ON THE ENERGY FUNCTIONAL ON FINSLER MANIFOLDS
AND APPLICATIONS TO STATIONARY SPACETIMES

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Abstract. In this paper we first study some global properties of the energy
functional on a non-reversible Finsler manifold. In particular we present a
fully detailed proof of the Palais–Smale condition under the completeness of
the Finsler metric. Moreover we define a Finsler metric of Randers type,
which we call Fermat metric, associated to a conformally standard stationary
spacetime. We shall study the influence of the Fermat metric on the causal
properties of the spacetime, mainly the global hyperbolicity. Moreover we
study the relations between the energy functional of the Fermat metric and
the Fermat principle for the light rays in the spacetime. This allows us to
obtain existence and multiplicity results for light rays, using the Finsler theory.
Finally the case of timelike geodesics with fixed energy is considered.

1. Introduction

In the recent years there has been an increasing interest in the study of Finsler
Geometry, both from the theoretical point of view and for the applications to many
fields of Physics. We mention the study of the multiplicity of geometrically distinct
closed geodesics, which presents different features with respect to Riemannian Ge-
ometry as shown by the Katok’s example (see [19]) and the study of the Zermelo
navigation problem which has led to a classification of Randers metrics with con-
stant flag curvature, see [6]. Finsler Geometry has also found many applications to
applied sciences as Biology, Classical and Quantum Optics, Relativity and Quan-
tum Gravity. We refer to the monographs [3],[4] and to the more recent papers
[11], [16], [29].

We recall some basic facts about Finsler manifolds and we refer to [5] for any
further information. Let \( M \) be a smooth, real, paracompact manifold of finite
dimension. A Finsler structure on \( M \) is a function \( F: TM \to [0, +\infty) \) which is
continuous on \( TM \), \( C^\infty \) on \( TM \setminus 0 \), vanishing only on the zero section, fiberwise
positively homogeneous of degree one, i.e. \( F(x, \lambda y) = \lambda F(x, y) \), for all \( x \in M \),

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$y \in T_xM$ and $\lambda > 0$, and which has fiberwise strictly convex square i.e. the matrix
\[ g_{ij}(x, y) = \left[ \frac{1}{2} \frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y) \right] \]
is positive definite for any $(x, y) \in TM \setminus 0$. The tensor
\[ g = g_{ij}(x, y)dx^i \otimes dx^j \]
(here and throughout the paper we adopt the Einstein summation convention) is the so called fundamental tensor of the Finsler manifold $(M, F)$; it is a symmetric section of the tensor bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$, where $\pi^*(T^*M)$ is the dual of the pulled-back tangent bundle $\pi^*TM$ over $TM \setminus 0$ ($\pi$ is the projection $TM \to M$).

**Remark 1.1.** We stress that, by homogeneity, $F^2$ is $C^1$ on $TM$ and it reduces to the square of the norm of a Riemannian metric if and only its second order fiber derivatives are continuous up to the zero section (see [35]).

**Remark 1.2.** Since $F$ is only positive homogeneous of degree 1, we have that, in general, $F(x, y) \neq F(x, -y)$. If for all $(x, y) \in TM$, $F(x, y) = F(x, -y)$, the Finsler metric $F$ is said reversible. The number $\lambda(x) = \max_{y \in T_xM} \{F(x, y) : F(x, y) = 1\}$ (see [32]) gives a measure of non reversibility for a Finsler metric.

The components $g_{ij}$ of the fundamental tensor define the formal Christoffel symbols $\gamma^i_{jk}$,
\[ \gamma^i_{jk}(x, y) := \frac{1}{2} g^{is} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right), \]
and the Cartan tensor
\[ A_{ijk}(x, y) := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4} \frac{\partial^3(F^2)}{\partial y^i \partial y^j \partial y^k}, \]
for all $(x, y) \in TM \setminus 0$. From Equation (3), we see that the components $A_{ijk}(x, y)$ are totally symmetric in $(i, j, k)$ and they are positively homogeneous of degree 0 in the $y$ variable.

The Chern connection $\nabla$ is the unique linear connection on $\pi^*TM$ whose connection 1-forms $\omega^j_i$ satisfy the following structural equations:
\[ dx^j \wedge \omega^j_i = 0 \quad \text{torsion free}, \quad (4) \]
\[ dg_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = \frac{2}{F} A_{ij}, \delta y^s \quad \text{almost } g\text{-compatibility}, \quad (5) \]
where $\delta y^s$ are the 1-forms on $\pi^*TM$ given as $\delta y^s := dy^s + N^s_j dx^j$, and
\[ N^j_i(x, y) := \gamma^i_{jk} y^k - \frac{1}{F} A_{ijk}^s y^r y^s \]
are the coefficients of the so called nonlinear connection on $TM \setminus 0$. The components of the Chern connection are given by:
\[ \Gamma^i_{jk}(x, y) = \gamma^i_{jk} - \frac{g^{ij}}{F} \left( A_{ijs} N^s_k - A_{jsk} N^s_i + A_{iks} N^s_j \right). \]
Clearly $\Gamma^i_{jk}(x, y)$ are defined on $TM \setminus 0$ and they are positively homogeneous of degree 0 with respect to $y$. 

Let $\gamma = \gamma(s)$ be a smooth regular curve on $M$, with velocity field $T = \dot{\gamma}$, and $W$ be a smooth vector field along $\gamma$. The Chern connection defines two different covariant derivatives $D_T W$ along $\gamma$:

$$D_T W = \left( \frac{dW_i}{dt} + W^j T^k \Gamma^i_{jk}(\gamma, T) \right) \frac{\partial}{\partial x^i} $$

with reference vector $T$, (7)

$$D_T W = \left( \frac{dW_i}{dt} + W^j T^k \Gamma^i_{jk}(\gamma, W) \right) \frac{\partial}{\partial x^i} $$

with reference vector $W$.

A geodesic of the Finsler manifold $(M, F)$ is a smooth regular curve $\gamma$ satisfying the equation

$$D_T \left( \frac{T}{F(\gamma, T)} \right) = 0,$$

with reference vector $T = \dot{\gamma}$. A curve $\gamma = \gamma(s)$ is said to have constant speed if $F(\gamma(s), \dot{\gamma}(s))$ is constant along $\gamma$. Constant speed geodesics satisfy the equation

$$D_T T = 0,$$

with reference vector $T = \dot{\gamma}$. The length of a piecewise smooth curve $\gamma: [a, b] \subset \mathbb{R} \to M$ with respect to the Finsler structure $F$ is defined by

$$L(\gamma) = \int_a^b F(\gamma(s), \dot{\gamma}(s)) ds.$$

Thus the distance between two arbitrary points $p, q \in M$ is given by

$$\text{dist}(p, q) = \inf_{\gamma \in C(p, q)} L(\gamma),$$

(9)

where $C(p, q)$ is the set of all piecewise smooth curves $\gamma: [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. The distance function (9) is nonnegative and satisfies the triangle inequality, but it is not symmetric as $F$ is non-reversible. Thus one has to distinguish the order of a pair of points in $M$ when speaking about distance. As a consequence, one is naturally led to the notions of forward and backward metric balls, spheres, Cauchy sequences and completeness (see [5, §6.2]). For instance: the forward metric ball $B^+_r(p)$ (resp. backward $B^-_r(p)$) of center $p \in M$ and radius $r \geq 0$ is given by all the points $x \in M$ such that $\text{dist}(p, x) < r$ (resp. $\text{dist}(x, p) < r$); a sequence $(x_n) \subset M$ is called forward (resp. backward) Cauchy sequence if for all $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that, for all $\nu \leq i \leq j$, $\text{dist}(x_i, x_j) \leq \varepsilon$ (resp. $\text{dist}(x_j, x_i) \leq \varepsilon$); $(M, F)$ is forward (resp. backward) complete if all forward (resp. backward) Cauchy sequences are convergent; $(M, F)$ is said forward (resp. backward) geodesically complete if every geodesic $\gamma: [a, b] \to M$ (resp. $\gamma: (b, a] \to M$) can be extended to a geodesic defined on the interval $[a, +\infty)$ (resp. $(-\infty, a]$). What is relevant here is that the topologies generated by the forward and the backward metric balls coincide with the underlying manifold topology; moreover a suitable version of Hopf-Rinow theorem holds (see [5, Theorem 6.6.1]) stating the equivalence of the notions of forward (or backward) completeness and the compactness of closed and forward (or backward) bounded subsets of $M$ and implying the existence of a geodesic connecting any pair of points in $M$ and minimizing the Finslerian distance. It is worth recalling that the two notions of completeness are not equivalent (see for example [5, §12.6.D]).

As in Riemannian Geometry, geodesics on a Finsler manifold $(M, F)$ satisfy a variational principle. First of all a curve is a geodesic for the Finsler metric $F$ if and
only if it minimizes the length between two sufficiently close points on the curve, see [5]. Moreover a smooth curve \( x: [a, b] \to M \) is a constant speed geodesic if and only if it is a stationary point of the energy functional

\[
J(x) = \frac{1}{2} \int_a^b F^2(x, \dot{x}) \, ds
\]

on the space of sufficiently smooth curves on the manifold \( M \) joining the points \( x(a) \) and \( x(b) \) (for more general boundary conditions see Section 2).

In this paper we shall study the main properties of the energy functional in the infinite dimensional setting of the Sobolev-Hilbert manifold of \( H^1 \) curves satisfying very general boundary conditions, containing the classical two points and periodic boundary conditions. In particular we shall present a fully detailed proof of the Palais–Smale condition for the energy functional. In the second part of the paper we present a new application of Finsler Geometry to General Relativity. In the class of conformally standard stationary spacetimes we define a Finsler metric of Randers type, which we call Fermat metric. The choice of this definition is due to the fact that this metric is strictly related to the Fermat principle of light rays in this class of spacetimes. We shall also show that the causal structure of a conformally stationary spacetime is influenced by the global properties of the Fermat metric. In particular the global hyperbolicity of the metric is strictly related to the completeness of the Fermat metric. Moreover the equivalence between the Fermat principle of light rays and the geodesic problem for the Fermat metrics allows one to obtain multiplicity results for light rays as an application of the analogous results in the Finsler setting. These results allow us to obtain a mathematical model of the gravitational lens effect on conformally stationary spacetimes. Finally analogous results for timelike geodesics on a stationary spacetime are presented.

2. The energy functional in Finsler Geometry

In this section we shall study the energy functional of a Finsler manifold \( (M, F) \) in the infinite dimensional setting of Hilbert-Sobolev manifolds. We recall that the infinite dimensional setting for the energy functional and the variational theory for geodesics on a Riemannian manifold was introduced by R. Palais in the paper [25] and extended by F. Mercuri to Finsler manifolds in the paper [23]. Here we shall prove in all the details that the critical points of the energy functional, defined on a manifold of curves satisfying boundary conditions (12), are smooth and they are exactly the geodesics satisfying (12) and parametrized with constant speed.

