BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR POSITIVE FUNCTIONALS

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Abstract. We introduce the so-called Bishop-Phelps-Bollobás property for positive functionals, a particular case of the Bishop-Phelps-Bollobás property for positive operators. First we show a version of the Bishop-Phelps-Bollobás theorem for positive elements and positive functionals in the dual of any Banach lattice. We also characterize the strong Bishop-Phelps-Bollobás property for positive functionals in a Banach lattice. We prove that any finite-dimensional Banach lattice has the the Bishop-Phelps-Bollobás property for positive functionals. A sufficient and a necessary condition to have the Bishop-Phelps-Bollobás property for positive functionals are also provided. As a consequence of this result, we obtain that the spaces $L^p(\mu)$ (1 $\leq p < \infty$), for any positive measure $\mu$, $C(K)$, and $M(K)$, for any compact and Hausdorff topological space $K$, satisfy the Bishop-Phelps-Bollobás property for positive functionals. We also provide some more clarifying examples.

1. Introduction

Bishop and Phelps proved that the set of norm attaining functionals on a Banach space is norm dense in the topological dual [8]. Bollobás proved a “quantitative version” of the Bishop-Phelps theorem [9]. In order to state that result we recall some notation. Given a Banach space $X$, we denote the unit sphere of $X$ by $S_X$ and the closed unit ball by $B_X$. The topological dual of $X$ is denoted by $X^*$. The following result can be found in [10, Theorem 16.1] and [11, Corollary 2.4] and it is known as the Bishop-Phelps-Bollobás theorem:

Let $\varepsilon > 0$ be arbitrary. If $x \in B_X$ and $x^* \in S_{X^*}$ are such that $|1 - x^*(x)| < \frac{\varepsilon^2}{2}$, then there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

In 2008 Acosta, Aron, García and Maestre [3] introduced the so-called Bishop-Phelps-Bollobás property for operators (see [3, Definition 1.1]) and provided some pairs of Banach spaces with that property. Afterwards a number of interesting results on that topic appeared. The survey paper [2] contains many of such results. Very recently the authors introduced the Bishop-Phelps-Bollobás property for positive operators between Banach lattices [5]. In order to recall such property we need some notions. The concepts in the first definition are standard and can be found, for instance, in [1].

Definition 1.1. An ordered vector space $X$ equipped with a vector space order, that is, an order relation $\leq$ on $X$ that is compatible with the algebraic structure of $X$. An ordered vector space is called a Riesz space if every pair of vectors has a least upper bound and a greatest lower bound. In a Riesz space $X$, given two elements $x$ and $y$ in $X$, we denote by $x \wedge y$, $x \vee y$, $|x|$, $x^+$, and $x^-$ the infimum of $x$ and $y$, the supremum of $x$ and $y$, the supremum of $x$ and $-x$, , the

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supremum of $x$ and $0$, and the supremum of $-x$ and $0$, respectively. A norm $\| \|$ on a Riesz space $X$ is said to be a lattice norm whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A normed Riesz space is a Riesz space equipped with a lattice norm. A normed Riesz space whose norm is complete is called a Banach lattice.

In case that $(\Omega, \mu)$ is a measure space, we denote by $L^0(\mu)$ the space of (equivalence classes of $\mu$-a.e. equal) real valued measurable functions on $\Omega$. We say that a Banach space $X$ is a Banach function space on $(\Omega, \mu)$ if $X$ is an ideal in $L^0(\mu)$ and whenever $x, y \in X$ and $|x| \leq |y|$ a.e., then $\|x\| \leq \|y\|$.

For any Banach lattice $X$, we will denote by $X^+$ the set of positive elements in $X$, that is, $X^+ = \{x \in X : 0 \leq x\}$. Recall that the dual of any normed Riesz space is a Banach lattice (see [7, Theorem 4.1]). We will use that in a Riesz space $X$, any element $x \in X$ satisfies $x = x^+ - x^-$, $|x| = x^+ + x^-$ and $x^+ \land x^- = 0$ (see [7, Theorem 1.5]).

If $X$ and $Y$ are Banach spaces, we denote by $L(X, Y)$ the space of all bounded and linear operators from $X$ to $Y$. A linear mapping $T : X \rightarrow Y$ between two ordered vector spaces is called positive if $x \geq 0$ implies $T(x) \geq 0$.

Now we recall the notion of Bishop-Phelps-Bollobás property for positive operators and introduce two new properties for functionals that will be useful in this paper.

**Definition 1.2** ([5, Definition 1.3]). Let $X$ and $Y$ be Banach lattices and $M$ a subspace of $L(X, Y)$. The subspace $M$ is said to have the Bishop-Phelps-Bollobás property for positive operators if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $S \in M$, such that $S \geq 0$, if $x_0 \in X$ satisfies $\|S(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and a positive operator $T \in M$ satisfying the following conditions

$$\|T(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$  

Acosta and Soleimani-Mourchehkhorti proved that the pair $(L_\infty(\mu), L_1(\nu))$ has the Bishop-Phelps-Bollobás property for positive operators, for any positive measures $\mu$ and $\nu$ (see [5, Theorem 1.6]). The same property holds for the pair $(c_0, L_1(\mu))$ for any positive measure $\mu$ (see [5, Theorem 1.7]). These results were extended in [6] to the pair $(c_0, Y)$ in case that $Y$ is a uniformly monotone Banach lattice and $(L_\infty(\mu), Y)$, whenever $Y$ is a uniformly monotone Banach lattice with a weak unit.

**Definition 1.3.** We say that a Banach lattice $X$ has the Bishop-Phelps-Bollobás property for positive functionals (we will write BPBp instead of Bishop-Phelps-Bollobás property) if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $x^* \in S_{X^*}$, such that $x^* \geq 0$, if $x_0 \in S_X$ satisfies $x^*(x_0) > 1 - \eta(\varepsilon)$, then there exist an element $y \in S_X$ and a positive functional $y^* \in S_{X^*}$ satisfying the following conditions

$$y^*(y) = 1, \quad \|y - x_0\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$  

A Banach lattice $X$ has the strong Bishop-Phelps-Bollobás property for positive functionals if it has the Bishop-Phelps-Bollobás property for positive functionals and additionally the element $y$ appearing in that definition is positive.

Trivially the strong Bishop-Phelps-Bollobás property for positive functionals implies the Bishop-Phelps-Bollobás property for positive functionals.

In Section 2 we obtain a general version of Bishop-Phelps-Bollobás property for Banach lattices where the elements in the Banach space and the functionals are positive (see Proposition 2.3). We
also prove an intrinsic characterization of the strong BPBp for positive functionals (see Theorem 2.9). In the case of the Bishop-Phelps-Bollobás property for positive functionals, we show that finite-dimensional Banach lattices always have such property (see Corollary 2.13). We also provide a sufficient and a necessary condition for the BPBp for positive functionals in Theorem 2.16.

