Stable, Spinning Embedded Vortices

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Abstract

The spinning vortex is a stationary generalization of the Nielsen-Olesen vortex involving a linear time dependence of the Goldstone boson. Here we show that this vortex can be embedded in models with $SU(2)_{\text{global}} \times U(1)_{\text{local}}$ symmetry. We also map the stability sector in parameter space and show that for sufficiently large spinning velocities the vortex is stable for any value of the Higgs self coupling parameter $\beta$. This is a significant improvement of stability compared to the semilocal vortex introduced by Vachaspati and Achucarro which is stable only for $\beta < 1$. This result may have significant implications for electroweak vortices.
1 Introduction

Topological defects form naturally in systems exhibiting spontaneous symmetry breaking. The universality of the concept of symmetry breaking makes topological defects fairly generic objects forming in both condensed matter and particle physics theories.

The Nielsen-Olesen vortex is a particularly interesting type of topological defect. It plays a crucial role in cosmology (gauged and global cosmic strings), in particle physics (electroweak strings, grand unification strings) and in condensed matter (superconductivity, superfluids, liquid crystals).

Topological Nielsen-Olesen vortices normally form in systems where the first homotopy group of the vacuum manifold is nontrivial. However, it was recently shown that this is a sufficient but not necessary condition for their formation. Vortices, dynamically stable to small perturbations may even form in the standard electroweak model for a certain range of parameters. Unfortunately, this range does not include the physically realized parameter values. Therefore, the following question arises: Can we find vortices that have better stability properties when embedded in theories with trivial \( \pi_1 \) (like the standard electroweak model)?

Previous attempts to address this question have shown that bound states may indeed improve the stability of embedded vortices but it was not clear if this improvement was enough to stabilize the electroweak vortex for the physically realized values of parameters. An alternative approach involves a generalization of the Nielsen-Olesen vortex and a subsequent embedding in extensions of the electroweak model (e.g. the two doublet model). This approach has led to an interesting new stable embedded vortex involving a combination of two Higgs doublets. The stability of this vortex is due to the presence of an extra global symmetry.

Here we follow a different approach. We consider a single scalar doublet and vortices with no bound states but with a linearly time dependent Goldstone boson (spinning vortices). In order to achieve this spinning state, the existence of a charged background is necessary. Such a background could be obtained for example by coupling to an external charged field with coherence length much larger than the width of the vortex, by a background of ions or by a superfluid phase transition in the early universe. Topologically stable spinning vortices have been shown to provide a description of...
the macroscopic properties of flux tubes in superconductors.\(^{18}\)

Embedded spinning vortices have some similarities to embedded Chern-Simons charged vortices\(^{21,22}\) but the fields of the former have different asymptotic behavior leading to significantly improved stability properties. Another interesting study of embedded charged vortices can be found in Ref.\(^{23}\) where a different ansatz from the one discussed here was used to show that a charged vortex can be stable in a theory with trivial \(\pi_1\).

Here, we show that spinning vortices, embedded in theories with trivial \(\pi_1\) have significantly better stability properties compared to the corresponding Nielsen-Olesen embedded vortices (semilocal strings\(^{10,24}\)).

\section{The Embedded Spinning Vortex}

Consider the \(SU(2)_{\text{global}} \times U(1)_{\text{local}}\) symmetric Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D_\mu \Phi|^2 - \frac{\lambda}{4} (|\Phi|^2 - \eta^2)^2 - A_\mu J^\mu \tag{1}
\]

where \(\Phi\) is a complex doublet scalar, \(D_\mu = \partial_\mu - ie A_\mu\), \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) and \(J_\mu = (\rho(\vec{x}), 0, 0, 0)\) represents a charged background density with \(\rho(\vec{x}) \to \rho_0\) as \(r \to \infty\).

Consider now the ansatz

\[
\Phi = \begin{pmatrix} 0 \\ f(r) e^{im\theta} e^{i\omega t} \end{pmatrix} \tag{2}
\]

\[
A_\theta = \frac{v(r)}{r}, \quad A_0 = \alpha(r) \tag{3}
\]

The difference between the embedded Nielsen-Olesen semilocal ansatz\(^{10}\) and the one of (2), (3) is the time dependent phase \(e^{i\omega t}\) which introduces a linear time dependence in the Goldstone boson. We have also allowed for the possibility of having \(A_0 \neq 0\).

