Mesoscopic Vortex–Meissner currents in ring ladders

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Abstract

Recent experimental progress has revealed Meissner and Vortex phases in low-dimensional ultracold atoms systems. Atomtronic setups can realize ring ladders, while explicitly taking the finite size of the system into account. This enables the engineering of quantized chiral currents and phase slips in between them. We find that the mesoscopic scale modifies the current. Full control of the lattice configuration reveals a reentrant behavior of Vortex and Meissner phases. Our approach allows a feasible diagnostic of the currents’ configuration through time-of-flight measurements.

The response of quantum coherent systems to an external perturbation may be implying subtle physical phenomena. A textbook example in this context is provided by the Meissner effect. Originally formulated in condensed matter, the Meissner effect explains how a superconductor (a phase coherent system) thicker than the penetration depth expels magnetic fields. In a so-called type II superconductor, however, the magnetic field can penetrate the superconductor, but it must be organized in a lattice of flux tubes (Abrikosov Vortex lattice) [1, 2]. Indeed, such phenomenon is intimately related to the Anderson–Higgs mechanism in relativistic Yang–Mills theories, and recent efforts have been devoted to understand whether Abrikosov vortices can occur in the Higgs field [3].

Ultracold atoms confined in optical lattices allow us to study the above problem in a novel and fruitful way [4]. Specifically, ladder structures have been considered, in which ultracold bosonic atoms can tunnel between two one-dimensional chains. An artificial gauge field [5–7] mimics the external magnetic field implied in the Meissner effect. The analog to the perfect ‘diamagnetism’ in a system of charged particles in a magnetic field arises from currents flowing along the legs (chiral currents). The Meissner phase is characterized as a specific current configuration involving a flow of particles only along the rungs of the ladder, while the Vortex phase also features currents between the rungs. Recently, many experiments of such systems have been realized [8–12] and the theory has been studied in [13–24]. It was understood, in particular, that the commensurate Meissner state undergoes a quantum phase transition to an incommensurate Vortex lattice for open boundary ladders. Recently, a very interesting Vortex–charge duality was shown [25].

To study the problem, here we are inspired by Atomtronics: optical circuits of very different spatial shapes and intensity for manipulation of cold atoms [26, 27]. Atomtronic quantum technology aims at devising a circuitry of a new type with atomic currents. At the same time, with closed conﬁnements, it can enable a new platform for cold-atom current-based quantum simulators to explore quantum many-body phases [28]. The Meissner/Vortex transition described above provides a striking example in which Atomtronics can demonstrate its full potential. In this paper, we study the physics implied by Meissner and Vortex phases at the mesoscopic scale in ladders with closed boundary conditions. We shall see that quantum phase slips [29–33] and flux quantization [34] play important roles in the physics of these system. The physical platform of the system will be
provided by a specific Atomtronic set-up made of two coupled bosonic condensates confined in 'on-top' ring-shaped optical potentials [35–37]. With our approach, we will explain the following: first, how the configuration of the current in the two phases are related to persistent currents flowing in the ring condensates. And secondly, how to measure the phases with absorption imaging of the condensate. Despite its noteworthiness, the qubit dynamics encoded into the system is discussed in the appendix E so as not to distract the reader from the main theme of the manuscript.

**Model**

Two rings A and B with each \( L \) sites are coupled via the rungs. A sketch of the model is shown in figure 1(a). The Hamiltonian \( \mathcal{H} = \mathcal{H}_A + \mathcal{H}_B + \mathcal{H}_I \) is given by

\[
\mathcal{H}_A = \sum_{m=1}^{L} \left( -t e^{i \phi_m} \hat{a}_m^\dagger \hat{a}_{m+1} + \text{h.c.} \right) + \frac{U}{2} \hat{n}_m \left( \hat{n}_m - 1 \right),
\]

\[
\mathcal{H}_B = \sum_{m=1}^{L} \left( -t e^{i \phi_m} \hat{b}_m^\dagger \hat{b}_{m+1} + \text{h.c.} \right) + \frac{U}{2} \hat{n}_m \left( \hat{n}_m - 1 \right),
\]

\[
\mathcal{H}_I = \sum_{m=1}^{L} \left[ -g \left( 1 - w \right) \hat{a}_m^\dagger \hat{b}_m + \frac{w}{2} \left( 1 + \gamma \right) \hat{a}_m^\dagger \hat{b}_{m+1} + \frac{w}{2} \left( 1 - \gamma \right) \hat{a}_m^\dagger \hat{b}_{m-1} \right] + \text{h.c.}
\]  

(1)

Operators \( \hat{a}_m^\dagger (\hat{a}_m) \) and \( \hat{b}_m^\dagger (\hat{b}_m) \) destroy (create) a boson at site \( m \) in rings A and B respectively and \( \hat{n}_m = \hat{a}_m^\dagger \hat{a}_m \) and \( \hat{n}_m = \hat{b}_m^\dagger \hat{b}_m \) are the particle number operators. The parameter \( t \) corresponds to the intra-ring coupling, \( g \) to the inter-ring coupling; ring-twist \( w \) and \( \gamma \) are parameters accounting for different coupling geometries between the two rings (as described in detail later on), \( U \) the on-site interaction and \( L \) the number of sites per ring. The operators are constrained by periodic boundary conditions for each ring \( \alpha \in \{ a, b \} \) with \( \hat{a}_{L+1} = \hat{a}_1 \). An artificial gauge field \( A_\alpha \) is introduced through the Peierls substitution \( t \to t e^{i \Omega_\alpha} \), where \( \Omega_\alpha = \phi_\alpha L = \frac{q}{\hbar} \int_C A_\alpha \text{d} \mathbf{r} \) is the total flux threading each ring, and \( \phi_\alpha \) the phase acquired when tunneling between neighboring sites. The artificial gauge field can be created in different ways [6, 38].

The phase term can be moved to the inter-ring coupling with the transformation \( \hat{a}_m = \hat{a}_m e^{-i \phi_m} \) [39, 40].

