ON BIHOM-ASSOCIATIVE DIALGEBRAS

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Abstract. The aim of this paper is to introduce and study BiHom-associative dialgebras. We give various constructions and study its connections with BiHom-Poisson dialgebras and BiHom-Leibniz algebras. Next we discuss the central extensions of BiHom-diassociative and we describe the classification of $n$-dimensional BiHom-diassociative algebras for $n \leq 4$. Finally, we discuss their derivations.

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1 Introduction

The associative dialgebras (also known as diassociative algebras) has been introduced by Loday in 1990 (see [6] and references therein) as a generalization of associative algebras. They are a generalization of associative algebras in the sense that they possess two associative multiplications and obey to three other conditions; when the two associative laws are equal we recover associative algebra. One of his motivation were to find an algebra whose commutator give rise to Leibniz algebra as it is the case in the relation between Lie and associative algebra. Another motivation come from the research of an obstruction to the periodicity in algebraic K-theory. Now, these algebras found their applications in classical geometry, non-commutative geometry and physics.

The centroid plays an important role in understanding forms of an algebra. It is an element in the classification of associative and diassociative algebras. They occurs naturally in the study of derivations of an algebra. The centroid and averaging operators are used in the deformation of algebra in order to generate another algebraic structure. The Nijenhuis operator on an associative algebra was introduced in [16] to study quantum bi-Hamiltonian systems while the notion Nijenhuis
operator on a Lie algebra originated from the concept of Nijenhuis tensor that was introduced by Nijenhuis in the study of pseudo-complex manifolds and was related to the well known concepts of Schouten-Nijenhuis bracket, the Frolicher-Nijenhuis bracket [1], and the Nijenhuis-Richardson bracket. The associative analog of the Nijenhuis relation may be regarded as the homogeneous version of Rota-Baxter relation[23].

BiHom-algebraic structures were introduced in 2015 by G. Graziani, A. Makhlouf, C. Menini and F. Panaite in [7] from a categorical approach as an extension of the class of Hom-algebras. Since then, other interesting BiHom-type algebraic structures of many Hom-algebraic structures has been intensively studied as BiHom-Lie colour algebras structures [17], Representations of BiHom-Lie algebras [30], BiHom-Lie superalgebra structures [23], \{σ, τ\}-Rota-Baxter operators, infinitesimal Hom-bialgebras and the associative (Bi)Hom-Yang-Baxter equation [22], The construction and deformation of BiHom-Novikov algebras [27], On n-ary Generalization of BiHom-Lie algebras and BiHom-Associative Algebras [3], Rota-Baxter operators on BiHom-associative algebras and related structures [19].

The goal of this paper is to introduce, classify and study structures, central extensions and derivations of BiHom-associative dialgebras. The paper is organized as follows. In section 2, we define BiHom-associative dialgebras, give some constructions using twisting, direct sum, elements of centroid, averaging operator, Nijenhuis operator and Rota-Baxter relation. We give a connection between BiHom-associative dialgebras and BiHom-Leibniz algebras. We introduce action of a BiHom-Leibniz algebra onto another and give a Leibniz structure on the semidirect structure. Then, we show that the semidirect sum of BiHom-Leibniz algebras associated to BiHom-associative dialgebras is the same that the BiHom-Leibniz algebra associated to the semidirect of BiHom-associative dialgebras. Finally, we introduce BiHom-associative dialgebras and show that any BiHom-associative dialgebra carries a structure of BiHom-Poisson dialgebra. In section 3, we introduce the notion of central extension of BiHom-associative dialgebras and define 2-cocycles and 2-coboundaries of BiHom-associative dialgebras with coefficients in a trivial BiHom-module. Then we establish relationship between 2-cocycles and central extensions. Section 4, is devoted to the classification of n-dimensional BiHom-associative dialgebras for n ≤ 4. We dedicated Section 5 to the derivations of BiHom-associative dialgebras.

2 Structure of BiHom-associative dialgebras

Definition 2.1. A BiHom-associative dialgebra is a 5-truple \((A, \alpha, \beta)\) consisting of a linear space \(A\) linear maps \(+, \cdot, \alpha, \beta\) : \(A \times A \rightarrow A\) and \(\alpha, \beta : A \rightarrow A\) satisfying, for all \(x, y, z \in A\) the following conditions :

\[
\begin{align*}
\alpha \circ \beta &= \beta \circ \alpha, \\
(x + y) + \beta(z) &= \alpha(x) + (y + z), \\
(x \cdot y) + \beta(z) &= \alpha(x) + (y \cdot z), \\
(x + y) \cdot \beta(z) &= \alpha(x) \cdot (y + z), \\
(x \cdot y) \cdot \beta(z) &= \alpha(x) \cdot (y \cdot z).
\end{align*}
\]

We called \(\alpha\) and \(\beta\) (in this order) the structure maps of \(A\).

Example 2.2. Any Hom-associative dialgebra [10] or any associative dialgebra is a BiHom-associative dialgebra by setting \(\beta = \alpha\) or \(\alpha = \beta = \text{id.}\)
Example 2.3. Let \((A,+,\cdot,\alpha,\beta)\) a BiHom-associative dialgebra. Consider the module of \(n \times n\) matrices \(M_n(D) = M_n(\mathbb{K}) \otimes D\) with the linear maps \(\alpha(A) = (\alpha(a_{ij}))\), \(\beta(A) = (\beta(a_{ij}))\) for all \(A \in M_n(D)\) and the products \((a \ast b)_{ij} = \sum_k a_{ik} b_{kj}\) and \((a \triangleright b)_{ij} = \sum_k a_{ik} \triangleright b_{kj}\). Then, \((M_n(D), \ast, \cdot, \alpha, \beta)\) is a BiHom-associative dialgebra.

Definition 2.4. A morphism \(f : (D,+,\cdot,\alpha,\beta)\) and \((D',\cdot',\cdot',\alpha',\beta')\) be a BiHom-associative dialgebras is a linear map \(f : D \to D'\) such that \(\alpha' \circ f = f \circ \alpha\), \(\beta' \circ f = f \circ \beta\) and \(f(x \ast y) = f(x) \cdot' f(y)\), \(f(x + y) = f(x) +' f(y)\), for all \(x, y \in D\).

Definition 2.5. A BiHom-associative dialgebra \((A,+,\cdot,\alpha,\beta)\) in which \(\alpha\) and \(\beta\) are morphism is said to be a multiplicative BiHom-associative dialgebra. If moreover, \(\alpha\) and \(\beta\) are bijective (i.e. automorphisms), then \((A,+,\cdot,\alpha,\beta)\) is said to be a regular BiHom-associative dialgebra.

We prove in the following proposition that any BiHom-associative dialgebra turn to another one via morphisms.

Theorem 2.6. Let \((D,+,\cdot,\alpha,\beta)\) be a BiHom-associative dialgebra and \(\alpha',\beta' : D \to D\) two morphisms of BiHom-associative dialgebras such that the maps \(\alpha,\alpha',\beta,\beta'\) commute pairwise. Then

\[D_{(\alpha',\beta')} = (D,+,\cdot,(\alpha' \otimes \beta'),\cdot' = (\alpha' \otimes \beta'),\alpha\alpha',\beta\beta')\]

is a BiHom-associative dialgebra.

Proof. We prove only one axiom and leave the rest to the reader. For any \(x,y,z \in D\),

\[(x \ast y) \ast \beta'(z) - \alpha\alpha'(x) \ast (y \ast \beta'(z)) = (x \Gamma y) \ast (\alpha' (x) +\beta'(y)) - \alpha\alpha'(x) +\beta'(y) + (\alpha' (x) +\beta'(y)) - \alpha\alpha'(x)\]

The left hand side vanishes by \((2.3)\). And, this ends the proof.

Corollary 2.7. Let \((D,+,\cdot,\alpha,\beta)\) be a multiplicative BiHom-associative dialgebra. Then

\[(D,+,\cdot,\circ(\alpha^n \otimes \beta^n),\cdot' = (\alpha^n \otimes \beta^n),\alpha^{n+1},\beta^{n+1})\]

is also a multiplicative BiHom-associative dialgebra.

Proof. It suffices to take \(\alpha' = \alpha^n\) and \(\beta' = \beta^n\) in Theorem \(2.6\).

Corollary 2.8. Let \((D,+,\cdot,\alpha)\) be a multiplicative Hom-associative dialgebra and \(\beta : D \to D\) an endomorphism of \(D\). Then

\[(D,+,\cdot,\circ(\alpha \otimes \beta),\cdot' = (\alpha \otimes \beta),\alpha^{2},\beta)\]

is also a Hom-associative dialgebra.

Proof. It suffices to take \(\alpha' = \alpha\) and replace \(\beta\) by \(Id_D\), and \(\beta'\) by \(\beta\) in Theorem \(2.6\).

Any regular Hom-associative dialgebra give rises to associative dialgebra as stated in the next corollary.

Corollary 2.9. If \((D,+,\cdot,\alpha,\beta)\) is a regular BiHom-associative dialgebra, then

\[(D,+,\cdot,\circ(\alpha^{-1} \otimes \beta^{-1}),\cdot' = (\alpha^{-1} \otimes \beta^{-1}))\]

is an associative dialgebra.
Proof. We have to take $\alpha' = \alpha^{-1}$ and $\beta' = \beta^{-1}$ in Theorem 2.6.

**Corollary 2.10.** Let $(D, +, \cdot)$ be an associative dialgebra and $\alpha : D \rightarrow D$ and $\beta : D \rightarrow D$ a pair of commuting endomorphisms of $D$. Then

$$(D, + \circ (\alpha \otimes \beta), + \circ (\alpha \otimes \beta), \alpha, \beta)$$

is a BiHom-associative dialgebra.

Proof. We have to take $\alpha = \beta = \text{Id}_D$ and replace $\alpha'$ by $\alpha$, and $\beta'$ by $\beta$ in Theorem 2.6.

**Definition 2.11.** Let $(D, +, \cdot, \alpha, \beta)$ be a BiHom-associative dialgebra. For any integers $k, l$, an even linear map $\theta : D \rightarrow D$ is called an element of $(\alpha^k, \beta^l)$-centroid on $D$ if

$$\begin{align*}
\alpha \circ \theta &= \theta \circ \alpha, \quad \beta \circ \theta = \theta \circ \beta, \\
\theta(x) + \alpha^k \beta^l(y) &= \theta(x) + \theta(y) = \alpha^k \beta^l(x) + \theta(y), \\
\theta(x) + \alpha^k \beta^l(y) &= \theta(x) + \theta(y) = \alpha^k \beta^l(x) + \theta(y),
\end{align*}$$

for all $x, y \in D$.

The set of elements of centroid is called centroid.

**Proposition 2.12.** Let $(A, +, \cdot, \alpha, \beta)$ a BiHom-associative dialgebra and $\phi : A \rightarrow A$ and $\psi : A \rightarrow A$ be a pair of commuting elements of centroid. Let us define

$$x \prec y := \phi(x) + y \quad \text{and} \quad x \succ y := \psi(x) + y.$$ 

Then, $(A, \prec, \succ, \alpha, \beta)$ is a BiHom-associative dialgebra if and only if $\text{Im}(\phi - \psi) \subseteq Z_1(A) := \{x \in A / x + y = 0, \forall y \in A\}$ and $\text{Im}(\phi - \psi) \subseteq Z_1(A) := \{x \in A / y + x = 0, \forall y \in A\}$.