In Section 3 we prove that the spaces $L^p(\mu)$, $C(K)$ and $M(K)$ satisfy the BPBp for positive functionals, for any positive measure $\mu$ and any compact and Hausdorff topological space $K$ (see Corollary 3.2). We also show that $\mathbb{R}^2$, endowed with an absolute norm, satisfies the assumptions in Theorem 2.16. However, the case of the three dimensional Banach lattice is different (see Examples 3.5 and 3.6). We also provide an example showing that not every Banach lattice has the Bishop-Phelps–Bollobás property for positive functionals (see Example 3.10).

We point out that throughout this paper we consider only real Banach spaces.

2. The results

The goal of this section is to provide classes of Banach lattices with the strong Bishop-Phelps-Bollobás property for positive functionals and the Bishop-Phelps-Bollobás property for positive functionals.

**Definition 2.1.** A real Banach lattice $X$ has the hereditary norm attaining property (HNAp in short) if it satisfies the following condition

$$x^* \in X^*, x \in X, x^*(x) = \|x^*\|\|x\| \Rightarrow x^{*+}(x^+) = \|x^{*+}\|\|x^+\|, \quad x^{*-}(x^-) = \|x^{*-}\|\|x^-\|.$$  

In the proof of the main results the following simple facts will be useful.

**Proposition 2.2.** Let $X$ be a Banach lattice.

i) If $x \in S_X$, $x^* \in S_{X^*}$ and $x^*(x) = 1$ then $|x^*|||x|| = 1$, therefore

$$x^{*+}(x^-) = 0 = x^{*-}(x^+)$$

ii) Assume that it is satisfied

$$x \in S_X, x^* \in S_{X^*}, x^*(x) = 1 \Rightarrow \|x^{*+}\|\|x^+\| + \|x^{*-}\|\|x^-\| \leq 1.$$  

Then $X$ has the hereditary norm attaining property.

**Proof.** For the proof of i) let us notice that

$$1 = x^*(x) = (x^{*+} - x^{*-})(x^+ - x^-) \leq x^{*+}(x^+) + x^{*-}(x^-) \leq (x^{*+} + x^{*-})(x^+ + x^-) = |x^*|||x|| \leq 1.$$  

Hence $|x^*|||x|| = 1$ and

$$x^{*+}(x^-) = 0 = x^{*-}(x^+).$$

Now we prove ii). Let us notice that it suffices to check that the condition in Definition 2.1 is satisfied for normalized elements. Let $x \in S_X$ and $x^* \in S_{X^*}$ be elements such that $x^*(x) = 1$. Then by using
the assumption we have that

\[ 1 = x^+(x) = (x^+ - x^-)(x^+ - x^-) \leq x^+(x^+) + x^-(x^-) \leq \|x^+\| \|x^+\| + \|x^-\| \|x^-\| \leq 1. \]

Hence \( x^+(x^+) = \|x^+\| \|x^+\| \) and \( x^-(x^-) = \|x^-\| \|x^-\| \). So \( X \) has the hereditary norm attaining property. \( \square \)

From Bishop-Phelps-Bollobás Theorem we will obtain the following general result.

**Proposition 2.3.** Let \( X \) be a Banach lattice and \( 0 < \varepsilon < 1 \). If \( x \in S_X \) is positive, \( x^+ \in S_{X^+} \) is positive and \( x^+(x) > 1 - \frac{\varepsilon^2}{2} \) there are elements \( y \in X^+ \cap S_X \) and \( y^* \in X^+ \cap S_{X^*} \) satisfying the following conditions

\[ y^*(y) = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon. \]

**Proof.** By applying the Bishop-Phelps-Bollobás Theorem there are elements \( z \in S_X \) and \( z^* \in S_{X^*} \) such that

\[ z^*(z) = 1, \quad \|z - x\| < \varepsilon \quad \text{and} \quad \|z^* - x^*\| < \varepsilon. \]  

Hence \( |z| \in S_X \) and \( |z^*| \in S_{X^*} \). In view of Proposition 2.2 we have that \( |z^*(|z|)| = 1 \). From (2.1), since \( x \) is positive and \( x^* \) is also positive we obtain that

\[ \|z - x\| < \varepsilon \quad \text{and} \quad \||z^* - x^*\| < \varepsilon. \]  

Hence the elements \( y = |z| \) and \( y^* = |z^*| \) satisfy the stated assertion. \( \square \)

Now we collect a few known facts in the following statement. They can be found in Theorems 1.5 and 1.7 and Exercise 2 in [7].

**Lemma 2.4.** Let \( X \) be a Banach lattice. The following statements hold.

1. If \( y, z \in X^+, y \wedge z = 0, x = y - z \Rightarrow y = x^+, z = x^-. \)
2. \( |x - y| = x \vee y - x \wedge y, \forall x, y \in X. \)
3. \( x \wedge y = \frac{1}{2}(x + y - |x - y|), \quad x \vee y = \frac{1}{2}(x + y + |x - y|), \quad \forall x, y \in X. \)
4. \( |x| \wedge |y| = \frac{1}{2}|x + y - |x - y||, \quad |x| \vee |y| = \frac{1}{2}|x + y + |x - y||, \quad \forall x, y \in X. \)
5. \( x, y \in X, x \perp y \Rightarrow sx \perp ty, \forall s, t \in \mathbb{R}. \)
6. \( x, y \in X^+, x \wedge y = 0 \Rightarrow |sx + ty| = |s||x| + |t||y|, \forall s, t \in \mathbb{R}. \)

As a consequence

\[ \|sx + ty\| \geq \max\{ |s||x||, |t||y|| \}. \]
Later we will also use the following result.

**Proposition 2.5.** Let $X$ be a Banach lattice and assume that $x, y \in X^+$ satisfy $x \neq 0$ and $x \wedge y = 0$. Then there is a positive functional $x^* \in S_{X^*}$ such that $x^*(x) = \|x\|$ and $x^*(y) = 0$.

**Proof.** In case that $y = 0$ the result follows from Hahn-Banach extension theorem and part i) of Proposition 2.2.

If $y \neq 0$ from Lemma 2.4 we obtain that $x$ and $y$ are linearly independent since

$$\|sx + ty\| \geq \max \{|s|\|x\|, |t|\|y\|\}, \quad \forall s, t \in \mathbb{R}. \quad (2.3)$$

We define the functional $y^*$ on span $\{x, y\}$ by

$$y^*(sx + ty) = s\|x\|, \quad s, t \in \mathbb{R}.$$ 

From (2.3) we have that $y^*$ is a continuous linear functional and satisfies

$$y^*(x) = \|x\|, \quad \|y^*\| = 1 \quad \text{and} \quad y^*(y) = 0.$$ 

We claim that $y^*$ is a positive functional. Otherwise there would be real numbers $a < 0$ and $b > 0$ such that $ax + by \geq 0$ and so $y \geq \frac{a}{b}x > 0$. Therefore

$$0 = x \wedge y \geq \min \left\{1, \frac{-a}{b}\right\}x > 0,$$

which is a contradiction. So $y^*$ is a positive functional on span $\{x, y\}$.