**Figure 1.** Two on-top ring lattice potentials loaded with bosons. Atoms can tunnel to the nearest neighbors in the ring as well as to sites on the other ring. Rings can be twisted in respect to each other to shift relative position of sites. Upper and lower ring are threaded by a flux. In (a), ring sites are aligned and inter-tunneling (denoted by red arrows) occurs only between adjacent sites. Optionally, diagonal inter-ring tunneling (dashed orange arrow) can be introduced. In (b), rings are twisted relative to each other, such that a site on ring A couples equally to two sites on ring B. The result is a triangular lattice. The red arrows indicate the possible tunneling between lattice sites. (c) Vortex–Meissner phase characterized by the quasi-momentum in a configuration with opposite flux in each ring. The arrows denote the direction and average value of the quasi-momentum. For weak inter-ring coupling \( g/t \) or a high flux \( \phi \), opposite, large average quasi-momentum in each ring (Vortex phase). (d) When inter-ring coupling is increased or flux is decreased: zero quasi-momentum for noninteracting rings or a small residual quasi-momentum with interaction (Meissner phase).
The phase winding in the ring is quantized to an integer winding number \( n = \frac{1}{2\pi} \oint_C \nabla \Theta(r) \, dr \) [41] (\( \Theta(r) \) being the phase of the condensate) and represents a topological quantity which is associated with the persistent current [42].

We now return to the ring-twist parameter \( w \) and \( \gamma \) mentioned in equation (1). These parameters are introduced in view of the new perspectives opened up by the Atomtronics quantum technology. We propose two ways to implement this parameter in experiment: either by engineering diagonal inter-ring tunneling \( (\gamma = 0, \text{ see figure 1(a) with dashed coupling}) \) or angular shift of the lattice sites of ring A relative to ring B \( (\gamma = 1) \). The minimal twist \( w = 0 \) is a simply connected ladder (see figure 1(a) without dashed coupling). For the first method the maximal twist \( w = 0.5 \) is achieved by configuring equal inter-ring tunneling rates for diagonal and direct coupling, while for the second configuration it realizes a triangular lattice (see figure 1(b) and [43–45]). Details on the implementation of the diagonal tunneling \( (\gamma = 0) \) is found in appendix A and for the relative angular shift \( (\gamma = 1) \) in appendix B.

**Configuration of currents**

We consider opposite flux in each ring \( \phi = \phi_A = -\phi_B \), which features the Meissner–Vortex transition mentioned in the introduction. We characterize this phase by looking at the quasi-momentum \( k \). The number of particles with quasi-momentum \( k \) in ring \( \alpha \) is given by

\[
\hat{n}_\alpha(k) = \hat{\alpha}_k^\dagger \hat{\alpha}_k = \frac{1}{L} \sum_{n,n'} e^{i(kn-n')\hat{\alpha}_n^\dagger \hat{\alpha}_n},
\]

(2)

We define the difference of center-of-mass quasi-momentum in rings A and B (called chiral momentum from now on)

\[
K_c = \sum_k k \langle \hat{n}_A(k) - \hat{n}_B(k) \rangle / N_p,
\]

(3)

where \( N_p \) is the number of particles and \( k \in (-\pi, \pi] \). The quasi-momentum \( k \) in a ring is quantized as \( k_n = 2\pi n / L \), where \( n \) is an integer. In our system \( n \) corresponds to the phase winding number as the state with quasi-momentum \( K_c \) changes its phase by \( \Theta = 2\pi n \) for one loop around the ring. The phase winding in the ring can be measured via time-of-flight expansion [34, 46, 47]. Chiral momentum \( K_c \) is the order parameter of the Vortex–Meissner phase, as for a noninteracting system it is zero in the Meissner phase (as both rings have the same quasi-momentum distribution), and non-zero in the Vortex phase with \( K_c \sim \phi \) when the inter-ring coupling \( g / t \) is small. We find that with interaction and a finite number of lattice sites, the chiral momentum of the Meissner phase becomes non-zero, but is small compared to the Vortex phase.

A sketch of the configuration and currents is shown in figures 1(c), (d). For small inter-ring coupling \( g / t \) or large flux per site \( \phi \), the two ring condensates are effectively decoupled and rotate independently of each other with the external flux, resulting in a large chiral momentum. The Vortex phase is defined by the ground state being the superposition of two degenerate states with counter-propagating quasi-momentum [17, 48]. As we impose periodic boundary conditions and the system is translational invariant, the density expectation value will not reveal the Vortex lattice along the legs as it has been seen in previous studies with open boundary conditions. Instead, the Vortex lattice structure emerges as specific spatial features in higher-order density–density and current–current correlations.

In the other limit with large \( g / t \) and small \( \phi \) (Meissner phase), the inter-ring tunneling locks the phase of the bosons in both rings. As a result, the bosons coherently cooperate to make the chiral momentum vanish.

Note that in [8, 15, 49] the Vortex–Meissner phase is characterized by the chiral current, which is the difference of the average currents in rings A and B. Then, the Meissner phase has maximal chiral current and Vortex diminishing chiral current. This is the natural definition in previous studies, where the system under study is a ladder with open boundaries. The flux affects the particle hopping along the rungs of the ladder and there is a protocol to measure the current in situ [8]. The momentum distribution of time-of-flight reflects this as well, with two peaks (proportional to \( \pm \phi \)) in Meissner phase, and four peaks (close to zero momentum) in Vortex phase.

However, for the ring ladder, the flux can be induced along the rings (legs of the ladder), which corresponds to a different gauge. Different gauge choices may lead to different time-of-flight images [50–52]. For open ladders, the effect of the gauge transformation in the time-of-flight expansion was studied in [53]. Below, we shall see how the ring ladder (equation (1)) can be read out using time-of-flight images.

We choose the chiral momentum equation (3) as the main observable. In a ring set-up, the canonical momentum is quantized in terms of the phase winding number \( n = k = 2\pi n / L \). The phase winding in rings has been studied extensively [34, 46, 47]. We can translate between the chiral current and the chiral momentum. In particular, the relation between our quasi-momentum distribution (in our gauge) and the chiral current
First, we consider the Meissner–Vortex ladder it decreases very quickly, for maximal ring-twist the Vortex phase with non-zero \( K \) plot number of sites intermediate ring-twist. In particular, for independent of ring-twist in \( t \) axis for intermediate ring-twist and \( \phi \approx 0.8\pi \). Line and open circle reference the cut through the phase diagram shown in figure 3. (b) Chiral momentum \( K_c \) against \( g/t \) for different values of interaction \( U/t \) for twist \( w = 0.2 \). The corresponding cut through the phase diagram for flux \( \phi = 0.8\pi \) is indicated as black line in (a). For \( U = 0 \), \( K_c \) is zero in the Meissner phase, whereas for \( U/t > 0 \), \( K_c \) is non-zero as other quasi-momentum modes are excited by scattering. The small steps in the profile for \( U/t > 0 \) result from the quantized phase winding for small rings. The Vortex phase is characterized by an increased chiral momentum. Numerical result with 12 sites per ring and 6 particles in total.