Let \( (M, F) \) be a forward or backward complete Finsler manifold and let us endow \( M \) with any complete Riemannian metric \( h \). Let \( N \) be a smooth submanifold of \( M \times M \). We consider the collection \( \Lambda_N(M) \) of curves \( x \) on \( M \) parameterized on the interval \([0, 1]\) with endpoints \((x(0), x(1))\) belonging to \( N \) and having \( H^1 \) regularity, that is, \( x \) is absolutely continuous and the integral \( \int_0^1 h(x)[\dot{x}, \dot{x}] \, ds \) is finite. It is well known that \( \Lambda_N(M) \) is a Hilbert manifold modeled on any of the equivalent Hilbert spaces of all the \( H^1 \) sections, with endpoints in \( TN \), of the pulled back bundle \( x^*TM \), \( x \) any regular curve in \( \Lambda_N(M) \), [20, Proposition 2.4.1]. In fact, the scalar product is given by

\[
\langle X, Y \rangle_1 = \int_0^1 h(x)[X, Y] \, ds + \int_0^1 h(x)[\nabla^h_x X, \nabla^h_y Y] \, ds,
\]  

(10)
for every $H^1$ sections, $X$ and $Y$ of $x^*TM$, $\nabla_x^h$ being the covariant derivative along $x$ associated to the Levi-Civita connection of the metric $h$.

Let us denote the function $F^2$ by $G$ and let us consider the energy functional

$$J(x) = \frac{1}{2} \int_0^1 G(x, \dot{x}) \, ds$$

(11)
of the Finsler manifold $(M, F)$, defined on the manifold $\Lambda_N(M)$. The functional $J$ is $C^{2-}$ on $\Lambda_N(M)$, i.e. it is $C^1$ with locally Lipschitz differential (see [23, Theorem 4.1]). A critical point $\gamma$ of $J$ is a curve $\gamma \in \Lambda_N(M)$ such that $dJ(\gamma) = 0$.

We first study the regularity of critical points for $J$. We shall show in all the details that, in spite of the loss of regularity of the Lagrangian function $G$ on the zero section, the $H^1$–critical points of $J$ are smooth curves.

We start by computing the differential of $J$ on $\Lambda_N(M)$ to show that a non constant critical point is a geodesic satisfying the boundary conditions

$$g(\gamma(0), \dot{\gamma}(0))[V, \dot{\gamma}(0)] = g(\gamma(1), \dot{\gamma}(1))[W, \dot{\gamma}(1)],$$

(12)
where $g$ is the fundamental tensor of the Finsler metric $F$ defined in (1) and $(V, W) \in T_{\gamma(0), \gamma(1)}N$.

**Remark 2.1.** Before the next Lemma, let us see what the boundary conditions (12) become in some particular cases:

(i) Let $\triangle$ be the diagonal in $M \times M$ and $N = \triangle$. From (1) and the Euler theorem for homogeneous functions, we know that $\partial_y G(x, y) = 2g(x, y)[\cdot, y]$, for any $(x, y) \in TM$. Hence, from $\dot{\gamma}(0) = \dot{\gamma}(1)$ and (12) we get

$$\partial_y G(\gamma(0), \dot{\gamma}(0)) = \partial_y G(\gamma(1), \dot{\gamma}(1)).$$

Since the map $y \mapsto \partial_y G(x, y)$ is an injective map (see the proof of Proposition 2.3 below), it has to be $\dot{\gamma}(0) = \dot{\gamma}(1)$.

(ii) Let $M_0$ and $M_1$ be two submanifolds of $M$ and $N = M_0 \times M_1$. In (12) take $W = 0$. Then, for any $V \in T_{\gamma(0)}M_0$ we get $g(\gamma(0), \dot{\gamma}(0))[V, \dot{\gamma}(0)] = 0$. Analogously taking $V = 0$, it has to be $g(\gamma(1), \dot{\gamma}(1))[W, \dot{\gamma}(0)] = 0$, for any $W \in T_{\gamma(1)}M_1$.

**Lemma 2.2.** Let $\gamma: [0, 1] \to M$ be a smooth regular curve and $\sigma: [0, 1] \times [-\varepsilon, \varepsilon] \to M$, $\sigma = \sigma(t, u)$ be a smooth regular variation of $\gamma$ (i.e. $\sigma(t, 0) = \gamma(t)$ for all $t \in [0, 1]$) with variation vector field $U = \partial_u \sigma$. Then

$$\partial_u (g(\sigma, T)[T, T]) = 2g(\sigma, T)[T, DU T],$$

(13)
where $T = \partial_t \sigma$ and the covariant derivative $DU T$ is with reference vector $T$ (see formula (7)).

**Proof.** From the symmetry of the $g_{i,j}$ we get

$$\partial_u (g_{ij}(\sigma, T)T^i T^j) =$$

$$= \partial_x^k (g_{ij}(\sigma, T)) U^k T^i T^j + \partial_y^k (g_{ij}(\sigma, T)) (\partial_u T)^k T^i T^j$$

$$+ 2g_{ij}(\sigma, T)T^i (\partial_u T)^j,$$

(14)

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1Though in that paper $J$ is defined on $\Lambda_{\triangle}(M)$, where $\triangle$ is the diagonal in $M \times M$, the $C^{2-}$ regularity of $J$ on $\Lambda_N(M)$ can be carried out along the same lines.
We are going to show that $\gamma$ vector field from Eq. (3)

Now we recall that $U$ the support of $\gamma$ which evaluated at $u$

This equation can be extended by density to any curve $\gamma \in \Lambda_N(M)$ satisfying (12) if and only if it is a (non constant) critical point of $J$.

Proof. Let $\gamma: [0, 1] \to M$ be a smooth curve in $\Lambda_N(M)$ and $Z \in T_{\gamma} \Lambda_N(M)$ be a smooth vector field along $\gamma$. Choose a smooth variation $\sigma: [0, 1] \times [-\epsilon, \epsilon] \to M$, $\sigma = \sigma(t, u)$ of $\gamma$ with variation vector field $U = \partial_u \sigma$ having endpoints in $TN$ and such that $U(t, 0) = Z(t)$ for all $t \in [0, 1]$. Moreover we set $T = \partial_t \sigma$. We will cover the support of $\gamma$ by a finite number of local charts $\{(V_\kappa, \varphi_\kappa)\}$ of the manifold $M$ so that the variation of $J$ will be computed using the systems of coordinates induced on $TM$. With abuse of notations we shall not change the symbols denoting the points, the vectors and the forms in such coordinate systems and we shall omit the sum symbol in the integrands. Since $G$ is $C^1$ on $TM$ (see Remark 1.1) and using the equality $\partial_u T = \partial_u U$, we get

\[
\frac{d}{du} J(\sigma) = \frac{1}{2} \int_0^1 (\partial_u G(\sigma, T)[U] + \partial_y G(\sigma, T)[\partial_u T])dt
\]

which evaluated at $u = 0$ gives

\[
dJ(\gamma)[Z] = \frac{1}{2} \int_0^1 (\partial_u G(\gamma, \dot{\gamma})[Z] + \partial_y G(\gamma, \dot{\gamma})[\dot{Z}])dt. \tag{18}
\]

This equation can be extended by density to any curve $\gamma \in \Lambda_N(M)$ and to any vector field $Z \in T_{\gamma} \Lambda_N(M)$. Now assume that $\gamma \in \Lambda_N(M)$ is a critical point of $J$. We are going to show that $\gamma$ is a smooth curve. Evaluating (18) on any smooth
vector field $Z$ with compact support in the interval $I_k = \gamma^{-1}(V_k) = (t_k, t_{k+1}) \subset [0,1]$ we get

$$\int_{I_k} (H + \partial_y G(\gamma, \dot{\gamma}))[\dot{Z}] dt = 0,$$

where $H = H(t)$ is the $T^*M$ valued function

$$H(t) = -\int_{t_k}^{t} (\partial_x G(\gamma, \dot{\gamma})) ds.$$

Last integration has only a local sense, since it consists of the integrals of the components of the covector $\partial_x G(\gamma, \dot{\gamma})$ along the curve $\gamma$. Moreover, equation (19) implies that there exists a constant covector $W \in \mathbb{R}^n$, with $n = \dim M$, such that

$$H(t) + \partial_y G(\gamma(t), \dot{\gamma}(t)) = W,$$

a. e. on $I_k$; since $H$ is continuous, the function $t \in I_k \mapsto \partial_y G(\gamma(t), \dot{\gamma}(t))$ is also continuous. Now fix $x \in M$ and consider the map $\mathcal{L}_x := y \in T_x M \setminus \{0\} \mapsto \partial_y G(x, y) \in T^*_x M$. Since $G$ vanishes only on the zero section and is positively homogeneous of degree 2 in $y$, by Euler’s theorem also $\partial_y G(x, y)$ is the map of constant value 0 if and only if $y = 0$. Hence $\mathcal{L}_x$ assumes values in $T^*_x M \setminus \{0\}$. Being $G$ fiberwise strictly convex on $T^*_M \setminus 0$, $\mathcal{L}_x$ is locally invertible on $T_x M \setminus \{0\}$. Moreover, as $\mathcal{L}_x$ is positively homogeneous of degree 1, it is a proper map and therefore it is a homeomorphism from $T_x M \setminus \{0\}$ onto $T^*_x M \setminus \{0\}$ (see [2, Theorem 1.7, p. 47]). Since the inverse of a homogeneous function of degree 1 is homogeneous of degree 1 and $\mathcal{L}_x(y) = 0$ if and only if $y = 0$, we obtain that $\mathcal{L}_x$ is a homeomorphism from $T_x M$ onto $T^*_x M$. Now consider the maps $\Phi := (x, y) \in TM \mapsto (x, \mathcal{L}_x(y)) \in T^*_M$ and $\Psi := (x, w) \in T^*_M \mapsto (x, \mathcal{L}_x^{-1}(w)) \in TM$. As $\partial_y G$ is positive definite on $TM \setminus 0$, from the inverse function theorem $\Phi$ is locally a homeomorphism on $TM \setminus 0$ and $\Phi^{-1} = \Psi$ on $T^*_M \setminus 0$. Therefore the map $(x, w) \in T^*_M \setminus 0 \mapsto \mathcal{L}_x^{-1}(w)$ is continuous and the continuity extends up to the zero section. In fact if $(x_n, w_n) \to (x, 0)$ then

$$\mathcal{L}_x^{-1}(w_n) = |w_n| \mathcal{L}_x^{-1}\left(\frac{w_n}{|w_n|}\right) \to 0,$$

where we have identified a neighborhood of $(x, 0)$ on $TM$ or $T^*_M$ with an open set of $\mathbb{R}^n \times \mathbb{R}^n$, and we have used the continuity of the map $\mathcal{L}^{-1}$ on $T^*_M \setminus 0$. Thus, we can state that the function $t \in I_k \mapsto \mathcal{L}^{-1}_{\gamma(t)} \circ \mathcal{L}_{\gamma(t)}(\dot{\gamma}(t)) = \mathcal{L}^{-1}_{\gamma(t)}(\partial_y G(\gamma(t), \dot{\gamma}(t))) = \dot{\gamma}(t)$ is continuous and $\gamma$ is a $C^1$ curve. From (20), we get that $\gamma$ satisfies the following equation a. e. on $I_k$

$$\frac{d}{dt} \partial_y G(\gamma, \dot{\gamma}) = \partial_x G(\gamma, \dot{\gamma}).$$

(21)

Hence we deduce that $\frac{d}{dt} \partial_y G(\gamma, \dot{\gamma})$ is continuous on $I_k$. This information and the fact that $G$ is fiberwise strictly convex imply that $\gamma$ is actually twice differentiable on every point $t$ where $\dot{\gamma}(t) \neq 0$ (see for instance [9, Proposition 4.2]). Now assume that $\gamma$ is not a constant curve and let $A_k \subset I_k$ be the open subset of the points $t \in I_k$ where $\dot{\gamma}(t) \neq 0$. From (21) we see that the energy $E(\gamma) := \partial_y G(\gamma, \dot{\gamma})[\dot{\gamma}] - G(\gamma, \dot{\gamma})$ is constant on every connected component of $A_k$. Since $G$ is positively homogeneous of degree 2, from the Euler’s theorem, we have that $E(\gamma) = G(\gamma, \dot{\gamma})$. Recalling that $F$ is zero only on the zero section and that the function $t \in I_k \mapsto G(\gamma(t), \dot{\gamma}(t))$ is continuous, we conclude that $G(\gamma(t), \dot{\gamma}(t))$ is constant (non zero) on every $I_k$. As
we can enlarge all the intervals $I_k$, except the last one, a small $\epsilon$, all the constants have to be the same and therefore $\gamma$ is a smooth regular curve.

