From 2D Integrable Systems to Self-Dual Gravity

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Abstract

We explain how to construct solutions to the self-dual Einstein vacuum equations from solutions of various two-dimensional integrable systems by exploiting the fact that the Lax formulations of both systems can be embedded in that of the self-dual Yang–Mills equations. We illustrate this by constructing explicit self-dual vacuum metrics on $\mathbb{R}^2 \times \Sigma$, where $\Sigma$ is a homogeneous space for a real subgroup of $SL(2, \mathbb{C})$ associated with the two-dimensional system.

1 Introduction

Ward [12] has observed that many integrable systems, particularly in two dimensions, may be obtained from the self-dual Yang–Mills (SDYM) equations by symmetry reduction. See [2] for a survey of such reductions. See also [8] for an account how reductions can be used as a framework for classification, and for a survey of applications of twistor theory.

It has been shown [5] that the SDYM equations with gauge group the volume-preserving diffeomorphisms SDiff$(M)$ of a four-manifold $M$ and translational symmetry in all four variables reduces to the self-dual (SD) Einstein vacuum equations on $M$. This result extends the work of Ashtekar et al. [1]. It also implies, [13], that solutions of the SDYM equations with two translational symmetries and gauge group SDiff($\Sigma$) for some two-manifold $\Sigma$ also determine solutions of the SD Einstein vacuum equations.

The aim of the present paper is to show that the correspondence between the Lax formulations of certain two-dimensional integrable systems and the SD Einstein equations enables us to construct SD vacuum metrics explicitly from solutions to various two-dimensional nonlinear integrable equations. We do this by considering $SL(2, \mathbb{C})$ SDYM fields invariant under the action of two translations of space-time. These fields are can be represented as solutions of various soliton equations in two dimensions. Self-dual vacuum metrics are recovered by representing the Lie algebra of (real forms of) $SL(2, \mathbb{C})$ as Hamiltonian vector fields on a two-dimensional homogeneous space for the gauge group.

Other approaches to self-dual gravity that reveal its connection with two-dimensional integrable systems have been given by Ward [13] and Q-Han Park [9].

In the next section we review briefly the classification of two-dimensional integrable systems arising from the $SL(2, \mathbb{C})$ SDYM equations. In section 3 we discuss the connection between the SDYM equations and self-dual gravity. Section 4 is devoted to the construction of normalised null tetrads and hence metrics on $\mathbb{R}^2 \times \Sigma$ from the SDYM Lax pairs for the two-dimensional integrable systems. In the last section we outline the twistor interpretation of the construction.

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2 Self-dual Yang-Mills and 2D integrable systems

Consider a Yang Mills vector bundle over a four-dimensional manifold \( \mathcal{M} \) (taken here to be \( \mathbb{C}^4 \) in general, or \( \mathbb{R}^4 \) when reality conditions are imposed) with connection one-form \( A = A_\mu(x^\nu)dx^\mu \in T^*\mathcal{M} \otimes LG \), where \( LG \) is the Lie algebra of some gauge group \( G \). The corresponding curvature \( F = F_{\mu\nu}dx^\mu \wedge dx^\nu \) is given by

\[
F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\mu, A_\nu],
\]

(2.1)

where \( D_\mu = \partial_\mu - A_\mu \) is the covariant derivative. The SDYM equations on a connection \( A \) are the self-duality conditions on the curvature under the Hodge star operation

\[
F = *F, \quad \text{or in index notation} \quad F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}.
\]

(2.2)

They are conformally invariant and are also preserved by the gauge transformations

\[
A \to g^{-1}Ag - g^{-1}dg, \quad g \in \text{Map}(\mathcal{M}, G).
\]

(2.3)

Let us introduce double-null coordinates \( w, \bar{w}, z, \bar{z} \), in which the metric on \( \mathcal{M} \) is \( ds^2 = dwd\bar{w} - dzd\bar{z} \). In these coordinates the SDYM equations may be rewritten as

\[
F_{w\bar{z}} = 0
\]

(2.4)

\[
F_{\bar{w}z} = 0
\]

(2.5)

\[
F_{w\bar{w}} - F_{zz} = 0,
\]

(2.6)

which are the compatibility conditions \([L, M] = 0\) for the linear system of equations \( L\Phi = 0, M\Phi = 0\) where the ‘Lax pair’, \( L \) and \( M \), are

\[
L = D_w - \lambda D_z, \quad M = D_z - \lambda D_{\bar{w}}
\]

(2.7)

for \( \lambda \in \mathbb{CP}^1 \) and \( \Phi \) an \( n \)-component column vector.

