CERTAIN STUDY OF GENERALIZED APOSTOL-BERNOULLI POLY-DAEHEE POLYNOMIALS AND ITS PROPERTIES

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Abstract. In this paper, we present a new type of generating function of generalized Apostol-Bernoulli poly Dahee polynomial (GABPDP). By using generating function of GABPDP, we discuss some special cases and useful identities of generalized Apostol-Bernoulli poly Dahee polynomials. We also drive implicit summation formulae of it.

1. Introduction

In the present article, we take following useful standard notation: \( \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{Z}^- \), \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) be set of all negative integer, natural, real and complex number.

We know well known classical poly-logarithm function \( L_{ik}(z) \) (see [5, 10]) is given by

\[
L_{ik}(z) = \sum_{n=0}^{\infty} \frac{z^n}{m^n}, \quad (z \in \mathbb{C}, k \in \mathbb{Z}).
\]  

(1.1)

For \( k = 1 \), \( L_{ik}(z) = -\log(1 - z) \).

Recently, many authors (see [1, 2, 4, 5, 6, 7, 8, 9, 12, 14, 16, 17]) have studied the Dahee polynomial, Bernoulli polynomial and Euler polynomial. Also studied second kind Bernoulli polynomial, poly Bernoulli polynomial and generalized Bernoulli, Euler and Genocchi polynomials. Which is defined as follows:

The \( n^\text{th} \) Dahee polynomial and Dahee number (see [18]) are defined as follows:

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{D}_n(\gamma) = (1 + x)^\gamma \frac{\log(1 + x)}{x}.
\]  

(1.2)

If we take \( \gamma = 0 \) then \( \mathcal{D}_n := \mathcal{D}_n(0) \) are the Dahee number.

The Euler polynomial \( \mathcal{E}_n(\gamma) \) and Euler number \( \mathcal{E}_n(0) \) are defined by the following generating function to be

\[
e^{\gamma x} \left( \frac{2}{e^x + 1} \right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{E}_n(\gamma), \quad (|x| < 2\pi).
\]  

(1.3)

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If we take $\gamma = 0$ then the Euler number $\mathcal{E}_n(0) := \mathcal{E}_n$ are defined by
\[ \frac{2}{e^x + 1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{E}_n, \quad (|x| < 2\pi). \]  

The Bernoulli polynomial and Bernoulli number is introduced by Datolli et al. (see [3]), which is defined by the following generating function to be
\[ e^{\gamma x} \left( \frac{x}{e^x - 1} \right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{B}_n(\gamma), \quad (|x| < 2\pi). \]  

If we take $\gamma = 0$ then the Euler number $\mathcal{B}_n(0) := \mathcal{B}_n$ are defined by
\[ \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{B}_n, \quad (|x| < 2\pi). \]  

The poly-Bernoulli polynomial $\mathcal{B}_n^{(k)}(\gamma)$ are defined by the following generating function to be
\[ e^{\gamma x} \left( \frac{L_{ik}(1 - e^{-x})}{e^x - 1} \right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{B}_n^{(k)}(\gamma). \]  

If we take $\gamma = 0$, then poly-Bernoulli number $\mathcal{B}_n^{(k)}(0) := \mathcal{B}_n^{(k)}$ are defined by
\[ \frac{L_{ik}(1 - e^{-x})}{e^x - 1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{B}_n^{(k)}. \]  

For $k = 1$ in (1.9) and (1.8), we get $\mathcal{B}_n^{(1)}(\gamma) := \mathcal{B}_n(\gamma)$, $\mathcal{B}_n^{(1)} := \mathcal{B}_n$.

