Interacting electrons in magnetic fields: Tracking potentials and Jastrow-product wavefunctions

Gábor Fáth and Stephen B. Haley

Institute of Theoretical Physics, University of Lausanne, Ch-1015 Lausanne, Switzerland
(15 January 1998)

The Schrödinger equation for an interacting spinless electron gas in a nonuniform magnetic field admits an exact solution in Jastrow product form when the fluctuations in the magnetic field track the fluctuations in the scalar potential. For tracking realizations in a two-dimensional electron gas, the degeneracy of the lowest Landau level persists, and the “tracking” solutions span the ground state subspace. In the context of the fractional quantum Hall problem, the Laughlin wave function is shown to be a tracking solution. Tracking solutions for screened Coulomb interactions are also constructed. The resulting wavefunctions are proposed as variational wave functions with potentially lower energy in the case of non-negligible Landau level mixing than the Laughlin function.

I. INTRODUCTION

Modern semiconductor technology has produced exceedingly pure materials, and nanostructures that have a number of interesting properties at low temperatures. A comprehensive review is given by Beenakker and van Houten. The Laughlin wavefunction attaches collective vorticity to the electron gas which efficiently reduces the probability of two electrons approaching each other, and the Coulomb energy is reduced when \( \nu \) is a simple odd fraction \( \nu = 1/(2m+1) \). It was soon realised that such vorticity can be generated by a singular gauge transformation which attaches infinitesimal flux-tubes to the particles, i.e., transforms electrons into composite objects with transmuted statistics. However, such singular gauge transformations by themselves are not capable of reproducing the full Laughlin wavefunction, since they do not give its correct radial dependence. The full Laughlin wavefunction is obtained by taking into account fluctuation effects around the mean field solution.

The Laughlin wavefunction is a trial state completely formed within the subspace of the lowest Landau level. It is believed to reflect the most important aspects of the FQHE ground state at odd fractions, \( \nu = 1/(2m+1) \), in the high magnetic field limit, where Landau level mixing can be neglected. Similarly, suggested generalizations to other fractional filling factors such as \( \nu = n/(2mn+1) \) are appropriately projected onto the lowest Landau level.

In the ground state it is a priori not clear how to take a strong electron-electron interaction into consideration, or alternatively a weaker magnetic field, when the mixing of the Landau levels is not negligible. In such cases the true many-body ground state also has components from higher Landau levels.

In this paper we develop exact eigenfunctions \( \Psi \) of the Schrödinger equation for an interacting electron gas in non-uniform magnetic fields. The initial insight and motivation is taken from the exactly solvable 1D constructions of Calegari and Sutherland, Polychronakos, and Haley in the absence of magnetic fields, and from
the 2D constructions for the Pauli Hamiltonian introduced by Aharonov and Casher [4], and further developed by Dubrovin and Novikov [5], for an electron in a periodic magnetic field. Our analysis is based on an ansatz that linearizes the vector Riccati equation in terms of a function $Q(\{r\}, t)$ defined by

$$
\Psi(\{r\}, t) = \exp[-Q(\{r\}, t)].
$$

In general, $Q = -\ln \Psi$ is a complex multivalued function. Substituting $\Psi$ from Eq. (2.3) into Eq. (2.1) leads to the nonlinear vector Riccati equation

$$
\sum_{i=1}^{N} \left[ \nabla_i \cdot (S_i - i \frac{2\pi}{\phi_0} A_i) - (S_i - i \frac{2\pi}{\phi_0} A_i) \cdot (S_i - i \frac{2\pi}{\phi_0} A_i) \right.
$$

$$
\left. + \gamma V_i \right] \Psi(\{r\}, t) = -i \gamma \hbar \frac{\partial}{\partial t} Q(\{r\}, t) \Psi(\{r\}, t) \quad (2.4)
$$

where $S_i = S_i(\{r\}, t) = \nabla_i Q(\{r\}, t)$. (2.5)

$A_i = A_i(\{r\}, t)$, and $V_i = V_i(\{r\}, t)$. The above form of the Riccati equation contains $\Psi$ as a multiplicative factor on both sides. In the following this factor will be divided out, even though such division is not legitimate at the zeros of $\Psi$. This subtlety results in the appearance of spurious delta-function potentials at the position of the zeros. However, as shown in section II, the correct physics is preserved.

Although the original Schrödinger equation is linear, second order equation for $\Psi$, Eq. (2.4) is a first order, nonlinear equation for the derivatives of $Q$. Given real vectors $A_i$, and real scalars $V_i$, Eq. (2.4) can, in principle, be solved for $Q(\{r\}, t)$. For time-independent potentials, the function $Q(\{r\}, t) = Q(\{r\}) + i(E/\hbar)t$, and Eq. (2.4) becomes the time-independent energy eigenvalue equation

$$
\sum_{i=1}^{N} \left[ \nabla_i \cdot (S_i - i \frac{2\pi}{\phi_0} A_i) - (S_i - i \frac{2\pi}{\phi_0} A_i) \cdot (S_i - i \frac{2\pi}{\phi_0} A_i) \right.
$$

$$
\left. + \gamma V_i \right] = \gamma E. \quad (2.6)
$$

In the following we restrict our attention to time-independent problems, but it will be noted that the tracking construction introduced in subsection II.B also linearizes Eq. (2.4).

In one dimension the transformation defined by Eq. (2.3) and (2.4) leads to a scalar Riccati equation that
is quite useful for obtaining the stationary, one-particle states for many scalar potentials in the absence of magnetic fields. In the many-particle case the major difficulty in solving Eq. (2.6) is that the quadratic term in $S_{ij}$ may contain interactions of order higher than those appearing in the scalar potential $V_i$. To see this, consider wave functions in a Jastrow product form,

$$\Psi(\{r\}) = \prod_i \prod_{j>i} \Omega(r_i - r_j) \cdot \prod_i \chi_i(r_i),$$  \hspace{1cm} (2.7)

where $\Omega(r_i - r_j)$ is a two-body function depending on the coordinate difference between the particles $i$ and $j$, and $\chi_i = \chi(r_i)$ is a one-body function involving only the coordinates of particle $i$. Due to the two-body factors $\Omega_{ij}$, the vector $S_i$ will be a sum over two-body functions, each of which depends on the coordinates of the $i$th electron and another one; thus the quadratic term in $S_i$ contains three body interactions.

