PARTIAL DEGREE FORMULAE FOR PLANE OFFSET CURVES

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Abstract. In this paper we present several formulae for computing the partial degrees of the defining polynomial of the offset curve to an irreducible affine plane curve given implicitly, and we see how these formulae particularize to the case of rational curves. In addition, we present a formula for computing the degree w.r.t the distance variable.

1. Introduction

Offset curves and surfaces are well-known geometric objects in the field of computer aided geometric design, possibly because they constitute a powerful tool in many applications (see [6], [11], [13]). On the other hand, offset construction is a real mathematical challenge. Even though one starts from a very simple curve/surface, the offset is usually much more complicated. Because of this fact, many authors try to deduce a priori information (on applied, algorithmic, or even theoretical aspects) of the offset from the original generating curve/surface. For instance, relevant results have been achieved in problems like: the determination of the genus of the offset (see [4]), deciding the rationality and parametrizing offsets (see [9], [12], [13], [14], [16]), implicitization techniques (see [9], [10], [19]), analyzing its topological type (see [11], [17]), studying analytic and algebraic properties (see [17], [8], [17]), etc.

An additional problem, not mentioned above and that is the central topic of this paper, is the computation of the degree of the offset. Results in this direction, for offset curves, can be found in [8] for the parametric case, in [15] for the implicit and parametric case, and in [2] for the implicit case. Note that the knowledge of the offset degree can be applied, for instance, for constructing ad hoc offset implicitization algorithms based on interpolation techniques.

All the contributions mentioned above deal with the problem of computing the degree of the offset curve; that is the total degree of its defining polynomial. In this paper, we complete this analysis providing formulae for the partial degree of the offset defining polynomial w.r.t. each variable, including the distance one. This extension of the work presented in [15] may have relevant implications in the improvement of interpolation-based algorithms for implicitizing, since with these additional information the interpolation space is reduced.

In order to formally state the problem, we consider a polynomial \(g(x_1, x_2, d)\) in the variables \(\{x_1, x_2, d\}\) such that for all values \(d_0\) of \(d\), but either none or finitely many exceptions, \(g(x_1, x_2, d_0)\) is the implicit equation of the offset at distance \(d_0\);
this polynomial is called the \textit{generic offset equation} and its existence and specialization properties are established in Section 2. In this situation, the problem consists in computing the partial degrees $\deg_{x_1}(g)$, $\deg_{x_2}(g)$, and $\deg_y(g)$. Concerning to the coordinate partial degrees, i.e. $\deg_{x_1}(g)$, $\deg_{x_2}(g)$, we present four different formulae; two of them for the implicit cases, and the two others for the parametric case. The distance degree formula is stated assuming that the input generator curve is given by means of its implicit equation.

The strategy we follow for developing the formulae is essentially the one used in \cite{15}. That is, we consider the intersection of the offset with a general vertical/horizontal line. Then, the partial degree is the number of intersection points. This number of intersection points is deduced from the intersection points of the original curve with an auxiliary curve, directly deduced from the input, and constructed ad hoc for each degree problem. Therefore, explicit knowledge on the offset is avoided. Note that the main difference, of the reasoning here and the reasoning in \cite{15}, is that the total degree of a curve is the number of intersections with a generic line but, for the partial degrees, generic vertical or horizontal lines need to be considered.

The structure of the paper is as follows. In Section 2 we introduce the notion of generic offset and generic offset equation, and we establish their main properties. In Section 3 we describe the theoretical strategy for computing the partial degree formulae. In Section 4 we introduce the auxiliary curve $S$ as well as the fake and non-fake intersection points. Finally, in Section 5 we apply these ideas to develop the partial degree formulae for the implicit case. The particularization of these formulae to the parametric case is done in Section 6. After that, the paper focuses on the distance degree formula. This is done in two sections. In Section 7 we show how to adapt the strategy for this special case, and in Section 8 the distance degree formula is deduced. The papers ends with an appendix (in page 28) where all the degrees (total and partial) are listed for a collection of curves.

2. The generic equation of the offset

We start recalling the classical and intuitive concept of offset curve. This notion will be formalized in this section. Let $C$ be a plane curve, and let $p \in C$. Let $L_N$ be the normal line to $C$ at $p$ (assume for now that this normal line is well defined). Let $q_1, q_2$ be the two points of $L_N$ at a fixed distance $d_0 \in \mathbb{C}^*$ of $p$. Then, the offset curve (or parallel curve) to $C$ at distance $d_0$, is the set $O_{d_0}(C)$ of the points $q_i$ obtained by means of this geometric construction.

As the distance $d_0$ varies, different offset curves are obtained. The idea is to have a global expression of the offset for all (or almost all) distances. This motivates the concept of \textit{generic equation of the offset} to $C$. This generic equation is a polynomial, depending on the variable distance $d$, such that for every (or almost every, see the examples below) value of $d$, the equation specializes to the equation of the offset at that particular distance.

Using this informal definition of generic offset equation, and using Gröbner basis techniques, one can see that if $C$ is the parabola $y^2 = x_1^2$, then the generic equation of its offset is:

$$g(x_1, x_2, d) = -48 d^2 x_1^4 - 32 d^2 x_1^2 x_2^2 + 48 d^4 x_1^2 + 16 x_1^6 + 16 x_2^2 x_1^4 + 16 d^3 x_2^2 - 16 d^6 - 40 x_2 x_1^4 - 32 x_1^2 x_2^3 + 8 d^2 x_1^2 - 32 d^2 x_2^3 + 32 d^4 x_2 + x_1^4 + 32 x_1^2 x_2^2 + 16 x_2^4 - 20 d^2 x_1^2 - 8 d^2 x_2^2 - 8 d^4 - 2 x_2 x_1^2 - 8 x_2^3 + 8 x_2 d^2 + x_2^2 - d^2.$$
In addition, and using again Gröbner basis techniques, one may check that for every distance the generic offset equation specializes properly. However, the generic offset equation of the circle \( y_1^2 + y_2^2 - 1 = 0 \) factors as the product of two circles of radius \( 1 + d \) and \( 1 - d \), that is:

\[
g(x_1, x_2, d) = (x_1^2 + x_2^2 - (1 + d)^2) \left( x_1^2 + x_2^2 - (1 - d)^2 \right).
\]

Observe that for \( d_0 = 1 \), this generic equation gives

\[
g(x_1, x_2, 1) = (x_1^2 + x_2^2 - 2^2) (x_1^2 + x_2^2 - 2^2) = (x_1^2 + x_2^2 - 2^2) (x_1 + ix_2) (x_1 - ix_2)
\]

which describes the union of a circle of radius 2, and two complex lines. This is not a correct representation of the offset at distance 1 to \( C \), which consists of the union of the circle of radius 2 and a point (the origin). Thus, in this example we see that the generic offset equation does not specialize properly for \( d_0 = 1 \). Nevertheless, for every other value of \( d_0 \) the specialization is correct.

In these examples we have introduced some of the notation that we will use in the sequel. The variables \( \bar{y} = (y_1, y_2) \) will be used for the equation of the curve \( C \), and \( \bar{x} = (x_1, x_2) \) will be used for the equation of the offset to \( C \), both for a particular distance or generically. The implicit equation of \( C \) is \( f(y_1, y_2) = 0 \) and the generic offset equation is \( g(x_1, x_2, d) = 0 \).

After these examples, we proceed to formally introduce the notions of offset and of generic offset equation. This can be done using a geometrical approach, by means of incidence diagrams (see [17]), or equivalently using results from Elimination Theory. Here we follow this second approach. For this purpose, let \( C \) be an irreducible algebraic plane curve given by the polynomial \( f(y_1, y_2) \in \mathbb{C}[y_1, y_2] \) such that \( f \) does not divide to \( f_1^2 + f_2^2 \). Note that this implies that the set of non-isotropic points of \( C \) is open and non-empty (see Proposition 2 in [17]); i.e. the set of points of \( C \) at which the non-zero normal vectors \( (n_1, n_2) \) satisfies that \( n_1^2 + n_2^2 \neq 0 \). Moreover, by Proposition 1 in [17], if \( C \) is real and irreducible this condition holds. Consider the following polynomial system:

\[
\begin{align*}
\begin{bmatrix}
b(\bar{y}, \bar{x}, d) : & f(y_1, y_2) = 0 \\
n(\bar{y}, \bar{x}) : & (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 = 0 \\
w(\bar{y}, u) : & -f_2(\bar{y})(x_1 - y_1) + f_1(\bar{y})(x_2 - y_2) = 0 \\
& u \cdot (f_1^2(\bar{y}) + f_2^2(\bar{y})) - 1 = 0
\end{bmatrix} & \equiv \mathcal{S}_1(d)
\end{align*}
\]

where \( f, b, n, w \in \mathbb{C}[\bar{y}, \bar{x}, d, u] \), with \( \bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \) and \( f_1 = \frac{\partial f}{\partial y_1} \).

Note that \( d \) is considered here as a variable, representing the distance. The second equation, \( b(\bar{y}, \bar{x}, d) \), represents a circle of radius \( d \) centered at the point \( \bar{y} \in C \), and the third one defines the normal line to \( C \) at \( \bar{y} \). The last equation excludes the possibility of \( \bar{y} \) being a singular (or, in general, isotropic) point of \( C \). In addition, observe that we have assumed that \( f \) does not divide to \( f_1^2 + f_2^2 \), and therefore \( \mathcal{S}_1(d) \) has always solutions.

First, we will establish the existence of the generic equation of the offset. Let

\[
I(d) = \langle f(\bar{y}), b(\bar{y}, \bar{x}, d), n(\bar{y}, \bar{x}), w(\bar{y}, u) \rangle
\]

be the ideal in \( \mathbb{C}[\bar{y}, \bar{x}, d, u] \) generated by the polynomials \( \{ f, b, n, w \} \). We denote by

\[
\Omega(d) = \mathbb{V}(I(d)) \subset \mathbb{C}^6
\]

the affine algebraic set defined by \( I(d) \); that is, \( \Omega(d) \) is the set of solutions in \( \mathbb{C}^6 \) of the system \( \mathcal{S}_1(d) \).
Now, for every particular \(d_0 \in \mathbb{C}^*\), let
\[
I(d_0) = < f(\tilde{y}), b(\tilde{y}, \tilde{x}, d_0), n(\tilde{y}, \tilde{x}), w(\tilde{y}, u) >
\]
be the ideal in \(\mathbb{C}[\tilde{y}, \tilde{x}, u]\) generated by \(\{ f, b(d_0), n, w\} \). And let
\[
\Omega(d_0) = V(I(d_0)) \subset \mathbb{C}^5
\]
be the affine algebraic set defined by \(I(d_0)\).

We consider the following two projection maps:
\[
\pi : \mathbb{C}^6 \to \mathbb{C}^3; (\tilde{y}, \tilde{x}, d, u) \mapsto (\tilde{x}, d) \quad \text{(non-specialized projection)}
\]
\[
\pi_0 : \mathbb{C}^5 \to \mathbb{C}^2; (\tilde{y}, \tilde{x}, u) \mapsto \tilde{x} \quad \text{(specialized projection)}
\]
In this situation, if one denotes by \(\mathcal{A}^*\) the Zariski closure of a set \(\mathcal{A}\), one has the following definition:

**Definition 2.1.** The offset to the curve \(\mathcal{C}\) at a distance \(d_0\) is
\[
\mathcal{O}_{d_0}(\mathcal{C}) = (\pi_0 (\Omega(d_0)))^* \subset \mathbb{C}^2
\]
The generic offset to the curve \(\mathcal{C}\) is
\[
\mathcal{O}_d(\mathcal{C}) = (\pi (\Omega(d)))^* \subset \mathbb{C}^3
\]

**Remark 2.2.** Note that this means that
\[
\mathcal{O}_d(\mathcal{C}) = V(\tilde{I}(d))
\]
where \(\tilde{I}(d) = I(d) \cap \mathbb{C}[\tilde{x}, d]\) is the \((\tilde{y}, u)\)-elimination ideal of \(I(d)\). Similarly
\[
\mathcal{O}_{d_0}(\mathcal{C}) = V(\tilde{I}(d_0))
\]
where \(\tilde{I}(d_0) = I(d_0) \cap \mathbb{C}[\tilde{x}]\) (see [2], Closure Theorem, p. 122).

The following result guarantees the existence of an equation for the generic offset.

**Lemma 2.3.** \(\mathcal{O}_d(\mathcal{C})\) is a surface in \(\mathbb{C}^3\).

**Proof.** This proof follows the reasoning of the proof of Lemma 1 in [17]. Let \(K\) be a component of \(\Omega(d)\), and let \((p, q, u_0, d_0) \in K\). Since \(w(p, u_0)=0\), \(p \in \mathcal{C}\) is non-isotropic. Moreover, \(q \in \mathcal{O}_{d_0}(\mathcal{C})\). Take \(P(t) = (x(t), y(t))\) to be a place of \(\mathcal{C}\) centered at \(p\) (\(P(t)\) is a local parametrization of \(\mathcal{C}\) by power series). Let \(N(t)\) be the associated normal vector, and let \(Q(t)\) be the lifting of \(P(t)\) to \(q \in \mathcal{O}_{d_0}(\mathcal{C})\) whose center is \(q\). That is,
\[
Q(t) = P(t) \pm d \frac{N(t)}{||N(t)||}
\]
The choice of sign is decided with the condition that \(Q(t)\) is centered at \(q\). Moreover, note that since \(p\) is non-isotropic, then \(Q(t)\) is also a local parametrization by power series. Then
\[
R(t, d) = \left( P(t), Q(t), d, \frac{1}{||N(t)||^2} \right)
\]
is a local parametrization of \(K\) at \((p, q, u_0, d_0)\). It follows that \(\dim K = 2\).

