Knot topology in QCD

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We consider topological structure of classical vacuum solutions in quantum chromodynamics. Topologically non-equivalent vacuum configurations are classified by non-trivial second and third homotopy groups for coset of the color group SU(3) (N = 3) under the action of maximal Abelian stability group. Starting with explicit vacuum knot configurations we study possible exact classical solutions. Exact analytic non-static knot solution in a simple CP^3 model in Euclidean space-time has been obtained. We construct an ansatz based on knot and monopole topological vacuum structure for searching new solutions in SU(2) and SU(3) QCD. We show that singular knot-like solutions in QCD in Minkowski space-time can be naturally obtained from knot solitons in integrable CP^3 models. A family of Skyrme type low energy effective theories of QCD admitting exact analytic solutions with non-vanishing Hopf charge is proposed.

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I. INTRODUCTION

Topological structure of classical solutions in SU(N) Yang-Mills theory implies numerous physical manifestations in such important phenomena in quantum chromodynamics (QCD) as the chiral symmetry breaking and confinement [1]. The most attractive mechanism of the confinement is based on the Meissner effect in dual color superconductor [2–4] where monopole vacuum condensate is generated dynamically due to quantum corrections [5–7]. Dyons represent alternative topological defects which may play an important role as well as monopoles in description of the confinement at zero and finite temperature [8]. Whereas the instanton and monopole solutions correspond to non-trivial topological Chern-Simons and monopole charges, the topological knot configurations with a non-zero Hopf charge represent another topological objects which become essential in various applications in standard QCD and effective Skyrme type theories of QCD in low energy region [9,10]. It has been found that knot solitons could be good candidates for description of glueball states which can be treated as excitations over the condensed vacuum [11,12].

The rich topological structure of QCD as a gauge theory is conditioned by the presence of non-trivial homotopy groups \( \pi_k(SU(N)/H) \), where the stability subgroup \( H \) determines possible coset spaces with different topological properties. The homotopy group \( \pi_1(SU(N)) = \mathbb{Z} \) describes topological classes of instanton field configurations corresponding to topological Pontryagin index [12–13]. It is well known that instantons realize tunneling between topologically non-equivalent vacuums and provide dominant contribution to chiral symmetry breaking. Another example of manifestation of non-trivial topology in QCD is provided by the second homotopy group \( \pi_2(SU(3)/U(1) \times U(1)) = \mathbb{Z} \times \mathbb{Z} \) which implies Weyl symmetric structure of vacuum and singular monopole solutions in SU(3) QCD [16,19].

A nice feature of quantum chromodynamics is that gauge connection (potential) allows natural implementation of the color vector \( \hat{n} \) in adjoint representation of SU(N) within the formalism of gauge invariant Abelian decomposition suggested first in [17,20–22] and developed further in [23,24]. The color vector \( \hat{n} \) corresponds to generators of the Cartan subalgebra of Lie algebra \( su(N) \) and gives a suitable tool for description of whole topological structure of the gauge theory. A crucial observation has been made that the classical vacuum in QCD can be explicitly constructed in terms of the color vector \( \hat{n} \) [26,27]. This immediately implies that classical vacuum is strongly degenerated and all topologically non-equivalent vacuums are classified by non-trivial homotopy groups \( \pi_{2,3}(SU(N)/H) \). In particular, in the case of SU(3) QCD it has been shown that classical vacuum possesses a non-trivial Weyl symmetric structure described by the second homotopy group \( \pi_2(SU(3)/U(1) \times U(1)) \) [19]. It should be stressed, that the color vector \( \hat{n} \) represents pure topological degrees of freedom, so that we have still a standard QCD. In low energy region, in effective QCD theories like a generalized Faddeev-Skyrme model [11], the vector \( \hat{n} \) becomes dynamical. The knowledge of the classical vacuum structure allows to study vacuum excitations in search of possible finite energy topological solutions and make further steps towards understanding fundamental properties of QCD at quantum level.

In the present paper we consider first the topological structure of the classical vacuum in SU(N), \( (N = 2,3) \) QCD and study possible manifestations of topological properties related with the homotopy groups \( \pi_{2,3}(SU(3)/U(1) \times U(1)) \) and \( \pi_{2,3}(SU(2)/U(1)) \).
ing with known exact knot solutions from the integrable sector of $CP^1$ models \cite{28,29} we will construct a vacuum with knot topology and obtain new analytic classical solutions in $CP^1$ model, standard QCD (in Euclidean and Minkowski space-time) and effective Skyrme type theory. The paper is organized as follows. In Section II we describe the general topological structure of the classical vacuum in $SU(2)$ and $SU(3)$ QCD. In Section III we consider a simple $CP^1$ model which can be treated as a restricted QCD with one field variable $\hat{n}$. Exact non-static knot like solution with a finite Euclidean action has been found. Section IV deals with an ansatz for possible topological solutions based on classical vacuum made of $\hat{n}$ with general topology. In Section V we present analytic singular knot like solutions in QCD in Minkowski space-time. A family of generalized Skyrme type effective theories admitting exact solutions with non-trivial Hopf numbers is proposed in Section VI.

II. TOPOLOGICAL STRUCTURE OF CLASSICAL VACUUM IN QCD

1. Topology of $SU(2)$ QCD vacuum

It has been shown that knot configurations providing minimums of the energy functional in Faddeev-Skyrme model may correspond to vacuums of QCD in maximal Abelian gauge \cite{26}. Later it has been proved that topologically non-equivalent classical vacuums in pure QCD can be constructed explicitly in terms of a color vector $\hat{n}^a$ ($a = 1, 2, 3$) \cite{27}. By this, the vacuum pure gauge fields $\hat{A}_\mu$ with different Chern-Simons numbers are in one-to-one correspondence with color fields $\hat{n}$ of respective Hopf charges.

