Critical parameters for non-hermitian Hamiltonians

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Abstract. We calculate accurate critical parameters for a class of non-hermitian Hamiltonians by means of the diagonalization method. We study three one-dimensional models and two perturbed rigid rotors with PT symmetry. One of the latter models illustrates the necessity of a more general condition for the appearance of real eigenvalues that we also discuss here.

1. Introduction

There has recently been interest in PT-symmetric Hamiltonians that exhibit real eigenvalues for a range of values of a potential parameter. Some of them are anharmonic oscillators [1–8] as well as models with Dirichlet [9–11] periodic and anti-periodic boundary conditions [12, 13].

Among the methods used for the study of such models we mention the WKB approximation [2, 3], the eigenvalue moment method [4, 6], the multiscale reference function analysis [5], the diagonalization method (DM) [7] and the orthogonal polynomial projection quantization (OPPQ) (an improved Hill-determinant method) [8].

For some particular values of the potential parameter the spectrum of those PT-symmetric Hamiltonians exhibits critical points where two real eigenvalues coalesce and emerge as complex conjugate eigenvalues. Such critical points are also known as exceptional points [14–17].

The purpose of this paper is the analysis of the critical points for a variety of simple models. The calculation is based on a well known simple and quite efficient application of the DM [14]. In section 2 we propose a somewhat more general condition for the existence of real eigenvalues (unbroken symmetry) [18, 19] that is suitable for models with degenerate states. In section 3 we present three one-dimensional examples already discussed earlier by other authors. In section 4 we outline the procedure for the calculation of critical points based on the DM. In section 5 we apply perturbation theory to one of the models and discuss the convergence of the perturbation series for the eigenvalues by comparison with the accurate results produced by the DM. In section 6...
we discuss a PT-symmetric perturbed planar rigid rotor that was studied earlier as an example with E2 algebra [13]. In section 7 we discuss a non-hermitian perturbed three-dimensional rigid rotor that was not treated before as far as we know. This most interesting model illustrates the generalized condition for real eigenvalues mentioned above. Finally, in section 8 we summarize the main results and draw conclusions.

2. PT Symmetry

It is well known that a wide class of non-hermitian Hamiltonians with unbroken PT symmetry exhibit real spectra [18,19]. In general, they are invariant under an antilinear or antiunitary transformation of the form $\hat{A}^{-1}\hat{H}\hat{A} = \hat{H}$. The antiunitary operator $\hat{A}$ satisfies [20]

$$\hat{A} (|f\rangle + |g\rangle) = \hat{A} |f\rangle + \hat{A} |g\rangle$$
$$\hat{A} c |f\rangle = c^* \hat{A} |f\rangle,$$

for any pair of vectors $|f\rangle$ and $|g\rangle$ and arbitrary complex number $c$, where the asterisk denotes complex conjugation. This definition is equivalent to

$$\langle \hat{A} f | \hat{A} g \rangle = \langle f | g \rangle^*$$

(2)

It follows from the antiunitary invariance mentioned above that $[\hat{H}, \hat{A}] = 0$. Therefore, if $|\psi\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $E$

$$\hat{H} |\psi\rangle = E |\psi\rangle,$$

(3)

we have

$$[\hat{H}, \hat{A}] |\psi\rangle = \hat{H}\hat{A} |\psi\rangle - \hat{A}\hat{H} |\psi\rangle = \hat{H}\hat{A} |\psi\rangle - E^* \hat{A} |\psi\rangle = 0.$$  

(4)

Consequently, $E$ is real if

$$\hat{H}\hat{A} |\psi\rangle = E\hat{A} |\psi\rangle,$$

(5)

that contains the condition of unbroken symmetry required by Bender et al [18,19]

$$\hat{A} |\psi\rangle = \lambda |\psi\rangle$$

(6)
as a particular case. Note that equation (5) applies to the case in which \( \hat{A} |\psi\rangle \) is a linear combination of degenerate eigenvectors of \( \hat{H} \) with eigenvalue \( E \).

If \( \hat{K} \) is an antilinear operator such that \( \hat{K}^2 = \hat{1} \) (for example, the complex conjugation operator) then it follows from \( \hat{K}^2 = \hat{1} \) that \( \hat{A}\hat{K} = \hat{U} \) is unitary (\( \hat{U}^\dagger = \hat{U}^{-1} \)). In other words, any antilinear operator \( \hat{A} \) can be written as a product of a unitary operator and the complex conjugation operation \( [20] \). In most of the non-hermitian models studied \( \hat{U}^{-1} = \hat{U} \) that results in \( \hat{A}^2 = \hat{1} \) (as in the case of the parity operator \( \hat{U} = \hat{P} \) that gives rise to PT symmetry) \( [18, 19] \).

