Abstract

We consider the extension of the method of Gauss-Newton from complex floating-point arithmetic to the field of truncated power series with complex floating-point coefficients. With linearization we formulate a linear system where the coefficient matrix is a series with matrix coefficients, and provide a characterization for when the matrix series is regular based on the algebraic variety of an augmented system. The structure of the linear system leads to a block triangular system. In the regular case, solving the linear system is equivalent to solving a Hermite interpolation problem. We show that this solution has cost cubic in the problem size. In general, at singular points, we rely on methods of tropical algebraic geometry to compute Puiseux series. With a few illustrative examples, we demonstrate the application to polynomial homotopy continuation.

1 Introduction

1.1 Preliminaries

A polynomial homotopy is a family of polynomial systems which depend on one parameter. Numerical continuation methods to track solution paths defined by a homotopy are classical, see e.g.: [3] and [27]. Studies of deformation methods in symbolic computation appeared in [10], [11], and [17]. In particular, the application of Padé approximants in [22] stimulated our development of methods to compute power series.

Keywords and phrases: Linearization, Gauss-Newton, Hermite interpolation, polynomial homotopy, power series, Puiseux series.

*This material is based upon work supported by the National Science Foundation under Grant No. 1440534. Date: 25 October 2017.
**Problem statement.** We want to define an efficient, numerically stable, and robust algorithm to compute a power series expansion for a solution curve of a polynomial homotopy. The input is a list of polynomials in several variables, where one of the variables is a parameter denoted by \( t \), and a value of \( t \) near which information is desired. The output of the algorithm is a tuple of series in \( t \) that vanish up to a certain degree when plugged in to either the original equations or, in special cases, a transformation of the original equations.

A power series for a solution curve forms the input to the computation of a Padé approximant for the solution curve, which will then provide a more accurate predictor in numerical path trackers. Polynomial homotopies define deformations of polynomial systems starting at generic instances and moving to specific instances. Tracking solution paths that start at singular solutions is not supported by current numerical polynomial homotopy software systems. At singular points we encounter series with fractional powers, Puiseux series.

**Background and related work.** As pointed out in [7], polynomials, power series, and Toeplitz matrices are closely related. A direct method to solve block banded Toeplitz systems is presented in [12]. The book [6] is a general reference for methods related to approximations and power series. We found inspiration for the relationship between higher-order Newton-Raphson iterations and Hermite interpolation in [24]. The computation of power series is a classical topic in computer algebra [10]. In [4], new algorithms are proposed to manipulate polynomials by values via Lagrange interpolation.

The Puiseux series field is one of the building blocks of tropical algebraic geometry [26]. For the leading terms of the Puiseux series, we rely on tropical methods [9], and in particular on the constructive proof of the fundamental theorem of tropical algebraic geometry [21], see also [23] and [28]. Computer algebra methods for Puiseux series in two dimensions can be found in [29].

**Our contributions.** Via linearization, rewriting matrices of series into series with matrix coefficients, we formulate the problem of computing the updates in Newton’s method as a block structured linear algebra problem. For matrix series where the leading coefficient is regular, the solution of the block linear system satisfies the Hermite interpolation problem. For general matrix series, where several of the leading matrix coefficients may be rank deficient, Hermite-Laurent interpolation applies. We characterize when these cases occur using the algebraic variety of an augmented system. To solve the block diagonal linear system, we propose to reduce the coefficient matrix to a lower triangular echelon form, and we provide a brief analysis of its cost.

The source code for the algorithm presented in this paper is archived at github via our accounts nbliss and janverschelde.

**Acknowledgments.** We thank the organizers of the ILAS 2016 minisymposium on Multivariate Polynomial Computations and Polynomial Systems, Bernard Mourrain, Vanni Noferini, and Marc Van Barel, for giving the second author the opportunity to present this work. In addition, we are grateful to the anonymous referee who supplied many helpful remarks.
1.2 Motivating Example: Padé Approximant

One motivation for finding a series solution is that once it is obtained, one can directly compute the associated Padé approximant, which often has much better convergence properties. Padé approximants are applied in symbolic deformation algorithms. In this section we reproduce Figure 1.1.1 in the context of polynomial homotopy continuation. Consider the homotopy

\[(1 - t)(x^2 - 1) + t(3x^2 - 3/2) = 0. \tag{1}\]

The function \[x(t) = \left(\frac{1 + t/2}{1 + 2t}\right)^{1/2}\] is a solution of this homotopy.

Its second order Taylor series at \(t = 0\) is \[s(t) = 1 - 3t/4 + 39t^2/32 + O(t^2)\]. The Padé approximant of degree one in numerator and denominator is \[q(t) = \frac{1 + 7t/8}{1 + 13t/8}\]. In Figure 1 we see that the series approximates the function only in a small interval and then diverges, whereas the Padé approximant is more accurate.

