Hypernuclear Gamma-Ray Spectroscopy and the Structure of p-shell Nuclei and Hypernuclei

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Summary. Information on $^7\Lambda Li$, $^9\Lambda Be$, $^{10}\Lambda B$, $^{12}\Lambda C$, $^{15}\Lambda N$, and $^{16}\Lambda O$ from the Ge detector array Hyperball is interpreted in terms of shell-model calculations that include both $\Lambda$ and $\Sigma$ configurations with p-shell cores. It is shown that the data puts strong constraints on the spin dependence of the $\Lambda N$ effective interaction.

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1 Introduction

The structure of Λ hypernuclei – i.e. many-body systems consisting of neutrons, protons, and Λ particles – is an interesting subject in its own right. However, the finer details of the structure of single-Λ hypernuclei, particularly the energy spacings of doublets formed by the coupling of a Λ in the lowest s orbit to a nuclear core state with non-zero spin, provide information on the spin dependence of the effective ΛN interaction. This is important because data on the free YN interaction are very sparse and essentially limited to spin-averaged s-wave scattering.

The spectroscopy of Λ hypernuclei has been reviewed recently by Hashimoto and Tamura [1]. The ‘workhorse’ reactions used to produce Λ hypernuclei have been the (K$^-$, π$^-$) (strangeness exchange) and (π$^+$, K$^+$) (associated production) reactions that convert a neutron into a Λ. The elementary n(K$^-$, π$^-$)Λ and n(π$^+$, K$^+$)Λ reactions have predominantly non-spin-flip character at the incident beam energies used.

The first information on Λ hypernuclei came from their production via K$^-$ mesons stopped in emulsion followed by their π$^-$-mesonic weak decay [2]. These studies provided Λ separation energies (B$_{Λ}$ values) up to A $\sim$ 15. These could be accounted for by a Λ-nucleus potential of Woods-Saxon shape with a depth of about 30 MeV. A number of ground-state spins were determined from angular correlation studies and weak-decay branching ratios, γ rays from excited states of $^4\Lambda$H and $^4\Lambda$He were seen, and so was proton emission from excited states of $^{12}\Lambda$C. Currently, counter experiments with stopped K$^-$ mesons are being performed at Frascati [3].

The momentum transfer to the hypernucleus is rather small in the forward direction in (K$^-$, π$^-$) reactions near the beam momenta of $\sim$ 800 MeV/c used at CERN and BNL and the cross sections for ΔL = 0 transitions are large. Because the cross sections are proportional to the spectroscopic factors for neutron removal from the target, the cross sections are largest when a high-spin neutron orbit is just filled and the produced Λ occupies the same orbit. Such transitions have been observed in selected nuclei up to $^{209}$Bi (see [4]).

The (π$^+$, K$^+$) reaction has been used at $p_π = 1.05$ GeV/c where the elementary cross section peaks strongly. The momentum transfer is high ($q \sim 350$ MeV/c) and the reaction selectively populates high-spin states. The cross sections are smaller than for the (K$^-$, π$^-$) reaction but the count rates for producing Λ hypernuclei can be comparable because more intense pion beams can be used. Transitions from nodeless high-spin neutron orbits can populate
the full range of nodeless bound, and just unbound, \( \Lambda \) orbitals and have been used to map out the spectrum of \( \Lambda \) single-particle energies for selected nuclei up to \( ^{208}\text{Pb} \) (see [1]). This information has provided a rather precise characterization of the \( \Lambda \)-nucleus potential. The best resolution, obtained at KEK using the SKS spectrometer and a thin \( ^{12}\text{C} \) target, is 1.45 MeV [4].

The free \( \Lambda \) decays into a nucleon and a pion with a lifetime of 263 ps. In a hypernucleus, the low-energy nucleon produced via this decay mode is Pauli blocked and the process \( \Lambda N \rightarrow NN \) rapidly dominates with increasing mass number. Nevertheless, the measured weak decay lifetimes of \( \Lambda \) hypernuclei remain around 200 ps. This means that particle-bound excited states of \( \Lambda \) hypernuclei normally decay electromagnetically. Then it is possible to make use of the excellent resolution of \( \gamma \)-ray detectors to measure the spacings of hypernuclear levels. The earliest measurements were made with NaI detectors but the superior (few keV) resolution of Ge detectors has been exploited since 1998 in the form of the Hyperball, a large-acceptance Ge detector array. A series of experiments on p-shell targets has been carried out at KEK and BNL using the \((\pi^{+}, K^{+}\gamma)\) and \((K^{-}, \pi^{-}\gamma)\) reactions, respectively [1]. As well as \( \gamma \)-ray transitions between bound states of the primary hypernucleus, \( \gamma \)-ray transitions are often seen from daughter hypernuclei formed by particle emission (most often a proton) from unbound states of the primary hypernucleus.

The results of these \( \gamma \)-ray experiments are interpreted in terms of nuclear structure calculations with a parametrized effective \( YN \) interaction as input.

2 The \( \Lambda N \) (YN) Effective Interaction

The hyperon-nucleon interaction involves the coupled \( \Lambda N \) and \( \Sigma N \) channels, as illustrated in Fig. 1. The diagrams in Fig. 1 make the point that the direct \( \Lambda N \)–\( \Lambda N \) interaction does not contain a one-pion exchange contribution because of isospin conservation (except for electromagnetic violations via \( \Lambda - \Sigma^0 \) mixing) while the coupling between the \( \Lambda N \) and \( \Sigma N \) channels does. For this reason, the \( \Lambda N \) interaction is relatively weak and there is reason to believe that the three-body interaction in a hypernucleus could be relatively important.

The free-space interactions are obtained as extensions of meson-exchange models for the NN interaction by invoking, e.g., a broken flavor SU3 symmetry. The most widely used model is the Nijmegen soft-core, one-boson-exchange potential model known as NSC97 [5]. The six versions of this model, labelled NSC97a..f, cover a wide range of possibilities for the strength of the central spin-spin interaction ranging from a triplet interaction that is stronger than the singlet interaction to the opposite situation. An extended soft-core version (ESC04) has recently been published [6]. Effective interactions for use in a nuclear medium are then derived through a G-matrix procedure [5, 6].

The \( \Lambda N \) effective interaction can be written (neglecting a quadratic spin-orbit component) in the form

\[ V_{\Lambda N}(r) = V_0(r) + V_\sigma(r) \mathbf{s}_N \cdot \mathbf{s}_A + V_\Lambda(r) \mathbf{l}_{NA} \cdot \mathbf{s}_A + V_N(r) \mathbf{l}_{NA} \cdot \mathbf{s}_N + V_T(r) \mathbf{s}_{12}, \]  \tag{1}
Fig. 1. Diagrams showing the important features of the coupled $ΛN - ΣN$ strangeness $-1$ interaction for isospin 1/2. The last diagram shows the two-pion exchange three-body interaction

where $V_0$ is the spin-averaged central interaction, $V_σ$ is the difference between the triplet and singlet central interactions, $V_Λ$ and $V_N$ are the sum and difference of the strengths of the symmetric spin-orbit (LS) interaction $I_{ΛN} \cdot (s_Λ + s_N)$ and antisymmetric spin-orbit (ALS) interaction $I_{ΛN} \cdot (s_Λ - s_N)$, and $V_T$ is the tensor interaction with ($C^2$ is a normalized spherical harmonic)

$$S_{12} = 3(\sigma_N \cdot \hat{r})(\sigma_Λ \cdot \hat{r}) - \sigma_N \cdot \sigma_Λ$$

$$= \sqrt{6} C^2(\hat{r}) \cdot [\sigma_N, \sigma_Λ]^2 .$$ (2)

For the $Λ$ in an s orbit, $I_{ΛN}$ is proportional to $I_N$ \[^7\]. The effective $ΛN$ and $ΣN$–$ΣN$ interactions can be written in the same way.

Each term of the potential in (1) can be written in the form

$$\langle L^k \cdot S^k = V_k(r) \langle \cdot \hat{r}^k [L^k, S^k]^0 \rangle ,$$

(3)

where $k$ is the spherical tensor rank of the orbital and spin operators and $\hat{k}^2 = 2k + 1$.

So-called YNG interactions, in which each term of the effective interaction is represented by an expansion in terms of a limited number of Gaussians with different ranges,

$$V(r) = \sum_i v_i e^{-r^2/β_i^2} ,$$

(4)

are often used for the central and spin-orbit components with the following form used for the tensor component,
\[ V_T(r) = \sum_i v_i r^2 e^{-r^2/\beta^2}. \] (5)

For example, the G-matrix elements from a nuclear matter calculation have been parametrized in this way, in which case the YNG interactions have a density dependence (through $k_T$).

Given the interaction in YNG, or some other, radial form, two-body matrix elements that define the interaction for a shell-model calculation can be calculated using a chosen set of single-particle radial wave functions. In the following, the procedure is sketched for harmonic oscillator radial wave functions in the case of equal mass particles. There are techniques to calculate the two-body matrix elements for any (e.g., Woods-Saxon) radial wave functions but the harmonic oscillator case illustrates where the important contributions come from and suggests ways in which the interaction can be parametrized in terms of the radial matrix elements themselves.

Separating the space and spin variables in (1) and (3) using (64)

\[ \langle l_1 l_2 L S | | l'_1 l'_2 L' S' \rangle^{JT} = \sum_k (-)^{L'+S+J} \left\{ \begin{array}{ccc} L & L' & k \\ S' & S & J \end{array} \right\} \times \hat{L} \langle l_1 l_2 L | | V_k(r) L^k | | l'_1 l'_2 L' \rangle \hat{S} \langle S || S^k || S' \rangle. \] (6)

Harmonic oscillator wave functions have the unique property that a transformation exists from the individual particle coordinates $r_1$, $r_2$ to the relative and center of mass coordinates $(r_1 - r_2)/\sqrt{2}$, $(r_1 + r_2)/\sqrt{2}$ of the pair $[8]$. This transformation and another application of (64) result in an expression in terms of radial integrals in the relative coordinate $r = |r_1 - r_2|$

\[ \langle l_1 l_2 L | | V_k(r) L^k | | l'_1 l'_2 L' \rangle = \sum_{N_c L_c l'} (-)^{l + l_c + k + L'} \hat{L} \hat{L}' \left\{ \begin{array}{ccc} L & l' & L' \\ k & L & l \end{array} \right\} \times \langle n l N_c L_c, L | | n_1 l_1 n_2 l_2, L \rangle \langle n' l' N_c L_c, L' | | n'_1 l'_1 n'_2 l'_2, L' \rangle \times \langle n | V_k(r) | n' \rangle \langle l || L^k || l' \rangle, \] (7)

where the number of quanta associated with coordinate $r$ is given by $q = 2n + l$ ($n = 0, 1, ..$) and energy conservation $q_1 + q_2 = q + Q_c$ fixes $n$ (similarly $n'$). The reduced matrix elements (see Appendix [A]) of the orbital and spin operators are listed in Table [2]. The radial integral can in turn be expressed in terms of Talmi integrals $I_p$

| Table 1. Two-particle reduced matrix elements of orbital and spin operators |
|-----------------------------|---------------------|---------------------|----------------------|
|                            | $s_N \cdot s_A$     | LS                  | ALS                  |
| $\hat{S}(S||S^k||S')$      | $\hat{S}(2S(S+1) - 3)/4$ | $\delta_{Sg} \hat{S} \sqrt{2}$ | $(-)^S(1 - \delta_{Sg}) \sqrt{3} \hat{S} \sqrt{20}/3$ |
| $\langle l || L^k || l' \rangle$ | $\sqrt{l(l+1)}$     | $\delta_{il} \sqrt{l(l+1)}$ | $\delta_{il} \sqrt{l(l+1)}$ |


\[ \langle nl|V(r)|n'l'\rangle = \sum_p B(nl,n'l';p) I_p. \] (8)

The harmonic oscillator radial relative wave function is a polynomial in \( r' \) times \( \exp(-r'^2) \) where \( r' = |r_1 - r_2|/\sqrt{2}b \) and \( b^2 = \hbar/m\omega \). Then,

\[ I_p = \frac{2}{\Gamma(p+3/2)} \int_0^\infty r^{2p} e^{-r^2} V(\sqrt{2}rb) r^2 dr. \] (9)

For a Gaussian potential, \( V(r) = V_0 \exp(-r^2/\mu^2) \), with \( \theta = b/\mu \),

\[ I_p = \frac{V_0}{(1 + 2\theta^2)^{p+3/2}}. \] (10)

For the case of a \( \Lambda \) in an \( s \) orbit attached to a light nucleus, the expressions for the matrix elements of each component of the interaction are shown in Table 2. In this simple case, the \( I_p \) are equal to the relative matrix elements in the angular momentum states denoted by \( p \) (the superfluous superscripts denote the interaction in even or odd relative states). For the nuclear \( p \) shell, there are just five \( p_Ns_\Lambda \) two-body matrix elements formed from \( p_{1/2}s_{1/2}(0^-,1^-) \) and \( p_{3/2}s_{1/2}(1^-,2^-) \) (alternatively, \( L = 1 \) and \( S = 0, 1 \)). This means that the five radial integrals \( V, \Delta, S_\Lambda, S_N, \) and \( T \) associated with each operator in Table 2 can be used to parametrize the \( \Lambda N \) effective interaction. In Appendix B, the \( p_Ns_\Lambda \) two-body matrix elements are given in terms of the parameters in both LS and \( jj \) coupling.

| \( V_{\Lambda N} \) | \( s_Ns_\Lambda \) | \( p_Ns_\Lambda \) | \( ^7Li \) values (MeV) |
|-------------------|----------------|----------------|-------------------|
| \( V_0 \) | \( I_0^s \) | \( \nabla = \frac{1}{2}(I_0^s + I_1^s) \) | \(-1.22\) |
| \( V_s \cdot s_\Lambda \) | \( I_0^\Delta \) | \( \Delta = \frac{1}{2}(I_0^\Delta + I_1^\Delta) \) | \( 0.480 \) |
| \( V_\Lambda I_\Lambda \cdot s_\Lambda \) | \( S_\Lambda = \frac{1}{2}I_1^\Lambda \) | \(-0.015 \) |
| \( V_N I_N \cdot s_N \) | \( S_N = \frac{1}{2}I_1^N \) | \(-0.400 \) |
| \( V_T S_{12} \) | \( T = \frac{1}{2}I_1^T \) | \( 0.030 \) |

A comprehensive program for the shell-model analysis of \( \Lambda \) binding energies for \( p \)-shell hypernuclei was set out by Gal, Soper, and Dalitz [7], who also included the three-body double-one-pion-exchange \( \Lambda N N \) interaction shown in Fig. 1. This interaction does not depend on the spin of the \( \Lambda \) and was characterized by a further five radial integrals. Dalitz and Gal went on to consider the formation of \( p \)-shell hypernuclear states via \((K^-,\pi^-)\) and \((K^-,\pi^0)\) reactions and the prospects for \( \gamma \)-ray spectroscopy based on the decay of these states [9]. Unfortunately, knowledge of the ground-state \( B_\Lambda \) values plus a few constraints from known ground-state spins was insufficient to provide definitive information on the spin-dependence of the \( \Lambda N \) interaction.
The most direct information on the spin dependence of the ΛN effective interaction comes from the spacing of $s_\Lambda$ doublets based on core states with non-zero spin. These spacings depend on the parameters $\Delta$, $S_\Lambda$, and $T$ that are associated with operators that involve the Λ spin. The energy separations of states based on different core states depend on $S_N$, but these separations can also be affected by the three-body interaction. Millener, Gal, Dover, and Dalitz [10] made estimates for $\Delta$, $S_\Lambda$, $S_N$, and $T$ using new information, particularly on γ-ray transitions in $^7_\Lambda$Li and $^9_\Lambda$Be [11], together with theoretical input from YN interaction models. These estimates were close to the values given in the right-hand column of Table 2 (the bracketed value for $V$ is not fitted) which fit the now-known energies of the four bound excited states of $^7_\Lambda$Li (see Sect. 4). An alternative set of parameters was proposed by Fetisov, Majling, Žofka, and Eramzhyan [12] who were motivated by the non-observation of a γ-ray transition from the ground-state doublet of $^{10}_\Lambda$B in the first experiment using Ge detectors [13].

Experiments with the Hyperball, starting in 1998 with $^7_\Lambda$(π+, K+γ) $^7_\Lambda$Li at KEK and $^{10}_\Lambda$(K−, π−γ) $^{10}_\Lambda$B at BNL, have provided the energies of numerous γ-ray transitions, together with information on relative intensities and lifetimes. The progress of the theoretical interpretation in terms of shell-model calculations has been summarized at HYP2000 [14] and HYP2003 [15]. By the latter meeting, Σ degrees of freedom were being included explicitly through the inclusion of both Λ and Σ configurations in the shell-model basis.

The most convincing evidence for the importance of Λ–Σ coupling comes from the s-shell hypernuclei and this is described in the following section. This is followed by a discussion of $^7_\Lambda$Li in Sect. 4 Because the LS structure of the p-shell core nuclei plays an important role in picking out particular combinations of the spin-dependent ΛN parameters, Sect. 5 is devoted to a general survey of p-shell calculations, spectra, and wave functions. This information is used in subsequent sections that are devoted to the remaining hypernuclei, up to $^{16}_\Lambda$O, for which data, particularly from γ rays, exists.

