Explicit formula of radiation fields of free waves with applications on channel of energy

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Abstract
In this work we give a few explicit formulas regarding the radiation fields of linear free waves. We then apply these formulas on the channel of energy theory. We characterize all the radial weakly non-radiative solutions in all dimensions and give a few new exterior energy estimates.

1 Introduction
1.1 Background and topics
The semi-linear wave equation
\[ \partial^2_t u - \Delta u = \pm |u|^{p-1} u, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}; \]
especially the energy critical case \( p = 1 + 4/(d-2) \), has been extensively studied by many mathematicians in the past few decades. Please see, for example, Kapitanski [18] and Lindblad-Sogge [26] for local existence and well-posedness; Ginibre-Soffer-Velo [15], Grillakis [16, 17], Kenig-Merle [22], Nakanishi [27, 28] and Shatah-Struwe [29, 30] for global existence, regularity, scattering and blow-up. Since the semi-linear wave equation can be viewed as a small perturbation of the homogeneous linear wave equation in many situations, especially when we consider the asymptotic behaviours of solutions as spatial variables or time tends to infinity, it is important to first understand the asymptotic behaviours of solutions to the homogenous linear wave equation, i.e. free waves. This work is concerned with two important tools to understand the asymptotic behaviours of free waves: radiation field and channel of energy. We first introduce a few necessary notations. Throughout this work we consider the homogenous linear wave equation with initial data in the energy space

\[
\begin{align*}
\partial^2_t u - \Delta u &= 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}; \\
u|_{t=0} &= u_0 \in \dot{H}^1(\mathbb{R}^d); \\
u_t|_{t=0} &= u_1 \in L^2(\mathbb{R}^d).
\end{align*}
\]

In this work we also use the notation \( S_L(u_0, u_1) \) to represent the free wave \( u \) defined above. If it is necessary to mention the velocity \( u_t \), we use the notation

\[
S_L(t) \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right) = \left( \begin{array}{c} u(\cdot, t) \\ u_t(\cdot, t) \end{array} \right) \in \dot{H}^1 \times L^2.
\]
It is well known that the linear wave propagation preserves the $\dot{H}^1 \times L^2$ norm, i.e. the energy conservation law holds. ($\nabla x, t u = (\nabla u, u_1)$)

$$\int_{\mathbb{R}^d} |\nabla x, t u(x, t)|^2 dx = \int_{\mathbb{R}^d} (|\nabla u_0|^2 + |u_1|^2) dx.$$ 

Now we make a brief review of radiation field and channel of energy method.

**Radiation field**  The asymptotic behaviour of free waves at the energy level can be characterized by the following theorem.

**Theorem 1.1** (Radiation filed). Assume that $d \geq 3$ and let $u$ be a solution to the free wave equation $\partial_t^2 u - \Delta u = 0$ with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. Then

$$\lim_{t \to \pm \infty} \int_{\mathbb{R}^d} \left( |\nabla u(x, t)|^2 - |u_r(x, t)|^2 + \frac{|u(x, t)|^2}{|x|^2} \right) dx = 0$$

and there exist two functions $G_{\pm} \in L^2(\mathbb{R} \times S^{d-1})$ so that

$$\lim_{t \to \pm \infty} \int_0^{\pi} \int_{S^{d-1}} \left| r^{d-1} \partial_r u(r \theta, t) - G_{\pm}(r + t, \theta) \right|^2 d\theta dr = 0;$$

$$\lim_{t \to \pm \infty} \int_0^{\pi} \int_{S^{d-1}} \left| r^{d-1} \partial_r u(r \theta, t) + G_{\pm}(r + t, \theta) \right|^2 d\theta dr = 0.$$

In addition, the maps $(u_0, u_1) \to \sqrt{2} G_{\pm}$ are a bijective isometries form $\dot{H}^1 \times L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R} \times S^{d-1})$.

This has been known for more than 50 years, at least in the 3-dimensional case. Please see Friedlander [11, 13], for example. The version of radiation field theorem given above and a proof for all dimensions $d \geq 3$ can be found in Duyckaerts et al. [7]. In addition, there is also a 2-dimensional version of radiation field theorem. The statement in dimension 2 can be given in almost the same way as in the higher dimensional case, except that the limit

$$\lim_{t \to \pm \infty} \int_{\mathbb{R}^2} \frac{|u(x, t)|^2}{|x|^2} dx = 0$$

no longer holds. A proof by Radon transform for all dimensions $d \geq 2$ can be found in Katayama [19], where the statement of the theorem is slightly different. Throughout this work we call the function $G_{\pm}$ radiation profiles and use the notations $T_{\pm}$ for the linear map $(u_0, u_1) \to G_{\pm}$.

**Channel of energy**  The second tool is the channel of energy method, which plays an important role in the study of wave equation in the past decade. This method is first introduced in 3-dimensional case by Duyckaerts-Kenig-Merle [3] and then in 5-dimensional case by Kenig-Lawrie-Schlag [20]. This method was used in the proof of soliton resolution conjecture of energy critical wave equation with radial data in all odd dimensions $d \geq 3$ by Duyckaerts-Kenig-Merle [5, 8]. It can also be used to show the non-existence of minimal blow-up solutions in a compactness-rigidity argument in the energy super or sub-critical case. Please see, for example, Duyckaerts-Kenig-Merle [6] and Shen [31]. Roughly speaking, the channel of energy method discusses the amount of energy located in an exterior region as time tends to infinity:

$$\lim_{t \to \pm \infty} \int_{|x|>|t|+R} |\nabla x, t u(x, t)|^2 dx.$$ 

Here the constant $R \geq 0$. Since the energy travels at a finite speed, the energy in the exterior region $\{ x : |x| > |t| + R \}$ decays as $|t|$ increases. Thus the limits above are always well-defined. We can also give the exact value of the limit in term of the radiation field:

$$\lim_{t \to \pm \infty} \int_{|x|>|t|+R} |\nabla x, t u(x, t)|^2 dx = 2 \int_{R} \int_{S^{d-1}} |G_{\pm}(s, \theta)|^2 d\theta ds. \quad (2)$$
We first introduce a few already known results. We start with the odd dimensions.

**Proposition 1.2** (see Duyckaerts-Kenig-Merle [1]). Assume that \( d \geq 3 \) is an odd integer. All solutions to \( \partial_t^2 u - \Delta u = 0 \) satisfies

\[
\sum_{\pm} \lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{x,t} u(x,t)|^2 \, dx = \int_{\mathbb{R}^d} |\nabla_x u(x,0)|^2 \, dx. \tag{3}
\]

As a result, we have

**Corollary 1.3.** Assume that \( d \geq 3 \) is odd. Then \( u \equiv 0 \) is the only free wave \( u \) satisfying

\[
\lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{x,t} u(x,t)|^2 \, dx = 0.
\]

In the contrast, if \( R > 0 \), the subspace of \( \dot{H}^1 \times L^2(\mathbb{R}^d) \) defined by

\[
P(R) = \left\{ (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : \lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_{x,t} S_L(u_0, u_1)(x,t)|^2 \, dx = 0 \right\}. \tag{4}
\]

does contain initial data \( (u_0, u_1) \) whose support is essentially bigger than \( \{ x : |x| \leq R \} \). The free waves \( u \) satisfying

\[
\lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_{x,t} u(x,t)|^2 \, dx = 0
\]
are usually called \((R\text{-weakly})\) non-radiative solutions. If the dimension is odd, these solutions are well-understood in the radial case:

**Theorem 1.4** (See Kenig et al [21], the proof uses radial Fourier transform). In any odd dimension \( d \geq 1 \), every radial solution \( u \) to (1) satisfies

\[
\max_{\pm} \lim_{t \to \pm \infty} \int_{r > |t| + R} |\nabla_{x,t} u(r,t)|^2 r^{d-1} \, dr \geq \frac{1}{2} \left\| \Pi_{P_{rad}(R)}^\perp (u_0, u_1) \right\|_{\dot{H}^1 \times L^2(r \geq R; r^{d-1} \, dr).} \tag{5}
\]

Here

\[
P_{rad}(R) \equiv \text{Span} \left\{ (r^{2k_1 - d}, 0), (0, r^{2k_2 - d}) : k_1, k_2 \in \mathbb{N}; 1 \leq k_1 \leq \frac{d + 2}{4}, 1 \leq k_2 \leq \frac{d}{4} \right\}. \tag{5}
\]

\( \Pi_{P_{rad}(R)}^\perp \) is the orthogonal projection from \( \dot{H}^1 \times L^2(r \geq R; r^{d-1} \, dr) \) onto the complement of the finite-dimensional subspace \( P_{rad}(R) \).

The case of even dimensions is much more complicated and subtle. Côte-Kenig-Schlag [11] shows that in general the inequality

\[
\sum_{\pm} \lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{x,t} u(x,t)|^2 \, dx \geq C \int_{\mathbb{R}^d} |\nabla_{x,t} u(x,0)|^2 \, dx
\]
does not hold for any positive constant \( C \) in even dimensions. But a similar inequality holds in the radial case for either initial data \((u_0, 0)\), if \( d = 0 \text{ mod } 4 \), or \((0, u_1)\), if \( d = 2 \text{ mod } 4 \). More precisely we have

\[
\lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{x,t} S_L(u_0, 0)(x,t)|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u_0(x)|^2 \, dx, \quad d = 4k, k \in \mathbb{N}; \tag{6}
\]

\[
\lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{x,t} S_L(0, u_1)(x,t)|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} |u_1(x)|^2 \, dx, \quad d = 4k + 2, k \in \mathbb{N}. \tag{7}
\]
In addition, Duyckaerts-Kenig-Merle [9] shows that the only non-radiative solution is still zero solution in even dimensions \( d \geq 4 \), i.e. Corollary 1.3 still holds for even dimensions \( d \geq 4 \), even in the non-radial case. Much less is known about the exterior energy estimate in the region \( \{x : |x| > R + |t|\} \) with \( R > 0 \). Duyckaerts at el. [2] proves the exterior energy estimate in dimension 4 and 6 if the initial data are radial: 

\[
\lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_x \cdot S_L(u_0,0)(x,t)|^2 \, dx \geq \frac{3}{10} \|\Pi_R^H u_0\|_{H^1((x \in \mathbb{R}^4 : |x| > R))}^2;
\]

\[
\lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_x \cdot S_L(0,u_1)(x,t)|^2 \, dx \geq \frac{3}{10} \|\Pi_R^H u_1\|_{L^2((x \in \mathbb{R}^6 : |x| > R))}^2.
\]

Here \( \Pi_R^H \) is the orthogonal projection from \( H^1(\{x \in \mathbb{R}^4 : |x| > R\}) \) onto the complement space of \( \text{span} \{x\} \). While \( \pi_R^H \) is the orthogonal projection from \( L^2(\{x \in \mathbb{R}^6 : |x| > R\}) \) onto the complement space of \( \text{span} \{x\} \).

