BOTT - BOREL - WEIL CONSTRUCTION FOR QUANTUM SUPERGROUP $U_q(gl(m|n))$

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Abstract
The finite dimensional irreducible representations of the quantum supergroup $U_q(gl(m|n))$ are constructed geometrically using techniques from the Bott - Borel - Weil theory and vector coherent states.

1 INTRODUCTION
Supersymmetry and quantum groups are two of the most important discoveries of mathematical physics in recent times. These two notions have been combined together in the theory of quantum supergroups, which has been under intensive investigation in the last few years. The origin of quantum supergroups may be traced back to the Perk - Schultz model in statistical mechanics, but the systematic study of these remarkable algebraic structures only began about six years ago. Since then much has been understood about both the structures and representations of the quantum supergroups, and their applications have also been widely explored, leading to significant advances in various areas of physics and mathematics, notably, integrable models and low dimensional topology. Amongst the quantum supergroups arising from ‘deformations’ of the universal enveloping algebras of basic classical Lie superalgebras, $U_q(gl(m|n))$ has been best studied, in particular, its representation theory was developed in [20]. The present paper aims to further develop the representation theory of $U_q(gl(m|n))$ using the techniques from Bott - Borel - Weil theory [5] and vector coherent states [10].

The Bott - Borel - Weil theorem [5] relates finite dimensional irreducible representations of compact Lie groups to cohomology groups of homogeneous vector bundles. Relevant to our investigations in this paper is the following situation: Let $G$ be a simple complex Lie group, which may be regarded as the complexification of some compact Lie group. Let $P \subset G$ be a parabolic subgroup. Given finite dimensional irrep $V_0$ of $P$ with a highest weight which is assumed to be integral and dominant with respect to $G$ as well, one forms the homogeneous vector bundle $G \times_P V_0 \to G/P$. The holomorphic sections of the vector bundle furnishes a finite dimensional irrep of $G$. The method of vector coherent states, which was developed to address specific problems in quantum
mechanics, provides a way to explicitly construct the holomorphic sections, and to realize the Lie algebra of $G$ as differential operators on $G/P$, thus putting the abstract Bott - Borel - Weil construction in a concrete form.

A supersymmetric version of Bott - Borel - Weil theorem was investigated by Penkov and Serganova [16][17], who also extensively developed the theory of homogeneous superspaces. It is well known that supermanifold geometry is much richer than ordinary geometry. This makes the Bott - Borel - Weil theory for Lie supergroups a very interesting, but also difficult, subject to study.

In recent years, the Bott - Borel - Weil theory has also been investigated for quantum groups [15][3][4]. In particular, the work of Biedenharn and Lohe [3][4] on the quantized universal enveloping algebra $U_q(gl(m))$ is closely related to vector coherent states. It has the virtue of being explicit and thus readily applicable to investigations of mathematical structures relevant to physics, e.g., quantum group tensor operators.

In this paper we will carry out the Bott - Borel - Weil construction for the quantum supergroup $U_q(gl(m|n))$. As preparations for treating the quantum supergroup, and also for the purpose of understanding the underlying geometrical structure of vector coherent states, we study them extensively for classical $GL(m|n)$. In the quantum case, we realize the module of any finite dimensional irreducible representation of $U_q(gl(m|n))$ in terms of vector valued functions on some supermanifold, and the generators of the algebra by difference operators acting on these functions.

The main body of the paper is divided into sections 2 and 3, respectively dealing with the Bott - Borel - Weil construction for classical and quantum $GL(m|n)$. The first two subsections of section 2 are of a review nature, summarizing properties of structural and representation theoretical features of the Lie superalgebra $gl(m|n)$. Some of the material covered can not be easily found in the mathematical physics literature, but is necessary for the remainder of the paper. Subsections 2.3 and 2.4 construct two types of vectors coherent states for the classical Lie supergroup $GL(m|n)$. One kind is associated with the subgroup $GL(m|n-1) \times GL(1)$, and the other provides a new method for studying irreducible components of Kac modules. Subsections 2.5 elucidates the geometrical structure of the vector coherent states, thus to position the results of the previous two subsections into the general framework of Bott - Borel - Weil theory for Lie supergroups.

Section 3 consists of three subsections. Subsection 3.1 investigates some structural and representation theoretical aspects of $U_q(gl(m|n))$. Subsection 3.2 constructs the vector coherent states for a special class of representations of the quantum supergroup, namely, the contragredient tensor irreps. Some rather technical results needed for the next subsection are also proved here. The last subsection presents the Bott - Borel - Weil construction of all the finite dimensional irreps of $U_q(gl(m|n))$.

2 CLASSICAL GL(m|n)

2.1 gl(m|n)

Let us begin by introducing the familiar formulation of Lie superalgebras given in [11][18]. For the purpose of studying the representation theory of Lie supergroups and their quantum analogues in geometrical terms, one needs to define supergroups within the framework of supermanifold theory, and to formulate Lie superalgebras accordingly.
However, we will postpone this until subsection 2.3 when we construct supersymmetric coherent states, where the theory of supermanifolds becomes indispensable. At the algebraic level, the formulation of [11][18] is much easier to handle.

Within this formulation, a Lie superalgebra is considered as a $\mathbb{Z}_2$ graded Lie algebra over the complex field $\mathbb{C}$, namely, a $\mathbb{Z}_2$ graded vector space endowed with a graded bracket. The underlying vector space of the Lie superalgebra $gl(m|n)$ has the standard homogeneous basis $\{e_{ab} \mid a, b \in I\}$, where the index set $I$ is $\{1, 2, ..., m + n\}$. Set $I' = I \backslash \{m + n\}$. Introduce the gradation index $[\ ] : I \to \mathbb{Z}_2$ such that $[a] = \begin{cases} 0, & a \leq m; \\ 1, & a > m. \end{cases}$

Let $gl(m|n)_\eta$, $\eta \in \mathbb{Z}_2$, be the vector space over $\mathbb{C}$ spanned by the $e_{ab}$ with $[a] + [b] \equiv \eta \ (mod \ 2)$. Then $gl(m|n)_0$ and $gl(m|n)_1$ are the even and odd subspaces of $gl(m|n)$ respectively. We will abuse the notation somewhat and define $[\ ] : gl(m|n)_0 \cup gl(m|n)_1 \to \mathbb{Z}_2$, $[x] = \begin{cases} 0, & x \in gl(m|n)_0; \\ 1, & x \in gl(m|n)_1. \end{cases}$ Then the $\mathbb{Z}_2$ graded Lie bracket for $gl(m|n)$ is defined by

$$[e_{ab}, e_{cd}] = \delta_{bc}e_{ad} - (-1)^{([a]+[b])([c]+[d])}e_{cb}\delta_{da}. \quad (1)$$

For convenience, we will regard $gl(m|n)$ as embedded in its universal enveloping algebra. Thus the graded bracket $[\ ,\ ]$ can be interpreted as the graded commutator

$$[x, y] = xy - (-1)^{|x||y|}yx. \quad (2)$$

Let $\mathfrak{h}$ be the Lie subalgebra generated by $\{e_{aa} \mid a \in I\}$, which is a Cartan subalgebra of $gl(m|n)$. Introduce a basis $\{\epsilon_a \mid a \in I\}$ for the dual vector space $\mathfrak{h}^*$ such that $\epsilon_a(e_{bb}) = \delta_{ab}$. Note that the bilinear form for $gl(m|n)$ defined by

$$e_{ab} \otimes e_{cd} \mapsto (-1)^{|a|}\delta_{bc}\delta_{ad},$$

is super invariant and nondegenerate. Its restriction to $\mathfrak{h}$ induces the following nondegenerate bilinear form

$$\langle \ , \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C},
\langle \epsilon_a, \epsilon_b \rangle = (-1)^{|a|}\delta_{ab}.$$ 

The positive and negative root spaces of $gl(m|n)$ with respect to $\mathfrak{h}$ are respectively given by

$$n^+ = \bigoplus_{a<b} \mathbb{C}e_{ab},
\quad n^- = \bigoplus_{a>b} \mathbb{C}e_{ab},$$

which are nilpotent super subalgebras of $gl(m|n)$. We will denote by $\mathfrak{b}^\pm$ the Borel subalgebras $\mathfrak{h} + n^\pm$ respectively.