Finally we check that span $\{x, y\}$ is a Riesz subspace of $X$. From c) and f) in Lemma 2.4 we have that

$$(ax + by) \wedge (cx + dy) = \frac{1}{2}(ax + by + cx + dy - |ax + by - cx - dy|)$$

$$= \frac{1}{2}((a + c)x + (b + d)y - |a - c|x - |b - d|y).$$

and

$$(ax + by) \vee (cx + dy) = \frac{1}{2}(ax + by + cx + dy + |ax + by - cx - dy|)$$

$$= \frac{1}{2}((a + c)x + (b + d)y + |a - c|x + |b - d|y).$$

Hence span $\{x, y\}$ is stable under suprema and infima. From [13, Theorem 39.2] there is a positive functional $x^* \in X^*$ such that

$$\|x^*\| = \|y^*\| = 1 \quad \text{and} \quad x^*|_{\text{span} \{x, y\}} = y^*$$

and the proof is finished. \qed

Firstly we recall the notion of uniformly monotone Banach lattice, which is well known, and introduce a new geometric property for Banach lattices.

**Definition 2.6.** A real Banach lattice $X$ is uniformly monotone (UM), if for every $0 < \varepsilon < 1$, there is $0 < \delta < \varepsilon$ satisfying the following property

$$x, y \in X^+, \|x + y\| \leq 1, \quad \|x\| > 1 - \delta \Rightarrow \|y\| < \varepsilon.$$
**Definition 2.7.** A real Banach lattice $X$ is uniformly monotone for orthogonal elements (UMOE in short), if for every $0 < \varepsilon < 1$ there is $0 < \delta < \varepsilon$ such that

$$x \in B_X, \|x^+\| > 1 - \delta \Rightarrow \|x^-\| < \varepsilon.$$

It is clear that any uniformly monotone Banach lattice is uniformly monotone for orthogonal elements. It is shown in [12, Theorem 6] that for Banach function spaces uniform monotonicity and uniform monotonicity for orthogonal elements are equivalent properties.

**Remark 2.8.** A real Banach lattice $X$ is uniformly monotone for orthogonal elements if and only if for every $0 < \varepsilon < 1$, there is $0 < \delta < \varepsilon$ such that

$$x, y \in X^+, x \wedge y = 0, \|x+y\| \leq 1, \|x\| > 1 - \delta \Rightarrow \|y\| < \varepsilon.$$

Now we can prove the following characterization.

**Theorem 2.9.** Let $X$ be a Banach lattice. Then $X$ is uniformly monotone for orthogonal elements if and only if it has the strong Bishop-Phelps-Bollobás property for positive functionals. Moreover the function $\eta$ in Definition [1.3] depends on the function $\delta$ satisfying the condition of uniformly monotonicity for orthogonal elements (see Definition 2.7).

**Proof.** Assume that $X$ is uniformly monotone for orthogonal elements with the function $\delta$. Let be $0 < \varepsilon < 1$, $x \in S_X$, $x^* \in S_{X^*}$ such that

$$x^* \geq 0 \quad \text{and} \quad x^*(x) > 1 - \min\left\{\frac{\varepsilon^2}{2}, \delta(\varepsilon)\right\}.$$

Since $x^* \geq 0$ we also have $x^*(|x|) \geq x^*(x) > 1 - \frac{\varepsilon^2}{2}$. By Bishop-Phelps-Bollobás theorem there are $y \in S_X$ and $y^* \in S_{X^*}$ satisfying

(2.4) \hspace{1cm} y^*(y) = 1, \quad \|y - x\| < \varepsilon \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.

From the inequality $x^*(x) = x^*(x^+) - x^*(x^-) > 1 - \delta(\varepsilon)$, since $x^* \geq 0$ we get that $x^*(x^+) > 1 - \delta(\varepsilon)$ and so $\|x^+\| > 1 - \delta(\varepsilon)$. By using that $X$ is uniformly monotone for orthogonal elements we get that $\|x^-\| < \varepsilon$ and so

$$\|x - |x|\| = 2\|x^-\| < 2\varepsilon.$$

From (2.4), we obtain that

$$\|y - x\| \leq \|y - |x|\| + \| |x| - x\| \leq \|y - x\| + 2\|x^-\| < 3\varepsilon.$$ 

In view of the previous inequality and (2.4) we proved that $X$ has the strong Bishop-Phelps-Bollobás property for positive functionals since $|y^*|$ attains its norm at $|y|$ and $\|y^* - x^*\| \leq \|y^* - x^+\| < \varepsilon$. Let us also notice that the function $\eta$ in Definition [1.3] depends only on the function $\delta$ appearing in the definition of uniformly monotonicity for orthogonal elements.

Assume now that $X$ has the strong Bishop-Phelps-Bollobás property for positive functionals with the function $\eta$. Let $0 < \varepsilon < 1$ and $x \in B_X$ such that $\|x^+\| > 1 - \eta(\varepsilon)$. It is known that $x^+ \wedge x^- = 0$. So by Proposition 2.3 there is a positive functional $x^* \in S_X$, such that $x^*(x) = \|x^+\| > 1 - \eta(\varepsilon)$. Therefore $x^*(\frac{x}{\|x\|}) > 1 - \eta(\varepsilon)$.

By the assumption there are elements $z \in S_X$ and $z^* \in S_{X^*}$ such that

(2.5) \hspace{1cm} z^* \geq 0, \quad z \geq 0, \quad z^*(z) = 1, \quad \left\|z - \frac{x}{\|x\|}\right\| < \varepsilon \quad \text{and} \quad \|z^* - x^*\| < \varepsilon.$
By (2.5) we conclude that
\[ \left\| x^+ \right\| = \left\| x^- \right\| \leq \left\| z - \frac{x}{\|x\|} \right\| < \varepsilon. \]
As we proved that \( \|x^-\| < \varepsilon \|x\| \leq \varepsilon \), \( X \) is uniformly monotone for orthogonal elements. \( \square \)

Our goal now is to provide classes of Banach spaces satisfying the Bishop-Phelps-Bollobás property for positive functionals.

Lemma 2.10. For any Banach lattice \( X \), the sets \( X^+ \) and \( X^- = \{ x \in X : x \leq 0 \} \) are closed.

