Computing A-Homotopy Groups Using Coverings and Lifting Properties

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Abstract. In classical homotopy theory, two spaces are homotopy equivalent if one space can be continuously deformed into the other. This theory, however, does not respect the discrete nature of graphs. For this reason, a discrete homotopy theory that recognizes the difference between the vertices and edges of a graph was invented, called A-homotopy theory. In classical homotopy theory, covering spaces and lifting properties are often used to compute the fundamental group of the circle. In this paper, we develop the lifting properties for A-homotopy theory. Using a covering graph and these lifting properties, we compute the fundamental group of the cycle $C_5$ and use this computation to show that $C_5$ is not contractible in this theory, even though the cycles $C_4$ and $C_4$ are contractible.

1. Introduction

A-homotopy theory is a discrete homotopy theory developed to investigate the invariants of graphs in a combinatorial way that respects the structure of graphs [5]. The first iteration of this theory is called Q-analysis, developed by Ron Atkin as a means to study the combinatorial “holes” of simplicial complexes [1, 2]. Q-analysis is used in several fields, including sociology and biology [4]. Atkin noticed that Q-analysis was the foundation of a general theory and gave a road map for constructing it [5]. Barcelo et al. developed this general theory for simplicial complexes and a related theory specifically for graphs in [4]. This related theory, named A-homotopy theory in honor of Atkin, was further developed by Babson et al. in [3].

Throughout algebraic topology, several tools are used to examine topological spaces [8, 10]. The fundamental group of a space is the set of continuous maps from the unit interval into the space under an equivalence. These continuous maps send both endpoints of the interval to the base point of the space. We compute the fundamental group of spaces using several different methods, including covering spaces and lifting properties. A covering space of a space $X$ is a space $\tilde{X}$ together with a continuous map $p : \tilde{X} \to X$ such that $p$ is locally a homeomorphism, that is, the space $\tilde{X}$ looks like the space $X$ locally. Given a covering space $p : \tilde{X} \to X$ and a continuous map $f : Y \to X$, a lift is a map $\tilde{f} : Y \to \tilde{X}$ which factors $f$ through the map $p$. In algebraic topology, the lifting properties tell us under what conditions these lifts exist.

In this paper, we develop an analogous approach to compute the A-homotopy fundamental group of a graph, using the existing definition of covering graphs found in [9]. We prove the Path Lifting Property (Theorem 3.8), the Homotopy Lifting Property (Theorem 3.9), and the Lifting Criterion (Theorem 3.16) for A-homotopy
theory. There are interesting conditions on the Homotopy Lifting Property (Theorem 3.9) and the Lifting Criterion (Theorem 3.16) because 3-cycles and 4-cycles are contractible in A-homotopy. To demonstrate the utility of covering graphs and lifting properties, we compute the A-homotopy fundamental group of the 5-cycle using a covering graph and lifting properties in a method analogous to the method used to compute the classical fundamental group of the circle.

This paper is organized as follows. In section 2, we give the basic definitions and theorems of A-homotopy theory currently found in the literature [3]. In section 3, we give the definition of covering graphs found in [9] and develop the Path Lifting Property (Theorem 3.8), the Homotopy Lifting Property (Theorem 3.9), and the Lifting Criterion (Theorem 3.16) for A-homotopy theory. In section 4, we use a covering graph and these lifting properties to compute the fundamental group of the 5-cycle.

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2. Background

In this section, we review the fundamental definitions of A-homotopy theory and claim no originality. But first, we need to consider some basic definitions and lemmas that are the building blocks of this discrete homotopy theory. The main objects that we consider are simple graphs. For a graph $G$, we denote the set of vertices of $G$ by $V(G)$, the set of edges by $E(G)$, and an edge between the vertices $v$ and $w$ by $\{v, w\}$. If $\{v, w\} \in E(G)$, then we say that the vertices $v$ and $w$ are adjacent. Some graphs we consider have one selected vertex called a distinguished vertex. We denote a graph $G$ with distinguished vertex $v$ by $(G, v)$.

The distinguished vertex may not be explicitly stated, when it is clear from context. In classical homotopy theory, we examine continuous maps from topological spaces to topological spaces. In A-homotopy theory, we need a mapping that respects the structure of graphs.

Definition 2.1. [3, Definition 2.1(2)] A graph homomorphism $f : G_1 \rightarrow G_2$ is a map of sets $V(G_1) \rightarrow V(G_2)$ such that, if $\{u, v\} \in E(G_1)$, then either $f(u) = f(v)$ or $\{f(u), f(v)\} \in E(G_2)$, that is, adjacent vertices in $G_1$ are mapped to the same vertex of $G_2$ or adjacent vertices of $G_2$.

This definition is slightly altered from the standard graph theory definition of a graph homomorphism [6, p. 3]. In Definition 2.1, adjacent vertices can always be mapped to the same vertex.

Definition 2.2. [4, Definition 5.1(4)] A based graph homomorphism $f : (G_1, v_1) \rightarrow (G_2, v_2)$ is a graph homomorphism $f : G_1 \rightarrow G_2$ that maps the distinguished vertex $v_1$ to the distinguished vertex $v_2$.

We frequently use the product of two spaces in algebraic topology. In A-homotopy theory, we use a standard discrete product found in graph theory.
Definition 2.3. [9, p. 74] The Cartesian product of the graphs \( G_1 \) and \( G_2 \), denoted \( G_1 \square G_2 \), is the graph with vertex set \( V(G_1) \times V(G_2) \). There is an edge between the vertices \( (u_1, u_2) \) and \( (w_1, w_2) \) if either \( u_1 = w_1 \) and \( \{u_2, w_2\} \in E(G_2) \) or \( u_2 = w_2 \) and \( \{u_1, w_1\} \in E(G_1) \).

By default, the distinguished vertex of the Cartesian product of two graphs \( G_1 \) and \( G_2 \) is \((v_1, v_2)\), the ordered pair with the distinguished vertices of each separate graph.

In classical homotopy theory, we continuously deform maps over the unit interval, and when forming the fundamental group, we map the unit interval into the space. In A-homotopy theory, in order to better distinguish between vertices and edges in the graphs that we examine, we replace the unit interval with graphs known as path of infinite length, denoted by \( I_{\infty} \), with vertices labeled by the integers. We now proceed to an introduction to A-homotopy theory.

Two maps \( f, g : A \to B \) are homotopic in algebraic topology if we can take the product of the space \( A \) with the unit interval and continuously deform the map \( f \) into the map \( g \) over time from 0 to 1 [8, p. 3]. In A-homotopy theory, we use the Cartesian product of a graph with a path \( I_n \) to deform one graph homomorphism into another graph homomorphism in a combinatorial way that keeps track of the vertices and edges of the graph.

Definition 2.4. [4, Definition 5.2(1)] Let \( f, g : (G_1, v_1) \to (G_2, v_2) \) be graph homomorphisms. If there exists a positive integer \( n \in \mathbb{N} \) and a graph homomorphism \( H : G_1 \square I_n \to G_2 \) such that
- \( H(v, 0) = f(v) \) for all \( v \in V(G_1) \),
- \( H(v, n) = g(v) \) for all \( v \in V(G_1) \), and
- \( H(v_1, i) = v_2 \) for all \( 0 \leq i \leq n \),
then \( f \) and \( g \) are A-homotopic, denoted \( f \simeq_A g \). The graph homomorphism \( H \) is called a graph homotopy from \( f \) to \( g \).

Just as in classical homotopy theory we seek to know when two spaces are homotopy equivalent, in A-homotopy theory we seek to know when two graphs are A-homotopy equivalent.

Definition 2.5. [4, Definition 5.2(2)] The graph homomorphism \( f : G_1 \to G_2 \) is an A-homotopy equivalence if there exists a graph homomorphism \( g : G_2 \to G_1 \) such that \( f \circ g \simeq_A 1_{G_2} \) and \( g \circ f \simeq_A 1_{G_1} \). In this case, the graphs \( G_1 \) and \( G_2 \) are A-homotopy equivalent.

Definition 2.6. A graph \( G \) is A-contractible if \( G \) is A-homotopy equivalent to the graph with a single vertex, called *, and no edge. For convenience, we will abuse the notation slightly and refer to this graph as *.

Proposition 2.7. [5, p. 47] The cycles \( C_3 \) and \( C_4 \) are A-contractible.

We leave this as an exercise for the interested reader. A full proof is included in [7, Propositions 3.6-3.7]. The results in [4, Proposition 5.12] imply that the cycle \( C_5 \) is not A-contractible. To prove this in a more direct way, we need to examine the A-homotopy invariants of the cycle. For example, it is shown in [7, Theorem 7.1] that the A-homotopy theory fundamental group of an A-contractible graph is equal to zero. Thus, if the fundamental group of a graph is not equal to zero, then the
graph cannot be A-contractible. In Section 4, we show that the A-homotopy theory fundamental group of $C_5$ is isomorphic to the group $\mathbb{Z}$, using classical homotopy inspired methods in a combinatorial way (see Theorem 4.8). This allows us to explore the question of why the cycles $C_3$ and $C_4$ are A-contractible and the cycles $C_k$, for $k \geq 5$, are not A-contractible. Then we need a more rigorous definition of the fundamental group of a graph in A-homotopy theory. The $n$-fold Cartesian product $I^n_{\infty}$, vertices labeled by $\mathbb{Z}^n$, features frequently in these definitions.

**Definition 2.8.** [3, Definition 3.1] A graph homomorphism $f : I^n_{\infty} \to G$ stabilizes in direction $\varepsilon i$ with $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$, if there exists an integer $m_0(f, \varepsilon i)$ such that:

- if $\varepsilon = +1$, then for all $m \geq m_0(f, +i)$,
  
  $f(a_1, \ldots, a_{i-1}, m, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, m_0(f, +i), a_{i+1}, \ldots, a_n),$

- if $\varepsilon = -1$, then for all $m \leq m_0(f, -i)$,
  
  $f(a_1, \ldots, a_{i-1}, m, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, m_0(f, -i), a_{i+1}, \ldots, a_n).

We always take $m_0(f, +i)$ to be the least integer and $m_0(f, -i)$ to be the greatest integer such that the previous statements are true. If $f$ is constant on the $i^{th}$-axis, then we take $m_0(f, +i) = m_0(f, -i) = 0$.

**Remark 2.9.** The graph homomorphisms $f : I^n_{\infty} \to G$ are not based graph homomorphisms, and we will only be considering these graph homomorphisms for $n = 0, 1, 2$.

The integer $m_0(f, \varepsilon i)$ gives us the point at which the graph homomorphism $f$ stabilizes on the $i^{th}$-axis in the $\varepsilon$ direction of that axis. When a graph homomorphism stabilizes in every direction, there is a finite region of the $n$-dimensional lattice with information that is not being repeated into infinite. We call the region of $I^n_{\infty}$, induced by the vertex set $\prod_{i=1}^{n}[m_0(f, -i), m_0(f, +i)]$, the **active region** for each graph homomorphism $f : I^n_{\infty} \to G$. For each path $f : I^n_{\infty} \to G$, we say that $f$ **starts** at $f(m_0(f, -1))$ and $f$ **ends** at $f(m_0(f, +1))$ when these integers exist.

**Definition 2.10.** [3, Definition 3.1] Let $C_n(G)$ be the set of graph homomorphisms from the infinite $n$-cube $I^n_{\infty}$ to the graph $G$ that stabilize in each direction $\varepsilon i$ for $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$. These graph homomorphisms are referred to as **stable graph homomorphisms**.

The set $C_n(G)$ consists of the graph homomorphisms from the graph $\ast$, with a single vertex $\ast$ and no edges, to the graph $G$. In order to better understand and discuss the graph homomorphisms of $C_n(G)$, we need the following tools.

**Definition 2.11.** [3, Definition 3.1] The **face map** $\alpha^n_{\varepsilon i} : C_n(G) \to C_{n-1}(G)$, with $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$, is defined by $f \mapsto \alpha^n_{\varepsilon i}(f)$, where

$$\alpha^n_{\varepsilon i}(f)(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_{i-1}, m_0(f, \varepsilon i), a_{i+1}, \ldots, a_n).$$

We refer to the map $\alpha^n_{\varepsilon i}(f)$ as the **face** of $f$ in the $\varepsilon i$ direction.

For each graph homomorphism $f \in C_n(G)$, the face $\alpha^n_{\varepsilon i}(f) : I^{n-1}_{\infty} \to G$ is a restriction of $f$ to $m_0(f, \varepsilon i)$ on the $i^{th}$-axis, that is, the face of $f$ in the $\varepsilon i$ direction. Thus, since $f$ is a stable graph homomorphism, each face $\alpha^n_{\varepsilon i}(f)$ is a stable graph homomorphism.
**Definition 2.12.** [3, Definition 3.1] The degeneracy maps $\beta^n_i : C_{n-1}(G) \to C_n(G)$ with $1 \leq i \leq n$ is defined by $f \mapsto \beta^n_i(f)$, where
\[
\beta^n_i(f)(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n).
\]

In general, these degeneracy maps $\beta^n_i$ repeat the graph homomorphisms $f : I^{n-1}_\infty \to G$ along the $i^{th}$-axis with $1 \leq i \leq n$, giving us a graph homomorphism from $I^n_\infty$ to $G$. For our purpose, we need only map between the sets $C_0(G)$ and $C_1(G)$, and between the sets $C_1(G)$ and $C_2(G)$ for each graph $G$.

\[
\begin{align*}
C_2(G) & \quad \overset{\alpha^2_{+2}}{\longrightarrow} \overset{\alpha^2_{-2}}{\longrightarrow} \overset{\alpha^2_{+1}}{\longrightarrow} \overset{\alpha_{1}^2}{\longrightarrow} \overset{\beta^n_1}{\longrightarrow} C_1(G) \\
& \quad \overset{\alpha^1_{+1}}{\longrightarrow} \overset{\alpha^1_{-1}}{\longrightarrow} \overset{\beta^n_0}{\longrightarrow} C_0(G)
\end{align*}
\]

Using these face and degeneracy maps, we can give a definition for a graph homotopy between two graph homomorphisms of $C_n(G)$.

**Definition 2.13.** [3, Definition 3.2] Let $f, g \in C_n(G)$. The graph homomorphisms $f$ and $g$ are $A$-homotopic, denoted $f \sim g$, if there exists a graph homomorphism $H \in C_{n+1}(G)$ such that for all $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$:

(a) $\alpha^n_\varepsilon(f) = \alpha^n_\varepsilon(g)$,
(b) $\alpha^{n+1}_\varepsilon(H) = \beta^n_0 \alpha^n_\varepsilon(f) = \beta^n_0 \alpha^n_\varepsilon(g)$,
(c) $\alpha^{n+1}_{-(n+1)}(H) = f$ and $\alpha^{n+1}_{n+1}(H) = g$.

The graph homomorphism $H : I^{n+1}_\infty \to G$ is called a graph homotopy from $f$ to $g$.

