Channel-Optimized Quantum Error Correction

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We develop a theory for finding quantum error correction (QEC) procedures which are optimized for given noisy channels. Our theory accounts for uncertainties in the noise channel, against which our QEC procedures are robust. We demonstrate via numerical examples that our optimized QEC procedures always achieve a higher channel fidelity than the standard error correction method, which is agnostic about the specifics of the channel. This demonstrates the importance of channel characterization before QEC procedures are applied. Our main novel finding is that in the setting of a known noise channel the recovery ancillas are redundant for optimized quantum error correction. We show this using a general rank minimization heuristic and supporting numerical calculations. Therefore, one can further improve the fidelity by utilizing all the available ancillas in the encoding block.

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I. INTRODUCTION

Quantum error correction is often considered the backbone of quantum information processing, since it converts what is essentially an analog information processor, subject to a continuum of errors, into a digital one, whose errors are discretized. The theory of quantum error correction was developed in analogy to classical coding for noisy channels [1–6]. These initial efforts focused on finding conditions and procedures for perfect recovery of quantum states passing through noisy channels. Recently, several authors considered error correction design as an optimization problem, with fidelity as the optimization target [7–10]. In this work we further develop the theory of optimal quantum error correction. As in [7–10], we consider the scenario where one has knowledge of the noise channel, and find correspondingly optimal codes.

That is, we assume that one has already performed a channel identification procedure, e.g., via quantum process tomography [11]. We show how, armed with a knowledge of the channel, one can design highly robust error correction procedures, whose fidelity is always at least as good as that of the "agnostic" codes of standard error correction [1–6].

More specifically, we present an indirect approach to fidelity maximization based on minimizing the error between the actual channel and the desired channel. This approach, like the previously developed approach to direct fidelity optimization, leads naturally to bi-convex optimization problems, namely, two semidefinite programs (SDPs) [12] which can be iterated between the recovery and encoding. For a given encoding the problem is convex in the recovery. For a given recovery, the problem is convex in the encoding. An important advantage of this approach is that noisy channels, which do not satisfy the standard assumptions for perfect correction [1–6], can be optimized for the best possible encoding and recovery.

The conventional fidelity optimization targets are the encoding and recovery operators. An important way in which the present work differs from previous studies is in the fact that we further add the distribution of the ancillas in the encoding and recovery to the optimization problem. This way, we utilize all possible degrees of freedom for optimization. As a consequence, we find a rather surprising result: in the optimized error correction procedure the fidelity is indifferent to the existence of the recovery ancillas. This result paves the way toward a more efficient utilization of the ancillas. Namely, we can use all the available ancilla qubits in the encoding to increase the fidelity.

Standard error correction schemes, as well as those produced by the aforementioned optimization methods which are tuned to specific errors, are often not robust to even small changes in the error channel. These errors can be mitigated by fault-tolerant methods which rely on several levels of code concatenation [13]. However, our method naturally enjoys a desirable robustness against error variations. We show a means to incorporate specific models of error channel uncertainty, resulting in highly robust error correction. Nevertheless, concatenated fault tolerant quantum error correction still enjoys a certain important advantage over the procedures we derive in this work, namely, it is robust also against imperfections in the encoding and recovery procedures, while here, as is the case for standard quantum error correction formulations, we assume these to be perfectly executed.

Since the number of optimization variables scales exponentially with the number of qubits used in the encoding and recovery operations, the computational effort required to solve any of the semidefinite program optimization (SDP) problems...
is similarly burdened. In order to reduce this effort we propose an approach based on optimization via the constrained least squares method. This alternative approach for solving the indirect optimization problem does not utilize semidefinite programming, and is significantly faster in our numerical simulations. Surprisingly, this method returns the exact same result as the SDP approach.

The organization of the paper is as follows. In Section II, we explain the problem formulation including the standard error correction model and state the direct and indirect optimization problems to be addressed. The indirect approach is explored in Section III. In Section IV, we investigate the optimal distribution of the ancillas between the recovery and encoding. Examples of the methods presented are given in Section V. Appendices A-E provide proofs and supporting material.

II. PROBLEM FORMULATION

A. Standard error correction model

Subject to the assumption that the initial system-bath state is classically correlated [14], the dynamics of an open quantum system can be represented in an elegant form known as the Kraus operator sum representation (OSR). In this representation, the noise $\mathcal{E}$ is described in terms of a completely-positive (CP) map: $\rho \rightarrow \sum_i A_i \rho A_i^{\dagger}$ [6]. Here $\rho$ is the initial system density matrix and the operators $A_i$, known as Kraus operators, or operation elements, satisfy the normalization relation $\sum_i A_i^{\dagger} A_i = I$ (identity). The standard error correction procedure involves CP encoding (C), error ($\mathcal{E}$), and recovery ($\mathcal{R}$) maps (or channels): $\rho_S \xrightarrow{\mathcal{C}} \rho_C \xrightarrow{\mathcal{E}} \sigma_C \xrightarrow{\mathcal{R}} \hat{\rho}_S$, as shown pictorially in the block diagram of Figure 1.

\[ \rho_S \xrightarrow{\mathcal{C}} \rho_C \xrightarrow{\mathcal{E}} \sigma_C \xrightarrow{\mathcal{R}} \hat{\rho}_S \]

FIG. 1: Standard representation of error correction.

Here $\rho_S$ is the $n_S \times n_S$ system state, $\rho_C$ is the $n_C \times n_C$ encoded state, $\sigma_C$ is the $n_C \times n_C$ perturbed encoded state, and $\hat{\rho}_S$ is the $n_S \times n_S$ recovered system state. Using the OSR:

\[ \hat{\rho}_S = \sum_{r,e} (R_r E_e C_C) \rho_S (R_r E_e C_C)^\dagger. \] (1)

The encoding elements $\{C_C\}_{c=1}^{m_C}$ and recovery elements $\{R_r\}_{r=1}^{m_R}$ are rectangular matrices, respectively $n_C \times n_S$ and $n_S \times n_C$, since they map between the system Hilbert space of dimension $n_S$ and the system-ancillas Hilbert space, the codespace, of dimension $n_C$. The error operation elements $\{E_e\}_{e=1}^{m_E}$ are square $(n_C \times n_C)$ matrices, and represent the effects of noise on the codespace. The number of elements, $m_C, m_E, m_R$ depend on the manner of implementation and basis representation [6]. More specifically, any OSR can be equivalently expressed, and consequently physically implemented, as a unitary with ancilla states [6, §8.23]. An example of this representation of the standard error correction model of Figure 1 is shown in the block diagram of Figure 2.

