NON-TORSION BRAUER GROUPS IN POSITIVE CHARACTERISTIC

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ABSTRACT. Unlike the classical Brauer group of a field, the Brauer-Grothendieck group of a singular scheme need not be torsion. We show that there exist integral normal projective surfaces over a large field of positive characteristic with non-torsion Brauer group. In contrast, we demonstrate that such examples cannot exist over the algebraic closure of a finite field.

1. INTRODUCTION

One way of extending the notion of the classical Brauer group of a field to any scheme $X$ is by defining the Brauer-Grothendieck group $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$. Just as for fields, this group is torsion for any regular integral noetherian scheme [5, Corollaire 1.8]. However, this no longer holds for singular schemes. Chapter 8 of [3] lists several counterexamples. For instance, if $R$ is the local ring of the vertex of a cone over a smooth curve of degree $d \geq 4$ in $\mathbb{P}^2_k$, then $\text{Br}(R)$ contains the additive group of $\mathbb{C}$. There are then affine Zariski open neighborhoods of the vertex with non-torsion Brauer group [3, Example 8.2.2]. However, the additive group of $k$ is torsion when $k$ has positive characteristic, so analogous constructions do not work there. There are also reducible varieties in arbitrary characteristic with non-torsion Brauer group [3, Section 8.1]. This leaves open the following question, communicated by Colliot-Thélène and Skorobogatov in an unpublished draft of [3]:

**Question 1.1.** If $X$ is an integral normal quasi-projective variety over a field $k$ of positive characteristic, is $\text{Br}(X)$ a torsion group?

To analyze this question, we use a result concerning the Brauer group of a normal variety $X$ with only isolated singularities $p_1, \ldots, p_n$. Suppose $X$ is defined over an algebraically closed field $k$ of arbitrary characteristic. Let $K$ be the function field of $X$, $R_i = \mathcal{O}_{X,p_i}$ be the local ring at each singularity, and $R_i^{\text{h}}$ its henselization. Then, $\text{Br}(X)$ is given by the exact sequence (see [3, Section 8.2], elaborating on [5, §1, Remarque 11 (b)])

$$0 \to \text{Pic}(X) \to \text{Cl}(X) \to \bigoplus_{i=1}^n \text{Cl}(R_i^{\text{h}}) \to \text{Br}(X) \to \text{Br}(K).$$

This sequence indicates that one way for $\text{Br}(X)$ to be large is for a singularity to have a large henselian local class group with divisors that do not extend globally to $X$. This

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idea is illustrated by a counterexample given by Burt Totaro, which shows that \( \text{Br}(X) \) is non-torsion for \( X \) a hypersurface of degree \( d \geq 3 \) in \( \mathbb{P}^4_k \) with a single node [3, Proposition 8.2.3]. Here \( k \) is any algebraically closed field with characteristic not 2.

We will show that counterexamples to Question 1.1 exist in dimension 2 if and only if \( k \) is not the algebraic closure of a finite field.

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## 2. A Surface Counterexample

The following construction is taken from [7], Example 1.27. Take a smooth cubic curve \( D \) in the projective plane and a quartic curve \( Q \) that meets it tranversally. Let \( Y = \text{Bl}_{q_1,\ldots,q_{12}} \mathbb{P}^2 \), where \( q_1,\ldots,q_{12} \) are the points of intersection. On \( Y \), the proper transform \( C \) of \( D \) satisfies \( C^2 = -3 \). Unlike rational curves, not all negative self-intersection higher genus curves may be contracted to yield a projective surface. Rather, the contraction might only exist as an algebraic space. However, in this case, \( C \) is contractible.

**Proposition 2.1.** There exists a normal projective surface \( X \) and a proper birational morphism \( Y \to X \) whose exceptional locus is exactly \( C \).

