A Map between \((q, h)\)-deformed Gauge Theories and ordinary Gauge Theories\(^*\)

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Abstract

We introduce a new map between a \((q, h)\)-deformed gauge theory and an ordinary gauge theory in a full analogy with Seiberg-Witten map.

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1 Introduction

In the past few years a huge amount of literature has been devoted to the study of quantum planes [1]. They provide a mathematical model for the system under consideration. All the properties of interest can be expressed in terms of noncommutative functions defined on these quantum spaces. Quantum groups [2] are generated by such functions. It was shown that the only quantum groups which preserve nondegenerate bilinear forms are $GL_{qp}(2)$ and $GL_{h^h}(2)$, [3]-[11]. They act on the $q$-plane (Manin plane), with relation $\hat{x}\hat{y} = q\hat{y}\hat{x}$ and on the $h$-plane (Jordanian plane), with relation $\hat{x}\hat{y} - \hat{y}\hat{x} = h\hat{y}^2$, respectively.

In a recent paper [12] we have constructed a map relating $q$-deformed gauge fields defined on the Manin plane and ordinary gauge fields in a full analogy with Seiberg-Witten map [13]. This work has been extended to $GL_q(N)$-covariant quantum hyperplane [14]. In the present letter we define a map which relates $(q, h)$-deformed gauge fields defined on the hybrid quantum space (given by $\hat{x}\hat{y} - q\hat{y}\hat{x} = h\hat{y}^2$) and the ordinary gauge fields. The product of functions on such a space is obtained by applying a singular transformation [11] on the Gerstenhaber star product [15].

2 $(q, h)$-deformed gauge theory versus ordinary gauge theory

To begin we consider the undeformed action

$$ S = -\frac{1}{4} \int d^4x \ F_{\mu\nu} F^{\mu\nu}, \quad (1) $$

where

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2) $$

$S$ is invariant with respect to infinitesimal gauge transformations:

$$ \delta_\lambda A_\mu = \partial_\mu \lambda. \quad (3) $$

To study the $(q, h)$-deformed analogue of this gauge theory let us first recall that the Manin plane, with the relation $\hat{X}\hat{Y} = q\hat{Y}\hat{X}$ and the hybrid plane with the relation $\hat{x}\hat{y} - q\hat{y}\hat{x} = h\hat{y}^2$ are related by a transformation [11]
\[
\begin{pmatrix}
\hat{X} \\
\hat{Y}
\end{pmatrix} = \begin{pmatrix}
1 & \alpha \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\hat{x} \\
\hat{y}
\end{pmatrix},
\]
\[
\begin{pmatrix}
\frac{\partial \hat{X}}{\partial \hat{Y}}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
-\alpha & 1
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \hat{x}}{\partial \hat{y}}
\end{pmatrix},
\] (4)

where \( \alpha = \frac{\hbar}{q^1} \).

The product of functions is defined via the Gerstenhaber star product [15]: Let \( A \) be an associative algebra and let \( D_i, E_i : A \rightarrow A \) be a pairwise derivations. Then the associative star product of \( a \) and \( b \) is given by

\[
a \star b = \mu \circ e^{\zeta \sum_i D_i \otimes E_i} a \otimes b,
\] (5)

where \( \zeta \) is a parameter and \( \mu \) the undeformed product given by

\[
\mu (f \otimes g) = fg.
\] (6)

On the Manin plane, we write this star product as:

\[
f \star g = \mu \circ e^{\frac{\theta i}{2} \left( X \frac{\partial}{\partial x} \otimes Y \frac{\partial}{\partial y} - Y \frac{\partial}{\partial y} \otimes X \frac{\partial}{\partial x} \right)} f \otimes g.
\] (7)

A straightforward computation gives then the following commutation relations

\[
X \star Y = e^{\frac{\theta i}{2}} XY, \quad Y \star X = e^{\frac{-\theta i}{2}} YX.
\] (8)

Whence

\[
X \star Y = e^{i\eta} Y \star X, \quad q = e^{i\eta}.
\] (9)

Thus we recover the commutation relations for the Manin plane.

