LOGARITHMIC SOBOLEV INEQUALITIES FOR DUNKL OPERATORS

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Abstract. In this paper we study several inequalities of log-Sobolev type for Dunkl operators. After proving an equivalent of the classical inequality for the usual Dunkl measure, we also study a number of inequalities for weighted probability Dunkl measures, with weights of the form $e^{-|x|^p}$. These are obtained using the method of $U$-bounds. Poincaré inequalities are obtained as consequences of the log-Sobolev inequality. The connection between Poincaré and log-Sobolev inequalities is further examined, obtaining in particular tight log-Sobolev inequalities.

1. Introduction

The logarithmic-Sobolev inequality (or log-Sobolev inequality, in short), on a general measure space $(\Omega, \mathcal{F}, \mu)$ with a quadratic form $Q$ defined on a suitable space of functions on $\Omega$, states that

$$\int f^2 \log f^2 \, d\mu \leq C Q(f) + D \int f^2 \, d\mu,$$

for some constants $C$ and $D$. If $D = 0$, we say that (1.1) is a tight log-Sobolev inequality. Although this inequality was used before, it was first explicitly recognised in Gross’s seminal paper [5]. His main result was the equivalence of log-Sobolev inequalities to hypercontractivity. More precisely, if $e^{-tH}$ is a Markov semigroup on $L^2(\mu)$, then the log-Sobolev inequality (1.1) is equivalent to the property that the semigroup $e^{-tH}$ is hypercontractive, i.e.,

$$\|e^{-tH}f\|_{L^q(\mu)} \leq e^{M(t)} \|f\|_{L^p(\mu)},$$

where $q(t) = 1 + (p - 1)e^{2t}$ and $M(t) = D \left( \frac{1}{p} - \frac{1}{q(t)} \right)$.

The log-Sobolev inequality in a rather general setting was the crucial tool used by Davies in [3] to obtain bounds on the heat kernel associated to a positive semigroup. More precisely, he showed that the log-Sobolev inequality (1.1) with sufficiently decaying constants is equivalent to the ultracontractivity property. This equivalence then allows one to deduce lower and upper bounds for the heat kernel. Recall that a semigroup $(P_t)_{t \geq 0}$ on $L^2(\mu)$ is called ultracontractive if for each $t > 0$ the operator $P_t$ is bounded from $L^2(\mu)$ to $L^\infty(\mu)$.

Among the many applications of the log-Sobolev inequality we also mention its use in statistical mechanics by providing bounds on the entropy. Indeed, the relative entropy functional is defined as

$$\text{Ent}(f) := \int f \log f \, d\mu - \int f \, d\mu \log \int f \, d\mu.$$
for $f \geq 0$, so the log-Sobolev inequality can be rephrased as

$$\text{Ent}(f^2) \leq C Q(f) + D \|f\|_{L^2(\mu)}^2.$$ 

An important feature of the log-Sobolev inequality, which makes it particularly useful in infinite dimensional settings, is the product property: if (1.1) holds on two measure spaces $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$, then a form of log-Sobolev also holds on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \times \mu_2)$. For more information about this and the use in statistical mechanics see the lecture notes [6].

Another important property of the log-Sobolev inequality which we exploit below is its connection with the Poincaré inequality. In general, the tight log-Sobolev inequality implies the Poincaré inequality. On the other hand, a non-tight log-Sobolev inequality, in the presence of a Poincaré inequality, can be improved to obtain a tight log-Sobolev inequality. See [2] for more information on this, and also for an excellent account of the log-Sobolev inequality.

Dunkl operators are differential-difference operators which generalise the usual partial derivatives by including difference terms defined in terms of a finite reflection group. Although originally introduced to study special functions with certain symmetries, they have found other applications, for example in mathematical physics where they have been used to study Calogero-Moser-Sutherland (CMS) models of interacting particles. A short introduction to the theory of Dunkl operators is given below in section 2. More information about applications to CMS models can be found in [11], and an overview of their use in probability theory is contained in [4].

Dunkl operators satisfy many useful properties, for instance they commute and they satisfy an integration by parts formula for a weighted measure. Moreover, with the help of an intertwining operator which connects Dunkl operators with the usual partial derivatives, one can ultimately define a Dunkl transform which is a generalisation of the usual Fourier transform and with which is shares many essential properties. On the other hand, certain elementary properties fail, most notably the Leibniz rule and the chain rule. This is the main source of difficulty in our study of log-Sobolev inequalities since these two results are prerequisites for virtually all the classical methods. Since no standard recipe applies, we study our results on a case by case basis where the main challenge is to bound the difference terms that arise as a particularity of Dunkl operators.

This paper is organised as follows. After a short introduction to Dunkl theory in section [2] we prove the main log-Sobolev inequality for the Dunkl measure $\mu_k$ with Dirichlet form

$$\int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_k$$

in section [3]. In the same section we also prove the log-Sobolev inequality for the Gaussian measure $e^{-|x|^2} \, d\mu_k$. More generally, we want to study the same problem for weighted Dunkl measure $e^{-U} \, d\mu_k$ for suitable $G$-invariant functions $U$ (usually of the form $|x|^p$). The main method for studying these weighted inequalities is provided by [7] where powerful machinery based on $U$-bounds was introduced. Thus we discuss $U$-bounds in section [4] which we then apply in section [5] to obtain the desired weighted log-Sobolev inequality. Finally, in section [6] we prove Poincaré inequalities for the Dunkl measure $\mu_k$ and then use this result, in section [7], to obtain tight log-Sobolev inequalities.
2. Introduction to Dunkl Theory

In this section we will present a very quick introduction to Dunkl operators. For more details see the survey papers [9] and [1].

A root system is a finite set $R \subseteq \mathbb{R}^N \setminus \{0\}$ such that $R \cap \alpha \mathbb{R} = \{-\alpha, \alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. Here $\sigma_\alpha$ is the reflection in the hyperplane orthogonal to the root $\alpha$, i.e.,

$$\sigma_\alpha x = x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

The group generated by all the reflections $\sigma_\alpha$ for $\alpha \in R$ is a finite group, and we denote it by $G$.