We shall consider the reality conditions for real ultra-hyperbolic spaces, recovered by imposing \( w = x - y, z = t + v, \bar{w} = x + y, \bar{z} = t - v \). (Reality conditions for Euclidean space are recovered by imposing \( \bar{w} = w \) and \( \bar{z} = -z \)) Solutions to (2.4–2.6) can be real for this choice of signature.

We fix the gauge group to be \( SL(2, \mathbb{C}) \) or one of its real subgroups. Conformal reduction of the SDYM equations involves the choice of the group \( H \) of conformal isometries of \( \mathcal{M} \). We shall restrict ourselves to the simplest case and suppose that a connection \( A \) is invariant under the flows of two independent translational Killing vectors \( X \) and \( Y \). These reductions are classified partially by the signature of the metric restricted to two-plane spanned by the translations.

1) Nondegenerate cases (\( H_1 \))

a) \( X = \partial_w - \partial_{\bar{w}}, \ Y = \partial_z - \partial_{\bar{z}} \).

\[
A_w = \frac{1}{4} \begin{pmatrix} \phi_t & -2\cos(\phi/2) \\ -2\cos(\phi/2) & -\phi_t \end{pmatrix}, \quad A_{\bar{w}} = \frac{i}{4} \begin{pmatrix} \phi_t & 2\cos(\phi/2) \\ 2\cos(\phi/2) & -\phi_t \end{pmatrix}, \quad A_z = \frac{1}{4} \begin{pmatrix} -\phi_x & -2\sin(\phi/2) \\ 2\sin(\phi/2) & \phi_x \end{pmatrix}, \quad A_{\bar{z}} = \frac{i}{4} \begin{pmatrix} -\phi_x & 2\sin(\phi/2) \\ -2\sin(\phi/2) & \phi_x \end{pmatrix}.
\]

(2.8)

The SDYM equations are satisfied in ultra-hyperbolic signature if \( \phi_{xx} + \phi_{tt} = \sin\phi \); the elliptic sine-Gordon equation.

b) \( G = SU(2), \ \ X = \partial_w, \ \ Y = \partial_{\bar{w}} \).

\[
A_z = 0, \quad A_w = \cos\phi \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
A_{\bar{w}} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad A_z = 1/2(\phi_v - \phi_t) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

(2.9)

The SDYM equations in ultra-hyperbolic signature yield \( \phi_{tt} - \phi_{vv} = 4\sin\phi \), the hyperbolic sine-Gordon equation.
2) **Partially degenerate case (H₂)**

We consider ultra-hyperbolic signature only with \(X = \partial_w - \partial_{\bar{w}}\) and \(Y = \partial_{\bar{z}}\).

\[
A_w = \begin{pmatrix} q & 1 \\ b & -q \end{pmatrix}, \quad A_{\bar{w}} = 0, \quad 2A_z = \begin{pmatrix} b_x & -2q_x \\ 2w & -b_x \end{pmatrix}, \quad A_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},
\]

where \(4w = q_{xxx} - 4qq_x - 2q_x^2 + 4q^2q_x\) and \(b = q_x - q^2\). The SDYM equations (with the definition \(u = -q_x\)) are equivalent to the Korteweg de Vries equation \(4u_z = u_{xxx} + 12uu_x\). The reduced Lax pair (2.7) yields the standard zero curvature representation of KdV [14].

b)

\[
A_w = \begin{pmatrix} 0 & \phi \\ \mp \phi & 0 \end{pmatrix}, \quad A_{\bar{w}} = 0, \quad A_z = i\left(\frac{|\phi|^2}{\phi_x} \pm \phi_x \right), \quad 2A_{\bar{z}} = \pm i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Here the upper (lower) sign corresponds to \(G = SU(2)\) (or \(SU(1,1)\)). SDYM become \(i\phi_z = -\phi_{xx} \mp 2|\phi|^2\phi\) which is the nonlinear Schrödinger equation with an attractive (respectively repulsive) self interaction [3].