Therefore \(\mathcal{O}_d(\mathcal{C})\) is defined by a polynomial in \(\mathbb{C}[\tilde{x}, d]\) (see [18], p.69, Th.3). Thus, we arrive at the following definition:

**Definition 2.4.** The generic offset equation is the defining polynomial of the surface \(\mathcal{O}_d(\mathcal{C})\). In the sequel, we denote by \(g(x_1, x_2, d) = 0\) the generic offset equation.

**Remark 2.5.**
For this purpose, we analyze the solutions of system $S$ the points in $C$ not known, we compute indirectly the number of points in $O$. Let therefore $L$ be the equation of a generic horizontal line. Let $\ell$ be the equation of a generic horizontal line and the offset. The following theorem gives the fundamental property of the generic offset.

**Theorem 2.6.** For all but finitely many exceptions, the generic offset equation specializes properly. That is, there exists a finite (possibly empty) set $\Upsilon \subset \mathbb{C}$ such that if $d_0 \not\in \Upsilon$, then

$$g(x_1, x_2, d_0) = 0$$

is the equation of $O_{d_0}(C)$.

**Proof.** Since $g(\bar{x}, d)$ defines the equation of $O_d(C)$, and

$$O_d(C) = \mathbf{V}(\bar{I}(d))$$

where $\bar{I}(d) = I(d) \cap \mathbb{C}[\bar{x}, d]$ is the $(\bar{y}, u)$-elimination ideal of $I(d)$ (see Remark 2.2), it follows that if $G(d)$ is a Gröbner basis of $I(d)$ w.r.t. an elimination ordering that eliminates $(\bar{y}, u)$, then up to multiplication by a non-zero constant, $G(d) \cap \mathbb{C}[\bar{x}, d] = \{g(\bar{x}, d)\}$ is a Gröbner basis of $\bar{I}(d)$. But then (see [5], exercise 7, page 284) there is a finite (possibly empty) set $\Upsilon \subset \mathbb{C}$ such that for $d_0 \not\in \Upsilon$, $G(d_0)$ specializes well to a Gröbner basis of $\bar{I}(d_0)$. It follows that, since $\bar{I}(d_0) = I(d_0) \cap \mathbb{C}[\bar{x}]$, then $G(d_0) \cap \mathbb{C}[\bar{x}] = \{g(\bar{x}, d_0)\}$ is a Gröbner basis of $\bar{I}(d_0)$. Thus, for $d_0 \not\in \Upsilon$, $g(\bar{x}, d_0)$ is the equation of $O_{d_0}(C) = \mathbf{V}(\bar{I}(d_0))$. □

**Remark 2.7.** Note that all the results in this section, though they have been presented for plane curves, extend naturally to the case of offsets to irreducible hypersurfaces (over algebraically closed fields of characteristic zero).

### 3. Strategy Description for the Partial Degree Formulae

First we deal with the problem of computing the partial degree in $x_i$ of the generic offset equation $g(\bar{x}, d)$. Let $\delta_i$ be the partial degree in $x_i$ of $g$. We will describe how to compute $\delta_1$. Then, simply exchanging the variables $x_1$ and $x_2$ allows to compute $\delta_2$. Also, we will exclude w.l.o.g. in our analysis the case where $C$ is a line. Note that, in particular, this implies that $\delta_i > 0$ in all cases.

When analyzing the offset total degree problem in our previous paper [15], the basic idea was to indirectly determine the number of intersection points between a generic line and the offset $O_d(C)$. Here, for the partial degree problem, we follow a similar strategy. However, in order to compute $\delta_1$, the generic line must be horizontal. Let therefore

$$\ell(\bar{x}, k) : x_2 - k = 0$$

be the equation of a generic horizontal line $L(k)$. Since the generic offset equation is not known, we compute indirectly the number of points in $O_d(C) \cap L(k)$, by counting the points in $C$ that, in a 1:1 correspondence, generate the points in $O_d(C) \cap L(k)$. For this purpose, we analyze the solutions of system $\mathcal{G}_1(d)$ lying on the line $L(k)$.

(1) Observe that the polynomial $g$ may be reducible (recall the example of the circle) but by construction it is always square-free. Moreover, $g$ is either irreducible or factors into two irreducible factors not depending only on $d$; this is so because, generically in $d$, the offset has at most two irreducible components (see [17], Theorem 1).

(2) It might happen that $g(\bar{x}, d)$ has a factor in $\mathbb{C}[d]$. In order to avoid this, and w.l.o.g., we will take the generic offset equation to be primitive w.r.t. $\bar{x}$. 

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**Proof.** Since $g(\bar{x}, d)$ defines the equation of $O_d(C)$, and

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That is, the solutions of the system:
\[
\begin{align*}
  b(\bar{y}, \bar{x}, d) : & \quad (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 = 0 \\
  n(\bar{y}, \bar{x}) : & \quad -f_2(\bar{y})(x_1 - y_1) + f_1(\bar{y})(x_2 - y_2) = 0 \\
  w(\bar{y}, u) : & \quad u \cdot (f_1(\bar{y}) - f_2(\bar{y})) - 1 = 0 \\
  \ell(\bar{x}, k) : & \quad x_2 - k = 0
\end{align*}
\]
The following result provides the theoretical foundation of our strategy, by establishing the 1:1 correspondence between the points in \( \mathcal{O}_d(\mathcal{C}) \cap \mathcal{L}(k) \), and the points in \( \mathcal{C} \) that generate them.

We recall that a ramification point of a curve is a point on the curve where at least one of the partial derivatives of the implicit equation vanishes. In our case, since we are analyzing the partial degree \( \delta_1 \), by abuse of notation, whenever we speak about ramification points we mean a ramification point where the partial derivative w.r.t. \( y_2 \) vanishes.

**Theorem 3.1.** There exists a non-empty Zariski open subset \( \Delta \) of \( \mathbb{C}^2 \) such that for \( (d_0, k_0) \in \Delta \):

1. There exist exactly \( \delta_1 \) solutions \( \Gamma = \{(p_i, q_i, u_i)\}_{i=1}^{\delta_1} \) of \( \mathcal{S}_2(d_0, k_0) \) satisfying that:
   a. \( q_1, \ldots, q_{\delta_1} \) are all different and \( \mathcal{L}(k_0) \cap \mathcal{O}_{d_0}(\mathcal{C}) = \{q_1, \ldots, q_{\delta_1}\} \).
   b. \( p_1, \ldots, p_{\delta_1} \) are different regular non-ramification points of \( \mathcal{C} \).
2. None of the points in \( \mathcal{C} \cap \mathcal{L}(k_0) \) is a ramification point of \( \mathcal{C} \).

**Proof.** Let us consider the generic offset equation as a polynomial in \( \mathbb{C}[x_2, d][x_1] \), by writing:
\[
g(x_1, x_2, d) = \sum_{i=0}^{\delta_1} g_i(x_2, d)x_1^i,
\]
where \( g_{\delta_1} \) is not identically zero. Observe that by assumption \( \delta_1 > 0 \). Thus, the set of solutions of \( g_{\delta_1}(k, d) = 0 \) is either empty, or a curve \( \Psi_1 \) in \( \mathbb{C}^2 \). We define \( \Delta_1 = \mathbb{C}^2 \setminus \Psi_1 \).

Besides, by Theorem 2.10 we know that there is only a finite set of bad distances, \( \Upsilon = \{d_1, \ldots, d_m\} \), such that for \( d_0 \notin \Upsilon \), the equation of \( \mathcal{O}_{d_0}(\mathcal{C}) \) is \( g(x_1, x_2, d_0) = 0 \). Let \( \Psi_2 \) be the union of the lines with equations \( d = d_i \) for \( d_i \in \Upsilon \). We define \( \Delta_2 = \Delta_1 \setminus \Psi_2 \).

Then, for \( (d_0, k_0) \in \Delta_2 \),
\[
g(x_1, d_0, k_0) = \sum_{i=1}^{\delta_1} g_i(d_0, k_0)x_1^i = 0
\]
is a polynomial in \( x_1 \) of degree \( \delta_1 \) (the leading coefficient does not vanish because of the construction of \( \Delta_1 \)). Now, since \( g \) is square-free (see Remark 2.4), \( \text{Res}_{x_1}(g(x_1, k, d)) \) is a non-identically zero polynomial in \( (k, d) \). Thus, it defines a curve \( \Psi_3 \) in the \( (k, d) \)-plane. We define \( \Delta_3 = \Delta_2 \setminus \Psi_3 \).

Let now \( \sigma = (\sigma_1, \sigma_2) \) be one of the finitely many singularities or vertical ramification points of \( \mathcal{C} \) (that is, one of the finitely many solutions of \( f = f_2 = 0 \); note that \( \mathcal{C} \) is irreducible). We compute the following resultant between the generic offset polynomial and the equation of a \( d \)-circle centered at \( \sigma \):
\[
R_\sigma(k, d) = \text{Res}_{x_1}(g(x_1, k, d), (x_1 - \sigma_1)^2 + (k - \sigma_2)^2 - d^2)
\]
This resultant can only vanish identically if both polynomials have a common factor in \( x_1 \). But the polynomial defining the circle is irreducible. Thus, this could only happen if, for every \( d_0 \notin \Upsilon \), \( \mathcal{O}_{d_0}(\mathcal{C}) \) contains a circle of radius \( d_0 \) centered at \( \sigma \). This would imply that \( \mathcal{C} \) is itself a circle centered at \( \sigma \), which is impossible since \( \sigma \in \mathcal{C} \). Thus, \( R_\sigma \) is not zero,
and it defines a curve in $\mathbb{C}^2$. Let $\Psi_4$ be the curve obtained as the union of such curves for all the possible points $\sigma$. We define $\Delta_4 = \Delta_3 \setminus \Psi_4$.

Now, observe that for $(d_0, k_0) \in \Delta_4$, no intersection point of $\mathcal{O}_{d_0}(C)$ and $\mathcal{L}(k_0)$ can be associated with a singularity or vertical ramification point of $C$.

Since $C$ has only finitely many vertical ramification points, we can exclude those values of $k$ such that the line $x_2 = k$ passes through one of those vertical ramification points. Let $\Psi_5$ be the finite union of such lines, and define $\Delta_5 = \Delta_4 \setminus \Psi_5$.

Take

$$\Delta = \Delta_5$$

Then, if $(d_0, k_0) \in \Delta$, because of the construction of $\Delta_2$, we know that $g(x_1, x_2, d_0)$ is the equation of $\mathcal{O}_{d_0}(C)$. Besides, the equation

$$g(x_1, d_0, k_0) = \sum_{i=1}^{\delta_1} g_i(d_0, k_0) x_i^1 = 0$$

has exactly $\delta_1$ different roots because of the construction of $\Delta_1$ and $\Delta_3$. Every solution of this equation represents an affine intersection point of $\mathcal{O}_{d_0}(C)$ and $\mathcal{L}(k_0)$. Moreover, because of the choice of $\Delta_4$, these points are associated to regular non-ramification affine points of $C$. This proves statement (1) of the theorem. Moreover, for $(d_0, k_0) \in \Delta$ the system $f(\bar{y}) = 0, f_2(\bar{y}) = 0, y_2 = k_0$ has no solutions, because of the construction of $\Delta_2$.

This proves statement (2). □

Remark 3.2.

1. In the sequel we assume that for $(d_0, k_0) \in \Delta$, $g(\bar{y}, d_0) = 0$ is the implicit equation of $\mathcal{O}_{d_0}(C)$. This can be assumed w.l.o.g., simply replacing $\Delta$ by $\Delta \setminus ([\mathbb{C} \setminus \mathcal{Y}] \times \mathbb{C})$ (see Theorem 2.3).

2. Note that besides the $\delta_1$ solutions mentioned in the theorem, the system $\mathcal{S}_2(d_0, k_0)$ may have other solutions. We will analyze in the next section the distinction between these two types of solutions of the system.

We have seen that, generically in $k$ and $d$, every point $q_j \in \mathcal{O}_d(C) \cap \mathcal{L}(k)$ is associated to a regular affine point $p_j \in C$, and this correspondence is a bijection. The number of such points is the offset partial degree $\delta_1$. The strategy now is to eliminate $x_1, x_2$ from the system $\mathcal{S}_2(d, k)$ in order to obtain information about $\delta_1$ through the solutions $(y_1, y_2)$ of the resulting system. This means that we switch our attention from the points $q = (x_1, x_2) \in \mathcal{O}_d \cap \mathcal{L}(k)$ to the associated points $p = (y_1, y_2) \in C$. In order to do that, we will identify these associated points as intersection points of $C$ with a certain auxiliary curve $\mathcal{S}$ (see Definition 4.1 below).

4. The Auxiliary Curve $\mathcal{S}$

This section is devoted to the study of the auxiliary curve mentioned at the end of the previous section. This curve is obtained computing a Gröbner basis to eliminate $x_1, x_2$ and $u$ in the system $\mathcal{S}_2(d, k)$. Doing this elimination, one arrives at the following definition:

**Definition 4.1.** Let $s$ be the polynomial:

$$s(\bar{y}, d, k) = (f_2^2 + f_3^2)(y_2 - k)^2 - f_3^2 d^2.$$  

For every $(d_0, k_0) \in \mathbb{C}^2$, the auxiliary curve $\mathcal{S}(d_0, k_0)$ to $C$ is the affine plane curve defined over $\mathbb{C}$ by the polynomial $s(\bar{y}, d_0, k_0)$.

