Electromagnetic properties of spin-3/2 Majorana particles

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The structure of the electromagnetic vertex function of spin-3/2 particles is analyzed in a general way, for the diagonal and off-diagonal couplings, of charged as well as neutral particles including the case of self-conjugate (Majorana) particles. The restrictions imposed by common principles such as electromagnetic gauge invariance and hermiticity are studied, and the implications due to the discrete space-time symmetries or the Majorana condition are deduced when they are applicable. In some cases certain features of the vertex function are analogous to the known ones for the spin-1/2 or spin-1 particles, but we find and discuss other particular properties which are related to the spin-3/2 Rarita-Schwinger representation. For example, in the diagonal Majorana case, the vertex function can contain a term of the form $\gamma_\mu \gamma_5$, which resembles the axial charge radius term for Majorana neutrinos, plus another one that resembles the vertex function for self-conjugate spin-1 particles, but with the particularity that the two terms may not appear independently of each other, but instead with a specific relative coefficient. In essence this is due to the requirement that the vertex function does not mix the genuine spin-3/2 degrees of freedom with the spurious spin-1/2 components of the Rarita-Schwinger representation. The analogous results for the other cases (off-diagonal and charged couplings) are discussed as well.

I. INTRODUCTION AND SUMMARY

Since electrically neutral particles do not couple to the photon at the tree-level, their electromagnetic properties can provide a window to higher order effects or sectors of the Standard Model (SM) and its extensions that may be difficult or impossible to probe directly. For example, it has been known for a long time that Majorana neutrinos can have neither electric nor magnetic dipole moments; it can only have the so-called axial charge radius. Results of this type can also be deduced, for example, for the transition moments between two different Majorana neutrinos\[1–5\], and similar results hold for spin-1 particles\[6\] and for Majorana particles of arbitrary spin\[7\]. By the same token, if departures from these results were observed experimentally, it would have implications for some important principles such as Lorentz and gauge invariance, CPT and crossing symmetry.

There has been considerable activity recently in the study of the effects of gravitinos in several cosmological contexts such as nucleosynthesis and inflation\[8–11\], as well as some areas of current phenomenological interest such as dark matter\[12–14\]. Many of these effects have to do with the electromagnetic properties of gravitinos. While the electromagnetic properties of neutrinos have been well studied, for example in the references mentioned, and for the spin-1 bosons they were studied in a general and comprehensive way in \[15\], for the spin-3/2 fermions the existing studies are geared toward specific cases and applications, such as the diagonal or transition vertices for charged baryons, for example\[15, 16\]. Thus it seems useful to look in a general way at the electromagnetic properties of spin-3/2 particles, to provide a systematic study that while being helpful for considering questions of fundamental and intrinsic interest, is also useful for phenomenological applications in the contexts mentioned. This work can also serve as a guide when testing the SM and its extensions, the possible departures from them or the breaking of some fundamental principles such as Lorentz or gauge invariance, and can also be relevant in other problems involving spin-3/2 particles such as spin-3/2 quark searches at the LHC\[17\] and others\[18, 19\]. In addition, although we limit ourselves here to the spin-3/2 to spin-3/2 electromagnetic vertex, it can also serve as a guide to consider the gravitino-neutrino radiative transition, which has also been of interest in the context of dark matter\[20, 21\].

Before embarking in the details we summarize the strategy that we follow and the main results. We denote by $j_\mu(Q,q)$ the matrix element of the electromagnetic current operator $J_\mu^{(em)}(0)$ between two spin-3/2 particles states of momentum $k$ and $k'$,

$$j_\mu(Q,q) \equiv \langle f'(k')|J_\mu^{(em)}(0)|f(k)\rangle ,$$

(1.1)
where it is convenient to introduce the variables

\[ q = k - k', \]
\[ Q = k + k', \]

and consider that matrix element as a function of them, as indicated in Eq. (1.1). The electromagnetic off-shell vertex function for spin-3/2 particles, \( \Gamma_{\alpha\beta\mu}(Q, q) \), is defined such that

\[ j_\mu(Q, q) = \bar{U}^\alpha(k') \Gamma_{\alpha\beta\mu}(Q, q) U^\beta(k), \]

where \( U^\alpha(k) \) is a Rarita-Schwinger (RS) spinor. The objective is to find the most general form of the vertex function consistent with Lorentz and electromagnetic gauge invariance, and other requirements that may be applicable such as the discrete space-time symmetries or the Majorana nature of the particles.

It is useful to remember that when \( f \) and \( f' \) represent electrically neutral particles, then electromagnetic gauge invariance implies that

\[ q^\mu \Gamma_{\alpha\beta\mu}(Q, q) = 0, \]

for any values of \( Q \) and \( q \). On the other hand, if they are charged, the relation analogous to Eq. (1.1) contains additional terms on the right-hand side involving the inverse propagators of the particles, reminiscent of the Ward identity in QED. Thus in that case, instead of Eq. (1.1), we get the weaker condition for the on-shell matrix element

\[ q^\mu j_\mu(Q, q) = 0. \]

In this paper we restrict ourselves to the on-shell matrix element \( j_\mu(Q, q) \), and therefore the condition as written in Eq. (1.3) applies uniformly for all the cases.

We can summarize our main result as follows. For the particular case in which the initial and final particle is the same and it is self-conjugate (the diagonal Majorana case), we find that the vertex function involves at most two independent terms, which can be written in the form

\[ \Gamma_{\alpha\beta\mu} = m F g_{\alpha\beta} \left( g_{\mu}^{-2} - q_{\mu} q_{\nu} \right) \gamma_5 + i G \left( q_{\mu} g_{\nu} q_{\lambda} - q_{\nu} g_{\mu} q_{\lambda} \right) q^\nu Q^\lambda, \]

where \( m \) is the mass and the two corresponding form factors, denoted here by \( F \) and \( G \), are real. The term proportional to \( \gamma_\mu \gamma_5 \) is reminiscent of the axial charge radius term for Majorana neutrinos\([1]\), while the other two terms resemble the result that was obtained in Ref.\([2]\) for the electromagnetic vertex function of self-conjugate spin-1 particles (Eq. (4.14) in that reference). An expression similar to Eq. (1.6) was obtained in Ref.\([3]\) for the vertex function in this same case. However, the noteworthy feature of our result quoted above, is the fact that the coefficients of the first two terms in Eq. (1.6) are not independent. Further conditions exist if some discrete symmetries hold. For example, if \( CP \) holds, then \( G = 0 \), so that only the \( F \) terms in Eq. (1.6) can be present. But if the \( \gamma_\mu \gamma_5 \) is present, then it must be accompanied by the second term in Eq. (1.6). In essence this result is due to the requirement that the vertex function does not mix the genuine spin-3/2 degrees of freedom with the spurious spin-1/2 components of the RS representation. We also obtain and discuss the corresponding results for the off-diagonal and for the charged particle cases.

Our plan in the rest of the paper is then as follows. In Section II we write down the most general form of the on-shell vertex function that is consistent with electromagnetic gauge invariance in terms of a set of form factors that we denote as \( a_i, b_i \). The resulting representation of the vertex function has the virtue that it involves simple combinations of the \( \gamma \) matrices, the momentum vectors, the metric and the Levi-Civita tensors, which makes it amenable for practical calculations of transition rates. However, as we point out there, the coefficients \( a_i, b_i \) so introduced must satisfy some relations, related to the use of the RS spinor representation, which are difficult to elucidate in general, and therefore that form of the vertex function is not the most convenient form for studying the implications of the various discrete symmetries and making the connection with the diagonal or transition electromagnetic moments of the particles. In Section III we write down another expression for the on-shell vertex function, in terms of two matrices which we denote by \( R_i, P_\mu \), and their products, which by construction satisfy the physical requirements related to gauge invariance and the use of the RS spinors that the vertex function must satisfy, without having to impose further conditions. As we show there, the various terms in this second representation of the vertex function have a simple interpretation in terms of the electromagnetic moments of the spin-3/2 particles. The formulas that express the relations between the two sets of form factors are given explicitly there as well, and some details of the derivation are given in the appendix. In Section IV we study the implications due to the discrete transformations, such as the \( C, P, T \) transformations and their products, and the conditions implied by the hermiticity of the interaction Lagrangian and crossing symmetry. We consider the diagonal and off-diagonal (transition) matrix element, and we include both the neutral and charged cases. We then consider in detail the Majorana case, for which some of the main results are summarized above, and in Section V we discuss the corresponding results for the off-diagonal and for the charged particle cases.
II. PARAMETRIZATION OF THE VERTEX FUNCTION

A. General form

The goal is to find the most general linearly independent set of tensors matrices $\Gamma^{(A)}_{\alpha\beta\mu}(Q,q)$, such that we can write

$$\Gamma_{\alpha\beta\mu}(Q,q) = \sum_A F_A \Gamma^{(A)}_{\alpha\beta\mu}(Q,q),$$

(2.1)

in a way that Eq. (1.5) is satisfied. The set of tensors $\Gamma^{(A)}_{\alpha\beta\mu}$ can be divided into two groups, according to whether they contain an even or odd number of powers of $q$ (equivalently, whether they are even or odd under $q \rightarrow -q$). Since the $F_A$ are functions only of $q^2$, this implies, corresponding to that classification, that $\Gamma_{\alpha\beta\mu}(Q,q)$ can then be expressed in the form

$$\Gamma_{\alpha\beta\mu}(Q,q) = X_{\alpha\beta\mu} + Y_{\alpha\beta\mu\nu}q^\nu,$$

(2.2)

where both $X$ and $Y$ are even under $q \rightarrow -q$. The transversality condition on $j_\mu$ implies that both pieces must be, separately, transverse; i.e.,

$$q^\mu U'^\alpha(k')X_{\alpha\beta\mu}U^\beta(k) = 0,$$

$$q^\mu q^\nu U'^\alpha(k')Y_{\alpha\beta\mu\nu}U^\beta(k) = 0.$$

(2.3)

The first condition implies that $X_{\alpha\beta\mu}$ must be of the form

$$X_{\alpha\beta\mu} = \tilde{g}_{\mu\nu} A_{\alpha\beta\nu},$$

(2.4)

where

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}.$$

(2.5)

