The coordinate transformation and the exact solutions of the Schrödinger equation with position-dependent effective mass

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Abstract

Using the coordinate transformation method, we solve the one-dimensional Schrödinger equation with position-dependent mass(PDM). The explicit expressions for the potentials, energy eigenvalues and eigenfunctions of the systems are given. The eigenfunctions can be expressed in terms of the Jacobi, Hermite and generalized Laguerre polynomials. All potentials for these solvable systems have an extra term \( V_m \) which produced from the dependence of mass on the coordinate, compared with that for the systems of constant mass. The properties of \( V_m \) for several mass functions are discussed.

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I. INTRODUCTION

In recent years, the study of quantum systems with position-dependent mass (PDM) has become one of the active subjects [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. These systems have wide applications in condensed matter physics, nuclear physics, semiconductor theory and other related fields. In the theoretical researches, many methods developed in the study of systems with constant mass have been generalized to the systems with PDM and a number of interesting results have been produced. Of those researches, the searching of exact solutions of the quantum wave equations (Schrödinger equation, Klein-Gordan equation and Dirac equation) for these systems is an important aspect. The factorization method [21], operator methods [22], coordinate transformation methods [23, 24, 25, 26], supersymmetric quantum mechanics [24, 27, 28], Lie Algebra approach [29, 30, 31], path integral approach [32] and so on that exactly solve the quantum wave equations with constant mass have been extended to the systems with PDM.

In the present work, we will use the coordinate transformation method to the PDM Schrödinger equations and obtain their solutions for several potentials. There are two kinds of coordinate transformations. The first one connects two different solvable potentials such as the Coulomb potential and harmonic oscillator, Rosen-Morse and generalized Scarf potentials [26]. The second one transforms the Schrödinger equation into the second order differential equation which has solutions of the special functions such as hypergeometric functions and confluent hypergeometric functions, and so may provide systematical studies to the exact solutions of the Schrödinger equation. When combined with the shape invariance, the coordinate transformation method provides an important way to classify the solvable potentials [25, 26, 28]. We will discuss the coordinate transformation method in the second sense in this paper.

The paper is organized as follows. In section II the coordinate transformation will be introduced and exact solutions of the one-dimensional Schrödinger equation with PDM for several potentials will be given. The eigenfunctions of these systems can be expressed in terms of Jacobi polynomials, Hermite polynomials and generalized Laguerre polynomials. In section III the effective potentials produced from the dependence of mass on the position of the particle are given and their properties are discussed for several mass functions. In the last section, some remarks and discussions will be made.
II. COORDINATE TRANSFORMATION, PDM SCHRODINGER EQUATION AND THEIR SOLUTIONS

When the mass of a particle depends on its position, the mass and momentum operator no longer commute, so there are several ways to define the kinetic energy operator of the quantum system [1]. In this paper, we will adopt the form of the kinetic energy introduced by BenDaniel and Duke [2], and the Hamiltonian with the position-dependent effective mass $M(\vec{r})$ and potential energy $V(\vec{r})$ reads

$$H = \vec{P} \cdot \frac{1}{2M(\vec{r})} \vec{P} + V(\vec{r}) = -\frac{\hbar^2}{2m_0} \left[ \nabla \cdot \frac{1}{m(\vec{r})} \nabla \right] + V(\vec{r}),$$ (1)

where $m_0$ is a constant mass and $m(\vec{r})$ is a dimensionless position-dependent mass. Using natural units ($m_0 = \hbar = 1$) and only considering one-dimensional system, we get the following Hamiltonian from Eq. (1)

$$H = -\frac{1}{2} \left[ \frac{d}{dx} \frac{1}{m(x)} \frac{d}{dx} \right] + V(x) = -\frac{1}{2m} \frac{d^2}{dx^2} + \frac{m'}{2m^2} \frac{d}{dx} + V(x),$$ (2)

where $m'$ denotes the first derivative of $m(x)$ with respect to the coordinate $x$. The PDM Schrödinger equation corresponding to the Hamiltonian (2) reads

$$\frac{d^2 \psi(x)}{dx^2} - \frac{m' \psi(x)}{m} + 2m \left[ E - V(x) \right] \psi(x) = 0.$$ (3)

Now, we make the following coordinate transformation to the eigenfunction $\psi(x)$ in Eq. (3)

$$\psi(x) = f(x)F(g(x)),$$ (4)

where $F(g)$ is some special function and satisfies the following second order differential equation

$$\frac{d^2 F}{dg^2} + Q(g) \frac{dF}{dg} + R(g) F(g) = 0.$$ (5)

$Q(g)$ and $R(g)$ in Eq. (5) are determined by the types of special functions. In this paper, we will consider three types of special functions. Substituting Eq. (1) into Eq. (3), one gets the equation

$$\frac{d^2 F}{dg^2} + \left[ \frac{g''}{(g')^2} + \frac{2 f'}{fg'} - \frac{m'}{mg'} \right] \frac{dF}{dg} + \left[ \frac{f''}{(f(g'))^2} - \frac{m' f'}{m (f(g'))^2} + \frac{2m \left[ E - V(x) \right]}{(g')^2} \right] F = 0.$$ (6)

