Research article

Existence, uniqueness and approximation of nonlocal fractional differential equation of sobolev type with impulses

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Abstract: This paper is concerned with the study of nonlocal fractional differential equation of sobolev type with impulsive conditions. An associated integral equation is obtained and then considered a sequence of approximate integral equations. By utilizing the techniques of Banach fixed point approach and analytic semigroup, we obtain the existence and uniqueness of mild solutions to every approximate solution. Then, Faedo-Galerkin approximation is used to establish certain convergence outcome for approximate solutions. In order to illustrate the abstract results, we present an application as a conclusion.

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1. Introduction

Research work in the area of fractional differential equation is multidisciplinary such as control systems, elasticity, circuit systems, heat transfer, fluid mechanics, signal analysis, traffic flow, pollution control, etc. It is considered as an alternative model to nonlinear differential equation. Fractional differential equations are a useful tool in modelling several events. In [1], controllability of Hilfer fractional neutral differential systems with infinite delay is obtained. An article obtained on
Neutral fractional stochastic partial differential equations with Clarke subdifferential is investigated using fractional calculus and fixed point theorems in [2]. Nisar et al. [3], in their publication, briefly discussed the analysis of controllability for nonlinear Hilfer neutral fractional derivatives via fractional calculus and Banach contraction principle. Numerous credible theoretical studies on fractional differential equation can be referred to books in [4–8] and the research articles are [9–15]. The fractional differential equation has many solutions with nonlocal conditions, impulsive type and sobolev type. In which, the nonlocal conditions is a generalization of the classical conditions, was motivated by the physical phenomena. The pioneering work on nonlocal condition is due to Byszewski [16]. Papers related on nonlocal conditions, we may refer [17–20].

Moreover, the theory of fractional impulsive differential equations has been entirely developed during the past decades. Since 1990’s many mathematician have derived lots of results on differential equations undergoing impulsive effects. It focuses on the analysis of dynamic processes that experience abrupt changes. Impulses have a relatively shorter time difference between changes than the overall length of the process. The following may act as motivation to investigate such systems using impulsive differential equations. Consider the simplest scenario for a person’s hemodynamic equilibrium. Some injectable medications, such as insulin, may be provided in the case of a de-compensation, such as high or low glucose levels. It is clear that the entrance of medications into the circulation and the body’s subsequent absorption are slow and ongoing processes. This circumstance might be seen as an impulsive activity that begins suddenly and lasts for a set amount of time. For detailed information about the impulsive fractional differential equation and its applications, we refer to the readers [21–25].

On the other hand, the Sobolev type differential system is typically seen in the mathematical structure of numerous physical events, such as fluid flow through fractured rocks, thermodynamics. Additionally, Sobolev type differential equations are utilised to describe the attributes of systems and processes in mathematical modelling and simulations. For more literature on sobolev type differential equation, see [26–30] and references therein. In addition, one of the most effective methods for determining out approximations of solutions to a given differential equation in an abstract space is the Faedo-Galerkin approach. The Faedo-Galerkin method may be used within a variational formulation in order to provide solutions of the equations under possibly weaker assumption on the data and may also prove a very useful tool for numerical approximation of equations. A detailed view on Faedo-Galerkin approximation we refer [31–35].

In accordance with the aforementioned literature survey, there are relatively few work outcomes that explore the existence and uniqueness of a mild solution to a Sobolev type fractional differential equation with Impulses applying a fixed point technique. This fact is the fundamental motivator behind our current progress. This article [36, 37], outlines the nonlocal Sobolev type fractional differential impulsive system as follows:

\[
\begin{align*}
\mathcal{D}_\sigma^\beta \left[ Mx(\sigma) \right] &= \mathcal{L}x(\sigma) + F(\sigma, x(\sigma), x(h(\sigma))), \sigma \in [0, T], \\
\Delta x(\sigma_i) &= I_i(x(\sigma_i)), \quad i = 1, 2, \ldots, q, \quad q \in \mathbb{N}, \\
g(x) &= \phi \in \mathbb{H}_1. 
\end{align*}
\]  

Where \( 0 < \beta < 1, \ T \in (0, \infty) \) \( \mathcal{D}_\sigma^\beta \) is the Caputo fractional derivative, \( 0 = \sigma_0 < \sigma_1 < \ldots < \sigma_q < \sigma_{q+1} = T \) are pre-fixed numbers, \( \Delta x|_{\sigma=\sigma_i} = x(\sigma_i^+) - x(\sigma_i^-) \) and \( x(\sigma_i^+) = \lim_{h \to 0^+} x(\sigma_i + h) \) and \( x(\sigma_i^-) = \lim_{h \to 0^-} x(\sigma_i + h) \) denote the right and left limits of \( x(\sigma) \) at \( \sigma = \sigma_i \), respectively. From (1.1), assume
\( \mathcal{L} : D(\mathcal{L}) \subset \mathbb{H}_1 \to \mathbb{H}_2 \) and \( \mathcal{M} : D(\mathcal{M}) \subset \mathbb{H}_1 \to \mathbb{H}_2 \) are closed (unbounded), positive and self-adjoint operators, where \( \mathbb{H}_1 \) and \( \mathbb{H}_2 \) are Hilbert spaces and the appropriate functions are \( \mathcal{F} : [0, T] \times \mathbb{H}_1 \to \mathbb{H}_1 \) and \( g : C([0, T], \mathbb{H}_1) \to \mathbb{H}_1 \), \( h : [0, T] \to [0, T] \), \( I_t : \mathbb{H}_1 \to \mathbb{H}_1 \).

This article is organized as:

Basic concepts and lemmas are covered in Section 2. In Section 3, the fixed point theorem is used to determine the existence and uniqueness of an approximate solution. In Section 4, the convergence of the approximate solutions is obtained. In Section 5, the convergence of approximate Faedo-Galerkin solutions is proved. Finally, we provide a theoretical application to assist in the effectiveness of our result.

2. Basic results

The upcoming segment recalls the necessary things to obtain the primary facts of our discussion.

Let \((\mathbb{H}_1, \| \cdot \|, < \cdot, \cdot >), (\mathbb{H}_2, \| \cdot \|, < \cdot, \cdot >)\) be Separable Hilbert spaces. Assume \( C([0, T], \mathbb{H}_1) \) from \([0, T]\) into \( \mathbb{H}_1 \) with \( |x|_{[0, T]} := \sup \{ ||x(\sigma)|| : \sigma \in [0, T] \} \) be a Banach space of continuous function and boundedness of linear operator \( L(\mathbb{H}_1) \) equipped with \( \|f\|_{L(\mathbb{H}_1)} = \sup \{ ||f(x)|| : ||x|| = 1 \} \).

**Definition 2.1.** [37] The R-L integral of order \( \beta > 0 \) is

\[
\mathcal{J}_\sigma^{\beta} \mathcal{F}(\sigma) = \frac{1}{\Gamma(\beta)} \int_0^\sigma (\sigma - s)^{\beta-1} \mathcal{F}(s) ds, \tag{2.1}
\]

where \( \mathcal{F} \in L^1((0, T), \mathbb{H}_1) \).

**Definition 2.2.** [37] The R-L derivative is

\[
^RLD_\sigma^\beta \mathcal{F}(\sigma) = D_\sigma^\beta \mathcal{J}_\sigma^{\beta-\delta} \mathcal{F}(\sigma), |\beta - \delta| \in (0, 1), \delta \in \mathbb{N}, \tag{2.2}
\]

where \( D_\sigma^\beta = \frac{d^\beta}{d\tau^\beta}, \mathcal{F} \in L^1((0, T), \mathbb{H}_1), \mathcal{J}_\sigma^{\delta-\beta} \mathcal{F} \in W^{\delta,1}((0, T), \mathbb{H}_1) \).

