SOME MORE WEAK HILBERT SPACES

GEORGE ANDROULAKIS, PETER G. CASAZZA, AND DENKA N. KUTZAROVA

Abstract: We give new examples of weak Hilbert spaces.

1. Introduction

The Banach space properties weak type 2 and weak cotype 2 were introduced and studied by V. Milman and G. Pisier [MP]. Later, Pisier [P1] studied spaces which are both of weak type 2 and weak cotype 2 and called them weak Hilbert spaces. Weak Hilbert spaces are stable under passing to subspaces, dual spaces, and quotient spaces. The canonical example of a weak Hilbert space which is not a Hilbert space is convexified Tsirelson space $T^2$ [CS, J1, J2, P1]. Tsirelson’s space was introduced by B.S. Tsirelson [T] as the first example of a Banach space which does not contain an isomorphic copy of $c_0$ or $\ell_p$, $1 \leq p < \infty$. Today, we denote by $T$ the dual space of the original example of Tsirelson since in $T$ we have an important analytic description of the norm due to Figiel and Johnson [FJ]. In [J1], Johnson introduced modified Tsirelson space $T_M$. Later, Casazza and Odell [CO] proved the surprising fact that $T_M$ is naturally isomorphic to the original Tsirelson space $T$. At this point, all the non-trivial examples of weak Hilbert spaces (i.e. those which are not Hilbert spaces) had unconditional bases and had subspaces which failed to contain $\ell_2$. A. Edgington [E] introduced a class of weak Hilbert spaces with unconditional bases which are $\ell_2$-saturated. That is, every subspace of the space contains a further subspace isomorphic to a Hilbert space but the space itself is not isomorphic to a Hilbert space. R. Komorowski [K] (or more generally Komorowski and Tomczak-Jaegermann [KT]) proved that there are weak

1991 Mathematics Subject Classification. 46B.

The second author was supported by NSF DMS 9706108.

The third author was partially supported by the Bulgarian Ministry of Education and Science under contract MM-703/97.
Hilbert spaces with no unconditional basis. In fact, they show that $T^2$ has such subspaces.

In another surprise, Nielsen and Tomczak-Jaegermann [NTJ] showed that all weak Hilbert spaces with unconditional bases are very much like $T^{(2)}$.

There are still many open questions concerning weak Hilbert spaces and $T^{(2)}$, due partly to the shortage of non-trivial examples in this area. For example, it is still a major open question in the field whether a Banach space for which every subspace has an unconditional basis (or just local unconditional structure - LUST) must be isomorphic to a Hilbert space. If there are such examples, they will probably come from the class of weak Hilbert spaces. It is an open question whether every weak Hilbert space has a basis, although Maurey and Pisier (see [M]) showed that separable weak Hilbert spaces have finite dimensional decompositions. Nielsen and Tomczak-Jaegermann have shown that weak Hilbert spaces that are Banach lattices have the property that every subspace of every quotient space has a basis. But it is unknown whether every weak Hilbert space can be embedded into a weak Hilbert space with a unconditional basis. In fact, it is unknown if a weak Hilbert space embeds into a Banach lattice of finite cotype. It turns out that this question is equivalent to the question of whether every subspace of a weak Hilbert space must have the GL-Property [CN] which is slightly weaker than having LUST. In this note we extend the list of non-trivial examples of weak Hilbert spaces by producing examples which are $\ell_2$-saturated but have no subspaces isomorphic to subspaces of the previously known examples.

2. Basic Constructions

If $F$ is a finite dimensional Banach space then let $d(F)$ denote the Banach-Mazur distance between $F$ and $\ell_2^{\dim F}$. The fundamental notion of this note is the one of the weak Hilbert space. Recall the following definition as one of the many equivalent ones (cf [PT] Theorem 2.1).

**Definition 2.1.** A Banach space $X$ is said to be a weak Hilbert space if there exist $\delta > 0$ and $C \geq 1$ such that for every finite dimensional subspace $E$ of $X$ there exists a subspace $F \subseteq E$ and a projection $P : X \rightarrow F$ such that $\dim F \geq \delta \dim E$, $d(F) \leq C$ and $\|P\| \leq C$. 
We need to recall the definition of the Schreier sets $S_n, n \in \mathbb{N}$ [AA]. For $F, G \subseteq \mathbb{N}$, we write $F < G$ when $\max(F) < \min(G)$ or one of them is empty, and we write $n \leq F$ instead of $\{n\} \leq F$.