Proof. We only prove axioms (2.3) and (2.5), the three other comes from BiHom-associativity. So for any $x, y, z \in A$,

$$(x \prec y) \prec \beta(z) - \alpha(x) \prec (y \succ z) = (\phi(x) + y) + \phi\beta(z) - \phi\alpha(x) + (y + \psi(z)) = (\phi(x) + y) + \phi\beta(z) - \phi\alpha(x) + (y + \psi(z))$$

and

$$(x \prec y) \succ \beta(z) - \alpha(x) \succ (y \succ z) = (\phi(x) + y) + \beta\psi(z) - \psi\alpha(x) + (y + \psi(z)) = (\phi(x) + y) + \beta\psi(z) - \psi\alpha(x) + (y + \psi(z))$$

A study of cancellation of the two equalities allows to conclude.

**Proposition 2.13.** Let $(A, \cdot, \alpha, \beta)$ be a BiHom-associative algebra, and $\phi : A \rightarrow A$ and $\psi : A \rightarrow A$ be a pair of commuting elements of centroid. Let us defined

$$x \triangleq y := \phi(x) \cdot y \quad \text{and} \quad x \triangleright y := \psi(x) \cdot y.$$ 

Then, $(A, \triangleq, \triangleright, \alpha, \beta)$ is a BiHom-associative dialgebra if and only if $\text{Im}(\phi - \psi)$ is contained in the set of isotropic vectors.
Suppose that \( f : \beta \rightarrow M \rightarrow \star \) and \( \star \). Then, for all \( m \),

\[
(x \cdot y) \cdot \beta(z) - \alpha(x) \cdot (y \cdot z) = (\phi(x)y)\beta(z) - \phi\alpha(x)(y\psi(z))
\]

\[
= (\phi(x)y)\beta(z) - \alpha\phi(x)(y\psi(z))
\]

\[
= (\phi(x)y)\beta(z) - (\phi(x)y)\beta\psi(z)
\]

\[
= (\phi(x)y)\beta(\phi - \psi)(z)
\]

\[
= \alpha\phi(x)(y(\phi - \psi)(z)).
\]

and

\[
(x \cdot y) \cdot \beta(z) - \alpha(x) \cdot (y \cdot z) = (\phi(x)y)\beta\psi(z) - \psi\alpha(x)(y\psi(z))
\]

\[
= (\phi(x)y)\beta\psi(z) - \alpha\phi(x)(y\psi(z))
\]

\[
= (\phi(x)y)\beta\psi(z) - (\psi(x)y)\beta\psi(z)
\]

\[
= [(\phi(x) - \psi(x))y]\beta\psi(z).
\]

This completes the proof.

\[\square\]

**Remark 2.14.** Proposition 2.13 may be seen as a consequence of Proposition 2.12.

**Proposition 2.15.** Let \((A, \cdot, \alpha, \beta)\) be a BiHom-associative algebra and \((M, \ast_L, \ast_R, \alpha_M, \beta_M)\) an A-BiHom-bimodule i.e. \(M\) is a vector space, \(\alpha_M : M \rightarrow M\) and \(\beta_M : M \rightarrow M\) are two linear maps, and \(\ast_L : A \rightarrow M\) and \(\ast_R : M \rightarrow A\) two bilinear maps such that

\[
\alpha(x) \ast_L (y \ast_L m) = (x \cdot y) \ast_L \beta_M(m)
\]

(2.10)

\[
\alpha(x) \ast_L (m \ast_R y) = (x \ast_L m) \ast_R \beta(y)
\]

(2.11)

\[
\alpha_M(m) \ast_R (x \cdot y) = (m \ast_R x) \ast_R \beta(y).
\]

(2.12)

Suppose that \( f : M \rightarrow A\) is a morphism of A-BiHom-bimodule i.e. \( f\) is linear such that \( \alpha \circ f = f \circ \alpha_M\), \( \beta \circ f = f \circ \beta_M\) and

\[
f(x \ast_L m) = x \cdot f(m)
\]

(2.13)

\[
f(m \ast_R x) = f(m) \cdot x.
\]

(2.14)

Then, \((M, \ast_L, \ast_R, \alpha_M, \beta_M)\) is a BiHom-associative dialgebra with

\[m \cdot n = f(m) \ast_R n\quad \text{and}\quad m \cdot n = m \ast_R f(n),\]

for all \(m, n \in M\).

**Proof.** We only prove axiom (2.6), the other being proved similarly. For any \(x, y, z \in A\),

\[
(m \circ n) \circ \beta_M(p) = (f(m) \ast_L n) \ast_R f \beta_M(p)
\]

\[
= (f(m) \ast_L n) \ast_R \beta f(p).
\]

By (2.11),

\[
(m \circ n) \circ \beta_M(p) = \alpha f(m) \ast_L (n \circ_R f(p))
\]

\[
= f \alpha_M(m) \ast_L (n \circ_R p)
\]

\[
= \alpha_M(m) \circ (n \circ_R p).
\]
Remark 2.16. Any \((\alpha^0,\beta^0)\)-element of centroid of a BiHom-associative algebra is a morphism of BiHom-bimodule.

Thanks to the above remark, we have what follows:

**Corollary 2.17.** Let \((A,\cdot,\alpha,\beta)\) be a BiHom-associative algebra and let \(\theta\) be an element of centroid on \(A\). Then, \((A,\cdot,\alpha,\beta)\) is a BiHom-associative dialgebra with
\[
x \triangleleft y = \theta(x) \cdot y \quad \text{and} \quad x \triangleright y = x \cdot \theta(y),
\]
for any \(x,y\in A\).

**Proposition 2.18.** Let \((D,\cdot,+,\cdot,\alpha,\beta)\) be a BiHom-associative dialgebra and \(R: D \to D\) a Rota-Baxter operator of weight \(0\) on \(D\) i.e. \(R\) is linear and \(\alpha \circ R = R \circ \alpha\), \(\beta \circ R = R \circ \beta\), and
\[
\begin{align*}
R(x) + R(y) &= R(R(x) \cdot y + x \cdot R(y)) \tag{2.16} \\
R(x) \cdot R(y) &= R(R(x) \cdot y + x \cdot R(y)) \tag{2.17}
\end{align*}
\]
Then, \((D,\cdot,+,\cdot,\alpha,\beta)\) is also a BiHom-associative algebra with
\[
\begin{align*}
x \triangleleft y &= R(x) \cdot y + x \cdot R(y), \tag{2.18} \\
x \triangleright y &= R(x) \cdot y + x \cdot R(y), \tag{2.19}
\end{align*}
\]
for all \(x,y\in D\).

**Proof.** We only prove axiom \((2.6)\), the other being proved in a similar way. Thus, For any \(x,y,z\in A\),
\[
(x \triangleleft y) \circ \beta(z) - \alpha(x) \circ (y \triangleright z) =
\begin{align*}
&= (x \triangleright R(y) + R(x) \cdot y + R(\beta(z) + R(R(x) \cdot y) + x \cdot R(y)) + \beta(z) \\
&- \alpha(x) \cdot R(R(y) + z + y \cdot R(z)) - \alpha(x) \cdot (R(y) + \beta(z) + R(\beta(z) + R(R(x) \cdot y)) + \beta(z) \\
&- \alpha(x) \cdot (R(y) + R(z)) - \alpha(x) \cdot (y + R(z) + R(x) \cdot (R(y) + z)).
\end{align*}
\]
The left hand side vanishes by axiom \((2.6)\). This ends the proof. \(\square\)

**Corollary 2.19.** Let \((D,\cdot,+,\cdot,\alpha,\beta)\) BiHom-associative dialgebra and \(R: D \to D\) a Rota-Baxter operator of weight \(0\) on \(D\). Then, \((D,\cdot,+,\cdot,\alpha,\beta)\) is a BiHom-associative algebra with \(x \cdot y = x \triangleleft y + x \triangleright y\).

**Corollary 2.20.** Let \((D,\cdot,+,\cdot,\alpha,\beta)\) be a BiHom-associative dialgebra and \(R: D \to D\) a Rota-Baxter operator of weight \(0\) on \(D\). Then, \((D,[\cdot,\cdot],\alpha,\beta)\) is a BiHom-Lie algebra with
\[
[x,y] = x \cdot y - \alpha^{-1} \beta(y) \cdot \alpha^{-1}(x),
\]
with \(x \cdot y = x \triangleleft y + x \triangleright y\).

As in the previous proposition, it is well known that a Nijenhuis operator on an associative algebra allows to define another associative algebra. In the next result, we try to establish an analog of this result for BiHom-associative dialgebras.
Proposition 2.21. Let \((D,+,\triangleright,\alpha,\beta)\) BiHom-associative dialgebra and \(N : D \rightarrow D\) a Nijenhuis operator on \(D\) i.e. \(N\) is linear and \(\alpha \circ N = N \circ \alpha\), \(\beta \circ N = N \circ \beta\), and

\[
N(x) + N(y) = N(N(x) + y + x + N(y) - N(x \cdot y)) \quad (2.20)
\]

Then, \((D,\triangleright,\alpha,\beta)\) is also a BiHom-associative algebra with

\[
\begin{align*}
\triangleright & = N(N(x) + y + x) - N(x \cdot y), \\
\triangleright & = N(N(x) + y + x) - N(x \cdot y), \\
\end{align*}
\]

for all \(x,y \in D\).

Proof. We only prove axiom (2.4) for the products \(\triangleright\) and \(\triangleright\). The others are leave to the reader.

\[
(x \triangleright y) \triangleright \beta(z) - \alpha(x) \triangleright (y \triangleright z) =
\]

\[
= N(N(x) + y + x + N(y) - N(x \cdot y)) + N(x) + \beta(z) + N(y) + N(x \cdot y) + N(\beta(z))
\]

By (2.20) and (2.21), we have

\[
(x \triangleright y) \triangleright \beta(z) - \alpha(x) \triangleright (y \triangleright z) =
\]

\[
= N\left(N(N(x) + y + x + N(y) - N(x \cdot y)) + N(x) + \beta(z) + N(y) + N(x \cdot y) + N(\beta(z))\right)
\]

Using again (2.20) and (2.21), it comes

\[
(x \triangleright y) \triangleright \beta(z) - \alpha(x) \triangleright (y \triangleright z) =
\]

\[
= N\left(N(N(x) + y + x + N(y) - N(x \cdot y)) + N(x) + \beta(z) + N(y) + N(x \cdot y) + N(\beta(z))\right)
\]

The left hand side vanishes by (2.4). □

Corollary 2.22. If \((D,+,\triangleright,\alpha)\) is a Hom-associative dialgebra and \(N : D \rightarrow D\) a Nijenhuis operator on \(D\), then \((D,\triangleright,\alpha)\) is also a Hom-associative algebra with

\[
\begin{align*}
\triangleright & = N(N(x) + y + x) - N(x \cdot y), \\
\triangleright & = N(N(x) + y + x) - N(x \cdot y), \\
\end{align*}
\]

for all \(x,y \in D\).
Corollary 2.23. If \((D,+,\cdot,\alpha,\beta)\) is an associative dialgebra and \(N:D\rightarrow D\) a Nijenhuis operator on \(D\), then \((D,\prec,\succ,\alpha,\beta)\) is also an associative algebra with

\[
\begin{align*}
  x\cdot y &= N(x) + y + x \cdot N(y) - N(x \cdot y), \\
  x\circ y &= N(x) \cdot y + x \cdot N(y) - N(x \cdot y),
\end{align*}
\]

for all \(x,y \in D\).

