GRAVITATIONAL COLLAPSES
AND HILBERTIAN REPULSION

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Abstract. If we take into account the Hilbertian gravitational repulsion, even a "dust" sphere with a zero fluido-dynamical pressure collapses to a body of a comparatively small, finite, volume.

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1. – In general relativity, according to well known arguments, even the presence of a fluido-dynamical pressure cannot prevent a massive body from collapsing to a space-time singularity if a given “point of no return” is passed. Now, McVittie [1] and the present authors [2] have proved with exact computations that a spherically-symmetric gravitational collapse under the action of a time-dependent pressure $p(t)$ ends up in a body with a finite extent. In this paper, we shall demonstrate a very general and very simple result: even with a zero pressure, a spherically-symmetric collapse, because of the decisive presence of the Hilbertian repulsive effect [3], [4], ends up in a body of a comparatively small, finite, volume.

2. – We start from a general expression of the spherically-symmetrical space-time interval $ds (\epsilon = G = 1)$ [5]:

\[ ds^2 = \exp [\nu(r, t)] dt^2 - \exp [\lambda(r, t)] dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \]

Then, we write the pertinent Einstein field equations with a mass tensor $T^k_j = \rho v^j v^k$, ($j, k = 0, 1, 2, 3$), where $\rho$ is the invariant mass density of a "dust" sphere, and $v^k$ is the four-velocity of a "dust" element:

\[ R^k_j - \frac{1}{2} \delta^k_j R + 8\pi \rho v^j v^k = 0 ; \]

(for an explicit evaluation of eqs. (2), see e.g. [6]). Eqs. (2) have as a consequence (the colon denotes a covariant derivative):

\[ (\rho v^j v^k)_k = 0 , \]

from which:

\[ v^k v_{j;k} = 0 : \]
the particles of the “dust” sphere move according to geodesic lines.

For the interval $d s_{\text{ext}}$ outside the sphere we choose the standard (Hilbert-Droste-Weyl) form of solution [3]:

$$ds_{\text{ext}}^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\phi^2),$$

where $m$ is the mass of our sphere (if $M$ is the mass in CGS units, we have $m = (G/c^2)M$).

Let us assume for simplicity that at the beginning of the collapse all the “dust” particles are at rest.

The surface shell of the sphere contracts itself as if the total mass were concentrated in $r = 0$. The other concentric spherical shells contract themselves as if the mass in $r = 0$ were correspondently reduced. By virtue of the Hilbertian repulsion [3], [4], the particles of the superficial shell arrive at $r = 2m$ with a zero velocity $dr/dt$ and zero acceleration $d^2r/dt^2$. The particles of a generic shell will arrive with zero velocity and zero acceleration at $r = 2\times$ (the relevant partial mass). Evidently, the collapse ends when the total mass of the sphere has filled up the spatial region $0 \leq r \leq 2m$.

**Conclusion.** – We see that, on account of Hilbert’s repulsive effect, the particles of the sphere cannot go beyond their “points of no return” $r = 2m$, $r = 2\times$ (the relevant partial mass) – and consequently the celebrated arguments, which we have mentioned in sect. 1, do not describe the real physical situation. In the final stage the collapsed body has a spatial extent, it has not been converted into a material point.

**APPENDIX**

*An instance of Hilbertian repulsion*

For reader’s convenience, we recall here the fundamental equations of the radial geodesics of Schwarzschild space-time, created by a point-mass $m$. Of course, Hilbert [3] starts from the $ds_{\text{ext}}$ of our eq. 5. He puts $\alpha \equiv 2m$. Consider Hilbert’s eqs. (53) and (54):

$$\frac{d^2r}{dt^2} - \frac{3\alpha}{2r(r-\alpha)}\left(\frac{dr}{dt}\right)^2 + \frac{\alpha(r-\alpha)}{2r^3} = 0;$$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{r-\alpha}{r}\right)^2 + A\left(\frac{r-\alpha}{r}\right)^3,$$

where the constant $A$ of this first integral is negative for the test-particles and zero for the light-rays.

Eq. (A1) tells us that the acceleration is negative or positive – i.e., that the gravitation acts in an attractive or in a repulsive way – according to the absolute value of the velocity. Thus, when
we have attraction; but when

we have repulsion.

Let us call $r^*$ the value of $r$ for which $\frac{d^2r}{dt^2} = 0$: attraction and repulsion counterbalance each other. At $r = r^*$ the velocity has its maximal value: $|\frac{dr}{dt}| = \frac{1}{\sqrt{3}} (r^* - \alpha)/r^*$.

For the light-rays ($A = 0$), eq. (A2) gives:

the light is repulsed everywhere; its velocity increases from zero at $r = \alpha$ to 1 at $r = \infty$.

We see from the above equations that test-particles and light-rays arrive at $r = \alpha$ with $\frac{dr}{dt} = 0$ and $\frac{d^2r}{dt^2} = 0$: the spatial surface $r = \alpha$ represents for them an insuperable barrier: a fact of paramount importance from the astrophysical standpoint.

It is remarkable that attraction and repulsion are linked in a physical way to the radial three-acceleration and the radial three-velocity. This means that the physical evolution parameter is the “Systemzeit” $t$ (von Laue) of Schwarzschild space-time, not the proper time of the test-particles, or the affine parameter of the light-rays. –

References

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[5] See, e.g., L.P. Eisenhart, Continuous Groups of Transformations (Dover Publs., New York) 1961, sect.58.
[6] L. Landau et E. Lifchitz, Théorie du Champ (Éditions Mir, Moscou) 1966, sect.97.
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