A parabolic subalgebra of $gl(m|n)$ is a proper subalgebra containing a Borel subalgebra. We call a parabolic subalgebra standard if it contains $\mathfrak{b}^+$. A feature distinct from those of ordinary Lie groups is that not all the parabolic subalgebras are Weyl group conjugate to standard ones.
Now every standard parabolic subalgebra is of the form

\[ p = b^+ + u_, \]

where \( u_ \) is a subalgebra of \( n^- \) generated by the elements of \( \{e_{a+1} \mid a \in \Theta \} \), where \( \Theta \) is a proper subset of \( I' \). Note that \( \bar{u}_- = n^- \setminus u_- \) is also a subalgebra of \( n^- \). For any Lie super subalgebra \( \mathfrak{f} \) of \( gl(m|n) \), we denote by \( U(\mathfrak{f}) \) its universal enveloping algebra. Then it follows from the Poincaré - Birkhoff - Witt theorem that

\[ U(gl(m|n)) = U(\bar{u}_-)U(p). \]

### 2.2 Irreducible representations

Let \( p \) be any standard parabolic subalgebra, and \( V_0(\lambda) \) be a finite dimensional irreducible \( p \) module. Since \( h \) is an abelian Lie subalgebra of \( p \), Lie’s theorem asserts that there exists at least one common eigenvector of all elements of \( h \) in \( V_0(\lambda) \). By noting the fact that \( U(p) \) can be decomposed into eigenspaces of \( h \) ( under the adjoint action ), we immediately see that the irreducible \( p \) module \( V_0(\lambda) \) must be of highest weight type, i.e., there exists a non - null \( v_+ \in V_0(\lambda) \) such that

\[ e_{ab}v_+ = 0, \quad e_{ab} \in p, \quad a < b, \]
\[ e_{aa}v_+ = \lambda_av_+, \quad e_{aa} \in p, \]

and \( V_0(\lambda) \) is cyclically generated by \( v_+ \).

Construct the vector space

\[ \bar{V}(\lambda) = U(gl(m|n)) \otimes_{U(p)} V_0(\lambda). \]

Then clearly \( \bar{V}(\lambda) = U(\bar{u}_-) \otimes V_0(\lambda) \), and \( \bar{V}(\lambda) \) furnishes a \( U(gl(m|n)) \) module with the action of \( U(gl(m|n)) \) defined in the standard way: for any \( x \in U(gl(m|n)) \), \( y \in U(\bar{u}_-) \), \( xy \) can be expressed in the form

\[ xy = \sum_t y'_t x'_t, \quad x'_t \in U(p), \quad y'_t \in U(\bar{u}_-). \]

Then

\[ x \circ (y \otimes v_0) = \sum_t y'_t \otimes x'_tv_0, \quad v_0 \in V_0(\lambda). \]

\( \bar{V}(\lambda) \) can be decomposed into a direct sum of weight spaces, i.e., eigenspaces of \( h \),

\[ \bar{V}(\lambda) = \bigoplus_{\omega \prec \lambda} \bar{V}^\omega, \quad \text{dim} \bar{V}^\omega < \infty, \quad (3) \]

where \( \omega \in h^* \), and \( hv = \omega(h)v, \forall v \in \bar{V}^\omega, \ h \in h \). The \( \prec \) is a partial ordering of the elements of \( h^* \) defined by \( \mu \prec \nu \) if \( \mu - \nu \in \bigoplus_{a<b} \mathbb{Z}_+ (\epsilon_a - \epsilon_b) \).

If \( W \) is a proper submodule of \( \bar{V}(\lambda) \), then \( W \cap V_0(\lambda) = 0 \), as every nonvanishing element of \( V_0(\lambda) \) cyclically generates \( \bar{V}(\lambda) \). Let \( M \) be the union of all the proper submodules of \( \bar{V}(\lambda) \). Then \( M \) is again a proper submodule, which is unique and is maximal in the sense that every other proper submodule of \( \bar{V}(\lambda) \) is a submodule of
M. We will set $M = 0$ in the case that no proper submodule of $V(\lambda)$ exists. More generally, we define

$$V(\lambda) = \bar{V}(\lambda)/M.$$  

It is a consequence of the maximality of $M$ that the quotient module $V(\lambda)$ is irreducible. Furthermore, $V(\lambda)$ admits a weight space decomposition similar to (3). Also, the canonical projection restricted to $V_0(\lambda)$ is one - to - one, and the image of $v_+$ is the unique maximal vector of $V(\lambda)$.

The irreducible $U(gl(m|n))$ module is finite dimensional if and only if

$$\lambda_a - \lambda_{a+1} \in \mathbb{Z}_+, \quad m \neq a \in \mathcal{I}.$$  

(4)

This is the familiar finite dimensionality condition obtained by Kac. To understand it within our framework, we use the Poincaré - Birkhoff - Witt theorem for $U(gl(m|n))$ again but in a different form to express it as

$$U(gl(m|n)) = U(f_-)U(gl(m) \oplus gl(n))U(f_+),$$

where $gl(m) \oplus gl(n) \subset gl(m|n)$ is the maximal even subalgebra, $f_+$ is the subalgebra spanned by all the odd raising elements $e_{\mu i}$, $i \leq m$, $\mu > m$, and $f_-$ is that spanned by $e_{\mu i}$, $i \leq m$, $\mu > m$. Note that both $U(f_\pm)$ are isomorphic to the Grassmann algebra with $mn$ generators, hence $\text{dim}(U(f_\pm)) = 2^{mn}$. Let $v^\lambda_+$ be the maximal vector of $V(\lambda)$. Consider $W_0 = U(gl(m) \oplus gl(n))v^\lambda_+$. As a $gl(m) \oplus gl(n)$ module, $W_0$ must be irreducible. Otherwise, a proper $gl(m) \oplus gl(n)$ submodule $W'_0$ of $W_0$ would generate a proper $gl(m|n)$ submodule $U(f_-)W'_0 \subset V(\lambda)$, contradicting the irreducibility of $V(\lambda)$. Now $W_0$ is finite dimensional if and only if $\lambda$ satisfies (4), and this in turn leads to our claim.

Recall that every finite dimensional irrep of $gl(m|n)$ is of highest weight type, and is uniquely determined by a highest weight $\lambda \in \mathfrak{h}^*$ satisfying (4). Therefore, for any given standard parabolic subalgebra $\mathfrak{p}$, the construction presented above yields all the finite dimensional irreps of $gl(m|n)$. It is worth noting that when $\mathfrak{p} = gl(m|n)\setminus f_-$, the construction coincides with that of Kac.

2.3 Vector coherent states: 1

From this subsection on, we need to formulate Lie superalgebras over supernumbers, following the treatment of DeWitt [7]. Introduce the infinite complex Grassmann algebra $\Lambda_\infty$ completed with Frechet topology. We will denote by $\mathbb{C}_c$ the commutative part, and by $\mathbb{C}_a$ the anti - commutative part of $\Lambda_\infty$, and represent $\mathbb{C}_a^m \times \mathbb{C}_c^n$ by $\mathbb{C}^{m|n}$.

We can now define a Lie superalgebra as a supervector space over $\Lambda_\infty$ endowed with a Lie super bracket. In particular, the Lie super bracket of $gl(m|n)$ is the same as that given in subsection 2.1. However, it is worth stressing that even elements of $gl(m|n)$ commute with all supernumbers, while odd elements anti - commute with $\mathbb{C}_a$. Our discussions on structures and representations of $gl(m|n)$ in the previous two subsections are still valid in this more general setting.

Let $\mathfrak{p}$ be the parabolic subalgebra spanned by the following elements

$$e_{ab}, \quad a, b \in \mathcal{I}, \quad a \leq b,$$

$$e_{dc}, \quad c, d \in \mathcal{I}', \quad c < d.$$
Note that the $gl(m|n-1) \oplus gl(1)$ subalgebra spanned by $\{e_{ab} \mid a, b \in \Gamma \} \cup \{e_{m+n+m+n}\}$ is contained in $\mathfrak{p}$. Let $V_0(\lambda)$ be a finite dimensional irreducible $\mathfrak{p}$ module with highest weight $\lambda \in \mathfrak{h}^*$ satisfying the condition (2), and denote by $\pi_0 : \mathfrak{p} \to gl(V_0(\lambda))$ the associated irrep of $\mathfrak{p}$. The induced module construction presented in the last subsection allows us to construct a finite dimensional irreducible $gl(m|n)$ module $V(\lambda)$ into which $V_0(\lambda)$ is canonically embedded. We can decompose $V(\lambda)$ into a direct sum of $e_{m+n+m+n}$ eigenspaces

$$V(\lambda) = \bigoplus_{k=0}^{L} V_k,$$

where $e_{m+n+m+n}$ takes eigenvalue $\lambda_m + k$ when acting on $V_k$. Note that $V_0 = V_0(\lambda)$.