The following theorem relates the solutions in Theorem 3.1 with the intersection points of $C$ and the auxiliary curve.
Theorem 4.2. Let $\Delta$ be as in Theorem 3.1, let $(d_0, k_0) \in \Delta$, and let $\Gamma$ be the set of $\delta_1$ solutions of $\mathcal{G}_2(d_0, k_0)$ appearing in Theorem 3.1. Then it holds that:

(a) If $(p, q, u_0) \in \Gamma$, then $p \in C \cap \mathcal{S}(d_0, k_0)$.

(b) If $p \in C \cap \mathcal{S}(d_0, k_0)$ and $p$ is not of ramification of $C$, there exist $q \in \mathbb{C}^2$ and $u_0 \in C$ such that $(p, q, u_0) \in \Gamma$.

Remark 4.3.

(1) The solution $(p, q, u_0)$ in statement (b) of Theorem 4.2 can be expressed as:

$$a_1 = \frac{(-f_1(p)b_2 + f_2(p)b_1 + f_1(p)k_0)}{f_2(p)}, \quad a_2 = k_0, \quad u_0 = \frac{1}{f_2^2(p) + f_2^2(p)}$$

where $p = (b_1, b_2)$ and $q = (a_1, a_2)$. Also note that, since $p$ is not of ramification of $C$, it follows that $f_2(p) \neq 0$. Moreover, since $f_2(p) \neq 0, d_0 \neq 0$ and $p \in \mathcal{S}(d_0, k_0)$, then $f_2^2(p) + f_2^2(p) \neq 0$.

(2) Note that $C \cap \mathcal{S}(d_0, k_0)$ may contain other points besides those appearing in the theorem. For example, every affine singularity of $C$ is also a point of $C \cap \mathcal{S}(d_0, k_0)$. But the theorem shows a 1:1 correspondence between $\Gamma$ and the points in $C \cap \mathcal{S}(d_0, k_0)$ that are not of ramification in $C$.

Proof.

(a) We consider the polynomials

$$v_1(\bar{y}) = -f_2^2(\bar{y}), \quad v_2(\bar{y}, \bar{x}) = f_1(\bar{y})(x_2 - y_2) + f_2(\bar{y})(x_1 - y_1), \quad v_3(\bar{y}, \bar{x}, \bar{k}) = (f_2^2(\bar{y}) + f_1^2(\bar{y}))(2y_2 - x_2 - k)$$

Then it can be easily checked that

$$s(\bar{y}, \bar{d}, \bar{k}) = v_1(\bar{y})b(\bar{y}, \bar{x}, \bar{d}) + v_2(\bar{y}, \bar{x})n(\bar{y}, \bar{x}) + v_3(\bar{y}, \bar{x}, \bar{k})\ell(\bar{y}, \bar{k})$$

Now, let $(p, q, u_0) \in \Gamma$. Then by Theorem 3.1(b), one has that $p \in C$. Moreover, because of the above description of the polynomial $s$, and taking into account that $(p, q, u_0)$ is a solution of $\mathcal{S}_2(d_0, k_0)$, one has that $p \in \mathcal{S}(d_0, k_0)$.

(b) Let $p = (b_1, b_2) \in C \cap \mathcal{S}(d_0, k_0)$ be such that $f_2(p) \neq 0$. Then we consider

$$q = (a_1, a_2) = \left(\frac{-f_1(p)b_2 + f_2(p)b_1 + f_1(p)k_0}{f_2(p)}, k_0\right)$$

and

$$u_0 = \frac{1}{f_2^2(p) + f_2^2(p)}$$

Note that $s(p, d_0, k_0) = (f_1^2(p) + f_2^2(p))(b_2 - k_0)^2 - f_2^2(p)d_0^2 = 0$ and $f_2(p)d_0^2 \neq 0$, and hence $f_1^2(p) + f_2^2(p) \neq 0$. Now, let us see that $(p, q, u_0) \in \Gamma$. Substituting $(p, q, u_0)$ in $\mathcal{S}_2(d_0, k_0)$ one sees that it is a solution of the system. Moreover, $p \in C$, it is regular and it is not of ramification. Furthermore, because of the vanishing of $f, b, n$ and $\ell$ at $(p, q, u_0)$, one has that $q \in \mathcal{L}(k_0) \cap \mathcal{O}_{d_0}(C)$. Therefore $(p, q, u_0) \in \Gamma$.

In Theorem 4.2 we have seen that (generically in $(d, k)$) there is a 1:1 correspondence between the $\delta_1$ points in $\Gamma$ and the points in $C \cap \mathcal{S}(d, k)$, where $f_2$ does not vanish. The advantage of this strategy is that, while the generic offset equation is not known, both $f$ and $s$ are known polynomials. Therefore we can use standard techniques, such as those provided by Bézout’s Theorem, to analyze the intersection points between the two plane curves. But, for our purposes, we have to ensure the following: first, we are going to consider all the intersection points of $C$ and $\mathcal{S}(d, k)$, so we have to treat the problem projectively. Thus, we consider the projective closures of the curves, and we denote them by $\overline{C}$ and $\overline{\mathcal{S}(d, k)}$, respectively. Secondly,
\( \mathfrak{C} \cap \overline{S(d, k)} \) may contain also points that are not associated to points in \( \Gamma \), and we need to distinguish them. This fact motivates the following definition.

**Definition 4.4.** Let \( \Delta \) be as in Theorem 3.1 and let \( (d_0, k_0) \in \Delta \).

1. The affine intersection points of \( \mathfrak{C} \) and \( \overline{S(d_0, k_0)} \) that are not of ramification of \( \mathfrak{C} \) are called non-fake points.
2. The remaining intersection points of \( \mathfrak{C} \) and \( \overline{S(d_0, k_0)} \) are called fake points.

We denote by \( \mathcal{F} \) the set of all fake points.

**Remark 4.5.** Observe that because of Theorems 3.1 and 4.2, for each \( (d_0, k_0) \in \Delta \) the number of non-fake points is precisely the partial degree \( \delta_1 \).

Although \( \mathcal{F} \) seems to depend on the choice of \( (d_0, k_0) \in \Delta \), in the next proposition we show that it is in fact invariant. Nevertheless, the set of non-fake points does depend on \( (d, k) \). Since we are working projectively, we denote by \( F, F_1, F_2 \) and \( S \) the homogenization w.r.t. a new variable \( y_3 \) of the polynomials \( f, f_1, f_2 \) and \( s \) respectively. We also denote \( \bar{y}_H = (y_1 : y_2 : y_3) \). Observe that:

\[
S = (F_2^2 + F_1^2)(y_2 - ky_3)^2 - F_2^2y_3^2d^2.
\]

**Proposition 4.6** (Invariance of the fake points). The set \( \mathcal{F} \) is finite, and does not depend on \( \{d, k\} \). Furthermore, \( p \in \mathcal{F} \) if and only if \( p \in \mathfrak{C} \) and either \( p \) is affine and singular or \( p \) is \( (1 : 0 : 0) \) or \( p \) is at infinity satisfying \( F_1^2(p) + F_2^2(p) = 0 \).

**Proof.**

Let \( p = (a : b : c) \in \mathcal{F} \). Then there exists \( (d_0, k_0) \in \Delta \) (as in Theorem 3.1), such that \( p \in \mathfrak{C} \cap \overline{S(d_0, k_0)} \) and either \( c \neq 0 \) and \( F_2(p) = 0 \) or \( c = 0 \). If \( c = 0 \), since \( S(p, d_0, k_0) = 0 \) one has that \( (F_1^2(p) + F_2^2(p))b = 0 \), and hence either \( p = (1 : 0 : 0) \) or \( p \) is at infinity and it is isotropic. On the other hand, if \( c \neq 0 \) and \( F_2(p) = 0 \), since \( p \in \overline{S(d_0, k_0)} \) one has that \( F_1(p)(b - k_0c) = 0 \). Now, because of the construction of \( \Delta \) (see how \( \Delta_4 \) is defined in the proof of Theorem 3.1), \( b - k_0c \neq 0 \). Therefore, \( p \) is affine and singular.

Conversely, if \( p \in \mathfrak{C} \) and it satisfies any of the three conditions in the statement of the proposition, then \( p \in \overline{S(d_0, k_0)} \). Thus, by Definition 4.4 the implication holds.

Finally, from the above characterization it follows that \( \mathcal{F} \) is finite. \( \square \)

**Remark 4.7.** Let \( p = (a : b : 1) \) be a non-fake point. Observe then that necessarily \( b - k_0 \neq 0 \), for every \( (d_0, k_0) \in \Delta \) (see the proof of Proposition 4.6).

In order to apply Bézout’s Theorem we need to prove that \( \mathfrak{C} \) and \( \overline{S(d_0, k_0)} \) do not have common components, and we have to analyze the multiplicity of intersection of \( \mathfrak{C} \) and \( \overline{S(d_0, k_0)} \) at the non-fake points. This is the content of the following proposition:

**Proposition 4.8** (Bézout’s Theorem preparation). There exists a non-empty open subset \( \bar{\Delta} \subset \Delta \), where \( \bar{\Delta} \) is as in Theorem 3.1, such that for every \( (d_0, k_0) \in \bar{\Delta} \) the following hold:

1. \( \deg(S(d_0, k_0)) = 2 \deg(C) \),
2. \( C \) and \( S(d_0, k_0) \) have no common component,
3. if \( p \) is a non-fake point, then \( \mult_p(C, S(d_0, k_0)) = 1 \),
4. Let \( S(\bar{y}_H, d, k) \) be considered as an element of \( (\mathbb{C}[\bar{y}_H])[d, k] \):

\[
S(\bar{y}_H, d, k) = Z_{2,0}(\bar{y}_H)d^2 + Z_{0,2}(\bar{y}_H)k^2 + Z_{0,1}(\bar{y}_H)k + Z_{0,0}(\bar{y}_H)
\]
where:
\[
\begin{aligned}
Z_{2,0}(\bar{y}_H) &= -F_2^2 y_3^2 \\
Z_{0,2}(\bar{y}_H) &= (F_2^2 + F_1^2)y_3^2 \\
Z_{0,1}(\bar{y}_H) &= -2(F_2^2 + F_1^2)y_2y_3 \\
Z_{0,0}(\bar{y}_H) &= (F_2^2 + F_1^2)y_2^2,
\end{aligned}
\]
and let \( \mathcal{J}_\alpha \) be the curve defined by \( Z_\alpha(\bar{y}_H) \). Then it holds that:
\[
\bigcap_\alpha \mathcal{C} \cap \mathcal{J}_\alpha \subset \mathcal{F}.
\]

Proof.

1. Let \( S = (F_2^2 + F_1^2)(y_1 - ky_3)^2 - F_2^2 y_3^2 d^2 \). The form \( F_2^2 y_3^2 d^2 \) has degree \( 2n \) in \( y \) for \( d \neq 0 \), and the form \( (F_2^2 + F_1^2)(y_2 - ky_3)^2 \) has degree less or equal than \( 2n \) in \( y \). Thus \( \deg_y(S) \leq 2n \). Now the degree could only drop if the two forms were identical, which is generically impossible, since \( d \) does not appear in the first one and \( k \) does not appear in the second one. Thus, our claim holds.

2. Let us see that for \( (d_0, k_0) \in \Delta \), \( \mathcal{C} \) and \( \overline{\mathcal{S}(d_0, k_0)} \) have no common components. Assume that they do. Then, since \( F \) is irreducible, there exists \( K(\bar{y}_H) \in \mathbb{C}[y_1, y_2, y_3] \) such that
\[
S(\bar{y}_H, d_0, k_0) = K(\bar{y}_H)F(\bar{y}_H).
\]
Now, we will see that then \( F_2 \) vanishes on almost all point of \( \mathcal{C} \). That implies that \( \mathcal{C} \) is a line, which is impossible by assumption. Indeed, if there were infinitely many points in \( \mathcal{C} \cap \overline{\mathcal{S}(d_0, k_0)} \) with \( F_2 \neq 0 \), this would imply infinitely many affine points in \( \mathcal{C} \cap \mathcal{S}(d_0, k_0) \) with \( f_2 \neq 0 \). Then Theorems 5.1 and 4.2 would give an infinite number of affine intersections between the line \( x_2 - k_0 = 0 \) and the offset, which is impossible; note that if \( \mathcal{O}_{d_0}(\mathcal{C}) \) contains a line, then \( \mathcal{C} \) is a line.

3. Let \( (d_0, k_0) \in \Delta \), and let \( p = (a, b) \) be a non-fake point. By definition, we know that \( p \) is an affine regular point of \( \mathcal{C} \). Therefore, there is only one branch of \( \mathcal{C} \) passing through \( p \). Let \( q \) be the point in \( \mathcal{O}_{d_0}(\mathcal{C}) \cap \mathcal{L}(k_0) \) associated with \( p \) (see Theorem 5.1 (1a) for the existence of \( q \)). Also, by Theorem 5.1 (1a), \( \text{mult}_q(\mathcal{O}_{d_0}(\mathcal{C}), \mathcal{L}(k_0)) = 1 \). Thus it is enough to prove that \( \text{mult}_p(\mathcal{C}, \overline{\mathcal{S}(d_0, k_0)}) = \text{mult}_q(\mathcal{O}_{d_0}(\mathcal{C}), \mathcal{L}(k_0)) \). The proof will proceed as follows:

1. First, we consider a place \( P(t) = (y_1(t), y_2(t)) \) of \( \mathcal{C} \) centered at \( p \), and we compute \( s(P(t)) \). Note that the order of this formal power series is \( \text{mult}_p(\mathcal{C}, \mathcal{S}(d_0, k_0)) \).