**Definition 2.3.** [37] The Caputo derivative is

\[
^cD_\sigma^\beta \mathcal{F}(\sigma) = \frac{1}{\Gamma(\delta - \beta)} \int_0^\sigma (\sigma - s)^{\delta-\beta-1} \mathcal{F}(s) ds, \delta - 1 < \beta < \delta, \tag{2.3}
\]

where \( \mathcal{F} \in C^{\delta-1}((0, T), \mathbb{H}_1) \cap L^1((0, T), \mathbb{H}_1) \),

\[
\mathcal{J}_\sigma^{\beta}(^cD_\sigma^\beta \mathcal{F}(\sigma)) = \mathcal{F}(\sigma) - \sum_{k=0}^{\delta-1} \frac{\sigma^k}{k!} \mathcal{F}^{(k)}(0) \text{ holds.}
\]

Operators \( \mathcal{L}, \mathcal{M} \) impose the following conditions:

(a) \( \mathcal{L} \) and \( \mathcal{M} \) are closed linear operators;
(b) \( D(\mathcal{M}) \subset D(\mathcal{L}) \) and \( \mathcal{M} \) is bijective;
(c) \( \mathcal{M}^{-1} : \mathbb{H}_2 \to D(\mathcal{M}) \subset \mathbb{H}_1 \) is compact.
Conditions (a)-(c) and closed graph theorem imply $\mathcal{L}\mathcal{M}^{-1} : \mathbb{H}_2 \to \mathbb{H}_2$ is the boundedness of the linear operator. Therefore, an infinitesimal generator $\mathcal{E} = \mathcal{L}\mathcal{M}^{-1}$ of semigroup $S(\sigma) := e^{\mathcal{L}\mathcal{M}^{-1}}$ and so $\max_{\sigma \in I} ||S(\sigma)||$ is finite. We have the following integral as per prior definition,

$$Mx(\sigma) = Mx(0) + \sum_{i=1}^{q} I_i(x(\sigma_i)) + \int_{0}^{\sigma} \left( (\sigma - s)^{\beta-1} \Gamma(\beta) \right) [Lx(s) + F(s, x(s))] ds, \quad \sigma \in [0, T].$$

(2.4)

The above (2.4) exists a.e. As a result, aforementioned equalization is equal to the impulsive differential equation of Sobolev type. Therefore, there exists $N_0 \geq 1$ a positive constant such that $||S(\sigma)|| \leq N_0$. Let the resolvent set of $\mathcal{E}$ be $\rho(\mathcal{E})$. Hence, $\mathcal{E}^\alpha$, $0 < \alpha \leq 1$ be the fractional power which is a closed linear operator and $D(\mathcal{E}^\alpha)$ is a subspace, in such a way its simple to show that it is a Banach space with supremum norm and is represented as $(\mathbb{H}_1)_\alpha$ with $(|| \cdot ||_\alpha)$. We have $(\mathbb{H}_1)_\eta \hookrightarrow (\mathbb{H}_1)_\alpha$, $0 < \alpha < \eta$ so the embedding is continuous. Then, we define $(\mathbb{H}_1)_{-\alpha} = ((\mathbb{H}_1)_\alpha)^*, \alpha > 0$, dual space of $(\mathbb{H}_1)_\alpha$, is a Banach space equipped with $||x||_{-\alpha} = ||\mathcal{E}^{-\alpha}x||, x \in (\mathbb{H}_1)_{-\alpha}$.

**Proposition 2.1.** [38] Assume $\mathcal{E}$ of $S(\sigma), \sigma \geq 0, 0 \in \rho(\mathcal{E})$ is an infinitesimal generator. We get

(i) For $\sigma > 0$, $\alpha \geq 0$, $S(\sigma)$ maps from $\mathbb{H}_1 \to D(\mathcal{E}^\alpha)$.

(ii) For each $x \in D(\mathcal{E}^\alpha)$, $S(\sigma)\mathcal{E}^\alpha x = \mathcal{E}^\alpha S(\sigma)x$.

(iii) Let $\left\| \frac{d}{d\sigma} S(\sigma) \right\| \leq N_j, j = 1, 2, \sigma > 0$, where $N_j$ is a positive constant.

(iv) A bounded operator $\mathcal{E}^\alpha S(\sigma), ||\mathcal{E}^\alpha S(\sigma)|| \leq N_0 \sigma^{-\alpha} e^{-\lambda \sigma}, \sigma > 0$.

(v) If $x \in D(\mathcal{E}^\alpha), \alpha \in (0, 1]$ implies $||S(\sigma)x - x|| \leq C_0 \sigma^\alpha ||\mathcal{E}^\alpha x||$.

**Remark 2.1.** [38] The boundedness of the linear operator $\mathcal{E}^{-\alpha}$ in $\mathbb{H}_1$ such that $D(\mathcal{E}^\alpha) = \text{Im}(\mathcal{E}^{-\alpha})$. Let’s denote $(\mathbb{H}_1)_\alpha(T) = C([0, T], (\mathbb{H}_1)_\alpha)$ be Banach space of all $(\mathbb{H}_1)_\alpha$-valued continuous function equipped with $||x||_{(\mathbb{H}_1)_\alpha(T)} = \sup_{\sigma \in [0, T]} ||x(\sigma)||_{\alpha},$ such that $x(\sigma)$ is continuous on $\sigma \neq \sigma_j$, left continuous at $\sigma = \sigma_j$ and right limit $x(\sigma_j^+) \exists$ exists for $j = 1, 2, ..., q$.

3. Existence of solutions

We inspect the existence of (1.1)–(1.3) as well as their uniqueness. The respective assumptions on $\mathcal{E}, \mathcal{F}, h, I(i = 1, 2, ..., q)$ is presented as:

(1) Let $\mathcal{E}$ be closed, positive definite and self-adjoint linear operator : $D(\mathcal{E}) \subset \mathbb{H}_2 \to \mathbb{H}_2$. A pure point spectrum $\mathcal{E}$ has

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m \leq ...,$$

with $\lambda_m \to \infty, m \to \infty$ and complete orthonormal system $\{\phi_j\}$,

$$\mathcal{E}\phi_j = \lambda_j\phi_j \text{ and } \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

(3.1)

where

$$\delta_{ij} = \begin{cases} 1, & j = 1 \\ 0, & \text{otherwise.} \end{cases}$$
Remark 3.1.

and PDF $\xi_\beta(\xi)$, i.e., $\xi_\beta(\xi) \geq 0$, $\int_0^\infty \xi_\beta(\xi) d\xi = 1$.

Remark 3.1. [38] Let $0 \leq \nu \leq 1$,

$$\int_0^\infty \xi^\nu \xi_\beta(\xi) d\xi = \int_0^\infty \xi^{\nu-\beta} \psi_\beta(\xi) d\xi = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \beta \nu)}.$$

Definition 3.1. [39] Let $x : [0, T] \rightarrow (H^1)_{s0}$ be a continuous function, if $x(0) = x_0$ and $x(\cdot)$ satisfies the following integral equation

$$x(\sigma) = \begin{cases} S_\beta(\sigma)[M][\chi](0) + \sum_{i=1}^q S_\beta(\sigma - \sigma_1)I_i(x(\sigma_i)) \
+ \int (\sigma - s)^{\beta-1}T_\beta(\sigma - s)F(s, x(s), x(h(s)))ds, \ \sigma \in [0, T], \end{cases}$$

is known as mild solution of (1.1)–(1.3), where

$$S_\beta(\sigma) = \int_0^\infty M^{-1}\zeta_\beta(\xi)S(\sigma^\beta \xi)d\xi,$$

$$T_\beta(\sigma) = \int_0^\infty M^{-1}\beta \xi \zeta_\beta(\xi)S(\sigma^\beta \xi)d\xi,$$

$$\zeta_\beta(\xi) = \frac{1}{\beta \xi^{\beta-1}} \psi_\beta(\xi^{\beta-1}) \geq 0,$$

$$\psi_\beta(\xi) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \xi^{-n\beta-1} \frac{\Gamma(n\beta + 1)}{n!} \sin(n\pi \beta), 0 < \xi < \infty,$$

and PDF $\zeta_\beta(\xi)$. i.e., $\zeta_\beta(\xi) \geq 0$, $\int_0^\infty \zeta_\beta(\xi) d\xi = 1$.

(2) The continuous mapping $\mathcal{F} : [0, \infty) \times (H^1)_{s0} \times (H^1)_{s0} \rightarrow H_2$ and $m_R : [0, \infty) \rightarrow (0, \infty)$ an increasing function exists, on $R > 0$ such that

$$\|\mathcal{F}(\sigma, z, w)\| \leq m_R(\sigma), \quad (3.2)$$

$$\|\mathcal{F}(\sigma_1, z_1, w_1) - \mathcal{F}(\sigma_2, z_2, w_2)\| \leq m_R(\sigma)[|\sigma_1 - \sigma_2|^{\theta_1} + \|z_1 - z_2\|_s], \quad (3.3)$$

for all $(\sigma, z, w), (\sigma_1, z_1, w_1), (\sigma_2, z_2, w_2) \in [0, \infty) \times (H^1)_{s0} \times (H^1)_{s0}$ where $B_R(H^1) = \{z \in H^1 : \|z\|_{H^1} \leq R\}$ and $\theta_1 \in (0, 1)$.

(3) Let nonlinear function $h : [0, T] \rightarrow [0, T]$ such that $h(\sigma) \leq \sigma$, $0 \leq \sigma \leq T$ and $3$ constants $L_h > 0$ such that

$$|h(\sigma_1) - h(\sigma_2)| \leq L_h|\sigma_1 - \sigma_2|, \quad \sigma_1, \sigma_2 \in [0, T]. \quad (3.4)$$

(4) There exist $\chi \in C([0, T], (H^1)_{s0})$ such that $g(\chi) = \phi$ and $\chi(\sigma)$ is locally Lipschitz continuous.

