COMPACT GROUP ACTIONS WITH THE ROKHLIN PROPERTY AND THEIR CROSSED PRODUCTS

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Abstract. We present a systematic study of the structure of crossed products and fixed point algebras by compact group actions with the Rokhlin property. Our main technical result is the existence of an approximate homomorphism from the algebra to its subalgebra of fixed points, which is a left inverse for the canonical inclusion. Upon combining this with known results regarding local approximations, we show that a number of classes characterized by inductive limit decompositions with weakly semiprojective building blocks, are closed under formation of crossed products by such actions. Similarly, in the presence of the Rokhlin property, if the algebra has any of the following properties, then so do the crossed product and the fixed point algebra: being a Kirchberg algebra, being simple and having tracial rank zero, having real rank zero, having stable rank one, and absorbing a strongly self-absorbing $C^*$-algebra. The $K$-theory and the Cuntz semigroup of crossed products by Rokhlin actions are also studied.

The methods of this paper unify, under a single conceptual approach, the work of a number of authors, who used rather different techniques. Our methods yield new results even in the well-studied case of finite groups actions with the Rokhlin property.

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1. Introduction

The Rokhlin property for discrete group actions on $C^*$-algebras first appeared in the late 1970’s and early 1980’s, in work of Fack and Maréchal [6], Kishimoto [21], and Herman and Jones [12] on cyclic group actions on UHF-algebras, and in the work of Herman and Ocneanu [13] on integer actions on UHF-algebras, although these authors did not use the term ‘Rokhlin property’.

In [16], Izumi provided a formal definition of the Rokhlin property for a finite group action. His classification theorems for such actions (both in [16] and [17])
are among the major results in the study of finite group actions with the Rokhlin property.

In a different direction, Izumi [16], Hirshberg and Winter [15], Phillips [29], Osaka and Phillips [26], and Pasnicu and Phillips [27], explored the structure of crossed products by finite group actions with the Rokhlin property on unital $C^*$-algebras, while Santiago [33] addressed similar questions in the non-unital case. The questions and problems addressed in each of these works are different, and consequently the approaches used by the above mentioned authors are substantially distinct in some cases.

In [15], Hirshberg and Winter also introduced the Rokhlin property for a compact group action on a unital $C^*$-algebra, and their definition coincides with Izumi’s in the case of finite groups. They showed that approximate divisibility and $\mathcal{D}$-stability, for a strongly self-absorbing $C^*$-algebra $\mathcal{D}$, are preserved under formation of crossed product by compact group actions with the Rokhlin property. Extending the results of [29], [26], and [27] to the case of arbitrary compact groups requires new insights, since the main technical tool in all of these works (Theorem 3.2 in [26]) seems not to have a satisfactory analog in the compact group case.

In this paper, we generalize the main results on finite group actions with the Rokhlin property of the above mentioned papers, to the case of compact group actions. Our contribution is two-fold. First, most of the results we prove here were known only in some special cases (mostly for finite or circle group actions; see [7] and [8] for the circle case), and some of them had not been noticed even in the context of finite groups. Second, our methods represent a uniform treatment of the study of crossed products by actions with the Rokhlin property, where the attention is shifted from the crossed product itself, to the algebra of fixed points.

Our results can be summarized as follows (the list is not exhaustive). We point out that (12) was first obtained, with different techniques, by Hirshberg and Winter as part (1) of Corollary 3.4 in [15].

**Theorem.** The following classes of separable $C^*$-algebras are closed under formation of crossed products and passage to fixed point algebras by actions of second-countable compact groups with the Rokhlin property:

1. Simple $C^*$-algebras (Corollary 2.25). More generally, the ideal structure can be completely determined (Theorem 2.24);  
2. $C^*$-algebras that are direct limits of certain weakly semiprojective $C^*$-algebras (Theorem 3.10). This includes UHF-algebras (or matroid algebras), AF-algebras, AI-algebras, AT-algebras, countable inductive limits of one-dimensional NCCW-complexes, and several other classes (Corollary 3.11);  
3. Kirchberg algebras (Corollary 3.20);  
4. Simple $C^*$-algebras with tracial rank zero (Theorem 3.16);  
5. $C^*$-algebras with nuclear dimension at most $n$, for $n \in \mathbb{N}$ (Theorem 2.13);  
6. $C^*$-algebras with decomposition rank at most $n$, for $n \in \mathbb{N}$ (Theorem 2.13);  
7. $C^*$-algebras with real rank zero (Proposition 3.21);  
8. $C^*$-algebras with stable rank one (Proposition 3.21);  
9. $C^*$-algebras with strict comparison of positive elements (Corollary 2.20);  
10. (Not necessarily simple) purely infinite $C^*$-algebras (Proposition 3.19);  
11. Stably finite $C^*$-algebras (Corollary 2.15);  
12. $\mathcal{D}$-absorbing $C^*$-algebras, for a strongly self-absorbing $C^*$-algebra $\mathcal{D}$ (Theorem 3.14);
Our work yields new results even in the case of finite groups. For example, in Theorem 2.18 we do not require the algebra $A$ to be simple, unlike in Theorem 3.13 of [16]. In addition, the classes of $C^*$-algebras considered in Theorem 3.10 may consist of simple $C^*$-algebras, unlike in Theorem 3.5 in [26] (we also do not impose any conditions regarding corners of our algebras).

Given that our results all follow as easy consequences of our main technical observation, Theorem 2.16, we believe that this paper unifies the work of a number of authors, who used rather different methods, under a single systematic and conceptual approach.

In this paper, we take $\mathbb{N} = \{1, 2, \ldots\}$.

Acknowledgements. The author is grateful to Chris Phillips for a number of helpful conversations regarding averaging processes. He also thanks Hannes Thiel for conversations on the Cuntz semigroup and local approximations. Finally, he wishes to thank Juan Pablo Lago for his support and encouragement.

2. The Rokhlin property and an averaging process

2.1. Definition and basic properties. We begin introducing some useful notation and terminology.

**Definition 2.1.** Let $A$ be a unital $C^*$-algebra. Let $\ell^\infty(\mathbb{N}, A)$ denote the set of all bounded sequences $(a_n)_{n \in \mathbb{N}}$ in $A$ with the supremum norm and pointwise operations. Then $\ell^\infty(\mathbb{N}, A)$ is a unital $C^*$-algebra, the unit being the constant sequence 1. Let $c_0(\mathbb{N}, A) = \{ (a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) : \lim_{n \to \infty} \|a_n\| = 0 \}$. Then $c_0(\mathbb{N}, A)$ is an ideal in $\ell^\infty(\mathbb{N}, A)$, and we denote the quotient $\ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ by $A_\infty$. We write $k_A : \ell^\infty(\mathbb{N}, A) \to A_\infty$ for the quotient map. We identify $A$ with the unital subalgebra of $\ell^\infty(\mathbb{N}, A)$ consisting of the constant sequences, and with a unital subalgebra of $A_\infty$ by taking its image under $k_A$. We write $A_\infty \cap A'$ for the relative commutant of $A$ inside of $A_\infty$.

If $\alpha : G \to \text{Aut}(A)$ is an action of $G$ on $A$, there are (not necessarily continuous) actions of $G$ on $A_\infty$ and on $A_\infty \cap A'$, both denoted by $\alpha_\infty$. We set

$$\ell^\infty_\alpha(\mathbb{N}, A) = \{ a \in \ell^\infty(\mathbb{N}, A) : g \mapsto (\alpha_\infty)g(a) \text{ is continuous} \},$$

and $A_{\infty, \alpha} = k_A(\ell^\infty_\alpha(\mathbb{N}, A))$. By construction, $A_{\infty, \alpha}$ is invariant under $\alpha_\infty$, and the restriction of $\alpha_\infty$ to $A_{\infty, \alpha}$, which we also denote by $\alpha_\infty$, is continuous.

If $G$ is a locally compact group, we denote by $Lt : G \to \text{Aut}(C_0(G))$ the action induced by left translation of $G$ on itself.

The following is essentially Definition 3.2 of [15].