One should stress, that the color vector $\hat{n}$ in $CP^1$ models (as well as in Faddeev-Skyrme theory) represents dynamic field variable whereas in QCD the vector field $\hat{n}$ contains only pure topological degrees of freedom. The most appropriate way how to implement the topological degrees of freedom of $\hat{n}$ into the gauge potential while keeping a standard QCD theory is provided by Cho-Duan-Ge gauge invariant Abelian projection \cite{17,20,21}

$$\hat{A}_\mu = A_\mu \hat{n} + \hat{C}_\mu + \hat{X}_\mu \equiv \hat{A}_\mu + \hat{X}_\mu,$$

where $A_\mu$ and $X_\mu$ are the Abelian and off-diagonal gauge potentials, $A_\mu$ is a restricted part of the gauge potential, and $C_\mu \equiv -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}$ is a magnetic potential. For simplicity we put the coupling constant $g$ equal to one. The vector $\hat{n}$ has a natural origin in the mathematical structure of the gauge theory, it is defined on the coset $G/H$ where the stability group $H$ is defined by Cartan subalgebra generators of the Lie algebra $\mathfrak{g}(G)$. Notice, that there is another type of Abelian decomposition proposed in \cite{23,25} which treats the color vector $\hat{n}$ as a part of the whole gauge potential. So that, such a decomposition leads to a theory different from the standard QCD already at classical level \cite{30}.

The magnetic field strength $\tilde{H}_{\mu\nu}$ constructed from the magnetic gauge potential $\tilde{C}_\mu$ defines the scalar magnetic field $H_{\mu\nu}$

$$\tilde{H}_{\mu\nu} = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu + \tilde{C}_\mu \times \tilde{C}_\nu \equiv H_{\mu\nu} \hat{n}. \quad (2)$$

The magnetic field $H_{\mu\nu}$ defines a closed differential 2-form $H = dx^\mu \wedge dx^\nu H_{\mu\nu}$ which implies the existence of dual magnetic potential $\tilde{C}_\mu$

$$H_{\mu\nu} = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu. \quad (3)$$

An explicit construction of the classical vacuum of QCD in terms of knot configurations of $\hat{n}$ has been found first in \cite{27}

$$\hat{A}_\mu^{\text{vac}} = -\tilde{C}_\mu \hat{n} + \tilde{X}_\mu. \quad (4)$$

This relation establishes connection between a pure gauge potential and color vector field $\hat{n}$ and implies that the classical vacuum configurations can be described by topologically non-equivalent classes of the color field $\hat{n}$. Namely, the topological classes of $\hat{n}$ are determined by two homotopy groups, $\pi_2(SU(2)/U(1))$ and $\pi_3(SU(2)/U(1)) = \pi_3(S^3)$. The first one describes monopole configurations, whereas the second homotopy describes Hopf mapping $\tilde{n} : S^3 \rightarrow S^2$ (we assume that the space $R^3$ is compactified to a three dimensional sphere $S^3$). So that, all topological non-equivalent classical vacuums are classified by Hopf, $Q_H$, and monopole, $g_m$, charges

$$Q_H = \frac{1}{32\pi^2} \int d^3x \epsilon^{ijk} \tilde{C}_i \tilde{H}_{jk},$$

$$g_m = \int_{S^2} \tilde{H}_{ij} \cdot \hat{n} \, d\sigma_{ij}. \quad (5)$$

One can show that Hopf number equals to the Chern-Simons number for vacuum gauge field configurations $A^{\text{vac}}_\mu$ constructed from $\hat{n}$.

To study possible exact solutions in QCD and QCD effective theories we will consider explicit expressions for the color vector $\hat{n}$ with a given knot topology. In particular, we will use known exact analytic knot solutions found in special integrable models. Let us recall first an explicit construction of a simple knot configuration of $\hat{n}$ as a mapping $S^3 \rightarrow S^2$ with unit Hopf charge. Surprisingly, such a simple construction leads directly to exact knot solutions found in $CP^1$ integrable models. Using stereographic projection it is convenient to parameterize the target space $S^2$ by a complex field $u \in C^4$

$$\hat{n} = \frac{1}{1 + uu^*} \left( \begin{array}{c} u + u^* \\ -i(u - u^*) \\ uu^* - 1 \end{array} \right), \quad (6)$$
A three-dimensional sphere $S^3$ is given by embedding into $R^4$ as follows
\[ |z_1|^2 + |z_2|^2 = 1 \] (7)
where $z_1, z_2$ are complex coordinates on the complex plane $C^2$. The Hopf mapping with the Hopf charge $Q_H = 1$ is determined by the following equation
\[ u = \frac{z_1}{z_2}. \] (8)

Starting with a given color vector $\hat{n}$ one can define the magnetic field $H_{\mu\nu}$ explicitly in terms of the complex field $u$
\[ H_{\mu\nu} = \frac{\epsilon^{abc} \hat{n}_a \partial_{\mu} \hat{n}_b \partial_{\nu} \hat{n}_c}{(1 + |u|^2)^2} \left( \partial_{\mu} u \partial_{\nu} u^* - \partial_{\nu} u \partial_{\mu} u^* \right). \] (9)

The dual magnetic potential $\tilde{C}_\mu$, (3) is written through the complex $SU(2)$ doublet $\zeta = (z_1, z_2)$ as follows
\[ \tilde{C}_\mu = -2i\zeta^\dagger \partial_\mu \zeta. \] (10)

For physical implications an explicit realization of Hopf mapping $\hat{n}$ depends on a specified model and topology of four-dimensional space-time. The three-dimensional sphere $S^3$ in the above definition of Hopf mapping can be related to the physical space-time by several ways. We will consider exact solutions in two cases: the first case, when the sphere $S^3$ is treated as embedding into four-dimensional Euclidean space-time $R^4$ (Section III, IV), and the second one, when $S^3$ is obtained from the real physical space $R^3$ by suitable compactification imposing boundary conditions at space infinity for physical fields (Section V, VI). In each case the different realizations of the sphere $S^3$ lead to different topological field configurations.

