Late time tails of the massive vector field in a black hole background

R.A. Konoplya and A. Zhidenko
Instituto de Física, Universidade de São Paulo
C.P. 66318, 05315-970, São Paulo-SP, Brazil

C. Molina
Escola de Artes, Ciências e Humanidades, Universidade de São Paulo
Av. Arlindo Bettio 1000, CEP 05828-000, São Paulo-SP, Brazil

We investigate the late-time behavior of the massive vector field in the background of the Schwarzschild and Schwarzschild-de Sitter black holes. For Schwarzschild black hole, at intermediately late times the massive vector field is represented by three functions with different decay laws $\Psi_0 \sim t^{-(\ell+3/2)} \sin mt$, $\Psi_1 \sim t^{-(\ell+5/2)} \sin mt$, $\Psi_2 \sim t^{-(\ell+1/2)} \sin mt$, while at asymptotically late times the decay law $\Psi \sim t^{-5/6} \sin (mt)$ is universal, and does not depend on the multipole number $\ell$. Together with previous study of massive scalar and Dirac fields where the same asymptotically late-time decay law was found, it means, that the asymptotically late-time decay law $\sim t^{-5/6} \sin (mt)$ does not depend also on the spin of the field under consideration. For Schwarzschild-de Sitter black holes it is observed two different regimes in the late-time decay of perturbations: non-oscillatory exponential damping for small values of $m$ and oscillatory quasinormal mode decay for high enough $m$. Numerical and analytical results are found for these quasinormal frequencies.

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I. INTRODUCTION

Black hole’s response to an external perturbation has been a subject of active investigations for recent ten years [1]. Now it is a well known fact that in the response signal of a black hole, the so-called oscillatory quasinormal frequencies dominate, which do not depend on excitations of the quasinormal modes, but only on a black hole parameters, thereby giving us the complete information about the geometry of a black hole in the fully non-linear general relativity theory [2].

As was shown for the first time by R. Price, the regime of quasinormal oscillations turns into decay with inverse power dependence on time for a Schwarzschild black hole [3]. Price showed that perturbations of the massless scalar and gravitational fields decay as $t^{-2(D-3)}$ at asymptotically late times $t \to \infty$. Bicak found that scalar massless field in the Reissner-Nordström background decays as $t^{-2(\ell+2)}$ for $|Q| < M$, and as $t^{-2(\ell+2)}$ for $|Q| = M$ [4]. The massless scalar field perturbations along null infinity and along future event horizon has the decay law $u^{-\ell+2}$ and $v^{-\ell+3}$ respectively, where $u$ and $v$ are outgoing and ingoing Eddington-Finkelstein coordinates [5]. In [6] it was shown that the charged scalar perturbations decay slower then a neutral ones at asymptotically late times, while, on the contrary, at the stage of quasinormal ringing, the neutral perturbations decay slower [7, 8]. Also, for Schwarzschild-de Sitter and Reissner-Nordström-de Sitter black holes, instead of power-law tails, the exponential tails were found [9].

The investigation of late time decay for massless fields was continued for $D$-dimensional black holes. Thus, tails of massless scalar, vector and gravitational fields for higher-dimensional Schwarzschild black holes have the decay law $t^{-2(D-2)}$ for odd $D > 3$, and for even $(D > 4)$ dimensions, the power law decay is $t^{-2(D+3D-8)}$ [10]. The late-time behavior for $D$-dimensional Gauss-Bonnet, Gauss-Bonnet-de Sitter and Gauss-Bonnet-anti-de Sitter black holes was elaborated in [11]. Late time tails were also observed perturbations around wormholes between the branes in a two brane Randall-Sundrum model [12]. In addition, the late-time tails can be a tool to probe the extra dimensions both at the ringing stage [13] and at the stage of late-time tails [14].

The late time behavior of massive fields is qualitatively different from massless ones: at late times the decay profile is oscillatory inverse power tail. For a massive scalar field with mass $m$ in the background of the Schwarzschild black hole with mass $M$, the perturbation decays as $t^{-6/5} \sin mt$ at intermediate late times $mM < mt < 1/(mM)^2$ [15], and as $t^{-5/5} \sin mt$ at asymptotically late times. The same $t^{-5/5}$ behavior was found for the massive scalar field perturbations of the Kerr black hole [16] (but with frequency which is slowly dependent on time $\sim t^{-5/5} \sin (\omega(t) \times t)$), of the dilaton black hole [20], and also, for massive Dirac perturbations of the Schwarzschild black hole [21]. Recently, it has been found in [22] that intermediate late time decay of massive scalar field in $D$-dimensional Schwarzschild black hole is proportional to $t^{D/2 - 1/2}$.