Let $k : R \to [0, \infty)$ be a $G$-invariant function, i.e., $k(\alpha) = k(g\alpha)$ for all $g \in G$ and all $\alpha \in R$. We will normally write $k_\alpha = k(\alpha)$ as these will be the coefficients in our Dunkl operators. We can write the root system $R$ as a disjoint union $R = R_+ \cup (-R_+)$, and we call $R_+$ a positive subsystem; this decomposition is not unique, but the particular choice of positive subsystem does not make a difference in the definitions below because of the $G$-invariance of the coefficients $k$.

From now on we fix a root system in $\mathbb{R}^N$ with positive subsystem $R_+$. We also assume without loss of generality that $|\alpha|^2 = 2$ for all $\alpha \in R$. For $i = 1, \ldots, N$ we define the Dunkl operator on $C^1(\mathbb{R}^N)$ by

$$T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} k_\alpha \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

We will denote by $\nabla_k = (T_1, \ldots, T_N)$ the Dunkl gradient, and $\Delta_k = \sum_{i=1}^N T_i^2$ will denote the Dunkl laplacian. Note that for $k = 0$ Dunkl operators reduce to partial derivatives, and $\nabla_0 = \nabla$ and $\Delta_0 = \Delta$ are the usual gradient and laplacian.

We can express the Dunkl laplacian in terms of the usual gradient and laplacian using the following formula:

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k_\alpha \left[ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} \right].$$

The weight function naturally associated to Dunkl operators is

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k_\alpha}.$$

This is a homogeneous function of degree

$$\gamma := \sum_{\alpha \in R_+} k_\alpha.$$

We will work in spaces $L^p(\mu_k)$, where $d\mu_k = w_k(x) \, dx$ is the weighted measure; the norm of these spaces will be written simply $\| \cdot \|_p$. With respect to this weighted measure we have the integration by parts formula

$$\int_{\mathbb{R}^N} T_i(f)g \, d\mu_k = -\int_{\mathbb{R}^N} fT_i(g) \, d\mu_k.$$
One of the main differences between Dunkl operators and classical partial derivatives is that the Leibniz rule does not hold in general. Instead, we have the following.

**Lemma 2.1.** If one of the functions \( f, g \) is \( G \)-invariant, then we have the Leibniz rule

\[ T_i(fg) = fT_ig + gT_if. \]

In general, we have

\[ T_i(fg)(x) = T_if(x)g(x) + f(x)T_ig(x) - \sum_{\alpha \in R_+} k_\alpha x_\alpha \left( f(x) - f(\sigma_\alpha x) \right) \left( g(x) - g(\sigma_\alpha x) \right). \]

A Sobolev inequality is available for the Dunkl gradient (see [12]):

**Proposition 2.2.** Let \( 1 \leq p < N \) and \( q = \frac{p(N + 2 \gamma)}{N + 2 \gamma - p} \). Then there exists a constant \( C > 0 \) such that we have the inequality

\[ \|f\|_q \leq C \|\nabla_k f\|_p \quad \forall f \in C_c^\infty(\mathbb{R}^N). \]

The theory of Dunkl operators is enriched by the construction of the Dunkl kernel, which acts as a generalisation of the classical exponential function. Using the Dunkl kernel it is then possible to define a Dunkl transform, which generalises the classical Fourier transform, with which it shares many important properties. The approaches in this paper are elementary and do not make any use of these notions, so we will not go into further details here; the interested reader can find a more complete account in the review papers recommended at the beginning of this section.

3. The main Log-Sobolev inequalities

To begin with, we have the following Dunkl equivalent of the classical log-Sobolev inequality.

**Theorem 3.1.** There exists a constant \( c \in \mathbb{R} \) such that for any \( \epsilon > 0 \) and for any \( f \in C_c^\infty(\mathbb{R}^N) \) we have

\[ \int_{\mathbb{R}^N} f^2 \log \left( \frac{f^2}{\int_{\mathbb{R}^N} f^2 \, d\mu_k} \right) \, d\mu_k \leq \epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_k + C(\epsilon) \int_{\mathbb{R}^N} f^2 \, d\mu_k, \]

where \( C(\epsilon) = \frac{N + 2 \gamma}{2} \left( \log \frac{1}{\epsilon} - c \right) \).

**Proof.** Fix \( f \in C_c^\infty(\mathbb{R}^N), \ f \neq 0. \) Then \( \frac{f^2}{\int_{\mathbb{R}^N} f^2 \, d\mu_k} \) is a probability measure, and so by Jensen’s inequality we have, for any \( \delta > 0, \)

\[ \int_{\mathbb{R}^N} f^2 \log \left( \frac{f^2}{\int_{\mathbb{R}^N} f^2 \, d\mu_k} \right) \, d\mu_k = \int_{\mathbb{R}^N} f^2 \, d\mu_k \cdot \int_{\mathbb{R}^N} f^2 \, d\mu_k - \log \left( \frac{\int_{\mathbb{R}^N} f^2 \, d\mu_k}{\int_{\mathbb{R}^N} f^2 \, d\mu_k} \right)^\delta \, d\mu_k \]

\[ \leq \frac{1}{\delta} \int_{\mathbb{R}^N} f^2 \, d\mu_k \cdot \log \int_{\mathbb{R}^N} \left( \frac{f^2}{\int_{\mathbb{R}^N} f^2 \, d\mu_k} \right)^{1+\delta} \, d\mu_k \]

\[ = \frac{\delta + 1}{\delta} \left\| f \right\|_2^2 \log \left( \frac{\left\| f \right\|_2^{2+2\delta}}{\left\| f \right\|_2^2} \right). \]

We then use the elementary inequality

\[ \log x \leq \epsilon x + \log \frac{1}{\epsilon} - 1, \]
Corollary 3.2. Let $2 < p < \infty$. Then, for any $\epsilon > 0$ and for any $f \in C_c^\infty(\mathbb{R}^N)$ such that $f \geq 0$, we have
\[
\int_{\mathbb{R}^N} f^p \log \frac{f^2}{\int_{\mathbb{R}^N} f^2 \, d\mu_k} \, d\mu_k \leq \epsilon \int_{\mathbb{R}^N} \nabla_k f \cdot \nabla_k (f^{p-1}) \, d\mu_k + C \left( \frac{2}{p} \right) \int_{\mathbb{R}^N} f^p \, d\mu_k,
\]
where $C(\epsilon)$ is as in the previous Theorem.

Proof. This follows from the previous Theorem and [3, Lemma 2.2.6].

