Strong Converse for Discrete Memoryless Networks with Tight Cut-Set Bounds

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Abstract

This paper proves the strong converse for any discrete memoryless network (DMN) with tight cut-set bound, i.e., whose cut-set bound is achievable. Our result implies that for any DMN with tight cut-set bound and any fixed rate tuple that resides outside the capacity region, the average error probabilities of any sequence of length-$n$ codes operated at the rate tuple must tend to 1 as $n$ grows. The proof is based on the method of types. The proof techniques are inspired by the work of Csiszár and Körner in 1982 which fully characterized the reliability function of any discrete memoryless channel (DMC) with feedback for rates above capacity.

Index Terms

Capacity region, cut-set outer bound, discrete memoryless network (DMN), method of types, strong converse.

I. INTRODUCTION

This paper considers a general network in which each node may send a message to any other node in the network. The network is assumed to be discrete memoryless and is referred to as discrete memoryless network (DMN) [1, Ch. 19]. A well-known outer bound on the capacity region of the DMN is the cut-set bound, developed by El Gamal in 1981 [2]. This bound states that for any cut-set $T$ of the network with nodes indexed by a set $T$, the sum of the rates of transmission of messages on one side of the cut is bounded above by the conditional mutual information between the input variables in $T$ and the output variables in $T^c \triangleq T \setminus T$ given the input variables in $T^c$. The DMN is a generalization of the well-studied discrete memoryless relay channel (DM-RC) [3]. It is known that the cut-set bound is not tight (achievable) in general [4], but it is tight (achievable) for several classes of DMNs, including the physically degraded DM-RC [3], the physically degraded DMN [5,6], the semi-deterministic DM-RC [7], the DM-RC with orthogonal sender components [8] and the linear deterministic multicast network [9] among others.

One potential drawback of the cut-set bound is the fact that it is only a weak converse for networks with tight cut-set bounds. This weak converse only guarantees that for any network with tight cut-set bound and any fixed rate tuple residing outside the capacity region, the average probabilities of decoding error of any sequence of length-$n$ codes operated at the rate tuple is bounded away from 0 as $n$ tends to infinity. In information theory, it is also important to establish a strong converse statement indicating that there is a sharp phase transition of the minimum achievable asymptotic error probability between rate tuples inside and outside the capacity region in the following sense: Any rate tuple inside the capacity region can be supported by some sequence of length-$n$ codes with asymptotic error probability being 0, and the asymptotic error probability of any sequence of length-$n$ codes operated at a rate tuple outside the capacity region equals 1. A strong converse indicates that for any fixed rate tuple residing outside the capacity region, the average error probabilities of any sequence of codes operated at the rate tuple must necessarily tend to 1.

A. Main Contribution

The main contribution of this work is a self-contained proof of the strong converse for DMNs with tight cut-set bounds. More precisely, our main result shows that for any given DMN, any rate tuple that can be supported by a sequence of codes with asymptotic error probability equal to $\varepsilon$ must belong to the region prescribed by the cut-set bound as long as $\varepsilon \in [0,1)$. Thus, this result establishes the strong converse for DMNs whose cut-set bounds are known to be tight. So for example, for the physically degraded DM-RC whose cut-set bound is tight (i.e., all rates below the cut-set bound are achievable in the sense that there exist codes with those rates and vanishing error probabilities), our strong converse result implies that the error probabilities of any sequence of length-$n$ codes operated at a rate above the capacity must necessarily tend to 1 as $n$ grows.

The proof of our main result is based on the method of types. The proof techniques are inspired by the work of Csiszár and Körner [10] which fully characterized the reliability function of any discrete memoryless channel (DMC) with feedback for rates above capacity (same as that of the corresponding DMC without feedback).

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B. Related Work

The papers that are most closely related to the present work are the ones by Behboodi and Piantanida who first conjectured the strong converse for DM-RCs [11] and DMNs [12] with tight cut-set bounds. Also see the thesis by Behboodi [13, App. C]. Unfortunately, it appears to the present authors that some steps of the proofs, which are based on the information spectrum method [14], are incomplete, as will be elaborated upon in Remark 1 after the main theorem is stated.

In our prior work [15], inspired by the work of Polyanskiy and Verdú [16], we leveraged properties of the conditional Rényi divergence to prove the strong converse for certain classes of multimessage multicast networks (MMNs) with tight cut-set bounds. These include the linear deterministic multicast network [9], the MMN consisting of independent DMCs [17] and the wireless erasure network [18], but excluding the following networks with tight cut-set bounds: the physically degraded DM-RC, the semi-deterministic DM-RC, and the DM-RC with orthogonal sender components. This work significantly strengthens our prior work by proving the strong converse for all DMNs with tight cut-set bounds including the four aforementioned networks left out by our prior work. See Remark 3 for a more detailed discussion.

C. Paper Outline

This paper is organized as follows. The notations used in this paper are described in the next subsection. Section II presents the problem formulation of the DMN and its ε-capacity region for ε ∈ [0, 1], followed by the main result in this paper — the strong converse for DMNs with tight cut-set bounds. The preliminaries for the proof of the main result are contained in Section III, which includes well-known results based on the method of types. Section IV presents the proof of the main result.