On comparing Eq. (5) with Eq. (6), we may have the following relations

$$Q(g) = \frac{g''}{(g')^2} + \frac{2 f'}{fg'} - \frac{m'}{mg'},$$ (7)
\[ R(g) = \frac{f''}{f(g')^2} = \frac{m'f'}{mf(g')^2} + \frac{2m [E - V(x)]}{(g')^2}. \]  

(8)

From Eqs. (7) and (8), we have

\[ \frac{f'}{f} = \frac{1}{2} \left( Qg' - \frac{g''}{g} + \frac{m'}{m} \right), \]  

(9)

\[ E - V(x) = \frac{(g')^2}{2m} \left[ R(g) - \frac{1}{2} \frac{dQ}{dg} - \frac{1}{4} Q^2 \right] + \frac{1}{4m} [G(g') - G(m)], \]  

(10)

where \( G(z) = \frac{z''}{2} - \frac{3}{2} \left( \frac{z'}{2} \right)^2 \). On making integration to Eq. (9), we get the transformation function \( f(x) \) to be of the form

\[ f(x) = \sqrt{\frac{m}{g'}} \exp \left( \frac{1}{2} \int_{g(x)}^{g} Q(g) dg \right), \]  

(11)

which has an extra factor \( \sqrt{m} \) compared with that for the constant mass \([25]\).

Now, it follows that the PDM Schrödinger equation can be exactly solved if the forms of functions \( Q(g) \) and \( R(g) \) (or the types of the special functions \( F(g) \)) are given for a mass \( m(x) \) that is a function of space coordinate \( x \), and if the right hand side of Eq. (10) can be divided into two parts: one depends on the quantum number \( n \), but is independent of \( x \), while another part depends on \( x \) rather than the quantum number \( n \). For the above division, the former is the eigenvalue \( E \) of the system and the latter is the corresponding potential \( V(x) \). Such system is exactly solvable with eigenfunction \( \psi(x) \), energy eigenvalue \( E \) and potential \( V(x) \).

In the following sections, we will take \( F(g) \) to be of the Jacobi, Hermite and generalized Laguerre polynomials, respectively. And for the convenience, we introduce the auxiliary function

\[ \mu(x) = \int_{x}^{x} \sqrt{m(x)} dx. \]  

(12)

\( G(g') \) in Eq.(10) can be rewritten into the following form in terms of \( \mu(x) \)

\[ G(g') = G(\mu') + (\mu')^2 G \left( \frac{dg}{d\mu} \right), \]  

(13)

where

\[ G \left( \frac{dg}{d\mu} \right) = \frac{d^3g}{d\mu^3} / \left( \frac{dg}{d\mu} \right) - \frac{3}{2} \left[ \frac{d^2g}{d\mu^2} / \left( \frac{dg}{d\mu} \right) \right]^2, \]  

(14)

which reduces to \( G(g') \) for the constant mass. Eqs. (10) and (13) show that there are a new term \( \frac{1}{4m}(G(\mu') - G(m)) \) and an extra factor \( \frac{1}{m} \) in each term compared with that for the
constant system. When \( m \) depends on the coordinate \( x \), the term \( \frac{1}{4m}(G(m') - G(m)) \) will be grouped into the potential \( V(x) \) of the system. In this sense, the PDM induces effective interaction between particles in the system, which is consistent with the concept of effective mass in the condensed matter physics and other related fields. We will denote \( V_m \) in the following sections as

\[
V_m = \frac{1}{4m}(G(m) - G(m')) = \frac{1}{8m} \left[ m'' - \frac{7}{4} \left( \frac{m'}{m} \right)^2 \right],
\]

which is attributed to the dependence of the mass \( m \) on \( x \).

A. Jacobi polynomial and solvable potentials

When we choose \( F(g) \) to be the Jacobi polynomial \( P_n^{(\alpha, \beta)}(g) \), the corresponding differential equation for \( P_n^{(\alpha, \beta)}(g) \) gives the expressions of \( Q(g), R(g) \) in Eq. (16)

\[
Q(g) = \frac{-\alpha + \beta}{1 - g^2} - \frac{(2 + \alpha + \beta)g}{1 - g^2},
\]

\[
R(g) = \frac{n(1 + \alpha + \beta + n)}{1 - g^2},
\]

where \( \alpha \) and \( \beta \) are parameters, \( n = 0, 1, 2, \cdots \). Substituting Eq. (16) into Eq. (10), we get

\[
E - V(x) = \frac{1}{4m}[G(g') - G(m)] + n(1 + \alpha + \beta + n) \frac{g^2}{2m(1 - g^2)}
\]

\[
+ \frac{2(2 + \alpha + \beta - (\alpha - \beta)^2)}{8m(1 - g^2)^2} \frac{g^2}{4m(1 - g^2)^2}
\]

\[
+ (\beta + \alpha)(\beta - \alpha) \frac{g^2 g'^2}{4m(1 - g^2)^2} + [1 - (1 + \alpha + \beta)^2] \frac{g^2 g'^2}{8m(1 - g^2)^2}.
\]