1.2 Main idea

According to [2] we may obtain exterior energy estimates conveniently from the radiation profiles \( G_{\pm} \). Please note that \( G_- \) and \( G_+ \) are not independent to each other. In fact the map \( T_+ \circ T_-^{-1} : G_- \to G_+ \) is a bijective isometry. If we could find explicit expressions of the maps

\[
T_+ \circ T_-^{-1} : G_- \to G_+; \quad T_-^{-1} : G_- \to (u_0,u_1); \quad S_L \circ T_-^{-1} : G_- \to u;
\]

then we would be able to

(a) Understand how the asymptotic behaviour in one time direction determines the behaviour in the other time direction. This is known in the odd dimensional case, as shown (although not stated explicitly) in the proof of Proposition 1.2 by Duyckaerts-Kenig-Merle [4]. In this work we will try to figure out the even dimensional case.

(b) Characterize (weakly) non-radiative solutions, especially in the radial case. We first determine all the radiation profiles \( G_- \) so that

\[
G_-(s,\theta) = G_+(s,\theta) = 0, \quad s > R \quad \Leftrightarrow \quad \lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_x \cdot u(x,t)|^2 \, dx = 0;
\]

then we may obtain all the non-radiative solutions (as well as their initial data) by applying the formula of \( T_-^{-1} \). In particular we prove that radial non-radiative solutions in the even dimension can be characterized in the same way as in the odd dimensions.

(c) Prove exterior energy estimates. We generalize the radial exterior energy estimates in 4 and 6 dimension to all even dimensions; we also prove a non-radial exterior energy estimate in the odd dimensions. In both applications (b) and (c) we follow the same roadmap:

- exterior energy \iff radiation profile \iff initial data.

1.3 Main results

Now we give the statement of our results. The details and proof can be found in subsequent sections.

**Theorem 1.5.** Let \( u \) be a finite-energy free wave with an even spatial dimension \( d \geq 2 \) and \( G_+, G_- \) be the radiation profiles associated with \( u \). Then we may give an explicit formula of the operator \( T_+ \circ T_-^{-1} : G_- \to G_+ \)

\[
G_+(s,\theta) = (-1)^{d/2} (\mathcal{H}G_-)(-s,-\theta)
\]

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Here $\mathcal{H}$ is the Hilbert transform in the first variable, i.e.\[
(\mathcal{H}G_\cdot)(-s, -\theta) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} G_\cdot(\tau, -\theta) \, d\tau.
\]

**Remark 1.6.** A similar but simpler argument shows that if $d$ is odd, then $T_+ \circ T^- : G_\cdot \to G_\cdot$ can be explicitly given by\[
G_+(s, \theta) = (-1)^{\frac{d+1}{2}} G_\cdot(-s, -\theta).
\]
This can also be verified by assuming that the initial data is smooth and compactly-supported, and considering the expression of $G_\cdot$ and $G_\cdot$ in terms of $(u_0, u_1)$ if $d$ is odd. Please refer to Duyckaerts-Kenig-Merle [4]. Since we have $\mathcal{H}^2 = -1$. We may write the odd and even dimensions in a universal formula\[
G_+(s, \theta) = ((-\mathcal{H})^{d-1} G_\cdot)(-s, -\theta).
\]

**Remark 1.7.** It has been proved in Section 3.2 of Duychaerts-Kenig-Merle [9] (in the language of Hankel and Laplace transforms) that the zero function is the only function $f \in L^2(\mathbb{R})$ satisfying\[
f(s) = 0, \ s > 0; \quad (\mathcal{H}f)(s) = 0, \ s < 0.
\]
It immediately follows that

**Corollary 1.8.** Assume $d \geq 2$. Let $\Omega$ be a region in $\mathbb{S}^{d-1}$. If a finite-energy solution $u$ to the homogeneous linear wave equation satisfies \[
\lim_{t \to \pm \infty} \int_{|t|}^{\infty} |\nabla_{t,x} u(r\theta, t)|^2 r^{d-1} \, d\theta \, dr = 0,
\]
then we have \[
\lim_{t \to \pm \infty} \int_{0}^{\infty} |\nabla_{t,x} u(r\theta, t)|^2 r^{d-1} \, d\theta \, dr = 0.
\]
This is an angle-localized version of Corollary 1.3.

**Applications on channel of energy** By following the idea described above, we obtain the following results about the channel of energy.

**Proposition 1.9** (Radial weakly non-radiative solutions). Let $d \geq 2$ be an integer and $R > 0$ be a constant. If initial data $(u_0, u_1) \in H^1 \times L^2$ are radial, then the corresponding solution to the homogeneous linear wave equation $u$ is $R$-weakly non-radiative, i.e.\[
\lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_{t,x} u(x,t)|^2 \, dx = 0,
\]
if and only if the restriction of $(u_0, u_1)$ in the region $\{x \in \mathbb{R}^d : |x| > R\}$ is contained in \[
P_{\text{rad}}(R) = \text{Span} \left\{(r^{2k_1-d}, 0), (0, r^{2k_2-d}) : 1 \leq k_1 \leq \left\lfloor \frac{d+1}{4} \right\rfloor, 1 \leq k_2 \leq \left\lfloor \frac{d-1}{4} \right\rfloor\right\}.
\]
Here the notation $\lfloor q \rfloor$ is the integer part of $q$. In particular, all radial $R$-weakly non-radiative solution in dimension 2 are supported in $\{x, t : |x| \leq |t| + R\}$.

**Remark 1.10.** If $d$ is odd, we have $\left\lfloor \frac{d+1}{4} \right\rfloor = \left\lfloor \frac{d+2}{4} \right\rfloor$ and $\left\lfloor \frac{d-1}{4} \right\rfloor = \left\lfloor \frac{d}{4} \right\rfloor$, thus our result here is the same as the already known result in odd dimension, as given in Theorem 1.4.
Proposition 1.11 (Radial exterior estimates in even dimensions). Let $d = 4k$ with $k \in \mathbb{N}$ and $R > 0$. If initial data $u_0 \in H^1(\mathbb{R}^d)$ are radial, then the corresponding solution $u$ to the homogenous linear wave equation with initial data $(u_0, 0)$ satisfies

$$\lim_{t \to \infty} \int_{|x| > R + |t|} |\nabla u(x, t)|^2 \, dx = \lim_{t \to \infty} \int_{|x| > R + |t|} |u_t(x, t)|^2 \, dx \geq \frac{1}{4} \|\Pi_{Q_k(R)} u_0\|^2_{H^1(\{x : |x| > R\})}.$$ 

Here $\Pi_{Q_k(R)}$ is the orthogonal projection from $H^1(\{x \in \mathbb{R}^d : |x| > R\})$ onto the complement of the $k$-dimensional linear space

$$Q_k(R) = \text{Span} \left\{ \frac{1}{|x|^{4k-2k_1}} : 1 \leq k_1 \leq k \right\}.$$ 

Similarly if the dimension $d = 4k + 2 \geq 2$ with $k \in \{0\} \cup \mathbb{N}$ and initial data $u_1 \in L^2(\mathbb{R}^d)$ are radial, the corresponding solution $u$ to the homogenous linear wave equation with initial data $(0, u_1)$ satisfies

$$\lim_{t \to \infty} \int_{|x| > R + |t|} |\nabla u(x, t)|^2 \, dx = \lim_{t \to \infty} \int_{|x| > R + |t|} |u_t(x, t)|^2 \, dx \geq \frac{1}{4} \|\Pi_{Q'_k(R)} u_1\|^2_{L^2(\{x : |x| > R\})}.$$ 

Here $\Pi_{Q'_k(R)}$ is the orthogonal projection from $L^2(\{x \in \mathbb{R}^d : |x| > R\})$ onto the complement of the $k$-dimensional linear space

$$Q'_k(R) = \text{Span} \left\{ \frac{1}{|x|^{4k+2-2k_1}} : 1 \leq k_1 \leq k \right\}.$$ 

Remark 1.12. Given $u_0 \in \dot{H}^1(\mathbb{R}^{4k})$ or $u_1 \in L^2(\mathbb{R}^{4k+2})$, the orthogonal projection of $u_0$ or $u_1$ onto the finite dimensional space $Q_k(R)$ or $Q'_k(R)$ gradually vanishes as $R \to 0^+$. Therefore if we make $R \to 0^+$ in Proposition [1,11] we immediately obtain (6) and (7).

Proposition 1.13 (Non-radial exterior energy estimates). Let $d \geq 3$ be an odd integer and $R > 0$ be a constant. Then the following inequality holds for all $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$.

$$\sum_{\pm} \lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_{t,x} S_L(t)(u_0, u_1)(x, t)|^2 \, dx = \left\| \Pi_{P(R)}(u_0, u_1) \right\|^2_{H^1 \times L^2(\mathbb{R}^d)}.$$ 

Here $\Pi_{P(R)}$ is the orthogonal projection from $\dot{H}^1 \times L^2(\mathbb{R}^d)$ onto the complement of the closed linear space

$$P(R) = \left\{ (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : \lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_{t,x} S_L(u_0, u_1)(x, t)|^2 \, dx = 0 \right\}.$$ 

Structure of this work This work is organized as follows. In section 2 we deduce an explicit formula of $T_{-1}^{-1}$ in all dimensions. Then in Section 3 we prove the explicit formula of $T_+ \circ T_{-1}^{-1}$ given in Theorem [1.5]. The rest of the paper is devoted to the applications in channel of energy. We characterize radial weakly non-radiative solutions in Section 4, prove radial exterior energy estimate for all even dimensions in Section 5 and finally give a short proof of non-radial exterior energy estimate in odd-dimensional space in Section 6. The appendix is concerned with Hilbert transform of a family of special functions, since the Hilbert transform is involved in the even dimensions.

Notations In this work we use the notation $C(d)$ for a nonzero constant determined solely by the dimension $d$. It may represent different constants in different places. This avoid the trouble of keeping track of the constants when unnecessary.
2 From Radiation Profile to Solution

Now we assume that $G_-(r, \theta)$ is smooth and compactly supported and give an explicit formula of the operator $T_0^{-1}$. We consider the odd dimensions first.

2.1 Odd dimensions

Lemma 2.1. Assume that $d \geq 3$ is odd. Let $G_-$ be a smooth function with $\text{supp} G_- \subset [-R, R] \times S^{d-1}$. Then $(u_0, u_1) = T_0^{-1}G_-$ satisfies

$$u_0(x) = \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} G_-(x_0 \cdot \omega, \omega) \, d\omega$$

$$u_1(x) = \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} G_-(x_0 \cdot \omega, \omega) \, d\omega (9)$$

Here the notation $G_{-}^{(k)}$ represents the partial derivative

$$G_{-}^{(k)}(s, \theta) = \frac{\partial^k G_{-}(s, \theta)}{\partial s^k}. (10)$$

Remark 2.2. This formula in 3-dimensional case was previously known. Please refer to Friedlander [12], for example.