**Proof.** The Picard group of \( Y \) is the free abelian group on \( H \), the pullback of a general line in \( \mathbb{P}^2 \), and the exceptional divisors \( E_1,\ldots,E_{12} \). Then, we claim that the line bundle \( L := \mathcal{O}_Y(4H - \sum_i E_i) = \mathcal{O}_Y(H + C) \) defines a basepoint-free linear system on \( Y \). Indeed, no point outside of the \( E_i \) can be a base point, and the proper transforms of both \( D \) and \( Q \) belong to the linear system. These don’t intersect on the exceptional locus by the transversality assumption. Therefore, the system defines a morphism from \( Y \) to projective space with image \( X' \). This morphism is birational because we have an injective map \( H^0(Y, \mathcal{O}_Y(H)) \hookrightarrow H^0(Y, L) \) and \( \mathcal{O}_Y(H) \) is the pullback of a very ample line bundle on \( \mathbb{P}^2 \).

The exceptional locus of the morphism \( Y' \to X' \) is precisely the union of the irreducible curves in \( Y \) on which \( L \) has degree zero. If \( C' \), an irreducible curve with \( C' \cdot (H + C) = 0 \), clearly \( C' \) is not supported on the \( E_i \), or the intersection would be positive. Therefore, \( H \cdot C' > 0 \), meaning \( C \cdot C' < 0 \). This means \( C' = C \), so the exceptional locus is the curve \( C \), which is mapped to a point. Thus, \( X' \) is a surface birational to \( Y \) and \( Y \to X' \) is an isomorphism away from \( C \). Finally, passing to the normalization \( X \) of \( X' \), we may assume \( X \) is a normal projective surface; the normalization will also be an isomorphism away from \( C \), and the image of \( C \) will again be a point in \( X \). \( \square \)

The resulting singularity \( p \) of \( X \) has minimal resolution with exceptional set exactly \( C \), a smooth elliptic curve. Singularities satisfying this condition are called **simple elliptic singularities**. Over \( \mathbb{C} \), such singularities are completely classified. A simple elliptic singularity with \( C^2 = -3 \) is known as an \( \tilde{E}_6 \) singularity, and is complex analytically isomorphic to a cone over \( C \) \([8]\). Here, we present a computation of the henselian local class group \( \text{Cl}(R^h) \) of the singularity that works in any characteristic.
Consider the pullback of the desingularization $Y \to X$ to a “henselian neighborhood”:

$$
\begin{array}{ccc}
Y^h & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec}(R^h) & \longrightarrow & X.
\end{array}
$$

The scheme $Y^h$ is regular and $Y^h \setminus C \cong \text{Spec}(R^h) \setminus \{m\}$, so we have an exact sequence $0 \to \mathbb{Z} \cdot [C] \to \text{Pic}(Y^h) \to \text{Cl}(R^h) \to 0$. Here, the first map is injective since $\mathcal{O}_{Y^h}(-C)$ has nonzero degree on $C$. It suffices, therefore, to compute $\text{Pic}(Y^h)$. To do so, we’ll first consider infinitesimal neighborhoods of $C$ in $Y$.

The sequence of infinitesimal neighborhoods $C = C_1 \subset C_2 \subset \cdots$ is defined by powers of the ideal sheaf $\mathcal{I}_C$. Notably, these $C_n$ are the same regardless of whether we consider them inside $Y$ or inside the henselian neighborhood $Y^h$. The normal bundle to $C$ in $Y$ gives obstructions to extending line bundles to successive neighborhoods, but we’ll show that all line bundles extend uniquely. The group $\lim_{\leftarrow n} \text{Pic}(C_n)$ in the proposition below is also the Picard group of the formal neighborhood of $C$ in $Y$.

**Proposition 2.2.** The restriction map $\lim_{\leftarrow n} \text{Pic}(C_n) \to \text{Pic}(C)$ is an isomorphism.

**Proof.** It’s enough to show that the maps $\text{Pic}(C_{n+1}) \to \text{Pic}(C_n)$ are all isomorphisms for $n \geq 1$. Each extension $C_n \subset C_{n+1}$ is a first-order thickening, since $C_n$ is defined in $C_{n+1}$ by the square-zero ideal sheaf $\mathcal{I}_C^n/\mathcal{I}_C^{n+1}$. Associated to such a thickening is a long exact sequence in cohomology [9, 0C6Q]

$$
\cdots \to H^1(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \text{Pic}(C_{n+1}) \to \text{Pic}(C_n) \to H^2(C, \mathcal{I}_C^n/\mathcal{I}_C^{n+1}) \to \cdots.
$$

We may take the outer cohomology groups to be over $C$ since the underlying topological spaces are the same. As sheaves of abelian groups on $C$, we have $\mathcal{I}_C^n/\mathcal{I}_C^{n+1} \cong \mathcal{O}_C(-nC)$, a multiple of the conormal bundle. But $C^2 = -3$ in $Y$ so this last bundle has degree $3n > 0$. Since $C$ is genus 1, the higher cohomology of $\mathcal{O}_C(-nC)$ vanishes and $\text{Pic}(C_{n+1}) \to \text{Pic}(C_n)$ is an isomorphism for all $n$.

