CHIRAL SYMMETRY BREAKING BY INSTANTONS

Dmitri Diakonov

Petersburg Nuclear Physics Institute, Gatchina, St.Petersburg 188350, Russia

1 Foreword

Instantons are certain configurations of the Yang–Mills potentials $A^a_\mu(x)$ satisfying the equations of motion $D^a_{\mu}F^{b}_{\mu\nu} = 0$ in euclidean space, i.e. in imaginary time. The solution has been found by Belavin, Polyakov, Schwartz and Tiupkin in 1975 [1]; the name "instanton" has been suggested by ’t Hooft in 1976 [2], who also made a major contribution to the investigation of the instantons properties.

In QCD instantons are the best studied non-perturbative effects, leading to the formation of the gluon condensate [3] and of the so-called topological susceptibility needed to cure the $U(1)$ paradox [2, 4]. The QCD instanton vacuum has been studied starting from the pioneering works in the end of the seventies [5, 6]; a quantitative treatment of the instanton ensemble, based on the Feynman variational principle, has been developed in ref. [7]. The most striking success of the QCD instanton vacuum is its capacity to provide a beautiful mechanism of the spontaneous chiral symmetry breaking [8, 9, 10]. Moreover, the instanton vacuum leads to a very reasonable effective chiral lagrangian at low energies, including the Wess–Zumino term, etc., which, in its turn gives a nice description of nucleons as chiral quark solitons [11].

It should be stressed that literally speaking instantons do not lead to confinement, though they induce a growing potential for heavy quarks at intermediate separations [12]; asymptotically it flattens out [3, 13]. However, it has been realized a decade ago [8, 9, 14], that it is chiral symmetry breaking and not confinement that determines the basic characteristics of light hadrons (one would probably need an explicit
confinement to get the properties of highly excited hadrons). Therefore, since the
instanton vacuum describes well the physics of the chiral symmetry breaking, one
would expect that instantons do explain the properties of light hadrons, both mesons
and baryons. Indeed, a detailed numerical study of dozens of correlation functions in
the instanton medium undertaken by Shuryak, Verbaarschot and Schäfer [15] (earlier
certain correlation functions were computed analytically in refs. [8, 9]) demonstrated
an impressing agreement with the phenomenology [16] and, more recently, with di-
rect lattice measurements [17]. As to baryons, the instanton-motivated chiral quark
soliton model [11] also leads to a very reasonable description of dozens of baryon
characteristics (for a review see ref.[18]).

More recently the instanton vacuum was studied in direct lattice experiments by
the so-called cooling procedure [19, 20, 21, 22, 17, 23], see also the proceedings of the
Lattice-94 meeting for a more complete list of references. It was demonstrated that
instantons and antiinstantons (I’s and I’s for short) are the only non-perturbative
gluon configurations surviving after a sufficient smearing of the quantum gluon fluc-
tuations. The measured properties of the I I medium appeared [21, 17, 23] to be close
to that computed from the variational principle [7] and to what had been suggested
by Shuryak in the beginning of the 80’s [6] from phenomenological considera-
tions.

Cooling down the quantum fluctuations above instantons kills both the one-gluon
exchange and the linear confining potential (a small residual string tension observed
in the cooled vacuum [21, 17] is probably due to the instanton-induced rising potential
at intermediate distances [12]). Nevertheless, in the cooled vacuum where only I’s
and I’s are left, the correlation functions of various mesonic and baryonic currents, as
well as the density-density correlation functions appear to be quite similar to those of
the true or ”hot” vacuum [17]. We consider it to be a remarkable confirmation of the
ideology which has been put forward quite some time ago [3, 5, 14]: instantons are
responsible for the basic properties of the QCD vacuum and of the low-mass hadrons,
while the confinement, fundamental as it is, does not readily manifest itself in those
properties.

These lectures are aimed as an introduction to understanding the above works.
2 Periodicity of the Yang–Mills potential energy

I start by explaining the physical meaning of instantons as classical tunneling trajectories in imaginary (euclidean) time. To that end I shall temporarily work in the $A_0^a = 0$ gauge, called Weyl or Hamiltonian gauge, and forget about fermions for a while. The remaining pure Yang–Mills or ”pure glue” theory is nonetheless nontrivial, since gluons are self-interacting. For simplicity I start from the $SU(2)$ gauge group.

The spatial YM potentials $A_i^a(x,t)$ can be considered as an infinite set of the coordinates of the system, where $i = 1, 2, 3, a = 1, 2, 3$ and $x$ are ”labels” denoting various coordinates. The YM action is

$$S = \frac{1}{4g^2} \int d^4x \ F_{\mu\nu}^a F^{a\mu\nu} = \int dt \left( \frac{1}{2g^2} \int d^3x \ E^2 - \frac{1}{2g^2} \int d^3x \ B^2 \right)$$  \hspace{1cm} (2.1)

where $E$ is the electric field strength,

$$E_i^a(x, t) = \dot{A}_i^a(x, t)$$  \hspace{1cm} (2.2)

(the dot stands for the time derivative), and $B$ is the magnetic field strength,

$$B_i^a(x, t) = \frac{1}{2} \epsilon_{ijk} \left( \partial_j A_k^a - \partial_k A_j^a + \epsilon^{abc} A_j^a A_k^c \right).$$  \hspace{1cm} (2.3)

Apparently, the first term in eq. (2.1) is the kinetic energy of the system of coordinates $\{A_i^a(x, t)\}$ while the second term is minus the potential energy being just the magnetic energy of the field. The simple and transparent form of eq. (2.2) is the advantage of the Weyl gauge.

Let us introduce an important quantity called the Pontryagin index or the four-dimensional topological charge of the YM fields:

$$Q_T = \frac{1}{32\pi^2} \int d^4x \ F_{\mu\nu}^a \tilde{F}^{a\mu\nu}, \hspace{1cm} \tilde{F}_{\mu\nu}^a \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{a}_{\alpha\beta}. \hspace{1cm} (2.4)$$

The integrand in eq. (2.4) is a full derivative of a four-vector $K_\mu$:

$$\frac{1}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\mu\nu} = \partial_\mu K_\nu, \hspace{1cm} K_\mu = \frac{1}{16\pi^2} \epsilon_{\mu\alpha\beta\gamma} \left( A_\alpha^a \partial_\beta A_\gamma^a + \frac{1}{3} \epsilon^{abc} A_\alpha^a A_\beta^b A_\gamma^c \right). \hspace{1cm} (2.5)$$

Therefore, assuming the fields $A_\mu$ are decreasing rapidly enough at spatial infinity, one can rewrite the 4-dimensional topological charge (2.4) as
\[
Q_T = \int d^4x(\partial_0 K_0 - \partial_i K_i) = \int dt \frac{d}{dt} \int d^3x K_0. \tag{2.6}
\]

Introducing the Chern–Simons number

\[
N_{CS} = \int d^3x K_0 = \frac{1}{16\pi^2} \int d^3x \epsilon_{ijk} \left( A_i^a \partial_j A_k^a + \frac{1}{3} \epsilon^{abc} A_i^a A_j^b A_k^c \right) \tag{2.7}
\]

we see from eq. (2.6) that \(Q_T\) can be rewritten as the difference of the Chern–Simons numbers characterizing the fields at \(t = \pm \infty\):

\[
Q_T = N_{CS}(+\infty) - N_{CS}(-\infty). \tag{2.8}
\]

The Chern–Simons characteristics of the field has an important property that it can change by integers under large gauge transformations. Indeed, under a general time-independent gauge transformation,

\[
A_i \rightarrow U^\dagger A_i U + iU^\dagger \partial_i U, \quad A_i \equiv A_i^a \frac{\tau^a}{2}, \tag{2.9}
\]

the Chern–Simons number transforms as follows:

\[
N_{CS} \rightarrow N_{CS} + N_W \tag{2.10}
\]

where \(N_W\) is the winding number of the gauge transformation (2.9):

\[
N_W = \frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \left[ (U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U) \right]. \tag{2.11}
\]

The \(SU(2)\) unitary matrix \(U\) of the gauge transformation (2.9) can be viewed as a mapping from the 3-dimensional space onto the 3-dimensional sphere of parameters \(S_3\). If at spatial infinity we wish to have the same matrix \(U\) independently of the way we approach the infinity (and this is what is usually assumed), then the spatial infinity is in fact one point, so the mapping is topologically equivalent to that from \(S_3\) to \(S_3\). This mapping is known to be non-trivial, meaning that mappings with different winding numbers are irreducible by smooth transformations to one another. The winding number of the gauge transformation is, analytically, given by eq. (2.11).

As it is common for topological characteristics, the integrand in (2.11) is in fact a full derivative. For example, if we take the matrix \(U(x)\) in a "hedgehog" form, \(U = \exp[i(\mathbf{r} \cdot \tau/\mathbf{r}) \cdot \mathbf{P}(|\mathbf{r}|)]\), eq. (2.11) can be rewritten as
\[ N_W = \frac{2}{\pi} \int dr \sin^2 P \frac{dP}{dr} = \frac{1}{\pi} \left[ P - \frac{\sin 2P}{2} \right]_0^\infty = \text{integer} \quad (2.12) \]

since \( P(r) \) both at zero and at infinity needs to be multiples of \( \pi \) if we wish \( U(r) \) to be unambiguously defined in the origin and at the infinity.

Let us return now to the potential energy of the YM fields,

\[ \mathcal{V} = \frac{1}{2g^2} \int d^3x (B_a^i)^2. \quad (2.13) \]

One can imagine plotting the potential energy surfaces over the Hilbert space of the coordinates \( A^a_i(x) \). It will be some complicated mountain country, like around Varenna. If the field happens to be a pure gauge, \( A_i = iU^\dagger \partial_i U \), the potential energy at such points of the Hilbert space is naturally zero. Imagine that we move along the "generalized coordinate" being the Chern–Simons number \( (2.7) \), fixing all other coordinates whatever they are. Let us take some point \( A^a_i(x) \) with the potential energy \( \mathcal{V} \). If we move to another point which is a gauge transformation of \( A^a_i(x) \) with a winding number \( N_W \), its potential energy will be exactly the same as it is strictly gauge invariant. However the Chern–Simons "coordinate" of the new point will be shifted by an integer \( N_W \) from the original one. We arrive to the conclusion first pointed out by Faddeev \[24\] and Jackiw and Rebbi \[25\] in 1976, that the potential energy of the YM fields is periodic in the particular coordinate called the Chern–Simons number.

One may wish to plot this periodic dependence of the YM potential energy on \( N_{CS} \). Putting such a problem in a situation where the potential energy depends also on an infinite number of other "coordinates", we imply that one is to find the minimal energy path, say, from \( N_{CS} = 0 \) to \( N_{CS} = 1 \), for a given value of \( N_{CS} \). However, the pure YM theory is scale-invariant at the classical level, so to solve the problem one has to fix the spatial size of the \( A_i \) fields somehow. The situation is different in the electro-weak theory where the scale invariance is explicitly broken at the classical level by the non-zero Higgs vacuum expectation value. Therefore the problem of finding the minimal energy pass in the mountain country of the YM Hilbert space is well defined. It was solved by Akiba, Kikuchi and Yanagida in 1988 \[26\], and the reader may satisfy his or her curiosity and have a view of the periodic dependence of the potential energy on the Chern–Simons number in that work (see also ref.\[27\] where this dependence is generalized to non-zero matter density).
The static configuration of the $A_i$ fields corresponding to the saddle point in the minimal-energy path and having exactly $N_{CS} = 1/2$ is called sphaleron. It was found many years ago by Dashen, Hasslacher and Neveu [28] and rediscovered in the context of the electro-weak theory by Manton and Klinkhammer [29]. The sphaleron is the "coordinate" of the top of the potential barrier separating two zero-potential points, $N_{CS} = 0$ and $N_{CS} = 1$. In the electro-weak theory the height of the barrier is of the order of $m_W/\alpha$ ($m_W$ is the mass of the $W$ boson and $\alpha$ is the gauge coupling constant). In an unbroken YM theory like QCD the classical energy barrier between topologically distinct vacua can be made infinitely small due to the scale invariance. However that does not mean that the barriers are easily penetrable: we shall calculate the transition amplitudes from one minimum to another in the next section.

3 Instanton configurations

In perturbation theory one deals with zero-point quantum-mechanical fluctuations of the YM fields near one of the minima, say, at $N_{CS} = 0$. The non-linearity of the YM theory is taken into account as a perturbation, and results in series in $g^2$ where $g$ is the gauge coupling. In that approach one is apparently missing a possibility for the system to tunnel to another minimum, say, at $N_{CS} = 1$. The tunneling is a typical non-perturbative effect in the coupling constant, and instantons have direct relation to the tunneling.

