Vertex-weighted Graphs and Freeness of \(\psi\)-graphical Arrangements

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Abstract

Let \(G\) be a simple graph on \(\ell\) vertices \(\{1, \ldots, \ell\}\) with edge set \(E_G\). The graphical arrangement \(A_G\) consists of hyperplanes \(\{x_i - x_j = 0\}\), where \(\{i, j\} \in E_G\). It is well known that three properties, chordality of \(G\), supersolvability of \(A_G\), and freeness of \(A_G\) are equivalent. Recently, Richard P. Stanley introduced \(\psi\)-graphical arrangement \(A_{G,\psi}\) as a generalization of graphical arrangements. Lili Mu and Stanley characterized the supersolvability of the \(\psi\)-graphical arrangements and conjectured that the freeness and the supersolvability of \(\psi\)-graphical arrangements are equivalent. In this paper, we will prove the conjecture.

Keywords: Hyperplane arrangement, Graphical arrangement, Ish arrangement, Coxeter arrangement, Free arrangement, Supersolvable arrangement, Chordal graph, Vertex-weighted graph, Multiarrangement

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1 Introduction

Let \(V\) be an \((\ell + 1)\)-dimensional vector space over a field \(K\) and \(\{z, x_1, \ldots, x_\ell\}\) a basis for the dual space \(V^*\). A central hyperplane arrangement (arrangement, for short) in \(V\) is a finite set of vector subspaces of codimension 1.

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Let $G$ be a simple graph with vertex set $V_G = \{1, \ldots, \ell\}$ and edge set $E_G$. Let $\psi : V_G \to 2^K$ be a map satisfying $|\psi(i)| < \infty$ for every vertex $i \in V_G$. The **graphical arrangement** $A_G$ and the $\psi$-**graphical arrangement** $A_{G,\psi}$ are defined by

$$A_G := \{ \{x_i - x_j = 0\} \mid \{i,j\} \in E_G\},$$

$$A_{G,\psi} := \{\{z = 0\}\} \cup \{\{x_i - x_j = 0\} \mid \{i,j\} \in E_G\} \cup \{\{x_i = az\} \mid 1 \leq i \leq \ell, a \in \psi(i)\},$$

where $\{\alpha = 0\} (\alpha \in V^*)$ stands for the hyperplane $\{v \in V \mid \alpha(v) = 0\}$. When $G$ is complete, the arrangement $A_G$ is known as the braid arrangement or the Coxeter arrangement of type $A_{\ell-1}$. The $\psi$-graphical arrangements were introduced by Richard P. Stanley [14] to investigate the number of chambers of the visibility arrangements of order polytopes.

For a central arrangement $\mathcal{A}$, the **intersection lattice** $L(\mathcal{A})$ is the set of intersections of hyperplanes in $\mathcal{A}$ with the order by reverse inclusion: $X \leq Y \iff X \supseteq Y$. We say that $\mathcal{A}$ is **supersolvable** if the lattice $L(\mathcal{A})$ is supersolvable as defined by Stanley [12].

Let $S$ be the symmetric algebra of the dual space $V^*$. Then $S$ can be identified with the polynomial ring $\mathbb{K}[z,x_1,\ldots,x_\ell]$. Let $\text{Der}(S)$ be the module of derivations of $S$:

$$\text{Der}(S) := \{\theta : S \to S \mid \theta \text{ is } \mathbb{K}\text{-linear},$$

$$\theta(fg) = f\theta(g) + \theta(f)g \text{ for any } f,g \in S\}.$$  

The module of logarithmic derivations $D(\mathcal{A})$ is defined to be

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}$$

$$= \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for any } H \in \mathcal{A}\},$$

where $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$ is the defining polynomial of $\mathcal{A}$ and $\alpha_H$ is a linear form such that $\ker(\alpha_H) = H$. We say that $\mathcal{A}$ is **free** if $D(\mathcal{A})$ is a free $S$-module. It is well known that a supersolvable arrangement is free [8, 10]. When $\mathcal{A}$ is free, the module $D(\mathcal{A})$ has a homogeneous basis $\{\theta_0, \ldots, \theta_\ell\}$. The tuple of degrees $\exp \mathcal{A} = (\deg \theta_0, \ldots, \deg \theta_\ell)$ is called the **exponents** of $\mathcal{A}$. 2
A graph is called chordal if every cycle of length at least four has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. A vertex $v$ is called simplicial if the subgraph induced by its neighbors is complete. We say that an ordering of the vertices $(v_1, \ldots, v_\ell)$ is a perfect elimination ordering if each $v_i$ is simplicial in the subgraph induced by the vertices $\{v_1, \ldots, v_i\}$.

**Theorem 1.1** (Fulkerson-Gross [7, Section 7]). Let $G$ be a simple graph. Then $G$ is chordal if and only if $G$ has a perfect elimination ordering.

These notions for graphs and properties for graphical arrangements are related by the following theorem.

**Theorem 1.2** (Stanley [13, Corollary 4.10], Edelman-Reiner [6, Theorem 3.3]). The following three conditions are equivalent:

1. $G$ is chordal.
2. $A_G$ is supersolvable.
3. $A_G$ is free.

We say that a perfect elimination ordering $(v_1, \ldots, v_\ell)$ of $G$ is a weighted elimination ordering if $\psi(v_i) \supseteq \psi(v_j)$ whenever $i < j$ and $\{v_i, v_j\} \in E_G$. Lili Mu and Stanley [9] characterized the supersolvability of $\psi$-graphical arrangements and conjectured that the freeness and the supersolvability of $\psi$-graphical arrangements are equivalent.

