Local Thermal Equilibrium States and Unruh Detectors in Quantum Field Theory

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Abstract

In the framework of local thermodynamic equilibrium by Buchholz, Ojima and Roos, a class $S_x$ of observables, whose members are supposed to model idealized measurements of thermal properties of given states at spacetime points $x$, plays a crucial role in determining and characterizing local equilibrium states in quantum field theory. Here it will be shown how elements from this space can be reproduced by a specific model of the idealized measurements modeled by an Unruh-de Witt detector.

1 Introduction

In [4] a method to identify states in relativistic quantum field theory that allow locally a thermodynamic interpretation without being in global thermodynamic equilibrium has been proposed. The basic procedure is the following: First one fixes at each spacetime point some (linear) space of observables, denoted by $S_x$, which model idealized measurements of thermal properties in arbitrarily small spacetime regions. Next one chooses a set of reference states with known thermal properties (in the cases investigated so far these have been mixtures of global equilibrium states). A given state is then called (locally) thermal at a spacetime point $x$ if there exists a reference state such that the expectation values of all elements in $S_x$ in the two states agree, i.e. as far as local measurements of (expectation values of) thermal parameters are concerned, the given state looks like the reference state. If a state fulfills this condition of thermality at $x$ one can then consistently assign to it the thermal parameters whose local measurement are modeled by elements of $S_x$ and among them the relations (e.g. equations of state) established for the reference states also hold true. As the reference state may vary from point to point, this yields for states fulfilling the thermality
condition in a region $\mathcal{O}$ an assignment of thermal parameters depending on $x \in \mathcal{O}$ (e.g. a time- and space-dependent temperature). For further details and applications of the formalism to models see [4], [3], [1] and [5].

If the set of reference states is fixed (e.g. as mixtures of global equilibrium states), there remains the question of which $\mathcal{S}_x$-spaces one should pick. In the models considered so far, the $\mathcal{S}_x$-spaces have been chosen as the spaces generated by so called “balanced derivatives” of Wick-square, i.e. by elements

$$\lim_{\zeta \to 0} \partial_\zeta^\mu (\phi(x + \zeta)\phi(x - \zeta) - \omega_\infty (\phi(x + \zeta)\phi(x - \zeta)))$$

($\mu$ a multiindex, $I$ the identity operator and $\omega_\infty$ the vacuum-state). Now a justification of this choice of $\mathcal{S}_x$-spaces is given in the models by the fact that it allows for a local assignment of the interesting thermodynamic properties to non-equilibrium-states in the above sense while still allowing non-trivial local-equilibrium states (i.e. states where the thermal parameters do vary in space and time). There is also a general discussion on the specification of a maximal space $T_x$ of “pointlike” observables that may be used to determine local thermal properties of states [4]; the above set $\mathcal{S}_x$ should be a proper subset of $T_x$.

As the concept of local thermodynamic equilibrium states under discussion is based on the idea of idealized measurements at a point, one could hope that a more detailed description of such an idealized measurement in a model could give some additional insight into the choice of $\mathcal{S}_x$ spaces. That this is indeed true will be shown in this article.

As a model for the measurement process some kind of Unruh-de Witt detector [11],[7], in this case a two-level system moving along a given trajectory in spacetime and interacting with the given quantum-field, is used. For large interaction-times and time-invariant states the (suitably normalized) probability for the transition of the detector system from its initial (ground- or excited state) to its respective other state due to interactions with the quantum field can be used to determine thermal properties of the quantum field. Namely, by the principle of detailed balancing in a thermal state one expects the transition rates from the ground- to the excited and from the excited- to the ground-state to be related by a Boltzmann factor $e^{-\beta E/\hbar}$ if the state of the field is a thermal state at inverse temperature $\beta$ and $E$ is the energy difference between the two levels of the detector system. In fact, such a relation between the two transition rates for all two-level monopole

\footnote{for curved spacetimes the “general covariant Wick-square” [8],[2] can be used for a similar definition}
detector “at rest” wrt. the thermal state exactly corresponds to the KMS condition for this state \[10\]. More generally, the dependence of the transition probabilities on \(E/\hbar\) corresponds to “spectral properties” of the field, with the principle of detailed balancing as a relation between the rates for \(E\) and \(-E\) as a special case.

