A Curve Flow on an Almost Hermitian Manifold  
Evolved by a Third Order Dispersive Equation  

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Abstract. We consider a curve flow for maps from a real line into a compact almost Hermitian manifold, which is governed by a third order nonlinear dispersive equation. This article shows short-time existence of a solution to the initial value problem for the equation. The difficulty comes from the lack of the Kähler condition on the target manifold, since the covariant derivative of the almost complex structure causes a loss of one derivative in our equation and thus the classical energy method breaks down in general. In the present article, we can overcome the difficulty by constructing a gauge transformation on the pull-back bundle for the map to eliminate the derivative loss essentially, which is based on the local smoothing effect of third order dispersive equations on the real line. 

Key Words and Phrases. Nonlinear dispersive partial differential equations, Vortex filament, Derivative loss, Smoothing effect, Energy method, Geometric analysis. 

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1. Introduction 

Let $N$ be a compact almost Hermitian manifold with an almost complex structure $J$ and a Hermitian metric $g$. Let $\nabla^N$ be the Levi-Civita connection with respect to the metric $g$. Consider the initial value problem for a third order nonlinear dispersive partial differential equation of the form 

\begin{equation} 
  u_t = a\nabla^2_x u_x + Ju_x u_x + bg(u_x, u_x)u_x 
  \text{ in } \mathbb{R} \times X, 
\end{equation} 

\begin{equation} 
  u(0, x) = u_0(x) \quad \text{in } X, 
\end{equation} 

where the unknown mapping $u = u(t, x) : \mathbb{R} \times X \to N$ describes the flow for the mapping from $X$ into $N$, $X$ denotes the real straight line $\mathbb{R}$ or the one-dimensional flat torus $\mathbb{R}/\mathbb{Z}$, and $u_0 = u_0(x) : X \to N$ is a given initial curve on $N$. Each term of the equation (1) is a vector field along $u$, where $u_t = du(\partial/\partial t)$, $u_x = du(\partial/\partial x)$, $du$ is the differential of $u$, $\nabla_x (=\nabla^N_{u_x})$ is the covariant derivative along $u$ in the direction $x$, $J_u$ is the almost complex structure at $u \in N$, and $a, b \in \mathbb{R}$ are constants. In particular, solutions to (1) with $a, b = 0$ are called one-dimensional Schrödinger flows.
The equation (1) is a generalization of two-sphere-valued physical models. Indeed, if \( N \) is the two-dimensional unit sphere \( S^2 \) with the canonical metric and the complex structure and if \( b = a/2 \), (1) can be formulated also as a system of third order nonlinear dispersive equations for \( \tilde{u}(t,x) : R \times X \rightarrow S^2 \subset R^3 \) of the form

\[
\tilde{u}_t = \tilde{u} \wedge \tilde{u}_{xx} + a \left[ \tilde{u}_{xxxx} + \frac{3}{2} \{ \tilde{u}_x \wedge (\tilde{u} \wedge \tilde{u}_x) \} \right],
\]

where \( \wedge \) is the exterior product in \( R^3 \). Originally in 1906, the equation (3) with \( a, b = 0 \), that is,

\[
\tilde{u}_t = \tilde{u} \wedge \tilde{u}_{xx},
\]

was first formulated by Da Rios [4] to model the motion of the vortex filament in \( R^3 \). In this physical model, the vortex filament at time \( t \) is modelled as the space curve \( \tilde{y}(t,\cdot) : X \rightarrow R^3 \) parameterized by the arc-length parameter \( x \), and \( \tilde{u}(t,x) := \tilde{y}_x(t,x) \in S^2 \) denotes the tangent vector of the vortex filament curve at the point \( \tilde{y}(t,x) \). In the same way, (3) was proposed by Fukumoto and Miyazaki [7] in 1991, where \( a \in R \) denotes the magnitude of the axial flow effect in the vortex filament. Both (3) and (4) are known to be completely integrable systems and appear also as a model equation for the continuous limit of ferromagnetic spin chain systems. See also, e.g., [8, 9, 14, 15, 16, 28] for their physical backgrounds.

In mathematical analysis, the existence problem of the solution to the initial value problem for \( S^2 \)-valued models has been studied extensively by many researchers since 1980’s. For example, Sulem, Sulem and Bardos [30] first showed time-local and global existence of a unique smooth solution to the IVP for (4), and Nishiyama and Tani [23, 31] established similar time-local and global existence result for (3). See, e.g., [20, 21, 23, 30, 31] and references therein.

On the other hand, by noting the geometric structure of \( S^2 \)-valued models such as (1), the relation between the existence problem of the solution to the IVP (1)–(2) and the geometric condition on \( (N,J,g) \) has been studied from some viewpoints of geometric analysis in recent fifteen years. Koiso [12, 13] first generalized (4) geometrically as the one-dimensional Schrödinger flow equation

\[
u_t = J_g \nabla_x u_x
\]

for \( u(t,x) : R \times R^2 \rightarrow N \) where \( N \) is a Kähler manifold. He introduced the framework of a kind of geometric classical energy method to show the short-time existence of a unique smooth solution to the IVP. He also showed the time-global existence when \( N \) is a locally Hermitian symmetric space.
(\nabla^N J \equiv \nabla^N R \equiv 0) \) by applying some conservation laws. See also [27] when \( X = \mathbb{R} \). After that, higher-dimensional Schrödinger flows for maps into Kähler manifolds were studied and short-time existence results were obtained in [5, 10, 17], based on the geometric classical energy method. Chang, Shatah and Uhlenbeck [1] established an essential understanding on the PDE structure of (5) for the mapping \( u(t, x) : \mathbb{R} \times \mathbb{R} \to N \) where \( N \) is a compact Riemann surface. More precisely, they constructed a good moving frame along \( u \) and reduced (5) to a complex-valued simpler semilinear Schrödinger equation, under the assumption that \( u(t, x) \) had a fixed base point as \( x \to +\infty \). See their pioneering work [1]. See also [19, 20, 21, 29] for the generalization to Schrödinger flows for maps into compact Kähler manifolds. When \( N \) is a compact complex Grassmannian, Terng and Uhlenbeck [33] showed the bi-Hamiltonian structure of (5) from the theory of completely integrable systems, which presented a global existence result for the IVP. As for the third order equation (1) with \( \alpha \neq 0 \), the author [24] proved a short-time existence theorem when \( N \) is a compact Kähler manifold. See also [25] for the PDE structure studied by the method of [1]. The author also showed the time-global existence result when \( \nabla^N J \equiv \nabla^N R \equiv 0 \) in [24, 26].