![FIG. 2: System-ancilla-bath representation of standard encoding-error-recovery model of error correction.](image)

In this case the encoding operation $\mathcal{C}$ is implemented by a unitary operator $U_C$ acting on the (tensor) product of the system state, $\rho_S$, and the encoding ancillas’ state, $|0_{CA}\rangle$, producing the encoded state $\rho_C = U_C (|0_{CA}\rangle \langle 0_{CA}| \otimes \rho_S) U_C^\dagger$. (The tensor ordering is arbitrary, but once established must remain fixed for consistency). If the encoding ancillas’ state has dimension $n_{CA}$, then the resulting codespace has dimension $n_C = n_S n_{CA}$. If, as is customary, we take $|0_{CA}\rangle$ as the $n_{CA}$-column vector with a one in the first element and zeros elsewhere (i.e., it is a tensor product of $\log_2 n_{CA}$ encoding ancillas, each in the state $|0\rangle = (1,0)^t$), then the OSR for $\mathcal{C}$ has the single $(m_C = 1) n_C \times n_S$ matrix element $C$ whose columns are the first $n_S$ columns of $U_C$, thus forming a set of orthonormal codewords, i.e.,

\[ U_C = [C \cdots], \text{ } C \text{ is } n_C \times n_S \] (2)

For the errors, $\mathcal{E}$, the ancillas’ states are not implemented by design, but rather, engendered by interaction with the bath, a term used to generically describe the physical environment. The error operation is thus equivalent to the unitary $U_E$ operating on the tensor product of $\rho_C$, the encoded state, and $\rho_B$, the bath state. The number of bath states may be very large, in principal infinite dimensional. However, it is always possible to represent $\mathcal{E}$ with a finite number of OSR elements with $m_E \leq n_B^2$ [6, Thm.8.3].

Finally, the recovery operation $\mathcal{R}$ can be implemented via the unitary $U_R$ operating on the (tensor) product of the perturbed encoded state, $\sigma_C$, and the (additional) recovery ancillas’ state $|0_{RA}\rangle$. If $|0_{RA}\rangle$ is an $n_{RA}$-column vector with a one in the first element and zeros elsewhere, then the OSR $\{R_r\}_{r=1}^{m_R}$ for $\mathcal{R}$ has $m_R = n_{CA} n_{RA}$ elements which consist of the first $n_C$ columns of $U_R$, i.e.,

\[ U_R = [R \cdots], \text{ } R = \begin{bmatrix} R_1 \\ \vdots \\ R_{m_R} \end{bmatrix}, \text{ } R \text{ is } n_S \times n_C \] (3)

The model represented in Figures 1 and 2 assumes that the encoding and recovery operations can be implemented much
faster than relevant time-scales associated with the bath. For a
detailed discussion of the validity of such a Markovian model
see [18]. Nevertheless, we will assume the model of Figures
1 and 2 for the remainder of this work, as complications asso-

Table I provides definitions of some frequently used symbols.

| Symbol | Definition |
|--------|------------|
| $n_S$  | dimension of the system space |
| $n_{CA}$ | dimension of the encoding ancillas space |
| $n_C$  | dimension of the (system + encoding ancillas) space, i.e., $n_C = n_S \times n_{CA}$ |
| $n_{RA}$ | dimension of the recovery ancillas space |
| $m_E$  | number of operation elements for error map |
| $m_R$  | number of operation elements for recovery map |

B. Performance measures

Assume that we are given the OSR elements of the error
channel $E$. This could be obtained, for example, from the out-
put of a quantum process tomography experiment [11]. The
error correction objective considered here is to design the en-
coding $C$ and the recovery $R$ so that, for a given error oper-
ation $E$, the map $\rho_S \to \hat{\rho}_S$ is as close as possible to a desired
$n_S \times n_S$ unitary logic gate $L_S$. Common measures of per-
formance between two quantum channels are typically based
on fidelity or distance [6], [20], [21], [22]. Here we will use the channel fidelity [7] between the error correction operation
$REC$ and the desired operation $L_S$:

$$f = \frac{1}{n_S^2} \sum_{r,e,c} |\text{Tr} \ L_S^\dagger R_r E_e C_c|^2. \quad (4)$$

where $0 \leq f \leq 1$ and from [5], [6, Thm.8.2], $f = 1$ if and only if there are constants $\delta_{rec}$ such that,

$$R_r E_e C_c = \delta_{rec} L_S, \quad \sum_{r,e,c} |\delta_{rec}|^2 = 1. \quad (5)$$

This suggests the indirect measure of fidelity, the “distance-
lke” error (using the Frobenius norm, $\|X\|_F^2 = \text{Tr} X^\dagger X$),

$$d = \sum_{r,e,c} \|R_r E_e C_c - \delta_{rec} L_S\|_F^2 = \sum_{r,e,c} \|RE(I_E \otimes C_c) - \Delta_e \otimes L_S\|_F^2 \quad (6)$$

where

$$\Delta_e \equiv [\delta_{rec}], \quad \dim \Delta_e = m_R \times m_E, \quad (7)$$

and $E$ is the $n_C \times n_C m_E$ rectangular “error matrix,”

$$E = [E_1 \cdots E_{m_E}], \quad (8)$$

$R$ is the $m_R n_S \times n_C$ matrix obtained by stacking the $m_R$
mats $R_r$ as in (3), and $I_E$ is the $m_E \times m_E$ identity. Hence, we have
$$\sum_{c} \text{Tr} \Delta_e^\dagger \Delta_e = \sum_{r,e,c} |\delta_{rec}|^2 = 1, \quad \text{and} \ R^\dagger R = \sum_{r} R^\dagger r R_r = I_C.$$  