**Theorem 1.3** (Stanley [14, Theorem 6] Mu-Stanley [9, Theorem 1, 2]). A $\psi$-graphical arrangement $A_{G,\psi}$ is supersolvable if and only if $(G, \psi)$ has a weighted elimination ordering.

We say that a path $v_1 \cdots v_k$ in $G$ is unimodal if there exists $i \in \{1, \ldots, k\}$ such that $\psi(v_1) \subseteq \cdots \subseteq \psi(v_i) \supseteq \cdots \supseteq \psi(v_k)$. Note that an edge $\{i, j\} \in E_G$ is unimodal if $\psi(i)$ and $\psi(j)$ are comparable, i.e., $\psi(i) \subseteq \psi(j)$ or $\psi(i) \supseteq \psi(j)$.

The main result in this paper is as follows:

**Theorem 1.4.** For $\psi$-graphical arrangements $A_{G,\psi}$, the following conditions are equivalent:

1. $(G, \psi)$ has a weighted elimination ordering.
(2) $A_{G,\psi}$ is supersolvable.

(3) $A_{G,\psi}$ is free.

(4) $G$ is chordal and does not contain the both of the following induced paths:

(i) An edge $\{v_1, v_2\}$ such that $\psi(v_1)$ and $\psi(v_2)$ are not comparable.

(ii) A path $v_1 \cdots v_k$ with $k \geq 3$ and $\psi(v_1) \supseteq \psi(v_2) = \cdots = \psi(v_{k-1}) \subsetneq \psi(v_k)$.

(5) $G$ is chordal and every induced path is unimodal.

The organization of this paper is as follows. In Section 2 we introduce a class of vertex-weighted graph over a poset in order to prove the equivalence (1) $\iff$ (4) $\iff$ (5) in Theorem 1.4. In Section 3 we will describe a relation between $\psi$-graphical arrangements and $N$-Ish arrangements, which are other deformation of braid arrangements. In Section 4 we will construct a basis for the logarithmic derivation module of the $\psi$-graphical arrangement when $(G, \psi)$ has a weighted elimination ordering. In Section 5 we will complete the proof of Theorem 1.4. In Section 6 we will introduce the multiarrangements corresponding to vertex-weighted graphs over nonnegative integers and give a characterization of their freeness.

2 Vertex-weighted graphs over a poset

Let $G = (V_G, E_G)$ be a simple graph with $\ell$ vertices and $P$ a poset. For a map $\psi: V_G \to P$, we call the pair $(G, \psi)$ a vertex-weighted graph over $P$. If $P$ is a singleton then the vertex-weighted graph $(G, \psi)$ may be identified with the graph $G$. Note that when $P$ is the poset consisting of finite subsets in $\mathbb{K}$, the pair $(G, \psi)$ defines the $\psi$-graphical arrangement $A_{G,\psi}$ as mentioned in Section 1.

We say that an ordering $(v_1, \ldots, v_{\ell})$ of the vertices in $(G, \psi)$ is a weighted elimination ordering if $(v_1, \ldots, v_{\ell})$ is a perfect elimination ordering and $\psi(v_i) \supseteq \psi(v_j)$ whenever $i < j$ and $\{v_i, v_j\} \in E_G$. For an induced subgraph $S$ of $G$, let $\psi_S$ denote the restriction of $\psi$ to $V_S$. We call the pair $(S, \psi_S)$ an induced subgraph of $(G, \psi)$. Note that a weighted elimination ordering of $(G, \psi)$ induces a weighted elimination ordering of any induced subgraph.
of \((G, \psi)\). A path \(v_1 \cdots v_k\) in \((G, \psi)\) is said to be \textbf{unimodal} if there exists \(i \in \{1, \ldots, k\}\) such that \(\psi(v_1) \leq \cdots \leq \psi(v_i) \geq \cdots \geq \psi(v_k)\).

**Proposition 2.1.** A path in \((G, \psi)\) is unimodal if and only if it contains none of the following paths:

(i) An edge \(\{v_1, v_2\}\) such that \(\psi(v_1)\) and \(\psi(v_2)\) are not comparable.

(ii) A path \(v_1 \cdots v_k\) with \(k \geq 3\) and \(\psi(v_1) > \psi(v_2) = \cdots = \psi(v_{k-1}) < \psi(v_k)\).

**Proof.** Since a path of type (i) or (ii) is not unimodal, a path containing at least either one of these paths is not unimodal. We will prove the converse. Let \(v_1 \cdots v_k\) be a non-unimodal path in \((G, \psi)\). We may assume that there is no edge of type (i), i.e., \(\psi(v_i)\) and \(\psi(v_{i+1})\) are comparable for every \(i \in \{1, \ldots, k-1\}\). Since the path \(v_1 \cdots v_k\) is not unimodal, there exist the minimum index \(i_0 \in \{1, \ldots, k-2\}\) satisfying \(\psi(v_{i_0}) > \psi(v_{i_0+1})\) and the minimum index \(i_2 \in \{i_0 + 2, \ldots, k\}\) satisfying \(\psi(v_{i_2}) < \psi(v_{i_2+1})\). Let \(i_1 \in \{i_0, \ldots, i_2-2\}\) be the maximum index satisfying \(\psi(v_{i_1}) > \psi(v_{i_1+1})\). The maximality of \(i_1\) and the minimality of \(i_2\) imply \(\psi(v_{i_1+1}) = \cdots = \psi(v_{i_2-1})\). Hence the path \(v_{i_1} \cdots v_{i_2}\) is of type (ii). \(\square\)

We will prove the following theorem, which is a generalization of Theorem 1.1. This theorem proves the equivalence (1) \(\Leftrightarrow\) (4) \(\Leftrightarrow\) (5) in Theorem 1.4.

