Understanding the Benefit of Being Patient in Payment Channel Networks

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Abstract—Scaling blockchain efficiency is crucial to its widespread usage in which the payment channel is one of the most prominent approaches. With payment channels and the network they construct, two users can move some transactions off-chain in a predetermined duration to avoid expensive and time-consuming on-chain settlements. Existing work is devoted to designing high-throughput payment channel networks (PCNs) or efficient PCN routing policies to reduce the fee charged by intermediate nodes. In this paper, we investigate the PCN routing from a different perspective by answering whether the routing fee of transactions can be saved through being a bit more patient. The key idea is to reorder the processing sequence of atomic transactions, other than to handle each of them separately and immediately. We present two mechanisms, one is periodic transaction processing assisted by a PCN broker and the other is purely strategic waiting. In the former, all the incoming transactions in a short time interval are processed collectively. We formulate an optimization model to minimize their total routing fee and derive the optimal permutation of processing transactions as well as the routing policy for each of them. A Shapley value based scheme is presented to redistribute the benefit of reordering among the transactions efficiently and fairly. In the latter, we model the waiting time of a strategic transaction on a single payment channel as the first passage time problem in queueing theory when the transaction value is higher than the edge capacity upon its arrival. By capturing the capacity dynamics, we are able to calculate the recursive expression of waiting time distribution that is useful to gauge a user’s cost of patience. Experimental results manifest that our cost redistribution mechanism can effectively save routing fees for all the transactions, and the waiting time distribution coincides with the model well.

I. INTRODUCTION

Since the advent of Bitcoin in 2008, we have witnessed the booming of various decentralized cryptocurrencies and the tremendous attentions they have gained. The historical transactions between cryptocurrency clients are recorded in a global and public data structure known as blockchain. It is envisioned that blockchain technology together with digital payment will embrace many new fields such as manufacturing, international trading, healthcare, etc [1] [2] [3]. Despite of their ever-going prosperity, cryptocurrencies suffer from poor scalability. For instance, Bitcoin [4] processes 7 transactions per second and Ethereum processes 15 transactions per second [5]. In contrast, visa network [6] can handle 1700 transactions per second, several orders of magnitude ahead of cryptocurrencies. The very low throughput hurdles its wide adoption. Scaling blockchain efficiency has become one of the most important issues that needs to be solved in order to ravel the transitional means of payment.

At present, there are several effective ways to improve the scalability of blockchain including payment channel [7], segregated witness [8] and shard chain [9] in which the payment channel is the most prominent one. It is an off-chain solution that two cryptocurrency users are allowed to deposit tokens on the blockchain for the transactions between them in a predetermined duration. When establishing a payment channel, both parties agree on the amount of deposit, also called capacity, which measures the maximum value of transfer from one to the other. In the course of a transaction, the sum of capacity in both directions remain the same but the unilateral capacities change. The users (or nodes interchangeably) that do not have a direct channel can chain multiple payment channels together into a payment channel network (PCN). Practical PCNs include Lightning network [7] in the Bitcoin system, Raiden network [10] in the Ethereum system and COMIT [11] in cross-chain applications that utilize the Hash-time Lock Contract (HTLC) scheme. A transaction can be routed from the sender to the receiver via intermediate nodes as long as the channel capacities on this route are sufficient, or will be processed on the public chain otherwise. As a return, the intermediate nodes will charge each transaction a certain amount of fee for the interconnections provided, and this amount is usually much lower than the overall cost on the public chain. Selecting the appropriate intermediate channels with the minimum cost has become an important issue in a blockchain PCN system.

The literature on route selection with the purpose of transaction cost reduction can be roughly categorized into two types. One is for an individual transaction. Zhang et al. [12] designed the Cheapay algorithm to find the cheapest available path with time and capacity constraints. A series of closely related works by Piatkovskyi et al. [16], Rohrer et al. [17] and Piatkovskyi et al. [18] presented new mechanisms which divided a transaction and transmitted on different paths to resist the capacity constraints. Ren et al. [14] proposed a new charging mechanism to maintain the balance of capacity, where the fee rate is defined as a reciprocal of the exponential function of the current capacity, so that more transactions can be processed successfully. The other is to design the mechanism for a sequence of transactions. Wang et al. in [13] proposed the Flash algorithm. By dividing the transaction into two categories according to the transaction amount, the large
transaction mainly focuses on the transaction fee cost, while the small transaction mainly focuses on the detection path cost. Varma and Maguluri utilized bilateral queues achieving as many transactions as possible can be processed successfully by using the concept of re-balancing on the chain [19]. Sivaraman et al. proposed the Spider network with the idea of packetizing transactions and adopting a multi-path transport protocol for high-throughput PCN routing [15]. However, the literature puts the emphasis on either the minimum cost or the high efficiency, while little attention is paid to their delicate tradeoff.

In this paper, we study the payment channel routing from a novel perspective: instead of pursuing the extreme efficiency, can the users save their transaction routing fees by being a bit more patient? Existing PCNs process the incoming transactions atomically and instantaneously [15]. Multiple transactions that arrive at different instants are routed on a first-come-first-serve (FCFS) basis. Two adverse effects might occur. One is that a transaction fails in the PCN and has to resort to the costly public chain if there is not a path with sufficient funds, the other is that the overall routing fee of a set of transactions (from/to different nodes) is usually high. As the underlying reason, PCNs are time evolving with dynamic edge capacities. For instance, Alice and Bob has built a payment channel. After a transaction from Alice to Bob, the capacity of Alice to Bob decreases and that of Bob to Alice increases, and their total capacity remains unchanged. Therefore, the FCFS processing is myopic in that a transaction possible misses the cheap paths created by the subsequent transactions, or intercepts their cheap paths unfortunately. Combined with the atomic transfer of transaction values in full, the FCFS means further increases the total routing fee. Inspired by this observation, we propose to reorder the processing of incoming transactions to reduce the routing fees, other than to process them immediately.

Our first mechanism stipulates (by a PCN broker) that the incoming transactions are handled together at a fixed duration periodically. We formulate an optimization problem to minimize their total atomic routing fee whose output is the permutation of orders of transactions and the corresponding routing policy for each of them. Some transactions gain more, some gain less, while some others might lose. Under certain rules, we present a coalitional game framework to incentivize the form of a grand coalition using the famous Shapley value as the cost re-distribution mechanism. This mechanism is shown to be individual rational, efficient and fair such that all the transactions in the coalition benefit from the reduction of total routing fee, and the higher success rate of PCN processing is achieved.

Our second mechanism is rather intuitive. A “patient” transaction can wait for the increase of the edge capacity if the initial capacity upon its arrival is below its transaction value. This simple approach is practical in the absence of the intervention by a PCN broker. However, computing how much time it should wait is a very challenging task even on a single PCN channel with bidirectional edges. We model the waiting time as the first passage time in queueing theory, that is, the first time that the initial capacity increases above the transaction value, given the stochastic transaction arrival processes on both directions of a single payment channel. A novel stochastic model is built to capture the dynamics of edge capacities. We compute the recursive expression of the distribution of waiting time that is useful for the transaction sender to gauge his cost of patience. Simulation results validate the accuracy of our model.

Our major contributions are briefly summarized as below.