Proof. It is satisfied that
\[ \|x^+ - y^+\| \leq \|x - y\|, \quad \forall x \in X, \]
so the mapping \( x \mapsto x^+ \) is continuous on \( X \). As a consequence, \( X^+ = \{ x \in X : 0 \leq x \} \) is closed. The same argument holds true for \( X^- \). \( \square \)

Proposition 2.11. Let \( X \) and \( Y \) be finite-dimensional Banach lattices. For every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( S \in S_{L(X,Y)} \) is a positive operator there is a positive and linear operator \( T \in S_{L(X,Y)} \) such that the following conditions hold:

i) \( \|T - S\| < \varepsilon \), and

ii) for all \( x_0 \in S_X \) satisfying \( \|S(x_0)\| > 1 - \delta \), there is \( u_0 \in S_X \) such that \( \|T(u_0)\| = 1 \) and such that \( \|u_0 - x_0\| < \varepsilon \).

Proof. The proof is similar to the proof of [Proposition 2.4] in [3]. The only difference is that in the current proof we assume that all operators are positive. Notice that because of Lemma 2.10, the set of positive (and bounded) operators between two Banach lattices is a closed subset of the space of bounded and linear operators between them. \( \square \)

So from this observation we conclude the next result.

Corollary 2.12. Assume that \( X \) and \( Y \) are finite-dimensional Banach lattices. Then \( L(X,Y) \) has the Bishop-Phelps-Bollobás property for positive operators.

Corollary 2.13. Every finite dimensional Banach lattice has Bishop-Phelps-Bollobás property for positive functionals.

In order to state another result for the Bishop-Phelps-Bollobás property for positive functionals, we introduce the following two notions.

Definition 2.14. A real Banach lattice \( X \) is strongly monotone (SM for short) if for every \( 0 < \varepsilon < 1 \) there is \( 0 < \delta < \varepsilon \) satisfying the following property
\[ x \in B_X, \|x^+\| > 1 - \delta \implies \exists b \in [0, 1]: \frac{x^+}{\|x^+\|} - bx^- \in S_X \quad \text{and} \quad \|bx^- - x^-\| < \varepsilon. \]

Definition 2.15. A real Banach lattice \( X \) is weakly monotone (WM for short) if for every \( 0 < \varepsilon < 1 \) there is \( 0 < \delta < \varepsilon \) such that
\[ x \in B_X, \|x^+\| > 1 - \delta \implies \exists y \in S_X : y^+ \in S_X \quad \text{and} \quad \|y - x\| < \varepsilon. \]

It is clear that any Banach lattice that is uniformly monotone for orthogonal elements is also strongly monotone. Strongly monotonicity implies weakly monotonicity.
**Theorem 2.16.** Let $X$ be a Banach lattice. If $X$ is strongly monotone and has the hereditary norm attaining property then $X$ has the Bishop-Phelps-Bollobás property for positive functionals. The Banach lattice $X$ is weakly monotone whenever it has the Bishop-Phelps-Bollobás property for positive functionals.

**Proof.** We begin by proving the first statement. Let be $0 < \varepsilon < 1$ and $\delta$ the function satisfying Definition 2.14. We can assume that $\delta(t) \leq t$ for every $t \in (0,1)$. Assume that $x^* \in S_{X^*}$ is positive and $x \in S_X$ are such that $x^*(x) > 1 - \frac{\delta(\varepsilon)}{2}$. By applying Bishop-Phelps-Bollobás theorem there are $y^* \in S_{X_*}$ and $y \in S_X$ satisfying also

\begin{equation}
\tag{2.6}
y^*(y) = 1, \quad \|y^* - x^*\| < \delta(\varepsilon) \quad \text{and} \quad \|y - x\| < \delta(\varepsilon),
\end{equation}

so

\begin{equation}
\tag{2.7}
\|y^+ - x^*\| = \|y^+ - x^+\| < \delta(\varepsilon).
\end{equation}

Therefore

$$
\|y^+\| > 1 - \delta(\varepsilon) \geq 1 - \varepsilon > 0
$$

and

\begin{equation}
\tag{2.8}
\|y^\ast - \|y^+ - x^\ast\| \leq \|y^\ast - x^\ast\| < \delta(\varepsilon).
\end{equation}

Hence

\begin{equation}
\tag{2.9}
\left\| \frac{y^+}{\|y^+\|} - y^\ast \right\| \leq \left| 1 - \|y^+\| \right| + \|y^\ast\| < 2\varepsilon.
\end{equation}

By Proposition 2.2 we know that $y^+(y^-) = y^-(y^+) = 0$. As a consequence we have that

$$
1 = y^*(y) = y^+(y^+) + y^-(y^-).
$$

So we obtain that

\begin{equation}
\tag{2.10}
\min\left\{\|y^+\|, \|y^\ast\|\right\} \geq y^+(y^+) = 1 - y^-y^- \geq 1 - \|y^\ast\| > 1 - \delta(\varepsilon) \geq 1 - \varepsilon > 0.
\end{equation}

Since $X$ is strongly monotone there is $b \in [0,1]$ such that

\begin{equation}
\tag{2.11}
\left\| \frac{y^+}{\|y^+\|} - by^- \right\| = 1 \quad \text{and} \quad \|by^- - y^-\| < \varepsilon,
\end{equation}

so

\begin{equation}
\left\| \frac{y^+}{\|y^+\|} - by^- - y \right\| \leq \left\| \frac{y^+}{\|y^+\|} - y^+ \right\| + \|y^- - by^-\|
\leq 1 - \|y^+\| + \varepsilon
\end{equation}

$$
< 2\varepsilon \quad \text{(by (2.10)).}
$$

If we put $z = \frac{y^+}{\|y^+\|} - by^-$ we have that $z \in S_X$ in view of (2.11) and we just checked that $\|z - y\| < 2\varepsilon$. By using also (2.6) we obtain that

\begin{equation}
\tag{2.12}
\|z - x\| \leq \|z - y\| + \|y - x\| < 3\varepsilon.
\end{equation}
We also have that
\[
\left\| \frac{y^+}{\|y^+\|} - x^* \right\| \leq \left\| \frac{y^+}{\|y^+\|} - y^* \right\| + \|y^* - x^*\| < 3\varepsilon \quad \text{(by (2.9) and (2.6))}.
\]
(2.13)

Since \(y^+(y^-) = 0\) it is also immediate that
\[
\frac{y^+}{\|y^+\|}(z) = \frac{y^+(y^+)}{\|y^+\|\|y^+\|}.
\]

By using that \(X\) has the hereditary norm attaining property we have that \(y^+(y^+) = \|y^+\|\|y^+\|\), so \(\frac{y^+}{\|y^+\|}(z) = 1\).

By taking into account also (2.12) and (2.13) we proved that \(X\) has the Bishop-Phelps-Bollobás property for positive functionals.

Now we prove the second assertion. Let assume that \(X\) has BPBp for positive functionals with the function \(\eta\) and \(0 < \eta(\varepsilon) < \varepsilon\) for every positive real number \(\varepsilon\).

