THE MOVABLE FAN OF THE HORROCKS–MUMFORD QUINTIC

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ABSTRACT. This paper explores the birational geometry of a general Horrocks–Mumford quintic threefold, describing the set of all minimal models up to marked isomorphism, the movable fan (the way in which the nef cones of all these models are arranged to form the movable cone), and the birational automorphism group. There are infinitely many models up to marked isomorphism, falling into just eight (unmarked-) isomorphism classes.

The Horrocks–Mumford bundle is a rank-2 vector bundle over $\mathbb{P}^4$, characterized by the fact that a general section vanishes along an Abelian surface. A general 2-dimensional subspace of its sections defines a quintic threefold with 100 nodes. Such threefolds (called Horrocks–Mumford quintics) have been studied by Aure [1] and by Borcea [3].

Borcea described the 14 projective small resolutions of a general Horrocks–Mumford quintic, which are all smooth Calabi–Yau threefolds, and also described an involution of the moduli space of general Horrocks–Mumford quintics which gives, for each such quintic, a birationally equivalent quintic. Up to isomorphism, there are just eight different threefolds among the resolutions of these two quintics, which lie in three different deformation families.

Here we describe the set of all minimal models of a general Horrocks–Mumford quintic up to marked isomorphism, the movable fan, and the action of the birational automorphism group of the quintic on this fan. It turns out that every one of these minimal models is isomorphic to one of the eight described by Borcea, although there are infinitely many up to marked isomorphism. We check that Morrison’s Movable Cone Conjecture [8] holds for these varieties. The movable cone has a surprising form; its boundary is once but not twice differentiable. In Section 2 we describe the simpler movable cone of determinantal quintics, of which Horrocks–Mumford quintics are a specialisation.

This paper gives an example of a method which translates the algebraic geometry of a variety into the real geometry and combinatorics of the cones, and then continues the analysis purely in those terms, finally recovering algebraic-geometric information from the description obtained.

Terminology: the movable fan. A convex cone in a real vector space is a convex subset closed under multiplication by positive real scalars.

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For a normal complex projective variety $X$, we say that two divisors $D_1$ and $D_2$ on $X$ are \(\mathbb{Q}\)-algebraically equivalent if some non-zero multiple of \((D_1 - D_2)\) is algebraically equivalent to 0. For Cartier divisors, this is the same as numerical equivalence. We define

\[
\tilde{N}^1(X) = \{ \text{divisors on } X, \text{modulo } \mathbb{Q}\text{-algebraic equivalence} \} \otimes \mathbb{R},
\]

a finite-dimensional real vector space, and write \(N^1(X)\) for the subspace generated by Cartier divisors. Intersection with a curve in $X$ gives a linear form on \(N^1(X)\), but this does not extend naturally to a linear form on \(\tilde{N}^1(X)\).

The nef cone of $X$, written \(\text{Amp}(X)\), is the closed convex cone in \(N^1(X)\) consisting of all Cartier \(\mathbb{R}\)-divisors that have non-negative intersection with every curve in $X$. It spans the space \(N^1(X)\), and its relative interior is the ample cone, \(\text{Amp}(X)\), the convex cone generated by ample divisors.

A divisor on $X$ is said to be movable if it is effective and moves in a linear system whose base locus is of codimension at least 2. The movable cone of $X$, written \(\text{Mov}_s(X)\), is the convex hull of the set of movable divisors on $X$; it is also a convex cone, which spans \(\tilde{N}^1(X)\), and its interior, written \(\text{Mov}(X)\), contains \(\text{Amp}(X)\).

If $Y$ is another normal projective variety, a birational equivalence between $X$ and $Y$ is called an isomorphism in codimension 1 if it gives an isomorphism between open subsets of $X$ and $Y$ whose complements are of codimension at least 2. Such a map gives an isomorphism of the divisor spaces \(\tilde{N}^1(X)\) and \(\tilde{N}^1(Y)\). Under this isomorphism, \(\text{Mov}_s(Y)\) is carried to \(\text{Mov}_s(X)\), but \(\text{Amp}(X)\) and \(\text{Amp}(Y)\) coincide only if the birational equivalence is a biregular isomorphism, and are otherwise disjoint. For Calabi–Yau threefolds, it is known (from the existence and termination of flops: see [7]) that the ample cones of all codimension 1 isomorphic models of $X$ form a locally finite disjoint cover of \(\text{Mov}(X)\). This arrangement of ample cones of various dimensions covering the single movable cone is called the movable fan.

1. Horrocks–Mumford Quintics

In this section we review the definitions and basic properties of the Horrocks–Mumford quintics. See [3], [4], and [1] for more details of these constructions.

We recall [5] that the Horrocks–Mumford bundle, \(\mathcal{F}\), is a rank-2 vector bundle on \(\mathbb{P}^4\), of degree 5. A generic section of \(\mathcal{F}\) vanishes on an Abelian surface, and \(\Gamma(\mathcal{F})\) is 4-dimensional.

If $s$ is a non-zero section of \(\mathcal{F}\), and $p = [s]$ the corresponding point in \(\mathbb{P}(\Gamma(\mathcal{F}))\), we will write $A_p$ for the zero locus of $s$.

**Definition.** Let $s_1$ and $s_2$ be two linearly independent sections of $\mathcal{F}$. Then $s_1 \wedge s_2$ is a non-zero section of $\wedge^2 \mathcal{F} \cong O(5)$, and its zero-locus is a quintic threefold $V$; we call such a threefold a Horrocks–Mumford quintic, or an HM quintic.