We show in Appendix A that there exists a recovery and
encoding pair, $R,C$, which achieves perfect error correction (equivalently $d = 0, f = 1$), iff for $c, c' = 1, \ldots, m_C$

$$(I_E \otimes C_{c'}^\dagger) E^\dagger (I_E \otimes C_c) = \Delta_e^\dagger \Delta_c \otimes I_S \quad (9)$$

This is a generalization to non-unitary CP encoding of the
Knill-Laflamme condition for perfect error correction with
unitary encoding [5]. In this latter case, $C$ has only a single
$n_C \times n_S$ matrix element $C$, $C^\dagger C = I_S$, whose $n_S$ columns
are the codewords. As $f$ and $d$ are explicitly dependent on the
channel elements, they are convenient for optimization. Con-
sider then the following optimization problems.

**Direct Fidelity Maximization**

maximize $f(R,C)$

subject to $R^\dagger R = I_C, \ C^\dagger C = I_S$  

**Indirect Fidelity Maximization**

minimize $d(R,C, \Delta_1, \ldots, \Delta_{m_C})$

subject to $R^\dagger R = I_C, \ C^\dagger C = I_S, \ \sum_c \|\Delta_c\|_F^2 = 1 \quad (11)$

Here $C$ is the $n_C \times n_S$ matrix obtained in (2). The direct approach
was used in [7, 9, 10, 15-17]. As shown in Appendix B, $f$ and $d$ are related as follows:

$$f(R,C) = \left(1 - \frac{d(R,C)}{2n_S}\right)^2$$  

$$\hat{d}(R,C) = \min \{d(R,C, \Delta_1, \ldots, \Delta_{m_C}) | \|\Delta_c\|_F^2 = 1, \forall c\} \quad (12)$$

This shows that minimizing the distance (11) is equivalent to
maximizing fidelity (10).

C. Robust error correction

An important advantage of the method presented here is that unlike the standard error correction model, it accounts for uncertainty in knowledge of the channel. Such uncertainty may exist for many reasons. For example, different runs of a tomography experiment can yield different error channels \{E_\alpha\}_{\alpha=1}^\ell. Or, a physical model of the error channel might be generated by a Hamiltonian \(H(\theta)\) dependent upon an uncertain set of parameters \(\theta\). In any case, not accounting for the uncertainties typically leads to non-robust error correction, in the sense that a small change in the error model can lead to poor performance of the error correction procedure.

One way to account for these Hamiltonian parametric uncertainties is to take a sample from the set of Hamiltonians, say, \(\{H_\beta\}_{\beta=1}^S\). Tracing out the \(S\) bath states will result in a set of error channels \(\{E_{\alpha,k}\}_{\alpha=1}^\ell, k=1, \ldots, S\), where each error channel has OSR elements \(E_{\alpha,k}^{\ell}\), \(k=1, \ldots, S\), where \(k\) is the largest of the number of OSR elements in each sample. In those samples with a smaller number we can set the corresponding OSR elements to zero.

Two standard measures of robustness are the average-case and worst-case. For the average-case, suppose that each OSR set \(E_\alpha\) is known to occur with probability \(p_\alpha\). Then define the average error channel by the OSR,

\[
E^\text{avg} = \{ \sqrt{p_\alpha} E_{\alpha,k} \mid \alpha = 1, \ldots, \ell, k = 1, \ldots, \kappa \} \tag{13}
\]

The average error channel in this form has \(n\ell\kappa\) OSR elements, potentially a very large number. However, this number is readily reduced to no more than \(m_E = n_C^2\) using a singular value decomposition [6, Thm.8.3]. Associated with \(E^\text{avg}\) is the average channel fidelity,

\[
f^\text{avg} = \sum_\alpha p_\alpha f_\alpha = \frac{1}{n_S} \sum_{r,e,c} |\text{Tr} R_i E_{e'}^{\text{avg}} C_c^\dagger|^2 \tag{14}
\]

where \(E_{e'}^{\text{avg}}, e' = 1, \ldots, \ell\kappa\) are the OSR elements of \(E^\text{avg}\) in (13).

For average-case robust error correction we replace \(f\) in (10) with \(f^\text{avg}\) in (14), and using the relationship (12), replace \(d\) in (11) with,

\[
d^\text{avg} = ||RE^\text{avg}(I_E \otimes C) - \Delta \otimes L_S||_F^2
\]

\[
E^\text{avg} = [E_1^{\text{avg}}, \ldots, E_{\ell\kappa}^{\text{avg}}] \tag{15}
\]

A similar formulation exists for worst-case error correction which was considered in [10]; we do not consider it any further here. The remainder of the paper concentrates on the average-case objective and development of the associated optimization algorithms. The examples presented in Sec. V show that this approach yields a high degree of robustness to uncertainty in the optimal codes.

We now discuss methods to approximately solve (obtain local solutions to) the indirect optimization problem (11).

III. INDIRECT FIDELITY MAXIMIZATION

We consider the encoding operator \(C\) as a unitary operator acting on both the encoding ancillas and the input qubit. Using the constraints in (11), we can express the distance measure (6) as

\[
d(R, C, \Delta) = ||RE(I_E \otimes C) - \Delta \otimes L_S||_F^2
\]

\[
= n_S + \text{Tr } E(I_E \otimes CC^\dagger) E^\dagger - 2\text{Re } \text{Tr } RE(\Delta^\dagger \otimes CL_S^\dagger)
\]

where \(\Delta\) is the single \(m_R \times m_E\) matrix in (7) with \(m_R = n_Cn_{RA}\) (note that in this case, since there is only a single \(\Delta_c\) matrix, we drop the subscript \(c\)).