By part (a), in order for the graph homomorphisms $f, g \in C_n(G)$ to be homotopic, they must stabilize to the same graph homomorphism of $C_{n-1}(G)$ in each $\varepsilon i$ direction for $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$, that is, they must have the same faces. By part (b), the graph homomorphism $H$ must stabilize in each $\varepsilon i$ direction for $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$ to the faces of $f$ and $g$ repeated along the $n^{th}$-axis. By part (c), the graph homomorphism $H$ must stabilize to $f$ in the negative direction of the $(n+1)^{st}$-axis and stabilize to $g$ in the positive direction of the $(n+1)^{st}$-axis.

Now that we have a way to compare graph homomorphisms from paths of different lengths to a graph $G$, we need an operation that combines the graph homomorphisms of $C_n(G)$.

**Definition 2.14.** Let $f$ and $g$ be graph homomorphisms of $C_1(G)$ with $\alpha^1_{-1}(f) = \alpha^1_{-1}(g)$. The concatenation of $f$ and $g$, denoted $f \cdot g$, is defined by
\[
(f \cdot g)(a) = \begin{cases} 
  f(a + m_0(f, -1)) & \text{for } a \geq 0, \\
  g(a + m_0(g, +1)) & \text{for } a \leq 0.
\end{cases}
\]

This operation essentially shifts the first graph homomorphism $f$ to stabilize in the negative direction on the $i^{th}$-axis at zero and shifts the second graph homomorphism $g$ to stabilize in the positive direction on the $i^{th}$-axis at zero. For this reason, the face of $f$ in the negative direction on the $i^{th}$-axis must be the same as the face of $g$ in the positive direction on the $i^{th}$-axis.
Proposition 2.15. If \( f, g \in C_1(G) \) with \( \alpha^n_1(f) = \alpha^n_1(g) \), then the concatenation \( f \cdot g \) is a graph homomorphism of \( C_1(G) \).

A full proof of this proposition can be found in [7, Proposition 4.15]

Lemma 2.16. For each \( f, g \in C_1(G) \) with \( \alpha^{-1}_1(f) = \alpha^{-1}_1(g) \), the concatenation \( f \cdot g \in C_1(G) \) stabilizes in the positive direction at \( m_0(f \cdot g, +1) = m_0(f, +1) - m_0(f, -1) \) and in the negative direction at \( m_0(f \cdot g, -1) = m_0(g, -1) - m_0(g, +1) \).

Proof. Let \( f, g \in C_1(G) \) be such that \( \alpha^{-1}_1(f) = \alpha^{-1}_1(g) \). Then by Proposition 2.15, the concatenation \( f \cdot g \) is a graph homomorphism. For \( i \geq 0 \), \( (f \cdot g)(i) = f(i + m_0(f, -1)) \). Since \( m_0(f, +1) - m_0(f, -1) \geq 0 \),

\[
(f \cdot g)(m_0(f, +1) - m_0(f, -1)) = f(m_0(f, +1) - m_0(f, -1) + m_0(f, -1)) = f(m_0(f, +1)).
\]

By Definition 2.8, \( m_0(f, +1) \) is the least integer such that \( f(m) = f(m_0(f, +1)) \) for all \( m \geq m_0(f, +1) \), so it follows that \( m_0(f, +1) - m_0(f, -1) \) is the least integer such that \( (f \cdot g)(i) = f(m_0(f, +1)) \) for all \( i \geq m_0(f, +1) - m_0(f, -1) \). Therefore, \( m_0(f \cdot g, +1) = m_0(f, +1) - m_0(f, -1) \). For \( i \leq 0 \), \( (f \cdot g)(i) = g(i + m_0(g, +1)) \). Since \( m_0(g, -1) - m_0(g, +1) \leq 0 \),

\[
(f \cdot g)(m_0(g, -1) - m_0(g, +1)) = g(m_0(g, -1) - m_0(g, +1) + m_0(g, +1)) = g(m_0(g, -1)).
\]

By Definition 2.8, \( m_0(g, -1) \) is the greatest integer such that \( g(m) = g(m_0(g, -1)) \) for all \( m \leq m_0(f, +1) \), so it follows that \( m_0(g, -1) - m_0(g, +1) \) is the greatest integer such that \( (f \cdot g)(i) = g(m_0(g, -1)) \) for all \( i \leq m_0(g, -1) - m_0(g, +1) \). Therefore, \( m_0(f \cdot g, -1) = m_0(g, -1) - m_0(g, +1) \). \( \square \)

Proposition 2.17. [3, Proposition 3.3] The homotopy relation \( \sim \) is an equivalence relation on \( C_1(G) \).

We leave the proof of this proposition to the interested reader. A full proof is included in [7, Proposition 4.18].

Definition 2.18. [3, Definition 3.4] Let \( v_0 \in G \) be a distinguished vertex of the graph \( G \). The set \( B_1(G, v_0) \subseteq C_1(G) \) is the subset of all graph homomorphisms from \( I^\infty \) to \( G \) that are equal to \( v_0 \) outside of a finite region of \( I^\infty \).

Definition 2.19. [3, Proposition 3.5] The fundamental group of the graph \( G \) with distinguished vertex \( v_0 \) is \( B_1(G, v_0)/\sim \).

This set \( B_1(G, v_0)/\sim \) is a group with the operation of concatenation, that is, if \( [\gamma_1], [\gamma_2] \in (B_1(G, v_0)/\sim) \), then \( [\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2] \in (B_1(G, v_0)/\sim) \) [7]. The identity of this group is the equivalence class of the path \( p_{v_0} : I^\infty \to G \) given by \( p_{v_0}(i) = v_0 \) for all \( i \in \mathbb{Z} \). Given the equivalence class of a path \( \gamma \), the inverse is the equivalence class of the path \( \bar{\gamma} \) given by \( \bar{\gamma}(i) = \gamma(-i) \) for all \( i \in \mathbb{Z} \).

Remark 2.20. The higher homotopy groups, \( B_n(G, v_0)/\sim \), are defined in [3].

3. LIFTING PROPERTIES

In topology, a covering space is a continuous map \( p : \tilde{X} \to X \) that preserves the local structure of the space. When considering a graph as a space, these covering spaces fail to recognize the structure of the graph, namely, the vertices and edges.
Thus there are covering graphs, that is, graph homomorphisms \( p : \tilde{G} \to G \) that preserve the local structures of the graphs. In topology, given a covering space \( p : \tilde{X} \to X \) and a continuous map \( f : Y \to X \), a lift is a map \( \tilde{f} : Y \to \tilde{X} \) which factor \( f \) through the space \( \tilde{X} \). There are lifting properties in topology that determine when a lift does or does not exist for covering spaces. While an analogous term and properties do not exist in the current literature for A-homotopy theory, we define a discrete version of lifts and develop the corresponding lifting properties in this section. The next three definitions give us a more precise idea of what covering graphs are.

**Definition 3.1.** Let \( G \) be a graph, and let \( v \in V(G) \). The closed neighborhood of \( v \), denoted \( N[v] \), is the set of vertices adjacent to \( v \) as well as \( v \) itself, more precisely,

\[
N[v] = \{ a \in V(G) \mid \{a, v\} \in E(G) \text{ or } a = v \}.
\]

**Definition 3.2.** Let \( G = (V, E) \) be a graph and \( V' \subseteq V \). The induced subgraph \( G[V'] \) is the graph with vertex set \( V' \) and edge set \( E' = \{ \{v, w\} \in E \mid v, w \in V' \} \), that is, all edges with vertices of \( V' \) as both endpoints.

**Definition 3.3.** [9] The graph homomorphism \( p : G_1 \to G_2 \) is a local isomorphism if \( p \) is onto and for each vertex \( v \in V(G_2) \) and each vertex \( w \in p^{-1}(v) \), the induced mapping \( p|_{N[w]} : N[w] \to N[v] \) is bijective.

**Remark 3.4.** While the restriction \( p|_{N[w]} : N[w] \to N[v] \) given in the previous definition is a bijection between the vertex sets \( N[w] \) and \( N[v] \), it is not necessarily a bijection between the edges of the induced subgraphs \( G_1(N[w]) \) and \( G_2(N[v]) \).

We define a different subgraph with the property \( p \) restricted to this subgraph is bijective on both vertices and edges. For \( x \in V(G_1) \), let \( N_x \) denote the subgraph of \( G_1 \) with vertex set \( V(N_x) = N[x] \) and edge set \( E(N_x) = \{ \{x, v\} \mid v \in N[x], v \neq x \} \).

If \( p : G_1 \to G_2 \) is a local isomorphism, then \( p \) induces a graph homomorphism from the subgraph \( N_x \) to the subgraph \( N_{p(x)} \) for each \( x \in V(G_1) \), that is, there is a graph homomorphism

\[
p|_{N_x} : N_x \to N_{p(x)},
\]

that is bijective on the vertices and edges of the subgraphs. This implies the following lemma.

**Lemma 3.5.** Let \( p : G_1 \to G_2 \) be a local isomorphism and \( x \in V(G_1) \). Then the graph homomorphism \( p|_{N_x} \) is invertible, and its inverse \( (p|_{N_x})^{-1} : N_{p(x)} \to N_x \) is a graph homomorphism.

These restrictions of local isomorphisms are useful when we discuss lifting properties.

**Definition 3.6.** [9] Let \( G \) and \( \tilde{G} \) be graphs, and let \( p : \tilde{G} \to G \) be a graph homomorphism. The pair \( (G, p) \) is a covering graph of \( G \) if \( p \) is a local isomorphism.

**Definition 3.7.** Let \( G \) be a graph, and let \( (\tilde{G}, \tilde{p}) \) be a covering graph of \( G \). Given a graph homomorphism \( f : K \to G \), a lift of \( f \) is a graph homomorphism \( \tilde{f} : K \to \tilde{G} \) such that \( p \circ \tilde{f} = f \).

**Theorem 3.8 (Path Lifting Property).** Let \( (\tilde{G}, \tilde{p}) \) be a covering graph of \( G \). For each \( f \in C_1(G) \) with \( f(m_0(f, -1)) = v_0 \in V(G) \) and each vertex \( \tilde{v}_0 \in p^{-1}(v_0) \), there exists a unique lift \( \tilde{f} \) of \( f \) starting at the vertex \( \tilde{v}_0 \).
Proof. Let $f \in C_1(G)$ with $f(m_0(f, -1)) = v_0 \in V(G)$, and suppose $\tilde{v}_0 \in p^{-1}(v_0)$. Define the map $\tilde{f} : I_\infty \to \tilde{G}$ by $\tilde{f}(i) = \tilde{v}_0$ for all $i \leq m_0(f, -1)$ and recursively by
\[
\tilde{f}(i) = (p|_{N_{\tilde{f}(i-1)}})^{-1}(f(i)) \quad \text{for} \quad i > m_0(f, -1).
\]

We must show that
\(1\) $\tilde{f}$ is well-defined,
\(2\) $\tilde{f}$ is a graph homomorphism,
\(3\) $\tilde{f}$ is a lift of $f$,
\(4\) $\tilde{f}$ is unique.

Since $\tilde{f}$ is defined to be constant for $i \leq m_0(f, -1)$ and defined recursively for $i > m_0(f, -1)$, in the following proofs of the four properties we address the case for $i \leq m_0(f, -1)$ separately and use induction to prove the properties for $i \geq m_0(f, -1)$.

The map $\tilde{f}$ is well-defined.

- By definition, $\tilde{f}(i) = \tilde{v}_0$ for all $i \leq m_0(f, -1)$. Thus $\tilde{f}(i)$ is well-defined for $i \leq m_0(f, -1)$.
- For $i \geq m_0(f, -1)$, we show that the correspondence $i \mapsto \tilde{f}(i)$ is well-defined by induction on $i$.

Base Case: By definition of $\tilde{f}$, $\tilde{f}(m_0(f, -1)) = \tilde{v}_0$ and
\[
\tilde{f}(m_0(f, -1) + 1) = (p|_{N_{\tilde{f}(m_0(f, -1) - 1)}})^{-1}(f(m_0(f, -1) + 1)) = (p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f, -1) + 1)).
\]

By Lemma 3.5, the inverse $(p|_{N_{\tilde{v}_0}})^{-1} : N_{p(\tilde{v}_0)} \to N_{\tilde{v}_0}$ exists. Since $\tilde{v}_0 \in p^{-1}(v_0)$, the domain of $(p|_{N_{\tilde{v}_0}})^{-1}$ is equal to $N_{\tilde{v}_0}$. Moreover, $f(m_0(f, -1) + 1) \in N_{\tilde{v}_0}$, since $f$ is a graph homomorphism. Thus $f(m_0(f, -1) + 1)$ is in the domain of $(p|_{N_{\tilde{v}_0}})^{-1}$, and therefore,
\[
(p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f, -1) + 1))
\]
is well-defined.

Inductive Hypothesis: For some $i > m_0(f, -1)$, suppose $\tilde{f}(i)$ is well-defined.

By definition, $\tilde{f}(i+1) = (p|_{N_{\tilde{f}(i)}})^{-1}(f(i+1))$. In order for $\tilde{f}(i+1)$ to be well-defined, we must verify that $\tilde{f}(i+1)$ is in the domain of $(p|_{N_{\tilde{f}(i)}})^{-1}$. By
the inductive hypothesis, \( \tilde{f}(i) = (p|_{N_{\tilde{f}(i-1)}})^{-1}(f(i)) \) is well-defined. Since \( p \) is a graph homomorphism,

\[
p|_{N_{\tilde{f}(i-1)}}(\tilde{f}(i)) = p|_{N_{\tilde{f}(i-1)}}((p|_{N_{\tilde{f}(i-1)}})^{-1}(f(i))) = f(i).
\]

By Lemma 3.5, the inverse \((p|_{N_{\tilde{f}(i)}})^{-1} : N_{p(\tilde{f}(i))} \to N_{\tilde{f}(i)} \) exists. Since \( p(\tilde{f}(i)) = f(i) \), the domain of \((p|_{N_{\tilde{f}(i)}})^{-1} \) is equal to \( N_{f(i)} \). Moreover, since \( f \) is a graph homomorphism, it follows that \( f(i+1) \in N[f(i)] \), so \( f(i+1) \) is in the domain of \((p|_{N_{\tilde{f}(i)}})^{-1} \). Therefore, \((p|_{N_{\tilde{f}(i)}})^{-1}(f(i+1)) \) is well-defined.

Thus by induction, the map \( \tilde{f} \) is well-defined for \( i \geq m_0(f,-1) \).

Hence, \( \tilde{f} \) is well-defined.

**The map \( \tilde{f} \) is a graph homomorphism.**

There is an edge \{\( i, i+1 \)\} \( \in E(G) \) for all \( i \in \mathbb{Z} \). Thus to show that \( \tilde{f} \) is a graph homomorphism, we must show that either \( \tilde{f}(i) = \tilde{f}(i+1) \) or \( \{\tilde{f}(i), \tilde{f}(i+1)\} \in E(G) \) for all \( i \in \mathbb{Z} \).

- By definition, \( \tilde{f}(i) = \tilde{v}_0 \) for all \( i \leq m_0(f,-1) \). Thus \( \tilde{f}(i) = \tilde{v}_0 = \tilde{f}(i+1) \) for all \( i < m_0(f,-1) \).
- For \( i \geq m_0(f,-1) \), we show that either \( \tilde{f}(i) = \tilde{f}(i+1) \) or \( \{\tilde{f}(i), \tilde{f}(i+1)\} \in E(G) \) by induction on \( i \).