**Definition 2.2.** Let $A$ be a unital $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha : G \to \text{Aut}(A)$ be a continuous action. We say that $\alpha$ has the Rokhlin property if there is an equivariant unital homomorphism

$$\varphi : (C(G), Lt) \to (A_{\infty, \alpha}, \alpha_\infty).$$
Remark 2.3. Definition 2.2 is formally weaker than Definition 3.2 in [11], since we do not require the map $\varphi$ to be injective. However, this condition is automatic: the kernel of $\varphi$ is a translation invariant ideal in $C(G)$, so it must be either $\{0\}$ or all of $C(G)$. It follows that the two notions are in fact equivalent.

Since unital completely positive maps of order zero are necessarily homomorphisms, it is easy to see that the Rokhlin property for a compact group action agrees with Rokhlin dimension zero in the sense of Definition 3.2 in [11]. In particular, the following is a consequence of Theorem 3.8 in [11].

Theorem 2.4. Let $A$ be a unital $C^*$-algebra let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be a continuous action of $G$ on $A$.

1. Let $B$ be a unital $C^*$-algebra, and let $\beta: G \to \text{Aut}(B)$ be a continuous action of $G$ on $B$. Let $A \otimes B$ be any $C^*$-algebra completion of the algebraic tensor product of $A$ and $B$ for which the tensor product action $g \mapsto (\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$ is defined. If $\alpha$ has the Rokhlin property, then so does $\alpha \otimes \beta$.

2. Let $I$ be an $\alpha$-invariant ideal in $A$, and denote by $\overline{\alpha}: G \to \text{Aut}(A/I)$ the induced action on $A/I$. If $\alpha$ has the Rokhlin property, then so does $\overline{\alpha}$.

Furthermore,

3. Let $(A_n, \iota_n)_{n \in \mathbb{N}}$ be a direct system of unital $C^*$-algebras with unital connecting maps, and for each $n \in \mathbb{N}$, let $\alpha^{(n)}: G \to \text{Aut}(A_n)$ be a continuous action such that $\iota_n \circ \alpha^{(n)} = \alpha^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $g \in G$. Suppose that $A = \varprojlim A_n$ and that $\alpha = \varprojlim \alpha^{(n)}$. If $\alpha^{(n)}$ has the Rokhlin property for infinitely many values of $n$, then $\alpha$ has the Rokhlin property as well.

It is not in general the case that the Rokhlin property for compact group actions is preserved by restricting to a closed subgroup. The reader is referred to Section 3.1 in [11] for a discussion about the interaction between Rokhlin dimension and restriction to closed subgroups.

The Rokhlin property is nevertheless preserved by passing to a subgroup in some special cases.

Proposition 2.5. Let $A$ be a unital $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. Let $H$ be a closed subgroup of $G$, and assume that at least one of the following holds:

1. The coset space $G/H$ is zero dimensional (for example, $[G : H] < \infty$).
2. We have $G = \prod_{i \in I} G_i$ or $G = \bigoplus_{i \in I} G_i$, and $H = G_j$ for some $j \in I$.
3. The subgroup $H$ is the connected component of $G$ containing its unit.

Then $\alpha|_H: H \to \text{Aut}(A)$ has the Rokhlin property.

Proof. This follows immediately from Proposition 3.11 in [11].

We now turn to dual actions. Recall that the Pontryagin dual of a compact abelian group is a discrete (abelian) group.

Definition 2.6. Let $\Gamma$ be a discrete abelian group, let $B$ be a $C^*$-algebra, and let $\beta: \Gamma \to \text{Aut}(B)$ be an action. We say that $\beta$ is approximately representable if there exists a unitary representation $u: \Gamma \to \mathcal{U}((M(B)\beta)_{\infty})$ such that

$$\beta_u(b) = u^*_s b u_s$$
for every $s \in \Gamma$ and every $b \in B$.

The proof of the following theorem is analogous to the proofs of Lemma 3.8 in [16], and Propositions 3.6 and 3.7 in [7], so we leave it to the reader.

**Theorem 2.7.** Let $A$ be a separable unital $C^*$-algebra, let $G$ be a second-countable compact abelian group, and set $\Gamma = \hat{G}$.

1. Let $\alpha: G \to \text{Aut}(A)$ be a strongly continuous action. Then $\alpha$ has the Rokhlin property if and only if its dual action $\hat{\alpha}: \Gamma \to \text{Aut}(A \rtimes_\alpha G)$ is approximately representable.

2. Let $\beta: \Gamma \to \text{Aut}(A)$ be an action. Then $\beta$ is approximately representable if and only if its dual action $\hat{\beta}: G \to \text{Aut}(A \rtimes_\beta \Gamma)$ has the Rokhlin property.

In particular, it follows from Theorem 2.7 that dual (and predual, whenever it exists) actions of actions with the Rokhlin property are approximately inner.

2.2. **Examples.** Compact group actions with the Rokhlin property are rare (and they seem to be even less common if the group is connected). In a forthcoming paper, we will show that there are many $C^*$-algebras of interest that do not admit any non-trivial compact group action with the Rokhlin property (such as the Cuntz algebra $\mathcal{O}_\infty$ and the Jiang-Su algebra $\mathcal{Z}$; see [14] for a stronger statement valid for compact Lie groups), while there are many $C^*$-algebras that only admit actions with the Rokhlin property of totally disconnected compact groups, such as the Cuntz algebras $\mathcal{O}_n$ for $n \geq 3$, UHF-algebras, AF-algebras, AI-algebras, etc. See [9] and [7] for some non-existence results of circle actions with the Rokhlin property.

Here, we shall construct a family of examples of compact group actions with the Rokhlin property on certain AH-algebras and on certain Kirchberg algebras, including $O_2$.

Given a second-countable compact group $G$, it is an easy exercise to check that the action $L_t: G \to \text{Aut}(C(G))$ has the Rokhlin property.

**Example 2.8.** Let $G$ be a second-countable compact group. For $n \in \mathbb{N}$, set $A_n = C(G) \otimes M_{2^n}$. Set $\alpha^{(n)} = L_t \otimes \text{id}_{M_{2^n}}: G \to \text{Aut}(A_n)$. Then $\alpha^{(n)}$ has the Rokhlin property by part (1) of Theorem 2.4. Fix a countable subset $X = \{x_1, x_2, x_3, \ldots\} \subseteq G$ such that $\{x_m, x_{m+1}, \ldots\}$ is dense in $G$ for all $m \in \mathbb{N}$. Given $n \in \mathbb{N}$, define a map $\iota_n : A_n \to A_{n+1}$ by

$$
\iota_n(f) = \begin{pmatrix}
  f & 0 \\
  0 & L_{t_{x_n}}(f)
\end{pmatrix}
$$

for every $f$ in $A_n$. Then $\iota_n$ is unital and injective. The direct limit $A = \lim_{\to} (A_n, \iota_n)$ is clearly a unital AH-algebra, and it is simple by Proposition 2.1 in [2].

It is easy to check that

$$
\iota_n \circ \alpha^{(n)}_g = \alpha^{(n+1)}_g \circ \iota_n
$$

for all $n \in \mathbb{N}$ and all $g \in G$, and hence $(\alpha^{(n)})_{n \in \mathbb{N}}$ induces a direct limit action $\alpha = \lim_{\to} \alpha^{(n)}$ of $G$ on $A$. Then $\alpha$ has the Rokhlin property by part (3) of Theorem 2.4.

In the example above, the $2^\infty$ UHF-pattern can be replaced by any other UHF or (simple) AF-pattern, and the resulting $C^*$-algebra is also a (simple) AH-algebra. If the group is a torus, then the direct limit algebra will be a simple $\mathcal{AT}$-algebra by Lin's classification results.
If the group is totally disconnected, the direct limit algebra will be an AF-algebra. For non-trivial groups, these AF-algebras will nevertheless not be UHF-algebras, even if a UHF-pattern is followed.

**Example 2.9.** Given a second-countable compact group $G$, let $A$ and $\alpha$ be as in Example 2.8. Then $$\alpha \otimes \text{id}_{O_\infty} : G \to \text{Aut}(A \otimes O_\infty)$$ has the Rokhlin property by part (1) of Theorem 2.4, and $A \otimes O_\infty$ is a Kirchberg algebra. One can obtain actions of $G$ on other Kirchberg algebras by following a different UHF or AF pattern in Example 2.8.

Using the absorption properties of $O_2$, we can construct an action of the circle on $O_2$ with the Rokhlin property.