Now we will chose appropriate \( g(x) \) to make RHS of Eq. (17) have a term that is independent of \( x \), but may contain \( n \). Once \( g(x) \) is determined, then we will get solvable potentials \( V(x) \) and its corresponding energy eigenvalue \( E \) from Eq. (17). Inserting \( g(x) \) into Eq. (11), we will obtain the explicit expression of \( f(x) \), and so the eigenfunction of the system is given by \( \psi(x) = f(x)P_n^{(\alpha, \beta)}(g) \) for the above potential \( V(x) \). Similar to those for the systems with constant mass, there are two cases that satisfy the above requirements, and each has several different functions \( g(x) \).

**Case 1:** \( g(x) \) satisfies the differential equation

\[
\frac{g'^2}{(1 - g^2)m(x)} = C,
\]

5
where $C$ is a constant. In this case, one has

$$G\left(\frac{dg}{d\mu}\right) = -C \left[ 1 + \frac{3g^2}{2(1 - g^2)} \right].$$  \hspace{1cm} (19)$$

Each solution $g(x)$ to Eq. (18) corresponds to an exactly solvable system. According to the procedures given above, we can get its potential, eigenfunction and energy eigenvalues. We will not give the details of calculations and just list the results. Note that the energy eigenvalues are chosen such that $E_{n=0} = 0$. The definitions of the parameters $s, \lambda$ are made so that the potential, eigenfunctions and energy eigenvalues can reduce to those for the system with constant mass when $m(x) = 1$, respectively \cite{24, 25}.

(i) $g(x) = i \sinh(a\mu(x)), C = -a^2$

$$E_n = \frac{1}{2} s^2 a^2 - \frac{1}{2} a^2(s - n)^2,$$

$$V(x) = \frac{1}{2} s^2 a^2 + \frac{1}{2} a^2(\lambda^2 - s^2 - s)\text{sech}^2(a\mu(x))$$

$$- \frac{1}{2} a^2 \lambda (2s + 1) \tanh(a\mu(x)) \text{sech}(a\mu(x)) + V_m,$$  \hspace{1cm} (20b)$$

$$\psi(x) = [m(x)]^{\frac{1}{4}} (\cosh(a\mu))^{-s} \exp \{(-\lambda \tan^{-1}[\sinh(a\mu(x))]\}$$

$$\times P_n^{(-i\lambda-s-\frac{1}{2},i\lambda-s-\frac{1}{2})}(i \sinh(a\mu(x))),$$  \hspace{1cm} (20c)$$

where $s = -\frac{1}{2}(\alpha + \beta + 1), \lambda = -\frac{1}{2}i(\beta - \alpha)$.

(ii) $g(x) = \cosh(a\mu(x)), C = -a^2$

$$E_n = \frac{1}{2} s^2 a^2 - \frac{1}{2} a^2(s - n)^2,$$

$$V(x) = \frac{1}{2} s^2 a^2 + \frac{1}{2} a^2(\lambda^2 + s^2 + s)\text{cosech}^2(a\mu(x))$$

$$- a^2 \lambda \left( s + \frac{1}{2} \right) \coth(a\mu(x)) \text{cosech}(a\mu(x)) + V_m,$$  \hspace{1cm} (21b)$$

$$\psi(x) = [m(x)]^{\frac{1}{4}} (\sinh(a\mu(x)))^{-s} \left( \frac{1 - \cosh(a\mu(x))}{1 + \cosh(a\mu(x))} \right)^{\frac{1}{2}}$$

$$\times P_n^{(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2})}(i \sinh(a\mu(x))),$$  \hspace{1cm} (21c)$$

where $s = -\frac{1}{2}(\alpha + \beta + 1), \lambda = -\frac{1}{2}(\beta - \alpha)$.

(iii) $g(x) = \cos(a\mu(x)), C = a^2$

$$E_n = -\frac{1}{2} s^2 a^2 + \frac{1}{2} a^2(s + n)^2,$$  \hspace{1cm} (22a)$$
\[ V(x) = - \frac{1}{2} s^2 a^2 + \frac{1}{2} a^2 (\lambda^2 + s^2 - s) \csc^2(a\mu(x)) \]
\[ - \frac{1}{2} a^2 \lambda (2s - 1) \cot(a\mu(x)) \csc(a\mu(x)) + V_m, \]  
(22b)

\[ \psi(x) = [m(x)]^{\frac{1}{4}} (\sinh(a\mu(x)))^{s} \left( \frac{1 + \cos(a\mu(x))}{1 - \cos(a\mu(x))} \right)^{\lambda} P_n^{(-\lambda + s - \frac{1}{2}, \lambda + s - \frac{1}{2})}(\cos(a\mu(x))), \]  
(22c)

where \( s = \frac{1}{2}(\alpha + \beta + 1), \lambda = \frac{1}{2}(\beta - \alpha). \)

