Multiply Degenerate Exceptional Points

and Quantum Phase Transitions

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Abstract

The realization of a genuine phase transition in quantum mechanics requires that at least one of the Kato’s exceptional-point parameters becomes real. A new family of finite-dimensional and time-parametrized quantum-lattice models with such a property is proposed and studied. All of them exhibit, at a real exceptional-point time $t = 0$, the Jordan-block spectral degeneracy structure of some of their observables sampled by Hamiltonian $H(t)$ and site-position $Q(t)$. The passes through the critical instant $t = 0$ are interpreted as schematic simulations of non-equivalent versions of the Big-Bang-like quantum catastrophes.
1 Introduction

Although the concept of phase transition originates from classical thermodynamics, it recently found a new area of applicability in the context of quantum mechanics where one may decide to work with the pseudo-Hermitian (the term used in mathematics, see reviews [1, 2]) alias $\mathcal{PT}$—symmetric (using the terminology of physicists, see review [3]), manifestly non-Hermitian Hamiltonians which are still capable of generating a stable, unitary evolution of the quantum system in question.

A key phenomenological novelty is that whenever our quantum Hamiltonian $H \neq H^\dagger$ varies with a real parameter (say, $H = H(\lambda)$), the “physical intervals” of the acceptability of $\lambda$ are, typically, different from the whole real axis. In the mathematical language of Kato [4] one can also say that for the analytic operator functions $H(\lambda)$ of the parameter, the whole interval (or rather a union of intervals) lies in an open complex set $\mathcal{D}$ and that the (in general, complex) points of its boundary $\partial \mathcal{D}$ coincide with the Kato’s exceptional points (EPs) as introduced in loc. cit.

The main mathematical feature of the EPs $\lambda^{(EP)}_j \in \partial \mathcal{D}$ is that the limiting operators $H^{(EP)} = \lim_{\lambda \to \lambda^{(EP)}_j} H(\lambda)$ cease to be tractable as physical Hamiltonians because even if the spectrum of the energies happens to remain real these operators cease to be diagonalizable. In a suitable basis they acquire a Jordan-blocks-containing canonical form [1, 5, 6]. Thus, even if one of values of $\lambda^{(EP)}_j$ is real, the limiting transition $\lambda \to \lambda^{(EP)}_j$ must still be perceived as a process during which certain eigenvectors of $H(\lambda)$ “parallelize” and become linearly dependent, resulting in the loss of the usual probabilistic tractability of the quantum system in question [7] - [13].

It is worth adding that in the vast majority of the standard applications of quantum mechanics the EP parameters $\lambda^{(EP)}_j$ are not real so that one cannot “cross” them when a parameter moves just along the real line. In such a case, as a rule, the boundaries of the domain of unitarity coincide with the boundaries of the Riemann surface $\mathcal{R}$ of the analyticity of the operator function $H(\lambda)$ so that one must speak about an end of the applicability of the Hamiltonian rather than about a quantum phase transition as realized within the same physical model.

In our present paper we intend to pay attention to the scenarios in which the EP values $\lambda^{(EP)}_j \in \partial \mathcal{D}$ are real and do not lie on any boundary of $\mathcal{R}$. In such an arrangement (observed already, in [14], in 1998) one may treat the EPs as the points of a true analytic realization of quantum phase transitions.

2 Pure quantum states in parallel representations

2.1 Three Hilbert space pattern

Whenever one reveals that the spectrum of a non-Hermitian operator of an observable of a quantum system (be it a Hamiltonian $H \neq H^\dagger$ or a position operator $Q \neq Q^\dagger$, etc) is all real and discrete, one feels tempted to expect that a smooth theoretical translation of all predictions into the language
of the textbook quantum mechanics may be performed \[15\].