The tunneling amplitude can be estimated as $\exp(-S)$, where $S$ is the action along the classical trajectory in imaginary time, leading from the minimum at $N_{CS} = 0$ at $t = -\infty$ to that at $N_{CS} = 1$ at $t = +\infty$ [31]. According to eq. (2.8) the 4-dimensional topological charge of such trajectory is $Q_T = 1$. To find the best tunneling trajectory having the largest amplitude one has thus to minimize the YM action (2.1) provided the topological charge (2.4) is fixed to be unity. This can be done using the following trick [1]. Consider the inequality

$$0 \leq \int d^4x \left( F_{\mu\nu}^a - \tilde{F}_{\mu\nu}^a \right)^2 = \int d^4x \left( 2F^2 - 2F\tilde{F} \right) = 8g^2S - 64\pi^2Q_T, \quad (3.1)$$

hence the action is restricted from below:
\[
S \geq \frac{8\pi^2}{g^2} Q_T = \frac{8\pi^2}{g^2}. \tag{3.2}
\]

Therefore, the minimal action for a trajectory with a unity topological charge is equal to \(8\pi^2/g^2\), which is achieved if the trajectory satisfies the \textit{self-duality} equation:

\[
F_{\mu\nu}^a = \tilde{F}_{\mu\nu}^a. \tag{3.3}
\]

Notice that any solution of eq. (3.3) is simultaneously a solution of the general YM equation of motion \(D_{\mu}^{ab} F_{b\mu\nu} = 0\): that is because the ”second pair” of the Maxwell equations, \(D_{\mu}^{ab} \tilde{F}_{b\mu\nu} = 0\), is satisfied identically.

To solve eq. (3.3) let us recall a few facts about the Lorentz group \(SO(3,1)\). Since we are talking about the tunneling fields which can only develop in imaginary time, it means that we have to consider the fields in euclidean space-time, so that the Lorentz group is just \(SO(4) = SU(2) \times SU(2)\). The gauge potentials \(A_\mu\) belong to the \((\frac{1}{2}, \frac{1}{2})\) representation of the \(SU(2) \times SU(2)\) group, while the field strength \(F_{\mu\nu}\) belongs to the \((1, 1)\) representation. In other words it means that one linear combination of \(F_{\mu\nu}\) transforms as a vector of the left \(SU(2)\), and another combination transforms as a vector of the right \(SU(2)\). These combinations are

\[
F_L^a = \eta_{\mu\nu}^a (F_{\mu\nu} + \tilde{F}_{\mu\nu}), \quad F_R^a = \bar{\eta}_{\mu\nu}^a (F_{\mu\nu} - \tilde{F}_{\mu\nu}), \tag{3.4}
\]

where \(\eta, \bar{\eta}\) are the so-called ’t Hooft symbols described in ref.[2], see also below. We see therefore that a self-dual field strength is a vector of the left \(SU(2)\) while its right part is zero. Keeping that experience in mind we look for the solution of the self-dual equation in the form

\[
A_\mu^a = \bar{\eta}_{\nu a}^\mu x_\nu \frac{1 + \Phi(x^2)}{x^2}. \tag{3.5}
\]

Using the formulae for the \(\eta\) symbols from ref.[2] one can easily check that the YM action can be then rewritten as

\[
S = \frac{8\pi^2}{g^2} \frac{3}{2} \int d\tau \left[ \frac{1}{2} \left( \frac{d\Phi}{d\tau} \right)^2 + \frac{1}{8} (\Phi^2 - 1)^2 \right], \quad \tau = \ln \left( \frac{x^2}{\rho^2} \right). \tag{3.6}
\]

This can be recognized as the action of the double-well potential whose minima lie at \(\Phi = \pm 1\), and \(\tau\) plays the role of ”time”, \(\rho\) is an arbitrary scale. The trajectory which tunnels from 1 at \(\tau = -\infty\) to \(-1\) at \(\tau = +\infty\) is
\[ \Phi = - \tanh \left( \frac{\tau}{2} \right), \quad (3.7) \]

and its action (3.6) is
\[ S = \frac{8\pi^2}{g^2}, \quad \text{as needed.} \]
Substituting the solution (3.7) into (3.5) we get
\[ A^a_\mu(x) = \frac{2\bar{\eta}_\mu \rho^2}{x^2(x^2 + \rho^2)}, \quad (3.8) \]

The correspondent field strength is
\[ F^a_{\mu\nu} = - \frac{4\rho^2}{(x^2 + \rho^2)^2} \left( \bar{\eta}_\nu^\mu - 2\bar{\eta}_\alpha^\mu \frac{x_\alpha x_\mu}{x^2} - 2\bar{\eta}_\nu^\beta \frac{x_\mu x_\beta}{x^2} \right), \quad F^a_{\mu\nu} F^a_{\mu\nu} = \frac{192\rho^4}{(x^2 + \rho^2)^4}, \quad (3.9) \]

and satisfies the self-duality condition (3.3).

The anti-instanton corresponding to tunneling in the opposite direction, from \( N_{CS} = 1 \) to \( N_{CS} = 0 \), satisfies the anti-self-dual equation, with \( \tilde{F} \to -\tilde{F} \); its concrete form is given by eqs. (3.8, 3.9) with the replacement \( \bar{\eta} \to \eta \).

Eqs. (3.8, 3.9) describe the field of the instanton in the singular Lorentz gauge; the singularity of \( A_\mu \) at \( x^2 = 0 \) is a gauge artifact: the gauge-invariant field strength squared is smooth at the origin. The formulae for instantons are more simple in the Lorentz gauge, and I shall use it further on.

The instanton field, eq. (3.8), depends on an arbitrary scale parameter \( \rho \) which we shall call the instanton size, while the action, being scale invariant, is independent of \( \rho \). One can obviously shift the position of the instanton to an arbitrary 4-point \( z_\mu \) – the action will not change either. Finally, one can rotate the instanton field in colour space by constant unitary matrices \( U \). For the \( SU(2) \) gauge group this rotation is characterized by 3 parameters, e.g. by Euler angles. For a general \( SU(N_c) \) group the number of parameters is \( N_c^2 - 1 \) (the total number of the \( SU(N_c) \) generators) minus \( (N_c - 2)^2 \) (the number of generators which do not effect the left upper \( 2 \times 2 \) corner where the standard \( SU(2) \) instanton (3.8) is residing), that is \( 4N_c - 5 \). These degrees of freedom are called instanton orientation in colour space. All in all there are
\[ 4 \ (\text{centre}) \ + \ 1 \ (\text{size}) \ + \ (4N_c - 5) \ (\text{orientations}) \ = \ 4N_c \quad (3.10) \]
so-called collective coordinates describing the field of the instanton, of which the action is independent.
It is convenient to introduce $2 \times 2$ matrices

$$\sigma^\pm = (\pm i \vec{\sigma}, 1), \quad x^\pm = x_\mu \sigma^\pm_\mu,$$  

such that

$$2i\tau^a \eta_{\mu a} = \sigma_\mu^+ \sigma_\nu^- - \sigma_\nu^+ \sigma_\mu^-; \quad 2i\tau^a \bar{\eta}_{\mu a} = \sigma_\mu^- \sigma_\nu^+ - \sigma_\nu^- \sigma_\mu^+,$$  

then the instanton field with arbitrary center $z_\mu$, size $\rho$ and colour orientation $U$ in the $SU(N_c)$ gauge group can be written as

$$A^a_\mu = A^a_\mu t^a = -i\rho^2 U[\sigma_\mu^- (x - z)^+ - (x - z)_\mu] U^+ \over (x - z)^2 [\rho^2 + (x - z)^2], \quad Tr(t^a t^b) = 1/2 \delta^{ab},$$

or

$$A^a_\mu = 2\rho^2 O^{ab} \eta_{\mu b}(x - z)_\mu \over (x - z)^2 [\rho^2 + (x - z)^2], \quad O^{ab} = Tr(U^+ t^a U \sigma^b), \quad O^{ab} O^{ac} = \delta^{bc}.$$  

Physically, one can think of instantons in two ways: on one hand it is a tunneling process occurring in imaginary time (this interpretation belongs to V.Gribov, 1976), on the other hand it is a localized pseudoparticle in the euclidean space (A.Polyakov, 1977 [31]).

### 4 Gluon condensate

The QCD perturbation theory implies that the fields $A^a_\mu(x)$ are performing quantum zero-point oscillations; in the lowest order these are just plane waves with arbitrary frequencies. The aggregate energy of these zero-point oscillations, $(\mathbf{B}^2 + \mathbf{E}^2)/2$, is divergent as the fourth power of the cutoff frequency, however for any state one has $\langle F_{\mu\nu}^2 \rangle = 2(\mathbf{B}^2 - \mathbf{E}^2) = 0$, which is just a manifestation of the virial theorem for harmonic oscillators: the average potential energy is equal to that of the kinetic (I am temporarily in the Minkowski space). One can prove that this is also true in any order of the perturbation theory in the coupling constant, provided one does not violate the Lorentz symmetry and the renormalization properties of the theory. Meanwhile, we know from the QCD sum rules phenomenology that the QCD vacuum possesses what is called *gluon condensate* [3].
\[
\frac{1}{32\pi^2} \langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle \simeq (200 \text{ MeV})^4 > 0. \tag{4.1}
\]

Instantons suggest an immediate explanation of this basic property of QCD. Indeed, instanton is a tunneling process, it occurs in imaginary time; therefore in Minkowski space one has \( E^a_i = \pm i B^a_i \) (this is actually the duality eq. (3.3)). Therefore, one gets a chance to explain the gluon condensate. In euclidean space the electric field is real as well as the magnetic one, and the gluon condensate is just the average action density. Let us make a quick estimate of its value.

Let the total number of \( I \)'s and \( \bar{I} \)'s in the 4-dimensional volume \( V \) be \( N \). Assuming that the average separations of instantons are larger than their average sizes (to be justified below), we can estimate the total action of the ensemble as the sum of individual actions (see eq. (3.2)):

\[
\langle F^2_{\mu\nu} \rangle V = \int d^4 x F^2_{\mu\nu} \simeq N \cdot 32\pi^2, \tag{4.2}
\]

hence the gluon condensate is directly related to the instanton density in the 4-dimensional euclidean space-time:

\[
\frac{1}{32\pi^2} \langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle \simeq \frac{N}{V} \equiv \frac{1}{\bar{R}^4}. \tag{4.3}
\]

In order to get the phenomenological value of the condensate one needs thus to have the average separation between pseudoparticles \( \bar{R} \)

\[
\bar{R} \simeq \frac{1}{200 \text{ MeV}} = 1 \text{ fm}. \tag{4.4}
\]

There is another point of view on the gluon condensate which I describe briefly. In principle, all information about field theory is contained in the partition function being the functional integral over the fields. In the euclidean formulation it is

\[
\mathcal{Z} = \int D A_\mu \exp \left( -\frac{1}{4g^2} \int d^4 x F^2_{\mu\nu} \right) \overset{T \to \infty}{\longrightarrow} e^{-ET}, \tag{4.5}
\]

where I have used that at large (euclidean) time \( T \) the partition function picks up the ground state or vacuum energy \( E \). For the sake of brevity I do not write the gauge fixing and Faddeev–Popov ghost terms. If the state is homogeneous, the energy can be written as \( E = \theta_{44} V^{(3)} \) where \( \theta_{\mu\nu} \) is the stress-energy tensor and \( V^{(3)} \) is the 3-volume of the system. Hence, at large 4-volumes \( V = V^{(3)} T \) the partition function
is $Z = \exp(-\theta_{44}V)$. This $\theta_{44}$ includes zero-point oscillations and diverges badly. A more reasonable quantity is the partition function, normalized to the partition function understood as a perturbative expansion about the zero-field vacuum:

$$\frac{Z}{Z_{P.T.}} = \exp\left[-(\theta_{44} - \theta_{44}^{P.T.})V\right].$$  \hspace{1cm} (4.6)

We expect that the non-perturbative vacuum energy density $\theta_{44} - \theta_{44}^{P.T.}$ is a negative quantity since we have allowed for tunneling: as usual in quantum mechanics, it lowers the ground state energy. If the vacuum is isotropical, one has $\theta_{44} = \theta_{\mu\mu}/4$. Using the trace anomaly,

$$\theta_{\mu\mu} = \frac{\beta(g^2)}{4g^4} \frac{(F_{\mu\nu}^a)^2}{32\pi^2} \simeq -b \frac{F_{\mu\nu}^4}{32\pi^2},$$  \hspace{1cm} (4.7)

where $\beta(g^2)$ is the Gell-Mann–Low function,

$$\beta(g^2) \equiv \frac{dg^2(M)}{\ln M} = -b \frac{g^4}{8\pi^2} - b' \frac{g^6}{2(8\pi^2)^2} - ..., \quad b = \frac{11}{3} N_c, \quad b' = \frac{34}{3} N_c^2,$$  \hspace{1cm} (4.8)

one gets [7]:

$$\frac{Z}{Z_{P.T.}} = \exp\left(\frac{b}{4} V \langle F_{\mu\nu}^2/32\pi^2 \rangle_{NP}\right).$$  \hspace{1cm} (4.9)

where $\langle F_{\mu\nu}^2 \rangle_{NP}$ is the gluon field vacuum expectation value which is due to non-perturbative fluctuations, i.e. the gluon condensate. The aim of any QCD-vacuum builder is to minimize the vacuum energy or, equivalently, to maximize the gluon condensate.

