Influence of stress gradients on the limit state of the plate weakened by a circular hole

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Abstract. The limit elastic state of a rectangular semi-infinite plate with a centrally-located circular hole has been considered. The tensile load is uniformly distributed across the width at the ends of the plate. The gradient plasticity condition governing the initial moment of yielding at the non-uniform stressed state is used as the criterion of the limit state. The analytical expressions have been derived which allow one to calculate the corresponding stresses and load.

1. Introduction
One of the most important issues in the building sector and engineering is the problem of the reduction of material consumption of the construction and at the same time the maintenance of its durability and operational safety. To solve this task, we need to improve calculation methods which would be able to register material nonlinearity and possible types of the real strained condition of the constructions and elements. From the analysis of experimental and theoretical data presented in [1-6] we see that the extreme elastic stresses, determined from the tests on uniaxial tension or pure shift, do not allow to solve accurately the issues connected with the spatially-heterogeneous distribution of stresses. Another problem is the usage of the yield criteria in calculating practice which applies the tension obtained by the tests on uniaxial (rarely biaxial) strain-compression as an extreme tension. Even few criteria existing nowadays which take into account the type of stress condition, do not register heterogeneity of the stress distribution (the type of stress condition in the locality of the point under consideration) [7-11].

In the present work, the effect of deformational constraint is used, what means that change in the strength property of the material for the description of the criterion at the limit state, which depends on the type of stress condition in the locality of the point under consideration, is also taken into consideration. The yield criterion (yield condition) is elaborated as a criterion at the limit state for plastic isotropic materials possessing a yield point. Actually authors are unaware of other results, which could provide the solution of such a kind problems in an analytical form.

2. Problem formulation and method of solution
Let us consider the uniform tension along the x-axis of a plate with the width 2b being incommensurably smaller than the length, caused by the distributed load q (Figure 1). A small circular hole at the plate center with the diameter 2a≪2b changes the stress state in its vicinity, which becomes non-uniform. G. Kirsch was the first to solve the given problem [12]. In the polar coordinates, the stress state has the following form:
\[
\sigma_r = \frac{q}{2} \left( 1 - \frac{a^2}{r^2} \right) + \frac{q}{2} \left( 1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2} \right) \cos 2\theta, \\
\sigma_\theta = \frac{q}{2} \left( 1 + \frac{a^2}{r^2} \right) - \frac{q}{2} \left( 1 + \frac{3a^4}{r^4} \right) \cos 2\theta, \\
\tau_{r\theta} = -\frac{q}{2} \left( 1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2} \right) \sin 2\theta, 
\]
where the radius \( r \) is counted from the plate center, and the polar angle \( \theta \) is laid off clockwise from the horizontal axis.

The influence of the hole has a local character and it is revealed most significantly at the points on the edge of the hole, wherein. \( \sigma_r = \tau_{r\theta} = 0 \), while the circumferential stresses at the points \( m \) and \( n \) take on the maximum values equal to \( \sigma_\theta = 3q \), and at the points \( p \) and \( s \) they even turn out to be negative \( \sigma_\theta = -q \) (Figure 1).

Figure 1. Scheme of the plate.

The yield condition for plastic isotropic materials possessing a yield plateau (which is characteristic for construction steels) was used as the criterion at the limit elastic state [13-16]. The criterion is based on the concept of shear strain constraint along the shear lines on the side of the less stressed volumes of the material, and it takes the increase in the effect of the limit elastic stresses in non-uniform fields into account. The degree of the influence of mutual adjacent volumes on the material is characterized by the vector, namely: the gradient of scalar stress field. At each point of the latter, the gradient vector field characterizes the direction and velocity of the stress variation. The higher the non-uniformity of the stress field, the larger mutual influence is exerted by the adjacent areas of the material.

The incremental plasticity condition can be written in the form of the Tresca - Saint Venant plasticity condition or Huber – Mises-Hencky plasticity condition considering the fact that the yield strength of the material is coupled in a particular way with a certain function in the stress gradient and its coordinate derivatives.

The beginning of the yielding process will be defined as the moment when the value of the maximum intensity of tangential stresses \( T_{\text{max}} \) reaches a certain increased value \( T_{gr} \) under the strain constraint. The higher non-uniformity of the stress distribution in the vicinity of the point under consideration, the larger will be the value’s elevation over the yield strength \( T_0 \) under the uniaxial stress state. This dependence is adopted as an asymptotic relationship:

\[
T_{gr} = T_0 + (T_m - T_0) \cdot \left( \frac{\text{grad}T}{T} \right) \cdot \left( \lambda_{r,s} + \text{grad}T/T \right), \tag{2}
\]
where \( T_m \) is the largest stress possible at the non-uniform stress state.
It has been adopted here that \( T_\text{m} = 1.5T_0 \), \( \lambda_{T,g} \) is a certain elastic material constant which could be determined experimentally and has the length dimension \([\text{m}^{-1}]\), \( \text{grad}T \) is the gradient modulus of the tangential stress intensity, what could be considered as the measure of the stress state non-uniformity.

In addition to the abovementioned linear-fractional function, other algebraic expressions and transcendental functions can be used. The utilization of irrational, exponential, trigonometric and logarithmic curves could be found in [17].

