Almost regularity conditions of spectral problems for a second order equation

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Abstract

In this paper the asymptotic distributions are exactly solved for linearly independent solutions considering problem of the second order and for the coefficients of asymptotic distribution the recurrent formulas are obtained. Further, using obtained recurrent formulas the necessary and sufficient conditions almost regularity of spectral problem for the equation of the second order is proved.

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1 Introduction

It is well known that spectral problems for ordinary differential equations (O.D.E) begin with G.D.Birkhoff’s work [1], where the regular boundary conditions were determined, it was established an expansion formula of function in series on eigen and joint elements (E.J.E) of relative operators, when the coefficients of equation and boundary conditions don’t depend on the complex parameter $\lambda$.

The following more general result belongs to Ya.D.Tamarkin [2], where a class of regular and strong regular spectral problems was studied, the theorems on expansion of functions in series on their root elements (i.e. on E.J.E) are proved, when the coefficients of equation and boundary conditions depend on the complex parameter $\lambda$.

In [4-6] for a spectral problem considered in [2], but for multipoint with discontinuous coefficients, the notion of regularity is given by M.L.Rasulov, distinct from [2] multiple expansion formula is obtained that it’s extremely important with point of view of solving of corresponding mixed problems for partial differential equations.

In M.V.Keldyshs article [7] it was introduced the theorem on multiple completeness of a system E.J.E. of a spectral problem which covers some non-regular cases. But in A.A.Shkalikov article [8] the wide subclass of non-regular problems (called almost regular) for which the system of root elements has a block-basisness property in determined spaces, is chosen. However, in [8] the almost regularity conditions of a spectral problems are connected with geometry of dominating addends of asymptotical expansion of characteristic determinant of Green’s function that doesn’t allows in the form of the problem to clarify, if it’s almost regular and if it’s, then of which order?

The laborious and cumbersome calculations of coefficients of asymptotical expansion of characteristic determinant in the case of variables of coefficients of equation, be practically impossible. Therefore, this question was open.

In this paper for a spectral problem of second order equation we obtained a necessary and sufficient condition of almost regularity of $m$ (arbitrary) order, distinct from [8], allowing immediately and elementarily to answer to the question of almost regularity of a spectral problem and determine
order of almost regularity. It’s proved that for equations with variable coefficients the almost regular problems of any order exist.

2 Statement of a problem and some preliminary remarks

Consider the spectral problem

$$l\left(\frac{d}{dx}, \lambda\right)y \equiv y'' + q(x)y = \lambda^2 y, \quad 0 < x < 1$$

(1)

$$U_i(y) \equiv \sum_{j=0}^1 \alpha_{ij}y^{(j)}(0) + \beta_{ij}y^{(j)}(1) = 0$$

(2)

where \(q(x)\) is a complex-valued function, \(\alpha_{ij}, \beta_{ij}\) are complex numbers such that

\[\alpha_{11}\alpha_{12}\beta_{11}\beta_{21}\alpha_{21}\alpha_{22}\beta_{21}\beta_{22} = 2, \lambda \text{ is a spectral parameter.}\]

It’s known [3] that if \(\alpha_{11}\beta_{21} - \beta_{11}\alpha_{21} \neq 0\), then the conditions (2) are regular by Birkhoff. If \(\alpha_{11}\beta_{21} - \beta_{11}\alpha_{21} = 0\), then without loss of generality we can assume that \(\alpha_{21} = \beta_{21} = 0\). Where in the last case, the boundary conditions (2) will be regular by Birkhoff only on fulfilment of one of the conditions

\[\alpha_{11}\beta_{20} + \beta_{11}\alpha_{20} \neq 0, \alpha_{11} = \beta_{11} = 0, \alpha_{10}\beta_{20} - \beta_{10}\alpha_{20} \neq 0.\]

This, among the various boundary conditions of the form (2) which are only non-regular by Birkhoff that they may be led to the form

$$U_1(y) = \alpha_{11}y'(0) + \alpha_{10}y(0) + \beta_{11}y'(1) + \beta_{10}y(1) = 0,$$

$$U_2(y) = \alpha_{20}y(0) + \beta_{20}y(1) = 0$$

(3)

and for whose coefficients

$$\alpha_{11}\beta_{20} + \beta_{11}\alpha_{20} = 0, |\alpha_{11}| + |\beta_{11}| > 0, |\alpha_{20}| + |\beta_{20}| > 0$$

(4)

is satisfied.

