FACTORIZATION HOMOLOGY AND CALCULUS À LA KONTSEVICH
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ABSTRACT. We use factorization homology over manifolds with boundaries in order to construct operations on Hochschild cohomology and Hochschild homology. These operations are parametrized by a colored operad involving disks on the surface of a cylinder defined by Kontsevich and Soibleman. The formalism of the proof extends without difficulties to a higher dimensional situation. More precisely, we can replace associative algebras by algebras over the little disks operad of any dimensions, Hochschild homology by factorization (also called topological chiral) homology and Hochschild cohomology by higher Hochschild cohomology. Note that our result works in categories of chain complexes but also in categories of modules over a commutative ring spectrum giving interesting operations on topological Hochschild homology and cohomology.

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Let $A$ be an associative algebra over a field $k$. A famous theorem by Hochschild Kostant and Rosenberg (see [HKR09]) suggests that the Hochschild homology of $A$ should be interpreted as the graded vector space of differential forms on the non commutative space “Spec$A$”. Similarly, the Hochschild cohomology of $A$ should be interpreted as the space of polyvector fields on Spec$A$.

If $M$ is a smooth manifold, let $\Omega_*(M)$ be the (homologically graded) vector space of de Rham differential forms and $V^*(M)$ be the vector space of polyvector fields (i.e. global sections of the exterior algebra on $TM$). This pair of graded vector spaces supports the following structure:

- The de Rham differential : $d : \Omega_*(M) \to \Omega_{*-1}(M)$.
- The cup product of vector fields $\cup : V^i(M) \otimes V^j(M) \to V^{i+j}(M)$.
- The Schouten-Nijenhuis bracket $[-, -] : V^i \otimes V^j \to V^{i+j-1}$.
- The cap product $\cap : \Omega_i \otimes V^j \to \Omega_{i-j}$ denoted by $\omega \otimes X \mapsto i_X \omega$.
- The Lie derivative $L_X : \Omega_i \otimes V^j \to \Omega_{i-1} \otimes V^j$ denoted by $\omega \otimes X \mapsto L_X \omega$. 


This structure satisfies some properties:

- The de Rham differential is indeed a differential, i.e. \( d \circ d = 0 \).
- The cup product and the Schouten-Nijenhuis bracket make \( V^*(M) \) into a Gerstenhaber algebra. More precisely, the cup product is graded commutative and the bracket satisfies the Jacobi identity and is a derivation in each variable with respect to the cup product.
- The cap product and the Lie derivative make \( \Omega^*(M) \) into a Gerstenhaber \( V^*(M) \)-module.

The Gerstenhaber module structure means that the following formulas are satisfied

\[
L_{[X,Y]} = [L_X, L_Y]
\]
\[
i_{[X,Y]} = [i_X, L_Y]
\]
\[
i_{X,Y} = i_X i_Y
\]
\[
L_{X,Y} = L_X i_Y + (-1)^{|X|} i_X L_Y
\]

where we denote by \([−, −]\), the (graded) commutator of operators on \( \Omega^*(M) \).

Finally we have the following formula called Cartan’s formula relating the Lie derivative, the exterior product and the de Rham differential:

\[
L_X = [d, i_X]
\]

Note that there is even more structure available in this situation. For example, the de Rham differential forms are equipped with a commutative differential graded algebra structure. However we will ignore this additional structure since it is not available in the non commutative case.

There is an operad \( \text{Calc} \) in graded vector spaces such that a \( \text{Calc} \)-algebra is a pair \((V^*, \Omega^*)\) together with all the structure we have just mentioned.

It turns out that any associative algebra gives rise to a \( \text{Calc} \)-algebra pair:

**0.1. Theorem.** Let \( A \) be an associative algebra over a field \( k \), let \( \text{HH}_*(A) \) (resp. \( \text{HH}^*(A) \)) denote the Hochschild homology (resp. cohomology) of \( A \), then the pair \((\text{HH}^*(A), \text{HH}_*(A))\) is an algebra over \( \text{Calc} \).

It is a natural question to try to lift this action to an action at the level of chains inducing the \( \text{Calc} \)-action in homology. This is similar to Deligne conjecture which states that there is an action of the operad of little 2-disks on Hochschild cochains of an associative algebra inducing the Gerstenhaber structure after taking homology.

Kontsevich and Soibelman in [KS09] have constructed a topological colored operad denoted \( K\mathcal{S} \) whose homology is \( \text{Calc} \). The purpose of this paper is to construct an action of \( K\mathcal{S} \) on the pair consisting of topological Hochschild cohomology and topological Hochschild homology.

More precisely, we prove the following theorem:

**0.2. Theorem.** Let \( A \) be an associative algebra in the category of chain complexes over a commutative ring or in the category of modules over a commutative symmetric ring spectrum. Then there is an algebra \((C, H)\) over \( K\mathcal{S} \) such that \( C \) is weakly equivalent to the (topological) Hochshchild cohomology of \( A \) and \( H \) is weakly equivalent to the (topological) Hochschild homology of \( A \).

We also prove a generalization of the above theorem to \( \mathcal{E}_d \)-algebras. Hochschild cohomology should be replaced by the derived endomorphisms of \( A \) seen as an \( \mathcal{E}_d \)-module over itself and Hochschild homology should be replaced by factorization homology (also called...
chiral homology). We construct obvious higher dimensional analogues of the operad $\mathcal{KS}$ and show that they describe the action of higher Hochschild cohomology on factorization homology.

The crucial ingredients in the proof is the swiss-cheese version of Deligne’s conjecture (see [Tho10] or [Gin13]) and a study of factorization homology on manifolds with boundaries as defined in [AFT12].

Note that one could imagine fancier versions of our main theorem using manifolds with corners instead of manifolds with boundaries (the relevant background can be found in [AFT12] and [Cal13]).

Plan of the paper.

- The first two sections are just background material about operads and model categories. We have proved the results whenever, we could not find a proper reference, however, this material makes no claim of originality.
- The third section is a definition of the little $d$-disk operad and the swiss-cheese operad. Again it is not original and only included to fix notations.
- The fourth and fifth sections are devoted to the construction of the operads $\mathcal{E}_d$ and $\mathcal{E}_d^\partial$. These are smooth versions of the little $d$ disk operad and the swiss-cheese operad.
- We show in the sixth section that $\mathcal{E}_d$ and $\mathcal{E}_d^\partial$ are weakly equivalent to the little $d$ disk operad and the swiss-cheese operad.
- In the seventh section we construct factorization homology of $\mathcal{E}_d$ and $\mathcal{E}_d^\partial$ algebras over a manifold (with boundary in the case of $\mathcal{E}_d^\partial$) and prove various useful results about it.
- In the eighth section, we construct a smooth analogue of the operad $\mathcal{KS}$ as well as its higher dimensional versions.
- Finally in the last section we construct an action of these operads on the pair consisting of higher Hochschild cohomology and factorization homology.

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Conventions. In this paper, we denote by $\mathbf{S}$ the category of simplicial set with its usual model structure. All our categories are implicitly assumed to be enriched in simplicial sets and all our functors are functors of simplicially enriched categories. We use the symbol $\simeq$ to denote a weak equivalence and $\cong$ to denote an isomorphism.

1. Colored operad

We recall the definition of a colored operad (also called a multicategory). In this paper we will restrict ourselves to the case of operads in $\mathbf{S}$ but the same definitions could be made in any symmetric monoidal category. Note that we use the word “operad” even when the operad has several colors. When we want to specifically talk about operads with only one color, we say “one-color operad”.

1.1. Definition. An operad in the category of simplicial sets consists of

- a set of colors $\text{Col}(\mathcal{M})$
for any finite sequence \( \{a_i\}_{i \in I} \) in \( \text{Col}(\mathcal{M}) \) indexed by a finite set \( I \), and any color \( b \), a simplicial set:
\[
\mathcal{M}(\{a_i\}_I; b)
\]
• a base point \(* \to \mathcal{M}(a; a)\) for any color \( a \)
• for any map of finite sets \( f : I \to J \), whose fiber over \( j \in J \) is denoted \( I_j \), compositions operations
\[
(\prod_{j \in J} \mathcal{M}(\{a_i\}_{i \in I_j}; b_j)) \times \mathcal{M}(\{b_j\}_{j \in J}; c) \to \mathcal{M}(\{a_i\}_{i \in I}; c)
\]

All these data are required to satisfy unitality and associativity conditions (see for instance \([\text{Lur11}]\) Definition 2.1.1.1.).

A map of operads \( \mathcal{M} \to \mathcal{N} \) is a map \( \text{Col}(\mathcal{M}) \to \text{Col}(\mathcal{N}) \) together with the data of images \( \mathcal{M}(\{a_i\}_I; b) \to \mathcal{N}(\{f(a_i)\}_I; f(b)) \) compatible with the compositions and units.

With the above definition, it is not clear that there is a category of operads since there is no set of finite sets. However it is easy to fix this by checking that the only data needed is the value \( \mathcal{M}(\{a_i\}_I; b) \) on sets \( I \) of the form \( \{1, \ldots, n\} \). The above definition has the advantage of avoiding unnecessary identification between finite sets.

1.2. Remark. Note that the last point of the definition can be used with an automorphism \( \sigma : I \to I \). Using the unitality and associativity of the composition structure, it is not hard to see that \( \mathcal{M}(\{a_i\}_I; b) \) supports an action of the group \( \text{Aut}(I) \). Other definitions of operads include this action as part of the structure.

1.3. Definition. Let \( \mathcal{M} \) be an operad. The underlying simplicial category of \( \mathcal{M} \) denoted \( \mathcal{M}^\text{u} \) is the simplicial category whose objects are the colors of \( \mathcal{M} \) and with
\[
\text{Map}_{\mathcal{M}}(m, n) = \mathcal{M}(\{m\}; n)
\]

1.4. Notation. Let \( \{a_i\}_{i \in I} \) and \( \{b_j\}_{j \in J} \) be two sequences of colors of \( \mathcal{M} \). We denote by \( \{a_i\}_{i \in I} \Join \{b_j\}_{j \in J} \) the sequence indexed over \( I \sqcup J \) whose restriction to \( I \) (resp. to \( J \)) is \( \{a_i\}_{i \in I} \) (resp. \( \{b_j\}_{j \in J} \)).

For instance if we have two colors \( a \) and \( b \), we can write \( a^\Join n \Join b^\Join m \) to denote the sequence \( \{a, \ldots, a, b, \ldots, b\}_{\{1, \ldots, n+m\}} \) with \( n \) a’s and \( m \) b’s.

Any symmetric monoidal category can be seen as an operad:

1.5. Definition. Let \((A, \otimes, I_A)\) be a small symmetric monoidal category enriched in \( \mathcal{S} \). Then \( A \) has an underlying operad \( UA \) whose colors are the objects of \( A \) and whose spaces of operations are given by
\[
UA(\{a_i\}_I; b) = \text{Map}_A(\bigotimes_{i \in I} a_i, b)
\]

1.6. Definition. We denote by \( \text{Fin} \) the category whose objects are nonnegative integers \( n \) and whose morphisms \( n \to m \) are maps of finite sets
\[
\{1, \ldots, n\} \to \{1, \ldots, m\}
\]
We allow ourselves to write \( i \in n \) when we mean \( i \in \{1, \ldots, n\} \).
The construction $A \mapsto \mathcal{U}A$ sending a symmetric monoidal category to an operad has a left adjoint that we define now. The underlying category of the left adjoint applied to $\mathcal{M}$ is $\mathbf{M}$. For this reason, we can safely use the letter $\mathbf{M}$ to denote that symmetric monoidal category.

