A counterexample to the conjecture:
Let $S$ be a singular inner function. Then $z \cdot S$ is onto $U$.

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Abstract
In this paper we give a counterexample to the conjecture: Let $S \in S\text{Inn}$. Then $z \cdot S$ is onto $U$.

1 The counterexample
We will generate recursively two infinite sequences. One sequence will contain only singular inner functions in $U$. The other sequence will contain only inner functions which are not singular inner functions. The functions which comprise the second sequence will each have exactly one simple zero. This zero will be located at the origin. We start with an arbitrary $S_0 \in S\text{Inn}$. Then $z \cdot S_0 \in \text{Inn} - S\text{Inn}$. Next:

$$S_1 = \exp \left( -\frac{1 + zS_0}{1 - zS_0} \right) \in S\text{Inn}.$$  

Then we have:

$$z \cdot S_1 = z \cdot \exp \left( -\frac{1 + zS_0}{1 - zS_0} \right) \in \text{Inn} - S\text{Inn}.$$  

Next:

$$S_2 = \exp \left( -\frac{1 + zS_1}{1 - zS_1} \right) = \exp \left( -\frac{1 + z \exp \left( -\frac{1 + zS_0}{1 - zS_0} \right)}{1 - z \exp \left( -\frac{1 + zS_0}{1 - zS_0} \right)} \right) \in S\text{Inn} \ldots ,$$
Our first sequence is \( \{S_n\}_{n=0}^{\infty} \) and our second sequence is \( \{z \cdot S_n\}_{n=0}^{\infty} \). The recursion that generates the first sequence (of the singular inner functions) is given by:

\[
S_{n+1} = \exp\left(-\frac{1 + z \cdot S_n}{1 - z \cdot S_n}\right), \quad n \in \mathbb{Z}_{\geq 0}.
\]

\( \{S_n\}_{n=0}^{\infty} \) (as a set of functions) is a normal family (Montel, 1906). Hence this family has at least one finite accumulation point, say \( S = \lim_{k \to \infty} S_{n_k} \).

We obtain a fixed-point equation for \( S \):

\[
S = \exp\left(-\frac{1 + z \cdot S}{1 - z \cdot S}\right).
\]

The normal family argument implies a local uniform convergence of \( \{S_{n_k}\}_{k=1}^{\infty} \) on \( U \). We conclude (Hurwitz Theorem) \( S \in SInn \). \( S \) is our candidate for the counterexample.

**Remark 1.1.** We will show later that

\[
\exp\left(-\frac{1 + z \cdot t}{1 - z \cdot t}\right)
\]

is a contraction map with respect to \( t \). In fact the absolute value of its \( t \)-derivative (for any fixed \( z \in U \)) is considerately smaller than one (smaller or equal to \( 4 \cdot e^{-2} \)). Hence by the result in the thesis of S. Banach

\[
\exp\left(-\frac{1 + z \cdot t}{1 - z \cdot t}\right)
\]

has a unique fixed-point \( t = S \).

**Theorem 1.2.** There exists a singular inner function \( S \in SInn \) such that the inner function \( z \cdot S \in Inn - SInn \) is injective and not surjective (onto \( U \)).

**Proof.**

We will consider the function \( S \) that was constructed above. Let us define the following holomorphic function which has the domain of definition which equals to the complex plane punctured at \( z = 1 \):

\[
f : \mathbb{C} - \{1\} \to \mathbb{C}, \quad f(w) = w \exp\left(\frac{1 + w}{1 - w}\right).
\]

Then \( f(z \cdot S(z)) = z \). For

\[
zS \exp\left(\frac{1 + zS}{1 - zS}\right) = z.
\]
This shows that the function $w(z) = z \cdot S(z)$ is injective. We use the fact that $f$ is a left inverse of $w$, i.e. $f(w(z)) = z$. Suppose that $w(z_1) = w(z_2)$. Then we have $z_1 = f(w(z_1)) = f(w(z_2)) = z_2$. On the other hand $w : U \to U$ is not a surjection (onto $U$). For if $w : U \to U$ were a surjection (onto $U$), then we could conclude that $w(z) = z \cdot S(z) \in \text{Aut}(U)$. This means that there is a $\theta \in [0, 2\pi)$ and an $a \in U$ so that