**Base Case:** By definition, \( \tilde{f}(m_0(f,-1)) = \tilde{v}_0 \in p^{-1}(v_0) \), and by part (1), \( \tilde{f}(m_0(f,-1) + 1) = (p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f,-1) + 1)) \). Therefore, since \( f(m_0(f,-1)) = v_0 \), it follows that

\[
\tilde{f}(m_0(f,-1)) = (p|_{N_{\tilde{v}_0}})^{-1}(v_0) = (p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f,-1))).
\]

Moreover, since \( f \) is a graph homomorphism, either

\[
f(m_0(f,-1)) = f(m_0(f,-1) + 1)
\]

or

\[
\{f(m_0(f,-1)), f(m_0(f,-1) + 1)\} \in E(G).
\]

Therefore, it follows that

\[
(p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f,-1))) = (p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f,-1) + 1))
\]

or

\[
\{(p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f,-1))), (p|_{N_{\tilde{v}_0}})^{-1}(f(m_0(f,-1) + 1))\} \in E(\tilde{G}),
\]

since \((p|_{N_{\tilde{v}_0}})^{-1} \) is a graph homomorphism. Thus either \( \tilde{f}(m_0(f,-1)) = \tilde{f}(m_0(f,-1) + 1) \) or \( \{\tilde{f}(m_0(f,-1)), \tilde{f}(m_0(f,-1) + 1)\} \in E(\tilde{G}) \).

**Inductive Hypothesis:** Suppose that for some \( i > m_0(f,-1) \), \( \tilde{f}(i-1) = \tilde{f}(i) \) or \( \{\tilde{f}(i-1), \tilde{f}(i)\} \in E(G) \).

By definition,

\[
\tilde{f}(i) = (p|_{N_{\tilde{f}(i-1)}})^{-1}(f(i)) \quad \text{and} \quad \tilde{f}(i+1) = (p|_{N_{\tilde{f}(i)}})^{-1}(f(i+1)).
\]
By Lemma 3.5, the inverses
\[(p|_{N_{f(i-1)}})^{-1} : N_{p(f(i-1))} \to N_{f(i-1)}\]
and
\[(p|_{N_{f(i)}})^{-1} : N_{p(f(i))} \to N_{f(i)}\]
exist. By the inductive hypothesis, either \(\tilde{f}(i-1) = f(i)\) or \(\{\tilde{f}(i-1), f(i)\} \in E(G)\), so \(\tilde{f}(i) \in N[\tilde{f}(i-1)] \cap N[f(i)]\). Since both \(p|_{N_{f(i-1)}}\)
and \(p|_{N_{f(i)}}\) are bijective, \(p(\tilde{f}(i)) \in N[p(\tilde{f}(i-1))] \cap N[p(f(i))]\). By part (1),
\(p(\tilde{f}(i)) = f(i)\). Thus, we can write
\[(p|_{N_{f(i-1)}})^{-1}(f(i)) = (p|_{N_{f(i)}})^{-1}(f(i)).\]

Since \(f\) is a graph homomorphism, \(f(i) = f(i+1)\) or \(f(i), f(i+1) \in E(G)\). Thus
\[(p|_{N_{f(i)}})^{-1}(f(i)) = (p|_{N_{f(i)}})^{-1}(f(i+1))\]
or
\[\{(p|_{N_{f(i)}})^{-1}(f(i)), (p|_{N_{f(i)}})^{-1}(f(i+1))\} \in E(\tilde{G}),\]
since \((p|_{N_{f(i)}})^{-1}\) is a graph homomorphism. Hence, \(\tilde{f}(i) = \tilde{f}(i+1)\) or
\(\{\tilde{f}(i), \tilde{f}(i+1)\} \in E(\tilde{G})\) for all \(i > m_0(f, -1)\) by induction on \(i\).
Thus \(\tilde{f}\) is a graph homomorphism.

**The map \(\tilde{f}\) is a lift of \(f\).**
- For all \(i \leq m_0(f, -1)\), the composition \(p \circ \tilde{f}\) is defined by \(p(\tilde{f}(i)) = p(\tilde{v}_0) = v_0\). Thus \(p(\tilde{f}(i)) = f(i)\) for all \(i \leq m_0(f, -1)\).
- For all \(i > m_0(f, -1)\),
  \[p(\tilde{f}(i)) = p((p|_{N_{f(i-1)}})^{-1}(f(i))) = f(i)\]
Thus \(p(\tilde{f}(i)) = f(i)\) for all \(i > m_0(f, -1)\).
Therefore, \(p \circ \tilde{f} = f\), and hence, the graph homomorphism \(\tilde{f}\) is a lift of \(f\).

**The map \(\tilde{f}\) is unique for each choice of \(\tilde{v}_0 \in p^{-1}(v_0)\).**

Let \(\tilde{g} : I_{\infty} \to \tilde{G}\) be a graph homomorphism such that \(\tilde{g}(m_0(\tilde{g}, -1)) = \tilde{v}_0\) and
\(p \circ \tilde{g} = f\).
- Since \(f(i) = v_0\) for all \(i \leq m_0(f, -1)\) and \(p \circ \tilde{g} = f\), it follows that
  \(p(\tilde{g}(i)) = v_0\) for all \(i \leq m_0(f, -1)\). By Lemma 3.5, \((p|_{N_{v_0}})^{-1} : N_{v_0} \to N_{\tilde{v}_0}\)
is a graph homomorphism, since \(\tilde{v}_0 \in p^{-1}(v_0)\). Thus \((p|_{N_{v_0}})^{-1}(p(\tilde{g}(i))) =
(p|_{N_{\tilde{v}_0}})^{-1}(v_0)\) for all \(i \leq m_0(f, -1)\). This implies that \(\tilde{g}(i) = \tilde{v}_0\) for all
\(i \leq m_0(f, -1)\). By definition \(\tilde{f}(i) = \tilde{v}_0\) for all \(i \leq m_0(f, -1)\), so \(\tilde{g}(i) = \tilde{f}(i)\)
for all \(i \leq m_0(f, -1)\).
- We now show that \(\tilde{g}(i) = \tilde{f}(i)\) for all \(i > m_0(f, -1)\) by induction on \(i\).

**Base Case:** By the previous case, \(\tilde{f}(m_0(f, -1)) = \tilde{v}_0 = \tilde{g}(m_0(f, -1))\).
**Inductive Hypothesis:** Suppose \(\tilde{g}(i) = \tilde{f}(i)\) for some \(i \geq m_0(f, -1)\).
Thus if a lift \( \tilde{\ell} \) of \( H \) exists, then a lift \( \tilde{G} \) of \( G \) is to build the lift \( \tilde{g} \) that lifts \( \tilde{f} \). Hence, \( \tilde{g} = \tilde{f} \), so the lift \( \tilde{f} \) of \( f \) is unique. 

We now use the Path Lifting Property (Theorem 3.8) to prove the Homotopy Lifting Property (Theorem 3.9). The fact that homotopy lifting does not hold for \( C_3 \) or \( C_4 \) is significant. As mentioned previously, the 3-cycle and 4-cycle are \( A \)-contractible, but the cycles on five or more vertices are not. In the next section, we use the Homotopy Lifting Property (Theorem 3.9) to show that the 5-cycle is not \( A \)-contractible.

**Theorem 3.9 (Homotopy Lifting Property).** Let \( G \) be a graph containing no 3-cycles or 4-cycles and \( (\tilde{G}, p) \) be a covering graph of \( G \). Given a homotopy \( H : K \Box I_n \rightarrow G \) from \( f \) to \( g \) and a lift \( \tilde{f} : K \rightarrow \tilde{G} \) of \( f \), there exists a unique homotopy \( \tilde{H} : K \Box I_n \rightarrow \tilde{G} \) that lifts \( H \).

The statement of this theorem can be summarized by the following diagram.

\[
\begin{array}{ccc}
K & \overset{\sim}{\rightarrow} & G \\
\downarrow{\tilde{f}} & & \downarrow{p} \\
\tilde{G} & \overset{\tilde{g}}{\sim} & G
\end{array}
\]

Here \( \tilde{G} \) is a cover of \( G \) by the graph homomorphism \( p \), the \( \sim \) between \( f \) and \( g \) represents the graph homotopy \( H \) from \( f \) to \( g \), and the \( \sim \) between \( \tilde{f} \) and \( \tilde{g} \) represents a lift \( \tilde{H} \) of \( H \), a graph homotopy between a lift \( \tilde{f} \) of \( f \) and a lift \( \tilde{g} \) of \( g \). Thus if a lift \( \tilde{H} \) of \( H \) exists, then a lift \( \tilde{g} \) of \( g \) exists as well. Now we proceed to the proof.

**Proof.** Let \( G, (\tilde{G}, p), H \) and \( \tilde{f} \) be as in the statement of the theorem. The strategy of this proof is to build the lift \( \tilde{H} \) inductively. For each \( Y \in V(K) \), we produce a lift of \( H \) restricted to \( N_Y \Box I_n \). First, we use the lift of \( f \) to construct a lift of \( H|_{N_Y(I_0)} \). Then we proceed by induction to define a lift of \( H|_{N_Y(I_{i+1})} \) for each \( 0 \leq i < n \), which agrees with the previous lift of \( H|_{N_Y(I_i)} \) on \( V(N_Y(I_{i-1})) \cap V(N_Y(I_i)) \). This produces a
lift of $H|_{N_{y,0} \cup \cdots \cup N_{y,n-1} \cup N_{y,n}}$, which we can then complete to a lift of $H|_{N_y \Box I_n}$. Once we have constructed a lift $\tilde{H}|_{N_y \Box I_n}$, we use it to build the lift $\tilde{H}$ by appealing to the uniqueness of the Path Lifting Property (Theorem 3.8).

Now we proceed to the construction of $\tilde{H}|_{N_y \Box I_n}$. Let $y \in V(K)$. Since $H$ is a graph homotopy from $f$ to $g$, it follows that $H|_{N_y \Box (0)} = f|_{N_y}$. Define $\tilde{H}|_{N_y \Box (0)} = f|_{N_y}$. Since $p$ is a covering map, the restriction

$$p|_{N_{\tilde{H}(y,0)}} : N_{\tilde{H}(y,0)} \to N_{p(\tilde{H}(y,0))}$$

is a bijection on the vertices and edges of these subgraphs. By definition of $H|_{N_y \Box (0)}$, it follows that $p(\tilde{H}(y,0)) = f(y) = H(y,0)$. Thus the inverse

$$(p|_{N_{\tilde{H}(y,0)}})^{-1} : N_{H(y,0)} \to N_{\tilde{H}(y,0)}$$

exists by Lemma 3.5. Since $H$ is a graph homomorphism, there is an inclusion of sets $H[N[y,0]] \subseteq N[H(y,0)]$ and, in particular, $H(y,1) \in N[H(y,0)]$. That is, $H(y,1)$ is in the domain of the inverse $(p|_{N_{\tilde{H}(y,0)}})^{-1}$. Define $\tilde{H}(y,1) = (p|_{N_{\tilde{H}(y,0)}})^{-1}(H(y,1))$. Since $\tilde{f}$ is a lift of $f$ and $H|_{N_y \Box (0)} = f|_{N_y}$, it follows that $\tilde{f}|_{N_y} = (p|_{N_{\tilde{H}(y,0)}})^{-1} \circ H|_{N_y \Box (0)}$. Thus we have defined $\tilde{H}|_{N_y \Box (0)}$, and it is a graph homomorphism because it is the composition of graph homomorphisms.

For the inductive step, assume that $H|_{N_{y,0} \cup \cdots \cup N_{y,i-1}}$ has a lift $\tilde{H}|_{N_{y,0} \cup \cdots \cup N_{y,i}}$ for some $0 \leq i < n$. Figure 1 illustrates the graph $N_{y,0} \cup \cdots \cup N_{y,i}$, in the case that the vertex $y$ has three adjacent vertices. The subgraph $N_{y,i}$ is shown in light blue, and the dashed edges shown in red are not included in the graph $N_{y,0} \cup \cdots \cup N_{y,i}$. Since $(y, i+1) \in N[y, i]$, it follows that $\tilde{H}(y, i+1)$ is defined.

Since $p$ is a covering map, the restriction $p|_{N_{\tilde{H}(y,i+1)}} : N_{\tilde{H}(y,i+1)} \to N_{H(y,i+1)}$ is a bijection on the vertices and edges of these subgraphs. Thus by Lemma 3.5, the inverse $(p|_{N_{\tilde{H}(y,i+1)}})^{-1} : N_{H(y,i+1)} \to N_{\tilde{H}(y,i+1)}$ exists and is a graph homomorphism. Define

$$\tilde{H}|_{N_{y,i+1}} = (p|_{N_{\tilde{H}(y,i+1)}})^{-1} \circ H|_{N_{y,i+1}}.$$

Since $H$ is a graph homomorphism, there is an inclusion

$$H(N[(y, i+1)]) \subseteq N[H(y, i+1)].$$

Thus $\tilde{H}|_{N_{y,i+1}}$ is well-defined. Since $\tilde{H}|_{N_{y,i+1}}$ is the composition of graph homomorphisms, it follows that $\tilde{H}|_{N_{y,i+1}}$ is a graph homomorphism. This is illustrated

![Figure 1](image-url)

**Figure 1.** The union of neighborhoods $N_{y,0} \cup \cdots \cup N_{y,i-1} \cup N_{y,i}$
by the following diagram.

Thus after a finite number of steps, the lift $\tilde{H}|_{N_y \sqcup \cdots \sqcup N_{y,n}}$ is defined.

Suppose $x \in N[y]$. Then $\{(x, i), (x, i + 1)\} \in E(N_y \sqcup I_n)$ for all $0 \leq i < n$. Hence, in order for $\tilde{H}|_{N_y \sqcup \cdots \sqcup N_{y,n}}$ to be extended to a graph homomorphism with domain $N_y \sqcup I_n$, we must show that $\tilde{H}(x, i) = \tilde{H}(x, i + 1)$ or $\{\tilde{H}(x, i), \tilde{H}(x, i + 1)\} \in E(\tilde{G})$ for all $0 \leq i < n$. By definition of $\tilde{H}|_{N_y \sqcup \cdots \sqcup N_{y,n}}$, 

$$\tilde{H}(x, i) = (p|_{N_{y,(y,i)}})^{-1} \circ H|_{N_{y,(y,i)}} (x, i) = (p|_{N_{y,(y,i+1)}})^{-1} \circ H(x, i)$$

and 

$$\tilde{H}(x, i + 1) = (p|_{N_{y,(y,i+1)}})^{-1} \circ H|_{N_{y,(y,i+1)}} (x, i + 1) = (p|_{N_{y,(y,i+1)}})^{-1} \circ H(x, i + 1).$$

That is, $\tilde{H}(x, i)$ is constructed using the graph homomorphism $(p|_{N_{y,(y,i)}})^{-1}$, and $\tilde{H}(x, i + 1)$ is constructed using the graph homomorphism $(p|_{N_{y,(y,i+1)}})^{-1}$. In order to show that $\tilde{H}(x, i) = \tilde{H}(x, i + 1)$ or $\{\tilde{H}(x, i), \tilde{H}(x, i + 1)\} \in E(\tilde{G})$ for all $0 \leq i < n$, we will examine the 4-cycle of $N_y \sqcup I_n$ shown in light blue in Figure 2.

