TAUT FOLIATIONS FROM DOUBLE-DIAMOND REPLACEMENTS

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Abstract. Suppose $M$ is an oriented 3-manifold with connected boundary a torus, and suppose $M$ contains a properly embedded, compact, oriented, surface $R$ with a single boundary component that is Thurston norm minimizing in $H_2(M, \partial M)$. We define a readily recognizable type of sutured manifold decomposition, which for notational reasons we call double-diamond taut, and show that if $R$ admits a double-diamond taut sutured manifold decomposition, then for every boundary slope except one, there is a co-oriented taut foliation of $M$ that intersects $\partial M$ transversely in a foliation by curves of that slope. In the case that $M$ is the complement of a knot $\kappa$ in $S^3$, the exceptional filling is the meridional one; in particular, restricting attention to rational slopes, it follows that every manifold obtained by non-trivial Dehn surgery along $\kappa$ admits a co-oriented taut foliation. As an application, we show that if $R$ is a Murasugi sum of surfaces $R_1$ and $R_2$, where $R_2$ is an unknotted band with an even number $2m \geq 4$ of half-twists, then every manifold obtained by non-trivial surgery on $\kappa = \partial R$ admits a co-oriented taut foliation.

1. Introduction

Taut foliations have long played an important role in the study of 3-manifolds, informed by the relationship among various geometric and algebraic properties. We present here the relevant definitions, known results, and conjectures. For simplicity of exposition, we assume all 3-manifolds are oriented and are not homeomorphic to $S^1 \times S^2$.

Definition 1.1. Call a 3-manifold foliar if it supports a co-oriented taut foliation.

The presence of a taut foliation (co-oriented or not) in a closed 3-manifold imposes restrictions on its fundamental group: it is not a non-trivial free product (indeed, $M$ is irreducible) \[33, 43, 45\], and it contains elements of infinite order \[20, 33, 18\]— indeed, the manifold is covered by $\mathbb{R}^3$ \[42\].

Another well-studied property of groups is orderability: that is, an ordering that respects the group operation. Focusing on transformations by group elements acting on the left:

Definition 1.2. A nontrivial group $G$ is left-orderable if its elements can be given a strict total ordering $<$ which is left invariant, meaning that $g < h$ implies $fg < fh$ for all $f, g, h \in G$.

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Another invariant of 3-manifolds is Heegaard Floer Homology, introduced by Ozsváth and Szabó [39, 40]. In particular, the Heegaard Floer homology of a rational homology sphere $Y$ satisfies

$$\text{Rank}(\widehat{HF}(Y; \mathbb{Z}/2)) \geq |H_1(M, \mathbb{Z})|.$$

If equality holds, $Y$ is called an L-space; elliptic manifolds, such as lens spaces, are examples [41]. In contrast, Ozsváth and Szabó proved foliar manifolds cannot be L-spaces [38, 2, 26, 27].

Boyer, Gordon, and Watson have conjectured that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable and also asked if the presence of a co-oriented taut foliation implies that a 3-manifold’s fundamental group is left-orderable [4], leading to the L-space Conjecture:

**Conjecture 1.3 (L-space Conjecture [40, 4, 25]).** Let $Y$ be an irreducible rational homology 3-sphere. Then the following three statements are equivalent:

1. $Y$ is not an L-space,
2. $\pi_1(Y)$ is left-orderable, and
3. $Y$ is foliar.

Namely, in the spirit of Haken, Waldhausen, and Thurston [21, 50, 48], these are three measures of what it means for a 3-manifold to be “large,” and the conjecture is that they are all equivalent. (We note that Gabai includes co-orientability in his original definition of taut.) It is known only that (3) implies (1). This paper addresses the conjecture that (1) implies (3).

**Definition 1.4.** Let $M$ be a 3-manifold with boundary a torus. A boundary slope is an isotopy class of curves in $\partial M$.

Following Boyer and Clay we introduce:

**Definition 1.5 ([3]).** A foliation $\mathcal{F}$ strongly realizes a slope if $\mathcal{F}$ intersects $\partial M$ transversely in a foliation by curves of that slope.

Note that if a foliation strongly realizes a compact slope, it may be capped off by disks to obtain a co-oriented taut foliation in the manifold obtained by Dehn filling along that slope.

Suppose $\partial M$ is a torus and $M$ contains a properly embedded, compact, oriented, Thurston norm minimizing [49] surface $R$ with a single boundary component. Call the isotopy class of $\partial R$ the longitude and any slope having geometric intersection number one with this longitude a meridian.

In this paper, we define an easily-recognized criterion that implies such a 3-manifold supports co-oriented taut foliations strongly realizing all boundary slopes except a single distinguished meridian $\nu$. Thus, the manifolds obtained by Dehn surgery along all slopes other than $\nu$ are foliar.

Focusing attention on manifolds obtained by Dehn surgery on knots, we introduce the following notation, definition, and conjecture:

**Notation 1.6.** Recall that a knot in a 3-manifold $P$ is a submanifold of the interior of $P$ homeomorphic to $S^1$. Given a knot $\kappa$ in a closed manifold $P$, let $N(\kappa)$ denote an open regular neighborhood of $\kappa$ with closure contained in the interior of $P$, and let $X_\kappa$ denote the knot complement.
\( P \setminus N(\kappa) \). In the case \( M = X_\kappa \), we show in the proof of Corollary 5.11 that the distinguished meridian indicated above is the meridian of \( \kappa \), namely the unique slope \( \mu \) whose elements bound disks in the closure of \( N(\kappa) \).

**Definition 1.7** (7). A knot \( \kappa \) in a 3-manifold is **persistently foliar** if, for each non-meridional boundary slope of \( X_\kappa \), there is a co-oriented taut foliation strongly realizing that slope.