2. Topology of $SU(3)$ QCD vacuum

The consideration of the topological vacuum structure of $SU(2)$ QCD can be generalized to the case of $SU(3)$ gauge group which possesses a more rich topology. The $SU(3)$ gauge invariant Abelian projection is defined as follows [17, 20]
\[ \tilde{A}_\mu = \hat{A}_\mu + \hat{X}_\mu, \]
\[ \hat{A}_\mu = A_\mu^a \hat{m}_r^a + C_\mu^a, \]
\[ C_\mu^a = -f^{abc} m_r^b \partial_\mu m_r^c \equiv - \left( \hat{m}_r \times \partial_\mu \hat{m}_r \right)^a, \] (11)
where two color octet fields $\hat{m}_r, r = (3, 8)$ correspond to Cartan subalgebra generators, i.e., two generators of the Abelian subgroups $U_3(1), U_8(1)$. The group manifold of $SU(3)$ can be described within the formalism of fiber bundle by several ways. One can realize $SU(3)$ manifold as a fiber bundle with the base $S^8$ and a fiber $S^3$. This is a generalization of the Hopf fibering $S^1 \rightarrow S^3 \rightarrow S^2$ to the case of $SU(3)$ group. Another possibility is to consider the group manifold $SU(3)$ as a fiber bundle over the coset $M^6 \equiv SU(3)/U(1) \times U(1)$ with a fiber as torus $T^2 = S^1 \times S^1$. The coset space $M^6$ itself can be treated as a bundle which is locally isomorphic to $CP^2 \times CP^1$. We will consider the last realization of $SU(3)$ manifold because of its importance in connection with the confinement phenomenon in QCD.

Due to presence of the non-trivial homotopy groups
\[ \pi_2(SU(3)/U(1) \times U(1)) = \pi_2(CP^2) \times \pi_2(CP^1) \]
\[ = Z \times Z, \]
\[ \pi_3(SU(3)/U(1) \times U(1)) = \pi_3(CP^1) = Z, \] (12)
one concludes that nonequivalent topological classes of color vector fields $\hat{n}_{3,8}$ are classified by the Hopf number $Q_H$ and two monopole charges $(m, a)$. As we will see later, the Hopf number is actually determined by two integer winding numbers as well.

Let us construct vacuum configurations in terms of independent complex field variables in analogy with the case of $SU(2)$ QCD. One needs to parameterize the coset $M^6$ by three complex coordinates. To do this one should express the color vectors $\hat{n}_{3,8}$ in terms of two complex triplet fields $\Psi, \bar{\Psi}$ which are projective coordinates on almost complex homogeneous manifold $M^6$. Let us first express the lowest weight vector $\hat{m}_8$ introducing a complex triplet field $\Psi$ that parameterizes the coset $CP^2 \sim SU(3)/SU(2) \times U(1)$ with a maximal stability subgroup
\[ \hat{m}_8 = -\frac{\sqrt{3}}{2} \bar{\Psi} \lambda^a \Psi, \]
\[ \bar{\Psi} \Psi = 1. \] (13)
The definition of the vector $\hat{m}_8$ is consistent with the normalization condition and symmetric $d$—product operation in the Lie algebra of $SU(3)$
\[ \hat{m}_8^2 = 1, \quad d^{abc} \hat{m}_8^a \hat{m}_8^b = -\frac{1}{3} \hat{m}_8^a. \] (14)

To construct a second Cartan vector $\hat{m}_3$ orthogonal to $\hat{m}_8$ it is convenient to define projection operators
\[ P_{\parallel}^{ab} = \hat{m}_8^a \hat{m}_8^b, \quad P_{\perp}^{ab} = \delta^{ab} - \hat{m}_3^a \hat{m}_3^b. \] (15)
With this the vector $\hat{m}_3$ can be parameterized as follows
\[ \hat{m}_3^a = P_{\perp}^{ab} \Phi \lambda^b \Phi = \Phi \lambda^a \Phi + \frac{1}{2} \bar{\Psi} \lambda^a \Psi, \] (16)
where we have introduced a second complex triplet field $\Phi$. The definition of color vectors $\hat{m}_{3,8}$ by Eqs. (13),(16) is invariant under local $\hat{U}(1) \times \hat{U}'(1)$ gauge transformations
\[ \Psi \rightarrow \exp[i\hat{\alpha}(x)]\Psi, \]
\[ \Phi \rightarrow \exp[i\hat{\alpha}'(x)]\Phi, \] (17)
which represent explicitly the dual Abelian magnetic symmetry in $SU(3)$ QCD. The dual magnetic potentials $\tilde{C}_\mu^r$ can be constructed explicitly by means of the complex fields

$$\tilde{C}_\mu^3 = 2i(\tilde{\Phi} \partial_\mu \Phi + \frac{1}{2} \tilde{\Psi} \partial_\mu \Psi),$$

$$\tilde{C}_\mu^8 = 2i(-\frac{\sqrt{3}}{2} \tilde{\Psi} \partial_\mu \Psi).$$

(18)

One can verify that $\tilde{m}_s$ satisfy the following relations

$$\tilde{m}_s^a \tilde{m}_s^a = \delta_{rs}, \quad \delta^{abc} \tilde{m}_s^a \tilde{m}_s^b = \delta_{rsq} \tilde{m}_q^s,$$

(19)

which imply the orthogonality condition $\bar{\Psi} \Phi = 0$ for the complex fields.