One of the realizations of such a conceptually appealing project has been proposed in nuclear physics (cf. \[16\]). According to our recent summaries \[17, 18\] of this possibility one simply has to make use of the simultaneous representation of the quantum states in three alternative Hilbert spaces $\mathcal{H}^{(F,S,P)}$. Out of the triplet the first space (viz., $\mathcal{H}^{(F)}$, often chosen in the form of $L^2(\mathbb{R})$) is most friendly. Naturally, whenever this space leaves our observables non-Hermitian, it must be declared unphysical. Fortunately, in the second, standard and physical space $\mathcal{H}^{(S)}$ the Hermitization of the same operators of observables may be comparatively easily achieved via the mere introduction of an amended, metric-mediated (i.e., often, non-local) inner product.

In the whole scheme, the latter space is finally (and, usually, constructively) declared unitarily equivalent to the third Hilbert space $\mathcal{H}^{(P)}$ with trivial metric which, as a rule, appears prohibitively complicated and inaccessible to any constructive fructification \[16\].

### 2.2 Five Hilbert space pattern

The generic technical nontriviality of the whole three-Hilbert-space (THS) parallel-representation recipe may force its users to apply the trick twice. More details may be found in \[19\]. In essence, the first, preparatory application yields just a tentative, simplified inner-product metric $\Theta_T$, the study of which may be motivated, e.g., by the generic difficulty of the verification of the reality of the spectrum or of the closed-form construction of any less elementary amendments of the tentative positive-definite metric. Subsequently, with $\Theta_T = \Theta_T^\dagger \neq I$ at our disposal, the final specification of the second, “sophisticated” metric $\Theta_S$ is expected to be guided by the redirection of emphasis from mathematics to phenomenology.

The following diagram characterizes the resulting five-Hilbert-space representation of a given
quantum system,

\[
\text{(initial stage)}
\]

- friendly space \( \mathcal{H}^{(F)} \)
- friendly observable \( F \neq F^\dagger \)
- false metric \( \Theta^{(F)} = I \)

| Step 1 (preparatory, aim: simplicity) | Step 2 (correct Dyson’s map, aim: real world) |
|---------------------------------------|---------------------------------------------|
| test space \( \mathcal{H}^{(T)} \)   | standard space \( \mathcal{H}^{(S)} \)     |
| \( F = F^\dagger = \Theta_T^{-1} F^\dagger \Theta_T \) | \( F = F^\dagger = \Theta_S^{-1} F^\dagger \Theta_S \) |
| trial \( \Theta_T = \Omega_T^1 \Omega_T \neq I \) | correct \( \Theta_S = \Omega_S^1 \Omega_S \neq I \) |
| spectral reality proof               | experimental predictions                    |

\( \leftarrow \) respective unitary equivalences \( \rightarrow \)

| Step 3 (mathematical reference) | Step 4 (physical reference) |
|---------------------------------|----------------------------|
| auxiliary space \( \mathcal{H}^{(A)}_{\text{math.}} \) | prohibited space \( \mathcal{H}^{(P)}_{\text{phys.}} \) |
| \( \tilde{f}_{\text{math.}} = \Omega_T F \Omega_T^{-1} = \tilde{f}^\dagger_{\text{math.}} \) | isospectral to \( F \) |
| (trivial metric)                | (trivial metric)            |

Typically, one starts from an observable Hamiltonian \( F = H \) or position \( F = Q \) (etc) which are all defined in \( \mathcal{H}^{(F)} \). One then moves towards the first nontrivial (though still unphysical) positive definite artificial metric candidate \( \Theta_T \) (say, to prove the reality of the spectrum). In the second step of construction one proceeds to the ultimate analysis of the standard representation of the system using a realistic, physical \( \Theta_S \) which is often known solely in an approximate form [20].

3 Exceptional points in a schematic model

In the context of our present paper the key purpose of the start of analysis from an operator of observable \( F \) which is defined in an unphysical Hilbert space \( \mathcal{H}^{(F)} \) (i.e., which is manifestly non-Hermitian there and which will be mostly sampled by the operator of position \( Q \) in what follows) is that such operators often possess the real exceptional points [21] – [23].