The second condition is solved by taking $Y_{\alpha\beta\mu\nu}$ to be the most general antisymmetric tensor in the indices $\mu,\nu$. We can choose to express $Y_{\alpha\beta\mu\nu}$ in terms of its dual and write

$$Y_{\alpha\beta\mu\nu} = \epsilon_{\mu\nu\lambda\rho} B_{\alpha\beta\lambda\rho}.$$

(2.6)

In this way we obtain that we can express the vertex function in the form

$$\Gamma_{\alpha\beta\mu} = \tilde{g}_{\mu\nu} A_{\alpha\beta\nu} + \epsilon_{\mu\nu\lambda\rho} q^\nu B_{\alpha\beta\lambda\rho},$$

(2.7)

The next step is to enumerate the possible tensor structures that can appear in $A_{\alpha\beta\nu}$ and $B_{\alpha\beta\lambda\rho}$. In doing so it is important to keep in mind that the spinors satisfy the Dirac equation as well as the auxiliary RS conditions

$$k^\alpha U_\alpha(k) = 0,$$

$$\gamma^\alpha U_\alpha(k) = 0,$$

(2.8)

with analogous relations for $U'_\alpha(k')$. The relations in Eq. (2.8) imply in particular the following,

- Terms containing a factor of $\gamma_\alpha$ or $\gamma_\beta$ do not appear.
- We need not consider any term involving the factors $q$ or $Q$ since any such factor can be eliminated by means of the Dirac equation satisfied by the $U$ and $U'$ spinors (together with judicious use of the anticommutation relations of the gamma matrices).
- The terms that contain a factor of $q_\alpha$ ($q_\beta$) are not independent of the analogous terms with $Q_\alpha$ ($Q_\beta$).
- When writing $A_{\alpha\beta\nu}$, the terms involving $q_\nu$ disappear when contracted with $\tilde{g}^{\mu\nu}$.

In this way we arrive at the following terms for $A_{\alpha\beta\nu}$,

$$A_{\alpha\beta\nu} = g_{\alpha\beta} \gamma_\nu(a_1 + a'_1 \gamma_5) + g_{\alpha\beta} Q_\nu (a_2 + a'_2 \gamma_5) + Q_\alpha Q_\beta \gamma_\nu (a_3 + a'_3 \gamma_5) + Q_\alpha Q_\beta Q_\nu (a_4 + a'_4 \gamma_5) + (g_{\alpha\beta} Q_\nu + g_{\nu\beta} Q_\alpha)(a_5 + a'_5 \gamma_5) + (g_{\alpha\nu} Q_\beta - g_{\nu\beta} Q_\alpha)(a_6 + a'_6 \gamma_5).$$
and proceeding in a similar way for \( B_{\alpha\beta\lambda\rho} \),
\[
B_{\alpha\beta\lambda\rho} = b_1 g_{\alpha\lambda} g_{\beta\rho} + b_2 g_{\alpha\lambda} Q_{\beta} Q_{\rho} + b_3 g_{\beta\lambda} Q_{\alpha} Q_{\rho} .
\]
In the discussions in the next sections, it will be more convenient to eliminate the appearances of \( q_{\alpha,\beta} \). Thus the final expressions for \( X_{\alpha\beta\mu} \) and \( Y_{\alpha\beta\mu\nu} q^\nu \) are
\[
X_{\alpha\beta\mu} = g_{\alpha\beta} \tilde{\gamma}_{\mu}(a_1 + a_1' \gamma_5) + g_{\alpha\beta} \bar{Q}_{\mu}(a_2 + a_2' \gamma_5) + q_{\alpha} q_{\beta} \bar{Q}_{\mu}(a_3 + a_3' \gamma_5) + q_{\alpha} q_{\beta} \bar{Q}_{\mu}(a_4 + a_4' \gamma_5) + (\bar{g}_{\mu} Q_{\beta} + \bar{g}_{\mu} q_{\alpha})(a_5 + a_5' \gamma_5) + (\bar{g}_{\mu} q_{\beta} - \bar{g}_{\mu} q_{\alpha})(a_6 + a_6' \gamma_5) ,
\]
\[
Y_{\alpha\beta\mu\nu} q^\nu = b_1 \epsilon_{\alpha\beta\mu\nu} q^\nu + b_2 g_{\alpha\mu} \lambda^\nu Q^\lambda + b_3 q_{\alpha} \epsilon_{\beta\mu\nu} q^\nu Q^\lambda ,
\]
where
\[
\tilde{\gamma}_{\mu} = \bar{g}_{\mu\nu} \gamma^\nu ,
\]
\[
\bar{Q}_{\mu} = \bar{g}_{\mu\nu} Q^\nu .
\]
Notice that we have not included the terms of the form
\[
\gamma_{\mu} \epsilon_{\alpha\beta\lambda\rho} q^\lambda Q^\rho
\]
\[
Q_{\mu} \epsilon_{\alpha\beta\lambda\rho} q^\lambda Q^\rho
\]
\[
\bar{g}_{\mu\nu} \epsilon_{\alpha\beta\nu\lambda} Q^\lambda
\]
Using the identity
\[
g_{\nu\mu} \epsilon_{\alpha\beta\lambda\rho} - g_{\nu\alpha} \epsilon_{\mu\beta\lambda\rho} - g_{\nu\beta} \epsilon_{\alpha\mu\lambda\rho} - g_{\nu\lambda} \epsilon_{\alpha\beta\mu\rho} - g_{\nu\rho} \epsilon_{\alpha\beta\lambda\mu} = 0 ,
\]
contracted with \( Q^\nu q^\lambda Q^\rho, \gamma^\nu q^\lambda Q^\rho \) or \( q^\nu q^\lambda Q^\rho \), those three terms can be expressed in terms of the ones we have already included. For similar reasons we have not included the terms involving gamma matrices of the form \( \epsilon \gamma_{\delta\lambda\rho} \gamma^\lambda \gamma^\rho \) or \( \epsilon \gamma_{\delta\lambda\rho} \gamma^\rho \). The former terms can be rewritten in terms of \( \sigma_{\gamma\delta} \gamma_5 \) (without the epsilon tensor) and therefore can be absorbed in the \( X \) terms that we have included above. The terms in which the epsilon tensor is contracted with one Dirac gamma matrix can be reduced by means of the identity [31]
\[
\epsilon_{\alpha\beta\gamma\lambda} \gamma^\lambda = \gamma_{\alpha} \gamma_{\beta} \gamma_{\gamma} \gamma_5 - (g_{\alpha\beta} \gamma_{\gamma} - g_{\alpha\gamma} \gamma_{\beta} + g_{\beta\gamma} \gamma_{\alpha}) \gamma_5 .
\]
The term with the three gamma matrices either yields zero when one gamma matrix is contracted with one of the spinors, or it can be reduced, by means the Dirac equation when a gamma matrix is contracted with \( q \) or \( Q \), to terms with only one Dirac gamma matrix (times \( \gamma_5 \)), which we already included in the \( X \) terms.

The form factors that appear in Eq. (2.9) must actually satisfy some conditions that follow from physical requirements. However, those conditions depend on whether we are considering charged or neutral particles, and also if we are considering the diagonal or the off-diagonal (transition) vertex. One set of conditions follows from the requirement that there should not be any kinematic singularities at \( q = 0 \). Another follows from the fact that we are dealing with a spin-3/2 particle. Both imply that there are certain kinematic relations between the form factors, none of which are taken into account in Eq. (2.9). We now derive both sets of conditions.

### B. Kinematic singularities at \( q = 0 \)

The singular terms are
\[
\Gamma_{\alpha\beta\mu}^{I(singular)} = \frac{1}{q^2} \left[ g_{\alpha\beta} q_{\mu} \bar{g}(a_1 + a_1' \gamma_5) + g_{\alpha\beta} q_{\mu} Q \cdot q(a_2 + a_2' \gamma_5) + q_{\mu} q_{\alpha} q_{\beta} \bar{g}(a_3 + a_3' \gamma_5) + q_{\alpha} q_{\beta} q_{\mu} Q \cdot q(a_4 + a_4' \gamma_5) + 2a_5 q_{\alpha} q_{\beta} q_{\mu} + 2a_5' q_{\alpha} q_{\beta} q_{\mu} 5 \right] .
\]
The conditions that the vertex function satisfies at \( q = 0 \) are different depending on whether it is zero at the tree level or not. As an example, let us consider the diagonal case and specifically an electrically neutral particle. In this case, there is no tree-level electromagnetic coupling and the vertex function is the result of the higher order corrections. Therefore, in all the relevant Feynman diagrams, the external photon line is attached to an internal line and the
momentum integrations ensure that all such contributions are finite as \( q \to 0 \). Remembering that \( Q \cdot q = 0 \) and that \( \not{q} = 0 \) between the spinors, the requirement that the singularity at \( q^2 = 0 \) is absent implies that the various coefficients \( a_i(q^2) \) must satisfy
\[
a'_1,3,5(0) = a_5(0) = 0. 
\]
(2.14)

While we could consider the nondiagonal case as well as the charged-particle cases in a similar way, we do not proceed any further in this directions since, as we will see in Section III A we will propose a more general and natural method to incorporate these requirements.

### C. Absence of spin-1/2 component

At this point it is useful to recall that the generators of the Lorentz group for the RS spinor are
\[
(S_{\mu\nu})_{\alpha\beta} = \left( \sigma_{\mu\nu} \right)_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} \sigma_{\mu\nu},
\]
(2.15)
where
\[
(S_{\mu\nu})_{\alpha\beta} \equiv i(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})
\]
(2.16)
are the spin-1 generators. It will be useful to also define the duals
\[
(\tilde{\Sigma}_{\mu\nu})_{\alpha\beta} \equiv \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} (\sigma_{\lambda\rho})_{\alpha\beta}.
\]
(2.17)

Notice that the terms \( a_6 \) and \( a'_6 \) in Eq. (2.9) involve the spin-1 generators \( S_{\mu\nu} \). The spin-1/2 generators will surface by rewriting the \( \gamma_\mu \) terms (e.g., the \( a_1 \) term), using the Gordon decomposition
\[
2m\bar{U}^\alpha(k')\gamma_\mu U^\beta(k) = \bar{U}^\alpha(k') \left[ Q_\mu - i\sigma_\mu q^\nu \right] U^\beta(k).
\]
(2.18)

However, without further considerations, that would result in the spin-1 generators \( S_{\mu\nu} \) and the spin-1/2 generators \( \frac{1}{2} \sigma_{\mu\nu} \) appearing separately in the vertex function, instead of the combination given in Eq. (2.15). In a physically sensible theory of the spin-3/2 particle, that cannot happen since it would lead to unitarity inconsistencies due to this mixing with the unphysical components of the RS field.