Many knots in \( S^3 \) are known to be persistently foliar, including all alternating and Montesinos knots with no L-space surgeries (8 and 9, 10, respectively), all fibered knots with fractional Dehn twist coefficient zero (44, 7), and all composite knots in which each of two summands is alternating, Montesinos, or fibered (7). (Among the alternating and Montesinos knots, only the \((2,n)\)-torus knots (11) and \((-2,3,n)\)-pretzel knots, where \( n \) is (odd and) positive (28, 11), have L-space surgeries.) As a direct derivative of the L-space conjecture, we have the following pair of conjectures about knots:

**Conjecture 1.8** (Classical L-space knot conjecture). A knot in \( S^3 \) is persistently foliar if and only if it has no non-trivial L-space or reducible surgeries.

**Conjecture 1.9** (L-space knot conjecture). A knot in an L-space is persistently foliar if and only if it has no non-trivial L-space or reducible surgeries.

In Section 3 we provide all essential background on spines and branched surfaces. In Section 4 we provide the necessary definitions and results from sutured manifold theory. Some familiarity with laminations and foliations is assumed, or would at least be helpful to the reader. In Section 5, we prove a technical theorem (Theorem 5.10) about a particular type of sutured manifold decomposition with associated distinguished meridian, which we call **double-diamond taut**. We defer the statement of this theorem and note the following direct consequence:

**Corollary 5.11**. Suppose the sutured manifold decomposition \((M|R, \partial M|_{\partial R}) \xrightarrow{\S} (M', \gamma')\) is double-diamond taut, with associated distinguished meridian \( \nu \). If the branched surface associated to this decomposition has no sink disk disjoint from \( \partial M \), then there are co-oriented taut foliations that strongly realize all boundary slopes except \( \nu \). Moreover, if \( M = X_\kappa \) for some knot \( \kappa \subset S^3 \), \( \kappa \) is persistently foliar.

We end with an application to knots in \( S^3 \). By Ghiggini (19), Ni (31, 32) and Juhasz (23, 24), non-fibered knots do not admit L-space surgeries; hence, if the L-space conjecture holds, all non-fibered knots that do not admit a reducible surgery are persistently foliar. If in addition the cabling conjecture holds, then by a result of Scharlemann (46) all non-fibered hyperbolic knots are persistently foliar. Although knots with Seifert surface obtained by plumbing Hopf bands have long been known to be fibered (17), knots obtained by plumbing a Seifert surface with a \((2,2n)\) torus link with \( |n| \geq 2 \) are never fibered (15). Thus we expect such knots to be persistently foliar, and indeed this is the case:

**Corollary 6.3**. Suppose \( \kappa \) is a knot in \( S^3 \) with minimal genus Seifert surface \( R \). If \( R \) is a plumbing of surfaces \( R_1 \) and \( R_2 \), where \( R_2 \) is an unknotted band with an even number \( 2m \geq 4 \) of
half-twists, then the decomposing disk for $R_2$ can be chosen to be double-diamond taut. Thus, $\kappa$ is persistently foliar.

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3. Spines and branched surfaces

For completeness we include the relevant standard definitions and results on spines and branched surfaces, as also found in [7].

Notation 3.1. Given any metric space $X$ and any subset $A \subset X$, the closed complement of $A$ in $X$, denoted $X|_A$, is the metric completion of $X \setminus A$.

Remark 3.2. The space $X|_A$ is obtained by “cutting” $X$ along $A$. Although other authors have used the notation $X_A$, we feel the inclusion of a vertical slash evokes the notion of “cutting.”

Remark 3.3. We introduce this notation to unify several related concepts and simplify the notation that arises in examples; however, the reader may prefer a different perspective. In the context of this paper, $X$ will be a 3-manifold and $A$ an embedded surface, or $X$ will be a surface and $A$ an embedded simple closed curve or arc. For these examples, $X$ is retrieved as a quotient space of $X|_A$ by identifying corresponding points of the two copies of $A$. Also, in these examples, $X|_A$ is homeomorphic to $X \setminus \text{int}N(A)$, where $N(A)$ is a regular neighborhood; furthermore, if $N(A)$ is $I$-fibered (see Definition 3.6, which extends in the obvious way to simple curves in surfaces) collapsing fibers gives a quotient map from $X \setminus \text{int}N(A)$ to a manifold homeomorphic to $X$. However, the definition above extends to laminations, for which regular neighborhoods do not apply, and our notation avoids the cumbersome notation of the regular neighborhood description.

Definition 3.4. A standard spine [6] is a space $\Sigma$ locally modeled on one of the spaces of Figure 1. A standard spine with boundary has the additional local models shown in Figure 2. The critical locus $\Gamma$ of $\Sigma$ is the 1-complex of points of $\Sigma$ where the spine is not locally a manifold. The critical locus is a stratified space (graph) consisting of triple points $\Gamma^0$ and arcs of double points $\Gamma^1 = \Gamma \setminus \Gamma^0$. The components of $\Sigma|_\Gamma$ are called the sectors of $\Sigma$.

![Figure 1](image-url) Local models of a standard spine at interior points.
Definition 3.5. A branched surface (with boundary) ([11]; see also [11, 34, 36]) is a space $B$ locally modeled on the spaces of Figure 3 (along with those in Figure 4); that is, $B$ is homeomorphic to a spine, with the additional structure of a well-defined tangent plane at each point. The branching locus $\Gamma$ of $B$ is the 1-complex of points of $B$ where $B$ is not locally a manifold; such points are called branching points. The branching locus is a stratified space (graph) consisting of triple points $\Gamma^0$ and arcs of double points $\Gamma^1 = \Gamma \setminus \Gamma^0$. The components of $B|_{\Gamma}$ are called the sectors of $B$.

Figure 3. Local model of a branched surface at interior points.

Definition 3.6 ([11]). An I-fibered neighborhood of a branched surface $B$ in a 3-manifold $M$ is a regular neighborhood $N(B)$ foliated by interval fibers that intersect $B$ transversely, as locally modeled by the spaces in Figure 5 at interior points; if the ambient manifold $M$ has non-empty boundary, all spines and branched surfaces are assumed to be properly embedded, with $N(B) \cap \partial M$ a union (possibly empty) of I-fibers. The surface $\partial N(B) \setminus \partial M$ is a union of two subsurfaces, $\partial_v N(B)$ and $\partial_h N(B)$, where $\partial_v N(B)$, the vertical boundary, is a union of sub-arcs of I-fibers, and $\partial_h N(B)$, the horizontal boundary, is everywhere transverse to the I-fibers.