(iv) \( g(x) = \sin[a\mu(x)], C = a^2 \)

\[ E_n = - \frac{1}{2} s^2 a^2 + \frac{1}{2} a^2 (s + n)^2, \]  
(23a)

\[ V(x) = - \frac{1}{2} s^2 a^2 + \frac{1}{2} a^2 (\lambda^2 + s^2 - s) \sec^2(a\mu(x)) \]
\[ - \frac{1}{2} \lambda a^2 (2s - 1) \sec(a\mu(x)) \tan(a\mu(x)) + V_m, \]  
(23b)

\[ \psi(x) = [m(x)]^{\frac{1}{4}} (\cos(a\mu(x)))^{s} \left( \frac{1 + \sin(a\mu(x))}{1 - \sin(a\mu(x))} \right)^{\lambda} P_n^{(-\lambda + s - \frac{1}{2}, \lambda + s - \frac{1}{2})}(\sin(a\mu(x))), \]  
(23c)

where \( s = \frac{1}{2}(\alpha + \beta + 1), \lambda = \frac{1}{2}(\beta - \alpha). \)

**Case 2:** \( g(x) \) satisfies the differential equation

\[ \frac{g'^2}{(1 - g^2)^2m(x)} = C, \]  
(24)

where \( C \) is a constant. Now, we have

\[ G \left( \frac{dg}{d\mu} \right) = -2C. \]  
(25)

We will consider four solutions of \( g \) to Eq. (24). The corresponding potential, eigenfunction and energy eigenvalue of the exactly solvable system for each \( g \) are listed as follows:

(i) \( g(x) = \tanh(a\mu(x)), C = a^2 \)

\[ E_n = \frac{1}{2} s^2 a^2 + \frac{\lambda^2}{2s^2} a^2 - \frac{1}{2} a^2 \left[ (s - n)^2 + \frac{\lambda^2}{(s - n)^2} \right], \]  
(26a)

\[ V(x) = \frac{1}{2} s^2 a^2 + \frac{\lambda^2}{2s^2} a^2 - \frac{1}{2} a^2 s(s + 1) \text{sech}^2(a\mu(x)) - \lambda a^2 \tanh(a\mu(x)) + V_m, \]  
(26b)
\[
\psi(x) = [m(x)]^{\frac{1}{4}}(\cosh(\alpha \mu(x)))^{s+n} \left(\frac{1 - \tanh(\alpha \mu(x))}{1 + \tanh(\alpha \mu(x))}\right)^{\frac{\lambda}{2}} P_n^{(\beta - \tilde{a}, s - n + \tilde{a})}(\tanh(\alpha \mu(x))),
\]
where
\[
\alpha = s - n + \tilde{a}, \quad \beta = s - n - \tilde{a}, \quad \tilde{a} = \frac{\lambda}{s - n}.
\]

(ii) \(g(x) = \coth(\alpha \mu(x)), \quad C = a^2\)
\[
E_n = \frac{1}{2}a^2s^2 + \frac{\lambda^2}{2s^2}a^2 + \frac{1}{2}a^2(s - 1) \csc^2(\alpha \mu(x)) - \lambda a^2 \cot(\alpha \mu(x)) + V_m,
\]
\[
\psi(x) = [m(x)]^{\frac{1}{4}}(\sinh(\alpha \mu(x)))^{n+s} \left(\coth(\alpha \mu(x)) - 1\right)^{\frac{\lambda}{2}} \coth(\alpha \mu(x))^{(s - n + \tilde{a}, s - n + \tilde{a})}(\coth(\alpha \mu(x))),
\]
where
\[
\alpha = -s - n + \tilde{a}, \quad \beta = -s - n - \tilde{a}, \quad \tilde{a} = \frac{\lambda}{s + n}.
\]

(iii) \(g(x) = -i \cot(\alpha \mu(x)), \quad C = -a^2\)
\[
E_n = -\frac{1}{2}a^2s^2 + \frac{\lambda^2}{2s^2}a^2 + \frac{1}{2}a^2(s - n)^2 + 1 \frac{a^2}{2} \left(\frac{\lambda^2}{s - n}\right)^2,
\]
\[
V(x) = -\frac{1}{2}a^2s^2 + \frac{\lambda^2}{2s^2}a^2 + \frac{1}{2}a^2s(s + 1) \csc^2(\alpha \mu(x)) - \lambda a^2 \cot(\alpha \mu(x)) + V_m,
\]
\[
\psi(x) = [m(x)]^{\frac{1}{4}}(\sin(\alpha \mu(x)))^{s-n} \exp[a\tilde{a}\mu(x)] P_n^{(s-n+i\tilde{a}, s-n-i\tilde{a})}(\alpha \mu(x)),
\]
where
\[
\alpha = s - n + i\tilde{a}, \quad \beta = s - n - i\tilde{a}, \quad \tilde{a} = \frac{\lambda}{s - n}.
\]

