A new upper bound for the largest complete \((k, n)\)-arc in \(PG(2, 71)\)

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**Abstract**

In this current work, we are presenting a new upper bound for some \(m_{6}(2, 71)\). Generally, in the projective plane in the two dimensional projective geometry of order \(q\) (briefly denoted by \(PG(2, q)\)) over a selected field \(F_q\) of a \(q\) number of components, a \((k, n)\)-arc can be defined as the set \(K\) of \(k\) number of points having mostly \(n\) number of points on any selected line of the plane. Therefore, this work can be done after determining what is \(k\) such that \(K\) becomes complete, as well as, these values are not included in a \((k+1, n)\)-arc. Particularly, finding a value that represents the largest existent value \(k\) for a complete \(K\), that can be written as \(m_{6}(2, q)\). In this current projective plane, the blocking set represents the complement of a \((k, n)\)-arc \(K\) in the two dimensional projective geometry of order \(q\) with \(s = q + 1 - n\).

**Keywords** — Upper Bound arc, Lower Bound arc, Maximal arc, PG(2,71), PG(2,q), Projective Geometry, Projective Plane.

**Mathematics subject classification:** 51E20

1 Introduction

In general, the points and the lines that are contained in any existent projective plane of the order \(q\) have the same number which is \(q^2 + q + 1\). In addition, \(q+1\) are distributed on every line where two different points are placed on each line. In other words, the point \(q\), is contained in \(q+1\) number of lines exactly, and a common point is placed in two distinct lines.

An \(s\)-fold blocking set \(B\) in the two dimensional projective geometry of order \(q\) can be defined as a set of points where every line has at least an \(s\) number of points of \(B\) and some of the lines contains exactly \(s\) number of points of \(B\). In fact, we consider the blocking set as the complement of a \((k, n)\)-arc \(K\) in \(PG(2, q)\) with \(s = q + 1 - n\). On the other hand, the lines are the smallest blocking sets, and any blocking sets has a line is going to be called trivial.

Blocking sets have been first studied in 1969 by Di Paola [8], where the author has calculated the minimum size of non-trivial blocking set in the two dimensional projective geometry of order \(q\), having orders of 4,5,7,8,9. The effective problem was the minimum size of a blocking set in an affine geometry of order \(q\). In 1981, Hill and Mason [9] studied multiple blocking sets in \(PG(2, q)\). Also, they noted that there exists a 2-blocking set of size \(3q\) when \(q\) is even, as well as a 3-blocking set of size \(4q\). In 1994, Ball [3] proved that (78,8)-arcs and (90,9)-arcs are the largest complete arcs in \(PG(2, 11)\); in \(PG(2, 13)\), there exists no (106,9)-arc, no (110,10)-arc and no (134,11)-arc. Also, in 1996, Ball [5] found a new lower bound of a triple blocking set in terms of the size in the Desarguesian projective plane \(PG(2, q)\) for \(q < 11\). After that, in the Desarguesian projective plane for \(q \geq 11\), Ball and Blokhuis [6] were able to obtain more than lower bounds of the double blocking set in terms of the size. In 2004, Daskalov [7] found the largest complete \((k, n)\)-arc in \(PG(2, 17)\) for \(n = 11, \ldots, 16\). In 2018, Alabudullah [1] proved a new lower bound for the smallest complete \((k, n)\)-arc in \(PG(2, q)\). Finally, in 2019, Yaseen and others [12] found a maximal arc in \(PG(2, q)\), where \(q\) is prime.

The main idea of this work is to obtain upper bounds for \(k\) of a \((k, n)\)-arc in the two dimensional projective geometry of order \(71\) for \(\frac{1}{2}(q + 3) < n < q - 1\).
Definition 1.1 (Hirschfeld [10], Mullen [11]).

(i) In the two dimensional projective geometry of order $q$, the projective plane over the field $\mathbb{F}_q$, a $(k,n)$-arc is a set $\mathcal{K}$ of $k$ points such that every line of the plane meets $\mathcal{K}$ in at most $n$ points, with equality for some line.