1.7. **Definition.** Let $\mathcal{M}$ be an operad, the objects of the free symmetric monoidal category $\mathbf{M}$ are given by

$$\text{Ob}(\mathbf{M}) = \bigsqcup_{n \in \text{Ob}(\text{Fin})} \text{Col}(\mathcal{M})^n$$

Morphisms are given by

$$\mathbf{M}((\{a_i\}_{i \in n}, \{b_j\}_{j \in m}) = \bigsqcup_{f: n \to m} \prod_{i \in m} \mathcal{M}(\{a_j\}_{j \in f^{-1}(i)}; b_i)$$

It is easy to check that there is a functor $\mathbf{M}^2 \to \mathbf{M}$ which on objects is

$$((\{a_i\}_{i \in n}, \{b_j\}_{j \in m}) \mapsto \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$$

1.8. **Proposition.** This functor can be extended to a symmetric monoidal structure on $\mathbf{M}$. □

Let $\mathbf{C}$ be a symmetric monoidal simplicial category.

An $\mathcal{M}$-algebra in $\mathbf{C}$ is a map of operads $\mathcal{M} \to \mathcal{U}\mathbf{C}$. By definition, an algebra over $\mathcal{M}$ induces a (symmetric monoidal) functor $\mathbf{M} \to \mathbf{C}$. We will use the same notation for the two objects and allow ourselves to switch between them without mentioning it. We denote by $\mathbf{C}[\mathcal{M}]$ the category of $\mathcal{M}$-algebras in $\mathbf{C}$.

**Right modules over operads.**

1.9. **Definition.** Let $\mathcal{M}$ be an operad. A **right $\mathcal{M}$-module** is a simplicial functor $R: \mathcal{M}^{\text{op}} \to \mathbf{S}$

1.10. **Remark.** If $\mathcal{O}$ is a one-color operad, it is easy to verify that the category of right modules over $\mathcal{O}$ in the above sense is isomorphic to the category of right modules over $\mathcal{O}$ in the usual sense (i.e. a right module over the monoid $\mathcal{O}$ with respect to the monoidal structure on symmetric sequences given by the composition product).

1.11. **Proposition.** Assume that $\mathbf{C}$ is cocomplete. Let $\alpha: \mathcal{M} \to \mathcal{N}$ a map of operads, the forgetful functor $\alpha^*: \mathbf{C}[\mathcal{N}] \to \mathbf{C}[\mathcal{M}]$ has a left adjoint $\alpha_!$.

For $A \in \mathbf{C}[\mathcal{M}]$, the value at the color $n \in \text{Col}(\mathcal{N})$ of $\alpha_!A$ is given by

$$\alpha_!A(n) = \mathcal{N}(\alpha(-), n) \otimes_{\mathcal{M}} A(-)$$

□

1.12. **Definition.** We keep the notations of the previous proposition. The $\mathcal{N}$-algebra $\alpha_!(A)$ is called the **operadic left Kan extension** of $A$ along $\alpha$.

2. **Homotopy theory of operads and modules**

In this section we collect a few facts about the homotopy theory in categories of algebras in a reasonable model category. The case of one-color operads has been extensively studied in [Fre09], however, we needed to apply these results with colored operads so we had to reprove some of the results.
2.1. **Definition.** An operad \( \mathcal{M} \) is said to be \( \Sigma \)-cofibrant if for any sequence of colors \( \{a_i\}_{i \in \mathbb{N}} \) and any color \( b \), the space \( \mathcal{M}(\{a_i\}; b) \) is a cofibrant object in \( S^\Sigma \) with its projective model structure for the \( \Sigma \)-action described in 1.2.

Similarly, a right module \( P \) over \( \mathcal{M} \) is \( \Sigma \)-cofibrant if for any sequence of colors \( \{a_i\}_{i \in \mathbb{N}} \), the \( \Sigma \)-simplicial set \( P(\{a_i\}) \) is cofibrant in \( S^\Sigma \).

2.2. **Remark.** For \( G \) a finite group, a \( G \)-simplicial set is cofibrant if the \( G \)-action is free. In this work, anytime, we claim that a simplicial set is \( G \)-cofibrant, we implicitly use this fact.

2.3. **Definition.** A weak equivalence between operads is a morphism of operad \( f : \mathcal{M} \to \mathcal{N} \) which satisfies:

- (Homotopical fully faithfulness) For each \( \{m_i\}_{i \in I} \) a finite set of colors of \( \mathcal{M} \) and each \( m \) a color of \( \mathcal{M} \), the map \( \mathcal{M}(\{m_i\}; m) \to \mathcal{N}(\{f(m_i)\}; f(m)) \)

is a weak equivalence.

- (Essential surjectivity) The underlying map of simplicial categories \( \mathcal{M} \to \mathcal{N} \) is essentially surjective (i.e. it is such when we apply \( \pi_0 \) to each space of maps).

2.4. **Remark.** The homotopy theory of simplicial operads with respect to the above definition of weak equivalences can be structured into a model category (see [CM11] or [Rob11]) but we will not use this fact in this work.

Note that the essential surjectivity condition is automatically satisfied if the map is an isomorphism on the set of colors.

**Model categories with a good theory of algebras.** Note that all our model categories are assumed to be simplicial and tensored over \( S \), we denote the tensor product \( S \times C \to C \) by the symbol \( \otimes \).

2.5. **Definition.** A cofibrantly generated symmetric monoidal simplicial model category \( (C, \otimes, I) \) has a good theory of algebras (resp. a good theory of algebras over \( \Sigma \)-cofibrant operads) if:

- For any operad \( \mathcal{M} \) (resp. \( \Sigma \)-cofibrant operad) in \( S \), the category \( C[\mathcal{M}] \) of \( \mathcal{M} \)-algebras in \( C \) has a model category structure for which weak equivalences and fibrations are created by the forgetful functors \( C[\mathcal{M}] \to C[\text{Col}(\mathcal{M})] \).

- If \( \alpha : \mathcal{M} \to \mathcal{N} \) is a map of operad (resp. \( \Sigma \)-cofibrant operads), the adjunction

\[
\alpha_! : C[\mathcal{M}] \rightleftarrows C[\mathcal{N}] : \alpha^*
\]

is a Quillen adjunction. Moreover, it is a Quillen equivalence if \( \alpha \) is a weak equivalence.

- For any operad \( \mathcal{M} \) (resp. \( \Sigma \)-cofibrant operad) in \( S \), the right adjoint \( C[\text{Col}(\mathcal{M})] \rightleftarrows C[\mathcal{M}] \) preserves cofibrations.

Let us mention two families of examples where these conditions are satisfied:

**Berger-Moerdijk model structure.**

2.6. **Theorem.** Let \( C \) be a left proper symmetric monoidal simplicial cofibrantly generated model category. Assume that \( C \) has a monoidal fibrant replacement functor and a cofibrant unit. Then \( C \) has a good theory of algebras over \( \Sigma \)-cofibrant operads.

**Proof.** The proof is essentially done in [BM05]. The idea is that \( H = \text{Sing}([0,1]) \) is a cocommutative monoid in \( S \), therefore for any fibrant \( \mathcal{M} \)-algebra \( A \), the object \( A^H \) is a path object in \( C[\mathcal{M}] \). \( \square \)
2.7. Remark. For instance $S$ and $\text{Top}$ obviously satisfy the conditions. If $R$ is a commutative ring, the category $\text{Ch}_{\geq 0}(R)$ with its projective model structure (i.e., the model structure for which weak equivalences are quasi-isomorphisms and fibrations are degreewise epimorphisms) satisfies the conditions.

If $C$ satisfies the conditions of the theorem, and $I$ is any small simplicial category. Then $\text{Fun}(I, C)$ with the objectwise tensor product and projective model structure also satisfies the conditions.

Algebras in categories of modules over a ring spectrum. In a symmetric monoidal category $C$, there is a symmetric monoidal structure on arrows of $C$ called pushout-product. If $f : A \to B$ and $g : C \to D$ are two maps, their pushout-product denoted $f \Box g$ is the obvious map:

$$A \otimes D \sqcup A \otimes C \to B \otimes C \to B \otimes D$$

The following definition is due to Lurie (see [Lur11]):

2.8. Definition. Let $C$ be a cofibrantly generated symmetric monoidal model category. A map $f : X \to Y$ is said to be a power cofibration if, for each $n$, the map $f^{\Box n}$ is a cofibration in $C^{\Sigma^n}$ with the projective model structure.

If $E$ is a commutative monoid in $\text{Spec}$, we define $\text{Mod}_E$ to be the category of right modules over $E$ equipped with the positive model structure (see [Sch07]). This category is a closed symmetric monoidal left proper simplicial model category. If $Z$ is a spectrum, we denote by $L_Z \text{Mod}_E$ the $Z$-Bousfield localization of $\text{Mod}_E$. The underlying category of $L_Z \text{Mod}_E$ is $\text{Mod}_E$. The cofibrations of $L_Z \text{Mod}_E$ are the cofibrations of $\text{Mod}_E$ and the weak equivalences are the $Z$-equivalences. The category $L_Z \text{Mod}_E$ is a symmetric monoidal simplicial model category (see [Bar10] for more details about Bousfield localizations).

2.9. Proposition. In the category $\text{Mod}_E$, any cofibration is a power cofibration. The same is true for the positive model structure of $L_Z \text{Mod}_E$ for any $Z$.

Proof. The appendix of [Per13] proves it in the case if $E$ is the sphere spectrum. To prove the result for $\text{Mod}_E$, it suffices to check it for generating cofibrations. Generating cofibrations in $\text{Mod}_E$ can be chosen of the form $f \otimes E$ where $f$ is a cofibration in $\text{Spec}$, therefore, the result follows from the case of $\text{Spec}$.

To take care of the $Z$ local case, it suffices to notice that, for any finite group $G$, we have the identity as model categories:

$$(L_Z \text{Mod}_E)^G = L_Z(\text{Mod}_E^G)$$

indeed in both cases the weak equivalences are the $Z$-equivalences and the generating cofibrations are the maps $G \otimes f$ where $f$ is a generating cofibration of $\text{Mod}_E$. Since cofibrations in $L_Z(\text{Mod}_E^G)$ are the cofibrations of $\text{Mod}_E^G$, the result follows from the non-localized case. $\square$

2.10. Proposition. Let $E$ be a commutative symmetric ring spectrum and $Z$ be any symmetric spectrum. Then the positive model structure on $\text{Mod}_E$ has a good theory of algebras. Similarly, the Bousfield localization $L_Z \text{Mod}_E$ with the positive model structure has a good theory of algebras.

Proof. The paper [EM06] only deals with modules over the sphere spectrum but it is easy to check that their proof can be adapted to this more general situation. The main ingredient of the proof of [EM06] is 2.9 which we have proved to be true in $L_Z \text{Mod}_E$. $\square$
Homotopy invariance of operadic coend. We want to study the homotopy invariance of coends of the form $P \otimes_{\mathcal{M}} A$ for $A$ an $\mathcal{M}$-algebra and $P$ a right module over $\mathcal{M}$.

The following definition is due to Muro (see [Mur13]).

2.11. Definition. Let $(\mathcal{C}, \otimes, \mathbb{I})$ be a symmetric monoidal model category. We say that an object $X$ of $\mathcal{C}$ is pseudo-cofibrant if tensoring with $X$ preserves cofibrations and trivial cofibrations.