Let $\{w^\alpha \mid \alpha = 1, 2, ..., d\}$ be a basis of $V(\lambda)$ such that $\{v^i = w^i \mid i = 1, 2, ..., d\}$ forms a basis of $V_0(\lambda)$, where $d = \dim V(\lambda)$, and $d_0 = \dim V_0(\lambda)$. Introduce the basis $\{\langle w^\alpha \mid | \alpha = 1, 2, ..., d\}$ for the dual vector space $V(\lambda)^*$ of $V(\lambda)$ such that

$$\langle w^\alpha \mid w^\beta \rangle = \delta_{\alpha\beta},$$

where $\langle | \rangle$ denotes the dual space pairing. Then the matrix elements of any $X \in gl(m|n)$ in the irrep $\pi : gl(m|n) \to gl(V(\lambda))$ furnished by the module $V(\lambda)$ are given by

$$\pi(X)_{\alpha\beta} = \langle w^\alpha \mid X w^\beta \rangle.$$

It is a standard result that $V(\lambda)^*$ carries a left $gl(m|n)$ module structure defined by

$$(X \circ \langle w_1 | w_2 \rangle = -\langle w_1 | X w_2 \rangle.$$

Introduce $(Z_\alpha)_{\alpha \in \Gamma} \in \mathbb{C}^{m|n-1}$. Set $Z_i = \theta_i$, $1 \leq i \leq m$, $Z_\mu = z_\mu$, $m < \mu < m + n$. Then the $z_\mu$ are even, while the $\theta_i$ are odd, and obey $\theta_i \theta_j = -\theta_j \theta_i$. Denote by $\Lambda_\infty[[Z]]$ the ring of polynomials in the variables $Z_\alpha$, and $\Lambda_\infty[[Z]]_L$ the subset of the polynomials of degree less or equal to $L$, which is clearly finite dimensional. Needless to say, it is assumed that $z_\mu$ commutes with all elements of $U(gl(m|n))$, while $\theta_i$ commutes with the even elements and anticommutes with the odd ones. We construct the linear space $V(\lambda)[[Z]] = \Lambda_\infty[[Z]] \otimes V(\lambda)$, and define the bilinear map $V(\lambda)^* \otimes V(\lambda)[[Z]] \to \Lambda_\infty[[Z]]$ by

$$\langle w_1 | p(Z) \otimes w_2 \rangle = (-1)^{|w_1||p(Z)} \langle w_1 | w_2 \rangle p(Z).$$

Set

$$g(Z) = \exp \left( \sum_{\alpha \in \Gamma} Z_\alpha \otimes e_{am+n} \right),$$

where the exponential should be understood as a formal power series at this stage. Now we define a linear map $\xi : V(\lambda) \to \Lambda_\infty[[Z]]_L \otimes V_0(\lambda)$ by

$$\xi_w(Z) = \sum_{i=1}^{d_0} (-1)^{|w_i|+|w|} \langle v^i | g(Z)(1 \otimes w) \rangle \otimes v^i.$$

We set $\xi_\alpha(Z) = \xi_w(\alpha)$. Denote the supervector space spanned by all the $\xi_\alpha(Z)$ by $V_2^\Lambda$, which will be called the space of vector coherent states. Then
Proposition 1 1. The $\xi_\alpha(Z)$, $\alpha = 1, 2, \ldots, d$, are linearly independent over $\Lambda_\infty$.

2. $V_Z^\lambda$ furnishes a $gl(m|n)$ module with the action defined by

$$X \circ \xi_w(Z) = \sum_{i=1}^{d_0} (-1)^{[iv_i][1+[v_i]+[w]]} \langle v_i | g(Z)(1 \otimes Xw) \rangle \otimes v^i,$$

or more compactly,

$$X \circ \xi_w(Z) = \xi_{Xw}(Z), \quad \forall X \in gl(m|n).$$

3. $V_Z^\lambda$ and $V(\lambda)$ are isomorphic as $gl(m|n)$ modules.

Proof: Part 1 is an immediate consequence of the irreducibility of $V(\lambda)$, while part 2 follows from part 1 and the following simple calculation

$$X \circ \xi_\alpha(Z) = \sum_{\beta} \pi(X)_{\beta\alpha} \xi_{\beta}(Z).$$

Observe that $1 \otimes v_+^\lambda$ is the maximal vector of $V_Z^\lambda$. It is easy to see that the weight of $1 \otimes v_+^\lambda$ is $\lambda$, hence our claim in part 3.

Now we have realized a finite dimensional irreducible $U_q(gl(m|n))$ module as a subset of $\Lambda_\infty[[Z]]_L \otimes V_0(\lambda)$. Under a suitable dualization, elements of $V_Z^\lambda$ may be interpreted as holomorphic sections of a homogeneous supervector bundle. We will give more details on this later. Here we present the explicit realization of $gl(m|n)$ in terms of differential operators on the superbundle.

Proposition 2 The irreducible representation $\pi : gl(m|n) \to End(V_Z^\lambda)$ of $gl(m|n)$ can be realized by

$$\pi(e_{ab}) = -(-1)^{[a][b]+1} Z_b \frac{\partial}{\partial Z_a} \otimes 1 + 1 \otimes \pi_0(e_{ab}),$$

$$\pi(e_{m+n,m+n}) = - \sum_{a \in I'} Z_a \frac{\partial}{\partial Z_a} \otimes 1 + 1 \otimes \pi_0(e_{m+n,m+n}),$$

$$\pi(e_{m+n,a}) = \sum_{b \in I'} \left\{ Z_b \otimes \pi_0(e_{ba}) + (-1)^{[a]} Z_a Z_b \frac{\partial}{\partial Z_b} \otimes 1 \right\} + Z_a \otimes \pi_0(e_{m+n,m+n}),$$

$$\pi(e_{a,m+n}) = \frac{\partial}{\partial Z_a} \otimes 1, \quad a, b \in I'.$$

Proof: The correctness of the realization for $e_{a,m+n}$ is clear. For the $e_{ab}, a, b \in I'$, we observe that

$$-(-1)^{[a][b]+1} Z_b \frac{\partial}{\partial Z_a} \otimes 1 + 1 \otimes e_{ab}$$

commutes with $g(Z)$. Hence

$$e_{ab} \circ \xi_\alpha(Z) = \sum_{i=1}^{d_0} \langle v_i | \left\{ -(-1)^{[a][b]+1} Z_b \frac{\partial}{\partial Z_a} \otimes 1 + 1 \otimes e_{ab} \right\} g(Z)(1 \otimes w) \rangle \otimes v^i(-1)^{[iv_i][[a]+[b]+1+[w]],}$$
which immediately yields what we want to prove. The case of \( e_{m+n,a} \) can be shown similarly. As to \( e_{m+n,a} \), by using

\[
g(Z)(1 \otimes e_{m+n,a})g(Z)^{-1} = 1 \otimes e_{m+n,a} + (-1)^{[a]}Z_a \sum_{b \in \Gamma} Z_b \frac{\partial}{\partial Z_b} \otimes 1
\]

\[
+ (-1)^{[a]}Z_a \otimes e_{m+n,m+n} + \sum_{b \in \Gamma} Z_b \otimes e_{b,a},
\]

we obtain

\[
e_{m+n,a} \circ \xi_a(Z) = \left\{ (-1)^{[a]}Z_a \sum_{b \in \Gamma} Z_b \frac{\partial}{\partial Z_b} \otimes 1 \right\} \xi_a(Z)
\]

\[
+ \sum_{i,j} (v^i)^j \left\{ (-1)^{[a]}Z_a \bar{\pi}_0(e_{m+n,m+n})_{ij} + \sum_{b \in \Gamma} Z_b \bar{\pi}_0(e_{b,a})_{ij} \right\} g(Z)(1 \otimes \omega^a))
\]

\[
\otimes v^i(-1)^{[n]([a]+[w])},
\]

from which the last formula of \( 11 \) can be read off.

Several features of this realization are worth observing. Repeated applications of \( \pi(e_{ab}) \), \( a < b \), \( a, b \in \Gamma \), to \( 1 \otimes v^\lambda \), the highest weight vector of \( V^\lambda_Z \), generate the irreducible module \( V^\lambda_Z \) automatically, rather than an analogue of \( V(\lambda) \). Also, the problem of studying the irreps of \( gl(m|n) \) may now be dealt with by studying the irreps of the chain of subalgebras

\[
gl(m|n) \supset gl(m|n-1) \supset ... \supset gl(m|1) \supset gl(m).
\]

This may provide a practical method for investigating the detailed structures of the finite dimensional irreps of \( gl(m|n) \).

### 2.4 Vector coherent states: 2

As we have already pointed out, Kac’ induced module construction is a special case of the method developed in subsection 2, with the parabolic subalgebra chosen to be

\[ \mathfrak{p} = gl(m) + gl(n) + \mathfrak{f}_+ \]

The vector coherent states associated with this parabolic subalgebra were studied by Le Blanc and Rowe\[12\]. Here we briefly outline the construction.

Let \( V_0(\lambda) \) be a finite dimensional irreducible \( \mathfrak{p} \) module with highest weight \( \lambda \). Following the general procedure of subsection 2, we construct \( \bar{V}(\lambda) \), which is isomorphic to \( U(\mathfrak{f}_-) \otimes V_0(\lambda) \), thus is a Kac module. As before, we denote by \( M \) the maximal proper submodule of \( \bar{V}(\lambda) \) in the atypical case, and set \( M = 0 \) in the typical case.