The next proposition asserts that the twist of the products of any BiHom-associative dialgebra by an averaging operator gives rise to another BiHom-associative dialgebra.

Proposition 2.24. Let \((D,+,\cdot,\alpha,\beta)\) be a BiHom-associative dialgebra and \(\theta:D\rightarrow D\) an injective averaging operator on \(D\) i.e. \(\theta\) is an injective linear map such that \(\alpha \circ \theta = \theta \circ \alpha, \beta \circ \theta = \theta \circ \beta\), and

\[
\begin{align*}
  \theta(x) + \theta(y) &= \theta(\alpha^k \beta^l(x) + \theta(y)) = \theta(\theta(x) + \alpha^k \beta^l(y)), \quad (2.24) \\
  \theta(x) \cdot \theta(y) &= \theta(\alpha^k \beta^l(x) \cdot \theta(y)) = \theta(\theta(x) + \alpha^k \beta^l(y)), \quad (2.25)
\end{align*}
\]

for any \(x,y \in D\). Then, \((D,\prec,\succ,\alpha,\beta)\) is also a BiHom-associative algebra with

\[
\begin{align*}
  x\cdot y &= \theta(x) + \alpha^k \beta^l(y)) \quad (2.26) \\
  x\circ y &= \alpha^k \beta^l(x) \cdot \theta(y), \quad (2.27)
\end{align*}
\]

for all \(x,y \in D\).

Proof. We only prove one identity, the others have a similar proof. For any \(x,y,z \in D\), one has:

\[
\begin{align*}
  \theta((x \cdot y) \cdot z) - \alpha(x) \cdot (y \cdot z)
  &= \theta(\theta(x) + \alpha^k \beta^l(z) - \alpha(x) \cdot (y \cdot z)) - \alpha(x) \cdot (y \cdot z) \cdot \theta(z)
  \\
  &= \theta(\theta(x) + \alpha^k \beta^l(y) \cdot z) - \alpha(x) \cdot (y \cdot z) \cdot \theta(z)
  \\
  &= (\theta(x) + \theta(y)) \cdot \beta^l(z) - \alpha(x) \cdot (y \cdot z) \cdot \theta(z).
\end{align*}
\]

Which vanishes by axiom (2.24), and the conclusion holds by injectivity. \(\square\)

At this moment, we introduce ideals for BiHom-associative dialgebra in order to give another construction of BiHom-associative dialgebras.

Definition 2.25. Let \((D,+,\cdot,\alpha,\beta)\) be a BiHom-associative dialgebra and \(D_o\) a subset of \(D\). We say that \(D_o\) is a BiHom-subalgebra of \(D\) if \(D_o\) is stable under \(\alpha\) and \(\beta\), and \(x+y, x \cdot y \in D_o\), for any \(x,y \in D_o\).

Example 2.26. If \(\varphi:D_1 \rightarrow D_2\) is a homomorphism of BiHom-associative dialgebras, the image \(\text{Im}\varphi\) is a BiHom-subalgebra of \(D_2\).

Definition 2.27. A two side BiHom-ideal of a BiHom-associative dialgebra \((D,+,\cdot,\alpha,\beta)\) is subspace \(I\) such that \(\alpha(I) \subseteq I, x \cdot y, y \cdot x \in I\) for all \(x \in D, y \in I\) with \(\cdot\) and \(\cdot\). Note that \(I\) is called the left and right BiHom-ideal if \(x \cdot y, x \cdot y + y \cdot x \in I\) are in \(I\), respectively, for all \(x \in D, y \in I\).

Example 2.28. i) Obviously \(I = \{0\}\) and \(I = D\) are two-sided ideals.
ii) If \(\varphi:D_1 \rightarrow D_2\) is a homomorphism of BiHom-associative dialgebras, the kernel \(\text{Ker}\varphi\) is a two sided ideal in \(D_1\).
iii) If \(I_1\) and \(I_2\) are two sided ideals of \(D\), then so is \(I_1 + I_2\).
In the below proposition, we prove that BiHom-associative dialgebras are closed under direct summation, and give a condition for which a linear map becomes a morphism.

**Proposition 2.29.** Let \((A, \cdot_A, \cdot_A, \alpha_A, \beta_A)\) and \((B, \cdot_B, \cdot_B, \alpha_B, \beta_B)\) be two BiHom-associative dialgebras. Then there exists a BiHom-associative dialgebra structure on \(A \oplus B\) with the bilinear maps \(\cdot, \odot : (A \oplus B)^{\otimes 2} \to A \oplus B\) given by

\[
(a_1 + b_1) \cdot (a_2 + b_2) = a_1 \cdot_A a_2 + b_1 \cdot_B b_2,
\]

\[
(a_1 + b_1) \odot (a_2 + b_2) = a_1 \odot_A a_2 + b_1 \odot_B b_2
\]

and the linear maps \(\alpha = \alpha_A + \alpha_B, \beta = \beta_A + \beta_B : A \oplus B \to A \oplus B\) given by

\[
(\alpha_A + \alpha_B)(a + b) = \alpha_A(a) + \alpha_B(b), \quad (\beta_A + \beta_B)(a + b) = \beta_A(a) + \beta_B(b), \quad \forall (a, b) \in (A \times B).
\]

Moreover, if \(\xi : A \to B\) is a linear map. Then \(\xi : (A, \cdot_A, \cdot_A, \alpha_A, \beta_A)\) to \((B, \cdot_B, \cdot_B, \alpha_B, \beta_B)\) is a morphism if and only if its graph \(\Gamma_\xi = \{(x, \xi(x)), x \in A\}\) is a BiHom-subalgebra of \((A \oplus B, \cdot, \odot, \alpha, \beta)\).

**Proof.** The proof of the first part of the proposition comes from a simple computation. Let us suppose that \(\xi : (A, \cdot_A, \cdot_A, \alpha_A, \beta_A) \to (B, \cdot_B, \cdot_B, \alpha_B, \beta_B)\) is a morphism of BiHom-associative dialgebras. Then

\[
(u + \xi(u)) \cdot (v + \xi(v)) = (u \cdot_A v + \xi(u) \cdot_B \xi(v)) = (u \cdot_A v + \xi(u) \cdot_A v)
\]

\[
(u + \xi(u)) \odot (v + \xi(v)) = (u \odot_A v + \xi(u) \odot_B \xi(v)) = (u \odot_A v + \xi(u) \odot_A v).
\]

Thus the graph \(\Gamma_\xi\) is closed under the operations \(\cdot\) and \(\odot\).

Furthermore since \(\xi \circ \alpha_A = \alpha_B \circ \xi,\) and \(\xi \circ \beta_A = \beta_B \circ \xi,\) we have

\[
(\alpha_A \oplus \alpha_B)(u, \xi(u)) = (\alpha_A(u), \alpha_B \circ \xi(u)) = (\alpha_A(u), \xi \circ \alpha_A(u)).
\]

and

\[
(\beta_A \oplus \beta_B)(u, \xi(u)) = (\beta_A(u), \beta_B \circ \xi(u)) = (\beta_A(u), \xi \circ \beta_A(u)),
\]

implies that \(\Gamma_\xi\) is closed \(\alpha_A \oplus \alpha_B\) and \(\beta_A \oplus \beta_B.\) Thus, \(\Gamma_\xi\) is a BiHom-subalgebra of \((A \oplus B, +, \cdot, \alpha, \beta).\)

Conversely, if the graph \(\Gamma_\xi \subset A \oplus B\) is a BiHom-subalgebra of \((A \oplus B, +, \cdot, \alpha, \beta)\) then we

\[
(u + \xi(u)) \cdot (v + \xi(v)) = (u + \xi(u) \cdot (v + \xi(v)) = (u + \xi(u) \cdot (v \cdot (v + \xi(v))) \in \Gamma_\xi
\]

\[
(u + \xi(u)) \odot (v + \xi(v)) = (u + \xi(u) \odot (v + \xi(v)) = (u + \xi(u) \odot (v \odot (v + \xi(v)))) \in \Gamma_\xi.
\]

Furthermore, \((\alpha_A \oplus \alpha_B)(\Gamma_\xi) \subset \Gamma_\xi,\) \((\beta_A \oplus \beta_B)(\Gamma_\xi) \subset \Gamma_\xi,\) implies

\[
(\alpha_A \oplus \alpha_B)(u, \xi(u)) = (\alpha_A(u), \alpha_B \circ \xi(u)) \in \Gamma_\xi, \quad (\beta_A \oplus \beta_B)(u, \xi(u)) = (\beta_A(u), \beta_B \circ \xi(u)) \in \Gamma_\xi,
\]

which is equivalent to the condition \(\alpha_B \circ \xi(u) = \xi \circ \alpha_A(u),\) i.e \(\alpha_B \circ \xi = \xi \circ \alpha_A.\) Similary, \(\beta_B \circ \xi = \xi \circ \beta_A.\)

Therefore, \(\xi\) is a morphism BiHom-associative dialgebras. \(\square\)

**Proposition 2.30.** Let \((D, +, \cdot, \alpha, \beta)\) be a BiHom-associative dialgebra and \(I\) be a two sided BiHom-ideal of \((D, +, \cdot, \alpha, \beta)\). Then, \((D/I, \cdot, \circ, \alpha, \beta)\) is a BiHom-associative dialgebra where

\[
\overline{x \cdot y} := \overline{x} \cdot \overline{y}, \quad \overline{x \circ y} := \overline{x} \circ \overline{y}, \quad \overline{\alpha(x)} := \overline{\alpha(x)}, \quad \overline{\beta(x)} := \overline{\beta(x)},
\]

for all \(\overline{x}, \overline{y} \in A/I.\)
Proof. We only prove left associativity, the other being proved similarly. For all \( x, y, z \in D/I \), we have

\[
\overline{x - y}^2 \overline{\beta(z)} - \overline{\alpha(x)} \overline{y + z} = \overline{x + y}^2 \overline{\alpha(z)} - \overline{\alpha(x)} \overline{y + z} = 0.
\]

Then, \((D/I, \overline{\alpha}, \overline{\beta})\) is BiHom-associative dialgebra. \(\square\)

Now, let us recall the definition of BiHom-Lie algebra.