The tangential stresses intensity in the polar coordinates is defined as

\[
T = \frac{1}{\sqrt{3}} \left( \sigma_\rho^2 + \sigma_\theta^2 - \sigma_\rho \sigma_\theta + 3\tau_\rho \theta \right)^{\frac{1}{2}}.
\] (3)

Substituting the stress state components (1) in relationship (3), after some mathematical treatment it could be found that

\[
T = \frac{q}{2\sqrt{3}} \left[ \cos 2\theta \cdot \left( -\frac{18a^6}{r^6} + \frac{12a^4}{r^4} - 16a^2 \right) + \cos^2 2\theta \cdot \left( 3 + \frac{27a^8}{r^8} - \frac{36a^6}{r^6} + \frac{34a^4}{r^4} - \frac{12a^2}{r^2} \right) \right]
\] (4)

\[
+ \sin^2 2\theta \cdot \left( \frac{27a^8}{r^8} - \frac{36a^6}{r^6} + \frac{6a^4}{r^4} + \frac{12a^2}{r^2} \right) + 1 + \frac{3a^4}{r^4} \right]^{\frac{1}{2}}.
\]

Assume that the gradient modulus of the tangential stress intensity has the following form:

\[
\text{grad}T = \left[ \left( \frac{\partial T}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial T}{\partial \theta} \right)^2 \right]^{\frac{1}{2}}.
\] (5)

After cumbersome transformations of the expressions (4) and (5) we finally obtain

\[
\text{grad}T = \frac{q}{4\sqrt{3}} \left[ \cos 2\theta \cdot \left( -\frac{18a^6}{r^6} + \frac{12a^4}{r^4} - 10a^2 \right) + \cos^2 2\theta \cdot \left( 3 + \frac{27a^8}{r^8} - \frac{36a^6}{r^6} + \frac{34a^4}{r^4} - \frac{12a^2}{r^2} \right) \right]
\] (6)

\[
+ \sin^2 2\theta \cdot \left( \frac{27a^8}{r^8} - \frac{36a^6}{r^6} + \frac{6a^4}{r^4} + \frac{12a^2}{r^2} \right) + 1 + \frac{3a^4}{r^4} \right]^{\frac{1}{2}} \cdot \left[ \cos 2\theta \cdot \left( \frac{108a^6}{r^6} - \frac{48a^4}{r^4} + \frac{9a^2}{r^2} \right) \right]
\]

\[
+ \cos^2 2\theta \cdot \left( -\frac{216a^8}{r^8} + \frac{216a^6}{r^6} - \frac{136a^4}{r^4} + \frac{24a^2}{r^2} \right) + \sin^2 2\theta \cdot \left( -\frac{216a^8}{r^8} + \frac{216a^6}{r^6} + \frac{24a^4}{r^4} - \frac{24a^2}{r^2} \right)
\]

\[
- \frac{12a^4}{r^4} \right) + 1 \left[ 2 \sin 2\theta \cdot \left( \frac{18a^6}{r^6} - \frac{12a^4}{r^4} + \frac{10a^2}{r^2} \right) + 2 \sin 4\theta \cdot \left( -\frac{40a^4}{r^4} + \frac{24a^2}{r^2} \right) \right] \right]^{\frac{1}{2}}.
\]

Obviously, the yielding starts at the cross-section at the points \( m \) and \( n \) at \( r=a, \theta=\frac{\pi}{2}, \frac{3\pi}{2} \), where the maximum value of the tangential stress intensity is equal to \( T_{\text{max}} = 3q \).

From the expression (6) the modulus of the intensity gradient at these points is
\[ \text{grad} T_{\text{max}} = \frac{8.5q}{\sqrt{3}a}, \]

which gives us the relative gradient

\[ \frac{\text{grad} T_{\text{max}}}{T_{\text{max}}} = \frac{17}{6a}. \]  

(7)

As it was determined earlier, considering the condition of the strain constraint \( T_{\text{max}} = T_{gr} \), from the expressions (7) and (2) with regard to \( T_m = 1.5T_0 \) the following dependence is obtained:

\[ T_{gr} = T_0 \frac{6a\lambda_{tg} + 34}{6a\lambda_{tg} + 17}, \]  

(8)

which is presented in the semi-logarithmic coordinates in Figure 2.

The computations were performed applying the numerical value \( \lambda_{tg} = 20.1587 \text{m}^{-1} \) obtained from the experimental data [2] for the problems of lateral bending [16,18]. The analysis of the last-mentioned expression shows that the value of the increased stress that corresponds to the beginning of plastic yielding depends neither on the load value nor on the absolute dimensions of the plate, but only on the ratio of the numerical values of the relative gradient and the material elastic constant.

\[ \frac{T_{gr}}{T_0}, \frac{q_{gr}}{q_0} \]

Figure 2. The graph of the increase in the load-bearing capacity of the plate with a hole.

According to the traditional approach \( T_{\text{max}} = T_0 \), that is \( \sqrt{3}q_0 = \frac{\sigma_0}{\sqrt{3}} \), resulting in \( q_0 = \frac{\sigma_0}{3} \). Due to the suggested approach \( T_{\text{max}} = T_{gr} \), that is \( \sqrt{3}q_{gr} = \frac{\sigma_0}{\sqrt{3}}, \frac{6a\lambda_{tg} + 34}{6a\lambda_{tg} + 17} \), whence it follows that
\[ q_{w} = \frac{\sigma_{0}}{3} \cdot \frac{6a\lambda_{s} + 34}{6a\lambda_{s} + 17}, \] from which it is evident that this value defines the increased load that causes the beginning of yielding.

3. Conclusion
The solution of the issues of the theory of durability with respect to the calculation of building structures in terms of the specific character of their work requires deeper development. The correctly chosen criterion of durability allows one to estimate the work of structures in the elastic stage more accurately, and to precise the resources of load bearing capacity. As the result, that allows to obtain more rational engineering decision.

The usage of the gradient yield condition in this work provides the analytical formulas which give satisfactory calculating results. It could considered as a basis for recording the influence of the heterogeneous stress condition on the deformation and transition of the material into the limit-state in calculating problems concerning the plates with round cut-outs.

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