Studies on the Lorenz model

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Abstract
We study the Lorenz model from the viewpoint of its accessible singularities and local index.

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1 Introduction
The Lorenz model

\[
\begin{align*}
\frac{dx}{dt} &= y - \sigma \varepsilon x, \\
\frac{dy}{dt} &= -xz + x - \varepsilon y, \\
\frac{dz}{dt} &= xy - \varepsilon bz
\end{align*}
\]

has been studied by Tabor and Weiss (1981) in detail [15]. In this paper, we study the phase space of (1) from the viewpoint of its accessible singularities and local index.

As the conditions which each accessible singular point can be resolved, we obtain the following

\[
\begin{align*}
\varepsilon(b - 1)(b - 2\sigma)(b + 3\sigma - 1) &= 0, \\
(b - 1)(b - 3\sigma + 1) &= 0, \\
\{b^2 - 5b - 2 - 3(b - 2)\sigma\}{\varepsilon^2}(b - 1)(7b - 15\sigma + 2) - 9 &= 0, \\
\varepsilon(b - 2\sigma)(b + 3\sigma - 1) &= 0.
\end{align*}
\]
These equations can be solved as follows:

\[ \{(\sigma, \varepsilon, b) = \left( \frac{1}{3}, \varepsilon, 0 \right), (1, -3, 2), (1, 3, 2), (2, 0, 1)\} \]

In each case, we study its first integrals, general solutions and phase space.

2 Accessible singularities

Let us review the notion of accessible singularity. Let \( B \) be a connected open domain in \( \mathbb{C} \) and \( \pi : \mathcal{W} \longrightarrow B \) a smooth proper holomorphic map. We assume that \( \mathcal{H} \subset \mathcal{W} \) is a normal crossing divisor which is flat over \( B \). Let us consider a rational vector field \( \tilde{\mathbf{v}} \) on \( \mathcal{W} \) satisfying the condition

\[ \tilde{\mathbf{v}} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})). \]

Fixing \( t_0 \in B \) and \( P \in \mathcal{W}_{t_0} \), we can take a local coordinate system \((x_1, \ldots, x_n)\) of \( \mathcal{W}_{t_0} \) centered at \( P \) such that \( H_{\text{smooth}} \) can be defined by the local equation \( x_1 = 0 \). Since \( \tilde{\mathbf{v}} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})) \), we can write down the vector field \( \tilde{\mathbf{v}} \) near \( P = (0, \ldots, 0, t_0) \) as follows:

\[ \tilde{\mathbf{v}} = \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{x_1 \partial x_2} + \cdots + a_n \frac{\partial}{x_1 \partial x_n}. \]

This vector field defines the following system of differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= a_1(x_1, x_2, \ldots, x_n, t), \\
\frac{dx_2}{dt} &= a_2(x_1, x_2, \ldots, x_n, t) \div x_1, \\
\vdots & \quad \vdots \\
\frac{dx_n}{dt} &= a_n(x_1, x_2, \ldots, x_n, t) \div x_1.
\end{align*}
\]

Here \( a_i(x_1, \ldots, x_n, t), \ i = 1, 2, \ldots, n, \) are holomorphic functions defined near \( P = (0, \ldots, 0, t_0) \).

**Definition 2.1.** With the above notation, assume that the rational vector field \( \tilde{\mathbf{v}} \) on \( \mathcal{W} \) satisfies the condition

\( (A) \quad \tilde{\mathbf{v}} \in H^0(\mathcal{W}, \Theta_{\mathcal{W}}(-\log \mathcal{H})(\mathcal{H})). \)
We say that \( \tilde{v} \) has an accessible singularity at \( P = (0, \ldots, 0, t_0) \) if

\[
x_1 = 0 \text{ and } a_i(0, \ldots, 0, t_0) = 0 \text{ for every } i, \ 2 \leq i \leq n.
\]

If \( P \in \mathcal{H}_{\text{smooth}} \) is not an accessible singularity, all solutions of the ordinary differential equation passing through \( P \) are vertical solutions, that is, the solutions are contained in the fiber \( \mathcal{W}_0 \) over \( t = t_0 \). If \( P \in \mathcal{H}_{\text{smooth}} \) is an accessible singularity, there may be a solution of \( (3) \) which passes through \( P \) and goes into the interior \( \mathcal{W} - \mathcal{H} \) of \( \mathcal{W} \).