$$w(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z}.$$ 

Since $w(a) = 0 = a \cdot S(a)$, we conclude that $a = 0$. So $z \cdot S(z) = e^{i\theta} \cdot z$. Hence $S(z) \equiv e^{i\theta}$, a constant. Using the fixed-point equation:

$$S = \exp \left( -\frac{1 + zS}{1 - zS} \right),$$

we deduce that:

$$e^{i\theta} = \exp \left( -\frac{1 + ze^{i\theta}}{1 - ze^{i\theta}} \right) \quad \forall z \in U \text{ for that fixed } \theta \in [0, 2\pi).$$

We plug into this equation $z = 0$, and obtain $e^{i\theta} = e^{-1}$. This is impossible since $|e^{i\theta}| = 1 > e^{-1}$. 

**Remark 1.3.** The function $w(z) = z \cdot S(z) : U \to U$ has a very interesting geometry. We list some of the properties of the mapping $w$:

1. $w$ is injective (hence a conformal mapping of $U$ onto the image of $w$, $\text{Im} \ w$).
2. $w$ is not surjective, i.e. $\text{Im} \ w \subset U$.
3. $w(0) = 0$ since $w(z) = z \cdot S(z)$.
4. $w \in \text{Imn} - S\text{Inn}$, so $\lim_{r \to 1-} w(re^{i\phi}) := w(e^{i\phi})$ exists almost everywhere on $0 \leq \phi < 2\pi$ with respect to the Lebesgue measure on $[0, 2\pi)$ and $\lim_{r \to 1-} |w(re^{i\phi})| = |w(e^{i\phi})| = 1$.
5. The image of $w$, $\text{Im} \ w$, is a simply-connected sub-domain of $U$. This follows because by (1) $w$ is a conformal mapping $U \to_{\text{onto}} \text{Im} \ w$.
6. By (5) and (2) we have $U - \text{Im} \ w = \bigcup_{i \in I} C_i$, where $I \neq \emptyset$ and where $\forall i \in I$, $C_i$ is a continuum (contains at least two points) that borders $\mathbb{T}$ exactly on an arc $(U - C_i)$ is simply-connected and if $i \neq j$ then $(U \cap C_i) \cap (U \cap C_j) = \emptyset$ $(i, j \in I)$.
7. Each point of $C_i$ is a cluster point of $w$ at a boundary point $p \in \mathbb{T}$, for which $\lim_{r \to 1-} w(rp)$ does not exist, or if it exists, or if it exists, then
\[ \lim_{r \to 1^-} |w(rp)| < 1. \]

(8) Since \( w \in \text{Inn} \), the aggregate of all the points \( p \in T \) we described in (7) has Lebesgue measure zero on \( T \). We note that \( \mathcal{M}(\{p \in T \text{ of (7)}\}) = 0 \) (\( \mathcal{M} \) is the Lebesgue measure), but \( |C_i| = 2^{80} \). This is not a contradiction, of course.

(9) By (1), \( w \) is a conformal mapping defined on \( U \). A theorem of C. Caratheodory asserts that if the boundary \( \partial (\text{Im} w) \) were a Jordan curve, then \( w \) had an extension to \( \partial U \) so that \( U \) were homeomorphic to \( \text{Im} w \). This can not happen here. Hence \( \partial (\text{Im} w) \) is not a Jordan curve.

Remark 1.4. Let us consider the function

\[ h(t) = \exp\left(\frac{-1 + z \cdot t}{1 - z \cdot t}\right) \text{ for a fixed } |z| < 1, \]

and in the unit disc \( |t| < 1 \). The derivative of \( h(t) \) with respect to \( t \) is:

\[ h'(t) = \frac{\partial}{\partial t} \exp\left(\frac{-1 + z \cdot t}{1 - z \cdot t}\right) = -\frac{2z}{(1 - z \cdot t)^2} \cdot \exp\left(\frac{-1 + z \cdot t}{1 - z \cdot t}\right). \]

One can deduce that there is a constant \( c, 0 < c < 1 \), such that

\[ \left| \frac{\partial}{\partial t} \exp\left(\frac{-1 + z \cdot t}{1 - z \cdot t}\right) \right| \leq c < 1. \]

Thus \( h(t) \) is a contraction.