3.1 Traditional studies starting from a Hamiltonian

In the traditional considerations one usually assumes that in the vicinity of a real Kato’s EP value of parameter \( \lambda_j^{(EP)} \in \partial D \subset \mathbb{R} \) the basis in the Hilbert space \( \mathcal{H}^{(F)} \) (where the superscript \( (F) \) stands
for “friendly” \(^{13}\)) is such that the whole diagonalizable part of a pre-determined Hamiltonian \(H(\lambda)\) is diagonalized. Thus, at \(\lambda_j^{(EP)}\) one may only pay attention to the non-diagonalizable part of the Hamiltonian. The latter operator may be also assumed to have acquired the standard canonical form of a direct sum of Jordan blocks. Naturally, one could further restrict attention just to one of the Jordan blocks (of matrix dimension \(N\)), considering its small vicinity in the following special form of the multiparametric finite- and tridiagonal-matrix toy model with an additional up-down symmetry of its matrix elements,

\[
\begin{pmatrix}
0 & 1 - \alpha & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\alpha & 0 & 1 - \beta & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & \beta & 0 & 1 - \gamma & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \gamma & 0 & 1 - \delta & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \delta & 0 & 1 - \gamma & 0 & \ldots & 0 & 0 & 0 \\
0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \delta & 0 & 1 - \gamma & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \gamma & 0 & 1 - \beta & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \delta & 0 & \gamma & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \beta & 0 & 1 - \alpha & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}.
\]

(2)

During our preparatory numerical experiments with the spectra of various Hamiltonian matrices of the generic tridiagonal form \(^{(2)}\) (cf. also Refs. \([24, 25]\)) it became clear that the transitions of quantum systems through their Jordan-block alias multiple-EP (or degenerate-EP) quantum-phase-transition points may prove to be a phenomenologically relevant and interesting process.

In fact, there emerged no true surprises in the context of mathematics where one simply observed differing patterns of behavior (and, in particular, of the complexification) of certain eigenvalues before and after the EP singularity. Unfortunately, once we started thinking about time \(t\) as parameter (with its EP value, say, at \(t = 0\)), we immediately imagined that certain purely formal nontriviality of the three-Hilbert-space (THS) formalism would force us to accept the restriction to adiabatically slow changes of the system in general and of the Hamiltonian operator \(H(t)\) in particular (cf. \([18]\) for details).

3.2 An amended strategy of analysis based on a given site operator \(\tilde{Q}\)

The unpleasant necessity of the slowness of changes of \(H(t)\) seems hardly compatible with the abrupt nature of the changes of the system near an EP singularity at \(t = 0\). For methodical reasons it seem reasonable, therefore, to replace the study of the time-dependent Hamiltonians \(H(t)\) (which combine the role of the operators of an observable energy with a partially independent role of the
generators of time evolution) by the formally similar but conceptually less confusing study of some other operators sharing the same perturbed-Jordan-block matrix form but representing, say, a non-Hermitian spin \( \Sigma(t) \) \([26] - [28]\) or, better, the observable \( Q(t) \) of discretized position alias site in a finite quantum lattice (cf. \([29] - [31]\)).

Our present strategy of the build-up of the theory initiated by the site operator \( \tilde{Q} \) might have been supported not only by the above-mentioned circumvention of the problems with adiabatic approximation (which are basically technical) but also by a few other, physics-oriented arguments. One of them is based on the observation \([32]\) that whenever one interprets the one-parametric family of eigenvalues \( q_n(t) \) of a site operator \( \tilde{Q} \) chosen in virtually any form of a perturbed \( N \)-dimensional Jordan block, then the unfolding pattern of these eigenvalues (very well sampled by their particular special case \( q_n(t) \sim c_n \sqrt{t} \) as derived, in Refs. \([33] - [35]\), for an exactly solvable discrete and \( PT \)-symmetric anharmonic oscillator) resembles the cosmological phenomenon of Big Bang. Indeed, in loc. cit., all of these eigenvalues stayed complex (i.e., unobservable) before the Big-Bang instant \( t = 0 \) while all of them became observable (i.e., real and non-degenerate and even, in absolute value, growing quickly) immediately after the Big-Bang time \( t = 0 \).