**Definition 2.31.** [7] A BiHom-Lie algebra \((L, [-, -], \alpha, \beta)\) is a 4-tuple in where \(L\) is linear space, \(\alpha, \beta : A \to A\), are linear maps and \([- , -] : L \otimes L \to L\) is a bilinear maps, such that, for all \(x, y, z \in L\):

\[
\alpha \circ \beta = \beta \circ \alpha,
\]

\[
\alpha([x, y]) = [\alpha(x), \alpha(y)], \text{ and, } \beta([x, y]) = [\beta(x), \beta(y)],
\]

\[
[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \text{ (BiHom-skew-symmetry),}
\]

\[
[\beta^2(x), [\beta(y), \alpha(z)] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0,
\]

(BiHom-Jacobi identity).

The maps \(\alpha\) and \(\beta\) (in this order) are called the structure maps of \(L\).

**Definition 2.32.** A morphism between two BiHom-Lie algebras \(f : (L, [-, -], \alpha, \beta) \to (L', [-, -]', \alpha', \beta')\) is a linear map \(f : L \to L'\) such that \(\alpha' \circ f = f \circ \alpha, \beta' \circ f = f \circ \beta\) and \(f([x, y]) = [f(x), f(y)]'\), for all \(x, y \in L\).

The following lemma asserts that the commutator of any BiHom-associative algebra gives rise to BiHom-Lie.

**Lemma 2.33.** [7] Let \((A, -, , \alpha, \beta)\) be a regular BiHom-associative algebra. Then

\[
L(A) = (A, [-, -], \alpha, \beta)
\]

is a regular BiHom-Lie algebra, where

\[
[x, y] = x \cdot y - \alpha^{-1} \beta(y) \cdot \alpha^{-1}(x),
\]

for any \(x, y \in A\).

**Proposition 2.34.** Let \((L, [-, -], \alpha, \beta)\) be a BiHom-Lie algebra and \(N : L \to L\) be a Nijenhuis operator on \(L\) i.e. \(\alpha \circ N = N \circ \alpha, \beta \circ N = N \circ \beta\) and

\[
[N(x), N(y)] = N([N(x), y] + [x, N(y)] - N([x, y]))
\]

for any \(x, y \in L\). Then, \((L, [-, -], N, \alpha, \beta)\) is a BiHom-Lie algebra with

\[
[x, y]_N = [N(x), y] + [x, N(y)] - N([x, y])
\]

for all \(x, y \in L\).

**Proof.** It follows from direct computation. \(\square\)
Corollary 2.35. Let $(A,\cdot,\alpha,\beta)$ be a BiHom-associative algebra and $N : A \to A$ be a Nijenhuis operator on $A$ i.e. $\alpha \circ N = N \circ \alpha$, $\beta \circ N = N \circ \beta$ and
\[ N(x) \cdot N(y) = N(N(x) \cdot y + x \cdot N(y) - N(x \cdot y)) \]
for any $x, y \in A$. Let us denote by $L(A)$ the BiHom-Lie algebra associated with $A$ as in Proposition 2.33. Then, $(A,\{\cdot,\cdot\}_N,\alpha,\beta)$ is a BiHom-Lie algebra.

Corollary 2.36. Let $(A,\cdot,\alpha,\beta)$ be a BiHom-associative algebra and $N : A \to A$ be a Nijenhuis operator on $A$ i.e. $\alpha \circ N = N \circ \alpha$, $\beta \circ N = N \circ \beta$ and
\[ N(x) \cdot N(y) = N(N(x) \cdot y + x \cdot N(y) - N(x \cdot y)) \]
for any $x, y \in A$. Then, $(A,\{-,\cdot\}_N,\alpha,\beta)$ is a BiHom-Lie algebra with
\[ [x, y] = x \cdot_N y - \alpha^{-1}(y) \cdot_N \beta^{-1}(x) \]
and
\[ x \cdot_N y = N(x) \cdot y + x \cdot N(y) - N(x \cdot y) \]
for all $x, y \in A$.

Proof. It is similar to the one of Proposition 2.21. And the Lemma 2.33 will end the proof. □

Remark 2.37. The BiHom-Lie algebra generated by Corollary 2.35 and Corollary 2.36 are equal.

Proposition 2.38. Let $(D,\cdot,\cdot,\cdot,\alpha,\beta)$ be a BiHom-associative dialgebra. Then, for all $x, y \in D$, the bracket
\[ [x, y] = [x, y]_L + [x, y]_R, \]
where
\[ [x, y]_L = x \cdot y - \alpha^{-1}(y) \cdot \beta^{-1}(x), \]
\[ [x, y]_R = x \cdot y - \alpha^{-1}(y) \cdot \beta^{-1}(x), \]
is a BiHom-Lie bracket if and only if
\[ \alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z), \quad (2.32) \]
\[ \alpha(x) \cdot (y + z) = (x + y) \cdot \beta(z). \quad (2.33) \]

Proof. It is essentially based on Lemma 2.33. That is, an expansion of BiHom-Jacobi identity leads to 48 terms including 8 terms which cancel pairwise by axiom (2.2), 4 terms cancel pairwise by axiom (2.3), 12 terms cancel pairwise by axiom (2.4), 6 terms cancel pairwise by axiom (2.5) and 6 terms cancel pairwise by axiom (2.6). For the 12 terms, 8 terms cancel pairwise by axiom (2.2) and 4 terms cancel pairwise by axiom (2.3). □

Definition 2.39. A (right) BiHom-Leibniz algebra is a 4-tuple $(L,\cdot,\cdot,\cdot,\alpha,\beta)$, where $L$ is a linear space, $\cdot,\cdot : L \times L \to L$ is a bilinear map and $\alpha,\beta : L \to L$ are linear maps satisfying
\[ [[x, y], \alpha(z)] = [[x, \beta(z)], \alpha(y)] + [\alpha(x), [y, \alpha(z)]], \quad (2.34) \]
for all $x, y, z \in L$. 

Example 2.40. Let \( L \) be a two-dimensional vector space and \( \{e_1, e_2\} \) be a basis of \( L \). Then, \((L, [-,-], \alpha, \beta)\) is a BiHom-Leibniz algebra with
\[
[e_1, e_2] = ae_1, [e_2, e_2] = be_1, \quad \alpha(e_2) = \beta(e_2) = e_1, a, b \in \mathbb{R}.
\]

Now, we introduce BiHom-Poisson dialgebras and study its connection with BiHom-associative dialgebras.

Definition 2.41. A BiHom-Poisson dialgebra is a BiHom-associative dialgebra \((P, +, \cdot, \alpha, \beta)\) and a BiHom-Leibniz algebra \((P, [-,-], \alpha, \beta)\) such that
\[
[x+y, \alpha \beta(z)] = \alpha(x) + [y, \alpha(z)] + [x, \beta(z)] + \alpha(y),
\]
\[
[x+y, \alpha \beta(z)] = \alpha(x) + [y, \alpha(z)] + [x, \beta(z)] + \alpha(y),
\]
\[
\{\alpha \beta(x), y+z\} = \beta(y) + [\alpha(x), z] + [\beta(x), y] + \beta(z) = [\alpha \beta(x), y+z],
\]
are satisfied for \(x, y, z \in P\).

Theorem 2.42. Let \((D, +, \cdot, \alpha, \beta)\) be a BiHom-associative dialgebra. Then,
\[
P(D) = (D, [-,-], +, \cdot, \alpha, \beta)
\]
is a BiHom-Poisson dialgebra, where \([x, y] = x \cdot y - y \cdot x\), for any \(x, y \in D\).

Proof. By Theorem 2.46, \(P(D)\) is a BiHom-Leibniz algebra. Moreover, for any \(x, y, z \in D\),
\[
[x+y, \alpha \beta(z)] - \alpha(x) + [y, \alpha(z)] - [x, \beta(z)] + \alpha(y) =
\]
\[
= (x+y) + \alpha \beta(z) - \alpha^{-1} \beta \alpha \beta(z) + \alpha \beta^{-1}(x+y) - \alpha(x) + (y+\alpha(z) - \alpha^{-1} \beta \alpha \beta(z) + \alpha \beta^{-1}(y))
\]
\[
= (x+y) + \alpha \beta(z) - \alpha^{-1} \beta \alpha \beta(z) + \alpha \beta^{-1}(x+y) + \alpha(y)
\]
\[
+ \alpha(x) + (\beta(z) + \alpha \beta^{-1}(y)) - (x + \beta(z)) + \alpha(y) + (\alpha^{-1} \beta \alpha \beta(z) + \alpha \beta^{-1}(x)) + \alpha(y).
\]
The last three axioms are proved analogously. This completes the proof. □

Theorem 2.43. Let \((P, +, \cdot, [-,-], \alpha, \beta)\) be a BiHom-Poisson dialgebra and \(\alpha', \beta' : D \rightarrow D\) two morphisms of BiHom-Poisson dialgebras such that the maps \(\alpha, \alpha', \beta, \beta'\) commute pairwise. Then
\[
P(\alpha', \beta') = (D, < := (\alpha' \otimes \beta'), \triangleright := (\alpha' \otimes \beta'), \{-,-\} := [-,-](\alpha' \otimes \beta'), \alpha \alpha', \beta \beta'),
\]
is a BiHom-Poisson dialgebra.

Proof. It is essentially based on the one of Theorem 2.46 □

Now, we introduce action of BiHom-Leibniz algebra on another one.

Definition 2.44. Let \(D\) and \(L\) be two BiHom-Leibniz algebras. An action of \(D\) on \(L\) consists of a pair of bilinear maps, \(D \times L \rightarrow L, (x,a) \mapsto [x,a]\) and \(L \times D \rightarrow [x,a]\), such that
\[
[[x, a], [a, \alpha(b)]] = [[x, a], a \beta(b),] - [[x, \beta(b)], \alpha(a)]
\]
(2.35)
\[
[[\alpha(a), [x, \alpha(b)]] = [[a, x], a \beta(b),] - [[a, \beta(b)], \alpha(x)]
\]
(2.36)
\[
[[\alpha(a), [b, \alpha(x)]] = [[a, b], a \beta(x),] - [[a, \beta(x)], [b, \alpha(x)]]
\]
(2.37)
\[
[[\alpha(a), [x, \alpha(y)]] = [[a, x], a \beta(y),] - [[a, \beta(y)], [x, \alpha(y)]]
\]
(2.38)
\[
[[\alpha(x), [a, \alpha(y)]] = [[x, a], a \beta(y),] - [[x, \beta(y)], [a, \alpha(y)]]
\]
(2.39)
\[
[[\alpha(x), [y, \alpha(a)]] = [[x, y], a \beta(a),] - [[x, \beta(a)], [y, \alpha(a)]]
\]
(2.40)
for all \(x, y \in D, a, b \in L\).
Lemma 2.45. Given a BiHom-Leibniz action of $D$ on $L$, we can consider the semidirect product

Leibniz algebra $L \rtimes D$, which consists of vector space $D \oplus L$ together with the Leibniz bracket given by

$$[(x, a), (y, b)] = ([x, y] + [x, b] + [a, y], [a, b])$$

(2.41)

for all $(x, a), (x, b) \in D \times L$.