### 3 The s-shell Λ Hypernuclei

The data on the s-shell hypernuclei is shown in Table 3. The $B_\Lambda$ values come from emulsion data [2] and the excitation energies from γ rays observed following the stopping of negative kaons in $^6_\Lambda$Li and $^7_\Lambda$Li [16].

The spin-spin component of the ΛN interaction contributes to the splitting of the $1^+$ and $0^+$ states of $^4_\Lambda$H and $^4_\Lambda$He. In the case of simple $s^3s_\Lambda$ configurations, the contribution is given by the radial integral of the spin-spin interaction in $s$ states (the $\Delta$ in Table 2) using

$$\sum_i s_i \cdot s_\Lambda = s_c \cdot s_\Lambda$$

$$= \frac{1}{2} [ S^2 - S_c^2 - S_\Lambda^2 ] .$$

(11)
Table 3. Data on the s-shell Λ hypernuclei

| Hypernucleus | Jπ (gs) | BΛ (MeV) | Jπ Ex (MeV) |
|--------------|---------|----------|-------------|
| $^3_1$H     | 1/2+    | 0.13(5)  |             |
| $^4_1$H     | 0+      | 2.04(4)  | 1+ 1.04(5)  |
| $^4_1$He    | 0+      | 2.39(3)  | 1+ 1.15(4)  |
| $^5_2$He    | 1/2+    | 3.12(2)  |             |

However, it has long been recognized as a problem to describe simultaneously the binding energies of the s-shell hypernuclei with a central ΛN interaction and that this problem might be solved by Λ–Σ coupling which strongly affects the 0+ states of the A = 4 hypernuclei. Recently, there has been a clear demonstration of these effects and it was found that the spin-spin and Λ–Σ coupling components of the NSC97e and NSC97f interactions give comparable contributions to the 1+–0+ doublet splitting [17]. Subsequent studies using a variational method with Jacobi-coordinate Gaussian-basis functions [18], Faddeev-Yakubovsky calculations [19], and stochastic variational calculations with correlated Gaussians [20] have confirmed and illustrated various aspects of the problem.

Akaishi et al. [17] calculated G-matrices for a small model space of s orbits only, writing two-component wave functions for either the 0+ or the 1+ states of $^4_1$He (or $^4_1$H) with isospin T = 1/2

$$|\text{3He}_0\rangle = \alpha s^3 s_\Lambda + \beta s^3 s_\Sigma.$$  

(12)

The Σ component is $2/3 \Sigma^+$ and $1/3 \Sigma^0$ for $^4_1$He ($2/3 \Sigma^-$ and $1/3 \Sigma^0$ for $^4_1$H). The off-diagonal matrix element

$$v(J) = \langle s^3 s_\Lambda, J|V|s^3 s_\Sigma, J \rangle$$  

(13)

can be derived from the ΛN–ΣN G matrix for 0s orbits, where V is used for the potential representing the G matrix interaction, by splitting one nucleon off from the s$^3$ configurations using the fractional parentage expansion

$$|s^3\rangle = \sum_{S(T)} \frac{1}{\sqrt{2}} (-)^{1+S} |s^2(TS), s(1/2 1/2)\rangle,$$  

(14)

where TS = 0 1 or 1 0. Coefficients of fractional parentage (cfp) specify how to construct a fully antisymmetric n-particle state from antisymmetric (n – 1)-particle states coupled to the nth particle [cf. (11)]. In this simple case, the magnitude of the cfp is determined by the symmetry with respect to T and S and the phase appears twice and cancels out in the problem at hand. Then, by recoupling on either side of (13) (see Appendix A),

$$v(J) = \frac{3}{2} \sum_{S\bar{S}} U(S 1/2 J 1/2, 1/2 \bar{S})^2 U(T 1/2 1/2, 1/2 1/2) \times U(T 1/2 1/2 1, 1/2 1/2) \langle s s_\Lambda, \bar{S}|V|s s_\Sigma, \bar{S} \rangle,$$  

(15)
where the factor of 3 appears because any one of the three s-shell nucleons can be chosen from an antisymmetric wave function. Specializing to the case of \( J = 0 \)

\[
v(0) = \frac{3}{2} \sqrt{3} V - \frac{1}{2} \sqrt{3} V \\
= \sqrt{V} + \frac{3}{4} \Delta,
\]

(16)

where

\[
\sqrt{V} = \frac{1}{4} \sqrt{3} V + \frac{3}{4} \sqrt{3} V, \quad \Delta = \sqrt{3} V - \sqrt{3} V
\]

(17)

Similarly,

\[
v(1) = \frac{1}{2} \sqrt{3} V + \frac{1}{2} \sqrt{3} V \\
= \sqrt{V} - \frac{1}{4} \Delta.
\]

(18)

Taking round numbers derived using the 20-range Gaussian interaction of [17] which represents NSC97f yields \( \sqrt{3} V = 4.8 \text{ MeV} \) and \( \sqrt{3} V = -1.0 \text{ MeV} \), which give \( \sqrt{V} = 3.35 \text{ MeV} \) and \( \Delta = 5.8 \text{ MeV} \). Then, \( v(0) = 7.7 \text{ MeV} \) and \( v(1) = 1.9 \text{ MeV} \). In a simple 2 \( \times \) 2 problem, the energy shifts of the \( \Lambda \)-hypernuclear states are given by \( \sim v(J)^2/\Delta E \) with \( \Delta E \sim 80 \text{ MeV} \) (and the admixture \( \beta \sim -v(J)/\Delta E \)). Thus, the energy shift for the 0\(^+\) state is \( \sim 0.74 \text{ MeV} \) while the shift for the 1\(^+\) state is small. The result is close to that for the NSC97f interaction in Fig. 1 of [17].

The same method can be used to obtain the singlet and triplet contributions of the \( \Lambda \) interaction for all the s-shell hypernuclei in the case of simple s-shell nuclear cores. The results are given in Table 4. The expressions in terms of \( \sqrt{V} \) and \( \Delta \) can be written down by inspection.

| Hypernucleus | \( J^\pi (gs) \) | \( \sqrt{3} V \) and \( 1 V \) | \( \sqrt{V} \) and \( \Delta \) |
|--------------|----------------|----------------------------|-----------------|
| \( ^3 \Lambda H \) | 1/2\(^+\) | 3/2 \( 1 V \) + 1/2 \( 3 V \) | 2\( \sqrt{V} \) - \( \Delta \) |
| \( ^3 \Lambda H \) | 3/2\(^+\) | 2 \( 3 V \) | 2\( \sqrt{V} \) + 1/2\( \Delta \) |
| \( ^4 \Lambda \) He, \( ^4 \Lambda H \) | 0\(^+\) | 3/2 \( 1 V \) + 3/2 \( 3 V \) | 3\( \sqrt{V} \) - 3/4\( \Delta \) |
| \( ^4 \Lambda \) He, \( ^5 \Lambda H \) | 1\(^+\) | 1/2 \( 1 V \) + 5/2 \( 3 V \) | 3\( \sqrt{V} \) + 1/4\( \Delta \) |
| \( ^5 \Lambda \) He | 1/2\(^+\) | \( 1 V \) + 3 \( 3 V \) | 4\( \sqrt{V} \) |

### 4 The \( ^7 \Lambda Li \) Hypernucleus

The first p-shell hypernucleus with particle-stable excited states that can be studied by \( \gamma \)-ray spectroscopy is \( ^7 \Lambda Li \) and it is of interest to compare the effects
of the ΛN spin-spin interaction and Λ–Σ coupling in $^{7}_{\Lambda}$Li with those in $^{4}_{\Lambda}$H and $^{4}_{\Lambda}$He.

The low-lying states of $^{7}_{\Lambda}$Li consist of a Λ in an s orbit coupled (weakly) to a $^{6}_{\Lambda}$Li core. Only the $1^+; 0 \ (J^\pi; T)$ ground state of $^{6}_{\Lambda}$Li is stable with respect to deuteron emission but the Λ brings in extra binding energy and the lowest particle-decay threshold for $^{7}_{\Lambda}$Li is $^{5}_{\Lambda}$He+d at 3.94(4) MeV derived from

$$S_d^{(7}_{\Lambda} \text{Li}) = S_d^{(6}_{\Lambda} \text{Li}) + B_A^{(7}_{\Lambda} \text{Li}) - B_A^{(5}_{\Lambda} \text{He}) = 1.475 + 5.58(3) - 3.12(2), \quad (19)$$

where the $B_A$ values (errors in parentheses) come from emulsion studies [2].

Figure 2 shows the spectrum of $^{7}_{\Lambda}$Li determined from experiments KEK E419 [21, 22] and BNL E930 [23] with the Hyperball detector. The four γ-rays seen in [21] – all except the $7/2^+ \rightarrow 5/2^+$ transition – show that the state based on the $0^+; 1$ state of $^{6}_{\Lambda}$Li is bound at an excitation energy of 3.88 MeV. Only the $5/2^+ \rightarrow 1/2^+$ transition was previously known from an experiment at BNL with NaI detectors [11]. Because the $3/2^+$ state is expected to be weakly populated in the $(\pi^+, K^+)$ reaction, much of the intensity of the 692-keV γ-ray transition comes from feeding via the γ-ray transition from the $1/2^+; 1$ level. The $7/2^+ \rightarrow 5/2^+$ doublet transition was seen in γγ coincidence with the

$$\begin{array}{cccccc}
3877 & 3940(40) & {\Lambda} \text{He} + d & 1/2^+; 1 & 98 & 0.60 & 3563 & 0^+; 1 \\
32 & 43 & \\
2521 & 83 & 17 & 7/2^+ & 0 & 0.08 & 2186 & 3^+; 0 \\
2050 & 96.2 & 3.8 & 5/2^+ & 74 & 1.23 & \\
692 & 100 & 3/2^+ & 6 & 0.13 & 0 & 1^+; 0 \\
0 & & {\Lambda} \text{Li} & 1/2^+ & 78 & 1.21 & \Lambda \Sigma \sigma(\pi^+, K^+) & {^{6}_{\Lambda} \text{Li}}
\end{array}$$

Fig. 2. The spectrum of $^{7}_{\Lambda}$Li determined from experiments KEK E419 and BNL E930 with the Hyperball detector. All energies are in keV. The solid arrows denote observed γ-ray transitions. The γ-ray branching ratios are theoretical and the dashed arrows correspond to unobserved transitions. For each state of $^{7}_{\Lambda}$Li, the calculated energy shifts due to Λ–Σ coupling and the calculated relative populations via the $(\pi^+, K^+)$ reaction are given [24]. The core states of $^{6}_{\Lambda}$Li are shown on the right.
$5/2^+ \rightarrow 1/2^+$ transition following $^3$He emission from highly-excited states of $^{10}_Λ^7$B (the s-hole region) produced via the $(K^-, π^-)$ reaction on $^{10}B$ [23].

Shell-model calculations for p-shell hypernuclei start with the Hamiltonian 

$$H = H_N + H_Y + V_{NY},$$

(20)

where $H_N$ is some empirical Hamiltonian for the p-shell core, the single-particle $H_Y$ supplies the $\sim 80$ MeV mass difference between $Λ$ and $Σ$, and $V_{NY}$ is the YN interaction. The two-body matrix elements of the YN interaction between states of the form $(pNsY)$ can be parametrized in the form given in Table 2 (see Appendix B). This form applies to the direct $ΛN$ interaction, the $ΛN$–$ΣN$ coupling interaction, and the direct $ΣN$ interaction for both isospin 1/2 and 3/2 (which is included in the calculations). The shell-model basis states are chosen to be of the form

$$|(p^noe^{Jc Te^YY}J T)\rangle,$$

(21)

where the hyperon is coupled in angular momentum and isospin to eigenstates of the p-shell Hamiltonian for the core. This is known as a weak-coupling basis and, indeed, the mixing of basis states in the hypernuclear eigenstates is generally very small. In this basis, the core energies are taken from experiment where possible and from the p-shell calculation otherwise.

For $^7_Λ^6$Li, the basis states are of the form $|p^2sΛ⟩$ and $|p^2sΣ⟩$. The $p^2$ wave functions for $^6$Li are close to the LS-coupling limit, as can be seen from Table 5 where wave functions are given for all three of Cohen and Kurath’s interactions [25] and two other interactions fitted to p-shell data. As discussed in

| $J^n \ (2S+1)L$ | fit69 | fit5 | CK616 | CK816 | CKPOT |
|----------------|-------|------|-------|-------|-------|
| $1^+_1; 0$ | $^3S$ | 0.9873 | 0.9906 | 0.9576 | 0.9484 | 0.9847 |
| | $^3D$ | −0.0422 | −0.0437 | −0.2777 | −0.3093 | −0.1600 |
| | $^3P$ | −0.1532 | −0.1298 | −0.0761 | −0.0703 | −0.0685 |
| $1^+_2; 0$ | $^3S$ | 0.0287 | −0.0347 | −0.2810 | −0.3082 | −0.1426 |
| | $^3D$ | −0.9007 | −0.9987 | −0.9591 | −0.9510 | −0.9673 |
| | $^3P$ | 0.4334 | 0.0708 | −0.0354 | 0.0259 | 0.2096 |
| $0^+_1; 1$ | $^1S$ | 0.9560 | 0.9909 | 0.9997 | 0.9999 | 0.9946 |
| | $^3P$ | 0.2935 | 0.1348 | 0.0247 | −0.0137 | 0.1036 |
| $2^+_1; 1$ | $^1D$ | 0.8760 | 0.9827 | 0.9486 | 0.9959 | 0.9839 |
| | $^3P$ | 0.4824 | 0.3148 | 0.1854 | 0.0905 | 0.1789 |
more detail in Sect. 5, the central interaction is attractive in spatially even (S and D) states and repulsive in (P) odd states. The $3^+; 0$ state in Fig. 2 is the lowest member of an $L = 2, S = 1$ ($^3D$) triplet completed by a $2^+; 0$ state at 4.31 MeV and a $1^+; 0$ state at 5.65 MeV; the $2^+; 1$ ($^1D$) state is at 5.67 MeV. The $L = 1$ admixtures are largely through the one-body spin-orbit interaction. The $p_1/2 - p_3/2$ splittings at $A = 5$ are small (0.14–1.29 MeV) for the Cohen and Kurath interactions (the $p$ states are unbound at $A = 5$ and the $p_{1/2}$ energy is poorly defined). The fit69 interaction has a larger splitting of 3.5 MeV while the fit5 interaction, the one used in the hypernuclear calculation, has an intermediate value of 1.8 MeV.

The structure of the core nucleus means that the $3/2^+$ and $7/2^+$ states are mainly $L = 0, S = 3/2$ and purely $L = 2, S = 3/2$, respectively. This accounts for their low population in the $(\pi^+, K^+)$ reaction which is dominantly non-spin-flip (the $^7$Li ground state has $L = 1, S = 1/2, J = 3/2$). The $1/2^+$ states are mainly $L = 0, S = 1/2$ while the $5/2^+$ state is $7/9 S = 1/2$ and $2/9 S = 3/2$ in the LS limit for the core. In this limit, it is easy to derive the contribution of each of the spin-dependent $\Lambda N$ parameters to the binding energies.

These contributions for the full shell-model calculation are given in Table 6 as the coefficients of each of the $\Lambda N$ effective interaction parameters. In the LS limit for the ground-state doublet, only $\Delta$ contributes while for the excited-state doublet all terms contribute. For the $1/2^+; 1$ state, there would be no contributions. However in the realistic case, $S_N$ contributes substantially for the predominantly $L = 0$ cases. This is because the associated operator $l_N \cdot s_N$ connects the $L = 0$ and $L = 1$ basis states of the core giving rise to a linear dependence on the amplitude of the $L = 1$ admixture. This admixture is quite sensitive to the model for the p-shell core (see Table 5). The hypernuclear shell-model states are very close to the weak-coupling limit. For the $5/2^+$ state, there is a 1.28% admixture based on the the $2^+; 0$ core state (because the $2^+$ and $3^+$ core states share the same $L$ and $S$). Otherwise, the intensity of the dominant basis state is > 99.7%.

