Critical exponents predicted by grouping of Feynman diagrams in $\varphi^4$ model

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October 26, 2018

Abstract

Different perturbation theory treatments of the Ginzburg–Landau phase transition model are discussed. This includes a criticism of the perturbative renormalization group (RG) approach and a proposal of a novel method providing critical exponents consistent with the known exact solutions in two dimensions. The usual perturbation theory is reorganized by appropriate grouping of Feynman diagrams of $\varphi^4$ model with $O(n)$ symmetry. As a result, equations for calculation of the two–point correlation function are obtained which allow to predict possible exact values of critical exponents in two and three dimensions by proving relevant scaling properties of the asymptotic solution at (and near) the criticality. The new values of critical exponents are discussed and compared to the results of numerical simulations and experiments.

Keywords: Ginzburg–Landau model, Feynman diagrams, renormalization group, critical exponents, quenched randomness.

1 Introduction

Phase transitions and critical phenomena is one of the most widely investigated topics in modern physics. Nevertheless, a limited number of exact and rigorous results is available [1]. Our purpose is to give a critical analysis of the conventional approach in calculation of critical exponents based on the perturbative renormalization group (RG) theory [2, 3, 4] and to propose a new method which provides results consistent with the known exact solutions. The usual RG theory is based on several assumptions which could seem to be plausible since the predicted values of critical exponents are well confirmed by some numerical results, particularly, by the estimations of the high–temperature series expansion [5, 6]. The basic hypothesis of RG theory is the existence of a certain fixed point for the RG transformation. The usual RG theory treatment of the Ginzburg–Landau model is based on the diagrammatic perturbation theory (Feynman diagrams). Straightforward application

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of the perturbation theory near criticality appears to be problematic in the case of the spatial dimensionality $d < 4$ because of the infrared (i.e., small wave vector $k$) divergence of the expansion terms. Wilson and Fisher \cite{2,7} have proposed a way to overcome this difficulty by expanding the Feynman diagrams of renormalized perturbation theory in double series of $\epsilon = 4 - d$ and $\ln k$ (as regards just the critical surface). From this the famous $\epsilon$–expansion of critical exponents has originated. The first results have been obtained in \cite{7,8,11}. Nowadays, an explicit $\epsilon$–expansion is available up to the fifth order \cite{11,12,13}. The $1/n$ expansion of critical exponents \cite{3,4,14} is based on a similar idea with the only essential difference that $1/n$ appears as an expansion parameter (at large $n$) instead of $\epsilon$. Alternatively, it has been proposed \cite{15} to expand the critical exponents in terms of the renormalized coupling constant at a fixed dimension $d = 3$. Later this method has been developed by several authors \cite{16,17,18,19,20,21}. Apart from the fundamental questions concerning the validity of the formal expansion in terms of $\ln k$ ($\ln k$ diverges at $k \rightarrow 0$!) and similar formal expansions which lie in the basis of the theory, a common problem for all these methods is that the resulting series for critical exponents are divergent (asymptotic), therefore, much efforts have been devoted to develop appropriate resummation techniques \cite{13,15,22,23,24}.

In spite of the claims about very accurate values of critical exponents predicted by the usual RG theory, we have revealed some serious problems concerning the validity of the basic assumptions of this theory. In particular, we have demonstrated that the standard RG treatment is contradictory and therefore cannot give correct values of critical exponents. Namely, based on a method which is mathematically correct and well justified in view of the conventional RG theory, we prove the nonexistence of the non–Gaussian fixed point predicted by this theory (Sect. 2). In Sect. 3 we prove that a correctly treated diagram expansion provides results which essentially differ from those of the perturbative (diagrammatic) RG theory.

Thus, it is worthwhile to search for some alternative analytical methods. One of such candidates could be the conformal field theory applied to three–dimensional systems \cite{25}. Note that in two dimensions this method allows to find the exact critical exponents, and even calculate the universal ratios of amplitudes. Some simple, but quite plausible models, like the fractal model of critical singularity proposed by Tseskis \cite{26}, also are interesting. We have proposed a novel analytical method of determination of critical exponents in the Ginzburg–Landau model (Sec. 4, 5), and have compared the predicted exact values of critical exponents to the results of numerical and real experiments (Sec. 6).

2 Critical analysis of the perturbative RG method

Here we consider the Ginzburg–Landau phase transition model within the usual renormalization group approach to show that this approach is contradictory. The Hamiltonian of this model in the Fourier representation reads

$$\frac{H}{T} = \sum_{k} \left( r_0 + c k^2 \right) \left| \varphi_k \right|^2 + u V^{-1} \sum_{k_1, k_2, k_3} \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \varphi_{-k_1 - k_2 - k_3},$$

(1)
where \( \varphi_k = V^{-1/2} \int \varphi(x) \exp(-ikx) \, dx \) are Fourier components of the scalar order parameter field \( \varphi(x) \), \( T \) is the temperature, and \( V \) is the volume of the system. In the RG field theory \([3, 4]\) Hamiltonian \( H \) is renormalized by integration of \( \exp(-H/T) \) over \( \varphi_k \) with \( \Lambda/s < k < \Lambda \), followed by a certain rescaling procedure providing a Hamiltonian corresponding to the initial values of \( V \) and \( \Lambda \), where \( \Lambda \) is the upper cutoff of the \( \varphi^4 \) interaction. Due to this procedure, additional terms appear in the Hamiltonian \( H \), so that in general the renormalized Hamiltonian contains a continuum of parameters. The basic hypothesis of the RG theory in \( d < 4 \) dimensions is the existence of a non–Gaussian fixed point \( \mu = \mu^* \) for the RG transformation \( R_s \) defined in the space of Hamiltonian parameters, i.e.,

\[
R_s \mu^* = \mu^* .
\]

The fixed–point values of the Hamiltonian parameters are marked by an asterisk \((r^*_0, c^*, \text{and} u^*, \text{in particular})\). Note that \( \mu^* \) is unambiguously defined by fixing the values of \( c^* \) and \( \Lambda \). According to the RG theory, the main terms in the renormalized Hamiltonian in \( d = 4 - \epsilon \) dimensions are those contained in \( H \) with \( r^*_0 \) and \( u^* \) of the order \( \epsilon \), whereas the additional terms are small corrections of order \( \epsilon^2 \).

Consider the Fourier transform \( G(k, \mu) \) of the two–point correlation (Green’s) function, corresponding to a point \( \mu \). Under the RG transformation \( R_s \) this function transforms as follows \([3]\)

\[
G(k, \mu) = s^{2-\eta} G(sk, R_s \mu) .
\]

Let \( G(k, \mu) \equiv G(k, \mu) \) \((\text{at} k \neq 0 \text{ and} V \rightarrow \infty)\) be defined within \( k \leq \Lambda \). Since Eq. (3) holds for any \( s > 1 \), we can set \( s = \Lambda/k \), which at \( \mu = \mu^* \) yields

\[
G(k, \mu^*) = a k^{-2+\eta} \quad \text{for} \quad k < \Lambda ,
\]

where \( a = \Lambda^{2-\eta} G(\Lambda, \mu^*) \) is the amplitude and \( \eta \) is the universal critical exponent. According to the universality hypothesis, the infrared behavior of the Green’s function is described by the same universal value of \( \eta \) at any \( \mu \) on the critical surface (with the only requirement that all parameters of Hamiltonian \( H \) are present), i.e.,

\[
G(k, \mu) = b(\mu) k^{-2+\eta} \quad \text{at} \quad k \rightarrow 0 ,
\]

where

\[
b(\mu) = \lim_{k \rightarrow 0} k^{2-\eta} G(k, \mu) .
\]

According to Eq. (3), which holds for any \( s = s(k) > 1 \) and for \( s = \Lambda/k \) in particular, Eq. (3) reduces to

\[
b(\mu) = \lim_{k \rightarrow 0} k^{2-\eta} s(k)^{2-\eta} G(sk, R_s \mu) = a ,
\]

if the fixed point \( \mu^* = \lim_{s \rightarrow \infty} R_s \mu \) exists. Let us define the function \( X(k, \mu) \) as \( X(k, \mu) = k^{-2}G^{-1}(k, \mu) \). According to Eqs. (4), (5), and (6), we have (for \( k < \Lambda \))

\[
X(k, \mu^*) = \frac{1}{a} k^{-\eta}
\]
and
\[ X(k, \mu) = \frac{1}{a} k^{-\eta} + \delta X(k, \mu), \tag{9} \]
where \( \mu \) belongs to the critical surface, and \( \delta X(k, \mu) \) denotes the correction–to-scaling term. From (8) and (9) we obtain the equation
\[ \delta X(k, \mu^* + \delta \mu) = X(k, \mu^* + \delta \mu) - X(k, \mu^*), \tag{10} \]
where \( \delta \mu = \mu - \mu^* \). Since Eq. (10) is true for any small deviation \( \delta \mu \) satisfying the relation
\[ \mu^* = \lim_{s \to \infty} R_s(\mu^* + \delta \mu), \tag{11} \]
we choose \( \delta \mu \) such that \( \mu^* \Rightarrow \mu^* + \delta \mu \) corresponds to the variation of the Hamiltonian parameters \( r_0^* \Rightarrow r_0^* + \delta r_0, c^* \Rightarrow c^* + \delta c \), and \( u^* \Rightarrow u^* + \epsilon \times \Delta \), where \( \Delta \) is a small constant. The values of \( \delta r_0 \) and \( \delta c \) are chosen to fit the critical surface and to meet the condition (11) at fixed \( c^* = 1 \) and \( \Lambda = 1 \). In particular, quantity \( \delta c \) is found \( \delta c = B \epsilon^2 + o(\epsilon^3) \) with some (small) coefficient \( B = B(\Delta) \), to compensate the shift in \( c \) of the order \( \epsilon^2 \) due to the renormalization (cf. [3]). The formal \( \epsilon \)–expansion of \( \delta X(k, \mu) \), defined by Eq. (10), can be obtained in the usual way from the perturbation theory. This yields
\[ \delta X(k, \mu) = \epsilon^2 \left[ C_1(\Delta) + C_2(\Delta) \ln k \right] + o(\epsilon^3) \quad \text{at} \quad k \to 0, \tag{12} \]
where \( C_1(\Delta) \) and \( C_2(\Delta) \) \((C_2 \neq 0)\) are coefficients independent on \( \epsilon \).

It is commonly accepted in the RG field theory to make an expansion like (12), obtained from the diagrammatic perturbation theory, to fit an asymptotic expansion in \( k \) powers, thus determining the critical exponents. In general, such a method is not rigorous since, obviously, there exist such functions which do not contribute to the asymptotic expansion in \( k \) powers at \( k \to 0 \), but give a contribution to the formal \( \epsilon \)–expansion at any fixed \( k \). Besides, the expansion coefficients do not vanish at \( k \to 0 \). Trivial examples of such functions are \( \epsilon^m \exp(-\epsilon k^{-\epsilon}) \) and \( \epsilon^m [1 - \tanh(\epsilon k^{-\epsilon})] \) where \( m \) is integer. Nevertheless, according to the general ideas of the RG theory (not based on Eq. (10)), in the vicinity of the fixed point the asymptotic expansion
\[ X(k, \mu) = \frac{1}{a} k^{-\eta} + b_1 k^{\epsilon + o(\epsilon^2)} + b_2 k^{2\epsilon + o(\epsilon)} + \ldots \tag{13} \]
is valid not only at \( k \to 0 \), but within \( k < \Lambda \). The latter means that terms of the kind \( \epsilon^m \exp(-\epsilon k^{-\epsilon}) \) are absent or negligible. Thus, if the fixed point does exist, then we can obtain correct \( \epsilon \)–expansion of \( \delta X(k, \mu) \) at small \( k \) by expanding the term \( b_1 k^{\epsilon + o(\epsilon^2)} \) (with \( b_1 = b_1(\epsilon, \Delta) \)) in Eq. (13) in \( \epsilon \) powers, and the result must agree with (12) at small \( \Delta \), at least. The latter, however, is impossible since Eq. (12) never agree with
\[ \delta X(k, \mu) = b_1(\epsilon, \Delta) \left[ 1 + \epsilon \ln k + o(\epsilon^2) \right] \tag{14} \]
obtained from (13) at \( k \to 0 \). Thus, in its very basics the perturbative RG method in \( 4 - \epsilon \) dimensions is contradictory. From this we can conclude that the initial assumption about existence of a certain fixed point, predicted by the RG field theory in \( 4 - \epsilon \) dimensions, is not valid.
3 A model with quenched randomness

Here we consider the Ginzburg–Landau phase transition model with $O(n)$ symmetry (i.e., the $n$–vector model) which includes a quenched randomness, i.e., a random temperature disorder. One of the basic ideas of the perturbative RG theory is that $n$ may be considered as a continuous parameter and the limit $n \to 0$ makes sense describing the self–avoiding random walk or statistics of polymers [3, 4]. We have proven rigorously that within the diagrammatic perturbation theory the quenched randomness does not change the critical exponents at $n \to 0$, which is in contrast to the prediction of the conventional RG theory formulated by means of the Feynman diagrams.