Proof.

$$[(x, a), [(y, b), a \beta(z, c)]] = \left[(\alpha(x), \alpha(a), ([y, \alpha(z)] + [y, \alpha(c)] + [b, \alpha(z)], [b, \alpha(c)])\right]$$

$$= \left((\alpha(x), [y, \alpha(z)] + [\alpha(x), [y, \alpha(c)] + [\alpha(x), [b, \alpha(z)] + [\alpha(x), [b, \alpha(c)]
\right.\left. + [\alpha(a), [y, \alpha(z)] + [\alpha(a), [y, \alpha(c)] + [\alpha(a), [b, \alpha(z)],
\alpha(a), [b, \alpha(c)]\right\}\right).$$

$$[((x, a), (y, b)), a \beta(z, c)] = \left([[(x, y) + [x, b] + [a, y]], [a, b]], (\alpha \beta(z), \alpha \beta(c))\right]$$

$$= \left(((x, y), [y, \beta(z)] + [x, \beta(z) + [a, \beta(z)], [a, \beta(c)]), (\alpha, \alpha)\right)$$

$$= \left([(x, \beta(z), [x, \beta(c)] + [y, \beta(z), [a, \beta(c)], (\alpha, \alpha)\right)$$

$$= \left([(x, \beta(z), [x, \beta(c)], [\alpha, \alpha] + [\alpha, \beta(z), [\alpha, \alpha] + [\alpha, \beta(c), [\alpha, \alpha]\right)$$

Using axioms in Definition 2.45 it follows that

$$[((x, a), (y, b)), a \beta(z, c)] = \left([(x, a), (y, b), \alpha, \alpha], [(y, b), \alpha, \alpha]\right).$$

Which proves the proposition. \(\square\)

Theorem 2.46. Let $(D, \cdot, +, \alpha, \beta)$ be a regular BiHom-associative dialgebra. Then the bracket defined by $[x, y] = x \cdot y - \alpha^{-1} \beta(y) \otimes \alpha^{-1}(x)$, defines a structure of BiHom-Leibniz algebra on $(D, \alpha, \beta)$. Denoted $\mathbf{Lb}(D)$.

Proof. For any $x, y, z \in D$, we have

$$[[x, y], \alpha \beta(z)] = (x + y - \alpha^{-1} \beta(y) - \alpha^{-1}(x)) + \alpha \beta(z)$$

$$= \alpha^{-1} \beta \alpha \beta(z) + (x + y - \alpha^{-1} \beta(y) - \alpha^{-1} \beta(z))$$

$$= (x + y + \alpha \beta(z) - (\alpha^{-1} \beta(y) + \alpha^{-1} \beta(z)) + \alpha \beta(z)$$

$$= -\beta^{2}(z) + \alpha \beta^{-1}(x) + \alpha \beta^{-1}(y) + \beta(z) + (y + \alpha^{2} \beta^{-2}(x)).$$

$$[[x, \beta(z), [x, \beta(c)] + [y, \beta(z), [a, \beta(c)], (\alpha, \alpha)\right)$$

$$= \left([(x, \beta(z), [x, \beta(c)], [\alpha, \alpha] + [\alpha, \beta(z), [\alpha, \alpha] + [\alpha, \beta(c), [\alpha, \alpha]\right)$$

$$= \left([(x, \beta(z), [x, \beta(c)], [\alpha, \alpha] + [\alpha, \beta(z), [\alpha, \alpha] + [\alpha, \beta(c), [\alpha, \alpha]\right)$$

By axioms in Definition 2.1 the conclusion holds. \(\square\)
In the relations contained in the below definition, we omitted the subscript for simplifying the typography.

**Definition 2.47.** Let $D$ and $L$ be dialgebras. An action of $D$ on $L$ consists of four linear maps, two of them denoted by the symbol $\cdot$ and other two by $\triangleright$,

$$\triangleright: D \otimes L \to L, \quad \triangleright: L \otimes D \to L,$$

such that the following 30 equalities hold:

1. $(x + a) \cdot (a + b) = a(x) + (a + b)$,
2. $(x + a) \cdot (a + b) = a(x) + (a + b)$,
3. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
4. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
5. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
6. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
7. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
8. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
9. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
10. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
11. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
12. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
13. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
14. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
15. $(x \cdot a) \cdot (a + b) = a(x) \cdot (a + b)$,
16. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
17. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
18. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
19. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
20. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
21. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
22. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
23. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
24. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
25. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
26. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
27. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
28. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
29. $(x + a) \cdot (a + b) = a(x) + (a + y)$,
30. $(x + a) \cdot (a + b) = a(x) + (a + y)$,

for all $x, y \in D, a, b \in L$. The action is called trivial if these four maps are trivial.

**Example 2.48.** i) Any BiHom-associative dialgebra may be seen as acting on itself. ii) Given a homomorphism $\varphi : D \to L$ of BiHom-associative dialgebras, then there is an action of $D$ on $L$ via the maps $x \cdot a := \varphi(x) \cdot a, x \triangleright a := \varphi(x) \triangleright a \triangleright a := \varphi(x)$ and $a \cdot x := a \cdot \varphi(x)$.

iii) If $\psi : L \to D$ is an isomorphism of BiHom-associative dialgebras, then there is an action of $D$ on $L$ via the maps $x \cdot a := \psi^{-1}(x) \cdot a, x \triangleright a := \psi^{-1}(x) \triangleright a$, $a \cdot x := a \cdot \psi^{-1}(x)$ and $a \cdot x := a \cdot \psi^{-1}(x)$.

iv) If $I$ is an ideal of $D$, then the left and the right product yield an action of $D$ on $I$.

**Lemma 2.49.** Given two regular BiHom-associative dialgebras $D$ and $L$ together with an action of $D$ on $L$, there is an action an action $Lb(D)$ on $Lb(L)$ given by

$$[x, a] = x \cdot a - a^{-1} \beta(a) + a \beta^{-1}(x),$$

$$[a, x] = a \triangleright x - a^{-1} \beta(x) + a \beta^{-1}(a),$$

for all $x \in Lb(D), a \in Lb(L)$.

**Proof.** For all $x \in Lb(D), a \in Lb(L)$,

$$[[x, a], \alpha(b)] = (x \cdot a - a^{-1} \beta(a) + a \beta^{-1}(x)) \cdot \alpha(b) - a^{-1} \beta(a) \cdot (x \cdot a - a^{-1} \beta(a) + a \beta^{-1}(x)) + \beta(a) \cdot (x \cdot a - a^{-1} \beta(a) + a \beta^{-1}(x)) + \beta(a) \cdot (x \cdot a - a^{-1} \beta(a) + a \beta^{-1}(x)) + \beta(a) \cdot (x \cdot a - a^{-1} \beta(a) + a \beta^{-1}(x)).$$
On the other hand,
\[
[[x,\beta(b)],\alpha(a)] + [\alpha(x),[a,\alpha(b)]] = \\
( x + \beta(b) - \alpha^{-1}\beta^2(b) + \alpha\beta^{-1}(x)) + \alpha(a) - \alpha^{-1}\beta\alpha(a) + \alpha\beta^{-1}(x + \beta(b) - \alpha^{-1}\beta^2(b) + \alpha\beta^{-1}(x)) \\
+ \alpha(x) + (a + \alpha(b) - \alpha^{-1}\beta\alpha(b) + \alpha\beta^{-1}(a)) - \alpha^{-1}\beta(a + \alpha(b) - \alpha^{-1}\beta\alpha(b) + \alpha\beta^{-1}(a)) + \alpha\beta^{-1}(a(x)) \\
= (x + \beta(b)) \alpha(a) - (\alpha^{-1}\beta^2(b) + \alpha\beta^{-1}(x)) + \alpha(a) - \beta(a) + (\alpha\beta^{-1}(x) + \alpha(b)) \\
+ \beta(a) + \beta^-1\beta^2(x)) + (\alpha(x) + (a + \alpha(b)) - \alpha(x) + \beta(b) + \alpha\beta^{-1}(a)) \\
- (\alpha^{-1}\beta(a) + \beta(b)) + \beta^{-1}\alpha^2(x) + (\alpha^{-1}\beta^2(b) + a) + \beta^{-1}\alpha^2(x).
\]
Using axioms (2.3), (2.5), it comes
\[
[[x,\beta(b)],\alpha(a)] + [\alpha(x),[a,\alpha(b)]] = \\
- (\alpha^{-1}\beta^2(b) + \alpha\beta^{-1}(x)) + \alpha(a) - \beta(a) + (\alpha\beta^{-1}(x) + \alpha(b)) \\
+ \alpha(x) + (a + \alpha(b)) + (\alpha^{-1}\beta^2(b) + a) + \alpha^2\beta^{-1}(x).
\]
By comparing, we get the attended result. The five other axioms are proved in the same way. □

**Lemma 2.50.** Let $D$ and $L$ be two regular BiHom-associative dialgebras together with an action of $D$ on $L$. There is a BiHom-associative dialgebra structure on $L \rtimes D$ which consists with vector space $L \oplus D$ and
\[
(a,x) \circ (b,y) = (a + b + a + y + x + b, x + y), \\
(a,x) \	riangleright (b,y) = (a + b + a + y + x + b, x + y),
\]
for any $(a,x),(b,y) \in L \times D$

**Proof.** For any $a,b,c \in L,x,y,z \in D$, one has
\[
((a,x) \circ (b,y)) \triangleright (b,c,z) - \alpha(a,x) \circ ((b,y) \triangleright (c,z)) = \\
= (a + b + a + y + x + b, x + y) \triangleright (b,c,z)) - (\alpha(a),\alpha(x)) \circ (b + c + b + z + y + c, y + z) \\
= ((a + b + a + y + x + b) + \beta(c)) + (a + b + a + y + x + b) + \beta(z) + (x + y) + \beta(c), (x + y) + \beta(z) \\
- (\alpha(a) + (b + c + b + z + y + c) + \alpha(a) + (y + z) + \alpha(a) + (b + c + b + z + y + c), \alpha(x) + (y + z)) \\
= ((a + b + \beta(c) + (a + y) + \beta(c) + (a + b) + \beta(c) + (a + y) + \beta(c) + (x + y) + \beta(c)) + (a + y) + \beta(c) + (x + y) + \beta(c)) + (a + y) + \beta(c) + (x + y) + \beta(c) \\
+ (x + y) + \beta(c) + (a + b) + \beta(c) - \alpha(a) + (b + c) - \alpha(a) + (b + c) - \alpha(a) + (y + z) - \alpha(a) + (x + y) + \beta(c)) \\
- (\alpha(x) + (b + c) - \alpha(x) + (y + c), (x + y) + \beta(c) - \alpha(x) + (y + z)).
\]
The left hand side vanishes by axiom (2.3) and axioms (02),(07),(12),(17),(22),(27) in Definition 2.47. The other axioms are proved in the same way. □

**Theorem 2.51.** Let $D$ and $L$ be two regular BiHom-associative dialgebras together with an action of $D$ on $L$. Then, $Lb(L \rtimes D) = Lb(L) \rtimes Lb(D)$.