Now let $Z \in T_0 \Lambda_N(M)$ be a smooth vector field along $\gamma$ and let $\sigma: [0, 1] \times [-\varepsilon, \varepsilon] \rightarrow M$, $\sigma = \sigma(t, u)$ be a smooth regular variation of $\gamma$ with variation vector field $U = \partial_1 \sigma$ having endpoints in $TN$ and such that $U(t, 0) = Z(t)$ for all $t \in [0, 1]$. Since $G(x, y) = g(x, y)[y, y]$ for any $(x, y) \in TM \setminus 0$, from (13) we get

$$\frac{d}{dt} J(\sigma) = \frac{1}{2} \int_0^1 \partial_u (g(\sigma, T)[T, T]) dt = \int_0^1 g(\sigma, T)[T, D_U T] dt,$$

(22)

where $T = \partial_0 \sigma$. On the other hand, as the variation $\sigma$ is smooth, it holds $D_U T = D_T U$ both with reference vector $T$ and hence, using this equality in (22) and evaluating at $u = 0$, we obtain

$$dJ(\gamma)[Z] = \int_0^1 g(\gamma, \dot{\gamma})[\dot{\gamma}, D_\gamma Z] dt,$$

(23)

where $D_\gamma Z$ has reference vector $\dot{\gamma}$. Moreover, arguing as in the proof of Lemma 2.2, one gets

$$\frac{d}{dt} (g(\gamma, \dot{\gamma})[\dot{\gamma}, Z]) = g(\gamma, \dot{\gamma})[D_\gamma \dot{\gamma}, Z] + g(\gamma, \dot{\gamma})[\dot{\gamma}, D_\gamma Z],$$

which, when applied to (23), gives us

$$0 = dJ(\gamma)[Z] = -\int_0^1 g(\gamma, \dot{\gamma})[D_\gamma \dot{\gamma}, Z] dt + g(\gamma(1), \dot{\gamma}(1))[\dot{\gamma}(1), Z(1)] - g(\gamma(0), \dot{\gamma}(0))[\dot{\gamma}(0), Z(0)].$$

(24)

Finally, by choosing an endpoints vanishing vector field $Z$ we see that $\gamma$ has to satisfy the equation $D_\gamma \dot{\gamma} = 0$. Thus, $\gamma$ is a constant speed geodesic satisfying the boundary conditions (12).

For the converse, we observe that if $\gamma$ is a constant non-zero speed geodesic satisfying the boundary conditions (12), then (24) holds and hence $\gamma$ is a critical point of $J$. \hfill \Box

3. ON THE PALAIS-SMALE CONDITION FOR THE ENERGY FUNCTIONAL

We prove now that the energy functional $J$ satisfies the Palais–Smale condition. We recall that a functional $J$ defined on a Banach manifold $X$ satisfies the Palais–Smale condition if every sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $\{J(x_n)\}_{n \in \mathbb{N}}$ is bounded and $\|dJ(x_n)\| \rightarrow 0$ contains a convergent subsequence.

The lack of regularity of the function $G = F^2$ on the zero section gives rise to some problems, for instance in the application of the mean value theorem, which do not occur in the proof of the Palais-Smale condition for the energy functional of a Riemannian manifold (see for instance [20]). In the paper [23] such problems are circumvented by using a sketched approximation argument. Here we give a fully detailed proof of the Palais–Smale condition. By a localization argument we will work on an open subset of $\mathbb{R}^n$. This allows us to reduce the technical aspects of the proof.

**Theorem 3.1.** Let $(M, F)$ be forward (resp. backward) complete and $N$ be a closed submanifold on $M \times M$ such that the first projection (resp. the second projection) of $N$ to $M$ is compact, then $J$ satisfies the Palais-Smale condition on $\Lambda_N(M)$. 


Proof. We prove the theorem in the forward complete case, being the backward one analogous. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence contained in \( \Lambda_N(M) \) on which \( J \) is bounded. Under the assumptions of Theorem 3.1, the manifold \( \Lambda_N(M) \) is a closed submanifold of the complete Hilbert manifold \( \Lambda(M) \) (see [20, Theorem 2.4.7]), which is the collection of all the \( H^1 \) curves in \( M \) parameterized on the interval \([0, 1]\) with scalar product as in Eq. (10). The differentiable manifold structure on \( \Lambda(M) \) is given by the charts \( \{(U_\omega, \exp_{\omega}^{-1})\}_{\omega \in C^\infty(M)} \), where \( \exp_{\omega}^{-1} \) is the inverse of the map \( \exp_{\omega}(\xi) = \exp_{\omega(t)}(\xi(t)) \), for all \( \xi \in H^1(O_\omega) \), being \( \exp \): \( TM \to M \) the exponential map of the Riemannian manifold \( (M, h) \) and \( O_\omega \) a neighborhood of the zero section in \( \omega^*TM \) (see [20, Theorem 2.3.12]).

First we prove that \( \{x_n\} \) converges uniformly. Pick a point \( \bar{p} \in p_1(N) \), where \( p_1 \) is the first projection of \( M \times M \). We evaluate the distance

\[
\text{dist}(\bar{p}, x_n(s)) \leq \text{dist}(\bar{p}, x_n(0)) + \text{dist}(x_n(0), x_n(s)) \\
\leq \text{dist}(\bar{p}, x_n(0)) + \int_0^1 F(x_n, \dot{x}_n) \, ds,
\]

for all \( s \in [0, 1] \), \( n \in \mathbb{N} \). Since \( p_1(N) \) is compact, there exists a constant \( K \) such that \( \text{dist}(\bar{p}, x_n(0)) \leq K \). By the Hölder inequality we get

\[
\text{dist}(\bar{p}, x_n(s)) \leq K + \left( \int_0^1 G(x_n, \dot{x}_n) \, ds \right)^{\frac{1}{2}} \leq K_1.
\]

Then by the Finslerian Hopf-Rinow theorem the supports of the curves \( x_n \) are contained in a compact subset \( C \) of \( M \). Hence there exist two positive constants \( c_1 \) and \( c_2 \) such that \( c_1|y|^2 \leq G(x, y) \leq c_2|y|^2 \), for every \( x \in C \) and for every \( y \in T_xM \). Here \( |\cdot| \) is the norm associated to the metric \( h \). Moreover, let \( d_{\text{dist}} \) be the distance associated to the Riemannian metric \( h \), then using the last inequality and again the Hölder’s one, we get

\[
\text{dist}_{h}(x_n(s_1), x_n(s_2)) \leq \int_{s_1}^{s_2} |\dot{x}_n| \, ds \leq \sqrt{s_2 - s_1}\left( \int_0^1 |\dot{x}_n|^2 \, ds \right)^{\frac{1}{2}} \\
\leq \frac{1}{c_1}\sqrt{s_2 - s_1}\left( \int_0^1 G(x_n, \dot{x}_n) \, ds \right)^{\frac{1}{2}} \leq K_2\sqrt{s_2 - s_1},
\]

with \( s_1 < s_2 \) in \([0, 1]\) and \( K_2 > 0 \). Hence \( \{x_n(t)\} \) is relatively compact for every \( t \in [0, 1] \) and uniformly Hölder. Therefore we can use the symmetric distance induced by \( h \) and the Ascoli-Arzela theorem to obtain a subsequence, which will be denoted again by \( \{x_n\} \), converging uniformly to a \( C^0 \) curve \( \bar{x} \) parameterized in \([0, 1]\) and having endpoints in \( N \).

Now we introduce the localization argument as in Appendix A.1 of [1]. Given any \( \eta > 0 \) small enough we have that the subset \( C = \{\exp_\bar{x}(s) : s \in [0, 1] ; v \in B(0, \eta) \subset T_{\bar{x}(s)}M\} \) is compact. Let \( \mu(p) \) be the injectivity radius of \( p \) in \((M, h)\) and \( \rho = \inf\{\mu(p) : p \in C\} \). As the injectivity radius is continuous (see [5, Proposition 8.4.1]), \( \rho > 0 \) and we can choose a \( C^\infty \) curve \( \omega \) in such a way that \( ||\bar{x} - \omega||_{\infty} < \min\{\eta, \rho/2\} \).

Let \( [0, 1] \ni t \to E(t) = (E_1(t), \ldots, E_r(t)) \) be a parallel orthonormal frame along \( \omega \), with \( r = \dim M \), \( P_t : \mathbb{R}^r \to T_{\omega(t)}M \) defined as \( P_t(v_1, \ldots, v_r) = v_1E(t) + \ldots + v_mE(t) \) and consider the Euclidean open ball of radius \( \rho \), which we name \( U \), and the map \( \varphi(t, v) = \exp_{\omega(t)} P_t(v) \). As \( \rho \) is smaller than the injectivity radius of \( \omega(t) \), the map
φ_t : U → M, defined as φ_t(v) = φ(t, v), is locally invertible and injective with invertible differential dφ_t(v), for every t ∈ [0, 1] and v ∈ U. By taking a smaller open in U that contains the closed ball of radius ρ/2 and it is contained in the closed ball of radius 2ρ/3, we can assume that all the continuous functions involved in the rest of the proof are uniformly bounded in [0, 1] × U or in \( \cup_{t \in [0,1]} \{ t \} \times \varphi(\{ t \} \times U) \), as for example the norms of dφ_t(v) and dφ(t, x), where φ(t, x) = φ_t^{-1}(x). Let \( \mathcal{O}_\omega \) be a neighborhood of \( \omega \) in \( H^1([0,1], M) \) such that the map

\[
φ_*^{-1} : \mathcal{O}_\omega \rightarrow H^1([0,1], U),
\]

defined as \( φ_*^{-1}(x)(t) = φ_t^{-1}(x(t)) \) is the map of a coordinate system centered at \( \omega \). Observe that the inverse of \( φ_*^{-1} \) is the map \( φ_* \), defined by \( φ_*(\xi)(t) = φ(t, \xi(t)) \).