(iv) \(g(x) = -i \tan(\alpha \mu(x)), \quad C = -a^2\)
\[
E_n = -\frac{1}{2}a^2s^2 + \frac{\lambda^2}{2s^2}a^2 + \frac{1}{2}a^2(s - n)^2 + 1 \frac{a^2}{2} \left(\frac{\lambda^2}{s - n}\right)^2,
\]
\[
V(x) = -\frac{1}{2}a^2s^2 + \frac{\lambda^2}{2s^2}a^2 + \frac{1}{2}a^2s(s + 1) \sec^2(\alpha \mu(x)) - \lambda a^2 \tan(\alpha \mu(x)) + V_m.
\]
\( \psi(x) = [m(x)]^{\frac{1}{4}} \left( \cos(a\mu(x)) \right)^{n-s} \exp[-a\bar{\alpha} \mu(x)] P_n^{(s-n+i\bar{\alpha},s-n-i\bar{\alpha})}(-i \tan(a\mu(x))), \)

where

\[
\alpha = s - n + i\bar{\alpha}, \quad \beta = s - n - i\bar{\alpha}, \quad \bar{\alpha} = \frac{\lambda}{s - n}.
\]

It is obvious that all above results reduce to those for the systems with constant mass when \( m(x) = 1 \) and \( \mu(x) = \sqrt{2} \). From above explicit expressions for the potentials, eigenfunctions and energy eigenvalues of the solvable systems, we see that the energy eigenvalues are the same as those for the systems with constant mass, but eigenfunctions and potentials do not so when the mass of the particle depends on \( x \). The effects of PDM to eigenfunctions and potentials are twofold: the argument of function \( \mu(x) \) and an extra factor containing \( m(x) \) in the eigenfunctions or a new term \( V_m \) in the potentials. In Section III, we will discuss the properties of \( V_m \) for several mass functions \( m(x) \). The above facts show that PDM will make the classes of solvable potentials more general than those for the constant mass.

B. Hermite polynomial and solvable potentials

When \( F(g) \) in Eq. (5) is the Hermite polynomial, i.e. \( F(g) = H_n(g) \), then \( Q(g) \) and \( R(g) \) in Eq. (5) have the following forms

\[
Q(g) = -2g, \quad R(g) = 2n. \quad (n = 0, 1, 2, \ldots)
\]

With above \( Q(g) \), \( R(g) \) and Eq. (10), one has

\[
E - V(x) = \frac{(g')^2}{2m} \left( 2n + 1 - g^2 \right) + \frac{1}{4m} [G(g') - G(m)].
\]

There are two cases of conditions with which we can obtain the potentials and energy eigenvalues of the solvable systems from Eq. (33).

**Case 1:** If \( g(x) \) satisfies the equation

\[
\frac{(g')^2}{m} = \omega,
\]

with \( \omega > 0 \), we chose

\[
g(x) = \sqrt{\omega} \mu(x).
\]
Inserting Eq. (37) into Eq. (35), we get the eigenvalue and potential of the system, respectively

\[ E_n = n\omega, \]  
(38a)

\[ V(x) = -\frac{1}{2}\omega + \frac{1}{2}\omega^2 [\mu(x)]^2 + V_m. \]  
(38b)

From Eqs. (34), (37) and (11), the transformation function reads

\[ f(x) = [m(x)]^{\frac{1}{4}} \exp(-\frac{1}{2}g^2), \]  
(39)

so the corresponding eigenfunction of the system with potential (38b) is

\[ \psi_n(x) = [m(x)]^{\frac{1}{4}} \exp(-\frac{1}{2}g^2)H_n(g). \]  
(40)

It is obvious that (38a) is the same as that for the harmonic oscillator up to a constant term. When \( m(x) = 1 \), Eqs. (38b) and (40) reduce to the potential function and eigenfunction of the harmonic oscillator, respectively.

**Case 2:** If \( g(x) \) is the solution of the differential equation

\[ \left(\frac{g'}{m}\right)^2 g^2 = \frac{4\omega^2}{(2n+1)^2}. \]  
(41)

where \( \omega > 0 \), we have

\[ g(x) = \sqrt{\frac{4\omega}{2n+1}} [\mu(x)]^{\frac{1}{2}}. \]  
(42)

Putting Eq. (42) into Eq. (35), we get

\[ E_n = 2\omega^2 - \frac{2\omega^2}{(2n+1)^2}, \]  
(43a)

\[ V(x) = 2\omega^2 - \frac{\omega}{2} [\mu(x)]^{-1} - \frac{3}{32} [\mu(x)]^{-2} + V_m. \]  
(43b)

