STREAM INSTABILITIES IN RELATIVISTICALLY HOT PLASMA

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ABSTRACT

The growth rates for Weibel and Buneman instabilities of relativistic ion beams in a relativistically hot electron background are derived analytically for general propagation angles. The Weibel instability perpendicular to the streaming direction is found to be the fastest growing mode and probably the first to appear. Oblique, quasiperpendicular modes grow almost as fast as the growth rate varies only moderately with angle, and they may distort or corrugate the filaments after the perpendicular mode saturates. The growth rate of the purely longitudinal (Buneman) mode is significantly smaller, contrary to the non-relativistic case. The results are consistent with simulations, which display aligned magnetic filaments and their subsequent disruption.

Key words: instabilities – magnetic fields – plasmas – shock waves

Online-only material: color figures

1. INTRODUCTION

The Weibel instability appears to be important for catalyzing shocks in unmagnetized plasmas.1 With the discovery of relativistic blast waves from gamma-ray bursts, it has received much attention (Medvedev & Loeb 1999; Brainerd 2000; Wiersma & Achterberg 2004; Lyubarsky & Eichler 2006; Achterberg & Wiersma 2007; Bret 2009; Bret et al. 2010; Yalinewich & Gedalin 2010; Rabink et al. 2011; Lemoine & Pelletier 2011) and appears to be confirmed by simulations. The conventional wisdom is that when two oppositely directed plasma streams collide, the Weibel instability generates small-scale magnetic fields; particle scattering off these magnetic fluctuations provides an isotropization mechanism necessary for the shock transition to form. Particle in cell simulations (Spitkovsky 2008a, 2008b) show that (1) magnetic filaments grow with a k vector that is perpendicular to the beam direction, (2) these filaments saturate at about 10% of the equipartition level and are eventually disrupted, and (3) the electrons are efficiently heated within the shock structure until their energy becomes comparable to the proton energy. The full shock transition occurs at a small scale of dozens of proton skin depths. The physical reason for the remarkably efficient electron heating has not been fully established yet; however, the transfer of energy to the electrons can already be seen in the linear instabilities, which are accompanied by induced electric fields that induce electron countercurrents, as per Lenz’s law (Blandford & Eichler 1987) and are discussed presently. It is necessary to understand the underlying physics before simulations can be scaled and generalized to real astrophysical phenomena.

An important point regarding the very first stages of relaxation is that the electron streaming is halted easily, whereas protons still plow on through an isotropic electron gas. The newly isotropized electron background effectivley suppresses the generation of magnetic field because of the induced electric field and attendant Lenz currents opposite to the proton currents (Blandford & Eichler 1987; Brainerd 2000; Wiersma & Achterberg 2004; Lyubarsky & Eichler 2006; Achterberg & Wiersma 2007). In the case of a Weibel unstable mode with the k vector exactly perpendicular to counterstreaming ion beams, the mode has a purely imaginary frequency in the electron frame. The magnitude of the Lenz currents then decreases with electron temperature. In the non-relativistic case, the fraction of electrons that resonate with the purely growing mode decreases with the increasing thermal velocity. In relativistically hot plasma, the electron inertia grows with the plasma temperature. In any case, both the magnitude and the spatial scale of the magnetic fluctuations depend on the “temperature” of the background electrons; the larger the temperature, the stronger the magnetic fluctuations.

Another interesting aspect of the simulation results is that current filaments aligned with the beams are formed initially, but eventually disrupted. The disruption may be due to nonlinear effects, but the filaments may also be disrupted by the eventually emergence of oblique modes, which one suspects would lead to a more complicated magnetic geometry and chaotic particle trajectories. The randomization of electron trajectories is of course connected to the issue of electron heating, as the rate of heating by electric fields is determined by electric resistivity, which is proportional to the electron scattering rate. So, oblique modes may play a role both in electron heating and filament disruption.