Table 7 re-expresses the same information in terms of energy differences between states and gives the actual energy contributions for the parameter

| $J^\pi; T$ | $\Lambda \Sigma$ | $\Delta$ | $S_\Lambda$ | $S_N$ | $T$ |
|-----------|-----------------|------|--------|------|-----|
| $1/2^+; 0$ | 78              | -0.975 | -0.025 | 0.242 | 0.080 |
| $3/2^+; 0$ | 6               | 0.486  | 0.013  | 0.253 | -0.205 |
| $5/2^+; 0$ | 74              | -0.796 | -1.165 | 0.980 | 1.177 |
| $7/2^+; 0$ | 0               | 0.500  | 1.000  | 1.000 | -1.200 |
| $1/2^+; 1$ | 98              | -0.002 | 0.002  | 0.453 | -0.005 |
Table 7. Energy spacings in $\Lambda^7$Li. $\Delta E_C$ is the contribution of the core level spacing. The first line in each case gives the coefficients of each of the $\Lambda N$ effective interaction parameters as they enter into the spacing while the second line gives the actual energy contributions to the spacing in keV.

| $J_i^\pi - J_f^\pi$ | $\Delta E_C$ | $\Delta E$ | $\Delta$ | $S_\Lambda$ | $S_N$ | $T$ | $\Delta E$ |
|---------------------|--------------|------------|----------|-------------|-------|-----|-----------|
| $3/2^+ - 1/2^+$     |              |            |          |             |       |     |           |
|                     | 0            | 72         | -1       | -4          | -9    | 693 |           |
| $5/2^+ - 1/2^+$     | 2186         | 4          | 77       | 17          | -288  | 33  | 2047      |
| $1/2^+ - 1/2^+$     | 0            | -20        | 418      | 0           | -82   | -3  | 3886      |
| $7/2^+ - 5/2^+$     | 3565         | -2          | 418      | 0           | -82   | -3  | 3886      |

set

$$\Delta = 0.430 \quad S_\Lambda = -0.015 \quad S_N = -0.390 \quad T = 0.030 . \quad (22)$$

This parameter set is chosen to reproduce the $\Lambda^7$Li spectrum which it does quite well, as can be seen by comparing the energies in the last column of Table 7 with the experimental energies at the left of Fig. 2. Note that an increase in one or both of the ‘small’ parameters $S_\Lambda$ and $T$ could reduce the excited-state doublet splitting to the experimental value of 471 keV. Also that these two parameters have to take small values if they have the same signs as in (22). Tighter constraints on these parameters come from the spectra of heavier p-shell hypernuclei (see later).

Returning to the LS limit, the coefficient of $\Delta$ for the ground-state doublet, derived from (11), is $3/2$. A similar evaluation using the LS structure of the members of the excited-state doublet gives $7/6$ for the coefficient of $\Delta$. In this case, the full expression for the splitting of the excited-state doublet is [9]

$$\Delta E = \frac{7}{6} \Delta + \frac{7}{3} S_\Lambda - \frac{14}{5} T . \quad (23)$$

This expression can be derived in a variety of ways using the results in Appendix A or Appendix B but perhaps most easily by multiplying the coefficients of $\Delta$, $S_\Lambda$ and $T$ for the $7/2^+$ state in Table 7 for which twice the $2^-$ two-body matrix element [66] or [67] enters because the angular momenta are stretched for the $7/2^+$ state, by $7/3$ because their contribution measures the shift from the centroid $2^+_\Lambda$ of the $7/2^+$ and $5/2^+$ levels.

The $\langle p_N s_A | V | p_N s_\Sigma \rangle$ matrix elements were calculated from the YNG interaction SC97f(S) of [17] using harmonic oscillator wave functions with $b = 1.7$ fm. These matrix elements were multiplied by 0.9 to simulate the $\Lambda-\Sigma$ coupling of SC97e(S) and thus the observed doublet splitting for $\Lambda^4$He (see [17]). In the same parametrization as for the AN interaction

$$\nabla = 1.45 \quad \Delta' = 3.04 \quad S'_\Lambda = -0.085 \quad S'_N = -0.085 \quad T' = 0.157 . \quad (24)$$
The YNG interaction has non-central components but the dominant feature is a strong central interaction in the $^{3}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!
\[ H \Psi = E \Psi \quad \Psi = \sum_i a_i \Phi_i , \]  

(26)

where \( \Phi \) could be expressed in \( jj \) coupling

\[ \Phi = |p_m^{(J_1 T_1)} p_n^{(J_2 T_2)}; JT \rangle \]  

(27)
or LS coupling (the Wigner supermultiplet scheme)

\[ \Phi = |p^n[f]KLSJT \rangle . \]  

(28)

The purpose of the present section is to illustrate the structure of p-shell nuclei in terms of the latter basis. This basis turns out to be very good in the sense that the wave functions for low-energy states are frequently dominated by one basis state, or just a few basis states. This aids in the physical interpretation of the structure. From the hypernuclear point of view, the contributions of \( \Delta \) and \( S_\Lambda \) depend only on the intensities of L and S in the total wave function and these can be obtained in the weak-coupling limit from a knowledge of the core wave function in an LS basis.

\( H \) is defined by two single-particle energies and 15 two-body matrix elements. In terms of the relative coordinates of a pair of nucleons, s, p and d states are possible for two p-shell nucleons. There are 6 central matrix elements (one in each relative state for \( S = 0, 1 \)) which are attractive in spatially even states and repulsive in odd states. The central part of the Millener–Kurath interaction \([28]\) (a single-range Yukawa interaction with \( b/\mu = 1.18 \), potential strengths in MeV) illustrates this point (the superscripts are \( 2T + 1 \) and \( 2S + 1 \)).

\[ V_{11}^{11} = 32.0 \quad V_{31}^{31} = -26.88 \quad V_{13}^{13} = -44.8 \quad V_{33}^{33} = 12.8 \]  

(29)

There are 6 vector matrix elements, 2 arising from spin-orbit interactions in triplet p and d states and 4 from antisymmetric spin-orbit (ALS) interactions that connect two-body states with \( S = 0 \) and \( S = 1 \) (these are not present in the free interaction for identical baryons). Finally, there are 3 tensor matrix elements in triplet p and d states and connecting triplet s and d, states.

The above approach is exemplified by the classic Cohen and Kurath \([25]\) fits to p-shell energy levels in terms of a constant set of single-particle energies and two-body matrix elements. The assumption of an A-independent interaction is a reasonable one for the p shell because the rms charge radii of p-shell nuclei are rather constant, as shown in Table 5. This is basically because the p-shell nucleons become more bound as more particles are added to the shell and the rms radii of the individual orbits tend to stabilize as nucleons are added.

Cohen and Kurath obtained three different interactions by fitting binding energies relative to \(^4\text{He}\) after the removal of an estimate for the Coulomb energy. These interactions were designated as \((8–16)2\text{BME}, (6–16)2\text{BME, and (8–16)POT}\) where the mass ranges fitted are specified and POT means that the 4 ALS matrix elements out of the 17 parameters were set to zero.
course of the hypernuclear studies described here (and for other reasons) many fits have been made to a modern data base of p-shell energy-level data. Only well determined linear combinations of parameters (considerably less than 17) defined by diagonalizing \( \frac{\partial \chi^2}{\partial x_i} \partial x_j \) have been varied where \( \chi^2 \) measures the deviation of the theoretical and experimental energies in the usual way and the \( x_i \) are the parameters. The fit69 and fit5 interactions in Table 5 are examples fitted to data on the \( A = 6 - 9 \) nuclei with only the central and one-body interactions varied for fit69 and with the tensor interaction and the one-body spin-orbit splitting fixed for fit5.

In the supermultiplet basis, \([f]\)KL label representations of SU3 \( \supset \) R3 in the orbital space and \([\tilde{f}]\beta \)TS label representations of SU4 \( \supset \) SU2 \( \times \) SU2 in the spin-isospin space. Actually, \([f] = [f_1 f_2 f_3] \) labels representations of U3 with \( f_1 \geq f_2 \geq f_3 \) and \( f_1 + f_2 + f_3 = n \) and can be represented pictorially by a Young diagram with \( f_1 \) boxes in the first row, \( f_2 \) in the second and \( f_3 \) in the third. For a totally antisymmetric wave function, \([\tilde{f}] \) must be the conjugate of \([f]\) and is obtained by interchanging the rows and columns of the Young diagram. The length of the rows is then restricted to four. In an oscillator

### Table 8. Root-mean-square charge radii (fm) of stable p-shell nuclei

|    | \(^6\)Li | \(^7\)Li | \(^9\)Be | \(^{10}\)B | \(^{11}\)B | \(^{12}\)C | \(^{13}\)C | \(^{14}\)C | \(^{15}\)N |
|----|---------|---------|--------|--------|--------|--------|--------|--------|--------|
| 2.57 | 2.41 | 2.52 | 2.45 | 2.42 | 2.47 | 2.44 | 2.56 | 2.52 | 2.59 |

### Table 9. Quantum numbers for p-shell nuclei

| U3 | SU3 | L | U4 | (TS) |
|----|-----|---|----|------|
| [2] (20) | 0.2 | [11] (01)(10) |
| [11] (01) | 1 | [2] (00)(11) |
| [3] (30) | 1.3 | [111] (\(\frac{1}{2} \frac{1}{2} \frac{1}{2}\)) |
| [21] (11) | 1.2 | [21] (\(\frac{1}{2} \frac{1}{2} \frac{1}{2}\)) |
| [111] (00) | 0 | [3] (\(\frac{1}{2} \frac{1}{2} \frac{1}{2}\)) |
| [4] (40) | 0,2,4 | [1111] (00) |
| [31] (21) | 1,2,3 | [21] (01)(10)(11) |
| [22] (02) | 0,2 | [22] (00)(11)(02)(20) |
| [211] (10) | 1 | [31] (01)(10)(11)(12)(21) |
| [41] (31) | 1,2,3,4 | [1] (\(\frac{1}{2} \frac{1}{2}\)) |
| [32] (12) | 1,2,3 | [22] (\(\frac{1}{2} \frac{1}{2} \frac{1}{2}\)) |
| [311] (20) | 0.2 | [311] (\(\frac{1}{2} \frac{1}{2} \frac{1}{2}\)) |
| [221] (01) | 1 | [32] (\(\frac{1}{2} \frac{1}{2} \frac{1}{2}\)) |
| [42] (22) | 0.2,3,4 | [11] (01)(10) |
| [411] (30) | 1.3 | [2] (00)(11) |
| [33] (03) | 1.3 | [222] (00)(11) |
| [321] (11) | 1.2 | [321] (01)(10)(11)(02)(20)(12)(21) |
| [222] (00) | 0 | [33] (01)(10)(12)(21)(03)(30) |
basis, there is one quantum per particle in the p-shell and \([f]\) labels also the symmetries of the quanta and the wave functions have an SU3 symmetry labelled by \((\lambda \mu) = (f_1 - f_2 f_2 - f_3)\) with \(K\) and \(L\) given by

\[
K = \mu, \mu - 2, \ldots, 1 \text{ or } 0 \\
L = K, K + 1, \ldots, K + \lambda \\
L = \lambda, \lambda - 2, \ldots, 1 \text{ or } 0 \text{ if } K = 0 .
\]

For convenience, the allowed quantum numbers for \(p^n\) configurations are given in Table 9. For \(12 - n\) particles, the \(L\) and TS quantum numbers are the same and \((\lambda \mu) \rightarrow (\mu \lambda)\).

In the following subsections, the spectra of selected p-shell nuclei for \(6 \leq A \leq 14\) are presented and discussed in relation to the supermultiplet structure of their p-shell wave functions (see the tabulations covering the energy levels of light nuclei [29] for more experimental information). Many of these nuclei form the nuclear cores of hypernuclei discussed in detail in later sections.

### 5.1 The Central Interaction

The central interaction gives the bulk of the binding energy in p-shell nuclei. It turns out to be essentially diagonal in the supermultiplet basis and can be represented by 5 SU4 invariants.

\[
H = 1.56 n - 1.79 \sum_{i<j} I_{ij} - 3.91 \sum_{i<j} P_{ij} + 0.59 \mathbf{L}^2 - 1.08 \mathbf{S}^2 + 0.59 \mathbf{T}^2 
\]

Here, the term linear in \(n\) includes the centroid energy of the \(p_3/2\) and \(p_1/2\) orbits at \(A = 5\) and takes care of the (constant) one-body terms that arise from \(\mathbf{L}^2\), \(\mathbf{S}^2\), and \(\mathbf{T}^2\). The two-body identity operator counts the number of pairs \(n(n-1)/2\) and the space-exchange operator counts the difference between the numbers of spatially symmetric and antisymmetric pairs \(n_s - n_a\) given by

\[
\langle [f] | \sum_{i<j} P_{ij} | [f]\rangle = \frac{1}{2} \sum_i f_i(f_i - 2i + 1) .
\]

A rule of thumb is that this can be read off the Young diagram by summing the number of pairs for each row and subtracting the number of pairs for each column. The relationship of the space-exchange operator to the quadratic Casimir operator for SU4 is also worth noting,

\[
\sum_{i<j} P_{ij} = 2i - \frac{1}{8} i^2 - \frac{1}{2} C(\text{SU4}) ,
\]

where

\[
C(\text{SU4}) = \frac{1}{2} \sum_{i<j} \mathbf{\sigma}_i \cdot \mathbf{\sigma}_j \mathbf{\tau}_i \cdot \mathbf{\tau}_j + \mathbf{S}^2 + \mathbf{T}^2 + \frac{9}{4} n .
\]
Since there are only 6 independent central matrix elements, there are only 6 independent operators to represent them. The remaining one would connect the space and spin-isospin spaces, e.g. $l_i \cdot l_j \tau_i \cdot \tau_j$. All the operators in (31) are SU3 scalars except for $L^2$ which transforms as a mixture of (0 0) and (2 2) tensors. An SU3 tensor expansion of the central interaction contains four scalars and two (2 2) tensors, so that the remaining operator has to have a (2 2) part to it. If the coefficient of this extra operator is zero, the Hamiltonian...
in (31) represents the entire effect of the central interaction throughout the shell (or for the range of nuclei fitted).

The decomposition in (31) comes from a fit to the $A = 10 - 12$ nuclei and the coefficient of the extra $(2,2)$ tensor is very small. Figure 3 shows the binding energies given by (31). The dashed lines show the energies for the full Hamiltonian in the cases of $^{10}\text{Be}$, $^{11}\text{B}$, and $^{12}\text{C}$. The gain is about 4 MeV in each case and is mostly due to turning on the spin-orbit interaction. The circled numbers give the differences in the expectation value of the space-exchange component ($n_s - n_a$) for successive spatial symmetries. Four times the coefficient of the space-exchange operator in (31) is $\sim 15$ MeV which is very close to the energy of the first $T=1$ states in $^{12}\text{C}$. Thus, it is evident that the SU4 invariant part of the central interaction gives a rather good account of the general structure of p-shell nuclei. The spin-orbit interaction, which transforms as $(1,1)$ mixes spatial symmetries and $L$ values. As will be seen, the interplay between the spin-orbit and tensor interactions can be very important and is quite subtle.

5.2 Structure of $^6\text{Li}, ^7\text{Li},$ and $^8\text{Be}$

The energy level schemes of these nuclei are shown in Fig. 4. All the levels shown can be accounted for by p-shell calculations. The lowest levels ($T=0$ for $^6\text{Li}$) have well-developed $\alpha + d$, $\alpha + t$, and $\alpha + \alpha$ cluster structures. For harmonic oscillator radial wave functions, coordinate transformations can be made on the states with maximal spatial symmetry so that all the quanta associated with the p-shell orbits reside on the relative coordinate between clusters formed from internal $0s$ wave functions (and the center of mass is in a $0s$ state). These states must transform as $(\lambda 0)$ where $\lambda$ is the number of quanta. Oscillator shell-model configurations beyond the p-shell are required to improve the radial behavior of the relative wave functions.

Wave functions for $^6\text{Li}$ have been given in Table 5. It can be seen that LS coupling is rather good and that the $3^+; 0$, $2^+; 0$ and $1^+_2; 0$ states form a triplet with $L=2$, and $S=1$. Both vector and tensor forces can contribute to the splitting of this triplet. The most natural explanation is that the splitting is mainly due to the one-body spin-orbit interaction, partly because the even-state spin-orbit interaction acts in relative $d$ states and the matrix elements are small in a G-matrix derived from a realistic NN interaction. This is not necessarily the case for a fitted interaction. For example, the Cohen and Kurath interactions have small one-body spin-orbit terms and substantial even-state and antisymmetric spin-orbit terms that act in part like a one-body spin-orbit interaction with a strength that depends linearly on $n$ and ensures that the p-hole states at $A=15$ are split by just over 6 MeV. The small quadrupole moment of $^6\text{Li}$ (experimentally $-0.082$ fm$^2$) provides a constraint on the balance of spin-orbit and tensor interactions. Writing

$$^{6}\text{Li}(gs) = a^3 S_1 + b^3 D_1 + c^1 P_1$$

(35)
Fig. 4. Energy-level schemes for $^6\text{Li}$, $^7\text{Li}$, and $^8\text{Be}$ with the dominant spatial symmetry $[f]$, equivalently ($\lambda$, $\mu$), indicated for groups of levels and L and S for particular levels. All energies are in MeV.

leads to

$$Q(^6\text{Li}) = e^0 b^2 \left( \frac{4}{\sqrt{5}} \alpha\beta + \gamma^2 - \frac{7}{10} \beta^2 \right), \quad (36)$$

where $b \sim 1.7$ fm is the oscillator parameter and $e^0 \sim 0.815$ is the isoscalar effective charge $(1 + \delta e_p + \delta e_n)/2$. The direct tensor interaction coupling the $^3S$ and $^3D$ states gives $\beta < 0$ while indirect coupling through the $^1P$ state via the spin-orbit interaction gives $\beta > 0$. Putting in numbers from Table 5 shows that a small negative value for $\beta$ is required. The B(M1; 2$^+ ; 1 \rightarrow 1^+ ; 0$) = $(8.3 \pm 1.5) \times 10^{-2}$ W.u. puts a similar restriction on $\beta$; briefly, the orbital contribution connecting $^3D$ to $^1P$ and the spin contribution connecting $^3P$ to $^1P$ are of the same sign while a spin contribution of the opposite sign from $^1D$ to $^3D$ cannot be too large if the B(M1) is to be reproduced. The interplay of spin-orbit and tensor interactions in leading to small but important wave function admixtures is a common feature in p-shell nuclei, most famously in the case of the very hindered $^{14}\text{C}(\beta^-)$ decay (see later).

