THE GREEN FUNCTION FOR THE STOKES SYSTEM WITH MEASURABLE COEFFICIENTS

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Abstract. We study the Green function for the stationary Stokes system with bounded measurable coefficients in a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \). We construct the Green function in \( \Omega \) under the condition (A1) that weak solutions of the system enjoy interior Hölder continuity. We also prove that (A1) holds, for example, when the coefficients are VMO. Moreover, we obtain the global pointwise estimate for the Green function under the additional assumption (A2) that weak solutions of Dirichlet problems are locally bounded up to the boundary of the domain. By proving a priori \( L^q \)-estimates for Stokes systems with BMO coefficients on a Reifenberg domain, we verify that (A2) is satisfied when the coefficients are VMO and \( \Omega \) is a bounded \( C^1 \) domain.

1. Introduction. We consider the Dirichlet boundary value problem for the stationary Stokes system

\[
\begin{aligned}
\mathcal{L}u + Dp &= f + D_{\alpha}f_\alpha & \text{in } \Omega, \\
\text{div } u &= g & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\tag{1.1}
\]

where \( \Omega \) is a domain in \( \mathbb{R}^n \). Here, \( \mathcal{L} \) is an elliptic operator of the form

\[
\mathcal{L}u = -D_\alpha(A_{\alpha\beta}D_\beta u),
\]

where the coefficients \( A_{\alpha\beta} = A_{\alpha\beta}(x) \) are \( n \times n \) matrix valued functions on \( \mathbb{R}^n \) with entries \( a_{ij}^{\alpha\beta} \) that satisfying the strong ellipticity condition; i.e., there is a constant \( \lambda \in (0, 1] \) such that for any \( x \in \mathbb{R}^n \) and \( \xi, \eta \in \mathbb{R}^{n \times n} \), we have

\[
\lambda |\xi|^2 \leq a_{ij}^{\alpha\beta}(x)\xi_j^\alpha \xi_i^\beta, \quad |a_{ij}^{\alpha\beta}(x)\xi_j^\alpha \eta_i^\beta| \leq \lambda^{-1} |\xi||\eta|.
\tag{1.2}
\]
We do not assume that the coefficients $A_{\alpha\beta}$ are symmetric. The adjoint operator $L^*$ of $L$ is given by

$$L^*u = -D_\alpha(A_{\beta\alpha}(x)^T D_\beta u).$$

We remark that the coefficients of $L^*$ also satisfy (1.2) with the same constant $\lambda$. There has been some interest in studying boundary value problems for Stokes systems with bounded coefficients; see, for instance, Giaquinta-Modica [14]. They obtained various interior and boundary estimates for both linear and nonlinear systems of the type of the stationary Navier-Stokes system.

Our first focus is to study of the Green function for the Stokes system with $L^\infty$ coefficients in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. More precisely, we consider a pair $(G(x,y), \Pi(x,y))$, where $G(x,y)$ is an $n \times n$ matrix valued function and $\Pi(x,y)$ is an $n \times 1$ vector valued function on $\Omega \times \Omega$, satisfying

$$\begin{cases}
L_xG(x,y) + D_x\Pi(x,y) = \delta_y(x)I \text{ in } \Omega, \\
\text{div}_xG(x,y) = 0 \text{ in } \Omega, \\
G(x,y) = 0 \text{ on } \partial\Omega.
\end{cases}$$

Here, $\delta_y(\cdot)$ is Dirac delta function concentrated at $y$ and $I$ is the $n \times n$ identity matrix. See Definition 2.1 for the precise definition of the Green function. We prove that if weak solutions of either

$$L^*u + Dp = 0, \quad \text{div } u = 0 \quad \text{in } B_R$$

or

$$L^*u + Dp = 0, \quad \text{div } u = 0 \quad \text{in } B_R$$

satisfy the following De Giorgi-Moser-Nash type estimate

$$[u]_{C^\mu(B_R/2)} \leq CR^{-n/2-\mu}\|u\|_{L^2(B_R)}, \quad (1.3)$$

then the Green function $(G(x,y), \Pi(x,y))$ exists and satisfies a natural growth estimate near the pole; see Theorem 2.3. It can be shown, for example, that if the coefficients of $L$ belong to the class of VMO (vanishing mean oscillations), then the interior Hölder estimate (1.3) above holds; see Theorem 2.5. Also, we are interested in the following global pointwise estimate for the Green function: there exists a positive constant $C$ such that

$$|G(x,y)| \leq C|x-y|^{2-n}, \quad \forall x, y \in \Omega, \quad x \neq y. \quad (1.4)$$

If we assume further that the operator $L$ has the property that the weak solution of

$$\begin{cases}
L^*u + Dp = f \quad \text{in } \Omega, \\
\text{div } u = g \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{cases}$$

is locally bounded up to the boundary, then we obtain the pointwise estimate (1.4) of the Green function. This local boundedness condition $A2$ is satisfied when the coefficients of $L$ belong to the class of VMO and $\Omega$ is a bounded $C^1$ domain. To see this, we employ the standard localization method and the global $L^q$-estimate for the Stokes system with Dirichlet boundary condition, which is our second focus in this paper.

Green functions for the linear equation and system have been studied by many authors. In [21], Littman-Stampacchia-Weinberger obtained the pointwise estimate of the Green function for elliptic equation. Grüter-Widman [15] proved existence and uniqueness of the Green function for elliptic equation, and the corresponding
results for elliptic systems with continuous coefficients were obtained in [6, 10]. Hofmann-Kim proved the existence of Green functions for elliptic systems with variable coefficients on any open domain. Their methods are general enough to allow the coefficients to be VMO. For more details, see [16]. We also refer the reader to [18, 23] and references therein for the study of Green functions for elliptic systems. Regarding the study of the Green function for the Stokes system with the Laplace operator, we refer the reader to [4, 18]. In those papers, the authors obtained the global pointwise estimate (1.4) for the Green function on a three dimensional Lipschitz domain. Mitrea-Mitrea [23] established regularity properties of the Green function for the Stokes system with Dirichlet boundary condition in a two or three dimensional Lipschitz domain. Recent progress may be found in the article of Ott-Kim-Brown [25]. This work includes a construction of the Green function with mixed boundary value problem for the Stokes system in two dimensions.

Our second focus in this paper is the global $L^q$-estimates for the Stokes systems of divergence form with the Dirichlet boundary condition. As mentioned earlier, the $L^q$-estimate for the Stokes system is the key ingredient in establishing the global pointwise estimate for the Green function. Moreover, the study of the regularity of solutions to the Stokes system plays an essential role in the mathematical theory of viscous fluid flows governed by the Navier-Stokes system. For this reason, the $L^q$-estimate for the Stokes system with the Laplace operator was discussed in many papers. We refer the reader to Galdi-Simader-Sohr [11], Maz’ya-Rossmann [22], and references therein. Recently, estimates in Besov spaces for the Stokes system are obtained by Mitrea-Wright [24]. In this paper, we consider the $L^q$-estimates for Stokes systems with variable coefficients in non-smooth domains. More precisely, we prove that if the coefficients of $L$ have small bounded mean oscillations on a Reifenberg flat domain $\Omega$, then the solution $(u, p)$ of the problem (1.1) satisfies the following $L^q$-estimate:

$$\|p\|_{L^q(\Omega)} + \|Du\|_{L^q(\Omega)} \leq C(\|f\|_{L^q(\Omega)} + \|f_\alpha\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}).$$

Moreover, we obtain the solvability in Sobolev space for the systems on a bounded Lipschitz domain. It has been studied by many authors that the $L^q$-estimates for elliptic and parabolic systems with variable coefficients on a Reifenberg flat domain. We refer the reader to Dong-Kim [8, 9] and Byun-Wang [3]. In particular, in [8], the authors proved $L^q$-estimates for divergence form higher order systems with partially BMO coefficients on a Reifenberg flat domain. Their argument is based on mean oscillation estimates and $L^\infty$-estimates combined with the measure theory on the “crawling of ink spots” which can be found in [20]. We mainly follow the arguments in [8], but the technical details are different due to the pressure term $p$. The presence of the pressure term $p$ makes the argument more involved.

The organization of the paper is as follows. In Section 2, we introduce some notation and state our main theorems, including the existence and global pointwise estimates for Green functions, and their proofs are presented in Section 4. Section 5 is devoted to the study of the $L^q$-estimate for the Stokes system with the Dirichlet boundary condition. In Appendix, we provide some technical lemmas.

2. Main results. Before we state our main theorems, we introduce some necessary notation. Throughout the article, we use $\Omega$ to denote a bounded domain in $\mathbb{R}^n$, where $n \geq 2$. For any $x = (x_1, \ldots, x_n) \in \Omega$ and $r > 0$, we write $\Omega_r(x) = \Omega \cap B_r(x)$, where
where \( B_r(x) \) is the usual Euclidean ball of radius \( r \) centered at \( x \). We also denote
\[
B_r^+(x) = \{ y = (y_1, \ldots, y_n) \in B_r(x) : y_1 > x_1 \}.
\]
We define \( d_x = \text{dist}(x, \partial \Omega) = \inf \{|x - y| : y \in \partial \Omega\} \). For a function \( f \) on \( \Omega \), we denote the average of \( f \) in \( \Omega \) to be
\[
(f)_{\Omega} = \int_\Omega f \, dx.
\]
We use the notation
\[
\text{sgn} z = \begin{cases} 
  z/|z| & \text{if } z \neq 0, \\
  0 & \text{if } z = 0.
\end{cases}
\]
For \( 1 \leq q \leq \infty \), we define the space \( L^q_0(\Omega) \) as the family of all functions \( u \in L^q(\Omega) \) satisfying \( (u)_{\Omega} = 0 \). We denote by \( W^{1,q}(\Omega) \) the usual Sobolev space and \( W^{1,q}_0(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in \( W^{1,q}(\Omega) \). Let \( f, f_\alpha \in L^2(\Omega)^n \) and \( g \in L^2_0(\Omega) \). We say that \( (u, p) \in W^{1,q}_0(\Omega)^n \times L^2_0(\Omega) \) is a weak solution of the problem
\[
\begin{cases}
\mathcal{L} u + Dp = f + D_\alpha f_\alpha & \text{in } \Omega, \\
\text{div } u = g & \text{in } \Omega,
\end{cases}
\]
if we have
\[
(2.1)
\]
and
\[
\int_\Omega A_{\alpha\beta} D_\beta u \cdot D_\alpha \varphi \, dx - \int_\Omega p \text{div } \varphi \, dx = \int_\Omega f \cdot \varphi \, dx - \int_\Omega f_\alpha \cdot D_\alpha \varphi \, dx
\]
for any \( \varphi \in C_0^\infty(\Omega)^n \). Similarly, we say that \( (u, p) \in W^{1,q}_0(\Omega)^n \times L^2_0(\Omega) \) is a weak solution of the problem
\[
\begin{cases}
\mathcal{L}^* u + Dp = f + D_\alpha f_\alpha & \text{in } \Omega, \\
\text{div } u = g & \text{in } \Omega,
\end{cases}
\]
(2.2)
if we have (2.1) and
\[
\int_\Omega A_{\alpha\beta} D_\beta \varphi \cdot D_\alpha u \, dx - \int_\Omega p \text{div } \varphi \, dx = \int_\Omega f \cdot \varphi \, dx - \int_\Omega f_\alpha \cdot D_\alpha \varphi \, dx
\]
for any \( \varphi \in C_0^\infty(\Omega)^n \).

**Definition 2.1** (Green function). Let \( G(x, y) \) be an \( n \times n \) matrix valued function and \( \Pi(x, y) \) be an \( n \times 1 \) vector valued function on \( \Omega \times \Omega \). We say that a pair \( (G(x, y), \Pi(x, y)) \) is a Green function for the Stokes system (SP) if it satisfies the following properties:

a) \( G(\cdot, y) \in W^{1,1}_0(\Omega)^{n \times n} \) and \( G(\cdot, y) \in W^{1,2}(\Omega \setminus B_R(y))^{n \times n} \) for all \( y \in \Omega \) and \( R > 0 \). Moreover, \( \Pi(\cdot, y) \in L^1_0(\Omega)^n \) for all \( y \in \Omega \).

b) For any \( y \in \Omega \), \( (G(\cdot, y), \Pi(\cdot, y)) \) satisfies
\[
\text{div } G(\cdot, y) = 0 \quad \text{in } \Omega
\]
and
\[
\mathcal{L} G(\cdot, y) + D\Pi(\cdot, y) = \delta_y I \quad \text{in } \Omega
\]
in the sense that for any \( 1 \leq k \leq n \) and \( \varphi \in C_0^\infty(\Omega)^n \), we have
\[
\int_\Omega a_{\alpha\beta} D_\beta G^{jk}(x, y) D_\alpha \varphi(x) \, dx - \int_\Omega \Pi^k(x, y) \text{div } \varphi(x) \, dx = \varphi^k(y).
\]
c) If \((u, p) \in W^{1,2}_0(\Omega)^n \times L^2_0(\Omega)\) is the weak solution of \((\text{SP}^*)\) with \(f, f_\alpha \in L^\infty(\Omega)^n\) and \(g \in L^2_0(\Omega\), then we have
\[
 u(x) = \int_{\Omega} G(y, x)^{\text{tr}} f(y) \, dy - \int_{\Omega} D_\alpha G(y, x)^{\text{tr}} f_\alpha(y) \, dy - \int_{\Omega} \Pi(x, y)g(y) \, dy.
\]

Remark 2.2. The \(L^2\)-solvability of the Stokes system with the Dirichlet boundary condition (see Section 3.1) and the part c) of the above definition give the uniqueness of a Green function. Indeed, if \((G(x, y), \Pi(x, y))\) is another Green function for \((\text{SP})\), then by the uniqueness of the solution, we have
\[
 \int_{\Omega} G(y, x)^{\text{tr}} f(y) \, dy - \int_{\Omega} \Pi(x, y)g(y) \, dy = \int_{\Omega} \tilde{G}(y, x)^{\text{tr}} f(y) \, dy - \int_{\Omega} \tilde{\Pi}(x, y)g(y) \, dy
\]
for any \(f \in C_0^\infty(\Omega)^n\) and \(g \in C_0^\infty(\Omega)\). Therefore, we conclude that \((G, \Pi) = (\tilde{G}, \tilde{\Pi})\) a.e. in \(\Omega \times \Omega\).

2.1. Existence of the Green function. To construct the Green function, we impose the following conditions.

(A0). There exist positive constants \(R_1\) and \(K_1\) such that the following holds: for any \(x_0 \in \partial \Omega\) and \(0 < r \leq R_1\), there is a coordinate system depending on \(x_0\) and \(r\) such that in the new coordinate system, we have
\[
 \Omega_r(x_0) = \{x \in B_r(x_0) : x_1 > \psi(x')\},
\]
where \(\psi : \mathbb{R}^{n-1} \to \mathbb{R}\) is a Lipschitz function with \(\text{Lip}(\psi) \leq K_1\).

(A1). There exist constants \(\mu \in (0, 1]\) and \(A_1 > 0\) such that the following holds: if \((u, p) \in W^{1,2}(B_R(x_0))^n \times L^2(B_R(x_0))\) satisfies
\[
 \begin{cases}
 \mathcal{L} u + Dp = 0 & \text{in } B_R(x_0), \\
 \text{div } u = 0 & \text{in } B_R(x_0),
\end{cases}
\]
where \(x_0 \in \Omega\) and \(R \in (0, d_{x_0}]\), then we have
\[
 [u]_{C^\mu(B_{R/2}(x_0))} \leq A_1 R^{-\mu} \left( \int_{B_{R}(x_0)} |u|^2 \, dx \right)^{1/2},
\]
where \([u]_{C^\mu(B_{R/2}(x_0))}\) denotes the usual Hölder seminorm. The statement is valid, provided that \(\mathcal{L}\) is replaced by \(\mathcal{L}^*\).