**Proof.** By lemma 2.49, $Lb(D)$ acts on $Lb(L)$, so it makes sense to consider the semidirect product Leibniz algebra $Lb(L) \rtimes Lb(D)$. It is clear that $Lb(L \rtimes D)$ and $Lb(L) \rtimes Lb(D)$ are egal as vector space, so we only need to verify that they share the same bracket. Let $(a,x),(b,y) \in L \times D$. If we use the bracket in $Lb(L) \rtimes Lb(D)$, we get:
\[
[(a,x),(b,y)] = ([(a,b) + [x, b] + [a,y]],[x,y]) = \\
(a + y - b + x + b + x + a + y - y + a, x + y - y + x).
\]
On the other hand, if we use the Leibniz bracket in $Lb(L times D)$ (Lemma 2.50), we get

$$\{(a,x),(b,y)\} = (a,x) \langle (b,y) - (b,y) \rangle (a,x)$$

$$= (a+b+x+b+a+y, x+y) - (b+a+y+a+b+x, y+x),$$

So the brackets are equal. $\square$

3 Central extensions

This section concerns the central extension of BiHom-associative dialgebras in relation with cocycles.

**Definition 3.1.** Let $(D_i, \cdot_i, \circ_i, \alpha_i, \beta_i), i = 1, 2, 3$ be three BiHom-associative dialgebras. The BiHom-associative dialgebra $D_2$ is called the extension of $D_1$ by $D_1$ if there are homomorphisms $\phi : D_1 \to D_2$ and $\psi : D_2 \to D_3$ such that the following sequence

$$0 \to D_1 \xrightarrow{\phi} D_2 \xrightarrow{\psi} D_3 \to 0$$

is exact.

**Definition 3.2.** An extension is called trivial if there exists a BiHom-ideal $I$ of $D_2$ complementary to $\text{Ker} \psi$ i.e.

$$D_2 = \text{Ker} \psi \oplus I$$

It may happen that there exist several extensions of $D_1$ by $D_1$. To classify extensions the notion of equivalent extensions is defined.

**Definition 3.3.** Two sequences

$$0 \to D_1 \xrightarrow{\phi} D_2 \xrightarrow{\psi} D_3 \to 0$$

and

$$0 \to D_1 \xrightarrow{\phi'} D_2 \xrightarrow{\psi'} D_3 \to 0$$

are equivalent extensions if there exists a associative dialgebra isomorphism $f : D_2 \to D_2'$ such that $f \circ \phi = \phi'$ and $\psi' \circ f = \psi$.

**Definition 3.4.** An extension

$$0 \to D_1 \xrightarrow{\phi} D_2 \xrightarrow{\psi} D_3 \to 0$$

is called central if the kernel of $\psi$ is contained in the center $Z(D_2)$ of $D_2$, i.e. $\text{Ker} \psi \subset Z(D)$. Now, we introduce 2-cocycle on BiHom-associative dialgebra with values in a BiHom-module.

**Definition 3.5.** Let $(D, \cdot, \circ, \alpha, \beta)$ be a BiHom-associative dialgebra and $(M, \alpha_M, \beta_M)$ a BiHom-module over the same field that $D$. A pair $\Theta = (\theta_1, \theta_2)$ of bilinear maps $\theta_1 : D \times D \to V$ and $\theta_2 : D \times D \to V$ is called a 2-cocycle on $D$ with values in $V$ if $\theta_1$ and $\theta_2$ satisfy

$$\theta_1(x + y, \beta(z)) = \theta_1(x, y + z), \quad (3.1)$$

$$\theta_1(x + y, \beta(z)) = \theta_1(x, y + z), \quad (3.2)$$

$$\theta_2(x + y, \beta(z)) = \theta_2(x, y + z), \quad (3.3)$$

$$\theta_2(x + y, \beta(z)) = \theta_2(x, y + z), \quad (3.4)$$

$$\theta_1(x + y, \beta(z)) = \theta_2(x, y + z), \quad (3.5)$$

for all $x, y, z \in D$. 
The set of all 2-cocycles on $D$ with values in $M$ is denoted $Z^2(D, M)$, which a vector space.

In the below lemma, we give a special type of 2-cocycles which are called 2-coboundaries.

**Lemma 3.6.** Let $\nu : D \rightarrow V$ be a linear map, and define $\varphi_1(x, y) = \nu(x \cdot y)$ and $\varphi_2(x, y) = \nu(x \cdot y)$. Then, $\Phi = (\varphi_1, \varphi_2)$ is a 2-cocycle on $D$.

**Proof.** We will prove one equality, the others being proved in the same way. For any $x, y, z \in D$, one has

$$\varphi_1(\alpha(x), y \cdot z) = \nu(\alpha(x) \cdot (y \cdot z)) = \nu((x \cdot y) + \beta(z)) = \nu(\alpha(x) \cdot (y \cdot z)) = \varphi_1(\alpha(x), y \cdot z).$$

This finishes the proof.

The set of all 2-cocycles is denoted by $B^2(D, M)$ and it is a subgroup of $Z^2(D, M)$. The group $H^2(D, M) = Z^2(D, M)/B^2(D, M)$ is said to be a second cohomology group of $D$ with values in $M$. Two cocycles $\Theta_1$ and $\Theta_2$ are said to be cohomologous cocycles if $\Theta_1 - \Theta_2$ is a coboundary.

**Theorem 3.7.** Let $(D, +, \cdot, \alpha_D, \beta_D)$ be a BiHom-associative dialgebra, $(M, \alpha_M, \beta_M)$ a BiHom-module, $\theta_1 : D \times D \rightarrow M$ and $\theta_2 : D \times D \rightarrow M$ be bilinear maps. Let us set $D_\Theta = D \oplus M$, where $\Theta = (\theta_1, \theta_2)$. For any $x, y \in D$, $v, w \in M$, let us define

$$(x + u) \triangleleft (y + v) = x \cdot y + \theta_1(x, y) \text{ and } (x + u) \triangleright (y + v) = x \cdot y + \theta_2(x, y).$$

Then, $(D_\Theta, \triangleleft, \triangleright, \alpha_D \otimes \alpha_M, \beta_D \otimes \beta_M)$ is a BiHom-associative dialgebra if and only if $\Theta$ is a 2-cocycle.

**Proof.** For any $x, y, z \in D, u, v, w \in M$, we have

$$((x + v) \triangleleft (y + w)) \triangleleft (\beta(z) + w) - (\alpha(x) + v) \triangleleft ((y + w) \triangleleft (z + w)) =$$

$$= ((x + v) \triangleleft (y + w)) \triangleright (\beta(z) + w) - (\alpha(x) + v) \triangleright ((y + z) + \theta_1(y, z)) =$$

$$= ((x + y) \cdot \beta(z) + \theta_1(x + y, \beta(z)) - (\alpha(x) + y \cdot z) - \theta_1(\alpha(x), y \cdot z).$$

The left hand vanishes by axioms (2.2) and (3.1). The other axioms are proved analogously.

**Lemma 3.8.** Let $\Theta$ be a 2-cocycle and $\Phi$ a 2-coboundary. Then, $D_{\Theta + \Phi}$ is a BiHom-associative dialgebra with

$$(x + u) \triangleleft (y + v) = x \cdot y + \varphi_1(x, y) + \theta_1(x, y),$$

$$(x + u) \triangleright (y + v) = x \cdot y + \varphi_2(x, y) + \theta_2(x, y).$$

Moreover, $D_\Theta \equiv D_{\Theta + \Phi}$.

**Proof.** First, we have to shown that $D_{\Theta + \Phi}$ is a BiHom-associative dialgebra. So, for any $x + u, y + v, z + w \in D \oplus M$,

$$((x + u) \triangleleft (y + v)) \triangleleft \beta(z + w) - \alpha(x + u) \triangleleft ((y + v)) \triangleleft (z + w)) =$$

$$= (x + y + \varphi_1(x, y) + \theta_1(x, y)) \triangleleft (\beta(z) + \beta(w)) - (\alpha(x) + \alpha(u)) \triangleleft (y + z + \varphi_1(y, z) + \theta_1(y, z)) =$$

$$= (x + y + \beta(z) + \varphi_1(x + y, \beta(z)) + \theta_1(x + y, \beta(z)) - \alpha(x) + y + \beta(z) + \varphi_1(\alpha(x), y + z) + \theta_1(\alpha(x), y + z).$$

The left hand side vanishes by (2.2) and (3.1). The proofs of the rest of axioms are leaved to the reader.
Next, the isomorphism $f : D_\Theta \to D_{\Theta + \Phi}$ is given by $f(x + v) = x + \nu(x) + v$. In fact, it is clear that $f$ is a bijective linear map and

$$f(\alpha_D + \alpha_M)(x + v) = f(\alpha_D(x) + \alpha_M(v))$$
$$= \alpha_D(x) + \nu \alpha_D(x) + \alpha_M(v)$$
$$= \alpha_D(x) + \alpha_M \nu(x) + \alpha_M(v)$$
$$= (\alpha_D + \alpha_M)(x + \nu(x) + v)$$
$$= (\alpha_D + \alpha_M) \circ f(x + v).$$

Thus, $f$ commutes $\sigma_D + \sigma_M$, and similarly with $\beta_D + \beta_M$. Then,

$$f((x + v) \triangleright (y + w)) = f(x + y + \theta_1(x, y))$$
$$= f(x + y) + f(\theta_1(x, y))$$
$$= x + y + \nu(x + y) + \theta_1(x, y)$$
$$= x + y + \psi_1(x, y) + \theta_1(x, y).$$

and

$$f(x + v) \trianglelefteq f(y + w) = (x + \nu(x) + v) \trianglelefteq (y + \nu(y) + w)$$
$$= (x + y) + \psi_1(x, y) + \theta_1(x, y).$$

□

**Corollary 3.9.** Let $\Theta_1, \Theta_2$ be two cohomologous 2-cocycles on a BiHom-associative dialgebra $D$, and $D_1, D_2$ be the central extensions constructed with these 2-cocycles, respectively. The central extensions $D_1$ and $D_2$ are equivalent extensions. In particular a central extension defined by a coboundary is equivalent with a trivial central extension.

The following theorem is proved Mutatis Mutandis as ([25], Theorem 4.1). So we omitted the proof.

**Theorem 3.10.** There exists one to one correspondence between elements of $H^2(D, M)$ and nonequivalent central extensions of associative dialgebra $D$ by $M$.

**4 Classification**

In this section, we give classification of BiHom-associative dialgebras in low dimension.