In $^7\text{Li}$, the lowest four states form the ground-state band and have $> 93\%$ purity of the indicated LS configurations. The first T = 3/2 has a similar
purity of [21] symmetry with \( L = 1 \) and \( S = 1/2 \). An interesting point is that the second \( 5/2^- \) state has a small width for decay into the \( \alpha + t \) channel despite its proximity to the first \( 5/2^- \) level which has a large decay width into this channel [29]. This means that the mixing matrix element between the [3] and lowest [21] symmetry \( 5/2^- \) states has to be small. Only a tensor interaction can connect the dominant components shown in Fig. 4 and this has to largely cancel with the spin-orbit contribution arising from a modest [21] \( ^2D \) component in the second state.

The hypernucleus \(^8\Lambda\)Li (also \(^8\Lambda\)Be) was frequently observed in emulsion studies [2] and analysis of the characteristic decay mode \(^8\Lambda\)Li \( \rightarrow \pi^- + ^4\text{He} + ^4\text{He} \) (37) established a ground-state spin-parity of \( 1^- \) and provided information on the mixing of configurations based on the ground-state and first-excited state of \(^7\text{Li} \). The configuration mixing is larger than usual because core states are close together and share the same \( L \) value. Both the ground-state spin of \(^8\Lambda\)Li and the mixing provide restrictions on the nature of the \( \Lambda N \) effective interaction. To be studied by \( \gamma \)-ray spectroscopy, the \( \Lambda = 8 \) hypernuclei have to be formed by particle emission from a heavier hypernucleus.

The lowest \( 0^+ \) and \( 2^+ \) states of \(^8\text{Be} \) form the core for bound states of \(^9\Lambda\)Be (discussed in detail later). The \( 0^+ \) state is unbound by 92 keV and has a width of \( \sim 6 \) eV while the \( 2^+ \) state has a width of \( \sim 1.5 \) MeV. In the p-shell model, these states have very pure [4] symmetry with a few percent of [31] symmetry with \( S = 1 \). Because the Gamow-Teller operator cannot change spatial quantum numbers, it is these small admixtures in the \( 2^+ \) wave function that account for the \( \beta \) decays of \(^8\text{Li} \) and \(^9\text{B} \). The near degeneracies of pairs of \('[31]' states with the same \( J^\pi \), different isospin, and similar space-spin wave functions lead to isospin mixing that is especially strong for the 16.63-MeV and 16.92-MeV \( 2^+ \) levels. The \( T = 1 \) analogs of these levels form the basis for the ground-state doublets of \(^9\Lambda\)Li and \(^9\Lambda\)B. The \(^9\Lambda\)Li hypernucleus has been studied recently via the \(^9\text{Be} (e,e'K^+)^9\Lambda\)Li reaction [30].

The ground-state binding energies of the nuclei in Fig. 4 increase rapidly with the number of particles because the Pauli principle permits up to two neutrons and two protons to correlate strongly in spatially even states and take advantage of the strong central interaction in relative \( s \) states, as quantified in Sect. 5.1.

### 5.3 Structure of \(^9\text{Be} \) and \(^{11}\text{C} \)

Partial energy level schemes of \(^9\text{Be} \) and \(^{11}\text{C} \) are given in Fig. 5. These nuclei are paired together because, with \( p^5 \) and \( p^7 \) configurations, they are related by a particle-hole symmetry reflected in the conjugate SU3 representations. Because the \( L \) values for the highest symmetry differ by steps of one (see Table 3), there are often two states with the same \( J \) value. To some extent,
\[ {^9\text{Be}} \]
- \( 14.48 \) \( L=1 \) \( S=1/2 \) \( 5/2^-; 3/2^- \)
- \( 14.39 \) \( L=2 \) \( S=1/2 \) \( 3/2^-; 3/2^- \)
- \( 11.81 \) \( L=1 \) \( S=3/2 \) \( 5/2^- \)
- \( 11.28 \) \( L=2 \) \( S=3/2 \) \( 7/2^- \)

\[ {^{11}\text{C}} \]
- \( 14.09 \) \( 9/2^- \)
- \( 12.58 \) \( 7/2^- \)
- \( 12.51 \) \( 1/2^-; 3/2^- \)
- \( 11.44 \) \( 5/2^- \)

\[
\begin{align*}
\text{Fig. 5.} & \quad \text{Energy-level schemes for } {^9}\text{Be and } {^{11}}\text{C with the dominant spatial symmetry} \quad [f], \quad \text{equivalently} \quad (\lambda, \mu), \quad \text{indicated for groups of levels and } L \text{ and } S \text{ for particular levels. All energies are in MeV. The bullets mark levels strongly populated in proton knockou}\nonumber
\text{(or pickup) from } {^{10}}\text{B and the } {^{14}}\text{N(p, } \alpha{)}{^{11}}\text{C reaction.} \\
\end{align*}
\]

these states can be organized into \( K = 3/2 \) and \( K = 1/2 \) bands. In fact, it was shown a long time ago \[31\] that shell-model states for the p-shell nuclei have a large overlap with states angular momentum projected from a Slater determinant made from the lowest Nilsson model states (restricted to the p-shell and with the same one-body spin-orbit interaction as in the shell-model calculation) for some deformation. The deformations varied smoothly with the number of nucleons and were prolate at the beginning of the shell and oblate at the end of the shell. Thus, the lowest states of \( ^6\text{Li}, ^7\text{Li}, \text{ and } ^{8}\text{Be} \) were obtained by filling the first \( K = 1/2 \) Nilsson orbit. The \( K = 3/2 \) orbit starts to fill at \( ^9\text{Be} \) and the second \( K = 1/2 \) orbit is relatively close in energy. The ground-state of \( ^{11}\text{C} \) (or \( ^{11}\text{B} \)) would have three nucleons (or a hole) in the \( K = 3/2 \) orbit.

The connection to SU3 symmetry is quite close because Elliott \[32\] showed that all the angular momentum states for a given SU3 representation could be projected out of a highest-weight state characterized by numbers of quanta \( N_z = a + \lambda + \mu \) and \( N_\perp = 2a + \mu \) with \( K_L = \mu, \mu - 2, \ldots, 1 \) or 0.
\[ \langle (\lambda \mu) K L L M \rangle = \frac{1}{a(K_L L)} P^L_{MK} \Phi(HW) \, . \] (38)

The highest-weight state is made up of asymptotic Nilsson orbits (no spin-orbit interaction in this case). Something closer to reality can be obtained by projecting from a product of the highest-weight state and an intrinsic-spin wave function \[ |(\lambda \mu) K J L S J M \rangle = \sum_L c(L) \langle (\lambda \mu) K L L S J M \rangle \, , \] (39)

so that a given state with good K in general contains a mixture of L values. In SU3 codes \[ 34 \], the basis of (38) is used with the states orthogonalized with respect to K\textsubscript{L}. The spin-orbit interaction can, and often does, mix L values to produce a good K\textsubscript{J}. It also mixes (\lambda \mu) and S values. For example, for the \textsuperscript{11}C ground state,

\[ |(1 3) K = 3/2 J = 3/2 \rangle = \sqrt{21/26} |L = 1 S = 1/2 \rangle - \sqrt{5/26} |L = 2 S = 1/2 \rangle \, . \] (40)

The CK816 interaction gives 0.7676 and \(-0.4833\) for the coefficients, meaning that K=3/2 accounts for 81.3% out of a total 82.3% \[ 43 \] symmetry. There is 13.5% \[ 421 \] symmetry in the wave function.

An important point to notice is that the \textsuperscript{9}Be ground state is not bound by much with respect to the neutron threshold (the \textsuperscript{9}B ground state is unbound by 185 keV with respect to proton emission). This is an effect of the Pauli principle (embodied in the supermultiplet symmetry) which strongly restricts the way in which an extra p-shell nucleon can interact with a fully occupied orbit (in the Nilsson sense). On the other hand, the 1s\textsubscript{0}d\textsubscript{0} states, which are near zero binding at this mass number, can couple to the \textsuperscript{8}Be core without restriction. In fact, the low-lying positive-parity (1\hbar\omega) states also have a good SU3 symmetry, namely (6 0) (typically \( > 85\% \)) obtained by coupling the (2 0) of the sd-shell nucleon to the (4 0) of the \textsuperscript{8}Be core.

The levels of \textsuperscript{9}Be in Fig. 5 marked by a bullet are strongly excited in proton knockout, or pickup, from \textsuperscript{10}B \[ 29 \]. The strength is governed by a spectroscopic factor which, by definition, is the square of the reduced matrix element of a creation operator connecting the two states involved. The J dependence of the reduced matrix element between basis states of the form \[ 28 \] is contained in a normalized 9\textit{j} symbol via \[ 64 \]. The reduced matrix element that remains is just \( \sqrt{n} \) times a one-particle coefficient of fractional parentage (cfp) which defines how to construct a fully antisymmetric \( n \)-particle state from antisymmetric \((n-1)\)-particle states coupled to the nth particle. Thus

\[ \langle (\lambda \mu) \kappa L S T || a^+ || (\lambda' \mu') \kappa' L' S' T' \rangle \]

\[ = \sqrt{n} \sqrt{\frac{n_f}{n_f}} \langle (\lambda' \mu') \kappa' L' (1 0) 1 || (\lambda \mu) \kappa L \rangle \langle \bar{f} | T' S' | \bar{1} 1/2 1/2 || \bar{f} | T S \rangle \, , \] (41)
where the Clebsch-Gordan coefficients for $SU_3 \supset R_3$ \cite{34,35} and $SU_4 \supset SU_2 \times SU_2$ \cite{35,36} result from applications of the Wigner-Eckart theorem for $SU_3$ and $SU_4$ and the weight factor $n_f/n_f'$ is the ratio of dimensions of representations of the symmetric groups $S_{n-1}$ and $S_n$ \cite{35}. The $[f']$ are found by removing one box from the Young diagram for $[f]$ in all allowed ways. Examples of the weight factors for $^9$Be and $^{10}$B are given in Table 10.

### Table 10. Weight factors for $^9$Be and $^{10}$B

| $^9$Be $\rightarrow$ $^8$Be + n | $\sqrt{n_f/n_f'}$ | $^{10}$B $\rightarrow$ $^9$Be + p | $\sqrt{n_f/n_f'}$ |
|----------------------------------|------------------|---------------------------------|------------------|
| $[41] \rightarrow [4]$          | $\sqrt{1}$       | $[42] \rightarrow [41]$        | $\sqrt{5}$       |
| $\rightarrow [31]$              | $\sqrt{3}$       |                                 | $\sqrt{6}$       |

Because states with different supermultiplet symmetry are widely separated, the one-particle removal strength is in general complex \cite{37}. The same is true for two-particle \cite{38} and three-particle \cite{39} removal but less so for the removal of an $\alpha$ particle \cite{40} because the removed $p^4$ configuration is an $SU_4$ scalar. Pickup reactions provide a powerful way of identifying predominantly p-shell states. Stripping reactions are also very useful but can strongly populate states in which particles reside in higher shells (usually the next shell).

Table 10 shows that one reason why the binding energy of $^9$Be is low with respect to $^8$Be is that 3/4 of the parentage of the $^9$Be ground state goes to highly-excited states of $^8$Be and $^8$Li. The weight factors for $^{10}$B show that the parentage is almost equally divided between states of $[41]$ and $[32]$ symmetry. In fact, the lowest three states seen strongly in knockout are mainly $[41]$ symmetry and the upper two states are mainly $[32]$ symmetry. The upper $7/2^-$ state has a spectroscopic strength that is a factor of two larger than that for the lower $7/2^-$ state \cite{29}. In the pure symmetry limit, this factor is $\sim 7$. The mixing of the two basis configurations needed to obtain the experimentally measured ratio is small. This is another case in which the balance between vector and tensor interactions in the mixing matrix element is important and different p-shell interactions tend to give rather different results for the ratio of strengths for the $7/2^-$ states.

A final observation for $^9$Be is that the 11.81-MeV $5/2^-$ state is fed very strongly in the $\beta^-$ decay of $^9$Li \cite{29} because it has largely the same spatial quantum numbers as the initial state. In fact, the B(GT) value is much larger than one would expect, perhaps because of difficulties in analysing the $\alpha + \alpha + n$ final state. The analogous $\beta^+$ decay of $^9$C \cite{29} has close to the strength expected from shell-model calculations.

As expected, the $^{11}$C ($^{11}$B) spectrum shows many similarities to the $^9$Be spectrum. The positive-parity states are now more bound with respect to the nucleon threshold and, indeed, $^{11}$Be has a $1/2^+$ ground state 0.32 MeV below the $1/2^-$ state. Because two particles can be promoted to the sd shell without
breaking up the $^8\text{Be}$ core, $(sd)^2$ states are found quite low in energy, starting with the 8.10-MeV $3/2^-$ level. One-neutron removal from $^{12}\text{C}$ is limited to the first two $3/2^-$ states and the first $1/2^-$ state ([44] → [43] is unique). However, triton removal from $^{14}\text{N}$ via the $^{14}\text{N}(p, \alpha)^{11}\text{C}$ reaction [41], and aided by the $^3\text{D}$ character of the $^{14}\text{N}$ ground state, strongly populates all the $T=1/2$ p-shell states included in Fig. 5 (for theory, see [39]).

5.4 Structure of $^{10}\text{B}$ and $^{10}\text{Be}$

Energy-level schemes of $^{10}\text{B}$ and $^{10}\text{Be}$ are given in Fig. 6. All the negative-parity states are shown. They are low in energy for the same reason that positive-parity states come low in $^9\text{Be}$. Now, low-lying $(sd)^2$ states are possible because two p-shell nucleons that are strongly affected by the Pauli principle

\[
\begin{align*}
9.60 & \quad \frac{2^+}{3^+} \quad 9.27 \quad \frac{4^-}{\text{K=0}} \\
9.40 & \quad \frac{2^+}{3^+} \quad 7.37 \quad \frac{3^-}{7.54 \quad \frac{2^+}{\text{K=2}}} \\
5.96 & \quad \frac{2^+}{\text{K=2}} \quad 6.26 \quad \frac{2^-}{6.18 \quad \frac{0^+}{\text{K=0}}} \\
3.37 & \quad \frac{2^+}{\text{K=0}} \quad 10\text{Be} \\
0 & \quad 0^+ \quad 6.87 \quad \frac{1^-}{7.56 \quad \frac{0^+;1}{7.95 \quad \frac{0^+;1}{\text{K=0}}}} \\
6.03 & \quad \frac{4^+}{6.56 \quad \frac{4^-}{5.92 \quad \frac{2^+}{6.13 \quad \frac{3^-}{5.18 \quad \frac{1^+}{\text{K=2}}}}}} \\
4.77 & \quad \frac{3^+}{5.11 \quad \frac{2^-}{5.95 \quad \frac{1^+}{\text{K=2}}}} \\
3.59 & \quad \frac{2^+}{2.15 \quad \frac{1^+}{1.74 \quad \frac{0^+;1}{0.72 \quad \frac{1^+}{0 \quad \frac{3^+}{\text{(22)}}}}}} \\
& \quad 1\omega \quad \frac{1^+}{2\omega} \quad \frac{2^+}{\text{(51)}} \quad \frac{2^+}{\text{(80)}}
\end{align*}
\]

\text{Fig. 6.} Energy-level schemes for $^{10}\text{B}$ (bottom) and $^{10}\text{Be}$ (top). All energies are in MeV. All states have mainly [42] spatial symmetry except for the 9.60-MeV $2^+$ level of $^{10}\text{Be}$, which has mainly [33] symmetry. The neutron and $\alpha$ thresholds in $^{10}\text{Be}$ are at 6.812 MeV and 7.410 MeV. The $\alpha$, deuteron, and proton thresholds in $^{10}\text{B}$ are at 4.461 MeV, 6.027 MeV, and 6.586 MeV.
can be promoted to the sd shell without breaking up the $|4\rangle$ symmetry for the first four p-shell nucleons. Shell-model calculations show that all the states (except one) have the highest spatial symmetry and are dominated by the leading SU3 symmetries, as indicated in Fig. 9.

The structure of the p-shell states of $^{10}\text{B}$ is interesting and is important for hypernuclear physics because $^{10}\text{B}$ forms the core for $^{11}\Lambda\text{B}$ which has been studied with the Hyperball detector. Six $\gamma$ rays were observed but not all of them can be placed in a decay scheme. Even for those that can be placed with reasonable certainty, there are some puzzles (see later).

