Nonlinear force-free configurations in cylindrical geometry

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We find a new family of solutions for force-free magnetic structures in cylindrical geometry. These solutions have radial power-law dependence and are periodic but non-harmonic in the azimuthal direction; they generalize the vacuum \( z \)-independent potential fields to current-carrying configurations.

Key words: astrophysical plasmas

1. Introduction

Force-free magnetic configurations satisfying the condition

\[ B = \kappa J, \]

where \( B \) is magnetic field and \( J \) is current density, are examples of magnetic structures that may represent the final stages of magnetic relaxation, or can be used as building blocks of plasma models (Lundquist 1951; Woltier 1958; Taylor 1974; Priest & Forbes 2000).

Particular linear examples of force-free equilibria, with spatially constant \( \kappa \), were considered by Chandrasekhar & Kendall (1957). The most often-used configurations are Lundquist fields in cylindrical geometry (Lundquist 1951) and spheromaks in spherical geometry (Bellan 2000).

Using the self-similar assumption, Lynden-Bell & Boily (1994) (see also Aly 1994) found nonlinear self-similar solutions in spherical geometry. Their model of axially symmetric twisted configurations has been widely used in astrophysical and space applications (e.g. Thompson, Lyutikov & Kulkarni 2002; Shibata & Magara 2011). In the spirit of Lynden-Bell & Boily (1994), in this paper we construct similar nonlinear magnetic configurations in cylindrical geometry.

2. Self-similar configuration in cylindrical geometry

Shafranov (1966) and Grad (1967) formulated what is known as the Grad–Shafranov equation, separating a complicated magnetic configuration in the set of nested/foliated flux surfaces, given by the condition that the flux function \( P \) is constant on the

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surface, and the encompassed current flow. Let us look for force-free equilibria that are independent of the coordinate $z$. The two Euler potentials $\alpha$ and $\beta$ (or, equivalently, the related Clebsh variables) are

$$\begin{cases} 
\alpha = z, \\
\beta = P(r, \phi), 
\end{cases}$$

while the magnetic field can be written as

$$B = \nabla P \times \nabla z + g(P)\nabla z,$$

where $g$ is some function.

Next we introduce a self-similar ansatz

$$P(r, \phi) = r^{-l}f(\phi), \\
g(P) = C|P|^p,$$

$$B = \nabla P \times \nabla z + C|P|^p\nabla z = \{f', \lf, C|f|^p\}r^{-(l+1)}.$$

The absolute value of $|P|$ in the nonlinear term ensures that magnetic field is real ($f(\phi)$ can become negative). Below, any appearance of $f$ to a non-integer power is to be understood to involve $\sqrt{f^2} = |f|$.

By dimensionality (equating radial powers in different terms in (1.1) with magnetic field given by (2.3)),

$$p = 1 + 1/l.$$  

The equation for $f$ becomes

$$lf'' + \beta f + C^2(1 + l)f^{(2+l)/l} = 0$$

(note that the component $B_z$ enters here as $B_z^2$. This justifies the use of $|P|$).

For vacuum fields $C = 0$, the above relations reproduce

$$P \propto r^{-m} \sin(m\phi), \\
B_r \propto r^{-(1+m)}f', \\
B_\phi \propto mr^{-(1+m)}f,$$

with integer $m$.

The first integral is

$$f'^2 + \beta f^2 + C^2|f|^{2(1+l)/l} = H_0.$$  

By redefining $f \to \sqrt{H_0}f$ and $C \to CH_0^{1/(2l)}$, the parameter $H_0$ can be set to unity,

$$f'^2 + \beta f^2 + C^2|f|^{2(1+l)/l} = 1.$$  

Equation (2.8) is the main equation describing nonlinear force-free structures in cylindrical geometry. It depends on one parameter – the current strength $C$. For a given $C$, the value of $l$ is then determined as an eigenvalue problem by requiring periodicity in $\phi$, as we describe next.

We can solve for $f$ in quadratures:

$$\phi = \int \left(\sqrt{1 - \beta f - C^2|f|^{2(1+l)/l}}\right)^{-1} df$$

(so that the integration constant in (2.8) is just a phase $\phi$ where $f = 0$).
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Figure 1. Dependence of the radial index l on the current parameter C for various harmonics $2\tilde{m} = 2, 4, 6, 8, 10$.

Periodicity in $\phi$ requires

$$\int_0^{f_{\text{max}}} (\sqrt{1 - f^2 - C^2 |f|^{2(1+l)/l}})^{-1} df = \frac{\pi}{2\tilde{m}},$$

(2.10)

where $\tilde{m} = 1, 2, \ldots$ is an integer azimuthal number (see a comment after (3.5) as to why odd solutions, $\propto 2\tilde{m} + 1$, in the denominator, are discarded). The value of $f_{\text{max}}$ satisfies

$$1 - \tilde{m}^2 f_{\text{max}}^2 - C^2 f_{\text{max}}^{2(1+l)/l} = 0.$$  (2.11)

For given $C$, the relations (2.10)–(2.11) constitute an eigenvalue problem on $l$. (For the vacuum no-current case $C = 0$, this reduces to $l = 2\tilde{m}$, an integer – as it should.) In practice, we follow the following procedure: for each $\tilde{m} = 1, 2, \ldots$, we assume some $l$ and find $C$ using the relations (2.10)–(2.11). Thus, for each $m$, there is a continuous relation $C(l)$. (Physically, of course, it is the current $C$ that determines the radial index $l$.)

Results are plotted in figures 2 and 3. In figure 2 we plot a particular solution for $l = 1$ and $\tilde{m} = 2$. The flux functions form a 'petal' pattern in azimuthal angle with the number of 'petals' equal to $2\tilde{m}$. There is a corresponding axial, unidirectional magnetic field $B_z$.

