Cut and compute: Quick cascades with multiple amplitudes

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Abstract

In an earlier paper [1], we have proposed a novel method to compute the decay width for a general 1 → n cascade decay where the propagators are off-shell and may be of different spins. Here, we extend our algorithm to accommodate those decays that are mediated by more than one such cascades. This generalizes our prescription and widens its applicability. We compute the three- and four-body toy decay chains where identical final states appear through different cascades. Here, we also provide the algorithm to calculate the interference terms. For four-body decays we discuss both symmetric and asymmetric cascades, providing the expressions for the detailed phase space structure in each case. We find that the results obtained with this algorithm have a very impressive agreement with those from standard softwares using a sophisticated Monte Carlo based phase space integration.

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1 Introduction

Field theoretic calculation of the decay width for 1 → n processes gets complicated as n increases, unless the Feynman diagram can be decomposed into smaller parts by cutting through the on-shell lines. While it is not allowed to cut the off-shell lines, one may ask whether it is possible to find some algorithm where the 1 → n diagrams can be effectively decomposed in a number of 1 → 2 subdiagrams with possible off-shell incoming and outgoing legs.

We have addressed this issue and come up with such an algorithm in our earlier paper [1], also tested against the standard numerical packages for simulating the multibody phase space. We have defined a modified $|\mathcal{M}|^2$ which is a matrix spanned over the bases of quantum numbers being carried by the off-shell propagators. We have shown that one can decompose the full cascade into several 1 → 2-body decays, which may contain off-shell particles as incoming or outgoing legs. Thus, one needs to compute only the 1 → 2-body decays, which is a standard textbook exercise. Then, following our proposal, all those 2-body decays can be combined, leading to the result for the full 1 → n-body decay. In essence, one has to remember two points: (i) In the calculation of the matrix element squared, $p^2$ for an off-shell field has to be replaced by its invariant mass squared and not its physical mass squared, and then one has to integrate over the entire range of the invariant mass, and (ii) For the spin sum, the completeness relation should contain the physical mass, and not the invariant mass.
In this paper, we extend our earlier proposal to include decays that proceed through more than one cascades leading to an identical final state. This necessitates the inclusion of interference diagrams, and we show how to take these diagrams systematically into account, even when the off-shell propagators have different spin (and other quantum numbers). For the help of the reader, we also compute some explicit examples of $1 \to 3$-body decays, and compare the results obtained using the software CalcHEP vis-a-vis our proposal. The results are in excellent agreement, but of course an implementation of our algorithm will be faster to execute. We have also shown how to deal with $1 \to 4$ decays, with topologically different cascades.

For a generic $1 \to n$-body decay with a single amplitude and multiple off-shell propagators, the algorithm can be found in Ref. [1], say in Eqs. (1) and (6). The essential trick is to write the $1 \to 2$ body amplitudes in terms of the invariant masses of the off-shell legs and then integrate over all such invariant masses, with energy-momentum conservation. A flowchart is given in Fig. 1.

![Figure 1: Schematic guidelines of the off-shell prescription.](image)

Let us also mention here that for particles with relatively large decay widths, the propagator should be written in the Breit-Wigner form and $(m_{ij}^2 - m^2)^2$ (where $m_{ij}$ is the invariant mass and $m$ is the rest mass) should be replaced by $(m_{ij}^2 - m^2)^2 + \Gamma^2 m^2$, where $\Gamma$ is the decay width. The $1 \to 2$ “decay width” $\tilde{\Gamma}$ is analogous to the actual decay width, $\Gamma$, but this is not a number; rather, this is a matrix in the basis of the quantum numbers carried by the off-shell particles. Obviously, so are the “amplitudes” $|M|^2$. This structure helps us to track the flow of those quantum numbers throughout the cascade. For a single cascade the footprints which need to be followed are structured in detail in Ref. [1].

In this paper, we generalize that idea, and also show how to take into account the full phase space, consistent with our factorized diagrams. Often there are multiple Feynman diagrams leading to the same final states, and for that we have to incorporate the interference amplitudes too. This has been
the main motivation of this paper. In Sections 2 and 3, we discuss the three-body and four-body
decays respectively, and conclude in Section 4. A lot of computational details have been relegated to
the Appendix.

2 3-particle final states

As an example, let us first consider the 3-body decays of the neutral Higgs boson $H \to b\bar{c}W^+$ through
off-shell top-quark and W-boson: $H(p) \xrightarrow{W^+(q)} W^+(p_1) \bar{c}(p_2) b(p_3)$. The relevant Feynman diagrams
are shown in Figs. 2 and 3. While the first amplitude is allowed in the SM itself, we have to introduce a
flavor-changing neutral current coupling for the Higgs boson to top and charm, given by $y_{ct}(\bar{t}c + \bar{c}t)H$,
to introduce the second amplitude. The Yukawa coupling $y_{ct}$ is fixed to such a value as to make the
amplitudes comparable in magnitude and hence emphasize the effect of the interference term. We
take the relevant quark mixing matrix elements, $V_{cb}$ and $V_{tb}$, and also the new Yukawa coupling $y_{ct}$,
to be real.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Cutting the off-shell propagator for the decay $H(p) \xrightarrow{W^+(q)} W^+(p_1)\bar{c}(p_2)b(p_3)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Cutting the off-shell propagator for the decay $H(p) \xrightarrow{t^*(k)} \bar{c}(p_2)W^+(p_1)b(p_3)$.}
\end{figure}

We will first discuss the individual contributions from these diagrams and then their interference
contribution. The amplitude squared for the decay $H(p) \xrightarrow{W^-(q)} W^+(p_1)\bar{c}(p_2)b(p_3)$ has been computed
in [1] and the result is

$$\Gamma_1(H \to W^+b\bar{c}) = \frac{N_cg^2m_W^4}{8m_Hv^2}|V_{cb}|^2 \int \left[ \frac{1}{\pi(m_{23}^2 - m_W^2)} \right] \int d^4 \phi_{PS} \int d^4 \phi_{PS} F_1,$$  (1)
where $N_c = 3$ is the colour factor, $g$ is the $SU(2)$ coupling, and $v$ is the vacuum expectation value of the Higgs field, given by $v = 2m_W/g$. Furthermore,

$$
\mathcal{F}_1 = \frac{16}{m_W^4} \left[ 2m_W^2 (p_1.p_2)(p_1.p_3) + 4m_W^2 (q.p_2)(q.p_3) - 2m_W^2 m_{23}^2 (p_2.p_3) - 2(q.p_1)(q.p_2)(p_1.p_3) \\
- 2(q.p_1)(q.p_3)(p_1.p_2) + 2(q.p_1)^2 (p_2.p_3) + m_{13}^2 (p_2.p_3) - 2m_W^2 (q.p_2)(q.p_3) \\
+ 2m_W^2 (q.p_1)^2 (p_2.p_3) - m_W^2 m_{23}^2 (p_2.p_3) (q.p_1)^2 + m_W^4 (p_3.p_2) \right].
$$

(2)

Note that we have used $q^2 = m_{23}^2$ and the factors of $m_W$ come from the polarization sum of the $W$-propagator.

Similarly, for the second amplitude, we get [1].

$$
\Gamma_2 (H \to b\bar{c}W^+) = \left( \frac{1}{4m_H} \right) \frac{N_c g^2 |y_{ct}|^2 |V_{tb}|^2}{8} \int \left[ \frac{1}{\pi} \left( \frac{d m_{13}^2}{(m_{13}^2 - m_t^2)^2} \right) \right] \int d^4 p_{W_0} \to \bar{c}t^* \int d^4 p_{ps} \to bW \mathcal{F}_2,
$$

(3)

where

$$
\mathcal{F}_2 = 16 ((p_2.k) - m_{c.m_t}) \left((k.p_3) + \frac{2(k.p_1)(p_3.p_1)}{m_W^2} \right) - 8 (k^2 - m_t^2) \left((p_2.p_3) + \frac{2(p_2.p_1)(p_3.p_1)}{m_W^2} \right).
$$

(4)