Now in order to be able to proceed to measurements taking place in a short time-interval, the idea is first not to look at the absolute transition probabilities but rather at their differences to those in a common reference state (i.e. to “remove the vacuum fluctuations”) and secondly to consider instead of the transition probabilities as function of \(E\) the sequence of moments of this function. When proceeding to arbitrarily short measurement times (while increasing the interaction coupling suitably) these moments will, in general, still diverge; however starting from the zeroth moment which stays finite one can subtract from the higher moments “perturbations” by the lower moments in such a way that the resulting (modified) moments all stay finite when sending the duration of measurement to zero. In this limit the modified moments are exactly what is measured by the balanced derivatives “in timelike direction”, i.e. by the balanced derivatives with \(\zeta\) tangential to the trajectory of the detector.

Finally it will be shown that although these balanced derivatives do not span the whole \(S_x\)-spaces, their linear combinations can be used to obtain the relevant thermodynamic observables that so far appeared in the model of the massless, neutral Klein-Gordon field \[3\].

2 Specification of the model

The quantum field considered is the massless neutral Klein-Gordon field on Minkowski spacetime \((\mathbb{R}^4, \eta)\), described by a star-algebra \(A\) generated by a unit element 1 and elements \(\phi(f)\) depending linearly on \(f \in C^\infty_0(\mathbb{R}^4)\) with the additional relations

- \(\phi(f)^* = \phi(\overline{f})\)
- \(\phi(\Box f) = 0\)
- \([\phi(f), \phi(g)] = E(f, g)1\)

where

\[
E(f, g) := \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} f(x)(G_{adv} - G_{ret})(x, x')g(x')dx'dx'
\]

\(^2\)see section 5 below for the notion of being at rest relative to a thermal state
and $G_{\text{adv}}$ and $G_{\text{ret}}$ are the advanced and retarded fundamental solutions of the Klein-Gordon equation, respectively.

States $\omega$ of the field are given by functionals on $\mathcal{A}$ which in addition fulfill for all $A \in \mathcal{A}$ and $f_1, \ldots, f_n \in C_0^\infty(\mathbb{R}^4)$,

- $\omega(A^*A) \geq 0$
- $\omega(1) = 1$
- $(f_1, \ldots, f_n) \mapsto \omega(\phi(f_1) \cdots \phi(f_n))$ is a distribution (depends continuously on the $f_i$).

There is a distinguished vacuum-state, denoted by $\omega_\infty$, and furthermore the states considered here are Gaussian Hadamard states, i.e. determined by their two-point function $C_0^\infty(\mathbb{R}^4) \times C_0^\infty(\mathbb{R}^4) \ni (f, g) \mapsto \omega(\phi(f)\phi(g))$ which is such that $(f, g) \mapsto \omega(\phi(f)\phi(g)) - \omega_\infty(\phi(f)\phi(g))$ is a regular distribution.

The physical picture of the measurements considered here is the following: An ensemble of quantum-mechanical detectors (two-level systems) moves along a (common, classical) trajectory $\gamma$, parametrized by proper time $\tau$, with each member initially in its ground-states. At some time the detectors are (smoothly) switched on, interact with the field for some time and are then switched off again. Finally the number of detectors in the excited state is determined, which yields the transition probability for a single detector (which is of course the same as the expectation value of the transition rate times the interaction duration).

Mathematically, the free two-level detector system is described in the Heisenberg picture by a two-dimensional complex Hilbert-Space $(\mathcal{H}_D, \langle \cdot, \cdot \rangle_{\mathcal{H}_D})$ spanned by the two orthonormal states $\psi_g$ and $\psi_e$ of the detector. These two states are assumed to be eigenstates with eigenvalues zero and $\epsilon$ of the detector-Hamiltonian $H_D$ and furthermore the existence of a (time-dependent) self-adjoint operator $\tau \mapsto M(\tau) := e^{i\tau H_D} M_0 e^{-i\tau H_D}$ such that $|\langle \psi_e, M_0 \psi_g \rangle| \neq 0$ is assumed.