$$S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}.$$  

If $n \in \mathbb{N} \cup \{0\}$ and $S_n$ has been defined,  

$$S_{n+1} = \{ \cup_1^n F_i : n \in \mathbb{N}, n \leq F_1 < F_2 < \cdots < F_n \text{ and } F_i \in S_n \text{ for } 1 \leq i \leq n \}.$$  

For $n \in \mathbb{N}$ a family of finite non-empty subsets $(E_i)$ of $\mathbb{N}$ is said to be $S_n$-admissible if $E_1 < E_2 < \cdots$ and $(\min(E_i)) \in S_n$. Also, $(E_i)$ is said to be $S_n$-allowable if $E_i \cap E_j = \emptyset$ for $i \neq j$ and $(\min(E_i)) \in S_k$.

Every Banach space with a basis can be viewed as the completion of $c_{00}$ (the linear space of finitely supported real valued sequences) under a certain norm. $(e_i)$ will denote the unit vector basis for $c_{00}$ and whenever a Banach space $(X, \| \cdot \|)$ with a basis is regarded as the completion of $(c_{00}, \| \cdot \|)$, $(e_i)$ will denote this (normalized) basis. If $x \in c_{00}$ and $E \subseteq \mathbb{N}$, $Ex \in c_{00}$ is the restriction of $x$ to $E$; $(Ex)_j = x_j$ if $j \in E$ and 0 otherwise. Also the support of $x$, $\text{supp}(x)$, (w.r.t. $(e_i)$) is the set $\{ j \in \mathbb{N} : x_j \neq 0 \}$. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function with $f(0) = 0$ then for $x \in c_{00}$ $f(x)$ will denote the vector $f(x) = (f(x_i))$ in $c_{00}$.

Let $(X, \| \cdot \|)$ be a Banach space with an unconditional basis. The norm of $X$ is 2-convex provided that  

$$\|(x^2 + y^2)^{1/2}\| \leq \| \|x\|^2 + \|y\|^2 \|^{1/2}$$

for all vectors $x, y \in X$. The 2-convexification of $(X, \| \cdot \|)$ is the Banach space $(X^{(2)}, \| \cdot \|^{(2)})$ with an unconditional basis, where $x \in X^{(2)}$ if and only if $x^2 \in X$ and  

$$\|x\|^{(2)} = \|x^2\|^{1/2}.$$  

Of course $\| \cdot \|^{(2)}$ is 2-convex. For $C > 0, 1 \leq p < \infty, n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$ we say that $(x_i)_{i=1}^n$ is $C$-equivalent to the unit vector basis of $\ell_p^n$ if there exist constants $A, B > 0$
with \( AB \leq C \) such that

\[
\frac{1}{A} \left( \sum |a_i|^p \right)^{1/p} \leq \| \sum a_i x_i \| \leq B \left( \sum |a_i|^p \right)^{1/p}
\]

for every sequence of scalars \((a_i)_{i=1}^n\). For \( C > 0 \), we say that \( X \) is an asymptotic \( \ell_p \) space (resp. asymptotic \( \ell_p \) space for vectors with disjoint supports) with constant \( C \) if for every \( n \) and for every sequence of vectors \((x_i)_{i=1}^n\) such that \((\text{supp} (x_i))_{i=1}^n\) is \( S_1 \)-admissible (resp. \( S_1 \)-allowable), we have that \((x_i)_{i=1}^n\) are \( C \)-equivalent to the unit vector basis of \( \ell_p^n \).

If \((\| \cdot \|)_n\) is a sequence of norms in \( c_{00} \) then \( \Sigma(\| \cdot \|_n) \) will denote the completion of \( c_{00} \) under the norm

\[
\| x \|_{\Sigma(\| \cdot \|_n)} = \sum_{n=1}^{\infty} \| x \|_n.
\]

Fix a sequence \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) of elements of \((0, 1)\) and \( \ell, u \in (0, 1) \) with \( 0 < \ell \leq \frac{\alpha_{n+1}}{\alpha_n} \leq u < 1 \) for all \( n \) and \( \sum_n \alpha_n = 1 \) (the existence of numbers \( \ell, u \in (0, 1) \) such that the last relationships are valid will always be assumed whenever a sequence \( (\alpha_n) \) will be considered in these notes).