Thus, we’ll consider the problem (1), (3) on fulfilment of the condition (4).
3 Connection between periodic properties of the potential \( q(x) \) of equation and almost regularity of any order of a spectral problem

Here under definite conditions on smoothness of the function \( q(x) \), we obtain a formula allowing us to calculate the coefficients of any member of asymptotic expansion on power \( \lambda \) (when \( |\lambda| \to \infty \)) of fundamental system of particular solutions (f.s.p.s) of the equation (1).

Let, \( q(x) \in C^m[0, 1] \), by the method of reduction to the systems of integral equations [3], it’s shown that for sufficiently large \( R > 0 \) at each of the domains \( C^+_R = \{ \lambda R \lambda > 0, |\lambda| > R \} \), \( C^-_R = \{ \lambda R \lambda < 0, |\lambda| > R \} \) of the equation has f.s.p.s, allowing the asymptotical representations

\[
\frac{d^{\nu} y_i(x, \lambda)}{dx^\nu} = \lambda^\nu \exp\left[(-1)^i \lambda x\right] \left[ \sum_{s=0}^{m} \lambda^{-s} g^{(s)}_{i\nu}(x) + \eta_{i\nu}(x, \lambda) \right], \ (i = 1, 2; \nu = 0, 1)
\]

where the functions \( \eta_{i\nu}(x, \lambda) \) are continuous at \( x \in [0, 1] \), analitical by \( \lambda \in C^+_R \) and satisfy the inequality

\[
|\eta_{i\nu}(x, \lambda)| \leq M |\lambda|^{-m-1}
\]

and \( g^{(s)}_{i\nu}(x) \) are determined by the recursion relations

\[
g^{(0)}_{i\nu}(x) = (-1)^\nu, \ g^{(1)}_{i\nu}(x) = (-\frac{1}{2})^{i(\nu+1)-1} \int_0^x q(\xi) d\xi, \ (i = 1, 2; \nu = 0, 1)
\]

\[
g^{(s)}_{i\nu}(x) = (-\frac{1}{2})^{i(\nu+1)-1} \int_0^x q(\xi) g^{(s-1)}_{i0}(\xi) d\xi + \sum_{j=0}^{s-2} (-\frac{1}{2})^{i-1(j+\nu+s-j)} \frac{d^{s-j-2}}{dx^{s-j-2}}
\]

\[
\times [q(x) g^{(j)}_{i0}(x)], \quad (i = 1, 2; \nu = 0, 1; s = 2, m)
\]

Further, using the formula (5) and estimation (6) for characteristic determinant of Green’s problem (1), (3) we obtain the expression

\[
\Delta(\lambda) = \delta_{-1}(\lambda)e^{-\lambda} + \delta_0(\lambda) + \delta_1(\lambda)e^\lambda,
\]

\[
\delta_k(\lambda) = \sum_{i=0}^{m} \lambda^{1-i} \delta_k^{(1-i)} + O\left(\frac{1}{\lambda^m}\right), \ (k = -1, 0, 1)
\]

where \( \delta_k^{(1-i)} \) \((k = -1, 0, 1; i = 0, 1, ...m)\) are constants given by the formulas

\[
\delta^{(1-i)}_{-1} = \sum_{j=0}^{i} [\alpha_{20}\beta_{11} g^{(j)}_{20}(0) g^{(i-j)}_{11}(1) - \beta_{20}\alpha_{11} g^{(j)}_{21}(0) g^{(i-j)}_{10}(1)] +
\]
\[
\alpha_{20} \beta_{10} - \beta_{20} \alpha_{10} \sum_{j=0}^{i-1} g^{(j)}_{20}(0) g^{(i-j-1)}_{10}(1),
\]

\[
\delta^{(1-i)}_0 = \sum_{j=0}^{i} \left[ \alpha_{20} \alpha_{11}(g^{(j)}_{20}(0) g^{(i-j)}_{11}(0) - g^{(j)}_{10}(0) g^{(i-j)}_{21}(0)) + \beta_{20} \beta_{11}(g^{(j)}_{20}(1) g^{(i-j)}_{11}(1) - g^{(j)}_{10}(1) g^{(i-j)}_{21}(1)) \right],
\]

\[
\delta^{(1-i)}_1 = \sum_{j=0}^{i} \left[ \beta_{20} \alpha_{11}(j) g^{(i-j)}_{11}(1) - \alpha_{20} \beta_{11} g^{(i-j)}_{10}(0) g^{(i-j)}_{21}(1) + (\alpha_{10} \beta_{20} - \beta_{10} \alpha_{20}) \sum_{j=0}^{i-1} g^{(j)}_{10}(0) g^{(i-j-1)}_{20}(1),
\]

\[
(i = 0, 1, 2, \ldots, m)
\]

