ON SOME RELATIONS FOR MELLIN TRANSFORMS OF HARDY’S FUNCTION

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To Professor Akio Fujii on the occasion of his retirement

Abstract. Some relations involving the Mellin and Laplace transforms of powers of the classical Hardy function

\[ Z(t) := \zeta\left(\frac{1}{2} + it\right)\left(\chi\left(\frac{1}{2} + it\right)\right)^{-1/2}, \quad \zeta(s) = \chi(s)\zeta(1 - s) \]

are obtained. In particular, we discuss some mean square identities and their consequences.

1. Definition of Hardy’s function

The familiar Riemann zeta-function

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (s = \sigma + it, \sigma > 1) \]

admits analytic continuation to \( \mathbb{C} \), having only a simple pole at \( s = 1 \). A vast literature exists on many aspects of zeta-function theory, such as the distribution of its zeros and power moments of \( |\zeta(\frac{1}{2} + it)| \) (see e.g., the monographs [9], [10], [38], [39] and [42]). It is within this framework that the classical Hardy function (see e.g., [9]) \( Z(t) \ (t \in \mathbb{R}) \) plays an important rôle. It is defined as

\[ Z(t) := \zeta\left(\frac{1}{2} + it\right)\left(\chi\left(\frac{1}{2} + it\right)\right)^{-1/2}, \]  

where \( \chi(s) \) comes from the well-known functional equation for \( \zeta(s) \) (see e.g., [9, Chapter 1]), namely \( \zeta(s) = \chi(s)\zeta(1 - s) \), so that

\[ \chi(s) = 2^s\pi^{s-1} \sin\left(\frac{1}{2}\pi s\right)\Gamma(1 - s), \quad \chi(s)\chi(1 - s) = 1. \]

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It follows that
\[
\chi \left( \frac{1}{2} + it \right) = \chi \left( \frac{1}{2} - it \right) = \chi^{-1} \left( \frac{1}{2} + it \right),
\]
so that \( Z(t) \in \mathbb{R} \) when \( t \in \mathbb{R} \), and \( |Z(t)| = |\zeta(\frac{1}{2} + it)| \). Thus the zeros of \( \zeta(s) \) on the “critical line” \( \Re s = 1/2 \) correspond to the real zeros of \( Z(t) \), which makes \( Z(t) \) an invaluable tool in the study of the zeros of the zeta-function on the critical line. Alternatively, if we use the symmetric form of the functional equation for \( \zeta(s) \), namely
\[
\pi^{-s/2} \zeta(s) \Gamma \left( \frac{1}{2} s \right) = \pi^{-(1-s)/2} \zeta(1-s) \Gamma \left( \frac{1}{2} (1-s) \right),
\]
then for \( t \in \mathbb{R} \) we obtain an equivalent form of (1.1). This is
\[
Z(t) = e^{i \theta(t)} \zeta \left( \frac{1}{2} + it \right), \quad e^{i \theta(t)} := \pi^{-it/2} \frac{\Gamma \left( \frac{1}{4} + \frac{1}{2} it \right)}{|\Gamma \left( \frac{1}{4} + \frac{1}{2} it \right)|}, \quad \theta(t) \in \mathbb{R}.
\]
Hardy’s original application of \( Z(t) \) was to show that \( \zeta(s) \) has infinitely many zeros on the critical line \( \Re s = 1/2 \) (see e.g., E.C. Titchmarsh [42]). Later A. Selberg (see [40] and [41]) obtained his famous result that a positive proportion of zeros of \( \zeta(s) \) lies on the critical line. His method was later used and refined by many mathematicians. The latest result is by S. Feng [5], who proved that at least 41.73% of the zeros of \( \zeta(s) \) are on the critical line and at least 40.75% of the zeros of \( \zeta(s) \) are simple and on the critical line.

There is also an important connection of \( Z(t) \) to the Riemann Hypothesis (RH, that all complex zeros of \( \zeta(s) \) have real parts 1/2). The function \( Z(t) \) has a negative local maximum \(-0.52625\ldots \) at \( t = 2.47575\ldots \). This is the only known occurrence of a negative local maximum, while no positive local minimum is known. The so-called Lehmer’s phenomenon (named after D.H. Lehmer, who in his works [35], [36] made significant contributions to the subject) is the fact that the graph of \( Z(t) \) sometimes barely crosses the \( t \)-axis. This means that the absolute value of the maximum or minimum of \( Z(t) \) between its two consecutive zeros is small. The Lehmer phenomenon shows the delicacy of the RH, and the possibility that a counterexample to the RH may be found numerically. For should it happen that, for \( t \geq t_0 \), \( Z(t) \) attains a negative local maximum or a positive local minimum, then the RH would be disproved. This assertion follows (see [18]) from the following proposition: If the RH is true, then the graph of \( Z'(t)/Z(t) \) is monotonically decreasing between the zeros of \( Z(t) \) for \( t \geq t_0 \).

The main aim of this article is to discuss the Mellin transforms of \( Z^k(t) \) when \( k \in \mathbb{N} \). The (modified) Mellin transform is introduced in Section 2, and some of its properties are given. New mean square results for \( M_1(s) \) and \( M_2(s) \) are presented in Section 3 and their proofs are given in Section 4. Section 5 is devoted to the Laplace transforms of \( Z^k(t) \), while some other relations and open problems are stated in Section 6.
2. The modified Mellin transform

First we recall that the Laplace and Mellin transforms of \( f(x) \) are commonly defined as \((\sigma, t \in \mathbb{R})\)

\[
\mathcal{L}[f(x)] = \int_0^\infty f(x)e^{-sx} \, dx \quad (s = \sigma + it),
\]

\[
\mathcal{M}[f(x)] = F(s) := \int_0^\infty f(x)x^{s-1} \, dx \quad (s = \sigma + it),
\]

provided that the integrals in question exist. Mellin and Laplace transforms play an important rôle in Analytic Number Theory. They can be viewed, by a change of variable, as special cases of Fourier transforms, and their properties can be deduced from the general theory of Fourier transforms (see e.g., E.C. Titchmarsh [41]).

