SYMOMETRY APPROACH TO INTEGRABILITY AND NON-ASSOCIATIVE ALGEBRAIC STRUCTURES.

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Chapter 1

Introduction

The symmetry approach to the classification of integrable PDEs has been developed since 1979 by: A. Shabat, A. Zhiber, N. Ibragimov, A. Fokas, V. Sokolov, S. Svinolupov, A. Mikhailov, R. Yamilov, V. Adler, P. Olver, J. Sanders, J.P. Wang, V. Novikov, A. Meshkov, D. Demskoy, H. Chen, Y. Lee, C. Liu, I. Khabibullin, B. Magadeev, R. Heredero, V. Marikhin, M. Foursov, S. Startsev, M. Balakhnev, and others. It is very efficient for PDEs with two independent variables and under additional assumptions can be applied to ODEs.

The basic definition of the symmetry approach is:

**Definition** 1.1. A differential equation is integrable if it possesses infinitely many higher infinitesimal symmetries.

The reader may ask: “Why should equations integrable in one sense or another have higher symmetries?”

A not rigorous but instructive answer is the following. Linear differential equations have infinitely many higher symmetries. Integrable nonlinear differential equations known as of today are related to linear equations by some transformations. The same transformations produce higher symmetries of the nonlinear equation from symmetries of the linear one.

It turns out that the existence of higher symmetries allows one to find all integrable equations from a beforehand prescribed class of equations. The first classification result in frames of the symmetry approach was:

**Theorem 1.1.** [1] *The nonlinear hyperbolic equation of the form

\[ u_{xy} = F(u) \]

possesses higher symmetries iff (up to scalings and shifts)

\[ F(u) = e^u, \quad F(u) = e^u + e^{-u}, \quad \text{or} \quad F(u) = e^u + e^{-2u}. \]
Several reviews [2]–[7] are devoted to the symmetry approach (see also [8]–[11]). In this book I mostly concentrate on the results which are not covered by these papers and books. I consider only ODEs and PDEs with two independent variables. For integrable multi-dimensional equations of KP type, most of the methods under consideration are not applicable.

The statements are formulated in the most simple form but often possible ways for generalization are pointed out. In the proofs only essential points are mentioned while for technical details references are given. The text contains many carefully selected examples, which give a perception on the subject. A number of open problems are suggested.

The author is not a scrabble in original references. Instead, some references to reviews, where an information of pioneer works can be found, are given.

The book addresses both experts in algebra and in classical integrable systems. It is suitable for PhD students and can serve as an introduction to classical integrability for scientists with algebraic inclinations.

The exposition is based on a series of lectures delivered by the author in USP (Sao Paulo, 2015).

The contribution of my collaborators A. Mikhailov, A. Meshkov, S. Svinolupov, and A. Shabat to results presented in this book is difficult to overestimate.

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1.1 List of basic notation

1.1.1 Constants, vectors and matrices

Henceforth, the field of constants is \( \mathbb{C} \); \( \mathbf{u} \) stands for \( N \)-dimensional vector, namely \( \mathbf{u} = (u^1, \ldots, u^N) \). Moreover, the standard scalar product \( \sum_{i=1}^{N} u^i v^i \) is denoted by \( \langle \mathbf{u}, \mathbf{v} \rangle \).

The associative algebra of order “m” square matrices is denoted by \( \text{Mat}_m \); the matrix \( \{u_{ij}\} \in \text{Mat}_m \) is denoted by \( \mathbf{U} \). The unity matrix is denoted by \( \mathbf{1} \) or \( \mathbf{1}_m \). The notation \( \mathbf{U}^t \) stands for the matrix transpose of \( \mathbf{U} \).
For the set of $n \times m$ matrices we use the notation $\text{Mat}_{n,m}$.

1.1.2 Derivations and differential operators

For ODEs the independent variable is denoted by $t$, whereas for PDEs we have two independent variables $t$ and $x$. Notation $u_t$ stands for the partial derivative of $u$ with respect to $t$. For the $x$-partial derivatives of $u$ the notation $u_x = u_1$, $u_{xx} = u_2$, etc., is used.

The operator $\frac{d}{dx}$ is often denoted by $D$. For the differential operator $L = \sum_{i=0}^{k} a_i D^i$ we define the operator $L^+$ as

$$L^+ = \sum_{i=0}^{k} (-1)^i D^i \circ a_i,$$

where $\circ$ means that, in this formula, $a_i$ is the operator of multiplication by $a_i$. By $L_t$ we denote

$$L_t = \sum_{i=0}^{k} (a_i)_t D^i.$$

1.1.3 Differential algebra

We denote by $\mathcal{F}$ a differential field. For our main considerations one can assume that elements of $\mathcal{F}$ are rational functions of finite number of independent variables $u_i$. However, very often we find some functions solving overdetermined systems of PDEs. In such a case we have to extend the basic field $\mathcal{F}$. We will avoid any formal description of such extensions hoping that in any particular case it is clear what we really need from $\mathcal{F}$.

The principle derivation

$$D \stackrel{\text{def}}{=} \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i},$$

(1.1)

generates all independent variables $u_i$ starting from $u_0 = u$.

When we speak of solutions (or common solutions) of ODEs and PDEs, we mean local solutions with generic initial date.

1.1.4 Algebra

We denote by $A(\circ)$ an $N$-dimensional algebra $A$ over $\mathbb{C}$ with an operation $\circ$. A basis of $A$ is denoted by $e_1, \ldots, e_N$, and the corresponding structural constants by
\( C_{jk}^i: \)
\[ e_j \circ e_k = C_{jk}^i e_i. \]

We denote by \( U \) the element
\[
U = \sum_{i=1}^{N} u_i e_i, \tag{1.2}
\]
In what follows we assume that the summation is carried out over repeated indices.

We will use the following notation:
\( \text{As}(X, Y, Z) = (X \circ Y) \circ Z - X \circ (Y \circ Z), \tag{1.3} \)
\( [X, Y, Z] = \text{As}(X, Y, Z) - \text{As}(Y, X, Z). \tag{1.4} \)

By \( \mathcal{G} \) and \( \mathcal{A} \) we usually denote a Lie and an associative algebra, respectively.

### 1.2 Infinitesimal symmetries

Consider a dynamical system of ODEs
\[
\frac{dy^i}{dt} = F_i(y^1, \ldots, y^n), \quad i = 1, \ldots, n. \tag{1.5}
\]

**Definition 1.2.** The dynamical system
\[
\frac{dy^i}{d\tau} = G_i(y^1, \ldots, y^n), \quad i = 1, \ldots, n \tag{1.6}
\]
is called an (infinitesimal) symmetry for (1.5) iff (1.5) and (1.6) are compatible.

Informally speaking, compatibility means that for any initial data \( y_0 \) there exists a common solution \( y(t, \tau) \) of equations (1.5) and (1.6) such that \( y(0, 0) = y_0. \)

More rigorously, it means that
\[
XY - YX = 0,
\]
where
\[
X = \sum F_i \frac{\partial}{\partial y^i}, \quad Y = \sum G_i \frac{\partial}{\partial y^i}. \tag{1.7}
\]

Consider now an evolution equation
\[
u_t = F(u, u_x, u_{xx}, \ldots, u_n), \quad u_i = \frac{\partial^i u}{\partial x^i} \tag{1.8}
\]

A higher (or generalized) infinitesimal symmetry (or a commuting flow) is an evolution equation

\[ u_\tau = G(u, u_x, u_{xx}, \ldots, u_m), \quad m > 1, \tag{1.9} \]

which is compatible with (1.8). Compatibility means that

\[ \frac{\partial}{\partial t} \frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{\partial u}{\partial t}, \tag{1.10} \]

where the partial derivatives are calculated in virtue of (1.8) and (1.9). In other words, for any initial value \( u_0(x) \) there exists a common solution \( u(x, t, \tau) \) of equations (1.8) and (1.9) such that \( u(x, 0, 0) = u_0(x) \). For a more rigorous definition in terms of evolution vector fields see [11] and/or Section 2.2.

**Remark 1.1.** Infinitesimal symmetries (1.9) with \( m \leq 1 \) correspond to one-parametric groups of point or contact transformations [11]. They are called classical symmetries. We do not consider them in this paper in spite of the fact that they are related to an important class of differential substitutions for evolution equations [12].

**Example 1.1.** Any equation (1.8) has the classical symmetry \( u_\tau = u_x \), which corresponds to the one-parametric group \( x \to x + \lambda \) of shifts.

**Example 1.2.** For any \( m \) and \( n \) the equation \( u_\tau = u_m \) is a symmetry for the linear equation \( u_t = u_n \). The symmetries for different \( m \) are compatible with each other. Thus, we have an infinite hierarchy of equations such that any of them is a symmetry for others.

**Example 1.3.** The Burgers equation

\[ u_t = u_{xx} + 2uu_x \tag{1.11} \]

has the following third order symmetry

\[ u_\tau = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u_x^2u_x. \tag{1.12} \]

**Example 1.4.** The simplest higher symmetry for the Korteweg–de Vries (KdV) equation

\[ u_t = u_{xxx} + 6uu_x \tag{1.13} \]

has the following form

\[ u_\tau = u_5 + 10uu_3 + 20u_1u_2 + 30u_2^2u_1. \tag{1.14} \]
Remark 1.2. The existence of higher symmetries is a strong indication that the equation (1.8) is integrable. One can propose the following “explanation” of this fact. According to Example 1.2, the linear equation has infinitely many higher symmetries. As a rule, an integrable nonlinear equation is related to a linear one by some transformation. The same transformation produces a hierarchy of higher symmetries for nonlinear equation starting from the symmetries of the corresponding linear equation.

For instance, the Burgers equation is integrable because of the Cole–Hopf substitution

$$u = \frac{v_x}{v},$$

which relates (1.11) to the linear heat equation \(v_t = v_{xx}\). Moreover, the same substitution maps the third order symmetry (1.12) of the Burgers equation to

$$v_\tau = v_{xxx},$$

e tc.

The transformation that reduces the KdV equation to the linear equation \(v_t = v_{xxx}\) is a non-linear generalization of the Fourier transform [13, 14]. It is much more nonlocal than the Cole-Hopf substitution. Nevertheless, the inverse transformation can be applied to all symmetries \(v_\tau = v_{2n+1}\) of the linear equation to produce an infinite hierarchy of symmetries of odd order for the KdV equation.

Naive symmetry test

By an example of fifth order equations we demonstrate how to state and to solve a simple classification problem for integrable polynomial homogeneous equations.

The differential equation (1.8) is said to be \(\lambda\)-homogeneous of weight \(\mu\) if it admits the one-parameter group of scaling symmetries

$$(x, t, u) \rightarrow (\tau^{-1}x, \tau^{-\mu}t, \tau^\lambda u).$$

For \(N\)-component systems with unknowns \(u^1, ..., u^N\) the corresponding scaling group has a similar form

$$(x,t,u^1,...,u^N) \rightarrow (\tau^{-1}x, \tau^{-\mu}t, \tau^{\lambda_1}u^1,...,\tau^{\lambda_N}u^N).$$

(1.16)

Theorem 1.2. [15] Scalar \(\lambda\)-homogeneous polynomial equation with \(\lambda > 0\) may possess a homogeneous polynomial higher symmetry only if

- Case 1: \(\lambda = 2\);
- Case 2: \(\lambda = 1\);
• Case 3: \(\lambda = \frac{1}{2}\).

For example, the KdV equation (1.13) is homogeneous of weight 3 for \(\lambda = 2\), its symmetry (1.14) has the same homogeneity. The mKdV equation
\[
\dot{u}_t = u_{xxx} + u^2u_x
\]
has the weight 3 for \(\lambda = 1\) and for the Ibragimov-Shabat equation
\[
\dot{u}_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x
\]
the weight is 3 and \(\lambda = \frac{1}{2}\).

The general form of a fifth order polynomial equation of homogeneity \(\lambda = 2\) is given by
\[
\dot{u}_t = u_5 + a_1uu_3 + a_2u_1u_2 + a_3u^2u_1,
\]
where \(a_i\) are constants. Let us find all equations (1.18) having a polynomial homogeneous seventh order symmetry of the form
\[
\dot{u}_\tau = u_7 + c_1uu_5 + c_2u_1u_4 + c_3u_2u_3 + c_4u^2u_3 + c_5uu_1u_2 + c_6u^3 + c_7u^3u_1.
\]

Compatibility condition (1.10) can be rewritten in the form \(F_\tau - G_t = 0\). When we eliminate \(\tau\) and \(t\)-derivatives in virtue of (1.18) and (1.19) from this defining equation, the left-hand side becomes a polynomial of sixth degree in variables \(u_1, \ldots, u_{10}\). Linear terms are absent. Equating coefficients of quadratic terms to zero, we find that
\[
c_1 = \frac{7}{5}a_1, \quad c_2 = \frac{7}{5}(a_1 + a_2), \quad c_3 = \frac{7}{5}(a_1 + 2a_2).
\]
The conditions arising from the cubic terms allow one to express \(c_4, c_5\) and \(c_6\) in terms of \(a_1, a_2, a_3\). Moreover, it turns out that
\[
a_3 = -\frac{3}{10}a_1^2 + \frac{7}{10}a_1a_2 - \frac{1}{5}a_2^2.
\]
The fourth degree terms give us an explicit dependence of \(c_7\) on \(a_1, a_2\) and also the main algebraic relation
\[
(a_2 - a_1)(a_2 - 2a_1)(2a_2 - 5a_1) = 0
\]
for the coefficients \(a_1\) and \(a_2\). Solving this equation we find that up to the scaling \(u \to \lambda u\) only four different integrable cases are possible: the linear equation \(\dot{u}_t = u_5\), equations
\[
\dot{u}_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1, \quad (1.20)
\]
\[
\dot{u}_t = u_5 + 10uu_3 + 25u_1u_2 + 20u^2u_1, \quad (1.21)
\]
and (1.14). In each of these cases all terms of the fifth and the sixth degrees in the defining equation are canceled automatically. Equations (1.20) and (1.21) are well known [16, 17].
Question 1.1. The problem solved above looks artificial. Why do we consider the fifth order equations with the seventh order symmetry?

Under the assumption that the right-hand side of equation (1.8) is polynomial and homogeneous with \( \lambda > 0 \), it was proved in [15] that it suffices to consider the following three cases:

- a second order equation with a third order symmetry;
- a third order equation with a fifth order symmetry;
- a fifth order equation with a seventh order symmetry.

Other integrable equations belong to the hierarchies of such equations. This statement looks very credible and without any additional restrictions for the right-hand side of equation. A proof in the non-polynomial case is absent and this statement has a status of a conjecture well-known for experts. In fact, no counterexamples to this conjecture are known.

In Section 2.2 an advanced symmetry test will be considered. No restrictions such as the polynomiality of the right-hand side of (1.8) or the fixation of symmetry order are imposed there.

1.3 First integrals and local conservation laws

In the ODE case the concept of a first integral (or integral of motion) is one of the basic notions. A function \( f(y^1, \ldots, y^n) \) is called a first integral for the system (1.5) if the value of this function does not depend on \( t \) for any solution \( \{y^1(t), \ldots, y^n(t)\} \) of (1.5). Since

\[
\frac{d}{dt} \left( f(y^1(t), \ldots, y^n(t)) \right) = X(f),
\]

where the vector field \( X \) is defined by (1.7), from the algebraic point of view a first integral is a solution of the first order PDE

\[
Xf(y^1, \ldots, y^n) = 0.
\]

In the case of evolution equations of the form (1.8) an integral of motion is not a function but a functional that does not depend on \( t \) for any solution \( u(x, t) \) of equation (1.8).

More rigorously, a local conservation law for equation (1.8) is a pair of functions \( \rho(u, u_x, \ldots) \) and \( \sigma(u, u_x, \ldots) \) such that

\[
\frac{\partial}{\partial t} \left( \rho(u, u_x, \ldots, u_p) \right) = \frac{\partial}{\partial x} \left( \sigma(u, u_x, \ldots, u_q) \right) \tag{1.22}
\]
for any solution $u(x, t)$ of equation (1.8). The functions $\rho$ and $\sigma$ are called a \textit{density} and a \textit{flux} of conservation law (1.22). It is easy to see that $q = p + n - 1$, where $n$ is the order of equation (1.8).

For solutions of solitonic type, which are decreasing at $x \to \pm \infty$, we get
\[
\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \rho \, dx = 0
\]
for any polynomial conserved density with zero constant term. This justifies the name \textit{conserved density} for the function $\rho$. Analogously, if $u(x, t)$ is a function periodic in $x$ with period $L$, then the value of the functional $I_i = \int_0^L \rho \, dx$ does not depend on time and therefore it is a constant of motion.

Suppose that functions $\rho$ and $\sigma$ satisfy (1.22). Then for any function $s(u, u_x, \ldots)$ the functions $\tilde{\rho} = \rho + \frac{\partial s}{\partial u}$ and $\tilde{\sigma} = \sigma + \frac{\partial s}{\partial u_x}$ satisfy (1.22) as well. We call the conserved densities $\rho$ and $\tilde{\rho}$ \textit{equivalent}. It is clear that equivalent densities define the same functional.

\textit{Example 1.5.} Any linear equation of the form $u_t = u_{2n+1}$ has infinitely many conservation laws with densities $\rho_k = u^{2k}$. Developing the arguments of Remark 1.2, one can say that each conserved density is expected to be common to all equations of odd order from a commutative hierarchy.

\textit{Exercise 1.1.} Check that for any $n \geq 1$ the equation $u_t = u_{2n}$ has only one conserved density $\rho = u$.

\textit{Example 1.6.} Functions
\[
\rho_1 = u, \quad \rho_2 = u^2, \quad \rho_3 = -u_x^2 + 2u^3
\]
are conserved densities of the Korteweg–de Vries equation (1.13). Indeed,
\[
\frac{\partial}{\partial t} \rho_1 = \frac{\partial}{\partial x}(u_2 + 3u^2),
\]
\[
\frac{\partial}{\partial t} \rho_2 = \frac{\partial}{\partial x}(2uu_{xx} - u_x^2 + 4u^3),
\]
\[
\frac{\partial}{\partial t} \rho_3 = \frac{\partial}{\partial x}(9u^4 + 6u^2u_{xx} + u_x^2 - 12uu_x^2 - 2uxu_3).
\]

Let us find all the conserved densities of the form $\rho(u)$ for equation (1.13). It is clear that the function $\sigma$ may depend on $u, u_x, u_{xx}$ only. Relation (1.22) has the form
\[
\rho'(u)(u_3 + 6uu_x) = \frac{\partial \sigma}{\partial u_{xx}} u_3 + \frac{\partial \sigma}{\partial u_x} u_{xx} + \frac{\partial \sigma}{\partial u} u_x.
\]
(1.23)
Since $u$ is an arbitrary solution of the KdV equation, the latter expression should be an identity in variables $u, u_x, u_{xx}, u_3$. Comparing the coefficients of $u_3$, we find
\[ \sigma = \rho'(u) u_{xx} + \sigma_1(u, u_x). \] Substituting it into (1.23) and equating the coefficients of \( u_{xx} \), we obtain
\[ \sigma_1 = -\frac{\rho''(u)}{2} u_x^2 + \sigma_2(u). \]
Taking it into account, we see that the coefficients of \( u_x^3 \) give rise to \( \rho''(u) = 0 \) i.e. \( \rho = c_2 u^2 + c_1 u + c_0 \). So, up to the trivial term \( c_0 \), the density is a linear combination of the above densities \( \rho_1 \) and \( \rho_2 \).

### 1.4 Transformations

#### 1.4.1 Point and contact transformations

Let \( x_1, \ldots, x_n \) be independent variables and \( u \) is the dependent variable. All symbols
\[ x_1, \ldots, x_n, \quad u, \quad \text{and} \quad u_\alpha, \quad (1.24) \]
where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( u_\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \), are regarded as independent. Let denote by \( \mathcal{F} \) the field of "all" functions depending on finite number of variables \( (1.24) \).

The total \( x_i \)-derivatives are given by
\[ D_i = \sum_\alpha u_{(\alpha_1, \ldots, \alpha_i+1, \ldots, \alpha_n)} \frac{\partial}{\partial u_{(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)}} \]
They are vector fields such that \( [D_i, D_j] = 0 \).

We consider transformations of the form
\[ \bar{x}_i = \phi_i, \quad \bar{u} = \psi, \quad \phi_i, \psi \in \mathcal{F} \quad (1.25) \]
The new derivatives are given by
\[ \bar{u}_{(\alpha_1, \ldots, \alpha_n)} = \bar{D}_1^{\alpha_1} \cdots \bar{D}_n^{\alpha_n} (\bar{u}), \]
where
\[ \bar{D}_i = \sum_{j=1}^n p_{ij} D_j, \quad p_{ij} \in \mathcal{F}. \quad (1.26) \]
To find \( p_{ij} \), we apply (1.26) to \( \bar{x}_k = \phi_k \) and obtain
\[ \sum_{j=1}^n p_{ij} D_j (\phi_k) = \delta_k^i. \quad (1.27) \]
In other words, the matrix \( P = \{p_{ij}\} \) is inverse for the generalized Jacobi matrix \( J \) with the entries \( J_{ij} = D_i(\phi_j) \).

Transformation (1.25) is called point transformation if the functions \( \psi \) and \( \phi_i \) depend only on \( x_1, \ldots, x_n, u \). Such a transformation is (locally) invertible if \( \psi \) and \( \phi_i \) are functionally independent.

There are invertible transformations more general than point ones.

**Contact transformation**

*Example 1.7.* Consider the Legendre transformation

\[
\bar{x}_i = \frac{\partial u}{\partial x_i}, \quad \bar{u} = u - \sum_{k=1}^{n} x_k \frac{\partial u}{\partial x_k}.
\]  

(1.28)

Notice that the functions \( \psi \) and \( \phi_i \) depend on first derivatives of \( u \). For such transformations the functions \( \frac{\partial \bar{u}}{\partial \bar{x}_i} \) usually depend on second derivatives of \( u \) and so on. In this case the transformation extended to derivatives of degree not greater than \( k \) is not invertible for any \( k \). However, for transformation (1.28), according to (1.27), we have

\[
\sum_j p_{ij} \frac{\partial^2 u}{\partial x_j \partial x_k} = \delta^i_k.
\]

One can verify that

\[
\frac{\partial \bar{u}}{\partial \bar{x}_i} = \sum_j p_{ij} D_j(u - \sum_k x_k \frac{\partial u}{\partial x_k}) = -x_i.
\]  

(1.29)

Hence transformation (1.28) extended to the first derivatives by (1.29) is invertible.

The Legendre transformation is an example of a contact transformation.

**Theorem 1.3.** [18] If a transformation extended to derivatives of degree not greater than \( k \) is invertible for some \( k \), then this is either a point or a contact transformation.

Let us consider contact transformations for the case \( n = 1 \) in more details. They have the following form

\[
\bar{x} = \phi(x, u, u_x), \quad \bar{u} = \psi(x, u, u_x), \quad \bar{u}_1 = \chi(x, u, u_x),
\]  

(1.30)

where

\[
\psi_{u_1}(\phi_u u_1 + \phi_x) = \phi_{u_1}(\psi_u u_1 + \psi_x).
\]  

(1.31)
Under the latter condition the function
\[ \chi = \frac{D(\psi)}{D(\phi)} = \frac{\psi_{u_2} u_1 + \psi_x u_1 + \psi_x}{\phi_{u_2} + \phi_x u_1 + \phi_x} \]
does not depend on \( u_2 \) and the transformation is invertible if the functions \( \phi, \psi \) and \( \chi \) are functionally independent.

**Remark 1.3.** Formally, the point transformations may be regarded as a particular case of contact transformations: if the functions \( \phi, \psi \) do not depend on \( u_x \), the contact condition (1.31) disappears.

For the Legendre transformation we have
\[ \phi = u_1, \quad \psi = u - u_1. \] (1.32)
The contact condition (1.31) is fulfilled and \( \chi = -x \).

The formulas (1.30) and (1.31) become simpler if we introduce a generating function \( F \):
\[ \psi(x, u, u_1) = F(x, u, \phi(x, u, u_1)). \]
It is possible if \( \phi_{u_1} \neq 0 \). We have
\[ \bar{u}_1 = F_\phi, \quad F_x + u_1 F_u = 0 \]
or
\[ \bar{u} = F(x, u, \bar{x}), \quad \bar{u}_1 = \frac{\partial F}{\partial \bar{x}}, \quad u_1 = \left( \frac{\partial F}{\partial u} \right)^{-1}. \] (1.33)

**Exercise 1.2.** Verify that for any function \( F(x, u, \bar{x}) \) the above formulas define a contact transformation.

Point and contact transformations (1.30) are important for classification of integrable evolution equations of the form
\[ u_t = F(x, u, u_x, \ldots, u_n) \]
since they preserve the form of the equation and transform symmetries to symmetries, i.e., map integrable equations to integrable ones. Indeed, consider a contact transformation of the form
\[ \bar{t} = t, \quad \bar{x} = \phi(x, u, u_x), \quad \bar{u} = \psi(x, u, u_x), \quad \bar{u}_1 = \chi(x, u, u_x), \]
where the contact condition (1.31) holds. Then the coefficients in the relations
\[ \bar{D}_t = \alpha D_t + \beta D, \quad \bar{D} = \gamma D_t + \delta D \]
can be found from the conditions
\[ \bar{D}_t(t) = 1, \quad \bar{D}_t(\phi) = 0, \quad \bar{D}(t) = 0, \quad \bar{D}(\phi) = 1. \]
We obtain
\[ \bar{D} = \frac{1}{D(\phi)} D, \quad \bar{D}_t = D_t - \frac{D_t(\phi)}{D(\phi)} D \]
and, therefore,
\[ \bar{u}_t = \left( \psi^+ - \frac{D(\psi)}{D(\phi)} \phi^+ \right)(F). \quad (1.33) \]
The variables \( x, u, u_x, \ldots \) in the right-hand side of the latter equation are supposed to be replaced by the new variables \( \bar{x}, \bar{u}, \ldots \). Due to condition (1.31) the differential operator
\[ \psi^+ - \frac{D(\psi)}{D(\phi)} \phi^+ \quad (1.34) \]
has zero order and the evolution equation has the form
\[ \bar{u} = \bar{F}(\bar{x}, \bar{u}, \bar{u}_1, \ldots, \bar{u}_n) \]
with the same \( n \).

**Exercise 1.3.** Verify that the heat equation \( u_t = u_x^2 \) transforms to
\[ \bar{u}_t = -\frac{1}{\bar{u}_2} \]
under the Legendre transformation (1.32).

**Remark 1.4.** If we restrict ourselves with the evolution equations, where the function \( F \) does not depend on \( x \) explicitly, then we still can use a subgroup of contact transformations with \( \phi = x + s(u, u_x) \).

### 1.4.2 Differential substitutions of Miura type

The famous Miura transformation [19]
\[ \bar{u} = u_1 - u^2 \]
relates the mKdV equation
\[ u_t = u_3 - 6u^2u_1 \]
and the KdV equation
\[ \bar{u}_t = \bar{u}_3 + 6\bar{u}\bar{u}_1. \]
The Miura substitution is not locally invertible: given a solution \( \bar{u} \) of the KdV equation, one has to solve the Riccati equation to find a solution of the mKdV equation.
Definition 1.3. A relation
\[ \bar{x} = \phi(x, u, u_2, \ldots, u_k), \quad \bar{u} = \psi(x, u, u_1, \ldots, u_k) \]  
(1.35)
is called a differential substitution from the equation
\[ u_t = F(x, u, u_1, \ldots, u_n) \]  
(1.36)
to the equation
\[ \bar{u}_t = G(x, \bar{u}, \bar{u}_1, \ldots, \bar{u}_n) \]  
(1.37)
if for any solution \( u(x, t) \) of equation (1.36) the function \( \bar{u}(t, \bar{x}) \) satisfies (1.37). The order of differential operator (1.34) is called the order of the differential substitution.

Remark 1.5. By definition, point and contact transformations are differential substitutions of order 0, while the order of the Miura substitution is equal to one.

The derivatives of \( \bar{u} \) are expressed through the functions \( \phi \) and \( \psi \) just as in Section 1.4.1. However, for the generic function \( F \) the right-hand side of (1.33) cannot be expressed through the variables \( \bar{x}, \bar{u}, \ldots \). This requirement imposes strong restrictions of \( F, \phi \) and \( \psi \).

Exercise 1.4. Verify that in the case of the Miura substitution for the MkDv equation, the right-hand side of (1.33) is expressed in terms of \( \bar{u}, \ldots, \bar{u}_3 \) as \( \bar{u}_3 + 6\bar{u}\bar{u}_1 \).

Proposition 1.1. Let \( \bar{D}_t(\bar{\rho}) = \bar{D}(\bar{\sigma}) \) be a conservation law for equation (1.37). Then \( D_t(\rho) = D(\sigma) \), where
\[ \rho = \bar{\rho} D(\phi), \quad \sigma = \bar{\sigma} + \bar{\rho} D_t(\phi), \]
is the conservation law for equation (1.36).

Group differential substitutions
In this section we consider differential substitutions associated with classical groups of symmetries [12].

Example 1.8. The Burgers equation
\[ \bar{u}_t = \bar{u}_{xx} + 2\bar{u} \bar{u}_x \]  
(1.38)
is related to the heat equation
\[ u_t = u_{xx} \]  
(1.39)
by the Cole-Hopf substitution
\[ \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = \frac{u_x}{u}. \] (1.40)

This substitution admits the following group-theoretical interpretation. The group \( G \) of dilatations \( u \rightarrow \tau u \) acts on the solutions of the heat equation. It is easily seen that any differential invariant of \( G \) is a function of the variables
\[ \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = \frac{u_x}{u}, \quad D\left(\frac{u_x}{u}\right), \ldots, \quad D^i\left(\frac{u_x}{u}\right), \ldots. \] (1.41)

Since the group \( G \) preserves the equation, the (total) \( t \)-derivative \( D_t \) sends invariants to invariants. Consequently \( D_t\left(\frac{u_x}{u}\right) \) must be a function of the variables (1.41). Thus the function \( \bar{u} = \frac{u_x}{u} \) must satisfy a certain evolution equation. A simple calculation shows that this latter equation coincides with (1.38).

**Remark 1.6.** The above reasoning shows that any evolution equation with the symmetry group \( u \rightarrow \tau u \) admits the Cole-Hopf transformation (1.40).

**Exercise 1.5.** The heat equation (1.39) admits the one-parameter group of shifts \( x \rightarrow x + \tau \). The simplest invariants of the group are
\[ \bar{t} = t, \quad \bar{x} = u, \quad \bar{u} = u_1. \] (1.42)

The derivation \( \bar{D} = \frac{1}{u_1} D \) generates a functional basis \( \bar{u}_i = \bar{D}^i(u_1) \) of all differential invariants. Therefore, the differential substitution (1.42) leads to an evolution equation for \( \bar{u} \). Verify that
\[ \bar{u}_t = \bar{u}^2 \bar{u}_{\bar{x}\bar{x}}. \]

In the examples considered above the equation (1.37) is the restriction of equation (1.36) to the set of differential invariants of a symmetry group \( G \). This equation (1.37) is called a quotienting of (1.36) by the group \( G \).

Consider evolution equations of the form
\[ u_t = F(u_1, u_2, \ldots, u_n). \] (1.43)

They admit the symmetry group \( u \rightarrow u + \tau \). The invariants
\[ \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u_1 \]
of this group define the differential substitution, which results the equation
\[ \bar{u}_t = D\left(\bar{F}(\bar{u}, \ldots, \bar{u}_{n-1})\right). \] (1.44)
Definition 1.4. Equation (1.44) is said to be obtained from the equation (1.43) by the potentiation.

Remark 1.7. Since any one-parametric group of point (contact) transformations is point (contact) equivalent to the group \( u \to u + \tau \), any quotienting by an one-parametric group is a composition of a point (contact) transformation and the potentiation.

Remark 1.8. The transformation \( u = \int \bar{u} \, dx \) from equation (1.44) to equation (1.43), inverse to the potentiation, is related to the fact that equation (1.44) has the conserved density \( \rho = \bar{u} \). If equation (1.43) possesses a conserved density of order not greater than 1, then it can be transformed to \( \rho = \bar{u} \) by a proper point or contact transformation. The resulting evolution equation has the form (1.44) and we may apply the transformation \( u = \int \bar{u} \, dx \) to obtain the corresponding equation of the form (1.43).

Since one-parameter groups of contact transformations are in one-to-one correspondence [11] with the infinitesimal symmetries of the form

\[ u_\tau = G(x, u, u_1), \quad (1.45) \]

the existence of such a symmetry for equation (1.36) guarantees the presence of the quotienting equation of the form (1.37), while the existence of a non-trivial conserved density of the form \( \rho(\bar{x}, \bar{u}, \bar{u}_1) \) for an equation of the form (1.37) gives rise to the existence of the corresponding equation (1.36).

Definition 1.5. Differential substitutions generated by infinitesimal symmetries or conservation laws of order not greater than 1 are called quasi-local transformations.

Question 1.2. When a quasi-local transformation preserves the higher symmetries?

The answer is evident: if (1.45) is a symmetry not only for equation (1.36) but also for the hierarchy of its symmetries, then the corresponding quotienting can be applied to the whole hierarchy. In other words, if the group \( G \) is a symmetry group for any equation from the hierarchy, then the corresponding substitution preserves the integrability in the sense of the symmetry approach.

Similarly, if all equations from a hierarchy of symmetries for equation (1.37) have the same conserved density \( \rho(\bar{x}, \bar{u}, \bar{u}_1) \), then the corresponding equation (1.36) has infinitely many symmetries.
Chapter 2

Symmetry approach to integrability

The symmetry approach to the classification of integrable PDEs with two independent variables is based on the existence of the higher symmetries and/or local conservation laws (see Introduction).

2.1 Description of some classification results

2.1.1 Hyperbolic equations

The first classification result obtained with the symmetry approach in 1979 was formulated in Theorem 1.1.

Open problem 2.1. A complete classification of integrable hyperbolic equations of the form

\[ u_{xy} = \Psi(u, u_x, u_y) \]  \hspace{1cm} (2.1)

is still an open problem. Some partial results were obtained in [20].

Example 2.1. The following equation [21]

\[ u_{xy} = S(u) \sqrt{1 - u_x^2} \sqrt{1 - u_y^2}, \]  \hspace{1cm} \text{where} \hspace{1cm} S'' - 2S^3 + cS = 0, \hspace{1cm} (2.2)

is integrable.

Example 2.2. The equation

\[ u_{xy} = Q(u) b(u_x) \bar{b}(u_y), \]  \hspace{1cm} \text{where} \hspace{1cm} Q'' - 2QQ' - 4Q^3 = 0,

and \( b, \bar{b} \) are solutions of the cubic equations

\[ (u_x - b)(b + 2u_x)^2 = 1, \hspace{1cm} (u_y - \bar{b})(\bar{b} + 2u_y)^2 = 1, \]
is integrable.

In [22] the following problem was solved. For hyperbolic equations (2.1) the symmetry approach assumes the existence of both $x$-symmetries of the form

$$ u_t = A(u, u_x, u_{xx}, \ldots), $$

and $y$-symmetries of the form

$$ u_\tau = B(u, u_y, u_{yy}, \ldots). $$

For example, the famous integrable sin-Gordon equation

$$ u_{xy} = \sin u $$

admits the symmetries

$$ u_t = u_{xxx} + \frac{1}{2} u_x^3, \quad u_\tau = u_{yyy} + \frac{1}{2} u_y^3. $$

It was assumed in [22] that both $x$ and $y$-symmetries are third order integrable evolution equations (maybe different). The corresponding classification result is formulated in Appendix 1.

Chapter 3 is devoted to a special class of integrable equations (2.1) named the Liouville type equations [23].

In [24] integrable hyperbolic systems of the form

$$ u_x = p(u, v), \quad v_y = q(u, v) $$

were investigated.

### 2.1.2 Evolution equations

Some necessary conditions for the existence of higher symmetries that do not depend on symmetry orders were found in [23, 2] for evolution equations of the form (1.8) (see Section 2.2). These conditions lead to an overdetermined system of PDEs for the right-hand side of (1.8). Solutions of this system are not always polynomial.

It was proved in [26] that the same conditions hold if the equation (1.8) possesses infinitely many local conservation laws. But the latter conditions are stronger than the conditions for symmetries. Moreover, there exist equations that have higher symmetries but have no higher conservation laws.

**Definition 2.1.** An equation is called $S$-integrable (in the terminology by F. Calogero) if it has infinitely many both higher symmetries and conservation laws. An equation is called $C$-integrable if it has infinitely many higher symmetries but finite number of higher conservation laws.
The simplest example of $C$-integrable equation is the Burgers equation (1.11).

**Remark 2.1.** Usually, the inverse scattering method can be applied to $S$-integrable equations while $C$-integrable equations can be reduced to linear equations by differential substitutions. However, to eliminate obvious exceptions, we need to refine Definition 2.1 (see Definition 2.13). Otherwise, the linear equation $u_t = u_{xxx}$ will still be $S$-integrable.

**Proposition 2.1.** (see Proposition 2.4 and Theorem 29 in [7]) A scalar evolution equation (1.8) of even order $n = 2k$ cannot possess infinitely many higher local conservation laws.

There are two types of classification results obtained in the frame of the symmetry approach: a “weak” version, where equations with conservation laws are listed and a “strong” version related to symmetries. The “weak” list contains $S$-integrable equations while the “strong” list consists of both $S$-integrable and $C$-integrable equations.

**Second order equations**

All nonlinear integrable equations of the form

$$u_t = F(x, t, u, u_1, u_2)$$

were listed in [27] and [28]. The answer is:

$$u_t = u_2 + 2uu_x + h(x),$$

$$u_t = u^2u_2 - \lambda xu_1 + \lambda u,$$

$$u_t = u^2u_2 + \lambda u^2,$$

$$u_t = u^2u_2 - \lambda x^2u_1 + 3\lambda xu.$$

This list is complete up to contact transformations. According to Proposition 2.1 all these equations are $C$-integrable. They are related to the heat equation $v_t = v_{xx}$ by group differential substitutions discussed in Section 1.3.2.

The first three equations possess local higher symmetries and form a list of integrable equations of the form

$$u_t = F(x, u, u_1, u_2)$$

obtained in [27].
Remark 2.2. It turns out that any integrable equation (2.3) has the form

\[ u_t = \frac{a_1 u_2 + a_2}{a_3 u_2 + a_4}, \quad a_i = a_i(x, u, u_1). \]

Any such equation can be reduced to a quasi-linear form

\[ u_t = a_1(x, u, u_1) u_2 + a_2(x, u, u_1) \]

by a proper contact transformation (1.30).