We should mention here that all works above assumed the target manifold \( N \) to be a Kähler manifold. Roughly speaking, the Kähler condition \( \nabla^N J \equiv 0 \) ensures that the equation behaves as symmetric hyperbolic systems and the geometric classical energy method developed in [5, 12, 13, 17, 24, 27] works to obtain short-time existence results. However, if the Kähler condition fails, \( \nabla^N J \neq 0 \), then (1) has a bad first order term which makes the classical energy method break down. Therefore it seems to be reasonable to investigate whether (1)–(2) can be solved or not even when \( N \) is an almost Hermitian manifold without the Kähler condition.

Remark 1. As is well known, compact almost Hermitian manifolds do not satisfy the Kähler condition in general. For instance, Hopf manifolds \( S^{2p+1} \times S^1 \), \( p \in \mathbb{N} \), are complex manifolds which never admit the Kähler condition. In addition, \( S^6 \) with the almost complex structure given by the Cayley exterior product in \( \mathbb{R}^7 \) is not a complex manifold and thus is not a Kähler manifold. See, e.g., [11].

In such non-Kähler cases, Chihara [2] first succeeded to overcome the difficulty of the derivative loss. Indeed, he recently showed short-time existence of a unique solution to the IVP for Schrödinger flow for maps from a general closed Riemannian manifold (or the Euclidean space \( \mathbb{R}^m \)) into a compact almost Hermitian manifold. His idea was to construct a gauge transformation acting on sections of the pull-back bundle to eliminate the bad first order term essentially.
The aim of the present article is to study the third order case \((a \neq 0)\) when \(N\) is a compact almost Hermitian manifold without assuming the Kähler condition. Roughly speaking, we can show the short time existence of a unique smooth solution to \((1)–(2)\) when \(X = \mathbb{R}\).

To state our results precisely, we here recall some classes related to \(N\)-valued mappings. For \(u : X \to N\), let \(G(u^{-1}TN)\) be the set of all sections of the pull-back bundle \(u^{-1}TN\). For a nonnegative integer \(m\), let \(H^m(X; TN)\) be the set of all sections \(V \in G(u^{-1}TN)\) satisfying

\[
||V||^2_{H^m} := \sum_{k=0}^{m} ||\nabla^k_x V||^2 = \sum_{k=0}^{m} \int_X g(\nabla^k_x V, \nabla^k_x V) dx < +\infty.
\]

In particular, \(H^0(X; TN)\) is written as \(L^2(X; TN)\). In addition, the class \(H^{m+1}(X; N)\) is defined by

\[
H^{m+1}(X; N) = \{u \in C(X; N); u_x \in H^m(X; TN)\}.
\]

Our main results is now stated as follows:

**Theorem 1.** Suppose that \(a \neq 0\) and \(X = \mathbb{R}\). Let \(m\) be a positive integer satisfying \(m \geq 4\). Then, for any \(u_0 \in H^{m+1}(X; N)\), there exists a constant \(T = T(a, b, N, ||u_0||_{H^4}) > 0\) such that \((1)–(2)\) possesses a unique solution \(u \in C([-T, T]; H^{m+1}(X; N))\).

Our idea of the proof of Theorem 1 comes from the theory of linear dispersive partial differential operators. Consider the initial value problem for linear partial differential equations of the form

\[
U_t + U_{xxx} + a(x)U_x + b(x)U = f(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{R},
\]

where \(a(x), b(x) \in \mathcal{B}^\infty(\mathbb{R})\), which is the set of all smooth functions on \(\mathbb{R}\) whose derivative of any order are bounded on \(\mathbb{R}\), \(U(t, x)\) is a complex-valued unknown function, and \(f(t, x)\) is a given function. Tarama [32] showed that the IVP for \((6)\) is \(L^2\)-well-posed if and only if there exists some constant \(C > 0\) such that

\[
\left|\int_x^y \text{Im} a(s) ds\right| \leq C|x - y|^{1/2}
\]

for any \(x, y \in \mathbb{R}\). The necessity is proved by the usual method of asymptotic solutions. In order to prove the sufficiency, he first constructed a nice pseudo-differential operator of order zero which was automorphic on \(L^2(\mathbb{R}; C)\) under the condition \((7)\), and eliminated \(\sqrt{-1} \text{Im} a(x) \partial_x\). This is one of the methods
of bringing out the local smoothing effect of $e^{-t\partial_t^3}$ on $\mathbf{R}$, and this smoothing effect breaks down on $\mathbf{R}/\mathbb{Z}$. See, e.g., [6]. Tarama also pointed out unofficially that if $\text{Im} \ a \in L^2(\mathbf{R}; \mathbf{R})$, then (7) holds and the proof of sufficiency becomes quite easier than the general case of (7). In this case, a gauge transformation defined by

$$U(x) \mapsto V(x) = U(x) \exp\left(\frac{1}{3} \int_{-\infty}^{x} \{\text{Im} \ a(y)\}^2 \, dy\right)$$

is automorphic on $L^2(\mathbf{R}; \mathbf{C})$, and (6) becomes

$$V_t + V_{xxx} - \{\text{Im} \ a(x)\}^2 V_{xx} + \{\tilde{a}(x) + \sqrt{-1} \, \text{Im} \ a(x)\} V_x + \tilde{b}(x) V = \tilde{f}(t, x)$$

with some $\tilde{a}, \tilde{b} \in \mathcal{B}^\infty(\mathbf{R})$ and $\tilde{f}$, where $\tilde{a}$ is a real-valued. The IVP for (9) is $L^2$-well-posed in the positive direction of $t$ since the second order term $\{\text{Im} \ a(x)\}^2 \partial_x^2$ dominates the seemingly bad first order term $\sqrt{-1} \, \text{Im} \ a(x) \partial_x$ essentially. In this special case, pseudodifferential calculus is not required.