- We propose a novel idea of reordering the transactions actively or opportunistically, other than the chronological processing in order to reduce the PCN routing cost.
- We present a periodic transaction processing scheme that yields an optimal transaction order, and formulate a coalitional game with Shapley value as the benefit redistribution mechanism.
- We present a novel transient queueing model to capture the capacity dynamics of a payment channel, and calculate the recursive expression on the waiting time distribution of a strategic transaction.

II. PROBLEM DESCRIPTION

In this section, we describe the mathematical model of payment channel networks (PCNs) and the motivation of being patient in PCNs by blockchain transactions.

A. Temporal Network Model

A payment channel network can be represented as a time-dependent directed graph $G_t = (N, E, C, W, B)$, where $N$ is the set of nodes and $E$ is the set of edges. Each node $n_i \in N$ represents a cryptocurrency account that has built one or more payment channel agreements with other nodes; each directed edge $e_{ij} \in E$ represents a payment channel from node $n_i$ to $n_j$. The edge $e_{ij}$ is associated with a 3-tuple $(c_{ij}(t), w_{ij}, b_{ij})$, where the first is the maximum amount of cryptocurrency coins $n_i$ can pay to $n_j$ at time $t$ and the second is the price per-unit of coin transfer charged by $n_j$ on this edge if it is a relay node. Let $b_{ij}$ be the (flat-rate) base fee of using the channel $e_{ij}$ as the relay, regardless the amount of transaction size. We denote by $X_k$ the $k$th transaction in the payment channel network that is expressed as a 4-tuple $X_k = (n_s(k), n_r(k), v_k, t_k)$. The sender and the receiver of $X_k$ are denoted by $n_s(k)$ and $n_r(k)$, respectively. Here, $v_k$ is the payment value measured by the number of coins and $t_k$ is the time instant that the transaction takes place. One primary difference between the PCN and the traditional communication network is that a payment channel between $n_i$ and $n_j$ contains two bidirectional edges whose sum of capacities is a constant. When a transaction from $n_i$ to $n_j$ has been processed successfully, the capacity of edge $e_{ij}$ decreases while that of edge $e_{ji}$ increases. For any pair of nodes $n_s$ and $n_r$ in the transaction $X_k$, a feasible path is an end-to-end path on $G_t$ whose edge capacities are sufficient to transfer the transaction value.

The business model of using payment channels is the following. If the nodes $n_s$ and $n_r$ form a payment channel
directly, there is no need for \( n_s \) to pay for using this channel when \( n_r \) is the receiver. If the transaction \( X_k \) is forwarded along a path \( \mathcal{P}_k(t) \) that starts at \( n_s \) and ends at \( n_r \), the intermediate nodes should be paid for the coin transfer. For instance, consider the path \( \mathcal{P}_k(t) = \{ n_1, n_2, n_3, n_4 \} \). The node \( n_1 \) will pay a certain amount fees to \( n_2 \) of using the channel \( e_{23} \), and \( n_2 \) needs to pay to \( n_3 \) for using the channel \( e_{34} \), but \( n_3 \) does not pay to the receiver \( n_4 \). Hence, the total routing fee that \( X_k \) need to pay can be expressed as:

\[
f = (w_{34} \cdot v_k + b_{34}) + \left( w_{23} \cdot (v_k + v_k w_{34} + b_{34}) + b_{23} \right)
\]

Formally, the total routing fee on the \( \mathcal{P}_k \) at graph \( G_t \) is given by:

\[
f(X_k, G_t) = \left\{ \begin{array}{ll}
x \\
\sum_{e_{ij} \in \mathcal{P}_k} (w_{ij} \cdot v_k(i,j) + b_{ij}) & \text{otherwise}
\end{array} \right.
\]

(1)

Here, \( V_k(i,j) \) represent the actual amount that \( X_k \) need to transfer through \( e_{ij} \), including \( v_k \) and the routing fees to be paid for subsequent edges. When the value of a transaction exceeds the edge capacity, it can be sent to the public chain [15] that will incur a fixed amount of payment plus a much longer confirmation time. Without loss of generality, we deem the processing cost of a transaction as a constant \( x \) in the public chain that is higher than the routing fee in the PCN [28].

In the hop-by-hop value transfer, the transaction fee need to transmit with the original transaction, which means \( f(X_k, G_t) + v_k \) is the actual value which \( n_s(k) \) need to pay at time \( t \). For the intermediate edges on \( \mathcal{P}_k(t) \), the capacity must be not less than the value and the total charge for remaining routes. the feasible flow paths should satisfy:

\[
\left\{ \begin{array}{ll}
V_k(i,j) \leq c_{ij}(t), \forall e_{ij} \in \mathcal{P}_k(t) \\
f(X_k, G_t) \leq x
\end{array} \right.
\]

(2)

We make a few assumptions to modeling our efforts while complying with the real-world blockchain PCNs. They are commonly used in the literature.

**Assumption 1:** (Information availability) The global information including the network topology, the channel capacities and edge prices, is foreknown to all the nodes [21] [22] [23].

**Assumption 2:** (Immediate Processing) The transaction over a payment channel network takes effect immediately if it is scheduled for processing [13] [24].

**Assumption 3:** (Indivisible Transaction) A transaction cannot be split into multiple transactions of smaller values.

The splitting of transactions may involve the multi-path routing problem that is usually more complicated. We only consider the indivisible transactions in this work, yet our problem and methodology are applicable to the divisible transactions.

### B. Transaction Reordering

The original purpose of constructing payment channel networks is to speed up the processing of Bitcoin transactions. The state-of-the-art efforts are devoted to designing high-throughput payment channel systems without sacrificing cryptocurrency security [7]. As a basic incentive to maintain the PCN, the payment channels may charge the sender of the transaction a small amount of routing fee. As a consequence, the user is inclined to selecting a feasible path that yields the minimum total routing fee [12].

Given a train of transactions sorted by their arrival times, the graph \( G_t \) changes after each successful transfer. In this sequential processing order, choosing the minimum cost path for each transaction might be myopic because the graph is dynamic. A transaction may have a chance to find another path with even lower cost if it can “wait” for some time. We hereby illustrate that being patient is “egoistic” or “altruistic”.

Fig.1 and Fig.2 shows the different routing fee that need to be paid for the same three transactions in different order. The three transactions are \( X_0 = (D, C, 3, 0), X_1 = (E, C, 1, 1), X_2 = (E, D, 2, 2) \). Without losing generality, base transaction fee is set as 1 and each edge charge 50% transaction amount. This setting has no influence for the problem which we want to illustrate. The red path (uniform dotted lines) represents the cheapest path through, the green path (uneven dotted line) represents the edges that change in the opposite direction. The number on the line indicates the capacity of the channel at the current time. Fig.1 shows the change in PCN topology when each transaction chooses to be processed as soon as it comes into being. For each transaction, the routing fee is \( f(X_0, G_0) = 5, f(X_1, G_1) = 0, f(X_2, G_2) = 5 \). If \( X_1 \) chooses to be processed at time \( \eta_1 > 2 \) and \( X_0 \) chooses to be processed at time \( \eta_0 > \eta_1 \), the sequential processing order is \( X_2, X_1, X_0 \), the change in PCN topology is shown in Fig.2. In this order, routing fee is \( f(X_0, G_{\eta_0}) = 0, f(X_1, G_{\eta_1}) = 1.5, f(X_2, G_{\eta_2}) = 2 \).