Figure 2. (a) Noninteracting Meissner–Vortex phase diagram showing flux per site \( \phi \) against inter-ring coupling \( g/t \) for three values of ring-twist \( w \) and for \( \gamma = 1 \). The Vortex phase area is denoted by the dots/crosses. There is a reflection symmetry at \( \phi = \pi \). Diagram depends strongly on ring-twist for \( \phi > 0.5\pi \). It shows a reentrant behavior along the \( g/t \) axis for intermediate ring-twist and \( \phi \approx 0.8\pi \). Line and open circle reference the cut through the phase diagram shown in figure 3. (b) Chiral momentum \( K_c \) against \( g/t \) for different values of interaction \( U/t \) for twist \( w = 0.2 \). The corresponding cut through the phase diagram for flux \( \phi = 0.8\pi \) is indicated as black line in (a). For \( U = 0 \), \( K_c \) is zero in the Meissner phase, whereas for \( U/t > 0 \), \( K_c \) is non-zero as other quasi-momentum modes are excited by scattering. The small steps in the profile for \( U/t > 0 \) result from the quantized phase winding for small rings. The Vortex phase is characterized by an increased chiral momentum. Numerical result with 12 sites per ring and 6 particles in total.

Figure 3. (a) Chiral momentum \( K_c \) for a large number of sites and zero interaction plotted against inter-ring coupling \( g/t \) and flux per site \( \phi = 0.8\pi \) for \( \gamma = 1 \). The corresponding cut through the phase diagram is shown as line in figure 2. For \( w = 0.2 \), reentrance of the chiral momentum is observed. (b) \( K_c \) plotted against ring-twist for different values of \( g/t \) and \( \phi \). The open circles in figure 2 show the corresponding positions in the phase diagram.

\[
\langle j_k \rangle = \frac{2t}{L} \sum_k \hat{n}_A(k) \sin(\phi - k) - \hat{n}_B(k) \sin(-\phi - k).
\]

Vortex/Meissner phases

First, we consider the Meissner–Vortex transition interaction \( U = 0 \). This allows us to consider a very large number of sites \( L \) so that the effect of the periodic boundary conditions can be disregarded. We investigate \( \gamma = 1 \), similar results for \( \gamma = 0 \) are presented in appendix A. The phase diagram for no interaction is presented in figure 2(a). We find two areas with Meissner phases, separated by the Vortex phase (dotted/crossed area). The area of the upper Meissner phase becomes smaller with increasing twist \( w \). The lower part of the Meissner phase has a positive chiral current and ordering characterized by \( k = 0 \), while the upper part displays negative chiral current and ordering \( k = \pi \). For the rectangular ladder (\( w = 0 \)), we find a reflection symmetry at \( \phi = \pi/2 \), which is broken for \( w \approx 0 \). By inspection, we find that for \( g/t \ll 1 \) or \( \phi \ll \pi/2 \) the phase diagram is nearly independent of ring-twist \( w \). Remarkably, by modifying only \( w \) it is possible to switch the phase of the system. In particular, for \( \pi > \phi > \pi/2 \) and \( w \approx 0.2 \) a patch of Meissner phase is enclosed by the Vortex state. This area becomes smaller with increasing \( w \). Therefore a reentrant behavior is found by increasing \( g \) for \( \phi \approx 0.8\pi \). It persists for non-zero interaction as seen in figure 2(b). For \( \gamma = 1 \) and \( \phi = \pi \) the dispersion is flat at the reentrance.

We plot the chiral momentum \( K_c \) for various cuts through the phase diagram in figure 3. In figure 3(a), we plot \( K_c \) against inter-ring coupling \( g \) for different ring-twist configurations and \( \phi = 0.8\pi \). While in a rectangular ladder it decreases very quickly, for maximal ring-twist the Vortex phase with non-zero \( K_c \) extends to nearly \( g/t = 8 \). This is due to destructive interference of inter-ring tunneling in triangular configuration. For intermediate ring-twist \( w = 0.2 \) the reentrance is observed: with increasing inter-ring coupling, the chiral momentum vanishes at \( g/t \approx 3 \). \( K_c \) resurfaces at intermediate couplings and then it is suppressed for larger \( g/t \).
In figure 3(b) $K_c$ is plotted against different degrees of ring-twist from rectangular ($w = 0$) to triangular ($w = 0.5$) configurations. Depending on inter-ring coupling and flux, the ring-twist $w$ can control the chiral momentum as well as it can drive the Meissner to Vortex phase. For $g/t = 8$ and $\phi = 0.8 \pi$ increasing $w$ reveals again the characteristic reentrant behavior of the Meissner–Vortex–Meissner transition. Next, we want to investigate the effect of a finite number of lattice sites $L$ on the ground state chiral momentum for zero ring-twist $w$. For small $L$, $K_c$ is not a continuous function anymore, but changes as a stair case for $U = 0$—figure 4(a). In particular, we note that an offset appears in $K_c$ for finite $L$ (similarly in the chiral current $(j_c)$). Such offset has genuine mesoscopic origins tracing back to persistent current flowing, and quasi-momentum quantization in the rings. When increasing $g/t$, we observe sharp transitions. Such behavior arises because of jumps between different values of quasi-momentum corresponding, in turn, to different winding numbers (a similar effect was evidenced in the Josephson current through a Luttinger liquid ring [54]). In between steps, $K_c$ does not define a plateau (strictly constant value between two steps), but decreases monotonously. The reason is the following: for small inter-ring coupling, each ring carries phase windings with opposite values (e.g. $n = 2$ in ring A, $n = -2$ ring B), well localized in each ring. By increasing the inter-ring coupling, the phase windings between the rings are mixed up, resulting in a suppression of the chiral momentum. In the Meissner phase, the phase winding is completely delocalized, thereby suppressing $K_c$. A similar behavior is seen in the chiral current as outlined in appendix C. The scenario emerging from the above results indicates that the Meissner–Vortex phases transition in our system occurs because of quantum phase slips [29]. In the Vortex phase, where the rings are effectively decoupled, upper and lower condensate have opposite phase windings. When increasing the inter-ring coupling, quantum phase slips occur between the opposing rotation states, which eventually cancel the phase winding difference in each ring to an average of zero.

Now we discuss the effect of non-zero interaction on $K_c$. For all values of interaction, a monotone decrease of the chiral momentum with $g/t$ is observed, with two sharp drops now instead of the discontinuous jump. As explained above, the sharp drops mark the transition between rotation states with different integer phase windings. The smooth transition heralds the appearance of superposition states of different topological phase winding numbers. To pinpoint at what value of inter-ring coupling quantum phase slips occur that change the winding number of the ground state, we refer to the phase of the two point correlation function $\Delta \Theta = \arg(\hat{a}_L^\dagger \hat{a}_{L/2+1}) - \arg(\hat{a}_L^\dagger \hat{a}_{L/2-1})$ in ring A. Indeed, this phase changes by $\Delta \Theta = 2\pi(1 - 1/L)$ whenever the ground state switches to a different phase winding. Figure 4(b) shows the phase jump of the two point correlations. This effect is best visible for an odd number of ring sites. The actual value of $\Delta \Theta$ is determined by the degree of superposition of two phase windings [55].