Let us now concentrate on the interference diagram. The heart of our proposal is to decompose every cascade into several $1 \to 2$ body decays irrespective of the length of the decay chain. We stick to our prime intention, write down the amplitude for every two body decay, and then join them as shown below. The two body decay amplitudes for $H(p) \to W^- (p_1) \bar{c}(p_2) b(p_3)$ through off-shell $W$ and $t$, as shown in Figs. 2 and 3 are written as:

$$
(M_{1})^{\mu\nu} (H \to W^- W^*) = g m_W \epsilon^{\mu*}_{(\lambda)} (p_1) \epsilon^{\alpha*}_{(\lambda')} (q),
$$

$$
(M_{2})_{\mu\nu} (W^- \to \bar{c}b) = \frac{g V_{cb}}{2\sqrt{2}} \epsilon^{\gamma}_{(\lambda')(q)} [\bar{W}^{(s_2)}(p_3) \gamma_{\nu} (1 - \gamma^5) u^{(s_3)} (p_2)],
$$

$$
(M_{3})_{b_2c_1} (H \to \bar{c}t^*) = -y_{ct} [\nu^{(s_3)}(p_2)]_{c_1} [\bar{W}^{(s)}(k)]_{b_2},
$$

$$
(M_{4})_{c_1 b_2}^{\nu} (t^* \to bW^+) = \frac{g}{2\sqrt{2}} V_{tb} [\bar{W}^{(s)}(p_3) \gamma_{\alpha} (1 - \gamma^5) u^{(s)}(k)]_{c_1 b_2} \epsilon_{(\lambda)}^{\nu*} (p_1).
$$

(5)

The interference term $[M_1 M_2]^\dagger M_3 M_4$ can be written as

$$
- \frac{N_c g^3 V_{tb} V_{cb} y_{ct} m_W}{8} \left[ \epsilon^{\mu*}_{(\lambda)} (p_1) \epsilon^{\alpha*}_{(\lambda')} (q) \epsilon_{(\lambda')} (q) [\bar{W}^{(s_2)}(p_3) \gamma_{\nu} (1 - \gamma^5) u^{(s_3)} (p_2)] \right]^\dagger \\
\times \left[ \nu^{(s_2)}(p_2) \bar{W}^{(s)}(k) \right]_{b_2 c_1} [\bar{W}^{(s)}(p_3) \gamma_{\alpha} (1 - \gamma^5) u^{(s)}(k)]_{c_1 b_2} \epsilon_{(\lambda)}^{\nu*} (p_1)
$$

$$
= - \frac{N_c g^3 V_{tb} V_{cb} y_{ct} m_W}{8} \left( -g^\mu + \frac{p_1^\mu p_1^\nu}{m_W^2} \right) \left( -g^\alpha + \frac{q^\alpha q^\nu}{m_W^2} \right) \\
\times \text{Tr} \left[ \gamma_{\nu} (1 - \gamma^5) (\not{p} + m_b) \gamma_{\alpha} (1 - \gamma^5) \right] \\
= N_c g^3 V_{tb} V_{cb} y_{ct} m_W \left\{ m_t \left[ \frac{2(p_1.p_3)(p_2.p_1)}{m_W^2} + \frac{2(p_3.q)(p_2.q)}{m_W^2} + (p_3.p_2) - \frac{m_{23}^2}{m_W^2} (p_3.p_2) \right] \right\}.
$$

(5)
Here, we have used the following polarization and spin sums:

\[
\sum_{\lambda} \epsilon^\mu_{(\lambda)}(k) \epsilon^\nu_{(\lambda)}(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m_W^2}, \quad \sum_s u^{(s)}(p) \bar{\pi}^{(s)}(p) = (\slashed{p} + m), \quad \sum_s v^{(s)}(p) \bar{\pi}^{(s)}(p) = (\slashed{p} - m).
\]  

(7)

One can easily check that this result is in complete agreement with that obtained using the full cascade without any decomposition.

This is, therefore, a good place to explain the interference algorithm. The entire matrix element squared computed in the canonical way without cutting any off-shell propagator is obviously a Lorentz and gauge scalar. Here, when one calculates the individual diagrams, the \(|M|^2\) for the cut diagrams need not be a scalar; this can carry Lorentz or spin indices. The index contractions, as has been explained in Ref. [1] are performed in such a way that both \(|\mathcal{M}|^2\)'s are matrices in the index space but the product is a scalar. Schematically speaking, the combined \(|\mathcal{M}|^2\) looks like \((|\mathcal{M}_1|^2)^\mu_\nu(|\mathcal{M}_2|^2)^\nu_\mu\), which ensures the “index flow” through the cut propagator. For the interference diagrams, the Lorentz and spin indices are to be contracted in such a way that both \((\mathcal{M}_1\mathcal{M}_2)^\dagger\) and \((\mathcal{M}_3\mathcal{M}_4)\) are matrices but their product is a scalar. The assignment can be followed in Eq. (6). In \(\mathcal{M}_1\mathcal{M}_2\), the contraction of the index \(\mu\) shows the “index flow” through the off-shell \(W\), and in \(\mathcal{M}_3\mathcal{M}_4\), the spin index \(b_1\) plays the same role. Note that \(\mathcal{M}_4\) is a matrix in both spin and Lorentz spaces. Another example of this “index flow” is shown in the next Section.

| \(\Gamma_{W^*}\) (GeV) | \(\Gamma_{\ell^*}\) (GeV) | \(\Gamma_{\text{int}}\) (GeV) | \(\Gamma\) (GeV) |
|----------------------|----------------------|----------------------|----------------------|
| 2.08 \times 10^{-7} (C1) | 2.49 \times 10^{-7} (C1) | 3.79 \times 10^{-7} (C1) | 8.36 \times 10^{-7} (C1) |
| 2.12 \times 10^{-7} (C2) | 2.41 \times 10^{-7} (C2) | 3.79 \times 10^{-7} (C2) | 8.32 \times 10^{-7} (C2) |
| 2.11 \times 10^{-7} (M) | 2.41 \times 10^{-7} (M) | 3.79 \times 10^{-7} (M) | 8.31 \times 10^{-7} (M) |

Table 1: Decay width of \(H \to W^+b\bar{\ell}\) calculated using CalcHEP v3.6.27 (C1), v3.6.23 (C2), and our algorithm with phase space integration numerically performed by Mathematica v10 (M).

After implementing the three body phase space, as discussed in Section A.1, we can add up all three contributions and write down the full partial decay width as \(\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_{\text{int}}\), where the interference contribution is

\[
\Gamma_{\text{int}}(H \to W^+b\bar{\ell}) = \frac{1}{m_H} \left[ \int \frac{1}{\pi} \left( \frac{\frac{d\Omega_{23}}{d\cos \theta}}{(m_{23}^2 - m_W^2)(m_{13}^2 - m_L^2)} \right) \right] \times \left[ \frac{1}{2} \int \frac{\beta d\cos \theta d\phi}{2\pi} \right] \times \left[ \frac{1}{2} \int \frac{\beta_{23} d\cos \theta_{23} d\phi_{23}}{2\pi} \right] \times 2\text{Re} \left( [\mathcal{M}_1\mathcal{M}_2]^\dagger\mathcal{M}_3\mathcal{M}_4] \right),
\]  

(8)

where the notations have been explained in the Appendix.
We have computed the total decay width $\Gamma$ for this process, using Mathematica v10 \[4\] to integrate the phase space numerically. The couplings we use are given by $V_{tb} = 1$, $V_{cb} = 0.04$, $y_{ct} = 0.04$, and the masses (in GeV) are $m_H = 125$ GeV, $m_c = 1.2$ GeV, $m_b = 4.23$ GeV, $m_W = 80.385$ GeV, and $m_t = 172.5$ GeV. The result has been compared with that obtained from CalcHEP v3.6.23 as well as v3.6.27 \[5\], which does the phase space integration with a numerical simulation. The comparison is shown in Table 1; we find an excellent agreement within the error margin.

### 3 4-particle final states

As an example of a $1 \rightarrow 4$ decay, let us consider the decay $H(p) \rightarrow q_1(p_1) \bar{q}_2(p_2) f_1(p_3) f_2(p_4)$ where $q_1$, $q_2$, $f_1$ and $f_2$ are four fermions, possibly quarks. The first amplitude proceeds through $H \rightarrow q_1 \bar{q}_1^*$, $\bar{q}_1^* \rightarrow \bar{q}_2 W^*$, $W^* \rightarrow f_1 \bar{f}_2$. To get the second amplitude, we introduce a hypothetical charged scalar $\Phi$ that replaces the $W$ boson in the first amplitude. The corresponding Feynman diagrams are shown in Figs. 4 and 5.