Finally, we recover the ultracontractivity property for the Dunkl heat semigroup. This was already established in [12] using properties of the heat kernel; using the new log-Sobolev approach, no a priori bounds on the heat kernel are necessary.

Corollary 3.3. The Dunkl heat semigroup $(H_t)_{t \geq 0}$ on $L^2(\mu_k)$ with generator $\Delta_k$ is ultracontractive. More precisely, for all $t > 0$ we have
\[
\|H_t f\|_\infty \leq C t^{-\frac{N+2\gamma}{2}} \|f\|_2,
\]
for a constant $C > 0$.

Proof. This follows from the previous Corollary and [3, Theorem 2.2.7].

In what follows, we want to prove similar results for probability measures
\[
d\mu_U := \frac{1}{Z} e^{-U} \, d\mu_k,
\]
where $U \geq 0$ is a $G$-invariant function and $Z = \int_{\mathbb{R}^N} e^{-U} \, d\mu_k$. To illustrate the method and to motivate the study of $U$-bounds in the next section, we first consider Gaussian weight in the following Theorem. This result will be further refined and generalised in the next sections, but the main lines of the proof will remain the same.

Theorem 3.4. Let $U = \|x\|^2$ and consider the probability measure $\mu_U := \frac{1}{Z} e^{-U} \mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} \, d\mu_k$. Then, there exist constants $C_1, C_2 > 0$ such that the following inequality holds for all $f \in C_c^\infty(\mathbb{R}^N)$:
\[
\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int_{\mathbb{R}^N} f^2 \, d\mu_U} \, d\mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + C_2 \int_{\mathbb{R}^N} f^2 \, d\mu_U.
\]
Proof. Plugging $\frac{1}{\sqrt{Z}} f e^{-U/2}$ in Theorem 3.1 we have
\[
\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \, d\mu_U} \, d\mu_U \leq \frac{1}{Z} \int_{\mathbb{R}^N} |\nabla_k (f e^{-U/2})|^2 \, d\mu_k \\
+ (C(\epsilon) + \log Z) \int_{\mathbb{R}^N} f^2 \, d\mu_U + \int_{\mathbb{R}^N} f^2 U \, d\mu_U.
\]

Since $U$ is $G$-invariant, we can use the Leibniz rule to compute
\[
\frac{1}{Z} \int_{\mathbb{R}^N} |\nabla_k (f e^{-U/2})|^2 \, d\mu_k = \int_{\mathbb{R}^N} |\nabla_k f - \frac{1}{2} f \nabla U|^2 \, d\mu_U \\
\leq 2 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + \frac{1}{2} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 \, d\mu_U \\
= 2 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + 2 \int_{\mathbb{R}^N} f^2 U \, d\mu_U,
\]
where, in the last line, we used the fact that $\nabla U = 2x$, so $|\nabla U|^2 = 4U$. Replacing this inequality in the above, we have
\[
\int_{\mathbb{R}^N} f^2 \log \frac{f^2}{\int f^2 \, d\mu_U} \, d\mu_U \leq 2\epsilon \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U \\
+ (C(\epsilon) + \log Z) \int_{\mathbb{R}^N} f^2 + (1 + 2\epsilon) \int_{\mathbb{R}^N} f^2 U \, d\mu_U. \tag{3.1}
\]

We now use the identity
\[
\nabla_k (f e^{-U/2}) = e^{-U/2} \nabla_k f - \frac{1}{2} f e^{-U/2} \nabla U \tag{3.2}
\]

\[
\int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U = \frac{1}{Z} \int_{\mathbb{R}^N} |\nabla_k (f e^{-U/2})|^2 \, d\mu_k + \frac{1}{4} \int_{\mathbb{R}^N} f^2 |\nabla U|^2 \, d\mu_U \\
+ \frac{1}{Z} \int_{\mathbb{R}^N} f \nabla U \cdot \nabla_k (f e^{-U/2}) e^{-U/2} \, d\mu_k. \tag{3.3}
\]

Keeping in mind that $\nabla U = 2x$, this equality implies that
\[
\int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U \geq \int_{\mathbb{R}^N} f^2 U \, d\mu_U + A, \tag{3.4}
\]

where
\[
A := \frac{1}{Z} \int_{\mathbb{R}^N} f \nabla U \cdot \nabla_k (f e^{-U/2}) e^{-U/2} \, d\mu_k.
\]

We now compute the quantity $A$. Firstly, by integration by parts we have
\[
A = -\sum_{i=1}^{N} \frac{1}{Z} \int_{\mathbb{R}^N} f e^{-U/2} T_i (2x_i f e^{-U/2}) \, d\mu_k(x).
\]
Using Lemma 2.1, we have
\begin{equation}
T_i(x_i,e^{-U/2}) = x_iT_i(e^{-U/2}) + f e^{-U/2} T_i(x_i) - \sum_{\alpha \in R_+} k_{\alpha} \alpha_i e^{-U(x_i)/2} \frac{(f(x) - f(\sigma_\alpha x))(x_i - (\sigma_\alpha x)_i)}{\langle \alpha, x \rangle}
\end{equation}

Thus
\begin{equation}
A = -A - 2(N + 2\gamma) \int_{\mathbb{R}^N} f^2 d\mu_U + 4 \sum_{\alpha \in R_+} k_{\alpha} \int_{\mathbb{R}^N} f(x)(f(x) - f(\sigma_\alpha x)) d\mu_U(x),
\end{equation}

and so
\begin{equation}
A = -N \int_{\mathbb{R}^N} f^2 d\mu_U - 2 \sum_{\alpha \in R_+} k_{\alpha} \int_{\mathbb{R}^N} f(x)f(\sigma_\alpha x) d\mu_U(x).
\end{equation}

Using the elementary inequality $2XY \leq X^2 + Y^2$, we obtain
\begin{equation}
A \geq -(N + 2\gamma) \int_{\mathbb{R}^N} f^2 d\mu_U.
\end{equation}

Replacing this in equation (3.6), we obtain
\begin{equation}
\int_{\mathbb{R}^N} f^2 \log f^2 \mu_U \leq \int_{\mathbb{R}^N} |\nabla_k f|^2 \mu_U + (N + 2\gamma) \int_{\mathbb{R}^N} f^2 d\mu_U.
\end{equation}

Finally, using this in (3.1), we have
\begin{equation}
\int_{\mathbb{R}^N} f^2 \log f^2 \mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \mu_U + C_2 \int_{\mathbb{R}^N} f^2 d\mu_U,
\end{equation}

for some constants $C_1, C_2 > 0$, as required.