Clearly if \( n \) is big enough, \( x_n \in φ_*(H^1([0,1], U)) \) and we call \( \xi_n = φ_*^{-1}(x_n) \). Hence, proving the strong convergence of \( \{ x_n \} \) is equivalent to proving the strong convergence of \( \{ \xi_n \} \) in \( H^1([0,1], U) \).

Now consider the orthogonal splitting

\[
H^1([0,1], \mathbb{R}^r) = H^1_0([0,1], \mathbb{R}^r) \oplus V,
\]

where \( V \) is the vector space of dimension \( 2r \), defined as \( V = \{ \zeta ∈ C^∞([0,1], \mathbb{R}^r) | \zeta'' - \zeta = 0 \} \). So, if \( n \in \mathbb{N} \) is big enough there exist \( \xi^0_n \in H^1_0([0,1], U) \) and \( \zeta_n \in V \) such that \( \xi_n = \xi^0_n + \zeta_n \). Considering \( J \) as defined on \( H^1([0,1], M) \), we have:

\[
d(J \circ φ_*)(\xi_n)[\xi_n - \xi_m] = d(J \circ φ_*)(\xi^0_n)[\xi^0_n - \xi^0_m] + d(J \circ φ_*)(\zeta_n)[\zeta_n - \zeta_m] = d(J(x_n))[dφ_*(\xi_n)[\xi^0_n - \xi^0_m]] + d(J \circ φ_*)(\zeta_n)[\zeta_n - \zeta_m] \rightarrow 0, \tag{25}
\]
as \( n, m \rightarrow \infty \). Indeed, the first term on the right-hand side of (25) goes to 0 as \( n, m \rightarrow \infty \) since \( \{ x_n \} \) is a Palais-Smale sequence for \( J \) on \( Λ_N(M) \), the norms of the operators \( dφ_*(\xi) \) are uniformly bounded and \( \{ \xi_n \} \) is a bounded sequence in \( H^1([0,1], U) \) (and hence also \( \{ \xi^0_n \} \) is a bounded sequence in \( H^1_0([0,1], U) \)). The fact that \( \{ \xi_n \} \) is a bounded sequence in \( H^1([0,1], U) \) follows from the inequality

\[
\int_0^1 |\dot{\zeta}_n|^2 ds = \int_0^1 |d\phi(s, x_n)((1, \dot{x}_n))|^2 ds \leq K_3 \int_0^1 (1 + h(x_n)[\dot{x}_n, x_n]) ds \leq K_3 + K_4 J(x_n) < K_5 < +\infty, \tag{26}
\]

where \( \phi(s, x) = φ_*^{-1}(x) \), for each \( s ∈ [0,1] \) and \( x ∈ φ_* (U) \), and \( K_3, K_4, K_5 \) are positive constants. The second term on the right-hand side of (25) goes also to 0 as it can be easily seen observing that \( \{ \zeta_n \} \) is a converging sequence in the \( C^1 \) norm (this follows from the \( C^0 \) convergence of \( \{ \xi_n \} \) and the smooth dependence of the solutions of the differential equation defining \( V \) on boundary data) and using (27) below, with \( \zeta_n - \zeta_m \) in place of \( \xi_n - \xi_m \), together with the fact that \( \{ \xi_n \} \) is bounded in \( H^1([0,1], U) \).

To complete the proof, we have to show that the sequence of curves \( \{ \xi_n \} \) is Cauchy in the \( H^1 \) norm. Notice that \( \dot{J} = J \circ φ_* \) is given by \( \dot{J}(\xi) = \frac{1}{2} \int_0^1 \dot{G}_s(\xi, \xi) ds \) for \( \xi ∈ H^1([0,1], U) \), where \( \dot{G}_s : U × \mathbb{R}^r → \mathbb{R} \) is defined as

\[
\dot{G}_s(x, y) = G(φ(s, x), dφ(s, x)((1, y))).
\]
Moreover, we assume that
\[ d\tilde{J}(\xi_n)[\xi_n - \xi_m] \]
\[ = \frac{1}{2} \int_0^1 \partial_s \tilde{G}_s(\xi_n, \dot{\xi}_n)[\xi_n - \xi_m]ds + \frac{1}{2} \int_0^1 \partial_y \tilde{G}_s(\xi_n, \dot{\xi}_n)[\xi_n - \dot{\xi}_n]ds \to 0, \]  
(27)
as \( m \) and \( n \) go to \( \infty \). Now consider the first integral in (27). We observe that, with the same abuse of notation as in the proof of Proposition 2.3,
\[ \partial_s \tilde{G}_s(x, y)[\cdot] = \partial_s G(\varphi(s, x), d\varphi(s, x)((1, y)))[d\varphi_s(x)[\cdot]] \]
\[ + \partial_y G(\varphi(s, x), d\varphi(s, x)((1, y))[\partial^2_x \varphi(s, x)((1, 0), \cdot) + d^2\varphi_s(x)[y, \cdot]]. \]
Moreover, as \( \partial_y G(x, y) \) and \( \partial_s G(x, y) \) are homogeneous in \( y \) of degree 1 and 2 respectively, using last equation, recalling that \( \{\xi_n\} \) is bounded in the \( C^0 \) norm and the fact that all the involved operators are uniformly bounded in norm we get
\[ \left| \int_0^1 \partial_s \tilde{G}_s(\xi_n, \dot{\xi}_n)[\xi_n - \xi_m]ds \right| \leq \]
\[ K_6 \int_0^1 (1 + |\dot{\xi}_n|^2)|\xi_n - \xi_m|ds + K_7 \int_0^1 (1 + |\dot{\xi}_n|^2)^{1/2}(1 + |\dot{\xi}_m|)|\xi_n - \xi_m|ds, \]  
(28)
for some positive constants \( K_6 \) and \( K_7 \). As by (26), \( \{\xi_n\} \) is bounded in the \( L^2 \) norm, and \( \{\xi_n\} \) is Cauchy in the \( C^0 \) norm, the right-hand side in (28) therefore and therefore the first integral in (27) goes to 0 as \( m, n \to \infty \).

Now we change the role of \( \xi_n \) and \( \xi_m \) considering \( d\tilde{J}(\xi_m)[\xi_n - \xi_m] \). Proceeding as in (25), we see that \( d\tilde{J}(\xi_m)[\xi_n - \xi_m] \to 0 \), as \( m, n \to \infty \). Therefore
\[ \int_0^1 (\partial_y \tilde{G}_s(\xi_n, \dot{\xi}_n)[\xi_n - \dot{\xi}_m] - \partial_y \tilde{G}_s(\xi_m, \dot{\xi}_m)[\xi_n - \dot{\xi}_m])ds \to 0, \]  
(29)
and \( \xi \) uniformly, using the facts that \( \partial_y G(x, y) \) is continuous on \( TM \) and positively homogeneous of degree 1 in \( y \), that a continuous function on a compact set is uniformly continuous and that \( \{\xi_n\} \) is uniformly bounded in the \( L^2 \) norm, the limit (29) gives also
\[ \int_0^1 (\partial_y \tilde{G}_s(\xi, \dot{\xi}_n) - \partial_y \tilde{G}_s(\xi, \dot{\xi}_m)))[\xi_n - \dot{\xi}_m]ds \to 0, \]  
(30)
as \( m, n \to \infty \).

Let us define \( \delta_i(s) = d\varphi(s, \xi(s))((1, \dot{\xi}(s))) \) for \( i \in \mathbb{N} \) and \( s \in [0, 1] \) and the following subsets of the interval \([0, 1] \). Let \( A_i \subset [0, 1] \) be the support of the \( L^2 \) function \( \varphi_i(s) = (h(\xi(s)) | \delta_i(s), \dot{\xi}(s)) \}^{1/2} \in \mathbb{R} \) for \( i \in \mathbb{N} \) and choose
\[ B_{nm} = \left\{ t \in A_n \cap A_m \left| \frac{\delta_m}{|\delta_m|} = -\frac{\delta_n}{|\delta_n|} \right. \text{a.e.} \right\}, \]
\[ C_{nm} = (A_n \cup A_m) \setminus B_{nm}, \]  
\[ D_{nm} = [0, 1] \setminus (A_n \cup A_m). \]