It is important that it is a renormalization-invariant quantity [1], meaning that its dependence on the ultraviolet cutoff $M$ and the bare charge $g^2(M)$ given at this cutoff is such that it is actually cutoff-independent:

The latter can be distinguished from the former by imposing a condition that it does not contain integration over singular Yang–Mills potentials; recall that the instanton potentials are singular at the origins.

To be more precise, the renorm-invariant quantity is $\langle \theta_{\mu\mu} \rangle$, see eq. (4.7); however if the coupling constant at the ultra-violet cutoff scale $g^2(M)$ is small enough, it is sufficient to use the beta function from one loop.
\[ \langle F_{\mu\nu}^2/32\pi^2 \rangle_{NP} = c \left[ M \exp \left( -\int_{g^2(M)} \frac{dg^2}{\beta(g^2)} \right) \right]^4 \simeq c' M^4 \exp \left[ -\frac{32\pi^2}{bg^2(M)} \right]. \] (4.10)

By definition of the QCD partition function \[ (4.5), \] the l.h.s. of eq. (4.10) is equal to \(-\frac{1}{V} d \ln Z / d [8\pi^2/g^2(M)]\). Applying the same differentiation operator to eq. (4.10) one gets a low-energy theorem \[ [32]: \]
\[
\frac{d^2 \ln (Z/Z_{P.T.})}{(d \left[ -\frac{8\pi^2}{g^2(M)} \right])^2} = \left\langle \int d^4x \frac{F_{\mu\nu}^2}{32\pi^2} \int d^4y \frac{F_{\mu\nu}^2}{32\pi^2} \right\rangle - \left\langle \int d^4x \frac{F_{\mu\nu}^2}{32\pi^2} \right\rangle^2
= \frac{4}{b} \left\langle \int d^4x \frac{F_{\mu\nu}^2}{32\pi^2} \right\rangle.
\] (4.11)

If the bare coupling \(g^2(M)\) is not chosen small enough, there are obvious corrections to this formula, following from the higher-order terms in the beta function.

This low-energy theorem has an instructive consequence for instantons, predicting the dispersion of the number of pseudoparticles in a given 4-dimensional volume [7]. Assuming the instanton ensemble is sufficiently dilute (corrections to this assumption will be discussed below) one can rewrite the low-energy theorem (4.11) as
\[
\langle N^2 \rangle - \langle N \rangle^2 = \frac{4}{b} \langle N \rangle = \frac{12}{11N_c} \langle N \rangle,
\] (4.12)
where \(N \equiv N_+ + N_-\) is the total number of \(I\)'s and \(\bar{I}\)'s.

Thus it follows from the renormalization properties of the Yang–Mills theory that the dispersion of the number of pseudoparticles is less than for a free gas for which one would get a Poisson distribution with \(\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle\). In particular, at \(N_c \to \infty\) the dispersion becomes zero, as it should be.

One concludes that some kind of interaction of instantons with each other is crucial to support the needed renormalization properties of the underlying theory: any cutoff of the integrals over instanton sizes "by itself" leads to the Poisson distribution and hence to the violation of the low energy theorem (4.12).

Differentiating \(\ln Z\) many times in respect to the bare coupling \(g^2(M)\), one can easily generalize eq. (4.12) to any moments of the distribution. In short, the distribution in the number of pseudoparticles should be (for large \(\langle N \rangle\))
\[
P(N) \sim \exp \left[ -\frac{b}{4} \left( \ln \frac{N}{\langle N \rangle} - 1 \right) \right].
\] (4.13)
5 One-instanton weight

The words "instanton vacuum" mean that one assumes that the QCD partition function is mainly saturated by an ensemble of interacting $I$’s and $\bar{I}$’s together with quantum fluctuations about them. Instantons are necessarily present in the QCD vacuum if only because they lower the vacuum energy in respect to the purely perturbative (divergent) one. The question is whether they give the dominant contribution to the gluon condensate, and to other basic quantities. To answer this question one has to compute the partition function (4.5) assuming that it is mainly saturated by instantons, and to compare the obtained gluon condensate with the phenomenological one. This work has been done a decade ago in ref. [7]; today direct lattice measurements confirm that the answer to the question is positive: the observed density of $I$’s and $\bar{I}$’s is in agreement with the estimate (4.4).

The starting point of this calculation is the contribution of one isolated instanton to the partition function (4.5) or the one-instanton weight. It has been computed by ’t Hooft [2], and generalized to arbitrary groups by Bernard in a very clearly written paper [33].

The general field can be decomposed as a sum of a classical field of an instanton $A_{\mu}(x, \xi)$ where $\xi$ is a set of $4N_c$ collective coordinates characterizing a given instanton (see eq. (3.13)), and of a presumably small quantum field $a_{\mu}(x)$:

$$A_{\mu}(x) = A_{\mu}^I(x, \xi) + a_{\mu}(x).$$

There is a subtlety in this decomposition due to the gauge freedom: an interested reader is addressed to ref. [7] where this subtlety is treated in detail.

The action is

$$\text{Action} = \frac{1}{4g^2} \int d^4x \, F_{\mu\nu}^2 = \frac{8\pi^2}{g^2} + \frac{1}{g^2} \int d^4x \, D_{\mu}F_{\mu\nu}a_{\nu}$$

$$+ \frac{1}{2g^2} \int d^4x \, a_{\mu}W_{\mu\nu}a_{\nu} + O(a^3). \tag{5.2}$$

Here the term linear in $a_{\mu}$ drops out because the instanton field satisfies the equation of motion. The quadratic form $W_{\mu\nu}$ has $4N_c$ zero modes related to the fact that the action does not depend on $4N_c$ collective coordinates. This brings in a divergence in the functional integral over the quantum field $a_{\mu}$ which, however, can
and should be qualified as integrals over the collective coordinates: centre, size and orientations. Formally the functional integral over $a_\mu$ gives

$$\frac{1}{\sqrt{\det W_{\mu\nu}(A^I)}},$$

which must be $i)$ normalized (to the determinant of the free quadratic form, i.e. with no background field), $ii)$ regularized (for example by using the Pauli–Villars method), and $iii)$ accounted for the zero modes. Actually one has to compute a ”quadrupole” combination,

$$\left[ \frac{\det' W \det(W_0 + M^2)}{\det W_0 \det(W + M^2)} \right]^{-\frac{1}{2}},$$

where $W_0$ is the quadratic form with no background field and $M^2$ is the Pauli–Villars mass playing the role of the ultra-violet cutoff; the prime reminds that the zero modes should be removed and treated separately. The resulting one-instanton contribution to the partition function (normalized to the free one) is $[2, 33]$

$$\frac{Z_{1-\text{inst}}}{Z_{P.T.}} = \int d^4z_\mu \int d\rho \int dU \ d_0(\rho),$$

$$d_0(\rho) = C(N_c) \left[ \frac{8\pi^2}{g^2(M)} \right]^{2N_c} (M\rho)^{\frac{N_c}{3}} \exp \left( -\frac{8\pi^2}{g^2(M)} \right).$$

The product of the last two factors is actually a combination of the cutoff $M$ and the bare coupling constant $g^2(M)$ given at this cutoff, which is cutoff-independent; it can be replaced by $(\Lambda_{QCD}/\rho)^{11N_c/3}$, see eq. (4.10). This is the way $\Lambda_{QCD}$ enters into the game; henceforth all dimensional quantities will be expressed through $\Lambda_{QCD}$, which is, of course, a welcome message. The numerical coefficient $C(N_c)$ depends explicitly on the number of colours; it also implicitly depends on the regularization scheme used. In the Pauli–Villars scheme exploited above $[33]$

$$C(N_c) = \frac{4.60 \exp(-1.68N_c)}{\pi^2(N_c - 1)! (N_c - 2)!}.$$  

If the scheme is changed, one has to change the coefficient $C(N_c) \to C'(N_c) = C(N_c) \cdot (\Lambda/\Lambda')^b$. One has: $\Lambda_{P.V.} = 1.09\Lambda_{MS} = 31.32\Lambda_{lat} = ...$

We have thus obtained the instanton weight in the one-loop approximation:
where $\beta(\rho) = 8\pi^2/g^2(\rho)$ is the one-loop inverse charge (not to be confused with the Gell-Mann–Low function!)

$$\beta(\rho) = b \ln \left( \frac{1}{\Lambda \rho} \right), \quad b = \frac{11}{3} N_c. \tag{5.8}$$

Note that the $\beta$ factor in the pre-exponent starts to "run" only at the 2-loop level, hence its argument is taken at the ultra-violet cutoff $M$.

In the 2-loop approximation the instanton weight is given by [34]

$$d_0(\rho) = \frac{C(N_c)}{\rho^5} \beta(M)^{2N_c} \exp \left[ -\beta(\rho) \right], \tag{5.7}$$

where $\beta(\rho)$ is the inverse charge to the two-loop accuracy:

$$\beta^\prime(\rho) = \beta(\rho) + \frac{b'}{2b} \ln \frac{2\beta(\rho)}{b}, \quad b' = \frac{34}{3} N_c^2. \tag{5.9}$$

Notice that both one- and two-loop eqs. (5.7, 5.9) formulae show that the integral over the instanton sizes $\rho$ in eq. (5.5) diverges as a high power of $\rho$ at large $\rho$: this is of course the consequence of asymptotic freedom. It means that individual instantons tend to swell. This circumstance plagued the instanton calculus for many years. If one attempts to cut the $\rho$ integrals "by hand", one violates the renormalization properties of the YM theory, as explained in the previous section. Actually the size integrals appear to be cut from above due to instanton interactions.

### 6 Instanton ensemble

To get a volume effect from instantons one needs to consider an $I\bar{I}$ ensemble, with their total number $N$ proportional to the 4-dimensional volume $V$. Immediately a mathematical difficulty arises: any superposition of $I$’s and $\bar{I}$’s is not, strictly speaking, a solution of the equation of motion, therefore, one cannot directly use the semiclassical approach of the previous section. There are two ways two overcome this difficulty. One is to use a variational principle [4], the other is to use the effective YM lagrangian in the instanton field [35].
The idea of the variational principle is to use a modified YM action for which a chosen $I\bar{I}$ ansatz is a saddle point. Exploiting the convexity of the exponent one can prove that the true vacuum energy is less than that obtained from the modified action. One can therefore use variational parameters (or even functions) to get a best upper bound for the vacuum energy. We call it the Feynman variational principle since the method was suggested by Feynman in his famous study of the polaron problem. The gauge theory is more difficult, though: one has not to lose either gauge invariance or the renormalization properties of the YM theory. These difficulties were overcome in ref. [7]. A decade later I still do not think one can do an analytical evaluation of the $I\bar{I}$ ensemble much better than in that paper: after all we are dealing with the "strong interactions", meaning that all dimensionless quantities are generally speaking of the order of unity – there are no small parameters in the theory. Therefore, one has to use certain numerical methods, and the variational principle is among the best. Today's direct lattice investigation of the $I\bar{I}$ ensemble seem to indicate that Petrov and I have obtained rather accurate numbers in this terrible problem.

The normalized (to perturbative) and regularized YM partition function takes the form of a partition function for a grand canonical ensemble of interacting pseudoparticles of two kind, $I$’s and $\bar{I}$’s:

$$\frac{Z}{Z_{PT}} = \sum_{N_+,N_-} \frac{1}{N_+! N_-!} N_+^{N_+} N_-^{N_-} \prod_n \int d^4 z_n d\rho_n dO \, d_0(\rho) \exp(-U_{int}), \quad (6.1)$$

where $d_0(\rho)$ is the 1-instanton weight (5.7) or (5.8). The integrals are over the collective coordinates of (anti)instantons: their coordinates $z$, sizes $\rho$ and orientations given by $SU(N_c)$ unitary matrices in the adjoint representation $O$; $dO$ means the Haar measure normalized to unity. The instanton interaction potential $U_{int}$ (to be discussed below) depends on the separation between pseudoparticles, $z_m - z_n$, their sizes $\rho_{m,n}$ and their relative orientations $O_m O_n^T$.