(ii) The maximum value of $k$ for a $(k,n)$-arc in the two dimensional projective geometry of order $q$ is denoted by $m_n(2,q)$. 

Theorem 1.2 (Ball [3]). Let $\mathcal{K}$ be a $(k,n)$-arc in the two dimensional projective geometry of order $q$, where $q$ is prime. 

(i) When $n \leq (q + 1)/2$, then $k \leq (n - 1)q + 1$.

Theorem 1.3 (Ball [4]). Let $\mathcal{B}$ be an $s$-blocking set in the two dimensional projective geometry of order $q$, where $q$ is prime and greater than 3.

(i) When $s < q^2/2$, then $|\mathcal{B}| \geq (s + \frac{1}{2})(q + 1)$.

(ii) When $s > q^2/2$, then $|\mathcal{B}| \geq (s + 1)q$.

Theorem 1.4 (Ball [5]). Let $\mathcal{B}$ be an $s$-blocking set in the two dimensional projective geometry of order $q$ that contains a line.

(i) When $(s - 1, q) = 1$, then $|\mathcal{B}| \geq q(s + 1)$.

(ii) When $(s - 1, q) > 1$ as well as $s \leq q^2/2 + 1$, then $|\mathcal{B}| \geq sq + q - s + 2$.

(iii) When $(s - 1, q) > 1$ as well as $s \geq q^2/2 + 1$, then $|\mathcal{B}| \geq s(q + 1)$.

Theorem 1.5 (Hirschfeld [10]). Let $\mathcal{B}$ be an $s$-blocking set in the two dimensional projective geometry of order $q$ that has no line. Then $\mathcal{B}$ has at least $sq + \sqrt{q} + 1$ number of points.

Lemma 1.6 (Ball [3]). For any set of $k$ number of points in the two dimensional projective geometry of order $q$, the below conditions hold:

\[
\sum_{i=0}^{q+1} \tau_i = q^2 + q + 1; \quad (1.1)
\]

\[
\sum_{i=1}^{q+1} i\tau_i = |\mathcal{B}|(q + 1); \quad (1.2)
\]

\[
\sum_{i=2}^{q+1} i(i-1)\tau_i = |\mathcal{B}|(|\mathcal{B}| - 1). \quad (1.3)
\]

Notation 1.7. For a $(k,n)$-arc $\mathcal{K}$ and an $(l,s)$-blocking set $\mathcal{B}$ in the two dimensional projective geometry of order $q$, let

\[
\tau_i \quad \text{represents the total number of the } i\text{-secants of } \mathcal{K};
\]

\[
l = q^2 + q + 1 - k,
\]

\[
s = q + 1 - n,
\]

\[
T \quad \text{represents the total number of the } s\text{-secants of } \mathcal{B};
\]

\[
r \quad \text{represents the length of longest secant},
\]

\[
\eta = (q^2 - 1)/4s.
\]
2 New Largest Bound

**Theorem 2.1.** In the two dimensional projective geometry of order 71, the \((k, n)\)-arc is not going to be existent for the given values of \(k\), below, giving corresponding upper bounds for \(m_n(2, 71)\).

| \(k\) | 2629 | 2701 | 2773 | 2845 | 2989 | 3133 | 3205 | 3277 | 3349 |
|---|---|---|---|---|---|---|---|---|---|
| \(n\) | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 |
| \(m_n(2, 71) \leq\) | 2628 | 2700 | 2772 | 2844 | 2988 | 3132 | 3204 | 3276 | 3348 |

**Proof** Since \(k=2629\) and \(n=38\), then \(l=2484\), \(s=34\) and

\[
|\mathcal{B}| = s(q + 1) + \frac{1}{2}(q + 1) \\
= 34 \times 72 + 36 \\
= 2484.
\]

This implies that \(|\mathcal{B}| = l\). Also,

\[
T = \frac{|\mathcal{B}|(q - 1)}{2s} \\
= \frac{2484 \times 70}{68}.
\]

\(T\) is not an integer, then there exists no \((2629, 38)\)-arc. So, \(m_{38}(2, q) \leq 2628\).