Substituting Eq. (42) into Eq. (11), one has the transformation function of the form

\[ f(x) = [m(x)\mu(x)]^{\frac{1}{4}} \exp(-\frac{1}{2}g^2), \]  
(44)

and the corresponding eigenfunction of the system reads

\[ \psi_n(x) = [m(x)\mu(x)]^{\frac{1}{4}} \exp(-\frac{1}{2}g^2)H_n(g). \]  
(45)
C. Generalized Laguerre polynomial and solvable potentials

When \( F(g) \) in Eq. (5) is the generalized Laguerre polynomial, i.e. \( F(g) = L_n^\alpha(g) \), \( Q(g) \) and \( R(g) \) in Eq. (5) will take the forms 33, 34, respectively

\[
Q(g) = \frac{\alpha + 1}{g} - 1, \quad R(g) = \frac{n}{g}, \quad (n = 0, 1, 2, \ldots, \alpha \neq -1, -2, -3, \ldots)
\]  
(46)

Substituting Eq. (46) into Eq. (10), we obtain the relation

\[
E - V(x) = \frac{(g')^2}{4mg} (2n + \alpha + 1) + \frac{(g')^2}{2mg^2} \left[ \frac{(\alpha + 1)}{2} - \frac{(\alpha + 1)^2}{4} \right] - \frac{(g')^2}{8m} + \frac{1}{4m} [G(g') - G(m)].
\]  
(47)

**Case 1:** When \( g(x) \) satisfies the equation

\[
\frac{(g')^2}{mg} = 4\omega,
\]  
(48)

with \( \omega > 0 \), we have

\[
g(x) = \omega [\mu(x)]^2.
\]  
(49)

Inserting Eq. (49) into Eq. (47), we get the energy eigenvalues and potential of the system

\[
E_n = 2n\omega,
\]  
(50a)

\[
V(x) = - \left( l + \frac{3}{2} \right) \omega + \frac{1}{2} \omega^2 [\mu(x)]^2 + \frac{1}{2} \frac{(l + 1)}{[\mu(x)]^2} + V_m,
\]  
(50b)

where

\[
l = \alpha - \frac{1}{2}. \quad (l \neq \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots)
\]  
(51)

In this case, the transformation function is

\[
f(x) = [m(x)]^{\frac{1}{2}} [\mu(x)]^{l+1} \exp(-\frac{1}{2} \omega \mu(x)^2).
\]  
(52)

With Eqs. (4) and (52), we obtain the corresponding eigenfunction of the system

\[
\psi_n(x) = [m(x)]^{\frac{1}{2}} [\mu(x)]^{l+1} \exp(-\frac{1}{2} \omega \mu(x)^2) L_n^{(l+\frac{1}{2})}(\omega \mu(x)^2).
\]  
(53)

If \( l = 0, 1, 2, \ldots \), and \( l \) is viewed as the angular momentum quantum number, then Eq. (50a) is the energy eigenvalues for the three-dimensional harmonic oscillator 35. When \( m(x) = 1 \),
Eqs. (50b) and (53) reduce to the potential and eigenfunction for the three-dimensional isotropic harmonic oscillator, respectively.

**Case 2:** If \( g(x) \) satisfies the equation

\[
\frac{(g')^2}{mg^2} = a^2,
\]

with \( a \neq 0 \), we choose

\[
g(x) = \exp[-a\mu(x)].
\]

With Eqs. (54), (55) and (47), we obtain the energy eigenvalues, potential of the system

\[
E_n = \frac{1}{2}a^2 s^2 - \frac{1}{2}a^2 (s - n)^2,
\]

\[
V(x) = \frac{1}{2}a^2 s^2 + \frac{a^2}{8} \exp[-2a\mu(x)] - \frac{1}{4}a^2 \exp[-a\mu(x)] + V_m,
\]

where

\[
s = n + \frac{1}{2} \alpha. \quad (s \neq 0, \pm \frac{1}{2}, \cdots, \pm \frac{n}{2}, \cdots)
\]

From Eqs. (46), (55), (4) and (11), we have the eigenfunction of the system

\[
\psi_n(x) = [m(x)]^\frac{1}{4} \exp[(n - s)a\mu(x)] \exp \left[-\frac{1}{2}e^{a\mu(x)}\right] L_n^{(2s-2n)}(\exp(-a\mu(x))).
\]

It is follows that the expression (56a) is the energy eigenvalue for the Morse potential with zero angular momentum. When \( m(x) = 1 \) and with the appropriate parameters \( a, b \) and \( c \), then (56b) and (58) are Morse potential and eigenfunction with zero angular momentum, respectively.

