Extended Lorentz-force-like equation and electromagnetic field tensor as consequence of a spinorial structure of space-time

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Abstract. We here show that the special, relativistic dynamical equation of the Lorentz-force-type, usually considered as a semi-empirical result, arises from the geometry of the Minkowski-Space-Time. By re-formulating this result in spinorial language, regarding spinors as being more fundamental objects than four-vectors, we obtain a set of dynamical Weyl spinors equations inducing this Lorentz-force-like-equation and geometrically obtain the spinorial form of the Electromagnetic Tensor. This representation alone actually reveals some properties of the free electromagnetic field that, within the context of the standard tensorial calculus, are known only by solving the second-order, partial-differential wave equation. Finally, we find that the spinorial equations of motion obtained, inducing the Lorentz-force-equation, do not only describe the evolution of the four-momentum but, surprisingly, also that of some additional degrees of freedom that may be associated to an intrinsic angular momentum.

1. The Geometrical Principle
Some years ago one of us [1] found for the Lorentz-force equation, traditionally considered as a semi-empirical result, a theoretical justification just as a consequence of the local geometry of Space-Time. To summarize briefly here, first realize that, the four-momentum of a certain particle, evaluated in some inertial frame at (proper) times $\tau$ and $\tau + \delta \tau$ respectively, must be related through an infinitesimal Lorentz-transformation (adopt the active point of view), as

$$\bar{p}(\tau + \delta \tau) = A(\tau)\bar{p}(\tau)$$

where $A(\tau)$ is an infinitesimal transformation given by

$$A(\tau) = I - (\delta \vec{\phi}(\tau) \cdot \vec{S} - \delta \vec{v}(\tau) \cdot \vec{K})$$

and $\vec{S}, \vec{K}$ are the generators of rotations and boosts respectively. The next step is to define two vector-fields $\vec{\epsilon}, \vec{\beta}$ such that

$$\delta \vec{v} = k\vec{\epsilon}(x)\delta \tau,$$
$$\delta \vec{\phi} = k\vec{\beta}(x)\delta \tau.$$

($k$ being some constant).
We then have, explicitly in matrix notation

\[
\begin{pmatrix}
p_0(\tau + \delta \tau) \\
p_1(\tau + \delta \tau) \\
p_2(\tau + \delta \tau) \\
p_3(\tau + \delta \tau)
\end{pmatrix} =
\begin{pmatrix}
p_0(\tau) \\
p_1(\tau) \\
p_2(\tau) \\
p_3(\tau)
\end{pmatrix} + k \delta \tau
\begin{pmatrix}
0 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\
-\epsilon_1 & 0 & \beta_3 & -\beta_2 \\
-\epsilon_2 & -\beta_3 & 0 & \beta_1 \\
-\epsilon_3 & \beta_2 & -\beta_1 & 0
\end{pmatrix}
\begin{pmatrix}
p_0(\tau) \\
p_1(\tau) \\
p_2(\tau) \\
p_3(\tau)
\end{pmatrix}
\]

(4)

If we now take the limit \( \delta \tau \to 0 \), then we obtain

\[
\frac{dp^0}{d\tau} = k \vec{p} \cdot \vec{\epsilon}
\]

(5)

\[
\frac{d\vec{p}}{d\tau} = k(\vec{\epsilon}p^0 + \vec{p} \times \vec{\beta})
\]

(6)

Finally, by identifying \( (c = 1) \)

\[
k \to q/m
\]

\[
\vec{\epsilon} \to \vec{E}
\]

\[
\vec{\beta} \to \vec{B}
\]

\[
t = \gamma \tau
\]

we see that (6) constitutes the Lorentz-Force-Equation

\[
\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}).
\]

(7)

Thus the Lorentz-force should no longer be considered as an empirical result but a consequence of the minkowskian local geometry of space-time. This result constitutes, as A. Bette once suggested, the Geometrical Principle [2].