4 The phase-transition interpretation of the real exceptional points

In our present paper let us finalize the definition of our family of models (with \( H \) replaced by \( Q \) in Eq. (2)) in such a way that all of the components of the input \( J \)-plet of variable parameters \( \tilde{\xi} = \{ \alpha, \beta, \ldots, \omega \} = \{ \xi_1, \xi_2, \ldots, \xi_J \} \) of Eq. (2) become either proportional to an absolute value of time (thus, we shall have the even-function-superscripted components \( \xi_j = \xi_j^{(e)}(t) = |t| \) defined at all real \( t \in \mathbb{R} \)) or proportional to the plain time (for the remaining, odd-function-superscripted components \( \xi_k = \xi_k^{(o)}(t) = t, t \in \mathbb{R} \)). This means that every eligible quantum model of our family (with \( J = \text{entier}(N/2) \) in general) may be naturally characterized by the respective superscripts in vector \( \xi \), i.e., by a word \( \varrho \) of length \( J \) in the two-letter alphabet \( \{ o, e \} \) (i.e., one has two eligible words \( \varrho_0 = (o) \) and \( \varrho_1 = (e) \) at \( J = 1 \), four candidates \( \varrho_0 = (oo), \varrho_1 = (oe), \varrho_2 = (eo) \) and \( \varrho_3 = (ee) \) at \( J = 2 \), etc).

Let us now add that in accord with the specific implementations of the three-Hilbert-space pattern of Refs. \([16, 18]\) the key technical task of its users should be seen to lie in the reconstruction of a suitable metric \( \Theta_S \) from a given site-position matrix \( Q^{(N)}_{(o)}(t) \). At the not too large and positive times \( t > 0 \) such a construction is, due to our choice of tridiagonal \( Q^{(N)}_{(o)}(t) \), recurrent and entirely routine \([36]\).

4.1 Spectra at times close to the Big Bang instant

At an illustrative matrix dimension \( N = 10 \), the first nontrivial sample of our present lattice-site operator (2) will contain five free parameters \( \alpha, \ldots, \varepsilon \) alias \( \xi_1, \xi_2, \ldots, \xi_5 \) which we decided to
Figure 1: The time-dependence of the real part of the spectrum of the perturbed-Jordan-block ten-by-ten matrix (3) with index $q_1 = (oooe)$. 

choose in the first nontrivial concrete form of quintuplet $t, t, t, t, \ |t|$. As we indicated above, this choice is encoded in the word $q_1 = (oooe)$ yielding the matrix

$$
Q_{(10)}(t) = \begin{pmatrix}
0 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t & 0 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t & 0 & 1-t & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t & 0 & 1-t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 1-|t| & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & |t| & 0 & 1-t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t & 0 & 1-t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 1-t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 1-t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\
\end{pmatrix}.
$$

This matrix represents one of many possible $t \neq 0$ perturbations of the 10 times 10 Jordan block to which it degenerates at the EP time $t = 0$.

At the small but non-vanishing times $t \neq 0$ the $t-$dependence of the real eigenvalues of matrix (3) is displayed in Fig. 1. Once we accept the physical interpretation of such a matrix as a site-position operator of a dynamical ten-point quantum lattice near its phase transition instant $t = 0$ of the Big-Bang type, we may conclude that at the positive times $t > 0$, i.e., after the Big Bang instant this operator has the whole spectrum real and, hence, it may be perceived as representing an observable quantity.

We should emphasize that in contrast to an analogous model with $q_0 = (ooooeo)$ (possessing just an empty real spectrum to the left from $t = 0$), there now exists a pair of eigenvalues $q(t)$ which still remain real also before the Big Bang. Thus, Fig. 1 may be perceived as a schematic sample of an EP-interrupted evolution in which just a simpler, two-site part of our ten-point quantum
lattice may be interpreted as observable at \( t < 0 \). In other words, one could read the message delivered by Fig. 1 as opening the possibility of an innovative, quantum cosmology resembling scenario in which a subspace-based, simpler, two-level observable Universe evolved from the left, i.e., during the previous Eon and towards its own \( t = 0 \) Big Crunch collapse.