As a step toward understanding the filament disruption and electron heating within the shock structure, one should consider the full spectrum of unstable modes in the presence of proton beams within the relativistically hot electron background and calculate how much of the energy is transferred from protons to electrons. Two electron heating mechanisms are possible in the linear stages of instability: (1) Lenz currents, mentioned above, and (2) Landau damping against hot electrons that move in resonance with the wave, feeling its longitudinal electric field along the direction of the wave’s phase velocity vector. The second happens only when the electrons are already hot enough to be in resonance with a wave of substantial phase velocity. These are the waves that make an oblique angle with the beam direction. The first can occur even for cold electrons,
because the phase velocity of the perpendicular Weibel unstable mode can vanish in the frame of the electrons. Thus, the relative importance of the two mechanisms hinges on the role of oblique waves and the effects of hot electrons on them.

The case of oblique modes has already been studied long ago for cold plasmas. Here, specific properties of relativistic streaming instabilities are dictated by the fact that the response of the relativistic particles to an external perturbation is highly anisotropic. The effective particle mass, namely, the ratio of force to acceleration, for longitudinal (with respect to the direction of the particle motion) perturbations is \( m_{\text{eff}} = \gamma m \) whereas for the transverse perturbations \( m_{\text{eff}} = m \gamma^2 \); here \( \gamma \) is the particle Lorentz factor. For this reason, a relativistic particle beam excites, in a cold plasma, obliquely propagating Langmuir waves much faster than the waves in the direction of the beam (Watson et al. 1960; Bludman et al. 1960) the growth rate of the resonant beam instability is \( \Gamma \propto m_{\text{eff}}^{-1/3} \) so that \( \Gamma \propto \gamma^{-1} \) for longitudinal modes and \( \Gamma \propto \gamma^{-1/3} \) for oblique modes, respectively. Fully electromagnetic analysis of a cold plasma and a cold beam system (Fainberg et al. 1970) confirmed that the obliquely propagating unstable modes are practically electrostatic and also revealed an instability of purely transverse perturbations with the growth rate \( \Gamma \propto m_{\text{eff}}^{-1/2} \propto \gamma^{-2/3} \), which is in fact the relativistic counterpart of the Weibel instability.

The fact that a relativistic beam is most unstable to the excitation of oblique waves has important implications for inertial fusion (e.g., Bret & Deutsch 2005), where the thermal electrons remain subrelativistic. In a relativistically hot plasma, however, electrons can Landau damp oblique waves, which travel at a significant fraction of \( c \), therefore one can expect that the physics of the stream instabilities significantly changes when the background plasma temperature approaches or exceeds \( m c^2 \). This should have important implications for relativistic shocks where the plasma is relativistically hot. Moreover, since strong electrostatic instabilities easily heat cold electrons, one can expect that the condition \( T_e \sim m c^2 \) is achieved already in the shock precursor. Unfortunately, streaming instabilities for the general propagation angle still mostly studied in the cold plasma limit (Bret 2009; Nakar et al. 2011; Rabinak et al. 2011).

The full spectrum of the unstable modes in the relativistically hot electron background has been considered only by Yalinewich & Gedalin (2010), who solved numerically the general dispersion equation and reported the results graphically for specific choices of parameters. They concluded that the Weibel instability is the fastest in this case but the longitudinal Buneman instability competes with the Weibel instability.

Here we use a more analytic approach, in the interest of generality. We present explicit solutions to the general dispersion equation describing a monochromatic beam in a relativistically hot electron background. We show that the electrons Landau damp the oblique modes, instability of purely transverse modes (Weibel instability) develops faster, contrary to the \( T_e \ll m c^2 \) case. The unstable oblique modes then take on more of a purely Weibel-like character with corresponding growth rate. We also show, contrary to Yalinewich & Gedalin (2010) claim, that the Buneman instability is significantly slower than the Weibel instability and should therefore not compete with it.