We make use of the idea of the gauge transformation (8) to our problem (1)–(2). Roughly speaking, if $u$ solves (1), $V_x^m u_x$ should formally satisfy the form

$$V_t - a V_x^3 - V_x J_u V_x - m (V_x J_u) V_x^m u_x = \text{harmless terms},$$

where $V_x J_u V_x$ is a composition of operators: $V_x J_u V_x W = V_x (J_u V_x W)$, and $(V_x J_u) := V_x J_u - J_u V_x$ is the covariant derivative of $J_u$ in the direction $x$. Remark that the first order term $m (V_x J_u) V_x$ cannot be controlled by the classical energy method since $(V_x J_u)$ behaves as a skew-symmetric operator on $L^2(\mathbf{R}; TN)$. To overcome this difficulty, we introduce the following gauge transformation acting on $L^2(\mathbf{R}; TN)$ defined by

$$V_x^m u_x(t, x) \mapsto V_x^m u_x(t, x) \exp\left(\frac{1}{3a} \int_{-\infty}^{x} g(u_x(t, y), u_y(t, y)) \, dy\right).$$

The commutator of the gauge transformation (11) with $a V_x^3$ generates a second order term $-g(u_x, u_y) V_x^2$ which dominates $m (V_x J_u) V_x$ essentially since $[(V_x J_u)] = O(g(u_x, u_y)^{1/2})$. See inequalities (59)–(62) for details.

In the next section, we prove Theorem 1. Our proof is based on a fourth order parabolic regularization of (1) and on the geometric energy method combining the gauge transformation (11). The assumption $m \geq 4$ comes from the requirement on the integer for our method of the gauge transformation to work.

Remark 2. After this work, by applying the idea of [2], Chihara and the author [3] succeeded to show Theorem 1 also when $X = \mathbf{R}/\mathbb{Z}$. Indeed, they...
constructed another gauge transformation on $L^2(R; TN)$ by taking a partition of unity on $N$ and by patching some sort of pseudodifferential operators together. Each of these pseudodifferential operators can be treated as a bounded pseudodifferential operator acting on vector-valued functions, and the commutator of the gauge transformation with $a V^3_x$ eliminates the seemingly bad first order term. See [3] for details. In fact, the above method in [3] is valid also in the present case $X = R$. In this sense, we should mention that the claim of Theorem 1 itself is now not new. However, the present article gives another proof which is simpler than that in [3], by making full use of the local smoothing effect of $e^{i a(x)}$ on $R$ which breaks down on $R/\mathbb{Z}$. Indeed, in the present article, we can construct the desired gauge transformation by using only a bounded real-valued function. Thus, neither patching together by the partition of unity on $N$ nor the pseudodifferential calculus is required in our proof. In other words, the above difference between [3] ($X = R/\mathbb{Z}$) and the present article ($X = R$) can be stated also as follows. In [3], roughly speaking, they regard $m(V_x u)$ in (10) as $\text{Im} a$ which has a potential in (6). The structure is known to be a sufficient condition that the IVP for (6) is $L^2$-well-posed (see Mizuhara [18]), and is applied effectively in [3]. On the other hand, in the present article, we give a simpler way to obtain the same result by applying a weaker structure of $m(V_x u)$ which corresponds to $\text{Im} a \in L^2(R; R)$ in (6).

2. Proof of Theorem 1

We first recall some basic notations and facts used below. Let $w: N \to R^d$ be an isometric embedding. We often identify an $N$-valued map $u$ with an $R^d$-valued function $w \circ u$ depending on the situation. In particular, we often use the equivalence between $u_x \in H^m(R; TN)$ and $dv(u_x) = (w \circ u)_x \in H^m(R; R^d)$, where $H^m(R; R^d)$ is the usual Sobolev space for $R^d$-valued functions on $R$. This equivalence follows from the Gagliardo-Nirenberg inequality. Thus $H^{m+1}(R; N)$ introduced in the previous section can be characterized by

$$H^{m+1}(R; N) = \{ u \in C(R; N); (w \circ u)_x \in H^m(R; R^d) \}.$$ 

Moreover, for an interval $I \subset R$, the class $C(I; H^{m+1}(R; N))$ consists of all $u \in C(I \times R; N)$ which satisfy $(w \circ u)_x \in C(I; H^m(R; R^d))$. The standard norm of $L^2(R; R^d)$ or $H^m(R; R^d)$ for $R^d$-valued functions is written respectively as $\| \cdot \|_{L^2(R; R^d)}$ or $\| \cdot \|_{H^m(R; R^d)}$, not to be confused with $\| \cdot \|$ or $\| \cdot \|_{H^m}$. See, e.g., [3, 5, 17, 24] for details. For further basic techniques of geometric analysis and Riemannian geometry, one can consult with [22]. Having them in mind, we start the proof of Theorem 1.

Proof of Theorem 1. The proof consists of the following four steps.
Step 1. Fourth order parabolic approximation.

For each $\varepsilon > 0$, we consider the IVP for a fourth order parabolic equation of the form

\begin{align}
(12) \quad &u_t = -\varepsilon \nabla_x^3 u_x + a \nabla_x^2 u_x + J_\varepsilon \nabla_x u_x + b g(u_x, u_x) u_x \quad \text{in } (0, T_\varepsilon) \times \mathbb{R}, \\
(13) \quad &u(0, x) = u_0(x) \quad \text{in } \mathbb{R},
\end{align}

where $u(t, x) : [0, T_\varepsilon] \times \mathbb{R} \to N$ is the unknown map and $u_0$ is the same initial data as that of (1)–(2). It is well-known that the IVP (12)–(13) admits a unique short-time solution $u \in C([0, T_\varepsilon]; H^{m+1}(\mathbb{R}; N))$ for each $\varepsilon > 0$. Indeed, we can show this fact by the mix of the standard contraction mapping argument and a kind of maximum principle in the same way as [2, 3, 10, 24]. The argument does not depend on whether or not $N$ is a Kähler manifold. Thus we omit the detail. See, e.g., [2, Section 2], [10, Section 2 and 3], or [24, Section 3] for details. By parameterizing the solution $u$ as $u^\varepsilon$ for each $\varepsilon \in (0, 1)$, we can construct a sequence $\{u^\varepsilon\}_{\varepsilon \in (0, 1)}$.

