Theory of electron motion in combined magnetic trap under conditions of gyromagnetic autoresonance

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Abstract. Using the Bogolyubov method, the solution of equations of electron motion in a combined mirror-type magnetic trap is obtained in cases when the electron approaches the cyclotron resonance region and when it passes through it. It is assumed that the synchronism between particle and the resonator standing wave is maintained by means of slow time variation of the guiding magnetic field. The problem is solved within the framework of the simplified model using the weakly relativistic approximation.

1. Introduction
Among various mechanisms of interaction between charged plasma particles and electromagnetic waves, the cyclotron autoresonance mechanism is special [1–8]. In this case, the initial condition of the particle-wave cyclotron resonance is automatically preserved during the entire period of particle motion, since this condition coincides with the integral of motion [8]. However, the electron motion in the cyclotron autoresonance regime can occur only under rather strict conditions, which, in fact, are not always fulfilled. Thereby, the autoresonance regime of motion is now understood to wider extent [9–11]: the “autoresonance” conditions are identified with the Veksler-McMillan “auto-phasing” conditions. When plasma electrons move in a combined magnetic trap, the cyclotron resonance is determined by the following condition:

\[ \frac{\Omega}{\gamma} = 1. \tag{1} \]

Here, \( \Omega \equiv \Omega(z, t) = eB(z, t)/mc\omega \) is the dimensionless cyclotron frequency, \( m, e > 0 \) are the electron mass and charge, respectively, \( c \) is the speed of light in vacuum, \( B(z, t) \) is the magnetic field at the trap axis (z axis), \( \omega \) is the standing wave frequency in the resonator, and \( \gamma \) is the relativistic factor (electron energy in units of the rest energy \( mc^2 \)).

Condition (1) is valid at the initial time, but it will obviously become violated during particle motion. However, it is possible to maintain the particle-wave synchronism by means of providing the corresponding variation of the guiding magnetic field. In this case, the changes in the magnetic field should strictly correspond to the relativistic changes in the particle energy. However, it is almost impossible to find the exact law describing the synchronous changes in the magnetic field, since the dependence of the \( \gamma \) energy, which is the solution of the equations of particle motion, on the \( \Omega \) frequency is very complicated. It is also almost impossible to realize the synchronizing profile of the magnetic field in experiment. Therefore, it is natural to study the particle motion in the magnetic field given a priori. It was shown in [11] that, when electron moves in both the high frequency (HF) field of the standing wave and the uniform magnetic field with amplitude slowly increasing with time, if the cyclotron resonance condition is fulfilled at the initial time, then, on average, the particle–wave
synchronism will be maintained. It was found that, in the weakly relativistic approximation, the electron energy oscillates around the mean value, which increases with time. Energy oscillations are associated with the fact that, as electron moves, the cyclotron resonance condition becomes violated [2], and energy increases due to a quasi-synchronous increase in the guiding magnetic field. This regime of motion is called the "gyromagnetic autoresonance" (GA) regime [11]. It was implemented in experiments at the facilities of the combined magnetic trap type [12–14]. The physical processes occurring in the GA regime were analyzed theoretically using the numerical methods [13, 14]. In [11], the guiding magnetic field was considered to be homogeneous. In this paper, the electron motion is calculated in the combined magnetic mirror-type trap, in which the synchronous regime is maintained in a certain region of the trap by slowly varying the guiding magnetic field with time. Using the Bogolyubov method [15], within the framework of the simplified model, in the weakly relativistic approximation, the law of transverse electron motion and its transverse velocity vector is calculated in cases, when electron approaches the cyclotron resonance region and passes through it.