4.2 Alternative changes of physics during the pass of our quantum lattices through the EP time \( t = 0 \)

In spite of a truly elementary form of our family of the Big-Bang-simulating toy models \( Q_{(\omega)}^{(N)}(t) \), their capability of simulation and variability of alternative Big-Crunch/Big-Bang quantum phase-transition scenarios seems truly inspiring. In particular, the \( N = 10 \) Fig. 1 obtained at \( \varrho = \varrho_1 = (ooeee) \) may be complemented by its analogue of Fig. 2 with \( \varrho_7 = (ooeee) \) where the six levels remain real before the Big Crunch instant, etc.

Naturally, a fully left-right symmetric picture would be obtained at \( \varrho_{31} = (eeee) \), representing a schematic ten-dimensional-Universe counterpart to the well known (though very differently supported and constructed) Penrose’s cyclic-cosmology pattern in which the structure of the Universe before and after the Big Bang singular time is not expected to be too different. In contrast, our present family of models may be understood as supporting a rather non-standard hypothesis of an “evolutionary” cosmology in which the complexity of the structure of the Universe may change during the EP singularity and, in principle, grow with time from some more elementary structures existing during the previous Eons.

Needless to add that within the similar “evolutionary cosmology” phenomenological speculations, the structure of the Universe during the newer Eons (including also, e.g., its spatial dimensionality, etc) could have been also enriched by certain newly emergent qualities, the observability (i.e., the reality of the corresponding eigenvalues) of which had been suppressed earlier, emerging only on a sufficiently sophisticated level of the iterative global cosmic evolution.

Once we now return back to the elementary mathematical level of the concrete study of our
schematic quantum models, we still have to address a number of multiple less ambitious questions like, e.g., the purely technical problem of the necessity of the modification of the definition of the “old” physical Hilbert spaces which had to exist and describe our quantum lattice before \( t = 0 \).

In fact, at the finite dimensions \( N < \infty \) such a modification remains mathematically more or less trivial. Indeed, at \( t < 0 \) one merely has to preserve (i.e., project out) just the vector space which is spanned by the eigenvectors of \( Q(t) \) with the real eigenvalues. The resulting algorithmic pattern will partially resemble Eq. (11) in having the very similar five-Hilbert space form of the diagram in which the left and right triplet of windows represents our quantum lattice alias discrete Universe before and after the Big Bang, respectively:

| Big – Bang instant plus its vicinity: | \[ Q(0) \] not diagonalizable |
|--------------------------------------|-----------------------------|
| \( t < 0 \) | \( t > 0 \) |
| before–Big–Bang map | after–Big–Bang map |
| less, \( N' < N \) observable sites | \( \neq \) |
| reduced space \( \mathcal{H}^{(R)} \) | all \( N \) sites |
| \( Q_R = Q_R^s = \Theta_R^{-1}Q_R^t\Theta_R \) | standard space \( \mathcal{H}^{(S)} \) |
| old – timer \( \Theta_R = \Omega_R^t\Omega_R \neq I \) | \( Q = Q^\dagger = \Theta_S^{-1}Q^t\Theta_S \) |
| \( \Delta = N - N' \) ghosts projected out | current metric \( \Theta_S = \Omega_S^t\Omega_S \neq I \) |
| \( \nabla \) respective unitary equivalences | \( \nabla \) |

Naturally, the extension of such a recipe and diagram to the infinite-dimensional Hilbert space limit \( N = \infty \) will lead to multiple further open questions in functional analysis. As long as such a step already lies fairly beyond the scope of our present paper, let us merely conjecture that in the infinite-dimensional Hilbert-space limit and in the “previous Eon” with \( t < 0 \), the necessary elimination of the “ghost-supporting” spurious subspace (in which the eigenvalues of \( Q(t) \) are not yet real) might proceed in an analogy with the Mostafazadeh’s elimination trick of Ref. [38].