The $(2\,2)$ representation of SU3 contains two $L = 2$ states [see Table 9 or (30)] and this is the only case in the p-shell for which the $K_L$ quantum number is required. For $^{10}\text{Be}$, $S = 0$ and the $K$ assignments are clear and understandable in terms of two particles in the $K = 3/2$ Nilsson orbit or one each in the $K = 3/2$ and $K = 1/2$ orbits (these orbits have $K_L = 1$ and $K_S = \pm 1/2$). For $^{10}\text{B}$, $S = 1$ and the $K_L = 0$ states are the 0.72-MeV $1^+$ state with $L = 0$ and the $L = 2$ triplet of states at 2.15, 3.59, and 4.77 MeV. The ground state has $K = 3$, and mostly $L = 2$, and is connected by a very strong E2 transition to the $4^+$ level at 6.03 MeV, there being a predicted but unobserved $5^+$ level at higher energy. The 5.92-MeV $2^+$ level is mainly $L = 2$ with $K_L = 2$ and in this sense is part of a triplet involving the $3^+$ ground state and a $1^+$ configuration predicted at higher energy. This triplet has the property of being very strongly split by the spin-orbit interaction while the $K_L = 0$ triplet remains much more compact. Electromagnetic transitions in $^{10}\text{B}$ have been investigated in great detail in the past [29] and it is from various selection rules that the $K$ quantum numbers can be assigned. In particular, strong isovector M1 transitions must connect states with the same $K_L$.

### 5.5 Structure of $^{12}\text{C}$, $^{13}\text{C}$, and $^{14}\text{N}$

The energy level schemes of $^{12}\text{C}$, $^{13}\text{C}$, and $^{14}\text{N}$ are given in Fig. 7. These nuclei are in a sense the particle-hole conjugates of the nuclei shown in Fig. 4. However, the effective spin-orbit interaction, indicated by the more than 6 MeV separation of the single-hole states of $^{15}\text{N}$ and $^{15}\text{O}$, is much larger. The larger spin-orbit interaction tends to break the supermultiplet symmetry. Nevertheless, the content of the highest symmetry in the “ground-state” bands is typically > 70% and often higher.

There are now an increasing number of “intruder” levels marked by dashed lines. In $^{12}\text{C}$, they include the Hoyle state at 7.65 MeV which is certainly not accounted for in shell-model calculations up to $2\hbar\omega$. The negative-parity states are, however, quite well accounted for in $1\hbar\omega$ shell-model calculations and have dominantly $|44\rangle$ symmetry and $(3\,3)$ SU3 symmetry. The $0^+; 2$ state is known to have a large, or even dominant, $(sd)^2$ component.

In $^{13}\text{C}$, the extra p-shell nucleon is not well bound with respect to $^{12}\text{C}$, the neutron threshold being at 4.95 MeV (cf. $^9\text{Be}$ vs. $^8\text{Be}$) and positive-parity states, again unhindered by the Pauli principle, appear at low energies. The
Fig. 7. Energy-level schemes for $^{12}\text{C}$, $^{13}\text{C}$, and $^{14}\text{N}$. All energies are in MeV. For $^{12}\text{C}$ and $^{13}\text{C}$, the club signs identify the members of the ground-state bands with the dominant symmetry indicated. For $^{14}\text{N}$, the bullets indicate a triplet of states with $^3D$ two-hole configurations. Dashed lines indicate non p-shell states. The lowest particle thresholds are $\alpha$ at 7.367 MeV in $^{12}\text{C}$, neutron at 4.946 MeV in $^{13}\text{C}$, and proton at 7.551 MeV in $^{14}\text{N}$.

8.86-MeV and 11.75-MeV levels are the lowest states with the $[432]$ symmetry of the 15.11-MeV $3/2^-$ state, while the 9.90-MeV level is the lowest $(sd)^2$ state.

In $^{14}\text{N}$, the lowest member of the marked group of predominantly $^3D$ two-hole states has become the ground state with the 3.95-MeV level being the predominantly $^3S$ state. The ground state is also predominantly two $p_{1/2}$ holes (there is an overlap of $\sqrt{20/27}$ with the $^3D$ configuration). The structure of the $^{14}\text{N}$ ground state is the important factor in the slowness of the $^{14}\text{C} \beta^-$ decay which is hindered by about six orders of magnitude compared with a strong allowed decay. Consider the following wave functions for the initial and final states in the $\beta^-$ decay

$$|^{14}\text{C}(0^+; 1)| = 0.7729 \ 1S + 0.6346 \ 3P$$

$$|^{14}\text{N}(1^+; 0)| = -0.1139 \ 3S + 0.2405 \ 1P - 0.9639 \ 3D$$

(42)

The Gamow-Teller matrix element is proportional to

$$\sqrt{3}a(1S) a(3S) + a(1P) a(3P)$$

(43)
and for the wave functions above the matrix element is $\simeq 0$. This is because the tensor interaction, essentially $\langle s|V_T|d\rangle$, was chosen to ensure the cancellation and kept fixed during a p-shell fit. This is another case where the spin-orbit interaction alone gives the wrong sign for the $^3S$ amplitude and a tensor interaction gives the opposite sign (see [42] for the history). Keeping the tensor interaction fixed leads to improvements in most of the cases for which the balance of tensor and vector interactions is important.

The above cancellation of the Gamow-Teller matrix element also plays an important role in the analogous M1 transition in $^{14}\Lambda N$. The absence of the normally dominant spin contribution to an isovector M1 transition leads to a rather small B(M1) dominated by the orbital contribution. This turns out to be important for understanding the properties of $^{15}\Lambda N$ which has been studied with the Hyperball.

Finally, the intruder positive-parity levels of $^{14}\Lambda N$ shown in Fig. 7 are of $(sd)^2$ character, as one would expect from the presence of the low-energy positive-parity states in $^{13}C$ and $^{13}N$, and in analogy to the $A = 10$ nuclei. The $2^+; 1$ levels have long been known to be of strongly mixed p-shell and $(sd)^2$ character.

6 The p-shell Hypernuclei

The structure of $^7\Lambda Li$ has already been discussed in Sect. 4 because, as an introduction to p-shell hypernuclei, it is a simple case with a $p^2 0^+ Li$ core that is amenable to hand calculation. This example was also used to compare and contrast the effects of $\Lambda-\Sigma$ coupling in the s-shell and p-shell hypernuclei.

Following the survey of p-shell structure in terms of the LS-coupling supermultiplet basis in Sect. 5 this section is devoted to presenting the results obtained with the Hyperball on heavier p-shell hypernuclei and giving interpretations in terms of the underlying p-shell structure and effective $YN$ interactions. The hypernuclei for which results have been obtained with the Hyperball in experiments at KEK and BNL are $^7\Lambda Li$, $^9\Lambda Be$, $^{10}\Lambda B$, $^{11}\Lambda B$, $^{12}\Lambda C$, $^{15}\Lambda N$, and $^{16}\Lambda O$. For $^{16}\Lambda O$, the calculation is a particle-hole calculation and for $^{15}\Lambda N$, the calculation is similar to that for $^7\Lambda Li$ in that there are two p-shell holes instead of two p-shell particles.

6.1 The Shell-Model Calculations

The Hamiltonian

$$H = H_N + H_Y + V_{NY},$$

and the weak-coupling basis were introduced in [20] and [21]. The formalism for the hypernuclear shell-model calculations is presented in Sect. 3.1 of [23] but some of the basic formulae are given here for completeness. The $YN$ interaction can be written in terms of products of two creation and two annihilation
operators with coefficients that are essentially the two-body matrix elements. The $a^+ a^+ a a$ product can be recoupled in any convenient order using any convenient coupling scheme. In the present case, it is convenient to write the operator in terms of $a^+ a$ pairs for the nucleons and hyperons so that we have a zero-coupled product of operators for separate spaces for which the matrix elements may be separated using the formulae in Appendix A. Formally, \( \alpha \) where

$$V = \sum_{\alpha} C(\alpha) \left( a_{J_\alpha}^+ \bar{a}_{J_\alpha} \right)^{J_\alpha T_\alpha} \left( a_{J_\alpha}^+ \bar{a}_{J_\alpha} \right)^{J_\alpha T_\alpha}^{00},$$  \tag{45}$$

where $\alpha$ stands for all the quantum numbers and the properly phased annihilation operators are given by

$$a_{j m \frac{1}{2} m_1} = (-)^{j - m + \frac{1}{2} - m_1} \bar{a}_{j - m \frac{1}{2} - m_1},$$  \tag{46}$$

and

$$C(\alpha) = \sum_K \left( \begin{array}{ccc} j_N & j_Y & K \\ j_N' & j_Y' & K \\ J_\alpha & J_\alpha & 0 \end{array} \right) \left( \begin{array}{ccc} 1/2 t_Y & T \\ 1/2 t_Y' & T \\ T & T \end{array} \right) \times \hat{K} \hat{T} \langle j_{N} j_{Y} t_{Y}; K T | V | \psi_{J_{\alpha} J_{\alpha} T_{\alpha}} \rangle \langle j_{N} j_{Y} t_{Y}; K T \rangle.$$  \tag{47}$$

Then

$$\langle \psi_{J_{\alpha} J_{\alpha} T_{\alpha}} | V_{N_{-Y_{-T}}^{T_{-J}}} | \psi_{J_{\alpha} J_{\alpha} T_{\alpha}} \rangle = \sum_{\alpha} C(\alpha) \left( \begin{array}{ccc} J_c' & J_c & J_c \\ J_f & J_f & J_f \end{array} \right) \left( \begin{array}{ccc} T_{\alpha} & T_{\alpha} & T_{\alpha} \\ 0 & 0 & 0 \end{array} \right) \hat{K} \hat{T} \langle \psi_{J_{\alpha} J_{\alpha} T_{\alpha}} | v_{J_{\alpha} J_{\alpha} T_{\alpha}} \rangle \langle \psi_{J_{\alpha} J_{\alpha} T_{\alpha}} \rangle.$$  \tag{48}$$

The basic input from the p-shell calculation is thus a set of one-body density-matrix elements between all pairs of nuclear core states that are to be included in the hypernuclear shell-model calculation. As noted in Sect. 4, experimental energies are used for the diagonal core energies where possible.

The one-body transition density that governs the cross section for the formation of a particular hypernuclear state is (see Sect. 3.2 of \[43\])

$$\langle p^{n-1} \psi_{J_{\alpha} J_{\alpha} T_{\alpha}} | V_{N_{-Y_{-T}}^{T_{-J}}} | p^n \psi_{J_{\alpha} J_{\alpha} T_{\alpha}} \rangle \Delta J_{1/2}$$  \tag{49}$$

An important result is that in the weak-coupling limit the total strength for forming the states in a weak-coupling multiplet (summing over $J_f j_L$) is proportional to the pickup spectroscopic factor from the target \[43\]. To see the consequences of the spin-flip characteristics of the reaction used to produce the hypernuclear states, it is useful to change the coupling from $(j_N j_L)$ to $(l_N l_L) \Delta L \Delta S \Delta J$ using \[62\].
6.2 The $^9\Lambda$Be Hypernucleus

The bound-state spectrum for $^9\Lambda$Be is shown in Fig. 8, which gives the $\gamma$-ray energies from an analysis of the BNL E930 data [44, 45], for the parameter set in (22) used for $^7\Lambda$Li. An earlier experiment with NaI detectors [11] observed a $\gamma$ ray at 3079(40) keV and put an upper limit of 100 keV on the doublet splitting.

Fig. 8. Energy levels of $^9\Lambda$Be and the $^8$Be core. The small shifts due to $\Lambda$–$\Sigma$ coupling are shown in the center. All energies are in keV. The measured $\gamma$-ray energies are 3024(3) and 3067(3) keV giving a doublet separation of 43(5) keV [45].

The breakdown of the doublet splitting is given in Table 11. In the LS limit for $^8$Be, the $2^+$ wave function has $L = 2$ and $S = 0$. Then, only the coefficient of $S_\Lambda$ survives and takes the value $-5/2$ as can be seen from an equation analogous to (11) with $S_c$ replaced by $L_c$. In the realistic case, the contributions of $S_\Lambda$ and $T$ work against those from $\Delta$ and the $\Lambda$–$\Sigma$ coupling (small in this case because the $\Sigma$ has to be coupled to $T = 1$ states of the core with a different symmetry from the $T = 0$ states). A similar thing happens for the excited-state doublet of $^7\Lambda$Li and the experimental results for both doublets restrict the combined effect of $S_\Lambda$ and $T$ to be small.

The parameter set chosen puts the $3/2^+$ state above the $5/2^+$ state but the order is not determined by this experiment. However, in the 2001 run of BNL E930 on a $^{10}$B target, only the upper level is seen following proton emission from $^{10}\Lambda$B. It can then be deduced that the $3/2^+$ state is the upper

| $0^+$ | $2^+$ | $3/2^+$ | $5/2^+$ |
|---|---|---|---|
| $^8$Be | $\Lambda\Sigma$ | $^9\Lambda$Be |

Table 11. Contributions from $\Lambda$–$\Sigma$ coupling and the spin-dependent components of the effective $\Lambda N$ interaction to the $3/2^+$, $5/2^+$ doublet spacing in $^9\Lambda$Be. The spectrum is shown on the right hand side of Fig. 8. As in Table 8, the first line gives the coefficient of each parameter and the second line gives the actual energy contributions in keV.

| $\Lambda\Sigma$ | $\Delta$ | $S_\Lambda$ | $S_N$ | $T$ | $\Delta E$ |
|---|---|---|---|---|---|
| $-0.033$ | $-2.467$ | $0.000$ | $0.940$ | |
| $-8$ | $-14$ | $37$ | $0$ | $28$ | $44$ |
Fig. 9. Proton decay of $^{10}_Λ$B to $^9_Λ$Be. Formation strengths for non-spin flip production in the $(K^- , \pi^-)$ reaction are given on the right for two p-shell models. Thresholds for particle decay of the $^{10}_Λ$B states are given on the left. All energies are in MeV.

member of the doublet via the following reasoning. Four states of $^9$B are strongly populated by neutron removal from $^{10}_Λ$B and the hypernuclear doublets based on these states are shown in Fig. 9. The structure factors which govern the population of these states are given at the right of the figure for two p-shell interactions. As discussed in Sect. 5.3, the relative neutron pickup strength to the two $7/2^-$ states which give rise to the $3^-/4^-$ doublets above the $^3_Λ$Be$^+ + p$ threshold is very sensitive to the non-central components of the p-shell interaction. Formation of the $3^-$ states is favored for the dominant $p_{3/2}$ removal by the coupling to get $\Delta L = 1$ and $\Delta S = 0$. The proton decay arises from $^9$B$(7/2^-) \to ^8$Be$(2^+) + p$ in the core. The $4^-$ states proton decay to $^9$Be$(5/2^+)$ and from the recoupling $(2^+ \times p_{3/2})7/2^- \times s_\Lambda \to (2^+ \times s_\Lambda)J_f \times p_{3/2}$, governed by

$$(-)^{3/2+J_f-3} U(3/2 2 3 1/2, 7/2, J_f),$$

one finds that the $3^-$ states proton decay to the $3/2^+$ and $5/2^+$ states in the ratio of 32 to 3. Overall, the $3/2^+$ state is favored by a factor of more than 3. The only caveat to this argument is that the uppermost $3^-$ state doesn’t $\alpha$ decay too much.
6.3 The $^{16}_{\Lambda}$O Hypernucleus

At small angles in the $^{16}$O($K^-, \pi^-)$ reaction used for BNL E930, $p^{-1}p_{\Lambda}$ $0^+$ states are strongly excited at about 10.6 and 17.0 MeV in excitation energy along with a broad distribution of $s^{-1}s_{\Lambda}$ strength centered near 25 MeV [46]. These levels can decay by proton emission (the threshold is at $\sim 7.8$ MeV) to $^{15}_{\Lambda}N$ via $s^4p^0(sd)s_{\Lambda}$ components in their wave functions. The low-lying states of $^{15}_{\Lambda}N$ shown in Fig. 10 can be populated by s-wave or d-wave proton emission and higher energy negative-parity states by p-wave emission.