Proof. Let $(u_0, u_1) = T_0^{-1}G_-$ and $u = S_L(u_0, u_1)$. Given a large time $t > 0$, we choose approximated data $(v_{0,t}, v_{1,t}) \approx (u(\cdot, -t), u_t(\cdot, -t))$ as below:

$$v_{1,t}(r, \theta) = r^{-\mu} G_-(r - t, \theta), \quad r > 0, \quad \theta \in S^{d-1}; \quad (11)$$

$$v_{0,t}(r, \theta) = -\chi(r/t) \int_{r}^{+\infty} r'^{-\mu} G_-(r' - t, \theta) d r', \quad r > 0, \quad \theta \in S^{d-1}. (12)$$

Here $\mu = (d - 1)/2$ and $\chi: \mathbb{R} \to [0, 1]$ is a smooth cut-off function satisfying

$$\chi(s) = \begin{cases} 1, & s > 1/2; \\ 0, & s < 1/4. \end{cases}$$

It is clear that the data $(v_{0,t}, v_{1,t})$ are smooth and compactly-supported in $\{x : |x| < R + t\}$. A straight-forward calculation shows that

$$\int_0^{\infty} \int_{S^{d-1}} |r^\mu v_{1,t}(r, \theta) - G_-(r - t, \theta)|^2 \, d\theta dr = 0;$$

$$\int_0^{\infty} \int_{S^{d-1}} |r^\mu \partial_r v_{0,t}(r, \theta) - G_-(r - t, \theta)|^2 \, d\theta dr \lesssim 1/t;$$

$$\int_{\mathbb{R}^d} (|\nabla v_{0,t}(x)|^2 - |\partial_t v_{0,t}(x)|^2) \, dx \lesssim 1/t.$$  

Thus by radiation field we have

$$\lim_{t \to +\infty} \|(v_{0,t}, v_{1,t}) - (u(\cdot, -t), u_t(\cdot, -t))\|_{H^1 \times L^2(\mathbb{R}^d)} = 0.$$  

Since the linear propagation operator $S_L(t)$ preserves the $H^1 \times L^2$ norm, we have

$$\lim_{t \to +\infty} \left\| \left( \begin{array}{c} u_0 \\ u_1 \\ \end{array} \right) - S_L(t) \left( \begin{array}{c} v_{0,t} \\ v_{1,t} \\ \end{array} \right) \right\|_{H^1 \times L^2(\mathbb{R}^d)} = 0. (13)$$
Next we use the explicit expression of linear propagation operator (see, for instance, Evans [10]) and write $v = S_L(v_0, v_1)$ in terms of $(v_0, v_1)$ when the initial are sufficiently smooth.

$$v(x, t) = c_d \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\mu - 1} \left( t^{d-2} \int_{\mathbb{S}^{d-1}} v_0(x + tw)dw \right) + c_d \cdot \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\mu - 1} \left( t^{d-2} \int_{\mathbb{S}^{d-1}} v_1(x + tw)dw \right)$$

$$= c_d t^\mu \int_{\mathbb{S}^{d-1}} \left[ ((w \cdot \nabla)^\mu v_0)(x + tw) + ((w \cdot \nabla)^{\mu - 1} v_1)(x + tw) \right] dw + \sum_{0 \leq k < \mu} A_{d,k} t^k \int_{\mathbb{S}^{d-1}} ((w \cdot \nabla)^k v_0)(x + tw)dw$$

$$+ \sum_{0 \leq k < \mu - 1} B_{d,k} t^{k+1} \int_{\mathbb{S}^{d-1}} ((w \cdot \nabla)^k v_1)(x + tw)dw.$$  

Here $c_d = \frac{1}{2(2\pi)^{d/2}}$, $A_{d,k}$, $B_{d,k}$ (and $A'_{d,k}$, $B'_{d,k}$ below) are all constants. We may differentiate and obtain

$$v_1(x, t) = c_d t^\mu \int_{\mathbb{S}^{d-1}} \left[ ((w \cdot \nabla)^{\mu + 1} v_0)(x + tw) + ((w \cdot \nabla)^{\mu} v_1)(x + tw) \right] dw$$

$$+ \sum_{1 \leq k \leq \mu} A'_{d,k} t^{k-1} \int_{\mathbb{S}^{d-1}} ((w \cdot \nabla)^k v_0)(x + tw)dw$$

$$+ \sum_{0 \leq k \leq \mu - 1} B'_{d,k} t^k \int_{\mathbb{S}^{d-1}} ((w \cdot \nabla)^k v_1)(x + tw)dw.$$  

Now we plug in $(v_0, v_1) = (v_{0,t}, v_{1,t})$ with large time $t$. We observe that

$$|(w \cdot \nabla)^k v_{j,t}(x + tw)| \lesssim t^{-\mu}, \quad j = 0, 1; \quad k \geq 0; \quad (13)$$

and ($r = |x + tw|$, $\theta = \frac{x + tw}{|x + tw|}$, $k = \mu - 1, \mu$)

$$((w \cdot \nabla)^{k+1} v_{0,t})(x + tw) = (w \cdot \theta)^{k+1} r^{-\mu} G^{(k)}_{+}(r - t, \theta) + O(t^{-\mu - 1});$$

$$((w \cdot \nabla)^k v_{1,t})(x + tw) = (w \cdot \theta)^k r^{-\mu} G^{(k)}_{-}(r - t, \theta) + O(t^{-\mu - 1}).$$

Thus

$$\begin{pmatrix} w_{0,t} \\ w_{1,t} \end{pmatrix} = S_L(t) \begin{pmatrix} v_{0,t} \\ v_{1,t} \end{pmatrix}$$

satisfies

$$w_{0,t} = c_d \int_{\mathbb{S}^{d-1}} (w \cdot \theta)^{\mu - 1} (1 + w \cdot \theta) G^{(\mu - 1)}_{-}(r - t, \theta) dw + O(1/t);$$

$$w_{1,t} = c_d \int_{\mathbb{S}^{d-1}} (w \cdot \theta)^{\mu} (1 + w \cdot \theta) G^{(\mu)}_{-}(r - t, \theta) dw + O(1/t).$$

Please note that the implicit constants in [13], $O(t^{-\mu - 1})$ and $O(1/t)$ above may depend on $x$ but remain to be uniformly bounded if $x$ is contained in a compact subset of $\mathbb{R}^d$. Next we observe the facts

$$\theta(\omega) = \omega + O(1/t); \quad r(\omega) - t = x_0 \cdot \omega + O(1/t);$$

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and further simplify the formula

\[ w_{0,t} = 2cd \int_{S^{d-1}} G^{(\mu-1)}(x \cdot \omega, \omega) \, d\omega + O(1/t); \]
\[ w_{1,t} = 2cd \int_{S^{d-1}} G^{(\mu)}(x \cdot \omega, \omega) \, d\omega + O(1/t). \]

Finally we make \( t \to +\infty \), utilize (12) and obtain

\[ u_0 = 2cd \int_{S^{d-1}} G^{(\mu-1)}(x \cdot \omega - \rho, \omega) \, d\omega; \]
\[ u_1 = 2cd \int_{S^{d-1}} G^{(\mu)}(x \cdot \omega - \rho, \omega) \, d\omega. \]

We plug in the value of \( cd \) and finish the proof. \( \square \)

**Remark 2.3.** An explicit formula of the free wave \( u = S_L T^{-1} G_- \) can be given by

\[ u(x, t) = \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} G^{(\mu-1)}(x \cdot \omega + t, \omega) \, d\omega. \]

This can be verified by a straight-forward calculation. One may check

- The function \( u \) above is a smooth solution to the homogenous linear wave equation;
- The initial data of \( u \) are exactly those given in Lemma 2.1.

We may differentiate and obtain

\[ u_t(x, t) = \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} G^{(\mu)}(x \cdot \omega + t, \omega) \, d\omega; \]
\[ \nabla u(x, t) = \frac{1}{(2\pi)^{d/2}} \int_{S^{d-1}} G^{(\mu)}(x \cdot \omega + t, \omega) \, d\omega \]

### 2.2 Even dimensions

The formula of \( T^{-1} \) in even dimensions are a little more complicated.

**Lemma 2.4.** Assume that \( d \geq 2 \) is even and \( G_- \in C_0^\infty(\mathbb{R} \times S^{d-1}) \). Then the operator \( T^{-1} \) is given explicitly by

\[ u_0(x) = \sqrt{2} \left( \frac{2\pi}{(2\pi)^{d/2}} \right)^{1/2} \int_0^\infty \int_{S^{d-1}} G^{(d/2-1)}(x \cdot \omega - \rho, \omega) \, d\omega d\rho; \]
\[ u_1(x) = \sqrt{2} \left( \frac{2\pi}{(2\pi)^{d/2}} \right)^{1/2} \int_0^\infty \int_{S^{d-1}} G^{(d/2)}(x \cdot \omega - \rho, \omega) \, d\omega d\rho. \]

**Proof.** Without loss of generality let us assume \( \text{supp} \, G_- \subset [-R_1, R_1] \times S^{d-1} \). It is sufficient to show that given any \( R_2 > 0 \), the formula above holds for almost everywhere \( x \in B(0, R_2) \). Let us use the notations \((u_0, u_1) = T^{-1} G_- \) and \( u = S_L(u_0, u_1) \). We consider the approximated data

\[ v_{1,t}(r\theta) = r^{-\mu} G_-(r - t, \theta), \quad r > 0, \, \theta \in S^{d-1}. \quad (14) \]
\[ v_{0,t}(r\theta) = -\chi(r/t) \int_r^{+\infty} r'^{-\mu} G_-(r' - t, \theta) \, dr', \quad r > 0, \, \theta \in S^{d-1}. \quad (15) \]
and
\[
\begin{pmatrix}
  w_{0,t} \\
  w_{1,t}
\end{pmatrix}
= S_L(t) \begin{pmatrix}
  v_{0,t} \\
  v_{1,t}
\end{pmatrix}.
\]

Here \( \chi \) is the center cut-off function as given in the previous subsection. A basic calculation shows
\[
\lim_{t \to +\infty} \| (v_{0,t}, v_{1,t}) - (u(\cdot, -t), u_t(\cdot, -t)) \|_{H^1 \times L^2(\mathbb{R}^d)} = 0.
\]

Thus
\[
\lim_{t \to +\infty} \| (w_{0,t}, w_{1,t}) - (u_0, u_1) \|_{H^1 \times L^2(\mathbb{R}^d)} = 0. \tag{16}
\]

Let us first recall the explicit formula of \( v = S_L(v_0, v_1) \) in the even dimensional case:
\[
v(x, t) = c_d \cdot \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{d-2}{2}} \left( t^{d-1} \int_{\mathbb{R}^d} \frac{v_0(x + ty)}{\sqrt{1 - |y|^2}} dy \right)
+ c_d \cdot \frac{1}{t} \frac{\partial}{\partial t} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{d-2}{2}} \left( t^{d-1} \int_{\mathbb{R}^d} \frac{v_1(x + ty)}{\sqrt{1 - |y|^2}} dy \right)
+ \sum_{0 \leq k < d/2} A_{d,k} t^{k} \int_{\mathbb{R}^d} \frac{(y \cdot \nabla)^k v_0(x + ty)}{\sqrt{1 - |y|^2}} dy
+ \sum_{0 \leq k < d/2-1} B_{d,k} t^{k+1} \int_{\mathbb{R}^d} \frac{(y \cdot \nabla)^k v_1(x + ty)}{\sqrt{1 - |y|^2}} dy.
\]