**Theorem 2.2.** Let \((G, \psi)\) be a vertex-weighted graph. Then the following are equivalent:

(1) \((G, \psi)\) has a weighted elimination ordering.

(2) \(G\) is chordal and does not contain induced paths in Proposition 2.1.

(3) \(G\) is chordal and every induced path is unimodal.

In order to prove Theorem 2.2 the following lemma is required.

**Lemma 2.3** (Dirac [5, Theorem 4]). Every chordal graph is complete or has at least two non-adjacent simplicial vertices.

**Proof of Theorem 2.2.** The equivalence (2) \(\Leftrightarrow\) (3) is clear from Proposition 2.1. In order to prove (1) \(\Rightarrow\) (2), assume that \((G, \psi)\) has a weighted elimination ordering. Then \(G\) is chordal by Theorem 1.1. Suppose that \((G, \psi)\) has a path mentioned in Proposition 2.1 as an induced path. This path has no
weighted elimination ordering, which is a contradiction since every induced subgraph of \((G, \psi)\) has a weighted elimination ordering. Therefore \(G\) has no induced paths in Proposition 2.1.

To prove \((2) \Rightarrow (1)\) suppose that \(G\) is chordal and does not contain induced paths in Proposition 2.1. Then \(\psi(u)\) and \(\psi(v)\) are comparable for every edge \(\{u, v\} \in E_G\). We will prove that \((G, \psi)\) has a weighted elimination ordering by induction on \(\ell\). If \(\ell = 1\) then the only ordering is a weighted elimination ordering. Assume that \(\ell \geq 2\). It suffices to show that there exists a simplicial vertex \(v\) in \(G\) such that \(\psi(v) \leq \psi(u)\) for any neighbor \(u\) of \(v\) since \(G \setminus \{v\}\) has a weighted elimination ordering by the induction hypothesis. Let \(S\) be a connected component of the subgraph of \(G\) induced by the vertices

\[
\{v \in V_G \mid \psi(v) \text{ is minimal in } \psi(V_G)\}
\]

Note that for any vertices \(u, v \in V_S\) we have that \(\psi(u) = \psi(v)\) by the minimality. Let

\[
N := \{v \in V_G \setminus V_S \mid \{v, u\} \in E_G \text{ for some } u \in V_S\}.
\]

We will prove that \(N\) is a clique of \(G\). If \(N\) were not a clique then there are non-adjacent vertices \(u, v \in N\). For any \(w \in V_S\), we have that \(\psi(u) > \psi(w) < \psi(v)\) by the minimality and comparability. Hence a shortest path from \(u\) to \(v\) in the induced subgraph \(S \cup \{u, v\}\) is a path of type (ii) in Proposition 2.1, which is a contradiction. Thus \(N\) is a clique.

Let \(F\) be the subgraph of \(G\) induced by \(V_S \cup N\). Since \(G\) is chordal, \(F\) is also chordal. Now we will prove that there exists a vertex \(v\) of \(S\) which is simplicial in \(F\). If \(F\) is complete, then every vertex of \(F\) is simplicial in \(F\). In particular, every vertex of \(S\) is simplicial in \(F\). If \(F\) is not complete, then by Lemma 2.3 there exist two non-adjacent vertices of \(F\) which are simplicial in \(F\). Since \(N\) is a clique, at least one of the vertices belongs to \(S\). Thus in the both cases, \(S\) has a vertex \(v\) which is simplicial in \(F\).

Since every neighbor of \(v\) in \(G\) belongs to \(F\), we have that \(v\) is also simplicial in \(G\). This is a desired simplicial vertex.

3 Relation to \(N\)-Ish arrangements

Abe and the authors \cite{3} introduced \(N\)-Ish arrangements, which are deformation of braid arrangements, to state the sharing property of the freeness of deleted Shi arrangements and deleted Ish arrangements.
Let \( N = (N_1, \ldots, N_\ell) \) be a tuple of finite subsets in \( K \). The tuple \( N \) is a **nest** if there exists a permutation \( w \) of \( \{1, \ldots, \ell\} \) such that \( N_{w(1)} \subseteq N_{w(2)} \subseteq \cdots \subseteq N_{w(\ell)} \). Define the \( N \)-Ish arrangement \( \mathcal{I}_N \) by

\[
\mathcal{I}_N := \{\{z = 0\}\} \cup \{\{x_i - x_j = 0\} \mid 1 \leq i < j \leq \ell\} \\
\quad \cup \{\{x_0 - x_i = az\} \mid 1 \leq i \leq \ell, a \in N_i\},
\]

where \( z, x_0, \ldots, x_\ell \) are coordinates of an \((\ell + 2)\)-dimensional vector space over \( K \).

**Theorem 3.1** ([3, Theorem 1.3]). The following conditions are equivalent:

1. \( N \) is a nest.
2. \( \mathcal{I}_N \) is supersolvable.
3. \( \mathcal{I}_N \) is free.

In [3], this theorem is formulated for fields of characteristic 0. However, the proof is independent of the field.