The mass of the beam particles is not specified here, but is intended in the context of proton beams within the shock structure. It is shown that the effects of Landau damping are comparable to or less than those of induced transverse electric fields, so that the general picture that is gaining acceptance—beam instability that is primarily Weibel in character, generating magnetic filaments of negligible phase velocity—more or less holds up even when oblique modes and Landau damping are included in the analysis. On the other hand, the oblique modes, as they become quasiperpendicular, increasingly resemble Weibel unstable electromagnetic modes and have a growth rate that is a bit slower but comparable to the latter. So, they are not entirely negligible and could eventually lead to a more complex magnetic structure.

Having in mind conditions at the shock front, one has to consider two counterstreaming proton beams propagating in hot electron background. Here we mostly concentrated on a single beam case because generalization to multi-beam systems is straightforward (see Section 6).

We consider here hot electrons but continue to assume cold ion beams. When the ion beams are warm, all instabilities are suppressed, but the Weibel branch is still unstable at sufficiently long wavelengths, where the electrostatic modes are superluminal and stable. So, we consider the cold ion beam case the most favorable for the electrostatic instabilities. We will see that even in this case, the Weibel instability dominates the oblique modes in a relativistically hot plasma.

The paper is organized as follows. In the next section, we present the dispersion equations for the monochromatic beam in the isotropic electron background. In Section 3 we study instability of longitudinal (along the beam direction) modes, the Buneman instability. The purely transverse mode corresponding to the Weibel instability is considered in Section 4. The stability of oblique modes is addressed in Section 5. Conclusions are presented in Section 6. In the Appendix, we present the permeabilities of the electron plasma with ultrarelativistic Maxwell’s distribution.

2. DISPERSION EQUATION

We consider the interaction between a relativistic “monochromatic” ion beam and a relativistic plasma. The principal contributions to the dielectric constant come from (1) the beam ions and (2) the plasma’s electrons, which are taken to be relativistically hot and isotropic. The total system is taken to be charge and current neutral, meaning that any additional ions (not in the beam) are assumed (1) to fill in what is needed for charge and current neutrality and yet (2) not contribute significantly to the dielectric constant. Alternatively, charge and current neutrality could be achieved with a second monochromatic ion counter-beam. In any case, the neglect of contribution from the other ions to the dielectric constant is justified for most directions of propagation because, as will be seen, the oblique and longitudinal unstable modes can resonate only with at most one of the beams rendering any other ions less significant. For the case of the purely transverse mode, we show that the generalization to the multibeam case is straightforward.

The stability analysis is reduced to solution of the dispersion equation

\[
\text{Det}[k^2 \delta_{ij} - k_i k_j - \omega^2 \epsilon_{ij}] = 0. \tag{1}
\]

In this paper, we take the speed of light to be unity. In the plasma-beam system, one can conveniently separate the dielectric tensor into the plasma and the beam parts, \( \epsilon_{ij} = \epsilon_{ij}^p + \epsilon_{ij}^b - \delta_{ij} \), and present the plasma dielectric tensor as (e.g., Lifshitz & Pitaevskii 1981)

\[
\epsilon_{ij}^p = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \epsilon_t + \frac{k_i k_j}{k^2} \epsilon_l, \tag{2}
\]

where \( \epsilon_t \) and \( \epsilon_l \) are the transverse and longitudinal dielectric permeabilities, respectively. The permeabilities of the Maxwell
plasma with highly relativistic temperatures are presented in the Appendix. We take a monochromatic beam of particles with mass $m_b$ and density $n_b$ moving in the $z$-direction with the velocity $v_b$. Then, the dielectric tensor of the beam is presented as

$$
e^b_{ij} = \delta_{ij} \left( 1 - \frac{\omega^2_{pb} + \omega^2_{pb}}{\omega^2_{pb} + \omega^2_{pb}} \right) - \frac{\omega^2_{pb}}{\omega^2_{pb} + \omega^2_{pb}} \left( k_i v_{bi} + k_j v_{bj} \right) \frac{\omega}{\omega - k_z v_b} + \frac{k^2 - \omega^2}{(\omega - k_z v_b)^2} v_i v_j.$$  

(3)

where

$$\omega^2_{pb} = 4\pi n_b e^2/m_b.$$  

(4)