**Case 3:** If \( g(x) \) is the solution to the differential equation

\[
\frac{(g')^2}{m} = 4\omega^2,
\]

where \( \omega > 0 \), we take

\[
g(x) = 2\omega\mu(x).
\]

If we make the replacements \( \omega = \frac{a}{n+l+1}, \alpha = 2l + 1 \) (\( l \neq -1, -\frac{3}{2}, -\frac{4}{2}, \cdots \)) and use Eqs. (60) and (47), we have

\[
E_n = \frac{a^2}{2(l+1)^2} - \frac{a^2}{2(n+l+1)^2},
\]

\[
V(x) = \frac{a^2}{2(l+1)^2} - \frac{a}{\mu(x)} + \frac{l(l+1)}{2\mu(x)^2} + V_m.
\]
In this case, the transformation function can be written as

\[ f(x) = [m(x)]^{\frac{1}{2}} (\mu(x))^{l+1} \exp \left[ -\frac{a}{n+l+1} \mu(x) \right], \quad (62) \]

so the eigenfunction of the system reads as

\[ \psi_n(x) = [m(x)]^{\frac{1}{2}} (\mu(x))^{l+1} \exp \left[ -\frac{a}{n+l+1} \mu(x) \right] L_n^{(2l+1)} \left( \frac{2a}{n+l+1} \mu(x) \right). \quad (63) \]

If \( a = Z \) (\( Z \) is the charge numbers of the particle), \( l = 0, 1, 2, \ldots \), and \( l \) is regarded as the angular momentum quantum number, then Eq. (61a) is just the energy eigenvalues for the three-dimensional Coulomb potential \( [35] \). When \( m(x) = 1 \), Eqs. (61b) and (63) reduce to the three-dimensional Coulomb potential and its eigenfunction, respectively.

### III. MASS FUNCTIONS AND POTENTIALS

In this section, we discuss the effective potentials \( V_m \) and their properties due to the dependence of mass on the coordinate for several mass functions.

**Example 1:** We take the effective mass function to be of the form

\[ m(x) = \left( \frac{b + x^2}{1 + x^2} \right)^2, \quad (64) \]

which has been used in many studies \([6, 11, 12]\). Substituting Eq. (64) into Eq. (11), we get

\[ \mu(x) = x + (b - 1) \arctan x. \quad (65) \]

With Eq. (15), the contribution to the potential from mass function is

\[ V_m = \frac{(b - 1)[3x^4 + 2(2 - b)x^2 - b]}{2(b + x^2)^4}. \quad (66) \]

It is seen that \( V_m = 0 \) when \( b = 1 \), which corresponds to the system of constant mass.

When \( 0 < b < 1 \) or \( 1 < b < 4 \), \( V_m \) has three extreme points

\[ x = 0, \quad x = \pm \left( b - 1 + \sqrt{\frac{1}{3}(2b^2 - 2b + 3)} \right)^{\frac{1}{2}}. \quad (67) \]

If \( b > 4 \), then there are five extreme points for \( V_m \), three of which has the same form as that in Eq. (67), the other two extreme points are

\[ x = \pm \left( b - 1 - \sqrt{\frac{1}{3}(2b^2 - 2b + 3)} \right)^{\frac{1}{2}}. \quad (68) \]
The characteristic curves for $V_m$ with different values of the parameter $b$ are depicted in Fig.1. It is seen that $V_m$ behaves like a barrier and it decrease as the parameter approaches to 1 from $b < 1$. While when $b > 1$, $V_m$ just looks like a well. In this sense, $V_m$ will bound the motion of the particle.

![Graphs showing characteristic curves for different values of $b$.](image)

**FIG. 1:** The potential $V_m$ for the mass function (64) and the different values of the parameter $b$: (a) $b = 0.5$(solid line), $b = 0.6$(dashed line), $b = 0.7$(dot-dashed line); (b) $b = 2.5$(solid line), $b = 3.0$(dashed line), $b = 3.5$(dot-dashed line); (c) $b = 9.0$(solid line), $b = 10.0$(dashed line), $b = 11.0$(dot-dashed line).

**Example 2:** The effective mass is \[9, 10, 20\]

$$m(x) = e^{-b|x|}, \quad (69)$$

where $b \geq 0$ to assure that mass is finite when $x \to \pm \infty$. Now, inserting Eq. (69) into (11), we get

$$\mu(x) = \begin{cases} 
-\frac{2}{b} e^{-\frac{x}{b}}, & (b \neq 0, \ x > 0) \\
\frac{2}{b} e^{\frac{x}{b}}, & (b \neq 0, \ x < 0) \\
x, & (b = 0)
\end{cases} \quad (70)$$

The extra interaction due to the mass is

$$V_m = -\frac{3}{32} b^2 e^{b|x|}, \quad (71)$$

which is monotonously changed as $|x|$ increases from 0. This $V_m$ has the typical characterization of Fig.2 for various values of the parameter $b$. It is a kind of barrier whose width decreases with increasing the parameter $b$. 
Example 3: The effective mass is of the form

$$m(x) = \frac{1}{b + x^2},$$  \hspace{1cm} (72)

where $b > 0$ to avoid its singularity. Now, we have

$$\mu(x) = \ln(x + \sqrt{b + x^2}).$$  \hspace{1cm} (73)