One of the basic properties of Mellin transforms is the inversion formula

\[
\frac{1}{2} \{f(x + 0) + f(x - 0)\} = \frac{1}{2\pi i} \int_{(\sigma)} F(s)x^{-s} \, ds = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} F(s)x^{-s} \, ds,
\]

where \( \int_{(\sigma)} \) denotes integration over the line \( \Re s = \sigma \). Formula (2.3) certainly holds if \( f(x)x^{\sigma-1} \in L^1(0, \infty) \), and \( f(x) \) is of bounded variation on every finite \( x \)-interval. Note that if \( G(s) \) denotes the Mellin transform of \( g(x) \) then, assuming \( f(x) \) and \( g(x) \) to be real-valued, we formally have

\[
\frac{1}{2\pi i} \int_{(\sigma)} F(s)G(s) \, ds = \int_0^\infty g(x)\left(\frac{1}{2\pi i} \int_{(\sigma)} F(s)x^{\sigma-it-1} \, ds\right) \, dx
\]

\[
= \int_0^\infty g(x)x^{2\sigma-1}\left(\frac{1}{2\pi i} \int_{(\sigma)} F(s)x^{-s} \, ds\right) \, dx = \int_0^\infty f(x)g(x)x^{2\sigma-1} \, dx.
\]

The relation (2.4) is a form of Parseval’s formula for Mellin transforms, and it offers various possibilities for mean square bounds. A condition under which (2.4) holds is that \( x^{\sigma}f(x) \) and \( x^{\sigma}g(x) \) belong to \( L^2((0, \infty), \, dx/x) \), where as usual

\[
L^p(a, b) := \left\{ f(x) \left| \int_a^b |f(x)|^p \, dx < \infty \right. \right\}.
\]

Our main object of study will be the function

\[
\mathcal{M}_k(s) := \int_1^\infty Z^k(x)x^{-s} \, dx \quad (k \in \mathbb{N}),
\]
where $\sigma = \Re s = \sigma(k)$ is so large that the integral in (2.5) converges absolutely, and henceforth $k \in \mathbb{N}$ will be fixed. This function, which appropriately can be called the modified Mellin transform of $Z^k(t)$, was introduced and investigated in [19], [20].

The reasons why we have defined somewhat differently $M_k(s)$ from the usual Mellin transforms are the following: the lower limit of integration $x = 1$ dispenses with potential convergence problems at $x = 0$, while the appearance of $x^{-s}$ instead of the familiar $x^{s-1}$ stresses the analogy with Dirichlet series where one has a sum of $f(n)n^{-s}$ and not $f(n)n^{s-1}$. Note that in the case when $k = 2m$ is even, in view of $|Z(t)| = |\zeta(1/2 + it)|$, we have

$$\mathcal{M}_{2m}(s) = \int_1^\infty |\zeta(1/2 + ix)|^{2m} x^{-s} \, dx.$$ 

The special cases $m = 1$ and $m = 2$ were investigated in several works, including [13], [15], [17], [21], [25], [26], [37] and [38].

The general modified Mellin transform $m[f(x)]$, of which $M_k(s)$ is a special case, is defined as

$$F^*(s) = m[f(x)] = \int_1^\infty f(x)x^{-s} \, dx \quad (s = \sigma + it),$$

provided that the integral in (2.6) converges. If $\bar{f}(x) = f(1/x)$ when $0 < x \leq 1$ and $\bar{f}(x) = 0$ otherwise, then

$$m[f(x)] = \mathcal{M} \left[ \frac{1}{x} \bar{f}(x) \right],$$

so that the properties of $m[f(x)]$ can be deduced from the properties of the ordinary Mellin transform $\mathcal{M}[f(x)]$ by the use of (2.7). In this way the author [13] proved several lemmas involving the modified Mellin transform. We present three of them, which will be used in the sequel.

**Lemma 1.** If $x^{-\sigma}f(x) \in L^1(1, \infty)$ and $f(x)$ is continuous for $x > 1$, then

$$f(x) = \frac{1}{2\pi i} \int_{(\sigma)} F^*(s)x^{s-1} \, ds, \quad F^*(s) = m[f(x)].$$

The inversion formula (2.8) is the analogue of (2.3) for modified Mellin transforms. The next lemma is the analogue of (2.4) for modified Mellin transforms.