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\(^1\)Those not shown here are fixed to their SM values.
According to our proposal the decay width for the diagram in Fig. 4 can be written as:

\[
\Gamma_1(H \to q_1\bar{q}_2f_1\bar{f}_2) = \frac{1}{m_H} \int \left[ \frac{1}{\pi} \left( \frac{dm_{12}^2}{(m_{12}^2 - m_{q_1}^2)^2} \right) \right] \left[ \frac{1}{\pi} \left( \frac{dm_{23}^2}{(m_{23}^2 - m_{f_1}^2)^2} \right) \right] \times \text{Tr} \left[ \bar{\Gamma}_1(H \to q_1\bar{q}_1^*)\bar{\Gamma}_2(\bar{q}_1^* \to \bar{q}_2W^*)\bar{\Gamma}_3(W^* \to f_1\bar{f}_2) \right],
\]

with

\[
\left[ \bar{\Gamma}_1(H \to q_1\bar{q}_1^*) \right]_{c_1b_2} = \int \frac{d^H_{PS} \rightarrow q_1\bar{q}_1^*}{2} \left[ |\mathcal{M}_1(H \to q_1\bar{q}_1^*)|^2 \right]_{c_1b_2},
\]

\[
\left[ \left[ \bar{\Gamma}_2(\bar{q}_1^* \to \bar{q}_2W^*) \right]_{b_2c_1} \right]_\mu^\nu = \int \frac{d^\bar{q}_2_{PS} \rightarrow \bar{q}_2W^*}{2} \left[ |\mathcal{M}_2(\bar{q}_1^* \to \bar{q}_2W^*)|^2 \right]_{b_2c_1},
\]

\[
\left[ \left[ \bar{\Gamma}_3(W^* \to f_1\bar{f}_2) \right]_{\mu}^\nu \right] = \int \frac{d^W_{PS} \rightarrow f_1\bar{f}_2}{2} \left[ |\mathcal{M}_3(W^* \to f_1\bar{f}_2)|^2 \right]_{\mu}^\nu,
\]

where the squared amplitudes are

\[
|\mathcal{M}_1|^2_{c_1b_2} = \frac{N_c g^2 m_{q_1}^2}{4 m_W^2} \left\{ \left[ \pi^{(s_1)}(p_1) \right]_{c_1} \left[ \pi^{(s_2)}(k_1) \right]_{b_2} \left[ \pi^{(s_2)}(k_1) \right]_{d_1} \left[ u^{(s_1)}(p_1) \right]_{d_1} \right\},
\]

\[
|\mathcal{M}_2|^2_{b_2c_1} = \frac{g^2 |V_{q_2q_1}|^2}{8} \left\{ \left[ \pi^{(s_3)}(k_1) \right]_{a_1} \left[ \gamma^\mu(1 - \gamma^5) \right]_{a_1a_2} \left[ \pi^{(s_3)}(p_2) \right]_{a_2} \times \left[ \pi^{(s_3)}(p_2) \right]_{b_1} \left[ \gamma^\alpha(1 - \gamma^5) \right]_{b_1b_2} \left[ \pi^{(s_3)}(k_1) \right]_{c_1} \right\} e^{(\lambda)}(k_2) e^{(\lambda)}(k_2),
\]

\[
|\mathcal{M}_3|^2_{\mu}^\nu = \frac{N_c g^2 |V_{f_1f_2}|^2}{8} \left\{ \left[ \pi^{(s_3)}(p_3) \gamma^\mu(1 - \gamma^5) \pi^{(s_6)}(p_4) \gamma^\beta(1 - \gamma^5) u^{(s_3)}(p_3) \right] \times e^{(\lambda)}(k_2) e^{(\lambda)}(k_2).\right\}
\]

We can now combine, at a step, two such squared amplitudes:

\[
|\mathcal{M}_{12}|^2_{\mu} = |\mathcal{M}_1|^2_{c_1b_2} \left| |\mathcal{M}_2|^2_{b_2c_1} \right|_{\nu} = \frac{N_c g^4 m_{q_1}^2 |V_{q_2q_1}|^2}{32 m_W^2} g_{\alpha\nu} \times \text{Tr} \left[ (k_1^4 - m_{q_1}^4)(p_1^4 + m_{q_1}^4)(k_1^4 - m_{q_1}^4)\gamma^\mu(1 - \gamma^5)(p_2^4 - m_{q_2}^4)\gamma^\alpha(1 - \gamma^5) \right] = \frac{N_c g^4 m_{q_1}^2 |V_{q_2q_1}|^2}{32 m_W^2} g_{\alpha\nu} \times \left[ 16 \left(k_1^4 - m_{q_1}^4\right) \left[m_1^{\mu} p_2^\alpha - g^{\alpha\mu}(k_1.p_2) + k_1^\alpha p_2^\mu + i(k_1)^\rho(p_2)^\delta e^{\rho\mu\delta\alpha}\right] \times +8 \left(m_{q_1}^2 - k_1^2\right) \left[p_1^{\mu} p_2^\alpha - g^{\alpha\mu}(p_1.p_2) + p_1^\alpha p_2^\mu + i(p_1)^\rho(p_2)^\delta e^{\rho\mu\delta\alpha}\right] \right],
\]

and with

\[
|\mathcal{M}_3|^2_{\mu} = N_c g^2 |V_{f_1f_2}|^2 g^{\nu\beta} \left( (p_3)_\beta (p_4)_{\mu} - g_{\beta\mu}(p_3.p_4) + (p_3)_{\mu}(p_4)_{\beta} + i(p_3)^\tau(p_4)^\lambda \epsilon_{\tau\beta\lambda\mu} \right),
\]
one gets $|\mathcal{M}|^2$ as:

$$
\operatorname{Tr} \left[ |\mathcal{M}_{12}|^2 |\mathcal{M}_3|^2 \right] = \frac{N_c g^6 m_q^2 |V_{q_1 q_2}|^2 |V_{f_1 f_2}|^2}{m_W^2} \left[ 2 (k_1 . p_1 - m_{q_1}^2) (k_1 . p_3) (p_2 . p_4) + (m_{q_1}^2 - k_1^2) (p_1 . p_3) (p_2 . p_4) \right].
$$

(14)

This expression completely agrees with the canonically computed expression, say as in Ref. [3]. The phase space is incorporated as shown in Eqs. (A.41), (A.42), (A.52), (A.53), (A.54) and (A.55).

The decay width is now trivial to write down:

$$
\Gamma_1 (H \to q_1 \overline{q}_2 f_1 \overline{f}_2) = \frac{1}{m_H} \int \left[ \frac{1}{\pi} \left( \frac{dm_{12}^2}{(m_{12}^2 - m_{q_1}^2)^2} \right) \right] \left[ \frac{1}{\pi} \left( \frac{dm_{23}^2}{(m_{23}^2 - m_{q_2}^2)^2} \right) \right] \times
$$

$$
\left( \frac{1}{2} \int \frac{\beta_1 d \cos \theta \, d\phi}{8\pi} \right) \left( \frac{1}{2} \int \frac{\beta_2 d \cos \theta_1 \, d\phi_1}{8\pi} \right) \left( \frac{1}{2} \int \frac{\beta_3 d \cos \theta_2 \, d\phi_2}{8\pi} \right) \times \operatorname{Tr} \left[ |\mathcal{M}_{12}|^2 |\mathcal{M}_3|^2 \right].
$$

(15)

The second diagram, as shown in Fig. 5, is analogous with $W$ replaced by a hypothetical charged scalar $\Phi$, whose coupling to any fermionic pair $pq$ is written as $y_{pq}$. As before,

$$
\Gamma_2 (H \to q_1 \overline{q}_2 f_1 \overline{f}_2) = \frac{1}{m_H} \int \left[ \frac{1}{\pi} \left( \frac{dm_{12}^2}{(m_{12}^2 - m_{q_1}^2)^2} \right) \right] \left[ \frac{1}{\pi} \left( \frac{dm_{23}^2}{(m_{23}^2 - m_{q_2}^2)^2} \right) \right] \times \operatorname{Tr} \left[ \hat{\Gamma}_1 (H \to q_1 \overline{q}_1 \Phi^*) \hat{\Gamma}_2 (\overline{q}_1 \Phi^*) \hat{\Gamma}_3 (\Phi^* \to f_1 \overline{f}_2) \right],
$$