![A rational versus a series approximation of a function](image)

Figure 1: Comparing a Padé approximant to a series approximation shows the promise of applying Padé approximants as predictors in numerical continuation methods.
1.3 Motivating Example: Viviani’s Curve

Viviani’s curve is defined as the intersection of the sphere \((x_1 + 2)^2 + x_2^2 + x_3^2 = 4\) and the cylinder \((x_1 + 1)^2 + x_2^2 = 1\) such that the surfaces are tangent at a single point; see Figure 2. Our methods will allow us to find the Taylor series expansion around any point on a 1-dimensional variety, assuming we have suitable starting information. For example, the origin \((0, 0, 0)\) satisfies both equations of Viviani’s curve. This is the point where the curve intersects itself, so the curve is singular there, meaning algebraically that the Jacobian drops rank, and geometrically that the tangent space does not have the expected dimension. If we apply our methods at this point, we obtain the following series solution for \(x_1, x_2, x_3\):

\[
\begin{align*}
-2t^2 &

2t - t^3 - \frac{1}{4}t^5 - \frac{1}{8}t^7 - \frac{5}{64}t^9 - \frac{7}{128}t^{11} - \frac{21}{512}t^{13} - \frac{33}{1024}t^{15}
\end{align*}
\]

This solution is plotted in Figure 3 for a varying number of terms. To check the correctness, we can substitute (2) into the original equations, obtaining series in \(O(t^{18})\). The vanishing of the lower-order terms confirms that we have indeed found an approximate series solution. Such a solution, possibly transformed into an associated Padé approximant, would allow for path tracking starting at the origin.

\footnote{Definition 2.1 makes this precise for general curves.}
2 The Problem and Our Solution

2.1 Problem Setup

For a polynomial system \( f = (f_1, f_2, \ldots, f_m) \) where each \( f_i \in \mathbb{C}[t, x_1, \ldots, x_n] \), the solution variety \( \mathbb{V}(f) \) is the set of points \( p \in \mathbb{C}^{n+1} \) such that \( f_1(p) = \cdots = f_m(p) = 0 \). Let \( f \) be a system such that the solution variety is 1-dimensional over \( \mathbb{C} \) and is not contained in the \( t = 0 \) coordinate hyperplane. We seek to understand \( \mathbb{V}(f) \) by treating the \( f_i \)'s as elements of \( \mathbb{C}((t))[x_1, \ldots, x_n] \), or in other words, polynomials in \( x_1 \ldots x_n \) with coefficients in the ring of formal Laurent series \( \mathbb{C}((t)) \). In this context we will denote the system by \( \tilde{f} \).

Our approach is to use Newton iteration on the system \( \tilde{f} \). Namely, we find some starting \( z \in \mathbb{C}((t))^n \) and repeatedly solve

\[
J_{\tilde{f}}(z) \Delta z = -\tilde{f}(z)
\]

for the update \( \Delta z \) to \( z \), where \( J_{\tilde{f}} \) is the Jacobian matrix of \( \tilde{f} \) with respect to \( x_1, \ldots, x_n \). This is a system of equations that is linear over \( \mathbb{C}((t)) \), so the problem is well-posed. Computationally speaking, one approach to solving it would be to overload the operators on (truncated) power series and apply basic linear algebra techniques. A main point of our paper is that this method can be improved upon.

Of course, applying Newton’s method requires a starting guess; here we must define what it means to be singular:

**Definition 2.1.** A point \( p \) on a \( d \)-dimensional component of a variety \( \mathbb{V}(f) \subset \mathbb{C}^n \) is *regular* if the Jacobian of \( f \) evaluated at \( p \) has rank \( n - d \). Points that are not regular are called *singular.*
In most cases the starting guess for Newton’s method can just be a point \( \tilde{p} = (p_1, \ldots, p_n) \) such that \( p = (0, p_1, \ldots, p_n) \) is in \( \mathcal{V}(f) \). However, if \( p \) is a singular point, this is insufficient. In addition, \( p \) could be a branch point (which we define later), in which case it is also not enough to use as the starting guess for Newton’s method.

We solve two problems in this paper. First, we find an effective way to perform the Newton step; the framework is established in Section 2.2 and our solution is laid out in Section 2.4. And second, we discuss the prelude to Newton’s method in Section 2.3 characterizing when techniques from tropical geometry are needed to transform the problem and obtain the starting guess.

