Quantum feedback control of a solid-state qubit

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We have studied theoretically the basic operation of a quantum feedback loop designed to maintain a desired phase of quantum coherent oscillations in a single solid-state qubit. The degree of oscillations synchronization with external harmonic signal is calculated as a function of feedback strength, taking into account available bandwidth and coupling to environment. The feedback can efficiently suppress the dephasing of oscillations if the qubit coupling to the detector is stronger than coupling to environment.

The principle of feedback control is used in a wide variety of physical and engineering problems. In particular, it can be applied in a straightforward way to tune the oscillation phase of a harmonic oscillator in order to achieve a desired synchronization with some reference oscillator. An intriguing and fundamental question is whether continuous feedback can be used to control quantum systems; for instance, if it is possible or not to tune the phase of quantum coherent (Rabi) oscillations in a qubit (two-level system).

At first sight the quantum feedback seems to be impossible, because according to the “orthodox” collapse postulated the quantum state is abruptly destroyed by the act of measurement. However, it was shown two decades ago, in particular by Leggett, in a typical solid-state setup the collapse of a qubit state should be considered as a continuous process rather than as instantaneous event. The reason is typically weak coupling between the quantum system and the detector and also the finite noise of the detector, so that it takes some time until acceptable signal-to-noise ratio is reached and the measurement can be regarded as completed.

While the Leggett’s theory as well as the majority of similar approaches can describe only ensembles of quantum systems, the theory describing the gradual collapse of a single solid-state qubit was developed only recently (A similar problem in quantum optics was solved much earlier – see, e.g. Refs.1 and references in1). Basically, the theory says that the evolution of a single quantum system due to continuous measurement is governed by the information continuously acquired from the detector. Similarly to classical probability, the Bayes formalism which naturally takes into account incomplete information from the detector, can still be applied to the density matrix of the measured quantum system; thus the formalism is called Bayesian.

In case of a poor detector the extra noise acting back onto the input disturbs the measured system stronger than the limit determined by the uncertainty principle; this leads to gradual decoherence of the measured system. In contrast, when measured with a good (quantum-limited) detector, the quantum system does not loose the coherence (even though the quantum state evolves randomly), moreover, its density matrix can be gradually purified which basially means acquiring as much information about the system as permitted by quantum mechanics.

Since the Bayesian formalism allows us to monitor the continuous evolution of a quantum system in a process of measurement, this naturally gives rise to a possibility of continuous feedback control of a quantum system. In this paper we will study the operation of a feedback loop proposed in Ref.2 and designed to maintain a desired phase of quantum coherent oscillations in a solid-state qubit. (Quantum feedback in quantum optics has been proposed and studied earlier – see, e.g., Refs.3–5). In particular, we will study the dependence of the loop operation on the feedback strength, available bandwidth, and dephasing due to environment.

As an example of the measurement setup (Fig. 1), we consider a qubit represented by a single electron in a double quantum dot (DQD), the location of which is measured by a quantum point contact (QPC) nearby in a way used in Ref.6. If the electron is in the dot 2 (state |2⟩) which is closer to QPC than dot 1, then the QPC tunnel barrier is higher and so the average current I2 through QPC is smaller than the average current I1 corresponding to the electron in the dot 1 (state |1⟩). Consequently, from the QPC current one gets information about the electron location. We consider a realistic case of weak response, ΔI ≡ I1 - I2 ≲ I0 ≡ (I1 + I2)/2. In this case the measurement time S1/2(ΔI)2, which is necessary to achieve signal-to-noise ratio equal to 1 (here S1 is the shot noise of the QPC current), is much larger than ε/I0, so the QPC current I(t) is continuous on the measurement timescale and we do not need to consider individual tunneling events in QPC.

The evolution of the qubit density matrix ρ during the measurement process is described within the Bayesian formalism by equations.

\[
\text{FIG. 1. Schematic of the quantum feedback loop maintaining the quantum oscillations in a qubit.}
\]
\[ \dot{\rho}_{11} = -\dot{\rho}_{22} = -2 \frac{H}{\hbar} \text{Im} \rho_{12} + \rho_{11} \rho_{22} \frac{2 \Delta I}{S_I} [I(t) - I_0], \quad (1) \]

\[ \dot{\rho}_{12} = \frac{i}{\hbar} \rho_{12} + \frac{H}{\hbar} (\rho_{11} - \rho_{22}) - (\rho_{11} - \rho_{22}) \frac{\Delta I}{S_I} [I(t) - I_0] \rho_{12} - \gamma \rho_{12}, \quad (2) \]

