A crucial challenge in network theory is to study how robust a network is when facing failures or attacks. In this work, we propose a novel methodology to measure the topological resilience and robustness of a network based on Information Theory quantifiers. This measure can be used with any probability distribution able to represent the network’s properties. In particular, we analyze the efficiency in capturing small perturbations in the network’s topology when using the degree and distance distributions. Theoretical examples and real networks are used to study the performance of this methodology. Although both cases show to be able to detect any single topological change, the distance distribution seems to be more consistent to reflect the network structural deviations. In all cases, the novel resilience and robustness measures computed by using the distance distribution reflect better the consequences of the failures, outperforming other methods.

A New Network Robustness Topology Measure based on Information Theory

Tiago A. Schieber, Laura Carpi, Alejandro C. Frery, Osvaldo A Rosso, Panos M. Pardalos, and Martín G. Ravetti

1Departamento de Engenharia de Produção, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil
2Departamento de Física e Engenharia Nuclear, Universitat Politècnica de Catalunya. Colom 11, Terrassa 08222, Barcelona, Spain
3Laboratório de Computação Científica e Análise Numérica (LaCCAN), Universidade Federal de Alagoas, Maceió, Alagoas, Brazil
4Instituto de Física, Universidade Federal de Alagoas, Maceió, Alagoas, Brazil
5Industrial and Systems Engineering, University of Florida, Gainesville, FL, USA

(Dated: November 3, 2014)

PACS numbers: 89.75.-k, 89.75.Fb, 89.70.Cf

A major challenge in network theory is to study how robust a network is when facing random failures or targeted attacks. Several approaches have been proposed to both, planning attack strategies and quantifying damages after failures. The most popular methodologies to measure robustness are those based on percolation theory (BC) [4–7]. However, these measures are not as sensitive as they should. Failures in links that do not disconnect the network or that do not modify its diameter may occur undetected by these measures. Depending on the network structure, it is possible to intentionally attack a great part of the network, to the point of dismantling it, and still remaining invisible to robustness measures based on changes in the biggest connected component.

In [8, 9] methodologies are proposed to study the evolution of networks based on Information Theory quantifiers. In particular, the Jensen-Shannon divergence proved to be very effective to measure changes in the network structure. In [9], a similar concept shows to be able to capture changes in several common network’s attributes.

In this work we propose a robustness structural measure based on Information Theory quantifiers, able to capture every single failure on the network topology. This general robustness measure analyzes the behavior of the topological structure of a network after failures of its components, disregarding the consequences of the dynamical process operating through it.

The quantification of a network structural resilience can be thought through the distance that a given topology is apart from itself after a failure. We assume that the resilience value ranges from 0, the maximum variation, to 1, unchanged characteristics, thus, a higher resilience value implies in smaller structural changes. In this work we consider the failure of links, implying in their remotion from the network. Node’s failures can be also analyzed considering the remotion of all links incident to them.
Let $G$ be an undirected network, defined by a set $V(G)$ of $N$ nodes, and a set $E(G)$ of $M$ links. A time-ordered sequence of failures $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ is defined as a set of $n$ links, where $f_i \in \mathcal{F}$ indicates that the link $i$ fails at instant $t$. To represent a node failure at instant $t$, all its incident links are removed at the same instant $t$. The resulting network after the failure of $n$ links, $G' = (V, E')$, presents the same number of nodes and a subset of edges $E' \subset E(G)$. Hence, a sequence of $n$ failures $\mathcal{F}$ in a network $G$ can be considered a discrete temporal process represented by a sequence of networks $S_n = (G_t)_{t \in \{1, 2, \ldots, n\}}$.

Considering a set $N_G$ of all networks that can be reached from $G$ after $n$ failures, a function $\sigma_G = N_G \rightarrow [0, 1]$ is a resilience function with respect to $G$. The distance between two networks will be computed as the divergence/distance between the probability distribution of selected network features. Changes in carefully chosen features will lead to sensitive indicators of significant changes. Without loss of generality, discrete distributions will be considered henceforth.