2.2 The Newton Step

Solving the Newton step (3) amounts to solving a linear system

\[
Ax = b
\]

over the field \( \mathbb{C}((t)) \). Our first step is linearization, which turns a vector of series into a series of vectors, and likewise for a matrix series. In other words, we refactor the problem and think of \( x \) and \( b \) as in \( \mathbb{C}^n((t)) \) instead of \( \mathbb{C}((t))^n \), and \( A \) as in \( \mathbb{C}^{n\times n}((t)) \) instead of \( \mathbb{C}((t))^{n\times n} \).

Suppose that \( a \) is the lowest order of a term in \( A \), and \( b \) the lowest order of a term in \( b \). Then we can write the linearized

\[
A = A_0 t^a + A_1 t^{a+1} + \ldots,
\]
\[
b = b_0 t^b + b_1 t^{b+1} + \ldots, \text{ and}
\]
\[
x = x_0 t^{b-a} + x_1 t^{b-a+1} + \ldots
\]

where \( A_i, b_i, x_i \in \mathbb{C}^n \). Expanding and equating powers of \( t \), the linearized version of (4) is therefore equivalent to solving

\[
A_0 x_0 = b_0
\]
\[
A_0 x_1 = b_1 - A_1 x_0
\]
\[
A_0 x_2 = b_2 - A_1 x_1 - A_2 x_0
\]
\[
\vdots
\]
\[
A_0 x_d = b_d - A_1 x_{d-1} - A_2 x_{d-2} - \cdots - A_d x_0
\]

for some \( d \). This can be written in block matrix form as

\[
\begin{bmatrix}
A_0 & A_1 & A_2 & \cdots & A_d \\
A_1 & A_0 & A_1 & \cdots & A_{d-1} \\
A_2 & A_1 & A_0 & \cdots & A_{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_d & A_{d-1} & A_{d-2} & \cdots & A_0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_d
\end{bmatrix}
= \begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_d
\end{bmatrix}.
\]
For the remainder of this paper, we will use $z$ and $\Delta z$ to denote vectors of series, while $x$ and $\Delta x$ will denote their linearized counterparts, that is, series which have vectors for coefficients.

**Example 1.** Let

$$f = (2t^2 + tx_1 - x_2 + 1, x_1^3 - 4t^2 + tx_2 + 2t - 1).$$

Starting with $z = (1, 1)$, the first Newton step $J_f(z)\Delta z = -f(z)$ can be written:

$$\begin{bmatrix} t & -1 \\ 3 & t \end{bmatrix} \Delta z = \begin{bmatrix} t + 2t^2 \\ 3t - 4t^2 \end{bmatrix}. \quad (11)$$

To put in linearized form, we have $a = 0, b = 1$,

$$A_0 = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$b_0 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}, \text{ and } b_1 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}. \quad (12)$$

Since $A_0$ is regular, we can solve in staggered form as in (8), which yields the next term:

$$\Delta x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t. \quad (14)$$

After another iteration, our series solution is

$$\begin{bmatrix} 1 - t \\ 1 + t + t^2 \end{bmatrix}. \quad (15)$$

In fact this is the entire series solution for $f$ — substituting (15) into $f$ causes both polynomials to vanish completely.

**Remark 1.** We constructed the example above so its solution is a series with finitely many terms, a polynomial. The solution of (4) can be interpreted as the solution obtained via Hermite interpolation. Observe that for a series

$$x(t) = x_0 + x_1 t + x_2 t^2 + x_3 t^3 + \cdots + x_k t^k + \cdots \quad (16)$$

its Maclaurin expansion is

$$x(t) = x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \frac{1}{3!}x'''(0)t^3 + \cdots + \frac{1}{k!}x^{(k)}(0)t^k + \cdots \quad (17)$$

where $x^{(k)}(0)$ denotes the $k$-th derivative of $x(t)$ evaluated at zero. Then:

$$x_k = \frac{1}{k!}x^{(k)}(0), \quad k = 0, 1, \ldots \quad (18)$$

Solving (4) up to degree $d$ implies that all derivatives up to degree $d$ of $x(t)$ at $t = 0$ match the solution. If the solution is a polynomial, then this polynomial will be obtained if (4) is solved up to the degree of the polynomial.
Figure 4: Lifting $x = 0$ to three different types of point. In general, the line $x = 0$ intersects the curve at regular points. If the curve intersects itself for $x = 0$, we are at a singular point. The curve turns at a branch point.

2.3 The Starting Guess, and Related Considerations

Our hope is that a solution $z(t)$ of $\tilde{f}$ parameterizes the curve in some neighborhood of a point $p \in V(f)$. In other words, if $\pi$ is the projection map of $V(f)$ onto the $t$-coordinate axis, then $z(t)$ should be a branch of $\pi^{-1}$.