**Example 2.10.** Let $G$ be a second-countable compact group, and let $A$ and $\alpha$ be as in Example 2.8. Use Theorem 3.8 in [18] to choose an isomorphism $\varphi : A \otimes O_2 \to O_2$, and define an action $\beta : G \to \text{Aut}(O_2)$ by $\beta_g = \varphi \circ (\alpha_g \otimes \text{id}_{O_2}) \circ \varphi^{-1}$ for $g \in G$. Then $\beta$ has the Rokhlin property by part (1) of Theorem 2.4.

More generally, the action constructed in Example 2.8 can be used to construct an action of $G$ on any $O_2$-absorbing $C^*$-algebra.

We mention, without proof, that methods similar to those used in [9] yield the following.

**Theorem 2.11.** Let $G$ be a second-countable locally compact group and let $A$ be a separable unital $C^*$-algebra such that $A \otimes O_2 \cong A$. Then the set of all actions of $G$ on the Rokhlin property is a dense $G_\delta$-set in the metric space $\text{Act}_G(A)$ of all strongly continuous actions of $G$ on $A$.

**2.3. First results on crossed product and the averaging process.** The following observation will be used repeatedly throughout this paper.

**Remark 2.12.** It is easy to check that actions with the Rokhlin property are saturated in the sense of Rieffel (see Definition 7.1.4 in [28]). In particular, the fixed point algebra and the crossed product by a compact group action with the Rokhlin property are Morita equivalent, and thus stably isomorphic whenever the original algebra is separable.

Combining the above observation with results from [10], we obtain estimates of the nuclear dimension and decomposition rank of crossed products by Rokhlin actions.

**Theorem 2.13.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha : G \to \text{Aut}(A)$ be an action with the Rokhlin property. Then

$$\dim_{\text{nuc}}(A^\alpha) = \dim_{\text{nuc}}(A \rtimes_\alpha G) \leq \dim_{\text{nuc}}(A), \quad \text{and}$$

$$\text{dr}(A^\alpha) = \text{dr}(A \rtimes_\alpha G) \leq \text{dr}(A).$$

**Proof.** The proof follows immediately from Theorem 3.3 and Theorem 3.4 in [10], since $\dim_{\text{Rok}}(\alpha) = 0$ (see also Remark 3.5 in [10]).

**Corollary 2.14.** Let $A$ be a unital AF-algebra, let $G$ be a second-countable compact group, and let $\alpha : G \to \text{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ and $A \rtimes_\alpha G$ are AF-algebras.
Rokhlin property. Then \(\alpha\) be a second-countable compact group, and let \(\alpha\) be a separable \(C^*\)-algebra with the Rokhlin property. Given a compact subset \(F \subseteq A\), a compact subset \(F_2 \subseteq A\alpha\), and \(\varepsilon > 0\), there exists a unital completely positive linear map \(\psi: A \to A\alpha\) such that

1. For all \(a, b \in F_1\), we have \(\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon\);
2. For all \(a \in F_2\), we have \(\|\psi(a) - a\| < \varepsilon\).

In other words, there exists an approximate homomorphism \((\psi_n)_{n \in \mathbb{N}}\) consisting of unital, completely positive linear maps \(\psi_n: A \to A\alpha\) for \(n \in \mathbb{N}\), such that \(\lim_{n \to \infty} \|\psi_n(a) - a\| = 0\) for all \(a \in A\alpha\).

Proof. Without loss of generality, we may assume that \(\|a\| \leq 1\) for all \(a \in F_1 \cup F_2\). For the compact set \(F = F_1 \cup F_2\), and the tolerance \(\varepsilon_0 = \frac{\varepsilon}{4}\), use Proposition 4.3 in [10] to find a positive number \(\delta > 0\), a finite subset \(K \subseteq G\), a partition of unity \((f_k)_{k \in K}\) of \(C(G)\), and a unital completely positive map \(\varphi: C(G) \to A\alpha\), such that

- \((a)\) If \(g\) and \(g'\) in \(G\) satisfy \(d(g, g') < \delta\), then \(\|\alpha_g(a) - \alpha_{g'}(a)\| < \varepsilon_0\) for all \(a \in F\).
- \((b)\) Whenever \(k\) and \(k'\) in \(K\) satisfy \(f_k f_{k'} \neq 0\), then \(d(k, k') < \delta\).
- \((c)\) For every \(g \in G\) and for every \(a \in F\), we have
  \[\|\alpha_g \left( \sum_{k \in K} \varphi(f_k)^{1/2} \varphi_k(a) \varphi(f_k)^{1/2} \right) - \sum_{k \in K} \varphi(f_k)^{1/2} \alpha_k(a) \varphi(f_k)^{1/2}\| < \varepsilon_0.\]
- \((d)\) For every \(a \in F\) and for every \(k \in K\), we have
  \[\|a \varphi(f_k) - \varphi(f_k) a\| < \frac{\varepsilon_0}{|K|}\] and \(\|a \varphi(f_k)^{1/2} - \varphi(f_k)^{1/2} a\| < \frac{\varepsilon_0}{|K|}\).
- \((e)\) Whenever \(k\) and \(k'\) in \(K\) satisfy \(f_k f_{k'} = 0\), then
  \[\|\varphi(f_k)^{1/2} \varphi(f_{k'})^{1/2}\| < \frac{\varepsilon_0}{|K|}.\]

Define a linear map \(\psi: A \to A\alpha\) by

\[\psi(a) = E \left( \sum_{k \in K} \varphi(f_k)^{1/2} \alpha_k(a) \varphi(f_k)^{1/2} \right).\]
for all $a \in A$. We claim that $\psi$ has the desired properties.

It is clear that $\psi$ is unital and completely positive. It follows from condition (c) above that

$$\|\psi(a) - \sum_{k \in K} \varphi(f_k)^{1/2} \alpha_k(a) \varphi(f_k)^{1/2}\| < \varepsilon$$

for all $a \in F$. We proceed to check that conditions (1) and (2) in the statement are satisfied.

Given $a, b \in F_1$, we use condition (c) at the second and fifth step (in the form of the observation above), conditions (a), (b), (d) and (e) at the third step, and the fact that $\varphi$ is unital and $(f_k)_{k \in K}$ is a partition of unity of $C(G)$ at the fourth step, to get

$$\psi(a)\psi(b) = E \left( \sum_{k \in K} \varphi(f_k)^{1/2} \alpha_k(a) \varphi(f_k)^{1/2} \right) E \left( \sum_{k' \in K} \varphi(f_{k'})^{1/2} \alpha_{k'}(b) \varphi(f_{k'})^{1/2} \right)$$

$$\approx \varepsilon_0 E \sum_{k \in K} \sum_{k' \in K} \varphi(f_k)^{1/2} \alpha_k(a) \varphi(f_k)^{1/2} \varphi(f_{k'})^{1/2} \alpha_{k'}(b) \varphi(f_{k'})^{1/2}$$

$$\approx \varepsilon_0 E \sum_{k \in K} \sum_{k' \in K} \varphi(f_k)^{1/2} \alpha_k(ab) \varphi(f_k)^{1/2}$$

$$= \sum_{k \in K} \varphi(f_k)^{1/2} \alpha_k(ab) \varphi(f_k)^{1/2}$$

$$\approx \varepsilon_0 E \left( \sum_{k \in K} \varphi(f_k)^{1/2} \alpha_k(ab) \varphi(f_k)^{1/2} \right)$$

$$= \psi(ab).$$

Hence $\|\psi(ab) - \psi(a)\psi(b)\| < 6\varepsilon_0 = \varepsilon$, and condition (1) is proved. For the second condition, let $a \in F_2 \subseteq A^{\alpha}$. Then

$$\psi(a) = E \left( \sum_{k \in K} \varphi(f_k)^{1/2} a \varphi(f_k)^{1/2} \right)$$

$$\approx \varepsilon_0 E \left( \sum_{k \in K} \varphi(f_k) a \right)$$

$$= E \left( \sum_{k \in K} \varphi(f_k) \right) a = a.$$ 

Thus $\|\psi(a) - a\| < \varepsilon$ for all $a \in F_2$, and the proof is complete. \qed

In some situations, we will need maps $A \to A^{\alpha}$ which leave $A^{\alpha}$ fixed, not just asymptotically fixed. The following variant of Theorem 2.16 will be convenient in those cases.