The Hamiltonian of the actually considered model is

$$H/T = \int \left[ (r_0 + \sqrt{u} f(x)) \varphi^2(x) + c (\nabla \varphi(x))^2 \right] dx$$

$$+ u V^{-1} \sum_{i,j,k_1,k_2,k_3} \varphi_i(k_1) \varphi_i(k_2) u_{k_1+k_2} \varphi_j(k_3) \varphi_j(-k_1-k_2-k_3)$$

where $\varphi(x)$ is an $n$–component vector with components $\varphi_i(x) = V^{-1/2} \sum_{k<\Lambda} \varphi_i(k) e^{ikx}$, depending on the coordinate $x$, and $f(x) = V^{-1/2} \sum_k f_k e^{ikx}$ is a random variable with the Fourier components $f_k = V^{-1/2} \int f(x) e^{-ikx} dx$. The only allowed configurations of the order parameter field $\varphi(x)$ are those corresponding to $\varphi_i(k) = 0$ at $k > \Lambda$. This is the limiting case $m \to \infty$ of the model where all configurations are allowed, but Hamiltonian (15) is completed by term $\sum_{i,k} (k/\Lambda)^{2m} |\varphi_i(k)|^2$.

The system is characterized by the two–point correlation function $G_i(k)$ defined by the equation

$$\langle \varphi_i(k) \varphi_j(-k) \rangle = \delta_{i,j} G_i(k) = \delta_{i,j} G(k) .$$

It is supposed that the averaging is performed over the $\varphi(x)$ configurations and then over the $f(x)$ configurations with a fixed (quenched) Gaussian distribution $P\{f_k\}$ for the set of Fourier components $\{f_k\}$, i.e., our random model describes a quenched randomness.

We have proven the following theorem.

**Theorem.** In the limit $n \to 0$, the perturbation expansion of the correlation function $G(k)$ in $u$ power series for the random model with the Hamiltonian (15) is identical to the perturbation expansion for the corresponding model with the Hamiltonian

$$H/T = \int \left[ r_0 \varphi^2(x) + c (\nabla \varphi(x))^2 \right] dx$$

$$+ u V^{-1} \sum_{i,j,k_1,k_2,k_3} \varphi_i(k_1) \varphi_i(k_2) u_{k_1+k_2} \varphi_j(k_3) \varphi_j(-k_1-k_2-k_3)$$

where $\bar{u}_k = u_k - \frac{1}{2} \left( |f_k|^2 \right)$.
For convenience, we call the model without the term $\sqrt{u}f(x)\varphi^2(x)$ the pure model, since this term simulates the effect of random impurities [3].

**Proof of the theorem.** According to the rules of the diagram technique, the formal expansion for $G(k)$ involves all connected diagrams with two fixed outer solid lines. In the case of the pure model, diagrams are constructed of the vertices $\Rightarrow<\ldots<\Rightarrow$, with factor $-uV^{-1}u_k$ related to any zigzag line with wave vector $k$. The solid lines are related to the correlation function in the Gaussian approximation $G_0(k) = 1/(2\tau_0 + 2ck^2)$. Summation over the components $\varphi_i(k)$ of the vector $\varphi(k)$ yields factor $n$ corresponding to each closed loop of solid lines in the diagrams. According to this, the formal perturbation expansion is defined at arbitrary $n$. In the limit $n \to 0$, all diagrams of $G(k)$ vanish except those which do not contain the closed loops. In such a way, for the pure model we obtain the expansion

$$G(k) = \frac{k}{-k} + \frac{\cdots}{\cdots-k} + \ldots. \quad (18)$$

In the case of the random model, the diagrams are constructed of the vertices $\Rightarrow<\ldots<\Rightarrow$ and $\cdots<\Rightarrow$. The factors $uV^{-1}\langle |f_k|^2 \rangle$ correspond to the coupled dotted lines and the factors $-uV^{-1}u_k$ correspond to the dashed lines. Thus, we have

$$G(k) = \frac{k}{-k} + \left[ \frac{k}{\cdots-k} + \frac{\cdots}{\cdots-k} \right] + \ldots. \quad (19)$$

In the random model, first the correlation function $G(k)$ is calculated at a fixed $\{f_k\}$ (which corresponds to connected diagrams where solid lines are coupled, but the dotted lines with factors $-\sqrt{u}V^{-1/2}f_k$ are not coupled), performing the averaging with the weight $P(\{f_k\})$ over the configurations of the random variable (i.e., the coupling of the dotted lines) afterwards. According to this procedure, the diagrams of the random model in general (not only at $n \to 0$) do not contain parts like $\Rightarrow\cdots\Rightarrow$, $\Rightarrow\cdots\cdots\Rightarrow$, etc., which would appear only if unconnected (i.e., consisting of separate parts) diagrams would be considered before the coupling of dotted lines.

It is evident from Eqs. (18) and (19) that all diagrams of the random model are obtained from those of the pure model if any of the zigzag lines is replaced either by a dashed or by a dotted line, performing summation over all such possibilities. Such a method is valid in the limit $n \to 0$, but not in general. The problem is that, except the case $n \to 0$, the diagrams of the pure model contain parts like $\Rightarrow\cdots\Rightarrow$, $\Rightarrow\cdots\cdots\Rightarrow$, $\Rightarrow\cdots\cdots\cdots\Rightarrow$, etc. If all the depicted here zigzag lines are replaced by the dotted lines, then we obtain diagrams which are not allowed in the random model, as explained before. At $n \to 0$, the only problem is to determine the combinatorial factors for the diagrams obtained by the above replacements. For a diagram constructed of $M_1$ vertices $\Rightarrow<\ldots<\Rightarrow$ and $M_2$ vertices $\cdots<$ the combinatorial factor is the number of possible different couplings of lines, corresponding to the given topological picture, divided by $M_1!M_2!$.

Our further consideration is valid also for the diagrams of free energy (at $n \to 0$ represented by the main terms containing single loop of solid lines) and of $2m$-point correlation function. We define that all diagrams which can be obtained from the
i-th diagram (i.e., the diagram of the i-th topology) of the pure model, belong to the i-th group. Obviously, all diagrams of the i-th group represent a contribution of order $u^l$, where $l$ is the total number of vertices $\cdots$ in the i-th diagram. The sum of the diagrams of the i-th group can be found by the following algorithm.

1. Depict the i-th diagram of pure model in an a priori defined way.

2. Choose any one replacement of the vertices $\cdots$ by $\cdots$ and perform the summation over all such possibilities. For any specific choice we consider only one of the equivalent $M_1!M_2!$ distributions of the numbered $M_1$ vertices $\cdots$ and $M_2$ vertices $\cdots$ over the fixed numbered positions instead of the summation over all these distributions with the weight $1/(M_1!M_2!)$. Thus, at this step the combinatorial factor for any specific diagram is determined as the number of possible distributions of lines (numbered before coupling) for one fixed location of vertices consistent with the picture defined in step 1.

3. The result of summation in step 2 is divided by the number of independent symmetry transformations (including the identical transformation) for the considered i-th diagram constructed of vertices $\cdots$, since the same (original and transformed) diagrams were counted as different.

Note that the location of any vertex $\cdots$ is defined by fixing the position of dashed line, the orientation of which is not fixed. According to this, the summation over all possible distributions of lines (numbered before coupling) for one fixed location of vertices yields factor $8^{M_1}4^{M_2/2}$. The i-th diagram of the pure model also can be calculated by such an algorithm. In this case we have $8^l$ line distributions, where $l = M_1 + M_2/2$ is the total number of vertices $\cdots$ in the i-th diagram. Obviously, the summation of diagrams of the i-th group can be performed with factors $8^l$ instead of $8^{M_1}4^{M_2/2}$, but in this case twice smaller factors must be related to the coupled dotted lines. The summation over all possibilities where zigzag lines are replaced by dashed lines with factors $-uV^{-1}u_k$ and by dotted lines with factors $\frac{1}{2}uV^{-1}\langle |f_k|^2 \rangle$, obviously, yields a factor $uV^{-1}\left(-u_k + \frac{1}{2}\langle |f_k|^2 \rangle\right) \equiv -uV^{-1}\tilde{u}_k$ corresponding to each zigzag line with wave vector $k$. Thus, the sum over the diagrams of the i-th group is identical to the i-th diagram of the pure model defined by Eq. (17). By this the theorem has proved not only for the two-point correlation function, but also for $2m$-point correlation function and free energy.

If, in general, the factor $\sqrt{u}$ in Eq. (15) is replaced by $\sqrt{u'}$, where $u'$ is an independent expansion parameter, then our analysis leads to the above relation between diagrams for $u \tilde{u}_k = u u_k - \frac{u'}{2}\langle |f_k|^2 \rangle$. According to this, at $n \to 0$ the pure and random models cannot be distinguished within the diagrammatic perturbation theory. If, in principle, critical exponents can be determined from the diagram expansions at $n \to 0$, as it is suggested in the usual RG theory, then the same critical exponents should be provided for both models at $n \to 0$. In such a way, we conclude that the RG method is not correct because the above condition is violated. As compared to our simple treatment of the random model, the RG treatment includes additional
Feynman diagrams because the Hamiltonian becomes more complicated after the renormalization. However, this does not enable to find the difference between both models: the original information, when one starts the perturbative renormalization of Hamiltonian (13), is contained in the Feynman diagrams we considered, but the renormalization by itself does not create new information about the model. Really, by renormalization we merely “forget” some information about the short-wave fluctuations to make that for the long-wave fluctuations easier accessible. Thus, our conclusion remains true.

4 Equations from reorganized perturbation theory

As we have already discussed in Sect. 3, it is not a rigorous method to make a formal expansion like (12) and to try calculate the critical exponents therefrom. We propose another treatment of the diagrammatic perturbation theory for the Ginzburg–Landau model defined by Eq. (17), where $\bar{u}_k = u_k$. The basic idea is to obtain suitable equations by appropriate grouping of the diagrams. Suitable are such equations which allow to find the asymptotic expansions at the critical point directly in $k$ power series, but not in terms of the formal parameter $\ln k$ (as in Eq. 12) which diverges at $k \to 0$.

4.1 Some fundamental problems of the perturbation theory

One of the problem which can arise in any perturbation theory is that the perturbation expansion alone does not define unambiguously the original function. For example, all expansion coefficients in the formal expansion of $\exp(-1/u)$ with $u$ considered as an expansion parameter at $u = +0$ are zero, whereas the original function is not zero. However, if quantities $A$ and $B$ have the same perturbation expansion in a power series of $u$, then $A(u) - B(u) = C(u)$ holds, where $C(u)$ is some function of $u$ with all the expansion coefficients equal to zero. Accordingly, the diagram technique principally allows, in the worst case (where $C(u) \neq 0$), to find $G(k)$ with an accuracy to some unknown function which tends to zero at $u \to 0$ faster than $u^l$ at any $l > 0$. We have shown in Sect. 5.1 that true (exact) critical exponents can be obtained neglecting this function.

Another problem is that the considered formal diagram expansions diverge. Nevertheless, the following way is possible, which lies in the basis of our diagrammatic treatment. First, we build up a perturbation expansion of quantity $A$ which is necessary to be found. Then we seek such a quantity $B$ (represented by converging sums and integrals), the perturbation expansion of which is identical to that we have built up. According to the above consideration, we have $A(u) = B(u) + C(u)$, where $C(u)$ is an insignificant correction (or zero). In this case it is not necessary that the perturbation sum converge if calculated in a straightforward way. Various manipulations with diagram blocks appearing in Sec. 4 are defined as constructions of corresponding formal (diverging) expressions which, however, provide correct expansion in terms of $u$. Various diagram representations of a given quantity yielding the same expansion in a power series of $u$ are defined as equivalent.
4.2 The diagram notation

Here we define some diagram notations appearing in Sec. 3.

Coupled diagram is defined as any diagram which does not contain uncoupled lines, i.e., any line starts from some kink and ends in the same or another kink, where "kink" means a merging point of solid and dashed lines of a vertex ▶ ◀ ▶ ◀.