On the other hand, the late time tails of fields corresponding to massive bosons have not been studied before, even for the simplest and a most interesting background geometry, namely, for the Schwarzschild black hole. In
the present paper we shall investigate the late time behavior of massive vector field, which shows quite non-trivial behavior also at the stage of quasinormal ringing [24]. The massive vector field obeys the Proca equation which can be reduced to a single wave-like equation for spherically symmetrical perturbations [24]. Therefore we are able to investigate the tail behavior for spherically symmetric perturbations both numerically and analytically. Yet, for higher multipoles \( \ell > 0 \) the Proca equations cannot be decoupled. Fortunately, in this case we also can investigate the tail behavior, because we need only the asymptotic form of the wave equations at large \( r \), rather than its exact form.

We found that at intermediate late times the three types of perturbations of the vector field decay with three different decay laws, while, at infinitely late times, these three types of decay approach a single universal decay law, which is independent of multipole number \( \ell \), and is the same for massive fields of other spin. Late time behavior for asymptotically de Sitter geometries essentially depends on the value of the mass of the field \( m \); at some threshold value of \( m \) the non-oscillatory decay changes into the oscillatory one.

The paper is organized as follows: in Sec. I we study, both analytically and numerically, the late time behavior for spherically symmetrical perturbations of Proca field in the background of Schwarzschild black holes. See II. is devoted to the analytical consideration of late time behavior for non-spherical perturbations. Finally, in Conclusion we summarize the obtained results.

II. SPHERICALLY SYMMETRIC PERTURBATIONS

We shall consider here the Schwarzschild black hole solution with a positive cosmological constant \( \Lambda \), i.e., Schwarzschild and Schwarzschild-de Sitter backgrounds in which the massive vector field propagates. The black hole metric is given by

\[
ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where

\[
h(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}.
\]

For self-consistence, let us briefly deduce the wave equation for spherically symmetric perturbations of the massive vector field for general form of the spherically symmetric static metric (1). The Proca equations have the form:

\[
F_{\mu\nu} - m^2 A^\mu = 0, \quad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.
\]

These equation can be reduced in the spherically symmetric case to the single equation of the form:

\[
f(r)B_{rr} - B_{tt} + \left(\frac{2f(r)}{r} + f'(r)\right)B_r + \left(\frac{2f'(r)}{r} - \frac{2f(r)}{r^2} + m^2\right)B = 0,
\]

where

\[
B = A_{r,t} - A_{t,r}.
\]

Introducing the usual “tortoise” radial coordinate \( r_\ast \) and a new wave function

\[
\Psi(r) = B(r)r,
\]

the wave equation can be re-written in the form [24]:

\[
\frac{\partial^2 \Psi(r_\ast, t)}{\partial r_\ast^2} - V(r(r_\ast))\Psi(r_\ast, t) = 0,
\]

with the effective potential

\[
V(r) = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) \left(\frac{2}{r^2} - \frac{6M}{r^3} + m^2\right).
\]

Positive values of the cosmological term \( \Lambda \) correspond to asymptotically de Sitter solutions.

The time evolution of the massive vector field can be described by the spectral decomposition method [23]. Thus,

\[
\Psi(r_\ast, t) = \int r_\ast'(G(r_\ast, r_\ast'; t)\Psi(t', 0) + G_t(r_\ast, r_\ast'; t)\Psi(r_\ast, 0)).
\]

For \( t > 0 \), the Green function obeys the equation

\[
\frac{\partial^2 G(r_\ast, r_\ast'; t)}{\partial r_\ast^2} - V(r)G(r_\ast, r_\ast'; t) = \delta(t)\delta(r_\ast - r_\ast').
\]

In order to find the Fourier’s component of the transformed Green function \( G(r_\ast, r_\ast'; t) \), we need the two linearly independent solutions \( \Psi_1, \Psi_2 \) to the homogeneous equation

\[
\frac{\partial^2 \Psi_i}{\partial r_\ast^2} + (\omega^2 - V)\Psi_i = 0, \quad i = 1, 2.
\]