The specific value of $g/t$, at which the steps appear, shifts with increasing interaction. However, we observe that the second step changes more with on-site interaction. The second step shifts from about $g/t \approx 1.2$ ($U = 0$) to $g/t \approx 0.84$ ($U = \infty$). Thus, the Meissner phase (minimal value of chiral momentum) appears at a lower values of coupling $g/t$ with increasing interaction (a similar effect was noticed in [19] also for ladders with open boundaries). We explain this phenomenon as following: the Meissner state has a small chiral momentum, with largest contribution with zero phase winding. However, the Vortex state consists of opposite, non-zero phase winding flows in rings A and B. These adverse flows have additional scattering mechanisms compared to the zero winding case, making the Vortex phase energetically more unfavorable compared to the Meissner phase with increasing interaction. We observe that in the Meissner phase ($g/t$ large) the absolute value of chiral momentum increases with interaction. We attribute this to scattering into higher momentum modes.

**Figure 4.** (a) Chiral momentum $K_c$ plotted against inter-ring coupling $g/t$ for different values of interaction $U/t$. Numerical result with $L = 13$ sites per ring, 6 particles in total. $\gamma = 1$, $w = 0$ and a total flux $\phi_0 L = -\phi_0 L = 3.2 \pi$ ($\phi_0 = 0.246 \pi$). The solid, light blue curve shows the chiral momentum for a large number sites. For small $g/t$, the $K_c$ depends quadratically on inter-ring coupling. For $L = 13$ and $g = 0$, the ground state has $2$ ($-2$) phase windings in ring A (B) and is in the Vortex phase. Chiral momentum and phase winding decrease with $g/t$ in two steps to nearly zero, which corresponds to the Meissner phase. With increasing $U/t$, second step appears at smaller $g/t$ and transition smears out. (b) Phase jump of two point correlations $\Delta \Theta = \arg(\hat{a}_L^\dagger \hat{a}_n) - \arg(\hat{a}_L^\dagger \hat{a}_n)$ in ring A. Whenever the phase jump becomes $\pi$ a transition between two different phase winding states occurs.
Diagnostics

The flow of the ultracold atoms confined in ring-shaped optical potentials can be read out through time-of-flight experiments [28]. We exploit such a feature to tell apart Vortex and Meissner phases. Figure 5 shows the momentum distribution as a result of a time-of-flight experiment for different values of interaction and inter-ring coupling and \(|\phi| \ L \ > \ \pi\). The width of the annulus is proportional to the phase winding of the condensate, which only assumes integer multiples and is a measure for the condensate flow. The width of the hole is proportional to the \(K_c\) parameter, as seen in figure 4(a). From small to large coupling the number of phase windings changes from \(n = 2\) (\(n = -2\)) in ring A (B) to \(n = 0\). In the Meissner phase, the phase winding is zero and we observe no hole in the time-of-flight. We observe that no hole in the time-of-flight of the system in the Meissner phase arises for \(|\phi| \ L \ < \ \pi\) (flux smaller than a single flux quantum): as a characteristic trait of the mesoscopic regime, the Vortex and Meissner phase can only be told apart above a threshold in \(\phi\). Interaction smears out the distribution, but the hole is still well visible. Thus, the time-of-flight experiments are a reliable way to determine the phase winding for two coupled rings. To gain the momentum distribution of an individual ring, the population of the other ring could be destroyed (e.g. by exciting it with a laser), before releasing the remaining ring to be measured. Observing the condensate from different angles or tracing the time-evolution of the expansion could realize a way to gain more information about the quasi-momentum distribution. Co-expanding the rings with an additional condensate as phase [46] can reveal the sign and superposition states of the phase winding.

Conclusion

We studied the currents and the different physical regimes that can be established in a Bose condensate confined in a ring ladder of mesoscopic scale. Such a system defines a specific atomtronic circuit with enhanced control [27, 35, 36]. Because of the closed geometry and the mesoscale of the system, the physics of the system can be related to notions like quantum phase slips and persistent currents, that cannot be even defined in the previous studies concerning the open ladders. Specifically: (i) by tuning the inter-ring distance, mesoscopic currents along the legs are responsible for an offset and quantized steps of the chiral current. With interaction, the smoothening of the steps reflects superpositions of different phase windings and quantum phase slips. (ii) The ring-twist \(w\) allows to vary continuously between rectangular and triangular ladder configurations (see figures 1(a), (b)). As function of \(w\), we found a reentrant behavior in the Meissner–Vortex phase diagram and in the chiral momentum (see figure 2). We note that the dispersion is flat at the reentrance for \(\gamma = 1\) and \(\phi = \pi\). This feature could be exploited to simulate Weyl semi-metals [56]. (iii) Finally, we demonstrated with our
approach that time-of-flight measurements (in the plane of the rings) provide a feasible way to expose the physics of the mesoscopic system implied in the current and phase winding of the condensate (see figure 5).

Recently, it has been shown that the current can reverse under specific conditions [18]. This poses an interesting subject to study with our set-up. The transition of the ring ladder between mesoscopic and macroscopic behavior with interaction could be studied using DMRG. In momentum space, the ring-twist realizes an inter-ring coupling which depends on the quasi-momentum \( k \) with \( g \propto 1 - w + w e^{ik} \). For an atom with fixed \( k \), we can identify each ring as an internal state of pseudo-spins [57]. This concept has been used to realize a supersolid [58]. With the twisted ring configuration, it could realize a nonlinear spin–orbit coupling to study new quantum phases.

In possible future work a ladder with three or more legs (increasing the number of rings) can be considered to study quantized edge currents analog to the quantum Hall effect [9, 10]. The entanglement inherent to interacting bosons could be used to create a protocol for quantum enhanced rotation sensing [42].

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Appendix A. Ring-twist by introducing diagonal tunneling

Figure A1. Schematics of the two rings with ring-twist \( w \) for (a) diagonal configuration (\( \gamma = 0 \)) and (b) physical twist \( \gamma = 1 \). For (a), diagonal coupling with strength \( w/2 \) is introduced. For (b), lattice sites on ring A are shifted relative to ring B, such that a site on ring A couples to two other sites on ring B with rate \( w \) and \( 1 - w \).

