Pseudoscalar Meson Decay Constants and Couplings, the Witten-Veneziano Formula beyond large $N_c$, and the Topological Susceptibility

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ABSTRACT: The QCD formulae for the radiative decays $\eta, \eta' \rightarrow \gamma\gamma$, and the corresponding Dashen–Gell-Mann–Oakes–Renner relations, differ from conventional PCAC results due to the gluonic $U(1)_A$ axial anomaly. This introduces a critical dependence on the gluon topological susceptibility. In this paper, we revisit our earlier theoretical analysis of radiative pseudoscalar decays and the DGMOR relations and extract explicit experimental values for the decay constants. This is our main result. The flavour singlet DGMOR relation is the generalisation of the Witten-Veneziano formula beyond large $N_c$, so we are able to give a quantitative assessment of the realisation of the $1/N_c$ expansion in the $U(1)_A$ sector of QCD. Applications to other aspects of $\eta'$ physics, including the relation with the first moment sum rule for the polarised photon structure function $g_1^\gamma$, are highlighted. The $U(1)_A$ Goldberger-Treiman relation is extended to accommodate $SU(3)$ flavour breaking and the implications of a more precise measurement of the $\eta$ and $\eta'$-nucleon couplings are discussed. A comparison with the existing literature on pseudoscalar meson decay constants using large-$N_c$ chiral Lagrangians is also made.

*This research is supported in part by PPARC grants PP/G/O/2002/00470 and PP/D507407/1.
1. Introduction

The phenomenology of the pseudoscalar mesons opens a window on many interesting aspects of the non-perturbative dynamics of QCD, including spontaneous chiral symmetry breaking, the electromagnetic and gluonic axial anomalies, the OZI rule, the gluon topological susceptibility, and so on. Nevertheless, until comparatively recently, phenomenological analyses did not take fully into account the role of the gluonic $U(1)_A$ anomaly in this sector, with the result that most of the existing determinations of the pseudoscalar meson decay constants and couplings are based on over-simplified theoretical formulae which miss the most interesting anomaly-sensitive physics.

In a previous paper [1] (see also [2]) we have analysed in detail the radiative decays of the neutral pseudoscalar mesons, $\pi^0, \eta, \eta' \to \gamma\gamma$, and provided a set of formulae describing these processes together with modified Dashen–Gell-Mann–Oakes–Renner (DGMOR) [3, 4] relations which fully include the effect of the anomaly and the gluon topological susceptibility in the flavour singlet sector as well as explicit flavour $SU(3)$ breaking. In this paper, we confront these formulae with experimental data to extract a set of results for the four pseudoscalar meson decay constants, $f_{0\eta'}, f_{0\eta}, f_{8\eta'}$, and $f_{8\eta}$.

Our approach, which is based on a straightforward generalisation of conventional PCAC to include the anomaly in a renormalisation group consistent way, is unique in that it is theoretically consistent yet does rely on using the $1/N_c$ expansion. The decay formulae and DGMOR relations we derive are valid for all $N_c$. At some point, however, the fact that the $\eta'$ is not a Nambu-Goldstone (NG) boson means that to make predictions from our formulae we need to augment the standard dynamical approximations of PCAC or chiral perturbation theory with a further dynamical input. Here, we make a judicious use of $1/N_c$ ideas to justify the use of the lattice calculation of the topological susceptibility in pure Yang-Mills theory as an input into our flavour singlet DGMOR relation. The self-consistency of this approach allows us to test the validity of $1/N_c$ methods in the $U(1)_A$ sector against real experimental data. In particular, the flavour singlet DGMOR relation is the finite-$N_c$ generalisation of the well-known Witten–Veneziano formula [5, 6], and we can use our results to study how well the large-$N_c$ limit is realised in real QCD. This is especially important in view of the extensive use of large $N_c$ in modern duality-based approaches to non-perturbative gauge theories such as the AdS-CFT correspondence.