Here \( \mathbb{B}_d \) is the unit ball in \( \mathbb{R}^d \) and \( c_d = (2\pi)^{-d/2} \) is a constant. The notations \( A_{d,k}, B_{d,k} \) (and \( A'_{d,k}, B'_{d,k} \) below) represent constants. We differentiate and obtain
\[
v_t(x, t) = c_d t^d/2 \int_{\mathbb{B}^d} \frac{(y \cdot \nabla)^{d/2+1} v_0(x + ty) + (y \cdot \nabla)^{d/2} v_1(x + ty)}{\sqrt{1 - |y|^2}} dy
+ \sum_{1 \leq k \leq d/2} A'_{d,k} t^{k-1} \int_{\mathbb{B}^d} \frac{(y \cdot \nabla)^k v_0(x + ty)}{\sqrt{1 - |y|^2}} dy
+ \sum_{0 \leq k < d/2-1} B'_{d,k} t^{k} \int_{\mathbb{B}^d} \frac{(y \cdot \nabla)^k v_1(x + ty)}{\sqrt{1 - |y|^2}} dy.
\]

We plug in \((v_0, v_1) = (v_{0,t}, v_{1,t})\) and observe
\[
|(y \cdot \nabla)^k v_{0,t}| \leq t^{-\frac{d-k}{2}} \quad \text{and} \quad |(y \cdot \nabla)^k v_{1,t}| \leq t^{-\frac{d-k}{2}}.
\]

This gives the approximation
\[
w_{0,t}(x) = c_d t^d/2 \int_{\mathbb{B}^d} \frac{(y \cdot \nabla)^{d/2} v_{0,t}(r\theta) + (y \cdot \nabla)^{d/2-1} v_{1,t}(r\theta)}{\sqrt{1 - |y|^2}} dy + O(t^{-1/2});
\]
\[
w_{1,t}(x) = c_d t^d/2 \int_{\mathbb{B}^d} \frac{(y \cdot \nabla)^{d/2+1} v_{0,t}(r\theta) + (y \cdot \nabla)^{d/2} v_{1,t}(r\theta)}{\sqrt{1 - |y|^2}} dy + O(t^{-1/2}).
\]

Here \( r = |x + ty|, \theta = \frac{x+ty}{|x+ty|} \). Furthermore, we observe \((k = d/2, d/2 - 1)\)
\[
\langle (y \cdot \nabla)^{k+1} v_{0,t}(r\theta) \rangle = (y \cdot \theta)^{k+1} r^{-\frac{d-k}{2}} G_{d/2}^k (r, \theta) + O(t^{\frac{d-k}{2}});
\]
\[
\langle (y \cdot \nabla)^{k} v_{1,t}(r\theta) \rangle = (y \cdot \theta)^k r^{-\frac{d-k}{2}} G_{d/2}^k (r, \theta) + O(t^{\frac{d-k}{2}});
\]
and write
\[
\begin{align*}
  w_{0,t}(x) &= c_d \cdot t^{d/2} \int_{\mathbb{R}^d} \frac{(y \cdot \theta)^{d/2-1} (y \cdot \theta + 1)r^{-d-1} G^{(d/2-1)}(r-t, \theta)}{\sqrt{1 - |y|^2}} dy + O(t^{-1/2}); \\
  w_{1,t}(x) &= c_d \cdot t^{d/2} \int_{\mathbb{R}^d} \frac{(y \cdot \theta)^{d/2} (y \cdot \theta + 1)r^{-d-1} G^{(d/2)}(r-t, \theta)}{\sqrt{1 - |y|^2}} dy + O(t^{-1/2}).
\end{align*}
\]

Next we observe that if \(|y| < 1 - \frac{R_1 + R_2}{t}\), then we have \(r \leq t|y| + |x| < t - R_1\) thus \(G^{(k)}(r-t, \theta) = 0\). As a result, we may restrict the domain of integral to
\[\mathbb{B}_t = \left\{ y \in \mathbb{R}^d : |y| \geq 1 - \frac{R_1 + R_2}{t} \right\}.\]

Because in the region we have
\[
\begin{align*}
  \theta &= \frac{y}{|y|} + O(1/t); & y \cdot \theta &= 1 + O(1/t); & r &= t + O(1).
\end{align*}
\]

We can simplify the formula
\[
\begin{align*}
  w_{0,t}(x) &= 2c_d \cdot t^{1/2} \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2-1)}(r-t, y/|y|)}{\sqrt{1 - |y|^2}} dy + O(t^{-1/2}); \\
  w_{1,t}(x) &= 2c_d \cdot t^{1/2} \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2)}(r-t, y/|y|)}{\sqrt{1 - |y|^2}} dy + O(t^{-1/2}).
\end{align*}
\]

Next we utilize the change of variables
\[
y = (1 - \rho/t)\omega, \quad (\rho, \omega) \in (0, R_1 + R_2) \times \mathbb{S}^{d-1},
\]
and the approximations
\[
r - t = x \cdot \omega - \rho + O(1/t); \quad \sqrt{1 - |y|^2} = (1 + O(1/t))\sqrt{2\rho/t}; \quad dy = (1 + O(1/t))t^{-1}d\rho d\omega;
\]
to obtain
\[
\begin{align*}
  w_{0,t}(x) &= \sqrt{2}c_d \cdot \int_0^{R_1+R_2} \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2-1)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}} d\omega d\rho + O(t^{-1/2}); \\
  w_{1,t}(x) &= \sqrt{2}c_d \cdot \int_0^{R_1+R_2} \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}} d\omega d\rho + O(t^{-1/2}).
\end{align*}
\]

Finally we recall (16), make \(t \to +\infty\) and conclude
\[
\begin{align*}
  u_0(x) &= \sqrt{2}c_d \cdot \int_0^{R_1+R_2} \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2-1)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}} d\omega d\rho; \\
  u_1(x) &= \sqrt{2}c_d \cdot \int_0^{R_1+R_2} \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2)}(x \cdot \omega - \rho, \omega)}{\sqrt{\rho}} d\omega d\rho.
\end{align*}
\]

This finishes the proof. \(\square\)

**Remark 2.5.** If \(d \geq 4\), the convergence (16) implies that \((w_{0,t}, w_{1,t})\) converges to \((u_0, u_1)\) in \(L^{\frac{d}{d-2}} \times L^2\) by Sobolev embedding. We may combine this convergence with the local uniform convergence given above to verify the identities above. This argument breaks down in dimension
2. We given another argument below in dimension 2. Given any test function \( \phi \in C_0^\infty(\mathbb{R}^2) \), integration by parts gives an identity

\[
\int w_{0,t}(x) \nabla \phi(x) \, dx = - \int \nabla w_{0,t}(x) \phi(x) \, dx.
\]

We recall the local uniform convergence of \( w_{0,t} \) given above and the \( L^2 \) convergence of \( \nabla w_{0,t} \to \nabla u_0 \), then obtain

\[
\int \left( \sqrt{2c_2} \cdot \int_0^\infty \int_{S^1} G_-(x \cdot \omega - \rho, \omega) \sqrt{\rho} \, d\omega \, d\rho \right) \nabla \phi(x) \, dx = - \int \nabla u_0(x) \phi(x) \, dx.
\]

This finishes the proof. Finally the author would like to mention that we have

\[
\lim_{|x| \to +\infty} \sqrt{2c_2} \cdot \int_0^\infty \int_{S^1} G_-(x \cdot \omega - \rho, \omega) \sqrt{\rho} \, d\omega \, d\rho = 0.
\]

**Corollary 2.6.** If \( G_- \in C_0^\infty(\mathbb{R} \times S^{d-1}) \), then \( u = S_L T_{-1}^- (G_-) \) is given by

\[
u(x,t) = \frac{\sqrt{2}}{(2\pi)^{d/2}} \int_0^\infty \int_{S^{d-1}} G_-^{(d/2-1)}(x \cdot \omega - \rho + t, \omega) \sqrt{\rho} \, d\omega \, d\rho.
\]

Thus

\[
u_t(x,t) = \frac{\sqrt{2}}{(2\pi)^{d/2}} \int_0^\infty \int_{S^{d-1}} \frac{d^{(d/2)}}{\sqrt{\rho}} G_-^{(d/2)}(x \cdot \omega - \rho + t, \omega) \sqrt{\rho} \, d\omega \, d\rho.
\]

**Proof.** A basic calculation shows that \( u(x,t) \) solves the free wave equation with initial data given in Lemma 2.4. \( \square \)

### 2.3 Universal formula

Now let us give a universal formula of \( T_{-1}^- \) for all dimensions. We first define two convolution operators \((1/\sqrt{\pi x}) \) is understood as zero if \( x < 0 \)

\[
Qf = \frac{1}{\sqrt{\pi x}} * f, \quad Q'f = \frac{1}{\sqrt{-\pi x}} * f.
\]

Their Fourier symbols are \( \frac{1 - i(\xi/|\xi|)}{2 \sqrt{\pi |\xi|}} \) and \( \frac{1 + i(\xi/|\xi|)}{2 \sqrt{\pi |\xi|}} \), respectively. Let us also use the notation \( \mathcal{D} = d/dx \) and recall that its Fourier symbol is \( 2\pi i \xi \). A simple calculation of symbols shows

\[
Q^2 \mathcal{D} = 1; \quad Q'^2 \mathcal{D} = -1; \quad QQ' \mathcal{D} = \mathcal{H}.
\]  \tag{17}

As a result, we may understand \( Q \) as \( \mathcal{D}^{-1/2} \) and rewrite \( u = S_L T_{-1}^- G_- \) in the form of

\[
u(x,t) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{S^{d-1}} \left( QG_-^{(d/2-1)} \right) (x \cdot \omega + t, \omega) \, d\omega
\]

\[
= \frac{1}{(2\pi)^d} \int_{S^{d-1}} \mathcal{D}^{\mu-1} G_- (x \cdot \omega + t, \omega) \, d\omega.
\]  \tag{18}

Here \( \mu = \frac{d-1}{2} \). This formula holds for both odd and even dimensions.
3 Between Radiation Profiles

In this section we give an explicit expression of the operator $T_+ \circ T_-^1$ in the even dimension case, without the radial assumption.