The higher order poly-Bernoulli polynomial $\mathcal{B}_n^{(k,r)}(\gamma)$ are defined by the following generating function to be
\[ e^{\gamma x} \left( \frac{L_{ik}(1 - e^{-x})}{e^x - 1} \right)^r = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{B}_n^{(k,r)}(\gamma). \]  

The poly-Daehee polynomials $\mathcal{D}_n^{(k)}(\gamma)$ (see [18]) are defined by
\[ \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} (1 + x)^\gamma = \sum_{n=0}^{\infty} \frac{\mathcal{D}_n^{(k)}(\gamma)}{n!} x^n \quad (k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}). \]  

If $\gamma = 0$, $\mathcal{D}_n^{(k)} = \mathcal{D}_n^{(k)}(0)$ are called the poly-Daehee numbers, which are defined as
\[ \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} = \sum_{n=0}^{\infty} \frac{\mathcal{D}_n^{(k)}}{n!} x^n \quad (k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}). \]  

The poly-Bernoulli polynomial $b_n^{(k)}(\gamma)$ of second kind (see [15]) are defined by the following generating function
\[(1 + x)^\gamma \left( \frac{L_k(1 - e^{-x})}{\log(1 + x)} \right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} b_n^{(k)}(\gamma). \quad (1.12)\]

If \( \gamma = 0 \), then \( b_n^{(k)}(0) := b_n^{(k)} \) are called poly-Bernoulli number of second kind.

The generalized Bernoulli, Euler and Genocchi polynomials of order \( a \in \mathbb{C} \) and \( |x| < 2\pi \) are defined by the following generating relation respectively:

\[
e^{\gamma x} \left( \frac{x}{e^x - 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} B_n^{(a)}(\gamma), \quad (1.13)\]

\[
e^{\gamma x} \left( \frac{2}{e^x + 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} E_n^{(a)}(\gamma), \quad (1.14)\]

\[
e^{\gamma x} \left( \frac{2x}{e^x + 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} G_n^{(a)}(\gamma). \quad (1.15)\]

If \( \gamma = 0 \) in generating function (1.11), (1.12) and (1.15), we get

\[
\left( \frac{x}{e^x - 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} B_n^{(a)}(0), \quad (1.16)\]

\[
\left( \frac{2}{e^x + 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} E_n^{(a)}(0), \quad (1.17)\]

\[
\left( \frac{2x}{e^x + 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} G_n^{(a)}(0), \quad (1.18)\]

where

\[
\mathcal{B}_n^{(a)}(0) := \mathcal{B}_n^{(a)}, \quad \mathcal{E}_n^{(a)}(0) := \mathcal{E}_n^{(a)}, \quad \mathcal{G}_n^{(a)}(0) := \mathcal{G}_n^{(a)} \quad (n \in \mathbb{N}_0),
\]

are the generalized Bernoulli, Euler and Genocchi numbers respectively.

It can be seen that if \( a = 1 \) in (1.11), (1.12) and (1.15) respectively, then

\[
\mathcal{B}_n^{(1)}(\gamma) := \mathcal{B}_n(\gamma), \quad \mathcal{E}_n^{(1)}(\gamma) := \mathcal{E}_n(\gamma), \quad \mathcal{G}_n^{(1)}(\gamma) := \mathcal{G}_n(\gamma) \quad (n \in \mathbb{N}_0).
\]

The generalized Apostol-Bernoulli polynomial \( \mathcal{B}_n^{(a)}(\gamma, \lambda) \) of order \( a \in \mathbb{C} \) (see [19]), defined by the following generating function to be

\[
e^{\gamma x} \left( \frac{x}{\lambda e^x - 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathcal{B}_n^{(a)}(\gamma; \lambda), \quad (|x + \ln \lambda| < 2\pi), \quad (1.19)\]

with

\[
\mathcal{B}_n^{(a)}(\gamma; 1) := \mathcal{B}_n^{(a)}(\gamma)
\]
The generalized Apostol-Euler polynomial $E_n^{a}(\gamma, \lambda)$ of order $a \in \mathbb{C}$ (see [20]), defined by the following generating function to be

$$e^{\gamma x} \left( \frac{2}{\lambda e^x + 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} E_n^{a}(\gamma; \lambda), \quad (|x + \ln \lambda| < 2\pi),$$

(1.20)

with

$E_n^{a}(\gamma; 1) := E_n^{a}(\gamma)$

and

$E_n^{a}(0; \lambda) := E_n^{a}(\lambda)$

which is known as Apostol-Euler number of order $a$.