2. The Geometrical Principle spinorially re-formulated: spinorial form of the Lorentz-force-equation

In order to obtain the spinorial form of the Lorentz-force-equation, we follow the same steps and note that the evolution of the dynamical, individual spinors from which any four-vector can be made, should obey to

\[
\xi^A(\tau + \delta \tau) = \delta t_A \xi^B(\tau) = \delta t_B A \epsilon^{BC} \xi_C,
\]

(8)

where \( \delta t_B^A \) is an infinitesimal element of \( SL(2, C) \) given by

\[
\delta t_B^A = I + \frac{1}{2}(\delta \vec{\alpha} \cdot \vec{\sigma})_B^A = I + \frac{1}{2}\begin{pmatrix}
\delta \alpha_3 & \delta \alpha_1 - i \delta \alpha_3 \\
\delta \alpha_1 + i \delta \alpha_3 & -\delta \alpha_3
\end{pmatrix},
\]

(9)

and \( \vec{\sigma} \) is the vector containing the three Pauli spin matrices, while \( \delta \vec{\alpha} = \delta \vec{\omega} + i \delta \vec{\theta} \) stands for the three complex infinitesimal parameters, in such a way that

\[
\delta \vec{\omega} = (\delta \omega_1, \delta \omega_2, \delta \omega_3) \to \text{infinitesimal parameters generating boosts},
\]

\[
\delta \vec{\theta} = (\delta \theta_1, \delta \theta_2, \delta \theta_3) \to \text{infinitesimal parameters generating rotations}.
\]
Again, if we associate the change in velocity to a vector field $\bar{\epsilon}$ and the infinitesimal rotation to $\bar{\beta}$

$$\delta \bar{\omega} \propto \bar{\epsilon}(\tau) \delta \tau,$$

$$\delta \bar{\theta} \propto \bar{\beta}(\tau) \delta \tau,$$

take the limit $\delta \tau \to 0$ and define the symmetric form

$$\phi^{AB} = -\frac{1}{2} (\delta \bar{\alpha} \cdot \bar{\sigma})_C B \epsilon^{AC},$$

we finally obtain the single-spinor dynamical equation

$$\dot{\xi}^A = k \phi^{AB} \xi_B,$$

where $\phi^{AB}$ is given by

$$\phi^{AB} = \frac{1}{2} \begin{pmatrix} -\epsilon_1 + \beta_2 & \epsilon_3 + i\beta_3 \\ \epsilon_3 + i\beta_3 & \epsilon_1 + \beta_2 \end{pmatrix},$$

Next, consider, as before, an electromagnetic field acting on a charged particle and let us calculate the derivative with respect to (proper) time of the four-momentum $p^{A A'} = \pi^A \bar{\pi}^A + \eta^A \bar{\eta}^A$ using, this time, the spinorial equation obtained for the evolution in time of the single spinors

$$\dot{\pi}^A = k \phi^{AB} \pi_B$$

$$\dot{\eta}^A = k \phi^{AB} \eta_B$$

where $k$ is the ratio $q/m$. In this way, we arrive at the expression

$$\dot{p}^{A A'} = \pi^A \bar{\pi}^{A'} + \pi^A \bar{\pi'^A} + \eta^A \bar{\eta'^A} + \eta^A \bar{\eta'^A} =$$

$$= (\epsilon^{AB} \phi^{A'B'} + \epsilon^{A'B'} \phi^{AB}) p_{B B'} =$$

$$q F^{A A' B B'} p_{B B'}$$

where

$$F^{A A' B B'} = \epsilon^{AB} \phi^{A'B'} + \epsilon^{A'B'} \phi^{AB}$$

is the spinorial form of the electromagnetic tensor. This expression, already known, it is obtained in textbooks by simple translation of the equation of motion $\dot{p}^\alpha = q F^{\alpha \beta} p_{\beta}$ to the spinorial language [3]. Instead of making so, we deduce it from the geometry, this time, of the Spinor Space.

The spinorial equation here deduced is fully equivalent to the Lorentz-Force-Equation (we shall see this later through an example). On the other side, here is a good place to point that all the same line of development could have been applied not only to the four-momentum, but also to any other dynamical four-vector. We therefore conclude that the evolution of any dynamical four-vector following a trajectory in Space-Time must obey to (14).
3. Some features of the free electromagnetic field obtained from its spinorial representation

We have already seen that, at the classical level, any field interacting with material particles must be spinorially given in terms of a symmetric form denoted by $\phi^{AB}$.