Step 2. Existence of a solution to (1)–(2).

We construct a solution to (1)–(2), which is the most important part in the present article. Let $\{u^\varepsilon\}_{\varepsilon \in (0, 1)}$ be the sequence of solutions to (12)–(13) constructed above. Then, we can show the following.

**Lemma 2.** There exists a constant $T > 0$ depending on $a$, $b$, $N$, $\|u_0\|_{H^4}$ and not on $\varepsilon \in (0, 1)$ such that the sequence $\{u^\varepsilon\}_{\varepsilon \in (0, 1)}$ is bounded in $L^\infty(0, T; H^m(\mathbb{R}; TN))$.

**Proof of Lemma 2.** Set $u = u^\varepsilon$ for ease of notation. The fundamental tool of the computation below is the formula of the integration by parts for the covariant derivative of the form

$$\int_{\mathbb{R}} g(\nabla_x W_1, W_2) dx = -\int_{\mathbb{R}} g(W_1, \nabla_x W_2) dx$$

for all $W_1, W_2 \in \Gamma(u^{-1} TN)$, which follows from the definition of the Levi-Civita connection on $N$ with the metric $g$.

We evaluate the square of $N_m(t)$ given by

$$N_m(t)^2 = \|u_x(t)\|_{H^{m-1}}^2 + \|A \nabla_x^m u_x(t)\|^2,$$

where $A = A(t, x)$ is a real-valued function defined by

$$A(t, x) = e^{K(t, x)},$$

$$K(t, x) = -\frac{1}{3a} \int_{-\infty}^x g(u_x(t, y), u_x(t, y)) dy.$$
The factor $1/3$ is not important. It slightly simplifies calculation. These real-valued functions $A(t,x)$ and $K(t,x)$ actually make sense, since

$$|K(t,x)| \leq \frac{1}{3|a|} \int_R g(u_0(t,y), u_3(t,y))dy \leq \frac{1}{3|a|} \|u_0\|^2$$

for $(t,x) \in [0,T_e] \times R$. The second inequality above follows from the energy inequality of the form

$$\frac{1}{2} \frac{d}{dt} \|u_3\|^2 = \int_R g(\nabla u_3, u_3)dx$$

$\geq -\varepsilon \int_R g(\nabla^4 u_3 + a \nabla^2 u_3 + \nabla_J \nabla u_3 + b \nabla [g(u_0, u_3)u_3], u_3)dx$

$= -\varepsilon \int_R g(\nabla^2 u_3, \nabla^2 u_3)dx - a \int_R g(\nabla^2 u_3, \nabla u_3)dx$

$- b \int_R g(u_3, u_3)g(u_3, \nabla u_3)dx$

$= -\varepsilon \|\nabla^2 u_3\|^2 - a \|g(u_3, \nabla u_3)\|_{L^2} dx - \frac{b}{4} \|g(u_3, u_3)\|^2_{L^2} dx$

$= -\varepsilon \|\nabla^2 u_3\|^2 \leq 0$.

We next set $r_0 = N_4(0)$. Note that this $r_0$ is independent of $\varepsilon \in (0,1)$. Moreover, we set

$$T_e^* = \sup \{ T > 0 \mid N_4(t) \leq 2r_0 \text{ for all } t \in [0,T] \}.$$

Then there exists a positive constant $C_1 > 1$ depending on $a$ and $r_0$ such that

$$C_1^{-1}N_m(t)^2 \leq \|u_3(t)\|^2_{H^m} \leq C_1N_m(t)^2 \text{ for all } t \in [0,T_e^*].$$

The above relation follows from the estimate

$$|e^{tK(t,x)}| \leq e^{\|u_0(t)\|^2/(3|a|)} \leq e^{\|u_0\|^2/(3|a|)}.$$

Let $C^k_b(R)$ be the space of functions possessing bounded and continuous derivatives up to order $k$ on $R$. Since $u \in C([0,T_e^*]; H^{m+1}(R; N))$ with $m \geq 4$, the Sobolev embedding $H^4(R) \subset C^3_b(R)$ implies that $A(t,\cdot), A^{-1}(t,\cdot) \in C^4_b(R)$, $A_4(t,\cdot) \in C_b(R)$, and that their $L^\infty$-norms are bounded uniformly in $t \in [0,T_e^*]$ by a constant $C_2 > 0$ depending only on $a, b, N, r_0$. Under this setting, we show the following key proposition.
Proposition 3. There exists a constant $C > 0$ depending on $a$, $b$, $N$, $r_0$ and not on $\epsilon \in (0, 1)$ such that

\begin{equation}
\frac{1}{2} \frac{d}{dt} N_m(t)^2 \leq CP(N_{m-1}(t))N_m(t)^2
\end{equation}

for all $t \in [0, T^*_C]$, where $P(s)$ is a monotone increasing function on $|s|$.