**Proposition 3.2.** Let \( G \) be the complete graph with \( \ell \) vertices \( \{1, \ldots, \ell\} \) and \( N = (N_1, \ldots, N_\ell) \) a tuple as above. Define a map \( \psi: V_G \to 2^K \) by \( \psi(i) = N_i \) for every \( i \in V_G \). Then \( \mathcal{A}_{G,\psi} \times \emptyset_1 \) and \( \mathcal{I}_N \) are affinely equivalent, where \( \emptyset_1 \) denotes the 1-dimensional empty arrangement.

**Proof.** The change of coordinates \( x_i \mapsto x_0 - x_i \) (\( 1 \leq i \leq \ell \)), \( x_0 \mapsto x_0 \), and \( z \mapsto z \) induces the equivalence. \( \square \)

Mu and Stanley stated the following lemma without proof.

**Lemma 3.3** (Mu-Stanley [9, Theorem 3]). If a \( \psi \)-graphical arrangement \( \mathcal{A}_{G,\psi} \) is free then \( \psi(i) \subseteq \psi(j) \) or \( \psi(i) \supseteq \psi(j) \) for all \( \{i, j\} \in E_G \).

In the rest of this section, we will give a proof of Lemma 3.3 with the characterization of the freeness of \( N \)-Ish arrangements.

A subarrangement \( B \) of an arrangement \( A \) is called a **localization** if

\[
B = A_X := \{H \in A \mid H \supseteq X\}
\]

for some \( X \in L(A) \). It is well known that every localization of a free arrangement is also free (see, for example, [10, Theorem 4.37]).
Proposition 3.4. For a ψ-graphical arrangement $\mathcal{A}_{G,\psi}$, let $S = (V_S, E_S)$ be an induced subgraph of $G$ and $\psi_S$ the restriction of $\psi$ to $V_S$. Then the $\psi_S$-graphical arrangement $\mathcal{A}_{S,\psi_S}$ and the graphical arrangement $\mathcal{A}_S$ are localizations of $\mathcal{A}_{G,\psi}$.

Proof. First we will prove that $\mathcal{A}_{S,\psi_S}$ is a localization of $\mathcal{A}_{G,\psi}$. Let $X := \bigcap_{H \in \mathcal{A}_{S,\psi_S}} H \in L(\mathcal{A}_{G,\psi})$. We will show that $\mathcal{A}_{S,\psi_S} = \mathcal{A}_X$. The inclusion $\mathcal{A}_{S,\psi_S} \subseteq \mathcal{A}_X$ is trivial. We will show the converse $\mathcal{A}_{S,\psi_S} \supseteq \mathcal{A}_X$. Take the vector $v \in X$ such that

$$z(v) = 0, \quad x_i(v) = \begin{cases} 0 & \text{if } i \in V_S, \\ i & \text{otherwise.} \end{cases}$$

Every hyperplane in $\mathcal{A}_X$ must contain the vector $v$. The arrangement $\mathcal{A}_{G,\psi}$ consists of hyperplanes of three types, $\{z = 0\}, \{x_i - x_j = 0\}$ and $\{x_i = az\}$. The hyperplane $\{z = 0\}$ is in both of $\mathcal{A}_{S,\psi_S}$ and $\mathcal{A}_X$. If $\{x_i - x_j = 0\} \in \mathcal{A}_X$ then $x_i(v) = x_j(v) = 0$. Hence $i, j \in V_S$ and $\{i, j\} \in E_S$ since $S$ is an induced graph of $G$. Therefore $\{x_i - x_j = 0\} \in \mathcal{A}_{S,\psi_S}$. If $\{x_i = az\} \in \mathcal{A}_X$ then $x_i(v) = az(v) = 0$. Therefore $i \in V_S$ and $\{x_i = az\} \in \mathcal{A}_{S,\psi_S}$. Hence $\mathcal{A}_{S,\psi_S} \supseteq \mathcal{A}_X$. Thus $\mathcal{A}_{S,\psi_S}$ is a localization of $\mathcal{A}_{G,\psi}$.

To verify that $\mathcal{A}_S$ is a localization, take the vector $v \in X := \bigcap_{H \in \mathcal{A}_S} H \in L(\mathcal{A}_{G,\psi})$ satisfying

$$z(v) = -1, \quad x_i(v) = \begin{cases} 0 & \text{if } i \in V_S, \\ i & \text{otherwise.} \end{cases}$$

By the similar argument we have $\mathcal{A}_S = \mathcal{A}_X$. Hence $\mathcal{A}_S$ is a localization. □

Proof of Lemma 3.3. Suppose that $\mathcal{A}_{G,\psi}$ is free and $\{i, j\} \in E_G$. Let $S$ be the subgraph induced by vertices $\{i, j\}$. By Proposition 3.3, $\mathcal{A}_{S,\psi_S}$ is a localization of $\mathcal{A}_{G,\psi}$ hence free. By Proposition 3.2, the arrangement $\mathcal{A}_{S,\psi_S}$ is affinely equivalent to the $N$-Ish arrangement $I_N$, where $N = (\psi(i), \psi(j))$. Theorem 3.1 asserts that $N$ is a nest. Therefore $\psi(i) \subseteq \psi(j)$ or $\psi(i) \supseteq \psi(j)$. □

4 Basis Construction

In this section, we construct a basis for the logarithmic derivation module of $\mathcal{A}_{G,\psi}$ when $(G, \psi)$ has a weighted elimination ordering.
Theorem 4.1. Suppose that \((G, \psi)\) has a weighted elimination ordering \((v_1, \ldots, v_\ell)\). We may assume that \(x_1, \ldots, x_\ell\) are the coordinates corresponding to \(v_1, \ldots, v_\ell\). Define the set \(C_{\geq k}\) by

\[
C_{\geq k} := \{i \mid k < i \leq \ell \text{ and there exists a path } v_k v_{j_1} v_{j_2} \cdots v_{j_n} v_i \text{ such that } k < j_1 < j_2 < \cdots < j_n < i\} \cup \{k\}.
\]