For a given state $\omega$ of the quantum field, the coupled detector-field system is described in the interaction picture in the Hilbert-space $\mathcal{H}_\omega \otimes \mathcal{H}_D$ where $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denotes the GNS representation [12, Chap. 4] of $\mathcal{A}$ belonging to $\omega$. The initial state $\Phi$ of the coupled system is taken to be $\Omega_\omega \otimes \psi_g$ and the time evolution of this state is determined by

$$i\partial_\tau \Phi(\tau) = H_{\text{int}}(\tau)\Phi(\tau) := [\chi(\tau)\phi(\gamma(\tau)) \otimes M(\tau)] \Phi(\tau)$$

where $\gamma : \mathbb{R} \to \mathbb{R}^4$ is the detector-trajectory as described above, $\chi \in \mathcal{S}(\mathbb{R})$ is the detector switching function (real-valued and normalized by the requirement $\int_\mathbb{R} \chi(\tau) d\tau = 1$) and $\mathbb{R}^4 \ni x \mapsto \phi(x)$ is related to $\pi_\omega(\phi(f))$ by
\( \pi_\omega(\phi(f)) = \int \phi(x)f(x)dx \) in the sense of quadratic forms on (a subset of) \( H_\omega \).

As one is interested in the transition probabilities of a (weakly) coupled detector without back-reaction effects a perturbative calculation seems appropriate; for a rigorous justification of this approach in a related situation see [6]. To first order perturbation-theory, the state of the detector-field system at time \( \tau \) is given by

\[
\Phi(\tau) = \Omega_\omega \otimes \psi_g - i \int_{-\infty}^\tau \chi(\tau') (\phi(\gamma(\tau'))\Omega_\omega) \otimes (M(\tau')\psi_g) d\tau'
\]

For the probability of finding the detector in the excited state \( \psi_e \) and the field in a state \( \Psi_n \) at large times, one then has

\[
|\langle \Psi_n \otimes \psi_e, \Phi(\infty) \rangle|^2 = \int_\mathbb{R} \int_\mathbb{R} \chi(\tau') \chi(\tau'') \omega(\phi(\gamma(\tau'))\phi(\gamma(\tau''))) \omega(\phi(\gamma(\tau''))\phi(\gamma(\tau''))) d\tau' d\tau''
\]

and by summing over a complete set of \( \Psi_n \) in \( H_\omega \) and using

\[
\langle \psi_e, M(\tau)\psi_g \rangle = \langle \psi_e, e^{itH_0} M_0 e^{-itH_0} \psi_g \rangle = e^{i\epsilon t} \langle \psi_e, M_0 \psi_g \rangle
\]

the probability of finding the detector in the excited and the field in any state is

\[
P_\omega(\epsilon) = |\langle \psi_e, M_0 \psi_g \rangle|^2 \int \int \chi(\tau') \chi(\tau'') e^{-i(\tau' - \tau'')} \times \ldots \\
\times \langle \psi_e, \psi_e, \chi(\tau') \chi(\tau'') \omega(\phi(\gamma(\tau'))\phi(\gamma(\tau''))) \rangle d\tau' d\tau''
\]

where the constant of proportionality \( m := |\langle \psi_e, M_0 \psi_g \rangle|^2 \) depends on details of the detector but not on the field configuration.

Now instead of comparing directly the transition probabilities \( P_\omega(\epsilon) \) and \( P_{\omega_0}(\epsilon) \) in two states, one can in principle compare their difference to the transition probability \( P_{\omega_{\text{ref}}} \) in a common reference state. Choosing \( \omega_{\text{ref}} = \infty \) as \( \omega_{\text{ref}} \) this amounts heuristically to “removing the vacuum fluctuations” and the resulting (difference in) transition probability is then

\[
P_{\omega}^{\text{ren}}(\epsilon) = m \int \int e^{-isF_\omega(\tau,s)} \chi(\tau + s/2) \chi(\tau - s/2) ds d\tau \tag{1}
\]