For given complex fields $\Psi = (\psi_1, \psi_2, \psi_3)$ and $\Phi = (\phi_1, \phi_2, \phi_3)$ one can introduce complex projective coordinates

$$u_1 = \frac{\psi_1}{\psi_2}, \quad u_2 = \frac{\psi_3}{\psi_2},$$

$$v_1 = \frac{\phi_1}{\phi_2}, \quad v_2 = \frac{\phi_3}{\phi_2}.$$  

(20)

With this one can obtain explicit parametrization for the color vector $\tilde{m}_8$

$$\tilde{m}_8 = \frac{\sqrt{3}}{2(1 + |u_1|^2 + |u_2|^2)} \begin{pmatrix} u_1 + u_1^* \\ i(u_1 - u_1^*) \\ |u_1|^2 - 1 \\ u_1^* u_2 + u_2 u_1 \\ -i(u_1 u_2 - u_2 u_1) \\ u_2 + u_2^* \\ -i(u_2 - u_2^*) \\ \frac{1}{\sqrt{3}} (1 + |u_1|^2 - 2|u_2|^2) \end{pmatrix}.\quad (21)$$

An explicit expression for the vector $\tilde{m}_3$ can be obtained by using Eqn. (16) in a similar manner. Notice, that due to the orthogonality condition $\bar{\Psi} \Phi = 0$ one has an additional constraint

$$1 + u_1^* v_1 + u_2^* v_2 = 0.$$  

(22)

So that, we can choose three independent complex coordinate functions, for instance, $u_1, u_2, v_1$, which contain the whole information on the topology of the space $SU(3)/U(1) \times U(1)$. This can be useful in search of essentially $SU(3)$ solutions with various combinations of Hopf and monopole charges corresponding to topologies of the fields $u_1, u_2, v_1$.

III. EXACT NON-STATIC SOLUTION IN $CP^1$ MODEL

The Lagrangian of a simple $CP^1$ model with $SU(2)$ triplet field $\hat{n}$ coincides formally with the Lagrangian of the restricted QCD with a vanishing "electric" gauge potential $A_\mu$

$$\mathcal{L} = -\frac{1}{4} H_{\mu\nu\rho}(\hat{n}).$$

(23)

We use four-dimensional spherical coordinate system with Euler angles $(0 \leq \theta \leq \pi, \quad 0 \leq \phi, \psi \leq 2\pi)$

$$x_1 = \rho \cos \frac{\theta}{2} \cos \phi,$$

$$x_2 = \rho \cos \frac{\theta}{2} \sin \phi,$$

$$x_3 = \rho \sin \frac{\theta}{2} \cos \psi,$$

$$x_4 = \rho \sin \frac{\theta}{2} \sin \psi.$$  

(24)

To define Hopf fibering one introduces complex coordinates $z_1, z_2$ in the complex plane $C^2$ ($R^4$)

$$z_1 = x_1 + ix_2,$$

$$z_2 = x_3 - ix_4.$$  

(25)

A three-dimensional sphere with the radius $\rho$ is defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \rho^2.$$  

(26)

The Hopf mapping with a unit Hopf charge is defined by

$$u = \frac{z_1}{z_2} = \cos \frac{\theta}{2} e^{i(\phi + \psi)}.$$  

(27)

Surprisingly, that simple configuration with a Hopf number $Q_H = 1$ represents an exact solution to the equation of motion for the Lagrangian $[24]$ when the field $\hat{n}$ is defined on three-dimensional sphere $S^3$ with a fixed radius $\rho = \text{const}$

$$\partial_\mu (\sqrt{g} \partial_\nu u H^{\mu\nu}) = 0.$$  

(28)

For higher Hopf numbers the solution is provided by the following ansatz

$$u = g(\theta)e^{i(m\phi + n\psi)}.$$  

(29)

Substituting the ansatz into Euler-Lagrange equations corresponding to the Lagrangian $\mathcal{L}$ one obtains the following solution $[31]$

$$g(\theta) = \sqrt[4]{1 + \frac{1}{c_2 + \frac{c_1 \log(m^2 + n^2 - (m^2 - n^2) \cos \theta)}{m^2 - n^2}}},$$

$$c_1 = \frac{m^2 - n^2}{\log|m^2| - \log|n^2|},$$

$$c_2 = \frac{\log[2n^2]}{\log|m^2| - \log|n^2|}.$$  

(30)
where the integration constants \((c_1, c_2)\) are fixed by the boundary conditions for the vector \(\hat{n}\)
\[
\hat{n}(\theta = \pm \pi) = (0, 0, \pm 1).
\] (31)

The corresponding Hopf charge is determined by two winding numbers \((m, n)\)
\[
Q_H = mn.
\] (32)

In the case \(m = n = 1\) the solution reduces to the special one, Eq. (27). Notice, the solution (30) can be treated as a "static" soliton in a sense that it does not depend on the radius \(\rho\). This solution coincides exactly with the solution obtained in [31] with the field \(\hat{n}\) defined on three-dimensional sphere of finite radius \(r_0\). In other words, the topology of the Euclidean space-time is chosen to be \(S^3 \times R^1\). The presence of the dimensional parameter \(r_0\) allows to overcome the restrictions of the Derrick theorem to existence of finite energy stable solutions. Notice, that solution (30) does not admit analytical continuation to the Minkowski space-time. Approximate solutions for similar knotted soliton on three-dimensional sphere in Fadeev-Skryme model had been constructed in [32].