In the paper [28] equations with weekly non-local symmetries were considered. The weekly non-locality of symmetries looks very natural if we assume that the right-hand side of the equation depends on \( t \). As a result, the list from the paper [27] was extended by the fourth equation.

Third order equations

A first result of the “weak” type for equations (1.8) is the following:

Theorem 2.1. [26] A complete list up to quasi-local transformations (see Definition 1.5) of equations of the form

\[ u_t = u_{xxx} + f(u, u_x, u_{xx}) \]  \hspace{1cm} (2.4)

with an infinite hierarchy of conservation laws can be written as:

\[ u_t = u_{xxx} + u u_x, \]
\[ u_t = u_{xxx} + u^2 u_x, \]
\[ u_t = u_{xxx} - \frac{1}{2} u_x^2 + (\alpha e^{2u} + \beta e^{-2u}) u_x, \]
\[ u_t = u_{xxx} - \frac{1}{2} P'' u_x + \frac{3}{8} (P - u_x^2)^2, \]  \hspace{1cm} (2.5)
\[ u_t = u_{xxx} - \frac{3}{2} \frac{u_x^2}{u_x} + P \]  \hspace{1cm} (2.6)

where \( P'''(u) = 0 \).

For the “strong” version see [29] and Appendix 3. A proof of the corresponding statement can be found in the survey [30].

For integrable third order equations

\[ u_t = F(u, u_x, u_{xx}, u_{xxx}), \]  \hspace{1cm} (2.7)

more general than (2.4), there are three possible types of \( u_{xxx} \)-dependence [4]:

24
1) \[ u_t = a u_{xxx} + b, \]

2) \[ u_t = \frac{a}{(u_{xxx} + b)^2}, \]

and

3) \[ u_t = \frac{2a u_{xxx} + b}{\sqrt{a u_{xxx}^2 + b u_{xxx} + c}} + d, \]

where the functions \( a, b, c \) and \( d \) depend on \( u, u_x, u_{xx} \). A complete classification of integrable equations of such type is not finished yet [31, 32].

**Conjecture 2.1.** All integrable third order equations (2.7) are related to the KdV equation or to the Krichever-Novikov equation (2.6) by differential substitutions of Cole-Hopf and Miura type [33].

**Fifth order equations**

All equations of the form

\[ u_t = u_5 + F(u, u_x, u_2, u_3, u_4), \]

possessing higher conservation laws were found in [34].

**Lemma 2.1.** A possible dependence of the function \( F \) on \( u_4, u_3 \) and \( u_2 \) for such equations is described by the following formula:

\[ u_t = u_5 + A_1 u_2 u_4 + A_2 u_4 + A_3 u_3^2 + (A_4 u_2^2 + A_5 u_2 + A_6) u_3 + A_7 u_2^2 + A_8 u_2 + A_9 u_2 + A_{10} u_2 + A_{11}, \]

where \( A_i = A_i(u, u_x) \).

The list of integrable cases contains both well-known equations (1.20), (1.21) and

\[ u_t = u_5 + 5(u_1 - u^2) u_3 + 5u_2^2 - 20uu_1 u_2 - 5u_1^3 + 5u_1^4, \]

(see [35]) as well as several new equations. One of them (missed in [36]) is given by

\[ u_t = u_5 + 5(u_2 - u_1^2 + \lambda_1 e^{2u} - \lambda_2 e^{-4u}) u_3 - 5u_1 u_2^2 + 15(\lambda_1 e^{2u} u_3 + 4\lambda_2 e^{-4u}) u_1 u_2 + u_1^5 - 90\lambda_2^2 e^{-4u} u_1^3 + 5(\lambda_1 e^{2u} - \lambda_2 e^{-4u})^2 u_1. \]
The “strong” version of this classification result (see Appendix 3) was published in [30].

At first glance, the problem of the classification of integrable equations

\[ u_t = u_n + F(u, u_x, u_{xx}, \ldots, u_{n-1}), \quad u_i = \frac{\partial^i u}{\partial x^i}. \]  

(2.8)

with arbitrary \( n \) seems to be far from a conclusive solution. This is not quite so. Each integrable equation together with all its symmetries form a so-called hierarchy of integrable equations. For the equations integrable by the inverse scattering method all the equations of the hierarchy possess the same \( L \)-operator. This fact lies in the basis of the commutativity of the equations in hierarchies (each equation of the hierarchy is a higher symmetry for all others). For another explanation of the commutativity see Remark 1.2. A general rigorous statement on “almost” commutativity of the symmetries for equation (2.8) can be found in [37].

Under the assumption that the right-hand side of equation (2.8) is polynomial and homogeneous, it was proved in the works [15, 38] that the hierarchy of any such equation contains an equation of second, third, or fifth order.

More references for the classification of scalar evolution equations can be found in the reviews [2, 3, 4, 7, 30]. Here, I would like also to mention the papers [39]–[42].

2.1.3 Systems of two equations

In [43, 44] the necessary conditions of integrability were generalized to the case of systems of evolution equations. However, component-wise computations in this case are very tedious. The only one serious classification problem has been solved [43, 44, 3]: all systems of the form

\[ u_t = u_2 + F(u, v, u_1, v_1), \quad v_t = -v_2 + G(u, v, u_1, v_1) \]  

(2.9)

possessing higher conservation laws were listed. In other words, the authors have found all \( S \)-integrable systems (2.9).

Besides the well-known NLS equation written as a system of two equations

\[ u_t = -u_{xx} + 2u^2v, \quad v_t = v_{xx} - 2v^2u, \]  

(2.10)

basic integrable models from a long list of such integrable models are:

- a version of the Boussinesq equation

\[ u_t = u_2 + (u + v)^2, \quad v_t = -v_2 + (u + v)^2; \]
• and the two-component form of the Landau-Lifshitz equation

\[
\begin{align*}
    u_t &= u_2 - \frac{2u_1^2}{u + v} - \frac{4(p(u, v) u_1 + r(u) v_1)}{(u + v)^2}, \\
v_t &= -v_2 + \frac{2v_1^2}{u + v} - \frac{4(p(u, v) v_1 + r(-v) u_1)}{(u + v)^2},
\end{align*}
\]

where \( r(y) = c_4 y^4 + c_3 y^3 + c_2 y^2 + c_1 y + c_0 \) and

\[
p(u, v) = 2c_4 u^2 v^2 + c_3 (uv^2 - vu^2) - 2c_2 uv + c_1 (u - v) + 2c_0.
\]

A complete list of integrable systems (2.9) up to transformations

\[
    u \to \Phi(u), \quad v \to \Psi(v)
\]

should contain more than 100 systems. Such a list has never been published. Instead, in [3] was presented a list complete up to “almost invertible” transformations [45].

**Remark 2.3.** All these equations have a fourth order symmetry of the form

\[
\begin{align*}
u_\tau &= u_{xxxx} + f(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}), \\
v_\tau &= -v_{xxxx} + g(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}).
\end{align*}
\]

Some of them have also a third order symmetry.

There are at least three reasons why the paper [46] on integrable systems of the form

\[
\begin{align*}
u_t &= u_{xx} + A_1(u, v) u_x + A_2(u, v) v_x + A_0(u, v), \\
v_t &= -v_{xx} + B_1(u, v) v_x + B_2(u, v) u_x + B_0(u, v)
\end{align*}
\]

was written.

**Reason 1.** There are \( C \)-integrable cases that are not in the Mikhailov-Shabat-Yamilov classification. One of the examples is the system

\[
\begin{align*}
u_t &= u_{xx} - 2uu_x - 2vv_x - 2uv_x + 2u^2 v + 2uv^2, \\
v_t &= -v_{xx} + 2vu_x + 2wu_x + 2vv_x - 2u^2 v - 2uv^2.
\end{align*}
\]

The system (2.13) was first discussed in [47]. It can be reduced to

\[
U_t = U_{xx}, \quad V_t = -V_{xx}
\]
by the following substitution of Cole-Hopf type:

\[ u = \frac{U_x}{(U + V)}, \quad v = \frac{V_x}{(U + V)}. \]

**Reason 2.** Systems (2.12) can be easily classified without any equivalence relations. The right-hand sides of such systems turn out to be polynomial.

**Reason 3.** Results of any serious classification problem should be verified independently. Only after that there is an assurance that nothing has been missed.

For classification of systems (2.12) the simplest naive version of the symmetry test was applied.

**Lemma 2.2.** If the system (2.12) has a fourth order symmetry (2.11), then the system is of the following form:

\[
\begin{align*}
    u_t & = u_{xx} + (a_{12} uv + a_1 u + a_2 v + a_0) u_x + (p_2 v + p_{11} u^2 + p_1 u + p_0) v_x + A_0(u, v), \\
    v_t & = -v_{xx} + (b_{12} uv + b_1 v + b_2 u + b_0) v_x + (q_2 u + q_{11} v^2 + q_1 v + q_0) u_x + B_0(u, v),
\end{align*}
\]

where \( A_0 \) and \( B_0 \) are polynomials of at most fifth degree.

The coefficients of the latter system satisfy an overdetermined system of algebraic equations. The most essential equations are

\[
\begin{align*}
    p_2(b_{12} - q_{11}) = 0, & \quad p_2(a_{12} - p_{11}) = 0, & \quad p_2(a_{12} + 2b_{12}) = 0, \\
    q_2(b_{12} - q_{11}) = 0, & \quad q_2(a_{12} - p_{11}) = 0, & \quad q_2(b_{12} + 2a_{12}) = 0, \\
    a_{12}(a_{12} - b_{12} + q_{11} - p_{11}) = 0, & \quad b_{12}(a_{12} - b_{12} + q_{11} - p_{11}) = 0, \\
    (a_{12} - p_{11})(p_{11} - q_{11}) = 0, & \quad (b_{12} - q_{11})(p_{11} - q_{11}) = 0, \\
    (a_{12} - p_{11})(a_{12} - b_{12}) = 0, & \quad (b_{12} - q_{11})(a_{12} - b_{12}) = 0.
\end{align*}
\]

As usual, such factorized equations lead to a tree of variants.

Solving the overdetermined system, we do not consider the so called triangular systems like the following:

\[ u_t = u_{xx} + 2uv_x, \quad v_t = -v_{xx} - 2vv_x. \]

Here, the second equation is separated and the first is linear with the variable coefficients defined by a given solution of the second equation.

The classification statement is formulated in Appendix 4.

**Open problem 2.2.** Find all systems (2.9) that have infinitely many symmetries.
2.2 Integrability conditions

For our aims the language of differential algebra \[48\] is the most adequate one.

2.2.1 Evolutionary vector fields, Fréchet and Euler derivatives

Consider evolution equations of the form (1.8). Suppose that the right-hand side of (1.8) as well as other functions in \(u, u_x, u_{xx}, \ldots\) belong to a differential field \(\mathcal{F}\). For our considerations one can assume that elements of \(\mathcal{F}\) are rational functions in a finite number of independent variables

\[ u_i = \frac{\partial^i u}{\partial x^i}. \]

When we are going to integrate a function with respect to one of its arguments or to take a radical of it, we have to extend the basic field \(\mathcal{F}\).

As usual in differential algebra, we have a principle derivation (1.1), which generates all independent variables \(u_i\) starting from \(u_0 = u\). This derivation is a formalization of the partial \(x\)-derivative, which acts on functions of the form \(g(u(x), \frac{\partial u}{\partial x}, \ldots)\). The vector field \(D\) defined by (1.1) is called the total \(x\)-derivative.

Remark 2.4. We very often use the fact that \(D(f) = 0, f \in \mathcal{F}\) implies \(f = \text{const}\).

Remark 2.5. The variable \(t\) in the local algebraic theory of evolution equations is considered as a parameter.

Amongst the main concepts related to dynamical system of ODEs (1.5), the finite-dimensional vector field (1.7) plays a key role. Analogously, the infinite-dimensional vector field

\[ D_F = \sum_{i=0}^{\infty} D^i(F) \frac{\partial}{\partial u_i} \]  \hspace{1cm} (2.14)

is associated with evolution equation (1.8). This vector field commutes with \(D\). We call vector fields of the form (2.14) evolutionary. The function \(F\) is called the generator of that evolutionary vector field. Sometimes we call (2.14) the total \(t\)-derivative with respect to (1.8) and denote it by \(D_t\).

The set of all evolutionary vector fields forms a Lie algebra over \(\mathbb{C}\): \([D_G, D_H] = D_K\), where

\[ K = H_s(G) - G_s(H). \]  \hspace{1cm} (2.15)

Henceforth, we use the following...
**Definition 2.2.** For any function \( a \in \mathcal{F} \) the Fréchet derivative is defined as the linear differential operator

\[
a_\ast = \sum_k \frac{\partial a}{\partial u_k} D^k.
\]

The order of the function \( a \) is defined as the order of the differential operator \( a_\ast \). We denote by \( a_\ast^+ \) the formally conjugate operator

\[
a_\ast^+ = \sum_k (-1)^k D^k \circ \frac{\partial a}{\partial u_k}.
\]

In the Introduction we defined a generalized symmetry of equation (1.8) as an evolution equation (1.9) that is compatible with (1.8). By definition, the compatibility means that \([D_F, D_G] = 0\). It can be written also in the form \( G_\ast(F) = F_\ast(G) \) or

\[
D_t(G) - F_\ast(G) = 0. \tag{2.16}
\]

Formula (2.15) defines a Lie bracket on our differential field \( \mathcal{F} \). An integrable hierarchy is nothing else but an infinite-dimensional commutative subalgebra of this Lie algebra.

**Proposition 2.2.** Suppose an operator \( \mathcal{R} \) satisfies the equation

\[
D_t(\mathcal{R}) = F_\ast \mathcal{R} - \mathcal{R} F_\ast. \tag{2.17}
\]

Then, for any symmetry (1.9) of the equation (1.8), the equation \( u_\tau = \mathcal{R}(G) \) is a symmetry of (1.8).

Hereinafter,

\[
D_t \left( \sum s_i D^i \right) = \sum D_t(s_i) D^i.
\]

**Proof.** Let us rewrite (2.17) as

\[
[D_t - F_\ast, \mathcal{R}] = 0. \tag{2.18}
\]

Now the statement follows from (2.16). \( \square \)

**Definition 2.3.** An operator \( \mathcal{R} : \mathcal{F} \to \mathcal{F} \) satisfying (2.17) is called a recursion operator for the equation (1.8).

---

1For the sake of brevity we often name generator \( G \) of symmetry simply by symmetry \( G \).

2Usually \( \mathcal{R} \) is a differential operator or a ratio of differential operators.
Euler operator

**Definition 2.4.** The Euler operator or the variational derivative of a function $a \in \mathcal{F}$ is defined as
\[
\frac{\delta a}{\delta u} = \sum_k (-1)^k D^k \left( \frac{\partial a}{\partial u_k} \right) = a^+_*(1).
\]

If a function $a$ is a total derivative $a = D(b)$, $b \in \mathcal{F}$ (we say that $a \in \text{Im } D$), then the variational derivative vanishes. Moreover, the vanishing of the variational derivative is almost a criterion that the function belongs to $\text{Im } D$.[49]

**Theorem 2.2.** For $a \in \mathcal{F}$ the variational derivative vanishes
\[
\frac{\delta a}{\delta u} = 0
\]
if and only if $a \in \text{Im } D + \mathbb{C}$.

**Lemma 2.3.** The following identities
\[
(ab)_* = ab_* + ba_*, \quad (D(a))_* = Da_* = D(a_*) + a_* D,
\]
\[
(D_t(a))_* = D_t(a_*) + a_* F_t, \quad (a_*(b))_* = Db_* + a_* b_*,
\]
\[
\left( \frac{\delta a}{\delta u} \right)_* = \left( \frac{\delta a}{\delta u} \right)_*^+, \quad \frac{\delta}{\delta a} (D_t(a)) = D_t \left( \frac{\delta a}{\delta u} \right) + F_t^+ \left( \frac{\delta a}{\delta u} \right)
\]
hold for any $a, b, F \in \mathcal{F}$.

**2.2.2 Pseudo-differential series**

Consider a skew field of (non-commutative) formal series of the form
\[
S = s_m D^m + s_{m-1} D^{m-1} + \cdots + s_0 + s_{-1} D^{-1} + s_{-2} D^{-2} + \cdots, \quad s_i \in \mathcal{F}. \quad (2.19)
\]

The number $m \in \mathbb{Z}$ is called the order of $S$ and is denoted by $\text{ord } S$. If $s_i = 0$ for $i < 0$ that $S$ is called a differential operator.

The product of two formal series is defined by the formula
\[
D^k \circ s D^m = s D^{m+k} + C^1_k D(s) D^{k+m-1} + C^2_k D^2(s) D^{k+m-2} + \cdots,
\]
where $k, m \in \mathbb{Z}$ and $C^j_n$ is the binomial coefficient
\[
C^j_n = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}, \quad n \in \mathbb{Z}.
\]

For the series this formula is extended by associativity.
Remark 2.6. For any series \( S \) and \( T \) we have \( \text{ord}(S \circ T - T \circ S) \leq \text{ord} S + \text{ord} T - 1 \).

The formally conjugated formal series \( S^+ \) is defined as
\[
S^+ = (-1)^m D^m \circ s_m + (-1)^{m-1} D^{m-1} \circ s_{m-1} + \cdots + s_0 - D^{-1} \circ s_{-1} + D^{-2} \circ s_{-2} + \cdots.
\]

Example 2.3. Let
\[
R = uD^2 + u_1 D, \quad S = -u_1 D^3, \quad T = uD^{-1};
\]
then
\[
R^+ = D^2 \circ u - D \circ u_1 = R,
\]
\[
S^+ = D^3 \circ u_1 = u_1 D^3 + 3u_2 D^2 + 3u_3 D + u_4,
\]
\[
T^+ = -D^{-1} u = -uD^{-1} + u_1 D^{-2} - u_2 D^{-3} + \cdots.
\]

For any series (2.19) one can uniquely find the inverse series
\[
T = t_m D^{-m} + t_{m-1} D^{-m-1} + \cdots, \quad t_k \in F
\]
such that \( S \circ T = T \circ S = 1 \). Indeed, multiplying \( S \) and \( T \) and equating the result to 1, we find that
\[
s_m t_m = 1, \quad \text{i.e., } t_m = \frac{1}{s_m}.
\]
Comparing the coefficients of \( D^{-1} \), we get
\[
ms_m D(t_m) + s_m t_{m-1} + s_{m-1} t_{m} = 0
\]
and therefore
\[
t_{m-1} = -s_{m-1} s_m - mD\left(\frac{1}{s_m}\right), \quad \text{etc.}
\]

Furthermore, we can find the \( m \)-th root of the series \( S \), i.e., a series
\[
R = r_1 D + r_0 + r_{-1} D^{-1} + r_{-2} D^{-2} + \cdots
\]
such that \( R^m = S \). This root is unique up to any number factor \( \varepsilon \) such that \( \varepsilon^m = 1 \).

Example 2.4. Let \( S = D^2 + u \). Assuming
\[
R = r_1 D + r_0 + r_{-1} D^{-1} + r_{-2} D^{-2} + \cdots,
\]
we compute
\[
R^2 = R \circ R = r_1^2 D^2 + (r_1 D(r_1) + r_1 r_0 + r_0 r_1) D + r_1 D(r_0) + r_0^2 + r_1 r_{-1} + r_{-1} r_1 + \cdots,
\]
and compare the result with \( S \). From the coefficients of \( D^2 \) we find \( r_1^2 = 1 \) or \( r_1 = \pm 1 \). Let \( r_1 = 1 \). Comparing coefficients of \( D \), we get \( 2r_0 = 0 \), i.e., \( r_0 = 0 \). From \( D^0 \) we obtain \( 2r_{-1} = u \), terms of \( D^{-1} r_{-2} = -\frac{u_1}{4} \), etc., i.e.
\[
R = S^{1/2} = D + \frac{u}{2} D^{-1} - \frac{u_1}{4} D^{-2} + \cdots.
\]
Definition 2.5. The residue of a formal series (2.19) by definition is the coefficient of $D^{-1}$:

\[\text{res} (S) \overset{\text{def}}{=} s_{-1}.\]

The logarithmic residue of $S$ is defined as

\[\text{res} \log S \overset{\text{def}}{=} s_{m-1} / s_m.\]

We will use the following important

Theorem 2.3. [50] For any two formal series $S, T$ the residue of the commutator belongs to $\text{Im} \ D$:

\[\text{res}[S, T] = D(\sigma(S, T)),\]

where

\[\sigma(S, T) = \sum_{i+j+1 > 0} \sum_{i \leq \text{ord}(T), \ j \leq \text{ord}(S)} C_{ij} \sum_{k=0}^{i+j} (-1)^k D^k s_j D^{i+j-k}(t_j).\]

Corollary 2.1. For any series $S$ and $T$

\[\text{res} (S - TST^{-1}) \in \text{Im} \ D.\]

Proof. It follows from the identity

\[S - TST^{-1} = [ST^{-1}, T].\]

\[\square\]

2.2.3 Formal symmetries

Definition 2.6. A pseudo-differential series

\[\Lambda = l_1 D + l_0 + l_{-1}D^{-1} + \cdots,\]

where $l_k = l_k(u, \ldots, u_{s_k}) \in \mathcal{F}$, is called a formal symmetry (or formal recursion operator) for equation (1.8) if $R = \Lambda$ satisfies the equation

\[D_t(R) = [F_*, R], \quad \text{where} \quad F_* = \sum_{i=0}^n \frac{\partial F}{\partial u_i} D^i. \quad (2.20)\]

\[\begin{alignedat}{1}\text{According to Proposition 2.2 any genuine operator that satisfies (2.20) maps higher symmetries of the equation (1.8) to higher symmetries.}\end{alignedat}\]
Proposition 2.3. Suppose a pseudo-differential series $R$ of order $k$ satisfies the equation (2.20). Then

1) $R^\dagger$ is a formal symmetry;

2) If $R_1$ and $R_2$ satisfy (2.20), then $R_1 \circ R_2$ satisfies (2.20);

3) $R^\dagger$ satisfies this equation for any $i \in \mathbb{Z}$;

4) Let $\Lambda$ be a formal symmetry. Then $R$ can be written in the form

$$R = \sum_{-\infty}^{k} a_i \Lambda^i, \quad k = \text{ord} R, \quad a_i \in \mathbb{C};$$

5) In particular, any formal symmetry $\bar{\Lambda}$ has the form

$$\bar{\Lambda} = \sum_{-\infty}^{1} c_i \Lambda^i, \quad c_i \in \mathbb{C}. \quad (2.21)$$

The coefficients of the formal symmetry can be found from equation (2.20).

Example 2.5. Let us consider equations of KdV type

$$u_t = u_3 + f(u, u_1) \quad (2.22)$$

and find a few coefficients $l_1, l_0, \ldots$ of the formal symmetry $\Lambda$. We substitute

$$F_* = D^3 + \frac{\partial f}{\partial u_1} D + \frac{\partial f}{\partial u}, \quad \Lambda = l_1 D + l_0 + l_{-1} D^{-1} + \cdots$$

into (2.20) and collect the coefficients of $D^3, D^2, \ldots$. We obtain

$$D^3 : \quad 3D(l_1) = 0; \quad D^2 : \quad 3D^2(l_1) + 3D(l_0) = 0; \quad D : \quad D^3(l_1) + 3D^2(l_0) + 3D(l_{-1}) + \frac{\partial f}{\partial u_1} D(l_1) = D_1(l_1) + l_1 D \left( \frac{\partial f}{\partial u_1} \right).$$

From the first equation it follows (see Remark 2.4) that $l_1$ is a constant and we set $l_1 = 1$. Now, from the second equation, it follows that $l_0$ is a constant and we choose $l_0 = 0$ (any constant is a trivial solution of equation (2.20)). It follows from the third equation that

$$D(l_{-1}) = D \left( \frac{1}{3} \frac{\partial f}{\partial u_1} \right),$$

and therefore

$$l_{-1} = \frac{1}{3} \frac{\partial f}{\partial u_1} + c_{-1}, \quad c_{-1} \in \mathbb{C}.$$
The constant of integration $c_{-1}$ can be set equal to zero without loss of generality (see formula (2.21)). Therefore

$$\Lambda = D + \frac{1}{3} \frac{\partial f}{\partial u_1} D^{-1} + \cdots .$$

Notice that a first obstacle for the existence of $\Lambda$ appears when we compare coefficients of $D^{-1}$. So the formal symmetry does not exist for any arbitrary function $f(u, u_1)$ in (2.22).

Remark 2.7. In general, we define the coefficients of $\Lambda$ from (2.20) step by step solving equations of the form $D(l_k) = S_k$, where $S_k \in \mathcal{F}$. This equation is resolvable only if $S_k \in \text{Im } D$ (see Theorem 2.2). So there are infinitely many obstacles for the existence of a formal symmetry.

Theorem 2.4. If equation (1.8) possesses an infinite sequence of higher symmetries

$$u_{r_i} = G_i(u, \ldots, u_{m_i}), \quad m_i \to \infty,$$

then the equation has a formal symmetry.

Proof. The main idea of the proof of Theorem 2.4 and the relation between the structure of the formal symmetry and higher symmetries can be illustrated by the following consideration [2]. Suppose that the equation (1.8) has one symmetry with a generator $G$. The function $G$ satisfies the equation (2.16). Let us compute the Fréchet derivative from the left-hand side of this equation. Using identities of Lemma 2.3, we retrieve the equation

$$D_t(G_*) + G_* F_* = D_G(F_*) + F_* G_*,$$

which can be rearranged in the form

$$D_t(G_*) - [F_*, G_*] = D_G(F_*) . \tag{2.23}$$

If the symmetry $G$ has a very large order $m$, then the order of left-hand side of (2.23) is much greater than the order of right-hand side. Therefore, the relations for several first coefficients of $G_*$ are exactly the same as for the first coefficients of the formal symmetry $\Lambda^m$. The first coefficients of a series of order $m$ that satisfies (2.20) coincide with the coefficients of the series $G_*^{1/2}$, which belong to $\mathcal{F}$. 

2.2.4 Conservation laws

The notion of first integrals, in contrast with infinitesimal symmetries, cannot be generalized to the case of PDEs. It is replaced by the concept of local conservation laws.
Definition 2.7. A function $\rho \in \mathcal{F}$ is called a density of a local conservation law of equation (1.8) if there exists a function $\sigma \in \mathcal{F}$ such that

$$D_t(\rho) = D(\sigma). \quad (2.24)$$

Equation (2.24) is evidently satisfied if $\rho = D(h)$ for any $h \in \mathcal{F}$. In this case $\sigma = D_t(h)$. Such “conservation laws” are called trivial.

Definition 2.8. Two conserved densities $\rho_1, \rho_2$ are called equivalent $\rho_1 \sim \rho_2$ if the difference $\rho_1 - \rho_2$ is a trivial density (i.e. $\rho_1 - \rho_2 \in \text{Im } D$).

Remark 2.8. Actually, a conserved density can be regarded as an equivalence class with respect to $\sim$. The Euler operator is well-defined on the equivalence classes.

Lemma 2.4. Any density $\rho$ is equivalent to a density $\bar{\rho}(u, \ldots, u_k)$ such that

$$\frac{\partial^2 \bar{\rho}}{\partial u_k^2} \neq 0.$$  

Proof. If $\rho$ is linear in the highest derivative, then we may reduce the differential order of $\rho$ by subtracting an element from $\text{Im } D$. \hfill $\square$

Definition 2.9. The number $k$ is called the order of the conserved density $\rho$. We denote $k$ by $\text{ord } (\rho)$.

Remark 2.9. It can be easily verified that the order of the differential operator $\left(\frac{\delta \rho}{\delta u}\right)_*$ is equal to $2 \text{ord } (\rho)$.

Question 2.1. Suppose that equation (1.8) and a function $\rho \in \mathcal{F}$ are given. How to verify whether a solution $\sigma \in \mathcal{F}$ for (2.24) exists or not?

The first way is to apply the Euler operator to relation (2.24). By Theorem 2.2 we obtain

$$\frac{\delta}{\delta u} \left( D_t(\rho) \right) = 0. \quad (2.25)$$

The left-hand side of the latter identity is known.

Unfortunately, we cannot find the function $\sigma$ by this method. However, there is the following straightforward inductive algorithm to do that. The left-hand side of (2.24) is given and the problem is

Question 2.2. How to solve an equation of the form $D(X) = S$ for given $S(u, \cdots, u_m)$?
It follows from (1.1) that $S$ must be linear in the highest derivative:

$$S = A(u, u_1, \ldots, u_{m-1}) u_m + B(u, u_1, \ldots, u_{m-1}).$$

If this is not true, then the equation has no solution. If this is true, we can reduce the order of $S$ subtracting from both sides of the equation a function of the form $D(r(u, u_1, \ldots, u_{m-1}))$ such that $S - D(r)$ has order less than $m$. For the function $r$ we can take any solution of the equation $\frac{\partial r}{\partial u_{m-1}} = A$. Thus if $X$ exists, it can be found in quadratures.

### 2.2.5 Formal symplectic operators

According to Lemma 2.3, $X = \frac{\delta \rho}{\delta u}$ satisfies the conjugate equation of (2.16):

$$D_t(X) + F^*_s(X) = 0. \quad (2.26)$$

**Definition 2.10.** Any solution $X \in \mathcal{F}$ of equation (2.26) is called a *cosymmetry*.

**Definition 2.11.** A pseudo-differential series

$$S = s_1 D + s_0 + s_{-1} D^{-1} + \cdots, \quad s_1 \neq 0$$

is called a formal symplectic operator\(^4\) for equation (1.8) if it satisfies the equation

$$D_t(S) + S F_s + F^*_s S = 0. \quad (2.27)$$

**Remark 2.10.** If $S$ is a formal symplectic operator, then $S^+$ is a formal symplectic operator as well. Therefore, we may additionally assume that $S^+ = -S$.

**Lemma 2.5.** The ratio $R = S^{-1}_1 S_2$ of any two series $S_1$ and $S_2$ that satisfy (2.27) satisfies equation (2.20).

**Theorem 2.5.** [26] If the equation $u_t = F$ possesses an infinite sequence of local conservation laws

$$D_t\left(\rho_i(u, \ldots, u_{m_i})\right) = D(\sigma_i), \quad \frac{\partial^2 \rho_i}{\partial u_{m_i}^2} \neq 0, \quad m_i \to \infty,$$

then the equation has a formal symmetry $\Lambda$ and a formal symplectic operator $R$ such that $S^+ = -S$ and

$$\Lambda^+ = -SAS^{-1}. \quad (2.28)$$

\(^4\)Relation (2.27) can be rewritten as $(D_t + F^*_s) \circ S = S(D_t - F_s)$. This means that a genuine operator $S : \mathcal{F} \to \mathcal{F}$ maps symmetries to cosymmetries. If the equation (1.8) is a Hamiltonian one, then the symplectic operator, which is inverse to the Hamiltonian operator, satisfies the equation (2.27) [51].

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Proof. For the full proof see [2]. Here, we mention only that the relation between (2.27) and (2.26) can be derived as follows. Using identities from Lemma 2.3, let us compute the Fréchet derivative from the left-hand side of equation (2.26), where $X = \frac{\delta \rho}{\delta u}$. The result can be represented as

\[ D_t(T) + TF^*_s + F^+_s T = Q, \]  

where

\[ T = \left( \frac{\delta \rho}{\delta u} \right)_s, \quad Q = - \sum_{k=1}^{2n} (-1)^k D^k \left( \frac{\delta \rho}{\delta u} (F_k)_s \right) \]

and $F_k$ are the coefficients of the Fréchet derivative $F_s$. □

**Proposition 2.4.** A scalar evolution equation of even order

\[ u_t = F(u, u_1, ..., u_{2n}) \]

cannot have a conserved density $\rho$ of order higher than $n$.

**Proof.** According to Remark 2.9, we have $\text{ord } T > 2n$. Comparing the coefficients of the highest power of $D$ in (2.29), we get $F_{2n} = 0$. □

**Remark 2.11.** This proposition is not true for systems of evolution equations.

### 2.2.6 Canonical densities and necessary integrability conditions

In this section, we formulate the necessary conditions for the existence of higher symmetries or conservation laws for equations of the form (1.8). In accordance with Theorems 2.4 and 2.5, such equations possess a formal symmetry. In Remark 2.7, we discussed obstructions to the existence of a formal symmetry, which are just integrability conditions. Below, we show that these conditions can be written in the form of conservation laws.

For equations (1.8) possessing a formal symmetry $\Lambda$ we define a sequence of canonical conserved densities.

**Definition 2.12.** The functions

\[ \rho_i = \text{res } (\Lambda^i), \quad i = -1, 1, 2, \ldots, \quad \text{and} \quad \rho_0 = \text{res } \log(\Lambda) \]  

are called canonical densities for the equation (1.8).

**Remark 2.12.** Despite the fact that the formal symmetry $\Lambda$ is not unique, the canonical densities are well defined. Namely, it follows from (2.21) that the canonical density $\bar{\rho}_i$ corresponding to any formal symmetry $\Lambda$ is a linear combination of the densities $\rho_j$, $j \leq i$ defined by $\Lambda$. 

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Theorem 2.6. If the equation (1.8) has a formal symmetry $\Lambda$, then the canonical densities (2.30) define local conservation laws

$$D_t(\rho_i) = D(\sigma_i), \quad \sigma_i \in \mathcal{F}, \quad i = -1, 0, 1, 2, \ldots, \quad (2.31)$$

for the equation (1.8).

Proof. If the formal symmetry $\Lambda$ satisfies the equation (2.20), then it follows from Proposition 2.3 that $\Lambda^k$, $k = -1, 1, 2, 3, \ldots$ also satisfy (2.20). Using Adler’s theorem 2.3, we get

$$D_t(\rho_k) = D_t(\text{res } \Lambda^k) = \text{res } ([F_*, \Lambda^k]) \in \text{Im } D, \quad k = -1, 1, 2, 3, \ldots$$

Moreover,

$$D_t(\rho_0) = \text{res } (D_t(\Lambda) \Lambda^{-1}) = \text{res } ([F_*, \Lambda] \Lambda^{-1}) = \text{res } ([F_*, \Lambda^{-1}, \Lambda]) \in \text{Im } D.$$

Theorem 2.7. Under the assumptions of Theorem 2.5 there exists a formal symmetry $\Lambda$ such that all even canonical densities $\rho_{2j}$ are trivial.

Proof. It follows from (2.28) that

$$(\Lambda^{2j})^+ = S \Lambda^{2j} S^{-1}.$$

Since $\text{res } (\Lambda^{2j}) = -\text{res } ((\Lambda^{2j})^+)$, the theorem follows from Corollary 2.1.

Example 2.6. The differential operator $\Lambda = D$ is a formal symmetry for any linear equation of the form $u_t = u_n$. Therefore, all canonical densities are equal to zero.

Example 2.7. The KdV equation (1.13) has a recursion operator

$$\hat{\Lambda} = D^2 + 4u + 2u_1 D^{-1}, \quad (2.32)$$

which satisfies equation (2.20). A formal symmetry for the KdV equation can be obtained as $\Lambda = \hat{\Lambda}^{1/2}$. The infinite commutative hierarchy of symmetries for the KdV equation is generated by the recursion operator:

$$G_{2k+1} = \hat{\Lambda}^k(u_1).$$

The first five canonical densities for the KdV equation (cf. Example 2.7) are

$$\rho_{-1} = 1, \quad \rho_0 = 0, \quad \rho_1 = 2u, \quad \rho_2 = 2u_1, \quad \rho_3 = 2u_2 + u^2.$$

We see that even canonical densities are trivial.
Example 2.8. The Burgers equation (1.11) has the recursion operator
\[ \Lambda = D + u + u_1 D^{-1} . \]
Functions \( G_n = \Lambda^n(u_1) \) are generators of symmetries for the Burgers equation. The canonical densities for the Burgers equation are
\[ \rho_{-1} = 1, \quad \rho_0 = u, \quad \rho_1 = u_1, \quad \rho_2 = u_2 + 2uu_1, \ldots . \]
Although \( \rho_0 \) is not trivial, all other canonical densities are trivial. This is in accordance with Proposition 2.4.

Now we can refine Definition 2.1 such that linear equations become \( C \)-integrable.

Definition 2.13. Equation (1.8) is called \( S \)-integrable if it has a formal symmetry that provides infinitely many linearly independent non-trivial canonical densities. An equation is called \( C \)-integrable if it has a formal symmetry such that only finite number of canonical densities are non-trivial and linearly independent.

The notions of \( S \) and \( C \)-integrability are well-defined due to Remark 2.12.

Suppose that equation (1.8) possesses a formal symmetry \( \Lambda \).

Question 2.3. How are the functions \( \rho_1, \sigma_i \) in (2.31) related to the right-hand side of equation (1.8)?

The coefficients of the formal symmetry \( \Lambda \) can be found directly from the linear equation (2.20). The first \( n - 1 \) coefficients \( l_1, l_0, \ldots, l_{3-n} \) of \( \Lambda \) coincide with the first \( n - 1 \) coefficients of the formal series \( (F_n)^{1/n} \). Indeed, \( u_\tau = F \) is a symmetry of equation (1.8) and we can use the main idea of the proof of Theorem 2.4. Carefully calculating the number of correct coefficients, we arrive at the ansatz
\[ \Lambda = (F_n)^{1/n} + \tilde{l}_{2-n} D^{2-n} + \tilde{l}_{1-n} D^{1-n} + \cdots . \]

Having the first \( n - 1 \) coefficients of \( \Lambda \), we can find \( n - 1 \) canonical densities \( \rho_{-1}, \rho_0, \ldots, \rho_{n-3} \) explicitly in terms of the coefficients
\[ F_i = \frac{\partial F}{\partial u_i} \]

of the Fréchet derivative
\[ F_* = F_n D^n + F_{n-1} D^{n-1} + \cdots + F_0. \]

Equating coefficients of \( D \) in equation (2.20), we find that the first unknown coefficient \( \tilde{l}_{2-n} \) of \( \Lambda \) can be found if and only if the first canonical density
\[ \rho_{-1} = F_n^{-\frac{1}{n}} \quad (2.33) \]
is a density of a local conservation law for equation (1.8), i.e., there exists such a function $\sigma_{-1} \in F$ that $D_t(\rho_{-1}) = D(\sigma_{-1})$.