(5) All the function $I_i : H^1 \rightarrow H^1$ $(i = 1, 2, ..., q)$ are continuous function such that

$$\|I_i(x)\|_s \leq O_i, \forall \alpha \in (0, 1),$$

$$\|I_i(x_1) - I_i(x_2)\|_s \leq N_i\|x_1 - x_2\|_s, \forall x_1, x_2 \in H^1,$$

where $O_i, N_i, i = 1, 2, ..., q$ are positive constants.
Proposition 3.1. [26] Let $S(\sigma)$ be a uniformly continuous semigroup and $\mathcal{E}$ be its infinitesimal generator. Then, $S_\beta(\sigma)$ and $T_\beta(\sigma)$ are boundedness of the linear operator such that

(i) $\|S_\beta(\sigma)x\| \leq W_1 N_0 \|x\|$ and $\|T_\beta(\sigma)x\| \leq \frac{W_1 N_0}{1 - \beta} \|x\|$, $x \in \mathbb{H}_1$.

(ii) The strong continuity of $\{T_\beta(\sigma), \sigma \geq 0\}$ and $\{S_\beta(\sigma), \sigma \geq 0\}$ $0 \leq \tau_1 < \tau_2 \leq T$, for $x \in \mathbb{H}_1$, we have $\|T_\beta(\tau_2)x - T_\beta(\tau_1)x\| \to 0$ and $\|S_\beta(\tau_2)x - S_\beta(\tau_1)x\| \to 0$ as $\tau_2 \to \tau_1$.

(iii) Suppose $S(\sigma)$, $\sigma \geq 0$ is compact, then $S_\beta(\sigma)$ and $T_\beta(\sigma)$ are compact operators.

(iv) For each $x \in \mathbb{H}_1$, we have $\mathcal{E}T_\beta(\sigma)x = \mathcal{E}^{1-\eta}T_\beta^{\eta}x$, $\sigma \in [0, T]$, $\eta \in (0, 1)$. We have $\|\mathcal{E}^\alpha T_\beta(\sigma)\| \leq \frac{W_1 N_0 \Gamma(1-\alpha)}{\Gamma(1 + \beta(1-\alpha))} \sigma^{-\alpha \beta}$, $\sigma \in [0, T]$, $\alpha \in (0, 1)$.

(v) For any $x \in \mathcal{F}_\alpha$ and fixed $\sigma \geq 0$, we have $\|S_\beta(\sigma)x\|_\alpha \leq W_1 N_0 \|x\|_\alpha$ and $\|T_\beta(\sigma)x\|_\alpha \leq \frac{W_1 N_0}{1 - \beta} \|x\|_\alpha$.

Arbitrary fixed point $T_0 > 0$ such that $0 < T < T_0 < \infty$,

$$\psi = \frac{W_1 N_0 \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} m_\alpha(T_0) \frac{T^{\beta(1-\alpha)}}{(1 - \alpha)} < 1.$$  

(3.6)

Let $\mathcal{H}_n$ be finite dimensional subspace, spanned by $\{\phi_0, \phi_1, ..., \phi_n\}$ and a projection operator $P^n : \mathbb{H}_1 \to \mathcal{H}_n$, $n = 0, 1, ...$. Assume $\mathcal{F}_n : [0, T] \times (\mathbb{H}_1)_\alpha \to \mathbb{H}_1$ and $I_{in} : \mathbb{H}_1 \to \mathbb{H}_1$, is defined by

$$\mathcal{F}_n(\sigma, x(\sigma), x(h(\sigma))) = \mathcal{F}(\sigma, P^n x(\sigma), P^n x(h(\sigma))), \quad (3.7)$$

$$I_{in}(x) = I_i(P^n x), \forall x \in \mathbb{H}_1, n = 0, 1, 2, ..., i = 1, 2, ... q, \quad (3.8)$$

and the operator $Q_n$ on $\mathbb{B}$ as follows

$$(Q_n x)(\sigma) = \begin{cases} S_\beta(\sigma) M_\chi(0) + \sum_{i=1}^{q} S_\beta(\sigma - \sigma_i) I_{in}(\sigma) \\ + \int_0^\sigma (\sigma - s)^{\beta - 1} T_\beta(\sigma - s) \mathcal{F}_n(s, x(s), x(h(s))) ds, \sigma \in [0, T]. \end{cases} \quad (3.9)$$

Theorem 3.1. Assume (1)–(5) holds, then $x_n \in \mathbb{B}_{\mathcal{E}}((\mathbb{H}_1)_\alpha(T))$ be a unique fixed point of $Q_n$ exists i.e., $Q_n x_n = x_n$ for each $n = 0, 1, 2, ...$ and $x_n$ fulfills the approximate integral equation,

$$x_n(\sigma) = \begin{cases} S_\beta(\sigma) M_\chi(0) + \sum_{i=1}^{q} S_\beta(\sigma - \sigma_i) I_{in}(x_n(\sigma_i)) \\ + \int_0^\sigma (\sigma - s)^{\beta - 1} T_\beta(\sigma - s) \mathcal{F}_n(s, x_n(s), x(h(s))) ds, \sigma \in [0, T]. \end{cases} \quad (3.10)$$

Proof. Let $Q_n : \mathbb{B}_{\mathcal{E}}((\mathbb{H}_1)_\alpha(T)) \to \mathbb{B}_{\mathcal{E}}((\mathbb{H}_1)_\alpha(T))$ is defined by

$$(Q_n x)(\sigma) = \begin{cases} S_\beta(\sigma) M_\chi(0) + \sum_{i=1}^{q} S_\beta(\sigma - \sigma_i) I_{in}(x(\sigma_i)) \\ + \int_0^\sigma (\sigma - s)^{\beta - 1} T_\beta(\sigma - s) \mathcal{F}_n(s, x(s), x(h(s))) ds, \sigma \in [0, T]. \end{cases} \quad (3.10)$$

We will demonstrate that $Q_n$ is well defined. This is sufficient to demonstrate that the map $\sigma \mapsto (Q_n x)(\sigma) : [0, T]$ into $(\mathbb{H}_1)_\alpha$ w.r.t. $\alpha$ norm is continuous.

Let $\sigma_1, \sigma_2 \in [0, T]$ with $\sigma_2 > \sigma_1$, we get
\[
\||Q_n x(\sigma_2) - Q_n x(\sigma_1)\||_\sigma = \||S_{\beta}(\sigma_2) - S_{\beta}(\sigma_1)\||_{M_Y(0)} \\
+ \sum_{i=1}^{q} \||S_{\beta}(\sigma_2 - \sigma_i) - S_{\beta}(\sigma_1 - \sigma_i)\|_{I_{i,n}(x(\sigma_i))} \||_{\sigma-1} \\
+ \|\int_{\sigma}^{\sigma_2} (\sigma_2 - s)^{\beta-1} T_{\beta}(\sigma_2 - s) F_n(s, x(s), x(h(s))) ds \|_{\sigma} \\
+ \|\int_{0}^{\sigma_1} (\sigma_2 - s)^{\beta-1} T_{\beta}(\sigma_2 - s) F_n(s, x(s), x(h(s))) ds \|_{\sigma} \\
- \|\int_{0}^{\sigma_1} (\sigma_1 - s)^{\beta-1} T_{\beta}(\sigma_1 - s) F_n(s, x(s), x(h(s))) ds \|_{\sigma} \\
\leq \||S_{\beta}(\sigma_2) - S_{\beta}(\sigma_1)\||_{M_Y(0)} \\
+ \sum_{i=1}^{q} \||S_{\beta}(\sigma_2 - \sigma_i) - S_{\beta}(\sigma_1 - \sigma_i)\|_{E^{\alpha-1} I_{i,n}(x(\sigma_i))} \||_{\sigma-1} \\
+ \int_{\sigma_1}^{\sigma} (\sigma_2 - s)^{\beta-1} \|E^{\alpha} T_{\beta}(\sigma_2 - s)\|_{F_n(s, x(s), x(h(s)))} ds \\
+ \int_{0}^{\sigma_1} (\sigma_1 - s)^{\beta-1} \|E^{\alpha} [T_{\beta}(\sigma_1 - s) - T_{\beta}(\sigma_2 - s)]\|_{F_n(s, x(s), x(h(s)))} ds \\
+ \int_{0}^{\sigma_1} [(\sigma_1 - s)^{\beta-1} - (\sigma_2 - s)^{\beta-1}] \|E^{\alpha} T_{\beta}(\sigma_2 - s)\|_{F_n(s, x(s), x(h(s)))} ds \\
\leq \||S_{\beta}(\sigma_2) - S_{\beta}(\sigma_1)\||_{M_Y(0)} + W_1 N_a \sum_{i=1}^{q} O_i \|E^{\alpha}\|((\sigma_2 - \sigma_1)) \\
+ \beta W_1 N_a \Gamma(2 - \alpha) m_R(T_0) \frac{m_R(T_0)}{\beta(1 - \alpha)} \frac{m_R(T_0)}{\Gamma(1 + \beta(1 - \alpha))} + \beta W_1 N_a \Gamma(2 - \alpha) m_R(T_0) \frac{m_R(T_0)}{\Gamma(1 + \beta(1 - \alpha))} \\
\times \int_{0}^{\sigma_1} (\sigma_1 - s)^{\beta-1} [(\sigma_1 - s)^{-\alpha \beta} - (\sigma_2 - s)^{-\alpha \beta}] ds + \beta W_1 N_a \Gamma(2 - \alpha) \frac{m_R(T_0)}{\Gamma(1 + \beta(1 - \alpha))} \\
\times m_R(T_0) \int_{0}^{\sigma_1} [(\sigma_1 - s)^{\beta-1} - (\sigma_1 - s)^{-\alpha-1}] (\sigma_2 - s)^{-\alpha \beta} ds. \quad (3.11)
\]