One may conclude that a wavefunction in the Jastrow product form of Eq. (2.7) can only be an eigenstate if the model in question contains an unphysical three-body potential. Boson Hamiltonians of this kind have been investigated in Ref. [9]. There it is argued that even though the three-body term is explicitly present in the Hamiltonian, it becomes irrelevant in the renormalization group sense; thus the Jastrow product, although it is not an exact eigenstate, describes correctly the low-energy, long-wavelength physics of the remaining two-body Hamiltonian. While three-body terms are required in the generic case, Jastrow products can be exact eigenstates of Hamiltonians without three-body terms, provided that special conditions are fulfilled, as shown in section II.B.

Well-known examples in one dimension are the Calogero-Sutherland models [10]. They form a class of integrable many-body systems, which describe particles with identical two-body potentials $V_e(r_i - r_j)$ in zero magnetic field $A_i \equiv 0$. Their ground state can be written in the product form of Eq. (2.7). For such systems the sum of the quadratic terms of the associated 1D Riccati equation reduces to a sum of two body interactions

$$\sum_i S_i^2 = \sum_{i} \sum_{j>i} F_{ij}. \hspace{1cm} (2.8)$$

The function $F_{ij} = F(r_i - r_j)$ is symmetric, and only depends on the coordinate difference between the $i$th and $j$th particles. This equation is a consequence of a more elementary functional equation (Sutherland’s equation [11]) assuring that the scattering of three particles can be factorized into two-particle scatterings. The class of models satisfying Eq. (2.8) is however limited: Typically, the two-body potential $V_e$ is constrained to diminish over short distances as $1/r_{ij}^2$. The Riccati equation, Eq. (2.6), provides an alternative description of the quantum mechanical system. However, care should be taken since not every solution of the Riccati equation yields a valid, single valued quantum mechanical wavefunction (we do not consider anyons here).

Assume that a solution $S_i$ of Eq. (2.6) is found. The function $Q$ can then be obtained by inverting Eq. (2.3), which is tantamount to the line integral

$$Q(\{r\}) = Q(\{r_0\}) + \int_{\{r_0\}}^{\{r\}} \{dr'\} \cdot \{S(\{r'\})\}, \hspace{1cm} (2.9)$$

where $\{S\} = \{S_1, S_2, \ldots, S_N\}$, and $\{dr\} = \{dr_1, dr_2, \ldots, dr_N\}$ are $DN$-component supervectors ($D$ is the dimension of the space), and $Q(\{r_0\})$ is a constant. In the many-particle case the line integral is path dependent whenever the $DN$ dimensional vector field $\{S\}$ is non-conservative, i.e., $\partial S_i^\alpha / \partial r_j^\beta - \partial S_j^\alpha / \partial r_i^\beta \neq 0$ for some $i,j = 1, \ldots, N$; $\alpha, \beta = 1, \ldots, D$. Since the wavefunction $\Psi$ must be single valued, the function $Q$ is determined only up to an additive term $i2\pi m$, with $m$ integer. Hence, we allow a set of well-defined, isolated, singularities of $\delta$-function type in any two-dimensional slice $(i\alpha,j\beta)$ of the $DN$ dimensional space. Restricting our attention only to solutions in the Jastrow-product form, it is enough to require a constraint on $\nabla_i \times S_i$, i.e., for slices in the same particle subspace with $i = j$, since then the special form in Eq. (2.7) assures that, at most, similar $\delta$-function type singularities appear in other slices with $i \neq j$.

In 3D we allow discrete loops described by vectors $c_k(s)$, parametrized by the length $s$, on which $\nabla_i \times S_i$ is proportional to a $\delta$-function. We write

$$\nabla_i \times S_i = -i2\pi \sum_k m_k \delta(r_i - c_k(s)) \partial_k c_k(s), \hspace{1cm} (2.10)$$

where $m_k$ is an integer, and $\partial_k c_k(s)$ is a unit vector tangent to loop $k$ at the point $s$. Although the singularity loops are independent of $r_i$, they may depend on the coordinates of the other particles. By virtue of (2.10), the value of the line integral in Eq. (2.9) is determined only up to the term $i2\pi \sum_k m_k n_k$, where the integer $n_k$ is the Gauss linking number (winding number in 2D) of the path linked by the $k$th singularity loop. Since $\sum m_k n_k$ is integer, Eq. (2.10) assures that the wavefunction $\Psi$ in Eq. (2.3) is single valued. In a 2D plane defined by the perpendicular unit vector $\hat{z}$, the vectors $c_k(s) \rightarrow c_k$ are discrete points, and $\partial_k c_k(s) \rightarrow \hat{z}$.

### B. Tracking Constructions

In this subsection we construct a class of exactly solvable many-body Hamiltonians in higher dimensions, which contain both scalar and vector potentials. The many-body eigenfunctions can be written in the product form (2.7), with each term related to terms of an appropriate decomposition of the vector potential. Our construction is motivated by the need to linearize the Schrödinger equation to eliminate the three-body interaction terms generated by the quadratic terms in $S_i$ and $A_i$ in (2.6). Linearization may be achieved by requiring that
\[ \sum_i \left( S_i - \frac{2\pi}{\phi_0} A_i \right)^2 = 0, \quad (2.11) \]

which is a sufficient condition for the factorizability of the wavefunction. Equation (2.11) is a stronger restriction than that in Eq. (2.9), since it requires \( F_{ij} = 0 \). In 1D with \( A_i \equiv 0 \), realizations of Eq. (2.11) are extremely limited since this equation implies that the wavefunction is a coordinate independent constant. It is the presence of the vector potential, and especially the higher dimensionality which make the constraint (2.11) plausible. We show in the following that Eq. (2.11) provides nontrivial, useful solutions to higher dimensional models containing vector potentials.