The cross section for the $0^+$ states drops rapidly with increasing angle while the $\Delta L = 1$ angular distribution rises to a maximum near 10° [43]. The population of the excited $1^-$ state is optimized by selecting pion angles near this maximum. The aim of the experiment was to observe $\gamma$-rays from the excited $1^-$ state to both members of the ground-state doublet and thus measure the doublet splitting. The doublet splitting is of interest because it depends strongly on the tensor interaction. For a pure $p_{1/2}s_{\Lambda}$ configuration, the combination of parameters governing the doublet splitting is [9]

\[
\begin{align*}
25.4 & \quad 0^+ \\
 s^1_{1/2}s_{1/2\Lambda} & = \sqrt{4/5}s^3p^{12}s_{\Lambda} + \sqrt{1/5}s^4p^{10}(02)(sd)s_{\Lambda} \\
17.1 & \quad 0^+ \\
p^{-1}_{5/2}p_{3/2\Lambda} + \varepsilon s^4p^{10}(sd)s_{\Lambda} \\
12.7 & \quad 3/2^+ \\
12.1 & \quad 1/2^+ \\
10.3 & \quad 1/2^+; 1 \\
10.6 & \quad 0^+ \\
p^{-1}_{1/2}p_{1/2\Lambda} + \varepsilon s^4p^{10}(sd)s_{\Lambda} \\
15_{\Lambda}O + \Lambda & \approx 12.5 \\
15_{\Lambda}N + p & \approx 6.5 \\
p^{-1}_{5/2}s_{1/2\Lambda} \\
0 & \quad 0^-, 1^-
\end{align*}
\]

Fig. 10. The energies of $1^-$ and $0^+$ states of $^{16}_{\Lambda}$O that are strongly populated in the $^{16}$O($K^-, \pi^-)$ reaction [46] are shown in the center. All energies are in MeV. The dominant components of the wave function are shown together with the smaller admixtures that permit proton emission to states of $^{15}_{\Lambda}N$. 

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**Note:** The provided text includes a mathematical expression that seems to be cut off and not fully visible. The full expression is likely intended to be a quantum mechanical wave function or a set of wave functions. The context suggests it relates to the states of $^{16}_{\Lambda}$O hypernucleus and their decay properties, which are a key aspect of the study described in the text.
\[ E(1^-) - E(0^-) = -\frac{1}{3}\Delta + \frac{4}{3}S_\Lambda + 8T. \] (51)

The measured values of the γ-ray energies \[47\] are 6533.9 keV and 6560.3 keV (with errors of \(\sim 2\) keV), giving 26.4 keV for the splitting of the ground-state doublet. Including recoil corrections of 1.4 keV to the γ-ray energies gives 6562 keV for the excitation energy of the 1\(^-\) state.

The breakdown of the contributions to the energy spacing in \(^{16}\)\(^\Lambda\)O from the shell-model calculation is given in Table 12 for the parameter set

\[ \Delta = 0.430 \quad S_\Lambda = -0.015 \quad S_N = -0.350 \quad T = 0.0287. \] (52)

These were obtained by starting with the parameter values in \(22\) and changing T to fit the measured ground-state doublet spacing of \(^{16}\)\(^\Lambda\)O and \(S_N\) to fit the excitation energy of the excited 1\(^-\) level. The most important feature of the ground-state doublet splitting is the almost complete cancellation between substantial contributions from \(T\) and \(\Delta\) (aided by \(\Lambda-\Sigma\) coupling). There is thus great sensitivity to the value of \(T\) if \(\Delta\) is fixed from other doublet spacings.

### Table 12. Energy spacings in \(^{16}\)\(^\Lambda\)O. \(\Delta E_C\) is the contribution of the core level spacing. The first line in each case gives the coefficients of each of the \(\Lambda N\) effective interaction parameters as they enter into the spacing while the second line gives the actual energy contributions to the spacing in keV

| \(J^\pi_i - J^\pi_f\) | \(\Lambda\Sigma\) | \(\Delta\) | \(S_\Lambda\) | \(S_N\) | \(T\) | \(\Delta E\) |
|-------------------|----------------|-------------|-------------|--------|------|-----------|
| \(1^- - 0^-\)     | \(-0.380\)     | 1.376       | -0.004      | 7.858  |      |           |
|                   | \(-0\)         | -161        | -21         | 1      | 226  | 27        |
| \(1^-_2 - 1^-_1\) | \(-0.240\)     | -1.252      | -1.492      | -0.720 |      |           |
|                   | 6176           | -103        | 19          | 522    | -21  | 6535      |
| \(2^- - 1^-_2\)   | 0.619          | 1.376       | -0.004      | -1.740 |      |           |
|                   | 81             | 266         | -21         | 1      | -50  | 292       |

Since Ref. \[17\] was published, another peak has been found at 6758 keV with a statistical significance of 3\(\sigma\). The most likely interpretation is that it corresponds to the \(2^- \rightarrow 1^-\) transition. The \(2^-\) level has to be excited by a weak spin-flip transition and it is possible that states based on nearby levels of \(^{15}\)O, shown in Fig. \[11\], could also be weakly excited. Accepting the first explanation puts the \(2^-\) state at 6786 keV and implies a splitting of 224 keV for the excited-state doublet. This is smaller than the 292 keV given in Table 12 for value of \(\Delta\) used for \(^7\)\(^\Lambda\)Li. Reducing \(\Delta\) from 0.43 MeV to 0.33 MeV reduces the doublet splitting to 238 keV. A scaling of two-body matrix elements as \(\sim A^{-0.3}\) is expected for heavier nuclei but for p-shell nuclei it is a more delicate question as could be anticipated from the discussion of Table 8. More evidence for a smaller value of \(\Delta\) in the latter half of the p shell comes from doublet splittings in \(^{15}\)\(^\Lambda\)N and \(^{11}\)\(^\Lambda\)B.
Fig. 11. Energy levels of $^{16}\Lambda\rm{O}$ and the $^{15}\Lambda\rm{N}$ core. The shifts due to $\Lambda-\Sigma$ coupling are shown in the center. All energies are in keV.

6.4 The $^{15}\Lambda\rm{N}$ Hypernucleus

As shown in Fig. [10], the high-energy $0^+$ states of $^{16}\Lambda\rm{O}$ populated strongly via the $(K^-,\pi^-)$ reaction at forward pion angles (and $2^+$ states at larger angles) populate states of $^{15}\Lambda\rm{N}$ by proton emission. Three $\gamma$-ray transitions, corresponding to the solid arrows in Fig. [12] have been observed [1, 48]. The measured energies are 2268, 1961, and 2442 keV. The 2268-keV line is very sharp without Doppler correction, indicating a long lifetime compared to the stopping time in the target, and is identified with the transition from the $1/2^+; 1$ level to the $3/2^+$ member of the ground-state doublet. The other two $\gamma$-ray lines are very Doppler broadened and therefore associated with states that have short lifetimes.

The excited-state doublet splitting is calculated to be 637 keV with the parameter set (52). This is much larger than the observed spacing of 481 keV, much like the situation for the excited-state doublet of $^{16}\Lambda\rm{O}$. The results in Fig. [12], Table [13], and Table [14] are calculated with the parameter set

$$\Delta = 0.330, \quad S_A = -0.015, \quad S_N = -0.350, \quad T = 0.0239, \quad (53)$$

where the value of $T$ has been adjusted to fit the observed (26 keV) ground-state doublet spacing of $^{16}\Lambda\rm{O}$.

Table [13] shows the difference between the contributions of $S_N$ for the mainly $p_{3/2}$ and $p_{1/2}^{-1}p_{3/2}$ core states. In LS coupling, the $1^+$ ground state is mainly $^3D$ [42] and the excited $1^+$ state is mainly $^3S$. Looked at in this way, the coefficients of $S_N$ for the last three states in the table arise mainly from the cross terms between the $L = 0$ and $L = 1$ components in the core wave functions. Small changes in the $\Lambda-\Sigma$ coupling interaction can be used to fine tune the energy of the $1/2^+; 1$ state with respect to the $T=0$ states.
Fig. 12. The spectrum of $^{15}_ΛN$ calculated from the parameters in (53). All energies are in keV. The levels of the $^{14}N$ core are shown on the left and the calculated lifetimes and shifts due to $Λ–Σ$ coupling on the right.

Table 13. Contributions of the spin-dependent $ΛN$ terms to the binding energies of the five lowest states of $^{15}_ΛN$ given as the coefficients of each of the $ΛN$ effective interaction parameters. In the $ΛΣ$ column, the gains in binding energy due to $Λ–Σ$ coupling are given in keV (same as in Fig. 12).

| $J^π; T$ | $ΛΣ$ | $Δ$ | $S_Λ$ | $S_N$ | $T$ |
|----------|------|-----|-------|-------|-----|
| $3/2^+; 0$ | $-56$ | $-0.283$ | $0.780$ | $1.800$ | $2.903$ |
| $1/2^+; 0$ | $-14$ | $0.457$ | $1.457$ | $1.824$ | $-6.053$ |
| $1/2^+; 1$ | $-105$ | $-0.922$ | $0.021$ | $1.816$ | $-0.063$ |
| $1/2^+; 0$ | $-70$ | $-0.915$ | $-0.084$ | $0.447$ | $0.091$ |
| $3/2^+; 0$ | $-9$ | $0.452$ | $0.046$ | $0.481$ | $-0.333$ |

The entries for the ground-state doublet of $^{15}_ΛN$ in Table 13 show a significant shift away from the $jj$-coupling limit with the result that the higher-spin member of the doublet is predicted to be the ground state in contrast to the usual case for p-shell hypernuclei, including $^{16}_ΛO$.

In the weak-coupling limit, the branching ratio for $γ$-rays from the $1/2^+; 1$ state is 2:1 in favor of the transition to the $3/2^+$ final state (the statistical factor from the sum over final states). However, the transition to the $1/2^+$ state is not observed despite the fact that the transition to the $3/2^+$ state is
Table 14. Energy spacings in $^{15}_A N$. $\Delta E_C$ is the contribution of the core level spacing. The first line in each case gives the coefficients of each of the $\Lambda N$ effective interaction parameters as they enter into the spacing while the second line gives the actual energy contributions to the spacing in keV. The first line of the table gives the coefficients for the ground-state doublet in the $jj$ limit.

| $J_i^+$, $J_f^*$ | $\Delta E_C$ | $\Lambda \Sigma$ | $\Delta$ | $S_A$ | $S_N$ | $T$ | $\Delta E$ |
|-----------------|--------------|------------------|--------|------|------|-----|----------|
| $1/2^+$, $3/2^+$ | 0.740        | -2.237           | 0.024  | -8   | -214 | 96  |          |
|                 | 0            | 42               | 244    | 33   | -5   | -71 | 2282     |
|                 | 0.262        | -0.752           | 0.016  | -2.966 |      |      |          |
|                 | 1.367        | 0.130            | 0.034  | -0.424 |      |      |          |
|                 | 0.474        | 0.025            | -1.335 | -0.271 |      |      |          |
|                 | 1635         | 96               | 156    | 0    | 467  | -6  | 2342     |

very clearly observed with over 700 counts. In addition, a lifetime estimate for the $1/2^+$; 1 level is 1.4 ps [48], which is very much longer than the 0.1 ps lifetime of $0^+$; 1 level in $^{14}N$. To understand these facts requires consideration of M1 transitions in $^{14}N$ and $^{15}N$ and this is the subject of the next subsection.

6.5 M1 transitions in $^{14}N$ and $^{15}N$

The effective M1 operator can be written

$$\mu = g_{l}^{(0)}l + g_{s}^{(1)}s + g_{p}^{(1)}p + g_{l}^{(0)}l_{\tau_{3}} + g_{s}^{(0)}s_{\tau_{3}} + g_{p}^{(0)}p_{\tau_{3}},$$

(54)

where $p = [Y^2, s]^1$. The values of the effective $g$ factors that fit the M1 properties of the single-hole states in $^{15}N$ and $^{15}O$, and the states of interest in $^{14}N$ are given, along with the bare $g$ factors, in Table 15.

The B(M1) value is given by

$$B(M1) = \frac{3}{4\pi} \frac{2J_f + 1}{2J_i + 1} M^2,$$

(55)

where

$$M = \langle f || |\mu(0)|| i \rangle + \langle T_f M_T 1 0 | T_f M_T \rangle \langle f || |\mu(1)|| i \rangle.$$

(56)

Table 15. Effective $g$ factors for M1 transitions at the end of the p-shell. See [49] for theoretical estimates.

|           | $g_{l}^{(0)}$ | $g_{s}^{(0)}$ | $g_{p}^{(0)}$ | $g_{l}^{(1)}$ | $g_{s}^{(1)}$ | $g_{p}^{(1)}$ |
|-----------|---------------|---------------|---------------|---------------|---------------|---------------|
| Bare      | 0.500         | 0.88          | 0             | 0.500         | 4.706         | 0             |
| Effective | 0.514         | 0.76          | 0             | 0.576         | 1.420         | 0.96          |
functions for the ground-state doublet will clearly lead to cancellations in the relevant M1 matrix elements because they bring in a large positive matrix element while the M1 matrix element between the large components is small. For the 1
\^\mu_N\rightarrow 0\^\mu_N\) transitions de-exciting the 1\^2\(^N\) state, the < \sigma\tau > matrix element is \sim 0 by construction \(13\) while in the latter it is very strong reflecting the allowed \(^3S\rightarrow ^3S\) nature of the transition. Also, the sign of the two matrix elements is different.

To see what this means for the M1 transitions de-exciting the 1/2\(^+\); 1 state of \(^{12}\Lambda_N\), the most important components of the shell-model wave functions for \(^{12}\Lambda_N\) are listed in Table 17. The small \(^{12}\Lambda_N\); 0 \times s_Λ admixtures in the wave functions for the ground-state doublet will clearly lead to cancellations in the relevant M1 matrix elements because they bring in a large positive matrix element while the M1 matrix element between the large components is small and negative.

The general expression for electromagnetic matrix elements between hypernuclear basis states is

\[
\langle (J_c T_c s_Y t_Y) J_f | M | (J_c' T_c' s_Y' t_Y') J_f' \rangle = \delta_{YY'} \left[ \begin{array}{c} J_c' \\Delta J \\Delta T \\Delta T \cr J_c \end{array} \right] \left[ \begin{array}{c} T_c' \\Delta T \\Delta T \cr T_c \end{array} \right] \langle J_c T_c | M_{\Delta J\Delta T} | J_c' T_c' \rangle + \delta_{cc'} \left[ \begin{array}{c} J_c' \\Delta J \\Delta T \\Delta T \cr J_c \end{array} \right] \left[ \begin{array}{c} T_c' \\Delta T \\Delta T \cr T_c \end{array} \right] \langle s_Y t_Y | M_{\Delta J\Delta T} | s_Y' t_Y' \rangle. \tag{57} \]

The two important \(g\) factors in the hyperonic sector are \(g_Λ = -1.226 \mu_N\) and \(g_{ΛΣ} = 3.22 \mu_N\) (the \(g\) factors for the Σ hyperons are included in the calculations). For hypernuclear doublet transitions in the weak-coupling limit,

\[
\mu = g_c J_c + g_Λ J_Λ = g_c J + (g_Λ - g_c) J_Λ \tag{58} \]

can be used to obtain a simple expression for the matrix element in terms of \(g_Λ - g_c\) as an overall multiplicative factor \(9\).

### Table 16

Contributions to the M1 matrix elements for \(^{14}\Lambda N\) M1 transitions; \(\mu\) is in \(\mu_N\) and \(B(M1)\) is in W.u. (the M1 Weisskopf unit is \(45/8\pi \mu_N^2\)).

| \(J_f^c T_f \) | \(J_f^i T_i \) | \(l \) or \(l\tau\) | \(s \) or \(s\tau\) | \(p \) or \(p\tau\) | \(\mu\) | \(B(M1)\) |
|---|---|---|---|---|---|---|
| \(^1\Lambda N\); 0 | \(^1\Lambda N\); 0 | 0.7461 | -0.3432 | 0 | 0.403 | 0.404 0.328 |
| \(^1\Lambda N\); 0 | 0\(^+\); 1 | -0.5010 | 0.0003 | 0.2556 | 0.025 | 0.026(1) 0.077 |
| \(^2\Lambda N\); 0 | 0\(^+\); 1 | -0.5590 | 3.4857 | 0.0304 | 3.50 | 3.0(9) 4.89 |
| \(^1\Lambda N\); 2\(^+\); 1 | 0.2282 | -4.1491 | 0.1653 | 1.13 | 0.99 | 1.65 |
| \(^2\Lambda N\); 2\(^+\); 1 | 0.1651 | 3.7665 | 0.1884 | 2.26 | 2.29 | 2.64 |
Table 17. Excitation energies and weak-coupling wave functions for $^{15}\Lambda N$

| $J^z T$ | $E_x$ (keV) | Wave function |
|---------|-------------|---------------|
| $3/2^+_1; 0$ | 0 | $0.9985 \times 1^+_1; 0 \times s_\Lambda + 0.0318 \times 1^+_2; 0 \times s_\Lambda + 0.0378 \times 2^+_1; 0 \times s_\Lambda$ |
| $1/2^+_1; 0$ | 96 | $0.9986 \times 1^+_1; 0 \times s_\Lambda + 0.0503 \times 1^+_2; 0 \times s_\Lambda$ |
| $1/2^+_1; 1$ | 2282 | $0.9990 \times 1^+_1; 1 \times s_\Lambda + 0.0231 \times 1^+_2; 1 \times s_\Lambda + 0.0206 \times 2^+_1; 1 \times s_\Lambda$ |
| $1/2^+_2; 0$ | 4122 | $-0.0261 \times 1^+_1; 1 \times s_\Sigma$ |
| $1/2^+_2; 0$ | 4624 | $-0.0333 \times 1^+_1; 0 \times s_\Lambda + 0.9984 \times 1^+_2; 0 \times s_\Lambda + 0.0363 \times 2^+_1; 0 \times s_\Lambda$ |

The important contributions for M1 decays from the $1/2^+; 1$ state in $^{15}\Lambda N$ are shown in Table 18. The strong cancellation resulting from the small $1^+_2 \times s_\Lambda$ admixtures is evident. Even the small $\Sigma$ admixtures contribute to the cancellation. The cancellation is stronger for the transition to the $1/2^+$ member of the ground-state doublet. The reason for this can be seen from Table 19. Namely, the largest contributions to the off-diagonal matrix elements come from $S_\Lambda$ and $T$ and add for the $1/2^+$ state and cancel for the $3/2^+$ state.