4. U-bounds

Looking back at the proof of the weighted log-Sobolev inequality in Theorem 3.4, we can see that inequality (3.6) was the key element. Inequalities of this form are called U-bounds (cf. [7]). In this section we will prove more general U-bounds by adapting our proof slightly, and these will later be used to deduce log-Sobolev inequalities.

**Proposition 4.1.** Let $p > 1$ and consider the function $U(x) = |x|^p$ and the probability measure $\mu_U := \frac{1}{Z} e^{-U} \mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} \mu_k$. We then have
\begin{equation}
\int_{\mathbb{R}^N} |f| \cdot |x|^{p-1} d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f| d\mu_U + D \int_{\mathbb{R}^N} |f| d\mu_U,
\end{equation}

for some constants $C, D > 0$.

**Proof.** In order to avoid a singularity that will arise at the origin, we first consider a function $f$ that vanishes on the unit ball. As before, we start with identity (3.2). Noting that in this case we have
\begin{equation}
\nabla U(x) = p|x|^{p-1} \nabla(|x|),
\end{equation}
the identity above now reads
\[ \nabla_k (f e^{-U}) = e^{-U} \nabla_k f - pf |x|^{p-1} e^{-U} \nabla(|x|). \]
Taking inner product with \( \nabla(|x|) \) and integrating on both sides, we have
\[
\frac{1}{Z} \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k (f e^{-U}) \, d\mu_k
= \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k f \, d\mu_k - p \int_{\mathbb{R}^N} |\nabla(|x|)|^2 f |x|^{p-1} \, d\mu_k.
\] (4.1)

We can use integration by parts on the left hand side to obtain
\[
\frac{1}{Z} \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k (f e^{-U}) \, d\mu_k = - \int_{\mathbb{R}^N} \Delta_k(|x|) f \, d\mu_k.
\]
Replacing this in (4.1), and using also the fact that \( |\nabla(|x|)| = 1 \) for \( x \neq 0 \), we have
\[
\int_{\mathbb{R}^N} f |x|^{p-1} \, d\mu_k = \frac{1}{p} \int_{\mathbb{R}^N} \nabla(|x|) \cdot \nabla_k f \, d\mu_k + \frac{1}{p} \int_{\mathbb{R}^N} \Delta_k(|x|) f \, d\mu_k
\leq \frac{1}{p} \int_{\mathbb{R}^N} \nabla_k f \, d\mu_k + \frac{1}{p} \int_{\mathbb{R}^N} \Delta_k(|x|) f \, d\mu_k.
\]
Finally, we have
\[
T_i^2(|x|) = T_i \left( \frac{x_i}{|x|} \right) = \frac{1}{|x|} - \frac{x_i^2}{|x|^3} + \sum_{\alpha \in \mathbb{R}_+} \kappa_\alpha \frac{\alpha_i^2}{|x|},
\]
so
\[
\Delta_k(|x|) = (N + 2\gamma - 1) \frac{1}{|x|}.
\]
Therefore, from the above we deduce that (recall that \( f \) vanishes on the unit ball)
\[
\int_{\mathbb{R}^N} f |x|^{p-1} \, d\mu_k \leq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla_k f| \, d\mu_k + \frac{N + 2\gamma - 1}{p} \int_{\mathbb{R}^N} |f| \, d\mu_k.
\]
Writing \( f = f_+ - f_- \), where \( f_+(x) = \max(f(x), 0) \) and \( f_-(x) = -\min(f(x), 0) \), we can apply this inequality to \( f_+ \) and \( f_- \) separately. Adding the two resulting inequalities, we have
\[
\int_{\mathbb{R}^N} |f| \cdot |x|^{p-1} \, d\mu_k \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f| \, d\mu_k + D_1 \int_{\mathbb{R}^N} |f| \, d\mu_k,
\]
where \( C_1 = \frac{1}{p} \) and \( D_1 = \frac{N + 2\gamma - 1}{p} \).

Consider now a function \( f \in L^1(d\mu_k) \). Let \( \varphi : \mathbb{R}_+ \to [0, 1] \) be a smooth function such that \( \varphi = 0 \) on \([0, 1]\) and \( \varphi = 1 \) on \([2, \infty)\). Assume also that \( |\varphi'| \leq 1 \) on \( \mathbb{R}_+ \). Let \( \phi(x) = \varphi(|x|) \).

The strategy now will be to split \( f = \phi f + (1 - \phi) f \); the first term vanishes on the unit ball so the above can be applied to it, while the second term has compact support and it will be easy to bound. We have
\[
\int_{\mathbb{R}^N} |f| \cdot |x|^{p-1} \, d\mu_k \leq \int_{\mathbb{R}^N} |\phi f| \cdot |x|^{p-1} \, d\mu_k + \int_{\mathbb{R}^N} |(1 - \phi) f| \cdot |x|^{p-1} \, d\mu_k
\leq C_1 \int_{\mathbb{R}^N} |\nabla_k (\phi f)| \, d\mu_k + D_1 \int_{\mathbb{R}^N} |\phi f| \, d\mu_k + 2^{p-1} \int_{\mathbb{R}^N} |f| \, d\mu_k
\leq C_1 \int_{\mathbb{R}^N} |\nabla_k f| \, d\mu_k + (C_1 + D_1 + 2^{p-1}) \int_{\mathbb{R}^N} |f| \, d\mu_k.
\]
Proof. We follow more closely the proof of (3.6). We have

$$ C, D$$

for some constants where

(4.3)

and

(4.2)

Let

Proposition 4.2. \( \square \)

This completes the proof.