Let \( (\theta_{\mu i})_{1 \leq i \leq m, 1 \leq \mu \leq m+n} \) be any point in \( \mathbb{C}^{mn[0]} \). Define

\[
g(\theta) = \exp \left( \sum_{i=1}^{m+m+n} \sum_{\mu=m+1}^{\mu_i} \theta_{\mu i} \otimes e_{\mu i} \right).
\]

We define a linear map \( \Psi : \bar{V}(\lambda) \rightarrow \Lambda_\infty[[\theta]] \otimes V_0(\lambda) \) by

\[
w \mapsto \xi_w(\theta) = \sum_{i=1}^{d_a} (-1)^{[n]([1+[w])} \langle v^i | g(\theta)(1 \otimes w) \rangle \otimes v^i.
\]
Then $\text{Im}\Psi$ furnishes a $\mathfrak{gl}(m|n)$ module with the module action

$$X \circ \xi_w(\theta) = \sum_{i=1}^{d_0} (-1)^{|v_i|(|X|+|w|)} \langle v^i | g(\theta)(1 \otimes Xw) \rangle \otimes v^i, \quad \forall X \in \mathfrak{gl}(m|n).$$

Furthermore,

**Proposition 3**

1. $\text{Ker}\Psi = M$, and $\text{Im}\Psi$ is irreducible.

2. When acting on $\text{Im}\Psi$, $\mathfrak{gl}(m|n)$ is realized by

$$\pi(e_{ij}) = -\sum_{\mu=m+1}^{m+n} \theta_{j\mu} \frac{\partial}{\partial \theta_{i\mu}} \otimes 1 + 1 \otimes \pi_0(e_{ij}),$$

$$\pi(e_{\mu
u}) = -\sum_{i=1}^{m} \theta_{i\nu} \frac{\partial}{\partial \theta_{i\mu}} \otimes 1 + 1 \otimes \pi_0(e_{\mu\nu}),$$

$$\pi(e_{\mu i}) = \sum_{j=1}^{m} \sum_{\nu=m+1}^{m+n} \theta_{\nu j} \theta_{\mu j} \frac{\partial}{\partial \theta_{\nu j}} \otimes 1$$

$$+ \sum_{j=1}^{m} \theta_{\mu j} \otimes \pi_0(e_{ji}) + \sum_{\nu=m+1}^{m+n} \theta_{\nu i} \otimes \pi_0(e_{\mu\nu}),$$

$$\pi(e_{i\mu}) = \frac{\partial}{\partial \theta_{i\mu}} \otimes 1, \quad i, j \leq m, \quad \mu, \nu > m, \quad (7)$$

where $\pi_0 : \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \to \mathfrak{gl}(V_0(\lambda))$ is the irrep of $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ afforded by $V_0(\lambda)$.

Results of this Proposition for the special case of $\mathfrak{gl}(m|1)$ were known to Dundi and Jarvis [8], and the general case were contained in some unpublished work of Bracken’s [2], and the work of Le Blanc and Rowe[12].

In the next subsection we will demonstrate that the underlying geometry of the vector coherent states of this subsection, and thus also Kac’ induced module construction, is very special.

### 2.5 Geometrical interpretation

The geometrical meaning of the vector coherent states for $\mathfrak{gl}(m|n)$ will be elucidated in this subsection. For this purpose, we need to consider supermanifolds more general than $\mathbb{C}^{m|n}$. We will follow DeWitt’s geometrical formulation, within which a supermanifold is, roughly speaking, a collection of open sets of $\mathbb{C}^{m|n}$ patched together, where a set is open if it is the pre - image of an open set in $\mathbb{C}^n$ under the natural projection of $\mathbb{C}^{m|n}$ onto $\mathbb{C}^n$.

As is well known, there are several distinct formulations of supermanifolds, but the most widely studied ones are those due respectively to DeWitt and to Berezin - Kostant - Leites. While DeWitt’s is suitable for theoretical physics, the sheaf theoretical approach of Berezin - Kostant - Leites has many mathematically appealing features. It has also been shown [4] that the DeWitt and Berezin - Kostant - Leites supermanifolds are the same in an appropriate sense. We may also mention that we will only deal with rather simple supermanifolds, and do not concern ourselves with
detailed functional analysis on them. Therefore the various approaches probably do not lead to any essential differences for us here.

Let $G$ be the component of the Lie supergroup $GL(m|n)$ connected to the identity. Needless to say, $G$ is a Lie supergroup in its own right, and has the Lie superalgebra $gl(m|n)$. Let $P \subset G$ be the Lie super subgroup with Lie superalgebra $p$, which is assumed to be a parabolic subalgebra of $gl(m|n)$.

We first consider the case where the parabolic subalgebra $p$ is generated by $\{e_{ab}, e_{m+n\cdot a}, e_{m+n\cdot m+n}, a, b \in I\}$. Note that $p \supset b^-$. Now $G/P$ (understood as the right coset space. A more appropriate notation may be $P\langle G,\rangle$) yields a homogeneous superspace $\mathbb{P}$, which coincides with the projective superspace $\mathbb{C}P^{m|n-1}$ defined in the following way. Let $(W_a)_{a \in I} = (\zeta_i; w_\mu)_{1 \leq i \leq m; 1 \leq \mu \leq m+n}$ be any point of the flat $m|n$ superspace $\mathbb{C}^{m|n}$. Consider the subspace $\mathbb{C}^{m|n}$ of $\mathbb{C}^{m|n}$ consisting of the points such that the bodies of the commuting coordinates of any given point do not vanish simultaneously. Define an equivalence relation $(W_a) \sim (W'_a)$, $(W_a), (W'_a) \in \mathbb{C}^{m|n}$, if $(W_a) = c(W'_a)$ for some $c \in \mathbb{C}_c$ with nonzero body. Then $\mathbb{C}P^{m|n-1} = \mathbb{C}^{m|n} / \sim$. This supermanifold can be covered by the following coordinate patches

$$U_\mu = \{(\theta_1^\mu, \ldots, \theta_m^\mu, z_{m+1}^\mu, \ldots, z_{m+n}^\mu) \mid \mu = m+1, \ldots, m+n, \theta_i^\mu = \zeta_i/w_\mu, \ z_\nu^\mu = w_\nu/w_\mu, \ \text{the body of } w_\mu \neq 0\}.$$  

The transition functions can also be easily obtained.

Consider a finite dimensional irreducible left $G$ module $V(\lambda)$ with highest weight $\lambda$, and denote the associated irrep by $T_\lambda$. Let $V(\lambda)^*$ be the dual $G$ module, and define $V_0(\lambda)^* \subset V(\lambda)^*$ to be the unique irreducible left $P$ submodule containing the lowest weight vector of $V(\lambda)^*$. Given a basis $\{\langle v'\rangle\}$ of $V_0(\lambda)^*$, we express the left action of $P$ by

$$p \ast \langle v_i \rangle = \sum_j L(p)_{ji} \langle v_j \rangle, \quad p \in P.$$  

There is also a natural right $P$ module structure on $V_0(\lambda)^*$ with

$$\langle v_i \rangle \ast p = \sum_j \langle v_j \rangle R(p)_{ij}, \quad R(p)_{ij} = L(p)_{ij}, \quad p \in P.$$  

Let us now define the following $V_0(\lambda)^*$ valued functions on $G$

$$\eta_w(g) = \sum_{i=1}^{d_0} (-1)^{[w_i]}(v_0 | T_\lambda(g)w) \otimes \langle v' \rangle, \quad g \in G, \ w \in V(\lambda),$$  

and denote by $O_\lambda(G/P)$ their linear span. The right translation of $G$ defines a module action on $O_\lambda(G/P)$

$$(g \ast \eta_w)(h) = \eta_w(hg),$$  

and the associated irrep of $G$ is isomorphic to $T_\lambda$.

A critical property of $O_\lambda(G/P)$ to be observed is that

$$\eta_w(pg) = \eta_w(g) R(p^{-1}), \quad \forall p \in P.$$  


This allows us to interpret elements of $O_\lambda(G/P)$ as global sections of the supervector bundle

$$G \times_P V_0^*(\lambda) \to G/P,$$

which is the quotient space $G \times V_0^*(\lambda)/\sim$, with the equivalence relation defined by

$$(pg, \eta) \sim (g, \eta R(p^{-1})), \quad p \in P.$$

Let $U \subset G$ be a neighbourhood of the identity $e \in G$. We consider $O_\lambda(G/P)$ restricted to $U$. Differentiating the associated irrep of $G$ yields an irrep of the Lie superalgebra $gl(m|n)$, which is regarded as the left invariant vector fields on $G$:

$$(X \circ \eta_w)(g) = \sum_{i=1}^{d_0} (-1)^{|\lambda|+|\mu|+(\lambda \eta w)} \langle v_i|T_\lambda(g)X \circ w \rangle \otimes \langle v_i |,
X \in gl(m|n), \ g \in U.$$

Recall the following decomposition of the Lie supergroup $G$,

$$G = PQ,$$

$$Q = \left\{ \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} \right\}.$$

For each $g \in U \subset G$, write $g = pq$, $p \in P \cap U$, $q \in Q \cap U$. Then $\eta_w(g) = \eta_w(q) R(p^{-1})$. Also, every elements of $Q \cap U$ can be expressed as $\exp(\sum_{a \in V} Z_a e_{am+n})$, $(Z_a) \in \mathbb{C}P^{m|n-1}$. Therefore, on $U$, the vector space $O_\lambda(G/P)$ is isomorphic to $V_2^\lambda$, with the isomorphism given by the duality between $V_0(\lambda)$ and $V_0(\lambda)^*$. By comparing the highest weights, we can see that this also defines a $gl(m|n)$ module isomorphism.