The Jensen-Shannon divergence between two probability distributions $P$ and $Q$ is defined as the Shannon entropy of the average minus the average of the entropies. This measure is proved to be the square of a metric between probability distributions $[10]$, and is defined as:

$$J^H(P, Q) = H\left(\frac{P + Q}{2}\right) - \frac{H(P) + H(Q)}{2},$$

being $H(P) = -\sum_{i, p_i > 0} p_i \ln p_i$, the entropy that measures the amount of uncertainty in a probability distribution.

It is possible, then, to define a structural resilience function of a network $G$ capable to quantify the network topological changes produced by the failures. To measure these changes we use the Jensen-Shannon divergence between a probability distribution function $P$ after a sequence $\mathcal{F}$ of $n$ failures as:

$$\sigma_{P,G,\mathcal{F}}(G') = 1 - \frac{J^H(P(G), P(G'))}{\ln 2},$$

where $G'$ is the network obtained from $G$ after the sequence of failures $\mathcal{F}$.

Given a resilience function $\sigma_{P,G,\mathcal{F}}$, the robustness of $G$ with respect to $(G_t)_{t \in \{1, 2, \ldots, n\}}$, with $G_0 = G$ is defined by:

$$R(\sigma_{G,\mathcal{F}}) = \prod_{t=1}^{n} \sigma_{G_{t-1},\mathcal{F}}(G_t).$$

The computation of the resilience of a network, when defined as in equation [1] could consider any probability distribution able to represent the network. This work considers the degree distribution, commonly used to characterize network’s structures, and the distance distribution that contains rich information about the graph structure. The degree and distance distributions are here defined for unweighted and undirected networks. Given a node $i$, its degree, represented by $k_i$, is the number of edges incident to it. Then, the degree distribution $P_{\deg}(k)$ is the fraction of nodes with degree $k$. The distance from the node $i$ to node $j$, $d_{ij}$, is the length of the minimum geodesic path from $i$ to $j$. If there is not a geodesic path from $i$ to $j$, $d_{ij}$ equals $\infty$. Then, the distance distribution $P_d(d)$ is the fraction of pairs of nodes at distance $d$. Both degree and distance distributions are discrete and take values on the sets $\{0, 1, \ldots, N - 1\}$ and $\{1, 2, \ldots, N - 1, \infty\}$, respectively.

To compare the performance of our methodology $(\sigma^{\deg}, \sigma^d)$ to measure structural resilience with commonly used methods based on the biggest connected component $(\sigma^{bc})$, and percolation $(\sigma^{\pi})$, we consider the most robust unweighted undirected graph, a complete graph. In that case, the simplest failure it can suffer is the deletion of a single link.

For a complete graph with $N$ nodes, $\sigma^{bc}$ is obtained by computing the fraction of nodes belonging to the biggest connected component. In this case, $\sigma^{bc}$ does not notice the removal of any link, as no disconnection is achieved. In fact, we can strategically remove $N^2/2 - 2N + 1$ links, leaving just the minimum spanning tree and the resilience will not be able to detect the attacks.

Percolation based measures correlate the resilience value of the network with the critical percolation threshold. There are different ways of computing this threshold, being the most common ones: number of links removed until increasing the diameter of the network, thus $\sigma^{\pi_d}$ indicates the percentage increase of the diameter, and the number of links removed until disconnecting the biggest connected component, in this case $\sigma^{\pi_c}$ is equivalent to $\sigma^{bc}$ indicating the fraction of nodes belonging to the biggest connected component. In this analysis, the deletion of a single link will increase the network diameter in one unit, but after the first attack, $\sigma^{\pi_d}$ will be unable to detect a second attack until removing $N - 2$ specific links. Once more, the network can be strategically attacked avoiding an increase in the network diameter. This is significative, as an objective could also be the improvement of the robustness of a network by including new links, given a certain budget. These methodologies are unable to properly guide in this purpose.