In the variational approach the interaction between instantons arise from $i$) the defect of the classical action, $ii$) the non-factorization of quantum determinants and $iii$) the non-factorization of jacobians when one passes to integration over the collective coordinates. All three factors are ansatz-dependent, but there is a tendency towards a cancellation of the ansatz-dependent pieces. Qualitatively, in any ansatz the interactions between $I$’s and $\bar{I}$’s resemble those of molecules: at large separations there is an attraction, at smaller separations there is a repulsion. It is very important
that the interactions depend on the relative orientations of instantons: if one averages over orientations (which is the natural thing to do if the $I\bar{I}$ medium is in a disordered phase; if not, one would expect a spontaneous breaking of both Lorentz and colour symmetries \cite{7}), the interactions seem to be repulsive at any separations.

In general, the mere notion of the instanton interactions is notorious for being ill-defined since instanton + antiinstanton is not a solution of the equation of motion. Such a configuration belongs to a sector with topological charge zero, thus it seems to be impossible to distinguish it from what is encountered in perturbation theory. The variational approach uses brute force in dealing with the problem, and the results appear to be somewhat dependent on the ansatz used. Thanks to the inequality for the vacuum energy mentioned above, we still get quite a useful information. However, recently a mathematically unequivocal definition of the instanton interaction potential has been suggested, based on analyticity and unitarity \cite{36,35}. This definition automatically cuts off the contribution of the perturbation theory. The first three leading terms for the interaction potential at large separations has been computed \cite{35}, at smaller separations one observes a strong repulsion \cite{37}, though the exact form is still unknown.

Summing up the discussion, I would say that today there exists no evidence that a variational calculation with the simplest sum ansatz used in ref. \cite{7} is qualitatively of even quantitatively incorrect, therefore I will cite the numerics from those calculations in what follows.

The main finding is that the $I\bar{I}$ ensemble stabilizes at a certain density related to the $\Lambda_{QCD}$ parameter (there is no other dimensional quantity in the theory!):

$$\langle F_{\mu\nu}^2/32\pi^2 \rangle \simeq \frac{1}{V} \langle Q_T^2 \rangle \simeq \frac{N}{V} \geq (0.75\Lambda_{\bar{M}\bar{S}})^4$$

(6.2)

which would require $\Lambda_{\bar{M}\bar{S}} \simeq 265 \text{ MeV}$ to get the phenomenological value of the condensate. It should be mentioned however that using more sophisticated variational Ansätze one can obtain a larger coefficient in eq. (6.2) and hence would need smaller values of $\Lambda$.

The average sizes $\bar{\rho}$ appear to be much less than the average separation $\bar{R}$. Numerically we have found for the $SU(3)$ colour:

$$\frac{\bar{\rho}}{\bar{R}} \simeq \frac{1}{3}$$

(6.3)
which coincides with what was suggested previously by Shuryak [6] from considering the phenomenological applications of the instanton vacuum. This value should be compared with that found from direct lattice measurements [17]: \( \bar{\rho}/\bar{R} \simeq 0.37 - 0.4 \), depending on where one stops the cooling procedure. The packing fraction, i.e. the fraction of the 4-dimensional volume occupied by instantons appears thus to be rather small, \( \pi^2 \bar{\rho}^4/\bar{R}^4 \simeq 1/8 \). This small number can be traced back to the "accidentally" large numbers appearing in the 4-dimensional YM theory: the 11\( N_c/3 \) of the Gell-Mann–Low beta function and the number of zero modes being 4\( N_c \). We have checked that the same variational principle applied to the 2-dimensional sigma models also possessing instantons, does not yield a small packing fraction. The 4-dimensional YM theory seems to be simpler from this angle. Meanwhile, it is exactly this small packing fraction of the instanton vacuum which gives an a posteriori justification for the use of the semi-classical methods. As I shall show in the next sections, it also enables one to identify adequate degrees of freedom to describe the low-energy QCD.

7 Chiral symmetry breaking in QCD

The QCD lagrangian with \( N_f \) massless flavours is known to posses a large global symmetry, namely a symmetry under \( U(N_f) \times U(N_f) \) independent rotations of left- and right-handed quark fields. This symmetry is called chiral. Instead of rotating separately the 2-component Weyl spinors one can make independent vector and axial \( U(N_f) \) rotations of the full 4-component Dirac spinors – the QCD lagrangian is invariant under these transformations too. Meanwhile, the axial transformation mixes states with different P-parities. Therefore, if that symmetry remains exact one would observe parity degeneracy of all states with otherwise the same quantum numbers. In reality the splittings between states with the same quantum numbers but opposite parities is huge. For example, the splitting between the vector \( \rho \) and the axial \( a_1 \) meson is \( (1200 - 770) \simeq 400 \text{ MeV} \); the splitting between the nucleon and its parity partner is even larger: \( (1535 - 940) \simeq 600 \text{ MeV} \). (Another possibility is that the nucleon is just massless, which looks even less pleasant.)

The splittings are too large to be explained by the small bare or current quark masses which break the chiral symmetry from the beginning. Indeed, the current masses of light quarks are: \( m_u \simeq 4 \text{ MeV}, \ m_d \simeq 7 \text{ MeV}, \ m_s \simeq 150 \text{ MeV} \). The only conclusion one can draw from these numbers is that the chiral symmetry of the
QCD lagrangian is broken down \textit{spontaneously}, and very strongly. Consequently, one should have light (pseudo) Goldstone pseudoscalar hadrons – their role is played by pions which indeed are by far the lightest hadrons.

The order parameter associated with chiral symmetry breaking is the so-called \textit{chiral} or \textit{quark condensate}:

\begin{equation}
\langle \bar{\psi}\psi \rangle \simeq -(250 \text{ MeV})^3. \tag{7.1}
\end{equation}

It should be noted that this quantity is well defined only for massless quarks, otherwise it is somewhat ambiguous. By definition, this is the quark Green function taken at one point; in momentum space it is a closed quark loop. If the quark propagator has only the "slash" term, the trace over the spinor indices implied in this loop would give an identical zero. Therefore, chiral symmetry breaking implies that a massless (or nearly massless) quark develops a non-zero dynamical mass (i.e. a "non-slash" term in the propagator). There are no reasons for this quantity to be a constant independent of the momentum; moreover, we understand that it should anyhow vanish at large momentum. The value of the dynamical mass at zero momentum can be estimated as one half of the \(\rho\) meson mass or one third of the nucleon mass, that is about \(M(0) \simeq 350 - 400\text{ MeV}\); this scale is also related to chiral symmetry breaking and should be emerge together with the condensate \((7.1)\).

One could imagine a world without confinement but with chiral symmetry breaking: it would not be drastically different from what we meet in reality. There would be a tightly bound light Goldstone pion, and relatively loosely bound \(\rho\) meson and nucleon with correct masses, which, however, would be possible to ionize from time to time. Probably the spectrum of the highly excited hadrons would be wrong, though even that is not so clear \[38\]. We see, thus, that the spontaneous chiral symmetry breaking is the main dynamical happening in QCD, which determines the face of the strong interactions world. In the next sections I explain why and how chiral symmetry is broken in the instanton vacuum, and why it is a most realistic picture. The forthcoming sections are based on our work with Petrov \[8, 9\].
8 Chiral symmetry breaking by definition

I start by writing down the QCD partition function. Functional integrals are well defined in euclidean space which is obtained by the following formal substitutions of Minkowski space quantities:

\[ ix_M = x_{E_4}, \quad x_{M_i} = x_{E_i}, \quad A_{M_0} = iA_{E_4}, \quad A_{M_i} = A_{E_i}, \]

\[ i\bar{\psi}_M = \psi^\dagger_{E_4}, \quad \gamma_{M_0} = \gamma_{E_4}, \quad \gamma_{M_i} = i\gamma_{E_i}, \quad \gamma_{M_5} = \gamma_{E_5}. \]  \(8.1\)

Neglecting for brevity the gauge fixing and Faddeev–Popov ghost terms, the QCD partition function can be written as

\[ Z = \int DA \psi \bar{D} \psi \exp \left[ -\frac{1}{4g^2} \int F_{\mu\nu}^2 + \sum_f N_f \int \psi^\dagger_f (i\nabla + im_f) \psi_f \right] \]

\[ = \int DA \psi \bar{D} \psi \exp \left[ -\frac{1}{4g^2} \int F_{\mu\nu}^2 \right] \prod_f \det(i\nabla + im_f). \]  \(8.2\)

The chiral condensate of a given flavour \( f \) is, by definition,

\[ \langle \bar{\psi}_f \psi_f \rangle_M = -i \langle \bar{\psi}^\dagger_f \psi_f \rangle_E = -\frac{1}{V} \frac{\partial}{\partial m_f} (\ln Z)_{m_f \to 0}. \]

The Dirac operator has the form

\[ i\nabla = \gamma_\mu (i\partial_\mu + A^{II}_\mu + a_\mu) \]  \(8.4\)

where \( A^{II}_\mu \) denotes the classical field of the \( II \) ensemble and \( a_\mu \) is a presumably small field of quantum fluctuations about that ensemble, which I shall neglect as it has little impact on chiral symmetry breaking. Integrating over \( DA_\mu \) in eq. \((8.2)\) means averaging over the \( II \) ensemble, therefore one can write

\[ Z = \overline{\det(i\nabla + im)} \]  \(8.5\)

where I temporarily restrict the discussion to the case of only one flavour for simplicity. Because of the \( im \) term the Dirac operator in \((8.5)\) is formally not hermitean; however the determinant is real due to the following observation. Suppose we have found the eigenvalues and eigenfunctions of the Dirac operator,

\[ i\nabla \Phi_n = \lambda_n \Phi_n, \]  \(8.6\)
then for any $\lambda_n \neq 0$ there is an eigenfunction $\Phi_n' = \gamma_5 \Phi_n$ whose eigenvalue is $\lambda_n' = -\lambda_n$. This is because $\gamma_5$ anticommutes with $i\nabla$. Owing to this the fermion determinant can be written as

$$det(i\nabla + im) = \prod_n (\lambda_n + im) = \sqrt{\prod_n (\lambda_n^2 + m^2)} = \exp \left[ \frac{1}{2} \sum_n \ln(\lambda_n^2 + m^2) \right]$$

$$= \exp \left[ \frac{1}{2} \int_{-\infty}^{\infty} d\lambda \, \nu(\lambda) \ln(\lambda^2 + m^2) \right], \quad \nu(\lambda) \equiv \sum_n \delta(\lambda - \lambda_n), \quad (8.7)$$

where I have introduced the spectral density $\nu(\lambda)$ of the Dirac operator $i\nabla$. Note that the last expression is real and even in $m$, which is a manifestation of the QCD chiral invariance.

Differentiating eq. (8.7) in $m$ and putting it to zero one gets according to the general eq. (8.3) a formula for the chiral condensate:

$$\langle \bar{\psi} \psi \rangle = -\frac{1}{V} \frac{\partial}{\partial m} \left[ \frac{1}{2} \int d\lambda \, \nu(\lambda) \ln(\lambda^2 + m^2) \right]_{m \to 0} = -\frac{1}{V} \left. \int_{-\infty}^{\infty} d\lambda \, \nu(\lambda) \frac{m}{\lambda^2 + m^2} \right|_{m \to 0} \quad (8.8)$$

where $\nu(\lambda)$ means averaging over the instanton ensemble together with the weight given by the fermion determinant itself. The latter, however, may be cancelled in the so-called quenched approximation where the back influence of quarks on the dynamics is neglected. Theoretically, this is justified at large $N_c$.

Naively, one would think that the r.h.s. of eq. (8.8) is zero at $m \to 0$. That would be correct for a finite-volume system with a discrete spectral density. However, if the volume goes to infinity faster than $m$ goes to zero (which is what one should assume in the thermodynamic limit) one should use instead

$$\frac{m}{\lambda^2 + m^2} \xrightarrow{\lambda \to 0} \text{sign}(m) \pi \delta(\lambda), \quad (8.9)$$

so that one gets

$$\langle \bar{\psi} \psi \rangle = -\frac{1}{V} \text{sign}(m) \pi \nu(0). \quad (8.10)$$

The chiral condensate is thus proportional to the averaged spectral density of the Dirac operator at zero eigenvalues. The appearance of the sign function is not accidental: it means that at small $m$ QCD partition functions depends on $m$ non-analytically:
\[ \ln Z = V(c_0 + \pi \nu(0)|m| + c_2 m^2 \ln(|m|) + ...). \] (8.11)

The fact that the partition function is even in \( m \) is the reflection of the original invariance of the QCD under \( \gamma_5 \) rotations; the fact that it is non-analytic in the symmetry-breaking parameter \( m \) is typical in the situation where symmetry is broken spontaneously.