3. Some simple one-dimensional examples

In this section we consider three examples of the Schrödinger equation

\[
\hat{H} \psi = E \psi \\
\hat{H} = \hat{p}^2 + \hat{V}(x),
\]

with eigenvalues \( E_0 < E_1 < \ldots \)

The first one \( [2, 4, 5] \)

\[
\hat{H} = \hat{p}^2 + i\hat{x}^3 + ia\hat{x},
\]

exhibits an infinite set of critical values \( 0 > a_0 > a_1 > \ldots > a_n > \ldots \) of \( a \) so that \( E_{2n} = E_{2n+1} \) at \( a = a_n \). Both eigenvalues are real when \( a > a_n \) and become complex conjugate numbers when \( a < a_n \). The eigenfunctions \( \psi_{2n} \) and \( \psi_{2n+1} \) are linearly dependent at the exceptional point \( a = a_n \) \( [14, 17] \).

The second example is \( [1, 3, 6] \)

\[
\hat{H} = \hat{p}^2 + \hat{x}^4 + ia\hat{x}.
\]

If \( \hat{P} \) denotes the parity operator we have \( \hat{P} \hat{H}(a) \hat{P} = \hat{H}(-a) \) so that \( E(-a) = E(a) \). Because of this property of the eigenvalues the crossings \( E_{2n} = E_{2n+1} \) take place at \( \pm a_n \), where \( 0 < a_0 < a_1 < \ldots < a_n < \ldots \). In this case the pair of coalescing eigenvalues become complex conjugate numbers when \( |a| > a_n \).
The third example is given by
\[ \hat{H} = \hat{p}^2 + ia\hat{x}, \] (10)
with the boundary conditions \( \psi(\pm 1) = 0 \). In this case we also find that the crossings take place at \( \pm a_n, a_n > 0 \) as in the preceding one. Because of physical reasons Rubinstein et al \[9\] considered only the half line \( a > 0 \).

4. Diagonalization method

In order to solve the Schrödinger equation (7) we resort to a matrix representation of the Hamiltonian operator \( H_{ij} = \langle i | \hat{H} | j \rangle \) in an appropriate orthonormal basis set \( \{ | j \rangle, j = 0, 1, \ldots \} \). We obtain the eigenvalues from the roots of the characteristic polynomial given by the secular determinant \( D(E, a) = | \hat{H} - EI | = 0 \), where \( \hat{H} \) is an \( N \times N \) matrix with elements \( H_{ij}, i, j = 0, 1, \ldots, N - 1 \) and \( I \) is the \( N \times N \) identity matrix. We look for those roots of the characteristic polynomial that converge as \( N \) increases. The characteristic polynomial gives us either \( E(a) \) or \( a(E) \).

In all the examples discussed here the critical parameters are given by \( a_n = a(e_n) \), where
\[ \left. \frac{da}{dE} \right|_{E=e_n} = 0, \] (11)
and \( E_{2n}(a_n) = E_{2n+1}(a_n) = e_n \). Therefore, we can obtain the critical parameters approximately from the set of polynomial equations \( \{ D(E, a) = 0, \partial D(E, a)/\partial E = 0 \} \) \[14\]. We look for pairs of roots \( (a_{n,N}, e_{n,N}) \) that converge as \( N \to \infty \).

The eigenvectors of the harmonic oscillator \( \hat{H}_0 = \hat{p}^2 + \hat{x}^2 \) are a suitable basis set for the first two examples \[8\] and \(9\), and for the third one \(10\) we choose
\[ \phi_n(x) = \langle x | n \rangle = \sin \left( \frac{n\pi(x + 1)}{2} \right), n = 1, 2, \ldots \] (12)

Before proceeding with the discussion of the examples we want to stress that the DM is a simple and most efficient approach for the accurate calculation of the eigenvalues and eigenfunctions of those PT-symmetric oscillators with eigenfunctions that vanish exponentially along the real \( x \) axis. In order to illustrate this point we compare the DM
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with the recently developed OPPQ [8]. As an example we choose the PT-symmetric oscillator \( \hat{H} = \hat{p}^2 + i\hat{x}^3 \) because Handy and Vrinteaum [8] showed OPPQ results of increasing order of accuracy for this model. Although both methods resort to the same Gaussian function and Hermite polynomials, Table [1] shows that the rate of convergence of the DM is noticeably greater. It is striking that the DM of order \( N \) appears to be nearly as accurate as the OPPQ of order \( N + 20 \).

Tables [2],[3] and [4] show the first critical parameters for the examples [8], [9] and [10] calculated with \( N \leq 300 \), \( N \leq 300 \) and \( N = 100 \) basis functions, respectively. With those results we carried out nonlinear regressions of the form

\[
a_n = b + ce_n^s,
\]

and obtained

\[
\begin{align*}
b &= -0.324 \pm 0.015 \\
c &= -1.9288 \pm 0.0083 \\
s &= 0.6751 \pm 0.0011,
\end{align*}
\]

for [8],

\[
\begin{align*}
b &= 0.407 \pm 0.010 \\
c &= 1.1540 \pm 0.0048 \\
s &= 0.7555 \pm 0.0010,
\end{align*}
\]

for [9] and

\[
\begin{align*}
b &= -0.00028 \pm 0.00016 \\
c &= 1.732092 \pm 9.5 \times 10^{-6} \\
s &= 0.9999951 \pm 9.4 \times 10^{-7},
\end{align*}
\]

for [10]. The parameters in equation [15] are in good agreement with the WKB ones [3] which suggests that even the first critical parameters for that model exhibit the large-\( e_n \)
asymptotic behaviour given exactly by the WKB method. The nonlinear regression appears to be most accurate for the example (10) where it seems that $a_n = 1.7321 e_n$. It seems that $e_n$ is always approximately between $E_{2n-1}(a = 0)$ and $E_{2n}(a = 0)$ and, therefore, increases asymptotically as $n^2$. Consequently, $a_n$ behaves approximately in the same way.