The self-energy block $\overbrace{\bullet \cdots \bullet}^{k}$ denotes the perturbation sum involving all connected diagrams of this kind, i.e., the sum of all specific self-energy blocks or diagrams which cannot be reduced to a linear chain like $\overbrace{\bullet \cdots \bullet}^{k}$ consisting of two or more blocks. The simplest specific self-energy blocks are

\[ k - V^{-1} \sum_q G_0(q) \] and \[ k - V^{-1} \sum_q u_k G_0(q) G_i(k - q). \]

Factors corresponding to the lines marked by crosses are omitted. Each case of topologically nonequivalent coupling of lines corresponds to one diagram.

Skeleton diagram is defined as a connected diagram, containing no parts like $\overbrace{\bullet \cdots \bullet}$, with factors $G(k)$ corresponding to the solid lines. For example, the simplest coupled skeleton diagrams are

\[ \overbrace{\bullet \cdots \bullet} = -u_0 V^{-1} \left( \sum_i G_i(q) \right)^2 \] and \[ \overbrace{\bullet \cdots \bullet} = -2V^{-1} \sum_i \sum_k u_k G_i(q) G_i(k - q). \]

Note that index $i$ can be removed, replacing $\sum_i$ with factor $n$. The simplest skeleton diagrams like $\overbrace{\bullet \cdots \bullet}$ with two outer lines (not coupled diagrams) are the same as the above given self-energy diagrams, but with $G(k)$ instead of $G_0(k)$.

Single block is defined as a connected diagram with two outer (broken) dashed lines which cannot be reduced to a linear chain like $\overbrace{\bullet \cdots \bullet}$ consisting of two or more blocks.

The single skeleton block $\overbrace{\bullet \cdots \bullet}^{k}$ denotes the perturbation sum involving all single blocks that belong to skeleton diagrams, i.e., the sum of all specific single skeleton blocks. The outer (broken) dashed lines or kinks in this case are marked by 1 and 2 (in general kinks are not depicted). Factor $-V^{-1} u_k$ corresponds to the pair of these lines. Each case of topologically nonequivalent coupling of lines with respect to fixed kinks 1 and 2, considered as nonequivalent, corresponds to one diagram of $\overbrace{\bullet \cdots \bullet}$. For instance,

\[ \overbrace{\bullet \cdots \bullet} = 16u_k V^{-2} \sum_i \sum_q G_i(q) G_i(k - q) u_{q-p} G_i(p) G_i(k - p). \]

Combinatorial factor corresponding to any specific diagram is not given explicitly, but is implied in the diagram itself. It can be calculated following the scheme in Sec. 3 (but without the replacements, since now we have only one kind of vertices).

4.3 Expansion of $G(k)$ in terms of skeleton diagrams

It is suitably to have a diagram expansion for $G(k)$ where the true correlation function $G(k)$ is related to solid lines instead of $G_0(k)$. To obtain this, first let us
Figure 1: An example of replacement, where the blocks of primary chain inside a diagram are replaced by the self–energy blocks, yielding a diagram of set $A$. The connecting solid lines of the primary chain are marked by 1. There is another (not primary) chain, marked by 2, which is built into a block of the primary chain.

Consider the quantity $\tilde{\Sigma}(k)$ defined by equation

$$G(k) = \frac{G_0(k)}{1 - 2\tilde{\Sigma}(k)G_0(k)}.$$  

(20)

It is well known [3] that terms of the perturbation expansion of $\tilde{\Sigma}(k)$ are diagrams of the self–energy block $\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$. Note only that the self–energy defined in Ref. [3] corresponds to $-2\tilde{\Sigma}(k)$.

The desired expansion with $G(k)$ instead of $G_0(k)$ is obtained by grouping of diagrams involved in the self–energy block. First we consider specific self–energy blocks, called the primal diagrams, from which no block of the kind $\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ can be extracted. Then we consider the set $A$ of diagrams obtained by extending the solid lines inside the primal diagrams by adding all possible numbers $m \in [0; \infty]$ of self–energy blocks to each of these lines. In particular, the diagram of set $A$, depicted on the right hand side of Fig. [1] is obtained from the primal diagram $\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ by adding two ($m = 2$) self–energy blocks to the inner solid line. Obviously, all diagrams of set $A$ are self–energy diagrams, i. e., they cannot be split in two blocks like $\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$. Besides, any specific self–energy block is contained in a perturbation sum represented by one of diagrams of set $A$. This is proved considering primary chains contained inside the specific self–energy blocks. A primary chain is defined as a linear chain of specific self–energy blocks which does not belong to (i. e., is not built into) a block of some other linear chain inside the diagram. If we replace the blocks of primary chains by $\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$, we obtain a diagram of set $A$, i. e., a diagram consisting of separate linear chains of the new blocks, the initial diagram being involved as a particular case (since $\bullet \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$ involves all specific blocks of such kind). A specific example of such a replacement is illustrated in Fig. [1]. Thus, set $A$ contains all specific self–energy blocks.

Because blocks are distributed independently over separate linear chains, summation over all possible lengths $N$ of linear chains yields a factor $\sum_{N} G_0(k) \left(2G_0(k) \times \frac{k}{\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc} \right)^N$ identical to the perturbation sum of $G(k)$ for each of the original solid lines (in a primal diagram) with wave vector $k$. Thus an
equivalent diagram representation of $\tilde{\Sigma}(k)$ (Sect. 4.1) is obtained if the perturbation sum over all diagrams of set $A$ is replaced by the perturbation sum over the primal diagrams in which $G_0(k)$ is replaced by $G(k)$. According to the definition in Sect. 4.2, diagrams of the new expansion are skeleton diagrams like $\begin{array}{c}
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of the kind ———— (constructed of odd number of solid lines by coupling) are principally impossible (in the opposite case parts like ———— could be extracted).

4.5 Summation of the simplest skeleton diagrams

Consider now a contribution to $D(G)$ (denoted by $D^{(0)}(G)$) of diagrams containing the simplest specific single skeleton blocks

$$\Sigma^{(0)}(k) = \frac{k}{k} \sum_{i,q} G_i(q) G_i(k - q)$$

(23)

exclusively. This contribution is not merely a formal sum since at small $u_k$ (at $r_0 > 0$) it can be obtained by straightforward summation, i. e.,

$$D^{(0)}(G) = \sum_{i,q} \ln \left[ 1 - 2\Sigma^{(0)}(q) \right] .$$

(24)

Equation (24) is true in any case if $D^{(0)}(G)$ is defined as a quantity having this diagram expansion. The sum (24) is calculated by the following method. To calculate the combinatorial factor for a cycle comprised of $N$ blocks, first we count all possible distributions of $N$ numbered vertices over $N$ sequentially numbered fixed sites along the cycle with all possible distributions of $4N$ uncoupled lines, which are then coupled in $2N$ lines. Then, the result is corrected taking into account that couplings obtained in such a way contain equivalent ones which differ merely by a diagram having been rotated as a whole or (and) transformed by a mirror–symmetry transformation, as well as in $2N$ ways transformed by a mirror–symmetry transformation of each ———— block separately. $2N \cdot 2^N$ such independent transformations exist.

4.6 Grouping of the skeleton diagrams

In this section we reorganize the perturbation expansion which is necessary for calculation of $D^*(G)$.

First, we have got a formal diagram equation performing manipulations similar to those in Sect. 4.5, but including all diagrams of the single skeleton block. A problem arises with more complicated diagrams than those considered in Sect. 4.5 because of additional symmetry related to different representations of a given diagram by specific cyclically coupled single skeleton blocks. In Fig. 8, an example of such a transformation is shown, which leads to a different representation of the diagram, retaining the couplings of lines unchanged. Two representations of a given coupled diagram with numbered vertices are considered as different or nonequivalent if they cannot be transformed into each other without using a transformation of this kind.

If we now consider the perturbation sum of cyclically coupled blocks ———— with the combinatorial factors calculated as in Sect. 4.5 without account for the
above considered additional symmetry, then a specific diagram of the $i$–th topology (or the $i$–th skeleton diagram) in the perturbation sum will have weight $m_i$, where $m_i$ is the number of possible different representations of this diagram by specific cyclically coupled single skeleton blocks. This leads to the following formal identity of perturbation sums

$$-\frac{1}{2} \sum_q \ln \left[ 1 - 2 \times \frac{q}{\bullet} \right] = \sum_i^i m_i C_i,$$

(25)

where $C_i$ denotes the $i$–th coupled skeleton diagram, the apostrophe ")" showing that diagram $\bullet \; \bullet \; \bullet$ is omitted.

Number $m_i$ can be identified with the number of linear chains consisting of dashed lines and blocks of the kind $\; \bullet \; \bullet \; \bullet$ (or a single dashed line) contained within diagram $C_i$, counting all such chains contained inside specific single skeleton blocks that comprise the cycle in a given representation and the chain representing the cycle itself. It holds because any linear chain contained in a single skeleton block can represent a cycle in one of the new representations of the diagram obtained by such a transformation as illustrated in Fig. 2. Note that linear chains can be built into each other in all possible ways (like in a fractal), and all these chains (of all levels) are counted as different.

We have avoided explicit counting of linear chains as follows. We have proven the following: If there exists a quantity $\Sigma(q, \zeta)$ which has the perturbation expansion $\Sigma(q, \zeta) = \sum_i \Sigma_i(q) \zeta^{n_i}$, where $\Sigma_i(q)$ is a specific single skeleton block of the $i$–th topology and $n_i$ is the number of linear chains contained in this block, and if there is a solution of the equation

$$D^*(G, \zeta) = -\frac{1}{2} \sum_q \ln [1 - 2 \Sigma(q, \zeta)] - \zeta \frac{\partial}{\partial \zeta} D^*(G, \zeta)$$

(26)

with the boundary condition

$$D^*(G, 0) = -\frac{1}{2} \sum_q \ln [1 - 2 \Sigma^{(0)}(q)],$$

(27)

then quantity $D^*(G, \zeta)$, calculated from this equation, has the perturbation expansion $D^*(G, \zeta) = \sum_i C_i \zeta^{m_i-1}$. Since Eqs. (24) and (27) define the solution unambiguously, it is necessary only to prove the identity of perturbation expansions in
the case if \( D^*(D, \zeta) \) has the expansion \( \sum_i C_i \zeta^{m_i-1} \). The identity of diagram expansions in (27) corresponding to \( m_i = 1 \) follows directly from the consideration in Sect. 4.3. The perturbation expansion of the logarithmic term in (26) contains diagrams constructed of specific single skeleton blocks \( \Sigma_i(q) \) supplied by factors \( \zeta^{n_i} \). At \( \zeta = 1 \) this expansion is identical to that on both sides of Eq. (25). Now, any diagram is supplied with an additional factor which is the product of factors of kind \( \zeta^{n_i} \) coming from all specific single skeleton blocks of this diagram, yielding the resulting additional factor \( \zeta^{m_i-1} \) (consistent with the definition of \( m_i \)) or, according to (25), the weight \( m_i \zeta^{m_i-1} \) of the \( i \)-th skeleton diagram. On the other hand, the same diagram appears in the perturbation expansion \( \sum_i C_i \zeta^{m_i-1} \) of \( D^*(G, \zeta) \) with the weight \( \zeta^{m_i-1} \) and in the perturbation expansion of \( \zeta \frac{\partial}{\partial \zeta} D^*(G, \zeta) \) with the weight \( (m_i - 1) \zeta^{m_i-1} \). This, obviously, leads to the identity of perturbation expansions in the left–hand side and in the right–hand side of (26), which proves the statement.

We have defined \( D^*(G) \) by

\[
D^*(G) = D^*(G, 1),
\]

(28)

which, obviously, provides the perturbation expansion of \( D^*(G) \) defined before. Based on Eqs. (26) to (28), we can calculate \( D^*(G) \) without explicit counting of the linear chains.

At the final step of our diagrammatic transformations, we make the summation over the linear chains contained in the single skeleton block following the method in Sect. 4.3, with the only difference that blocks \( \quad \) instead of \( \quad \), are considered.

As a result, the new perturbation expansion of \( \Sigma(q, \zeta) \) is

\[
\Sigma(q, \zeta) = q + \zeta q + \zeta^2 \left\{ q + q + q + q + q \right\} + \ldots,
\]

(29)

where the perturbation expansion of the waved line represents the sum over the linear chains of all possible lengths \( N \), i. e., the sum of the geometrical progression,

\[
q \cdots q \quad \text{def} = -u_q V^{-1}/[1 - 2\Sigma(q, \zeta)] = -u_q V^{-1} \sum_{N \geq 0} [2\Sigma(q, \zeta)]^N.
\]

(30)

According to this procedure, the new diagrams do not contain parts like \( \quad \) and/or \( \quad \), i. e., they are single blocks and skeleton diagrams with respect to both solid and waved lines. By this the grouping of diagrams is completed. Such a grouping of diagrams is unique in the sense that it allows to analyze all diagrams simultaneously, considering asymptotic equations related to critical phenomena.