To manifest the implications of Eq. (2.11) we decompose the complex vector \( S_i \) as \( S_i = \Re S_i + i \Im S_i \), and write out the real and imaginary parts of Eq. (2.11), respectively as

\[
\sum_i \left[ (\Re S_i)^2 - (\Im S_i - \frac{2\pi}{\phi_0} A_i)^2 \right] = 0, \\
\sum_i \Re S_i \cdot (\Im S_i - \frac{2\pi}{\phi_0} A_i) = 0. \quad (2.12)
\]

For a system of identical particles, each term in Eq. (2.12) is zero, and consequently the two real vectors \( \Re S_i \) and \( \Im S_i - \frac{2\pi}{\phi_0} A_i \) are orthogonal to each other and have the same length at any point of the configuration space \( \{ r \} \). Evidently, if only \( A_i \) is given, this restriction still allows an infinite number of possible solutions for the complex vector \( S_i \). To parametrize these solutions we formally decompose \( A_i \) into the sum of two components \( A_i = A_i^\parallel + A_i^\perp \), with each term real. Equation (2.11), or equivalently the equations in (2.12), are satisfied by the ansatz

\[
S_i = \frac{2\pi}{\phi_0} \left[ \frac{\hat{n} \times A_i^\parallel}{D_i(\hat{n})} + i A_i^\parallel \right], \quad D_i(\hat{n}) = |\hat{n} \times A_i^\parallel|, \quad (2.13)
\]

where \( \hat{n} \) is a unit vector with arbitrary direction, and the unit vector \( A_i^\parallel = A_i^\parallel / |A_i^\parallel| \). Equation (2.13) completely specifies the vector \( S_i \) in terms of \( \hat{n} \) and the decomposition of \( A \). Using the ansatz (2.13), the Riccati Eq. (2.6) reduces to

\[
\sum_{i=1}^N \left[ -\mu_B \nabla_i \cdot \left( \frac{\hat{n} \times A_i^\parallel}{D_i(\hat{n})} + i A_i^\parallel \right) + V_i \right] = E, \quad (2.14)
\]

where \( \mu_B = e\hbar/2m^* \) is the Bohr magneton. In general, the first term in Eq. (2.14) is a complicated nonlinear function of the angle between \( \hat{n} \) and \( A_i^\parallel \). Considerable simplification is achieved when \( \hat{n} \) and the coordinate system can be chosen such that \( D_i(\hat{n}) \) and \( \hat{n} \) are constants. In 3D these conditions are restrictive, but in 2D they can always be satisfied (See Section II). Assuming that \( D_i(\hat{n}) \) and \( \hat{n} \) are constant, Eq. (2.14) reduces to the linear equations

\[
\sum_{i=1}^N \left[ \frac{\mu_B}{D_i(\hat{n})} \hat{n} \cdot (\nabla_i \times A_i^\parallel) + V_i \right] = E, \quad (2.15)
\]

and

\[
\sum_{i=1}^N \nabla_i \cdot A_i^\parallel = 0. \quad (2.16)
\]

Since the rhs of Eq. (2.15) is a coordinate independent constant \( E \), Eq. (2.13) can only hold if the spatial variations of the component of the magnetic field \( B_i^\parallel = \nabla_i \times A_i^\parallel \) in the direction \( \hat{n} \) track those of the scalar potential. Although Eqs. (2.13) and (2.16) give a restriction only on the sum of \( N \) terms, in case of identical particles they hold identically for each particle \( i \) and each vector potential \( A_i^\parallel \) is in the Coulomb gauge, \( \nabla_i \cdot A_i^\parallel = 0 \).

By virtue of the single valuedness condition Eq. (2.10), whose rhs is purely imaginary, it follows using Eq. (2.13) that \( B_i^\parallel = \nabla_i \times A_i^\parallel \) is necessarily restricted to the form

\[
B_i^\parallel = -\phi_0 \sum_k m_k (r_i - c_k) \partial_s c_k(s), \quad (2.17)
\]

with \( m_k \) an integer. Thus the construction (2.13) excludes regular magnetic fields in \( B_i^\parallel \), only allowing singular flux tubes proportional to \( \delta \)-functions. It is evident that if we already possess a solution of the Schrödinger equation, we can construct a new Hamiltonian, with the associated new eigenfunction, by adding a term \( \delta A_i^\parallel \) to the vector potential and the associated term \( \delta \Im S_i \) to the logarithmic derivative of the wavefunction, providing that they satisfy Eq. (2.13). The reader recognizes that the imaginary part of Eq. (2.13) is related to gauge transformations generated by real functions \( \Lambda(r_i) \), which modify the vector potential as \( \delta A_i^\parallel = i s \nabla_i \Lambda_i \), with \( \phi_0 \Lambda_i = \phi_0 (s_i) \Lambda_i \). Note that Eqs. (2.13) and (2.17) also allow singular gauge transformations, generated by multivalued functions \( \Lambda_i \). Singular gauge transformations may lead to more substantial modifications in the model than regular transformations by adding singular flux tubes to the system and thus eventually transmuting particle statistics.

The single valuedness condition Eq. (2.10) also implies a restriction on the type-I component \( A_i^\parallel \). Using Eq. (2.13) in \( \nabla_i \times \Re S_i = 0 \), with \( D_i(\hat{n}) \) and \( \hat{n} \) constant, and \( \nabla_i \cdot A_i^\parallel = 0 \), gives the constraint

\[
(\hat{n} \cdot \nabla_i) A_i^\parallel = 0. \quad (2.18)
\]

Thus \( A_i^\parallel \) must be constant in the \( \hat{n} \) direction. However, since \( \nabla_i \times A_i^\parallel \) is not restricted, regular magnetic fields are allowed in \( B_i^\parallel \).