2. Basic equations

The magnetic field $\mathbf{B}_0$ of the combined mirror-type trap is considered in the paraxial approximation, the field axis is directed along the $z$ axis, and it is assumed that it also slowly varies with time:

$$
\mathbf{B}_0 = \left\{ -\frac{x \partial B}{2 \partial x}, -\frac{y \partial B}{2 \partial y}, B(z, t) \right\}.
$$

We represent the real HF field of the standing wave resonator $E, B$ in a simplified form:

$$
E = \{ E \sin k z \cos \omega t, E \sin k z \sin \omega t, 0 \},
$$

$$
B = \{ -E \cos k z \cos \omega t, -E \cos k z \sin \omega t, 0 \},
$$

where $E, k, \omega$ are the amplitude, wave number and wave frequency, respectively. Fields given by formulas (3) and (4), approximating the field in the resonator, are the exact solutions of the Maxwell equations. We introduce the dimensionless coordinates $(X, Y, Z) = k(x, y, z)$ and dimensionless time $\tau = \omega t$, as well as new complex variables $\zeta = X + iY$ and $\zeta = X + iY$; here, dot indicates differentiation with respect to dimensionless time $\tau$. Then the equations of electron motion can be represented in the following form:

$$
\ddot{\zeta} = -\frac{\gamma}{\omega} (\sin Z - i\dot{Z} \cos Z) e^{i \tau} + \frac{1}{\omega} \left( \Omega \dot{\zeta} + \frac{\dot{\zeta}}{\gamma} \right),
$$

$$
\dot{Z} = i \frac{\omega}{2 \gamma} \cos Z \left( \dot{\zeta} e^{-i \tau} - \dot{\zeta}^* e^{i \tau} \right) + \frac{i \omega}{4 \gamma} \cos Z \left( \zeta e^{i \tau} - \zeta^* e^{-i \tau} \right) + \frac{\gamma}{2 \omega} \dot{Z} \sin Z \left( \dot{\zeta} e^{-i \tau} + \dot{\zeta}^* e^{i \tau} \right)
$$

Here, $g = eE/m\omega$ is the dimensionless parameter determined by the electric field strength in the resonator, $\Omega' \equiv \frac{\partial \Omega(Z, \tau)}{\partial Z}$, symbol $(...)^*$ marks complex conjugate quantity, and $\gamma = \left\{ 1 - |\zeta|^2 - Z^2 \right\}^{-1/2}$ is the relativistic factor. Next, we introduce the following change of variables:

$$
\zeta = \eta \exp \left\{ i \int_{\tau_0}^{\tau} dt \frac{\partial a(Z(\tau'), \tau)}{2 \gamma(Z(\tau'))} \right\},
$$

$$
\dot{\eta} + \left\{ \left( \frac{\partial}{\partial \tau} \right)^2 \eta \right\} \left( \frac{\partial}{\partial \tau} \right) \dot{\eta} - \frac{\dot{\eta}}{\gamma} \left( \sin Z - i\dot{Z} \cos Z \right) e^{i \varphi} + \frac{\gamma}{2 \omega} \sin Z \left\{ \left| \dot{\eta} \right|^2 + i \frac{\partial}{\partial \tau} \dot{\eta} \right\} e^{-i \varphi} + \left[ \left| \dot{\eta} \right|^2 + \left( \frac{\partial}{\partial \tau} \right)^2 \left| \eta \right|^2 \right] e^{i \varphi}
$$

Here, the phase is expressed as follows:

$$
\varphi(\tau) = \tau - \int_{\tau_0}^{\tau} dt \frac{\partial a(Z(\tau'), \tau)}{2 \gamma(Z(\tau'))} \equiv \tau - \zeta(\tau).
$$

Equation (8) has the form of the oscillator equation with a slowly varying complex frequency, and its right-hand side is the driving force. The expression in the second line is a consequence of the relativistic electron motion. In contrast to the nonrelativistic motion considered in [16], in the relativistic case, Eq. (8) describes the forced oscillations of a system of two coupled oscillators.
3. Separation of “fast” and “slow” motion

