Spectrum of the Hermitian Wilson-Dirac Operator for a Uniform Magnetic Field in Two Dimensions

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\textbf{Abstract}

It is shown that the eigenvalue problem for the hermitian Wilson-Dirac operator of for a uniform magnetic field in two dimensions can be reduced to one-dimensional problem described by a relativistic analog of the Harper equation. An explicit formula for the secular equations is given in term of a set of polynomials. The spectrum exhibits a fractal structure in the infinite volume limit. An exact result concerning the index theorem for the overlap Dirac operator is obtained.
The effects of magnetic fields on two-dimensional systems have attracted continual interests from condensed matter to elementary particle physics. Magnetic fields not only bundle energy spectrum but also rearrange the eigenvalues as they continuously changes, giving rise to rich band structure. The most impressive is the energy spectrum of tightly bounded electron in a uniform magnetic field. It is well-known that the energy spectrum of this system has a fractal structure known as the butterfly diagram [1]. The key ingredient that underlies such variety is the topological nature of magnetic fields.

Relativistic analogue of the Hamiltonian of tightly bounded electron is the hermitian Wilson-Dirac operator (HWDO), which attracts renewed interests in the context of overlap Dirac operator (ODO) describing chiral gauge theories on the lattice [2]. Magnetic field also affects the spectrum of HWDO, giving rise to interesting phenomena such as chiral anomaly [3]. We also expect the spectrum has a fractal structure similar to the butterfly diagram.

In this paper we investigate the spectrum of HWDO for a uniform magnetic field in two dimensions. We show that the two dimensional system can be converted to one dimensional problem for an arbitrary uniform magnetic field and, as the consequence, the spectrum can be characterized by a set of polynomials, which enables us not only to understand the fractal structure of the spectrum but also to compute the exact index of the ODO.

The spectral flow of HWDO for the one-parameter family of link variables was investigated in Ref. [4] and the connection to chiral anomaly was elucidated. Recently, one of the present author reanalyzed the same system [5] and gave some exact results on the spectrum for a particular set of uniform magnetic fields, for which the index of the ODO can be obtained rigorously. The purpose of this paper is to extend them for an arbitrary uniform magnetic field.

We consider a two-dimensional square lattice of unit lattice spacing and of sides $L$. The HWDO $H$ is defined by

$$H\psi(x) = \sigma_3 \left\{ (2 - m)\psi(x) - \sum_{\mu=1,2} \left( \frac{1}{2} \sigma_\mu U_\mu(x) \psi(x + \hat{\mu}) + \frac{1}{2} \sigma_\mu U_\mu^*(x - \hat{\mu}) \psi(x - \hat{\mu}) \right) \right\} , \quad (1)$$

where $\sigma_\mu \ (\mu = 1, 2, 3)$ are the Pauli matrices and $\hat{\mu}$ stands for the unit vector along the $\mu$-th axis. The link variables $U_\mu(x)$ and the lattice fermion field $\psi(x)$ are subject to periodic boundary conditions. In the context of ODO $m$ must be chosen to satisfy $0 < m < 2$ in order for the correct continuum limit to be achieved. We adopt $m = 1$ unless otherwise specified.

We are interested in the spectrum of $H$ for the link variables of the form

$$U_1(x) = \exp\left[-it\frac{2\pi}{L}x_1\delta(x_1,L-1)\right], \quad U_2(x) = \exp\left[it\frac{2\pi}{L^2}x_1\right] , \quad (2)$$

where $t$ is an arbitrary parameter and $x_\mu$ stands for the periodic lattice coordinates defined by $x_\mu = x_\mu$ for $0 \leq x_\mu < L$ and $x_\mu + L = x_\mu$. For $t = Q$ being an integer, the magnetic field $F_{12}(x) = -i U_1(x)U_2(x+\hat{1})U_1^*(x+\hat{2})U_2^*(x)$ becomes a constant and the flux per plaquette...
normalized by $2\pi$ is given by

$$\alpha \equiv \frac{1}{2\pi} F_{12}(x) = \frac{Q}{L^2}. \quad (3)$$

Hence $Q$ is nothing but the topological charge $[6]$. The parameter $t$ connects continuously the link variables with various uniform magnetic fields belonging to different topological sectors classified by the integer topological charge $[3, 4, 5]$.