If $D_t(\rho_{-1}) \notin \text{Im } D$, then the formal symmetry does not exist and consequently the equation (1.8) cannot have an infinite sequence of higher symmetries or conservation laws.

If $\sigma_{-1} \in F$ exists, then the coefficient $\tilde{l}_{2-n}$ can be expressed explicitly in terms of the coefficients $F_n, ..., F_0$ and $\sigma_{-1}$. Similarly, the next coefficient $\tilde{l}_{1-n}$ can be found (as an element of $F$) if and only if $D_t(\rho_0) = D(\sigma_0)$ for some $\sigma_0 \in F$. In this case $\tilde{l}_{1-n}$ can be explicitly expressed in terms of $F_n, ..., F_0, \sigma_{-1}, \sigma_0$, etc.

Example 2.9. Consider evolution equations of second order $u_t = F(u, u_1, u_2)$.

Computations described above show that the three first canonical densities can be written in the form

$$\rho_{-1} = F_2^{-1/2}, \quad \rho_0 = F_2^{-1/2} \sigma_{-1} - F_2^{-1} F_1,$$

$$\rho_1 = \rho_{-1} F_0 - \frac{\rho_0^2}{4 \rho_{-1}} + \frac{\rho_0 \sigma_{-1}}{2} - \frac{\rho_{-1} \sigma_0}{2}.$$

Remark 2.13. It follows from Theorems 2.4, 2.5, 2.6 and 2.7 that if we are going to find equations (1.8) with higher symmetries, we have to use conditions (2.31) only, while for equations with higher conservation laws we may additionally assume that $\rho_{2j} = D(\theta_j) + c_j$, where $\theta_j \in F$ and $c_j \in \mathbb{C}$. Thus the necessary conditions, which we employ for conservation laws are stronger than the ones for symmetries.

Using the ideas of [39, 53], one can derive a recursive formula for the whole infinite chain of the canonical conserved densities $\rho_i$. For equations of the form (2.4) such a formula was obtained in [30]. It has the following form:

$$\rho_{n+2} = \frac{1}{3} \left[ \sigma_n - \delta_{n,0} f_0 - f_1 \rho_n - f_2 \left( D(\rho_n) + 2 \rho_{n+1} + \sum_{s=0}^{n} \rho_s \rho_{n-s} \right) \right] - \sum_{s=0}^{n+1} \rho_s \rho_{n+1-s}$$

$$- \frac{1}{3} \sum_{0 \leq s + k \leq n} \rho_s \rho_k \rho_{n-s-k} - D \left[ \rho_{n+1} + \frac{1}{2} \sum_{s=0}^{n} \rho_s \rho_{n-s} + \frac{1}{3} D(\rho_n) \right], \quad n \geq 0,$$

(2.34)

where the first two elements of the sequence $\rho_i$ read as

$$\rho_0 = -\frac{1}{3} f_2, \quad \rho_1 = \frac{1}{9} f_2^2 - \frac{1}{3} f_1 + \frac{1}{3} D(f_2).$$

(2.35)
Here, $\delta_{i,j}$ is the Kronecker delta, $f_i = \frac{\partial f}{\partial u_i}$, where $i = 0, 1, 2$. The density $\rho_{-1} = 1$ disappears in the above recursive formula.

Given an equation of the form (2.4), one can verify on a computer as many integrability conditions (2.31), (2.34) as the computer allows.

Moreover, using several first integrability conditions (see, for instance, the next section), one can solve the following classification problem: find all equations of the prescribed type, which have a formal symmetry. A full classification result includes:

1) A complete list of integrable equations that satisfy the necessary integrability conditions;
2) A confirmation of integrability for each equation from the list;
3) A constructive description of transformations that bring a given integrable equation to one from the list;
4) The number of necessary conditions, which should be verified for a given equation to establish its integrability.

A proof of a classification result contains Items 3) and 4). For Item 2) one can find a Lax representation or a transformation that links the equation with an equation known to be integrable. The existence of an auto-Bäcklund transformation with an arbitrary parameter is also a proper justification of integrability.

### 2.2.7 Classification of integrable KdV-type equations

To demonstrate how the necessary conditions are efficient, we solve in this section a simple classification problem [52, 40].

Consider evolution equations of the form (2.22). From (2.35) it follows that for such equations $\rho_0 = \sigma_0 = 0$ and

$$D_t \left( \frac{\partial f}{\partial u_1} \right) = D(\sigma_1),$$

where $\sigma_1$ is a function depending on $u, u_1, \ldots, u_3$.

**Example 2.10.** For the mKdV equation

$$u_t = u_3 + 6u^2 u_1$$

(2.37)

the conservation law (2.36) reads as follows:

$$D_t(u^2) = D(2uu_2 - u_1^2 + 3u^4).$$

Very often complete up to a class of admissible transformations.
Applying the Euler operator to both sides of (2.36) and using (2.25), we obtain

\[
0 = \delta \frac{\partial}{\partial u} D_t \left( \frac{\partial f}{\partial u_1} \right) = 3u_4 \left( u_2 \frac{\partial^4 f}{\partial u_1^4} + u_1 \frac{\partial^4 f}{\partial u_1^3 \partial u} \right) + O(3),
\]

where \( O(3) \) denotes terms of order not greater than 3. It should be identity in the variables \( u, u_1, \ldots, u_4 \). Equating the coefficient of \( u_4 \) to zero and employing the fact that \( f \) is independent of \( u_2 \), we get

\[
f(u, u_1) = \mu u_1^3 + A(u) u_1 + B(u) u_1 + C(u)
\]

with some constant \( \mu \). It can be readily verified that for such a function \( f \) the condition (2.38) is equivalent to the following system of ODEs:

\[
\mu A' = 0, \quad B'' + 8\mu B' = 0, \quad (B'C)' = 0, \quad AB' + 6\mu C' = 0.
\]

The next necessary integrability condition (2.34) reads

\[
D_t \left( \frac{\partial f}{\partial u} \right) = D(\sigma_2),
\]

which implies

\[
\frac{\delta}{\delta u} D_t \left( \frac{\partial f}{\partial u} \right) = 0.
\]

The latter condition leads to the following additional equations:

\[
A' = 0, \quad AC'' = 0, \quad (C'' + 2\mu C')' = 0, \quad (C'')' = 0.
\]

In the case \( \mu \neq 0 \) solving ODEs obtained above, we determine the functions \( A, B \) and \( C \). As a result, up to a scaling \( u \rightarrow \text{const} \ u \), we arrive at the equations

\[
u_t = u_{xxx} - \frac{1}{2} u_x^3 + (c_1 e^{2u} + c_2 e^{-2u} + c_3) u_x
\]

and

\[
u_t = u_{xxx} + c_1 u_x^2 + c_2 u_x^2 + c_3 u_x + c_4,
\]

where \( c_i \) are arbitrary constants.

If \( \mu = 0 \), then solving the above system of ODEs for the functions \( A, B, C \), we find that the equation has the form

\[
u_t = u_{xxx} + c_0 u_x^2 + (c_1 u^2 + c_2 u + c_3) u_x + c_4 u + c_5,
\]

where

\[
c_0 c_1 = 0, \quad c_0 c_2 = 0, \quad c_1 c_1 = 0, \quad c_4 c_2 = 0, \quad c_1 c_5 = 0.
\]
By the third integrability condition (see formula \(2.34\) for \(\rho_3\)), we find the additional relations
\[c_0c_4 = 0, \quad c_2c_5 = 0.\]
In the case \(c_0 \neq 0\) we arrive at a particular case of equation \(2.40\). If \(c_0 = 0\), two cases are possible: a) \(c_4 = c_5 = 0\) and b) \(c_1 = c_2 = 0\). In the case a) we get
\[u_t = u_{xxx} + (c_1u^2 + c_2u + c_3)u_x.\]  \(2.41\)
In the case b) we have a linear equation, which has a formal symmetry \(\Lambda = D\). Integrability of equations \(2.39\)–\(2.41\) has to be determined by different means. Equation \(2.41\) can be reduced to the KdV or to the mKdV equation by scalings and a shift of \(u\). Equation \(2.40\) is related to \(2.41\) by the potentiation (see Definition 1.4). Equation \(2.39\) was found in [54]. It is interconnected with the KdV equation by a differential substitution of Miura type [33].

### 2.2.8 Equations of Harry-Dym type

The Harry-Dym equation
\[u_t = u^3 u_{xxx}\]  \(2.42\)
is one of well-known integrable evolution equations. Equations of the form
\[u_t = f(u) u_3 + Q(u, u_1, u_2), \quad f'(u) \neq 0\]  \(2.43\)
are called \textit{equations of Harry-Dym type}.

The simplest integrability condition for the equation \(1.8\) (see formula \(2.33\) has the form
\[D_t(\rho^{-1}) = D(\sigma),\]
where
\[\rho^{-1} = \left(\frac{\partial F}{\partial u_n}\right)^{-\frac{1}{n}}\]
In this section we show that this condition allows one to reduce the function \(f\) for any integrable equation of the form \(2.43\) to 1 by the quasi-local transformations (see Definition 1.5).

**Theorem 2.8.** [2]. Any integrable equation of the form can be reduced by a potentiation and point transformations to the form
\[v_t = v_3 + G(v_1, v_2).\]  \(2.44\)

**Proof.** First of all we make the point transformation \(\tilde{u} = f(u)^{-1/3}\) to bring the equation \(2.43\) to the form
\[\tilde{u}_t = \tilde{u}_3 + \tilde{Q}(\tilde{u}, \tilde{u}_1, \tilde{u}_2).\]
For such an equation we have $\rho_{-1} = \tilde{u}$, and therefore for any integrable equation of this form the function $\tilde{u}$ is a conserved density:

$$\tilde{u}_t = D \left( \tilde{u}_2 \frac{\tilde{u}_1}{\tilde{u}_3} + \Psi(\tilde{u}, \tilde{u}_1) \right).$$

The second step is the potentiation $D\tilde{u} = \hat{u}$. As the result, we obtain

$$\hat{u}_t = \frac{\hat{u}_3}{\hat{u}_1^3} + \Psi(\hat{u}_1, \hat{u}_2).$$

The last step is the point transformation

$$\hat{t} = t, \quad \hat{x} = v, \quad \hat{u} = x.$$ (2.45)

For this transformation (see Section 1.4) we have

$$\hat{u}_1 = \frac{1}{v_1}, \quad \hat{u}_2 = -\frac{v_2}{v_1^4}, \quad \hat{u}_3 = -\frac{v_3}{v_1^4} + \frac{3v_2^2}{v_1^8}, \quad \hat{u}_t = -\frac{v_t}{v_1}.$$  Using this formulas, one can check that any equation of the form

$$\hat{u}_t = \frac{\hat{u}_3}{\hat{u}_1^3} + \Psi(\hat{u}_1, \hat{u}_2)$$

transforms to an equation of the form (2.44). \hfill \square

**Example 2.11.** For the Harry-Dim equation (2.42) we take $\tilde{u} = \frac{1}{u}$ to get the equation

$$\tilde{u}_t = D_x \left( \frac{\tilde{u}_2}{\tilde{u}_3^3} - \frac{3 \tilde{u}_1^2}{2 \tilde{u}_4} \right).$$

Applying the transformation (2.45), we obtain the Swartz-KdV equation

$$v_t = v_3 - \frac{3v_2^2}{2v_1},$$

which is a particular case of the Krichever-Novikov equation (2.6).

### 2.2.9 Integrability conditions for non-evolution equations

In this section we generalize [55] the main concepts of the symmetry approach such as the formal recursion operator and the canonical conserved densities to the case of non-evolutionary equations of the form

$$q_{tt} = F(q, q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_m), \quad n > m.$$ (2.46)

Such type equations were excluded from consideration in [24], where only evolution equations have been investigated.
Remark 2.14. Equation (2.46) can be rewritten as a system of two evolution equations
\[ q_t = p, \quad p_t = F(q, q_1, q_2, \ldots, q_n, p, p_1, \ldots, p_m). \]

However, the matrix coefficient of leading derivatives in the linearization operator is a Jordan block whereas in the paper [4, Section 3.2.1] this coefficient was supposed to be diagonalizable.

For equation (2.46) all mixed derivatives of \( q \) containing at least two time differentiations can be expressed in terms of
\[ q, \quad q_x, \quad q_{xx}, \quad \ldots, q_i, \quad \ldots, \quad q_t, \quad q_{1t} = q_{xt}, \quad q_{2t} = q_{xxt}, \quad \ldots, \quad q_{it}, \quad \ldots \quad (2.47) \]
in virtue of (2.46). The derivatives (2.47) are regarded as independent variables.

Equation of the form
\[ q_\tau = G(q, q_1, q_2, \ldots, q_r, q_1t, q_2t, \ldots, q_st) \quad (2.48) \]
compatible with (2.46) is called infinitesimal (local) symmetry of equation (2.46). Compatibility implies that the function \( G \) satisfies (cf. with (2.16)) the equation
\[ L(G) = 0, \]
where
\[ L = D_t^2 - \sum_{i=0}^{n} \frac{\partial F}{\partial q_i} D_x^i - \left( \sum_{i=0}^{m} \frac{\partial F}{\partial q_{ti}} D_x^i \right) D_t = D_t^2 - (M + N D_t) \]
is the linearization operator for equation (2.46).

In order to rewrite consistency conditions of (2.46) and (2.48) in terms of conservation laws of (2.46) one can use a formal Lax representation of the problem. The linearization of equations (2.46), (2.48) gives rise to the compatibility problem
\[ \phi_{tt} = (M + N D_t) \phi, \quad \phi_\tau = (A + B D_t) \phi \]
or, equivalently,
\[ \Phi_t = F_s \Phi, \quad \Phi_\tau = G_s \Phi, \quad \Phi = \begin{pmatrix} \phi \\ \phi_t \end{pmatrix}, \quad F_s = \begin{pmatrix} 0 & 1 \\ M & N \end{pmatrix} \quad (2.49) \]
where
\[ G_s = \begin{pmatrix} A & B \\ \hat{A} & \hat{B} \end{pmatrix}, \quad \hat{A} = A_t + B M, \quad \hat{B} = B_t + B N + A. \quad (2.50) \]
The cross differentiation of equations (2.49) yields
\[ D_t(G_s) = [F_s, G_s] + D_\tau(F_s) \]
where $F$, $G$ are matrix differential operators. The crucial step in the symmetry approach (see proof of Theorem 2.4) is to consider instead of above equation one as follows

$$D_t(R) = [F, R],$$  \hspace{1cm} (2.51)

where $R$ is a pseudo-differential series with matrix coefficients. We call $R$ matrix formal symmetry.

Denoting as before $R_{11} = A$, $R_{12} = B$, we can rewrite (2.51) as follows

$$A_{tt} - N A_t + [A, M] + (2 B_t + [B, N]) M + B M_t = 0,$$ \hspace{1cm} (2.52)

$$B_{tt} + 2 A_t + [B, M] + [A, N] + ([B, N] + 2 B_t) N + B N_t - N B_t = 0.$$ \hspace{1cm} (2.53)

Identities (2.52), (2.53) mean that the scalar pseudo-differential series $R = A + B D_t$ is related to the linearization $\mathcal{L}$ of equation (2.46) by

$$\mathcal{L} (A + B D_t) = (\bar{A} + B D_t) \mathcal{L},$$ \hspace{1cm} (2.54)

where $\bar{A} = A + 2 B_t + [B, N]$. A pseudo-differential series $R = A + B D_t$, whose components

$$A = \sum_{i=0}^{n} a_i D^i, \hspace{1cm} B = \sum_{i=0}^{m} b_i D^i$$

satisfy (2.52) and (2.53), is called scalar formal symmetry for equation (2.46). If $A$ and $B$ are differential operators (or ratios of differential operators), condition (2.54) implies the fact that the recursion operator $\mathcal{R}$ maps symmetries of equation (2.46) to symmetries.

**Remark 2.15.** Let $\mathcal{R}_1 = A_1 + B_1 D_t$ and $\mathcal{R}_2 = A_2 + B_2 D_t$ be two scalar formal recursion symmetries. Then the product $\mathcal{R}_3 = \mathcal{R}_1 \mathcal{R}_2$, in which $D_t^2$ is replaced by $(M + N D_t)$ is also a scalar formal symmetry.

An operator $S = P + Q D_t$ is said to be symplectic (see (2.27)) if

$$\mathcal{L}_* S + \bar{S} \mathcal{L} = 0, \hspace{1cm} \text{where} \hspace{1cm} \bar{S} = \bar{P} + \bar{Q} D_t.$$  

If $S$ can be applied to symmetries, then it maps symmetries to cosymmetries. In the symmetry approach $P$ and $Q$ are supposed to be formal pseudo-differential symbols.

The operator equations for the components $P$ and $S$ of formal symplectic operator $S = P + Q D_t$ have the following form

$$P_{tt} + N^+ P_t + 2 Q_t M + Q M_t = M^+ P - PM - (QN + N^+ Q) M - N^+ P,$$

$$Q_{tt} + 2 P_t + 2 Q_t N + N^+ Q_t = M^+ Q - Q M - (QN + N^+ Q) N - (PN + N^+ P) - (N^+ Q + Q N_t).$$

The linearization $P + Q D_t$ of the variational derivative of any conserved density for equation (2.46) satisfies these equations up to a "small" rest (cf. [26]).
Theorem 2.9. i) If equation (2.46) possesses an infinite sequence of higher symmetries of the form
\[ q_{r_i} = G_i(q, q_1, \ldots, q_{r_i}, q_t, q_{t1}, q_{t2}, \ldots, q_{ts_i}), \]
then there exists a scalar formal symmetry of the form
\[ R = (a_0 + a_{-1}D^{-1} + \ldots) + (b_{-1}D^{-1} + b_{-2}D^{-2} + \ldots) D_t, \tag{2.55} \]
where \( a_i, b_i \) are some functions of variables (2.47).

ii) If equation (2.46) possesses an infinite sequence of local higher conservation laws \( D_t \rho_i = D \sigma_i \), where \( \rho_i, \sigma_i \) are functions of variables (2.47), then there exists a formal symmetry (2.55) and a formal symplectic operator of the form
\[ S = (p_0 + p_{-1}D^{-1} + \ldots) + (q_{-1}D^{-1} + q_{-2}D^{-2} + \ldots) D_t. \]

Let \( R \) be a formal symmetry of the form (2.55). It is easy to find an operator
\[ R^{-1} = (\alpha_{-1}D^{-1} + \alpha_{-2}D^{-2} \ldots) + (\beta_{-2}D^{-2} + \beta_{-3}D^{-3} + \ldots) D_t \]
such that \( RR^{-1} = R^{-1}R = 1 \). Recall that we eliminate the term \( D_t^2 \) in the product of scalar formal symmetries in virtue of the relation \( L = 0 \). Operator \( R^{-1} \) is uniquely defined, the coefficient \( \beta_{-2} \) is equal to \((b_{-1})^{-1}\).

In Section 2.2.6 we have used the residues of the powers of a formal symmetry to define the canonical conserved densities. Here, we apply a different construction.

Theorem 2.10. Let \( R \) be a formal symmetry of the form (2.55). Then there exists a unique representation of total derivative operators \( D \) and \( D_t \) in the form
\[ D_x = \sum_{-\infty}^{2} \rho_i R^i, \quad D_t = \sum_{-\infty}^{3} \sigma_i R^i. \tag{2.56} \]
Functions \( \rho_i \) and \( \sigma_i \) are densities and fluxes of some (maybe trivial) conservation laws
\[ (\rho_i)_x = (\sigma_i)_t \tag{2.57} \]
for equation (2.46).

Example 2.12. The following formulas define the simplest integrability conditions (2.57) for equations of the form
\[ q_{ut} = q_{xxx} + F(q, q_x, q_t, q_{xx}, q_{xt}) : \tag{2.58} \]
\[ \rho_1 = u_2, \quad \rho_2 = v_2 + \frac{2}{3} \sigma_1, \quad \rho_3 = 6\sigma_2 - u_2\sigma_1 + 9u_1 - 3u_2v_2 - \frac{1}{3} u_2^3, \]
48
\[ \rho_4 = 6\sigma_3 - 9u_2\sigma_2 + 3\sigma_1^2 + 27u_1u_2 - u_4^2 + 81v_1 - 9u_2^2v_2 - 27v_2^2, \]
\[ \rho_5 = -2\sigma_4 - 18\sigma_1\sigma_2 + 27(\sigma_1)_1 + 3\sigma_1^2u_2 + 3\sigma_3u_2 + 9\sigma_1u_2v_2 + \sigma_1u_2^2 - 27\sigma_1u_1, \]

where
\[ u_1 = \frac{\partial F}{\partial q_t}, \quad v_1 = \frac{\partial F}{\partial q_x}, \quad u_2 = \frac{\partial F}{\partial q_{xt}}, \quad v_2 = \frac{\partial F}{\partial q_{xx}}. \]

The conditions mean that \( \rho_i \) are densities of local conservation laws for equation (2.58). In other words, for any \( \rho_i \) there exists a function \( \sigma_i \) depending on variables (2.47).

Open problem 2.3. Find all integrable Boussinesq type equations of the form
\[ q_{tt} = q_{xxxx} + F(q, q_x, q_{xx}). \]

2.3 Recursion and Hamiltonian quasi–local operators

Recursion and Hamiltonian operators establish additional relations between higher symmetries and conserved densities.

A recursion operator is an operator \( \mathcal{R} \) that satisfies (2.17) and therefore maps symmetries to symmetries. The simplest symmetry for any equation (1.8) is \( u_\tau = u_x \). The usual way to get generators of all the other symmetries is to act on \( u_x \) by a recursion operator.

Lemma 2.6. Let \( \mathcal{R} \) be a recursion operator. Then the operator \( \mathcal{R}^+ \) maps cosymmetries to cosymmetries.

Proof. It follows from (2.18) that \( [D_t + F^+_t, \mathcal{R}^+] = 0 \). Using Definition 2.10, we arrive at the statement of the lemma.

The main problem is that for almost all integrable models the recursion operator is non-local (see, for instance, (2.32)) and we can apply it only to very special functions from \( \mathcal{F} \) to get a function from \( \mathcal{F} \).

2.3.1 Quasi-local recursion operators

In this section we consider non-formal recursion operators, which generate hierarchies of symmetries for integrable evolution equations (see Proposition 2.2).

Most of known recursion operators have the following special non-local structure:
\[ \mathcal{R} = R + \sum_{i=1}^{k} G_i D^{-1} \circ g_i, \quad g_i, G_i \in \mathcal{F}, \quad (2.59) \]
where $R$ is a differential operator. Without loss of generality we assume that the functions $G_1, \ldots, G_k$ (as well as $g_1, \ldots, g_k$) are linearly independent over $\mathbb{C}$. Operators of the form (2.59) are called quasi-local or weakly nonlocal.

**Remark 2.16.** It is easy to see that any expression of the form $fD^{-1} \circ g$ can be written as

$$fD^{-1} \circ g = (aD - b)^{-1}, \quad \text{where} \quad a = \frac{1}{fg}, \quad b = \frac{D(f)}{gf^2}.$$ 

Thus, the quasi-local ansatz is a non-commutative analog of the partial fraction decomposition of a rational function. Any ratio $AB^{-1}$ of differential operators can be written in a quasi-local form (2.59). However, the functions $g_i$ and $G_i$ belong to a differential extension of the basic differential field $\mathcal{F}$ such that $B$ admits a factorization into first order multipliers. The assumption $g_i, G_i \in \mathcal{F}$ is important for further considerations.

**Lemma 2.7.** Let

$$\mathcal{R} = \mathcal{R} + \sum_{i=1}^{k} \tilde{G}_i D^{-1} \circ \tilde{g}_i \quad (2.60)$$

be a quasi-local operator. Then the product $\mathcal{R} \circ \mathcal{R}$ is quasi-local if $g_i \tilde{G}_j = D(a_{ij}), \ a_{ij} \in \mathcal{F}$ for any $i, j$.

**Proof.** It suffices to prove that $G_i D^{-1} \circ g_i \tilde{G}_j D^{-1} \circ \tilde{g}_j$ is quasi-local. It follows from identities

$$G_i D^{-1} \circ g_i \tilde{G}_j D^{-1} \circ \tilde{g}_j = G_i D^{-1}(D \circ a_{ij} - a_{ij} D) D^{-1} \circ \tilde{g}_j = G_i a_{ij} D^{-1} \circ \tilde{g}_j - G_i D^{-1} \circ a_{ij} \tilde{g}_j.$$ 

\[\Box\]

**Definition 2.14.** An operator $\mathcal{R}$ of the form (2.59) is called a quasi-local recursion operator for equation (1.8) if

1) $\mathcal{R}$, considered as a pseudo-differential series, satisfies (2.17);

2) the functions $G_i$ are generators of some symmetries for (1.8);

3) the functions $g_i$ are variational derivatives of conserved densities.

**Example 2.13.** It is easy to see that the recursion operator (2.32) for the KdV equation is quasi-local with $k = 1$, $G_1 = \frac{u_x}{2}$ and $g_1 = \frac{\delta u}{\delta u} = 1$.

**Remark 2.17.** Maybe it is reasonable [56] to add the hereditary property [57] of the operator $\mathcal{R}$ to 1–3.
Remark 2.18. Substituting (2.59) into (2.17) and equating the non-local terms, we obtain that

\[ \sum_{i=1}^{k} (D_t - F_*) (G_i) D^{-1} \circ g_i + \sum_{i=1}^{k} G_i D^{-1} \circ (D_t + F_*) (g_i) = 0. \]

This “almost” implies that the functions \( G_i \) are symmetries and the functions \( g_i \) are cosymmetries.

Question 2.4. Why applying a quasi-local recursion operator to local symmetries, we obtain local symmetries?

Question 2.5. Why the product of two quasi-local recursion operators is a quasi-local recursion operator?

Suppose that equation (1.8) is a member of a commutative hierarchy (see Remark 1.2 and Example 1.5). This means that

i) Any equation of the hierarchy is a higher symmetry for all the other equations;

ii) Each conserved density of any equation from the hierarchy is a conserved density for all the other equations.

In this case the expression \( \mathcal{R}(g) \), where \( g \) is any symmetry of equation (1.8), belongs to \( \mathcal{F} \). It follows from Item ii) and

Lemma 2.8. \([11]\) The product of the right-hand side \( F \) of equation (1.8) and the variational derivative of any conserved density for equation (1.8) belongs to \( \text{Im } D \).

A different choice of integration constants gives rise to an additional linear combination of the symmetries \( G_1, \ldots, G_k \). Probably a quasi-local ansatz for finding a recursion operator was used for the first time in \([58]\).

Let \( \mathcal{R} \) and \( \bar{\mathcal{R}} \) given by (2.59) and (2.60) be quasi-local recursion operators. Then the non-local terms in the product \( \mathcal{R} \circ \bar{\mathcal{R}} \) has the form

\[ \sum_{i=1}^{k} \mathcal{R}(G_i) D^{-1} \circ \bar{g}_i + \sum_{i=1}^{k} G_i D^{-1} \circ \bar{\mathcal{R}}(g_i). \]

To obtain this formula one can use Lemma 2.7 and the fact that for any differential operator \( S \) and \( p, q \in \mathcal{F} \) the function \( p S(q) + q S^+(p) \) belongs to \( \text{Im } D \).

We see that the operator \( \mathcal{R} \circ \bar{\mathcal{R}} \) satisfies the requirements 1) (see Proposition 2.3) and 2) of Definition 2.14. It follows from Lemma 2.6 that the functions \( g_i \) for the product are cosymmetries (but not necessarily variational derivatives of some conserved densities). However, for a particular equation one can try to prove that any cosymmetry is a variational derivative of a conserved density. Briefly outline how to prove it for the KdV equation.
1. Using that for the KdV equation the operator $\frac{\partial}{\partial u}$ maps any cosymmetry to a cosymmetry reducing its order by 2, prove that any cosymmetry has an even order;

2. Derive from (2.26) that any cosymmetry $S$ has the form $S = c u_{2k} + O(2k-1)$;

3. Employ the fact that for any $k$ the KdV equation has a conserved density of the form $\rho_k = u_k^2 + O(k-1)$ to reduce the order of cosymmetry: $S_1 = S + (-1)^{k+1} \frac{c}{2} \frac{\partial \rho_k}{\partial u}$ has a lower order;

4. Use the induction over $k$.

**Open problem 2.4.** Prove the same statement for the Krichever–Novikov equation.

**Remark 2.19.** The function $u_1$ is a cosymmetry for the linear equation $u_t = u_{xxx}$, which is not a variational derivative of a conserved density.

Thus the set of all quasi-local recursion operators for the KdV equation form a commutative (see Proposition 2.3) associative algebra $A_{rec}$ over $\mathbb{C}$. It is possible to prove that this algebra is generated by the operator (2.32). In other words, $A_{rec}$ is isomorphic to the algebra of all polynomials in one variable.

It turns out [59] that it is not true for integrable models such as the Krichever–Novikov and the Landau–Lifshitz equations.

### 2.3.2 Recursion operators for Krichever-Novikov equation

Consider the Krichever-Novikov equation (2.6). This equation is the most interesting among integrable equations of the form (2.4) because of two reasons.

**Reason 1.** All integrable equations (2.4) can be obtained by some transformations and limit procedures from (2.6).

**Reason 2.** All algebraic structures related to equation (2.6) are the most generic and sophisticated.

Denote by $G_1$ the right-hand side of (2.6). The fifth order symmetry of (2.6) is given by

$$G_2 = u_5 - 5 \frac{u_4 u_2}{u_1} - \frac{5}{2} u_3^2 - \frac{25}{8} u_3 u_2^2 - \frac{45}{6} u_3^2 - \frac{5}{3} P u_3 + \frac{25}{9} P u_2^2 + \frac{5}{3} P^2 u_2 - \frac{5}{18} u_1 P - \frac{5}{9} u_1 P''.$$
The three simplest conserved densities of (2.6) are

\[ \rho_1 = -\frac{1}{2} \frac{u_2}{u_1^2} - \frac{3}{3} \frac{P}{u_1}, \quad \rho_2 = \frac{1}{2} \frac{u_2}{u_1^2} - \frac{3}{8} \frac{u_3}{u_1} - \frac{5}{6} \frac{P u_2}{u_1^2} + \frac{1}{18} \frac{P^2}{u_1} - \frac{5}{9} \frac{P''}{u_1}, \]

\[ \rho_3 = u_1^2 + 3 \frac{u_2}{u_1} - \frac{19}{2} \frac{u_2 u_2}{u_1^2} + \frac{7}{3} \frac{P u_2}{u_1^2} + \frac{35}{9} \frac{P' u_3}{u_1^2} + \frac{45}{8} \frac{u_3^2}{u_1^2} - \frac{259}{36} \frac{u_3^2 P}{u_1^2} + \frac{35}{18} \frac{P^2 u_2}{u_1^2}, \]

\[ -\frac{14}{9} \frac{P'' u_2}{u_1^2} + \frac{1}{27} \frac{P^3}{u_1^2} - \frac{14}{27} \frac{P'' P}{u_1^2} - \frac{7}{27} \frac{P^2}{u_1^2} - \frac{14}{9} \frac{P^{(IV)} u_2}{u_1^2}. \]

In the paper [58] the fourth order quasi–local recursion operator

\[ \mathcal{R}_1 = D^4 + a_1 D^3 + a_2 D^2 + a_3 D + a_4 + G_1 D - 1 \circ \frac{\delta \rho_1}{\delta u} + u_2 D - 1 \circ \frac{\delta \rho_2}{\delta u}, \]

was found. Here, the coefficients \( a_i \) are given by

\[ a_1 = -4 \frac{u_2}{u_1}, \quad a_2 = 6 \frac{u_2}{u_1^2} - 2 \frac{u_3}{u_1} - \frac{4}{3} \frac{P}{u_1^2}, \]

\[ a_3 = -2 \frac{u_4}{u_1} + 8 \frac{u_3 u_2}{u_1^2} - 6 \frac{u_2^3}{u_1^3} + 4 \frac{P u_2}{u_1^3} - \frac{2}{3} \frac{P'}{u_1}, \]

\[ a_4 = \frac{u_5}{u_1} - 2 \frac{u_2^2}{u_1^2} + 8 \frac{u_3^2 u_2}{u_1^3} - \frac{4}{3} \frac{u_5 u_2}{u_1^3} - \frac{3 u_2}{u_1} + \frac{4}{9} \frac{P^2}{u_1} + \frac{4}{3} \frac{P u_2}{u_1} + \frac{10}{9} \frac{P''}{u_1} - \frac{8}{3} \frac{P' u_2}{u_1}. \]

The following statement can be verified directly.

**Theorem 2.11.** There exists a quasi–local recursion operator for (2.6) of the form

\[ \mathcal{R}_2 = D^6 + b_1 D^5 + b_2 D^4 + b_3 D^3 + b_4 D^2 + b_5 D + b_6 - \]

\[ \frac{1}{2} u_2 D - 1 \circ \frac{\delta \rho_3}{\delta u} + G_1 D - 1 \circ \frac{\delta \rho_2}{\delta u} + G_2 D - 1 \circ \frac{\delta \rho_1}{\delta u}, \]  

(2.61)

where

\[ b_1 = -6 \frac{u_2}{u_1}, \quad b_2 = -9 \frac{u_2}{u_1} - 2 \frac{P}{u_1^2} + 21 \frac{u_2}{u_1}, \]

\[ b_3 = -11 \frac{u_4}{u_1} + 60 \frac{u_3 u_2}{u_1^2} + 14 \frac{P u_2}{u_1^2} - 5 \frac{u_3^2}{u_1} - 3 \frac{P'}{u_1}, \]

\[ b_4 = -4 \frac{u_5}{u_1} + 38 \frac{u_3 u_2}{u_1^2} + 22 \frac{u_4}{u_1^2} + 99 \frac{u_4}{u_1} - 155 \frac{u_3 u_2}{u_1^2} + 34 \frac{P u_3}{u_1} - 44 \frac{P u_2}{u_1^2} \]

\[ + \frac{4}{3} \frac{P^2}{u_1} + 12 \frac{P u_2}{u_1} - \frac{P''}{u_1}, \]  

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The operators $R_1$ and $R_2$ are related by the elliptic curve

$$R_2^2 = R_1^3 - g_2 R_1 - g_3,$$

where

$$g_2 = \frac{16}{27} \left( (P')^2 - 2P''P + 2P^{(IV)}P \right),$$

$$g_3 = \frac{256}{243} \left( -\frac{1}{3} (P')^3 - \frac{3}{2} (P')^2 P^{(IV)} + P' P'' P''' + 2P^{(IV)}P''P - P(P')^2 \right).$$

**Remark 2.20.** It is easy to verify that $g_2$ and $g_3$ are constants for any polynomial $P(u)$ such that $\deg P \leq 4$. Under Möbius transformations of the form

$$u \rightarrow \frac{\alpha u + \beta}{\gamma u + \delta},$$

in equation (2.6) the polynomial $P(u)$ changes according to the same rule as in the differential $\omega = \frac{du}{\sqrt{P(u)}}$. The expressions $g_2$ and $g_3$ are invariants with respect to the Möbius group action.

**Remark 2.21.** The ratio $R_3 = R_2 R_1^{-1}$ satisfies equation (2.20). It belongs to the skew field of differential operator fractions $[60]$. However, this operator is not quasi-local and it is unclear how to apply it even to the simplest symmetry generator $u_x$.  

\[ b_5 = -\frac{2 u_6}{u_1} + 29 \frac{u_4 u_3}{u_1^2} + 80 P \frac{u_2^3}{u_1^3} + \frac{23}{3} P' u_3 - 104 \frac{u_2 u_3^2}{u_1^3} - 70 \frac{u_4 u_3^2}{u_1^4} + 241 \frac{u_2^3 u_3}{u_1^4} + 14 \frac{u_5 u_2}{u_1^5} \\
+ \frac{20}{3} P \frac{u_4}{u_1^3} - \frac{170}{3} P \frac{u_2 u_3}{u_1^3} + \frac{4 P'' P}{3 u_1^3} - 22 P \frac{u_2^2}{u_1^3} + 2 P'' \frac{u_2}{u_1} - \frac{16}{3} P^2 \frac{u_2}{u_1^3} - 108 \frac{u_5}{u_1^5}, \\
\]

\[ b_6 = \frac{u_7}{u_1^7} - \frac{6 u_2 u_6}{u_1^4} + \frac{8}{9} P^2 \frac{u_2^2}{u_1^3} - 195 \frac{u_3^2 u_2}{u_1^4} + 6 P \frac{u_2^3}{u_1^4} + \frac{142}{3} P \frac{u_2}{u_1^4} + \frac{28}{9} P' \frac{u_2}{u_1^3} + \frac{101}{3} P^3 \frac{u_2}{u_1^3} + 10 \frac{u_4 u_3^2}{u_1^3} \\
+ \frac{34}{3} P \frac{u_4 u_2}{u_1^4} - 72 \frac{u_5}{u_1^3} - \frac{28}{9} P''' u_2 + \frac{38}{3} P'' \frac{u_2}{u_1^3} - \frac{19}{3} P' \frac{u_4}{u_1^4} - \frac{122}{3} P'' \frac{u_3}{u_1^4} - \frac{10 u_2}{u_1^4} + 22 u_3^3 \\
- \frac{178}{3} P \frac{u_3 u_2^2}{u_1^4} + \frac{14}{9} P^{(IV)} \frac{u_2}{u_1^4} + \frac{113}{3} P' \frac{u_4 u_2}{u_1^4} - \frac{2}{3} P \frac{u_5}{u_1^4} - \frac{17}{3} P'' \frac{u_3}{u_1^4} - \frac{4}{3} P^2 \frac{u_3}{u_1^4} - 89 u_4 u_3^2 \\
+ 236 \frac{u_3^4 u_2}{u_1^4} - 13 \frac{u_5 u_3}{u_1^4} + 25 \frac{u_5 u_2^2}{u_1^4} - \frac{7 P^2}{9 u_1^4} - \frac{8 P^3}{27 u_1^4} - \frac{4 P^{(IV)}P}{9 u_1^4}. \\
\]
2.3.3 Hamiltonian operators and bi-Hamiltonian structure for the Krichever–Novikov equation

Most of known integrable equations (1.8) can be written in a Hamiltonian form

\[ u_t = \mathcal{H} \left( \frac{\delta \rho}{\delta u} \right), \]

where \( \rho \) is a conserved density and \( \mathcal{H} \) is a Hamiltonian operator. The analog of the operator identity (2.17) for Hamiltonian operators is given by

\[ (D_t - F_\nu) \mathcal{H} = \mathcal{H} (D_t + F_\nu^+), \]

which means that \( \mathcal{H} \) maps cosymmetries to symmetries.