We choose the coordinate system such that $k_z = 0$. Then, the dispersion equation for the plasma-beam system is reduced, after some algebra, to the form

$$\left[ \varepsilon_l - \frac{\omega^2_{pb}}{\gamma_b^3 (\omega - k_z v_b)^2} \right] \left[ k^2 - \omega^2 \varepsilon_l + \frac{\omega^2}{\gamma_b^3} \right] = \frac{\omega^2_{pb} k^2 v_b^2}{\gamma_b^3 (\omega - k_z v_b)^2} \left( 1 - \varepsilon_l \left( 1 - \frac{\omega^2}{k^2} \right) - \frac{\omega^2}{k^2} \varepsilon_l \right).$$  

(5)

If the plasma is cold, $\varepsilon_l = \varepsilon_l = 1 - \omega_\mathrm{pb}^2/\omega^2$, $\omega_\mathrm{pb}^2 = 4\pi e^2 n/m_e$. Equation (5) reproduces the result of Fainberg et al. (1970).

For all practical purposes we can safely assume that

$$\frac{\omega_p}{\sqrt{\gamma_b^3}} \ll \frac{\omega_p}{\gamma_b^3}. $$  

(6)

For electron beams, this implies $n_b/\gamma_b \ll n_p/\gamma_T$ whereas for proton beams, which is of special interest in the context of relativistic shocks, this condition is fulfilled even for $n_b \sim n_p$, $\gamma_b \sim \gamma_T$.

### 3. BUNEMAN INSTABILITY

First we consider instability of the wave propagating along the beam, $\vec{k} = (0, 0, k)$. In this case the dispersion equation (5) is reduced to

$$\varepsilon_l = \frac{\omega^2_{pb}}{\gamma_b^3 (\omega - k_z v_b)^2}.$$  

(7)

The instability occurs due to the resonance of particles with waves, so that one can write $\omega = k_z v_b + \delta \omega; \delta \omega \ll \omega$. For a highly relativistic beam, $\omega \approx k_z v_b$, therefore we have to use Equation (A9) for $\varepsilon_l$. Then, the dispersion equation is reduced to the cubic equation

$$\delta \omega^2 \left[ \delta \omega + \xi (k - k_0) - \frac{k_0}{2\gamma^2_b} \right] = \frac{\omega^2_{pb} \omega_0^3}{12 \omega^2_{pb} \gamma_b^5 \gamma_T^2}.$$  

(8)

The maximal growth rate is achieved at $k = k_0[1 + 1/(2\xi \gamma_b^2)]$ and is equal to

$$\Gamma_{\mathrm{Buneman}} = \frac{3^{1/6}}{2^{5/3}} \left( \frac{\omega_p}{\omega_T} \right)^{1/3} \frac{\omega_0}{\gamma_b \gamma_T^{1/3}} = \frac{3^{1/6} \omega_b^{2/3} \omega_p^{1/3} \sqrt{\ln 2} \gamma^2_T}{2^{5/3} \gamma_b^{1/2} \gamma_T^{5/6}}.$$  

(9)

The Buneman regime occurs at the condition $\gamma_b > \gamma_T$. When the beam velocity is less than the plasma thermal velocities, $\gamma_b \ll \gamma_T$, the beam is still unstable, but the growth rate decreases. According to Lominadze & Mikhailovskii (1979), the maximal growth rate in this case is estimated as

$$\Gamma \approx \frac{\omega_b \sqrt{2 \ln 2}}{\gamma_b^{3/2}}.$$  

(10)

### 4. WEIBEL INSTABILITY

Now let us consider the instability of transverse waves, $\vec{k} = (k, 0, 0)$. In this case one can neglect the terms with $\omega_\mathrm{pb}$ in the left-hand side of the dispersion equation (5). The right-hand side becomes comparable to the left-hand side only at $\omega \ll k$. In this limit, the plasma dielectric permeabilities (A4) and (A5) are reduced to

$$\varepsilon_l = 1 + \frac{\omega^2_{pb}}{\gamma_T k^2}; \quad \varepsilon_l = 1 - \frac{\omega^2_{pb}}{\gamma_T k^2} + i \frac{\omega_p^2}{4\gamma_T k}.$$  