A. Optimal Recovery

Since only the last term in (16) depends on the recovery matrix \(R\), minimizing \(d(R, C, \Delta)\) with respect to \(R\) is equivalent to maximizing the last term. In Appendix A, we show that this maximization results in

\[
\max_{R^{\dagger}R = I_C} \text{Re } \text{Tr } RE(\Delta^\dagger \otimes CL_S^\dagger) = \text{Tr } \sqrt{E(\Gamma \otimes CC^\dagger) E^\dagger}, \tag{17}
\]

where the \(m_E \times m_E\) matrix \(\Gamma\) is defined as,

\[
\Gamma = \Delta^\dagger \Delta, \tag{18}
\]

and the associated \(n_{Cn_{RA}} \times n_C\) optimal recovery matrix is,

\[
R = [v_1 \ldots v_{n_C}] [u_1 \ldots u_{n_C}]^\dagger \tag{19}
\]

where \(v_i, u_i, i = 1, \ldots, n_C\) are, respectively, the right and left singular vectors in the singular value decomposition of the \(n_{Cn_{RA}}\) matrix \(E(\Delta^\dagger \otimes CL_S^\dagger)\), with the singular values, as usual, in descending order. Thus, to obtain the optimal recovery, we need first to find \(\Gamma\) which maximizes (17) – this is equivalent to minimizing \(d\) over \(R\). Following this we need to determine \(\Delta\) satisfying (18).

To find \(\Gamma\), observe that \(\Gamma \geq 0\) by definition (18), and the constraint \(||\Delta||_F = 1\) from (11) is equivalent to \(\text{Tr } \Gamma = 1\). Hence, optimal recovery can be obtained by first solving for \(\Gamma\) from,

\[
\text{maximize } \text{Tr } \sqrt{E(\Gamma \otimes CC^\dagger) E^\dagger}
\]

subject to \(\Gamma \geq 0, \text{Tr } \Gamma = 1\) \tag{20}

In Appendix D it is shown that the optimal \(\Gamma\) is the solution of an equivalent SDP.

The next step is to use (18) to obtain \(\Delta\) from \(\Gamma\). The fol-
lowing choice adheres to the given dimensions:

\[
\begin{cases}
\frac{n_{CA}}{m_E} \leq \frac{n_{CA}}{m_R} = \frac{m_E}{m_R} \\
\frac{n_{RA}}{m_{CA}} = m_E
\end{cases}
\]  
\Rightarrow \begin{cases}
\Delta = \sqrt{F} \\
R \text{ is tall } (n_S m_E \times n_C)
\end{cases}
\]
\[\begin{cases}
\Delta = \left[ \begin{array}{cc}
0_{n_{CA} - m_E \times m_E} \\
\sqrt{F}
\end{array} \right] \\
R \text{ is unitary } (n_C \times n_C)
\end{cases}
\]  
(21)

Clearly the choice of \( \Delta \) is not unique. In fact, the result does not change if \( \Delta \) is multiplied by a unitary, i.e., \( \Delta \rightarrow U \Delta \). This is exactly the unitary freedom in choosing the OSR elements [6]. Interestingly, however, from many numerical calculations we observe that the following holds:

\[
\begin{cases}
\text{rank}(\Gamma) = n_{CA} \text{ if } n_{CA} \leq m_E \\
\text{rank}(\Gamma) = m_E \text{ if } n_{CA} > m_E
\end{cases}
\]
(22)

Since the \( m_E \times m_E \) matrix \( \Gamma \) is Hermitian (= \( \Delta^\dagger \Delta \)), and \( \Delta \) is \( m_R \times m_E \) with \( m_R = n_{CA} n_{RA} \), it follows that if (22) is true then, \( n_{RA} = 1 \).

(23)

If, in the optimized error correction, the dimension of the recovery ancillas space is one, then the optimal recovery matrix \( R \) is always a unitary – recovery ancillas are redundant in maximizing the fidelity. Note that we started with a generic \( n_{RA} \) parameter, and the properties of the optimal solution led us to the above conclusion. Although we do not have a rigorous proof that the recovery ancillas are redundant, a compelling heuristic argument is offered in Section IV along with supporting numerical results.

B. Optimal Encoding

For a given \( R \) and \( \Delta \), the optimal encoding \( C \) can be found by solving (11) for \( C \), that is,

\[
\begin{align*}
\text{minimize} & \quad d(R, C, \Delta) = ||RE(I_E \otimes C) - \Delta \otimes L_S||_F^2, \\
\text{subject to} & \quad C^\dagger C = I_S
\end{align*}
\]
(24)

As shown in Appendix E, the optimal encoding \( C \) is given by,

\[
C = UV^\dagger
\]
(25)

The difference between the two algorithms is in computing the optimal recovery (Steps 1). In Step 1 of Algorithm-1, no iterations are required; the optimal recovery is achieved by solving the SDP (20). For Step 1 of Algorithm-2, an optimal recovery is the result of some number of iterations involving the constrained least-squares problem (26). Although at present a proof is not available, in every case we have tried the optimal fidelity in both recovery algorithms converges to the same result. Additionally, the total CPU-time in MATLAB to compute the optimal recovery in Algorithm-2 (including the iterations) is significantly less than the CPU-time for the recovery step in Algorithm-1 using YALMIP [28] to call the solver SDPT3 [29].
| Algorithm-1 | Algorithm-2 |
|-------------|-------------|
| Initialize \( R \) and \( C \) | Initialize \( R \) and \( C \) |
| Repeat | Repeat |
| 1. **Optimal recovery** | 1. **Optimal recovery** – Repeat a-c |
| (a) solve (20) for \( \Gamma \) | (a) solve (26) for \( \Delta \) |
| (b) \( \Gamma \to \Delta \) via (21) | (b) \( \Delta \to \Gamma \) via (19) |
| (c) \( \Delta \to R \) via (19) | (c) Until \( d \) stops decreasing |
| 2. **Optimal encoding** | 2. **Optimal encoding** |
| (a) solve (24) for \( C \) | (a) solve (24) for \( C \) |
| Until \( d \) stops decreasing | Until \( d \) stops decreasing |

### IV. DIMENSION OF THE RECOVERY ANCILLAS SPACE

In our formalism, the dimension of the Recovery ancillas’ space, i.e., the required number of recovery ancilla qubits, is determined by the rank of the \( m_E \times m_E \) matrix \( \Gamma \).