**Figure 2.** The union of neighborhoods $N_{(y,0)} \cup \cdots \cup N_{(y,n)}$

We denote this 4-cycle subgraph by $C_{x,i}$. Since $H$ is a graph homomorphism and $G$ contains no 3-cycles or 4-cycles, we have the following nine cases of how $H$ maps $C_{x,i}$ to $G$, illustrated in Figure 3. The label ‘-’ means that $H$ maps the pair of vertices to the same vertex in $G$. The label $a$ means that $H$ maps the pair of vertices to adjacent vertices in $G$. In cases (8) and (9), the pair of vertices being mapped to the same vertex are circled in red.

For cases (1)-(8), there is an inclusion of sets $H(C_{x,i}) \subseteq N[H(y, i)]$, and $H(x, i) = H(x, i + 1)$ or $\{H(x, i), H(x, i + 1)\} \in E(G)$ for all $0 \leq i < n$. Thus the subgraph $C_{x,i}$ is mapped by $H$ into the domain of the inverse $(p|_{N_{H(y,i)}})^{-1} : N_{H(y,i)} \to N_{\tilde{H}(y,i)}$. 
Since $\tilde{H}|_{C_{x,i}} = (p|_{N\tilde{H}(y,i)})^{-1} \circ H|_{C_{x,i}}$, it follows that
\[
\tilde{H}(x,i) = \tilde{H}(x,i + 1) \quad \text{or} \quad \{\tilde{H}(x,i), \tilde{H}(x, i + 1)\} \in E(\tilde{G}).
\]

For case (9), there is an inclusion of sets $H(C_{x,i}) \subseteq N[H(y, i + 1)]$, and $H(x,i) = H(x,i + 1) \cup \{H(x, i), H(x, i + 1)\} \in E(G)$ for all $0 \leq i < n$. Thus the subgraph $C_{x,i}$ is mapped by $H$ into the domain of the inverse $(p|_{N\tilde{H}(y,i+1)})^{-1} : N\tilde{H}(y,i+1) \rightarrow N\tilde{H}(y,i+1)$. Since $\tilde{H}|_{C_{x,i}} = (p|_{N\tilde{H}(y,i+1)})^{-1} \circ H|_{C_{x,i}}$, it follows that
\[
\tilde{H}(x,i) = \tilde{H}(x,i + 1) \quad \text{or} \quad \{\tilde{H}(x,i), \tilde{H}(x, i + 1)\} \in E(\tilde{G}).
\]

Thus we can extend the graph homomorphism $\tilde{H}|_{N(y,0) \cup \cdots \cup N(y,n)}$ to $\tilde{H}|_{N_y \square I_n}$.

The restriction $H|_{\{y\} \square I_n}$ is a graph homomorphism from $I_n$ to $G$ and can be written as $H_y : I_n \rightarrow G$. By the Uniqueness of Path Lifting (3.8), the lift $\tilde{H}_y : I_n \rightarrow \tilde{G}$ is unique with $\tilde{H}_y(0) = \tilde{H}(y,0) = \tilde{f}(y)$. Since each graph homomorphism $H_x : I_n \rightarrow G$ must have a unique lift $\tilde{H}_x : I_n \rightarrow \tilde{G}$ for all $x \in N[y]$ with $\tilde{H}_x(0) = \tilde{H}(x,0) = \tilde{f}(x)$, the lift $\tilde{H}|_{N_y \square I_n}$ must be unique for each $y \in V(K)$. Since $\tilde{H}_x$ is unique for each $x \in V(K)$ and is a restriction of the graph homomorphism $\tilde{H}|_{N_y \square I_n}$.

**Figure 3.** The cases of how $H$ maps $C_{x,i}$ to $G$
for each \( y \in V(K) \) such that \( x \in N[y] \), the graph homomorphisms \( \tilde{H}|_{N(y) \cap I_0} \) must form a unique lift \( \tilde{H} \) of the homotopy \( H \).

\[ \square \]

We now use the Path Lifting Property (3.8) and the Homotopy Lifting Property (3.9) to prove the general Lifting Criterion (3.16), but first we need the following definition and lemmas.

**Definition 3.10.** Let \( f : (K, y_0) \to (G, x_0) \) be a graph homomorphism. The induced map \( f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim \) is defined by \( f_*([\gamma]) = [f \circ \gamma] \), where \([\gamma]\) is an equivalence class of \( B_1(K, y_0)/\sim \).

**Lemma 3.11.** If \( f : (K, y_0) \to (G, x_0) \) is a graph homomorphism, then the induced map \( f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim \) is well-defined.

**Proof.** Let \( f : (K, y_0) \to (G, x_0) \) be a graph homomorphism, and let the induced map \( f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim \) be defined by \( f_*([\gamma]) = [f \circ \gamma] \), where \([\gamma] \in B_1(K, y_0)/\sim \). Suppose \( \gamma_1, \gamma_2 \in B_1(K, y_0) \) such that \( \gamma_1 \sim \gamma_2 \). Thus there exist a graph homomorphism \( H_1 \in C_2(K) \) such that

1. \( \alpha_{-1}^1(\gamma_1) = \alpha_{-1}^1(\gamma_2) \) and \( \alpha_{+1}^1(\gamma_1) = \alpha_{+1}^1(\gamma_2) \),
2. \( \alpha_{-1}^2(\gamma_1) = \beta_{-1}^1(\gamma_1) = \beta_{+1}^1(\gamma_2) \) and \( \alpha_{+1}^2(\gamma_1) = \beta_{+1}^1(\gamma_1) = \beta_{+1}^1(\gamma_2) \),
3. \( \alpha_{-2}^1(H_1) = \gamma_1 \) and \( \alpha_{+2}^1(H_1) = \gamma_2 \).

We need to show that \( f_*([\gamma_1]) = f_*([\gamma_2]) \), that is, \( f \circ \gamma_1 \sim f \circ \gamma_2 \). Thus we must define a map \( H_2 : I_\infty^G \to G \) and show that \( H_2 \) is well-defined, a graph homomorphism, and is a graph homotopy from \( f \circ \gamma_1 \) to \( f \circ \gamma_2 \). Define \( H_2 : I_\infty^G \to G \) by \( H_2 = f \circ H_1 \). Since \( H_2 \) is a composition of the graph homomorphisms \( H_1 : I_\infty^G \to K \) and \( f : K \to G \), it follows that \( H_2 \) is a graph homomorphism. We now show that \( H_2 \) is a graph homotopy by verifying conditions (a)-(c) of Definition 2.13.

(a) By part (1), we have that

\[ \gamma_1(m_0(\gamma_1, -1)) = \gamma_2(m_0(\gamma_2, -1)) \]

and

\[ \gamma_1(m_0(\gamma_1, +1)) = \gamma_2(m_0(\gamma_2, +1)). \]

Since \( f \) is a graph homomorphism, it follows that

\[ f(\gamma_1(m_0(\gamma_1, -1))) = f(\gamma_2(m_0(\gamma_2, -1))) \]

and

\[ f(\gamma_1(m_0(\gamma_1, +1))) = f(\gamma_2(m_0(\gamma_2, +1))). \]

Therefore, \( \alpha_{-1}^1(f \circ \gamma_1) = \alpha_{-1}^1(f \circ \gamma_2) \) and \( \alpha_{+1}^1(f \circ \gamma_1) = \alpha_{+1}^1(f \circ \gamma_2) \).

(b) By part (2), \( H_1(m_0(H_1, -1), j) = \gamma_1(m_0(\gamma_1, -1)) = \gamma_2(m_0(\gamma_2, -1)) \) and \( H_1(m_0(H_1, +1), j) = \gamma_1(m_0(\gamma_1, +1)) = \gamma_2(m_0(\gamma_2, +1)) \) for all \( j \in \mathbb{Z} \). Since \( f \) is a graph homomorphism, it follows that

\[ f(H_1(m_0(H_1, -1), j)) = f(\gamma_1(m_0(\gamma_1, -1))) = f(\gamma_2(m_0(\gamma_2, -1))) \]

and

\[ f(H_1(m_0(H_1, +1), j)) = f(\gamma_1(m_0(\gamma_1, +1))) = f(\gamma_2(m_0(\gamma_2, +1))) \]

for all \( j \in \mathbb{Z} \). Thus \( \alpha_{-1}^2(H_2) = \beta_{+1}^1(\gamma_1) = \beta_{+1}^1(\gamma_2) \) and \( \alpha_{+1}^2(H_1) = \beta_{+1}^1(\gamma_1) = \beta_{+1}^1(\gamma_2) \), since \( H_2 = f \circ H_1 \).
(c) By part (3), \( H_1(i, m_0(H_1, -2)) = \gamma_1(i) \) and \( H_1(i, m_0(H_1, +2)) = \gamma_2(i) \) for all \( i \in \mathbb{Z} \). Since \( f \) is a graph homomorphism, it follows that
\[
f(H_1(i, m_0(H_1, -2))) = f(\gamma_1(i))
\]
and
\[
f(H_1(i, m_0(H_1, +2))) = f(\gamma_2(i))
\]
for all \( i \in \mathbb{Z} \). Therefore, \( \alpha^2_{-2}(H_2) = f \circ \gamma_1 \) and \( \alpha^2_{+2}(H_2) = f \circ \gamma_2 \), since \( H_2 = f \circ H_1 \).

Thus \( H_2 \) is a graph homotopy from \( f \circ \gamma_1 \) to \( f \circ \gamma_2 \), so \( f \circ \gamma_1 \sim f \circ \gamma_2 \). Therefore, \( f_* \) is well-defined.

When a path \( f : I_\infty \to G \) maps a sequence of consecutive vertices to the same vertex in \( G \), this section is called padding. The General Padding Lemma (3.12) states that a path with padding is homotopic to that same path with the padding removed.

**Lemma 3.12 (General Padding Lemma).** Let \( f \in C_1(G) \). Define \( f' \in C_1(G) \) by
\[
f'(i) = \begin{cases} f(i - m) & \text{for } i \geq b + m, \\ f(b) & \text{for } b - n \leq i \leq b + m, \\ f(i + n) & \text{for } i \leq b - n, \end{cases}
\]
for some \( n, m \in \mathbb{N} \) and \( b \in \mathbb{Z} \) with \( m_0(f, -1) < b < m_0(f, +1) \). Then \( f \sim f' \).

A full proof of this lemma can be found in [7, Lemma 5.3]. We use the General Padding Lemma (3.12) in the proof of the following lemma.

**Lemma 3.13.** If \( f : (K, y_0) \to (G, x_0) \) is a graph homomorphism, then the induced map \( f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim \) is a group homomorphism.

**Proof.** Let \( f : (K, y_0) \to (G, x_0) \) be a graph homomorphism, and let the induced map \( f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim \) be defined by \( f_*([\gamma]) = [f \circ \gamma] \), where \([\gamma] \in B_1(K, y_0)/\sim \). Suppose \( \gamma_1, \gamma_2 \in B_1(K, y_0) \). Since \( B_1(K, y_0) \) is closed with respect to concatenation, it follows that \( \gamma_1 \cdot \gamma_2 \in B_1(K, y_0) \). We need to show that \( f_*([\gamma_1 \cdot \gamma_2]) = f_*([\gamma_1]) \cdot f_*([\gamma_2]) \), that is, \( f \circ (\gamma_1 \cdot \gamma_2) \sim (f \circ \gamma_1) \cdot (f \circ \gamma_2) \). The concatenation \( (f \circ \gamma_1) \cdot (f \circ \gamma_2) \) is defined by
\[
((f \circ \gamma_1) \cdot (f \circ \gamma_2))(i) = \begin{cases} (f \circ \gamma_1)(i + m_0(f \circ \gamma_1, -1)) & \text{for } i \geq 0, \\ (f \circ \gamma_2)(i + m_0(f \circ \gamma_2, +1)) & \text{for } i \leq 0, \\ (f \circ \gamma_1)(i + m_0(f \circ \gamma_1, -1)) & \text{for } i \geq 0, \\ (f \circ \gamma_2)(i + m_0(f \circ \gamma_2, +1)) & \text{for } i \leq 0. \end{cases}
\]

Similarly, the concatenation \( \gamma_1 \cdot \gamma_2 \) is defined by
\[
(\gamma_1 \cdot \gamma_2)(i) = \begin{cases} \gamma_1(i + m_0(\gamma_1, -1)) & \text{for } i \geq 0, \\ \gamma_2(i + m_0(\gamma_2, +1)) & \text{for } i \leq 0. \end{cases}
\]

Thus the composition \( f \circ (\gamma_1 \cdot \gamma_2) \) is defined by
\[
f((\gamma_1 \cdot \gamma_2)(i)) = \begin{cases} f(\gamma_1(i + m_0(\gamma_1, -1))) & \text{for } i \geq 0, \\ f(\gamma_2(i + m_0(\gamma_2, +1))) & \text{for } i \leq 0. \end{cases}
\]
Since $f$ might possibly map vertices to $x_0$ after $\gamma_1$ stabilizes at $m_0(\gamma_1, -1)$, it follows that $m_0(f \circ \gamma_1, -1) \geq m_0(\gamma_1, -1)$. Thus $m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1) \geq 0$, which implies that

$$f((\gamma_1 \cdot \gamma_2)(m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1))) = f(\gamma_1(m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1) + m_0(\gamma_1, -1))) = f(\gamma_1(m_0(f \circ \gamma_1, -1))).$$

Since $m_0(f \circ \gamma_1, -1)$ is the greatest integer such that $(f \circ \gamma_1)(m) = f(\gamma_1(m_0(f \circ \gamma_1, -1))$ for all $m \leq m_0(f \circ \gamma_1, -1)$, it follows that $f \circ (\gamma_1 \cdot \gamma_2)$ maps all vertices between 0 and $m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1)$ to $v_0$.

Since $f$ might possibly map vertices to $x_0$ before the end of $\gamma_2$ at $m_0(\gamma_2, +1)$, it follows that $m_0(f \circ \gamma_1, +1) \leq m_0(\gamma_1, +1)$. Thus $m_0(f \circ \gamma_1, +1) - m_0(\gamma_1, +1) \leq 0$, which implies that

$$f((\gamma_1 \cdot \gamma_2)(m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1))) = f(\gamma_2(m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1) + m_0(\gamma_2, +1))) = f(\gamma_2(m_0(f \circ \gamma_2, +1))).$$

Since $m_0(f \circ \gamma_2, +1)$ is the least integer such that $(f \circ \gamma_2)(m) = f(\gamma_2(m_0(f \circ \gamma_2, +1))$ for all $m \geq m_0(f \circ \gamma_2, +1)$, it follows that $f \circ (\gamma_1 \cdot \gamma_2)$ maps all vertices between $m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1)$ and 0 to $v_0$.