It should be noted that the actual grouping of skeleton diagrams cannot be extended trivially to include even the simplest \( \varphi^6 \) term. If we include, e. g., the \( \varphi^6 \)
vertex ————, subblocks of the kind ———— are possible (———, in particular), therefore, the cyclical coupling of single skeleton blocks ———— (see Sect. 4.4) yield diagrams which are not skeleton diagrams anymore. For example, the diagram ———— represents one of the diagrams involved in ————. The cyclical coupling of two such blocks is not a skeleton diagram because it can be split in two blocks ———— with two outer solid lines.

4.7 Equations for calculation of the correlation function

In this section equations are considered from which \( \Sigma(q, \zeta) \) and \( G(k) \) can, in principle, be calculated. Although this quantity is not defined unambiguously by the perturbation expansion, we can use any of possible functions having the expansion (29). Only function \( \vartheta(k) \) in Eq. (21) can be finally affected by the specific choice. We have defined \( \Sigma(q, \zeta) \) by equations

\[
\Sigma(q, \zeta) = \Sigma^{(0)}(q) + \int_0^{u-p} e^{-t_1} dt_1 \int_0^{u-p} e^{-t_2} B(q, \zeta, t_1 t_2) dt_2 ,
\]

(31)

\[
B(q, \zeta, t) = \sum_{m=1}^{\infty} \frac{\zeta m^m}{(m!)^2} \Sigma^{(m)}(q, \zeta) ,
\]

(32)

where \( \Sigma^{(m)}(q, \zeta) \) represents the sum of diagrams of the \( m \)-th order \((m \geq 0)\) in (29), and \( p \) is a constant having the value \( 0 < p < 1/2 \). Term \( \Sigma^{(0)}(q) \) in (31) is separated to ensure that the boundary condition \( \Sigma(q, 0) = \Sigma^{(0)}(q) \) is satisfied when the first diagram is retained in (29). Here and in our further considerations expansion in the vicinity of the point \( u = +0 \) is used. Equations (31) and (32) yield function \( \Sigma(q, \zeta) \) which has the perturbation expansion equivalent (i.e., the same, if represented in terms of \( u \)) to (29), and to \( \Sigma(q, \zeta) = \sum_i \Sigma_i(q) \zeta^{n_i} \), since

\[
\int_0^{u-p} e^{-t} m dt = m! + o(u^{-p} \exp(-u^{-p})) \quad \text{and} \quad B(q, \zeta, t) \sim \zeta^u t \quad \text{(for } 0 < t \leq u^{-2p}) \]

hold at \( u \to 0 \) and \( 0 < p < 1/2 \). The latter ensures convergence of integrals (31) at \( u \to 0 \). This relation holds because the actual expansion parameter in (32) at \( u \to 0 \) is \( \zeta^u t \) and this sum converges absolutely, as discussed below. At large \( m \) the number of terms of the \( m \)-th order increases approximately as \( m^\gamma 6^m m! \) with some constant \( \gamma \) and these terms behave as \( -(-b)^m \) where \( b > 0 \). According to this consideration terms in (32) can be approximated as \( \text{const} \cdot m^\gamma (-b \zeta t)^m / m! \) with new \((6 \text{ times larger}) \) value of \( b \). Consequently, the perturbation sum of \( \Sigma(q, \zeta) \) diverges at any values of parameters, whereas (32) converges absolutely at any given values of \( u, \zeta \) and \( t \). Therefore, we can find \( B(q, \zeta, t) \) from (32) with any desired accuracy (the larger are \( \zeta \) and \( t \), the larger is the number of terms to be counted). Then we can find \( \Sigma(q, \zeta) \) from (31).

The Dyson equation for \( G(k) \) following from Eqs. (20) and (21) is

\[
\frac{1}{2G(k)} = r_0 + c k^2 - \frac{\partial D(G)}{\partial G_i(k)} + \vartheta(k) .
\]

(33)
The same equation with term \( \vartheta(k) \) neglected has been obtained in Ref. [27].

Our further analysis is limited to the case \( u(x) = u\delta(x) \) or \( u_k = u \), where \( u > 0 \). In this case, from (33), (22), and (28) we obtain

\[
\frac{1}{2G(k)} = r_0 + ck^2 + 2u\tilde{G} + R(k) + \vartheta(k),
\]

where

\[
\tilde{G} = \langle \varphi^2(x) \rangle = V^{-1}n \sum_k G(k),
\]

\[
R(k) = -\frac{\partial D^*(G, 1)}{\partial G_i(k)}. \tag{36}
\]

All terms in Eq. (34) are well defined. \( D^*(G, \zeta) \) and, consequently, \( R(k) \) is defined by Eqs. (26) to (32). According to the definition, \( \vartheta(k) \) is a quantity which have to be included to obtain an exact equation which is satisfied by the exact correlation function \( G(k) \) given by the statistical integrals. We know that \( \vartheta(k) \) does not contribute to the formal expansion of \( G(k) \) in \( u \) power series. It means that \( \lim_{u \to 0} \vartheta(k, u)u^{-\tau} = 0 \) holds for any positive \( \tau \). Our equations have an obvious physical solution \( \Sigma(k, \zeta) \simeq \Sigma^{(0)}(k), D(G) \simeq D^{(0)}(G), \) and \( G(k) \simeq G_0(k) \) at \( r_0 > 0 \) and \( u \to 0 \), which agree with the true (exact) \( G(k) \). Analytic continuation to arbitrary \( r_0 \) value is possible if one starts with a finite volume \( V \) and consider the thermodynamic limit \( V \to \infty \) afterwards.

5 Asymptotic solution and critical exponents

5.1 Determining the critical exponents at \( u \to 0 \)

We will show in this section how the true asymptotic solution at \( k \to 0 \) (at the critical point \( T = T_c \)) can be found from the simplified equations where \( u \) tends to zero. The right-hand side of (34) vanishes at \( k = 0 \) and \( V \to \infty \) at the critical point and, thus, we have

\[
\frac{1}{2G(k)} = ck^2 + R(k) - R(0) + \vartheta(k) - \vartheta(0) \quad \text{at} \quad T = T_c. \tag{37}
\]

The asymptotic of the correlation function at \( k \to 0 \) in this case is \( G(k) \simeq a k^{-\lambda} \) where \( a \) is a constant and \( \lambda = 2 - \eta, \) \( \eta \) being the critical exponent. The case of the spatial dimensionality \( d < 4 \) is considered. The correlation function is well described by \( G(k) = a k^{-\lambda} \) within some critical region \( k < k_{\text{crit}}(u) \), where \( k_{\text{crit}}(u) \) tends to zero at \( u \to 0 \), since at \( u = 0 \) the Gaussian approximation with \( \lambda = 2 \) is the solution of (37) for any \( k \). Let us define the effective value of \( \eta \) at some \( k = \tilde{k} \) by

\[
\eta(\tilde{k}, u) = -\partial \left( \ln \left[ k^{-2}G^{-1}(k) \right] \right) /\partial (\ln k) \bigg|_{k=\tilde{k}}. \tag{38}
\]

Then, \( k_{\text{crit}}(u) \) is defined by

\[
|\eta(k_{\text{crit}}(u), u) - \eta| = \varepsilon, \tag{39}
\]
where $\varepsilon$ is a sufficiently small constant. According to the universality hypothesis, we have $\lim_{k \to 0} \eta(k, u) = \eta$ for any positive $u$. We have also $\lim_{k \to 0} \eta(u^l k_{\text{crit}}(u), u) = \eta$ at $r > 0$, since the critical exponent is determined at $k \ll k_{\text{crit}}(u)$ if $u \to 0$, and, therefore, corresponds to the asymptotic solution at $k \to 0$. Based on a non-perturbative analysis provided in Appendix B, we have shown that

$$\lim_{u \to 0} \left( u^s / k_{\text{crit}}(u) \right) = 0 ,$$

holds at large enough $s$. Quantity $k_{\text{crit}}(u)$ can be related to the region where the correlation function is well approximated by the asymptotic expansion $G(k) = \sum_l b_l k^{-\lambda_l}$ of any given number of terms. In this case $k_{\text{crit}}(u)$ is defined by the condition that the approximation error at $k = k_{\text{crit}}(u)$ corresponds to the variation of the smallest $\lambda_l$ by some small $\varepsilon$. Following the method in Appendix B, the statement (40) holds in this case too, and coefficients $c_l(u)$ in the asymptotic expansion $1/G(k) = \sum_l c_l(u) k^{2\lambda_l - \lambda_l}$ (where $l \geq 0$ and $\lambda_0 \equiv \lambda$) meet the condition $\lim_{u \to 0} (u^s / c_l(u)) = 0$ at large enough $s_l$.

According to the discussed property of $c_l$, the following is true. If we assume that, at $k = u^l k_{\text{crit}}(u)$ and $u \to 0$, $\omega(k, u) = \vartheta(k) - \vartheta(0)$ is either compatible with some term $c_l k^{2\lambda_l - \lambda_l}$, or is much larger than all these terms, then $\lim_{u \to 0} \left[ u^l / \omega(u^l k_{\text{crit}}(u), u) \right] = 0$ holds for large enough values of $\tau$. On the other hand, the basic property of $\omega(k, u)$ is that $\lim_{u \to 0} \left( u^l / \omega(k, u) \right) = \infty$ holds for any $\tau$ at any arbitrarily small, but fixed $k$ (see the end of Sect. 4.7). From this and Eq. (40) we conclude the following: if the above assumption is true, then, at $u \to 0$, $| \omega(k, u) |$ decreases faster than any negative power of $k$ if $k$ is increased within some region of infinitely small, but larger than $u^l k_{\text{crit}}(u)$ values of $k$. On the other hand, the only essential singularity of the correlation function is at $k = 0$, which means that the above assumption leads to unphysical conclusions regarding behavior of $G(k)$, calculated from Eq. (37), unless the sharp decrease of $| \omega(k, u) |$ is compensated by the corresponding variation in $R(k) - R(0)$. However, our further analysis strongly supports the idea that $R(k) - R(0)$ is a well defined smooth function of $k$ which behaves like some power of $k$ at $k \to 0$. Thus, the compensation is impossible and the discussed here assumption is false, i.e., the opposite is true: at $k = u^l k_{\text{crit}}(u)$ and $u \to 0$ the term $\omega(k, u)$ is negligible compared to any of corrections to scaling in the asymptotic expansion of $1/G(k)$.

In principle, critical exponents can be found by calculating $G(k)$ within $k \in [u^l k_{\text{crit}}(u); \Lambda]$ directly from Eq. (37) where the term $\vartheta(k) - \vartheta(0)$ is neglected, followed by extrapolation of the results to smaller values of $k$ in the form $G(k) = \sum_l b_l k^{-\lambda_l}$. At $u \to 0$ this yields true (exact) correlation function, since Eq. (37) is exact (according to the definition), term $\vartheta(k) - \vartheta(0)$ is negligible, and the lower marginal value of the considered interval is infinitesimal compared to $k_{\text{crit}}(u)$. According to the above analysis, this method yields exact critical exponents $\lambda_l$. 

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5.2 Scaling properties of the main terms at $T = T_c$

It is impossible to calculate precisely all terms in Eq. (32). However, this is an unique feature of our reorganized diagram expansion that all terms have common scaling properties where the order of diagram does not appear as a relevant parameter. Our claims are based on the proof of these scaling properties at $d = 2, 3$ for separate terms and also for the whole sum (32) not cutting the series. To simplify the notation, $n$ is considered as a fixed parameter not included in the list of arguments.

We introduce a lower limit of the wave vector $k_{\text{min}}$ to simulate a finite–size effect (as if the linear size of the system would be $L = 2\pi/k_{\text{min}}$) in calculation of $R(k)$, which ensures the convergence of any $k$–space integral. Finally, we consider the limit $k_{\text{min}} \to 0$ in the equation for $R(k)$.

Since $\eta > 0$ holds for $d < 4$, the term $ck^2$ in (37) gives a small correction which is by factor $k\eta$ smaller than the main term $R(k) - R(0)$. Term ”1” in (26), (27), and (30) causes a small correction to $R(k)$ as well. This can be checked easily by a direct calculation as in Ref. [28], if only the first term in (29) is retained. We prove here that a selfconsistent solution of our equations can be found where this condition is satisfied with account for all terms.