where \( \varepsilon \) and \( H \) are, respectively, the energy asymmetry and tunneling strength of the qubit [the qubit Hamiltonian is \( \mathcal{H}_{\text{q}} = (\varepsilon / 2)(c_1^\dagger c_2 - c_1^\dagger c_1) + H(c_1^\dagger c_2 + c_2^\dagger c_1) \)], and \( \gamma = \gamma_d + \gamma_c \) is the dephasing rate due to the detector nonideality (\( \gamma_d \)) and coupling with the environment (\( \gamma_c \)). Theoretically, \( \gamma_d = 0 \) when qubit is measured by a QPC; however, if instead of QPC we use a single-electron transistor (SET), then dephasing \( \gamma_d \) is usually quite significant (except the case when the SET operates in a cotunneling regime). Notice that the ensemble dephasing rate \( \Gamma = \gamma + (\Delta I)^2 / S_I \) is larger than \( \gamma \) because of different evolution of the ensemble members due to random \( I(t) \). Individual realizations can be simulated using the formula

\[ I(t) - I_0 = (\rho_{11} - \rho_{22}) \Delta I / 2 + \xi(t), \quad (3) \]

where \( \xi(t) \) is the pure white noise with spectral density \( S_{\xi} = S_I \). If Eqs. (1) and (2) are averaged over \( \xi(t) \) (we use Stratonovich definition for stochastic differential equations), then we get usual ensemble-averaged equations for qubit evolution (terms proportional to \( \Delta I \) will disappear and \( \gamma \) will be replaced by \( \Gamma \)).

It is natural to characterize the effect of extra dephasing \( \gamma_d \) by the detector ideality (efficiency) \( \eta \equiv 1/[1 + \gamma_d S_I / (\Delta I)^2] \). One can show that \( \eta = (\hbar / 2 \epsilon_d)^2 \) where \( \epsilon_d \equiv (\epsilon / \epsilon_0)^{1/2} \) where \( \epsilon_0 \) is the usual (output) energy sensitivity and \( \epsilon \) is a similar quantity characterizing back-action to the input. So, an ideal case \( \eta = 1 \) corresponds to a detector with quantum-limited sensitivity.

To realize a feedback loop (Fig. 1), we can monitor the qubit evolution using the detector current \( I(t) \) plugged into Eqs. (1) and (2). Then the qubit state is compared with the desired state, and the difference signal is used to control the qubit parameters \( H \) and/or \( \varepsilon \). In the example studied in this paper the feedback loop is designed to stabilize the quantum oscillations of the state of a symmetric qubit (\( \varepsilon = 0 \)), so the desired evolution is \( \rho_{11}(t) = 1 - \rho_{22}(t) = [1 + \cos(\Omega t)] / 2, \rho_{12}(t) = \rho_{21}(t) = \frac{1}{2} \sin(\Omega t) / \hbar \), where the frequency is \( \Omega = (4 \hbar^2 + \varepsilon^2)^{1/2} / h = 2 H / \hbar \). As a difference (“error”) signal we use the phase difference \( \Delta \phi = |\phi - \phi_0| < \pi \) between the desired value \( \phi_0(t) = \Omega t \mod 2 \pi \) and the monitored value \( \phi(t) \equiv \arctan(2 \text{Im} \rho_{12}(t) / (\rho_{11}(t) - \rho_{22}(t))) \).

This difference is used to control the qubit parameter \( H \) (changing the barrier height of DQD); here we study a linear control: \( H_{fb} = (1 - F_{\text{FB}}) \Delta \phi \). One can see that for a moderate value of \( F = 0.5 \) the synchronization is already very good [the ideal case would be \( K_{2}(\tau) = \cos(\Omega \tau) / 2 \)].

The correlation function \( K_{2}(\tau) \) of the qubit quantum oscillations for \( C = 1 \) and feedback factors \( F = 0 \) (thin solid line), \( 0.05 \) (thick solid line), and \( 0.5 \) (dashed line). Nondecaying oscillations are due to synchronization by the feedback.