The threefold $V$ is determined by the line $L$ through the two points $[s_1]$ and $[s_2]$ in \(\mathbb{P}(\Gamma(\mathcal{F}))\), and is covered by the pencil of surfaces \(\{ A_p \mid p \in L \}\), giving a rational map $V \twoheadrightarrow L$. We can resolve the indeterminacy of this rational map by taking

\[
\tilde{V} = \{ (x, p) \in V \times L \mid x \in A_p \}. \tag{1}
\]
In [5, §5], it is shown that the fivefold
\[ Z := \{ (x, p) \in \mathbb{P}^4 \times \mathbb{P}(\Gamma(F)) \mid x \in A_p \} \]
is non-singular. If \( P_1 \) and \( P_2 \) are two planes in \( \mathbb{P}(\Gamma(F)) \) such that \( L = P_1 \cap P_2 \), then \( \tilde{V} = Z \cap (\mathbb{P}^4 \times P_1) \cap (\mathbb{P}^4 \times P_2) \), and by Bertini’s theorem, this is non-singular for generic \( P_1 \) and \( P_2 \), so \( \tilde{V} \) is non-singular for generic \( L \).

If \( x \in V \), then the projection \( \tilde{V} \to V \) is an isomorphism at \( x \) unless both \( s_1 \) and \( s_2 \) vanish at \( x \), in which case \( \{x\} \times L \subset \tilde{V} \) is a rational curve contracted by the projection. Thus the singular locus of \( V \) is \( A[s_1] \cap A[s_2] \), which for generic \( s_1 \) and \( s_2 \) consists of 100 nodes.

**Definition.** Let \( \epsilon = e^{2\pi i/5} \), and define three elements of \( \text{SL}_5 \mathbb{C} \) by
\[
(\sigma x)_i = x_{i+1}, \quad (\tau x)_i = \epsilon^i x_i, \quad (\iota x)_i = x_{-i}.
\]
The Heisenberg group \( H_5 \) is the group of order 125 generated by \( \sigma \) and \( \tau \).

The centre of \( H_5 \) is \( \mu_5 \), the group of 5th roots of 1, and \( H_5/\mu_5 \cong \mathbb{Z}_5^2 \). Recall from [5] that the action of \( \langle H_5, \iota \rangle \) on \( \mathbb{P}^4 \) preserves each surface \( A_p \), and so preserves \( V \) and can be lifted to an action on \( \tilde{V} \). Since \( \mu_5 \) acts trivially on \( \mathbb{P}^4 \), the 100 nodes of \( V \) form \( 4 \cdot H_5 \)-orbits of 25 points, which pair up into 2 \( \langle H_5, \iota \rangle \)-orbits of 50 points.

## 2. Determinantal Quintics

This section considers determinantal quintics, of which Horrocks–Mumford quintics are a special case.

Let \((a_{ijk})\) be complex numbers, for \( i, j, k \in \mathbb{Z}_5 \), and consider the matrix of linear forms
\[
M_{ij}(x) = \sum_k a_{ijk} x_k.
\]
If \( \det M(x) \) does not vanish identically, the equation \( \det M(x) = 0 \) defines a quintic threefold \( W \) in \( \mathbb{P}^4 \). Say that \( M \) is nice if \( W \) is regular in codimension 1, \( \text{rank } M(x) \geq 3 \) everywhere, and \( \text{rank } M(x) = 4 \) at all but finitely many points of \( W \).

Let
\[
X = \{ (x, y) \in \mathbb{P}^4 \times \mathbb{P}^4 \mid \forall i, \sum_{j,k} a_{ijk} x_k y_j = 0 \}.
\]
The projection \( \pi_1 \) takes \( X \) to \( W \), with \( \pi_1^{-1}\{x\} = \{x\} \times \mathbb{P}(\ker M(x)) \), so if \( M \) is nice, \( \pi_1 \) is a birational morphism, contracting finitely many rational curves on \( X \).

We can construct two more matrices of linear forms from \((a_{ijk})\):
\[
M'_{ij}(x) = \sum_k a_{jki} x_k, \quad M''_{ij}(x) = \sum_k a_{kij} x_k.
\]
and the corresponding threefolds $W'$, $X'$, $W''$, and $X''$. Note that $W'$ is the image of $X$ under the other projection $\pi_2$; if $M'$ and $M''$ are also nice, we get a sequence of birational maps, all isomorphisms in codimension 1:

$$
\begin{array}{cccccccc}
\ldots & \pi_1 & X & \pi_2 & X' & \pi_1 & X'' & \pi_2 & \ldots \\
W & \xrightarrow{\phi} & W' & \xrightarrow{\phi'} & W'' & \xrightarrow{\phi''} & W \\
\end{array}
$$

(2)

Let $H$, $H'$, and $H''$ be hyperplane sections of $W$, $W'$, and $W''$ respectively. We can define an infinite sequence of divisors on $W$:

$$
\begin{align*}
D_{-2} &= \phi''\phi'H' \\
D_{-1} &= \phi''H'' \\
D_0 &= H \\
D_1 &= \phi^*H' \\
D_2 &= \phi^*\phi'^*H'' \\
\vdots
\end{align*}
$$

**Lemma 1.** For any $i \in \mathbb{Z}$, the divisor $D_{i-1} + D_{i+1}$ is linearly equivalent to $4D_i$. All the divisors $\pi^*_i D_i$ on $X$ are Cartier. The cone in $N^1(X)$ generated by $\pi^*_i D_i$ and $\pi^*_i D_{i+1}$ is the nef cone of a marked model of $X$, isomorphic to $X$, $X'$, or $X''$ (according to the value of $i$ modulo 3).