Let $\pi^A\tilde{\pi}^A$ be the spinorial representation, in the usual way \(^1\), of the four-momentum associated to a certain photon of energy ($\hbar = 1$) $\omega$ and spacial momentum $\vec{k}$. Now, we shall make the following Ansatz relating the spinor $\pi^A$ to the symmetric spinor $\phi^{AB}$:

$$\phi^{AB} = \pi^A \pi^B,$$

explicitely given by

$$\phi^{AB} = \pi^A \pi^B = \begin{pmatrix} (\pi^0)^2 & \pi^0 \pi^1 \\ \pi^0 \pi^1 & (\pi^1)^2 \end{pmatrix}.$$ 

In turn, the components $(\pi^0, \pi^1)$, according to the results of the previous section, should be related to two vector fields $\vec{\epsilon}, \vec{\beta}$ through the equality

$$\begin{pmatrix} (\pi^0)^2 & \pi^0 \pi^1 \\ \pi^0 \pi^1 & (\pi^1)^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -[\epsilon_1 + \beta_2] + i[\epsilon_2 - \beta_1] & \epsilon_3 + i\beta_3 \\ \epsilon_3 + i\beta_3 & [\epsilon_1 - \beta_2] + i[\epsilon_2 + \beta_1] \end{pmatrix}$$ (16)

Now, since in this case \(^2\)

$$\phi^{AB} \phi_{AB} = \pi^A \pi^B \pi_A \pi_B = 0,$$

the associated vector fields $\vec{\epsilon}, \vec{\beta}$ must obey to

$$|\vec{\epsilon}|^2 - |\vec{\beta}|^2 + 2i\vec{\epsilon} \cdot \vec{\beta} = 0.$$

Since the cartesian components of $\vec{\epsilon}, \vec{\beta}$ are real, the only one possibility left is

$$|\vec{\epsilon}| = |\vec{\beta}|,$$
$$\vec{\epsilon} \cdot \vec{\beta} = 0.$$

Let now $(o^A\bar{o}^A', o^A\bar{i}^A', i^A\bar{o}^A', i^A\bar{i}^A')$ be the chosen basis in $S(2, C) \bigotimes S(2, C)$, the vector-space which $\pi^A\pi^A'$ belongs to and let us carry out the usual change of basis \[^4\]

$$t^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} o^A \bar{o}^{A'} + i^A \bar{i}^{A'} \\ o^A \bar{i}^{A'} + i^A \bar{o}^{A'} \end{pmatrix},$$
$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} o^A \bar{i}^{A'} - i^A \bar{o}^{A'} \\ o^A \bar{o}^{A'} - i^A \bar{i}^{A'} \end{pmatrix}.$$ 

Now, it is easy to see that $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$ is perpendicular to the direction of propagation of the photon $\vec{k}$. For this purpose, define the four-vector $\epsilon^{AA'} = F^{AA'B'B} t_{BB'}$, where $F^{AA'B'B}$ is

\(^1\) We here adopt the usual, standard notation introduced by R. Penrose.
\(^2\) Note that the standard, tensorial representation of the free electromagnetic field provides $F^{\alpha \beta} F_{\alpha \beta} = constant$. However, its spinorial form goes further by setting this constant to 0, so this representation contains more information than the standard one.
the skew-symmetric form given by (15) associated to $\phi^{AB} = \pi^A \pi^B$. The result is the null-time-component-four-vector

$$\epsilon^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_3 & \epsilon_1 + i\epsilon_2 \\ \epsilon_1 - i\epsilon_2 & -\epsilon_3 \end{pmatrix},$$

whose physical components are $\epsilon^\mu = (0, \epsilon_1, \epsilon_2, \epsilon_3)$. Next, the relation

$$\epsilon^{AA'} \pi_\bar{A} \bar{A}' = -\bar{\epsilon} \cdot \bar{k} = t_{BB'} \left( \epsilon^{AA'} \pi_\bar{A} \bar{A}' \pi_\bar{A}' \bar{A} + \epsilon^{A'A} \pi_\bar{A} \bar{A}' \pi_\bar{A}' \bar{A} \right) = 0$$

yields to the conclusion

$$\bar{\epsilon} \perp \bar{k}.$$ (18)