Equation (2.13) is the main result of our paper. It completely determines the decomposition of the magnetic flux density, and if a physical problem, defined by giving
III. TRACKING IN TWO DIMENSIONS

In this section we restrict the interacting electron gas to two dimensions, and develop several examples of tracking solutions. In 2D, with both \( \mathbf{S}_i \) and \( \mathbf{A}_i \) restricted to the \( x-y \) plane, we seek tracking solutions with \( \mathbf{u} = \dot{\mathbf{z}} \). Noting that \( D_i(\dot{z}) = 1 \), in 2D the vector \( \mathbf{S}_i \) is defined by the equation

\[
\mathbf{S}_i = \nabla_i Q = \frac{2\pi}{\phi_0} \left[ -\dot{\mathbf{z}} \times \mathbf{A}_i^I + i\mathbf{A}_i^{II} \right].
\]

The tracking constraint (2.15) becomes

\[
\sum_{i=1}^{N} \left[ \mu_B B_i^I(\mathbf{r}_i) + V(\mathbf{r}_i) \right] = E,
\]

with \( B_i^I = \dot{\mathbf{z}} \cdot (\nabla_i \times \mathbf{A}_i^I) \). The constraints (2.16) - (2.18) are \( \nabla_i \cdot \mathbf{A}_i^I = 0 \),

\[
B_i^{II} = \dot{\mathbf{z}} \cdot (\nabla_i \times \mathbf{A}_i^{II}) = -\phi_0 \sum_{k} m_k \delta(\mathbf{r}_i - \mathbf{c}_k),
\]

and \( \partial \mathbf{A}_i^I / \partial z = 0 \), respectively.

In the following we assume that the particles are spinless fermions, and that the system is subject to an applied uniform magnetic field \( \mathbf{B}_a = \dot{\mathbf{z}} B_a \), with \( B_a > 0 \). The corresponding vector potential in the symmetric gauge, written in polar coordinates \( \mathbf{r} = (r, \varphi) \), is

\[
\mathbf{A}_a = \frac{1}{2} B_a r \dot{\varphi}.
\]

We now develop a few interesting solutions of the tracking relation Eq. (3.4), relevant to the fractional quantum Hall effect, which satisfy the constraints following Eq. (3.2), and construct the associated eigenfunctions.

A. Non-Interacting Electrons

When \( \mathbf{A} = \mathbf{A}_a \) is the only potential which appears in the Hamiltonian, i.e., the scalar potential \( V = 0 \), the one-electron states form highly degenerate Landau levels. Restricting our attention to filling factors \( \nu = N\phi_0 / B_a \leq 1 \), we consider exclusively the lowest Landau level (LLL), which has the one-particle energy \( E/N = \mu B_a \). The single particle (\( N = 1 \)) Schrödinger equation with vector potential \( \mathbf{A} = \mathbf{A}_a \) has eigenfunctions with anti-analytic prefactors of the form

\[
\psi_m(\mathbf{r}) = c_m Z^m \exp \left( -\frac{1}{4} l_m^2 Z \bar{Z} \right), \quad Z = x + iy = re^{i\varphi},
\]

where \( l_m^2 = eB_a / \hbar = (2\pi / \phi_0) B_a \), and \( c_m \propto l_a^{-(m+3/2)} \) is a normalization constant. These one-particle basis functions \( \psi_m \) with definite canonical angular momentum \( m\hbar \)
(m a nonnegative integer) span the LLL. Substitution of \( \psi_m \) into Eq. (2.22) yields the corresponding energy eigenvalue \( E_m = \mu_B B_a \). Using \( \psi_m \) and \( A_a \) in Eq. (2.22) gives the circulating current density

\[
j_m = \mu_B |\psi_m|^2 \left( \frac{2m}{r} - l_a^2 r^2 \right) \dot{\hat{r}}
\]

Since \( c_m^2 \propto B_a^{m+3/2} \), the current \( j_m \) → 0 in the limit \( B_a \) → 0, as expected when there is no force on the electron.

We now show that any basis state \( \psi_m \) is a tracking solution corresponding to a \( \delta \) function potential \( V(r, m) \). Using the formalism of section II with \( N = 1 \), we decompose the vector potential \( A_a \) as \( A_a = A^I + A^{II} \) with

\[
A^I = A_a - a(r, m), \quad A^{II} = a(r, m),
\]

where the particle index has been omitted, and \( a \) is a singular vector potential defined by

\[
a(r, m) = \frac{\phi_0 m}{2\pi r} \dot{\hat{r}}.
\]

The auxiliary vector potential \( a(r, m) \) attaches a fictitious flux tube of density

\[
b(r, m) = \dot{\hat{r}} \cdot \nabla \times a = m\phi_0 \delta(r),
\]

to the origin of the coordinate system, but since the flux tubes are attached with opposite sign in \( A^I \) and \( A^{II} \), they cancel out in the Hamiltonian and have no direct physical effect.

To form the corresponding tracking eigenfunction we solve Eq. (3.1), with \( A^I \) and \( A^{II} \) given by Eq. (2.7). The imaginary part gives \( 3Q = m\phi_0 \), while the real part yields \( 3Q = 1/4l_a^2 - m\ln r \). Adding these results and using Eq. (2.19) gives the wavefunction \( \bar{\psi}_m \) of Eq. (3.3). The current density of the tracking solution can be calculated using Eq. (2.23), which yields Eq. (3.6). In the analysis it is seen that the \( A^I \) piece is responsible for the the radial part of the polynomial prefactor in \( \psi_m \), while the \( A^{II} \) piece gives the angular part.