The relation of our analysis of the low-energy decays $\eta'(\eta) \to \gamma\gamma$ to high-energy physics in the form of the first moment sum rule for the polarised photon structure function $g_1^p$ [7, 8] is also discussed. The dependence of this sum rule on the photon momentum encodes many aspects of non-perturbative QCD physics and its full measurement is now within the reach of planned high-luminosity $e^+e^-$ colliders [9]. Our methods can also be readily extended to a variety of other reactions involving the pseudoscalar mesons, including $\eta'(\eta) \to V\gamma$, where $V$ is one of the flavour singlet vector mesons $\rho, \omega, \phi$, $\eta'(\eta) \to \pi^+\pi^-\gamma$, a variety of electro and photoproduction reactions such as $\gamma p \to p\eta'(\eta)$ and $\gamma p \to p\phi$, and $\eta$ and $\eta'$ production in $pp$ collisions, $pp \to pp\eta'(\eta)$.

The latter are relevant for determining the couplings $g_{\eta NN}$ and $g_{\eta' NN}$ which occur in generalisations of the Goldberger-Treiman relation [10]. In this paper, we present a
new version of the $U(1)_A$ Goldberger-Treiman relation [11, 12, 13] incorporating flavour octet-singlet mixing in the $\eta - \eta'$ sector. We are then able to use our results for the decay constants to show how the physical interpretation of this relation, which is closely related to the ‘proton spin’ problem in high-energy QCD (i.e. the first moment sum rule for the polarised proton structure function $g_1^p$), depends critically on the experimental determination of $g_{\eta' NN}$.

Finally, in an appendix, we give a brief comparison of the similarities and differences between our approach and an alternative theoretically consistent framework for describing $\eta$ and $\eta'$ physics, viz. the large-$N_c$ chiral Lagrangian formalism of refs.[14, 15, 16] (see also [17, 18]). A comprehensive review of the phenomenology of the pseudoscalar mesons, incorporating the chiral Lagrangian results, is given in ref.[19]

2. Radiative decay and DGMOR formulae

In this section, we present the radiative decay and DGMOR formulae and discuss how to extract phenomenological quantities from them, emphasising their $1/N_c$ dependence. A brief review of their derivation in the language of ‘$U(1)_A$ PCAC’ is given in section 3, but we refer to the original papers [1, 2, 20] for full details, including a more precise formulation using functional methods. Throughout, we consider the case of three light flavours, $n_f = 3$.

The assumptions made in deriving these formulae are the standard ones of PCAC. The formulae are based on the zero-momentum chiral Ward identities. It is then assumed that the decay ‘constants’ ($f^{a\alpha}(k^2)$ in the notation below) and couplings ($g_{\pi^a\gamma\gamma}(k^2)$) are approximately constant functions of momentum in the range from zero, where the Ward identities are applied, to the appropriate physical particle mass. It is important that this approximation is applied only to pole-free quantities which have only an implicit dependence on the quark masses. This is equivalent to assuming pole-dominance of the propagators for the relevant operators by the (pseudo-)NG bosons, and naturally becomes exact in the chiral limit. Notice that these dynamical approximations are equally necessary in the chiral Lagrangian approach, where they are built in to the basic structure of the model – the effective fields are chosen to be in one-to-one correspondence with the NG bosons and pole-dominance is implemented through the low-momentum expansion with momentum-independent parameters corresponding to the decay constants.

The relations for the $\pi^0$ decouple from those for the $\eta$ and $\eta'$ in the realistic approximation of $SU(2)$ flavour invariance of the quark condensates, i.e. $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$. The decay formula for $\pi^0 \to \gamma\gamma$ is the standard one,

$$f_{\pi}g_{\pi\gamma\gamma} = a^3_{\text{em}} \frac{\alpha_{\text{em}}}{\pi} \frac{m_\pi^2}{m_\pi^2}$$

with $a^3_{\text{em}} = \frac{1}{3}N_c$, while the DGMOR formula is simply

$$f_{\pi}^2 m_{\pi}^2 = - (m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle)$$

The coupling $g_{\pi\gamma\gamma}$ is defined as usual from the decay amplitude:

$$\langle \gamma\gamma | \pi^0 \rangle = - i g_{\pi\gamma\gamma} \epsilon^{\lambda\rho\sigma\delta} p_1^\rho p_2^\sigma \epsilon^\lambda (p_1) e^\delta (p_2)$$
in obvious notation [1]. The r.h.s. of eq.(2.1) arises from the electromagnetic \( U(1)_A \) anomaly and is of course sensitive to the number of colours, \( N_c \). The decay constant has a simple theoretical interpretation as the coupling of the pion to the flavour triplet axial current, \( \langle 0|J^a_{3\mu}|\pi \rangle = ik_\mu f_\pi \), which is conserved in the chiral limit.