First, let us consider the dominant behavior of $\Sigma(q, \zeta)$ calculated at $G(k) = a k^{-\lambda}$. A property of the asymptotic solution is such that

$$\Sigma(q, \zeta) = a^2 \times q^{d-2\lambda} \Psi(\Lambda/q, k_{\text{min}}/q, \zeta, \lambda, d, u)$$

holds, where $\Psi$ is a function of the given arguments only. Such a solution is possible since each term in (32) has the scaled form like (41) if (31) is substituted by (41) neglecting ”1”. It becomes obvious after the following manipulations: all the sums are replaced by integrals (the standard procedure), $V^{-1} \sum_k \Rightarrow (2\pi)^{-d} \int d^d k$, and all the wave vectors in (29) or (32) are normalized to the current value of $q$. This produces factor $a q^{-\lambda}$ corresponding to each of the solid lines, factor $a^{-2} q^{-2\lambda} - d$ corresponding to each of the waved lines, and factor $q^d$ corresponding to each integration for any diagram of (29). As a consequence, the resulting factor is $a^2 q^{d-2\lambda}$ irrespective of the order of diagram $m$ (there are $2m + 2$ solid lines, $m$ waved lines, and $m + 1$ integrations). Thus, $a^2 q^{d-2\lambda}$ appears as a common prefactor for the whole sum (32), and the problem reduces to summation of scaling functions depending merely on $\Lambda/q, k_{\text{min}}/q, \zeta, \lambda, d, \text{and } u$, which leads to (11). In this case arguments $\Lambda/q$ and $k_{\text{min}}/q$ represent the upper and the lower limits of integration for normalized wave vectors related to the solid lines.

Formally we could allow other kind of solutions, but the true result must coincide with the asymptotic expansion $\Sigma(q, \zeta) = \sum_m \zeta^m \Sigma^{(m)}(q)$ at $\zeta \to 0$ (consistent with the boundary condition $\Sigma(q, 0) = \Sigma^{(0)}(q)$), which shows that (11) is the only possibility. If the correction term ”1” in Eq. (30) is omitted, then any of the expansion coefficients has the scaled form with factor $a^2 q^{d-2\lambda}$ multiplied by some function of $\Lambda/q, k_{\text{min}}/q, \lambda, d, \text{and } u$. This is proved by induction over $m$: it holds at $m = 0$; if it holds for terms up to the $m$–th order, then it holds for terms of the $(m + 1)$–th order, calculated from diagrams with no more than $m + 1$ waved lines (expanded in terms of $\zeta$). Thus, the asymptotic solution at $\zeta \to 0$ is unambiguous and has the...
scaled form \((41)\), which is a property of the solution as well at finite \(\zeta\).

It is purposeful to seek a solution for \(\partial \Sigma(q, \zeta)/\partial G_i(k)\) in form
\(V^{-1}a k^{-\lambda} Y(q/k, k/k, \Delta/k, k_{min}/k, \zeta, \lambda, d, u)\) using Eqs. \((31)\) and \((32)\), where \(\Sigma(q, \zeta)\) is considered as a quantity which is already known. The latter can be represented in the form \(a^2 k^{d-2\lambda} \Psi(q/k, \Delta/k, k_{min}/k, \zeta, \lambda, d, u)\) obtained from \((41)\) by changing variables \(q \rightarrow k \cdot (q/k)\). This yields a selfconsistent equation for the unknown function \(Y\). The arguments of this function only are contained therein, since all terms \(\partial \Sigma^{(m)}(q, \zeta)/\partial G_i(k)\) in the sum for \(\partial B(q, \zeta, t)/\partial G_i(k)\), obtained from \((22)\), have the form \(V^{-1}a k^{-\lambda}\) multiplied by some function of these arguments. In a similar way, we can find from the closed equations \((23)\) and \((24)\) a selfconsistent solution \(\partial D^*(G, \zeta)/\partial G_i(k) = a^{-1} k^\lambda \phi(\Lambda/k, k_{min}/k, \zeta, \lambda, d, u)\). The analysis of the asymptotic solution at \(\zeta \rightarrow 0\) shows that discussed above are the true solutions meeting \((27)\).

Consider now the limit \(k_{min} \rightarrow 0\). The nonzero lower limit of integration has been introduced to avoid the divergence of \(k\)-space integrals in \((29)\) considering the formal expansion in terms of \(\zeta\). If the selfconsistent solution of Eqs. \((31)\) and \((32)\) is considered, then convergence of integrals at \(k_{min} = 0\) is ensured since these equations provide the solution of physical problem where the existence of thermodynamic limit \((k_{min} \rightarrow 0)\) is doubtless, and this conclusion agree with the formal analysis of our equations at \(k_{min} = 0\): \(\Sigma(q, \zeta)\) diverges at \(q \rightarrow 0\), but this term appears in the denominator of \((10)\), which ensures the convergence of \(k\)-space integrals in \((29)\). A selfconsistent solution with exponentially diverging \(\Sigma(q, \zeta)\) at \(q \rightarrow 0\) is not possible, which ensures the convergence in \((26)\). Thus, we can set \(k_{min} = 0\), which at \(\zeta = 1\) leads directly to the relation \(a k^{-\lambda} R(k) = \phi^*(\Lambda/k, \lambda, d, u)\), where \(\phi^*\) is a function exclusively of the given arguments. Since \(R(0)\) is a constant, the function \(\phi^*\) can be represented as \(\phi^*(\Lambda/k, \lambda, d, u) = a \Lambda^{-\lambda} R(0) \cdot (\Lambda/k)^\lambda + 1/[2 \phi(\Lambda/k, \lambda, d, u)]\). We get
\[
G(k) = a k^{-\lambda} \phi(\Lambda/k, \lambda, d, u)
\]
by substituting this into \((37)\).

According to the consideration in Sect. 5.1, the true value of \(\eta\) can be found from our equations in the limit \(u \rightarrow 0\). In this case behavior of \(G(k)\) within some region \(k \sim u^* k_{crit}(u)\) and extrapolation to smaller values of \(k\) have to be considered. It follows from the definition of \(k_{crit}(u)\) that a stable solution in the form \(G(k) \simeq ak^{-\lambda}\) does exist at \(k \sim u^* k_{crit}(u)\). It means that \(\phi\) does not depend on \(\Lambda/k\) (or \(k\)) within this region at \(u \rightarrow 0\). Thus, the value of argument \(\Lambda/k\) in \((12)\) can be replaced by \(\Lambda u^{-r} k_{crit}(u)^{-1}\). On the other hand, universal positive value of \(\eta\) is obtained from Eq. \((37)\) at \(u \rightarrow 0\), which means that a finite limit \(\lim_{u \rightarrow 0} \phi(\Lambda u^{-r} k_{crit}(u)^{-1}, \lambda, d, u) = b(\lambda, d)\) exists. So, we have an asymptotic equation
\[
G(k) = a k^{-\lambda} = a k^{-\lambda} \times b(\lambda, d)
\]
from which, in principle, the universal value of \(\lambda\) can be found (i.e., \(b(\lambda, d) = 1\)).
5.3 Scaling properties at $T = T_c$ including correction terms

The analysis of the previous section can be extended by including corrections to scaling. As it was mentioned, correction $\varepsilon(k) \sim k^n$ is introduced by $ck^2$ in (37). As regards term $"1"$ in (29), (27), and (30), it yields (performing the same analysis as for the main terms) a correction $\delta(k) \sim k^{2\lambda - d}$ in (37) corresponding to the contribution of the correction term (with the factor $q^{2\lambda - d}$) in equation

$$1 - 2\Sigma(q, \zeta) \simeq -2\Sigma(q, \zeta) \times \left(1 - q^{2\lambda - d} / \left[2a^2 \Psi (\Lambda/q, k_{\min}/q, \zeta, \lambda, d, u)\right]\right). \quad (44)$$

Here, and in equations (29) to (32) $q, \Lambda/q$, and $k_{\min}/q$ are considered as independent variables, considering the limit $q \to 0$ at any given $\Lambda/q$, and $k_{\min}/q$. Similarly, in equations for $R(k)$ and $G(k)$ $k, \Lambda/k$, and $k_{\min}/k$ are considered as independent variables. Finally, we set $k_{\min}/k \to 0$ to fit the thermodynamic limit, and then we consider the limit $u \to 0$ with the simultaneous tending of $\Lambda/k$ to infinity (as in Sect. 5.2) to get the asymptotic solution at a fixed $\Lambda$. The solution can be expanded in terms of $\varepsilon(k)$ and $\delta(k)$ to yield an asymptotic expansion at $k \to 0$

$$G(k) = \sum_l b_l k^{-\lambda_l} \quad (45)$$

where

$$\lambda_l = \lambda - n_l \cdot \eta - m_l \cdot (2\lambda - d) \quad (46)$$

with $n_l, m_l = 0, 1, 2, ...$ corresponding to the correction of order $\varepsilon^{n_l} \delta^{m_l}$, $b_l$ being expansion coefficients.

If the right hand side of Eq. (38) is substituted by (44), we get

$$R(k) = \sum_l \phi^{\ast}_l (\Lambda/k, \lambda, d, u) \ k^{2\lambda - \lambda_l} \quad (47)$$

where $\phi^{\ast}_l (\Lambda/k, \lambda, d, u)$ are functions of the given arguments. They depend also on the set of coefficients $b_l$. This is obtained by finding selfconsistent solutions similarly as in Sect. 5.2 with the only difference that any given number of corrections to the scaling is included.

Since $R(0)$ is constant, $\phi^{\ast}_l$ behaves as $\text{const} \cdot (\Lambda/k)^{2\lambda - \lambda_l}$ at $k \to 0$, and herefrom it follows that $R(0)$ can be represented as $R(0) = \sum_l R_l \Lambda^{2\lambda - \lambda_l}$ with $R_l$ depending on $\lambda, d, u$, and coefficients $b_l$. Based on the same logic as in Sect. 5.2, functions $\tilde{\phi}_l (\Lambda/k, \lambda, d, u) = \phi^{\ast}_l (\Lambda/k, \lambda, d, u) - R_l (\Lambda/k)^{2\lambda - \lambda_l}$ may be replaced by $\tilde{\phi}_l (\Lambda u^{-r} k_{\text{crit}}^{-1}(u), \lambda, d, u)$ at $u \to 0$ to obtain the asymptotic solution (45) at a fixed $\Lambda$. This replacement is justified by the following argument. If the amplitude $a (\equiv b_0)$ is considered as known (fixed) quantity and $k \sim u^{r} k_{\text{crit}}(u) \to 0$, then $\tilde{\phi}_l (\Lambda/k, \lambda, d, u)$ tends to some function $\phi_l (\lambda, d, u)$ and $\phi_l (\Lambda/k, \lambda, d, u) - \tilde{\phi}_l (\lambda, d, u)$ tends to zero faster than any positive power of $k$, since only in this case exponents in the right–hand side of (37) are not affected by short–wave fluctuations and are the same as those in the left–hand side of the equation.

To show this precisely, we prove the following statement: if in the considered limit $\tilde{\phi}_l (\Lambda/k, \lambda, d, u)$ tends to $\phi_l (\lambda, d, u)$, then the tending is faster than $(\Lambda/k)^{-\sigma}$,
where \( \sigma \) is any finite and positive constant. Really, if \( G(k) \) in the right hand side of Eq. (37) is replaced by \( G(k) - \delta G(k) \), where

\[
\delta G(k) = G(k) \cdot (k/\Lambda')^m / [1 + (k/\Lambda')^m],
\]

\( \Lambda' < \Lambda \), and \( m \to \infty \), then this is equivalent to the shift of the upper limit of wave vector magnitude from \( \Lambda \) to \( \Lambda' \) at a fixed amplitude \( a \). On the other hand, the quantity \( \delta G(k) \) can be treated as any other correction term. If for arbitrary \( (l-\text{th}) \) correction \( \phi_l \) tends to \( \tilde{\phi}_l \), then at \( k \sim u^r k_{\text{crit}}(u) \to 0 \) the quantity \( \delta G(k) \) produces in the right hand side of Eq. (37) a correction term of order \( k^{\Lambda+m} \) where \( m \to \infty \). Thus, the shift of \( \Lambda \) produces a correction smaller than \( k^\sigma \) at any finite \( \sigma \), which proves the statement.

According to physical arguments, the condition that \( \tilde{\phi}_l \) tends to \( \phi_l \), obviously, is satisfied. It means that the main contribution to \( \phi_l (\Lambda/k, \lambda, d, u) \) in the equation for \( 1/G(k) \) at \( k \to 0 \) is provided by the integration over small wave vectors (therefore the result is almost independent on \( \Lambda \)), i.e., the critical behavior is governed by the long–wave fluctuations. Besides, this condition means that the asymptotic solution in approximation \( G(k) = \sum_{l=0}^n b_l k^{-\lambda_l} \), including \( m \) correction–to–scaling terms \((m=0, 1, 2, \text{etc.})\), is stable with respect to \((\text{i.e., is not changed by})\) higher order corrections, which is reasonable and expected.

As regards corrections, we allow a possibility that any of corresponding expansion coefficients can be zero, our analysis is correct in this case.