In order to satisfy the tracking equation Eq. (3.2) with energy \( E = \mu_B B_a \), we must introduce the scalar potential

\[
V(r, m) = \mu_B b(r, m) = m\mu_B \phi_0 \delta(r).
\]

Since the wavefunction \( \psi_m \) has a zero at the origin for \( m > 0 \), such a singular potential is absolutely harmless and it does not contribute to the physical properties of the system. The appearance of the delta function is due to the transformation (2.23), which transforms algebraic zeros of \( \Psi \) into logarithmic singularities of \( Q \), and the subsequent division of Eq. (2.24) by the wavefunction to obtain the time independent Riccati equation Eq. (2.23). Since \( V^2 \bar{\psi}_m \) is regular, the tracking solutions found also satisfy the original Schrödinger equation without the above \( \delta \)-function potential.

It is of interest to point out that another tracking solution, associated with the decomposition

\[
A^I = A_a - a(r, m), \quad A^{II} = a(r, m),
\]

yields the same LLL energy eigenvalue \( E_m = \mu_B B_a \) and current density Eq. (3.6), but with basis function \( \bar{\psi}_m \) whose prefactor is analytic. The form \( \bar{\psi}_m \), which is the one used by Laughlin [13] corresponds to a vector potential \( A = A_a - 2a(r, m) \). Hence there is a (negative) net flux tube attached to the origin of the coordinate system, which the moving electron observes and picks up an Aharonov-Bohm phase.

Now consider the N-electron problem. The construction (2.7) can be generalized, using the particle-dependent decomposition

\[
A^I_j = A_a - a_i(r_i, m_i), \quad A^{II}_j = a_i(r_i, m_i),
\]

with \( a \) defined as in Eq. (2.7). We find that \( Q = 1/4l_a^2 \sum_i Z_i \dot{Z}_i - \sum_i m_i \ln Z_i \), which leads to the eigenfunction

\[
\Psi_{\{m\}} = \bar{Z}_{1}^{m_1} \bar{Z}_{2}^{m_2} \cdots \bar{Z}_{N}^{m_N} \exp \left( -\frac{1}{4l_a^2} \sum_i Z_i \dot{Z}_i \right).
\]

In the case of a fully filled LLL, \( N = 1 \), for a circular droplet with unit area, the particle number is \( N = B_a/\phi_0 \), and the one-electron states are filled with increasing \( m \) quantum numbers up to \( N - 1 \). The total canonical angular momentum \( M = \sum_{i=1}^{N} m_i = 0 + 1 + \ldots + (N - 1) = N(N - 1)/2 \) is a conserved quantity. The angular momentum, however, can be arbitrarily distributed among the particles; thus any linear superposition

\[
\Psi_M = \sum_{\{m\}} c_{\{m\}} \Psi_{\{m\}},
\]

subject to the constraint \( \sum_i m_i = M \), is a valid solution. The N-electron wavefunction must be antisymmetric with respect to the exchange of the particles. This requires that \( c_{\{m\}} \) be the totally antisymmetric tensor of \( N \) indices, i.e., we have to form a Slater determinant. A closed form is achieved by forming the Vandermonde determinant

\[
\sum_{\{m\}} c_{\{m\}} \prod_i \bar{Z}_i^{m_i} = \det \begin{vmatrix} 1 & \bar{Z}_1 & \bar{Z}_1^2 & \cdots & \bar{Z}_1^{N-1} \\ 1 & \bar{Z}_2 & \bar{Z}_2^2 & \cdots & \bar{Z}_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{Z}_N & \bar{Z}_N^2 & \cdots & \bar{Z}_N^{N-1} \end{vmatrix} = \prod_{i=1}^{N} \prod_{j>i}^{N} (\bar{Z}_j - \bar{Z}_i).
\]

Thus the N-electron wavefunction for the fully filled LLL takes the form

\[
\Psi_M = \prod_{i=1}^{N} \prod_{j>i}^{N} (\bar{Z}_j - \bar{Z}_i) \exp \left( -\frac{1}{4l_a^2} \sum_i Z_i \dot{Z}_i \right),
\]

where \( 2M = N(N - 1) \).
B. Interacting Particles - The Laughlin Wave Function

The above discussion remains equally valid if the system is subject to internal fields created by the particles themselves. Our general technique to find solutions to the many body problem is to solve the one-body problem, form product wave functions from these solutions, and finally antisymmetrize them for fermions (or symmetrize for bosons).

Consider the addition of a Chern-Simons vector potential, whose value at the position of the \( i \)th electron is determined by the positions of the other particles as

\[
a_{cs}(\mathbf{r}_i, \tilde{\phi}) = \frac{\tilde{\phi}}{2\pi} \sum_{j \neq i}^N \mathbf{z} \times \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \tag{3.17}
\]

with \( \tilde{\phi} = \phi_{cs}/(\phi_0 N) \) the number of flux quanta per particle arising from \( a_{cs} \). The Chern-Simons vector potential \( a_{cs} \) attaches fictitious flux tubes of density

\[
b_{cs}(\mathbf{r}_i, \tilde{\phi}) = \nabla_i \times a_{cs}(\mathbf{r}_i) = \frac{\tilde{\phi}}{2\pi} \phi_{0} \sum_{j \neq i}^N \delta(\mathbf{r}_i - \mathbf{r}_j) \tag{3.18}
\]

to the particles. These composite particles remain fermions if \( \tilde{\phi} \) is even, while their statistics transmute to bosons when \( \tilde{\phi} \) is odd.