In the discussion below we sometimes find it convenient to write $g$ for $ia$ and consider $g$ complex. Figure 1 shows $E_n(g)$, $n = 1, 2, 3, 4$ for the example (10) for $g$ real and purely imaginary. We will discuss this case with more detail in the next section.

5. Perturbation theory

Delabaere and Trinh [2] derived the exact Rayleigh-Schrödinger series asymptotic to the eigenvalues of the Hamiltonian (8) for large $a$. For the three examples discussed in section 5 it is also possible to obtain a perturbation series for small $a$. In all of them the Taylor series for $E_n$ about $a = 0$ exhibits a finite nonzero radius of convergence (see, for example, page 111 in reference [22] and references therein). The three Hamiltonians are PT symmetric when $g$ is imaginary and (9) and (10) are Hermitian when $g$ is real. The perturbation series for the eigenvalues of either (9) or (10) reads

$$E_n(g) = \sum_{j=0}^{\infty} E_{n,j} g^{2j}. \quad (17)$$

We can calculate the coefficients $E_{n,j}$ approximately for the former and exactly for the latter. By means of a variety of well known methods [22] we easily obtain

$$E_n = \frac{1}{2} b_n + \frac{(2 b_n - 15) g^2}{12 b_n^2} + \frac{(b_n^2 - 105 b_n + 495) g^4}{18 b_n^4} + \frac{(2 b_n^3 - 825 b_n^2 + 23400 b_n - 95625) g^6}{36 b_n^6} + \ldots, \quad (18)$$

where $b_n = n^2 \pi^2 / 2$. The radius of convergence of the perturbation series for both $E_{2n-1}$ and $E_{2n}$, $n = 1, 2, \ldots$ cannot be greater than $a_n$ because the two eigenvalues coalesce at the exceptional branch points $g = \pm ia_n$.

Figure 2 shows the first four eigenvalues of the problem (10) when $g = ia$ calculated by means of the DM and by perturbation theory of order 20. We appreciate that there
is a good agreement between both approaches for the first two eigenvalues for almost all the values of \(-a_1 < a < a_1\) except close to the crossings where perturbation theory is expected to fail. The situation appears to be quite similar for the fourth eigenvalue but the behaviour of the perturbation series for the third eigenvalue strongly suggests that its radius of convergence may be considerably smaller than \(a_2\).

If \(g = \pm ia_n\) were the singularities closest to the origin, one could obtain them from the perturbation coefficients \(E_{n,j}\) as follows:

\[
a_n = \lim_{k \to \infty} \left| \frac{(1/2 - k) E_{2n-1,k}}{(k+1)E_{2n-1,k+1}} \right|^{1/2} = \lim_{k \to \infty} \left| \frac{(1/2 - k) E_{2n,k}}{(k+1)E_{2n,k+1}} \right|^{1/2}.
\]

(19)

Table 5 shows that \(a_1(k) = \left| \frac{(1/2 - k) E_{1,k}}{(k+1)E_{1,k+1}} \right|^{1/2}\) already converges towards the result in Table 4 as \(k\) increases, and we obtain identical results with the coefficient \(E_{2,k}\) as expected. However, the sequences with \(E_{n,k}\) do not converge when \(n > 2\) which suggests that there may be other branch points on the complex \(g\)-plane closest to the origin. For example, \(E_3(g)\) exhibits branch points at \(g_c = \pm 11.48088661 + 26.24188126i\) and also at \(g^*_c\) that are closer to the origin than \(g_2 = ia_2\) (\(|g_c| = 28.64344759 < a_2\)). As already mentioned above, equation (19) is only suitable for a branch point on the imaginary axis and therefore does not converge in the latter case. The branch points at \(g_c\) and \(g^*_c\) account for the behaviour of the perturbation series for \(E_3(g)\) in Fig. 2 discussed above.

6. Non-hermitian perturbed planar rigid rotor

In this section we consider a simple model with periodic boundary conditions that we prefer to treat separately from those in section 3.

Bender and Kalveks studied the eigenvalues of

\[
- \psi''(\theta) + g \cos(\theta) \psi(\theta) = E \psi(\theta),
\]

(20)

with periodic \(\psi(\theta + 2\pi) = \psi(\theta)\) and anti-periodic \(\psi(\theta + 2\pi) = -\psi(\theta)\) boundary conditions. This equation is a particular case of

\[
\hat{H} \psi = E \psi,
\]
\[
\hat{H} = \hat{J}^2 + V(g, \theta), \quad \hat{J} = -i \frac{d}{d\theta},
\]
when \(V(g, \theta) = g \cos(\theta)\).