### 3 SDYM and self-dual gravity

Let \(\mathcal{M}\) be a four-dimensional complex manifold (for example the complexification of some real slice \(\mathcal{M}_\mathbb{R}\)) and let \(g\) be a holomorphic metric on \(\mathcal{M}\) (for example the complexification of a real metric on \(\mathcal{M}_\mathbb{R}\)). The following theorem states that the self-duality equations on the curvature can be expressed in terms of the consistency condition for a Lax pair of vector fields.

**Theorem 3.1 (Mason & Newman 1989 [3]).** Let \(V_a = (W, \bar{W}, Z, \bar{Z})\) be four independent holomorphic vector fields on a four-dimensional complex manifold \(\mathcal{M}\) and let \(\nu\) be a nonzero holomorphic 4-form. Put

\[
L = W - \lambda \bar{Z}, \quad M = Z - \lambda \bar{W}.
\]

Suppose that for every \(\lambda \in \mathbb{C}P^1\)

\[
[L, M] = 0
\]

(3.13)

\[
\mathcal{L}_L \nu = -\mathcal{L}_M \nu = 0
\]

(3.14)

Here \(\mathcal{L}_V\) denotes a Lie derivative. Then \(\sigma_a = f^{-1}V_a\), where \(f^2 = \nu(W, \bar{W}, Z, \bar{Z})\), is a normalised null-tetrad for a half-flat metric (i.e. with vanishing Ricci tensor and self-dual Weyl tensor). Every half-flat metric arises in this way.

The covariant metric is conveniently expressed in terms of the dual frame \(e_V\)

\[
g = f^2(e_W \circ e_{\bar{W}} - e_Z \circ e_{\bar{Z}}),
\]

where

\[
e_W = f^{-2}\nu(..., W, Z, \bar{Z}) \quad \text{and} \quad e_{\bar{W}} = f^{-2}\nu(W, ..., Z, \bar{Z})
\]

\[
e_Z = f^{-2}\nu(W, \bar{W}, ..., \bar{Z}) \quad \text{and} \quad e_{\bar{Z}} = f^{-2}\nu(W, \bar{W}, Z, ...).
\]

The operators \(L\) and \(M\) determine a basis of ASD two-forms on \(\mathcal{M}\)

\[
\alpha = f^2 e_W \wedge e_Z, \quad \omega = f^2(e_W \wedge e_{\bar{W}} - e_Z \wedge e_{\bar{Z}}), \quad \tilde{\alpha} = f^2 e_{\bar{W}} \wedge e_{\bar{Z}}.
\]

(3.16)

We note that \(-i(\alpha - \tilde{\alpha}), i\omega\) and \(\alpha + \tilde{\alpha}\) are nondegenerate symplectic forms, which (together with three compatible complex structures) endow \(\mathcal{M}\) with a complexified hyper-Kähler structure.
4 Self-dual metrics on $\mathbb{R}^2 \times \Sigma$

We connect the self-duality equations on a Yang-Mills field and those on a four-dimensional metric by considering gauge potentials that take values in a Lie algebra of vector fields on some manifold. Theorem (3.1) reveals one such connection: $W, \bar{W}, Z$ and $\bar{Z}$ are generators of the group of volume-preserving (holomorphic) diffeomorphisms of $(M, \nu)$. We make the identification: $W = D_w, \bar{W} = D_{\bar{w}}, Z = D_z, \bar{Z} = D_{\bar{z}}$. By comparing (3.13) with (2.7), we see that the half flat equation is a reduction of the SDYM with this gauge group by translations along the four coordinate vectors $\partial_w, \partial_{\bar{w}}, \partial_z, \partial_{\bar{z}}$.

