Valuation Semantics for First-Order Logics of Evidence and Truth

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Abstract
This paper introduces the logic \(QLET_F\), a quantified extension of the logic of evidence and truth \(LET_F\), together with a corresponding sound and complete first-order non-deterministic valuation semantics. \(LET_F\) is a paraconsistent and paracomplete sentential logic that extends the logic of first-degree entailment (\(FDE\)) with a classicality operator \(\circ\) and a non-classicality operator \(\bullet\), dual to each other: while \(\circ A\) entails that \(A\) behaves classically, \(\bullet A\) follows from \(A\)’s violating some classically valid inferences. The semantics of \(QLET_F\) combines structures that interpret negated predicates in terms of anti-extensions with first-order non-deterministic valuations, and completeness is obtained through a generalization of Henkin’s method. By providing sound and complete semantics for first-order extensions of \(FDE\), \(K3\), and \(LP\), we show how these tools, which we call here the method of anti-extensions + valuations, can be naturally applied to a number of non-classical logics.

Keywords Logics of evidence and truth · Paraconsistency · Information · First-order valuation semantics

1 Introduction

The main aim of this paper is to introduce a quantified extension of the logic of evidence and truth \(LET_F\), introduced in Rodrigues, Bueno-Soler & Carnielli...
[1], together with a corresponding sound and complete first-order valuation semantics. The latter are a development of the two-valued non-deterministic semantics for sentential logics proposed and investigated by Loparic et al. [2–5] from the 1970s onward in order to provide adequate semantics for some non-classical logics. \( LET_F \) is a paraconsistent and paracomplete sentential logic that extends the logic of first-degree entailment (\( FDE \)), also known as Belnap-Dunn 4-valued logic, with a classicality and a non-classicality operator, \( \circ \) and \( \bullet \), dual to each other: while \( \circ A \) entails that \( A \) behaves classically, \( \bullet A \) follows from \( A \)’s violating some classically valid inferences. A sound and complete valuation semantics for \( LET_F \) was presented in [1], and a Kripke-style semantics in [6].

\( LET_s \) have been conceived in order to formalize the deductive behavior of positive and negative evidence, which can be either conclusive or non-conclusive. According to this proposal, conclusive evidence is assumed to behave classically, and so is subjected to classical logic, whereas non-conclusive evidence may be incomplete or contradictory – and, in the case of \( LET_F \), is subjected to \( FDE \). Now, it is well-known that \( FDE \) can be interpreted as an information-based logic (e.g. [7–9]). In [1, 6], it has been argued that \( LET_F \) can also be interpreted along the same lines, and this interpretation can be naturally extended to \( LET_F \)’s first-order extension \( QLET_F \) to be investigated here (see Section 2 below).

The semantics of \( QLET_F \) combines structures that interpret negated predicates in terms of anti-extensions [10, 11] with first-order non-deterministic valuations [12, 13], and completeness is obtained via a Henkin-style proof. These tools, which for convenience we call the method of anti-extensions + valuations, are required for handling the non-deterministic character of \( QLET_F \), and can be naturally applied to a number of non-classical logics. In order to illustrate the generality of this method, in Section 5 we show how the semantics of \( QLET_F \) can be adapted to provide sound and complete semantics for first-order versions of \( FDE \), Kleene’s \( K3 \), and the logic of paradox \( LP \).

The remainder of this paper is structured as follows. In Section 2 we briefly present the interpretation of \( LET_s \) in terms of information, as well as the motivations for adding the classicality operator \( \circ \) to \( FDE \).\(^1\) Readers who are exclusively concerned with the technical aspects of the formal semantics and the completeness proofs may skip directly to Sections 3, 4, and 5, where the main results are presented. In Section 3 we present both a natural deduction system and a corresponding valuation semantics for \( QLET_F \). Section 4 contains detailed proofs of soundness and completeness for \( QLET_F \), along with other metatheoretical results, such as compactness and a few versions of the Löwenheim-Skolem theorem. In Section 5 we provide sound and complete semantics for the first-order versions of \( FDE \), \( K3 \), and \( LP \), which result from applying the method of anti-extensions + valuations mentioned above. Section 6 wraps up the text with some historical remarks about valuation semantics and their generality.

\(^1\)Parts of Section 2 have already appeared in other papers by the authors, but we have decided to include them here to make the paper as self-contained as possible.
2 On the Intended Interpretation of $QLET_F$

In this section, we explain the interpretation of $QLET_F$ in terms of positive and negative information, as well as the motivations for adding the classicality operator $\odot$ to $FDE$. We first present the information-based interpretation of $FDE$ proposed by Belnap and Dunn, and then discuss how it can be extended to $QLET_F$.

2.1 On Positive and Negative Information

The logic of first-degree entailment ($FDE$) appears in Anderson and Belnap [14] as the first-order system $LEQ_1$. In the 1970s, Belnap and Dunn published a series of papers with a four-valued semantics for the sentential fragment of $LEQ_1$, along with a corresponding interpretation in terms of information [7, 8, 17]. The resulting logic is what is usually referred to in the literature as the Belnap-Dunn logic [18], or simply $FDE$ [19].

Belnap’s proposal in [7] takes $FDE$ as a logic to be used by a computer that receives information from different sources, which may be either inconsistent or incomplete. The semantic values $T$, $F$, $Both$, and $None$ allow the following four scenarios to be expressed with respect to a given sentence $A$:

1. $A$ holds, $\neg A$ does not hold: $\nu(A) = T$,
2. $\neg A$ holds, $A$ does not hold: $\nu(A) = F$,
3. Neither $A$ nor $\neg A$ holds: $\nu(A) = None$,
4. Both $A$ and $\neg A$ hold: $\nu(A) = Both$.

The values $T$ and $F$ are not to be understood as the standard truth values of classical logic, but are rather explained by Belnap as ‘told true’ and ‘told false’ signs in the sense that a computer ‘has been told’ that $A$ is true, and that it ‘has been told’ that $A$ is false, respectively [7, p. 38]. Likewise, $Both$ means that the computer has been told that $A$ is both true and false, while $None$ means that nothing about $A$ has been told to the computer.

Positive information is represented by $A$, whereas negative information is represented by $\neg A$ (more about this briefly). Moreover, the presence of negative (positive) information $A$ does not rule out the presence of positive (negative) information $A$. The value $Both$ thus represents scenarios where there is both positive and negative information with respect to $A$, and the value $None$ represents scenarios where there is no information at all about $A$. These inconsistent and incomplete scenarios can be thought of as databases that contain, respectively, contradictory information $A$ and $\neg A$, and no information about $A$.

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2 The logic $LEQ_1$ is the $\rightarrow$-free fragment of the system $EQ$, which, in turn, is a quantified version of the system $E$ of entailment (see, e.g., [15, 16]). Since $\rightarrow$ is taken to express $E$’s entailment relation, $A \rightarrow B$ is called a first-degree entailment when $A$ and $B$ are formulas in which no connectives other than $\neg$, $\land$, and $\lor$ occur. A first-degree entailment is valid just in case it is a tautological entailment, that is, $A$’s disjunctive normal form $A_1 \lor \cdots \lor A_m$ and $B$’s conjunctive normal form $B_1 \land \cdots \land B_n$ are such that each $A_i$ and $B_j$ share a sentential letter or a negated sentential letter, for every $i$ and $j$ [16, pp. 14ff.]. $LEQ_1$ is equivalent to the logic $QFDE$ presented in Section 5.1 below.
A notion of information that fits with the above interpretation of $\text{FDE}$ is the so-called general definition of information as well-formed meaningful data (see, e.g., [20, Section 4.3]), or simply meaningful data (see [21]).\(^3\) A linguistic version of this notion is presented by Dunn in the following passage:

[Information is] what is left from knowledge [defined as justified true belief] when you subtract justification, truth, belief, and any other ingredients such as reliability that relate to justification. . .

[Information] is something like a Fregean “thought,” i.e., the “content” of a belief that is equally shared by a doubt, a concern, a wish, etc. It might be helpful to say that it is what philosophers call a “proposition,” but that term itself would need explanation [22, p. 581].

Propositions, conceived of as the kind of thing that can be either true or false, seem to fit well the concept of information as meaningful data, but we acknowledge that the concept of proposition has problems that could be transferred to the concept of information. In order to explain positive and negative information, we can avoid these problems by moving from propositions to sentences, that is, instead of saying that positive (negative) information is the proposition that $A$ ($\neg A$), we talk about information conveyed by sentences. Thus, we say that positive information $A$ is conveyed by the sentence ‘$A$’, and that negative information $A$ is conveyed by the sentence ‘$\neg A$’.

2.2 The Classicality Operator: Reliable and Unreliable Information

Let us now see how the above interpretation can be extended to $\text{LET}$s, which are extensions of $\text{FDE}$ equipped with the operator $\circ$.

First off, classical negation can be recovered in $\text{LET}$s for sentences that occur in the scope of $\circ$, that is:

$A, \neg A \not\equiv B$, for some $A$ and $B$, but $\circ A, A, \neg A \vdash B$, for every $A$ and $B$;

$B \not\equiv A \lor \neg A$, for some $A$ and $B$, but $\circ A, B \vdash A \lor \neg A$ for every $A$ and $B$.

The connective $\circ$ is called a classicality operator. It divides the sentences of the language into two groups: the ones subjected to classical logic, and those subjected to a paracomplete and paraconsistent logic.\(^4\) For example, in $\text{LET}_F$, being $i_1$ an atom or a negated atom, given $\{\circ l_1, \ldots, \circ l_n\}$, all sentences formed with $\{l_1, \ldots, l_n\}$ over $\{\neg, \lor, \land\}$ behave classically.\(^5\)

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\(^3\)Note that according to the general definition of information, a piece of information is not required to be true. Therefore, the veridicality thesis, advocated by Floridi [20, pp. 93ff.], is not taken to hold. The veridicality thesis is discussed by Dunn [22, 23] and Fetzer [21]. For a discussion of the notion of data upon which this definition is based, see Floridi [24, Section 1.3].

\(^4\)LETs are a development of logics of formal inconsistency (LFI s) (see, e.g., [12, 25, 26]), which, in turn, are a generalization of da Costa’s work on paraconsistency [27, 28]. To the extent that LETs recover both explosion and excluded middle, they are logics of formal inconsistency and undeterminedness (LFIUs) [29, 30]. As far as we can tell, the idea of recovering at once explosion and excluded middle appeared for the first time in Loparic and da Costa [5].

\(^5\)This result is stated and proven as Fact 31 of [1], and it also holds for $\text{QLET}_F$. 

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As already mentioned, LETs have been originally conceived to express the deductive behavior of conclusive and non-conclusive evidence, but they can also be interpreted in terms of reliable and unreliable information (see [6] and [31, Section 2.2.1]). According to this interpretation, $\circ A$ expresses that the information conveyed by $A$, either positive or negative, is reliable, and it is assumed that reliable information is subjected to classical logic. Thus, whenever $\circ A$ holds, $A$ cannot be contradictory (on pain of triviality).\(^6\)

The motivation for adding the connective $\circ$ to $FDE$ is to be able to represent, in addition to the four scenarios expressed by $FDE$, two more scenarios that specifically concern reliable information. More precisely, when $\circ A$ does not hold, and so when the information conveyed by $A$ is not known to be reliable, we have the four scenarios mentioned above on page 3. But when $\circ A$ does hold, those four scenarios are narrowed down to just two: either the information $A$ or the information $\neg A$ is reliable, but not both. The resulting six scenarios are expressed in terms of the two-valued semantics adopted here as follows:

No reliable information about $A$: $v(\circ A) = 0$:

1. Only positive information $A$: $v(A) = 1$, $v(\neg A) = 0$;
2. Only negative information $A$: $v(A) = 0$, $v(\neg A) = 1$;
3. No positive nor negative information $A$: $v(A) = 0$, $v(\neg A) = 0$;
4. Both positive and negative information $A$: $v(A) = 1$, $v(\neg A) = 1$.\(^7\)

Either $A$ or $\neg A$ is reliable: $v(\circ A) = 1$:

5. Reliable information $A$: $v(A) = 1$, $v(\neg A) = 0$;
6. Reliable information $\neg A$: $v(A) = 0$, $v(\neg A) = 1$.

To the extent that the connective $\circ$ indicates the presence of reliable information, it allows distinguishing circumstances in which only positive (negative), though non-reliable, information $A$ is available – viz., $v(A) = 1$ and $v(\circ A) = 0$ – from circumstances in which there is positive (negative) reliable information $A$ – viz., $v(A) = v(\circ A) = 1$. In other words, unlike in $FDE$, in LET$_F$ (and QLET$_F$) we are able to distinguish, among the six scenarios above, 1 from 5, as well as 2 from 6. The connective $\bullet$ works in a dual manner to $\circ$: while $\circ A$ implies that $A$ behaves classically, a non-classical behavior of $A$ implies $\bullet A$ (see Proposition 4 below), and the intuitive interpretation of $\bullet A$ is that there is no information that $A (\neg A)$ is reliable.

It is worth noting that the idea of using the connective $\circ$ (and its dual $\bullet$) to represent reliable and unreliable information is not new. It has appeared in Carnielli et al. [33], where the logic $LFI1$ was proposed as a system suitable for dealing with possibly inconsistent databases. In contrast to LET$_F$ (and to QLET$_F$), $LFI1$ complies with the closed world assumption, according to which if there is no information $A$ in a given database, a query on $A$ is answered $\neg A$ (cf. [34]). The main idea of [33] is that of a database in which conflicting information $A$ and $\neg A$ is represented in the formal

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\(^6\)For an account of the notion of evidence of LETs, as well as the connections between information and evidence, see [32, Sections 1 and 2].

\(^7\)A proof of the equivalence between the two-valued and the four-valued semantics of $FDE$ is given in [1, Section 2.2].
system by $\bullet A$, whose intuitive meaning is that $A$ is contradictory. In de Amo et al. [35] we find the proposal of a deductive query language based on $LFI_1$, in which $\diamond A$ and $\bullet A$ mean respectively that $A$ is sure and $A$ is controversial. The logic $LFI_1$ can be seen as an extension of $QLET_F$ suitable for closed world databases, as it merely adds to the latter both excluded middle and the inference schema $\bullet A \vdash A \land \neg A$ (see [10]). Even though there are significant differences between $LFI_1$ and $QLET_F$, the intuitive readings of $\diamond$ and $\bullet$ in both [33] and [35] are indeed very similar to the ones proposed here. It is also worth noting that although the intuitive interpretation of $QLET_F$ has been presented as a further development of the interpretation of $FDE$, as the underlying logic of possibly inconsistent and incomplete databases, it does not apply only to databases. Rather, it applies to any context of reasoning where people deal with information, reliable and unreliable. It is not the purpose of this paper to provide concrete applications of the logic $QLET_F$, e.g., in database management systems, though we do believe that deductive query languages for inconsistent and incomplete databases based on $QLET_F$ could be designed and implemented.

3 The Logic $QLET_F$

The logical vocabulary of $QLET_F$ is composed by the unary connectives $\neg$, $\diamond$, $\bullet$, the binary connectives $\land$ and $\lor$, the quantifiers $\forall$ and $\exists$, the identity symbol $=\equiv$, the individual variables from $\mathcal{V} = \{v_i : i \in \mathbb{N}\}$, and parentheses. From now on we shall specify the non-logical vocabulary of a first-order language by means of its signature, which is a pair $\mathcal{S} = (\mathcal{C}, \mathcal{P})$ such that $\mathcal{C}$ is an infinite set of individual constants and $\mathcal{P}$ is a set of predicate letters. Each element $P$ of $\mathcal{P}$ is assumed to have a corresponding finite arity, and $=\equiv \in \mathcal{P}$ is a binary predicate. Given a signature $\mathcal{S}$, its cardinality is the cardinality of the set $\mathcal{C} \cup \mathcal{P}$.