For the ladder with ring-twist two different Hamiltonians are proposed. In this section, the ring-twist is engineered by a diagonal inter-ring tunneling with \( \gamma = 0 \) as defined in the main text. This corresponds to a strong next-nearest neighbor inter-ring tunneling. The set-up is sketched in figure A1(a). To create the lattice in the lab, we propose the following procedure. First, a two ring lattice is created by a blue-detuned laser. The ring sites of each lattice are vertically aligned, which has been successfully demonstrated in the lab [36]. Next, a second tightly focused red-detuned single ring lattice is projected in between the two rings. This lattice is chosen such that it modifies the tunneling rate between the two blue-detuned lattice rings. By modifying this second potential, nearest-neighbor and next-nearest neighbor inter-tunneling rates between the two lattice rings can be modified. This red-detuned lattice is used to reduce the nearest-neighbor inter-ring tunneling, and increase next-nearest neighbor tunneling. Then, we define the ring-twist as the difference of nearest and next-nearest neighbor inter-ring tunneling contributions. The new ring–ring interaction Hamiltonian is then

\[
H_1 = \sum_{m=1}^{L} - g \left[ (1 - w) \hat{a}_{m}^{\dagger} \hat{b}_{m} + \frac{w}{2} \hat{a}_{m}^{\dagger} \hat{b}_{m+1} + \frac{w}{2} \hat{a}_{m-1}^{\dagger} \hat{b}_{m} \right] + \text{h.c.}
\]

In this Hamiltonian there are now three inter-ring tunneling contributions. Our calculations show that the same reentrant behavior is found in nearly the same parameter space as in the \( \gamma = 1 \) configuration. The dispersion relation (derivation for case without ring-twist found here [20]) is

\[
E_{\pm} = -2t \cos(k) \cos(\phi) \pm \sqrt{[g(1 - w(1 - \cos(k)))]^2 + (2t \sin(k) \sin(\phi))^2}.
\]

The negative branch is flat and independent of quasi-momentum \( k \) for \( g = \frac{2t}{k} \) and \( \phi = \pi \) when \( 0 < w < 0.5 \). The dispersion relation in that region is plotted in figure A2. This corresponds to massless particles and can be used to simulate Weyl semi-metals [56]. The Meissner phase is characterized by a single minimum of quasi-momentum \( k \) of the dispersion relation at \( k = 0 \) or \( k = \pi \). The Vortex phase has a two degenerate minima at

\[
\begin{align*}
H_{1} &= \sum_{m=1}^{L} - g \left[ (1 - w) \hat{a}_{m}^{\dagger} \hat{b}_{m} + \frac{w}{2} \hat{a}_{m}^{\dagger} \hat{b}_{m+1} + \frac{w}{2} \hat{a}_{m-1}^{\dagger} \hat{b}_{m} \right] + \text{h.c.} \\
E_{\pm} &= -2t \cos(k) \cos(\phi) \pm \sqrt{[g(1 - w(1 - \cos(k)))]^2 + (2t \sin(k) \sin(\phi))^2}.
\end{align*}
\]
The positive branch is for \( f < 0.5 \pi \), the negative for \( f > 0.5 \pi \). Whenever the above equation is real-valued, the system is in the Vortex phase. The occupation number is
\[
\begin{aligned}
n_k &= -\frac{1}{2}\left(\cos(\phi_k)\right)
\end{aligned}
\]
with
\[
\begin{aligned}
\phi_k &= \frac{1}{2} \arctan \left( \frac{2(1 - w \cos(\phi_k))}{2r \sin(\theta_k) \sin(\phi_k)} \right).
\end{aligned}
\]
For a system with infinite number of sites, chiral momentum and current can be calculated by plugging the value for \( k_{\text{min}} \), \( n_{\text{A}}(k) \), and \( n_{\text{B}}(k) \) into equations (3) and (4) respectively. The phase diagram can be calculated from the minima of the dispersion relation. The lower transition line is given by
\[
\begin{aligned}
g_c/t &= \frac{1}{(1 - 2w)w} \left[ 1 - 6w + 8w^2 + (1 - 2w) \cos(2\phi) \right] \\
&\pm \sqrt{2} \sqrt{(1 - 2w)^2 \cos(\phi)^2 [1 - 6w + (1 + 2w) \cos(2\phi)]^2},
\end{aligned}
\]
for solutions with imaginary part zero.

The resulting phase diagram is plotted in figure A3(a). Chiral momentum against inter-ring coupling \( g/t \) is plotted in figure A4(a) and against ring-twist \( w \) in figure A5(a).
Appendix B. Ring-twist created by relative shift of the rings

We discuss the realization of the twist \( w \) by introducing a relative shift of one ring lattice to the other, which corresponds to \( \gamma = 1 \). A cartoon of the configuration is shown in figure A1(b). The Hamiltonian is

\[
\mathcal{H}_t = \sum_{m=1}^{L} -g[(1-w)\hat{a}_m^\dagger \hat{b}_m + w\hat{a}_{m+1}^\dagger \hat{b}_m] + \text{h.c.},
\]
and the dispersion relation

\[
E_{\pm} = -2t \cos(k) \cos(\phi) \\
\quad \pm \sqrt{g^2(1-2w(1-w))(1-\cos(k)) + (2t\sin(k)\sin(\phi))^2}.
\]

The Meissner phase is characterized by a single minimum of quasi-momentum \( k \) of the dispersion relation at \( k = 0 \) or \( k = \pi \). The Vortex phase has a two degenerate minima at \( \pm k(t=1) \)

\[
k/\pi = \arccos\left[\frac{1}{4\sin(\phi)^2} \left( g^2(1-w) w \right) \right] \\
\quad \pm \sqrt{\cos(\phi)^2(g^2(1-w)^2 + 4\sin(\phi)^2(g^2w^2 + 4\sin(\phi)^2))}.
\]

The positive branch is for \( \phi < 0.5\pi \), the negative for \( \phi > 0.5\pi \). Whenever the above equation is real-valued, the system is in the Vortex phase. The occupation number is \( n_A(k) = N_p/2 \cos^2(\theta_k), \) \( n_B(k) = N_p/2 \sin^2(\theta_k), \) with

\[
\theta_k = \frac{1}{2} \arctan\left(\frac{\sqrt{1-2w(1-w)(1-\cos(k))}}{2t\sin(k)\sin(\phi)}\right)
\]

For a system with infinite number of sites, chiral momentum and current can be calculated by plugging the value for \( k_{\text{min}} \) into equations (3) and (4) respectively. The transition between Meissner and Vortex phase can be calculated from the minima of the dispersion relation, and is given by

\[
\cos(\phi_\pm) = -\frac{g}{4t} + \sqrt{1 + \left(\frac{g}{4t}\right)^2(1-2w)^2},
\]

\[
\cos(\phi_\pm) = \frac{g}{4t}(1-2w) - \sqrt{1 + \left(\frac{g}{4t}\right)^2}.
\]

The phase diagram is plotted in figure A3(b). Chiral momentum against inter-ring coupling \( g/t \) is plotted in figure A4(b) and against ring-twist \( w \) in figure A5(b).
Both rectangular and triangular ladder configurations ($w = 0$ and $w = 0.5$) may be realized by Laguerre–Gauss beams as outlined in [37].