The generalized Apostol-Genocchi polynomial $G_n^{a}(\gamma, \lambda)$ of order $a \in \mathbb{C}$ (see [21]), defined by the following generating function to be

$$e^{\gamma x} \left( \frac{2x}{\lambda e^x + 1} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} G_n^{a}(\gamma; \lambda), \quad (|x + \ln \lambda| < 2\pi),$$

(1.21)

with

$G_n^{a}(\gamma; 1) := G_n^{a}(\gamma)$

and

$G_n^{a}(0; \lambda) := G_n^{a}(\lambda)$

which is known as Apostol-Genocchi number of order $a$.

The generalized Apostol-Bernoulli polynomial $B_{n,a}^{[m-1]}(\eta, \lambda)$ of order $a \in \mathbb{C}$ (see [23]), defined by the following generating function to be

$$e^{\eta x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} B_{n,a}^{[m-1]}(\eta; \lambda)$$

(1.22)

If $\eta = 0$, the generalized Apostol-Bernoulli number defined by:

$$\left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} B_{n,a}^{[m-1]}(\lambda)$$

(1.23)
2. Generalized Apostol-Bernoulli Poly-Daehee Polynomials

In this section, we define a generalization and unification of generalized Apostol-Bernoulli poly-Daehee polynomials (GABPDP) and define a certain useful properties and implicit formulae of Apostol-Bernoulli poly-Daehee Polynomials. which is defined as follows:

For $\gamma, \eta \in \mathbb{R}, n, m \in \mathbb{N}$ and $a \in \mathbb{C}$ we have define the following generating function for generalized Apostol-Bernoulli poly Daehee polynomials (GABPDP):

$$(1 + x)\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})}e^{\eta x} \left(\frac{x^m}{\lambda e^x - \sum_{l=0}^{n-1} x^l \frac{1}{l!}}\right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda). \quad (2.1)$$

If $\gamma = \eta = 0$, then $\mathfrak{D}_{n,a}^{[k,m-1]}(0,0; \lambda) := \mathfrak{D}_{n,a}^{[k,m-1]}(\lambda)$ are called generalized Apostol-Bernoulli poly-Daehee number (GABPDN).

2.1. Special Cases. In this section, we discuss some particular cases of generalized Apostol-Bernoulli poly-Daehee polynomials:

1. If $m = 1$, then equation (2.1) become Apostol-Bernoulli poly-Daehee polynomials with generating function

$$(1 + x)\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})}e^{\eta x} \left(\frac{x}{\lambda e^x - 1}\right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{(k)}(\gamma, \eta; \lambda).$$

2. If $m = 1, \lambda = 1$ and $a = 1$, then equation (2.1) become Bernoulli poly-Daehee polynomials with generating function

$$(1 + x)\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})}e^{\eta x} \left(\frac{x}{e^x - 1}\right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{(k)}(\gamma, \eta).$$

3. If $m = 1, k = 1$, then equation (2.1) become Bernoulli based Dahee polynomials with generating function

$$(1 + x)\gamma \frac{\log(1 + x)}{x}e^{\eta x} \left(\frac{x}{\lambda e^x - 1}\right)^a = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{(k)}(\gamma, \eta).$$

4. If $m = 1, \eta = 0$ and $a = 0$, then (2.1) become poly Dahee polynomials with generating function

$$(1 + x)\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{(k)}(\gamma).$$

5. If $m = 1, k = 1, \eta = 0$ and $a = 0$, then (2.1) become $n^{th}$ Dahee polynomials with generating function

$$(1 + x)\gamma \frac{\log(1 + x)}{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{(k)}(\gamma).$$
Theorem 2.1. Let $\gamma, \eta \in \mathbb{R}$, $n, m \in \mathbb{N}_0$ then GABPDP satisfy the following relation:

$$\mathbb{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathbb{D}_{n-j}^{(k)}(\gamma) \mathbb{D}_{j}^{[m-1,a]}(\eta; \lambda).$$ \hspace{1cm} (2.2)