**Egoistic Waiting.** Egoistic waiting means that waiting can reduce the user’s own routing cost. If \( X_0 \) chooses not to wait, it must be processed in public chain and the cost is \( x \). Through waiting, \( X_0 \) can be processed after \( X_1 \) and \( X_2 \), there is an available path due to the successful processing of \( X_1 \) and \( X_2 \). The routing cost decrease for \( X_0 \) through waiting.

**Altruistic Waiting.** Altruistic waiting is defined as the waiting of one user will cause its own routing cost to increase or remain unchanged, but other users’ routing cost may decrease, leading to a decrease in the total cost. If \( X_1 \) is processed at time \( \eta_1 \), the capacity of \( E \to C \) is sufficient and the routing fee is 0. After waiting, \( X_1 \) can only choose \( E \to B \to C \) due to the successful processing of \( X_2 \), the routing fee of \( X_1 \) increase to 1.5. However, the routing fee of \( X_2 \) decrease to 2 because the processing of \( X_1 \). The total cost of \( X_1 \) and \( X_2 \) decrease.

By reordering the transactions (other than first-come-first-serve (FCFS)), either some of the transactions can lower down their routing fees charged by the PCN, or these transactions as a whole can save a certain amount of routing fees. In order to take this advantage, there needs an incentive mechanism to encourage the users to be more patient. We hereby consider two scenarios, in which the former requires the intervention
of the broker of the payment channel network, and the latter is completely compatible to the existing payment channel networks.

![Fig. 1: The process of $X_0 = (D, C, 3, 0), X_1 = (E, C, 1, 1), X_2 = (E, D, 2, 2), b_{ij} = 1, w_{ij} = 0.5 \ \forall e_{ij} \in E.$](image)

1) Periodic Transaction Processing (PTP): The broker of the PCN processes all the incoming transactions cyclically after a fixed duration $T$. An optimal reordering policy will be implemented to minimize the total transaction cost, and the total routing fee will be redistributed among these transactions. As the underlying principle, no transaction will receive a higher routing fee in the reordered processing than in the FCFS processing.

2) Strategic Patience (SP): When a myopic user finds that the cheapest payment channel does not have the sufficient capacity to transfer his transaction value, he may resort to an expensive channel or the public chain. While a strategic user can predict how much time he can wait until the capacity of the cheap channel is greater than his transaction value, given the arrival patterns of the transactions at both directions of the channel.

III. Coalition Mechanism Design

In this section, we formulate a cooperative game and formally define the benefit distribution mechanism that is compatible to the optimal transaction reordering principle.

A. Cooperative game in PCN

Cooperative game (or coalition game narrowly) is a mathematical theory in revealing the behaviors of rational players in a cooperative setting. The players make agreements among themselves to form coalitions that affect their strategies and utilities, as opposed to the non-cooperative games. In what follows, we formulate the benefit (or cost) redistribution mechanism as a cooperative game, namely $CG$.

- **Player**: A user who initiates a transaction is regarded as a player. If not mentioned explicitly, the transaction is equivalent to the user so that the set of players are expressed as $X := \{X_0, X_1, \cdots, X_{N-1}\}$ with the arrival times $t_n \leq T, \ \forall 0 \leq n \leq N - 1$.

- **Worth function**: A function $\psi(S)$ where $S$ is an arbitrary non-empty subset of $X$.

We denote $\psi(S)$ as the worth function, which measures the benefit produced by coalitions. The redistribution mechanism in the coalition is Shapley value function.

**Definition 1 (Grand coalition)**: If every player chooses to join in coalition, the coalition is called as grand coalition.

B. Worth of $S$

The calculation of the worth function $\psi(S)$ is special in our PCN coalitional game. First of all, $\psi(S)$ is an outcome of minimum cost routing problem that demands an appropriate algorithm to generate this value. Second, the coalition is expressed in the form of reordering the transactions so that the sequence number of processing this coalition as a whole

Algorithm 1 Framework of calculating the cheapest path.

**Input**: Transaction, $X = (n_s, n_r, v)$; Topology of PCN, $G_0$; Output: Transaction fee, $f$; Path, $P$; New topology, $G$;

1: $G_1 := G_0$ for all edges with $(c_{ij} < v)$ removed;
2: $N = G_1$.nodes; $E = G_1$.edges;
3: $R := \{n_r\}; S := N \setminus n_r$;
4: Initialize result as $\{(fee_i, path_i)\}$,
   for every $e_{ij} \in E$, $fee_i := w_{ij} \cdot (v + fee_i)$, $path_i := [n_j, n_i]$;
   otherwise, $fee_i := x_{path_i}$, $path_i := [n_i]$.
5: $w_{n_i,j} := 0, b_{n_i,j} := 0, \text{for each } n_i \in n_s \text{neighbor}$;
6: while result was updated and $S$ is not empty do
7:   $n_i := \arg\min_{n_k \in R} fee_k$ $\forall e_{ij} \in E$.
8:   for $n_j$ in $n_i$.pre do
9:     if $fee_i + w_{ij} \cdot (v + fee_i) + b_{ij} < fee_j \text{ and } v + w_{ij} \cdot (v + fee_i) \leq c_{ij}$ then
10:        $fee_j := fee_i + w_{ij} \cdot (v + fee_i) + b_{ij}$;
11:        $path_j := path_i$.append($n_j$);
12:        $R$.append($n_j$); $S$.remove($n_j$);
13:    end if
14:   end for
15: end while
16: $fee := fee_s; path := path_s$;
17: for $e_{ij} \in path$ do
18:    $c_{ij} := c_{ij} - (v + fee_i)$;
19:    $c_{ji} := c_{ji} + (v + fee_i)$;
20: end for
21: return $f := fee_s, P := path_s, G := G_0$.
needs to be decided. Third, there might exist the “free-riding” phenomenon in the game. To facilitate the form of physically meaningful coalition(s), a set of rules are enforced.

**Rule 1:** The sequence order of a coalition $S \in X$ is determined by that of the earliest transaction in $S$. Denote by $\eta_S$ the order of coalition $S$, and by $\eta_k$ the order of $X_k$ in the original set of transactions. There are

$$\eta_k = \arg\min_j \{t_k | X_k \notin S\}; \quad (3)$$

$$\eta_S = \arg\min_j \{t_j | X_j \in S\}. \quad (4)$$

For instance, given four transactions $X_0, X_1, X_2, X_3$, where $X_1$ and $X_3$ form a coalition $S$, $\eta_0 = 0, \eta_1 = 1, \eta_3 = 3$. In other word, the processing sequence is $X_0, S, X_2$ accordingly. This is to say, the coalition does not affect the processing of the transactions who arrive earlier than all the coalitional players.