By means of the unitary operator \(\hat{U}\) that produces the transformation
\[
\hat{U}^\dagger \theta \hat{U} = \theta + \pi, \quad \hat{U}^\dagger \hat{J} \hat{U} = \hat{J}
\]
we can construct the antiunitary operator \(\hat{A} = \hat{U} \hat{T} = \hat{T} \hat{U}\), where \(\hat{T}\) is the time-inversion operator, as indicated in section 2. Since \(A^{-1} \hat{H} \hat{A} = \hat{H}\) when \(g = ia\) is purely imaginary we expect real eigenvalues for some real values of \(a\).

Here we consider only periodic boundary conditions and transform equation (20) into the Mathieu equation [21] by means of the transformations \(\theta = 2x, E_{BK} = E/4\) and \(g_{BK} = g/2\), so that
\[
\varphi''(x) + [E - 2g \cos(2x)] \varphi(x) = 0,
\]
where \(\varphi(x) = \psi(2x)\). The even and odd solutions to this equation can be expanded in the Fourier series
\[
\varphi_e(x) = \sum_{m=0}^{\infty} A_{2m} \cos(2mx),
\]
\[
\varphi_o(x) = \sum_{m=1}^{\infty} B_{2m} \sin(2mx),
\]
respectively, where the coefficients \(A_{2m}\) and \(B_{2m}\) can be calculated by means of simple three-term recurrence relations [21]. We can efficiently calculate accurate eigenvalues from either the secular determinant, as discussed in section 4, or the truncation conditions \(A_{2N} = 0\) and \(B_{2N} = 0\) for sufficiently large values of \(N\). We denote \(E_{e,n}, n = 0, 1, \ldots\) and \(E_{o,n}, n = 1, 2, \ldots\) the eigenvalues of the even and odd solutions, respectively. Obviously, \(E_{e,n} = E_{o,n} = 4n^2, n = 1, 2, \ldots\), when \(g = 0\).

The results of Bender and Kalveks [13] suggest that pairs of eigenvalues \((E_{e,2n}, E_{e,2n+1})\) and \((E_{o,2n+1}, E_{o,2n})\), \(n = 0, 1, \ldots\) coalesce at \(\pm a_{e,n}\) and \(\pm a_{o,n}\), respectively, when \(g = ia\). Tables 6 and 7 show the critical parameters for the even and odd solutions, respectively, to the Mathieu equation (22). They approximately follow a straight line of the form \(a_n = 0.582e_n + 3.66\). Once again we appreciate that both \(e_n\) and \(a_n\) increase asymptotically as \(n^2\).
7. Non-hermitian perturbed three-dimensional rigid rotor

An even more interesting example of rigid rotor is provided by

$$\hat{H} = \hat{L}^2 - g \cos(\theta),$$

(24)

where $\hat{L}^2$ is the square of the dimensionless quantum-mechanical angular-momentum operator. This Hamiltonian is invariant under the antiunitary transformation $\hat{A} = \hat{U} \hat{T}$ discussed above when $g$ is purely imaginary.

In order to apply the DM we resort to the set of eigenvectors $|l, m\rangle$ of $\hat{L}^2$ and $\hat{L}_z$:

$$\hat{L}^2 |l, m\rangle = l(l + 1) |l, m\rangle,$$

$$\hat{L}_z |l, m\rangle = m |l, m\rangle,$$

(25)

where $l = 0, 1, \ldots$ and $m = 0 \pm 1, \pm 2, \ldots, \pm l$ are the angular momentum and magnetic quantum numbers, respectively. Every eigenvector $|\psi\rangle$ of $\hat{H}$ can be expanded as

$$|\psi\rangle = \sum_{i=0}^{N} c_i |M + i, m\rangle,$$

(26)

where $M = |m|$ and the coefficients satisfy the recurrence relation [22] (and references therein)

$$A_i c_{i-1} + B_i c_i + A_{i+1} c_{i+1} = 0,$$

$$A_i = -g \left[ \frac{i(i + 2M)}{4(i + M)^2 - 1} \right]^{1/2}, \quad B_i = (i + M)(i + M + 1) - E.$$  

(27)

There is also a simple recurrence relation for the secular determinants [22] but we do not need it here because we can efficiently obtain $E(g)$ from the roots of $c_N = 0$ for sufficiently large $N$.

We denote $E_{M,n}$, $M,n = 0, 1, \ldots$ the eigenvalues of $\hat{H}$ so that $E_{M',n'} = E_{M,n}$ when $M + n = M' + n'$ and $g = 0$. The eigenvectors $|\psi_{m,n}\rangle$ with $m = \pm M$ are degenerate. In the coordinate representation the basis set of eigenvectors of $\hat{L}^2$ and $\hat{L}_z$ are the spherical harmonics $\langle \theta, \phi | l, m \rangle = Y^m_l(\theta, \phi)$ that satisfy $\hat{A} |l, m\rangle = (-1)^l |l, -m\rangle$.