A generalization of the above formulae to the case of several flavours is simple \[21\]. Consider \( N_f \) quark flavours with a most general mass matrix

\[ \psi^\dagger \left( im_L \frac{1 + \gamma_5}{2} + im_R \frac{1 - \gamma_5}{2} \right) \psi. \] (8.12)

The fermion determinant can be written as

\[
\det(i\nabla + im) = \exp \left[ \frac{1}{2} \int d\lambda \nu(\lambda) \ln \det N_f(\lambda^2 + mLm_R) \right]
= \exp \left[ -\frac{1}{2} \int d\lambda \nu(\lambda) \int \frac{dt}{t} \text{Tr}_{N_f} \exp \left( -t(\lambda^2 + mLm_R) \right) \right]
\] (8.13)

where \( \lambda^2 \) is a \( N_f \times N_f \) matrix proportional to the unit matrix. Let us expand \( \nu(\lambda) \) at small \( \lambda \), and leave only the \( \nu(0) \) term. Integrating (8.13) in \( \lambda \) we get the correspondent non-analytic term in the partition function:

\[
\ln Z \sim -\frac{\sqrt{\pi}}{2} \nu(0) \int_0^\infty \frac{dt}{t^{3/2}} \text{Tr}_{N_f} \left[ \exp(-tm_Lm_R) \right] = \pi \nu(0) \text{Tr}_{N_f}(m_Lm_R)^{1/2}. \] (8.14)

This expression is non-analytic in the mass matrix; differentiating it in respect to the masses one finds the phases of the condensate.

In the next sections I shall present three different derivations of the fact that instantons indeed break chiral symmetry.

9 Chiral symmetry breaking by instantons: qualitative derivation

The key observation is that the Dirac operator in the background field of one (anti) instanton has an exact zero mode with \( \lambda = 0 \) \[22\]. It is a consequence of the general Atiah–Singer index theorem; in our case it is guaranteed by the unit Pontryagin index.
or the topological charge of the instanton field. These zero modes are 2-component Weyl spinors: right-handed for instantons and left-handed for antiinstantons. Explicitly, the zero modes are \((\alpha = 1...N_c)\) is the colour and \(i,j,k = 1,2\) are the spinor indices):

\[
[\Phi_R(x-z_1)]_i^\alpha = \phi(x-z_1, \rho_1)(x-z_1)^i_1 U_{1k}^\alpha \epsilon^{jk},
\]

\[
[\Phi_L(x-z_2)]_i^\alpha = \phi(x-z_2, \rho_2)(x-z_2)^i_2 U^{\alpha}_{2k} \epsilon^{jk},
\]

\[
\phi(x, \rho) = \frac{\rho}{\pi(2\pi^2)^{1/2}(\pi^2 + \rho^2)^{3/2}}.
\]

Here \(z_{1\mu}, \rho_1, U_1\) are the center, size and orientation of an instanton and \(z_{2\mu}, \rho_2, U_2\) are those of an antiinstanton, respectively, \(\epsilon^{jk}\) is the \(2 \times 2\) antisymmetric matrix.

For infinitely separated \(I\) and \(\bar{I}\) one has thus two degenerate states with exactly zero eigenvalues. As usual in quantum mechanics, this degeneracy is lifted through the diagonalization of the hamiltonian, in this case the hamiltonian is the full Dirac operator. The two "wave functions" which diagonalize the "hamiltonian" are the sum and the difference of the would-be zero modes, one of which is a 2-component left-handed spinor, and the other is a 2-component right-handed spinor. The resulting wave functions are 4-component Dirac spinors; one can be obtained from another by multiplying by the \(\gamma_5\) matrix. As the result the two would-be zero eigenstates are split symmetrically into two 4-component Dirac states with non-zero eigenvalues equal to the overlap integral between the original states:

\[
\lambda = \pm |T_{12}|, \quad T_{12} = \int d^4 x \Phi_1^\dagger (-i \partial) \Phi_2 \xrightarrow{R_{12} \rightarrow \infty} -2\frac{\rho_1 \rho_2}{R_{12}^4} \text{Tr}(U_1^\dagger U_2 R_{12}^+). \]

We see that the splitting between the would-be zero modes fall off as the third power of the distance between \(I\) and \(\bar{I}\); it also depends on their relative orientation. The fact that two levels have eigenvalues \(\pm \lambda\) is in perfect agreement with the \(\gamma_5\) invariance mentioned in the previous section.

When one adds more \(I\)'s and \(\bar{I}\)'s each of them brings in a would-be zero mode. After the diagonalization they get split symmetrically in respect to the \(\lambda = 0\) axis. Eventually, for an \(II\) ensemble one gets a continuous band spectrum with a spectral density \(\nu(\lambda)\) which is even in \(\lambda\) and finite at \(\lambda = 0\).

One can make a quick estimate of \(\nu(0)\): Let the total number of \(I\)'s and \(\bar{I}\)'s in the 4-dimensional volume \(V\) be \(N\). The spread \(\Delta\) of the band spectrum of the would-be zero modes is given by their average overlap \((9.2)\):
where $\bar{\rho}$ is the average size and $\bar{R} = (N/V)^{-1/4}$ is the average separation of instantons. Note that the spread of the would-be zero modes is parametrically much less than $1/\bar{\rho}$ which is the typical scale for the non-zero modes. Therefore, neglecting the influence of the non-zero modes is justified if the packing fraction of instantons is small enough.

From eq. (8.10) one gets an estimate for the chiral condensate induced by instantons:

$$\langle \bar{\psi} \psi \rangle = -\frac{\pi}{V} \nu(0) \simeq -\frac{\pi N}{V \Delta} \sim -\frac{1}{R^2 \bar{\rho}}. \quad (9.4)$$

Note that the chiral condensate appears to be proportional to the square root of the instanton density $N/V$: again it is as it should be for the order parameter of spontaneous symmetry breaking.

It is amusing that the physics of the spontaneous breaking of chiral symmetry resembles the so-called Mott–Anderson conductivity in disordered systems. Imagine random impurities (atoms) spread over a sample with final density, such that each atom has a localized bound state for an electron. Due to the overlap of those localized electron states belonging to individual atoms, the levels are split into a band, and the electrons become delocalized. That means conductivity of the sample. In our case the localized zero quark modes of individual instantons randomly spread over the volume get delocalized due to their overlap, which means chiral symmetry breaking.

There is a difference with the Mott–Anderson conductivity, though. In the case of atoms, the wave functions have an exponential falloff, so that the overlap integrals are exponentially small at large separations. It means that, if the density of impurities is small enough, there might be a phase transition to an insulator state. In our case the eigenvalue is exactly zero, and the zero modes decay as a power – see eq. (9.1). The overlap integrals are also decreasing as a power of the separation (see eq. (9.2)), and chiral symmetry breaking occurs at any instanton densities – at least in the quenched approximation, when one neglects the back influence of quarks on the dynamics of instantons. However, at non-zero temperatures the density of instantons decrease $[41]$, and the fermion zero modes have an exponential falloff $[42]$: therefore one can expect a Mott–Anderson phase transition to an insulator state, meaning the restoration of chiral symmetry. It should be noted that chiral symmetry may be restored as due to the back influence of quarks on the instanton ensemble – it is an $O(N_f/N_c)$ effect then.
It has been studied recently in refs. [43] where a formation of instanton molecules has been suggested as a mechanism of chiral symmetry restoration at high temperatures. In fact, the restoration mechanism may be different for different numbers of colours and flavours.

I should mention that the idea that instantons can break chiral symmetry has been discussed previously (see refs. [44, 45, 4, 5]) however the present mechanism and a consistent formalism has been suggested and developed in papers [8, 9].

Recently there have been much interesting work done generalizing this mechanism of chiral symmetry breaking to other configurations [46] and studying general properties of the spectral density of the Dirac operator for various ensembles [17, 48].

10 Derivation II: quark propagator

Having explained the physical mechanism of chiral symmetry breaking as due to the delocalization of the would-be zero fermion modes in the field of individual instantons, I shall indicate how to compute observables in the instanton vacuum. The main quantity is the quark propagator in the instanton vacuum, averaged over the instanton ensemble. This quantity has been calculated in refs. [8, 10]. In particular, Pobylitsa [10] has derived a closed equation for the averaged quark propagator, which can be solved as a series expansion in the formal parameter $N\bar{\rho}^4/VN_c$ which numerically is something like 1/250.

The result of refs. [8, 10] is that in the leading order in the above parameter the quark propagator has the form of a massive propagator with a momentum-dependent dynamical mass:

$$S(p) = \frac{\not{p} + iM(p^2)}{p^2 + M(p^2)} , \quad M(p^2) = c\sqrt{\frac{\pi^2 N\bar{\rho}^2}{VN_c}}F(p\bar{\rho}), \quad (10.1)$$

where $F(z)$ is a combination of the modified Bessel functions and is related to the Fourier transform of the zero mode (9.1): it is equal to one at $z = 0$ and decreases rapidly with the momentum, measured in units of the inverse average size of instantons (see next section); $c$ is a constant of the order of unity which depends slightly on the approximation used in deriving the propagator. Note that the dynamical quark mass is non-analytical in the instanton density (similar to the chiral condensate it is an order parameter for spontaneous symmetry breaking).
Fixing the average density by the empirical gluon condensate (see section 4) so that \( \bar{R} \simeq 1 \text{ fm} \) and fixing the ratio \( \bar{\rho}/\bar{R} = 1/3 \) from our variational estimate, we get the value of the dynamical mass at zero momentum,

\[
M(0) \simeq 350 \text{ MeV}
\]

while the quark condensate is

\[
\langle \bar{\psi}\psi \rangle = i \int \frac{d^4p}{(2\pi)^4} \text{Tr} S(p) = -4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M(p)}{p^2 + M^2(p)} \simeq -(255 \text{ MeV})^3.
\]

Both numbers, (10.2) and (10.3), appear to be close to their phenomenological values.

Using the above small parameter one can also compute more complicated quantities like 2- or 3-point mesonic correlation functions of the type

\[
\langle J_A(x)J_B(y) \rangle, \quad \langle J_A(x)J_B(y)J_c(z) \rangle, \quad J_A = \bar{\psi} \Gamma_A \psi
\]

where \( \Gamma_A \) is a unit matrix in colour but an arbitrary matrix in flavour and spin. Instantons influence the correlation function in two ways: \( i \) the quark and antiquark propagators get dressed and obtain the dynamical mass, as in eq. (10.1), \( ii \) quark and antiquark may scatter simultaneously on the same pseudoparticle; that leads to certain effective quark interactions. These interactions are strongly dependent on the quark-antiquark quantum numbers: they are strong and attractive in the scalar and especially in the pseudoscalar and the axial channels, and rather weak in the vector and tensor channels. I shall derive these interactions in the next section, but already now we can discuss the pseudoscalar and the axial isovector channels. These are the channels where the pion shows up as an intermediate state.

Since we have already obtained chiral symmetry breaking by studying a single quark propagator in the instanton vacuum, we are doomed to have a massless Goldstone pion in the appropriate correlation functions. However, it is instructive to follow how does the Goldstone theorem manifest itself in the instanton vacuum. It appears that technologically it follows from a kind of detailed balance in the pseudoscalar channel (such kind of equations are encountered in perturbative QCD where there is a delicate cancellation between real and virtual gluon emission). However, since we have a concrete dynamical realization of chiral symmetry breaking we can not only
check the general Ward identities of the PCAC (which work of course) but we are in
a position to find quantities whose values do not follow from general relations. One of
the most important quantities is the $F_\pi$ constant: it can be calculated as the residue
of the pion pole. We get:

$$F_\pi = \frac{\text{const}}{\bar{\rho}} \left( \frac{\bar{\rho}}{\bar{R}} \right)^2 \sqrt{\ln \frac{\bar{R}}{\bar{\rho}}} \approx 100 \text{ MeV} \quad \text{vs.} \quad 93 \text{ MeV (exper.}). \quad \text{(10.5)}$$

This is a very instructive formula. The point is, $F_\pi$ is surprisingly small in the strong
interactions scale which, in the instanton vacuum, is given by the average size of
pseudoparticles, $1/\bar{\rho} \approx 600 \text{ MeV}$. The above formula says that $F_\pi$ is down by the
packing fraction factor $(\bar{\rho}/\bar{R})^2 \approx 1/9$. It can be said that $F_\pi$ measures the diluteness
of the instanton vacuum! However it would be wrong to say that instantons are in a
dilute gas phase – the interactions are crucial to stabilize the medium and to support
the known renormalization properties of the theory, therefore they are rather in a
liquid phase, however dilute it may turn to be.