Theorem 2.3. Let \(\Omega\) be a domain in \(\mathbb{R}^n\) with \(\text{diam}(\Omega) \leq K_0\), where \(n \geq 3\). Assume conditions (A0) and (A1). Then there exist Green functions \((G(x, y), \Pi(x, y))\) and \((G^*(x, y), \Pi^*(x, y))\) for \((\text{SP})\) and \((\text{SP}^*)\), respectively, such that the following identity:
\[
 G(x, y) = G^*(y, x)^{\text{tr}}, \quad \forall x, y \in \Omega, \quad x \neq y.
\]
Also, for any \(x, y \in \Omega\) satisfying \(0 < |x - y| < d_y/2\), we have
\[
 |G(x, y)| \leq C|x - y|^{2^{-n}}.
\]
Moreover, for any \(y \in \Omega\) and \(R \in (0, d_y]\), we obtain
i) \(\|G(\cdot, y)\|_{L^2(\Omega \setminus B_{R}(y))} + \|D_{\alpha}G(\cdot, y)\|_{L^2(\Omega \setminus B_{R}(y))} \leq CR^{2-n}/2\).
ii) \(|\{x \in \Omega : |G(x, y)| > t\}| \leq Ct^{-n/(n-2)}\) for all \(t > d_y^{1-n}\).
iii) \(|\{x \in \Omega : |D_{\alpha}G(x, y)| > t\}| \leq Ct^{-n/(n-1)}\) for all \(t > d_y^{1-n}\).
iv) \(\|G(\cdot, y)\|_{L^{\infty}(B_{R}(y))} \leq C_q R^{2-n+2/q}\), where \(q \in [1, n/(n-2)]\).
v) \(\|D_{\alpha}G(\cdot, y)\|_{L^1(B_{R}(y))} \leq C_q R^{1-n+2/q}\), where \(q \in [1, n/(n-1)]\).
In the above, \( C = C(n, \lambda, K_0, K_1, R_1, \mu, A_1), \) \( C_q = C_q(n, \lambda, K_0, K_1, R_1, \mu, A_1, q), \) and \( C_{y,q} = C_{y,q}(n, \lambda, K_0, K_1, R_1, \mu, A_1, q, d_q). \) The same estimates are also valid for \( (G^*(x,y), \Pi^*(x,y)). \)

**Remark 2.4.** Let \( (u, p) \in W^{1,2}_0(\Omega)^n \times L^2_0(\Omega) \) be the weak solution of the problem
\[
\begin{aligned}
\mathcal{L} u + Dp &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega.
\end{aligned}
\]
Then by the property c) of Definition 2.1 and the identity (2.5), we have the following representation for \( u: \)
\[
u(x) := \int_\Omega G(x,y)f(y)\,dy - \int_\Omega D_\alpha G(x,y)f_\alpha(y)\,dy.
\]
Also, the following estimates are easy consequences of the identity (2.5) and the estimates i) – v) in Theorem 2.3 for \( G^*(\cdot, x): \)
\[
a) \|G(x, \cdot)\|_{L^{2n/(n-2)}(\Omega \setminus B_R(x))} + \|DG(x, \cdot)\|_{L^2(\Omega \setminus B_R(x))} \leq CR^{2/(n-2)},
\]
\[
b) \|G(x, \cdot)\|_{L^2(\Omega \setminus B_R(x))} \leq Ct^{n/2} \quad \text{for all } t > d^2_x - R.
\]
\[
c) \|D_s G(x, \cdot)\|_{L^2(\Omega \setminus B_R(x))} \leq Ct^{n/2} \quad \text{for all } t > d^2_x - R.
\]
\[
d) \|DG(x, \cdot)\|_{L^2(\Omega \setminus B_R(x))} \leq C_R^{2-n/2} \quad \text{for all } t > d^2_x - R.
\]

In the theorem and the remark below, we show that if the coefficients have a vanishing mean oscillation (VMO), then the condition (A1) holds.

**Theorem 2.5.** Suppose that the coefficients of \( \mathcal{L} \) belong to the class of VMO; i.e. we have
\[
\lim_{\rho \to 0} \omega_\rho(A_{\alpha\beta}) := \lim_{\rho \to 0} \sup_{x \in \mathbb{R}^n, s \leq \rho} \int_{B_s(x)} |A_{\alpha\beta} - (A_{\alpha\beta})_{B_s(x)}| = 0.
\]
If \( (u, p) \in W^{1,2}(B_R(x_0))^n \times L^2(B_R(x_0)) \) satisfies (2.3) with \( x_0 \in \Omega \) and \( 0 < R \leq \min\{d_{x_0}, 1\}, \) then for any \( \mu \in (0, 1), \) the estimate (2.4) holds with the constant \( A_1 \) depending only on \( n, \lambda, \mu, \) and the VMO modulus of the coefficients.

**Remark 2.6.** In the above theorem, the constant \( \min\{d_{x_0}, 1\} \) is interchangeable with \( \min\{d_{x_0}, c\} \) for any fixed \( c \in (0, \infty), \) possibly at the cost of increasing the constant \( A_1. \) Setting \( c = \text{diam } \Omega, \) the condition (A1) holds with the constant \( A_1 \) depending on \( n, \lambda, \text{diam } \Omega, \mu, \) and the VMO modulus \( \omega_\rho \) of coefficients.

The following corollary is immediate consequence of Theorem 2.3 and Remark 2.6.

**Corollary 2.7.** Let \( \Omega \) be a Lipschitz domain in \( \mathbb{R}^n, \) where \( n \geq 3. \) Suppose the coefficients of \( \mathcal{L} \) belong to the class of VMO. Then there exists the Green function for (SP) and it satisfies the assertions in Theorem 2.3.

**2.2. Global estimate of the Green function.** We impose the following assumption to obtain the global pointwise estimate for the Green function.

(A2). There exists a constant \( A_2 > 0 \) such that if \( (u, p) \in W^{1,2}_0(\Omega)^n \times L^2_0(\Omega) \) satisfies
\[
\begin{aligned}
\mathcal{L} u + Dp &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{aligned}
\]
where \( f \in L^\infty(\Omega)^n \), then \( u \in L^\infty(\Omega)^n \) with the estimate
\[
\|u\|_{L^\infty(\Omega_{R/2}(x_0))} \leq A_2 \left( R^{-n/2} \|u\|_{L^2(\Omega_R(x_0))} + R^2 \|f\|_{L^\infty(\Omega_R(x_0))}\right)
\]
for any \( x_0 \in \Omega \) and \( 0 < R < \text{diam} \Omega \). The statement is valid, provided that \( L \) is replaced by \( L^n \).

**Theorem 2.8.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with \( \text{diam}(\Omega) \leq K_0 \), where \( n \geq 3 \). Assume conditions (A0), (A1), and (A2). Let \((G(x,y), \Pi(x,y))\) be the Green function for (SP) in \( \Omega \) as constructed in Theorem 2.3. Then we have the global pointwise estimate for \( G(x,y) \):
\[
|G(x,y)| \leq C|x-y|^{2-n}, \quad \forall x, y \in \Omega, \quad x \neq y,
\]
where \( C = C(n, \lambda, K_0, K_1, R_1, A_2) \).

From the global \( L^q \)-estimates for the Stokes systems in Section 5, we obtain an example of the condition (A2) in the theorem below. The proof of the theorem follows a standard localization argument; see Section 4.4 for the details. Similar results for elliptic systems are given for the Dirichlet problem in [18] and for the Neumann problem in [5].

**Theorem 2.9.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with \( \text{diam}(\Omega) \leq K_0 \), where \( n \geq 3 \). Assume the condition (A0) with a sufficiently small \( K_1 \), depending only on \( n \) and \( \lambda \). If the coefficients of \( L \) belong to the class of VMO, then the condition (A2) holds with the constant \( A_2 \) depending only on \( n \), \( \lambda \), \( K_0 \), \( R_1 \), and the VMO modulus of the coefficients.

By combining Theorems 2.8 and 2.9, we immediately obtain the following result.

**Corollary 2.10.** Let \( \Omega \) be a bounded \( C^1 \) domain in \( \mathbb{R}^n \), where \( n \geq 3 \). Suppose that the coefficients of \( L \) belong to the class of VMO. Then there exists the Green function for (SP) and it satisfies the global pointwise estimate (2.7).

3. Some auxiliary results.

3.1. \( L^2 \)-solvability. In this subsection, we consider the existence theorem for weak solutions of the Stokes system with measurable coefficients. For the solvability of the Stokes system, we impose the following condition.

(D) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), where \( n \geq 2 \). There exist a linear operator \( B : L^2_0(\Omega) \to W^{1,2}_0(\Omega)^n \) and a constant \( A > 0 \) such that
\[
\text{div} \, Bg = g \quad \text{in} \quad \Omega \quad \text{and} \quad \|Bg\|_{W^{1,2}_0(\Omega)} \leq A\|g\|_{L^2(\Omega)}.
\]

**Remark 3.1.** It is well known that if \( \Omega \) is a Lipschitz domain with \( \text{diam}(\Omega) \leq K_0 \), which satisfies the condition (A0), then for any \( 1 < q < \infty \), there exists a bounded linear operator \( B_q : L^q_0(\Omega) \to W^{1,q}_0(\Omega)^n \) such that
\[
\text{div} \, B_q g = g \quad \text{in} \quad \Omega, \quad \|D(B_q g)\|_{L^q(\Omega)} \leq C\|g\|_{L^q(\Omega)},
\]
where the constant \( C \) depends only on \( n, q, K_0, K_1, \) and \( R_1 \); see e.g., [1]. We point out that if \( \Omega = B_R(x) \) or \( \Omega = B_R^+(x) \), then
\[
\|D(B_q g)\|_{L^q(\Omega)} \leq C\|g\|_{L^q(\Omega)},
\]
where \( C = C(n, q) \).
Lemma 3.2. Assume the condition (D). Let

\[ q = \frac{2n}{n+2} \quad \text{if } n \geq 3 \quad \text{and} \quad q = 2 \quad \text{if } n = 2. \]

For \( f \in L^q(\Omega)^n \), \( f_\alpha \in L^2(\Omega)^n \), and \( g \in L^2_0(\Omega) \), there exists a unique solution \((u,p) \in W^{1,2}_0(\Omega)^n \times L^2_0(\Omega)\) of the problem

\[
\begin{aligned}
\mathcal{L}u + Dp &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\div u &= g \quad \text{in } \Omega.
\end{aligned}
\]

(3.2)

Moreover, we have

\[ \|p\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)} \leq C \left( \|f\|_{L^q(\Omega)} + \|f_\alpha\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \right), \tag{3.3} \]

where \( C = C(n,\lambda,A) \) if \( n \geq 3 \) and \( C = C(\lambda,A,|\Omega|) \) if \( n = 2 \). In the case when \( \Omega = B_R(x) \) or \( \Omega = B_R^+(x) \), if \( f \in L^2(\Omega)^n \), then we have

\[ \|p\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)} \leq C' \left( R\|f\|_{L^2(\Omega)} + \|f_\alpha\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \right), \tag{3.4} \]

where \( C' = C'(n,\lambda) \).

Proof. We mainly follow the argument given by Maz'ya-Rossmann [22, Theorem 5.2]. Also see [25, Theorem 3.1]. Let \( H(\Omega) \) be the Hilbert space consisting of functions \( u \in W^{1,2}_0(\Omega)^n \) such that \( \div u = 0 \) and \( H^+(\Omega) \) be orthogonal complement of \( H(\Omega) \) in \( W^{1,2}_0(\Omega)^n \). We also define \( P \) as the orthogonal projection from \( W^{1,2}_0(\Omega)^n \) onto \( H^+(\Omega) \). Then, one can easily show that the operator \( B = P \circ B : L^2_0(\Omega) \rightarrow H^+(\Omega) \) is bijective. Moreover, we obtain for \( g \in L^2_0(\Omega) \) that

\[ \div Bg = g \quad \text{in } \Omega, \quad \|Bg\|_{W^{1,2}_0(\Omega)} \leq A\|g\|_{L^2(\Omega)}. \tag{3.5} \]

Now, let \( f, f_\alpha \in L^2(\Omega)^n \) and \( g \in L^2_0(\Omega) \). Then from the above argument, there exists a unique function \( w := Bg \in H^+(\Omega) \) such that \((3.5)\) is satisfied. Also, by the Lax-Milgram theorem, one can find the function \( v \in H(\Omega) \) that satisfies

\[
\int_\Omega A_{\alpha\beta}D_\beta \varphi \cdot D_\alpha v \, dx = \int_\Omega f \cdot \varphi \, dx - \int_\Omega f_\alpha \cdot D_\alpha \varphi \, dx - \int_\Omega A_{\alpha\beta}D_\beta w \cdot D_\alpha \varphi \, dx
\]

for all \( \varphi \in H(\Omega) \). By setting \( \varphi = v \) in the above identity, and then, using Hölder’s inequality and the Sobolev inequality, we have

\[ \|Dv\|_{L^2(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|f_\alpha\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \right), \]

where \( q = 2 \) if \( n = 2 \) and \( q = 2n/(n+2) \) if \( n \geq 3 \). Therefore, the function \( u = v + w \) satisfies \( \div u = g \) in \( \Omega \) and the following identity:

\[
\int_\Omega A_{\alpha\beta}D_\beta u \cdot D_\alpha \varphi \, dx = \int_\Omega f \cdot \varphi \, dx - \int_\Omega f_\alpha \cdot D_\alpha \varphi \, dx, \quad \forall \varphi \in H(\Omega). \tag{3.6}
\]

Moreover, we have

\[ \|Du\|_{L^2(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|f_\alpha\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \right). \tag{3.7} \]

To find \( p \), we let

\[
\ell(\phi) = \int_\Omega A_{\alpha\beta}D_\beta u \cdot D_\alpha (B\tilde{\phi}) \, dx - \int_\Omega f \cdot B\tilde{\phi} \, dx + \int_\Omega f_\alpha \cdot D_\alpha (B\tilde{\phi}) \, dx,
\]

where \( \phi \in L^2(\Omega) \) and \( \tilde{\phi} = \phi - (\phi)_{\Omega} \in L^2_0(\Omega) \). Since

\[ \|B\tilde{\phi}\|_{W^{1,2}_0(\Omega)} \leq A\|\tilde{\phi}\|_{L^2(\Omega)} \leq C(n,A)\|\phi\|_{L^2(\Omega)}, \]

\( \ell \) is a bounded linear functional on \( L^2(\Omega) \). Therefore, there exists a function \( p_0 \in L^2(\Omega) \) so that
\[
\int_\Omega p_0 \hat{\phi} \, dx = \ell(\hat{\phi}), \quad \forall \hat{\phi} \in L^2_0(\Omega),
\]
and thus, \( p = p_0 - (p_0)_{\Omega} \in L^2_0(\Omega) \) also satisfies the above identity. Then by using the fact that \( B(L^2_0(\Omega)) = H^1(\Omega) \), we obtain
\[
\int_{\Omega} A_{\alpha\beta} D_\beta u \cdot D_\alpha \varphi \, dx - \int_{\Omega} p \, \text{div} \varphi \, dx = \int_{\Omega} f \cdot \varphi \, dx - \int_{\Omega} f_\alpha \cdot D_\alpha \varphi \, dx \tag{3.8}
\]
for all \( \varphi \in H^1(\Omega) \). From (3.6) and (3.8), we find that \((u, p)\) is the weak solution of the problem (3.2). Moreover, by setting \( \varphi = Bp \) in (3.8), we have
\[
\|p\|_{L^2(\Omega)} \leq C \left( \|Du\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|f_\alpha\|_{L^2(\Omega)} \right),
\]
and thus, we get (3.3) from (3.7).

To establish (3.4), we observe that
\[
\|u\|_{L^2(\Omega)} \leq C(n)R \|Du\|_{L^2(\Omega)}, \quad \forall u \in W^{1,2}_0(\Omega),
\]
provided that \( \Omega = B_R(x) \) or \( \Omega = B_R^n(x) \). By using the above inequality and (3.1), and following the same argument as above, one can easily show that the estimate (3.4) holds. The lemma is proved. \( \Box \)

3.2. Interior estimates. In this subsection we derive some interior estimates of \( u \) and \( p \). We start with the following Caccioppoli type inequality that can be found, for instance, in [7, 14].

**Lemma 3.3.** Assume that \((u, p) \in W^{1,2}(B_R(x_0))^n \times L^2(B_R(x_0))\) satisfies
\[
\begin{cases}
\mathcal{L} u + Dp = 0 & \text{in } B_R(x_0), \\
\text{div } u = 0 & \text{in } B_R(x_0),
\end{cases}
\]
where \( x_0 \in \mathbb{R}^n \) and \( R > 0 \). Then we have
\[
\int_{B_{R/2}(x_0)} |p - (p)_{B_{R/2}(x_0)}|^2 \, dx + \int_{B_{R/2}(x_0)} |Du|^2 \, dx \leq CR^{-2} \int_{B_R(x_0)} |u|^2 \, dx,
\]
where \( C = C(n, \lambda) \).