Here, in the last step, we used the fact that

Assume first that \( p \) and let \( \epsilon > 0 \). Then, using Hölder’s inequality with coefficients \( \frac{p}{2(p-1)} \) and \( \frac{p-2}{2(p-1)} \), and then Young’s inequality with the same coefficients, we have

Thus

(4.3)

Assume first that \( p > 2 \) and let \( \epsilon > 0 \). Then, using Hölder’s inequality with coefficients \( \frac{p}{2(p-1)} \) and \( \frac{p-2}{2(p-1)} \), and then Young’s inequality with the same coefficients, we have

$$ \leq \frac{p}{2(p-1)} \epsilon \int_{\mathbb{R}^N} f^2 \, d\mu_U + \frac{p-2}{2(p-1)} \epsilon \int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \, d\mu_U. $$
Thus, by choosing $\epsilon > 0$ small enough such that
\[
1 > \frac{(p - 2)[p^2 + p(N + 2\gamma - 2)]}{p^2(p - 1)}\epsilon,
\]
we obtain inequality (4.2) for some constants $C, D > 0$.

The case $1 < p < 2$ requires more care. The inequality (4.3) holds for all $p > 1$, and so taking $1_{B_1^c}f$ in this result, where $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$ and $B_1^c = \mathbb{R}^N \setminus B_1$, we obtain
\[
\int_{B_1^c} f^2 |x|^{2(p-1)} \, d\mu_U \leq \frac{4}{p^2} \int_{B_1^c} |\nabla_k f|^2 \, d\mu_U + \frac{2 \left[p^2 + p(N + 2\gamma - 2)\right]}{p^2} \int_{B_1^c} f^2 |x|^{p-2} \, d\mu_U.
\]
Note that, since $p < 2$,
\[
\int_{B_1^c} f^2 |x|^{p-2} \, d\mu_U \leq \int_{B_1^c} f^2 \, d\mu_U,
\]
and so
\[
\int_{B_1^c} f^2 |x|^{2(p-1)} \, d\mu_U \leq \frac{4}{p^2} \int_{B_1^c} |\nabla_k f|^2 \, d\mu_U + \frac{2 \left[p^2 + p(N + 2\gamma - 2)\right]}{p^2} \int_{B_1^c} f^2 \, d\mu_U.
\]
On the other hand, since $p > 1$, we have
\[
\int_{B_1} f^2 |x|^{2(p-1)} \, d\mu_U \leq \int_{B_1} f^2 \, d\mu_U.
\]
Adding the last two inequalities, we obtain
\[
\int_{\mathbb{R}^N} f^2 |x|^{2(p-1)} \, d\mu_U \leq \frac{4}{p^2} \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + \frac{2 \left[p^2 + p(N + 2\gamma - 2)\right]}{p^2} \int_{\mathbb{R}^N} f^2 \, d\mu_U,
\]
which concludes the proof of the Proposition.

\[
\square
\]

5. Log-Sobolev inequalities for weighted measures

In this section we use the $U$-bounds obtained above to deduce log-Sobolev inequalities. To begin with, from Proposition 4.2 we obtain a generalised $L^2(\mu_U)$ log-Sobolev inequality for the weighted probability measure $d\mu_U = \frac{1}{Z} e^{-U(x)} \, d\mu_k$, for $U(x) = |x|^p$ for $1 \leq p \leq 2$. Similarly, from Proposition 4.3 we obtain a log-Sobolev inequality in $L^1(\mu_U)$ for general $1 < p < \infty$. The approach in both these results is similar to that of Theorem 3.1 first employing Jensen’s inequality to take the logarithm outside the integral, and then using the classical Sobolev inequality. $U$-bounds will be used to control residual terms arising from the introduction of a weight.

**Proposition 5.1.** Let $U(x) = |x|^p$ for $1 < p \leq 2$ and consider the probability measure $d\mu_U = \frac{1}{Z} e^{-U} \, d\mu_k$, where $Z = \int_{\mathbb{R}^N} e^{-U} \, d\mu_k$. Let $s = \frac{2p-1}{p}$. Then we have the inequality
\[
\int_{\mathbb{R}^N} f^2 \left[ \log \frac{\int_{\mathbb{R}^N} f^2 \, d\mu_U}{\int_{\mathbb{R}^N} f^2 \, d\mu_U} \right]^s \, d\mu_U \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + C_2 \int_{\mathbb{R}^N} f^2 \, d\mu_U,
\]
for some constants $C_1, C_2 > 0$. 
Proof. Consider the function \( h = \frac{1}{\sqrt{Z}} f e^{-U/2} \). Then \( \int_{\mathbb{R}^N} f^2 \, d\mu_U = \int_{\mathbb{R}^N} h^2 \, d\mu_k \), and
\[
(5.1) \quad \int_{\mathbb{R}^N} f^2 \left| \log \frac{f^2}{\int f^2 \, d\mu_U} \right|^s \, d\mu_U = \int_{\mathbb{R}^N} h^2 \left| \log \frac{h^2}{\int h^2 \, d\mu_k} + U + \log Z \right|^s \, d\mu_k
\]
\[
\leq \int_{\mathbb{R}^N} h^2 \left| \log \frac{h^2}{\int h^2 \, d\mu_k} \right|^s \, d\mu_k + \int_{\mathbb{R}^N} h^2 U^s \, d\mu_k + |\log Z|^s \int_{\mathbb{R}^N} h^2 \, d\mu_k,
\]
where in the last inequality we used the fact that since \( s \in (0, 1) \) we have \( (X+Y)^s \leq X^s + Y^s \) for \( X, Y > 0 \).

Before we start the usual procedure of applying Jensen’s inequality, we note that the function \( x |\log x|^s \) is bounded on \((0, 1)\), and let
\[
D = \sup_{x \in (0, 1)} x |\log x|^s < \infty.
\]
Consider now the function \( \log^{+} x := \max\{0, \log x\} \). Then the above observation implies that
\[
x |\log x|^s \leq x (\log^{+} x)^s + D \quad \text{for all } x > 0.
\]
With this in mind, we have
\[
(5.2) \quad \int_{\mathbb{R}^N} h^2 \left| \log \frac{h^2}{\int h^2 \, d\mu_k} \right|^s \, d\mu_k = \int_{\mathbb{R}^N} h^2 \, d\mu_k \cdot \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \, d\mu_k} \left| \log \frac{h^2}{\int h^2 \, d\mu_k} \right|^s \, d\mu_k
\]
\[
\leq \int_{\mathbb{R}^N} h^2 \, d\mu_k \cdot \left[ \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \, d\mu_k} \left( \log^{+} \left( \frac{h^2}{\int h^2 \, d\mu_k} \right)^s \right) \, d\mu_k + D \right].
\]