The proposed methodology, for both probability distributions $(P_{\deg}, P_d)$, detects the removal of any single link, independently of which one is removed from the graph. The resilience values of $\sigma^{\deg}$ and $\sigma^d$ as a function of $N$ are presented in Table [2].

Among the measures here considered, only $\sigma^{\deg}$ and $\sigma^d$ are capable of capturing the removal of any link. There are interesting differences between $P_{\deg}$ and $P_d$ still to
be analyzed. The computational complexity of obtaining the degree distribution is linear plus a constant cost to update it after any link removal. For obtaining $P_3$ the best known algorithm to optimally solve all-pairs shortest paths problem requires $O(N^2 \log N)$ in time complexity \[11\], and the computational cost of the PDF update will depend on the link removed. However new algorithms as the ANF or HyperANF (algorithms based on HyperLogLog counters) offer an extremely fast and precise approach \[12–15\], obtaining very good approximations of the distance PDF for graphs with millions of nodes in a few seconds.

Another important comparison to be made is the correlation of the PDFs with the structural characteristics of the network. The average degree, mean degree and the minimum and maximum degree are immediately obtained from the degree distribution. Furthermore, the efficiency, the diameter of the graph, the average path length, the fraction of disconnected pairs of nodes and other distance related features are easily obtained from the distance distribution. The resilience value, when using the distance distribution, is able to detect changes also perceived by $\sigma_{bc}$ and $\sigma_{\pi d}$ are able to do, dominating their performance.

A further aspect to note is that the disconnection of the graph into two or more clusters, is undetected by the degree distribution, however the distance distribution computes the fraction of disconnected pairs of nodes. Consider the graph in Figure 1, the four resilience functions are used to analyze three possible link failures, $\ell_i$, $\ell_j$ and $\ell_r$. $\sigma_{bc}$ only detects the disconnection of the BC, $\sigma_{\pi d}$ detects $\ell_j$, and presents a special case after removing the link $\ell_i$, that disconnection creates two clusters and consequently the diameter gets smaller, indicating falsely an increase in the resilience value. The use of the degree distribution detects each link failure, but as mentioned before it fails in handling the graph disconnection. The measure based on the distance distribution captures in a more appropriated way all the network failures. This small example captures important advantages and disadvantages of each resilience measure, the use of the distance distribution appears as more adequate for the analysis.

We test the proposed methodology on two real networks, the Dolphin Social Network \[10\] and the Western States Power Grid of the United States network \[17\].

The Dolphin network is an undirected social network of bottlenose dolphins (genus Tursiops). The nodes are the bottlenose dolphins of a dolphin community from New Zealand, where an edge indicates a frequent association. The dolphins were observed between 1994 and 2001. It presents, $N = 62$, $M = 159$, an average degree of 5.13, an average path length of 3.357, and a clustering coefficient of 0.258.

The Power Grid Network is the undirected unweighted representation of the topology of the Western States Power Grid of the United States, compiled by Duncan Watts and Steven Strogatz. It presents, $N = 4941$, $M = 6594$, an average degree of 2.67, an average path length of 18.99, and a clustering coefficient of 0.103.

For both cases 10% of their links were randomly removed and the resilience and robustness measures here proposed were computed. For each case 30 independent experiments were performed, and Figure 2 depicts the outcomes. As it is possible to see, both measures are able to detect changes in the networks.

Regarding the resilience measures, each value indicates how affected the topologies are after a failure, the error bar shows the standard deviation for each value. Close to one value with small variability implies a strong resilience to a random selection of links. On the contrary, large variability indicates that certain links are critical, generating bigger changes in the topology, see Figures (2-a), (2-c), (2-e) and (2-g). Robustness measures give cumulative information about the evolution of the state of the network. Large variability values imply that a careful scheduling of the attacks could lead to a deeper impact on the network structure, Figures, (2-b), (2-d), (2-f) and (2-h).