(11)

Now the dispersion equation (5) is reduced to the cubic equation

$$\left( 1 + \frac{\omega^2_{pb}}{\gamma_T k^2} \right) \left( k^2 - i \frac{\omega_p^2}{4\gamma_T k} \right) = -\frac{\omega^2_{pb} \omega_p^2}{\gamma_T \gamma_b \omega^2}.$$  

(12)

The unstable solutions to this equation are purely imaginary, which implies aperiodic instability. The growth rate has a broad maximum (see Figure 1)

$$\Gamma_{\mathrm{Weibel}} \approx \frac{\omega_b}{\sqrt{\gamma_b}}.$$  

(13)

at

$$k \sim \left( \frac{\omega^2_{pb} \omega_p}{\gamma_T \gamma_b^{1/2}} \right)^{1/3}.$$  

(14)

In the long and small wavelength limits the growth rate is given by

$$\Gamma = \left( \frac{4 \omega^2_{pb} \omega_b}{\pi \omega_p^2 \gamma_b} \right)^{1/3} k; \quad k \ll \left( \frac{\omega^2_{pb} \omega_b}{\gamma_T \gamma_b^{1/2}} \right)^{1/3};$$  

(15)

$$\Gamma = \frac{\omega_b}{\sqrt{\gamma}} \left( \frac{\omega_p^2}{\gamma_T k^2 + \omega_p^2} \right)^{1/2}; \quad k \gg \left( \frac{\omega^2_{pb} \omega_p}{\gamma_T \gamma_b^{1/2}} \right)^{1/3}.$$  

(16)

Comparing the Buneman and the Weibel instabilities, one finds that the ratio of the growth rates very weakly depends on the mass and density ratios of the beam and plasma particles:

$$\frac{\Gamma_{\mathrm{Buneman}}}{\Gamma_{\mathrm{Weibel}}} \approx \frac{3^{1/6}}{2^{5/3}} \left( \frac{\omega_p}{\omega_T} \right)^{1/3} \sqrt{\ln 2} \gamma^2_T \gamma_b^{1/2} \gamma_T^{5/6}$$  

$$= \frac{3^{1/6}}{2^{5/3}} \left( \frac{m_b n_p}{m_e n_T} \right)^{1/6} \sqrt{\ln 2} \gamma_b^{1/2} \gamma_T^{5/6}. $$  

(17)

Therefore, in the relativistic case, $\gamma_T, \gamma_b \gg 1$, the Weibel instability dominates the Buneman one even for the proton beam unless the density of the beam is extraordinarily small. The physical reason is that, in the relativistic case, the effective longitudinal mass significantly exceeds the effective transverse mass. The claim of Yalinewich & Gedalin (2010) that the growth rates of two modes are comparable may be due to an error in their numerical solution to the dispersion equation. For example, for two counterstreaming proton beams of equal density with parameters $n_b = n_p/2$, $\gamma_b = 100$, $\gamma_T = 20$, they found $\Gamma_{\mathrm{Buneman}}/\Gamma_{\mathrm{Weibel}} = 0.65$ whereas our result (even though the above expressions are found for a single beam, generalization to the multi-beam case is straightforward; see Section 6) is $\Gamma_{\mathrm{Buneman}}/\Gamma_{\mathrm{Weibel}} = 0.02$. 