Besides, it follows from the recurrence relation (27) that $c_{i,M,n}$ is either real or imaginary when $i$ is even or odd, respectively. Therefore, $c^*_{i,M,n} (-1)^{M+i} = (-1)^M c_{i,M,n}$ and $\hat{A} |\psi_{m,n}\rangle = (-1)^M |\psi_{-m,n}\rangle$. We clearly see that in this case $\hat{A} |\psi_{m,n}\rangle \neq \lambda |\psi_{m,n}\rangle$ but
the eigenvalue $E_{M,n}$ is real because $\hat{H}\hat{A}|\psi_{m,n}\rangle = E_{M,n}\hat{A}|\psi_{m,n}\rangle$ in agreement with the more general condition for real eigenvalues developed in section 2.

Fig. 3 shows the eigenvalues $E_{M,n}$ for $M = 0, 1, 2, 3$ and $n = 0, 1, 2$. It suggests that pairs of eigenvalues $(E_{M,2n}, E_{M,2n+1})$ coalesce at $a = \pm a_{M,n}$ when $g = ia$. Tables 8, 9, 10 and 11 show several critical parameters for $M = 0, 1, 2, 3$, respectively. In this case we also find a linear relationship $a_{M,n} = b + ce_{M,n}$ between the critical parameters, where $c \approx 1.18$, and that they increase asymptotically as $n^2$.

8. Conclusions

It appears to be clear from the results obtained throughout this paper that the DM is a remarkably simple and efficient tool for the calculation of eigenvalues and eigenvectors of a wide class of PT-symmetric models. In fact, the DM appears to converge more rapidly than more elaborate approaches and seems to be particularly useful for the calculation of critical parameters and exceptional points.

The condition for real eigenvalues developed in section 2 appears to be more general than the one invoked in earlier studies of the PT-symmetric Hamiltonians. This fact is plainly illustrated by the perturbed rigid rotator (24) for which the commonly used condition for unbroken symmetry (6) does not hold but the eigenvalues are real as long as the more general condition (5) applies.

Present numerical investigation suggests that both critical parameters for the three models (10), (20) and (24) behave asymptotically as $n^2$. We may be tempted to conjecture that this is a general property of such systems but that is not the case. The analysis of the exactly solvable models with piecewise constant potentials proposed by Znojil and collaborators [10] reveals a different behaviour. In the case of the potential $V(x) = iZx/|x|$, $-1 < x < 1$, with Dirichlet or periodic boundary conditions at $x = \pm 1$, the critical parameters $e_n$ and $Z_n$ appear to behave asymptotically as $n^2$ and $n$, respectively.
Acknowledgments

We thank Dr. M. Znojil for useful comments and suggestions that helped to improve this paper.

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Table 1. Convergence of the DM and OPPQ for $\hat{H} = \hat{p}^2 + i\hat{x}^3$

| $N$ | $E_0$ | $E_1$ | $E_2$ | $E_3$ |
|-----|-------|-------|-------|-------|
| 20  | 1.15638348063027 | 1.15720107946295 | 4.10944159217725 | 3.85785039690029 |
| 40  | 1.15626708286738 | 1.15626701076546 | 4.10922836311577 | 4.10917078909004 |
| 60  | 1.15626707198883 | 1.15626707203003 | 4.10922875272617 | 4.10922884747775 |
| 80  | 1.15626707198881 | 1.15626707198786 | 4.10922875280961 | 4.10922875282249 |
| 100 | 1.15626707198881 | 1.15626707198881 | 4.10922875280965 | 4.10922875280956 |

Figure 1. First four eigenvalues $E_n(g)$ for the model \((10)\) with $g = a$ (dashed line) and $g = ia$ (solid line)
Figure 2. First four eigenvalues $E_n(g)$ for the model (11) with $g = 2\alpha$ calculated by the diagonalization method (solid line) and perturbation theory (dashed line).

Figure 3. Eigenvalues $E_{M,n}(\alpha)$ of the rigid rotor (24).
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Table 2. Critical parameters for the oscillator (8)

| $n$ | $e_n$            | $a_n$            |
|-----|------------------|------------------|
| 0   | 1.28277353565056613093 | −2.61180935658887732269 |
| 1   | 4.18138810077014360384  | −5.37587963413369849339 |
| 2   | 7.47676353160394726567  | −7.81513358112177472963 |
| 3   | 11.03766256181169489101  | −10.07564704682307238859  |
| 4   | 14.80256612165608800708  | −12.21531517192134682450  |
| 5   | 18.73495127980953607811  | −14.26484986999511696653  |
| 6   | 22.81035758069971715940  | −16.24312145518034341186  |
| 7   | 27.01113795189653614640  | −18.16282077195707147474  |
| 8   | 31.32389750099022726315  | −20.03302515800852425790  |
| 9   | 35.73808590590511219720429  | −21.86052509995057604960  |
| 10  | 40.24515595690885890435  | −23.65057685885184848462  |
| 11  | 44.83802791988671972061  | −25.40736007787328669330  |
| 12  | 49.51073146222982552736  | −27.13427141318176849625  |
| 13  | 54.25815672832834563937  | −28.83412125570701326472  |
| 14  | 59.07587562778958582850  | −30.50927027533271429436  |
| 15  | 63.96001004047454636742  | −32.16172704057811933141  |
| 16  | 68.9071323790276889544  | −33.7932195834531855763  |
| 17  | 73.91418907962735035  | −35.405249007491647104  |
| 18  | 78.9784407229043496  | −36.9991304044614914  |
**Table 3.** Critical parameters for the oscillator (9)