D. Notation

The sets of real and non-negative real numbers are denoted by \( \mathbb{R} \) and \( \mathbb{R}_+ \) respectively. We use \( \Pr \{ \mathcal{E} \} \) to represent the probability of an event \( \mathcal{E} \), and we let \( \mathbb{1} \{ \mathcal{E} \} \) be the characteristic function of \( \mathcal{E} \). We use a capital letter (e.g., \( X \)) to denote a random variable, and use the corresponding small letter (e.g., \( x \)) and calligraphic letter (e.g., \( \mathcal{X} \)) to denote the realization and the alphabet of the random variable respectively. We use \( X^n \) to denote a random tuple \( (X_1, X_2, \ldots, X_n) \), where the components \( X_k \) have the same alphabet \( \mathcal{X} \). We let \( p_X \) and \( p_{Y|X} \) denote the probability mass distribution of \( X \) and the conditional probability mass distribution of \( Y \) given \( X \) respectively for any discrete random variables \( X \) and \( Y \). We let \( p_X p_{Y|X} \) denote the joint distribution of \( (X, Y) \), i.e., \( p_X p_{Y|X}(x, y) = p_X(x) p_{Y|X}(y|x) \) for all \( x \) and \( y \). For simplicity, we drop the subscript of a notation if there is no ambiguity. For any discrete random variable \( (U, X, Y, Z) \) distributed according to \( p_{U,X,Y,Z} \), we let \( H_{p_{U,X,Y,Z}}(X|Z) \) or more simply \( H_{p_{X,Z}}(X|Z) \) denote the entropy of \( X \) given \( Z \), and let \( I_{p_{U,X,Y,Z}}(X;Y|Z) \) or more simply \( I_{p_{X,Y,Z}}(X;Y|Z) \) denote the mutual information between \( X \) and \( Y \) given \( Z \). The \( L_1 \)-distance between two distributions \( p_X \) and \( q_X \) on the same discrete alphabet \( \mathcal{X} \), denoted by \( \|p_X - q_X\|_{L_1} \), is defined as \( \|p_X - q_X\|_{L_1} \triangleq \sum_{x \in \mathcal{X}} |p_X(x) - q_X(x)| \).

II. DISCRETE MEMORYLESS NETWORK AND MAIN RESULT

We consider a general network that consists of \( N \) nodes. Let
\[
\mathcal{I} \triangleq \{1, 2, \ldots, N\}
\]
be the index set of the nodes. The \( N \) terminals exchange information in \( n \) time slots as follows. Node \( i \) chooses message \( W_{i,j} \) according to the uniform distribution from the alphabet
\[
W_{i,j} \triangleq \{1, 2, \ldots, \lfloor 2^{nR_{i,j}} \rfloor \}
\]
and sends \( W_{i,j} \) to node \( j \) for each \( (i, j) \in \mathcal{I} \times \mathcal{I} \), where \( R_{i,j} \) characterizes the rate of message \( W_{i,j} \) and all the messages are mutually independent. For each \( k \in \{1, 2, \ldots, n\} \) and each \( i \in \mathcal{I} \), node \( i \) transmits \( X_{i,k} \in \mathcal{X}_i \), a function of \( W_{i,\ell} \) \( \ell \in \mathcal{I} \) and \( Y_{i,k}^{k-1} \), and receives \( Y_{i,k} \in \mathcal{Y}_i \) in the \( k \)th time slot where \( \mathcal{X}_i \) and \( \mathcal{Y}_i \) are some alphabets that possibly depend on \( i \). After \( n \) time slots, node \( j \) declares \( \hat{W}_{i,j} \) to be the transmitted \( W_{i,j} \) based on \( \{W_{i,\ell} : \ell \in \mathcal{I}\} \) and \( Y_j^n \) for each \( (i, j) \in \mathcal{I} \times \mathcal{I} \).

To simplify notation, we use the following conventions for each \( T \subseteq \mathcal{I} \): For any random tuple
\[
(X_1, X_2, \ldots, X_N) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_N,
\]
we let
\[
X_T \triangleq (X_i : i \in T)
\]
be the subtuple of \( (X_1, X_2, \ldots, X_N) \) and \( \mathcal{X}_T \) be the alphabet of \( X_T \). Similarly, for any \( k \in \{1, 2, \ldots, n\} \) and any random tuple
\[
(X_{1,k}, X_{2,k}, \ldots, X_{N,k}) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_N,
\]
we let
\[
X_{T,k} \triangleq (X_{i,k} : i \in T)
\]
be the subtuple of \((X_{1,k}, X_{2,k}, \ldots, X_{N,k})\). For any \(N^2\)-dimensional random tuple 
\[(W_{1,1}, W_{1,2}, \ldots, W_{N,N}) \in W_{1,1} \times W_{1,2} \times \cdots \times W_{N,N},\]
we let 
\[W_{T \times T^e} \triangleq (W_{i,j} | (i, j) \in T \times T^e)\]
be the subtuple of \((W_{1,1}, W_{1,2}, \ldots, W_{N,N})\). The following six definitions formally define a DMN and its capacity region.

**Definition 1:** A discrete network consists of \(N\) finite input sets \(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\), \(N\) finite output sets \(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_N\) and a transition matrix \(q_{Y|X}\). The discrete network is denoted by \((\mathcal{X}_I, \mathcal{Y}_I, q_{Y|X}|_{X_I})\). For every \(T \subseteq I\), the marginal distribution \(q_{Y|X}|_{X_I}\) is defined as 
\[q_{Y|X}|_{X_I}(y|_{X_I}) \triangleq \sum_{y \in Y^T} q_{Y|X}|_{X_I}(y|_{X_I})\]
for all \(x_I \in \mathcal{X}_I\) and all \(y_{T^e} \in \mathcal{Y}_{T^e}\).

**Definition 2:** Let \((\mathcal{X}_I, \mathcal{Y}_I, q_{Y|X}|_{X_I})\) be a discrete network. An \((n, R_{I \times I})\)-code, where \(R_{I \times I}\) denote the \(N^2\)-dimensional rate tuple \((R_{I,1}, R_{I,2}, \ldots, R_{I,N}, N)\), for \(n\) uses of the network consists of the following:

1. A message set \(W_{i,j}\) at node \(i\) for each \((i, j) \in I \times I\) as defined in (1). Message \(W_{i,j}\) is uniform on \(W_{i,j}\).
2. An encoding function 
\[f_{i,k} : W_{(i) \times I} \times Y_{i}^{k-1} \rightarrow \mathcal{X}_i\]
for each \(i \in I\) and each \(k \in \{1, 2, \ldots, n\}\), where \(f_{i,k}\) is the encoding function at node \(i\) in the \(k\)th time slot such that \(^1\)
\[X_{i,k} = f_{i,k}(W_{(i) \times I}, Y_{i}^{k-1}).\]
3. A decoding function 
\[\varphi_i : W_{(i) \times I} \times Y_{i}^n \rightarrow W_{I \times \{i\}}\]
for each \(i \in I\), where \(\varphi_i\) is the decoding function for \(W_{I \times \{i\}}\) at node \(i\) such that 
\[\hat{W}_{I \times \{i\}} = \varphi_{i}(W_{(i) \times I}, Y_{i}^n)\].

