On unbounded operators and applications \(\ast\)\(\dagger\)

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract
Assume that \(Au = f\), (1) is a solvable linear equation in a Hilbert space \(H\), \(A\) is a linear, closed, densely defined, unbounded operator in \(H\), which is not boundedly invertible, so problem (1) is ill-posed. It is proved that the closure of the operator \((A^*A + \alpha I)^{-1}A^*\), with the domain \(D(A^*)\), where \(\alpha > 0\) is a constant, is a linear bounded everywhere defined operator with norm \(\leq 1\). This result is applied to the variational problem
\[
F(u) := ||Au - f||^2 + \alpha||u||^2 = \min,
\]
where \(f\) is an arbitrary element of \(H\), not necessarily belonging to the range of \(A\). Variational regularization of problem (1) is constructed, and a discrepancy principle is proved.

1. Introduction
The main results of this paper are formulated as Theorems 1 and 2 and proved in Sections 1 and 3, respectively. In Section 1 we formulate Theorem 1 which deals with a linear, unbounded, closed, densely defined operator \(A\). In Section 2 this operator is assumed not boundedly invertible and the problems arising in the study of variational regularization of the solution to the equation
\[
Au = f,
\]
are studied, where \(A : H \to H\) is a linear, unbounded, closed, densely defined, not boundedly invertible operator on a Hilbert space \(H\) with domain \(D(A)\) and range \(R(A)\). Since \(A\) is densely defined and closed, its adjoint \(A^*\) is a closed, densely defined linear operator. The operators \(T = A^*A\) and \(Q = AA^*\) are nonnegative, selfadjoint, densely defined in \(H\) operators (see [1]), the operator \(T_\alpha := T + \alpha I\), \((I\) is the identity operator and \(\alpha > 0\) is a constant) is boundedly invertible, i.e., its inverse is a bounded linear operator, defined on all of \(H\), with norm \(\leq \frac{1}{\alpha}\). It is easy to check that the operator \(A^*Q_\alpha^{-1}\) is bounded, defined on all of \(H\), and \(||A^*Q_\alpha^{-1}|| \leq \frac{1}{2\sqrt{\alpha}}\). We assume in Section 2 that the operator \(A\) is not boundedly invertible, in which case problem (1) is ill-posed.

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We are interested in the operator $S := S_{\alpha} := T_{\alpha}^{-1}A^*$ defined on a dense set $D(A^*)$. The reasons for our interest will be explained soon. The product of an unbounded closed operator ($A^*$ in our case) and a bounded operator ($T_{\alpha}^{-1}$ in our case) is not necessarily closed, in general, as a simple example shows: if $A = A^* \geq 0$ is unbounded selfadjoint (and consequently closed) operator and $B = (I + A)^{-1}$ is a bounded operator, then the operator $BA$ with the domain $D(A)$ is not closed. Its closure is a bounded operator defined on all of $H$. This closure is uniquely defined by continuity.

**Lemma 1.** If $A$ is a linear, closed, densely defined, unbounded operator in $H$, and $B$ is a bounded linear operator such that $R(B^*) \subset D(A)$, then the operator $BA^*$ with the domain $D(A^*)$ is closable.

**Proof.** To prove the closability of $BA^*$ one has to prove that if $u_n \to 0$ and $BA^*u_n \to w$, then $w = 0$. Let $h$ be arbitrary. Then $B^*h$ belongs to $D(A)$. Therefore

$$(w, h) = \lim(BA^*u_n, h) = \lim(u_n, AB^*h) = 0.$$ 

Thus, $w = 0$. 

In our case $B = T_{\alpha}^{-1} = B^*$ and $R(T_{\alpha}^{-1}) \subset D(A)$. By Lemma 1, the operator $T_{\alpha}^{-1}A^*$ is closable. The operator $AT_{\alpha}^{-1}$ is bounded, defined on all of $H$, with norm $\leq \frac{1}{2\sqrt{\alpha}}$. Indeed, by the polar decomposition one has $A = UT_{1/2}^T$, where $U$ is an isometry, so $||U|| \leq 1$. Thus, $||AT_{\alpha}^{-1}|| \leq ||T_{1/2}^{-1}|| = \sup_{s \geq 0} \frac{s^{1/2}}{s+\alpha} = \frac{1}{2\sqrt{\alpha}}$.