Therefore we demand that in a physically sensible theory the form factors that appear in Eq. (2.9) must satisfy further relations in such a way that only the spin-3/2 generators appear and not the spin-1 and spin-1/2 separately.

As an illustration, let us consider the identification of the dipole moment terms in the diagonal case. In addition to Eq. (2.18), we will use
\[
Q_\mu \bar{U}^\alpha(k')\gamma_5 U^\beta(k) = \bar{U}^\alpha(k')i\sigma_\mu q^\nu \gamma_5 U^\beta(k).
\]
(2.19)
ext together with the identity
\[
\sigma_\mu \gamma_5 = i\tilde{\sigma}_\mu
\]
(2.20)
where
\[
\tilde{\sigma}_\mu = \frac{1}{2} \epsilon_{\mu\rho\lambda\nu} \sigma^{\lambda\nu}.
\]
(2.21)

Then, using Eq. (2.18) to rewrite the \( a_1 \) term, and Eq. (2.19) and (2.20) to rewrite the \( a'_2 \) term, we obtain
\[
X_{\alpha\beta} = a_0 g_{\alpha\beta} Q_\mu - \left( \frac{a_1}{2m} \right) g_{\alpha\beta} i\sigma_\mu q^\nu + a'_2 g_{\alpha\beta} \epsilon_{\mu\nu\lambda\rho} q^\nu \sigma^{\lambda\rho}
+ a_6 (g_{\mu\alpha} g_{\beta\nu} - g_{\mu\beta} g_{\alpha\nu}) q^\nu + OT,
\]
(2.22)
where we have defined
\[
a_0 = a_2 + \frac{a_1}{2m},
\]
(2.23)
and OT stands for the other terms, which are not relevant for the present discussion.
This form suggests what must happen. Namely, the parameters \(a_4\) and \(a_6\) must be related at \(q \to 0\) such that those terms combine into one term proportional to the tensor matrix \((\tilde{\Sigma}_{\mu\nu})_{\alpha\beta}\) defined above. Similarly the parameters \(a'_4\) and \(b_1\) must be related such that their corresponding terms combine into one proportional to \((\tilde{\Sigma}_{\mu\nu})_{\alpha\beta}\). The two combinations obtained in this way will be identified with the magnetic and electric dipole terms, respectively. Thus, we assert that the following relations must hold

\[
\begin{align*}
  a_6 &= \frac{a_1}{m} + O(q^2) \\
  a'_4 &= \frac{i}{2}b_1 + O(q^2),
\end{align*}
\]

so that

\[
\begin{align*}
  a_6(g_{\mu\alpha}g_{\beta\nu} - g_{\mu\beta}g_{\alpha\nu})q^\nu - \frac{a_1}{2m}g_{\alpha\beta}i\sigma_{\mu\nu}q^\nu &= \frac{a_1}{m}(-i\Sigma_{\mu\nu})_{\alpha\beta}q^\nu + O(q^2), \\
  -\frac{a'_4}{2}g_{\alpha\beta}\epsilon_{\mu\nu\lambda\rho}q^\nu\sigma^{\lambda\rho} + \frac{b_1}{2}\epsilon_{\mu\nu\lambda\rho}q^\nu(\delta^\lambda\alpha\delta^\rho\beta - \delta^\lambda\beta\delta^\rho\alpha) &= \frac{b_1}{2}(-i\tilde{\Sigma}_{\mu\nu})_{\alpha\beta}q^\nu + O(q^2),
\end{align*}
\]

and therefore

\[
\Gamma_{\alpha\beta\mu} = a_0g_{\alpha\beta}Q_\mu + \frac{a_1}{m}(-i\Sigma_{\mu\nu})_{\alpha\beta}q^\nu + \frac{b_1}{2}(-i\tilde{\Sigma}_{\mu\nu})_{\alpha\beta}q^\nu + O(q^2) + OT.
\]

The identification of the electric charge and dipole moments is obvious in this form, but it is evident that it is not practical to find all the appropriate relations between the coefficients in this way.

### D. Nonrelativistic limit

To motivate our strategy for taking into account all the issues mentioned in Sections 1.1B and 1.1C it is useful to consider the nonrelativistic limit. In the case of the spin-1/2 particles, if we consider the set of all the possible bilinears \(\omega'\Gamma\omega\), where \(\omega\) and \(\omega'\) are the two-component spinors corresponding to the initial and final particle states, it is not possible to construct any higher rank three-dimensional tensors other than the vector \(\omega^i\sigma_i\omega\). This follows from the relations that the spin matrices \(s^i \equiv \frac{\sigma^i}{2}\) satisfy,

\[
\begin{align*}
  [s^i, s^j] &= i\epsilon^{ijk}s^k, \\
  \{s^i, s^j\} &= \frac{1}{2}\delta^{ij},
\end{align*}
\]

which is of course a reflection of the fact that, apart from the identity matrix, the Pauli matrices span the space of \(2 \times 2\) matrices. Therefore, the particle can have only electric and magnetic dipole moments.

For a spin-1 particle, the spin matrices

\[(S^i)^j_k = -i\epsilon^{ijk}\]

satisfy Eq. (2.27), but not Eq. (2.28). Therefore, the antisymmetric products do not yield linearly independent matrices, but the symmetric combinations do. Thus in this case, in addition to the dipole moments, the particle can have quadrupole moments involving terms of the form \(\xi^i\{S^i, S^j\}\xi\), where \(\xi\) and \(\xi'\) stand for the space part of the spin-1 polarization vectors, written as column matrices. The spin-1 particle can have no higher moments because, together with the \(S^i\), we have already exhausted the number of independent \(3 \times 3\) matrices. This is reflected in the fact that the symmetrized product of the spin-1 matrices is given by

\[
\{S^i, S^j\}S^k + (i \leftrightarrow k) + (j \leftrightarrow k) = 2\left(S^i\delta^{jk} + S^j\delta^{ki} + S^k\delta^{ij}\right),
\]

which is the generalization of the familiar relation \((S^i)^3 = S^i\).

This suggests that the generalization to the spin-3/2 is to consider the symmetrized products of the spin-3/2 matrices

\[
(S^i)^{jk} = \frac{1}{2}\delta^{ijk} + (S^i)^{jk},
\]

which are the three-dimensional versions of the generators defined in Eq. (2.15). Then, as we have already seen in Eq. (2.19), the bilinears involving \(\Sigma^i\) are related to the dipole moments, while the quadrupole and octupole moments are
associated with the symmetrized combinations of the products $\Sigma^i \Sigma^j$ and $\Sigma^i \Sigma^j \Sigma^k$, respectively. As we have already mentioned, this requires that the coefficients that appear in Eq. (2.20) be related in such a way that the spin-1/2 and spin-1 spin matrices do not appear separately, but rather in the combinations given by the $\Sigma^i$ and their symmetrized products. A bonus of these requirements, is that the spin-1/2 mixing problem goes away as we now show.

First it is useful to point out the following relation,

$$\sigma^i (\Sigma^i)^j = \frac{1}{2} \sigma^i \sigma^j,$$  \hspace{1cm} (2.32)

which is readily verified by using the definition of the $(\Sigma^i)^j$ together with $\sigma^i \sigma_i = \sigma^i \sigma^i + 2i \epsilon^{ijk} \sigma^k$. This relation is easily generalized, by induction for example, to the product of any number of $\Sigma$ matrices,

$$\sigma^i (\Sigma^i \Sigma^j \ldots \Sigma^n)^k = \frac{1}{2^n} \sigma^i \sigma^j \ldots \sigma^n \sigma^k.$$  \hspace{1cm} (2.33)

Now consider the spinor $\phi^i$ of a spin-3/2 particle in its rest frame, constructed as a linear combination of products of the form

$$\phi^i \sim \xi^i \omega,$$  \hspace{1cm} (2.34)

where $\xi^i$ and $\omega$ stand for the spin-1 and spin-1/2 spinors defined above. Such a spinor satisfies

$$\sigma^i \phi^i = 0,$$  \hspace{1cm} (2.35)

which is just the nonrelativistic limit of the RS condition expressing the fact that $\phi^i$ has no spin-1/2 component of the products $\xi^i \omega$. Eq. (2.33) then implies

$$\sigma^i (\Sigma^i \Sigma^j \ldots \Sigma^n)^k \phi^k = 0,$$  \hspace{1cm} (2.36)

which means that the spinor $(\Sigma^i)^k \phi^k$, in or general $(\Sigma^i \Sigma^j \ldots \Sigma^n)^k \phi^k$ also satisfies Eq. (2.35) and therefore it is guaranteed to represent a spin-3/2 spinor without any spin-1/2 component.

This then yields the following recipe. If we write the transition amplitude in the form

$$M = \phi^{i1} T^{jk} \phi^k,$$  \hspace{1cm} (2.37)

the above results imply that, if $T$ is constructed from the $(\Sigma^i)^k$ and its products, then $T$ does not have matrix elements between the spin-1/2 and spin-3/2 components. In particular, if the initial spinor $\phi^i$ satisfies Eq. (2.35), then $T$ has matrix elements only if the final spinor also satisfies that same condition; there is no spin-1/2 mixing problem.

The covariant version of this argument is based on the following identities which generalize the relations given in Eq. (2.20) and (2.33),

$$\gamma^\alpha (\Sigma_{\mu \nu})_{\alpha \beta} = \frac{1}{2} \sigma_{\mu \nu} \gamma^\beta,$$  

$$\gamma^\alpha (\Sigma_{\mu_1 \nu_1} \ldots \Sigma_{\mu_n \nu_n})_{\alpha \beta} = \frac{1}{2^n} \sigma_{\mu_1 \nu_1} \ldots \sigma_{\mu_n \nu_n} \gamma^\beta.$$  \hspace{1cm} (2.38)

with the understanding that the products are defined by

$$(\Sigma_{\mu_1 \nu_1} \Sigma_{\mu_2 \nu_2})_{\alpha \beta} = (\Sigma_{\mu_1 \nu_1})_{\alpha \gamma} (\Sigma_{\mu_2 \nu_2})_{\beta \gamma}.$$  \hspace{1cm} (2.39)

Denoting the transition amplitude by $\bar{U}^{\alpha \beta} T_{\alpha \beta} U^\beta$, these imply that if the initial spinor satisfies the RS condition, then the final spinor $T_{\alpha \beta} U^\beta$ also satisfies the RS condition if $T_{\alpha \beta}$ is constructed from the $\Sigma_{\mu \nu}$ and their products.