\[ q_{(old)} = \Omega R Q_R \Omega_R^{-1} = q_{(old)}^\dagger \]

\[ \text{extinct physics} \]

\[ \rightarrow \text{quantum phase transition} \rightarrow \text{contemporary physics} . \]

\[ \text{Hermitian reference} \]

\[ \text{previous Eon space } \mathcal{H}^{(P)}_{(old)} \]

\[ q_{(old)} = \Omega R Q_R \Omega_R^{-1} = q_{(old)}^\dagger \]

\[ \text{extinct physics} \]

\[ \rightarrow \text{quantum phase transition} \rightarrow \text{hermitian reference} \]

\[ \text{this Eon space } \mathcal{H}^{(P)}_{(now)} \]

\[ “our” \text{ coordinates } q_{(now)} = q_{(now)}^\dagger \]

\[ \text{contemporary physics} . \]
5 Discussion

Our present paper was inspired by the recent enormous popularity of the building of quantum models in which the unitary evolution is guaranteed and controlled, paradoxically, by non-Hermitian Hamiltonians $H \neq H^\dagger$. On this background our attention was shifted from the traditional $\mathcal{PT}$-symmetric alias pseudo-Hermitian Hamiltonians (such that one has $H^\dagger = \mathcal{P}HP^{-1} \neq H$ in terms of an indeterminate pseudometric $\mathcal{P}$) to the other, less prominent operators of observables. We emphasized that the choice of some other observables might enrich, first of all, our insight in some less understood processes in which the quantum system in question is forced to move through an exceptional-point singularity.

5.1 Considerations inspired by the open problems in physics

For the sake of definiteness we paid attention to certain specific time-dependent site-position operators $Q(t)$ which were chosen, for the sake of mathematical simplicity, in the form of finite matrices. In parallel, in a way emphasizing their phenomenological appeal we choose our $N$ by $N$ toy-model matrices $Q^{(N)}(t)$ in such a form that at the EP value of time $t = 0$ they degenerated to an $N-$dimensional and quantum-phase-transiton inducing Jordan-block non-diagonalizable matrix $Q^{(N)}(0)$.

We emphasized that any operator $Q$ representing an observable quantity admits in fact the same mathematical treatment as the most often considered energy operator alias Hamiltonian. In particular, once we wish to declare any operator observable in $\mathcal{H}^{(S)}$, the equivalent requirement of the Hermiticity of its image in $\mathcal{H}^{(P)}$ must be imposed, say, in the form

$$q = \Omega_S Q \Omega_S^{-1} = q^\dagger.$$

This condition may be also equivalently reformulated, inside the second physical space $\mathcal{H}^{(S)}$, as follows,

$$Q^{\dagger}(t) \Theta_S(t) = \Theta_S(t) Q(t). \quad (5)$$

In full analogy with the standard THS recipe, one can start, therefore, from any family of the benchmark-model forms of the operator $Q$ and perform the reconstruction of the admissible metrics $\Theta_S$ via Eq. (5).

In the context of physics we revealed that after an appropriate further sophistication and development our initial idea of studying the phase-transition pass through the Big-Bang-resembling EP singularity of $Q^{(N)}(t)$ at $t = 0$ could find multiple future applications even in quantum cosmology. We conjectured that the traditional alternative scenarios of “nothing before Big Bang” and of the “cyclic repetition of Big Bangs” (e.g., in the form as proposed by Penrose [37]) could be, in this light, complemented by the slightly subtler possibilities and speculations about having “Darwin-like evolution” and various “physics-structure jumps” at the subsequent Big Bangs.
Considerations inspired by the open problems in mathematics

Naturally, all of the above-sampled phenomenological speculations suffer from the insufficiently realistic and oversimplified finite-dimensional alias discrete-lattice nature of our toy-model observables $Q^{(N)}(t)$. In this sense, the main opening mathematical challenge (not to be addressed here at all) may be now seen in the study of $N \to \infty$ extensions of the discrete-lattice models.