(16)

with

$$
\left[ \hat{\Gamma}_1 (H \to q_1 \overline{q}_1) \right]_{c_1 b_2} = \int \frac{d_{PS}^{H \to q_1 \overline{q}_1}}{2} \langle |\mathcal{M}_1 (H \to q_1 \overline{q}_1)|^2 \rangle^*_{c_1 b_2},
$$

$$
\left[ \hat{\Gamma}_2 (\overline{q}_1 \Phi^*) \right]_{b_2 c_1} = \int \frac{d_{PS}^{\overline{q}_1 \Phi^*}}{2} \langle |\mathcal{M}_2 (\overline{q}_1 \Phi^*)|^2 \rangle^*_{b_2 c_1},
$$

$$
\left[ \hat{\Gamma}_3 (\Phi^* \to f_1 \overline{f}_2) \right] = \int \frac{d_{PS}^{\Phi^* \to f_1 \overline{f}_2}}{2} |\mathcal{M}_3 (\Phi^* \to f_1 \overline{f}_2)|^2.
$$

(17)

These functions contain the squared amplitudes for the $1 \to 2$-body processes:

$$
[|\mathcal{M}_1|^2]_{c_1 b_2} = \frac{N_c g^2 m_q^2}{4 m_W^2} \left\{ \left[ u^{(s_1)} (p_1) \right]_{c_1} \left[ v^{(s_2)} (k_1) \right]_{b_1} \left[ \pi^{(s_1)} (p_3) \right]_{d_1} \right\},
$$

$$
[|\mathcal{M}_2|^2]_{b_2 c_1} = \langle y_{q_1 q_2} \rangle^2 \left\{ \left[ u^{(s_2)} (p_2) \right]_{b_2} \left[ v^{(s_2)} (k_1) \right]_{d_1} \left[ \pi^{(s_1)} (p_4) \right]_{a_1} \right\},
$$

$$
[|\mathcal{M}_3|^2] = 4 N_c \left| y_{f_1 f_2} \right|^2 \left\{ \left[ v^{(s_1)} (p_3) \right]_{a_2} \left[ u^{(s_2)} (p_4) \right]_{a_1} \left[ \pi^{(s_1)} (p_3) \right]_{n_1} \right\}.
$$

(18)

As the scalar propagator does not carry any polarization index, $|\mathcal{M}_3|^2$ is only a number. Combining all the squared amplitudes, we get

$$
|\mathcal{M}_1|^2 |\mathcal{M}_2|^2 |\mathcal{M}_3|^2 = \frac{N_c g^2 m_q^2 |V_{q_1 q_2}|^2 |V_{f_1 f_2}|^2}{m_W^2} \left\{ \left[ (p_3 . p_4) - m_{f_1} m_{f_2} \right] \times
$$

(19)
The decay width, therefore, is

$$
\Gamma_2(H \rightarrow q_1 \bar{q}_2 f_1 \bar{f}_2) = \frac{1}{m_H} \int \left[ \frac{1}{\pi} \left( \frac{d \Gamma_2}{(m_{12}^2 - m_{q_1}^2)^2} \right) \right] \frac{1}{\pi} \left( \frac{d \Gamma_2}{(m_{23}^2 - m_{q_2}^2)^2} \right) \times \left( \frac{1}{2} \int \frac{d \cos \theta_1 d \phi_1}{8\pi} \right) \left( \frac{1}{2} \int \frac{d \cos \theta_2 d \phi_2}{8\pi} \right) \times \text{Tr} |M_1|^2 |M_2|^2 |M_3|^2 .
$$

The interference term has to be calculated following the “index flow” algorithm as discussed for the $1 \rightarrow 3$ decays. The spin indices are assigned as shown in Figs. 4 and 5. The $1 \rightarrow 2$ amplitudes are as follows:

$$(M_1)_{b_2 a_1} (H \rightarrow q_1 \bar{q}_1) = - \frac{g m_{q_1}}{2 m_W} [\pi^{(s_1)}(p_1)]_{b_2} [\nu^{(s)}(k_1)]_{a_1},$$

$$[(M_2)_{a_1 b_2}]^\mu_\nu (\bar{q}_1 \rightarrow \bar{q}_2 W^*) = \frac{g\sqrt{\frac{1}{2}}}{} V_{q_1 q_2} [\pi^{(s)}(k_1)\gamma^\mu(1 - \gamma^5)\nu^{(s_2)}(p_2)]_{a_1 b_2} \epsilon^{(s)}_\nu(k_2),$$

$$(M_3)^\nu_\mu (W^* \rightarrow f_1 \bar{f}_2) = \frac{g\sqrt{\frac{1}{2}}}{} f_{f_1 f_2} \epsilon^{(s)}_\mu(k_2) [\pi^{(s)}(p_3)\gamma^\nu(1 - \gamma^5)\nu^{(s_4)}(p_4)],$$

$$(M_4)_{c_1 d_1} (H \rightarrow q_1 \bar{q}_1) = - \frac{g m_{q_1}}{2 m_W} [\pi^{(s_1)}(p_1)]_{c_1} [\nu^{(s)}(k_1)]_{d_1},$$

$$(M_5)_{d_1 c_1} (\bar{q}_1 \rightarrow \bar{q}_2 \Phi^*) = - y_{q_1 q_2} [\pi^{(s)}(k_1)]_{d_1} [\nu^{(s_2)}(p_2)]_{c_1},$$

$$(M_6)_{d_1 c_1} (\Phi^* \rightarrow f_1 \bar{f}_2) = - y_{f_1 f_2} [\pi^{(s_3)}(p_3)\nu^{(s_4)}(p_4)].$$

Combining all these contributions, the interference term comes out to be

$$\text{Tr} |M_3|^2 M_2^\dagger |M_1|^2 M_4^\dagger M_5 M_6$$

$$= \frac{N_c^2 g^4 m_{q_1}^2}{32} \frac{m_W^2}{V_{q_1 q_2} y_{q_1 q_2} y_{f_1 f_2} (\pi^{(s_3)}(p_3)\nu^{(s_4)}(p_4)) \times \left( \epsilon^{(s)}_\mu(k_2)\pi^{(s)}(p_3)\gamma^\nu(1 - \gamma^5)\nu^{(s_4)}(p_4) \right) \dagger \times \left( \pi^{(s)}(k_1)\gamma^\mu(1 - \gamma^5)\nu^{(s_2)}(p_2) \right)_{a_1 b_2} \epsilon^{(s)}_\nu(k_2) \dagger \times \left( \pi^{(s_1)}(p_1) \right)_{b_2} [\nu^{(s)}(k_1)]_{a_1} \left( \pi^{(s)}(p_1) \right)_{c_1} [\nu^{(s)}(k_1)]_{d_1} \left( \pi^{(s)}(k_1) \right)_{d_1} [\nu^{(s_2)}(p_2)]_{c_1}$$

$$= - \frac{N_c^2 g^4 m_{q_1}^2}{32} \frac{m_W^2}{V_{q_1 q_2} y_{q_1 q_2} y_{f_1 f_2} \text{Tr} \left( \varphi_1 + m_{f_1}(\varphi_1 - m_{f_2})\gamma^\mu(1 - \gamma^5) \right) \times \text{Tr} \left( \varphi_2 - m_{q_2} \gamma_\mu(1 - \gamma^5)(k_1 - m_{q_1})(\varphi_1 + m_{q_1})(k_1 - m_{q_1}) \right)$$

$$= \frac{N_c^2 g^4 m_{q_1}^2}{2 m_W^2} \frac{m_W^2}{V_{q_1 q_2} y_{q_1 q_2} y_{f_1 f_2} (m_{q_1}^3 + m_{q_1} k_1^2 - 2 m_{q_1} (p_1 k_1)) (m_{f_2} (p_2 p_3) - m_{f_1} (p_2 p_4)) - m_{q_2} (m_{q_1}^2 - k_1^2) \times"
\[ \{m_f(2p_3) - m_f(p_1p_4)\} + 2m_{q_2}(m^2_{q_1} - p_1.k_1) \{m_f(k_1p_3) - m_f(k_1p_4)\} \] .