| $n$ | $e_n$         | $a_n$         |
|-----|--------------|--------------|
| 0   | 3.17338956654721488704 | 3.1690361416747275234 |
| 1   | 11.32761640743725703756 | 7.62596008108023132512 |
| 2   | 21.47216949764589814716 | 12.11537100311929607154 |
| 3   | 33.02428793244591467473 | 16.61105709045349074831 |
| 4   | 45.70317143857586670043 | 21.10901685823201530899 |
| 5   | 59.33696104179223837682 | 25.60805225319570978500 |
| 6   | 73.80750220757362981500 | 30.10768065951909870293 |
| 7   | 89.0276216454863021248  | 34.6076704707909122127  |
| 8   | 104.9298551095159538    | 39.10789674411541992    |
| 9   | 121.460151413651        | 43.6082861575648        |
| 10  | 138.5740516383          | 48.10879285437          |
Critical parameters for non-hermitian Hamiltonians

Table 4. Critical parameters for the box model (10)

| $n$ | $e_n$                | $a_n$                |
|-----|----------------------|----------------------|
| 1   | 7.1085995967646      | 12.3124556722597     |
| 2   | 30.70746876678       | 53.18689607587       |
| 3   | 70.9578499846        | 122.902601369        |
| 4   | 127.862609648        | 221.464536296        |
| 5   | 201.42215453         | 348.87340541         |
| 6   | 291.6365885          | 505.1293887          |
| 7   | 398.5059476          | 690.2325483          |
| 8   | 522.030247           | 904.182911           |
| 9   | 662.209493           | 1146.98049           |
| 10  | 819.043691           | 1418.625286          |
| 11  | 992.532842           | 1719.11731           |
| 12  | 1182.67695           | 2048.4566            |
| 13  | 1389.476009          | 2406.643039          |
| 14  | 1612.93003           | 2793.67675           |
| 15  | 1853.0390            | 3209.5577            |
| 16  | 2109.80293           | 3654.28585           |
| 17  | 2383.22182           | 4127.8613            |
Table 5. Critical parameter $a_1$ from the perturbation series

| $k$ | $a_1(k)$         |
|-----|-----------------|
| 9   | 12.31814954     |
| 19  | 12.31354496     |
| 29  | 12.31290237     |
| 39  | 12.31269737     |
| 49  | 12.31260686     |
| 59  | 12.31255909     |
| 69  | 12.31253084     |
| 79  | 12.31251277     |
| 89  | 12.31250051     |
| 99  | 12.31249181     |
Table 6. Critical parameters for the even states of the Mathieu equation \(^{(22)}\)

| \(\epsilon_n\)                  | \(a_n\)                  |
|--------------------------------|--------------------------|
| 2.088 69890274969540742210705005 | 1.46876861378514199230729308986 |
| 27.319127640344351613697285995  | 16.4711658922636564062419622945 |
| 80.6582642367217733231182880374 | 47.8059657025975746007950854808 |
| 162.107021116501331382763597087 | 95.4752727072182593469528060868 |
| 271.665574614890515399359662310 | 159.479212669357057187230627715 |
| 409.333979844643194402422763806 | 239.817810495650789094138995905 |
| 575.112259376089614140747231520 | 336.491073930202402676797136666 |
| 769.000424132277697815886582932 | 449.499006061556590915787589874 |
| 990.9984800354536142042914292  | 578.841608335703329386650346074 |
| 1241.10643057248550485513720070 | 724.518881510280902738995966517 |
| 1519.3242779292387338728350008  | 886.530826016874701071963928710 |
| 1825.6520235451618732787374825  | 1064.87744211774801292171298635 |
| 2160.08966840670531841142295477 | 1259.55872998069003603099135522 |
| 2522.63721321244381656497120868 | 1470.57468971764989052881373681 |
| 2913.29465847091131070672526973 | 1697.92532140597167537370468979 |
| 3332.06200456108886585906171039 | 1941.61062510068817839368674171 |
| 3778.93925177117865219161016434 | 2201.63060084195557145589058358 |
| 4253.92640324228639948323784  | 2477.98524865972002792020985438 |
| 4757.02345039561656754162219537 | 2770.67456857674245292672357391 |
| 5288.23040212502649969662284873 | 3079.69856061061467026098084225 |
Table 7. Critical parameters for the odd states of the Mathieu equation \(^{(22)}\)