Let $(D, \cdot, \triangleright, \alpha, \beta)$ be an $n$-dimensional BiHom-associative dialgebra, $\{e_i\}$ be a basis of $D$. For any $i, j \in \mathbb{N}, 1 \leq i, j \leq n$, let us put

$$e_i \triangleright e_j = \sum_{k=1}^{n} a_{ij}^k e_k, \quad e_i \triangleright e_j = \sum_{k=1}^{n} b_{ij}^k e_k, \quad \alpha(e_j) = \sum_{k=1}^{n} a_{kj} e_k, \quad \beta(e_j) = \sum_{k=1}^{n} b_{kj} e_k.$$
The axioms in Definition 2.1 are respectively equivalent to

\[
\begin{align*}
\beta_{kj} \alpha_{pk} - \alpha_{ji} \beta_{pj} &= 0, \\
\gamma_{i}^{p} \beta_{qk} \gamma_{pq}^{r} - \alpha_{pi} \gamma_{jk}^{q} \gamma_{pq}^{r} &= 0, \\
\gamma_{i}^{p} \beta_{qk} \gamma_{pq}^{r} - \alpha_{pi} \delta_{jk}^{q} \gamma_{pq}^{r} &= 0, \\
\gamma_{i}^{p} \beta_{qk} \delta_{pq}^{r} - \alpha_{pi} \delta_{jk}^{q} \delta_{pq}^{r} &= 0, \\
\delta_{i}^{p} \beta_{qk} \gamma_{pq}^{r} - \alpha_{pi} \delta_{jk}^{q} \gamma_{pq}^{r} &= 0, \\
\delta_{i}^{p} \beta_{qk} \delta_{pq}^{r} - \alpha_{pi} \delta_{jk}^{q} \delta_{pq}^{r} &= 0.
\end{align*}
\] (4.1) (4.2) (4.3) (4.4) (4.5) (4.6)

### 4.1 One dimensional

There is only one 1-dimensional BiHom-associative dialgebra; the nul (or trivial) BiHom-associative dialgebra.

### 4.2 Two dimensional

| Algebras | Multiplications | Morphisms \( \alpha, \beta \) |
|----------|-----------------|-----------------------------|
| \( \mathcal{Alg}_1 \) | \( e_1 \cdot e_2 = ae_1, \) <br>\( e_2 \cdot e_1 = be_1, \) <br>\( e_1 \cdot e_2 = ce_1, \) <br>\( e_2 \cdot e_1 = de_1, \) <br>\( e_2 \cdot e_2 = fe_1. \) | \( \alpha(e_2) = e_1, \) <br>\( \beta(e_2) = e_1. \) |
| \( \mathcal{Alg}_2 \) | \( e_1 \cdot e_2 = ae_1, \) <br>\( e_2 \cdot e_1 = ae_1, \) <br>\( e_2 \cdot e_2 = e_1, \) <br>\( e_1 \cdot e_2 = e_1, \) <br>\( e_2 \cdot e_1 = e_1. \) | \( \alpha(e_2) = e_1, \) <br>\( \beta(e_2) = e_1. \) |
| \( \mathcal{Alg}_3 \) | \( e_1 \cdot e_2 = ae_1, \) <br>\( e_1 \cdot e_2 = be_1, \) <br>\( e_2 \cdot e_1 = ce_1, \) <br>\( e_2 \cdot e_2 = de_1. \) | \( \alpha(e_2) = e_1, \) <br>\( \beta(e_2) = e_1. \) |
| \( \mathcal{Alg}_4 \) | \( e_1 \cdot e_2 = e_1, \) <br>\( e_2 \cdot e_1 = e_1, \) <br>\( e_2 \cdot e_2 = ae_1, \) <br>\( e_1 \cdot e_2 = be_1, \) <br>\( e_2 \cdot e_1 = ce_1, \) <br>\( e_2 \cdot e_2 = de_1. \) | \( \alpha(e_2) = e_1, \) <br>\( \beta(e_2) = e_1. \) |

**Remark 4.1.** In two dimensional, all of the BiHom-associative dialgebras are Hom-associative dialgebras i.e. \( \alpha = \beta. \)

### 4.3 Three dimensional

| Algebras | Multiplications | Morphisms \( \alpha, \beta \) |
|----------|-----------------|-----------------------------|
| \( \mathcal{Alg}_1 \) | \( e_1 \cdot e_2 = e_1, \) <br>\( e_2 \cdot e_1 = e_1, \) <br>\( e_2 \cdot e_2 = ae_1, \) <br>\( e_2 \cdot e_3 = be_1, \) <br>\( e_3 \cdot e_2 = fe_1, \) | \( \alpha(e_2) = e_1, \) <br>\( \beta(e_2) = e_1, \) <br>\( \beta(e_3) = be_3. \) |
### 4.4 Four dimensional

| Algebras | Multiplications | Morphisms $\alpha, \beta$. |
|----------|-----------------|-----------------------------|
| $\mathcal{A}lg_1$ | $e_1 + e_1 = e_4$, $e_2 + e_1 = e_4$, $e_3 + e_1 = e_4$, $e_4 + e_1 = e_4$, $e_1 + e_2 = e_4$, $e_2 + e_2 = ce_4$, $e_3 + e_2 = e_4$, $e_4 + e_2 = e_4$, $\alpha(e_2) = be_2$, $\beta(e_3) = e_2$, $\beta(e_4) = e_3$, | |
| $\mathcal{A}lg_2$ | $e_1 + e_2 = e_4$, $e_2 + e_1 = e_4$, $e_3 + e_1 = e_4$, $e_3 + e_2 = e_4$, $\alpha(e_2) = e_2$, $\beta(e_3) = e_2$, $\beta(e_4) = e_3$, | |
| $\mathcal{A}lg_3$ | $e_1 + e_4 = e_4$, $e_2 + e_1 = e_4$, $e_2 + e_2 = be_4$, $e_2 + e_3 = be_4$, $e_3 + e_1 = ce_4$, $e_3 + e_2 = e_4$, $\alpha(e_2) = e_2$, $\beta(e_3) = e_2$, $\beta(e_4) = e_3$, | |
| $\mathcal{A}lg_4$ | $e_1 + e_4 = e_4$, $e_2 + e_1 = e_4$, $e_2 + e_2 = be_4$, $e_2 + e_3 = be_4$, $e_3 + e_1 = ce_4$, $e_3 + e_2 = e_4$, $\alpha(e_2) = e_2$, $\beta(e_3) = e_2$, $\beta(e_4) = e_3$, | |
| $\mathcal{A}lg_5$ | $e_1 + e_2 = e_1$, $e_2 + e_1 = e_1$, $e_2 + e_2 = e_1$, $e_2 + e_3 = e_1$, $e_3 + e_2 = e_1$, $\alpha(e_2) = e_1$, $\beta(e_2) = e_1$, $\beta(e_3) = be_3$, | |
| Algebras | Multiplications | Morphisms $\alpha, \beta$. |
|----------|-----------------|-----------------------------|
| $\text{Alg}_5$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_1 \cdot e_2 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_6$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_1 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_7$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_3) = e_3, \beta(e_2) = e_1, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_8$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_3) = e_3, \beta(e_3) = e_2, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_9$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_3) = e_3, \beta(e_3) = e_2, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_{10}$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_3) = e_3, \beta(e_3) = e_2, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_{11}$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_3) = e_3, \beta(e_3) = e_2, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_{12}$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_3) = e_3, \beta(e_3) = e_2, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_{13}$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_3) = e_3, \beta(e_3) = e_2, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
| $\text{Alg}_{14}$ | $e_1 \cdot e_4 = e_4, e_2 \cdot e_3 = e_4, e_2 \cdot e_3 = e_4$ | $\alpha(e_2) = e_2, \beta(e_2) = e_1, \alpha(e_3) = e_3, \beta(e_3) = e_2, \alpha(e_4) = e_4, \beta(e_4) = e_3$ |
Example 5.2. We consider the 2-dimensional BiHom-associative algebra with a basis \{e_1, e_2\}. For \(\mu(e_1, e_1) = -e_1\), \(\mu(e_1, e_2) = -e_2\), \(\mu(e_2, e_1) = 0\), \(\mu(e_2, e_2) = e_2\) and \(\alpha(e_1) = e_1\), \(\alpha(e_2) = -e_2\), \(\beta(e_1) = e_1\), \(\beta(e_2) = e_2\). A direct computation gives that: \(D(e_1) = d_{22} e_1\), \(D(e_2) = d_{22} e_2\), \(\alpha^x(e_1) = \frac{\alpha_1 \beta_{21}}{\beta_{21}} e_1 + \frac{\alpha_2}{\beta_{21}} e_2\), \(\alpha^x(e_2) = \alpha_2 e_1\), \(\beta'(e_1) = \frac{\alpha_1}{\beta_{21}} e_2\), \(\beta'(e_2) = \beta_2 e_1 + \beta_2 e_2\).

Definition 5.3. Let \((D, +, \cdot, \alpha, \beta)\) be a BiHom-associative dialgebra. A linear map \(D : D \to D\) is called an \((\alpha^k, \beta^l)\)-derivation of \(D\) if it satisfies

1. \(D \circ \alpha = \alpha \circ D\) and \(D \circ \beta = \beta \circ D\);
2. \(D(x \cdot y) = \alpha^k \beta^l(x) \cdot D(y) + D(x) \cdot \alpha^k \beta^l(y)\);
3. \(D(x + y) = \alpha^k \beta^l(x) + D(x) + \alpha^k \beta^l(y)\),

for \(x, y \in D\).

We denote by \(\text{Der}(D) := \bigoplus_{k \geq 0} \bigoplus_{l \geq 0} \text{Der}(\alpha^k, \beta^l)(D)\), where \(\text{Der}(\alpha^k, \beta^l)(D)\) is the set of all \((\alpha^k, \beta^l)\)-derivations of \(D\).

Proposition 5.4. For any \(D \in \text{Der}(\alpha^*, \beta^*)(A)\) and \(D' \in \text{Der}(\alpha'^*, \beta'^*)(A)\), we have \([D, D'] \in \text{Der}(\alpha'^* \circ \sigma, \beta'^* \circ \sigma')(A)\).
Proof. For $x,y \in A$, we have

$$[D, D'] \circ \mu(x, y) = D \circ D' \circ \mu(x, y) - D' \circ D \circ \mu(x, y)$$

$$= D(\mu(D'(x), \alpha^x \beta^y(y)) + \mu(\alpha^x \beta^y(x), D'(y)))$$

$$- D'(\mu(D(x), \alpha^x \beta^y(y)) + \mu(\alpha^x \beta^y(x), D(y)))$$

$$= \mu(D \circ D'(x), \alpha^{x+\beta^y} \beta^{x+\beta^y}(y)) + \mu(\alpha^x \beta^y \circ D'(x), D \circ \alpha^x \beta^y(y))$$

$$+ \mu(D \circ \alpha^x \beta^y(x), \alpha^{x+\beta^y} \beta^{x+\beta^y}(y), D \circ D'(y))$$

$$- \mu(D'(x), \alpha^{x+\beta^y} \beta^{x+\beta^y}(y)) - \mu(\alpha^{x+\beta^y} \beta^{x+\beta^y}(x), D'(x), D' \circ \alpha^x \beta^y(y))$$

$$- \mu(D' \circ \alpha^x \beta^y(x), \alpha^x \beta^y D(y)) - \mu(\alpha^{x+\beta^y} \beta^{x+\beta^y}(x), D' \circ D(y)).$$

Since $D$ and $D'$ satisfy $D \circ \alpha = \alpha \circ D, D' \circ \alpha = \alpha \circ D'$, $D \circ \beta = \beta \circ D, D' \circ \beta = \beta \circ D'$. We obtain $\alpha^x \beta^y \circ D' = \alpha^x \beta^y \circ \alpha^x \beta^y = \alpha^x \beta^y \circ D$. Therefore, we arrive at

$$[D, D'] \circ \mu(x, y) = \mu(\alpha^{x+\beta^y} \beta^{x+\beta^y}(x), [D, D'](y)) + \mu([D, D'](x), \alpha^{x+\beta^y} \beta^{x+\beta^y}(y)).$$

Furthermore, it is straightforward to see that

$$[D, D'] \circ \alpha = D \circ D' \circ \alpha - D' \circ D \circ \alpha$$

$$= \alpha \circ D \circ D' - \alpha \circ D' \circ D = \alpha \circ [D, D'].$$

$$[D, D'] \circ \beta = D \circ D' \circ \beta - D' \circ D \circ \beta$$

$$= \beta \circ D \circ D' - \beta \circ D' \circ D = \beta \circ [D, D'].$$

which yields that $[D, D'] \in \text{Der}_{(\alpha^x \beta^y \circ \alpha^x \beta^y)}(A)$ with $\mu = \ldots \ldots$.

\[ \square \]

**Proposition 5.5.** The space $\text{Der}_{(\alpha^x \beta^y)}(A)$ is an invariant of the triple BiHom-associative algebra $A$.