**Proof.** Let \( r \in (0, R] \) and denote \( B_r = B_r(x_0) \). By Remark 3.1, there exists \( \phi \in W^{1,2}_0(B_r) \) such that \( \text{div } \phi = p - (p)_{B_r} \) in \( B_r \)
and
\[
\|\phi\|_{L^{2n/(n-2)}(B_r)} \leq C \|D\phi\|_{L^2(B_r)} \leq C \|p - (p)_{B_r}\|_{L^2(B_r)},
\]
where \( C = C(n) \). Since \( \mathcal{L} u + D(p - (p)_{B_r}) = 0 \) in \( B_r \), (3.9) by testing with \( \phi \) in (3.9), we get
\[
\int_{B_r} |p - (p)_{B_r}|^2 \, dx \leq C_1 \int_{B_r} |Du|^2 \, dx, \quad \forall r \in (0, R], \tag{3.10}
\]
where \( C_1 = C_1(n, \lambda) \). From the above inequality, it remains us to show that
\[
\int_{B_{R/2}} |Du|^2 \, dx \leq CR^{-2} \int_{B_R} |u|^2 \, dx. \tag{3.11}
\]
Let $0 < \rho_1 < \rho_2 \leq R$ and $\delta \in (0, 1)$. Let $\eta$ be a smooth function on $\mathbb{R}^d$ such that
\[
0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } B_{\rho_1}, \quad \text{supp } \eta \subset B_{\rho_2}, \quad |D\eta| \leq C(d)(\rho_2 - \rho_1)^{-1}.
\]
Then by applying $\eta^2 u$ as a test function to
\[
\mathcal{L}u + D(p - (p)_{B_{\rho_2}}) = 0 \quad \text{in } B_R
\]
and using the fact that $\text{div } u = 0$, we have
\[
\int_{B_R} A_{\alpha\beta} \eta D\beta u \eta D\alpha u \, dx = -2 \int_{B_R} A_{\alpha\beta} \eta D\beta u \eta D\alpha u \, dx + 2 \int_{B_R} (p - (p)_{B_{\rho_2}}) \eta D\eta u \, dx,
\]
and thus, by the ellipticity condition, Hölder's inequality, and Young's inequality, we obtain
\[
\int_{B_{\rho_1}} |Du|^2 \, dx \leq \frac{C_\delta}{(\rho_2 - \rho_1)^2} \int_{B_{\rho_2}} |u|^2 \, dx + \frac{\delta}{C_1} \int_{B_{\rho_2}} |p - (p)_{B_{\rho_2}}|^2 \, dx,
\]
where $C_\delta = C_\delta(n, \lambda, \delta)$, and $C_1$ is the constant in (3.10). From this together with (3.10), it follows that
\[
\int_{B_{\rho_1}} |Du|^2 \, dx \leq \frac{C_\delta}{(\rho_2 - \rho_1)^2} \int_{B_{\rho_2}} |u|^2 \, dx + \delta \int_{B_{\rho_2}} |Du|^2 \, dx. \tag{3.12}
\]
Let us set
\[
\delta = \frac{1}{8}, \quad \rho_k = \frac{R}{2} \left(2 - \frac{1}{2^k}\right), \quad k = 0, 1, 2, \ldots.
\]
Then by (3.12), we have
\[
\int_{B_{\rho_k}} |Du|^2 \, dx \leq \frac{C4^k}{R^2} \int_{B_{r_{k+1}}} |u|^2 \, dx + \delta \int_{B_{r_{k+1}}} |Du|^2 \, dx, \quad k \in \{0, 1, 2, \ldots\},
\]
where $C = C(n, \lambda)$. By multiplying both sides of the above inequality by $\delta^k$ and summing the terms with respect to $k = 0, 1, \ldots$, we obtain
\[
\sum_{k=0}^{\infty} \delta^k \int_{B_{\rho_k}} |Du|^2 \, dx \leq C \sum_{k=0}^{\infty} (4\delta)^k \int_{B_{r_{k+1}}} |u|^2 \, dx + \sum_{k=1}^{\infty} \delta^k \int_{B_{r_k}} |Du|^2 \, dx.
\]
By subtracting the last term of the right-hand side in the above inequality, we obtain the desired estimate (3.11). The lemma is proved. \hfill \Box

Lemma 3.4. Assume that the condition $(A1)$ holds. Let $(u, p) \in W^{1,2}(B_R(x_0))^n \times L^2(B_R(x_0))$ satisfy
\[
\begin{cases}
\mathcal{L}u + Dp = 0 & \text{in } B_R(x_0), \\
\text{div } u = 0 & \text{in } B_R(x_0),
\end{cases}
\tag{3.13}
\]
where $x_0 \in \Omega$ and $R \in (0, d_{x_0}]$. Then we have
\[
\int_{B_r(x_0)} |Du|^2 \, dx \leq C_1 \left(\frac{r}{s}\right)^{n-2+2\mu} \int_{B_s(x_0)} |Du|^2 \, dx, \quad 0 < r < s \leq R, \tag{3.14}
\]
where $C_1 = C_1(n, \lambda, A_1)$. Moreover, we get
\[
||u||_{L^\infty(B_R(x_0))} \leq C_2 R^{-n} \|u\|_{L^1(B_R(x_0))}, \tag{3.15}
\]
where $C_2 = C_2(n, \mu, A_1)$. The statement is valid, provided that $\mathcal{L}$ is replaced by $\mathcal{L}^*$.\n
Proof. To prove (3.14), we only need to consider the case $0 < r \leq s/4$. Also, by replacing $u - (u)_{B_r(x_0)}$ if necessary, we may assume that $(u)_{B_s(x_0)} = 0$. Since $(u - (u)_{B_{2r}(x_0), p})$ is a weak solution of (3.13), we get from Lemma 3.3 that

$$
\int_{B_r(x_0)} |Du|^2 \, dx \leq Cr^{-2} \int_{B_{2r}(x_0)} |u - (u)_{B_{2r}(x_0)}|^2 \, dx.
$$

By (A1), the Poincaré inequality, and the above inequality, we have

$$
\int_{B_r(x_0)} |Du|^2 \, dx \leq Cr^{-2+2\mu} |u|_{C^0(B_r(x_0))}^2 \leq Cr^{-2+2\mu} |u|_{C^0(B_{s/2}(x_0))}^2,
$$

which establishes (3.14).

We observe that (A1) and a well known averaging argument yield

$$
\|u\|_{L^\infty(B_{R/2}(x_0))} \leq C \left( \int_{B_R(x_0)} |u|^2 \, dx \right)^{1/2},
$$

(3.16)

for any $R \in (0, d_{x_0}]$, where $C = C(n, \mu, A_1)$. For the proof that (3.16) implies (3.15), we refer to [13, pp. 80-82].

**Lemma 3.5.** Let $\Omega$ be a domain in $\mathbb{R}^n$ with $\operatorname{diam}(\Omega) \leq K_0$, where $n \geq 3$. Assume conditions (A0) and (A1). Let $(u, p) \in W^{1,2}_0(\Omega)^n \times L^2(\Omega)$ be a solution of the problem

$$
\begin{aligned}
\mathcal{L}u + Dp &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{aligned}
$$

where $f \in L^\infty(\Omega)^n$. Then for any $x_0 \in \Omega$ and $R \in (0, d_{x_0}]$, $u$ is continuous in $B_R(x_0)$ with the estimate

$$
|u|_{C^{0,1}(B_{R/2}(x_0))} \leq C \left( R^{-n/2+1-\mu_1} \|Du\|_{L^2(\Omega)} + \|f\|_{L^\infty(\Omega)} \right),
$$

(3.17)

for any $q \in \left( \frac{n}{2}, \frac{n}{2-n} \right)$, where $\mu_1 := 2 - n/q$ and $C = C(n, \lambda, \mu, A_1, q)$. Moreover, if $f$ is supported in $B_R(x_0)$, then we have

$$
\|u\|_{L^\infty(B_{R/2}(x_0))} \leq CR^2 \|f\|_{L^\infty(B_{R}(x_0))},
$$

(3.18)

where $C = C(n, \lambda, K_0, K_1, R_1, \mu, A_1)$. The statement is valid, provided that $\mathcal{L}$ is replaced by $\mathcal{L}^\ast$.

**Proof.** Let $x \in B_{R/2}(x_0)$ and $0 < s \leq R/2$. We decompose $(u, p)$ as $(u_1, p_1) + (u_2, p_2)$, where $(u_2, p_2) \in W^{1,2}_0(\Omega)^n \times L^2(\Omega)$ satisfies

$$
\begin{aligned}
\mathcal{L}u_2 + Dp_2 &= f \quad \text{in } B_s(x), \\
\text{div } u_2 &= 0 \quad \text{in } B_s(x).
\end{aligned}
$$

And then $(u_1, p_1) \in W^{1,2}(\Omega)^n \times L^2(\Omega)$ satisfies

$$
\begin{aligned}
\mathcal{L}u_1 + Dp_1 &= 0 \quad \text{in } B_s(x), \\
\text{div } u_1 &= 0 \quad \text{in } B_s(x).
\end{aligned}
$$

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From the estimate (3.3) and Hölder’s inequality, it follows that
\[ \|Du_2\|_{L^q(B_1(x))} \leq C\|f\|_{L^{2q/(n+2)}(B_1(x))} \leq Cs^{n/2-1+\mu_1}\|f\|_{L^q(B_R(x_0))}, \tag{3.19} \]
where \( q \in \left(\frac{n}{2}, \frac{n}{2-\mu}\right) \), \( \mu_1 = 2 - n/q \), and \( C = C(n, \lambda, q) \). For \( 0 < r < s \), we obtain by Lemma 3.4 that
\[
\int_{B_r(x)} |Du|^2 \, dx \leq C \left( \frac{r}{s} \right)^{n-2+2\mu} \int_{B_s(x)} |Du_1|^2 \, dx + C \int_{B_1(x)} |Du_2|^2 \, dx
\]
where \( C = C(n, \lambda, A_1) \). Therefore we get from (3.19) and (3.20) that
\[
\int_{B_r(x)} |Du|^2 \, dx \leq C \left( \frac{r}{s} \right)^{n-2+2\mu} \int_{B_s(x)} |Du|^2 \, dx + Cs^{n-2+2\mu_1}\|f\|_{L^q(B_R(x_0))}^2.
\]
Then by [12, Lemma 2.1, p. 86], we have
\[
\int_{B_r(x)} |Du|^2 \, dx \leq C \left( \frac{r}{R} \right)^{n-2+2\mu_1} \int_{B_1(x)} |Du|^2 \, dx + C r^{n-2+2\mu_1}\|f\|_{L^q(B_R(x_0))}^2
\]
for any \( x \in B_{R/2}(x_0) \) and \( r \in (0, R/2) \). From this together with Morrey-Campanato’s theorem, we prove (3.17).

To see (3.18), assume \( f \) is supported in \( B_R(x_0) \). Notice from the Sobolev inequality that
\[ \|u\|_{L^2(B_R(x_0))} \leq C(n)R\|Du\|_{L^2(\Omega)}. \]
Then we obtain by (3.17) and the above estimate that
\[ \|u\|_{L^\infty(B_{R/2}(x_0))} \leq CR^{\mu_1}\|u\|_{C^{\mu_1}(B_{R/2}(x_0))} + CR^{n/2}\|u\|_{L^2(B_R(x_0))} \]
\[ \leq CR^{-n/2+1}\|Du\|_{L^2(\Omega)} + CR^2\|f\|_{L^\infty(B_R(x_0))}, \]
and thus, we get the desired estimate from the inequality (3.3). The lemma is proved.

4. Proofs of main theorems. In the section, we prove main theorems stated in Sections 2.1 and 2.2.

4.1. Proof of Theorem 2.3.

4.1.1. Averaged Green function. Let \( y \in \Omega \) and \( \varepsilon > 0 \) be fixed, but arbitrary. Fix an integer \( 1 \leq k \leq n \) and let \((u_\varepsilon, \pi_\varepsilon) = (v_{\varepsilon,y,k}, \pi_{\varepsilon,y,k})\) be the solution in \( W_0^{1,2}(\Omega)^n \times L_0^2(\Omega)\)

\[
\begin{cases}
L u + Dp = \frac{1}{|\Omega_\varepsilon(y)|}1_{\Omega_\varepsilon(y)} e_k & \text{in } \Omega, \\
\div u = 0 & \text{in } \Omega,
\end{cases}
\]
where \( e_k \) is the \( k \)-th unit vector in \( \mathbb{R}^n \). We define the averaged Green function \((G_\varepsilon(\cdot, y), \Pi_\varepsilon(\cdot, y))\) for (SP) by setting
\[ G_\varepsilon^{jk}(\cdot, y) = v_{\varepsilon,y,k}^j \quad\text{and}\quad \Pi_\varepsilon^k(\cdot, y) = \pi_{\varepsilon,y,k}. \tag{4.1} \]
Then \((G_\varepsilon(\cdot, y), \Pi_\varepsilon(\cdot, y))\) satisfies
\[
\int_{\Omega} a^{ij\alpha\beta}_{\varepsilon} D_\beta G_{\varepsilon}^{jk}(\cdot, y) D_\alpha \varphi^i \, dx - \int_{\Omega} \Pi_{\varepsilon}^k(\cdot, y) \text{div} \, \varphi \, dx = \int_{\Omega_{\varepsilon}(y)} \varphi^k \, dx \tag{4.2}
\]
for any \(\varphi \in W_0^{1,2}(\Omega)^n\). We also obtain by (3.3) that
\[
\|\Pi_{\varepsilon}(\cdot, y)\|_{L^2(\Omega)} + \|DG_{\varepsilon}(\cdot, y)\|_{L^2(\Omega)} \leq C\varepsilon^{(2-n)/2}, \quad \forall \varepsilon > 0, \tag{4.3}
\]
where \(C = C(n, \lambda, K_0, K_1, R_1)\). The following lemma is an immediate consequence of Lemma 3.3.

**Lemma 4.1.** Let \(y \in \Omega\) and \(\varepsilon > 0\).

(i) For any \(x_0 \in \Omega\) and \(R \in (0, d_{x_0}]\) satisfying \(B_R(x_0) \cap B_{\varepsilon}(y) = \emptyset\), we have
\[
\int_{B_{R/2}(x_0)} |DG_{\varepsilon}(x, y)|^2 \, dx \leq CR^{-2} \int_{B_R(x_0)} |G_{\varepsilon}(x, y)|^2 \, dx,
\]
where \(C = C(n, \lambda)\).

(ii) Let \(R \in (0, 2d_y/3]\) and \(\varepsilon \in (0, R/4)\). Then we have
\[
\int_{B_R(y) \setminus B_{R/2}(y)} |DG_{\varepsilon}(x, y)|^2 \, dx \leq CR^{-2} \int_{B_{3R/2}(y) \setminus B_{R/4}(y)} |G_{\varepsilon}(x, y)|^2 \, dx,
\]
where \(C = C(n, \lambda)\).

With the preparations in the previous section, we obtain the pointwise estimate of the averaged Green function \(G_{\varepsilon}(\cdot, y)\).

**Lemma 4.2.** There exists a constant \(C = C(n, \lambda, K_0, K_1, R_1, \mu, A_1) > 0\) such that for any \(x, y \in \Omega\) satisfying \(0 < |x - y| < d_y/2\), we have
\[
|G_{\varepsilon}(x, y)| \leq C|x - y|^{2-n}, \quad \forall \varepsilon \in (0, |x - y|/3). \tag{4.4}
\]

**Proof.** Let \(y \in \Omega\), \(R \in (0, d_y)\), and \(\varepsilon \in (0, R/2)\). We denote \(v_x\) to be the \(k\)-th column of \(G_{\varepsilon}(\cdot, y)\). Assume that \((u, p) \in W_0^{1,2}(\Omega)^n \times L_0^2(\Omega)\) is the solution of
\[
\begin{align*}
\mathcal{L}^\varepsilon u + Dp &= f \quad \text{in} \ \Omega, \\
\text{div} \, u &= 0 \quad \text{in} \ \Omega, \tag{4.5}
\end{align*}
\]
where \(f^i(x) = 1_{B_R(y)} \text{sgn}(u^i(x))\) and \(f = (f^1, \ldots, f^n) \in L^\infty(\Omega)^n\). Then by testing with \(v_x\) in (4.5), we have
\[
\int_{\Omega} A_{\alpha\beta} D_\beta v_x \cdot D_\alpha u \, dx = \int_{B_R(y)} f \cdot v_x \, dx.
\]
Similarly, we set \(\varphi = u\) in (4.2) to obtain
\[
\int_{\Omega} A_{\alpha\beta} D_\beta v_x \cdot D_\alpha u \, dx = \int_{B_{\varepsilon}(y)} u^k \, dx.
\]
From the above two identities, we get
\[
\int_{B_R(y)} f \cdot v_x \, dx = \int_{B_{\varepsilon}(y)} u^k \, dx, \tag{4.6}
\]
and thus, by (3.18), we derive
\[
\|G_{\varepsilon}(\cdot, y)\|_{L^1(B_R(y))} \leq CR^2, \quad R \in (0, d_y), \quad \varepsilon \in (0, R/2), \tag{4.7}
\]
where \(C = C(n, \lambda, K_0, K_1, R_1, \mu, A_1)\).
Moreover, we derive the following uniform \( L^1 \)-estimates uniformly in \( \epsilon > 0 \).