Henceforth, we implicitly assume the usual definitions of such syntactic notions as term, formula, bound/free occurrence of a variable, sentence etc. – but with the proviso that formulas with void quantifiers are not allowed. Given a signature $\mathcal{S}$, we shall denote the set of terms generated by $\mathcal{S}$ by $\text{Term}(\mathcal{S})$. Likewise, the set of formulas and the set of sentences generated by $\mathcal{S}$ will be denoted by $\text{Form}(\mathcal{S})$ and $\text{Sent}(\mathcal{S})$.

For the sake of simplicity, the deductive systems and the formal semantics of the logics discussed below will be formulated exclusively in terms of sentences. This is the reason why we have assumed right from the outset that languages must always have an infinite stock of individual constants – for otherwise we could be prevented from applying some quantifier rules due to the lack of enough constants. However, none of the following definitions and results depend essentially on this decision (see Remark 25).

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8Hereafter $x$, $x_1$, $x_2$ will be used as metavariables ranging over $\mathcal{V}$, $c$, $c_1$, $c_2$, … as metavariables ranging over $\mathcal{C}$, $t$, $t_1$, $t_2$, … as metavariables ranging over $\text{Term}(\mathcal{S})$, and $A$, $B$, $C$, … as metavariables ranging over $\text{Form}(\mathcal{S})$. Given $t, t_1, t_2 \in \text{Term}(\mathcal{S})$, we will use the notation $t(t_2/t_1)$ to denote the result of replacing every occurrence of $t_1$ in $t$ (if any) by $t_2$. Similarly, $A(t/x)$ will denote the formula that results by replacing every free occurrence of $x$ in $A$ by $t$. Springer
Definition 1 Let $S$ be a signature, $c \in C$, and $A, B, C \in \text{Sent}(S)$. The logic $QLET_r$ is defined over $S$ by the following natural deduction rules:

$$
\frac{A}{A \land B} \quad \text{\text{\text{-I}}}
$$

$$
\frac{A \land B}{A} \quad \text{\text{\text{-E}}}
$$

$$
\frac{A \lor B}{A} \quad \text{\text{\text{-I}}}
$$

$$
\frac{B \lor A}{A \lor B} \quad \text{\text{\text{-I}}}
$$

$$
\frac{A \lor B}{C} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg A}{\neg (A \land B)} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg B}{\neg (A \land B)} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg (A \land B)}{C} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg (A \lor B)}{\neg A} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg (A \lor B)}{\neg B} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg A}{\neg \neg A} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg A}{A} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\circ A}{\neg A} \quad \text{\text{\text{EXP}}}^\circ
$$

$$
\frac{\circ A}{A \lor \neg A} \quad \text{\text{\text{PEM}}}^\circ
$$

$$
\frac{\circ A}{\bullet A} \quad \text{\text{\text{Cons}}}
$$

$$
\frac{\circ A \lor \bullet A}{\bullet A} \quad \text{\text{\text{Comp}}}
$$

$$
\frac{B \lor A(c/x)}{B \lor \forall x A} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\forall x A}{A(c/x)} \quad \text{\text{\text{-E}}}
$$

$$
\frac{A(c/x)}{\exists x A} \quad \text{\text{\text{I}}}
$$

$$
\frac{\exists x A}{C} \quad \text{\text{\text{-E}}}
$$

$$
\frac{\neg A(c/x)}{\neg \forall x A} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\forall x A}{C} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg A(c/x)}{\exists x A} \quad \text{\text{\text{-I}}}
$$

$$
\frac{\neg \exists x A}{\neg A(c/x)} \quad \text{\text{\text{-I}}}
$$

$$
\frac{c = c}{I}
$$

$$
\frac{c_1 = c_2}{A(c_1/x)} \quad \text{\text{\text{E}}}
$$

$$
\frac{A(c_2/x)}{A'} \quad \text{\text{\text{AV}}}
$$
In rules $\forall E$, $\neg \land E$, $\exists E$, and $\neg \forall E$, the hypotheses enclosed within brackets get discharged upon the application of the corresponding rule. This is indicated by the occurrence of the numerical index $i$ both in the rule label and in the hypotheses being discharged. For instance, $\neg A(c/x)$ gets discharged upon the application $\neg \forall E$. The conclusion of each rule depends on whatever undischarged hypotheses its premises depend on, while any undischarged hypothesis depends on itself. For example, if $C$ is the conclusion of an application of $\exists E$, then it depends on whatever sentences $\exists x A$ and the previous occurrence of $C$ depend, except for $A(c/x)$.

In $\forall I$, $c$ must not occur in $A$ or $B$, nor in any hypothesis on which $B \lor A(c/x)$ depends; and in $\neg \exists I$, $c$ must not occur in $A$ nor in any hypothesis on which $\neg A(c/x)$ depends. In $\exists E$ and $\neg \forall E$, $c$ must occur neither in $A$ or $C$, nor in any hypothesis on which $C$ depends, except $A(c/x)$ ($\neg A(c/x)$). Finally, in $\forall I$, $A'$ denotes any alphabetic variant of $A$.\footnote{A formula is an alphabetic variant of another if they only differ in (some of) their bound variables. See [36, pp. 126-7] for details.}

Given a signature $S$ and $\Gamma \cup \{A\} \subseteq \text{Sent}(S)$, the definition of a deduction of $A$ from $\Gamma$ in $QLET_F$ is the usual one [see, e.g., 37 Ch. 2]. It suffices to say here that a derivation $\Theta$ is a tree of labeled sentences in which each node either is an element of the set of premises $\Gamma$ or results from preceding nodes by the application of one of the rules above, and whose bottommost sentence is the conclusion of $\Theta$. We shall use the notation $\Gamma \vdash_A S$ to express that there exists a derivation in $QLET_F$ from the premises in $\Gamma$ and whose conclusion is $A$, omitting the subscript $S$ if there is no risk of confusion.

**Proposition 2** The usual universal generalization rule:

$$\frac{A(c/x)}{\forall x A}$$

(where $c$ occurs neither in $A$ nor in any hypothesis on which $A(c/x)$ depends) can be derived in $QLET_F$.

**Proof** It suffices to consider the following derivation:

$$\frac{A(c/x)}{\forall x A \lor A(c/x)} \quad \forall I$$

$$\frac{\forall x A \lor \forall x A}{\forall x A \lor \forall x A} \quad \forall I$$

$$\frac{[\forall x A]_1 [\forall x A]_1}{[\forall x A]_1 \lor E_1}$$

\[ \square \]

**Proposition 3** Consider the following eight rules:

$$\frac{\forall x \neg A}{\neg \exists x A} \quad \frac{\neg \exists x A}{\forall x \neg A} \quad \frac{\exists x \neg A}{\neg \forall x A} \quad \frac{\neg \forall x A}{\exists x \neg A}$$

$$\frac{\forall x A}{\neg \exists x \neg A} \quad \frac{\neg \exists x \neg A}{\forall x A} \quad \frac{\exists x A}{\neg \forall x \neg A} \quad \frac{\neg \forall x \neg A}{\exists x A}$$

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1. Each one of these rules can be derived in $QLET_F$.
2. The first four rules, together with $\forall I, \forall E, \exists I,$ and $\exists E,$ are sufficient for deriving $\neg\forall I, \neg\forall E, \neg\exists I,$ and $\neg\exists E$.

Proof Left to the reader. \qed

**Proposition 4** The following inferences hold in $QLET_F$:

1. $A, \neg A \vdash \bullet A$.
2. $\vdash A \lor \neg A \lor \bullet A$.

Proof Left to the reader. \qed

The inferences (1) and (2) are dual, respectively, to rules $PEM^o$ and $EXP^o$. While (1) means that inconsistency implies $\bullet A$, (2) means that incompleteness implies $\bullet A$.

### 3.1 First-Order Valuation Semantics for $QLET_F$

**Definition 5** Let $\mathcal{S}$ be a signature. A $\mathcal{S}$-structure $\mathfrak{A}$ is a pair $(\mathcal{D}, \mathcal{I})$ such that $\mathcal{D}$ is a non-empty set (the domain of $\mathfrak{A}$) and $\mathcal{I}$ is an interpretation function such that:

1. For every constant $c \in \mathcal{C}$, $\mathcal{I}(c) \in \mathcal{D}$;
2. For every $n$-ary predicate $P \in \mathcal{P}$, $\mathcal{I}(P)$ is a pair $\langle P^+_A, P^-_A \rangle$ such that $P^+_A \cup P^+_A \subseteq \mathcal{D}^n$.
3. $\langle a, a \rangle = \{ (a, a) : a \in \mathcal{D} \}$.

Given an $\mathcal{S}$-structure $\mathfrak{A} = (\mathcal{D}, \mathcal{I})$, we shall write $c^\mathfrak{A}$ and $P^\mathfrak{A}$ instead of respectively $\mathcal{I}(c)$ and $\mathcal{I}(P)$.

According to the definition above, individual constants are interpreted as elements of the domain $\mathcal{D}$ of $\mathfrak{A}$, while predicate letters are interpreted as pairs of relations over $\mathcal{D}$: each predicate letter $P$, including $\equiv$, is assigned both an extension, $P^\mathfrak{A}$, and an anti-extension, $P^-\mathfrak{A}$, where $P^\mathfrak{A}$ and $P^-\mathfrak{A}$ are intended to express, respectively, the presence of positive and negative evidence (or information) for the atomic sentences of the relevant language.

Notice that, given an $n$-ary predicate letter $P$, although $P^\mathfrak{A} \cup P^-\mathfrak{A}$ must be a subset of $\mathcal{D}^n$, there are no constraints to the effect that $P^\mathfrak{A} \cup P^-\mathfrak{A} = \mathcal{D}^n$, nor to the effect that $P^\mathfrak{A} \cap P^-\mathfrak{A} = \emptyset$. As will become clear below, this means that it is not required that exactly one of $P(c_1, \ldots, c_n)$ and $\neg P(c_1, \ldots, c_n)$ receive a designated value, for every constant $c_1, \ldots, c_n$: it may be that neither $P(c_1, \ldots, c_n)$ nor $\neg P(c_1, \ldots, c_n)$ holds in $\mathfrak{A}$, or that both $P(c_1, \ldots, c_n)$ and $\neg P(c_1, \ldots, c_n)$ do. Notice further that although $\equiv$ is also interpreted as a pair of relations, as any other predicate letter, its extension, $\equiv^\mathfrak{A}$, must be the identity relation on $\mathcal{D}$, which is meant to ensure that $\equiv$ satisfies the most basic properties of identity. Nonetheless, nothing prevents there existing some $a \in \mathcal{D}$ such that $(a, a) \in \equiv_{-\mathfrak{A}}$, in which case both $c \equiv c$ and $\neg(c \equiv c)$ will receive a designated value, for some $c$ (we return to the interpretation of identity in Remark 26).
Definition 6 Let \( S = \langle C, \mathcal{P} \rangle \) be a signature and let \( \mathfrak{A} \) be an \( S \)-structure. The **diagram signature** \( S_{\mathfrak{A}} \) of \( \mathfrak{A} \) is the pair \( \langle C_{\mathfrak{A}}, \mathcal{P} \rangle \) such that \( C_{\mathfrak{A}} = C \cup \{ \bar{a} : a \in D \} \); that is, \( S_{\mathfrak{A}} \) is the signature that results from \( S \) by introducing a new individual constant \( \bar{a} \) for each element \( a \) of the domain. The language generated by \( S_{\mathfrak{A}} \) will be called the **diagram language** of \( \mathfrak{A} \), and we shall use the notation \( \mathfrak{A} \) to denote the \( S_{\mathfrak{A}} \)-structure that is just like \( \mathfrak{A} \) except that \( \bar{a}_{\mathfrak{A}} = a \), for every \( a \in D \).

The definition of a \( QLET_F \)-structure resembles very much the corresponding definition in classical first-order logic, except for the interpretation given to the predicate letters in terms of extensions and anti-extensions. Unlike classical logic, however, specifying a \( QLET_F \)-structure is not sufficient to determine the semantic values of every sentence of the relevant language. This is a consequence of the fact that in \( QLET_F \) some sentential connectives, viz. \( \neg, \circ, \text{ and } \bullet \), are non-deterministic, which means that the semantic value of, say, \( A \) does not always determine the semantic values of \( \neg A, \circ A, \) and \( \bullet A \). For instance, even when \( A \) holds in a certain structure \( \mathfrak{A} \), there are circumstances in which \( \circ A \) holds in \( \mathfrak{A} \), and circumstances in which \( \circ A \) does not hold in \( \mathfrak{A} \). In order to comply with the non-deterministic nature of \( \neg, \circ, \) and \( \bullet \), it is thus necessary to supplement a structure with a valuation function that is meant to interpret formulas whose semantic values are not determined by a \( QLET_F \)-structure alone.

Definition 7 Let \( S = \langle C, \mathcal{P} \rangle \) be a signature and \( \mathfrak{A} \) be an \( S \)-structure. A mapping \( v : \text{Sent}(S_{\mathfrak{A}}) \rightarrow \{0, 1\} \) is an **\( \mathfrak{A} \)-valuation** if it satisfies the following conditions:

1. \( v(P(c_1, \ldots, c_n)) = 1 \) iff \( \langle c_{1\mathfrak{A}}, \ldots, c_{n\mathfrak{A}} \rangle \in P_{+\mathfrak{A}} \), for every \( c_1, \ldots, c_n \in C_{\mathfrak{A}} \);
2. \( v(\neg P(c_1, \ldots, c_n)) = 1 \) iff \( \langle c_{1\mathfrak{A}}, \ldots, c_{n\mathfrak{A}} \rangle \in P_{-\mathfrak{A}} \), for every \( c_1, \ldots, c_n \in C_{\mathfrak{A}} \);
3. \( v(A \land B) = 1 \) iff \( v(A) = 1 \) and \( v(B) = 1 \);
4. \( v(A \lor B) = 1 \) iff \( v(A) = 1 \) or \( v(B) = 1 \);
5. \( v(\neg(A \land B)) = 1 \) iff \( v(\neg A) = 1 \) or \( v(\neg B) = 1 \);
6. \( v(\neg(A \lor B)) = 1 \) iff \( v(\neg A) = 1 \) and \( v(\neg B) = 1 \);
7. \( v(\neg\neg A) = 1 \) iff \( v(A) = 1 \);
8. If \( v(\circ A) = 1 \), then \( v(A) = 1 \) iff \( v(\neg A) = 0 \);
9. \( v(\circ A) = 1 \) iff \( v(\bullet A) = 0 \);
10. \( v(\forall x A) = 1 \) iff \( v(A(\bar{a}/x)) = 1 \), for every \( a \in D \);
11. \( v(\exists x A) = 1 \) iff \( v(A(\bar{a}/x)) = 1 \), for some \( a \in D \);
12. \( v(\neg\forall x A) = 1 \) iff \( v(\neg A(\bar{a}/x)) = 1 \), for some \( a \in D \);
13. \( v(\neg\exists x A) = 1 \) iff \( v(\neg A(\bar{a}/x)) = 1 \), for every \( a \in D \);
14. If \( A' \) is an alphabetic variant of \( A \), then \( v(A') = v(A) \);
15. Let \( A \in \text{Form}(S_{\mathfrak{A}}) \) be such that no variables other than \( x \) are free in \( A \), and let \( c_1, c_2 \in C_{\mathfrak{A}} \). If \( c_{1\mathfrak{A}} = c_{2\mathfrak{A}} \) and \( v(A(c_1/x)) = v(A(c_2/x)) \), then \( v(\#A(c_1/x)) = v((\#A(c_2/x)) \) (where \# \in \{\neg, \circ, \bullet\}).