2. Second, we use \( P(t) \) to obtain a place \( Q(t) \) of \( \mathcal{O}_{d_0}(\mathcal{C}) \) centered at \( q \), and we obtain \( \ell(Q(t), k_0) \). Note that the order of this formal power series is \( \text{mult}_q(\mathcal{O}_{d_0}(\mathcal{C}), \mathcal{L}(k_0)) \).

3. Finally we prove that \( \text{ord}(\ell(Q(t), k_0)) = \text{ord}(s(P(t))) \).

Let
\[
\begin{aligned}
f_1(P(t)) &= v_1 + \alpha t + \cdots \\
f_2(P(t)) &= v_2 + \beta t + \cdots
\end{aligned}
\]
for some \( v_1, v_2, \alpha, \beta \in \mathbb{C} \), where \( f_1(p) = v_1, f_2(p) = v_2 \). This means that the tangent vector to \( \mathcal{C} \) at \( p \) is \((-v_2, v_1)\) and so, there exists \( \lambda \) such that the place \( P(t) \) can be expressed in the form:
\[
P(t) : \begin{cases} y_1 = a - \lambda v_2 t + \cdots \\
y_2 = b + \lambda v_1 t + \cdots
\end{cases}
\]
The notation \( T_0 = \sqrt{v_1^2 + v_2^2} \) and \( T_1 = v_1 \alpha + v_2 \beta \) will be used in the rest of the proof. Note that, since \( (d_0, k_0) \in \Delta \), and \( p \) is non-fake, then \( v_2, T_0 \) and \( b - k_0 \) are all not zero (see Remark 5.7). Now, substituting \( P(t) \) into the polynomial \( s(y_1, y_2, d_0, k_0) \) leads to a
power series, whose zero-order term coefficient $A_0$ must vanish (because $p \in S(d_0, k_0)$).

This term is:

$$A_0 = (v_1^2 + v_2^2)(b - k_0)^2 - d_0^2 v_2^2 = T_0^2 (b - k_0)^2 - d_0^2 v_2^2$$

Therefore we get that:

$$T_0^2 = \frac{-(d_0 v_2)^2}{(b - k_0)^2}$$

The coefficient of the first-order term $A_1$ of $s(P(t))$ is:

$$A_1 = 2(-d_0^2 v_2 + T_0^2 (b - k_0) \lambda v_1 + (v_1 + v_2 \beta)(b - k_0)^2).$$

Next, using $P(t)$, we generate a place $Q(t)$ of $\mathcal{O}_{d_0}(C)$ centered at $q$. If $(y_1, y_2)$ is a regular point in $C$, the associated point $(x_1, x_2)$ in $\mathcal{O}_{d_0}(C)$ is given by:

$$(x_1, x_2) = (y_1, y_2) \pm d_0 \frac{(f_1, f_2)}{\sqrt{f_1^2 + f_2^2}}$$

Moreover, since $v_1^2 + v_2^2 \neq 0$, the power series

$$f_1^2(P(t)) + f_2^2(P(t)) = (v_1^2 + v_2^2) + 2(v_1 \alpha + v_2 \beta)t + \cdots$$

has order zero (it is a unit), and hence

$$\frac{1}{\sqrt{f_1^2(P(t)) + f_2^2(P(t))}} = \frac{1}{\sqrt{v_1^2 + v_2^2}} - \frac{v_1 \alpha + v_2 \beta}{(v_1^2 + v_2^2)^{3/2}}t + \cdots$$

So:

$$\begin{align*}
\frac{f_1}{\sqrt{f_1^2 + f_2^2}} &= \frac{v_1}{T_0} + \left(\frac{\alpha}{T_0} - \frac{T_1 v_1}{T_0^3}\right) t + \cdots \\
\frac{f_2}{\sqrt{f_1^2 + f_2^2}} &= \frac{v_2}{T_0} + \left(\frac{\beta}{T_0} - \frac{T_1 v_2}{T_0^3}\right) t + \cdots
\end{align*}$$

Therefore $Q(t)$ is one of the two places:

$$Q(t) = (x_1(t), x_2(t)) = P(t) \pm d_0 \frac{(f_1(P(t)), f_2(P(t)))}{\sqrt{f_1^2(P(t)) + f_2^2(P(t))}},$$

and so:

$$\begin{align*}
x_1(t) &= \left(a \pm \frac{d_0 v_1}{T_0}\right) + \left(-\lambda v_2 \pm \frac{d_0 \alpha}{T_0} + \frac{d_0 T_1 v_1}{T_0^3}\right) t + \cdots \\
x_2(t) &= \left(b \pm \frac{d_0 v_2}{T_0}\right) + \left(\lambda v_1 \pm \frac{d_0 \beta}{T_0} + \frac{d_0 T_1 v_2}{T_0^3}\right) t + \cdots
\end{align*}$$

Substituting $Q(t)$ in the line $\mathcal{L}(k_0)$ one has:

$$x_2(t) - k_0 = \left(b \pm \frac{d_0 v_2}{T_0} - k_0\right) + \left(\lambda v_1 \pm \frac{d_0 \beta}{T_0} + \frac{d_0 T_1 v_2}{T_0^3}\right) t + \cdots = B_0 + B_1 t + \cdots$$

Now, since $\text{mult}_q(\mathcal{O}_{d_0}(C), \mathcal{L}(k_0)) = 1$, one has that

$$B_0 = \left(b \pm \frac{d_0 v_2}{T_0} - k_0\right) = 0,$$

and

$$B_1 = \left(\lambda v_1 \pm \frac{d_0 \beta}{T_0} + \frac{d_0 T_1 v_2}{T_0^3}\right) \neq 0$$

Therefore

$$\pm T_0 = \frac{d_0 v_2}{b - k_0}$$

Substituting the above equality in $B_1$ one gets

$$B_1 = \frac{1}{T_0} \left(\mp \lambda v_1 \left(\frac{d_0 v_2}{b - k_0}\right)^3 \pm 2 \frac{d_0 v_1}{b - k_0} \left(\frac{d_0 v_2}{b - k_0}\right)^2 \mp d_0 T_1 v_2\right) = \pm\frac{1}{T_0} \left(\mp \lambda v_1 \left(\frac{d_0 v_2}{b - k_0}\right)^3 \pm 2 \frac{d_0 v_1}{b - k_0} \left(\frac{d_0 v_2}{b - k_0}\right)^2 \mp d_0 T_1 v_2\right)$$
Note that this result does not depend on the previous choice of sign. And using the same equality in $A_1$ gives:

$$A_1 = 2\left(-d_0^2\beta v_2 + \left(\frac{d_0v_2}{b-k_0}\right) (b-k_0)\lambda v_1 + T_1(b-k_0)^2\right) =$$

$$\frac{2}{b-k_0} \left(-d_0^2\beta v_2(b-k_0) + d_0v_2\lambda v_1 + T_1(b-k_0)^3\right)$$

We observe that the term in parenthesis in $A_1$ and $B_1$ coincides. Since $B_1 \neq 0$, one has that $A_1 \neq 0$ and $\text{mult}_p(C, S(d_0,k_0)) = 1$.

(4) Since we have assumed that $f$ does not divide to $f_1^2 + f_2^2$ (in particular $f_1^2 + f_2^2 \neq 0$), and that $C$ is not a line (in particular $f_2 \neq 0$), all $\mathcal{J}_n$ are algebraic curves. Now

$$\bigcap_{\alpha} \overline{\mathcal{C} \cap \mathcal{J}_n} \subset \overline{\mathcal{C} \cap \mathcal{J}(2,0)}$$

and by Proposition 4.9 $\overline{\mathcal{C} \cap \mathcal{J}(2,0)} \subset \mathcal{F}$.

(5) Let $p = (0:0:1)$ and $A(d,k) = S(p,d,k)$. If either $p \in \mathcal{F}$ or $p \notin \mathcal{C}$, then no further restriction on $\Delta$ is required. Now, let $p \in \mathcal{C}$ and $p \notin \mathcal{F}$. Then by Proposition 4.9 $P$ is not a singularity of $\mathcal{C}$. Now, if $F_2(p) \neq 0$, then $A$ is not constant. Moreover, if $F_2(p) = 0$, then $F_1(p) \neq 0$ and $A$ is not constant either. Let $\Psi$ be the curve in $\mathbb{C}^2$ defined by $A$. Then in $\Delta \setminus \Psi$ statement (5) holds. Indeed, if $p \in \big((\mathcal{C} \cap S(d_0,k_0)) \setminus \mathcal{F}\big)$, then $p \in \mathcal{C}$, $A(d_0,k_0) = 0$ and $p \notin \mathcal{F}$. Thus $(d_0,k_0) \in \Psi$.

5. Cornerstone Theorem

Later, in Section 4 when analyzing the problem of the degree in $d$ of the generic offset, we will find another situation which involves the intersection of $\mathcal{C}$ with an auxiliary curve that plays the role that $S$ plays here, and a concept of fake and non-fake intersection points with properties analogous to those described in the previous results. The next result shows how those properties of an auxiliary curve can be used to establish a degree formula. We will give a general formulation in order to apply this same result to both situations. In the statement of the next theorem we use the following terminology: let $\bar{u} = (u_1, u_2)$. Then, if $h \in \mathbb{C}[y_1, y_2, y_3, \bar{u}]$, we denote by $\text{PP}_u(h)$ the primitive part of $h$ w.r.t. $\bar{u}$, and by $\text{Res}_{y_3}(h_1, h_2)$ the resultant of $h_1, h_2 \in \mathbb{C}[y_1, y_2, y_3, \bar{u}]$ w.r.t. $y_3$. Recall that $\bar{y}_H = (y_1 : y_2 : y_3)$.

**Theorem 5.1 (Cornerstone Theorem).** Let $\mathcal{D}$ be an irreducible affine plane curve, not being a line, and let $Z(\bar{y}_H, \bar{u}) \in \mathbb{C}[\bar{y}_H, \bar{u}]$ be homogeneous in $\bar{y}_H$ and depending on $y_3$. Let us suppose that there exists an open set $\Xi \subset \mathbb{C}^2$ such that, for $\bar{w} \in \Xi$ the following hold:

1. $\deg_{\bar{y}_H}(Z(\bar{y}_H, \bar{w})) = \deg_{\bar{y}_H}(Z(\bar{y}_H, \bar{u}))$. Let $\bar{Z}(\bar{w})$ be the plane curve defined by $Z(\bar{y}_H, \bar{w})$ (note that $Z(\bar{y}_H, \bar{w})$ is non-constant).
2. $\bar{Z}(\bar{w})$ and $\mathcal{D}$ do not have common components.
3. Let

$$\mathcal{G} = \bigcap_{\bar{u} \in \Xi} \overline{Z(\bar{u}) \cap \mathcal{D}}.$$  

Then, for every $p \in \overline{(\mathcal{Z}(\bar{w}) \cap \mathcal{D}) \setminus \mathcal{G}}$, $\text{mult}_p(\mathcal{D}, \mathcal{Z}(\bar{w})) = 1$. 

(4) Let \( Z(\bar{y}_H, \bar{u}) \) be considered as an element of \((\mathbb{C}[\bar{y}_H])[\bar{u}]\), so that one has:

\[
Z(\bar{y}_H, \bar{u}) = \sum_{\alpha} Z_{\alpha}(\bar{y}_H)\bar{u}^{\alpha}
\]

for some \( Z_{\alpha}(\bar{y}_H) \in \mathbb{C}[\bar{y}_H] \). If \( Z_{\alpha}(\bar{y}_H) \) is not constant, let \( J_{\alpha} \) be the curve it defines. Then it holds that:

\[
\bigcap_{\alpha} (D \cap J_{\alpha}) \subset \mathcal{G}.
\]

(5) \((0 : 0 : 1) \notin (\mathbb{Z}(\bar{\omega}) \cap D) \setminus \mathcal{G}\n\

Then, there exists a non-empty open subset \( \Xi^* \subset \Xi \) such that for \( \bar{\omega} \in \Xi^* \):

\[
\text{Card}(\mathbb{Z}(\bar{\omega}) \cap D \setminus \mathcal{G}) = \deg_{\{y_1, y_2\}}(PP_{\bar{\omega}}(\text{Res}_{y_3}(G(\bar{y}_H), Z(\bar{y}_H, \bar{u})))).
\]

where \( G \) is the form defining the projective closure \( D \) of \( D \).