\[
F_\omega(\tau,s) := \omega(\phi(\gamma(\tau + s/2)) \phi(\gamma(\tau - s/2))) - \ldots \\
\ldots - \omega_{\infty}(\phi(\gamma(\tau + s/2)) \phi(\gamma(\tau - s/2)))
\]
Assuming $\omega$ to be a Hadamard state, the integrand is smooth and compactly supported and therefore $\epsilon \mapsto P^{\text{ren}}_\omega(\epsilon)$ is a rapidly decreasing, smooth function. Thus all moments of this function are defined and we turn to the analysis of these moments in order to obtain a means for local investigation of states.

3 Convolution and moments

As already mentioned above, in the case of global equilibrium and for $\chi$ approaching a constant function, the dependence of the transition probabilities on $\epsilon$ approaches a function that gives information about the thermal properties of the state under consideration. Disregarding for a moment the $\tau$-integration in (1), more rapid switching of the detector can be seen to disturb this function by convolution with a function that becomes wide as $\chi$ becomes narrow, as is of course to be expected due to time-energy uncertainty. There is however a way to get around this, if one is only interested in the moments of this function and knows the moments of $\chi$.

To see this, consider two rapidly decaying functions $f, h \in S(\mathbb{R})$. Denote the $k$-th moment of a function $f \in S(\mathbb{R})$ by

$$M_k[f] := \int_{\mathbb{R}} t^k f(t) dt$$

and assume $M_0[h] = 1$ (i.e. the integral over $h$ is one). Then by a direct computation one has for the convolution $f \ast h$:

$$M_k[f \ast h] = \sum_{j=0}^{k} \binom{k}{j} M_j[f] M_{k-j}[h]$$

For the special case $k = 0$ this relation gives $M^0[f \ast h] = M[f]$, so the zeroth moment of $f$ is always known once the zeroth moment of $f \ast h$ is known. Noting that on the rhs of equation (2) the $k$-th moment of $f$ is multiplied by one and the further terms involve only lower moments of $f$, one can therefore determine the moments of $f$ from those of $f \ast h$ and $h$ by the recursion

$$M_k[f] = M_k[f \ast h] - \sum_{j=0}^{k-1} \binom{k}{j} M_j[f] M_{k-j}[h]$$

starting from $M^0[f]$.
4 Modified moments of transition rates and elements of $S_x$

Now returning to (1), for general $\chi$ the application of the idea of the preceding section to the transition-probability is complicated by the $\tau$-integration. This problem can however be overcome by choosing $\chi$ to be a Gaussian of width $\sigma$ centered at $\tau_0$:

$$\chi_{\sigma,\tau_0} : \tau \mapsto \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\tau-\tau_0)^2}{2\sigma^2}}$$

Then $\chi_{\sigma,\tau_0}(\tau + s/2) \chi_{\sigma,\tau_0}(\tau - s/2)$ factorizes into $\chi_{\sigma}/\sqrt{2}\tau_0(\tau)\chi_{\sqrt{2}\sigma,0}(s)$ and the moments of $\epsilon \mapsto P_{\text{ren}}(\omega)$ are:

$$\mathcal{M}_k[P_{\text{ren}}] = m \int_{\mathbb{R}} \int_{\mathbb{R}} F_\omega(\tau,s) \chi_{\sigma}/\sqrt{2}\tau_0(\tau) e^{-i\epsilon s} \chi_{\sqrt{2}\sigma,0}(s) d\epsilon d\tau$$

Now identifying the Fourier transform of $s \mapsto \sqrt{2\pi} \int_{\mathbb{R}} F_\omega(\tau,s) \chi_{\sigma}/\sqrt{2}(\tau) d\tau$ with $f$ and the Fourier transform of $\sqrt{2\pi} \chi_{\sqrt{2}\sigma}$ with $h$, this is the situation of the preceding section and by the recursive procedure described there, one can obtain the $k$-th moments of $f$ which will be called $P^k_{\omega}$. Explicitly they are given by

$$P^k_{\omega} = m \int_{\mathbb{R}} \int_{\mathbb{R}} (-i\partial_s)^k F_\omega(\tau,s)|_{s=0} \chi_{\sigma}/\sqrt{2}\tau_0(\tau) d\tau$$