Proof of Proposition 3. Set $V = AV^\mu u_\xi$ for short. A simple computation shows

\begin{equation}
\frac{1}{2} \frac{d}{dt} N_m(t)^2 = \frac{1}{2} \frac{d}{dt} \|u_\xi(t)\|^2_{H^m} + \frac{1}{2} \frac{d}{dt} \|V(t)\|^2
\end{equation}

\begin{align*}
&= \frac{1}{2} \frac{d}{dt} \sum_{k=0}^{m-1} \|\nabla_x^k u_\xi(t)\|^2 + \frac{1}{2} \frac{d}{dt} \|V(t)\|^2 \\
&= \sum_{k=0}^{m-1} \int_R g(\nabla_x^k u_\xi, \nabla_x^k u_\xi) dx + \int_R g(V, V) dx \\
&= \int_R g(V, u_\xi) dx + \sum_{k=1}^{m-1} \int_R g(\nabla_x^k u_\xi, \nabla_x^k u_\xi) dx \\
&\quad + \int_R g(V, V) dx \\
&= -\epsilon \|\nabla_x^2 u_\xi\|^2 + \sum_{k=1}^{m-1} \int_R g(\nabla_x^k u_\xi, \nabla_x^k u_\xi) dx + \int_R g(V, V) dx.
\end{align*}

Note that the last equality above follows from (14). Hence, in order to evaluate the right hand side of (16), we start from the computation of $\nabla_x^k u_\xi$ for $1 \leq k \leq m - 1$ and $\nabla_x V$ below.

Let $1 \leq k \leq m$. By acting $\nabla_x^{k+1}$ on (12), we have

\begin{equation}
\nabla_x^{k+1} u_t = -\epsilon \nabla_x^{k+4} u_\xi + a \nabla_x^{k+3} u_\xi + \nabla_x^{k+1} J \nabla_x u_\xi + b \nabla_x^{k+1}[g(u_\xi, u_\xi) u_\xi].
\end{equation}

We compute each term of (17). At first, the definition of the Levi-Civita connection implies that

\begin{equation}
\nabla_x^{k+1} u_t = \nabla_x^{k+1} u_\xi + \sum_{l=0}^{k-1} \nabla_x^l [R(u_\xi, u_t) \nabla_x^{k-l-1} u_\xi].
\end{equation}

By substituting (12) into the second term of the right hand side of (18), we deduce
where \( Q_{1,k} \) is the covariant derivative of \( x_{J+1} \) along \( u_{k} \) which is just the product of \( x_{J} \) and \( u_{k} \), and \( x_{J+1} \) is defined by
\[
(x_{J+1})_{x} = x_{J} \cdot u_{k}.
\]

Thirdly, we compute \( \nabla_{x}^{k+1}u_{x} \), where the difference with the case of Kähler manifolds appears. Indeed, a simple computation shows that
\[
\nabla_{x}^{k+1}[g(u_{x}, u_{x})u_{x}] = 2g(\nabla_{x}^{k+1}u_{x}, u_{x})u_{x} + g(u_{x}, u_{x})\nabla_{x}^{k+1}u_{x} + Q_{4,k},
\]
where \( (\nabla_{x}J) \) is the covariant derivative of \( J \) along \( u \) in the direction \( x \) defined by
\[
(\nabla_{x}J)_{x} = \nabla_{x}J_{x} - J_{x}W \quad \text{for } W \in \Gamma(u^{-1}TN).
\]

(\( \nabla_{x}J \)) is also a \((1,1)\)-tensor field along \( u \). We use the notation like \( (\nabla_{x}J) \) not to be confused with \( \nabla_{x}x \) which is just the product of \( \nabla_{x} \) and \( x \). Moreover, by regarding \( (\nabla_{x}J) \) and \( \nabla_{x}^{k+1}u_{x} \) as a \((1,1)\)-tensor field and a \((1,0)\)-tensor field respectively, we deduce
\[
\sum_{l=1}^{k} \nabla_{x}^{l}(\nabla_{x}J)\nabla_{x}^{k+1-l}u_{x} = k(\nabla_{x}J)\nabla_{x}u_{x} + \sum_{l=1}^{k} \frac{l!}{j!(l-j)!} (\nabla_{x}^{j+1}J)\nabla_{x}^{k+1-j}u_{x}.
\]
Here, for \( j \in \mathbb{N} \), \((\nabla^j_x J)\) is the \((j+1)\)-th covariant derivative of \( J \) defined inductively by
\[
(\nabla^{j+1}_x J)W = \nabla_x(\nabla^j_x J)W - (\nabla^j_x J)\nabla_x W \quad \text{for} \quad W \in \Gamma(u^{-1}TN),
\]
and \((\nabla^1_x J) = (\nabla_x J)\). It follows from (21) and (22) that
\[
(23) \quad \nabla^{k+1}_x J = \nabla_x \nabla^{k}_x J + k(\nabla_x J)\nabla^{k}_x J + Q_{5,k},
\]
where
\[
Q_{5,k} = \sum_{l=1}^k \frac{1}{j!} (\nabla^{j+1}_x J)\nabla^{k+1-j}_x J.
\]
Consequently, by substituting (19), (20) and (23) into (17), we deduce
\[
(24) \quad (\nabla_1 + \varepsilon \nabla^4_x - a \nabla^3_x - \nabla_x J \nabla_x J - k(\nabla_x J)\nabla_x \nabla^{k}_x J + e F_{1,k} - F_{2,k} = F_{3,k},
\]
\[
(25) \quad F_{1,k} = Q_{1,k},
\]
\[
(26) \quad F_{2,k} = k(\nabla_x J)\nabla^{k}_x J - a R(u_x, \nabla_x J \nabla^{k}_x J)u_x + 2 bg(\nabla_x J \nabla^{k}_x J, u_x) u_x + b g(u_x, u_x)\nabla_x \nabla^{k}_x J,
\]
\[
(27) \quad F_{3,k} = -a Q_{2,k} - Q_{3,k} + Q_{4,k} + Q_{5,k}.
\]
Thus we obtain
\[
(28) \quad (\nabla_1 + \varepsilon \nabla^4_x - a \nabla^3_x - \nabla_x J \nabla_x J - k(\nabla_x J)\nabla_x \nabla^{k}_x J + e F_{1,k} + F_{3,k}) - a R(u_x, \nabla_x J \nabla^{k}_x J)u_x + 2 bg(\nabla_x J \nabla^{k}_x J, u_x) u_x + e F_{1,k} + F_{3,k}.
\]
We next compute \( \nabla_1 V \), where \( V = \nabla^m_x u_x \). By acting \( A \) on (28) with \( k = m \), we obtain
\[
(29) \quad A(\nabla_1 + \varepsilon \nabla^4_x - a \nabla^3_x - \nabla_x J \nabla_x J - m(\nabla_x J)\nabla_x \nabla^{m}_x J + e F_{1,m} + F_{3,m}).
\]
In the computation below, we use a notation denoting the modulo-\( L^2 \)-bounded operator in some sense. More precisely, for an operator \( A = A(t, x) \) acting on \( \Gamma(u^{-1}TN) \), we use the notation \( A \equiv 0 \) if there exists a positive constant \( C_3 > 0 \) depending on \( a, b, N \) and \( r_0 \) such that
\[
\| AW(t) \| \leq C_3 \| W(t) \| \quad \text{for all} \quad W(t, \cdot) \in L^2(R; TN)
\]
on \([0, T^*_x] \). In particular, we can easily check that
\[
(30) \quad \frac{\partial^2}{\partial \tau^2} A, \frac{\partial^2}{\partial \tau^2} (A^{-1}), A_t \equiv 0 \quad \text{for} \quad \tau = 0, 1, 2, 3, 4.
\]
These properties (30) are used below without any comments.
Now we continue the computation of (29). By a simple computation, we can check that