Then the homogeneous derivations

\[
\theta_E = \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i} + z \frac{\partial}{\partial z},
\]

\[
\theta_k = \sum_{i \in C_{\geq k}} \left( \prod_{j \in E_{<k}} (x_j - x_i) \prod_{a \in \psi(v_k)} (x_i - az) \right) \frac{\partial}{\partial x_i} \quad (1 \leq k \leq \ell)
\]

form a basis for \(D(A_G, \psi)\), where \(E_{<k} = \{j \mid 1 \leq j < k, \{v_j, v_k\} \in E_G\}\).

Proof. First we will prove that the derivations belong to \(D(A_G, \psi)\). It is known that the Euler derivation \(\theta_E\) belongs to the logarithmic derivation module of any central arrangement. Since \(\theta_k\) does not contain \(\frac{\partial}{\partial z}\), we have \(\theta_k(z) = 0 \in zS\).

Suppose \(\{x_s - x_t = 0\} \in A_G, \psi\), i.e., \(\{v_s, v_t\} \in E_G\) \((s < t)\). We will verify \(\theta_k(x_s - x_t) \in (x_s - x_t)S\). From the definition of \(C_{\geq k}\), we can immediately check that \(s \in C_{\geq k}\) implies \(t \in C_{\geq k}\). Thus we only need to consider the cases of (i) \(s, t \in C_{\geq k}\), (ii) \(s \notin C_{\geq k}\) and \(t \in C_{\geq k}\), and (iii) \(s, t \notin C_{\geq k}\). When (i) \(s, t \in C_{\geq k}\), then

\[
\theta_k(x_s - x_t) = \prod_{j \in E_{<k}} (x_j - x_s) \prod_{a \in \psi(v_k)} (x_s - az) - \prod_{j \in E_{<k}} (x_j - x_t) \prod_{a \in \psi(v_k)} (x_t - az)
\]

\[
\equiv \prod_{j \in E_{<k}} (x_j - x_s) \prod_{a \in \psi(v_k)} (x_s - az) - \prod_{j \in E_{<k}} (x_j - x_t) \prod_{a \in \psi(v_k)} (x_t - az) \pmod{x_s - x_t}
\]

\[
= 0.
\]

When (ii) \(s \notin C_{\geq k}\) and \(t \in C_{\geq k}\), there is a path \(v_kv_{j_1} \cdots v_{j_n} v_t\) such that \(k < j_1 < \cdots < j_n < t\). Since \(\{v_{j_n}, v_t\}, \{v_s, v_t\} \in E_G\) and \(v_t\) is simplicial in
the subgraph induced by the vertices \( \{ v_j \mid j \leq t \} \), we have \( \{ v_s, v_n \} \in E_G \). Continuing this process, we can see that \( \{ v_s, v_k \} \in E_G \), hence \( s \in E_{<k} \). Thus

\[
\theta_k(x_s - x_t) = - \prod_{j \in E_{<k}} (x_j - x_t) \prod_{a \in \psi(v_k)} (x_t - az) \in (x_s - x_t)S.
\]

When (iii) \( s, t \notin C_{\geq k} \), then \( \theta_k(x_s - x_t) = 0 \). Therefore, we conclude that \( \theta_k(x_s - x_t) \in (x_s - x_t)S \) for any \( \{ x_s - x_t = 0 \} \in \mathcal{A}_{G,\psi} \).

Suppose that \( \{ x_s - bz = 0 \} \in \mathcal{A}_{G,\psi} \), i.e., \( b \in \psi(v_k) \). If \( s \in C_{\geq k} \), then there is a path \( v_kv_{j_1} \cdots v_{j_n}v_s \) such that \( k < j_1 < \cdots < j_n < s \), which implies \( \psi(v_k) \supseteq \psi(v_{j_1}) \supseteq \cdots \supseteq \psi(v_{j_n}) \supseteq \psi(v_s) \). Thus \( b \in \psi(v_k) \), so

\[
\theta_k(x_s - bz) = \prod_{j \in E_{<k}} (x_j - x_s) \prod_{a \in \psi(v_k)} (x_s - az) \in (x_s - bz)S.
\]

If \( s \notin C_{\geq k} \), then \( \theta_k(x_s - bz) = 0 \). Therefore \( \theta_k(x_s - bz) \in (x_s - bz)S \) for \( \{ x_s - bz = 0 \} \in \mathcal{A}_{G,\psi} \). Consequently, \( \theta_k \in D(\mathcal{A}_{G,\psi}) \) for \( 1 \leq k \leq \ell \). Finally, we show that \( \theta_E, \theta_1, \ldots, \theta_\ell \) form a basis for \( D(\mathcal{A}_{G,\psi}) \). Notice that if \( 1 \leq i < k \), then \( i \notin C_{\geq k} \), so \( \theta_k(x_i) = 0 \). Thus the determinant of the coefficient matrix of \( \theta_E, \theta_1, \ldots, \theta_\ell \) can be calculated as follows:

\[
\begin{vmatrix}
\theta_E(z) & \theta_1(z) & \cdots & \theta_\ell(z) \\
\theta_E(x_1) & \theta_1(x_1) & \cdots & \theta_\ell(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
\theta_E(x_\ell) & \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell)
\end{vmatrix} = \begin{vmatrix}
z & 0 & \cdots & 0 \\
x_1 & \theta_1(x_1) & \cdots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
x_\ell & \theta_1(x_\ell) & \cdots & \theta_\ell(x_\ell)
\end{vmatrix}
\]

\[
= z \prod_{k=1}^{\ell} \theta_k(x_k)
\]

\[
= z \prod_{k=1}^{\ell} \left( \prod_{j \in E_{<k}} (x_j - x_k) \prod_{a \in \psi(v_k)} (x_k - az) \right)
\]

\[
= z \prod_{v_j, v_k \in E_G} (x_j - x_k) \prod_{1 \leq k \leq \ell} (x_k - az)
\]

\[
= Q(\mathcal{A}_{G,\psi}).
\]