By partial integration using the decay-properties of $\chi_{\sigma,\tau_0}$ and the regularity of $F_\omega$ this can be expressed as

$$P^k_{\omega} = m \int_{\mathbb{R}} (-i\partial_s)^k F_\omega(\tau,0)|_{s=0} \chi_{\sigma}/\sqrt{2}\tau_0(\tau) d\tau$$

For this expression one can proceed to the limit $\sigma \to 0$, which shows that the $P^k_{\omega}$ are objects which can be measured over arbitrarily short time-intervals. In the limit one has

$$\lim_{\sigma \to 0} P^k_{\omega} = m(-i\partial_s)^k F_\omega(\tau_0,0)|_{s=0}$$

Defining for a (sufficiently differentiable) function $f : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ and $u_1, \ldots, u_n \in \mathbb{R}^4$ the balanced derivative $\partial^{(u_1,\ldots,u_n)} f(x)$ by

$$\partial^{(u_1,\ldots,u_n)} f(x) = \partial_{t_1} \ldots \partial_{t_n} f\left(x + \sum_{i=1}^n t_i u_i , x - \sum_{j=1}^n t_j u_j\right)|_{t_1=\ldots=t_n=0}$$

7
one can rewrite (4) for inertial detectors with $\gamma(\tau_0) = x$ and $\dot{\gamma}(\tau_0) = u$ as

$$\lim_{\sigma \to 0} P^k_\omega = m(-i/2)^k \mathcal{O}^{(u,\ldots,u)} w(x)$$

$$w(x, y) := \omega(\phi(x)\phi(y)) - \omega_\infty(\phi(x)\phi(y))$$

in the sense of distrib.

The definition (5) slightly generalizes the definition for $\mathcal{O}^{(\mu_1,\ldots,\mu_n)}$ from (4) which is recovered by choosing $u_i = \eta^{\mu_i\mu_i} e_{\mu_i}$ with $e_0, \ldots, e_3$ the basis vectors of $\mathbb{R}^4$, so the expectation values of elements $m(-i/2)^k \mathcal{O}^{(0,\ldots,0)} : \phi^2 :$ from the $S_x$-spaces in the state $\omega$ used there can be interpreted as measurements of $P^k_\omega$ in the limit of arbitrarily short interaction of detector and field. A prominent example from this class is the Wick-square itself, which gives the expectation-value of the local temperature squared.

5 Moving detectors and comparison with the full $S_x$-space

As established in the last section, the measurement of balanced derivatives $\mathcal{O}^{(u,\ldots,u)} : \phi^2(x) :$ can be described by a limiting process involving measurements carried out on an ensemble of detectors moving through the spacetime point $x$ with a four-velocity $u$. These balanced derivatives however only generate a subset of the $S_x$-spaces and important (local thermal) observables like the (thermal) stress-energy tensor are not among them.

Now global equilibrium states on Minkowski spacetime are not invariant under the full Poincaré group but single out a set of inertial frames that only differ by rotations and translations and physically correspond to the observers being at rest with respect to the “gas” described by the thermal state. As thermality properties like the principle of detailed balancing hold only for those systems coupled to the field which are at rest in these inertial frames, for the investigation of an equilibrium state with an unknown associated rest frame one should use not one detector, but detectors with all possible velocities smaller than the velocity of light relative to a given one. The detector behaving according to the principle of detailed balancing then indicates the rest frame of the given state, and starting from this one can then check whether the readings of the other detectors are compatible with the interpretation of being in relative motion to a thermal state.