Thus there is potentially padding in $f \circ (\gamma_1 \cdot \gamma_2)$ from the vertex $m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1))$ to the vertex $m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1))$. Therefore, $f \circ (\gamma_1 \cdot \gamma_2) \sim (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by the General Padding Lemma (3.12), and it follows that $f_*$ is a group homomorphism.

\begin{lemma}
Let $(\tilde{G}, p)$ with $p : (\tilde{G}, \tilde{x}_0) \rightarrow (G, x_0)$ be a covering graph of $G$ and $f : (K, y_0) \rightarrow (G, x_0)$ be a graph homomorphism. Given a lift $\tilde{f} : (K, y_0) \rightarrow (\tilde{G}, \tilde{x}_0)$ of $f$, $p_\ast \circ \tilde{f}_\ast = f_\ast$.
\end{lemma}

\begin{proof}
Let $(\tilde{G}, p)$ with $p : (\tilde{G}, \tilde{x}_0) \rightarrow (G, x_0)$ be a covering graph of $G$, let $f : (K, y_0) \rightarrow (G, x_0)$ be a graph homomorphism, and let $\tilde{f} : (K, y_0) \rightarrow (\tilde{G}, \tilde{x}_0)$ be a lift of $f$. For all $[\gamma] \in B_1(K, y_0)/\sim$,

$$(p_\ast \circ \tilde{f}_\ast)([\gamma]) = p_\ast(\tilde{f}_\ast([\gamma])) = p_\ast([f \circ \gamma]) = [p \circ (f \circ \gamma)] = [(p \circ \tilde{f}) \circ \gamma] = [f \circ \gamma] = f_\ast([\gamma]).$$

Therefore, $p_\ast \circ \tilde{f}_\ast = f_\ast$.
\end{proof}

\begin{definition}
A graph $G$ is connected if for each $v, w \in V(G)$, there exists a stable graph homomorphism $f \in C_1(G)$ such that $f(m_0(f, -1)) = v$ and $f(m_0(f, +1)) = w$.

While this is not the standard definition of a connected graph found in e.g. [11], it is equivalent.
\end{definition}
**Theorem 3.16 (Lifting Criterion).** Let $G$ be a connected graph, let $(\tilde{G}, p)$ be a covering graph of $G$, and let $f : (K, I_0) \to (G, x_0)$ be a stable graph homomorphism. If $G$ contains neither 3-cycles nor 4-cycles, then there is a lift $\tilde{f} : (K, y_0) \to (\tilde{G}, \tilde{x}_0)$ of $f$ if and only if $f_* (B_1(K, y_0)/ \sim) \subseteq p_* (B_1(\tilde{G}, \tilde{x}_0)/ \sim)$.

**Proof.** Let $G$ be a connected graph with no 3-cycles or 4-cycles, $(\tilde{G}, p)$ be a covering graph of $G$, and $f : (K, y_0) \to (G, x_0)$ be a stable graph homomorphism.

Suppose a lift $\tilde{f} : (K, y_0) \to (\tilde{G}, \tilde{x}_0)$ of $f$ exists. Then $p \circ \tilde{f} = f$, which implies that $p_* \circ \tilde{f}_* = f_*$ by Lemma 3.14. Let $[\gamma] \in B_1(K, y_0) / \sim$. Thus $f_* ([\gamma]) = (p_* \circ \tilde{f}_*) ([\gamma]) = p_* (\tilde{f}_* ([\gamma])) \in p_* (B_1(\tilde{G}, \tilde{x}_0) / \sim)$, since $\tilde{f}_* ([\gamma]) = [\tilde{f} \circ \gamma] \in B_1(\tilde{G}, \tilde{x}_0) / \sim$. Therefore, $f_* (B_1(K, y_0) / \sim) \subseteq p_* (B_1(\tilde{G}, \tilde{x}_0) / \sim)$.

Conversely, suppose $f_* (B_1(K, y_0) / \sim) \subseteq p_* (B_1(\tilde{G}, \tilde{x}_0) / \sim)$. Let $y \in V(K)$. Since $G$ is connected, there is a stable graph homomorphism $\gamma_y : I_\infty \to K$ with $\gamma_y (m_0 (\gamma_y, -1)) = y_0$ and $\gamma_y (m_0 (\gamma_y, +1)) = y$. Thus $f \circ \gamma_y : I_\infty \to G$ is a stable graph homomorphism with $f (\gamma_y (m_0 (\gamma_y, -1))) = x_0$ and $f (\gamma_y (m_0 (\gamma_y, +1))) = f (y) \in V(G)$. Hence, by the *Path Lifting Property* (3.8), there is a unique lift $\tilde{f}_\gamma : I_\infty \to \tilde{G}$ with $\tilde{f}_\gamma (m_0 (\gamma_y, -1)) = \tilde{x}_0 \in p^{-1} (x_0)$. Define $\tilde{f} : K \to \tilde{G}$ by $\tilde{f} (y) = f (\gamma_y (m_0 (\gamma_y, +1))) \in p^{-1} (f (y))$.

We must show that

1. $\tilde{f}$ is well-defined,
2. $\tilde{f}$ is a graph homomorphism,
3. $\tilde{f}$ is a lift of $f$.

The map $\tilde{f}$ is well-defined.

We must show that $\tilde{f} (y)$ does not depend on the choice of $\gamma_y$. Suppose $\gamma'_y : I_\infty \to K$ is another stable graph homomorphism with $\gamma'_y (m_0 (\gamma'_y, -1)) = y_0$ and $\gamma'_y (m_0 (\gamma'_y, +1)) = y$. Then $f \circ \gamma'_y : I_\infty \to G$ is a stable graph homomorphism with $f (\gamma'_y (m_0 (\gamma'_y, -1))) = x_0$ and $f (\gamma'_y (m_0 (\gamma'_y, +1))) = f (y)$. Recall that $\overline{\gamma_y} : I_\infty \to K$ is defined by $\overline{\gamma_y} (i) = \gamma_y (-i)$ for all $i \in \mathbb{Z}$. Therefore, the concatenation $\overline{\gamma_y} \cdot \gamma'_y : I_\infty \to K$...
$I_\infty \to K$ is defined by

$$\overline{\gamma_y} \cdot \gamma'_y(i) = \begin{cases} 
\tau_y(i + m_0(\overline{\gamma_y}, -1)) & \text{for } i \geq 0, \\
\gamma'_y(i + m_0(\gamma'_y, +1)) & \text{for } i \leq 0,
\end{cases}$$

for $\nu = 0, 1, 2, \ldots$.

Since $\overline{\gamma_y}(m_0(\overline{\gamma_y}, -1)) = \gamma_y(-m_0(\overline{\gamma_y}, -1)) = \gamma_y(m_0(\gamma_y, +1)) = y = \gamma'_y(m_0(\gamma'_y, +1))$, it follows that $\alpha_{\nu}^{-1}(\overline{\gamma_y}) = \alpha_{\nu}^{-1}(\gamma'_y)$. Thus by Proposition 2.15, $\overline{\gamma_y} \cdot \gamma'_y$ is a graph homomorphism. By Lemma 2.16, $\overline{\gamma_y} \cdot \gamma'_y$ stabilizes in the negative direction at $m_0(\overline{\gamma_y}, \gamma'_y, -1) = m_0(\gamma'_y, -1) - m_0(\gamma'_y, +1)$ and in the positive direction at $m_0(\overline{\gamma_y}, \gamma'_y, +1) = m_0(\overline{\gamma_y}, +1) - m_0(\gamma_y, -1) = -m_0(\gamma_y, -1) + m_0(\gamma_y, +1)$. Therefore,

$$\overline{\gamma_y} \cdot \gamma'_y(m_0(\overline{\gamma_y}, \gamma'_y, -1)) = \overline{\gamma_y} \cdot \gamma'_y(m_0(\gamma'_y, -1) - m_0(\gamma'_y, +1)) = \gamma'_y(m_0(\gamma'_y, -1) - m_0(\gamma'_y, +1) + m_0(\gamma_y, +1)) = \gamma'_y(m_0(\gamma'_y, -1)) = y_0 \quad \text{and}$$

$$\overline{\gamma_y} \cdot \gamma'_y(m_0(\overline{\gamma_y}, \gamma'_y, +1)) = \overline{\gamma_y} \cdot \gamma'_y(-m_0(\gamma_y, -1) + m_0(\gamma_y, +1)) = \gamma_y(m_0(\gamma_y, -1) - m_0(\gamma_y, +1) + m_0(\gamma_y, +1)) = \gamma_y(m_0(\gamma_y, -1)) = y_0.$$

Thus $[\overline{\gamma_y} \cdot \gamma'_y] \in B_1(K, y_0)/\sim$, namely, a ‘loop’ in the graph $K$ based at the distinguished vertex $y_0$. Since $f_*$ is a group homomorphism by Lemma 3.13,

$$f_*([\overline{\gamma_y} \cdot \gamma'_y]) = [f(\overline{\gamma_y} \cdot \gamma'_y)] = [\overline{f \gamma_y} \cdot f \gamma'_y].$$

Therefore, $[\overline{f \gamma_y} \cdot f \gamma'_y] \in f_*(B_1(K, y_0)/\sim) \subseteq p_*(B_1(\tilde{G}, \tilde{x}_0)/\sim)$. Thus there exists an equivalence class $[g] \in B_1(\tilde{G}, \tilde{x}_0)/\sim$ such that $p_*([g]) = [\overline{f \gamma_y} \cdot f \gamma'_y]$. Hence, $[pg] = [\overline{f \gamma_y} \cdot f \gamma'_y]$, which implies that $pg \sim \overline{f \gamma_y} \cdot f \gamma'_y$. Therefore, it follows that there exists a graph homotopy $\tilde{H} : I^2_\infty \to G$ from $pg$ to $\overline{f \gamma_y} \cdot f \gamma'_y$. The graph homomorphism $g : I_\infty \to \tilde{G}$ is a lift of $pg$. By the Path Lifting Property (3.8), there is a unique lift

$$\overline{\gamma_y} \cdot f \gamma'_y : I_\infty \to \tilde{G}$$

of $\overline{f \gamma_y} \cdot f \gamma'_y$ with $\overline{\gamma_y} \cdot f \gamma'_y(m_0(\overline{\gamma_y} \cdot f \gamma'_y, -1)) = \tilde{x}_0$. Since $G$ contains neither 3-cycles nor 4-cycles, the Homotopy Lifting Property (3.9) holds. Thus there exists a lifted homotopy $\tilde{H} : I^2_\infty \to \tilde{G}$ from $g$ to $\overline{\gamma_y} \cdot f \gamma'_y$. Since $[g] \in B_1(\tilde{G}, \tilde{x}_0)/\sim$, it follows that

$$\overline{\gamma_y} \cdot f \gamma'_y(m_0(\overline{\gamma_y} \cdot f \gamma'_y, -1)) = \overline{\gamma_y} \cdot f \gamma'_y(m_0(\overline{\gamma_y} \cdot f \gamma'_y, +1)) = \tilde{x}_0$$

as well. By definition of concatenation, $\overline{f \gamma_y} \cdot f \gamma'_y : I_\infty \to G$ is first defined by $f \gamma'_y$ followed by $\overline{\gamma_y}$. By definition of inverses, $\overline{f \gamma_y}$ is defined by $f \gamma_y$ in reverse.
Therefore, by the uniqueness of the Path Lifting Property (3.8), the first part of $f_{\gamma_y} \cdot f_{\gamma_y}'$ is the lift $\tilde{f}_{\gamma_y}$ of $f_{\gamma_y}'$ followed by the lift $\tilde{f}_{\gamma_y}$ of $f_{\gamma_y}$ in reverse with the common vertex $\tilde{f}_{\gamma_y}'(m_0(\gamma_y'+1)) = \tilde{f}_{\gamma_y}(m_0(\gamma_y, +1))$. Thus $\tilde{f}(y)$ is not dependent on the choice of path $\gamma_y$ starting at $y_0$ and ending at $y$. Therefore, $\tilde{f}$ is well-defined.

**The map $\tilde{f}$ is a graph homomorphism.**

Suppose $x \in N[y]$, the closed neighborhood of $y$. The map $\tilde{f}$ is a graph homomorphism if either $\tilde{f}(y) = \tilde{f}(x)$ or $\{\tilde{f}(y), \tilde{f}(x)\} \in E(\tilde{G})$. Define $\beta : I_\infty \to G$ by

$$\beta(i) = \begin{cases} 
\gamma_y(i) & \text{for } i \leq m_0(\gamma_y, +1), \\
x & \text{for } i > m_0(\gamma_y, +1).
\end{cases}$$

Since $\gamma_y$ is a stable graph homomorphism and $x \in N[y]$, the map $\beta$ is a stable graph homomorphism with $m_0(\beta, +1) = m_0(\gamma_y, +1) + 1$. Therefore, $\tilde{f}(x) = \tilde{f}_\beta(m_0(\beta, +1))$. Since $\beta(m_0(\beta, +1) - 1) = \beta(m_0(\gamma_y, +1)) = \gamma_y(m_0(\gamma_y, +1)) = y$ and $f$ is a graph homomorphism, $f(\beta(m_0(\beta, +1) - 1)) = f(\gamma_y(m_0(\gamma_y, +1)))$. Thus

$$\tilde{f}_\gamma(m_0(\gamma_y, +1)) = \tilde{f}_\beta(m_0(\beta, +1) - 1),$$

which implies that either

$$\tilde{f}_\gamma(m_0(\gamma_y, +1)) = \tilde{f}_\beta(m_0(\beta, +1))$$

or

$$\{\tilde{f}_\gamma(m_0(\gamma_y, +1)), \tilde{f}_\beta(m_0(\beta, +1))\} \in E(\tilde{G}).$$

Therefore, $\tilde{f}(y) = \tilde{f}(x)$ or $\{\tilde{f}(y), \tilde{f}(x)\} \in E(\tilde{G})$, and hence, $\tilde{f}$ is a graph homomorphism.

**The map $\tilde{f}$ is a lift of $f$, that is, $p \circ \tilde{f} = f$.**

Since $\tilde{f}_\gamma : I_\infty \to \tilde{G}$ is a lift of $f_{\gamma_y}$ and $p \circ \tilde{f}_\gamma = f_{\gamma_y}$, it follows that

$$p \circ \tilde{f}(y) = p(\tilde{f}_\gamma(m_0(\gamma_y, +1)))$$

$$= f(\gamma_y(m_0(\gamma_y, +1)))$$

$$= f(y)$$

for all $y \in V(K)$. Thus $p \circ \tilde{f} = f$, and the graph homomorphism $\tilde{f} : K \to \tilde{G}$ is lift of $f$. $\square$

In the next section, we use these lifting properties to show that the fundamental group of the cycle $C_5$ is isomorphic to $\mathbb{Z}$ in a combinatorial way.

4. **An Application to $C_5$**

As the 3-cycle and 4-cycle are A-contractible, the 5-cycle is the best candidate for a graph with behavior similar to the circle in classical homotopy theory. Like the fundamental group of the circle in traditional homotopy theory, the A-homotopy fundamental group of the 5-cycle is isomorphic to $\mathbb{Z}$. We compute this using the lifting properties established in the previous section, concluding our question of why the cycles $C_3$ and $C_4$ are A-contractible, while cycles on five or more vertices are not. For the sake of concreteness, the proofs in this section are only for the 5-cycle, but they could be generalized to show that the fundamental group of all $k$-cycles with $k \geq 5$ is isomorphic to $\mathbb{Z}$.
Definition 4.1. Let $C_5$ be a 5-cycle with vertices labeled $[0]$, $[1]$, $[2]$, $[3]$, and $[4]$, and let $p_5 : I_\infty \to C_5$ be the graph homomorphism defined by $p_5(i) = [i \mod 5]$ for all $i \in \mathbb{Z}$.