If \( \Lambda = 0 \), we require that the function \( \Psi_1 \) should represent purely in-going waves at the event horizon \( r = r_+ \): \( \Psi_1 \sim e^{-i\omega r}r_\ast \), as \( r_\ast \to -\infty \). The other function \( \Psi_2 \) should damp exponentially at spatial infinity: \( \Psi_2 \sim e^{-\sqrt{\omega^2 - m^2}r}r_\ast \), as \( r_\ast \to +\infty \). On the other hand, if \( \Lambda > 0 \), the spacetime have a new feature, the cosmological horizon \( r = r_c \). In this case, the region of interest is the block \( r_+ < r < r_c \). The effective potential decays exponentially to zero at both limits, and therefore we require that \( \Psi_2 \sim e^{-i\omega r}r_\ast \), as \( r_\ast \to +\infty \).
To check the analytical results which will be found below, we used a numerical characteristic integration scheme, based in the light-cone variables \(u = t - r_\ast\) and \(v = t + r_\ast\). In the characteristic initial value problem, initial data are specified on the two null surfaces \(u = u_0\) and \(v = v_0\). The discretization scheme applied, used for example in \([5, 23]\), is

\[
\Psi(N) = \Psi(W) + \Psi(E) - \Psi(S) - \Delta^2 V(S) \frac{\Psi(W) + \Psi(E)}{8} + O(\Delta^4),
\]

where we have used the definitions for the points: \(N = (u + \Delta, v + \Delta), W = (u + \Delta, v), E = (u, v + \Delta)\) and \(S = (u, v)\).

In the asymptotically de Sitter scenario, we also employ a semi-analytic approach to investigate the perturbative behavior of the vector field. As will be seen, there is a region of the parameter space of the geometry where the late-time perturbations are dominated by quasinormal mode behavior. At this regime, we use the WKB scheme, based in the light-cone variables

\[
\psi = \psi(u, v) = \frac{1}{\sqrt{-2V_0}} e^{-\frac{i\omega}{\sqrt{-2V_0}}} (L_2 - L_3 - L_4 - L_5 - L_6 = n + \frac{1}{2}),
\]

where \(V_0\) is the height of the effective potential, and \(V_0''\) is the second derivative with respect to the tortoise coordinate of the potential at the maximum. \(L_2, L_3, L_4, L_5\) and \(L_6\) are presented in \([13]\).

A. Schwarzschild black hole background

1. Intermediate late time behavior

At intermediate late times, when \(r \ll t \ll M/(Mm)^2\), one can neglect the effect of backscattering of the field from asymptotically far regions and, following \([16]\), use the approximation \(M \ll r \ll M/(Mm)^2\). One can expand the wave equation into power series in \(M/r\). Neglecting terms of order \(O((M/m)\Delta)\), we obtain

\[
\frac{d^2 \Psi}{dr^2} + \left(\omega^2 - m^2 + \frac{4Mm^2}{r} - \frac{2}{r^2}\right) \Psi = 0.
\]

Provided that the observer and the initial data are situated far from the black hole we can further approximate

\[
\frac{d^2 \Psi}{dr^2} + \left(\omega^2 - m^2 - \frac{2}{r^2}\right) \Psi = 0.
\]

The last equation is similar to Eq. (15) in \([16]\), therefore we can easily find the branch cut contribution to the Green function for the above equation:

\[
G^C(r_\ast, r_\ast', t) = \frac{8}{9\pi A^2} \int_0^m \psi(r_\ast, k)\psi((r_\ast', k)^3) e^{-i\omega t} d\omega
\]

where \(k = \sqrt{m^2 - \omega^2}\), and \(A\) is a normalization constant.