Another set of the open mathematical problems emerges even at the finite matrix dimensions $N < \infty$. One of the most important ones concerns the problem of the influence of perturbations on any toy-model based qualitative result. In the conclusion of this paper let us now pay more attention to this fairly important subproblem.

Our basic inspiration has been provided by the experience covered by the Trefethen’s and Embree’s book [39], with its extreme emphasis on the constructive considerations and with its rather persuasive recommendation that the influence of the perturbations should be always characterized in the language of the so called pseudospectra [40].

An easy and compact introduction in the concept of the pseudospectrum may be provided here either by the reference to loc. cit. or to Fig. 3. In the latter picture one sees that the purely real spectrum (i.e., the real-eigenvalue positions $z_n \in \mathbb{R}$ of the infinitely high peaks of the norm $|R(z)|$ of the resolvent) may be perceived as very well approximated by the $0 < \varepsilon \ll 1$ pseudospectra (which are defined, according to loc. cit., as the - in general, multiply connected - open complex domains $J_\varepsilon$ of $z$ in which $|R(z)| > 1/\varepsilon$).
In contrast to the after Big Bang picture as provided by Fig. 3, a conceptually much more challenging problem emerges at the negative, pre-Big-Bang times at which some of the eigenvalues of matrix $Q^{(N)}(\tau)(t)$ remain non-real. As we already mentioned above, we are not going to address this challenge directly but rather we intend to recall the Trefethen’s and Embree’s advice.

Although, in their book, the pseudospectra are predominantly recommended for an estimate of the influence of a generic perturbation, the role of their study in our present context will be different. Indeed, in the light of the diagram of Eq. (1) one of the key tasks of the constructive approach to the quantitative description of our present versions of the quantum phase transitions lies in the possibility of a clear separation of the “quantum observables before Big Bang” (i.e., of a subspace which is spanned by the real-eigenvalue eigenvectors of $Q(t)$) from the perpendicular subspace of the unobservable, non-real-eigenvalue “ghosts”.

In practice, naturally, the separation of this type (which, incidentally, resembles strongly the Gupta-Bleuler trick known from textbooks on quantum electrodynamics) must be performed numerically. This implies that one must specify the domains of applicability of a feasible separation of this type.

In this context, our present final recommendation is to use the inspection of the pseudospectra for the purpose. For illustration, let us first recall Figs. 4 (with its equivalent form 5) in which one sees an entirely distinct numerical separation of the states with the real eigenvalues from the ghosts. In such a dynamical scenario (with $\varrho = \varrho_{19} = (eeoeo)$) one may expect that the projector-operator elimination of the ghosts from the physical before-the-Big-Bang reduced Hilbert space.
Figure 5: The real function of Fig. 4 in its two-dimensional representation. The lines marking the constant norm are the boundaries of the complex domains called “pseudospectra” [39].

Figure 6: A rearrangement of the pseudospectra of Fig. 5 at another index, $\rho = (eoooe)$.
Figure 7: Another set of the before-the-Big-Bang pseudospectra, with $\varrho = (eoeoe)$.

$H^{(R)}$ will be numerically feasible and straightforward.

In contrast, the inspection of another, $\varrho = \varrho_{17} = (eoeeoe)$ toy model may be expected to indicate an emergence of the technical and numerical difficulties because the related Fig. 6 shows that up to the very small $\varepsilon$s, the pseudospectral domains $J_\varepsilon$ of $z$ are shared by more peaks and do not sufficiently clearly separate each of the two real eigenvalues from its two complex conjugate neighbors. At the same time, the four “remote” complex eigenvalues still seem to be separated in a sufficiently clear manner.

From the perturbation-influence-estimate point of view, by far the worst situation is encountered at $\varrho = (eoeoe)$ where our last Fig. 7 indicates that and why the separation of the eight-dimensional ghost subspace may be expected to be truly difficult.