From the formula ((10) and (7)) we find:

\[
\delta^{(1-i)}_{-1} = (-1)^i \delta^{(1-i)}_1, \quad (i = 1, 1, 2, \ldots m)
\]

From the formulas (11), (8), (9) according to A.A. Shkalikov’s definition [8] we have Definition 1. Let \( q(x) \in C^m[0, 1] \), the boundary forms \( U_i(i = 1, 2) \) have the form (3) and \( |\alpha_{11}| + |\beta_{11}| > 0, |\alpha_{20}| + |\beta_{20}| > 0 \). Then the spectral problem (1), (2) is called almost regular of the order \( m \geq 0 \), if

\[
\delta^{(1)}_{-1} = \delta^{(0)}_{-1} = \ldots = \delta^{(2-m)}_{-1} = 0, \quad \delta^{(2-m)}_{-1} \neq 0
\]

Remark 1. The almost regular problem of zero order \( (m = 0, \delta^{(1)}_{-1} = 0) \) is regular by Tamarkin-Rasulov and corresponding to it the boundary conditions (2) are regular by Birkhoff. This follows from the expansion

\[
\delta^{(1)}_{-1} = -(\alpha_{20} \beta_{11} + \beta_{20} \alpha_{11}),
\]

that is obtained by the substitution of (7) in (11). Substituting (7) in (11) when \( i = 1 \) we obtain

\[
\delta^{(0)}_{-1} = -\frac{1}{2} (\alpha_{20} \beta_{11} + \beta_{20} \alpha_{11}) \int_0^1 q(\xi) d\xi - (\alpha_{10} \beta_{20} - \beta_{10} \alpha_{20})
\]

from (13), (14) and definition 1 it’s obvious that the problem (1), (2) will be almost regular of the first order, if

\[
\alpha_{20} \beta_{11} + \beta_{20} \alpha_{11} = 0, \quad \alpha_{10} \beta_{20} - \beta_{10} \alpha_{20} \neq 0
\]
at the same time for almost regularity of this problem, of the order \(m \geq 2\), it’s necessary that
\[
\alpha_{20}\beta_{11} + \beta_{20}\alpha_{11} = 0, \quad \alpha_{10}\beta_{20} - \beta_{10}\alpha_{20} = 0 \tag{16}
\]

But on fulfilment of (16), the expression for the number \(\delta_{-1}^{(1-i)} (i \geq 2)\) is simplified and takes the form
\[
\delta_{-1}^{(1-i)} = \alpha_{20}\beta_{11} \sum_{j=0}^{i} (-1)^j [g_{10}^{(j)}(0)g_{11}^{(i-j)}(1) - g_{11}^{(j)}(0)g_{10}^{(i-j)}(1)] \tag{17}
\]
\[(i = 2, 3, \ldots),\]
where it must be \(\alpha_{20}\beta_{11} \neq 0\). Since it’s easy to see that on non-fulfilment of this inequality, the problem (1), (2) becomes to the cauchy problem and isn’t normal in sense of [8]. As it’s obvious from (17) the order of almost regularity higher than first, doesn’t depend on coefficients of the boundary conditions (2) and may be connected only with the coefficient \(q(x)\) of the equation (1). From (17) after simple transformations we obtain
\[
\delta_{-1}^{(-1)} = \frac{1}{2}\alpha_{20}\beta_{11}[q(1) - q(0)],
\]
\[
\delta_{-1}^{(-2)} = \frac{1}{4}\alpha_{20}\beta_{11}\{[q'(1) + q'(0)] + [q(1) - q(0)] \int_0^1 q(\xi)d\xi\},
\]
\[
\delta_{-1}^{(-3)} = \frac{1}{8}\alpha_{20}\beta_{11}\{[q''(1) - q''(0)] [q'(1) + q'(0)] \int_0^1 q(\xi)d\xi +
+ 2[q(1) - q(0)] \int_0^1 q(\xi) \cdot g_{10}''(\xi)d\xi + 2[q^2(1) - q^2(0)], \tag{18}
\]
and so on. The regularity which we can note from these formulas, is generalized in the following proposition:

Lemma 1. Let (16) be satisfied, \(\alpha_{20}\beta_{11} \neq 0\) and \(\delta_{-1}^{(-1)} = \delta_{-1}^{(-2)} = \ldots = \delta_{-1}^{(-k)} = 0\). Then in order to be \(\delta_{-1}^{(-k-1)} = 0\), it’s necessary and sufficient that
\[
q^{(k)}(0) = (-1)^kq^{(k)}(1), \quad (k \geq 0) \tag{19}
\]
For the proof of this statement we must establish the following lemma, which has an independent value in sense of simplification of the recursion relations (7).