Edgington defined a sequence of norms \((\| \cdot \|_{E,n})\) on \( c_{00} \) by

\[
\| x \|_{E,0} = \| x \|_\infty, \quad \| x \|_{E,n+1}^2 = \sup \{ \sum_i \| E_i x \|_{E,n}^2 : (E_i)_i \text{ is } S_1\text{-admissible} \}.
\]

Then Edgington defined the norm \( \| \cdot \|_{E_n} \) by

\[
\| x \|_{E_n} = \left( \sum_n \alpha_n \| x \|_{E,n}^2 \right)^{1/2}.
\]

Let \( E_\alpha \) denote the completion of \( c_{00} \) with respect to \( \| \cdot \|_E \). It is shown in [E] that \( E_\alpha \) is a weak Hilbert space which is not isomorphic to \( \ell_2 \), yet it is \( \ell_2 \)-saturated. It is easy to see that the spaces constructed by Edgington are asymptotic \( \ell_2 \) spaces for vectors with disjoint supports. The main theorem that we prove in these notes (Theorem 3.1) shows that such spaces are weak Hilbert spaces.

Let \((| \cdot |_n)_{n \in \mathbb{N}}\) denote the sequence of the Schreier norms on \( c_{00} \):

\[
|x|_n = \sup_{S \in \mathcal{S}_n} \sum_{j \in S} |x_j|
\]
Then the weak Hilbert space $E_\alpha$ that was constructed by Edgington is the 2-convexification of $\Sigma(\alpha_n|.|_n)$. One can see that $\Sigma(\alpha_n|.|_n)$ is an asymptotic $\ell_1$ space for vectors with disjoint supports which is $\ell_1$-saturated, yet not isomorphic to $\ell_1$. In these notes we give examples of sequences of norms that can replace $(|.|_n)$ in $\Sigma(\alpha_n|.|_n)$ to obtain asymptotic $\ell_1$ spaces for vectors with disjoint supports which are $\ell_1$-saturated yet not isomorphic to $\ell_1$. The 2-convexification of each of these spaces will give $\ell_2$ saturated weak Hilbert spaces which are not isomorphic to $\ell_2$.

**Definition of the spaces $V$, $W$, $V'$ and $W'$:** Let $(\theta_n)_{n \in \mathbb{N}}$ be a sequence of real numbers in $(0,1)$ with $\lim_n \theta_n = 0$ (this assumption will always be valid whenever a sequence $(\theta_n)$ will be considered in these notes) and let $s \in \mathbb{N}$. The asymptotic $\ell_1$ spaces $V = T_M(\theta_n, S_n)_n$ and $W = T_{M(s)}(\theta_n, S_n)_n$ were introduced in [AD] and [ADKM] as the completion of $c_{00}$ under the norms:

$$
\|x\|_V = \|x\|_\infty \vee \sup_n \sup \{\theta_n \sum_i \|E_i x\|_V : (E_i) \text{ is } S_n \text{ allowable}\},
$$

$$
\|x\|_W = \|x\|_\infty \vee \sup_{n \leq s} \sup \{\theta_n \sum_i \|E_i x\|_W : (E_i) \text{ is } S_n \text{ allowable}\}
\vee \sup_{n \geq s+1} \sup \{\theta_n \sum_i \|E_i x\|_W : (E_i) \text{ is } S_n \text{ admissible}\},
$$

respectively. These norms can also be defined as limits of appropriate sequences. For $x \in c_{00}$ let

$$
\|x\|_{V,0} = \|x\|_{W,0} = \|x\|_\infty
$$

and for $m \in \mathbb{N}$ define:

$$
\|x\|_{V,m+1} = \|x\|_\infty \vee \sup_n \sup \{\theta_n \sum_i \|E_i x\|_{V,m} : (E_i) \text{ is } S_n \text{ allowable}\},
$$

$$
\|x\|_{W,m+1} = \|x\|_\infty \vee \sup_{n \leq s} \sup \{\theta_n \sum_i \|E_i x\|_{W,m} : (E_i) \text{ is } S_n \text{ allowable}\}
\vee \sup_{n \geq s+1} \sup \{\theta_n \sum_i \|E_i x\|_{W,m} : (E_i) \text{ is } S_n \text{ admissible}\},
$$

Then

$$
\|x\|_V = \lim_m \|x\|_{V,m}, \quad \|x\|_W = \lim_m \|x\|_{W,m}.
$$
Then one can construct the spaces $V' = \sum(\alpha_n \|\cdot\|_{V,n})$, and $W' = \sum(\alpha_n \|\cdot\|_{W,n})$. We show that $V'$ and $W'$ are $\ell_1$-saturated asymptotic $\ell_1$ spaces for vectors with disjoint supports.