\[
L_t = \frac{L}{C_0}
\]

and

\[
\varepsilon A V_x^4 = \varepsilon V_x^4 A + \varepsilon [A, V_x^4],
\]

\[
-\alpha A V_x^3 = -\alpha V_x^3 A - \alpha [A, V_x^3]
\]

which implies

\[
-\alpha V_x^3 A + \alpha (A_{xxx} + 3 A_{xx} \varepsilon_x + 3 \varepsilon_x^2)
\]

and

\[
-\alpha V_x^3 A + 3 \alpha (A_{xx} \varepsilon_x + A_x \varepsilon_x^2).
\]

Furthermore, by using the trivial equality \(1 = A^{-1} A\), we deduce that

\[
A_{xx} V_x = A_{xx} V_x A^{-1} A
\]

and

\[
A_x V^2_x = A_x V^2_x A^{-1} A
\]

By substituting (34) and (35) into (33), we obtain

\[
-\alpha A V_x^3 \equiv -\alpha V_x^3 A + 3 \alpha A_x A^{-1} V_x^2 A + 6 \alpha A_x (A^{-1})_x A V_x A + 3 \alpha A_{xx} A^{-1} V_x A.
\]

We next evaluate the term \(-A V_x J V_x = -V_x J V_x A - [A, V_x J V_x]\). For the second term of the right hand side of the above, a simple computation gives

\[
V_x A = \alpha V_x + \alpha,
\]

\[
J V_x A = A J V_x + A_x J,
\]

\[
V_x J V_x A = A_x J V_x + A V_x J V_x + A_{xx} J + A_x (V_x J) + A_x J V_x
\]

and

\[
- [A, V_x J V_x] = -A V_x J V_x + V_x J V_x A
\]

which implies

\[
= 2 A_x J V_x + A_{xx} J + A_x (V_x J)
\]

\[
= 2 A_x J V_x A^{-1} A + A_{xx} J + A_x (V_x J)
\]
\[= 2A_x(A^{-1})_xJA + 2A_xA^{-1}JV_xA + A_{xx}J + A_x(V_xJ)\]
\[\equiv 2A_xA^{-1}JV_xA.\]

Thus we obtain

\[\text{(37)} \quad -A\nabla_xJ\nabla_x\equiv -\nabla_xJ\nabla_xA + 2A_xA^{-1}JV_xA.\]

In the same way, we deduce that

\[\text{(38)} \quad -mA(\nabla_xJ)\nabla_x\equiv -m(\nabla_xJ)\nabla_xA,\]
\[\text{(39)} \quad -bAg(u_x,u_x)\nabla_x\equiv -bg(u_x,u_x)\nabla_xA,\]

and

\[\text{(40)} \quad -aAR(u_x,V_xV_x^m)u_x = -aR(u_x,V_xAV_x^m)u_x + aR(u_x,A_xV_x^m)u_x\]
\[\equiv -aR(u_x,V_xAV_x^m)u_x,\]
\[\text{(41)} \quad 2bAg(V_xV_x^m)u_x = 2bg(V_xAV_x^m)u_x - 2bg(A_xV_x^m)u_x\]
\[\equiv 2bg(V_xAV_x^m)u_x.\]

By combining (31), (32), (36), (37), (38), (39), (40) and (41), we obtain that

\[\text{(42)} \quad (\nabla_t + \varepsilon\nabla_x^4 - a\nabla_x^3 - \nabla_xJ\nabla_x - m(\nabla_xJ)\nabla_x - bg(u_x,u_x)\nabla_x)V\]
\[= -3aA_xA^{-1}\nabla_x^2V - 2A_xA^{-1}JV_xV - aR(u_x,\nabla_xV_x\nabla_x)u_x\]
\[+ 2bg(\nabla_xV_x)u_x - 3aA_xA^{-1}\nabla_xV_xV - 6aA_x(A^{-1})_x\nabla_xV_x\]
\[+ \varepsilon A\delta_{F_1,m} - \varepsilon[A,\nabla_x^3]\nabla_x^mV_xu_x + F_3^{'m},\]

where the term \(F_3^{'m}\) has the estimate of the form

\[\text{(43)} \quad \|F_3^{'m}\| \leq C(a,b,N,r_0)P(N_{m-1})N_m.\]

Here, \(C(a,b,N,r_0)\) is a positive constant depending on \(a, b, N\) and \(r_0\), and \(P(s)\) is a monotone increasing function on \(|s|\). Here and hereafter, we use the same letter \(C(x,\beta,\ldots)\) and \(P(s)\) denoting different constants which depend on \(x,\beta,\ldots\) and different monotone increasing functions on \(|s|\) respectively.