For the fixed function \( h \) the measure \( \frac{h^2}{\int h^2 \, d\mu_k} \, d\mu_k \) is a probability measure and so we can apply Jensen’s inequality to the concave function \( (\log^{+} t)^s \) in the below as follows
\[
\int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \, d\mu_k} \left( \log^{+} \frac{h^2}{\int h^2 \, d\mu_k} \right)^s \, d\mu_k = \frac{1}{\delta^s} \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \, d\mu_k} \left( \log^{+} \left( \frac{h^2}{\int h^2 \, d\mu_k} \right)^\delta \right)^s \, d\mu_k
\]
\[
\leq \frac{1}{\delta^s} \left( \log^{+} \int_{\mathbb{R}^N} \left( \frac{h^2}{\int h^2 \, d\mu_k} \right)^{1+\delta} \, d\mu_k \right)^s
\]
\[
= \frac{(1+\delta)^s}{\delta^s} \left( \log^{+} \frac{\|h\|_2^2}{\|h\|_2^2 \delta} \right)^s.
\]

By standard elementary means we can show that there exists a decreasing function \( \overline{C}(\epsilon) \) defined for all \( \epsilon > 0 \) such that the inequality
\[
|\log^{+} x|^s \leq \epsilon x + \overline{C}(\epsilon)
\]
holds for all \( x > 0 \). Applying this in the previous inequality, we obtain
\[
(5.3) \quad \int_{\mathbb{R}^N} \frac{h^2}{\int h^2 \, d\mu_k} \left( \log^{+} \frac{h^2}{\int h^2 \, d\mu_k} \right)^s \, d\mu_k \leq \frac{(1+\delta)^s}{\delta^s} \left( \epsilon \frac{\|h\|_2^2}{\|h\|_2^2 \delta} + \overline{C}(\epsilon) \right)
\]
\[
\leq \frac{(1+\delta)^s}{\delta^s} \frac{1}{\delta^2} \frac{\|\nabla h\|^2_2 + \overline{C}(\epsilon) \|h\|_2^2}{\|h\|_2^2}.
\]
Proof. The proof is similar to that of the previous result except for in this case we rely on Proposition 5.2.

Finally, from Proposition 4.2 and a relabelling of the constants, we obtain
\[
\int \log \left( \frac{f}{f^2 d\mu_U} \right) d\mu_U \leq \epsilon \int |\nabla_k f|^2 d\mu_U + C(\epsilon) \int f^2 d\mu_U,
\]
where \(C(\epsilon) = A_1 C(\epsilon) + A_2 \epsilon + A_3\), for some constants \(A_1, A_2, A_3, \alpha > 0\). Choosing small \(\epsilon > 0\) we have \(C(\epsilon) > 0\). This concludes the proof. \(\square\)

Proposition 5.2. Let \(U(x) = |x|^p\) with \(1 \leq p < \infty\) and consider the finite measure \(d\mu_U = e^{-U} d\mu_k\). Let \(s = \frac{p-1}{p}\). Then we have the inequality
\[
\int f \left( \log \frac{f}{f d\mu_U} \right)^s d\mu_U \leq C_1 \int |\nabla_k f|^2 d\mu_U + C_2 \int |f|^2 d\mu_U,
\]
for some constants \(C_1, C_2 > 0\).

Proof. The proof is similar to that of the previous result except for in this case we rely on the Sobolev inequality
\[
\|h\|_q \leq C \|\nabla h\|_1
\]
where \(q = \frac{N+2\gamma}{N+2\gamma-1}\), and the \(U\)-bound of Proposition 4.1. \(\square\)

6. Poincaré inequalities

In this section we discuss Poincaré inequalities for Dunkl operators. These will be used in the next section to improve some of our previous log-Sobolev inequalities, but are also of independent interest.

We note that by solving the equation \(C(\epsilon) = 0\), Theorem 8.1 implies the following tight log-Sobolev inequality
\[
\int f^2 \log \frac{f^2}{f^2 d\mu_k} d\mu_k \leq C \int |\nabla_k f|^2 d\mu_k,
\]
(6.1)
which holds for a constant $C > 0$ and for all $f \in C_c^\infty(\mathbb{R}^N)$. A simple density argument shows that (6.1) holds in fact for any $f \in L^2(\mu_k)$ for which we also have $\nabla_k f \in L^2(\mu_k)$.

Using a standard argument (see for example [2]), this inequality implies the following Poincaré inequality.

**Theorem 6.1.** Let $R > 0$ and consider the ball $B_R := \{ x \in \mathbb{R}^N : |x| \leq R \}$. There exists a constant $C > 0$ independent of $R$ such that for any $f \in L^2(B_R, \mu_k)$ with $\nabla_k f \in L^2(B_R, \mu_k)$, we have the inequality

$$\int_{B_R} \left| f - \frac{1}{\mu_k(B_R)} \int_{B_R} f \, d\mu_k \right|^2 \, d\mu_k \leq CR^2 \int_{B_R} |\nabla_k f|^2 \, d\mu_k.$$  

**Proof.** To simplify the notation, let $\tilde{\mu}_R := \frac{1}{\mu_k(B_R)} \mu_k$ be the Dunkl probability measure on the ball $B_R$. Note that it is enough to prove the Theorem for $f$ that satisfies the additional assumption $\int f \, d\tilde{\mu}_R = 0$, in which case the inequality takes the form

$$(6.2) \quad \int f^2 \, d\tilde{\mu}_R \leq CR^2 \int |\nabla_k f|^2 \, d\tilde{\mu}_R.$$  

To obtain the general case it is then enough to replace $f$ by $f - \int f \, d\tilde{\mu}_R$ in (6.2).

For any $\epsilon > 0$ consider the function $g = 1 + \epsilon f$. A Taylor expansion shows that

$$g^2 \log \frac{g^2}{g^2 \, d\tilde{\mu}_R} = 2\epsilon f + 3\epsilon^2 f^2 - \epsilon^2 \int f^2 \, d\tilde{\mu}_R + o(\epsilon^2),$$

and thus

$$(6.3) \quad \int g^2 \log \frac{g^2}{g^2 \, d\tilde{\mu}_R} \, d\tilde{\mu}_R = 2\epsilon^2 \int f^2 \, d\tilde{\mu}_R + o(\epsilon^2),$$

as $\epsilon \to 0$.