In summary, the form factors that appear in the vertex function, for example in Eq. (2.20), must be related such that the spin-1/2 and spin-1 matrices do not appear separately in the amplitude but only in the combinations given by $\Sigma_{\mu \nu}$ and their products. Specifically, apart from the electric monopole term, the vertex function must be expressible in terms of the following matrices

$$\Sigma_{\mu \nu}, \quad \bar{\Sigma}_{\mu \nu},$$  

$$\Sigma_{\mu \nu} \Sigma_{\lambda \rho}, \quad \Sigma_{\mu \nu} \bar{\Sigma}_{\lambda \rho},$$  

$$\Sigma_{\mu \nu} \bar{\Sigma}_{\lambda \rho} \Sigma_{\sigma \tau}, \quad \Sigma_{\mu \nu} \Sigma_{\lambda \rho} \Sigma_{\sigma \tau}.$$  \hspace{1cm} (2.40)

This program is carried out in the next section.
III. PHYSICAL PARAMETRIZATION OF THE VERTEX FUNCTION

A. Physical form factors

It is convenient to define

\[ R_\mu = \sum_{\nu} q^\nu, \]
\[ P_\mu = \sum_{\nu} q^\nu. \]  \hspace{1cm} (3.1)

Then, remembering what we have discussed in the previous section, an economical way to incorporate the conditions that the vertex function must satisfy is to write it in the form

\[ \Gamma_{\alpha\beta\mu} = A_1 \tilde{Q}_\mu g_{\alpha\beta} + i A_2 \tilde{Q}_\mu g_{\alpha\beta} \gamma_5 + i B_1 (R_\mu)_{\alpha\beta} + i B_2 (P_\mu)_{\alpha\beta} + C_1 \tilde{Q}_\mu (R \cdot R)_{\alpha\beta} + C_2 \tilde{Q}_\mu (P \cdot R)_{\alpha\beta} + \left( iC_3 [R_\mu, P_\nu]_{\alpha\beta} + C_4 \{R_\mu, P_\nu\}_{\alpha\beta} + C_5 \{P_\mu, R_\nu\}_{\alpha\beta} \right) Q^\nu + iD_1 \{R_\mu, R \cdot R\}_{\alpha\beta} + D_2 \{R_\mu, R \cdot P\}_{\alpha\beta} + i E_1 \{R_\mu, R \cdot P\}_{\alpha\beta} + i E_2 (R^\lambda P_\mu R_\lambda)_{\alpha\beta}, \]  \hspace{1cm} (3.2)

where \( \tilde{Q}_\mu \) is defined in Eq. (2.10). This form ensures that it satisfies the requirements considered in Section II C. The antisymmetric and symmetric combinations in terms of the commutators and anticommutators have been chosen such that the various terms have simple transformation properties under the various discrete symmetries, as will be discussed below.

The tree-level electromagnetic coupling of the RS field contributes only to \( A_1 \) and \( B_1 \). All the other terms in Eq. (3.2) can arise only through the higher order corrections. Therefore, using the argument stated in Section III B, the coefficients must satisfy

\[ A_2(0) = C_1(0) = C_2(0) = 0, \]  \hspace{1cm} (3.3)

which justifies why we do not include the terms involving \( P \cdot R \) and \( [P_\mu, R_\nu] Q^\nu \) in Eq. (3.2). The cubic terms

\[ (P_\mu R \cdot R)_{\alpha\beta}, (R_\mu R P_\mu)_{\alpha\beta}, (R^\lambda R_\mu R_\lambda)_{\alpha\beta}, (R^\lambda R_\mu P_\lambda)_{\alpha\beta}, (R^\lambda R_\mu L_\lambda)_{\alpha\beta}, \]  \hspace{1cm} (3.9)

are redundant due to the relations

\[ P_\mu R \cdot R = R \cdot R P_\mu = R^\lambda P_\mu R_\lambda + q^2 P_\mu, \]
\[ R^\lambda R_\mu R_\lambda = R \cdot R P_\mu - q^2 P_\mu, \]
\[ P^\lambda R_\mu R_\lambda = R_\mu R \cdot P - q^2 P_\mu. \]  \hspace{1cm} (3.10)
which are easily derived from Eq. (3.7) (and using Eq. (3.5)), and the relation

\[(R^\Lambda R_\mu R_\lambda)_{\alpha\beta} = \frac{1}{2}(R_\mu R \cdot R + R \cdot RR_\mu)_{\alpha\beta} - q^2(R_\mu)_{\alpha\beta}, \quad (3.11)\]

which also follows from Eq. (3.5).

In some other cases, a given term becomes redundant when the RS conditions are taken into account. For example, using the well-known identities for the product of two Levi-Civita tensors, it is straightforward to show that

\[(P \cdot P)_{\alpha\beta} = (R \cdot R)_{\alpha\beta} - \frac{q^2}{2}(\Sigma_{\mu\nu}\Sigma^{\mu\nu})_{\alpha\beta}. \quad (3.12)\]

while, on the other hand, a little algebra shows that

\[(\Sigma_{\mu\nu}\Sigma^{\mu\nu})_{\alpha\beta} = 9g_{\alpha\beta} + 2i\sigma_{\alpha\beta} \quad (3.13)\]

Thus, using the relation

\[i\sigma_{\alpha\beta} = g_{\alpha\beta} - \gamma_{\alpha\gamma}\gamma_{\gamma\beta}, \quad (3.14)\]

the RS auxiliary condition implies that the term involving \((P \cdot P)_{\alpha\beta}\) does not yield a new contribution. Similarly, as shown in Eq. (A1), the terms \((R \cdot PR_\mu)_{\alpha\beta}\) and \((R_\mu R \cdot P)_{\alpha\beta}\) differ by a quantity that vanishes between the spinors and therefore only their anticommutator \((E_1)\) is included in Eq. (3.2). Incidentally, notice also that terms with explicit factors of \(\gamma_5\) in Eq. (3.2) would be redundant. For example, \(R_\mu\gamma_5\) is contained in \(R_\mu R \cdot P\), as can be seen either from Eq. (A1) or Eq. (3.2) below.

Eq. (3.2) can be rewritten in terms of the form factors introduced in Eq. (2.9) by expanding the products of the \(\Sigma\) matrices and following the rules stated in Section II A. For example, let us consider in some detail the \(C_1\) term in Eq. (3.2). After a little bit of algebra, we find

\[\left(\Sigma_{\mu\sigma}\Sigma^\mu_{\tau}\right)_{\alpha\lambda} = \frac{5}{4}g_{\tau\gamma}g_{\alpha\beta} + 3g_{\sigma\alpha}g_{\gamma\beta} + g_{\sigma\alpha}g_{\tau\beta} + \frac{1}{2}\gamma_{\sigma}\gamma_{\tau}g_{\alpha\beta} \quad (3.15)\]

and therefore

\[(R \cdot R)_{\alpha\beta} = \frac{7}{4}q^2g_{\alpha\beta} + 4q_aq_{\beta} + L_{1\alpha\beta}, \quad (3.16)\]

where

\[L_{1\alpha\beta} = -(q_a\gamma_5 + q_\beta\gamma_a). \quad (3.17)\]

In a similar fashion we find for the term involving \(C_2\)

\[(R \cdot P)_{\alpha\beta} = \frac{5}{4}q^2a_{\alpha\gamma_5} + L_{2\alpha\beta} \quad (3.18)\]

where

\[L_{2\alpha\beta} = \frac{i}{2}q^2\gamma_{\alpha}\gamma_5. \quad (3.19)\]

In all the cases, the terms that contain a factor of \(L_{1\alpha\beta}\) or \(L_{2\alpha\beta}\) drop out when they appear between the RS spinors due to the RS condition and the use of the relation given in Eq. (2.33). For example, consider a term of the form

\[\bar{U}^{(\Lambda)}(L_{1}O)_{\alpha\beta}U^\beta, \quad (3.20)\]

where \(O\) is any matrix built from products of \(R_\mu\) and/or \(P_\mu\). The \(\gamma_\alpha\) term of \(L_1\) gives zero by the RS condition satisfied by the \(U^\alpha\) spinor while the term with \(\gamma_\beta\) can be pushed all the way to the right-hand side by the use of Eq. (2.33), which then yields zero by the RS condition on the \(U^\beta\) spinor. A similar argument holds for the terms of the form \(OL_1\) and the analogous terms with \(L_1\) replaced by \(L_2\), and we summarize this by writing

\[\bar{U}^{(\Lambda)}(L_{1,2}O)_{\alpha\beta}U^\beta = \bar{U}^{(\Lambda)}(OL_{1,2})_{\alpha\beta}U^\beta = 0. \quad (3.21)\]
Therefore, any term that contains a factor of \( L_1 \) or \( L_2 \) will not contribute in the expression for the matrix element of the vertex function.