In the \( \eta, \eta' \) sector, however, explicit \( SU(3) \) flavour breaking means there is mixing. The decay constants therefore form a \( 2 \times 2 \) matrix:

\[
\begin{pmatrix}
    f^{0\eta'} & f^{0\eta} \\
    f^{8\eta'} & f^{8\eta}
\end{pmatrix}
\]  

(2.4)

where the index \( a \) is an \( SU(3) \times U(1) \) flavour index (including the singlet \( a = 0 \) ) and \( \alpha \) denotes the physical particle states \( \pi^0, \eta, \eta' \). These four decay constants are independent [14, 15, 21, 22], in contradiction to the original phenomenological parametrisations which erroneously expressed \( f^{\alpha\alpha} \) as a diagonal decay constant matrix times an orthogonal \( \eta - \eta' \) mixing matrix, giving a total of only three independent parameters. Sometimes the four decay constants are expressed in terms of two constants and two mixing angles but, while we also express our results in this form in section 4, there is no particular reason to parametrise in this way.

The other new feature in the \( \eta, \eta' \) sector is the presence of the gluonic \( U(1)_A \) anomaly in the flavour singlet current. This means that even in the chiral limit the \( \eta' \) is not a true NG boson and therefore the direct analogue of the PCAC formula (2.1) does not hold. Nonetheless, we can still write an analogous formula but with an extra term related to the gluon topological susceptibility.

The decay formulae are [1]:

\[
f^{0\eta'} g_{\eta'\gamma\gamma} + f^{0\eta} g_{\eta\gamma\gamma} + \sqrt{6} A g G_{\gamma\gamma} = a^0_{em} \frac{\alpha_{em}}{\pi}
\]  

(2.5)

\[
f^{8\eta'} g_{\eta'\gamma\gamma} + f^{8\eta} g_{\eta\gamma\gamma} = a^8_{em} \frac{\alpha_{em}}{\pi}
\]  

(2.6)

where \( a^0_{em} = \frac{2\sqrt{2}}{3\sqrt{3}} N_c \) and \( a^8_{em} = \frac{1}{3\sqrt{3}} N_c \), and the corresponding DGMOR relations are:

\[
(f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_{\eta}^2 = -\frac{2}{3}(m_u\langle \bar{u}u \rangle + m_d\langle \bar{d}d \rangle + m_s\langle \bar{s}s \rangle) + 6A
\]  

(2.7)

\[
(f^{8\eta'})^2 m_{\eta'}^2 + (f^{8\eta})^2 m_{\eta}^2 = -\frac{\sqrt{2}}{3}(m_u\langle \bar{u}u \rangle + m_d\langle \bar{d}d \rangle - 2m_s\langle \bar{s}s \rangle)
\]  

(2.8)

\[
(f^{8\eta'})^2 m_{\eta'}^2 + (f^{8\eta})^2 m_{\eta}^2 = -\frac{1}{3}(m_u\langle \bar{u}u \rangle + m_d\langle \bar{d}d \rangle + 4m_s\langle \bar{s}s \rangle)
\]  

(2.9)

\[\]  