It is known [CO] that if $\theta_n = \delta^n$ for some $\delta \in (0, 1)$ then one can replace the “allowable” by “admissible” in the definition of $\|\cdot\|_V$ to obtain an equivalent norm for $V$. For this choice of $(\theta_n)$ the variant of the norm $\|\cdot\|_{V,m+1}$ by replacing “allowable” by “admissible” can be minorized and majorized up to a uniform multiplicative constant by the norms $\|\cdot\|_{V,m}$ and $\|\cdot\|_{V,m+1}$ respectively. This is enough to conclude that the new norms lead to an equivalent norm for $V'$ (see [B]).

3. The main Theorems

The following results is the main tool of our paper for constructing weak Hilbert spaces.

**Theorem 3.1.** If $X$ is an asymptotic $\ell_2$ space for vectors with disjoint supports then $X$ is a weak Hilbert space.

**Proof** Since the ideas needed for the proof of Theorem 3.1 exist in the literature, we will just outline the proof. Recall that the fast growing hierarchy is a sequence of functions on the natural numbers $(g_n)$ which is defined inductively by: $g_0(n) = n + 1$, and for $i \geq 0$, $g_{i+1}(n) = g^n_i(n)$ where $g^n$ is the n-fold iteration of $g$ and $g^0 = I$.

**Step I**

If $X$ is asymptotic-$\ell_2$ with constant $C$ for vectors with disjoint supports, then for every $i \geq 0$ any $g_i(n)$ normalized disjointly supported vectors with supports after $n$ are $C^i$-equivalent to the unit vector basis of $\ell_2$.

We proceed by induction on $i$ with the case $i = 0$ being trivial. So, assume Step I holds for some $i \geq 0$ and let $\{x_k : 1 \leq k \leq g_{i+1}(n)\}$ be a sequence of disjointly supported vectors in $X$ with supports after $n$. For $1 \leq j \leq n$ let

$$E_j = \{k : g_{i+1}^{j-1}(n) \leq k \leq g_{i+1}^j - 1\}.$$
Then,
\[ \left\| \sum_{k=1}^{g_{n+1}(n)} x_k \right\| \approx C \left( \sum_{j=1}^{n} \left\| \sum_{k \in E_j} x_k \right\|^2 \right)^{1/2} \]

Applying the induction hypotheses to each sum on the right we continue this equivalence as
\[ C_{n+1} \approx \left( \sum_{k=1}^{g_{n+1}(n)} \left\| x_k \right\|^2 \right)^{1/2} \].

**Step II**

If \( X \) is asymptotic-\( \ell_2 \) with constant \( C \) for vectors with disjoint supports then every \( n \)-dimensional subspace of \( X \) supported after \( n \) is \( 8C^3 \)-isomorphic to a Hilbert space and \( 8C^3 \)-complemented in \( X \).

If \( E \) is a \( 5n \)-dimensional subspace of \( X \) supported after \( n \), then by a result of Johnson (See Proposition V.6 of [CS]) there is a subspace \( G \) of \( X \) spanned by \( \leq g_3(n) \) disjointly supported vectors supported after \( n \) and an operator \( V : E \to G \) with \( \|Vx - x\| \leq \frac{1}{7}\|x\| \), for all \( x \in E \). Now, by Step I, we have that \( E \) is \( 2C^3 \)-isomorphic to a Hilbert space. It follows [12] that every \( 5n \)-dimensional space of \( X^* \) supported after \( n \) is \( 4C^3 \)-isomorphic to a Hilbert space and \( 4C^3 \)-complemented in \( X^* \). Therefore, every \( n \)-dimensional subspace of \( X \) supported after \( n \) is \( 8C^3 \)-isomorphic to a Hilbert space and \( 8C^3 \)-complemented in \( X \).

**Step III**

Every asymptotic-\( \ell_2 \) space for vectors with disjoint supports is a weak Hilbert space.

If \( E \) is a \( 2n \)-dimensional subspace of \( X \), let \( F =: E \cap (\text{span}_{k \geq n} e_k) \). Then \( F \) is supported after \( n \) and \( \text{dim } F \geq n \) implies \( F \) is \( K \)-isomorphic to a Hilbert space and \( K \)-complemented in \( X \) by Step II, where \( K = 8C^3 \). It follows from Definition 2.7 that \( X \) is a weak Hilbert space. \( \square \)

The 2-convexification of certain Tsirelson spaces for obtaining weak Hilbert spaces was first used in [ADKM]. More generally we have the following:
Corollary 3.2. If $X$ is an asymptotic $\ell_1$ space for vectors with disjoint supports then $X^{(2)}$ is a weak Hilbert space.