Table 18. Important contributions for M1 decays from the $1/2^+; 1$ state in $^{15}\Lambda N$

| $J^z$ | $E_x$ (keV) | Wave function |
|-------|-------------|---------------|
| $1/2^+_1; 1 \rightarrow 3/2^+_1; 0$ | large component | $0.9979 \times 0.9988 \times (-0.251) -0.250$ |
| $1^+_2 \times s_\Lambda$ admixture | $0.0318 \times 0.9988 \times (2.957) +0.095$ |
| $\Sigma$ admixture | +0.011 |
| Partial sum | $-0.137$ |
| $1/2^+_1; 1 \rightarrow 1/2^+_2; 0$ | large component | $0.9983 \times 0.9988 \times (-0.251) -0.250$ |
| $1^+_2 \times s_\Lambda$ admixture | $0.0545 \times 0.9988 \times (2.957) +0.161$ |
| $\Sigma$ admixture | +0.008 |
| Partial sum | $-0.081$ |

Table 19. Coefficients of the AN interaction parameters in the off-diagonal matrix elements between the $1^+_1; 0 \times s_\Lambda$ and $1^+_2; 0 \times s_\Lambda$ basis states in $^{15}\Lambda N$ and the $1^-$ states in $^{16}\Lambda O$. The second line gives the energy contributions in MeV

| $J^z$ | $\Delta$ | $S_\Lambda$ | $S_N$ | $T$ | ME |
|-------|-------|--------|-------|-----|-----|
| $1/2^+$ | 0.1275 | -0.1275 | 0.4851 | -4.0664 |
| $3/2^+$ | -0.0637 | 0.0637 | 0.4851 | 2.0332 |
| $1^-$ | 0.0421 | 0.0019 | -0.1698 | -0.0972 | -0.223 |
| $1^-$ | -0.0210 | -0.0010 | -0.1698 | 0.0486 | -0.143 |
| $1^-$ | 0.4714 | -0.4714 | 0.1412 | 0.0338 | 0.196 |
Finally, the M1 transition data for $^{16}_{\Lambda}$O and $^{15}_{\Lambda}$N are collected in Table 20, mainly to emphasize the weakness of the M1 transitions from the $1/2^+; 1$ level of $^{15}_{\Lambda}$N.

Table 20. M1 transition strengths in $^{16}_{\Lambda}$O and $^{15}_{\Lambda}$N

| $J_f^\pi; T_f$ | $J_i^\pi; T_i$ | $E_\gamma$ (keV) | B(M1) (W.u.) | $\gamma$ branch (%) | lifetime |
|----------------|----------------|-----------------|--------------|---------------------|---------|
| 0$^+_1; 1/2$   | 1$^+_2; 1/2$   | 6562            | 0.336        | 72.5                | 0.24 fs |
| 1$^-_1; 1/2$   | 1$^+_2; 1/2$   | 6535            | 0.129        | 27.5                |         |
| 0$^-_1; 1/2$   | 1$^+_1; 1/2$   | 26              | 0.176        | weak                | 10 ns   |
| 3/2$^+_1; 0$   | 1/2$^+_1; 1$   | 2268            | $4.55 \times 10^{-3}$ | 86        | 0.51 ps |
| 1/2$^+_1; 0$   | 1/2$^+_1; 1$   | 2172            | $8.89 \times 10^{-4}$ | 14        |         |
| 3/2$^+_1; 0$   | 1/2$^+_1; 0$   | 96              | 0.240        | weak/$\gamma$      | 150 ps  |
| 1/2$^+_1; 1$   | 3/2$^+_1; 0$   | 2442            | 1.133        | 96.9                | 1.9 fs  |
| 1/2$^+_1; 0$   | 1/2$^+_1; 0$   | 1961            | 1.080        | 97.4                | 3.8 fs  |

6.6 The $^{10}_{\Lambda}$B, $^{12}_{\Lambda}$C, and $^{13}_{\Lambda}$C Hypernuclei

It was noted in Sect. 5.3 that $^9$Be/$^9$B and $^{11}$B/$^{11}$C have similar structure, as is evident from Fig. 5. The hypernuclei $^{10}_{\Lambda}$B and $^{12}_{\Lambda}$C will have $2^-/1^-$ ground-state doublets with $1^-$ as the ground state (this is known experimentally for $^{12}_{\Lambda}$B [2]). However, there are considerable differences in how these levels can be studied experimentally. In $^{10}_{\Lambda}$B, only the states of the ground-state doublet are particle-stable because, as Fig. 9 shows, the neutron threshold is at 2.00 MeV while the proton threshold is at 9.26 MeV in $^{12}_{\Lambda}$C. Fig. 9 also shows that the $2^-$ state of $^{10}_{\Lambda}$B is populated by non-spin-flip transitions from the $3^+$ ground state of $^{10}$B. The resulting $\gamma$-ray transition was first searched for in $^{10}$B without success, an upper limit of 100 keV being put on the doublet spacing (in BNL E930, the transition was also looked for and not found at roughly the same limit). In $^{12}_{\Lambda}$C, it is the $1^-$ ground state that is populated by non-spin-flip transitions from a $^{12}$C target and the doublet spacing is best investigated by looking for transitions from higher bound states of $^{12}_{\Lambda}$C. This approach was tried in KEK E566 and the data is still under analysis.

The similarity of the contributions from the spin-dependent $\Lambda N$ interaction to the two ground-state doublets is shown in Table 21 for a calculation using the parameters of (22) and the fitted p-shell interaction used for Fig 5. If the parameters of (53) are used the ground-state doublet spacings for $^{10}_{\Lambda}$B and $^{12}_{\Lambda}$C drop to 121 keV and 150 keV, respectively.

The most notable point to be taken from Table 21 is that the $\Lambda-\Sigma$ coupling increases the doublet spacing in $^{12}_{\Lambda}$C and reduces it in $^{10}_{\Lambda}$B. The reason for this is that spin-spin matrix element for the $\Lambda N$ interaction depends on an isoscalar one-body density-matrix element of the nuclear spin operator for the core while the corresponding matrix element for $\Lambda-\Sigma$ coupling depends
Table 21. Coefficients of the \( \Lambda N \) interaction parameters for the \( 2^-/1^- \) ground-state doublet separations of \( ^{10}_\Lambda B \) and \( ^{12}_\Lambda C \). The energy contributions from \( \Lambda - \Sigma \) coupling and the doublet splitting \( \Delta E \) are in keV

| \( \Lambda \Sigma \) | \( \Delta \) | \( S_\Lambda \) | \( S_N \) | \( T \) | \( \Delta E \) |
|------------------|---------------|-------------|-------------|-------|-------------|
| \( ^{12}_\Lambda C \) | 58 | 0.540 | 1.44 | 0.046 | -1.72 | 191 |
| \( ^{10}_\Lambda B \) | -15 | 0.578 | 1.41 | 0.013 | -1.07 | 171 |

on an isovector one-body density-matrix element of the nuclear spin operator for the core [see (48)]. The isoscalar and isovector matrix elements are both large but they have opposite relative sign for the two hypernuclei (this is a type of particle-hole symmetry for the \( K = 3/2 \) Nilsson orbit). The coupling matrix elements are broken down in Table 22. The “diagonal” matrix elements involving the \( 3/2^- \) core states contain a contribution of 1.45 MeV from \( \bar{V}' \) (24) and the contribution from \( \Delta' \) produces the shifts from this value. If it were not from the contribution to the energy shifts from the \( 1/2^- \times \Sigma \) configuration (the \( 1/2^- \) and \( 3/2^- \) core states both have \( L = 1 \)), there would be a much larger effect on the relative ground-state doublet spacings in \( ^{10}_\Lambda B \) and \( ^{12}_\Lambda C \).

Table 22. Matrix elements (in MeV) coupling \( \Sigma \) configurations with the members of the \( 3/2^- \times \Lambda \) ground-state doublets in \( ^{10}_\Lambda B \) and \( ^{12}_\Lambda C \). The energy shifts caused by these couplings are given in keV

| \( J^\pi \) | \( 3/2^- \times \Sigma \) | \( 1/2^- \times \Sigma \) | \( \Delta \Sigma \) shift |
|----------|----------------|----------------|----------------|
| \( ^{10}_\Lambda B \) | 1^- | 0.55 | 1.47 | 34 |
|          | 2^- | 1.95 |    | 49 |
| \( ^{12}_\Lambda C \) | 1^- | 1.92 | -1.35 | 98 |
|          | 2^- | 1.13 |  | 40 |

Apart from the effect of \( \Lambda - \Sigma \) coupling, several of the coefficients in Table 21 are sensitive to the model of the p-shell core. For example, the ground states of the core nuclei tend to be characterized by a good \( K \) value and this involves a mixing of \( L \) values as noted in, and following, (40). For \( L = 1 \), the coefficient of \( \Delta \) contributing to the doublet spacing is \( 2/3 \) whereas for \( L = 2 \) the coefficient is \(-2/5 \). For the wave function in (40), the coefficient is the \( 6/13 \sim 0.46 \). The CK816 interaction gives a coefficient close to this value and the results of a calculation for \( ^{12}_\Lambda C \) with this interaction and the parameter set (22) are shown in Fig. 13 and Table 23.

The non-observation of the ground-state doublet spacing in \( ^{10}_\Lambda B \) is an important problem. A number of p-shell interactions give a smaller coefficient for \( \Delta \) (due to more \( L \) mixing) or larger coefficient of \( T \), both of which lead to a reduction in the doublet spacing. A better understanding of how the pa-
Fig. 13. The spectrum of $^{12}_\Lambda C$ calculated from the parameters in (22). The levels of the $^{11}C$ core are shown on the left and the calculated shifts due to $\Lambda$–$\Sigma$ in the center. All energies are in keV.

Table 23. Energy spacings in $^{12}_\Lambda C$. $\Delta E_C$ is the contribution of the core level spacing. The first line in each case gives the coefficients of each of the $\Lambda N$ effective interaction parameters as they enter into the spacing while the second line gives the actual energy contributions to the spacing in keV.

| $J_i^-$ | $J_f^-$ | $\Delta E_C$ | $\Lambda \Sigma$ | $\Delta$ | $S_\Lambda$ | $S_N$ | $T$ | $\Delta E$ |
|--------|--------|-------------|-----------------|--------|---------|------|---|--------|
| $2_1^-$ | $1^-_1$ | 0.463 | 1.518 | 0.030 | -2.078 |
| 0 | 54 | 199 | -23 | -12 | -62 | 143 |
| $1_2^- - 1_1^-$ | 0.315 | 1.150 | -1.104 | 0.635 |
| 2000 | 45 | 136 | -17 | 430 | 19 | 2548 |
| $1_3^- - 1_1^-$ | 0.372 | -0.385 | -1.647 | 0.561 |
| 4804 | 64 | 160 | 6 | 642 | 17 | 5536 |

The parameters vary with mass number and, indeed, whether the parametrization in use is sufficient are also important questions.

There have been many experiments using a $^{12}C$ target [1] and many show excitation strength in the region of the excited $1^-$ states. For example, in the $(\pi^+,K^+)$ reaction with a thin carbon target, the second $1^-$ state is found at 2.5 MeV [4]. Table [23] shows that the $S_N$ parameter is mainly responsible for raising the excitation energy above the core spacing of 2 MeV. Recently, excited states of $^{12}_\Lambda B$ have been observed with better resolution via the $(e,e'K^+)$ reaction [50].
Table 24. Energy spacings in $^{13}\Lambda C$. Coefficients of the $\Lambda N$ effective interaction parameters for the $1/2^+$ ground state and $3/2^+$ excited state are given followed by the difference and the actual energy contributions in keV.

| $J^\pi$ | $\Delta E_C$ | $\Delta \Sigma$ | $\Delta$ | $S_\Lambda$ | $S_N$ | $T$ | $\Delta E$ |
|--------|--------------|-----------------|---------|-------------|------|-----|--------|
| $1/2^+$ | 27           | $-0.016$        | 0.016   | 2.421       | $-0.049$ |     |        |
| $3/2^+$ | 18           | $-0.045$        | $-1.455$| 1.430       | $-0.929$ |     |        |
| $3/2^+ - 1/2^+$ | 4439 | $-0.029$        | $-1.471$| $-0.991$    | $-0.880$ |     |        |
|         | 9            | $-12$           | 22      | 386         | $-28$ | 4803 |        |

A similar effect of $S_N$ is seen in Table 24 for $^{13}\Lambda C$ where the excited $3/2^+$ state built on the 4.44-MeV $2^+$ state of $^{12}\Lambda C$ (cf. Fig. 8) is seen at 4.880(20) MeV in a γ-ray experiment using NaI detectors [51] and at 4.85(7) MeV in KEK E336 via the $(\pi^+, K^+)$ reaction [11]. Note that, as for $^9\Lambda Be$, the effects of $\Lambda - \Sigma$ coupling are small. In contrast to $^9\Lambda Be$, the coefficients of $S_N$ are large. This is because the $^{12}\Lambda C$ core states, while still having dominantly [44] spatial symmetry, have substantial [431] components with $S = 1$ (a low 68% [44] and 25% [431] in the ground state for the WBP interaction [52] used for Table 24 but typically ~ 79% [44] for the Cohen and Kurath interactions).

The 4.88-MeV γ-ray was actually a by-product of an experiment [51] designed to measure the spacing of $1/2^-$ and $3/2^-$ states at ~ 11 MeV in $^{13}\Lambda C$ ($B(\Lambda) = 11.67$ MeV is the lowest particle threshold). To a first approximation, the two states are pure $p_\Lambda$ single-particle states. In this case, and with harmonic oscillator wave functions, the spacing produced by the interaction of the $p_\Lambda$ interacting with the filled s shell is related to $S_\Lambda$ (with a coefficient of $-6$) because both depend on the same Talmi integral $I_1$ [10]. However, as noted above, the $^{12}\Lambda C$ core is not by any means pure $L=0, S=0$ which means that components of the $\Lambda N$ effective interaction other than the $\Lambda$ spin-orbit interaction, particularly the tensor interaction, play important roles in the small spacing of 152 keV [14]. In addition, the loose binding of the $p_\Lambda$ orbit is important (the harmonic oscillator approximation is not good), as is configuration mixing produced by the quadrupole-quadrupole component in the $p_N p_\Lambda$ interaction [14, 43].

6.7 The $^{11}_\Lambda B$ Hypernucleus

The $^{11}_\Lambda B$ hypernucleus has a rather complex spectrum because the $^{10}B$ core has many low-lying p-shell levels, as shown in Fig. 6. The γ-decay properties of these levels have been very well studied [29]. Furthermore, the lowest particle threshold (proton) in $^{11}_\Lambda B$ is at 7.72 MeV which means that the hypernuclear states based on the p-shell states of $^{10}B$ up to 6 MeV or so are expected to be particle stable and thus could be seen via their γ decay if they could be populated strongly enough.
Fig. 14. The spectrum of $^{11}_{\Lambda}$B based on the six observed $\gamma$-ray transitions. All energies are in keV. The placements of the 264-keV, 1482-keV, and 2477-keV transitions are well founded. The placements of the other three $\gamma$-rays are more speculative. The formation factors for the ($\pi^+$, $K^+$) reaction on the left and the lifetimes on the right are from the shell-model calculation.