Lemma 2. For any natural \(s\) for the function \(g_{10}^{(s)}(x)\) the following representation is valid.
\[
g_{10}^{(s)}(x) = 2^{-s} \sum_{\nu=1}^{s_0} [k_1 + \ldots + k_\nu = s + 1 - 2\nu \sum_{\nu,k_1,\ldots,k_\nu} \cdot q^{(k_1)}(x)\ldots q^{(k_\nu)}(x) +
\]
+ k_1 + \ldots + k_\nu = s + 1 - 2\nu \sum \alpha_{s,k_1,\ldots,k_{\nu-1}}^{(\nu)} \cdot q^{(k_1)}(x) q^{(k_{\nu-1})}(x) \ldots q^{(s-2\nu-k_1-\ldots-k_{\nu-1})}(x) \]

where \( s_0 = \left[ \frac{s+1}{2} \right] \) is real part of \( \frac{s+1}{2} \),

\[
q_i(x) = 2^i \int_0^x q(\xi) g_{10}^{(i)}(\xi) d\xi, \quad (i = 0, 1, \ldots)
\]

and the natural numbers \( \alpha_{s}^{(\nu)} \ldots \), are determined by the formulas

\[
\alpha_{s,k_1,\ldots,k_{\nu}}^{(\nu)} = \sum_{j=k_{\nu}+1}^{s-2(\nu-1)-k_1-\ldots-k_{\nu}} C_{s-2(\nu-1)-k_1-\ldots-k_{\nu-2-j}}^{k_{\nu-1}} \alpha_{s,k_1,\ldots,k_{\nu-2,j}}^{(\nu)} \alpha_{s,k_1}^{(1)} = 1
\]

Proof: It’s obvious that under the designation of (21) we can represent (7) (when \( \nu = 0 \)) in the form of

\[
g_{10}^{(s)}(x) = 2^{-s} \sum_{i=0}^{s-1} q_i^{(s-1-i)}(x),
\]

whence we have

\[
q_s(x) = 2^s q(x) g_{10}^{(s)}(x) = q(x) \sum_{i=0}^{s-1} q_i^{(s-1-i)}(x).
\]

From (23) subject to (24) we obtain

\[
g_{10}^{(s)}(x) = 2^{-s} \left\{ [q_{s-1}(x) + q_0^{(s-1)}(x)] + \sum_{i_1=0}^{s-1} [q_{i_1}^{(s-2-i_1)}(x)] \right\} =
\]

\[
= 2^{-s} \left\{ [q_{s-1}(x) + q_0^{(s-1)}(x)] + \sum_{i_1=0}^{s-1} q(x) \sum_{i_2=0}^{i_1-1} [q_{i_2}^{(i_1-1-i_2)}(x)] \right\} =
\]

\[
= 2^{-s} \left\{ [q_{s-1}(x) + q_0^{(s-1)}(x)] + \sum_{i_1=0}^{s-1} \sum_{i_2=0}^{i_1-1} \sum_{k_1=0}^{s-3-i_1-2} C_{s-i_1-2}^{k_1} q(k_1)(x) \times q_{i_2}^{(s-3-i_2-k_1)}(x) \right\}
\]

\[
= 2^{-s} \left\{ [q_{s-1}(x) + q_0^{(s-1)}(x)] + \sum_{k_1=0}^{s-3} \sum_{i_2=0}^{s-3-k_1} \sum_{i_1=i_2+1}^{s-3-k_1} C_{s-i_1-2}^{k_1} q(k_1)(x) q_{i_2}^{(s-3-i_2-k_1)}(x) \right\}
\]