Definition 3.7. The complementary regions of $B$ are the components of $M \setminus \text{int}N(B)$; the complement of $B$ is the union of the complementary regions.
Figure 5. Local models for $N(B)$ (at interior points).

**Definition 3.8.** A surface is *carried by* $B$ if it is contained in $N(B)$ and is everywhere transverse to the one-dimensional foliation of $N(B)$. A surface is *fully carried by* $B$ if it carried by $B$ and has nonempty intersection with every I-fiber of $N(B)$. A lamination $\mathcal{L}$ is *carried by* $B$ if each leaf of $\mathcal{L}$ is carried by $B$, and *fully carried* if, in addition, each I-fiber of $N(B)$ has nonempty intersection with some leaf of $\mathcal{L}$. A foliation is *fully carried by* $B$ if there is a Denjoy splitting of $\mathcal{F}$ that is fully carried by $B$.

Let $\pi$ be the retraction of $N(B)$ onto the quotient space obtained by collapsing each fiber to a point. The branched surface $B$ is obtained, modulo a small isotopy, as the image of $N(B)$ under this retraction. We will freely identify $B$ with this image and the core of each component of vertical boundary with its image in $\Gamma$. Double points of the branching locus are *cusps* with *cusp direction* pointing inward from the vertical boundary if $B$ is viewed as the quotient of $N(B)$ obtained by collapsing the vertical fibers to points.

**Notation 3.9.** *Cusp directions* will be indicated by arrows, as in Figures 3 and 4. Call a sector $S$ of $B$ a *source* (*sink*) if all cusp directions along $\partial S$ point out of (into) $S$.

**Definition 3.10.** A branched surface $B$ in a closed 3-manifold $M$ is called an *essential* branched surface if it satisfies the following conditions:

1. $\partial_h N(B)$ is incompressible in $M \setminus \text{int}(N(B))$, no component of $\partial_h N(B)$ is a sphere, and $M \setminus \text{int}(N(B))$ is irreducible.
2. There is no monogon in $M \setminus \text{int}(N(B))$; i.e., no disk $D \subset M \setminus \text{int}(N(B))$ with $\partial D = D \cap N(B) = \alpha \cup \beta$, where $\alpha \subset \partial_h N(B)$ is in an interval fiber of $\partial_v N(B)$ and $\beta \subset \partial_h N(B)$.
3. There is no Reeb component; i.e., $B$ does not carry a torus that bounds a solid torus in $M$.

In the spirit of earlier definitions of Gabai, Oertel, Sullivan and others, we introduce:

**Definition 3.11.** A branched surface is *taut* if it is co-oriented, has taut sutured manifold complement (see Definition 4.3), and through every sector there is a closed oriented curve that is positively transverse to $B$.

Observe that a taut branched surface is, in particular, essential; furthermore, if a taut branched surface fully carries a lamination, this lamination is a sub-lamination of a taut foliation (see
Corollary 4.9 below). However, in practice, it can be difficult to determine whether an essential branched surface fully carries a lamination. In [29, 30], Li defines the notion of *laminar*, a very useful criterion that is sufficient (although not necessary) to guarantee that an essential branched surface fully carries a lamination. We recall the necessary definitions here.

**Definition 3.12** ([29, 30]). Let $B$ be a branched surface in a 3-manifold $M$. A *sink disk* is a disk sector of $B$ that is a sink. A *half sink disk* is a sink disk which has nonempty intersection with $\partial M$.

![Figure 6. A sink disk.](image)

Sink disks and half sink disks play a key role in Li’s notion of laminar branched surface. A sink disk or half sink disk $D$ can be eliminated by splitting $D$ open along a disk in its interior; these trivial splittings must be ruled out:

**Definition 3.13** ([29, 30]). Let $D_1$ and $D_2$ be the two disk components of the horizontal boundary of a $D^2 \times I$ region in $M \setminus \text{int}(N(B))$. If the projection $\pi : N(B) \rightarrow B$ restricted to the interior of $D_1 \cup D_2$ is injective, i.e., the intersection of any $I$-fiber of $N(B)$ with $\text{int}(D_1) \cup \text{int}(D_2)$ is either empty or a single point, then we say that $\pi(D_1 \cup D_2)$ forms a *trivial bubble* in $B$.

**Definition 3.14** ([29, 30]). An essential branched surface $B$ in a compact 3-manifold $M$ is called *laminar* if it satisfies the following conditions:

1. $B$ has no trivial bubbles.
2. $B$ has no sink disk or half sink disk.

**Theorem 3.15** ([29, 30]). Suppose $M$ is a compact and orientable 3-manifold.

(a) Every laminar branched surface in $M$ fully carries an essential lamination.

(b) Any essential lamination in $M$ that is not a lamination by planes is fully carried by a laminar branched surface.

## 4. Sutured manifold decompositions

We assume the reader is familiar with the basics of co-oriented taut foliations. Precise definitions and terminology as used here can be found in [7]. The basics of Gabai’s theory of sutured manifolds [13, 16, 17] play a key role in this paper, and in this section we give some necessary sutured manifold background. Where noted, we use the definitions of [13] as stated in [5].

**Definition 4.1** (Definition 2.6 of [13, 5]). A pair $(M, \gamma)$ is a *sutured manifold* if $M$ is a compact, oriented 3-manifold and $\gamma \subset \partial M$ is a compact subsurface such that

$$\gamma = A(\gamma) \cup T(\gamma),$$
where \( A(\gamma) \cap T(\gamma) = \emptyset, A(\gamma) \) is a disjoint union of annuli and \( T(\gamma) \) a disjoint union of tori. The components of \( A(\gamma) \) are called annular sutures, the components of \( T(\gamma) \) are called toral sutures, and the closure of \( \partial M \setminus \gamma \) is denoted by \( R(\gamma) \). The surface \( R(\gamma) \) comes with a transverse orientation; let \( R_+ (\gamma) \) be the subsurface oriented outward from \( M \) and \( R_- (\gamma) \) be the inwardly oriented subsurface. For each annular suture \( A \), it is assumed that one component of \( \partial A \) is also a component of \( \partial R_+ (\gamma) \), the other a component of \( \partial R_-(\gamma) \).