Now, we need only compare $\lim_{\leftarrow n} \text{Pic}(C_n)$ to $\text{Pic}(Y_h)$. Using the Artin approximation theorem [1, Theorem 3.5], the map $\text{Pic}(Y^h) \to \lim_{\leftarrow n} \text{Pic}(C_n)$ is injective with dense image. However, the topology of the latter group is discrete in this setting because each group of the inverse limit is $\text{Pic}(C)$. Therefore, the map is surjective also and $\text{Pic}(Y^h) \cong \text{Pic}(C)$.

**Theorem 2.3.** Let $k$ be an algebraically closed field that is not the algebraic closure of a finite field and $X$ be the surface defined in the proof of Proposition 2.1. Then, $\text{Br}(X)$ is non-torsion.

**Proof.** From the above, we have the identification $\text{Cl}(R^h) \cong \text{Pic}(Y^h)/\mathbb{Z} \cdot \mathcal{O}_{Y^h}(C)
\cong \text{Pic}(C)/\mathbb{Z} \cdot \mathcal{O}_C(C)$. Since $\deg_C \mathcal{O}_C(C) = 3$, the class group is then an extension of $\mathbb{Z}/3$ by $\text{Pic}^0(C) \cong C(k)$, where $C(k)$ is the group of $k$-rational points of the elliptic curve $C$ with a chosen identity point. Since $k \neq \overline{\mathbb{F}}_p$, $C(k)$ has infinite rank [4, Theorem 10.1]. Note that
in contrast, \( C(k) \) is torsion for an elliptic curve \( C \) over \( \mathbb{F}_p \), because every point is defined over \( \mathbb{F}_{p^m} \) for some \( m \).

However, the global class group of \( X \) is quite small: \( \text{Cl}(Y) = \text{Pic}(Y) \cong \mathbb{Z}^{13} \) since \( Y \) is the blow up of \( \mathbb{P}^2 \) in 12 points and \( \text{Cl}(X) \cong \text{Cl}(Y) / \mathbb{Z} \cdot [C] \). Therefore, the cokernel of the restriction map \( \text{Cl}(X) \rightarrow \text{Cl}(R^h) \) in (1) contains non-torsion elements, so \( \text{Br}(X) \) does too.

To complement the above result, we also prove:

**Theorem 2.4.** Suppose that \( X \) is an integral normal surface over the algebraic closure \( k = \overline{\mathbb{F}}_p \) of a finite field. Then \( \text{Br}(X) \) is torsion.

*Proof.* The strategy is similar to the previous theorem. Here, the crucial fact is that all possible “building blocks” of the henselian local class group - abelian varieties over \( k \), the additive group of \( k \), and the multiplicative group of \( k \) - are torsion.

Since the singularities of a normal surface are isolated, we may apply the exact sequence (1). The group \( \text{Br}(K) \) is always torsion, so if we can prove \( \oplus_{i=1}^n \text{Cl}(R^h_i) \) is as well, the result follows. Let \( Y \rightarrow X \) be a desingularization. We focus on the base change \( \pi : Y^h \rightarrow \text{Spec}(R^h) \) to the henselian local ring at just one singular point \( p \). Let \( E = \pi^{-1}(p) \) be the scheme-theoretic fiber. We may choose \( Y \) such that \( E_{\text{red}} \) is a union of irreducible curves \( F_j \), which are smooth and meet pairwise transversely, with no three containing a common point.