(22)

After incorporating the four-body phase space structure as shown in Section A.2.2, the contribution to the decay width from the interference diagram is

\[
\Gamma_{\text{int}}(H \to q_1q_2f_1f_2) = \frac{1}{m_H} \int \left[ \frac{1}{\pi} \left( \frac{dm^2_{12}}{(m^2_{12} - m^2_{q_1})^2} \right) \right] \left[ \frac{1}{\pi} \left( \frac{dm^2_{23}}{(m^2_{23} - m^2_{W})^2} \right) \right] \times \\
\left( \frac{1}{2} \int \frac{\beta_1 \, d \cos \theta \, d \phi \, \beta_2 \, d \cos \theta \, d \phi_1 \, \beta_3 \, d \cos \theta_2 \, d \phi_2} {8 \pi} \right) \times \\
\left[ 2 \text{Re}(M_6M_3^\dagger M_1^\dagger M_4^\dagger M_5) \right].
\]

(23)

The total decay width for the process is given by

\[
\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_{\text{int}}.
\]

(24)

As an example, we consider the decay \( H \to t\bar{b}l\bar{c} \), keeping the top mass intentionally light enough for this decay to be kinematically possible but making sure that \( 2m_t > m_H \). The comparison with CalcHEP, as shown in Table 2, is quite impressive. For evaluation, we have used \( V_{tb} = 1, V_{cb} = 0.04, y_{tb} = y_{cb} = 1, m_t = 125 \text{ GeV}, m_f = 65 \text{ GeV}, \) and \( m_W = 80.385 \text{ GeV}, \) varying the mass of \( \Phi \). Note that we could have had two amplitudes even without the introduction of \( \Phi \) through a symmetric cascade: \( H \to W^+W^-*, W^+ \to t\bar{b}, W^- \to l\bar{c} \), by suitable adjusting the masses. Such a symmetric cascade has already been discussed in Ref. [1].

| \( m_t \) (GeV) | \( m_\Phi \) (GeV) | \( \Gamma \) \( (\text{C}) \) (10^{-8}) | \( \Gamma \) \( (\text{M}) \) (10^{-8}) |
|---------------|---------------|----------------|----------------|
| 65            | 200           | 1.33 \times 10^{-8} | 1.47 \times 10^{-8} |
| 65            | 150           | 4.36 \times 10^{-8} | 4.87 \times 10^{-8} |
| 65            | 100           | 2.51 \times 10^{-7} | 2.79 \times 10^{-7} |

Table 2: Comparison of the decay width for the process \( H \to t\bar{b}l\bar{c} \) with CalcHEP v3.6.23 (denoted by C) and Mathematica v10 (denoted by M) using our algorithm. All entries are in GeV.

4 Summary and conclusion

In this paper we extend and generalize our algorithm to extend its applicability to processes with multiple amplitudes and hence with interference contributions in squared amplitudes. One can decompose the entire chain in several \( 1 \to 2 \) decays with off-shell particles as incoming and/or outgoing legs. The algorithm for “index flow” for the interference diagrams has been exemplified by a \( 1 \to 3 \) and a \( 1 \to 4 \) decay process. The detailed calculation of the phase space has been discussed in the Appendix.

We find an impressive agreement with the results obtained with the software CalcHEP that calculates the phase space by Monte Carlo simulation. This shows that one can reach about the same level of accuracy with much less computer time. The present paper completes our discussion on tree-level decays.
Another advantage of the algorithm is the way one can reduce it to a limited number of elementary vertices. There are only six such types: $\phi\phi$, $VVV$, $\phi\phi V$, $\phi VV$, $f\bar{f}\phi$ and $f\bar{f}V$, where $\phi$, $V$, and $f$ stand for any generic spin-0, spin-1, and spin-$\frac{1}{2}$ particle. Once one specifies the particle content of each vertex and the momentum flow, the entire cascade can be built up using those $1 \to 2$ diagrams as building blocks. Following this technique, we plan to automatize the algorithm in near future.

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APPENDIX

A Phase space decomposition

Our proposal is based on the decomposition of a long cascade to several $1 \to 2$ subprocesses and then putting them together following the algorithm proposed. Here, we show how the full phase space may look like when we decompose that in terms of two-body phase spaces. Interference diagrams are also included in the discussion. We adopt some toy decays without specifying the quantum numbers for the off-shell propagators.

A.1 Three-body decay

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{three-body-decay.png}
\caption{Representative figure of the three-body decay, $A \to BDE$.}
\end{figure}

Let us consider a toy three-body decay: $A(p) \to B(p_1)C^*(k) \to B(p_1)D(p_2)E(p_3)$, and also assume that $A$ is a scalar while particles $B$ to $E$ can have any spin. According to our proposal the decay
The width functions are

$$\Gamma(A \to BDE) = \frac{f_s}{m_A} \int \left[ \frac{1}{\pi} \left( \frac{dm_{23}^2}{(m_{23}^2 - m_{23}^2)^2} \right) \right] \text{Tr} \left[ [\tilde{\Gamma}_1(A \to BC^*)] \tilde{\Gamma}_2(C^* \to DE) \right]. \quad (A.1)$$

Here, \(k^2 \equiv m_{23}^2\), \(f_s\) is the symmetry factor, and \(m_i (i = A, B, C, D, E)\) is the mass of the \(i\)th particle.

The width functions are

\[
[\tilde{\Gamma}_1]_{ij} = \int \frac{d\phi_B}{2} |M_1(A \to BC^*)|^2 \frac{1}{8\pi} \frac{\beta_1}{2} \left[ |M_1(A \to BC^*)|^2 \right]_{ij},
\]

\[
[\tilde{\Gamma}_2]_{ij} = \int \frac{d\phi_C}{2} |M_2(C^* \to DE)|^2 \frac{1}{8\pi} \frac{\beta_2}{2} \left[ |M_2(C^* \to DE)|^2 \right]_{ij}, \quad (A.2)
\]

where the boost factors are

\[
\beta_1 = \sqrt{1 - \frac{(m_B^2 + m_C^2)^2}{m_A^2} - \frac{(m_B^2 - m_{23}^2)^2}{m_A^4}},
\]

\[
\beta_2 \equiv \beta = \sqrt{1 - \frac{2(m_B^2 + m_E^2)}{m_{23}^2} - \frac{(m_B^2 - m_{23}^2)^2}{m_{23}^4}}. \quad (A.4)
\]

The indices \(i\) and \(j\) are Lorentz or spin indices for spin-1 or spin-\(\frac{1}{2}\) particles respectively. For scalar propagators there are no such indices, the respective \(\tilde{\Gamma}\) is a number rather than a matrix.

As the phase space measure \(d^3p/2E\) is Lorentz invariant, every two-body phase space can be computed in a reference frame where the decaying particle is considered to be at rest. These subspaces are to be joined using proper boost factors.

Considering \(A\) to be at rest, the four-momenta of \(B\) and \(C^*\) can be written as:

\[
p_1 = \frac{m_A}{2} \left( 1 + \frac{m_B^2}{m_A^2} - \frac{m_{23}^2}{m_A^2}, 0, 0, -\beta \right), \quad (A.5)
\]

\[
k = \frac{m_A}{2} \left( 1 - \frac{m_B^2}{m_A^2} + \frac{m_{23}^2}{m_A^2}, 0, 0, \beta \right). \quad (A.6)
\]

The boost factor from the rest frame of \(C^*\) towards the rest frame of \(A\) is

\[
\gamma = \frac{k_0}{\sqrt{m_{23}^2}} = \frac{m_A}{2\sqrt{m_{23}^2}} \left( 1 - \frac{m_B^2}{m_A^2} + \frac{m_{23}^2}{m_A^2} \right) \cdot \frac{m_A}{2\sqrt{m_{23}^2}} = \frac{m_A^2}{2m_{23}^2} \beta. \quad (A.7)
\]

Similarly, in the rest frame of \(C^*\), the four-momenta of \(D\) and \(E\) are

\[
\hat{p}_2 = \frac{\sqrt{m_{23}^2}}{2} \left( 1 + \frac{m_D^2}{m_{23}^2}, \beta_{23} \sin \theta_{23}, 0, \beta_{23} \cos \theta_{23} \right), \quad (A.8)
\]

\[
\hat{p}_3 = \frac{\sqrt{m_{23}^2}}{2} \left( 1 - \frac{m_D^2}{m_{23}^2}, -\beta_{23} \sin \theta_{23}, 0, -\beta_{23} \cos \theta_{23} \right), \quad (A.9)
\]
and hence in the rest frame of $A$ they are

$$p_2 = \frac{\sqrt{m_{23}^2}}{2} \begin{pmatrix} \gamma \left( 1 + \frac{m_2^2}{m_{23}^2} - \frac{m_3^2}{m_{23}^2} \right) + \gamma \beta \beta_{23} \cos \theta_{23} \\
\beta_{23} \sin \theta_{23} \\
0 \\
\gamma \beta \left( 1 + \frac{m_2^2 - m_3^2}{m_{23}^2} \right) + \gamma \beta_{23} \cos \theta_{23} \end{pmatrix},$$

(A.10)

$$p_3 = \frac{\sqrt{m_{23}^2}}{2} \begin{pmatrix} \gamma \left( 1 - \frac{m_2^2}{m_{23}^2} + \frac{m_3^2}{m_{23}^2} \right) - \gamma \beta \beta_{23} \cos \theta_{23} \\
-\beta_{23} \sin \theta_{23} \\
0 \\
\gamma \beta \left( 1 - \frac{m_2^2 + m_3^2}{m_{23}^2} \right) - \gamma \beta_{23} \cos \theta_{23} \end{pmatrix}.