For \( x \in \mathbb{H}_1 \), we have,

\[
[S(\sigma_2^\beta \xi) - S(\sigma_1^\beta \xi)] = \int_{\sigma_1}^{\sigma_2} \frac{d}{d\sigma} S(\sigma^\beta \xi) x d\sigma = \int_{\sigma_1}^{\sigma_2} \beta \xi (\sigma_{\beta}^{-1} E S(\sigma^\beta \xi)) d\sigma.
\]
Thus, we get

\[
\int_0^\infty M^{-1} \xi_\beta(\xi) ||S(\sigma_2^\beta \xi) - S(\sigma_1^\beta \xi)|| M^* \chi(0) || d\xi \leq \int_0^\infty M^{-1} \xi_\beta(\xi) \left( \int_{\sigma_1}^{\sigma_2} \left| \frac{d}{ds} S(\sigma^\beta \xi) \right| d\sigma \right) M^* ||\chi(0)|| d\xi
\]

\[
\leq \int_0^\infty M^{-1} \xi_\beta(\xi) [N_1(\sigma_2 - \sigma_1)] M^* ||\chi(0)|| d\xi
\]

\[
\leq R_1(\sigma_2 - \sigma_1),
\]

(3.12)

where \( R_1 = N_1 W_1 ||M|| ||\chi(0)|| \).

From (3.11), we have

\[
\int_0^\sigma (\sigma - s)^{\beta - 1} [(\sigma_2 - s)^{-\beta\alpha} - (\sigma_1 - s)^{-\beta\alpha}] ds \leq vd_{i_1}^{-1} (1 - h)^{-p_1(1 - \nu) - 1}(\sigma_2 - \sigma_1)^{p_1(1 - \nu)}, \quad (3.13)
\]

where \( h = [1 - \left(\frac{1 - \beta}{\beta} \right)^{1/\nu}] \), \( p_1 = 1 - \beta \alpha \), \( v = \frac{(1 - \beta)}{1 - \beta \alpha} \) and \( 0 < d_1 \leq 1 \).

\[
\int_0^\sigma [(\sigma_1 - s)^{\beta - 1} - (\sigma_2 - s)^{\beta - 1}](\sigma_2 - s)^{-\beta \alpha} ds \leq \frac{N_1 + a}{\sigma} b_1^{\sigma - 1} (1 - h_1)^{-\beta(1 - \alpha) - 1}(\sigma_2 - \sigma_1)^{\beta(1 - \alpha)}, \quad (3.14)
\]

where \( h_1 = (1 - \left(\frac{1}{\beta} \right)^{1/\nu}) \), \( 0 < b_1 \leq 1 \) and \( N_1 + a \) is some positive constant with \( ||E^{\sigma + 1} S(\sigma)|| \leq N_1 + a \sigma^{-1 - \alpha}, \forall \sigma > 0 \). Thus, from the inequalities (3.11)–(3.14), (2).

We conclude that \( \sigma \mapsto \mathcal{F}_n(\sigma, x(\sigma)) \) map is uniformly Hölder’s continuous. We justify

\[
Q_n(\mathbb{R}(\mathcal{H}_1)_a(T)) \subseteq \mathcal{B}_R((\mathcal{H}_1)_a(T)). \quad \text{Let } x \in \mathcal{B}_R((\mathcal{H}_1)_a(T)), \quad 0 \leq \sigma \leq T. \quad \text{We get}
\]

\[
\|Q_n x(\sigma)\|_a \leq \|S_\beta(\sigma) M \chi(0)\|_a + \sum_{i=1}^q \|S_\beta(\sigma - \sigma_i) I_{\sigma_i}(x(\sigma_i))\|_a
\]

\[
+ \left\| \int_0^\sigma (\sigma - s)^{\beta - 1} \mathcal{F}_n(\sigma - s) \mathcal{F}_n(s, x(s), x(h(s))) ds \right\|_a
\]

\[
\leq W_1 ||M|| ||N_0|| \chi(0)\|_a + W_1 N_\alpha \sum_{i=1}^q O_i + \frac{\beta W_1 N_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^\sigma (\sigma - s)^{\beta(1 - \alpha) - 1} m_\alpha(T_0) ds
\]

\[
\leq W_1 ||M|| ||N_0|| \chi(0)\|_a + W_1 N_\alpha \sum_{i=1}^q O_i + \frac{W_1 N_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} m_\alpha(T_0) \frac{T^{\beta(1 - \alpha)}(1 - \alpha)}{1 - \alpha}, \quad (3.15)
\]

We may now take \( R \) as a positive integer such that,

\[
R = W_1 ||M|| ||N_0|| \chi(0)\|_a + W_1 N_\alpha \sum_{i=1}^q O_i + \frac{W_1 N_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} m_\alpha(T_0) \frac{T^{\beta(1 - \alpha)}(1 - \alpha)}{1 - \alpha}.
\]
Therefore, we deduce that \( Q_n(\mathbb{B}_R((\mathbb{H}_1)_{\alpha}(T))) \subseteq \mathbb{B}_R((\mathbb{H}_1)_{\alpha}(T)) \). Lastly, we demonstrate \( Q_n \) is a strict contraction map. For \( x_1, x_2 \in \mathbb{B}_R((\mathbb{H}_1)_{\alpha}(T)) \), \( 0 \leq \sigma \leq T \).

\[
\|(Q_n x_1)(\sigma) - (Q_n x_2)(\sigma)\|_\alpha \leq \int_0^\sigma (\sigma - s)^{\beta - 1} E^\alpha T_{\beta}(\sigma - s) ds \times \|\mathcal{F}_n(s, x_1, x_1(h(s))) - \mathcal{F}_n(s, x_2, x_2(h(s)))\|_\alpha \\
+ \sum_{i=1}^q \|S_{\beta}(\sigma - \sigma_i)\|_\alpha \|I_{i,n}(x_1(\sigma_i)) - I_{i,n}(x_2(\sigma_i))\|_\alpha \\
\leq \frac{W_1 N_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} m_\alpha(T_0) \frac{T^{\beta(1 - \alpha)}}{(1 - \alpha)} \|x_1 - x_2\|_{(\mathbb{H}_1)_{\alpha}(T)} \\
+ W_1 N_\alpha \sum_{i=1}^q N_i \|x_1 - x_2\|_{T;\alpha} \leq \land \|x_1 - x_2\|_{(\mathbb{H}_1)_{\alpha}(T)}. 
\] (3.16)

In Eq (3.16), \( \land = \frac{W_1 N_\alpha \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} m_\alpha(T_0) \frac{T^{\beta(1 - \alpha)}}{(1 - \alpha)} + W_1 N_\alpha \sum_{i=1}^q N_i < 1 \).

As a result, \( Q_n \) is determined to be a contraction mapping. Thus, a unique \( x_n \in \mathbb{B}_R((\mathbb{H}_1)_{\alpha}(T)) \) exists such that \( Q_n x_n = x_n \).

\[\Box\]

**Lemma 3.1.** Assume (1)-(5) holds.

(i) Let \( \chi(0) \in D(\mathcal{E}_\nu) \), \( \alpha \in (0, 1) \) implies \( x_n(\sigma) \in D(\mathcal{E}_\nu) \forall 0 < \sigma \leq T, \nu \in [0, 1) \).

(ii) If \( \chi(0) \in D(\mathcal{E}_\nu) \), implies \( x_n(\sigma) \in D(\mathcal{E}_\nu) \forall 0 \leq \sigma \leq T, \nu \in [0, 1) \).