**Proof.** (Here $\sim$ denotes linear equivalence.) In [3, p. 24], it is shown that $D_{-1} + D_1$ (there called $\Delta + \Delta'$) is the intersection of $W$ with a quartic. Therefore $D_{-1} + D_1 \sim 4H = 4D_0$. Applying the same result at other points in the sequence (2) gives $D_{i-1} + D_{i+1} \sim 4D_i$ for all $i$.

Since $\pi^*_i D_0 = \pi^*_i H$ and $\pi^*_i D_1 = \pi^*_2 H'$ are Cartier, this equation then implies that $\pi^*_i D_i$ is Cartier for all $i$.

$\pi^*_i H$ and $\pi^*_2 H'$ generate the nef cone of $X$; again, we can translate this along the sequence (2) to give the last statement of the theorem. \hfill \Box

Solving the difference equation $D_{i-1} - 4D_i + D_{i+1} = 0$, we get

$$
D_i = \frac{1}{2}(D_0 + \frac{1}{\sqrt{3}}(D_1 - 2D_0))(2 + \sqrt{3})^i + \frac{1}{2}(D_0 - \frac{1}{\sqrt{3}}(D_1 - 2D_0))(2 - \sqrt{3})^i,
$$

so the limits of the rays generated by $D_i$ as $i \to \pm \infty$ are generated by $D_0 \pm \frac{1}{\sqrt{3}}(D_1 - 2D_0)$; these rays then constitute the boundary of $\overline{\text{Mov}}(X) \cap N^1(X)$. (For generic $M$, $X$ is smooth, so $N^1(X) = \overline{N^1(X)}$ and these rays are the boundary of $\overline{\text{Mov}}(X)$.)

Now let $y \in \mathbb{C}^5$, and consider the matrix $M_y$, where

$$(M_y)_{i,j}(x) = x_{i+j}y_{i-j}, \quad i, j \in \mathbb{Z},$$
and the corresponding varieties and maps

\[ \cdots W_y \xrightarrow{\phi_y} W_y' \xrightarrow{\phi_y'} W_y'' \xrightarrow{\phi_y''} W_y \cdots. \]  

(3)

Because \( M_\mu = M_{\iota y} \), this sequence can also be written as

\[ \cdots W_y \xrightarrow{\phi_y'} W_y'' \xrightarrow{\phi_y''} W_y \cdots. \]  

(4)

From [3, attributed there to R. Moore] we see that, if \( V \) is a sufficiently general HM quintic, and \( y \) is any node of \( V \), then \( V = W_y \); also that there is an automorphism of the space of HM quintics, \( \beta \), such that \( \beta V = W_y' \).

Therefore, (3) and (4) can also be written as

\[ \cdots V \xrightarrow{\phi'_y} \beta V \xrightarrow{\phi''_y} V \cdots. \]  

(5)

3. THE SMALL RESOLUTIONS OF \( V \)

From now on all threefolds we consider will be isomorphic in codimension 1 to a fixed sufficiently general HM quintic \( V \). We can therefore identify corresponding divisors in any two of these threefolds. Any such divisor determines a point in the real vector space \( N = N^1(V) = N^1(\tilde{V}) \).

Choose representatives \( y_1, \ldots, y_4 \) (respectively, \( z_1, \ldots, z_4 \)) of the four \( H_5 \)-orbits of nodes of \( V \) (respectively, of \( \beta^{-1}V \)), such that \( \iota y_1 = y_3, \iota y_2 = y_4, \iota z_1 = z_3, \) and \( \iota z_2 = z_4 \). Then \( V \) fits into six different sequences of the form (5):

\[ \cdots \beta V \xrightarrow{\phi''_{y_i}} V \xrightarrow{\phi'_{y_i}} \beta V \xrightarrow{\phi''_{y_i}} V \cdots, \quad i = 1, 2; \]

\[ \cdots \beta^{-1} V \xrightarrow{\phi'_{z_i}} V \xrightarrow{\phi''_{z_i}} V \cdots, \quad i = 1, \ldots, 4. \]

This gives us 12 divisors on \( V \):

\( \Delta_i \), the strict transform under \( \phi_{y_i} \) of a hyperplane section of \( \beta V \).

\( \nabla_{\iota} \), the strict transform under \( \phi''_{z_i} \) of a hyperplane section of \( \beta^{-1}V \).

\( \nabla'_{\iota} \), the strict transform under \( \phi_{z_i} \) of a hyperplane section of \( V \).

We also have \( H \), a hyperplane section of \( V \), and \( A \), one of the pencil of Abelian surfaces on \( V \). By Lemma 3 we have \( \Delta_i + \iota \Delta_i \sim \nabla_{\iota} + \nabla'_{\iota} \sim 4H \).

We write \( \Lambda_{\iota} \) for the rational curve on \( \tilde{V} \) that is contracted to the node \( y_{\iota} \) on \( V \). Noting from [2] that a singular member of the pencil of surfaces \( \{A_p\} \) is an elliptic scroll, we let \( \Gamma \) be a line in one of these scrolls.
The following intersection numbers on $\tilde{V}$ are computed in [3] ($\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are there called $L_1, L_3, L_2, L_4$—note the different order):

\[
\begin{array}{cccccccccccc}
H & A & \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \nabla_1 & \nabla_2 & \nabla_3 & \nabla_4 & \nabla_1' & \nabla_2' & \nabla_3' & \nabla_4' \\
\Gamma & 1 & 0 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\
\Lambda_1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \\
\Lambda_2 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
\Lambda_3 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
\Lambda_4 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]

(6)

Corollary 2. \{H, A, \Delta_1, \Delta_2\} is a basis for $N$; the group $H_5$ acts trivially on $N$; in particular, the classes of $\Delta_i$ and $\Lambda_i$ do not depend on the choice of $y_i$ within its $H_5$-orbit. The class of $\Gamma$ does not depend on which of the singular fibres it is chosen from.