Therefore it follows from Saito’s criterion \textbf{[11]} (see also \textbf{[10] Theorem 4.19]) that \( \theta_E, \theta_1, \ldots, \theta_\ell \) form a basis for \( D(\mathcal{A}_{G,\psi}) \). \( \square \)
Combining the proof of Theorem 4.1 and following Ziegler’s theorem, we may derive that $A_{G,\psi}$ is supersolvable if $(G,\psi)$ has a weighted elimination ordering. This is another proof of the sufficiency in Theorem 1.3.

**Theorem 4.2** (Ziegler [19, Theorem 6.6]). Let $A$ be a central arrangement. Then $A$ is supersolvable if and only if there is a basis for $D(A)$ such that the coefficient matrix of the basis is lower triangular for some choice of coordinates.

As a corollary of Theorem 4.1, we may give the exponents of $A_{G,\psi}$, which was stated by Mu and Stanley without proof.

**Corollary 4.3** (Mu-Stanley [9, Proposition 2]). Suppose that $(G,\psi)$ has a weighted elimination ordering $(v_1,\ldots,v_\ell)$ and let $E_{<k}$ be as in Theorem 4.1. Then $A_{G,\psi}$ is supersolvable with exponents $\{1,|E_{<1}|+|\psi(v_1)|,\ldots,|E_{<\ell}|+|\psi(v_\ell)|\}$.

5 The proof of Theorem 1.4

We will first prove the following lemma.

**Lemma 5.1.** Let $\ell \geq 3$ and $A = A_{G,\psi}$ a $\psi$-graphical arrangement defined by the following graph $(G,\psi)$:

```
 1 -- 2 -- 3 -- ... -- \ell - 2 -- \ell - 1 -- \ell
```

with $\psi(1) \supset \psi(2) = \cdots = \psi(\ell - 1) \subset \psi(\ell)$.

Then $A$ is not free.

The following theorem is required:

**Theorem 5.2** (Orlik-Terao [10, Theorem 4.46], Terao [15]). Let $A$ be an arrangement and $H_0 \in A$. Let $(A, A', A'')$ be the triple with respect to $H_0$, i.e., $A' = A \setminus \{H\}, A'' = \{H \cap H_0 \mid H \in A'\}$. If $A$ and $A'$ are free then $A''$ is also free and $\exp A'' \subset \exp A'$.

**Proof of Lemma 5.1.** We will prove the assertion by induction on $\ell$. Suppose that $\ell = 3$. Let $H = \{x_2 - x_3 = 0\}$ and $(A, A', A'')$ the triple with respect to $H$. Then the ordering $(1,2,3)$ is a weighted elimination ordering for $A'$. Hence $A'$ is supersolvable and free with the exponents
\[ \exp(\mathcal{A}') = (1, |\psi(1)|, |\psi(2)| + 1, |\psi(3)|) \] by Corollary 4.3. Assume that \( \mathcal{A} \) is free. Then Theorem 5.2 asserts that \( \mathcal{A}'' \) is also free and \( \exp \mathcal{A}'' \subset \exp \mathcal{A}' \). Then \( \psi(1) \subseteq \psi(3) \) or \( \psi(1) \supseteq \psi(3) \) by Lemma 3.3. By Lemma 4.3, the exponents of \( \mathcal{A}'' \) are \( (1, |\psi(1)| + 1, |\psi(3)|) \) or \( (1, |\psi(1)|, |\psi(3)| + 1) \) respectively. In the both case, we have that \( \exp \mathcal{A}'' \not\subset \exp \mathcal{A}' \). This contradiction concludes that \( \mathcal{A} \) is not free.

Suppose that \( \ell \geq 4 \) and that the statement is true for \( \ell - 1 \). Let \( H = \{ x_{\ell-1} - x_\ell = 0 \} \) and \( (\mathcal{A}, \mathcal{A}', \mathcal{A}'') \) the triple with respect to \( H \). Since the ordering \( (1, 2, \ldots, \ell) \) is a weighted elimination ordering for \( \mathcal{A}' \), the arrangement \( \mathcal{A}' \) is free. If \( \mathcal{A} \) is free, then \( \mathcal{A}'' \) is also free by Theorem 5.2. Contrary, by the induction hypothesis, we have that \( \mathcal{A}' \) is not free, which is a contradiction. Hence \( \mathcal{A} \) is not free.

\[ \square \]

**Proof of Theorem 1.4.** By Theorem 1.3, the equivalence (1) \( \iff \) (2) holds. The implication (2) \( \Rightarrow \) (3) is well-known. Theorem 2.2 shows (1) \( \iff \) (4) \( \iff \) (5). The rest part of the proof is (3) \( \Rightarrow \) (4). Suppose that \( \mathcal{A}_{G, \psi} \) is free. By Proposition 3.4, the graphical arrangement \( \mathcal{A}_{G} \) is free. Therefore \( G \) is chordal by Theorem 1.2. By Lemma 3.3, Proposition 3.4 and Lemma 5.1, \( (G, \psi) \) has no induced paths in (4). Thus the proof has been completed. \[ \square \]

## 6 Characterization of the freeness of the \( \psi \)-graphical multiarrangements

In this section, we introduce the \( \psi \)-graphical multiarrangements, which are determined by a vertex-weighted graph over \( \mathbb{Z}_{\geq 0} \), and give a characterization of the freeness of the \( \psi \)-graphical multiarrangements. First, we introduce some definitions of multiarrangements.