**Proof.** We get a unique \( x_n \in \mathbb{B}_R((\mathbb{H}_1)_{\alpha}(T)) \) that satisfies (3.10) by using the previous theorem. In [38], for \( \sigma > 0, 0 \leq \nu < 1 \) we get \( S(\sigma) : \mathbb{H}_1 \rightarrow D(\mathcal{E}_\nu), D(\mathcal{M}_\nu) \subseteq D(\mathcal{M}_\nu) \). Also \( S(\sigma)x \in D(\mathcal{E}_\nu) \). As a outcome, of all these facts, \( D(\mathcal{E}_\nu) \subseteq D(\mathcal{E}_\nu), 0 \leq \nu \leq 1 \).

\[\Box\]

**Lemma 3.2.** If (1)-(5) holds.

(i) Let \( \chi(0) \in D(\mathcal{E}_\nu) \), \( \alpha \in (0, 1), 0 < \sigma_0 \leq T, \) then a constant \( S_{\sigma_0} \) exist,

\[
\|x_n(\sigma)\|_\nu \leq S_{\sigma_0}, \ 0 \leq \nu < 1, \ \sigma \in [\sigma_0, T], \text{ independent of } n.
\]

(ii) Let \( \chi(0) \in D(\mathcal{E}_\nu) \), then a constant \( S_0 > 0 \) exists,

\[
\|x_n(\sigma)\|_\nu \leq S_0, \ 0 \leq \nu < 1, \ \sigma \in [0, T], \text{ independent of } n.
\] (3.17)

**Proof.** Let \( \chi(0) \in D(\mathcal{M}_\nu) \). In (3.10), we apply \( \mathcal{E}_\nu \) on both sides, we have

\[
\|\mathcal{E}_\nu x_n(\sigma)\| \leq \|\mathcal{E}_\nu MS_{\beta}(\sigma)\chi(0)\| + \sum_{i=1}^q \|S_{\beta}(\sigma - \sigma_i)\mathcal{E}_\nu I_i(x(\sigma_i))\| \\
+ \int_0^\sigma (\sigma - s)^{\beta - 1}\|\mathcal{E}_\nu T_{\beta}(\sigma - s)\| \|\mathcal{F}_n(s, x_n(s), x(h(s)))\| ds \\
\leq N_\nu W_1\|\mathcal{M}\|\sigma_0^{\nu} \|\chi(0)\| + N_\nu W_1 \beta N_\nu W_1 \Gamma(2 - \nu) \frac{m_\alpha(T_0) T^{\beta(1 - \nu)}}{(1 + \beta(1 - \nu))} \leq S_{\sigma_0}. 
\] (3.18)
Again, if \( \chi(0) \in D(\mathcal{E}) \Rightarrow \chi(0) \in D(\mathcal{E}^\nu) \), \( 0 \leq \nu \leq 1 \) and we get

\[
\| x_n(\sigma) \|_{\nu} \leq N_0 W_1 \| M \| \| \chi(0) \|_{\nu} + N_0 W_1 \sum_{i=1}^{q} O_i + \frac{\beta N_0 W_1 \Gamma(2-\nu)}{\Gamma(1+\beta(1-\nu))} m_R(T_0) \frac{T^{\beta(1-\nu)}}{\beta(1-\nu)} \leq S_0.
\] (3.19)

4. Convergence of solutions

Now, to investigate the convergence of solution \( x_n(\sigma) \) of the approximate integral Eq (3.10) to a unique solution \( x(\cdot) \) of the Eq (3.5).

**Theorem 4.1.** Suppose (1)–(5) holds, \( \chi(0) \in D(\mathcal{A}^\alpha) \), \( \alpha \in (0, 1) \). Then,

\[
\lim_{m \to \infty} \sup_{n \geq m, \sigma \in [\sigma_0, T]} \| x_n(\sigma) - x_m(\sigma) \|_\alpha = 0, \text{ for each } 0 < \sigma_0 \leq T.
\] (4.1)

**Proof.** Let \( n \geq m \), we have

\[
\| \mathcal{F}_n(\sigma, x_n(\sigma), x_n(h(s))) - \mathcal{F}_m(\sigma, x_m(\sigma), x_m(h(s))) \| \\
\leq \| \mathcal{F}_n(\sigma, x_n(\sigma), x_n(h(s))) - \mathcal{F}_n(\sigma, x_m(\sigma), x_m(h(s))) \| \\
+ \| \mathcal{F}_n(\sigma, x_m(\sigma), x_m(h(s))) - \mathcal{F}_m(\sigma, x_m(\sigma), x_m(h(s))) \| \\
\leq 2m_R(T_0) \| x_n(\sigma) - x_m(\sigma) \|_\alpha + m_R(T_0) \| (P^\nu - P^m) x_m(\sigma) \|_\alpha. \\
\] (4.2)

For \( 0 < \alpha < \nu < 1 \), we get

\[
\| \mathcal{E}^\alpha(P^\nu - P^m) x_m(\sigma) \| \leq \| \mathcal{E}^{\alpha-\nu}(P^\nu - P^m) \mathcal{E}^\nu x_m(\sigma) \| \\
\leq \frac{1}{\lambda_m^{\nu-\alpha}} \| x_m(\sigma) \|_\nu. 
\] (4.3)

Thus, from (4.2), (4.3) we obtain

\[
\| \mathcal{F}_n(\sigma, x_n) - \mathcal{F}_m(\sigma, x_m) \| \leq 2m_R(T_0) \| x_n(\sigma) - x_m(\sigma) \|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}} \| x_m(\sigma) \|_\nu.
\]

Similarly, , we estimate

\[
\| I_{t,n}(x_n(\sigma_i)) - I_{t,m}(x_m(\sigma_i)) \| \leq N_i \| x_n(\sigma_i) - x_m(\sigma_i) \|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}} \| \mathcal{E}^\nu x_m(\sigma) \|_\nu.
\]

We choose \( \sigma_0' \) such that \( 0 < \sigma_0' < \sigma_0 < T \),
Thus, from (4.4)–(4.6) we conclude

\[ \|x_n(\sigma) - x_m(\sigma)\|_\alpha \leq \left( \int_0^{\sigma_0'} + \int_{\sigma_0'}^\sigma \right) (\sigma - s)^{\beta - 1} ds \|E^{\alpha} T_\beta(\sigma - s)\| \]

\[ \times \left[ \|F_n(s, x_n(s), x_n(h(s))) - F_m(s, x_m(s), x_m(h(s)))\| \right. \]

\[ + \sum_{i=0}^q \|S_\beta(\sigma - \sigma_i)\| \|I_{n,\theta}(x_n(\sigma_i)) - I_{m,\theta}(x_m(\sigma_i))\|_\alpha. \]  

(4.4)

1\(^{st}\) integral of inequality (4.4),

\[ \int_0^{\sigma_0'} (\sigma - s)^{\beta - 1} \|E^{\alpha} T_\beta(\sigma - s)\| \times \|F_n(s, x_n(s), x_n(h(s))) - F_m(s, x_m(s), x_m(h(s)))\| ds \]

\[ \leq \frac{2\beta W_1 N_a \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} 2m(T_0) \left( (T - \sigma_0')^{\beta(1 - \alpha) - 1} \sigma_0' \right). \]  

(4.5)

2\(^{nd}\) integral of (4.4) is evaluated as

\[ \int_{\sigma_0'}^\sigma (\sigma - s)^{\beta - 1} \|E^{\alpha} T_\beta(\sigma - s)\| \times \|F_n(s, x_n(s), x_n(h(s))) - F_m(s, x_m(s), x_m(h(s)))\| ds \]

\[ \leq \frac{\beta W_1 N_a \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} 2m(T_0) \left( U_{\sigma_0'/T_\beta(1 - \alpha)} m + \int_{\sigma_0'}^\sigma (\sigma - s)^{\beta(1 - \alpha) - 1} \|x_n - x_m\|_{(E^{\alpha})_{s\lambda}} ds \right). \]  

(4.6)