**Rule 2:** In $S$, the sequence of players should be rearranged so as to minimize the total routing fee. Consider a grand coalition $S = \{X_0, X_1, \cdots, X_{K-1}\}$, the PCN graph is redefined as $G_k$ before processing the transaction $X_k$. The optimal routing for a particular $X_k$ in the current topology of PCN is expressed below.

$$\min \sum_{(i,j)\in E} f_{ij}(X_k) \gamma_{ij}(X_k) \quad (5)$$

s.t. $\gamma_{ij}(X_k) := \begin{cases} 1, & \text{if } X_k \text{ is routed along } e_{ij}; \\ 0, & \text{else}. \end{cases}$ \quad (6)

$$\sum_{j\in N} \gamma_{ij}(X_k) - \sum_{j\in N} \gamma_{ji}(X_k) =: \begin{cases} 1, & n_i = n_s(k); \\ -1, & n_i = n_r(k); \\ 0, & \text{else.} \end{cases} \quad (7)$$

$$\gamma_{ij}(X_k)(v_k + f_{ij}(X_k)) \leq c_{ij}(\eta_k), \quad \forall e_{ij} \in E; \quad (8)$$

$$f_{ij}(X_k) = \gamma_{ij}(X_k) \sum_{h\in N} \gamma_{jh}(X_k)(w_{jh}v_k + f_{jh}(X_k) + b_{jh}) \quad (9)$$

$$f_{ij}(X_k) = 0, \quad \text{if } n_i = n_r(k); \quad (10)$$

$$\sum_{(i,j)\in E} f_{ij}(X_k) \gamma_{ij}(X_k) \leq \xi. \quad (11)$$

The objective is the sum of the routing fees on all the edges. The binary variable $\gamma_{ij}(X_k)$ in Eq.(6) indicates the usage of an edge, Eq.(7) represents the flow balance conditions, and Eq.(8) presents the maximum amount of value that can be processed. Eq.(9)~(10) refers to the transaction fee calculations, i.e. $f_{ij}(X_k)$ represents the routing fee that $X_k$ need to pay to other nodes after the edge $e_{ij}$. Actually, the main constraints are about the selection of feasible paths, and we need to find the cheapest one among the set of feasible paths. If such a path leads to a higher routing cost than $\xi$, the public chain will be selected for the value transfer.

Algorithm 1 is used to calculate the cheapest path for a single transaction and to update the PCN topology(Eq.5).

Algorithm 2 is designed to calculate the minimum routing fee that coalition need to pay. According to Algorithm 2, we can find that we should ensure as much as possible transactions can be processed successfully firstly. Under this premise, the value that can be transferred should be as much as possible.

We can get

$$\psi(S) = \sum_{X_k \in S} f(X_k, G_{\eta_k}) - f(X_k, G_{\eta_S}) \quad (12)$$

There are some properties about $\psi(S)$.

- **Cohesive:** A coalitional game $< X, \psi >$ with transferable payoff is cohesive if

$$\psi(X) \geq n \sum_{k=1}^{n} \psi(S_k)$$

for every partition $\{S_1, \cdots, S_n\}$ of $X$.

- **Weak superadditivity:** In a cooperation game $< X, \psi >$, for any $S, S_1, S \subset X, S_1 \subset S$, then there must be $\psi(S_1) \leq \psi(S)$, it is said that the characteristic function $\psi$ satisfies weak superadditivity.

The total benefit of coalition will increase because of the existence of altruistic waiting user. It is necessary for egoistic waiting user share part of their benefit with altruistic waiting user to compensate them for their losses. But there are some egoistic waiting users who want to monopolize the benefit and not join the coalition. This kind of players is called as Free-rider. In the social sciences, the free-rider problem is a type of market failure that occurs when those who benefit from resources, public goods (such as public roads or hospitals), or services of a communal nature do not pay for them[1] or under-pay. Free riders are a problem because while not paying for the good (either directly through fees or tolls or indirectly through taxes), they may continue to access or use it. We use an example to illustrate this problem and establish a constrain to avoid this kind of phenomenon.

![Fig. 3: The process of $X_2 = (A, C, 6, 2), X_0 = (C, E, 1, 0), X_1 = (C, B, 3, 1), b_{ij} = 0, w_{ij} = 0.5 \ \forall e_{ij} \in E$](image)

As Fig.3 shows, there are three transactions, $X_0 = (C, E, 1, 0), X_1 = (C, B, 3, 1), X_2 = (A, C, 6, 2)$, to simplify, we set $b_{ij} = 0$ and all of the $w_{ij} = 0.5$. If each transaction choose not to cooperate, the benefit of three transactions are 0, 0, 0 separately. If $X_2$ and $X_0$ choose to cooperate, according the priority mechanism, they should be processed firstly and the processing order is $X_2, X_0, X_1$. The benefit are $\xi - 0.5, \xi, 0$.
for \(X_0, X_1\) and \(X_2\). \(X_0\) is egoistic user and according the common sense, he should share some benefit with \(X_2\). \(X_1\) also benefit but he does not share his earning with \(X_2\) because of not joining in coalition. We call \(X_1\) Free-rider. \(X_2\) is the altruistic waiting user and he is not paid for his service to free-rider. Free-rider will lead to the instability of the coalition, so we need to set up an effective mechanism to resist this phenomenon.

**Definition 2 (Free-rider solution):** The topology of PCN is isomeric. The path of the player who does not join in S is checked and the cheaper path which is created by S is prohibited from using them.

Free-rider solution can protect the payoff of \(S\) and to avoid loss of income, middle nodes will choose obey it.

**Algorithm 2 Calculating the minimum routing fee of coalition.**

**Input:** Coalition, \(S = \{X_0, X_1, \ldots, X_n\}\); Topology of PCN, \(G_0\);

**Output:** \(\{\langle \text{fee}, \text{path} \rangle \} \) for all \(X_i \in S\);

1. \(X_{\text{list}} = \{ \text{Req}_i | \text{Req}_i \in \text{Permutations}(S) \} \);
2. \(G \leftarrow G_0\);
3. for all \(\text{Req}_i\) do
4. for all \(X_j \in \text{Req}_i\) do
5. \(f, P, G = \text{Algorithm 1}(X_j, G)\);
6. \(\text{result}_{\text{all}}.\text{append}(f, P)\);
7. end for
8. end for
9. \(\text{result} \leftarrow r_i \in \text{result}_{\text{all}}\) with largest number of successful transactions.
10. if Number of successful transactions is same then
11. \(\text{result} \leftarrow r_i \in \text{result}_{\text{all}}\) with more successful transfer amount.
12. end if
13. return \(\text{result}\);

**C. Benefit Redistribution via Shapley Value**

The reordering of the players may cause some of them pay less routing fee and while some others pay more. In order to incentive the players to form a grand coalition, the redistribution of the benefit is inevitable. It needs to be fair to all the players, and yields a unique payoff vector known as the value of the coalitional game [31].

**Definition 3:** A benefit redistribution mechanism is an operator \(\phi\) on a payment channel network \(< X, \psi >\) that allocates a cost vector \(\phi(X, \psi) = (\phi_0, \phi_1, \ldots, \phi_{N-1})\) in \(\mathbb{R}^N\) for all the players.

We hereby design a benefit redistribution mechanism with the following desirable properties among the players.

**Property 1 (Rationality):** If \(\phi_i(S) \geq \psi(\{X_i\})\), it is **individual rationality.** If \(\sum_{X_i \in S} (\phi_i(S)) = \psi(S)\), it is **group rationality.**

Individual rationality motivates the players to join the coalition and cooperate accordingly. Group rationality requires that the profit assigned equals the profit received from the coalition.

**Property 2 (Balanced contribution):** A value \(\phi\) satisfies the **balanced contributions property** if for every coalitional game with transferable benefit \(< X, \psi >\) we have

\[
\phi_i(X) - \phi_i(X \setminus \{X_i\}) = \phi_j(X) - \phi_j(X \setminus \{X_i\})
\]

where \(X_i \in X\) and \(X_j \in X\).