(A.11)

This is sufficient to perform the phase space integration.

A.2 Four-body decay

For the decay $A(p) \rightarrow 1(p_1)2(p_2)3(p_3)4(p_4)$, let us consider both symmetric and asymmetric cascades.

A.2.1 Symmetric Decay

To start with let us first discuss the symmetric decay chain leading to four-body final state. Let us consider decay of the particle $A$ as: $A(p) \rightarrow B^*(q_{12})C^*(q_{34})$, followed by decays of off-shell propagators: $B^*(q_{12}) \rightarrow 1(p_1)2(p_2), C^* \rightarrow 3(p_3)4(p_4)$.
Using our prescription decay width can be written as:

\[
\Gamma(A \to 1234) = \frac{1}{m_A} \int \left[ \frac{1}{\pi} \left( \frac{d\ell_{12}^2}{(m_{12}^2 - m_B^2)^2} \right) \right] \int \left[ \frac{1}{\pi} \left( \frac{d\ell_{34}^2}{(m_{34}^2 - m_C^2)^2} \right) \right] f_s 
\]

\[
\text{Tr} \left[ \tilde{\Gamma}_1(A \to B^* C^*) \tilde{\Gamma}_2(B^* \to 12) \tilde{\Gamma}_3(C^* \to 34) \right].
\]

(A.12)

The \( \tilde{\Gamma} \) functions are expressed as:

\[
\left[ \tilde{\Gamma}_1(A \to B^* C^*) \right]_{ij}^i = \int \frac{d_{PS}^{A \to B^* C^*}}{2} \left[ |\mathcal{M}_1(A \to B^* C^*)|^2 \right]_{ij},
\]

(A.13)

\[
\left[ \tilde{\Gamma}_2(B^* \to 12) \right]_{jk}^j = \int \frac{d_{PS}^{B^* \to 12}}{2} \left[ |\mathcal{M}_2(B^* \to 12)|^2 \right]_{jk},
\]

(A.14)

\[
\left[ \tilde{\Gamma}_3(C^* \to 34) \right]_{ik}^k = \int \frac{d_{PS}^{C^* \to 34}}{2} \left[ |\mathcal{M}_3(C^* \to 34)|^2 \right]_{ik},
\]

(A.15)

where,

\[
\beta \left( \frac{m_{12}^2}{m_A^2}, \frac{m_{34}^2}{m_A^2} \right) = \sqrt{1 - \frac{2 (m_{12}^2 + m_{34}^2)}{m_A^2}} + \frac{(m_{12}^2 - m_{34}^2)^2}{m_A^4},
\]

(A.16)

\[
\beta_{12} \left( \frac{m_1^2}{m_{12}^2}, \frac{m_2^2}{m_{12}^2} \right) = \sqrt{1 - \frac{2 (m_1^2 + m_2^2)}{m_{12}^2}} + \frac{(m_1^2 - m_2^2)^2}{m_{12}^4},
\]

(A.17)

\[
\beta_{34} \left( \frac{m_3^2}{m_{34}^2}, \frac{m_4^2}{m_{34}^2} \right) = \sqrt{1 - \frac{2 (m_3^2 + m_4^2)}{m_{34}^2}} + \frac{(m_3^2 - m_4^2)^2}{m_{34}^4}.
\]

(A.18)

Combining all the contributions, we find

\[
\Gamma(A \to 1234) = \frac{f_s}{m_A} \int \left[ \frac{1}{\pi} \left( \frac{d\ell_{12}^2}{(m_{12}^2 - m_B^2)^2} \right) \right] \int \left[ \frac{1}{\pi} \left( \frac{d\ell_{34}^2}{(m_{34}^2 - m_C^2)^2} \right) \right]
\]

\[
\frac{1}{23} \int \frac{\beta}{8\pi} \frac{d\cos\theta \, d\phi}{2\pi} \int \frac{\beta_{12}}{8\pi} \frac{d\cos\theta_{12} \, d\phi_{12}}{2\pi} \int \frac{\beta_{34}}{8\pi} \frac{d\cos\theta_{34} \, d\phi_{34}}{2\pi} \left[ |\mathcal{M}_1(A \to B^* C^*)|^2 \right]_{ij} |\mathcal{M}_2(B^* \to 12)|^2 \left[ |\mathcal{M}_3(C^* \to 34)|^2 \right]_{ik}.
\]

(A.19)

In the centre-of-mass (CM) frame we have following momenta:

\[
q_{12} = \frac{m_A}{2} \left( 1 + \frac{m_{12}^2}{m_A^2} - \frac{m_{34}^2}{m_A^2}, 0, 0, \beta \right),
\]

(A.20)
Thus after including boost factors one can write down the momenta of 1, 2, 3 and 4, in the CM frame,
\[ q_{34} = \frac{m_A}{2} \left( 1 - \frac{m_{12}^2}{m_A^2} + \frac{m_{34}^2}{m_A^2}, 0, 0, -\beta \right). \] (A.21)

The necessary boost factors from the rest frame of \( q_{12} \) towards the CM frame can be written as:
\[ \gamma_1 = \frac{q_{12}^0}{\sqrt{m_{12}^2}} = \frac{m_A}{2\sqrt{m_{12}^2}} \left( 1 + \frac{m_{12}^2}{m_A^2} - \frac{m_{34}^2}{m_A^2} \right), \] (A.22)
\[ \gamma_1\beta_1 = \frac{q_{12}^0}{\sqrt{m_{12}^2}} |p_{12}| = \frac{m_A}{2\sqrt{m_{12}^2}} \beta. \] (A.23)

One can similarly write the same from the rest frame of \( q_{34} \) as:
\[ \gamma_2 = \frac{q_{34}^0}{\sqrt{m_{34}^2}} = \frac{m_A}{2\sqrt{m_{34}^2}} \left( 1 - \frac{m_{12}^2}{m_A^2} + \frac{m_{34}^2}{m_A^2} \right), \] (A.24)
\[ \gamma_2\beta_2 = \frac{q_{34}^0}{\sqrt{m_{34}^2}} |p_{34}| = \frac{m_A}{2\sqrt{m_{34}^2}} \beta. \] (A.25)

In the rest frame of \( q_{12} \) the momenta components of the particles 1 and 2 can be given as:
\[ \hat{p}_1 = \frac{\sqrt{m_{12}^2}}{2} \left( 1 + \frac{m_{12}^2}{m_{12}^2} - \frac{m_{34}^2}{m_{12}^2}, \beta_{12} \sin \theta_{12}, 0, \beta_{12} \cos \theta_{12} \right), \] (A.26)
\[ \hat{p}_2 = \frac{\sqrt{m_{12}^2}}{2} \left( 1 - \frac{m_{12}^2}{m_{12}^2} + \frac{m_{34}^2}{m_{12}^2}, -\beta_{12} \sin \theta_{12}, 0, -\beta_{12} \cos \theta_{12} \right). \] (A.27)

The momenta of 3 and 4, in the rest frame of \( q_{34} \), can be given as:
\[ \hat{p}_3 = \frac{\sqrt{m_{34}^2}}{2} \left( 1 + \frac{m_{34}^2}{m_{34}^2} - \frac{m_{34}^2}{m_{34}^2}, \beta_{34} \sin \theta_{34} \cos \phi_{34}, \beta_{34} \sin \theta_{34} \sin \phi_{34}, \beta_{34} \cos \theta_{34} \right), \] (A.28)
\[ \hat{p}_4 = \frac{\sqrt{m_{34}^2}}{2} \left( 1 - \frac{m_{34}^2}{m_{34}^2} + \frac{m_{34}^2}{m_{34}^2}, -\beta_{34} \sin \theta_{34} \cos \phi_{34}, -\beta_{34} \sin \theta_{34} \sin \phi_{34}, -\beta_{34} \cos \theta_{34} \right). \] (A.29)