Here we review the notion of local index. Let \( v \) be an algebraic vector field with an accessible singular point \( \overrightarrow{p} = (0, \ldots, 0) \) and \( (x_1, \ldots, x_n) \) be a coordinate system in a neighborhood centered at \( \overrightarrow{p} \). Assume that the system associated with \( v \) near \( \overrightarrow{p} \) can be written as

\[
\frac{d}{dt} Q \left( \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right) = \frac{1}{x_1} \left( Q \left( \begin{array}{ccc} a_1 & & \\ & a_2 & \\ & & \ddots \\ & & & a_n \end{array} \right) Q^{-1} \left( \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right) \right) + \left( \begin{array}{c} x_1 f_1(x_1, x_2, \ldots, x_n, t) \\ x_2 f_2(x_1, x_2, \ldots, x_n, t) \\ \vdots \\ x_n f_n(x_1, x_2, \ldots, x_n, t) \end{array} \right),
\]

\( (f_i \in \mathbb{C}(t)[x_1, \ldots, x_n], \ Q \in GL(n, \mathbb{C}(t)), \ a_i \in \mathbb{C}(t)) \)

where \( f_1 \) is a polynomial which vanishes at \( \overrightarrow{p} \) and \( f_i, i = 2, 3, \ldots, n \) are polynomials of order at least 2 in \( x_1, x_2, \ldots, x_n \). We call ordered set of the eigenvalues \( (a_1, a_2, \ldots, a_n) \) local index at \( \overrightarrow{p} \).

We remark that we are interested in the case with local index

\[
(1, a_2/a_1, \ldots, a_n/a_1) \in \mathbb{Z}^n.
\]

If each component of \( (1, a_2/a_1, \ldots, a_n/a_1) \) has the same sign, we may resolve the accessible singularity by blowing-up finitely many times. However, when different signs appear, we may need to both blow up and blow down.

In order to consider the phase spaces for the system \( \Pi \), let us take the compactification \([z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3\) of \((x, y, z) \in \mathbb{C}^3\) with the natural embedding

\[
(x, y, z) = (z_1/z_0, z_2/z_0, z_3/z_0).
\]

Moreover, we denote the boundary divisor in \( \mathbb{P}^3 \) by \( \mathcal{H} \). Extend the regular vector field on \( \mathbb{C}^3 \) to a rational vector field \( \tilde{v} \) on \( \mathbb{P}^3 \). It is easy to see that \( \mathbb{P}^3 \) is covered by four copies of \( \mathbb{C}^3 \):

\[
U_0 = \mathbb{C}^3 \ni (x, y, z),
\]

\[
U_j = \mathbb{C}^3 \ni (X_j, Y_j, Z_j) \ (j = 1, 2, 3),
\]

3
via the following rational transformations

\[
X_1 = \frac{1}{x}, \quad Y_1 = \frac{y}{x}, \quad Z_1 = \frac{z}{x},
\]
\[
X_2 = \frac{x}{y}, \quad Y_2 = \frac{1}{y}, \quad Z_2 = \frac{z}{y},
\]
\[
X_3 = \frac{x}{z}, \quad Y_3 = \frac{y}{z}, \quad Z_3 = \frac{1}{z}.
\]

The following Lemma shows that this rational vector field $\tilde{v}$ has five accessible singular points on the boundary divisor $\mathcal{H} \subset \mathbb{P}^3$.

**Lemma 2.1.** The rational vector field $\tilde{v}$ has five accessible singular points:

\[
\begin{align*}
P_1 &= \{ (X_1, Y_1, Z_1) | X_1 = Y_1 = Z_1 = 0 \}, \\
P_2 &= \{ (X_2, Y_2, Z_2) | X_2 = Y_2 = Z_2 = 0 \}, \\
P_3 &= \{ (X_3, Y_3, Z_3) | X_3 = Y_3 = Z_3 = 0 \}, \\
P_4 &= \{ (X_2, Y_2, Z_2) | X_2 = Y_2 = 0, \ Z_2 = \sqrt{-1} \}, \\
P_5 &= \{ (X_2, Y_2, Z_2) | X_2 = Y_2 = 0, \ Z_2 = -\sqrt{-1} \}.
\end{align*}
\]

Next let us calculate its local index at the points $P_i$ ($i = 1, 2, 3$).

| Singular point | Type of local index |
|---------------|---------------------|
| $P_1$         | $(0, \sqrt{-1}, -\sqrt{-1})$ |
| $P_2$         | $(0, 0, 0)$          |
| $P_3$         | $(0, 0, 0)$          |

We see that there are no solutions which pass through $P_i$, ($i = 1, 2, 3$), respectively.