Choosing the gauge

\[
\lim_{r \to \infty} A_0 = 0, \quad \partial_\mu A^\mu = 0 \tag{4}
\]

we obtain the equations of motion for \(f(r), v(r)\) and \(\alpha(r)\) as follows
\[ f'' + \frac{f'}{r} + (\omega_0 - \alpha)^2 f - \frac{(m - v)^2}{r^2} f - \beta (f^2 - 1) f = 0 \quad (5) \]

\[ v'' - \frac{v'}{r} + 2 f^2 (m - v) = 0 \quad (6) \]

\[ \alpha'' + \frac{\alpha'}{r} + 2 f^2 (\omega_0 - \alpha) = -2 \rho(r) \quad (7) \]

where ' denotes derivative with respect to the azimuthal radius \( r \), we have rescaled \( f \rightarrow \eta f, \ r \rightarrow \sqrt{2} r/(\eta e), \ v \rightarrow v/e, \ \omega_0 \rightarrow e\eta \omega_0/\sqrt{2}, \ \alpha \rightarrow \eta \alpha/\sqrt{2}, \ \rho(r) \rightarrow (e^2 \eta^3/\sqrt{2}) \rho(r) \) and defined \( \beta \equiv 2\lambda/e^2 \).

The boundary conditions to be imposed on \( f, v \) and \( \alpha \) in order to solve the system of (5), (6) and (7) are

\[ r \rightarrow 0 \Rightarrow f \rightarrow 0, \ v \rightarrow 0, \ \alpha' \rightarrow 0 \quad (8) \]

\[ r \rightarrow \infty \Rightarrow f' \rightarrow 0, \ v \rightarrow m, \ \alpha \rightarrow 0 \quad (9) \]

These boundary conditions automatically fix the value of \( \omega_0 \) given the value of the parameter \( \rho_0 \) (see equation (23) below).

The \( U(1) \) charge \( Q_v \) of the configuration (2), (3) may be obtained from the current \( J_{\nu} = \frac{1}{2} i ((D\nu \Phi)^\dagger \Phi - \Phi^\dagger D\nu \Phi) \). It is easily shown that

\[ Q_v = \int_V d^2 x J^0 = \int_V d^2 x f^2 (\omega_0 - \alpha) \simeq f_\infty^2 \omega_0 V \quad (10) \]

where \( f_\infty \) is the asymptotic value of \( f \) and \( V \) is a large cutoff volume in two dimensions expressing the range of the background. The angular momentum \( \vec{M}_v \) of the configuration may also be found as

\[ \vec{M}_v = \int_V d^2 x \vec{r} \times (\vec{E}_v \times \vec{B}) = -2 \int_V d^2 x f(r)^2 (\omega_0 - \alpha) v(r) \hat{e}_z \simeq -2 m Q_v \hat{e}_z \quad (11) \]

where \( E_v \) is the electric field induced by the charge of the vortex. The charge \( Q_{bg} \) and angular momentum \( \vec{M}_{bg} \) of the background charge density may also be obtained and shown to cancel the divergences of the corresponding vortex quantities

\[ Q_{bg} = \int_V d^2 x \rho(r) \simeq \rho_0 V \simeq -Q_v \quad (12) \]

\[ \vec{M}_{bg} = \int_V d^2 x \vec{r} \times (\vec{E}_{bg} \times \vec{B}) = -2 \int_V d^2 x \rho(r) v(r) \hat{e}_z \simeq -2 m Q_{bg} \hat{e}_z \quad (13) \]
where $E_{bg}$ is the electric field due to the background charge density $\rho(r)$. The total charge $Q_{\text{tot}} = Q_v + Q_{bg}$ and total angular momentum $\mathbf{M}_{\text{tot}} = \mathbf{M}_v + \mathbf{M}_{bg}$ are in general non-zero, finite and conserved.

The connection between background charge and vortex angular momentum shown in (10)-(13) implies that injecting charge into the vacuum in the presence of strings is equivalent to spinning up the vortices. This property has been used previously\cite{20} to construct a cosmological model where large scale coherent velocity fields are generated \textit{non-gravitationally} by global topological vortices which spin up due to the injection of charge provided by a temporary late time superfluid phase.

### 3 Stability

Consider now the following small perturbation to the ansatz (2), (3)

$$\delta \Phi = \begin{pmatrix} g(r) e^{i \theta} \\ \delta f(r, \theta) \end{pmatrix} \quad (14)$$

$$\delta A_\mu(r, \theta) \quad (15)$$

The energy of the perturbed configuration decouples as follows

$$E = E_0(f, v, \alpha) + \delta E(\delta f, \delta A_\mu) + E_1(g) \quad (16)$$

where

$$E_0 = \int_V d^2x \left[ f'^2 + \alpha'^2 + \frac{v^2}{2 r^2} + (\omega_0 - \alpha)^2 f^2 + \frac{(m - v)^2}{r^2} f^2 + \frac{\beta}{2} (f^2 - 1)^2 + \rho \alpha \right] \quad (17)$$