Note that $p_5$ does not stabilize in either direction. Let $[i-1,i+1]$ denote the subgraph of $I_\infty$ with vertex set $V([i-1,i+1]) = \{i-1,i,i+1\}$ and edge set $E([i-1,i+1]) = \{i-1,i,i+1\}$ for all $i \in \mathbb{Z}$. The relative graph homomorphism $p_5|_{N[i]} = p_5|[i-1,i+1]$ is bijective for all $i \in \mathbb{Z}$. Thus $p_5$ is a local isomorphism, and the pair $(I_\infty,p_5)$ forms a covering graph of $C_5$.

If $\alpha : I_\infty \to C_5$ is a stable graph homomorphism with $\alpha(m_0(\alpha,-1)) = [0]$, then by the Path Lifting Property (Theorem 3.8) there is a unique graph homomorphism $\bar{\alpha} : I_\infty \to I_\infty$ with $\bar{\alpha}(m_0(\bar{\alpha},-1)) = \bar{x}$ for each $\bar{x} \in p_5^{-1}([0])$ such that the diagram

\[
\begin{array}{ccc}
(I_\infty,\bar{x}) & \xrightarrow{p_5} & (C_5,[0]) \\
\downarrow{\bar{\alpha}} & & \\
I_\infty & \xrightarrow{\alpha} & (C_5,[0])
\end{array}
\]

commutes, that is, $p_5 \circ \bar{\alpha} = \alpha$.

Lemma 4.2 (Path Lift). Let $\alpha \in B_1(C_5,x_0)$, and let the pair $(I_\infty,p_5)$ be as in Definition 4.1. Suppose $\bar{x}_0 \in p_5^{-1}(x_0)$. Then a lift $\bar{\alpha} : I_\infty \to I_\infty$ of $\alpha$ is defined for all $i \leq m_0(\alpha,-1)$ by $\bar{\alpha}(i) = \bar{x}_0$ and for all $i > m_0(\alpha,-1)$ recursively by

\[
\bar{\alpha}(i) = \begin{cases} 
\bar{x}_0 & \text{if } \alpha(i) = \alpha(i-1) + [1], \\
\bar{\alpha}(i-1) & \text{if } \alpha(i) = \alpha(i-1), \\
\bar{x}_0 & \text{if } \alpha(i) = \alpha(i-1) - [1]. 
\end{cases}
\]

Proof. Let $\alpha \in B_1(C_5,x_0)$ and the pair $(I_\infty,p_5)$ be as defined previously. Suppose $\bar{x}_0 \in p_5^{-1}(x_0)$ and by the Path Lifting Property (3.8), there is a unique lift $\bar{\alpha} : I_\infty \to I_\infty$ defined by $\bar{\alpha}(i) = \bar{x}_0$ for all $i \leq m_0(\alpha,-1)$, and recursively by $\bar{\alpha}(i) = (p_5|_{N[\alpha(i)-1]})^{-1}(\alpha(i))$ for all $i > m_0(\alpha,-1)$, so we only need to compute $(p_5|_{N[\bar{\alpha}(i)-1]})^{-1}(\alpha(i))$ for $i > m_0(\alpha,-1)$. By definition of $I_\infty$, it follows that the subgraph $N[\bar{\alpha}(i)-1] = [\bar{\alpha}(i-1) - 1,\bar{\alpha}(i-1) + 1]$. Therefore, $p_5|_{N[\bar{\alpha}(i)-1]}$ is a graph homomorphism from the subgraph with vertex set $\{\bar{\alpha}(i-1)-1,\bar{\alpha}(i-1),\bar{\alpha}(i-1)+1\}$ to the subgraph with vertex set $\{p_5(\bar{\alpha}(i-1)-1),p_5(\bar{\alpha}(i-1)),p_5(\bar{\alpha}(i-1)+1)\}$. By definition of $p_5$ and since $p_5 \circ \bar{\alpha} = \alpha$,

\[
p_5(\bar{\alpha}(i-1)-1) = \begin{cases} 
\lfloor(\bar{\alpha}(i-1) - 1) \mod 5 \rfloor & \text{if } \bar{\alpha}(i-1) = \alpha(i-1), \\
\lfloor(\bar{\alpha}(i-1) \mod 5) - 1 \rfloor & \text{if } \bar{\alpha}(i-1) = \alpha(i-1) - 1, \\
\lfloor(\bar{\alpha}(i-1) \mod 5) + 1 \rfloor & \text{if } \bar{\alpha}(i-1) = \alpha(i-1) + 1. 
\end{cases}
\]

and

\[
p_5(\bar{\alpha}(i-1)) = \alpha(i-1),
\]
and

\[ p_5(\tilde{\alpha}(i - 1) + 1) = [\tilde{\alpha}(i - 1) + 1 \mod 5] \]
\[ = [\tilde{\alpha}(i - 1) \mod 5] + [1] \]
\[ = p_5(\tilde{\alpha}(i - 1)) + [1] \]
\[ = \alpha(i - 1) + [1]. \]

Thus \( (p_5|_{N_{\tilde{\alpha}(i-1)}})^{-1}(\alpha(i-1) - [1]) = \tilde{\alpha}(i-1) - 1, (p_5|_{N_{\tilde{\alpha}(i-1)}})^{-1}(\alpha(i-1) + [1]) = \tilde{\alpha}(i-1) + 1. \) Therefore, \( \tilde{\alpha} \) is defined by \( \tilde{\alpha}(i) = \bar{x}_0 \) for all \( i \leq m_0(\alpha, -1) \) and recursively by

\[ \tilde{\alpha}(i) = \begin{cases} 
\alpha(i - 1) + 1 & \text{if } \alpha(i) = \alpha(i - 1) + [1], \\
\alpha(i - 1) & \text{if } \alpha(i) = \alpha(i - 1), \\
\alpha(i - 1) - 1 & \text{if } \alpha(i) = \alpha(i - 1) - [1]. 
\end{cases} \]

for all \( i > m_0(\alpha, -1). \) \( \square \)

We also need to propose a representative for each of the equivalence classes of \( B_1(\mathcal{C}_5, [0]) / \sim. \)

**Definition 4.3.** Let the map \( \gamma_n : I_\infty \to \mathcal{C}_5 \) be defined for each \( n \geq 0 \) by

\[ \gamma_n(i) = \begin{cases} 
[0] & \text{for } i \leq 0, \\
[i \mod 5] & \text{for } 0 \leq i \leq 5n, \\
[0] & \text{for } i \geq 5n, 
\end{cases} \]

and for each \( n \leq 0 \) by

\[ \gamma_n(i) = \begin{cases} 
[0] & \text{for } i \leq 0, \\
[(-i) \mod 5] & \text{for } 0 \leq i \leq -5n, \\
[0] & \text{for } i \geq -5n. 
\end{cases} \]

When \( n = 0, \gamma_n \) is the constant map at \([0]\). For \( n > 0 \), the graph homomorphism \( \gamma_n \) starts at \([0]\) and wraps around \( \mathcal{C}_5 \) in a clockwise direction \( n \) times. Similarly, for \( n < 0 \), the graph homomorphism \( \gamma_n \) starts at \([0]\) and wraps around \( \mathcal{C}_5 \) in a counterclockwise direction \( n \) times. Given these \( \gamma_n \) representatives, we need lifts \( \tilde{\gamma}_n \). If \( n \geq 0 \), then

\[ \gamma_n(i) = \begin{cases} 
[i \mod 5] & \text{for } i \leq 0, \\
\alpha(i - 1) + [1] & \text{for } 0 \leq i \leq 5n, \\
\alpha(i - 1) & \text{for } i \geq 5n, 
\end{cases} \]

for all \( 0 < i \leq 5n \), and \( \gamma_n(i) = [0] \) otherwise. Similarly, if \( n \leq 0 \), then

\[ \gamma_n(i) = \begin{cases} 
[(-i) \mod 5] & \text{for } i \leq 0, \\
\alpha(i - 1) - [1] & \text{for } 0 \leq i \leq -5n, \\
\alpha(i - 1) & \text{for } i \geq -5n. 
\end{cases} \]
for all $0 < i \leq -5n$, and $\gamma_n(i) = [0]$ otherwise. Thus by Lemma 4.2, the lift of $\gamma_n$ starting at 0 is $\tilde{\gamma}_n : I_\infty \to I_\infty$ defined by

$$
\tilde{\gamma}_n(i) = \begin{cases} 
0 & \text{for } i \leq 0, \\
 i & \text{for } 0 \leq i \leq 5n, \\
5n & \text{for } i \geq 5n,
\end{cases}
$$

and

$$
\tilde{\gamma}_n(i) = \begin{cases} 
0 & \text{for } i \leq 0, \\
 -i & \text{for } 0 \leq i \leq -5n, \\
5n & \text{for } i \geq -5n,
\end{cases}
$$

We also need to know how the representatives $\gamma_n$ relate to each other. To do this, we first need the Shifting Lemma (4.4). This lemma states that a path is homotopic to that same path shifted down to start at an earlier vertex and to that same path shifted up to start at a later vertex.

**Lemma 4.4 (Shifting Lemma).** Let $f \in C_1(G)$ and $n \in \mathbb{Z}$. Define $f_n \in C_1(G)$ by $f_n(i) = f(i - n)$, that is, $f$ shifted by $n$. Then $f \sim f_n$.

A full proof of this lemma can be found in [7, Lemma 5.4]. Now we use the Shifting Lemma (4.4) to show how the representatives $\gamma_n$ relate to each other.

**Lemma 4.5.** Let $\gamma_n, \gamma_{-n} \in B_1(C_5, [0])$ be as defined in Definition 4.3 for $n \in \mathbb{Z}$. Then $\gamma_{-n} \sim \overline{\gamma}_n$, whose equivalence class is the inverse of $[\gamma_n]$.

**Proof.** Suppose $n \geq 0$. By Definition 4.3,

$$
\overline{\gamma}_n(i) = \gamma_n(-i) = \begin{cases} 
[0] & \text{for } -i \leq 0, \\
((-i) \mod 5) & \text{for } 0 \leq -i \leq 5n, \\
[0] & \text{for } -i \geq 5n,
\end{cases}
$$

$$
= \begin{cases} 
[0] & \text{for } i \leq -5n, \\
((-i) \mod 5) & \text{for } -5n \leq i \leq 0, \\
[0] & \text{for } i \geq 0,
\end{cases}
$$

By the construction of $\gamma_n$,

$$
\gamma_{-n}(i + 5n) = \begin{cases} 
[0] & \text{for } i + 5n \leq 0, \\
((-i - 5n) \mod 5) & \text{for } 0 \leq i + 5n \leq 5n, \\
[0] & \text{for } i + 5n \geq 5n,
\end{cases}
$$

$$
= \begin{cases} 
[0] & \text{for } i \leq -5n, \\
((-i) \mod 5) & \text{for } -5n \leq i \leq 0, \\
[0] & \text{for } i \geq 0,
\end{cases}
$$

for all $i \in \mathbb{Z}$. Therefore, the graph homomorphism $\overline{\gamma}_n$ is $\gamma_{-n}$ shifted by $-5n$. Thus it follows by the Shifting Lemma (4.4) that $\gamma_{-n} \sim \overline{\gamma}_n$ for $n \geq 0$. Suppose $n \leq 0$. 


By Definition 4.3, 
\[ \gamma_n(i) = \begin{cases} 
0 & \text{for } -i \leq 0, \\
[i \mod 5] & \text{for } 0 \leq -i \leq -5n, \\
0 & \text{for } -i \geq -5n,
\end{cases} \]
\[ = \begin{cases} 
0 & \text{for } i \leq 5n, \\
[i \mod 5] & \text{for } 5n \leq i \leq 0, \\
0 & \text{for } i \geq 0.
\end{cases} \]

By construction of \(\gamma_n\), 
\[ \gamma_n(i-5n) = \begin{cases} 
0 & \text{for } i-5n \leq 0, \\
[i-5n \mod 5] & \text{for } 0 \leq i-5n \leq -5n, \\
0 & \text{for } i-5n \geq -5n,
\end{cases} \]
\[ = \begin{cases} 
0 & \text{for } i \leq 5n, \\
[i \mod 5] & \text{for } 5n \leq i \leq 0, \\
0 & \text{for } i \geq 0.
\end{cases} \]

for all \(i \in \mathbb{Z}\). Therefore, the graph homomorphism \(\gamma_n\) is \(\gamma_n\) shifted by \(5n\). Thus it follows by the Shifting Lemma (4.4) that \(\gamma_n \sim n\) for \(n \leq 0\). Therefore, \(\gamma_n \sim n\) for all \(n \in \mathbb{Z}\). \(\square\)

We need one last lemma before proceeding to the proof that \((B_1(C_5, [0]) / \sim) \cong \mathbb{Z}\).

**Definition 4.6.** Let \(\tilde{f} : I_\infty \to I_\infty\) be a stable graph homomorphism. For \(i \in \mathbb{Z}\), the value \(\tilde{f}(i)\) is increasing if \(\tilde{f}(i) < \tilde{f}(i + 1)\) and is decreasing if \(\tilde{f}(i) > \tilde{f}(i + 1)\) and is constant if \(\tilde{f}(i) = \tilde{f}(i + 1)\).

**Lemma 4.7.** If \(\tilde{f} : I_\infty \to I_\infty\) is a stable graph homomorphism with \(\tilde{f}(m_0(f, -1)) = 0\) and \(\tilde{f}(m_0(f, +1)) = 5n\), then \([\tilde{f}] = [\gamma_n]\), where \(\gamma_n\) is a lift of \(\gamma_n : I_\infty \to C_5\).