Let fix \(0 < \varepsilon < 1\), \(0 < \eta = \eta(\frac{\varepsilon}{2}) < \frac{\varepsilon}{2}\) and \(x \in B_X\) such that \(\|x^+\| > 1 - \eta > 1 - \frac{\varepsilon}{2} > 0\). From Proposition 2.5 there is a positive functional \(x^* \in S_{X^*}\) such that
\[
x^*(x^+) = \|x^+\| > 1 - \eta \quad \text{and} \quad x^*(x^-) = 0.
\]
Hence \(x^*(\frac{x}{\|x\|}) = x^*(\frac{x^+}{\|x^+\|}) > 1 - \eta\). Since \(X\) has the BPBp for positive functionals with the function \(\eta\), there are a positive functional \(y^* \in S_{X^*}\) and \(y \in S_X\) satisfying also
\[
y^*(y) = 1, \quad \text{and} \quad \left\| y - \frac{x}{\|x\|} \right\| < \frac{\varepsilon}{2}.
\]
(2.14)

Since \(y^*\) is a positive functional we obtain that \(y^*(y^+) = 1\) and so \(\|y^+\| = 1\). From (2.14) we deduce that
\[
\|y - x\| \leq \left\| y - \frac{x}{\|x\|} \right\| + \left\| \frac{x}{\|x\|} - x \right\| < \frac{\varepsilon}{2} + 1 - \|x\| \leq \frac{\varepsilon}{2} + 1 - \|x^+\| < \varepsilon.
\]
Therefore \(X\) is weakly monotone. \(\square\)

Now we make a sketch of some results known until now

\[
\text{UM} \Rightarrow \text{UMOE} \Rightarrow \text{SM} \Rightarrow \text{WM}
\]

On Banach function spaces \(\text{UM} \Leftrightarrow \text{UMOE}\)

\[
\text{strong BPBp for positive functionals} \Rightarrow \text{BPBp for positive functionals}
\]

\[
\text{strong BPBp for positive functionals} \Leftrightarrow \text{UMOE}
\]

\[
\text{finite dimensional} \Rightarrow \text{BPBp for positive functionals}
\]
SM + HNAP ⇒ BPBP for positive functionals ⇒ WM

3. Examples

Our goal now is to provide examples of spaces satisfying the sufficient condition in Theorem 2.16. Later we will exhibit more examples showing that the converse implications that we wrote above does not hold. We also show with these examples that the Bishop-Phelps-Bollobás property for positive operators is not trivially satisfied for any Banach lattice.

Proposition 3.1. The following Banach lattices are strongly monotone and have the hereditary norm attaining property:

i) $L_p(\mu)$ for any positive measure $\mu$ and $1 \leq p < \infty$.

ii) $C(K)$ for any compact and Hausdorff topological space $K$.

iii) $\mathcal{M}(K)$, the space of regular Borel measures on a compact and Hausdorff topological space $K$.

In fact, $L_p(\mu)$ for $1 \leq p < \infty$ and $\mathcal{M}(K)$ are uniformly monotone.

Proof. i) For $1 < p < \infty$ the space $L_p(\mu)$ is uniformly convex and so uniformly monotone.

For $p = 1$ we check that $L_1(\mu)$ is also uniformly monotone. If $\varepsilon > 0$, $f, g \in L_1(\mu)$ are positive elements such that

$$\|f\|_1 + \|g\|_1 = \|f + g\|_1 \leq 1 \quad \text{and} \quad \|f\|_1 > 1 - \varepsilon \Rightarrow \|g\|_1 < \varepsilon.$$

Now we show that $L_p(\mu)$ has the hereditary norm attaining property. If $p > 1$, since the dual of $L_p(\mu)$ is identified with $L_q(\mu)$ where $1/p + 1/q = 1$, let $f \in S_{L_p(\mu)}$ and $g \in S_{L_q(\mu)}$. Since $f^+$ and $f^-$ have disjoint support we have that

$$1 = \|f\|_p^p = \|f^+\|_p^p + \|f^-\|_p^p.$$

From Hölder inequality it follows that

$$\|g^+\|_q \|f^+\|_p + \|g^-\|_q \|f^-\|_p \leq 1.$$

The previous inequality implies the hereditary norm attaining property in view of Proposition 2.2.

In case that $p = 1$, if $x^* \in S_{L_1(\mu)^*}$ and $f \in S_{L_1(\mu)}$ we have that

$$\|x^+\| \|f^+\|_1 + \|x^-\| \|f^-\|_1 \leq \|f^+\|_1 + \|f^-\|_1 = 1,$$

so by using again Proposition 2.2 $L_1(\mu)$ also has the hereditary norm attaining property.

ii) It is clear that

$$\|f\| = \max\{\|f^+\|, \|f^-\|\}, \quad \forall f \in C(K).$$

So, if $0 < \varepsilon < 1$, $f \in B_{C(K)}$ and $\|f^+\| > 1 - \varepsilon > 0$, the element $\frac{f^+}{\|f^+\|} - f^- \in S_{C(K)}$, so $C(K)$ satisfies Definition 2.14 with $\delta(\varepsilon) = \varepsilon$ and $b = 1$.

Now we check that $C(K)$ has the hereditary norm attaining property. We use that the topological dual of $C(K)$ is identified with $\mathcal{M}(K)$, the space of regular and Borel measures on $K$, and for the dual norm we have that

$$\|\mu\| = \|\mu^+\| + \|\mu^-\|, \quad \forall \mu \in \mathcal{M}(K).$$

As a consequence we obtain that

$$\|x^+\| \|x^+\| + \|x^-\| \|x^-\| \leq \|x^+\| + \|x^-\| = 1, \quad \forall x \in S_{C(K)}, x^* \in S_{C(K)^*}.$$
Hence $C(K)$ has the hereditary norm attaining property.

iii) From (3.1) it follows that $M(K)$ is uniformly monotone, so it is strongly monotone. By using again that $M(K)$ is an $L$-space and an argument similar to (3.2) it is immediate to obtain that $M(K)$ also has the hereditary norm attaining property.

\[ \square \]

**Corollary 3.2.** For any positive measure $\mu$ and $1 \leq p < \infty$ the space $L_p(\mu)$ has the strong Bishop-Phelps-Bollobás property for positive functionals. The same property holds for $M(K)$ for any compact and Hausdorff topological space $K$. The space $C(K)$ has the Bishop-Phelps-Bollobás property for positive functionals.

**Proof.** It suffices to use Proposition 3.1, Theorems 2.9 and 2.16. \[ \square \]

**Definition 3.3.** A norm $| |$ on $\mathbb{R}^N$ is called absolute if it satisfies that

\[ |(x_i)| = |(|x_i|)|, \quad \forall (x_i) \in \mathbb{R}^N. \]

An absolute norm $| |$ is said to be normalized if $|e_i| = 1$ for every $1 \leq i \leq N$, where $\{e_i : 1 \leq i \leq N\}$ is the canonical basis of $\mathbb{R}^N$.