The characteristics of the spectrum of HWDO found in $[6]$ can be summarized as: (1) The eigenvalues at integral values of $t$ are separated by several gaps and form clusters. (2) For noninteger $t$ the $H$ has in general $2L^2$ distinct eigenvalues and rearrangements of eigenvalues between the clusters occur in a characteristic way as $t$ increases continuously from an integer to the next integer. By carefully inspecting the spectral flows one realizes that for special integer values of $t = r$ with $r$ being an arbitrary divisor of $L$ each eigenvalue is exactly $r$-ply degenerate. This can be proven by noting that $H$ can be block diagonalized into $r \times 2sL \times r$ matrices each describing a one-dimensional lattice system of degrees of freedom $2sL$. In what follows we will generalize the argument of Ref. $[5]$ concerning the characterization of matrices each describing a one-dimensional lattice system of degrees of freedom $2sL$. (3) The spectrum is an odd functions in $t$. Hence, it suffices to analyze the eigenvalue spectrum for $0 \leq t \leq L^2/2$.

We first analyze the spectrum for $t$ being an integer multiple of $L$. Let $L$ be a product of two positive integers $r$ and $s$. Then $t$ can be expressed as $t = nL^2/r = nsL$, where $n$ is some integer coprime with $r$. Then we have $U_1(x) = 1$ and $U_2(x) = \exp[2\pi in\pi_1/r]$. Since the link variable is independent of $x_2$ and periodic in $x_1$ with a period $r$, we can simplify the eigenvalue problem for $H$ by the following Fourier transformation

$$\varphi(y; p, q) = \frac{1}{sL} \sum_{l=0}^{s-1} \sum_{x_2=0}^{L-1} e^{-iql - ipx_2} \psi(rl + y, x_2), \quad (4)$$

where $y$ ranges between $0$ and $r - 1$. The Fourier momenta $p$ and $q$ are given by

$$p = \frac{2\pi}{L}k, \quad q = \frac{2\pi}{s}j. \quad (k = 0, 1, \cdots, L - 1, \quad j = 0, 1, \cdots, s - 1) \quad (5)$$

$H$ is then block-diagonalized into $sL \times 2r$ hermitian matrices $h(p, q)$ given by

$$h(p, q) = \begin{pmatrix} B(p, q) & C(p, q) \\ C^\dagger(p, q) & -B(p, q) \end{pmatrix}, \quad (6)$$

where the first (second) row acts on the upper (lower) component of $\varphi(y; p, q)$. $B(p, q)$ and $C(p, q)$ are defined by

$$\begin{align*}
(B(p, q))_{y, y'} &= -\frac{1}{2} \delta^{(q)}_{y+1, y'} + \left\{ 1 - \cos \left( p + \frac{2\pi ny}{r} \right) \right\} \delta^{(q)}_{y, y'} - \frac{1}{2} \delta^{(q)}_{y, y' + 1}, \\
(C(p, q))_{y, y'} &= \frac{1}{2} \delta^{(q)}_{y+1, y'} + \sin \left( p + \frac{2\pi ny}{r} \right) \delta^{(q)}_{y, y'} - \frac{1}{2} \delta^{(q)}_{y, y' + 1}. \quad (7)
\end{align*}$$

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The $\delta^{(q)}_{y',y}$ is the Kronecker $\delta$-symbol for $0 \leq y, y' \leq r - 1$ and satisfies the twisted boundary conditions

$$
\delta^{(q)}_{y',y} = e^{-iq\delta_{y,0}}, \quad \delta^{(q)}_{r,y'} = e^{iq\delta_{y,0}}.
$$

We thus find that the original two-dimensional system of degrees of freedom $2L^2$ is decomposed into $sL$ one-dimensional systems of degrees of freedom $2r$, each described by $h(p, q)$. The corresponding eigenvalue problem is just a relativistic analog of the Harper equation [8].

The eigenvalues of $H$ are determined by the secular equation $\det(h(p, q) - \lambda) = 0$. It takes the form

$$
\det(h(p, q) - \lambda) = f_r^{(n)}(\lambda; p) - \frac{(-1)^{r-1}}{2^{r-4}} \sin^2 \frac{rp}{2} \sin^2 \frac{q}{2} = 0,
$$

where $f_r^{(n)}(\lambda; p) = \lambda^{2r} + \cdots$ is a polynomial of order $2r$ and is defined by

$$
f_r^{(n)}(\lambda; p) = \det(h(p, 0) - \lambda).
$$

The $q$ dependent term in (8) can be easily found from the the explicit form of $h(p, q)$. For $n$ and $r$ being coprime $h(p, 0)$ and $h(0, 0)$ are related by an orthogonal transformation and $f_r^{(n)}(\lambda; p)$ then becomes independent of $p$. We simply write it as $f_r^{(n)}(\lambda)$. Explicit forms of $f_r^{(1)}(\lambda)$ are given in Ref. [8] for $r = 1, \ldots, 6$.