**Theorem 2.17.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. Given a compact subset $F \subseteq A$ and $\varepsilon > 0$, there exists a unital linear map $\psi: A \to A^{\alpha}$ such that

1. For all $a, b \in F$, we have

$$\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon \quad \text{and} \quad \|\psi(a^*) - \psi(a)^*\| < \varepsilon;$$
(2) We have $\psi(a) = a$ for all $a \in A^\alpha$.

In other words, there exists an approximate homomorphism $(\psi_n)_{n \in \mathbb{N}}$ consisting of unital linear maps $\psi_n: A \to A^\alpha$ for $n \in \mathbb{N}$, such that $\psi_n(a) = a$ for all $a \in A^\alpha$.

**Proof.** The argument is similar to the one exhibited in the proof of [Theorem 2.16](#), so we only give a sketch of it.

Without loss of generality, we may assume that $\|a\| \leq 1$ for all $a \in F$. For the compact set $F$ and the tolerance $\varepsilon_0 = \frac{1}{8}$, use Proposition 4.3 in [10] to find a positive number $\delta > 0$, a finite subset $K \subseteq G$, a partition of unity $(f_k)_{k \in K}$ of $C(G)$, and a unital completely positive map $\varphi: C(G) \to A$, such that conditions (a) through (e) listed in the proof of [Theorem 2.16](#) are satisfied with these choices. Define a unital linear map $\psi: A \to A^\alpha$ by

$$\psi(a) = E \left( \sum_{k \in K} \varphi(f_k) \alpha_k(a) \right)$$

for all $a \in A$. We claim that $\psi$ has the desired properties. It is immediate that $\psi(a) = a$ for all $a \in A^\alpha$, using the properties of the conditional expectation $E$. Condition (1) in the statement can be checked using similar arguments as in [Theorem 2.16](#). We leave the details to the reader. \hfill \Box

Our first application of [Theorem 2.16](#) is to the maps induced by the canonical inclusion $A^\alpha \hookrightarrow A$ at the level of $K$-theory ([Theorem 2.18](#)) and Cuntz semigroup ([Theorem 2.21](#)). In [Corollary 2.22](#) we relate the $K$-theory and Cuntz semigroup of the crossed product to that of the original algebra.

**Theorem 2.18.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. Then the canonical inclusion $\iota: A^\alpha \to A$ induces an injective map $K_*(\iota): K_*(A^\alpha) \to K_*(A)$.

**Proof.** The result for $K_1$ follow from the result for $K_0$, by tensoring with any unital nuclear $C^*$-algebra $B$ satisfying the UCT with $K$-theory $(0, \mathbb{Z})$, and using the Künneth formula. We will therefore only prove the theorem for $K_0$.

Since the action $\alpha \otimes \text{id}_{M_n}$ of $G$ on $M_n(A)$ has the Rokhlin property for every $n$ in $\mathbb{N}$ by part (1) of [Theorem 2.4](#), it is enough to show that if $p$ and $q$ are two projections in $A^\alpha$ that are Murray-von Neumann equivalent in $A$, then they are Murray-von Neumann equivalent in $A^\alpha$. To prove this, choose a partial isometry $s \in A$ with $s^* s = p$ and $ss^* = q$. Use [Theorem 2.16](#) to find a unital completely positive map $\psi: A \to A^\alpha$ such that, in particular,

$$\|\psi(s)^* \psi(s) - p\| < 1, \quad \|\psi(s) \psi(s)^* - q\| < 1.$$

By Lemma 2.5.3 in [22], there exists a partial isometry $t$ in $A^\alpha$ such that $t^* t = p$ and $tt^* = q$. This finishes the proof. \hfill \Box

We need to briefly review the definition of the Cuntz semigroup. The material to be recalled, and much more, can be found in [3].

**Definition 2.19.** Let $A$ be a $C^*$-algebra and let $a$ and $b$ be positive elements in $A$. We say that $a$ is **Cuntz subequivalent** to $b$, and denote this by $a \preceq b$, if there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $A$ such that

$$\lim_{n \to \infty} \|x_n^* b x_n - a\| = 0.$$
We say that \( a \) is Cuntz equivalent to \( b \), and denote this by \( a \sim b \), if \( a \preceq b \) and \( b \preceq a \).

It can be shown that \( \preceq \) is a preorder on the set of positive elements of \( A \), from which it follows that \( \sim \) is an equivalence relation. We denote by \([a]\) the Cuntz equivalence class of the positive element \( a \in A \).

**Definition 2.20.** The Cuntz semigroup of a \( C^* \)-algebra \( A \), denoted by \( \text{Cu}(A) \), is defined to be the set of Cuntz equivalence classes of positive elements of \( A \otimes \mathcal{K} \). Addition in \( \text{Cu}(A) \) is given by direct sum. Moreover, \( \text{Cu}(A) \) becomes an ordered semigroup when equipped with the order given by \([a] \leq [b] \) if \( a \preceq b \), for \( a, b \in (A \otimes \mathcal{K})_+ \).

If \( A \) and \( B \) are \( C^* \)-algebras, any homomorphism \( \varphi: A \to B \) induces an order-preserving map \( \text{Cu}(\varphi): \text{Cu}(A) \to \text{Cu}(B) \), given by

\[
\text{Cu}(\varphi)([a]) = ([\varphi \otimes \text{id}_\mathcal{K}](a))
\]

for every positive element \( a \in A \).

**Theorem 2.21.** Let \( A \) be a unital, separable \( C^* \)-algebra, let \( G \) be a second-countable compact group, and let \( \alpha: G \to \text{Aut}(A) \) be an action with the Rokhlin property. Then the canonical inclusion \( \iota: A^0 \to A \) induces an order-embedding \( \text{Cu}(\iota): \text{Cu}(A^0) \to \text{Cu}(A) \).

**Proof.** Let \( x \) and \( y \) be positive elements in \( A^0 \otimes \mathcal{K} \) such that \( x \leq y \) in \( A \otimes \mathcal{K} \). By Rørdam’s Lemma (Proposition 2.4 in [32]), given \( \varepsilon > 0 \) there exist \( k \in \mathbb{N} \), a positive number \( \delta > 0 \) and \( s \in A \otimes M_k \) such that \( (x - \varepsilon)_+ = s^*s \) and \( ss^* \) belongs to the hereditary subalgebra of \( A \otimes M_k \) generated by \( (y - \delta)_+ \). Note that the action \( \alpha \otimes \text{id}_{M_k} \) of \( G \) on \( A \otimes M_k \) has the Rokhlin property by part (1) of Theorem 2.4 and that \( M_k(A)^{\alpha \otimes \text{id}_{M_k}} \) can be canonically identified with \( M_k(A^\alpha) \). Let \( (\psi_n)_{n \in \mathbb{N}} \) be a sequence of unital completely positive maps \( A \to A^\alpha \) as in the conclusion of Theorem 2.16. For \( n \in \mathbb{N} \), we denote by \( \psi_n^{(k)}: M_k(A) \to M_k(A^\alpha) \) the tensor product of \( \psi_n \) with \( \text{id}_{M_k} \). Since \( ss^* = (x - \varepsilon)_+ \), we have

\[
\lim_{n \to \infty} \left\| \psi_n^{(k)}(s^*\psi_n^{(k)}(s) - (x - \varepsilon)_+) \right\| = 0.
\]

We can therefore find a sequence \( (t_m)_{m \in \mathbb{N}} \) in \( M_k(A^\alpha) \) such that

\[
\| t_m^*t_m - (x - \varepsilon)_+ \| < \frac{1}{m} \quad \text{and} \quad \| t_m^*t_m - ss^* \| < \frac{1}{m}
\]

for all \( m \in \mathbb{N} \). We deduce that

\[
\left( x - \varepsilon - \frac{1}{m} \right)_+ \leq [t_m^*t_m] = [t_m^*t_m]
\]

in \( \text{Cu}(A^\alpha) \). Taking limits as \( m \to \infty \), and using Rørdam’s Lemma again, we conclude that

\[
[x - \varepsilon]_+ \leq [ss^*] \leq [y]
\]

in \( \text{Cu}(A^\alpha) \). Since \( \varepsilon > 0 \) is arbitrary, this implies that \( x \leq y \) in \( \text{Cu}(A^\alpha) \), as desired. This finishes the proof. \( \square \)

**Corollary 2.22.** Let \( A \) be a unital, separable \( C^* \)-algebra, let \( G \) be a second-countable compact group, and let \( \alpha: G \to \text{Aut}(A) \) be an action with the Rokhlin property.
(1) There is a canonical identification of $K_*(A \rtimes_\alpha G)$ with an order-subgroup of $K_*(A)$;
(2) There is a canonical identification of $\text{Cu}(A \rtimes_\alpha G)$ with a sub-semigroup of $\text{Cu}(A)$.

