes the proof for this case as \(Z\) must be covered by a hexagonal web of circles.

Now suppose that \(X\) is of type \((\infty, 2, 2)\) so that \(X\) is a sphere. In this case the inverse stereographic projection of a hexagonal web of lines in the plane is a hexagonal web of circles in \(X\).

Finally suppose that \(X\) is of type \((\lambda, 4, 3)\) so that \(Z\) is a Darboux cyclide. Recall from Remark 11 that our definition of Darboux cyclide coincides with the definition in [32]. Thus in this case it follows from [32, Theorem 18] together with Remark 2 and Corollary 3 that \(Z\) is covered by a hexagonal web of circles.

We concluded the proof, since by Theorem 1 we considered all possible types \((\lambda, n, d)\) of \(X\) such that \(\lambda \geq 3\). \(\square\)
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