Proof. In [1, Proposition 2.2] it is proved that $\dim N = 4$. The multiplication table above shows that $H, A, \Delta_1, \Delta_2$ are linearly independent, so they form a basis. All are fixed by $H_5$, so $H_5$ acts trivially. The intersections of $\Gamma$ with these basis elements do not depend on the singular fibre used, so the numerical class of $\Gamma$ does not.

Convention. Following [3], we will use the basis

\[
\{A, 2H - \Delta_1, 2H - \Delta_2, H\}
\]

for $N$. The dual basis for $N^\vee$ is \{\(\frac{1}{4}(\Lambda_1 + \Lambda_3), \frac{1}{2}(\Lambda_1 - \Lambda_3), \frac{1}{2}(\Lambda_2 - \Lambda_4), \Gamma\}\}. The components of a divisor $D$ will be written $(d_1, d_2, d_3, d_4)$.

Corollary 3. The birational automorphisms of $V$ that act trivially on $N$ are exactly the elements of $H_5/\mu_5$.

Proof. Any such automorphism fixes $H$, and so comes from a projective automorphism of $V \subset \mathbb{P}^4$. These are exactly $\langle H_5, \iota \rangle/\mu_5$, and $\iota$ does not act trivially.

We can now draw a picture of a neighbourhood of the ray generated by $H$ in $N$ (Figure 1), showing the structure of the movable fan there: taking the affine slice of the fan given by $(\Gamma + \Lambda_1 + \Lambda_3) \cdot D = 2$, we find that the four planes given by $\Lambda_i$ divide the neighbourhood up in such a way that the vertex figure is a cuboctahedron (as do any four planes in general position through a point). The 12 vertices of this cuboctahedron are on the lines joining $2H$ to the 12 divisors constructed out of the determinantal description ($\Delta$s and $\nabla$s) (see Figure 2).

Let $\nabla$ be any of these 12 divisors. Coming as it does from pulling back an ample divisor on a model of $V$ with Picard number 1, it generates a 1-dimensional cone—a ray—in the movable fan. [2] in section 3 shows that there is a marked model, not necessarily minimal, of $V$, that is simultaneously a partial small resolution of the two marked models corresponding to $H$ and $\nabla$, so there is a cone in the fan whose closure contains both rays.

Therefore the 12 rays generated by the $\Delta$s and $\nabla$s are neighbours in the fan to the ray generated by $H$, and the cuboctahedral arrangement extends as far as these neighbouring rays (Figure 3). Since the marked models corresponding to these rays are all isomorphic, ignoring the marking, either to $V$ or to $\beta V$, there will be a similar cuboctahedral figure.
about each of them, and so about every ray of the fan that can be reached from $H$ along 2-dimensional cones of the fan.

In the next two sections we will describe the nef cones of the resolutions of $V$—that is, the 4-dimensional cones of the fan whose closure contains $H$. This will lead to a complete description of the movable fan. These nef cones are the duals to the cones of curves described in [3], but our notation and point of view are different, and we need several details not found there.

Whenever we refer to the cuboctahedral neighbourhood of a ray in the fan, we will use terms such as top and left, assuming that the cuboctahedron is oriented as in Figures 1 and 2.

4. The Nef Cone of $\tilde{V}$.

The nef cone of $\tilde{V}$ lies in the sector of the cuboctahedral figure given by the conditions $\Lambda_1 \cdot D \geq 0$; it contains the square face of the cuboctahedron spanned by $\nabla_1', \nabla_2', \nabla_3'$, and $\nabla_4'$—that is, the bottom square face in the diagram.

In [3, p. 31] it is shown that $\phi'_z$ (there called $\phi_{\nabla'}$) lifts to a biregular automorphism of $\tilde{V}$, acting by translation on each Abelian surface $A_p$, and that $\phi'_{z_1}$ and $\phi'_{z_2}$ generate a free
abelian group of rank 2; in terms of the basis (7), this group acts as the group of matrices
\[
\begin{pmatrix}
1 & 2x & 2y & x^2 + y^2 \\
0 & 1 & 0 & x \\
0 & 0 & 1 & y \\
0 & 0 & 0 & 1
\end{pmatrix} \left| 2x, 2y, x - y \in \mathbb{Z} \right.
\]

Now we have enough information to describe the nef cone of $\tilde{V}$.

**Theorem 4.** (i) $\overline{\text{Ampl}(\tilde{V})}$ is generated by $A$ and all classes of the form $(x^2 + y^2, x, y, 1)$ with $2x, 2y,$ and $x - y$ all integers. In the affine slice given by $d_4 = 1$, this is a regular square tiling of the paraboloid $d_1 - d_2^2 - d_3^2 = 0$ (see Figure 3).

(ii) The automorphism group of $\tilde{V}$ acts transitively on the vertices and on the faces of this tiling.

(iii) $(\text{Aut} \tilde{V})/(H_5/\mu_5)$ is isomorphic to the group of all translation and order-2 rotation symmetries of this tiling.
Proof. (i) The set of vertices given is the orbit of $H$ under the group generated by $\phi_{z_1}$ and $\phi_{z_2}$. We described one of the faces (that given by $\Lambda_1$) above; taking this information around by the action of the group gives the whole tiling. This tells us everything about the cone within the convex hull of the tiling, and on the other hand the boundary of the tiling is just the ray generated by $A$, which certainly is nef, so $\overline{\text{Amp}}(\tilde{V})$ is as claimed.

(ii) follows from the description of the cone in (i)—the translations $\phi_{z_i}$ are enough to give transitivity of the action on faces and on vertices.