**Lemma 2.** The operator $S := T_{\alpha}^{-1}A^*$ with domain $D(A^*)$ has the closure $\overline{S} = S^{**}$, which is a bounded operator defined on all of $H$, with norm $\leq \frac{1}{2\sqrt{\alpha}}$, and $S^* = AT_{\alpha}^{-1}$ is a bounded operator defined on all of $H$.

**Proof.** The operator $S$ is densely defined. By Lemma 1 it is closable, so the operator $S^*$ is densely defined. Let us prove that $S^* = AT_{\alpha}^{-1}$. Let $h \in H$ be arbitrary. We have

$$(T_{\alpha}^{-1}A^*u, h) = (A^*u, T_{\alpha}^{-1}h) = (u, AT_{\alpha}^{-1}h).$$

This implies that $D(S^*) = H$ and $(T_{\alpha}^{-1}A^*)^* = AT_{\alpha}^{-1}$. We have used the relation $R(T_{\alpha}^{-1}) \subset D(A)$. Let us check this relation. Let $g \in R(T_{\alpha}^{-1})$, then $g = T_{\alpha}^{-1}h$ and $h = T_{\alpha}g$. Thus, $g \in D(T) \subset D(A)$, as claimed. Lemma 2 is proved.

From Lemmas 1 and 2 we obtain the following result.

**Theorem 1.** Let $A$ be a linear, closed, densely defined, unbounded operator in a Hilbert space. Then the operator $S = T_{\alpha}^{-1}A^*$ with domain $D(A^*)$ admits a unique closed extension $\overline{S}$ defined on all of $H$, with the norm $\leq \frac{1}{2\sqrt{\alpha}}$.

**Why should one be interested in the above theorem?**

The answer is: because of its crucial role in the study of equation (1) and of variational regularization for equation (1). The corresponding theory is developed in Section 2 and proofs are given in Section 3.

2. Variational regularization

**Assumption A:** We assume throughout that $A$ is linear, unbounded, densely defined operator in $H$, and that $A$ is not boundedly invertible, so problem (1) is ill-posed. We
assume that equation (1) is solvable, possibly nonuniquely, that \( f \neq 0 \), and denote by \( y \) its unique minimal-norm solution, \( y \perp N \), where \( N = N(A) \).

This assumption is not repeated below, but is a standing one throughout the rest of this paper.

Assume that \( \| f_\delta - f \| \leq \delta \), where \( f_\delta \) is the "noisy" data, which are known for some given small \( \delta > 0 \), while the exact data \( f \) are unknown. The problem is to construct a stable approximation \( u_\delta \) to \( y \), given the data \( \{ A, \delta, f_\delta \} \). Stable approximation means that \( \lim_{\delta \to 0} \| u_\delta - y \| = 0 \). Variational regularization is one of the methods for constructing such an approximation.

If \( A \) is bounded, this method consists of solving the minimization problem

\[
F(u) := F_{\alpha,\delta} = \| Au - f_\delta \|^2 + \alpha \| u \|^2 = \min,
\]

and choosing the regularization parameter \( \alpha = \alpha(\delta) \) so that \( \lim_{\delta \to 0} u_\delta = y \), where \( u_\delta := u_{\alpha(\delta),\delta} \). It is well known and easy to prove that if \( A \) is bounded, then problem (2) has a unique solution, \( u_{\alpha,\delta} = T_{\alpha}^{-1}A^*f_\delta \), which is a unique global minimizer of the quadratic functional (2), and this minimizer solves the equation \( T_{\alpha}u_{\alpha,\delta} = A^*f_\delta \). The last equation does not make sense, in general, if \( A \) is unbounded, because \( f_\delta \) may not belong to \( D(A^*) \). This is the difficulty arising in the case of unbounded \( A \). In this case it is not a priori clear if the global minimizer of functional (2) exists. We prove that this minimizer exists for any \( f_\delta \in H \), that it is unique, and that there is a function \( \alpha = \alpha(\delta) > 0 \), \( \lim_{\delta \to 0} \alpha(\delta) = 0 \), such that \( \lim_{\delta \to 0} u_{\alpha(\delta),\delta} = y \), so the element \( u_\delta := u_{\alpha(\delta),\delta} \) is a stable approximation of the unique minimal-norm solution to (1). Theorem 1 allows one to define the element \( T_{\alpha}^{-1}A^*f_\delta \) for any \( f_\delta \), and not only for those \( f_\delta \) which belong to \( D(A^*) \). We also prove for unbounded \( A \) a discrepancy principle in the following form. Let \( u_{\delta,\alpha} \) solve (2). Consider the equation for finding \( \alpha = \alpha(\delta) \):