The result of subsection 2.4. can be interpreted similarly, by choosing the parabolic subalgebra $p$ to be $gl(m) + gl(n) + \mathfrak{f}$. We will not repeat the details here, but merely point out that in this case, the homogeneous superspace $G/P$ is the flat superspace $\mathbb{C}^{m|n}$, which does not have a body.

3 QUANTUM GL(m|n)

3.1 $U_q(gl(m|n))$ and its irreps

We will consider Jimbo’s version of the quantum supergroup $U_q(gl(m \mid n))$ over $A_\infty$. Fix $q \in \mathbb{C}^\times$, the body of which is assumed to be nonzero and not a root of unity. $U_q(gl(m|n))$ is a $\mathbb{Z}_2$-graded unital algebra generated by $\{K_a, K_a^{-1}, a \in I; E_{b+1}, E_{b+1,b}, b \in I\}$, subject to the following relations

$$K_a K_a^{-1} = 1, \quad K_a^\pm K_b^\pm = K_b^\mp K_a^\pm,$$
$$K_a E_{b+1} K_a^{-1} = q_a^{\mp \delta_{a,b+1}} E_{b+1},$$
$$[E_{a+1}, E_{b+1}] = \delta_{ab} (K_a K_{a+1} - K_{a-1} K_{a+1})/(q_a - q_a^{-1}),$$
$$(E_{m+1})^2 = 0,$$
$$E_a E_{a+1} = E_{a+1} E_a,$$
$$E_{a+1} E_{b+1} = E_{b+1} E_{a+1}, \quad |a - b| \geq 2,$$
$$S_a^{+} = S_a^{-} = 0, \quad a \neq m,$$
$$\{E_{m-1,m+2}, E_{m+1,m} \} = \{E_{m+2,m-1}, E_{m+1,m} \} = 0,$$  (10)
where \( q_a = q^{(-1)[a]} \),

\[
S_{a a+1}^{(+)} = (E_{aa+1})^2 E_{a+1 a+1} - (q + q^{-1}) E_{aa+1} E_{a+1 a+1} E_{aa+1} + E_{a+1 a+1} (E_{aa+1})^2,
\]

\[
S_{a a+1}^{(-)} = (E_{aa+1})^2 E_{a+1 a+1} - (q + q^{-1}) E_{aa+1} E_{a+1 a+1} E_{aa+1} + E_{a+1 a+1} (E_{aa+1})^2,
\]

and \( E_{m-1 m+2} \) and \( E_{m+2 m-1} \) are the \( a = m - 1, b = m + 1 \), cases of the following elements

\[
E_{ab} = E_a E_b - q_c^{-1} E_c E_a, \\
E_{ba} = E_b E_a - q_c E_c E_a, \quad a < c < b.
\]

The \( \mathbb{Z}_2 \) grading of the algebra is specified such that the elements \( K_a^{±1}, \forall a \in I \), and \( E_{bb+1}, E_{b+1 b}, b \neq m \), are even, while \( E_{m m+1} \) and \( E_{m+1 m} \) are odd. It is well known that \( U_q(gl(m|n)) \) has the structure of a \( \mathbb{Z}_2 \) graded Hopf algebra, with a co - multiplication

\[
\Delta(E_{aa+1}) = E_{aa+1} \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{aa+1}, \\
\Delta(E_{a+1 a}) = E_{a+1 a} \otimes 1 + K_a^{-1} K_{a+1} \otimes E_{a+1 a}, \\
\Delta(K_a^{±1}) = K_a^{±1} \otimes K_a^{±1},
\]

co - unit

\[
\epsilon(E_{aa+1}) = \epsilon(E_{a+1 a}) = 0, \quad \forall a \in I', \\
\epsilon(K_b^{±1}) = 1, \quad \forall b \in I,
\]

and anti - pode

\[
S(E_{aa+1}) = -E_{aa+1} K_a^{-1} K_{a+1}, \\
S(E_{a+1 a}) = -K_a^{-1} K_{a+1} E_{a+1 a}, \\
S(K_a^{±1}) = K_a^{±1} \otimes K_a^{±1}.
\]

The quantum supergroup \( U_q(gl(m|n)) \) has various \( \mathbb{Z}_2 \) graded Hopf subalgebras, which are useful for analysing the representation theory. From equation (10), we can easily see that

\( \{K_a \mid a \in I\} \) generate an abelian Lie algebra

\[
[K_a, K_b] = K_a K_b - K_b K_a = 0.
\]

\( \{K_a, a \in I; E_{b b+1}, E_{b+1 b}, \mid m + n - 1 > b \in I'\} \) generate a subalgebra \( U_q(gl(m|n-1) \oplus gl(1)) \).

\( \{K_a \mid a \in I\} \cup \{E_{b b+1} \mid b \in I'\} \) generate a subalgebra \( U_q(b_+) \), which is isomorphic to the quantized universal enveloping algebra of \( b_+ \).

Fix any subset \( \Theta \subset I' \), then \( \{K_a^{±1} \mid a \in I\} \cup \{E_{b b+1} \mid b \in I'\} \cup \{E_{c+1 c} \mid c \in \Theta\} \) generate a (\( \Theta \) dependent) subalgebra \( U_q(p) \), which is isomorphic to the quantized universal enveloping algebra of a standard parabolic subalgebra of \( gl(m|n) \).
Properties of the $E_{ab}$ were studied extensively in [20]. We recall some of them below, which will be used repeatedly in the Bott - Borel - Weil construction.

**Lemma 1** 1. Assume $a < b$, then

\[
[E_{ab}, E_{cc+1}] = 0,
\]

\[
[E_{ba}, E_{c+1c}] = 0, \; a \neq c, \; c + 1, \; \& \; b \neq c, \; c + 1,
\]

\[
[E_{ab}, E_{c+1c}] = \delta_{b+c}E_{ac}K_cK^{-1}_{c+1}q_c^{-1} - \delta_{a+c}(-1)^{\delta_m}E_{c+1b}K^{-1}_cK^{-1}_{c+1},
\]

\[
[E_{ba}, E_{cc+1}] = \delta_{a+c}E_{b+c}K_cK^{-1}_{c+1}q_c^{-1} - \delta_{b+c+1}(-1)^{\delta_m}E_{ca}K^2_cK^{-1}_{c+1},
\]

\[a \neq c, \; c + 1, \; \text{or} \; b \neq c, \; c + 1.\]

2.

\[
[E_{ab}, E_{ba}] = (K_aK^{-1}_b - K^{-1}_{b}K_a)/(q_a - q_a^{-1}),
\]

\[
[E_{ac}, E_{cb}] = \begin{cases} 
E_{ab}K_cK^{-1}_cq_b^{-1}, & a > b > c, \\
E_{ab}K^{-1}_cK^{-1}_bq_b, & b > a > c, \\
E_{ab}K^{-1}_cK^{-1}_bq_b^{-1}, & a < b < c,
\end{cases}
\]

\[
E_{ca}E_{cb} = (-1)^{(\lfloor a \rfloor + \lfloor c \rfloor)(\lfloor b \rfloor + \lfloor c \rfloor)}q_cE_{cb}E_{ca},
\]

\[
E_{bc}E_{ac} = (-1)^{(\lfloor a \rfloor + \lfloor c \rfloor)(\lfloor b \rfloor + \lfloor c \rfloor)}q_c^{-1}E_{ac}E_{bc}, \; a < b < c, \; \text{or} \; b > a > c,
\]

\[
[E_{ca}, E_{cb}] = [E_{ac}, E_{bc}] = 0, \; a < c < b, \; \text{or} \; a > b > c.
\]

3. Assume that $a < b, c < d$, and no two of $a, b, c$ and $d$ are equal. Introduce the sets $S(a, b) = \{a, a + 1, ..., b\}$, and $S(c, d) = \{c, c + 1, ..., d\}$.

If $S(a, b) \cap S(c, d) = \emptyset$, $S(a, b)$, or $S(c, d)$,

then $[E_{ab}, E_{cd}] = [E_{ab}, E_{dc}] = [E_{ba}, E_{dc}] = [E_{ba}, E_{dc}] = 0$.

Define

\[
E_{ab} = E_{ac}E_{cb} - q_cE_{cb}E_{ac},
\]

\[
E_{ba} = E_{bc}E_{ca} - q_c^{-1}E_{cb}E_{ac}, \; a < c < b.
\]

Then the $E_{ab}$ are related to $E_{ab}$ with the help of the anti-pode

\[
S(E_{ab}) = -E_{ab}K^{-1}_aK_b, \quad a < b,
\]

\[
S(E_{ba}) = -K_aK^{-1}_bE_{ba}, \quad a < b.
\]

Define the bilinear adjoint action $Ad : U_q(gl(m|n)) \otimes U_q(gl(m|n)) \rightarrow U_q(gl(m|n))$ by

\[
Ad_x(y) = \sum_x (-1)^{|x(2)|}|y|x(1)YS(x(2)),
\]

then $U_q(gl(m|n)) \rightarrow Ad_{U_q(gl(m|n))}$, $x \mapsto Ad_x$, yields an algebra homomorphism, where Sweedler’s sigma notation is employed.