The remaining cases are proved similarly.

**Theorem 2.2.** In the two dimensional projective geometry of order 71, the \((k, n)\)-arc is not going to be existent for the given values of \(k\) below. Therefore, the upper bound for \(m_n(2, 71)\) is established in the corresponding cases.

| \(k\) | 2917 | 3061 | 3565 | 3637 | 3781 | 3997 | 4069 | 4213 | 4357 | 4429 | 4573 | 4645 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(n\) | 42 | 44 | 45 | 51 | 52 | 54 | 57 | 58 | 60 | 62 | 63 | 65 | 66 |
| \(m_n(2, 71) \leq\) | 2916 | 3060 | 3564 | 3636 | 3780 | 3996 | 4068 | 4212 | 4356 | 4428 | 4572 | 4644 |

**Proof** Since \(k=2917\) and \(n=42\), then \(l=2196\), \(s=30\) and

\[
|\mathcal{B}| = s(q + 1) + \frac{1}{2}(q + 1) \\
= 30 \times 72 + 36 \\
= 2196.
\]

This implies that \(|\mathcal{B}| = l\). Then

\[
T = \frac{|\mathcal{B}|(q - 1)}{2s} \\
= \frac{2196 \times 70}{60} \\
= 2562.
\]
So, finding a (2917, 42)-arc is equivalent to finding a \{2196, 30\}-blocking set \(B\). If \(r = 72\), then the blocking set \(B\) has a line and Theorem 1.4 implies that \(|B| \geq 2201\), and this contradicts the theorem. If \(67 \leq r \leq 71\), and if the lines through only a point are considered on just the given longest secant and not in \(B\), then consequently, \(B\) will have not less than \(30 + 71 + r\) number of points. This obviously contradicts that \(|B| = 2196\).

Now, if we take the 30-secants intersection along with the longest secant, equation 2.13 \[2\] gives

\[
\eta \geq (r - 30)(72 - r).
\]

The lower bounds for \(\tau_{30}\) are calculated according to (2.1) as shown in the following table.

| \(r\) | 66 | 65 | 64 | 63 | 62 | 61 | 60 | 59 | 58 | 57 | 56 | 55 |
|------|----|----|----|----|----|----|----|----|----|----|----|----|
| \(\eta \geq\) | 216 | 245 | 272 | 297 | 320 | 341 | 360 | 377 | 392 | 405 | 416 | 425 |
| \(r\) | 54 | 53 | 52 | 51 | 50 | 49 | 48 | 47 | 46 | 45 | 44 | 43 |
| \(\eta \geq\) | 432 | 437 | 440 | 441 | 440 | 437 | 432 | 425 | 416 | 405 | 392 | 377 |
| \(r\) | 42 | 41 | 40 | 39 | 38 | 37 | 36 | 35 | 34 | 33 | 32 |
| \(\eta \geq\) | 360 | 341 | 320 | 297 | 272 | 245 | 215 | 185 | 152 | 117 | 80 |

This table shows that all values of \(r\) for \(r = 66, \ldots, 32\) give a contradiction. This is because the value of \(\eta\) is 42.

However, for \(r = 30\) and \(r = 31\), equations (1.1), (1.2) and (1.3) become as the following:

\[
\begin{align*}
\tau_{30} + \tau_{31} & = 5113, \\
30\tau_{30} + 31\tau_{31} & = 158112, \\
870\tau_{30} + 930\tau_{31} & = 4820220.
\end{align*}
\]

As there is no solution of this system, so no \(\{2196, 30\}\)-blocking set exists and hence no (2917, 42)-arc exists.

The other bounds are established similarly.

\[\blacksquare\]

**Conclusion**

To sum up, the non-existence of bunch of \((k, n)\)-arcs that are in the two dimensional projective geometry of the order 71 is proved as well as a new upper bound in the same projective geometry order is found as \(m_n(2, 71) \leq 36(2n - 3)\) for \(\frac{1}{4}(q + 3) < n < q - 1\).

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