**Lemma 2.** If $F^*(s) = m[f(x)]$, $G^*(s) = m[g(x)]$, and $f(x), g(x)$ are real-valued, continuous functions for $x > 1$, such that

$$x^{1/2-\sigma}f(x) \in L^2(1, \infty), \quad x^{1/2-\sigma}g(x) \in L^2(1, \infty),$$

then

$$f(x)g(x) = \frac{1}{2\pi i} \int_{(\sigma)} F^*(s)G^*(s)x^{s-1} \, ds.$$
then
\begin{equation}
\int_1^\infty f(x)g(x)x^{1-2\sigma} \, dx = \frac{1}{2\pi i} \int_{(\sigma)} F^*(s)G^*(s) \, ds.
\end{equation}

**Lemma 3.** Suppose that \( g(x) \) is a real-valued, integrable function on \([a, b]\), a subinterval of \([2, \infty)\), which is not necessarily finite. Then
\begin{equation}
\int_0^T \left| \int_a^b g(x)x^{-s} \, dx \right|^2 \, dt \leq 2\pi \int_a^b g^2(x)x^{1-2\sigma} \, dx \quad (s = \sigma + it, T > 0, a < b).
\end{equation}

### 3. Mean square identities with \( M_k(s) \)

To formulate our results on some mean square identities with \( M_k(s) \), we need the definition of the function \( E_k(T) \), the error term in the asymptotic formula for the \( 2k \)-th moment of \( |\zeta(\frac{1}{2} + it)| \). Namely, for any fixed \( k \in \mathbb{N} \), we expect (since the lower bound of integration in (2.5) is unity, it is convenient to have it also in the integral in (3.1))
\begin{equation}
\int_1^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt = TP_k(\log T) + E_k(T)
\end{equation}
to hold, where it is generally assumed that
\begin{equation}
P_k(y) = \sum_{j=0}^{k^2} a_{j,k}y^j
\end{equation}
is a polynomial in \( y \) of degree \( k^2 \) (the integral in (3.1) is unconditionally \( \gg_k T \log^{k^2} T \); see e.g., [9, Chapter 9] and [39]). The function \( E_k(T) \) is to be considered as the error term in (3.1), namely one supposes that
\begin{equation}
E_k(T) = o(T) \quad (T \to \infty).
\end{equation}

So far the formulas (3.1)–(3.3) are known to hold only for \( k = 1 \) and \( k = 2 \) (see [9] and [10] for a detailed account). For higher moments one has the bound
\begin{equation}
\int_1^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll T^{\frac{k}{2}(k+2)} \log^{C_k} T \quad (2 \leq k \leq 6),
\end{equation}
and more complicated bounds for \( k > 6 \). A comprehensive account is to be found in Chapter 8 of [9]. As usual, \( f \ll g \) (same as \( f = O(g) \)), means that \( |f(x)| \leq Cg(x) \)
for some $C > 0$ and $x \geq x_0$, and $f \ll g$ means that the $\ll$-constant depends on $\varepsilon$.
Therefore in view of the existing knowledge on the higher moments of $|\zeta(\frac{1}{2} + it)|$, embodied essentially in (3.4), at present the really important cases of (3.1) are $k = 1$ and $k = 2$. Plausible heuristic arguments for the values of the coefficients $a_{j,k}$ in the general case were given by Conrey et al. [4], by using methods from Random Matrix Theory (see also Keating–Snaith [32]).

For $k = 1$ the relation (3.1) becomes (see Chapter 15 of [9]) the well-known mean square formula

\begin{equation}
\int_1^T |\zeta(1/2 + it)|^2 dt = T \left( \log \frac{T}{2\pi} + 2\gamma - 1 \right) + E(T),
\end{equation}

where one usually writes $E(T) \equiv E_1(T)$, and $\gamma = -\Gamma'(1) = 0.5772156649\ldots$ is Euler’s constant. It is known that $E_1(T) \ll T^{131/416+\varepsilon}$ (see Huxley-Ivić [8] and N. Watt [43]) and on the other hand we have $E_1(T) = \Omega_{\pm}(T^{1/4})$. Also we have (see Hafner-Ivić [6], [7])

\begin{equation}
\int_1^T E(t) dt = \pi T + G(T), \quad G(T) = O(T^{3/4}), \quad G(T) = \Omega_{\pm}(T^{3/4}).
\end{equation}

For $k = 2$ we have

\begin{equation}
E_2(T) = \int_1^T |\zeta(1/2 + it)|^4 dt - TP_4(\log T).
\end{equation}

Note that $P_4(x)$ is a polynomial of degree four in $x$ with leading coefficient $1/(2\pi^2)$ (see e.g., [9, Chapter 4]). Its coefficients were evaluated independently by J.B. Conrey [3] and the author [11]. We have $E_2(T) \ll T^{2/3} \log^9 T$ and $E_2(T) = \Omega_{\pm}(T^{1/2})$ (see [23] and [38]). It is conjectured that $E_1(T) \ll T^{1/4+\varepsilon}$ and that $E_2(T) \ll T^{1/2+\varepsilon}$.

Here and later $\varepsilon (> 0)$ will denote arbitrarily small constants, not necessarily the same ones at each occurrence. As usual, $f = \Omega_{\pm}(g)$ means that $\limsup f/g > 0$ and $\liminf f/g < 0.$ We also have (see [10])

\begin{equation}
\int_1^T E^2(t) dt = DT^{3/2} + O(T \log^4 T)
\end{equation}

with $D = 2(2\pi)^{-1/2}\zeta^4(3/2)/(3\zeta(3))$, and (see [23] and [38])

\begin{equation}
\int_1^T E_2^2(t) dt \ll T^2 \log^{22} T.
\end{equation}

With this notation we can formulate our mean square results for $M_k(s)$. 

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THEOREM 1. For $\sigma > 1$ we have

$$\frac{1}{\pi} \int_0^\infty |M_1(\sigma + it)|^2 \, dt = \frac{2\gamma - \log(2\pi)}{2\sigma - 2} + \frac{1}{(2\sigma - 2)^2} + \left(2\gamma - 1 - \log(2\pi)\right)$$

$$+ 2\pi \sigma + 2\sigma(2\sigma - 1) \int_1^\infty G(x)x^{-1-2\sigma} \, dx,$$

where the integral on the right-hand side of (3.10) converges absolutely for $\sigma > 3/8$.