**Proof.**

We denote by \( R(y_1, y_2, \bar{u}) = \text{Res}_{y_3}(G, Z) \); observe that, since \( G \) is irreducible and \( D \) is not a line, \( G \) depends on \( y_3 \), moreover \( Z \) depends also on \( y_3 \) by hypothesis. Let \( R(y_1, y_2, \bar{u}) \) factor as

\[
R(y_1, y_2, \bar{u}) = M(y_1, y_2)N(y_1, y_2, \bar{u})
\]

where \( M \) and \( N \) are the content and primitive part of \( R \) w.r.t. \( \bar{u} \), respectively. Then \( M \) and \( N \) are homogeneous polynomials in \( y_1, y_2 \), and \( M \in \mathbb{C}[y_1, y_2], N \in \mathbb{C}[\bar{u}][y_1, y_2] \). This implies that \( M \) factors over \( \mathbb{C} \) in linear factors, namely:

\[
M = \prod_{i=1}^{r}(\beta_iy_1 - \alpha_iy_2)
\]

We observe that the leading coefficient \( L \) of \( Z \) w.r.t. \( y_3 \) is a non-zero polynomial in \( \mathbb{C}[\bar{u}][y_1, y_2] \). If \( L \) does not depend on \( \bar{u} \) or any coefficient of \( L \) w.r.t. \( \{y_1, y_2\} \) is a non-zero constant we take \( \Psi = \emptyset \), otherwise we take \( \Psi \) as the intersection of all curves in \( \mathbb{C}^2 \) defined by each non-constant coefficient of \( L \) w.r.t. \( \{y_1, y_2\} \). Let \( \Xi_1 = \Xi \setminus \Psi \). Since \( G \) does not depend on \( \bar{u} \), for every \( \bar{\omega} \in \Xi_1 \), both leading coefficients of \( G \) and \( Z(\bar{y}_H, \bar{\omega}) \) w.r.t. \( y_3 \) do not vanish. In particular, this implies that the resultant specializes properly; i.e. if \( Z_0(\bar{y}_H) = Z(\bar{y}_H, \bar{\omega}) \) and \( R_0(y_1, y_2) = \text{Res}_{y_3}(G, Z_0) \), then for \( \bar{\omega} \in \Xi_1 \):

\[
R_0 = M(y_1, y_2)N(y_1, y_2, \bar{\omega}).
\]

By Lemma 18 in \cite{15}, and because of \( \Xi_1 \) and hypothesis (1), we observe that \( R \) and \( R_0 \) have the same degree. Hence the degree of \( N(y_1, y_2, \bar{u}) \) and \( N_0 = N(y_1, y_2, \bar{\omega}) \) is also the same. Moreover, since \( N_0 \) is a homogeneous polynomial, it can be factored as

\[
N_0 = \prod_{j=1}^{s}(\beta_jy_1 - \alpha_jy_2).
\]

Thus

\[
R_0 = M \cdot N_0 = \prod_{i=1}^{r}(\beta_iy_1 - \alpha_iy_2) \prod_{j=1}^{s}(\beta_jy_1 - \alpha_jy_2)
\]

In this situation, for \( \bar{\omega} \in \Xi \) let \( \mathcal{B}_0 = (\mathbb{Z}(\bar{\omega}) \cap D) \setminus \mathcal{G} \). Then, since \( \deg(N) = \deg(N_0) \), the proof ends if we find a non-empty open subset \( \Xi^* \subset \Xi \) such that \( \text{Card}(\mathcal{B}_0) = \deg(N_0) \) for \( \bar{\omega} \in \Xi^* \).

We start the construction of \( \Xi^* \). First, we prove that there exists a non-empty open subset \( \Xi_2 \subset \Xi_1 \) such that, if \( \bar{\omega} \in \Xi_2 \), then \( \gcd(N_0, M) = 1 \). Indeed, first we observe that \( \gcd(N, M) = 1 \), since otherwise \( N \) would have a factor depending on \( \{y_1, y_2\} \), and \( N(y_1, y_2, \bar{u}) \) is primitive w.r.t. \( \bar{u} \). Now, for each factor \( (\beta_iy_1 - \alpha_iy_2) \) of \( M \), we consider the
polynomial $N(\alpha_i, \beta, \bar{u})$. This polynomial is not identically zero because $\gcd(N, M) = 1$. Then $\Xi_2 = \Xi_1 \setminus (\Gamma_1 \cup \cdots \cup \Gamma_m)$, where $\Gamma_i$ is the curve in $\mathbb{C}^2$ defined by $N(\alpha_i, \beta, \bar{u})$.

Now, we prove the existence of a non-empty open subset $\Xi_3 \subset \Xi_2$ such that for $\omega \in \Xi_3$ the projective lines $\mathcal{L}_i$, defined by the equations $\beta_j y_1 - \alpha_j y_2 = 0$, do not contain points of $\mathcal{B}_2$; recall that $\beta_1 y_1 - \alpha_1 y_2$ is a factor of $M$. For this purpose, observe that $\mathcal{L}_i$ meets $\mathcal{F}$ in a finite number of points; recall that by assumption $\mathcal{D}$ is irreducible and it is not a line. Let $[\mathcal{F} \cap \mathcal{L}_i] \setminus \mathcal{G} = \{P_1^i, \ldots, P_k^i\}$. Now, consider the polynomials $Z(P_j^i, \bar{u})$. These polynomials are not identically zero, because otherwise it would imply that all coefficients of $Z(y_H, \bar{u})$ w.r.t. $\bar{u}$ vanish at $P_j$, and by hypothesis (5), that $P_{j_1}^i \in \bigcap_{\alpha} (\mathcal{F} \cap \mathcal{F}_\alpha) \subset \mathcal{G}$, which is impossible. Then, if $\Psi_j^i$ is the curve in $\mathbb{C}^2$ defined by $Z(P_j^i, \bar{u})$, Let

$$\Xi_3 = \Xi_2 \setminus \bigcup_{i=1}^r \bigcup_{j=1}^{k_i} \Psi_j^i.$$  

Let us see that $\Xi_3$ satisfies the requirements. Let $\omega \in \Xi_3$, and assume that there exists $P \in (\mathcal{F} \cap \overline{\mathcal{L}}(\omega) \cap \mathcal{F}) \setminus \mathcal{G}$. Then, $P \in (\mathcal{L}_i \cap \mathcal{F}) \setminus \mathcal{G}$. Therefore there exists $j_i$ such that $P = P_j^i$, and because $P \in \overline{\mathcal{L}}(\omega)$ one has that $Z(P_j^i, \omega) = 0$, which is a contradiction since $\omega \notin \Psi_j^i$.

Finally the last open subset is constructed. Let $W(y_1, y_2)$ be the leading coefficient of $G(y_H)$ w.r.t. $y_2$. Note that $W \in \mathbb{C}[y_1, y_2]$ is homogeneous. Then, we choose a non-empty Zariski open subset $\Xi_4 \subset \Xi_3$ such that for every $\omega \in \Xi_4$ it holds that $\gcd(N_0, W) = 1$.

For this purpose, let $W$ factor as

$$W = \prod_{k=1}^m (\sigma_i y_1 - \nu_i y_2).$$

We consider the polynomials $N(\nu_i, \sigma_i, \bar{u})$. These polynomials are not identically zero, because otherwise it would imply (note that $N$ is homogeneous in $y_1, y_2$) that $N$ has a factor, namely $(\sigma_i y_1 - \nu_i y_2)$, and $N$ is primitive w.r.t. $\bar{u}$. Then, we consider

$$\Xi_4 = \Xi_4 \setminus (\Psi_1 \cup \cdots \cup \Psi_m),$$

where $\Psi_i$ is the curve in $\mathbb{C}^2$ defined by $N(\nu_i, \sigma_i, \bar{u})$. Let us see that $\Xi_4$ satisfies the requirements. Let us assume that $\omega \in \Xi_4$, and that there exists a factor $\Lambda = \beta_j' y_1 - \alpha_j' y_2$ of $N_0 = N(y_1, y_2, \omega)$ such that $\gcd(\Lambda, W) \neq 1$. Then, there exists $i \in \{1, \ldots, m\}$ such that $\Lambda = \sigma_i y_1 - \nu_i y_2$. Thus $N(\nu_i, \sigma_i, \omega) = 1$. That is, $\omega \in \Psi_i$, which is a contradiction.

Now, we take $\Xi' = \Xi_4$, and we prove that for every $\omega \in \Xi'$, $\Card(\mathcal{B}_\omega) = \deg(N_0)$:

(a) Let us see that if $P = (a : b : c) \in \mathcal{G} \setminus \{(0 : 0 : 1)\}$ then $(by_1 - ay_2)$ divides $M$. Indeed: $P \in \overline{\mathcal{L}(\omega)} \cap \mathcal{F}$ for every $\omega \in \Xi'$. Thus, $R_0(a, b, \bar{u}) = 0$ for every $\omega \in \Xi'$. Since the resultant specializes properly in $\Xi'$, because of $\Xi_1$, then $R(a, b, \bar{u}) = M(a, b)N(a, b, \bar{u})$ vanishes on $\Xi'$. Moreover, $N(a, b, \bar{u})$ cannot vanish on $\Xi'$, since otherwise it would imply that $(by_1 - ay_2)$ divides $N$, and $N$ is primitive w.r.t. $\bar{u}$. Thus, $M(a, b) = 0$.

(b) Let us see that every linear factor of $N_0$ (for every $\omega \in \Xi'$) generates a point in $\mathcal{B}_\omega$. Indeed: let $(by_1 - ay_2)$ divide $N_0$ then, because of $\Xi_4$, there exists $c$ such that $(a : b : c) \in \overline{\mathcal{L}(\omega)} \cap \mathcal{F}$. Note that $(a : b : c) \neq (0 : 0 : 1)$. Now, taking into account (a), and because of $\Xi_4$, one has that $(a : b : c) \in \mathcal{B}_\omega$.

(c) Let us see that every point in $\mathcal{B}_\omega$ (for every $\omega \in \Xi'$) generates a factor in $N_0$. Indeed, let $P = (a : b : c) \in \mathcal{B}_\omega$, then by hypothesis (5) $A = (by_1 - ay_2) \neq 0$. Thus, $A$ divides $R_0$, and because of $\Xi_3$, $A$ does not divide $M$. Therefore, $A$ divides $N_0$. 


(d) Now the result follows from Lemma 19 in [15], from (b), (c), from hypothesis (4), and because \( \gcd(M, N_0) = 1 \) in \( \Sigma^* \).

6. Partial degree formulae for the implicit case

Using the previous results, we derive the first two partial degree formulae for offset curves. For the first formula we observe that, by Proposition 4.8, and by Bézout’s Theorem, we know that for \( (d_0, k_0) \in \Delta \) (with \( \Delta \) as in Theorem 3.1)

\[
\deg(\mathcal{C}) \deg(S(d_0, k_0)) = \sum_{p \in \mathcal{C} \cap S(d_0, k_0)} \mult_p(\mathcal{C}, S(d_0, k_0)) = \\
\sum_{p \in \mathcal{F}} \mult_p(\mathcal{C}, S(d_0, k_0)) + \sum_{p \in (\mathcal{C} \cap S(d_0, k_0)) \setminus \mathcal{F}} \mult_p(\mathcal{C}, S(d_0, k_0))
\]

Moreover, since there are \( \delta_1 \) non-fake points (see Remark 4.5), and for each of them the multiplicity of intersection is one, the following formula holds.

**Theorem 6.1 (First partial degree formula).** Let \( \bar{\Delta} \) be as in Theorem 4.8. For every \( (d_0, k_0) \in \bar{\Delta}, \) it holds that:

\[
\delta_1 = \deg_{x_1}(O_{d}(\mathcal{C})) = 2 (\deg(\mathcal{C}))^2 - \sum_{p \in \mathcal{F}} \mult_p(\mathcal{C}, S(d_0, k_0))
\]

The above formula is, although algorithmically applicable, mainly of theoretical interest, and probably not so useful in practice, because it requires an explicit description of the inequalities defining the open set \( \Delta \).

In order to overcome this difficulty, we present a second formula that uses a univariate resultant and gcd computations. This formula is a direct consequence of Theorem 5.1. Recall that \( PP_u(h) \) is the primitive part of \( h \) w.r.t. \( u \), and \( \text{Res}_{y_3}(h_1, h_2) \) is the resultant of \( h_1, h_2 \in \mathbb{C}[y_1, y_2, y_3, u] \) w.r.t. \( y_3 \). Recall also that \( \bar{y}_H = (y_1 : y_2 : y_3) \). The second partial degree formula is then the following:

**Theorem 6.2 (Second partial degree formula).**

\[
\delta_1 = \deg_{x_1}(O_{d}(\mathcal{C})) = \deg_{y_1, y_2, y_3}(PP_{d, k}(\text{Res}_{y_3}(F(\bar{y}_H), S(\bar{y}_H, d, k))))
\]

We recall that \( F \) is the homogeneous implicit equation of \( \mathcal{C} \), and \( S \) is the homogenization of the polynomial introduced in Definition 4.7.

**Proof of Theorem.** In order to prove the theorem, we apply Theorem 5.1. Let \( D = \mathcal{C}, Z(\bar{y}_H, \bar{u}) = S(\bar{y}_H, d, k) \), where \( \bar{u} = (d, k) \), and \( \Xi = \Delta \), where \( \Delta \) as in Proposition 4.8.

We check that all the hypothesis are satisfied:

- \( \mathcal{C} \) is irreducible and it is not a line by assumption.
- \( S \) can be written as
  \[
  S = ((F_1^2 + F_2^2)k^2 - F_2^2d^2)y_3^2 - 2k(F_1^2 + F_2^2)y_3 + (F_1^2 + F_2^2)y_2^2
  \]
  Thus, since \( F_1^2 + F_2^2 \) and \( F_2^2 \) are not identically zero, \( S \) depends on \( y_3 \).
- (1) and (2) in Theorem 6.1 follow from (1) and (2) in Proposition 4.8.
- Let us see that
  \[
  \mathcal{F} = \bigcap_{(d, k) \in \Delta} \left[ S(d, k) \cap C \right].
  \]
  Indeed, the left-right inclusion follows from Definition 4.4 and Proposition 4.6. Now, let \( p \in \bigcap_{(d, k) \in \Delta} \left[ S(d, k) \cap C \right] \). Then, \( p \in C \cap S(p, d, k) \) vanishes on \( \Delta \). Thus \( S(p, d, k) \) is identically zero. So, \( p \in \bigcap_{(d, k) \in \Delta} \left[ C \cap J_\alpha \right] \), where \( J_\alpha \) is as in Proposition 4.8. Then, by Proposition 4.8 (4), one has that \( p \in \mathcal{F} \).
In this situation, hypothesis (3), (4) and (5) in Theorem 5.1 follows from Proposition 4.8(3), (4) and (5), respectively. Then, Theorem 5.1 implies that there exists a non-empty open $\Delta^* \subset \tilde{\Delta}$ such that for $(d_0, k_0) \in \Delta^*$

$$\text{Card}(S(d_0, k_0) \cap C \setminus \mathcal{F}) = \deg_{(y_1, y_2)} (PP_{(d, k)} (Res_{y_3}(F(\tilde{y}_H), S(\tilde{y}_H, d, k))))$$

Now the theorem follows from Remark 4.5 and Proposition 4.6. □

7. Partial degree formulae for the parametric case

The formulae derived in the previous sections are valid for the implicit representation of any irreducible algebraic plane curve. In this section, we will present a simpler formula, adapted to the case of rational algebraic plane curves given parametrically. This formula only requires the computation of the degree of three univariate gcds, directly related to the parametrization.