A shell-model calculation for $^{11}_{\Lambda}$B was made using the p-shell interaction of Barker [53] who made some changes to one of the Cohen and Kurath interactions [25] to improve the description of electromagnetic transitions in $^{10}$B. The strengths for formation via non-spin-flip transitions and the electromagnetic matrix elements for decay were calculated for all the bound p-shell hypernuclear states of $^{11}_{\Lambda}$B (i.e., up to the states based on the 5.92-MeV 2$^+$; 0 level of $^{10}$B). The $\gamma$-ray cascade was followed from the highest levels, summing the direct formation strength and the feeding by $\gamma$ rays from above. The conclusion was that perhaps as many as eight transitions would contain enough intensity to be seen in an experiment with the Hyperball. The formation strengths on the left side of Fig. 14 show that the most strongly formed excited state is expected to be the 3/2$^+$ level based on the 5.16-MeV 2$^+$; 1 state of $^{10}$B, followed by a number of states based on the low-lying 1$^+$; 0 and 0$^+$; 1 states. The lowest 1/2$^+$; 0 level, originally predicted at 1.02 MeV, acts as a collection point for the $\gamma$-ray cascade. The predicted $\gamma$ width at this energy corresponds to a lifetime of $\sim$ 250 ps (the 1$^+$ state of $^{10}$B has to decay by an E2 transition.
Table 25. Contributions of the spin-dependent ΛN terms to the binding energies of the eight levels of $^{11}_ΛB$ shown in Fig. 14 given as the coefficients of each of the ΛN effective interaction parameters. The theoretical excitation energies and the gains in binding energy due to $Λ−Σ$ coupling are given in keV.

| $J^π;T$ | $E_0$ | $ΛΣ$ | $Δ$ | $S_Δ$ | $S_N$ | $T$ |
|---------|-------|-------|-----|-------|-------|-----|
| 5/2⁺;0  | 0     | 66    | −0.616 | −1.377 | 1.863 | 1.847 |
| 7/2⁺;0  | 266   | 11    | 0.409  | 1.090  | 1.890 | −1.512 |
| 1/2⁺;0  | 968   | 71    | −0.883 | −0.116 | 0.746 | 0.243 |
| 3/2⁺;0  | 1442  | 12    | 0.403  | 0.094  | 0.872 | −0.194 |
| 1/2⁺;1  | 1970  | 93    | −0.007 | 0.008  | 1.543 | −0.013 |
| 3/2⁺;0  | 2241  | 46    | −0.266 | 0.754  | 1.536 | −1.264 |
| 1/2⁺;0  | 2554  | 35    | 0.333  | −1.333 | 1.674 | 2.639 |
| 3/2⁺;1  | 5366  | 103   | −0.203 | −1.293 | 1.519 | 0.598 |

and has a lifetime of 1 ns) which is comparable with the expected lifetime for weak decay.

In the subsequent experiment, KEK E518, six $γ$ rays were seen [54, 45]. Figure 14 shows an attempt to construct a level scheme for $^{11}_ΛB$ from a combination of the experimental information and the results of the shell-model calculation. The theoretical energies and the contributions from $YN$ effective interactions are given in Table 25 for the parameter set (53). Apart from the $γ$-ray energies, the experimental information includes relative intensities and some estimate of the lifetimes from the degree of Doppler broadening.

The strongest $γ$ ray in the spectrum was found at 1483 keV and it is very sharp implying a long lifetime. Despite the unexpectedly high energy, it is natural to associate this $γ$ ray with the de-excitation of the lowest $1/2⁺;0$ level. The 2477-keV $γ$ ray shows up after the Doppler-shift correction and it too has a natural assignment in Fig. 14. It is a 1.1 W.u. isovector M1 transition between states with $L=2$ and $K_L=0$. The 264-keV line is now known to be due to the ground-state doublet transition (0.2 W.u.), having been seen following proton emission from $^{12}C$ [55] (this is the reason for showing a calculation using the parameter set (53) with $Δ = 0.33$ MeV). The placement of the other three $γ$ transitions in Fig. 14 is speculative, although the intensities and lifetimes match the theoretical estimates quite well. The $1/2⁺;1 → 1/2⁺;0$ transition is 2.0 W.u. M1 transition between states with $L=0$.

The most glaring discrepancy is that the shell-model calculation greatly underestimates the excitation energies of the two doublets based on the $1⁺;0$ levels of $^{10}B$. From Table 25 it can be seen that $S_N$ does raise the energies of these doublets with respect to the ground-state doublet but not nearly enough. The shell-model calculation is in fact quite volatile with respect to the p-shell wave functions for the $1⁺;0$ core levels. There is also mixing of the members of these two doublets and this is evident from the difference between the coefficients of $S_N$ for the doublet members.
7 Summary and Outlook

The era of Hyperball experiments at KEK and BNL between 1998 and 2005 has provided accurate energies for about 20 γ-ray transitions in p-shell hypernuclei, the number in each hypernucleus being five for $^7\Lambda$Li, two for $^9\Lambda$Be, six for $^{11}\Lambda$B, three for $^{15}\Lambda$N, and three for $^{16}\Lambda$O. Data from the last experiment, KEK 566, on a $^{12}\Lambda$C target using the upgraded Hyperball-2 detector array is still under analysis but there is evidence for one γ-ray transition in $^{12}\Lambda$C and two γ-rays from $^{11}\Lambda$B have been seen following proton emission from the region of the $p_\Lambda$ states of $^{12}\Lambda$C [55]. Several electromagnetic lifetimes have been measured by the Doppler shift attenuation method or lineshape analysis, and many estimates of, or limits on, lifetimes have been made based on the Doppler broadening of observed γ rays. In addition, two measurements of γγ coincidences have been made, for the $7/2^+ \rightarrow 5/2^+ \rightarrow 1/2^+$ cascade in $^7\Lambda$Li (471-keV and 2050-keV γ rays) and the $3/2^+ \rightarrow 1/2^+ \rightarrow 3/2^+$ cascade in $^{15}\Lambda$N (2442-keV and 2268-keV γ rays).

With the exception of transitions in $^{11}\Lambda$B that most likely involve levels based on the two lowest $1^+$ states of $^{10}\Lambda$B, the γ-ray data can be accounted for by shell-model calculations that include both Λ and Σ configurations with p-shell cores. The spin-dependence of the effective ΛN interaction appears to be well determined. The singlet central interaction is more attractive than the triplet as evidenced by the value $\Delta = 0.43$ MeV needed to fit the 692-keV ground-state doublet separation in $^7\Lambda$Li (and the 471-keV excited-state doublet spacing). In $^7\Lambda$Li, the contribution from Λ–Σ coupling is $\sim 12\%$ of the contribution from the ΛN spin-spin interaction in contrast to the $0^+, 1^+$ spacings in the $A = 4$ hypernuclei, where the contributions are comparable in magnitude. The ΛN interaction parameters do exhibit a dependence on nuclear size. For example, the spacings of the excited-state doublets in $^{16}\Lambda$O ($1^-, 2^-$) and $^{15}\Lambda$N ($1/2^+, 3/2^+$), and the ground-state doublet in $^{11}\Lambda$B ($5/2^+, 7/2^+$) require $\Delta \sim 0.32$ MeV. Given $\Delta$, the tensor interaction strength $T$ is well determined ($\sim 0.025$ MeV) by the ground-state doublet ($0^-, 1^-$) spacing in $^{16}\Lambda$O because of the sensitivity provided by a strong cancellation involving $T$ and $\Delta$. The Λ-spin-dependent spin-orbit strength $S_\Lambda$ is constrained to be very small ($\sim -0.015$ MeV) by the excited-state doublet spacing in $^{9}\Lambda$Be ($3/2^+, 5/2^+$). Finally, substantial effects of the nuclear-spin-dependent spin-orbit parameter $S_N \sim -0.4$ MeV, which effectively augments the nuclear spin-orbit interaction in changing the spacing of core levels in hypernuclei, are seen in almost all the hypernuclei studied. The small value of $S_\Lambda$ and the substantial value for $S_N$ mean that the effective LS and ALS interactions have to be of equal strength and opposite sign. The parametrization of the effective ΛN interaction includes some three-body effects (see later) but, if interpreted in terms of YN potential models, the value for $\Delta$ picks out NSC97e,f [5]. As noted in Sect. 3 and Sect. 4, these YN models have the correct combination of spin-spin and Λ–Σ coupling strengths to account for data on $^4\Lambda$He ($^4\Lambda$H) and $^7\Lambda$Li. They also have weak odd-state tensor interactions that give a small positive
value for $T \sim 0.05$ MeV. The LS interaction, which gives rise to $S_+ = (S_A + S_N)/2$, has roughly the correct strength but the ALS interaction is only about one third as strong as the LS interaction, although with the correct relative sign. For the newer ESC04 interactions [6], the ALS interaction is a little stronger and the other components seem comparable to those of the favored NSC97 interactions, except for differences in the odd-state central interaction. The attractive odd-state central interaction of the ESC04 models is favored by some data on $p\Lambda$ states over the overall repulsive interaction for the NSC97 models. The most recent quark-model baryon-baryon interactions of Fujiwara and collaborators [56] also have trouble explaining the small doublet splitting in $^9\Lambda$Be.

The mass dependence of the interaction parameters has been studied by calculating the two-body matrix elements from YNG interactions using Woods-Saxon wavefunctions. This approach requires the assignment of binding energies for the p-shell nucleons and the $s_\Lambda$ orbit. For the nucleons, binding energy effects are not so easy to deal with because, as emphasized in Sect. 5, the nuclear parentage is widely spread because of the underlying supermultiplet structure of p-shell nuclei (the allowed removal of a nucleon generally involves more than one symmetry [f] for the core and states with different symmetries are widely separated in energy). Perhaps the best that can be done is to take an average binding energy derived from the spectroscopic centroid energy for the removed nucleons. This changes rapidly for light systems up to $^8\Lambda$Be, beyond which it remains rather constant.

Any description of the absolute binding energies of p-shell hypernuclei (the $B_\Lambda$ values) requires the consideration of binding-energy effects and the introduction of three-body interactions, real as in Fig. 1 or effective from many-body theory for a finite shell-model space. The empirical evidence for this need is that the $B_\Lambda$ values for p-shell hypernuclei don’t grow as fast as $n\nabla$, requiring a repulsive term quadratic in $n$ (the number of p-shell nucleons) [10]. Also, a description of $\Lambda$ single-particle energies over the whole periodic table requires that the single-particle potential have a density dependence [57], as might arise from the the zero-range three-body interaction in a Skyrme Hartree-Fock calculation. Much of the effect of a three-body interaction is included in the parametrization of the effective two-body $\Lambda N$ interaction. If two or one s-shell nucleons are involved, the three-body interaction contributes to the $\Lambda$ single-particle energy or the effective two-body $\Lambda N$ interaction, respectively. This leaves only the $p^2s_\Lambda$ terms to consider. The real three-body interactions derived from meson exchange present a problem for shell-model calculations in that they possess singular short-range behavior. In the two-body case, this is where the G-matrix or a purely phenomenological treatment come in. For the three-body case, there are too many independent matrix elements to parametrize, although Gal, Soper, and Dalitz [7] have introduced a five parameter representation of the two-pion-exchange three-body interaction.
Given a set of three-body matrix elements, it is certainly possible to include them in the shell-model calculations [7]. Another useful extension of the shell-model codes would be to use the complete 1\hbar\omega space for the non-normal-parity levels of p-shell hypernuclei. Simple calculations for \(p^n p_\Lambda\) configurations have been done [43] but it is important to include the configurations involving 1\hbar\omega states of the core nucleus coupled to an \(s_\Lambda\). This is necessary to permit the exclusion of spurious center-of-mass states from the shell-model basis and to provide the amplitudes for nucleon emission leaving low-lying states of the daughter hypernucleus with a \(\Lambda\) in the \(s_\Lambda\) orbit, as indicated in Fig. 10. The configuration mixing also redistributes the strength from from \(p^np_\Lambda\) states strongly formed in strangeness-exchange or associated-production reactions. In calculating matrix elements involving the \(p_\Lambda\) orbit, it is important to include binding-energy effects by using realistic radial wave functions because the \(p_\Lambda\) orbit becomes bound only at \(A \sim 12\) and the rms radius of the \(\Lambda\) orbit can be \(\sim 4.5\) fm compared with \(\sim 2.8\) fm for the p-shell nucleon.

The next generation of hypernuclear \(\gamma\)-ray spectroscopy experiments using a new Hyperball-J and the \((K^-, \pi^-)\) reaction is being prepared for J-PARC, starting perhaps early in 2009. The day-one experiment [58] will be run at \(p_K = 1.5\) GeV/c. The spin-flip amplitudes are strong in the elementary interaction between 1.1 GeV/c and 1.5 GeV/c and the cross sections for spin-flip vs non-spin-flip strength will be checked by using a \(^4\)He target and monitoring the \(\gamma\) ray from the \(1^+\) excited state of \(^4\)\(^1\)He. Also, the intention is to make a precise measurement of the lifetime of the first-excited \(3/2^+\) state of \(^7\)\(^1\)Li using the Doppler shift attenuation method. For \(^{10}\)\(^3\)B, the ground-state doublet spacing will be determined unless it is smaller than 50 keV. For \(^{11}\)\(^3\)B, the power of a larger and more efficient detector array will be used to sort out the complex level scheme by the use of \(\gamma\gamma\) coincidence measurements. Finally, a \(^{19}\)\(^3\)F target will be used to measure the ground-state doublet spacing in \(^{19}\)\(^3\)F.

The measurement on \(^{19}\)\(^3\)F represents the start of a program of \(\gamma\)-ray spectroscopy on sd-shell nuclei. This will require shell-model calculations for both \(0\hbar\omega\) and \(1\hbar\omega\) sd-shell hypernuclear states. In much of the first half of the sd shell, supermultiplet symmetry, SU3 symmetry, and LS coupling are still rather good symmetries. As a result, there are the same opportunities as in the p shell to emphasize certain spin-dependent components of the effective \(\Lambda N\) interaction by a judicious choice of target. Now there are more two-body matrix elements – 8 for \((sd)s_\Lambda\) – and more sensitivity to the range structure of the \(\Lambda N\) effective interaction.

The experiments just outlined represent the start of a very rich experimental program using Hyperball-J at J-PARC (see Ref. [1]). It should be possible to go to all the way to rather heavy nuclei where the \(p_\Lambda\) orbit is below the lowest particle-decay threshold (this is true for the special case of \(^{12}\)\(^3\)C [51]).
A Basics of Racah Algebra for SU2

The Wigner-Eckart theorem is used in the form

\[ \langle J f M_f | T k q | J_i M_i \rangle = \langle J_i M_i k q | J_f M_f \rangle \langle J_f | T k | J_i \rangle . \]

(59)

The elements of the unitary transformation that defines the recoupling of three angular momenta are given by

\[ \langle (j_1 j_2) J_{12}, J_3 | J \rangle = \sum_{J_{12} J_3} U(j_1 j_2, j_3 J) \langle (j_1 j_2 J_{12}) | J_3 \rangle \langle J_3 | J \rangle \],

(60)

where the U-coefficient is simply related to the W-coefficient and 6j symbol

\[ U(abcd, ef) = \hat{e} \hat{f} W(abcd, ef) = \hat{e} \hat{f} (-)^{a+b+c+d} \{ \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \} . \]

(61)

The reduced matrix elements of a coupled operator consisting of spherical tensor operators that operate in different spaces, e.g. different shells or orbital and spin spaces, are given by

\[ \langle J_1 J_2; J'_1 J' \rangle \langle R^{k_1} S^{k_2} | x \rangle \langle R^{l_1} S^{l_2} | y \rangle = \sum_{J_{12}} \langle J_1 J_2 | T^{k_1} S^{k_2} | J'_{12} J' \rangle \langle J'_1 J'_2 | x \rangle \langle J'_1 J'_2 | y \rangle . \]

(64)

where \( \Gamma \) represents all the angular momentum type quantum numbers such as JT or LST; \( x \) and \( y \) represent the other labels necessary to specify the states spanning a space.
B Two-body Matrix Elements of the ΛN Interaction

Here, the two-body $p_Ns_A$ matrix elements of the ΛN effective interaction in $(1)$ are given in both LS and $jj$ coupling in terms of the parameters $V$, $\Delta$, $S_A$, $S_N$, and $T$ from Table[2]. Actually, the results are given in terms of $S_+$ and $S_-$, the radial matrix elements associated with the symmetric and antisymmetric spin-orbit interactions, respectively, so that $S_A = S_+ + S_-$ and $S_N = S_+ - S_-$. Note that in Ref. [7] the matrix elements are defined to give contributions to the Λ binding energy $B_\Lambda$ and that the order of angular momentum coupling is $(SL)J$ rather than $(LS)J$. In LS coupling,

$$\langle 3P_0|V|3P_0 \rangle = V + \frac{1}{4}\Delta - 2S_+ - 6T$$
$$\langle 3P_1|V|3P_1 \rangle = V + \frac{1}{4}\Delta - S_+ + 3T$$
$$\langle 3P_2|V|3P_2 \rangle = V + \frac{1}{4}\Delta + S_+ - \frac{3}{5}T$$
$$\langle 1P_1|V|1P_1 \rangle = V - \frac{3}{4}\Delta$$
$$\langle 3P_1|V|1P_1 \rangle = -\sqrt{2}S_-$$. \hspace{1cm} (66)

In $jj$ coupling,

$$\langle p_{3/2} 1^-|V|p_{3/2} 1^- \rangle = V - \frac{5}{12}\Delta - \frac{1}{3}S_+ + T - \frac{4}{3}S_-$$
$$\langle p_{3/2} 2^-|V|p_{3/2} 2^- \rangle = V + \frac{1}{4}\Delta + S_+ - \frac{3}{5}T$$
$$\langle p_{1/2} 0^-|V|p_{1/2} 0^- \rangle = V + \frac{1}{4}\Delta - 2S_+ - 6T$$
$$\langle p_{1/2} 1^-|V|p_{1/2} 1^- \rangle = V - \frac{1}{12}\Delta - \frac{2}{3}S_+ + 2T + \frac{4}{3}S_-$$
$$\langle p_{3/2} 1^-|V|p_{1/2} 1^- \rangle = \sqrt{\frac{2}{3}} \{ \Delta - S_+ - S_- + 3T \} \hspace{1cm} (67)$$

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