1Notice that compared to refs.[1, 2, 20, 9] we have changed the normalisation of the flavour singlets by a factor \( 1/\sqrt{2\pi} = 1/\sqrt{6} \). The normalisations here are \( \text{tr}T^aT^b = \frac{1}{2}\delta_{ab} \) for generators \( T^a \) (\( a = 0, 1, \ldots, 8 \)) of \( SU(3) \times U(1) \), i.e. \( T^a = \frac{i}{2}\lambda^a \) (\( i = 1, \ldots, 8 \)) where \( \lambda^a \) are the Gell-Mann matrices, and \( T^0 = \frac{1}{\sqrt{3}}1 \) for the flavour singlet. The \( d \)-symbols are defined by \( \{ T^a, T^b \} = d_{abc}T^c \) and include \( d_{000} = d_{013} = d_{033} = d_{333} = \frac{1}{2}, d_{323} = d_{332} = -d_{333} = \frac{1}{2} \).
There are several features of these equations which need to be explained. First, because of the anomaly, the decay constants in the flavour singlet sector can not be identified as the couplings of the \( \eta, \eta' \) to the singlet axial-vector current. In particular, the object \( F_{\eta'} \) defined as \( \langle 0|J_{\mu 5}^0|\eta' \rangle = i k_\mu F_{\eta'}^0 \) is not a renormalisation group invariant and so is not to be identified as a physical decay constant. The derivation of the radiative decay and DGMOR formulae in the next section makes it clear that \( F_{\eta'} \) plays no role – certainly it is not \( f_{\eta'}^0 \). Instead, our decay constants are defined in terms of the couplings of the \( \pi, \eta, \eta' \) to the pseudoscalar currents through the relations \( f_a \alpha \langle 0|\phi^5_a|\eta' \rangle = d_{abc} \langle \phi^c \rangle \) (for notation, see section 3). This coincides with the usual definition except in the flavour singlet case.

The coefficient \( A \) appearing in the flavour singlet equations is the non-perturbative constant that determines the topological susceptibility in QCD. Recall that the topological susceptibility \( \chi(0) \) is defined as

\[
\chi(0) = \int d^4 x \ i \langle 0|T Q(x) Q(0)|0 \rangle
\]

where \( Q = \frac{\alpha_s}{8\pi} \text{tr} G^{\mu\nu} \tilde{G}_{\mu\nu} \) is the gluon topological charge. The anomalous chiral Ward identities determine the dependence of \( \chi(0) \) on the quark masses and condensates up to a single non-perturbative parameter \([23, 1]\), viz:

\[
\chi(0) = - A \left( 1 - A \sum_{q=u,d,s} \frac{1}{m_q \langle q q' \rangle} \right)^{-1}
\]

or alternatively,

\[
\chi(0) = \frac{-A m_u m_d m_s \langle \bar{u}u \rangle \langle \bar{d}d \rangle \langle \bar{s}s \rangle}{m_u m_d m_s \langle \bar{u}u \rangle \langle \bar{d}d \rangle \langle \bar{s}s \rangle - A (m_u m_d \langle \bar{u}u \rangle \langle \bar{d}d \rangle + m_u m_s \langle \bar{u}u \rangle \langle \bar{s}s \rangle + m_d m_s \langle \bar{d}d \rangle \langle \bar{s}s \rangle)}
\]  

Notice how the well-known result that \( \chi(0) = 0 \) if any of the quark masses is zero is realised in this expression.

The final new element in the flavour singlet decay formula is the coupling parameter \( g_{G\gamma\gamma} \). This is unique to our approach. It takes account of the fact that, because of mixing with the pseudoscalar gluon operator \( Q \) due to the anomaly, the physical \( \eta' \) is not a NG boson. \( g_{G\gamma\gamma} \) is not a physical coupling, although it may reasonably be thought of as the coupling of the photons to the gluonic component of the \( \eta' \). (A justification for this picture is given later.) Its precise field-theoretic definition is given in section 3 and refs.[1, 2, 20]. We should emphasise, however, that an analogous parameter must appear in any correct current algebra analysis involving the \( \eta' \), such as the \( 1/N_c \) chiral Lagrangian approach of refs.[14, 15, 16]. Phenomenologically, it may be found to be relatively small (we test this numerically in section 4), but any formulae which leave out such a term completely are inevitably theoretically inconsistent.

Of course, in addition to these formulae, we also have the DGMOR relations for the remaining pseudoscalars. For example, for the \( K^+ \),

\[
f_K^2 m_K^2 = -(m_u \langle \bar{u}u \rangle + m_s \langle \bar{s}s \rangle)
\]