Now we go back to the estimate for the third term of (16). By using (42), we deduce that
\[ \int_R g(\nabla_x V, V) dx \]

\[(44) \quad \int_R \left\{ a\partial_x^4 + a\partial_x^3 + V^2 \right\} dx \]

\[(45) \quad + \int_R g(m(\nabla_x J) \nabla_x V, V) dx \]

\[(46) \quad + \int_R g((bg(u_x, u_x) - 6aA_x(A^{-1})_x - 3aA_{xx}A^{-1}) \nabla_x V, V) dx \]

\[(47) \quad + \int_R g(-aR(u_x, \nabla_x V) u_x, V) dx \]

\[(48) \quad + \int_R g(2bg(\nabla_x V, u_x) u_x, V) dx \]

\[(49) \quad + \int_R g(-3aA_x A^{-1} \nabla_x^2 V, V) dx \]

\[(50) \quad + \int_R g(-2A_x A^{-1} \nabla_x V, V) dx \]

\[(51) \quad + \int_R g(\varepsilon \Delta F_{1,m} - \varepsilon [A, \nabla_x^4 \nabla_x^2 u_x, V) dx \]

\[(52) \quad + \int_R g(F_{3,m}', V) dx. \]

For the integral (44), the repeatedly use of integration by parts implies

\[(53) \quad \int_R g(-a\partial_x^4 V, V) dx = -a||\nabla_x V||^2, \]

\[(54) \quad \int_R g(a\partial_x^3 V, V) dx = -a \int_R g(\nabla_x^2 V, \nabla_x V) dx = 0, \]

\[(55) \quad \int_R g(V_x \nabla_x V, V) dx = -\int_R g(J \nabla_x V, \nabla_x V) dx = 0. \]

Also for (47) and (48), the integration by parts again shows that

\[(56) \quad (47) = -a \int_R g(R(u_x, \nabla_x V) u_x, V) dx + \frac{a}{2} \int_R g(R(u_x, V) u_x, \nabla_x V) dx \]

\[+ a \int_R g(R(u_x, V) \nabla_x u_x, V) dx + \frac{a}{2} \int_R g(R(\nabla_x u_x, V) u_x, V) dx \]

\[+ \frac{a}{2} \int_R g((\nabla_x R)(u_x, V) u_x, V) dx. \]
\begin{align*}
&= a \int_R g(R(u_x, V) \nabla_x u_x, V) dx + \frac{a}{2} \int_R g((\nabla_x R)(u_x, V) u_x, V) dx \\
&\leq C(a, N, r_0) \| V \|^2,
\end{align*}

(57) \hspace{1cm} (48) = b \int_R g(g(\nabla_x V, u_x) u_x, V) dx - b \int_R g(g(V, u_x) u_x, \nabla_x V) dx \\
&- b \int_R g(g(V, u_x) \nabla_x u_x, V) dx - b \int_R g(g(V, \nabla_x u_x) u_x, V) dx \\
&= -2b \int_R g(g(V, \nabla_x u_x) u_x, V) dx \\
&\leq C(b, r_0) \| V \|^2.

Note that, in order to obtain the second equality of (56), we use the fact that

\[ g(R(W_1, W_2) W_3, W_4) = g(R(W_3, W_4) W_1, W_2) \]

holds for any \( W_j \in \Gamma(u^{-1} TN), \ j = 1, 2, 3, 4 \), which is the fundamental property of the Riemannian curvature tensor on \( N \). In the same way, we can show that

\begin{align*}
(46) &= -\frac{1}{2} \int_R g([bg(u_x, u_x) - 6aA_x(A^{-1})_x - 3aA_x A^{-1}]_x V, V) dx \\
&\leq C(a, b, r_0) \| V \|^2.
\end{align*}

The estimate for rest terms is important and is demonstrated as follows. For (49), a simple computation shows that \( 3aA_x A^{-1} = 3aK_x = -g(u_x, u_x) \), and thus the integration by parts, the Sobolev embedding and the Hölder inequality show that

\begin{align*}
(49) &= \int_R g(g(u_x, u_x) \nabla_x^2 V, V) dx \\
&= -\int_R g(g(u_x, u_x) \nabla_x V, \nabla_x V) dx - 2 \int_R g(g(\nabla_x u_x, u_x) \nabla_x V, V) dx \\
&\leq -\| (g(u_x, u_x))^{1/2} \nabla_x V \|^2 + C(r_0) \| V \|^2.
\end{align*}

Note that the estimate for the last inequality above is obtained similarly to that of (46).

In order to estimate (45), it is important that there exists a constant \( C_4 > 0 \) depending only on \( N \) such that

\[ \|(\nabla_x J)(x) \| \leq C_4(g(u_x(x), u_x(x)))^{1/2} \]
uniformly in \( x \in \mathbb{R} \). By noting this and by using the Schwarz inequality, we deduce

\[
\begin{align*}
(45) \leq & \, m \| (V_x J) V_x V \| \| V \|
\leq & \, m C_4 \| (g(u_x, u_x))^{1/2} V_x V \| \| V \|
\leq & \, \rho \| (g(u_x, u_x))^{1/2} V_x V \|^2 + \frac{m^2 C_4^2}{4 \rho} \| V \|^2
\end{align*}
\]

for any \( \rho > 0 \).

The estimate for (50) is similar to the above argument. Indeed, we deduce

\[
\begin{align*}
(50) = & \, \frac{2}{3 a} \int_{\mathbb{R}} g(g(u_x, u_x)) J V_x V, V) \, dx \\
\leq & \, \frac{2}{3 |a|} \int_{\mathbb{R}} g((g(u_x, u_x))^{1/2} V_x V, (g(u_x, u_x))^{1/2} J V) \, dx \\
\leq & \, \rho \| (g(u_x, u_x))^{1/2} V_x V \|^2 + \left( \frac{2}{3 |a|} \right)^2 \frac{1}{4 \rho} \| (g(u_x, u_x))^{1/2} J V \|^2
\end{align*}
\]

for any \( \rho > 0 \).