2.12. Proposition. We have:

- Cofibrant objects are pseudo-cofibrant.
- The unit $\mathbb{I}$ is pseudo-cofibrant.
- A tensor product of pseudo-cofibrant objects is pseudo-cofibrant.
- If $\mathcal{C}$ is a simplicial symmetric monoidal model category, objects of the form $K \otimes \mathbb{I}$, where $K$ is any simplicial set, are pseudo-cofibrant.
- If $X \to Y$ is a cofibration and $X$ is pseudo-cofibrant, then $Y$ is pseudo-cofibrant.

Proof. Only the last claim is not entirely trivial. It follows easily from an application of the pushout product axiom (see [Mur13] for a proof).

2.13. Lemma. Let $\mathcal{C}$ be a symmetric monoidal model category with a good theory of algebras (resp. with a good theory of algebras over $\Sigma$-cofibrant operads). Let $\mathcal{M}$ be an operad (resp. $\Sigma$-cofibrant operad) and let $\mathcal{M}$ be the free symmetric monoidal category on $\mathcal{M}$. Let $A : \mathcal{M} \to C$ be an algebra. If $A$ is cofibrant, then for each object $m$ of $\mathcal{M}$, the value of $A$ at $m$ is pseudo-cofibrant. Moreover, if the unit of $\mathcal{C}$ is cofibrant, the value of $A$ at each object of $\mathcal{M}$ is cofibrant.

Proof. By assumption, $A_0 \to A$ is a cofibration in $\mathcal{C}[\text{Col}(\mathcal{M})]$ where $A_0$ is the initial $\mathcal{M}$-algebra. Let $m$ be a color of $\mathcal{M}$. $A_0(m)$ is $\mathcal{M}(\emptyset; m) \otimes \mathbb{I}$ which is pseudo-cofibrant and even cofibrant if $\mathbb{I}$ is cofibrant. This implies that $A(m)$ is pseudo-cofibrant and even cofibrant if $\mathbb{I}$ is cofibrant. For a general object $x$ of $\mathcal{M}$, $A(x)$ is a tensor product of $A(m)$’s for various colors $m$ of $\mathcal{M}$ hence it is pseudo-cofibrant and cofibrant if $\mathbb{I}$ is cofibrant.

2.14. Proposition. Let $\mathcal{C}$ be a symmetric monoidal model category with a good theory of algebras (resp. with a good theory of algebras over $\Sigma$-cofibrant operads). Let $\mathcal{M}$ be an operad (resp. $\Sigma$-cofibrant operad) and let $\mathcal{M}$ be the free symmetric monoidal category on $\mathcal{M}$. Let $A : \mathcal{M} \to C$ be an algebra. Then

1. Let $P : \mathcal{M}^{\text{op}} \to \mathcal{S}$ be a right module (resp. $\Sigma$-cofibrant right module). Then $P \otimes_{\mathcal{M}} -$ preserves weak equivalences between cofibrant $\mathcal{M}$-algebras.
2. If $A$ is a cofibrant algebra, the functor $- \otimes_{\mathcal{M}} A$ is a left Quillen functor from right modules over $\mathcal{M}$ to $\mathcal{C}$.
3. Moreover the functor $- \otimes_{\mathcal{M}} A$ preserves all weak equivalences between right modules (resp. $\Sigma$-cofibrant right modules).

Proof. For $P$ any simplicial functor $\mathcal{M}^{\text{op}} \to \mathcal{S}$, we denote by $\mathcal{M}_P$ the operad whose colors are $\text{Col}(\mathcal{M}) \sqcup \infty$ and whose spaces of operations are as follows:

$\mathcal{M}_P(\{m_1, \ldots, m_k\}, n) = \mathcal{M}(\{m_1, \ldots, m_k\}, n)$ if $\infty \notin \{m_1, \ldots, m_k, n\}$

$\mathcal{M}_P(\{m_1, \ldots, m_k\}; \infty) = P(\{m_1, \ldots, m_k\})$ if $\infty \notin \{m_1, \ldots, m_k\}$

$\mathcal{M}_P(\{\infty\}; \infty) = *$

$\mathcal{M}_P(\{m_1, \ldots, m_k\}, n) = \emptyset$ in any other case

It is easy to see that there is an operad map $\alpha_P : \mathcal{M} \to \mathcal{M}_P$. Moreover by 1.11 we have $\text{ev}_\infty(\alpha_P)_! A = P \otimes_{\mathcal{M}} A$. 

□
where $ev_\infty$ denotes the functor that evaluate an $\mathcal{M}_P$-algebra at the color $\infty$.

Proof of the first claim. If $A \rightarrow B$ is a weak equivalence between cofibrant $\mathcal{M}$-algebras, then $(\alpha_P)_!A$ is weakly equivalent to $(\alpha_P)_!B$ since $(\alpha_P)_!$ is a left Quillen functor. To conclude the proof, we observe that the functor $ev_\infty$ preserves all weak equivalences.

Proof of the second claim. To show that $P \mapsto P \otimes_M A$ is left Quillen it suffices to check that it sends generating (trivial) cofibrations to (trivial) cofibrations.

For $m \in \text{Ob}(M)$, denote by $\iota_m$ the functor $S \mapsto \text{Fun}(\text{Ob}(M), S)$ sending $X$ to the functor sending $m$ to $X$ and everything else to $\emptyset$. Denote by $F_M$ the left Kan extension functor

$$F_M : \text{Fun}(\text{Ob}(M)\text{op}, S) \rightarrow \text{Fun}(M\text{op}, S)$$

We can take as generating (trivial) cofibrations the maps of the form $F_{M^I} (F_{M^J})$ for $I$ (resp. $J$), the generating cofibrations (resp. trivial cofibrations) of $S$. We have:

$$F_{M^I} \otimes_M A \cong I \otimes A(m)$$

Since $A$ is cofibrant as an algebra its value at each object of $M$ is pseudo-cofibrant (see 2.13). Moreover, the left tensoring $S \times C$ is a Quillen bifunctor by hypothesis, therefore $F_{M^I} \otimes_M A$ consists of cofibrations. Similarly, $F_{M^J} \otimes_M A$ consists of trivial cofibrations.

Proof of the third claim. Let $P \rightarrow Q$ be a weak equivalence between functors $M^{\text{op}} \rightarrow S$. This induces a weak equivalence between operads $\beta : \mathcal{M}_P \rightarrow \mathcal{M}_Q$. It is clear that $\alpha_Q = \beta \circ \alpha_P$, therefore $(\alpha_Q)_! A = \beta_! (\alpha_P)_! A$. We apply $\beta^*$ to both side and get

$$\beta^* \beta_! (\alpha_P)_! A = \beta^* (\alpha_Q)_! A$$

Since $(\alpha_P)_! A$ is cofibrant and $\beta^*$ preserves all weak equivalences, the adjunction map $(\alpha_P)_! A \rightarrow \beta^* \beta_! (\alpha_P)_! A$ is a weak equivalence by definition of a Quillen equivalence. Therefore the obvious map

$$(\alpha_P)_! A \rightarrow \beta^* (\alpha_Q)_! A$$

is a weak equivalence.

If we evaluate this at the color $\infty$, we find a weak equivalence

$$P \otimes_M A \rightarrow Q \otimes_M A$$

Operadic vs categorical homotopy left Kan extension. Proposition 1.11 insures that for a map of operad $\alpha : \mathcal{M} \rightarrow \mathcal{N}$, the operadic left Kan extension $\alpha_!$ applied to an algebra $A$ over $\mathcal{M}$ coincides with the left Kan extension of the functor $A : \mathcal{M} \rightarrow C$. We call the latter the categorical left Kan extension of $A$.

It is not clear that the derived functors of these two different left Kan extension coincide. Indeed, in the case of the derived operadic left Kan extension, we take a cofibrant replacement of the $\mathcal{M}$-algebra $A$ as an algebra and in the case of the categorical left Kan extension we take a cofibrant replacement of the functor $A : \mathcal{M} \rightarrow C$ in the category of functors with the projective model structure. However, it turns out that in good cases, the two constructions coincide.

2.15. Proposition. Assume $C$ has a good theory of algebras (resp. a good theory of algebras over $\Sigma$-cofibrant operads) and assume that $C$ has a cofibrant unit. Let $\mathcal{M} \xrightarrow{\alpha} \mathcal{N}$ be a morphism of simplicial operads (resp. $\Sigma$-cofibrant operads). Let $A$ be an algebra over $\mathcal{M}$. The derived operadic left Kan extension $\mathbb{L}_\alpha(A)$ is weakly equivalent to the homotopy left Kan extension of $A : \mathcal{M} \rightarrow C$ along the induced map $\mathcal{M} \rightarrow \mathcal{N}$.
Proof. Let $QA \to A$ be a cofibrant replacement of $A$ as an $\mathcal{M}$-algebra. The value at $n$ of the homotopy left Kan extension of $A$ can be computed as the geometric realization of the Bar construction

$$B_\bullet(\mathcal{N}(\alpha-, n), \mathcal{M}, QA)$$

By 2.13, since $\mathcal{I}_C$ is cofibrant, $QA$ is objectwise cofibrant. Therefore, the bar construction is Reedy-cofibrant and computes the categorical left Kan extension of $A$.

We can rewrite this simplicial object as

$$B_\bullet(\mathcal{N}(\alpha-, n), \mathcal{M}, \mathcal{M}) \otimes \mathcal{M} QA$$

The geometric realization is

$$|B_\bullet(\mathcal{N}(\alpha-, n), \mathcal{M}, \mathcal{M})| \otimes \mathcal{M} QA$$

It is a classical fact that the map

$$|B_\bullet(\mathcal{N}(\alpha-, n), \mathcal{M}, \mathcal{M})| \to \mathcal{N}(\alpha-, n)$$

is a weak equivalence of functors on $\mathcal{M}$. Therefore by 2.14, the Bar construction is weakly equivalent to $\alpha_! QA$ which is exactly the derived operadic left Kan extension of $A$. □

Note that in $L_Z\mathcal{M}od_E$, the unit is not cofibrant. Nevertheless the result is also true:

2.16. Proposition. Let $A$ be an object of $L_Z\mathcal{M}od_E[\mathcal{M}]$. The derived operadic left Kan extension $\alpha_!(A)$ is weakly equivalent to the homotopy left Kan extension of $A : \mathcal{M} \to \mathcal{C}$ along the induced map $\mathcal{M} \to \mathcal{N}$.

Proof. We can consider the bar construction as a simplicial object of $L_Z\mathcal{M}od_E$, the $Z$-Bousfield localization of the category $\mathcal{M}od_E$ with the absolute model structure (see [Sch07]). In that category, the unit is cofibrant, therefore, the previous argument applies and shows that the bar construction is Reedy cofibrant. Since the weak equivalences are the same in $L_Z\mathcal{M}od_E$ and $L_Za\mathcal{M}od_E$, the bar construction computes the derived categorical left Kan extension of $A$. The rest of the argument of the previous proposition works. □

3. The little $d$-disk operad

In this section, we give a traditional definition of the little $d$-disk operad $\mathcal{D}_d$ as well as a definition of the swiss-cheese operad $\mathcal{SC}_d$ which we denote $\mathcal{D}_{d-1}$. The swiss-cheese operad, originally defined by Voronov (see [Vor99] for a definition when $d = 2$ and [Tho10] for a definition in all dimensions), is a variant of the little $d$-disk operad which describes the action of an $\mathcal{D}_d$-algebra on an $\mathcal{D}_{d-1}$-algebra.