Later we need to consider (10) for $n$ and $r$ not necessarily being coprime. In general $f_r^{(n)}(\lambda; p)$ depends on $p$ and satisfies the factorization property

$$
f_r^{(n)}(\lambda; p) = \prod_{j=0}^{s'-1} \left( f_r^{(n')}(\lambda) - \frac{(-1)^{r'-1}}{2^{r'-4}} \sin^2 \frac{r'p}{2} \sin^2 \frac{\pi j}{s'} \right),
$$

where $n' = n/s'$ and $r' = r/s'$ with $s'$ being the greatest common divisor of $n$ and $r$. The easiest way to show this is to consider the case of $r = L$ and $t = nL^2/r = n'L^2/r' = n's'L$. Since $s = 1$, the only allowed value of $q$ is 0. The secular equation (8) then becomes $f_r^{(n)}(\lambda; p) = 0$. On the other hand the same set of eigenvalues must be reproduced by the expressions (8) with the substitution $r, n \rightarrow r', n'$ and $q \rightarrow 2\pi j/s'$ for $j = 0, \ldots, s' - 1$. The identity (11) then follows from these.

We have shown that the eigenvalues of $H$ for $t$ being any integer multiple of $L$ are completely characterized by a set of functions $f_r^{(n)}(\lambda)$. We now extend the results to an arbitrary integer $t$. Let $n$ and $s$ be positive coprime integers such that $t/L = n/s$, then we may find a positive integer $r$ satisfying $t = nr$ and $L = rs$. Denoting $x_2$ by two nonnegative integers $y$ and $l$ ($0 \leq y < s$, $0 \leq l < r$) as $x_2 = sl + y$, we see that the link variables (2) are independent of $l$ or, equivalently, periodic in $y$ with a period $s$. We then define new variables $\Psi(z; q)$ by

$$
\Psi(z; q) = \frac{1}{L} \sum_{y=0}^{s-1} \sum_{l=0}^{r-1} e^{-i(py+ql)}\psi(x_1, sl + y),
$$

where $\psi(x_1, sl + y) = \psi(x_1, x_2)$.
where $q$, $p$ and $z$ are defined by

$$
q = \frac{2\pi j}{r}, \quad p = \frac{2\pi nk}{s} + \frac{2\pi j}{L}, \quad z = kL + \frac{\pi}{r} \quad (j = 0, \ldots, r-1, \ k = 0, \ldots, s-1) \quad (13)
$$

Since the shift $z \rightarrow z + sL$ simply corresponds to $k \rightarrow k + s$ or, equivalently, to $p \rightarrow p + 2\pi n$, we may regard $\Psi(z; q)$ as functions of $z$ with a period $sL$. Here a miracle occurs. In the new variables $\Psi(z; q)$ defined on the one-dimensional periodic lattice, $H$ is again block diagonalized into $r$ hermitian $2sL \times 2sL$ matrices of the form (6) and (7). The concrete expressions of $B$ and $C$ are obtained from (7) by making the following substitutions

$$
y, r, p, q \rightarrow z, sL, q/s, 0 \quad (14)
$$

The secular equations for the eigenvalues of $H$ at $t = nr$ are then given by

$$
f_{sL}^{(n)}(\lambda; q/s) = 0.
$$

Noting the factorization formula (11), we find that all the eigenvalues of $H$ at $t = nr$ are determined by the following set of secular equations

$$
0 \leq (-1)^{r's^2-1}2^{r's^2-4}f_{r's^2}^{(n')}(\lambda) \leq 1 \quad (16)
$$

Each interval contains exactly $rs'$ eigenvalues. Furthermore, the intervals themselves form roughly $2r's^2/n'$ clusters. This explains the characteristic features of the spectral flows of $H$ found by the numerical investigation [3]. We can also count the multiplicity of the eigenvalues from (13). In particular the multiplicity at $t = rn$ with $n$ and $r$ being coprime is exactly $r$. In Fig. 1 the spectrum of $H$ for uniform magnetic field is shown. One easily see that the feather-shape gaps form a fractal pattern. It looks quite different from the butterfly diagram [1]. This is due the special choice of $m = 1$ and the butterfly-like gaps appear for $m \neq 1$. The spectrum for $m = 1/2$ is shown in Fig. 2 for comparison.