Let

$$P(t) = \begin{pmatrix} X(t) \\ Y(t) \\ W(t) \end{pmatrix}$$

be a proper rational parameterization of a plane curve $C$, where

$$\gcd(X, Y, W) = 1.$$ 

As a normal vector associated to $P(t)$ we consider $(N_1(t), N_2(t))$, where

$$\begin{cases} N_1(t) = -(W(t)Y'(t) - W'(t)Y(t)) \\ N_2(t) = W(t)X'(t) - W'(t)X(t) \end{cases}$$

Now, substituting in system $S_2(d, k)$ the variables $\bar{y}$ by the parametrization and the partial derivatives $f_i$ by the normal vector components $N_i$, and clearing up denominators, one may apply a similar strategy to derive the partial degree formulae. More precisely, the auxiliary curve $S$ is replaced here by a univariate polynomial $\hat{S}(t)$ that takes values in the parameter space, namely

$$\hat{S}(t) = (N_1^2 + N_2^2)(Wk - Y)^2 - d^2W^2N_2^2.$$ 

A similar argument to the implicit case, based on the genericity of $k$ and $d$, shows that the partial offset degree is the degree of the primitive part of $\hat{S}$ w.r.t. $(d, k)$. That is:

$$\delta_1 = \deg_{x_1}(O_d(C)) = \deg_{t}(PP_{(k, d)} ((N_1^2 + N_2^2)(Wk - Y)^2 - d^2W^2N_2^2))$$

Collecting the coefficients of $\hat{S}$ w.r.t. $(d, k)$ one deduces that the content is given by the following gcd:

$$\Theta(t) = \gcd(W^2 \gcd(N_1, N_2)^2, (N_1^2 + N_2^2)Y \gcd(W, Y))$$

Since the degree of $\hat{S}$ equals $2(\max(\deg(Y), \deg(W)) + \max(\deg(N_1), \deg(N_2)))$, one gets the following second formula:

$$\delta_1 = 2(\max(\deg(Y), \deg(W)) + \max(\deg(N_1), \deg(N_2))) - \deg_t(\Theta(t))$$
8. Strategy description for the distance degree formula

Since the generic offset equation $g$ also depends on $d$, it is natural to complete this degree analysis by studying the degree of $g$ in $d$. We denote it by $\delta_d$. We begin recalling that, for all but a finite (possibly empty) set of values of $d$, the generic offset equation specializes properly (see Theorem 2.6). This implies that there are infinitely many values $d_0$ such that $g(\bar{x}, d_0) = 0$ is the equation of $O_{d_0}(C)$ and, simultaneously, $g(\bar{x}, -d_0) = 0$ is the equation of $O_{-d_0}(C)$. But, because of the symmetry in the construction, the offsets $O_{d_0}(C)$ and $O_{-d_0}(C)$ are exactly the same. Thus, it follows that for infinitely many values of $d_0$ it holds that up to multiplication by a non-zero constant:

$$g(\bar{x}, d_0) = g(\bar{x}, -d_0).$$

Hence, we have proved the following proposition:

**Proposition 8.1.** The generic offset equation belongs to $\mathbb{C}[\bar{x}][d^2]$. That is, it only contains even powers of $d$. In particular, $\delta_d$ is even.

**Remark 8.2.** In the sequel we denote $\delta_d = 2\mu$, where $\mu \in \mathbb{N}$.

Now, the strategy is slightly different to the one described in Section 3, but follows a similar structure. Essentially, it consists in the following steps:

1. First, we recall that
   $$n(\bar{y}, \bar{x}) = -f_2(\bar{y})(x_1 - y_1) + f_1(\bar{y})(x_2 - y_2),$$
   and let
   $$N(\bar{y}_H, \bar{x}) = -F_2(\bar{y}_H)(x_1 y_3 - y_1) + F_1(\bar{y}_H)(x_2 y_3 - y_2)$$
   be the homogenization of $n(\bar{y}, \bar{x})$ w.r.t. $\bar{y}$. For $\bar{\tau} = (\tau_1, \tau_2) \in \mathbb{C}^2$ we denote by $N(\bar{\tau})$ the curve defined by $N(\bar{y}_H, \tau)$ (observe that there exists an open subset of values of $\bar{\tau}$ such that $N(\bar{\tau})$ is indeed a curve). Let $\overline{N(\bar{\tau})}$ denote the projective closure of $N(\bar{\tau})$. This curve $\overline{N(\bar{\tau})}$ will play the role of the curve $\overline{Z(\bar{y}_H, \bar{u})}$ used in the Cornerstone Theorem 5.1.

2. Secondly, we consider the system
   $$\begin{align*}
   F(\bar{y}_H) &= 0 \\
   N(\bar{y}_H, \bar{x}) &= 0
   \end{align*}$$
   and we analyze its solutions; this is done in Theorem 8.3.

3. Based on this analysis, the notion of $d$-fake and non $d$-fake points are introduced.

4. Next, the invariance of the set of $d$-fake points is established in Proposition 8.5.

5. In order to apply Bézout’s Theorem, we state Proposition 8.9 which is similar to Proposition 4.8.

6. Finally, we apply the cornerstone Theorem.

The second step is the content of the following theorem (compare to Theorem 3.1 and Theorem 4.2).

**Theorem 8.3.** There exists a non-empty Zariski open subset $U$ of $\mathbb{C}^2$, such that for $\bar{\tau} = (\tau_1, \tau_2) \in U$

1. Let $\hat{p}$ be an affine regular point of $C$. If $\hat{p}$ is the origin or it is isotropic in $C$, then it is not a solution of $\mathcal{S}_3(\bar{\tau})$. 

(2) There exist exactly $\mu$ solutions (see Remark 2) \( \hat{T}(\bar{\tau}) = \{ \bar{p}_i \}_{i=1,\ldots,\mu} \) of \( \Theta_3(\bar{\tau}) \) satisfying that \( \bar{p}_1, \ldots, \bar{p}_\mu \) are different affine and non-isotropic points of \( \mathcal{C} \).

(3) For every \( \bar{p}_i = (a_i : b_i : 1) \in \hat{T}(\bar{\tau}) \), let
\[
\bar{d}_i^2 = (a_i - \tau_1)^2 + (b_i - \tau_2)^2.
\]
Then \( d_1, \ldots, d_\mu \) are all different and non-zero.

(4) For every \( \bar{p}_i \in \hat{T}(\bar{\tau}) \), and its corresponding \( d_i \) introduced in (3), it holds that \( \bar{\tau} \in \mathcal{O}_{d_i}(\mathcal{C}) \), and it is the point on the offset generated by \( \bar{p}_i \).

**Proof.** The open set \( U \) is constructed in a finite number of steps, as follows:

(i) Since \( g \) is primitive w.r.t \( d, g(\bar{x}, 0) \) cannot be identically zero. Let \( \Psi_0 \) be the zero set in \( \mathcal{C}^2 \) of \( g(\bar{x}, 0) \). And let \( U_0 = \mathcal{C}^2 \setminus (\mathcal{C} \cup \Psi_0) \).

(ii) The next open subset ensures that \( \deg_{\Psi_0}(N) \) stays invariant when specializing \( \bar{x} \). First, observe that none of \( F_1, F_2 \) cannot be identically zero because \( \mathcal{C} \) is irreducible and it is not a line. Now, we introduce the polynomial \( Z_i(y_1, y_2) \) as the leading coefficient of \( F_i \) w.r.t. \( y_3 \) if \( F_i \) depends on \( y_3 \), and otherwise \( Z_i = F_i \). Let \( \hat{A}(\bar{x}, y_1, y_2) \) be the leading coefficient of \( N \) w.r.t. \( y_3 \). Then \( A \) is either \( -Z_2x_1 + Z_1x_2 \) or \( -Z_2x_1 \) or \( Z_1x_2 \). In any case, it is clear that there exists an open subset of \( U_0 \), say \( U_1 \), such that for \( \bar{\tau} \in U_1 \), \( A(\bar{\tau}, y_1, y_2) \) does not vanish.

(iii) Let \( T(\bar{x}) = \text{Dis}_d(g(\bar{x}, d)) \). Note that \( g \) is square-free and primitive w.r.t \( d \), and hence \( T \) is not identically zero. Let \( \Psi_2 \) be the curve defined by \( T \) in \( \mathcal{C}^2 \) if \( T \) is not constant and \( \Psi_2 = \emptyset \) otherwise. Then we consider the open subset \( U_2 = U_1 \setminus \Psi_2 \).

Now, let \( \bar{\tau} \in U_2 \). Then \( g(\bar{\tau}, d) \) has exactly \( \delta_d \) roots because of \( U_1 \), being all different because of \( U_2 \). Proposition 2 implies that these roots can be grouped in pairs, with elements in each pair differing only by multiplication by \(-1\). Let \( \Theta(\bar{\tau}) = \{ d_1, \ldots, d_\mu \} \) be a collection of \( \mu \) roots of \( g(\bar{\tau}, d) \) where each \( d_i \) is from one of these pairs. Also, observe that because of \( U_0 \), \( d_i \neq 0, \forall i = 1, \ldots, \mu \).

(iv) Now, let \( Y \) be the set in Theorem 4. Also consider the finite (possibly empty) set \( \mathcal{T} \) of values of \( d \) such that for \( d_0 \in \mathcal{T}, \mathcal{O}_{d_0}(\mathcal{C}) \) has a special component (see section 5 in [17]). Let \( \Psi_3 = \cup_{d_0 \in (Y \cup \mathcal{T})} \mathcal{O}_{d_0}(\mathcal{C}) \), and take \( U_3 = U_2 \setminus \Psi_3 \).

(v) Recall that \( \mathcal{O}_{d}(\mathcal{C}) = (\pi(\Omega(d)))^* \). Let \( \mathcal{M} = \mathcal{O}_{d}(\mathcal{C}) \setminus \pi(\Omega(d)) \). Let us see that, if \( \mathcal{M} \neq \emptyset \), then \( \dim(\mathcal{M}) \leq 1 \). For this purpose, let:
\[
\Omega(d) = \Gamma_1 \cup \cdots \cup \Gamma_s
\]
where \( \Gamma_i \) are the irreducible components of \( \Omega(d) \). Let \( \mathcal{O}_i = (\pi(\Gamma_i))^* \). Then, since
\[
\mathcal{M} = \mathcal{O}_{d}(\mathcal{C}) \setminus \pi(\Omega(d)) = (\cup_{i=1}^s \Gamma_i)^* \setminus \pi(\cup_{i=1}^s \Gamma_i) = \cup_{i=1}^s (\pi(\Gamma_i))^* \setminus \cup_{i=1}^s (\pi(\Gamma_i))^* \subset \bigcup_{i=1}^s (\pi(\Gamma_i))^* \setminus (\pi(\Gamma_i))
\]
if \( \dim(\mathcal{M}) > 1 \), there exists \( i \in \{1, \ldots, s\} \) such that \( \dim(\pi(\Gamma_i)^* \setminus \pi(\Gamma_i)) > 1 \). Consider now the rational map \( \pi : \Gamma_i \rightarrow \pi(\Gamma_i)^* \); note that both closed sets are irreducible. By Theorem 7(ii) in [18], page 76, there exists a non-empty open subset \( U \) of \( \pi(\Gamma_i)^* \) such that the dimension of the fiber is invariant. Hence \( \mathcal{M} \subset \pi(\Gamma_i)^* \setminus U \), which is a contradiction, because \( \dim(\pi(\Gamma_i)^* \setminus U) \leq 1 \). Now, we consider the projection
\[
\pi_x : \mathcal{C}^3 \rightarrow \mathcal{C}^2; (\bar{x}, d) \mapsto \bar{x}
\]
Then \( \Psi_4 = (\pi_4(\mathcal{M}))^* \) is either empty or \( \dim(\Psi_4) \leq 1 \). Let us define \( U_4 = U_3 \setminus \Psi_4 \).

(vi) Consider the following results:
\[
R_i(\bar{x}) = \text{Res}_d \left( g(\bar{x}, d), \frac{\partial g}{\partial x_i}(\bar{x}, d) \right)
\]
for \( i = 1, 2 \). Note that \( \frac{\partial g}{\partial x_i} \) cannot be identically zero, because \( C \) is not a line. Also observe that \( R_i \) cannot be identically zero, since this would imply that \( \frac{\partial g}{\partial x_i}(\bar{x}, d) \) and \( g(\bar{x}, d) \) have a common factor of positive degree in \( d \). This factor cannot depend only on \( d \) because of the definition of the generic offset equation. Thus, this would imply that for \( d \not\in \Upsilon \) (the set in Theorem 2.6), the offset has infinitely many ramification points, and this is impossible since the offset cannot have multiple components, and it cannot be a line because \( C \) is not a line. Let \( \Phi_i \) be the zero set of \( R_i(\bar{x}) \) in \( \mathbb{C}^2 \). Then, we could take places of \( 0 \) and \( \bar{\tau} \).