The Poisson bracket corresponding to the Hamiltonian operator \( \mathcal{H} \) is defined by

\[ \{f, g\} = \frac{\delta f}{\delta u} \mathcal{H} \left( \frac{\delta g}{\delta u} \right). \]

Remark 2.22. Actually, the Poisson bracket has to be defined between functionals \( \int f \, dx \) and \( \int g \, dx \), i.e., between equivalence classes of \( f \) and \( g \) (see Remark 2.8).

Since the Poisson bracket is defined on the vector space of equivalence classes \( \mathcal{F}/\text{Im}D \), the Leibniz rule has no sense for (2.63). The skew-symmetricity and the Jacobi identity for (2.63) are required. In terms of elements of \( \mathcal{F} \) this means that

\[ \{f, g\} + \{g, f\} \in \text{Im} D, \]

\[ \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} \in \text{Im} D. \]

Therefore, besides (2.62) the Hamiltonian operator for equation (1.8) should satisfy some identities (see for example, [11]) equivalent to (2.64), (2.65).

Lemma 2.9. If operators \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) satisfy (2.62), then \( \mathcal{R} = \mathcal{H}_2 \mathcal{H}_1^{-1} \) satisfies (2.20).

As a rule, the Hamiltonian operators are local (i.e. differential) or quasi–local operators. In the latter case

\[ \mathcal{H} = H + \sum_{i=1}^{m} G_i D_i^{-1} \tilde{G}_i, \]

where \( H \) is a differential operator and \( G_i, \tilde{G}_i \) are fixed generators of symmetries [61, 62].

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The KdV equation possesses two local Hamiltonian operators

\[ \mathcal{H}_1 = D, \quad \mathcal{H}_2 = D^3 + 4uD + 2u_x. \]

The first example

\[ \mathcal{H}_0 = u_xD^{-1}u_x \]

of a quasi-local Hamiltonian operator for the Krichever-Novikov equation \(^{(2.6)}\) was found in \([58]\) (see also \([59]\)).

The recursion operators \( \mathcal{R}_i \) for \((2.6)\) presented above appear to be the ratios

\[ \mathcal{R}_1 = \mathcal{H}_1\mathcal{H}_0^{-1}, \quad \mathcal{R}_2 = \mathcal{H}_2\mathcal{H}_0^{-1} \]

of the following quasi-local Hamiltonian operators

\[ \mathcal{H}_1 = \frac{1}{2}(u_x^2D^3 + D^3 \circ u_x^2) + (2u_{xxx}u_x - \frac{9}{2}u_{xx}^2 - \frac{2}{3}P)D + D \circ (2u_{xxx}u_x - \frac{9}{2}u_{xx}^2 - \frac{2}{3}P) \]

\[ + G_1D^{-1} \circ G_1 + u_xD^{-1} \circ G_2 + G_2D^{-1} \circ u_x, \]

\[ \mathcal{H}_2 = \frac{1}{2}(u_x^2D^5 + D^5 \circ u_x^2) + (3u_{xxx}u_x - \frac{19}{2}u_{xx}^2 - P)D^3 + D^3 \circ (3u_{xxx}u_x - \frac{19}{2}u_{xx}^2 - P) \]

\[ + hD + D \circ h + G_1D^{-1} \circ G_2 + G_2D^{-1} \circ G_1 + u_xD^{-1} \circ G_3 + G_3D^{-1} \circ u_x, \]

where

\[ h = u_{xxxxx}u_x - 9u_{xxxx}u_{xx} + \frac{19}{2}u_{xxxx}^2 \frac{2}{3u_x} (5P - 39u_{xx}^2) + \frac{u_x^2}{u_x} (5P - 9u_{xx}^2) + \frac{2P^2}{3u_x^2} + u_x^2P'' \]

and \( G_3 = \mathcal{R}_1(G_1) = \mathcal{R}_2(u_x) \) is the generator of the seventh order symmetry for \((2.6)\):

\[ G_3 = u_7 - \frac{7}{3} \frac{u_2u_6}{u_1} - \frac{7}{6} \frac{u_5}{u_1} (2P + 12u_3u_1 - 27u_2^2) - \frac{21}{2} \frac{u_4^2}{u_1^2}u_2(2P - 11u_2^2) \]

\[ - \frac{7}{3} \frac{u_4}{u_1} (2P'u_1 - 51u_2u_3) + \frac{49}{2} \frac{u_3^3}{u_1^2} + \frac{7}{8} \frac{u_3^2}{u_1^2} (22P - 417u_2^2) + \frac{2499}{24} \frac{u_2^4}{u_1^3} \]

\[ + \frac{91}{3} \frac{P'u_3}{u_1^2} - \frac{595}{6} \frac{P}{u_1} \frac{u_2^2}{u_3} - \frac{35}{16} u_3 (2P''u_1^4 - P^2) - \frac{1575}{16} \frac{u_2^6}{u_1^4} + \frac{1813}{24} \frac{u_2^4}{u_1^4} + P' \]

\[ - \frac{203}{6} \frac{u_3^3}{u_1^3} + \frac{49}{3} \frac{u_2^2}{u_1^2} (6P''u_1^4 - 5P^2) - \frac{7}{9} \frac{u_2^2}{u_1^2} (2P''u_1^4 - 5PP') + \frac{7}{54} \frac{P^3}{u_1}. \]

The operators \( \mathcal{H}_i, \ i = 1, 2, \) were found in \([59]\). It was verified that they satisfy \((2.62)\). It is easy to verify that the Poisson brackets \((2.63)\) corresponding to \( \mathcal{H}_i \) satisfy identity \((2.64)\).
Open problem 2.5. Prove that the Poisson brackets (2.63) corresponding to $\mathcal{H}_i$ are compatible and satisfy the Jacobi identity (2.65).

Remark 2.23. Very recently in [56] it was proved that $\mathcal{H}_1$ satisfies the Jacobi identity and compatible with $\mathcal{H}_0$. 
Chapter 3

Integrable hyperbolic equations of Liouville type

The (open) Toda lattices

\[(u_i)_{xy} = \sum_j A^i_j \exp(u_j), \quad (3.1)\]

where \(A^i_j\) is the Cartan matrix of a simple Lie algebra [63], provide examples of Liouville type systems.

For the Lie algebra of type \(A_1\) the system coincides with the famous Liouville equation

\[u_{xy} = \exp u. \quad (3.2)\]

The Liouville equation possesses the following remarkable properties:

• **1.** It has a local formula for the general solution

\[u(x, y) = \log \left( \frac{2f'(x)g'(y)}{(f(x) + g(y))^2} \right);\]

• **2.** It admits a group of classical symmetries

\[x \to \phi(x), \quad y \to \psi(y), \quad u \to u - \log \phi'(x) - \log \psi'(y)\]

depending on two arbitrary functions of one variable;

• **3.** It possesses the generalized first integrals

\[w = u_{xx} - \frac{1}{2} u_x^2, \quad \bar{w} = u_{yy} - \frac{1}{2} u_y^2;\]
4. It has a non-commutative hierarchy of higher infinitesimal symmetries of the form
\[ u_{\tau} = (D_x + u_x) P(w, w_x, \ldots, w_n) + (D_y + u_y) Q(\bar{w}, \bar{w}_y, \ldots, \bar{w}_m), \]
where \( P \) and \( Q \) are arbitrary functions, \( n \) and \( m \) are arbitrary integers;

5. It has a terminated sequence of the Laplace invariants.

These are typical features of the so called equations of Liouville type (or, which is the same, Darboux integrable equations) [64]–[68].

### 3.1 Generalized integrals

Consider hyperbolic equations of the form
\[ u_{xy} = F(x, y, u, u_x, u_y). \] (3.3)

The corresponding total \( x \) and \( y \)-derivatives are given by the recursive formulas
\[
D = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} \bar{D}^{i-1}(F) \frac{\partial}{\partial \bar{u}_i},
\]
and
\[
\bar{D} = \frac{\partial}{\partial y} + \sum_{i=0}^{\infty} \bar{u}_{i+1} \frac{\partial}{\partial \bar{u}_i} + \sum_{i=1}^{\infty} D^{i-1}(F) \frac{\partial}{\partial u_i},
\]
where
\[ u_0 = \bar{u}_0 = u, \quad u_1 = u_x, \quad \bar{u}_1 = u_y, \quad u_2 = u_{xx}, \quad \bar{u}_2 = u_{yy}, \ldots. \] (3.4)

Although at first glance it seems that \( D \) is defined in terms of \( \bar{D} \) and vice versa, the vector fields are actually well defined by these formulas. One can verify that \([D, \bar{D}] = 0\) in virtue of (3.3).

**Example 3.1.** For the Liouville equation (3.2) we have
\[ \bar{D} = \frac{\partial}{\partial \bar{y}} + \sum_{i=0}^{\infty} \bar{u}_{i+1} \frac{\partial}{\partial \bar{u}_i} + \exp(u) \left( \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2} + (u_2 + u_1^2) \frac{\partial}{\partial u_3} + \cdots \right). \]

It is easy to verify that \( \bar{D}\left(u_2 - \frac{1}{2}u_1^2\right) = 0\).

**Definition 3.1.** A function \( W(x, y, u, u_1, \ldots, u_p) \) is called \( y \)-integral for equation (3.3) if \( \bar{D}(W) = 0 \). The number \( p \) is called the order of \( W \). Analogously, a function \( W(x, y, \bar{u}_1, \ldots, \bar{u}_p) \) such that \( D(W) = 0 \) is called \( x \)-integral.
It is obvious that for any \( y \)-integral \( w \) and any function \( S \) the expression
\[
W = S \left( x, w, D(w), \ldots, D^k(w) \right)
\]
(3.5)
is a \( y \)-integral as well.

**Proposition 3.1.** \cite{20} i) Any \( y \)-integral is a function of the variables \( x, y, u, u_1, u_2, \ldots, u_k, \ldots \).

ii) Any \( y \)-integral \( W \) has the form \((3.5)\), where \( w \) is an integral of minimal possible order. The minimal integral is defined uniquely up to any transform \( w \to \phi(x, w) \).

iii) If the order \( n \) of minimal integral is greater than 1, then there exists a minimal integral \( w \) such that
\[
\frac{\partial^2 w}{\partial u^2_n} = 0. \]

**Proof.** Let a function \( W = W(x, y, u, u_1, \ldots, u_n, \bar{u}_1, \ldots, \bar{u}_m) \) be a \( y \)-integral of equation \((3.3)\). Then
\[
\left( \frac{\partial}{\partial y} + \bar{u}_1 \frac{\partial}{\partial u} + \bar{u}_2 \frac{\partial}{\partial \bar{u}_1} + \cdots + \bar{u}_{m+1} \frac{\partial}{\partial \bar{u}_m} \right) W + \left( F \frac{\partial}{\partial u_1} + D(F) \frac{\partial}{\partial u_2} + \cdots + D^{n-1}(F) \frac{\partial}{\partial u_n} \right) W = 0.
\]

Since \( F, D(F), \ldots, D^{n-1}(F) \) depend on \( x, y, u, \bar{u}_1, u_1, u_2, \ldots, u_n \), we get
\[
\frac{\partial W}{\partial \bar{u}_m} = 0, \quad \frac{\partial W}{\partial \bar{u}_{m-1}} = 0, \quad \cdots \quad \frac{\partial W}{\partial \bar{u}_1} = 0.
\]

Then any \( y \)-integral is a function that depends on the variables \( x, y, u, u_1, u_2, \ldots, u_k, \ldots \) only.

Denote by \( w \) a \( y \)-integral of the smallest order \( n \). Let \( W = W(x, y, u, u_1, \ldots, u_m), m \geq n \) be another \( y \)-integral. It can be rewritten in the form
\[
W = W(x, y, u, u_1, \ldots, u_{n-1}, w, w_1, \ldots, w_{m-n}).
\]

For any values of \( x, w, w_1, \ldots, w_{m-n} \) this function is a \( y \)-integral of order less than \( n \). Therefore, it is a constant and \( W \) does not depend on \( y, u, u_1, \ldots, u_{n-1} \).

To prove iii) let us differentiate the relation
\[
\bar{D} w = \left( \frac{\partial}{\partial y} + \bar{u}_1 \frac{\partial}{\partial u} + F \frac{\partial}{\partial u_1} + D(F) \frac{\partial}{\partial u_2} + \cdots + D^{n-1}(F) \frac{\partial}{\partial u_n} \right) w = 0
\]
by \( u_n \) twice to obtain
\[
(\bar{D} + F_{u_1}) \frac{\partial w}{\partial u_n} = 0, \quad (\bar{D} + 2F_{u_1}) \frac{\partial^2 w}{\partial u^2_n} = 0.
\]

\(^1\)The similar statements are true for \( x \)-integrals.
This implies that
\[ \frac{\partial^2 w}{\partial u_n^2} \left( \frac{\partial w}{\partial u_n} \right)^{-2} \]
is a \( y \)-integral and therefore it is a function of \( x \) and \( w \). Hence
\[ \frac{\partial w}{\partial u_n} H(x, w) = g(x, y, u_1, \ldots, u_{n-1}) \]
for some functions \( H \) and \( g \) and
\[ W = \int H \, dw \]
is an integral linear in \( u_n \).

Definition 3.2. An equation of the form  is called **Darboux integrable** if it has both \( x \) and \( y \) integrals.

Some of linear hyperbolic equations are Darboux integrable.

Example 3.2. The first order integrals for the wave equation
\[ u_{xy} = 0 \]
are given by
\[ w = u_1, \quad \bar{w} = \bar{u}_1. \]

Example 3.3. The Euler-Poisson equation
\[ u_{xy} = \frac{u_y - u_x}{x - y} \]
possesses minimal integrals of second order
\[ w = \frac{u_2}{x - y}, \quad \bar{w} = \frac{\bar{u}_2}{x - y}. \]

### 3.2 Laplace invariants for hyperbolic operators

Consider a linear hyperbolic operator
\[ L_0 = \frac{\partial^2}{\partial x \partial y} + a_0(x, y) \frac{\partial}{\partial x} + b_0(x, y) \frac{\partial}{\partial y} + c_0(x, y). \] (3.6)

It is easy to verify that
\[ L_0 = \left( \frac{\partial}{\partial x} + b_0 \right) \left( \frac{\partial}{\partial y} + a_0 \right) - h_1 = \left( \frac{\partial}{\partial y} + a_0 \right) \left( \frac{\partial}{\partial x} + b_0 \right) - k_0, \]
where
\[ h_1 = \frac{\partial a_0}{\partial x} + b_0 a_0 - c_0, \quad k_0 = \frac{\partial b_0}{\partial y} + a_0 b_0 - c_0. \] (3.7)
Definition 3.3. The functions (3.7) are called main Laplace invariants of the operator $L_0$.

Lemma 3.1. Operators $L_0$ and $\bar{L}$ of the form (3.6) are related by a gauge transform

$$\bar{L} = \alpha(x, y) L_0 \alpha(x, y)^{-1}$$

for some function $\alpha$ iff their main Laplace invariants coincide.

As demonstrated below, the functions (3.7) can be considered as subsequent terms of a sequence. That is why the first of these functions is denoted by $h_1$ (not by $h_0$).

Definition 3.4. The functions $h_1$ and $k_0$ are called the main Laplace invariants of the operator (3.6).

The equation $L_0(V) = 0$ is equivalent to the system

$$\left( \frac{\partial}{\partial y} + a_0 \right) V = V_1, \quad \left( \frac{\partial}{\partial x} + b_0 \right) V_1 = h_1 V.$$  

If $h_1 \neq 0$, we can find $V$ from the second equation and substitute into the first one. Therefore, $V_1$ satisfies a hyperbolic equation $L_1(V_1) = 0$, where

$$L_1 = \frac{\partial^2}{\partial x \partial y} + a_1(x, y) \frac{\partial}{\partial x} + b_1(x, y) \frac{\partial}{\partial y} + c_1(x, y).$$

The coefficients and the Laplace invariants of the new hyperbolic operator $L_1$ are given by

$$a_1 = a_0 - (\log h_1)_y, \quad b_1 = b_0, \quad c_1 = a_1 b_0 + (b_0)_y - h_1,$$

$$h_2 = (a_1)_x - (b_0)_y + h_1, \quad k_1 = h_1.$$

Remark 3.1. The invariant $h_2$ can be rewritten in terms of the main invariants of the operator (3.6) only:

$$h_2 = 2 h_1 - k_0 - (\log h_1)_{xy}.$$  

If $h_2 \neq 0$, we may continue. As a result, we get a chain of hyperbolic operators

$$L_i = \frac{\partial^2}{\partial x \partial y} + a_i \frac{\partial}{\partial x} + b_i \frac{\partial}{\partial y} + c_i,$$

where $i \in \mathbb{N}$ and

$$a_i = a_{i-1} - (\log h_i)_y, \quad b_i = b_0, \quad c_i = a_i b_0 + (b_0)_y - h_i,$$
The initial linear hyperbolic equation $L_0(V) = 0$ can be also rewritten as

$$
\left( \frac{\partial}{\partial x} + b_0 \right) V = V_{-1}, \quad \left( \frac{\partial}{\partial y} + a_0 \right) V_{-1} = k_0 V.
$$

If $k_0 \neq 0$, then $V_{-1}$ satisfies the following equation

$$
\left( \frac{\partial^2}{\partial x \partial y} + a_{-1} \frac{\partial}{\partial x} + b_{-1} \frac{\partial}{\partial y} + c_{-1} \right) V_{-1} = 0,
$$

etc. We get the chain of operators

$$
L_{-i} = \frac{\partial^2}{\partial x \partial y} + a_{-i} \frac{\partial}{\partial x} + b_{-i} \frac{\partial}{\partial y} + c_{-i}, \quad i \in \mathbb{N},
$$

where

$$
a_{-i} = a_0, \quad b_{-i} = b_{-(i-1)} - (\log k_{-(i-1)})_x, \quad c_{-i} = a_0 b_{-i} + (a_0)_x - k_{-(i-1)},
$$

$$
h_{1-i} = k_{-(i-1)}, \quad k_{-i} = h_{-i} = 2h_{-(i-1)} - h_{-(i-2)} - (\log h_{-(i-1)})_{xy}.
$$

**Definition 3.5.** The functions $h_i, \ i \in \mathbb{Z}$ defined by (3.8), (3.9) are called the Laplace invariants of the operator (3.6).

The sequence of the Laplace invariants is uniquely defined by the following recursive formula

$$
h_i = 2h_{i-1} - h_{i-2} - (\log h_{i-1})_{xy}, \quad i \in \mathbb{Z}
$$

and by the initial data

$$
h_1 = \frac{\partial a_0}{\partial x} + a_0 b_0 - c_0, \quad h_0 = \frac{\partial b_0}{\partial y} + a_0 b_0 - c_0.
$$

Remarkably, (3.10) is nothing else but the integrable infinite A-Toda lattice.

If one of the Laplace invariants vanishes, then the next invariant is not defined and the sequence of invariants terminates. If it terminates in both directions: $h_r = h_{-s} = 0$ for $r \geq 1, s \geq 0$, then the sequence becomes finite and (3.10) turns out to be the open Toda lattice [63]. In this case the equation $L_0(V) = 0$ can be solved explicitly (see, for example, [23]).
3.3 Nonlinear hyperbolic equations of Liouville type

The linearization operator for (3.3) is given by the formula

\[ L = D\bar{D} - \frac{\partial F}{\partial u_1} D - \frac{\partial F}{\partial \bar{u}_1} \bar{D} - \frac{\partial F}{\partial u}. \]  

(3.12)

This is a linear hyperbolic operator of the form (3.6), where the partial derivatives are replaced by the total derivative operators \( D \) and \( \bar{D} \) and the coefficients become functions of \( x, y \) and finite number of variables (3.4).

In accordance with (3.11), we define the main Laplace invariants for equation (3.3) as

\[ H_1 \overset{\text{def}}{=} -D \left( \frac{\partial F}{\partial u_1} \right) + \frac{\partial F}{\partial u} \frac{\partial F}{\partial \bar{u}_1}, \quad H_0 \overset{\text{def}}{=} -\bar{D} \left( \frac{\partial F}{\partial \bar{u}_1} \right) + \frac{\partial F}{\partial u} \frac{\partial F}{\partial u_1} + \frac{\partial F}{\partial u}. \]

The invariants \( H_i \) for \( i > 1 \) and for \( i < 0 \) are determined from

\[ D\bar{D}(\log H_i) = -H_{i+1} - H_{i-1} + 2H_i, \quad i \in \mathbb{Z}. \]

For the Liouville equation we have \( H_0 = H_1 = \exp u \). It is easy to verify that \( H_2 = H_{-1} = 0 \).

Definition 3.6. We call (3.3) an equation of Liouville type if there exists \( r \geq 1 \) and \( s \geq 0 \) such that

\[ H_r = H_{-s} \equiv 0. \]

Notice that this is a much more constraining property than the existence of the generalized integrals used by Darboux since Definition 3.6 involves only the right-hand side of the equation.

Remark 3.2. For linear hyperbolic equations the Laplace invariants coincide with the Laplace invariants of the corresponding linear operator (see Section 3.2). In particular equations form Examples 3.2, 3.3 are Liouville integrable.

Example 3.4. For the equation

\[ u_{xy} = \frac{1}{u} \sqrt{1 - u^2_x} \sqrt{1 - u^2_y} \]  

(3.13)

we have \( H_2 = H_{-1} = 0 \).

Example 3.5. For the equation

\[ u_{xy} = uu_y \]  

(3.14)

the invariants \( H_3 \) and \( H_0 \) are equal to zero.
Example 3.6. For the equation
\[ u_{xy} = -\frac{2k}{x+y}u_y \sqrt{u_y}, \quad k \in \mathbb{N} \] (3.15)
we have \( H_{k+1} = H_{-k} = 0 \).

Theorem 3.1. [69, 70] The equation (3.3) has non-trivial both \( x \) and \( y \)-integrals iff it is an equation of Liouville type.

Proposition 3.2. For any Liouville type equation (3.3) there exist functions \( \psi(x, y, u, u_1, \ldots, u_p) \) and \( \bar{\psi}(x, y, u, \bar{u}_1, \ldots, \bar{u}_p) \) such that
\[
\frac{\partial F}{\partial u_1} = \bar{D} \log \psi(x, y, u, u_1, \ldots, u_p), \quad \frac{\partial F}{\partial \bar{u}_1} = D \log \bar{\psi}(x, y, u, \bar{u}_1, \ldots, \bar{u}_p).
\]

Remark 3.3. Another statement of this kind is the following:
\[
\frac{\partial F}{\partial u_1} \frac{\partial F}{\partial \bar{u}_1} + \frac{\partial F}{\partial u} = \bar{D} \phi(x, y, u, u_1, \ldots, u_s) = D \bar{\phi}(x, y, u, \bar{u}_1, \ldots, \bar{u}_s)
\]
for some functions \( \phi \) and \( \bar{\phi} \).

Theorem 3.2. For any Liouville type equation (3.3) the evolution equation
\[ u_\tau = \mathcal{M} \left[ Q(x, w, D^k(w)), \ldots, D^k(w) \right], \quad k \geq 0, \] (3.16)
where
\[ \mathcal{M} = \bar{\psi} \frac{1}{H_1} D \circ \frac{1}{H_2} \cdots D \circ \frac{1}{H_{r-1}} D \circ \psi \frac{1}{H_{r-1}} \cdots \frac{1}{H_1} \cdots \frac{1}{H_1} \circ \frac{1}{H_0} \circ \psi, \] (3.17)
w is the minimal \( y \)-integral and \( Q \) is an arbitrary function, is an infinitesimal symmetry.

Remark 3.4. For a generic function \( Q \) the evolution equation (3.16) is not integrable.

For a generalization of Theorem 3.2 to the case of the Darboux integrable multi-component systems see [71].

Theorem 3.3. For any Liouville type equation all coefficients of the differential operator
\[ \mathcal{L} = \frac{\bar{\psi}}{\psi} H_{0} H_{-1} \cdots H_{1-s} D \circ \frac{1}{H_{1-s}} \circ D \cdots \frac{1}{H_0} \circ D \circ \frac{1}{H_1} \cdots D \circ \frac{1}{H_{r-1}} D \circ \frac{\psi \bar{H}_1 \cdots H_{r-1}}{\psi}, \] (3.18)
are \( y \)-integrals.
Example 3.7. For the Liouville equation \((3.2)\) we have
\[
\mathcal{L} = \exp(u) D \circ \exp(-u) D \circ \exp(-u) D \circ \exp(u) = D^3 + 2wD + w_x,
\]
where \(w = u_{xx} - \frac{1}{2} u_x^2\). The operator \(\mathcal{M}\) is given by
\[
\mathcal{M} = \exp(-u) D \circ \exp(u) = D + u_x.
\]
If \(Q(x, w, \ldots) = w\), then the corresponding symmetry is the integrable evolution equation
\[
u_x = u_{xxx} - \frac{1}{2} u_x^3.
\]

An attack to the problem of a complete classification of the Darboux integrable equations has been made in [23]. The proof of the classification statement consisted of more than 150 pages and has been not published. However, O. Kaptsov pointed out to an omission in the classification. In Appendix 2 Darboux integrable equations known to the author are collected.

### 3.4 Integrable multi-component systems of Liouville type

Consider multi-component systems of the form
\[
\vec{u}_{xy} = \vec{F}(x, y, \vec{u}, \vec{u}_x, \vec{u}_y) \quad \vec{u} = (u^1, \ldots, u^N).
\]
(3.19)

For the system (3.19) the coefficients of the linearization operator (3.12) are \(N \times N\)-matrices.

Evidently, most part of the definitions, constructions and statements about the Liouville type equations presented above can be generalized to the case of systems (3.19). However a serious problem arises in the definition of the Laplace invariants.

The linearization operator (3.12) becomes an operator of the form
\[
L = D\bar{D} + aD + b\bar{D} + c
\]
(3.20)

with matrix coefficients. A straightforward generalization of all definitions to the matrix case looks as follows. The main Laplace invariants are defined by the formulas
\[
H_1 = D(a) + b a - c, \quad H_0 = \bar{D}(b) + a b - c.
\]
Now they are \(N \times N\)-matrices. The matrices \(H_i\) for \(i > 1\) are recurrently determined from the following system of equations
\[
\bar{D}H_i - H_i a_{i-1} + a_i H_i = 0,
\]
(3.21)
\[ H_{i+1} = 2H_i + D(a_i - a_{i-1}) + [b, a_i - a_{i-1}] - H_{i-1}, \]  

(3.22)

where \( a_0 = a \). Obviously, in the scalar case these formulas coincide with the corresponding formulas from (3.8).

Suppose that the matrices \( H_i \) for \( i \leq k \) and \( a_i \) for \( i \leq k - 1 \) are already given. Then we derive \( a_k \) from (3.21) and after that find \( H_{k+1} \) from (3.22). However if \( \det H_k = 0 \), then \( a_k \) does not exist at all or it is defined not uniquely but up a matrix \( \alpha \) such that \( \alpha H_k = 0 \). In the last case, the existence and properties of next Laplace invariants could essentially depend on the choice of \( \alpha \).

The degeneration \( \det H_k = 0 \) for some \( k \) is typical for the open Toda lattices.

\textit{Example 3.8.} Consider the \( A_2 \)-Toda lattice:

\[ u_{xy} = -2 \exp u + \exp v, \quad v_{xy} = \exp u - 2 \exp v. \]

The linearization operator has the form

\[ D\bar{D} + \begin{pmatrix} 2 \exp u & -\exp v \\ -\exp u & 2 \exp v \end{pmatrix}. \]

In this case

\[ h = \begin{pmatrix} -2 \exp u & \exp v \\ \exp u & -2 \exp v \end{pmatrix} \]

is a non-degenerate matrix. Using (3.21), we get

\[ a_1 = \frac{1}{3} \begin{pmatrix} -4u_y + v_y & -2u_y + 2v_y \\ 2u_y - 2v_y & u_y - 4v_y \end{pmatrix}, \]

and

\[ h_1 = \begin{pmatrix} \exp u - 2 \exp v & 2 \exp u - \exp v \\ -\exp u + 2 \exp v & -2 \exp u + \exp v \end{pmatrix}. \]

We see that \( \det h_1 = 0 \).

It is not difficult to prove (72) the following general statement (compare with Theorem 3.1):

\textbf{Theorem 3.4.} \textit{Suppose that (3.19) has non-degenerate \( y \) and \( x \)-integrals}

\[ W(x, y, \bar{u}, \bar{u}_x, \ldots, \bar{u}_p), \quad \bar{W}(x, y, \bar{u}, \bar{u}_y, \ldots, \bar{u}_p); \]

\textit{then} \( \det H_r = \det H_{-s} = 0 \) \textit{for some} \( r \leq \bar{p} \) \textit{and} \( s \leq p - 1 \).

According to the theorem, for systems of Toda type some of Laplace invariants must be degenerate and we are confronted with the question of how to well define a chain of Laplace invariants.
Let us set
\[ Z_k = H_k H_{k-1} \cdots H_1. \]
It follows from (3.21) that
\[ Z_k (\bar{D} + a) = (\bar{D} + a_k) Z_k. \tag{3.23} \]
If the matrices \( H_i \) (and hence \( Z_k \)) for \( i \leq k \) and \( a_i \) for \( i \leq k - 1 \) are already given, then we define \( a_k \) from (3.23) and after that find \( H_{k+1} \)
\[ H_{k+1} = (D + b) a_k - (\bar{D} + a_k) b + H_k. \tag{3.24} \]
The latter relations are equivalent to (3.22). The matrix \( a_k \) is defined up to arbitrary matrix \( \alpha \) such that \( \alpha Z_k = 0 \).

**Theorem 3.5.** [23, 73] Suppose that \( H_i \) for \( i \leq k \) are already known and for \( i < k \) the following conditions
\[ (\bar{D} + a) \left( \ker Z_i \right) \subset \ker Z_i \tag{3.25} \]
and
\[ (D - b') \left( \ker Z_i^t \right) \subset \ker Z_i^t \tag{3.26} \]
are fulfilled. Then \( a_k \) exists iff condition (3.25) is fulfilled for \( i = k \). Further, \( Z_{k+1} \) does not depend on the arbitrary matrix \( \alpha \), which appears in the general formula for \( a_k \), iff condition (3.26) holds for \( i = k \).

**Proof.** It follows from (3.23) that (3.25) with \( i = k \) is a necessary condition for the existence of the \( a_k \). The sufficiency follows from the Kronecker-Capelli theorem. The formula (3.22) implies the condition (3.26). \( \square \)

**Remark 3.5.** In the case \( a = b = 0 \) in (3.20), conditions (3.25), (3.26) are fulfilled iff (see [73]) the vector spaces \( \ker Z_k \) and \( \ker Z_k^t \) admit bases, consisting of vectors from \( \ker \bar{D} \) and \( \ker D \), respectively.

**Example 3.9.** For the open \( A_3 \)-Toda lattice
\[
\begin{align*}
(u_1)_{xy} &= 2 \exp u_1 - \exp u_2, \\
(u_2)_{xy} &= -\exp u_1 + 2 \exp u_2 - \exp u_3, \\
(u_3)_{xy} &= -\exp u_2 + 2 \exp u_3,
\end{align*}
\]
all matrices \( Z_k \) are uniquely defined and rank \( Z_k = 4 - k \). In particular, \( Z_4 = 0 \). The vector \( e_1 = (1, 1, 1)^t \) forms a basis of \( \ker Z_2 \). A basis of \( \ker Z_3 \) can be chosen as follows: \( e_1 \) and \( e_2 = (1, 0, -1)^t \). For bases of \( \ker Z_2^t \) and \( \ker Z_3^t \) can be taken \( f_1 = (3, 4, 3)^t \) and \( f_1, f_2 = (1, 0, -1)^t \). Thus for this Toda lattice, the vector spaces \( \ker Z_k \) and \( \ker Z_k^t \) admit constant bases and therefore conditions (3.25), (3.26) hold.

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Example 3.10. For the $C_3$-Toda lattice

\[
\begin{align*}
(u_1)_{xy} &= 2 \exp u_1 - \exp u_2, \\
(u_2)_{xy} &= -\exp u_1 + 2 \exp u_2 - \exp u_3, \\
(u_3)_{xy} &= -2 \exp u_2 + 2 \exp u_3,
\end{align*}
\]

the matrices $Z_k$ are uniquely defined and \( \text{rank } Z_1 = 3, \text{ rank } Z_2 = 2, \text{ rank } Z_3 = 2, \text{ rank } Z_4 = 1, \text{ rank } Z_5 = 1, \text{ and } Z_6 = 0. \) All vector spaces \( \ker Z_k \) and \( \ker Z_k^t \) admit constant bases.

Example 3.11. In the case of $D_3$-Toda lattice

\[
\begin{align*}
(u_1)_{xy} &= 2 \exp u_1 - \exp u_2 - \exp u_3, \\
(u_2)_{xy} &= -\exp u_1 + 2 \exp u_2, \\
(u_3)_{xy} &= -\exp u_1 + 2 \exp u_3,
\end{align*}
\]

we have \( \text{rank } Z_k = 4 - k. \) All vector spaces \( \ker Z_k \) and \( \ker Z_k^t \) admit constant bases.

**Conjecture 3.1.** \cite{23} For any open Toda lattice (3.1), the indices \( i \) such that the rank of \( Z_i \) is decreased, coincide with the exponents of the corresponding simple Lie algebra and the index \( h \) such that \( Z_h = 0 \) is equal to the Coxeter number.

For the classical simple Lie algebras the conjecture was proved in \cite{74}. Later A.M. Guryeva verified it for the exceptional Lie algebras. In the following example we present formulas for \( Z_k \) from \cite{74} in the $A_n$-case.

Example 3.12. We rewrite the $A_n$-Toda lattice with the Cartan matrix

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 2
\end{pmatrix}
\]

in the form

\[
D\bar{D} \mathbf{u} = A \mathbf{u} \mathbf{c},
\]

where

\[
\mathbf{u} = (u_1, u_2, u_3, \ldots, u_{n-1}, u_n)^t, \quad \mathbf{c} = (1, 1, 1, \ldots, 1, 1)^t,
\]

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\[ U = \text{diag}\left( \exp(u_1), \exp(u_2), \ldots, \exp(u_n) \right). \]

The linearization operator is given by
\[ L = D\overline{D} - AU. \]

For this operator we have
\[ Z_k = AJ^{1-k}S_k (J^t)^{1-k}, \quad k = 1, 2, \ldots, n. \]

Here,
\[ S_k = \text{diag}\left\{ 0, 0, \ldots, 0, \exp\left( \sum_{i=1}^{k} u^i \right), \exp\left( \sum_{i=2}^{k+1} u^i \right), \ldots, \exp\left( \sum_{i=n-k}^{n-1} u^i \right), \exp\left( \sum_{i=n-k+1}^{n} u^i \right) \right\} \]
and
\[
J = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
0 & 1 & 1 & \ldots & 1 & 1 \\
& \ddots & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & \ldots & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}.
\]

It is clear that \( S_{n+1} = 0. \)

Notice that \( \text{rank } Z_k = n - k - 1, \quad k = 1, 2, \ldots. \)

Thus the numbers \( k \) such that \( \text{rank } Z_{k+1} < \text{rank } Z_k \) coincide with the exponents \( 1, 2, \ldots, n \) for the Lie algebra of \( A_n \), and the number \( k = n + 1 \) such that \( Z_k = 0 \) is equal to the Coxeter number.

**Remark 3.6.** It would be interesting to understand an algebraic meaning of the matrices, which appear in Example 3.12 and generalize them to any simple Lie algebra.

For \( i > 0 \) the invariants \( H_{-i} \) and the matrices \( Z_{-i} = H_{-i}H_{-(i-1)} \cdots H_0 \) are given by the formulas
\[
D(Z_{1-i}) - Z_{1-i}b + b_{-i}Z_{1-i} = 0,
\]
\[
H_{-i} = 2H_{1-i} + \tilde{D}(b_{-i} - b_{1-i}) + [a, b_{-i} - b_{1-i}] - H_{2-i}.
\]

The conditions (3.25) and (3.26) have to be replaced by
\[
(D + b) \left( \ker Z_{-i} \right) \subset \ker Z_{-i}, \quad (\tilde{D} - a^t) \left( \ker Z_{-i}^t \right) \subset \ker Z_{-i}^t.
\]

The termination of the sequence \( Z_i \) can be taken as a definition of Liouville type systems [3.19].
Definition 3.7. Suppose that for a system of hyperbolic equations of the form (3.19) all conditions (3.25) and (3.26) are fulfilled and there exist \( r \geq 1 \) and \( s \geq 0 \) such that \( Z_r = Z_{-s} \equiv 0 \); then (3.19) is called a system of Liouville type.

In the scalar case Theorem 3.1 shows that the equation (3.3) is Darboux integrable (see Definition 3.2) iff it is an equation of Liouville type. For the multi-component systems this is not true. In [73] an example of a system, which is Darboux integrable but is not a system of Liouville type has been constructed.

Open problem 3.1. Prove that any system of Liouville type is Darboux integrable.

Using Definition 3.7 let us classify all Liouville type systems of the form

\[
\begin{cases}
(u_1)_{xy} = 2 \exp u_1 + k_1 \exp u_2, \\
(u_2)_{xy} = k_2 \exp u_1 + 2 \exp u_2
\end{cases}
\]  

(3.27)

with non-degenerate \((k_1 k_2 \neq 4)\) and non-diagonal Cartan matrix.

It is easy to verify that for any \( k_1, k_2 \) we have rank \( Z_1 = 2 \), rank \( Z_2 = 1 \). Further, \( Z_3 = 0 \) iff \( k_1 = k_2 = -1 \) (the open \( A_2 \)-Toda lattice).

Let \( Z_3 \neq 0 \), then condition (3.25) with \( i = 3 \) is fulfilled iff \( k_1 = -1 \) or \( k_2 = -1 \). Without loss of generality, we set \( k_1 = -1 \). Then \( Z_4 = 0 \) iff \( k_2 = -2 \) (\( C_2 \)-Toda lattice).

If \( k_2 \neq -2 \), then rank \( Z_5 = 1 \) and condition (3.25) with \( i = 5 \) holds iff \( k_2 = -3 \). In this case \( Z_6 = 0 \) (\( G_2 \)-Toda lattice).

Thus we proved that all Liouville type systems (3.27) are exhausted by the Toda lattices corresponding to the simple Lie algebras of rank 2.

Exercise 3.1. Prove a similar statement for the case of rank 3.