It is natural to think that there are two scenarios for the starting point $p \in V(f)$, namely that it is a regular point or it is singular. And indeed, when $p$ is singular, tropical methods are required. Intuitively speaking, when at a singular point, knowing just the point itself is insufficient to determine the series; higher-derivative information is required. Observe the second frame of Figure 4.

The point $p$ being regular, however, is not enough. Consider the third frame of Figure 4. Here $x = 0$ cannot be lifted because the origin is a branch point of the curve. In other words, the derivative at $p$ in terms of $t$ is undefined, so a Taylor series in $t$ is impossible without a transformation of the problem.

The proper way to check if Newton’s method can be applied directly to $p$, or whether tropical methods are needed, is by checking if $p$ is a singular point of $V(f) \cap V(t)$. Setting $f_{\text{aug}} = (t, f_1, \ldots, f_n)$, we have $V(f_{\text{aug}}) = V(f) \cap V(t)$. We can thus use $V(f_{\text{aug}})$ to distinguish the first frame of Figure 4 from the latter two. This is summarized and proven in the following.

**Proposition 2.2.** Let $p = (0, p_1, \ldots, p_n) \in V(f)$, and set $\tilde{p} = (p_1, \ldots, p_n)$. Then $p$ is a regular point of $V(f_{\text{aug}})$ if and only if for every step of Newton’s method applied to $x(t) := \tilde{p}$, $a = 0$ and $A_0$ has full rank.

**Proof.** ($\Rightarrow$) By definition, $p$ is a regular point of $f_{\text{aug}}$ if and only if $J_{f_{\text{aug}}}(p)$ has full rank. But note that $J_{f_{\text{aug}}}$ is

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\frac{df_1}{dt} & \frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\
\frac{df_2}{dt} & \frac{df_2}{dx_1} & \cdots & \frac{df_2}{dx_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{df_m}{dt} & \frac{df_m}{dx_1} & \cdots & \frac{df_m}{dx_n}
\end{bmatrix}
$$

(19)
and $J_f$ is
\[
\begin{bmatrix}
\frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\
\frac{df_2}{dx_1} & \cdots & \frac{df_2}{dx_n} \\
\vdots & & \vdots \\
\frac{df_m}{dx_1} & \cdots & \frac{df_m}{dx_n}
\end{bmatrix}.
\]
(20)

So $J_{f_{\text{aug}}}$ has full rank at $p$ if and only if $J_\tilde{f}|_{t=0}$ has full rank at $\tilde{p}$. Thus it suffices to show that after each Newton step, $a = 0$ and $x(0) = \tilde{p}$ remain true, so that $A_0 = J_f(x(0)) = J_\tilde{f}(\tilde{p})|_{t=0}$ continues to have full rank.

We clearly have $a \geq 0$ at every step, since the Newton iteration cannot introduce negative exponents. At the beginning, $a = 0$ and $x(0) = \tilde{p}$ hold trivially. Inducting on the Newton steps, if $a = 0$ and $x(0) = \tilde{p}$ at some point in the algorithm, then the next $A_0$, namely $J_f(x(0)) = J_\tilde{f}(\tilde{p})|_{t=0}$, is the same matrix as in the last step, hence it is again regular and $a$ is 0. Since $f(x(0)) = f(\tilde{p})|_{t=0} = 0$, $b$ must be strictly greater than 0. Thus the next Newton update $\Delta x$ must have positive degree in all components, leaving $x(0) = \tilde{p}$ unchanged.

$(\Leftarrow)$ If $p$ is a singular point of $V(f_{\text{aug}})$, then on the first Newton step $A_0 = J_f(\tilde{p})|_{t=0}$ must drop rank by the same argument as above comparing (19) and (20).

To summarize the cases:

**Lemma 2.3.** There are three possible scenarios for $V(f_{\text{aug}})$:

1. $\exists p \in V(f_{\text{aug}})$ regular,
2. $\exists p \in V(f_{\text{aug}})$ singular, or
3. $\nexists p \in V(f_{\text{aug}})$

In the first case, we can simply use $\tilde{p} = (p_1, p_2, \ldots, p_n)$ to start the Newton iteration. In the second, we must defer to tropical methods in order to obtain the necessary starting $z$, which will lie in $\mathbb{C}[[t]]^n$. In the final case, we also defer to tropical methods, which provide a starting $z$ that will have negative exponents. A change of coordinates brings the problem back into one of the first two cases, and we can apply our method directly. It is important to reiterate that $p$ may be a regular point of $V(f)$ but a singular point of $V(f_{\text{aug}})$, as is the case in the third frame of Figure 4. The following example also demonstrates this behavior.