**Theorem 3.1.** Assume that $d \geq 2$ is an even integer. The operator $T_+ \circ T_-^1$ can be explicitly given by the formula

$$G_+(s, \theta) = (T_+ T_-^1 G_-)(s, \theta) = (-1)^{d/2}(\mathcal{H} G_-)(-s, -\theta)$$

Here $\mathcal{H}$ is the Hilbert transform in the first variable, i.e.

$$(\mathcal{H} G_-)(-s, -\theta) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{G_-(\tau, -\theta)}{-\tau - s} \, d\tau.$$

**Proof.** Since $T_+ \circ T_-^1$ is a bijective isometry from $L^2(\mathbb{R} \times S^{d-1})$ to itself. We only need to prove this formula for smooth and compactly supported data $G_-$. Without loss of generality let us assume $\text{supp} G_- \subset [-R_1, R_1] \times S^{d-1}$. Let us also fix a positive constant $R_2 > 0$. If $(s, \theta) \in (-R_2, R_2) \times S^{d-1}$, then we may apply Corollary 2.6 and obtain

$$(t + s)^{\frac{d-1}{2}} \partial_t u((t + s) \theta, t) = \sqrt{2c_d(t + s)^{\frac{d-1}{2}}} \int_0^{\infty} \int_{S^{d-1}} G_{d/2}^{(d/2)}((t + s)\theta \cdot \omega - \rho + t, \omega) \frac{d\omega d\rho}{\sqrt{\rho}}$$

Let $M \gg R_1 + R_2 + 1$ be a large constant, we may split the integral above into two parts

$$J_1 = \sqrt{2c_d(t + s)^{\frac{d-1}{2}}} \int_0^{\infty} \int_{\theta \cdot |\omega| < -1 + M/t} G_{d/2}^{(d/2)}((t + s)\theta \cdot \omega - \rho + t, \omega) \frac{d\omega d\rho}{\sqrt{\rho}};$$

$$J_2 = \sqrt{2c_d(t + s)^{\frac{d-1}{2}}} \int_0^{\infty} \int_{\theta \cdot |\omega| \geq -1 + M/t} G_{d/2}^{(d/2)}((t + s)\theta \cdot \omega - \rho + t, \omega) \frac{d\omega d\rho}{\sqrt{\rho}};$$

We may find an upper bound of $J_2$. In this region we have

$$(t + s)\theta \cdot \omega + t \geq M - R_2 \implies G_-(t + s)\theta \cdot \omega - \rho + t) = 0, \text{if } \rho < M/2.$$ Thus we may integrate by parts and obtain

$$J_2 = C(d)(t + s)^{\frac{d-1}{2}} \int_0^{\infty} \int_{\theta \cdot |\omega| \geq -1 + M/t} G_-(t + s)\theta \cdot \omega - \rho + t, \omega) \frac{d\omega d\rho}{\rho^{\frac{d+1}{2}}}$$

Thus when $t$ is sufficiently large

$$|J_2| \lesssim t^{\frac{d-1}{2}} \int_{\theta \cdot \omega \geq -1 + M/t} \frac{G_-(t + s)\theta \cdot \omega - \rho + t, \omega) \frac{d\omega d\rho}{\rho^{\frac{d+1}{2}}}$$

$$\lesssim t^{\frac{d-1}{2}} \int_{\theta \cdot \omega \geq -1 + M/t} \frac{G_-(t + s)\theta \cdot \omega - \rho + t, \omega) \frac{d\omega d\rho}{\rho^{\frac{d+1}{2}}}$$

$$\lesssim t^{\frac{d-1}{2}} \int_{\theta \cdot \omega \geq -1 + M/t} \frac{1}{|t\theta \cdot \omega + t|^{\frac{d+1}{2}}} d\omega$$

$$\lesssim 1/M.$$ In the integral region of $J_1$, we have the approximation $\omega = -\theta + O(t^{-1/2})$. Thus we have

$$J_1 = \sqrt{2c_d} t^{\frac{d-1}{2}} \int_0^{\infty} \int_{\theta \cdot \omega < -1 + M/t} G_{d/2}^{(d/2)}((t + s)\theta \cdot \omega - \rho + t, -\theta) \frac{d\omega d\rho}{\sqrt{\rho}} + O(t^{-1/2}).$$
Next we utilize the change of variables (please refer to figure 1 for a geometrical meaning)

\[
\omega = (-1 + \rho'/t)\vartheta + \sqrt{(\rho'/t)(2 - \rho'/t)}\varphi, \quad \rho' \in [0, M], \varphi \in \mathbb{S}^{d-2} = \{\varphi \in \mathbb{S}^{d-1}: \varphi \perp \theta\}.
\]

\[
d\omega = [1 + O(1/t)](2\rho'/t)^{d/2 - 1}d\mathbb{S}^{d-2}(\varphi) \cdot \frac{d\rho'}{\sqrt{2\rho't}} = [1 + O(1/t)](2\rho')^{d/2 - 1}t^{-1/2}d\mathbb{S}^{d-2}(\varphi)d\rho'.
\]

and obtain

\[
J_1 = \frac{1}{2\pi^{d/2}} \int_0^\infty \int_0^M \int_{\mathbb{S}^{d-2}} G^{(d/2)}(\rho' - \rho - s, -\theta)\rho^{d/2 - 1}d\rho'd\rho + O(t^{-1/2}).
\]

We observe that the integrand is independent of \(\varphi\) and integrate by parts

\[
J_1 = \frac{(-1)^{d/2 - 1}}{\pi} \int_0^M \int_{|\varphi|}^{M - \varphi} \frac{G_{\varphi}^{\prime}(\tau - s, -\theta)}{\sqrt{\tau^2 - \varphi^2}}d\varphi d\tau + O(t^{-1/2}).
\]

We next change the variables \(\tau = \rho' - \rho, \eta = \rho' + \rho\), and write

\[
J_1 = \frac{(-1)^{d/2 - 1}}{\pi} \int_{-\infty}^{\infty} \int_{|\tau|}^{M - \tau} G_{\tau}^{\prime}(\tau - s, -\theta) \ln(2M - \tau + \sqrt{4M^2 - 4M\tau}) - \ln |\tau| d\tau + O(t^{-1/2})
\]

\[
= \frac{(-1)^{d/2 - 1}}{\pi} \int_{-R_1 - R_2}^{R_1 + R_2} G_{\tau}^{\prime}(\tau - s, -\theta) \ln(2M - \tau + \sqrt{4M^2 + 4M\tau}) - \ln(4M) d\tau + O(t^{-1/2}).
\]

The integrals above can be split into two parts:

\[
I_1 = \int_{-R_1 - R_2}^{R_1 + R_2} G_{\tau}^{\prime}(\tau - s, -\theta) \ln(2M - \tau + \sqrt{4M^2 + 4M\tau}) d\tau
\]

\[
= \int_{-R_1 - R_2}^{R_1 + R_2} G_{\tau}^{\prime}(\tau - s, -\theta) \ln(2M - \tau + \sqrt{4M^2 - 4M\tau}) - \ln(4M) d\tau
\]

\[
= \int_{-R_1 - R_2}^{R_1 + R_2} G_{\tau}^{\prime}(\tau - s, -\theta)O(1/M)d\tau = O(1/M);
\]

where

\[
\int_{-R_1 - R_2}^{R_1 + R_2} G_{\tau}^{\prime}(\tau - s, -\theta) \ln(2M - \tau + \sqrt{4M^2 + 4M\tau}) d\tau
\]

\[
= \int_{-R_1 - R_2}^{R_1 + R_2} G_{\tau}^{\prime}(\tau - s, -\theta) \ln(2M - \tau + \sqrt{4M^2 - 4M\tau}) - \ln(4M) d\tau
\]
and

\[
I_2 = - \lim_{\epsilon \to 0^+} \int_{\epsilon < |\tau| < R_1 + R_2} \frac{G_+(\tau - s, -\theta)}{\tau} d\tau
\]

\[
= \lim_{\epsilon \to 0^+} \int_{\epsilon < |\tau| < R_1 + R_2} \frac{G_+(\tau - s, -\theta)}{\tau} d\tau
\]

\[
= -\pi(HG_+)(-s, -\theta).
\]

In summary we have

\[
J_1 = (-1)^{d/2}(HG_-)(-s, -\theta) + O(1/M) + O(t^{-1/2}).
\]

Now we may combine \(J_1\) and \(J_2\)

\[
(t + s) \frac{d-1}{2} \partial_t u((t + s)\theta, t) = (-1)^{d/2}(HG_-)(-s, -\theta) + O(1/M) + O(t^{-1/2}).
\]

Because the implicit constants in \(O\)’s do not depend on \(s \in [-R_2, R_2]\) or \(\theta \in \mathbb{S}^{d-1}\), we may make \(t \to +\infty\) then \(M \to +\infty\) to conclude

\[
\lim_{t \to +\infty} \int_{-R_2}^{R_2} \int_{\mathbb{S}^{d-1}} \left((t + s) \frac{d-1}{2} \partial_t u((t + s)\theta, t) - (-1)^{d/2}(HG_-)(-s, -\theta)\right)^2 d\theta ds = 0.
\]

This finishes the proof. \(\square\)

4 Radial Weakly Non-radiative Solutions

In this section we prove Proposition 1.9. First of all, we briefly show that any initial data in \(P_{\text{rad}}(R)\) leads to a \(R\)-weakly non-radiative solution. By linearly we only need to consider the case \((u_0, u_1) = (r^{2k_1-d}, 0)\) or \((u_0, u_1) = (0, r^{2k_2-d})\). If \((u_0, u_1) = (r^{2k_1-d}, 0)\), then a basic calculation shows that if we choose \(C_1, C_2, \ldots, C_{k_1-1}\) inductively, the solution

\[
u_{k_1}(x, t) = \frac{1}{|x|^{d-2k_1}} + \frac{C_1 t^2}{|x|^{d-2k_1+2}} + \cdots + \frac{C_{k_1-1} t^{2k_1-2}}{|x|^{d-2}}
\]

solves the linear wave equation with initial data \((|x|^{2k_1-d}, 0)\) in the region \(\mathbb{R}^d \setminus \{0\}\). By finite speed of propagation, we have

\[
S_L(u_0, u_1)(x, t) = u_{k_1}(x, t), \quad |x| > R + |t|.
\]

A simple calculation shows that this is indeed a non-radiative solution. The case \((u_0, u_1) = (0, r^{2k_2-d})\) can be dealt with in the same manner by considering the solution

\[
u_{k_2}(x, t) = \frac{t}{|x|^{d-2k_2}} + \frac{C_1 t^3}{|x|^{d-2k_2+2}} + \cdots + \frac{C_{k_2-1} t^{2k_2-1}}{|x|^{d-2}}.
\]

Thus it is sufficient to show initial data of any non-radiative solution are contained in the space \(P_{\text{rad}}(R)\). We first consider the odd dimensions.