THEOREM 2. For $\sigma > 1$ we have

$$\frac{1}{\pi} \int_0^\infty |M_2(\sigma + it)|^2 \, dt = \sum_{j=0}^5 \frac{c_j}{(\sigma - 1)^j} + (2\sigma - 1) \int_1^\infty E_2(x)x^{-2\sigma} \, dx,$$

where the constants $c_j$ can be evaluated explicitly, and the integral on the right-hand side of (3.11) converges absolutely for $\sigma > 3/4$.

Corollary. We have

$$\lim_{\sigma \to 1+0} \left\{ \frac{1}{\pi} \int_0^\infty |M_1(\sigma + it)|^2 \, dt - \frac{2\gamma - \log(2\pi)}{2\sigma - 2} - \frac{1}{(2\sigma - 2)^2} \right\}$$

$$= 2\pi + 2\gamma - 1 - \log(2\pi) + 2 \int_1^\infty G(x)x^{-3} \, dx$$

and

$$\lim_{\sigma \to 1+0} \left\{ \frac{1}{\pi} \int_0^\infty |M_2(\sigma + it)|^2 \, dt - \sum_{j=0}^5 \frac{c_j}{(\sigma - 1)^j} \right\} = \int_1^\infty E_2(x)x^{-2} \, dx.$$

Remark. Theorem 1 could have been formulated, analogously to (3.11), as

$$\frac{1}{\pi} \int_0^\infty |M_1(\sigma + it)|^2 \, dt = C_0 + \frac{C_1}{\sigma - 1} + \frac{C_2}{(\sigma - 1)^2} + (2\sigma - 1) \int_1^\infty E(x)x^{-2\sigma} \, dx,$$

where the integral on the right-hand side converges absolutely for $\sigma > 5/8$. This follows by the Cauchy-Schwarz inequality for integrals from (3.8), but (3.10) is more precise.

For $k \geq 3$ let us define

$$\theta_k := \inf \left\{ a_k : \int_{-\infty}^\infty |M_k(\sigma + it)|^2 \, dt < \infty \text{ for } \sigma > a_k \right\}.$$  

It is clear that $\theta_k$ always exists (e.g., since $\zeta(\frac{1}{2} + it) \ll |t|^{1/6}$) and that it has an intrinsic connection with power moments of $|\zeta(\frac{1}{2} + it)|$. Of course, the above
definition makes sense for \( k = 1, 2 \) as well, but in these cases we have much more precise information in view of Theorem 1 and Theorem 2.

**THEOREM 3.** We have

\[
\theta_k \geq 1 \quad (\forall k), \quad \theta_k \leq \frac{3}{4} + \frac{k}{8} \quad (3 \leq k \leq 6),
\]

and

\[
\int_1^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll \varepsilon \, T^{2\theta_k - 1 + \varepsilon} \quad (k \geq 3).
\]

4. Proofs of the theorems

We begin with (2.9) of Lemma 2, which in case when

\[ f(x) = g(x) = Z^k(x) \]

reduces to

\[
\int_1^\infty |\zeta(\frac{1}{2} + ix)|^{2k} x^{1-2\sigma} \, dx = \frac{1}{2\pi} \int_\infty^{-\infty} |\mathcal{M}_k(\sigma + it)|^2 \, dt = \frac{1}{\pi} \int_0^\infty |\mathcal{M}_k(\sigma + it)|^2 \, dt,
\]

since \( \mathcal{M}_k(s) = \mathcal{M}_k(\bar{s}) \). To evaluate the left-hand side of (4.1) note that differentiation of (3.1) yields

\[ |\zeta(\frac{1}{2} + it)|^{2k} = P_{k^2}(\log t) + P'_{k^2}(\log t) + E'_k(t), \]

with \( P_{k^2} \) given by (3.2). Hence, initially for \( \Re \sigma \geq \sigma_1(k) \), we have

\[
\int_1^\infty |\zeta(\frac{1}{2} + ix)|^{2k} x^{1-2\sigma} \, dx = \int_1^\infty x^{1-2\sigma} \, d \left\{ x P_{k^2}(\log x) + E_k(x) \right\}
\]

\[
= \int_1^{\infty} (P_{k^2}(\log x) + P'_{k^2}(\log x)) x^{1-2\sigma} \, dx - E_k(1) + (2\sigma - 1) \int_1^{\infty} E_k(x)x^{-2\sigma} \, dx.
\]
But for $\Re \sigma > 1$ change of variable $\log x = t$ gives

$$
\int_1^\infty (P_k^2(\log x) + P_k'(\log x)) x^{1-2\sigma} \, dx \\
= \int_1^\infty \left\{ \sum_{j=0}^{k^2} a_{j,k} \log^j x + \sum_{j=0}^{k^2-1} (j+1)a_{j+1,k} \log^j x \right\} x^{1-2\sigma} \, dx \\
= \int_0^\infty \left\{ \sum_{j=0}^{k^2} a_{j,k} t^j + \sum_{j=0}^{k^2-1} (j+1)a_{j+1,k} t^j \right\} e^{-(2\sigma-2)t} \, dt \\
= \frac{a_{k^2,k}(k^2)!}{(2\sigma-2)^{k^2+1}} + \sum_{j=0}^{k^2-1} (a_{j,k}j! + a_{j+1,k}(j+1)!)(2\sigma-2)^{-j-1}.
$$

(4.3)