**Example 2** (Viviani, continued). In Section 1.3 we introduced the example of Viviani’s curve. If we translate by a substitution so that setting $x_1 = 0$ gives not the singular point at the origin, but instead the highest and lowest points on the curve, the system becomes
\[
f = (x_1^2 + x_2^2 + x_3^2 - 4, (x_1 - 1)^2 + x_2^2 - 1).
\]
(21)
When $x_1 = 0$ we obtain the two points $(0,0,2)$ and $(0,0,-2)$, which are both regular points. For the augmented system $f_{aug}$, the Jacobian $J_{f_{aug}}$ is

$$
\begin{bmatrix}
1 & 0 & 0 \\
2x_1 & 2x_2 & 2x_3 \\
2x_1 - 2 & 2x_2 & 0
\end{bmatrix}
$$

which at the point $p = (0,0,2)$ becomes

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 4 \\
-2 & 0 & 0
\end{bmatrix}
$$

This matrix drops rank, hence $p$ is a singular point of $f_{aug}$ and we are in the second case of Lemma 2.3. Following Lemma 2.3 we defer to tropical methods to begin, obtaining the transformation $x_1 \rightarrow 2t^2$ and the starting term $z = (2t, 2)$.

Now the first Newton step can be written:

$$
\begin{bmatrix}
4t & 0 \\
4t & 0
\end{bmatrix}
\Delta z = -\begin{bmatrix}
4t^2 + 4t^4 \\
4t^4
\end{bmatrix}.
$$

Note that $J_{\tilde{f}}(z)$ is now invertible over $\mathbb{C}((t))$. Its inverse begins with negative exponents of $t$:

$$
\begin{bmatrix}
0 & 1/4 \\
1/4 t^{-1} & -1/4 t^{-1}
\end{bmatrix}
$$

To linearize, we first observe that $a = 0$ and $b = 2$, so $x$ will have degree at least $b - a = 2$. The linearized block form of (24) is then

$$
\begin{bmatrix}
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 4 \\
0 & 0 & 4 & 0 & 0 & 0
\end{bmatrix}
\Delta x =
\begin{bmatrix}
-4 \\
0 \\
0 \\
0 \\
-4 \\
-4
\end{bmatrix}.
$$

Whether we solve (24) over $\mathbb{C}((t))$ or solve (26) in the least squares sense, we obtain the same Newton update

$$
\Delta x = \begin{bmatrix} 0 \\ -1 \end{bmatrix} t^2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^3,
$$

or in non-linearized form,

$$
\Delta z = \begin{bmatrix} -t^3 \\ -t^2 \end{bmatrix}.
$$

Substituting $z + \Delta z = (2t - t^3, 2 - t^2)$ into (21) produces $(x_1^6 + x_1^4, x_1^6)$, and we have obtained the desired cancellation of lower-order terms.

The matrix in (26) we call a Hermite-Laurent matrix, because its correspondence with Hermite-Laurent interpolation.
Figure 5: The banded block structure of a generic Hermite-Laurent matrix for $n = 5$ at the left, with at the right its lower triangular echelon form.

### 2.4 A Lower Triangular Echelon Form

When we are in the regular case of Lemma 2.3 and the condition number of $A_0$ is low, we can simply solve the staggered system (8). When this is not possible, we are forced to solve (9). Figure 5 shows the structure of the coefficient matrix (9) for the regular case, when $A_0$ is regular and all block matrices are dense. The essence of this section is that we can use column operations to reduce the block matrix to a lower triangular echelon form as shown at the right of Figure 5, solving (9) in the same time as (8).

The lower triangular echelon form of a matrix is a lower triangular matrix with zero elements above the diagonal. If the matrix is regular, then all diagonal elements are nonzero. For a singular matrix, the zero rows of its echelon form are on top (have the lowest row index) and the zero columns are at the right (have the highest column index). Every nonzero column has one pivot element, which is the nonzero element with the smallest row index in the column. All elements at the right of a pivot are zero. Columns may need to be swapped so that the row indices of the pivots of columns with increasing column indices are sorted in decreasing order.

**Example 3.** (Viviani, continued). For the matrix series in (26), we have the following reduction:

$$
\begin{pmatrix}
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 4 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 4 \\
0 & 0 & 4 & 0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 4 & 4 & 0
\end{pmatrix}.
$$

(29)

Because of the singular matrix coefficients in the series, we find zeros on the diagonal in the echelon form.