In order to understand the relationship with two-dimensional integrable systems, we look at this in a slightly different way. Let $(\Sigma, \Omega_\Sigma)$ be a two-dimensional symplectic manifold and let SDiff($\Sigma$) be the group of canonical transformations of $\Sigma$. Consider the SDYM equations with the gauge group $G$ in a slightly different way. Let $(\Sigma, \Omega_\Sigma)$ be a two-dimensional symplectic manifold and let SDiff($\Sigma$) be the group of canonical transformations of $\Sigma$. Although it has been observed that SDiff($\Sigma$) is a subgroup of such defined SL($\infty$), it seems that SL($n, \mathbb{C}$) is a subgroup of such defined SL($\infty$) only for $n = 2$. In this case we can take the linear action of SL(2,$\mathbb{R}$) on $\mathbb{R}^2$ or a Möbius action of SU(2) and SU(1, 1) on $\mathbb{C}P^1$ or $D$ (the Poincaré disc) respectively. We shall restrict ourselves to real vector fields, which will imply that our SD metrics will have ultra-hyperbolic signature (Euclidean examples can also be obtained in a similar way).

To be more explicit we write down the Hamiltonian corresponding to the matrix

$$A_\mu = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in LSL(2, \mathbb{C}).$$

In the three cases we have

$$\Sigma = \mathbb{R}^2, \quad \Omega_\Sigma = dm \wedge dn, \quad H_\mu = \left( \frac{b^2}{2} + amn - \frac{c^2}{2} \right), \quad (4.19)$$

$$\Sigma = \mathbb{C}P^1, \quad \Omega_\Sigma = \frac{id\xi \wedge d\bar{\xi}}{(1 + \xi \bar{\xi})^2}, \quad H_\mu = -i \frac{\xi \bar{b} - \xi b - 2a}{1 + \xi \bar{\xi}}, \quad (4.20)$$

$$\Sigma = D, \quad \Omega_\Sigma = \frac{id\xi \wedge d\bar{\xi}}{(1 - \xi \bar{\xi})^2}, \quad H_\mu = -i \frac{\bar{\xi} \bar{b} - \xi b - 2a}{1 - \xi \bar{\xi}}. \quad (4.21)$$

The form of the null tetrad (3.16) and the hyper-Kähler structure (3.17) obtained after the two-dimensional reductions of SDYM is as follows:

(i) $H_1$ $(X = \partial_w, Y = \partial_{\bar{w}})$, $\nu = dz \wedge d\bar{z} \wedge \Omega_\Sigma$

$$f^2 = \nu(W, \bar{W}, Z, \bar{Z}) = \Omega_\Sigma(W, \bar{W}) = \{H_w, H_{\bar{w}}\} = F_{w\bar{w}}. \quad (4.22)$$

In the last formula $F_{w\bar{w}}$ is a function rather than a matrix. This follows from the identification (via (4.19)-(4.21)) of $2 \times 2$ matrices in the Lie algebra of SL(2,$\mathbb{C}$) and Hamiltonians. Let $d_S$ stand for

\[1\] We only require the representation of $A_\mu$ by volume-preserving vector fields on $\Sigma$; Hamiltonians are defined up to the addition of a function of the (residual) space variables, but different choices of such functions do not change the metric.
the exterior derivative on $\Sigma$.

\[ e_W = f^{-2}(\Omega_\Sigma(W, Z) dz + \Omega_\Sigma(W, \bar{Z})d\bar{z} + \Omega_\Sigma(\ldots, \bar{W})) \]

\[ e\bar{W} = f^{-2}([H_{\bar{w}}, H_z] dz - [H_{\bar{w}}, H_z] d\bar{z} + d\Sigma H_w) \]

\[ e_z = dz \]

\[ e_{\bar{z}} = d\bar{z} \]

\[ \alpha = -[H_{\bar{w}}, H_z] dz \land d\bar{z} - d\Sigma H_{\bar{w}} \land dz \]

\[ \omega = ([H_z, H_\bar{w}] - [H_w, H_z]) dz \land d\bar{z} + \Omega_\Sigma + d\Sigma H_z \land dz + d\Sigma H_{\bar{w}} \land d\bar{z} \]

\[ \tilde{\alpha} = [H_w, H_z] dz \land d\bar{z} + d\Sigma H_w \land d\bar{z}. \]

The gauge freedom is used to set $A_z$ (and hence $H_z$) to 0.

\[ ds^2 = \frac{1}{[H_w, H_{\bar{w}}]} \left( - ([H_{\bar{w}}, H_z] [H_w, H_z]) dz^2 - ([H_w, H_{\bar{w}}])^2 d\bar{z} \right) \]

\[ -(\partial_t H_w \partial_t H_\bar{w}) dz^2 - (\partial_j H_w \partial_j H_\bar{w}) d\xi^2 - ((\partial_t H_w \partial_j H_\bar{w}) + (\partial_j H_w \partial_j H_\bar{w})) d\xi d\bar{\xi} \]

\[ + (\partial_t H_w [H_w, H_\bar{w}] + \partial_t H_w [H_\bar{w}, H_z]) dz \land d\xi + (\partial_j H_w [H_w, H_z] + \partial_j H_w [H_\bar{w}, H_\bar{z}]) dz \land d\xi \].