(2.13)
If we assume exact $SU(2)$ flavour symmetry, we can then use eqs.(2.2) and (2.13) to eliminate the quark masses and condensates in favour of $f_\pi, f_K, m_\pi^2$ and $m_K^2$ in the DGMOR formulae for the $\eta$ and $\eta'$. We find:

\begin{align}
(f^{0\eta'})^2m_{\eta'}^2 + (f^{0\eta})^2m_\eta^2 &= \frac{1}{3}(f_\pi^2m_\pi^2 + 2f_K^2m_K^2) + 6A \tag{2.14} \\
f^{0\eta'}f^{8\eta'}m_{\eta'}^2 + f^{0\eta}f^{8\eta}m_\eta^2 &= \frac{2\sqrt{2}}{3}(f_\pi^2m_\pi^2 - f_K^2m_K^2) \tag{2.15} \\
(f^{8\eta'})^2m_{\eta'}^2 + (f^{8\eta})^2m_\eta^2 &= -\frac{1}{3}(f_\pi^2m_\pi^2 - 4f_K^2m_K^2) \tag{2.16}
\end{align}

A quick look at the full set of decay and DGMOR formulae (2.5), (2.6) and (2.14)–(2.16) now shows that we have five equations for six parameters, viz. $f^{0\eta'}, f^{0\eta}, f^{8\eta'}, f^{8\eta}, A$ and $g_{G\gamma\gamma}$, assuming that $f_\pi, f_K$ and the physical masses are known along with the experimental values for the couplings $g_{\eta'\gamma\gamma}$ and $g_{\eta\gamma\gamma}$. It is not surprising that this set is under-determined. In particular, the necessary presence of the unphysical coupling $g_{G\gamma\gamma}$ in the flavour singlet decay equation essentially removes its predictivity. At this stage, the best we can do is to evaluate the singlet decay constants and the gluonic coupling $g_{G\gamma\gamma}$ as functions of the topological susceptibility parameter $A$. This is done in section 5. In order to make more progress, we therefore need a further, dynamical input.

As yet, everything we have done has been entirely independent of the $1/N_c$ expansion. The $1/N_c$ expansion (or the OZI limit – see ref.[24] for a careful discussion of the differences) is known to give a good approximation to many aspects of the dynamics of QCD and could provide the required extra input, but its application to the $U(1)_A$ sector needs to be handled with great care. For example, in the chiral limit, the mass of the $\eta'$, which arises due to the anomaly, is formally $m_{\eta'}^2 = O(1/N_c)$; however, numerically (allowing for the quark masses going to zero) this is of the same order of magnitude as a typical meson such as the $\rho$ and is certainly not small. Generally, it is not clear that quantities which are formally suppressed in $1/N_c$ are in fact numerically suppressed in real QCD, so we must be extremely careful in applying $1/N_c$ methods here.

Despite these caveats, we will see that $1/N_c$ can play a useful role in analysing the decay formulae and DGMOR relations. Conventional large $N_c$ counting gives the following orders for the various quantities: $f^{aa} = O(\sqrt{N_c})$ for all the decay constants; the $\eta$ and $\eta'$ couplings $g_{\eta\gamma\gamma} = O(\sqrt{N_c})$ but $g_{G\gamma\gamma} = O(1)$; $m_{\eta'}^2$ and $m_\eta^2$ are both $O(1)$, but note that $m_{\eta'}^2 = O(1/N_c)$ in the chiral limit – the numerically dominant contribution to its mass from the anomaly is formally $1/N_c$ suppressed relative to the $O(1)$ contribution from the explicit chiral symmetry breaking quark masses; the condensates $\langle \bar{q}q \rangle = O(N_c)$; the anomaly coefficients $a_g^{\alpha\alpha} = O(N_c)$; while finally the coefficient in the topological susceptibility is $A = O(1)$.

Referring to eqs.(2.11) or (2.12), we therefore see that at large $N_c$, $\chi(0) \simeq -A = O(1)$. Moreover, it is clear from looking at planar diagrams that at leading order in $1/N_c$, $\chi(0)$ in QCD coincides with the topological susceptibility in pure Yang-Mills theory, $\chi(0)|_{YM}$. 


It follows that

\[ A = \chi(0)|_{YM} + O(1/N_c) \quad (2.17) \]

a result that plays an important role in the Witten-Veneziano formula (see below).