Thus, from (4.4)–(4.6) we conclude

\[ \|x_n(\sigma) - x_m(\sigma)\| \leq D_1 \sigma_0' + D_2 \frac{\lambda_{m - \sigma}}{\lambda_{m - \sigma}} + D_3 \|x_n(\sigma) - x_m(\sigma)\|_\alpha + D_4 \int_{\sigma_0'}^\sigma (\sigma - s)^{\beta(1 - \alpha) - 1} \|x_n - x_m\|_{(E^{\alpha})_{s\lambda}} ds, \]

where

\[ D_1 = \frac{2\beta W_1 N_a \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} 2m(T_0)(T - \sigma_0')^{\beta(1 - \alpha) - 1}, \]

\[ D_2 = \frac{\beta W_1 N_a \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} 2m(T_0) \frac{U_{\sigma_0'/T_\beta(1 - \alpha)}}{\beta(1 - \alpha)} + W_1 N_a \sum_{i=1}^q N_i, \]

\[ D_3 = W_1 N_a \sum_{i=1}^q N_i, \]

\[ D_4 = \frac{\beta W_1 N_a \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} 2m(T_0). \]
Since $1 - W_1N_{\sigma} \sum_{i=1}^{q} N_{\tau} > 0$, we have

$$\|x_n(\sigma) - x_m(\sigma)\|_a \leq \frac{1}{1 - D_3} \left[ D_1\sigma_0' + \frac{D_2}{\lambda_m^\alpha} + D_3 \int_{\sigma_0}^{\sigma} (\sigma - s)^{q(1-\alpha)-1} \|x_n - x_m\|_{(E_{1,\alpha})_a} ds \right].$$

By lemma 5.6.7 in [38], we have that there exist a constant $K$ such that

$$\|x_n(\sigma) - x_m(\sigma)\|_a \leq \frac{1}{1 - D_3} \left[ D_1\sigma_0' + \frac{D_2}{\lambda_m^\alpha} \right] K.$$

Taking supremum over $[\sigma_0, T]$ and let $m \to \infty$, we obtain

$$\lim_{m \to \infty} \sup_{n\geq m, \sigma_0 \leq \sigma \leq T} \|x_n(\sigma) - x_m(\sigma)\|_a \leq \frac{D_1}{(1 - D_3)} \sigma_0' K.$$

Because $\sigma_0'$ is arbitrary, the right side of the expression could be made as tiny as required simply reducing $\sigma_0'$.

**Proposition 4.1.** Assume $\chi(0) \in D(\mathcal{E})$. Then

$$\lim_{m \to \infty} \sup_{\sigma \in [0, T]} \|x_n(\sigma) - x_m(\sigma)\|_a = 0.$$

**Theorem 4.2.** Assume (1)–(5) holds, $\chi(0) \in D(\mathcal{E}^\alpha)$. Then, $\exists x_n(\sigma) \in (H_1)_a(T)$ a unique function satisfying,

$$x_n(\sigma) = \begin{cases} S_\beta(\sigma)M_\chi(0) + \int_0^{\sigma} (\sigma - s)^{q-1} T_\beta(\sigma - s) F_n(s, x_n(s), x_n(h(s))) ds \\ + \sum_{i=1}^{q} S_\beta(\sigma - \sigma_i) I_{\alpha}(x_n(\sigma_i)), & \sigma \in [0, T], \end{cases}$$

(4.7)

and $x \in (H_1)_a(T)$ satisfying

$$x(\sigma) = \begin{cases} S_\beta(\sigma)M_\chi(0) + \int_0^{\sigma} (\sigma - s)^{q-1} T_\beta(\sigma - s) F(s, x(s), x(h(s))) ds \\ + \sum_{i=1}^{q} S_\beta(\sigma - \sigma_i) I_{\alpha}(x(\sigma_i)), & \sigma \in [0, T]. \end{cases}$$

(4.8)

such that $x_n$ converges to $x$ in $(H_1)_a(T)$ i.e., $x_n \to x$ as $n \to \infty$.

**Proof.** Suppose $\chi(0) \in D(\mathcal{E})$. From previous preposition, there is a $x \in (H_1)_a(T)$ such that $\lim_{n \to \infty} x_n(\sigma) = x(\sigma)$. Since $x_n \in B_{\mathcal{E}}((H_1)_a(T)) \forall n$, we get $x \in B_{\mathcal{E}}((H_1)_a(T)) \forall \sigma_0 \in (0, T],$

$$\|F_n(\sigma, x_n(\sigma), x_n(h(s))) - F(\sigma, x(\sigma), x(h(s)))\| = \|F(\sigma, P^\alpha x_n(\sigma), x_n(h(s))) - F(\sigma, x(\sigma), x(h(s)))\| \
\leq 2m_R(T_0)\|x_n(\sigma) - x(\sigma)\|_a + \|(P^\alpha - I)x(\sigma)\|_a.$$

Taking supremum over $[0, T]$, we have

$$\sup_{\sigma \in [0, T]} \|F_n(\sigma, x_n(\sigma)) - F(\sigma, x(\sigma))\| \leq 2m_R(T_0)\|x_n - x\|_{(H_1)_a(T)} + \|(P^\alpha - I)x\|_{(H_1)_a(T)} \to 0, \text{ as } n \to \infty.$$
Thus, we get
\[
x(\sigma) = \begin{cases} 
S_\beta(\sigma)Mx_0 + \int_0^\sigma (\sigma - s)^{\beta-1}T_\beta(\sigma - s)F(s, x(s), x(h(s))) ds \\
+ \sum_{i=1}^q S_\beta(\sigma - \sigma_i)I_i(x(\sigma_i)), \ \sigma \in [0, T].
\end{cases}
\]

Now, let \(\chi(0) \in D(\mathcal{E}^\alpha)\). Since \(\mathcal{E}^\alpha x_n(\sigma)\) converges to \(\mathcal{E}^\alpha x(\sigma)\) for each \(\sigma \in (0, T]\) and \(x_n(0) = x(0) = \chi(0)\), Then, \(\mathcal{E}^\alpha x_n(\sigma)\) converges to \(\mathcal{E}^\alpha x(\sigma)\) in \(\mathcal{H}_1\). Furthermore, we have that \(x_n \in B_R((\mathcal{H}_1)_R(T))\) for each \(n\). Also \(x \in B_R((\mathcal{H}_1)_R(T))\). From previous theorem, we get
\[
\lim_{n \to \infty} \sup_{\sigma \in [\sigma_0, T]} \|x_n(\sigma) - x(\sigma)\|_a = 0.
\]

Also, we have
\[
\sup_{\sigma \in [0, T]} \|F_n(\sigma, x_n(\sigma), x_n(h(s))) - F(\sigma, x(\sigma), x(h(s)))\| \\
\leq 2m_R(T_0)\|x_n - x\|_{(\mathcal{H}_1)_R(T)} + \|(P^\alpha - I)x\|_{(\mathcal{H}_1)_R(T)} \to 0 \text{ as } n \to \infty.
\]

Therefore, \(0 < \sigma_0 < \sigma\), Eq (3.10) can be reframed as
\[
x_n(\sigma) = \begin{cases} 
S_\beta(\sigma)Mx_0 + \int_0^\sigma (\sigma - s)^{\beta-1}T_\beta(\sigma - s)F_n(s, x_n(s), x_n(h(s))) ds \\
+ \sum_{i=1}^q S_\beta(\sigma - \sigma_i)I_i(x_n(\sigma_i)), \ \sigma \in [0, T].
\end{cases}
\]

we estimate the 1st integral of (4.9) as
\[
\left\| \int_0^\sigma (\sigma - s)^{\beta-1}T_\beta(\sigma - s)F_n(s, x_n(s), x_n(h(s))) ds \right\| \leq \frac{N_0 W_1}{\Gamma(\beta)} 2m_R(T_0) \frac{\sigma^\beta_0}{\beta}.
\]

Thus, we deduce that
\[
\|x_n(\sigma) - S_\beta(\sigma)Mx_0 - \sum_{i=1}^q S_\beta(\sigma - \sigma_i)I_i(x_n(\sigma_i)) - \int_0^\sigma (\sigma - s)^{\beta-1}T_\beta(\sigma - s)F_n(s, x_n(s), x_n(h(s))) ds \| \leq \frac{N_0 W_1}{\Gamma(\beta)} 2m_R(T_0) \frac{\sigma^\beta_0}{\beta}.
\]