Proof. Recall that two Morita equivalent separable $C^*$-algebras have canonically isomorphic $K$-groups and Cuntz semigroup. The result then follows from Remark 2.12 together with Theorem 2.18 or Theorem 2.21. □

Recall that an ordered semigroup $S$ is said to be almost unperforated if for every $x, y \in S$, if there exists $n \in \mathbb{N}$ such that $(n+1)x \leq ny$, then $x \leq y$. Since almost unperforation passes to sub-semigroups (with the induced order), the following is immediate.

**Corollary 2.23.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. If $\text{Cu}(A)$ is almost unperforated, then so are $\text{Cu}(A \rtimes_\alpha G)$ and $\text{Cu}(A \rtimes_\alpha G)$.

Under the presence of the Rokhlin property, the ideal structure of crossed products and fixed point algebras can be completely determined: ideals are induced by invariant ideals in the original algebra.

**Theorem 2.24.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property.

(1) If $I$ is an ideal in $A^\alpha$, then there exists an $\alpha$-invariant ideal $J$ in $A$ such that $I = J \cap A^\alpha$.
(2) If $I$ is an ideal in $A \rtimes_\alpha G$, then there exists an $\alpha$-invariant ideal $J$ in $A$ such that $I = J \rtimes_\alpha G$.

Proof. (1). Let $I$ be an ideal in $A^\alpha$. Then $J = AIA$ is an $\alpha$-invariant ideal in $A$. We claim that $J \cap A^\alpha = I$. It is clear that $I \subseteq J \cap A^\alpha$. For the reverse inclusion, let it is enough to show that if $a_1, \ldots, a_m, b_1, \ldots, b_m$ belong to $A$, $x_1, \ldots, x_m$ belong to $I$, and $\sum_{j=1}^m a_j x_j b_j$ belongs to $A^\alpha$, then $\sum_{j=1}^m a_j x_j b_j$ is the limit of elements in $I$.

Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n : A \to A^\alpha$ as in the conclusion of Theorem 2.16. Then

$$\sum_{j=1}^m a_j x_j b_j = \lim_{n \to \infty} \psi_n \left( \sum_{j=1}^m a_j x_j b_j \right) = \lim_{n \to \infty} \sum_{j=1}^m \psi_n(b_j) x_j \psi_n(a_j).$$

Since $\sum_{j=1}^m \psi_n(b_j) x_j \psi_n(a_j)$ belongs to $I$, the claim is proved.

(2). This follows from (1) together with the fact that $\alpha$ is saturated (see Remark 2.12). We omit the details. □

**Corollary 2.25.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. If $A$ is simple, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

In particular, we deduce that strict comparison of positive elements is preserved under formation of crossed products and passage to fixed point algebras by the actions here considered.
**Corollary 2.26.** Let \( A \) be a simple, unital, separable \( C^* \)-algebra, let \( G \) be a second-countable compact group, and let \( \alpha : G \to \text{Aut}(A) \) be an action with the Rokhlin property. If \( A \) has strict comparison of positive elements, then so do \( A^\alpha \) and \( A \rtimes \alpha G \).

**Proof.** By Lemma 6.1 in \cite{35}, strict comparison of positive elements for a simple \( C^* \)-algebra is equivalent to almost unperforation of its Cuntz semigroup. Since \( A^\alpha \) and \( A \rtimes \alpha G \) are simple by Corollary 2.25, the result follows from Corollary 2.23. \( \square \)

3. Structure of crossed products and fixed point algebras

3.1. Generalized local approximations. We now turn to the study of preservation of other structural properties that have been studied in the context of Elliott’s classification program. In order to provide a conceptual approach, it will be necessary to introduce some convenient terminology.

**Definition 3.1.** Let \( \mathcal{C} \) be a class of separable \( C^* \)-algebras and let \( A \) be a \( C^* \)-algebra.

1. We say that \( A \) is an (unital) approximate \( \mathcal{C} \)-algebra, if \( A \) is isomorphic to a direct limit of \( C^* \)-algebras in \( \mathcal{C} \) (with unital connecting maps).
2. We say that \( A \) is a (unital) local \( \mathcal{C} \)-algebra, if for every finite subset \( F \subseteq A \) and for every \( \varepsilon > 0 \), there exist a \( C^* \)-algebra \( B \) in \( \mathcal{C} \) and a not necessarily injective (unital) homomorphism \( \varphi : B \to A \) such that \( \text{dist}(a, \varphi(B)) < \varepsilon \) for all \( a \in F \).
3. We say that \( A \) is a generalized (unital) local \( \mathcal{C} \)-algebra, if for every finite subset \( F \subseteq A \) and for every \( \varepsilon > 0 \), there exist a \( C^* \)-algebra \( B \) in \( \mathcal{C} \) and a sequence \( (\varphi_n)_{n \in \mathbb{N}} \) of (unital) completely positive contractive maps \( \varphi_n : B \to A \) that \( \text{dist}(a, \varphi_n(B)) < \varepsilon \) for all \( a \in F \) and for all \( n \) sufficiently large.

**Remark 3.2.** The term ‘local \( \mathcal{C} \)-algebra’ is sometimes used to mean that the local approximations are realized by injective homomorphisms. For example, in \cite{34} Thiel says that a \( C^* \)-algebra \( A \) is ‘\( \mathcal{C} \)-like’, if for every finite subset \( F \subseteq A \) and for every \( \varepsilon > 0 \), there exist a \( C^* \)-algebra \( B \) in \( \mathcal{C} \) and an injective homomorphism \( \varphi : B \to A \) such that \( \text{dist}(a, \varphi(B)) < \varepsilon \) for all \( a \in F \). Finally, we point out that what we call here ‘approximate \( \mathcal{C} \)’ is called ‘\( \mathcal{A}^C \)’ in \cite{34}.

The Rokhlin property is related to the above definition in the following way. Note that the approximating maps for \( A^\alpha \) that we obtain in the proof are not necessarily injective, even if we assume that the approximating maps for \( A \) are.

**Proposition 3.3.** Let \( \mathcal{C} \) be a class of \( C^* \)-algebras, let \( A \) be a \( C^* \)-algebra, let \( G \) be a second-countable group, and let \( \alpha : G \to \text{Aut}(A) \) be an action with the Rokhlin property. If \( A \) is a (unital) local \( \mathcal{C} \)-algebra, then \( A^\alpha \) is a generalized (unital) local \( \mathcal{C} \)-algebra.

**Proof.** Let \( F \subseteq A^\alpha \) be a finite subset, and let \( \varepsilon > 0 \). Find a \( C^* \)-algebra \( B \) in \( \mathcal{C} \) and a (unital) homomorphism \( \varphi : B \to A \) such that \( \text{dist}(a, \varphi(B)) < \frac{\varepsilon}{2} \) for all \( a \in F \). Let \( (\psi_n)_{n \in \mathbb{N}} \) be a sequence of unital completely positive maps \( \psi_n : A \to A^\alpha \) as in the conclusion of Theorem 2.16. Then \( (\psi_n \circ \varphi)_{n \in \mathbb{N}} \) is a sequence of (unital) completely positive contractive maps \( B \to A^\alpha \) as in the definition of generalized local \( \mathcal{C} \)-algebra. \( \square \)
Let $C$ be a class of $C^*$-algebras. It is clear that any (unital) approximate $C$-algebra is a (unital) local $C$-algebra, and that any (unital) local $C$-algebra is a generalized (unital) local $C$-algebra.

While the converses to these implications are known to fail in general, the notions in Definition 3.1 agree under fairly mild conditions on $C$; see Proposition 3.9.

**Definition 3.4.** Let $C$ be a class of $C^*$-algebras. We say that $C$ has (unital) approximate quotients if whenever $A \in C$ and $I$ is an ideal in $A$, the quotient $A/I$ is a (unital) approximate $C$-algebra, in the sense of Definition 3.1.