\[ \frac{d}{dt} \Delta \phi = -\sin \phi \frac{\Delta I}{S_I} \left( \Delta I / 2 \cos \phi + \xi \right) - \frac{2FH}{\hbar} \Delta \phi, \quad (4) \]

which assumes the absence of \( 2\pi \) phase slips (good or moderate synchronization). For weak coupling (\( C / 8 \ll 1 \)) we can neglect the first term in parentheses and av-
average the random term over \(\sin \phi\) assuming almost harmonic evolution that leads to the simplified equation

\[
\frac{d}{dt} \Delta \phi = \dot{\xi} - \frac{2FH}{\hbar} \Delta \phi, \tag{5}
\]

where \(\dot{\xi}(t)\) is the white noise with spectral density \(S_\xi = (\Delta I)^2/2S\). This equation describes a particle diffusion in the parabolic potential (we again assume \(|\Delta \phi| < \pi\)). The corresponding Fokker-Planck equation has an exact solution which is used to calculate the correlation function \(K_z(\tau) \approx (\cos[\Delta \phi(t) - \Delta \phi(t + \tau)]) \cos \Omega \tau/2\). In this way we obtain the analytical expression

\[
K_z(\tau) = \cos \frac{\Omega \tau}{2} \exp \left[ \frac{C}{16F} \left( e^{-2F\hbar \tau/\hbar} - 1 \right) \right], \tag{6}
\]

which fits well the Monte-Carlo results when \(C/8 \ll 1\) and \(C/16F \ll 1\) (weak coupling and moderate or good synchronization). As an example, the dots in Fig. 3 show the numerically calculated (using the least-mean-square fit) asymptotic amplitude \(A_{Kz}\) of \(K_z(\tau)\) oscillations (at \(\tau \to \infty\)) as a function of the feedback factor \(F\) for three values of the coupling \(C\), while solid lines show the corresponding analytical curves \(A_{Kz} = \exp(-C/16F)/2\).

The correlation function \(K_I(\tau) \equiv \langle I(t + \tau)I(t)\rangle\) of the detector current \(I(t)\) is somewhat similar to \(K_z(\tau)\), however, it also has the decaying contribution due to correlation \(K_{\bar{z}z}\) and a \(\delta\)-function contribution due to the detector noise. The analytical result for the same regime as above,

\[
K_I(\tau) = \frac{S_I}{2} \left[ \delta(\tau) + \frac{(\Delta I)^2}{4} \frac{\cos(\Omega \tau)}{2} \left( 1 + e^{-2F\hbar \tau/\hbar} \right) \right] \times \exp \left[ \frac{C}{16F} \left( e^{-2F\hbar \tau/\hbar} - 1 \right) \right], \tag{7}
\]

also agrees well with the Monte Carlo results.

The spectral density \(S_I(\omega)\) of the detector current can be obtained as a Fourier transform of \(K_I(\tau)\). While in absence of feedback the quantum oscillations in the qubit can provide only a moderate peak of \(S_I(\omega)\) around frequency \(\Omega\) (the peak height cannot be larger than 4 times the noise pedestal\(^\ddagger\)), the feedback synchronization leads to the appearance of a \(\delta\)-function at the frequency of desired oscillations. (In principle the desired frequency can differ a little from \(\Omega\); however, in this case the performance of the feedback loop worsens.)

Besides the correlation function and spectral density, we have studied one more characteristic, \(D\), of the synchronization degree. We define \(D\) as the average scalar product of the unity-length vector on the Bloch sphere corresponding to the desired state and the vector corresponding to the actual state of the qubit. The equivalent definition is \(D = 2(\text{Tr} \rho_\rho) - 1\), where \(\rho_\rho\) is the density matrix of the desired pure state. [The so-called fidelity is equal to either \((D + 1)/2\) or \(\sqrt{(D + 1)/2}\), depending on the definition\(\ddagger\).] Perfect synchronization corresponds to \(D = 1\). It is simple to show that in the limit of weak coupling and for symmetric distribution of \(\Delta \phi\) (unshifted desired frequency), \(A_{Kz}\) coincides with \(D^2/2\). Notice, however, that at moderate coupling, \(D^2/2\) (see dashed lines in Fig. 4) is significantly closer to the analytical result than \(A_{Kz}\).

Upper solid line in Fig. 4 shows the dependence of \(D\) on the feedback factor \(F\) for \(C = 1\) and \(\tau_a = 0\). One can see that \(D\) is proportional to \(F\) for small \(F\) (\"soft\" onset of the synchronization) and \(D\) is asymptotically approaching 1 at large \(F\). The analytical result \(D = \exp(-C/32F)\) (dashed line in Fig. 4) is very close to the numerical results at moderate and good synchronization.