We assume that $\Omega(Z, \tau)$, $Z$ and $\dot{Z}$ are the “slowly” varying quantities, that is, the particle, making the fast transverse vibrations, slowly moves along the slowly varying magnetic field (compared to these oscillations). In this case, $\tau$ is the “fast” time, and $\theta = g \tau$ is the “slow” time; so, the cyclotron frequency $\Omega$ and the instantaneous frequency $v$ of the driving force are functions of the slow time:

$$\Omega(Z(\theta), \theta), \quad v(\theta) = \frac{d\phi}{d\tau} = 1 - \Omega/2\gamma \cong 1 - \Omega (1 - W)/2 \quad (10)$$

Here, $Z = Z(\theta)$, $\dot{Z} = g \frac{dz}{d\theta}$ and $\frac{d\theta}{d\tau} = g \frac{d\theta}{d\theta}$. In relation (10), we took into account that, in the weakly relativistic approximation, $\gamma \cong 1 + W$, where $W = |\xi^2|/2 + \dot{Z}^2/2$ is the particle kinetic energy. Assuming that $\eta = \eta_1 + \eta_2$, from Eq. (8), we obtain the set of equations for two weakly coupled oscillators with slowly varying frequency, which move under the effect of a periodic force. It follows from these equations that the oscillator coupling occurs due to both the relativistic effects and the fact that the magnetic field of the mirror trap varies with time. In the weakly relativistic approximation, this coupling is mainly determined by the following expression:

$$\frac{g}{\gamma} \frac{d\theta}{d\theta} - \frac{a}{\gamma^2} \dot{\gamma}. \quad (11)$$

As already noted, of itself, the condition of cyclotron resonance (1) cannot be satisfied during the entire time of the electron motion. According to condition (1), in order to “force” maintaining the particle-wave resonance conditions, it is required that

$$\frac{d}{d\tau} \frac{\dot{\gamma}}{\gamma} = \frac{1}{\gamma} \left(\frac{d\theta}{d\tau} + \frac{dz}{d\dot{\theta}}\right) - \frac{a}{\gamma^2} \dot{\gamma} \cong 0. \quad (12)$$

This is a condition for maintaining the "autoresonance", or more precisely, the synchronous regime of particle motion provided by the corresponding space and time variations of the mirror magnetic field profile. Comparing formulas (11) and (12), it is easy to see that expression (11) can be considered as the condition for the “forced” maintenance of the particle-wave resonance in the interaction region, where the spatial changes in the mirror magnetic field can be neglected. This is possible, if the velocity of the magnetic field convective transport along the axis is much less than the rate of the field time-changes. Thus, solving the equations of electron motion (8) in the synchronous regime, we can believe that the infinitesimal order of expression (11) is at least $g^2$. Thus, in the first approximation, we can consider the oscillators in Eq. (8) as the independent oscillators with weak coupling, which will manifest itself only in the second (or the next) approximation.

4. Solution of equations of transverse motion for a particle passing through the resonance region

When studying particle motion both in the resonance and near-resonance regions (when the particle approaches the resonance region), according to the Bogolyubov method [15], the solution of Eq. (8) for the $\eta_1$ component should be tried in the following form:

$$\eta_1 = a_1 \cos \left(\frac{p}{q} \phi + \psi_1\right) + g U_{11} (\theta, a_1, \phi, \frac{p}{q} \phi + \psi_1) + g^2 U_{12} (\theta, a_1, \phi, \frac{p}{q} \phi + \psi_1) + \cdots \quad (13)$$

Here, the $a_1$ amplitude and the $\psi_1$ phase are functions of time and can be determined from the following set of differential equations:

$$\frac{da_1}{d\tau} = g A_{11} (\theta, a_1, \psi_1) + g^2 A_{12} (\theta, a_1, \psi_1) + \cdots, \quad (14)$$

$$\frac{d\psi_1}{d\tau} = \frac{\Omega}{2\gamma} - \frac{p}{q} v (\tau) + g B_{11} (\theta, a_1, \psi_1) + g^2 B_{12} (\theta, a_1, \psi_1) + \cdots$$

The frequency $v (\theta)$ is given by formula (10), and $p$ and $q$ are some prime numbers. In a similar way, we can try solution of the equation for the $\eta_2$ component.