### 5.4 Asymptotic solution at \( T \to T_c \)

In this section we have extended our scaling analysis to describe the critical behavior of the model when approaching the critical point from higher temperatures, i.e., at positive \( \Delta = T - T_c \to 0 \).

The critical exponents cannot be affected by short–wave fluctuations. According to this idea, the contribution of sufficiently large \( k \) may be neglected in equations for \( 1/G(0) \) and \([1/G(k)] - [1/G(0)]\)

\[
\frac{1}{2G(0)} = \frac{dr_0}{dT} \cdot \Delta + \frac{2u}{V} \sum_k [G(k) - G^\ast(k)] + R(0) - R^\ast(0),
\]

\[
\frac{1}{2G(k)} - \frac{1}{2G(0)} = R(k) - R(0) + ck^2
\]

obtained from Eq. (34) omitting the irrelevant correction \( \delta(k) \) and assuming that \( r_0(T) = r_0(T_c) + (dr_0/dT) \cdot \Delta \) is the only parameter in (37) which depends on temperature. In Eq. (45), \( G^\ast(k) \) is the value of \( G(k) \) at \( T = T_c \) and \( R^\ast(0) \) is the value of \( R(0) \) calculated at \( G(k) = G^\ast(k) \). This equation represents the condition that \( 1/G(0) \) vanishes at \( T = T_c \).

Considering the solution at \( T = T_c \), we have concluded that terms in the right–hand side of equation for \( 1/(2G(k)) \), calculated at a fixed \( G(k) \) (i.e., at fixed amplitudes \( b_l \) in (34)), are not sensitive to a variation in the upper limit of the wave vector magnitude \( \Lambda \) at \( u \to 0 \) if \( \Lambda/k \sim u^{-r} k_{\text{crit}}^{-1}(u) \) holds, where \( r \) is any positive constant. In this case unambiguous solution insensitive to the short–wave fluctuations is obtained, based on the asymptotic expansion (34), if the main amplitude
$b_0$ is considered as a known (fixed) quantity. The asymptotic solutions at $T = T_c$ and $T > T_c$ join at $k \sim 1/\xi$, where $\xi \sim \Delta^{-\nu}$ is the correlation length, therefore, the solution of Eqs. (48) and (49) at $T \to T_c$ for $k \sim 1/\xi$ is insensitive to variation of $\Lambda$ within some region $\Lambda \sim \xi^{-1} u^{-\nu} k_{\text{crit}}^{-1}(u)$ (at $u \to 0$), if calculations are performed at fixed $G^*(k)$ (i.e., at fixed $b_0$). It is supposed that $k_{\text{crit}}(u)$ is determined at a fixed upper integration limit $\Lambda'$, and $\Lambda$ is smaller than, but comparable with $\Lambda'$. The latter condition is satisfied for values of $\Delta$ which are smaller than, but comparable with $B u^{r/\nu} \Delta_{\text{crit}}(u)$, where $B$ is an appropriate constant and $\Delta_{\text{crit}}(u) \sim [A(u) k_{\text{crit}}(u)]^{1/\nu}$ is the width of the critical region inside of which the correlation function is described by $G(k) = \xi^\lambda g(k\xi)$ with the relative error not exceeding some small given value. According to our definition $\xi \simeq A(u) \Delta^{-\nu}$ holds, and the above relation for $\Delta_{\text{crit}}(u)$ is true since the width of the critical region for $\xi^{-1}$ is proportional to $k_{\text{crit}}(u)$ due to the joining of asymptotic solutions at $k \sim 1/\xi$. We can conclude from this discussion that the region $k > C(u)/\xi$ with $C(u) = u^{-\nu} k_{\text{crit}}^{-1}(u)$ corresponds to negligible short-wave fluctuations if the solution inside the asymptotic region $k \sim 1/\xi$ is considered at $u \to 0$ and $\Delta \sim B u^{r/\nu} \Delta_{\text{crit}}(u)$. In such a way, we may neglect the short-wave fluctuations by formally setting $\Lambda = \hat{C}(u) \Delta^\nu$ (i.e., $G^*(k) = 0$ and $\Delta_{\text{crit}} = 0$ at $k > \hat{C}(u) \Delta^\nu$) in the right-hand side of Eqs. (48) and (49) where $G^*(k)$ has been calculated before this procedure at the true (constant) upper limit $\Lambda'$, and $\hat{C}(u) = C(u)/A(u)$. According to the above discussion, this method provides correct correlation function at $k \sim 1/\xi$, which means that it yields true critical exponents.

In this case the asymptotic solution can be found in the form

$$ G(k) = \sum_{l \geq 0} \Delta^{-\gamma + \gamma_l} g_l(k \Delta^{-\nu}) , \quad (50) $$

where exponents $\gamma_l$ are related to those given by (48) in the way predicted by the scaling hypothesis, i.e., $g_l(y)$ behave like $g_l(y) \simeq b_l y^{-\lambda_l}$ at $y \to \infty$ to yield (48) at $\Delta \to 0$. Thus, the exponents in (50) are

$$ \gamma = \lambda \nu = (2 - \eta) \nu , \quad (51) $$

$$ \gamma_l = n_l \delta_l + m_l \delta_2 , \quad (52) $$

where

$$ \delta_1 = 2\nu - \gamma > 0 , \quad (53) $$

$$ \delta_2 = 2\gamma - d\nu > 0 . \quad (54) $$

Here the main term is given by $l = 0$ and $\gamma$ is the susceptibility exponent. The $l$-th term with $l > 0$ represents the correction of order $\varepsilon^{n_l} \delta^{m_l}$, where $\varepsilon(\Delta) = \Delta^{\delta_1}$ and $\delta(\Delta) = \Delta^{\delta_2}$.

This result is obtained by a scaling analysis similar to that we have made at $T = T_c$. The only difference is that wave vectors are normalized to $\Delta^\nu$, but not to the current value of $q$ (in equation for $\Sigma(q, \zeta)$) or $k$ (in equation for $G(k)$). Besides, the lower limit of integration may be set $k_{\text{min}} = 0$ from the very beginning since there is no singularity at $k = 0$. In such a way, retaining only the main
term, we prove the following scaled form for the relevant quantities calculated at 
Λ = ˆC(u): Σ(k, ζ) = 2γ + dνΨ(k', ζ), ∂Σ(q, ζ)/∂G_i(k) = 2γ Y(q', k', ζ), 
∂D^*(G, ζ)/∂G_i(k) = 2γ r(k', ζ), and R(k) = 2γ r(k'), where k' = kΔ−ν, and at 
any fixed u, d, n, and g_0(kΔ−ν) the scaling functions depend on the given arguments 
only. Including corrections produced by ck^2 in Eq. (34) (correction of order Δδ_1) 
and "1" in Eq. (30) (correction of order Δδ_2), we obtain the asymptotic expansion 
in the scaled form

\[ R(k) = \sum_{l \geq 0} \Delta^{\gamma + \delta_1} r_l(k') \]

with exponents γ_l defined by Eq. (52). If Eqs. (48) and (49) are substituted by (55) 
at Λ = ˆC(u)Δν, then we obtain a solution where no other correction exponents 
appear.

5.5 Possible values of critical exponents

By using the equations and scaling relations obtained in previous sections, here we 
derive our central result – the set of possible values for critical exponents.

It is reasonable to assume that γ > 1, which leads to the conclusion that 
\((dr_0/dT) \Delta\) in Eq. (48), where Λ = ˆC(u)Δν, is compensated by one of the terms 
coming from the asymptotic expansion

\[ 2u V^{-1} \sum_{k < C(u)\Delta^\nu} G(k) - G^*(k) = \sum_{l \geq 0} B_l \Delta^{2\gamma + \delta_2} . \]

It means that

\[ \gamma_m = 1 - \gamma + \delta_2 \]

holds at some m ≥ 0. Condition γ_m < δ_2 follows herefrom, since γ > 1. Thus, with 
account for (52) we have γ_m = m δ_1, and Eq. (57) becomes

\[ m \delta_1 - \delta_2 + \gamma = 1 . \]

We need one more relation to determine the values of critical exponents. We obtain 
this relation from calculation of specific heat C_V assuming the well known hyper-
scaling hypothesis

\[ \alpha + d\nu = 2 . \]

The singular part of the specific heat behaves like Δ−α and it can be related to the 
singular part of  ̃G, i. e., to ̃G − ̃G^* (where ̃G^* is the value of ̃G at T = T_c), as follows

\[ C_V \propto \frac{\partial}{\partial \Delta} \left[ V^{-1} \sum_{k < \Lambda'} (G(k) - G^*(k)) \right] , \]

where Λ' is constant. The latter relation follows from thermodynamics, taking into 
account that

\[ \frac{\partial}{\partial r_0} \left( \frac{F}{T} \right) = -\frac{\partial \ln Z}{\partial r_0} = V \langle \varphi^2(x) \rangle \equiv V \tilde{G} \]

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holds where $F$ is the free energy and $Z$ is the statistical sum, consistent with the definition of Hamiltonian $[7]$ where $r_0$ is the only parameter depending on temperature and the dependence is linear.

According to the universality hypothesis, critical exponents do not depend on the coupling constant $u$, therefore the exponents obtained by our method at $u \to 0$ can be used in calculation of the singular part of $C_V$ at a finite $u$. Coefficients $B_l$ in (56) can be changed, not changing $G(k)$ at $k \sim 1/\xi$, if the solution at $\Lambda = \Lambda' = \text{const}$ instead of the formal solution at $\Lambda = \hat{C}(u)\Delta^\nu$ is considered. For instance, a variation in $B_l$ due to the change in $G(k)$ at $k \sim \hat{C}(u)\Delta^\nu \gg 1/\xi$ (at $u \to 0$) can be compensated by a contribution coming from $R(0) - R^*(0)$ due to the integration over $\hat{C}(u)\Delta^\nu < k < \Lambda'$. Thus, the true values of the amplitudes in the expansion of $G - G^*$ are unknown, and we allow all the possibilities.

Thus, consider a finite $u$. Our basic idea is that the contribution to (60) provided by the summation over $k > C/\xi$ or $k > C\Delta^\nu$ at $C \to \infty$ (we consider the limit $\Delta \to 0$ at a given $C$, which then is tended to infinity) with the true correlation functions $G(k)$ and $G^*(k)$ (calculated at $\Lambda = \Lambda'$) cannot change the critical exponent $\alpha$, because the opposite would mean a violation of scaling relations for critical exponents. However, we allow that a logarithmic correction can be caused by this contribution. Thus, we can replace the summation limit $k < \Lambda'$ in (61) by $k < C\Delta^\nu$. The first non–vanishing (i. e., having nonzero amplitude at $C = \infty$) singular term in the resulting asymptotic expansion represents the leading singularity of $C_V$. Formally, a constant contribution also is defined as singular (with $V \to \infty$) limit $\Delta \to 0$ at a given $C$, which then is tended to infinity) with the true correction functions $G(k)$ and $G^*(k)$ (calculated at $\Lambda = \Lambda'$) cannot change the critical exponent $\alpha$, because the opposite would mean a violation of scaling relations for critical exponents. However, we allow that a logarithmic correction can be caused by this contribution. Thus, we can replace the summation limit $k < \Lambda'$ in (61) by $k < C\Delta^\nu$. The first non–vanishing (i. e., having nonzero amplitude at $C = \infty$) singular term in the resulting asymptotic expansion represents the leading singularity of $C_V$. Formally, a constant contribution also is defined as singular (with $\alpha = 0$) in the case if there is a jump of $C_V$ at $T = T_c$ from one constant value to another, or if a refined analysis reveals logarithmic singularity. Thus, according to (16), (50), (57), and (60), all possible values of $\alpha$ are given by

$$\alpha = \gamma_m - \gamma_i$$

where $i \geq 0$ is integer. In this case $g_l(y) = b_l y^{-\lambda_i}$ (cf. Eqs. (54) and (53)) tends to zero at $y \to \infty$ and the tending is faster than $y^{-\sigma}$ where $\sigma < d$, since the opposite would mean that the critical exponent $\alpha$ is changed due to the contribution of $k > C\Delta^\nu$. In the marginal case when $g_l(y) = b_l y^{-\lambda_i} \sim y^{-d}$ holds this contribution yields a logarithmic correction, i. e., $C_V \sim \Delta^{-\alpha} \ln \Delta$. In such a way our theory provides an explanation of the known logarithmic singularity of the specific heat at $n = 1$ and $d = 2$.