In addition the terms involving \( \sigma_{\lambda \rho} \) must be reduced using the Gordon identities,

\[
\tilde{U}^{\alpha}(R_{\mu})_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left(\tilde{Q}_\mu - M \gamma_\mu\right) U^\beta = \tilde{U}^{\alpha}\left(\tilde{Q}_\mu + M \gamma_\mu\right) U^\beta,
\]

\[
\tilde{U}^{\alpha}(P_{\mu})_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left(\tilde{Q}_\mu + \Delta \gamma_\mu\right) U^\beta = \tilde{U}^{\alpha}\left(\tilde{Q}_\mu - \Delta \gamma_\mu\right) U^\beta,
\]

where \( \gamma_\mu \) and \( \tilde{Q}_\mu \) are defined in Eq. (2.10) and

\[
M = m + m', \quad \Delta = m - m',
\]

with \( m \) and \( m' \) denoting the mass of the initial and final particle, respectively. Thus we can write

\[
\tilde{U}^{\alpha}(R_{\mu})_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left[\left\{ i \left(\tilde{g}_{\mu \alpha} q_\beta - \tilde{g}_{\mu \beta} q_\alpha\right) - \frac{i}{2} g_{\alpha \beta} \left(\tilde{Q}_\mu - M \tilde{\gamma}_\mu\right)\right\} U^\beta,
\]

\[
\tilde{U}^{\alpha}(P_{\mu})_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left[\left\{ i \epsilon_{\alpha \beta \mu \nu} q^\nu - \frac{1}{2} g_{\alpha \beta} \left(\tilde{Q}_\mu + \Delta \tilde{\gamma}_\mu\right)\right\} U^\beta,
\]

where we have used Eq. (A3) and also

\[
g_{\mu \alpha} q_\beta - g_{\mu \beta} q_\alpha = \tilde{g}_{\mu \alpha} q_\beta - \tilde{g}_{\mu \beta} q_\alpha.
\]

Using Eq. (3.16) and (3.18), and remembering Eq. (3.21), we then obtain

\[
\tilde{U}^{\alpha}(R \cdot R)_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left[\frac{7}{4} q^2 g_{\alpha \beta} + 4 q_\alpha q_\beta\right] U^\beta,
\]

\[
\tilde{U}^{\alpha}(R \cdot P)_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left[-\frac{5}{4} q^2 g_{\alpha \beta} \gamma_5\right] U^\beta,
\]

(3.26)

The cubic terms in \( R \) and \( P \) as well as the terms involving the commutator or anticommutator of \( R \) and \( P \) can be reduced in similar fashion. The details are given in the appendix and the final results are summarized below. First for the cubic terms

\[
\tilde{U}^{\alpha}(R_{\mu}, R_{\nu})_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left[\frac{7}{4} q^2 (R_{\mu})_{\alpha \beta} - 4 i q_\alpha q_\beta (\tilde{Q}_\mu - M \tilde{\gamma}_\mu) + 4 i q^2 (\tilde{g}_{\mu \alpha} q_\beta - \tilde{g}_{\mu \beta} q_\alpha)\right] U^\beta,
\]

\[
\tilde{U}^{\alpha}(R_{\mu}, P_{\nu})_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left[\frac{5}{4} q^2 g_{\alpha \beta} (\tilde{Q}_\mu + \Delta \tilde{\gamma}_\mu)\right] U^\beta,
\]

\[
\tilde{U}^{\alpha}(P_{\mu}, R_{\nu})_{\alpha \beta} U^\beta = \tilde{U}^{\alpha}\left[\frac{3}{4} q^2 (P_{\mu})_{\alpha \beta} - 2 q_\alpha q_\beta (\tilde{Q}_\mu + \Delta \tilde{\gamma}_\mu)\right] U^\beta,
\]

(3.27)

and finally for the terms involving the commutator or anticommutator of \( R \) and \( P \),

\[
\tilde{U}^{\alpha}(R_{\mu}, P_{\nu})_{\alpha \beta} Q^\nu U^\beta = \tilde{U}^{\alpha}\left[\alpha_{\beta \mu \nu} \lambda q^\lambda - q_\beta \epsilon_{\alpha \mu \lambda} q^\nu \tilde{Q}_\lambda + \frac{i}{2} g_{\alpha \beta} \left(M (q^2 - \Delta^2) \tilde{\gamma}_\mu - \Delta M \tilde{Q}_\mu\right)\right] U^\beta,
\]

\[
\tilde{U}^{\alpha}(P_{\mu}, R_{\nu})_{\alpha \beta} Q^\nu U^\beta = \tilde{U}^{\alpha}\left[\frac{1}{2} q^2 g_{\alpha \beta} \tilde{Q}_\mu \tilde{\gamma}_\mu - q_\beta \epsilon_{\alpha \beta \mu \nu} q^\nu \tilde{Q}_\lambda + \frac{i}{2} g_{\alpha \beta} \left(\tilde{Q}_\mu - \Delta \tilde{\gamma}_\mu\right)\right] U^\beta,
\]

\[
\tilde{U}^{\alpha}(R_{\mu}, P_{\nu})_{\alpha \beta} Q^\nu U^\beta = \tilde{U}^{\alpha}\left[\frac{1}{2} q^2 g_{\alpha \beta} \tilde{Q}_\mu \tilde{\gamma}_\mu + 2 q_\alpha \epsilon_{\beta \mu \lambda} q^\nu \tilde{Q}_\lambda + 2 q_\beta \epsilon_{\alpha \mu \lambda} q^\nu \tilde{Q}_\lambda \right. \\
\left. + \left((\Delta^2 - q^2) \epsilon_{\alpha \beta \mu \nu} q^\nu - 2 i q_\alpha q_\beta (\tilde{Q}_\mu + \Delta \tilde{\gamma}_\mu)\right)\right] U^\beta,
\]

(3.28)
These formulas allow us to determine by inspection the relations between the set of form factors introduced in Eq. (2.9) and those introduced in Eq. (3.2). Thus,

\[ a_1 = \frac{1}{2} MB_1 - \frac{7}{4} q^2 M D_1, \]
\[ a'_1 = \frac{1}{2} i \Delta B_2 - \frac{1}{2} (q^2 - \Delta^2) MC_3 - \frac{5}{4} i q^2 \Delta E_1 - \frac{3}{8} i q^2 \Delta E_2, \]
\[ a_2 = A_1 + \frac{1}{2} B_1 + \frac{7}{4} q^2 C_1 + \frac{7}{4} q^2 D_1, \]
\[ a'_2 = i A_2 - \frac{1}{2} B_2 - \frac{5}{4} i q^2 C_2 + \frac{1}{2} M \Delta C_4 - \frac{i}{2} q^2 C_4 - \frac{i}{2} q^2 C_5 - \frac{5}{4} i q^2 E_1 - \frac{3}{8} i q^2 E_2, \]
\[ a_3 = -4 M D_1, \]
\[ a'_3 = -2 i \Delta C_5 - 2 i \Delta E_2, \]
\[ a_4 = 4 C_1 + 4 D_1, \]
\[ a'_4 = -2 i C_5 - 2 i E_2, \]
\[ a_5 = 4 i q^2 D_2, \]
\[ a'_5 = 0, \]
\[ a_6 = -B_1 - \frac{15}{2} q^2 D_1, \]
\[ a'_6 = i (q^2 - M^2) C_4 + \frac{5}{2} i q^2 E_1, \]
\[ b_1 = -B_2 - (q^2 - \Delta^2) (C_4 + C_5) - \frac{3}{4} q^2 E_2, \]
\[ b_2 = -i C_3 + 2 C_4 - C_5, \]
\[ b_3 = i C_3 + 2 C_4 - C_5. \] (3.29)

Apart from the immediate result that \( a'_5 = 0 \), other less obvious relations that follow from the above formulas are

\[ a'_3 = \Delta a'_4, \]
\[ \frac{1}{2} M a_6 = a_1 + \frac{1}{2} q^2 a_3. \] (3.30) (3.31)

By considering Eq. (3.29) in the diagonal case, in which \( \Delta = 0 \), it is also straightforward to verify that the conditions in Eq. (2.14) and (2.24) are indeed satisfied identically by these formulas. For example, with \( M = 2m \), Eq. (3.31) precisely has the form stated in Eq. (2.24). In conclusion, the parametrization given in Eq. (3.2) satisfies the conditions that the vertex function must satisfy, as elaborated in Sections II B and II C.

B. Multipole moment interactions

Equation (3.2) also has the virtue that it allows us to identify the contributions from the various multipole moment terms systematically. Since the correspondence between such terms and the traditional definitions of the multipole moments using the static and the nonrelativistic limit is not very useful in the context of the applications that we envisage (where neither limit may be applicable), we proceed in a different direction. Making reference to the discussion in Section II D, we write the amplitude \( M \) in the presence of an external potential in the form

\[ M = -U^\alpha (\mathcal{M})_{\alpha\beta} U^\beta, \] (3.32)

where

\[ (\mathcal{M})_{\alpha\beta} = \Gamma_{\mu\alpha\beta} A^\mu (q). \] (3.33)

Notice that, since we have constructed \( \Gamma_{\mu\alpha\beta} \) such that it is transverse, we can take \( A^\mu (q) \) to be transverse since the longitudinal part does not contribute. Introducing the field strength

\[ F_{\mu\nu} (q) = i (q_\mu A_\nu (q) - q_\nu A_\mu (q)), \] (3.34)
we have the following relations

\[ R \cdot A(q) = \frac{i}{2} \Sigma_{\mu \nu} F_{\mu \nu} \]

\[ P \cdot A(q) = \frac{i}{2} \Sigma_{\mu \nu} \tilde{F}_{\mu \nu} = \frac{i}{2} \tilde{\Sigma}_{\mu \nu} F_{\mu \nu} \]

\[ q^2 \tilde{g}_{\mu \nu} A'(q) = iF_{\mu \nu} q^\nu. \]  

(3.35)

It is then evident that \( \mathcal{M} \) has the following expansion in powers of \( q \),

\[ \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_{\mu \nu} F_{\mu \nu}(q) + i\mathcal{M}_{\lambda \mu \nu} q^\lambda F_{\mu \nu}(q) + \mathcal{M}_{\lambda \rho \sigma} q^\lambda q^\rho q^\sigma F_{\mu \nu}(q), \]  