The property of balanced contributions addresses the fairness between any pair of transactions in X. If we start with a set of two transactions \((X, \psi) = (\{X_1, X_2\}, \psi)\), the gain from cooperation is \(\psi(X) - \psi(X_1) - \psi(X_2)\). Thus, the egalitarian solution is

\[
\phi_i(X, \psi) = \psi(\{i\}) + \frac{1}{2}[\psi(X) - \psi(\{X_1\}) - \psi(\{X_2\})].
\]

**Property 3 (Symmetry):** If \(\phi(S \cup X_i) = \phi(S \cup X_j)\), for all \(S \in X \setminus \{X_i, X_j\}\), then \(\phi_i(S) = \phi_j(S)\).

The symmetry property requires that if two players contribute the same to every subset of other players, they should receive the same amount of cost.

**Property 4 (Dummy):** If \(i\) is a dummy player in coalition \(S\) then \(\phi(S) = \psi(\{X_i\})\).

In our game, if a player does not contribute to the reduction of routing fee, i.e. \(\psi(S) + \psi(\{X_i\}) = \psi(S \cup X_i)\), the payoff of this player of joining the coalition is identical to that of not joining it.

**Property 5 (Additivity):** For any two game \((S, \psi_1)\) and \((S, \psi_2)\) we have \(\phi_i(\psi_1 + \psi_2) = \phi_i(\psi_1) + \phi_i(\psi_2)\) for all \(i \in S\), where \(\psi_1 + \psi_2\) is the game defined by \((\psi_1 + \psi_2)(S) = \psi_1(S) + \psi_2(S)\) for every coalition \(S\).

Additivity ensures that even if charging rate of some edges changes, our redistribution mechanism is still available.

The **Shapley value** is the unique value that satisfies all five properties. Then, it is defined as follows. [30]

**Definition 4 (Marginal contribution):** The marginal contribution of player \(i\) to any coalition \(S\) with \(i \notin S\) in the game \(< X, \psi >\) to be

\[
\Delta_i(S) = \psi(S \cup \{i\}) - \psi(S)
\]

**Definition 5 (Shapley value):** The Shapley value \(\phi\) is defined by the condition

\[
\phi_i(S) = \sum_{S_i \subseteq S} \frac{|S| - |S_1|!(|S_1| - 1)!}{|S|!} \Delta_i(S_1 \cup \{X_i\}), \forall X_i \in S.
\]

where \(|S|\) and \(|S_1|\) represent the numbers of players in \(S\) and \(S_1\), respectively. Actually, Shapley value represents the expected marginal contribution over all orders of this player to the set of players who precede him.

**Lemma 1:** \(\psi\) satisfies weak superadditivity and the corresponding Shapley value satisfies the individual rationality.

**Proof:** Here, \(\psi\) satisfies weak superadditivity, then:

\[
\psi(S \cup i) \geq \psi(S) + \psi(\{i\}), \forall S \in X, i \in X \setminus S
\]

Therefore \(\Delta_i(S) \geq \psi(\{i\})\). According Eq(13), we can get that \(\phi_i(S)\) satisfies individual rationality. This lemma is proved.
D. Operation Procedure

We have shown the cooperation mechanism and the redistribution function in the last three subsections. In this subsection, we address the operation procedure of PTP. Let us briefly introduce the implementation process of this mechanism.

- Step1: Before $T$, each transaction $X_i$ need to choose joining in the coalition or not at $t_i$. No matter they join or not, they need to submit information to system, including their sender, receiver, value. The arrival time is automatically recorded by the system.

- Step2: At time $T$, system first calculate the routing fee for each transaction based on the first-come-first-serve. This result can get through the information they submit.

- Step3: Then, system calculate the priority of each transaction through Eq(4) and Eq(3).

- Step4: According to the result of priority, system begin to process the transactions. For coalition $S$, it will reorder the players to get the minimum routing fee they need to pay. Algorithm 2 is used to solve this problem.

- Step5: System need to distribution the payoff to each player in $S$ using $\phi_i(S)$.

- Step6: System initialize the time to 0 until the next $T$ arrives, repeating from step1.

Remark: All operations can be done through smart contracts, so it does not break the principle of decentralization. The system need to store topology of PCN at initial time to defend free-rider.

IV. STOCHASTIC WAITING TIME

In this section, we formulate the stochastic model of capacity dynamics on a simplified PCN and calculate the waiting time distribution of a strategic user.

A. Stochastic Capacity Model

When there is absence of the intervention of the PCN broker, the cost redistribution will be difficult to implement because the utility transfer among users cannot be enforced. Now we will show that an individual “patient” transaction can achieve a lower cost by waiting for the feasibility of a payment channel when the capacity of this channel is below the transaction value in the very beginning. We consider a simplified PCN with only two nodes forming a payment channel in Fig.4 that can be easily generalized to the network with parallel payment channels. Node $A$ (resp. node $B$) is in charge of processing the left-side (resp. right-side) transactions on edge $e_{AB}$ (resp. edge $e_{BA}$). The successful transactions from $A$ to $B$ increase the capacity $e_{BA}$ and decrease the capacity $e_{AB}$, and vice versa. When a transaction finds the capacity $c_{AB}$ insufficient, it can keep patient until $c_{AB}$ is larger than its transaction value. An interesting question is how much time a transaction needs to wait before the successful processing, given the stochastic arrivals of other transactions on both sides of the channel.

Capacity Dynamics. The time axis is set to $t = 0$ when a transaction arrives to node $A$ and will be transferred to node $B$ through edge $e_{AB}$. If the initial capacity $u$ is no less than the transaction value $v$, it is processed immediately. Otherwise, it will wait for until $c_{AB}$ is greater than $v$. Without loss of generality, we denote this transaction as the tagged transaction $X$. Denote by $U_i$ the capacity of edge $e_{AB}$ at time $t$, there exists

$$U_i = u + \sum_{i=0}^{N_1(t)} v_{2i} - \sum_{i=0}^{N_2(t)} v_{1i}$$

(14)

where $v_{1i}$ indicates the value of the $i^{th}$ transaction from $A$ to $B$ and $v_{2i}$ indicates that of the $i^{th}$ transaction from $B$ to $A$. Here, $N_1(t)$ and $N_2(t)$ are the numbers of transactions on edge $e_{AB}$ and $e_{BA}$ by time $t$ respectively. We make the following assumptions.

- The arrival process of transactions on edge $e_{AB}$ is the Poisson process $\{N_1(t) : t \geq 0\}$ with parameter $\lambda_1$, and that on edge $e_{BA}$ is the Poisson process $\{N_2(t) : t \geq 0\}$ with parameter $\lambda_2$.

- The transaction values on both directions, i.e. $v_{1i}$ and $v_{2i}$, are independent and identically distributed (i.i.d) with the probability density function $g(v)$.

The assumption of Poisson arrival is commonly adopted in decentralized payment systems [27], and the bilateral transactions are deemed to have the same distribution of values but with different arrival rates. Therefore, the evolution of $\{U_i : t \geq 0\}$ is a compound Poisson process.