2 Historical background and more conclusions

It will be convenient to recall notions and results from [5]. We start with:

**Definition 89.0.193** ([5], page 517) An inner function \( f(z) \) is called a refined inner function (an R-inner function) if \( \lim_{r \to 1^-} |f(re^{i\theta})| = 1 \) for all \( e^{i\theta} \) on \( |z| = 1 \), except possibly for a set of capacity zero on \( |z| = 1 \). We will freely use the induced notions: R-singular inner function, R-continuous inner function, R-Blaschke product and an R-Blaschke sequence (the zero set of an R-Blaschke product).

The reason for making this definition was a use that was made in [5], page 520, in a result of A. J. Lohwater. [4]. This result is quoted in the book [2]. Here is that result:
Theorem 5.14 ([2], page 109) Let $w = f(z)$ be analytic and bounded, $|f(z)| < 1$ in $|z| < 1$, and let the radial limit of the modulus $|f(re^{i\theta})|$ be 1 for all $e^{i\theta}$ on $|z| = 1$, except possibly for a set of capacity zero on $|z| = 1$. Then unless $f(z)$ reduces to a finite Blaschke product or to a constant, every value of $|w| < 1$ is assumed infinitely often by $f(z)$ with at most one exception.

Using this theorem it was proved in [5], page 520 that:

**Theorem 89.0.199** ([5], page 520) There is an $R$-Blaschke sequence which is not the zero set (multiplicities are taken into account) of any function of the form $S(z) - \alpha$, where $S(z)$ is an $R$-singular inner function and $\alpha \in \mathbb{U} - \{0\}$.

This last theorem is equivalent to:

**Proposition 95.0.225** ([5], page 568) Let $T(z)$ be an $R$-singular inner function. Then $z \cdot T(z)$ is a $\mathbb{U} \rightarrow \mathbb{U}$ surjection.

This last result, makes it natural to ask if the condition on $T(z)$ of being an $R$-singular inner function is necessary to conclude that $z \cdot T(z)$ is a $\mathbb{U} \rightarrow \mathbb{U}$ surjection. Our Theorem 1.2 implies that if $T(z)$ is only assumed to be an inner function then $z \cdot T(z)$ need not be a surjection $\mathbb{U} \rightarrow \mathbb{U}$. So some condition is indeed necessary.

**Remark 2.1.** It is important to say that Proposition 95.0.227 in [5], is faulty. The author is grateful to the reviewer of [5], which found an error in the proof given in [5]. The reviewer which did an excellent job is Professor R. Mortini. Thus Proposition 95.0.227 will have to be erased in a later edition of [5].

We may add now the following:

**Corollary 2.2.** The singular inner function $S(z)$, that was constructed within the proof of Theorem 1.2 above, is such that the exceptional set on $T$ of those $e^{i\theta}$ for which the modulus $|S(re^{i\theta})|$ does not tend to 1 when $r \rightarrow 1^-$, has a positive capacity.

The author found out recently that a very similar notion to the notion of R-inner etc... was in fact used (in another context) many years before by A. Beurling. He used it in [1]. We refer the reader to page 31 of the book [3]:
Theorem 3.2.1 (Beurling's theorem) Let $f$ be a function in $\mathcal{D}$, the Dirichlet space. Then there exists $E \subset \mathbb{T}$ with outer capacity (corresponding to the logarithmic capacity) $c^*(E) = 0$ such that, if $\xi \in \mathbb{T} - E$, then $f^*(\xi) := \lim_{r \to 1} f(r\xi)$ exists, and $f(z) \to f^*(\xi)$ as $z \to \xi$ inside each region $|z - \xi| < \kappa(1 - |z|)$.

"A property is said to hold quasi-everywhere (q.e.) on $\mathbb{T}$, if it holds everywhere on $\mathbb{T} - E$ where $c^*(E) = 0$. Thus Beurling’s theorem can be summarized by saying that each $f \in \mathcal{D}$ has non-tangential limits quasi-everywhere on $\mathbb{T}$.”

Thus it might have been better to use ”quasi-inner” after Beurling, instead of ”refined-inner” or ”R-inner”.

References

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