The term ‘approximate quotients’ has been used in [26] with a considerably stronger meaning. Our weaker assumptions still yield an analog of Proposition 1.7 in [26], thanks to results of Thiel [34]; see Proposition 3.9.

We need to recall a definition due to Loring. The original definition appears in [24], while in Theorem 3.1 in [4] it is proved that weak semiprojectivity is equivalent to a condition that is more reminiscent of semiprojectivity. For the purposes of this paper, the original definition is better suited.

**Definition 3.5.** A $C^*$-algebra $A$ is said to be weakly semiprojective (in the unital category) if given a $C^*$-algebra $B$ and given a (unital) homomorphism $\psi: A \to B\infty$, there exists a (unital) homomorphism $\phi: A \to \ell^\infty(N, B)$ such that $\kappa_B \circ \phi = \psi$. In other words, the following lifting problem can always be solved:

$$
\begin{array}{c}
\ell^\infty(N, B) \\
\downarrow \phi \\
A \\
\downarrow \psi \\
B_{\infty}.
\end{array}
$$

The proof of the following observation is left to the reader. It states explicitly the formulation of weak semiprojectivity that will be used in our work, specifically in Proposition 3.9.

**Remark 3.6.** Using the definition of the sequence algebra $B_{\infty}$, it is easy to show that if $A$ is a weakly semiprojective $C^*$-algebra, and $(\psi_n)_{n \in \mathbb{N}}$ is an asymptotically $*$-multiplicative sequence of linear maps $\psi_n: A \to B$ from $A$ to another $C^*$-algebra $B$, then there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of homomorphisms $\varphi_n: A \to B$ such that

$$
\lim_{n \to \infty} \| \varphi_n(a) - \psi_n(a) \| = 0
$$

for all $a \in A$. If each $\psi_n$ is unital and $A$ is weakly semiprojective in the unital category, then $\varphi_n$ can also be chosen to be unital.

We proceed to give some examples of classes of $C^*$-algebras that will be used in Theorem 3.10. We need a definition first, which appears as Definition 11.2 in [5].

**Definition 3.7.** A $C^*$-algebra $A$ is said to be one-dimensional noncommutative cellular complex, or one-dimensional NCCW-complex for short, if there exist finite dimensional $C^*$-algebras $E$ and $F$, and unital homomorphisms $\varphi, \psi: E \to F$, such that $A$ is isomorphic to the pull back $C^*$-algebra

$$
\{(a, b) \in E \oplus C([0, 1], F): b(0) = \varphi(a) \text{ and } b(1) = \psi(a)\}.
$$

It was shown in Theorem 6.2.2 of [5] that one-dimensional NCCW-complexes are semiprojective (in the unital category).
Examples 3.8. The following are examples of classes of weakly semiprojective $C^*$-algebras (in the unital category) which have approximate quotients.

1. The class $C$ of matrix algebras. The (unital) approximate $C^*$-algebras are precisely the matroid algebras (UHF-algebras).
2. The class $C$ of finite dimensional $C^*$-algebras. The (unital) approximate $C^*$-algebras are precisely the (unital) AF-algebras.
3. The class $C$ of interval algebras, this is, algebras of the form $C([0,1]) \otimes F$, where $F$ is a finite dimensional $C^*$-algebra. The (unital) approximate $C^*$-algebras are precisely the (unital) AI-algebras.
4. The class $C$ of circle algebras, this is, algebras of the form $C(T) \otimes F$, where $F$ is a finite dimensional $C^*$-algebra. The (unital) approximate $C^*$-algebras are precisely the (unital) AT-algebras.
5. The class $C$ of one-dimensional NCCW-complexes. We point out that certain approximate $C^*$-algebras have been classified, in terms of a variant of their Cuntz semigroup, by Robert in [31].

The following result is well-known for several particular classes.

Proposition 3.9. Let $C$ be a class of separable $C^*$-algebras which has (unital) approximate quotients (see Definition 3.4). Assume that the $C^*$-algebras in $C$ are weakly semiprojective (in the unital category). For a separable (unital) $C^*$-algebra $A$, the following are equivalent:

1. $A$ is an (unital) approximate $C^*$-algebra;
2. $A$ is a (unital) local $C^*$-algebra;
3. $A$ is a generalized (unital) local $C^*$-algebra.

Proof. The implications (1) ⇒ (2) ⇒ (3) are true in full generality. Weak semiprojectivity of the algebras in $C$ implies that any generalized local approximation by $C^*$-algebras in $C$ can be perturbed to a genuine local approximation by $C^*$-algebras in $C$ (see Remark 3.6), showing (3) ⇒ (2).

For the implication (2) ⇒ (1), note that since $C$ has approximate quotients, every a local $C$-algebra is AC-like, in the sense of Definition 3.2 in [34] (see also Paragraph 3.6 there). It then follows from Theorem 3.9 in [34] that $A$ is an approximate $C$-algebra.

For the unital case, one uses Remark 3.6 to show that (3) ⇒ (2) when units are considered. Moreover, one checks that in the proof of Theorem 3.9 in [34], if one assumes that the building blocks are weakly semiprojective in the unital category, then the conclusion is that a unital AC-like algebra is a unital AC$^*$-algebra. We omit the details. □

The following is the main application of our approximations results.

Theorem 3.10. Let $C$ be a class of separable weakly semiprojective $C^*$-algebras (in the unital category), and assume that $C$ has (unital) approximate quotients. Let $A$ be a (unital) local $C$-algebra, let $G$ be a second-countable group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. Then $A$ is a (unital) approximate $C$-algebra.

Proof. This is an immediate consequence of Proposition 3.3 and Proposition 3.9. □
Theorem 3.10 allows for far more flexibility than Theorem 3.5 in [26], since we do not assume our classes of $C^*$-algebras to be closed under direct sums or by taking corners, nor do we assume that our algebras are semiprojective. In particular, the class $C$ of weakly semiprojective Kirchberg algebras satisfies the assumptions of Theorem 3.10 but appears not to fit into the framework of flexible classes discussed in [26].

Corollary 3.11. Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha : G \to \text{Aut}(A)$ be an action with the Rokhlin property.

(1) If $A$ is a UHF-algebra, then $A^\alpha$ is a UHF-algebra and $A \rtimes_\alpha G$ is a matroid algebra. If $G$ is finite, then $A \rtimes_\alpha G$ is also a UHF-algebra.

(2) If $A$ is an AF-algebra, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

(3) If $A$ is an AI-algebra, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

(4) If $A$ is an AT-algebra, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

(5) If $A$ is a direct limit of one-dimensional NCCW-complexes, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

Proof. Since the classes in Examples 3.8 have approximate quotients and contain only weakly semiprojective $C^*$-algebras, the claims follow from Theorem 3.10. □

See Corollary 2.14 for an alternative proof of (2) above.

3.2. Further structure results. We now turn to preservation of classes of $C^*$-algebras that are not necessarily defined in terms of an approximation by weakly semiprojective $C^*$-algebras. The classes we study can all be dealt with using Theorem 2.16.

The following is Definition 1.3 in [36].

Definition 3.12. A unital, separable $C^*$-algebra $D$ is said to be strongly self-absorbing, if it is infinite dimensional and the map $D \to D \otimes_{\min} D$, given by $d \mapsto d \otimes 1$ for $d \in D$, is approximately unitarily equivalent to an isomorphism.

It is a consequence of a result of Effros and Rosenberg that strongly self-absorbing $C^*$-algebras are nuclear, so that the choice of the tensor product in the definition above is irrelevant. The only known examples of strongly self-absorbing $C^*$-algebras are the Jiang-Su algebra $Z$, the Cuntz algebras $O_2$ and $O_\infty$, UHF-algebras of infinite type, and tensor products of $O_\infty$ by such UHF-algebras. It has been conjectured that these are the only examples of strongly self-absorbing $C^*$-algebras. See [36] for the proofs of these and other results concerning strongly self-absorbing $C^*$-algebras.

The following is a useful criterion to determine when a unital, separable $C^*$-algebra absorbs a strongly self-absorbing $C^*$-algebra tensorially. The proof is a straightforward combination of Theorem 2.2 in [36] and Choi-Effros lifting theorem, and we shall omit it.