Set

\[
X_a = -E_aE_{m+n}K^{-1}_aK_{m+n},
\]

\[
Y_a = E_{m+n,a}, \quad a \in \mathcal{Y},
\]

and denote by $\mathcal{X}$ the linear span of $\{X_a \mid a \in \mathcal{Y}\}$, and by $\mathcal{Y}$ that of $\{Y_a \mid a \in \mathcal{Y}\}$. Then
Lemma 2 \( \mathcal{X} \) and \( \mathcal{Y} \) are closed under the adjoint action \( Ad \) of the the subalgebra \( U_q(gl(m|n-1) \oplus gl(1)) \subset U_q(gl(m|n)) \),

\[
\begin{align*}
Ad_{K_a} X_b &= q_a^{\delta_{ab}} X_b, \\
Ad_{e_c} X_b &= \delta_{c+1,b} X_c, \\
Ad_{f_c} X_b &= \delta_{c,b} X_{c+1}, \\
Ad_{K_a} Y_b &= q_a^{-\delta_{ab}} Y_b, \\
Ad_{e_c} Y_b &= -q_{c+1} \delta_{c,b} Y_{c+1}, \\
Ad_{f_c} Y_b &= -q_{c+1} \delta_{a+1,b} Y_c, \\
Ad_{K_{m+n}} X_b &= q_{m+n}^{-1} X_b, \\
Ad_{K_{m+n}} Y_b &= q_{m+n} Y_b,
\end{align*}
\]

where \( e_c = E_{c+1} \), \( f_c = E_{c+1} \), \( a, b, c \in \mathcal{Y}, \ c < m + n - 1 \).

Regarded as \( U_q(gl(m|n-1) \oplus gl(1)) \) modules, both \( \mathcal{X} \) and \( \mathcal{Y} \) are irreducible, and are dual to each other in the following sense: the bilinear pairing \( \mathcal{Y} \otimes \mathcal{X} \to \Lambda_\infty \) defined by \( \langle Y_a, X_b \rangle = \delta_{ab} \) identifies \( \mathcal{Y} \) with the dual vector space \( \mathcal{X}^* \) of \( \mathcal{X} \). This also defines a \( U_q(gl(m|n-1) \oplus gl(1)) \) module isomorphism

\[
(Ad_u(Y_a), X_b) = (-1)^{|u|} (Y_a, Ad_{S(u)}(X_b)), \ u \in U_q(gl(m|n-1) \oplus gl(1)).
\]

Therefore, \( \bar{C} := \sum_{a \in \mathcal{Y}} (-1)^{|[a|]} S^2(Y_a) \otimes X_a \), defines an invariant of the quantum subgroup

\[
Ad_u(\bar{C}) = \sum_{a \in \mathcal{Y}} (-1)^{|[a|]+|b|} Ad_{u(2)}(S^2(Y_a)) \otimes Ad_{u(1)}(X_a)
\]

\[
= \epsilon(u)^{-1} \bar{C}, \ u \in U_q(gl(m|n-1) \oplus gl(1)).
\]

From this fact we can easily deduce that

Lemma 3 The \( \mathcal{C} \) defined by

\[
\mathcal{C} = \sum_{a \in \mathcal{Y}} (-1)^{|b|} Y_a \otimes S^{-1}(X_a),
\]

satisfies

\[
[\Delta'(u), \mathcal{C}] = 0, \ \forall u \in U_q(gl(m|n-1) \oplus gl(1)), \quad (11)
\]

where \( \Delta' \) is the opposite co - multiplication.

The finite dimensional irreducible representations of \( U_q(gl(m|n)) \) defined over the complex field were studied systematically in [20]. The main conclusions still apply to the present situation. We have

Every finite dimensional irreducible \( U_q(gl(m|n)) \) module admits a basis, relative to which the \( K_a \) are diagonal.

Every finite dimensional irreducible \( U_q(gl(m|n)) \) module is of highest weight type and is uniquely ( up to isomorphisms ) characterized by a highest weight.
An irreducible \( U_q(gl(m|n)) \) module \( V(\lambda) \) with the maximal vector \( v_+^\lambda \)

\[
E_{a \ a+1} v_+^\lambda = 0, \quad a \in I', \\
K_b v_+^\lambda = q_b^\lambda v_+^\lambda, \quad b \in I.
\]

is finite dimensional iff \( \lambda \) satisfies \( \lambda_a - \lambda_{a+1} \in \mathbb{Z}_+, \ a \neq m. \)

When \( V(\lambda) \) is finite dimensional, it has the same weight space decomposition as that of the corresponding irreducible \( gl(m|n) \) module with highest weight \( \lambda \).

### 3.2 A realization of \( U_q(gl(m|n)) \) on projective superspace

Before embarking on the Bott - Borel - Weil construction for the quantum supergroup \( U_q(gl(m|n)) \), we consider first a simple realization for it in terms of difference operators on the projective superspace \( \mathbb{C}P^{m|n-1} \).

Let \( z \) be a variable living in \( \mathbb{C}_c \). Define a difference operator \( \nabla_z \) on analytic functions by

\[
\nabla_z f(z) = \frac{f(qz) - f(q^{-1}z)}{z(q - q^{-1})}.
\]

Then

\[
\nabla_z(f(z)h(z)) = \nabla_z f(z) q^{dz} h(z) + q^{-dz} f(z) \nabla_z h(z) = \nabla_z f(z) q^{dz} h(z) + q^{dz} f(z) \nabla_z h(z),
\]

where

\[
d_z = z \frac{d}{dz}
\]

is the scaling operator. For a Grassmannian variable \( \theta \), we denote

\[
\nabla_{\theta} = \frac{d}{d\theta}, \\
d_{\theta} = \theta \frac{d}{d\theta}.
\]

Consider \((W_a)_{a \in I} = (\zeta_i; w_\mu) \in \mathbb{C}^{m|n}. \) It is clear that

\[
[\nabla_{w_\mu}, \nabla_{w_\nu}] = 0, \\
[\nabla_{w_\mu}, \nabla_{\zeta_i}] = 0, \\
(\nabla_{\zeta_i}, \nabla_{\zeta_j}) = 0, \quad \forall i, j, \mu, \nu.
\]

The quantum supergroup \( U_q(gl(m|n)) \) can be realized in terms of the difference operators. Direct calculations can easily establish that the following operators satisfy the defining relations (10) of \( U_q(gl(m|n)) \)

\[
E_{a \ a+1} = - W_{a+1} \nabla_{W_a}, \\
E_{a+1 \ a} = -(-1)^{[a+1][|a|+1]} W_a \nabla_{W_{a+1}}, \quad a \in I', \\
K_b^{\pm 1} = q_b^{\pm c} \frac{d W_b}{d W_b}, \quad b \in I,
\]  

(12)
where \( c \in \mathbb{C} \) is an arbitrary but fixed even number. The homogeneous polynomials in the components of \((W_a)\) of a given degree furnishes an irreducible module over \( U_q(gl(m|n))\) in this realization. Such irreducible modules can all be obtained through repeatedly tensoring the contragredient vector module with itself.

To turn this realization of \( U_q(gl(m|n))\) into one on \( \mathbb{C}P^{m|n-1}\), we consider, e.g., the coordinate patch \( U_{m+n} \subset \mathbb{C}P^{m|n-1}\). Let 
\[
(Z_a)_{a \in \mathbf{I}'} = (\theta_i; z_\mu)_{1 \leq i \leq m; m+1 \leq \mu \leq m+n-1},
\]
\[
\theta_i = \zeta_i / w_{m+n}, \quad 1 \leq i \leq m,
\]
\[
z_\mu = w_\mu / w_{m+n}, \quad m+1 \leq \mu \leq m+n-1.
\]

Set \( \nabla_{Z_a} = \nabla_a \). Then 
\[
d_{W_a} = d_a,
\]
\[
W_a \nabla_{W_b} = Z_a \nabla_b,
\]
\[
W_{m+n} \nabla_{W_b} = \nabla_b,
\]
\[
W_a \nabla_{W_{m+n}} = Z_a W_{m+n} \nabla_{W_{m+n}}, \quad \forall a, b \in \mathbf{I}',
\]
where \( W_{m+n} \nabla_{W_{m+n}} \) can be expressed as 
\[
W_{m+n} \nabla_{W_{m+n}} = \frac{q^{d_{W_{m+n}}} - q^{-d_{W_{m+n}}}}{q - q^{-1}}.
\]

Recall that \( \prod_{a \in \mathbf{I}} (K_a)^{(-1)^{|a|}} \) is a central element of \( U_q(gl(m|n))\). In an irrep realized by degree \( k \) homogeneous polynomials in \((W_a)\), it takes the eigenvalue \( q^{(m-n)c-k} \). Thus in this irrep, \( K_{m+n} \) and also \( W_{m+n} \nabla_{W_{m+n}} \) can be expressed in terms of the \( K_a, a \in \mathbf{I}' \).