Space of rectilinear embeddings. Let $D$ denote the open disk of dimension $d$, $D = \{x \in \mathbb{R}^d, \|x\| < 1\}$.

3.1. Definition. Let $U$ and $V$ be connected subsets of $\mathbb{R}^d$, let $i_U$ and $i_V$ denote the inclusion into $\mathbb{R}$. We say that $f : U \to V$ is a rectilinear embedding if there is an element $L$ in the subgroup of $\text{Aut}(\mathbb{R}^d)$ generated by translation and dilations with positive factor such that

$$i_V \circ f = L \circ i_U$$

We extend this definition to disjoint unions of open subsets of $\mathbb{R}^d$.

3.2. Definition. Let $U_1, \ldots, U_n$ and $V_1, \ldots, V_m$ be finite families of connected subsets of $\mathbb{R}^d$. The notation $U_1 \sqcup \ldots \sqcup U_n$ denotes the coproduct of $U_1, \ldots, U_n$ in the category of topological spaces. We say that a map from $U_1 \sqcup \ldots \sqcup U_n$ to $V_1 \sqcup \ldots \sqcup V_m$ is a rectilinear embedding if it satisfies the following properties:
(1) Its restriction to each component can be factored as $U_i \to V_j \to V_1 \sqcup \ldots \sqcup V_m$ where the second map is the obvious inclusion and the first map is a rectilinear embedding $U_i \to V_j$.

(2) The underlying map of sets is injective.

We denote by $\text{Emb}_{\text{lin}}(U_1 \sqcup \ldots \sqcup U_n, V_1 \sqcup \ldots \sqcup V_m)$ the subspace of $\text{Map}(U_1 \sqcup \ldots \sqcup U_n, V_1 \sqcup \ldots \sqcup V_m)$ whose points are rectilinear embeddings.

Observe that rectilinear embeddings are stable under composition.

The $d$-disk operad.

3.3. Definition. The linear $d$-disk operad, denoted $\mathcal{D}_d$, is the operad in topological spaces whose $n$-th space is $\text{Emb}_{\text{lin}}(D \sqcup^{\ell n}, D)$ with the composition induced from the composition of rectilinear embeddings.

There are variants of this definition but they are all equivalent to this one. In the above definition $\mathcal{D}_d$ is an operad in topological spaces. By applying the functor $\text{Sing}$, we get an operad in $S$. We use the same notation for the topological and the simplicial operad.

The swiss-cheese operad. As before, we denote by $D$, the $d$-dimensional disk and by $H$ the $d$-dimensional half-disk

$$H = \{x = (x_1,\ldots,x_d), \|x\| < 1, x_d \geq 0\}$$

3.4. Definition. The linear $d$-dimensional swiss-cheese operad, denoted $\mathcal{D}_d^\partial$, has two colors $z$ and $h$ and its mapping spaces are

$$\mathcal{D}_d^\partial(z \oslash^{\ell n} z, z) = \text{Emb}_{\text{lin}}(D \sqcup^{\ell n}, D)$$
$$\mathcal{D}_d^\partial(z \oslash^{\ell n} h \oslash^{\ell m} h, h) = \text{Emb}_{\text{lin}}(D \sqcup^{\ell n} \sqcup H \sqcup^{\ell m}, H)$$

where the $\partial$ superscript means that we restrict to embeddings preserving the boundary.

3.5. Proposition. The full suboperad of $\mathcal{D}_d^\partial$ on the color $z$ is isomorphic to $\mathcal{D}_d$ and the full suboperad on the color $h$ is isomorphic to $\mathcal{D}_{d-1}$.

Proof. Easy. □

3.6. Proposition. The evaluation at the center of the disks induces weak equivalences

$$\mathcal{D}_d(n) \xrightarrow{\cong} \text{Conf}(n, D)$$
$$\mathcal{D}_d^\partial(z \oslash^{\ell n} h \oslash^{\ell m}, h) \xrightarrow{\cong} \text{Conf}(m, \partial H) \times \text{Conf}(n, H - \partial H)$$

Proof. These maps are Hurewicz fibration whose fibers are contractible. □

4. Homotopy pullback in $\text{Top}_W$

The material of this section can be found in [And10]. We have included it mainly for the reader’s convenience and also to give a proof of 4.5 which is mentioned without proof in [And10].
**Homotopy pullback in $\text{Top}$.** Let us start by recalling the following well-known proposition:

4.1. **Proposition.** Let

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow g \\
Y & \rightarrow & Z
\end{array} \]

be a diagram in $\text{Top}$. The homotopy pullback of that diagram can be constructed as the space of triples $(x, p, y)$ where $x$ is a point in $X$, $y$ is a point in $Y$ and $p$ is a path from $f(x)$ to $g(y)$ in $Z$. □

**Homotopy pullback in $\text{Top}_W$.** Let $W$ be a topological space. There is a model structure on $\text{Top}_W$ the category of topological spaces over $W$ in which cofibrations, fibrations and weak equivalences are reflected by the forgetful functor $\text{Top}_W \rightarrow \text{Top}$. We want to study homotopy pullbacks in $\text{Top}_W$.

We denote a space over $W$ by a single capital letter like $X$ and we write $p_X$ for the structure map $X \rightarrow W$. Let $I = [0, 1]$, for $Y$ an object of $\text{Top}_W$, we denote by $Y^I$ the cotensor in the category $\text{Top}_W$. Concretely, $Y^I$ is the space of paths in $Y$ whose image in $W$ is a constant path.

4.2. **Definition.** Let $f : X \rightarrow Y$ be a map in $\text{Top}_W$. We denote by $Nf$ the following pullback in $\text{Top}_W$:

\[ \begin{array}{ccc}
Nf & \xrightarrow{p_f} & Y^I \\
\downarrow & & \downarrow \text{id} \\
X & \xrightarrow{f} & Y
\end{array} \]

Concretely, $Nf$ is the space of pairs $(x, p)$ where $x$ is a point in $X$ and $p$ is a path in $Y$ whose value at 0 is $f(x)$ and lying over a constant path in $W$.

We denote by $p_f$, the map $Nf \rightarrow Y$ sending a path to its value at 1.

4.3. **Proposition.** Let

\[ \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow g \\
Y & \rightarrow & Z
\end{array} \]

be a diagram in $\text{Top}_W$ in which $X$ and $Z$ are fibrant (i.e. the structure maps $p_X$ and $p_Z$ are fibrations) then the pullback of the following diagram in $\text{Top}_W$ is a model for the homotopy pullback:

\[ \begin{array}{ccc}
Nf & \xrightarrow{p_f} & Y^I \\
\downarrow & & \downarrow \text{id} \\
Y & \xrightarrow{p_f} & Z
\end{array} \]

Concretely, this proposition is saying that the homotopy pullback is the space of triple $(x, p, y)$ where $x$ is a point in $X$, $y$ is a point in $Y$ and $p$ is a path in $Z$ between $f(x)$ and $g(y)$ lying over a constant path in $W$. 
Proof of the proposition. The proof is similar to the analogous result in $\text{Top}$, it suffices to check that the map $p_f : Nf \to Z$ is a fibration in $\text{Top}_W$ which is weakly equivalent to $X \to Z$. Since the category $\text{Top}_W$ is right proper, a pullback along a fibration is always a homotopy pullback. □

From now on when we talk about a homotopy pullback in the category $\text{Top}_W$, we mean the above specific model.

4.4. Remark. The map from the homotopy pullback to $Y$ is a fibration. If $X$, $Y$, $Z$ are fibrants, the homotopy pullback can be computed in two different ways but they are clearly isomorphic. In particular, the map from the homotopy pullback to $X$ is also a fibration.

Comparison of homotopy pullbacks in $\text{Top}$ and in $\text{Top}_W$. For a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
| & | & | \\
Y & \rightarrow & Z
\end{array}
\]

in $\text{Top}$ (resp. $\text{Top}_W$), we denote by $\text{hpb}(X \to Z \leftarrow Y)$ (resp. $\text{hpb}_W(X \to Z \leftarrow Y)$) the above model of homotopy pullback in $\text{Top}$ (resp. $\text{Top}_W$).

Note that there is an obvious inclusion

$$\text{hpb}_W(X \to X \leftarrow Y) \to \text{hpb}(X \to Z \leftarrow Y)$$

which sends a path (which happens to be constant in $W$) to itself.

4.5. Proposition. Let $W$ be a topological space and $X \to Y \leftarrow Z$ be a diagram in $\text{Top}_W$ in which the structure maps $Z \to W$ and $Y \to W$ are fibrations, then the inclusion

$$\text{hpb}_W(X \to Y \leftarrow Z) \to \text{hpb}(X \to Y \leftarrow Z)$$

is a weak equivalence.

Proof. ¹ Let us consider the following commutative diagram

\[
\begin{array}{ccc}
\text{hpb}_W(X \to Y \leftarrow Z) & \xrightarrow{\text{hpb}_W(X \to Y \leftarrow Z)} & X \\
| & | & | \\
\text{hpb}_W(Y \to Y \leftarrow Z) & \xrightarrow{\text{hpb}_W(Y \to Y \leftarrow Z)} & Y \\
| & | & | \\
W & \rightarrow & W^I
\end{array}
\]

The map $\text{hpb}(Y \to Y \leftarrow Z) \to W^I$ sends a triple $(y, p, z)$ to the image of the path $p$ in $W$. The map $W \to W^I$ sends a point in $W$ to the constant map at that point. All other maps should be clear.

It is straightforward to check that each square is cartesian.

The category $\text{Top}_W$ is right proper. This implies that a pullback along a fibration is always a homotopy pullback.

Now we make the following three observations:

1. The map $\text{hpb}(Y \to Y \leftarrow Z) \to W^I$ is a fibration. Indeed it can be identified with the obvious map $Y^I \times_Y Z \to W^I \times_W W$ and $Y^I \to W^I$, $Z \to W$ and $Y \to W$ are fibrations. This implies that the bottom square is homotopy cartesian.

¹The following proof is due to Ricardo Andrade
The middle row of the diagram hopb\(_W(Y \to Y \leftarrow Z) \to Y\) is a fibration because of remark 4.4. A priori it is a fibration in Top\(_W\) but this is equivalent to being a fibration in Top. This implies that the big horizontal rectangle is homotopy cartesian.

If we combine (2) and (3) we find that the top left-hand side square is homotopy cartesian. If we combine that with (1), we find that the big horizontal rectangle is homotopy cartesian. The map \(W \to W^I\) is a weak equivalence. Therefore the map

\[
\text{hopb}_W(X \to Y \leftarrow Z) \to \text{hopb}(X \to Y \leftarrow Z)
\]

is a weak equivalence as well. □

5. Embeddings between structured manifolds

This section again owes a lot to [And10]. In particular, the definition 5.3 can be found in that reference. We then make analogous definitions of embedding spaces for framed manifolds with boundary.

**Topological space of embeddings.** There is a topological category whose objects are \(d\)-manifolds possibly with boundary and mapping object between \(M\) and \(N\) is \(\text{Emb}(M, N)\), the topological space of smooth embeddings with the weak \(C^1\) topology. The reader should look at [Hir76] for a definition of this topology. We want to emphasize that this topology is metrizable, in particular \(\text{Emb}(M, N)\) is paracompact.