#### A. Rank minimization of \( \Gamma \)

In this section, we study the rank of \( \Gamma \) through a heuristic argument by noting the similarity between our problem and the so called “Rank Minimization Problem” (RMP) \[24\]:

\[
\text{minimize } \rank(X) \\
\text{subject to } X \in \mathcal{X}
\]

The matrix \( X \) is the optimization variable and \( \mathcal{X} \) is a convex set denoting the constraints.

Although several special cases of the RMP have well-known solutions, in general the RMP is known to be computationally intractable. However, there are a number of heuristic approaches to solving this problem. Restate (20) as follows,

\[
\begin{align*}
\text{minimize} & \quad \Tr(\Gamma) \\
\text{subject to} & \quad \Gamma \geq 0, \quad \sqrt{\E(\Gamma \otimes CC^\dagger) \E^\dagger} \geq \text{const.}
\end{align*}
\]

where the constant is the maximum which arose in (20). A well known heuristic for RMP when \( X \) is positive semidefinite \[25–27\] is to replace the rank objective with \( \Tr(X) \) and solve,

\[
\begin{align*}
\text{minimize} & \quad \Tr(X) \\
\text{subject to} & \quad X \in \mathcal{X}, \ X \geq 0
\end{align*}
\]

By comparing (29) with (30), we can view our problem in (20) as an RMP that minimizes the rank of \( \Gamma \). Thus, the rank of the optimal \( \Gamma \) is the smallest possible consistent with not changing the rank of our objective matrix, \( \sqrt{\E(\Gamma \otimes CC^\dagger) \E^\dagger} \). Noting that \( \rank(CC^\dagger) = n_S \) and \( \rank(E) = n_C \) and with a straightforward linear algebra analysis we find that this property holds if

\[
\begin{align*}
\rank(\Gamma) & \geq \nCA \quad \text{if } \nCA \leq m_E \\
\rank(\Gamma) & = m_E \quad \text{if } \nCA > m_E.
\end{align*}
\]

That is, in the first case, if \( \rank(\Gamma) < m_E \), \( \rank(\sqrt{\E(\Gamma \otimes CC^\dagger) \E^\dagger}) \) decreases by decreasing the rank of \( \Gamma \). But if \( \rank(\Gamma) \geq \nCA \), \( \rank(\sqrt{\E(\Gamma \otimes CC^\dagger) \E^\dagger}) = n_C \), and it does not depend on \( \rank(\Gamma) \). In the second case, \( \Gamma \) should be full rank. Therefore the rank of the optimal \( \Gamma \) is

\[
\begin{align*}
\rank(\Gamma_{\text{opt}}) & = \nCA \quad \text{if } \nCA \leq m_E \\
\rank(\Gamma_{\text{opt}}) & = m_E \quad \text{if } \nCA > m_E,
\end{align*}
\]

which agrees with (22). Note that the same argument also applies in the average case (15) with \( E \) replaced by \( E^{\text{avg}} \).

#### B. Numerical result for randomly generated error maps

Here, we examine the result above for randomly generated error maps. Namely, we find the rank of the optimal \( \Gamma \) for each random map by applying the indirect optimization method. The error map is modeled as shown in Fig. 2 as a unitary \( U_E \) acting on the joint codespace-bath Hilbert space. The unitary \( U_E \) arises from a randomly selected \( m_{EN_C} \times m_{EN_C} \) time-independent Hamiltonian \( H_E \), i.e., \( U_E = e^{-iH_E} \) (we work in units where \( \hbar = 1 \)). The unitary evolution operator generated
by this Hamiltonian at time $t = 1$ is

$$U_E = \exp(-iH_E) = \begin{bmatrix} E_1 & \cdots \\ E_2 & \cdots \\ \vdots \\ E_{mE} & \cdots \end{bmatrix} \tag{33}$$

That is, we pick the first $n_C$ columns of the matrix $U_E$. Here, $E_1 \ldots E_m$ are the $n_C \times n_C$ OSR elements of the error operation, and from (8), $E = [E_1 \cdots E_{mE}]$.

Figure 3 presents the channel fidelity vs. the number of iterations in Algorithm 1 for 100 random error maps. In this experiment, the system is a single qubit and one qubit is used as an encoding ancilla, i.e., $n_S = 2$, $n_{CA} = 2$. Each error map has 4 OSR elements, i.e., $m_E = 4$, and is generated using a $16 \times 16$ random Hamiltonian matrix according to (33). Therefore, the matrix $\Gamma$ in (20) is $4 \times 4$. Figure 4 shows the histogram of the rank of $\Gamma$ vs. the number of iterations. This histogram indicates that after 20 iterations in the optimization algorithm, the rank of $\Gamma$ is always two, which is equal to $n_{CA}$. In fact, those $\Gamma$ that are not rank 2 after 10 iterations are associated to the error maps with lower rate of fidelity convergence.

Figure 5, which shows the singular values of the same $\Gamma$ matrices, is included for comparison of the magnitude of the singular values. In all cases, the nonzero singular values are of the order of $10^{-1}$. The numerical precision of all the results is $10^{-8}$. We repeated the experiment for more than 1000 random maps with different dimensions (only 100 are shown), and the result holds for all of them. Namely, after sufficiently many iterations in Algorithm 1, the rank of the optimal $\Gamma$ is the same as the dimension of the encoding ancillas space, i.e., $\text{rank}(\Gamma_{\text{opt}}) = n_{CA}$.

V. EXAMPLES

We now apply the methods developed above to the goal of preserving a single qubit ($n_S = 2$) using a $q_C$-qubit ($n_C = 2^{q_C}$) codespace. In these examples, the error channel $E$ consists of single-qubit errors occurring independently on all qubits with probability $p$. We examine two cases of bit-flip and bit-phase-flip errors.