When $k = 1$ we have, by (3.5),

$$
P_1(y) = y + 2\gamma - 1 - \log(2\pi),
$$

hence for $\Re \sigma > 1$

$$
\int_1^\infty (P_1(\log x) + P_1'(\log x)) x^{1-2\sigma} \, dx \\
= \int_1^\infty (\log x + 2\gamma - \log(2\pi)) x^{1-2\sigma} \, dx \\
= \frac{2\gamma - \log(2\pi)}{2\sigma - 2} + \frac{1}{(2\sigma - 2)^2}.
$$

(4.4)

Next note that

$$
E_1(1) \equiv E(1) = -P_1(0) = \log(2\pi) + 1 - 2\gamma.
$$

(4.5)

Finally integration by parts yields, on using (3.6),

$$
\int_1^\infty E_1(x)x^{-2\sigma} \, dx \\
= \int_1^x E(y) \, dy \cdot x^{-2\sigma} \bigg|_1^\infty + 2\sigma \int_1^\infty \int_1^x E(y) \, dy \cdot x^{-1-2\sigma} \, dx \\
= 2\sigma \int_1^\infty (\pi x + G(x)) x^{-1-2\sigma} \, dx \\
= \frac{2\pi \sigma}{2\sigma - 1} + 2\sigma \int_1^\infty G(x)x^{-1-2\sigma} \, dx,
$$

(4.6)
and the last integral converges absolutely for \( \sigma > 3/8 \) in view of the \( O \)-bound in (3.6). The assertion of Theorem 1 follows then from (4.1)-(4.6).

For \( k = 2 \) write

\[
P_4(y) = \sum_{j=0}^{4} a_{j,4} y^j = \sum_{j=0}^{4} A_j y^j, \quad A_4 = 1/(2\pi^2),
\]

so that \( E_2(1) = -P_4(0) = -A_0 \) by (3.7). From (4.2) and (4.7) we infer that, for \( \sigma > 1 \),

\[
\int_1^\infty (P_4(\log x) + P_4'(\log x)) x^{1-2\sigma} \, dx = \sum_{j=1}^{5} \frac{B_j}{(\sigma - 1)^j}
\]

with

\[
B_5 = \frac{3}{8\pi^2}, \quad B_j = A_{j-1}(j-1)!2^{-j} + A_j j!2^{-j} \quad (j = 1, 2, 3, 4).
\]

This clearly gives (3.11) of Theorem 2 with

\[
c_0 = A_0, \quad c_j = A_{j-1}(j-1)!2^{-j} + A_j j!2^{-j} \quad (j = 1, 2, 3, 4), \quad c_5 = \frac{3}{8\pi^2}.
\]

The integral on the right-hand side of (3.11) converges absolutely for \( \sigma > 3/4 \), since by the Cauchy-Schwarz inequality for integrals and (3.2) we have

\[
\int_X^{2X} E_2(x)x^{-2\sigma} \, dx \leq \left\{ \int_X^{2X} E_2^2(x) \, dx \int_X^{2X} x^{-4\sigma} \, dx \right\}^{1/2} \leq \varepsilon X^{3/2-2\sigma + \varepsilon} \leq X^{-\varepsilon}
\]

for \( \sigma > 3/4 + \varepsilon \).

To prove Theorem 3 recall that it was stated after (3.1) that unconditionally

\[
\int_1^T |\zeta(\frac{1}{2} + it)|^{2k} dt \gg_k T(\log T)^k.
\]

This implies that \( \mathcal{M}_k(s) \) diverges for \( s = 1 \), hence \( \theta_k \geq 1 \) must hold. On the other hand we use (3.4) to deduce that, for \( 3 \leq k \leq 6 \),

\[
\int_X^{2X} |\zeta(\frac{1}{2} + ix)|^{2k} x^{1-2\sigma} \, dx \ll X^{1-2\sigma} X^{(k+2)/4}(\log X)^{C_k}
\]

and \( 1 - 2\sigma + (k + 2)/4 < 0 \) for \( \sigma > 3/4 + k/8 \). This means that, in this range for \( \sigma \), the integral

\[
\int_1^\infty |\zeta(\frac{1}{2} + it)|^{2k} x^{1-2\sigma} \, dx
\]

converges. But then (4.1) holds and hence \( \theta_k \leq 3/4 + k/8 \) for \( 3 \leq k \leq 6 \), proving Theorem 3. The conjecture \( \theta_k = 1 (\forall k) \) is clearly equivalent to the Lindelöf hypothesis that \( \zeta(\frac{1}{2} + it) \ll \varepsilon |t|^\varepsilon \) (see (7.2) of [9] and Theorem 13.2 of [42]).
5. The Laplace Transform of $Z^k(t)$

Let

\begin{equation}
\mathcal{L}_k(s) := \int_{1}^{\infty} Z^k(x) e^{-sx} \, dx \quad (\sigma = \Re s > 0, \ k \in \mathbb{N})
\end{equation}

denote the (modified) Laplace transform of $Z^k(x)$. Analogously to the modified Mellin transform (2.5), this differs from the standard definition of the Laplace transform in the lower bound of integration which is in (5.1) unity, and not zero like in (2.1). This is convenient because, for $c > 0$ and $\sigma \geq \sigma_0(k)$, we obtain

\begin{equation}
\mathcal{L}_k(s) = \frac{1}{2\pi i} \int_{(c)} \Gamma(w) \left( \int_{1}^{\infty} (sx)^{-w} Z^k(x) \, dx \right) \, dw
\end{equation}