**Proof.** Let \(\tilde{f} : I_\infty \to I_\infty\) be a stable graph homomorphism with \(\tilde{f}(m_0(f, -1)) = 0\) and \(\tilde{f}(m_0(f, +1)) = 5n\) with \(n \in \mathbb{Z}\). Although the path \(\tilde{f}\) starts at 0 and ends at 5n, \(\tilde{f}\) may increase, decrease, or remain constant from the vertex \(m_0(f, -1)\) to the vertex \(m_0(f, +1)\). In contrast, for \(n \geq 0\), \(\gamma_n\) increases constantly from starting at 0 to ending at 5n, and for \(n \leq 0\), \(\gamma_n\) decreases constantly from starting at 0 to ending at 5n. We show that \(\tilde{f}\) is homotopic to \(\gamma_n\) by first showing that \(\tilde{f}\) is homotopic to a path \(\tilde{f}'\) that has no negative increasing values and no positive decreasing values. Since \(\tilde{f}'\) starts at 0 as well, if \(n \geq 0\), no negative increasing values implies that \(\tilde{f}'\) has no negative values at all, and no positive decreasing values implies that \(\tilde{f}'\) is constant or increasing from 0 to 5n. If \(n \leq 0\), no positive decreasing values implies that \(\tilde{f}'\) has no positive values at all, and no negative increasing values implies that \(\tilde{f}'\) is constant or decreasing from 0 to 5n. Then we use the General Padding Lemma (3.12) to show that this path \(\tilde{f}'\) is homotopic to \(\gamma_n\).
Define $H : I_\infty \sqcup I_\infty \to I_\infty$ for all $j \leq 0$ by $H(i, j) = \tilde{f}(i)$, and recursively for all $j > 0$ by

$$H(i, j) = \begin{cases} 
H(i, j - 1) - 1 & \text{if } 0 \leq H(i + 1, j - 1) < H(i, j - 1), \\
H(i, j - 1) & \text{if } 0 \leq H(i, j - 1) \leq H(i + 1, j - 1), \\
H(i, j - 1) + 1 & \text{if } H(i, j - 1) < H(i + 1, j - 1) \leq 0, \\
H(i, j - 1) & \text{if } H(i + 1, j - 1) \leq H(i, j - 1) \leq 0.
\end{cases}$$

First, we must confirm that these are all of the cases. Define $H_j : I_\infty \to I_\infty$ by $H_j(i) = H(i, j)$ for all $i, j \in \mathbb{Z}$. The first case is if $H_{j-1}(i)$ is a positive decreasing value. The second case is if $H_{j-1}(i)$ is a non-negative increasing or constant value. The third case is if $H_{j-1}(i)$ is a non-positive decreasing or constant value. These are all possible cases. Note that the second and fourth cases overlap when $H_{j-1}(i) = 0$ and is a constant value. The map $H$ is well-defined, however, since $H(i, j) = H(i, j - 1)$ in both cases.

We now need to show that $H$ is a graph homomorphism. By the definitions of $I_\infty$ and the Cartesian product, there are edges $\{(i, j), (i + 1, j)\}, \{(i, j), (i, j + 1)\} \in E(I_\infty \sqcup I_\infty)$ for all $i, j \in \mathbb{Z}$. Thus $H$ is a graph homomorphism if either $H(i, j) = H(i + 1, j)$ or $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$, and either $H(i, j) = H(i, j + 1)$ or $\{H(i, j), H(i, j + 1)\} \in E(I_\infty)$. Since $H(i, j) = \tilde{f}(i)$ for all $j \leq 0$ and $\tilde{f}$ is a graph homomorphism, we only need to examine $H$ for $j \geq 0$. Let $j \geq 0$.

**First consider $H(i, j)$ and $H(i + 1, j)$**.

Since $H$ is defined recursively for $j > 0$, we show that either $H(i, j) = H(i + 1, j)$ or $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$ by induction on $j$.

Base case: For $j = 0$, $H(i, j) = H(i, 0) = \tilde{f}(i)$ and $H(i + 1, j) = H(i + 1, 0) = \tilde{f}(i + 1)$. Since $\{i, i + 1\} \in E(I_\infty)$ and $\tilde{f}$ is a graph homomorphism, either $\tilde{f}(i) = \tilde{f}(i + 1)$ or $\{\tilde{f}(i), \tilde{f}(i + 1)\} \in E(I_\infty)$. Thus $H(i, 0) = H(i + 1, 0)$ or $\{H(i, 0), H(i + 1, 0)\} \in E(I_\infty)$.

**Inductive Hypothesis:** Assume $H(i, j - 1) = H(i + 1, j - 1)$ or $\{H(i, j - 1), H(i + 1, j - 1)\} \in E(I_\infty)$ for some $j > 0$.

We examine the four cases for how $H(i, j)$ is defined, and for each of these cases, the four cases for how $H(i + 1, j)$ is defined.

1. Suppose $0 \leq H(i + 1, j - 1) < H(i, j - 1)$. By definition of $H$, $H(i, j) = H(i, j - 1) - 1$ in this case. By the inductive hypothesis, since $H(i + 1, j - 1) < H(i, j - 1)$, it follows that $H(i + 1, j - 1) = H(i, j - 1) - 1 = H(i, j)$. We now examine the four cases for how $H(i + 1, j)$ is defined in this case.
   (a) Suppose $0 \leq H(i + 2, j - 1) < H(i + 1, j - 1)$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1) - 1$. Thus $H(i + 1, j) = H(i, j) - 1$, and $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$.
   (b) Suppose $0 \leq H(i + 1, j - 1) \leq H(i + 2, j - 1)$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1)$. Thus $H(i + 1, j) = H(i, j)$.
   (c) Suppose $H(i + 1, j - 1) < H(i + 2, j - 1) \leq 0$. Since $0 \leq H(i + 1, j - 1)$, this is a contradiction.
(d) Suppose $H(i + 2, j - 1) \leq H(i + 1, j - 1) = 0$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1)$, so $H(i + 1, j) = H(i, j)$.

(2) Suppose $0 \leq H(i, j - 1) \leq H(i + 1, j - 1)$. By definition of $H$, $H(i, j) = H(i, j - 1)$ in this case. By the inductive hypothesis, since $H(i, j - 1) \leq H(i + 1, j - 1)$, it follows that $H(i + 1, j - 1) = H(i, j - 1)$ or $H(i + 1, j - 1) = H(i, j - 1) + 1$. Thus $H(i + 1, j) = H(i, j)$ or $H(i + 1, j - 1) = H(i, j) + 1$.

We now examine the four cases for how $H(i + 1, j)$ is defined in this case.

(a) Suppose $0 \leq H(i + 2, j - 1) < H(i + 1, j - 1)$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1) - 1$. Thus $H(i + 1, j) = H(i, j) - 1$ or $H(i + 1, j) = H(i, j)$, which implies that $H(i + 1, j) = H(i, j)$ or $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$.

(b) Suppose $0 \leq H(i + 1, j - 1) \leq H(i + 2, j - 1)$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1)$. Thus $H(i + 1, j) = H(i, j)$ or $H(i + 1, j) = H(i, j) + 1$, which implies that $H(i + 1, j) = H(i, j)$ or $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$.

(c) Suppose $H(i + 1, j - 1) < H(i + 2, j - 1) \leq 0$. Since $0 \leq H(i + 1, j - 1)$, this is a contradiction.

(d) Suppose $H(i + 2, j - 1) \leq H(i + 1, j - 1) = 0$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1)$. Thus $H(i + 1, j) = H(i, j)$ or $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$.

Therefore, $H(i, j) = H(i + 1, j)$ or $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$.

(3) Suppose $H(i, j - 1) < H(i + 1, j - 1) \leq 0$. By definition of $H$, $H(i, j) = H(i, j - 1) + 1$ in this case. By the inductive hypothesis, since $H(i, j - 1) < H(i + 1, j - 1)$, it follows that $H(i + 1, j - 1) = H(i, j - 1) + 1 = H(i, j)$.

We now examine the four cases for how $H(i + 1, j)$ is defined in this case.

(a) Suppose $0 \leq H(i + 2, j - 1) < H(i + 1, j - 1)$. Since $H(i + 1, j - 1) \leq 0$, this is a contradiction.

(b) Suppose $0 = H(i + 1, j - 1) \leq H(i + 2, j - 1)$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1)$. Thus $H(i + 1, j) = H(i, j)$.

(c) Suppose $H(i + 1, j - 1) < H(i + 2, j - 1) \leq 0$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1) + 1$. Thus $H(i + 1, j) = H(i, j) + 1$, which implies that $\{H(i, j), H(i + 1, j)\} \in E(I_\infty)$.

(d) Suppose $H(i + 2, j - 1) \leq H(i + 1, j - 1) = 0$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1)$. Thus $H(i + 1, j) = H(i, j)$.

(4) Suppose $H(i + 1, j - 1) \leq H(i, j - 1) \leq 0$. By definition of $H$, $H(i, j) = H(i, j - 1)$ in this case. By the inductive hypothesis, since $H(i + 1, j - 1) \leq H(i, j - 1)$, it follows that $H(i + 1, j - 1) = H(i, j - 1)$ or $H(i + 1, j - 1) = H(i, j - 1) - 1$. Thus $H(i + 1, j) = H(i, j)$ or $H(i + 1, j - 1) = H(i, j) - 1$.

We now examine the four cases for how $H(i + 1, j)$ is defined in this case.

(a) Suppose $0 \leq H(i + 2, j - 1) < H(i + 1, j - 1)$. Since $H(i + 1, j - 1) \leq 0$, then this is a contradiction.

(b) Suppose $0 = H(i + 1, j - 1) \leq H(i + 2, j - 1)$. By definition of $H$, $H(i + 1, j) = H(i + 1, j - 1)$. Since $H(i + 1, j - 1) \leq H(i, j - 1) \leq 0$, it
follows that \( H(i+1, j-1) = H(i, j-1) = 0 \). Therefore, \( H(i+1, j) = 0 = H(i, j) \).

(c) Suppose \( H(i+1, j-1) < H(i+2, j-1) \leq 0 \). By definition of \( H \),
\[
H(i+1, j) = H(i+1, j-1) + 1.
\]
Thus
\[
H(i+1, j) = H(i, j) + 1
\]
or
\[
H(i+1, j) = H(i, j) - 1 + 1 = H(i, j),
\]
which implies that \( H(i, j) = H(i+1, j) \) or \( \{H(i, j), H(i+1, j)\} \in E(I_\infty) \).

(d) Suppose \( H(i+2, j-1) \leq H(i+1, j-1) \leq 0 \). By definition of \( H \),
\[
H(i+1, j) = H(i+1, j-1) - 1.
\]
Thus
\[
H(i+1, j) = H(i, j) - 1
\]
which implies that \( H(i, j) = H(i+1, j) \) or \( \{H(i, j), H(i+1, j)\} \in E(I_\infty) \).

Therefore, \( H(i, j) = H(i+1, j) \) or \( \{H(i, j), H(i+1, j)\} \in E(I_\infty) \) for all \( i, j \in \mathbb{Z} \).

Next consider \( H(i, j) \) and \( H(i, j+1) \).

For each \( i \in \mathbb{Z} \) and \( j \geq 0 \), we show that either \( H(i, j) = H(i, j+1) \) or \( \{H(i, j), H(i, j+1)\} \in E(I_\infty) \) directly by examining the four possible cases which define \( H(i, j+1) \).

(1) Suppose \( 0 \leq H(i+1, j) < H(i, j) \). By definition of \( H \), \( H(i, j+1) = H(i, j) - 1 \). Thus \( \{H(i, j), H(i, j+1)\} \in E(I_\infty) \).

(2) Suppose \( 0 \leq H(i, j) \leq H(i+1, j) \). By definition of \( H \), \( H(i, j+1) = H(i,j) \).

(3) Suppose \( H(i, j) < H(i+1, j) \leq 0 \). By definition of \( H \), \( H(i, j+1) = H(i, j) + 1 \). Thus \( \{H(i, j), H(i, j+1)\} \in E(I_\infty) \).

(4) Suppose \( H(i+1, j) \leq H(i, j) \leq 0 \). By definition of \( H \), \( H(i, j+1) = H(i, j) \).

Therefore, \( H(i, j) = H(i, j+1) \) or \( \{H(i, j), H(i, j+1)\} \in E(I_\infty) \) for all \( i, j \in \mathbb{Z} \).

Thus \( H \) is a graph homomorphism.

We now show that \( H \) is stable. Recall that \( H_j : I_\infty \rightarrow I_\infty \) is defined by \( H_j(i) = H(i, j) \) for all \( i, j \in \mathbb{Z} \). Since \( H \) is a graph homomorphism, the restriction \( H_j \) is a graph homomorphism. Since \( \bar{f} \) is a stable graph homomorphism, the difference between \( m_0(\bar{f},-1) \) and \( m_0(\bar{f},1) \) is finite. Thus there are a finite number of \( m \in \mathbb{Z} \) with \( m_0(\bar{f},-1) \leq m \leq m_0(\bar{f},1) \).

(1) Suppose \( H_j(m) = 0 \). By definition of \( H \), either \( 0 = H_j(m) \leq H_j(m+1) \) and \( H_{j+1}(m) = H_j(m) \), or \( H_j(m+1) \leq H_j(m) = 0 \) and \( H_{j+1}(m) = H_j(m) \).

Thus if \( H_j(m) = 0 \), then \( H_{j+1}(m) = 0 \). This also implies that if \( H_0(m) = \bar{f}(m) > 0 \), then \( H_j(m) \geq 0 \) for all \( j \geq 0 \), and if \( H_0(m) = \bar{f}(m) < 0 \), then \( H_j(m) \leq 0 \) for all \( j \geq 0 \).

(2) Suppose \( H_j(m) > 0 \). By definition of \( H \), either \( 0 \leq H_j(m+1) < H_j(m) \) and \( H_{j+1}(m) = H_j(m) - 1 \), or \( 0 \leq H_j(m) \leq H_j(m+1) \) and \( H_{j+1}(m) = H_j(m) \).

Thus \( H_j(m) \) is constant or decreasing as \( j \) increases.

(3) Suppose \( H_j(m) < 0 \). By definition of \( H \), either \( H_j(m) < H_j(m+1) \leq 0 \) and \( H_{j+1}(m) = H_j(m) + 1 \), or \( H_j(m+1) \leq H_j(m) \leq 0 \) and \( H_{j+1}(m) = H_j(m) \).

Thus \( H_j(m) \) is constant or increasing as \( j \) increases.
Observe that if there exists \( j \geq 0 \) such that \( H_{j+1}(i) = H_j(i) \) for all \( i \in \mathbb{Z} \), then \( H \) stabilizes in the positive direction on the \( 2^{nd} \)-axis, that is, the integer \( m_0(H, +2) \) exists. For each \( j \geq 0 \), \( H \) does not stabilize at \( j \) in the positive direction in the \( 2^{nd} \)-axis if and only if there exists some \( m \in \mathbb{Z} \) with \( m_0(f, -1) \leq m \leq m_0(f, +1) \) such that \( H_j(m) \neq H_{j+1}(m) \). We now count how many times it is possible for \( H_j(m) \neq H_{j+1}(m) \) for \( j \geq 0 \) and \( m_0(f, -1) \leq m \leq m_0(f, +1) \). There are at most \( m_0(f, +1) - m_0(f, -1) \) choices for \( m \in \mathbb{Z} \) with \( m_0(f, -1) \leq m \leq m_0(f, +1) \). By parts (1)-(3), for each such \( m \), there are at most \( |f(m)| \) times that \( H_j(m) \neq H_{j+1}(m) \). This implies that \( H \) is not stable in the positive direction on the \( 2^{nd} \)-axis at a maximum of \( j = \sum_m |f(m)| < \infty \). Therefore, the integer \( m_0(H, +2) \) exists.

We now show that \( H \) is a graph homotopy from \( f \) to \( \alpha^2_{+2}(H) \) by verifying conditions (a)-(c) of Definition 2.13.

(a) We use induction on \( j \) to show that \( H_j(m_0(H_j, -1)) = 0 \) for all \( j \geq 0 \).

Basis Case: By construction of \( H \), \( H_0 = \tilde{f} \). Since \( \tilde{f}(m_0(\tilde{f}, -1)) = 0 \), it follows that \( H_0(m_0(H_0, -1)) = 0 \).