As noted in [17], a branched surface gives rise to a sutured manifold. If \( B \) is a co-oriented branched surface in \( M \), and \( \partial M \) is a union of tori, then the regions \( (\partial M \setminus \text{int} N(B), \partial_h N(B) \cap \partial M) \) are products. Setting \( \gamma = \partial_e N(B) \cup (\partial M \setminus \text{int} N(B)) \), the pair \( (M \setminus \text{int} N(B), \gamma) \) is a sutured manifold, with \( R_+ (\gamma) \) (respectively, \( R_- (\gamma) \)) consisting of the components of \( \partial_h N(B) \) with co-orientation pointing into (respectively, out of) \( N(B) \).

**Definition 4.2** ([13]). A sutured manifold \( (M, \gamma) \) is a product sutured manifold if \( (M, \gamma) \) is homeomorphic to \( (F \times [0,1], \partial F \times [0,1]) \), where \( F \) is a compact, not necessarily connected, surface with nonempty boundary.

**Definition 4.3** ([13, 5]). A sutured manifold \( (M, \gamma) \) is taut if \( M \) is irreducible and \( R(\gamma) \) is both incompressible and Thurston norm minimizing [49] in \( H_2 (M, \gamma) \).

In particular, product sutured manifolds are taut, as are sutured manifolds of the form \( (M, \gamma) = (X_\kappa|_R, \partial X_\kappa|_{\partial R}) \), for \( R \) a (co-oriented) minimal genus Seifert surface for \( \kappa \), or more generally of the form \( (M|_R, \partial M|_{\partial R}) \), where \( \partial M \) is a torus and \( R \) is a properly embedded, co-oriented, compact, Thurston norm minimizing surface with a single boundary component. In the latter cases we will denote \( R_+ (\gamma) \) simply by \( R_+ \) and, similarly, \( R_- (\gamma) \) simply by \( R_- \).

**Definition 4.4** (Definition 3.1 of [13, 5]). Let \( (M, \gamma) \) be a sutured manifold and let \( S \subset M \) be a compact, properly imbedded, transversely oriented surface. Suppose that one of the following holds for each component \( \lambda \) of \( S \cap \gamma \):

1. \( \lambda \) is a properly imbedded, nonseparating arc in \( A(\gamma) \).
2. \( \lambda \) is a simple closed curve in a component \( A \) of \( A(\gamma) \) parallel to and coherently oriented with each component of \( \partial A \).
3. If \( T \) is a component of \( T(\gamma) \) and \( \lambda \subset T \), then \( \lambda \) is an essential circle in \( T \) and every component of \( S \cap T \) is parallel to and coherently oriented with \( \lambda \).

Further suppose that, if \( \alpha \) is a circle component of \( S \cap R(\gamma) \), \( \alpha \) does not bound a disk in \( R(\gamma) \) nor is \( \alpha \) the boundary of a disk component \( D \) of \( S \). Finally, if the component \( \alpha \) of \( S \cap R(\gamma) \) is a properly imbedded arc, suppose that it is not boundary compressible in \( R(\gamma) \). Then \( S \) is called a decomposing surface for \( (M, \gamma) \).

A decomposing surface \( S \) defines a sutured manifold decomposition

\[
(M, \gamma) \xrightarrow{S} (M', \gamma')
\]

where

\[
M' = M|_S
\]
and

\[ \gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+ (\gamma)), \]
\[ R'_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+)|_{A(\gamma)}, \quad \text{and} \]
\[ R'_-(\gamma') = ((R_-(\gamma) \cap M') \cup S'_-)|_{A(\gamma)}, \]

where \( S'_+ (S'_-) \) is that component of \( \partial N(S) \cap M' \) whose normal vector points out of (into) \( M' \).

This sutured manifold decomposition is called taut if both \((M, \gamma)\) and \((M', \gamma')\) are taut.

**Definition 4.5** (Definition 3.4 of [13]). A sutured manifold \((M_0, \gamma_0)\) is decomposable if there is a sequence of taut sutured manifold decompositions

\[(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \ldots \xrightarrow{S_n} (M_n, \gamma_n)\]

such that \((M_n, \gamma_n)\) is a product sutured manifold. If each component of \((M_n, \gamma_n)\) is homeomorphic to \((D^2 \times [0, 1], \partial D^2 \times [0, 1])\), then the sequence is called a taut sutured manifold hierarchy.

**Notation 4.6** (Construction 4.16 of [17]). Suppose \(n \geq 1\) and

\[(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \ldots \xrightarrow{S_n} (M_n, \gamma_n)\]

is a sequence \(S\) of sutured manifold decompositions. Let \(B^G(S) = \langle S_1, \ldots, S_n \rangle\) denote the co-oriented branched surface with underlying co-oriented spine \(S_1 \cup \ldots \cup S_n\).

These branched surfaces \(B^G(S)\) will play a key role in the constructions found in this paper. The superscript \(G\) is used both to remind the reader that this is the branched surface described by Gabai and to distinguish this family of branched surfaces from the branched surfaces \(B(S')\) described in Section 4.

Note that, by definition of decomposing surface, \(B^G(S)\) has no trivial bubbles. It follows immediately from Theorem 4.7 below that \(B^G(S)\) is taut if \(S\) is a sequence of taut sutured manifold decompositions.

We will use extensively the following result of Gabai:

**Theorem 4.7** (Theorems 3.13, 4.2 and 5.1 of [13]). Suppose \((M, \gamma)\) is taut. Then \((M, \gamma)\) is decomposable, with a choice \(S\) of taut sutured manifold hierarchy such that Construction 4.17 of [17] applied to \(B^G(S)\) yields a co-oriented taut foliation fully carried by \(B^G(S)\).