We propose different options to create a generic ring-twist $w$: first, spatial light modulators (SLM) or digital micromirror devices (DMD) can create arbitrary potentials by shaping wavefronts. By using two devices and two red-detuned light fields with different wavelength to circumvent interference one can envision the following procedure. Beam propagation is along $z$-direction. In a plane orthogonal to $z$, two ring lattices at positions $z_a$ and $z_b$ are created. Each ring $\alpha$ is created by one of the SLMs with wavelengths $\lambda_\alpha$ by tightly focusing the light at position $z_\alpha$. The light pattern created by the SLM is programmed such that the lattice sites of ring $A$ are shifted relative to ring $B$ as depicted in the cartoon figure B1.

However, the on focused light intensity of the image at $z_a$ will influence the potential the atoms see in the plane of $z_a$ as the two focal points are close together. One can now change the potential at $z_a$ with the SLM, until a ring lattice shaped potential with the additional stray light from the beam focused at $z_b$ is restored. The change of the potential at $z_a$ will of course influence the total intensity at $z_a$. Now, the same procedure has to be done for the plane $z_b$. This will be an iterative process for both focal points until a reasonable two ring pattern is formed. This procedure has to be pre-calculated for each value of the ring-twist. However as of now, no such experimental realization exists and the general convergence of this special iterative process has to be proven.

Alternatively, the ring-twist could be created with holographic methods [39]. It has been shown that using an SLM spiral pattern in direction of beam propagation can be created. The ring-twist of the two rings corresponds to a relative shift of lattice sites in rings A and B. Using the same holographic method and a red-detuned laser, a ring lattice tube could be arranged with a spiral pattern in $z$-direction. By imprinting an intensity modulation in direction of beam propagation (e.g. by focusing a blue-detuned laser in the plane orthogonal to propagation), separated rings are formed. The spiral winding realizes the ring-twist.

Finally, the two rings could be created concentrically in the same plane, with two different radii. This way, creating and addressing the rings becomes simpler as only a single SLM or DMD is required to create any potential shape in two dimensions. However, as the two rings have different radii, an asymmetry in the two rings are introduced in the intra-ring hopping. This asymmetry could be corrected by modifying each ring potential separately.

In the discussion so far the ring-twist parameter $w$ is a linear parameter in the lattice Hamiltonian. We investigate how this linear parameters relates to an actual twisted two lattice ring configuration. The tunneling can be calculated with the WKB approximation [37]

$$ g = 4 \sqrt{\frac{\hbar}{2m}} \frac{V^{1/4}}{\sqrt{s}} e^{-\frac{\sqrt{2mV}}{\pi \hbar}}, $$

where $s$ is the distance between two sites. We plot the tunneling rates with that function and our set-up in the attached figure B1. $a$ is the distance between the lattice sites of a ring, $g/t$ the ratio of inter-ring tunneling and intra-ring tunneling, $d$ the inter-ring distance, $x/a$ the relative shift in space of lattice sites between rings A and B. $x/a = 0$ or $x/a = 1$ corresponds to both rings aligned, $x/a = 0.5$ corresponds to a triangular configuration. We assume Rubidium 87 atoms and rings with $L = 12$ sites, radius $R = 8 \mu$m. A comparable lattice configuration has been realized in [36]. We assume a potential barrier of $V = 20E_{\text{recoil}}$.

As seen in the left figure, the linear parameter ring-twist $w$ changes nonlinearly with the physical shift of the rings. However, the change occurs in a continuous and monotonous fashion and covers nearly all possible values of ring-twist. By knowing this function, the physical shift of the rings can be related to our linear parameter $w$.

In the right graph, the ratio $g/t$ is plotted against the physical ring shift $x/a$. We notice that the inter-ring tunneling rate changes by a small factor. This means, that while physical twisting the rings, the inter-ring tunneling changes as well by a small amount. There are two ways to address this: either the change in tunneling

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**Figure B1.** (a) Ring-twist $w$ as defined in the paper plotted against the relative shift $x/a$ of rings A and B in units of lattice constant $a$. (b) Ratio inter-ring coupling to intra-ring coupling $g/t$ plotted against the relative shift $x/a$ of rings A and B in units of lattice constant $a$. 
rate has to be accounted for in the evaluation of the experiment, or the inter-ring distance has to be tuned accordingly to keep the tunneling rate constant.

**Appendix C. Mesoscopic currents**

In figure C1, we study the chiral current (as defined in equation (4)) for mesoscopic and macroscopic number of sites. We find that for large number of sites the current is zero for small inter-ring couplings, while we observe a persistent current for a finite number of sites. The sign of the persistent chiral current changes with the total flux in the rings \( \Omega = \phi L \).

**Appendix D. Dispersion relation**

Periodic boundary conditions quantize the quasi-momentum to \( k_n = \frac{2 \pi n}{L} \), with integer \( n \). As an example, we show the allowed quasi-momenta for \( L = 12 \) with cross symbols in figure D1 with \( \gamma = 1 \) for zero and maximal ring-twist. The dispersion relation with intermediate ring-twist is plotted as full line in figure D2. The Meissner state is characterized by a single minimum at quasi-momentum \( k = 0 \) or \( k = \pi \) and the Vortex phase is characterized by two degenerate minima.

**Appendix E. Qubit dynamics of the ring ladder**

It has been shown, that for sufficiently large \( L \), the dynamics of the rectangular ladder system \( w = 0 \) is governed by an effective two-level system. Here, we study the configuration leading to the qubit dynamics. In the present case of moderate \( L \), we find for \( \phi_A = -\phi_B = \pi/2 \) the energy gap is well defined for a wide parameter range. However, the first and second excited states results to be degenerate. This feature may render controlled addressing of only the two lowest levels as an effective qubit difficult. For this parameter regime, the system is...
always in a Vortex state. Here, we report that \( f + p f = » L L L A A \) results de

ning a working point for a two-level system as well. In figure E1 the average current in either ring, current covariance, energy gap and quality factor (as defined in [47]) is studied in dependence of \( f W = LL AB \), for various values of interaction \( U / t \) and inter-ring coupling \( g / t \).