In figure 1 we plot the curves $l(C)$ for various $2\tilde{m} = 2, 4, 6, 8, 10$. Each curve starts at a point $\{C = 0, l = 2\tilde{m}\}$. For non-zero current $C > 0$, the radial dependence becomes more shallow, $l < 2\tilde{m}$.

In figure 3 we plot values of $C$ as a function of azimuthal number $m$ for different values of $l = 0.25, \ldots, 2$. Dashed lines are for convenience only; they connect points corresponding to the same radial parameter $l$. 
3. Analysis of the solutions

In a formulation of force-free fields in the form

$$\text{curl} \, \mathbf{B} = \kappa \mathbf{B},$$  \hfill (3.1)

the value of $\kappa$ is

$$\kappa = C \frac{1 + l}{lr f^{1/\eta}}.$$ \hfill (3.2)

It is constant on flux surfaces $P$, (2.3).
The current density (we incorporate factors of $4\pi/c$ into the definition of magnetic field) is

\[
\begin{align*}
    j_r & = C \frac{1 + l}{l} r^{-(l+2)} \partial_\phi (|f|)^{(1+l)/l}, \\
    j_\phi & = C (l + 1) r^{-(l+2)} (|f|)^{(1+l)/l}, \\
    j_z & = r^{-(l+2)} (f'' + \hat{l} f) = -C^2 \frac{1 + l}{l} r^{-(l+2)} (|f|)^{(2+l)/l},
\end{align*}
\]

The total axial current is

\[
I_z = \int_{r_0}^\infty r \, dr \int_0^{2\pi} d\phi j_z = -\frac{r_0^{-l}}{l} \int_0^{2\pi} d\phi (f'' + \hat{l} f),
\]

where $r_0$ is the inner boundary. The total axial current vanishes if the following two conditions are satisfied:

\[
\begin{align*}
    \int_0^{2\pi} f \, d\phi & = 0, \\
    f'(2\pi) & = f'(0).
\end{align*}
\]

All the solutions considered here satisfy these conditions: the second one requires even azimuthal numbers, $2\hat{m}$. Generally, there is a larger family of self-similar force-free equilibria with non-zero total axial current.

There is a non-zero toroidal current

\[
\begin{align*}
    j_\phi & = C (l + 1) r^{-2-l} (|f|)^{(1+l)/l}, \\
    \int_0^{2\pi} d\phi j_\phi & \neq 0.
\end{align*}
\]

The radial current density integrated over $\phi$ satisfies

\[
\int_0^{2\pi} d\phi \partial_\phi (|f|)^{(1+l)/l} = (|f|)^{(1+l)/l} |_{0}^{2\pi} = 0.
\]

### 4. Discussion

In this paper we make analytical progress with the highly nonlinear problem(s) of magnetohydrodynamics (Lynden-Bell & Boily 1994). We find a class of nonlinear self-similar force-free equilibria in cylindrical geometry. The solutions we find all connect to the vacuum case, in which case the flux function is $P_{vac} \propto r^{-m} \sin(m\phi)$. Structures with vanishing total axial current require even values of $m$ (hence $m \to 2\hat{m}$). For non-zero distributed current with the current parameter $C$ the radial dependence changes to $r^{-l}$, with $l < 2\hat{m}$, while remaining periodic in $\phi$ at $2\hat{m}$. Solutions for a given $m$ resemble vacuum solutions $\propto \sin(2m\phi)$, but they are not exactly harmonic in the nonlinear case.

For very large currents the solutions asymptote to $l \approx 0$, but never reach this limit. The case $l = 0$ corresponds to $B_r \propto 1/r$. Mathematically, this is the analogue of the split monopole case in spherical geometry – the split monopole case can be achieved in spherical geometry (with a corresponding anti-monopole in the opposite hemisphere), but is not possible in cylindrical geometry.
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REFERENCES

Aly, J. J. 1994 Asymptotic formation of a current sheet in an indefinitely sheared force-free field: an analytical example. *Astron. Astrophys.* **288**, 1012–1020.
Bellan, P. M. 2000 *Spheromaks: A Practical Application of Magnetohydrodynamic Dynamos and Plasma Self-Organization*. World Scientific.
Chandrasekhar, S. & Kendall, P. C. 1957 On force-free magnetic fields. *Astrophys. J.* **126**, 457–+.
Grad, H. 1967 Toroidal containment of a plasma. *Phys. Fluids* **10** (1), 137–154.
Lundquist, S. 1951 On the stability of magneto-hydrostatic fields. *Phys. Rev.* **83**, 307–311.
Lynden-Bell, D. & Boily, C. 1994 Self-similar solutions up to flashpoint in highly wound magnetostatics. *Mon. Not. R. Astron. Soc.* **267**, 146.
Priest, E. & Forbes, T. 2000 *Magnetic Reconnection*. Cambridge University Press.
Shafranov, V. D. 1966 Plasma equilibrium in a magnetic field. *Rev. Plasma Phys.* **2**, 103–+.
Shibata, K. & Magara, T. 2011 Solar flares: magnetohydrodynamic processes. *Living Rev. Sol. Phys.* **8** (1), 6.
Taylor, J. B. 1974 Relaxation of toroidal plasma and generation of reverse magnetic fields. *Phys. Rev. Lett.* **33**, 1139–1141.
Thompson, C., Lyutikov, M. & Kulkarni, S. R. 2002 Electrodynamics of magnetars: implications for the persistent X-ray emission and spin-down of the soft gamma repeaters and anomalous X-ray pulsars. *Astrophys. J.* **574** (1), 332–355.
Woltjer, L. 1958 *Proc. Natl Acad. Sci. USA* **44**, 489.