Here we used the well-known Mellin inversion formula (see e.g., the Appendix of [8])

\[ e^{-x} = \frac{1}{2\pi i} \int_{(c)} \Gamma(w)x^{-w} \, dw \quad (\Re x > 0, \ c > 0). \]

Therefore by the inversion formula for modified Mellin transforms (see (2.8) of Lemma 1) one has

\begin{equation}
\Gamma(s)\mathcal{M}_k(s) = \int_{1}^{\infty} \mathcal{L}_k\left(\frac{1}{x}\right)x^{-1-s} \, dx \quad (\sigma \geq \sigma_0(k)).
\end{equation}

This relation was used by the author in [17] to show that $\mathcal{M}_2(s)$ has meromorphic continuation to $\mathbb{C}$. Its poles are $s = 1$ of order two, and simple poles at $s = -1, -3, -5, \ldots$. This result was independently proved also by M. Lukkarinen [37]. In [19] the author proved that $\mathcal{M}_1(s)$ has analytic continuation which is regular for $\Re s > 0$, and M. Jutila [30] showed that $\mathcal{M}_1(s)$ is even entire. The functions $\mathcal{M}_3(s)$ and $\mathcal{M}_4(s)$ were investigated in [19],[20],[21] and [27].

From (2.9) of Lemma 2 and (5.3) we obtain then

\begin{equation}
\int_{1}^{\infty} \mathcal{L}_k^2\left(\frac{1}{x}\right)x^{-1-2\sigma} \, dx = \frac{1}{\pi} \int_{0}^{\infty} |\Gamma(\sigma + it)|^2 |\mathcal{M}_k(\sigma + it)|^2 \, dt \quad (\sigma \geq \sigma_2(k) (> 0)).
\end{equation}

Since, for fixed $\sigma > 0$, we have

\[ |\Gamma(\sigma + it)|^2 \sim 2\pi t^{2\sigma - 1}e^{-\pi t} \quad (t \rightarrow +\infty) \]

by Stirling’s formula for the gamma-function, this means that the integrals

\[ \int_{1}^{\infty} \mathcal{L}_k^2\left(\frac{1}{x}\right)x^{-1-2\sigma} \, dx, \quad \int_{1}^{\infty} t^{2\sigma - 1}e^{-\pi t}|\mathcal{M}_k(\sigma + it)|^2 \, dt \]
both converge or both diverge for a given $\sigma (> 0)$.

One can, of course, consider also the classical Laplace transform

$$L_k(s) := \int_0^\infty Z^k(x)e^{-sx}\,dx \quad (k \in \mathbb{N}, \Re s > 0).$$

E.C. Titchmarsh’s well-known monograph [42, Chapter 7] provides a discussion of $L_{2m}(s)$ when $s = \sigma$ is real and $\sigma \to 0+$, especially detailed in the cases $m = 1$ and $m = 2$. Indeed, a classical result of H. Kober [33] says that, as $\sigma \to 0+$,

$$L_2(2\sigma) = \frac{\gamma - \log(4\pi \sigma)}{2 \sin \sigma} + \sum_{n=0}^N c_n \sigma^n + O_N(\sigma^{N+1})$$

for any given integer $N \geq 1$, where the $c_n$’s are effectively computable constants and $\gamma$ is Euler’s constant. For complex values of $s$ the function $L_4(s)$ was studied by F.V. Atkinson [1], and more recently by M. Jutila [27], who noted that Atkinson’s argument gives

$$L_4(s) = -ie^{\frac{1}{2}is}(\log(2\pi) - \gamma + (\frac{\pi}{2} - s)i) + 2\pi e^{-\frac{1}{2}is}\sum_{n=1}^\infty d(n) \exp(-2\pi ine^{-is}) + \lambda_1(s)$$

in the strip $0 < \Re s < \pi$, where the function $\lambda_1(s)$ is holomorphic in the strip $|\Re s| < \pi$. Moreover, in any strip $|\Re s| \leq \theta$ with $0 < \theta < \pi$, we have

$$\lambda_1(s) \ll_\theta (|s| + 1)^{-1}.$$ 

For $L_4(\sigma)$ F.V. Atkinson [2] obtained the asymptotic formula

$$L_4(\sigma) = \frac{1}{\sigma} \left( A \log^4 \frac{1}{\sigma} + B \log^3 \frac{1}{\sigma} + C \log^2 \frac{1}{\sigma} + D \log \frac{1}{\sigma} + E \right) + \lambda_4(\sigma),$$

where $\sigma \to 0+$,

$$A = \frac{1}{2\pi^2}, \quad B = \pi^{-2}(2\log(2\pi) - 6\gamma + 24\zeta'(2)\pi^{-2})$$

and

$$\lambda_4(\sigma) \ll \varepsilon \left( \frac{1}{\sigma} \right)^{\frac{13}{14} + \varepsilon}.$$ 

He also indicated how, by the use of estimates for Kloosterman sums, one can improve the exponent $\frac{13}{14}$ in (5.7) to $\frac{8}{9}$. This is of historical interest, since it is one
of the first instances of an application of Kloosterman sums to analytic number theory. Atkinson in fact showed that \((\sigma = \Re e s > 0)\)

\[(5.8) \quad L_4(s) = 4\pi e^{-\frac{1}{2} s} \sum_{n=1}^{\infty} d_4(n) K_0(4\pi i \sqrt{n} e^{-\frac{1}{2} s}) + \phi(s),\]

where \(d_4(n)\) is the divisor function generated by \(\zeta^4(s)\), \(K_0\) is the familiar Bessel function, and the series in (5.8) as well as the function \(\phi(s)\) are both analytic in the region \(|s| < \pi\).