In order to understand the appearance of the fractal structure of the spectrum we consider the infinite volume limit. The magnetic flux $\alpha$ defined by (3) may be an arbitrary real number by considering the limit $t, L \rightarrow \infty$ with $\alpha = t/L^2$ fixed. In particular the eigenvalues at $\alpha = n/r$ with $n$ and $r$ being coprime positive integers form $2r$ bands given by the inequality

$$
0 \leq (-1)^{r-1}2^{r-4}f_r^{(n)}(\lambda) \leq 1 \quad (17)
$$

where $n$ specifies how these bands cluster each other. Roughly speaking, $n$ if $2n < r$ or $r - n$ if $2n > r$ stands for the number of near lying bands. An irrational flux is realized as an appropriate limit $r, n \rightarrow \infty$. This implies that the finite number of bands for a rational
flux split into an infinite number of tiny bands (maybe a Cantor set). Though the spectrum appears smoothly varying with the flux due to the low resolution of the plot, such tremendous splittings and focusings of the bands take place continually during a small change of the flux.

As was argued in Ref. [5], it is possible to find the index of the ODO $D$ defined by

$$D = 1 + \sigma_3 \frac{H}{\sqrt{H^2}}.$$  \hfill (18)

The index of $D$ is given by

$$\text{index} D = \text{Tr} \sigma_3 \left( 1 - \frac{1}{2} D \right) = -\frac{1}{2} \text{Tr} \frac{H}{\sqrt{H^2}},$$  \hfill (19)

where Tr implies the sum over the lattice coordinates as well as the trace over the spin indices. Since the trace on the RHS of this expression equal the number of positive eigenvalues of $H$ minus the number of negative eigenvalues, we can find index$D$ for constant field strength configurations by counting the root asymmetry of the secular equations (15), where by root asymmetry of a polynomial equation we mean the number of positive roots minus the number of negative roots. In general the origin $\lambda = 0$ lies out side of the intervals defined by the inequalities (16) as can be seen from Fig. 1. If this is satisfied, it is possible to relate index$D$ with the root asymmetry of $f_{r's'}^{(n')} (\lambda) = 0$. We thus obtain for $t = nr = n'r's'^2$

$$\text{index} D = -\frac{1}{2} r's'\sigma_3^{(n')} ,$$  \hfill (20)

where $\sigma_3^{(n)}$ stands for the root asymmetry of $f_{r}^{(n)}(\lambda) = 0$. 

Figure 1: The spectrum of $H$ for uniform magnetic fields ($L = 23, m = 1$).
To understand $\sigma_r^{(n)}$ the roots of $f_r^{(1)}(\lambda) = 0$ are plotted in Fig. [3]. As is easily seen, the root with minimum absolute value for each $r$ monotonically increases as $r$ and changes the sign from minus to plus at $r = 4$. We thus find $\sigma_r^{(1)} = 0$ for $r \leq 3$ and $\sigma_r^{(1)} = 2$ for $r \geq 4$. In the case $n > 1$ the behaviors of $\sigma_r^{(n)}$ are rather complicated for $r < 4n$. However, we know $\sigma_r^{(n)} = 0$ for $r = 2n$, $3n$ and $\sigma_r^{(n)} = 2n$ for $r = 4n$, $5n$, $\cdots$ from the factorization relation (11). In fact it is possible to show $\sigma_r^{(n)} = 2n$ for sufficiently large $r$ ($\geq 4n$) rigorously by directly evaluating the spectral asymmetry of the hermitian matrix (6) for $p = q = 0$ in the large $r$ limit [9]. We thus obtain for $r's^2 \geq 4n'$ or equivalently for $t = nr \leq L^2/4$

$$\text{index}D = -rs'n' = -rn = -\frac{1}{2\pi} \sum_x F_{12}(x) ,$$

(21)

where use has been made of (9). This is the index theorem for the ODO (18) in the abelian gauge background in two dimensions [4, 5].

We have argued that the HWDO can be block-diagonalized for an arbitrary uniform magnetic field and the spectrum is described by a relativistic analog of the Harper equation. We have found that the polynomials characterizing the spectrum possess a remarkable factorization property, from which we can understand the fractal nature of the spectrum and count the degeneracy of the eigenvalue. The root asymmetry of these polynomials turned out to be related with the index of the ODO. We have established the index theorem for the uniform magnetic field $|F_{12}(x)| \leq \pi/2$. The bound is of course not optimal and depends on the choice of the parameter $m$. 

Figure 2: The spectrum of $H$ for uniform magnetic fields ($L = 23$, $m = 1/2$).
Figure 3: Roots of $f_r^{(1)}(\lambda) = 0$ for $1 \leq r \leq 70$. The two curves indicate the known bound $|\lambda| \geq \sqrt{1 - (2 + \sqrt{2}) \sin \frac{\pi}{r}}$.

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