5.1. **Remark.** If one is only interested in the homotopy type of this topological space. One could work with the \(C^r\)-topology for any \(r\) (even \(r = \infty\)) instead of the \(C^1\)-topology. The choice of taking the weak (as opposed to strong topology) however is a serious one. The two topologies coincide when the domain is compact. However the strong topology does not have continuous composition maps

\[
\text{Emb}(M, N) \times \text{Emb}(N, P) \to \text{Emb}(M, P)
\]

when \(M\) is not compact.

**Embeddings between framed manifolds.** For a manifold \(M\) possibly with boundary, we denote by \(\text{Fr}(TM) \to M\) the principal \(\text{GL}(d)\)-bundle of frames of the tangent bundle of \(M\).

5.2. **Definition.** A framed \(d\)-manifold is a pair \((M, \sigma_M)\) where \(M\) is a \(d\)-manifold and \(\sigma_M\) is a smooth section of the principal \(\text{GL}(d)\)-bundle \(\text{Fr}(TM)\).

If \(M\) and \(N\) are two framed \(d\)-manifolds, we define a space of framed embeddings denoted by \(\text{Emb}_f(M, N)\) as in [And10]:

5.3. **Definition.** Let \(M\) and \(N\) be two framed \(d\)-dimensional manifolds. The topological space of framed embeddings from \(M\) to \(N\), denoted \(\text{Emb}_f(M, N)\), is given by the following homotopy pullback in the category of topological spaces over \(\text{Map}(M, N)\):

\[
\begin{array}{ccc}
\text{Emb}_f(M, N) & \to & \text{Map}(M, N) \\
\downarrow & & \downarrow \\
\text{Emb}(M, N) & \to & \text{Map}_{\text{GL}(d)}(\text{Fr}(TM), \text{Fr}(TN))
\end{array}
\]
The right hand side map is obtained as the composition

$$\text{Map}(M, N) \to \text{Map}_{\text{GL}(d)}(M \times \text{GL}(d), N \times \text{GL}(d)) \cong \text{Map}_{\text{GL}(d)}(\text{Fr}(TM), \text{Fr}(TN))$$

where the first map is obtained by taking the product with $\text{GL}(d)$ and the second map is induced by the identification $\text{Fr}(TM) \cong M \times \text{GL}(d)$ and $\text{Fr}(TN) \cong N \times \text{GL}(d)$.

It is not hard to show that there are well defined composition maps

$$\text{Emb}_f(M, N) \times \text{Emb}_f(N, P) \to \text{Emb}_f(M, P)$$

allowing the construction of a topological category $f\text{Man}_d$ (see [And10]).

5.4. Remark. Taking a homotopy pullback in the category of spaces over $\text{Map}(M, N)$ is not strictly necessary. Taking the homotopy pullback of the underlying diagram of spaces would have given the same homotopy type by 4.5. However, this definition has the psychological advantage that any point in the space $\text{Emb}_f(M, N)$ lies over a point in $\text{Map}(M, N)$ in a canonical way. If we had taken the homotopy pullback in the category of spaces, the resulting object would have had two distinct maps to $\text{Map}(M, N)$, one given by the upper horizontal arrow and the other given as the composition $\text{Emb}_f(M, N) \to \text{Emb}(M, N) \to \text{Map}(M, N)$.

**Embeddings between framed manifolds with boundary.** If $N$ is a manifold with boundary, $n$ a point of the boundary, and $v$ is a vector in $TN_n - T(\partial N)_n$, we say that $v$ is pointing inward if it can be represented as the tangent vector at 0 of a curve $\gamma : [0, 1) \to N$ with $\gamma(0) = n$.

5.5. Definition. A $d$-manifold with boundary is a pair $(N, \phi)$ where $N$ is a $d$-manifold with boundary in the traditional sense and $\phi$ is an isomorphism of $d$-dimensional vector bundles over $\partial N$

$$\phi : T(\partial N) \oplus \mathbb{R} \to TN_{|\partial N}$$

which is required to restrict to the canonical inclusion $T(\partial N) \to TN_{|\partial N}$, and which is such that for any $n$ on the boundary, the point $1 \in \mathbb{R}$ is sent to an inward pointing vector through the composition

$$\mathbb{R} \to T_n(\partial N) \oplus \mathbb{R} \xrightarrow{\phi_n} T_nN$$

In other words, our manifolds with boundary are equipped with smooth family of inward pointing vector at each point of the boundary. We require maps between manifolds with boundary to preserve the direction defined by these vectors:

5.6. Definition. Let $(M, \phi)$ and $(N, \psi)$ be two $d$-manifolds with boundary, we define the space $\text{Emb}(M, N)$ to be the topological space of smooth embeddings from $M$ into $N$ sending $\partial M$ to $\partial N$, preserving the splitting of the tangent bundles along the boundary $T(\partial M) \oplus \mathbb{R} \to T(\partial N) \oplus \mathbb{R}$. The topology on this space is the weak $C^1$-topology.

In particular, if $\partial M$ is empty, $\text{Emb}(M, N) = \text{Emb}(M, N - \partial N)$. If $\partial N$ is empty and $\partial M$ is not empty, $\text{Emb}(M, N) = \emptyset$.

We now introduce framings on manifolds with boundary. We require a framing to interact well with the boundary.

5.7. Definition. Let $(N, \phi)$ be a $d$-manifold with boundary. We say that a section $\sigma_N$ of $\text{Fr}(TN)$ is compatible with the boundary if for each point $n$ on the boundary of $N$ there is a splitting-preserving isomorphism

$$T_n(\partial N) \oplus \mathbb{R} \xrightarrow{\phi_n} T_nN \xrightarrow{\sigma_N} \mathbb{R}^{d-1} \oplus \mathbb{R}$$
whose restriction to the $\mathbb{R}$-summand is multiplication by a positive real number.

A framed $d$-manifold with boundary is a $d$-manifold with boundary together with the datum of a compatible framing.

5.8. **Definition.** Let $M$ and $N$ be two framed $d$-manifolds with boundary. We denote by $\text{Map}^\partial_{\text{GL}(d)}(\text{Fr}(TM),\text{Fr}(TN))$ the topological space of $\text{GL}(d)$-equivariant maps sending $\text{Fr}(TM|_{\partial M})$ to $\text{Fr}(TN|_{\partial N})$ and preserving the framings that are compatible with the boundary.

5.9. **Definition.** Let $M$ and $N$ be two framed $d$-manifolds with boundary. The topological space of framed embeddings from $M$ to $N$, denoted $\text{Emb}_f(M,N)$, is the following homotopy pullback in the category of topological spaces over $\text{Map}((M,\partial M),(N,\partial N))$.

$$
\begin{array}{ccc}
\text{Emb}_f(M,N) & \xrightarrow{\text{Map}((M,\partial M),(N,\partial N))} & \text{Map}^\partial_{\text{GL}(d)}(\text{Fr}(TM),\text{Fr}(TN)) \\
\downarrow & & \downarrow \\
\text{Emb}(M,N) & \xrightarrow{\text{Map}^\partial_{\text{GL}(d)}} & \text{Map}((M,\partial M),(N,\partial N))
\end{array}
$$

Concretely, a point in $\text{Emb}_f(M,N)$ is a pair $(\phi,p)$ where $\phi : M \to N$ is an embedding of manifolds with boundary and $p$ is the data at each point $m$ of $M$ of a path between the two trivializations of $T_mM$ (the one given by the framing of $M$ and the one given by pulling back the framing of $N$ along $\phi$). These paths are required to vary smoothly with $m$. Moreover if $m$ is a point on the boundary, the path between the two trivializations of $T_mM$ must be such that at any time, the first $d-1$-vectors are in $T_m\partial M \subset T_mM$ and the last vector is a positive multiple of the inward pointing vector which is part of our definition of a manifold with boundary.

6. **Homotopy type of spaces of embeddings**

We want to analyse the homotopy type of spaces of embeddings described in the previous section. None of the result presented here are surprising. Some of them are proved in greater generality in [Cer61]. However the author of [Cer61] is working with the strong topology on spaces of embeddings and for our purposes, we needed to use the weak topology.

As usual, $D$ denotes the $d$-dimensional open disk of radius 1 and $H$ is the upper half-disk of radius 1.

We will make use of the following two lemmas.

6.1. **Lemma.** Let $X$ be a topological space with an increasing filtration by open subsets $X = \bigcup_{n \in \mathbb{N}} U_n$. Let $Y$ be another space and $f : X \to Y$ be a continuous map such that for all $n$, the restriction of $f$ to $U_n$ is a weak equivalence. Then $f$ is a weak equivalence.

**Proof.** It suffices to show that the induced map $f_* : [K,X] \to [K,Y]$ is an isomorphism for all finite $CW$-complexes.

Since $f_{|U_1}$ is a weak equivalence, the composition $[K,U_1] \to [K,X] \to [K,Y]$ is surjective this forces $[K,X] \to [K,Y]$ to be surjective.

Let $a,b$ be two points in $[K,X]$ whose image in $[K,Y]$ are equal, let $\alpha,\beta$ be continuous maps $K \to X$ representing $a$ and $b$ and such that $f \circ \alpha$ is homotopical to $f \circ \beta$. Since the topological space $K$ is compact, $\alpha$ and $\beta$ are maps $K \to U_n$ for some $n$. The composite $U_n \to X \xrightarrow{f} Y$ is a weak equivalence, thus $\alpha$ is homotopical to $\beta$ in $U_n$. This implies that $\alpha$ is homotopical to $\beta$ in $X$ or equivalently that $a = b$. \hfill $\Box$
6.2. Lemma. (Cerf) Let $G$ be a topological group and let $p : E \to B$ be a map of $G$-topological spaces. Assume that for any $x \in B$, there is a neighborhood of $x$ on which there is a section of the map

$$G \to B$$

$$g \mapsto g.x$$

Then if we forget the action, the map $p$ is a locally trivial fibration. In particular, if $B$ is paracompact, it is a Hurewicz fibration.

Proof. See [Cer62].

Let $\text{Emb}^*(D,D)$ (resp. $\text{Emb}^{\partial,*}(H,H)$) be the topological space of self embeddings of $D$ (resp. $H$) mapping 0 to 0.

6.3. Proposition. The “derivative at the origin” map

$$\text{Emb}^*(D,D) \to \text{GL}(d)$$

is a Hurewicz fibration and a weak equivalence. The analogous result for the map

$$\text{Emb}^*(H,H) \to \text{GL}(d-1)$$

also holds.

Proof. Let us first show that the derivative map

$$\text{Emb}^*(D,D) \to \text{GL}(d)$$

is a Hurewicz fibration.

The group $\text{GL}(d)$ acts on the source and the target and the derivative map commutes with this action. We use lemma 6.2, it suffices to show that for any $u \in \text{GL}(d)$, we can define a section of the multiplication by $u$ map

$$\text{GL}(d) \to \text{GL}(d)$$

which is trivial.

Now we show that the fibers are contractible. Let $u \in \text{GL}(d)$ and let $\text{Emb}^u(D,D)$ be the space of embedding whose derivative at 0 is $u$, we want to prove that $\text{Emb}^u(D,D)$ is contractible. It is equivalent but more convenient to work with $\mathbb{R}^d$ instead of $D$. Let us consider the following homotopy:

$$\text{Emb}^u(\mathbb{R}^d,\mathbb{R}^d) \times (0,1] \to \text{Emb}^u(\mathbb{R}^d,\mathbb{R}^d)$$

$$(f,t) \mapsto \left(x \mapsto \frac{f(tx)}{t}\right)$$

At $t = 1$ this is the identity of $\text{Emb}^u(D,D)$. We can extend this homotopy by declaring that its value at 0 is constant with value the linear map $u$. Therefore, the inclusion $\{u\} \to \text{Emb}^u(D,D)$ is a deformation retract.