(3.36)

where, with the exception of \( \mathcal{M}_0 \), as noted below, the various coefficients depend on \( q \) only through the \( q^2 \) dependence of the form factors \( A, B, C, D, E \). By a straightforward calculation we find,

\[ \mathcal{M}_0 = \begin{cases} A_1 Q_\mu A^\mu(q) & \text{(charged diagonal case)} \\ \frac{iA_1}{q^2} Q_\mu g_{\lambda \nu} q^\lambda F_{\mu \nu}(q) & \text{(otherwise)} \end{cases} \]

\[ \mathcal{M}_{\mu \nu} = -\frac{1}{2} (B_1 \Sigma_{\mu \nu} + B_2 \tilde{\Sigma}_{\mu \nu}) \]

\[ \mathcal{M}_{\lambda \mu \nu} = \frac{iA_2}{q^2} Q_\mu g_{\lambda \nu} \gamma_5 - \frac{1}{2} \left( iC_3 [\Sigma_{\mu \nu}, \tilde{\Sigma}_{\lambda \rho}] + C_4 \{\Sigma_{\mu \nu}, \tilde{\Sigma}_{\lambda \rho}\} + C_5 \{\Sigma_{\mu \nu}, \Sigma_{\lambda \rho}\} \right) Q^\rho, \]

\[ \mathcal{M}_{\lambda \rho \sigma} = -\frac{D_1}{4} \{\Sigma_{\mu \nu}, \{\Sigma_{\sigma \lambda}, \tilde{\Sigma}_{\rho \mu}\}\} + \frac{iD_2}{4} [\Sigma_{\mu \nu}, \{\Sigma_{\sigma \lambda}, \tilde{\Sigma}_{\rho \mu}\}] \]

\[ - \frac{E_1}{4} \{\Sigma_{\mu \nu}, \Sigma_{\sigma \lambda} \tilde{\Sigma}_{\rho \mu} \} - \frac{E_2}{4} (\Sigma_{\sigma \lambda} \tilde{\Sigma}_{\mu \nu} \Sigma_{\rho \mu} + (\lambda \leftrightarrow \rho)) \]

\[ \mathcal{M}_{\lambda \rho \sigma \tau} = \frac{C_1}{2q^2} \{\Sigma_{\rho \sigma}, \tilde{\Sigma}_{\tau \mu}\} Q_\mu g_{\lambda \nu} + \frac{C_2}{2q^2} \{\Sigma_{\rho \sigma}, \Sigma_{\tau \mu}\} Q_\mu g_{\lambda \nu}, \]  

(3.37)

where, in writing the \( C_2 \) and \( E_1 \) terms we have taken into account Eq. (3.8).

In the charged diagonal case, the term \( \mathcal{M}_0 \) is just the electric charge interaction with the external potential. But in the off-diagonal case, or in the diagonal case if the particle is electrically neutral, so that \( A_1(0) = 0 \), the term in \( \mathcal{M}_0 \) actually represents an additional contribution to \( \mathcal{M}_{\lambda \mu \nu} \) of the form

\[ \mathcal{M}'_{\lambda \mu \nu} = \frac{iA_1}{q^2} Q_\mu g_{\lambda \nu}. \]  

(3.38)

Equations (3.36) and (3.37) together represent a clear separation of the various multipole moment interactions in terms of the form factors defined in Eq. (3.32).

IV. IMPLICATIONS OF THE DISCRETE SYMMETRIES

The form factors that appear in the vertex function may satisfy additional requirements depending on whether the interaction Lagrangian is invariant under the various discrete space-time symmetries, and/or further conditions depending, for example, on whether the particle is self-conjugate (Majorana condition) or not. We consider the implications of all these conditions here.

Let us consider the \( f(k) \to f'(k') + \gamma \) transition amplitude where \( f, f' \) are RS fermions. Using the definition of the vertex function in Eq. (1.3) the amplitude is given by

\[ M(f(k) \to f'(k') + \gamma) = \bar{U}^\alpha(k') \Gamma_\alpha \beta \mu(k, k') U^\beta(k) \varepsilon^\mu, \]  

(4.1)

where, in this section, we will consider the fermion momenta as the independent variables in the vertex function.

First of all, independently of whether or not any of the discrete transformations may be symmetries of the Lagrangian, the hermiticity of the Lagrangian and the substitution rule (crossing) yield the following two relations,

\[ M(f'(k') \to f(k) + \gamma) = \bar{U}^\beta(k) \Gamma_\alpha \beta \mu(k, k') U^\alpha(k') \varepsilon^\mu, \]  

(4.2)

\[ M(f'(k') \to \bar{f}(k) + \gamma) = \bar{U}^\alpha(k) \Gamma_\alpha \beta \mu(-k', -k') U^\beta(k') \varepsilon^\mu, \]  

(4.3)
\[ \delta C \quad \delta P \quad \delta T \quad \delta \text{CP} \quad \delta \text{CP}^T \quad \delta \mu \]

\begin{tabular}{cccccccc}
\hline
i & + & + & - & - & - & - & - \\
\gamma_5 & + & - & - & - & - & - & - \\
\gamma_\alpha & - & + & - & + & + & + & - \\
\gamma_\alpha \gamma_5 & + & - & - & - & + & + & + \\
\sigma_{\mu\nu} & - & + & - & - & + & + & - \\
\sigma_{\mu\nu} \gamma_5 & + & - & + & - & - & - & - \\
S_{\mu\nu} & - & + & - & + & - & - & + \\
\tilde{S}_{\mu\nu} & - & - & + & + & + & + & + \\
\Sigma_{\mu\nu} & - & - & + & + & + & + & + \\
\tilde{\Sigma}_{\mu\nu} & - & - & - & + & - & - & + \\
\hline
\end{tabular}

**TABLE I:** Transformation rules under the discrete transformations of the various quantities that appear in the vertex function.

\[ \{ \Sigma_{\mu\nu}, \Lambda_\mu \} \quad \delta C \quad \delta P \quad \delta T \quad \delta \text{CP} \quad \delta \text{CP}^T \quad \delta \mu \]

\begin{tabular}{cccccccc}
\hline
\{ \Sigma_{\mu\nu}, \Lambda_\mu \} & + & + & + & + & + & + & + \\
\{ \Sigma_{\mu\nu}, \tilde{\Lambda}_\mu \} & - & - & - & + & + & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\{ \Sigma_{\mu\nu}, \tilde{\Xi}_\mu \} & + & - & + & - & - & - & - \\
\hline
\end{tabular}

**TABLE II:** Transformation rules under the discrete transformations of the various products of \( \Sigma_{\mu\nu} \) and \( \tilde{\Sigma}_{\mu\nu} \) that appear in the vertex function.

respectively, where

\[ \bar{\Gamma}_{\alpha\beta\mu}(k, k') = \gamma^0 \Gamma^{\dagger}_{\alpha\beta\mu}(k, k') \gamma^0, \]

\[ \Gamma^C_{\alpha\beta\mu}(k, k') = C^{-1} \Gamma_{\alpha\beta\mu}(k, k') C, \]

with \( C \) being the matrix with the property \( C^{-1} \gamma^\dagger \gamma C = -\gamma \) (e.g., \( C = i\gamma_2 \gamma_0 \) in the Dirac representation of the \( \gamma \) matrices). The vertex function \( \Gamma^C_{\alpha\beta\mu} \) in Eq. (4.3) is obtained from \( \Gamma_{\alpha\beta\mu} \) by multiplying every quantity that appears in \( \Gamma_{\alpha\beta\mu} \) by its \( C \) parity phase \( \delta_C \) as given in Tables I and II.

We now envisage calculating the amplitude using a transformed interaction Lagrangian that is obtained from the original one by replacing each field by its transformed counterpart. For example, for the parity transformation, the photon and the RS fields would be replaced by

\[ A^\mu \rightarrow \Lambda^\mu_{P\mu} A^\nu (\Lambda^{-1})^\mu_{P\nu}, \]

\[ \psi^\mu \rightarrow \eta_P A^\mu_{P\gamma} \gamma^0 \psi^\nu (\Lambda^{-1})^\mu_{P\nu}, \]

where \( \eta_P \) is a phase factor. Other relevant fields are similarly replaced by their corresponding parity-transformed counterpart. Then, the one-photon transition amplitude obtained in this way, which we denote by \( M^P(f(k) \rightarrow f'(k') + \gamma) \), is given by

\[ M^P(f(k) \rightarrow f'(k') + \gamma) = \eta_P \eta_P^* U^\alpha_{P\gamma}(k') \Gamma^P_{\alpha\beta\mu}(k, k') U^\beta_{P\gamma}(k) \varepsilon^\mu, \]

where

\[ \Gamma^P_{\alpha\beta\mu}(k, k') = \gamma_0 \Gamma_{\alpha\beta\mu}(k, k') \gamma_0, \]

and it is obtained from \( \Gamma_{\alpha\beta\mu}(k, k') \) by multiplying every quantity that appears in \( \Gamma_{\alpha\beta\mu} \) by its parity phase \( \delta_P \) as given in Tables I and II.
TABLE III: Transformation rules under the discrete transformations of the various matrices that appear in the parametrization of vertex function given in Eq. (3.2). In addition, it must be remembered that for $T$ and $H$ each form factor is replaced by its complex conjugate.

|          | $\delta_C$ | $\delta_P$ | $\delta_T$ | $\delta_{CP}$ | $\delta_{CPT}$ | $\delta_H$ |
|----------|------------|------------|------------|---------------|----------------|------------|
| $Q_\mu$  | -          | +          | -          | -             | +              | +          |
| $i\tilde{Q}_\mu\gamma_5$ | -          | -          | +          | +             | +              |            |
| $iR_\mu$ | -          | +          | -          | -             | +              | +          |
| $iP_\mu$ | -          | +          | -          | +             | +              | +          |
| $i[R_\mu, P_\nu]Q^\nu$ | -          | +          | -          | +             | +              | +          |
| $(R \cdot P)\tilde{Q}_\mu \{R_\mu, P_\nu\}Q^\nu \{P_\mu, R_\nu\}$ | -          | +          | -          | +             | +              | +          |

If the Lagrangian is invariant under the given transformation, then the two amplitudes should be equal and therefore

$$\Gamma_{\alpha\beta\mu}(k, k') = \Gamma_{\alpha\beta\mu}(k, k'),$$  \hspace{1cm} (4.8)

which results in conditions on the form factors. Similarly, for time-reversal and charge conjugation,

$$M^T(f(k) \rightarrow f'(k') + \gamma) = -\eta T\eta T^\dagger U^{\alpha\beta}(k')\Gamma^{T}_{\alpha\beta\mu}(-k, -k')U^\beta(k)e^{i\epsilon},$$ \hspace{1cm} (4.9)

$$M^C(f'(k') \rightarrow f(k) + \gamma) = -\eta C\eta C^\dagger U^{\alpha\beta}(k)\Gamma^{C}_{\alpha\beta\mu}(-k, -k')U^\beta(k)e^{i\epsilon},$$ \hspace{1cm} (4.10)

respectively, where

$$\Gamma^{T}_{\alpha\beta\mu}(k, k') = (C\gamma_5)\Gamma^{*}_{\alpha\beta\mu}(-k, -k')(C\gamma_5),$$ \hspace{1cm} (4.11)

and $\Gamma^{C}_{\alpha\beta\mu}$ has been defined in Eq. (4.4). $\Gamma^{T}_{\alpha\beta\mu}(k, k')$ is obtained from $\Gamma_{\alpha\beta\mu}(k, k')$ by multiplying every quantity that appears in $\Gamma_{\alpha\beta\mu}(k, k')$ by its corresponding $\delta_T$ phase.