Moreover, we assume that \( \frac{\delta_m}{|\delta_m|} \neq -\frac{\delta_n}{|\delta_n|} \) a.e. in \( A_n \cap A_m \setminus B_{nm} \). We observe that the interval \([0, 1] \) is the union of the sets \( B_{nm}, C_{nm} \) and \( D_{nm} \), for every \( n \) and \( m \); moreover on \( B_{nm} \) we have \( \delta_n = -\lambda_{nm}\delta_n \), with \( \lambda_{nm} = \frac{|\delta_m|}{|\delta_n|} \). The subsets \( B_{nm} \) and \( D_{nm} \) are precisely the instants where the mean value theorem cannot be applied.
because of the lack of smoothness of \( G \) on the null section. Applying the mean value theorem for every \( s \in \mathcal{C}_{nm} \) and using the fact that
\[
\partial_{yy} \tilde{G}_s(x, y)[\cdot, \cdot] = \partial_{yy} G(\varphi(s, x), d\varphi(s, x)[(1, y)]) \left[ d\varphi_s(x)[\cdot], d\varphi_s(x)[\cdot] \right]
\]
is positive definite and \( \partial_{yy} G(x, y) \) is positive homogeneous of degree 0 in \( y \), we get the existence of a positive constant \( K_{\delta} \) such that
\[
\int_{\mathcal{C}_{nm}} (\partial_y \tilde{G}_s(\xi, \xi_m) - \partial_y \tilde{G}_s(\bar{\xi}, \bar{\xi}_m)) |\xi_n - \xi_m| ds
= \int_{\mathcal{C}_{nm}} \partial_{yy} \tilde{G}_s(\bar{\xi}, \bar{\xi}_m)(\varphi_s(\bar{\xi}), \varphi_s(\bar{\xi}_m)) |\xi_n - \xi_m| ds
\geq K_{\delta} \int_{\mathcal{C}_{nm}} |\xi_n - \xi_m|^2 ds. \tag{31}
\]
where \( \varphi : \mathcal{C}_{nm} \to \mathbb{R} \) is a function assuming values in \([0, 1]\). We pass now to estimate the functions \( (\partial_y \tilde{G}_s(\bar{\xi}, \xi_n) - \partial_y \tilde{G}_s(\bar{\xi}, \xi_m)) |\xi_n - \xi_m| \) over the subsets \( B_{nm} \). To this end, we observe that
\[
(\partial_y \tilde{G}_s(\bar{\xi}, \xi_n) - \partial_y \tilde{G}_s(\bar{\xi}, \xi_m)) |\xi_n - \xi_m|
= \left( \partial_y G(\varphi_s(\bar{\xi}), d\varphi(s, \bar{\xi})[(1, \xi_n)])
- \partial_y G(\varphi_s(\bar{\xi}), d\varphi(s, \bar{\xi})[(1, \xi_m)]) \right) |d\varphi_s(\bar{\xi})[\xi_n - \xi_m]|
= \left( \partial_y G(\varphi_s(\bar{\xi}), d\varphi(s, \bar{\xi})[(1, \xi_n)])
- \partial_y G(\varphi_s(\bar{\xi}), d\varphi(s, \bar{\xi})[(1, \xi_m)]) \right) |d\varphi(s, \bar{\xi})[(0, \xi_n)] - d\varphi(s, \bar{\xi})[(0, \xi_m)]|
= \left( \partial_y G(\varphi_s(\bar{\xi}), d\varphi(s, \bar{\xi})[(1, \xi_n)])
- \partial_y G(\varphi_s(\bar{\xi}), d\varphi(s, \bar{\xi})[(1, \xi_m)]) \right) |d\varphi(s, \bar{\xi})[(1, \xi_n)] - d\varphi(s, \bar{\xi})[(1, \xi_m)]|.
\]
Therefore, recalling that \( \delta_m = -\lambda_{nm}\delta_n \) over the subsets \( B_{nm} \), we get
\[
\int_{B_{nm}} (\partial_y \tilde{G}_s(\bar{\xi}, \xi_n) - \partial_y \tilde{G}_s(\bar{\xi}, \xi_m)) |\xi_n - \xi_m| ds
= \int_{B_{nm}} (\partial_y G(\varphi_s(\bar{\xi}), \delta_n) - \partial_y G(\varphi_s(\bar{\xi}), \delta_m)) |\delta_n - \delta_m| ds
= \int_{B_{nm}} (1 + \lambda_{nm}) \partial_y G(\varphi_s(\bar{\xi}), \delta_n) |\delta_n| ds
+ \int_{B_{nm}} \left( 1 + \frac{1}{\lambda_{nm}} \right) \partial_y G(\varphi_s(\bar{\xi}), \delta_m) |\delta_m| ds.
\]
By Euler’s theorem the above integrals are equal to
\[
\int_{B_{nm}} 2(1 + \lambda_{nm}) G(\varphi_s(\bar{\xi}), \delta_n) ds + \int_{B_{nm}} 2 \left( 1 + \frac{1}{\lambda_{nm}} \right) G(\varphi_s(\bar{\xi}), \delta_m) ds
\]
and then, by homogeneity, we get
\[\int_{B_{nm}} 2(1 + \lambda_{nm})G(\varphi_s(\xi), \delta_n) ds + \int_{B_{nm}} 2 \left(1 + \frac{1}{\lambda_{nm}}\right)G(\varphi_s(\xi), \delta_m) ds \geq K_9 \left(\int_{B_{nm}} |\delta_n|^2 ds + \int_{B_{nm}} |\delta_m|^2 ds\right),\] (32)
where \(K_9\) is a positive constant independent of \(n\) and \(m\). Now observe that, by linearity
\[|\delta_n - \delta_m|^2 = |d\varphi(s, \xi)\langle(0, \xi_n - \xi_m)\rangle|^2 \geq K_{10} |\xi_n - \xi_m|^2,\]
where \(K_{10}\) is the minimum value of the function \(|d\varphi(s, \xi(s))\langle(0, v)\rangle|^2\) over the compact set \([0, 1] \times S^{r-1}\), where \(S^{r-1}\) is the \((r-1)\)-dimensional sphere. Therefore we obtain
\[\int_{B_{nm}} |\dot{\xi}_n - \dot{\xi}_m|^2 ds \leq \frac{1}{K_{10}} \int_{B_{nm}} |\delta_n - \delta_m|^2 ds \leq \frac{2}{K_{10}} \left(\int_{B_{nm}} |\delta_n|^2 ds + \int_{B_{nm}} |\delta_m|^2 ds\right).\] (33)
Over the subset \(D_{nm}\) both \(\delta_n\) and \(\delta_m\) are zero, hence
\[\int_{D_{nm}} |\dot{\xi}_n - \dot{\xi}_m|^2 ds \leq \frac{1}{K_{10}} \int_{D_{nm}} |\delta_n - \delta_m|^2 ds = 0,\] (34)
for all \(n\) and \(m\). From (30), summing up (31), (32), (33) and (34) and recalling that the interval \([0, 1] = B_{nm} \cup C_{nm} \cup D_{nm}\), we finally get
\[\int_0^1 |\dot{\xi}_n - \dot{\xi}_m|^2 ds \rightarrow 0\]
as \(n, m \rightarrow \infty\). \(\square\)

With the Palais-Smale condition in hand, infinite dimensional Lusternik and Schnirelman theory becomes available (see [26]); so we can obtain existence and multiplicity results about the number of critical points of \(J\), depending on \(N\) and the topology of \(M\), for example in the non-contractible case. We consider here the case of geodesics joining two different submanifolds of \(M\) (compare also with [22, Theorem 6]).

**Proposition 3.2.** Let \((M, F)\) be a forward or backward complete Finsler manifold and let \(M_1\) and \(M_2\) be two closed submanifolds of \(M\) such that \(M_1\) or \(M_2\) is compact. Then there exists a geodesic \(\gamma\) connecting \(M_1\) and \(M_2\) and satisfying (12). Moreover, if the manifold \(M\) is non-contractible and \(M_1, M_2\) are contractible then there exist infinitely many geodesics \(\gamma_n\) connecting \(M_1\) and \(M_2\), satisfying (12) and such that \(\lim_n J(\gamma_n) = +\infty\) (according to Theorem 3.1, in the forward case such geodesics start from the compact submanifold while, in the backward case, they arrive to it).

**Proof.** Existence is a standard application of the Deformation Lemma (see [26]). For the multiplicity result we recall that, given a topological space \(X\), the Lusternik-Schnirelman category of \(X\), is a homotopy invariant defined as the minimum number, denoted by \(\text{cat} X\), of closed contractible subsets of \(X\) which cover \(X\). Let \(C^0_{M_1 \times M_2}(M)\) be the space of the continuous curves having endpoints in \(M_1 \times M_2\). The inclusion of \(\Lambda_{M_1 \times M_2}(M)\) in \(C^0_{M_1 \times M_2}(M)\) is a homotopy equivalence (see [17],
Remark 4.1. Let $\Omega(M)$ be the space of based loops in $M$. Since $M_1$ and $M_2$ are contractible, $C^0_{M_1 \times M_2}(M)$ has the same homotopy type as $M_1 \times M_2 \times \Omega(M)$, moreover $\text{cat} \Omega(M) = \infty$ (see [12, Proposition 3.2 and Corollary 3.2]), hence

$$\text{cat}(\Lambda_{M_1 \times M_2}(M)) = \infty$$

as well. By Theorem 7.2 of [26] and Theorem 3.1, $J$ has infinitely many critical points $\gamma_n$ which are geodesics connecting $M_1$ to $M_2$ and satisfying (12). Finally, $\sup_{n \in \mathbb{N}} J(\gamma_n) = +\infty$ otherwise would be possible to retract the manifold $\Lambda_{M_1 \times M_2}(M)$ onto a sublevel of the functional $J$. This would be a contradiction, since the sublevels of a $C^1$ functional defined on a Banach manifold, bounded from below and satisfying the Palais-Smale condition have finite Lusternik-Schnirelman category.

$\square$  

Remark 3.3. We point out that the above multiplicity result does not guarantee, in general, that the infinitely many geodesics are geometrically distinct (they might cover the same closed geodesic, as on the standard sphere).

Remark 3.4. For the two endpoints boundary conditions, the above multiplicity result can also be obtained by using Morse theory and a finite dimensional approximation of the path space $\Lambda_{\{p\} \times \{q\}}(M)$, see [24, Ch. III §3.16,17] (cf. [34] for the Finsler case). However, for general boundary conditions the infinite dimensional approach is very useful. In particular for periodic boundary conditions, in contrast to the finite dimensional approximation, the free loop space carries, for a non-reversible Finsler metric, a canonical $S^1$-action leaving the energy functional invariant.

4. The Fermat Metric

In this section we present some applications of Finsler Geometry to the study of the causal structure of a conformally stationary spacetime. We first recall the definition of a Finsler manifold of Randers type, then we introduce a Randers metric, that we call Fermat metric, which is related to the Fermat principle for lightlike geodesics in a conformally stationary spacetime.

4.1. Randers metrics. Let $h$ be a Riemannian tensor and $\omega$ be a one-form on $M$. A Randers metric $F$ is defined as follows:

$$F(x, y) = \sqrt{h(x)[y, y]} + \omega(x)[y], \quad \|\omega\|_x < 1,$$

(35)

where $\|\omega\|_x = \sup_{v \in T_x M, 0 < \|v\|_x \leq 1} \frac{|\omega(x)[v]|}{\sqrt{h(x)[v, v]}}$. Remarkably enough, the condition $\|\omega\|_x < 1$ for all $x \in M$, not only implies that $F$ is positive but also that it has fiberwise strongly convex square (see [5, §11.1]).

Remark 4.1. We observe that if the Riemannian metric $(M, h)$ is complete and

$$\|\omega\|_x = \sup_{x \in M} \|\omega\|_x < 1,$$

(36)

the Randers manifold $(M, F)$ is forward and backward complete. In fact, let $\{x_n\}$ be, for instance, a forward Cauchy sequence for $(M, F)$, then for any $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ with $\nu \leq i \leq j$, $\text{dist}(x_i, x_j) < \varepsilon$. By definition of distance, there exists a curve $\gamma_{ij}$ connecting $x_i$ to $x_j$, such that

$$\varepsilon > \int_{\gamma_{ij}} F(\gamma_{ij}, \gamma_{ij}) \geq (1 - \|\omega\|) \int_{\gamma_{ij}} \sqrt{h(\gamma_{ij})[\gamma_{ij}, \gamma_{ij}]} \geq (1 - \|\omega\|) \text{dist}^h(x_i, x_j),$$

where
where \( \text{dist}^h \) is the distance associated to the Riemannian metric \( h \). Being \((M, h)\) complete, \( \{x_n\} \) converges.

4.2. The Fermat metric of a conformally standard stationary spacetime.

A Fermat principle in General Relativity is a variational characterization of the light rays joining an event with the worldline of an observer in the spacetime. A spacetime is given by a Lorentzian manifold whose metric tensor satisfies the Einstein equations together with a time orientation, while light rays are given by the lightlike geodesics of the Lorentzian manifold. In the recent years there has been a great amount of work about the Fermat principle in General Relativity, because it allows one to obtain a mathematical description of the gravitational lens effect in Astrophysics, see [15, 28].

A Lorentzian manifold \((M, g)\) is a smooth, connected spacelike manifold \( M \) endowed with a symmetric non-degenerate tensor field \( g \) of type \((0, 2)\) having index 1. A geodesic of \((M, g)\) is a smooth curve \( \gamma : [a, b] \to M \) satisfying the equation

\[
\nabla_\gamma \dot{\gamma} = 0,
\]

where \( \nabla_\gamma \) is the covariant derivative along \( \gamma \) associated to the Levi-Civita connection of the metric \( g \) (we refer to [8] for all the needed background material on Lorentzian geometry). As in the Riemannian case, a geodesic has to satisfy the conservation law

\[
g(\gamma)(\dot{\gamma}, \dot{\gamma}) = E_\gamma = \text{const.,}
\]

which corresponds to energy conservation in Lagrangian mechanics. According to the sign of \( E_\gamma \), a geodesic is said timelike if \( E_\gamma < 0 \), lightlike if \( E_\gamma = 0 \), spacelike if \( E_\gamma > 0 \) or \( \dot{\gamma}(s) = 0 \) for all \( s \in [a, b] \). This partition of the set of geodesics is known as the causal character of a geodesic. Such a terminology is used also for any vector in any tangent space and for any piecewise smooth curve if and only if its tangent vector field has the same character at any point where it is defined.