We have restricted our analysis to $\gamma > 1$ and $\alpha > 1 - \gamma$, which is very reasonable assumption in view of the known results. From (57) we obtain $1 - \gamma = \gamma_m - \delta_2$, which, in this case, yields $\alpha = \gamma_m - \gamma_i > \gamma_m - \delta_2$ or $\gamma_i < \delta_2$. Then, combining (58) with (52) and (59), with account for definitions (53), (54), and (52), the following set of possible values for critical exponents is obtained

$$\gamma = \frac{d + 2j + 4m}{d(1 + m + j) - 2j}; \quad \nu = \frac{2(1 + m) + j}{d(1 + m + j) - 2j},$$

where $m$ may have a natural value starting with 1 and $j$ is integer equal or larger than $-m$. This result is obtained by proving all the relevant scaling properties not
cutting the perturbation series (32), based on equations which allow to find the exact critical exponents (Sec. 5.1). Thus, according to these arguments, Eq. (63) represents all possible values for the exact critical exponents at \(d = 2\) and \(d = 3\) in the cases where the second–order phase transition with spontaneous long–range ordering takes place.

Our analysis is not valid at \(d \geq 4\) since \(\delta_1\) and \(\delta_2\) (Eqs. (53) and (54)) are positive, i. e., \(\Delta^\delta_1\) and \(\Delta^\delta_2\) are small corrections, merely at \(d < 4\). Besides, the analysis in Appendix B is not true at \(d = 4\). Solutions at natural \(n\) (the dimensionality of the order parameter) only have a meaning, since our method is strongly based on the proof that in the relevant asymptotical region at \(u \to 0\) the solution of our equations agree with the exact correlation function defined by the statistical integrals. Although the formal perturbation expansion exists at arbitrary \(n\), such a method of proof would be meaningless at a not natural \(n\), since the exact correlation function is not defined in this case, and we cannot guarantee that a formal solution at arbitrary \(n\) has all the correct properties (e. g., existence of the secon–order phase transition and scaling relation (59)) which have been assumed to derive Eq. (63). Our predictions do not refer also to the case \(n = 0\). This case is exceptional in view of our analysis, since the term \(2\Sigma(q, \zeta)\) in the denominator in Eq. (30) which appears as the main term at \(n \geq 1\) vanish at \(n = 0\).

In general, different values of \(j\) and \(m\) can correspond to different (natural) \(n\), i. e., \(j = j(n)\) and \(m = m(n)\). It is easy to verify that at \(j = 0\) and \(m = 3\) Eq. (63) reproduces the known [1] exact results in two dimensions. The known exact exponents for the spherical model [1] \((n = \infty)\) are obtained at \(j(n)/m(n) \to \infty\). Although the derivations are true for \(d < 4\), Eq. (63) provides correct result \(\nu = 1/2\) and \(\gamma = 1\) also at \(d = 4\). It is reasonable to consider \(d\) as a continuous parameter. This leads to the conclusion that \(m = 3\) and \(j = 0\) are the correct values for the case \(n = 1\) not only at \(d = 2\), but also at \(d = 3\). In the latter case we have \(\gamma = 5/4\) and \(\nu = 2/3\). The nearest values of \(\gamma\) and \(\nu\) provided by Eq. (63), e. g., at \(j = 1\) and \(m = 3\) or at \(j = 1\) and \(m = 4\) are then the most probable candidates for the case \(n = 2\). It is interesting to note that our prediction for the singularity of specific heat \(\alpha = 0\) for the Ising model \((n = 1)\) agree with that made by Tseskis [26] based on a fractal model.

6 Comparison with Monte–Carlo and experimental results and discussion

It is commonly believed that all more or less correct Monte Carlo (MC) simulations confirm the values of critical exponents obtained from the perturbation expansions based on the renormalization group. This is not true. We have found that some kind of MC simulations at the critical point, namely, the MC simulations of fractal configurations of Ising model [29] and the MC simulations of the energy density [30] for the XY model in reality do not confirm the results of the RG theory, but provide the values of critical exponents which are very close to those we predicted.

The MC simulations of Ref. [29] allows to determine the fractal dimensionality
Figure 3: Fractal dimensionality $D$ of the three dimensional Ising model at the critical point simulated by Monte Carlo method (MCS means Monte Carlo steps). The upper and lower dashed lines indicate the theoretical values expected from the known and from our critical exponents, respectively.

$D$ (the largest cluster in the relevant configuration has the volume $L^D$ where $L$ denotes the linear size of the system) which is related to the critical exponents by $\gamma = \nu(2D - d)$ or, which is the same, $\eta = 2 - \gamma/\nu = d + 2 - 2D$. In our opinion, this method is better than other more convenient simulation methods, since it provides the value of $\eta$ as a result of direct simulation, i.e., there are no fitting parameters. Besides, the result is relatively insensitive to the precise value of the critical coupling (temperature). In Fig. 3 we have shown the average values of $D$ (the averaging is is made over the MC steps from 1 to 10 (except the initial point), from 11 to 20, and so on) calculated from the MC data of Ref. [29] by measuring deviation from the line $D = 2.48$ in Fig. 8 (of Ref. [29]). If properly treated, these simulation data confirm the value of $\eta$ about 1/8 (or $D = 2.4375$) consistent with our prediction $\gamma = 5/4$ and $\nu = 2/3$, as it is evident from Fig. 3. The value $D = 2.46 \pm 0.01$ reported in Ref. [29] seems to be determined from the upper MC points (Fig. 8 in Ref. [29]) only which are closer to the known theoretical prediction $D = 2.48$.

As regards the MC simulations of the energy density $E$ of $XY$ model [30] at the critical point, the true picture can be reconstructed from the simulated values listed in Tab. I of Ref. [31]. Since all the values of $E$ are of comparable accuracy, it is purposeful to use the least–squares method to find the optimum value of $1/\nu$ by fitting the MC data to the prediction of the finite–size scaling theory

$$E(L) = E_0 + E_1 L^{\frac{1}{\nu} - d},$$

(64)

where $E(L)$ is the energy density at the critical temperature $T_\lambda$ depending on the linear size of the system $L$. The standard deviation of the simulated data points from the analytical curve (64) can be easily calculated for any given value of $1/\nu$ with the parameters $E_0$ and $E_1$ corresponding to the least–squares fit. The result
Figure 4: The standard deviation $\sigma$ vs the value of $1/\nu$ used in the least–squares fit of the finite–size scaling curve to the simulated results including 11 data points (solid curve) and 9 data points (dashed curve). Minimum of the solid curve, shown by a vertical dashed line, corresponds to the best fit $1/\nu = 1.4457$ which is close to our theoretical value $13/9$ indicated by a vertical dotted line. Other vertical dashed line indicates the value $1.487$ proposed by authors of Ref. [30].

is shown in Fig. 4. The thick solid curve is calculated including all 11 data points ($L=10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 80$), whereas the dashed line – including 9 data points (except $L=10, 15$) used for the fitting in Ref. [30]. Minimum of the solid curve, shown by a vertical dashed line, corresponds to the best fit $1/\nu = 1.4457$ which comes very close to our theoretical value $13/9$ (provided by (63) at $j = 1$ and $m = 3$) indicated by a vertical dotted line. We have estimated the statistical error of this MC result about $\pm 0.007$ by comparing the best fits for several random data sets. Different data sets have been generated from the original one by omitting some data points with $10 < L < 80$. We have found it unreasonable to omit the data points with two smaller sizes, as it has been proposed in Ref. [30], since the result in this case becomes very poorly defined, i. e., the dashed curve in Fig. 4 has a very broad minimum. Besides, there is no reason to omit the smallest sizes, since the analytical curve (64) excellently fit all the data points and the standard deviation for 11 data points is even smaller than that for 9 data points (see Fig. 4). The possible systematical error due to the inaccuracy in the critical temperature $T_\lambda = 2.2017 \pm 0.0005$ (the error bars are taken from the source of this estimation [31]) used in the simulations [30] has been evaluated $\pm 0.017$ by comparing the simulation results at $T_\lambda$ values 2.2012, 2.2017, and 2.2022. In this case the values of the energy density at a slightly shifted temperature have been calculated from the specific heat data given in Tab. I of Ref. [30]. In such a way, our final estimate from the original MC data of Ref. [30] is $1/\nu = 1.446 \pm 0.025$ in a good agreement with our theoretical value $13/9 = 1.444...$ and in a clear disagreement with the usual (RG) prediction...
about 1.492. One can only wonder where the value 1.487 proposed in Ref. [30] comes from. It does not correspond neither to the best fit for 11 data points nor to that for 9 data points, as it is evident from Fig. 4. The values of $1/\nu$ and $\alpha/\nu$ cannot be determined independently from the discussed here energy density data. One of them have to be calculated from the scaling relation $\alpha/\nu + d = 2/\nu$. If authors of Ref. [30] were able to determine $1/\nu$ with $\pm 0.081$ accuracy, then they should be able to find $\alpha/\nu$ with $\pm 0.162$ accuracy. In this aspect, the estimate $\alpha/\nu = -0.0258 \pm 0.0075$ given by the authors looks more than strange.

In Fig. 5 we have shown our fits to the MC data for the energy density $E(L) = 2.0108 - 2.0286 L^{-14/9}$ and for the specific heat $c(L) = 7.360 - 6.990 L^{-1/9}$. They do not look worse than those in Ref. [30], but our fit for $c(L)$ seems to be better.

One believes that the value of critical exponent $\nu$ about 0.67, predicted by the RG theory at $n = 2$, is well confirmed by very accurate measurements of the superfluid fraction $\rho_s/\rho = y$ in $^4$He. This is not true, since in reality these experiments provide a good evidence that the effective critical exponent $\nu(t) = \partial (\ln y)/\partial (\ln t)$ remarkably increases when the reduced temperature $t = (T_\lambda - T)/T_\lambda$ (where $T_\lambda$ is the critical temperature) is decreased below $10^{-5}$. According to Ref. [32], $\rho_s/\rho$ is given by

$$\rho_s/\rho = y(t) = k_0 (1 + k_1 t) (1 + D_\rho t^\Delta) t^\zeta \times (1 + \delta(t)),$$

(65)

where $k_0$, $k_1$, $D_\rho$, and $\zeta$ are the fitting parameters, $\Delta = 0.5$ is supposed to be the correction–to-scaling exponent, and $\delta(t)$ is the measured relative deviation from the expected theoretical expression obtained by setting $\delta(t) = 0$. The percent deviation discussed in Ref. [32] is 100 times $\delta(t)$. From Eq. (65) we obtain

$$\nu_{eff}(t) = \zeta + \frac{k_1 t}{1 + k_1 t} + \frac{\Delta D_\rho t^\Delta}{1 + D_\rho t^\Delta} + \frac{1}{1 + \delta(t)} \times \frac{\partial \delta(t)}{\partial (\ln t)}.$$

(66)

For the values of $t$ as small as $t < 10^{-5}$ and for $\delta(t) \ll 1$ Eq. (66) with the fitting parameters $\zeta = 0.6705$, $k_0 = 2.38$, $k_1 = -1.74$, and $D_\rho = 0.396$ used in Ref. [32]...
reduces to
\[ \nu_{eff}(t) \simeq \zeta + \partial \delta(t)/\partial(\ln t) . \] (67)

The second term in this equation is proportional to the slope of the percent deviation plot 100\( \delta(t) \) vs \( \ln t \) or \( \lg t \) (the decimal logarithm) in Figs. 2 and 3 of Ref. [32]. We have read the experimental data from Fig. 2 in Ref. [32] within the region \( t < 10^{-4} \) and have depicted them in Fig. 6. Almost all the data points with a reasonable accuracy fit the smooth curve \( \delta(t) \) vs \( \lg t \) (dashed line) having a maximum at about \( \lg t = -5.5 \). It means that \( \partial^2 \delta(t)/\partial(\ln t)^2 \) is negative within some region around the maximum, i. e., according to (67) the effective critical exponent \( \nu_{eff}(t) \) increases if \( t \) is decreased. We have roughly estimated and have shown by straight line the slope of this curve at \( t = t^* = 5 \cdot 10^{-7} \) (\( t^* \) value is indicated in Fig. 6 by vertical dashed line). From this we obtain \( \partial \delta(t)/\partial(\ln t) \approx 0.025 \). This result depends on the shift in the experimentally determined \( T_\lambda \) value. To obtain a more reliable estimate, we have performed the same manipulations with the data depicted in Fig. 3 of Ref. [32] corresponding to \( T_\lambda \) shifted by \( \pm 20nK \), and have obtained the values of \( \partial \delta(t)/\partial(\ln t) \) about 0.03 and 0.015, respectively. Our final result 0.0233±0.0083 for this derivative at \( t = t^* \) has been obtained by averaging over the three above discussed estimates (0.015, 0.025, and 0.03) with the error bars large enough to include all these values. According to this, from Eq. (67) with \( \zeta = 0.6705 \) we obtain \( \nu_{eff}(t^*) = 0.694 \pm 0.009 \) which, again, is in a good agreement with the value \( \nu = 9/13 \simeq 0.6923 \) provided by Eq. (63) at \( j = 1 \) and \( m = 3 \) and in a disagreement with the RG predictions.
7 Conclusions

We have proposed a novel method in critical phenomena (Sect. 4, 5) which is based on the grouping of Feynman diagrams in $\varphi^4$ model with $O(n)$ symmetry. As a result, equations for calculation of the two–point correlation function have been obtained containing an infinite, but converging perturbation sum. It has been shown that these equations allow, in principle, to find the exact critical exponents. In distinction to the usual renormalization group approach, our predictions are based not on evaluation of some of the first terms in the perturbation expansion, but on the proof of relevant scaling properties for the whole sum, which is possible due to the actually proposed reorganization of the perturbation theory.