The proof for $H$ is similar.

6.4. Proposition. Let $M$ be a manifold (possibly with boundary). The map

$$\text{Emb}(D,M) \to \text{Fr}(TM)$$

is a weak equivalence and a Hurewicz fibrations. Similarly the map

$$\text{Emb}(H,M) \to \text{Fr}(T\partial M)$$

is a weak equivalence and a Hurewicz fibration.
Proof. The fact that these maps are Hurewicz fibrations will follow again from lemma 6.2. We will assume that $M$ has a framing because this will make the proof easier and we will only apply this result with framed manifolds. However the result remains true in general.

Let us do the proof for $D$. The derivative map

$$
\text{Emb}(D, M) \to \text{Fr}(TM) \cong M \times \text{GL}(d)
$$

is equivariant with respect to the action of the group $\text{Diff}(M) \times \text{GL}(d)$. It suffices to show that for any $x \in \text{Fr}(TM)$, the “action on $x$” map

$$
\text{Diff}(M) \times \text{GL}(d) \to M \times \text{GL}(d)
$$

has a section in a neighborhood of $x$. Clearly it is enough to show that for any $x$ in $M$, the “action on $x$” map

$$
\text{Diff}(M) \to M
$$

has a section in a neighborhood of $x$.

We can restrict to neighborhoods $U$ such that $U \subset \bar{U} \subset V \subset M$ in which $U$ and $V$ are diffeomorphic to $\mathbb{R}^d$.

Let us consider the group $\text{Diff}^c(V)$ of diffeomorphisms of $V$ that are the identity outside a compact subset of $V$. Clearly we can prolong one of these diffeomorphism by the identity and there is a well define inclusion of topological groups

$$
\text{Diff}^c(V) \to \text{Diff}(M)
$$

Now we have made the situation local. It is equivalent to construct a map

$$
\phi : D \to \text{Diff}^c(\mathbb{R}^d)
$$

with the property that $\phi(x)(0) = x$.

Let $f$ be a smooth function from $\mathbb{R}^d$ to $\mathbb{R}$ which is such that

- $f(0) = 1$
- $\|\nabla f\| \leq \frac{1}{2}$
- $f$ is compactly supported

We claim that

$$
\phi(x)(u) = f(u)x + u
$$

satisfies the requirement which proves that

$$
\text{Emb}(D, M) \to \text{Fr}(TM)
$$

is a Hurewicz fibration. The case of $H$ is similar.

Now let us prove that this derivative maps are weak equivalences.

We have the following commutative diagram

$$
\begin{array}{ccc}
\text{Emb}(D, M) & \to & \text{Fr}(TM) \\
\downarrow & & \downarrow \\
M & \to & M
\end{array}
$$

Both vertical maps are Hurewicz fibration, therefore it suffices to check that the induced map on fibers is a weak equivalence. We denote by $\text{Emb}^m(D, M)$ the subspace consisting of those embeddings sending 0 to $m$. Hence all we have to do is prove that for any point $m \in M$ the derivative map $\text{Emb}^m(D, M) \to \text{Fr}T_mM$ is a weak equivalence. If $M$ is $D$ and $m = 0$, this is the previous proposition. In general, we pick an embedding $f : D \to M$ centered at $m$. Let $U \subset \text{Emb}^m(D, M)$ be the subspace of embeddings mapping $D_n$ to the image of $f$ (where $D_n \subset D$ is the subspace of points of norm at most $1/n$). Clearly $U_n$ is
open in $\text{Emb}^m(D, M)$ and $\bigcup_n U_n = \text{Emb}^m(D, M)$, by 6.1 it suffices to show that the map $U_n \to \text{Fr}(T_m M)$ is a weak equivalence for all $n$.

Clearly the inclusion $U_1 \to U_n$ is a deformation retract for all $n$, therefore, it suffices to check that $U_1 \to \text{Fr}(T_m M)$ is a weak equivalence. Equivalently, it suffices to prove that $\text{Emb}^0(D, D) \to \text{GL}(d)$ is a weak equivalence and this is exactly the previous proposition. □

This result extends to disjoint union of copies of $H$ and $D$ with a similar proof.

6.5. Proposition. The derivative map
$$\text{Emb}(D^p \sqcup H^q, M) \to \text{Fr}(T\text{Conf}(p, M - \partial M)) \times \text{Fr}(T\text{Conf}(q, \partial M))$$
is a weak equivalence and a Hurewicz fibration.

In the case of framed embeddings, we have the following result:

6.6. Proposition. The evaluation at the center of the disks induces a weak equivalence
$$\text{Emb}_f(D^p \sqcup H^q, M) \to \text{Conf}(p, M - \partial M) \times \text{Conf}(q, \partial M)$$

Proof. To simplify notations, we restrict to studying $\text{Emb}_f(H, M)$, the general case is similar. By definition 5.9 and proposition 4.5, we need to study the following homotopy pullback:
$$\text{Map}((H, \partial H), (M, \partial M))$$

$$\text{Emb}(H, M) \xrightarrow{\text{Map}_0^\partial} \text{Map}_{\text{GL}(d-1)}(\text{Fr}(TH), \text{Fr}(TM))$$

This diagram is weakly equivalent to

$$\partial M$$

$$\text{Fr}(T(\partial M)) \xrightarrow{\text{Fr}(T(\partial M))} \text{Fr}(T(\partial M))$$

where the bottom map is the identity. Therefore, $\text{Emb}_f(H, M) \simeq \partial M$. □

6.7. Proposition. Let $M$ be a $d$-manifold with compact boundary and let $S$ be a compact $(d - 1)$-manifold without boundary. The “restriction to the boundary” map
$$\text{Emb}(S \times [0, 1), M) \to \text{Emb}(S, \partial M)$$
is a Hurewicz fibration and a weak equivalence.

Proof. Note that an embedding between compact connected manifold without boundary is necessarily a diffeomorphism. Therefore the two spaces in the proposition are empty unless $S$ is diffeomorphic to a disjoint union of connected components of $\partial M$.

Let us assume that $S$ and $\partial M$ are connected and diffeomorphic. The general case follows easily from this particular case.

We first prove that this map is a Hurewicz fibration. We use the criterion 6.2. The map
$$\text{Emb}(S \times [0, 1), M) \to \text{Emb}(S, \partial M)$$
is obviously equivariant with respect to the obvious right action of $\text{Diff}(S)$ on both sides. Therefore, for any $f \in \text{Emb}(S, \partial M)$, we need to define a section of the “action on $f$” map
$$\text{Diff}(S) \to \text{Emb}(S, \partial M)$$
but this map is by hypothesis a diffeomorphism.
Now let us prove that each fiber is contractible. Let $\alpha$ be a diffeomorphism $S \to \partial M$. We need to prove that the space $\text{Emb}^\alpha(S \times [0, 1), M)$ consisting of embeddings whose restriction to the boundary is $\alpha$ is contractible.

Let us choose one of these embeddings $\phi : S \times [0, 1) \to M$ and let’s denote its image by $C$. For $n > 0$, let $U_n$ be the subset of $\text{Emb}^\alpha(S \times [0, 1), M)$ consisting of embeddings $f$ with the property that $f(S \times [0, \frac{1}{n}]) \subset C$. By definition of the weak $C^1$-topology, $U_n$ is open in $\text{Emb}^\alpha(S \times [0, 1), M)$, moreover $\text{Emb}^\alpha(S \times [0, 1), M) = \bigcup_n U_n$, therefore by 6.1, it is enough to prove that $U_n$ is contractible for all $n$.

Let us consider the following homotopy:

$$H : \left[0, 1 - \frac{1}{n}\right] \times U_n \to U_n$$

$$(t, f) \mapsto ((s, u) \mapsto f(s, (1 - t)u))$$

It is a homotopy between the identity of $U_n$ and the inclusion $U_1 \subset U_n$. Therefore $U_1$ is a deformation retract of each of the $U_n$ and all we have to prove is that $U_1$ is contractible. But each element of $U_1$ factors through $C = \text{Im} \phi$, hence all we need to do is prove the lemma when $M = S \times [0, 1)$ and $\alpha = \text{id}$. It is equivalent and notationally simpler to do it for $S \times \mathbb{R}_{\geq 0}$.

For $t \in (0, 1]$, let $h_t : S \times \mathbb{R}_{\geq 0} \to S \times \mathbb{R}_{\geq 0}$ be the diffeomorphism sending $(s, u)$ to $(s, tu)$

Let us consider the following homotopy

$$(0, 1] \times \text{Emb}^{\text{id}}(S \times \mathbb{R}_{\geq 0}, S \times \mathbb{R}_{\geq 0}) \to \text{Emb}^{\text{id}}(S \times \mathbb{R}_{\geq 0}, S \times \mathbb{R}_{\geq 0})$$

$$(t, f) \mapsto h_{1/t} \circ f \circ h_t$$

At time 1, this is the identity of $\text{Emb}^{\text{id}}(S \times [0, +\infty), S \times [0, +\infty))$. At time 0 it has as limit the map

$$(s, u) \mapsto \left(s, u \frac{\partial f}{\partial u}(s, 0)\right)$$

that lies in the subspace of $\text{Emb}^{\text{id}}(S \times [0, +\infty), S \times [0, +\infty))$ consisting of element which are of the form $(s, u) \mapsto (s, a(s)u)$ for some smooth function $a : S \to \mathbb{R}_{\geq 0}$. This space is obviously contractible and we have shown that it is deformation retract of $\text{Emb}^{\text{id}}(S \times [0, +\infty), S \times [0, +\infty))$. \hfill $\square$

**6.8. Proposition.** Let $M$ be a framed $d$-manifold with compact boundary. The “restriction to the boundary” map

$$\text{Emb}_f(S \times [0, 1), M) \to \text{Emb}_f(S, \partial M)$$

is a weak equivalence.

**Proof.** There is a restriction map comparing the pullback diagram defining $\text{Emb}_f(S \times [0, 1), M)$ to the pullback diagram defining $\text{Emb}_f(S, \partial M)$. Each of the three maps is a weak equivalence (one of them because of the previous proposition) therefore, the homotopy pullbacks are equivalent. \hfill $\square$

We are now ready to define the operads $\mathcal{E}_d$, $\mathcal{E}_d^{\partial}$.

**6.9. Definition.** The operad $\mathcal{E}_d$ of little $d$-disks is the simplicial operad whose $n$-th space is $\text{Emb}_f(D_{\leq n}, D)$.

Note that there is an inclusion of operads

$$\mathcal{D}_d \to \mathcal{E}_d$$

\footnote{The following was suggested to us by Søren Galatius}
6.10. **Proposition.** This map is a weak equivalence of operads.

**Proof.** It is enough to check it degreewise. The map

\[ D_d \to \text{Conf}(n, D) \]

is a weak equivalence which factors through \( \mathcal{E}_d(n) \) by 6.6, the map \( \mathcal{E}_d(n) \to \text{Conf}(n, D) \) is a weak equivalence. \[ \square \]

6.11. **Definition.** The operad \( \mathcal{E}_d^\partial \) is a colored operad with two colors \( z \) and \( h \) and with

\[
\mathcal{E}_d^\partial(z^\boxplus n ; z) = \mathcal{E}_d(n),
\]

\[
\mathcal{E}_d^\partial(z^\boxplus n \boxplus h^\boxplus m ; h) = \text{Emb}_f(D^{\boxplus n} \sqcup H^{\boxplus m}, H).
\]

6.12. **Proposition.** The obvious inclusion of operads

\[ D_d^\partial \to \mathcal{E}_d^\partial \]

is a weak equivalence of operads.