3.5 Differential substitutions and Liouville type equations

In this section we consider differential substitutions (see Section 1.4.2) of the form

\[ \hat{u} = P(x, u, u_1, \ldots, u_k), \]

which connect two evolution equations

\[ u_t = f(x, u, u_1, \ldots, u_n) \]

and

\[ \hat{u}_t = g(x, \hat{u}, \hat{u}_1, \ldots, \hat{u}_n). \]
Suppose we have a Liouville type equation (3.3) with minimal $y$-integral
\[ \hat{u} = w(x, u, u_1, \ldots, u_p). \]
Let \[ u_\tau = f(x, u, u_1, \ldots, u_n) \]
be a symmetry for (3.3).
Since the total derivatives $D_y$ and $D_\tau$ commute, we have
\[ D_y D_\tau (\hat{u}) = D_\tau D_y (\hat{u}) = 0 \]
i.e. $D_\tau (\hat{u})$ is a $y$-integral as well. According to Proposition 3.1, we have
\[ \hat{u}_a = Q(x, \hat{u}, \hat{u}_1, \ldots, \hat{u}_n) \]
for some function $Q$. Thus, the minimal $y$-integral of any hyperbolic equation (3.3) defines a differential substitution from $x$-symmetries of (3.3) to some evolution equations. Similarly, the minimal $x$-integral defines a differential substitution from $y$-symmetries.

**Example 3.13.** The Liouville equation has a symmetry
\[ u_\tau = u_{xxx} - \frac{1}{2} u^3. \]
The minimal $y$-integral
\[ \hat{u} = u_{xx} - \frac{1}{2} u^2 \]
defines the differential substitution from (3.28) to the KdV-equation
\[ \hat{u}_\tau = \hat{u}_{xxx} + 3 \hat{u}_x. \]

**Example 3.14.** Consider the Liouville type equation (3.14). Its minimal $y$ and $x$-integrals are given by
\[ w = u_x - \frac{1}{2} u^2, \quad \tilde{w} = \frac{u_{yyu}}{u_y} - \frac{3}{2} \frac{u_{yyu}^2}{u_y^2}. \]
The first of these substitutions is just the Miura transformation.
General formulas from Theorems 3.2 and 3.3 give us
\[ \mathcal{M} = D^2 + uD + u_x, \quad \tilde{\mathcal{M}} = u_y, \quad \mathcal{L} = D^3 + 2wD + w_x. \]
The simplest $x$ and $y$-symmetries are given by
\[ u_\tau = \mathcal{M}w = u_{xxx} - \frac{3}{2} u^2 u_x \]
and

\[ u_\tau = \mathcal{M} \bar{w} = u_{yyy} - \frac{3}{2} u_y^2 \]

are well-known integrable evolution equations. It is easy to verify that

\[ w_\tau = w_{xxx} + 3 w w_x \]

and \( \bar{w}_\tau = \bar{w}_{yyy} + 3 \bar{w} \bar{w}_y \).

Moreover, for any \( x \)-symmetry of equation (3.14) given by

\[ u_\tau = \left( D^2 + u D + u_x \right) G(x, w, w_x, \ldots, w_n) \]  

(3.29)

the minimal \( y \)-integral

\[ w = u_x - \frac{1}{2} u^2 \]

defines the differential substitution from \( 3.29 \) to an equation of the form \( w_\tau = Q(x, w, w_x, \ldots) \).

Let us find the function \( Q \) explicitly. We have

\[ w_\tau = \left( D - u \right) u_\tau = \left( D - u \right) \left( D^2 + u D + u_x \right) G(x, w, w_x, \ldots, w_n) = \left( D^3 + 2 w D + w_x \right) G(x, w, w_x, \ldots, w_n) = \mathcal{L} G(x, w, w_x, \ldots, w_n). \]

Arguing as above, we prove the following

**Proposition 3.3.** Let \( w \) be a minimal \( y \)-integral of an equation of Liouville type and let

\[ u_\tau = \mathcal{M} G(x, w, w_x, \ldots), \]

(3.30)

be a symmetry of this equation. Then equation (3.30) is related to an evolution equation

\[ \hat{u}_\tau = w_* \mathcal{M} \left( G(x, \hat{u}, \hat{u}_x, \ldots) \right) \]

(3.31)

by the differential substitution

\[ \hat{u} = w(x, u, u_x, \ldots, u_k). \]

(3.32)

Here, \( w_* \) is the Fréchet derivative of the minimal \( y \)-integral \( w \). The possible freedom \( w \to f(x, w) \) in the choice of the minimal integral corresponds to a point transformation in equation (3.31).

**Remark 3.7.** In particular, this proposition states that the coefficients of the differential operator \( w_* \mathcal{M} \) are \( y \)-integrals.
Remark 3.8. For all known examples the operator $w_* \mathcal{M}$ coincides with the operator $\mathcal{L}$ given by (3.18). If this true, it follows from (3.17), (3.18) that the formula
\[ w_* = \frac{\bar{\psi}}{\psi} H_0 H_{-1} \cdots H_{1-s} D \cdot \frac{1}{H_{1-s}} D \cdots \frac{1}{H_0} D \cdot \frac{1}{\psi} \]
must be valid. This formula was proved in [23] under the assumption that the order $k$ of minimal integral is equal to $\text{ord} \mathcal{L} - \text{ord} \mathcal{M}$.

3.5.1 Differential substitutions of first order

We associate the hyperbolic equation
\[ u_{xy} = -\frac{P_u}{P_x} u_y \]
with any differential substitution of the form
\[ v = P(x, u, u_x). \]

It is clear that $P$ is the minimal $y$-integral for (3.33). It is easy to verify that $H_0 = 0$ for this equation.

Observation 3.1. For all known differential substitutions of the form (3.33) is an equation of the Liouville type (i.e. $H_r = 0$ for some $r > 0$).

Theorem 3.6. [20, Lemma 4.1], [75] Equation (3.33) is a Liouville type equation iff (up to a transformation of the form $u \to f(x,u)$) the function $P(x, u, u_x)$ is given by
\[ u_x = \alpha(x, P) u^2 + \beta(x, P) u + \gamma(x, P) \]
for some functions $\alpha, \beta$ and $\gamma$. Equation (3.33) with $P$ defined by (3.34) has the $x$-integral
\[ \bar{W} = \frac{u_{yyy}}{u_y} - \frac{3 u_{yy}^2}{2 u_y^2}. \]

Example 3.15. For the Miura substitution we have $u_x = P + \frac{1}{2} u^2$.

Example 3.16. For well-known differential substitution [33]
\[ v = u_x + \exp(u) + \exp(-u) \]

\[ \text{If } H_0 = 0, \text{ then } w_* = \frac{\bar{\psi}}{\psi} D(\frac{1}{\psi}). \]
the corresponding hyperbolic equation is given by

\[ u_{xy} = \left( \exp(-u) - \exp(u) \right) u_y. \] (3.35)

The function \( P \) for (3.35), after the transformation \( u \to \ln u \), satisfies the following relation of the form (3.34):

\[ u_x = -u^2 + Pu - 1. \]

Example 3.17. For the Cole-Hopf substitution (1.15) we have \( u_x = Pu \).

### 3.5.2 Integrable operators

Consider the set \( F \) of all functions depending on

\[ u_0 = u, \quad u_1 = u_x, \quad u_2 = u_{xx}, \quad \ldots. \]

For any \( f(u, u_1, \ldots, u_n) \) we denote

\[ f_\ast \overset{\text{def}}{=} \sum_{i=0}^{n} \frac{\partial f}{\partial u_i} D^i. \]

The Lie bracket

\[ [f, g] \overset{\text{def}}{=} g_\ast (f) - f_\ast (g) \] (3.36)

equips \( F \) with a structure of Lie algebra (see (2.15)). Bracket (3.36) corresponds to the commutator of the flows for evolution equations \( u^1_t = f \) and \( u^2_t = g \). Therefore, given a Liouville type equation, the set of all symmetries of the form

\[ u_\tau = S(x, u, u_1, u_2, \ldots) \] (3.37)

is a Lie subalgebra in \( F \) (as usual, we identify symmetries and their right-hand sides).

The operator \( L = D^3 + 2uD + u_1 \), corresponding to the Liouville equation (see Example 3.7), possesses the following remarkable property. Its image is a Lie subalgebra in \( F \). Namely, it is easy to verify that for any \( f, g \in F \)

\[ [L(f), L(g)] = L \left( D(f) g - D(g) f + g_\ast L(f) - f_\ast L(g) \right). \]

Since \( L \) is injective, the last formula defines a new Lie bracket

\[ [f, g]_1 \overset{\text{def}}{=} D(f) g - D(g) f + g_\ast L(f) - f_\ast L(g) \]

on \( F \).

A differential operator \( L \) is called integrable \[56\] if \( \text{Im} L \) is a Lie subalgebra in \( F \).
Remark 3.9. It can be shown that for any Hamiltonian operator $\mathcal{H} : \mathcal{F} \to \mathcal{F}$ (see (11)) its image is a subalgebra in $\mathcal{F}$. Therefore, integrable operators can be regarded as a non-skewsymmetric generalization of the Hamiltonian operators.

For all known Liouville type equations, the operator $\mathcal{L}$ given by (3.18) is integrable. We show that, in connection with the above results, it looks rather natural.

Let us denote by $\mathcal{F}_w$ a Lie algebra of functions depending on variables $x, w, w_1, \ldots$ with respect to bracket (3.36). It follows from Theorem 3.3 that all coefficients of $\mathcal{L}$ are functions of $x, w, w_1, \ldots$, where $w$ is the minimal $Y$-integral. Thus we have a differential operator $\mathcal{L} : \mathcal{G}_w \to \mathcal{G}_w$. Let us show that under some assumptions $\text{Im} \mathcal{L}$ is a Lie subalgebra in $\mathcal{G}_w$.

Indeed, consider the set $\mathcal{S}$ of all symmetries of the form (3.30) of a given Liouville type equation. For all known examples, $\mathcal{S}$ coincides with the set of all symmetries of the form (3.37). In this case, $\mathcal{S} \subset \mathcal{F}$ is a Lie subalgebra with respect to bracket (3.36). It was shown in Proposition 3.3 that the image of the symmetry (3.30) under substitution (3.32) is given by $w_\tau = Q$, where

$$Q = w_* \mathcal{M} \left( G(x, w, w_1, \ldots) \right).$$

Hence the set of all functions of the form (3.38) is expected to be a subalgebra with respect to (3.36). To conclude the speculation, we recall that, according to Remark 3.8, $w_* \mathcal{M} = \mathcal{L}$ for all known equations of Liouville type.

Example 3.18. Consider equation [76]

$$u_{xy} = \frac{1}{6u + y} B^2(B - 1)B(B - 1)^2 + \frac{1}{6u + x} B^2(B - 1)B(B - 1)^2, \quad (3.39)$$

where $B = B(u_x)$ and $\bar{B} = \bar{B}(u_y)$ are solutions of cubic equations

$$\frac{1}{3} B^3 - \frac{1}{2} B^2 = u_x, \quad \frac{1}{3} \bar{B}^3 - \frac{1}{2} \bar{B}^2 = u_y. \quad (3.40)$$

For this equation we have $H_3 = H_{-2} = 0$. The minimal $y$ and $x$-integrals $w$ and $\bar{w}$ are given by

$$w = D \left\{ \ln \left[ u_2 - \frac{B^4(B - 1)^2}{6u + y} - \frac{B^2(B - 1)^4}{6u + x} \right] - \ln B(B - 1) \right\} -$$

$$- \left[ \left( \frac{1}{6u + y} + \frac{1}{6u + x} \right) B - \frac{1}{6u + x} \right] B(B - 1),$$

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\[ \bar{w} = \bar{D} \left\{ \ln \left[ \bar{u}_2 - \frac{\bar{B}^4 (\bar{B} - 1)^2}{6u + x} - \frac{\bar{B}^2 (\bar{B} - 1)^4}{6u + y} \right] - \ln \bar{B} (\bar{B} - 1) \right\} - \left[ \left( \frac{1}{6u + y} + \frac{1}{6u + x} \right) B - \frac{1}{6u + x} \right] B (B - 1). \]

The functions \( \psi \) and \( \bar{\psi} \) (see Proposition 3.2) are defined by

\[ \psi = u_2 - \frac{B^4 (B - 1)^2}{6u + y} - \frac{B^2 (B - 1)^4}{6u + x}, \quad \bar{\psi} = \bar{u}_2 - \frac{\bar{B}^4 (\bar{B} - 1)^2}{6u + x} - \frac{\bar{B}^2 (\bar{B} - 1)^4}{6u + y}. \]

The Laplace invariants for equation (3.39) have the form

\[ H_1 = \frac{B(\bar{B} - 1)}{B(B - 1)} \left[ \frac{\bar{B} - 1}{(6u + y)(B - 1)^2} - \frac{\bar{B}}{(6u + x)B^2} \right] \psi, \]

\[ H_0 = \frac{B(B - 1)}{B(B - 1)} \left[ \frac{B - 1}{(6u + x)(B - 1)^2} - \frac{B}{(6u + y)B^2} \right] \bar{\psi}, \]

\[ H_2 = \frac{2(6u + y)(6u + x)}{B(B - 1) [(6u + x)B^2 (B - 1) - (6u + y)B (B - 1)^2]^2} \psi \bar{\psi}, \]

\[ H_{-1} = \frac{2(6u + y)(6u + x)}{B(B - 1) [(6u + y)B^2 (B - 1) - (6u + x)B (B - 1)^2]^2} \bar{\psi} \psi. \]

The corresponding operator \( \mathcal{L} \) can be factorized:

\[ \mathcal{L} = (D + w)(D + w)(D + 2w)(D + 3w). \quad (3.41) \]

One can verify that the operator \( \mathcal{L} \) is integrable.

Let \( \mathcal{L} \) be a differential operator with coefficients from \( \mathcal{F}_w \). It is not very difficult to check whether its image is a Lie subalgebra. This condition turns out to be rather rigid.

For example, let us consider operators "similar" to (3.41). Namely, the coefficients of (3.41) are polynomials in \( w, w_1, w_2, \ldots \). Operator (3.41) is homogeneous if we assign a weight 1 to \( D \) and \( i + 1 \) to \( w_i \). We present here a complete list of integrable operators \( \mathcal{L} \) of orders 2-6 with the above homogeneity property. It turns out that all of them can be decomposed into factors of first order.

- Operator of second order:
  \[ \mathcal{L}_1^{(2)} = D (D + w); \]

- Operator of third order:
  \[ \mathcal{L}_1^{(3)} = D (D + w) (D + w); \]
• Operators of fourth order:

\[ \mathcal{L}^{(4)}_1 = D (D + w) (D + w) (D + w), \]
\[ \mathcal{L}^{(4)}_2 = D (D + w) (D + w) (D + 2w); \]

• Operators of fifth order:

\[ \mathcal{L}^{(5)}_1 = D (D + w) (D + w) (D + w) (D + w), \]
\[ \mathcal{L}^{(5)}_2 = D (D + w) (D + w) (D + 2w) (D + 3w); \]

• Operators of sixth order:

\[ \mathcal{L}^{(6)}_1 = D (D + w) (D + w) (D + w) (D + w) (D + w), \]
\[ \mathcal{L}^{(6)}_2 = D (D + w) (D + w) (D + w) (D + w) (D + 2w), \]
\[ \mathcal{L}^{(6)}_3 = D (D + w) (D + w) (D + 2w) (D + 3w) (D + 3w), \]
\[ \mathcal{L}^{(6)}_4 = D (D + w) (D + w) (D + 2w) (D + 3w) (D + 4w). \]

Note that the operator \( \mathcal{L}^{(5)}_2 \) coincides with (3.41).

In the paper [77] these examples were prolonged by infinite sequences of integrable operators of arbitrary order. In [78] some examples of integrable operators with matrix coefficients were constructed.
Chapter 4

Integrable non-abelian equations

4.1 ODEs on free associative algebra

We consider ODE systems of the form

\[ \frac{dx_\alpha}{dt} = F_\alpha(x), \quad x = (x_1, \ldots, x_N), \quad (4.1) \]

where \( x_i(t) \) are \( m \times m \) matrices, \( F_\alpha \) are (non-commutative) polynomials with constant scalar coefficients. As usual, a symmetry is defined as an equation

\[ \frac{dx_\alpha}{d\tau} = G_\alpha(x) \quad (4.2) \]

compatible with (4.1).

Manakov top

Let \( N = 2, \; x_1 = u, \; x_2 = v \). The following system

\[ u_t = u^2 v - v u^2, \quad v_t = 0 \quad (4.3) \]

has infinitely many symmetries for arbitrary size of matrices \( u \) and \( v \).

Many important multi-component integrable systems can be obtained as reductions of (4.3).

For instance, if \( u \) is \( m \times m \) matrix such that \( u^t = -u \), and \( v \) is a constant diagonal matrix, then (4.3) is equivalent to the \( m \)-dimensional Euler top. The integrability of this model by the inverse scattering method was established by S.V. Manakov in [79].
Consider the cyclic reduction

\[
\begin{bmatrix}
0 & u_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & u_2 & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & u_{m-1} \\
u_m & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
, \quad
\begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & J_m \\
J_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & J_2 & 0 & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & J_{m-1} & 0 \\
\end{bmatrix}
, \quad
\]

where \( u_k \) and \( J_k \) are matrices of lower size. Then (4.3) is equivalent to the non-abelian Volterra chain

\[
\frac{d}{dt} u_k = u_k u_{k+1} J_{k+1} - J_{k-1} u_{k-1} u_k, \quad k \in \mathbb{Z}_k.
\]

If we assume \( m = 3 \), \( J_1 = J_2 = J_3 = \text{Id} \) and \( u_3 = -u_1 - u_2 \) then the latter system yields

\[
u_t = u^2 + uv + vu, \quad v_t = -v^2 - uv - vu.
\]

It turns out that for any \( i, j \in \mathbb{N} \) system (4.3) has first integrals of the form \( \text{tr} \ P_{ij} \), where \( P_{ij} \) is a non-commutative homogeneous polynomial of degree \( i \) in \( v \) and degree \( j \) in \( u \). For example,

\[
P_{2,2} = 2v^2 u^2 + vu, \quad P_{3,2} = v^3 u^2 + v^2 uv, \quad P_{2,3} = v^2 u^3 + vu v^2.
\]

It follows from the property \( \text{tr} (uv - vu) = 0 \) that the polynomials \( P_{ij} \) are defined up to any cyclic permutations in factors of their monomials.

There exists the following integrable generalization [80] of system (4.3) to the case of \( N \) matrices for arbitrary \( N \):

\[
\frac{dx_\alpha}{dt} = \sum_{\beta \neq \alpha} x_\alpha x_\beta^2 - x_\beta x_\alpha^2 + \sum_{\beta \neq \alpha} \frac{x_\beta x_\alpha^2 - x_\alpha x_\beta^2}{(\lambda_\alpha - \lambda_\beta)c_\beta} + \sum_{\beta \neq \alpha} \frac{x_\beta x_\alpha^2 - x_\alpha x_\beta^2}{(\lambda_\alpha - \lambda_\beta)c_\alpha}
\]

Here,

\[
\sum_{\alpha=1}^{N} x_\alpha = C,
\]

where \( C \) is a constant matrix.

**Non-abelian systems**

To check that (4.2) is a symmetry of (4.1) for arbitrary size of matrices \( x_i \), one only need the associativity of the matrix product. Therefore, the compatibility of (4.1) and (4.2) is valid even if \( x_1, \ldots, x_N \) are generators of the free associative algebra \( \mathcal{A} \). We call systems on \( \mathcal{A} \) *non-abelian systems*.

However, in the non-abelian case the following questions arise...
• What do formulas like (4.1) and (4.2) mean?

• How to define the functional \( tr \) for first integrals?

**Definition 4.1.** A linear map \( d : \mathcal{A} \to \mathcal{A} \) is called derivation if it satisfies the Leibniz rule: 
\[
d(xy) = xd(y) + d(x)y.
\]

If we fix \( d(x_i) = F_i(x) \), where \( x_i \) are generators of \( \mathcal{A} \), then \( d(z) \) is uniquely defined for any element \( z \in \mathcal{A} \) by the Leibniz rule. It is clear that the polynomials \( F_i \) can be taken arbitrarily.

Instead of the dynamical system (4.1) we consider the derivation \( D_t : \mathcal{A} \to \mathcal{A} \) such that 
\[
D_t(x_i) = F_i.
\]
The compatibility of (4.1) and (4.2) means that the corresponding derivations \( D_t \) and \( D_\tau \) commute: 
\[
D_t D_\tau - D_\tau D_t = 0.
\]

To introduce the concept of a first integral, we need an analog of the trace, which is not defined for the algebra \( \mathcal{A} \). As a matter of fact, in our calculations we use only two properties of the trace, namely its linearity and the possibility to perform cyclic permutations in monomials. Let us define an equivalence relation for elements of \( \mathcal{A} \) in a standard way.

**Definition 4.2.** We say that two elements \( f_1 \) and \( f_2 \) of \( \mathcal{A} \) are equivalent, and we denote it by \( f_1 \sim f_2 \), iff \( f_1 \) can be obtained from \( f_2 \) by cyclic permutations of factors in its monomials. We denote by \( tr f \) the equivalence class of an element \( f \).

**Definition 4.3.** An element \( tr h \), where \( h \in \mathcal{A} \), is said to be a first integral of the system (4.1) if 
\[
D_t(h) \sim 0.
\]

There is an obvious similarity between Definition 4.3 and the definition for conserved densities in the theory of evolutionary PDEs (see Remark 2.8). In both cases, first integrals and conserved densities are defined as elements of equivalence classes, the difference is in the choice of the equivalence relation.

**Remark 4.1.** Any commutator is equivalent to zero and vise versa if \( f_1 \sim f_2 \), then there exist elements \( a_i, b_i \) such that 
\[
f_1 - f_2 = \sum [a_i, b_i].
\]
Therefore, we may regard \( tr f \) as an element of the quotient vector space \( \mathcal{T} = \mathcal{A}/[\mathcal{A}, \mathcal{A}] \). It is clear that the derivation \( D_t \) is well defined on \( \mathcal{T} \).

From the viewpoint of the symmetry approach, system (4.1) is integrable if it possesses infinitely many linearly independent symmetries. Note that if \( x_i \) are matrices of a fixed size, then the existence of an infinite set of symmetries is impossible.

Another criterion of integrability is the existence of an infinite set of first integrals.

**Example 4.1.** [81] The following non-abelian system 
\[
\begin{align*}
u_t &= v^2, \\
v_t &= u^2
\end{align*}
\]
possesses infinitely many symmetries and first integrals. F. Calogero has observed that in the matrix case the functions $x_i = \lambda_i^{1/2}$, where the $\lambda_i$ are eigenvalues of the matrix $u - v$, satisfy the following integrable system:

$$x_i'' = -x_i^5 + \sum_{j \neq i} \left[ (x_i - x_j)^{-3} + (x_i + x_j)^{-3} \right].$$

4.1.1 Quadratic non-abelian systems

In the case of arbitrary $N$ a homogeneous quadratic non-abelian system has the form

$$(u^i)_t = \sum_{j,k} C^i_{jk} u^j u^k, \quad (4.4)$$

where $u^i, i = 1, \ldots, N$ are non-commutative variables. We denote the set of coefficients $C^i_{jk}, i,j,k = 1, \ldots, N$ by $C$.

**Remark 4.2.** One can associate a $N$-dimensional algebra with structural constants $C^i_{jk}$ with any system (4.4).

The class of systems (4.4) is invariant with respect to the group $GL_N$ of linear transformations

$$\hat{u}^i = \sum_j s^i_j u^j.$$ 

**Definition 4.4.** A function $F(C)$ is called a $GL_N$ semi-invariant of degree $k$ if

$$F(\hat{C}) = (\det S)^k F(C),$$

where $S$ is the matrix with entries $s^i_j$. Semi-invariants of degree 0 are called invariants.

**Definition 4.5.** A row-vector $V(C)$ is called a vector $GL_N$ semi-invariant of degree $k$ if

$$V(\hat{C}) = (\det S)^k V(C) S.$$ 

Vector semi-invariants of degree 0 are called vector invariants.

**Lemma 4.1.** i) The product $I V$ of a semi-invariant $I$ and a vector semi-invariant $V$ of degrees $k_1$ and $k_2$ is a vector semi-invariant of degree $k_1 + k_2$.

ii) Let $V_i, i = 1, \ldots, N$ be vector semi-invariants of degrees $k_i$. Then the determinant of the matrix constituted of the vectors $V_i$ is a (scalar) semi-invariant of degree $1 + \sum k_i$.

**Open problem 4.1.** Construct series of simple algebras with the structural constants $C^i_{jk}$ such that equation (4.4) has a symmetry of the form

$$(u^i)_t = \sum_{j,k,m} B^i_{jkm} u^j u^k u^m.$$
4.1.2 Two-component non-abelian systems

Consider non-abelian systems

\[ u_t = P(u, v), \quad v_t = Q(u, v), \quad P, Q \in \mathcal{A} \]  \hspace{1cm} (4.5)

on the free associative algebra \( \mathcal{A} \) with generators \( u \) and \( v \).

Define the involution \( \star \) on \( \mathcal{A} \) by the formulas

\[ u^\star = u, \quad v^\star = v, \quad (a b)^\star = b^\star a^\star, \quad a, b \in \mathcal{A}. \] \hspace{1cm} (4.6)

Two systems related to each other by a linear transformation of the form

\[ \hat{u} = \alpha u + \beta v, \quad \hat{v} = \gamma u + \delta v, \quad \alpha \delta - \beta \gamma \neq 0 \] \hspace{1cm} (4.7)

and by the involution (4.6) are called equivalent.

**Definition 4.6.** A system, which is equivalent to a system of the form

\[ u_t = P(u, v), \quad v_t = Q(v) \]

is called triangular. If \( Q(v) = 0 \), then the system is called strongly triangular.

**Remark 4.3.** From the algebraic point of view the triangularity is equivalent to the fact that the corresponding algebra (see Remark 4.2) has one-dimensional double-side ideal.

**Open problem 4.2.** Describe all non-equivalent non-triangular systems (4.5) which possess infinitely many symmetries and/or first integrals.

The simplest class of such systems are quadratic systems of the form

\[
\begin{align*}
    u_t &= \alpha_1 u u + \alpha_2 u v + \alpha_3 v u + \alpha_4 v v, \\
    v_t &= \beta_1 v v + \beta_2 v u + \beta_3 u v + \beta_4 u u.
\end{align*}
\]  \hspace{1cm} (4.8)

**Equivalence**

The group \( \text{GL}_2 \) of transformations (4.7) acts on polynomials of the eight coefficients \( \alpha_i, \beta_i \) of the system (4.8). The corresponding infinitesimal action of the Lie algebra
$gl_2$ is defined by the following vector fields:

\[
X_{11} = \alpha_1 \frac{\partial}{\partial \alpha_1} - \alpha_4 \frac{\partial}{\partial \alpha_4} + \beta_2 \frac{\partial}{\partial \beta_2} + \beta_3 \frac{\partial}{\partial \beta_3} - 2\beta_4 \frac{\partial}{\partial \beta_4},
\]

\[
X_{22} = \beta_1 \frac{\partial}{\partial \beta_1} - \beta_4 \frac{\partial}{\partial \beta_4} + \alpha_2 \frac{\partial}{\partial \alpha_2} + \alpha_3 \frac{\partial}{\partial \alpha_3} - 2\alpha_4 \frac{\partial}{\partial \alpha_4},
\]

\[
X_{12} = -\beta_4 \frac{\partial}{\partial \alpha_1} + (\alpha_1 - \beta_3) \frac{\partial}{\partial \alpha_2} + (\alpha_1 - \beta_2) \frac{\partial}{\partial \alpha_3}
\]
\[+ (\alpha_2 + \alpha_3 - \beta_1) \frac{\partial}{\partial \alpha_4} + (\beta_2 + \beta_3) \frac{\partial}{\partial \beta_1} + \beta_4 \frac{\partial}{\partial \beta_2} + \beta_4 \frac{\partial}{\partial \beta_3},
\]

\[
X_{21} = -\alpha_4 \frac{\partial}{\partial \beta_1} + (\beta_1 - \alpha_3) \frac{\partial}{\partial \beta_2} + (\beta_1 - \alpha_2) \frac{\partial}{\partial \beta_3} + (\beta_2 + \beta_3 - \alpha_1) \frac{\partial}{\partial \beta_4}
\]
\[+ (\alpha_2 + \alpha_3) \frac{\partial}{\partial \alpha_1} + \alpha_4 \frac{\partial}{\partial \alpha_2} + \alpha_4 \frac{\partial}{\partial \alpha_3}.
\]

The eight-dimensional vector space spanned by the coefficients $\alpha_i, \beta_i$ can be decomposed into a direct sum of three subspaces invariant with respect to this action. The first one is four-dimensional and spanned by

\[x_1 = \alpha_4, \quad x_2 = \alpha_2 + \alpha_3 - \beta_1, \quad x_3 = \beta_2 + \beta_3 - \alpha_1, \quad x_4 = \beta_4;\]

another two are two-dimensional and spanned by

\[x_5 = \alpha_1 + \beta_3, \quad x_6 = \beta_1 + \alpha_3;\]

and

\[x_7 = \alpha_1 + \beta_2, \quad x_8 = \beta_1 + \alpha_2.\]

Denote these vector spaces by $W_1, W_2$ and $W_3$, respectively.

**Lemma 4.2.** The vectors $V_1 = (x_5, x_6)$ and $V_2 = (x_7, x_8)$ are vector invariants.

The (scalar) semi-invariants of the transformation group (4.7) have rather simple form in the variables $x_i$. There exists only one semi-invariant of degree 1:

\[I_0 = x_5 x_7 - x_5 x_8,\]

In the original variables

\[I_0 = \alpha_1 \alpha_2 - \alpha_1 \alpha_3 - \alpha_3 \beta_2 - \beta_1 \beta_2 + \alpha_2 \beta_3 + \beta_1 \beta_3.\]

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Remark 4.5. The following polynomials are semi-invariants of degree 2:
\[ I_1 = x_2^2x_3^2 + 4x_1x_3^3 + 4x_4^3x_1 + 18x_1x_2x_3x_4 - 27x_1^2x_4^2, \]
\[ I_2 = (x_5 - x_7)^3x_1 - (x_5 - x_7)(x_6 - x_8)x_2 - (x_5 - x_7)(x_6 - x_8)^2x_3 + (x_6 - x_8)^3x_4, \]
\[ I_3 = (x_3^3 - x_5^3)x_1 + (x_7^3x_8 - x_5^3x_6)x_2 + (x_7^3x_8 - x_5^3x_6)x_3 + (x_6^3 - x_3^3)x_4, \]
\[ I_4 = (x_2^2 - x_7^2)x_2^2 + (x_6^2 - x_8^2)x_2^2 + 3(x_2^2 - x_7^2)x_1x_3 + 3(x_2^2 - x_8^2)x_2x_4 + 9(x_7x_8 - x_5x_6)x_1x_4. \]

Remark 4.4. The semi-invariant \( I_1 \) is symmetric with respect to the involution (4.0) and the other four semi-invariants are skew-symmetric.

Proposition 4.1. Any invariant \( J \) of the transformation group (4.7) is a function of the four functionally independent invariants \( J_i = \frac{I_i}{I_0^3}, i = 1, 2, 3, 4 \).

Proof. Invariants are solutions of the PDE system \( X_{ij}(J) = 0 \), \( i, j = 1, 2 \). It is easy to verify that the vector fields \( X_{ij} \) are linearly independent over functions. Therefore, there exist only four functionally independent invariants. A straightforward computation of the rank for the Jacobi matrix shows that \( J_i, i = 1, 2, 3, 4 \) are functionally independent. \( \square \)

Remark 4.5. Actually there exist 9 linearly independent over \( \mathbb{C} \) semi-invariants of degree 2. But only five of them are functionally independent. Moreover there exist 7 linearly independent over \( \mathbb{C} \) vector semi-invariants of degree 1. Each semi-invariant of degree 2 can be obtained by the construction of Lemma 4.1 applying to vector semi-invariants of degree 1 and vector invariants.

Example 4.2. The vectors
\[ V_3 = \left( -\beta_4(\alpha_3 - \alpha_2)^2 + (\beta_3 - \beta_2)(\alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2\beta_2 + \alpha_3\beta_2), \right. \]
\[ \left. \alpha_4(\beta_3 - \beta_2)^2 - (\alpha_3 - \alpha_2)(\beta_1\beta_2 - \beta_1\beta_3 - \beta_2\alpha_2 + \beta_3\alpha_3) \right) \]
and
\[ V_4 = \left( (\beta_2 - \beta_3)\alpha_4\beta_4 - (\alpha_2 - \alpha_3)(\alpha_1 - \beta_1)\beta_4 + \beta_3(\alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2\beta_2 + \alpha_3\beta_2), \right. \]
\[ \left. -(\alpha_2 - \alpha_3)\alpha_4\beta_4 + (\beta_2 - \beta_3)(\beta_2 - \alpha_1)\alpha_4 - \alpha_3(\beta_1\beta_2 - \beta_1\beta_3 - \beta_2\alpha_2 + \beta_3\alpha_3) \right) \]
are vector semi-invariants of degree 1.

Proposition 4.2. System (4.8) is triangular iff \( V_3 = V_4 = 0 \).

Proposition 4.3. System (4.8) is strongly triangular iff the vectors \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) and \( (\beta_4, \beta_3, \beta_2, \beta_1) \) are linearly dependent.
Examples

Some experiments with non-triangular systems (4.8) having symmetries of degree $m$ have been made in [81]. One of the results is:

**Theorem 4.1.** Any non-triangular equation (4.8) possessing a symmetry of the form

\[
\begin{align*}
    u_t &= \gamma_1 u u u + \gamma_2 u u v + \gamma_3 u v u + \gamma_4 v u u + \gamma_5 v v v + \gamma_6 v u v + \gamma_7 v v u + \gamma_8 v v v, \\
    v_t &= \delta_1 u u u + \delta_2 u u v + \delta_3 u v u + \delta_4 v u u + \delta_5 v v v + \delta_6 v u v + \delta_7 v v u + \delta_8 v v v
\end{align*}
\]

is equivalent to one of the following:

\[\begin{align*}
    a) : \begin{cases}
        u_t &= u u - u v, \\
        v_t &= v v - u v + v u,
    \end{cases} & \quad b) : \begin{cases}
        u_t &= u v, \\
        v_t &= v u,
    \end{cases} & \quad c) : \begin{cases}
        u_t &= u u - u v, \\
        v_t &= v v - u v,
    \end{cases} \\
    d) : \begin{cases}
        u_t &= -u v, \\
        v_t &= v v + u v - v u,
    \end{cases} & \quad e) : \begin{cases}
        u_t &= u v - u v, \\
        v_t &= u u + u v - v u,
    \end{cases} & \quad f) : \begin{cases}
        u_t &= v v, \\
        v_t &= u u.
    \end{cases}
\end{align*}\]

**Remark 4.6.** Possibly equation a) has only one symmetry while equations b)-f) have infinitely many. It is proved for equations b), c), and f).

It is a remarkable fact that a requirement of the existence of just one cubic symmetry selects a finite list of equations with no free parameters (or more precisely, all possible parameters can be removed by linear transformations (4.7)).

A list of six equations with quartic symmetries was presented in [81]. As mentioned by A. Odesskii, two of them are equivalent. The five non–equivalent equations are given by

\[\begin{align*}
    u_t &= -u v, \\
    v_t &= v v + u v, &
    u_t &= -v u, \\
    v_t &= v v + u v, &
    u_t &= u u - u v - 2 u v, \\
    v_t &= v v - u v - 2 u v, &
    u_t &= u u - 2 v u, \\
    v_t &= v v - 2 v u, &
    u_t &= u u - 2 u v, \\
    v_t &= v v - 4 v u.
\end{align*}\]

Using computer algebra system CRACK [82], T.Wolf verified that it is a complete list of non–triangular systems that have quartic but no cubic symmetries.

**Open problem 4.3.** Describe non-equivalent non-triangular systems (4.8) that possess infinitely many symmetries and/or first integrals.
4.2 PDEs on free associative algebra

In this section we consider the so called non-abelian evolution equations, which are natural generalizations of the evolution matrix equations.

4.2.1 Matrix integrable equations

The matrix Burgers equation is given by

$$U_t = U_{xx} + 2UU_x,$$  \hspace{1cm} (4.9)

where $U(x,t)$ is unknown $m \times m$-matrix. The matrix KdV equation has the following form

$$U_t = U_{xxx} + 3(UUU_x + U_xU).$$  \hspace{1cm} (4.10)

It is known that this equation has infinitely many higher symmetries for arbitrary $m$. All of them can also be written in the matrix form. The simplest higher symmetry of (4.10) is given by

$$U_\tau = U_{xxxx} + 5(UUU_{xxx} + U_xU) + 10(U_xU_{xx} + U_{xx}U_x) + 10(U^2U_x + UUU_xU + U_xU^2).$$

For $m = 1$ this matrix hierarchy coincides with the usual KdV hierarchy.

In general, we may consider matrix equations of the form

$$U_t = F(U, U_1, \ldots, U_n), \quad U_i = \frac{\partial^i U}{\partial x^i},$$

where $F$ is a (non-commutative) polynomial. As usual in this chapter, the criterion of integrability is the existence of matrix higher symmetries

$$U_\tau = G(U, U_1, \ldots, U_m).$$

Such equations can be regarded as matrix generalizations of scalar integrable equations [1, 8].

Below we present a list of integrable matrix equations. The main goal is to demonstrate that the matrix KdV equation is not a single accident. Many of known integrable models have matrix generalizations [83–85].

In particular, the mKdV equation (2.37) has two different matrix generalizations:

$$U_t = U_{xxx} + 3U^2U_x + 3UU_xU^2,$$  \hspace{1cm} (4.11)

and (see [85])

$$U_t = U_{xxx} + 3[U, U_{xx}] - 6UU_xU.$$
The matrix generalization of the NLS equation (2.10) is given by

\[ U_t = U_{xx} - 2UVU, \quad V_t = -V_{xx} + 2VUV. \]  

(4.12)

The following integrable matrix system of the derivative NLS type

\[ U_t = U_{xx} + 2UVU_x, \quad V_t = -V_{xx} + 2V_xUV \]  

(4.13)

was found in [86].