is the energy of the unperturbed solution which is identical to the energy of the \textit{topologically stable} charged spinning vortex obtained by substituting the complex doublet in the ansatz (2) by a complex singlet. The term $\delta E(\delta f, \delta A_\mu)$ is identical to the one obtained by perturbing the topologically stable charged spinning vortex and therefore it is non-negative. Thus, the term that determines the stability of the embedded spinning vortex is $E_1(g)$ which to second order in $g$ is

$$E_1 = \int_V d^2x \left[ g'^2 + \alpha^2 g^2 + \frac{(n - v)^2}{r^2} g^2 + \beta(f^2 - 1)g^2 \right] \quad (18)$$
Stability is obtained for that range of parameters \((\rho_0, \beta)\) for which \(E_1(g)\) is positive definite. Stability also depends on the detailed functional form of \(\rho(r)\) around the core of the vortex which in turn can be obtained by specifying the dynamics of the background charge density. In order to keep our analysis simple and general we will not assign any special dynamics to \(\rho(r)\). Instead we will study two opposite extreme forms of it. Most dynamically obtained forms of \(\rho(r)\) are expected to be between these two extreme cases.

The first extreme is that of a soft background which completely adjusts itself to the shape of the core thus neutralizing the charge density induced by the spin of the vortex. For such a background

\[
\omega_0 f^2(r) = -\rho(r), \quad \alpha = 0 \tag{19}
\]

The second extreme is that of a hard background which is not affected by the presence of the vortex. In this case

\[
\rho(r) = \rho_0, \quad \alpha \neq 0 \tag{20}
\]
everywhere within the range of the background.

We first study the stability in a soft background. In this case \(E_1\) is non-negative \textit{iff} the following eigenvalue equation has only non-negative eigenvalues \(\omega^2\)

\[
-g'' - g' \frac{g'}{r} + \frac{(v-n)^2}{r^2} g + \beta(f^2 - 1)g = \omega^2 g \tag{21}
\]

In what follows we consider \(n = 0\) in order to examine eigenstates of lowest energy. Notice that both parameters \(\rho_0\) and \(\beta\) enter in (21) since the unperturbed fields \(f\), and \(v\) depend on \(\rho_0\). In particular \(\rho_0\) determines the spinning velocity \(\omega_0\) which in turn specifies the asymptotic behavior of \(f\) as

\[
\lim_{r \to \infty} f^2 = (1 + \frac{\omega_0^2}{\beta}) \equiv \gamma^2 \tag{22}
\]

where from (19) \(\omega_0\) is the solution of

\[
\gamma^2 \omega_0 = -\rho_0 \tag{23}
\]

We have solved the system of (5), (6) ((7) is trivial for a soft background) with boundary conditions (8), (9), using a relaxation scheme based on Gaussian elimination\[25\]. Fig. 1 shows the potential of the Schroedinger-like
eigenvalue equation (21) (dotted line) with $\beta = 15$, $\rho_0 = 2$, compared with the corresponding potential (continuous line) obtained in the case of an embedded non-spinning vortex (semilocal string, $\beta = 15$, $\rho_0 = 0$). Clearly the potential in the case of the spinning vortex is more positive and thus less receptive to negative eigenvalues.

The improvement of stability may also be seen by a rescaling $r \to r/\gamma$, $f \to \gamma f$ which transforms (5) and (6) (with $\alpha = 0$) to exactly the Nielsen-Olesen equations. The potential of the Schroedinger-like equation (21) becomes

$$V(r) = \frac{v^2}{r^2} + \beta(f^2 - 1/\gamma^2) \quad (24)$$

The case of the non-spinning (semilocal) vortex is obtained in the limit $\gamma \to 1$ (i.e. $\omega_0 \to 0$). In this limit the stability problem has been solved by Hindmarsh[26] (see also [27, 28]) who found that there are no negative eigenvalues for $0 \leq \beta \leq 1$ (implying stability) while for $\beta > 1$ there are negative eigenvalues and the embedded vortex is unstable. This instability is the main reason that the embedded electroweak vortex is unstable for the physically realized values of parameters of the standard electroweak model[12].

In the case of the spinning vortex we have $\omega_0 \neq 0$ or $\gamma^2 > 1$ and the potential (24) is less receptive to negative eigenvalues. In fact, as $|\omega_0| \to \infty$ (i.e. $|\rho_0| \to \infty$), $V(r)$ is positive definite and the vortex is stable for all values of $\beta$.

The obtained solution for $f$, $v$ with winding number unity$^3$ ($m = 1$) was used to solve the eigenvalue equation (21). A fourth order Runge-Kutta algorithm[25] was implemented for the solution.