In order to do analysis for the accessible singularities $P_4$, $P_5$, we need to replace a suitable coordinate system because each point has multiplicity of order 2.

At first, let us do the Painlevé test. To find the leading order behaviour of a singularity at $t = t_1$ one sets

\[
\begin{align*}
x &\propto \frac{a}{(t - t_1)^m}, \\
y &\propto \frac{b}{(t - t_1)^n}, \\
z &\propto \frac{c}{(t - t_1)^p},
\end{align*}
\]

from which it is easily deduced that

\[
m = 1, \quad n = 2, \quad p = 2
\]

and

\[
a = \pm 2\sqrt{-1}, \quad n = \mp 2\sqrt{-1}, \quad p = -2.
\]
Each order of pole \((m, n, p)\) suggests a suitable coordinate system to do analysis for the accessible singularities \(P_4, P_5\), which is explicitly given by

\[
(X, Y, Z) = \left(\frac{1}{x^m}, \frac{y}{x^n}, \frac{z}{x^p}\right).
\]

In this coordinate, the singular points \(P_4, P_5\) are given as follows:

\[
\begin{align*}
P_4 &= \{(X, Y, Z) = \left(0, \frac{\sqrt{-1}}{2}, \frac{1}{2}\right)\}, \\
P_5 &= \{(X, Y, Z) = \left(0, -\frac{\sqrt{-1}}{2}, \frac{1}{2}\right)\}.
\end{align*}
\]

Next let us calculate its local index at each point.

| Singular point | Type of local index          |
|----------------|-------------------------------|
| \(P_4\)       | \((-\frac{\sqrt{-1}}{2}, -2\sqrt{-1}, -\sqrt{-1})\) |
| \(P_5\)       | \((\frac{\sqrt{-1}}{2}, 2\sqrt{-1}, \sqrt{-1})\) |

Now, we try to resolve the accessible singular points \(P_4, P_5\).

**Step 0:** We take the coordinate system centered at \(P_4\):

\[
p = X, \quad q = Y - \frac{\sqrt{-1}}{2}, \quad r = Z - \frac{1}{2}.
\]

**Step 1:** We make the linear transformation:

\[
p^{(1)} = p, \quad q^{(1)} = q - \sqrt{-1}r, \quad r^{(1)} = r.
\]

In this coordinate, the system (11) is rewritten as follows:

\[
\frac{d}{dt} \begin{pmatrix} p^{(1)} \\ q^{(1)} \\ r^{(1)} \end{pmatrix} = \frac{1}{p^{(1)}} \begin{pmatrix} \frac{\sqrt{-1}}{2} & 0 & 0 \\ 0 & -2\sqrt{-1} & 0 \\ 0 & 0 & -\sqrt{-1} \end{pmatrix} \begin{pmatrix} p^{(1)} \\ q^{(1)} \\ r^{(1)} \end{pmatrix} + \ldots.
\]

By considering the ratio \(\left(1, -\frac{2\sqrt{-1}}{\sqrt{-1}}, -\frac{\sqrt{-1}}{\sqrt{-1}}\right) = (1, 4, 2)\), we obtain the resonances \((4, 2)\). This property suggests that we will blow up four times to the direction \(q^{(1)}\) and two times to the direction \(r^{(1)}\).

**Step 2:** We blow up at the point \(P_4 = \{(p^{(1)}, q^{(1)}, r^{(1)}) = (0, 0, 0)\}:

\[
p^{(2)} = p^{(1)}, \quad q^{(2)} = \frac{q^{(1)}}{p^{(1)}}, \quad r^{(2)} = \frac{r^{(1)}}{p^{(1)}}.
\]
where $g(7)$

In this coordinate, the system (1) is rewritten as follows:

$$
\begin{align*}
p^{(3)} &= p^{(2)}, \\
qu^{(3)} &= \frac{q^{(2)} - \frac{5}{3}(b - 1)}{p^{(2)}}, \\
r^{(3)} &= \frac{r^{(2)} - \sqrt{-1}\xi(b - 2\sigma)}{p^{(2)}}.
\end{align*}
$$