A. 3-qubit bit-flip errors

In this example, we consider the independently occurring bit-flip error as the noise channel, where the bit-flip operator
the average case, we computed an optimized encoding and
compute the optimization twice, once for each range. For
and

for the standard recovery [6] in the range
qubits followed by the standard recovery.

over the error channels corresponding to
ery. For the average case, we computed an optimized encod-

is X = [0 1 0]. We used qC = 2 encoding ancilla qubits.
There are 2^3 = 8 OSR error elements for 3-qubit encoding:

\[
\{ E_i \}_{i=1}^{8} = A_{i_1} \otimes A_{i_2} \otimes A_{i_3}, \ i_1, i_2, i_3 \in \{1, 2\} \\
A_1 = \sqrt{(1 - p)} I \ (\text{no error}) \\
A_2 = \sqrt{p} X \ (\text{bit-flip error})
\]

Figure 6 shows \( f \) vs. bit-flip probability \( p \) in the range
\( p \leq 0.9 \) for the standard 3-qubit code, optimal recovery at
each \( p \), average-case recovery over the \( p \) range, and no recov-
ery. For the average case, we computed an optimized encod-
ing and recovery for the single channel obtained by averaging
over the error channels corresponding to \( p = 0, 0.1, ..., 0.9 \) as
defined in (15). We then applied this encoding and recovery to
each of these 10 channels, thus producing the 10 fidelity values
shown. Note that the optimal recovery can be achieved
equivalently by either the constrained least squares method
(Algorithm-2) or the convex optimization method (Algorithm-
1). Interestingly, the standard 3-qubit code not only pro-
vides optimal recovery for the range \( p \leq 0.5 \), it is optimal for
both recovery and encoding in this range. For \( p > 0.5 \)
the standard code is clearly no longer optimal. Only in this
range does the optimal recovery outperform the standard code,
a phenomenon similar to what was reported for amplitude-
damping errors in [7]. Analysis of our optimal encoding recov-
ery results reveals the following simple picture. The opti-
mal code is the standard 3-qubit code for the entire \( p \) range,
\( i.e., |0\rangle = |000\rangle \) and \( |1\rangle = |111\rangle \). The optimal recovery is
the standard recovery [6] in the range \( 0 \leq p \leq 0.5 \). In the
range \( 0.5 \leq p \leq 0.8 \) the optimal recovery is a bit-flip on all
qubits followed by the standard recovery.

Figure 7 shows channel fidelity \( f \) in two ranges: \( p < 0.5 \) and
\( 0.5 < p \leq 0.9 \). Unlike the previous case, here we
compute the optimization twice, once for each range. For
the average case, we computed an optimized encoding and
recovery for the single channel obtained by averaging over
the error channels corresponding to \( p = 0, 0.1, ..., 0.4 \). We
then applied this encoding and recovery to each of these 5
channels, thus producing the 5 fidelity values shown in the
range \( 0 \leq p \leq 0.5 \). We then repeated this procedure for
\( p = 0.5, 0.6, ..., 0.9 \). For \( p < 0.5 \), the standard, optimal,
average-case, all coincide. For \( p > 0.5 \), the optimal and
average-case codes coincide and divert again from the stan-
dard. The optimal encoding and recovery are the same as in
Figure 6, \( i.e., \) the standard 3-qubit code, with standard recov-
eries in the range \( 0 \leq p \leq 0.5 \), and bit-flips preceding standard
recovery in the range \( 0.5 \leq p \leq 0.9 \). We conclude from the
examples in Figures 6 and 7 that optimal encoding and recov-
ery has no advantage over standard encoding and recovery
for low bit-flip probabilities \( p < 0.5 \), and thus increasing the
codespace would be required to improve fidelity. For large
errors \( p > 0.5 \), optimization is more effective in that it iden-
tifies an optimal recovery. In both cases the achieved opti-
mal fidelity is independent of the number of recovery ancillas
used, hence in all examples shown in Figures 6 and 7 there are
no additional recovery ancillas required. It is striking that the
average case fidelity matches the optimal in Figure 7, but not
in Figure 6. This is entirely due to the range of \( p \) values over
which the average is performed. The lesson is that the more
information is available about the noise channel, the more ro-
ust the encoding and recovery will be: in Figure 7 we know
that the probability is in the range \( [0, 0.5] \) or \( [0.5, 0.9] \), while
in Figure 6 we only know that it is in the range \( [0, 0.9] \). Absent
such information, robustness may still be attainable by exper-
imenting with tuning the encoding and recovery over a range
of channels.
FIG. 8: Channel fidelity $f$ vs. bit-phase flip probability $p$ for 3 qubit code and 0, 1, or 2 recovery ancillas, with optimal encoding and recovery.

B. Bit-Phase flip error

In this example, the noise channel consists of bit-phase flip errors $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ occurring independently with probability $p$. We do not allow for more than three to occur simultaneously (i.e., we consider weight-3 errors). We examine two cases: 1. Considering a fixed number of encoding ancillas, we compare the fidelity using different numbers of recovery ancillas. 2. We fix the total number of available ancilla qubits, and compare the fidelity for various distributions of encoding and recovery ancillas.

1. 5-qubit bit-phase flip error

In this example, the bit-phase flip errors occur independently on the input qubit and 4 ancillas. There are 26 error OSR elements: 1 for no error, 5 for a single error, 10 for double errors, and 10 for triple errors. Thus the matrix $\Gamma$ in (20) is $26 \times 26$ and the rank of $\Gamma_{\text{opt}}$ is equal to $n_{CA} = 16$, meaning that the optimal distribution of ancillas is having all four in the encoding block and none in the recovery block.

Figure 8 shows $f$ vs. bit-phase flip error probability $p$ for the optimal encoding/recovery in the case of zero, one and two recovery ancillas. The result shows that all cases yield the same fidelity. Therefore, the fidelity of the system is independent of the number of recovery ancillas.

2. bit-phase flip errors with a fixed number of ancillas

In this example, we consider six ancilla qubits that can be used either in the encoding block or in the recovery block. We compare the fidelity for the following distributions: four encoding ancillas and two recovery ancillas, five encoding ancillas and one recovery ancilla, and six encoding ancillas with no recovery ancilla. Figure 9 shows that the channel fidelity increases significantly by using the ancillas in the encoding instead of the recovery. Thus the most efficient use of ancillas is achieved when they are all used for encoding.