In figure E1 (a), we observe a change from small to large average current across the optimal working point \( \Omega = \pi \). At that point in figure E1 (b), the current–current correlation (the so-called covariance) displays a positive maximum, which indicates that the direction of the currents in the both rings are positively correlated. Thus, the two levels are now provided by co-propagating currents in the two rings. This is in contrast the other working point, where the flux and current in each ring has opposite sign.

Appendix F. Artificial gauge field

The artificial gauge field can be generate by following different protocols: two photon Raman transitions, suitable gradient laser intensity, steering and suitable shaking of the potentials.

For the potential shaking: if the shaking is fast compared to the Hamilton dynamics, it will generate an effective, time-independent tunneling parameter. For sinusoidal shaking, it is possible to control the sign of the tunneling. When the driving breaks time-reversal symmetry, complex tunneling constants are possible, which can be engineered into artificial gauge fields [60]. Artificial gauge fields also have been realized using laser assisted tunneling [6, 8]. All such protocols can be be applied to the two coupled rings if the two rings are sufficiently distant apart.

Another option to construct an artificial gauge field is the rotation of the lattice. Experimentally this can be realized by stirring the condensate, e.g. by a moving a barrier through the ring with a constant velocity. This method has been experimentally realized with a single continuous ring [34]. For two rings, the barrier in each
ring would need opposite velocity in each ring. Experimentally, the difficulty lies in applying the barriers with opposite velocity in each ring separately, without disturbing the other. This could be done by separating the ring potentials at large enough distance, such that a laser for each individual barrier can be independently realized in each ring without influencing the other ring. After preparing the desired winding state, the rings are brought slowly into contact. For very high flux, this method may not work well.

We now proceed to describe the creation of the artificial gauge field in a single ring. In the rotating frame, the Hamiltonian of a particle in a ring becomes (while neglecting the barrier contribution)

$$\mathcal{H} = \frac{\hat{p}^2}{2m} - \Omega_{\text{rot}} \hat{L}_z,$$

with $\hat{p}$ the momentum operator, $\hat{L}_z$ the angular momentum operator, $m$ the mass and $\Omega_{\text{rot}}$ the angular velocity of the stirring by the external force. For a one-dimensional ring, this equation can be rewritten with an effective gauge field $qA(R)$ as

$$\mathcal{H} = (\hat{p} - qA)^2 - V_{\text{centr}},$$

$$qA = \Omega_{\text{rot}} R m,$$

$$V_{\text{centr}} = \frac{\Omega_{\text{rot}}^2 R^2 m}{2},$$

where $V_{\text{centr}}(R)$ is centrifugal potential. The resulting force of the gauge field is the Coriolis force.

Akin to the magnetic flux threading a superconducting loop, we define the Coriolis flux threading the system

$$\mathcal{H} = - q A \oint C \frac{d\mathbf{r}}{\hbar} = 2\pi m q \Omega_{\text{rot}} R^2 / \hbar,$$

where $\omega$ is the angular velocity of the barrier, $m$ the mass of the particles and $R$ the radius of the ring. In a ring, the angular momentum of a particle is quantized in integer number $n$ due to the periodic boundary condition and given by

$$k_n = \frac{2\pi n}{L},$$

with $L = 2\pi R$ the length of the ring. The energy of the particle is then

$$E = \left( \frac{\hbar k_n - qA}{2m} \right)^2 = \frac{\hbar^2}{2m} \left( \frac{2\pi n \Omega_{\text{rot}} R^2}{\hbar} \right)^2.$$

The ground state energy of a particle in a ring changes with flux $\Omega$ with a period of $2\pi$. The angular velocity of the condensate in the non-rotating frame is given by $\omega = \frac{2\pi n R^2}{m'}$ and is quantized in integer multiples of

$$\omega_0 = \frac{\hbar}{L m R^2}.$$

For a lattice system, the ring is discretized into $L$ sites and the intra-ring tunneling becomes

$$\mathcal{H} = \sum_{m=1}^{L} (-t e^{i\phi/L} \hat{a}_m^\dagger \hat{a}_{m+1} + \text{h.c.}) + \text{h.c.}.$$

The phase shift for tunneling between neighboring sites is given by

$$\phi = \frac{2\pi m \omega R^2}{\hbar L} = \frac{2\pi m v R}{\hbar L},$$

where $v$ the velocity of the rotation. For the two ring set-up proposed in [36] with a ring radius $5 \mu m$ and loaded with Rubidium atoms, the condensate rotates in multiples of the frequency $f_0 = 4.73$ Hz.

Appendix G. Time-of-flight

The atoms are released from the trap and expand freely for a certain amount of time. The momentum distribution $n(k)$ corresponds then to the particle density of the expanded gas. We calculate it with the Fourier transform of the one-body density matrix $\rho_{11}(x, x') = \langle \hat{\psi}^\dagger(x) \hat{\psi}(x') \rangle$ and get

$$n(k) = \int dx \int dx' \langle \hat{\psi}^\dagger(x) \hat{\psi}(x') \rangle e^{ik(x-x')}.$$  

To account for the lattice structure, we express the equation in terms of Wannier functions $w_j(x) = w(x - x_j)$, which are localized at the location of lattice site $x_j$. The boson operator becomes $\hat{\psi}(x) = \sum_j w_j(x) \hat{c}_j$, where $\hat{c}_j$ is
the annihilation operator of the Bose–Hubbard operator for site \( j \) and the sum runs over all sites of both rings. With this, equation (G1) is recast to

\[
n(k) = \ii \tilde{w}(k) \iota \sum_{\mathbf{r}_j \mathbf{r}_j'} e^{i k (\mathbf{r}_j - \mathbf{r}_j')} \langle \hat{c}_j^\dagger \hat{c}_{j'} \rangle,
\]

where \( \tilde{w}(k) \) is the Fourier transform of the Wannier function.