Next we show that the assumptions in Theorem 2.16 implying the Bishop-Phelps-Bollobás Theorem for functionals are satisfied by $\mathbb{R}^2$ with an absolute norm. Later we will exhibit examples showing that this is not the case for $\mathbb{R}^3$.

**Proposition 3.4.** Let $| |$ be an absolute norm on $\mathbb{R}^2$. Then $X = (\mathbb{R}^2, | |)$ has the hereditary norm attaining property. The space $X$ is also strongly monotone.

**Proof.** Firstly we prove that $X$ has the hereditary norm attaining property. Let $x = (a, b) \in S_X$ and $x^* = (u, v) \in S_{X^*}$ such that $x^*(x) = 1$. In case that $x \geq 0, x \leq 0, x^* \geq 0$ or $x^* \leq 0$ the condition

\[ \|x^+\| \|x\| + \|x^-\| \|x\| \leq 1 \]

is trivially satisfied.

Otherwise we have that

\[ (3.3) \quad ab < 0 \quad \text{and} \quad uv < 0. \]

Since

\[ (3.4) \quad 1 = x^*(x) = ua + vb \leq |ua| + |vb| \leq 1, \]

we obtain that

\[ (3.5) \quad ua > 0 \quad \text{and} \quad vb > 0. \]

In case that $a > 0$, from (3.3) and (3.5) we have that $b < 0, u > 0$ and $v < 0$, so

\[ x^+ = (a, 0), \quad x^- = (0, -b), \quad x^+ = (u, 0) \quad \text{and} \quad x^- = (0, -v). \]

In view of (3.4) we get that

\[ \|x^+\| \|x\| + \|x^-\| \|x\| = |u| |a| + |v| |b| \leq 1. \]

If $a < 0$, from (3.3) we have that $b > 0, v > 0$ and $u < 0$, hence

\[ x^+ = (0, b), \quad x^- = (-a, 0), \quad x^+ = (0, v) \quad \text{and} \quad x^- = (-u, 0). \]
By using again (3.4) we obtain that
\[ \|x^+\| \|x^+\| + \|x^−\| \|x^−\| = |v| |b| + |u| |a| \leq 1. \]

By Proposition 2.2 we proved that \( X \) has the hereditary norm attaining property.

From [4, Lemma 2.5] it follows that \( X \) is also strongly monotone. \( \square \)

**Example 3.5.** There is a norm on \( \mathbb{R}^3 \) which is absolute and normalized, but it does not satisfy the hereditary norm attaining property.

**Proof.** Let \( |(r, s)| = \max\{|r|, |s|, \frac{2}{3}(|r| + |s|)\} \).

Clearly the previous norm is a normalized absolute norm on \( \mathbb{R}^2 \) and satisfies
\[ 1 = \left| \left( \frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2} \right) \right| = \left| \left( \frac{\varepsilon_1}{3}, \frac{\varepsilon_2}{3} \right) \right|, \quad \forall \varepsilon_i \in \{\pm 1\}, \ i = 1, 2. \]

Define the mapping \( || \) on \( \mathbb{R}^3 \) by
\[ ||(r, s, t)|| = |(r, s)| + |t|, \]
that is an absolute and normalized norm on \( \mathbb{R}^3 \). Let \( X \) be \( \mathbb{R}^3 \), endowed with the previous norm and the usual order.

We consider the elements \( x = \frac{1}{4}(3, -3, 0) \) and \( x^* = \frac{1}{3}(2, -2, 3) \). We know that
\[ x \in S_X, \quad x^+ = \frac{3}{4}e_1 \quad \text{and} \quad x^- = \frac{3}{4}e_2 \]
since \( \|x^+\| = \|x^-\| = \frac{3}{4} \). Now we check that \( x^* \in B_X \). Assume that \( (r, s, t) \in B_X \) and so
\[ \frac{2}{3}(|r| + |s|) + |t| \leq 1. \]

Hence
\[ x^*(r, s, t) = \frac{2}{3}(r - s) + t \]
\[ \leq \frac{2}{3}(|r| + |s|) + |t| \]
\[ \leq 1. \]

Since \( \|e_3\| = 1, \|x^*\| \geq x^*(e_3) = 1 \). As a consequence \( x^* \in S_X \) and we clearly have
\[ x^*(x) = 1, \quad x^{*+} = \frac{1}{3}(2, 0, 3) \quad \text{and} \quad x^{*-} = \frac{1}{3}(0, 2, 0). \]

Since \( \|x^{*+}\| \leq \|x^*\| = 1 \) and \( x^{*+}(e_3) = 1 \) we have that \( \|x^{*+}\| = 1 \). It is also satisfied that \( \|x^{*-}\| = \frac{2}{3} \).

As a consequence we obtain that
\[ \|x^{*+}\| \|x^+\| + \|x^{*-}\| \|x^-\| = \frac{3}{4} + \frac{2}{3} \times \frac{3}{4} = \frac{5}{4} > 1, \]
so \( X \) does not satisfy the hereditary norm attaining property. \( \square \)

**Example 3.6.** There is a three dimensional Banach lattice \( X \) which is not strongly monotone.
Proof. We consider the space $X = \mathbb{R}^3$ endowed with the usual order and the absolute norm whose closed unit ball is the convex hull of the following set

$$\{(x, y, 0) : x^2 + y^2 \leq 1\} \cup \{(x, 0, z) : x^2 + z^2 \leq 1\} \cup \{(0, y, z) : y^2 + z^2 \leq 1\} \cup \{(r, s, t) : r, s, t \in \{1, -1\}\}.$$ 

Firstly we will check that the dual norm of an element $x^* \in X^*$ is given by

$$\|x^*\| = \max \left\{ \|P_{ij}(x^*)\|_2, i, j \in \{1, 2, 3\}, \frac{\|x^*\|_1}{\sqrt{2}} \right\},$$

where we identified the dual of $X$ as the set $\mathbb{R}^3$ and $P_{ij}$ is the projection on $\mathbb{R}^3$ given by $P_{ij}(x^*) = x^*(i)e_i + x^*(j)e_j, \quad (1 \leq i, j \leq 3)$.

Since $\{(x, y, 0) : x^2 + y^2 \leq 1\} \cup \{(x, 0, z) : x^2 + z^2 \leq 1\} \cup \{(0, y, z) : y^2 + z^2 \leq 1\} \subset B_X$, then it is immediate to check that $$\|x^*\| = \max \left\{ \|P_{ij}(x^*)\|_2 : i, j \in \{1, 2, 3\} \right\}.$$ From the inclusion $$\left\{ \frac{1}{\sqrt{2}}(r, s, t) : r, s, t \in \{1, -1\} \right\} \subset B_X,$$ it follows that $$\|x^*\| \geq \sqrt{\frac{\|x^*\|_1}{2}}.$$ As a consequence we obtain that $$\|x^*\| \geq \max \left\{ \|P_{ij}(x^*)\|_2, i, j \in \{1, 2, 3\}, \frac{\|x^*\|_1}{\sqrt{2}} \right\},$$ and the reverse inequality is trivially satisfied.