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5. INSTABILITY OF OBLIQUE MODES

Now let us consider oblique modes. In this case one can also neglect the terms with \( \omega_b \) in the left-hand side of the dispersion equation (5). The right-hand side becomes comparable with the left-hand side only for modes in resonance with the beam, i.e., if \( \omega \approx k_z = k \cos \theta \). Therefore, one can substitute \( \omega = k \cos \theta \) into all the terms with the exception of the resonance denominator, which yields

\[
\varepsilon_l (1 - \cos^2 \theta \varepsilon_l) = \frac{\omega_p^2 \sin^2 \theta}{\gamma (\omega - kv_b \cos \theta)^2} [1 - \sin^2 \theta \varepsilon_l - \cos^2 \theta \varepsilon_l],
\]

(18)

where one can use the permeabilities in the form of Equations (A4) and (A5) with \( \omega = k \cos \theta \):

\[
\varepsilon_l = 1 + \frac{\omega_p^2}{k^2 \gamma} \left( 1 - \cos \theta \ln \cot \frac{\theta}{2} + \frac{\pi}{2} i \cos \theta \right)
\]

(19)

\[
\varepsilon_t = 1 - \frac{\omega_p^2}{2k^2 \gamma} \left( \sin^2 \theta \ln \cot \frac{\theta}{2} + \cos \theta - \frac{\pi}{2} i \sin^2 \theta \right).
\]

(20)

Note that these expression diverge at \( \theta = 0 \) and \( \theta = \pi/2 \); therefore these specific cases should be considered separately; this has been done in Sections 3 and 4.
The polarization of the unstable modes is shown in Figure 3. In Figure 4, the growth rate is presented as a function of $k$ at a fixed angle $\theta = \pi/4$. In Figure 5, we plot the maximal growth rate as a function of the angle between the wave and the beam. One sees that the largest growth rate is achieved for the transverse mode, i.e., for the Weibel instability.

6. DISCUSSION AND CONCLUSIONS

We found the full set of unstable modes for the monochromatic relativistic beam propagating through a relativistically hot electron background. This configuration is of relevance to the relativistic, collisionless shock structure where two oppositely directed proton beams plow on through an isotropic electron gas. Even though we concentrated on the single beam system, the results are easily generalized to the multi-stream case. The oblique and longitudinal modes are resonant; therefore, each beam excites the appropriate wave independently of others. In the case of the Weibel instability, one has to substitute, for

Now the solution to the dispersion equation is written as

$$\omega = k \cos \theta \pm \frac{\omega_b}{\sqrt{\gamma_b}} \Phi \left( \frac{\omega^2_{p} \gamma_{T} k^2}{\gamma_b}, \theta \right); \quad (21)$$

$$\Phi = \sin \theta \sqrt{\frac{1 - \sin^2 \theta E_l - \cos^2 \theta E_t}{E_l(1 - \cos^2 \theta E_t)}}. \quad (22)$$

The growth rate of the instability is presented as

$$\Gamma = \frac{\omega_b}{\sqrt{\gamma_b}} \text{Im} \Phi \left( \frac{\omega^2_{p} \gamma_{T} k^2}{\gamma_b}, \theta \right). \quad (23)$$

One sees that the dependence on the beam parameters is split from the dependence on the plasma parameters; therefore, we can find the universal dependence of the growth rate on the parameters by normalizing $\Gamma$ by $\omega_b/\sqrt{\gamma_b}$ and $k$ by $\omega_p/\sqrt{\gamma_T}$. This dependence is presented in Figure 2.
the term $\omega_b^2/\gamma_b$ in the right-hand side of the dispersion equation (12), the sum over all the beams $\sum_{b,j} \omega_b^2/\gamma_b$; this yields the same substitution in the expressions for the growth rate (see also Lyubarsky & Eichler 2006).

Our analysis confirms that the fastest growing mode is that with the wavevector perpendicular to the beam direction (Weibel instability), with the growth rate given by Equation (13). The purely longitudinal (Buneman) mode is strongly suppressed with respect to the transverse modes because the response of a relativistic particle to an external force is highly anisotropic; the transverse perturbations are more easily excited than longitudinal ones. The reason that the oblique Buneman-like modes are suppressed, relative to the case of a relativistic beam in a non-relativistic plasma where they grow much faster than even the perpendicular Weibel instability (Watson et al. 1960; Bludman et al. 1960; Fainberg et al. 1970; Bret & Deutsch 2005), is that the resonance waves are subluminal, $\omega/k = v_b \cos \theta$. Therefore, in a relativistically hot plasma, they are suppressed by Landau damping. The reason, on the other hand, the oblique Weibel modes grow more slowly than the perpendicular Weibel instability, which is also suppressed by the electrons, is that the projection of the beam anisotropy on the $k$ vector is reduced by $\sin \theta$.