**Definition 3:** A discrete network \((\mathcal{X}_I, \mathcal{Y}_I, q_{Y|X}|_{X_I})\), when used multiple times, is called a disrete memoryless network (DMN) if the following holds for any \((n, R_{I \times I})\)-code:

Let \(U_{I \times I} = (W_{I \times I}, X_I^{k-1}, Y_{I}^{k-1})\) be the collection of random variables that are generated before the \(k\)th time slot. Then, for each \(k \in \{1, 2, \ldots, n\}\) and each \(T \subseteq I\),
\[\Pr\{U_{I \times I} = u_{I \times I}, X_{I,k} = x_{I,k}, Y_{T^e,k} = y_{T^e,k}\} = \Pr\{U_{I \times I} = u_{I \times I}, X_{I,k} = x_{I,k}\} q_{Y|X}|_{X_I}(y_{T^e,k}|_{X_I,k})\]
holds for all \(u_{I \times I} \in U_{I \times I}, x_{I,k} \in X_I\) and \(y_{T^e,k} \in \mathcal{Y}_{T^e}\).

**Definition 4:** For an \((n, R_{I \times I})\)-code, we can calculate the average probability of decoding error defined as 
\[\Pr\left\{\bigcup_{i \in I} \{\varphi_{i}(W_{(i) \times I}, Y_{i}^n) \neq W_{I \times \{i\}}\}\right\}.
We call an \((n, R_{I \times I})\)-code with average probability of decoding error no larger than \(\varepsilon_n\) an \((n, R_{I \times I}, \varepsilon_n)\)-code.

**Definition 5:** A rate tuple \(R_{I \times I} \in \mathbb{R}_+^{N^2}\) is \(\varepsilon\)-achievable if there exists a sequence of \((n, R_{I \times I}, \varepsilon_n)\)-codes such that \(\limsup_{n \to \infty} \varepsilon_n \leq \varepsilon\).

Without loss of generality, we assume that \(R_{i,i} = 0\) for all \(i \in I\) in the rest of this paper.

**Definition 6:** The \(\varepsilon\)-capacity region, denoted by \(C_{\varepsilon}\), of the DMN is the closure of the set consisting of every \(\varepsilon\)-achievable rate tuple \(R_{I \times I}\) with \(R_{i,i} = 0\) for all \(i \in I\). The capacity region is defined to be 0-capacity region \(C_0\).

The following theorem is the main result in this paper.

**Theorem 1:** Let \((\mathcal{X}_I, \mathcal{Y}_I, q_{Y|X}|_{X_I})\) be a DMN. Define
\[\mathcal{R}_{\text{cut-set}} \triangleq \bigcup_{p_{X|I} \subseteq \mathbb{R}_+^{N^2}} \left\{R_{I \times I} \in \mathbb{R}_+^{N^2} \middle| \sum_{(i,j) \in T \times T^e} R_{i,j} \leq I_{p_{X|I} q_{Y|X}|_{X_I}}(X_T; Y_{T^e}|X_T), \forall i \in I \right\}.\]
Remark 1: The authors in [12,13] conjectured that the strong converse holds for general DMNs with tight cut-set bounds and they employed information spectrum techniques. However, the fourth equality of the chain of equalities after equation (C.8) in [13] need not hold, which implies that the first step of their proof in [12, Section IV.B] is incomplete. Consequently, their proof has a gap. Our proof of Theorem 1 does not use information spectrum methods. Rather, we use the method of types to establish a strong converse for DMNs with tight cut-set bounds.

Remark 2: We observe from Theorem 1 that the cut-set bound characterized by (3) is a universal outer bound on \( C_\varepsilon \) for all \( 0 \leq \varepsilon < 1 \), which implies the strong converse for the class of DMNs whose cut-set bounds are achievable. As mentioned in Section I, the class includes the physically degraded DM-RC [3], the physically degraded DMN [5,6], the semi-deterministic DM-RC [7], the DM-RC with orthogonal sender components [8] and the linear deterministic multicast network [9] among others.

Remark 3: Theorem 1 establishes the strong converse for any DMN with tight cut-set bound under the multiple unicast demand where each node has a unique message to send to each other node. By slightly changing the definition of multiple unicast codes to multimessage multicast codes and modifying the definition of \( \varepsilon \)-achievable rates, we can follow almost the same steps as in the proof of Theorem 1 and derive the strong converse for DMNs with tight cut-set bounds under the multimessage multicast demand where each source node sends a single message and each destination node wants to recover all the source messages. This strong converse result under the multimessage multicast demand strengthens our prior strong converse result [15] established for some classes of multicast networks with tight cut-set bounds. To be more explicit, our prior strong converse result specialized to the multiple unicast demand scenario states that

\[
C_\varepsilon \subseteq R_{\text{out}}
\]

for all \( \varepsilon \in [0,1) \), where

\[
R_{\text{out}} \triangleq \bigcap_{T \subseteq I} \bigg\{ \sum_{(i,j) \in T} R_{i,j} \leq I_{p_x q_{Y|X^T}} \left( X^T; Y^T | X^T \right), \quad \sum_{i \in I} R_{i,j} = 0 \text{ for all } i \in I \bigg\}.
\]

Comparing (2) to (5), we observe that the union and intersection are swapped and consequently, \( R_{\text{cut-set}} \subseteq R_{\text{out}} \) with strict inequality for many classes of networks. Thus, Theorem 1 is considerably stronger than the main theorem in [15]. In particular, Theorem 1 establishes the strong converse for the following four networks: the physically degraded DM-RC, the physically degraded DMN, the semi-deterministic DM-RC, and the DM-RC with orthogonal sender components. Strong converses for these important networks were not proved in our previous paper [15]. The proof of Theorem 1 is based on the method of types [19], which is completely different compared to the Rényi divergence approach in our prior work [15]. It seems challenging (to the authors) to use other standard strong converse proof techniques to prove Theorem 1 such as the Rényi divergence approach [15,16], the information spectrum method [14] and the blowing-up lemma [19, Ch. 5] [20].