Proof. Using equation (2.1), we obtain

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \mathbb{D}_{n,a}^{(k)}(\gamma, \eta; \lambda) = (1 + x)^\gamma \frac{\log(1 + x)}{L_k(1 - e^{-x})} e^{\eta x} \left( \frac{x^m}{\lambda x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a = \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathbb{D}_{n}^{(k)}(\gamma) \right) \left( \sum_{j=0}^{\infty} \frac{x^j}{j!} \mathbb{D}_{j}^{[m-1,a]}(\eta; \lambda) \right) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^{n+j}}{n!j!} \mathbb{D}_{n}^{(k)}(\gamma) \mathbb{D}_{j}^{[m-1,a]}(\eta; \lambda).$$

Replacing $n$ by $n-j$ and using series arrangement technique, we obtain

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \mathbb{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left( \sum_{j=0}^{n} \binom{n}{j} \mathbb{D}_{n-j}^{(k)}(\gamma) \mathbb{D}_{j}^{[m-1,a]}(\eta; \lambda) \right).$$

By equating both side with same power of $x^n$ we get desired result. \hfill \Box

Theorem 2.2. Let $n, m \in \mathbb{N}_0$, $\gamma, \eta \in \mathbb{R}$ then GABPDP satisfy following relation:

$$\mathbb{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) = \frac{\mathbb{D}_{n+1,a}^{[k,m-1]}(\gamma + 1, \eta; \lambda) - \mathbb{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda)}{n + 1}.$$ \hspace{1cm} (2.3)

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \mathbb{D}_{n,a}^{(k)}(\gamma + 1, \eta; \lambda) - \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathbb{D}_{n,a}^{(k)}(\gamma, \eta; \lambda) = (1 + x)^{\gamma+1} \frac{\log(1 + x)}{L_k(1 - e^{-x})} e^{\eta x} \left( \frac{x^m}{\lambda x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a - (1 + x)^\gamma \frac{\log(1 + x)}{L_k(1 - e^{-x})} e^{\eta x} \left( \frac{x^m}{\lambda x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a = (1 + x)^\gamma \frac{\log(1 + x)}{L_k(1 - e^{-x})} e^{\eta x} \left( \frac{x^m}{\lambda x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a \left( 1 + x - 1 \right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \mathbb{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda)$$

Replacing $n$ by $n+1$ left hand side and equating both side the same power of $x^{n+1}$, we get desired result (2.3). \hfill \Box
Theorem 2.3. Let $\gamma, \eta, \omega \in \mathbb{R}$, $n, j \in \mathbb{N}_0$ and $m \in \mathbb{N}$ then generalized Apostol-Bernoulli poly-Daehee polynomial satisfy following relation:

$$\mathfrak{B}\mathfrak{D}_{n,a}^{[k,m-1]}(\gamma + \omega, \eta; \lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathfrak{B}\mathfrak{D}_{n-j,a}^{[k,m-1]}(\gamma, \eta; \lambda)(\omega)_j \quad (2.4)$$

Proof. we know that

$$(\omega)_j = \omega(\omega - 1)(\omega - 2)\ldots(\omega - j + 1)$$

By using above relation and replace $\gamma$ by $\gamma + \omega$ in equation (2.1), we get