Step 4: We blow up along the curve $\{(p^{(3)}, q^{(3)}, r^{(3)})|p^{(3)} = 0, q^{(3)} = \frac{\sqrt{-1}}{9}(\xi^2(b - 1)(7b - 15\sigma + 2) - 9)\}$:

$$
\begin{align*}
p^{(4)} &= p^{(3)}, \\
qu^{(4)} &= \frac{q^{(3)} - \frac{\sqrt{-1}}{9}(\xi^2(b - 1)(7b - 15\sigma + 2) - 9)}{p^{(3)}}, \\
r^{(4)} &= r^{(3)}.
\end{align*}
$$

Step 5: We blow up along the curve $\{(p^{(4)}, q^{(4)}, r^{(4)})|p^{(4)} = 0, q^{(4)} = \frac{4}{3}\xi(b - 1)q^{(4)} - \frac{2}{27}\xi(b + 2)(\xi^2(b - 1)(7b - 15\sigma + 2) - 9)\}$:

$$
\begin{align*}
u &= p^{(4)}, \\
v &= \frac{q^{(4)} - \left(\frac{4}{3}\xi(b - 1)r^{(4)} - \frac{2}{27}\xi(b + 2)(\xi^2(b - 1)(7b - 15\sigma + 2) - 9)\right)}{p^{(4)}}, \\
w &= r^{(4)}.
\end{align*}
$$

In this coordinate, the system (1) is rewritten as follows:

$$
\begin{align*}
\frac{du}{dt} &= g_1(u, v, w), \\
\frac{dv}{dt} &= \frac{4\xi \xi^3(b - 1)(b - 2\sigma)(b + 3\sigma - 1)}{9} - \frac{2\xi^2 18(b - 1)(b - 3\sigma + 1)w}{27} \\
&\quad - \frac{2\xi^2 \{b^2 - 5b - 2 - 3(b - 2)\sigma\} \{\xi^2(b - 1)(7b - 15\sigma + 2) - 9\}}{27} + g_2(u, v, w), \\
\frac{dw}{dt} &= -\frac{\sqrt{-1} \xi^2(b - 2\sigma)(b + 3\sigma - 1)}{3} + g_3(u, v, w),
\end{align*}
$$

where $g_i(u, v, w) \in \mathbb{C}[u, v, w]$ $(i = 1, 2, 3)$.

Each right-hand side of the system (7) is a polynomial if and only if

$$
\begin{align*}
\xi(b - 1)(b - 2\sigma)(b + 3\sigma - 1) &= 0, \\
(b - 1)(b - 3\sigma + 1) &= 0, \\
\{b^2 - 5b - 2 - 3(b - 2)\sigma\} \{\xi^2(b - 1)(7b - 15\sigma + 2) - 9\} &= 0, \\
\xi(b - 2\sigma)(b + 3\sigma - 1) &= 0.
\end{align*}
$$

These equations can be solved as follows:

$$
(\sigma, \xi, b) = \left(\frac{1}{3}, \xi, 0\right), (1, -3, 2), (1, 3, 2), (2, 0, 1).
$$
3 The case of \((\sigma, \varepsilon, b) = (\frac{1}{3}, \varepsilon, 0)\)

\[
\begin{cases}
\frac{dx}{dt} = y - \frac{\varepsilon}{3} x, \\
\frac{dy}{dt} = -xz + x - \varepsilon y, \\
\frac{dz}{dt} = xy.
\end{cases}
\]

(10)

This system is equivalent to the 3-rd order ordinary differential equation:

\[
\frac{d^3x}{dt^3} = -\frac{\varepsilon}{3}x^3 - x^2 \frac{dx}{dt} + 4\varepsilon \left( \frac{dx}{dt} \right)^2 + \frac{1}{x} \frac{dx}{dt} \frac{d^2x}{dt^2} - \frac{4\varepsilon}{3} \frac{d^2x}{dt^2}.
\]

(11)

This system does not appear in the Chazy polynomial class.

**Theorem 3.1.** The phase space \(\mathcal{X}\) for the system (10) is obtained by gluing three copies of \(\mathbb{C}^3\):

\[
U_j \cong \mathbb{C}^3 \ni \{(x_j, y_j, z_j)\}, \quad j = 0, 1, 2
\]

via the following birational transformations:

\[
\begin{align*}
0) & \quad x_0 = x, \quad y_0 = y, \quad z_0 = z, \\
1) & \quad x_1 = \frac{1}{x}, \quad y_1 = \left( y - \frac{\varepsilon}{3} x - \sqrt{-1}z + \frac{\sqrt{-1}(5\varepsilon^2 + 9)}{9} \right)x + \frac{4\varepsilon}{9}(3z + \varepsilon^2 - 3) x, \\
& \quad z_1 = z - \frac{1}{6}(3x - 4\sqrt{-1}\varepsilon)x, \\
2) & \quad x_2 = \frac{1}{x}, \quad y_2 = \left( y - \frac{\varepsilon}{3} x + \sqrt{-1}z - \frac{\sqrt{-1}(5\varepsilon^2 + 9)}{9} \right)x + \frac{4\varepsilon}{9}(3z + \varepsilon^2 - 3) x, \\
& \quad z_2 = z - \frac{1}{6}(3x + 4\sqrt{-1}\varepsilon)x.
\end{align*}
\]