Letting \(n \to \infty\) and getting
\[
\|x(\sigma) - S_\beta(\sigma)Mx_0 - \sum_{i=1}^q S_\beta(\sigma - \sigma_i)I_i(x(\sigma_i)) - \int_0^\sigma (\sigma - s)^{\beta-1}T_\beta(\sigma - s)F(s, x(s), x(h(s))) ds \| \leq \frac{N_0 W_1}{\Gamma(\beta)} 2m_R(T_0) \frac{\sigma^\beta_0}{\beta}.
\]
Since, $\sigma_0$ is arbitrary, we deduce $x(\sigma)$ satisfies the integral Eq (3.5). Now, we shall show the uniqueness. Let $x_1$ and $x_2$ be the two solutions of integral Eq (3.5). Thus, we have

$$\|x_1(\sigma) - x_2(\sigma)\|_a \leq \int_0^\sigma (\sigma - s)^{\beta-1} \|\mathcal{E}_a^\alpha T_\beta(\sigma - s)\| \|\mathcal{F}(s, x_1(s), x_1(h(s))) - \mathcal{F}(s, x_2(s), x_2(h(s)))\| ds + \sum_{i=1}^q \|S_\beta(\sigma - \sigma_i)\| \|I_n(x_1(\sigma_i)) - I_n(x_2(\sigma_i))\|$$

$$\leq \frac{\beta N_0 W_1 \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^\sigma (\sigma - s)^{\beta(1 - \alpha) - 1} 2m_R(T_0) \|x_1 - x_2\|_{(H^1)_a} ds$$

$$+ N_0 W_1 \sum_{i=1}^q N_i \|x_1 - x_2\|_{(H^1)_a(\sigma)}.$$

Taking supremum on $[0, \sigma]$ and obtaining

$$\|x_1 - x_2\|_{(H^1)_a(T)} \leq \frac{\beta N_0 W_1 \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \int_0^\sigma (\sigma - s)^{\beta(1 - \alpha) - 1} 2m_R(T_0) \|x_1 - x_2\|_{(H^1)_a(\sigma)} ds$$

$$+ N_0 W_1 \sum_{i=1}^q N_i \|x_1 - x_2\|_{(H^1)_a(\sigma)}.$$

From Gronwall’s inequality and the fact that

$$\|x_1(\sigma) - x_2(\sigma)\| \leq \frac{1}{\beta_0^a} \|x_1 - x_2\|_{(H^1)_a(\sigma)}.$$

We deduce that $x_1 = x_2$ on $[0, T]$.

5. **Faedo-Galerkin approximation**

Additionally, the convergence findings were accomplished using the Faedo-Galerkin approximation technique.

There is a unique $x \in (H^1)_a(T), T \in (0, T_0)$, that fulfils the integral equation,

$$x(\sigma) = \begin{cases} S_\beta(\sigma) M_{\mathcal{X}}(0) + \int_0^\sigma (\sigma - s)^{\beta-1} T_\beta(\sigma - s) \mathcal{F}(s, x(s), x(h(s))) ds \\ + \sum_{i=1}^q S_\beta(\sigma - \sigma_i) I_n(x(\sigma_i)), \quad \sigma \in [0, T]. \end{cases}$$

An approximate integral equation has an unique solution $x_n \in (H^1)_a(T),

$$x_n(\sigma) = \begin{cases} S_\beta(\sigma) M_{\mathcal{X}}(0) + \int_0^\sigma (\sigma - s)^{\beta-1} T_\beta(\sigma - s) \mathcal{F}_n(s, x_n(s), x_n(h(s))) ds \\ + \sum_{i=1}^q S_\beta(\sigma - \sigma_i) I_{n,n}(x_n(\sigma_i)), \quad \sigma \in [0, T]. \end{cases}$$
The Faedo-Galerkin Approximation is produced by applying the projection on (5.2) as $v_n(\sigma) = P^nx_n(\sigma)$.

$$P^nx_n(\sigma) = v_n(\sigma) = \begin{cases} S_\beta(\sigma)MP^x\chi(0) + \int_0^\sigma (\sigma - s)^{\beta - 1}T_\beta(\sigma - s)P^x\mathcal{F}(s, P^nx_n(s), P^nx_n(h(s)))ds \\ + \sum_{i=1}^q S_\beta(\sigma - \sigma_i)P^nx_n(\sigma_i) \end{cases}$$

$$v_n(\sigma) = \begin{cases} S_\beta(\sigma)MP^x\chi(0) + \int_0^\sigma (\sigma - s)^{\beta - 1}T_\beta(\sigma - s)P^x\mathcal{F}(s, v_n(s), v_n(h(s)))ds \\ + \sum_{i=1}^q S_\beta(\sigma - \sigma_i)P^nI_{\lambda_n}(v_n(\sigma_i)), \sigma \in [0, T]. \end{cases}$$ (5.3)

Let solution $x(\cdot)$ of (5.1) and $v_n(\cdot)$ of (5.3) have the following representation:

$$x(\sigma) = \sum_{i=0}^\infty \alpha_i(\sigma)\phi_i, \alpha_i(\sigma) = (x(\sigma), \phi_i), i = 0, 1, 2, ..., (5.4)$$

$$v_n(\sigma) = \sum_{i=0}^n \alpha^n_i(\sigma)\phi_i, \alpha^n_i(\sigma) = (v_n(\sigma), \phi_i), i = 0, 1, 2, ... (5.5)$$

Using (5.5) in (5.3) and taking inner product with $\phi_i$, we obtain a system of fractional order integro-differential equation of the form.

$$\frac{d^{\beta}}{d\sigma^{\beta}}\alpha^n_i(\sigma) + \lambda_i\alpha^n_i(\sigma) = \mathcal{F}^n_i(\sigma, \alpha^n_0(\sigma), \alpha^n_1(\sigma), ..., \alpha^n_n(\sigma)), \Delta^\alpha_i(\sigma_k) = I^n_i(\alpha^n_i(\sigma_k)), k = 1, 2, ...q, \alpha^n_i(0) = Z_i, (\ast)$$

Where,

$$\mathcal{F}^n_i(\sigma, \alpha^n_0(\sigma), \alpha^n_1(\sigma), ..., \alpha^n_n(\sigma)) = \left( M^{-1}\mathcal{F}\left( \sigma, \sum_{i=0}^n \alpha^n_i(\sigma)\phi_i, \sum_{i=0}^n \alpha^n_i(\sigma)\phi_i \right), \phi_i \right),$$

$$I^n_i = \left( I_k\left( \sum_{k=1}^q \sum_{i=1}^n \alpha^n_i(\sigma_k)\phi_i \right), \phi_i \right),$$

$$Z_i = (\chi(0), \phi_i), i = 1, 2, ..., n.$$ We also have the following convergence theorem.

**Theorem 5.1.** If the hypothesis (1)–(5) holds. The results follows:

(i) If $x_0 \in D(\mathcal{E}^{\alpha})$, then for any $\sigma_0 \in (0, T]$

$$\lim_{m \to \infty} \sup_{n \geq m, \sigma \in [\sigma_0, T]} ||\mathcal{E}^{\alpha}[v_n(\sigma) - v_m(\sigma)]|| = 0. \quad (5.6)$$

(ii) If $x_0 \in D(\mathcal{E})$, then

$$\lim_{m \to \infty} \sup_{n \geq m, \sigma \in [0, T]} ||\mathcal{E}^{\alpha}[v_n(\sigma) - v_m(\sigma)]|| = 0. \quad (5.7)$$
Proof. If \( n \geq m, 0 \leq \alpha < \nu \). Then, we have
\[
\|v_n(\sigma) - v_m(\sigma)\|_\alpha = \|P^n x_n(\sigma) - P^m x_m(\sigma)\|_\alpha \\
\leq \|P^n [x_n(\sigma) - x_m(\sigma)]\|_\alpha + \|(P^n - P^m)x_n\|_\alpha \\
\leq \|x_n(\sigma) - x_m(\sigma)\|_\alpha + \frac{1}{\lambda_{m,n}^{\nu-a}}\|x_m(\sigma)\|_\nu.
\]
By the Theorem 4.1 and Proposition 4.1, we have that \( x_n \to x_m \) and \( \lambda_m \to \infty \) as \( m \to \infty \).

Theorem 5.2. Suppose (1)–(5) is fulfilled, \( x_0 \in D(\mathcal{E}^\alpha) \), unique function \( v_n \in (\mathbb{H}_a(T))_n \) exists, satisfying the following equation:
\[
v_n(\sigma) = \begin{cases} 
S_{\beta}(\sigma) M \mathcal{P} x_0 + \int_0^\sigma (\sigma - s)^{\beta-1} T_{\beta}(\sigma - s) P^\nu F(s, v_n(s)) ds \\
+ \sum_{i=1}^q S_{\beta}(\sigma - \sigma_i) P^\nu I_{\alpha}(v_n(\sigma_i)), \ & \sigma \in [0, T].
\end{cases} \tag{5.8}
\]

Proof. For \( x_0 \in D(\mathcal{E}^\alpha) \) and \( \sigma \in [0, T] \). We have
\[
\|v_n(\sigma) - x(\sigma)\|_\alpha = \|P^n x_n(\sigma) - P^\nu x(\sigma) + P^\nu x(\sigma) - x(\sigma)\|_\alpha \\
\leq \|P^n (x_n(\sigma) - x(\sigma))\|_\alpha + \|(P^n - I)x(\sigma)\|_\alpha.
\]
We have \( v_n \to x \) as \( n \to \infty \) according to the Theorem 4.2. As a result, the Theorem 4.2 leads to the conclusion. The preceding theorem can be used to show \( \alpha_n^\nu \) to \( \alpha_i^\nu \)’s convergence.