A time orientation on a Lorentzian manifold is determined by a timelike vector field \( Y \) on \( M \), i.e. for any \( p \in M \), \( Y(p) \) is a timelike vector. A piecewise smooth lightlike (resp. future-pointing) curve \( \gamma : [a, b] \to M \) is said to be future-pointing (resp. past-pointing) if \( g(\gamma(s))[\dot{\gamma}(s), Y(\gamma(s))]<0 \) (resp. \( g(\gamma(s))[\dot{\gamma}(s), Y(\gamma(s))]>0 \)) for all \( s \in [a, b] \) where \( \dot{\gamma}(s) \) is defined. The notion of being future-pointing (resp. past-pointing) and non-spacelike can be extended to a continuous curve \( \gamma : [a, b] \to M \) requiring that for any \( s_0 \in [a, b] \) there is a convex normal neighborhood \( U \subset M \) of \( \gamma(s_0) \) and an interval \( J \subset [a, b] \) containing \( s_0 \) such that for any \( s_1, s_2 \in J \), with \( s_1 < s_2 \), a smooth future-pointing (resp. past-pointing) curve connecting \( \gamma(s_1) \) to \( \gamma(s_2) \) and contained in \( U \) exists. From now on, non-spacelike curves are assumed to be future-pointing.

A conformally standard stationary Lorentzian manifold is a manifold \( M \) which splits as a product \( M = M_0 \times \mathbb{R} \), where \( M_0 \) is endowed with a Riemannian metric \( g_0 \), with a vector field \( \delta \) and a positive function \( \beta \). Moreover, there exists a positive function \( \varphi \) on \( M \), such that the Lorentzian metric \( g \) on \( M \) is given by

\[
g(x, t)[(y, \tau), (y, \tau)] = \varphi(x, t)(g_0(x)[y, y] + 2g_0(x)[\delta(x), y] \tau - \beta(x) \tau^2),
\]

for any \((x, t) \in M_0 \times \mathbb{R} \) and \((y, \tau) \in T_x M_0 \times \mathbb{R} \). A conformally standard stationary Lorentzian manifold is time oriented by the timelike Killing vector field \( \partial_t \) and a piecewise smooth non-spacelike curve \( \gamma = (x, t) \) is future-pointing if \( t > 0 \) where \( \dot{\gamma} \) exists.

Since lightlike geodesics and causal properties - as global hyperbolicity - of a conformally stationary spacetime are invariant under conformal changes of the metric tensor \( g \) (see for example [8, 31]), we can assume that \( g \) is given by \( g/\varphi(x, t) \).
Indeed, rather than the metric in (37), we will consider the metric

\[ g(x, t)([y, \tau], [y, \tau]) = g_0(x)[y, y] + 2g_0(x)[\delta(x), y] \tau - \beta(x)\tau^2. \]

We introduce now the Fermat metric associated to a standard stationary Lorentzian manifold. Let \( z_0 = (x_0, t_0) \in \mathcal{M} \) be an event in \( \mathcal{M} \) and let \( \mathbb{R} \ni s \to \gamma(s) = (x_1, s) \in \mathcal{M} \) be a timelike vertical curve, that is an integral curve of the timelike vector field \( \partial_t \). Let \([0, 1] \ni s \to z(s) = (x(s), t(s)) \in \mathcal{M} \) be a lightlike curve joining \( z_0 \) and \( \gamma \). Concretely the lightlike curve \( z \) satisfies

\[ g_0(x)[\dot{x}, \dot{x}] + 2g_0(x)[\delta(x), \dot{x}][t - \beta(x)t^2 = 0, \]

and the boundary conditions \( x(0) = x_0, x(1) = x_1, t(0) = t_0 \). The arrival time \( T(z) \) of the lightlike curve \( z \) is given by \( t(1) \). From (39), assuming that the lightlike curve is future-pointing, solving with respect to \( t \) and integrating we obtain:

\[ t(s) = t_0 + \int_0^s \left( \bar{g}_0(x)[\delta(x), \dot{x}] + \sqrt{\bar{g}_0(x)[\delta(x), \dot{x}]^2 + \bar{g}_0(x)[\dot{x}, \dot{x}]} dv, \]

where \( \bar{g}_0 = g_0/\beta \). So the arrival time \( T(z) \) is given by

\[ T(z) = t_0 + \int_0^1 \left( \bar{g}_0(x)[\delta(x), \dot{x}] + \sqrt{\bar{g}_0(x)[\delta(x), \dot{x}]^2 + \bar{g}_0(x)[\dot{x}, \dot{x}]} ds. \]

**Definition 4.2.** The Fermat metric associated to a standard stationary Lorentzian manifold \( (\mathcal{M}, g) \) (as in (38)) is the Randers metric \( F \) on \( \mathcal{M}_0 \) given by

\[ F(x, y) = \bar{g}_0(x)[\delta(x), y] + \sqrt{\bar{g}_0(x)[\delta(x), y]^2 + \bar{g}_0(x)[y, y]} \]

for every \( (x, y) \in T\mathcal{M}_0 \), being \( \bar{g}_0 = g_0/\beta \) (cf. (35)): here the Riemannian metric \( h \) is given by \( h(x)[y, y] = \bar{g}_0(x)[\delta(x), y]^2 + \bar{g}_0(x)[y, y] \).

**Remark 4.3.** The Fermat metric associated to a conformally standard stationary spacetime as in (37) will be the Fermat metric associated to the standard stationary spacetime \( (\mathcal{M}, g/\varphi(x, t)) \).

**Remark 4.4.** Observe that the arrival time \( T(z) \) of a future-pointing lightlike curve \( z = (x, t) \) is equal, up to the initial instant of time \( t_0 \), to the length of its spatial projection \( x \) with respect to the Fermat metric \( F \).

The Fermat metric \( F \) allows one to reduce Fermat’s principle for light rays on a standard stationary spacetime to a variational principle involving only the spatial projections of the lightlike curves. We recall that the relativistic Fermat principle for lightlike geodesics states that among all lightlike curves \( z : [0, 1] \to \mathcal{M} \) connecting some event \( p \in \mathcal{M} \) with some timelike curve \( \gamma \) on \( \mathcal{M} \), lightlike geodesics are, up to reparameterization, critical points of the arrival time, which is the functional \( z \to \gamma^{-1}(z)(1) \). The property of lightlike geodesics (light rays) of being stationary points of the arrival time is classically known as Fermat’s principle. The first one to formulate Fermat’s principle in General Relativity in the above generality was I. Kovner in [21], but a rigorous proof was given by V. Perlick in [27]. Some special versions of Fermat’s principle for static, stationary and conformally stationary Lorentzian manifolds, contained in several books about General Relativity, can be deduced from the above general version. The Finslerian reduction of the principle for a standard stationary spacetime consists in proving that a future-pointing lightlike curve \([0, 1] \ni s \to (\tilde{x}(s), \tilde{t}(s)) \in \mathcal{M} \) joining \((x_0, t_0)\) with \( \gamma(s) = (x_1, s) \) is a lightlike geodesic of \((\mathcal{M}, g/\beta)\) (and up to reparameterization of \((\mathcal{M}, g)\)) if and only if its spatial component \( \tilde{x} \) is a geodesic of the Fermat metric.
Theorem 4.5 (Fermat’s principle). Let \((M, g)\) be a standard stationary spacetime and \((x_0, t_0) \in M\), \(\mathbb{R} \ni s \to \gamma(s) = (x_1, s) \in M\), \(x_1 \in M_{t_0}\). A curve \([0, 1] \ni x \to z(s) = (x(s), t(s)) \in M\) is a future-pointing lightlike geodesic of \((M, g/\beta)\) if and only if \(x(s)\) is a geodesic for the Fermat metric \(F\), parameterized to have constant Riemannian speed \(h(x)\dot{x} = \tilde{g}_0(x)\delta(x, \dot{x})^2 + \tilde{g}_0(x)[\dot{x}, \dot{x}]\), and \(t(s)\) is given by (40).

Proof. Using the Levi-Civita connection \(\nabla\) of the metric \(\tilde{g}_0\), the Euler-Lagrange equations of the functional (41) can be written as

\[-\nabla_x \left( \frac{\dot{x} + \tilde{g}_0[\delta, \dot{x}] \dot{\delta}}{\sqrt{h(\dot{x}, \dot{x})}} \right) + \frac{\tilde{g}_0[\delta, \dot{x}] (\nabla \delta)^* [\dot{x}] \sqrt{h(\dot{x}, \dot{x})}}{\sqrt{h(\dot{x}, \dot{x})}} + (\nabla \delta)^* [\dot{x}] - \nabla \delta [\dot{x}] = 0, \] (42)

where \((\nabla \delta)^*\) is the adjoint with respect to \(\tilde{g}_0\) of \(\nabla \delta\) and \((\nabla \delta)[\dot{x}] = \nabla_x \delta\). Hence if \(x\) is parameterized to have constant Riemannian speed \(h(\dot{x}, \dot{x})\), we get:

\[\nabla_x \dot{x} = -\nabla_x (\tilde{g}_0[\delta, \dot{x}] \dot{\delta}) + \tilde{g}_0[\delta, \dot{x}] (\nabla \delta)^* [\dot{x}] + \sqrt{h(\dot{x}, \dot{x})} ((\nabla \delta)^* [\dot{x}] - \nabla \delta [\dot{x}])\]

\[= -\frac{d}{ds} (\tilde{g}_0[\delta, \dot{x}] \dot{\delta}) + \tilde{g}_0[\delta, \dot{x}] ((\nabla \delta)^* [\dot{x}] - \nabla \delta [\dot{x}]) + \sqrt{h(\dot{x}, \dot{x})} ((\nabla \delta)^* [\dot{x}] - \nabla \delta [\dot{x}])\]

\[= F(x, \dot{x}) \Omega[\dot{x}] - \frac{d}{ds} (\tilde{g}_0[\delta, \dot{x}] \dot{\delta}), \] (43)

where \(\Omega[\dot{x}] = (\nabla \delta)^* [\dot{x}] - (\nabla \delta)[\dot{x}]\). Lightlike geodesics of \((M, g/\beta)\) are critical points of the energy functional

\[(x, t) \mapsto \frac{1}{2} \int_0^1 (\tilde{g}_0[\delta, \dot{x}] + 2\tilde{g}_0[\delta, \dot{x}] \dot{t}^2) ds, \]

so they satisfy the Euler-Lagrange equations

\[\begin{cases}
\nabla_x \dot{x} = \dot{t} \Omega[\dot{x}] - \frac{dt}{ds} \delta, \\
\tilde{g}_0[\delta, \dot{x}] + C = \dot{t},
\end{cases}\] (44)

where \(C\) is a constant. By the second equation in (44), \(\frac{dt}{ds} = \frac{d}{ds}(\tilde{g}_0[\delta, \dot{x}])\) and recalling that a future-pointing lightlike curve has to satisfy the equation

\[\dot{t} = F(x, \dot{x}),\] (45)

we get (43). Finally integrating (45) we get that \(t(s)\) is given by (40). The reciprocal is analogous. \qed

Remark 4.6. We point out that the name Fermat metric has been used in some paper to denote the Riemannian metric \(\tilde{g}_0\) (see [28, §4.2] and the references therein). We think that our definition is more appropriate because, as for the Fermat principle in classical optics, arrival times of lightlike curves and in particular light rays are measured as lengths with respect to a metric, in this case a Finsler one.