Note that the contribution to the Green function is dominated by low-frequencies at late times. In the large \(t\) limit the effective contribution to the above integral is from \(\sqrt{m^2 - \omega^2} = O(\sqrt{m/t})\). This is stipulated by rapidly oscillating term \(e^{-i\omega t}\) so, that there is cancellation between the positive and negative parts of the integrand. Finally, similar to \([16]\), for \(t \gg m^{-1}\) we obtain for a fixed radius, and \(r_\ast, r_\ast' \ll t\):

\[
G^C(r_\ast, r_\ast', t) = \frac{\sqrt{2 m^{3/2}(r_\ast r_\ast')^2 t^{-5/2} \cos(mt - (5\pi/4))}}{6!}
\]

Therefore one can conclude that at intermediate times...
the field $\Psi$ decays at a fixed radius according to the law

$$\Psi \sim t^{-5/2} \sin mt,$$  

and at the black hole event horizon, the decay is the same

$$\Psi \sim v^{-5/2} \sin mt.$$  

This analytically obtained result in the regime of immediately late times is confirmed by numerical characteristic integration with high accuracy, what is illustrated in Figs. 1 and 2. There one can see that at sufficiently late times the numerical envelope approaches the analytical law $\Psi \propto t^{-5/6}$. The difference between numerical envelope and the line $t^{-5/6}$ is decreasing as time increases and is about 1% for $t \sim 10000$ (see Fig. 1).

2. Asymptotic late-time behavior

In [17], it was shown that at asymptotically late times the law of decay is stipulated by the behavior of the wave equation at large $r$. Thus, in the region $r/M \gg 1$, expanding the wave equation in powers of $M/r$, we obtain

$$\frac{d^2 \psi}{dx^2} + \left( \frac{1}{4} + \frac{a}{x} - \frac{b^2 - 1/4}{x^2} \right) \psi = 0,$$  

(20)

where we used a new variable $x = 2r \sqrt{m^2 - \omega^2}$, and $a$, $b$ are constants depending on $m$, $M$, and $\omega$. Following the papers [18] and [17], one can show that the monopole massive vector field at asymptotically late times $t \to \infty$ undergoes oscillatory inverse power law decay

$$\Psi \sim t^{-5/6} \sin mt, \quad t \to \infty.$$  

(21)

This power-law envelope is shown in Figs. 1 and 2. In the range $mM \gg 1/(mM)^2$, the smaller the value $mM$ is, the later the $t^{-5/6}$ tail begins to dominate. In the range $mM \gg mM$, on the contrary, the larger the value $mM$ is, the later $t^{-5/6}$ tail begins to dominate. The numerical investigation show that, both in the intermediate and late time regime, the oscillation period can be perfectly approximated by

$$\text{period} = \frac{2\pi}{m}$$  

(22)

This is illustrated in Fig. 3.

B. Schwarzschild-de Sitter black hole background

The introduction of a positive cosmological constant changes drastically the late-time tails. Direct numerical integration shows two distinct late-time decay regimes. We have observed that the qualitative aspects of the field decay are independent of the value of the cosmological constant, in the non-extreme regime.

Figure 3: Numerical value of $t$ in which the wave function is zero. Strait lines imply that period of oscillations is a constant. For late times, the lines $t_{\text{zero}}(i) = 2\pi/m$ are excellent approximations of the numerical values.

For fields with small values of mass, we observe that the late-time decay is exponential and non-oscillatory:

$$\Psi \sim e^{k_1 t}.$$  

(23)

The exponential index $k_1$ approaches zero as the field mass tends to zero. This decay mode is similar to the tail observed in the massless vector perturbation in Schwarzschild-de Sitter black hole backgrounds.

For $m$ greater than a critical value, the decay is dominated by quasinormal modes. In this regime, it is possible to use the well known semi-analytical WKB approach to calculate the quasinormal frequencies. In this approach, by the way, the convergence is very good. We present some results in table 1B where it is compared frequencies calculated directly from the characteristic integration and first and sixth order WKB formulas.

The transition from non-oscillatory to oscillatory decay is shown in Fig. 4. We also observe that, the larger the cosmological constant, the greater is the critical mass $m$ where the oscillatory regime dominates. For high values of $m$, the exponential envelope index $k_2$ approaches a constant value. These observations are illustrated in Fig. 5.
values of the parameters $m$ and $\Lambda$. The black hole mass is set to $M = 1$.

| $\Lambda$ | $m$ | $\text{Re}(\omega_0)$ | $\text{Im}(\omega_0)$ | $\text{Re}(\omega_0)$ | $\text{Im}(\omega_0)$ | $\text{Re}(\omega_0)$ | $\text{Im}(\omega_0)$ |
|----------|-----|------------------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| $10^{-4}$ | 0.5 | 0.476680 $\times 10^{-3}$ | 0.4771778 $\times 10^{-3}$ | 4.582987 $\times 10^{-3}$ | 0.4773061 $\times 10^{-3}$ | 4.569051 $\times 10^{-3}$ |
| $10^{-3}$ | 0.5 | 0.453757 $\times 10^{-2}$ | 0.4526006 $\times 10^{-2}$ | 1.368657 $\times 10^{-2}$ | 0.4534351 $\times 10^{-2}$ | 1.333422 $\times 10^{-2}$ |

One interesting limit of the Schwarzschild-de Sitter black hole geometry is its near extreme limit, when the event and cosmological horizons are very close, and the cosmological constant approach its maximum value $(1/9M^2)$.