Proof: Let $(X, \|\cdot\|)$ be an asymptotic $\ell_1$ space for vectors with disjoint supports. Then there exists $C > 0$ such that for every sequence of vectors $(x_i)$ with $(\text{supp } x_i)$ being $S_1$-allowable, we have that $C \sum \|x_i\| \leq \|\sum x_i\|$. It suffices to prove that $X^{(2)}$ is an asymptotic $\ell_2$ space for vectors with disjoint support. Let $(y_i)$ be a sequence of vectors in $X^{(2)}$ with $(\text{supp } y_i)$ being $S_1$-allowable. Then

$$C^{1/2}(\sum \|y_i\|_{(2)}^2)^{1/2} = C^{1/2}(\sum \|y_i^2\|)^{1/2} \leq \sum \|y_i^2\|^{1/2} = \|(\sum y_i^2)^{1/2}\|.$$ 

Also,

$$\|(\sum y_i^2)^{1/2}\|_{(2)} = \sum \|y_i^2\|^{1/2} \leq \sum \|y_i\|_{(2)}^2 = \sum \|y_i\|_{(2)}^{1/2}.$$ 

The spaces $V, W, V'$ and $W'$ are asymptotic $\ell_1$ spaces for vectors with disjoint supports. Indeed, this is obvious for $V$ and $W$. To see this for $V'$ let $n \in \mathbb{N}$ and vectors $(x_i)_{i=1}^n$ with disjoint supports with $n \leq x_i$ for all $i$. Then:

$$\|\sum_{i=1}^n x_i\| = \sum_{m=1}^{\infty} \alpha_m \|\sum_{i=1}^n x_i\|_{V,m} \geq \sum_{m=1}^{\infty} \alpha_m \theta_1 \sum_{i=1}^n \|x_i\|_{V,m-1} \geq \theta_1 u \sum_{i=1}^n \|x_i\|_{V'}$$

The proof for $W'$ is similar. Thus $V^{(2)}, W^{(2)}, V''^{(2)}$ and $W''^{(2)}$ are weak Hilbert spaces.

Proposition 3.3. The spaces $V$ and $W$ do not contain an isomorph of $\ell_1$. The spaces $V'$ and $W'$ are $\ell_1$-saturated without being isomorphic to $\ell_1$.

The following Lemma will be used in the proof of Proposition 3.3.

Lemma 3.4. For every $m \in \mathbb{N}$ the completion of $(c_{00}, \|\cdot\|_{V,m})$ is a $c_0$-saturated space.

In order to prove this Lemma we need a result of Fonf along with the notion of the boundary.
Definition 3.5. A subset $B$ of the unit sphere of the dual of a Banach space $X$ is called a boundary for $X$ if for every $x \in X$ there exists $f \in B$ such that $f(x) = \|x\|$.

Theorem 3.6. ([F1], see also [F2], [H] [DGZ]) Every Banach space with a countable boundary is $c_0$-saturated.

Proof of Lemma 3.4 Define inductively on $i \leq m$ the sets $K^i$ of the unit ball of the dual of $(c_0, \|\cdot\|_{V,m})$. Let $K^0 = \{\pm e_n : n \in \mathbb{N}\}$. For $i < m$ if $K^i$ has been defined then let $K^{i+1} = K^i \cup \{\theta_k(f_1 + \cdots + f_r) : F_j \in K^m, \text{ for all } j, (\text{supp } f_j)^r_{j=1} \text{ is } S_k \text{ allowable } k = 1, 2, \ldots \}$. Then $K^m$ is a norming set for $(c_0, \|\cdot\|_{V,m})$:

$$\|x\|_{V,m} = \sup\{|f(x)| : f \in K^m\}.$$ 

It is easy to see that $K^m \cup \{0\}$ is a pointwise closed set since each $S_k$ is pointwise closed and $\lim_k \theta_k = 0$. The previous Theorem of Fonf finishes the proof of the Lemma. \qed

Proof of Proposition 3.3 The statement for $V$ and $W$ is obvious since they are reflexive [AD], [ADKM].