(ii) $H_2$, $(X = \partial_w - \partial_\bar{w}, Y = \partial_\bar{z})$, $\nu = dx \land dz \land \Omega_\Sigma$,

\[ f^2 = [H_w - H_{\bar{w}}, H_z] = F_{w\bar{z}}, \]

\[ e_W = f^{-2}([H_{\bar{z}}, H_{\bar{w}}] dx + [H_z, H_\bar{w}] dz - d\Sigma H_z) \]

\[ e\bar{W} = f^{-2}(-[H_{\bar{z}}, H_{\bar{w}}] dx - [H_z, H_\bar{w}] dz + d\Sigma H_z) \]

\[ e_z = dz \]

\[ e_{\bar{z}} = f^{-2}([H_w, H_{\bar{w}}] dx + [H_w - H_{\bar{w}}, H_z] dz - d\Sigma (H_w - H_{\bar{w}})) \]

\[ \alpha = [H_z, H_{\bar{w}}] dx \land dz + d\Sigma H_z \land dz \]

\[ \omega = ([H_w, H_z] - [H_w, H_{\bar{w}}]) dx \land dz + d\Sigma H_z \land dx - d\Sigma (H_w - H_{\bar{w}}) \land dz \]

\[ \tilde{\alpha} = [H_w, H_z] dx \land dz + \Omega_\Sigma + d\Sigma H_w \land dx - d\Sigma H_z \land dz. \]

We can perform a further gauge transformation to set $H_{\bar{w}} = 0$ in which case

\[ ds^2 = \left( \frac{[H_z, H_{\bar{z}}]^2}{[H_w, H_{\bar{w}}]} + [H_w, H_z] \right) dz^2 - \frac{[H_z, H_{\bar{z}}]^2}{[H_w, H_z]} d\xi^2 - \frac{[H_{\bar{w}}, H_z]^2}{[H_w, H_{\bar{w}}]} d\bar{\xi}^2 \]

\[ + \left( \partial_t H_w + 2 [H_z, H_z] \partial_j H_\bar{w} \right) d\xi d\bar{\xi} + \left( \partial_j H_w + 2 [H_{\bar{z}}, H_z] \partial_j H_{\bar{z}} \right) d\xi d\bar{\xi} \]

\[ - 2 \partial_t H_w \partial_j H_{\bar{w}} d\xi d\bar{\xi} + [H_z, H_\bar{w}] dx \land dz - \partial_t H_z dx d\xi - \partial_j H_{\bar{z}} dx d\xi. \]

Reductions by $X = \partial_w$, $Y = \partial_\bar{z}$ are not considered because the resulting metric turns out to be degenerate everywhere as a direct consequence of the SDYM equations. Equation (2.4) becomes now $[X_{H_{\bar{w}}}, X_{H_z}] = 0$ which, in the case of finite dimensional sub-algebras of LSDiff($\Sigma$), implies linear dependence of $X_{H_{\bar{w}}}$ and $X_{H_z}$.

The construction naturally applies to the complex four-manifolds. We start from the SDYM equations on $\mathbb{C}^4$ with gauge group $SL(2, \mathbb{C})$. Then we perform one of the possible reductions to
\( \mathbb{C}^2 \). Let \( \Sigma^2_\mathbb{C} \) be a two-dimensional complex manifold, for example \( \mathbb{C}P^1 \times \mathbb{C}P^1_* \). \( SL(2, \mathbb{C}) \) acts on one Riemann sphere by a Möbius transformation, and the other by the inverse:

\[
(\xi, \tilde{\xi}) \mapsto \left( \frac{A\xi + B}{C\xi + D}, \frac{D\tilde{\xi} - C}{-B\tilde{\xi} + A} \right).
\]

Here \( \xi \) and \( \tilde{\xi} \) are independent complex coordinates on \( \mathbb{C}P^1 \) and \( \mathbb{C}P^1_* \). The action preserves the symplectic form \( \Omega_{\Sigma^2} = (1 + \xi \tilde{\xi})^{-2}(d\xi \wedge d\tilde{\xi}) \) defined on the complement of \( 1 + \xi \tilde{\xi} = 0 \). All results of this section may be extended to the complex case by replacing \( \xi \) by the independent coordinate \( \tilde{\xi} \).