The Krichever-Novikov equation (2.6) with \( P = 0 \) is called the Schwartz KdV equation. Its matrix generalization is given by

\[ U_t = U_{xxx} - \frac{3}{2}U_{xx}U^{-1}U_{xx}. \]  

(4.14)

The Krichever-Novikov equation (2.6) with the generic \( P \) has probably no matrix generalizations.

The matrix Heisenberg equation has the form

\[ U_t = U_{xx} - 2U_x(U + V)^{-1}U_x, \quad V_t = -V_{xx} + 2V_x(U + V)^{-1}V_x. \]  

(4.15)

One of the most renowned hyperbolic matrix integrable equations is the principle chiral \( \sigma \)-model

\[ U_{xy} = \frac{1}{2}(U_xU^{-1}U_y + U_yU^{-1}U_x). \]  

(4.16)

The following, maybe, new integrable system

\[
\begin{cases}
U_t = \lambda_1 U_x + (\lambda_2 - \lambda_3) W^t V^t, \\
V_t = \lambda_2 V_x + (\lambda_3 - \lambda_1) U^t W^t, \\
W_t = \lambda_3 W_x + (\lambda_1 - \lambda_2) V^t U^t
\end{cases}
\]

is a matrix generalization of the 3-wave model. In contrast with the previous equation it contains the matrix transposition denoted by \( ^t \).

Let \( e_1, \ldots, e_N \) be a basis of some associative algebra \( \mathcal{B} \) and \( U \) be the element (1.2). Then, any of the matrix equations presented above gives rise to an integrable system for unknown functions \( u_1, \ldots, u_N \) associated with \( \mathcal{B} \). Indeed, only the associativity of the product in \( \mathcal{B} \) is needed to verify that symmetries of a matrix equation remain symmetries of the corresponding system for \( u_i \).

Interesting examples of integrable multi-component systems are produced by the Clifford algebras and by the group algebras of associative rings.

However, the most fundamental so called non-abelian equations correspond to free associative algebras (cf. Section 3.1).
4.2.2 Non-abelian evolution equations

In order to formalize the concept of matrix equations, let us consider evolution equations on a free associative algebra $A$. In the case of one-field non–abelian equations the generators of $A$ are

$$U, \quad U_1 = U_x, \quad \ldots, \quad U_k = \frac{\partial^k U}{\partial x^k}, \quad \ldots$$  \hspace{1cm} (4.17)

Since $A$ is assumed to be a free algebra, no algebraic relations between the generators $U_i$ are allowed. All definitions can be easily generalized to the case of several non-abelian variables.

The formula

$$U_t = F(U, U_1, \ldots, U_n), \quad F \in A$$  \hspace{1cm} (4.18)

does not mean that we consider an element of non-associative algebra depending on time $t$. As usual, formula (4.18) defines a derivation $D_t$ of $A$, which commutes with the basic derivation

$$D = \sum_0^\infty U_{i+1} \frac{\partial}{\partial U_i}.$$  

It is easy to check that $D_t$ is defined by the vector field

$$D_t = \frac{\partial}{\partial t} + \sum_0^\infty D'(F) \frac{\partial}{\partial U_i}.$$  

A generalization of the symmetry approach to differential equations on free associative algebras requires proper definitions for concepts such as symmetry, conservation law, Fréchet derivative, and formal symmetry.

As in the scalar case, a symmetry is an evolution equation

$$U_\tau = G(U, U_1, \ldots, U_m),$$

such that the vector field

$$D_G = \sum_0^\infty D'(G) \frac{\partial}{\partial u_i}$$

commutes with $D_t$. The polynomial $G$ is called the symmetry generator.

The condition $[D_t, D_G] = 0$ is equivalent to $D_t(G) = D_G(F)$. The latter relation can be rewritten as

$$G_s(F) - F_s(G) = 0,$$

where the Fréchet derivative $H_s$ for any $H \in A$ can be defined as follows.
For any \( a \in \mathcal{A} \) we denote by \( L_a \) and \( R_a \) the operators of left and right multiplication by \( a \):
\[
L_a(X) = aX, \quad R_a(X) = Xa, \quad X \in \mathcal{A}.
\]
The associativity of \( \mathcal{A} \) is equivalent to the identity \([L_a, R_b] = 0\) for any \( a \) and \( b \).
Moreover,
\[
L_{ab} = L_a L_b, \quad R_{ab} = R_b R_a, \quad L_{a+b} = L_a + L_b, \quad R_{a+b} = R_a + R_b.
\]

**Definition 4.7.** We denote by \( \mathcal{O} \) the associative algebra generated by all operators of left and right multiplication by elements \((4.17)\). This algebra is called the *algebra of local operators*.

Let us now extend the set of generators \((4.17)\) by adding non-commutative symbols \( V_0, V_1, \ldots \) and let us prolong the derivation \( D \) by \( D(V_i) = V_{i+1} \).

Given \( H(U, U_1, U_2, \ldots, U_k) \in \mathcal{A} \), we find
\[
L_H = \left. \frac{\partial}{\partial \varepsilon} H(U + \varepsilon V_0, U_1 + \varepsilon V_1, U_2 + \varepsilon V_2, \ldots) \right|_{\varepsilon = 0}
\]
and represent this expression as \( H_s(V_0) \), where \( H_s \) is a linear differential operator of order \( k \), whose coefficients belong to \( \mathcal{O} \). For example, \((U_2 + UU_1)_s = D^2 + L_U D + R_{U_1}\).

In contrast with the definition of the symmetry, which is a straightforward generalization of the corresponding scalar notion, the definition of a conserved density has to be essentially modified.

Recall that in the scalar case a conserved density is a function \( \rho \in \mathcal{F} \) such that \( D_t(\rho) = D(\sigma) \) for some \( \sigma \in \mathcal{F} \). Equivalent densities define the same functional, whose values do not depend on time (see Section 1.3). Here, the equivalence relation is defined as follows: \( \rho_1 \sim \rho_2 \) iff \( \rho_1 - \rho_2 = D(s) \), \( s \in \mathcal{F} \). In other words, a conserved density is an equivalence class in \( \mathcal{F} \) such that \( D_t \) takes it to zero equivalence class.

In the non-abelian case we hold the same line. The following elementary operations define an equivalence relation:

- Addition of elements of the form \( D(s) \), where \( s \in \mathcal{A} \), to the element \( \rho \in \mathcal{A} \);
- Cyclic permutation of factors in any monomial of the polynomial \( \rho \).

Two densities \( \rho_1, \rho_2 \in \mathcal{A} \) related to each other through a finite sequence of the elementary operations are called *equivalent*. An equivalent definition is \( \rho_1 \sim \rho_2 \) iff
\[
\rho_1 - \rho_2 \in \mathcal{A}/(\text{Im} D + [\mathcal{A}, \mathcal{A}]).
\]
It is clear that in the scalar case this definition coincides with Definition 2.8.
Definition 4.8. The equivalence class of an element $\rho$ is called the trace of $\rho$ and is denoted by $\text{tr}\, \rho$.

A motivation of the definition is that in the matrix case the functional

$$I(U) = \int_{-\infty}^{\infty} \text{trace}(\rho(U, U_x, \ldots)) \, dx$$

is well defined on the equivalence classes if $\rho(0, 0, \ldots) = 0$ and $U(x, t) \to 0$ as $x \to \pm \infty$.

Definition 4.9. The equivalence class $\rho$ is called conserved density for equation (4.18) if $D_t(\rho) \sim 0$.

Poisson brackets (2.63) are defined on the vector space $\mathcal{A}/(\text{Im} \, D + [\mathcal{A}, \mathcal{A}])$.

A general theory of Poisson and double Poisson brackets on algebras of differential functions was developed in [87]. An algebra of (non-commutative) differential functions is defined as a unital associative algebra $D$ with a derivation $D$ and commuting derivations $\partial_i$, $i \in \mathbb{Z}_+$ such that the following two properties hold:

1). For each $f \in D$, $\partial_i(f) = 0$ for all but finitely many $i$;
2). $[\partial_i, D] = \partial_{i-1}$.

Formal symmetry

At least for non-abelian equations of the form (4.18), where

$$F = U_n + f(U, U_1, \ldots, U_{n-1})$$

all definitions and results concerning the formal symmetry (see Section 3.1) can be easily generalized.

Definition 4.10. A formal series

$$\Lambda = D + l_0 + l_{-1}D^{-1} + \cdots, \quad l_k \in \mathcal{O}$$

is called a formal symmetry for equation (4.18), (4.19) if it satisfies equation

$$D_t(\Lambda) - [F_\ast, \Lambda] = 0.$$

Remark 4.7. Stress that the coefficients of both $F_\ast$ and $\Lambda$ belong not to $\mathcal{A}$ but to the associative algebra $\mathcal{O}$ of local operators (see Definition 4.7 above).

For example, in the case of non-abelian Korteweg-de Vries equation (4.18) one can take $\Lambda = \mathcal{R}^{1/2}$, where $\mathcal{R}$ is the following recursion operator for (4.10) (see [84]):

$$\mathcal{R} = D^2 + 2(L_U + R_U) + (L_U + R_U)D^{-1} + (L_U - R_U)D^{-1}(L_U - R_U)D^{-1}.$$ 

In the scalar case, this recursion operator coincides with the usual one (see (2.32)).

Analogs of Theorems 2.4, 2.5 can be proved by reasonings similar to the ones used for the original statements.
Chapter 5

Integrable systems and non–associative algebras

One of the most remarkable observations by S. Svinolupov is the discovery of the fact that polynomial multi-component integrable equations are closely related to the well-known nonassociative algebraic structures such as left-symmetric algebras, Jordan algebras, triple Jordan systems, etc. This connection allows one to clarify the nature of known vector and matrix generalizations (see, for instance [88, 89, 90]) of classical scalar integrable equations and to construct some new examples of this kind [91].

5.1 Notation and definitions of algebraic structures

Let $\mathcal{A}$ be an $N$-dimensional algebra over $\mathbb{C}$ with a product $\circ$, $e_1, \ldots, e_N$ be a basis in $\mathcal{A}$. By $C_{jk}^i$ we denote the structural constants of $\mathcal{A}$. For any triple system with an operation $\{\cdot, \cdot, \cdot\}$ we denote by $B_{jkm}^i$ the structural constants:

$$\{e_j, e_k, e_m\} = \sum_{i=1}^{N} B_{jkm}^i e_i.$$

We denote by $U$ the element (1.2). Hereinafter, we use the notation (1.3), (1.4).

5.1.1 Left-symmetric algebras

Definition 5.1. Algebras with identity $[X, Y, Z] = 0$ are called left-symmetric [92].

Example 5.1. Examples of left-symmetric algebras:
1. Any associative algebra is left-symmetric.

2. The operation
   \[ x \circ y = \langle x, c \rangle y + \langle x, y \rangle c, \]  
   where \( \langle \cdot, \cdot \rangle \) is the scalar product, \( c \) is a given vector, defines a left-symmetric algebra.

3. Let \( \mathfrak{A} \) be associative algebra. Assume that \( R : \mathfrak{A} \to \mathfrak{A} \) satisfies the modified classical Yang-Baxter equation
   \[ R([R(x), y] - [R(y), x]) = [x, y] + [R(x), R(y)]. \]
   Then the multiplication
   \[ x \circ y = [R(x), y] - (xy + yx) \]
   is left-symmetric.

### 5.1.2 Jordan algebras

**Definition 5.2.** An algebra \( \mathcal{A} \) is called Jordan if the following identities
\[ X \circ Y = Y \circ X, \quad X^2 \circ (Y \circ X) = (X^2 \circ Y) \circ X \]  
hold.

**Example 5.2.** Examples of simple Jordan algebras:

1. The set of all \( m \times m \) matrices with respect to the operation
   \[ X \circ Y = XY + YX; \]  
2. The set of all symmetric \( m \times m \) matrices with the same operation [5.3];
3. The set of all \( N \)-dimensional vectors with respect to the operation
   \[ x \circ y = \langle x, c \rangle y + \langle y, c \rangle x - \langle x, y \rangle c, \]  
   where \( \langle \cdot, \cdot \rangle \) is the scalar product, \( c \) is a given vector;
4. The special Jordan algebra \( H_3(O) \) of dimension 27.
5.1.3 Triple Jordan systems

**Definition 5.3.** A triple system $T$ is called Jordan if the following identities hold
\[ \{X, Y, Z\} = \{Z, Y, X\}, \]
\[ \{X, Y, \{V, W, Z\}\} - \{V, W, \{X, Y, Z\}\} = \{\{X, Y, V\}, W, Z\} - \{V, \{Y, X, W\}, Z\}. \]

**Example 5.3.** Examples of simple triple Jordan systems:

a) The set of all $N \times N$ matrices with respect to the operation
\[ \{X, Y, Z\} = \frac{1}{2} (XY + ZYX); \quad (5.5) \]

b) The set of all skew-symmetric $N \times N$ matrices with the operation (5.5);

c) The set of all $N$-dimensional vectors with the operation
\[ \{x, y, z\} = \langle x, y \rangle z + \langle y, z \rangle x - \langle x, z \rangle y; \quad (5.6) \]

d) The set of all $N$-dimensional vectors with respect to
\[ \{x, y, z\} = \langle x, y \rangle z + \langle y, z \rangle x. \quad (5.7) \]

**Remark 5.1.** There is the following generalization of the product (5.7). The vector space of all $n \times m$-matrices is a triple Jordan system with respect to the operation
\[ \{X, Y, Z\} = XY^tZ + ZY^tX, \quad (5.8) \]
where “$t$” stands for the transposition.

There are close relations between Jordan algebras and triple systems.

**Proposition 5.1.** For any triple Jordan system $\{X, Y, Z\}$ the product
\[ X \circ Y = \{X, C, Y\}, \quad (5.9) \]
where $C$ is an arbitrary fixed element, defines a Jordan algebra.

**Proposition 5.2.** For any Jordan algebra with a product $\circ$ the formula
\[ \{X, Y, Z\} = (X \circ Y) \circ Z + (Z \circ Y) \circ X - Y \circ (X \circ Z) \quad (5.10) \]
defines a triple Jordan system.

**Proposition 5.3.** For each triple Jordan system $\{X, Y, Z\}$ and any element $C$ the formula
\[ \sigma(X, Y, Z) = \{X, \{C, Y, C\}, Z\} \]
defines a triple Jordan system.

**Proof.** To prove the statement, we apply Proposition 5.2 to multiplication (5.9) and take into account the identities of Definition 5.3. \qed
5.2 Jordan KdV systems

Consider equations of the form (4.18), where $F$ is a non-commutative and non-associative polynomial. Suppose that $F$ is fixed but the multiplication in the algebra is unknown.

For example, let us consider the KdV equation

$$U_t = U_{xxx} + 3 U \circ U_x,$$

where $\circ$ is a multiplication in some algebra $A$. The main question is:

**Question 5.1.** For which algebras $A$ is this equation integrable?

If $U$ is given by (1.2), then equation (5.11) is equivalent to the following $N$-component system

$$u^i_t = u^i_{xxx} + \sum_{k,j} C^i_{jk} u^k u^j_x, \quad i,j,k = 1, \ldots, N.$$

(5.12)

Vice versa any system of the form (5.12) can be written as (5.11) for a proper algebra $A$. Systems (5.12) seem to be a natural multi-component generalization of the KdV equation (1.13).

**Remark 5.2.** Let $I$ be a double–sided ideal in $A$. Choose a basis such that $e_{M+1}, \ldots, e_N$ span $I$. For such a basis we have $C^i_{jk} = 0$ for $i \leq M$ and $j > M$ or $k > M$. This means that the equations for $u_1, \ldots, u_M$ constitute a closed subsystem of the same form (5.12) and the whole system has a “triangular” form.

Any linear transformation of $\vec{u} = (u_1, \ldots, u_N)$ preserves the class of systems (5.12). A reasonable description of integrable cases has to be invariant under these transformations.

The system (5.12) is homogeneous (1.16) with $\mu = 3, \lambda_i = 2$. Therefore, without loss of generality we may assume that the polynomial higher symmetries for (5.12) are also homogeneous. Any such a symmetry has the following form

$$U_\tau = U_m + A_1(U, U_{n-2}) + A_2(U_1, U_{n-3}) + \cdots.$$  

(5.13)

Here, the quadratic terms are defined by unknown bi-linear operations $A_i$, the cubic terms are defined by three-linear operations and so on.

**Theorem 5.1.** [93] Suppose that the algebra $A$ is finite-dimensional and commutative. Then equation (5.11) has a higher symmetry of the form (5.13), where $m \geq 5$ iff $A$ is a Jordan algebra.

---

1In a weak version of the symmetry approach one can assume that operations in (5.13) are expressed in terms of the multiplication $\circ$ in $A$ like $A_1(X, Y) = c_1 X \circ Y + c_2 Y \circ X$ and so on.
The original proof of Theorem 5.1 was done for the system (5.12) by straightforward computations in terms of structural constants. It turns out that the computations can be performed in terms of algebraic operations, which define the equation and the symmetry.

In the following theorem we do not assume that the algebra \( \mathcal{A} \) is commutative or finite-dimensional. For simplicity, we assume that equation (5.11) has a symmetry (5.13) of fifth order. For the KdV equation such a symmetry is given by (1.14).

**Theorem 5.2.** Suppose that the algebra \( \mathcal{A} \) possesses the following property: if for any \( Z \) we have

\[
(X \circ Y - Y \circ X) \circ Z = 0,
\]

then \( X \circ Y - Y \circ X = 0 \). Equation (5.11) has a homogeneous symmetry (5.13) of fifth order iff \( \mathcal{A} \) is a Jordan algebra.

**Proof.** The symmetry can be written in the form

\[
U_\tau = U_5 + 5 A_1(U, U_3) + 5 A_2(U_1, U_2) + 5 B_1(U, U, U_1),
\]

where \( A_1 \) and \( A_2 \) are unknown bi-linear operations and \( B_1 \) is a three-linear operation. The symmetry condition

\[
D_t(D_\tau(U)) = D_\tau(D_t(U))
\]

is an identity with respect to the independent variables \( U_5, U_4, \ldots, U_0 = U \). Let us perform the scaling \( U_i \to z_i U_i, \ z_i \in \mathbb{C} \) and collect coefficients of different monomials in \( z_i \). Comparing the coefficients of \( z_5 z_1 \), we get \( A_1(X, Y) = X \circ Y \). The terms that involve \( z_4 z_2 \) gives us \( A_2(X, Y) = X \circ Y + Y \circ X \). The terms with \( z_3 z_1 z_0 \) implies identities

\[
B_1(X, X, Y) = \frac{1}{2} \left( 2 X \circ (X \circ Y) + (X \circ X) \circ Y \right)
\]

and

\[
\left( X \circ Y - Y \circ X \right) \circ Z = 0.
\]

So, we expressed the symmetry in terms of multiplication in the algebra \( \mathcal{A} \). Moreover, it follows from the assumption of the theorem that \( \mathcal{A} \) is commutative. Now, we obtain the Jordan identity from the coefficients of \( z_2 z_0 \). The remaining identities are satisfied in virtue of (5.2).

**Remark 5.3.** It is easy to see that any simple algebra satisfies the requirement of Theorem 5.2.
**Exercise 5.1.** Prove that equation (5.11) has a higher homogeneous symmetry of the form (5.13), where $m \geq 5$ iff the identities

$$(X \circ X) \circ (X \circ Y) = X \circ \left( (X \circ X) \circ Y \right),$$

$$2(X \circ Y) \circ (X \circ Y) - 2X \circ \left( Y \circ (X \circ Y) \right) + (X \circ X) \circ (Y \circ Y) - \left( (X \circ X) \circ Y \right) \circ Y = 0,$$

and (5.14) hold.

According to Remark 5.2, the most interesting non-triangular integrable equations (5.11) correspond to simple Jordan algebras. Classification of such algebras can be found in [94]. All simple Jordan algebras are described in Example 5.2.

### Integrable Jordan non-triangular KdV equations

1) The matrix KdV equation

$$U_t = U_{xxx} + UU_x + U_x U,$$

where $U$ is an $m \times m$-matrix, is generated by (5.3). It coincides with (4.10) up to a scaling of $U$;

2) The matrix KdV equation under the reduction $U^t = U$;

3) The vector KdV equation [91]:

$$u_t = u_{xxx} + \langle c, u \rangle u_x + \langle c, u_x \rangle u - \langle u, u_x \rangle c,$$

where $c$ is a given constant vector, corresponds to the product (5.4).

**Open problem 5.1.** Find solitonic and finite-gap solutions for the KdV equations related to the special Jordan algebra $H_3(O)$

### 5.3 Left-symmetric algebras and Burgers type systems

Consider the equation

$$U_t = U_2 + 2 U \circ U_1 + B(U, U, U),$$

where $B$ is a three-linear operation. In the finite-dimensional case it is equivalent to a system of evolution equations of the form

$$u_t^i = u_{xx}^i + 2C_{jk}^i u_j^k u_x^l + B_{jkm}^i u^k u^j u^m,$$
where \( i, j, k = 1, \ldots, N \).

The system (5.17) is homogeneous (1.16) with \( \mu = 2, \lambda_i = 1 \). The Burgers equation (1.11) has a homogeneous symmetry (1.12) of third order. The general ansatz of such symmetry in the case of equations (5.16) is given by

\[
U_\tau = U_3 + 3A_1(U, U_2) + 3A_2(U_1, U_1) + 3B_1(U, U, U_1) + C_1(U, U, U, U) \quad (5.18)
\]

Denote \( A(X, Y) \) by \( X \circ Y \).

**Theorem 5.3.** Equation (5.16) possesses a symmetry of the form (5.18) iff

\[
B(X, X, X) = X \circ (X \circ X) - (X \circ X) \circ X,
\]

and \( \circ \) is a left-symmetric product. The operations in (5.18) have the following form:

\[
\begin{align*}
A_1(X, Y) &= X \circ Y, \\
A_2(X, X) &= X \circ X, \\
B_1(X, X, Y) &= X \circ (X \circ Y) + Y \circ (X \circ X) - (Y \circ X) \circ X, \\
C_1(X, X, X, X) &= X \circ (X \circ (X \circ X)) - X \circ ((X \circ X) \circ X) + (X \circ X) \circ (X \circ X) - ((X \circ X) \circ X) \circ X.
\end{align*}
\]

In contrast with Jordan algebras, there is no complete classification of finite-dimensional simple left-symmetric algebras.

Any associative algebra is left-symmetric. The system, which corresponds to the case of the matrix algebra, is the matrix Burgers equation (4.9). The vector Burgers equation [91]

\[
u_t = u_{xx} + 2\langle u, u_x \rangle c + 2\langle c, u \rangle u_x + \langle u, u \rangle \langle c, u \rangle c - \langle u, u \rangle \langle c, c \rangle u,
\]

where \( c \) is a constant vector, comes from the left-symmetric algebra with the product (5.1).
Chapter 6

Integrable models associated with triple systems.

6.1 Integrability and triple Jordan systems

Here, we consider some classes of polynomial integrable systems of evolution equations with cubic non-linear terms. They are generalizations of the following famous scalar integrable equations: the modified Korteweg-de Vries equation (2.37), the nonlinear Schrödinger equation (2.10), and the nonlinear derivative Schrödinger equation

\[ u_t = u_{xx} + 2(u^2 v)_x, \quad v_t = -v_{xx} - 2(v^2 u)_x. \]

6.1.1 MKdV-type systems

Consider systems of the form

\[ u_i^t = u_{xxx}^i + \sum_{j,k,m} B_{jkm}^i u_j u_k u_m^x, \quad i, j, k = 1, \ldots, N. \]  (6.1)

Let \( T \) be a triple system with basis \( e_1, \ldots, e_N \), such that

\[ \{ e_j, e_k, e_m \} = \sum_i B_{jkm}^i e_i. \]

If \( U = \sum_k u^k e_k \), then the algebraic form of the equation is given by

\[ U_t = U_{xxx} + B(U, U_x, U). \]  (6.2)

The triple systems \( B(X, Y, Z) \) such that \( B(X, Y, Z) = B(Z, Y, X) \) are in one-to-one correspondence with the systems (6.1).
Equations \((6.2)\) are homogeneous \((1.16)\) with \(\mu = 3, \lambda = 1\). They are also invariant with respect to the discrete involution \(U \rightarrow -U\). Without loss of generality we assume that all symmetries enjoy the same properties.

**Theorem 6.1.** For any triple Jordan system \(\{\cdot, \cdot, \cdot\}\), the equation \((6.2)\), where
\[
B(X, Y, Z) = \{X, Z, Y\} + \{Z, X, Y\},
\]
has a fifth order symmetry of the form
\[
U_\tau = U_5 + B_1(U, U, U_3) + B_2(U, U_1, U_2) + B_3(U_1, U_1, U_1) + C(U, U, U, U_1). \tag{6.3}
\]

The converse of this statement was proved by I. Shestakov and VS.

**Theorem 6.2.** The equation \((6.2)\) has a fifth order symmetry of the form \((6.3)\) iff
\[
B(X, Y, Z) = \{X, Z, Y\} + \{Z, X, Y\}, \tag{6.4}
\]
where \(\{\cdot, \cdot, \cdot\}\) is a triple Jordan system.

**Proof.** The compatibility condition
\[
0 = (U_\tau)_t - (U_\tau)_t = P(U, U_1, ..., U_5) \tag{6.5}
\]
of \((6.2)\) and \((6.3)\) leads to a defining polynomial \(P\) that should be identically zero. After the scaling \(U_i \rightarrow z_i U_i\) in \(F\) all coefficients of different monomials in \(z_0, ..., z_5\) have to be identically zero. Equating the coefficient of \(z_0 z_1 z_5\) to zero, we find that
\[
B_1(X, X, Y) = B(X, Y, X). \tag{6.6}
\]
The coefficient of \(z_0 z_2 z_4\) leads to
\[
B_2(X, Y, Z) = 2B(X, Y, Z) + 2B(X, Z, Y). \tag{6.7}
\]
All other terms with \(z_5\) and \(z_4\) disappear by virtue of \((6.6)\) and \((6.8)\). Comparing the coefficients of \(z_1 z_2 z_3\), we obtain
\[
B_3(X, X, X) = B(X, X, X), \tag{6.8}
\]
while the coefficients of \(z_3 z_1 z_0^3\) give rise to
\[
C(X, X, X, Y) = B(X, B(X, Y, X), X) + \frac{1}{2}B(X, Y, B(X, X, X)). \tag{6.9}
\]
Thus the symmetry \((6.3)\) is expressed in terms of the triple system \(B\). All fifth order identities \(I_i = 0, i = 1, 2, 3, 4\) for the triple system \(B\) come from coefficients of the monomials \(z_0^3 z_1 z_3, z_0^3 z_2^2, z_0^2 z_1^2 z_2^2\) and \(z_0 z_1^4\). They are defined by the formulas
\[
I_1(X, Y, Z) = 2B(X, Z, B(X, X, Y)) - 3B(X, Z, B(X, Y, X)) + B(Y, Z, B(X, X, X)),
\]
100
I_2(X, Y) = 2B(X, Y, B(X, X, Y)) - 3B(X, Y, B(X, Y, X)) + B(Y, Y, B(X, X, X)),

I_3(X, Y, Z) = -2B(X, Y, B(X, Y, Z)) + 6B(X, Y, B(X, Z, Y)) - B(X, Y, B(Y, X, Z)) - B(X, Y, B(Z, X, Y)) + 4B(X, Z, B(X, Y, Y)) - 2B(X, Z, B(Y, X, Y)) - 2B(X, B(Y, Y, Z), X) + 2B(X, B(Z, Y, Y), X) - 2B(Y, Y, B(X, X, Z)) + 3B(Y, Y, B(X, Z, X)) - 4B(Y, Z, B(X, X, Y)) + 2B(Y, Z, B(X, Y, X)) - 2B(Y, B(X, X, Y), Z) - 2B(Y, B(X, Z, X), Y) - 2B(Z, Y, B(X, X, Y)) + 3B(Z, Y, B(X, Y, X)) - 2B(Z, B(X, Y, X), Y),

and

I_4(X, Y) = B(X, Y, B(Y, Y, Y)) + 2B(Y, Y, B(X, Y, Y)) - B(Y, Y, B(Y, X, Y)) - 2B(Y, B(X, Y, Y), Y).

It is clear that I_2(X, Y) = I_1(X, Y, Y). Using the method of undetermined coefficients, we will show that the identity I_4 = 0 is a consequence of the identities I_2 = 0 and I_3 = 0. First, introduce the polarizations of these identities. Let

- J_2(X, Y, Z, U, V) be the coefficient of \( k_1 k_2 k_3 \) in \( I_2(k_1 X + k_2 U + k_3 V, Y, Z) \);
- J_3(X, Y, Z, U, V) be the coefficient of \( k_1 k_2 k_3 k_4 \) in \( I_3(k_1 X + k_2 U, k_3 Y + k_4 V, Z) \);
- J_4(X, Y, Z, U, V) be the coefficient of \( k_1 k_2 k_3 k_4 \) in \( I_4(X, k_1 Y + k_2 Z + k_3 U + k_4 V) \).

Consider the following ansatz

\[
Z = J_4(X, Y, Z, U, V) - \sum_{\sigma \in S_4} b_\sigma J_2(\sigma(X), \sigma(Y), \sigma(Z), \sigma(U), \sigma(V)) - \sum_{\sigma \in S_4} c_\sigma J_3(\sigma(X), \sigma(Y), \sigma(Z), \sigma(U), \sigma(V)),
\]

where \( \sigma \) is a permutation of the set \( \{ X, Y, Z, U, V \} \). To take into account the identity \( B(X, Y, Z) = B(Z, Y, X) \), we fix the ordering

\[ U < V < X < Y < Z \]

and replace all expressions of the form \( B(P, Q, R) \) by \( B(R, Q, P) \) if \( P > R \). After that, equating the coefficients of similar terms in the relation \( Z = 0 \), we obtain an overdetermined system of linear equations for the coefficients \( b_\sigma \) and \( c_\sigma \). Solving this system by computer, we find that

\[
J_4(X, Y, Z, U, V) = \frac{1}{6} \left( J_2(U, X, V, Y, Z) + J_2(U, X, Y, V, Z) + J_2(U, X, Z, V, Y) + J_2(V, X, U, Y, Z) - J_3(U, V, X, Y, Z) - J_3(U, V, X, Z, Y) - J_3(U, V, Y, X, Z) - J_3(U, Y, V, X, Z) - J_3(V, U, X, Y, Z) - J_3(V, U, X, Z, Y) - J_3(V, U, Y, X, Z) - J_3(V, Y, U, X, Z) - J_3(X, U, V, Y, Z) - J_3(X, U, V, Z, Y) - J_3(X, U, Z, V, Y) - J_3(X, Y, U, Z, V) - J_3(Y, U, X, Z, V) \right).
\]
Consider a triple system
\[ \{X, Y, Z\} = \frac{1}{2} \left( B(Y, Z, X) + B(Y, X, Z) - B(X, Y, Z) \right). \]

It is easy to verify that
\[ B(X, Y, Z) = \{X, Z, Y\} + \{Z, X, Y\}. \tag{6.10} \]

Let us prove that the identities \( J_2 = J_3 = 0 \) are equivalent to one identity \( J = 0 \), where (see Definition 5.3 of Jordan triple system)
\[ J(X, Y, Z, U, V) = \{X, Y, \{U, V, Z\}\} - \{\{X, Y, U\}, V, Z\} - \{\{U, V, Z\}, X\} + \{\{U, X, V\}, Z\}. \]

The identity \( J = 0 \) is supposed to be rewritten in terms of the triple system \( B \) by means of (6.10). By the same method of undetermined coefficients we have verified that the identity \( J = 0 \) follows from \( J_2 = J_3 = 0 \) and vice versa each of the identities \( J_2 \) and \( J_3 \) follows from \( J = 0 \). For example,
\[ J_2(X, Y, Z, U, V) = -J(U, X, V, Z, Y) + J(U, X, Y, Z, V) - J(U, Y, X, Z, V) + \]
\[ J(V, X, Y, Z, U) - J(V, Y, U, Z, X) - J(V, Y, X, Z, U) - \]
\[ J(X, U, V, Z, Y) + J(X, U, Y, Z, V) - J(X, V, U, Z, Y) + \]
\[ J(Y, U, V, Z, X) + J(Y, V, U, Z, X) + J(Y, V, X, Z, U). \]

The formulas, which express \( J_3 \) through \( J \) and \( J \) through \( J_2, J_3 \), are much more complicated.

Besides the above fifth order identities there exist two identities of order 7. The coefficient of \( z_0^5 z_2^2 \) in the polynomial \( P \) yields
\[ B(X, B(X, Y, X), B(X, X, X)) - B(X, B(X, Y, B(X, X, X)), X) = 0 \]
while the coefficient of \( z_0^5 z_1^2 \) leads to
\[ 2B[X, Y, B[X, X, B[X, Y, X]]] - 2B[X, Y, B[X, Y, B[X, X, X]]] - 3B[X, Y, B[X, B[X, Y, X], X]] + \]
\[ 4B[X, B[X, Y, X], B[X, X, X]] + 2B[X, B[X, Y, X], B[X, X, X]] - 2B[X, B[X, Y, B[X, X, X]], X] + \]
\[ 3B[X, B[X, Y, B[X, X, X]], X] - 4B[X, B[X, B[X, Y, X], Y], X] - \]
\[ B[X, B[Y, Y, B[X, X, X]], X] + B[B[X, X, X], Y, B[X, Y, X]] = 0. \]

Using the method of undetermined coefficients, one can check that both identities follows from \( J = 0 \).

**Remark 6.1.** Since equation (6.2) is expressed through \( B(X, Y, X) \), it follows from (6.10) that all equations that have the fifth order symmetry are described by Theorem 6.1.
The integrable vector systems corresponding to the operations (5.6) and (5.7) are given by
\[ u_t = u_{xxx} + \langle u, u \rangle u_x \] (6.11)
and
\[ u_t = u_{xxx} + \langle u, u \rangle u_x + \langle u, u_x \rangle u, \] (6.12)
respectively.

The triple Jordan system (5.5) generates the matrix mKdV equation (4.11).

### 6.1.2 NLS-type systems

Multi-component generalizations (see \[88, 96\]) of the nonlinear Schrödinger equation (2.10) are systems of $2N$ equations of the following form
\[
\begin{align*}
   u_i^t &= u_{xx}^i + 2 \sum_{j,k,m} b_{jkm}^i u_j v_k u_m, \\
   v_i^t &= -v_{xx}^i - 2 \sum_{j,k,m} b_{jkm}^i v_j u_k v_m,
\end{align*}
\] (6.13)
where $i = 1, \ldots, N$, and $b_{jkm}^i$ are constants. In terms of the corresponding triple system it can be written in the form
\[
\begin{align*}
   U_i &= U_{xx} + 2B(U,V,U), \\
   V_i &= -V_{xx} - 2B(V,U,V).
\end{align*}
\] (6.14)

The triple systems $B(X,Y,Z)$ such that $B(X,Y,Z) = B(Z,Y,X)$ are in one-to-one correspondence with the systems (6.13). The systems (6.14) are homogeneous (1.16) with $\mu = 2, \lambda_i = 1$.

**Remark 6.2.** The system (6.14) has the following additional scaling symmetry $U \rightarrow \lambda U, V \rightarrow \lambda^{-1} V$. Therefore, we can assume without loss of generality that all symmetries of this equation are homogeneous and invariant with respect to the same scaling.

**Theorem 6.3.** \[96\] The equation (6.14) has a third order symmetry of the form
\[
\begin{align*}
   U_\tau &= U_{xxx} + 3B_1(U,V,U_x) + 3B_3(U,V_x), \\
   V_\tau &= U_{xxx} + 3B_2(V,U,V_x) + 3B_4(V,U_x),
\end{align*}
\]
iff $B_1(X,Y,Z) = B_2(X,Y,Z) = B(X,Y,Z), B_3(X,Y,X) = B_4(X,Y,X) = 0$ and $B(\cdot, \cdot, \cdot, \cdot)$ is a triple Jordan system.

The above examples of simple triple Jordan systems from Section 4.1.3 provide several interesting vector and matrix integrable systems.

**Example 6.1.** The triple system (5.5) produces the matrix NLS-system (4.12) up to a scaling.
Example 6.2. The well-known vector Schrödinger equation \[97\]
\[
\begin{align*}
    u_t &= u_{xx} + 2\langle u, v \rangle u, \\
    v_t &= -v_{xx} - 2\langle v, u \rangle v
\end{align*}
\] (6.15)
corresponds to the triple Jordan system \([5.7]\).

Example 6.3. One more integrable vector nonlinear Schrödinger equation
\[
\begin{align*}
    u_t &= u_{xx} + 2\langle u, v \rangle u - \langle u, u \rangle v, \\
    v_t &= -v_{xx} - 2\langle v, u \rangle v + \langle v, v \rangle u
\end{align*}
\] (6.16)
found in \([98]\) corresponds to the triple Jordan system \([5.6]\).

6.1.3 Derivative NLS-type systems

The derivative NLS equation is given by
\[
\begin{align*}
    u_t &= u_{xx} + 2(u^2)v_x, \\
    v_t &= -v_{xx} + 2(v^2)u_x.
\end{align*}
\]
Consider its following generalization:
\[
\begin{align*}
    U_t &= U_{xx} + B(U,V,U)_x, \\
    V_t &= -V_{xx} + B(V,U,V)_x
\end{align*}
\] (6.17)
where \(B(X,Y,Z) = B(Z,Y,X)\). The systems \([6.17]\) are homogeneous \([1.16]\) with \(\mu = 1, \lambda_i = \frac{1}{2}\).

**Theorem 6.4.** A system \([6.17]\) has a third order polynomial symmetry of the form
\[
\begin{align*}
    U_\tau &= U_{xxx} + P(U,V,U_x,V_x,U_{xx},V_{xx}), \\
    V_\tau &= V_{xxx} + Q(U,V,U_x,V_x,U_{xx},V_{xx}).
\end{align*}
\] (6.18)
iff \(B(\cdot,\cdot,\cdot)\) is a triple Jordan system.

Matrix and vector examples of integrable derivarive NLS type systems can be constructed using formulas \([5.5]-[5.7]\).

