A nonlinear observer to estimate unknown parameters during the synchronization of chaotic systems

L. Torres

Abstract—This paper proposes an Extended-Kalman-Filter-like observer for parameter estimation during synchronization of chaotic systems. The exponential stability of the observer is guaranteed by a persistent excitation condition. This approach is shown to be suitable for various classical chaotic systems and several simulations are presented accordingly.

I. INTRODUCTION

Synchronization phenomena are commonly found in natural sciences as well as in engineering. Because indeed nature is rich in connections, interactions and communications, it is often found that nonlinear systems are made of several subsystems which interact with each other.

Several works have thus been carried out on synchronization of chaotic systems. Among them, one can cite the pioneering work of [1], and other very important contributions in [2]-[3].

Synchronization by means of state observers has been early discussed in [4]. In this work, state observers are presented as slave systems which synchronize with a master system. In subsequent works, such as in [6]-[5], state observers have been used to further deal with a parameter-mismatching between the synchronizing systems. It indeed appears that the estimation of parameters during synchronization can be addressed by the use of adaptive observers. Such observers usually require some specific excitation (so-called persistent excitation) for a guarantee of convergence, and an idea in those works was that chaotic behaviours can be expected to provide such an excitation.

In the present paper, following the same idea, we propose to use a new observer for the purpose of parameter estimation and synchronization of chaotic systems: this observer combines the well-known Extended Kalman Filter (EKF) [9], with the ‘high gain’ technique originally proposed in [11] for so-called uniformly observable systems. In fact, the proposed observer derives from an observer presented in [10] which is not specific to such uniformly observable systems, in the sense that it can be used for systems with singular inputs w.r.t. observability, provided that some excitation condition is satisfied. This observer design further requires a specific structure of the underlying state-space representation.

The purpose here is to show that for various classical chaotic systems such an appropriate structure can indeed be obtained when extending the state vector with the parameters to be estimated, by means of an immersion methodology presented in [10]. It is then checked by simulations that the observer convergence is obtained, achieving synchronization with parameter estimation.

Section 2 presents the proposed observer, while section 3 provides some illustrative examples. Section 4 concludes the paper.

II. EKF-LIKE OBSERVER

The proposed observer is based on a state-space representation with the structure below:

\[\dot{\hat{x}} = A(u,y)\hat{x} + B(u,z)\]
\[y = C(u)\hat{x} + D(u)\]  

where the involved matrices read as:

\[A(u,y) = \begin{pmatrix} 0 & A_{12}(u,y) & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{q-1,q}(u,y) \end{pmatrix}\]

\[B(u,z) = \begin{pmatrix} B_1(u,z_1) \\ B_2(u,z_2) \\ \vdots \\ B_{q-1}(u,z_1,\ldots,z_{q-1}) \\ B_q(u,z) \end{pmatrix}\]

and

\[C(u) = \begin{pmatrix} C_1(u) & 0 & \cdots & 0 \end{pmatrix},\]

with \(z = \text{col}(z_1,\ldots,z_q) \in \mathbb{R}^N\), \(z_i \in \mathbb{R}^{N_i}\), \(A_i \in \mathbb{R}^{N_i \times N_i}\) and \(z_i \in \mathbb{R}^{N_i}\), \(A_{i-1} \in \mathbb{R}^{N_i \times N_i}\), for \(i = 2,\ldots,q\). From this structure, it can be seen that the system is not necessarily uniformly observable, namely its observability depends on the inputs.

For the observer convergence, a specific excitation condition will then be required, defined in terms of inputs as follows:

Definition 1 (Locally regular inputs [12], [13], [10]):

An input function \(u\) is said to be locally regular for a system (I) if there exist \(\alpha > 0\), \(\lambda_0 > 0\) such that for all \(\lambda \geq \lambda_0\) all \(t \geq \frac{1}{\alpha}\) and all \(x_0 \in \mathbb{R}^N\):

\[\int_{t-\frac{1}{\alpha}}^t \Phi_{u,x_0}(\tau,t)^TC(u)C(u)\Phi_{u,x_0}(\tau,t)d\tau \geq \alpha \lambda \Lambda^{-2}(\lambda)\]

where

\[\Lambda(\lambda) = \begin{pmatrix} \lambda N_1 & 0 \\ \lambda^2 I_{N_2} & \ddots \\ 0 & \lambda^3 I_{N_q} \end{pmatrix}\]

The author is with the Engineering Institute, UNAM.
mailto:forteres@ingen.unam.mx
and
\[ \frac{d\Phi_{u,x_0}(\tau,t)}{d\tau} = A(u(\tau),y_{u,x_0}(\tau))\Phi(\tau,t)_{u,x_0}, \]
\[ \Phi(t,t) = I_N, \]
with \( y_{u,x_0}(t) \) denoting the output trajectory of system \( \Pi \) under input \( u \) when starting from state \( x_0 \) at \( t = 0 \) and \( I_n \) denoting the identity matrix in \( R^n \).