**Lemma 4.3.** Let \( y \in \Omega, R \in (0,d_y) \), and \( \epsilon > 0 \). Then we have

\[
\| G_\epsilon(\cdot,y) \|_{L^{2/(n-2)}(\Omega \setminus B_R(y))} + \| D G_\epsilon(\cdot,y) \|_{L^2(\Omega \setminus B_R(y))} \leq CR^{(2-n)/2}. \tag{4.8}
\]

Also, we obtain

\[
\{ x \in \Omega : |G_\epsilon(x,y)| > t \} \leq Ct^{-n/(n-2)}, \quad \forall t > d_y^{2-n}, \tag{4.9}
\]

\[
\{ x \in \Omega : |D G_\epsilon(x,y)| > t \} \leq Ct^{-n/(n-1)}, \quad \forall t > d_y^{1-n}. \tag{4.10}
\]

Moreover, we derive the following uniform \( L^q \) estimates:

\[
\| G_\epsilon(\cdot,y) \|_{L^q(B_R(y))} \leq C q R^{-n+n/q}, \quad q \in [1,n/(n-2)), \tag{4.11}
\]

\[
\| D G_\epsilon(\cdot,y) \|_{L^q(B_R(y))} \leq C q R^{-1+n+n/q}, \quad q \in [1,n/(n-1)), \tag{4.12}
\]

\[
\| \Pi_\epsilon(\cdot,y) \|_{L^q(\Omega)} \leq C_{y,q}, \quad q \in [1,n/(n-1)). \tag{4.13}
\]

In the above, \( C = C(n,\lambda,\alpha_0,\alpha_1,\alpha_2,\alpha_3) \), \( C_q = C_q(n,\lambda,\alpha_0,\alpha_1,\alpha_2,\alpha_3,q) \), and \( C_{y,q} = C_{y,q}(n,\lambda,\alpha_0,\alpha_1,\alpha_2,\alpha_3,q,d_y) \).

**Proof.** Recall the notation (4.1). We first prove the estimate (4.8). From the obvious fact that \( d_y/3 \) and \( d_y \) are comparable to each other, we only need to prove the estimate (4.8) for \( R \in (0,d_y/3) \). If \( \epsilon \geq R/12 \), then by (4.3) and the Sobolev inequality, we have

\[
\| G_\epsilon(\cdot,y) \|_{L^{2/(n-2)}(\Omega \setminus B_R(y))} + \| D G_\epsilon(\cdot,y) \|_{L^2(\Omega \setminus B_R(y))} \leq C \| D G_\epsilon(\cdot,y) \|_{L^2(\Omega)} \leq C R^{(2-n)/2}. \tag{4.14}
\]

On the other hand, if \( \epsilon \in (0,R/12) \), then by setting \( \varphi = \eta^2 v_\epsilon \) in (4.2), where \( \eta \) is a smooth function satisfying

\[
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } \mathbb{R}^n \setminus B_R(y), \quad \eta \equiv 0 \text{ on } B_{R/2}(y), \quad |D\eta| \leq CR^{-1},
\]

we have

\[
\int_{\Omega} \eta^2 |Dv_\epsilon|^2 \, dx \leq C \int_{\Omega} |D\eta|^2 |v_\epsilon|^2 \, dx + C \int_D |\pi_\epsilon - (\pi_\epsilon)_D|^2 \, dx, \tag{4.15}
\]

where \( D = B_R(y) \setminus B_{R/2}(y) \). By Remark 3.1, there exists a function \( \phi_\epsilon \in W^{1,2}_0(D)^n \) such that

\[
\text{div } \phi_\epsilon = \pi_\epsilon - (\pi_\epsilon)_D \text{ in } D, \quad \| D\phi_\epsilon \|_{L^2(D)} \leq C \| \pi_\epsilon - (\pi_\epsilon)_D \|_{L^2(D)},
\]

where \( C = C(n) \). Therefore, by setting \( \varphi = \phi_\epsilon \) in (4.2), we get from Lemma 4.1 (ii) that

\[
\int_D |\pi_\epsilon - (\pi_\epsilon)_D|^2 \, dx \leq C \int_D |Dv_\epsilon|^2 \, dx \leq CR^{-2} \int_{B_{3R/4}(y) \setminus B_{R/4}(y)} |v_\epsilon|^2 \, dx. \tag{4.16}
\]
Then by combining (4.15) and (4.16), we find that
\[ \int_\Omega \eta^2 |Dv_\varepsilon|^2 \, dx \leq CR^{-2} \int_{B_{R/2}(y) \setminus B_{R/4}(y)} |v_\varepsilon|^2 \, dx \leq CR^{2-n}, \] (4.17)
where we used Lemma 4.2 in the last inequality. Also, by using the fact that
\[ \|\eta v_\varepsilon\|_{L^{2n/(n-2)}(\Omega)} \leq C\|D(\eta v_\varepsilon)\|_{L^2(\Omega)} \leq C\|\eta Dv_\varepsilon\|_{L^2(\Omega)} + C\|D\eta v_\varepsilon\|_{L^2(\Omega)}, \]
the inequality (4.17) implies
\[ \|G_\varepsilon(\cdot, y)\|_{L^{2n/(n-2)}(\Omega \setminus B_R(y))} + \|DG_\varepsilon(\cdot, y)\|_{L^2(\Omega \setminus B_R(y))} \leq CR^{(2-n)/2}. \]
This together with (4.14) gives (4.8) for \( R \in (0, d_y/3]. \)

Now, let \( A_t = \{ x \in \Omega : |G_\varepsilon(x, y)| > t \} \) and choose \( t = R^{2-n} > d_y^{2-n} \). Then by (4.8), we have
\[ |A_t \setminus B_R(y)| \leq t^{-2n/(n-2)} \int_{A_t \setminus B_R(y)} |G_\varepsilon(x, y)|^{2n/(n-2)} \, dx \leq C t^{n/(n-2)}. \]
From this inequality and the fact that \( |A_t \cap B_R(y)| \leq CR^n = Ct^{-n/(n-2)} \), we get (4.9). Let us fix \( q \in (1, n/(n-2)) \). Note that
\[ \int_{B_R(y)} |G_\varepsilon(x, y)|^q \, dx = \int_{B_R(y) \cap A_t} |G_\varepsilon(x, y)|^q \, dx + \int_{B_R(y) \cap A_t} |G_\varepsilon(x, y)|^q \, dx \]
\[ \leq CR^{2-n/q} + \int_{A_t} |G_\varepsilon(x, y)|^q \, dx, \] (4.18)
where \( t = R^{2-n} > d_y^{2-n} \). From (4.9) it follows that
\[ \int_{A_t} |G_\varepsilon(x, y)|^q \, dx = q \int_0^\infty s^{q-1} \{ x \in \Omega : |G_\varepsilon(x, y)| > \max(t, s) \} \, ds \]
\[ \leq C_q t^{-n/(n-2)} \int_0^t s^{q-1} \, ds + C_q \int_t^\infty s^{q-1-n/(n-2)} \, ds \]
\[ \leq C_q R^{(2-n)/q}, \] (4.19)
where \( C_q = C_q(n, \lambda, K_0, K_1, R_1, \mu, A_1, q) \). Therefore, by combining (4.18) and (4.19), we obtain (4.11). Moreover, by utilizing (4.8), and following the same steps as in the above, we get (4.10) and (4.12).

It only remains to establish (4.13). From Hölder’s inequality, we only need to prove the inequality with \( q \in (1, n/(n-1)) \). Let \( q' = q/(q-1) \), and denote
\[ w := \text{sgn}(\varepsilon) |x|^{q-1}. \]
Then we have
\[ w \in L^{q'}(\Omega), \quad n < q' < \infty. \]
Therefore by Remark 3.1 and the Sobolev inequality, there exists a function \( \phi \in W_0^{1,q'}(\Omega)^n \) such that
\[ \text{div} \phi = w - (w)_\Omega \quad \text{in} \quad \Omega, \]
\[ \|\phi\|_{L^{\infty}(\Omega)} \leq C\|D\phi\|_{L^q(\Omega)} \leq C\|w\|_{L^{q'}(\Omega)}. \] (4.20)

We observe that
\[ \int_\Omega \pi \text{div} \phi \, dx = \int_\Omega \pi (w - (w)_\Omega) \, dx = \int_\Omega \pi w \, dx = \int_\Omega |w|^{q'} \, dx. \] (4.21)
By setting \( \varphi = \phi \) in (4.2), we get from (4.20) and (4.21) that
\[
\int \Omega |w|^q \, dx \leq C(1 + \|Dv_\varepsilon\|_{L^q(\Omega)})\|w\|_{L^{q'}(\Omega)}.
\] (4.22)

Notice from (4.8) and (4.12) that
\[
\|Dv_\varepsilon\|_{L^q(\Omega)} \leq C_{y,q}
\]
for all \( \varepsilon > 0 \), where \( C_{y,q} = C_{y,q}(n, \lambda, K_0, K_1, R_1, \mu, A_1, q, d_y) \). This together with (4.22) gives (4.13). The lemma is proved. \( \square \)

4.1.2. Construction of the Green function. Let \( y \in \Omega \) be fixed, but arbitrary. Notice from Lemma 4.3 and the weak compactness theorem that there exist a sequence \( \{\varepsilon_\rho\}_{\rho=1}^\infty \) tending to zero and functions \( G(\cdot, y) \) and \( \hat{G}(\cdot, y) \) such that
\[
G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup G(\cdot, y) \quad \text{weakly in } W^{1,2}(\Omega \setminus \overline{B}_{d_y/2}(y))^{n \times n},
\]
\[
G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup \hat{G}(\cdot, y) \quad \text{weakly in } W^{1,q}(B_{d_y}(y))^{n \times n},
\] (4.23)
where \( q \in (1, n/(n-1)) \). Since \( G(\cdot, y) \equiv \hat{G}(\cdot, y) \) on \( B_{d_y}(y) \setminus \overline{B}_{d_y/2}(y) \), we shall extend \( G(\cdot, y) \) to entire \( \Omega \) by setting \( G(\cdot, y) \equiv \hat{G}(\cdot, y) \) on \( \overline{B}_{d_y/2}(y) \). By applying a diagonalization process and passing to a subsequence, if necessary, we may assume that
\[
G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup G(\cdot, y) \quad \text{weakly in } W^{1,2}(\Omega \setminus \overline{B}_R(y))^{n \times n}, \quad \forall R \in (0, d_y].
\] (4.24)

Indeed, if we consider a sequence \( \{R_i\}_{i=1}^\infty \) satisfying \( R_i \in (0, d_y] \) and \( R_i \searrow 0 \), then for each \( i \in \{1, 2, \ldots\} \), there exists a subsequence of \( \{G_{\varepsilon_\rho}(\cdot, y)\} \), denoted by \( \{G_{\varepsilon_{\rho_i}}(\cdot, y)\} \), such that
\[
\{G_{\varepsilon_{\rho_{i+1}}} (\cdot, y)\} \subset \{G_{\varepsilon_{\rho_{i}}} (\cdot, y)\}
\]
and
\[
G_{\varepsilon_{\rho_{i}}}(\cdot, y) \rightharpoonup G(\cdot, y) \quad \text{weakly in } W^{1,2}(\Omega \setminus \overline{B}_R(y))^{n \times n} \quad \text{as } j \to \infty.
\]

Taking the subsequence as \( \{G_{\varepsilon_{\rho_{i}}}(\cdot, y)\} \), we see that (4.24) holds. By (4.13), there exists a function \( \Pi(\cdot, y) \in L_0^q(\Omega)^n \) such that, by passing to a subsequence,
\[
\Pi_{\varepsilon_\rho}(\cdot, y) \rightharpoonup \Pi(\cdot, y) \quad \text{weakly in } L^q(\Omega)^n.
\] (4.25)

We shall now claim that \( (G(x, y), \Pi(x, y)) \) satisfies the properties a) - c) in Definition 2.1 so that \( (G(x, y), \Pi(x, y)) \) is indeed the Green function for (SP). Notice from (4.24) that for any \( \zeta \in C_0^\infty(\Omega) \) satisfying \( \zeta \equiv 1 \) on \( B_R(y) \), where \( R \in (0, d_y) \), we have
\[
(1 - \zeta)G_{\varepsilon_\rho}(\cdot, y) \rightharpoonup (1 - \zeta)G(\cdot, y) \quad \text{weakly in } W^{1,2}(\Omega)^{n \times n}.
\]

Since \( W_0^{1,2}(\Omega) \) is weakly closed in \( W^{1,2}(\Omega) \), we have \( (1 - \zeta)G(\cdot, y) \in W_0^{1,2}(\Omega)^{n \times n} \), and thus the property a) is verified. Let \( \eta \) be a smooth cut-off function satisfying \( \eta \equiv 1 \) on \( B_{d_y/2}(y) \) and \( \text{supp} \eta \subset B_{d_y}(y) \). Then by (4.2), (4.23) – (4.25), we obtain
for $\varphi \in C_0^\infty(\Omega)^n$ that
\[ \varphi^k(y) = \lim_{\rho \to \infty} \int_{\Omega_\rho(y)} \varphi^k \]
\[ = \lim_{\rho \to \infty} \left( \int_\Omega a_{ij}^{ij} D_\beta G_{\varepsilon^k}(\cdot, y) D_\alpha(\eta \varphi^i) + \int_\Omega a_{ij}^{ij} D_\beta G_{\varepsilon^k}(\cdot, y) D_\alpha((1 - \eta) \varphi^i) \right) \]
\[ - \lim_{\rho \to \infty} \int_\Omega \Pi_{\varepsilon^k}(\cdot, y) \text{div} \varphi \]
\[ = \int_\Omega a_{ij}^{ij} D_\beta G_{\varepsilon^k}(\cdot, y) D_\alpha \varphi^i - \int_\Omega \Pi^k(\cdot, y) \text{div} \varphi. \]

Similarly, we get
\[ \int_\Omega \phi(x) \text{div}_x G(x, y) \, dx = 0, \quad \forall \phi \in C^\infty(\Omega). \]

From the above two identity, the property b) is satisfied. Finally, if $(u, p) \in W^{1,2}_0(\Omega)^n \times L^2_0(\Omega)$ is the weak solution of the problem (SP), then by setting $\varphi$ to be the $k$-th column of $G_{\varepsilon^k}(\cdot, y)$ in (2.2) and setting $\varphi = u$ in (4.2), we have (see e.g., Eq. (4.6))
\[ \int_{\Omega_\rho(y)} u = \int_\Omega G_{\varepsilon^k}(\cdot, y)^{tr} f - \int_\Omega D_\alpha G_{\varepsilon^k}(\cdot, y)^{tr} f_\alpha - \int_\Omega \Pi_{\varepsilon^k}(\cdot, y) g. \] (4.26)

By letting $\rho \to \infty$ in the above identity, we find that $(G(x, y), \Pi(x, y))$ satisfies the property c) in Definition 2.1.

Next, let $y \in \Omega$ and $R \in (0, d_y]$. Let $v$ and $v_\varepsilon$ be the $k$-th column of $G(\cdot, y)$ and $G_\varepsilon(\cdot, y)$, respectively. Then for any $g \in C_0^\infty(B_R(y))^n$, we obtain by (4.11) and (4.23) that
\[ \left| \int_{B_R(y)} v \cdot g \, dx \right| = \lim_{\rho \to \infty} \left| \int_{B_\rho(y)} v_\varepsilon \cdot g \, dx \right| \leq C_q R^{2 - n + q'/q} \|g\|_{L^{q'}(B_R(y))}, \]
where $q \in [1, n/(n - 2)]$ and $q' = q/(q - 1)$. Therefore, by a duality argument, we obtain the estimate iv) in Theorem 2.3. Similarly, from Lemma 4.3, (4.23), and (4.24), we have the estimates i) and v) in the theorem. Also, ii) and iii) are deduced from i) in the same way as (4.9) and (4.10) are deduced from (4.8). Therefore, $G(x, y)$ satisfies the estimates i) – v) in Theorem 2.3. For $x, y \in \Omega$ satisfying $0 < |x - y| < d_y/2$, set $r := |x - y|/4$. Notice from the property b) in Definition 2.1 that $(G(\cdot, y), \Pi(\cdot, y))$ satisfies
\[ \begin{cases} \mathcal{L} G(\cdot, y) + D\Pi(\cdot, y) = 0 & \text{in } B_r(x), \\ \text{div} G(\cdot, y) = 0 & \text{in } B_r(x). \end{cases} \]

Then by Lemma 3.4 and Hölder’s inequality, we have
\[ |G(x, y)| \leq C r^{(2 - n)/2} \|G(\cdot, y)\|_{L^{2n/(n - 2)}(B_{2r}(x))} \]
\[ \leq C r^{(2 - n)/2} \|G(\cdot, y)\|_{L^{2n/(n - 2)}(\Omega) \setminus B_r(y)}. \]

This together with the estimate i) in Theorem 2.3 implies
\[ |G(x, y)| \leq C |x - y|^{2 - n}, \quad 0 < |x - y| < d_y/2. \]

**Lemma 4.4.** For each compact set $K \subset \Omega \setminus \{y\}$, there is a subsequence of $\{G_{\varepsilon^k}(\cdot, y)\}$ that converges to $G(\cdot, y)$ uniformly on $K$. 
Proof. Let $x \in \Omega$ and $R \in (0, d_x]$ satisfying $B_R(x) \subset \Omega \setminus \{y\}$. Notice that there exists $\varepsilon_B > 0$ such that for $\varepsilon < \varepsilon_B$, we have

$$
\begin{align*}
\mathcal{L} G_\varepsilon(\cdot, y) + D \Pi_\varepsilon(\cdot, y) &= 0 \quad \text{in } B_R(x), \\
\mathrm{div} G_\varepsilon(\cdot, y) &= 0 \quad \text{in } B_R(x).
\end{align*}
$$

By (A1) and (4.8), $\{G_\varepsilon(\cdot, y)\}_{\varepsilon \leq \varepsilon_B}$ is equicontinuous on $\overline{B_{R/2}(x)}$. Also, it follows from Lemma 3.4 that $\{G_\varepsilon(\cdot, y)\}_{\varepsilon \leq \varepsilon_B}$ is uniformly bounded on $\overline{B_{R/2}(x)}$. By the Arzelà-Ascoli theorem, we obtain the desired conclusion. 