In the near extreme limit, an analytic expression can be obtained for the quasinormal frequencies, which dominate the late-time decay if the mass $m$ is large enough.

Following the work presented in [26, 27], we have

$$\text{Re}(\omega_n) = \frac{\Lambda(1 - 9M^2\Lambda)}{3} \sqrt{\frac{m^2 - 1}{\Lambda}},$$

$$\text{Im}(\omega_n) = \frac{\Lambda(1 - 9M^2\Lambda)}{3} \left(n + \frac{1}{2}\right),$$

where $n = 0, 1, 2, \ldots$. (24)

The above formula is very well confirmed numerically: Fig. 6 shows a comparison of the values obtained by direct numerical integration, WKB formula and from Eq. (24).
III. PERTURBATIONS OF HIGHER MULTipoles

In this section we shall consider only the case of Schwarzschild black hole geometry. It is known from the work of Galtsov, Pomerantsseva and Chizhov [28] that the perturbation equations for massive vector field in the Schwarzschild background cannot be decoupled for general case of arbitrary multipolarity $\ell$. Yet, far from black hole the equations can be decoupled [28], what allows us judge about asymptotically late-time behavior. As in the paper [28] the perturbation equations are not shown in explicit form, but rather as an operator equations, we shall have to repeat here some of the derivations of [28], in order to proceed our analysis of tail behavior.

The Proca equation reads

$$F_{\mu\nu} - m^2 A^\mu = 0, \quad F^{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \quad (25)$$

Using the Newman-Penrose formalism, we go over to the tetrad projections of the field $F_{\mu\nu}$

$$\Phi_0 = F_{\mu\nu} l^\mu n^\nu, \quad \Phi_1 = \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + m^\mu m^\nu), \quad (26)$$

$$\Phi_2 = F_{\mu\nu} m^\mu m^\nu, \quad (27)$$

and of the vector-potential

$$A^\mu = A_1 n^\mu + A_0 l^\mu - A_m^\mu m^\mu - A_n m^\mu m^\nu. \quad (28)$$

After projecting of the equations (15) onto the tetrad’s vectors and expansions of $\Phi_i$ and $A_0$ into spin-weighted spherical harmonics [30]

$$\Phi_0 = \sum_{\ell=1, |m| \leq \ell} e^{-i\omega t + im\phi} (-1)^{\ell} Y_{\ell m}(r) \Psi_0^{\ell \ell}(r), \quad (29)$$

$$\Phi_1 = \sum_{\ell=0, |m| \leq \ell} e^{i\omega t + im^2} (0 Y_{\ell m}) \Psi_1^{\ell \ell}(r), \quad (30)$$

$$\Phi_2 = \sum_{\ell=1, |m| \leq \ell} e^{i\omega t + im\phi} (-1)^{\ell} Y_{\ell m}(r) \Psi_2^{\ell \ell}(r), \quad (31)$$

$$A_m = \sum_{\ell=1, |m| \leq \ell} e^{-i\omega t + im\phi} (-1)^{\ell} Y_{\ell m}(r) \Psi_3^{\ell \ell}(r), \quad (32)$$

$$A_{m^*} = \sum_{\ell=1, |m| \leq \ell} e^{-i\omega t + im\phi} (-1)^{\ell} Y_{\ell m}(r) \Psi_4^{\ell \ell}(r), \quad (33)$$

$$A_n = \sum_{\ell=0, |m| \leq \ell} e^{-i\omega t + im\phi} (0 Y_{\ell m}) \Psi_5^{\ell \ell}(r), \quad (34)$$

$$A_l = \sum_{\ell=0, |m| \leq \ell} e^{-i\omega t + im\phi} (0 Y_{\ell m}) \Psi_6^{\ell \ell}(r), \quad (35)$$