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{B}\mathfrak{D}_{n,a}^{[k,m-1]}(\gamma + \omega, \eta; \lambda) = (1 + x)^{\gamma + \omega} \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} e^{\eta x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a (1 + x)^\omega$$

$$= (1 + x)^{\gamma} \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} e^{\eta x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a \left( \sum_{j=0}^{\infty} (\omega)_j \frac{x^j}{j!} \right)$$

$$= \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{B}\mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) \right) \left( \sum_{j=0}^{\infty} (\omega)_j \frac{x^j}{j!} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^{n+j}}{n! j!} \mathfrak{B}\mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda)(\omega)_j.$$
\[ L_{ik}(1 - e^{-x}) \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}^a } \]

\[ = \left( \sum_{j=0}^{\infty} \frac{x^j}{j!} \mathcal{B}_j^k \right) \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) \right) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^{n+j}}{n!j!} \mathcal{B}_j^k \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda). \]

By replacing \( n \) to \( n-j \) and arranging the terms in the above relation, we obtain

\[ (1 + x)^{\gamma} \frac{\log(1 + x)}{(e^x - 1)} e^{\eta x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}^a } \right) = \sum_{n=0}^{\infty} x^n \left( \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_j^k \mathfrak{D}_{n-j,a}^{[k,m-1]}(\gamma, \eta; \lambda) \right). \]

(2.6)

Again, solve for r.h.s. using (1.2) and (1.6), we get

\[ (1 + x)^{\gamma} \frac{\log(1 + x)}{(e^x - 1)} e^{\eta x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}^a } \right) = \sum_{n=0}^{\infty} x^n \left( \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_j^k \mathfrak{D}_{n-j,a}^{[k,m-1]}(\gamma, \eta; \lambda) \right). \]

(2.7)

Therefore, by comparing the equations (2.6) and (2.7), we get the desired result (2.5).

\[ \square \]

**Theorem 2.5.** Let \( \gamma, \eta, \lambda \in \mathbb{R} \), \( n, j \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \) the following relation hold true:

\[ \mathcal{B}^{[m-1]}_{n,a}(\eta; \lambda) = \sum_{j=0}^{\infty} \binom{n}{j} b^j_k(-\gamma) \mathfrak{D}_{n-j,a}^{[k,m-1]}(\gamma, \eta; \lambda). \]

(2.8)
Proof. By using (1.22) and (2.1),
\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{[m-1]}(\eta; \lambda) = e^{\eta x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)^a
\]
\[
= (1 + x)^{-\gamma} \left( \frac{L_{ik}(1 - e^{-x})}{\log(1 + x)} \right) \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda)
\]
\[
= \left( \sum_{j=0}^{\infty} \frac{x^j}{j!} b_j^{(k)} (-\gamma) \right) \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) \right)
\]
Therefore, by using series arrangement technique and equating the coefficient of \(x^n\) both side in the above relation, we get required result (2.8). \(\square\)

3. Implicit and summation formulae for GABPDP

In this section, we drive a some useful implicit and summation formula of generalized Apostol-Bernoulli based poly daehee polynomial (GABPDP).

**Theorem 3.1.** Let \(\gamma, \eta, \lambda \in \mathbb{R}, n, p, q, b, c \in \mathbb{N}_0, m \in \mathbb{N}\) and \(a \in \mathbb{C}\), the following implicit summation formula for generalized Apostol-Bernoulli poly-Daehee polynomials holds true:
\[
\mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta, \lambda) = \sum_{b,c} \left( \frac{b}{n} \right) \left( \frac{c}{q} \right) (\eta - \omega)^{n+q} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \omega; \lambda) \quad (3.1)
\]