Theorem 5.3. Suppose (1)–(5) holds. Then,
(i) If \( x_0 \in D(\mathcal{E}^\alpha) \), then for any \( 0 < \sigma_0 \leq T \)
\[
\lim_{n \to \infty} \sup_{\sigma \in [\sigma_0, T]} \left[ \sum_{i=0}^n \lambda_i^{2\alpha} (\alpha_i(\sigma) - \alpha_i^\nu(\sigma))^2 \right] = 0. \tag{5.9}
\]
(ii) If \( x_0 \in D(\mathcal{E}) \), then
\[
\lim_{n \to \infty} \sup_{\sigma \in [0, T]} \left[ \sum_{i=0}^n \lambda_i^{2\alpha} (\alpha_i(\sigma) - \alpha_i^\nu(\sigma))^2 \right] = 0. \tag{5.10}
\]

Proof. The system (*) determines the \( \alpha_i^n \)’s. It can be easily investigated that
\[
\mathcal{E}^\alpha [x(\sigma) - v(\sigma)] = \mathcal{E}^\alpha \left[ \sum_{i=0}^\infty (\alpha_i(\sigma) - \alpha_i^\nu(\sigma)) \phi_i \right] \\
= \sum_{i=0}^\infty \lambda_i^{\alpha} (\alpha_i(\sigma) - \alpha_i^\nu(\sigma)) \phi_i.
\]
Thus,
\[
\|\mathcal{E}^\alpha [x(\sigma) - v(\sigma)]\|^2 \geq \sum_{i=0}^n \lambda_i^{2\alpha} (\alpha_i(\sigma) - \alpha_i^\nu(\sigma))^2.
\]
As a result, the Theorems 5.1 and 5.2 lead to the conclusion.
6. Applications

Let the fractional impulsive differential system of sobolev type is of the form:

\[ ^cD^\beta_\sigma[w(\sigma, x) - w_{x,i}(\sigma, x)] + \frac{\partial^2 w(\sigma, x)}{\partial x^2} = f(\sigma, w(\sigma, x)), \quad x \in S, \quad \sigma \in [0, T], \]

\[ \Delta w(\sigma, x) = \frac{2w(\sigma_i, x)}{2 + w(\sigma_i, x)}, \quad i = 1, 2, ..., q, \quad x \in (0, \pi), \]

\[ w(0, x) = w_0(x), \]

\[ w(\sigma, 0) = w(\sigma, \pi) = 0, \quad 0 \leq \sigma \leq T, \quad 0 < T < \infty. \]

Where \(^cD^\beta_\sigma\) is Caputo derivative, \(\beta \in (0, 1)\). Suppose \(w(\sigma)(x) = w(\sigma, x)\) and \(f(\sigma, \cdot) = F(\sigma, \cdot)\). Let \(\Delta w(\sigma_i, x) = w(\sigma_i^+, x) - w(\sigma_i^-, x)\), where \(w(\sigma_i^+, x)\) and \(w(\sigma_i^-, x)\) respectively the right and the left hand limit of \(w\) at \((\sigma, x) = (\sigma_i, x)\).

Now, we take \(H_1 = \mathbb{H}_1 = L^2(0, \pi)\) and consider the operator \(L, M\) on domains and ranges contained in \(L^2(0, \pi)\) defined by

\[ M y = y - y'' \]

\[ L y = -y'' \]

An infinitesimal generator of an analytic semigroup is denoted by \(E = L M^{-1}\), such that

\[ E y = -y'', \]

with the domain

\[ D(E) = \{ y \in \mathbb{H}_1 : y, \ y' \text{ are absolutely continuous } y'' \in \mathbb{H}_1, \ y(0) = y(\pi) = 0 \} \]

If we take \(\beta = \frac{1}{2}\), then \(D(E^{\frac{1}{2}})\) which is denoted by \(\beta_\frac{1}{2}\) is the Banach space endowed with the norm,

\[ ||x||_{\frac{1}{2}} = ||E^{\frac{1}{2}}x||, \quad x \in D(E^{\frac{1}{2}}). \]

Also, for \(\sigma \in [0, T]\), we define \(\beta_\frac{1}{2} = \{ y : y \text{ is a map from } [0, T] \text{ into } \beta_\frac{1}{2} \equiv x(\sigma) \text{ is continuous at } \sigma \neq \sigma_i \text{ left continuous at } \sigma = \sigma_i \text{ and right limit } x(\sigma_i^+) \text{ exists for } i = 1, 2, ..., q\}.\)

The spectrum of \(E\) is given by \(E y = -y'' = \alpha y\). The general solution \(y\) of \(E y = \alpha y\) is

\[ y(x) = C \cos(\sqrt{\alpha}x) + D \sin(\sqrt{\alpha}x). \]

Using the boundary conditions \(y(0) = y(\pi) = 0\), we obtain \(C = 0, \ \alpha = \alpha_n = n^2, \ n \in \mathbb{N}\). Thus for each \(n \in \mathbb{N}\), the solution is given by \(y_n(x) = D \sin nx\). If we take \(D = \frac{\sqrt{n}}{\sqrt{2}}\), then \(y_n y_m > 0, \ n \neq m \) and \(y_n y_m = 1, \ n = m\). Thus \(E\) has pure point spectrum and eigenvalues \(y_n\) are orthonormal.

Suppose, \(I_s(w(\sigma, x)) = \frac{2w(\sigma, x)}{2 + w(\sigma, x)}\). Let us define \(y(\sigma)(x) = w(\sigma, x)\) and \(I_s(w(\sigma, x)) = I_s(x(\sigma_i))(y)\) then \(I_s(y(\sigma)) = \frac{2y(\sigma)}{2y(\sigma)}\). For \(y_1, y_2 \in D(A^{\frac{1}{2}})\), we have

\[ ||I_s(y_1) - I_s(y_2)||_{\frac{1}{2}} \leq ||y_1 - y_2||_{\frac{1}{2}}. \]
\[ I_i(y_i) \parallel_2 \leq \|y_i\|_2. \]

Now, we define
\[
\begin{align*}
 f(\sigma, x(\sigma)) &= \mathcal{F}(\sigma, w(\sigma, x)), \\
 I_i(w(\sigma_i, x)) &= I_i(x(\sigma_i))(y), \\
 \phi(\sigma)(x) &= g(\sigma, x),
\end{align*}
\]

then problem (6.1)–(6.4) reduces to
\[
\begin{align*}
 \mathcal{D}_\sigma^\beta [Mx(\sigma)] &= \mathcal{L}x(\sigma) + \mathcal{F}(\sigma, x(\sigma), x(h(\sigma))), \quad \sigma \in [0, T], \\
 \Delta x(\sigma_i) &= I_i(x(\sigma_i)), \quad i = 1, 2, \ldots, q, \quad q \in \mathbb{N}, \\
 g(x) &= \phi.
\end{align*}
\]

It is easy to see that the operator \(E\) fulfills (1). Also, by Hölder continuity of \(h\), fulfills (3). Then, from the definitions, it can be easily shown that \(\chi\) and \(I_i\) are satisfies (4) and (5).

Now we prove that \(\mathcal{F}\) satisfies the condition (2):
\[
\begin{align*}
 &\|\mathcal{F}(\sigma_1, z_1, w_1) - \mathcal{F}(\sigma_2, z_2, w_2)\|_{L^2} \\
 \leq &L \left[ \int_0^\pi \left| \mathcal{F}(\sigma_1(x, \sigma), z_1(x, \sigma), w_1(x, \sigma)) - \mathcal{F}(\sigma_2(x, \sigma), z_2(x, \sigma), w_2(x, \sigma)) \right|^2 dx \right]^{1/2} \\
 \leq &L \left[ \int_0^\pi \left| (\sigma_1(x, \sigma) - \sigma_2(x, \sigma)) + \|z_1(x, \sigma) - z_2(x, \sigma)\| \right|^2 dx \right]^{1/2} \\
 \leq &2L[|\sigma_1 - \sigma_2| + \|z_1 - z_2\|_{L^2}].
\end{align*}
\]

Hence (2) holds.

Thus, all the assumptions of Theorem 5.2 are satisfied. So, Theorem 5.2 guarantees the existence of Faedo-Galerkin approximations and their convergence to the unique solution of (6.1)–(6.4).

7. Conclusions

The Faedo-Galerkin approximation outcomes for Caputo fractional impulsive derivative of Sobolev type with nonlocal condition are the main subject of this paper. The major ideas were developed by utilising the analytic semigroup and the Banach fixed point theorem. Finally, we give examples to back up our abstract conclusion. In future, it might be used to find the generalization in fractional differential equations.

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Conflict of interest

The authors declare no conflicts of interest.

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