Since the offset has infinitely many ramification points, it follows that \( \bar{\tau} \) is a regular point of \( \mathcal{O}_{d_0}(C) \). Otherwise one has

\[
g(\bar{\tau}, d_0) = \frac{\partial g}{\partial x_i}(\bar{\tau}, d_0) = 0
\]

for \( i = 1, 2 \). This means that \( R_i(\bar{\tau}) = 0 \) for \( i = 1, 2 \), contradicting the construction of \( U_5 \).

(vii) Let \{\( \bar{p}_1, \ldots, \bar{p}_\mu \)\} be the isotropic affine and regular points of \( C \). This is a finite set because \( C \) is irreducible. For \( i = 1, \ldots, r \), let \( \gamma_i \) be the normal line to \( C \) at \( \bar{p}_i \).

Let \( U_6 = U_5 \setminus \bigcup_{i=1}^{r} \gamma_i \).

(viii) If \( (0 : 0 : 1) \in \mathbb{C} \) and it is regular, let \( \Psi \) be the zero set in \( \mathbb{C}^2 \) of

\[
N(0 : 0 : 1, x) = -f_2(0, 0)x_1 + f_1(0, 0)x_2
\]

and define \( U_7 = U_6 \setminus \Psi \); note that, since \( (0 : 0 : 1) \) is regular in \( C \).

Let us see that \( U = U_7 \) satisfies the requirements. Let \( \bar{\tau} \in U \), and let \( d_i \in \Theta(\bar{\tau}) \) (see the construction of \( U_2 \)). Then \( g(\bar{\tau}, \pm d_i) = 0 \). Thus \( (\bar{\tau}, \pm d_i) \) \( \in \mathcal{O}_d(\mathbb{C}) \) because of \( U_5 \). Moreover, because of \( U_4 \), \( \bar{\tau} \not\in \pi_\tau(\{(M)\}^+) \). Hence, \( (\bar{\tau}, \pm d_i) \in \pi(\Omega(d)) \). Thus, there exist \( \bar{p}_i \in \mathbb{C} \) and \( u_0 \in \mathbb{C} \) such that \( (\bar{p}_i, \bar{\tau}, u_0) \) is a solution of \( \mathcal{O}_d(\mathbb{C}) \). In particular, this implies that \( \bar{p}_i \) is a solution of \( \mathcal{O}_d(\mathbb{C}) \), and that \( \bar{p}_i \) generates \( \bar{\tau} \) in \( \mathcal{O}_d(\mathbb{C}) \). Let \( \Gamma = \{\bar{p}_1, \ldots, \bar{p}_\mu\} \). Observe that \( \bar{p}_i \in \mathbb{C} \) and it is affine. Moreover, since \( (\bar{p}_i, \bar{\tau}, u_0) \) is a solution of \( \mathcal{O}_d(\mathbb{C}) \), then \( \bar{p}_i \) is non-isotropic on \( C \). Now, since \( d_i \neq d_j \) for \( i \neq j \) (see the construction of \( U_2 \)), and since \( \bar{p}_i \) belongs to a circle of radius \( d_i \) and centered at \( \bar{\tau} \), one concludes that \( \bar{p}_i \neq \bar{p}_j \). So, statement (1) and (4) hold. Statement (3) follows from the construction of \( U_2 \).

The existence part of Statement (2) follows from the construction of \( U_6 \) and \( U_7 \). It remains only to prove that, for \( \bar{\tau} \in U \), \( \bar{\Gamma}(\bar{\tau}) \) contains all the affine and non-isotropic solutions of \( \mathcal{O}_d(\mathbb{C}) \). Suppose that \( \bar{\tau} \) is an affine non-isotropic point of \( C \) such that \( N(\bar{p}, \bar{\tau}) = 0 \) and \( \bar{p} \not\in \bar{\Gamma}(\bar{\tau}) \). Because of \( U_2 \), it follows that \( \bar{p} \) generates \( \bar{\tau} \in \mathcal{O}_{d_i}(C) \) for some \( d_i \in \Theta(\bar{\tau}) \). Then, we could take places of \( \mathcal{C} \) at both \( \bar{p} \) and \( \bar{p}_i \) and lift them to places of the offset at \( \bar{\tau} \). Since \( \mathcal{O}_{d_i}(C) \) has no special component, these two places cannot lift to the same place of the offset. But if they lift to different places, it follows that \( \bar{\tau} \) is not regular in \( \mathcal{O}_{d_i}(C) \), and this contradicts the construction with \( U_5 \).

In the next definition we extend the terminology of fake and non-fake points to this degree problem.

**Definition 8.4.** Let \( U \) be as in Theorem 2.6. We denote:

\[
d\mathcal{F} = \bigcap_{\bar{\tau} \in U} \left( \overline{N(\bar{\tau})} \cap \mathbb{C} \right)
\]

The points of the set \( d\mathcal{F} \) are called \( d \)-fake points. For \( \bar{\tau} \in U \), the points in \( \left( \overline{N(\bar{\tau})} \cap \mathbb{C} \right) \setminus d\mathcal{F} \) are called non-\( d \)-fake points.

The next step in the strategy consists in showing the invariance of the set of \( d \)-fake points. This is established in the next proposition (compare to Proposition 4.4).
Proposition 8.5 (Invariance of the $d$-fake points). Let $U$ be as in Theorem 8.3. The set $d\mathcal{F}$ is finite. Moreover,
\[ d\mathcal{F} = \text{Sing}_a(\overline{\mathcal{C}}) \cup \text{Iso}_\infty(\overline{\mathcal{C}}) \]
where $\text{Sing}_a(\overline{\mathcal{C}})$ is the affine singular locus of $\overline{\mathcal{C}}$ and $\text{Iso}_\infty(\overline{\mathcal{C}})$ is the set of isotropic points at infinity of $\overline{\mathcal{C}}$; that is, the set of points of $\mathcal{C}$ that satisfy $y_3 = 0$ and $F_1^2 + F_2^2 = 0$.

Proof.
Let $p = (a : b : c) \in d\mathcal{F}$. Then $p \in \overline{\mathcal{C}}$ and $N(p, \overline{\tau}) = 0$ for every $\overline{\tau} \in U$. Thus, considering $N(\overline{y_\mu}, \overline{x}) \in \mathbb{C}[\overline{y_\mu}[\overline{x}]]$, one has that:
\[-F_2(p)c = 0, \quad F_1(p)c = 0, \quad F_2(p)a - F_1(p)b = 0.\]
If $c \neq 0$, then $p$ is affine and $F_1(p) = F_2(p) = 0$. Thus $p \in \text{Sing}_a(\overline{\mathcal{C}})$. If $c = 0$, then using Euler’s identity
\[ F_1(p)a + F_2(p)b = \text{deg}(F)F(p) = 0 \]
From this relation and $F_2(p)a - F_1(p)b = 0$ one has that $F_1^2(p) + F_2^2(p) = 0$. Thus $p \in \text{Iso}_\infty(\overline{\mathcal{C}})$. Therefore $d\mathcal{F} \subseteq \text{Sing}_a(\overline{\mathcal{C}}) \cup \text{Iso}_\infty(\overline{\mathcal{C}})$.

Conversely, let $p \in \text{Sing}_a(\overline{\mathcal{C}}) \cup \text{Iso}_\infty(\overline{\mathcal{C}})$. If $p = (a : b : c) \in \text{Sing}_a(\overline{\mathcal{C}})$, then $p \in \overline{\mathcal{C}}$ and for every $\overline{\tau} \in U$ one has $N(p, \overline{\tau}) = 0$. Thus, $p \in d\mathcal{F}$. If $p \in \text{Iso}_\infty(\overline{\mathcal{C}})$, then $p \in \mathcal{C}$, $c = 0$, and $F_1^2(p) + F_2^2(p) = 0$. Using Euler’s identity as before one has $F_1(p)a + F_2(p)b = 0$. From these relations one gets $N(p, \overline{\tau}) = F_2(p)a - F_1(p)b = 0$ for all $\overline{\tau} \in U$. Thus, $p \in d\mathcal{F}$.

The finiteness of $d\mathcal{F}$ follows from the equality $d\mathcal{F} = \text{Sing}_a(\overline{\mathcal{C}}) \cup \text{Iso}_\infty(\overline{\mathcal{C}})$.

Remark 8.6.
(1) The proof of Proposition 8.5 shows that if $p$ is a point at infinity of $\mathcal{C}$, and for some $\overline{\tau} \in U$, $p \in \overline{N(\overline{\tau})} \cap \overline{\mathcal{C}}$, then $p \in \text{Iso}_\infty(\overline{\mathcal{C}})$.
(2) From the definition of $d\mathcal{F}$ it follows that for any non empty open subset $U \subseteq \mathbb{C}$, one has
\[ d\mathcal{F} = \bigcap_{\overline{\tau} \in U} \overline{N(\overline{\tau})} \cap \overline{\mathcal{C}} = \bigcap_{\overline{\tau} \in U} \overline{N(\overline{\tau})} \cap \overline{\mathcal{C}} \]

Proposition 8.7 (Characterization of the $d$-fake points). Let $U$ be as in Theorem 8.3. With the notation of Theorem 8.3, for each $\overline{\tau} \in U$, it holds that
(1) $\overline{N(\overline{\tau})} \cap \overline{\mathcal{C}} = \overline{\hat{\Gamma}(\overline{\tau})} \cup d\mathcal{F}$
(2) $\overline{\hat{\Gamma}(\overline{\tau})} \cap d\mathcal{F} = \emptyset$

Proof.
Let $\overline{\tau} \in U$.
(1) Let $p = (a : b : c) \in \overline{N(\overline{\tau})} \cap \overline{\mathcal{C}}$. If $c = 0$, then by Remark 8.6 (1), one has $p \in \text{Iso}_\infty(\overline{\mathcal{C}})$, and by Proposition 8.5 $p \in d\mathcal{F}$. If $c \neq 0$ and $p \in \text{Sing}_a(\overline{\mathcal{C}})$, then again by Proposition 8.5 $p \in d\mathcal{F}$. If $c \neq 0$ and $p \notin \text{Sing}_a(\overline{\mathcal{C}})$, then $p$ is an affine regular point of $\mathcal{C}$. By Theorem 8.3, then $p \in \overline{\hat{\Gamma}(\overline{\tau})}$. Thus, in any case, $p \in d\mathcal{F} \cap \overline{\hat{\Gamma}(\overline{\tau})}$. The reverse inclusion is trivial.
(2) This follows from Proposition 8.5.

Remark 8.8. Proposition 8.7 shows that if $\overline{\tau} \in U$, then the set of non $d$-fake points is precisely $\overline{\hat{\Gamma}(\overline{\tau})}$. In particular,
\[ \text{Card}((\overline{N(\overline{\tau})} \cap \overline{\mathcal{C}}) \setminus d\mathcal{F}) = \text{Card}(\overline{\hat{\Gamma}(\overline{\tau})}) = \mu = \frac{\delta d}{2} \]

The next proposition gathers the information we need when applying Bezout’s Theorem to the curves $\overline{\mathcal{C}}$ and $\overline{N(\overline{\tau})}$ (compare to Proposition 4.8).
Proposition 8.9. There exists a non-empty open subset $\tilde{U} \subset U$, where $U$ is as in Theorem 8.3, such that for every $\tilde{\tau} \in \tilde{U}$ the following hold:

1. $\deg(\mathcal{N}(\tilde{y}_H, \tilde{x}))$ does not depend on $\tilde{x}$,
2. $C$ and $N(\tilde{\tau})$ have no common component,
3. if $\tilde{p}$ is a non $d$-fake point, then $\text{mult}_{\tilde{p}}(\overline{C, N(\tilde{\tau})}) = 1$.
4. Let $N(\tilde{y}_H, \tilde{x})$ be considered as an element of $(\mathbb{C}[\tilde{y}_H])[\tilde{x}]$:
   \[ N(\tilde{y}_H, \tilde{x}) = Z_{1,0}(\tilde{y}_H)x_1 + Z_{0,1}(\tilde{y}_H)x_2 + Z_{0,0}(\tilde{y}_H) \]
   where:
   \[ \begin{cases} 
   Z_{1,0}(\tilde{y}_H) = -F_2y_3 \\
   Z_{0,1}(\tilde{y}_H) = F_1y_3, \\
   Z_{0,0}(\tilde{y}_H) = F_2y_1 - F_1y_2, 
   \end{cases} \]
   and let $J_\alpha$ be the zero in $\mathbb{C}^2$ set of $Z_\alpha(\tilde{y}_H)$. Then it holds that:
   \[ \bigcap_\alpha (\overline{C} \cap J_\alpha) \subset DF. \]

5. $(0 : 0 : 1) \notin \overline{(\mathcal{N}(\tilde{\tau}) \cap \overline{C}) \setminus DF}$

Proof.