Let us consider the scheme for solving Eq. (8) for the $\eta_1$ component in the first approximation in the case, when particle passes through the resonance region defined by condition (12). After differentiating expression (13) with allowance for expressions (14) and (10), equating terms of the same infinitesimal order and using the principle of harmonic balance, from Eq. (8), in the case of the main resonance ($p = q = 1$), we obtain the following equations determining the $A_{11}$ and $B_{11}$ quantities for the set of equations (14):

$$...$$
\[ a_1 \frac{d}{\gamma} B_{11} - \frac{\partial A_{11}}{\partial \psi_1} \left( \frac{n}{\gamma} - 1 \right) = \frac{1}{\gamma} \sin Z \cos \psi_1, \quad \text{(15)} \]

\[ A_{11} \frac{d}{\gamma} + \frac{a_1 d A_{11}}{2 \frac{d}{\gamma} + a_1 \frac{\partial B_{11}}{\partial \psi_1} \left( \frac{n}{\gamma} - 1 \right) = \frac{1}{\gamma} \sin Z \sin \psi_1. \]

In this case, the \( U_{11} \) functions are zero: \( U_{11} = 0 \). Set of equations (15) has the following solution:

\[ A_{11} = \gamma^{-1} \sin Z \sin \psi_1 - a_1 \frac{d}{\gamma} + (g/\gamma) \sin Z \sin \psi_1, \quad B_{11} = (1/a_1 \gamma) \sin Z \cos \psi_1. \quad \text{(16)} \]

Thus, the set of equations (14) takes the form:

\[ \frac{da_1}{d\tau} = -a_1 \frac{d}{\gamma} + (g/\gamma) \sin Z \sin \psi_1, \quad \frac{d\psi_1}{d\tau} = \frac{a}{\gamma} - 1 + (g/a_1 \gamma) \sin Z \cos \psi_1. \quad \text{(17)} \]

Using the substitution

\[ L_1 = -ia_1 \exp(i \psi_1), \]

we can find the solution in the following form:

\[ L_1(\tau) = \frac{\gamma}{\gamma_0} \exp \left( \gamma \int_{\tau_0}^{\tau} \gamma^{-1/2}(\tau) \sqrt{\Omega(\tau)/\gamma_0} \sin Z(\tau) \cdot \exp \left[ -i \int_{\tau_0}^{\tau} \gamma^{-1/2}(\tau) \right. \right. \]

so that

\[ \int_{\tau_0}^{\tau} \gamma^{-1/2}(\tau) \left. \right]} \exp \left[ i \int_{\tau_0}^{\tau} \gamma^{-1/2}(\tau) \right] \}

In the case of the main resonance, expression (18) makes it possible to find the solution of Eq. (8) in the first approximation: \( \eta_1 = a_1 \cos (\varphi + \psi_1) \). Similarly, we find the solution of Eq. (8) for the other component: \( \eta_2 = a_2 \cos (\varphi + \psi_2) \). Carrying out rather cumbersome calculations and taking into account the initial conditions, we find the law of transverse motion for the particle passing through the resonance region:

\[ \zeta(\tau) = \sqrt{\Omega_0} \gamma(\tau)/\gamma_0 \Omega(\tau) \right] \int_{\tau_0}^{\tau} \gamma^{-1/2}(\tau) \right] \frac{d}{\gamma} + (g/\gamma) \sin Z(\tau) \cdot \exp \left[ -i \int_{\tau_0}^{\tau} \gamma^{-1/2}(\tau) \right. \]

as well as the expression for the particle complex transverse velocity:

\[ \dot{\zeta}(\tau) = \frac{\gamma_0 \sqrt{\gamma} \Omega(\tau)/\gamma_0 \gamma(\tau) \right] \exp \left( i \left[ \int_{\tau_0}^{\tau} \gamma^{-1/2}(\tau) \right. \right. \]