Remark 4: The proof of Theorem 1 implies that for any fixed rate tuple \( R_{I \times I} \) lying outside the cut-set bound \( R_{\text{cut-set}} \), the average probabilities of correct decoding of any sequence of \( (n,R_{I \times I}) \)-codes tend to 0 exponentially fast. See (65) in the proof for the derived upper bound on the non-asymptotic probability of correct decoding. In other words, we have proved an exponential strong converse for networks with tight cut-set bounds (cf. Oohama’s works in [21] and [22] that established exponential strong converse for broadcast channels). We leave the exact characterization of the strong converse exponent to future work.

Remark 5: The proof of Theorem 1 is inspired by two works which are based on the method of types [19]. First, Tan showed in [23] that the proof techniques used for analyzing the reliability functions of DMCs with feedback can be applied to DM-RCs. Second, Csiszár and Körner [10] fully characterized the reliability functions of any DMC with feedback for rates above capacity. We use those ideas in the proof of Theorem 1.

Remark 6: Consider a 3-node DM-RC where the source is indexed by 1, the relay is indexed by 2 and the destination is indexed by 3. The capacity of the DM-RC is unknown in general. However, if there exists a noiseless feedback link which carries \( Y_{k-1}^k \) to node 2 in each time slot \( k \), then the capacity of the resultant DM-RC with feedback coincides with the cut-set bound [1, Sec. 17.4], which is intuitively true because the feedback link transforms the DM-RC into a physically degraded DM-RC. Consequently, Theorem 1 implies that the DM-RC with feedback to the relay satisfies the strong converse property. In addition, inserting two noiseless feedback links which carry \( Y_{k-1}^k \) and \( Y_{(k-1)}^k \) to node 1 in each time slot \( k \) does not further increase the capacity of the DM-RC with feedback, and hence the strong converse property also holds under this setting.
III. Method of Types

The following definitions and results are standard [19, Ch. 2]. The type of a sequence $x^n \in \mathcal{X}^n$, denoted by $\phi_X^{[x^n]}$, is the empirical distribution of $x^n$, i.e.,

$$
\phi_X^{[x^n]}(a) = \frac{N(a|x^n)}{n}
$$

for all $a \in \mathcal{X}$ where $N(a|x^n)$ denotes the number of occurrences of the symbol $a$ in $x^n$. The set of all possible types of sequences in $\mathcal{X}^n$ is denoted by

$$
\mathcal{P}_n(\mathcal{X}) \triangleq \left\{ \phi_X^{[x^n]} \mid x^n \in \mathcal{X}^n \right\}.
$$

Similarly, the set of all possible types of sequences in $\mathcal{Y}^n$ conditioned on a type $r_X \in \mathcal{P}_n(\mathcal{X})$ is denoted by

$$
\mathcal{P}_n(\mathcal{Y}|r_X) \triangleq \left\{ s_Y|x \mid \text{There exists an } (x^n, y^n) \text{ such that } \phi_X^{[x^n]} = r_X \text{ and } \phi_X^{[x^n, y^n]} = r_X s_Y|x \right\}.
$$

For a given type $r_X \in \mathcal{P}_n(\mathcal{X})$, the type class of $r_X$ is defined as

$$
\mathcal{T}^{(n)}_{r_X} \triangleq \left\{ x^n \in \mathcal{X}^n \mid \phi_X^{[x^n]} = r_X \right\}.
$$

A well-known upper bound on the number of types is

$$
|\mathcal{P}_n(\mathcal{X})| \leq (n + 1)^{|\mathcal{X}|}.
$$

We will frequently use the following fact without explicit explanation: For each $r_X \in \mathcal{P}_n(\mathcal{X})$, each $s_Y|x \in \mathcal{P}_n(\mathcal{Y}|r_X)$ and each transition matrix $q_{Y|x}$, the following equality holds for any $(x^n, y^n) \in \mathcal{T}^{(n)}_{r_X} s_Y|X$:

$$
\prod_{k=1}^n q_{Y|x}(y_k|x_k) = \prod_{x,y} q_{Y|x}(y|x) ^ {r_X(x) s_Y|x(y|x)} = 2^{-n(H_{r_X} + D(s_Y|x||q_{Y|x}|r_X))}.
$$

IV. Proof of Theorem 1

In this section, we will show that for all $\varepsilon \in [0, 1)$

$$
\mathcal{C}_\varepsilon \subseteq \mathcal{R}_{\text{cut-set}}
$$

where $\mathcal{R}_{\text{cut-set}}$ is as defined in (2). It suffices to show that for any $R_{I \times I} \notin \mathcal{R}_{\text{cut-set}}$ and any sequence of $(n, R_{I \times I}, \varepsilon_n)$-codes,

$$
\lim_{n \to \infty} \varepsilon_n = 1.
$$

To this end, we fix a rate tuple $R_{I \times I} \notin \mathcal{R}_{\text{cut-set}}$ and a sequence of $(n, R_{I \times I}, \varepsilon_n)$-codes.

A. Relating $R_{I \times I}$ to the Cut-Set Bound

Since $R_{I \times I} \notin \mathcal{R}_{\text{cut-set}}$ and $\mathcal{R}_{\text{cut-set}}$ is closed, we can always find a positive number denoted by $\delta > 0$ such that for any distribution $r_{XZ}$ defined on $\mathcal{X}_Z$, there exists a $V_{r_{XZ}} \subseteq I$ that satisfies

$$
\sum_{(i,j) \in V_{r_{XZ}}} R_{i,j} \geq I_{r_{XZ}} q_{V_{r_{XZ}}|X} (X_{V_r}; Y_{V_r}|X_{V_r}) + \delta,
$$

where the shorthand notation $V_r$ is used to denote $V_{r_{XZ}}$.