Thus $\bar{\beta}$ must be either perpendicular or parallel to the direction of propagation $\bar{k}$. After a brief calculus in terms of the explicit components $(\pi^0, \pi^1)$ we find that also

$$\bar{\beta} \perp \bar{k}.$$ (19)

The vector fields $\bar{\epsilon}, \bar{\beta}$ can be then identified as the electric and magnetic fields respectively associated to a plane-wave with wave-vector $\bar{k}$. We therefore end this section by concluding that the spinor-object $\pi^A$ does not only provides a spinorial representation of the four-momentum associated to a massless particle (for instance, a photon), but also contains some additional information about the features of the associated field.

4. Extended Lorentz-force-like equation

Consider now a massive particle, its four-momentum being spinorially represented as [4]

$$p^{AA'} = \pi^A \pi^{A'} + \eta^A \bar{\eta}^{A'}.$$

The two individual spinors appearing in this expression suggested A. Bette [2] to define three additional, normalized four-vectors

$$s^{AA'} = \frac{1}{\sqrt{2m}} (\pi^A \pi^{A'} - \eta^A \bar{\eta}^{A'}),$$
$$v^{AA'} = \frac{1}{\sqrt{2m}} (\pi^A \bar{\eta}^{A'} + \eta^A \pi^{A'}),$$
$$w^{AA'} = \frac{i}{\sqrt{2m}} (\pi^A \pi^{A'} - \eta^A \bar{\eta}^{A'}),$$

which keep orthogonal to the four-momentum along the trajectory. The spinorial equation of motion (11) describes then not only the evolution in time of the four-momentum, but also that of these three dynamical four-vectors, i.e., if we work out the expressions for $s^{AA'}, v^{AA'}, w^{AA'}$ we find that they obey to

$$\dot{s}^{AA'} = k F^{AA'BB'} s_{BB'},$$
$$\dot{v}^{AA'} = k F^{AA'BB'} v_{BB'},$$
$$\dot{w}^{AA'} = k F^{AA'BB'} w_{BB'},$$

(20)
hence we shall name it “Spinorial Extended Lorentz-Force-like Equation”. On the other side, note that $s^AA'$, $v^AA'$ and $w^AA'$ become purely spacial in the particle’s proper frame and constitute therefore a rigid system attached to the particle. These four-vectors do have a really fundamental physical meaning: they can be identified, as we shall see in brief through an example, with components of an intrinsic angular momentum or spin of the particle. The Spinorial Extended Lorentz-Force-like Equation reveals then itself as being the fundamental equation describing the evolution of all physical quantities linked to a dynamical system.

5. Application of the extended Lorentz-force-like equation to a simple case: homogeneous constant magnetic field

Next we illustrate with an example the use of the extended Lorentz-force-like equations. To do that, suppose a massive ($m = 1$), charged particle in the presence of an external, homogeneous, constant magnetic field directed along the $z$-axis. By applying the standard Lorentz-force-equation and solving the first order coupled system for the evolution of the momenta we get

$$p_z(\tau) = \text{constant}, \quad (21)$$

$$p_\mu(\tau) = A_1 \cos(qB\tau) + A_2 \sin(qB\tau), \quad (22)$$

$$p_\nu(\tau) = A_2 \cos(qB\tau) - A_1 \sin(qB\tau), \quad (23)$$

where $A_1$ and $A_2$ are to be determined from the initial conditions. In this case, the equations to solve are

$$\dot{\pi}^0 = -i\frac{qB}{2} \pi^0,$$

$$\dot{\pi}^1 = i\frac{qB}{2} \pi^1,$$

(of course the same holds for $\dot{\eta}^A$) whose solution is

$$\pi^0(\tau) = \pi^0(\tau = 0) \exp \left( -i\frac{qB}{2} \tau \right),$$

$$\pi^1(\tau) = \pi^1(\tau = 0) \exp \left( i\frac{qB}{2} \tau \right),$$

$$\eta^0(\tau) = \eta^0(\tau = 0) \exp \left( -i\frac{qB}{2} \tau \right),$$

$$\eta^1(\tau) = \eta^1(\tau = 0) \exp \left( i\frac{qB}{2} \tau \right),$$

and therefore

$$p^{00'} = |\pi^0(\tau = 0)|^2 + |\eta^0(\tau = 0)|^2 = \text{constant},$$

$$p^{01'} = \pi^0(\tau = 0)\bar{\pi}^1(\tau = 0) \exp(-iqB\tau) + \eta^0(\tau = 0)\bar{\eta}^1(\tau = 0) \exp(-iqB\tau) = A(\tau = 0) \exp(-iqB\tau),$$