one can exclude the functions $\Psi_3, \Psi_4, \Psi_5, \Psi_6$ and obtain the three equations for the three scalar functions $\Psi_0, \Psi_1, \Psi_2$:

$$(D_0 D_0^+ - U) \Psi_0 + \sqrt{2} \lambda r^{-2} \Psi_1 = 0, \quad (36)$$

$$(D_0^+ D_0 - U) \Psi_2 + \sqrt{2} \lambda r^{-2} \Psi_1 = 0, \quad (37)$$

$$r^3 (D_0^+ r^{-4} D_0^+ + D_1 r^{-4} D_0^+ r) \Psi_1 - 2 U \Psi_1 +$$

$$+ 2 \sqrt{2} (r^2 - 2 M r)^{-1} (\Psi_0 + \Psi_2) = 0. \quad (38)$$

Here we used the operators $D_j, j = 0, 1$:

$$D_j = \partial_r - \frac{i \omega r}{r - 2 M} + j \frac{2 (r - M)}{r (r - 2 M)} \quad (39)$$

$$D_j^+ = \partial_r - \frac{i \omega r}{r - 2 M} + j \frac{2 (r - M)}{r (r - 2 M)}. \quad (40)$$

Here $\lambda = \ell (\ell + 1)$, and

$$U = (\lambda^2 + \mu^2 r^2) (r^2 - 2 M r)^{-1}. \quad (41)$$

Using new notations:

$$x = \frac{r}{2 M} - 1, \quad \mu = 2 \sqrt{2} M \lambda, \quad w = 2 M \omega \quad (41)$$

we obtain the system of equations

$$\Psi_{0,xx} + \left[ w^2 \left( 1 + \frac{1}{x} \right)^2 - \frac{\mu^2}{2} \left( 1 + \frac{1}{x} \right) - \frac{i w}{x^2} - \frac{\lambda^2}{x (x + 1)} \right] \Psi_0 + \frac{\sqrt{2} \lambda}{x (x + 1)} \Psi_1 = 0 \quad (42)$$
\[ \Psi_{2,xx} + \left[ w^2 \left( 1 + \frac{1}{x} \right)^2 - \frac{\mu^2}{x^2} \left( 1 + \frac{1}{x} \right) + \frac{\lambda^2}{x(x+1)} \right] \Psi_2 + \frac{\sqrt{2} \lambda}{x(x+1)} \Psi_1 = 0 \]  

(43)

\[ \Psi_{1,xx} + \frac{1}{x(x+1)} \Psi_{1,x} + \left[ w^2 \left( 1 + \frac{1}{x} \right)^2 - \frac{\mu^2}{x^2} \left( 1 + \frac{1}{x} \right) + \frac{x - 2x^2}{x^2(x+1)^2} - \frac{\lambda^2}{x(x+1)} \right] \Psi_1 + \frac{\sqrt{2} \lambda}{x(x+1)}(\Psi_0 + \Psi_2) = 0. \]  

(44)

To check these equations one can consider here spherically symmetric perturbations, for which \( \Psi_0 = \Psi_2 = 0 \), while \( \Psi_1 \) is non-zero and obeys the wave equation \( \Box \Psi = 0 \) with \( \ell = 0 \). After coming back to \( r \)-coordinate, Eq. (44) reduce to Eq. (8).

A. Intermediate late-time behavior

Now, let us consider phenomenologically interesting region \( n M \ll 1 \), what corresponds to the notion of \textit{"small"} perturbations, i.e. perturbations where the mass of the field \( m \) is much smaller then the mass of the black hole. Again, at intermediate late times (\( r \ll t \ll M/(Mm)^2 \)), we neglect the effect of backscattering of the field from asymptotically far region, and use the approximation \( M \ll r \ll M/(Mm)^2 \). For an observer and an initial data situated far from the black hole, from the above Eqs. (12)-(14), we obtain the following equations

\[ \Psi_{0,xx} + \left( w^2 - \frac{\mu^2}{2} - \frac{\lambda^2}{x^2} \right) \Psi_0 + \frac{\sqrt{2} \lambda}{x^2} \Psi_1 = 0 \]  

(45)

\[ \Psi_{2,xx} + \left( w^2 - \frac{\mu^2}{2} - \frac{\lambda^2}{x^2} \right) \Psi_2 + \frac{\sqrt{2} \lambda}{x^2} \Psi_1 = 0 \]  