\[
\| Au_{\alpha,\delta} - f_\delta \| = C\delta, \quad C > 1, \quad \| f_\delta \| > C\delta,
\]

where \( C \) is a constant. Equation (3) is the discrepancy principle. We prove that equation (3) determines \( \alpha(\delta) \) uniquely, \( \alpha(\delta) \to 0 \) as \( \delta \to 0 \), and \( u_\delta := u_{\delta,\alpha(\delta)} \to y \) as \( \delta \to 0 \). This justifies the discrepancy principle for choosing the regularization parameter (see [2] for various forms of the discrepancy principle).

Let us formulate the results.

**Theorem 2.** For any \( f \in H \) the functional \( F(u) = \| Au - f \|^2 + \alpha \| u \|^2 \) has a unique global minimizer \( u_{\alpha} = A^*Q_{\alpha}^{-1}f \), where \( Q = AA^* \), \( Q_{\alpha} := Q + \alpha I \), \( \alpha > 0 \) is a constant, and

\[
A^*Q_{\alpha}^{-1}f = T_{\alpha}^{-1}A^*f,
\]

where \( T_{\alpha}^{-1}A^* \) is the closure of the operator \( T_{\alpha}^{-1}A^* \) defined on \( D(A^*) \). If \( f \in R(A) \), then

\[
\lim_{\alpha \to 0} \| u_{\alpha} - f \| = 0,
\]

where \( u_{\alpha} \) is the unique global minimizer of \( F(u) \) and \( y \) is the minimal-norm solution to (1). If \( \| f_\delta - f \| \leq \delta \) and \( u_{\delta,\alpha} = A^*Q_{\alpha}^{-1}f_\delta \), then there exists an \( \alpha(\delta) > 0 \) such that

\[
\lim_{\delta \to 0} \| u_{\delta} - f \| = 0, \quad \lim_{\delta \to 0} \alpha(\delta) = 0, \quad u_\delta := u_{\alpha(\delta),\delta}.
\]
Equation (3) is uniquely solvable for $\alpha$, and for its solution $\alpha(\delta)$ equation (6) holds.

In Section 3 proof of Theorem 2 is given.

3. Proof of Theorem 2

3.1. For any $h \in D(A)$ let $u_\alpha := A^*Q_\alpha^{-1}$. One has

$$F(u_\alpha + h) = F(u_\alpha) + ||Ah||^2 + \alpha||h||^2 + 2\Re[(Au_\alpha - f, Ah) + \alpha(u_\alpha, h)],$$

and

$$(Au_\alpha - f, Ah) + \alpha(u_\alpha, h) = (QQ_\alpha^{-1}f - f, Ah) + \alpha(u_\alpha, h) = -\alpha(Q_\alpha^{-1}f, Ah) + \alpha(u_\alpha, h) = -\alpha[A^*Q_\alpha^{-1}f + u_\alpha, h] = 0.$$ (7)

From (7) and (8) it follows that $u_\alpha$ is the unique global minimizer of $F(u)$.