The results are consistent with simulations, which display beam-aligned filaments that persist for tens of ion inertial periods and eventually disrupt. The details of the disruption remains an open question. We have shown that, even if there are no faster instabilities, the oblique Weibel mode will grow on a somewhat longer timescale—provided the linear approximation remains valid. This would introduce wiggles into the aligned magnetic filaments and would probably lead to their eventual disruption. However, the effect of trapping within the aligned filaments on the growth of oblique Weibel modes, and the existence of competing instabilities, remains open questions suitable for future work.

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APPENDIX

DIELECTRIC PERMEABILITIES OF RELATIVISTICALLY HOT PLASMA

Generally, the longitudinal and transverse permeabilities are written as (e.g., Lifshitz & Pitaevskii 1981)

$$\varepsilon_{il} = 1 + \frac{4\pi e^2}{k^2} \int \frac{k \cdot \partial f(p)}{\omega + i0 - k \cdot v} \frac{d^3 p}{\omega^2} \quad (A1)$$

$$\varepsilon_{il} = 1 + \frac{4\pi e^2}{\omega^2} \int \left( v - \frac{k \cdot v}{k^2} k \right) \frac{\partial f(p)}{\partial p} \frac{d^3 p}{\omega + i0 - k \cdot v} \quad (A2)$$

We assume that the plasma electrons have Maxwell’s distribution

$$f(p) = \frac{n}{8\pi (m_e \gamma_T)^3} \exp \left( -\frac{\gamma^2}{\gamma_T} \right) \quad (A3)$$

with ultrarelativistic temperatures, $\gamma_T \gg 1$. Transforming to the spherical coordinates in the momentum space, one can easily perform integration over the angles in Equations (A1) and (A2). Integration over $p$ could be performed only in the massless limit, i.e., by substituting $v = 1$; thus, one obtains (Silin 1960; see also Lifshitz & Pitaevskii 1981)

$$\varepsilon_{il} = 1 + \frac{\omega_p^2}{k^2 \gamma_T} \left[ 1 + \frac{\omega}{2k} \ln \left( \frac{\omega + i0 - k}{\omega + k} \right) \right] \quad (A4)$$

$$\varepsilon_{il} = 1 + \frac{\omega_p^2}{4\omega k \gamma_T} \left[ \left( 1 - \frac{\omega^2}{k^2} \right) \ln \left( \frac{\omega + i0 - k}{\omega + k} \right) - \frac{2\omega}{k} \right], \quad (A5)$$

where

$$\omega_p^2 = \frac{4\pi e^2 n}{m_e} \quad (A6)$$

Note that the expression (A4) diverges at $\omega \to k$ although the exact expression (A1) is not singular. The reason is that when
obtaining Equation (A4), we neglected small deviations of \( v \) from the speed of light in the denominator \( \omega - k \cdot v \). Therefore, the expression (A4) is valid only at the condition
\[
|\omega - k| \gg \frac{\omega}{2\gamma^2 T}, \quad (A7)
\]
so that it does not describe longitudinal waves with the phase velocity close to the speed of light and in particular subluminal modes crucially important for the beam instability (Lominadze & Mikhailovskiĭ 1979).