If we take \( \rho = 1/4 \), then (59), (60) and (61) yield that

\[
(45) + (49) + (50) \leq -\frac{1}{2} \| (g(u_x, u_x))^{1/2} V_x V \|^2 + C(a, N, r_0) \| V \|^2,
\]

which completes the elimination of the derivative loss arising from the seemingly bad first order term.

For (52), it is easy to show that

\[
(52) \leq C(a, b, N, r_0) P(N_{m-1}) N_m \| V \|,
\]

by the Sobolev embedding and the Hölder inequality.

For (51), repeatedly use of integration by parts and the Schwarz inequality show that

\[
(51) \leq \frac{\varepsilon}{2} \| V_x^2 V \|^2 + C(a, b, N, r_0) P(N_{m-1}) N_m^2.
\]

Combining (53) and (64), we deduce

\[
(44) + (51) \leq -\frac{\varepsilon}{2} \| V_x^2 V \|^2 + C(a, b, N, r_0) P(N_{m-1}) N_m^2.
\]
Consequently, the energy estimate
\[ (66) \quad \int_R g(\nabla_x V, V)dx + \frac{\varepsilon}{2}\|\nabla_x^2 V\|^2 + \frac{1}{2} ||(g(u_x(t), u_x(t)))^{1/2}\nabla_x V\|^2 \leq C(a, b, N, r_0) P(N_{m-1})N_m^2 \]
follows from (56), (57), (58), (63), and (65).

On the other hand, we can easily obtain the estimate of the form
\[ (67) \quad \sum_{k=1}^{m-1} \int_R g(\nabla_x^k u_x, \nabla_x^k u_x)dx + \frac{\varepsilon}{2} \sum_{k=1}^{m-1}\|\nabla_x^{k+2} u_x\|^2 \leq C(a, b, N, r_0) P(N_{m-1})N_m^2, \]
if we note (28).

Finally, by adding (66) and (67) into (16), we conclude that the estimate of the form
\[ \frac{1}{2} \frac{d}{dt}N_m^2 + \frac{\varepsilon}{2} \left(\|\nabla_x^2 V\|^2 + \sum_{k=0}^{m-1}\|\nabla_x^{k+2} u_x\|^2 \right) + \frac{1}{2} ||(g(u_x, u_x))^{1/2}\nabla_x V\|^2 \leq C(a, b, N, r_0) P(N_{m-1})N_m^2 \]
holds on $[0, T_\varepsilon^*)$, which completes the proof of Proposition 3. \hfill \square

Lemma 2 follows immediately from Proposition 3 in the following way. At first, we set $m = 4$. Then, (15) implies that
\[ (N_4(t))^2 \leq r_0^2 \exp(2C(a, b, N, r_0)t) \]
for $t \in [0, T_\varepsilon^*)$. In particular, if we set $t = T_\varepsilon^*$, then this becomes
\[ 4r_0^2 = (N_4(T_\varepsilon^*))^2 \leq r_0^2 \exp(2C(a, b, N, r_0)T_\varepsilon^*), \]
which implies $T_\varepsilon^* \geq 2C(a, b, N, r_0)/\log 4$. Hence, if we set $T = 2C(a, b, N, r_0)/\log 4$, then $T_\varepsilon^* \geq T$ and $\{u^\varepsilon\}_{\varepsilon \in (0, 1)}$ is a bounded sequence in $L^\infty(0, T; H^4(R; TN))$. Note that $T$ actually depends on $a, b, N$ and $\|u_0\|_{H^4}$, and not on $\varepsilon \in (0, 1)$. Then, by the energy inequality (15), the inductive use of the Gronwall inequality for $m = 5, 6, \ldots$ shows that $\{u^\varepsilon\}_{\varepsilon \in (0, 1)}$ is a bounded sequence in $L^\infty(0, T; H^m(R; TN))$, which completes the proof of Lemma 2. \hfill \square

We can construct a solution to (1)–(2) by Lemma 2. Indeed, if we set $v^\varepsilon = w \circ u^\varepsilon$, then Lemma 2 implies that $\{v^\varepsilon\}_{\varepsilon \in (0, 1)}$ is a bounded sequence in $L^\infty(0, T; H^m(R; R^d))$, and thus the standard compactness argument implies that there exists a subsequence $\{v^j\}_{j \in N}$ of $\{v^\varepsilon\}_{\varepsilon > 0}$ and $v$ such that
for any $M > 0$. In particular, $v \in C([0, T] \times \mathbb{R}; w(N))$ follows from the above. Furthermore, we can easily check that

$$u(x, t) = w^{-1} \circ v \in C([0, T] \times \mathbb{R}; N)$$

solves the original IVP (1)–(2). In addition, since $u \in L^\infty(0, T; H^m(R; \mathbb{R}^d)) \cap C([0, T]; H^{m-1}(R; \mathbb{R}^d))$, the solution $u$ satisfies

$$u_x \in L^\infty(0, T; H^m(R; TN)) \cap C([0, T]; H^{m-1}(R; TN)).$$

**Step 3. Uniqueness of the solution.** Let $u, v \in C([0, T] \times \mathbb{R}; N)$ be solutions to (1)–(2) satisfying (68) and $u(0, x) = v(0, x)$. Then $z = w \circ u - w \circ v$ is well-defined as an $\mathbb{R}^d$-valued function. From the classical energy estimate for $z$ in $H^1(\mathbb{R}^d)$, we can show that $z = 0$, and hence the uniqueness follows. In the energy estimate, whether $N$ is a Kähler manifold or not is not essential, but the antisymmetry of $Ju_T N$ plays a crucial part. Indeed, the proof in [24, Section 5] also valid in the present case. Thus we omit the detail.

**Step 4. Continuity in time of $V_x^m u_x$ in $L^2$.** We have already proved the existence of a unique solution $u \in C([0, T] \times \mathbb{R}; N)$ with (68). Thus the proof of $V_x^m u_x \in C([0, T]; L^2(R; TN))$ is left. In order to complete this, it suffices to show that $dw u(A V_x^m u_x)$ belongs to $C([0, T]; L^2(R; \mathbb{R}^d))$. We omit the detail, since the proof is same as that in [3, pp. 402–403].

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