Definition 8 Let \( S \) be a signature. An **\( S \)-interpretation** is a pair \( \langle \mathfrak{A}, v \rangle \) such that \( \mathfrak{A} \) is an \( S \)-structure and \( v \) is an \( \mathfrak{A} \)-valuation. A sentence \( A \) is said to **hold in** the interpretation \( \langle \mathfrak{A}, v \rangle \) \( (\mathfrak{A}, v \vdash A) \) if and only if \( v(A) = 1 \); and a set of sentences \( \Gamma \) is said to hold in \( \langle \mathfrak{A}, v \rangle \) \( (\mathfrak{A}, v \vdash \Gamma) \) if and only if every element of \( \Gamma \) holds in \( \langle \mathfrak{A}, v \rangle \). \( \Gamma \) is said to have a **model** if it holds in some interpretation. Finally, \( A \) is a **semantic**
consequence of $\Gamma$ ($\Gamma \models A$) if and only if $\mathcal{A}, v \models A$ whenever $\mathcal{A}, v \models \Gamma$ for every interpretation $(\mathcal{A}, v)$.

Notice that a valuation assigns either 1 or 0 to each sentence of the diagram language of $\mathcal{S}$, which, of course, includes all sentences in Sent($\mathcal{S}$). Resorting to diagram languages is required to make sure that the quantifiers range over all the objects of the domain of $\mathcal{A}$. Since $\forall$ and $\exists$ are given a substitutional interpretation – i.e., the semantic value of a formula $\forall x A$ depends on the semantic values of all the substitution instances of $A$ – we need to extend the original language with a new individual constant for each element of the domain $\mathcal{D}$, and to extend the interpretation function of the original structure accordingly. Concerning this point, it is also worth emphasizing that though, as per clauses (10)-(13), the quantifiers are given a substitutional interpretation, the fact that we have defined valuations as functions whose domain is the set of sentences of the diagram language of $\mathcal{A}$ – in which each individual $a$ of $\mathcal{D}$ has a canonical name $\overline{a}$ – means that our substitutional $\forall$ and $\exists$ are equivalent to their objectual analogues with respect to the sentences of the language – which, in view of our decision to disregard open formulas, means that there are no significant differences between either interpretation of the quantifiers.

Notice further that clause (14) explicitly requires that any two formulas that differ only in some of their bound variables must be assigned the same value by a valuation. This clause is the counterpart of rule $AV$, without which formulas such as $\circ \forall x A$ and $\circ \forall y A(y/x)$ (where $y$ does not occur in $A$) cannot be proven to be equivalent. Finally, clause (15) is required for similar reasons: had it been missing, nothing would prevent $\circ A(c_1/x)$ and $\circ A(c_2/x)$ from being assigned different values by a valuation, even though $A(c_1/x)$ and $A(c_2/x)$ had the same value and $c_1$ and $c_2$ were interpreted as the same individual of the domain.

Remark 9 Back in Section 2 we invoked the notion of a database to clarify the intended, information-based, interpretation of $QLET_F$. Let us say a few words about how that notion, taken in the broadest sense, relates to the semantic concepts just defined – viz., structures, valuations, and interpretations. First off, recall that in $QLET_F$, $v(A) = 1$ is meant to express that there is positive information $A$, and $v(\neg A) = 1$, that there is negative information $A$, while $v(A) = 0$ and $v(\neg A) = 0$ indicate respectively that there is no positive information $A$ and that there is no negative information $A$. Moreover, $v(\circ A) = 1$ is meant to express that the information $A$, either positive or negative, is reliable, and $v(\circ A) = 0$ that no information about the reliability of $A$ is available. Because in $QLET_F$ valuations are always associated to a certain structure, they are thus meant to indicate the availability of positive and

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10 If void quantifiers were allowed, there would be sentences that are intuitively equivalent but which could receive different semantic values even in the presence of clause (14) – e.g., $\circ \forall x P.c$ and $\circ P.c$. That is the reason why we excluded “formulas” in which void quantifiers occur from the set of formulas. An alternative approach would be to allow for void quantifiers, but extend the definition of alphabetic variants in such a way that formulas that differ by the presence of one or more void quantifiers would also count as alphabetic variants of one another.

11 Later on, in Proposition 10, we will prove a generalization of clause (15).
negative information \textit{relative to a certain structure}. It is the combination of these two elements, valuations and structures – which together make up an interpretation – that we take to be the formal counterpart of what we previously called a ‘database’.

Since databases are taken here to correspond to interpretations, clauses (1)-(15) of Definition 7 represent certain ‘closure constraints’ for pieces of information, either positive, negative, reliable, or unreliable, to be included in a given database \( X \). For instance, if \( X \) includes positive information \( A \) and positive information \( B \), then clause (3) tells us that it also includes positive information \( A \land B \); and if \( X \) includes negative information \( A \lor B \), then clause (6) tells us that it includes both negative information \( A \) and negative information \( B \). Similar observations apply to all the remaining clauses, but it is worth commenting a bit more on how this interpretation fares with (8)-(13).

First, neither (8) nor any of the remaining semantic clauses establish sufficient conditions for \( \circ A \) to be included in a database \( X \), though it imposes some constraints on the database if it does include \( \circ A \) – namely, \( X \) cannot contain both positive and negative information \( A \), nor can it fail to contain either. That there are no sufficient conditions for including \( \circ A \) in a database expresses the idea that marking some pieces of information as reliable is not something that can be done from within a given database, and corresponds to the fact that \( QLET_F \) does not contain introduction rules for \( \circ \). Clause (9), on the contrary, only establishes sufficient, though not necessary, conditions for having \( \bullet A \) included in a database. It follows from (8) and (9) that if \( X \) either includes both positive and negative information \( A \) or if it includes no information about \( A \) (i.e., it includes neither \( A \) nor \( \neg A \)), then \( \bullet A \) belongs to \( X \) (see Proposition 4).

Second, clauses (10)-(13) state necessary and sufficient conditions for \( X \) to include pieces of information that concern all or some individuals among the putative objects the database is supposed to store information about. For instance, in order for \( X \) to include positive information \( \forall x A \) it is both necessary and sufficient that it includes positive information \( A(\bar{a}/x) \), for all individuals \( a \); and for \( X \) to include negative information \( \forall x A \) it is necessary and sufficient that it includes negative information \( A(\bar{a}/x) \), for at least one individual \( a \). It is also worth noting that in view of the proposal of understanding interpretations as the formal counterparts of databases in the way just indicated, we may rephrase the definition of logical consequence in Definition 8 as preservation of information in the following way: \( \Gamma \models A \) if and only if every database that includes every piece of information in \( \Gamma \) also includes the information \( A \).

Let us move on now to the task of proving the completeness of \( QLET_F \).

### 4 Soundness and Completeness of \( QLET_F \)

We start by establishing that \( QLET_F \) is sound with respect to the class of all \( QLET_F \)-structures, leaving the proof of its completeness to the next subsection.\(^{\text{12}}\)

\(^{\text{12}}\)The proof of the completeness of \( QLET_F \) below is based on the one presented in [13] for the logic \( QmbC \), but requires a few adjustments in order to comply with specificities of \( QLET_F \). In fact, the
4.1 Soundness

**Proposition 10** Let $\mathcal{S} = \langle C, \mathcal{P} \rangle$ be a signature and let $\langle \mathcal{A}, v \rangle$ be an $\mathcal{S}$-interpretation. Let $A \in \text{Form}(\mathcal{S}_A)$ be such that $A$ has at most $x$ free, and $c_1, c_2 \in C_A$. If $c_1^\mathcal{A} = c_2^\mathcal{A}$, then $v(A(c_1/x)) = v(A(c_2/x))$.

*Proof* The result follows by induction on the complexity of $A$ and uses clause (15) of Definition 7.

It is worth noting at this point that if the language of $QLET_F$ did not include the connectives $\circ$ and $\bullet$, then the proof of the lemma above would not require clause (15) of Definition 7. In effect, it can be proven, without using that clause, that:

**Proposition 11** Let $A \in \text{Form}(\mathcal{S}_A)$ be such that (i) no variables other than $x$ are free in $A$ and (ii) $A$ is $\circ$- and $\bullet$-free. Let $c_1, c_2 \in C_A$. If $c_1^\mathcal{A} = c_2^\mathcal{A}$, then $v(A(c_1/x)) = v(A(c_2/x))$.

*Proof* The result follows by induction on the complexity of $A$: If $A$ is an atomic formula, then the result follows immediately from Definition 7(1). If $A$ has the form $\neg B$, then there are a few cases: If $B$ is an atomic formula, then the result follows from Definition 7(2); if $B$ has the form $\neg C$, then the result follows from the induction hypothesis (IH) and Definition 7(7); if $B$ has the form $A \land B$ or $A \lor B$, then the result follows from (IH) and Definition 7(5) and (6); and if $B$ has the form $\forall x C$ or $\exists x C$, then the result follows from (IH) and Definition 7(12) and (13). The remaining cases (viz., $A = B \land C, A = B \lor C, A = \forall x B$, and $A = \exists x B$) are immediate consequences of (IH) and the corresponding clauses in Definition 7—i.e., clauses (3), (4), (10), and (11), respectively.

Moreover, if the language of $QLET_F$ did not include $\circ$ and $\bullet$, clause (14) would be unnecessary as well, for it would then be provable by a straightforward induction on the complexity of $A$.

The following technical result, which will be used in the proof of the soundness of $QLET_F$, is an immediate consequence of Proposition 10 above:

**Corollary 12** Let $\mathcal{S} = \langle C, \mathcal{P} \rangle$ be a signature such that $c \in C$. Let $\mathcal{A} = \langle D, \mathcal{T} \rangle$ be an $\mathcal{S}$-structure and assume that $a \in D$. Let $\mathcal{A}' = \langle D, \mathcal{T}' \rangle$ be the $\mathcal{S}$-structure such that $\mathcal{T}'$ is just like $\mathcal{T}$ except that $\mathcal{T}'(c) = a$. Let $v$ be an $\mathcal{A}$-valuation and let $v'$ be an $\mathcal{A}'$-valuation that agrees with $v$ on all sentences of $\text{Sent}(\mathcal{S})$ in which $c$ does not occur. If no variables other than $x$ are free in $A \in \text{Form}(\mathcal{S})$ and $c$ does not occur in $A$, then $v(A(a/x)) = v'(A(c/x))$.

*Proof* Since $\overline{a}^{\mathcal{A}'} = c^{\mathcal{A}'}$, it follows from Proposition 10 that $v'(A(\overline{a}/x)) = v'(A(c/x))$. But since $c$ does not occur in $A$, $v(A(\overline{a}/x)) = v'(A(c/x))$. 

Semantics and the completeness proof of $QmbC$ in [13] turn out to be special cases of the semantics and the proof presented here.
Theorem 13 (Soundness Theorem) Let S be a signature and \( \Gamma \cup \{ A \} \subseteq \text{Sent}(S) \). If \( \Gamma \vdash A \), then \( \Gamma \models A \).

Proof Let \( \Theta \) be a derivation of \( A \) from \( \Gamma \) in \( \text{QLET}_F \) and let \( n \) be the number of nodes in \( \Theta \). If \( n = 1 \), then either \( A \in \Gamma \) or \( A \) is the result of an application of \( \text{Comp} \) or \( \text{I} \). If \( A \in \Gamma \), then \( \Gamma \models A \), since \( \models \) is reflexive. If \( A = oB \lor B \) results from an application of \( \text{Comp} \), then \( v(oB) = 1 \) if and only if \( v(B) \neq 1 \), for every \( S \)-interpretation \( \langle \mathfrak{A}, v \rangle \) (by Definition 7(8)). Hence, \( v(oB) = 1 \) or \( v(B) = 1 \). By Definition 7(4), \( v(oB \lor B) = 1 \). Thus, \( \mathfrak{A}, v \models oB \lor B \), and therefore \( \models A \). If, on the other hand, \( A = c \Rightarrow c \) results from an application of \( \text{I} \), then \( \langle c^\mathfrak{A}, c^\mathfrak{A} \rangle \in \models_+ \), for every structure \( \mathfrak{A} \) (by Definition 7(5)). By Definition 7(1), \( v(c \Rightarrow c) = 1 \), for every \( \mathfrak{A} \)-valuation \( v \). Hence, \( \mathfrak{A}, v \models c \Rightarrow c \), and therefore \( \models A \).

Suppose that \( n > 1 \) and that the result holds for every derivation \( \Theta' \) with fewer nodes than \( \Theta \). We shall prove that \( \Gamma \models A \). Since \( n > 1 \), \( A \) results from an application of one of the rules of \( \text{QLET}_F \) other than \( \text{Comp} \) and \( \text{I} \). In these cases, the result follows almost immediately from the lemmas above together with the corresponding clauses of Definition 7. For instance:

3. Let \( A = C \lor \forall x B \) and suppose that it results from an application of rule \( \forall I \) to \( C \lor B(c/x) \). Hence, there is a derivation \( \Theta' \) of \( C \lor B(c/x) \) from \( \Gamma \) in \( \text{QLET}_F \) such that \( \Theta' \) has fewer nodes than \( \Theta \). Let \( \Gamma_0 \subseteq \Gamma \) be set of hypotheses on which \( C \lor B(c/x) \) depends on \( \Theta' \). Clearly, \( \Theta' \) is a derivation of \( C \lor B(c/x) \) from \( \Gamma_0 \) and, given the restrictions upon \( \forall I \), \( c \) occurs neither in \( B, C \), nor in any element of \( \Gamma_0 \). By (IH), \( \Gamma_0 \models C \lor B(c/x) \). Let \( \langle \mathfrak{A}, v \rangle \) be an arbitrary \( S \)-interpretation and suppose that \( \mathfrak{A}, v \models \Gamma_0 \). Let \( a \) be an arbitrary element of \( D \) and consider then an interpretation \( \langle \mathfrak{A}', v' \rangle \) such that \( \mathfrak{A}' \) is just like \( \mathfrak{A} \) except that \( \mathfrak{A}'(c) = a \) and \( v' \) agrees with \( v \) on all sentences in which \( c \) does not occur. Since \( c \) does not occur in \( \Gamma_0 \), \( v(D) = 1 \) if and only if \( v'(D) = 1 \), for every \( D \in \Gamma_0 \). Thus, \( \mathfrak{A}', v' \models \Gamma_0 \), and therefore \( \mathfrak{A}', v' \models C \lor B(c/x) \) (by Corollary 12, \( v(C^\mathfrak{A}/x) \lor B(a/x)) = 1 \).

Now, since \( C(a/x) = C \), it follows that \( v(C \lor B(a/x)) = 1 \). By Definition 7(4), \( v(C) = 1 \) or \( v(B(a/x)) = 1 \). But since \( a \) was arbitrary, it follows that \( v(C) = 1 \) or \( v(B(a/x)) = 1 \), for every \( a \in D \). By Definition 7(10), \( v(C) = 1 \) or \( v(\forall x B) = 1 \), and by Definition 7(4) again, \( v(C \lor \forall x B) = 1 \). That is: \( \mathfrak{A}, v \models A \).