In order to find \( \epsilon_{\parallel}^{pl} \) at \( \omega \approx k \), we follow the procedure proposed by Lominadze & Mikhailovskiĭ (1979). Let us first find the longitudinal wave with the phase velocity equal to the speed of light, \( \omega_0 = k_0 \). In this case the dispersion equation, \( \epsilon_{\parallel}^{pl}(\omega_0, k_0) = 0 \), with the exact permeability (A1) yields
\[
\omega_0^2 = k_0^2 = \frac{\omega^2}{\gamma^2 T} \ln 2\gamma^2. \quad (A8)
\]
Now we could present \( \epsilon_{\parallel}^{pl} \) in the vicinity of the point \( \omega_0, k_0 \) as a Taylor expansion in small \( \omega - \omega_0 \) and \( k - k_0 \):
\[
\epsilon_{\parallel}^{pl}(\omega, k) = \left( \frac{\partial \epsilon_{\parallel}^{pl}}{\partial \omega} \right)_0 (\omega - \omega_0) + \left( \frac{\partial \epsilon_{\parallel}^{pl}}{\partial k} \right)_0 (k - k_0). \quad (A9)
\]
The zero subscript means that the derivatives are taken at \( k = k_0 \), \( \omega = \omega_0 \). Straightforward calculation yields
\[
\left( \frac{\partial \epsilon_{\parallel}^{pl}}{\partial \omega} \right)_0 = \frac{12\omega^2\gamma^2 T}{\omega^3} \quad (A10)
\]
and \( \left( \frac{\partial \epsilon_{\parallel}^{pl}}{\partial k} \right)_0 \) can be represented as
\[
\left( \frac{\partial \epsilon_{\parallel}^{pl}}{\partial k} \right)_0 = - (1 - \xi) \left( \frac{\partial \epsilon_{\parallel}^{pl}}{\partial \omega} \right)_0, \quad (A11)
\]
where \( \xi \) is some function of temperature that decreases as \( \gamma^2 T \) (see Figure 6). Note that \( (\partial \epsilon_{\parallel}^{pl}/\partial \omega)_0 > (\partial \epsilon_{\parallel}^{pl}/\partial k)_0 \) therefore it immediately follows from Equation (A9) that the dispersion curve, \( \epsilon_{\parallel}^{pl} = 0 \), enters the region \( \omega < k \) at \( k > k_0 \). This means that the longitudinal waves become subluminal at \( k > k_0 \); these waves could be resonantly excited by a relativistic particle beam, see Section 3.

REFERENCES

Achterberg, A., & Wiersma, J. 2007, A&A, 475, 1
Blandford, R. D., & Eichler, D. 1987, Phys. Rep., 154, 1
Bludman, S. A., Watson, K. M., & Rosenbluth, M. N. 1960, Phys. Fluids, 3, 747
Brainerd, J. J. 2000, ApJ, 538, 628
Bret, A. 2009, ApJ, 699, 990
Bret, A., & Deutsch, C. 2005, Phys. Plasmas, 12, 082704
Bret, A., Dieckmann, M. E., & Gremillet, L. 2010, Ann. Geophys., 28, 2127
Fainberg, Ya. B., Shapiro, V. D., & Shevchenko, V. I. 1970, Sov. Phys.—JETP, 30, 528
Lemoine, M., & Pelletier, G. 2011, MNRAS, 417, 1148
Lifshitz, E. M., & Pitaevskii, L. P. 1981, Physical Kinetics (Course of Theoretical Physics; Oxford: Pergamon)
Lominadze, D. G., & Mikhailovskiĭ, A. B. 1979, Sov. Phys.—JETP, 49, 483
Lyubarsky, Y., & Eichler, D. 2006, ApJ, 647, 1250
Medvedev, M., & Loeb, A. 1999, ApJ, 526, 697
Nakar, E., Bret, A., & Milosavljevic, M. 2011, ApJ, 738, 93
Rabin, I., Katz, B., & Waxman, E. 2011, ApJ, 736, 157
Silin, V. P. 1960, Sov. Phys.—JETP, 11, 1136
Spitkovsky, A. 2008a, ApJ, 682, L5
Spitkovsky, A. 2008b, ApJ, 673, L39
Watson, K. M., Bludman, S. A., & Rosenbluth, M. N. 1960, Phys. Fluids, 3, 741
Weibel, E. S. 1959, Phys. Rev. Lett., 2, 83
Wiersma, J., & Achterberg, A. 2004, A&A, 428, 365
Yalinewich, A., & Gedalin, M. 2010, Phys. Plasmas, 17, 062101