These transition functions satisfy the condition:

\[
dx_i \wedge dy_i \wedge dz_i = dx \wedge dy \wedge dz \quad (i = 1, 2).
\]

**Theorem 3.2.** Let us consider a system of first order ordinary differential equations in the polynomial class:

\[
\begin{align*}
\frac{dx}{dt} &= f_1(x, y, z), \\
\frac{dy}{dt} &= f_2(x, y, z), \\
\frac{dz}{dt} &= f_3(x, y, z).
\end{align*}
\]
We assume that

(A1) \( \text{deg}(f_i) = 2 \) with respect to \( x, y, z \).

(A2) The right-hand side of this system becomes again a polynomial in each coordinate system \( (x_i, y_i, z_i) \) \( (i = 1, 2) \).

Then such a system coincides with the system (10).

## 4 Modified Lorenz model

In this section, we present 4-parameter family of modified Lorenz model explicitly given by

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{\varepsilon}{3} x + y + \frac{1}{72} (8(9\alpha_1 - \varepsilon\alpha_3) + \sqrt{-1}(24\alpha_2 + \alpha_3^2 - 16\varepsilon^2)), \\
\frac{dy}{dt} &= -xz - \frac{1}{72} (24\alpha_2 + \alpha_3^2 - 8\varepsilon^2) x - \varepsilon y + \frac{1}{6}(\alpha_3 - 4\sqrt{-1}\varepsilon) z \\
&\quad + \frac{1}{432} (24\alpha_2\alpha_3 + \alpha_3^3 - 432\alpha_1\varepsilon + 64\alpha_3\varepsilon^2 - 2\sqrt{-1}\varepsilon(120\alpha_2 + 5\alpha_3^2 - 16\varepsilon^2)), \\
\frac{dz}{dt} &= xy + \frac{1}{72} (12(6\alpha_1 - \varepsilon\alpha_3) + \sqrt{-1}(24\alpha_2 + \alpha_3^2)) x - \frac{1}{6}(\alpha_3 - 4\sqrt{-1}\varepsilon)y \\
&\quad - \frac{1}{432} (\alpha_3 - 4\sqrt{-1}\varepsilon)(12(6\alpha_1 - \alpha_3\varepsilon) + \sqrt{-1}(24\alpha_2 + \alpha_3^2)),
\end{align*}
\]

where \( \alpha_i, \varepsilon \) are complex parameters.

**Theorem 4.1.** The phase space \( X \) for the system (13) is obtained by gluing three copies of \( \mathbb{C}^3 \):

\[ U_j \cong \mathbb{C}^3 \ni \{(x_j, y_j, z_j)\}, \quad j = 0, 1, 2 \]

via the following birational transformations:

\[
\begin{align*}
0) \quad x_0 &= x, \quad y_0 = y, \quad z_0 = z, \\
1) \quad x_1 &= \frac{1}{x}, \quad y_1 = -\left((y - \varepsilon \frac{3}{3} x - \sqrt{-1}z + \alpha_1) x + \frac{4\varepsilon}{9}(3z + \alpha_2)\right)x, \\
&\quad z_1 = z - \frac{1}{6}(3x - \alpha_3)x, \\
2) \quad x_2 &= \frac{1}{x}, \quad y_2 = -(y - \varepsilon \frac{3}{3} x + \sqrt{-1}z + \frac{1}{36}(36\alpha_1 + 24\sqrt{-1}\alpha_2 + \sqrt{-1}\alpha_3^2 - 48\sqrt{-1}\varepsilon^2) x \\
&\quad + \frac{4\varepsilon}{9}(3z + \frac{1}{3}(3\alpha_2 + 2\sqrt{-1}\alpha_3\varepsilon + 8\varepsilon^2))x, \\
&\quad z_2 = z - \frac{1}{6}(3x - \alpha_3 + 8\sqrt{-1}\varepsilon)x.
\end{align*}
\]
These transition functions satisfy the condition:

\[ dx_i \wedge dy_i \wedge dz_i = dx \wedge dy \wedge dz \quad (i = 1, 2). \]

**Theorem 4.2.** Let us consider a system of first order ordinary differential equations in the polynomial class:

\[ \frac{dx}{dt} = f_1(x, y, z), \quad \frac{dy}{dt} = f_2(x, y, z), \quad \frac{dz}{dt} = f_3(x, y, z). \]

We assume that

(A1) \( \text{deg}(f_i) = 2 \) with respect to \( x, y, z \).