Given a general $n$-by-$m$ dimensional matrix $A$, the lower triangular echelon form $L$ can be described by one $n$-by-$n$ row permutation matrix $P$ which swaps
the zero rows of \( A \) and a sequence of \( m \) column permutation matrices \( Q_k \) (of dimension \( m \)) and multiplier matrices \( U_k \) (also of dimension \( m \)). The matrices \( Q_k \) define the column swaps to bring the pivots with lowest row indices to the lowest column indices. The matrices \( U_k \) contain the multipliers to reduce what is at the right of the pivots to zero. Then the construction of the lower triangular echelon form can be summarized in the following matrix equation:

\[
L = PAQ_1Q_2U_2 \cdots Q_mU_m.
\]

(30)

Similar to solving a linear system with a LU factorization, the multipliers are applied to the solution of the lower triangular system which has \( L \) as its coefficient matrix.

3 Some Preliminary Cost Estimates

Working with truncated power series is somewhat similar to working with extended precision arithmetic. In this section we make some observations regarding the cost overhead.

3.1 Cost of one step

First we compare the cost of computing a single Newton step using the various methods introduced. We let \( d \) denote the degree of the truncated series in \( A(t) \), and \( n \) the dimension of the matrix coefficients in \( A(t) \) as before.

The staggered system. In the case that \( a \geq 0 \) and the leading coefficient \( A_0 \) of the matrix series \( A(t) \) is regular, the equations in (8) can be solved with \( O(n^3) + O(dn^2) \) operations. The cost is \( O(n^3) \) for the decomposition of the matrix \( A_0 \), and \( O(dn^2) \) for the back substitutions using the decomposition of \( A_0 \) and the convolutions to compute the right hand sides.

The big block matrix. Ignoring the triangular matrix structure, the cost of solving the larger linear system (9) is \( O((dn)^3) \).

The lower triangular echelon version. If the leading coefficient \( A_0 \) in the matrix series is regular (as illustrated by Figure 5), we may copy the lower triangular echelon form \( L_0 = A_0Q_0U_0 \) of \( A_0 \) to all blocks on the diagonal and apply the permutation \( Q_0 \) and column operations as defined by \( U_0 \) to all other column blocks in \( A \). The regularity of \( A_0 \) implies that we may use the lower triangular echelon form of \( L_0 \) to solve (9) with substitution. Thus with this quick optimization we obtain the same cost as solving the staggered system (8).

In general, \( A_0 \) and several other matrix coefficients may be rank deficient, and the diagonal of nonzero pivot elements will shift towards the bottom of \( L \). We then find as solutions vectors in the null space of the upper portion of the matrix \( A \).
3.2 Cost of computing $D$ terms

Assume that $D = 2^k$. In the regular case, assuming quadratic convergence, it will take $k$ steps to compute $2^k$ terms. We can reuse the factorization of $A_0$ at each step, so we have $O(n^3)$ for the decomposition plus

$$O(2n^2 + 4n^2 + 8n^2 + \cdots + 2^{k-1}n^2) = O(2^kn^2)$$

(31)

for the back substitutions. Putting these together, we find the cost of computing $D$ terms to be $O(n^3) + O(Dn^2)$.

4 Computational Experiments

Our power series methods have been implemented in PHCpack [33] and are available to the Python programmer via phcpy [34]. To set up the problems we used the computer algebra system Sage [32], and for tropical computations we used Gfan [8] and Singular [13] via the Sage interface.

4.1 The Problem of Apollonius

Figure 6: Singular configuration of Apollonius circles. The input circles are filled in, the solution circles are dark gray. Because the input circles mutually touch each other, three of the solution circles coincide with the input circles.

The classical problem of Apollonius consists in finding all circles that are simultaneously tangent to three given circles. A special case is when the three circles are mutually tangent and have the same radius; see Figure 6. Here the solution variety is singular – the circles themselves are double solutions. In this
figure, all have radius 3, and centers $(0, 0)$, $(2, 0)$, and $(1, \sqrt{3})$. We can study this configuration with power series techniques by introducing a parameter $t$ to represent a vertical shift of the upper circle. We then examine the solutions as we vary $t$. This is represented algebraically as a solution to

\[
\begin{cases}
  x_1^2 + x_2^2 - r^2 - 2r - 1 = 0 \\
  x_1^2 + x_2^2 - r^2 - 4x_1 - 2r + 3 = 0
\end{cases}
\]  

(32)

Because we are interested in power series solutions of (32) near $t = 0$, we use $t$ as our free variable. To simplify away the $\sqrt{3}$, we substitute $t \rightarrow \sqrt{3}t$, $x_1 \rightarrow \sqrt{3}x_1$, and the system becomes

\[
\begin{cases}
  x_1^2 + 3x_2^2 - r^2 - 2r - 1 = 0 \\
  x_1^2 + 3x_2^2 - r^2 - 4x_1 - 2r + 3 = 0 \\
  3t^2 + x_1^2 - 6tx_2 + 3x_2^2 - r^2 + 6t - 2x_1 - 6x_2 + 2r + 3 = 0
\end{cases}
\]