3.2. Let us prove (4). If (4) holds on a dense in $H$ linear subset $D(A^*)$, then it holds on all of $H$ by continuity because $A^*Q_\alpha^{-1}$ is a bounded linear operator, defined on all of $H$, with norm $\leq \frac{1}{2\sqrt{\alpha}}$, so that the closure of the operator $T_\alpha^{-1}A^*$ defined on $D(A^*)$, is a bounded operator, defined on all of $H$, with norm $\leq \frac{1}{2\sqrt{\alpha}}$. Indeed, let $f \in D(A^*)$, $g := Q_\alpha^{-1}f$, so $Q_\alpha g = f$ and $g \in D(A^*AA^*)$. Therefore equation (4) is equivalent to $A^*Q_\alpha g = T_\alpha A^*g$, or $A^*AA^*g + \alpha A^*g = A^*AA^*g + \alpha A^*g$, which is an identity. If $f \in D(A^*)$ (so that $g \in D(A^*AA^*)$), then the above formulas are justified and one can go back from the identity $A^*AA^*g + \alpha A^*g = A^*AA^*g + \alpha A^*g$, valid for any $g \in D(A^*AA^*)$, define $f = Q_\alpha g$, (this $f$ belongs to $D(A^*)$ because $Q_\alpha g \in D(A^*)$), and get (4).

Note that if $A$ were bounded, then one would have the identity

$$A^*\phi(Q) = \phi(T)A^*, \quad T = A^*A, \quad Q = AA^*,$$ (9)

valid for any continuous function $\phi$. Indeed, if $\phi$ is a polynomial, then (9) is obvious (for example, if $\phi(s) = s$, then (9) becomes $A^*(AA^*) = (A^*A)A^*$). If $\phi$ is a continuous function on the interval $[0, ||A||^2]$, then it is a limit (in the sup-norm on this bounded interval) of a sequence of polynomials (Weierstrass’ theorem), so (9) holds. In our problem $A$ is unbounded, so are $Q$ and $T$, and $\phi(s) = \frac{1}{s + \alpha}$ (with $\alpha = const > 0$) is a continuous function on an infinite interval $[0, \infty)$. Linear unbounded operators do not form an algebra, in general, because of the difficulties with domain of definition of the product of two unbounded operators (the product may have the trivial domain $\{0\}$). That is why formula (4), which is a particular case of (9) for bounded operators, has to be proved independently of this formula.

3.3 Let us prove (5). If $f \in R(A)$, then $f = Ay$, where $y \perp N$ is the minimal-norm solution to (1). We have $u_\alpha - y = T_\alpha^{-1}Ty - y = -\alpha T_\alpha^{-1}y$ and

$$\lim_{\alpha \to 0} ||\alpha T_\alpha^{-1}y||^2 = \lim_{\alpha \to 0} \int_0^{\infty} \frac{\alpha^2}{(\alpha + s)^2} d(Es y, y) = ||P_N y||^2 = 0,$$

where $E_s$ is the resolution of the identity of the selfadjoint operator $T$ and $P_N$ is the orthogonal projector onto $N = N(A)$, so $||P_N y|| = 0$ because $y \perp N$. 
3.4. Let us prove (6). We have

\[ ||u_\delta - y|| \leq ||u_\delta - u_\alpha|| + ||u_\alpha - y|| := I_1 + I_2.\]

We have already proved that \( \lim_{\delta \to 0} I_2 = 0 \), because \( \lim_{\delta \to 0} \alpha(\delta) = 0 \). Let us estimate \( I_1 \):

\[ ||u_\delta - u_\alpha|| = ||A^*Q_{\alpha(\delta)}^{-1}(f_\delta - f)|| \leq \frac{\delta}{2\sqrt{\alpha}}.\]

Thus, if \( \lim_{\delta \to 0} \alpha(\delta) = 0 \) and \( \lim_{\delta \to 0} \frac{\delta}{2\sqrt{\alpha}} = 0 \), then (6) holds.

3.5. Finally, let us prove

The discrepancy principle:

Equation (3) is uniquely solvable for \( \alpha \) and its solution \( \alpha(\delta) \) satisfies (6).