(A2) The right-hand side of this system becomes again a polynomial in each coordinate system \((x_i, y_i, z_i)\) \((i = 1, 2)\).

Then such a system coincides with the system (13).

## 5 The case of \((\sigma, \varepsilon, b) = (2, 0, 1)\)

\[
\begin{cases}
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = -xz + x, \\
\frac{dz}{dt} = xy.
\end{cases}
\]

(15)

**Proposition 5.1.** This system has

(16) \[ I := x^2 - 2z \]

as its first integral.

This system can be solved by reduction to 2nd-order ordinary differential equation

\[
\begin{cases}
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = -\frac{1}{2}x^3 + (1 + \frac{I}{2})x,
\end{cases}
\]

(17)

or equivalently

(18) \[ \frac{d^2x}{dt^2} = -\frac{1}{2}x^3 + (1 + \frac{I}{2})x, \]

which is a special case of equation Ince-VIII.
6 The case of \((\sigma, \varepsilon, b) = (1, 3, 2)\)

\[
\begin{align*}
\frac{dx}{dt} &= y - 3x, \\
\frac{dy}{dt} &= -xz + x - 3y, \\
\frac{dz}{dt} &= xy - 6z.
\end{align*}
\]

(19)

**Proposition 6.1.** This system has

\[
I := e^{6t}(x^2 - 2z)
\]

as its first integral.

Owing to its first integral, this system can be reduced to the system

\[
\begin{align*}
\frac{dx}{dt} &= y - 3x, \\
\frac{dy}{dt} &= -\frac{1}{2}x^3 - 3y + \frac{e^{-6t}(I + 2e^{6t})}{2}x.
\end{align*}
\]

(20) (21)

By making the change of variables

\[
\begin{align*}
X &= \frac{\sqrt{-1}}{2}x, \\
Y &= \frac{2x + \sqrt{-1}x^2 - 2y}{2x},
\end{align*}
\]

(22)

we obtain

\[
\begin{align*}
\frac{dX}{dt} &= X^2 - XY - 2X, \\
\frac{dY}{dt} &= Y^2 - 3XY - 2Y - \frac{I}{2}e^{-6t}.
\end{align*}
\]

(23)

**Proposition 6.2.** After a series of explicit blowing-ups at ten points including seven infinitely near points, the phase space \(X\) for this system can be obtained by gluing four copies of \(\mathbb{C}^2 \times \mathbb{C}\):

\[
U_j \times \mathbb{C} \cong \mathbb{C}^2 \times \mathbb{C} \ni \{(x_j, y_j, t)\}, \quad j = 0, 1, 2, 3
\]
via the following birational transformations:

\[
\begin{align*}
0) \quad & x_0 = x, \quad y_0 = y, \\
1) \quad & x_1 = \frac{1}{x}, \quad y_1 = \left( yx + \frac{I}{4} e^{-6t}x + \frac{I}{2} e^{-6t} \right) x, \\
2) \quad & x_2 = \frac{1}{x}, \quad y_2 = \left( (y - 2x)x - \frac{I}{4} e^{-6t}x + \frac{I}{2} e^{-6t} \right) x, \\
3) \quad & x_3 = xy, \quad y_3 = \frac{1}{y}.
\end{align*}
\]

Here, for notational convenience, we have renamed \( X, Y \) to \( x, y \).

We remark that the phase space \( \mathcal{X} \) is not a rational surface of type \( E_7^{(1)} \) (see Figure 1).

It is still an open question whether integrability status of this system is known or not.

7 The case of \( (\sigma, \varepsilon, b) = (1, -3, 2) \)

\[
\begin{align*}
\frac{dx}{dt} &= y + 3x, \\
\frac{dy}{dt} &= -xz + x + 3y, \\
\frac{dz}{dt} &= xy + 6z
\end{align*}
\]

Proposition 7.1. This system has

\[
I := e^{-6t}(x^2 - 2z)
\]

as its first integral.
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