We are primarily interested in the case that \((M, \gamma)\) is the complement of an I-fibered neighbourhood of a taut co-oriented branched surface \(B\) that fully carries a lamination \(\mathcal{L}\). In this setting, we are interested in applying Theorem 4.7 to extend \(\mathcal{L}\) to a co-oriented taut foliation. Construction 4.17 of [17] guarantees that this is always possible. This follows from the fact that there is a branched surface \(B'\), obtained from \(B\) by splitting along at most finitely many compact surfaces of contact, such that \(B'\) fully carries \(\mathcal{L}\) and \((B', \mathcal{L})\) satisfies the following noncompact extension property.

**Definition 4.8** (Noncompact extension property; see [8] for details). Suppose \(B\) fully carries a lamination \(\mathcal{L}\). A component \(A\) of \(\partial_v N(B)\) satisfies the noncompact extension property relative to
the pair \((B, \mathcal{L})\) if there is a copy of \([0, \infty) \times [0, 1]\) properly embedded in \(N(B) \cap M|_L\) with \(\{0\} \times [0, 1]\) contained in an I-fiber of \(N(B)\) and containing an I-fiber of \(A\), and \([0, \infty) \times \{0, 1\}\) contained in leaves of \(\mathcal{L}\).

\((B, \mathcal{L})\) satisfies the noncompact extension property if each component of \(\partial_v N(B)\) does.

**Corollary 4.9** ([8]). Suppose \(B\) is a co-oriented branched surface that fully carries a lamination \(\mathcal{L}\). If \(B\) is taut, then there is a co-oriented taut foliation that contains \(\mathcal{L}\) as a sublamination. Indeed, if \(S\) is the taut sutured manifold hierarchy

\[
(M \setminus \text{int} N(B), \gamma) \overset{S_1}{\to} (M_1, \gamma_1) \overset{S_2}{\to} (M_2, \gamma_2) \cdots \overset{S_n}{\to} (M_n, \gamma_n),
\]

where \(\gamma = \partial_v N(B) \cup (\partial M \setminus \text{int} N(B))\), then the branched surface obtained by applying Gabai’s Construction 4.16 to \(S\), starting with \(B\), yields a taut branched surface that fully carries a taut foliation containing \(\mathcal{L}\) as a sublamination.

The following proposition shows that if the complementary region containing a torus boundary component of \(M\) is of a particular type and also has vertical boundary whose components all satisfy the noncompact extension property, then \(\mathcal{L}\) has extensions to taut, co-oriented foliations realizing every slope, except one, on that boundary component.

**Proposition 4.10** ([8]). Let \(\partial_0 M\) denote a torus boundary component of \(M\). Suppose \(B\) is a taut co-oriented branched surface that fully carries a lamination \(\mathcal{L}\) and is disjoint from \(\partial_0 M\). If the complementary region \((Y, \partial_v Y)\) of \(N(B)\) containing \(\partial_0 M\) is homeomorphic to

\[
(\partial_0 M \times [0, 1], V_1 \cup \cdots \cup V_{2n}),
\]

where \(\partial_0 M \times \{0\} = \partial M\) and \(V_1, \ldots, V_{2n}\) are disjoint essential annuli in \(\partial_0 M \times \{1\}\) that satisfy the noncompact extension property relative to \((B, \mathcal{L})\), then there are co-oriented taut foliations extending \(\mathcal{L}\) that strongly realize all boundary slopes on \(\partial_0 M\) except the one isotopic in \(Y\) to the core of any \(V_i\).

### 5. Modifying sutured manifold decompositions

Some sutured manifold hierarchies are better than others. In this section, we focus on the case that there is a sutured manifold hierarchy

\[
(M_0, \partial M_0) \overset{R}{\to} (M_1, \gamma_1) \overset{S}{\to} (M_2, \gamma_2) \cdots \overset{S_n}{\to} (M_n, \gamma_n)
\]
in which \(R\) is a minimal Seifert surface, and \(S\) is double-diamond taut (see Definition 5.8). In this case, it is possible to construct an associated branched surface that fully carries co-oriented taut foliations that strongly realize all boundary slopes except one. For simplicity of exposition, we assume in what follows that \(\partial M_0\) is connected.

Roughly speaking, the branched surface \(B^G(S)\) of Gabai is modified as follows. Before beginning the hierarchy, a boundary-parallel torus, which we denote by \(T\), is added, separating \(M\) into two components. The component homeomorphic to \(M_0\) then replaces \(M_0\) in the sutured manifold hierarchy above. The union of surfaces \(T \cup R \cup S \cdots \cup S_n\) is smoothed to a branched surface \(B(S')\) in two steps. First, the forthcoming special property of \(S\) (the double-diamond condition)
is used to smooth $T \cup R \cup S$, in such a way that the resulting branched surface is laminar and has taut sutured manifold complement. This complement consists of two components: one is isomorphic to the complement of the branched surface $(R, S)$ and the other is isomorphic to $(T^2 \times [0, 1], (V_1 \cup V_2) \times \{0\})$, where $T^2 \times \{0\} = T, T^2 \times \{1\} = \partial M_0$, and $V_1$ and $V_2$ are disjoint annuli with slope a meridian $\nu$, which in the case of a knot complement is necessarily the usual meridian. In the second step, the addition of the surfaces $S_1, \ldots, S_n$ and the smoothing to $B(S')$ proceed as described in Gabai’s Construction 4.16; see Notation 4.6.

The branched surface $B(S')$ fully carries a lamination $\mathcal{L}$ and has complement the disjoint union of $(M_n, \gamma_n)$ with $(T^2 \times [0, 1], (V_1 \cup V_2) \times \{0\})$. It follows that $\mathcal{L}$ admits extensions to a family of foliations strongly realizing all boundary slopes except $\nu$.

**Notation 5.1.** Let $M = M_0$ be a compact oriented 3-manifold with $\partial M$ a torus. Let $T$ be a boundary parallel torus properly embedded in $M$, and let $X$ and $Y$ denote the components of $M \setminus T$, with $X$ the component homeomorphic to $M$. Associated to a sequence $S$ of sutured manifold decompositions

$$(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \gamma_n),$$

(where we assume that $S_i \cap Y = \partial S_i \times I$, for each $i$) is the isomorphic sequence $S'$

$$(X, T) \xrightarrow{S_1'} (X_1, \gamma_1') \xrightarrow{S_2'} \cdots \xrightarrow{S_n'} (X_n, \gamma_n'),$$

where $S'_i = S_i \cap X$. Let $\Sigma(S)$ denote the co-oriented spine $S_1 \cup \cdots \cup S_n$, and let $\Sigma(S')$ denote the unoriented spine $T \cup S'_1 \cup \cdots \cup S'_n$.