4. Let \( A = \neg B(c/x) \) and suppose that it results from an application of \( \neg \exists x \) to \( \neg \exists x B \). Hence, there is a derivation \( \Theta' \) of \( \neg \exists x B \) from \( \Gamma \) in \( \text{QLET}_F \) such that \( \Theta' \) has fewer nodes than \( \Theta \). By (IH), \( \Gamma \models \neg \exists x B \). Suppose that \( \mathfrak{A}, v \models \Gamma \). Thus, \( v(\neg(\exists x B)) = 1 \). By Definition 7(13), \( v(\neg B(a/x)) = 1 \), for every \( a \in D \). Let \( b \in D \) be such that \( c^\mathfrak{A} = b \). Therefore, \( v(\neg(B(b/x)) = 1 \). Since \( b^\mathfrak{A} = c^\mathfrak{A} \), it follows by Proposition 10 that \( v(\neg B(c/x)) = 1 \). Therefore, \( \mathfrak{A}, v \models A \).

5. Let \( A = B(c_2/x) \) and suppose that it results from an application of \( \neg E \) to \( c_1 \Rightarrow c_2 \) and \( B(c_1/x) \). Hence, there are derivations \( \Theta_1 \) and \( \Theta_2 \) of, respectively, \( c_1 \Rightarrow c_2 \) and \( B(c_1/x) \) from \( \Gamma \). By (IH), \( \Gamma \models c_1 \Rightarrow c_2 \) and \( \Gamma \models B(c_1/x) \). Suppose that \( \mathfrak{A}, v \models \Gamma \). Thus, \( v(c_1 \Rightarrow c_2) = v(B(c_1/x)) = 1 \). As a result, \( c_1^\mathfrak{A} = c_2^\mathfrak{A} \) (by Definition 5(5)). Since \( v(B(c_1/x)) = 1 \), it follows by Proposition 10 that \( v(B(c_2/x)) = 1 \).
The proof of the remaining cases is left to the reader.

4.2 Completeness

Let us now prove the completeness of $QLET_F$. As usual, the proof will be divided up in two separate steps. First, we shall prove that, given a set of sentences $\Gamma$ that does not prove a certain sentence $A$, $\Gamma$ can be extended to a set $\Delta$ that (i) still does not prove $A$, (ii) is closed under $\vdash$, and (iii) has witnesses for every universal and existential sentence – in the sense that if $B(c/x) \in \Delta$, for every constant $c$, then $\forall x B \in \Delta$, and if $\exists x B \in \Delta$, then there is an individual constant $c$ such that $B(c/x) \in \Delta$. Second, we shall prove that, given a set $\Delta$ satisfying (i)-(iii), it is possible to construct a structure $\mathfrak{A}$ and define an $\mathfrak{A}$-valuation $v$ such that all and only the sentences belonging to $\Delta$ hold in $\mathfrak{A}$ and $v$.

Definition 14 Let $\mathcal{S} = (\mathcal{C}, \mathcal{P})$ be a signature and $\Delta \subseteq \text{Sent}(\mathcal{S})$. $\Delta$ is a Henkin set if and only if for every $B \in \text{Sent}(\mathcal{S})$ and $x \in \forall$ (i) $\Delta \vdash \exists x B$ iff $\Delta \vdash B(c/x)$, for some $c \in \mathcal{C}$; and (ii) $\Delta \vdash \forall x B$ iff $\Delta \vdash B(c/x)$, for every $c \in \mathcal{C}$.

Definition 15 Let $\mathcal{S}$ be a signature and $\Delta \cup \{A\} \subseteq \text{Sent}(\mathcal{S})$. $\Delta$ is a regular set if and only if: (i) $\Delta$ is non-trivial, i.e., $\Delta \not\vdash A$, for some $A \in \text{Sent}(\mathcal{S})$; (ii) $\Delta$ is closed, i.e., if $\Delta \vdash A$, then $A \in \Delta$, for every $A \in \text{Sent}(\mathcal{S})$; and (iii) $\Delta$ is a disjunctive set, i.e., if $\Delta \vdash A \lor B$, then $\Delta \vdash A$ or $\Delta \vdash B$.

Lemma 16 Let $\mathcal{S}$ be a signature and $\Delta \cup \{A\} \subseteq \text{Sent}(\mathcal{S})$. If $\Delta$ is a regular Henkin set, then:

1. $B \land C \in \Delta$ iff $B \in \Delta$ and $C \in \Delta$;
2. $B \lor C \in \Delta$ iff $B \in \Delta$ or $C \in \Delta$;
3. $\neg(B \land C) \in \Delta$ iff $\neg B \in \Delta$ or $\neg C \in \Delta$;
4. $\neg(B \lor C) \in \Delta$ iff $\neg B \in \Delta$ and $\neg C \in \Delta$;
5. $\neg\neg B \in \Delta$ iff $B \in \Delta$;
6. $oB \in \Delta$ iff $\bullet B \notin \Delta$;
7. If $\neg B \in \Delta$, then $B \in \Delta$ iff $\neg B \notin \Delta$;
8. If $B'$ is an alphabetic variant of $B$, then $B' \in \Delta$ iff $B \in \Delta$;
9. $\forall x B \in \Delta$ iff $B(c/x) \in \Delta$, for every $c \in \mathcal{C}$;
10. $\exists x B \in \Delta$ iff $B(c/x) \in \Delta$, for some $c \in \mathcal{C}$;
11. $\neg\forall x B \in \Delta$ iff $\neg B(c/x) \in \Delta$, for some $c \in \mathcal{C}$;
12. $\neg\exists x B \in \Delta$ iff $\neg B(c/x) \in \Delta$, for every $c \in \mathcal{C}$.

Proof (1)-(8) are straightforward consequences of the assumption that $\Delta$ is a regular set together with the rules of $QLET_F$. (9) and (10) follow immediately from rules $\forall E$ and $\exists I$ and the hypothesis that $\Delta$ is a Henkin set. As for (11) and (12), they follow respectively from (10) and (9) and the fact that $\neg\forall x B$ and $\exists x \neg B$, and $\neg\exists x B$ and $\forall x \neg B$, are derivable from one another in $QLET_F$.

We can now prove that if $\Gamma$ is such that $\Gamma \not\vdash A$, it can be extended to a regular Henkin set $\Delta$ such that $\Delta \not\vdash A$. 

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Lemma 17 Let $S = \langle C, \mathcal{P} \rangle$ be a signature and $\Gamma \cup \{A\} \subseteq \text{Sent}(S)$. If $\Gamma \not\models A$, then there is a signature $S^+ = \langle C^+, \mathcal{P} \rangle$ and a regular Henkin set $\Delta \subseteq \text{Sent}(S^+)$ such that $C \subseteq C^+$, $\Gamma \subseteq \Delta$, and $\Delta \not\models A$.

Proof The result follows by transfinite induction on the cardinality $\kappa$ of $S$, which, of course, coincides with that of $\text{Sent}(S)$. Let $C^+ = C \cup \{c_{\alpha} : \alpha < \kappa\}$ be such that $\{c_{\alpha} : \alpha < \kappa\} \cap (C \cup \mathcal{P}) = \emptyset$, and let us adopt a fixed enumeration $\langle B_{\alpha} \rangle_{\alpha < \kappa}$ of the sentences in $\text{Sent}(S^+)$.\(^{13,14}\) The following definition, which is taken from [38, p. 464] (see Remark 29 below), will be useful:

\[ \Gamma \vdash \Pi \text{ iff } \Gamma \vdash A_1 \lor \ldots \lor A_n, \text{ for some finite subset } \{A_1, \ldots, A_n\} \text{ of } \Pi \]

where $n \geq 1$, and $\Gamma$ and $\Pi$ are subsets of $\text{Sent}(S^+)$ such that $\Pi \neq \emptyset$ (if $n = 1$, then $A_1 \lor \ldots \lor A_n = A_1$). Note that $\Gamma \not\models \Pi$ iff $\Gamma \not\models A_1 \lor \ldots \lor A_n$, for every $n \geq 1$ and every $A_1, \ldots, A_n \in \Pi$. Hence, if $\Gamma \not\models \Pi$, then $\Gamma \not\models B$, for every $B \in \Pi$.

Now, consider the families $\langle \langle \lambda \alpha \rangle_{\alpha < \kappa} \rangle$ and $\langle \langle \lambda \alpha \rangle_{\alpha < \kappa} \rangle$ of subsets of $\text{Sent}(S^+)$ defined as follows:

1. $\Gamma_0 = \Gamma$ and $\Pi_0 = \{A\}$
   $$
   \begin{cases}
   \Gamma_\alpha & \text{if } \Gamma_\alpha, B_\alpha \not\models \Pi_\alpha \\
   \Gamma_\alpha \cup \{B_\alpha\} & \text{if } \Gamma_\alpha, B_\alpha \not\models \Pi_\alpha \text{; and } B_\alpha \neq \exists x C, \text{ for every } x \in \mathcal{V} \text{ and } C \in \text{Form}(S^+) \\
   \end{cases}
   $$
2. $\Gamma_{\alpha+1} = \Gamma_\alpha \cup \{B_\alpha, C(c_{j_\alpha}/x)\}$ if $\Gamma_\alpha, B_\alpha \not\models \Pi_\alpha$; and $B_\alpha \neq \exists x C, \text{ for some } x \in \mathcal{V} \text{ and } C \in \text{Form}(S^+)$
3. $\Gamma_\lambda = \bigcup_{\alpha < \lambda} \Gamma_\alpha$ if $\lambda$ is a limit ordinal
   $$
   \begin{cases}
   \Pi_\alpha & \text{if } \Gamma_\alpha, B_\alpha \not\models \Pi_\alpha \\
   \Pi_\alpha \cup \{B_\alpha\} & \text{if } \Gamma_\alpha, B_\alpha \not\models \Pi_\alpha \text{; and } B_\alpha \neq \forall x C, \text{ for every } x \in \mathcal{V} \text{ and } C \in \text{Form}(S^+) \\
   \end{cases}
   $$
4. $\Pi_{\alpha+1} = \Pi_\alpha \cup \{B_\alpha \lor C(c_{j_\alpha}/x)\}$ if $\Gamma_\alpha, B_\alpha \not\models \Pi_\alpha$; and $B_\alpha \neq \forall x C, \text{ for some } x \in \mathcal{V} \text{ and } C \in \text{Form}(S^+)$
5. $\Pi_\lambda = \bigcup_{\alpha < \lambda} \Pi_\alpha$ if $\lambda$ is a limit ordinal

In (2.1) and (2.3), $j_\alpha$ is the least ordinal $\nu < \kappa$ such that $c_{j_\nu}$ does not occur in $B_\alpha$, nor in any element of $\Gamma_\alpha \cup \Pi_\alpha$.\(^{15}\) Finally, let $\Delta = \bigcup_{\alpha < \kappa} \Gamma_\alpha$. Clearly, $\Gamma \subseteq \Delta$. We shall prove that $\Delta$ is a regular Henkin set such that $\Delta \not\models A$. It suffices to prove the following facts:

---

\(^{13}\)Lower-case Greek letters are used here as variables ranging over ordinal numbers.

\(^{14}\)The proof thus makes an essential use of the Well-Ordering Theorem, or, equivalently, of the Axiom of Choice.

\(^{15}\)Notice that, for every $0 < \alpha < \kappa$, only finitely many constants in $C^+ \setminus C$ occur in the new formulas introduced in $\langle \Gamma_{\alpha+1} \cup \Pi_{\alpha+1} \rangle \setminus (\Gamma_\alpha \cup \Pi_\alpha)$. Since $\kappa$ is an infinite cardinal, it is always possible to pick a new constant $c_{j_\nu}$ not occurring in $\Gamma_\alpha \cup \Pi_\alpha \cup \{B_\alpha\}$ in items (2.1) and (2.3). Hence, the construction is always possible.
1. For every $\alpha < \beta < \kappa$, $\Gamma_\alpha \subseteq \Gamma_\beta$ and $\Pi_\alpha \subseteq \Pi_\beta$: This is an immediate consequence of the construction.

2. For every $\alpha < \kappa$, $\Gamma_\alpha \not\models \Pi_\alpha$: The proof proceeds by transfinite induction on $\alpha$.
   By the initial hypothesis, $\Gamma_0 \not\models \Pi_0$. Suppose that $\Gamma_\alpha \not\models \Pi_\alpha$ (IH), and let us prove that the result holds for $\alpha + 1$.
   For the sake of contradiction, assume that $\Gamma_{\alpha+1} \models \Pi_{\alpha+1}$. There are two cases: either (i) $\Gamma_\alpha, B_\alpha \not\models \Pi_\alpha$ or (ii) $\Gamma_\alpha, B_\alpha \not\models \Pi_\alpha$. If (i) $\Gamma_\alpha, B_\alpha \not\models \Pi_\alpha$, then $\Gamma_{\alpha+1} = \Gamma_\alpha$. If (i.a) $B_\alpha \not\models \forall x C$, then $\Pi_{\alpha+1} = \Pi_\alpha \cup \{B_\alpha\}$.
   Since we are reasoning under the hypotheses that $\Gamma_{\alpha+1} \models \Pi_{\alpha+1}$ and $\Gamma_\alpha, B_\alpha \not\models \Pi_\alpha$, it follows that $\Gamma_\alpha \models A_1 \lor \ldots \lor A_n \lor B_\alpha$, for some $A_1, \ldots, A_n \in \Pi_\alpha$, and $\Gamma_\alpha, B_\alpha \models A'_1 \lor \ldots \lor A'_m$, for some $A'_1, \ldots, A'_m \in \Pi_\alpha$. Let $B = A_1 \lor \ldots \lor A_n \lor A'_1 \lor \ldots \lor A'_m$. Given that $\Gamma_\alpha, B \models B$ and $\Gamma_\alpha, B_\alpha \models B$ (by applying rule $\forall I$), we may infer $\Gamma_\alpha, B \lor B_\alpha \models B$. But since $\Gamma_\alpha \models A_1 \lor \ldots \lor A_n \lor B_\alpha$, it then follows by $\forall I$ that $\Gamma_\alpha \models B \lor B_\alpha$. Hence, $\Gamma_\alpha \models B$, and so $\Gamma_\alpha \models \Pi_\alpha$, which contradicts (IH).

   If, on the other hand, (i.b) $B_\alpha = \forall x C$, then $\Pi_{\alpha+1} = \Pi_\alpha \cup \{B_\alpha \lor C(c_{i_\alpha}/x)\}$.
   Since we are reasoning under the hypothesis that $\Gamma_{\alpha+1} \models \Pi_{\alpha+1}$ and $\Gamma_{\alpha+1} = \Gamma_\alpha$, it follows that $\Gamma_\alpha \models A_1 \lor \ldots \lor A_n \lor B_\alpha \lor C(c_{i_\alpha}/x)$, for some $A_1, \ldots, A_n \in \Pi_\alpha$.
   Let $B = A_1 \lor \ldots \lor A_n$. Since $c_{i_\alpha}$ does not occur in $\Gamma_\alpha \cup \{B, B_\alpha, C\}$, we can apply rule $\forall I$ to obtain $\Gamma_\alpha \models B \lor B_\alpha$:

\[
\frac{(B \lor B_\alpha) \lor C(c_{i_\alpha}/x)}{\forall I} \quad \frac{(B \lor B_\alpha) \lor \forall x C}{\forall x C = B_\alpha} \quad \frac{B \lor B_\alpha}{B \lor B_\alpha} \quad \frac{[B_{\alpha}]_1}{\forall I} \quad \frac{\forall I}{\forall E_1}
\]

But this result, when combined with the fact that $\Gamma_\alpha, B \models \Pi_\alpha$ and (i) above, entails $\Gamma_\alpha \models \Pi_\alpha$, which contradicts (IH).