(46)

\[ \Psi_{1,xx} + \left( w^2 - \frac{\mu^2}{2} - \frac{\lambda^2 + 2\lambda^2}{x^2} \right) \Psi_1 + \frac{\sqrt{2} \lambda}{x^2} (\Psi_0 + \Psi_2) = 0. \]  

(47)

The above equations can be easily diagonalized by linear \( x \)-independent transformations of the \( \Psi_i \)-functions.

\[ \tilde{\Psi}_{0,xx} + \left( w^2 - \frac{\mu^2}{2} - \frac{\lambda^2}{x^2} \right) \tilde{\Psi}_0 = 0 \]  

(48)

\[ \tilde{\Psi}_{2,xx} + \left( w^2 - \frac{\mu^2}{2} - \frac{1 + 2\lambda^2}{x^2} \right) \tilde{\Psi}_2 = 0 \]  

(49)

\[ \tilde{\Psi}_{1,xx} + \left( w^2 - \frac{\mu^2}{2} - \frac{1 + 2\lambda^2 + \sqrt{1 + 4\lambda^2}}{x^2} \right) \tilde{\Psi}_1 = 0 \]  

(50)

The above equations can be re-written in the following way:

\[ \tilde{\Psi}_{0,xx} + \left[ w^2 - \frac{\mu^2}{2} - \frac{\ell(\ell + 1)}{x^2} \right] \tilde{\Psi}_0 = 0, \quad \ell = 1, 2, 3, \ldots \]  

(51)

\[ \tilde{\Psi}_{2,xx} + \left[ w^2 - \frac{\mu^2}{2} - \frac{\ell(\ell - 1)}{x^2} \right] \tilde{\Psi}_2 = 0, \quad \ell = 1, 2, 3, \ldots \]  

(52)

\[ \tilde{\Psi}_{1,xx} + \left[ w^2 - \frac{\mu^2}{2} - \frac{\ell(\ell + 3) + 2}{x^2} \right] \tilde{\Psi}_1 = 0, \quad \ell = 0, 1, 2, \ldots \]  

(53)

One can see that \( \ell(\ell + 1) \) in Eq. (51) runs values 2, 6, 12, 20, \ldots, \( \ell(\ell - 1) \) in Eq. (52) runs the values 0, 2, 6, 12, 20, \ldots, while \( \ell(\ell + 3) + 2 \) runs values 2, 6, 12, 20, \ldots . Therefore we can finally re-write the equations in the following unique form:

\[ \tilde{\Psi}_{j,xx} + \left[ w^2 - \frac{\mu^2}{2} - \frac{\ell_j(\ell_j + 1)}{x^2} \right] \tilde{\Psi}_j = 0, \]  

(54)

where \( j = 0, 1, 2, \ldots \), and

\[ \ell_0 = \ell, \quad \ell = 1, 2, \ldots \]  

(55)

\[ \ell_1 = \ell + 1, \quad \ell = 0, 1, 2, \ldots \]  

(56)

\[ \ell_2 = \ell - 1, \quad \ell = 1, 2, \ldots \]  

(57)

Repeating analysis of the section II, we can easily see that the intermediate late-time behavior will be

\[ \tilde{\Psi}_0 \sim t^{-(\ell+3/2)} \sin mt, \quad \ell = 1, 2, \ldots \]  

(58)

\[ \tilde{\Psi}_1 \sim t^{-(\ell+1/2)} \sin mt, \quad \ell = 0, 1, 2, \ldots \]  

(59)

\[ \tilde{\Psi}_2 \sim t^{-(\ell+1/2)} \sin mt, \quad \ell = 1, 2, \ldots \]  

(60)