Theorem 3.13. Let $A$ be a separable, unital $C^*$-algebra, and let $D$ be a strongly self-absorbing $C^*$-algebra. Then $A$ is $D$-stable if and only if for every $\varepsilon > 0$, for every finite subset $F \subseteq A$, and every finite subset $E \subseteq D$, there exists a unital completely positive map $\varphi : D \to A$ such that

\begin{enumerate}
  \item $\|a\varphi(d) - \varphi(d)a\| < \varepsilon$ for all $a \in F$ and for all $d \in E$;
  \item $\|\varphi(de) - \varphi(d)\varphi(e)\| < \varepsilon$ for every $d, e \in E$.
\end{enumerate}
The following result was obtained as part (1) of Corollary 3.4 in [15], using completely different methods. We include a proof here to illustrate the generality of our approach.

**Theorem 3.14.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. Let $D$ be a strongly self-absorbing $C^*$-algebra and assume that $A$ is $D$-stable. Then $A^\alpha$ and $A \rtimes_\alpha G$ are $D$-stable as well.

**Proof.** Since $D$-stability is preserved under Morita equivalence by Corollary 3.2 in [36], it is enough to prove the result for $A^\alpha$.

Let $\varepsilon > 0$, and let $F \subseteq A^\alpha$ and $E \subseteq D$ be finite subsets of $A$ and $D$, respectively. Use Theorem 3.13 to choose a unital, completely positive map $\varphi: D \to A$ such that

1. $\|a\varphi(d) - \varphi(d)a\| < \varepsilon$ for all $a \in F$ and all $d \in E$;
2. $\|\varphi(de) - \varphi(d)\varphi(e)\| < \varepsilon$ for every $d, e \in E$.

Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n: A \to A^\alpha$ as in the conclusion of Theorem 2.16. Since $\lim_{n \to \infty} \psi_n(a) = a$ for all $a \in F$, we deduce that

$$\limsup_{n \to \infty} \|a\psi_n(\varphi(d)) - \psi_n(\varphi(d))a\| \leq \|a\varphi(d) - \varphi(d)a\| < \varepsilon$$

for all $a \in F$ and all $d \in E$. Likewise,

$$\limsup_{n \to \infty} \|\psi_n(\varphi(de)) - \psi_n(\varphi(d))\psi_n(\varphi(e))\| \leq \|\varphi(de) - \varphi(d)\varphi(e)\| < \varepsilon$$

for all $d, e \in E$. We conclude that for $n$ large enough, the unital completely positive map

$$\psi_n \circ \varphi: D \to A^\alpha$$

satisfies the conclusion of Theorem 3.14, showing that $A^\alpha$ is $D$-stable. \qed

Similar methods allow one to prove that the property of being approximately divisible is inherited by the crossed products and the fixed point algebra of a compact group action with the Rokhlin property. (This was first obtained by Hirshberg and Winter as part (2) of Corollary 3.4 in [15].) Our proof is completely analogous to that of Theorem 3.14 (using a suitable version of Theorem 3.13), so for the sake of brevity, we shall not present it here.

Our next goal is to show that Rokhlin actions preserve the property of having tracial rank zero in the simple, unital case.

We will need a definition of tracial rank zero. What we reproduce below is not Lin's original definition, but it follows from Proposition 3.8 in [23] the two notions are equivalent.

**Definition 3.15.** Let $A$ be a simple, unital $C^*$-algebra. We say that $A$ has tracial rank zero if for every finite subset $F \subseteq A$, for every $\varepsilon > 0$, and for every non-zero positive element $x \in A$, there exist a projection $p \in A$, a finite dimensional $C^*$-algebra $B$, and a unital homomorphism $\varphi: B \to A$, such that

1. $\|ap - pa\| < \varepsilon$ for all $a \in F$;
2. $\text{dist}(pap, \varphi(B)) < \varepsilon$ for all $a \in F$;
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$. 


We will need the following notation. For $t \in (0, \frac{1}{2})$, we denote by $f_t: [0, 1] \to [0, 1]$ the continuous function that takes the value 0 on $[0, t]$, the value 1 on $[2t, 1]$, and is linear on $[t, 2t]$.

**Theorem 3.16.** Let $A$ be a unital, separable, simple $C^\ast$-algebra of tracial rank zero, let $G$ be a second-countable compact group, and let $\alpha: G \to \text{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ is a unital, separable, simple $C^\ast$-algebra of tracial rank zero. If $G$ is finite, then the same holds for the crossed product $A \rtimes_\alpha G$.

**Proof.** Let $F \subseteq A^\alpha$ be a finite subset, let $\varepsilon > 0$ and let $x \in A^\alpha$ be a non-zero positive element. Without loss of generality, we may assume that $\|a\| \leq 1$ for all $a \in F$, and that $\varepsilon < 4$. Find $t \in (0, \frac{1}{2})$ such that $(x-t)_+$ is not zero. Set $y = (x-t)_+$. Then $y$ belongs to $A^\alpha$ and moreover $f_t(x) = yf_t(x) = y$.

Using that $A$ has tracial rank zero, find a finite dimensional $C^\ast$-algebra $B$, a unital homomorphism $\varphi: B \to A$, a projection $q \in \overline{yAy}$ and a partial isometry $s \in A$ such that

- $\|ap - pa\| < \frac{x}{4}$ for all $a \in F$;
- $\text{dist}(pap, \varphi(B)) < \frac{x}{4}$ for all $a \in F$;
- $1-p = ss^*$ and $q = ss^*$.

Since $f_t(x)$ is a unitary for $\overline{yAy}$, it follows that $q = f_t(x)qf_t(x)$. Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n: A \to A^\alpha$ as in the conclusion of Theorem 2.10. We have

(a) $\limsup_{n \to \infty} \|\psi_n(p)a - \alpha\psi_n(p)\| < \frac{x}{4}$ for all $a \in F$;
(b) $\limsup_{n \to \infty} \text{dist}(\psi_n(p)\alpha\psi_n(a), (\psi_n \circ \varphi)(B)) < \frac{x}{4}$ for all $a \in F$;
(c) $\lim_{n \to \infty} \|\psi_n(p)^*\psi_n(p) - \psi_n(p)\| = 0$;
(d) $\lim_{n \to \infty} \|1 - \psi_n(p) - \psi_n(s)^*\psi_n(s)\| = 0$;
(e) $\lim_{n \to \infty} \|\psi_n(q)^*\psi_n(q) - \psi_n(q)\| = 0$;
(f) $\lim_{n \to \infty} \|\psi_n(q) - \psi_n(s)\psi_n(s)^*\| = 0$;
(g) $\lim_{n \to \infty} \|\psi_n(q) - f_t(x)\psi_n(q)f_t(x)\| = 0$;

With $r_n = f_t(x)\psi_n(q)f_t(x)$ for $n \in \mathbb{N}$, it follows from conditions (e) and (g) that

(h) $\lim_{n \to \infty} \|r_n r_n - r_n\| = 0$.

Find $\delta_1 > 0$ such that whenever $e$ is an element in a $C^\ast$-algebra $C$ such that $\|e^*e - e\| < \delta$, then there exists a projection $f$ in $C$ such that $\|e - f\| < \frac{x}{4}$. Find $\delta_2 > 0$ such that whenever $\rho: B \to A^\alpha$ is a unital positive linear map which is $\delta_2$-multiplicative on the unit ball of $B$, there exists a homomorphism $\pi: B \to A^\alpha$ such that $\|\rho - \pi\| < \frac{x}{4}$. Set $\delta = \min\{\delta_1, \delta_2\}$.

Choose $n \in \mathbb{N}$ large enough so that the quantities in conditions (a), (b) and (g) are less than $\frac{x}{4}$, the quantities in (c) and (h) are less than $\delta$, the quantities in (d) and (f) are less than $1 - \frac{x}{4}$, and so that $\psi_n \circ \varphi$ is $\delta$-multiplicative on the unit ball of $B$. Since $r_n$ belongs to $\overline{xA^\alpha x}$ for all $n \in \mathbb{N}$, by the choice of $\delta$ there exist a projection $e$ in $\overline{xA^\alpha x}$ such that $\|e - r_n\| < \frac{x}{4}$, and a projection $f \in A^\alpha$ such that $\|f - \psi_n(p)\| < \frac{x}{4}$. Let $\pi: B \to A^\alpha$ be a homomorphism such that $\|\pi - \psi_n \circ \varphi\| < \frac{x}{4}$.