B. Simplifying the Correct Decoding Probability by Using the Discrete Memoryless Property

Fix a natural number $n$ and let $p_{W_{I \times I}, X^n, Y^n, W_{I \times I}}$ be the probability distribution induced by the $(n, R_{I \times I}, \varepsilon_n)$-code. Unless specified otherwise, the probabilities are evaluated according to $p_{W_{I \times I}, X^n, Y^n, W_{I \times I}}$ in the rest of the proof. Consider the probability of correct decoding

$$
1 - \varepsilon_n = \frac{1}{|W_{I \times I}|} \sum_{w_{I \times I} \in W_{I \times I}} \Pr \left\{ \bigcap_{i \in I} \left\{ \varphi_i \left( w_{\{i\} \times I} \times I^n \right) = w_{I \times I} \right\} \bigg| W_{I \times I} = w_{I \times I} \right\}.
$$
In order to simplify the RHS of (16), we write for each $w_{I \times I} \in \mathcal{W}_{I \times I}$

$$
\Pr \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{I \times I}, Y_i^n) = w_{I \times I} \} \bigg| W_{I \times I} = w_{I \times I} \right\} 
= \sum_{y_{I}^{n} \in \mathcal{Y}_{I}^{n}} \prod_{i=1}^{n} p_{Y_i}^{(1)} \mathbb{1} \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{I \times I}, Y_i^n) = w_{I \times I} \} \right\} 
= \sum_{y_{I}^{n} \in \mathcal{Y}_{I}^{n}} \prod_{i=1}^{n} p_{Y_i|X_{I}}(y_{I}^{n-1}, w_{I \times I}) (y_{I}^{n-1}) \mathbb{1} \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{I \times I}, Y_i^n) = w_{I \times I} \} \right\}
\overset{(a)}{=} \sum_{y_{I}^{n} \in \mathcal{Y}_{I}^{n}} \prod_{i=1}^{n} p_{Y_i|X_{I}}(y_{I}^{n-1}, w_{I \times I})(y_{I}^{n-1}) \mathbb{1} \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{I \times I}, Y_i^n) = w_{I \times I} \} \right\},
$$

where (a) follows from Definitions 2 and 3.

C. Further Simplifying the Correct Decoding Probability by Using the Method of Types

With a slight abuse of notation, we let

$$
f_i(w_{i \times I}, Y_i^n) \triangleq (f_i,1(w_{i \times I}, Y_i^n), f_i,2(w_{i \times I}, Y_i^n), \ldots, f_i,n(w_{i \times I}, Y_i^n))
$$

for each $i \in I$. For each $w_{I} \in \mathcal{W}_{I}$, each type $r_{X_I} \in \mathcal{P}_n(\mathcal{X}_{I})$ and each conditional type $s_{Y_{I}|X_{I}} \in \mathcal{P}_n(\mathcal{Y}_{I}|r_{X_I})$, we define

$$
\mathcal{A}(w_{I \times I}; r_{X_I}, s_{Y_{I}|X_{I}}) \triangleq \left\{ y_{I}^{n} \in \mathcal{Y}_{I}^{n} \left| \{(f_i(w_{i \times I}, Y_i^n) : i \in I), y_{I}^{n}\} \in \mathcal{T}_{r_{X_I}, s_{Y_{I}|X_{I}}}^{(n)} \right. \right\}
$$

and define for each $T \subseteq I$ and each $w_{T^{-} \times I} \in \mathcal{W}_{T^{-} \times I}$

$$
\mathcal{F}_T(w_{T^{-} \times I}; r_{X_T}, s_{Y_{T^{-}}|X_{T}}) \triangleq \left\{ y_{T^{-}}^{n} \in \mathcal{Y}_{T^{-}}^{n} \left| \{(f_i(w_{i \times I}, Y_i^n) : i \in I), y_{T^{-}}^{n}\} \in \mathcal{T}_{r_{X_T}, s_{Y_{T^{-}}|X_{T}}}^{(n)} \right. \right\}.
$$

Note that the set $\mathcal{A}(w_{I \times I}; r_{X_I}, s_{Y_{I}|X_{I}})$ in (21) also plays a crucial role in the proof of the upper bound on the reliability functions for DM-RCs in [23]. Following (19) and adopting the shorthand notation $\mathcal{A}(w_{I \times I}; r, s)$ to denote the set in (21), since the collection of sets $\{\mathcal{A}(w_{I \times I}; r, s) | r \in \mathcal{P}_n(\mathcal{X}_{I}), s \in \mathcal{P}_n(\mathcal{Y}_{I}|r_{X_I})\}$ forms a partition on $\mathcal{Y}_{I}^{n}$ for each $w_{I \times I} \in \mathcal{W}_{I \times I}$, the probability of correct decoding conditioned on $\{W_{I \times I} = w_{I \times I}\}$ can be expressed as

$$
= \sum_{y_{T^{-}}^{n} \in \mathcal{Y}_{T^{-}}^{n}} \prod_{k=1}^{n} p_{Y_{I}}(y_{I}, k | f_i, k(w_{i \times I}, Y_i^{k-1}) : i \in I) \mathbb{1} \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{I \times I}, Y_i^{n}) = w_{I \times I} \} \right\}
= \sum_{r_{X_I} \in \mathcal{P}_n(\mathcal{X}_{I})} \sum_{s_{Y_{I}|X_{I}} \in \mathcal{P}_n(\mathcal{Y}_{I}|r_{X_I})} \sum_{y_{I}^{n} \in \mathcal{A}(w_{I \times I}; r_{X_I}, s_{Y_{I}|X_{I}})} \prod_{k=1}^{n} p_{Y_{I}}(y_{I}, k | f_i, k(w_{i \times I}, Y_i^{k-1}) : i \in I)
\times \mathbb{1} \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{I \times I}, Y_i^{n}) = w_{I \times I} \} \right\}.
$$

D. Bounding the Correct Decoding Probability in Terms of $\mathcal{F}_T(w_{T^{-} \times I}; r, s)$

Fix any arbitrary $T \subseteq I$. Define

$$
a_T \triangleq H_{r_{X_T}, s_{Y_{T^{-}}|X_{T}}}(Y_T | X_T) + D(s_{Y_{T^{-}}|X_{T}} || q_{Y_{T^{-}}|X_{T}} | r_{X_T})
$$