(iii) Any automorphism $\eta$ of $\tilde{V}$ must preserve the pencil of Abelian surfaces, since it must, from the shape of the cone, preserve the class $A$. It must preserve the individual surfaces in that pencil, since the general member of the pencil is not isomorphic to any other member. Since the general surface in the pencil has no automorphisms other than those all Abelian surfaces have—translation and reflection about a point—$\eta$ must act in one of these two ways. If the former, it commutes with other translations, in particular $\psi_{z_1}$ and $\psi_{z_2}$, and so acts on the tiling as a translation. If the latter, $\eta \iota$ is a translation. This shows that $(\text{Aut} \tilde{V})/\{\text{automorphisms acting trivially on } N \}$ is generated by translations and $\iota$. But by Corollary 3, the automorphisms that act trivially on $N$ are exactly $H_5$.

Corollary 5. The cubic intersection form on $\tilde{V}$ can be written as

$$D^3 = 5 \Gamma \cdot D ( (\Gamma \cdot D)^2 + 3Q(D, D)), \quad (9)$$

where $Q(D, E) = d_1e_4 + d_4e_1 - 2d_2e_2 - 2d_3e_3$ in the basis $[7]$. 

Proof. The cubic form must be preserved by all automorphisms, and Γ and Q generate the ring of polynomial forms preserved by the automorphism group, so \( D^3 \) must be a linear combination of \( \Gamma^3 \) and \( \Gamma Q \). Suppose \( D^3 = a\Gamma^3 + b\Gamma Q \).

Then we can work out the values of \( \Gamma \) and \( Q \) on \( H \) and \( A \), and

\[
5 = H^3 = a(\Gamma \cdot H)^3 + b(\Gamma \cdot H)Q(H, H) = a \cdot 1^3 + b \cdot 1 \cdot 0 = a,
\]

\[
10 = H^2 \cdot A = a(\Gamma \cdot H)(\Gamma \cdot A)^3 + \frac{2}{3} b(\Gamma \cdot H)Q(A, H) + \frac{1}{3} b(\Gamma \cdot A)Q(H, H) = a \cdot 1^2 \cdot 0 + \frac{2}{3} b \cdot 1 \cdot 1 + \frac{1}{3} b \cdot 0 \cdot 1 = \frac{2}{3} b,
\]

give \( a = 5, b = 15 \). □

5. Second and Third Resolutions.

Since \( \text{Aut} \ ˜V \) acts transitively on the faces of \( \overline{\text{Amp}}( ˜V) \), all models of \( V \) that can be obtained by making one flop from \( ˜V \) are isomorphic. So let us consider the variety \( ˜V_1 \) obtained by flopping in the \( \Lambda_1 \) face, so that \(-\Lambda_1 \) is effective on \( ˜V_1 \).

The nef cone can be found easily in this case:

**Theorem 6.** An affine slice of \( \overline{\text{Amp}}( ˜V_1) \) is the span of the five divisors

\[
H, \ \nabla_1^{'}, \ \nabla_2^{'}, \ \ H^{'}, \ \ \Delta_1,
\]

where \( H' = \nabla_1^{} + \nabla_2^{} - H \). It is a square pyramid, the base being the face at which it meets \( \overline{\text{Amp}}( ˜V) \) (see Figure 4). \( (\text{Aut} \ ˜V_1)/(H_5/\mu_5) \cong \mathbb{Z}_2 \). acting by a rotation of order 2 on the nef cone.

![Figure 4](image-url)

**Figure 4.** An affine slice of \( \overline{\text{Amp}}( ˜V_1) \).

Proof. We are now considering the lower triangular faces of the cuboctahedron of Figure 3. Taking any one of the edges of the flopping face, the description of the movable fan inside the cuboctahedron centred on that edge tells us that half the boundary of the pyramid...
coincides with part of the boundary of $\text{Imp}(\tilde{V}_1)$; these four descriptions put together give the whole of the boundary of the pyramid, so the nef cone must be the pyramid described.

The automorphism group must fix the only square face of the pyramid, so $\text{Aut} \tilde{V}_1$ consists exactly of those elements of $\text{Aut} \tilde{V}$ that fix the flopping face.

The apex of the pyramid, $\Delta_1$, gives a contraction to $\beta V$, and also has a cuboctohedral neighbourhood. We can see that the cone just described must fit into the top square face of this cuboctohedron, since the bottom face is occupied by a model $\cong \beta \tilde{V}$, and the top face is the only other face fixed by an automorphism of the movable fan of order 2.

Although there are two $\text{Aut} \tilde{V}_1$-orbits of triangular faces of this pyramid, the whole situation is symmetric (the two are swapped by changing our arbitrary choice of $y_1$ and $y_2$ at the beginning of section 3), so we only need to analyse one of the flops. Let us consider $\tilde{V}_2$, obtained by flopping $\tilde{V}$ in the planes defined by $\Lambda_1$ and $\Lambda_2$.

**Theorem 7.** An affine slice of $\text{Imp}(\tilde{V}_2)$ is the span of the eight divisors

$$H, \quad \Delta_1, \quad \Delta_4, \quad \nabla_1', \quad \nabla_2', \quad \nabla_3,$$

$$(\nabla_1' + \nabla_3 - H), \quad (\Delta_1 + \nabla_3 - H), \quad (\Delta_4 + \nabla_3 - H).$$

We will call this shape a lozengoid (see Figure 5). $(\text{Aut} \tilde{V}_2)/(H_5/\mu_5) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, acting as the group of orientation-preserving symmetries of this polyhedron.