$V'$ is $\ell_1$ saturated: Let $(x_i)$ be an arbitrary block basis of $V'$. It is enough to construct a normalized (in $V'$) block basis $(v_i)$ of $(x_i)$ and an increasing sequence of positive integers $1 = p_1 < p_2 < p_3 < \cdots$ such that

$$\sum_{m=p_i}^{p_{i+1}-1} \alpha_m \|v_i\|_{V,m} \geq \frac{1}{2}$$

for all $i$. Once this is done then for $(\lambda_i) \in c_0$

$$\| \sum_{i=1}^{n} \lambda_i v_i \|_{V'} = \sum_{m=1}^{\infty} \alpha_m \| \sum_{i=1}^{n} \lambda_i v_i \|_{V,m} = \sum_{j=1}^{\infty} \sum_{m=p_j}^{p_{j+1}-1} \alpha_m \| \sum_{i=1}^{n} \lambda_i v_i \|_{V,m} \geq \frac{1}{2} \sum_{j=1}^{\infty} |\lambda_j|$$
which shows that \((v_i)\) is equivalent to the unit vector basis of \(\ell_1\). In order to choose such \((v_i)\) and \((p_i)\) we use that for every \(m \in \mathbb{N}\) the norms \(\|\cdot\|_{V,m}\) and \(\|\cdot\|_{V,m+1}\) are not equivalent. Thus for every \(m, M, K \in \mathbb{N}\) there is \(u\) in the span of \((x_i)\) with

\[
M \leq u, \quad \|u\|_{V,m} < \frac{1}{4}, \quad \text{and} \quad \|u\|_{V,m+1} \geq K.
\]

Then

\[
\sum_{i=1}^{m} \alpha_i \|u\|_{V,i} < \frac{1}{4} \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_i \|u\|_{V,i} \geq \alpha_{m+1} K.
\]

Let \(v = \frac{u}{\|u\|_{V'}}\). By taking \(K\) large enough we can assume that

\[
\sum_{i=1}^{m} \alpha_i \|v\|_{V,i} < \frac{1}{4}.
\]

Also choose \(m' > m\) with

\[
\sum_{i=m'+1}^{\infty} \alpha_i \|v\|_{V,i} < \frac{1}{4}.
\]

Thus

\[
\sum_{i=m+1}^{m'} \alpha_i \|v\|_{V,i} \geq \frac{1}{2}.
\]

It only remains to show that for every \(m \in \mathbb{N}\) the norms \(\|\cdot\|_{V,m}\) and \(\|\cdot\|_{V,m+1}\) are not equivalent on the span of \((x'_i)\). By the previous Lemma there is a block sequence \((y_i)\) of \((x_i)\) such that \(\|y_i\|_{V,m} = 1\) for all \(i\) and \(((y_i), \|\cdot\|_{V,m})\) is 2-equivalent to the unit vector basis of \(c_0\).

For \(n \in \mathbb{N}\) let \(k \in \mathbb{N}\) with \(n \leq y_{k+1} < y_{k+2} < \cdots < y_{k+n}\). Thus

\[
\left\| \sum_{i=k+1}^{k+n} y_i \right\|_{V,m} \leq 2
\]

yet

\[
\left\| \sum_{i=k+1}^{k+n} y_i \right\|_{V,m+1} \geq \theta_1 n.
\]

This proves the result.

\(W'\) is \(\ell_1\)-saturated: Similar.
$V'$ is not isomorphic to $\ell_1$: If the statement were false then the basis of $V'$ would be isomorphic to the unit vector basis of $\ell_1$ (since every normalized unconditional basic sequence in $\ell_1$ is equivalent to the usual unit basis of $\ell_1$, [LP]). Observe that for $x \in c_{00}$,

$$
\|x\| = \sum_m \alpha_m \|x\|_{V,m} \leq \sup_m \|x\|_{V,m} = \|x\|_V.
$$

By [ADKM] the norm of $V$ can become arbitrarily smaller than the $\ell_1$ norm on certain vectors.

$W'$ is not isomorphic to $\ell_1$: Similar. \hfill \Box

**Remark 3.7.** Note that a space $X$ with an unconditional basis contains $\ell_1$ if and only if $X^{(2)}$ contains $\ell_2$. Since $V$ and $W$ do not contain an isomorph of $\ell_1$, we obtain that $V^{(2)}$ and $W^{(2)}$ are weak Hilbert spaces which do not contain an isomorph of $\ell_2$. Since the spaces $V'$ and $W'$ are $\ell_1$ saturated without being isomorphic to $\ell_1$, we obtain that the spaces $V'^{(2)}$, and $W'^{(2)}$ are $\ell_2$-saturated weak Hilbert spaces which are not isomorphic to $\ell_2$.