denoting

\[
\alpha_{s,s-1}^{(1)} = \alpha_{s,0}^{(1)} = 1, \quad \alpha_{s,k_1,i_2}^{(2)} = \sum_{i_1=i_2+1}^{s-2-k_1} C_{s-i_1-2}^{k_1} \alpha_{s,i_1}^{(1)} = \sum_{i_1=i_2+1}^{s-2-k_1} C_{s-i_1-2}^{k_1} = C_{s-i_1-2}^{k_1}
\]
from the last formula we have
\[ g_{10}^{(s)}(x) = 2^{-s} \left\{ \left[ \alpha^{(1)}_{s, s-1} q_{s-1}(x) + \alpha^{(1)}_{s, 0} q_{0}^{(s-1)}(x) \right] + \sum_{k_1 + i_2 \leq s-3} \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) \right\} \]
\[ \times q_{i_2}^{(s-3-i_2-k_1)}(x) \]  
\[ = 2^{-s} \left\{ \left[ \alpha^{(1)}_{s, s-1} q_{s-1}(x) + \alpha^{(1)}_{s, 0} q_{0}^{(s-1)}(x) \right] + \right. \]
\[ \left. \sum_{k_1 + i_2 = s-3} \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) q_{i_2}(x) + \sum_{k_1 = 0}^{s-4} \alpha^{(2)}_{s, k_1, 0} \cdot q^{(k_1)}(x) \right\} \]
\[ \times q_{0}^{(s-3-k_1)}(x) \]  
\[ \sum_{k_1 + i_2 \leq S-4} \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) q_{i_2}^{(s-3-i_2-k_1)}(x) \}

using the formulas (23), (24) we transform the last addend in the right hand side of (25)
\[ \sum_{k_1 + i_2 \leq S-4} \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) q_{i_2}^{(s-3-i_2-k_1)}(x) = \]
\[ = \sum_{k_1 + i_2 \leq S-4} \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) \left[ q^{(i_2-1-i_3)}(x) \sum_{i_3 = 0}^{i_2} q_{i_3}^{(i_2-1-i_3)}(x) \right]^{(s-4-k_1-i_2)} = \]
\[ = \sum_{k_1 = 0}^{s-5} \sum_{i_2 = 1}^{s-4-k_2} \sum_{i_3 = 0}^{s-4-k_1-i_2} \sum_{i_2 = i_3 + 1}^{s-4-k_1-i_2} C^{k_2}_{s-4-k_1-i_2} \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) q^{(k_2)}(x) \]
\[ \times q_{i_3}^{(s-5-k_1-k_2-i_3)}(x) = \sum_{k_1 = 0}^{s-5} \sum_{k_2 = 0}^{s-5-k_1-k_2} \sum_{i_3 = 0}^{s-4-k_2-i_2} \sum_{i_2 = i_3 + 1}^{s-4-k_1-i_2} C^{k_2}_{s-4-k_1-i_2} \]
\[ \times \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) q^{(k_2)}(x) q_{i_3}^{(s-5-k_1-k_2-i_3)}(x) = \]
\[ = \sum_{k_1 + k_2 + i_3 \leq S-5} \alpha^{(3)}_{s, k_1, k_2, i_3} q^{(k_1)}(x) q^{(k_2)}(x) q_{i_3}^{(s-5-k_1-k_2-k_2-i_3)}(x) \]

where
\[ \alpha^{(3)}_{s, k_1, k_2, i_3} = \sum_{i_2 = i_3 + 1}^{s-4-k_1-k_2} C^{k_2}_{s-4-k_1-i_2} \alpha^{(2)}_{s, k_1, i_2}. \]

substituting (26) in (25) we obtain
\[ g_{10}^{(s)}(x) = 2^{-s} \left\{ \left[ \alpha^{(1)}_{s, s-1} q_{s-1}(x) + \alpha^{(1)}_{s, 0} q_{0}^{(s-1)}(x) \right] + \right. \]
\[ \left. + \left[ \sum_{k_1 + i_2 = s-3} \alpha^{(2)}_{s, k_1, i_2} q^{(k_1)}(x) q_{i_2}(x) + \sum_{k_1 \leq s-4} \alpha^{(2)}_{s, k_1, 0} q^{(k_1)}(x) q_{0}^{(s-3-k_1)}(x) \right] + \right. \]
\[ \left. + \left[ \sum_{k_1 + k_2 + i_3 = s-5} \alpha^{(3)}_{s, k_1, k_2, i_3} q^{(k_1)}(x) q^{(k_2)}(x) q_{i_3}(x) \right] + \right. \]
\[ \left. + \sum_{k_1 + k_2 \leq s-6} \alpha^{(3)}_{s, k_1, k_2} q^{(k_1)}(x) q^{(k_2)}(x) q_{0}^{(s-5-k_1-k_2)}(x) \right\} + \]
\[
+ \sum_{k_1+k_2+i_3 \leq s-6} \alpha_{s,k_1,k_2,i_3}^{(3)} q^{(k_1)}(x)q^{(k_2)}(x)q_{i_3}^{(s-5-k_1-k_2-i_3)}(x) \}
\]