By calculating three-point correlation functions in the instanton vacuum we are
able to determine, e.g. the charge radius of the Goldstone excitation:

$$\sqrt{\bar{r}_\pi^2} \approx \frac{\sqrt{N_c}}{2\pi F_\pi} \approx (340 \text{ MeV})^{-1} \quad \text{vs.} \quad (310 \text{ MeV})^{-1} \quad \text{(exper.)}. \quad \text{(10.6)}$$

Let me note that all quantities exhibit the natural behaviour in the number of
colours $N_c$:

$$\langle F_{\mu\nu}^2 \rangle \sim \frac{N}{V} = O(N_c), \quad \langle \bar{\psi}\psi \rangle = O(N_c), \quad F_\pi^2 = O(N_c),$$

$$\bar{\rho} = O(1), \quad M(0) = O(1), \quad \sqrt{r_\pi^2} = O(1), \quad \text{etc.} \quad \text{(10.7)}$$

A systematic numerical study of various correlation functions in the instanton
vacuum has been performed by Shuryak, Verbaarschot and Schaefer \[15\], see also
Shuryak’s lectures at this School. In all cases considered the results agree well or very
well with experiments and phenomenology. As I already mentioned in the introd-
uction, similar conclusions have been recently obtained from direct lattice measurements
\[14\]. I think that one can conclude that instantons are explaining the basic properties
of the QCD ground state and that of light hadrons.
Derivation III: Nambu–Jona-Lasinio model

The idea of the first two derivations of chiral symmetry breaking by instantons, presented above, is: "Calculate quark observables in a given background gluon field, then average over the ensemble of fields", in our case the ensemble of $I$'s and $\bar{I}$'s. The idea of the third derivation is the opposite: "First average over the $I\bar{I}$ ensemble and obtain an effective theory written in terms of interacting quarks only. Then compute observables from this effective theory". This approach is in a sense more economical; it has been developed in refs. [9, 40].

Quark interaction arises when two or more (anti) quarks happen to scatter over the same pseudoparticle; averaging over its positions and orientations results in a four- (or more) fermion interaction term whose range is that of the average size of instantons. The most essential way how instantons influence quarks is, of course, via the zero modes. Since each massless quark flavour has its own zero mode, it means that the effective quark interactions will be actually $2N_f$ fermion ones. They are usually referred to as 't Hooft interactions as he was the first to point out the quantum numbers of these effective instanton-induced interactions [2]. In case of two flavours they are four-fermion interactions, and the resulting low-energy theory resembles the old Vaks–Larkin–Nambu–Jona-Lasinio model [49] which is known to lead to chiral symmetry breaking. In this section I derive this model from instantons, following refs. [9, 40]. Recently it has been revisited in ref. [50].

The starting point is the quark Green function in the field of one instanton. It can be written as a sum over all eigenfunctions of the correspondent Dirac operator $\Phi_n(x)$ – see eq. (8.6):

$$S'_I(x,y) \equiv \langle \psi(x)\psi^\dagger(y) \rangle = -\sum_n \frac{\Phi_n(x)\Phi_n^\dagger(y)}{\lambda_n + im}$$

$$= -\frac{\Phi_0(x)\Phi_0^\dagger(y)}{im} + S''_I(x,y)$$

(11.1)

where $\Phi_0$ is the zero mode (9.1) (henceforth I omit the subscript 0), and $S'$ is the sum over non-zero modes, which is finite in the chiral limit $m \rightarrow 0$. For the simplicity of the derivation we replace it by the free Green function $S_0(x,y)$, though the exact propagator in the field of one instanton is known. Thus instead of the exact propagator we write
This approximate Green function is correctly taking into account the zero mode, that is the low-momentum part, and at large momentum it reduces to the free Green function, as it should. Therefore, it is an interpolation of the exact propagator; at momenta $p \sim 1/\rho$ the numerics will be not exact. However, phenomena related to chiral symmetry breaking correspond to lower momenta, and the use of the simplification (11.2) is therefore theoretically justified.

We have now to build the Green function (and other quantities) in the field of $N_+ I$’s and $N_- \bar{I}$’s and to average over their ensemble. To that end we use the following mathematical trick. Consider a fermion action

$$\exp \left( -A^I[\psi^\dagger, \psi] \right) = \prod_f \exp \left( \int d^4 x \psi^\dagger f i \not\partial \psi_f \right) \prod_{I} \left( im_f - V^I[\psi^\dagger f, \psi_f] \right),$$

where $\Phi^I$ is the zero mode in the field of the $I$th pseudoparticle. In case of $N_f > 1$ one has to take here a product of the $(im_f - V^I)$ factors for all flavours.

The action (11.3) has the following properties: $i$) the correspondent partition function normalized to the free one is equal to $im$, as it should be in case of one instanton; $ii$) the Green function, computed with this action coincides with that of eq. (11.2). It means that this action correctly describes quarks in the field of a given instanton at low and at large momenta, and interpolates in between.

In the field of $N_+ I$’s and $N_- \bar{I}$’s the fermion action is

$$\exp \left( -A[\psi^\dagger, \psi] \right) = \prod_f \exp \left( \int d^4 x \psi^\dagger f i \not\partial \psi_f \right) \prod_{I} \left( im_f - V^I[\psi^\dagger f, \psi_f] \right).$$

To get the QCD partition function one has to integrate over the quark fields and average over the ensemble of instantons:

$$Z_{QCD} = \int D\psi D\psi^\dagger \langle \exp(-A[\psi^\dagger, \psi]) \rangle$$

where $\langle ... \rangle$ denotes averaging over the ensemble. It was shown in ref. [7] (see also [50]) that the $I\bar{I}$ ensemble can be described by an effective one-particle distribution in the instanton sizes, which can be found from a variational principle:
\[ d(\rho) = \text{const} \rho^{b-5} \exp \left( -\frac{\rho^2}{\bar{\rho}^2} \frac{b-4}{2} \right), \quad b = \frac{11}{3} N_c - \frac{2}{3} N_f. \quad (11.6) \]

At large \( N_c \) it is \( \delta \)-peaked around the average \( \bar{\rho} \). Therefore, one can replace the averaging over ensemble by independent averaging over positions and orientations of individual pseudoparticles. Thus eq. (11.3) becomes

\[ Z_{QCD} = \int D\psi D\psi^\dagger \exp \left( \int d^4x \psi^\dagger \not{\partial} \psi \right) \]

\[ \cdot \prod_{I(f)} \left( \frac{m_f - V[\psi^\dagger, \psi]}{N_f} \right)^{N_+} \left( \frac{m_f + V[\psi^\dagger, \psi]}{N_f} \right)^{N_-} \quad (11.7) \]

where the bar means averaging over individual pseudoparticles. For simplicity I shall replace all sizes by their average \( \bar{\rho} \). I shall also consider only the chiral limit \( m_f \to 0 \).

It is natural to introduce non-local 2\( N_f \) fermion vertices:

\[ Y_\pm = (-)^{N_f} \int d^4 z_I(\ell) \int dU_I(\ell) \prod_{I(f)} V^I(\ell)[\psi^\dagger, \psi] \quad (11.8) \]

where \( V^I(\ell) \) depends on the (anti) instanton centers \( z_\mu \) and orientations \( U \) through the zero modes \( \Phi^I(\ell) \), see eq. (11.3) and (9.1). The partition function (11.7) can be written as

\[ Z_{QCD} = \int D\psi_f D\psi^\dagger_f \exp \left( \int d^4x \psi^\dagger \not{\partial} \psi \right) \]

\[ \cdot \int \frac{d\lambda_\pm}{2\pi} \int d\Gamma_\pm \exp \left[ i\lambda_\pm(Y_+ - \Gamma_+) + N_+ \ln \frac{\Gamma_+}{V} + (+ \to -) \right]. \quad (11.9) \]

Indeed, integrating over \( \lambda_\pm \) one gets \( \delta(Y - \Gamma) \), and after integrating over \( \Gamma \) one recovers eq. (11.7). In the thermodynamic limit \( N_\pm, V \to \infty, N/V \) fixed, integration over \( \Gamma_\pm \) and \( \lambda_\pm \) can be performed by the saddle point method. We integrate first over \( \Gamma_\pm \):

\[ Z_{QCD} = \int \frac{d\lambda_\pm}{2\pi} \exp \left[ N_+ \left( \ln \frac{N_+}{i\lambda_+ V} - 1 \right) + (+ \to -) \right] \]

\[ \cdot \int D\psi_f D\psi^\dagger_f \exp \left( \int d^4x \psi^\dagger \not{\partial} \psi + i\lambda_+ Y_+ + i\lambda_- Y_- \right). \quad (11.10) \]

Because of the non-locality it is more convenient to write the vertices (11.8) in the momentum space. Let us decompose the 4-component Dirac spinors describing quark fields into left- and right-handed 2-component Weyl spinors which we denote as
where \( f = 1 \ldots N_f \) stand for flavour, \( \alpha = 1 \ldots N_c \) stand for colour and \( i = 1, 2 \) stand for spin indices. Let us introduce the formfactor functions \( F(k) \) which are related to the Fourier transforms of the zero modes \( \psi_{L(R)}(z) \) and are attributed to each fermion entering the vertex \( z = k\rho/2 \):

\[
F(k) = 2z[I_0(z)K_1(z) - I_1(z)K_0(z)] = \frac{6}{k^3 \rho^3}, \quad F(0) = 1. \tag{11.12}
\]

The \( N_f \) fermion vertices can be written as:

\[
Y^+_{N_f} = \int dU \prod_{n=1}^{N_f} \int \frac{d^4k_n}{(2\pi)^4} 2\pi \rho F(k_n) \int \frac{d^4l_n}{(2\pi)^4} 2\pi \rho F(l_n)(2\pi)^4 \delta(k_1 + \ldots + k_{N_f} - l_1 - \ldots - l_{N_f}) \cdot U^\alpha_{\gamma_n} U^\beta_{\delta_n} \epsilon^\gamma_{j_n, j_n}(k_n) \psi^\alpha_{L_{fn}}(l_n); \tag{11.13}
\]

for the vertices \( Y^- \) induced by \( \bar{\psi}'s \) one has to replace left-handed Weyl spinors by right-handed ones. The integral \( \int dU \) means averaging over instanton orientations in colour space. In particular, one has:

\[
\int dU = 1, \quad \int dU U^\alpha_{\gamma} U^\beta_{\delta} = \frac{1}{N_c} \delta^\alpha_{\delta} \delta^\beta_{\gamma}, \quad \text{etc.} \tag{11.14}
\]

To get the \( 2N_f \) vertices in a closed form one has to perform explicitly integration over the instanton orientations. I present below the results \cite{9} for \( N_f = 1, 2 \) and for any \( N_f \) but \( N_c \to \infty \).

\( N_f = 1 \)

In this case the ”vertex” \( Y^{+1} \) is just a mass term:

\[
Y^{+1} = \int \frac{d^4k}{(2\pi)^4} \psi^\dagger(k)(2\pi \rho F(k))^2 \frac{1 \pm \gamma_5}{2} \psi(k). \tag{11.15}
\]

One has to plug it into the eq. \( 11.10 \), integrate over fermions, and find the saddle-point values of \( \lambda_{\pm} \). If the \( CP \) symmetry is conserved, one has \( N_+ = N_- = N/2 \), and the saddle-point values satisfy \( \lambda_+ = \lambda_- \). Then the \( \gamma_5 \) term in \( Y^{+} \) get cancelled, and eq. \( 11.13 \) gives a momentum-dependent mass

\[
M(k) = M(0) F^2(k), \quad M(0) = \lambda(2\pi \rho)^2, \tag{11.16}
\]

31
where $M(0)$ or $\lambda$ is found from the equation \[8, 9\] (called sometimes self-consistency or gap equation):

$\frac{4N_c}{N/V} \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{k^2 + M^2(k)} = 1.$ \hspace{1cm} (11.17)

Let me mention that exactly the same gap equation (11.17) has been obtained \[8\] in another approach: by first finding the quark propagator in the instanton vacuum and then averaging over the instanton ensemble. It is seen from eq. (11.17) that the dynamically generated mass is of the order of $M(0) \sim \sqrt{N/(V N_c)} \bar{\rho}$. Knowing the form of $M(k)$ given by eq. (11.12) and using the "standard" values $N/V = (1 \text{ fm})^{-4}$, $\bar{\rho} = (1/3) \text{ fm}$ we find numerically $M(0) \simeq 350 \text{ MeV}$. If one neglects $M^2$ in the denominator of eq. (11.17) \[10\] one gets $M(0) \simeq 420 \text{ MeV}$. This deviation indicates the accuracy of the "zero mode approximation" used in this derivation: it is about 15%.