VI. CONCLUSION

We have presented an optimization approach to quantum error correction that yields codes which achieve robust performance, when tuned to a specific noise channel. An important aspect of developing optimal codes which are tuned to a class of errors, or are robust over a range of errors, is that the optimized performance levels may be sufficient for the intended purposes. Hence, no further increases in codespace dimension may be necessary. This cannot be known without performing the optimization.

We also showed that the fidelity of such a system is independent of the number of the recovery ancillas. This is entirely due to the structure of the error correction optimization problem, for which we found that a unitary recovery operator maximizes the fidelity of the system. However, the fidelity increases significantly by increasing the dimension of the encoding ancillas space. Therefore, in the optimal quantum error correction scheme, one should use all the available ancilla qubits in the encoding block.

Although not further developed here, the resulting codes, unlike standard codes, have support over all basis states. Some of the recovery structure is revealed via the indirect approach. This in turn leads to a method for approximating optimal recovery involving only a singular value decomposition, making it potentially useful in evaluating very large blocks of encoding to see if further performance improvement is possible.

We stress that there is an important difference between the standard error correction schemes [1–6] and the approach pre-
sented here. While in the standard case only the class of errors should be known, in our method the exact form of the noise map is required for optimization. In general, the noise map can be identified using quantum process tomography [11]. In most cases this extra knowledge is equivalent to identifying the probability of the error, which can also be found using our method. In order to identify the probability in a particular error model, one should calibrate the fidelity of the system using a fixed pair of recovery and encoding operators. Once the relation between the fidelity associated to this pair and the error probability is known, a measurement of the fidelity yields the probability.

It thus appears that the effectiveness of optimization is dependent upon the structure of the error operation, a result seemingly heralded by Feynman [30]:

“In a machine such as this there are very many other problems due to imperfections. . . . there may be small terms in the Hamiltonian besides the ones we’ve written. . . . At least some of these problems can be remedied in the usual way by techniques such as error correcting codes . . . But until we find a specific implementation for this computer, I do not know how to proceed to analyze these effects. However, it appears that they would be very important in practice. This computer seems to be very delicate and these imperfections may produce considerable havoc.”

Determining the “specific implementation” is currently an on-going research effort. Analyzing the “effects” however will undoubtedly be accomplished by a combination of physical modeling and/or system identification (e.g., process tomography and parameter estimation). This leads to an intriguing prospect: to integrate the results found here within a complete “black-box” error correction scheme, that takes quantum state or process tomography as input and iterates until it finds an optimal error correcting encoding and recovery.

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APPENDIX A: PROOF OF EQUATIONS (9), (17), (19)

The $n_C \times n_C n_{RA}$ matrix $W = E(\Delta^\dagger \otimes C)$ has a maximum rank of $n_C$. Hence a singular value decomposition is of the form $W = USV^\dagger$, $S = [S_0 \ 0]$, with $S_0$ an $n_C \times n_C$ diagonal matrix containing the $n_C$ singular values. If $V$ is partitioned as $V = [V_1 \ V_2]$ with $V_1$ being $n_{RA}n_C \times n_C$ then the objective function in (17) becomes,

$$\text{Re} \ \text{Tr} \ RW = \text{Re} \ \text{Tr} \ S_0 X, \ X = V_1^\dagger RU$$ (A1)
Since \( \|X\| \leq 1 \), then \( \text{Re } \text{Tr} S_0 X \leq \text{Tr} S_0 \). Equality occurs if and only if \( X = I_C \), or equivalently, \( R = V_I U^\dagger \), which is precisely the result in (19). This also establishes that the optimal objective function is \( \text{Tr} S_0 \) which, by definition, is equal to \( \text{Tr} \sqrt{WW^\dagger} \), thus

\[
\max_{R^\dagger R = I_C} \text{Re } \text{Tr} RW = \text{Tr} \sqrt{WW^\dagger} \tag{A2}
\]

which establishes (17).

Condition (9) follows directly from (5) by multiplying both sides by their respective conjugate (with indices \( c \) and \( c' \)) which also eliminates \( R \) because \( R^\dagger R = I_C \). This immediately establishes that (9) is a necessary condition for (5). To prove sufficiency, first expand (6) to get,

\[
d = \sum_c \text{Tr} (I_E \otimes C_c^\dagger) E^\dagger E (I_E \otimes C_c) \\
+ \text{Tr} \Gamma_c \otimes I_S - 2 \text{Re } \text{Tr} RE(\Delta^\dagger_c \otimes C_c L_S^\dagger) \tag{A3}
\]

\[
\Gamma_c = \Delta^\dagger_c \Delta_c
\]

From (A2), we get,

\[
\min_{R^\dagger R = I_C} d = \sum_c [\text{Tr} (I_E \otimes C_c^\dagger) E^\dagger E (I_E \otimes C_c) \\
+ \text{Tr} \Gamma_c \otimes I_S - 2 \text{Re } \text{Tr} RE(\Delta^\dagger_c \otimes C_c L_S^\dagger)]
\]

\[
W = \sum_c E(\Delta^\dagger_c \otimes C_c L_S^\dagger) \tag{A4}
\]

Using (9) we get, \( \text{Tr} \sqrt{WW^\dagger} = \sum_c E(\Delta^\dagger_c \otimes C_c L_S^\dagger) E^\dagger \). This, together with repeated uses of (9) shows that \( \min_{R^\dagger R = I_C} d = 0 \). Since \( d \) is a norm, and is zero, then so is its argument, which by definition establishes (5) and thus shows sufficiency of (9).