By repeating the above mentioned transformation once more \( S_0 \) times, it’s evident that we obtain the formulas (20), (22) that it’s easily confirmed also by induction. Remark 2. The natural numbers \( \alpha_{s,k_1,\ldots,k_\nu}^{(\nu)} \) appearing in the formula (20), have the property
\[
\alpha_{s,k_1,\ldots,k_\nu}^{(\nu)} = \alpha_{s-k_\nu,k_1,\ldots,k_{\nu-1}}^{(\nu)} \quad \text{(27)}
\]
which will be usefull to prove lemma 1. Proof of lemma 1. We use the mathematical induction method. When \( k = 0 \) and \( n = 1 \) the validity of statement of lemma follows from the first two equalities of the formula (18). Let for some \( i \geq 4 \) it be valid for all \( k \leq i - 3 \). We proof the validity of statement of lemma when \( k = i - 2 \), i.e. let the conditions (16) be satisfied, \( \alpha_{20} \cdot \beta_{11} \neq 0 \) and
\[
\delta_{1-1}^{(-1)} = \ldots = \delta_{1-1}^{(2-i)} \quad \text{(28)}
\]
and we establish that \( \delta_{1-1}^{(1-i)} = 0 \), iff the following equality holds
\[
q^{(i-2)}(1) = (-1)^{i-2}q^{(i-2)}(0), \quad \text{(29)}
\]
from the fulfilment of (28) by virtue of our supposition it follows that
\[
q^{(k)}(1) = (-1)^k q^{(k)}(0), \quad (k = 0, 1, \ldots, i - 3). \quad \text{(30)}
\]
It’s lasy to see that by using the designation (21) we can represent in the form of
\[
(\alpha_{20} \beta_{11})^{-1} \delta_{1-1}^{(1-i)} = 2[g^{(i)}_{10}(1) + (-1)^{i+1}g^{(i)}_{10}(0)] + \sum_{j=0}^{i-1}(-1)^{j+1}2^{j+i-j}g^{(j)}_{10}(0)q_{i-j-1}(1), \quad \text{(31)}
\]
substituting (20) in (31) we find
\[
2^{i-1}(\alpha_{20} \beta_{11})^{-1} \delta_{1-1}^{(1-i)} = \sum_{\nu=1}^{i_0} \sum_{k_1+\ldots+k_\nu=i+1-2\nu} \alpha^{(\nu)}_{i,k_1,\ldots,k_\nu-1} \times
\]
\[
\times [q^{(k_1)}(1)\ldots q^{(k_{\nu-1})}(1)q_{k\nu}(1) + (-1)^{i+1}q^{(k_1)}(0)\ldots q^{(k_{\nu-1})}(0) \times q_{k\nu}(0)] + \]
\[
+ \sum_{k_1+\ldots+k_\nu \leq 1-2\nu} \alpha^{(\nu)}_{i,k_1,\ldots,k_\nu-1,0}[q^{(k_1)}(1)\ldots q^{(k_{\nu-1})}(1)q^{(1-2\nu-k_1-\ldots-k_{\nu-1})}(1) + q^{(k_1)}(0)\ldots q^{(k_{\nu-1})}(0)q^{(1-2\nu-k_1-\ldots-k_{\nu-1})}(0)] - q_{i-1}(1)+
\]
\[
\sum_{j=2}^{i-1} (-1)^{j+1} q_{i-j-1}(1) \sum_{\nu=1}^{j+1-2\nu} \alpha_{j,k_1,\ldots,k_\nu}^{(\nu)} \times q^{(k_1)}(0) \ldots q^{(k_{\nu-1})}(0) q_{k\nu}(0) + \\
\sum_{k_1+\ldots+k_\nu \leq j-2\nu} \alpha_{j,k_1,\ldots,k_{\nu-1},0}^{(\nu)} q^{(k_1)}(0) \ldots q^{(k_{\nu-1})}(0) q^{(j-2\nu-k_1-\ldots-k_{\nu-1})}(0),
\]

where we denote by \(i_0\) and \(j_0\), integer parts of the numbers \(i+\frac{2}{2}\) and \(i+\frac{1}{2}\), respectively. Subject to (30) and (21) \(q_{k\nu}(0) = 0\) (see (21)) from (32) we have