Now consider the flavour singlet DGMOR relation (2.7) or (2.14). Each term is \( O(N_c) \) apart from \( A \), which is \( O(1) \). Naively, we might think that this sub-leading term would be small so could be neglected. However, we know that it contributes at the same order as the sub-leading term in \( (f_0^N)^2m_{\eta'}^2 \) given by the large anomaly-induced \( O(1/N_c) \) contribution to the \( \eta' \) mass squared. So the topological susceptibility contribution to this relation is crucial, even though it is formally suppressed in \( 1/N_c \). However, we may reasonably expect that the further \( O(1/N_c) \) corrections to \( A \) are genuinely small and that a good numerical approximation to eqs.(2.7) or (2.14) is obtained by keeping only the terms up to \( O(1) \). This means that we may sensibly approximate the parameter \( A \) by \( \chi(0)|_{YM} \) in the DGMOR relation.

This is the crucial simplification. A reliable estimate of \( \chi(0)|_{YM} \) at \( O(1) \) is available from lattice calculations in pure Yang-Mills theory [25]. This extra dynamical input allows us to determine the four decay constants from eqs.(2.6) and (2.14)–(2.16). We can then analyse the flavour singlet decay formula (2.5) to determine \( g_{G\gamma\gamma} \) and see how important this new term actually is numerically.

We already know that the \( Ag_{G\gamma\gamma} \) contribution to eq.(2.12) is suppressed by one power of \( 1/N_c \) compared to the other terms in the formula. Moreover, \( g_{G\gamma\gamma} \) is renormalisation group invariant (see refs.[2, 20]). Now, in previous work we have developed an intuition as to when it is likely to be reliable to assume that \( 1/N_c \) suppressed terms are actually numerically small. The argument is based on the idea that violations of the OZI rule are associated with the \( U(1)_A \) anomaly, so that we can identify OZI-violating quantities by their dependence on the anomalous dimension associated with the non-trivial renormalisation of \( J_{\mu5}^0 \) due to the anomaly. RG non-invariance can therefore be used as an indicator of which quantities we expect to show large OZI violations. In this case, \( g_{G\gamma\gamma} \) is RG invariant, so we would expect the OZI rule to be good. This means that since it is OZI suppressed (essentially, higher order in \( 1/N_c \)) relative to the \( \eta' \) decay coupling \( g_{\eta'^{\mu}\gamma\gamma} \), it should be numerically smaller as well.\(^2\)

We will test this conjecture with the experimental data in section 4. The issue is an important one. If \( g_{G\gamma\gamma} \) can be neglected, then the naive current algebra formulae will turn out phenomenologically to be a good approximation to data, even though they are theoretically inconsistent. This would apply not just to the radiative pseudoscalar decays, but to a whole range of current algebra processes involving the \( \eta' \) (see, for example, refs.[19, 20]). We have also used this conjecture in our analysis of the closely related ‘proton spin’ problem in polarised deep inelastic scattering where it plays an important role in our prediction of the first moment of the polarised proton structure function \( q_1^p \) [26].

Finally, to close this section, we show in detail how these DGMOR relations are related to the well-known Witten-Veneziano formula for the mass of the \( \eta' \), which is derived in

\(^2\)To be precise, the conjecture is that the contribution \( \sqrt{6}Ag_{G\gamma\gamma} \) in the decay formula will be small compared to the dominant term \( f_0^{\eta'}g_{\eta'^{\mu}\gamma\gamma} \). Note that as defined the dimensions of \( g_{G\gamma\gamma} \) and \( g_{\eta'^{\mu}\gamma\gamma} \) are different, so they can not be directly compared.
the large-$N_c$ limit of QCD. In fact, this is simply the $N_c \to \infty$ limit of the flavour singlet DGMOR formula, which we emphasise is valid for all $N_c$. To see this in detail, recall that the Witten-Veneziano formula for non-vanishing quark masses is [6]

$$m_{q'}^2 + m_q^2 - 2m_K^2 = -\frac{6}{f_\pi^2} \chi(0)_{YM}$$

(2.18)

Of course, only the $m_{q'}^2$ term on the l.h.s. is present in the chiral limit. Now add the DGMOR formulae (2.14) and (2.16). We find

$$(f_0^{q'})^2 m_{q'}^2 + (f_0^q)^2 m_q^2 + (f_8^{q'})^2 m_{q'}^2 - 2f_K^2 m_K^2 = 6A$$

(2.19)

To reduce this to its Witten-Veneziano approximation, we impose the large-$N_c$ limit to identify the full QCD topological charge parameter $A$ with $-\chi(0)_{YM}$ according to eq.(2.17). We then set the ‘mixed’ decay constants $f_{0q}$ and $f_{8q}$ to zero and all the remaining decay constants $f_0^{q'}$, $f_8^{q'}$ and $f_K$ equal to $f_\pi$. With these approximations, we recover eq.(2.18).