It has already been shown that with such an excitation, and under the usual technical (Lipschitz) assumption for a high gain design, one can obtain asymptotic estimation of the state for system \( \Pi \) with some high-gain Kalman-like observer [10].

In the present paper, a modified version will be instead considered, recovering equations of a classical Extended-Kalman-Filter type but with a guarantee of convergence. This is done in the same spirit as the so-called high-gain EKF proposed in [8] or discussed in [14] for the particular case of uniformly observable structures, but extended to our case of possibly non uniformly observable systems:

**Theorem 1:** Under the excitation condition of Definition \( \Pi \) with \( u \) bounded and making \( A(u,y) \) bounded, if \( B \) is globally Lipschitz in \( z \) uniformly in \( u \), then an exponential observer can be designed as follows: For every \( \sigma > 0 \), there exist \( \lambda, \gamma > 0 \) such that the system:
\[
\begin{align*}
\dot{z} &= A(u,y)\dot{z} + B(u,z) - \lambda(\gamma)SC^T(u)(y - y); \\
\dot{y} &= C(u)\dot{z} + D(u) \\
\dot{S} &= \lambda(\gamma)S + [A(u,y) + dB\dot{u}\dot{z}]S \\
&+ S[A(u,y) + dB\dot{u}\dot{z}]^T - SC^T(u)(C(u)S)
\end{align*}
\]  \hspace{1cm} (3)
with \( \lambda(\gamma) \) as in \( \Pi \), \( dB \) \( \frac{1}{\lambda} \Lambda^{-1}(\lambda)\frac{\partial B}{\partial z} \Lambda(\lambda) \), and initial conditions \( \dot{z}(0) \in R^N, S(0) \geq 0 \), ensures:
\[ ||z(t) - \hat{z}(t)|| \leq \mu e^{-\sigma t}, \mu > 0, \forall t \geq \frac{1}{\lambda}. \]

This result can be established with similar arguments as in [10] on the one hand, and [8] on the other hand, basically using two technical results:

**Lemma 1:** Consider \( S \) as in theorem \( \Pi \) with the same assumptions holding true.
Then \( \exists \gamma > 0 \) and \( \alpha_1, \alpha_2 > 0 \) such that \( \forall \lambda \geq 0 \) and \( \forall t \geq \frac{1}{\lambda^2} \), \( \alpha_1 S_N \leq S^{-1}(t) \leq \alpha_2 S_N \).

From this, a Lyapunov function can be built for the error system:

**Lemma 2:** Consider \( S \) as in theorem \( \Pi \) with the same assumptions holding true.
Then if \( e := \hat{z} - z \) stands for the observation error, \( V(t,e) := e^T\Lambda^{-1}(\lambda)S^{-1}(t)\Lambda^{-1}(\lambda)e \) is a Lyapunov function for the error system, satisfying \( \dot{V} \leq -\alpha(\lambda)V \) for \( \lambda \) large enough and a strictly positive increasing function \( \alpha \), along the trajectories of \( e \).

From this, and standard Lyapunov arguments, the result of Theorem \( \Pi \) is obtained.

In short, lemma \( \Pi \) can be established by inspection of solutions for \( S \) in equation (3), and lemma \( \Pi \) by direct computation of \( \dot{V} \) along trajectories of \( e \), while full details of the proof are presented in [7].

### III. Application to Various Chaotic Systems

In this section it will be shown how the observer of Theorem \( \Pi \) can be applied to estimate states and parameters in a few typical chaotic systems. For each system, the conception of its observer begins with the inclusion of its parameters and states into an extended state vector of a new system.

It is then shown how this new system can be turned into the suitable form for the proposed observer design, by using the immersion procedure described in [10]. The observer gains are finally tuned by the parameters of the adaptive law given in Theorem 1, \( \lambda > 0 \) and \( \gamma > 0 \). Both parameters allow to tune the observer rate of convergence.

As a first example, a Unified Chaotic System (UCS) is considered with a single unknown parameter, whereas the second example takes into account the Lorenz and Rössler systems with two unknown parameters. In both cases, the design procedure is meticulously described and some figures are given to expose simulation results.