Concerning the use of the degree distribution, it is possible to see in Figures (2-b) and (2-f) that the robustness measure shows a smooth behavior, this happened as the distribution is unable to detect clusters disconnections.

| $N$ nodes | $\sigma_{bc}^G$ | $\sigma_{\pi d}^G$ | $\sigma_{bc}^G$ | $\sigma_{\pi d}^G$ |
|-----------|----------------|------------------|----------------|------------------|
| 1,000     | $10^{-3}$      | $10^{-10}$       | 1.0            | 1.0              |
| 10,000    | $10^{-4}$      | $10^{-8}$        | 1.0            | 0.97             |
| 100,000   | $10^{-5}$      | $10^{-10}$       | 1.0            | 0.93             |
| 1,000,000 | $10^{-6}$      | $10^{-13}$       | 1.0            | 0.94             |
| 10,000,000| $10^{-7}$      | $10^{-15}$       | 1.0            | 0.94             |

**TABLE I: Sensitivity values for each resilience metric when a link is removed from a complete graph with $N$ nodes.**

**FIG. 1:** Computation of the structural resiliences for three different edge removal: $\ell_i$, $\ell_j$ and $\ell_r$, respectively. The * indicates a special situation, as the network is disconnected the diameter of the network changes from 3 to 1.
This is not the case for the distance distribution, where the fraction of disconnected pairs of nodes is acknowledged, Figures (2.d) and (2.h). The large decrease in the $R_δ$ values usually represents cluster disconnections from the network. As we are analyzing average values, the disconnection may happen only in a fraction of the 30 independent averaged experiments.

About the efficiency of the attacks, the Dolphin network seems to be more susceptible to these failures than the Power Grid network. This is possible to see by analyzing Figures (2.a), (2.c), (2.e) and (2.g), the resilience values for Dolphin, shows large variability and large average values, implying that certain attacks may cause a larger structural change. The robustness measures, in particular with the distance distribution, Figures (2.d) and (2.h) also show big leaps when the link removal is around 6% and 9% for Dolphin and 3% and 6% for the Power Grid.

Summarizing, we propose a new methodology to measure the resilience and robustness of a network to component failures or targeted attacks. These structural measures are based on the use of Information Theory quantifiers, and they can be used with any probability distribution representing the network.

The results here presented are significant, as the method efficiently works with disconnected networks. This is not a small issue as all the most common ways to analyze the network resilience, are defined for connected ones. The method, when using the distance distribution, is able to acknowledge the fraction of disconnected pairs of nodes, thus, the resilience and robustness measures are able to detect these changes.

The generalization for weighted and directed networks is straightforward, as there are many different ways to obtain a degree and distance distribution for those cases. Besides its computational CPU time, the measures based on the distance distribution are more consistent, in the sense that their values are correlated with the consequences of the failures in the network topology.

ACKNOWLEDGMENTS

Research partially supported by FAPEMIG, CNPq and grant RSF 14-41-00039.

[Also at Instituto Tecnológico de Buenos Aires (ITBA), Ciudad Autónoma de Buenos Aires, Argentina]

[Electronic address: martin.ravetti@dej.ufmg.br] Also at Departament de Física Fonamental, Universitat de Barcelona, Barcelona, Spain

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FIG. 2: Resilience and robustness under random failures for Dolphin and Power Grid networks. At each time step, a random edge is disconnected from the network and $\sigma^\text{deg}$, $\sigma^\delta$, $R^\text{deg}$, and $R^\delta$ functions are computed. The experiment is independently executed 30 times and the average with the standard error are presented for each case. For Dolphin network: (a) shows $\sigma^\text{deg}$, (b) $R^\text{deg}$, (c) $\sigma^\delta$ and (d) $R^\delta$. For Power Grid network: (e) shows $\sigma^\text{deg}$, (f) $R^\text{deg}$, (g) $\sigma^\delta$ and (h) $R^\delta$. 