A **multiarrangement** \( (\mathcal{A}, m) \) is a pair of an arrangement \( \mathcal{A} \) with a map \( m : \mathcal{A} \to \mathbb{Z}_{\geq 0} \), which is called multiplicity, and its defining polynomial is

\[ Q(\mathcal{A}, m) = \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}. \]

When \( m(H) = 1 \) for any \( H \in \mathcal{A} \), then \( (\mathcal{A}, m) \) is just a hyperplane arrangement and sometimes called a simple arrangement.

The module of logarithmic derivations \( D(\mathcal{A}, m) \) of a multiarrangement is defined by

\[ D(\mathcal{A}, m) := \{ \theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^{m(H)} S \text{ for any } H \in \mathcal{A} \}. \]

We say that \( (\mathcal{A}, m) \) is **free** if \( D(\mathcal{A}, m) \) is a free \( S \)-module.
Definition 6.1. Let \((G, \psi)\) be a vertex-weighted graph over \(\mathbb{Z}_{\geq 0}\) with vertex set \(V_G = \{1, \ldots, \ell\}\) and edge set \(E_G\). Note that \(\psi\) is a map from \(V_G\) to \(\mathbb{Z}_{\geq 0}\). The \(\psi\)-graphical multiarrangement \(M_{G, \psi}\) is defined by the following defining polynomial:

\[
Q(M_{G, \psi}) = \prod_{\{i, j\} \in E_G} (x_i - x_j) \prod_{1 \leq i \leq \ell} x_i^{\psi(i)}.
\]

We can regard a \(\psi\)-graphical multiarrangement as a graphical multiarrangement, which is a graphical arrangement with multiplicity, by the coordinates change \(x_i \mapsto x_0 - x_i (1 \leq i \leq \ell)\). Several partial results for characterizing the freeness of graphical multiarrangements have been found, for example, see [2, 4, 17].

The freeness of the \(\psi\)-graphical multiarrangements is characterized as follows similarly to Theorem 1.4:

Theorem 6.2. Let \(M_{G, \psi}\) be a \(\psi\)-graphical multiarrangement. Then the following conditions are equivalent:

1. \((G, \psi)\) has a weighted elimination ordering.
2. \(D(M_{G, \psi})\) has a basis whose coefficient matrix is lower triangular.
3. \(M_{G, \psi}\) is free.

To prove this theorem, we need the following theorems and lemmas.

Let \(\mathcal{A}\) be a central arrangement and fix \(H_0 \in \mathcal{A}\). The Ziegler restriction \((\mathcal{A}^{H_0}, m^{H_0})\) of \(\mathcal{A}\) onto \(H_0\) is defined by \(\mathcal{A}^{H_0} = \{ H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\} \}\), \(m^{H_0}(X) = |A_X \setminus \{H_0\}|\) for \(X \in \mathcal{A}^{H_0}\).

Theorem 6.3 (Ziegler [20, Theorem 11]). Let \(\mathcal{A}\) be an arrangement and \(H_0 \in \mathcal{A}\). If \(\mathcal{A}\) is free and \(\{\theta_E, \theta_1, \ldots, \theta_\ell\}\) is a basis for \(D(\mathcal{A})\), where \(\theta_i(\alpha_{H_0}) = 0\) for \(1 \leq i \leq \ell\), then the Ziegler restriction \((\mathcal{A}^{H_0}, m^{H_0})\) is free and \(\{\theta_1|_{H_0}, \ldots, \theta_\ell|_{H_0}\}\) is a basis for \(D(\mathcal{A}^{H_0}, m^{H_0})\).

Theorem 6.4 (Yoshinaga [18, Theorem 2.2]). Suppose \(\ell \geq 3\). Let \(\mathcal{A}\) be an arrangement in \((\ell + 1)\)-dimensional vector space and \(H_0 \in \mathcal{A}\). Then \(\mathcal{A}\) is free if and only if the Ziegler restriction \((\mathcal{A}^{H_0}, m^{H_0})\) is free and \(A_X\) is free for any \(X \in L(\mathcal{A}^{H_0}) \setminus \{\bigcap_{H \in \mathcal{A}} H\}\).
Lemma 6.5 (Abe [11, Lemma 3.8]). Let \((A, m)\) be a multiarrangement and \(X \in L(A)\). If \((A, m)\) is free, then the localization \((A_X, m_X)\) is free, where \(m_X(H) = m(H)\) for \(H \in A_X\).