From Theorem 3.1 we have that

$$\int g^2 \log \frac{g^2}{g^2 \, d\tilde{\mu}_R} \, d\tilde{\mu}_R \leq \delta \int |\nabla_k g|^2 \, d\tilde{\mu}_R + (C(\delta) + \log(\mu_k(B_R))) \int g^2 \, d\tilde{\mu}_R,$$

holds for all $\delta > 0$. However, using the fact that $\mu_k(B_R) = c'R^{N+2\gamma}$ for a constant $c' > 0$, we find that $\delta = c''R^2$, for a constant $c'' > 0$, solves the equation $C(\delta) + \log(\mu_k(B_R)) = 0$. Therefore, we have the tight log-Sobolev inequality

$$(6.4) \quad \int g^2 \log \frac{g^2}{g^2 \, d\tilde{\mu}_R} \, d\tilde{\mu}_R \leq c''R^2 \int |\nabla_k g|^2 \, d\tilde{\mu}_R.$$  

Combining (6.3) and (6.4), and letting $\epsilon \to 0$, we have obtained (6.2), as required. \qed

**Remark.** This Poincaré inequality corresponds to the classical Neumann-Poincaré inequality. A Dirichlet-Poincaré inequality for Dunkl operators was also proved in [13]. Namely, we have the result:

$$\int_\Omega |f|^2 \, d\mu_k \leq C(\Omega) \int_\Omega |\nabla_k f|^2 \, d\mu_k,$$

which holds on any bounded domain $\Omega \subset \mathbb{R}^N$ for a constant $C(\Omega) > 0$ and for all $f \in C_c^\infty(\mathbb{R}^N)$.

We can now use the previous result together with the $U$-bounds proved above to obtain a Poincaré inequality for the weighted measure $\mu_U$. 

Proposition 6.2. Let $p > 1$ and consider the weighted probability measure $d\mu_U = \frac{1}{Z} e^{-U} d\mu_k$, where $U(x) = |x|^p$ and $Z = \int_{\mathbb{R}^N} e^{-U} d\mu_k$. We then have the Poincaré inequality

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f \, d\mu_U \right|^2 \, d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U$$

which holds for all functions $f \in L^2(\mu_k)$ such that $\nabla_k f \in L^2(\mu_k)$, with a constant $C > 0$ independent of $f$.

Proof. We first note that for any constant $\zeta \in \mathbb{R}$ we have

$$\left| \int_{\mathbb{R}^N} f \, d\mu_U - \zeta \right| \leq \int_{\mathbb{R}^N} |f - \zeta| \, d\mu_U \leq \left( \int_{\mathbb{R}^N} |f - \zeta|^2 \, d\mu_U \right)^{\frac{1}{2}},$$

so by the triangle inequality for the $L^2(\mu_k)$ norm

$$\int_{\mathbb{R}^N} \left| f - \int_{\mathbb{R}^N} f \, d\mu_U \right|^2 \, d\mu_U \leq 4 \int_{\mathbb{R}^N} |f - \zeta|^2 \, d\mu_U.$$

Thus, it is enough to prove the inequality

$$\int_{\mathbb{R}^N} |f - \zeta|^2 \, d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U$$

for some $\zeta \in \mathbb{R}$.

Let $R > 0$ and let $B_R = \{|x| \leq R\}$. We will prove (6.5) with $\zeta = \frac{1}{\mu_k(B_R)} \int_{B_R} f \, d\mu_k$ for large enough $R$. Firstly, we have

$$\int_{B_R} |f - \zeta|^2 \, d\mu_U \leq \frac{1}{Z} \int_{B_R} \left| f - \frac{1}{\mu_k(B_R)} \int_{B_R} f \, d\mu_k \right|^2 \, d\mu_k$$

$$\leq \frac{C}{Z} R^2 \int_{B_R} |\nabla_k f|^2 \, d\mu_k$$

$$\leq CR^2 e^{R^p} \int_{B_R} |\nabla_k f|^2 \, d\mu_U.$$

Here we used the Poincaré inequality of Theorem 6.1 and the bounds $e^{-R^p} \leq e^{-U} \leq 1$ on $B_R$.

On the other hand, we can use the $U$-bounds of Proposition 6.2 to the remaining integral as follows

$$\int_{\mathbb{R}^N \setminus B_R} |f - \zeta|^2 \, d\mu_U \leq R^{-2(p-1)} \int_{|x| > R} |f(x) - \zeta|^2 \, d\mu_U$$

$$\leq CR^{-2(p-1)} \int_{|x| > R} |\nabla_k f|^2 \, d\mu_U + DR^{-2(p-1)} \int_{|x| > R} |f - \zeta|^2 \, d\mu_U.$$

But $R$ was an arbitrary positive number so we are free to choose it such that $DR^{-2(p-1)} < 1$. Then we have

$$\int_{\mathbb{R}^N \setminus B_R} |f - \zeta|^2 \, d\mu_U \leq \frac{CR^{-2(p-1)}}{1 - DR^{-2(p-1)}} \int_{|x| > R} |\nabla_k f|^2 \, d\mu_U.$$

Adding the inequalities (6.6) and (6.7), we obtain (6.5), and therefore, by the observation above, the Proposition is proved. \qed
7. Tight log-Sobolev inequalities

We now have all the ingredients to obtain tight log-Sobolev inequalities. The first is a tight version of the log-Sobolev inequality for Gaussian measure from Theorem 3.4.

**Theorem 7.1.** Let \( U(x) = |x|^2 \) and consider the probability measure \( d\mu_U = \frac{1}{Z} e^{-U} \, d\mu_k \), where \( Z = \int_{\mathbb{R}^N} e^{-U} \, d\mu_k \). Then there exists a constant \( C > 0 \) such that we have the inequality

\[
\int_{\mathbb{R}^N} f^2 \log \left( \frac{f^2}{\int f^2 \, d\mu_U} \right) \, d\mu_U \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U.
\]

In order to prove this result we will need the following inequality, known as Rothaus’s lemma.