**Definition 5.2.** Two 1-manifolds properly embedded in a common surface intersect efficiently if any intersections are transverse and no isotopy through properly embedded 1-manifolds reduces the number of points of intersection.

**Definition 5.3.** Let $M$ be an oriented 3-manifold with connected boundary $\partial M$ a torus, and suppose $M$ contains a properly embedded, compact, oriented, Thurston norm minimizing surface $R$ with a single boundary component. Call a decomposing surface $S$ for $(M|_R, \partial M|_{\partial R})$ tight if

1. the images of the 1-manifolds $\alpha_+ = S \cap R_+ \cap \alpha_- = S \cap R_-$ under the quotient map $M|_R \to M$ intersect efficiently in $R$, and
2. the sutured manifold decomposition $(M|_R, \partial M|_{\partial R}) \xrightarrow{S} (M', \gamma')$ is taut.

**Definition 5.4 ([15]).** A decomposing surface $D$ is called a product disk if $D$ is a disk and $|D \cap A(\gamma)| = 2$.

**Notation 5.5.** Let $S$ be a tight decomposing surface for $(M|_R, \partial M|_{\partial R})$. Let $\partial_1 S, \ldots, \partial_s S$ be the components of $\partial S$ that have nonempty intersection with $A(\gamma)$. Each such component can be written as a concatenation of arcs that lie alternately in $R_+, \partial M$, and $R_-$. Letting $*$ denote concatenation, and denoting the (even) number of arcs of $\partial_i S$ on $\partial M$ by $2n_i$, we write:

$$\partial_i S = \alpha_{i,1} * \tau_{i,1} * \cdots * \alpha_{i,2n_i} * \tau_{i,2n_i}, \quad 1 \leq i \leq s,$$

where $\alpha_{i,2j-1} \subset R_+, \alpha_{i,2j} \subset R_-$, and $\tau_{i,k} \subset \partial M$, for $j = 1, \ldots, n_i$ and $k = 1, 2, \ldots, 2n_i$. 


Definition 5.6. We call the arcs \(\tau_{i,k}\), as well as the corresponding arcs in the spine \(\Sigma(S')\), transition arcs.

Definition 5.7. Given a framing on \(\partial M\) with longitude given by the oriented curve \(\partial R\) and arbitrary fixed meridian \(\mu_0\), we may talk of the sign of the transition \(\tau_{i,k}\) relative to \(\mu_0\). This sign is defined as follows. Orient \(\tau_{i,k}\) so that it is a path from \(R^+\) to \(R^-\). Associated to the framing \(\partial M = S^1 \times S^1\), with \(S^1 \times \{1\}\) representing the preferred longitude and \(\{1\} \times S^1\) representing the preferred meridian, is a retraction \(r_{\mu_0} : S^1 \times S^1 \rightarrow S^1\) given by \(r_{\mu_0}(z, w) = z\). Define a transition arc to be positive (negative) rel \(\mu_0\) if \(\tau_{i,k}\) can be isotoped in \(\partial M\) rel endpoints so that the restriction of \(r_{\mu_0}\) to \(\tau_{i,k}\) is orientation preserving (reversing). If the meridian \(\mu_0\) is understood, refer to the associated retraction as \(r\) and to a transition arc as being positive or negative. This designation of positive or negative is independent of the choice of orientation chosen on \(R\).

For an alternative viewpoint that is often useful, observe that associated to a transition arc \(\tau_{i,k}\) (oriented as above), there are exactly two choices of meridian with representatives disjoint from \(\tau_{i,k}\). These are obtained from the two simple closed curves obtained by pasting \(\tau_{i,k}\) with one of the two components of \(\partial R|_{\partial \tau_{i,k}}\). For either such choice of meridian, \(v = r(\tau_{i,k})\) is a proper arc of \(S^1 \times \{1\}\), and \(\tau_{i,k}\) is positive or negative, respectively, if \(v\) inherits positive or negative orientation along \(\partial R\) from \(\tau_{i,k}\), as illustrated in Figure 7. The transition arc is positive with respect to one choice and negative with respect to the other.

![Positive and negative transition models.](image)

Definition 5.8. Suppose that \(S\) is a tight decomposing surface for \((M|R, \partial M|_{\partial R})\) that is not a product disk. For simplicity of notation, identify arcs in \(R^+ \cup R^-\) with their images in \(R\) under the quotient map \(M|_R \rightarrow M\). Suppose that \(s \geq 1\) and there exist \(i, j\) such that \(\alpha_{i,j}\) and \(\alpha_{i,j+1}\) are isotopic through proper embeddings in \(R\), and let \(\Delta\) be the component of \(R|_{\alpha_{i,j}\cup\alpha_{i,j+1}}\) that contains \(\tau_{i,j}\). If \(\Delta\) is a source when considered as a sector of the branched surface \((R, S)\), then call \(S\) and the sutured manifold decomposition \((M|R, \partial M|_{\partial R}) \xrightarrow{S} (M', \gamma')\) double-diamond taut (with respect to \(\alpha_{i,j} \ast \tau_{i,j} \ast \alpha_{i,j+1}\)).

In particular, a double-diamond taut decomposition is taut. In addition, a double-diamond taut decomposition determines a distinguished meridian, which we denote by \(\nu\), defined as follows.
Definition 5.9. Suppose $S$ is double-diamond taut with respect to $\alpha_{i,j} \ast \tau_{i,j} \ast \alpha_{i,j+1}$. Define the distinguished meridian $\nu$ on $\partial M$ to be the isotopy class of

$$\nu_{i,j} = \tau_{i,j} \cup v_{i,j},$$

where $v_{i,j}$ in the unique arc in $\partial R \cap \Delta$ joining the endpoints of $\tau_{i,j}$.