In our approach we do not restrict the Euclidean four-dimensional space-time to the topology \(S^3 \times R^1\). To find a non-static solution to equations of motion of QCD in Euclidean space-time \(R^4_E\) we apply the same ansatz which is used in [31]
\[
u = f(\rho, \theta) \exp[i(m\phi + n\psi)].
\] (33)

After proper changing variable \(g(\rho, \theta) = \frac{1}{1 + f^2(\rho, \theta)}\) one results in a linear partial differential equation
\[
\rho \partial_\rho(\rho \partial_\rho g) + \frac{4}{A \sin \theta} \partial_\theta(A \sin \theta \partial_\theta g) = 0,
\]
\[
A \equiv m^2 \sec^2 \frac{\theta}{2} + n^2 \csc^2 \frac{\theta}{2}.
\] (34)

The equation admits separation of variables
\[
g(\rho, \theta) = R(\rho) Y(\theta).
\] (35)

A solution for the radial function is given by
\[
R(\rho) = C_1 \sin[w^2 \log(\mu \rho)] + C_2 \cos[w^2 \log(\mu \rho)],
\] (36)

where \(w^2\) is the separation constant and \(\mu\) is a free mass dimensional parameter. The equation for the angular function \(Y(\theta)\)
\[
\frac{4}{A \sin \theta} \partial_\theta(A \sin \theta \partial_\theta Y(\theta)) - w^2 Y(\theta) = 0.
\] (37)

can be reduced to the Heun type equation. Notice, the angular part of our solution, \(Y(\theta)\), is the same as in [31], however, the radial function \(R(\rho)\) is different. An essential difference is that our solution has a dimensional mass scale parameter \(\mu\) which is completely arbitrary contrary to the case of the solution with a fixed radius \(r_0\) in the \(CP^1\) model on the space with topology \(S^3 \times R^1\).

IV. KNOT ANSATZ FOR SOLUTIONS IN QCD IN EUCLIDEAN SPACE-TIME

One method of constructing solutions is based on deformation of vacuum configurations using an appropriate ansatz with suitable trial functions. Since the classical vacuum can be described in terms of the \(CP^1\) vector field \(\hat{n}\) which possesses non-trivial topological properties, an interesting possibility arises: to obtain new solutions starting with known exact Hopfion solutions in \(CP^1\) integrable model. Another advantage of using the field \(\hat{n}\) with non-trivial topology in studying vacuum excitations appears when one considers \(SU(3)\) QCD which has more rich topological structure. In this section we apply \((m, n)\)-family of known static solutions \(\hat{n}\) [31] to construction of new classical solutions in Euclidean \(SU(2)\) QCD.

We define a following ansatz with three radial trial functions \(f_i(\rho)\)
\[
\vec{A}_\mu = -f_1(\rho)\vec{C}_\mu \hat{n} - f_2(\rho)\hat{n} \times \partial_\mu \hat{n} + f_3(\rho)\partial_\mu \hat{n},
\] (38)

where \(\hat{n}\) describes Hopf mapping with Hopf charge \(Q_H = mn\) and it is defined by the complex function
\[
u = \exp[i(m\phi + n\psi)]|g(\theta; m, n)\).
\] (39)

We consider a simple case when the function \(g(\theta; m, n)\) is defined for equalled winding numbers \(m = n\) [29]
\[
g(\theta; m, n) = \cot \frac{\theta}{2}.
\] (40)

Substituting the ansatz into the full equations of motion of pure QCD
\[
\vec{D}_\mu F^{\mu
u} = 0
\] (41)
one can obtain solutions with finite and infinite energy. We select the most interesting solutions:
1. 't Hooft instanton with Chern-Simons number \(N_{CS} = 1\): \(f_1(\rho) = 0, f_2(\rho) = f_3(\rho)\) all equations of motion reduce to one non-trivial ordinary differential equation:
\[
\nu^2 f_1'' + rf_1' - 4f_1(f_1 - 1)(2f_1 - 1).
\] (42)

The equation has a simple solution
\[
f_1 = \frac{a^2}{a^2 + \rho^2}
\] (43)

which leads to gauge equivalent representation of known 't Hooft instanton. One can check that the Chern-Simons number for the corresponding gauge potential equals exactly to the Hopf number. Notice, that there are several gauge equivalent representations for the known 't Hooft instanton which can be obtained within the ansatz [38].

2. Infinite energy solution with the vector field \(\hat{n}\) with unit Hopf charge: the solution is obtained by using constant valued trial functions \(f_{1,2,3}\). Direct solving all equations of motion leads to the following solution
\[
f_1 = \frac{1}{2}, f_2 = 1 \pm \frac{\sqrt{2}}{4}, f_3 = \pm \frac{\sqrt{2}}{4}.
\] (44)
The corresponding energy density (density of the Euclidean action) takes the form

$$E(\rho) = \frac{3 \sin \theta}{8 \rho}$$ (45)

which coincides exactly with the expression for the energy density of the knot solution with unit Hopf charge on three-dimensional sphere with a finite radius $\rho$. In this case it becomes evident that an exact solution of the $CP^1$ model on three dimensional sphere turns into the singular solution of the Yang-Mills theory.

3. Singular solutions with the vector field $\hat{n}$ with the Hopf charge $Q_H = m^2$ ($m = n$): the ansatz includes two constant valued trial functions $f_2 = 1$, $f_3 = 0$. In that case all equations of motion reduce to one ordinary differential equation

$$m(r^2 f''_1 + rf'_1 - 4(f_1 - 1)) = 0$$ (46)

which has the following solution with an arbitrary parameter $a$ of length dimension

$$f_1 = 1 + c_1 \cosh(2 \log(\frac{r}{a})) + c_2 \sinh(2 \log(\frac{r}{a})).$$ (47)

The energy density of that solution reads

$$E(\rho) = m^2 \rho^3 \sin \theta \frac{\theta}{a^4} \left( (c_1 + c_2)^2 + \frac{a^8 (c_1 - c_2)^2}{r^8} \right).$$ (48)

In special cases $c_1 = \pm c_2$ the solution takes a simple polynomial form.