| \(\epsilon_n\) | \(a_n\) |
|-----------------|---------|
| 11.1904735991293865896020980123 | 6.92895475876018147964342787950 |
| 50.4750161557597516452005364504 | 30.0967728375875542000339071418 |
| 117.868924160843684783814608183 | 69.5987932768953947914148570394 |
| 213.372568637479279993815862834 | 125.435411314308272709560436718 |
| 336.986043950205287207567051913 | 197.606678692480922034682411560 |
| 488.709384475887730016940247407 | 286.112608761678078262070275163 |
| 668.5426056416296776276359437 | 390.9532062955967798894940683 |
| 876.485715432799125784653813063 | 512.128473373035028002129394632 |
| 1112.53871831587949363459807537 | 649.638411028231983524090563574 |
| 1376.70161704521717624727857705 | 803.48301982783526685397002241 |
| 1668.97441338489968248739617901 | 973.66230010589300632623663802 |
| 1989.35710852120412013890118346 | 1160.17625207096144146935385017 |
| 2337.84970328115567018520806164 | 1363.0248758594750618204515605 |
| 2714.4521982588903861462924172 | 1582.20817156403242791167234121 |
| 3119.16459389224590002123285708 | 1817.72613924973691186689894477 |
| 3551.98689051091222035608251051 | 2069.57877896343363081378199317 |
| 4012.91908836792421411986633224 | 2337.76609073971215299835857492 |
| 4501.96118766068778900529756079 | 2622.28807406462195641810433326 |
| 5019.11318854547174684532365214 | 2923.14473057813385462524614741 |
| 5564.37509114759075160843417379 | 3240.33605867580166867491717218 |
| 6137.74689556870672683751212500 | 3573.86205890991029411196172754 |
| 6739.22860189215699095301722791 | 3923.72273129028543611909139321 |
| 7368.82021018690440596500439191 | 4289.91807582487536028648347609 |
Table 8. Critical parameters $\epsilon_{0,n}$ and $a_{0,n}$ for the rigid rotor [24]

| $\epsilon_{0,n}$       | $a_{0,n}$                  |
|------------------------|----------------------------|
| 1.11850860747789604879129584124 | 1.89945169187324547365901350058 |
| 9.18271110777602614314692313478     | 11.446937313504414112902268409  |
| 24.2743374650550706797661994079    | 29.1570364187843312750194317104  |
| 46.3934021737006494552531157256    | 55.0338230301496191682912997241  |
| 75.5399201827232487162615410925    | 89.0777654885162317220462901731  |
| 111.713900380652159264985603687    | 131.288974351497399022703273189  |
| 154.915347674365981794080626693    | 181.667486575132082638246802870  |
| 205.144264889286269484450927321    | 240.213317402561182509835277416  |
| 262.400653742394747193838743959     | 306.926474080239226003553463095  |
| 326.684515329643806117627299256    | 381.806960425731499957841447183  |
| 397.995850380696676312538960541    | 464.854778613145237343458945801  |
| 476.334659399023202928517978051    | 556.069929958740540140414381845  |
| 561.700942742735329186201508058    | 655.452415299711583081531438128  |
| 654.094700673266377544747681781    | 763.002235190629698442777914673  |
| 753.51593385805509695023538647     | 878.719390011577165280746746226  |
| 859.96464102895999602185575728     | 1002.60388003069997387686020878  |
| 973.440823717829191452457184709    | 1134.65570544194626000993141955  |
| 1093.9448154292133694845436910     | 1274.8748663886694863623150054  |
| 1221.47561457673730003885489988    | 1423.26136297886278858803945222  |
| 1356.03422287637955815251177760    | 1579.81519529525700149353590512  |
| 1497.62030649039627961298586330    | 1744.53636340218999765413768412  |
| 1646.23386545750814120158073731    | 1917.4248673503704616935562365  |
| 1801.87489981019236211810598331    | 2098.4807071802518235307967219  |
| 1964.54340957565684861430612511    | 2287.70388292446187324632564614  |
| 2134.23939477686596016391844190    | 2485.09439460958034550662389867  |
| 2310.9628554333590341463022204    | 2690.65224225745820774568590806  |
| 2494.71379156175787158357548067    | 2904.37742588620969831042658859  |
| $e_{1,n}$       | $a_{1,n}$       |
|----------------|----------------|
| 4.55877886725924641484810680290 | 5.41369967947421154076411805664 |
| 16.1375907539446796948176665321 | 19.03665393666410365977084445332 |
| 34.7430624620380644472582121023 | 40.8287653735067375076502059995 |
| 60.3758571013293211953590679065 | 70.7886035789264845417420851348 |
| 93.0360844044641996314893055308 | 108.915912141697487819521883860 |
| 132.723772524583231646115964443 | 155.210615993212798290721507998 |
| 179.438930647130853022234048182 | 209.67268643303952841747656225 |
| 233.181562239157762078784983517 | 272.302110446390569333909532764 |
| 293.95166873105347695957684378 | 343.098881384146395386662172917 |
| 361.749250740754368699280246494 | 422.062995536098442607803489884 |
| 436.574308535710155676251436297 | 509.194450686347541286810153760 |
| 518.42684222412309140965417532 | 604.493245438036181898256235869 |
| 607.306851839808768157712103699 | 707.959378871064297125726282783 |
| 703.21433781672878552754182737 | 819.592850356874342990995371324 |
| 806.149298832917264169042646291 | 939.393659452681323113486417084 |
| 916.111736170419252202468288054 | 1067.36180583827770608337708586 |
| 1033.10164936940069761597136643 | 1203.49728927679401104146044559 |
| 1157.11903840568460024469637942 | 1347.80010958951204915746935249 |
| 1288.16390325671987308020948612 | 1500.27026663922414661704954168 |
| 1426.23624390197230137796623183 | 1660.90776031895581341708604925 |
| 1571.3360632299333370614104097 | 1829.7125905441926566147073681 |
| 1723.4633525033125098536155945 | 2006.68475724713587692441388188 |
| 1882.61812042837198303912549092 | 2191.82426037315622876377594946 |
| 2048.80036408515554263447292272 | 2385.13109987744749454382949289 |
| 2222.01008346219172139310274467 | 2586.60527572308025767686172501 |
| 2402.24727854928656430083016851 | 2796.24678787933020022750968876 |
| 2589.5119493378457410429640333 | 3014.0556363204372517182298588 |
| $e_{2,n}$     | $a_{2,n}$     |
|--------------|--------------|
| 10.3208166747903646568973037932 | 10.428855015906556880861532857 |
| 25.4207623327887544466386812765 | 28.15827403903332761901432188657 |
| 47.5418883505162195939968735032 | 54.0402932178718093860340915287 |
| 76.689141324321657490533380555 | 88.0863457427007731538106071227 |
| 112.863428447014039279577496571 | 130.29859327871353720215414528 |
| 156.065011915265846305454988719 | 180.67768292235052591381170392 |
| 206.29398803933216027102550914 | 239.223862208215672094273506007 |
| 263.550397974137409898781499507 | 305.93724193146725274206268463 |
| 327.834261238895647696546524794 | 380.817877474579788921392992095 |
| 399.145587817865853131006047948 | 463.865798896565719257861484418 |
| 477.484383113449595517060626653 | 555.081023588338983061868169447 |
| 562.850650174688796531422680455 | 654.463562138981667213774784976 |
| 655.244390776764559156247145443 | 762.013421271343462077625746903 |
| 754.665605975802226624019198276 | 877.730605405071181726830864340 |
| 861.114296408136089048224877344 | 1001.61511753266849630211224248 |
| 974.590462458325714246783438069 | 1133.66695973221802998481790522 |
| 1095.09410435669169203890272295 | 1273.88613347875288396645703396 |
| 1222.62522223754679315385270273 | 1422.27263983948977169496533497 |
| 1357.183816174788868488820532514 | 1578.82647959970741927897388563 |
| 1498.76988620408725276163261494 | 1743.547653345992907523155490638 |
| 1647.38343233693679094718818617 | 1916.43616152210473017073444202 |
| 1803.02445456967522723386079799 | 2097.49200446830527099050583774 |
| 1965.69295288932481713767155384 | 2286.71518244784416497102175928 |
| 2135.38892727740046871358920607 | 2484.10569566659182256233911612 |
| 2312.1123771239876092876125824 | 2689.66354428692976622447303147 |
| 2495.86330417142194483897972208 | 2903.38872843796560070724063761 |
| 2686.64170663122979954448053178 | 3125.28124822310130537164659655 |