**Proof.** We are now looking at the left-hand-side square face of the cuboctohedron in Figure 3. This immediately gives five vertices: $H$, $\Delta_1$, $\Delta_4$, $\nabla_1'$, and $\nabla_3$. The cone also fits

![Figure 5. An affine slice of $\text{Imp}(\tilde{V}_2)$.](image-url)
into a side square face of the equivalent cuboctahedron centred at \( \nabla_1' \); since \( \tilde{V}_2 \) is also a resolution of \( \nabla_1' \), there is a symmetry that interchanges \( H \) and \( \nabla_1' \), which must also swap \( \Delta_1 \) and \( \Delta_4 \), and this determines its effect completely; it takes \( \nabla_3 \) to \( \nabla_2' + \nabla_3 - H \).

Now, looking from the point of view of the vertex \( \Delta_1 \), the pyramidal cone of \( \tilde{V}_1 \), as discussed above, fits into the top square face of the cuboctahedron, so \( \tilde{V}_2 \), one flop away, fits into an upper triangular face. Similarly, the pyramid that sits in the top face of \( H \)'s cuboctahedron is in a lower triangular face from the point of view of \( \nabla_3 \), and \( \tilde{V}_2 \), two flops away, must again be in an upper triangular face. Symmetry shows that the same holds for \( \Delta_4 \) and \( \nabla_2' + \nabla_3 - H \). This means that we must have a symmetry of the given slice of \( \text{Amp}(\tilde{V}_2) \) any of these to any other, which forces the cone to be the shape described.

Since this involves the upper triangular faces of the cuboctahedron, it completes the job of describing cones that meet \( H \)—that is, nef cones of resolutions of \( V \). But now every face of any of these cones borders another cone of one of these three types, so we have in fact described every cone in the fan:

**Theorem 8.** The minimal models of general HM quintics fall into three families, the families of \( \tilde{V} \), \( \tilde{V}_1 \), and \( \tilde{V}_2 \), that we will call, after their nef cones, tiling type, pyramid type and lozengoid type respectively. All the models satisfy Morrison’s nef cone conjecture [9]; that is, the action of the biregular automorphism group of each variety on its nef cone has a rational polyhedral fundamental domain.

Any given HM quintic \( V \) has infinitely many marked minimal models in each family, but up to isomorphism has only two minimal models of tiling type, two of pyramid type, and four of lozengoid type.

**Proof.** Theorems 4, 6, and 7 construct the three types of cone, and the construction describes what is on the other side of any face of any of these cones—there are square faces where tilings meet pyramids, triangular faces where pyramids meet lozengoids and quadrilateral faces where lozengoids meet other lozengoids, twisted by a quarter-turn. (This is hard to visualize, since these faces are not square on the picture (Figure 5), but of course these are affine pictures and we have chosen different affine slice for the different models.) Every minimal model must be a finite sequence of flops away from \( \tilde{V} \), so we have described all minimal models.

The nef cone conjecture is clear, and a fundamental domain is easily seen from the descriptions in the three cases.

For the last statement, the two models of tiling type are the resolutions of \( V \) and \( \beta V \), those of pyramid type are one flop from each of these (all such flops being isomorphic because \( \text{Aut}(\tilde{V}) \) acts transitively on the faces of \( \text{Amp}(\tilde{V}) \)); lozengoid-type models come from flopping two adjacent faces of \( \text{Amp}(\tilde{V}) \), which can be chosen in two ways because of the lack of automorphisms that act as order-4 rotations on the nef cone of \( \tilde{V} \).

6. **Combinatorics of the Movable Fan.**

From the theorems of the last two sections we can deduce a complete combinatorial description of the movable fan of \( V \), since they give the shapes of the cones and the way the
cones fit together at each ray. In this section we aim to give an understanding of the combinatorics of the movable fan as a polyhedral complex, disregarding the linear structure, and of the action on that complex of the birational automorphism group. So we will allow ourselves to squash and fold the cones. We will talk in terms of affine slices, making it a problem about 3-dimensional solids.

The first thing we can do to simplify matters is to glue the pyramids to the faces of the tilings. The result is a solid, no longer convex, with twice the vertices of the old square tiling, arranged like a square tiling again (at $45^\circ$ to the old one), but with half the vertices protruding and with a diagonal inward fold (marked by dotted lines in Figure 6) across each face.

![Figure 6. A stellation of a square tiling.](image)

This reduces the problem to two sorts of solid, these stellated tilings and the lozengoids. Now the lozengoid of Figure 5 can be regarded as a cube distorted in such a way that two opposite faces (the top and bottom faces in Figure 5) become folded outwards along their diagonals (see Figure 7).

The way these cones are put together sets the dotted ridges on the lozengoids into the dotted furrows on the tilings, so we can flatten these folds to obtain a complex of ordinary cubes and square-tiled paraboloids.

Consider what has happened to the vertex figure (Figure 3) in this process: gluing the pyramids to the tilings has the effect of deleting the four edges around the bottom face of the cuboctahedron, leaving four downward-pointing spikes, which are flattened out when the dotted edges on the solids are flattened. The result is a square antiprism (Figure 8). A square tiling pokes into each of the two square faces and a cube pokes into each triangular face.