**Proof.** Similar to 6.10. \[ \square \]

7. **Factorization homology**

In this section, we define factorization homology of \( \mathcal{E}_d \)-algebras and \( \mathcal{E}_d^\partial \)-algebras. The paper \([AFT12]\) defines factorization homology of manifolds with various kind of singularities. The only originality of this section is the language of model categories as opposed to \( \infty \)-categories.

Let \( \mathfrak{M} \) be the set of framed \( d \) manifolds whose underlying manifold is a submanifold of \( \mathbb{R}^\infty \). Note that \( \mathfrak{M} \) contains at least one element of each diffeomorphism class of framed \( d \)-manifold.

7.1. **Definition.** We denote by \( f\text{Man}_d \) an operad whose set of colors is \( \mathfrak{M} \) and with mapping objects:

\[
f\text{Man}_d(\{M_1, \ldots, M_n\}, M) = \text{Emb}_f(M_1 \sqcup \ldots \sqcup M_n, M)
\]

As usual, we denote by \( f\text{Man}_d \) the free symmetric monoidal category on the operad \( f\text{Man}_d \).

We can see \( D \subset \mathbb{R}^d \subset \mathbb{R}^\infty \) as an element of \( \mathfrak{M} \). The operad \( \mathcal{E}_d \) is the full suboperad of \( f\text{Man}_d \) on the color \( D \). The category \( \mathbf{E}_d \) is the full subcategory of \( f\text{Man}_d \) on objects of the form \( D^{\boxplus n} \) with \( n \) a nonnegative integer.

Similarly, we define \( \mathfrak{M}^\partial \) to be the set of submanifold of \( \mathbb{R}^\infty \) possibly with boundary. \( \mathfrak{M}^\partial \) contains at least one element of each diffeomorphism class of framed \( d \)-manifold with boundary.

7.2. **Definition.** We denote by \( f\text{Man}_d^\partial \) the operad whose set of colors is \( \mathfrak{M}^\partial \) and with mapping objects:

\[
f\text{Man}_d^\partial(\{M_1, \ldots, M_n\}, M) = \text{Emb}_f^\partial(M_1 \sqcup \ldots \sqcup M_n, M)
\]

We denote by \( f\text{Man}_d^\partial \) the free symmetric monoidal category on the operad \( f\text{Man}_d^\partial \).

The suboperad \( \mathcal{E}_d^\partial \) is the full suboperad of \( f\text{Man}_d^\partial \) on the colors \( D \) and \( H \).

From now on, we assume that \( \mathbf{C} \) is a cofibrantly generated symmetric monoidal simplicial model category with a good theory of algebras over \( \Sigma \)-cofibrant operads.
7.3. Definition. Let $A$ be an object of $C[\mathcal{E}_d]$. We define factorization homology with coefficients in $A$ to be the derived operadic left Kan extension of $A$ along the map of operads $\mathcal{E}_d \rightarrow f\text{Man}_d$.

We denote by $\int_M A$ the value at the manifold $M$ of factorization homology. By definition, $M \mapsto \int_M A$ is a symmetric monoidal functor.

We have $\int_M A = \text{Emb}(-, M) \otimes_{\mathcal{E}_d} QA$ where $QA \rightarrow A$ is a cofibrant replacement in the category $C[\mathcal{E}_d]$. We use the fact that the operad $\mathcal{E}_d$ is $\Sigma$-cofibrant and that the right module $\text{Emb}(-, M)$ is $\Sigma$-cofibrant.

We can define factorization homology of an object of $f\text{Man}_d^\partial$ with coefficients in an algebra over $\mathcal{E}_d^\partial$.

7.4. Definition. Let $(B, A)$ be an algebra over $\mathcal{E}_d^\partial$ in $C$. Factorization homology with coefficients in $(B, A)$ is the derived operadic left Kan extension of $(B, A)$ along the obvious inclusion of operads $\mathcal{E}_d^\partial \rightarrow f\text{Man}_d^\partial$. We write $\int_M (B, A)$ to denote the value at $M \in f\text{Man}_d^\partial$ of the induced functor.

Again, we have $\int_M (B, A) = \text{Emb}^\partial(-, M) \otimes_{\mathcal{E}_d^\partial} Q(B, A)$ where $Q(B, A) \rightarrow (B, A)$ is a cofibrant replacement in the category $C[\mathcal{E}_d^\partial]$. We use the fact that $\mathcal{E}_d^\partial$ is $\Sigma$-cofibrant and that $\text{Emb}^\partial(-, M)$ is $\Sigma$-cofibrant as a right module over $\mathcal{E}_d^\partial$.

Factorization homology as a homotopy colimit. In this section, we show that factorization homology can be expressed as the homotopy colimit of a certain functor on the poset of open sets of $M$ that are diffeomorphic to a disjoint union of disks. Note that this result in the case of manifolds without boundary is proved in [Lur11]. We assume that $C$ is a symmetric monoidal simplicial cofibrantly generated model category with a good theory of algebras over $\Sigma$-cofibrant operads and satisfying proposition 2.15. As we have shown, proposition 2.15 is satisfied if $C$ has a cofibrant unit or if $C$ is $L_Z \text{Mod}_E$.

We will rely heavily on the following theorem:

7.5. Theorem. Let $X$ be a topological space and $U(X)$ be the poset of open subsets of $X$. Let $\chi : A \rightarrow U(X)$ be a functor from a small discrete category $A$. For a point $x \in X$, denote by $A_x$ the full subcategory of $A$ whose objects are those that are mapped by $\chi$ to open sets containing $x$. Assume that for all $x$, the nerve of $A_x$ is contractible. Then the obvious map:

$$\text{hocolim} \chi \rightarrow X$$

is a weak equivalence.

Proof. See [Lur11] Theorem A.3.1. p. 971. □

Let $M$ be an object of $f\text{Man}_d$. Let $D(M)$ the poset of subset of $M$ that are diffeomorphic to a disjoint union of disks. Let us choose for each object $V$ of $D(M)$ a framed diffeomorphism $V \cong D^{\text{in}}$ for some uniquely determined $n$. Each inclusion $V \subset V'$ in $D(M)$ induces a morphism $D^{\text{in}} \rightarrow D^{\text{in}'}$ in $\mathcal{E}_d$ by composing with the chosen parametrization. Therefore each choice of parametrization induces a functor $D(M) \rightarrow \mathcal{E}_d$. Up to homotopy this choice is unique since the space of automorphisms of $D$ in $\mathcal{E}_d$ is contractible.

In the following we assume that we have one of these functors $\delta : D(M) \rightarrow \mathcal{E}_d$. We fix a cofibrant algebra $A : \mathcal{E}_d \rightarrow C$.

7.6. Lemma. The obvious map:

$$\text{hocolim}_{V \in D(M)} \text{Emb}_f(-, V) \rightarrow \text{Emb}_f(-, M)$$

is a weak equivalence in $\text{Fun}(\mathcal{E}_d, S)$. 
Proof. It suffices to prove that for each \( n \), there is a weak equivalence in spaces:
\[
\text{hocolim}_{V \in \mathbf{D}(M)} \text{Emb}_f(D^{i_n}, V) \simeq \text{Emb}_f(D^{i_n}, M)
\]
We can apply theorem 7.5 to the functor:
\[
\mathbf{D}(M) \to \mathbf{U}(\text{Emb}_f(D^{i_n}, M))
\]
sending \( V \) to \( \text{Emb}_f(D^{i_n}, V) \subset \text{Emb}_f(D^{i_n}, M) \). For a given point \( \phi \) in \( \text{Emb}_f(D^{i_n}, M) \), we have to show that the poset of open sets \( V \in \mathbf{D}(M) \) such that \( \text{im}(\phi) \subset V \) is contractible. But this poset is filtered, thus its nerve is contractible. \( \square \)

7.7. Corollary. We have:
\[
\int_M A \simeq \text{hocolim}_{V \in \mathbf{D}(M)} \int_{\delta(V)} A
\]
Proof. By 2.15, we know that \( \int_M A \) is weakly equivalent to the Bar construction \( B(\text{Emb}_f(-, M), E_d, A) \). Therefore we have:
\[
\int_M A \simeq B(\ast, \mathbf{D}(M), B(\text{Emb}_f(-, -), E_d, A))
\]
The right hand side is the realization of a bisimplicial object and its value is independent of the order in which we do the realization. \( \square \)

7.8. Corollary. There is a weak equivalence:
\[
\int_M A \simeq \text{hocolim}_{V \in \mathbf{D}(M)} A(\delta(V))
\]
Proof. By 7.7 the left-hand side is weakly equivalent to:
\[
\text{hocolim}_{V \in \mathbf{D}(M)} \int_{\delta(V)} A
\]
Let \( U \) be an object of \( E_d \). The object \( \int_U A \) is the coend:
\[
\text{Emb}_f(-, U) \otimes_{E_d} A
\]
Yoneda’s lemma implies that this coend is isomorphic to \( A(U) \). Moreover, this isomorphism is functorial in \( U \). Therefore we have the desired identity. \( \square \)

We want to use a similar approach for manifolds with boundaries. Let \( M \) be an object of \( f\text{Man}_{d-1} \) and let \( M \times [0, 1) \) be the object of \( f\text{Man}_d^\partial \) whose framing is the direct sum of the framing of \( M \) and the obvious framing of \( [0, 1) \). We identify \( \mathbf{D}(M) \) with the poset of open sets of \( M \times [0, 1) \) of the form \( V \times [0, 1) \) with \( V \) an open set of \( M \) that is diffeomorphic to a disjoint union of disks. As before we can pick a functor \( \delta : \mathbf{D}(M) \to E_d^\partial \).

7.9. Lemma. The obvious map:
\[
\text{hocolim}_{V \in \mathbf{D}(M)} \text{Emb}_f(-, V \times [0, 1)) \to \text{Emb}_f(-, M \times [0, 1))
\]
is a weak equivalence in \( \text{Fun}((E_d^\partial)^{\text{op}}, S) \).
Proof. It suffices to prove that for each \( p, q \), there is a weak equivalence in spaces:
\[
\text{hocolim}_{V \in \mathbf{D}(M)} \text{Emb}_f(D^{i_p} \sqcup H^{i_q}, V \times [0, 1)) \simeq \text{Emb}_f(D^{i_p} \sqcup H^{i_q}, M \times [0, 1))
\]
It suffices to show, by 7.5, that for any \( \phi \in \text{Emb}(D^{i_p} \sqcup H^{i_q}, M \times [0, 1)) \), the poset \( \mathbf{D}(M)_\phi \) (which is the subposet of \( \mathbf{D}(M) \) on open sets \( V \) that are such that \( V \times [0, 1) \subset M \times [0, 1) \) contains the image of \( \phi \)) is contractible. But it is easy to see that \( \mathbf{D}(M)_\phi \) is filtered. Thus it is contractible. \( \square \)
7.10. **Proposition.** Let \((B, A) : E_d^\partial \to C\) be a cofibrant \(E_d^\partial\)-algebra, then we have:

\[
\int_{M \times [0,1]} (B, A) \simeq \text{hocolim}_{V \in D(M)} (B, A)(\delta(V))
\]

**Proof.** The proof is a straightforward modification of 7.8. \(\square\)

There is a morphism of operad \(\mathcal{E}_{d-1} \to \mathcal{E}_d^\partial\) sending the unique color of \(\mathcal{E}_{d-1}\) to \(H\). Indeed \(H\) is diffeomorphic to the product of the \((d - 1)\)-dimensional disk with \([0, 1]\). Hence, for \((B, A)\) an algebra over \(\mathcal{E}_d^\partial\), \(A\) has an induced \(\mathcal{E}_{d-1}\)-structure.