We shall see now how the Fermat metric has not only a clear variational meaning, but it plays a basic role also in the study of causal properties of a conformally standard stationary spacetime. We recall some basic definitions and properties about causality (our main references about that are [8, 18]). A Lorentzian manifold \((M, g)\) is said strongly causal if every \(p \in M\) has arbitrarily small neighborhoods such that no non-spacelike curve that leaves one of these neighborhoods ever returns. A non-spacelike curve \(\gamma: (a, b) \to M\) is said future inextendible (resp. past inextendible)
if the limit \( \lim_{s \to b^-} \gamma(s) \) (resp. \( \lim_{s \to a^+} \gamma(s) \)) does not exist. It is said \textit{inextendible} if it is both future and past inextendible. Two non-spacelike continuous curves are considered equivalent if one is the reparameterization of the other. Henceforth, whenever the domain of the parameter is not specified, we will be regarding the equivalence class of the curve. For any \( p \in \mathcal{M} \), let \( J^+(p) \subset \mathcal{M} \) (resp. \( J^-(p) \subset \mathcal{M} \)) be the subset of the points \( q \) in \( \mathcal{M} \) such that there exists a non-spacelike curve \( \gamma: [a, b] \to \mathcal{M} \) with \( \gamma(a) = p \) and \( \gamma(b) = q \) (resp. \( \gamma(a) = q \) and \( \gamma(b) = p \)). A Lorentzian manifold \( (\mathcal{M}, g) \) is said \textit{globally hyperbolic} if it admits a \textit{Cauchy surface} i.e. a subset \( S \) which every inextendible timelike curve intersects exactly once. It can be proved that \( (\mathcal{M}, g) \) is globally hyperbolic if and only if it is strongly causal and for all \( p, q \in \mathcal{M} \) the set \( J^+(p) \cap J^-(q) \) is compact (see [18, Proposition 6.6.3 and Proposition 6.6.8]).

For future references, we show here, in the case of a conformally standard stationary Lorentzian manifold, a fact that is cited, without proof, in several references (see for example [18, p. 213]).

**Lemma 4.7.** Let \( (\mathcal{M}, g) \) be a conformally standard stationary Lorentzian manifold and \( p, q \in \mathcal{M} \) two causally connected points. Then there is a piecewise lightlike geodesic connecting \( p \) and \( q \).

**Proof.** As causality is invariant by conformal transformations we can assume that the metric is standard stationary as in (38). Let \( \gamma: [0, 1] \to \mathcal{M} \) be a non-spacelike curve joining \( p \) and \( q \) given by \( \gamma(s) = (x(s), t(s)) \). To find a piecewise smooth lightlike geodesic connecting \( p \) and \( q \) is equivalent to finding a piecewise smooth Fermat geodesic joining \( x(0) \) and \( x(1) \) and having length equal to \( t(1) - t(0) \). As convex neighborhoods always exist in Finsler geometry (see [36]), the support of \( x \) can be covered by a finite number of them. So we can assume, without loss of generality, that \( x(0) \) and \( x(1) \) are in the same convex neighborhood. We will show that there exist piecewise smooth geodesics from \( x(0) \) to \( x(1) \) having length \( s \) for every \( s \geq \text{dist}(x(0), x(1)) \), and then the result follows, since \( t(1) - t(0) = \int_0^1 \dot{x}(s)ds \geq \int_0^1 F(x, \dot{x})ds \geq \text{dist}(x(0), x(1)) \), where the first inequality comes from the inequality \( g_0(x)[\dot{x}, \dot{x}] + 2g_0(x)[\dot{x}, \dot{x}]i - \beta(x)i^2 \leq 0 \) which says that \( \gamma \) is a non-spacelike curve. First, observe that there is a minimal geodesic from \( x(0) \) to \( x(1) \) with Fermat length equal to \( \text{dist}(x(0), x(1)) \), because they are contained in a convex neighborhood. Then we can choose two sequences of points \( \{x_i\} \) and \( \{y_j\} \) in such a way that the distance between one element of the first sequence and another of the second one is always bigger than a small enough \( \varepsilon > 0 \). Making a sufficient number of “zig zags”, the length of the piecewise geodesic can be made as big as needed. In the first “zig zag” where the piecewise geodesic length becomes bigger than \( s \), we can move back the last point along the last piece of geodesic. As the variation of the length is continuous, we can construct, in this way, a piecewise geodesic with length \( s \). \[ \square \] \[ \square \]

In the next theorem, we show that on a conformally standard stationary Lorentzian manifold, global hyperbolicity is strictly related to the Fermat metric completeness. To the authors’ knowledge, this link between global hyperbolicity and the completeness of the Fermat metric does not appear elsewhere in literature. We are going to use the following notation for \( p_0 = (x_0, t_0) \in \mathcal{M} \): \( C^+(p_0, \mu) = \cup_{s \in [0, \mu]} B^+(s \cdot x_0) \times \{t_0 + s\} \) and \( C^-(p_0, \mu) = \cup_{s \in [0, \mu]} B^-(s \cdot x_0) \times \{t_0 - s\} \), where \( B^\pm_s(x_0) \) is the closure of \( B^\pm_s(x_0) \) in \( \mathcal{M}_0 \).
Theorem 4.8. Let \((\mathcal{M}, g)\) be a conformally standard stationary Lorentzian manifold and let \(\tilde{t} \in \mathbb{R}\). Then the following propositions hold:

1. If the Fermat metric on \(\mathcal{M}_0\) defined in 4.2 is forward (or resp. backward) complete then \(J^+(p_0) = C^+(p_0, +\infty)\) and \(J^-(p_0) = C^-(p_0, +\infty)\) for every \(p_0 = (x_0, t_0) \in L\), the balls \(B^+_{\tilde{t}}(x_0)\) (resp. \(B^-_{\tilde{t}}(x_0)\)) are compact and \((\mathcal{M}, g)\) is globally hyperbolic;

2. If \((\mathcal{M}, g)\) is globally hyperbolic with Cauchy surface \(S = \mathcal{M}_0 \times \{\tilde{t}\}\) then the Fermat metric on \(\mathcal{M}_0\) is forward and backward complete.

Proof. Again we can assume that \(g\) is as in (38). We begin with proving (1), assuming that \(F\) is forward complete (the proof in the backward case is analogous). Compactness of the balls \(B^+_{\tilde{t}}(x_0)\) is a consequence of the Finslerian Hopf-Rinow theorem. Now assume that \((x_1, t_1) \in B^+_{\tilde{t}}(x_0) \times \{t_0 + s\}\) for a certain \(s \in [t_0, +\infty)\).

By applying the Finslerian Hopf-Rinow theorem we can choose a Finslerian minimal geodesic \(x\) from \(x_0\) to \(x_1\) with speed equal to 1 and length not greater than \(s\). Considering the lightlike geodesic \(\lambda \rightarrow (x(\lambda), \lambda + t_0)\) with \(\lambda \in [0, L(x)]\), where \(L(x)\) is the Fermat length of \(x\), and then the timelike curve \(\lambda \rightarrow (x(L(x)), \lambda)\) with \(\lambda \in [t_0 + L(x), t_0 + s]\), we see that \((x_1, t_1) \in J^+(p_0)\). If \(q = (x_1, t_1) \in J^+(p_0)\), then by Lemma 4.7 there exists a piecewise smooth lightlike geodesic \(\gamma(s) = (x(s), t(s))\) which connects \(p_0\) to \(q\), such that \(\text{dist}(x_0, x_1) \leq L(x)\), hence \(q \in B^+_{\tilde{t}}(x_0) \times \{t_0 + L(x)\}\). Analogously one can prove the other equality. Now since \((\mathcal{M}, g)\) admits the coordinate \(t\) as a global time function it is stably causal and then strongly causal (see for instance [8, p. 64 and p. 73]). Furthermore, if \(p = (\tilde{x}, \tilde{t})\) and \(q = (\tilde{x}, \tilde{t})\) are points in \(\mathcal{M}\), then we can assume that \(\tilde{t} > \tilde{t}\) otherwise \(J^+(\tilde{t}) \cap J^-(\tilde{t})\) is empty. Moreover we have

\[
J^+(\tilde{t}) \cap J^-(\tilde{t}) = \bigcup_{s \in [0,1]} \left( \bar{B}^+_{s(\tilde{t} - \tilde{t})}(\tilde{x}) \cap \bar{B}^-_{1-s(\tilde{t} - \tilde{t})}(\tilde{x}) \right) \times \{\tilde{t} + s(\tilde{t} - \tilde{t})\},
\]

which is compact or empty. This can be shown as follows. Take a sequence \(\{(x_n, t_n)\} \subset J^+(\tilde{t}) \cap J^-(\tilde{t})\); as \(\{t_n\}\) moves in a compact set, we can extract a convergent subsequence. If \(\tilde{r} = \sup_n \{t_n - \tilde{t}\}\) and \(\tilde{r} = \sup_n \{\tilde{t} - t_n\}\), then \(\{x_n\}\) is contained in the subset

\[
\Bar{B}^+_{\tilde{t}}(\tilde{x}) \cap \Bar{B}^-_{\tilde{t}}(\tilde{x}),
\]

which is compact because it is the intersection of a compact subset and a closed subset. Therefore we can extract a subsequence such that \((x_n, t_n)\) converges to \((x_0, t_0)\). Now set \(\tilde{r}_n = \text{dist}(\tilde{x}, x_n)\), \(\tilde{r}_n = \text{dist}(x_n, \tilde{x})\), \(\tilde{r}_0 = \text{dist}(\tilde{x}, x_0)\) and \(\tilde{r}_0 = \text{dist}(x_0, \tilde{x})\). We know that \(t_n - \tilde{t} \geq \tilde{r}_n\) and \(\tilde{t} - t_n \geq \tilde{r}_n\) and, as a consequence, we have \(t_0 - \tilde{t} \geq \tilde{r}_0\) and \(\tilde{t} - t_0 \geq \tilde{r}_0\). Hence it follows that

\[
(x_0, t_0) \in \left( \bar{B}^+_{s_0(\tilde{t} - \tilde{t})}(\tilde{x}) \cap \bar{B}^-_{1-s_0(\tilde{t} - \tilde{t})}(\tilde{x}) \right) \times \{\tilde{t} + s_0(\tilde{t} - \tilde{t})\},
\]

with \(s_0 = \frac{\tilde{r}_n - \tilde{r}_0}{\tilde{t} - \tilde{t}}\), and then it belongs to \(J^+(\tilde{t}) \cap J^-(\tilde{t})\). Therefore \((\mathcal{M}, g)\) is globally hyperbolic.