(24) to simplify notation. In order to simplify the RHS of (23), we consider the innermost product. In particular, we consider the following chain of equalities for each $r_{X_I} \in \mathcal{P}_n(\mathcal{X}_{I})$, each $s_{Y_{I}|X_{I}} \in \mathcal{P}_n(\mathcal{Y}_{I}|r_{X_I})$, each $w_{I \times I} \in \mathcal{W}_{I \times I}$, and each
\[ y^n_i \in A(w_{I \times T}; r, s): \]
\[ \prod_{k=1}^{n} p_{Y_{T,k}} | X_{I,k} (y_{T,k}(w_{i}(I) \times I, y_{i}^{k-1}) : i \in I)) \]
\[ = \prod_{k=1}^{n} p_{Y_{T,k}} | X_{I,k} (y_{T,k}(w_{i}(I) \times I, y_{i}^{k-1}) : i \in I))p_{Y_{T,k}} | X_{I,k} (y_{T,k}(w_{i}(I) \times I, y_{i}^{k-1}) : i \in I), y_{T,k}) \]
\[ = 2^{-n \alpha T} \prod_{k=1}^{n} p_{Y_{T,k}} | X_{I,k} (y_{T,k}(w_{i}(I) \times I, y_{i}^{k-1}) : i \in I), y_{T,k}) \]
\[ (25) \]
\[ (26) \]
\[ (27) \]

where (b) follows from Definition 3 and the fact that \( y^n_i \in A(w_{I \times T}; r, s) \) recall the definition of \( A(w_{I \times T}; r, s) \) in (21).

Following (23) and letting \( F_T(w_{T \times T}) \) denote the set in (22), we consider the following chain of inequalities for each \( r_{X_I} \in \mathcal{P}_n(X_I) \) and each \( s_{Y_I} | X_I \in \mathcal{P}_n(Y_I | r_{X_I}) \):

\[ \sum_{w_{I \times T} \in W_{I \times T}} \sum_{y^n_I \in A(w_{I \times T}; r, s)} \prod_{k=1}^{n} p_{Y_{T,k}} | X_{I,k} (y_{T,k}(w_{i}(I) \times I, y_{i}^{k-1}) : i \in I))1 \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{i}(I) \times I, y^n_i) = w_{I \times \{i\}} \} \right\} \]
\[ (28) \]
\[ \leq 2^{-n \alpha T} \sum_{w_{I \times T} \in W_{I \times T}} \sum_{y^n_I \in F_T(w_{T \times T}; I, r, s)} \prod_{k=1}^{n} p_{Y_{T,k}} | X_{I,k} (y_{T,k}(w_{i}(I) \times I, y_{i}^{k-1}) : i \in I), y_{T,k}) \]
\[ \times 1 \left\{ \bigcap_{i \in T^c} \{ \varphi_i(w_{i}(I) \times I, y^n_i) = w_{I \times \{i\}} \} \right\} \]
\[ (29) \]
\[ = 2^{-n \alpha T} \sum_{w_{I \times T} \in W_{I \times T}} \sum_{y^n_I \in F_T(w_{T \times T}; I, r, s)} 1 \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{i}(I) \times I, y^n_i) = w_{I \times \{i\}} \} \right\} \]
\[ (30) \]
\[ = 2^{-n \alpha T} \sum_{w_{T \times T} \in W_{T \times T}} \sum_{y^n_T \in F_T(w_{T \times T}; I, r, s)} \sum_{w_{T \times T} \in W_{T \times T}} 1 \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{i}(I) \times I, y^n_i) = w_{I \times \{i\}} \} \right\} \]
\[ (31) \]
\[ \leq 2^{-n \alpha T} \sum_{w_{T \times T} \in W_{T \times T}} \sum_{y^n_T \in F_T(w_{T \times T}; I, r, s)} 1 \left\{ F_T(w_{T \times T}; I, r, s) \right\} \]
\[ (32) \]
\[ = 2^{-n \alpha T} \sum_{w_{T \times T} \in W_{T \times T}} |F_T(w_{T \times T}; I, r, s)| \]
\[ (33) \]

where

(c) follows from the definitions of \( A(w_{I \times T}; r, s) \) and \( F_T(w_{T \times T}; I, r, s) \) in (21) and (22) respectively.

(d) follows from the fact that for each \( w_{T \times T} \in W_{T \times T} \) (which is a projection of \( W_{T \times T} \)) and each \( y^n_T \in Y^n_T \),

\[ \sum_{w_{T \times T} \in W_{T \times T}} 1 \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{i}(I) \times I, y^n_i) = w_{I \times \{i\}} \} \right\} \]
\[ \leq \sum_{w_{I \times T} \in W_{I \times T}} 1 \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{i}(I) \times I, y^n_i) = w_{I \times \{i\}} \} \right\} \]
\[ = 1 \]
\[ (34) \]
\[ (35) \]

(the last equality is a consequence of the simple fact that \( \varphi_i \) is a function that outputs values in \( W_{I \times \{i\}} \) for each \( i \in I \)).
E. Bounding the Size of $\mathcal{F}_T(w_{T^c} \times I; r, s)$

For each $r_{XZ} \in \mathcal{P}_n(\mathcal{X}_Z)$ and each $s_{Y[Z]|X_Z} \in \mathcal{P}_n(\mathcal{Y}_Z | r_{XZ})$, we let $u_{X_{T^c}, Y_{T^c}}$ denote the marginal type induced by $r_{XZ} s_{Y[Z]|X_Z}$ in order to obtain an upper bound on $|\mathcal{F}_T(w_{T^c} \times I; r, s)|$ as follows. For each $r_{XZ} \in \mathcal{P}_n(\mathcal{X}_Z)$, each $s_{Y[Z]|X_Z} \in \mathcal{P}_n(\mathcal{Y}_Z | r_{XZ})$ and each $w_{T^c \times I} \in \mathcal{W}_{T^c \times I}$, since

$$\sum_{y_{T}^n \in \mathcal{F}_T(w_{T^c \times I}; r, s)} \prod_{k=1}^{n} u_{Y_{T^c} \mid X_{T^c}} (y_{T^c, k} | (f_i, k(w_i) \times I, y_i^{k-1}) : i \in T^c) \leq 1, \quad (36)$$

it follows that

$$\sum_{y_{T}^n \in \mathcal{F}_T(w_{T^c \times I}; r, s)} \prod_{k=1}^{n} u_{Y_{T^c} \mid X_{T^c}} (y_{T^c, k} \mid (x_{T^c} - y_{T^c})); X_{T^c}) \leq 1 \quad (37)$$