   Now, if (ii) $\Gamma_\alpha, B_\alpha \not\models \Pi_\alpha$, then $\Pi_{\alpha+1} = \Pi_\alpha$ and either $\Gamma_{\alpha+1} = \Gamma_\alpha \cup \{B_\alpha\}$ or $\Gamma_{\alpha+1} = \Gamma_\alpha \cup \{\exists x C, C(c_{i_\alpha}/x)\}$. If (ii.a) $\Gamma_{\alpha+1} = \Gamma_\alpha \cup \{B_\alpha\}$, then $\Gamma_\alpha, B_\alpha \models \Pi_\alpha$, by the assumption that $\Gamma_{\alpha+1} \models \Pi_{\alpha+1}$. But this result contradicts (ii). Suppose then that (ii.b) $B_\alpha = \exists x C$, and so that $\Gamma_{\alpha+1} = \Gamma_\alpha \cup \{B_\alpha, C(c_{i_\alpha}/x)\}$. Thus, there exists $A_1, \ldots, A_n \in \Pi_\alpha$ such that $\Gamma_\alpha, B_\alpha, C(c_{i_\alpha}/x) \models A_1 \lor \ldots \lor A_n$. Let $B = A_1 \lor \ldots \lor A_n$. Since $c_{i_\alpha}$ does not occur in $\Gamma_\alpha \cup \{B, B_\alpha, C\}$, we can apply rule $\exists E$ to obtain $\Gamma_\alpha, B_\alpha \models B$:

\[
\frac{\exists x C}{B \lor B_\alpha} \quad \frac{B_\alpha}{B \lor B_\alpha} \quad \frac{B}{\exists E_1}
\]

But this result entails $\Gamma_\alpha, B_\alpha \models \Pi_\alpha$, which contradicts (ii).

   Finally, by assuming that $\Gamma_\alpha \not\models \Pi_\alpha$, for every $\alpha < \lambda$ (IH), let us prove that $\Gamma_\lambda \not\models \Pi_\lambda$, for every limit ordinal $\lambda < \kappa$. For the sake of contradiction, suppose then that $\Gamma_\lambda \models \Pi_\lambda$. By the compactness of $\models$, the definitions of $\Gamma_\lambda$ and $\Pi_\lambda$, and by item 1 above, there exists $\alpha < \lambda$ such that $\Gamma_\alpha \models \Pi_\alpha$, which contradicts (IH). Hence, $\Gamma_\lambda \not\models \Pi_\lambda$. This concludes the proof of the fact that $\Gamma_\alpha \not\models \Pi_\alpha$, for every $\alpha < \kappa$. 
3. For every \( \alpha < \kappa \), \( \Delta \not\subseteq \Pi_\alpha \) (in particular, \( \Delta \not\subseteq A \)): Suppose that \( \Delta \vdash \Pi_\alpha \), for some \( \alpha < \kappa \). By the compactness of \( \vdash \), the definition of \( \Delta \), and by item 1 above, \( \Gamma_\beta \vdash \Pi_\beta \), for some \( \beta \) such that \( \alpha \leq \beta < \kappa \). But this contradicts item 2.

4. If \( \Delta \vdash C \), then \( C \in \Delta \): Suppose that \( \Delta \vdash C \) and that \( C \notin \Delta \). Let \( \alpha < \kappa \) be such that \( C = B_\alpha \). Since \( B_\alpha \notin \Delta \), \( B_\alpha \notin \Gamma_{\alpha+1} \). Hence, \( \Gamma_\alpha , B_\alpha \vdash \Pi_\alpha \), and so \( \Delta , B_\alpha \vdash \Pi_\alpha \). Therefore, \( \Delta \vdash \Pi_\alpha \), which contradicts item 3 above.

5. If \( \Delta \not\vdash C \lor D \), then \( \Delta \not\vdash C \) or \( \Delta \not\vdash D \): Suppose that \( \Delta \not\vdash C \lor D \) and that \( \Delta \not\vdash C \) and \( \Delta \not\vdash D \). Thus, \( C \notin \Delta \) and \( D \notin \Delta \). Let \( \alpha , \beta < \kappa \) be such that \( B_\alpha = C \) and \( B_\beta = D \). Hence, \( B_\alpha \notin \Gamma_{\alpha+1} \) and \( B_\beta \notin \Gamma_{\beta+1} \), and so \( \Gamma_\alpha , B_\alpha \vdash \Pi_\alpha \) and \( \Gamma_\beta , B_\beta \vdash \Pi_\beta \). Thus, there exists \( A_1, \ldots , A_n \in \Pi_\alpha \) and \( A'_1, \ldots , A'_m \in \Pi_\beta \) such that \( \Gamma_\alpha , B_\alpha \vdash A_1 \lor \cdots \lor A_n \) and \( \Gamma_\beta , B_\beta \vdash A'_1 \lor \cdots \lor A'_m \). Let \( B = A_1 \lor \cdots \lor A_n \lor A'_1 \lor \cdots \lor A'_m \), and let \( y \) be the maximum of \( \alpha \) and \( \beta \). Since \( \Gamma_\alpha , \Gamma_\beta \subseteq \Delta \), it follows that \( \Delta , B_\alpha \vdash B \) and \( \Delta , B_\beta \vdash B \). By rule \( \lor E \), \( \Delta , B_\alpha \lor B_\beta \vdash B \). Since, by the initial hypothesis, \( \Delta \not\vdash B_\alpha \lor B_\beta \), it then follows that \( \Delta \not\vdash B_\alpha \) and so \( \Delta \not\vdash \Pi_y \). But this result contradicts item 3 above.

6. For every \( C \in \text{Sent}(S^+) \), \( \Delta \not\vdash \forall x C \) if and only if \( \Delta \not\vdash C(c/x) \), for every \( c \in C^+ \): It suffices to prove that if \( \Delta \not\vdash C(c/x) \), for every \( c \in C^+ \), then \( \Delta \not\vdash \forall x C \), since the other direction is an immediate consequence of rule \( \forall E \). Let \( \alpha < \kappa \) be such that \( \forall x C = B_\alpha \) and suppose that \( \Delta \not\vdash B_\alpha \). Hence, \( \Gamma_{\alpha+1} \not\vdash B_\alpha \), and so \( B_\alpha \notin \Gamma_{\alpha+1} \). It then follows from the definition of \((\Gamma_\alpha)_{\alpha < \kappa}\) that \( \Gamma_\alpha , B_\alpha \vdash \Pi_\alpha \) and \( \Pi_{\alpha+1} = \Pi_\alpha \cup \{ B_\alpha \lor C(c_j/x) \} \). Suppose that \( \Delta \vdash C(c_j/x) \). Thus, \( \Delta \not\vdash B_\alpha \lor C(c_j/x) \), and so \( \Delta \not\vdash \Pi_{\alpha+1} \), which contradicts item 3 above. Therefore, \( \Delta \not\vdash C(c/x) \), for at least one \( c \in C^+ \).

7. For every \( C \in \text{Sent}(S^+) \), \( \Delta \not\vdash \exists x C \) if and only if \( \Delta \not\vdash C(c/x) \), for some \( c \in C^+ \): As in item 6, we shall only prove the (contrapositive of the left-to-right direction, since the other direction follows immediately by applying rule \( \exists I \). Let \( \alpha < \kappa \) be such that \( \exists x C = B_\alpha \) and suppose that \( \Delta \not\vdash C(c/x) \), for every \( c \in C^+ \). In particular, \( \Delta \not\vdash C(c_j/x) \). Hence, \( \Gamma_{\alpha+1} \not\vdash C(c_j/x) \), and so \( C(c_j/x) \notin \Gamma_{\alpha+1} \). By construction, \( \Gamma_{\alpha+1} = \Gamma_\alpha \) and \( \Gamma_\alpha , B_\alpha \vdash \Pi_\alpha \). By the monotonicity of \( \vdash \), it then follows that \( \Delta , B_\alpha \vdash \Pi_\alpha \). Suppose that \( \Delta \vdash B_\alpha \). By the transitivity of \( \vdash \) and the assumption that \( \Delta \vdash \exists x C \), \( \Delta \vdash \Pi_\alpha \), which contradicts item 3 above. Hence, \( \Delta \not\vdash B_\alpha \) (i.e., \( \Delta \not\vdash \exists x C \)).

Remark 18 Had we been exclusively concerned with enumerable signatures (and so with denumerable languages) the proof of Lemma 17 could be somewhat simplified by replacing \((\Pi_\alpha)_{\alpha < \kappa}\) by the following sequence \((A_n)_{n \in \mathbb{N}}\) of sentences of \( S^+ \):

\[
A_0 = A; \\
A_n = A_n \quad \text{if } \Gamma_n , B_n \not\vdash A_n \\
A_n \lor B_n \quad \text{if } \Gamma_n , B_n \vdash A_n; \text{ and } B_n \not\vdash \forall x C, \text{ for every } x \in \mathcal{V} \text{ and } C \in \text{Form}(S^+) \\
A_n \lor B_n \lor C(c_{j_n}/x) \quad \text{if } \Gamma_n , B_n \vdash A_n; \text{ and } B_n = \forall x C, \text{ for some } x \in \mathcal{V} \text{ and } C \in \text{Form}(S^+) 
\]

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where is \( j_n \) is the least natural number \( k \) such that \( c_k \) does not occur in \( \{ \Gamma_n, A_n, B_n \} \).

As a result, item 1 would give place to: For every \( m < n, \Gamma_m \subseteq \Gamma_n \) and \( A_m \vdash A_n \). This modified version of item 1, along with corresponding reformulations of items 2-7, could then be proven in essentially the same way as above. We opted for the more general, transfinite, version of Lemma 17 in order to be able to state and prove the Löwenheim-Skolem theorems for \( QLET_F \) in their full force (see Corollaries 23 and 24 below).

**Lemma 19** Let \( \mathcal{S} \) be a first-order signature, \( \Gamma \subseteq \text{Sent}(\mathcal{S}) \), and \( c_1, c_2, c_3 \in \mathcal{C} \). Then:

1. \( \Gamma \vdash c_1 \equiv c_1 \).
2. If \( \Gamma \vdash c_1 \equiv c_2 \), then \( \Gamma \vdash c_2 \equiv c_1 \); and
3. If \( \Gamma \vdash c_1 \equiv c_2 \) and \( \Gamma \vdash c_2 \equiv c_3 \), then \( \Gamma \vdash c_1 \equiv c_3 \).

**Proof** (1)-(3) result immediately from applying rules \( \equiv I \) and \( \equiv E \). \( \square \)

The only remaining step to finish the proof of the completeness of \( QLET_F \) is to show that, given a regular Henkin set \( \Delta \) that does not prove \( A \), one can construct a canonical model \( \mathfrak{A} \) and a valuation \( \nu \) in \( \mathfrak{A} \) such that all elements of \( \Delta \) (and no others) hold in \( \mathfrak{A} \) and \( \nu \). Since \( A \notin \Delta \), this result will then be enough to conclude that \( \Delta \not\models A \) – which, when combined with Lemma 17, suffices for proving the completeness of \( QLET_F \).

Lemma 20 below has a rather long proof and so it might be worthy detailing its structure. The proof comprises three different parts: in the first part we define a canonical structure \( \mathfrak{A} \) that interprets the non-logical symbols of the relevant language. As usual, the domain of \( \mathfrak{A} \) will be composed by the terms of the language (or rather, by certain equivalence classes thereof), while the non-logical symbols will be interpreted in terms of the derivability-from-\( \Delta \) relation. As a result, for each constant \( c \in \mathcal{C} \), the corresponding diagram language will include a new individual constant \( [c] \), for each constant of the original language. In the second part we define a valuation \( \nu \) such that all and only the elements of \( \Delta \) hold in \( \mathfrak{A} \) and \( \nu \). Finally, in the third part, we prove that \( \nu \), thus defined, is indeed a valuation in \( \mathfrak{A} \), which comes down to showing that it satisfies all clauses of Definition 7.

**Lemma 20** Let \( \mathcal{S} = \langle \mathcal{C}, \mathcal{P} \rangle \) be a signature and \( \Delta \cup \{ A \} \subseteq \text{Sent}(\mathcal{S}) \). If \( \Delta \) is a regular Henkin set such that \( \Delta \not\models A \), then there is an \( \mathcal{S} \)-interpretation \( \langle \mathfrak{A}, \nu \rangle \) such that \( \mathfrak{A}, \nu \models \Delta \) and \( \mathfrak{A}, \nu \not\models A \).

**Proof** Let \( \sim \) be the relation on \( \mathcal{C} \) defined by: \( c_1 \sim c_2 \) iff \( \Delta \vdash c_1 \equiv c_2 \). For each \( c \in \mathcal{C} \), let \( [c] = \{ c' \in \mathcal{C} : c' \sim c \} \). Notice that since \( \sim \) is an equivalence relation (by Lemma 19), \( [c_1] = [c_2] \) if and only if \( c_1 \sim c_2 \). Now, define the \( \mathcal{S} \)-structure \( \mathfrak{A} = \langle \mathcal{D}, \mathcal{I} \rangle \) as follows:

1. \( \mathcal{D} = \{ [c] : c \in \mathcal{C} \} \);
2. For every \( c \in \mathcal{C} \), \( c^{\mathfrak{A}} = [c] \);

\[ \text{Thus, } \mathcal{D} \text{ is the partition of } \mathcal{C} \text{ generated by } \sim. \]
3. For every $n$-ary predicate letter $P \in \mathcal{P}$, and $c_1, \ldots, c_n \in C$

3.1. \(\langle [c_1], \ldots, [c_m] \rangle \in P^\mathfrak{A} \text{ iff } \Delta \vdash P(c_1, \ldots, c_n)\); and

3.2. \(\langle [c_1], \ldots, [c_m] \rangle \in P^\mathfrak{B} \text{ iff } \Delta \vdash \neg P(c_1, \ldots, c_n)\).

The following facts guarantee that $\mathfrak{A}$ is indeed an $S$-structure:

1. For every $P \in \mathcal{P}_n$, the definition of $P^\mathfrak{A} = (P^\mathfrak{A}_+, P^\mathfrak{A}_-)$ does not depend on the representatives $c_1, \ldots, c_n$.

2. \(\langle [c], [c] \rangle \in =^\mathfrak{A}_+, \text{ for every } c \in C\).

In order to prove (1), it suffices to show that $\Delta \vdash P(c_1, \ldots, c_n)$ if and only if $\Delta \vdash P(c_1', \ldots, c_n')$, whenever $c_i \sim c_i'$, for every $1 \leq i \leq n$ (and similarly for $\neg P(c_1, \ldots, c_n)$). But this is an immediate consequence of the application of rule $\equiv E$. (2) follows immediately by rule $== I$ from clause (3) in the definition of $\mathfrak{A}$ and the definition of $\sim$.