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**References**

[1] Meissner W and Ochsenfeld R 1933 Naturwissenschaften 21 787
[2] Rickayzen G 1969 *Superconductivity* ed R D Park (New York: Interscience, Marcel Dekker)
[3] Sudbø A 2013 *Superconductivity: Discoveries and Discoverers* (Berlin: Springer) pp 129–31
[4] Bloch I 2005 *Nat. Phys.* 1 23
[5] Eckardt A, Weiss Cand Holthaus M 2005 *Phys. Rev. Lett.* 95 260404
[6] Dalibard J, Gerbier F, Juzeliūnas G and Oehberg P 2011 *Rev. Mod. Phys.* 83 1523
[7] Tang S, Schweikhard V and Cornell E A 2006 *Phys. Rev. Lett.* 97 240402
[8] Atala M, Aidaelsburger M, Lohse M, Barreiro J T, Paredes B and Bloch I 2014 *Nat. Phys.* 10 588
[9] Mancini M et al 2015 *Science* 349 1510
[10] Stuhl B, Lu H-I, Aycock L, Genkina D and Spielman I 2015 *Science* 349 1514
[11] An F A, Meier E J and Gadday B 2017 *Sci. Adv.* 3 e1602685
[12] Livi L et al 2016 *Phys. Rev. Lett.* 117 220401
[13] Kardar M 1986 *Phys. Rev. B* 33 3125
[14] Granato E 1990 *Phys. Rev. B* 42 42797
[15] Orignac E and Giarmarchi T 2001 *Phys. Rev. B* 64 144515
[16] Crépin F, Llourence N, Roux G and Simon P 2011 *Phys. Rev. B* 84 054517
[17] Tokuno A and Georges A 2014 *New J. Phys.* 16 073003
[18] Greschner S, Piraud M, Heidrich-Meisner F, McCallouohan L, Schollwöck U and Vekua T 2015 *Phys. Rev. Lett.* 115 190402
[19] Piraud M, Heidrich-Meisner F, McCallouohan I P, Greschner S, Vekua T and Schollwöck U 2015 *Phys. Rev. B* 91 140406
[20] Keley A and Oktel M 2015 *Phys. Rev. A* 91 013629
[21] Di Dio M, Citro R, De Palma S, Orignac E and Chiofalo M-L 2015 *Eur. Phys. J. Spec. Top.* 224 525
[22] Orignac E, Citro R, Di Dio M, De Palma S and Chiofalo M-L 2016 *New J. Phys.* 18 055017
[23] Guo C and Poletti D 2016 *Phys. Rev. A* 94 033610
[24] Guo C and Poletti D 2017 *Phys. Rev. B* 96 165409
[25] Greschner S and Vekua T 2017 *Phys. Rev. Lett.* 119 073401
[26] Seaman B, Krämer M, Anderson D and Holland M 2007 *Phys. Rev. A* 75 023615
[27] Amico L, Birkl G, Boshier M and Kwek L-C 2016 *New J. Phys.* 19 020201
[28] Amico L, Osterloh A and Cataliotti F 2005 *Phys. Rev. Lett.* 95 063621
[29] Matveev K, Larkin A and Glazman L 2002 *Phys. Rev. Lett.* 89 096802
[30] Rastelli G, Pop I M and Hong K 2013 *Phys. Rev. B* 87 174513
[31] Pop I-M, Protopopov I, Lencoc F, Peng Z, Pannevier B, Buisson O and Guichard W 2010 *Nat. Phys.* 6 589
[32] Astiatiev O, Joffe L, Kafanov S, Pathkin Y A, Arutyunov K Y, Shahar D, Cohen O and Tsai J 2012 *Nature* 484 355
[33] Roscilde T, Faulkner M F, Bramwell S T and Holdsworth P C W 2016 *New J. Phys.* 18 075003
[34] Wright K C, Blakestad R B, Lobb C J, Phillips W D and Campbell G K 2013 *Phys. Rev. Lett.* 110 025302
[35] Li T, Kelkar H, Medellin D and Raizen M 2008 *Opt. Express* 16 5465
[36] Amico L, Aghaharyan D, Aukstol D, Crepaz H, Dumke R and Kwek L-C 2014 *Sci. Rep.* 4 4289
[37] Aghaharyan D, Amico L and Kwek L-C 2013 *Phys. Rev. A* 88 063627
[38] Goldman N, Juzeliūnas G, Oehberg P and Spielman I B 2014 *Rept. Prog. Phys.* 77 126401
[39] Osterloh A, Amico L and Eckern U 2000 *Nucl. Phys. B* 588 331
[40] Amico L, Osterloh A and Eckern U 1998 *Phys. Rev. B* 58 R1703
[41] Kashurnikov V A, Podlivaev A I, Prokofev N V and Svistunov B V 1996 *Phys. Rev. B* 53 13091
[42] Ragole S and Taylor J M 2016 *Phys. Rev. Lett.* 117 203002
[43] Mishra T, Pai R V, Mukerjee S and Paramekanti A 2013 *Phys. Rev. B* 87 174504
[44] Anisimovas E, Račiūnas M, Sträter C, Eckardt A, Spielman I and Juzeliūnas G 2016 *Phys. Rev. A* 94 063632
[45] Longhi S 2014 *Opt. Lett.* 39 5892
[46] Eckel S, Jedrzejewski F, Kumar A, Lobb C and Campbell G 2014 *Phys. Rev. X* 4 031052
[47] Aghaharyan D, Cominotti M, Rizzi M, Rossini D, Hekking F, Minguzzi A, Kwek L-C and Amico L 2015 *New J. Phys.* 17 045023
[48] Wei R and Mueller E J 2014 *Phys. Rev. A* 89 063617
[49] Cha M-C and Shin J-G 2011 *Phys. Rev. A* 83 055602
[50] Moller G and Cooper N R 2010 *Phys. Rev. A* 82 063625
[51] Lin Y-J, Compton R, Jiménez-Garcia K, Phillips W, Porto J and Spielman I 2011 *Nat. Phys.* 7 531
[52] Kennedy C J, Burton W C, Chung W C and Ketterle W 2015 *Nat. Phys.* 11 859–64
[53] Greschner S, Piraud M, Heidrich-Meisner F, McCallouohan L, Schollwöck U and Vekua T 2016 *Phys. Rev. A* 94 063628
[54] Fazio R, Hekking F and Odintsov A 1995 *Phys. Rev. Lett.* 74 1843
[55] Dangshita I and Polkovnikov A 2010 *Phys. Rev. B* 82 094404
[56] Nie W, Mei F, Amico L and Kwek L-C 2017 *Phys. Rev. E* 96 020106
[57] Li J, Huang W, Sletynas B, Burchesky S, Top F C, Su E, Lee J, Jamison A O and Ketterle W 2016 *Phys. Rev. Lett.* 117 185301
[58] Li J-R, Lee J, Huang W, Burchesky S, Shteynas B, Top F Ç, Jamison A O and Ketterle W 2017 Nature 543 91
[59] Latychevskaia T and Fink H-W 2016 Sci. Rep. 6 26312
[60] Struck J, Ölschläger C, Weinberg M, Hauke P, Simonet J, Eckardt A, Lewenstein M, Sengstock K and Windpassinger P 2012 Phys. Rev. Lett. 108 225304