\[2^{i-1}(\alpha_{20,11})^{-1} \delta^{(1-i)}_{-1} = \sum_{\nu=1}^{i_0} \sum_{k_1+\ldots+k_\nu = i+1-2\nu} \alpha_{i,k_1,\ldots,k_{\nu-1}}^{(\nu)} \times \]

\[\times q^{(k_1)}(1) \ldots q^{(k_{\nu-1})}(1) q_{k\nu}(1) + \sum_{\nu=2}^{i_0} \sum_{k_1+\ldots+k_\nu = i+1-2\nu} \alpha_{i,k_1,\ldots,k_\nu,0}^{(\nu)} \times \]

\[\times q^{(k_1)}(1) \ldots q^{(k_{\nu-1})}(1) q^{(i-2\nu-k_1-\ldots-k_{\nu-1})}(1) \times \]

\[\times [1 + (-1)^{i+1}(-1)^{-2\nu}] + \alpha_{i,0}^{(1)} [q^{(i-2)}(1) + (-1)^{i+1} q^{(i-2)}(0)] - \]

\[- q_{i-1}(1) + \sum_{j=2}^{i-1} (-1)^{j+1} q^{(1)}_{i-j-1} \sum_{\nu=1}^{j_0} \sum_{k_1+\ldots+k_{\nu-1} \leq j-2\nu} \alpha_{j,k_1,\ldots,k_{\nu-1},0}^{(\nu)} \times \]

\[\times q^{(k_1)}(0) \ldots q^{(k_{\nu-1})}(0) q^{(j-2\nu-k_1-\ldots-k_{\nu-1})}(0) = \]

\[= \alpha_{i,0}^{(1)} [q^{(i-2)}(1) + (-1)^{i+1} q^{(i-2)}(0)] + \]

\[+ \sum_{\nu=2}^{i_0} \sum_{k_1+\ldots+k_\nu = i+1-2\nu} \alpha_{i,k_1,\ldots,k_\nu}^{(\nu)} q^{(k_1)}(1) \ldots q^{(k_{\nu-1})}(1) q_{k\nu}(1) + \]

\[+ \sum_{j=2}^{i-1} (-1)^{j+1} q^{(1)}_{i-j-1} \sum_{\nu=1}^{j_0} \sum_{k_1+\ldots+k_{\nu} \leq j-2\nu} (-1)^{j+2\nu} \alpha_{j,k_1,\ldots,k_{\nu-1},0}^{(\nu)} \times \]

\[\times q^{(k_1)}(1) \ldots q^{(k_{\nu-1})}(1) q^{(j-2\nu-k_1-\ldots-k_{\nu-1})}(1) \]

(33)

By virtue of the formula (22), \(\alpha_{i,0}^{(1)} = 0\) and when \(k_1 + \ldots + k_\nu = i + 1 - 2\nu\) we have;

\[\alpha_{i,k_1,\ldots,k_{\nu}}^{(\nu)} = \sum_{j=i+2-2\nu-k_1-\ldots-k_{\nu-1}}^{i-2(\nu-1)-k_1-\ldots-k_{\nu-1}} C_{i-2(\nu-1)-k_1-\ldots-k_{\nu-1}-j}^{k_{\nu-1}} \times \]

\[\times \alpha_{i,k_1,\ldots,k_{\nu-1},0}^{(\nu-1)} = C_{k_{\nu-1}}^{k_{\nu-1}} \alpha_{i,k_1,\ldots,k_{\nu-1},0}^{(\nu-1)} \]

\[= \alpha_{i,k_1,\ldots,k_{\nu-1},0}^{(\nu-1)} = \alpha_{i,k_1,\ldots,k_{\nu-1},0}^{(\nu-1)}. \]
substituting these values in the right hand side of (33) we obtain

\[ 2^{i-1}(\alpha_{20}\beta_{11})^{-1}\delta_{-1}^{(1-i)} = [q^{(i-2)}(1) + (-1)^{i+1}q^{(i-2)}(0)] + \]

\[ + \sum_{\nu=2}^{i_0} \sum_{k_1+...+k_{\nu}=i+1-2\nu} \alpha^{(\nu-1)}_{i,k_1,...,k_{\nu-2},i-2(\nu-1)-k_1-...-l_{\nu-1}} \times \]