(33)

Call this system $f$. Now we examine the system at $(t, x_1, x_2, r) = (0, 1, 1, 1) = p$. The Jacobian $J_f$ at $p$ is

\[
\begin{bmatrix}
  0 & 2 & 6 & -4 \\
  -2 & 6 & -4 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

(34)

so $f$ — and by extension $f_{\text{aug}}$ — is singular at $p$, and we are in the second case of Lemma 2.3. Tropical methods give two possible starting solutions, which rounded for readability are $(t, 1, 1 + 0.536t, 1 + 0.804t)$ and $(t, 1, 1 + 7.464t, 1 + 11.196t)$. We will continue with the second; call it $z$. For the first step of Newton’s method, $A$ is

\[
\begin{bmatrix}
  2 & 6 & -4 \\
  -2 & 6 & -4 \\
  0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
  0 & 44.785 & -22.392 \\
  0 & 44.785 & -22.392 \\
  0 & 38.785 & -22.392
\end{bmatrix} t
\]

(35)

and $b$ is

\[
\begin{bmatrix}
  41.785 \\
  41.785 \\
  0
\end{bmatrix} t^2.
\]

(36)

From these we can construct the linearized system

\[
\begin{bmatrix}
  A_0 & A_0 \\
  A_1 & A_0
\end{bmatrix} \Delta x = \begin{bmatrix}
  b_0 \\
  0
\end{bmatrix}.
\]

(37)

Solving in the least squares sense, we obtain two more terms of the series, so in total we have

\[
\begin{cases}
  x_1 = 1 \\
  x_2 = 1 + 7.464t + 45.017t^2 + 290.992t^3 \\
  r = 1 + 11.196t + 77.971t^2 + 504.013t^3.
\end{cases}
\]

(38)
By comparison, the series we obtain from the other possible starting solution is

\[
\begin{align*}
  x_1 &= 1 \\
  x_2 &= 1 + 0.536t - 0.0177t^2 + 0.00777t^3 \\
  r &= 1 + 0.804t + 0.0297t^2 - 0.013t^3.
\end{align*}
\]

(39)

From these, we get a good idea of what happens near \( t = 0 \): the first solution circle grows rapidly (corresponding to the larger coefficients in (38)), while the other stays small (corresponding to the smaller coefficient in (39)). This is illustrated in Figure 7 which shows the solutions of the system at \( t = 0.13 \).

![Figure 7: Solution to (32) for \( t = 0.13 \). The largest circles correspond to power series solutions with larger coefficients than the coefficients of the power series solutions for the smaller circles.](image)

This example demonstrates the application of power series solutions in polynomial homotopies. Current numerical continuation methods cannot be applied to track the solution paths defined by the homotopy in (32), because at \( t = 0 \), the start solutions are double solutions. The power series solutions provide reliable predictions to start tracking the solution paths defined by (32).

4.2 Tangents to Four Spheres

Our next example is that of finding all lines mutually tangent to four spheres in \( \mathbb{R}^3 \); see [14], [25], [30], and [31]. If a sphere \( S \) has center \( c \) and radius \( r \), the condition that a line in \( \mathbb{R}^3 \) is tangent to \( S \) is given by

\[
\|m - c \times t\|^2 - r^2 = 0,
\]

(40)

where \( m = (x_0, x_1, x_2) \) and \( t = (x_3, x_4, x_5) \) are the moment and tangent vectors of the line, respectively. For four spheres, this gives rise to four polynomial
equations; if we add the equation $x_0x_3 + x_1x_4 + x_2x_5 = 0$ to require that $t$ and $m$ are perpendicular and $x_3^2 + x_4^2 + x_5^2 = 1$ to require that $\|t\| = 1$, we have a system of 6 equations in 6 unknowns which we expect to be 0-dimensional.

Figure 8: A singular configuration of four spheres. The input spheres mutually touch each other and the tangent lines common to all four input spheres occur with multiplicity.

If we choose the centers to be $(+1, +1, +1), (+1, -1, -1), (-1, +1, -1)$, and $(-1, -1, +1)$ and the radii to all be $\sqrt{2}$, the spheres all mutually touch and the configuration is singular; see Figure 8. In this case, the number of solutions drops to three, each of multiplicity 4.