Lemma 6.6 (Terao [16, Lemma 1], Ziegler [21, Theorem 1.5]). Let \((A, m)\) be a multiarrangement in a vector space \(V\) over a field \(K\). For any field extension \(F/K\), \((A, m)\) is free if and only if \((A_F, m_F)\) is free, where \(A_F\) denotes the arrangement in \(V \otimes_K F\) consisting of hyperplanes \(H \otimes_K F\) for \(H \in A\) and 
\[m_F(H \otimes_K F) := m(H).\]

Lemma 6.7. Suppose a graph \(G\) is a path \(v_1 \cdots v_k\) with \(k \geq 3\) and \(\tilde{\psi} : V_G \to 2^K\) satisfies \(\tilde{\psi}(v_1) \supseteq \tilde{\psi}(v_2) = \cdots = \tilde{\psi}(v_{k-1}) \subseteq \tilde{\psi}(v_k)\). Let \(H_0 = \{z = 0\}\). Then \((A_{G, \tilde{\psi}})_X\) is free for any \(X \in L(A_{H_0}) \setminus \{\bigcap_{H \in A_{G, \tilde{\psi}}} H\}\).

Proof. Since \((A_{G, \tilde{\psi}})_X\) is a subarrangement of the \(\tilde{\psi}\)-graphical arrangement \(A_{G, \tilde{\psi}}\) containing \(H_0\), we may put \(A_{S, \tilde{\psi}'} = (A_{G, \tilde{\psi}})_X\), where \(S\) is a subgraph of \(G\) and \(\tilde{\psi}' : V_S \to 2^K\). Moreover, we may assume that \(S\) is connected.

Let \(x_1, \ldots, x_k\) be the coordinates corresponding to the vertices \(v_1, \ldots, v_k\). If \(A_{S, \tilde{\psi}'}\) does not contain the hyperplanes of the form \(\{x_i = az\}\), then \(A_{S, \tilde{\psi}'}\) is a simple graphical arrangement of the chordal graph \(S\) with the hyperplane \(H_0\). Hence \(A_{S, \tilde{\psi}'}\) is free by Theorem 1.2.

Assume that \(A_{S, \tilde{\psi}'}\) contains at least one hyperplane \(\{x_{i_0} = a_0z\}\), i.e., there exist a vertex \(v_{i_0} \in V_S\) and an element \(a_0 \in \tilde{\psi}'(v_{i_0})\). Since \(H_0 = \{z = 0\}\), we have \(\{x_{i_0} = az\} \in A_{S, \tilde{\psi}'}\) for any \(a \in \tilde{\psi}(v_i)\). Moreover, since \(S\) is connected, we have \(\{x_i = az\} \in A_{S, \tilde{\psi}'}\) for any \(v_i \in V_S\) and \(a \in \tilde{\psi}(v_i)\). Therefore \(\tilde{\psi}' = \tilde{\psi}_S\), where \(\tilde{\psi}_S = \tilde{\psi}|_{V_S}\). Since \(X \neq \bigcap_{H \in A_{G, \tilde{\psi}}} H\), we have \(A_{S, \tilde{\psi}'} = (A_{G, \tilde{\psi}})_X \subseteq A_{G, \tilde{\psi}}\), namely \(S \subsetneq G\). Hence \((S, \tilde{\psi}_S)\) satisfies the condition in Theorem 1.4 (4). Therefore \(A_{S, \tilde{\psi}'} = A_{S, \tilde{\psi}_S}\) is free. \(\square\)

Proof of Theorem 6.3. By Lemma 6.6 we may assume that the ground field \(K\) of \(M_{G, \psi}\) is infinite. Then there exists an injection \(\iota : \mathbb{Z}_{>0} \to K\). Define \(\tilde{\psi} : V_G \to 2^K\) by \(\tilde{\psi}(i) = \{\iota(1), \ldots, \iota(\psi(i))\}\). Then \(M_{G, \psi}\) is the Ziegler restriction of the \(\psi\)-graphical arrangement \(A_{G, \tilde{\psi}}\) on to the hyperplane \(\{z = 0\}\).

Suppose that \((G, \psi)\) has a weighted elimination ordering. Then \((G, \tilde{\psi})\) also has a weighted elimination ordering. Hence, by Theorem 4.1, \(D(A_{G, \tilde{\psi}})\) has a basis \(\{\theta_E, \theta_1, \ldots, \theta_\ell\}\) whose coefficient matrix is lower triangular and \(\theta_i(z) = 0\) for \(1 \leq i \leq \ell\). Therefore, we can see \((1) \Rightarrow (2)\) from Theorem 6.3.
The implication (2) ⇒ (3) is obvious.

Let us show (3) ⇒ (1). Suppose that $\mathcal{M}_{G,\psi}$ is free. It is easy to verify that the graphical arrangement $A_G$ is a localization of $\mathcal{M}_{G,\psi}$ by Proposition 3.4. By Lemma 6.5, we have $A_G$ is free, and hence $G$ is chordal by Theorem 1.2.

Assume that there exists an induced path $P = v_1 \cdots v_k$ of $G$ with $k \geq 3$ and $\psi(v_1) > \psi(v_2) = \cdots = \psi(v_{k-1}) < \psi(v_k)$. Note that $\mathcal{M}_{P,\psi_P}$ is a localization of $\mathcal{M}_{G,\psi}$ by Proposition 3.4. Hence it follows from Lemma 6.5 that $\mathcal{M}_{P,\psi_P}$ is free. However, Using Lemma 6.7 and Theorem 6.4, we have $\mathcal{M}_{P,\psi_P}$ is not free since $A_{P,\tilde{\psi}_P}$ is not free by Theorem 1.4. This is a contradiction. Therefore $(G, \psi)$ does not contain such induced paths.

Since $\mathbb{Z}_{\geq 0}$ is totally ordered, we conclude that $(G, \psi)$ has a weighted elimination ordering by Theorem 2.2.

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