It is clear that for every $0 < r < 1$ the element $z = \frac{1}{2} \left( (r, \sqrt{1 - r^2}, 0) + \frac{1}{\sqrt{2}}(1, 1, -1) \right)$ belongs to $B_X$. Let choose a sequence $(r_n)$ of real numbers such that $0 < r_n < \frac{1}{\sqrt{2}}$ and $(r_n) \to \frac{1}{\sqrt{2}}$.

Put $y = \frac{1}{\sqrt{2}}(1, 1, -1)$ and consider for $n \in \mathbb{N}$ the element $z_n$ given by

$$z_n = \frac{1}{2} \left( (r_n, \sqrt{1 - r_n^2}, 0) + y \right)$$

that belongs to $B_X$ and satisfies

$$z_n^+ = \frac{1}{2} \left( r_n + \frac{1}{\sqrt{2}}, \sqrt{1 - r_n^2} + \frac{1}{\sqrt{2}}, 0 \right) \quad \text{and} \quad z_n^- = \frac{1}{2} \left( 0, 0, \frac{1}{\sqrt{2}} \right) = \frac{1}{2\sqrt{2}}e_3.$$ 

Since $(r_n)$ converges to $\frac{1}{\sqrt{2}}$ we obtain that

$$\lim (z_n^+) = \frac{1}{\sqrt{2}}(1, 1, 0).$$
Since the element $\frac{1}{\sqrt{2}}(1, 1, 0)$ belongs to $B_X$, the element $x^* = \frac{1}{\sqrt{2}}(e_1^* + e_2^*) \in B_{X^*}$ and 
\[
\frac{1}{\sqrt{2}}((e_1^* + e_2^*)\left(\frac{1}{\sqrt{2}}(1, 1, 0)\right)) = 1,
\]
then $\frac{1}{\sqrt{2}}(1, 1, 0) \in S_X$.

**Claim.** For every $\alpha \in \mathbb{R}^+$ it is satisfied that 
\[
\left\| \frac{z_+^n}{\|z_+^n\|} + \alpha e_3 \right\| > 1, \quad \forall n \in \mathbb{N}.
\]

We prove the claim. Let us fix $n \in \mathbb{N}$.

Firstly let us notice that $\|z_+^n\| = \|z_+^n\|_2$. Since $z_n(1)$ and $z_n(2)$ are positive and $z_n(1) \neq z_n(2)$, we conclude that $\|z_+^n\|_1 < \sqrt{2}\|z_+^n\|_2$. So we can choose a real number $u_n$ such that 
\[
0 < u_n < \min\{z_n(1), z_n(2), \sqrt{2}\|z_+^n\|_2 - \|z_+^n\|_1\}.
\]

If we write $x_n^* = \left(\frac{z_n(1)}{\|z_+^n\|_2}, \frac{z_n(2)}{\|z_+^n\|_2}, u_n\right)$, in view of (3.6) we have that
\[
x_n^* = \max\left\{ \frac{\|z_+^n\|_2}{\|z_+^n\|_2}, \frac{1}{\|z_+^n\|_2} \|z_n(1), u_n\|_2, \frac{1}{\|z_+^n\|_2} \|z_n(2), u_n\|_2, \frac{1}{\sqrt{2}} \|z_+^n\|_2 \|(z_n(1), z_n(2), u_n)\|_1 \right\}.
\]

Since $0 < u_n < \min\{z_n(1), z_n(2)\}$ and $z_n \in B_X$, the first three numbers in the previous expression are less or equal to 1. By the choice of $u_n$ we also have that $\|z_+^n\|_1 + u_n < \sqrt{2}\|z_+^n\|_2$, so the last number in the expression of $\|x_n^*\|$ is also less or equal to 1. As a consequence $\|x_n^*\| \leq 1$ and
\[
x_n^* \left(\frac{z_+^n}{\|z_+^n\|_2} + \alpha e_3\right) = x_n^* \left(\frac{z_+^n}{\|z_+^n\|_2}\right) + \frac{u_n}{\|z_+^n\|_2} \alpha = 1 + \frac{u_n}{\|z_+^n\|_2} \alpha > 1
\]
in case that $\alpha > 0$. So we checked that 
\[
\left\| \frac{z_+^n}{\|z_+^n\|_2} + \alpha e_3 \right\| > 1
\]
for $\alpha \in \mathbb{R}^+$. Since the norm of $X$ is absolute and $z_+(3) = 0$, the same condition is also satisfied for $\alpha < 0$.

Since $\lim(\|z_+^n\|) = 1$ and $\|z_-^n\| = \frac{1}{2\sqrt{2}}$ for any natural number $n$, we conclude that the space $X$ is not strongly monotone. \hfill \Box

In view of Proposition 2.13, Examples 3.5 and 3.6 show that none of the sufficient conditions in Theorem 2.16 are necessary conditions.

For the next example we need some auxiliary results.

**Lemma 3.7.** Let $\| \|$ be an absolute and normalized norm on $\mathbb{R}^2$. Then 
\[
0 \leq s < t, \quad 0 \leq u < v \Rightarrow |(s, u)| < |(t, v)|.
\]

**Proof.** Let us choose $0 < \varepsilon < \min\{1 - \frac{s}{t}, 1 - \frac{u}{v}\}$ and so 
\[
s < (1 - \varepsilon)t \quad \text{and} \quad u < (1 - \varepsilon)v.
\]

As a consequence 
\[
|(s, u)| \leq |(1 - \varepsilon)t, (1 - \varepsilon)v| = (1 - \varepsilon)|(t, v)| < |(t, v)|.
\]
\hfill \Box
**Lemma 3.8.** Let $| |$ be an absolute and normalized norm on $\mathbb{R}^2$. Assume that $c, d, e \in \mathbb{R}_0^+$, $d \neq e$ and $|(c, d)| = 1 = |(c, e)|$. Then $c = 1$.