4.1 Odd dimensions

Assume that \(u = S_L(u_0, u_1)\) is a radial \(R\)-weakly non-radiative solution. Let \(G_- = T_-(u_0, u_1)\). By radial assumption \(G_-\) is independent of the angle \(\omega \in \mathbb{S}^{d-1}\). Let us first consider smooth functions \(G_-\). We may calculate \((r > R, e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d)\)

\[
u_0(e_1) = (2\pi)^{-\mu} \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(r \omega_1) d\omega = \frac{\sigma_{d-2}}{(2\pi)^\mu} \int_{-1}^{1} G_-^{(\mu-1)}(r \omega_1)(1 - \omega_1^2)^{\mu-1} d\omega_1.
\]

\[
= (2\pi)^{-\mu} \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(r \omega_1) d\omega = \frac{\sigma_{d-2}}{(2\pi)^\mu} \int_{-1}^{1} G_-^{(\mu-1)}(r \omega_1)(1 - \omega_1^2)^{\mu-1} d\omega_1.
\]

\[
= (2\pi)^{-\mu} \int_{\mathbb{S}^{d-1}} G_-^{(\mu-1)}(r \omega_1) d\omega = \frac{\sigma_{d-2}}{(2\pi)^\mu} \int_{-1}^{1} G_-^{(\mu-1)}(r \omega_1)(1 - \omega_1^2)^{\mu-1} d\omega_1.
\]
Here $\omega_1$ is the first variable of $\mathbb{R}^d \supset S^{d-1}$, $\sigma_{d-2}$ is the area of the sphere $S^{d-2}$. We may integrate by parts and rescale

$$u_0(re_1) = \frac{(-1)^{\mu-1} \sigma_{d-2}}{(2\pi)^{\mu-1}} \int_{-1}^{1} G_-(r\omega_1) \left[ \partial_{\omega_1}^{\mu-1} (1 - \omega_1^2)^{\mu-1} \right] d\omega_1$$

$$= \sum_{k=0}^{\lfloor (\mu-1)/2 \rfloor} \frac{A_{d,k}}{r^{\mu-1}} \int_{-1}^{1} G_-(r\omega_1) \omega_1^{\mu-1-2k} d\omega_1$$

$$= \sum_{k=0}^{\lfloor (d-3)/4 \rfloor} \frac{A_{d,k}}{r^{d-2-2k}} \int_{-1}^{1} G_-(r\omega_1) \omega_1^{d/2-2k} d\omega_1$$

$$= \sum_{k=1}^{\lfloor (d+1)/4 \rfloor} \frac{A_{d,k}}{r^{d-2k}} \int_{-1}^{1} G_-(r\omega_1) \omega_1^{d+1/2-2k} d\omega_1$$

Here $A_{d,k}$’s are nonzero constants. Similarly we have

$$u_1(re_1) = \frac{(2\pi)^{-\mu}}{(2\pi)^{\mu}} \int_{S^{d-1}} G_0^{(\mu)}(r\omega_1) d\omega$$

$$= \frac{\sigma_{d-2}}{(2\pi)^{\mu}} \int_{-1}^{1} G_0^{(\mu)}(r\omega_1)(1 - \omega_1^2)^{\mu-1} d\omega_1$$

$$= \frac{(-1)^{\mu} \sigma_{d-2}}{(2\pi)^{\mu}} \int_{-1}^{1} G_-(r\omega_1) \left[ \partial_{\omega_1}^{\mu} (1 - \omega_1^2)^{\mu-1} \right] d\omega_1$$

$$= \sum_{k=0}^{\lfloor (\mu-2)/2 \rfloor} \frac{B_{d,k}}{r^{\mu}} \int_{-1}^{1} G_-(r\omega_1) \omega_1^{\mu-2-2k} d\omega_1$$

$$= \sum_{k=1}^{\lfloor (d-1)/4 \rfloor} \frac{B_{d,k}}{r^{d-2k}} \int_{-1}^{1} G_-(r\omega_1) \omega_1^{d-1/2-2k} d\omega_1.$$

Here $B_{d,k}$’s are nonzero constants. Since smooth functions are dense in $L^2([-R, R])$, we have

**Proposition 4.1.** There exist constants \( \{A_{d,k}\}_{1 \leq k \leq \lfloor (d+1)/4 \rfloor}, \ {B_{d,k}\}_{1 \leq k \leq \lfloor (d-1)/4 \rfloor} \), so that for any \( G_- \in L^2(\mathbb{R}) \) supported in \([-R, R]\), the initial data \( (u_0, u_1) = T_{\omega}^{-1} G_- \) satisfy \( r > R \)

$$u_0(r) = \sum_{k=1}^{\lfloor (d+1)/4 \rfloor} \left( A_{d,k} \int_{-R}^{R} G_-(s) s^{d+1/2-2k} ds \right) r^{-d+2k};$$

$$u_1(r) = \sum_{k=1}^{\lfloor (d-1)/4 \rfloor} \left( B_{d,k} \int_{-R}^{R} G_-(s) s^{d-1/2-2k} ds \right) r^{-d+2k}.$$

This clearly shows that if \( u = S_L(u_0, u_1) \) is a radial \( R \)-weakly non-radiative solution, then \( (u_0, u_1) \in P_{rad}(R) \).

### 4.2 Even dimensions

The even dimensions involve Hilbert transform, thus are much more difficult to handle with. The general idea is the same. If the initial data \( (u_0, u_1) \) are radial, then \( G_+(s) = T_{\mu}^{-1} (u_0, u_1) \) is independent to the angle. We also have \( G_+(s) = (-1)^{d/2}HG_-(-s) \). Thus \( S_L(u_0, u_1) \) is \( R \)-weakly non-radiative if and only if \( G_- \) is contained in the space

$$P_{rad} = \{ G_- \in L^2(\mathbb{R}) : G_-(s) = 0, s > R; (HG_-)(s) = 0, s < -R \}.$$

Now recall the operators \( Q, Q' \) and \( D \) defined in Subsection 2.3. We claim
**Lemma 4.2.** \( Q\mathcal{P}_{\text{rad}} = H^{1/2}(-R, R) \). Here \( \dot{H}^{1/2}_0(-R, R) \) is the completion of \( C^\infty_0(-R, R) \) equipped with the \( H^{1/2}(\mathbb{R}) \) norm.

**Proof.** In order to avoid technical difficulties, we use an approximation technique. Given any \( G_- \in \mathcal{P}_{\text{rad}}, \) we may utilize a local smoothing kernel to generate a sequence \( G_k \), so that

(a) \( G_k \in \mathcal{P}_{\text{rad}}(R + 1/k); \)

(b) \( G_k \in H^n(\mathbb{R}) \) for all \( n \geq 0 \) thus \( G_k \in C^\infty(\mathbb{R}) \).

(c) \( G_k \) converges to \( G_- \) in \( L^2(\mathbb{R}) \).

Let us consider the properties of the function \( g_k = QG_k \in C^\infty(\mathbb{R}) \). According to part (a), \( G_k(s) = 0 \) if \( s > R + 1/k \). We may use the convolution expression of \( Q \) to obtain that \( g_k \) vanishes in the interval \( (R + 1/k, +\infty) \). Similarly \( g_k = QH^1G_k \) vanishes in the interval \( (-\infty, -R - 1/k) \). We recall that \( Q' : L^2(\mathbb{R}) \to H^{1/2}(\mathbb{R}) \) is an isometry up to a constant. Thus \( g_k \to g = QG_- \) in \( H^{1/2}(\mathbb{R}) \). This verifies \( g \in \dot{H}^{1/2}_0(-R, R) \). We also need to show that given any \( g \in \dot{H}^{1/2}_0(-R, R) \), then \( Q^{-1}g \in \mathcal{P}_{\text{rad}} \). It is sufficient to consider \( g \in C^\infty_0(-R, R) \) by smooth approximation. A simple calculation of Fourier symbols shows that \( Q^{-1} = -Q^D \) and \( H^1Q^{-1} = QD \). A combination of these identities with the convolution expressions of \( Q \) and \( Q' \) immediately verifies \( Q^{-1}g \in \mathcal{P}_{\text{rad}} \).

We also need to use the following explicit formula of \( T_- \) for radial data

**Lemma 4.3.** Assume \( G \in C^\infty(\mathbb{R}) \) so that \( |G(s)| \lesssim |s|^{-3/2} \) for \( |s| \gg 1 \). Then the corresponding radial free wave \( u = S_L^*T_-^*G \) satisfies

\[
  u(r, t) = C(d) \cdot r^{1-d/2} \int_{-1}^{1} QG(r\omega_1 + t) P_d(w_1)(1 - w_1^2)^{-1/2} d\omega_1. 
\]

(19)

Here \( P_d \) is an even or odd polynomial of degree \( d/2 - 1 \) defined by

\[
  \left( \frac{\partial}{\partial w_1} \right)^{d/2} (1 - w_1^2)^{d/2} = P_d(w_1)(1 - w_1^2)^{-1/2}. 
\]

**Proof.** If \( G \in C^\infty_0(\mathbb{R}) \), we use the polar coordinates and integrate by parts:

\[
  u(r, t) = C(d) \int_{-\infty}^{\infty} \int_{\mathbb{S}^{d-1}} \frac{G^{(d/2-1)}(r\omega_1 - \rho + t)}{\sqrt{\rho}} d\omega_1 d\rho \\
  = C(d) \int_{0}^{\infty} \int_{-1}^{1} \frac{G^{(d/2-1)}(r\omega_1 - \rho + t)}{\sqrt{\rho}} (1 - w_1^2)^{d/2} d\omega_1 d\rho \\
  = C(d) \cdot r^{1-d/2} \int_{0}^{\infty} \int_{-1}^{1} \frac{G(r\omega_1 - \rho + t)}{\sqrt{\rho}} P_d(w_1)(1 - w_1^2)^{-1/2} d\omega_1 d\rho \\
  = C(d) \cdot r^{1-d/2} \int_{-1}^{1} QG(r\omega_1 + t) P_d(w_1)(1 - w_1^2)^{-1/2} d\omega_1. 
\]

This verifies the formula if \( G \in C^\infty_0(\mathbb{R}) \). In order to deal with profile \( G \) without compact support, we use standard smooth cut-off techniques. More precisely, we may choose \( G_k \in C^\infty_0(\mathbb{R}) \) so that \( G_k \to G \) in \( L^2(\mathbb{R}) \) and

\[
  |G_k(s) - G(s)| \begin{cases} 
  = 0, & s < k; \\
  \lesssim |s|^{-3/2} & s \geq k. 
\end{cases}
\]
Thus we have \( \|QG - QG_k\|_{L^\infty} \leq 1/k \). This means we have the uniform convergence for all \((r, t)\) in any compact subset of \(\mathbb{R}^+ \times \mathbb{R}^2\):

\[
 u_k(r, t) = \frac{C(d)}{r^{d/2-1}} \int_{-1}^{1} QG_k(r \omega + t) P_d(w_1)(1 - w_1^2)^{-1/2} d\omega_1
\]

Combining this with the convergence \( u_k \to u \) in \( \dot{H}^1 \) we finish the proof.

\[ \square \]

**Remark 4.4.** If \( d \geq 4 \) and \( G \in L^2(\mathbb{R}) \), then formula (19) still holds. This follows standard smooth approximation and/or cut-off techniques. Let \( G_k \in C_0^\infty(\mathbb{R}) \) so that \( G_k \to G \) in \( L^2(\mathbb{R}) \). Thus \( QG_k \to QG \) in \( \dot{H}^{1/2}(\mathbb{R}) \). Finally we observe the fact \( P_d(w_1)(1 - w_1^2)^{-1/2} \in \dot{H}^{-1/2}(\mathbb{R}) \), obtain a locally uniform convergence \( u_k(r, t) \to u(r, t) \) and conclude the proof.