**Proof.** We know following series manipulation formulae
\[
\sum_{N=0}^{\infty} g(N) \left( \frac{\gamma + \eta}{N!} \right)^N = \sum_{n,m=0}^{\infty} \frac{\gamma^n \eta^m}{n! m!} g(n + m) \quad (3.2)
\]
Now, replacing \(x\) by \(x + \mu\) in (2.1), we get
\[
(1 + (x + \mu))^\gamma \frac{\log(1 + (x + \mu))}{L_{ik}(1 - e^{-(x+\mu)})} e^{\eta(x+\mu)} \left( \frac{(x + \mu)^m}{\lambda e^{(x+\mu)} - \sum_{l=0}^{m-1} \frac{(x+\mu)^l}{l!}} \right)^a
\]
\[
= \sum_{b,c} \frac{x^b \mu^c}{b! c!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta, \lambda) \quad (3.3)
\]
Again, replacing \(\eta\) by \(\omega\) in (3.3) and equate with (3.3), we have
\[
e^{(\eta-\omega)(x+\mu)} \sum_{b,c} \frac{x^b \mu^c}{b! c!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \omega; \lambda) = \sum_{b,c} \frac{x^b \mu^c}{b! c!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) \quad (3.4)
\]
Using (3.2) in the l.h.s. of (3.5), we get
\[
\sum_{N=0}^{\infty} \frac{[(\eta - \omega)(x + \mu)]^N}{N!} \sum_{b,c=0}^{\infty} \frac{x^b \mu^c}{b! \ c!} \mathfrak{D}_{n,a+b+c}^{[k,m-1]}(\gamma, \omega; \lambda) = \sum_{b,c=0}^{\infty} \frac{x^b \mu^c}{b! \ c!} \mathfrak{D}_{n,a+b+c}^{[k,m-1]}(\gamma, \eta; \lambda).
\]

(3.5)

Using (3.2) in the l.h.s. of (3.5), we get
\[
\sum_{n,q=0}^{b,c} \frac{(\eta - \omega)^{n+q} x^n \mu^q}{n! q!} \sum_{b,c=0}^{\infty} \frac{x^b \mu^c}{b! \ c!} \mathfrak{D}_{n,a+b+c}^{[k,m-1]}(\gamma, \omega; \lambda) = \sum_{b,c=0}^{\infty} \frac{x^b \mu^c}{b! \ c!} \mathfrak{D}_{n,a+b+c}^{[k,m-1]}(\gamma, \eta; \lambda)
\]

(3.6)

Using l.h.s. series arrangement method after that replacing \(b\) by \(b - n\) and \(c\) by \(c - q\) in (3.6), we have
\[
\sum_{b,c=0}^{\infty} \sum_{n,q=0}^{b,c} \frac{(\eta - \omega)^{n+q} x^n \mu^q}{n! q!} \mathfrak{D}_{n,a+b+c}^{[k,m-1]}(\gamma, \omega; \lambda) = \sum_{b,c=0}^{\infty} \frac{x^b \mu^c}{b! \ c!} \mathfrak{D}_{n,a+b+c}^{[k,m-1]}(\gamma, \eta; \lambda)
\]

(3.7)

Therefore, by equating the coefficient of same power of \(x^b\) and \(x^c\) in (3.7), we get the desired result. \(\square\)

**Theorem 3.2.** For \(\gamma, \eta, \omega, \lambda \in \mathbb{R}, n \in \mathbb{N}_0, m \in \mathbb{N}\) and \(a, b \in \mathbb{C}\), then \(GABPDP\) satisfy the following relation:
\[
\mathfrak{D}_{n,a+b}^{[k,m-1]}(\gamma, \eta + \omega; \lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathfrak{D}_{n-j,a}^{[k,m-1]}(\gamma, \eta; \lambda) \mathfrak{D}_{j,b}^{[m-1]}(\omega, \lambda).
\]

(3.8)

**Proof.** Replacing \(\eta\) by \(\eta + \omega\) and \(a\) by \(a + b\) in equation (2.1), we obtain
\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a+b}^{[k,m-1]}(\gamma, \eta + \omega; \lambda) = (1 + x)\gamma \log(1 + x) L_{ik}(1 - e^{-x}) e^{(\eta + \omega)x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{\pi}} \right)^{a+b}
\]

\[
= \left( 1 + x \right)^\gamma \log(1 + x) e^{\eta x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{\pi}} \right)^a \left( e^{\omega x} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{\pi}} \right)^b \right)
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \mathfrak{D}_{n,a}^{[k,m-1]}(\gamma, \eta; \lambda) \right) \left( \sum_{j=0}^{\infty} \frac{x^j}{j!} \mathfrak{D}_{j,b}^{[m-1]}(\omega; \lambda) \right).
\]