### 4.1 Solitonic metrics

We can now establish the connection between the integrable systems reviewed in section 2 and self-dual vacuum metrics. We do so by expressing the Hamiltonians above in terms of solutions to various soliton equations. From a given solution of a two-dimensional nonlinear equation we can generate a null tetrad \((3.16)\).

1) NLS

\[
W = \partial_x + (\overline{\phi}\xi^2 + \phi)\partial_\xi + (\phi\overline{\xi}^2 + \overline{\phi})\partial_{\overline{\xi}}
\]

\[
\tilde{W} = \partial_x
\]

\[
\tilde{Z} = -i\xi\partial_\xi + i\overline{\xi}\partial_{\overline{\xi}}
\]

\[
Z = \partial_x + i(-\overline{\phi}\xi^2 + 2|\phi|^2\xi + \phi_\xi)\partial_\xi - i(-\phi\overline{\xi}^2 + 2|\phi|^2\overline{\xi} + \overline{\phi}_\xi)\partial_{\overline{\xi}}
\]

\[
f^2 = \frac{2\Re{(\overline{\xi}\phi)}}{1 + |\xi|^2}
\]

2) KdV

\[
W = \partial_x + (qm + n)\partial_m + (bm - qn)\partial_n
\]

\[
\tilde{W} = \partial_x
\]

\[
\tilde{Z} = m\partial_n
\]

\[
Z = \partial_x + (\frac{b_x}{2} m - q_x n)\partial_m + (wm - \frac{b_y}{2} n)\partial_n
\]

\[
f^2 = -m(q + mn)
\]

where \( b = q_x - q^2 \) and \( 4w = q_{xxx} - 4qq_{xx} - 2q_x^2 + 4q^2q_x \).

3) SG; elliptic case.

\[
W = \partial_x + \frac{1}{4}(\phi m - 2\cos(\phi/2)n)\partial_m + \frac{1}{4}(-\phi n - 2\cos(\phi/2)m)\partial_n
\]

\[
\tilde{W} = \partial_x + \frac{1}{4}(\phi m + 2\cos(\phi/2)n)\partial_m + \frac{1}{4}(-\phi n + 2\cos(\phi/2)m)\partial_n
\]

\[
\tilde{Z} = \partial_t + \frac{1}{4}(-\phi x m - 2\sin(\phi/2)n)\partial_m + \frac{1}{4}(\phi x n - 2\sin(\phi/2)m)\partial_n
\]

\[
Z = \partial_t + \frac{1}{4}(-\phi x m + 2\sin(\phi/2)n)\partial_m + \frac{1}{4}(\phi x n + 2\sin(\phi/2)m)\partial_n
\]

\[
f^2 = (\sin(\phi)mn)
\]

4) SG; hyperbolic case

\[
W = (-i\xi^2e^{-i\phi} + ie^{i\phi})\partial_\xi + (i\xi^2e^{i\phi} - ie^{-i\phi})\partial_{\overline{\xi}}
\]

\[
\tilde{W} = (-i\xi^2 + i)\partial_\xi + (i\xi^2 - i)\partial_{\overline{\xi}}
\]
\[ \bar{Z} = \partial_{\bar{z}} \]

\[ Z = \partial_z - i(\partial_z \phi)\xi \partial_{\xi} + i(\partial_{\bar{z}} \phi)\bar{\xi} \partial_{\bar{\xi}} \]

\[ f^2 = \frac{4 \sin \phi(|\xi|^2 - 1)}{|\xi|^2 + 1}. \]