$$p^{11'} = |\pi^1(\tau = 0)|^2 + |\eta^1(\tau = 0)|^2 = \text{constant},$$

where $A(\tau = 0)$ is a complex number whose value is to be determined from the initial conditions,
so $p^{01'}$ can be written in the form

$$p^{01'}(\tau) = A_1 \cos(qB\tau) + A_2 \sin(qB\tau) + i [A_2 \cos(qB\tau) - A_1 \sin(qB\tau)]$$

(24)

being $A_1 = \text{Re}\{A(\tau = 0)\}$ and $A_2 = \text{Im}\{A(\tau = 0)\}$. We get then for the components of the four-momentum

$$\sqrt{2}p^{00'} = E + p_z = \text{constant}$$

$$\sqrt{2}p^{11'} = E - p_z = \text{constant}$$

$$\implies E, p_z = \text{constant},$$

(25)

which, as expected, coincides with the well-known solutions. For the time-evolution of the four-vectors $s^\alpha, v^\alpha, w^\alpha$ we find

$$\sqrt{2}s^{00'} = \frac{\sqrt{2}}{m} [\pi^0(\tau = 0)^2 - |\eta^0(\tau = 0)|^2] = s^0 + s^z$$

$$\sqrt{2}s^{11'} = \frac{\sqrt{2}}{m} [\pi^1(\tau = 0)^2 - |\eta^1(\tau = 0)|^2] = s^0 - s^z$$

$$\implies s^0, s^z = \text{constant},$$

(26)

$$\sqrt{2}v^{00'} = \frac{\sqrt{2}}{m} [\pi^0(\tau = 0)|\eta^0(\tau = 0)|^2 + c.c.] = v^0 + v^z$$

$$\sqrt{2}v^{11'} = \frac{\sqrt{2}}{m} [\pi^1(\tau = 0)|\eta^1(\tau = 0)|^2 + c.c.] = v^0 - v^z$$

$$\implies v^0, v^z = \text{constant},$$

(27)

$$\sqrt{2}w^{00'} = \frac{\sqrt{2}}{m} [\pi^0(\tau = 0)|\eta^0(\tau = 0)|^2 + c.c.] = w^0 + w^z$$

$$\sqrt{2}w^{11'} = \frac{\sqrt{2}}{m} [\pi^1(\tau = 0)|\eta^1(\tau = 0)|^2 + c.c.] = w^0 - w^z$$

$$\implies w^0, w^z = \text{constant},$$

(28)

$$\sqrt{2}s^{01'} = s_x + is_y = B(\tau = 0)e^{-qB\tau} \implies \begin{cases} s_x = B_1 \cos(qB\tau) + B_2 \sin(qB\tau) \\ s_y = B_2 \cos(qB\tau) - B_1 \sin(qB\tau) \end{cases},$$

(29)

$$\sqrt{2}v^{01'} = v_x + iv_y = C(\tau = 0)e^{-qB\tau} \implies \begin{cases} v_x = C_1 \cos(qB\tau) + C_2 \sin(qB\tau) \\ v_y = C_2 \cos(qB\tau) - C_1 \sin(qB\tau) \end{cases},$$

(30)

$$\sqrt{2}w^{01'} = w_x + iw_y = D(\tau = 0)e^{-qB\tau} \implies \begin{cases} w_x = D_1 \cos(qB\tau) + D_2 \sin(qB\tau) \\ w_y = D_2 \cos(qB\tau) - D_1 \sin(qB\tau) \end{cases}$$

(31)