We now consider separately the application of these relations to various cases.

### A. Diagonal case $f = f'$

In this case, we can summarize the effect of performing the discrete transformations $C$, $P$ and $T$ by saying that the amplitude for the process $f(k) \rightarrow f(k') + \gamma$, calculated with the transformed Lagrangian, is obtained by making the substitutions

$$\Gamma_{\alpha\beta\mu}(k, k') \xrightarrow{C} -\Gamma^{C}_{\alpha\beta\mu}(-k', -k),$$

$$\Gamma_{\alpha\beta\mu}(k, k') \xrightarrow{P} \Gamma^{P}_{\alpha\beta\mu}(k, k'),$$

$$\Gamma_{\alpha\beta\mu}(k, k') \xrightarrow{T} -\Gamma^{T}_{\alpha\beta\mu}(-k, -k').$$ \hspace{1cm} (4.12)

From these, the transformation rules for the combined transformations follow, for example,

$$\Gamma_{\alpha\beta\mu}(k, k') \xrightarrow{CP} -\Gamma^{CP}_{\alpha\beta\mu}(-k', -k).$$ \hspace{1cm} (4.13)

If the Lagrangian is invariant under a given transformation, then the arrow symbol in the corresponding relation is replaced by the equals sign, which results in conditions on the form factors.

In order to ease the application of such conditions to the parametrization given in Eq. (3.2), the transformation rules of the various terms that appear in that equation are displayed in Table III. For example, starting from Eq. (3.2), the expression for $\Gamma^{CP}_{\alpha\beta\mu}(-k', -k)$ is obtained by multiplying every term in that equation by its corresponding phase $\delta_{CP}$ as given in Table III.

As an example, let us consider $CP$. If the relevant interaction Lagrangian is $CP$ symmetric, then only the terms with $\delta_{CP}$ odd in Table III can appear in the vertex function. Therefore, only $A_1$, $B_1$, $C_3$ and $D_1$ may be nonzero in that case.
Independently of any such conditions, Eq. (4.2) implies that the vertex function in this case satisfies
\[
\Gamma^H_{\alpha\beta\mu}(k,k') = \Gamma_{\alpha\beta\mu}(k,k') ,
\] (4.14)
where \(\Gamma^H_{\alpha\beta\mu}\) is defined by
\[
\Gamma^H_{\alpha\beta\mu}(k,k') \equiv \bar{\Gamma}_{\beta\alpha\mu}(k',k) ,
\] (4.15)
with \(\bar{\Gamma}_{\alpha\beta\mu}(k,k')\) defined in Eq. (4.4). \(\Gamma^H_{\alpha\beta\mu}(k,k')\) is obtained from \(\Gamma_{\alpha\beta\mu}(k,k')\) in Eq. (3.2) by multiplying every quantity that appears there by the phase \(\delta_H\) given in Table III (and replacing each form factor by its complex conjugate). Thus, Eq. (4.14) implies that all the form factors \(A,B,C,D,E\) in Eq. (3.2) are real.

1. Self-conjugate (Majorana) particles

In this case, Eq. (4.3) implies the additional relation
\[
\Gamma_{\alpha\beta\mu}(k,k') = \Gamma_C^\alpha(-k',-k) .
\] (4.16)
irrespectively of the discrete symmetries that may exist. As can be seen from Table III the vertex function consists only of the \(C_3\) and \(D_2\) terms. In terms of the parametrization defined in Eq. (2.9), Eq. (3.29) then implies that the vertex function in this case is of the form
\[
\Gamma_{\alpha\beta\mu} = -q^2mC_3g_{\alpha\beta\gamma_5} + iq\bar{C}_3\left(q_\alpha\epsilon_{\beta\mu\lambda}\gamma^\lambda - q_\beta\epsilon_{\alpha\mu\lambda}\gamma^\lambda\right) + 4iq^2D_2(g_{\mu\alpha}q_\beta + g_{\mu\beta}q_\alpha) ,
\] (4.17)
where we have used the fact that \(m' = m\) in the present case.

The term proportional to \(\gamma_\mu\gamma_5\), which is related to the axial charge radius, is reminiscent of the analogous result that holds for Majorana neutrinos \[1\]. The other two terms in Eq. (4.17) resemble the result that was obtained in Ref. \[6\] for the electromagnetic vertex function of self-conjugate spin-1 particles [Eq. (4.14) in that reference]. The noteworthy feature of the result obtained here for the spin 3/2 case is the fact that the coefficients of the first two terms in Eq. (4.17) above are given in terms of the same form factor. Ultimately this is related to the requirement that the vertex function acting on the initial spinor, or on the final spinor, does not yield a spinor with a spin 1/2 component as we have discussed in the previous section, which the parametrization in terms of \(C_3\) and \(D_2\) automatically satisfies.

While the above result is general, further conditions exist if some discrete symmetries hold. For example, if \(CP\) holds, then \(D_2 = 0\) as already stated in Section IV A, so that only the \(C_3\) terms in Eq. (4.17) can be present.

B. Off-diagonal case \(f \neq f'\)

In the off-diagonal case, both the hermiticity condition and crossing only relate the amplitude of one process to the amplitude of a different process and therefore they do not lead by themselves to any restriction on the form factors. The restrictions arise only if some discrete symmetries hold. For example, if \(CP\) is valid, then we obtain the relation
\[
-\eta_{CP}\eta_{CP}'^{*}\Gamma_{\alpha\beta\mu}^{CP}(-k,-k') = \Gamma^H_{\alpha\beta\mu}(k,k') ,
\] (4.18)
where \(\eta_{CP}\) and \(\eta_{CP}'\) are the phases in the \(CP\) transformation rule of the fermion fields. This implies that the group of form factors that appear in Eq. (3.2) are divided in two groups. The first group consists of
\[
A_1, B_1, C_3, D_1 ,
\] (4.19)
which are the coefficients of the terms with \(\delta_{CP}\) odd according to Table III and therefore must be relatively real. The second group consists of the remaining coefficients, which are associated with the terms that have \(\delta_{CP}\) even in Table III and therefore must be relatively imaginary with respect to the coefficients included in the first group.
1. Self-conjugate (Majorana) particles

In this case, Eq. (4.2) and (4.3) together imply that

\[ \Gamma_{\alpha\beta\mu}^{C}(-k, -k') = \Gamma_{\alpha\beta\mu}^{H}(k, k'), \]  

(4.20)

independently of the discrete symmetries. Thus, the form factors are again divided in two groups,

\[ C_3, D_2 = \text{real} \]  

(4.21)

while all the others are imaginary.

The discrete symmetries would yield additional conditions. Considering CP as an example once more, if it holds, then the same argument that lead to Eq. (4.18), in this case leads to

\[ -\eta \Gamma_{\alpha\beta\mu}^{CP}(-k, -k') = \Gamma_{\alpha\beta\mu}^{H}(k, k'), \]  

(4.22)

where \( \eta \) is +1 or −1 according to whether the two Majorana fermions have the same or opposite CP parity, respectively.

Thus, if the two fermions have the same CP parity (\( \eta = +1 \)), then

\[ A_1, B_1, C_3, D_1 = \text{real}, \]  

(4.23)

while the other coefficients are imaginary. Thus, Eq. (4.21) and (4.23) together imply that in this case only the \( C_3 \) is nonzero. Conversely, if the fermions have the opposite CP parity, then only \( D_2 \) is nonzero. Once more, this is reminiscent of the analogous result for Majorana neutrinos[1] in which the transition electromagnetic moment is of the form \( \sigma_{\mu\nu}\gamma_5 \) of \( \sigma_{\mu\nu} \) when the neutrinos have the same or opposite CP phase, respectively.

V. DISCUSSION

Although we have focused our attention on the Majorana case, our results have application to the charged cases as well. Let us consider the diagonal vertex function, assuming exact \( P \) and CP symmetry as would be applicable to a baryon for example. Since, as we have already seen, CP in the diagonal case implies that only \( A_1, B_1, D_1 \) and \( C_3 \) are non-zero, the additional \( P \) symmetry implies that only the first of those three are non-zero. The vertex function then reduces to the form

\[ \Gamma_{\alpha\beta\mu} = g_{\alpha\beta}\gamma_{\mu}a_1 + g_{\alpha\beta}Q_{\mu}a_2 + q_{\alpha}q_{\beta}\gamma_{\mu}a_3 + q_{\alpha}Q_{\mu}a_4 + (g_{\mu\alpha}q_{\beta} - g_{\mu\beta}q_{\alpha})a_6, \]  

(5.1)

where, remembering that \( M = 2m \) in this case,

\[
\begin{align*}
a_1 &= -mB_1 - \frac{7}{2}q^2mD_1, \\
a_2 &= A_1 + \frac{1}{2}B_1 + \frac{7}{4}q^2D_1, \\
a_3 &= -8mD_1, \\
a_4 &= 4D_1, \\
a_6 &= -B_1 - \frac{15}{2}q^2D_1.
\end{align*}
\]  

(5.2)

Of course we can choose to write the vertex function in terms of any three independent parameters. Choosing \( a_{1,2,3} \), so that

\[
\begin{align*}
a_4 &= \frac{1}{2m}a_3, \\
a_6 &= \frac{1}{m}\left(a_1 + \frac{1}{2}q^2a_3\right),
\end{align*}
\]  

(5.3)

then yields

\[ \Gamma_{\alpha\beta\mu} = \left[g_{\alpha\beta}\gamma_{\mu} - \frac{i}{m}(S_{\mu\nu}q^{\nu})_{\alpha\beta}\right]a_1 + g_{\alpha\beta}Q_{\mu}a_2 + q_{\alpha}q_{\beta}\left[\gamma_{\mu} - \frac{1}{2m}Q_{\mu}\right]a_3. \]  

(5.4)
where the $S_{a\nu}$ are the spin-1 generators of the Lorentz group defined in Eq. \eqref{2.16}.