(recall the definition of $\mathcal{F}_T(w_{T^c} \times I; r, s)$ in (22)), which implies that

$$\sum_{y_{T}^n \in \mathcal{F}_T(w_{T^c \times I}; r, s)} 2^{-n H_{U_{T^c}, Y_{T^c}} (Y_{T^c} \mid X_{T^c})} \leq 1, \quad (38)$$

which then implies that

$$|\mathcal{F}_T(w_{T^c} \times I; r, s)| \leq 2^{-n H_{U_{T^c}, Y_{T^c}} (Y_{T^c} \mid X_{T^c})}. \quad (39)$$

Combining (33), (24) and (39) and using the fact due to (24) that

$$\left| \frac{W_{(T \times T^c) \times I}}{W_{I \times I}} \right| = \prod_{(i,j) \in T \times T^c} \left[ \frac{1}{2^{n R_{i,j}}} \right] \leq 2^{-n \sum_{(i,j) \in T \times T^c} R_{i,j}}, \quad (40)$$

we have for each $r_{XZ} \in \mathcal{P}_n(\mathcal{X}_Z)$ and each $s_{Y[Z]|X_Z} \in \mathcal{P}_n(\mathcal{Y}_Z | r_{XZ})$

$$\frac{1}{|W_{I \times I}|} \sum_{w_{I \times Z} \in \mathcal{W}_{I \times Z}} \sum_{y_{T}^n \in \mathcal{F}_T(w_{I \times Z}; r, s)} \prod_{k=1}^{n} p_{Y_{I \times Z} | X_{I \times Z}} (y_{I \times Z, k} | (f_i, k(w_i) \times I, y_i^{k-1}) : i \in I) \left\{ \bigcap_{i \in I} \{ \varphi_i(w_i) : w_{I \times \{i\}} = w_{I \times \{i\}} \} \right\} \leq 2^{-n \sum_{(i,j) \in T \times T^c} R_{i,j} - \frac{1}{2} n_{Y[Z]|X_Z} + D(s_{Y[Z]|X_Z} || q_{Y[Z]|X_Z})}, \quad (41)$$

Note that (41) resembles [10, Eq. (5)] in the proof of the reliability functions for DMCs with feedback.

F. Bounding the Correct Decoding Probability in Terms of $\mathcal{A}(w_{I \times Z}; r, s)$

We now bound the LHS of (41) in another way for each $r_{XZ} \in \mathcal{P}_n(\mathcal{X}_Z)$ and each $s_{Y[Z]|X_Z} \in \mathcal{P}_n(\mathcal{Y}_Z | r_{XZ})$ as follows:

$$\frac{1}{|W_{I \times I}|} \sum_{w_{I \times Z} \in \mathcal{W}_{I \times Z}} \sum_{y_{T}^n \in \mathcal{F}_T(w_{I \times Z}; r, s)} \prod_{k=1}^{n} p_{Y_{I \times Z} | X_{I \times Z}} (y_{I \times Z, k} | (f_i, k(w_i) \times I, y_i^{k-1}) : i \in I) \left\{ \bigcap_{i \in I} \{ \varphi_i(w_i) : w_{I \times \{i\}} = w_{I \times \{i\}} \} \right\} \leq \frac{1}{|W_{I \times I}|} \sum_{w_{I \times Z} \in \mathcal{W}_{I \times Z}} \sum_{y_{T}^n \in \mathcal{F}_T(w_{I \times Z}; r, s)} \prod_{k=1}^{n} p_{Y_{I \times Z} | X_{I \times Z}} (y_{I \times Z, k} | (f_i, k(w_i) \times I, y_i^{k-1}) : i \in I) \quad (42)$$

\[
= \frac{1}{|W_{I \times I}|} \sum_{w_{I \times Z} \in \mathcal{W}_{I \times Z}} \sum_{y_{T}^n \in \mathcal{F}_T(w_{I \times Z}; r, s)} \prod_{x_{I \times Z}} q_{Y_{I \times Z} | X_{I \times Z}} (y_{I \times Z} | x_{I \times Z}) p(x_{I \times Z} | y_{I \times Z}) \quad (43)
\]

\[
= \frac{-n (H_{I \times Z} + D(s_{Y[Z]|X_Z} || q_{Y[Z]|X_Z}))}{|W_{I \times I}|} \sum_{w_{I \times Z} \in \mathcal{W}_{I \times Z}} |\mathcal{A}(w_{I \times Z}; r, s)|, \quad (44)
\]

where (e) follows from the definition of $\mathcal{A}(w_{I \times Z}; r, s)$ in (21) and Definition 3.
G. Bounding the Size of $A(w_{I\times I}; r, s)$

For each $r_{X_k} \in P_n(\mathcal{X}_k)$, each $s_{Y_k|X_k} \in P_n(\mathcal{Y}_k|r_{X_k})$ and each $w_{I\times I} \in W_{I\times I}$, since

$$
\sum_{y_k^n \in A(w_{I\times I}; r, s)} \prod_{k=1}^n s_{Y_k|X_k}(y_k^i | (f_{i,k}(w_{(i)\times I}, y_{i}^{k-1}) : i \in I)) \leq 1,
$$

it follows that

$$
\sum_{y_k^n \in A(w_{I\times I}; r, s)} \prod_{x_k \in I} s_{Y_k|X_k}(y_k^i | x_k)^{H(r_{X_k}; y_k^i | x_k)} \leq 1
$$

(recall the definition of $A(w_{I\times I}; r, s)$ in (21)), which implies that

$$
\sum_{y_k^n \in A(w_{I\times I}; r, s)} 2^{-nH(r_{X_k}; y_k^i | x_k)} (Y_k|X_k) \leq 1,
$$

which then implies that

$$
|A(w_{I\times I}; r, s)| \leq 2^{nH(r_{X_k}; y_k^i | x_k)} (Y_k|X_k).
$$

Combining (44) and (48), we have for each $r_{X_k} \in P_n(\mathcal{X}_k)$ and each $s_{Y_k|X_k} \in P_n(\mathcal{Y}_k|r_{X_k})$

$$
\frac{1}{|W_{I\times I}|} \sum_{w_{I\times I} \in W_{I\times I}} \sum_{y_k^n \in A(w_{I\times I}; r, s)} \prod_{k=1}^n p_{Y_k,i|X_k}(y_k^i | (f_{i,k}(w_{(i)\times I}, y_{i}^{k-1}) : i \in I)) 1 \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{(i)\times I}, y_{i}^{n}) = w_{I\times I} \} \right\}
\leq 2^{-nD(s_{Y_k|X_k}||Q_{Y_k|X_k}|r_{X_k})}.
$$