We have considered a simple ansatz with spherically symmetric trial functions. It would be interesting to apply the ansatz with axially symmetric trial functions to study new finite energy solutions, especially to the search of essentially $SU(3)$ instanton like solutions.

V. SINGULAR SOLUTIONS IN QCD IN MINKOWSKI SPACE-TIME

It is known that Wu-Yang monopole represents a solution to classical equations of motion in $SU(2)$ Yang-Mills theory. Within the formalism of the restricted QCD the Wu-Yang monopole is given by the unit color vector directed along the radius

$$\hat{n}^i = \frac{x^i}{r} \quad (i = 1, 2, 3).$$ (49)

The solution is singular, and it satisfies the equations of motion $\partial^\mu H_{\mu \nu} = 0$ everywhere except the origin. The unit monopole charge of Wu-Yang monopole is provided by the nontrivial homotopy group $\pi_2(SU(2)/U(1))$.

Let us consider a vector $\hat{n}$ with a natural knot topology induced by the definition of the three-dimensional sphere obtained by compactification of the space $R^3$ to three-sphere $S^3$ by identifying all points at infinity. To do this we apply stereographic projection of $S^3$ to $R^3$

$$x_1 = \frac{2a^2}{a^2 + r^2} Y, \quad x_2 = \frac{2a^2}{a^2 + r^2} Y, \quad x_3 = \frac{2a^2}{a^2 + r^2} Z, \quad r^2 \equiv X^2 + Y^2 + Z^2,$$ (50)

where $X, Y, Z$ are Cartesian coordinate in $R^3$. The three-dimensional sphere $S^3$ is defined by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2,$$ (51)

where we keep the radius $a$ of the sphere as a free parameter which will be useful in further. One can pass from the Cartesians coordinate system $X, Y, Z$ to the standard spherical coordinates $r, \theta, \phi$. The complex coordinates $z_1, z_2$ can be written as follows

$$z_1 = x_1 + ix_2 = \frac{2ar \cos \theta - i(a^2 - r^2)}{a^2 + r^2}, \quad z_2 = x_3 + ix_4 = \frac{a(2ar \cos \theta - i(a^2 - r^2))}{a^2 + r^2}.$$ (52)

It is convenient to define the Hopf mapping by

$$u = \frac{z_2}{z_1} = \frac{2ar \sin \theta - i(a^2 - r^2)}{2ar \sin \theta}.$$ (53)

One can write down an explicit expression for the color vector $\hat{n}$ as follows

$$\hat{n} = \frac{1}{(a^2 + r^2)^2} \left( 4ar(2ar \cos \phi \cos \theta + (a^2 - r^2) \sin \phi) \sin \theta \right) \left( 4ar((-a^2 + r^2) \cos \phi + 2ar \sin \phi \cos \theta) \sin \theta \right).$$ (54)

Notice, that magnetic field components have simple expressions and correspond to magnetic helical vortex configuration

$$H_{r\theta} = \frac{32a^3 r^2 \sin \theta}{(a^2 + r^2)^3}, \quad H_{\theta \varphi} = -\frac{8a^2 r^2 \sin(2\theta)}{(a^2 + r^2)^2}, \quad H_{r \varphi} = -\frac{8a^2 r(a^2 - r^2)(1 - \cos(2\theta))}{(a^2 + r^2)^3}.$$ (55)

Surprisingly, even though the magnetic field is axially symmetric, the energy density $E$ is spherically symmetric [28], and the knot configuration has a finite energy

$$E = \int d^3 V E = \int r^2 \sin \theta dr d\theta d\phi \frac{256a^4}{(a^2 + r^2)^4} = \frac{32\pi^2}{a}.$$ (56)
The solution was first found as an exact solution in the Nicole model with a Lagrangian \[ \mathcal{L} \simeq -((\partial_a \hat{n})^2)^{3/2}, \] and later as a special solution with winding numbers \( m = n = 1 \) in the integrable Aratyn-Ferreira-Zimerman (AFZ) model defined by the Lagrangian \[ \mathcal{L}_{AFZ} = -\frac{1}{4}(H_{\mu \nu \rho}^2)^{3/4}. \]

We introduce the following axially symmetric ansatz for the restricted gauge potential

\[ A_0^a = 0, \]
\[ \hat{A}_r^a = P(r, \theta)\hat{n}_a^\perp + \hat{C}_r^a, \]
\[ \hat{A}_\theta^a = Q(r, \theta)\hat{n}_a^\perp + \hat{C}_\theta^a, \]
\[ \hat{A}_\phi^a = R(r, \theta)\hat{n}_a^\perp + \hat{C}_\phi^a, \]

where \( P, Q, R \) are trial functions. Due to the dual \( U(1) \) symmetry one can impose a gauge condition on the trial functions. It is suitable to choose a condition \( P(r, \theta) = 0 \). The ansatz implies the following expression for the scalar magnetic field \( (m, n = r, \theta, \phi) \)

\[
\begin{align*}
H_{mn} &= H_{mn}^a \hat{n}_a^\perp, \\
H &= dx^m \wedge dx^n H_{mn} = \\
dr \wedge d\theta \left( \frac{32a^3 \sin^2 \theta}{(a^2 + r^2)^3} + Q_r \right) + \\
dr \wedge d\phi \left( \frac{16a^2 r^2 (a^2 - r^2) \sin^2 \theta}{(a^2 + r^2)^3} + R_r \right) + \\
d\theta \wedge d\phi \left( - \frac{8a^2 r^2 \sin(2\theta)}{(a^2 + r^2)^3} + R_\theta \right). 
\end{align*}
\]