**Lemma 7.2** ([10]). Recall that

\[
\text{Ent}(g) := \int_{\mathbb{R}^N} g \log g \, d\mu_U - \int_{\mathbb{R}^N} g \, d\mu_U \log \int_{\mathbb{R}^N} g \, d\mu_U,
\]

for \( g \geq 0 \). Then, for all \( f \) measurable with \( \int_{\mathbb{R}^N} f \, d\mu_U = 0 \), we have the inequality

\[
\text{Ent}((f + c)^2) \leq \text{Ent}(f^2) + 2 \int_{\mathbb{R}^N} f^2 \, d\mu_U,
\]

for all \( c \in \mathbb{R} \).

**Proof of Theorem 7.1.** By Rothaus’s lemma we have

\[
\text{Ent}(f^2) \leq \text{Ent} \left( \left( f - \int f \, d\mu_U \right)^2 \right) + 2 \int_{\mathbb{R}^N} \left( f - \int f \, d\mu_U \right)^2 \, d\mu_U.
\]

Furthermore, from Theorem 3.4 we have

\[
\text{Ent}(f^2) \leq C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + (2 + C_2) \int_{\mathbb{R}^N} \left( f - \int f \, d\mu_U \right)^2 \, d\mu_U.
\]

Finally, using the Poincaré inequality of Proposition 6.2 we obtain

\[
\text{Ent}(f^2) \leq (C_1 + C(2 + C_2)) \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U,
\]

which is exactly what we wanted to prove. \( \square \)

The following inequality is also a tight version of the generalised log-Sobolev inequality of Proposition 5.1 and it is obtained from this result in a manner very similar to the proof that we have just seen.

**Theorem 7.3.** Let \( 1 \leq p \leq 2 \) and \( s = \frac{2p-1}{p} \). Let \( U(x) = |x|^p \) and consider the probability measure \( d\mu_U = \frac{1}{Z} e^{-U} \, d\mu_k \), where \( Z = \int_{\mathbb{R}^N} e^{-U} \, d\mu_k \). Let also

\[
\Phi(x) = x(\log(x + 1))^s.
\]

Then we have the inequality

\[
\int_{\mathbb{R}^N} \Phi(f^2) \, d\mu_U - \Phi \left( \int_{\mathbb{R}^N} f^2 \, d\mu_U \right) \leq C \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U,
\]
for a constant $C > 0$.

As before, we need the following generalisation of Rothaus’s lemma.

**Lemma 7.4** \([\textbf{7.4}]\). Let $\Phi$ be as in the statement of the Theorem and define, for $g \geq 0$,

\[
\text{Ent}_\Phi(g) := \int_{\mathbb{R}^N} \Phi(g) \, d\mu_U - \Phi \left( \int_{\mathbb{R}^N} g \, d\mu_U \right).
\]

Then there exist constants $A_1, B_1 > 0$ such that for any $f$ with $\int_{\mathbb{R}^N} f \, d\mu_U = 0$ we have

\[
\text{Ent}_\Phi(f^2) \leq A_1 \text{Ent}_\Phi(f^2) + B_1 \int_{\mathbb{R}^N} f^2 \, d\mu_U
\]

for all $c \in \mathbb{R}$.

**Proof of Theorem 7.3** The proof of this goes along the same lines as the proof of Theorem \([\textbf{7.1}]\) although in this case we cannot apply Proposition \([\textbf{5.1}]\) directly. Instead, we note that

\[
\text{Ent}_\Phi(g) = \int_{\mathbb{R}^N} g \left( \log(1 + g) \right)^s - \left( \log \left( 1 + \int_{\mathbb{R}^N} g \right) \right)^s \, d\mu_U
\]

\[
\leq \int_{\mathbb{R}^N} g \log \frac{g + 1}{g \, d\mu_U + 1}^s \, d\mu_U,
\]

where we used the inequality $(X + Y)^s \leq X^s + Y^s$ which holds for all $X, Y \geq 0$ since $s \in [0, 1]$. We compute the integral on the right hand side separately over $X = \{ x : g(x) \geq \int_{\mathbb{R}^N} g \, d\mu_U \}$ and $X = \mathbb{R} \setminus X$. On $X$ we have

\[
1 \leq \frac{g + 1}{g \, d\mu_U + 1} \leq \frac{g}{g \, d\mu_U},
\]

so

\[
\int_X g \left| \log \frac{g + 1}{g \, d\mu_U + 1} \right|^s \, d\mu_U \leq \int_{\mathbb{R}^N} g \left| \log \frac{g}{g \, d\mu_U} \right|^s \, d\mu_U.
\]

On the other hand, on $X$ we have

\[
1 \leq \frac{g \, d\mu_U + 1}{g + 1} \leq 1 + \frac{g \, d\mu_U}{g},
\]

so

\[
\int_X g \left| \log \frac{g + 1}{g \, d\mu_U + 1} \right|^s \, d\mu_U = \int_X g \left( \log \frac{g \, d\mu_U + 1}{g + 1} \right)^s \, d\mu_U
\]

\[
\leq \int_X g \left( \frac{g \, d\mu_U}{g} \right)^s \, d\mu_U \leq \int_{\mathbb{R}^N} g \, d\mu_U,
\]

where we first used the inequality $\log(1 + x) \leq x$ for all $x \geq 0$, and then the fact that $s \leq 1$, so $\left( \frac{g \, d\mu_U}{g} \right)^s \leq \frac{g \, d\mu_U}{g}$. Thus

\[
\text{Ent}_\Phi(g) \leq \int_{\mathbb{R}^N} g \left| \log \frac{g}{g \, d\mu_U} \right|^s \, d\mu_U + \int_{\mathbb{R}^N} g \, d\mu_U.
\]
We can now apply the same strategy as before. First, by Proposition 5.1 we have
\[ \text{Ent}_b(g^2) \leq C_1 \int_{\mathbb{R}^N} |\nabla_k g|^2 \, d\mu_U + (C_2 + 1) \int_{\mathbb{R}^N} g^2 \, d\mu_U. \]
Taking \( g = f - \int_{\mathbb{R}^N} f \, d\mu_U \) in this inequality and applying the previous Lemma, we have
\[ \text{Ent}_b(f^2) \leq A_1 C_1 \int_{\mathbb{R}^N} |\nabla_k f|^2 \, d\mu_U + (A_1(C_2 + 1) + B_1) \int_{\mathbb{R}^N} \left( f - \int_{\mathbb{R}^N} f \, d\mu_U \right)^2 \, d\mu_U. \]
Finally, using the Poincaré inequality of Proposition 6.2 the proof is complete. \( \square \)

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