Whereas for a global equilibrium state, whose rest-frame is usually known a priori this discussion might sound rather odd, the hydrodynamical description of gases by a velocity field varying in space and time can be rephrased as the statement that at each point the state looks like a thermal state.
with reference frames at different points being in relative motion to each other. As this dependence of the frames on space-time is not known a priori but rather one of the informations an LTE-formalism should yield, it seems therefore sensible to not just consider one detector with a worldline passing through a spacetime point \( x \) but the set of all detectors passing through it. By the above procedure, one then obtains as local thermal observables the balanced derivatives \( \delta^{(\mu, \nu)} : \phi^2 : (x) \) for all time unit vectors \( u \). By multiplying with scalars, the requirement of normalization for \( u \) can be dropped, and forming linear combinations one furthermore has

\[
\begin{align*}
\frac{1}{4} \delta^{(e_j, e_j)} : \phi^2 : &= \delta^{(e_0, e_0 + \frac{1}{2} e_j, e_0 + \frac{1}{2} e_j)} : \phi^2 : + \delta^{(e_0 - \frac{1}{2} e_j, e_0 - \frac{1}{2} e_j)} : \phi^2 : - \ldots \\
&\quad \ldots - \delta^{(e_0, e_0)} : \phi^2 : \\
2 \delta^{(e_0, e_j)} : \phi^2 : &= \delta^{(e_0, e_0 + \frac{1}{2} e_j, e_0 + \frac{1}{2} e_j)} : \phi^2 : - \delta^{(e_0 - \frac{1}{2} e_j, e_0 - \frac{1}{2} e_j)} : \phi^2 : \\
\delta^{(e_j, e_k)} : \phi^2 : &= \delta^{(e_0, e_0 + \frac{1}{2} e_j + \frac{1}{2} e_k, e_0 + \frac{1}{2} e_j + \frac{1}{2} e_k)} : \phi^2 : - \ldots \\
&\quad \ldots - \delta^{(e_0 + \frac{1}{2} e_j - \frac{1}{2} e_k, e_0 + \frac{1}{2} e_j - \frac{1}{2} e_k)} : \phi^2 : - \ldots \\
&\quad \ldots - \delta^{(e_0, e_0 - \frac{1}{2} e_k, e_0 - \frac{1}{2} e_k)} : \phi^2 : + \delta^{(e_0, e_0 + \frac{1}{2} e_k, e_0 + \frac{1}{2} e_k)} : \phi^2 : 
\end{align*}
\]

where \( k \neq j, k, j = 1, 2, 3 \) and \( e_0, \ldots, e_3 \) are the basis-vectors of \( \mathbb{R}^4 \) which shows that balanced derivatives \( \delta^{(\mu, \nu)} : \phi^2 : \) for \( \mu, \nu = 0, \ldots, 4 \) can be expressed as linear combinations of balanced derivatives \( \delta^{(u, u)} : \phi^2 : \) with different, timelike \( u \) (so that e.g. the expectation values of the thermal stress-energy tensor at \( x \) can be determined as a linear combination of measurement results of detectors moving through \( x \) with different velocities).

In the models considered so far it was also possible to locally determine the entropy current and the particle density by introducing an approximation process in the \( S_x \)-spaces adapted to the notion of local thermality used. For a detailed description and motivation see [4]; at a technical level this approximation criterion boils down to the following: Given a function \( V^+ \ni \beta \mapsto \Xi(\beta) \in \mathbb{R} \) describing the (known) dependence of a macroscopic observable \( \Xi \) on the (parameters \( \beta \) labeling) global equilibrium states, try to approximate \( \Xi \) in the seminorms

\[
\sup_{\beta \in B} |\Xi(\beta)| =: \|\Xi\|_B
\]

(\( B \) a bounded set contained inside the open forward lightcone \( V^+ \)) by the “thermal functions” of the elements in \( S_x \). The thermal function of an element from \( S_x \) is defined as the mapping that associates to \( \beta \) the expectation value of this element in the global-equilibrium state belonging to \( \beta \); those
associated to $\vartheta^\mu : \phi^2 : (x)$ will be denoted by $\Theta^\mu$.