After substituting the ansatz into the equations of motion \( \check{D}_\mu F_{\mu \nu} = 0 \) one can find that all nine equations reduce to three linear partial differential equations

\[
\begin{align*}
(a^2 + r^2)^3 (\sin Q_r + \cos Q_r) + 32a^3 r^2 \sin(2\theta) &= 0, \\
(a^2 + r^2)^4 Q_{rr} + 64a^3 r^2 (a^2 - 2r^2) \sin \theta &= 0, \\
(a^2 + r^2)^4 (\sin \theta (r^2 R_{rr} + R_{\theta \theta}) - \cos \theta R_\theta) - 32a^3 r^4 (-5a^2 + r^2) \sin^3 \theta &= 0.
\end{align*}
\]

The fact that we have finally linear differential equations is caused by Abelian structure of the chosen ansatz. The solution to the first two equations is given as follows

\[
Q = c_1 + c_2 \frac{r}{\sin \theta} - \frac{4ar^2 (a^2 - r^2)}{(a^2 + r^2)^2} \sin \theta - 4 \arctan \left( \frac{a}{r} \right) \sin \theta.
\]

A solution to the equation for the function \( R \) can be found as a sum of a general solution \( R_0(r, \theta) \) to the homogeneous part of the equation and a special solution \( R_1(r, \theta) \) to the inhomogeneous equation

\[
\begin{align*}
R(r, \theta) &= R_0(r, \theta) + R_1(r, \theta), \\
R_1(r, \theta) &= \left( \frac{c_{01}}{r} + c_{02} r^2 + \frac{8a^2 r^2}{(a^2 + r^2)^2} \right) \sin^2 \theta.
\end{align*}
\]

The homogeneous equation for \( R_0(r, \theta) \) is separable and the general solution is given by

\[
R_0(r, \theta) = \sum_w K(r; w) Y(\theta; w),
\]

\[
K(r; w) = a_{wr}^w (1 + \sqrt{1 + 4w})/2 + b_{wr}^w (1 + \sqrt{1 + 4w})/2,
\]

\[
Y(\theta; w) = c_w 8F_1[p_+, -p_-, 1/2; \cos^2 \theta] +
\]

\[
d_w \cos \theta 2F_1[p_-, p_+, 3/2; \cos^2 \theta],
\]

\[
p_{\pm} = (1 \pm \sqrt{1 + 4w})/4,
\]

where \( w \) is a constant of separation. The hypergeometric functions \( 2F_1(a, b, c; z) \) in the above equations reduce to trigonometric polynomials for integer and semi-integer values of \( p_{\pm} \), i.e., if we impose the following conditions

\[
w = l(l + 1), \quad l = 0, 1, 2, \ldots ,
\]

\[
c_l = 0, \quad \text{for } l = 2, 4, 6, \ldots ,
\]

\[
d_l = 0, \quad \text{for } l = 1, 3, 5, \ldots .
\]

In general, the solution represents a class of singular infinite energy field configurations due to singularities at the origin, \( r = 0 \), or at space infinity, \( r = \infty \). Let us consider special solutions which have singularities at the origin and the energy density \( E(r) \) falling down at infinity as \( \left( \frac{1}{r} \right)^{(l+1)\alpha} \) \( (l = 0) \):

(i) \( c_1 = c_2 = c_{01} = c_{02} = b_w = a_w = d_w = 1 \): the solution represents the Wu-Yang monopole.

(ii) \( c_1 = c_2 = c_{02} = b_w = 0, a_w = d_w = 1 \): a singular monopole solution with two non-zero magnetic components \( H_{\theta \phi}, H_{r \phi} \).

(iii) if \( c_2 \neq 0 \): all components of the magnetic field \( H_{mn} \) are not vanishing. In this case one has an additional singularity along \( OZ \) axis.

Notice, since the obtained exact solutions are singular, they are not of much interest by themselves, especially in the pure QCD. However, the singular solutions can be used in constructing finite energy solutions in extended theories. One well-known example is given by the Skyrmion, which can be treated as a dressed Wu-Yang monopole solution [12]. Another interesting direction is to consider a more general configuration for the vector \( \hat{n} \) in search of possible finite energy monopole like configurations in QCD and \( CP^1 \) model. This will be considered in a separate paper [34].

VI. SKYRME TYPE MODELS WITH KNOT SOLITONS

The knot soliton given by the \( CP^1 \) field [53] represents an exact solution with a unit Hopf charge in the
integrable models \cite{28,29} with Lagrangians \cite{37,58} containing fractional degrees of the kinetic terms. These models are far from realistic physical theories like the standard and effective low energy QCD. An interesting question arises, is there a Skyrme type model of effective low energy QCD which admits a finite energy exact solution with the knot configuration \cite{53}. One interesting model was found in \cite{33} where a generalized Faddeev-Skyrme model was suggested as a possible low energy effective theory of QCD. In particular, it had been found that the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} H_{\mu\nu}^2 + \xi (\partial_\mu \hat{n})^4 \]  

admits existence of analytic solutions with zero energy in the critical case \( \xi = 1 \). It turns out that there is another family of Skyrme type models with an additional scalar field which admits exact analytic solutions for non-vanishing Hopf numbers.