The waiting time of the transaction $X$ is actually the duration between 0 and the instant that the capacity $c_{AB}$ is greater than $v$ for the first time. Thus, the waiting time can be modeled as the first passage time of $U_i$ to $v$ in queuing theory. Formally, we denote $t$ as the waiting time that has

$$\hat{t} = \arg \min_{t} \{ t \mid U_t - \sum_{i=0}^{N_1(t)} v_{1i} + \sum_{i=0}^{N_2(t)} v_{2i} \geq v \}$$

(15)

Given the stochastic arrival of transactions on the edges $e_{AB}$ and $e_{BA}$, $\hat{t}$ is a random variable by nature.

Before calculating the distribution of $\hat{t}$, we provide the precondition of waiting. The sum of $c_{AB}$ and $c_{BA}$ has been decided upon the construction of the payment channel so that the transaction $X$ cannot be processed in this channel if this sum is below $v$. We provides the following lemma with regard to the expected waiting time but omitting the proof due to its simplicity.

Lemma 2: The expected waiting time $E[\hat{t}|u]$ is calculated by

$$E[\hat{t}] = \left\{ \begin{array}{ll} \frac{v-u}{(\lambda_2-\lambda_1)\mu} & \lambda_2 > \lambda_1, \\ \infty & \text{otherwise}. \end{array} \right.$$

(16)

B. Computing Waiting Time Distribution

The expected waiting time overlooks the stochastic behavior of transaction arrivals and the random transaction values,
which is not sufficient to quantify the characteristic of the waiting time. We are more interested in how the chance of the successful processing increases over the waiting time. Hereby we analyze the probability distribution of the waiting time.

Denote by $\Phi(v_k, u, t)$ the probability of a transaction being processed by time $t$

$$\Phi(v_k, u, t) = \Pr\{\tilde{t} \leq t \mid U_0 = u, v = v_k\}. \quad (17)$$

where $u$ is the initial capacity observed by $X$.

Fig. 5: Example of arrival transactions on time-slot.

Computing $\Phi(v_k, u, t)$ is a very challenging task. We use a timeline in Fig.5 to illustrate the possible events that inspires our basic idea. One can observe that the first passage event must occur at the instant of processing a transaction on edge $e_{BA}$. Then, we can compute $\varphi(r, n)$ which is the probability that the first passage event takes place at the arrival of the $n^{th}$ transaction on edge $e_{BA}$, where $r = v_k - u$. The calculation of $\Phi(v_k, u, t)$ is thus transformed into the union of $\varphi(r, n)$ for all $n$ mutually exclusive events that happen before time $t$.

During the inter-arrival time of $(n - 1)^{th}$ and $n^{th}$ transaction from $B$ to $A$, we need to scrutinize the number of transaction arrivals on edge $e_{AB}$ and the distribution of their total value. Formally, we provide the following theorem on the distribution of waiting time.

**Theorem 1**: The distribution of the waiting time $\Phi(v_k, u, t)$ is expressed as an iterative equation:

$$\Phi(v_k, u, t) = \sum_{n=1}^{\infty} \sum_{j=0}^{n} e^{-\lambda_2 t} \frac{\lambda_2^j}{j!} \varphi(r, n) \varphi(r, n - 1);$$

$$\varphi(r, n) = \sum_{m=0}^{\infty} p_m \int_0^\infty q_n dG^m(z) + \varphi(r, 1); \quad (18)$$

$$\varphi(r, 1) = \sum_{m=0}^{\infty} p_m \int_0^\infty [1 - G(r + z)] dG^m(z); \quad (19)$$

$$q_n = \int_0^{u+z} \varphi(u + z - x, n - 1) dG(x); \quad (20)$$

$$p_m = \frac{\lambda_1^m \lambda_2}{(\lambda_1 + \lambda_2)^{m+1}}; \quad (21)$$

where $G(x)$ is the probability distribution function of $x$, $G^m(z)$ represents the convolution of $G(v_1)$ by $m$ times.

**Proof**: Our proof is carried out in two steps.

**Step 1: Equivalence between $\Phi(v_k, u, t)$ and $\varphi(r, n)$**. Define $U(m)$ the capacity of edge $e_{AB}$ when there are $m$ transaction arrivals on edge $e_{BA}$. Define $T_k$ the inter-arrival time of two consecutive arrivals, $X_{k-1}$ and $X_k$, on edge $e_{AB}$. Then, the probability of the first passage event upon the arrival of $Y_m$ is given by:

$$U(m) = \sum_{i=0}^{m} v_{2i} - \sum_{i=0}^{m} v_{2i} \quad (24)$$

$$\varphi(r, n) = \Pr\{U(m) \geq r, \exists m \leq n\}. \quad (25)$$

The first passage event is the union of mutually exclusive events that the first passage happens at the $n^{th}$ transaction arrival on edge $e_{BA}$. Accordingly, the waiting time distribution is expressed as

$$\Phi(v_k, u, t) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} T_j < t | \lambda_2 \varphi(r, n) - \varphi(r, n - 1)). \quad (26)$$

Since the arrival processes are Poisson, the number of transaction arrivals on each edge is given by:

$$\Pr\{N_1(t) = m\} = \frac{\lambda_1^m e^{-\lambda_1 t}}{m!}, \quad (27)$$

$$\Pr\{N_2(t) = m\} = \frac{\lambda_2^m e^{-\lambda_2 t}}{m!}. \quad (28)$$

The inter-arrival time of transactions obeys the memoryless exponential distribution so that we can write down the sum of $n$ random variables in the form of Erlang distribution

$$\Pr\{T_n \leq t | \lambda = 1 - e^{-\lambda t}\} \quad (29)$$

$$\Pr\{\sum_{i=1}^{n} T_i \leq t | \lambda_2 = \sum_{j=n}^{\infty} e^{-\lambda_2 t} \frac{(\lambda_2 t)^j}{j!}. \quad (30)$$

**Step 2: Calculation of $\varphi(r, n)$**. According to the full probability formula, we can obtain

$$F_1 = \Pr\{U(m) \geq r, \exists m \leq n | T_1 = t, \sum_{i=1}^{y} v_{1i} = z, v_{21} = x\} \quad (31)$$

$$\begin{align*}
\varphi(r, n) &= \Pr(N_1(t) = y) \int_0^\infty \Pr(T_1 = t) \int_0^\infty \Pr(v_{1i} = z) \\
&= \sum_{y=0}^{\infty} \frac{\lambda_1^y}{y!} e^{-\lambda_1 t} \int_0^\infty \lambda_2 e^{-\lambda_2 t} dt \int_0^\infty \Pr(v_{1i} = z) \int_0^\infty F_1 d(G(x)) dx \\
&= \sum_{y=0}^{\infty} \frac{\lambda_2^y}{y!} \int_0^\infty F_1 d(G(x)) dx \\
&= \int_0^\infty F_1 d(G(x)) dx \\
&= \int_0^\infty F_1 d(G(x)) dx \\
&= \int_0^\infty F_1 d(G(x)) dx \\
&= \int_0^\infty F_1 d(G(x)) dx \\
&= \int_0^\infty F_1 d(G(x)) dx
\end{align*}$$

where $G^y(z)$ represents the convolution of $G(v_1)$ by $y$ times.

We next derive the iterative formula as below:

- $n = 1: \int_0^\infty F_1 d(G(x)) = \int_0^\infty d(G(x)) = 1 - G(r + z)$;
that if a grand coalition is formed, each user's benefit will be distributed through Shapley value. After reordering, the routing fee decrease 72% and the success rate of the number of transactions increase 25%.