6.2 Some integrable systems corresponding to new classes of triple systems

6.2.1 Equations of potential mKdV type

Consider multi-component generalizations of known integrable equation (see \([2.40]\) and Example \([3.7]\))
\[
\begin{align*}
    u_t &= u_{xxx} + 3u_x^3.
\end{align*}
\] (6.19)
This equation has the following fifth order symmetry

\[ u_\tau = u_{xxxxx} + 15u_x^2u_{xxx} + 15u_xu_{xx}^2 + \frac{27}{2}u_x^5. \]

The multi-component generalizations of \((6.19)\) have the form

\[ U_t = U_{xxx} + 3B(U_x, U_x, U_x), \quad (6.20) \]

where

\[ B(X, Y, Z) = B(\sigma(X), \sigma(Y), \sigma(Z)), \quad \sigma \in S_3. \]

**Theorem 6.5.** Equation \((6.20)\) has a symmetry of the form

\[ U_\tau = U_5 + 15B_1(U_1, U_1, U_3) + 15B_2(U_2, U_2, U_1) + \frac{27}{2}C(U_1, U_1, U_1, U_1) \]

iff the symmetric triple system \(B(\cdot, \cdot, \cdot)\) satisfies the identity

\[ B(X, Y, B(Y, Y, Z)) + B(Y, Y, B(X, Y, Z)) + B(Y, Z, B(X, Y, Y)) - 3B(X, Z, B(Y, Y, Y)) = 0. \]

**Remark 6.3.** The symmetry is defined by

\[ B_1(X, X, Y) = B(X, X, Y), \quad B_2(X, X, Y) = B(X, X, Y), \]

\[ C(X, X, X, X) = B(X, X, B(X, X, X)). \]

### 6.2.2 Systems of Olver–Sokolov type

A multi-component generalization of the matrix system \((4.13)\) reads as follows

\[ U_t = U_{xx} + 2B(U, V, U_x), \quad V_t = -V_{xx} + 2B(V_x, U, V), \quad (6.21) \]

where \(B(\cdot, \cdot, \cdot)\) is a triple system. The systems of the form \((6.21)\) are homogeneous with the same weights as in Section 6.1.3.

**Theorem 6.6.** A system \((6.21)\) has a third order polynomial symmetry of the form \((6.18)\) iff \(B(\cdot, \cdot, \cdot)\) satisfies the identities

\[ B\left(X, Y, B(X, V, U)\right) = B\left(X, B(Y, X, V), U\right), \]

\[ B\left(X, B(Y, Z, V), Z\right) = B\left(B(X, Y, Z), V, Z\right), \quad (6.22) \]

\[ B\left(X, Y, B(Z, Y, V)\right) = B\left(B(X, Y, Z), Y, V\right). \]
Remark 6.4. The latter identity means that the multiplication

\[ X \circ Y = B(X, Z, Y) \]

is associative for any \( Z \). Moreover, varying \( Z \), we obtain a vector space of compatible associative products \[100\].

In the paper \[101\] the associative triple systems with identities

\[ B \left( X, Y, B(Z, U, V) \right) = B \left( X, B(Y, Z, U), V \right) = B \left( B(X, Y, Z), U, V \right) \] (6.23)

were considered. It is clear that (6.23) implies (6.22).

Since the triple matrix product

\[ B(X, Y, Z) = XYZ, \quad X, Y, Z \in \text{Mat}_n \]

satisfies (6.23), the system (4.13) belongs to the class described in Theorem 6.6.

One more example of a triple system satisfying (6.23) is given by

\[ B(X, Y, Z) = XY^t Z, \quad X, Y, Z \in \text{Mat}_{n,m}. \]

As far as I know, triple systems (6.22) have never been considered by algebraists.

Open problem 6.1. Find all simple triple systems (6.22).
Chapter 7

Integrability and deformations of non-associative structures

7.1 Inverse element in triple Jordan systems and rational integrable equations

7.1.1 Inverse element as a solution of PDE-system

The function \( y = \frac{1}{x} \) can be defined as the homogeneous solution of the differential equation
\[
y' + y^2 = 0.
\]

It turns out that an inverse element for triple Jordan systems can be defined in a similar way.

Let \( \{X, Y, Z\} \) be a triple system, \( e_1, \ldots, e_N \) be its basis, \( U = \sum_{k} u_k e_k \). Our aim is to define an element

\[
\phi(U) = \sum_{k=1}^{N} \phi_k(u_1, \ldots, u_N) e_k
\]

inverse for \( U \) as a solution of a proper system of PDEs.

Proposition 7.1. The following overdetermined system

\[
\frac{\partial \phi}{\partial u_k} = -\{\phi, e_k, \phi\},
\]

where \( k = 1, \ldots, N \), is compatible iff \( \{X, Y, Z\} \) is a triple Jordan system.
Definition 7.1. For any triple Jordan system any homogeneous \(\phi\) solution \((7.1)\) of the system \((7.2)\) is called the inverse element for the element \(U\).

Remark 7.1. It can be easily seen that any homogeneous solution of \((7.2)\) satisfies the identity

\[
\phi(U) = \{\phi(U), U, \phi(U)\}. \tag{7.3}
\]

Example 7.1. For the matrix triple Jordan system \((5.5)\) the homogeneous solution \(\phi(U)\) of \((7.2)\) is just the matrix inverse \(U^{-1}\).

Example 7.2. For the triple Jordan system \((5.6)\) the homogeneous solution for \((7.2)\) is given by

\[
\phi(u) = \frac{u}{|u|^2}.
\]

The following algebraic definition of \(U^{-1}\) is well known in the theory of triple Jordan systems. Let us define a linear operator \(P_X\) by the formula \(P_X(Y) = \{X, Y, X\}\). If \(P_U\) is non-degenerate, then by definition \(U^{-1} = P_U^{-1}(U)\).

Proposition 7.2. If \(P_U\) is non-degenerate, then

\[
\phi(U) = P_U^{-1}(U) \tag{7.4}
\]

satisfies \((7.2)\).

Remark 7.2. It follows from \((7.3)\) and from Proposition \(7.2\) that if \(P_U\) is non-degenerate, then there exists a unique homogeneous solution of \((7.2)\).

Example 7.3. For the triple Jordan system \((5.7)\) the operator \(P_u\) is degenerate for any \(u\) and the formula \((7.4)\) does not work. The general solution of \((7.2)\) is given by

\[
\phi(u) = \frac{c}{2\langle c, u \rangle},
\]

where \(c\) is arbitrary constant vector. This formula is a special case of

\[
\phi(U) = \frac{1}{2} C (U^t C)^{-1},
\]

which gives a solution in the case of \((5.8)\).

Remark 7.3. We see that in Example \((7.3)\) there are many homogeneous solutions of \((7.2)\).

Open problem 7.1. Find \(\phi(U)\) for the triple Jordan system of \(m \times m\) skew-symmetric matrices for odd \(m\).

\(^1\)This means that \(\sum u_i \frac{\partial \phi}{\partial u_i} = -\phi.\)
7.1.2 Several classes of integrable rational Jordan models

In all “rational” integrable models described below \( \phi(U) \) denotes an arbitrary solution of the system (7.2).

Let us introduce the following notation

\[
\alpha_U(X, Y) = \{X, \phi(U), Y\}
\]

and

\[
\sigma_U(X, Y, Z) = \{X, \{\phi(U), Y, \phi(U)\}, Z\}.
\]

**Remark 7.4.** According to Propositions 5.1 and 5.3 for any fixed \( U \) the operation \( \alpha_U(X, Y) \) is a multiplication in a Jordan algebra while \( \sigma_U(X, Y, Z) \) defines a triple Jordan system. The coefficients \( u_1, \ldots, u_N \) of the element \( U \) can be regarded as deformation parameters of these Jordan structures.

**Class 1**

For any triple Jordan system consider the hyperbolic equation

\[
U_{xy} = \alpha_U(U_x, U_y). \tag{7.5}
\]

In the matrix case, (7.5) coincides with the equation of the principal chiral field (4.16). For this reason we will call (7.5) the Jordan chiral field equation.

It is easy to verify that (7.5) admits the following Lax representation

\[
\Psi_x = \frac{2}{(1 - \lambda)} L_{U_x} \Psi, \quad \Psi_y = \frac{2}{(1 + \lambda)} L_{U_y} \Psi.
\]

As usual, we denote by \( L_X \) the operator of left multiplication. Note that this formula gives us a Lax representation for the matrix \( \sigma \)-model (4.16):

\[
\Psi_x = \frac{1}{(1 - \lambda)} M \Psi, \quad \Psi_y = \frac{1}{(1 + \lambda)} N \Psi,
\]

where \( \Psi \) is a matrix and

\[
M \Psi = -U_x U^{-1} \Psi - \Psi U^{-1} U_x, \quad N \Psi = -U_y U^{-1} \Psi - \Psi U^{-1} U_y,
\]

which is different from the standard one.
Class 2

The following evolution equation

\[ U_t = U_{xxx} - 3\alpha U(U_x, U_{xx}) + \frac{3}{2}\sigma_U(U_x, U_{xx}, U_x) \]  

(7.6)

has infinitely many higher symmetries for any triple Jordan systems. The matrix and two vector equations, which correspond to the triple systems (5.5), (5.6) and (5.7), have the following form:

\[
\begin{align*}
U_t &= U_{xxx} - \frac{3}{2} U_x U^{-1} U_{xx} - \frac{3}{2} U_{xx} U^{-1} U_x + \frac{3}{2} U_x U^{-1} U_x U^{-1} U_x, \\
u_t &= u_{xxx} - 3 \frac{\langle u, u_{xx} \rangle}{|u|^2} u_{xx} - 3 \frac{\langle u, u_{xx} \rangle}{|u|^2} u_x + 3 \frac{\langle u, u_{xx} \rangle}{|u|^2} u - 3 \frac{|u_x|^2}{|u|^2} u_x + 6 \frac{\langle u, u_x \rangle^2}{|u|^4} u_x - 3 \frac{\langle u, u_x \rangle |u_x|^2}{|u|^4} u,
\end{align*}
\]

and

\[
\begin{align*}
\nu_t &= u_{xxx} - \frac{3}{2} \langle c, u_{xx} \rangle u_{xx} - \frac{3}{2} \langle c, u_x \rangle u_x + \frac{3}{2} \langle c, u_x \rangle^2 u_x.
\end{align*}
\]

Class 3

The following equations

\[ V_t = V_{xxx} - \frac{3}{2} \alpha_V(V_{xx}, V_{xx}) \]

of the Schwartz-KdV type have infinitely many symmetries for any triple Jordan system. They are related to the equations of Class 2 by the potentiation \( U = V_x \). The matrix equation is given by (4.14). The two vector Schwartz-KdV equations have the form

\[
\begin{align*}
u_t &= u_{xxx} - 3 \frac{\langle u_x, u_{xx} \rangle}{|u_x|^2} u_{xx} + 3 \frac{|u_{xx}|^2}{|u_x|^2} u_x,
\end{align*}
\]

and

\[
\begin{align*}
u_t &= u_{xxx} - \frac{3}{2} \langle c, u_{xx} \rangle u_{xx}.
\end{align*}
\]

Class 4

The scalar representative of this class is the Heisenberg model

\[
\begin{align*}
&u_t = u_{xx} - \frac{2}{u + v} u_x^2, \\
v_t = -v_{xx} + \frac{2}{u + v} v_x^2.
\end{align*}
\]

The following coupled system

\[
\begin{align*}
U_t &= U_x - 2\alpha U + V(U_x, U_x), \\
V_t &= -V_{xx} + 2\alpha U + V(V_x, V_x).
\end{align*}
\]
is a Jordan generalization of equation (4.15). It has the following third order symmetry:

\[ U_t = U_{xxx} - 6\sigma U_{x} V(U_x, U_{xx}) + 6\sigma U_{V} V(U_x, U_{xx}), \quad V_t = V_{xxx} - 6\alpha U_{x} V(V_x, V_{xx}) + 6\sigma U_{V} V(V_x, V_{xx}). \]

One of the two vector equations is given by

\[
\begin{align*}
\begin{aligned}
\dot{u}_t &= u_{xxx} - 4 \langle u_x, u + v \rangle \frac{|u + v|^2}{u + v} u_x + 2 \frac{|u_x|^2}{|u + v|^2} (u + v), \\
\dot{v}_t &= -v_{xxx} + 4 \langle v_x, u + v \rangle \frac{|u + v|^2}{u + v} v_x - 2 \frac{|v_x|^2}{|u + v|^2} (u + v).
\end{aligned}
\end{align*}
\]

Open problem 7.2. Find Lax representations in the Tits–Kantor–Koecher superstructural algebra \[99\] for equations from Classes 2-4.

7.2 Deformations of non-associative algebras and integrable systems of geometric type

7.2.1 Geometric description of deformations

Let \( E \) be the Euclidean connection on an \( N \)-dimensional manifold \( \mathcal{M} \), \( u = (u_1, \ldots, u_N) \) be the local coordinates on \( \mathcal{M} \). Denote by \( E_{jk}^i(u) \) the components of \( E \). Let us consider a connection \( \Gamma \) with the components \( \Gamma_{jk}^i(u) = E_{jk}^i(u) + C_{jk}^i(u) \), where \( C_{jk}^i(u) \) are components of a tensor field \( C \) on \( \mathcal{M} \).

**Definition 7.2.** \[102\] The connection \( \Gamma \) is called a covariantly constant deformation of the Euclidean connection if the deformation tensor \( C \) is covariantly constant with respect to \( \Gamma \).

It follows from the standard formulas for recalculating the curvature and torsion under the deformation of a connection (see, for example, \[103\]) that both the curvature tensor of \( \Gamma \) and the torsion can be expressed in terms of the deformation tensor \( C \) only:

\[
T_{jk}^i = C_{jk}^i - C_{kj}^i, \quad (7.7)
\]

\[
P_{mjk} = \sum_r C_{rm}^i C_{jk}^r - C_{rm}^i C_{kj}^r + C_{kr}^i C_{jm}^r - C_{jr}^i C_{km}^r. \quad (7.8)
\]

**Remark 7.5.** Since the tensor \( C \) is covariantly constant, then \( \mathcal{M} \) is a space of covariantly constant curvature and torsion.
Rewriting in terms of the Euclidean connection $E$ the fact that $C$ is a covariantly constant tensor, we obtain

$$\nabla_m (C^i_{jk}) = \sum_r C^i_{rk} C^r_{mj} + C^i_{jr} C^r_{mk} - C^i_{mr} C^r_{jk}. \quad (7.9)$$

Here, we denote by $\nabla_m$ the covariant $u^m$-derivative with respect to $E$. The relations (7.9) will be regarded as an overdetermined system of first order PDE’s with respect to unknown functions $C^i_{jk}(u)$. We intend to investigate the compatibility conditions for (7.9). Of course, they are independent of the choice of coordinate system. For calculation, it is natural to use the coordinate system in which all components of $E$ are identically zero. With this preferred local coordinates, (7.9) takes the form of overdetermined system of PDEs for functions $C^i_{jk}(u^1, \ldots, u_N)$:

$$\frac{\partial C^i_{jk}}{\partial u^m} + C^i_{rk} C^r_{mj} + C^i_{jr} C^r_{mk} - C^i_{mr} C^r_{jk} = 0. \quad (7.10)$$

7.2.2 Algebraic description of deformations

Let $\mathcal{V}$ be a vector space with a basis $e_1, \ldots, e_N$. The tensor $C(u)$ gives rise to the $N$-parameter family of multiplications on $\mathcal{V}$. Namely, the products of basis vectors are defined by the formula

$$e_j \circ e_k = \sum_i C^i_{jk}(u) e_i. \quad (7.11)$$

In terms of the product (7.11) the deformation equation (7.10) takes the form

$$\partial_X (Y \circ Z) = (X \circ Y) \circ Z + Y \circ (X \circ Z) - X \circ (Y \circ Z).$$

Here and below, for any $X = \sum x^i e_i$, we denote by $\partial_X$ the vector field $\sum x^i \frac{\partial}{\partial u^i}$. Note that (7.11) and (7.8) can be rewritten in the following compact form

$$T(X, Y) = X \circ Y - Y \circ X,$$

$$R(X, Y, Z) = [Y, Z, X], \quad X, Y, Z \in \mathcal{V},$$

where $[\cdot, \cdot, \cdot]$ is defined by (1.3) and (1.4). The notation $T(X, Y)$ means the value of a tensor $T$ on vectors $X$ and $Y$.

**Theorem 7.1.** The system (7.10) is compatible iff for any $u = (u^1, \ldots, u^N)$ the product (7.11) satisfies the following identity [102, 104]:

$$[V, X, Y \circ Z] - [V, X, Y] \circ Z - Y \circ [V, X, Z] = 0. \quad (7.12)$$
Theorem 7.2  a) The class of algebras with identity (7.12) contains:
1) Associative algebras;
2) Left-symmetric algebras;
3) Lie algebras;
4) Jordan algebras;
5) LT-algebras. Any commutative algebra with identity (7.12) is an LT-algebra.

b) Let $C^i_{jk}(u)$ be the solution of system (7.10) with an initial data $C^i_{jk}(0)$. If $C^i_{jk}(0)$ are structural constants of an algebra from one of the classes 1–5, then the algebra with multiplication (7.11) belongs to the same class for any $u$. In other words, all these classes of algebras are invariant with respect to the deformation (7.10).

Open problem 7.3. Suppose that $C^i_{jk}(u)$ satisfies (7.10). Prove that for small $u$ the algebra with the structural constants $C^i_{jk}(u)$ is isomorphic to the algebra with the structural constants $C^i_{jk}(0)$.

Proposition 7.3. Let $\{X, Y, Z\}$ be a triple Jordan system and $\phi(u)$ be a solution of (7.2). Then the structural constants of the product

$$X \circ Y = \{X, \phi, Y\}$$

satisfy the system (7.10).

7.2.3 Equations of geometric type
Consider evolution systems of the form

$$u_i^t = u_{xxx}^i + \alpha_{jk}^i(u) u_x^j u_{xx}^k + \gamma_{jks}^i(u) u_x^j u_x^k u_x^s, \quad i = 1, \ldots, N. \quad (7.13)$$

For such systems unknown coefficients are not constants but functions of $u^1, \ldots, u^N$. Here and below, we assume that the summation is carried out over repeated indices.

It is convenient to rewrite (7.13) in the following way

$$u_i^t = u_{xxx}^i + 3\alpha_{jk}^i u_x^j u_{xx}^k + \left(\frac{\partial \alpha_{km}^i}{\partial u^j} + 2\alpha_{jr}^i\alpha_{km}^r - \alpha_{rj}^i\alpha_{km}^r + \beta_{jkm}^i\right) u_x^j u_x^k u_x^m. \quad (7.14)$$

The class of systems (7.14) is invariant under the arbitrary point transformations $u \to \Phi(u)$, where $u = (u^1, \ldots, u^N)$. It is easy to see that under such a change of coordinates, $\alpha_{jk}^i$ and $\beta_{jkm}^i$ are transformed just as components of an affine connection $\Gamma$ and a tensor $\beta$, respectively.
Example 7.4. In the case $N = 1$ the equation (7.14) has the form

$$u_t = u_{xxx} + 3\alpha(u) u_x u_{xx} + \left(\alpha'(u) + \alpha(u)^2 + \beta(u)\right) u_x^3.$$  

Using the symmetry approach (see Section 2.2), one can verify that this equation possesses higher symmetries iff $\beta' = 2\alpha\beta$. By a proper point transformation of the form $u \rightarrow \Phi(u)$ the function $\alpha$ can be reduced to zero (for $N = 1$ any affine connection is flat) and the function $\beta$ becomes a constant. The equation $u_t = u_{xxx} + \text{const} u_x^3$ is known to be integrable. It is related to the mKdV equation by a potentiation.

Without loss of generality we assume that the tensor $\beta$ is symmetric:

$$\beta(X, Y, Z) = \beta(Y, X, Z) = \beta(X, Z, Y)$$

for any vectors $X, Y, Z$.

**Question 7.1.** For which affine connections $\Gamma$ and which tensors $\beta$ is the equation (7.14) integrable?

Let $R$ and $T$ be the curvature and the torsion tensors of $\Gamma$. In order to formulate the classification results, we introduce the following tensor:

$$\sigma(X, Y, Z) = \beta(X, Y, Z) - \frac{1}{3} \delta(X, Y, Z) + \frac{1}{3} \delta(Z, X, Y), \quad (7.15)$$

$$\delta(X, Y, Z) = T(X, T(Y, Z)) + R(X, Y, Z) - \nabla_X(T(Y, Z)). \quad (7.16)$$

Using Bianchi’s identity $R(X, Y, Z) + R(Y, Z, X) + R(Z, X, Y) = 0$, one can check that

$$\sigma(X, Y, Z) = \sigma(Z, Y, X). \quad (7.17)$$

**Remark 7.6.** Taking into account (7.17), the symmetricity of $\beta$ and the identity $\delta(X, Y, Z) = -\delta(X, Z, Y)$, one can check that

$$\beta(X, Y, Z) = \frac{1}{3} \left(\sigma(X, Y, Z) + \sigma(Y, Z, X) + \sigma(Z, X, Y)\right). \quad (7.18)$$

**Theorem 7.3.** The equation (7.14) possesses a higher symmetry of the form

$$u_x = u_n + G(u, u_x, \cdots, u_{n-1}), \quad n > 3,$$

iff the following identities for the tensors $T, R$ and $\sigma$ hold:

$$\nabla_X(R(Y, Z, V)) = R(Y, X, T(Z, V)), \quad (7.19)$$

---

2This theorem was proved by S. Svinilupov and V. Sokolov (unpublished) and was formulated in the survey [105] dedicated to the memory of Sergey Svinilupov.
∇_X \left( \nabla_Y (T(Z, V)) - T(Y, T(Z, V)) - R(Y, Z, V) \right) = 0, \quad (7.20)

∇_X \left( \sigma(Y, Z, V) \right) = 0, \quad (7.21)

T(X, \sigma(Y, Z, V)) + T(Z, \sigma(Y, X, V)) + T(Y, \sigma(X, V, Z)) + T(V, \sigma(X, Y, Z)) = 0, \quad (7.22)

\sigma(X, \sigma(Y, Z, V), W) - \sigma(W, V, \sigma(X, Y, Z)) +
\sigma(Z, Y, \sigma(X, V, W)) - \sigma(X, V, \sigma(Z, Y, W)) = 0, \quad (7.23)

and relation (7.15), where the tensors \( \beta \) and \( \delta \) are eliminated by means of (7.16) and (7.18).

Remark 7.7. The identities (7.17) and (7.23) mean that \( \sigma^j_{km}(u) \) are the structural constants of a triple Jordan system for any \( u \).

Remark 7.8. In the case \( T = 0 \) only the following conditions remain:

∇_X \left( \sigma(Y, Z, V) \right) = 0, \quad (7.24)

R(X, Y, Z) = \sigma(X, Z, Y) - \sigma(X, Y, Z), \quad (7.25)

(7.17) and (7.23). This means that we are dealing with a symmetric space and a special covariantly constant deformation of a triple Jordan system.

It can be verified that for equations (7.6) we have \( T = 0 \) and the integrability conditions of Remark 7.8 are satisfied. More generally, any equation of the form

\[ U_t = U_{xxx} - 3 A(U, U_x, U_{xx}) + \frac{3}{2} B(U, U_x, U_x), \quad (7.26) \]

where the product \(-A_U(\cdot, \cdot, \cdot)\) is defined as the deformation (7.10) of any Jordan algebra and \( B_U \) is the corresponding triple system (5.10), satisfy the conditions of Remark 7.8.

The following two equations

\[ u_t = u_{xxx} + \frac{3}{2} \left( P(u, u_x)(c - |c|^2 u) \right)_x + 3 \mu |c|^2 P(u, u_x) u_x, \]

where \( \mu = -\frac{1}{2} \) or \( \mu = 0 \), \( u \) is an \( N \)-dimensional vector, \( c \) is a given constant vector, and

\[ P(u, u_x) = \left| u_x + \frac{\langle c, u_x \rangle}{1 - \langle c, u \rangle} u \right|^2, \]

satisfy the integrability conditions but do not belong to the class (7.26).
Theorem 7.4. For any triple Jordan system \( \{\cdot, \cdot, \cdot\} \) with the structural constants \( s^i_{jkm} \), there exists a unique (up to point transformations) solution of equations (7.17), (7.23), (7.24) and (7.25) such that

\[
\sigma^i_{jkm}(0) = s^i_{jkm}.
\] (7.27)

In the case \( T \neq 0 \), a generalization of the symmetric spaces arises. We do not know if such affine connected spaces have been considered by geometers.
Chapter 8

Vector integrable evolution equations

In Sections 5.2–6.1 we have seen that many examples of integrable models related to non-associative algebras are polynomial either matrix or vector equations. A first attempt for classification of such equations was made in [106].

8.1 Integrable polynomial homogeneous vector systems

In this section we consider several types of vector polynomial \( \lambda \)-homegeneous systems (see Theorem 1.2). By analogy with the scalar case \( \lambda \) is supposed to be 2,1, or \( \frac{1}{2} \). Vector NLS type systems (6.15) and (6.16) give us examples of such integrable systems. Here, we present several interesting vector systems from [106], which have higher symmetries.

Example 8.1. The vector analogue of the Ibragimov-Shabat equation (1.17) is given by

\[
\begin{align*}
    u_t &= u_{xxx} + 3\langle u, u \rangle u_{xx} + 6\langle u, u_x \rangle u_x + 3\langle u, u \rangle^2 u_x + 3\langle u_x, u_x \rangle u, \\
    v_t &= -v_{xxx} + 2\beta \langle u, v \rangle v_x + 2\beta \langle v, u_x \rangle v + \alpha \beta \langle u, v \rangle^2 v
\end{align*}
\]

Example 8.2. There are two following systems of the derivative NLS type with two vectors \( u \) and \( v \):

\[
\begin{align*}
    u_t &= u_{xx} + 2\alpha \langle u, v \rangle u_x + 2\alpha \langle u, v_x \rangle u - \alpha \beta \langle u, v \rangle^2 u, \\
    v_t &= -v_{xx} + 2\beta \langle u, v \rangle v_x + 2\beta \langle v, u_x \rangle v + \alpha \beta \langle u, v \rangle^2 v
\end{align*}
\]
\[
\begin{aligned}
\begin{cases}
    u_t &= u_{xx} + 2\alpha \langle u, v \rangle u_x + 2\beta \langle u, v_x \rangle u + \beta (\alpha - 2\beta) \langle u, v \rangle^2 u, \\
    v_t &= -v_{xx} + 2\alpha \langle u, v \rangle v_x + 2\beta \langle v, u_x \rangle v - \beta (\alpha - 2\beta) \langle u, v \rangle^2 v.
\end{cases}
\end{aligned}
\]

Here, \(\alpha\) and \(\beta\) are arbitrary constants. One of them can be normalized by a scaling.

In the above examples \(\lambda = \frac{1}{2}\).

**Example 8.3.** Four systems with a vector \(u\) and a scalar function \(u\), where \(\lambda = 2\), were found in [106]:

\[
\begin{aligned}
\begin{cases}
    u_t &= u_{xxx} + uu_x - \langle u, u_x \rangle, \\
    u_t &= u_{xxx} + uu_x + u_x u, \\
    u_t &= u_{xxx} + 3uu_x + 3\langle u, u_x \rangle, \\
    u_t &= u_{xxx} + uu_x + uu_x - \langle u, u_x \rangle, \\
    u_t &= -2uu_x - uu_x.
\end{cases}
\end{aligned}
\]

System (8.1) is just (5.15), where \(c = (1, 0, \ldots, 0)\). System (8.2) has been considered in [108]. It is a vector generalization of the Ito equation. Systems (8.3) and (8.4) are vector generalizations of the corresponding scalar systems from the paper [109].

**Open problem 8.1.** Find a Lax representation for systems (8.1) - (8.4) in the frames of general approach of [110].

**Remark 8.1.** The fifth order symmetry

\[
\begin{aligned}
\begin{cases}
    u_x &= u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x - 10\langle u, u_{xxx} \rangle - 15\langle u_x, u_{xx} \rangle - 10u_x \langle u, u \rangle - 20u\langle u, u_x \rangle, \\
    u_{xx} &= -9u_{xxxx} - 30u u_{xxx} - 45u_x u_{xx} - (35u_{xx} + 20u^2 + 5\langle u, u \rangle) u_x \\
    u_{xxx} &= -(10u_{xxx} + 20uu_x + 5\langle u, u_x \rangle) u
\end{cases}
\end{aligned}
\]

(8.5)
of system (8.3) is nothing but a vector generalization of the Kaup-Kuperschmidt equation. Indeed, if the vector part is absent (i.e., \( u = 0 \)), then system (8.3) becomes trivial and (8.5) turns out to be the Kaup-Kuperschmidt equation (1.21).

A big list of integrable systems with one scalar and one vector unknown functions and \( \lambda = 1 \) can be found in [111].

8.2 Symmetry approach to integrable vector equations

It turns out that for vector models the symmetry approach can be developed [112] in the same universality as for the scalar equations (see Section 2.2). In particular, we do not assume anymore that the right-hand side of the equation is polynomial.

Example 8.4. The following vector Harry Dym equation

\[
\mathbf{u}_t = \langle \mathbf{u}, \mathbf{u} \rangle^{3/2} \mathbf{u}_{xxx},
\]

where \( \mathbf{u} \) is an \( N \)-component vector, has infinitely many symmetries and conservation laws for any \( N \).

Equations (6.11), (6.12) and (8.6) belong to the following class of vector equations:

\[
\mathbf{u}_t = f_n \mathbf{u}_n + f_{n-1} \mathbf{u}_{n-1} + \cdots + f_1 \mathbf{u}_1 + f_0 \mathbf{u}, \quad \mathbf{u}_i = \frac{\partial^i \mathbf{u}}{\partial x^i}.
\]

Here, the (scalar) coefficients \( f_i \) depend on scalar products between vectors \( \mathbf{u}, \mathbf{u}_x, \ldots, \mathbf{u}_{n-1} \).

The crucial point is that we regard the variables

\[
\mathbf{u}[i,j] = \langle \mathbf{u}_i, \mathbf{u}_j \rangle, \quad i, j = 0, 1, \ldots \quad j \geq i
\]

as independent. We denote by \( \mathcal{F} \) a field of functions depending on variables (8.8).

It is clear that in the case of the standard scalar product in \( N \)-dimensional vector space all such equations (8.7) are isotropic, i.e. they are invariant with respect to the group \( SO(N) \).
Examples of integrable non-isotropic vector equations

**Example 8.5.** The integrable vector KdV equation (5.15) (see [91] and Section 5.2).

**Example 8.6.** Consider the equation [113]:

\[
\begin{align*}
    u_t &= \left( u_{xx} + \frac{3}{2}(u_x, u_x) u \right)_x + \frac{3}{2}(u, R(u)) u_x, \\
    |u| &= 1,
\end{align*}
\]  

(8.9)

where \( R = \text{diag}(r_1, \ldots, r_N) \) is a constant matrix. If \( N = 3 \), then (8.9) is a commuting flow of the Landau-Lifshitz equation.

In the case of Example 8.5 we have to take for \( F \) functions of a finite number of independent variables

\[
\langle u_i, u_j \rangle, \quad j \geq i, \quad \langle c, c \rangle, \quad \langle u_i, c \rangle.
\]  

(8.10)

**Open problem 8.2.** Solve several simple classification problems for such type equations.

For equations like (8.9) we may consider \((X, Y) = \langle X, R(Y) \rangle\) as a new scalar product and take for \( F \) functions of independent variables

\[
\begin{align*}
    u[i, j] &= \langle u_i, u_j \rangle, \\
    v[i, j] &= (u_i, u_j), \\
    i, j &= 0, 1, \ldots \quad j \geq i.
\end{align*}
\]  

(8.11)

**Remark 8.2.** It is clear that linear transformations of the scalar products

\[
\begin{align*}
    \tilde{u}[i, j] &= c_1 u[i, j] + c_2 v[i, j], \\
    \tilde{v}[i, j] &= c_3 u[i, j] + c_4 v[i, j], \\
    c_1 c_4 \neq c_2 c_3
\end{align*}
\]

preserve the class of all integrable equations of such type.

The symmetry approach we are developing in the next sections can be easily extended to equations (8.7) with coefficients that depend on variables (8.11).

### 8.2.1 Canonical densities

Consider the equations (8.7) with coefficients from \( F \), where \( F \) is a field of functions, which depend on variables (8.8), (8.10) or (8.11).

**Theorem 8.1.** ([112]). If the equation (8.7) possesses an infinite series of vector commuting flows of the form

\[
\begin{align*}
    u_r &= g_m u_m + g_{m-1} u_{m-1} + \cdots + g_1 u_1 + g_0 u, \\
    g_i &\in F,
\end{align*}
\]  

(8.12)

then
\( i) \). there exists a formal Lax pair \( L_t = [A, L] \), where
\[
L = a_1 D + a_0 + a_{-1} D^{-1} + \cdots, \quad A = \sum_{i=0}^{n} f_i D^i.
\] (8.13)

Here, \( f_i \) are the coefficients of equation (8.7) and \( a_i \in \mathcal{F} \).

\( ii) \). The following functions
\[
\rho_{-1} = \frac{1}{a_1}, \quad \rho_0 = \frac{a_0}{a_1}, \quad \rho_i = \text{res} L^i, \quad i \in \mathbb{N}
\] (8.14)

are conserved densities for the equation (8.7).

The conservation laws with densities (8.14) are called canonical.

\textbf{Proof.} Idea of the proof. \( i) \). Let us rewrite the equation (8.7) and its commuting flow (8.12) in the form
\[
u_t = A(u), \quad \nu = B(u), \quad \text{where } B = \sum_{i=0}^{m} g_i D^i.
\] (8.15)

The compatibility of (8.15) leads to the operator identity
\[
B_t - [A, B] = A_r.
\]

For large \( m \) we may “ignore” the right-hand side, since it has a small order comparing with the other terms. In other words, the operator \( L = B \) satisfies \( L_t = [A, L] \) approximately. Then the series of first order \( L_m = B^{1/m} \) is an approximate solution as well. The gluing of the first order approximate solutions corresponding to different commuting flows into an exact formal Lax operator \( L \) is similar to the scalar case \([2]\).

\( ii) \). It follows from Adler’s theorem \([2,3]\).

\textbf{Theorem 8.2.} ([12]). If the equation (8.7) possesses an infinite series of conserved densities from \( \mathcal{F} \), then

\( i) \). there exist a formal Lax operator \( L \) and a series \( S \) of the form
\[
S = s_1 D + s_0 + s_{-1} D^{-1} + s_{-2} D^{-2} + \cdots,
\]
such that
\[
S_t + A^+ S + SA = 0, \quad S^+ = -S, \quad L^+ = -S^{-1}LS,
\]
where the upper index + stands for the formal conjugation.
ii). Under the conditions of Item i) the densities (8.14), with $i = 2k$, are of the form $\rho_{2k} = D(\sigma_k)$ for some functions $\sigma_k$.

Proof. For the proof see [2, 112].

Remark 8.3. In contrast with (2.20) the differential operator $A$ in (8.13) is not the Fréchet derivative of the right-hand side of (8.7). For this reason we avoid the names “formal symmetry” or “formal recursion operator” for the series $L$.

8.2.2 Euler operator and Fréchet derivative

To define the canonical conserved densities we have considered differential operators and pseudo-differential series with scalar coefficients from $F$. However, similar to the matrix case (see Section 4.2.2), it is not sufficient for constructing of Hamiltonian and recursion operators, and we should change the ring of coefficients.

Let $F$ be a field of functions that depend on (8.8). Denote by $R_{i,j}$ an $F$-linear operator that acts on vectors by the rule

$$ R_{i,j}(v) = \langle u_j, v \rangle u_i. $$

It is easy to see that

$$ R_{i,j}R_{p,q} = \langle u_j, u_p \rangle R_{i,q}, \quad R^i_{i,j} = R_{j,i}, \quad \text{tr} \, R_{i,j} = \langle u_i, u_j \rangle, $$

$$ D \circ R_{i,j} = R_{i,j} D + R_{i+1,j} + R_{i,j+1}. $$

Denote by $\mathcal{O}$ the algebra over $F$, generated by operators $R_{i,j}$ and by the identity operator.

The Fréchet derivative of an element from $F$ is a differential operator with coefficients from $\mathcal{O}$. Such differential operators are called local. For instance, the Fréchet derivative of the right-hand side $F$ of an equation (8.7) is equal to

$$ F_\ast = \sum_k f_k D^k + \sum_{i,j,k} \frac{\partial f_k}{\partial u_{i,j \Uparrow k}} \left( R_{k,i} D^j + R_{k,j} D^i \right), \quad (8.16) $$

where $i, j, k = 0, \ldots, n$.

The Euler operator (or the variational derivative) is given by

$$ \delta \overline{u} = \sum_{i \leq j} (-D)^i \circ u_j \left( \frac{\partial}{\partial u_{i,j \Uparrow i}} \right) + (-D)^j \circ u_i \left( \frac{\partial}{\partial u_{i,j \Uparrow j}} \right). $$

Possibly most of the Hamiltonian structures for vector integrable equations are non-local. For example, the Hamiltonian operator $H$ and the symplectic operator $T$ for the vector MKdV-equation

$$ u_t = u_{xxx} + \langle u, u \rangle u_x $$

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are given by
\[ H(w) = D(w) + \langle u, D^{-1} \circ u \rangle w - \langle u, D^{-1} \circ w \rangle u, \]
\[ T(w) = D(w) + u D^{-1} \circ \langle u, w \rangle. \]

It is easy to see that these operators, written as pseudo-differential series, have coefficients from \( O \).

**Remark 8.4.** One can define a formal symmetry as a pseudo-differential series with coefficients from \( O \) that satisfies (2.20) and develop the symmetry approach related to the existence of such a formal symmetry. But the Fréchet derivative (8.16) is much more complicated than the operator \( A \) defined by (8.13). Moreover Theorem 8.1 is true for non-isotropic equations also while the definition of \( O \) essentially depends on the choice of \( F \) (see Examples 8.3 and 8.6).