Thus the essential properties of the space $E_\alpha$ constructed by Edgington are shared by $V'^{(2)}$ and $W'^{(2)}$.

**Theorem 3.8.** Let $(\theta_n) \subset (0, 1)$ with $\lim_n \theta_n^{1/n} = 1$, $s \in \mathbb{N}$, and $\beta = (\beta_n) \subset (0, 1)$ with $\sum_n \beta_n = 1$ and $0 < \inf \frac{\beta_{n+1}}{\beta_n} \leq \sup \frac{\beta_{n+1}}{\beta_n} < 1$. Then $V'^{(2)}$ and $W'^{(2)}$ are not isomorphic to $E_\beta$.

**Proof:** Let $T : X \to E_\beta$ be an isomorphism where $X$ is either $V'^{(2)}$ or $W'^{(2)}$. Since $T$ is an isomorphism there exists $C > 0$ such that

$$
\frac{1}{C} \|Tx\|_{E_\beta} \leq \|x\|_X \leq C \|Tx\|_{E_\beta}
$$

for all $x \in c_{00}$. Also, by [E] (proof of Theorem 7) there exists $\delta > 0$ such that

$$
\|Tx\|_{E_\beta} \leq C \|Tx\|_{T^{(2)}(\delta, S_1)}.
$$

Thus for $x \in c_{00}$

$$
\|x\|_X \leq C^2 \|Tx\|_{T^{(2)}(\delta, S_1)}. \tag{1}
$$
Since the unit vector basis \((e_i)\) of \(X\) is weakly null, we can select a subsequence \((e_{k_i})\) of \((e_i)\), a block sequence \((u_i)\) in \(T^{(2)}(\delta, S_1)\) with non-negative coefficients and a number \(K > 0\) such that:
\[
\|T(e_{k_i}) - u_i\|_{T^{(2)}(\delta, S_1)} < \frac{\varepsilon}{2^i} \quad \text{and} \quad \frac{1}{K} \leq \|u_i\|_{T^{(2)}(\delta, S_1)} \leq K \quad \text{for all } i,
\]
where \(\varepsilon > 0\) will be chosen later. Let \(n \in \mathbb{N}\) to be selected later. Let \((x_i)_{i \in I} \subset (0, 1)\) for some \(I \in S_n\) so that
\[
\sum_{i \in I} x_i^2 = 1, \quad \text{and} \quad \|\sum_{i \in I} x_i^2 u_i^2\|_{T^{(2)}(\delta, S_1)} \leq \delta^n + \varepsilon
\]
([OTW] Theorem 5.2 (a)). Then
\[
\|T\left(\sum_{i \in I} x_i e_{k_i}\right)\|_{T^{(2)}(\delta, S_1)} \leq \|\sum_{i \in I} x_i u_i\|_{T^{(2)}(\delta, S_1)} + \sum_{i \in I} x_i \|Te_{k_i} - u_i\|_{T^{(2)}(\delta, S_1)}
\]
\[
\leq \|\sum_{i \in I} x_i^2 u_i^2\|_{T^{(2)}(\delta, S_1)} \leq \varepsilon
\]
\[
\leq K \|\sum_{i \in I} x_i^2 \frac{u_i^2}{\|u_i\|_{T^{(2)}(\delta, S_1)}}\|_{T^{(2)}(\delta, S_1)}^{1/2} \leq \varepsilon
\]
\[
\leq K(\delta^n + \varepsilon)^{1/2} + \varepsilon.
\]

On the other hand if \(Y = V'\) when \(X = V'^{(2)}\) or \(Y = W'\) when \(X = W'^{(2)}\) then
\[
\|\sum_{i \in I} x_i e_{k_i}\|_X = \|\sum_{i \in I} x_i^2 e_{k_i}\|_Y^{1/2} \geq \left(\sum_{m=1}^{\infty} \beta_m \theta_n \sum_{i \in I} x_i^2\right)^{1/2} = \sqrt{\theta_n}
\]
Therefore ([I]) gives
\[
\sqrt{\theta_n} \leq C^2(K(\delta^n + \varepsilon)^{1/2} + \varepsilon).
\]
But since \(\lim_n \theta_n^{1/n} = 1\), \(n\) and \(\varepsilon\) can be chosen so that this inequality fails. \(\square\)

**References**

[AA] D.E. Alspach and S.A. Argyros, *Complexity of weakly null sequences*, Dissertationes Mathematicae 321, 1992.