4.1.3. Proof of the identity (2.5). For any $x \in \Omega$ and $\sigma > 0$, we define the averaged Green function $\{G^*_\sigma(\cdot, x), \Pi^*_\sigma(\cdot, x)\}$ for (SP$^*$) by letting its $l$-th column to be the unique weak solution in $W_0^{1,2}(\Omega)^n \times L^2_0(\Omega)$ of the problem

$$
\begin{align*}
\mathcal{L}^* \mathbf{u} + Dp &= \frac{1}{|\Omega_\sigma(x)|} \Omega_\sigma(x) \mathbf{e}_l \quad \text{in } \Omega, \\
\mathrm{div} \mathbf{u} &= 0 \quad \text{in } \Omega,
\end{align*}
$$

where $\mathbf{e}_l$ is the $l$-th unit vector in $\mathbb{R}^n$. Then by following the same argument as in Sections 4.1.1 and 4.1.2, there exist a sequence $\{\sigma_\nu\}_{\nu \geq 1}$ tending to zero and the Green function $\{G^*(\cdot, x), \Pi^*(\cdot, x)\}$ for (SP$^*$) satisfying the counterparts of (4.23), (4.24), (4.25), and Lemma 4.4.

Now, let $x, y \in \Omega$ and $x \neq y$. We then obtain for $\varepsilon \in (0, d_y]$ and $\sigma \in (0, d_x]$ that

$$
\int_{B_\varepsilon(y)} (G^*_\sigma)^{kl}(\cdot, x) = \int_{\Omega} a^i_{\sigma,\beta} D_\beta (G^*_\sigma)^{jl}(\cdot, y) D_\alpha ((G^*_\sigma)^{il}(\cdot, x)) = \int_{B_\varepsilon(x)} G^*_\varepsilon(\cdot, y).
$$

(4.27)

We define

$$
I_{\rho,\nu}^{kl} := \int_{B_\rho(y)} (G^*_\sigma)^{kl}(\cdot, x) = \int_{B_\sigma(x)} G^*_\varepsilon(\cdot, y).
$$

Then by the continuity of $G^*_\varepsilon(\cdot, y)$ and Lemma 4.4, we have

$$
\lim_{\rho \to \infty} \lim_{\nu \to \infty} I_{\rho,\nu}^{kl} = \lim_{\rho \to \infty} G^*_\varepsilon(x, y) = G^{lk}(x, y).
$$

Similarly, we get

$$
\lim_{\rho \to \infty} \lim_{\nu \to \infty} I_{\rho,\nu}^{kl} = \lim_{\rho \to \infty} \int_{\Omega_\sigma(y)} (G^*)^{kl}(\cdot, x) = (G^*)^{kl}(y, x).
$$

We have thus shown that

$$
G^{lk}(x, y) = (G^*)^{kl}(y, x), \quad \forall x, y \in \Omega, \quad x \neq y,
$$

which gives the identity (2.5). Therefore, we get from (4.27) that

$$
G^*_\varepsilon(x, y) = \lim_{\nu \to \infty} \int_{B_{\sigma_\nu}(x)} G^*_\varepsilon(\cdot, y) = \lim_{\nu \to \infty} \int_{B_\varepsilon(y)} (G^*_\sigma)^{kl}(\cdot, x)
$$

$$
= \int_{B_\varepsilon(y)} (G^*)^{kl}(\cdot, x) = \int_{B_\varepsilon(y)} G^*(x, \cdot), \quad \varepsilon \in (0, d_y],
$$

and

$$
\lim_{\varepsilon \to 0} G^*_\varepsilon(x, y) = G^{lk}(x, y), \quad \forall x, y \in \Omega, \quad x \neq y.
$$

(4.28)

The theorem is proved.
4.2. Proof of Theorem 2.5. The proof is based on \( L^n \)-estimates for Stokes systems with VMO coefficients. In this proof, we assume that \( x_0 \in \Omega \) and \( 0 < R \leq \min \{ dx_0, 1 \} \), and denote \( B_r = B_r(x_0) \) for \( r > 0 \).

Lemma 4.5. Let \( q > n \), \( 0 < \rho < r \leq R \leq 1 \), and \((v, b) \in W^{1,q}(B_r)^n \times L^q(B_r)\) satisfy
\[
\begin{align*}
\mathcal{L} v + Db &= 0 \quad \text{in } B_r, \\
\text{div } v &= 0 \quad \text{in } B_r,
\end{align*}
\]
where the coefficients of \( \mathcal{L} \) belong to the class of VMO. Then we have
\[
\|Dv\|_{L^q(B_r)} + \frac{1}{r-\rho} \|v\|_{L^n(B_r)} \leq C \left( \|Dv\|_{L^{q/(n+q)}(B_r)} + \frac{1}{r-\rho} \|v\|_{L^{q/(n+q)}(B_r)} \right),
\]
where \( C \) depends on \( n, \lambda, q \), and the VMO modulus of the coefficients.

Proof. Let \( \tau = (\rho + r)/2 \) and \( \eta \) be a smooth function in \( \mathbb{R}^2 \) such that
\[
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{\rho}, \quad \text{supp } \eta \subset B_r, \quad |D\eta| \leq C(r-\rho)^{-1},
\]
Denote \( b_0 = (b)_{B_r} \) and observe that \((\eta v, \eta(b - b_0))\) satisfies
\[
\begin{align*}
\mathcal{L}(\eta v) + D(\eta(b - b_0)) &= (b - b_0)D\eta - A_{\alpha\beta}D\beta v D_{\alpha} \eta - D_{\alpha}(A_{\alpha\beta}D\beta \eta v) \quad \text{in } B_r, \\
\text{div}(\eta v) &= D\eta \cdot v \quad \text{in } B_r, \\
\eta v &= 0 \quad \text{on } \partial B_r.
\end{align*}
\]
By Corollary 5.3 with scaling, we have
\[
\|Dv\|_{L^q(B_r)} \leq \frac{C}{r-\rho} \left( \|b - b_0\|_{L^{q/(n+q)}(B_r)} + \|Dv\|_{L^{q/(n+q)}(B_r)} + \|v\|_{L^n(B_r)} \right),
\]
where \( C \) depends on \( n, \lambda, q \), and the VMO modulus of the coefficients. Note that
\[
\|v\|_{L^n(B_{r_1})} \leq \frac{C}{r_1} \left( \|b - b_0\|_{L^{q/(n+q)}(B_{r_1})} + C\|Dv\|_{L^{q/(n+q)}(B_{r_1})} \right)
\]
for \( 0 < r_1 \leq r \). Combining the above two estimates we have
\[
\|Dv\|_{L^q(B_r)} + \frac{1}{r-\rho} \|v\|_{L^n(B_r)} \leq \frac{C}{r-\rho} \left( \|b - b_0\|_{L^{q/(n+q)}(B_r)} + \|Dv\|_{L^{q/(n+q)}(B_r)} + \frac{1}{r-\rho} \|v\|_{L^{q/(n+q)}(B_r)} \right).
\]
(4.30)

Set \( s = nq/(n+q) \) and \( \tilde{b} = \text{sgn}(b - b_0)\|b - b_0\|^{-s-1} \in L^{s/(s-1)}(B_r) \). There exists \( \phi \in W_{0,1}^{s/(s-1)}(B_r)^n \) such that (see Remark 3.1)
\[
\text{div } \phi = \tilde{b} - (\tilde{b})_{B_r} \quad \text{in } B_r, \quad \|D\phi\|_{L^{s/(s-1)}(B_r)} \leq C(n, q)\|\tilde{b}\|_{L^{s/(s-1)}(B_r)}.
\]
Using \( \phi \) as a test function, we obtain
\[
\int_{B_r} |b - b_0|^s dx = \int_{B_r} (b - b_0) \text{div } \phi dx = \int_{\Omega} A_{\alpha\beta}D\beta v \cdot D_{\alpha} \phi dx,
\]
which implies that
\[
\|b - b_0\|_{L^s(B_r)} \leq C(n, \lambda, q)\|Dv\|_{L^s(B_r)} \|b - b_0\|^{-s-1}_{L^r(B_r)}.
\]
From this together with (4.30), we get the desired estimate. \qed
Now we are ready to prove Theorem 2.5. Let \((u, p) \in W^{1, 2}(B_R)^n \times L^2(B_R)\) satisfy (2.3). Let \(q > n, 0 < r \leq R, \) and \(p = r/4.\) Set
\[
q_i = \frac{nq}{n + q_i}, \quad r_i = \rho + \frac{ri}{4m}, \quad i \in \{0, \ldots, m\},
\]
where \(m\) is the smallest integer such that \(m \geq n(1/2 - 1/q).\) Then by applying Lemma 4.5 iteratively, we see that \((u, p) \in W^{1, q}(B_R)^n \times L^q(B_R)\) and
\[
\|Du\|_{L^q(B_R)} + \frac{4m}{r} \|u\|_{L^q(B_R)} \leq \left(\frac{Cm}{r}\right)^m \left(\|Du\|_{L^q(B_R)} + \frac{4m}{r} \|u\|_{L^q(B_R)}\right).
\]
Using Hölder’s inequality and Lemma 3.3, we have
\[
\|Du\|_{L^q(B_{r/4})} + \frac{1}{r} \|u\|_{L^q(B_{r/4})} \leq \left(\frac{Cmr}{r}\right)^m \left(\|Du\|_{L^q(B_{r/2})} + \frac{1}{r} \|u\|_{L^q(B_{r/2})}\right) \leq \frac{C \rho^{n(1/2 - 1/2)}}{r} \|u\|_{L^2(B_r)}.
\]
By the Sobolev inequality with scaling, we get
\[
[u]_{C^{1 - n/q}(B_{r/4}(x_0))} \leq C r^{-1 + n/q} \left(\int_{B_r(x_0)} |u|^2 \, dx\right)^{1/2},
\]
where \(C\) depends on \(n, \lambda, q,\) and the VMO modulus of the coefficients. Since the above inequality holds for all \(x_0 \in \Omega\) and \(0 < r \leq R \leq \min\{d_{x_0}, 1\},\) we conclude that
\[
[u]_{C^{1 - n/q}(B_{r/2}(x_0))} \leq C r^{-1 + n/q} \left(\int_{B_{2r}(x_0)} |u|^2 \, dx\right)^{1/2}.
\]
This completes the proof of Theorem 2.5.

4.3. Proof of Theorem 2.8. For \(y \in \Omega\) and \(\varepsilon > 0,\) let \((G_\varepsilon(\cdot, y), \Pi_\varepsilon(\cdot, y))\) be the averaged Green function on \(\Omega\) as constructed in Section 4.1.1, and let \(G_\varepsilon^{k}(\cdot, y)\) be the \(k\)-th column of \(G_\varepsilon(\cdot, y).\) Recall that \((G_\varepsilon^{k}(\cdot, y), \Pi_\varepsilon^k(\cdot, y))\) satisfies
\[
\begin{cases}
  \mathcal{L}G_\varepsilon^{k}(\cdot, y) + D\Pi_\varepsilon^{k}(\cdot, y) = g_k & \text{in } \Omega, \\
  \text{div } G_\varepsilon^{k}(\cdot, y) = 0 & \text{in } \Omega,
\end{cases}
\]
where
\[
g_k = \frac{1}{|\Omega_{\varepsilon}(y)|} 1_{\Omega_{\varepsilon}(y)} e_k.
\]
By (A2), we obtain for any \(x_0 \in \Omega\) and \(0 < r < \text{diam } \Omega\) that
\[
\|G_\varepsilon^{k}(\cdot, y)\|_{L^\infty(\Omega_{r/2}(x_0))} \leq A_2 \left(r^{-n/2} \|G_\varepsilon^{k}(\cdot, y)\|_{L^2(\Omega_{r/2}(x_0))} + r^2 \|g_k\|_{L^\infty(\Omega_{r/2}(x_0))}\right) .
\]
Applying a standard argument (see, for instance, [13, pp. 80-82]), we have
\[
\|G_\varepsilon^{k}(\cdot, y)\|_{L^\infty(\Omega_{r/2}(x_0))} \leq C \left(r^{-n} \|G_\varepsilon^{k}(\cdot, y)\|_{L^1(\Omega_{r}(x_0))} + r^2 \|g_k\|_{L^\infty(\Omega_{r}(x_0))}\right) ,
\]
where \(C = C(n, A_2).\) We remark that if \(B_r(x_0) \cap B_\varepsilon(y) = \emptyset,\) then
\[
\|G_\varepsilon^{k}(\cdot, y)\|_{L^\infty(\Omega_{r/2}(x_0))} \leq C r^{-n} \|G_\varepsilon^{k}(\cdot, y)\|_{L^1(\Omega_{r}(x_0))} .
\] (4.31)
Next, let $y \in \Omega$ and $R \in (0, \text{diam} \Omega)$. Assume that $f \in L^\infty(\Omega)^n$ with $\text{supp} \, f \subset \Omega_{R(y)}$. Let $(u, p) \in W^{1,2}_0(\Omega)^n \times L^2_0(\Omega)$ be the weak solution of the problem

$$
\begin{aligned}
\mathcal{L}u + Dp &= f \quad \text{in } \Omega, \\
\text{div} \, u &= 0 \quad \text{in } \Omega.
\end{aligned}
$$

By (A2), the Sobolev inequality, and (3.3), we have

$$
\|u\|_{L^\infty(\Omega_{R/2}(y))} \leq 2 \left( R^{-n/2} \|u\|_{L^2(\Omega_{R(y)})} + R^2 \|f\|_{L^\infty(\Omega_{R(y)})} \right)
$$

$$
\leq CR^2 \|f\|_{L^\infty(\Omega_{R(y)})},
$$

where $C = C(n, \lambda, K_0, K_1, R_1, A_2)$. Using this together with the fact that (see, for instance, (4.26))

$$
\int_{\Omega_{\varepsilon}(y)} u^k \, dx = \int_{\Omega_{\varepsilon}(y)} G^{ik}_\varepsilon(\cdot, y) f^i \, dx,
$$

we have

$$
\left| \int_{\Omega_{\varepsilon}(y)} G^{ik}_\varepsilon(\cdot, y) f^i \, dx \right| \leq CR^2 \|f\|_{L^\infty(\Omega_{R(y)})}
$$

for all $0 < \varepsilon < R/2$ and $f \in L^\infty(\Omega_{R(y)})^n$. Taking

$$
f^i(x) = 1_{\Omega_{\varepsilon}(y)} \text{sgn}(G^{ik}_\varepsilon(x, y)),
$$

we have

$$
\|G^{ik}_\varepsilon(\cdot, y)\|_{L^1(\Omega_{R(y)})} \leq CR^2, \quad \forall \varepsilon \in (0, R/2). \tag{4.32}
$$

Now we are ready to prove the theorem. Let $x, y \in \Omega$ and $x \neq y$ and take $R = 3r = 3|x - y|/2$. Then by (4.31) and (4.32), we obtain for $\varepsilon \in (0, r)$ that

$$
|G(x, y)| \leq C r^{-n} \|G(x, y)\|_{L^1(\Omega_{\varepsilon}(x))} \leq CR^{-n} \|G(x, y)\|_{L^1(\Omega_{\varepsilon}(y))} \leq CR^{2-n},
$$

where $C = C(n, \lambda, K_0, K_1, R_1, A_2)$. Therefore, by letting $\varepsilon \to 0$ and using (4.28), we obtain that

$$
|G(x, y)| \leq C |x - y|^{2-n}.
$$

The theorem is proved. \qed

4.4. **Proof of Theorem 2.9.** Let $(u, p) \in W^{1,2}_0(\Omega)^n \times L^2_0(\Omega)$ be the weak solution of (2.6). By Corollary 5.3, $u$ is Hölder continuous. To prove the theorem, we first consider the localized estimates for Stokes systems as below.

For $y \in \Omega$ and $r > 0$, we denote $B_r = B_r(y)$ and $\Omega_r = \Omega_r(y)$.