Thus after including boost factors one can write down the momenta of 1, 2, 3 and 4, in the CM frame, as:
\[ p_1 = \frac{\sqrt{m_{12}^2}}{2} \begin{bmatrix} \gamma_1 \left( 1 + \frac{m_{12}^2}{m_{12}^2} - \frac{m_{34}^2}{m_{12}^2} \right) + \gamma_1 \beta_1 \beta_{12} \sin \theta_{12} \\ 0 \\ \gamma_1 \beta_1 \left( 1 + \frac{m_{12}^2}{m_{12}^2} - \frac{m_{34}^2}{m_{12}^2} \right) + \gamma_1 \beta_{12} \cos \theta_{12} \end{bmatrix}, \] (A.30)
\[ p_2 = \frac{\sqrt{m_{12}^2}}{2} \begin{bmatrix} \gamma_1 \left( 1 - \frac{m_{12}^2}{m_{12}^2} + \frac{m_{34}^2}{m_{12}^2} \right) - \gamma_1 \beta_1 \beta_{12} \cos \theta_{12} \\ -\beta_{12} \sin \theta_{12} \\ \gamma_1 \beta_1 \left( 1 - \frac{m_{12}^2}{m_{12}^2} + \frac{m_{34}^2}{m_{12}^2} \right) - \gamma_1 \beta_{12} \cos \theta_{12} \end{bmatrix}, \] (A.31)
respectively.

These details are sufficient to perform the phase space integration and to compute the decay width for the full cascade.

\subsection*{A.2.2 Asymmetric Decay}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{asymmetric_cascade}
\caption{Representative diagram for the asymmetric four-body cascade $A \to 1234$.}
\end{figure}

Let us consider the identical decay $A \to 1234$ here but with a different cascade topology; this is an asymmetric cascade (Fig. 8), as both the off-shell propagators are appearing in the same chain: $A(p) \to 1(p_1)B^*(k_1)$, $B^*(k_1) \to 2(p_2)C^*(k_2)$, $C^*(k_2) \to 3(p_3)4(p_4)$.

According to our proposal, we write the decay width as

$$
\Gamma(A \to 1234) = \frac{f_s}{m_A} \int \left[ \frac{1}{\pi} \left( \frac{d\theta_1}{(m_{12}^2 - m_{13}^2)^2} \right) \right] \int \left[ \frac{1}{\pi} \left( \frac{d\theta_2}{(m_{23}^2 - m_{13}^2)^2} \right) \right] \times \text{Tr} \left[ \tilde{\Gamma}_1(A \to B^*1)\tilde{\Gamma}_2(B^* \to C^*2)\tilde{\Gamma}_3(C^* \to 34) \right],
$$

where the $\tilde{\Gamma}$ functions are given as:

$$
\left[ \tilde{\Gamma}_1(A \to B^*1) \right]_{ij} = \int \frac{dA_{B^*1}}{2} \left[ |M_1(A \to B^*1)|^2 \right]_{ij}
$$

$$
= \frac{1}{2} \int \frac{d\phi}{2\pi} \frac{d\phi}{2\pi} \frac{d\phi}{2\pi} \frac{d\phi}{2\pi} \left[ |M_1(A \to B^*1)|^2 \right]_{ij},
$$

(A.34)
\[
\begin{align*}
\left[ \tilde{\Gamma}_2 (B^* \to C^* 2) \right]_k^j &= \int \frac{d^{B^* \to C^* 2}}{2} \left[ |M_2 (B^* \to C^* 2)| \right]^j_k
= \frac{1}{2} \int \frac{\beta_2 \cos \theta_1 \, d\phi_1}{8\pi} \left[ |M_2 (B^* \to C^* 2)| \right]_k^j, \\
\left[ \tilde{\Gamma}_3 (C^* \to 34) \right]^i_k &= \int \frac{d^{C^* \to 34}}{2} \left[ |M_3 (C^* \to 34)| \right]^i_k
= \frac{1}{2} \int \frac{\beta_3 \cos \theta_2 \, d\phi_2}{8\pi} \left[ |M_3 (C^* \to 34)| \right]^i_k.
\end{align*}
\] (A.36)

The boost factors are written as:

\[
\begin{align*}
\tilde{\beta}_1 \left( \frac{m_1^2}{m_A^2}, \frac{m_{12}^2}{m_A^2} \right) &= \sqrt{1 - \frac{2 (m_1^2 + m_{12}^2)}{m_A^2} + \frac{(m_1^2 - m_{12}^2)^2}{m_A^2}}, \\
\tilde{\beta}_2 \left( \frac{m_2^2}{m_{12}^2}, \frac{m_{23}^2}{m_{12}^2} \right) &= \sqrt{1 - \frac{2 (m_2^2 + m_{23}^2)}{m_{12}^2} + \frac{(m_2^2 - m_{23}^2)^2}{m_{12}^2}}, \\
\tilde{\beta}_3 \left( \frac{m_3^2}{m_{23}^2}, \frac{m_4^2}{m_{23}^2} \right) &= \sqrt{1 - \frac{2 (m_3^2 + m_4^2)}{m_{23}^2} + \frac{(m_3^2 - m_4^2)^2}{m_{23}^2}}.
\end{align*}
\] (A.38)

To explain the phase space structure we define $S_A$, $S_B$ and $S_C$ to be the rest frames of $A$, $B^*$ and $C^*$ respectively, see Fig. 9.

In $S_A$, the components of momenta of 1 and $B^*$ are given as:

\[
\begin{align*}
p_1 &= \frac{m_A}{2} \left( 1 + \frac{m_1^2}{m_A^2} - \frac{m_{12}^2}{m_A^2}, 0, 0, -\beta_1 \right), \\
k_1 &= \frac{m_A}{2} \left( 1 - \frac{m_1^2}{m_A^2} + \frac{m_{12}^2}{m_A^2}, 0, 0, \beta_1 \right).
\end{align*}
\] (A.41)
The same for 2 and \(C^*\) in \(S_B\) are given as:

\[
\hat{p}_2 = \frac{\sqrt{m^2_{12}}}{2} \left( 1 + \frac{m^2_2}{m^2_{12}} - \frac{m^2_3}{m^2_{12}}, -\bar{\beta}_2 \sin \theta_1, 0, -\bar{\beta}_2 \cos \theta_1 \right),
\]

\[
\hat{k}_2 = \frac{\sqrt{m^2_{12}}}{2} \left( 1 - \frac{m^2_2}{m^2_{12}} + \frac{m^2_3}{m^2_{12}}, \bar{\beta}_2 \sin \theta_1, 0, \bar{\beta}_2 \cos \theta_1 \right),
\]

and similarly for 3 and 4 in \(S_C\) the momenta are

\[
\hat{p}_3 = \frac{\sqrt{m^2_{23}}}{2} \left( 1 + \frac{m^2_2}{m^2_{23}} - \frac{m^2_3}{m^2_{23}}, \bar{\beta}_3 \sin \theta_2 \cos \phi_2, \bar{\beta}_3 \sin \theta_2 \sin \phi_2, \bar{\beta}_3 \cos \theta_2 \right),
\]

\[
\hat{p}_4 = \frac{\sqrt{m^2_{23}}}{2} \left( 1 - \frac{m^2_2}{m^2_{23}} + \frac{m^2_3}{m^2_{23}}, -\bar{\beta}_3 \sin \theta_2 \cos \phi_2, -\bar{\beta}_3 \sin \theta_2 \sin \phi_2, -\bar{\beta}_3 \cos \theta_2 \right).
\]

The necessary boost factors from \(S_A\) to \(S_B\) are

\[
\beta' = \frac{|k^0_1|}{k^0_1} \hat{z}, \quad \gamma' = \frac{k^0_1}{\sqrt{m^2_{12}}}, \quad \gamma' \beta' = \frac{m_A \bar{\beta}_1}{2 \sqrt{m^2_{12}}},
\]

and the Lorentz transformation matrix from \(S_B\) to \(S_A\) is

\[
A' = \begin{pmatrix}
\gamma' & 0 & 0 & \gamma' \beta' \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\gamma' \beta' & 0 & 0 & \gamma'
\end{pmatrix}.
\]