Now, define the mapping $\ast : Term(\mathcal{S}_\mathfrak{A}) \rightarrow Term(S)$ as follows: if $t \in \mathcal{V} \cup \mathcal{C}$, then $t^\ast = t$, and if $t = [c]$, for some $c \in C$, then $t^\ast$ is an arbitrary element $c_\ast$ of $[c]$. The mapping $\ast$ can then be naturally extended to the formulas of $\mathcal{S}_\mathfrak{A}$ as follows:

1. If $B = P(t_1, \ldots, t_n)$, then $B^\ast = P(t_1^\ast, \ldots, t_n^\ast)$;
2. If $B = \#C$ ($\# \in \{\neg, \land, \lor\}$), then $B^\ast = \#C^\ast$;
3. If $B = C \# D$ ($\# \in \{\land, \lor\}$), then $B^\ast = C^\ast \# D^\ast$;
4. If $B = Q \times C$ ($Q \in \{\mathcal{V}, \mathcal{G}\}$), then $B^\ast = Q \times C^\ast$.

Notice that (†) for every $c \in C_\mathfrak{A}$, $[c] \in P^\mathfrak{A}_+$ and only if $c^\mathfrak{A} \in P^\mathfrak{A}_+$ (and likewise for $P^\mathfrak{A}_-$). For if $c \in C$, then $c^\ast = c$ and $c^\mathfrak{A} = [c]$. Now, let $c = [c']$, for some $c' \in C$, and let $c_\ast$ be the constant in $[c']$ chosen to define the function $\ast$. Thus, $c^\ast = c_\ast$ and $\Delta \vdash c_\ast = c'$. By rule $= E$, $\Delta \vdash P(c^\ast)$ if and only if $\Delta \vdash P(c')$. By clause (3) in the definition of $\mathfrak{A}$ it follows that $[c^\ast] \in P^\mathfrak{A}_+$ if and only if $[c'] \in P^\mathfrak{A}_+$. Since $c^\mathfrak{A} = [c']$, it then follows that $[c^\ast] \in P^\mathfrak{A}_+$ if and only if $c^\mathfrak{A} \in P^\mathfrak{A}_+$. Of course, the same result holds for predicates of any arity, including $\equiv E$.

Notice further that (††) $B^\ast(c/x) \in \Delta$ if and only if $B^\ast(c^\mathfrak{A}/x) \in \Delta$, for every $c \in C$ and every $B \in Form(\mathcal{S}_\mathfrak{A})$ that has at most $x$ free. In effect, $c^\mathfrak{A} = [c]$, and so $c^\mathfrak{A}^\ast = [c^\ast]$ is the constant $c_\ast$ of $[c]$ chosen to define $\ast$. Thus, $\Delta \vdash c_\ast = c^\ast$. By rule $= E$, it follows that $\Delta \vdash B^\ast(c/x)$ if and only if $\Delta \vdash B^\ast(c_\ast/x)$. Therefore, $B^\ast(c/x) \in \Delta$ if and only if $B^\ast(c^\mathfrak{A}/x) \in \Delta$.

Finally, define the mapping $\nu : Sent(\mathcal{S}_\mathfrak{A}) \rightarrow \{0, 1\}$ by: $\nu(B) = 1$ if and only if $B^\ast \in \Delta$. Given $B \in Sent(\mathcal{S}_\mathfrak{A})$, $\nu$ assigns to $B$ the value 1 if the corresponding formula $B^\ast$ of $Sent(S)$ belongs to $\Delta$ (and 0 otherwise).\(^{17}\) Because $\Delta \notin \Delta$ (by hypothesis), $\mathfrak{A}$ and $\nu$ are such that $\mathfrak{A}, \nu \models \Delta$ and $\mathfrak{A}, \nu \notmodels A$, as required.

\(^{17}\)The mapping $\ast$ is necessary to ensure that the sentences of the diagram language of $S$ which do not belong to $Sent(S)$ get assigned one of the values 1 or 0 by $\nu$. Since the domain of $\nu$ is $Sent(\mathcal{S}_\mathfrak{A})$, we are prevented from defining it directly as the characteristic function of $\Delta$ (all of whose elements belong to $Sent(S)$). By making use of $\ast$, we are nonetheless able to define $\nu$ in terms of $membership-in-\Delta$, for the values of the elements of $Sent(\mathcal{S}_\mathfrak{A}) \setminus Sent(S)$ are then determined by whether their $\ast$-translations belong to $\Delta$. So, for example, the value of $P([c])$, which is a sentence of the diagram language of $S$, is 1 if and only if its $\ast$-translation $Pc_\ast$ belongs to $\Delta$, where $c_\ast$ is the constant in $[c]$ chosen to define $\ast$.  

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We haven’t finished yet, however. For we are still required to show that $v$, as defined above, is indeed an $\mathfrak{A}$-valuation – given that nothing so far guarantees that it satisfies all clauses of Definition 7. In the remainder of this proof, we shall prove that $v$ satisfies at least some of those clauses, leaving the remaining cases to the reader.

1. Let $B \in \text{Sent}(\mathcal{S}_\mathfrak{A})$ be the formula $P(c_1, \ldots, c_n)$. Then:

$$
\begin{align*}
\text{if} & \quad P(c_1, \ldots, c_n) \in \Delta \\
\text{if} & \quad P(c_1^*, \ldots, c_n^*) \in \Delta \\
\text{if} & \quad \Delta \vdash P(c_1^*, \ldots, c_n^*) \\
\text{if} & \quad \{c_1^*, \ldots, [c_n^*]\} \in P^\Delta_+ \\
\text{if} & \quad (c_1^*, \ldots, c_n^*) \in P^\Delta_+ \quad (\dagger) \text{above}
\end{align*}
$$

3. Let $B = (C \land D)$. Then:

$$
\begin{align*}
\text{if} & \quad (C \land D)^* \in \Delta \\
\text{if} & \quad C^* \in \Delta \text{ and } D^* \in \Delta \quad \text{Lemma 16(1)} \\
\text{if} & \quad v(C) = 1 \text{ and } v(D) = 1
\end{align*}
$$

8. Suppose that $B = \circ C$ and that $v(B) = 1$. Hence, $(\circ C)^* \in \Delta$, and so $C^* \in \Delta$. By Lemma 16(7), $C^* \in \Delta$ if and only if $\neg C^* \notin \Delta$. Thus, $v(C) = 1$ if and only if $v(\neg C) = 0$.

10. Let $B = \forall x C$. Then:

$$
\begin{align*}
\text{if} & \quad (\forall x C)^* \in \Delta \\
\text{if} & \quad \forall x C^* \in \Delta \\
\text{if} & \quad C^*(c/x) \in \Delta, \text{ for every } c \in C \quad \text{Lemma 16(9)} \\
\text{if} & \quad C^*(c/x^*) \in \Delta, \text{ for every } c \in C \quad (\dagger \dagger) \\
\text{if} & \quad C(c/x^*) \in \Delta, \text{ for every } c \in C \\
\text{if} & \quad C(a/x)^* \in \Delta, \text{ for every } a \in D \\
\text{if} & \quad v(C(a/x)) = 1, \text{ for every } a \in D
\end{align*}
$$

11. Let $B = \exists x C$ and let $c \in C$ be such that $\exists x C^* \in \Delta$ if and only if $C^*(c/x) \in \Delta$, which we know to exist due to Lemma 16(10). We then have:

$$
\begin{align*}
\text{if} & \quad (\exists x C)^* \in \Delta \\
\text{if} & \quad \exists x C^* \in \Delta \\
\text{if} & \quad C^*(c/x) \in \Delta \\
\text{if} & \quad C^*(c/x^*) \in \Delta \quad (\dagger \dagger) \\
\text{if} & \quad C(c/x^*) \in \Delta \\
\text{if} & \quad C(a/x)^* \in \Delta, \text{ for some } a \in D \\
\text{if} & \quad v(C(a/x)) = 1, \text{ for some } a \in D
\end{align*}
$$

15. Let $B \in \text{Form}(\mathcal{S}_\mathfrak{A})$ be such that $x$ is the only variable free in $B$, and let $c_1, c_2 \in \mathcal{C}_\mathfrak{A}$. Suppose that $c_1^* = c_2^*$ and that $v(B(c_1/x)) = v(B(c_2/x))$. Since $c_1^* = c_2^*$, it follows that $\{c_1^*, c_2^*\} \in \Delta$, and so $\{c_1^*, [c_2^*]\} \in \Delta$ by $(\dagger)$ above). Hence, $[c_2^*] = [c_2^*]$, and therefore $\Delta \vdash c_1^* = c_2^*$. Let $\# \in \{\neg, \circ, \bullet\}$. Thus:

$$
\begin{align*}
\text{if} & \quad \#(B(c_1/x))^* \in \Delta \\
\text{if} & \quad \#(B(c_1/x))^* \in \Delta \\
\text{if} & \quad \#(B(c_2/x))^* \in \Delta \quad \Delta \vdash c_1^* = c_2^* \\
\text{if} & \quad \#(B(c_2/x))^* \in \Delta \\
\text{if} & \quad \#(B(c_2/x))^* \in \Delta
\end{align*}
$$

Therefore, $v(\#B(c_1/x)) = v(\#B(c_2/x))$.}

Having proved lemmas 17 and 20, proving the completeness of $QLET_F$ is relatively straightforward:
Theorem 21 (Completeness Theorem) Let $\mathcal{S} = \langle \mathcal{C}, \mathcal{P} \rangle$ be a signature and $\Gamma \cup \{A\} \subseteq \text{Sent}(\mathcal{S})$. If $\Gamma \vdash A$, then $\Gamma \models A$.

Proof Suppose that $\Gamma \not\vdash A$. By Lemma 17, there is a signature $\mathcal{S}^+ = \langle \mathcal{C}^+, \mathcal{P} \rangle$ and a set $\Delta \subseteq \text{Sent}(\mathcal{S}^+)$ such that $\mathcal{C} \subseteq \mathcal{C}^+$, $\Gamma \subseteq \Delta$, and $\Delta$ is a regular Henkin set that does not prove $A$. By Lemma 20, there exists an $\mathcal{S}^+$-interpretation $\langle \mathfrak{A}, v \rangle$ such that $\mathfrak{A}, v \models \Delta$ and $\mathfrak{A}, v \not\models A$. Let $\mathfrak{A}_0$ be the $\mathcal{S}$-reduct of $\mathfrak{A}$ and $v_0$ be the restriction of $v$ to $\text{Sent}(\mathcal{S}_0)$. Clearly, $\mathfrak{A}_0, v_0 \models B$ if and only if $\mathfrak{A}, v \models B$, for every $B \in \text{Sent}(\mathcal{S})$. As a result, $\mathfrak{A}_0, v_0 \models \Gamma$ (since $\mathfrak{A}, v \models \Gamma \subseteq \Delta$) and $\mathfrak{A}_0, v_0 \not\models A$. Therefore, $\Gamma \not\models A$.

Compactness and Löwenheim-Skolem theorems are immediate consequences of the soundness and the completeness of $\text{QLET}_F$:

Corollary 22 (Compactness Theorem) Let $\Gamma \cup \{A\} \subseteq \text{Sent}(\mathcal{S})$. Then:

1. $\Gamma \models A$ if and only if there is a finite subset $\Gamma_0$ of $\Gamma$ such that $\Gamma_0 \models A$;
2. $\Gamma$ has a model if and only if every finite subset $\Gamma_0$ of $\Gamma$ has a model.

Proof (1) is an immediate consequence of the soundness and completeness theorems and the fact that derivations are finite. As for (2), we shall only prove the right-to-left direction. Suppose that every finite subset $\Gamma_0$ of $\Gamma$ has a model and that $\Gamma$ does not have a model. Hence, $\Gamma \not\models A$, for every sentence $A$, and so there is a sentence $B$ such that $\Gamma \models \neg B \land B \land \neg B$. By Corollary 22, it then follows that some finite $\Gamma_0 \subseteq \Gamma$ is such that $\Gamma_0 \models \neg B \land B \land \neg B$. Therefore, $\Gamma_0$ is trivial and does have a model, which contradicts the initial hypothesis.

Corollary 23 (Downward Löwenheim-Skolem Theorem) Let $\mathcal{S} = \langle \mathcal{C}, \mathcal{P} \rangle$ be a signature whose cardinality is $\kappa$, and suppose that $\Gamma \subseteq \text{Sent}(\mathcal{S})$. If $\Gamma$ has a model, then $\Gamma$ has a model whose cardinality is less than or equal to $\kappa$.

Proof Since $\Gamma$ has a model, it is non-trivial (by soundness). Hence, there is a sentence $A$ such that $\Gamma \not\models A$. By Lemma 17, there is a signature $\mathcal{S}^+ = \langle \mathcal{C}^+, \mathcal{P} \rangle$, with $|\mathcal{C}^+| = \kappa$, and a set $\Delta \subseteq \text{Sent}(\mathcal{S}^+)$ such that $\Delta$ is a regular Henkin set, $\Gamma \subseteq \Delta$, and $\Delta \not\models A$. By Lemma 20, there is an $\mathcal{S}^+$-structure $\mathfrak{A}^+$ whose domain $\mathcal{D}^+$ is the set $\{c \in \mathcal{C}^+: c \in \mathcal{C}^+\}$, and there is a valuation $v^+$ in $\mathfrak{A}^+$ such that $\mathfrak{A}^+, v^+ \models \Delta$. Since $\mathcal{D}^+$ is a partition of $\mathcal{C}^+$, $|\mathcal{D}^+| \leq |\mathcal{C}^+|$, and so $|\mathcal{D}^+| \leq \kappa$. Finally, let $\mathfrak{A}$ be the $\mathcal{S}$-reduct of $\mathfrak{A}^+$ and $v$ be the restriction of $v^+$ to $\text{Sent}(\mathcal{S}_0)$. Since $\Gamma \subseteq \Delta$ and the domain $\mathcal{D}$ of $\mathfrak{A}$ is equal to $\mathcal{D}^+$, it follows that $\mathfrak{A}, v \models \Gamma$ and $|\mathcal{D}| \leq \kappa$.

Corollary 24 (Upward Löwenheim-Skolem Theorem) Let $\mathcal{S} = \langle \mathcal{C}, \mathcal{P} \rangle$ be a signature whose cardinality is $\kappa$, and let $\Gamma \subseteq \text{Sent}(\mathcal{S})$ be such that $\Gamma \models \forall x \forall y \forall z (x = y)$. If $\Gamma$ has an infinite model, then $\Gamma$ has a model of cardinality $\lambda$, for every $\lambda \geq \kappa$.

Proof Let $\lambda \geq \kappa$ and let $\mathcal{S}^+ = \langle \mathcal{C}^+, \mathcal{P} \rangle$ be such that $\mathcal{C}^+ = \mathcal{C} \cup \{c_\alpha : \alpha < \lambda\}$. Consider the set:

$$\Delta = \Gamma \cup \{c_\alpha \models \neg c_\beta : \alpha, \beta < \lambda \text{ and } \alpha \neq \beta\}$$
Since $\Gamma$ has a model, so does $\Delta$. For let $\langle \mathfrak{A}, v \rangle$ be an infinite $\mathcal{S}$-interpretation such that $\mathfrak{A}, v \models \Gamma$, and consider an arbitrary finite subset $\Delta_0$ of $\Delta$. Define $\mathfrak{A}^+ = \langle \mathcal{D}, \mathcal{I}^+ \rangle$ to be the extension of $\mathfrak{A}$ such that:

1. For every $\alpha, \beta < \lambda$ such that $c_\alpha$ and $c_\beta$ occur in $\Delta_0$, if $\alpha \neq \beta$, then $\mathcal{I}^+(c_\alpha) \neq \mathcal{I}^+(c_\beta)$;
2. For every $\alpha$ such that $c_\alpha$ does not occur in $\Delta_0$, $\mathcal{I}^+(c_\alpha)$ is a fixed element $a$ of $\mathcal{D}$;
3. $\mathcal{I}^+(\vec{=} \vec{=}) = \mathcal{I}(\vec{=} \vec{=})$.