**APPENDIX B: RELATION BETWEEN FIDELITY \( f \) AND DISTANCE \( d \)**

The problem is,

\[
\text{minimize } d = \sum_c \|RE(I_E \otimes C_c) - \Delta_c \otimes L_S\|_F^2 \\
\text{subject to } \sum_c \|\Delta_c\|_F^2 = 1
\]

Form the Lagrangian,

\[
L = d + \lambda \sum_c \text{Tr} \Delta^\dagger_c \Delta_c - 1 \tag{B2}
\]

with \( \lambda \) the Lagrange multiplier. Then, \( \nabla_{\delta_{rec}} L = 0 \) when \( (n_S + \lambda)\delta_{rec} = \text{Tr} R_c E_c C_c L_S \). To enforce the constraint \( \sum_c \|\Delta_c\|_F^2 = 1 \) requires that \( (n_S + \lambda)^2 = \sum_{r,c,e} |\text{Tr} R_r E_c C_c L_S|^2 \). Hence,

\[
\Delta_c = \frac{\bar{\Delta}_c/\sqrt{\sum_c \|\Delta_c\|_F^2}}{\delta_{rec} = \text{Tr}(R_c E_c C_c L_S)^{\dagger}/n_S} \tag{B3}
\]

Observe that \( \sum_c \|\Delta_c\|_F^2 = f \). This together with \( \sum_c R_r^\dagger R_r = I_C \), \( \sum_c C_c^\dagger C_c = I_S \) gives the optimal distance as given implicitly by (12). Note also that with no constraint, \( \lambda = 0 \), the \( \bar{\Delta}_c \) are the optimal least-squares (unconstrained) solution.

**APPENDIX C: UNITARY FREEDOM IN EQUATION (17)**

In (17), \( \Gamma = \Delta^\dagger \Delta \) remains unchanged if \( \Delta \) is multiplied by a unitary. This unitary freedom is exactly the unitary freedom in describing the error map OSR. To see this, recall again from [6, Thm.8.2] that two error maps with OSR elements \( E = [E_1 \ldots E_{m_E}] \) and \( F = [F_1 \ldots F_{m_E}] \) are equivalent if and only if \( E_j = \sum_i W_{ij} F_j \) where the \( m_E \times m_E \) matrix \( W \) is unitary. Equivalently from (A3), \( E = F(W \otimes I_C) \). Substituting this for \( E \) into the left hand side of (17) gives,

\[
\text{Re } \text{Tr} RE(\Delta^\dagger \otimes CL_S^\dagger) = \text{Re } \text{Tr} RF((\Delta')^\dagger \otimes CL_S^\dagger) \tag{C1}
\]

with \( \Delta' = \Delta W^\dagger \). Hence, \( \Delta^\dagger \Delta' = W \Delta^\dagger \Delta W^\dagger = \Gamma \), which establishes the claim.

**APPENDIX D: SOLVING EQUATION (20) VIA AN SDP**

Problem (20) is of the form,

\[
\begin{align*}
\text{maximize } & \text{Tr} \sqrt{F(\Gamma)} \\
\text{subject to } & \Gamma \geq 0, \text{Tr} \Gamma = 1 \tag{D1}
\end{align*}
\]

where \( F(\Gamma) \) is linear in \( \Gamma \). Consider the relaxed problem,

\[
\begin{align*}
\text{maximize } & \text{Tr} Y \\
\text{subject to } & F(\Gamma) - Y^2 \geq 0, \Gamma \geq 0, \text{Tr} \Gamma = 1 \tag{D2}
\end{align*}
\]

This is an SDP in \( \Gamma \) and \( Y \) with Lagrangian,

\[
L(\Gamma, Y, P, Z) = -\text{Tr} Y - \text{Tr} P(F(\Gamma) - Y^2) - \text{Tr} Z \Gamma + \lambda (\text{Tr} \Gamma - 1) \tag{D3}
\]

The dual function is,

\[
g(P, \lambda, Z) = \inf_{Y, Z} L(\Gamma, Y, P, Z) = \begin{cases} 
\inf_Y \text{Tr}(PY^2 - Y) - \lambda, & Z = \lambda I - A(P) \\
-\infty, & \text{otherwise}
\end{cases} \tag{D4}
\]

with \( A(P) = \frac{2}{n} \text{Tr} PF(\Gamma) \), which is not dependent on \( \Gamma \) because \( F(\Gamma) \) is linear in \( \Gamma \). Performing the indicated infimum gives \( Y = (1/2)P^{-1} \) and \( g = -(\lambda + (1/4) \text{Tr} P^{-1}) \). The dual optimization associated with (D2) is to maximize \( g \), or equivalently, minimize its negative, i.e.,

\[
\begin{align*}
\text{minimize } & \lambda + \frac{1}{4} \text{Tr} P^{-1} \\
\text{subject to } & P > 0, \lambda I - A(P) \geq 0 \tag{D5}
\end{align*}
\]

This is an SDP in the dual variables \( P, \lambda \). For this problem strong duality holds [12]. Consequently, at optimality of (D2) and (D5) the complementary slackness condition is \( P_{\text{opt}}(F(\Gamma_{\text{opt}}) - Y_{\text{opt}}^2) = 0 \). Since \( P_{\text{opt}} > 0 \), we have
\[ Y_{\text{opt}} = \sqrt{F(\Gamma_{\text{opt}})} \] This establishes that solving the SDP (D2) is equivalent to solving the original problem (D1).

**APPENDIX E: SOLVING FOR C IN EQUATION (24)**

The problem is,

\[
\begin{align*}
\text{minimize} & \quad d = \sum_c \| RE(I_E \otimes C) - \Delta \otimes L_S \|_F^2 \\
\text{subject to} & \quad C^\dagger C = I_S
\end{align*}
\]  

(E1)

Form the Lagrangian,

\[ L = d + \text{Tr} \ P(C^\dagger C - I_S) \]  

(E2)

with \( P \) the Lagrange multiplier. Then, \( \nabla_C L = 0 \) when \( C = \bar{C}(I_S + P)^{-1} \) with \( \bar{C} \) as defined in (24). To enforce the constraint \( C^\dagger C = I_S \) requires that \( (I_S + P)^2 = \bar{C}^\dagger \bar{C} \). Hence, \( C = \bar{C}(\bar{C}^\dagger \bar{C})^{-1/2} \). The actual computation of \( C \) is done using the SVD (25)-(26). Note that with no constraint, \( P = 0 \), and \( \bar{C} \) is the optimal least-squares (unconstrained) solution.