**Step 1.** Let $n/(n-1) < q \leq t$, $0 < \rho < r < \tau$, and $\eta$, $\zeta$ be smooth functions in $\mathbb{R}^n$ satisfying

$$
0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_\rho, \quad \text{supp} \, \eta \subset B_\tau, \quad |D\eta| \leq C(r - \rho)^{-1},
$$

$$
0 \leq \zeta \leq 1, \quad \zeta \equiv 1 \text{ on } B_\rho, \quad \text{supp} \, \zeta \subset B_\tau, \quad |D\zeta| \leq C(\tau - r)^{-1}.
$$

Then $(\eta u, \eta p)$ is the weak solution of the problem

$$
\begin{aligned}
\mathcal{L}(\eta u) + D(\eta p) &= \eta f + pD\eta - \Psi - D\Phi, \quad \text{in } \Omega, \\
\text{div} \, \eta u &= D\eta \cdot u \quad \text{in } \Omega, \\
\eta u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
$$

where

$$
\Psi = A_{\alpha \beta} D\beta u D\alpha \eta \quad \text{and} \quad \Phi = A_{\alpha \beta} D\beta \eta u.
$$
By Corollary 5.3, we have
\[
\|\eta p - (\eta p)\Omega\|_{L^q(\supp \eta; \Omega)} + \|Du\|_{L^q(\Omega)_{\rho}} \\
\leq \frac{C}{r - \rho} \left( \|p\|_{L^{q/(n+q)}(\Omega_{\rho})} + \|Du\|_{L^{q/(n+q)}(\Omega_{\rho})} \right) \\
+ C \left( r^{1+n/q} \|f\|_{L^\infty(\Omega)} + \frac{1}{r - \rho} \|u\|_{L^\infty(\Omega)} \right).
\]
Using the fact that
\[
\|p\|_{L^{q/(n+q)}(\Omega_{\rho})} = \|\zeta p - (\zeta p)\Omega + (\zeta p)\Omega\|_{L^{q/(n+q)}(\Omega_{\rho})} \\
\leq \|\zeta p - (\zeta p)\Omega\|_{L^{q/(n+q)}(\supp \zeta; \Omega)} + C(n, q) r^{1+n/q} |(\zeta p)|,
\]
we have
\[
\|\eta p - (\eta p)\Omega\|_{L^q(\supp \eta; \Omega)} + \|Du\|_{L^q(\Omega)_{\rho}} \\
\leq \frac{C}{r - \rho} \left( \|\zeta p - (\zeta p)\Omega\|_{L^{q/(n+q)}(\supp \zeta; \Omega)} + \|Du\|_{L^{q/(n+q)}(\Omega_{\rho})} \right) \\
+ C \left( r^{1+n/q} \|f\|_{L^\infty(\Omega)} + \frac{1}{r - \rho} \|u\|_{L^\infty(\Omega)} \right),
\]
where \(C\) depends on \(n, \lambda, K_0, R_1, q\), and the VMO modulus of the coefficients.

**Step 2.** Let \(t > n\) and \(0 < \rho < r < \text{diam}\ \Omega\). Set
\[
t_i = \frac{nt}{n + t_i}, \quad r_i = \rho + (r - \rho) i/m, \quad i \in \{0, \ldots, m + 1\},
\]
where \(m\) is the smallest integer such that \(t_m \leq 2\). Let \(\eta_{r_i}, \ i \in \{0, \ldots, m\}\), be smooth functions in \(\mathbb{R}^n\) satisfying
\[
0 \leq \eta_{r_i} \leq 1, \quad \eta_{r_i} \equiv 1 \text{ on } B_{r_i}, \quad \supp \eta_{r_i} \subset B_{r_{i+1}}, \quad |D\eta_{r_i}| \leq \frac{C(n, t)}{r - \rho}.
\]
Applying (4.33) iteratively, we have
\[
\|Du\|_{L^q(\Omega_{\rho})} \\
\leq C_m \left( \frac{m}{r - \rho} \right)^m \left( \|\eta_{r_m} p - (\eta_{r_m} p)\Omega\|_{L^q(\supp \eta_{r_m}; \Omega)} + \|Du\|_{L^q(\Omega_{r_m})} \right) \\
+ \sum_{i=1}^m C_i \left( \frac{m}{r - \rho} \right)^i r^{1+n/t_i-1} |\eta_{r_i} p| \Omega \\
+ \sum_{i=1}^m C_i \left( \frac{m}{r - \rho} \right)^{i-1} \left( r^{1+n/t_{i-1}} \|f\|_{L^\infty(\Omega_{r_i})} + \frac{m}{r - \rho} \|u\|_{L^{t_i-1}(\Omega_{r_i})} \right).
\]
Hence, by Hölder’s inequality we obtain
\[
\|Du\|_{L^q(\Omega_{\rho})} \leq C_0 \left( \frac{r}{r - \rho} \right)^m r^{n(1/t-1/2)} \left( \|p\|_{L^2(\Omega_{\rho})} + \|Du\|_{L^2(\Omega_{\rho})} \right) \\
+ C_0 \left( \frac{r}{r - \rho} \right)^m r^{1+n/t} \|f\|_{L^\infty(\Omega_{\rho})} + C_0 \left( \frac{r}{r - \rho} \right)^m r^{-1} \|u\|_{L^{t}(\Omega_{\rho})},
\]
where \(C_0\) depends on \(n, \lambda, K_0, R_1, \) and the VMO modulus of the coefficients. By taking \(\rho = r/2\), we have
\[
\|Du\|_{L^q(\Omega_{r/2})} \leq C_0 r^{n(1/t-1/2)} \left( \|p\|_{L^2(\Omega_{r/2})} + \|Du\|_{L^2(\Omega_{r/2})} \right) \\
+ C_0 (r^{1+n/t} \|f\|_{L^\infty(\Omega_{r/2})} + r^{-1} \|u\|_{L^{t}(\Omega_{r/2})}).
\]
We apply Caccioppoli’s inequality (see, for instance, [17]) to the above estimate to get
\[
\|Du\|_{L^q(\Omega_{r/4})} \leq C_0 (r^{1+n/q} \|f\|_{L^\infty(\Omega_r)} + r^{-1} \|u\|_{L^q(\Omega_r)}). \tag{4.34}
\]

**Step 3.** We extend \(u\) to \(\mathbb{R}^n\) by setting \(u \equiv 0\) on \(\mathbb{R}^n \setminus \Omega\). For \(y \in \Omega\) and \(0 < r < \text{diam} \Omega\), we obtain by (4.29) and (4.34) that
\[
r^{-1} \|u\|_{L^q(B_{r/4})} + \|Du\|_{L^q(B_{r/4})} \leq C (r^{1+n/q} \|f\|_{L^\infty(\Omega_r)} + r^{-1} \|u\|_{L^q(B_r)}).
\]
Using this together with the Sobolev inequality, we have
\[
\|u\|_{L^\infty(B_{r/4})} \leq C \|f\|_{L^\infty(\Omega_r)} + r^{-n/2} \|u\|_{L^q(\Omega_r)}.
\]

Since the above estimate holds for any \(y \in \Omega\) and \(0 < r < \text{diam} \Omega\), by using a standard argument (see, for instance, [13, pp. 80-82]), we derive
\[
\|u\|_{L^\infty(\Omega_{r/2})} \leq C (r^{2} \|f\|_{L^\infty(\Omega_r)} + r^{-n/2} \|u\|_{L^2(\Omega_r)}).
\]
This completes the proof of Theorem 2.9. \(\square\)

5. **\(L^q\)-estimates for the Stokes systems.** In this section, we consider the \(L^q\)-estimate for the solution to
\[
\begin{cases}
\mathcal{L}u + Dp = f + D_\alpha f_\alpha & \text{in } \Omega, \\
div u = g & \text{in } \Omega.
\end{cases} \tag{5.1}
\]
We let \(\Omega\) be a domain in \(\mathbb{R}^n\), where \(n \geq 2\). We denote
\[
U := |p| + |Du| \quad \text{and} \quad F := |f| + |f_\alpha| + |g|, \tag{5.2}
\]
and we abbreviate \(B_R = B_R(0)\) and \(B_R^+ = B_R^+(0)\), etc.

5.1. **Main results.**

**(A3 (\(\gamma\))).** There is a constant \(R_0 \in (0, 1]\) such that the following hold.

(a) For any \(x \in \overline{\Omega}\) and \(R \in (0, R_0]\) so that either \(B_R(x) \subset \Omega\) or \(x \in \partial \Omega\), we have
\[
\int_{B_R(x)} |A_{\alpha \beta} - (A_{\alpha \beta})_{B_R(x)}| \leq \gamma.
\]

(b) \((\gamma\text{-Reifenberg flat domain})\) For any \(x \in \partial \Omega\) and \(R \in (0, R_0]\), there is a spatial coordinate systems depending on \(x\) and \(R\) such that in this new coordinate system, we have
\[
\{y : x_1 + \gamma R < y_1\} \cap B_R(x) \subset \Omega_R(x) \subset \{y : x_1 - \gamma R < y_1\} \cap B_R(x).
\]

**Theorem 5.1.** Assume the condition \((\text{D})\) in Section 3.1 and \(\text{diam}(\Omega) \leq K_0\). For \(2 < q < \infty\), there exists a constant \(\gamma > 0\), depending only on \(n, \lambda, \gamma, q, q\), such that, under the condition \((\text{A3 (\(\gamma\))})\), the following holds: if \((u, p) \in W^{1, q}_0(\Omega)^n \times L^q(\Omega)\) satisfies (5.1), then we have
\[
\|p\|_{L^q(\Omega)} + \|Du\|_{L^q(\Omega)} \leq C (\|f\|_{L^q(\Omega)} + \|f_\alpha\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}), \tag{5.3}
\]
where \(C = C(n, \lambda, K_0, q, A, R_0)\).

**Remark 5.2.** We remark that \(\gamma\text{-Reifenberg flat domains with a small constant } \gamma > 0 \text{ satisfy the condition (D). Indeed, } \gamma\text{-Reifenberg flat domains with sufficiently small } \gamma \text{ are John domains (and NTA-domains) that satisfy the condition (D). We refer to [2, 1, 19] for the details.}

Since Lipschitz domains with a small Lipschitz constant are Reifenberg flat, we obtain the following result from Theorem 5.1.
Corollary 5.3. Let $\Omega$ be a domain in $\mathbb{R}^n$ with $\text{diam}(\Omega) \leq K_0$, where $n \geq 2$. Assume that the coefficients of $\mathcal{L}$ belong to the class of VMO. For $1 < q < \infty$, there exists a constant $L = L(n, \lambda, q) > 0$ such that, under the condition (A0) with $R_1 \in (0, 1]$ and $K_1 \in (0, L]$, the following holds: if $q_1 \in (1, \infty)$, $q_1 \geq \frac{q_0}{q}$, $f \in L^{q_1}(\Omega)^n$, $f_\alpha \in L^q(\Omega)$, and $g \in L^q_0(\Omega)$, there exists a unique solution $(u, p) \in W^{1,q}_0(\Omega)^n \times L^q_0(\Omega)$ of the problem (5.1). Moreover, we have

$$
\|p\|_{L^q(\Omega)} + \|D u\|_{L^q(\Omega)} \leq C \left( \|f\|_{L^{q_1}(\Omega)} + \|f_\alpha\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)} \right),
$$

where the constant $C$ depends on $n$, $\lambda$, $K_0$, $R_1$, $q$, and the VMO modulus of the coefficients.

Proof. It suffices to prove the corollary with $f = (f^1, \ldots, f^n) = 0$. Indeed, by the solvability of the divergence equation in Lipschitz domains, there exist $\Phi_i \in W^{1,q_1}_0(\Omega)^n$ such that

$$
\text{div} \Phi = f^i - (f^i)_\Omega \quad \text{in} \quad \Omega, \quad \|D \Phi_i\|_{L^{q_1}(\Omega)} \leq C \|f^i\|_{L^{q_1}(\Omega)},
$$

where $C = C(n, \lambda, K_0, R_1, q)$. If we define $\Phi_\alpha = (\Phi^1_\alpha, \ldots, \Phi^n_\alpha)$ by

$$
\Phi^i_\alpha(x) = \varphi^i_\alpha(x) + \frac{(f^i)_\Omega}{n} x_\alpha,
$$

then we have that

$$
\sum_{\alpha=1}^n D_\alpha \Phi_\alpha = f
$$

and

$$
\|\Phi_\alpha\|_{L^q(\Omega)} \leq C \|D \Phi_\alpha\|_{L^{q_1}(\Omega)} \leq C \|f\|_{L^{q_1}(\Omega)}.
$$

Due to Lemma 3.2, it is enough to consider the case $q \neq 2$.

Case 1. $q > 2$. Let $\gamma = \gamma(n, \lambda, q)$ and $M = M(n, q)$ be constants in Theorem 5.1 and [11, Theorem 2.1], respectively. Set $L = \min \{\gamma, M\}$. If $K_1 \in (0, L]$, then by Theorem 5.1, the method of continuity, and the $L^q$-solvability of the Stokes systems with simple coefficients (see [11, Theorem 2.1]), there exists a unique solution $(u, p) \in W^{1,q}_0(\Omega)^n \times L^q_0(\Omega)$ of the problem (5.1) with $f = 0$.

Case 2. $1 < q < 2$. We use the duality argument. Set $q_0 = \frac{q}{q-1}$, and let $L = L(n, \lambda, q_0)$ and $M = M(n, q)$ be constants from Case 1 and [11, Theorem 2.1], respectively. Assume that $K_1 \leq L$ and $(u, p) \in W^{1,q}_0(\Omega)^n \times L^q_0(\Omega)$ satisfies (5.1) with $f = 0$. For $h_\alpha \in L^{q_0}(\Omega)^n$, there exists $(v, \pi) \in W^{1,q_0}_0(\Omega)^n \times L^{q_0}_0(\Omega)$ such that

$$
\begin{aligned}
\mathcal{L}^* v + D \pi &= D_\alpha h_\alpha \quad \text{in} \quad \Omega, \\
\text{div} v &= 0 \quad \text{in} \quad \Omega,
\end{aligned}
$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. Then we have

$$
\int_\Omega D_\alpha u \cdot h_\alpha \, dx = - \int_\Omega A_{\alpha\beta} D_\beta u \cdot D_\alpha v \, dx + \int_\Omega \pi \text{div} u \, dx
$$

$$
= \int_\Omega f_\alpha \cdot D_\alpha v \, dx + \int_\Omega \pi g \, dx,
$$

which implies that

$$
\left| \int_\Omega D_\alpha u \cdot h_\alpha \, dx \right| \leq C \left( \|f_\alpha\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)} \right) \|h_\alpha\|_{L^{q_0}(\Omega)},
$$

where $C$ is a constant depending on $n, \lambda, K_0, R_1, q, q_0$, and $M$. Therefore, the solution $(u, p)$ is unique.
where the constant $C$ depends on $n$, $\lambda$, $K_0$, $R_1$, $q$, and the VMO modulus of the coefficients. Since $h_\alpha$ was arbitrary, it follows that

$$\|Du\|_{L^q(\Omega)} \leq C(\|f_\alpha\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}).$$

(5.4)

To estimate $p$, let $w \in L^q(\Omega)$ and $w_0 = w - (w)_{\Omega}$. Then by Remark 3.1, there exists $\phi \in W^{1,q}_0(\Omega)$ such that

$$\text{div } \phi = w_0 \quad \text{in } \Omega, \quad \|\phi\|_{W^{1,q}(\Omega)} \leq C\|w_0\|_{L^q(\Omega)}.$$

By testing $\phi$ in (5.1), it is easy to see that

$$\left| \int_\Omega pw \, dx \right| = \left| \int_\Omega pw_0 \, dx \right| \leq C \left( \|Du\|_{L^q(\Omega)} + \|f_\alpha\|_{L^q(\Omega)} \right) \|w_0\|_{L^q(\Omega)} \leq C \left( \|Du\|_{L^q(\Omega)} + \|f_\alpha\|_{L^q(\Omega)} \right) \|w\|_{L^q(\Omega)}.$$  

This together with (5.4) yields

$$\|p\|_{L^q(\Omega)} + \|Du\|_{L^q(\Omega)} \leq C(\|f_\alpha\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}).$$

Using the above $L^q$-estimate, the method of continuity, and the $L^q$-solvability of the Stokes systems with simple coefficients, there exists a unique solution $(u, p) \in W^{1,q}_0(\Omega)^n \times L^q_0(\Omega)$ of the problem (5.1) with $f = 0$. \hfill \Box

5.2. Auxiliary results.

**Lemma 5.4.** Recall the notation (5.2). Suppose that the coefficients of $\mathcal{L}$ are constants. Let $k$ be a constant.

(a) If $(u, p) \in W^{1,2}(B_R)^n \times L^2(B_R)$ satisfies

$$\begin{cases} \mathcal{L}u + Dp = 0 & \text{in } B_R, \\ \text{div } u = k & \text{in } B_R, \end{cases}$$

then there exists a constant $C = C(n, \lambda)$ such that

$$\|U\|_{L^\infty(B_{R/2})} \leq CR^{-n/2}\|U\|_{L^2(B_R)} + C|k|. \quad (5.5)$$

(b) If $(u, p) \in W^{1,2}(B_R^+)^n \times L^2(B_R^+)$ satisfies

$$\begin{cases} \mathcal{L}u + Dp = 0 & \text{in } B_R^+, \\ \text{div } u = k & \text{in } B_R^+, \\ u = 0 & \text{on } B_R \cap \{x_1 = 0\}, \end{cases}$$

then there exists a constant $C = C(n, \lambda)$ such that

$$\|U\|_{L^\infty(B_{R/2}^+)} \leq CR^{-n/2}\|U\|_{L^2(B_R^+)} + C|k|. \quad (5.6)$$

**Proof.** The interior and boundary estimates for Stokes systems with variable coefficients were studied by Giaquinta [14]. The proof of the assertion (a) is the same as that of [14, Theorem 1.10, pp. 186–187]. See also the proof of [14, Theorem 2.8, p. 207] for the boundary estimate (5.6). We note that in [14], he gives the complete proofs for the Neumann problem and mentioned that the method works for other boundary value problem. Regarding the Dirichlet problem, we need to impose a normalization condition for $p$ because $(u, p + c)$ satisfies the same system for any constant $c \in \mathbb{R}$. By this reason, the right-hand sides of the estimates (5.5) and (5.6)
Theorem 5.5. Let $2 < \nu < q < \infty$ and $\nu' = 2\nu/(\nu - 2)$. Assume $(u, p) \in W^{1,q}_0(\Omega)^n \times L_0^q(\Omega)$ satisfies

\[
\begin{aligned}
\mathcal{L} u + Dp &= f + D_\alpha f_\alpha \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega,
\end{aligned}
\]

where $f, f_\alpha \in L^2(\Omega)^n$ and $g \in L_0^q(\Omega)$.