As above, we decompose the vector potential \( A \) as

\[
\begin{align*}
A^I &= A_a - a(\mathbf{r}, m) - a_{cs}(\mathbf{r}, \tilde{\phi}), \\
A^H &= a(\mathbf{r}, m) + a_{cs}(\mathbf{r}, \tilde{\phi}). \tag{3.19}
\end{align*}
\]

Again the flux tubes appear with opposite sign in \( A^I \) and \( A^H \), so they cancel out from the Hamiltonian. The associated eigenfunction is formed by solving Eq. (3.1) using Eq. (3.19). Equation (3.2) is satisfied with energy \( E = N \mu_B B_{\phi} \), provided that we introduce the fictitious scalar potential

\[
V(\mathbf{r}, m, \tilde{\phi}) = \mu_B \phi_{0} [m \delta(\mathbf{r}_i) + \tilde{\phi} \sum_{j \neq i}^N \delta(\mathbf{r}_i - \mathbf{r}_j)]. \tag{3.20}
\]

Recalling the discussion after Eq. (3.10), this singular potential is spurious and has no real effect on the problem. Again, the wavefunctions to be found below are valid solutions of the original Schrödinger equation without \( V(\mathbf{r}, m, \tilde{\phi}) \).

The Chern-Simons contribution gives

\[
\Re Q_{cs} = -\tilde{\phi} \sum_{i} \tan^{-1} \left( \frac{y_i - y_j}{x_i - x_j} \right), \\
\Im Q_{cs} = -\tilde{\phi} \sum_{i} \ln |\mathbf{r}_i - \mathbf{r}_j|,
\]

which, combined with the other contributions calculated in the free particle case, gives

\[
Q = \frac{1}{4\mu_a} \sum_{i} Z_i \tilde{Z}_i - \sum_{i} m_i \ln \tilde{Z}_i - \tilde{\phi} \sum_{i} \sum_{j>i} \ln (\tilde{Z}_i - \tilde{Z}_j).
\]

Following the anti-symmetrization procedure in Eq. (3.15), the wavefunction takes the form

\[
\Psi_L = \prod_{i} \prod_{j>i} (\tilde{Z}_j - \tilde{Z}_i)^{\tilde{\phi} + 1} \Psi_M
\]

\[
= \prod_{i} \prod_{j>i} (\tilde{Z}_j - \tilde{Z}_i)^{\tilde{\phi} + 1} \exp \left( -\frac{1}{4\mu_a} \sum_{i} Z_i \tilde{Z}_i \right), \tag{3.21}
\]

where \( \Psi_M \) is the free electron wave function of Eq. (3.16).

This is the Laughlin wavefunction, used to describe the state of the electrons for fractional filling factors \( 1/(\tilde{\phi} + 1) \) (with \( \tilde{\phi} \) even). The distinction between type-I and type-II potentials is that the type-I piece generates the radial part of the Laughlin prefactor, while the type-II piece generates the angular part. It is the type-I vector potential in the above construction that keeps the particles further away from each other, and increases the effective filling factor to \( \nu_{eff} = 1 \).

Our derivation of the Laughlin wavefunction deserves some interpretation. It is well-known that without the repulsive interaction of the particles any wavefunction with an anti-analytic prefactor is an eigenstate in the LLL. The state in Eq. (3.21) is only one of the many degenerate ground states. Switching on the weak Coulomb interaction slightly mixes up the different Landau levels. The Laughlin function \( \Psi_L \) ceases to be an exact ground state but in the high field limit it is believed to become a good variational ground state, better, in general, than other wavefunctions with anti-analytic prefactors.

Usually, the Laughlin wavefunction is derived in a field theoretic framework beyond the mean field level. Starting either from a fermionic or bosonic field theory, fictitious Cherns-Simons flux tubes are attached to the particles. Comparing these theories, based on singular gauge transformations, to our decomposition of the vector potential into type-I and type-II components, it is obvious that the former theories only take into account the singular type-II piece. As a consequence, these mean-field theories are not able alone to account for the radial component of the Laughlin prefactor. The full Laughlin prefactor only emerges when fluctuations of the fictitious gauge field beyond the mean field level are taken into account. In contrast, our construction correctly accounts for fluctuations and produces the full Laughlin wave function. In this sense it goes beyond the mean field theories.

Another recent theory due to Rajaraman and Sondhi rather artificially attaches complex flux tubes to the electrons, using a complex Chern-Simons-type vector potential \( a_{cs} - i\mathbf{z} \times a_{cs} \). The imaginary term in the vector potential allows them to obtain the complete \( \Psi_L \) at the mean field level. The Aharonov-Bohm phase picked up by an electron moving in the presence of such a complex vector potential is \( Q_{AB} = -i/(2\pi/\phi_0) \int (a_{cs} - i\mathbf{z} \times a_{cs}) \, dr \).
In our construction a moving particle observes real type-I and type-II Chern-Simons vector potentials, $A^I = -\mathbf{a}_{\text{cs}}$. The type-I component gives the modification $3Q = (2\pi/\phi_0) \int \mathbf{a}_{\text{cs}} \cdot d\mathbf{r}$ while the type-II component gives $\mathcal{R}Q = (2\pi/\phi_0) \int (\hat{\mathbf{z}} \times \mathbf{a}_{\text{cs}}) \cdot d\mathbf{r}$. Since $\mathcal{R}Q + i\bar{3}Q = -Q_{\text{AB}}$, the two approaches give the same mathematical wave function. However the physics is different: our approach does not attach bare flux tubes to the particles. The full wavefunction is constructed by assigning a different role to the two components of the vector potential. We use the Chern-Simons vector potential $\mathbf{a}_{\text{cs}}$ only as an auxiliary tool in deriving new solutions of the Hamiltonian, in which the total vector potential is $\mathbf{A}$, due only to the uniform applied field. On the other hand, the complex vector potential used in Ref. 22 is an unrealistic characteristic which remains in the Hamiltonian.