If two players choose to cooperate with each other, the benefits are shown in Table I. If there are there players choose to join in the coalition, the benefit is shown in Table II.

The result shows under this topology, the worth function of coalition and the re-distribution function fully meet the properties desired.

B. Success ratio of successful transactions in PCNs.

We will show the performance of coalesional mechanism under different network parameters. The performance metric is success rate including success rate of transaction number and transaction value in PCN. Success rate of transaction number is the number of transactions which is processed successfully in PCN over the number of total transactions. Success rate of transaction value is the total amount of payments which is processed successfully in PCN over the total amount of payments generated. The transaction fee of trading on the public chain is much greater than that of trading on the PCN. Therefore, we ignore routing fees on PCN and focus on how many and how much of transactions can be successfully processed in PCNs through our mechanism. Also, we study the performance under different conditions, including different graph size and different number of transactions. 50 qualified network topologies are generated randomly, 50 sets of transactions are generated under each network topology, and each set of transactions includes some transactions. We compare the maximum and the minimum number and value of transactions which are processed successfully in PCNs and calculate the average for all sets of transactions under all topology as the result.

In our topologies, the number of nodes is set to 15, the number of edges is set to 70, the capacity of channel is randomly selected in the range of 10 coins and 15 coins. Considering that most of the transaction value is relatively small, we randomly select the transaction value in the range of 1 coin to 12 coins. Each node can choose its own charging standard, and we choose it randomly from 1% to 5%. Base routing fee is 0 for all channels. Each set of transactions includes 4 transactions.

Fig.7 shows the result when the nodes change. With the increase of the number of nodes, the growth rates of successful transaction number and transaction value in PCN shows a downward trend. But the growth rate of successful transaction number is still more than 12%, the growth rate of successful transaction value is more than 7% when the nodes of PCN increases to 19. The results manifest that the number and value of successful transactions of our proposed mechanism are significantly improved in the case of different number of nodes. In the case of the same number of edges, increasing the number of nodes (reducing network density) will lead to a decrease in the growth rate of the number and the amount of transactions which are processed successfully.

\[ \psi(x) = 10.08 \]

We can compute the benefits for the individual transaction.

The first row of Table III shows everyone's benefit if no one chooses to cooperate, and the second row shows that if a grand coalition is formed, each user's benefit will be distributed through Shapley value. After reordering, the routing fee decrease 72% and the success rate of the number of transactions increase 25%.

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The density of undirected graph. The density \( \rho \) of undirected graph is defined as:

\[
\rho = \frac{k}{j(j-1)}
\]

where the value of \( \rho \) ranges from 0 to 1. The higher the value is, the denser the network is. Fig.8 shows the performance result of changing the number of nodes when \( \rho = 0.3 \). The growth rate is relatively stable with the increase in the number of nodes. This proves the success rate is relatively stable with the same density, regardless of the number or the value of successful transactions in PCN.

The performance of coalitional mechanism under different number of edges is researched. Fig.9 shows in the case of the same number of nodes, growth rate of transaction number and transaction value both increase with the increase of the number of edges.

Number of transactions is also an important factor of performance for coalitional mechanism. In the above experiments, the number of transactions is set to 4. The number of transactions per unit time is much more than 4 in real system. Therefore we want to explore how the performance of the mechanism will change with the increase of the number of transactions under the same network size and related charges. Fig.10 shows with the increase of the number of transactions, the growth rate of the number and the value of successful transactions in PCN increases. This means although the number of transactions in experiment is set to 4, it still proves the effectiveness of our proposed mechanism. Because in real system, the more transactions, the more the number and value of successful transactions in PCN will increase.

C. Experiment results of stochastic waiting time.

We assume that the transaction size obeys the exponential distribution with the parameter 2 coin, the transaction from A to B is a Poisson process with rate 1, and from B to A is a Poisson process with rate 2. The transaction amount to be transmitted is 10 coins, and the current capacity is 9 coins.

**TABLE I**: The benefit of individual transaction in a two-player coalition.

| \( S = \{X_0, X_1\} \) | \( \psi(S) = 9.68 \) | \( \psi(X_2) = 0 \) | \( \psi(X_3) = 0 \) | \( \phi_{X_0} = 4.84 \) | \( \phi_{X_1} = 4.84 \) |
| \( S = \{X_0, X_2\} \) | \( \psi(S) = 1.2 \) | \( \psi(X_1) = 0 \) | \( \psi(X_2) = 0 \) | \( \phi_{X_0} = 0.6 \) | \( \phi_{X_2} = 0.6 \) |
| \( S = \{X_0, X_3\} \) | \( \psi(S) = 0 \) | \( \psi(X_1) = 0 \) | \( \psi(X_2) = 0 \) | \( \phi_{X_0} = 0 \) | \( \phi_{X_3} = 0 \) |
| \( S = \{X_1, X_2\} \) | \( \psi(S) = 0.4 \) | \( \psi(X_0) = 0 \) | \( \psi(X_2) = 0 \) | \( \phi_{X_1} = 0.2 \) | \( \phi_{X_2} = 0.2 \) |
| \( S = \{X_1, X_3\} \) | \( \psi(S) = 0 \) | \( \psi(X_0) = 0 \) | \( \psi(X_2) = 0 \) | \( \phi_{X_1} = 0 \) | \( \phi_{X_2} = 0 \) |
| \( S = \{X_2, X_3\} \) | \( \psi(S) = 0 \) | \( \psi(X_0) = 0 \) | \( \psi(X_2) = 0 \) | \( \phi_{X_2} = 0 \) | \( \phi_{X_3} = 0 \) |

**TABLE II**: The benefit of individual transaction in a three-player coalition.

| \( S = \{X_0, X_1, X_2\} \) | \( \psi(S) = 10.08 \) | \( \psi(X_3) = 0 \) | \( \phi_{X_0} = 5.04 \) | \( \phi_{X_1} = 4.64 \) | \( \phi_{X_2} = 0.4 \) |
| \( S = \{X_0, X_1, X_3\} \) | \( \psi(S) = 9.68 \) | \( \psi(X_2) = 0 \) | \( \phi_{X_0} = 4.84 \) | \( \phi_{X_2} = 4.84 \) | \( \phi_{X_3} = 0 \) |
| \( S = \{X_0, X_2, X_3\} \) | \( \psi(S) = 1.2 \) | \( \psi(X_1) = 0 \) | \( \phi_{X_0} = 0.6 \) | \( \phi_{X_1} = 0.6 \) | \( \phi_{X_3} = 0 \) |
| \( S = \{X_1, X_2, X_3\} \) | \( \psi(S) = 0.4 \) | \( \psi(X_0) = 0 \) | \( \phi_{X_1} = 0.2 \) | \( \phi_{X_2} = 0.2 \) | \( \phi_{X_3} = 0 \) |

**TABLE III**: Benefit redistribution of FCFS and coalition.

| \( \psi(X_0) = 0 \) | \( \psi(X_1) = 0 \) | \( \psi(X_2) = 0 \) | \( \psi(X_3) = 0 \) | \( \phi_{X_0} = 0.4 \) | \( \phi_{X_2} = 0 \) |

Fig. 7: Growth rate of successful transactions in PCN w.r.t. number of nodes.

Fig. 8: Growth rate of successful transactions in PCN w.r.t. number of edges.