$N_f = 2$

In this case averaging over the instanton orientations gives a nontrivial 4-fermion interaction:

$Y_2^\pm = \frac{2N_c^2}{N/V} \int \frac{d^4k_1d^4k_2d^4l_1d^4l_2}{(2\pi)^{12}} \sqrt{M(k_1)M(k_2)M(l_1)M(l_2)}$

$\times \frac{\epsilon_{f_1f_2g_1g_2}}{2(N_c^2 - 1)} \left[ 2N_c - \frac{1}{2N_c} \left( \psi_{L_{f_1}}(k_1)\psi_{L_{g_1}}(l_1)\psi_{L_{f_2}}(k_2)\psi_{L_{g_2}}(l_2) \right) \right.$

$\left. + \frac{1}{8N_c} \left( \psi^\dagger_{L_{f_1}}(k_1)\sigma_{\mu\nu}\psi_{L_{g_1}}^\dagger(l_1)\right)\left( \psi^\dagger_{L_{f_2}}(k_2)\sigma_{\mu\nu}\psi_{L_{g_2}}^\dagger(l_2) \right) \right]$ \hspace{1cm} (11.18)

For the antiinstanton-induced vertex $Y^\mp$ one has to replace left handed components by right-handed. In eq. (11.18) I have included the factor $\lambda_+$ and, moreover, fixed it from the saddle-point equation. With the normalization of eq. (11.18) the dynamical mass $M(k)$ satisfies exactly the same gap equation (11.17) as in the case of $N_f = 1$. Note that the second (tensor) term is negligible at large $N_c$. Using the identity

$2\epsilon_{f_1f_2g_1g_2} = \delta_{f_1g_1}^f \delta_{f_2g_2}^f - (\tau^A)^{f_1}_{g_1} (\tau^A)^{f_2}_{g_2}$ \hspace{1cm} (11.19)

one can rewrite the leading (first) term of eq. (11.18) as

$(\psi^\dagger \psi)^2 + (\psi^\dagger \gamma_5 \psi)^2 - (\psi^\dagger \tau^A \psi)^2 - (\psi^\dagger \tau^A \gamma_5 \psi)^2$ \hspace{1cm} (11.20)
which resembles closely the Nambu–Jona-Lasinio model. It should be stressed though that in contrast to that at hoc model the interaction (11.18) i) violates explicitly the $U_A(1)$ symmetry, ii) has a fixed interaction strength and iii) contains an intrinsic ultraviolet cutoff due to the formfactor function $M(k)$. This model is known to lead to chiral symmetry breaking, at least at large $N_c$ when the use of the mean field approximation to the model is theoretically justified.

Any $N_f$

At arbitrary $N_f$ the leading term at $N_c \to \infty$ can be written as a determinant of a $N_f \times N_f$ matrices composed of quark bilinears:

$$ Y_{N_f}^{(+)} \sim \infty \left( \frac{2V}{N} \right)^{N_f-1} \int d^4x \det_{N_f} J^{(\pm)}, $$

$$ J_{fg}^{(\pm)}(x) = \int \frac{d^4k d^4l}{(2\pi)^8} e^{i(k-L,x)} \sqrt{M(k)M(l)} \psi_f^\dagger(k) \frac{1 \pm \gamma_5}{2} \psi_g(l). $$

Again, one can prove that at least at large $N_c$, this interaction leads to the spontaneous chiral symmetry breaking, with the dynamical mass determined by the gap equation (11.17), and the chiral condensate given by eq. (10.3). The bosonization of these interactions has been performed in ref. [9]; it paves the way to studying analytically various correlation functions in the instanton vacuum.

A separate issue is the application of these ideas to hadrons made of heavy quarks [12] and of light and heavy ones [51].

12 QCD at still lower energies

Using the packing fraction of instantons $\bar{\rho}/\bar{R} \simeq 1/3$ as a new algebraic parameter one observes that all degrees of freedom in QCD can be divided into two categories: i) those with masses $\geq 1/\bar{\rho}$ and ii) those with masses $\ll 1/\bar{\rho}$. If one restricts oneself to low-energy strong interactions such that momenta are $\ll 1/\bar{\rho} \approx 600$ MeV, one can neglect the former and concentrate on the latter. There are just two kind of degrees of freedom whose mass is much less than the inverse average size of instantons: the (pseudo) Goldstone pseudoscalar mesons and the quarks themselves which obtain a dynamically-generated mass $M \sim (1/\bar{\rho})(\bar{\rho}^2/R^2) \ll 1/\bar{\rho}$. Thus in the domain of momenta $k \ll 1/\bar{\rho}$ QCD reduces to a remarkably simple though nontrivial theory of
massive quarks interacting with massless or nearly massless pions. It is given by the partition function \[8, 9\]

\[
Z_{QCD}^{\text{low mom.}} = \int D\psi D\psi^\dagger \exp \left[ \int d^4x \psi^\dagger \left( i \partial + iMe^{i\pi^A\tau_A\gamma_5} \right) \psi \right]. \tag{12.1}
\]

Notice that there is no kinetic energy term for the pions, and that the theory is not a renormalizable one. The last circumstance is due to the fact that it is an effective low-energy theory; the ultraviolet cutoff is actually \(1/\bar{\rho}\).

There is a close analogy with solid state physics here. The microscopic theory of solid states is QED: it manages to break spontaneously the translational symmetry, so that a Goldstone excitation emerges, called the phonon. Electrons obtain a ”dynamical mass” \(m^*\) due to hopping from one atom in a lattice to another. The ”low energy” limit of solid state physics is described by interactions of dressed electrons with Goldstone phonons. These interactions are more or less fixed by symmetry considerations apart from a few constants which can be deduced from experiments or calculated approximately from the underlying QED. Little is left of the complicated dynamics at the atom scale.

What Petrov and I have attempted, is a similar path: one starts with the fundamental QCD, finds that instantons stabilize at a relatively low density and that they break chiral symmetry; what is left at low momenta are just the dynamically massive quarks and massless pions. One needs two scales to describe strong interactions at low momenta: the ultra-violet cutoff, whose role is played by the inverse instanton size, and the dynamical quark mass proportional to the square root of the instanton density. If one does not believe our variational calculations of these quantities one can take them from experiment.

If one integrates off the quark fields in eq. (12.1) one gets the effective chiral lagrangian:

\[
S_{\text{eff}}[\pi] = -N_c \text{Tr} \ln \left( i \beta + iMU^\gamma_5 \right),
\]

\[
U = \exp(i\pi^A\tau^A), \quad U^\gamma_5 = \exp(i\pi^A\tau^A\gamma_5), \quad L_\mu = iU^\dagger \partial_\mu U. \tag{12.2}
\]

One can expand eq. (12.2) in powers of the derivatives of the pion field and get \[8, 9\]:

\[
S_{\text{eff}}[\pi] = \frac{F_\pi^2}{4} \int d^4x \, \text{Tr} L_\mu^2 - \frac{N_c}{192\pi^2} \int d^4x \left[ 2\text{Tr}(\partial_\mu L_\mu)^2 + \text{Tr} L_\mu L_\nu L_\mu L_\nu \right]
\]

34
\[ \frac{N_c}{240\pi^2} \int d^5x \, \epsilon_{\alpha\beta\gamma\delta\epsilon} \, \text{Tr} L_{\alpha} L_{\beta} L_{\gamma} L_{\delta} L_{\epsilon} + \ldots \]  

(12.3)

The first term here is the old Weinberg chiral lagrangian with

\[ F_{\pi}^2 = 4N_c \int \frac{d^4k}{(2\pi)^4} \frac{M^2(k)}{[k^2 + M^2(k)]^2}; \]

(12.4)

the second term are the four-derivative Gasser–Leutwyler terms (with coefficients which turn out to agree with those following from the analysis of the data); the last term in eq. (12.3) is the so-called Wess–Zumino term. Note that the \( F_{\pi} \) constant diverges logarithmically at large momenta; the integral is cut by the momentum-dependent mass at \( k \sim 1/\bar{\rho} \), so that one gets the same expression as in a different approach described in section 10, see eq. (10.5).

An ideal field of application of the low-momentum partition function (12.1) is the quark-soliton model of nucleons [11] – actually the model has been derived from this partition function. The size of the nucleon \( \sim (250 \text{ MeV})^{-1} \) is much larger than the size of instantons \( \sim (600 \text{ MeV})^{-1} \); hence the low-momentum theory (12.1) seems to be justified. Indeed, the computed static characteristics of baryons like formfactors, magnetic moments, etc., are in a good accordance with the data (for a review see ref. [18]). What is not yet computed, are the nucleon structure functions at a low normalization point, however given the previous experience, I can bet it would go through the data points.

13 How instantons may help confinement

Our analytical calculations sketched in these lectures, the extensive numerical studies of the instanton vacuum by Shuryak and collaborators and the recent direct lattice measurements – all point out that instantons play a crucial role in determining the world of light hadrons, including the nucleon. Confinement has not much to do with it – contrary to what has been a common wisdom a decade ago and in what many people still believe. Nevertheless, confinement is a property of QCD, and one needs to understand the confinement mechanism. What can be said today is that confinement must be "soft": it should destroy neither the successes of the perturbative description of high-energy processes (no "string effects" there) nor the successes of instantons at
low momenta. At the moment I can think of two possible scenarios:

A) Instantons have a micro-structure, like merons;
B) Confinement is due to monopoles which are massless because of instantons.

Let me stress that, contrary to the case of matter where objects exist by themselves, in field theory of the vacuum one has first of all to create the objects (like monopoles) which could bring in confinement. Therefore, only such objects can give a sizeable effect whose mass is effectively zero, so that one does not loose energy to create them. Note that in a sense instantons have zero mass since they have finite action, and action is mass times the (infinite) observation time. Merons like instantons have also finite action (if one takes care to cut them both at large and small distances), therefore there is nothing wrong in principle with merons. The only field-theoretical example of confinement we know today is the famous Polyakov’s example in the 2+1 dimensional Georgi–Glashow model, and it is an example of a meron type. Moreover, a meron pair resembles one instanton, so a confinement mechanism based on merons might fit in well into the successful instanton vacuum.

However, there is a more popular version of confinement on the market, as due to the Mandelstam–’t Hooft monopole condensation mechanism. According to this mechanism, monopole-like particles are somehow formed out of the gluon fields, they develop in time, interact and annihilate in pairs, and the crucial thing about them is that they form a quantum-mechanical condensate characterized by a macroscopic wave function – like the Cooper pairs of electrons in the superconductor. The confinement of colour electric charges is then due to the dual Meissner effect.

Contrary to the case of electrons which exist in abundance in any sample so that one needs just a small attraction to bind them into a condensate, monopoles have to be first of all created, and that costs energy. Therefore, this mechanism has a chance only if the mass of the monopoles is effectively zero. However the monopole mass can be estimated as $M_{mon} \sim 4\pi/g^2 \cdot (\text{inverse size})$. “$1/g^2$” appears here because the monopole should carry a unit magnetic flux. Therefore, in the weak coupling regime there is no chance for monopoles to condense. This is a point where instantons may help.

Let us write down the effective action for gluons in the background field of one instanton:

$\mathcal{M}_{mon} \sim 4\pi/g^2 \cdot (\text{inverse size})$. “$1/g^2$” appears here because the monopole should carry a unit magnetic flux. Therefore, in the weak coupling regime there is no chance for monopoles to condense. This is a point where instantons may help.

Let us write down the effective action for gluons in the background field of one instanton:

\[ \mathcal{M}_{mon} \sim \frac{4\pi}{g^2} \cdot (\text{inverse size}) \]

This section is based on our speculations with Victor Petrov. The reader is kindly asked not to be too severe to these qualitative considerations.
\[ Z = \int D\alpha \exp \left( -\int \frac{F_{\mu\nu}^2}{4g^2} \right) \int d^4z \, d\rho \, d\alpha \exp \left( -\frac{2\pi^2\rho^2}{g^2} F_{\mu\nu}(z) O^{ab} \bar{\eta}^{\mu\nu b} + \ldots \right). \]  

(13.1)

This effective action has been first suggested in ref. [5] to describe the leading dipole-dipole interactions of instantons, then it has been re-derived in a more general form in ref. [34]. Later on Yung showed [53] that its domain of applicability is wider than anticipated; recently this effective action has been used to derive the \( \bar{I}I \) interaction potential up to the next-to-next-to leading order [35]. This effective action reproduces also the instanton field itself as an expansion in \( \rho^2 / (x - z)^2 \). For an anti-instanton one has to replace \( \bar{\eta} \rightarrow \eta \).