The following argument is due to Artin [2]. Suppose \( G \) is the free abelian group of divisors supported on \( E \), and consider the map \( \alpha : \text{Pic}(Y^h) \rightarrow \text{Hom}(G, \mathbb{Z}) \) given by \( L \mapsto (D \mapsto D \cdot L) \). This restricts to a map \( \alpha|_G : G \rightarrow \text{Hom}(G, \mathbb{Z}) \) that is injective because the intersection matrix of the curves \( F_j \) is negative definite. Since \( G \) and \( \text{Hom}(G, \mathbb{Z}) \) are free abelian groups of equal rank, \( G \rightarrow \text{Hom}(G, \mathbb{Z}) \) also has finite cokernel. This allows us to find an effective Cartier divisor \( Z = \sum_j r_j F_j \) with all \( r_j > 0 \) such that \( \alpha(Z) = \alpha(-H) \) for an ample line bundle \( H \) on \( Y^h \) [2, p. 491]. If we restrict this Cartier divisor to the scheme associated to \( Z \), the resulting line bundle \( \mathcal{O}_Z(-Z) \) has positive degree on every irreducible component of \( Z \), so it is ample. We’ll examine infinitesimal neighborhoods of the closed subscheme \( Z \) in \( Y^h \).

As before, for every \( n \geq 1 \), there is an exact sequence

\[
\cdots \rightarrow H^1(Z, \mathcal{O}_Z(-nZ)) \rightarrow \text{Pic}(Z_{n+1}) \rightarrow \text{Pic}(Z_n) \rightarrow H^2(Z, \mathcal{O}_Z(-nZ)) \rightarrow \cdots.
\]

Because \( \dim(Z) = 1 \), the last group is always zero. Since \( \mathcal{O}_Z(-Z) \) is ample, the first group is zero for \( n \gg 0 \) by Serre vanishing, which holds on any projective scheme [6, Theorem II.5.2]. Therefore, the inverse limit \( \varprojlim_n \text{Pic}(Z_n) \) is constructed as a finite series of extensions of \( \text{Pic}(Z) \) by finite-dimensional \( k \)-vector spaces. Applying Artin approximation, we have that \( \text{Pic}(Y^h) \rightarrow \varprojlim_n \text{Pic}(E_n) \) is injective with dense image. However, for large \( n \), the scheme \( E_n \) is nested between two infinitesimal neighborhoods of \( Z \), where all restrictions of Picard groups are surjective (use a similar exact sequence to the above,
e.g. [9, 09NY]). It follows that \( \lim_{\leftarrow n} \text{Pic}(E_n) \cong \lim_{\leftarrow n} \text{Pic}(Z_n) \) and that both have the discrete topology, so \( \text{Pic}(Y^h) \cong \lim_{\leftarrow n} \text{Pic}(Z_n) \).

Next, let \( \bar{Z} \) be the disjoint union of the schemes \( r_j F_j \), where \( r_j F_j \) is the subscheme of \( Y^h \) cut out by the ideal sheaf of \( F_j \) to the power \( r_j \). Then \( f : \bar{Z} \to Z \) is a finite map that is an isomorphism away from the finite set of intersection points and such that \( \mathcal{O}_Z \subset f_* \mathcal{O}_{\bar{Z}} \). It follows (see [9, 0C1M, 0C1N]) that \( \text{Pic}(Z) \) is a finite sequence of extensions of \( \text{Pic}(\bar{Z}) \) by quotients of \( (k,+) \) or \( (k,*) \). Lastly, \( \text{Pic}(\bar{Z}) = \bigoplus_j \text{Pic}(r_j F_j) \), where each summand is built from finite-dimensional \( k \)-vector spaces and \( \text{Pic}(F_j) \cong \mathbb{Z} \oplus \text{Pic}^0(F_j) \). Since the \( \text{Pic}^0(F_j) \) are groups of \( k \)-points of abelian varieties over \( k \), they are all torsion.

Taken together, all of this implies that \( G \) and \( \text{Pic}(Y^h) \) have equal rank. Since \( G \to \text{Hom}(G,\mathbb{Z}) \) is injective, the first map in the excision sequence of class groups \( 0 \to G \to \text{Pic}(Y^h) \to \text{Cl}(R^h) \to 0 \) is injective also. Therefore, the quotient \( \text{Cl}(R^h) \) is a torsion group, as desired. \( \square \)

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