It is easily seen that this vertex configuration determines a unique simply connected polyhedral complex, because there is no choice about how the figures at two adjacent vertices
are oriented with respect to one another. But we can construct such a simply-connected polyhedral complex with this configuration at every vertex as a uniform hyperbolic tiling (see [4], [6]) as follows:

The Coxeter graph \( \circ \circ \circ \circ \circ \) describes a hyperbolic reflection group \( C \) with a tetrahedral fundamental cell: the nodes of the graph correspond to the faces of the cell, and the angle between two faces is \( \pi/8 \) if the two nodes are joined by a double line, and \( \pi/4 \) if the two nodes are not joined. The marked graph \( \bullet \circ \circ \circ \circ \bullet \) represents a hyperbolic tiling with \( C \) acting transitively on its vertices: the unique vertex of the tiling that is in the fundamental cell lies on the faces of the cell corresponding to unfilled nodes in the marked graph, and not on the faces corresponding to filled nodes. \( C \)-orbits of \( k \)-dimensional faces of the tiling are described by \( k \)-node subgraphs of \( \bullet \circ \circ \circ \circ \bullet \) having at least one marked node in each connected component (for example, deleting an end node gives \( \bullet \circ \circ \circ \circ \bullet \), a square tiling, and deleting a middle node gives \( \bullet \circ \circ \circ \bullet \), a square prism), so it is possible to work out the vertex configuration of this tiling: it is the same square antiprism as produced above.

Therefore we can identify the birational automorphism group of \( V \):

**Theorem 9.** Bir(\( V \))/(\( H_5/\mu_5 \)) is isomorphic to a subgroup of index 8 in the Coxeter group whose graph is \( \circ \circ \circ \circ \circ \circ \).

**Proof.** Recall from Corollary 3 that \( H_5/\mu_5 \) is the group of birational automorphisms that act trivially on \( N \): therefore Bir(\( V \))/(\( H_5/\mu_5 \)) is the group of all symmetries of the movable fan arising from birational automorphisms. The construction above identifies the group of all symmetries of the fan with a subgroup of finite index in the group of symmetries of the uniform tiling \( T \) given by \( \circ \circ \circ \circ \circ \circ \), which is larger than the Coxeter group.
there are no automorphisms of $\text{Bir}(V)$.

In fact, the subgroup in question is of index 8 in $C$—a factor of 2 comes from the fact that the creased face diagonals must be preserved (alternatively, because the two classes of vertices corresponding to $V$ and $\beta V$ must be preserved) and a further factor of 4 because there are no automorphisms of $\tilde{V}$ that induce rotations of order 4 or reflections.

7. The Boundary of the movable Cone.

The aim of this section is to demonstrate that, although by the results of the last section the movable fan of $V$ is combinatorially similar to a tiling of hyperbolic 3-space, the cone $\text{Mov}_c(V)$ itself is more subtle, by proving this theorem:

\textbf{Theorem 10.} The boundary of the cone $\text{Mov}_c(V)$ is not twice differentiable at any point in $\partial \text{Mov}_c(V)$.

The description of the nef cones in sections 4 and 5 allows us to find all the effective divisors on the boundary of the movable cone, since every such divisor is nef on some model. But the only rays in any of the nef cones described above that are in the boundary of $\text{Mov}_c(V)$ are those corresponding to Abelian fibrations on tiling type models. These rays must then be dense on the boundary, and we can investigate the shape of the boundary through them. So let us see how we can produce examples of these points.

From $\tilde{V}$ we can flop the four classes of curves $\Lambda_1, \ldots, \Lambda_4$ to get a pyramid type model (the top of the cuboctahedron), from which we can flop one more class, say $\Lambda'$, to get a model $V' \cong \beta \tilde{V}$. We know that $\Lambda' \cdot \nabla_i = 0$ for $i = 1, \ldots, 2$, and $\Lambda' \cdot H = 1$, which determine $\Lambda'$; it can easily be checked that $\Lambda' = 3(\Lambda_1 + \Lambda_3) + \Gamma$.

Let $A'$ be the class of an Abelian fibre on $\tilde{V}$, and $\Gamma'$ be the class of a line in a singular fibre of the Abelian fibration given by $A'$. Recall that in Corollary 3 we defined an $\text{Aut} \tilde{V}$-invariant quadratic form $Q$ by $Q(D, E) = d_1 e_4 + d_4 e_3 - 2d_2 e_2 - 2d_3 e_3$, in the basis $(\tilde{e})$.

Let $Q'$ be the $\text{Aut} V'$-invariant quadratic form defined in the same way with reference to $V'$.

To get from $\tilde{V}$ to this model we have the flopped the curves in each of the five classes $\Lambda_1, \ldots, \Lambda_4$, and $\Lambda'$, so we know $A' \cdot \Lambda_1 = \cdots = A' \cdot \Lambda_4 = A' \cdot \Lambda' = -1$, which determines $A'$ exactly:

$$A' = 5H - A = (-1, 0, 0, 5). \quad (10)$$

We can also do this process in reverse, flopping a face of $\text{Amp}(\tilde{V})$, say that given by $L_1$, and four more faces to get a model $V'' \cong \beta \tilde{V}$. Then $V$ bears the relation to $V''$ that $V'$ bears to $\tilde{V}$, so $A = 5\Delta_1 - A''$; that is,

$$A'' = 5\Delta_1 - A = (-1, -5, 0, 10). \quad (11)$$
Therefore we have
\[ Q(A', A') = -10, \quad Q(A'', A'') = -70, \]
\[ \Gamma \cdot A' = 5, \quad \Gamma \cdot A'' = 10, \]
so
\[ 5Q(A', A') + 2(\Gamma \cdot A')^2 = 0, \quad (12) \]
\[ 5Q(A'', A'') + 11(\Gamma \cdot A'')^2 = 0. \quad (13) \]

This is enough to prove Theorem 10.

**Proof.** These two points \( A' \) and \( A'' \) can be moved by the automorphisms of \( \tilde{V} \) to get infinitely many points on the boundary of \( \text{Mov}_+ (V) \), accumulating at \( A \). The equations (12) and (13) are preserved by the automorphisms of \( \tilde{V} \), so there are points accumulating at \( A \) on two different quadric cones tangent at \( A \).

These two sets of points constrain the second derivative, if it were defined, to take two different values, so it cannot be defined.