Note that $\nu_{i,j}$ is a simple closed curve that has geometric intersection number one with $\partial R$ and hence is a meridian. Since $S$ is tight and $\alpha_{i,j}$ and $\alpha_{i,j+1}$ are isotopic, the transition arcs $\tau_{i,j-1}, \tau_{i,j},$ and $\tau_{i,j+1}$ have the same sign rel $\nu$. For each transition arc $\tau_{i,k}$, let $\nu_{i,k}$ denote the representative $\tau_{i,k} \cup r(\tau_{i,k})$ of $\nu$. It will follow from Corollary 5.11 that if $M = X_\kappa$ for a knot $\kappa$, then $\nu = \mu$, the meridian of $\kappa$.

In the following theorem statement and proof, we use Notation 5.1. In particular, $T$ is a boundary parallel torus properly embedded in $M$. We also refer to the curve on $T$ isotopic to $\nu$ as the distinguished meridian and use the same notation for it.

Theorem 5.10 (Double-diamond replacement). Let $S_0$ be a sequence of taut sutured manifold decompositions

$$(M, \partial M) \xrightarrow{\text{R}} (M_1, \gamma_1) \xrightarrow{\text{S}} (M_2, \gamma_2)$$

such that $\partial M$ is a torus, $\partial R$ is connected and nonempty, and $S$ is double-diamond taut with respect to

$$\alpha_{i,j} \ast \tau_{i,j} \ast \alpha_{i,j+1} \subset \partial S.$$

Let $\Delta$ be the component of $R \mid_{\alpha_{i,j} \cup \alpha_{i,j+1}}$ that contains $\tau_{i,j}$, and suppose each sector of $\Sigma(S_0') \cap \Sigma(S_0)$ except $\Delta$ inherits its co-orientation from that of $\Sigma(S_0)$. Give $\Delta$ the opposite orientation from $R$. There is a choice of orientations on the sectors of $T$ in $\Sigma(S_0')$ so that $\Sigma(S_0')$ admits a smoothing to a (co-oriented) taut branched surface $B(S_0')$ that has complement comprised of two sutured manifold components: one is $B^G(S_0)$ and the other is $(\partial M \times [0,1], V_1 \cup V_2)$, where $\partial M \times \{0\} = \partial M$ and $V_1$ and $V_2$ are disjoint annuli of slope $\nu$, the distinguished meridian determined by $S$.

Proof. In order to simplify diagrams, we enhance arrow notation for cusp directions with diamond notation, as originally introduced by Wu [52] and depicted in Figure 8.

```

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (a) at (0,0) {\includegraphics[width=0.5\textwidth]{figure8.png}};
\end{tikzpicture}
\caption{Diamond notation}
\end{figure}
```

The desired co-orientations on the sectors of $T$, along with the resulting smoothing and the paths of the sutures that result, are indicated for negative transitions in Figure 8 ; the construction for positive transitions is given by the mirror image. A schematic treatment for negative transitions
is provided in Figure 10 and for positive transitions in Figure 11. (The reason for the mysterious term “double-diamond” should now be clear!) The local models for co-orientations and associated smoothings in neighborhoods of $\nu_{i,j-1}, \nu_{i,j}$ and $\nu_{i,j+1}$ are highlighted in Figures 10 and 11. It is straightforward to check that the resulting co-orientation on $\Sigma(S'_0)$ admits a smoothing to a co-oriented branched surface, which we denote by $B(G(S_0))$.

The local smoothings at $\nu_{i,j-1}$ and $\nu_{i,j}$ (marked C in Figures 10, 11 and 12) each introduce a cusp of slope $\nu$ on $T$ with sink direction pointing into $R$ and $S$, resulting in a suture on the boundary of the complementary region $Y$ containing $\partial M$. As shown in Figures 9 and 12, the remaining complementary region is isotopic to the complement of $B(G(S_0))$. The complementary regions of $B(S'_0)$ are therefore taut sutured manifolds.

It remains only to show that through every sector of $B(S'_0)$ there is a closed oriented curve that is positively transverse to $B(G(S_0))$. For points in $\Delta$, such a transversal is provided by a curve $\alpha$ parallel to $\nu$, appropriately oriented. It follows from Theorem 4.7 and Definition 5.3 that $B(G(S_0))$ is taut; therefore, through any point in a sector disjoint from $T$ and $\Delta$ there is a closed positive transversal to $B(G(S_0))$. It is possible that this transversal passes through $\Delta$, in which case such intersections may be removed by inserting copies of $\alpha$. Finally, let $\Gamma$ be any sector of $B(S'_0)$ that is contained in $T$. Without loss of generality, we may assume that a positive transversal to $\Gamma$ points into a complementary region disjoint from $\partial M$. Let $\beta_1$ be an arc properly embedded in the complement of $B(S'_0)$ that runs from $\Gamma$ to $S_-$ (viewed as pushed slightly off the negative side of $S$), let $\beta_2$ be a simple arc in $S_-$ from the endpoint of $\beta_1$ to a point in a sector of $T$ adjoining a meridional cusp, and let $\beta_3$ be a simple arc properly embedded in the complementary region containing $\partial M$ that connects $\beta_3$ to $\beta_1$. The simple closed curve $\beta = \beta_1 \ast \beta_2 \ast \beta_3$ is the desired transversal. (See Figure 13.)

We have now established the desired taut branched surface, but in contrast to Gabai’s branched surface $B(G(S_0))$, it is not inherent in the construction of $B(S'_0)$ that it fully carries a lamination. However, an easily checked technical condition on $B(G(S_0))$ suffices to ensure that it does. We then apply Corollary 4.9 and Proposition 4.10 to show this lamination extends to taut foliations that strongly realize all boundary slopes except the distinguished meridian $\nu$.