Now the irrep can be realized solely in terms of the new variables \((Z_a)\). We collect the results into

**Proposition 4** The polynomials in components of \((Z_a)_{a \in \mathbf{I}'} \in U_{m+n} \subset \mathbb{C}P^{m|n-1}\) of degrees less or equal to \( k \) furnishes an irreducible \( U_q(gl(m|n)) \) module with the generators realized in terms of difference operators on \( U_{m+n} \) by
\[
K_a = q^c q_a^{-d_a}, \quad a \in \mathbf{I}',
\]
\[
E_{a+1} = -Z_{a+1} \nabla_a,
\]
\[
E_{a+1} = (-1)^{|a+1|(|a|+1)} W_a \nabla_{a+1}, \quad a + 1 \in \mathbf{I}',
\]
\[
K_{m+n} = q^{-k} q^{d_a} \prod_{a \in \mathbf{I}'} q_{m+n}^{-d_a},\quad (13)
\]
\[
E_{m+n-1} = -\nabla_{m+n-1},
\]
\[
E_{m+n} = -z_{m+n-1} \frac{q^{-c} K_{m+n} - q^c K_{m+n}^{-1}}{q_{m+n} - q_{m+n}^{-1}}.\quad (14)
\]

Such a realization can be constructed for each coordinate patch of \( \mathbb{C}P^{m|n-1}\), and the various realizations are related by coordinate changes on the projective superspace.

The \( E_{a+1}, E_{a+1} a < m + n - 1, \) and \( K_b, b \in \mathbf{I} \) given by equation (13) realize the \( U_q(gl(m|n-1) \oplus gl(1)) \) subalgebra of \( U_q(gl(m|n)) \). We will denote this realization by \( \Upsilon(U_q(gl(m|n-1) \oplus gl(1))) \), and introduce the algebra homomorphism
\begin{align*}
\Upsilon : U_q(gl(m|n-1)\oplus gl(1)) & \rightarrow \Upsilon(U_q(gl(m|n-1)\oplus gl(1))) \text{ defined by } \Upsilon(E_{a+1}) = E_{a+1}, \\
& \text{etc.}
\end{align*}

We now construct the tensor operator \( \Upsilon \) of the subalgebra in this representation, the components of which will be denoted by \( y_a \). Note that the complicated factor multiplying \( z_{m+n-1} \) in the expression of \( \E_{m+n} \) commutes with the subalgebra, thus we can ignore it without affecting the tensorial properties of the tensor operator. A simple calculation gives

\[ y_a = (-1)^{|a|+1} Z_a q^{-\sum_{b=a+1}^{m+n-1} \{(1)^{b+1}+d_b\}}, \quad a \in \Upsilon. \]

The corresponding \( C \) operator, which will be denoted by \( \mathcal{O} \), is given by

\[ \mathcal{O} = \sum_{a \in \Upsilon} (-1)^{|a|+1} y_a \otimes E_{a+m+n}. \tag{15} \]

An immediate consequence of Lemma 3 is that

**Lemma 4**

\[ [(\Upsilon \otimes id) \Delta'(u), \mathcal{O}] = 0, \quad \forall u \in U_q(gl(m|n-1)\oplus gl(1)), \tag{16} \]

where \( \Delta' \) represents the opposite co-product.

Let us re-write \( \mathcal{O} \) as

\[ \mathcal{O} = \mathcal{O}' (q^{-d_{m+n-1}} \otimes 1) + z_{m+n-1} \otimes E_{m+n-1}^{m+n}. \]

Then \( \mathcal{O}' \) does not depend on \( z_{m+n-1} \), and

\[ \mathcal{O}' (z_{m+n-1} \otimes E_{m+n-1}^{m+n}) = q^{-1} (z_{m+n-1} \otimes E_{m+n-1}^{m+n}) \mathcal{O}'. \]

Introduce the formal power series

\[ g(Z) = \exp_q(\mathcal{O}), \]
\[ \exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}. \tag{17} \]

The \( g(Z) \) is well behaved when acting on \( \Lambda_\infty[[Z]] \otimes V \) if the \( U_q(gl(m|n)) \) module \( V \) is finite dimensional. Consider the action of \( \mathcal{O}^k \) on \( 1 \otimes v \in \Lambda_\infty[[Z]] \otimes V \), where \( v \) is an arbitrary element of \( V \). An easy induction can establish that

\[ \left[ \mathcal{O}' (q^{-d_{m+n-1}} \otimes 1) + z_{m+n-1} \otimes E_{m+n-1}^{m+n} \right]^k (1 \otimes v) \]
\[ = \sum_{l=0}^{k} \frac{[k]_q!}{[k-l]_q! [l]_q!} (\mathcal{O}')^{k-l} (z_{m+n-1} \otimes E_{m+n-1}^{m+n} v), \]

which in turn leads to

\[ \exp_q(\mathcal{O}) (1 \otimes v) = \exp_q(\mathcal{O}') \exp_q(z_{m+n-1} \otimes E_{m+n-1}^{m+n}) (1 \otimes v). \]

The action of powers of \( \mathcal{O}' \) on \( z_{m+n-1}^k \otimes v \) can be similarly simplified. Such a process can be continued, and we eventually arrive at

**Lemma 5**

\[ \exp_q(\mathcal{O}) (1 \otimes v) = \exp_q(\mathcal{O}_1) \exp_q(\mathcal{O}_2) ... \exp_q(\mathcal{O}_{m+n-1}) (1 \otimes v), \quad \forall v \in V, \tag{18} \]

where

\[ \mathcal{O}_a = \tilde{Z}_a \otimes E_{a+m+n}, \]
\[ \tilde{Z}_a = q^m \sum_{b=a+1}^{m+n-1} (-1)^{|b+1|} Z_a, \quad a \in \Upsilon. \]
3.3 Bott - Borel - Weil construction

With the preparations of the last two subsections, we can now carry out the Bott - Borel - Weil construction for $U_q(gl(m|n))$. Let $U_q(p)$ be the parabolic subalgebra of $U_q(gl(m|n))$ generated by

\[ K_a^\pm, \ a \in I; \ E_{b+1}, \ E_{c+1}, \ b, \ c \in I', \ c < m + n - 1, \]

subject to the appropriate relations of (10). Then from results of [20] we can deduce that

\[ U_q(gl(m|n)) = U(q^{-}) U_q(p), \]

where $U(q^{-})$ is the subalgebra generated by $\{ E_{m+a} | a \in I' \}$.

The induced module construction of subsection 2.2 can be generalized to the present case without much difficulty. Given any irreducible $U_q(p)$ module $V_0$, we construct the infinite dimensional $U_q(gl(m|n))$ module

\[ \bar{V} = U_q(gl(m|n)) \otimes_{U_q(p)} V_0. \]

When $\bar{V}$ is irreducible, we set $V = \bar{V}$. If it is not irreducible, then the given condition that $V_0$ is an irreducible $U_q(p)$ module leads to the existence of a unique maximal submodule $M$ of $\bar{V}$. Now $V = \bar{V}/M$ is irreducible.

When $V_0$ is finite dimensional, it admits a unique maximal vector $v_+$ such that

\[ E_{a+1} v_+ = 0, \ a \in I'; \]

\[ K_b v_+ = q^{\lambda_b} v_+, \ b \in I, \]

and the $\lambda_a$ must satisfy the condition that $\lambda_a - \lambda_{a+1} \in \mathbb{Z}_+$, $m \neq a < m + n - 1$. If the highest weight satisfies the further condition that $\lambda_{m+n-1} - \lambda_{m+n} \in \mathbb{Z}_+$, then the resultant irreducible $U_q(gl(m|n))$ module $V$ is finite dimensional. In this way, we can construct all the finite dimensional irreducible $U_q(gl(m|n))$ modules.

We will denote by $V(\lambda)$ the finite dimensional irreducible $U_q(gl(m|n))$ module with highest weight $\lambda$. Decompose it into eigenspaces of $K_{m+n}$,

\[ V(\lambda) = \bigoplus_{k=0}^{L} V^{(k)}(\lambda), \ L < \infty, \]

where

\[ K_{m+n} v = q^{k+\lambda_{m+n}} v, \ \forall v \in V^{(k)}(\lambda). \]

We choose a basis for each subspace $V^{(k)}(\lambda)$. Then the union of all of them furnishes a basis $\{ w^\alpha \ | \ \alpha = 1, 2, ..., dim V(\lambda) \}$ for $V(\lambda)$. We order the $w^\alpha$ in such a way that $v^i = w^i, \ i = 1, 2, ..., dim V^{(0)}(\lambda)$, are the basis elements of $V^{(0)}(\lambda)$. Let $\{ \langle w^\alpha \ | \ \rangle \}$ be the dual basis of $V(\lambda)^*$, i.e., $\langle w^\alpha | w^\beta \rangle = \delta_{\alpha \beta}$.