| $e_{3,n}$          | $a_{3,n}$          |
|-------------------|-------------------|
| 18.39326169754919346793973788 | 16.8966533642743226378461806280 |
| 37.02612766486838864684719643139 | 38.7837061748124744181847563089 |
| 62.6669859858232821773304748336 | 68.774058689106779587798103556 |
| 95.3306215019736927494865693118 | 106.915119263007675368240263857 |
| 135.020028981874691707132974862 | 153.2171948543871052086541377321 |
| 181.736151824585196120205263944 | 207.683669335464793838124652103 |
| 235.479364255103716530898139084 | 270.3159290352207287532388027826 |
| 296.249838077915056335119725000 | 341.114629238351091770190392543 |
| 364.047660655340315855748376441 | 420.080112725581765701401257790 |
| 438.872879915491110916811406148 | 507.21257298754482348482273980 |
| 520.725523743932161740530113278 | 602.512125938872178124628573208 |
| 609.605609143244375059516152739 | 705.97884426605700033807188021 |
| 705.51314688384280562496171499 | 817.6127760733132832143973505231 |
| 808.448144007708646910693123011 | 937.413953057538062250373844424 |
| 918.410605243078958607010397392 | 1065.3823977445896328397480722 |
| 1035.40053383702430388114116904 | 1201.51821260599229426362249599 |
| 1159.41793206278032717306053303 | 1345.82114958635061001956413623 |
| 1290.46280153831843226127329649 | 1498.29147690625339213241310920 |
| 1428.53514343169852409564937760 | 1658.92911448450732063468893814 |
| 1573.63495859625721769067726115 | 1827.73406726311001705562366049 |
| 1725.76224766315409747995008471 | 2004.7063390716901625781758106 |
| 1884.91701110119625582112832348 | 2189.84593290656915762534485743 |
| 2051.09924926310976011661273016 | 2383.15285115035342904111463156 |
| 2224.30896241500183043304906063 | 2584.62709570470366379109882661 |
| 2404.54615075871058072339672777 | 2794.26866810660184529365303827 |
| 2591.81081444781812982947488569 | 3012.0775690765198814361352503 |
| 2786.10295359939116923987354036 | 3238.0538012349091542379786847 |

Table 11. Critical parameters $e_{3,n}$ and $a_{3,n}$ for the rigid rotor \footnote{24}