Put \( d_A \xi = d\xi + i\xi \partial_z \phi \, dz \). Then we have

\[
ds^2 = \frac{1}{1 + \xi^2} \left( [(1 - \xi^2)^2 \cot \phi + i(1 - \xi^2)]d_A \xi \otimes d_A \xi + 2 \sin \phi \, dz \otimes d\bar{z} + (\cot \phi(1 - \xi^2)(1 - \xi^2) + i[1 + \xi^2](1 - \xi^2) \right. \\
- (1 - \xi^2)(1 + \xi^2)]d_A \xi \otimes d_A \xi + [(1 - \xi^2)^2 \cot \phi - i(1 - \xi^2)]d_A \xi \otimes d_A \xi \right) .
\]

If one takes a solution describing the interaction of a half kink and a half anti-kink (two topological solitons travelling in \( z - \bar{z} \) direction and increasing from 0 to \( \pi \) as \( z + \bar{z} \) goes from \(-\infty \) to \( \infty \)) then the singularity in \( \sin \phi = 0 \) may be absorbed by a conformal transformation of \( z + \bar{z} \) \( \mathbb{R} \).

From the Yang-Mills point of view, the solutions that we have obtained are metrics on the total space of \( \mathcal{E} \), the \( \Sigma \)-bundle associated to the Yang-Mills bundle. Therefore it is of interest to consider the effect of gauge transformations. First notice that diffeomorphisms of \( \mathbb{R}^2 \times \Sigma \) given by

\[ x^a \rightarrow x^a + \epsilon X_F(x^a) \]

yield \( H_\mu \rightarrow H_\mu + \epsilon (\{ H_\mu, F \} + \partial_a F) \) which is an infinitesimal form of the full gauge transformation \( \mathcal{P} \). Here \( \mu \) is an index on \( \mathcal{M} \), whereas \( a \) is an index on \( \mathcal{M} = \mathbb{R}^2 \times \Sigma \). The vector field \( X_F \) is Hamiltonian with respect to \( \Omega_\Sigma \), with Hamiltonian \( F = F(x^a) \).

If \( (2.27) \) preserves the Kähler structure of \( \Sigma \) then \( H_\mu \) transforms under (a real form of) \( SL(2, \mathbb{C}) \) and therefore our construction remains ‘invariant’.

## 5 Final remarks

### 5.1 The relationship between the twistor correspondences

To finish, we explain how our construction ties in with the twistor correspondences for the self-duality equations. We consider only the complex case of the SDYM equations with two commuting symmetries \( X, Y \). The \( SL(2, \mathbb{C}) \) SDYM connection defines, by the Ward construction \( \mathcal{P} \), a holomorphic vector bundle over the (non-deformed) twistor space, \( E_W \rightarrow \mathcal{P} \). It is convenient \( \mathcal{P} \) to use the bundle \( \mathcal{E}^5_W \) - associated to \( E_W \) by the representation of \( SL(2, \mathbb{C}) \) as holomorphic canonical transformations of the complex symplectic manifold \( \Sigma^2_\mathbb{C} \).

On the other hand, the SD vacuum metric corresponds to a deformed twistor space \( \mathcal{P}_\mathcal{M} \). In this paper we have explained how the quotient of \( \mathcal{E} \) by lifts of \( X, Y \) is, by theorem \( (2.2) \), equipped with a half-flat metric \( \mathfrak{D} \). To give a more complete picture we can obtain the deformed twistor space directly from \( \mathcal{E}^5_W \) and show that this is the twistor space of \( \mathcal{M} \). Consider the following chain of correspondences:

\[
\begin{array}{cccc}
\mathcal{E}^5_W & \mathcal{F} & \mathcal{E} & \mathcal{F}_\mathcal{M} \\
\mathcal{P}_\mathcal{M} & \mathcal{P} & \mathbb{C}^4 & \mathcal{M} & \mathcal{P}_\mathcal{M} \\
\end{array}
\]

Here \( \mathcal{F} \) and \( \mathcal{F}_\mathcal{M} \) are the standard projective spin bundles fibre over \( \mathbb{C}^4 \) and \( \mathcal{M} \) respectively. The space \( \mathcal{F}^5_\mathcal{E} \), the pullback of the spin bundle \( \mathcal{F} \) to the total space of the bundle \( \mathcal{E} \), fibres over all the spaces in the above diagram. Taking the quotient by lifts of \( X, Y \) we project \( \mathcal{F}^5_\mathcal{E} \) to \( \mathcal{F}_\mathcal{M} \).