Finite available bandwidth of the detector current \(I(t)\) (finite averaging time \(\tau_a\) in our formalism) worsens the performance of the quantum feedback loop. The solid lines in Fig. 4 show the dependence of the synchroniza-
tion degree \(D(F)\) for \(\tau_a = 0, 1/3, \) and \(2/3,\) where \(T = 2\pi/\Omega\) is the oscillation period. Obviously, a significant information loss occurs when \(\tau_a\) becomes comparable to \(T,\) leading to a decrease of \(D.\) The curves \(D(F)\) saturate at large \(F\) allowing us to introduce the dependence \(D_{\text{max}}(\tau).\) Calculations for the parameters of Fig. [3] show pretty good synchronization, \(D_{\text{max}} = 0.993,\) for \(\tau_a = T/30,\) while \(D_{\text{max}} = 0.98, 0.92,\) and 0.57 for \(\tau_a = T/10, T/3,\) and \(2T/3,\) respectively.

The main potential practical importance of the quantum feedback is the ability to suppress the effect of the qubit dephasing caused by interaction with the environment (see Fig. [3]). This can be used, for example, for qubit initialization in a solid-state quantum computer. Solid lines in Fig. [3] show the dependence \(D(F)\) for several magnitudes of the dephasing due to environment, \(d_c = 0, 0.1,\) and 0.5, where \(d_c \equiv \gamma_c/[(\Delta I)^2/4S_I]\) is the ratio between the qubit coupling to the environment and to the detector (we still assume an ideal detector). First of all, we see that the feedback still maintains the qubit phase synchronization for infinitely long time. However, for finite \(d_c\) the degree of synchronization \(D\) saturates at a level less than unity. We have studied numerically the dependence \(D_{\text{max}}(d_c)\) for \(C = 1/2, 1,\) and 2 (while \(\tau_a = 0\) and \(\eta = 1\)) and found a linear dependence at small \(d_c: D_{\text{max}} \simeq 1 - 0.5d_c.\) [A little better formula \(D_{\text{max}} \simeq 1 - 0.5d_c/(1 + d_c)\) works reasonably well up to \(d_c < 1.\)] This means that the feedback loop can efficiently suppress the qubit dephasing due to the coupling to the environment if this coupling is much weaker than the qubit coupling to a nearly ideal detector.

Notice that the solid lines shown in Figs. [3] and [5] are calculated assuming the feedback control of the tunnel matrix element \(H_{fb} = H[1 - F \times \Delta \phi]\) even when \(H_{fb}\) becomes negative (this is also an assumption for the analytical results). To eliminate this unphysical assumption we have also performed numerical calculations with restrictions \(H_{fb} > 0\) and \(H_{fb} > H/2.\) This leads to rather minor modifications of the presented curves (dashed and dotted lines in Fig. [5] show the results for \(d_c = 0\) and \(\tau_a = 0\).) However, important difference is that \(D(F)\) goes down at large \(F,\) so the optimum \(D_{\text{max}}\) is achieved at some finite value of \(F.\) More detailed study of this problem will be presented elsewhere.

Besides the discussed feedback based on \(\Delta \phi\) calculation, we have also studied a “direct” feedback loop in which \(H_{fb}(t)/H = 1 - F[2(I_a(t) - I_0)/\Delta I - \cos \Omega(t - \tau_a/2)]\) (we call it also a “naive” feedback because this control formula is easily designed from the naive assumption that the detector current directly follows the evolution of \(\rho_{11}\)). Direct feedback is much simpler for experimental realization since it does not require real-time solution of the Bayesian equations (direct feedback in quantum optics has been studied in Refs. [14], [15]). Quite surprisingly for us, the direct feedback can also provide a good phase synchronization of quantum oscillations if \(F/C\) is close to 1/4 (see dotted line in Fig. [3]). However, it requires more careful choice of \(F\) and \(\tau_a\) than for the Bayesian feedback, and also suffers more significantly from the restriction on \(H_{fb}\) variation. The results in more detail will be discussed elsewhere.

Experimentally, besides the realization of quantum feedback control of a DQD continuously measured by a QPC, one can also think about the qubit based on a single-Cooper-pair box measured by a single-electron transistor (see discussion in Ref. [14]). This realization can be preferable because of a rapid progress of metallic single-electronics technology. However, the problems are high output impedance of the single-electron transistor and its nonideality as a quantum detector. The third potential realization can be based on SQUIDs. For any realization the major problem is the bandwidth: the feedback should be at least faster than the qubit dephasing. Because of that, the quantum feedback of a solid-state qubit should probably be attempted only after the realization of recently proposed Bell-type two-detector correlation experiment [16], which would show the possibility of quantum monitoring, the first step to quantum feedback control.

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The feedback considered in Ref. [15] is somewhat similar and in our notations would correspond to $H_{fb} = |1 - F \times \text{sign}(\Delta \phi)|H$. Unfortunately, the dependence of the feedback loop operation on the feedback strength was not studied in Ref. [15].

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