Thus the boost factors from \(S_B\) to \(S_C\) are

\[
\beta'' = \frac{|k^0_2|}{k^0_2} (\sin \theta_1 \hat{x} + \cos \theta_1 \hat{z}), \quad \gamma'' \beta'' = \frac{\sqrt{m^2_{12}}}{2} \frac{\bar{\beta}_2}{\sqrt{m^2_{23}}}.
\]

Using the velocity addition theorem we can write down the components of the boost from \(S_A\) to \(S_C\) as:

\[
\beta_x = \frac{\beta'' x}{\gamma' (1 + \beta' \beta'')}, \quad \beta_y = \frac{\beta'' y}{\gamma' (1 + \beta' \beta'')}, \quad \beta_z = \frac{\beta'' z}{1 + \beta' \beta''}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2_x - \beta^2_z}}.
\]

Now combining all these we can write down the Lorentz transformation matrix from \(S_C\) to \(S_A\):

\[
A = \begin{pmatrix}
\gamma & \gamma \beta_x & 0 & \gamma \beta_z \\
\gamma \beta_x & 1 + (\gamma - 1) \frac{\beta^2_x}{\beta^2_z} & 0 & (\gamma - 1) \frac{\beta_x \beta_z}{\beta^2_z} \\
0 & 0 & 1 & 0 \\
\gamma \beta_z & (\gamma - 1) \frac{\beta_x \beta_z}{\beta^2_z} & 0 & 1 + (\gamma - 1) \frac{\beta^2_z}{\beta^2_z}
\end{pmatrix}.
\]
This helps us to compute the momenta of 2 and \(C^*\) in the rest frame of \(A\), i.e., \(S_A\), as:

\[
p_2 = \frac{\gamma' \sqrt{m^2_{21}}}{2} \delta \left( \begin{array}{c} \gamma' \left( 1 + \frac{m^2_2}{m^2_{12}} - \frac{m^2_3}{m^2_{12}} \right) - \gamma' \beta_2 \beta_2 \cos \theta_1 \\ \beta_2 \sin \theta_1 \\ 0 \end{array} \right),
\]

\[
p_3 = \frac{\sqrt{m^2_{23}}}{2} \delta \left( \begin{array}{c} \gamma \left[ \left( 1 + \frac{m^2_2}{m^2_{23}} - \frac{m^2_3}{m^2_{23}} \right) + \beta_x \beta_3 \sin \theta_2 \cos \phi_2 + \beta_z \beta_3 \cos \theta_2 \right] \\ \beta_3 \sin \theta_2 \sin \phi_2 \\ 0 \end{array} \right),
\]

\[
p_4 = \frac{\sqrt{m^2_{23}}}{2} \delta \left( \begin{array}{c} \gamma \left[ \left( 1 - \frac{m^2_2}{m^2_{23}} + \frac{m^2_3}{m^2_{23}} \right) - \beta_x \beta_3 \sin \theta_2 \cos \phi_2 - \beta_z \beta_3 \cos \theta_2 \right] \\ -\beta_3 \sin \theta_2 \sin \phi_2 \\ 0 \end{array} \right),
\]

and, that for 3 and 4 as:

\[
p_3 = \frac{\sqrt{m^2_{23}}}{2} \delta \left( \begin{array}{c} \gamma \left[ \left( 1 + \frac{m^2_2}{m^2_{23}} - \frac{m^2_3}{m^2_{23}} \right) + \beta_x \beta_3 \sin \theta_2 \cos \phi_2 + \beta_z \beta_3 \cos \theta_2 \right] \\ \beta_3 \sin \theta_2 \sin \phi_2 \\ 0 \end{array} \right),
\]

\[
p_4 = \frac{\sqrt{m^2_{23}}}{2} \delta \left( \begin{array}{c} \gamma \left[ \left( 1 - \frac{m^2_2}{m^2_{23}} + \frac{m^2_3}{m^2_{23}} \right) - \beta_x \beta_3 \sin \theta_2 \cos \phi_2 - \beta_z \beta_3 \cos \theta_2 \right] \\ -\beta_3 \sin \theta_2 \sin \phi_2 \\ 0 \end{array} \right),
\]

respectively. One can now compute the integration numerically using all these momenta in the \(S_A\) frame.

### A.3 Interference of amplitudes of four-body decay through symmetric and asymmetric channels

So far we have discussed the solitary contributions to the four body phase space from either symmetric or asymmetric cascade. We have mentioned repeated that it is indeed possible to have more than one cascade decay chain leading to same final states. Now if both the cascades are either symmetric or asymmetric then the phase space for the interference diagrams would be trivial. If one of them is symmetric and other one is asymmetric, then we can have two possible structures depending on the positions of the external particles: (i) same for both cascades, (ii) shuffled among themselves. We have discussed the earlier case in detail in text.

Here, we are providing a brief sketch of structures of the interference term for the latter scenario. Note that unlike Figs. (8) and (9), here in Figs. (10) and (11), the off-shell propagators are different and also the positions of the external particles.
To explain this possibility let us consider the following four-body decay: $A(p) \rightarrow 1(p_1)2(p_2)3(p_3)4(p_4)$ which can be achieved through two different cascades. One of them is symmetric: $A(p) \rightarrow B^*(k_1)C^*(k_2)$, $B^*(k_1) \rightarrow 1(p_1)2(p_2)$, $C^*(k_2) \rightarrow 3(p_3)4(p_4)$, see Fig. (10) and the other one is asymmetric: $A(p) \rightarrow 4(p_4)X^*(k_3)$, $X^*(k_3) \rightarrow 2(p_2)Y^*(k_4)$, $Y^*(k_4) \rightarrow 3(p_3)1(p_1)$, see Fig. (11).

The interference contribution to the decay width can be given as:

$$\Gamma_{int}(A \rightarrow 1234) = \frac{1}{m_A} \int \left[ \frac{1}{\pi} \left( \frac{dm_{12}^2}{(m_{12}^2 - m_B^2)(m_{13}^2 - m_C^2)} \right) \int \left[ \frac{1}{\pi} \left( \frac{dm_{34}^2}{(m_{34}^2 - m_C^2)(m_{123}^2 - m_X^2)} \right) \right] \right] \times \left[ \frac{\bar{\beta}}{8\pi} \frac{d\cos \theta d\phi}{2} \int \left[ \frac{\bar{\beta}_{12}}{8\pi} \frac{d\cos \theta_{12} d\phi_{12}}{2} \int \left[ \frac{\bar{\beta}_{34}}{8\pi} \frac{d\cos \theta_{34} d\phi_{34}}{2} \int \frac{M_1(A \rightarrow B^* C^*) M_2(B^* \rightarrow 1 2) M_3(C^* \rightarrow 3 4) \dagger}{M_4(A \rightarrow 4 X^*) M_5(X^* \rightarrow 2 Y^*) M_6(Y^* \rightarrow 3 4) \dagger} \right] \right] \right], \quad (A.56)
where the necessary kinematical variables are \( k_1^2 \equiv m_{12}^2 = (p_1 + p_2)^2 \), \( k_2^2 \equiv m_{34}^2 = (p_3 + p_4)^2 \), \( k_3^2 \equiv m_{123}^2 = (p_1 + p_2 + p_3)^2 \) and \( k_4^2 \equiv m_{13}^2 = (p_1 + p_3)^2 \). Now one can use the explicit forms of the momentum variables, like \( p_1 \), \( p_2 \), and \( p_3 \) as given in Eqns. (A.30), (A.31) and (A.32) respectively, and then \( m_{13}^2 \) and \( m_{123}^2 \) can be written in terms of \( m_{12}^2 \), \( m_{34}^2 \) and, \( \theta's \) and \( \phi's \) to perform phase space integration.

### B Interaction couplings

Here, we have tabulated the interactions that we have used in our toy examples:

| Interactions | Couplings |
|--------------|-----------|
| \( Hq\bar{q} \) | \(-i g m_q / 2 m_W \) |
| \( HW^{\mu} W^{\nu} \) | \( ig m_W \eta^{\mu \nu} \) |
| \( W q_1 q_2 \) | \( i g \sqrt{2} V_{q_1 q_2} \) |
| \( W f_1 \bar{f}_2 \) | \( i g \sqrt{2} \) |
| \( \Phi q_1 \bar{q}_2 \) | \(-i g_{q_1 q_2} \) |
| \( \Phi f_1 \bar{f}_2 \) | \(-i g_{f_1 f_2} \) |

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