Note that the author [11] applied a result on the fourth moment of \(|\zeta(\frac{1}{2} + it)|\), obtained jointly with Y. Motohashi [22]–[24] (see also [38]), to establish that

\[\lambda_4(\sigma) \ll \sigma^{-1/2} \quad (\sigma \to 0+).\]

This is essentially best possible, as shown by the author in [14], who obtained a refinement of (5.6) by means of the spectral theory of the non-Euclidean Laplacian (see Y. Motohashi's monograph [38] for a comprehensive account).

For \(k \geq 5\) not much is known about \(L_k(s)\), even when \(s = \sigma \to 0+\). This is not surprising, and is analogous to the situation with \(M_k(s)\), since not much is known (cf. (3.4)) about upper bounds for the \(k\)-th moment of \(|\zeta(\frac{1}{2} + it)|\) when \(k \geq 5\).

6. Further discussion and some open problems

There is a natural connection between \(M_k(s)\) and the moments of \(|\zeta(\frac{1}{2} + it)|\). For example, the author [19] proved that

\[(6.1) \quad \int_T^{2T} |\zeta(\frac{1}{2} + it)|^6 dt \ll_{\varepsilon} T^{2\sigma-1} \int_1^{T^{1+\varepsilon}} |M_3(\sigma + it)|^2 dt + T^{1+\varepsilon} \quad (\frac{1}{2} < \sigma \leq 1),\]

provided that \(M_3(s)\) can be continued analytically to \(\Re s \geq \sigma\) (and that is the catch!). Heuristically, we should be able to have \(\sigma = 3/4 + \varepsilon\), and then the integral on the right-hand side of (6.1) should be \(\ll_{\varepsilon} T^{1/2+\varepsilon}\), giving the bound \(O_{\varepsilon}(T^{1+\varepsilon})\), which is a weak form of the sixth moment. Note that (see [13, eq. (4.7)]) for the eighth moment we have (since \(Z_2(s) \equiv M_4(s)\))

\[(6.2) \quad \int_T^{2T} |\zeta(\frac{1}{2} + it)|^8 dt \ll_{\varepsilon} T^{2\sigma-1} \int_1^{T^{1+\varepsilon}} |M_4(\sigma + it)|^2 dt + T^{1+\varepsilon} \quad (\frac{1}{2} < \sigma \leq 1),\]

and an analogue of (6.1) and (6.2) holds also for the mean square and fourth power of \(|\zeta(\frac{1}{2} + it)|\). In these cases, however, the results are not of particular interest, since we have precise information which has been obtained by other methods. Note that \(M_4(s)\), unlike \(M_3(s)\), is known to possess analytic continuation to the region
\[ \sigma > \frac{1}{2}, \text{ where it is regular except for a pole of order five at } s = 1 \text{ (see Y. Motohashi [37]).} \]

The (nontrivial) bounds for the sixth moment of \(|\zeta(\frac{1}{2} + it)|\) are intricately connected to the problem of the analytic continuation of \( \mathcal{M}_3(s) \) to the region \( \sigma \leq 1 \). This, in turn, depends on the asymptotic evaluation of the integral

\[
F_k(T) := \int_1^T Z^k(t) \, dt
\]

when \( k = 3 \). The author in [16] proved that

\[
F_1(T) = \int_1^T Z(t) \, dt = O(T^{1/4+\varepsilon}),
\]

which was improved to \( F_1(T) = O(T^{1/4}) \) by M. Korolev [34], who also proved that \( F(T) = \Omega_{\pm}(T^{1/4}) \). M. Jutila [28], [31] gave a different proof of the same results by establishing precise formulas for \( F_1(T) \). In [19] it was proved that, for \( k = 1, 2, 3, 4 \), we have

\[
F_k(2T) - F_k(T) = \int_T^{2T} Z^k(t) \, dt
\]

\[
= 2\pi \sqrt{\frac{2}{k}} \sum_{(\frac{T}{2\pi})^{k/2} \leq n \leq (\frac{T}{2\pi})^{k/2}} d_k(n)n^{-\frac{k}{2}+\frac{1}{3}} \cos(k\pi n^2 + \frac{1}{8}(k-2)\pi) +
\]

\[
+ \ldots + O(\varepsilon(T^{k/4+\varepsilon})),
\]

where + . . . denotes terms similar to the one on the right-hand side of (6.4), with the similar cosine term, but of a lower order of magnitude. It was also indicated that actually the terms standing for + . . . may be omitted. Here \( d_k(n) \) is the divisor function generated by \( \zeta^k(s) \) (so that \( d_1(n) \equiv 1, d_2(n) \equiv d(n) = \sum_{\delta|n} 1 \)). The interesting case of (6.4) is \( k = 3 \) (since \( k = 1 \) is solved, and for \( k = 2, 4 \) we have the well-known even moments), when the exponential sum in (6.4) can be estimated. This in turn furnishes the following result on \( \mathcal{M}_3(s) \) (see [19, Theorem 5]): we have

\[
\mathcal{M}_3(s) = \int_1^\infty Z^3(x)x^{-s} \, dx = V_1(s) + V_2(s),
\]

say, where \( V_2(s) \) is regular for \( \sigma > 3/4 \), and for \( \sigma > 1 \) the function

\[
V_1(s) = (2\pi)^{1-s} \sqrt{\frac{2}{3}} \sum_{n=1}^\infty d_3(n)n^{-\frac{4}{3} - \frac{3}{2}s} \cos(3\pi n^\frac{2}{3} + \frac{1}{8}\pi)
\]
Mellin transforms of Hardy’s function

is regular. In connection with this, one may naturally pose the following problems (see [19], [20] for the first one):