Investigating the global equilibrium states for the neutral, massless Klein-Gordon field (labeled by a timelike four-vector $\beta$ already used above), the functions belonging to the particle density and the entropy current are found to be, respectively, a solution of the wave-equation and a gradient of such a solution (wrt. the parameter $\beta$) \cite{4}, \cite{3}. That such functions can indeed be approximated in the above sense is established in \cite{3}. The approximation given there only needs thermal functions $l_\mu \Theta^\mu$ corresponding to $\vartheta^{(l,...,l)} : \phi^2 :$ with $l$ lightlike\footnote{In this section $\mu$ is always assumed to be a multi-index and the Einstein summation convention is used}, whereas here an approximation with thermal functions $u_\mu \Theta^\mu$ of elements $\vartheta^{(u,...,u)} : \phi^2 :$ with $u$ timelike is desired.

Relying on these results, it is thus sufficient to show that the functions $l_\mu \Theta^\mu$ for given, lightlike $l$ can be approximated in the seminorms \cite{3} by functions $u_\mu \Theta^\mu$ with timelike vectors $u$.

For the massless, neutral Klein-Gordon field, the thermal functions $u_\mu \Theta^\mu$ of elements $\vartheta^{(u,...,u)} : \phi^2 :$ are proportional to \cite{4}

$u_\mu \Theta^\mu = \left( u^\kappa \frac{\partial}{\partial \beta_\kappa} \right)^m \frac{1}{\beta^2}$

for $m := |\mu|$ even, and zero otherwise. This $m$-fold directional derivative is given by a sum starting with the term $m!2^n(u^\beta)^m(\beta^2)^{-m-1}$ and further terms proportional to $(u^2)^k(u^\beta)^l(\beta^2)^{-n}(k, n \in \mathbb{N} \setminus \{0\}, l \in \mathbb{N})$ that vanish for $u$ lightlike. Now for given lightlike $l$ choose $u = l + \delta e_0$. The first term in the sum is then itself a sum of $m!2^n(l^\beta)^m(\beta^2)^{-m-1}$ and functions bounded on $B$ times powers of $\delta$ and the remaining summands also give functions bounded on $B$ multiplied by powers of $\delta$ greater than zero (because of the terms $(u^2)^k, k \in \mathbb{N} \setminus \{0\}$). For given $B,l$ and $\epsilon$ one can therefore choose $\delta$ such that $(l + \delta e_0)_\mu \Theta^\mu$ approximates $l_\mu \Theta^\mu$ in the $\| \cdot \|_B$-norm up to an error $\epsilon$.

Summarizing, for the model of the massless, neutral Klein-Gordon field the relevant local thermodynamic observables can all be obtained by linear combinations and limiting processes involving the balanced derivatives from section four, whose operational significance has been established.

As far as the case of a massive Klein-Gordon field is concerned, the results up to and including the determination of the thermal stress-energy tensor using detectors in relative motion do not depend on the assumption of a vanishing mass; the question whether one can still approximate local observables for the particle density and the entropy current in the case of a
massive field is however more difficult to answer, as in this situation one
has on the one hand to do the approximation in a different topology (for a
definition and reasons for its use see [9]) and on the other hand the thermal
functions of the generators of $S_x$ are more complicated.

6 Conclusion

Starting from an Unruh-de Witt detector model, it was shown that some
modified moments characterizing the dependence of the transition rate of
this detector on the separation of its levels survive the limit of going to
arbitrarily short detector-field interactions (rescaling the strength of the
interaction in the natural way). The value of these moments obtained in
the limit coincides for geodesic detectors in Minkowski spacetime with the
expectation values of the balanced derivatives defined in the context of non-
equilibrium thermodynamics. Like other observables defining properties of
quantum fields at a point (for example the stress-energy tensor), the objects
$P^k_\omega$ have to be carefully defined in order to remain meaningful in the point-
limit. Here this definition involves

- Looking at differences between transition probabilities in different states
  instead of looking at the transition probabilities itself.

- “Decoupling” time and frequency domain by choosing Gaussian switching
  functions, choosing moments of the transition probability and re-
  moving the perturbing effects of lower on higher moments by the re-
  cursion-relation (3).

The last point bears some similarity to a renormalization-group procedure:
When going to higher moments, the high-frequency (small scale) properties
of the field are emphasized and at the same time the low-frequency part is
discarded in order to be able to proceed to a pointlike limit.

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