Now, extend $v$ to a valuation $v^+$ such that for every $c_1, \ldots, c_n \in \mathcal{C}$ and $c_1', \ldots, c_n' \in \mathcal{C}^+$, and for every sentence $A \in \text{Sent}(\mathcal{S})$, if $\mathcal{I}^+(c_i) = \mathcal{I}^+(c_i')$, then $v^+(A(c_1'/c_1; \ldots; c_n'/c_n)) = v(A)$. Clearly, $\Delta_0$ holds in $\langle \mathfrak{A}^+, v^+ \rangle$, since $\langle a, b \rangle \in \vec{=} \vec{=}$ if and only if $a \neq b$. Therefore, every finite subset of $\Delta$ has a model, and, by Corollaries 22(2) and 23, $\Delta$ has a model whose cardinality $\lambda'$ is less than or equal to $\lambda$. But because $\Delta$ includes every sentence $c_\alpha = c_\beta$ and $\Delta \models \forall x \forall y (x = y)$, $\lambda'$ must be equal to $\lambda$ (for if $\mathfrak{A}, v \models \alpha(c_1 = c_2) \land c_1 \neq c_2$, then $\mathfrak{A}, v \models c_1 = c_2$, and so $\mathcal{I}(c_1) \neq \mathcal{I}(c_2)$).

Remark 25 On extended valuations

So far we have defined all relevant syntactic and semantic notions with respect to sentences, completely disregarding open formulas. In particular, while presenting the natural deduction system for $QLET_F$ we have replaced the more traditional quantifier rules by corresponding rules in which constants play the roles of variables or terms. This choice led us to assume that every language has an infinite stock of individual constants to ensure that there will always be enough constants to meet the restrictions upon some quantifier rules (viz., $\forall I, \exists E, \neg \forall E, \text{ and } \neg \exists I$).

Now, although focusing on sentences brings some significant technical simplifications, we could have adopted a more traditional approach, formulating the natural deduction system for $QLET_F$ with the usual rules, and defining the semantic consequence relation to include both open and closed formulas. This could be achieved by making use of extended valuations [see 12, Def. 7.3.10], which can be defined as follows: Given an $\mathcal{S}$-interpretation $\langle \mathfrak{A}, v \rangle$, the extension of $v$ in $\mathfrak{A}$ is the mapping $\overline{v}: \text{Form}(\mathcal{S}\mathfrak{A}) \times \mathcal{D}^\mathcal{V} \rightarrow \{0, 1\}$ such that:

$$\overline{v}(A, s) = v(A(s(x_1), \ldots, s(x_n)/x_1, \ldots, x_n))$$

where $s$ is an assignment of elements of $\mathcal{D}$ to the individual variables and all variables free in $A \in \text{Form}(\mathcal{S})$ are among $x_1, \ldots, x_n$.

Hence, the extension $\overline{v}$ of $v$ assigns to an open formula $A(x_1, \ldots, x_n)$ the value assigned by $v$ to the sentence $A(s(x_1), \ldots, s(x_n))$, which results from $A$ by replacing the variables $x_1, \ldots, x_n$ by the constants $s(x_1), \ldots, s(x_n)$ of the corresponding diagram language.

This strategy allows us to mimic the definitions usually found in traditional formulations of a referential semantics for, say, first-order classical logic. In particular, it can be proven that each clause of Definition 7 can be rewritten in terms of extended

\[\text{Notice that the cardinality of } S^+ \text{ is } \lambda \text{ rather than } \kappa.\]
valuations. For instance, clause (4), together with the definition of $\overline{v}$, allows us to prove:

$$\overline{v}(B \lor C, s) = 1 \iff \overline{v}(B, s) = 1 \text{ or } \overline{v}(C, s) = 1$$

while clause (10) allows us to prove:

$$\overline{v}(\forall x B, s) = 1 \iff \overline{v}(B, s^a_x) = 1, \text{ for every } a \in D$$

(where $s^a_x$ is the assignment that differs from $s$ at most by assigning $a$ to $x$).

Had we chosen to adopt this strategy and defined the semantic consequence relation accordingly, we would also be capable of proving all the results above, though the corresponding definitions and proofs would become much more cumbersome. That this can be done suffices to ensure that nothing in this paper hinges on our choice to focus entirely on sentences, and that we could have done without the assumption that languages must always have infinitely many individual constants.

**Remark 26** On the non-classical identity of $QLET_F$

An identity sentence $a \equiv b$ in $QLET_F$ behaves classically when $\sigma(a \equiv b)$ holds. When $\sigma(a \equiv b)$ does not hold, it admits paraconsistent and paracomplete scenarios. Let us take a closer look at this point.

There are four scenarios of absence of reliable information (i.e., when $v(\sigma(a \equiv b)) = 0$),

1. $v(a \equiv b) = 0, v(a \not\equiv b) = 0$ (incomplete information),
2. $v(a \equiv b) = 0, v(a \not\equiv b) = 1$ (only negative information),
3. $v(a \equiv b) = 1, v(a \not\equiv b) = 0$ (only positive information),
4. $v(a \equiv b) = 1, v(a \not\equiv b) = 1$ (contradictory information),

and two scenarios of reliable information (i.e. when $v(\sigma(a \equiv b)) = 1$),

5. $v(a \equiv b) = 1, v(a \not\equiv b) = 0$,
6. $v(a \equiv b) = 0, v(a \not\equiv b) = 1$.

Scenarios (5) and (6) are classical, so $\langle I(a), I(b) \rangle$ must be either in the extension or in the anti-extension of $\equiv$, not both. The point in question is how to express the non-classical scenarios (1) (no information at all) and (4) (conflicting information). Let us see how they are represented in the semantics of $QLET_F$.

Let $a, b,$ and $c$ be the names ‘Hesperus’, ‘Phosphorus’, and ‘Venus’, and $P$ the predicate ‘is a planet’. Now consider the following hypothetical scenarios:

(i) $\{\neg Pa, \neg Pb, Pc\}$.

There is the information that neither Hesperus nor Phosphorus is a planet, and Venus is a planet. Nothing is said about whether or not they are the same object. Thus, if $I(a) = \bar{a}$ and $I(b) = \bar{b}$, the pair $\langle \bar{a}, \bar{b} \rangle$ is not in $\equiv\_\_$ and, of course, not in $\equiv\_\_\_$.  

(ii) $\{a \neq b, \neg Pa, \neg Pb, Pc\}$.

Scenario (ii) is like to (i), except that we have the additional information that Hesperus and Phosphorus are not the same object. Thus, the pair $\langle \bar{a}, \bar{b} \rangle$ is in $\equiv\_\_$.

Regarding scenarios (i) and (ii), note that according to the semantic clauses of identity, $\langle I(a), I(b) \rangle$ belongs to the anti-extension $\equiv\_\_$ if and only if $a \not\equiv$...
b holds. The anti-extension \( \equiv \) does not contain every pair \( \langle \bar{a}, \bar{b} \rangle \) such that \( \mathcal{I}(a) \neq \mathcal{I}(b) \), but only the pairs \( \langle \bar{a}, \bar{b} \rangle \) such that \( v(a \neq b) = 1 \).

(iii) \( \{a \equiv b, \neg Pa, \neg Pb, Pc\} \).

Scenario (iii) is also like (i), except that now we have the additional information that Hesperus and Phosphorus are indeed the same object. Thus, \( \mathcal{I}(a) \) and \( \mathcal{I}(b) \) have to have the same denotation, say, \( \bar{a} \), and the pair \( \langle \bar{a}, \bar{a} \rangle \) is, of course, in \( \equiv_{+} \) (as well for every other object in the domain).

(iv) \( \{a \equiv b, \neg Pa, \neg Pb, a = c, b = c, Pc\} \).

This is a contradictory scenario that adds to scenario (ii) the information that both Hesperus and Phosphorus are in fact the planet Venus, and so the same object.

Here we have contradictory information about their identity, which means that \( a \) and \( b \) denote one and the same object, say \( \bar{a} \), but also that these names denote different objects. To express this scenario we make \( \mathcal{I}(a) = \mathcal{I}(b) = \bar{a} \), and the pair \( \langle \bar{a}, \bar{a} \rangle \) is in both \( \equiv_{+} \) and \( \equiv_{-} \).

Some additional remarks are in order here.

The basic idea of the intuitive interpretation in terms of information is that sentences may merely provide information about objects and their properties, which of course does not mean that these sentences are true, nor that the putative objects they refer to exist. In the classical scenarios (5) and (6) we may assume that the names denote ‘real objects’, that is, the sentences \( a \equiv b \) and \( a \equiv b \) talk about objects that indeed exist in the world. On the other hand, it might well be that in the non-classical scenarios the object that the interpretation assigns to a name does not exist.

It is not difficult to imagine, for example, a database that contains information about an individual that, in fact, does not exist. The domain, in this case, reflects this situation and has an object that corresponds to that name, i.e., that is the pseudo-denotation of the name.

5 First-Order FDE and some of its Extensions

Now that we have proven soundness and completeness theorems for \( QLET_F \), we will show in this section how \( QLET_F \) can be modified to yield first-order versions of \( FDE \) and some of its extensions, namely, Kleene’s \( K3 \), the logic of paradox \( LP \), and classical logic, to be called here \( QFDE \), \( QK3 \), and \( QLP \), respectively.

We start by presenting a natural deduction system and a corresponding first-order valuation semantics for \( QFDE \), which result from slight modifications of those for \( QLET_F \). We shall also indicate how the completeness proof above can be adapted to the case of \( QFDE \) and, in Section 5.2, to those of \( QK3 \) and \( QLP \).

5.1 First-Order FDE

The logical vocabulary of \( QFDE \) is the same as that of \( QLET_F \), except that \( \circ \) and \( \bullet \) are no longer included in the set of logical primitives. We will continue to make use of
first-order signatures to specify the non-logical vocabulary of a first-order language, and adopt the same notational conventions as before.

**Definition 27** The logic $QFDE$ is obtained by dropping rules $EXP^*$, $PEM^*$, $Cons$, $Comp$, and $AV$ from $QLET_F$ (see Definition 1).

Notice that $QFDE$ includes all $\circ$- and $\bullet$-free rules of $QLET_F$, except for $AV$. As it turns out, the absence of the $\circ$ and $\bullet$ allows us to prove that any two alphabetically variant sentences are deductively equivalent.

Recall that while presenting the semantics of $QLET_F$ in Section 3 it was necessary to supplement a structure $\mathfrak{A}$ with a valuation $\nu$ in order to ensure that all sentences in which $\circ$ or $\bullet$ occur get assigned a semantic value – this was done because their values are not always determined by the values of their subformulas. In $QFDE$, however, valuations are no longer necessary. Thus, although the notion of a first-order structure remains the same as before (see Definition 5), interpretations, in the sense of Definition 8, could be dispensed with in the case of $QLET_F$. Nonetheless, in order to preserve the notation used in the preceding sections and to demonstrate the generality of the method of anti-extensions + valuations, we shall continue to talk as if sentences get assigned one of the values 1 or 0 by a valuation in $QFDE$, but this time to each structure $\mathfrak{A}$ there will correspond a single valuation $\nu_{\mathfrak{A}}$, which is the valuation induced by $\mathfrak{A}$.

**Definition 28** Let $\mathcal{S} = \langle \mathcal{C}, \mathcal{P} \rangle$ be a signature and $\mathfrak{A}$ an $\mathcal{S}$-structure. The mapping $\nu_{\mathfrak{A}} : Sent(\mathcal{S}_\mathfrak{A}) \rightarrow \{0, 1\}$ is the valuation induced by $\mathfrak{A}$ if it satisfies clauses (1)-(7) and (10)-(13) of Definition 7 (where $\nu$ is replaced everywhere by $\nu_{\mathfrak{A}}$).

Given $A \in Sent(\mathcal{S})$ and an $\mathcal{S}$-structure $\mathfrak{A}$, we shall say that $A$ holds in $\mathfrak{A}$ ($\mathfrak{A} \models A$) if and only if $\nu_{\mathfrak{A}}(A) = 1$; and that $A$ is a semantic consequence of $\Gamma$ in $QFDE$ if and only if $\mathfrak{A} \not\models A$ whenever $\mathfrak{A} \not\models B$, for every $B \in \Gamma$.

By suitable modifications of the definitions and proofs in Section 4, it can be proven that $QFDE$ is sound and complete with respect to the class of all $QFDE$-structures. Specifically, the proof of (the $QFDE$-analogue of) Proposition 11 (and so of Corollary 12) is the same as before. Hence, except for absence of proofs for the rules involving $\circ$ and $\bullet$, the soundness proof for $QFDE$ remains the same as that for $QLET_F$. As for completeness, there are no significant changes either. In particular, the proof of (the $QFDE$-analogue of) Lemma 20 differs from the one presented above only in that we are not required to show that $\nu_{\mathfrak{A}}$, where $\mathfrak{A}$ is the canonical structure of a regular Henkin set, satisfies clauses (8), (9), (14), and (15).

**Remark 29** A sequent calculus for quantified $FDE$ is found in Anderson and Belnap [14], and natural deduction systems in Priest [11, pp. 331ff.] and Sano and Omori [38, p. 463]. The deductive systems in [14] and [11] are equivalent to the one above, but that of [38], as far as we can tell, is not. It seems to us not only that (i) there is a
gap in the completeness proof of [38], but also that (ii) the natural deduction system presented therein is in fact incomplete. Specifically, the system in [38] cannot prove:

\[ \forall x (B \lor A) \vdash B \lor \forall x A, \]  

where \( x \) is not free in \( B \). Items (i) and (ii) bring to light some interesting points about first-order extensions of \( FDE \).

Let us take a look at (i). In [38], Sano and Omori introduce some formal systems that extend the first-order version of \( FDE \) (\( QFDE \)), which they call \( BD \) logic. They present a natural deduction system for \( QFDE \), where the introduction rule for the universal quantifier is

\[ \frac{A(c/x)}{\forall x A} \]  

instead of

\[ \frac{B \lor A(c/x)}{B \lor \forall x A} \]  

(with the usual restrictions). The semantics is similar to the one above except for the use of relations instead of functions. [38, Section 4] offer a general method for proving the completeness of \( BD \) and some of its extensions, but the proof of the result corresponding to Lemma 17 above [38, Lemma 4.2] is not entirely convincing. They adopt the following notation [38, p. 464]:

\[ \text{iff} \quad \exists \]  

for any calculus \( \mathcal{R} \) such that \( BD \subseteq \mathcal{R} \). They then proceed with a Lindenbaum construction, defining a sequence \( \langle \Gamma_n, \Pi_n \rangle \), and claim, without presenting a proof, that for every \( n \), \( \Gamma_n \not\vdash_{\mathcal{R}} \Pi_n \). The sensible point of their strategy is to show that \( \Gamma_{n+1} \not\vdash_{\mathcal{R}} \Pi_{n+1} \) when \( \Gamma_n, \forall x B \vdash \Pi_n \). In this case, \( \Gamma_{n+1} = \Gamma_n \) and \( \Pi_{n+1} = \Pi_n \cup \{ \forall x B, B(c/x) \} \), where \( c \) is a new constant. If we were to fill in the gaps in their proof, we could assume that (a) \( \Gamma_n \vdash_{\mathcal{R}} \Pi_n \cup \{ \forall x B, B(c/x) \} \) to obtain (b) \( \Gamma_n \vdash_{\mathcal{R}} \Pi_n \), which yields a contradiction with the induction hypothesis. However, there is no obvious way to get from (a) to (b) by applying rule \( \forall I' \) instead of \( \forall I \).