7.11. **Proposition.** Let \((B, A)\) be an \(\mathcal{E}_d^\partial\)-algebra, then we have a weak equivalence:

\[
\int_{M \times [0,1]} (B, A) \simeq \int_M A
\]

**Proof.** Let \(\delta' : D(M) \to E_{d-1}\) be defined as before. Then \(\delta\) can be take to be the composite of \(\delta'\) and the map \(E_{d-1} \to E_d^\partial\).

Now we prove the proposition. Because of the previous proposition, the left hand side is weakly equivalent to \(\text{hocolim}_{V \in D(M)} (B, A)(\delta(V))\). But \((B, A)(\delta(V))\) is \(A(\delta'(V))\). Therefore, by 7.8 \(\text{hocolim}_{V \in D(M)} (B, A)(\delta(V))\) is weakly equivalent to \(\int_M A\). \(\square\)

8. **\(\mathcal{K}\mathcal{S}\) and its higher versions.**

In this section, we recall the definition of the operad \(\mathcal{K}\mathcal{S}\) defined in [KS09]. We construct an equivalent version of that operad as well as higher dimensional analogues of it.

8.1. **Definition.** Let \(D\) be the 2-dimensional disk. An injective continuous map \(D \to S^1 \times (0, 1)\) is said to be **rectilinear** if it can be factored as

\[
D \xrightarrow{l} \mathbb{R} \times (0, 1) \to \mathbb{R} \times (0, 1)/\mathbb{Z} = S^1 \times (0, 1)
\]

where the map \(l\) is rectilinear and the second map is the quotient by the \(\mathbb{Z}\)-action.

We say that an embedding \(S^1 \times [0, 1) \to S^1 \times [0, 1)\) is rectilinear if it is of the form \((z, t) \mapsto (z + z_0, at)\) for some fixed \(z_0 \in S^1\) and \(a \in (0, 1)\).

We denote by \(\text{Emb}_\text{lin}^\partial(S^1 \times [0, 1) \sqcup D^\text{lin}, S^1 \times [0, 1)\) the topological space of injective maps whose restriction to each disk and to \(S^1 \times [0, 1)\) is rectilinear.

8.2. **Definition.** The Kontsevich-Soibelman’s operad \(\mathcal{K}\mathcal{S}\) has two colors \(a\) and \(m\) and its spaces of operations are as follows

\[
\mathcal{K}\mathcal{S}(a^{\text{lin}}; a) = \mathcal{D}_2(n)
\]

\[
\mathcal{K}\mathcal{S}(a^{\text{lin}} \boxplus m; m) = \text{Emb}_\text{lin}^\partial(S^1 \times [0, 1) \sqcup D^\text{lin}, S^1 \times [0, 1)\)
\]

Any other space of operations is empty.

Now we define generalizations of \(\mathcal{K}\mathcal{S}\).

8.3. **Definition.** Let \(S\) be a \((d - 1)\)-manifold with framing \(\tau\). We define \(S^\partial_\tau \text{Mod}\) to be the operad with two colors \(a\) and \(m\) and spaces of operations are as follows

\[
S^\partial_\tau \text{Mod}(a^{\text{lin}}; a) = \mathcal{E}_d(n)
\]

\[
S^\partial_\tau \text{Mod}(a^{\text{lin}} \boxplus m; m) = \text{Emb}_\text{lin}^\partial(S \times [0, 1) \sqcup D^\text{lin}, S \times [0, 1)\)
\]

The category \(S^\partial_\tau \text{Mod}\) is the category whose objects are disjoint unions of copies of \(S \times [0, 1)\) and \(D\).
8.4. Proposition. Let $S$ be a compact connected $(d-1)$-manifold. Let $N$ be a manifold with a boundary diffeomorphic to $S$ and let $M$ be an object of $S^\tau_C\text{Mod}$ which can be expressed as a disjoint union

$$M = P \sqcup Q$$

in which one of the first factor is of the form $S \times [0,1) \sqcup D^{jn}$ and the other is a disjoint union of disks. Then the restriction maps

$$\text{Emb}_f(M, N) \to \text{Emb}_f(P, N)$$

is a fibration.

Proof. The category $S^\tau_C\text{Mod}$ is a symmetric monoidal category. One can consider the category $\text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)$. It is a symmetric monoidal category for the convolution tensor product. The Yoneda’s embedding:

$$S^\tau_C\text{Mod} \to \text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)$$

is a symmetric monoidal functor. By the enriched Yoneda’s lemma, the space $\text{Emb}_f(M, N)$ can be identified with the space of natural transformations

$$\text{Map}_{\text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)}(\text{Emb}_f(-, M), \text{Emb}_f(-, N))$$

and similarly for $\text{Emb}_f(P, N)$ and $\text{Emb}_f(Q, N)$. The category $\text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)$ is a symmetric monoidal model category in which fibrations and weak equivalences are objectwise.

In fact, more generally, if $A$ is a small simplicial symmetric monoidal category, the category of simplicial functors to simplicial sets $\text{Fun}(A, S)$ with the projective model structure and the Day tensor product is a symmetric monoidal model category (this is proved in [Isa09] proposition 2.2.15). It is easy to check that in this model structure, a representable functor is automatically cofibrant (this comes from the characterization in terms of lifting against trivial fibrations together with the fact that trivial fibration in $S$ are epimorphisms). Moreover, we have the identity

$$\text{Emb}_f(-, M) \cong \text{Emb}_f(-, P) \otimes \text{Emb}_f(-, Q)$$

This immediately implies that

$$\text{Emb}_f(-, P) \to \text{Emb}_f(-, M)$$

is a cofibration in $\text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)$. But the category $\text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)$ is also a model category enriched in $S$, therefore, the induced map

$$\text{Map}_{\text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)}(\text{Emb}_f(-, M), \text{Emb}_f(-, N))$$

$$\to \text{Map}_{\text{Fun}(S^\tau_C\text{Mod}^{\text{op}}, S)}(\text{Emb}_f(-, P), \text{Emb}_f(-, N))$$

is a fibration by the pushout-product property. □

Note that a linear embedding preserves the framing on the nose. Therefore, there is a well defined inclusion

$$K_S \to (S^1)^\tau_C\text{Mod}$$

8.5. Proposition. This map is a weak equivalence.

Proof. There is a restriction map

$$S^\tau_C\text{Mod}(a^{\text{fin}} \oplus m; m) \to \text{Emb}_f(D^{jn}, S \times [0,1))$$

This map is a fibration by 8.4. Its fiber over a particular configuration of disks is the space of embeddings of $S \times [0,1)$ into the complement of that configuration. By 6.8, this space is weakly equivalent to $\text{Emb}_f(S, S)$. 
We have a diagram
\[
\begin{array}{ccc}
\Emb_{\text{lin}}(S^1 \times [0, 1] \sqcup D^n, S^1 \times [0, 1]) & \longrightarrow & \Emb_{\text{lin}}(S^1 \times [0, 1] \sqcup D^{\text{lin}}, S^1 \times [0, 1]) \\
\downarrow & & \downarrow \\
\Emb_{\text{lin}}(D^{\text{lin}}, S^1 \times (0, 1)) & \longrightarrow & \Emb_{f}(D^{\text{lin}}, S^1 \times (0, 1))
\end{array}
\]
Both vertical maps are fibrations. The bottom map is a weak equivalence since both sides are weakly equivalent to Conf \((n, S^1 \times (0, 1))\). To prove that the upper horizontal map is a weak equivalence, it suffices to check that it induces an equivalence on each fiber. The map induced on the fibers is weakly equivalent to the inclusion
\[S^1 \rightarrow \Emb_{f}(S^1, S^1)\]
It is well-known that this map is a weak equivalence. □

9. Action of the higher version of \(KS\)

Let \((B, A)\) be an algebra over the operad \(\mathcal{E}^\partial_d\) in the category \(\mathcal{C}\). Let \(M\) be a framed \((d - 1)\)-manifold and \(\tau\) be the product framing on \(TM \oplus \mathbb{R}\).

9.1. Theorem. The pair \((B, \int_M A)\) is weakly equivalent to an algebra over the operad \(M_{\tau} \odot \text{Mod}\).

Proof. The construction \(\int_{\text{lin}}(B, A)\) is a simplicial functor \(f\text{Man}^\partial_d \rightarrow \mathcal{C}\). Hence, \(\int_{\text{lin}}(B, A)\) is a functor from the full subcategory of \(f\text{Man}^\partial_d\) spanned by disjoint unions of copies of \(D\) and \(M \times [0, 1]\) to \(\mathcal{C}\). Moreover this functor is symmetric monoidal. The operad \(M_{\tau} \odot \text{Mod}\) has a map to the endomorphism operad of the pair \((D, M \times [0, 1])\) in the symmetric monoidal category \(f\text{Man}^\partial_d\), therefore \((\int_D(B, A), \int_{M \times [0, 1]}(B, A))\) is an algebra over \(M_{\tau} \odot \text{Mod}\). To conclude, we use the fact that \(\int_D(B, A) \cong B\) by Yoneda’s lemma and \(\int_{M \times [0, 1]}(B, A) \cong \int_M A\) by 7.11. □

This theorem is mainly interesting because of the following theorem due to Thomas (see [Tho10]):

9.2. Theorem. Let \(A\) be an \(\mathcal{E}_{d-1}\)-algebra in \(\mathcal{C}\), then there is an algebra \((B', A')\) over \(\mathcal{E}^\partial_d\) such that \(B'\) is weakly equivalent to \(\text{HH}_{\mathcal{E}_d}(A)\) and \(A'\) is weakly equivalent to \(A\).

This has the following immediate corollary:

9.3. Corollary. We keep the notations of 9.1. The pair \((\text{HH}_{\mathcal{E}_d}(A), \int_M A)\) is weakly equivalent to an algebra over the operad \(M_{\tau} \odot \text{Mod}\).

The previous theorem has the following interesting corollary:

9.4. Theorem. Let \((M, \tau)\) be a framed \((d - 1)\)-dimensional and \(N\) be a \((d - 1)\)-connected manifold. The pair \((\text{Map}(S^d, N)^{-TN}, \Sigma_+ \text{Map}(M, N))\) is weakly equivalent to an algebra over \(M_{\tau} \odot \text{Mod}\).

Proof. Let \(R = \Sigma_+ \Omega^d N\). \(R\) is an \(\mathcal{E}_d\)-algebra in \(\text{Spec}\). It is proved in [Kle06] that
\[\text{HH}_{\mathcal{E}_d}(R) \cong \text{Map}(S^d, N)^{-TN}\]
Similarly, it is proved in [Fra12] that
\[\int_M R \cong \Sigma_+ \text{Map}(M, N)\]
The result is then a direct corollary of 9.3. □
9.5. Remark. This result remains true if $N$ is a Poincaré duality space.

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