Now we are ready to give an expression of \( G^{-1} \). We claim that it is sufficient to consider the case \( QG \in C_0^\infty(-R, R) \). In fact, we may choose \( G_k \in \mathcal{P}_{rad}(R) \) so that \( QG_k \in C_0^\infty(-R, R) \) so that

\[
 QG_k \to QG \text{ in } \dot{H}^{1/2}(-R, R) \quad \iff \quad G_k \to G \text{ in } L^2(\mathbb{R})
\]

Now we observe a few important facts: the embedding \( \dot{H}^{1/2}_0(-R, R) \hookrightarrow L^p(-R, R) \) for all \( 1 \leq p < +\infty \) and

\[
 \frac{P_d(w_1)}{\sqrt{1 - w_1^2}} \in L^p(\mathbb{R}) \quad \Rightarrow \quad W_d(\sigma) \in L^p(\mathbb{R}), \quad p \in (1, 2).
\]

As a result, if the identity

\[
 u_k(r, t) = \frac{C(d)}{r^{d/2}} \int_{-1}^{1} QG_k(s) W_d \left( \frac{s - t}{r} \right) ds, \quad k \geq 1
\]

holds, then we may make \( k \to +\infty \) in the identity above and verify that a similar identity holds for \( u \) and \( G_- \). In fact the left hand side converges in the space \( \dot{H}^1(\mathbb{R}^d) \) for any given time \( t \), while the right hand side converges uniformly for \((r, t)\) in any compact subset of \(\mathbb{R}^+ \times \mathbb{R}^2\). Now we assume \( g = QG \in C_0^\infty(-R, R) \). Then \( G_- = Q^{-1} g = -Q^Dg \) satisfies the assumption of Lemma 4.3. As a result we have

\[
 u(r, t) = C(d) \cdot r^{1-d/2} \int_{-1}^{1} Q Q' Dg (r \omega + t) P_d(w_1)(1 - w_1^2)^{-1/2} d\omega_1
\]

\[
 = C(d) \cdot r^{1-d/2} \int_{-1}^{1} H g (r \omega + t) P_d(w_1)(1 - w_1^2)^{-1/2} d\omega_1
\]

\[
 = \frac{C(d)}{r^{d/2-1}} \int_{-\infty}^{\infty} g(r \sigma + t) W_d(\sigma) d\sigma.
\]
According to Lemma 4.5, we have already obtained the proof of Proposition 1.11. The image of radial data in the form of Lemma 5.1. In this section we prove Proposition 1.11. It suffices to consider the case where $d/2 - 2$ in the interval $(-1, 1)$.

**Proof of Proposition 1.11** According to Lemma 4.5, we have already obtained

$$u(r, t) = \frac{C(d)}{r^{d/2}} \int_{-R}^{R} Q' G_-(s) W_d \left( \frac{s - t}{r} \right) ds.$$ 

Here $Q'G_- \in \dot{H}^{1/2}_0(-R, R) \hookrightarrow L^p(-R, R)$ for all $1 < p < +\infty$. If we also have $r > |t| + R$, then

$$\left| \frac{s - t}{r} \right| < 1, \quad \forall s \in (-R, R).$$

If $d = 2$, Lemma 4.6 immediately gives $u(r, t) \equiv 0$ if $r > |R| + t$ since we always have $W_2(\frac{\cdot}{r^2}) = 0$. In higher dimensional case $d \geq 4$, then Lemma 1.10 guarantees that

$$W_d(s) = \sum_{l=1}^{[d/4]} A_l s^{\frac{d}{2}-2l}, \quad -1 < s < 1.$$ 

is a polynomial. We plug this in the expression of $u$ and obtain

$$u(r, t) = C(d) \sum_{l=1}^{[d/4]} \frac{A_l}{r^{d-2l}} \int_{-R}^{R} Q' G_-(s)(s - t)^{\frac{d}{2}-2l} ds, \quad r > R + |t|. \quad (20)$$

This immediately gives $(u_0, u_1) \in P_{rad}(R)$.

**5 Exterior Energy Estimates of Even Dimensions**

In this section we prove Proposition 1.11. It suffices to consider the case $d = 4k$. The proof of $d = 4k + 2$ are almost the same. Again we switch to the space of radiation profiles $G_- \in L^2(\mathbb{R} \times S^{d-1})$. We start by

**Lemma 5.1.** The image of radial data in the form of $(u_0, 0)$ can be characterized by

$$\{ \mathcal{T}_{-}(u_0, 0) : u_0 \in H_{rad}^{1}(\mathbb{R}^d) \} = \{ G_- \in L^2(\mathbb{R}) : \mathcal{H} G_-(-s) = -G_-(s) \}$$

$$= \left\{ \left( \frac{G(s) - \mathcal{H} G(-s)}{2} : G \in L^2(\mathbb{R}) \right) \right\}$$. 

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Proof. First of all, if \( u_0 \in \hat{H}^{1}_{\text{rad}}(\mathbb{R}^d) \), then free wave \( u = S_L(u_0, u_1) \) is radial and satisfies
\[
u(x, t) = u(x, -t); \quad u_t(x, t) = -u_t(x, -t).
\]
Then \( G_-, G_+ \) are radial, i.e. independent of \( \omega \) and satisfy \( G_+(s) = -G_-(s) \). We may apply Theorem 1.5 and obtain \( G_+(s) = \mathcal{H}G_-(s) \). As a result, \( G_- \) satisfies the identity \( \mathcal{H}G_-(-s) = -G_-(s) \). Next, let us assume \( G_- \) satisfies this identity. Then we have
\[
G_-(s) = \frac{G_-(s) - \mathcal{H}G_-(s)}{-2} \in \left\{ \frac{G(s) - \mathcal{H}G(s)}{-2} : G \in L^2(\mathbb{R}) \right\}.
\]
Finally, if \( G_-(s) = \frac{G(s) - \mathcal{H}G(s)}{-2} \), we show there exists \( u_0 \in \hat{H}^{1}_{\text{rad}}(\mathbb{R}^d) \), so that \( G_- = T_-(u_0, 0) \).
In fact, we consider radial initial data \((u_0, u_1) = T_-(0, 0)\) and free wave \( u = S_L(u_0, u_1) \). We may reverse the time and obtain \( u(x, -t) = S_L(u_0, -u_1)(x, t) \). Thus
\[
T_-(u_0, -u_1)(s) = -T_+(u_0, u_1)(s) = -\mathcal{H}G(-s)
\]
Therefore we have
\[
T_-(2u_0, 0)(s) = G(s) - \mathcal{H}G(-s) = 2G_-(s)
\]
and complete the proof. \(\square\)

The key observation is the following

**Lemma 5.2.** Given \( g \in L^2(\mathbb{R}^+) \), there exists a function \( G \) with \( \|G\|_{L^2(\mathbb{R})} \leq 2\|g\|_{L^2(\mathbb{R}^+)} \) so that
\[
G(s) - \mathcal{H}G(-s) = 2g(s), \quad s > 0.
\]

**Proof.** Let us first find a function \( G \) with \( \|G\|_{L^2(\mathbb{R})} \leq 2\|g\|_{L^2} \) so that
\[
G(s) - \frac{G(s) + \mathcal{H}G(-s)}{2} = g(s), \quad s > 0.
\]
We define a linear bounded operator \( T \) from \( L^2(\mathbb{R}^+) \) to itself \(1\)
\[
(TG)(s) = \frac{G(s) + \mathcal{H}G(-s)}{2} = \frac{G(s)}{2} - \frac{1}{2\pi} \int_0^\infty \frac{G(\tau)}{s + \tau} d\tau, \quad s > 0.
\]
We may further rewrite it as
\[
TG = \frac{G}{2} - \frac{1}{2\pi} L^2 G.
\]
Here \( L \) is the Laplace transform
\[
L G(s) = \int_0^\infty G(\tau) e^{-\tau s} d\tau,
\]
which is self-adjoint operator in \( L^2(\mathbb{R}^+) \) with an operator norm \( \sqrt{\pi} \). More details about the Laplace transform can be found in Lax [25]. As a result, we have
\[
\|TG\|^2_{L^2(\mathbb{R}^+)} = \frac{1}{4} (G - (1/\pi) L^2 G, G - (1/\pi) L^2 G)
\]
\[
= \frac{1}{4} \|G\|^2_{L^2} + \frac{1}{4\pi} \|L^2 G\|^2_{L^2} - \frac{1}{4\pi} (G, L^2 G) - \frac{1}{4\pi} (L^2 G, G)
\]
\[
\leq \frac{1}{4} \|G\|^2_{L^2} + \frac{1}{4\pi} \|L^2 G\|^2_{L^2} - \frac{1}{2\pi} (L^2 G, L^2 G)
\]
\[
= \frac{1}{4} \|G\|^2_{L^2} - \frac{1}{4\pi} \|L^2 G\|^2_{L^2}.
\]

\(1\)When we apply the Hilbert transform, we extend the domain of \( G \) to \( \mathbb{R} \) by assuming \( G(s) = 0 \) if \( s < 0 \).
Thus the operator norm of $T$ is less or equal to $1/2$. This means that the function
\[ G = \sum_{j=0}^{\infty} T^j g \in L^2(\mathbb{R}^+) \]
satisfies the equation $G - TG = g$ and $\|G\|_{L^2(\mathbb{R}^+)} \leq 2\|g\|_{L^2(\mathbb{R}^+)}$. Finally we naturally extend the domain of $G$ to $\mathbb{R}$ by defining $G(s) = 0$ if $s < 0$. We have
\[ \frac{G(s) - HG(-s)}{2} = \begin{cases} g(s), & s > 0; \\ (-1/2)HG(-s), & s < 0. \end{cases} \]
Therefore we may find an upper bound of the $L^2$ norm
\[ \left\| \frac{G(s) - HG(-s)}{2} \right\|_{L^2(\mathbb{R})}^2 \leq \|g\|^2_{L^2(\mathbb{R})} + \frac{1}{4}\|HG\|^2_{L^2(\mathbb{R})} \leq 2\|g\|^2_{L^2(\mathbb{R})}. \]
\[ \square \]

**Proof of Theorem 1.11** Let $G_- = T_-(u_0, 0)$ and $g(s)$ be its cut-off version:
\[ g(s) = \begin{cases} G_-(s), & s > R; \\ 0, & s < R. \end{cases} \]
Then radiation field implies that the free wave $u = S_L(u_0, 0)$ satisfies
\[ \lim_{t \to -\infty} \int_{|x| > R + |t|} |\nabla u(x, t)|^2 dx = \lim_{t \to -\infty} \int_{|x| > R + |t|} |u_t(x, t)|^2 dx = \sigma_{4k-1} \|g\|^2_{L^2(\mathbb{R}^+)} + |\bar{u}_0|^2_{H^1(\mathbb{R}^{4k})} \leq 4\sigma_{4k-1} \|g\|^2_{L^2(\mathbb{R}^+)}.
\]
Here again $\sigma_{4k-1}$ is the area of the sphere $S^{4k-1}$. According to Lemma 5.1 and Lemma 5.2, there exists a function $\bar{u}_0 \in H^1_{rad}(\mathbb{R}^{4k})$, so that
\[ T_-(\bar{u}_0, 0)(s) = g(s), \quad s > 0; \quad \|\bar{u}_0\|^2_{H^1(\mathbb{R}^{4k})} \leq 4\sigma_{4k-1} \|g\|^2_{L^2(\mathbb{R}^+)}.
\]
Therefore $T_-(u_0 - \bar{u}_0, 0)$ vanishes if $s > R$. A combination of this fact with the time symmetry gives
\[ \lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_t S_L(u_0 - \bar{u}_0, 0)(x, t)|^2 dx = 0. \]
As a result, we may apply Proposition 1.13 and conclude $u_0 - \bar{u}_0 \in \mathcal{Q}_L \cap \{T_- \in L^2(\mathbb{R} \times S^{d-1}) :\ \text{supp}G_- \subset [-R, R] \times S^{d-1}\}$. A combination of this inequality and identity 21 immediately verifies the conclusion of Proposition 1.11 in the negative time direction. The positive time direction follows the time symmetry.