C. Coulomb Tracking

In previous subsections we analyzed problems with a uniform applied magnetic flux density $B_a$. Now we consider an additional one-body, non-uniform flux density $B_1(\mathbf{r})$, giving the total flux density

$$B(\mathbf{r}) = B_a - B_1(\mathbf{r}).$$

Choosing the associated vector potential in the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, allows us to interpret $B(\mathbf{r})$ as generated by a type-I vector potential. In general, non-uniform perturbations of this kind lift the degeneracy of the LLL, and the one-electron eigenfunctions cannot be found analytically. However, according to Eq. (3.22), if $B(\mathbf{r})$ is tracked by a scalar potential $V_1(\mathbf{r})$, i.e.,

$$\mu_B B_1(\mathbf{r}) = V_1(\mathbf{r}),$$

the degeneracy of the LLL is preserved. Since any nonzero spatial average components of the flux density and scalar potential only have a trivial level shifting effect on the LLL structure, we assume that the mean values of $B_1$ and $V_1$ are zero in the following treatment. Although the LLL energy does not change, the basis functions spanning this level pick up an extra prefactor

$$\psi_m(\mathbf{r}) = e^{-\mathcal{R}Q_1} \psi_m(\mathbf{r}),$$

where $\mathcal{R}Q_1(\mathbf{r})$ is the solution of the real part of Eq. (3.1) with $\mathbf{A}^I = -B_1(\mathbf{r})$.

This result for the single electron states was pointed out in Ref. 22, although all explicit analysis was based on a Pauli spinor Hamiltonian with no scalar potential. However, we proceed further and suppose that the perturbing magnetic field is due to a two-body, type-I vector potential in a system of $N$ electrons. Our goal is to construct a Hamiltonian, which involves realistic two-body scalar potentials, such as the screened Coulomb interaction,

$$V_1(\mathbf{r}_i) = g_c e^2 \sum_{j \neq i} e^{-|\mathbf{r}_i - \mathbf{r}_j|/\xi}, \quad 0 \leq g_c \leq 1,$$

whose ground state can be constructed exactly. The interaction is characterized by an exponential screening parameter $\xi$ and a linear strength parameter $g_c$, with nominal value $g_c = 1$. According to the tracking equation Eq. (3.22) this requires the presence of a screened magnetic monopole-like field

$$B_1(\mathbf{r}_i) = \frac{g_m e^2}{\mu_B} \sum_{j \neq i} \frac{e^{-|\mathbf{r}_i - \mathbf{r}_j|/\xi}}{|\mathbf{r}_i - \mathbf{r}_j|},$$

with strength $g_m = g_c$. With this proviso the degeneracy of the LLL remains intact and the ground state wavefunctions have the form

$$\Psi(\mathbf{r}) = e^{-\mathcal{R}Q_1} \Psi(\mathbf{r}),$$

where $\mathcal{R}Q_1(\{\mathbf{r}\})$ is again the solution of the real part of Eq. (3.1) with $\nabla \times \mathbf{A}^I = -B_1(\mathbf{r}_i)$, and $\Psi$ is the wave function in the absence of $V_1$. In the present case, with $B_1$ given by (3.26), one obtains

$$\mathcal{R}Q_1(\{\mathbf{r}\}) = \alpha \xi \sum_i \sum_{j>i} \left[ \text{Ei} \left( -\frac{|\mathbf{r}_i - \mathbf{r}_j|}{\xi} \right) - \ln \left( \frac{|\mathbf{r}_i - \mathbf{r}_j|}{\xi} \right) \right],$$

where $\text{Ei}(r)$ is the exponential integral function $\text{Ei}(r) = -\int_{-r}^\infty dt \exp(-t)/t$, and $\alpha = 2m^* e^2 g_c / \hbar^2$. In the strict Coulomb limit $\xi \to \infty$, Eq. (3.28) reduces to

$$\mathcal{R}Q_1(\{\mathbf{r}\}) = -\alpha \sum_i \sum_{j>i} |\mathbf{r}_i - \mathbf{r}_j|.$$

Assuming tacitly that the ground state structure does not change drastically in the presence of a finite electron-electron interaction, i.e., the system remains in the Laughlin liquid phase and does not cross over into a Wigner solid, we pick up the Laughlin wavefunction from the set of degenerate wavefunctions of the LLL, and set $\Psi = \Psi_L$ in Eq. (3.27). In the pure Coulomb case, using Eq. (3.29), this leads to the Coulomb tracking wave function

$$\Psi_c = \prod_i^{N} \prod_j^{N} (\tilde{Z}_j - \tilde{Z}_i)^{\phi+1} \exp \left( \alpha \sum_i \sum_{j>i} |\mathbf{r}_i - \mathbf{r}_j| - \frac{1}{4} \alpha \sum_i |\mathbf{r}_i|^2 \right).$$

The magnetic flux density in Eq. (3.26) may appear rather artificial, but it can be viewed as a smeared out analog of the standard Chern-Simons flux tube. In comparison with the Laughlin wavefunction the extra Jastrow factor $\exp(-\mathcal{R}Q_1)$ pushes the particles further...
apart, decreasing even more the probability of two particles approaching each other, and thus decreasing the Coulomb energy. However, this factor does not modify the effective filling factor: since $\Re Q_1$ appears in the exponent, it does not change the polynomial prefactor. In a realistic situation with a Coulomb interaction between the particles, $\Psi_C$ may be a useful variational ground state wavefunction. If one uses Eq. (3.28), the corresponding screened Coulomb tracking function depends on two parameters $\xi$ and $g_c$, which can be adjusted variationally, independently of their nominal values in Eq. (3.28).

FIG. 1. For the model parameter space shown, $g_c$ and $g_m$ are the strengths of the Coulomb and tracking magnetic field interactions, respectively. Exact tracking solutions in the space of two-body couplings exist along the tracking line, $g_m = g_c$. Laughlin’s wavefunction $\Psi_L$ is an exact solution at $g_m = g_c = 0$. The real physical problem is characterized by $g_c = 1$, and $g_m = 0$. Any point on the tracking line could be used as a variational trial state.