In section 4, when we find the explicit experimental values for all these quantities in real QCD, we will be able to judge how good an approximation the large-$N_c$ Witten-Veneziano formula is to our general DGMOR relation.

3. Theory

We now sketch the ‘$U(1)_A$ PCAC’ derivation of the decay formulae and DGMOR relations. For a more precise treatment in terms of functional chiral Ward identities and a complete renormalisation group analysis, as well as an effective Lagrangian formulation, we refer to our original papers [1, 2, 20].

The starting point is the $U(1)_A$ chiral anomaly equation in pure QCD with $n_f$ flavours of massive quarks, viz\(^3\)

$$\partial^\mu J^a_{\mu 5} = M_{ab} \phi_b^5 + \sqrt{2n_f} Q \delta_{a0}$$

(3.1)

where the axial vector current is $J^a_{\mu 5} = \bar{q} \gamma_\mu \gamma_5 T^a q$ and the pseudoscalar is $\phi_5^a = \bar{q} \gamma_5 T^a q$. The corresponding chiral Ward identities for the two-point Green functions of interest are therefore (in momentum space):

$$ik^\mu \langle J^a_{\mu 5} Q \rangle - \sqrt{2n_f} \delta_{a0} \langle Q Q \rangle - M_{ac} \langle \phi_c^5 Q \rangle = 0$$

(3.2)

\(^3\)We use the following $SU(3)$ notation for the quark masses and condensates:

$$\begin{pmatrix}
  m_u & 0 & 0 \\
  0 & m_d & 0 \\
  0 & 0 & m_s
\end{pmatrix} = \sum_{a=0,3,8} m^a T^a$$

and

$$\begin{pmatrix}
  \langle \bar{u} u \rangle & 0 & 0 \\
  0 & \langle \bar{d} d \rangle & 0 \\
  0 & 0 & \langle \bar{s} s \rangle
\end{pmatrix} = \sum_{a=0,3,8} \langle \phi^a \rangle T^a$$

where $\langle \phi^a \rangle$ is the VEV of $\phi^a = \bar{q} T^a q$. It is also very convenient to introduce the compressed notation

$$M_{ab} = d_{abc} m^c \\
\Phi_{ab} = d_{abc} \langle \phi^c \rangle$$
\[ i k^\mu \langle J^a_{\mu \nu} \phi_b^\nu \rangle - \sqrt{2n_f} \delta_{ab} \langle Q \phi_b^\nu \rangle - M_{ac} \langle \phi_c^\nu \phi_b^\nu \rangle = \Phi_{ab} \]  

(3.3)

Since there are no massless pseudoscalar mesons in the theory, the terms in these identities involving the axial current vanish at zero momentum because of the explicit factors of \( k^\mu \). The zero-momentum chiral Ward identities simply comprise the remaining terms. In particular, we have

\[ M_{ac} M_{bd} \langle \phi_b^\nu \phi_d^\nu \rangle = -(M\Phi)_{ab} + (2n_f) \langle Q \phi_b^\nu \rangle \delta_{ab} \delta_{b0} \]  

(3.4)

We can also derive the result quoted in eq.(2.11) for the topological susceptibility. In this notation,

\[ \chi(0) \equiv \langle Q Q \rangle = -A \frac{1}{1 - (2n_f) A(M\Phi)_{00}^{-1}} \]  

(3.5)