Let us consider the following Lagrangian

\[ \mathcal{L}(\phi, \hat{n}) = -\mu^2 (\partial_\mu \phi)^2 - \frac{\beta}{4} \phi^2 (\partial_\mu \hat{n})^2 - \frac{\nu}{32} \frac{1}{\phi^2} H_{\mu\nu}^2. \]  

(67)

The term proportional to \( \frac{1}{\phi^2} (\partial_\mu \hat{n})^4 \) can be added to the Lagrangian as well. We will not consider this case since the structure of the exact solution is not affected in principal by adding such a term. Up to change of variables the first two terms in \( \mathcal{L} \) coincide with the respective first two terms in the original Skyrme model. The scalar field \( \phi \) can be interpreted as a fluctuation of the length of the field \( \hat{n} \).

The exact solution in AFZ model \cite{58} in the case of equalled winding numbers \( m = n \) has the following form \cite{28} (in toroidal coordinates \( \eta, \xi, \phi \))

\[ u(\eta, \xi, \phi) = \frac{1}{\sinh \eta} \exp[i \mu (\xi + \phi)]. \]  

(68)

The Hopf charge of the solution is given by \( Q_H = m^2 \). The solution with Hopf charge \( Q_H = 1 \) coincides with the knot configuration \cite{59} in spherical coordinates. We have found for the case \( m = 1, \ldots, 5 \) that for each topological charge \( Q_H \) there is a special Lagrangian \cite{67} with constrained parameters \( \mu^2, \beta, \nu \) which admits exact finite energy solutions. The solution to Euler-Lagrange equations for the scalar field \( \phi \) is the same for various \( Q_H \)

\[ \phi = \frac{a}{\sqrt{a^2 + r^2}}. \]  

(69)

The equation of motion for the field \( u \) has the following solution

\[ u = \frac{e^{i m \phi}}{2 a r \sin \theta} \left( \frac{2 a r \cos \theta - i (a^2 - r^2)}{(2 a r \cos \theta - i (a^2 - r^2))^{m-1}} \right)^{m-1} \]  

(70)

under the assumption that the initial parameters \( \mu, \beta, \nu \) in the Lagrangian satisfy certain constraints. The solution \( \mathcal{L} \) coincides with the solution \( \mathcal{L} \) given in toroidal coordinates up to symmetry \( z_1 \leftrightarrow z_2 \) in the definition of the Hopf mapping.

For the case of a unit Hopf charge, \( Q_H = 1 \), one has a constraint

\[ \frac{3}{8} \mu^2 + \beta - \frac{\nu}{a^2} = 0. \]  

(71)

In the special case of \( \nu = 0 \) one has a model without any dimensional parameters while possessing a static solution which includes a dimensional parameter \( a \). The solution represents a saddle point configuration with spherically symmetric energy density

\[ \mathcal{E}_0 = \frac{a^2 \mu^2 (r^2 - 3a^2)}{(a^2 + r^2)^3} \nu. \]  

(72)

This implies a finite total energy

\[ E_1 = \frac{4 \pi^2 \nu}{a}. \]  

(74)

For Hopf charges \( Q_H > 1 \) corresponding to winding numbers \( m = 2, 3, 4, 5 \) the solution \( \mathcal{L} \) satisfies the equations of motion if only the following constraints are imposed:

\[ \beta = 0, \quad \mu^2 - \frac{8 m^2 \nu}{3 a^2} = 0. \]  

(75)

The energy density is

\[ \mathcal{E}_m = \frac{8 m^2 \nu (r^2 + 3a^2)}{3(a^2 + r^2)^3} \nu. \]  

(76)

The total energy of the solution corresponding to each model with a Lagrangian specified by parameters \( \beta, \mu, \nu \) is proportional to the Hopf charge

\[ E_m = \frac{4 \pi^2 \nu}{a} Q_H. \]  

(77)

For the case \( m \neq n \) we expect that knot solutions in such models will have a total energy expressed by the same equation \( \mathcal{L} \). A detailed study of knot solutions in a generalized Skyrme-Faddeev model with the length parameter \( a \) determined by equations of motion is presented in \cite{39}.  


VII. DISCUSSION

In standard approach to description of the classical vacuum in QCD in terms of pure gauge connection all topologically non-equivalent vacuums are classified by Chern-Simons number. Using an explicit construction of the classical vacuum in terms of the $CP^1$ vector field $\hat{n}$ we have demonstrated that classical vacuum possesses a more rich topological structure. Namely, all non-trivial topological vacuums are described by two homotopy groups $\pi_2(\text{SU}(N)/H)$. In [19] we have studied the Weyl symmetric structure of the classical QCD vacuum described by the second homotopy group which determines the monopole charge. In the present paper we consider the knot topology of the classical vacuum determined by the third homotopy group. A natural question arises on possible physical manifestations of such topological vacuum structure. We have considered excitations over the vacuum with a non-trivial Hopf charge in search of new analytic classical solutions in the $CP^1$ model, standard QCD and Skyrme type low energy effective theory of QCD. Our approach of studying new solutions can be useful in constructing essentially SU(3) topological solitons like instantons and monopoles. It is interesting to study possible physical applications of the proposed Skyrme type model which can be relevant to the low energy effective theory of QCD like the Faddeev-Skyrme model [11].

Certainly, since QCD is essentially a quantum theory, it is of great importance to find physical effects of the non-trivial topological structure at quantum level where the degeneracy of the classical vacuum should disappear. One possible effect includes a non-trivial Weyl symmetric structure of minimums in the two-loop effective potential in the presence of monopole condensates. It would be interesting as well to study implications of knot and monopole configurations, in particular, the effects related with contribution of all possible non-trivial topological field configurations to physical quantities like Wilson loop functional and effective action. One of principal unresolved problems is how to construct in pure QCD (without introducing any additional scalar fields) finite energy monopole field which is the main ingredient part of the mechanism of confinement based on dual color superconductivity model. Some of these issues will be considered in the forthcoming paper [34].

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