First, we apply a series arrangement method after that comparing both sides to \(x^n\), we get required result (3.8). \(\square\)
Theorem 3.3. For $\gamma, \eta \in \mathbb{R}, n \in \mathbb{N}_0, m \in \mathbb{N}$ and $a \in \mathbb{C}$, the generalized Apostol-Bernoulli poly-Daehee Polynomials satisfy following relation:

$$B_{D}^{\left[k,m-1\right]}(\gamma, \eta; \lambda) = \sum_{j=0}^{n} \binom{n}{j} B_{D}^{\left[k,m-1\right]}(\gamma; \eta - \omega) B_{j, b}^{\left[m-1\right]}(\omega, \lambda)$$  \hspace{1cm} (3.9)

Proof. By using (2.1), we can write

$$(1 + x)^\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right) e^{(\eta-\omega)x+\omega x}$$

$$= \left( (1 + x)^\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} e^{(\eta-\omega)x} \right) \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right)$$

$$= \left( \sum_{n=0}^{\infty} D_{h,a}^{\left[k,m-1\right]}(\gamma; \eta - \omega) \right) \left( \sum_{j=0}^{\infty} B_{j, b}^{\left[m-1\right]}(\omega, \lambda) \right)$$

By using series arrangement technique and comparing both sides $x^n$, we get desired result (3.9). \hfill \Box

Theorem 3.4. Let $\gamma, \eta, \lambda \in \mathbb{R}, n, j \in \mathbb{N}_0, m \in \mathbb{N}$ and $a \in \mathbb{C}$, then GABPDP satisfy following relation:

$$B_{D}^{\left[k,m-1\right]}(\gamma, \eta + 1; \lambda) = \sum_{j=0}^{n} \binom{n}{j} B_{D}^{\left[k,m-1\right]}(\gamma, \eta; \lambda)$$  \hspace{1cm} (3.10)

Proof. Using (2.1) and replacing $\eta$ by $\eta + 1$, we obtain

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} D_{n, a}^{\left[k,m-1\right]}(\gamma, \eta + 1; \lambda) = (1 + x)^\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} \left( \frac{x^m}{\lambda e^x - \sum_{l=0}^{m-1} \frac{x^l}{l!}} \right) e^{(\eta+1)x}$$

$$= \left( (1 + x)^\gamma \frac{\log(1 + x)}{L_{ik}(1 - e^{-x})} e^{\eta x} \right) x^e$$

$$= \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} D_{n, a}^{\left[k,m-1\right]}(\gamma, \eta; \lambda) \right) \left( \sum_{j=0}^{\infty} \frac{x^j}{j!} \right).$$

By using series arrangement technique and comparing both side $x^n$, we get required result (3.10). \hfill \Box
In this paper, we defined the relation between generalized Apostol-Bernoulli polynomial and poly Daehee polynomial known as generalized Apostol-Bernoulli poly-Daehee polynomial (GABPDP) with generating function (2.1). Which is very useful to us, because polynomial plays a very important role to use as a solution of different kind of differential equations, also polynomial use to represent a different kind of characteristic linear dynamic system. We also discuss some useful identities and their implicit summation formulae of generalized Apostol-Bernoulli poly-Daehee polynomials (GABPDP).

4. Conclusion

In this paper, we defined the relation between generalized Apostol-Bernoulli polynomial and poly Daehee polynomial known as generalized Apostol-Bernoulli poly-Daehee polynomial (GABPDP) with generating function (2.1). Which is very useful to us, because polynomial plays a very important role to use as a solution of different kind of differential equations, also polynomial use to represent a different kind of characteristic linear dynamic system. We also discuss some useful identities and their implicit summation formulae of generalized Apostol-Bernoulli poly-Daehee polynomials (GABPDP).

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