The two processes for obtaining new boundary rays from old can be iterated; the plots in Figure 10 are of two different orthogonal projections of an affine slice of some such rays, showing the subtle shape of the boundary.

8. THE MOVABLE CONE CONJECTURE.

Finally, we can check explicitly that \( V \) satisfies Morrison’s movable cone conjecture [8]:

**Theorem 11.** There is a rational polyhedral fundamental domain for the action of \( \text{Bir}(V) \) on \( \text{Mov}_+ (V) \).

**Proof.** First consider the hyperbolic uniform tiling constructed in section 8 by distorting the movable fan of \( V \). The standard fundamental domain for the action of the reflection group on this tiling is a tetrahedron with a vertex at the centre of one cell in each orbit—the centres of two cubes and two square-tiled paraboloids (where the ‘centre’ of a paraboloid is the unique point it contains that is in the boundary of hyperbolic 3-space). The thick-line triangles on Figure 9 show how eight such fundamental domains can be combined into a single fundamental domain for the action of \( \text{Bir}(V) \).

Its vertices are seven in number:

- \( v_0 \), the centre of the paraboloid drawn (in this view, the point at infinity vertically downwards),
- \( v_a, v_b, v_c, \) and \( v_d \), the centres of the square-tiled paraboloids meeting the one drawn at the points \( a, b, c, \) and \( d \),
- \( v_e \) and \( v_f \), the centres of the cubes meeting the paraboloid drawn at \( e \) and \( f \).

The facets of this fundamental domain are the triangles \( v_b v_d v_e, \ v_a v_d v_e, \ v_a v_d v_f, \) and \( v_c v_d v_f \), and the quadrilaterals \( v_0 v_b v_d v_e, \ v_0 v_d v_e v_b, \) and \( v_0 v_c v_f v_d. \)
Now consider the movable fan of $V$. For each of the points $v_i$, the centre of a square-tiled paraboloid or a cube, let $\hat{v}_i$ be a generator of the central ray of the corresponding square-tiled paraboloid or lozengoid in the movable fan, all chosen in the same affine slice $\Pi$. It is clear from symmetry that the three quadrilaterals $\hat{v}_0\hat{v}_b\hat{v}_d\hat{v}_e$, $\hat{v}_0\hat{v}_a\hat{v}_c\hat{v}_b$, and $\hat{v}_c\hat{v}_d\hat{v}_f\hat{v}_a$ are planar, and the triangles $\hat{v}_b\hat{v}_d\hat{v}_e$, $\hat{v}_a\hat{v}_d\hat{v}_f$, and $\hat{v}_c\hat{v}_d\hat{v}_f$ are certainly planar, so there is a (possibly non-convex) polyhedron with these seven faces.

The cone on this polyhedron will be a rational polyhedral fundamental domain for the action of $\text{Bir}(V)$ on $\text{Mov}_+(V)$ as long as it is convex. Examination of the shape of the polyhedron shows that it will be convex as long as the three quadrilaterals are convex and the line $\hat{v}_e\hat{v}_f$ passes through the interior of $\hat{v}_0\hat{v}_a\hat{v}_d$. We will therefore check these facts.

Choosing the right copy of the fundamental domain, we have, for some positive scalars $\{\lambda_i\}$, the following expressions (the expression for $\lambda_e\hat{v}_e$ is the sum of the vertices of the lozengoid described in Theorem 5; $A'$ was described in section 7, and the remainder follow by translating these two by appropriate symmetries of $\text{Amp}(\tilde{V})$)

\[
\begin{align*}
\lambda_0\hat{v}_0 &= A, \\
\lambda_a\hat{v}_a &= A' = 5H - A \\
\lambda_b\hat{v}_b &= 5\nabla_1' - A \\
\lambda_c\hat{v}_c &= 5\nabla_2' - A \\
\lambda_d\hat{v}_d &= 5\Delta_1 - A \\
\lambda_e\hat{v}_e &= H + \Delta_1 + \Delta_4 + \nabla_1' + \nabla_3 \\
&\quad + (\nabla_1' + \nabla_3 - H) + (\Delta_1 + \nabla_3 - H) + (\Delta_4 + \nabla_3 - H) \\
&= 10H + 10\nabla_1' - 6A \\
\lambda_f\hat{v}_f &= 10H + 10\nabla_2' - 6A
\end{align*}
\]
Now the convexity of the quadrilaterals follows from
\[\lambda_b \hat{v}_b + \lambda_c \hat{v}_c = 5\nabla_1' + 5\nabla_2' - 2A = 5\Delta_1 + 3A = \lambda_a \hat{v}_a + 4\lambda_0 \hat{v}_0,\]
\[2\lambda_a \hat{v}_a + 2\lambda_b \hat{v}_b = 10H + 10\nabla_1' - 4A = \lambda_e \hat{v}_e + 2\lambda_0 \hat{v}_0,\]
\[2\lambda_a \hat{v}_a + 2\lambda_c \hat{v}_c = 10H + 10\nabla_2' - 4A = \lambda_f \hat{v}_f + 2\lambda_0 \hat{v}_0,\]

while the condition on \( \hat{v}_e, \hat{v}_f \) follows from
\[\lambda_e \hat{v}_e + \lambda_f \hat{v}_f = 20H + 10\nabla_1' + 10\nabla_2' - 12A\]
\[= 10\Delta_1 + 20H - 2A\]
\[= 4\lambda_a \hat{v}_a + 2\lambda_d \hat{v}_d + 4\lambda_0 \hat{v}_0.\]

All these equalities can be checked from the table [6].
Figure 10. Two views of part of the boundary of \( \text{Mov}_c(V) \).
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