1. See step (ii) in the proof of Theorem 8.3.
2. Let us consider $n$ as a polynomial in $\mathbb{C}[y_1, y_2][x_1, x_2]$. If $n$ and $f$ have a common factor, one has that $f_1 = f_2 = 0$ for every point of $C$, which is a contradiction since $C$ is irreducible.
3. Let
   \[ P(t) = (y_1(t), y_2(t)) \]
   with
   \[ \begin{cases} 
   y_1 = a_0 + a_1t + \cdots \\
   y_2 = b_0 + b_1t + \cdots 
   \end{cases} \]
   be a place of $C$ centered at $\tilde{p}$. Then the multiplicity of intersection $\text{mult}_{\tilde{p}}(\overline{C, N(\tilde{\tau})})$ is equal to the order of $n(P(t), \tilde{\tau})$. Let now
   \[ \begin{cases} 
   f_1(P(t)) = a_0 + a_1t + \cdots \\
   f_2(P(t)) = b_0 + b_1t + \cdots 
   \end{cases} \]
   Note that $a_0^2 + b_0^2 \neq 0$ because $\tilde{p}$ is non $d$-fake. Besides, since the point $\tilde{\tau} = (\tau_1, \tau_2)$ is generated by $\tilde{p}$ in $\mathcal{O}_{d_1}(C)$, one has:
   \[ \begin{align*}
   \tau_1 &= a_0 + d_1 \frac{a_0}{\sqrt{(a_0^2 + b_0^2)}} \\
   \tau_2 &= b_0 + d_1 \frac{b_0}{\sqrt{(a_0^2 + b_0^2)}} 
   \end{align*} \]
   Substituting the above expressions in $n$ we arrive at:
   \[ n(P(t), \tilde{\tau}) = \left( -a_0b_1 + a_1d_1 \frac{b_0}{\sqrt{(a_0^2 + b_0^2)}} + b_0a_1 - b_1d_1 \frac{a_0}{\sqrt{(a_0^2 + b_0^2)}} \right) t + \cdots \]
   (the order zero term vanishes identically.) Now, we will suppose that we have $\text{mult}_{\tilde{p}}(\overline{C, N}) > 1$ and we will arrive at a contradiction. This would imply that
   \[ -a_0b_1 + a_1d_1 \frac{b_0}{\sqrt{(a_0^2 + b_0^2)}} + b_0a_1 - b_1d_1 \frac{a_0}{\sqrt{(a_0^2 + b_0^2)}} = 0 \]
From this one gets:
\((-\alpha_1\beta_0 + \beta_1\alpha_0)d_i = -\sqrt{(\alpha_0^2 + \beta_0^2)(\alpha_0 b_1 - \beta_0 a_1)}\)

Now observe that \((a_1, b_1)\) is a tangent vector to \(C\) at \(\hat{p}\), and \((\alpha_0, \beta_0)\) is a normal at the same point. Thus \(a_1\alpha_0 + b_1\beta_0 = 0\). Thus, if \(-\alpha_1\beta_0 + \beta_1\alpha_0 = 0\), since \(\alpha_0^2 + \beta_0^2 \neq 0\), one obtains:
\[
\begin{cases}
\alpha_0 b_1 - \beta_0 a_1 = 0 \\
\beta_0 b_1 + \alpha_0 a_1 = 0
\end{cases}
\]

It follows that \(a_1 = b_1 = 0\), which is a contradiction, since \(\hat{p}\) is regular in \(C\). Thus, we have shown that \(-\alpha_1\beta_0 + \beta_1\alpha_0 \neq 0\). And therefore
\[d_i = \sqrt{(\alpha_0^2 + \beta_0^2)} \frac{\alpha_0 b_1 - \beta_0 a_1}{\alpha_1\beta_0 - \beta_1\alpha_0} \]

Now, as in the proof of Proposition 8.1, we can offset the place \(P(t)\) to get a place of \(O_d(C)\) centered at \(\hat{p}\):
\[O(t) = (O_1(t), O_2(t)) = \left(y_1(t) + d_i \frac{f_1(t)}{\sqrt{f_1^2(t) + f_2^2(t)}}, y_2(t) + d_i \frac{f_2(t)}{\sqrt{f_1^2(t) + f_2^2(t)}}\right)\]

Substituting the above expressions by \(y_1(t), y_2(t), f_1(t), f_2(t)\) and \(d_i\), one has, after simplifying the expression:
\[O_1(t) = \tau_1 + (a_1\alpha_0 + b_1\beta_0) \frac{\alpha_0}{\alpha_0^2 + \beta_0^2} t + \cdots\]

Similarly
\[O_2(t) = \tau_2 + (a_1\alpha_0 + b_1\beta_0) \frac{\beta_0}{\alpha_0^2 + \beta_0^2} t + \cdots\]

Since \(a_1\alpha_0 + b_1\beta_0 = 0\), this would imply that \(\hat{p}\) is not regular in \(O_d(C)\), contradicting the construction of the open set \(U\).

(4) If \(p = (a : b : c) \in \bigcap_{i=1}^{n} (C_i \cap J_i)\), then \(cF_1(p) = 0\) and \(cF_2(p) = 0\). If \(c \neq 0\), it follows that \(F_1(p) = F_2(p) = 0\). If \(c = 0\), then \(F_1^2(p) + F_2^2(p) = 0\) follows by Remark 8.6 (1). In either case, by Proposition 8.3, \(p \in dF\).

(5) This follows from statement (1) in Theorem 8.4. 

\section{9. Degree formulae for the distance}

As a consequence of the results in the previous section, we derive the following formula for computing \(\delta_d\).

\begin{theorem}[Degree formula for the distance] \label{degree_formula_caption}
\[
\delta_d = \deg_d(O_d(C)) = 2 \deg_{\{y_1, y_2\}} \left(PP_{\{x_1, x_2\}}(\text{Res}_{y_3}(F(y_H), N(y_H, \bar{x})))\right)
\]

We recall that \(F\) is the homogeneous implicit equation of the curve, and \(N\) is the polynomial introduced after Remark 8.2.
\end{theorem}

\textbf{Proof of Theorem.}

In order to prove the theorem, we apply Theorem 6.4. Let \(D = C, Z(y_H, \bar{u}) = N(y_H, \bar{x})\), where \(\bar{x} = (x_1, x_2)\), and \(\Xi = \bar{U}\), where \(\bar{U}\) is as in Proposition 8.9. We check that all the hypothesis are satisfied:

- \(C\) is irreducible and it is not a line by assumption.
- \(N\) can be written as
  \[N = (-F_2x_1 + F_1x_2)y_3 + (y_1F_2 - y_2F_1)\]
  Thus, since \(F_1\) and \(F_2\) are not identically zero, \(S\) depends on \(y_3\).
- (1) and (2) in Theorem 6.4 follow from (1) and (2) in Proposition 8.9.
The equality $dF = \bigcap_{\xi \in \mathcal{C}} \overline{\mathcal{N}(\xi)} \cap \overline{\mathcal{C}}$ follows from Remark 8.6(2).

In this situation, hypothesis (3), (4) and (5) in Theorem 5.1 follows from Proposition 8.9(3), (4) and (5).

Then, Theorem 5.1 implies that there exists a non-empty open $U^* \subset \hat{U}$ such that for $\bar{r} \in U^*$

$$\text{Card}(\overline{\mathcal{N}(\bar{r})} \cap \overline{\mathcal{C}}) \setminus dF = \deg_{(y_1, y_2)}(PP_{\bar{r}}(\text{Res}_{y_1}(F(\bar{y}_H), N(\bar{y}_H, \bar{x}))))$$

Now the theorem follows from Remark 8.3.

\[\square\]

**Appendix: Table of Offset degrees**

In the following table we list, for some curves, the total degree $d$ w.r.t $\{x_1, x_2\}$ of the generic offset equation $g(x_1, x_2, d)$, its partial degrees $\delta_1$ and $\delta_2$ w.r.t $x_1$ and $x_2$, respectively, and the degree $\delta_d$ w.r.t $d$.

| Curve $\mathcal{C}$ | Equation $f(y_1, y_2) = 0$ | $\delta$ | $\delta_1$ | $\delta_2$ | $\delta_d$ |
|----------------------|----------------------------|--------|-----------|-----------|----------|
| Circle               | $y_1^2 + y_2^2 - r^2 = 0$ | 4      | 4         | 4         | 4        |
| Parabola             | $y_2 - a + by_1 + cy_1^2 = 0$ | 6      | 6         | 4         | 6        |
| Ellipse              | $y_1^2/a^2 + y_2^2/b^2 - 1 = 0$ | 8      | 8         | 8         | 8        |
| Hyperbola            | $y_1^2/a^2 - y_2^2/b^2 - 1 = 0$ | 8      | 8         | 8         | 8        |
| Hyperbola            | $y_1y_2 - 1 = 0$ | 8      | 6         | 6         | 6        |
| Cubic Cusp           | $y_1^2 - y_2^2 = 0$ | 8      | 8         | 6         | 6        |
| Folium               | $y_1^2 + y_2^2 - 3y_1y_2 = 0$ | 14     | 14        | 14        | 14       |
| Conchoid             | $(y_1 - 1)(y_1^2 + y - 2^2) + y_1^2 = 0$ | 8      | 8         | 6         | 6        |
| A cubic              | $y_1^2 + y_2^2 - y_1y_2 - 1 = 0$ | 18     | 18        | 18        | 18       |
| Epitrochoid          | $y_2^2 + 2y_1y_2^2 - 34y_2^2 + y_1^2 - 34y_2^2 + 96y_1 - 63 = 0$ | 10     | 10        | 10        | 8        |
| Cardioid             | $(y_1^2 + 4y_2 + y_2^2)^2 - 16y_1^2 - 16y_2^2 = 0$ | 8      | 8         | 8         | 6        |
| Rose (three petals)  | $(y_1^2 + y_2^2)^2 + y_1(3y_2^2 - y_1^2) = 0$ | 14     | 14        | 12        | 12       |
| Ramphoid Cusp        | $y_1^2 + y_1y_2^2 - 2y_1^2y_2 - y_1y_2 - y_1^2 + y_2^2 = 0$ | 14     | 14        | 10        | 14       |
| Lemniscate           | $(y_1^2 + y_2^2)^2 - 2(y_1^2 - y_2^2)^2 = 0$ | 12     | 12        | 12        | 12       |
| Scarabeus            | $(y_1^2 + y_2^2)(y_1^2 + y_2^2 + y_1)^2 - (y_1^2 - y_2^2)^2 = 0$ | 18     | 18        | 18        | 14       |

**References**

[1] Alcázar J.G., Sendra J.R. (2006). Local Shape of Offsets to Rational Algebraic Curves. Technical Report SFB2006, RICCAM, Austria 2006 (to appear).
[2] Anton F., Emiris I., Mourrain B., Teillaud M. (2005), The offset to an algebraic curve and an application to conics. O. Gervasi et al. (Eds.): ICCSA 2005, LNCS 3480, pp. 683-696.
[3] Arrondo E., Sendra J., Sendra J. R. (1997). Parametric Generalized Offsets to Hypersurfaces. Journal of Symbolic Computation vol. 23, pp. 267–285.
[4] Arrondo E., Sendra J., Sendra J. R. (1999). Genus Formula for Generalized Offset Curves, Journal of Pure and Applied Algebra vol. 136, no. 3, pp. 199–209.
[5] Cox D., Little J. and O’Shea D. (1997). Ideals, Varieties, and Algorithms. Springer-Verlag, New York.
[6] Farin G., Hoschek J., Kim M.-S. (2002). Handbook of Computer Aided Geometric Design. North-Holland.
[7] Farouki R. T., Neff C. A. (1990). Analytic properties of plane offset curves, Computer Aided Geometric Design vol. 7, pp. 83-99.
[8] Farouki R. T., Neff C. A. (1990). Algebraic properties of plane offset curves, Computer Aided Geometric Design vol. 7, pp. 101-127.
[9] Hoffmann C. M. (1990). Algebraic and Numerical Techniques for Offsets and Blends, Computation of Curves and Surfaces, Dahmen W., Gasca M., Michelli C. A. (eds.), pp. 499–528. Kluwer Academic Publishers, Dordrecht.
[10] Hoffmann C. M. (1993). Geometric and Solid Modeling. Morgan Kaufmann Publ., Inc.
[11] Hoschek J., Lasser D. (1993). *Fundamentals of Computer Aided Geometric Design*. A.K. Peters Wellesley MA., Ltd.

[12] Lü W. (1995). *Offset-Rational Parametric Plane Curves*, Computer Aided Geometric Design vol. 12, pp. 601–617.

[13] Peternell M., Pottmann H. (1998). A *Laguerre Geometric Approach to Rational Offsets*. Computer Aided Geometric Design vol. 15, pp. 223–249.

[14] Pottmann H. (1995). *Rational Curves and Surfaces with Rational Offsets*. Computer Aided Geometric Design vol. 12, pp. 175–192.

[15] San Segundo F., Sendra J.R. (2005). *Degree Formulae for Offset Curves*. Journal of Pure and Applied Algebra vol. 195, pp. 301–335.

[16] Sendra J., Sendra J. R. (2000). *Rationality Analysis and Direct Parametrization of Generalized Offsets to Quadrics*. Applicable Algebra in Engineering, Communication and Computing vol. 11, no. 2, pp. 111–139.

[17] Sendra J., Sendra J. R. (2000). *Algebraic Analysis of Offsets to Hypersurfaces*. Mathematische Zeitschrift vol. 234, pp. 697–719.

[18] Shafarevich R. I., *Basic Algebraic Geometry*, Springer, 1977, 2nd edition, 1994.

[19] Wang D. (2003). *Implicitization and Offsetting via Regular Systems*. Geometric Computation, Chen F., Wang D. (eds.), pp. 156–176. World Scientific, Singapore New Jersey.

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