**Corollary 5.11.** Let $S_0$ be a sequence of taut sutured manifold decompositions

$$(M, \partial M) \xrightarrow{R} (M_1, \gamma_1) \xrightarrow{S} (M_2, \gamma_2)$$

such that $\partial M$ is a torus, $\partial R$ is connected and nonempty, and $S$ is double-diamond taut with respect to

$$\alpha_{i,j} \ast \tau_{i,j} \ast \alpha_{i,j+1} \subset \partial S.$$

If $B(G(S_0))$ has no sink disks disjoint from $\partial M$, then there are co-oriented taut foliations that strongly realize all boundary slopes except $\nu$, the distinguished meridian determined by $S$. Moreover, if $M = X_\kappa$ for some knot $\kappa$ in an L-space, $\kappa$ is persistently foliar.
Proof. Let $B(S'_0)$ be the taut, and hence essential, branched surface obtained from $B^G(S_0)$ by double-diamond replacement, as described in Theorem 5.10. Since $B^G(S_0)$ contains no trivial bubbles, neither does $B(S'_0)$. We show that $B(S'_0)$ contains no sink disks, and hence is laminar.

The sectors of $B(S'_0)$ that are contained in $T$ correspond to digon complementary regions of $B^G(S_0) \cap \partial M$ and hence are not sink disks. We therefore restrict attention to the sectors of $B(S'_0)$ that are not contained in $T$. These are in bijective correspondence with, and isomorphic to, those in $B^G(S_0)$, with arrows replaced by diamonds at $\alpha_{i,j} \cup \alpha_{i,j+1}$. In addition, outward pointing arrows are introduced along boundary arcs on $T$ except for those adjacent to one of the two meridian cusps, marked $C$ in Figures 10 and 12. Any sector with boundary that has nonempty intersection with $T$ therefore has outward pointing arrow along $T$; in particular, this is true for the sectors that have nonempty intersection with $\alpha_{i,j} \cup \alpha_{i,j+1}$. Moreover, by assumption, any sector that has boundary disjoint from $T$ is not a sink disk. Thus, $B(S'_0)$ is laminar, and therefore fully carries a lamination $\mathcal{L}$.

Since any leaf of $\mathcal{L}$ that passes through the I-fibered neighbourhood of the branch $S$ is asymptotic to $T$, it is noncompact. Hence the conditions of Proposition 4.10 are satisfied. It therefore follows
from Corollary 4.9 and Proposition 4.10 that $M$ supports co-oriented taut foliations that strongly realize all boundary slopes except $\nu$. In the case $M = X_\kappa$ for a knot $\kappa$ in an L-space, it must be the case that $\nu = \mu$, the meridian of $\kappa$, since by [38, 2, 26, 27] an L-space supports no co-oriented taut foliation; hence $\kappa$ is persistently foliar.

□

Remark 5.12. If $S$ is a tight product disk and $\alpha_1 = \alpha_{1,1}$ and $\alpha_2 = \alpha_{1,2}$ are isotopic through proper embeddings in $R$, then after isotoping $D$ so that $\alpha_1 = \alpha_2$, $D$ determines a meridional
annulus, and hence $\kappa$ is composite, as illustrated in Figure 14. For some results on persistently foliar composite knots, see [7].

Remark 5.13. Definition 5.3 admits a straightforward generalization to sutured manifold decompositions of greater length; namely, to appropriately constrained decompositions of the form

$$(M, \partial M) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} (M_n, \gamma_n) \xrightarrow{S} (M', \gamma').$$

This generalization then leads naturally to a generalization of Theorem 5.10. As these generalizations require significantly more cumbersome notation, and have yet to prove useful in explicit applications, we leave their statements as exercises for the interested reader.

6. Application to Murasugi sums

Definition 6.1 ([12]). Let $n \geq 1$. The oriented surface $R$ in $S^3$ is a Murasugi sum along $P$ of oriented surfaces $R_1$ and $R_2$ if

1. there is a sphere $S$ in $S^3$ such that $P = S \cap R$ is a $2n$-gon embedded in $R$ with boundary $\partial P = \alpha_1 \ast \beta_1 \ast \cdots \alpha_n \ast \beta_n$ such that each $\alpha_i$ is a subarc of $\partial R$ and each $\beta_i$ is properly embedded in $R$, and

2. if $B_1$ and $B_2$ are the two embedded 3-balls with common boundary $S$, then for each $i$, $R_i$ is homeomorphic to the subsurface $B_i \cap R$. 

Figure 13. Positive closed transversals for $\Delta$ and the sectors of $T$ in $B(S')$.

Figure 14. If $S$ is a tight product disk and $\alpha_1 = \alpha_{1,1}$ and $\alpha_2 = \alpha_{1,2}$ are isotopic through proper embeddings in $R$, then $\kappa$ is composite.
When \( n = 2 \), a Murasugi sum is also known as a plumbing (see Figure 15). For a nice history of the definitions of plumbing and Murasugi sum, see [37].

**Theorem 6.2 ([12 [14]).** Suppose \( R \subset S^3 \) is a Murasugi sum along \( P \) of oriented surfaces \( R_1 \) and \( R_2 \). Then \( R \) is a minimal genus Seifert surface for \( \partial R \) if and only if each \( R_i, i = 1, 2 \), is a minimal genus Seifert surface for \( \partial R_i \).

**Corollary 6.3.** Suppose \( \kappa \) is a knot in \( S^3 \) with minimal genus Seifert surface \( R \). If \( R \) is a plumbing of surfaces \( R_1 \) and \( R_2 \), where \( R_2 \) is an unknotted band with an even number \( 2m \geq 4 \) of half-twists, then the decomposing disk for \( R_2 \) can be chosen to be double-diamond taut. Thus, \( \kappa \) is persistently foliar.

**Proof.** The decomposing disk \( D \) dual to the band \( R_2 \) is double-diamond taut, as indicated in Figure 15. Moreover, \( B^G(S_0) = \langle R, D \rangle \) has no sink disks disjoint from \( \partial X_\kappa \). The result therefore follows immediately from Corollary 5.11. The relevant parts of \( B^G(S_0) \) and \( B(S'_0) \) are shown in Figure 16.

\[ \square \]

**Figure 15.** \( B(S'_0) \) for a positive plumbed band.

**Figure 16.** Comparison of the sutures in the complements of \( B^G(S_0) \) and \( B(S'_0) \).
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