The effective potential $V_m$ is

$$V_m = -\frac{2b + x^2}{8(b + x^2)}. \hspace{1cm} (74)$$

This $V_m$ has the behavior of Fig.3 for various values of the parameter $b$. It is a typically potential well whose width increases as the parameter $b$ is increased.
**Example 4:** The effective mass is as follows\(^4, 5\)

\[ m(x) = 1 + \tanh(bx), \quad (75) \]

where \(a\) is a real parameter. Inserting Eq. \((75)\) into Eq. \((11)\), one has

\[
\mu(x) = \begin{cases} 
\frac{\sqrt{2}}{b} \tanh^{-1} \sqrt{\frac{1}{2} (1 + \tanh(bx))}, & (b \neq 0) \\
\frac{x}{b}, & (b = 0)
\end{cases} \quad (76)
\]

Similarly, we get the potential produced from the dependence of mass on \(x\)

\[ V_m = -\frac{1}{32} b^2 \text{sech}(a x) \left[ 7 \cosh(b x) + \sinh(b x) \right] \left[ \cosh(2b x) - \sinh(2b x) \right]. \quad (77) \]

The curves of \(V_m\) for various values of the parameter \(b\) are displayed in Fig.4. It can be seen that these \(V_m\) look like the semi-infinite potential barriers whose widths decrease as the parameter \(b\) increases.

![FIG. 4: The potential \(V_m\) for the mass function \((75)\) and the parameters \(b = 0.15\) (solid line), \(b = 0.2\) (dashed line), \(b = 0.3\) (dot-dashed line), respectively.](image)

With all above mass functions, we see that the dependence of the mass on the position of the particle will affect the behaviors of the system through two ways: the argument of the part in the potential that has the same form as that for the system with constant mass is \(\mu(x)\) instead of \(x\) and an extra term \(V_m\). These effects also increase the number of the solvable potentials for the same energy eigenvalue.

**IV. REMARKS AND DISCUSSIONS**

In this paper, we use the coordinate transformation method to study the exact solutions of the PDM Schrodinger equation for several potentials. The eigenstates of all these systems
with PDM can be expressed in terms of three kinds of special functions. We also give the explicit expressions of the potentials $V_m$ for several mass functions that are used in some physically interested problems and study their properties. All these results will reduce to those for the systems with constant mass (see, for example [25, 26, 31]) if we set the mass functions $m(x)$ to be constants in all equations in the above sections. It should be noted that there are two functions $m(x)$ and $g(x)$ in the PDM case, while only one function $g(x)$ is concerned in the cases of constant mass. So, the classes with the same form of the energy eigenvalues but with different potentials related by coordinate transformation are enlarged.

In our above discussions, we use the Hamiltonian of the symmetric form [1]. If we adopt the Hamiltonian

$$H = \frac{1}{4} \left( m^\alpha \vec{P} m^\beta \cdot \vec{P} m^\gamma + m^\gamma \vec{P} m^\beta \cdot \vec{P} m^\alpha \right) + V(\vec{r}),$$

with the condition $\alpha + \beta + \gamma = -1$, then all the results above also hold provided that we replace $V(x)$ by $V_{eff}$ in both Eq. [1] and other related relations, here

$$V_{eff}(x) = V(x) + \frac{1}{2} m'' - \frac{1}{2} m' \left( \alpha (\alpha + \beta + 1) + \beta + 1 \right) \frac{m''}{m^3},$$

$$m' = \frac{dm}{dx} \quad \text{and} \quad m'' = \frac{d^2 m}{dx^2}.$$  

This is so due to the fact that the Hamiltonian (78) can be rewritten as

$$H = -\frac{1}{2} \left[ \frac{d}{dr} \frac{1}{m(x)} \frac{d}{dr} \right] + V_{eff}(x),$$

for the one-dimensional system.

Also, for the three-dimensional systems with PDM and spherical symmetry, the solution of the system can be written as the product of angular and radial parts. The radial Schrodinger equation for the Hamiltonian [1] takes the form

$$-\frac{1}{2} \left[ \frac{d}{dr} \frac{1}{m} \frac{d}{dr} \right] \phi(r) + \left[ V(r) + \frac{1}{2m} \frac{l(l+1)}{r^2} - \frac{m'}{2m^2} \right] \phi(r) = E \phi(r),$$

where $m' = \frac{dm}{dr}$, $R(r) = \frac{\phi(r)}{r}$ is the radial wave function, $E$ and $l$ are the energy eigenvalue and angular momentum quantum number of the system, respectively. Eq. (81) has the same form with the Schrodinger equation for the Hamiltonian (80), so the results in this paper can also be applied to the spherically symmetrical systems with PDM upon some modifications.

We known that the special functions can have some generalizations, such as the q-deformed forms [34], so the corresponding differential equations have more general forms than Eq. (3). The coordinate transformation method can in principle apply to this generalized case and may give a more general classification to the solvable potentials.
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