We can also write the product of functions as

\[
f \star g = f e^{\frac{i}{2} \int \Theta e^{\Theta} (x,y) \bar{\Theta} g}
\] (10)
where $\Theta^{kl}(X,Y) = \eta XY \epsilon^{kl}$ with $\epsilon^{12} = -\epsilon^{21} = 1$.

Using (4) we can define the product of functions on the hybrid space as

$$f \star g = \mu \circ e^{\frac{i}{2}[\sigma \Theta^{kl}(x,y) \otimes (\nu \Theta^{kl}(x,y) - (\nu \Theta^{kl}(x,y) - (\nu \Theta^{kl}(x,y) - \nu \Theta^{kl}(x,y))]} (f \otimes g). \tag{11}$$

A direct computation gives

$$x \star y = e^{\frac{i\eta}{2}xy} \left( e^{\frac{i\eta}{2} - 1} \right) \alpha y^2, \quad y \star x = e^{-\frac{i\eta}{2}yx} \left( e^{-\frac{i\eta}{2} - 1} \right) \alpha y^2, \tag{12}$$

whence

$$x \star y = e^{i\eta} y \star x + \left( e^{i\eta} - 1 \right) \alpha y^2. \tag{13}$$

Thus we recover the commutation relations for the hybrid plane $x \star y = qy \star x + hy^2$ with $q = e^{i\eta}$.

Expanding to first nontrivial order in $\eta$ and $h$ ($\eta, h << 1$) we find

$$f \star g = f g + i \frac{\eta}{2} \left( x \frac{\partial f}{\partial x} + \alpha y \frac{\partial f}{\partial x} \right) \left( y \frac{\partial g}{\partial y} - \alpha y \frac{\partial g}{\partial x} \right)$$

$$- i \frac{\eta}{2} \left( y \frac{\partial f}{\partial y} - \alpha y \frac{\partial f}{\partial x} \right) \left( x \frac{\partial g}{\partial x} + \alpha y \frac{\partial g}{\partial x} \right)$$

$$= f g + i \frac{\eta}{2} \left( qy - ihy^2 \right) \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \left( \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \right). \tag{14}$$

We can also write (11) as

$$f \star g = f \ e^{\frac{i}{2} \sigma \Theta^{kl}(x,y) \tilde{\partial}_l g}. \tag{15}$$

Here the antisymmetric matrix $\Theta^{kl}(x,y) = \eta (x + \alpha y) y e^{kl} = (\eta xy - ihy^2) \epsilon^{kl}$ with $\epsilon^{12} = -\epsilon^{21} = 1$.

The $(q,h)$-deformed infinitesimal gauge transformations are defined by
\[ \delta_{\lambda} \hat{A}_\mu = \partial_\mu \hat{\lambda} + i \left[ \hat{A}_\mu, \hat{\lambda} \right]_\star = \partial_\mu \hat{\lambda} + i \hat{\lambda} \star \hat{A}_\mu - i \hat{A}_\mu \star \hat{\lambda}, \]
\[ \delta_{\lambda} \hat{F}_{\mu\nu} = i \hat{\lambda} \star \hat{F}_{\mu\nu} - i \hat{F}_{\mu\nu} \star \hat{\lambda}. \]  

(16)

To first order in \( \theta \), the above formulas for the gauge transformations read

\[ \delta_{\lambda} \hat{A}_\mu = \partial_\mu \hat{\lambda} - \frac{1}{2} \hat{\theta}^{\rho\sigma} (x, y) \left( \partial_\rho \hat{\lambda} \partial_\sigma A_\mu - \partial_\rho A_\mu \partial_\sigma \hat{\lambda} \right), \]
\[ \delta_{\lambda} \hat{F}_{\mu\nu} = -\frac{1}{2} \hat{\theta}^{\rho\sigma} (x, y) \left( \partial_\rho \hat{\lambda} \partial_\sigma F_{\mu\nu} - \partial_\rho F_{\mu\nu} \partial_\sigma \hat{\lambda} \right). \]  