The function with the lowest powers of $q$, such as what might arise from a tree-level vertex, corresponds to setting $a_3 = 0$. In any case, the terms with $Q_\mu$ could be rewritten using the Gordon identities given in Eq. \eqref{3.22}, if desired. But regardless of that, this equation reveals that the electromagnetic vertex in the RS representation cannot consist solely of the $g_{a\beta}q_\mu$ term. The underlying reason is that while each of the couplings $g_{a\beta}Q_\mu$ and $(\Sigma_{\mu\nu}q^\nu)_{a\beta}$ independently satisfy a relation similar to Eq. \eqref{2.38}, and therefore the vertex can contain one or the other or both, the couplings $g_{a\beta}Q_\mu$ and $\sigma_{\mu\nu}q_{a\beta}$ do not satisfy any such relations. Thus in contrast to the spin-1/2 case, in the spin-3/2 case the $\gamma_\mu$ coupling term must be accompanied by the appropriate spin coupling term, as shown in Eq. \eqref{5.4}, exposing the fact that such is the combination that is actually hidden inside the $Q_\mu g_{a\beta}$ and/or $\Sigma_{\mu\nu}q^\nu$ couplings. This indicates that the minimal substitution prescription in the RS representation, which leads to a pure $\gamma_\mu$ coupling at the tree-level, can lead to inconsistencies (e.g., nonunitarity problems\cite{18, 19}) related to the nondecoupling of the spurious spin-1/2 components of the RS representation.

VI. CONCLUSIONS

We have studied the structure of the electromagnetic vertex function of spin-3/2 particles using the Rarita-Schwinger description of such particles, including both the diagonal and off-diagonal (transition) cases. We considered the cases in which the particles are charged or electrically neutral as well as the particular case that they are self-conjugate (Majorana type).

In Section \[\text{II}\] we considered the general form of the on-shell vertex function that is consistent with electromagnetic gauge invariance in terms of simple combinations of the $\gamma$ matrices, the momentum vectors, the metric and the Levi-Civita tensors. While that representation is simple and useful for practical calculations of transition rates, it is not practical for taking into account the physical requirement that the vertex function does not mix the genuine spin-3/2 degrees of freedom with the spurious spin-1/2 components of the RS representation. Thus, in Section \[\text{III}\] we considered another expression for the on-shell vertex function, in terms of the matrices $R_\mu$, $P_\mu$, and their products. We showed that, without having to impose further conditions, this construction of the vertex function satisfies the physical requirements related to gauge invariance and the use of the RS spinors, avoiding the spin-1/2 - spin-3/2 mixing problem to which we have alluded above. The formulas that give the relations between the form factors of the two representations were given explicitly there as well. Finally in Section \[\text{IV}\] we studied the implications due to the discrete transformations, such as the $C, P, T$ transformations and their products, and the conditions implied by the hermiticity of the interaction Lagrangian and crossing symmetry, for both the diagonal and off-diagonal cases. We considered in some detail the diagonal Majorana case, and showed that the vertex function can contain a term of the form $\gamma_\mu \gamma_5$, which resembles the axial charge radius term for Majorana neutrinos\cite{1}, plus another one that resembles the vertex function for self-conjugate spin-1 particles\cite{6}, but with the particularity that the two terms appear with a specific relative coefficient and not independently. In essence this result is due to the requirement that the vertex function does not mix the genuine spin-3/2 degrees of freedom with the spurious spin-1/2 components of the RS representation, which the expression for the vertex function in terms of the matrices $R_\mu$, $P_\mu$ automatically satisfies. The analogous results for the off-diagonal vertex, as well as the charged particles cases, were discussed as well.

The analysis that we have presented can be a useful in several ways. On one hand, it can serve as a guide to parametrize the electromagnetic couplings of spin-3/2 particles in a way that is general, model-independent and consistent. On the other hand, the results of our analysis can be used to test deviations from fundamental physical principles, such as gauge invariance and crossing symmetry, in the context of the processes described by the electromagnetic couplings of such particles, in particular Majorana ones. In addition it can serve as a guide to study the spin-3/2 to spin-1/2 transition vertex, for example the gravitino-neutrino radiative transition which can be of interest in cosmological contexts.

Appendix A: Reduction formulas for products of $R$ and $P$

Eq. \eqref{3.16} and \eqref{3.18} can in turn be used to find the expressions for the terms in Eq. \eqref{3.2} that involve three factors of $R$, $P$ and $Q$,

\begin{align*}
(R_\mu R \cdot R)_a & = \frac{7}{4} q^2 (R_\mu)_{a\beta} + 2 q_\alpha q_\beta \sigma_{\mu \nu} q^\nu + 4 i (q^2 g_{\mu \alpha} - q_{\mu} q_\alpha) q_\beta + (R_\mu L_1)_{a\beta}, \\
(R \cdot RR_\mu)_{a\beta} & = \frac{7}{4} q^2 (R_\mu)_{a\beta} + 2 q_\alpha q_\beta \sigma_{\mu \nu} q^\nu - 4 i q_\alpha (q^2 g_{\mu \beta} - q_{\mu} q_\beta) + (L_1 R_\mu)_{a\beta}, \\
(R_\mu R \cdot P)_{a\beta} & = -\frac{5}{4} q^2 (R_\mu)_{a\beta} \gamma_5 + (R_\mu L_2)_{a\beta},
\end{align*}
\[(R \cdot PR_\mu)_{\alpha\beta} = -\frac{5}{4}q^2 (R_\mu)_{\alpha\beta} \gamma_5 + (L_2 R_\mu)_{\alpha\beta},\]
\[(R^\lambda P_\mu R_\lambda)_{\alpha\beta} = \frac{3}{4}q^2 (P_\mu)_{\alpha\beta} - 2iq_\alpha q_\beta \sigma_{\mu\nu} \gamma_5 q^\nu + (P_\mu L_1)_{\alpha\beta}\] (A1)

The terms that multiply \(C_{3,4,5,6}\) in Eq. (3.2) can be expressed explicitly in similar fashion,

\[\{R_\mu, P_\nu\}_{\alpha\beta} = q_\beta \epsilon_{\mu\nu\alpha\lambda} q^\lambda - q_\alpha \epsilon_{\mu\nu\beta\lambda} q^\lambda - \frac{1}{2}\left(2\tilde{g}_\mu^{\gamma\lambda} \sigma_{\lambda\nu} - q_\nu \sigma_{\mu\lambda} q^\lambda\right)q_{\alpha\beta} \gamma_5,\]
\[\{R_\mu, P_\nu\}_{\alpha\beta} = -\frac{i}{2}q^2 \tilde{g}_{\mu\nu} g_{\alpha\beta} \gamma_5 + i\epsilon_{\nu\alpha\beta\rho} q^\rho \sigma_{\mu\lambda} q^\lambda + (q_\beta g_{\mu\alpha} - q_\alpha g_{\mu\beta}) \sigma_{\rho\lambda} q^\lambda \gamma_5 - q_\beta \epsilon_{\mu\nu\alpha\lambda} q^\lambda - q_\alpha \epsilon_{\mu\nu\beta\lambda} q^\lambda,\] (A2)

where use has been made of the commutator of the \(\sigma_{\mu\nu}\) as well as their anticommutator

\[\{\sigma_{\mu\nu}, \sigma_{\lambda\rho}\} = 2(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) + 2i\epsilon_{\mu\nu\lambda\rho} \gamma_5,\] (A3)

which in particular yields

\[\{\sigma_{\mu\nu} q^\nu, \sigma_{\lambda\rho} q^\rho\} = 2q^2 \tilde{g}_{\mu\lambda}.\] (A4)

In obtaining some of these formulas, we have found useful the following expression for \(P_\mu\),

\[\langle P_\mu \rangle_{\alpha\beta} = i \left(\epsilon_{\mu\nu\alpha\beta} - \frac{1}{2}g_{\alpha\beta} \sigma_{\mu\nu} \gamma_5\right) q^\nu.\] (A5)

For example, this easily gives the relation

\[q^\alpha (P_\mu)_{\alpha\beta} = -\frac{1}{2}i\sigma_{\mu\nu} \gamma_5 q^\nu q_\beta.\] (A6)

The latter formula, together with the relation

\[\tilde{g} \sigma_{\mu\nu} \gamma_5 = \sigma_{\mu\nu} \gamma_5 \tilde{g}\] (A7)

and

\[\gamma^\alpha (P_\mu)_{\alpha\beta} = -\frac{1}{2}i\sigma_{\mu\nu} \gamma_5 \gamma^\nu q_\beta,\] (A8)

which follows from Eq. (2.35), allows us to confirm explicitly that

\[(L_1 P_\mu)_{\alpha\beta} = (P_\mu L_1)_{\alpha\beta},\] (A9)

which in turn implies that \((R \cdot PR_\mu)_{\alpha\beta} = (P_\mu R \cdot R)_{\alpha\beta}\) as stated in Eq. (3.11).

In obtaining the last formula quoted in Eq. (3.28), we have used the Gordon identity to write

\[\tilde{U}^{\rho\alpha} i\epsilon_{\nu\alpha\beta\rho} q^\sigma \sigma_{\mu\lambda} q^\lambda Q^\nu U^\beta = \tilde{U}^{\rho\alpha} \left[-\epsilon_{\nu\beta\sigma\rho} q^\lambda Q^\nu (Q_\mu - M_{\gamma\mu})\right] U^\beta\]

which, using the identity in Eq. (2.11) and proceeding as indicated there, can be reduced further to the form

\[\tilde{U}^{\rho\alpha} i\epsilon_{\nu\alpha\beta\rho} q^\sigma \sigma_{\mu\lambda} q^\lambda Q^\nu U^\beta = \tilde{U}^{\rho\alpha} \left[q_\alpha \epsilon_{\beta\nu\mu\lambda} q^\lambda Q^\nu + q_\beta \epsilon_{\alpha\mu\lambda} Q^\nu + (\Delta^2 - q^2) \epsilon_{\alpha\beta\nu\mu} q^\nu\right] U^\beta.\]

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