The points on the continuous line in Fig. 2 represent critical values of $\rho_0$ (for fixed $\beta$) such that for $\rho_0 > (\rho_0)_{crit}$ there are no negative eigenvalues, while for $\rho_0 < (\rho_0)_{crit}$ we found negative eigenvalues. Thus sector I in Fig. 2 is the sector of stability in a soft background while for parameter values in sectors II and III the embedded spinning vortex is unstable. In the limit $\rho_0 \to 0$ we recover the semilocal vortex case (stability for $\beta < 1$, instability for $\beta > 1$) as expected. Also, sector I is consistent with the expected stability for all positive values of $\beta$ in the limit $\rho_0 \to \infty$.

We now consider the opposite extreme background form: the hard back-
ground. In this case \( \alpha \neq 0 \) and the eigenvalue equation (21) becomes

\[
- g'' - \frac{g'}{r} + \frac{(v - n)^2}{r^2} g + \beta (f^2 - 1) g + \alpha^2 g = \omega^2 g
\]

The potential of (25) has an additional positive contribution by the term \( \alpha^2 \) which may lead to the immediate conclusion that stability is further enhanced in this case. This additional positive definite contribution is obtained for any form of background density \( \rho(r) \) which is different from the soft background form defined in equation (19). Any such background will excite the \( A_0 \) component of the gauge field and will in general further improve the stability of the embedded spinning vortex. In order to test this conjecture we have explicitly solved the eigenvalue equation (25) for the opposite extreme case of the hard background defined in equation (20). Using a similar method as in the case of the soft background we have solved the unperturbed system (5), (6), (7) and used the solution to study (25). The stability sector is shown in Fig. 2 as the union of sectors I and II. In the parameter sector III we found negative eigenvalues which imply that the embedded spinning vortex with a hard background is unstable for parameters in this sector. It is clear from Fig. 2 that as expected, a hard background improves further the stability of the spinning vortex. This may also be seen in Fig. 1 where we plot the potentials for the eigenvalue equation (25) in the cases of semilocal (continuous line), spinning in soft background (dotted line) and spinning in hard background (dashed line) vortices.

Clearly, the stability has improved dramatically compared to the non-spinning embedded vortex (semilocal string) in both types of background considered. Since the background types are opposite extremes of each other we are led to the conclusion that a background charge density inducing spin is a generically effective way to stabilize the embedded vortex.

A simplified way to understand this enhancement of stability for the spinning vortex is that the centrifugal force introduced by the rotation of the Goldstone boson tends to prevent the tilt of the field \( \Phi \) towards the upper component thus stabilizing the vortex.

4 Conclusion

In conclusion we have shown that it is always possible to stabilize a vortex embedded in an \( SU(2)_{\text{global}} \times U(1)_{\text{local}} \) theory by introducing the appropriate
amount of angular momentum in the Goldstone boson. This angular momentum may be introduced by a charged background density $\rho(r)$ which is also needed for long range neutrality. The role of $\rho(r)$ can be played either by ions (in condensed matter systems) or by a coupling to a dynamical charged field with coherence length much larger than the width of the vortex.

Some exciting questions remain to be addressed:

- What are the stability properties of the spinning vortex when embedded in the standard electroweak model?

- Can embedded global vortices (i.e. vortices involving no gauge fields) be stabilized by the introduction of spin through a charged background?

Stable, spinning electroweak vortices could be effective in a non-gravitational generation of large scale coherent velocities provided that a background charge density is generated by a late superfluid phase transition. Such a prospect would give more realistic dimensions to the mechanism proposed in Ref. [20].

The second question is also particularly interesting in view of the fact that non-spinning embedded global vortices are always unstable to small perturbations [29]. Global spinning vortices are important in correctly modeling many features of condensed matter systems [7, 8] (e.g. superfluids). These issues are currently under investigation.

Acknowledgments
I would like to thank Robert Brandenberger, Manuel Barriola and Tanmay Vachaspati for useful comments on the manuscript. This work was supported by a CfA Postdoctoral Fellowship.

Figure Captions

Figure 1: The potential of the Schroedinger-like stability equation for semilocal, spinning in soft background and spinning in hard background vortices. Notice that the potential favors negative eigenvalues less, when spin is introduced.

Figure 2: The stability map in parameter space. The spinning vortex in a soft background is stable for points in sector I but unstable for points in sectors II and III. In the case of a hard background stability occurs in both
sectors I and II while instability occurs in sector III. For $\rho_0 = 0$ we recover the semilocal string. $\rho_0$ is in units of $e^2 \eta^3 / \sqrt{2}$.

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