The proof follows the one in [2], p.22. One has

\[ g(\alpha, \delta) := ||Au_{\alpha, \delta} - f_\delta||^2 = ||QQ_{\alpha}^{-1}f_\delta - f_\delta||^2 = \alpha^2 \int_0^\infty \frac{d(E_s f_\delta, f_\delta)}{(s + \alpha)^2} = C^2 \delta^2, \quad (10)\]

where \( E_s \) is the resolution of the identity of the selfadjoint operator \( Q \). The function \( g(\alpha) := g(\alpha, \delta) \) for a fixed \( \delta > 0 \) is continuous, strictly increasing on \([0, \infty)\) and \( g(\infty) > C^2 \delta^2 \) while \( g(0) \leq \delta^2 \), as we will prove below. Thus, there exists a unique \( \alpha(\delta) > 0 \), such that \( g(\alpha(\delta), \delta) = C^2 \delta^2 \), and \( \lim_{\delta \to 0} \alpha(\delta) = 0 \) because \( g(\alpha, \delta) > 0 \) for \( \alpha \neq 0 \) and any \( \delta \in [0, \delta_0) \), provided that \( ||f|| \neq 0 \), which we assume. Here \( \delta_0 > 0 \) is a sufficiently small number.

Let us prove the two inequalities: \( g(\infty) > C^2 \delta^2 \) and \( g(0) \leq \delta^2 \). We have

\[ g(\infty) = \int_0^\infty \frac{d(E_s f_\delta, f_\delta)}{(s + \alpha)^2} = ||f_\delta||^2 > C^2 \delta^2, \]

because of the assumption \( ||f_\delta|| > C\delta \). Also

\[ g(0) = ||P_{N(Q)}f_\delta||^2 \leq \delta^2. \]

Indeed, \( f_\delta = f + (f_\delta - f) \). The element \( f \in R(A) \), so \( f \perp N(A^*) = N(Q) \). Therefore \( P_{N(Q)}f_\delta = P_{N(Q)}(f_\delta - f) \). Consequently,

\[ ||P_{N(Q)}f_\delta|| \leq ||P_{N(Q)}(f_\delta - f)|| \leq ||f_\delta - f|| \leq \delta. \]

Let us prove the limiting relation \( \lim_{\delta \to 0} ||u_\delta - f|| = 0 \). We have

\[ F_{\alpha(\delta)}(u_\delta) = ||Au_\delta - f_\delta||^2 + \alpha(\delta)||u_\delta||^2 \leq F_{\alpha(\delta)}(y) \leq \delta^2 + \alpha(\delta)||y||^2. \quad (11)\]

Since \( ||Au_\delta - f_\delta||^2 = C^2 \delta^2 \geq \delta^2 \) we conclude from (11) that \( ||u_\delta|| \leq ||y|| \) for all \( \delta \in [0, \delta_0) \).

Thus we may assume that \( u_\delta \to z \) as \( \delta \to 0 \), where \( \to \) denotes weak convergence in \( H \). Since \( \lim_{\delta \to 0} f_\delta = f \), we conclude from \( ||Au_\delta - f_\delta|| = C\delta \) that \( \lim_{\delta \to 0} ||Au_\delta - f|| = 0 \). This implies \( Az = f \). Indeed, for any \( h \in D(A^*) \) one has

\[ (f, h) = \lim_{\delta \to 0} (Au_\delta, h) = (z, A^*h). \]
Therefore $Az = f$. Since $||u_\delta|| \leq ||y||$, we have $\lim_{\delta \to 0} ||u_\delta|| \leq ||y||$. From $u_\delta \rightrightarrows z$ we obtain $||z|| \leq \lim_{\delta \to 0} ||u_\delta|| \leq ||y||$. Thus, $||z|| \leq ||y||$. Since the minimal-norm solution to (1) is unique, it follows that $z = y$. Thus, $u_\delta \rightrightarrows y$ and $\lim_{\delta \to 0} ||u_\delta|| \leq ||y|| \leq \lim_{\delta \to 0} ||u_\delta||$.

This implies $\lim_{\delta \to 0} ||u_\delta|| = ||y||$. Consequently, $\lim_{\delta \to 0} ||u_\delta - y|| = 0$, because $||u_\delta - y||^2 = ||u_\delta||^2 + ||y||^2 - 2 \Re(u_\delta, y) \to 0$ as $\delta \to 0$.

Theorem 2 is proved. □

References

[1] T. Kato, Perturbation Theory for Linear Operators, Springer Verlag, New York, 1984.

[2] A. G. Ramm, Inverse Problems, Springer, New York, 2005.