\( \mathbb{C}^4 \) denotes also the general case of \( G = SDiff(\Sigma^2_\mathbb{C}) \). For this we work with \( \mathcal{E}^5_W \) rather than the principal Ward bundle, since the latter has infinite-dimensional fibres. The notation is such that the upper index of a space stands for the complex dimension of that space.
Taking the quotient by the twistor distribution, $\mathcal{F}_E^7$ also projects to the Ward bundle $\mathcal{E}_W^5$. By definition it projects to $\mathcal{E}$ and it could equivalently have been defined as the pullback of $\mathcal{E}$ to $\mathcal{F}$. The compatibility of these projections is a consequence of the commutativity of the diagram

\[
\begin{array}{ccc}
\mathbb{C}P^1 \times \mathbb{C}^4 \times \Sigma \subsetneq \mathcal{F}_E^7 & \xrightarrow{\pmatrix{X \\ Y}} & \mathcal{F}_M^5 \\
\downarrow & & \downarrow \\
\mathcal{E}_W^5 & \xrightarrow{\pmatrix{X \\ Y}} & \mathcal{P}_M.
\end{array}
\]

which follows from the integrability the the distribution spanned by (lifts of) $X, Y, L, M$, and from the fact that $(X, Y)$ commute with $(L, M)$.

### 5.2 Global issues

In order to obtain a compact space one might attempt the following:

- choose the gauge group to be $SU(2)$ so that the fibre space is compact, and
- Compactify $\mathbb{R}^2$ after the reduction.

We restrict the rate of decay of $A_\mu$ by the requirement that $A_\mu$ should be smoothly extendible to $S^2$ in the split signature case. Other possibilities are to restrict to the class of rapidly decreasing soliton solution of corresponding integrable equation. If we have reduced from a Euclidean signature solution to the SDYM equations, then it is more natural to compactify $\mathbb{R}^2$ in such a way as to obtain a Riemann surface of genus greater than one as it is only for such a compactifications that one can have existence of nontrivial solutions, [4].

However, we still have singularities in the metrics corresponding to (1.23) and (1.25), even if we can eliminate those from the Yang-Mills connection. We are left with singularities associated with sets on which the tetrad becomes linearly dependent. This reduces to the proportionality (or vanishing) of the Higgs fields on $\Sigma$, which generically occurs on a real co-dimension one subset of each fibre (and hence co-dimension one in the total space). In the above formulae this set is given by the vanishing of $f$. The Weyl curvature $C_{abcd}$ blows up as $f$ goes to zero. Calculation of curvature invariants show that these lead to genuine singularities that cannot be eliminated by a change of frame or coordinates. For example

\[
C_{abcd}C^{abcd} = \sum_{i=-3}^{3} C_i f^{2i},
\]

where $C_i = C_i(x^a)$ are generally non-vanishing regular functions on $\mathcal{M}$, which explicitly depend on Yang-Mills curvature $F_{\mu\nu}$ and (derivatives of) Hamiltonians (1.13, 1.15). Those singularities appear (for purely topological reasons) because each vector in the tetrad $(\tilde{W}, \tilde{Z}, \tilde{Z})$ has at least one zero, when restricted to $\Sigma = S^2$.

One can also obtain Euclidean metrics as above by using reductions of the SDYM equations from Euclidean space, but we will still be unable to avoid these same co-dimension one singularities.

### 5.3 Other reductions

We have focused in this article on the familiar $1+1$ soliton equations. However, it is clear from the discussion of section §3 [4] that the construction will extend to any symmetry reduction of the SDYM equations to systems in two dimensions with gauge group contained in $SL(2, \mathbb{C})$, in particular when the symmetry imposed consists of two translations as for the Euclidean signature examples mention previously. However, one can also use the same device to embed examples using any other two-dimensional symmetry subgroup of the conformal group. In particular, with cylindrical symmetry, one obtains the Ernst equations (the two symmetry reduction of the full, non self-dual four-dimensional Einstein vacuum equations) and this can similarly be embedded into the self-dual (but not vacuum) equations.
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