In the approximation under consideration, the particle kinetic energy is determined by the following formula:

\[ W = \frac{K^2}{z} = \frac{\gamma_0}{\gamma_0 \gamma_0} W_0 = g \frac{\gamma}{\gamma} \frac{d}{\gamma} \left[ \dot{\zeta}_0 G \cdot \exp(2i \varphi) + \dot{\zeta}_0 \cdot \exp(-2i \varphi) \right] + g^2 \frac{\gamma}{\gamma} |G|^2. \quad \text{(22)} \]

If the initial transverse particle velocity has random distribution, then the changes in the electron energy will be determined by integral (21). To calculate integral (21), it is necessary to know the law of the magnetic field variation along the trap axis \( \Omega(\tau, Z(\tau)) \), as well as the law of particle motion \( Z(\tau) \).

Let us estimate energy acquired by electron as it passes through the resonance region. We suppose that electron enters the resonance region at time \( \tau_0 \equiv \tau_r \), so that \( \Omega_0/\gamma_0 = 1 \). If the resonance is maintained during the time interval \( \Delta_\tau = \tau - \tau_r \), then, within this time interval, relation (1) will be true: \( \Omega/\gamma \equiv \Omega_0/\gamma_0 = 1 \). Next, we suppose that electron moves uniformly at a certain mean velocity \( \beta_z \), so that \( Z(\tau) = Z(\tau_r) + \beta_z (\tau - \tau_r) \). Assuming that the resonance region is considerably small, we will use the following expression: \( \sin Z(\tau) \equiv \sin Z(\tau_r) + \beta_z (\tau - \tau_r) \cos Z(\tau_r) \). If, at time of entering the resonance region, the transverse velocity components are equal to zero on average, then the energy acquired by electron during the resonance interaction with the resonator field will be determined by the following approximate formula

\[ W_r \equiv (g^2/2)(\Delta_r \sin Z_r + (\beta_z \Delta_{\tau}^2 / 2) \cos Z_r)^2. \quad \text{(23)} \]

Here \( \Delta_r \) is the time interval, during which the guiding magnetic field increases in order to maintain the synchronism between electron and the HF resonator field. It can be seen that the acquired energy strongly depends on the electron location on the resonator axis at time of entering the resonance region.
5. Longitudinal motion of particle

The longitudinal motion of electron is described by Eq. (6), which, in the first approximation, takes the following form

\[\ddot{Z} = \frac{ig}{2\gamma} \sqrt{\frac{\gamma_0 \Omega(\tau)}{a_0 \gamma(\tau)}} \cos Z \left[ \hat{\xi}_0 e^{-i(\tau-2\xi)} - \hat{\xi}_0^* e^{i(\tau-2\xi)} \right] + \]
\[+ \frac{\alpha_1}{4\gamma} \left\{ \hat{\xi}_0 \hat{\xi}_0^* e^{2i\xi} - \hat{\xi}_0^* \hat{\xi}_0 e^{-2i\xi} + \frac{2i\gamma_0}{\alpha_1} \hat{\xi}_0 \right\}^2 (1 - \cos 2\xi) \]. \hspace{1cm} (24)

It can be seen that, on average, the longitudinal acceleration of electron is quite small, therefore, when deriving estimate (23), it was quite reasonable to assume the uniform longitudinal motion of electron in the resonance region.

6. Conclusions

Formulas (19) and (20) were obtained that determine the transverse coordinates and components of the electron velocity vector in the combined magnetic trap in cases, when electron approaches the resonance region and passes through it. Based on these formulas, some phenomena can be analyzed occurring in the combined magnetic trap in the GA regime. Energy determined from expression (23) is considerably overestimated, since, if the guiding magnetic field variations are rather slow, the resonance condition (1) will be not always satisfied during the entire autoresonance motion, but only at certain instants of time.

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