For a grand canonical ensemble of \( I \)'s and \( \bar{I} \)'s one writes an effective action

\[ Z^{\bar{I}} = \sum_{N_+, N_-} \frac{1}{N_+!} \frac{1}{N_-!} \left( \int d^4z \ldots \exp(...\bar{\eta}) \right)^{N_+} \left( \int d^4z \ldots \exp(...\eta) \right)^{N_-} \]

\[ = \int D\alpha \exp \left\{ -\int \frac{F_{\mu\nu}^2}{4g^2} - \int d^4z \, d\rho \, d\alpha \exp \left[ -\frac{2\pi^2\rho^2}{g^2} (F\bar{\eta}) + e^{-\frac{2\pi^2\rho^2}{g^2}} (FO\bar{\eta}) \right] \right\}. \]

(13.2)

If one expands the exponents (in the exponent) in powers of \( F_{\mu\nu} \) one observes that the linear term is zero owing to integrations over orientations, and the quadratic term actually corresponds to the renormalization of the gauge coupling due to the instanton medium:

\[ \frac{1}{4g^2} \rightarrow \frac{1}{4g^*^2} = \frac{1}{4g^2} - \frac{1}{4(N_c^2 - 1) g^4} \int d\rho \, d(\rho)(2\pi\rho)^4 \]

\[ = \frac{1}{4g^2} \left( 1 - \frac{(2\pi)^4}{g^2(N_c^2 - 1) V^4} \right). \]

(13.3)

We see that if the packing fraction is large enough the effective coupling \( g^*^2 \) for long range fields blows up. It means that if the instanton packing is close to some critical value, one does not need much energy to create a monopole in such medium; that is the necessary condition for their condensation. To get an accurate estimate of the critical density is not so easy, though. To that end one needs to have a good understanding of the usual perturbative renormalization of the charge by instantons: what is the precise argument of the running coupling constant \( g^2 \) in eq. (13.3)? Using
the numbers obtained in ref. [7] we get that the density is about half that of the critical, but the uncertainty of this estimation is high: it could be close to the critical as well.

Closing this section, I would like to mention that the monopole condensation of Mandelstam and 't Hooft is probably not what we in fact need – the confinement mechanism should be probably more subtle. The essence of that mechanism is the Landau–Ginzburg or the Higgs effect — but for dual Yang–Mills potentials. 't Hooft has elaborated it in some detail [54]: all fields are classified in respect to the maximal abelian $U(1) \times U(1)$ subgroup of the $SU(3)$ colour group, and monopoles have magnetic charges in respect to those $U(1)$ subgroups. If they condense all particles which carry electric charges in respect to those $U(1)$ subgroups are confined. Particles which happen to be neutral are not, though. For example, two gluons out of eight are "photons" of these $U(1)$ subgroups, so they are neutral and are not confined, instead they may obtain a "magnetic" mass which is of the order of the string tension, that is about 420 $MeV$, maybe up to a factor of 2 heavier. Probably such objects should show up as resonances in usual particle production, but we do not know of two such additional states. Even worse, quark-antiquark pairs belonging to the colour octet representation but having colour $T_3$ and $Y$ zero are also neutral in respect to the $U(1)$ subgroups, so they should also exist and be observable. There are two such additional states for each set of meson quantum numbers. There would be also five additional types of baryon states which are not colour singlets but which are neutral in respect to the both $U(1)$ subgroups. And of course nothing prevents monopoles themselves from getting into an experimentalist’s detector, if only they do not, in addition, carry electric charges in respect to the $U(1)$ subgroups.

Therefore, I think that what we actually need in QCD is not condensation of monopoles but rather a pre-condensation, something of the kind of the Berezinsky–Kosterlitz–Thouless phase, characterized by large anomalous dimensions of the monopole (and probably also gluon) fields. To obtain that one also needs massless or effectively massless monopoles, and that is where instantons might help.

**Acknowledgements.** These lectures have been written up while visiting the European Centre for Theoretical Studies (ECT*) in Trento. I acknowledge the support of the ECT* and of the I.N.F.N. My special gratitude is to Victor Petrov with whom we worked together for many years on the topics discussed in these lectures.
References

[1] A.Belavin, A.Polyakov, A.Schwartz and Yu.Tyupkin, *Phys. Lett.* 59 (1975) 85; A.Polyakov, *Nucl. Phys.* B120 (1977) 429

[2] G.’t Hooft, *Phys. Rev. Lett.* 37 (1976) 8; *Phys. Rev.* D14 (1976) 3432; Erratum: *ibid.* D18 (1978) 2199

[3] M.Shifman, A.Vainshtein and V.Zakharov, *Nucl.Phys.* B147 (1979) 385

[4] D.Diakonov, *The U(1) problem and instantons*, in: *Gauge Theories of the Eighties*, Lecture Notes in Physics, Springer-Verlag (1983) p.207

[5] C.Callan, R.Dashen and D.Gross, *Phys. Rev.* D17 (1978) 2717

[6] E.Shuryak, *Nucl. Phys.* B203 (1982) 93, 116, 140

[7] D.Diakonov and V.Petrov, *Nucl. Phys.* B245 (1984) 259

[8] D.Diakonov and V.Petrov, *Phys. Lett.* 147B (1984) 351; *Sov. Phys. JETP* 62 (1985) 204, 431; *Nucl.Phys.* B272 (1986) 457

[9] D.Diakonov and V.Petrov, *Spontaneous breaking of chiral symmetry in the instanton vacuum*, preprint LNPI-1153 (1986), published (in Russian) in: Hadron matter under extreme conditions, Kiew (1986) p.192; D.Diakonov and V.Petrov, *Diquarks in the instanton picture*, in: *Quark Cluster Dynamics*, Lecture Notes in Physics, Springer-Verlag (1992) p.288

[10] P.Pobylitsa, *Phys. Lett.* 226B (1989) 387

[11] D.Diakonov and V.Petrov, *Sov. Phys. JETP. Lett.* 43 (1986) 75; D.Diakonov, V.Petrov and P.Pobylitsa, *Nucl. Phys.* B306 (1988) 809; D.Diakonov, V.Petrov and M.Praszalowicz, *Nucl. Phys.* B323 (1989) 53

[12] D.Diakonov, V.Petrov and P.Pobylitsa, *Phys. Lett.* 226B (1989) 372

[13] E.Shuryak, *Nucl. Phys.* B328 (1989) 85

[14] D.Diakonov, in: *Skyrmions and Anomalies*, World Scientific (1987) p. 27
[15] E.Shuryak and J.Verbaarschot, *Nucl. Phys.* **B410** (1993) 55;  
T.Schäfer, E.Shuryak and J.Verbaarschot, *Nucl. Phys.* **B412** (1994) 143;  
T.Schäfer and E.Shuryak, *Phys. Rev.* **D50** (1994) 478

[16] E.Shuryak, *Rev. Mod. Phys.* **65** (1993) 1

[17] M.-C.Chu, J.Grandy, S.Huang and J.Negele, *Phys. Rev. Lett.* **70** (1993) 225;  
*Phys. Rev.* **D49** (1994) 6039

[18] K.Goeke, A.Blotz, E.Nikolov, D.Diakonov, M.Polyakov, V.Petrov, A.Gorski, W.Broniowski, M.Praszalowicz and G.Ripka, in: *Many-Body Physics*, World Scientific (1994) p. 73

[19] M.Teper, *Phys. Lett.* **162B** (1985) 357

[20] E.-M.Ilgenfritz, M.Laursen, M.Müller-Preussker, G.Shierholz and H.Schiller,  
*Nucl. Phys.* **B268** (1986) 693

[21] M.Polikarpov and A.Veselov, *Nucl. Phys.* **B297** (1988) 34

[22] M.Campostrini, A.Di Giacomo, M.Maggiore, H.Panagopoulos and E.Vicari,  
*Nucl. Phys.* **B329** (1990) 683

[23] C.Michael and P.Spencer, Helsinki preprint HU TFT 95-21, Liverpool preprint LTH-346 (1995), [hep-lat/9503018](http://arxiv.org/abs/hep-lat/9503018)

[24] L.D.Faddeev, *Looking for multi-dimensional solitons* in: Non-local Field Theories, Dubna (1976)

[25] R.Jackiw and C.Rebbi, *Phys. Rev. Lett.* **37** (1976) 172

[26] T.Akiba, H.Kikuchi and T.Yanagida, *Phys. Rev.* **D38** (1988) 1937

[27] D.Diakonov, M.Polyakov, J.Schaldach, P.Sieber and K.Goeke, *Phys. Rev.* **D49** (1994) 6864

[28] R.Dashen, B.Hasslacher and A.Neveu, *Phys. Rev.* **D10** (1974) 4138

[29] N.Manton, *Phys. Rev.* **D28** (1983) 2019;  
F.R.Klinkhammer and N.Manton, *Phys. Rev.* **D30** (1984) 2212
[30] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Pergamon (1965) Ch.VII

[31] A. Polyakov, *Nucl. Phys.* **B120** (1977) 429

[32] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, *Nucl. Phys.* **B191** (1981) 301

[33] C. Bernard, *Phys. Rev.* **D19** (1979) 3013

[34] A. Vainshtein, V. Zakharov, V. Novikov and M. Shifman, *Sov. Phys. Uspekhi* **136** (1982) 553

[35] D. Diakonov and M. Polyakov, *Nucl. Phys.* **B389** (1993) 109

[36] D. Diakonov and V. Petrov, in: *Proc. of the 26th LNPI Winter School*, Leningrad (1991)

[37] D. Diakonov and V. Petrov, *Phys. Rev.* **D50** (1994) 266

[38] D. Diakonov and V. Petrov, *Rotating chiral solitons lie on linear Regge trajectories*, preprint LNPI-1394 (1988) (unpublished)

[39] T. Banks and A. Casher, *Nucl. Phys.* **B169** (1980) 103

[40] D. Diakonov, Habilitation thesis (LNPI, 1986) (unpublished)

[41] D. Diakonov and A. Mirlin, *Phys. Lett.* **203B** (1988) 299

[42] V. V. Khoze and A. V. Yung, *Z. Phys.* **C50** (1991) 155

[43] M.-E. Ilgenfritz and E. Shuryak, *Phys. Lett.* **325B** (1994) 263; T. Schaefer, E. Shuryak and J. Verbaarschot, *Phys. Rev.* **D51** (1995) 1267

[44] D. G. Caldi, *Phys. Rev. Lett.* **39** (1977) 121

[45] R. D. Carlitz, *Phys. Rev.* **D17** (1978) 3225; R. D. Carlitz and D. B. Creamer, *Ann. Phys. (N.Y.)* **118** (1979) 429

[46] A. Gonzalez-Arroyo and Yu. Simonov, *Fermionic zero modes for dyons and chiral symmetry breaking in QCD*, preprint FTUAM-95-16
[47] H.Leutwyler and A.Smilga, *Phys. Rev.* **D** (1992);  
A.Smilga and J.Verbaarschot, *Phys. Rev.* **D51** (1995) 829

[48] J.Verbaarschot and I.Zahed, *Phys. Rev. Lett.* **70** (1993) 3852;  
J.Verbaarschot, *Nucl. Phys.* **B427** (1994) 534; *Phys. Rev. Lett.* **72** (1994) 2531

[49] V.G.Vaks and A.I.Larkin, *ZhETF* **40** (1961) 282;  
Y.Nambu and G.Jona-Lasinio, *Phys. Rev.* **122** (1961) 345

[50] D.Diakonov, M.Polyakov and C.Weiss, *Hadronic matrix elements of gluon operators in the instanton vacuum*, in preparation

[51] S.Chernyshev, M.Nowak and I.Zahed, *Heavy mesons in a random instanton gas*, preprint SUNY-NTG-94-37

[52] D.Diakonov, *Instanton vacuum and confinement in QCD*, lectures at the 3d Petersburg Winter School in QCD, February 26 – March 11, (1995) (unpublished)

[53] A.V.Yung, *Do instanton-induced amplitudes break unitary bound?*, preprint LNPI-1617 (1990)

[54] G.’t Hooft, *Nucl. Phys.* **B190 [FS3]** (1981) 455