Concerning (ii), let us call \( QFDE' \) the first-order extension of \( FDE \) with rule \( \forall I' \) instead of \( \forall I \). As far as we can see, \( QFDE' \) is incomplete with respect to the standard semantics for quantified \( FDE \), an issue which is closely related to the validity of (1). A sketch of a proof that \( QFDE' \) does not prove (1) is as follows. Let us call \( QFDE'_{G} \) the \( \rightarrow \)-free fragment of López-Escobar’s refutability calculus [39]. It is straightforward to prove that if \( \Gamma \vdash A \) does not hold in \( QFDE'_{G} \), then it does not hold in \( QFDE' \). Now, define a notion of generalized subformula in such a way that \( \neg A \) and \( \neg B \) are generalized subformulas of \( \neg (A \land B) \), \( \neg (A \lor B) \) of \( \neg \forall x A \), and so on. Since cut-elimination holds for \( QFDE'_{G} \), it is easy to see that all formulas in a cut-free derivation in \( QFDE'_{G} \) are generalized subformulas of the endsequent of the derivation. So, if (1) were valid in \( QFDE'_{G} \), it would be provable with the positive rules only, but every proof-search ends with a topsequent which is not an axiom. It is

19In [38, p. 466], we read only that “[for every \( n \)] it is easy to see that \( \Gamma_n \not\vdash_{\mathcal{R}} \Pi_n \)”.  

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also worth noting that if (1) were provable with the positive fragment of $QFDE'_G$, it would be provable in intuitionistic logic, but it is not.\footnote{It seems to us, however, that the natural way of constructively extending $FDE$ to a first-order logic is given by $QFDE'$. Of course, a corresponding adequate semantics for $QFDE'$ would not be given by $FDE$'s standard semantics, found here and in [11, 38], but rather, we conjecture, by a Kripke-style semantics with variable domains.}

### 5.2 On some Extensions of QFDE

In this section we show how the semantics of $QFDE$ can be adapted to obtain sound and complete semantics for first-order versions with identity of the logics $LP$, $K3$, and classical logic, called here, respectively, $QK3$, $QLP$, and $QCL$. Our main purpose is to illustrate the generality of the method of anti-extensions + valuations, showing how it can be easily applied to a number of non-classical logics, and even to classical logic.

In fact, it should be no surprise by now that the definitions and results presented in Sections 3 and 4 can be straightforwardly modified to yield sound and complete natural deduction systems not only for $QFDE$, but also for $QK3$, $QLP$, and $QCL$. One has to simply add either excluded middle or explosion (or both) to the rules of $QLET_F$. As for the semantics, it suffices to impose some further conditions on the relations between the extensions and anti-extensions of predicate letters.\footnote{A similar approach, but with a classical treatment of identity, is found in Beall and Logan [40, Ch. 16]; see Remark 34 below.}

**Definition 30** Consider the following two rules:

\[
\begin{align*}
\frac{A \lor \neg A}{\text{PEM}} & \quad \frac{A}{\text{EXP}}
\end{align*}
\]

1. $QLP$ results from adding $\text{PEM}$ to the rules of $QFDE$;
2. $QK3$ results from adding $\text{EXP}$ to the rules of $QFDE$;
3. $QCL$ results from adding both $\text{PEM}$ and $\text{EXP}$ to the rules of $QFDE$.

**Definition 31** Let $S$ be a signature and let $\mathcal{A} = (D, I)$ be a $QFDE$-structure. Then:

- $\mathcal{A}$ is a $QLP$-structure if and only if $P^\mathcal{A}_+ \cup P^\mathcal{A}_- = D^n$, for every $n$-ary predicate letter $P$ of $S$;
- $\mathcal{A}$ is a $QK3$-structure if and only if $P^\mathcal{A}_+ \cap P^\mathcal{A}_- = \emptyset$, for every predicate letter $P$ of $S$;
- $\mathcal{A}$ is a $QCL$-structure if and only if it satisfies both (E1) and (E2).

If $L$ is one of $QLP$, $QK3$, or $QCL$, then an $L$-interpretation is a pair $\langle \mathcal{A}, v_\mathcal{A} \rangle$, where $\mathcal{A}$ is an $L$-structure and $v_\mathcal{A}$ is the valuation induced by $\mathcal{A}$ – i.e., mapping from $\text{Sent}(S)$ to $\{0, 1\}$ that satisfies clauses (1)-(7) and (10)-(13) of Definition 7.

This way of presenting classical predicate logic is unusual and filled with redundancies – $QCL$ could be more simply described as the logic resulting from adding $\text{EXP}$ and $\text{PEM}$ to the positive fragment of $QFDE$ with the usual introduction...
rule for \( \forall \) (i.e., \( \forall I' \)). Moreover, the semantics, and so the metatheoretical results used above for proving completeness, could also be simplified. As with \( QFDE \), valuations are not strictly necessary, since structures alone suffice to determine the semantic values of all formulas – and in the case of \( QCL \) anti-extensions are not required. Our point in adapting the semantics of \( QLET_F \) to \( QFDE \), \( K3 \), \( LP \), and \( QCL \), however, is to show that the method of anti-extensions + valuations can be easily applied to a number of non-classical logics, and even to classical logic.

**Lemma 32** Let \( \mathfrak{A} \) be a \( QFDE \)-structure. Then:

1. If \( \mathfrak{A} \) is a \( QLP \) or a \( QCL \)-structure, then \( v_{\mathfrak{A}}(A) = 1 \) or \( v_{\mathfrak{A}}(\neg A) = 1 \), for every \( A \in \text{Sent}(\mathcal{S}_3) \);
2. If \( \mathfrak{A} \) is a \( QK3 \)- or a \( QCL \)-structure, then \( v_{\mathfrak{A}}(A) \neq 1 \) or \( v_{\mathfrak{A}}(\neg A) \neq 1 \), for every \( A \in \text{Sent}(\mathcal{S}_3) \).

**Proof** Both results follow by straightforward inductions on the complexity of \( A \). The atomic cases of the proofs of (1) and (2) depend respectively on conditions (E1) and (E3), and on (E2) and (E3) of Definition 31.

**Theorem 33** Let \( \mathcal{L} \) be one of \( QLP \), \( QK3 \), or \( QCL \). Then, \( \mathcal{L} \) is sound and complete with respect to the class of all \( \mathcal{L} \)-structures.

**Proof** Lemma 32(1) is all we need to prove that rule \( PEM \) is valid in both \( QLP \) and \( QCL \), while the validity of \( EXP \) in \( QK3 \) and \( QCL \) follows from 32(2). By adapting the proof of the soundness of \( QLET_F \) (Theorem 13), it can then be easily proven that all three systems are sound with respect to the class of corresponding structures. Proving their completeness is equally straightforward: it suffices to ensure that the canonical structure defined in the proof of Lemma 20 satisfies the corresponding restriction in Definition 31. But this is an immediate consequence of the presence, in each case, of either \( PEM \) or \( EXP \) (or both) in the deductive system.\(^{22}\)

**Remark 34** Beall & Logan [40] (from now on B&L) also contains first-order versions with identity of \( FDE \), \( K3 \), and \( LP \). Though their semantics is based on the notions of truth in a case and falsity in a case, it is essentially the relational semantics introduced by Dunn [8]. Taking \( c \) to stand for a case, our valuations and their cases are related in the following way:

\[
\begin{align*}
\begin{array}{ll}
\text{c} & \models_1 A \iff v(A) = 1, \\
\text{c} & \not\models_1 A \iff v(A) = 0, \\
\text{c} & \models_0 A \iff v(\neg A) = 1, \\
\text{c} & \not\models_0 A \iff v(\neg A) = 0.
\end{array}
\end{align*}
\]

Now, except for their adoption of the following clause for identity in [40, p. 204],

\[^{22}\text{In the case of } QLP, \text{ for example, given a regular Henkin set } \Delta, \text{ it follows by } PEM \text{ that } \Delta \vdash_{QLP} P(c_1, \ldots, c_n) \lor \neg P(c_1, \ldots, c_n), \text{ and so that either } \Delta \vdash_{QLP} P(c_1, \ldots, c_n) \text{ or } \Delta \vdash_{QLP} \neg P(c_1, \ldots, c_n). \text{ Therefore, if } \mathfrak{A} = \langle \mathcal{D}, \mathcal{I} \rangle \text{ is } \Delta \text{'s canonical structure, then for every } a_1, \ldots, a_n \in \mathcal{D}, \text{ either } (a_1, \ldots, a_n) \in P^\Delta_+ \text{ or } (a_1, \ldots, a_n) \in P^\Delta_-.\]

\[\text{Springer}\]
B&L’s semantics are equivalent to the semantics for $QFDE$, $QK3$, and $QLP$ presented above. However, when taken together with $= = \{\langle a, b \rangle : a \in D, b \in D \text{ and } a \neq b\}$, which they also assume to hold, the above clause implies that $= = \text{satisfies items (E1) and (E2) of Definition 31.}$ As a result, their first-order versions of $FDE$, $K3$, and $LP$ validate the following instances of excluded middle and explosion:

$$\begin{align*}
(a) & \vdash a \implies b \lor a \equiv b, \\
(b) & a \implies b, a \not\equiv b \not\vdash A
\end{align*}$$

This fact is also evident from their systems of tableaux rules [40, p. 213], for in either of the three logics we obtain a closed tableau for (a) and (b). Thus, B&L’s treatment of identity, according to which $= = \text{is a classical predicate,}$ is fundamentally different from ours, according to which $= = \text{allows for both gaps and gluts (see Remark 26 above).}$ Treating identity as a classical predicate within a non-classical logic is, in principle, a perfectly legitimate strategy. However, with regard to these three logics, we think that, in each case, to treat identity in accordance with the behavior of negation would be more in line with their spirit.

### 6 Final Remarks

The so-called Suszko’s thesis [41] asserts that every Tarskian and structural logic admits of a two-valued semantics. A proof of this result for sentential logics can be found in [42, pp. 72-73], and is in fact very simple. Given a (possibly infinite) multi-valued semantics for a Tarskian and structural logic $\mathcal{L}$, a two-valued semantics for $\mathcal{L}$ is defined as follows: if a formula $A$ receives a designated valued in a multi-valued interpretation $I$, the value 1 is assigned to $A$ in a two-valued interpretation $I'$, otherwise $A$ is assigned the value 0 in $I'$. Semantic consequence is then defined as preservation of the value 1, instead of preservation of a designated value.

An analogous result has been obtained by Loparic and da Costa in [5, pp. 121-122], where they present a general notion of valuation semantics. Given a consequence relation $\vdash$ and a language $L$, a function $e : L \rightarrow \{0, 1\}$ is an evaluation if $e$ satisfies the following clauses:

1. If $A$ is an axiom, then $e(A) = 1$;
2. If $e$ assigns the value 1 to all the premises of an application of an inference rule, then it also assigns 1 to its conclusion;
3. For some formula $A$, $e(A) = 0$.

It is also necessary that a Lindenbaum construction can be carried out for $\vdash$, which requires that $\vdash$ has to be Tarskian and compact. Let a set $\Delta$ be $A$-saturated when $\Delta \not\vdash A$ and for every $B \not\in \Delta$, $\Delta \cup \{B\} \vdash A$. Now, assuming that $\Gamma \not\vdash A$:

1. There is an $A$-saturated set $\Delta$, such that $\Gamma \subseteq \Delta$;
2. $\Delta \vdash B$ iff $B \in \Delta$;
(vi) The characteristic function $c$ of $\Delta$ is an evaluation.

Since (iv) and (v) are immediate consequences of the Lindenbaum construction, it suffices to prove (vi). Clearly, $c$ satisfies (i) and (iii) above. As for (ii), suppose $c$ assigns the value 1 to the premises of a derivation $\Delta_0 \vdash B$, $\Delta_0 \subseteq \Delta$. Since, by (v), $B \in \Delta$, it then follows that $c(B) = 1$.

Now, define a valuation as an evaluation that is the characteristic function of some $A$-saturated set. The collection of all valuations so defined is an adequate valuation semantics for $\vdash$. Soundness follows from the definition of evaluations (the set of valuations is a proper subset of the set of evaluations), and completeness from the fact that $c$ assigns 1 to all the sentences of $\Gamma$, while assigning 0 to $A$.\textsuperscript{23} Apparently, provided appropriate conditions for the construction of an $A$-saturated set, this result could be extended to first-order logics as well.

Valuation semantics were proposed by Loparic, Alves, and da Costa for the paraconsistent logics of da Costa’s $Cn$ hierarchy [2–4], which are ‘ancestors’ of the logics of formal inconsistency and logics of evidence and truth. The problem they had at hand was to provide semantics for paraconsistent logics that are not finitely-valued. They then came up with the idea of generalizing classical two-valued semantics in such a way that the axioms and rules were ‘mirrored’ by the semantic clauses in terms of 0s and 1s. The value 0 assigned to a formula $\alpha$ can be read as ‘$\alpha$ does not hold’ and 1 as ‘$\alpha$ holds’ – note that this is the basic idea of the general notion of valuation as defined above. Later, valuation semantics were proposed for several non-classical sentential logics, including minimal and intuitionistic logic, $FDE$, Nelson’s $N4$, and logics of formal inconsistency and undeterminedness [1, 5, 25, 31, 43]. First-order valuation semantics were also proposed for da Costa’s quantified $Cn$ hierarchy [44], and for some logics of formal inconsistency [12, 13].

As we have seen above, when a non-deterministic semantics is extended to first-order the crucial point is how to handle its extended non-deterministic character. The results presented here suggest anti-extensions + valuations as a general method for providing first-order valuation semantics for non-classical logics. Of course, these tools can sometimes be simplified, as we have just seen in Section 5. Valuations can be dispensed with in the case of $QFDE$, $QLP$, and $QK3$, but are indispensable in the case of $QLET_{F}$, along with several other logics of formal inconsistency (e.g., $QmbC$ [13]). Classical logic is a limiting case, since standard Tarskian structures (where anti-extensions are just the complement of extensions) are enough to provide an adequate semantics. In all these cases, however, the semantics are nothing but special cases of the general method described here.

\textsuperscript{23}Note that the set of all evaluations for a given consequence relation $\vdash$ does not suffice for providing a semantics. Consider, e.g., the semantics of classical logic, which is a special case of a valuation semantics, and let $T$ be the set of all classical theorems. The characteristic function $c$ of $T$ is an evaluation, but for all atoms $p$, neither $p$ nor $\neg p$ is in $T$, so $c(p) = 0$ and $c(\neg p) = 0$, even though $c(p \lor \neg p) = 1$. It is also worth noting that the notion of an $A$-saturated set provides a method for proving completeness for any logic for which a Lindenbaum construction can be carried out. We thank Andrea Loparic for some conversations that clarified the general notion of valuation semantics.
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