Proof. Let $\sigma: (A, \cdot_A, \cdot_{A'}, \cdot_{A''}, \cdot_{A'''} \beta') \to (B, \cdot_B, \cdot_{B'}, \cdot_{B''}, \cdot_{B'''} \beta')$ be a triple BiHom-associative algebra isomorphism and let $D$ be a $(\alpha^x, \beta')$-derivation of $A$. Then for any $x, y, z \in B$. We have:

$$\sigma_D \circ \sigma^{-1} \circ ((x \cdot_B (y)) \cdot_B (z)) = \sigma_D \circ ((\sigma^{-1}(x) \cdot_{A'} (y)) \cdot_{A'} (z))$$

$$= \sigma_D \circ \sigma^{-1}(x) \cdot_{A'} \sigma^{-1}(y) \cdot_{A'} \sigma^{-1}(z)$$

$$= \sigma_D \circ \sigma^{-1}(x) \cdot_{A'} \sigma^{-1}(y) \cdot_{A'} \sigma^{-1}(z) + \sigma_D(\sigma^{-1}(x) \cdot_{A'} \sigma^{-1}(y) \cdot_{A'} \sigma^{-1}(z))$$

$$= \sigma_D(\sigma^{-1}(x) \cdot_{A'} \sigma^{-1}(y) \cdot_{A'} \sigma^{-1}(z)) + \sigma_D(\sigma^{-1}(x) \cdot_{A'} \sigma^{-1}(y) \cdot_{A'} \sigma^{-1}(z))$$

$$+ \sigma_D(\sigma^{-1}(x) \cdot_{A'} \sigma^{-1}(y) \cdot_{A'} \sigma^{-1}(z)) + \sigma_D(\sigma^{-1}(x) \cdot_{A'} \sigma^{-1}(y) \cdot_{A'} \sigma^{-1}(z))$$

Thus $\sigma \circ D \circ \sigma^{-1}$ is a $(\alpha^x, \beta')$-derivation of $B$, hence the mapping $\psi: \text{Der}_{(\alpha^x \beta')}(A) \to \text{Der}_{(\alpha^x \beta')}(B), \ D \mapsto \sigma_D \circ \sigma^{-1}$ is an isomorphism of triple BiHom-associative algebras.

In fact, it is easy to see that $\psi$ is linear. Moreover let $D_1, D_2, D_3$ be derivations of $A$:

$$\alpha^x \beta^y \circ \psi(D_1 \cdot_{tr} D_2) \cdot_{tr} D_3 =$$

$$= \alpha^x \beta^y \circ \psi(D_1 \cdot_{tr} D_2) \cdot_{tr} D_3 + \alpha^x \beta^y \circ \psi(D_2 \cdot_{tr} D_3) \cdot_{tr} D_1$$

$$= \alpha^x \beta^y \circ \psi(D_1 \cdot_{tr} D_2) + \alpha^x \beta^y \circ \psi(D_2 \cdot_{tr} D_3) + \alpha^x \beta^y \circ \psi(D_3 \cdot_{tr} D_1)$$

$$= \alpha^x \beta^y \circ \psi((\psi(D_1) + \psi(D_2)) \cdot_{tr} D_3) + \alpha^x \beta^y \psi(D_3)$$

Thus $\sigma \circ D \circ \sigma^{-1}$ is a $(\alpha^x, \beta')$-derivation of $B$, hence the mapping $\psi: \text{Der}_{(\alpha^x \beta')}(A) \to \text{Der}_{(\alpha^x \beta')}(B)$, $D \mapsto \sigma_D \circ \sigma^{-1}$ is an isomorphism of triple BiHom-associative algebras.

In fact, it is easy to see that $\psi$ is linear. Moreover let $D_1, D_2, D_3$ be derivations of $A$:

$$\alpha^x \beta^y \circ \psi(D_1 \cdot_{tr} D_2) \cdot_{tr} D_3 =$$

$$= \alpha^x \beta^y \psi(tr(D_1)(D_2 + D_3)) + \alpha^x \beta^y \psi(tr(D_2)(D_1 + D_3))$$

$$= \alpha^x \beta^y \psi(D_1) + \alpha^x \beta^y \psi(D_3)$$

Then $\alpha^x \beta^y \psi((\psi(D_1) + \psi(D_2)) \cdot_{tr} D_3) = \alpha^x \beta^y ((\psi(D_1) + \psi(D_2)) \cdot_{tr} \psi(D_3)).$
References

[1] A. Frolicher, A. Nijenhuis, *Theory of vector valued differential forms*, Part I. Indag Math, 1956, 18: 338-360

[2] A. Zahari and A. Makhlouf, Structure and Classification of Hom-Associative Algebras, Acta et commentationes universitatis Tartuensis de mathematica, vol 24 (1) 2020.

[3] A. Kitouni, A. Makhlouf, S. Silvestrov, *On n-ary Generalization of BiHom-Lie algebras and BiHom-Associative Algebras*, arXiv:1812.00094, 2018.

[4] A. Majumdar, and G. Mukherjee, (2002). Deformation theory of dialgebras. K-theory, 27(1):33-60.

[5] A. P. Pozhidaev, (2008). Dialgebras and related triple systems. Siberian Mathematical Journal, 49(4):696-708.

[6] Basri, W., Rakhimov, I., Rikhsiboev, I., et al. (2015). Four-dimensional nilpotent diassociative algebras. Journal of Generalized Lie Theory and Applications, 9(1):1-7.

[7] Graziani, A. Makhlouf, C. Menini and F. Panaite, *BiHom-Associative Algebras, BiHom-Lie Algebras and BiHom-Bialgebras*, Symmetry, Integrability and Geometry: Methods and Applications SIGMA 11 (2015), 086, 34 pages.

[8] H. Adimi, T. Chtioui, S. Mabrouk, S. Massoud, *Construction of BiHom-post-Lie algebras*, arXiv:math.RA/2001.02308.

[9] I. BAKAYOKO and M. BANGOURA, *Bimodules and Rota-Baxter relations*, J. Appl. Mech. Eng 4:178, doi:10.4172/2168-9873.1000178, 2015.

[10] I. BAKAYOKO and M. BANGOURA, *Left-Hom-symmetric and Hom-Poisson dialgebras*, Konuralp Journal of Mathematics, 3 No.2, 42-53, 2015.

[11] I. Rikhsiboev, I. Rakhimov and W. Basri, (2014). Diassociative algebras and their derivations. In Journal of Physics: Conference Series, volume 553, pages 1?9. IOP Publishing.

[12] I. M. Rikhsiboev, I. Rakhimov, and W. Basri (2010). Classification of 3-dimensional complex diassociative algebras. Malaysian Journal of Mathematical Sciences, 4(2):241-254.

[13] J. Li, L. Chen, B. Sun, *BiHom-Nijenhuis operators and extensions of BiHom-Lie superalgebras*.

[14] J. M. Casas, R. F. Casado, E. Khmaladze and M. Ladra, More on crossed modules of Lie, Leibniz, associative and diassociative algebras, available as arXiv:1508.01147v1 (05.08.2015).

[15] J.-L. Loday, Dialgebras. Dialgebras and related operads, pp. 7-66, Lecture Notes in Math., 1763, Springer, Berlin, 2001.

[16] J. Carinena, J. Grabowski, G. Marmo, *Quantum bi-Hamiltonian systems*, Internat J. Modern Phys A, 2000, 15: 4797-4810.

[17] K. Abdaoui, B. H. Abdelkader and A. Makhlouf, *BiHom-Lie colour algebras structures*, arXiv 1706.02188v1[math. RT] 6 Juin 2017.
[18] L. Lin and Y. Zhang, $F [x, y]$ as a dialgebra and a Leibniz Algebra, Comm. Algebra 38(9) (2010), 3417-3447.

[19] L. Liu, A. Makhlouf, C. Menini, F. Panaite, Rota-Baxter operators on BiHom-associative algebras and related structures, arXiv:math.QA/1703.07275.

[20] L. Ling, A. Makhlouf, Claudia M. and Florin P., BiHom-Novikov algebras and infinitesimal BiHom-bialgebras, arXiv 1903.08145v1[Math. QA] 18 Mars 2019.

[21] L. Ling, A. Makhlouf, Claudia M. and Florin P., BiHom-pre-Lie algebras, BiHom-Leibniz algebras and Rota-Baxter operators on BiHom-Lie algebras, arXiv 1706.00457v2[Math. QA] 2 Fevrier 2020.

[22] L. Liu, A. Makhlouf, C. Menini, F. Panaite, $\{\sigma, \tau\}$-ota-Baxter operators, infinitesimal Hom-bialgebras and the associative (Bi)Hom-Yang-Baxter equation, Canad. Math. Bull., DOI: 10.4153/CMB-2018-028-8.

[23] P. Leroux, Contraction of Nijenhuis operators and Dendriform trialgebras, February 2004.

[24] P. Kolesnikov, and V. Y. Voronin, (2013). On special identities for dialgebras. Linear and Multilinear Algebra, 61(3):377-391.

[25] S. Isamiddin and Rakhimov, On central Extensions of Associative Dialgebras, J. Physics : conf. Ser. 697 (2016).

[26] Salazar-Diaz, O. Velasquez, R., and Wills-Toro, L. A. (2016). Construction of dialgebras through bimodules over algebras. Linear and Multilinear Algebra, pages 1-22.

[27] S. Guo, X. Zhang, S. Wang, The construction and deformation of BiHom-Novikov algebras, J. Geom. Phys. 132 (2018), 460-472.

[28] S. Wang, S. Guo, BiHom-Lie superalgebra structures, arXiv:1610.02290v1 (2016).

[29] X. LI, BiHom-Poisson algebra and its application, International Journal of Algebra, Vol 13, 2019, no 2, 73-81.

[30] Y. Cheng, H. Qi, Representations of BiHom-Lie algebras, arXiv:1610.04302v1(2016).