(i) Suppose that (A3 (γ)) (a) holds at $0 \in \Omega$ with $\gamma > 0$. Then, for $R \in (0, \min(R_0, d_0)]$, where $d_0 = \text{dist}(0, \partial \Omega)$, $(u, p)$ admits a decomposition

\[ (u, p) = (u_1, p_1) + (u_2, p_2) \quad \text{in } B_R, \]

and we have

\[
\begin{aligned}
(U_{1})_{B_R}^{1/2} &\leq C \left( \gamma'^{1/\nu'} (U')_{B_R}^{1/\nu} + (F^2)_{B_R}^{1/2} \right), \\
\|U_2\|_{L^{\infty}(B_{R/2})} &\leq C \left( \gamma'^{1/\nu'} (U')_{\Omega_R}^{1/\nu} + (U^2)_{\Omega_R}^{1/2} + (F^2)_{\Omega_R}^{1/2} \right),
\end{aligned}
\]

where $C = C(n, \lambda, \nu)$.

(ii) Suppose that (A3 (γ)) (a) and (b) hold at $0 \in \partial \Omega$ with $\gamma \in (0, 1/2)$. Then, for $R \in (0, R_0]$, $(u, p)$ admits a decomposition

\[ (u, p) = (u_1, p_1) + (u_2, p_2) \quad \text{in } \Omega_R, \]

and we have

\[
\begin{aligned}
(U_{1})_{\Omega_R}^{1/2} &\leq C \left( \gamma'^{1/\nu'} (U')_{\Omega_R}^{1/\nu} + (F^2)_{\Omega_R}^{1/2} \right), \\
\|U_2\|_{L^{\infty}(\Omega_{R/4})} &\leq C \left( \gamma'^{1/\nu'} (U')_{\Omega_R}^{1/\nu} + (U^2)_{\Omega_R}^{1/2} + (F^2)_{\Omega_R}^{1/2} \right),
\end{aligned}
\]

where $C = C(n, \lambda, \nu)$.

Here, we define $U_i$ in the same way as $U$ with $p$ and $u$ replaced by $p_i$ and $u_i$, respectively.

Proof. The proof is an adaptation of that of [8, Lemma 8.3]. To prove assertion (i), we denote

\[ \mathcal{L}_0 u = -D_\alpha (A^0_{\alpha\beta} D_\beta u), \]

where $A^0_{\alpha\beta} = (A_{\alpha\beta})_{B_R}$. By Lemma 3.2, there exists a unique solution $(u_1, p_1) \in W^{1,2}_0(B_R)^n \times L_0^2(B_R)$ of the problem

\[
\begin{aligned}
\mathcal{L}_0 u_1 + Dp_1 &= f + D_\alpha f_\alpha + D_\alpha h_\alpha \quad \text{in } B_R, \\
\text{div } u_1 &= g - (g)_{B_R} \quad \text{in } B_R,
\end{aligned}
\]

where

\[ h_\alpha = (A^0_{\alpha\beta} - A_{\alpha\beta}) D_\beta u. \]

We also get from (3.4) that (recall $R \leq R_0 \leq 1$)

\[ \|U_1\|_{L^{2}(B_R)} \leq C \left( \|h_\alpha\|_{L^{2}(B_R)} + \|F\|_{L^{2}(B_R)} \right), \]

where $C = C(n, \lambda)$. Therefore, by using the fact that

\[ \|h_\alpha\|_{L^{2}(B_R)} \leq C \|A^0_{\alpha\beta} - A_{\alpha\beta}\|_{L^1(B_R)} \|Du\|_{L^{\nu}(B_R)} \leq C \gamma^{1/\nu'} \|B_R\|^{1/\nu'} \|Du\|_{L^{\nu}(B_R)}, \]

contain the $L^2$-norm of $p$. For more detailed proof, one may refer to [7]. Their methods are general enough to allow the coefficients to be measurable in one direction and gives more precise information on the dependence of the constant $C$. \hfill \Box
we obtain (5.7). To see (5.8), we note that \((u, p) - (u_1, p_1)\) satisfies
\[
\begin{cases}
\mathcal{L}_0 u_2 + D p_2 = 0 & \text{in } B_R, \\
\text{div } u_2 = (g)_{B_R} & \text{in } B_R.
\end{cases}
\]
Then by Lemma 5.4, we get
\[
||U_2||_{L^\infty(B_{R/2})} \leq C(U_2^2)_{B_R}^{1/2} + C(|g|^2)_{B_R}^{1/2},
\]
and thus, we conclude (5.8) from (5.7).

Next, we prove assertion (ii). Without loss of generality, we may assume that (A3(\gamma)) \((b)\) holds at 0 in the original coordinate system. Define \(\mathcal{L}_0\) as above. Let us fix \(y := (\gamma R, 0, \ldots, 0)\) and denote
\[
B_R^\gamma := B_R \cap \{x_1 > \gamma R\}.
\]
Then we have
\[
B_{R/2} \cap \{x_1 > \gamma R\} \subset B_{R/2}^+(y) \subset B_R^\gamma.
\]
Take a smooth function \(\chi\) defined on \(\mathbb{R}\) such that
\[
\chi(x_1) \equiv 0 \text{ for } x_1 \leq \gamma R, \quad \chi(x_1) \equiv 1 \text{ for } x_1 \geq 2\gamma R, \quad |\chi'| \leq C(\gamma R)^{-1}.
\]
We then find that \((\hat{u}(x), \hat{p}(x)) = (\chi(x_1)u(x), \chi(x_1)p(x))\) satisfies
\[
\begin{cases}
\mathcal{L}_0 \hat{u} + D \hat{p} = \mathcal{F} & \text{in } B_R^\gamma, \\
\text{div } \hat{u} = \mathcal{G} & \text{in } B_R^\gamma, \\
\hat{u} = 0 & \text{on } B_R \cap \{x_1 = \gamma R\},
\end{cases}
\]
where we use the notation \(\mathcal{G} = D\chi \cdot u + \chi g\) and
\[
\mathcal{F} = \chi f + \chi D_\alpha f_\alpha + p D\chi + D_\alpha(\alpha_0^\alpha D_\beta((1 - \chi)u) - (\alpha_0^\alpha - A_{\alpha\beta}) D_\beta u) + (\chi - 1) D_\alpha(A_{\alpha\beta} D_\beta u).
\]
Let \((\hat{u}_1, \hat{p}_1) \in W^{1,2}_0(B_{R/2}^+(y)) \times L^2_0(B_{R/2}^+(y))\) satisfy
\[
\begin{cases}
\mathcal{L}_0 \hat{u}_1 + D \hat{p}_1 = \mathcal{F} & \text{in } B_{R/2}^+(y), \\
\text{div } \hat{u}_1 = \mathcal{G} - (\mathcal{G})_{B_{R/2}^+(y)} & \text{in } B_{R/2}^+(y), \\
\hat{u}_1 = 0 & \text{on } \partial B_{R/2}^+(y).
\end{cases}
\tag{5.11}
\]
Then by testing with \(\hat{u}_1\) in (5.11), we obtain
\[
\int_{B_{R/2}^+(y)} A_{\alpha\beta} D_\beta \hat{u}_1 \cdot D_\alpha \hat{u}_1 \, dx
\]
\[
= \int_{B_{R/2}^+(y)} f \cdot (\chi \hat{u}_1) - f_\alpha \cdot D_\alpha(\chi \hat{u}_1) + p D\chi \cdot \hat{u}_1 \, dx
\]
\[
+ \int_{B_{R/2}^+(y)} -A_{\alpha\beta} D_\beta((1 - \chi)u) \cdot D_\alpha \hat{u}_1 + (A_{\alpha\beta} - A_{\alpha\beta}) D_\beta u \cdot D_\alpha \hat{u}_1 \, dx
\]
\[
+ \int_{B_{R/2}^+(y)} -A_{\alpha\beta} D_\beta u \cdot D_\alpha((\chi - 1)\hat{u}_1) \, dx + \hat{p}_1(D\chi \cdot u + \chi g) \, dx.
\tag{5.12}
\]
Note that
\[
|D\chi(x_1)| + |D(1 - \chi(x_1))| \leq C(x_1 - \gamma R)^{-1}, \quad \forall x_1 > \gamma R.
\]
Therefore, we obtain by Lemma 6.1 that
\[ \|D(\chi \hat{u}_1)\|_{L^2(B_{R/2}^+(y))} + \|D((1 - \chi) \hat{u}_1)\|_{L^2(B_{R/2}^+(y))} \leq C \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))}, \tag{5.13} \]
and hence, we also have
\[ \|D\chi \cdot \hat{u}_1\|_{L^2(B_{R/2}^+(y))} \leq C \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))}. \tag{5.14} \]
From (5.14) and Hölder’s inequality, we get
\[ \int_{B_{R/2}^+(y)} pD\chi \cdot \hat{u}_1 \, dx \leq \|p\|_{L^2(B_{R/2}^+(y))} \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))} \leq C \gamma^{1/\nu'} R^{n/\nu'} \|p\|_{L^\nu(\Omega_R)} \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))}. \tag{5.15} \]
Then, by applying (5.13)–(5.15), and the fact that (recall \( R \leq R_0 \leq 1 \))
\[ \|\hat{u}_1\|_{L^2(B_{R/2}^+(y))} \leq C(n) \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))} \]
to (5.12), we have
\[ \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))} \leq \varepsilon \|\hat{p}_1\|_{L^2(B_{R/2}^+(y))} + C \|F\|_{L^2(\Omega_R)} + C \varepsilon K, \forall \varepsilon > 0, \]
where
\[ K := \gamma^{1/\nu'} R^{n/\nu'} \|p\|_{L^\nu(\Omega_R)} + \|D((1 - \chi)u)\|_{L^2(B_{R/2}^+(y))} + \|Du\|_{L^2(B_{R/2}^+(y))} \cap \{x_1 < 2\gamma R\} + \|(A^0_{\alpha\beta} - A_{\alpha\beta})Du\|_{L^2(B_{R/2}^+(y))}. \]
Similarly, we have
\[ \|\hat{p}_1\|_{L^2(B_{R/2}^+(y))} \leq C \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))} + C \|F\|_{L^2(\Omega_R)} + CK. \]
Therefore, from the above two inequality, we conclude that
\[ \|\hat{p}_1\|_{L^2(B_{R/2}^+(y))} + \|D\hat{u}_1\|_{L^2(B_{R/2}^+(y))} \leq C \|F\|_{L^2(\Omega_R)} + CK, \tag{5.16} \]
where \( C = C(n, \lambda, \nu) \). Now we claim that
\[ \mathcal{K} \leq C \gamma^{1/\nu'} R^{n/\nu'} \|U\|_{L^\nu(\Omega_R)}. \tag{5.17} \]
Observe that by Hölder’s inequality and Lemma 6.2, we have
\[ \|Du\|_{L^2(B_{R/2}^+(y)) \cap \{x_1 < 2\gamma R\}} \leq C(n, \nu) \gamma^{1/\nu'} R^{n/\nu'} \|Du\|_{L^\nu(\Omega_R)}. \tag{5.18} \]
We also have
\[ \|(A^0_{\alpha\beta} - A_{\alpha\beta})Du\|_{L^2(B_{R/2}^+(y))} \leq C \left( \int_{B_R} \left| A^0_{\alpha\beta} - A_{\alpha\beta} \right| \, dx \right)^{1/\nu'} \|Du\|_{L^\nu(\Omega_R)}. \]
where \( C = C(n, \lambda, \nu) \). To estimate \( \|D((1 - \chi)u)\|_{L^2(B_{R/2}^+(y))} \), we recall that \( \chi - 1 = 0 \) for \( x_1 \geq 2\gamma R \). For any \( y' \in B'_R \), let \( \hat{y}_1 = \hat{y}_1(y') \) be the largest number such that \( \hat{y} = (\hat{y}_1, y') \in \partial \Omega \). Since \( |\hat{y}_1| \leq \gamma R \), we have
\[ x_1 - \hat{y}_1 \leq x_1 + \gamma R \leq 3\gamma R, \forall x_1 \in [\gamma R, 2\gamma R], \]
and thus, we obtain
\[ |D\chi(x_1)| \leq C(x_1 - \hat{y}_1), \forall x_1 \in [\gamma R, 2\gamma R]. \]
Therefore, we find that
\[
\int_{\gamma R} |D((1 - \chi)u)(x_1, y')|^2 \, dx_1 \leq \int_{\gamma_1} |D((1 - \chi)u)(x_1, y')|^2 \, dx_1
\]
\[
\leq C \int_{\gamma_1} |Du(x_1, y')|^2 \, dx_1, \tag{5.19}
\]
where \( r = r(y') = \min (2\gamma R, \sqrt{R^2 - |y'|^2}) \). We then get from (5.19) that
\[
\|D((1 - \chi)u)\|_{L^2(B^+_R(y))} \leq C\gamma^{1/\nu'} R^{n/\nu'} \|Du\|_{L^\nu(\Omega_R)},
\]
where \( C = C(n, \nu) \). From the above estimates, we obtain (5.17), and thus, by combining (5.16) and (5.17), we conclude
\[
\|\hat{p}_1\|_{L^2(B^+_R(y))} + \|D\hat{u}_1\|_{L^2(B^+_R(y))} \leq C \left( \gamma^{1/\nu'} R^{n/\nu'} \|U\|_{L^\nu(\Omega_R)} + \|F\|_{L^2(\Omega_R)} \right), \tag{5.20}
\]
where \( C = C(n, \lambda, \nu) \).

Now, we are ready to show the estimate (5.9). We extend \( \hat{u}_1 \) and \( \hat{p}_1 \) to be zero in \( \Omega_R \setminus B^+_R(y) \). Let \((u_1, p_1) = (\tilde{u}_1 + (1 - \chi)u, \tilde{p}_1 + (1 - \chi)p)\). Since \((1 - \chi)u\) vanishes for \( x_1 \geq 2\gamma R \), by using the second inequality in (5.19) and Hölder’s inequality as in (5.18), we see that
\[
\|D((1 - \chi)u)\|_{L^2(B^+_R/2)} \leq C(n) \gamma^{1/\nu'} R^{n/\nu'} \|Du\|_{L^\nu(\Omega_R)}.
\]
Moreover, it follows from Hölder’s inequality that
\[
\|(1 - \chi)p\|_{L^2(B^+_R/2)} \leq C(n) \gamma^{1/\nu'} R^{n/\nu'} \|p\|_{L^\nu(\Omega_R)}.
\]
Therefore, we conclude (5.9) from (5.20).

Next, let us set \((u_2, p_2) = (u, p) - (u_1, p_1)\). Then, it is easily seen that \((u_2, p_2) = (0, 0)\) in \( \Omega_R \setminus B^+_R \) and \((u_2, p_2)\) satisfies
\[
\begin{cases}
\mathcal{L}_0 u_2 + Dp_2 = 0 & \text{in } B^+_R(y), \\
\text{div } u_2 = (\mathcal{G})_{B^+_R(y)} & \text{in } B^+_R/2(y), \\
u_2 = 0 & \text{on } B^{R/2}(y) \cap \{x_1 = \gamma R\}.
\end{cases}
\]
By Lemma 5.4, we get
\[
\|U_2\|_{L^\infty(B^+_R/2)} \leq CR^{-n/2} \left( \|U_2\|_{L^2(\Omega_R)} + \|\mathcal{G}\|_{L^2(B^+_R(y))} \right),
\]
and thus, from (5.18) and (5.9), we obtain (5.10). This completes the proof of the theorem.