We claim that the projection $f$ and the homomorphism $\pi: B \to A^\alpha$ satisfy the conditions in Definition 3.15.
Given \(a \in F\), the estimate
\[
\|a f - fa\| \leq \|a \psi_n(p) - \psi_n(p)a\| + 2\|\psi_n(p) - f\| < \frac{3\varepsilon}{4} < \varepsilon
\]
suggests that condition (1) is satisfied. In order to check condition (2), given \(a \in F\), find \(b \in B\) with \(\|b\| \leq 1\), such that
\[
\|\psi_n(p)a \psi_n(p) - \psi_n(\phi(b))\| < \frac{\varepsilon}{4}.
\]
Then
\[
\|faf - \pi(b)\| \leq \|afa - \psi_n(p) a \psi_n(p)\| + \|\psi_n(p)a \psi_n(p) - \psi_n(\phi(b))\|
+ \|\psi_n(\phi(b)) - \pi(b)\|
< 2\|f - \psi_n(p)\| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon,
\]
so condition (2) is also satisfied. To check condition (3), it is enough to show that \(1 - f\) is Murray-von Neumann equivalent (in \(A^\alpha\)) to \(e\). We have
\[
\|(1 - f) - \psi_n(s)^* \psi_n(s)\| \leq \|f - \psi_n(p)\| + \|\psi_n(p) - \psi_n(s)^* \psi_n(s)\|
< \frac{\varepsilon}{4} + 1 - \frac{\varepsilon}{4} = 1,
\]
and likewise \(\|e - \psi_n(s) \psi_n(s)^*\| < 1\). By Lemma 2.5.3 in [22], \(1 - f\) and \(e\) are Murray-von Neumann equivalent in \(A^\alpha\), and the proof is complete. \(\square\)

We believe that a condition weaker than the Rokhlin property ought to suffice for the conclusion of Theorem 3.16. In view of Theorem 2.8 in [29], we presume that fixed point algebras by a suitable version of the tracial Rokhlin property for compact group actions would preserve the class of simple \(C^*\)-algebras with tracial rank zero.

We now turn to pure infiniteness in the non-simple case. The following is Definition 4.1 in [19].

**Definition 3.17.** A \(C^*\)-algebra \(A\) is said to be **purely infinite** if the following conditions are satisfied:

1. There are no non-zero characters (this is, homomorphisms onto the complex numbers) on \(A\), and
2. For every pair \(a, b\) of positive elements in \(A\), with \(b\) in the ideal generated by \(a\), there exists a sequence \((x_n)_{n \in \mathbb{N}}\) in \(A\) such that \(\lim_{n \to \infty} \|x_n^* b x_n - a\| = 0\).

The following is Theorem 4.16 in [19] (see also Definition 3.2 in [19]).

**Theorem 3.18.** Let \(A\) be a \(C^*\)-algebra. Then \(A\) is purely infinite if and only if for every non-zero positive element \(a \in A\), we have \(a \oplus a \lesssim a\).

We use the above result to show that, in the presence of the Rokhlin property, pure infiniteness is inherited by the fixed point algebra and the crossed product.

**Proposition 3.19.** Let \(A\) be a unital, separable \(C^*\)-algebra, let \(G\) be a second-countable compact group, and let \(\alpha : G \to \text{Aut}(A)\) be an action with the Rokhlin property. If \(A\) is purely infinite, then so are \(A^\alpha\) and \(A \rtimes_\alpha G\).
Proof. By Remark 2.12 and Theorem 4.23 in [19], it is enough to prove the result for $A^\alpha$. Let $a$ be a nonzero positive element in $A^\alpha$. Since $A$ is purely infinite, by Theorem 4.16 in [19] (here reproduced as Theorem 3.18), there exist sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $A$ such that

(a) $\lim_{n \to \infty} \|x_n^* a x_n - a\| = 0$;
(b) $\lim_{n \to \infty} \|x_n^* a y_n\| = 0$;
(c) $\lim_{n \to \infty} \|y_n^* a x_n\| = 0$;
(d) $\lim_{n \to \infty} \|y_n^* a y_n - a\| = 0$.

Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n : A \to A^\alpha$ as in the conclusion of Theorem 2.16. Easy applications of the triangle inequality yield

(a') $\lim_{n \to \infty} \|\psi_n(x_n)^* a \psi_n(x_n) - a\| = 0$;
(b') $\lim_{n \to \infty} \|\psi_n(x_n)^* a \psi_n(y_n)\| = 0$;
(c') $\lim_{n \to \infty} \|\psi_n(y_n)^* a \psi_n(x_n)\| = 0$;
(d') $\lim_{n \to \infty} \|\psi_n(y_n)^* a \psi_n(y_n) - a\| = 0$.

Since $\psi_n(x_n)$ and $\psi_n(y_n)$ belong to $A^\alpha$ for all $n \in \mathbb{N}$, we conclude that $a \oplus a \preceq a$ in $A^\alpha$. It now follows from Theorem 4.16 in [19] (here reproduced as Theorem 3.18) that $A^\alpha$ is purely infinite, as desired. □

Recall that a $C^*$-algebra is said to be a Kirchberg algebra if it is purely infinite, simple, separable and nuclear.

Corollary 3.20. Let $A$ be a unital Kirchberg algebra, let $G$ be a second-countable compact group, and let $\alpha : G \to \text{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ and $A \rtimes_\alpha G$ are Kirchberg algebras.

Proof. It is well-known that $A^\alpha$ and $A \rtimes_\alpha G$ are nuclear and separable. Simplicity follows from Theorem 2.24 and pure infiniteness follows from Proposition 3.19 □

In the following proposition, the Rokhlin property is surely stronger than necessary for the conclusion to hold, although some condition on the action must be imposed. We do not know, for instance, whether finite Rokhlin dimension with commuting towers preserves real rank zero and stable rank one.

Proposition 3.21. Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second-countable compact group, and let $\alpha : G \to \text{Aut}(A)$ be an action with the Rokhlin property.

(1) If $A$ has real rank zero, then so do $A^\alpha$ and $A \rtimes_\alpha G$.
(2) If $A$ has stable rank one, then so do $A^\alpha$ and $A \rtimes_\alpha G$.

Proof. By Remark 2.12, Theorem 3.3 in [30], and Theorem 2.5 in [1], it is enough to prove the proposition for $A^\alpha$. Since the proofs of both parts are similar, we only prove the first one.

Let $a$ be a self-adjoint element in $A^\alpha$ and let $\varepsilon > 0$. Since $A$ has real rank zero, there exists an invertible self-adjoint element $b$ in $A$ such that $\|b - a\| < \frac{\varepsilon}{2}$. Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence of unital completely positive maps $A \to A^\alpha$ as in the
conclusion of Theorem 2.16. Then $\psi_n(b)$ is self-adjoint for all $n \in \mathbb{N}$. Moreover,
\[
\lim_{n \to \infty} \|\psi_n(b)\psi_n(b^{-1}) - 1\| = \lim_{n \to \infty} \|\psi_n(b^{-1})\psi_n(b) - 1\| = 0 \quad \text{and}
\lim_{n \to \infty} \|\psi_n(a) - a\| = 0.
\]
Choose $n$ large enough so that $\|\psi_n(b)\psi_n(b^{-1}) - 1\| < 1$ and $\|\psi_n(b^{-1})\psi_n(b) - 1\| < 1$, and also so that $\|\psi_n(a) - a\| < \frac{\varepsilon}{2}$. Then $\psi_n(b)\psi_n(b^{-1})$ and $\psi_n(b^{-1})\psi_n(b)$ are invertible, and hence so is $\psi_n(b)$. Finally,
\[
\|a - \psi_n(b)\| \leq \|a - \psi_n(a)\| + \|\psi_n(a) - \psi_n(b)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]
which shows that $A^a$ has real rank zero.

3.3. The non-unital case. In view of the results in [33], one may wish to generalize the results in this paper to actions of compact groups on not necessarily unital $C^*$-algebras. The definition of the Rokhlin property for a compact group action on an arbitrary $C^*$-algebra should be along the lines of Definition 3.1 in [25]. Using the right definition, one should be able to prove a theorem analogous to Theorem 2.16 using the techniques from [11] and [33]. Once this is achieved, most of the results in this paper would then carry over to the (separable) non-unital setting as well.

We intend to explore this direction in a future project.

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