Figure 1 illustrates schematically the situation in the space of model parameters: Exact ground state wavefunctions can be constructed along the straight line denoting models with equal strengths of the Coulomb interaction and its tracking two-body magnetic field. Laughlin’s wavefunction $\Psi_L$ refers to the special point $g_m = g_c = 0$, while the full Coulomb tracking wavefunction $\Psi_C$ refers to the point $g_m = g_c = 1$. A priori, $\Psi_L$ and $\Psi_C$ are equally good (or bad) representations of the wave function for the the real physical case $g_c = 1$, $g_m = 0$. Improvement on the FQHE ground state could be achieved by considering tracking solutions where the common value of the coupling constants $g_m = g_c = g$ is used as a variational parameter, which refers to moving along the tracking line of the figure.

The prefactor of $\Psi_C$, including the real factor $\exp(-\Re Q_1)$, is not completely anti-analytic; thus it is not entirely in the lowest Landau level. Nevertheless, it has presumably an extensive overlap with the LLL due to the dominating anti-analytic polynomial factor. In a situation where the Coulomb energy is comparable to the magnetic energy, Landau level mixing is not negligible, and $\Psi_C$ could be tested directly as a variational ground state. Note that the form of the Jastrow correction factor $\exp(-\Re Q_1)$ taken from Eq. (3.28) or (3.29) is different from the form used recently by Price et al. It would be interesting to calculate numerically the variational upper bound on the ground state energy set by the tracking solution and compare it directly to that of Ref. 24.

IV. CONCLUSIONS

In summary, we analyzed a system of $N$ electrons interacting with externally applied non-uniform scalar and vector potentials, and with each other through two-body potentials. Our approach was based on transforming the Schrödinger equation into a nonlinear Riccati equation and then linearizing the latter with an appropriate ansatz. We showed that when the magnetic flux density tracks the spatial dependence of a scalar potential, exact analytic solutions of the Schrödinger equation can be obtained in any dimension. These “tracking” solutions, which form a Jastrow product, are characterized by a unit vector $\hat{n}$ and a decomposition of the vector potential. In the 2D examples analyzed here, the tracking solutions constitute the degenerate ground states of the system. In particular, the tracking solution corresponding to the 2D electron gas in a homogeneous transverse magnetic field was found to be the Laughlin wavefunction whenever the vector potential decomposition is based on the Chern-Simons vector potential $a_{ss}$. Our construction, however, does not attach singular flux-tubes to the particles, since $a_{ss}$ does not appear in the Hamiltonian.

Adding a realistic repulsive two-body potential, such as the screened Coulomb potential, a tracking solution exists only if there is a two-body magnetic field present. Nevertheless, the resulting tracking wave function may be useful in a variational calculation to provide further insight into the nature of the fractional quantum Hall ground state in the presence of strong electron-electron interactions and Landau level mixing.

In three dimensions there are many possible tracking solutions not considered here. Furthermore, the tracking constructions that linearize the Riccati eigenfunction also linearize the time-dependent Riccati equation, creating another domain of exactly solvable problems. We leave these interesting problems to future study.
ACKNOWLEDGMENTS

Discussions with P. Erdős and J-J. Loeffel are appreciated, and the partial support of the Swiss National Science Foundation through grant No. 20-46676.96 is acknowledged.

a On leave from the Research Institute for Solid State Physics, Budapest, Hungary.
b Permanent Address: Department of Electrical and Computer Engineering, University of California, Davis, CA 95616.

1 C. W. J. Beenakker and H. van Houten, in Solid State Physics: Advances in Research and Applications, edited by H. Ehrenreich and D. Turnbull (Academic, New York, 1991), Vol. 44., p. 1
2 J. K. Jain, Phys. Rev. Lett. 63, 199 (1989); Adv. Phys. 41, 105 (1992).
3 A. Lopez, and E. Fradkin, Phys. Rev. B 44, 52462 (1991).
4 B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B 47, 7312 (1993).
5 L. Brey, Phys. Rev. B 50, 11861 (1994).
6 D. B. Chklovskii, Phys. Rev. B 51, 9895 (1995).
7 N. Read, Surface Science 361/362, 7 (1996)
8 S. H. Simon, J. Phys. Condens. Matter 8, 10127 (1996).
9 P. Kopietz and G. E. Castilla, Phys. Rev. Lett. 78, 314 (1997).
10 R. B. Laughlin, Phys. Rev. B 27, 3383 (1983).
11 R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
12 F. Calogero, J. Math. Phys. 12, 419 (1971).
13 B. Sutherland, Phys. Rev. A 4, 2019 (1971); Phys. Rev. Lett. 34, 1085 (1975).
14 A. P. Polychronakos, Phys. Rev. Lett. 69, 703 (1992)
15 S. B. Haley, Am. J. Phys. 65, 237 (1997).
16 Y. Aharonov and A. Casher, Phys. Rev. A 19, 2461 (1979).
17 B. A. Dubrovin and S. P. Novikov, Sov. Phys. JETP 52, 511 (1980).
18 M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. 71, 313 (1981); 94, 313 (1983).
19 C. L. Kane, S. Kivelson, D. H. Lee, and S. C. Zhang, Phys. Rev. B 43, 3255 (1991).
20 S. M. Girvin and T. Jach, Phys. Rev. B 29, 5617 (1984).
21 S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. 62, 82 (1989); S. C. Zhang, Int. J. Mod. Phys. B6, 25 (1992).
22 I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products, (Academic, New York, 1994), 5th ed. p.1144.
23 R. Rajaraman and S. L. Sondhi, Int. J. Mod. Phys. B10, 793 (1996); see also R. K. Ghosh and S. Rao, cond-mat/9703217.
24 R. Price, P. M. Platzman and Song He, Phys. Rev. Lett. 70, 339 (1993).