Based on this scaling analysis, we have derived a set of possible values for the exact critical exponents (63). A disagreement with the actually accepted values of the critical exponents in three dimensions has been revealed. However, we argue that our result is correct since, in distinction to the usual treatment criticised in Sec. 2 and 3, our method is faultless from the mathematical point of view. Some assumptions have been made, but they look innocent and have been well motivated. Besides, our method, being equally valid in two and three dimensions, reproduces the known exact critical exponents at $d = 2$.

A comparison of results has been made in Sect. 6, showing that in some cases, at least, our predictions are in accurate agreement with properly treated MC simulation data as well as with experiments. More comparison with MC data is in progress.

In summary, we conclude the following.

1. The conventional method of the perturbative RG theory is contradictory and, therefore, cannot give correct values of critical exponents.

2. Our equations, derived in Sec. 4 by grouping of the Feynman diagrams, allow to find the exact critical exponents for $\varphi^4$ model with $O(n)$ symmetry ($n \geq 1$) by proving all the relevant scaling properties of the asymptotic solutions at $T = T_c$ and $T \rightarrow T_c$, not cutting the perturbation series. These scaling properties have been proven in Sec. 5.

3. In the cases of the second–order phase transition with spontaneous long–range ordering, all possible values for the exact critical exponents at $d = 2, 3$ and $n = 1, 2, 3$, etc. are given by Eq. (63).

4. At $m = 3$ and $j = 0$ our result (63) reproduces the known exact critical exponents in two dimensions at $n = 1$ ($\gamma = 7/4$ and $\nu = 1$). Based on the idea that $d$ may be considered as a continuous parameter, we conclude that $\gamma = 5/4$ and $\nu = 2/3$ are the true (exact) values at $d = 3$ and $n = 1$.

5. The comparison with Monte–Carlo data in Sec. 6 well confirms the hypothesis that $\gamma = 17/13$ and $\nu = 9/13$ (corresponding to $m = 3$, $j = 1$) are the true values of the critical exponents at $d = 3$ and $n = 2$. 

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Appendix A. Large–order estimation

Let us consider coupled diagrams, constructed of vertices \( \blacktriangleleft \blacktriangledown \blacktriangleleft \), which are skeleton diagrams with respect to both solid and waved lines (i. e., do not contain parts \( \blacktriangledown \) and/or \( \blacktriangleleft \blacktriangledown \blacktriangledown \)). At large \( m \) most of the coupled skeleton diagrams of the \((m+1)\)-th order can be constructed by coupling vertex \( \blacktriangleleft \blacktriangledown \blacktriangledown \) to a diagram \( \blacktriangledown \) of the \( m \)-th order with four outer lines obtained by breaking two of \( 2m \) solid lines in a coupled skeleton diagram of the \( m \)-th order. Since at large \( m \) the lines in most cases of coupled diagrams are arranged randomly (i. e., the symmetry elements are not important) this yields approximately \( 6m^2 \) (relevant to \( m \cdot (2m - 1) \) possible ways to break the solid lines, and 3 different ways to couple the vertex to get a skeleton diagram) topologically nonequivalent diagrams of the \((m + 1)\)-th order corresponding to a given diagram of the \( m \)-th order. However, different diagrams of the \( m \)-th order yield the same diagram of the \((m+1)\)-th order. Really, we can subtract any of the \( m + 1 \) vertices \( \blacktriangleleft \blacktriangledown \blacktriangledown \) from a (random) coupled diagram of the \((m+1)\)-th order and obtain a diagram \( \blacktriangledown \) of the \( m \)-th order. \( m + 1 \) reverse procedures represent \( m + 1 \) different ways of how to obtain this coupled diagram of the \((m+1)\)-th order from diagrams of the \( m \)-th order. Thus, the total number of the above defined coupled skeleton diagrams of the \((m+1)\)-th order exceeds the number of corresponding diagrams of the \( m \)-th order approximately \( 6m \) times, or the number of diagrams at large \( m \) increases approximately as \( 6^m m! \).

Let us consider other possibilities to construct coupled skeleton diagrams of the \((m+1)\)-th order. We can couple a vertex \( \blacktriangleleft \blacktriangledown \blacktriangledown \) to a diagram obtained by breaking two solid lines in a non skeleton diagram. The simplest of such diagrams \( \blacktriangledown \) and \( \blacktriangledown \) are constructed of two above defined skeleton diagrams of orders \( m_1 \) and \( m_2 \) with \( m_1 + m_2 = m \). The number of such diagrams relative to the number of the diagrams \( \blacktriangledown \) obtained from the skeleton diagrams of the \( m \)-th order tends to zero at \( m \to \infty \) as \( (m^2_1 m^2_2 m_1! m_2!) / (m^2 m!) \). More complicated diagrams like \( \blacktriangledown \blacktriangledown \blacktriangledown \ldots \blacktriangledown \) (where any of the neighbouring blocks can be connected by waved lines instead of the solid lines) also can be included. However, this gives only a second–order correction. Other kind of diagrams among those obtained by breaking two solid lines in a non skeleton diagram, e. g., \( \blacktriangledown \), are not included because the coupling of vertex in this case does not yield a skeleton diagram. In such a way, the number of possible constructions is increased insignificantly if non skeleton diagrams of the \( m \)-th order are included. So, factor \( 6m \) is corrected only slightly. For a more accurate estimation this factor is replaced by \( 6m + o(1) \), which allows to conclude that the number of diagrams of the \( m \)-th order in Eq. (29) increases with \( m \) like \( \text{const} \cdot m^\gamma 6^m m! \), where \( \gamma \) is a constant. Note that the diagrams of (29) are obtained from the actually discussed coupled skeleton diagrams by breaking a waved line, replacing this line by a (broken) dashed line.

Consider now a random high–order diagram \( \Sigma_{m,i}(q, \zeta) \) of Eq. (29), i. e., the \( i \)-th diagram among those of the \( m \)-th order. Calculation of the sum over wave vectors and order–parameter–components for such a diagram is analogous to finding
of the statistical sum of a random lattice formed by the lines of this diagram. Two conditions must be fulfilled: first, the sum of the wave vectors coming into any of the kinks is zero, the wave vectors of the outer lines being fixed (q and \(-q\)), and second, the same order–parameter–component is related to solid lines of a closed loop. These conditions represent certain interaction between the lines of the diagram. The “interaction energy” is zero if they are satisfied, and \(\infty\) otherwise. Besides, each line has its own “energy”. If normalized to \(T\), this energy is equal to minus logarithm of the absolute value of the factor related to the corresponding line in the diagram notation. This analogy can be used in calculation of \(|\Sigma_{m,i}(q, \xi)|\). In a typical case of large \(m\) this yields \(\Sigma_{m,i}(q, \xi) \sim (-1)^mb^m\), where \(b^m\) with \(b > 0\) represents the absolute value and \((-1)^m\) – the sign of this term. This result remains true (with new value of \(b\)) if the combinatorial factor is included, since the latter is \(4 \cdot 8^m\) for any random diagram having no symmetry elements (see Sect. 4.2).

**Appendix B. Non–perturbative analysis**

For convenience, here we consider the one–component case \(n = 1\), since the specific (natural) value of \(n\) is irrelevant in the present scaling analysis. The Hamiltonian of the considered model

\[
H/T = \int \left[r_0\varphi^2(x) + c(\nabla\varphi(x))^2 + u\varphi^4(x)\right] dx
\]

in the Fourier representation is

\[
H/T = \sum_k \left(r_0 + c|k|^2\right)|\varphi_k|^2 + uV^{-1}\sum_{k_1,k_2,k_3}\varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{-k_1-k_2-k_3}.
\]

Summation in Eq. (B2) is performed within \(k < k_0\) (here \(k_0 = \Lambda\)). Points in the \(k\)–space are separated by a distance \(2\pi/L\), where \(L\) is the linear size of the system with volume \(V = L^d\) at \(d < 4\). By a formal change of variables \(\Psi_p = u^{\alpha}c^{-d\alpha/2}\varphi_k\), \(p = e^{2\alpha}u^{-\alpha}k\), \(p_0 = e^{2\alpha}u^{-\alpha}k_0\), and \(R = r_0c^{d\alpha}u^{-2\alpha}\), where \(\alpha = 1/(4-d)\) we obtain

\[
H/T = \sum_p \left(R + p^2\right)|\Psi_p|^2 + V_1^{-1}\sum_{p_1,p_2,p_3}\Psi_{p_1}\Psi_{p_2}\Psi_{p_3}\Psi_{-p_1-p_2-p_3}
\]

where \(V_1 = L^d_1\) with \(L_1 = L \cdot c^{-2\alpha}u^\alpha\). In Eq. (B3) the summation is performed within \(p < p_0\). Points are separated by a distance \(2\pi/L_1\). According to equations (B2) and (B3), we have (fluctuation modes with \(k > k_0\) are excluded)

\[
G(k) = \langle |\varphi_k|^2\rangle = c^{d\alpha}u^{-2\alpha}\langle |\Psi_p|^2\rangle = c^{d\alpha}u^{-2\alpha}g(p, p_0, R)
\]

where, at a fixed \(d\) and \(V_1 \to \infty\), \(g\) is a function of the given arguments only. The latter is true since the only parameters contained in Hamiltonian (B3) are \(p_0\), \(R\), and \(V_1\). For the same reason, the critical value of \(R\) depends merely on \(p_0\), and the correlation function at the critical point is

\[
G^\ast(k) = c^{d\alpha}u^{-2\alpha}g\ast(p, p_0).
\]
It follows from Eq. (B5) that the width of the critical region for \( p \) is \( p_{\text{crit}} = \phi(p_0) \) where \( \phi(z) \) is a single–argument function. Here both \( d \) and the accuracy parameter \( \varepsilon \) in Eq. (B3) are considered as fixed quantities. According to the relation between \( p \) and \( k \), we have \( p_{\text{crit}} = c^{2\alpha} u^{-\alpha} k_{\text{crit}} \), which yields

\[
k_{\text{crit}} = c^{-2\alpha} u^{\alpha} \phi \left( c^{2\alpha} u^{-\alpha} k_0 \right).
\]

(B6)

Similarly, by using Eq. (B5) we obtain an exact scaling relation for the asymptotic expansion at \( k \to 0 \), i.e.,

\[
G^a(k) = \sum_l b_l k^{-\lambda_l},
\]

(B7)

where \( \lambda_0 \equiv \lambda = 2 - \eta \) and terms with \( l > 0 \) are corrections to scaling with the amplitudes represented in the scaled form

\[
b_l = B_l \left( c^{2\alpha} u^{-\alpha} k_0 \right) \cdot c^{\alpha(d-2\lambda_l)} u^{\alpha(\lambda_l-2)}.
\]

(B8)

It is evident from Eq. (B6) that \( k_{\text{crit}} \) can exponentially (like \( \exp(-u^{-\sigma}) \) with \( \sigma > 0 \)) tend to zero at \( u \to 0 \) only if the function \( \phi(z) \) decreases exponentially at \( z \to \infty \). Let us now consider the behavior of \( k_{\text{crit}} \) at \( k_0 \to \infty \) at fixed \( c \) and \( u \). We conclude immediately: if \( k_{\text{crit}} \) decreases exponentially at \( u \to 0 \), then this quantity behaves in the same way at \( k_0 \to \infty \). Physically, this means that the width of the critical region is strongly affected by short–wave fluctuations and the effect dramatically (exponentially) increases with shortening of the wavelength. The latter is rather unphysical, since the critical behavior is well known to be governed by long–wave fluctuations. Thus, \( k_{\text{crit}} \) cannot decrease exponentially (or, in general, faster than \( u^s \) at any \( s > 0 \)) at \( u \to 0 \), which means that \( \lim_{u \to 0} (u^s / k_{\text{crit}}(u)) = 0 \) holds at large enough \( s \). Based on Eq. (B8), the same conclusion can be made for the amplitudes \( b_l \).

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