Now we show (2). We can assume without loss of generality that \(\tilde{t} = 0\). We will prove that \((\mathcal{M}_0, F)\) is forward complete showing that every constant speed geodesic \(x: [0,b) \rightarrow M_0\) can be extended to \(b\). Assume that \(x\) has been parameterized with speed equal to 1. Let \(\{s_n\} \subset [0,b)\) be a sequence converging to \(b\). We consider the lightlike curve \(\gamma: [0,b) \rightarrow M\) such that \(\gamma(s) = (x(s), -b + s)\). Then \((x(\tilde{s}), 0) \in J^+(x(0), -b)\) for every \(\tilde{s} \in [0, b)\), because we can consider the lightlike
curve \( \gamma(s) \) with \( s \in [0, \bar{s}] \) and then the timelike curve \((x(s), -b + s)\) with \( s \in [\bar{s}, b] \).
Since in a globally hyperbolic manifold the intersection of the future or the past of a point with a Cauchy surface is compact (see for instance [18, Proposition 6.6.6]), the sequence \( x(s_n) \) is contained in a compact subset and converges in contradiction with the fact that \( x \) is inextendible. Finally, arguing as above, we can show that \((M_0, F)\) is also backward complete. \( \square \ \square \)

From Proposition 3.2, we obtain the following result, which gives a more geometrical interpretation of previous results [13, 31] because, apart from the non-triviality of the topology of the spacetime, it rests only on the completeness of the Randers metric \( F \).

**Proposition 4.9.** Let \((M, g)\) be a conformally standard stationary Lorentzian manifold and consider a point \((\bar{x}, \bar{g}_0)\) and the timelike curve \( \gamma(s) = (\bar{x}, s) \). Assume that \((M_0, F)\) is forward or backward complete, then there exists a future-pointing light ray joining \((\bar{x}, \bar{g}_0)\) and \( \gamma(s) \). Moreover, assume that \( M_0 \) is non-contractible, then there exist infinitely many lightlike geodesics \( z_n = (x_n, t_n) \) joining the point \((\bar{x}, \bar{g}_0)\) with the curve \( \gamma(s) \) and having arrival time \( T(x_n) \to +\infty \), as \( n \to \infty \).

**Remark 4.10.** We observe that, since the multiple geodesics found in the previous theorem have different arrival time, they are also geometrically distinct. However we cannot conclude, in general, that their spatial projections \( x_n \) are geometrically distinct.

**Remark 4.11.** Proposition 4.9 can be generalized to lightlike geodesics joining two submanifolds in \((M, g)\) as in [30]. Moreover, as a closed geodesic exists on every compact Finsler manifold, we can also obtain the existence of at least one non-trivial spatially periodic lightlike geodesic, whenever \( M_0 \) is compact.

**Remark 4.12.** A fully analogous result can be proved for past-pointing light rays by using the reversed Fermat metric
\[
F^*(x, y) = -\tilde{g}_0(x)[\delta(x), y] + \sqrt{\tilde{g}_0(x)[\delta(x), y]^2 + \tilde{g}_0(x)[y, y]}
\]
and the arrival time functional \( T^*(z) = t_0 - \int_0^1 F^*(x, \dot{x})ds \). The reversed metric is related to the negative solution of Eq. (39). We point out that multiplicity results about lightlike geodesics connecting a point and a timelike curve in a spacetime are important in the study of the gravitational lensing (see for instance [15, 28]), that is, the deflection of light rays due to the gravitational field of a galaxy. According to gravitational lensing, the above result for past-pointing light rays can be interpreted as follows: \((M, g)\) is a conformally stationary spacetime having a non trivial topology, the point \((\bar{x}, \bar{g}_0)\) represents the position and the time in which an observer receives the light signals, that is, the lightlike geodesics emitted from a source whose trajectory in the spacetime is the curve \( \gamma \). The fact that there exist infinitely many lightlike geodesics connecting \((\bar{x}, \bar{g}_0)\) to \( \gamma \) means that the observer sees, at the same instant of time, many images of the same source.

**Remark 4.13.** In view of the importance of the Fermat metric completeness in the statement of Proposition 4.9, it is natural to ask under what conditions on \( g_0, \beta \) and \( \delta \), the Fermat metric \( F \) is forward or backward complete. In the paper [33], it is proved that a conformally standard stationary spacetime is globally hyperbolic, with Cauchy surface \( M_0 \times \{0\} \), and then by Theorem 4.8 its Fermat metric is
forward and backward complete, if \( g_0 \) is complete and \( \beta \) and \( |\delta|_0^2 \) have at most quadratic growth at infinity i.e. there exist constants \( c_1, c_2, c_3, c_4 \geq 0 \) such that
\[
|\delta(x)|_0^2 \leq c_1 \text{dist}_0^2(x, x_0) + c_2, \\
\beta(x) \leq c_3 \text{dist}_0^2(x, x_0) + c_4,
\]
where \( x_0 \) is any fixed point in \( M_0 \), \( |\cdot|_0 \) is the norm associated to the metric \( g_0 \) and \( \text{dist}_0 \) is the distance on \( M_0 \) induced by the metric \( g_0 \). On the other hand, we can obtain a condition for the Fermat metric completeness directly from Remark 4.1. In fact, it is enough to show that \( g_0/\beta \) is complete and \( ||\omega|| < 1 \). Using the Cauchy-Schwarz inequality \( g_0(y, y) \geq g_0(\delta, y)^2/|\delta|_0^2 \), we obtain a sufficient condition for \( ||\omega|| < 1 \) as
\[
\sup_{x \in M_0} \frac{|\delta(x)|_0}{\sqrt{|\delta(x)|_0^2 + \beta(x)}} < 1. \tag{47}
\]

4.3. Timelike geodesics with fixed energy in stationary spacetimes. In this subsection we reconsider the Fermat metric on a one-dimensional higher manifold in order to prove multiplicity of timelike geodesics with fixed energy on a standard stationary spacetime \((M, g)\), where \( g \) is given by (38). Observe that, as timelike geodesics are not invariant under conformal changes of the metric, we are now obliged to consider only standard stationary spacetimes. The idea is to use a Kaluza-Klein model without the electromagnetic field (see [10] for an existence result of solutions for the relativistic Lorentz force equation based on Kaluza-Klein).

More precisely, we seek for timelike geodesics \( z : [0, 1] \to M \) connecting a point \((x_0, t_0) \in M \) with a timelike curve \( \gamma(s) = (x_1, s) : \mathbb{R} \to M \) and having a priori fixed energy \( E_z = g(z)[\dot{z}, \dot{z}] = -E < 0 \), for all \( s \in [0, 1] \).

We extend the Riemannian manifold \( M_0 \) to the manifold \( N_0 = M_0 \times \mathbb{R} \) endowed with the metric \( n_0 = g_0 + du^2 \), where \( u \) is the natural coordinate on \( \mathbb{R} \), and we associate to the manifold \( N_0 \) the one dimensional higher Lorentzian manifold \((\overline{N}, n)\), with the metric \( n \) defined as
\[
n(x, u, t)(y, v, \tau)(y, v, \tau) = g_0(x)[y, y] + v^2 + 2g_0(x)[\delta(x), y]\tau - \beta(x)\tau^2. \tag{48}
\]
Since \( \partial_u \) is a Killing vector field for the metric \( n \), geodesics \( \zeta(s) = (x(s), u(s), t(s)) \) in \((\overline{N}, n)\) have to satisfy also the conservation law
\[
n[\dot{\zeta}, \partial_u] = \text{const.},
\]
which implies that the \( u \) component of a geodesic is an affine function. Moreover the projection \( z(s) = (x(s), t(s)) \) on \( M \) of \( \zeta \) is a geodesic for \((M, g)\). In particular lightlike geodesics for the metric \( n \) satisfy the following equation
\[
g_0[\dot{x}, \dot{x}] + 2g_0[\delta, \dot{x}]\dot{\tau} - \beta\dot{\tau}^2 = -\dot{u}^2 = \text{const}.
\]
Thus in order to find timelike geodesics \( z = (x, t) \) in \((M, g)\) with fixed energy \(-E < 0\) it is enough to find lightlike geodesics in \((\overline{N}, n)\) whose \( u \) component has derivative equal to \( \sqrt{E} \). Fermat’s principle in Subsection 4.2 can be restated in \((\overline{N}, n)\), reducing lightlike geodesics on \((\overline{N}, n)\) to geodesics for the Fermat metric \( \bar{F} \) on the manifold \( N_0 \), where \( \bar{F} \) is given by
\[
\bar{F}((x, u), (y, v)) = \sqrt{g_0[y, y] + v^2/\beta(x) + g_0[\delta(x), y]^2 + g_0[\delta(x), y]}, \tag{49}
\]
for all \( ((x, u), (y, v)) \in T_{N_0} \). We recall that \( \bar{g}_0 = g_0/\beta \). Therefore for any value of energy \(-E < 0\) we obtain the following result, which improves previous results
about timelike geodesics with fixed energy on standard stationary Lorentzian manifolds as in [7], where \( \delta = 0 \), and [14], where only some ranges of values for \( E \) are allowed.

**Proposition 4.14.** Let \((\mathcal{M}, g)\) be a standard stationary Lorentzian manifold. Assume that \((\mathcal{M}_0, F)\) is forward or backward complete and moreover assume that \(\mathcal{M}_0\) is non-contractible, then there exist infinitely many timelike geodesics \( z_n = (x_n, t_n) \) connecting the point \((\bar{x}, t_0) \in \mathcal{M}\) with the timelike curve \( \gamma(s) = (\bar{x}, s) \), parameterized on the interval \([a, b]\), having fixed energy \( -E \) and diverging arrival time.

**Proof.** Observe that if \( \{(x_n, u_n)\} \subset \mathcal{N}_0 \) is a forward Cauchy sequence for the Randers metric \( \bar{F} \) defined at (49), then \( \{x_n\} \subset M \) is a forward Cauchy sequence for the Fermat metric \( F \) on \( M_0 \) defined in 4.2. Hence \( x_n \) converges and \( \beta \) is bounded on \( \{x_n\} \). Thus also \( u_n \) is a Cauchy sequence in \( \mathbb{R} \) and therefore \( \{(x_n, u_n)\} \) converges, i.e. \((\mathcal{N}_0, \bar{F})\) is forward complete. Then apply Proposition 3.2 to the Randers manifold \((\mathcal{N}_0, \bar{F})\) and to the functional \( J((x, u)) = \int_a^b \bar{F}^2((x, u), (\dot{x}, \dot{u})) ds \) defined on the manifold

\[
\Lambda_{\{(\bar{x}, aE^{1/2})\} \times \{(\bar{x}, bE^{1/2})\}}(\mathcal{N}_0)
\]

(here the curves are parametrized on \([a, b]\)) and use Fermat’s principle on the manifold \((\mathcal{N}, n)\), between the point \((\bar{x}, aE^{1/2}, t_0)\) and the curve \( s \to (\bar{x}, bE^{1/2}, s) \). □ □

The case \( E = 1 \) is the most interesting one, because timelike geodesics with \( E = 1 \) correspond to test particles, freely falling in the gravitational field \( g \), parameterized with respect to the *proper time* (see [18]). In such a case, fixing the interval of parameterization is equivalent to fixing the arrival proper time of the trajectory.

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