(17)

To ensure that an ordinary gauge transformation of \( A \) by \( \lambda \) is equivalent to \((q, h)\)-deformed gauge transformation of \( \hat{A} \) by \( \hat{\lambda} \) we consider the following relation [13]

\[ \hat{A} (A) + \delta_{\lambda} \hat{A} (A) = \hat{A} (A + \delta \lambda A). \]  

(18)

We first work the first order in \( \theta \)

\[ \hat{A} = A + A' (A) \]
\[ \hat{\lambda} (\lambda, A) = \lambda + \lambda' (\lambda, A). \]  

(19)

Expanding (18) in powers of \( \theta \) we find

\[ A'_\mu (A + \delta \lambda A) - A'_\mu (A) - \partial_\mu \lambda' = \theta^{kl} (x, y) \partial_k A_\mu \partial_l \lambda. \]  

(20)

The solutions are given by

\[ \hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\rho\sigma} (x, y) (A_\rho F_{\sigma\mu} + A_\rho \partial_\sigma A_\mu), \]
\[ \hat{\lambda} = \lambda + \frac{1}{2} \theta^{\rho\sigma} (x, y) A_\sigma \partial_\rho \lambda. \]  

(21)
The $q$-deformed curvature $\hat{F}_{\mu\nu}$ is given by

$$
\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i \left[ \hat{A}_\mu, \hat{A}_\nu \right] \star,
$$

$$
= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i \hat{A}_\mu \star \hat{A}_\nu + i \hat{A}_\nu \star \hat{A}_\mu.
$$

(22)

Using (21) we find

$$
\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma}(x) (F_{\rho\mu}F_{\nu\sigma} - A_\rho \partial_\sigma F_{\mu\nu})
$$

$$
- \frac{1}{2} \partial_\mu \theta^{\rho\sigma}(x) (A_\rho F_{\sigma\nu} + A_\rho \partial_\sigma A_\nu)
$$

$$
+ \frac{1}{2} \partial_\nu \theta^{\rho\sigma}(x) (A_\rho F_{\sigma\mu} + A_\rho \partial_\sigma A_\mu),
$$

(23)

which we can write as

$$
\hat{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} + o(\theta^2),
$$

(24)

where $f_{\mu\nu}$ is the quantum correction linear in $\theta$. The quantum analogue of the action (1) is given by

$$
\hat{S} = - \frac{1}{4} \int d^4 x \, \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}.
$$

(25)

For functions $f$, $g$ that vanish rapidly enough at infinity, the measure is defined such that

$$
\int f \star g = \int g \star f = \int f \, g.
$$

(26)

From this equation we see that the kinetic part the action is the same as its commutative version. So the free field propagators in commutative and non commutative spaces are the same.

The deformed action (25) reads
\[ \tilde{S} = S + S_\theta, \quad (27) \]

where \( S \) is the undeformed action (1) and \( S_\theta \) is the correction linear in \( \theta \).

Now, we make the following remarks. First for \( h = 0 \) (\( \theta^{kl}(x, y) = \eta xy\epsilon^{kl} \)), the action (25) is just the first order \( q \)-deformed action.

Second, for \( \eta = 0 \) (\( \theta^{kl}(x, y) = -ihy^2\epsilon^{kl} \)), the action (25) is the Jordanian \( h \)-deformed action.

For \( h = 0 \) and \( q = 1 \ (\eta = 0) \) this action coincides with the classical action (1) as it should be.

To conclude, we have obtained a more general Seiberg-Witten map in \((q, h)\)-deformed quantum plane which reduces to the known ones at some limits. The method developed in this letter can be applied to various quantum deformed models [16]-[19].

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