#### A. Example 1: Unified Chaotic System

A system which unifies the Lorenz, Chen and Lü systems is proposed in [16]. This system is called Unified Chaotic System and it is expressed by the following equations:
\[
\begin{align*}
\dot{x}_1 &= (25\alpha + 10)(x_2 - x_1) \\
\dot{x}_2 &= (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2 \\
\dot{x}_3 &= x_1x_2 - (\alpha + \frac{8}{3})x_3
\end{align*}
\] \hspace{1cm} (4)

System \( \Pi \) admits:
* The Lorenz attractor if \( 0 \leq \alpha \leq 0.8 \).
* The Chen attractor if \( \alpha = 0.8 \).
* The Lü attractor if \( 0.8 \leq \alpha \leq 1 \).

In what follows, an observer with the structure given by Eq. \( \Pi \) is proposed in order to estimate the parameter \( \alpha \) and the states \( x_2 \) and \( x_3 \) from the available state \( x_1 \). To do this task, system \( \Pi \) must be changed into the adequate form given by \( \Pi \).

Thus, fixing \( z_1 = x_1, z_2 = (25\alpha + 10), z_3 = (25\alpha + 10)x_2 \) and \( z_4 = (25\alpha + 10)x_3 \), the following system is the desired version of \( \Pi \) to design the observer:
\[
\begin{align*}
\dot{z} &= \begin{bmatrix} 0 & -y & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -y \\
0 & 0 & 0 & 0 \\
\end{bmatrix} z \\
&+ \begin{bmatrix} 0 \\
(28 - 35\alpha\frac{10}{25})y_{z_2} + (29\alpha\frac{10}{25} - 1)z_3 \\
(y_{z_3} - \frac{1}{(x_2 - 10/25)} + \frac{3}{2})z_4 \\
\end{bmatrix}
\end{align*}
\]
For the simulation tests, the observer was tuned with $S(0) = I_d$, $\lambda = 5$ et $\gamma = 2$.

The results of the tests are presented in Fig. 1 where it can be appreciated the good performance of the observer to estimate different values of $\alpha$ ($\alpha = 0$, $\alpha = 0.8$, $\alpha = 1$).

![Estimation of $\alpha$ for the Unified Chaotic System](image)

**Fig. 1. Estimation of $\alpha$ for the Unified Chaotic System**

### B. Example 2: Lorenz and Rössler systems with two unknown parameters

So-called Lorenz and Rössler equations are well-known chaotic systems with typical attractors in nonlinear sciences. They are expressed by the ordinary differential equations summarized in Table II. Here, those famous systems are treated as benchmarks in order to show how the proposed observer can work as a parameter estimator for chaotic systems. The considered set of parameters for each system, as well its initial state, are given in Tables II and III respectively. Notice that the excitation condition of Definition II is parameterized by the input, and the initial conditions, namely it makes sense also for Lorenz and Rössler systems with no input. Assuming that $\theta_3$ is known and that only one state variable is available, the aim is to reconstruct the unknown parameters and the states. This purpose can be achieved by transforming each system into the appropriate structure of system I. In both cases, the transformation gives 5-dimensional systems which are obtained by means of the following immersion maps:

- **Lorenz System:** if $y = x_1$ then
  
  $$x \rightarrow z = [x_1 \ \theta_1 \ x_1 \ x_2 \ \theta_1 x_2 \ \theta_1 x_3]$$
  
  (5)

- **Rössler System:** if $y = x_2$ then
  
  $$x \rightarrow z = [x_2 \ \theta_1 \ x_1 \ x_3 \ \theta_2]$$
  
  (6)

The mapping (5) gives system (7) below, while the mapping (6) gives system (8). In accordance with Theorem 1, an observer for any of the systems in Table II can be conceived through the transformed systems (7) and (8).

| SYSTEM | INITIAL CONDITIONS FOR SYSTEMS |
|--------|--------------------------------|
| Lorenz | $x_1(0) = 1, x_2(0) = 1, x_3(0) = 1$ |
| Rössler | $x_1(0) = 4, x_2(0) = 4, x_3(0) = 0.1$ |

The observer tuning parameters, $\lambda$ and $\gamma$, as well as the initial conditions for each case, are given in Table IV and Table V. The Fig. 2 illustrates the good estimation realized by the Lorenz observer, while Fig. 3 shows the estimation results of the Rössler system.