[AD] S.A. Argyros and I. Deliyanni, *Examples of asymptotic \(\ell_1\) spaces*, Trans. Amer. Math. Soc. 349 (1997), 973-995.

[ADKM] S.A. Argyros, I. Deliyanni, D.N. Kutzarova and A. Manoussakis, *Modified mixed Tsirelson spaces*, preprint.
[B] S.F. Bellenot, Tsirelson subspaces and $\ell_p$, J. Funct. Analysis 69 (1986), 207-228.

[CN] P.G. Casazza and N.J. Nielsen, A Gaussian Average Property of Banach spaces, Illinois J. Math. 41 No 4 (1997), 559-576.

[CO] P.G. Casazza and E. Odell, Tsirelson’s space and minimal subspaces, Longhorn notes, University of Texas (1982-83), 61-72.

[CS] P.G. Casazza and T. Shura, Tsirelson’s space, Lecture Notes in Math. 1363, Springer 1989.

[DGZ] R. Deville, G. Godefroy and V. Zizler, Smoothness and renormings in Banach spaces, Pitman Monographs Surveys Pure Appl. Math. Vol. 64, Longman Sci. Tech., 1993.

[E] A. Edgington, Some more weak Hilbert spaces Studia Math. 100 (1991), 1-11.

[FJ] T. Fiegel and W.B. Johnson, A uniformly convex Banach space which contains no $\ell_p$, Compositio Mathematica, Vol. 29, Fasc 2 (1974), 191-196.

[F1] V.P. Fonf, Weakly extremely properties of Banach spaces, Mat. Zametki 45(6) (1989) 83-92 (Russian).

[F2] V.P. Fonf, On exposed and smooth points of convex bodies in Banach spaces, Bull. London Math. Soc. 28 (1996) 51-58.

[H] P. Hajek, Smooth norms that depend locally on finitely many coordinates, Proc. Amer. Math. Soc. 123 (1995), 3817-3821.

[J1] W.B. Johnson, A reflexive Banach space which is not sufficiently Euclidean, Studia Math. 55 (1976), 201-205.

[J2] W.B. Johnson, Banach spaces all of whose subspaces have the approximation property, Seminaire d’ Analyse Fonct. Expose 16 (1979-80). Ecole Polytechnique, Paris.

[K] R. Komorowski, Weak Hilbert spaces without unconditional bases, Proc. Amer. Math. Soc. 120 (1994), 101-107.

[KT] R. Komorowski and N. Tomczak-Jaegermann, Banach spaces without local unconditional structure, Israel J. Math 89 (1995), 205-226.

[LP] J. Lindenstrauss and A. Pełczyński, Absolutely summing operators in $L_p$-spaces and their applications, Studia Math. 29 (1968) 275-326.

[M] V. Mascioni, On Banach spaces isomorphic to their duals, Houston J. Math. 19 (1993), 27-38.

[MP] V.D. Milman and G. Pisier, Banach spaces with a weak cotype 2 property, Israel J. Math. 54 (1986), 139-158.

[NTJ] N.J. Nielsen and N. Tomczak-Jaegermann, On subspaces of Banach spaces with Property (H) and weak Hilbert spaces,

[OTW] E. Odell, N. Tomczak-Jaegermann and R. Wagner, Proximity to $\ell_1$ and Distortion in Asymptotic $\ell_1$ spaces, preprint.
[P1] G. Pisier, Weak Hilbert spaces, Proc. London Math. Soc. 56 (1988), 547-579.
[T] B.S. Tsirelson, Not every Banach space contains $\ell_p$ or $c_0$, Funct. Anal. Appl. 8 (1974), 138-141.

DEPARTMENT OF MATHEMATICS, MATH. SCI. BLDG., UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA MO 65211
E-mail address: giorgis@math.missouri.edu

DEPARTMENT OF MATHEMATICS, MATH. SCI. BLDG., UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA MO 65211
E-mail address: pete@casazza.math.missouri.edu

CURRENT: DEPT. MATH. & STAT., MIAMI UNIVERSITY, OXFORD, OHIO, PERMANENT: INSTITUTE OF MATHEMATICS, BULGARIAN ACADEMY OF SCIENCES, 1113 SOFIA, BULGARIA
E-mail address: denka@math.acad.bg