Similar to the classical case, we define a linear map $\Xi : V(\lambda) \to \Lambda_{\infty}[Z] \otimes V_0(\lambda)$ by

\[ \Xi_w(Z) = \sum_{i=1}^{d_0} (-1)^{[v^i]_1} \langle v^i | e^{\exp(O)}(1 \otimes w) \rangle \otimes v^i, \]

where

\[ d_0 = dim V^{(0)}(\lambda), \]
where \(\exp_q(\mathcal{O})\) is given in [17]. Since \(V(\lambda)\) is finite dimensional, \(\exp_q(\mathcal{O})(1 \otimes w)\) is well defined for all \(w \in V(\lambda)\).

Using Lemma 3, we can easily see that the elements of

\[
\{\Xi_a(Z) = \Xi_{w^a}(Z) | \alpha = 1, 2, \ldots, dim V(\lambda)\}
\]

are linearly independent. Their linear span \(V_2^\lambda \subset A_\infty[[Z]]_L \otimes V_0(\lambda)\) yields a \(U_q(gl(m|n))\) module with the module action defined by

\[
u \circ \Xi_w(Z) = \Xi_{w^\nu}(Z), \quad \forall \nu \in U_q(gl(m|n)),
\]

which is isomorphic to \(V(\lambda)\) itself.

The quantum supergroup \(U_q(gl(m|n))\) can now be realized in terms of operators acting on \(V_2^\lambda\). We have

**Proposition 5** Let \(\pi\) denote the irreducible representation of \(U_q(gl(m|n))\) afforded by the irreducible module \(V_2^\lambda\). Then

\[
\begin{align*}
\pi(E_{a a+1}) &= -Z_a \nabla_a \otimes 1 + q_a^{-a} q_{a+1}^{a+1} \otimes \pi_0(E_{a a+1}), \\
\pi(E_{a a+1}) &= -Z_a \nabla_a \otimes q_a^{-a} q_{a+1}^{a+1} + 1 \otimes \pi_0(E_{a a+1}), \quad a < m + n - 1, \\
\pi(K_b) &= q_b^{-b} \otimes \pi_0(K_b), \quad b \leq m + n - 1, \\
\pi(K_{m+n}) &= \sum_{n=m}^{m+n-2} q_n^{-a} \otimes \pi_0(K_{m+n}), \\
\pi(E_{m+n-1 m+n}) &= \nabla_{m+n-1} \otimes 1, \\
\pi(E_{m+n m+n+1}) &= \frac{1}{q_{m+n-1} - q_{m+n-1}^{-1}} \left\{ (z_{m+n-1} \otimes \pi_0(K_{m+n-1})) \pi(K_{m+n-1}^{-1}) - (z_{m+n-1} \otimes \pi_0(K_{m+n-1}^{-1}) \pi(K_{m+n-1}) \right\} \\
&+ \sum_{a=1}^{m+n-2} \tilde{Z}_a q_{m+n}^{-a+1} \sum_{b=1}^{a} q_b^{-b} \otimes \pi_0(E_{m+n-1} K_{m+n-1}^{-1} K_{m+n-1}),
\end{align*}
\]

where \(\pi_0\) denotes the irreducible representation of \(U_q(p)\) afforded by \(V^{(0)}(\lambda)\).

**Proof:** We consider the \(U_q(gl(m|n-1) \oplus gl(1))\) subalgebra first. Observe that

\[
\begin{align*}
(\Upsilon \otimes id) \Delta'(E_{a a+1}) (1 \otimes w) &= 1 \otimes E_{a a+1} w, \\
(\Upsilon \otimes id) \Delta'(E_{a a+1}) (1 \otimes w) &= 1 \otimes E_{a a+1} w, \quad \forall w \in V(\lambda), \quad a < m + n - 1.
\end{align*}
\]

By applying Lemma 4, we obtain

\[
E_{a a+1} \circ \Xi_w(Z) = \sum_{i=1}^{d_0} (-1)^{[v^i]} (1+\delta_{a m}+[w]) \langle v^i | e^{\exp_q(\mathcal{O})} (\infty \otimes \mathcal{E}_{a a+1} \otimes \Xi_w(Z)) \rangle \otimes \Xi_w(Z)
\]

which immediately yields the expression for \(\pi(E_{a a+1})\). In exactly the same way, we can obtain the realizations of the other generators of this subalgebra.
The realization of $E_{m+n-1 \cdot m+n}$ can be easily obtained from Lemma 3. To prove the formula for $E_{m+n \cdot m+n-1}$, however, considerable effort is required. We will need the following technical results

$$\left[ \exp_q (\mathcal{O}_{m+n-1}) , \ 1 \otimes E_{m+n \cdot m+n-1} \right]$$

$$= \frac{1}{q_{m+n} - q^{-1}_{m+n}} \left[ (z_{m+n-1} \otimes K_{m+n-1}) \exp_q (\mathcal{O}_{m+n-1}) (1 \otimes K_{m+n}^{-1}) \right.$$

$$\left. - (z_{m+n-1} \otimes K_{m+n-1}^{-1}) \exp_q (\mathcal{O}_{m+n-1}) (1 \otimes K_{m+n}) \right], \tag{20}$$

$$\exp_q (\mathcal{O}_a) , \ 1 \otimes E_{m+n \cdot m+n-1}$$

$$= (\tilde{Z}_aq_{m+n}^{-1} \otimes E_{a \cdot m+n-1}) \exp_q (\mathcal{O}_a) (1 \otimes K_{m+n-1}K_{m+n}^{-1}), \tag{21}$$

which can be proven by repeatedly applying Lemma 4.

Set

$$\eta = \sum_{i=1}^{d_0} \langle v^i | \exp_q (\mathcal{O}_1) \ldots \exp_q (\mathcal{O}_{m+n-2}) \left[ \exp_q (\mathcal{O}_{m+n-1}) , \ 1 \otimes E_{m+n \cdot m+n-1} \right] (1 \otimes w) \rangle$$

$$\otimes v^i (-1)^{[v^i] (1+[w])}$$

$$\zeta = \sum_{i=1}^{d_0} \langle v^i | \exp_q (\mathcal{O}_1) \ldots \exp_q (\mathcal{O}_{m+n-2}) , \ 1 \otimes E_{m+n \cdot m+n-1} \exp_q (\mathcal{O}_{m+n-1}) (1 \otimes w) \rangle$$

$$\otimes v^i (-1)^{[v^i] (1+[w])}.$$

Then

$$E_{m+n \cdot m+n-1} \circ \Xi_w = \eta + \zeta.$$

By using (20), we can re-write $\eta$ as

$$\eta = \frac{(z_{m+n-1} \otimes \pi_0 (K_{m+n-1}) \pi (K_{m+n}^{-1}) - (z_{m+n-1} \otimes \pi_0 (K_{m+n-1}^{-1}) \pi (K_{m+n}))}{q_{m+n-1} - q^{-1}_{m+n-1}} \Xi_w.$$

Lemma 4 asserts that $E_{a \cdot m+n-1}$ (anti)commutes with all $E_{c \cdot m+n}$ if $c < a$. Thus by using (21) we arrive at

$$\zeta = \sum_{i=1}^{d_0} \sum_{a=1}^{m+n-2} \langle v^i | (\tilde{Z}_aq_{m+n}^{-1} \otimes E_{a \cdot m+n-1} K_{m+n-1}) \exp_q (\mathcal{O}_1) \ldots \exp_q (\mathcal{O}_a)$$

$$\times (1 \otimes K_{m+n}^{-1}) \exp_q (\mathcal{O}_{a+1}) \ldots \exp_q (\mathcal{O}_{m+n-1}) (1 \otimes w) \rangle \otimes v^i (-1)^{[v^i] (1+[w])}$$

$$= \sum_{i=1}^{d_0} \sum_{a=1}^{m+n-2} \langle v^i | (\tilde{Z}_aq_{m+n}^{-1} \sum_{b=1}^{m+n-2} d_b \otimes E_{a \cdot m+n-1} K_{m+n-1}K_{m+n}^{-1}) \exp_q (\mathcal{O}_1) \ldots \exp_q (\mathcal{O}_a)$$

$$\times (1 \otimes \pi_0 (E_{a \cdot m+n-1} K_{m+n-1}K_{m+n}^{-1})) \Xi_w.$$
CONCLUSION

As we have explained, the construction of the vector coherent states for the supergroup \( GL(m|n) \) at the classical level can be regarded as one manifestation of a supersymmetric generalization of the celebrated Bott - Borel - Weil theorem, and the vector coherent states themselves may be interpreted as holomorphic sections of homogeneous super-vector bundles. It should be possible to give the quantum coherent states a similar interpretation within a yet to be fully developed framework of noncommutative geometry. The results on the quantum Bott - Borel - Weil construction should feedback concrete useful information, improving this framework itself. In a forthcoming publication, we will develop a global version of the quantum Bott - Borel - Weil construction, the connection of which with the results presented here will also be clarified. In doing so, results of references [19] [15] [14] [9] on quantum fiber bundles will come to play.

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