If the initial conditions are such that $p_x(\tau = 0) = p_y(\tau = 0) = 0$ the particle keeps on moving with constant velocity along the z-axis and the (orbital) angular momentum is therefore null. However, if the particle is to have an intrinsic angular momentum, although no Thomas precession occurs, we should expect its intrinsic magnetic moment to precess around the z-axis, due to the presence of the magnetic field. This is precisely the behaviour found for $\vec{s}, \vec{v}, \vec{w}$: they do precess due solely to the presence of the applied magnetic field and can be therefore identified as being components of an intrinsic angular momentum. To analyze the situation in the rest
frame of the particle, one sets $\vec{p} = 0$. Doing so, we find that the temporal evolution of the spinors in such frame attends to

$$\pi^A(\tau) = \left( \begin{array}{c} |\pi^0(\tau = 0)e^{i\phi_0}e^{i\frac{qB}{2}}| \\ |\pi^1(\tau = 0)e^{i\phi_1}e^{i\frac{qB}{2}}| \end{array} \right), \eta^A(\tau) = \left( \begin{array}{c} |\pi^0(\tau = 0)e^{i\xi_0}e^{i\frac{qB}{2}}| \\ |\pi^1(\tau = 0)e^{i\xi_1}e^{i\frac{qB}{2}}| \end{array} \right),$$

(33)

where the phase-factors $\phi_0, \phi_1, \xi_0, \xi_1$ must satisfy, in this frame, the relation $(\phi_1 - \xi_1) = (\phi_0 - \xi_0 + (2n + 1)\pi)$. For the momenta of the null directions associated to massless particles in this inertial frame we find

$$\vec{p}_x = \left( \begin{array}{c} |\pi^0| \cos(qB\tau + \delta \phi), |\pi^0| \sin(qB\tau + \delta \phi), \sqrt{2} \left[ |\pi^0|^2 - |\pi^1|^2 \right] \end{array} \right),$$

(34)

$$\vec{p}_y = -\left( \begin{array}{c} |\pi^0| \cos(qB\tau + \delta \phi), |\pi^0| \sin(qB\tau + \delta \phi), \sqrt{2} \left[ |\pi^0|^2 - |\pi^1|^2 \right] \end{array} \right),$$

(35)

with $\delta \phi = (\phi_0 - \phi_1)$. We see in (34) and (35) that the $z$-component remains constant in time, while the $x$- and $y$-components do evolve. This causes, in this particular case, the rotation of the axis containing $\vec{p}_x$ and $\vec{p}_y$ around the $z$-axis. With use of the equations (13),(15) one obtains

$$\vec{s} = \frac{\vec{p}_x}{m},$$

(36)

$$v_x = \frac{\sqrt{2}}{m} \left\{ |\pi^0|^2 \cos(qB\tau + \phi_0 - \xi_1) - |\pi^1|^2 \cos(qB\tau + \xi_0 - \phi_1) \right\},$$

$$v_y = \frac{\sqrt{2}}{m} \left\{ |\pi^0|^2 \sin(qB\tau + \phi_0 - \xi_1) - |\pi^1|^2 \sin(qB\tau + \xi_0 - \phi_1) \right\},$$

(37)

$$v_z = \frac{2\sqrt{2}}{m} |\pi^0| \cos(\phi_0 - \xi_0),$$

$$w_x = \frac{\sqrt{2}}{m} \left\{ |\pi^0|^2 \sin(qB\tau + \phi_0 - \xi_1) - |\pi^0|^2 \sin(qB\tau + \phi_0 - \xi_1) \right\},$$

$$w_y = \frac{\sqrt{2}}{m} \left\{ |\pi^0|^2 \cos(qB\tau + \phi_0 - \xi_1) - |\pi^1|^2 \cos(qB\tau + \xi_0 - \phi_1) \right\},$$

(38)

$$w_z = -\frac{2\sqrt{2}}{m} |\pi^0| \sin(\phi_0 - \xi_0).$$

It is straightforward to check that $\vec{s}, \vec{v}, \vec{w}$ keep instantaneously, at any proper time $\tau$, orthogonal one to each other while undergoing the precession around the $z$-axis and so it shall occur with any linear combination of them, just as expected from the theoretic predictions above.

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