\[ \times q^{(k_1)}(1)...q^{(k_{\nu-1})}(1)q_{k\nu}(1) - \sum_{j=2}^{i-1} q_{i-j-1}(1) \times \]

\[ \times q^{(j-2\nu-k_1-...-l_{\nu-1})}(1) \]

we prove that

\[ \sum_{\nu=2}^{i_0} \sum_{k_1+...+k_{\nu}=i+1-2\nu} \alpha^{(\nu-1)}_{i,k_1,...,k_{\nu-2},i-2(\nu-1)-k_1-...-k_{\nu-1}} \times \]

\[ \times q^{(k_1)}(1)...q^{(k_{\nu-1})}(1)q_{k\nu}(1) = \]

\[ = \sum_{j=2}^{i-1} q_{i-j-1}(1) \sum_{\nu=1}^{j-2\nu} \sum_{l_1+...+l_{\nu-1}} \alpha^{(\nu)}_{j,l_1,...,l_{\nu-1},0} \times \]

\[ \times q^{(l_1)}(1)...q^{(l_{\nu-1})}(1)q^{(j-2\nu-l_1-...-l_{\nu-1})}(1) \]

It’s easy to see that the orders of derivatives and the indices \( q(1) \) in left and right hand sides of the equality (35) takes the same values. Therefore it’s sufficient to show the equality of corresponding coefficients. Let’s fix some numbers \( \nu = m \) and \( k_1 = p_1, ..., k_\mu = p_\mu \) such that \( 2 \leq \mu \leq i_0, \ p_1 + ... + p_\mu = i + 1 - 2\mu \). In the left hand side of (35) the coefficient of production \( q^{(p_1)}(1)...q^{(p_{\mu-1})}(1)q_{p\mu}(1) \) will be the number

\[ \alpha^{(\mu-1)}_{i,p_1,...,p_{\mu-2},i-2(\mu-1)-p_1-...-p_{\mu-1}}, \]  

and in the right hand side, such production arises when \( j = i - p_m - 1, \ \nu = \mu - 1, \ l_1 = p_1, ..., l_{\mu-2} = p_{\mu-2}, \ j-2(\mu-1)-l_{\mu-1}-...-l_{\mu-2} = p_{\mu-1} \) (i.e. \( p_{\mu-1} = i - p_\mu - 1 - 2(\mu - 1) - p_1 - ... - p_{\mu-2} \)). In addition, the coefficient of this production will be the number

\[ \alpha_{i-p_\mu-1,p_1,...,p_{\mu-2},0} = \alpha^{(\mu-1)}_{2(\mu-1)+p_1+...+p_{\mu-1},p_1,...,p_{\mu-2},0} \]
The last equality is valid in connection with fact that $p_1 + \ldots + p_{\mu-1} = i + 1 - 2\mu$ and the right hand side of (35) are equal to the number (36) by virtue of Remark 1. Then from (34) we have

$$2^{i-1}(\alpha_{20}\beta_{11})^{-1}\delta_{-1}^{(1-i)} = q^{(i-2)}(1) + (-1)^{i+1}q^{(i-2)}(0)$$

(38)

from (38) it follows the validity of statement that $\delta_{-1}^{(1-i)} = 0$ iff (29) satisfied Lemma 1 is proved. From lemma 1, definition 1 and the formulas (13), (14) the validity of the following basic statement immediately follows.

Theorem. Let $q(x) \in C^m[0, 1]$, the boundary forms $U_i$ ($i = 1, 2$) have the form (3) and $|\alpha_{11}| + |\beta_{11}| > 0$, $|\alpha_{20}| + |\beta_{20}| > 0$. Then for almost regularity of the order $m \geq 0$ of the spectral problem (1), (2), it’s necessary and sufficient that: $\alpha_{11}\beta_{20} + \beta_{11}\alpha_{20} \neq 0$, when $m = 0$ (regularity) $\alpha_{11}\beta_{20} + \beta_{11}\alpha_{20} = 0$, $\alpha_{10}\beta_{20} - \beta_{10}\alpha_{20} \neq 0$, when $m = 1$; $\alpha_{11}\beta_{20} + \beta_{11}\alpha_{20} = 0$, $\alpha_{10}\beta_{20} - \beta_{10}\alpha_{20} = 0$, $\alpha_{11}\beta_{20} \neq 0$, $q^{(i)}(0) = (-1)^iq^{(i)}(1)$, $q^{(m-2)}(0) = (-1)^{m-2}q^{(m-2)}(1)$, $m \geq 2$. 

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