1. Does there exist a constant $0 < c_3 < 1$ such that

$$F_3(T) = O(T^{c_3})? \tag{6.5}$$

Note that $c_3 = 1 + \varepsilon$ is trivial, since by the Cauchy-Schwarz inequality for integrals we easily obtain a better result, namely

$$\left| \int_0^T Z^3(t) \, dt \right| \leq \left( \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt \int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt \right)^{1/2} \ll T(\log T)^{5/2}$$

on using the well-known elementary bounds (see e.g., [9])

$$\int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt \ll T \log T, \quad \int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T \log^4 T.$$  \tag{6.6}

2. What is the least lower bound for $c_3$? It seems reasonable to conjecture that (6.5) holds with $c_3 = 3/4 + \varepsilon$ but does not hold for $c_3 < 3/4$, but so far no positive lower bounds for $c_3$ are known.

3. Does there exist a constant $0 < c_5 \leq 9/8$ such that

$$F_5(T) = O(T^{c_5})? \tag{6.6}$$

Namely (6.6) holds certainly with $c_5 = 9/8 + \varepsilon$ in view of [9, eq. (8.57)], and any value $c_5 \leq 9/8$ would be non-trivial and very interesting.

4. What is the least lower bound for $c_5$? Is it perhaps $c_5 = 1$, or is there enough cancelation in the terms of $Z^5(t)$ to produce a smaller exponent than unity? Naturally, similar questions could be asked for any odd $k > 3$, but they are quite difficult in the general case.

Note that by (2.10) of Lemma 2 we have, taking $f(x) = Z(x), g(x) = Z^4(x),

$$\int_1^\infty Z^5(x)x^{1-2\sigma} \, dx = \frac{1}{2\pi i} \int_{(\sigma)} M_1(s)\overline{M_4(s)} \, ds \tag{6.7}$$

for $\sigma (> 1)$ sufficiently large. Since integration by parts shows that (cf. (6.3))

$$\int_1^\infty Z^5(x)x^{1-2\sigma} \, dx = (2\sigma - 1) \int_1^\infty F_5(x)x^{-2\sigma} \, dx, \tag{6.8}$$

then if $F_5(x) \ll_{\varepsilon} x^{c_5+\varepsilon}$, this implies that the left-hand side of (6.8) converges for $\sigma > \frac{1}{2}(1 + c_5)$, and in this range the integral on the right-hand side of (6.7)
converges as well. The integral on the right-hand side of (6.7) can be dealt with by several techniques. One way is to use the Cauchy-Schwarz inequality, the defining relations for \(M_k\) and Lemma 3. However, so far I have not been able to improve on \(c_5 \leq 9/8 + \varepsilon\).

It may be also mentioned that in [19] it was shown that, if \(k = 1, 2, 3, 4\) and \(c > 1\) is fixed, then for \(U \gg x\) and \(\varepsilon > 0\) sufficiently small,

\[
(6.9) \quad Z^k(x) = \frac{1}{2\pi i} \int_{c-iU}^{c+iU} x^{s-1}M_k(s)\, ds + O_{\varepsilon,k}(x^{c-1}U^{-\varepsilon/2}).
\]

One may use (6.9) for various estimates involving moments of \(Z(t)\). For example, take \(U = C_1X, X/2 \leq x \leq 5X/2\), let \(\varphi(x) (\geq 0)\) be a smooth function supported in \([X/2, 5X/2]\) such that \(\varphi(x) = 1\) when \(X \leq x \leq 2X\) and \(\varphi^{(r)}(x) \ll_r X^{-r}\) for \(r \in \mathbb{N}\). Then (6.9) yields

\[
\int_{X/2}^{5X/2} \varphi(x)Z^k(x)\, dx = \frac{1}{2\pi i} \int_{c-iC_1X}^{c+iC_1X} M_k(s) \left( \int_{X/2}^{5X/2} \varphi(x)x^{s-1}\, dx \right)\, ds + O_{\varepsilon}(X^{c-\varepsilon/2}).
\]

However, for any \(r (\in \mathbb{N})\) repeated integration by parts yields

\[
\int_{X/2}^{5X/2} \varphi(x)x^{s-1}\, dx = (-1)^r \int_{X/2}^{5X/2} \varphi^{(r)}(x) \frac{x^{s+r-1}}{s(s+1)\cdots(s+r-1)}\, dx \ll_r \frac{x^\sigma-1}{|t|^r} \ll x^{-A}
\]

for any given large \(A > 0\) provided that \(|t| \geq X^\varepsilon\) if \(r = [(A + \sigma)/\varepsilon]\). This gives, on writing \(\sigma (> 1)\) in place of \(c\), for \(k = 1, 2, 3, 4\),

\[
(6.10) \quad \int_{X/2}^{5X/2} \varphi(x)Z^k(x)\, dx \ll_{\varepsilon} X^\sigma \max_{|t|\leq X^\varepsilon} |M_k(\sigma + it)| + X^{\sigma-\varepsilon/2}.
\]

The bound (6.10) shows essentially that the integral on the left-hand side is bounded by \(X^{\sigma+\varepsilon}\), if \(M_k(s)\) can be continued analytically to \(\Re s \geq \sigma\). This is in fact another way of seeing how the power moments of \(Z^k(t)\) and \(M_k(s)\) are connected. Probably (6.10) holds for \(k > 4\) as well.
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