Next we introduce an extra parameter $t$ to the equations so that the radii of the spheres are $\sqrt{2} + t$. This results in a 1-dimensional system $F$, which we omit for succinctness. $F$ is singular at $t = 0$, so we are once again in the second case of Lemma 2.3. Tropical and algebraic techniques — in particular, the tropical basis [8] in Gfan [20] and the primary decomposition in Singular [13].
— decompose $F$ into three systems, one of which is

$$f = \begin{cases} 
  x_0 &= 0 \\
  x_3 &= 0 \\
  x_4^2 + x_2 x_5 + x_3^2 &= 0 \\
  x_1 x_4 + x_2 x_5 &= 0 \\
  x_1 x_2 - x_2 x_4 + x_1 x_5 &= 0 \\
  x_1^2 + x_2^2 - 1 &= 0 \\
  2t^4 + 4t^2 + x_2 x_5 &= 0 \\
  x_2^2 x_4 - x_2 x_5 + x_1 x_5^2 - x_4 &= 0 \\
  x_2^2 - x_2 - x_5 &= 0. 
\end{cases}$$

(41)

Using our methods we can find several solutions to this, one of which is

$$\begin{cases} 
  x_0 &= 0 \\
  x_1 &= 2t + 4.5t^3 + 30.9375t^5 + 299.3906t^7 + 3335.0889t^9 + 40316.851t^{11} \\
  x_2 &= 1 - 2t^6 - 11t^8 - 986.5t^{10} - 11503t^{12} \\
  x_3 &= 0 \\
  x_4 &= 2t - 3.5t^3 - 23.0625t^5 - 193.3594t^7 - 2019.3486t^9 - 23493.535t^{11} \\
  x_5 &= -4t^2 - 10t^4 - 64t^6 - 614t^8 - 6818t^{10} - 82283t^{12}. 
\end{cases}$$

Substituting back into $f$ yields series in $O(t^{12})$, confirming the calculations. This solution could be used as the initial predictor in a homotopy beginning at the singular configuration.

In contrast to the small Apollonius circle problem, this example is computationally more challenging, as covered in [14], [25], [30], and [31]. It illustrates the combination of tropical methods in computer algebra with symbolic-numeric power series computations to define a polynomial homotopy to track solution paths starting at multiple solutions.

### 4.3 Series Developments for Cyclic 8-Roots

A vector $u \in \mathbb{C}^n$ of a unitary matrix $A$ is biunimodular if for $k = 1, 2, \ldots, n$: $|u_k| = 1$ and $|v_k| = 1$ for $v = A u$. The following system arises in the study [15] of biunimodular vectors:

$$f(x) = \begin{cases} 
  x_0 + x_1 + \cdots + x_{n-1} = 0 \\
  i = 2, 3, 4, \ldots, n-1 : \sum_{j=0}^{n-1} \prod_{k=j}^{i-1} x_k \mod n = 0 \\
  x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0. 
\end{cases}$$

(42)

Cyclic 8-roots has solution curves not reported by Backelin [3]. Note that because of the last equation, the system has no solution for $x_0 = 0$, or in other words $\mathcal{V}(f_{\text{aug}}) = \emptyset$. Thus we are in the third case of Lemma 2.3.

In [1, 2], the vector $v = (1, -1, 0, 1, 0, 0, -1, 0)$ gives the leading exponents of the series. The corresponding unimodular coordinate transformation $x = z^M$
is

\[
M = \begin{bmatrix}
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

(43)

Solving the transformed system with \(z_0\) set to 0 gives the leading coefficient of the series.

After 2 Newton steps, invoked in PHCpack with \texttt{phc -u}, the series for \(z_1\) is

\[
(-1.25000000000000E+00 + 1.25000000000000E+00*i) *z0^2 \\
+ ( 5.00000000000000E-01 - 2.37676980513323E-17*i) *z0 \\
+ (-5.00000000000000E-01 - 5.00000000000000E-01*i);
\]

After a third step, the series for \(z_1\) is

\[
( 7.12500000000000E+00 + 7.12500000000000E+00*i) *z0^4 \\
+(-1.52745512076048E-16 - 4.25000000000000E+00*i) *z0^3 \\
+(-1.25000000000000E+00 + 1.25000000000000E+00*i) *z0^2 \\
+( 5.00000000000000E-01 - 1.45255178343636E-17*i) *z0 \\
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
\]

Bounds on the degree of the Puiseux series expansion to decide whether a point is isolated are derived in [18]. While the explicit bounds (which can be computed without prior knowledge of the degrees of the solution curves) are large, the test of whether a point is isolated can still be performed efficiently with our quadratically convergent Newton’s method.

In a future work, we plan to apply the power series methods to the cyclic \(16\)-roots problem, the \(16\)-dimensional version of this polynomial system, for which the tropical prevariety was computed recently [19].

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