H. Relating the Bounds on Correct Decoding Probability to the Cut-Set Bound

Defining

$$
\alpha_T(r, s) \triangleq 2^{-\left( \sum_{(i,j) \in T \times T^c} (R_{i,j} - I_{X_I}\times Y_{T^c} ; X_k | X_T) + D(s_{Y_k|X_k}||Q_{Y_k|X_k}|r_{X_k}) \right)}
$$

and

$$
\beta(r, s) \triangleq 2^{-nD(s_{Y_k|X_k}||Q_{Y_k|X_k}|r_{X_k})},
$$

we obtain from (41) and (49) that for each $r_{X_k} \in P_n(\mathcal{X}_k)$ and each $s_{Y_k|X_k} \in P_n(\mathcal{Y}_k|r_{X_k})$.

$$
\frac{1}{|W_{I\times I}|} \sum_{w_{I\times I} \in W_{I\times I}} \sum_{y_k^n \in A(w_{I\times I}; r, s)} \prod_{k=1}^n p_{Y_k,i|X_k}(y_k^i | (f_{i,k}(w_{(i)\times I}, y_{i}^{k-1}) : i \in I)) 1 \left\{ \bigcap_{i \in I} \{ \varphi_i(w_{(i)\times I}, y_{i}^{n}) = w_{I\times I} \} \right\}
\leq \min\{\alpha_T(r, s), \beta(r, s)\}.
$$

Combining (16), (19) and (23) and using the fact that (52) holds for each $r_{X_k} \in P_n(\mathcal{X}_k)$, each $s_{Y_k|X_k} \in P_n(\mathcal{Y}_k|r_{X_k})$ and any arbitrary $T \subseteq I$, we conclude that

$$
1 - \varepsilon_n \leq \sum_{r_{X_k} \in P_n(\mathcal{X}_k)} \sum_{s_{Y_k|X_k} \in P_n(\mathcal{Y}_k|r_{X_k})} \min\{\alpha_T(r, s), \beta(r, s)\}
$$

where the set $V_r \subseteq I$ was carefully chosen to depend on $r_{X_k} \in P_n(\mathcal{X}_k)$ so that (15) holds. Note that (53) resembles [10, Eq. (7)]. Let $\xi > 0$ be a positive constant to be specified later. It then follows from (53) that

$$
1 - \varepsilon_n \leq \sum_{r_{X_k} \in P_n(\mathcal{X}_k)} \sum_{s_{Y_k|X_k} \in P_n(\mathcal{Y}_k|r_{X_k})} \min\{\alpha_T(r, s), \beta(r, s)\}
\times \left( 1 \left\{ D(s_{Y_k|X_k}||Q_{Y_k|X_k}|r_{X_k}) \geq \xi \right\} + 1 \left\{ D(s_{Y_k|X_k}||Q_{Y_k|X_k}|r_{X_k}) < \xi \right\} \right).
$$
I. Bounding the Correct Decoding Probability in Two Different Ways

Recalling that \( \delta > 0 \) was chosen such that (15) holds, we choose \( \xi > 0 \) to be a positive constant such that the following statement holds for all \( T \subseteq I \):

\[
|I_{g_{X_z}, Y_z}(X_T; Y_T' | X_{T'}) - I_{h_{X_z}, Y_z}(X_T; Y_T' | X_{T'})| \leq \delta/2
\]

(55)

for all distributions \( g_{X_z, Y_z} \) and \( h_{X_z, Y_z} \) defined on \((X_z, Y_z)\) that satisfy

\[
\|g_{X_z, Y_z} - h_{X_z, Y_z}\|_{L_2} \leq \sqrt{2\xi / \log e}.
\]

(56)

The existence of such a \( \xi > 0 \) is guaranteed by the fact that the mapping \( p_{X_z, Y_z} \rightarrow I_{p_{X_z, Y_z}}(X_T; Y_T' | X_{T'}) \) is continuous with respect to the \( L_1 \)-distance for all \( T \subseteq I \). Following (54), we consider the following two chains of inequalities for each \( r_{X_z} \in P_n(X_z) \) and each \( s_{Y_z | X_z} \in P_n(Y_z | r_{X_z}) \):

\[
\min\{\alpha_{V_z}(r, s), \beta(r, s)\} \times 1 \{ D(s_{Y_z | X_z} \| q_{Y_z | X_z} | r_{X_z}) \geq \xi \} \\
\leq \beta(r, s) \times 1 \{ D(s_{Y_z | X_z} \| q_{Y_z | X_z} | r_{X_z}) \geq \xi \} \\
\leq 2^{-n\xi}
\]

(57)

and

\[
\min\{\alpha_{V_z}(r, s), \beta(r, s)\} \times 1 \{ D(s_{Y_z | X_z} \| q_{Y_z | X_z} | r_{X_z}) < \xi \} \\
\leq \alpha_{V_z}(r, s) \times 1 \{ \| r_{X_z} s_{Y_z | X_z} - r_{X_z} q_{Y_z | X_z} \|_{L_1} < \sqrt{2\xi / \log e} \} \\
\leq 2^{-n\xi}
\]

(58)

where \( r_{X_z} \) follows from Pinsker’s inequality. Combining (54), (58) and (63) followed by using the fact due to (10) that

\[
|P_n(X_z \times Y_z)| \leq (n + 1)|X_z||Y_z|
\]

(64)

we obtain

\[
1 - \varepsilon_n \leq (n + 1)|X_z||Y_z|2^{-n\min\{\xi, \delta/2\}}
\]

(65)

(analogous to the last inequality in [10]) which implies (14). Since (65) holds for any sequence of \( (n, R_{I \times I}, \varepsilon_n) \)-codes with \( R_{I \times I} \notin R_{\text{cut-set}} \), it follows that (13) holds for all \( \varepsilon \in [0, 1] \).

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