## 6 Non-radial Exterior Energy Estimates

In this section we give a short proof of Proposition 1.13. We start by

**Lemma 6.1.** Let $d \geq 3$ be an odd integer. Then
\[ \sum_{j=0}^{\infty} \lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_t S_L(u_0, u_1)(x, t)|^2 dx = 2 \int_{|x| > R} \int_{S^{d-1}} |T_- (u_0, u_1)(s, \theta)|^2 d\theta ds. \]
In particular, we have (see [4] for the definition of $P(R)$)
\[ T_-(P(R)) = \mathcal{P}(R) = \{G_- \in L^2(\mathbb{R} \times S^{d-1}) : \text{supp}G_- \subset [-R, R] \times S^{d-1}\}. \]
Proof. Let \( u \) be the solution of linear wave equation with initial data \((u_0, u_1)\). Then by radiation field (Theorem\(^\text{[1,1]}\)) we have

\[
\lim_{t \to -\infty} \int_{|x|>|t|+R} |\nabla_{t,x} u|^2 \, dx = 2 \int_{R}^{\infty} \int_{S^{d-1}} |G_-(s, \theta)|^2 \, d\theta \, ds;
\]

\[
\lim_{t \to -\infty} \int_{|x|<|t|-R} |\nabla_{t,x} u|^2 \, dx = 2 \int_{-\infty}^{-R} \int_{S^{d-1}} |G_-(s, \theta)|^2 \, d\theta \, ds.
\]

In addition, we may apply the energy conservation law, Proposition\(^\text{[1.2]}\) and obtain

\[
\lim_{t \to -\infty} \int_{|x|<|t|} |\nabla_{t,x} u|^2 \, dx = \int_{\mathbb{R}^d} (|\nabla u_0|^2 + |u_1|^2) \, dx - \lim_{t \to -\infty} \int_{|x|>|t|-R} |\nabla_{t,x} u|^2 \, dx
\]

\[
= \lim_{t \to +\infty} \int_{|x|>|t|+R} |\nabla_{t,x} u|^2 \, dx.
\]

Combining these identities we have

\[
\sum_{\pm} \lim_{t \to -\infty} \int_{|x|>|t|+R} |\nabla_{t,x} u(x, t)|^2 \, dx = 2 \int_{|s|>R} \int_{S^{d-1}} |G_-(s, \theta)|^2 \, d\theta \, ds.
\]

Finally \((u_0, u_1) \in P(R)\) is equivalent to saying

\[
\int_{|s|>R} \int_{S^{d-1}} |G_-(s, \theta)|^2 \, d\theta \, ds = 0,
\]

namely \(\text{supp} G_- \subset [-R, R] \times S^{d-1}\). This finishes the proof. \(\square\)

Now we are ready to prove Proposition\(^\text{[1.3]}\). Since \(\sqrt{2}\mathcal{T}_-\) is a bijective isometry from \(\dot{H}^1 \times L^2(\mathbb{R}^d)\) to \(L^2(\mathbb{R} \times S^{d-1})\). We have

\[
\Pi_{\mathcal{T}^{-1}}(u_0, u_1) = \mathcal{T}^{-1} \Pi_{\mathcal{T}_-} P(R) \mathcal{T}_-(u_0, u_1).
\]

We next use the expression of \(P(R) = \mathcal{T}_-(P(R))\):

\[
\left\| \Pi_{\mathcal{T}^{-1}(P(R))}(u_0, u_1) \right\|^2_{\dot{H}^1 \times L^2} = 2 \left\| \Pi_{\mathcal{T}_-} P(R) \mathcal{T}_-(u_0, u_1) \right\|^2_{L^2(\mathbb{R} \times S^{d-1})}
\]

\[
= 2 \int_{|s|>R} \int_{S^{d-1}} |\mathcal{T}(u_0, u_1)(s, \theta)|^2 \, d\theta \, ds.
\]

Combining this with \(\text{[22]}\) we finish the proof.

7 Appendix

In this section we prove Lemma\(^\text{[4,6]}\). We first prove this lemma for two special cases, i.e. \(P(x) = 1\) and \(P(x) = 1 - x^2\). We start with \(P(x) = 1\). A straightforward calculation gives

\[
\pi W(s) = \text{p.v.} \int_{-1}^{1} \frac{(1-x^2)^{-1/2} - (1-s^2)^{-1/2}}{x-s} \, dx
\]

\[
= \text{p.v.} \int_{-1}^{1} \frac{(1-s^2)^{-1/2} - (1-x^2)^{-1/2}}{s-x} \, dx + \int_{-1}^{1} \frac{(1-x^2)^{-1/2} - (1-s^2)^{-1/2}}{s-x} \, dx
\]

\[
= (1-s^2)^{-1/2} \ln \frac{1+s}{1-s} + \int_{-1}^{1} \frac{(1-s^2) - (1-x^2)}{(1-s^2)(1-x^2) + \sqrt{1-x^2}} \, dx
\]

\[
= (1-s^2)^{-1/2} \ln \frac{1+s}{1-s} + \frac{s}{\sqrt{1-s^2}} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2} \sqrt{1-s^2 + \sqrt{1-s^2}}} \, dx.
\]
Next we apply the change of variables $x = \frac{2z}{1+z^2}$. We have
\[
\sqrt{1-x^2} = \frac{1-z^2}{1+z^2}, \quad dx = \frac{2(1-z^2)}{(1+z^2)^2} dz
\]

Thus
\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}(\sqrt{1-x^2} + \sqrt{1-s^2})} dx = \int_{-1}^{1} \frac{2dz}{1-z^2 + \sqrt{1-s^2}(1+z^2)}
\]
\[
= \frac{2}{s} \int_{0}^{1} \left( \frac{1}{1+\sqrt{1-z^2} - z} + \frac{1}{1+\sqrt{1-z^2} + z} \right) dz
\]
\[
= \frac{2}{s} \ln \left| \frac{1+\sqrt{1-z^2} + 1}{1+\sqrt{1-z^2} - 1} \right|
\]
\[
= \frac{1}{s} \ln \left| \frac{1+s}{1-s} \right|
\]

This immediately gives $W(x) = 0$. Next we consider the case $P(x) = 1 - x^2$. In this case we calculate the Hilbert transform of $\sqrt{1-x^2}$
\[
\pi W(s) = \text{p.v.} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{s-x} dx
\]
\[
= \text{p.v.} \int_{-1}^{1} \frac{\sqrt{1-s^2}}{s-x} dx + \int_{-1}^{1} \frac{\sqrt{1-x^2} - \sqrt{1-s^2}}{s-x} dx
\]
\[
= \sqrt{1-s^2} \ln \left| \frac{1+s}{1-s} \right| + \int_{-1}^{1} \frac{(1-x^2) - (1-s^2)}{(s-x)(\sqrt{1-x^2} + \sqrt{1-s^2})} dx
\]
\[
= \sqrt{1-s^2} \ln \left| \frac{1+s}{1-s} \right| + s \int_{-1}^{1} \frac{1}{\sqrt{1-x^2} + \sqrt{1-s^2}} dx
\]
\[
= \sqrt{1-s^2} \ln \left| \frac{1+s}{1-s} \right| + \pi s + s \int_{-1}^{1} \left( \frac{1}{\sqrt{1-x^2} + \sqrt{1-s^2}} - \frac{1}{\sqrt{1-x^2}} \right) dx
\]
\[
= \sqrt{1-s^2} \ln \left| \frac{1+s}{1-s} \right| + \pi s \sqrt{1-s^2} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2} + \sqrt{1-s^2}} dx
\]
\[
= \pi s
\]

Here we use the integral (23) again.

**Induction** Now we are ready to prove Lemma 4.6 by an induction. It is clear that we only need to show the Hilbert transform of $f_{\kappa}(x) = x^\kappa(1-x^2)^{-1/2}$ is a polynomial of degree $\kappa - 1$ in the interval $(-1,1)$. The cases of $\kappa = 0, 2$ have been done. Now let us consider the case of $f_1(x) = x(1-x^2)^{-1/2}$. We observe that $(s \in (-1,1))$
\[
\mathcal{H}f_1 = \mathcal{H} \frac{d}{dx} \left( -\sqrt{1-x^2} \right) = -\frac{d}{ds} \mathcal{H}(\sqrt{1-x^2}) = -1.
\]

This prove the case $\kappa = 1$. Now let us assume that the cases $\kappa = 0, 1, 2, \cdots, n$ are done and consider the case $\kappa = n + 1$. Here $n \geq 2$. We have
\[
x^{n+1}(1-x^2)^{-1/2} = -x^{n-1}(1-x^2)^{1/2} + x^{n-1}(1-x^2)^{-1/2}.
\]
The Hilbert transform of the second term in the right hand side has been known to be a polynomial of degree $n - 2$. Thus we only need to consider the first term. We have
\[
\frac{d}{ds} \mathcal{H} \left( x^{n-1}(1-x^2)^{1/2} \right) = \mathcal{H} \frac{d}{dx} \left( x^{n-1}(1-x^2)^{1/2} \right) \\
= \mathcal{H} \left\{ [-nx^n + (n-1)x^{n-2}] (1-x^2)^{-1/2} \right\}
\]
This is a polynomial of degree $n - 1$ by induction hypothesis. A simple integration then finish the proof of case $\kappa = n + 1$.

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2 Generally speaking, the derivative with respect to $s$ is in the weak sense. But since the derivative is known to be a polynomial in $(-1, 1)$, we can integrate as usual.
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