Now, we recall the maximal function theorem. Let
\[
\mathcal{R} = \{B_r(x) : x \in \mathbb{R}^n, r \in (0, \infty)\}.
\]
For a function \( f \) on a set \( \Omega \subset \mathbb{R}^n \), we define its maximal function \( M(f) \) by
\[
M(f)(x) = \sup_{B \in \mathcal{R}, x \in B} \int_B |f(y)|1_\Omega \, dy.
\]
Then for \( f \in L^q(\Omega) \) with \( 1 < q \leq \infty \), we have
\[
\|M(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\Omega)},
\]
where \( C = C(n, q) \). As is well known, the above inequality is due to the Hardy-Littlewood maximal function theorem. Hereafter, we use the notation

\[
A(s) = \{ x \in \Omega : U(x) > s \}, \\
B(s) = \{ x \in \Omega : \gamma^{-1/\nu'} (\mathcal{M}(F^2)(x))^{1/2} + (\mathcal{M}(U^\nu')(x))^{1/\nu} > s \}.
\]

With Theorem 5.5 in hand, we get the following corollary.

**Corollary 5.6.** Suppose that (A3(\( \gamma \))) holds with \( \gamma \in (0, 1/2) \), and 0 \( \in \overline{\Omega} \). Let \( 2 < \nu < q < \infty \) and \( \nu' = 2\nu/(\nu - 2) \). Assume \((u, p) \in W_0^1,\delta(\Omega)^n \times L_0^2(\Omega)\) satisfies

\[
\left\{ \begin{array}{l}
\mathcal{L} u + Dp = D_\alpha f + f \quad \text{in } \Omega, \\
\text{div } u = g \quad \text{in } \Omega,
\end{array} \right.
\]

where \( f, f_\alpha \in L^2(\Omega)^n \) and \( g \in L^2(\Omega) \). Then there exists a constant \( \kappa = \kappa(n, \lambda, \nu) > 1 \) such that the following holds: If

\[
|\Omega_{R/32} \cap A(\kappa s)| \geq \gamma^{2/\nu'} |\Omega_{R/32}|, \quad R \in (0, R_0], \quad s > 0,
\]

then we have

\[
\Omega_{R/32} \subset B(s).
\]

**Proof.** By dividing \( U \) and \( F \) by \( s \), we may assume \( s = 1 \). We prove by contradiction. Suppose that there exists a point \( x \in \Omega_{R/32} = B_{R/32}(0) \cap \Omega \) such that

\[
\gamma^{-1/\nu'} (\mathcal{M}(F^2)(x))^{1/2} + (\mathcal{M}(U^\nu')(x))^{1/\nu} \leq 1.
\]

(5.22)

In the case when \( \text{dist}(0, \partial \Omega) \geq R/8 \), we note that

\[
x \in B_{R/32} \subset B_{R/8} \subset \Omega.
\]

Due to Theorem 5.5 (i), we can decompose \((u, p) = (u_1, p_1) + (u_2, p_2)\) in \( B_{R/8} \) and then, by (5.22), we have

\[
(U_1^2)^{1/2}_{B_{R/8}} \leq C_0 (\gamma^{1/\nu'} (U^\nu')^{1/\nu}_{B_{R/8}} + (F^2)^{1/2}_{B_{R/8}}) \leq C_0 \gamma^{1/\nu'}
\]

and

\[
||U_2||_{L^\infty(B_{R/8})} \leq C_0 (\gamma^{1/\nu'} (U^\nu')^{1/\nu}_{B_{R/8}} + (U^2)^{1/2}_{B_{R/8}} + (F^2)^{1/2}_{B_{R/8}}) \leq C_0,
\]

where \( C_0 = C_0(n, \lambda, \nu) \). From these inequalities and Chebyshev’s inequality, we get

\[
|B_{R/32} \cap A(\kappa)| = |\{ x \in B_{R/32} : U(x) > \kappa \}| \\
\leq |\{ x \in B_{R/32} : U_1 > \kappa - C_0 \}| \\
\leq C(n) \frac{C_0^2}{(\kappa - C_0)^2} \gamma^{2/\nu'} |B_{R/32}|,
\]

(5.23)

which contradicts with (5.21) if we choose \( \kappa \) sufficiently large.

We now consider the case \( \text{dist}(0, \partial \Omega) < R/8 \). Let \( y \in \partial \Omega \) satisfy \( |y| = \text{dist}(0, \partial \Omega) \). Then we have

\[
x \in \Omega_{R/32} \subset B_{R/4}(y).
\]

By Theorem 5.5 (ii), we can decompose \((u, p) = (u_1, p_1) + (u_2, p_2)\) in \( \Omega_{R}(y) \) and then, by (5.22), we have

\[
(U_1^2)|_{\Omega_{R}(y)} \leq C_0 \gamma^{1/\nu'} \quad \text{and} \quad ||U_2||_{L^\infty(\Omega_{R/4}(y))} \leq C_0.
\]

From this, and by following the same steps used in deriving (5.23), we get

\[
|\Omega_{R/32} \cap A(\kappa)| \leq C(n) \frac{C_0^2}{(\kappa - C_0)^2} \gamma^{2/\nu'} |\Omega_{R/32}|,
\]

(5.24)
where \(\gamma\) is a constant to be chosen later and \(\kappa = \kappa(n, \lambda, \nu)\) be the constant in Corollary 5.6. Since
\[
|A(ks)| \leq C_0(ks)^{-1}\|U\|_{L^2(\Omega)}
\]
for all \(s > 0\), where \(C_0 = C_0(n, K_0)\), we get
\[
|A(ks)| \leq \gamma^{2/\nu'}|B_{R_0/32}|,
\]
provided that
\[
s \geq \frac{C_0}{k \gamma^{2/\nu'}|B_{R_0/32}|}\|U\|_{L^2(\Omega)} := s_0.
\]
Therefore, from (5.24), Corollary 5.6, and Lemma 6.3, we have the following upper bound of the distribution of \(U\);
\[
|A(ks)| \leq C_1\gamma^{2/\nu'}|B(\gamma)| \quad \forall s > s_0,
\]
where \(C_1 = C_1(n)\). Using this together with the fact that
\[
|A(ks)| \leq (\kappa s)^{-2}\|U\|_{L^2(\Omega)}^2, \quad \forall s > 0,
\]
we have
\[
\|U\|_{L^q(\Omega)}^q = q \int_0^\infty |A(s)|s^{q-1}ds = q\kappa^q \int_0^\infty |A(ks)|s^{q-1}ds \\
= q\kappa^q \int_0^{s_0} |A(ks)|s^{q-1}ds + q\kappa^q \int_{s_0}^\infty |A(ks)|s^{q-1}ds \\
\leq C_2\gamma^{2(2-q)/\nu'}\|U\|_{L^2(\Omega)}^q + C_3\gamma^{2/\nu'}\int_0^\infty |B(s)|s^{q-1}ds,
\]
where \(C_2 = C_2(n, \lambda, K_0, q, R_0)\) and \(C_3 = C_3(n, \lambda, q)\). The Hardy-Littlewood maximal function theorem implies that
\[
\|U\|_{L^q(\Omega)}^q \leq C_4\gamma^{2(2-q)/\nu'}\|U\|_{L^2(\Omega)}^q + C_4\gamma^{2/\nu'}\|U\|_{L^q(\Omega)}^q,
\]
where \(C_4 = C_4(n, \lambda, q)\). Notice from Lemma 3.2 and Hölder’s inequality that
\[
\|U\|_{L^q(\Omega)}^q \leq C_5\|F\|_{L^q(\Omega)}^q,
\]
where \(C_5 = C_5(n, \lambda, K_0, q, A)\). Combining the above two estimates and taking \(\gamma = \gamma(n, \lambda, q) \in (0, 1/2)\) sufficiently small, we conclude (5.3).

\[\square\]

6. Appendix. In this section, we provide some lemmas.

**Lemma 6.1.** Let \(f \in W_0^{1,2}(I)\), where \(I = (0, R)\). Then we have
\[
\|x^{-1}f(x)\|_{L^2(I)} \leq C\|Df\|_{L^2(I)},
\]
where \(C > 0\) is a constant.

**Proof.** We first note that (6.1) holds for any \(f \in C^\infty([0, R])\) satisfying \(Df(0) = 0\); see [8, Lemma 7.9]. Suppose that \(f \in W_0^{1,2}(I)\) and \(\{f_n\}\) is a sequence in \(C_0^\infty([0, R])\) such that \(f_n \to f\) in \(W_1^{1,2}(I)\). Then by the Sobolev embedding theorem, \(f_n \to f\)
in $C([0,R])$. Since the estimates (6.1) is valid for $f_n$, we obtain by Fatou’s lemma that
\[
\int_0^R |x^{-1}f(x)|^2 \, dx = \int_0^R \lim_{n \to \infty} |x^{-1}f_n(x)|^2 \, dx \\
\leq \liminf_{n \to \infty} \int_0^R |x^{-1}f_n(x)|^2 \, dx \\
\leq C \liminf_{n \to \infty} \int_0^R |Df_n(x)|^2 \, dx = \int_0^R |Df(x)|^2 \, dx,
\]
which establishes (6.1).

**Lemma 6.2.** Suppose that $(A3(\gamma)) \ (b)$ holds at $0 \in \partial \Omega$ with $\gamma \in \left(0, \frac{1}{2}\right)$. Then for $R \in (0,R_0]$, we have
\[
|\Omega_R| \geq CR^n,
\]
and
\[
|\Omega_R \cap \{x : x_1 < 2\gamma R\}| \leq C\gamma|\Omega_R|,
\]
where $C = C(n)$.

**Proof.** Note that
\[
|\Omega_R \cap \{x : x_1 < 2\gamma R\}| \leq 2^n \gamma R^n.
\]
Let us fix $a \in \left(\frac{1}{2}, 1\right)$ and
\[
Q = \left\{x : |x_1| < aR, |x_i| < \sqrt{\frac{1-a^2}{d-1}}R, \ i = 2, \ldots, n\right\}.
\]
Then we have
\[
Q \cap \{x : x_1 > R/2\} \subset \Omega_R,
\]
and hence, we obtain
\[
\left(\frac{a-1}{2}\right)^{(n-1)/2} \frac{(1-a^2)^{(n-1)/2}}{n-1} R^n = |Q \cap \{x : x_1 > R/2\}| \leq |\Omega_R|,
\]
which implies (6.2). By combining (6.2) and (6.4), we get (6.3).

The following lemma is a result from the measure theory on the “crawling of ink spots” which can be found in [20, 26]. See also [3].

**Lemma 6.3.** Suppose that $(A3(\gamma)) \ (b)$ holds with $\gamma \in \left(0, \frac{1}{2}\right)$. Let $A$ and $B$ are measurable sets satisfying $A \subset B \subset \Omega$, and that there exists a constant $\varepsilon \in (0,1)$ such that the following hold:

(i) $|A| < \varepsilon|B_{R_0/32}|.$

(ii) For any $x \in \overline{\Omega}$ and for all $R \in (0,R_0/32]$ with $|B_R(x) \cap A| \geq \varepsilon|B_R|$, we have $\Omega_R(x) \subset B$.

Then we get
\[
|A| \leq C\varepsilon|B|,
\]
where $C = C(n)$. 
Proof. We first claim that for a.e. $x \in A$, there exists $R_x \in (0, R_0/32)$ such that

$$|A \cap B_{R_x}(x)| = \varepsilon |B_{R_x}|$$

and

$$|A \cap B_R(x)| < \varepsilon |B_R|, \quad \forall R \in (R_x, R_0/32). \quad (6.5)$$

Note that the function $\rho = \rho(r)$ given by

$$\rho(r) = \frac{|A \cap B_r(x)|}{|B_r|} = \int_{B_r(x)} 1_A(y) \, dy$$

is continuous on $[0, R_0]$. Since $\rho(0) = 1$ and $\rho(R_0/32) < \varepsilon$, there exists $r_x \in (0, R_0/32)$ such that $\rho(r_x) = \varepsilon$. Then we get the claim by setting

$$R_x := \max\{r_x \in (0, R_0) : \rho(r_x) = \varepsilon\}.$$ 

Hereafter, we denote by

$$\mathcal{U} = \{B_{R_x}(x) : x \in A'\},$$

where $A'$ is the set of all points $x \in A$ such that $r_x$ exists. Then by the Vitali lemma, we have a countable subcollection $G$ such that

(a) $Q \cap Q' = \emptyset$ for any $Q, Q' \in G$ satisfying $Q \neq Q'$.
(b) $A' \subset \bigcup\{B_{5R}(x) : B_R(x) \in G\}$.
(c) $|A| = |A'| \leq 5^n \sum_{Q \in G} |Q|.$

By the assumption (i) and (6.5), we see that

$$|A \cap B_{5R}(x)| < \varepsilon |B_{5R}| = \varepsilon 5^n |B_{R}|, \quad \forall B_{R}(x) \in G.$$ 

Using this together with the assumption (ii) and Lemma 6.2, we have

$$|A| = \big| \bigcup\{B_{5R}(x) \cap A : B_R(x) \in G\} \big| \leq \sum_{B_R(x) \in G} |B_{5R}(x) \cap A|$$

$$< \varepsilon 5^n \sum_{B_R(x) \in G} |B_R(x)| \leq \varepsilon C(n) \sum_{B_R(x) \in G} |B_R(x) \cap \Omega|$$

$$= \varepsilon C(n) \big| \bigcup\{B_R(x) \cap \Omega : B_R(x) \in G\} \big|$$

$$\leq \varepsilon C(n) |B|,$$

which completes the proof. \qed

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REFERENCES

[1] Gabriel Acosta, Ricardo G. Durán and María A. Muschietti, Solutions of the divergence operator on John domains, Adv. Math., 206 (2006), 373–401.
[2] Hiroaki Aikawa, Martin Boundary and Boundary Harnack Principle for Non-smooth Domains, Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 2005.
[3] Sun-Sig Byun and Lihe Wang, Elliptic equations with BMO coefficients in Reifenberg domains, Comm. Pure Appl. Math., 57 (2004), 1283–1310.
[4] TongKeun Chang and Hi Jun Choe, Estimates of the Green’s functions for the elasto-static equations and Stokes equations in a three dimensional Lipschitz domain, Potential Anal., 30 (2009), 85–99.
[5] Jongkeun Choi and Seick Kim, Neumann functions for second order elliptic systems with measurable coefficients, Trans. Amer. Math. Soc., 365 (2013), 6283–6307.
[6] Georg Dolzmann and Stefan. Müller, Estimates for Green’s matrices of elliptic systems by $L^p$ theory, Manuscripta Math., 88 (1995), 261–273.
Hongjie Dong and Doyoon Kim, *Lq*-estimates for stationary Stokes system with coefficients measurable in one direction, *arXiv:1604.02690v2*.

Hongjie Dong and Doyoon Kim, Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains, *J. Funct. Anal.*, 261 (2011), 3279–3327.

Hongjie Dong and Doyoon Kim, The Conormal Derivative Problem for Higher Order Elliptic Systems with Irregular Coefficients, volume 581 of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, 2012.

Martin Fuchs, The Green matrix for strongly elliptic systems of second order with continuous coefficients, *Z. Anal. Anwendungen*, 5 (1986), 507–531.

Giovanni Paolo Galdi, Christian G. Simader and Hermann Sohr, On the Stokes problem in Lipschitz domains, *Ann. Mat. Pura Appl.*, 167 (1994), 147–163.

Mariano Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems, volume 105 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 1983.

Mariano Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

Mariano Giaquinta and Giuseppe Modica, Nonlinear systems of the type of the stationary Navier-Stokes system, *J. Reine Angew. Math.*, 330 (1992), 173–214.

Michael Grüter and Kjell-Ove Widman, The Green function for uniformly elliptic equations, *Manuscripta Math.*, 37 (1982), 303–342.

Steve Hofmann and Seick Kim, The Green function estimates for strongly elliptic systems of second order, *Manuscripta math.*, 124 (2007), 139–172.

Kyungkeun Kang, On regularity of stationary Stokes and Navier-Stokes equations near boundary, *J. Math. Fluid Mech.*, 6 (2004), 78–101.

Kyungkeun Kang and Seick Kim, Global pointwise estimates for Green’s matrix of second order elliptic systems, *J. Differential Equations*, 249 (2010), 2643–2662.

Carlos E. Kenig and Tatiana Toro, Harmonic measure on locally flat domains, *Duke Math. J.*, 87 (1997), 509–551.

Nicola V. Krylov and Mikhail V. Safonov, A property of the solutions of parabolic equations with measurable coefficients, *Izv. Akad. Nauk SSSR Ser. Mat.*, 44 (1980), 161–175.

Walter Littman, Guido Stampacchia and Hans F. Weinberger, Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup. Pisa*, 17 (1963), 43–77.

Vladimir Gilelevich Maz'ya and Jürgen Rossmann, *Lp* estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains, *Math. Nachr.*, 280 (2007), 751–793.

Dorina Mitrea and Irina Mitrea, On the regularity of Green functions in Lipschitz domains, *Comm. Partial Differential Equations*, 36 (2011), 304–327.

Marius Mitrea and Matthew Wright, Boundary value problems for the Stokes system in arbitrary Lipschitz domains, *Astérisque*, 344 (2012), viii+241.

Katharine A. Ott, Seick Kim and Russell Murray Brown, The Green function for the mixed problem for the linear Stokes system in domains in the plane, *Math. Nachr.*, 288 (2015), 452–464.

Mikhail V. Safonov, Harnack’s inequality for elliptic equations and Hölder property of their solutions, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 96 (1980), 272–287.

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