### 8.2.3 Vector isotropic equations of KdV-type

Consider vector equations of the form
\[ u_t = u_{xxx} + f_2 u_{xx} + f_1 u_x + f_0 u, \tag{8.17} \]
where coefficients \( f_i \) of the equation are scalar functions of the following six independent variables:
\[ \langle u, u \rangle, \quad \langle u, u_x \rangle, \quad \langle u_x, u \rangle, \quad \langle u, u_{xx} \rangle, \quad \langle u_x, u_{xx} \rangle, \quad \langle u_{xx}, u_{xx} \rangle. \tag{8.18} \]

**Theorem 8.3.** [112] For equations (8.17) the canonical densities are defined by the following recurrence formula:
\[
\rho_{n+2} = \frac{1}{3} \left[ \sigma_n - f_0 \delta_{n,0} - 2 f_2 \rho_{n+1} - f_2 D \rho_n - f_1 \rho_n \right]
- \frac{1}{3} \left[ f_2 \sum_{s=0}^{n} \rho_s \rho_{n-s} + \sum_{0 \leq s+k \leq n} \rho_s \rho_k \rho_{n-s-k} + 3 \sum_{s=0}^{n+1} \rho_s \rho_{n-s+1} \right]
- D \left[ \rho_{n+1} + \frac{1}{2} \sum_{s=0}^{n} \rho_s \rho_{n-s} + \frac{1}{3} D \rho_n \right], \quad n \geq 0, \tag{8.19} \]
where \( \delta_{i,j} \) is the Kronecker delta and \( \rho_0, \rho_1 \) are defined by
\[
\rho_0 = -\frac{1}{3} f_2,
\rho_1 = \frac{1}{9} f_2^2 - \frac{1}{3} f_1 + \frac{1}{3} D(f_2). \]
Remark 8.5. The same formulas for the canonical densities hold for the non-isotropic case. The most general recurrence formula for equations (8.7) with \( n = 3 \) is presented in Section 8.2.7.

Open problem 8.3. Perform a complete classification of integrable equations (8.17) in isotropic or/and non-isotropic cases.

Some particular results were obtained in [114, 115]. Here, we formulate one of them.

Shift-invariant equations

Consider equations of the form

\[
\mathbf{u}_t = \mathbf{u}_{xxx} + f_2 \mathbf{u}_{xx} + f_1 \mathbf{u}_x, \tag{8.20}
\]

where \( f_i \) depend only on \( \langle \mathbf{u}_x, \mathbf{u}_x \rangle, \langle \mathbf{u}_x, \mathbf{u}_{xx} \rangle, \langle \mathbf{u}_{xx}, \mathbf{u}_{xx} \rangle \). It is clear that such equations are invariant with respect to translations of the form \( \mathbf{u} \rightarrow \mathbf{u} + \mathbf{c} \).

Theorem 8.4. [114] Any equation (8.20) with infinite sequence of higher symmetries or local conservation laws belongs to the following list:

\[
\begin{align*}
\mathbf{u}_t &= \mathbf{u}_{xxx} + \frac{3}{2} \left( \frac{a^2 u_{[1,2]}^2}{1 + a u_{[1,1]}} - a u_{[2,2]} \right) \mathbf{u}_x, \\
\mathbf{u}_t &= \mathbf{u}_{xxx} - 3 \frac{u_{[1,2]}}{u_{[1,1]}} \mathbf{u}_{xx} + \frac{3}{2} \frac{u_{[2,2]}}{u_{[1,1]}} \mathbf{u}_x, \\
\mathbf{u}_t &= \mathbf{u}_{xxx} - 3 \frac{u_{[1,2]}}{u_{[1,1]}} \mathbf{u}_{xx} + \frac{3}{2} \left( \frac{u_{[2,2]}}{u_{[1,1]}} \frac{u_{[1,2]^2}}{u_{[1,1]}^2(1 + a u_{[1,1]})} \right) \mathbf{u}_x, \\
\mathbf{u}_t &= \mathbf{u}_{xxx} - \frac{3}{2} (p + 1) \frac{u_{[1,2]}}{p u_{[1,1]}} \mathbf{u}_{xx} + \frac{3}{2} (p + 1) \left( \frac{u_{[2,2]}}{u_{[1,1]}} - \frac{a u_{[1,2]}^2}{p^2 u_{[1,1]}} \right) \mathbf{u}_x.
\end{align*}
\]

Here, \( a \) is an arbitrary constant and \( p = \sqrt{1 + a u_{[1,1]}} \). Notice that, if \( a = 0 \), the last equation of the list is reduced to

\[
\mathbf{u}_t = \mathbf{u}_{xxx} - 3 \frac{u_{[1,2]}}{u_{[1,1]}} \mathbf{u}_{xx} + 3 \frac{u_{[2,2]}}{u_{[1,1]}} \mathbf{u}_x.
\]

Remark 8.6. We verified that each of these equations possesses a symmetry of fifth order and non-trivial local conservation laws. To prove their integrability auto-Bäcklund transformations of first order with a spectral parameter were found in [114].

\(^1\)We call a conservation law \( D_t(\rho) = D(\sigma) \) local if \( \rho, \sigma \in \mathcal{F} \).
8.2.4 Auto-Bäcklund transformations

An auto-Bäcklund transformation of the first order is defined by
\[ u_x = h v_x + f u + g v, \]
where \( u \) and \( v \) are solutions of the same vector equation. The functions \( f, g \) and \( h \) are (scalar) functions of variables
\[ u_{[0,0]} = \langle u, u \rangle, \quad v_{[i,j]} \overset{def}{=} \langle v_i, v_j \rangle, \quad w_i \overset{def}{=} \langle u, v_i \rangle, \quad i, j \geq 0. \]

Example 8.7. The auto-Bäcklund transformation for the vector Swartz-KdV equation
\[ u_t = u_{xxx} - 3 u_{[1,2]} u_{[1,1]} + 3 u_{[2,2]} u_{[1,1]} + u_{xx} \]
is given by
\[ u_x = 2 \mu v_x^2 \langle u - v, v_x \rangle (u - v) - \mu v_x^2 |u - v|^2 v_x, \]
where \( \mu \) is an arbitrary (spectral) parameter. The superposition formula
\[ z = u + (\mu - \nu) \frac{\nu (u - v')^2 (u - v) - \mu (u - v)^2 (u - v')}{(\mu (u - v) - \nu (u - v'))^2}, \]
corresponding to this auto-Bäcklund transformation connects 4 different solutions
\[ v' \xrightarrow{\mu} z \quad \nu \]
of the vector Schwartz-KdV equation. It defines a known integrable vector discrete model.

8.2.5 Equations on the sphere

Isotropic equations

Consider the equations (8.17) on the sphere \( \langle u, u \rangle = 1 \). In this case, the coefficients are functions of only three independent variables
\[ u_{[1,1]} = \langle u_x, u_x \rangle, \quad u_{[1,2]} = \langle u_x, u_{xx} \rangle, \quad u_{[2,2]} = \langle u_{xx}, u_{xx} \rangle \quad (8.21) \]
instead of six variables \(8.18\). Differentiating the constraint \(\langle u, u \rangle = 1\), one can express the remaining scalar products through \(8.21\). Moreover, the relation \(\langle u, u_t \rangle = 0\) implies that

\[
f_0 = f_2 u_{[1,1]} + 3 u_{[1,2]}\]

and any equation \(8.17\) on the sphere has the following form

\[
u_t = u_3 + f_2 u_2 + f_1 u_1 + (f_2 u_{[1,1]} + 3 u_{[1,2]}) u, \quad |u| = 1. \tag{8.22}
\]

**Theorem 8.5.** [112] Any equation \(8.22\) with an infinite sequence of higher symmetries or local conservation laws belongs to the following list:

\[
u_t = u_{xxx} - 3 \frac{u_{[1,2]}}{u_{[1,1]}} u_{xx} + 3 \frac{u_{[2,2]}}{u_{[1,1]}} \left( u_{[1,1]} + a u_{[1,1]} \right) u_x,
\]

\[
u_t = u_{xxx} + 3 \frac{a^2 u_{[1,2]}^2}{u_{[1,1]}} - a (u_{[2,2]} - u_{[1,1]}^2) u_x + 3 u_{[1,2]} u,
\]

\[
u_t = u_{xxx} - 3 \frac{u_{[1,2]}}{u_{[1,1]}} u_{xx} + 3 \frac{u_{[2,2]}}{u_{[1,1]}} u_x,
\]

\[
u_u = u_{xxx} - 3 \frac{u_{[1,2]}}{u_{[1,1]}} u_{xx} + 3 \frac{u_{[2,2]}}{u_{[1,1]}} u_x,
\]

\[
u_t = u_{xxx} - 3 \frac{(q + 1) u_{[1,2]}}{2 q u_{[1,1]}} u_{xx} + 3 \frac{(q - 1) u_{[1,2]}}{2 q} u
\]

\[+ \frac{3}{2} \left( \frac{(q + 1) u_{[2,2]}}{u_{[1,1]}} - \frac{(q + 1) a u_{[1,2]}}{2} q^2 u_{[1,1]} + u_{[1,1]} (1 - q) \right) u_x,
\]

\[
u_t = u_3 + 3 u_{[1,1]} u_1 + 3 u_{[1,2]} u,
\]

\[
u_t = u_3 + 3 \frac{u_{[1,1]}}{2} u_1 + 3 u_{[1,2]} u,
\]

where \(a\) and \(c\) are arbitrary constants, \(q = \sqrt{1 + a u_{[1,1]}}\).

### 8.2.6 Equations with two scalar products

Suppose that the coefficients of the equation \(8.17\) are functions of the variables \(8.11\), where \(i \leq j \leq 2\). Under the assumption \(\langle u, u \rangle = 1\) and up to transformations \(\bar{v}[i, j] = v[i, j] + \text{const} v[i, j]\), all equations that have an infinite series of higher symmetries have been found [112, 116]. The equations of the classification list that have rational coefficients are:

\[
u_t = u_3 + \left( 3 \frac{u_{[1,1]}}{2} + v_{[0,0]} \right) u_1 + 3 u_{[1,2]} u_0,
\]

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In the case

\[ \mathbf{u}_t = \mathbf{u}_3 - 3 \frac{u_{[2]}}{u_{[1]}} \mathbf{u}_2 + \frac{3}{2} \left( \frac{u_{[2]}}{u_{[1]}} + \frac{u_{[1]}}{u_{[2]}} + \frac{v_{[1]}}{u_{[1]}} \right) \mathbf{u}_1, \]

\[ \mathbf{u}_t = \mathbf{u}_3 - 3 \frac{v_{[1]}}{v_{[0]}} \mathbf{u}_2 + 3 \left( \frac{v_{[0]}}{v_{[1]}} - 2 \frac{v_{[1]}}{v_{[0]}} \right) \mathbf{u}_1 + 3 \left( \frac{u_{[2]}}{v_{[0]}} - \frac{v_{[1]}}{v_{[0]}} \right) \mathbf{u}, \]

\[ \mathbf{u}_t = \mathbf{u}_3 - 3 \frac{v_{[0]}}{v_{[1]}} \left( \mathbf{u}_2 + u_{[1]} \mathbf{u} \right) + 3 u_{[1]} \mathbf{u} + 3 \left( - \frac{u_{[2]}}{v_{[0]}} + \frac{(u_{[2]})^2}{v_{[0]}(v_{[0]} + u_{[1]})} \right) + \left( \frac{v_{[0]}}{v_{[1]}} - \frac{v_{[0]} v_{[1]}}{v_{[0]}^2} \right) \mathbf{u}_1. \]

where \( q = u_{[1]} + v_{[0]} + a, \ a \) is an arbitrary constant,

\[ \mathbf{u}_t = \mathbf{u}_3 - 3 \frac{v_{[0]}}{v_{[1]}} \mathbf{u}_2 - 3 \left( \frac{v_{[0]}}{v_{[0]}} + a - \frac{5}{2} \frac{v_{[1]}}{v_{[0]}} \right) \mathbf{u}_1 + 3 \left( \frac{u_{[1]}}{v_{[0]}} - \frac{v_{[0]}}{v_{[0]}} \right) \mathbf{u}, \]

\[ \mathbf{u}_t = \mathbf{u}_3 - 3 \frac{v_{[0]}}{v_{[1]}} \left( \mathbf{u}_2 + u_{[1]} \mathbf{u} \right) + 3 u_{[1]} \mathbf{u} + \frac{3}{2} \left( - \frac{u_{[2]}}{v_{[0]}} + \frac{(u_{[2]})^2}{v_{[0]}(v_{[0]} + u_{[1]})} \right) + \frac{(v_{[0]})^2}{v_{[0]}^2} + \frac{v_{[0]}^2}{v_{[0]}} \mathbf{u}_1. \]

The first of these equations is just \(^{(3.1)}\). To justify the integrability for all equations auto-Bäcklund transformations with the spectral parameter were found.

Vector hyperbolic equations on the sphere with integrable third-order symmetries

In \(^{[117]}\) vector hyperbolic equations of the form

\[ \mathbf{u}_{xy} = h_0 \mathbf{u} + h_1 \mathbf{u}_x + h_2 \mathbf{u}_y, \quad \mathbf{u}^2 = 1 \]

on the sphere were considered. Here, \( h_i \) are some scalar-valued functions depending on two different scalar products \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) between vectors \( \mathbf{u}, \mathbf{u}_x \) and \( \mathbf{u}_y \). All such equations that have integrable vector \( x \) and \( y \)-symmetries were found (cf. Section \(^{2.1.1}\)).

Example 8.8. A hyperbolic integrable equation on the sphere is given by

\[ \mathbf{u}_{xy} = \frac{\mathbf{u}_x}{(\mathbf{u}, \mathbf{u})} \left( \langle \mathbf{u}, \mathbf{u}_y \rangle + \sqrt{1 + \langle \mathbf{u}, \mathbf{u} \rangle (\mathbf{u}_x, \mathbf{u}_x) - 2 \phi \right) - \langle \mathbf{u}_x, \mathbf{u}_y \rangle \mathbf{u}, \]

where \( \phi = \sqrt{\langle \mathbf{u}, \mathbf{u}_y \rangle^2 + \langle \mathbf{u}, \mathbf{u} \rangle (1 - \langle \mathbf{u}_y, \mathbf{u}_y \rangle). \)

In the case \( N = 2 \) this equation is equivalent to (cf. \(^{[2.2]}\))

\[ u_{xy} = \text{sn}(u) \sqrt{u_x^2 + 1} \sqrt{u_y^2 + 1}. \]
8.2.7 Equations with constant vector

In Section 5.2 the integrable vector KdV equation (5.15) appears. The simplest conserved densities for this equation are given by

\[ \rho_1 = (c, u), \quad \rho_2 = c^2 u^2 - 2(c, u)^2, \]

and

\[ \rho_3 = 6(c, u_x)^2 - 3c^2 u_x^2 + 3c^2 u^2(c, u) - 4(c, u)^3. \]

This equation belongs to the class of equations of the form

\[ u_t = f_3 u_{xxx} + f_2 u_{xx} + f_1 u_x + f_0 u + h c. \]  

(8.23)

In this case the coefficients depend on \((u_i, u_j), (u_i, c), i \leq j.\) (8.24)

Without loss of generality we may assume that \((c, c) = 1.\)

All statements of theorem 8.1 can be easily generalized to the case of equations of the form

\[ u_t = f_n u_n + f_{n-1} u_{n-1} + \cdots + f_1 u_1 + f_0 u + h c = A(u) + h c, \]  

(8.25)

where the coefficients \(f_i\) and \(h\) depend on (8.24).

**Theorem 8.6.** Equations (8.25), which have infinitely many symmetries, possess a formal Lax operator

\[ L = a_1 D_x + a_0 + a_{-1} D_x^{-1} + \cdots \]

with scalar coefficients such that

\[ L_t = [A, L]. \]

For equation (8.23) the recurrent formula for densities of canonical conservation laws

\[ D_t \rho_i = D_x \theta_i, \quad i = -1, 0, 1, 2, \ldots \]

is given by

\[ \rho_{n+2} = \frac{a}{3} \left( \theta_n - f_0 \delta_{n,0} - 2 a f_2 \rho_{n+1} - f_2 \frac{d}{dx} \rho_n - f_1 \rho_n \right) \]

\[ - \frac{a}{3} f_2 \sum_{i+j=n} \rho_i \rho_j - \frac{1}{3} a^{-2} \sum_{i+j+k=n} \rho_i \rho_j \rho_k - a^{-1} \sum_{i+j+n+1} \rho_i \rho_j \]

\[ - a^{-2} D a \rho_{n+1} - \frac{1}{2} a^{-2} D \sum_{i+j=n} \rho_i \rho_j - \frac{1}{3} a^{-2} D^2 \rho_n, \quad n \geq 0, \]
where \( f_3 = a^{-3} \), \( \delta_{i,j} \) is the Kronecker delta, and

\[
\rho_{-1} = a,
\]

\[
\rho_0 = -\frac{1}{3} a^3 f_2 - D \ln a,
\]

\[
\rho_1 = \frac{a}{3} \theta_{-1} + a^{-1} \rho^0_0 - \frac{1}{3} a^2 f_1 + \frac{1}{3} a^{-3} (D a)^2 + 2 a^{-2} \rho_0 D a - a^{-1} D \rho_0 - \frac{1}{3} a^{-2} D^2 a.
\]

**Remark 8.7.** Formula (8.19) is valid for integrable equations (8.7) with \( n = 3 \) in the cases, when the coefficients depend on one or two scalar products. If \( f_3 = a = 1 \) it coincides with (8.19).

Although the recurrent formula for the canonical densities are obtained, no classification results for equations (8.23) are known.
Chapter 9

Appendices

9.1 Appendix 1. Hyperbolic equations with integrable third order symmetries

Theorem 9.1.

\[ u_{xy} = c_1e^u + c_2e^{-u}; \quad u_{xy} = f(u)\sqrt{u_x^2 + 1}, \quad \text{where} \quad f'' = cf; \]

\[ u_{xy} = \sqrt{u_x\sqrt{u_y^2 + 1}}; \quad u_{xy} = \sqrt{P(u) - \mu \sqrt{u_x^2 + 1}\sqrt{u_y^2 + 1}}; \]

\[ u_{xy} = 2uu_x; \quad u_{xy} = 2u_x\sqrt{u_y}; \quad u_{xy} = u_x\sqrt{u_y^2 + 1}; \]

\[ u_{xy} = \sqrt{u_xu_y}; \quad u_{xy} = \frac{u_x(u_y + c)}{u}, \quad c \neq 0; \quad u_{xy} = (c_1e^u + c_2e^{-u})u_x; \]

\[ u_{xy} = \frac{u_y\eta}{\sinh(u)}(\eta e^u - 1); \quad u_{xy} = \frac{2u_y\eta}{\sinh(u)}(\eta \cosh(u) - 1); \]

\[ u_{xy} = \frac{2\eta\xi}{\sinh(u)}((\eta\xi + 1)\cosh(u) - \xi - \eta); \quad u_{xy} = \frac{u_y}{u}(\eta - 1) + cu(\eta + 1); \]

\[ u_{xy} = \frac{2u_y}{u}(\eta - 1); \quad u_{xy} = \frac{2\eta\xi}{u}(\eta - 1)(\xi - 1); \quad u_{xy} = \frac{u_xu_y}{u} - 2u^2u_y; \]

\[ u_{xy} = \frac{u_x(u_y + c)}{u} - uu_y; \quad u_{xy} = \sqrt{u_y} + cu_y; \quad u_{xy} = cu. \]

Here, \( \mu \) is a solution of the equation \( 4\mu^3 - g_2\mu - g_3 = 0 \), \( c, c_1, c_2, g_2, g_3 \) are constants, and \( \xi = \sqrt{u_y + 1}, \ \eta = \sqrt{u_x + 1} \).
9.2 Appendix 2. Scalar hyperbolic equations of Liouville type

In this section nonlinear equations (3.3) of Liouville type known to the author are collected [23, 118]. A list is presented up to the involution $x \leftrightarrow y$ and transformations of the form

$$x \rightarrow \zeta(x), \quad y \rightarrow \xi(y), \quad u \rightarrow \theta(x, y, u).$$

**Class 1.** Equations of this class have the form

$$u_{xy} = -\frac{W_y}{W_u}. \quad (9.1)$$

Here the function $W(x, y, u_x)$ is defined from the equation

$$u_x = q_0(x, y) + \sum_{i=1}^{n} \alpha_i(y) q_i(x, W),$$

where $\alpha_i, q_i$ are arbitrary functions. □

**Class 2.** (see Section 3.5.1)

$$u_{xy} = -\frac{P_u}{P_{ux}} u_y, \quad (9.2)$$

where

$$u_x = \alpha(x, P)u^2 + \beta(x, P)u + \gamma(x, P),$$

and $\alpha, \beta, \gamma$ are arbitrary functions.

**Remark 9.1.** The following well known equation:

$$u_{xy} = e^n u_y$$

gives us an example of equation from Class 2.

**Class 3.** Equations of the form

$$u_{xy} = A_n(x, y) \sqrt{u_x u_y}, \quad n \geq 1,$$

where $A_n(x, y)$ is defined as follows, are Liouville integrable. Let

$$h_0 = \frac{1}{4} A_n^2, \quad h_1 = \frac{1}{4} A_n^2 - \frac{\partial^2}{\partial x \partial y} \ln A_n. \quad (9.3)$$
Define $h_i$, $i > 1$ by the formula

$$h_{k+1} = 2h_k - h_{k-1} - \frac{\partial^2}{\partial x \partial y} \ln h_k, \quad k = 1, \ldots, \quad (9.4)$$

Then, by definition, $A_n$ is any solution of the equation $h_n = 0$. In particular, Class 3 contains equations with

$$A_n = \frac{2n\lambda}{\lambda(x + y) - xy}$$

and their degenerations with

$$A_n = \frac{2n}{(x + y)}. \quad \square$$

Apart from these three series of Liouville type equations the following equations of the form

$$u_{xy} = A(x, y, u) B(u_x) \bar{B}(u_y) \quad (9.5)$$

are known: (3.2), (3.13), (3.14),

$$u_{xy} = \left( \frac{1}{u - x} + \frac{1}{u - y} \right) u_x u_y; \quad (9.6)$$

$$u_{xy} = f(u) b(u_x), \quad \text{where} \quad f f'' - f'^2 = 0, \quad bb' + u_1 = 0; \quad (9.7)$$

$$u_{xy} = f(u) b(u_x) \bar{b}(u_y), \quad \text{where} \quad (\ln f)'' - f'^2 = 0, \quad bb' + u_1 = 0, \quad \bar{b}b' + \bar{u}_1 = 0; \quad (9.8)$$

$$u_{xy} = \frac{1}{u} b(u_x) \bar{b}(u_y), \quad \text{where} \quad bb' + c\bar{b} + u_1 = 0, \quad \bar{b}b' + c\bar{b} + \bar{u}_1 = 0; \quad (9.9)$$

$$u_{xy} = \frac{1}{(x + y)} b(u_x) \bar{b}(u_y), \quad \text{where} \quad \bar{b}' = \bar{b}^2 + \bar{b}^2, \quad \bar{b}' = \bar{b}^3 + \bar{b}^2; \quad (9.10)$$

$$u_{xy} = \frac{1}{u} b(u_x) \bar{b}(u_y), \quad \text{where} \quad bb' + b - 2u_1 = 0, \quad \bar{b}b' + \bar{b} - 2\bar{u}_1 = 0; \quad (9.11)$$

The equations of the form more complicated than (9.5) are [118] [76]

$$u_{xy} = \left( \frac{u_y}{u - x} + \frac{u_y}{u - y} \right) u_x + \frac{u_y}{u - x} \sqrt{u_x}; \quad (9.12)$$

$$u_{xy} = 2 \left[ (u + Y)^2 + u_y + (u + Y) \sqrt{(u + Y)^2 + u_y} \right] \times \left[ \frac{\sqrt{u_x + u_y}}{u - x} - \frac{u_x}{\sqrt{(u + Y)^2 + u_y}} \right]; \quad (9.13)$$

where $Y = Y(y)$ is an arbitrary function, and (3.39).
9.3 Appendix 3. Integrable scalar evolution equations

Admissible point transformations

Let us describe point transformations we use in the classification of equations (2.8).

1) The transformations
\[ \tilde{u} = \phi(u); \]

2) the scalings
\[ \tilde{x} = ax, \quad \tilde{t} = a^n t; \]

3) the Galilean transformation
\[ \tilde{x} = x + ct; \]

4) If the function \( F \) is independent of \( u \), then the transformation
\[ \tilde{u} = u + c_1 x + c_2 t \]
is admissible;

5) if \( F(\lambda u, \lambda u_1, \ldots, \lambda u_{n-1}) = \lambda F(u, u_1, \ldots, u_{n-1}) \), then for arbitrary constants \( a \) and \( b \) the transformation
\[ \tilde{u} = u \exp(at + bx) \]
is applicable.

The equations related by the above transformations are called equivalent. It is important to note that the classification is purely algebraic. We are not interested in properties of the solutions of the studied equations such as being real and the functions and constants that are involved in the transformations can be complex. For instance, the equations \( u_t = u_3 - u_1^3 \) and \( u_t = u_3 + u_1^3 \) are regarded as equivalent.

\[ ^1 \text{Hereinafter, once transformation rules for some of the variables } t, x, \text{ or } u \text{ are not indicated in the formulas, this means that the corresponding variables are not changed.} \]
9.3.1 Third order equations

Theorem 9.2. [29, 2, 30] Up to transformations of the form 1)–5) each non-linear equation

\[ u_t = u_{xxx} + F(u, u_1, u_2). \]

possessing infinitely many higher symmetries belongs to the list

\[
\begin{align*}
  u_t & = u_{xxx} + uu_x, \\
  u_t & = u_{xxx} + u^2 u_x, \\
  u_t & = u_{xxx} + u^2, \\
  u_t & = u_{xxx} - \frac{1}{2} u_x^3 + (c_1 e^{2u} + c_2 e^{-2u}) u_x, \\
  u_t & = u_{xxx} - \frac{3u_x u_{xx}^2}{2(u_x^2 + 1)} + c_1 (u_x^2 + 1)^{3/2} + c_2 u_x^3, \\
  u_t & = u_{xxx} - \frac{3u_x^2}{2u_x} + \frac{3}{u_x} - \frac{3}{2} \varphi(u) u_x^3, \\
  u_t & = u_{xxx} - \frac{3u_x^2}{2(u_x^2 + 1)} - \frac{3}{2} \varphi(u) u_x (u_x^2 + 1), \\
  u_t & = u_{xxx} - \frac{3u_x^2}{2u_x}, \\
  u_t & = u_{xxx} - \frac{3u_x^2}{4u_x} + c_1 u_x^{3/2} + c_2 u_x^2, \\
  u_t & = u_{xxx} - \frac{3}{4} u_x^2 + 3 u_{xx} u_x^{-1} (\sqrt{u_x + 1} - u_x - 1) - 6 u^{-2} u_x (u_x + 1)^{3/2} + 3 u^{-2} u_x (u_x + 1) (u_x + 2), \\
  u_t & = u_{xxx} - \frac{3}{4} u_x^2 - \frac{3}{4} u_x (u_x + 1) \text{cosh} u + 3 \frac{u_x \sqrt{u_x + 1}}{\sinh u} - 6 \frac{u_x (u_x + 1)^{3/2} \text{cosh} u}{\sinh^2 u} + 3 \frac{u_x (u_x + 1)(u_x + 2)}{\sinh^2 u} + u_x^2 (u_x + 3), \\
  u_t & = u_{xxx} + 3 u_x^2 u_{xx} + 9 u u_x^2 + 3 u^3 u_x, \\
  u_t & = u_{xxx} + 3 u u_{xx} + 3 u_x^2 + 3 u^2 u_x.
\end{align*}
\]

(9.14)

Here, \( \varphi(u) \) is any solution of the equation \((\varphi')^2 = 4\varphi^3 - g_2\varphi - g_3\), and \( c_1, c_2, g_2, g_3 \) are arbitrary constants.
Remark 9.2. Quite often instead of equations (9.14) and (9.15) one considers point equivalent to them equations (2.6) and (2.5). If $P' \neq 0$, then one can make the transformation $u = f(v)$, where $(f')^2 = P(f)$, in equations (2.6) and (2.5). Then for $v$ we get equations (9.14) and (9.15), respectively.

9.3.2 Fifth order equations

Theorem 9.3. Suppose nonlinear equation

$$u_t = u_5 + F(u, u_x, u_2, u_3, u_4)$$

satisfies two conditions:

1) there exists an infinite sequence of higher symmetries

$$u_{t_i} = G_i(u, \ldots, u_{n_i}), \quad i = 1, 2, \ldots, \quad n_{i+1} > n_i > \cdots > 5; \quad (9.17)$$

2) there exist no symmetries (9.17) of orders $1 < n_i < 5$.

Then the equation is equivalent to one in the list
\[ u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1, \]
\[ u_t = u_5 + 5uu_3 + \frac{25}{2}u_1u_2 + 5u^2u_1, \]
\[ u_t = u_5 + 5u_1u_3 + \frac{5}{3}u_1^3, \]
\[ u_t = u_5 + 5u_1u_3 + \frac{15}{4}u_2^2 + \frac{5}{3}u_1^3, \]
\[ u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1, \]
\[ u_t = u_5 + 5(u_2 - u^2)u_3 - 5u_1u_2^2 + u_1^5, \]
\[ u_t = u_5 + 5(u_2 - u_1^2 + \lambda_1e^{2u} - \lambda_2^2e^{-4u})u_3 - 5u_1u_2^2 + 15(\lambda_1e^{2u} + 4\lambda_2^2e^{-4u})u_1u_2 \\
\quad + u_1^5 - 90\lambda_2^2e^{-4u}u_1^3 + 5(\lambda_1e^{2u} - \lambda_2^2e^{-4u})^2u_1, \]
\[ u_t = u_5 + 5(u_2 - u_1^2 - \lambda_1^2e^{2u} + \lambda_2e^{-u})u_3 - 5u_1u_2^2 - 15\lambda_1^2e^{2u}u_1u_2 \\
\quad + u_1^5 + 5(\lambda_1^2e^{2u} - \lambda_2e^{-u})^2u_1, \quad \lambda_2 \neq 0, \]
\[ u_t = u_5 - \frac{5uu_4}{u_1} + \frac{5u_2u_3}{u_1^2} + 5\left(\frac{\mu_1}{u_1} + \mu_2u_1^2\right)u_3 - 5\left(\frac{\mu_1}{u_1} + \mu_2u_1^2\right)u_2^2 \\
\quad - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2u_1^2 + \mu_2u_1^5, \]
\[ u_t = u_5 - \frac{5uu_4}{u_1} - \frac{15u_2^2}{4u_1} + \frac{65}{4}u_2^2u_3 + 5\left(\frac{\mu_1}{u_1} + \mu_2u_1^2\right)u_3 - \frac{135u_4^2}{16u_1^3} \\
\quad - 5\left(\frac{7\mu_1}{4u_1^2} - \frac{\mu_2u_1}{2}\right)u_2^2 - 5\frac{\mu_1^2}{u_1} + 5\mu_1\mu_2u_1^2 + \mu_2^2u_1^5, \]

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\[ u_t = u_5 - \frac{5}{2} \frac{u_2 u_4}{u_1} - \frac{5}{4} u_1^2 + \frac{5}{2} u_2 u_3 + \frac{5}{2} u_2 u_3 + 5 \frac{u_2 u_3}{2 \sqrt{u_1}} - 5(u_1 - 2 \mu u_1^{1/2} + \mu^2) u_3 - \frac{35 u_4^2}{16 u_1^2} \]
\[ - \frac{5}{3} \frac{u_3^3}{u_1^{3/2}} + 5 \left( \frac{3\mu^2}{4u_1} - \frac{\mu}{\sqrt{u_1}} + \frac{1}{4} \right) u_2^2 + \frac{5}{3} u_1^3 - 8\mu u_1^{5/2} + 15\mu^2 u_2^2 - \frac{40}{3} \mu^3 u_1^{3/2}, \]

\[ u_t = u_5 + \frac{5}{2} \frac{f - u_1}{f^2} u_2 u_4 + \frac{5}{4} \frac{2 f - u_1}{f^2} u_3^2 + 5 \mu (u_1 + f)^2 u_3 + \frac{5}{2} \frac{4 u_1^2 - 8 u_1 f + f^2}{f^4} u_2 u_3 + \]
\[ \frac{5}{16} \frac{2 - 9 u_1^3 + 18 u_1^2 f}{f^6} u_4^2 + \frac{5}{4} \frac{\mu}{f^2} u_2 + \frac{4}{f^2} f^2 (f u_1 + f^2 - 1), \]

\[ u_t = u_5 + \frac{5}{2} \frac{f - u_1}{f^2} u_2 u_4 + \frac{5}{4} \frac{2 f - u_1}{f^2} u_3^2 - 5 \omega (f^2 + u_1) u_3 + \frac{5}{4} \frac{4 u_1^2 - 8 u_1 f + f^2}{f^4} u_2 u_3 + \]
\[ \frac{5}{16} \frac{2 - 9 u_1^3 + 18 u_1^2 f}{f^6} u_4^2 + \frac{5}{4} \frac{\mu}{f^2} \frac{u_3^2 - 2 u_1^2 f - 11 u_1 f^2 - 2}{f^2} u_2^2 - \frac{5}{2} \omega' \left( u_1^3 - 2 u_1 f + 5 f^2 \right) u_1 u_2 \]
\[ + 5 \omega^2 u_1 f^2 (3 u_1 + f) (f - u_1), \]

\[ u_t = u_5 + \frac{5}{2} \frac{f - u_1}{f^2} u_2 u_4 + \frac{5}{4} \frac{2 f - u_1}{f^2} u_3^2 + \frac{5}{4} \frac{4 u_1^2 - 8 u_1 f + f^2}{f^4} u_2 u_3 + \]
\[ + \frac{5}{16} \frac{2 - 9 u_1^3 + 18 u_1^2 f}{f^6} u_4^2 + 5 \omega \frac{2 u_1^3 + 1 u_1^2 f - 2 u_1 f^2 + 1}{f^2} u_2^2 \]
\[ - 10 \omega u_3 (3 u_1 f + 2 u_1^2 + 2 f^2) - 10 \omega' (2 f^2 + u_1 f + u_1^2) u_1 u_2 \]
\[ + 20 \omega^2 u_1 (u_1^3 - 1) (u_1 + 2 f), \]

\[ u_t = u_5 + \frac{5}{2} \frac{f - u_1}{f^2} u_2 u_4 + \frac{5}{4} \frac{2 f - u_1}{f^2} u_3^2 - \frac{5}{4} \frac{4 f^2 + u_1^2}{\omega^2} u_3 \]
\[ + \frac{5}{4} \frac{4 u_1^2 - 8 u_1 f + f^2}{f^4} u_2 u_3 + \frac{5}{16} \frac{2 - 9 u_1^3 + 18 u_1^2 f}{f^6} u_4^2 \]
\[ - 10 \omega (3 u_1 f + 2 u_1^2 + 2 f^2) u_3 - \frac{5}{4} \frac{11 u_1 f^2 + 2 u_1^2 f + 2 - 5 u_1^3}{\omega^2 f^2} u_2^2 \]
\[ + 5 \omega \frac{2 u_1^3 + u_1^2 f - 2 u_1 f^2 + 1}{f^2} u_2^2 + 5 c \omega' u_1^3 + 5 f^2 - 2 u_1 f u_1 u_2 \]
\[ - 10 \omega' (2 f^2 + u_1 f + u_1^2) u_1 u_2 + 20 \omega^2 u_1 (u_1^3 - 1) (u_1 + 2 f) \]
\[ + 40 \frac{c u_1 f^3 (3 u_1 + f)}{\omega} + 5 \frac{c u_1 f^2 (3 u_1 + f) (f - u_1)}{\omega^4}, \quad c \neq 0. \]

Here, \( \lambda_1, \lambda_2, \mu, \mu_1, \mu_2, \) and \( c \) are parameters, the function \( f(u_1) \) solves the algebraic equation
\[ (f + u_1)^2 (2 f - u_1) + 1 = 0, \quad (9.18) \]

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and \( \omega(u) \) is any nonconstant solution to the differential equation
\[
\omega'^2 = 4 \omega^3 + c.
\] (9.19)

### 9.4 Appendix 4. Systems of two equations

**Theorem 9.4.** [3, 46] Any nonlinear nontriangular system \((2.12)\), having a symmetry \((2.11)\), up to scalings of \(t, x, u, v\), shifts of \(u\) and \(v\), and the involution
\[u \leftrightarrow v, \quad t \leftrightarrow -t\]

belongs to the following list:

\[
\begin{align*}
\{ & u_t = u_{xx} + (u + v)u_x + uv_x, \\
& v_t = -v_{xx} + (u + v)v_x + vu_x, \\
& u_t = u_{xx} - 2(u + v)u_x - 2uv_x + 2u^2v + 2uv^2 + \alpha u + \beta v + \gamma, \\
& v_t = -v_{xx} + 2(u + v)v_x + 2uv_x - 2u^2v - 2uv^2 - \alpha u - \beta v - \gamma, \\
& u_t = u_{xx} + vu_x + uv_x, \\
& v_t = -v_{xx} + vu_x + u_x,
\}
\]

\[
\begin{align*}
\{ & u_t = u_{xx} + 2vu_x + 2uv_x + 2u^2 + \alpha u + \beta v + \gamma, \\
& v_t = -v_{xx} - 2v^2v_x - u_x,
\}
\]

\[
\begin{align*}
\{ & u_t = u_{xx} + u_x + (u + v)^2 + \beta(u + v) + \gamma, \\
& v_t = -v_{xx} + u_x - (u + v)^2 - \beta(u + v) - \gamma,
\}
\]

\[
\begin{align*}
\{ & u_t = u_{xx} + (u + v)u_x + 4\alpha v_x + \alpha(u + v)^2 + \beta(u + v) + \gamma, \\
& v_t = -v_{xx} + (u + v)v_x + 4\alpha u_x - \alpha(u + v)^2 - \beta(u + v) - \gamma,
\}
\]

\[
\begin{align*}
\{ & u_t = u_{xx} + 2u^2v_x + 2\beta uvu_x + \alpha(\beta - 2\alpha)u^3v^2 + \gamma u^2v + \delta u, \\
& v_t = -v_{xx} + 2\alpha u^2u_x + 2\beta uvu_x - \alpha(\beta - 2\alpha)u^2v^3 - \gamma uv^2 - \delta v,
\}
\]

\[
\begin{align*}
\{ & u_t = u_{xx} + 2uvu_x + (\alpha + u^2)v_x, \\
& v_t = -v_{xx} + 2uvu_x + (\beta + v^2)u_x,
\}
\]

\[
\begin{align*}
\{ & u_t = u_{xx} + 2\alpha uvu_x + 2\alpha u^2v_x - \alpha \beta u^4v^2 + \gamma u, \\
& v_t = -v_{xx} + 2\beta v^2u_x + 2\beta uvu_x + \alpha \beta u^2v^3 - \gamma v,
\}
\]
Here, \( \alpha, \beta, \gamma, \delta \) are arbitrary constants. We omitted the term \((cu_x, cv_x)^t\) on the right-hand sides of all systems. It is a Lie symmetry, corresponding to the invariance of our classification problem with respect to the shift of \( x \).
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