Induction Hypothesis: Suppose \( H_j(m_0(H_j, -1)) = 0 \) for some \( j \geq 0 \).

Then \( 0 = H_j(m_0(H_j, -1)) \leq H_j(m_0(H_j, -1) + 1) \), or \( H_j(m_0(H_j, -1) + 1) \leq H_j(m_0(H_j, -1)) = 0 \), which implies that

\[
H_{j+1}(m_0(H_j, -1)) = H_j(m_0(H_j, -1)) = 0
\]

by definition of \( H \). Thus by induction, \( H_j(m_0(H_j, -1)) = 0 \) for all \( j \geq 0 \). Therefore, \( \alpha^1_{-1}(\alpha^2_{+2}(H))(*) = \alpha^2_{+2}(H)(m_0(H_{m_0(H), +2}, -1)) = 0 \), which implies that \( \alpha^1_{-1}(\tilde{f}) = \alpha^1_{+1}(\alpha^2_{+2}(H)) \).

We now use induction on \( j \) to show that \( H_j(m_0(H_j, +1)) = 5n \) for all \( j \geq 0 \).

Basic Case: By construction of \( H \), \( H_0 = \tilde{f} \). Since \( \tilde{f}(m_0(\tilde{f}, +1)) = 5n \), it follows that \( H_0(m_0(H_0, +1)) = 5n \).

Induction Hypothesis: Suppose \( H_j(m_0(H_j, +1)) = 5n \) for some \( j \geq 0 \).

Then it follows that \( H_j(m_0(H_j, +1) + 1) = 5n \). Therefore,

\[
H_{j+1}(m_0(H_j, +1)) = H_j(m_0(H_j, +1)) = 5n
\]

by definition of \( H \). Thus by induction, \( H_j(m_0(H_j, +1)) = 5n \) for all \( j \geq 0 \). Therefore, \( \alpha^1_{-1}(\alpha^2_{+2}(H))(*) = \alpha^2_{+2}(H)(m_0(H_{m_0(H), +2}, +1)) = 5n \), which implies that \( \alpha^1_{+1}(\tilde{f}) = \alpha^1_{+1}(\alpha^2_{+2}(H)) \).

(b) This condition is a consequence of the inductive arguments in part (a). By part (a), \( H(m_0(H, -1), j) = 0 = \alpha^1_{-1}(\tilde{f})(*) = \alpha^1_{-1}(\alpha^2_{+2}(H))(*) \) for all \( j \in \mathbb{Z} \), and similarly, \( H(m_0(H, +1), j) = 5n = \alpha^1_{+1}(\tilde{f})(*) = \alpha^1_{+1}(\alpha^2_{+2}(H))(*) \) for all \( j \in \mathbb{Z} \). Therefore, \( \alpha^2_{+1}(H) = \beta^1_{+1}(\alpha^1_{+1}(\tilde{f}) = \beta^1_{+1}(\alpha^1_{+1}(\alpha^2_{+2}(H))) \) and \( \alpha^2_{+1}(H) = \beta^1_{+1}(\alpha^1_{+1}(\tilde{f}) = \beta^1_{+1}(\alpha^1_{+1}(\alpha^2_{+2}(H))) \).

(c) By construction of \( H \), \( \alpha^2_{+2}(H) = \tilde{f} \). Trivially, \( \alpha^2_{+1}(H) = \alpha^2_{+2}(H) \).

Thus \( H \) is a homotopy from \( \tilde{f} \) to \( \alpha^2_{+2}(H) \), so \( \tilde{f} \sim \alpha^2_{+2}(H) \). By definition of \( H \), the face \( \alpha^2_{+2}(H) \) has no positive decreasing value and no negative increasing values.
Since $\alpha_{i-1}^1(\alpha_{i+2}^2(H))(\ast) = 0$ and $\alpha_{i-1}^1(\alpha_{i+2}^2(H))(\ast) = 5n$, it follows that $\alpha_{i+2}^2(H)$ must be increasing or constant from 0 to 5n. Thus by the General Padding Lemma (3.12), $\alpha_{i+2}^2(H) \sim \gamma_n$. Therefore, $f \sim \gamma_n$ for all $n \in \mathbb{Z}$. \hfill \Box

We conclude this section by computing the fundamental group of the 5-cycle.

**Theorem 4.8.** The fundamental group of $C_5$ is $(B_1(C_5, [0]) / \sim, \cdot) \cong (\mathbb{Z}, +)$.

**Proof.** Define $\varphi : \mathbb{Z} \rightarrow B_1(C_5, [0]) / \sim$ by $n \mapsto [\gamma_n]$, the homotopy class of the stable graph homomorphism $\gamma_n : I_\infty \rightarrow C_5$ defined in Definition 4.3. We now show that this map $\varphi$ is an isomorphism.

**Group Homomorphism:** We show that $\varphi(n + m) = \varphi(n) \cdot \varphi(m)$ for all $n, m \in \mathbb{Z}$.

- **Case 1:** Suppose $n, m \geq 0$. The concatenation $\gamma_n \cdot \gamma_m$ is defined by

$$
(\gamma_n \cdot \gamma_m)(i) = \begin{cases} 
\gamma_n(i + m_0(\gamma_n, -1)) & \text{for } i \geq 0, \\
\gamma_m(i + m_0(\gamma_m, +1)) & \text{for } i \leq 0,
\end{cases}
$$

$$
= \begin{cases} 
\gamma_n(i + 0) & \text{for } i \geq 0, \\
\gamma_m(i + 5m) & \text{for } i \leq 0,
\end{cases}
$$

$$
= \begin{cases} 
[0] & \text{for } i \geq 5n, \\
[i \mod 5] & \text{for } 0 \leq i \leq 5n, \\
[(i + 5m) \mod 5] & \text{for } -5m \leq i \leq 0, \\
[0] & \text{for } i \leq -5m,
\end{cases}
$$

Thus $(\gamma_n \cdot \gamma_m)(i + 5m) = \gamma_{n+m}(i)$, and $\gamma_n \cdot \gamma_m \sim \gamma_{n+m}$ by the Shifting Lemma (4.4).

- **Case 2:** Suppose $n, m < 0$. The concatenation $\gamma_n \cdot \gamma_m$ is defined by

$$
(\gamma_n \cdot \gamma_m)(i) = \begin{cases} 
\gamma_n(i + m_0(\gamma_n, -1)) & \text{for } i \geq 0, \\
\gamma_m(i + m_0(\gamma_m, +1)) & \text{for } i \leq 0,
\end{cases}
$$

$$
= \begin{cases} 
\gamma_n(i + 0) & \text{for } i \geq 0, \\
\gamma_m(i - 5m) & \text{for } i \leq 0,
\end{cases}
$$

$$
= \begin{cases} 
[0] & \text{for } i \geq -5n, \\
[(-i) \mod 5] & \text{for } 0 \leq i \leq -5n, \\
[((-i) + 5m) \mod 5] & \text{for } 5m \leq i \leq 0, \\
[0] & \text{for } i \leq 5m,
\end{cases}
$$

Thus $\gamma_n \cdot \gamma_m(i + 5m) = \gamma_{n+m}(i)$, and $\gamma_n \cdot \gamma_m \sim \gamma_{n+m}$ by the Shifting Lemma (4.4).
• **Case 3:** Suppose \( n \geq 0, m < 0 \). By Lemma 4.5, \( \gamma_n \sim \gamma_{-n} \) and \( \gamma_m \sim \gamma_{-m} \). By Case 1, if \( n + m \geq 0 \), then \( \gamma_n = \gamma_{n+m-m} \sim \gamma_{n+m} \cdot \gamma_{-m} \). By Case 2, if \( n + m < 0 \), then \( \gamma_m = \gamma_{n+n+m} \sim \gamma_{n} \cdot \gamma_{n+m} \). Thus

\[
\gamma_n \cdot \gamma_m \sim \gamma_n \cdot \gamma_{-m} \sim \gamma_{n+m} \cdot \gamma_{-m} \sim \gamma_{n+m} \quad \text{if } n + m \geq 0,
\]

and

\[
\gamma_n \cdot \gamma_m \sim \gamma_{-n} \cdot \gamma_m \sim \gamma_{-n} \cdot \gamma_{n+m} \sim \gamma_{n+m} \quad \text{if } n + m < 0.
\]

• **Case 4:** Suppose that \( n < 0, m \geq 0 \). Again by Lemma 4.5, \( \gamma_n \sim \gamma_{-n} \) and \( \gamma_m \sim \gamma_{-m} \). By Case 2, if \( n + m < 0 \), then \( \gamma_n = \gamma_{n+m-m} \sim \gamma_{n+m} \cdot \gamma_{-m} \). By Case 1, if \( n + m \geq 0 \), then \( \gamma_m = \gamma_{n+n+m} \sim \gamma_{n} \cdot \gamma_{n+m} \). Thus

\[
\gamma_n \cdot \gamma_m \sim \gamma_{-n} \cdot \gamma_{-m} \sim \gamma_{n+m} \cdot \gamma_{-m} \sim \gamma_{n+m} \quad \text{if } n + m < 0,
\]

and

\[
\gamma_n \cdot \gamma_m \sim \gamma_{-n} \cdot \gamma_m \sim \gamma_{-n} \cdot \gamma_{n+m} \sim \gamma_{n+m} \quad \text{if } n + m \geq 0.
\]

Therefore, \( \varphi(n+m) = [\gamma_{n+m}] = [\gamma_n] \cdot [\gamma_m] = \varphi(n) \cdot \varphi(m) \) for all \( n, m \in \mathbb{Z} \).

**Surjective:** We show that if \( [f] \in B_1(C_5, [0]) \), then there exists \( n \in \mathbb{Z} \) such that \( \varphi(n) = [f] \).

Let \( [f] \in B_1(C_5, [0]) \). Then \( f \) is a stable graph homomorphism with

\[
f(m_0(f, -1)) = f(m_0(f, +1)) = [0].
\]

Hence, there exists a unique lift \( \tilde{f} : I_\infty \to I_\infty \) with \( \tilde{f}(m_0(f, -1)) = 0 \) and \( f = p \circ \tilde{f} \).

Since \( f(m_0(f, +1)) = [0] \), it follows that \( p(f(m_0(f, +1))) = [0] \), so

\[
\tilde{f}(m_0(f, +1)) \mod 5 = 0.
\]

Thus there exists \( n \in \mathbb{Z} \) such that \( \tilde{f}(m_0(f, +1)) = 5n \). Hence, by the Lemma 4.7, we have that \( \tilde{f} \sim \gamma_n \), which implies that there exists a graph homotopy \( H : I_\infty^2 \to I_\infty \) from \( \tilde{f} \) to \( \gamma_n \). Since \( H \) and \( p_5 \) are graph homomorphisms, the composition \( p_5 \circ H : I_\infty^2 \to C_5 \) is a graph homomorphism. We now show that \( p_5 \circ H \) is a graph homotopy from \( f \) to \( \gamma_n \) by verifying conditions (a)-(c) of Definition 2.13.

(a) By the definitions of \( \tilde{f} \) and \( \gamma_n \),

\[
\tilde{f}(m_0(\tilde{f}, -1)) = \gamma_n(m_0(\gamma_n, -1)) = 0
\]

and

\[
\tilde{f}(m_0(\tilde{f}, +1)) = \gamma_n(m_0(\gamma_n, +1)) = 5n.
\]

Since \( p_5 \) is a graph homomorphism,

\[
p_5(\tilde{f}(m_0(\tilde{f}, -1))) = p_5(\gamma_n(m_0(\gamma_n, -1))) = [0]
\]

and

\[
p_5(\tilde{f}(m_0(\tilde{f}, +1))) = p_5(\gamma_n(m_0(\gamma_n, +1))) = [0].
\]

Therefore, \( \alpha_{-1}(f) = \alpha_{-1}(\gamma_n) \) and \( \alpha_{+1}(f) = \alpha_{+1}(\gamma_n) \).

(b) Since \( H \) is a graph homotopy from \( f \) to \( \gamma_n \),

\[
\alpha_{-1}(H)(j) = H(m_0(H, -1), j) = 0
\]

and

\[
\alpha_{+1}(H)(j) = H(m_0(H, +1), j) = 5n
\]
Therefore, \( p \) injective: We show that if \( H \) satisfies the Topology Lifting Property (3.9), there is a graph homotopy \( \tilde{\gamma} \) such that
\[
(p \circ H)(m_0(H, -1), j) = [0] = (p \circ \tilde{f})(m_0(\tilde{f}, -1)) = (p \circ \tilde{\gamma}_n)(m_0(\gamma_n, -1))
\]
and
\[
(p \circ H)(m_0(H, +1), j) = [0] = (p \circ \tilde{f})(m_0(\tilde{f}, +1)) = (p \circ \tilde{\gamma}_n)(m_0(\gamma_n, +1))
\]
for all \( j \in \mathbb{Z} \). Therefore, \( \alpha^2_{1,1}(p \circ H) = \beta^1_1 \alpha^1_{1,1} = \beta^1_1 \alpha^1_{1,1}(\gamma_n) \) and \( \alpha^2_{1,1}(p \circ H) = \beta^1_1 \alpha^1_{1,1}(f) = \beta^1_1 \alpha^1_{1,1}(\gamma_n) \).

(c) Since \( H(i, m_0(H, -2)) = \tilde{f}(i) \) and \( H(i, m_0(H, +2)) = \tilde{\gamma}_n(i) \) for all \( i \in \mathbb{Z} \), it follows that \( p \circ H(i, m_0(H, -2)) = p \circ \tilde{f}(i) \) and \( p \circ H(i, m_0(H, +2)) = p \circ \tilde{\gamma}_n(i) \) for all \( i \in \mathbb{Z} \). Thus \( \alpha^2_{1,2}(p \circ H) = f \) and \( \alpha^2_{1,2}(p \circ H) = \gamma_n \).

Therefore, \( p \circ H \) is a homotopy from \( f \) to \( \gamma_n \), so it follows that \( \gamma_n = [\gamma_n] \). Hence, \( \phi(n) = [f] \).

Injective: We show that if \( \phi(n) = \phi(m) \), then \( n = m \).

Let \( \phi(n) = \phi(m) \). Then \( \gamma_n = [\gamma_m] \), which implies that \( \gamma_n \sim \gamma_m \). Therefore, there exists a graph homotopy \( H : I^2_\infty \rightarrow C_5 \) from \( \gamma_n \) to \( \gamma_m \). By the Homotopy Lifting Property (3.9), there is a graph homotopy \( \tilde{H} : I^2_\infty \rightarrow I_\infty \) from \( \tilde{\gamma}_n \) to \( \tilde{\gamma}_m \). Thus \( \tilde{\gamma}_n \sim \tilde{\gamma}_m \), and it follows that \( \alpha^1_{1,1}(\tilde{\gamma}_n) = \alpha^1_{1,1}(\tilde{\gamma}_m) \). Therefore, \( \tilde{\gamma}_n(m_0(\gamma_n, +1)) = \tilde{\gamma}_m(m_0(\gamma_m, +1)) \). Hence it follows that \( 5n = 5m \), which implies that \( n = m \).

Thus \( \phi \) is an isomorphism, and \( (B_1(C_5, [0])/\sim) \cong \mathbb{Z} \).

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