**Proof.** Since the norm $| |$ is absolute and normalized we have that 
$$c = |(c, 0)| \leq |(c, d)| \leq 1.$$ 
Assume that $d < e$. 
Assume that $c < 1$. For any $d \leq y \leq e$ we have that 
$$1 = |(c, d)| \leq |(c, y)| \leq |(c, e)| = 1,$$ 
so 
$$\{(c, y) : y \in [d, e]\} \subset S_{\mathbb{R}^2}.$$ 
Since the unit ball is convex, it contains the segment whose extreme points are $(c, e)$ and $(1, 0)$. Since $0 < d < e$ there is a positive real number $x$ such that $(x, d)$ belongs to that segment and so $|(x, d)| \leq 1$. By using that the norm is absolute and $c < x$ we have that 
$$1 = |(c, d)| \leq |(x, d)| \leq 1,$$ 
so $|(x, d)| = 1$. 
By using that $0 \leq d < e$ and $c < x < 1$ the element 
$$z = \frac{1}{3} ( (c, d) + (x, d) + (c, e) ) = \left( \frac{2c + x}{3}, \frac{2d + e}{3} \right),$$ 
satisfies that 
$$1 = |(c, d)| \leq \left| \left( \frac{2c + x}{3}, \frac{2d + e}{3} \right) \right| \leq 1,$$ 
and so $u = \left( \frac{2c + x}{3}, \frac{2d + e}{3} \right)$ belongs to the unit sphere. Since 
$$c < \frac{2c + x}{3} \quad \text{and} \quad d < \frac{2d + e}{3},$$ 
and $|(c, d)| = 1$ the element $u$ does not belongs to the unit sphere in view of Lemma 3.7. Hence $c = 1$ as we wanted to show. \hfill $\square$

**Lemma 3.9.** Let $| |$ be an absolute and normalized norm on $\mathbb{R}^2$. Assume that 
$$0 = \max\{x \in \mathbb{R} : |(x, 1)| = 1\} = \max\{y \in \mathbb{R} : |(1, y)| = 1\}.$$ 
Then 
$$|s| < |t| \Rightarrow |(r, s)| < |(r, t)| \quad \text{and} \quad |(s, r)| < |(t, r)|.$$ 

**Proof.** Since the norm is absolute, it suffices to prove the statement for $0 \leq s < t$. Notice that if $r = 0$, the assertion stated in the lemma is satisfied. Otherwise, by using again that the norm is absolute, it is enough to prove the statement for $r > 0$. 
Since the norm is absolute we know that $|(r, s)| \leq |(r, t)|$. In case that $|(r, s)| = |(r, t)|$ it is also satisfied 
$$\left| \left( \frac{r}{|(r, s)|}, \frac{s}{|(r, s)|} \right) \right| = 1 = \left| \left( \frac{r}{|(r, s)|}, \frac{t}{|(r, s)|} \right) \right|.$$ 
By Lemma 3.8 we obtain that $r = |(r, s)|$, that is, 
$$1 = \left| \left( 1, \frac{t}{r} \right) \right|.$$
By assumption we get \( t = 0 \), which is impossible, so we proved that \(|(r, s)| < |(r, t)|\). From the same argument we also obtain that \(|(s, r)| < |(t, r)|\).

**Example 3.10.** There is a Banach lattice isomorphic to \( \ell_2 \) that is not weakly monotone. So this space does not have the Bishop-Phelps-Bollobás property for positive functionals.

**Proof.** Let \((\alpha_n)\) be a sequence of positive real numbers that is strictly increasing and converges to 1. For every natural number \( n \), consider the absolute norm \(|\cdot|_n\) on \( \mathbb{R}^2 \) whose closed unit ball is the set \( B_n \) given by

\[
B_n = \text{co} \left\{ \pm e_1, \pm e_2, \left( \alpha_n, \pm \frac{1}{2} \right), \left( -\alpha_n, \pm \frac{1}{2} \right) \right\}.
\]

The norm \(|\cdot|_n\) clearly satisfies the assumption of Lemma 3.9.

We write \( X = \ell_2 \) as a Riesz space. Since \(|\cdot|_n\) is an absolute and normalized norm for each natural number \( n \) we have that

\[
\max\{|x_{2n-1}|, |x_{2n}|\} \leq |(x_{2n-1}, x_{2n})|_n \leq |x_{2n-1}| + |x_{2n}|, \quad \forall n \in \mathbb{N}.
\]

So we can define the norm on \( X \) given by

\[
\|x\| = \|\{(x_{2n-1}, x_{2n})\}_n\|_2, \quad (x \in X).
\]

Clearly the previous norm is equivalent to the usual norm on \( \ell_2 \) and it is a lattice norm on \( X \). For a nonempty subset \( C \subset \mathbb{N} \) we denote by \( P_C \) the operator \( P_C : \ell_2 \to \ell_2 \) given by

\[
P_C(x) = \sum_{n \in C} x(n)e_n, \quad (x \in \ell_2).
\]

In case that \( C = \emptyset \) we simply take \( P_C = 0 \).

Since we can apply Lemma 3.9 to the norm \(|\cdot|_k\) for any natural number \( k \) we have that

\[
|(x_{2k-1}, 0)|_k < |(x_{2k-1}, x_{2k})|_k, \quad \forall x \in X, x_{2k} \neq 0.
\]

Of course, the analogous inequality holds changing the role of the coordinates. Since the norm of \( \ell_2 \) is strictly monotone, for any natural number \( n \) we conclude that

\[
\|P_{\mathbb{N}\setminus\{n\}}(x)\| < \|x\|, \quad \forall x \in X, x_n \neq 0.
\]

Since \( X \) is a Banach lattice, as a consequence of the previous inequality, we obtain that

\[
(3.7) \quad x \in X, A \subset \mathbb{N}, P_A(x) \neq 0 \Rightarrow \|(I - P_A)(x)\| = \|P_{\mathbb{N}\setminus A}(x)\| < \|x\|.
\]

Notice that for \( x \in X \), if we denote by \( A \) and \( N \) the sets

\[
A = \{ n \in \mathbb{N} : x_n > 0 \} \quad \text{and} \quad B = \{ n \in \mathbb{N} : x_n < 0 \},
\]

then we have that

\[
x^+ = P_A(x) \quad \text{and} \quad x^- = -P_B(x).
\]

By taking into account (3.7) we deduce that

\[
(3.8) \quad x \in X, \|x\| = \|x^+\| \Rightarrow x \geq 0.
\]

Finally, we show that \( X \) is not weakly monotone. In order to do this, we define the sequence \((z_n)\) given by

\[
z_n(2n - 1) = \alpha_n, \quad z_n(2n) = -\frac{1}{2} \quad \text{and} \quad z_n(k) = 0 \text{ if } k \in \mathbb{N}\setminus\{2n - 1, 2n\}.
\]
For each natural number \( n \), we clearly have that \( z_n \in X \) and \( \|z_n\|^2 = \left|\left(\alpha_n, -\frac{1}{2}\right)\right|_n = 1 \). We also have that
\[
\|z^+_n\| = \alpha_n \quad \text{and} \quad \|z^-_n\| = \frac{1}{2}.
\]
If \( x \in X \) and \( \|x\| = \|x^+\| = 1 \), by (3.8) we obtain that \( x = x^+ \), that is, \( x^- = 0 \). Hence
\[
\frac{1}{2} = \|z^-_n\| = \|x^- - z^-_n\| \leq \|x - z_n\|, \quad \forall n \in \mathbb{N}.
\]
Since \( \lim(\|z^+_n\|) = \lim(\alpha_n) = 1 \), the space \( X \) is not weakly monotone.

In view of Theorem 2.16 the space \( X \) cannot have the Bishop-Phelps-Bollobás property for positive functionals. \( \square \)

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