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References

[1] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7 (2010) 1191.

[2] E. Caliceti and S. Graffi, in “Nonselfadjoint operators in quantum physics: mathematical aspects.” (F. Bagarello et al, Eds., Wiley, Hoboken, 2015, p. 183), in print.

[3] C. M. Bender, Rep. Prog. Phys. 70 (2007) 947.

[4] T. Kato, Perturbation theory for linear operators. (Springer-Verlag, Berlin, 1966).

[5] M. V. Keldysh, Russian Math. Surveys 26 (1971) 15.

[6] D. Borisov, Acta Polytechnica 54 (2014) 93.

[7] W. D. Heiss, J. Phys. A: Math. Theor. 45 (2012) 444016.

[8] Y. N. Joglekar, C. Thompson, D. D. Scott and G. Vemuri, Eur. Phys. J. Appl. Phys. 63 (2013) 30001.

[9] A. Tanaka, S. W. Kim and T. Cheon, Phys. Rev. E 89 (2014) 042904.

[10] J. Fuchs, J. Main, H. Cartarius and G. Wunner, J. Phys. A: Math. Theor. 47 (2014) 125304.

[11] F. Bagarello and F. Gargano, Phys. Rev. A (in print), arXiv:1402.6201.

[12] I. Rotter, J. Phys. A: Math. Theor. 42 (2009) 153001.

[13] N. Moiseyev, Non-Hermitian Quantum Mechanics. (Cambridge University Press, Cambridge, 2011).

[14] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.

[15] A. Turbiner, private communication (May 2000).

[16] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.

[17] M. Znojil, Phys. Rev. D 78 (2008) 085003.

[18] M. Znojil, SIGMA 5 (2009) 001 (arXiv overlay: 0901.0700).

[19] M. Znojil and H. B. Geyer, Fort. d. Physik 61 (2013) 111.

[20] A. Mostafazadeh, J. Phys. A: Math. Gen. 39 (2006) 10171.

[21] M. Znojil, Ann. Phys. (NY) 336 (2013) 98.

[22] S. Longhi, Europhys. Lett. 106 (2014) 3400.
[23] A. V. Smilga, Int. J. Theor. Phys. (in print), arXiv:1409.8450.

[24] M. Znojil, J. Phys. A: Math. Theor. 45 (2012) 444036.

[25] G. Lévai, F. Ruzicka and M. Znojil, Int. J. Theor. Phys. 53 (2014) 2875.

[26] M. Znojil, J. Phys. A: Math. Gen. 39 (2006) 441.

[27] M. Znojil, Phys. Lett. A 353 (2006) 463.

[28] M. Znojil, J. Phys. A: Math. Gen. 39 (2006) 4047.

[29] M. Znojil, Phys. Rev. D. 80 (2009) 045022.

[30] M. Znojil, Phys. Rev. D. 80 (2009) 045009.

[31] M. Znojil, Phys. Lett. A 375 (2011) 3176.

[32] M. Znojil, J. Phys.: Conf. Ser. 343 (2012) 012136.

[33] M. Znojil, J. Phys. A: Math. Theor. 40 (2007) 4863.

[34] M. Znojil, J. Phys. A: Math. Theor. 40 (2007) 13131.

[35] M. Znojil, J. Phys. A: Math. Theor. 41 (2008) 244027.

[36] M. Znojil, J. Phys. A: Math. Theor. 45 (2012) 085302.

[37] V. G. Gurzadyan and R. Penrose, Eur. Phys. J. Plus 128 (2013) 22.

[38] A. Mostafazadeh, Phil. Trans. R. Soc. A 371 (2013) 20120050.

[39] L. N. Trefethen and M. Embree, Spectra and pseudospectra - the behavior of nonnormal matrices. (Princeton University Press, Princeton, 2005).

[40] D. Krejcirik, P. Siegl, M. Tater and J. Viola, Pseudospectra in non-Hermitian quantum mechanics. arXiv:1402.1082.