### Table I

| SYSTEM | Chaotic Systems |
|--------|----------------|
| Lorenz | $x_1 = \theta_1(x_2 - x_1)$ |
| Rössler | $x_1 = -x_2 - x_3$ |

### Table II

| SYSTEM | PARAMETERS FOR SYSTEMS |
|--------|------------------------|
| Lorenz | $\theta_1 = 10, \theta_2 = 45.6, \theta_3 = 4$ |
| Rössler | $\theta_1 = 0.4, \theta_2 = 0.1, \theta_3 = 14$ |

### Table III

| SYSTEM | INITIAL CONDITIONS FOR OBSERVERS, EXAMPLE 2 |
|--------|---------------------------------------------|
| Lorenz | $\lambda = 1, \gamma = 10$ |
| Rössler | $\lambda = 1, \gamma = 2, \|A(u, y)\|$ |

### Table IV

| SYSTEM | INITIAL CONDITIONS FOR OBSERVERS, EXAMPLE 2 |
|--------|---------------------------------------------|
| Lorenz | $z_1(0) = 10, z_2(0) = 15, z_3(0) = 190$ |
| Rössler | $z_1(0) = 760, z_2(0) = 190.$ |

| SYSTEM | INITIAL CONDITIONS FOR OBSERVERS, EXAMPLE 2 |
|--------|---------------------------------------------|
| Lorenz | $z_1(0) = 2, z_2(0) = 0.2, z_3(0) = 2$ |
| Rössler | $z_4(0) = 0.2, z_5(0) = 0.2.$ |
IV. CONCLUSIONS AND FUTURE WORKS

In this work an approach to the problem of state and parameter reconstruction in several chaotic systems has been proposed. The approach is subject to a persistent excitation condition on the one hand, in this case seemingly provided by the chaotic behaviors of the considered systems, and to the existence of a suitable coordinate transformation on the other hand. This change of coordinates has been shown to be easily achieved for the various considered examples, by following an immersion procedure described in [10]. From this, synchronization with parameter estimation could be obtained for all those systems.

REFERENCES

[1] L. M. Pecora, and T. L. Carroll, “Synchronization in chaotic systems,” *Phys. Rev. A*, vol. 64, pp. 821–824, 1990.
[2] E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic Synchronization*, Applications to living systems. World Scientific, 2002.
[3] J. H. Park, “Chaos synchronization of a chaotic system via nonlinear control,” *Chaos, Solitons and Fractals*, vol. 25, pp. 579–584, 2005.
[4] H. Nijmeijer, and I. Mareels, “An observer look at synchronization,” *IEEE Trans. Circuits Syst. I*, vol. 44, no. 10, pp. 882–890, 1997.
[5] A. Loria, E. Panteley, and A. Zavala-Rio “Adaptive Observers With Persistence of Excitation for Synchronization of Chaotic Systems,” *IEEE Trans. Circuits Syst. I*, vol. 56, no. 12, pp. 2703–2716, 2009.
[6] G. Besançon, J. De Leon-Morales, and J. Huerta-Guevara, “On adaptive observers for state affine systems,” *International Journal of Control*, vol. 79, no. 6, pp. 581–591, 2006.
[7] L. Torres, *Modèles et observateurs pour les systèmes d’écoulement sous pression. Extension aux systèmes chaotiques*. Université de Grenoble, 2011.
[8] J. P. Gauthier and I. A. K. Kupka, *Deterministic observation theory and applications*. Cambridge Univ. Press, 2001.
[9] A. Gelb, *Applied Optimal Estimation*. MIT Press, Cambridge, 1984.
[10] G. Besançon and A. Ticlea, “An immersion-based observer design for rank-observable nonlinear systems,” *IEEE Trans. on Automatic Control*, vol. 52, no. 1, pp. 83–88, 2007.
[11] J. Gauthier, H. Hammouri, and S. Othman, “A simple observer for nonlinear systems - applications to bioreactors,” *IEEE Trans. on Automatic Control*, vol. 37, no. 6, pp. 875–880, 1992.
[12] G. Bornard, F. Celle-Couenne, and G. Gilles, “Observability and observers,” in *Nonlinear Systems - T.1, Modeling and Estimation*. Chapman & Hall, London., 1995, pp. 173–216.
[13] G. Besançon, “Further results on high gain observers for nonlinear systems,” in *Proc. 38th IEEE Conf. on Decision and Control*, Phoenix, AZ, USA, 1999.
[14] N. Boizot and E. Busvelle, *Nonlinear observers and applications*. Springer, LNCIS 363 (G. Besançon Ed.), 2007, ch. Adaptive-gain observers and applications.
[15] J. L. Hindmarsh and R. M. Rose, “A Model of Neuronal Bursting using Three Coupled First Order Differential Equations,” *Proc. R. Soc. Lond.*, vol. 221, no. 1222, pp. 87–102, 1984.
[16] J. Lu and X. Wu and L. Lü, “Synchronization of a unified chaotic system and the application in secure communication,” *Physic Letters A*, vol. 305, pp. 365–370, 2002.