B. Asymptotic late-time behavior

If in the regime \( x \gg 1 \), we take into consideration sub-dominant terms of order \( \omega/x \) in Eq. (12)-(14), we come to the following equations:
\[ \Psi_{0,xx} + \left[ w^2 \left( 1 + \frac{2}{x} \right) - \frac{\mu^2}{2} \left( 1 + \frac{1}{x} \right) - \frac{\lambda^2}{x^2} \right] \Psi_0 + \frac{\sqrt{2}\lambda}{x^2} \Psi_1 = 0 \]  
(61)

\[ \Psi_{2,xx} + \left[ w^2 \left( 1 + \frac{2}{x} \right) - \frac{\mu^2}{2} \left( 1 + \frac{1}{x} \right) - \frac{\lambda^2}{x^2} \right] \Psi_2 + \frac{\sqrt{2}\lambda}{x^2} \Psi_0 = 0 \]  
(62)

\[ \Psi_{1,xx} + \left[ w^2 \left( 1 + \frac{2}{x} \right) - \frac{\mu^2}{2} \left( 1 + \frac{1}{x} \right) - \frac{\lambda^2 + 2}{x^2} \right] \Psi_1 + \frac{\sqrt{2}\lambda}{x^2} (\Psi_0 + \Psi_2) = 0. \]  
(63)

Following [28], one can introduce a new variable \( z = 2\sqrt{m^2 - \omega^2} r \), then one can see that the system of equations [61,63,10] has a solution [28]:

\[ \tilde{\Psi}_j = C_j W_{p,j}(z), \quad j = 0, 1, 2. \]  
(64)

where \( W_{p,j} \) is the Whittaker function [29]\

\[ q = (2\omega^2 - m^2)M/\sqrt{m^2 - \omega^2}, \]  
(65)

and for \( p \), as was shown in [28], one has the following system of equations:

\[ (p^2 - \lambda^2 - (1/4))C_0 + \sqrt{2}\lambda C_1 = 0, \]  
(66)

\[ (p^2 - \lambda^2 - (1/4))C_2 + \sqrt{2}\lambda C_1 = 0, \]  
(67)

\[ (p^2 - \lambda^2 - (9/4))C_1 + \sqrt{2}\lambda (C_2 + C_0) = 0. \]  
(68)

Therefore, one immediately has values of \( p \):

\[ p = \ell + (1/2) + \sigma, \quad \sigma = 0, \pm 1. \]  
(69)

As was shown in [17] for massive scalar field, the asymptotically late-time behavior of the perturbations governed by the wave equation which looks at asymptotically large \( r \) as a Whittaker equation, the decay law is universal and does not depend on \( \ell \). Thus, in a similar fashion with [17], we find that the decay at asymptotically late times \( t \to \infty \) again is

\[ \tilde{\Psi}_j \sim t^{-5/6} \sin m t, \quad j = 0, 1, 2. \]  
(70)

IV. CONCLUSIONS

In the present paper we have considered evolution of perturbations of massive vector field at late times. For spherically symmetric case, the perturbation equations can be reduced to a single wave-like equation, and therefore can be analyzed numerically with characteristic integration method. In this case, time-domain picture of a signal has good agreement with analytical predictions for the considered range of times.

General vector field perturbations can be represented by three scalar functions \( \Psi_i, \quad i = 0, 1, 2 \). At intermediate late times \( r \ll t \ll M/(Mt)^2 \), these three functions decay according to different laws which depend on a multipole number \( \ell \), \( \tilde{\Psi}_0 \sim t^{-(\ell+3/2)} \), \( \tilde{\Psi}_1 \sim t^{-(\ell+5/2)} \), \( \tilde{\Psi}_2 \sim t^{-(\ell+1/2)} \). On the contrary, at asymptotically late times \( t \to \infty \), the decay law is universal, i.e. the same for all three functions and does not depend on \( \ell \):

\[ \tilde{\Psi}_j \sim t^{-5/6} \sin m t. \quad s = 0, 1/2, 1. \]  
(71)

The latter law is obtained analytically and confirmed numerically with high accuracy.

For Schwarzschild-de Sitter black hole it is observed quite non-trivial tail behavior. In the considered asymptotically de Sitter geometries, the power-law oscillatory tails are replaced by purely exponential tails, if \( m \) is small, or by oscillatory quasinormal decay, if the parameter \( m \) is large enough. Although there is some universality for the tail behavior in the context of Schwarzschild geometry, this is not so in non-asymptotically flat backgrounds.

An interesting question, which was beyond our study, is whether the asymptotically flat background universality of tail behavior keeps in higher dimensions, and for a more general then Schwarzschild backgrounds? Yet, for Kerr black hole there is no hope to separate variables in the perturbed Proca equations, while for a Reissner-Nordström black hole such a separation is certainly possible for spherically symmetric perturbations, and charged Proca field should be considered instead. We hope that future investigations will clarify all these points.
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