Fig. 9: Growth rate of successful transactions in PCN w.r.t. number of transactions.

Fig. 10: Growth rate of successful transactions in PCN w.r.t. number of transactions.

Fig. 11: Results of model and simulation.
We validate Eq.18 and Fig.11 show the result. In Fig.11, the points represent the simulation results, the black line represents the model result. Horizontal axis is the time, and the vertical axis is cumulative probability function of $\hat{\ell}$. The simulation results are basically matched with the theoretical results. The error is mainly caused by numerical integration. For the integral calculation of the original function, we use the numerical integration method instead of the Newton-Leibniz formula. Considering the running time of program, it is necessary to select the appropriate numerical integration step, which will introduce errors.

Fig. 12 shows the results of $\Phi(v, u, t)$ with different parameters. Here, $x$—coordinate is the probability result of $\Phi(v, u, t)$, and left $y$—axis is the difference between the amount to be transferred and the current capacity $(v - u)$, right $y$—axis is different limited time $(t)$. The dotted line shows that with the increase of the transmission amount, the probability of reaching the requirement before 3 decreases continuously. The solid line illustrates with the increase of limited time, the probability increase from 0.25 to 0.8.

![Fig. 12: Result of probability w.r.t. t and probability w.r.t. v - u.](image)

VI. CONCLUSION

In this paper, we study how to find the payment channel routing with minimum cost in payment channel network. We focus on this problem from the point of view of whether we can get lower transaction fees by waiting patiently instead of pursuing maximum efficiency. A periodic transaction processing scheme is designed and it re-orders the transactions to get an minimum routing fee. The benefit is re-distributed by Shapley value which ensures the fairness. Also, The capacity dynamics of a payment channel is captured by a transient queuing model and the waiting time distribution of a strategic transaction is obtained.

REFERENCES

[1] Shae, Zonyin, and Jeffrey JP Tsai. “On the design of a blockchain platform for clinical trial and precision medicine.” 2017 IEEE 37th international conference on distributed computing systems (ICDCS). IEEE, 2017.
[2] Karlsson, Kolbeinn, et al. “Vegvisir: A partition-tolerant blockchain for the internet-of-things.” 2018 IEEE 38th International Conference on Distributed Computing Systems (ICDCS). IEEE, 2018.
[3] Huang, Yaodong, et al. “Resource allocation and consensus on edge blockchain in pervasive edge computing environments.” 2019 IEEE 39th International Conference on Distributed Computing Systems (ICDCS). IEEE, 2019.
[4] https://bitcoin.org/en/bitcoin-core/
[5] http://www.ethereum.org/
[6] Visa acceptance for retailers. https://usa.visa.com/run-your-business/small-business-tools/retail.html.
[7] Poon, Joseph, and Thaddeus Dryja. “The bitcoin lightning network: Scalable off-chain instant payments.” (2016).
[8] Singh, Amritraj, et al. “Public blockchains scalability: An examination of sharding and segregated witness.” Blockchain Cybersecurity, Trust and Privacy. Springer, Cham, 2020. 203-232.
[9] Nguyen, Lan N., et al. “OptChan: optimal transactions placement for scalable blockchain sharding.” 2019 IEEE 39th International Conference on Distributed Computing Systems” (ICDCS). IEEE, 2019.
[10] Network, Raiden. “What is the raiden network.” (2018).
[11] Hosp, Dr. Toby Hoenisch, and Paul Kittiwongsunthorn. “COMIT-Cryptographically-secure Off-chain Multi-asset Instant Transaction Network.” arXiv preprint arXiv:1810.02174 (2018).
[12] Zhang, Yuhui, Dejun Yang, and Guoliang Xue. "Cheapay: An optimal algorithm for fee minimization in blockchain-based payment channel networks." ICC 2019-2019 IEEE International Conference on Communications (ICC). IEEE, 2019.
[13] Wang, Peng, et al. “Flash: efficient dynamic routing for offchain networks.” Proceedings of the 15th International Conference on Emerging Networking Experiments And Technologies. 2019.
[14] Ren, Alvin Heng Jun, et al. “Optimal Fee Structure for Efficient Lightning Networks.” 2018 IEEE 24th International Conference on Parallel and Distributed Systems (ICPADS). IEEE, 2018.
[15] Sivaraman, Vibhaaakshmi, et al. ”High Throughput Cryptocurrency Routing in Payment Channel Networks.” 17th USENIX Symposium on Networked Systems Design and Implementation (NSDI 20). 2020.
[16] Piatkivskiy, Dmytro, and Mariusz Nowostawski. “Split payments in payment networks.” Data Privacy Management, Cryptocurrencies and Blockchain Technology. Springer, Cham, 2018. 67-75.
[17] Rohrer, Elias, Jann-Frederik Laß, and Florian Tschorsch. “Towards a concurrent and distributed route selection for payment channel networks.” Data Privacy Management, Cryptocurrencies and Blockchain Technology. Springer, Cham, 2017. 411-419.
[18] Piatkivskiy D, Nowostawski M. Split payments in payment networks[MI]/Data Privacy Management, Cryptocurrencies and Blockchain Technology. Springer, Cham, 2018: 67-73.
[19] Varma, Sushil Mahavir, and Siva Theju Maguluri. “Throughput Optimal Routing in Blockchain Based Payment Systems.” arXiv preprint arXiv:2001.05299 (2019).
[20] Maleki, Sasan, et al. “The Shapley value for a fair division of group discounts for coordinating cooling loads.” PloS one 15.1 (2020): e0227049.
[21] Rohrer, Elias, Julian Malliaris, and Florian Tschorsch. “Discharged Payment Channels: Quantifying the Lightning Network’s Resilience to Topology-Based Attacks.” 2019 IEEE European Symposium on Security and Privacy Workshops (EuroS & PW). IEEE, 2019.
[22] Engelmann, Felix, et al. “Towards an economic analysis of routing in payment channel networks.” Proceedings of the 1st Workshop on Scalable and Resilient Infrastructures for Distributed Ledgers. 2017.
[23] Egger, Christoph, Pedro Moreno-Sanchez, and Matteo Maffei. “Atomic multi-channel updates with constant collateral in bitcoin-compatible payment-channel networks.” Proceedings of the 2019 ACM SIGSAC Conference on Computer and Communications Security. 2019.
[24] Khan, Nida. “Lightning network: A comparative review of transaction fees and data analysis.” International Congress on Blockchain and Applications, Springer, Cham, 2019.
[25] Hoenisch, Philipp, and Ingo Weber. “Addr-based routing for payment channel networks.” International Conference on Blockchain. Springer, Cham, 2018.
[26] Béres, Ferenc, Istvan Andras Seres, and András A. Benzcár. “A cryptocurrency traffic analysis of Bitcoins lightning network.” arXiv preprint arXiv:1911.09432 (2019).
[27] Cordi, Christopher Neal. “Simulating high-throughput cryptocurrency payment channel networks.” (2017).
[28] https://ycharts.com/indicators/bitcoin_average_transaction_fee.
[29] Faust, Katherine. “Comparing social networks: size, density, and local structure.” Metodoloski zvezki 3.2 (2006): 185.
[30] Winter, Eyal. “The shapley value.” Handbook of game theory with economic applications 3.2 (2002): 2025-2054.
[31] Osborne, Martin J., and Ariel Rubinstein. A course in game theory. MIT press, 1994.