The physical mesons \( \eta^\alpha = (\pi^0, \eta^0, \eta') \) couple to the pseudoscalar operators \( \phi_5^a \) and \( Q \), so their properties may be deduced from the two-point Green functions above. In the PCAC approximation, as explained in section 2, the zero-momentum Ward identities are sufficient. In order to make the correspondence between the QCD operators and the physical mesons as close as possible, it is convenient to redefine linear combinations such that the propagator (two-point Green function) matrix is diagonal and properly normalised. We therefore define operators \( \eta^\alpha \) and \( G \) such that

\[ \begin{pmatrix} \langle Q Q \rangle & \langle Q \phi_5^b \rangle \\ \langle \phi_5^2 Q \rangle & \langle \phi_5^b \phi_5^b \rangle \end{pmatrix} \rightarrow \begin{pmatrix} \langle G G \rangle & 0 \\ 0 & \langle \eta^\alpha \eta^\beta \rangle \end{pmatrix} \]  

(3.6)

This is achieved by taking

\[ G = Q - \langle Q \phi_5^b \rangle (\langle \phi_5 \phi_5 \rangle)_{ab}^{-1} \phi_5^b \]  

(3.7)

which reduces at zero momentum to

\[ G = Q + 2n_f A \Phi_{0b}^{-1} \phi_5^b \]  

(3.8)

and defining

\[ \eta^\alpha = f^T_{\alpha \beta} \Phi_{0b}^{-1} \phi_5^b \]  

(3.9)

With this choice, the \( \langle G G \rangle \) propagator at zero momentum is simply

\[ \langle G G \rangle = -A \]  

(3.10)

and we demand that the \( \langle \eta^\alpha \eta^\beta \rangle \) propagator has the canonical normalisation

\[ \langle \eta^\alpha \eta^\beta \rangle = \frac{1}{k^2 - m_{\eta^\alpha}^2} \delta^{\alpha\beta} \]  

(3.11)

This normalisation fixes the decay constants introduced in eq.(3.9). The DGMOR relations then follow immediately from the zero-momentum chiral Ward identity (3.4) and the expression (3.5) for the topological susceptibility. We find

\[ f^{\alpha \beta} m_{\alpha\beta}^2 f^{T\beta \nu} = \Phi_{ac} (\langle \phi_5 \phi_5 \rangle)_{cd}^{-1} \Phi_{db} \]  

\[ = -(M\Phi)_{ab} + (2n_f) A \delta_{a0} \delta_{b0} \]  

(3.12)
Unwrapping the condensed SU(3) notation then shows that this matrix equation is simply the set of DGMOR relations (2.7)–(2.9).

The next step is the PCAC calculation of $\eta^\alpha \rightarrow \gamma\gamma$. We implement PCAC by the identification

$$\partial_\mu J_{\mu 5}^{a} \rightarrow f^{a\alpha} m_{\alpha\beta}^{2} \eta^\beta + \sqrt{2n_f} G \delta_{a0}$$

which follows from the anomaly eq.(3.1) and the definitions of the fields $G$ and $\eta^\alpha$. To be precise, the $\rightarrow$ notation in eq.(3.13) means that the identification can be made for insertions of the operators into zero-momentum Green functions and matrix elements only. Notice that it is not valid at non-zero momentum, in particular for on-shell matrix elements. It is the natural generalisation of the familiar PCAC relation $\partial_\mu J_{\mu 5}^{3} \rightarrow f_\pi m_\pi^{2} \pi$, defining the phenomenological pion field $\pi$.

The other input is the full axial anomaly for QCD coupled to electromagnetism, viz.

$$\partial_\mu J_{\mu 5}^{a} = M_{ab} \phi_{b}^{0} + \sqrt{2n_f} Q \delta_{a0} + a_{em}^{a} \alpha_{8} \pi F_{\mu\nu} \bar{F}_{\mu\nu}$$

where $a_{em}^{a}$ are the anomaly coefficients given in section 2.

Implementing the PCAC relation (3.13) together with the full anomaly equation, we therefore find

$$ik\langle \gamma\gamma | J_{\mu 5}^{a} | 0 \rangle = f^{a\alpha} m_{\alpha\beta}^{2} \langle \gamma\gamma | \eta^\beta | 0 \rangle + \sqrt{2n_f} \langle \gamma\gamma | G | 0 \rangle \delta_{a0} + a_{em}^{a} \alpha_{8} \pi \langle \gamma\gamma | F_{\mu\nu} \bar{F}_{\mu\nu} | 0 \rangle$$

$$= f^{a\alpha} m_{\alpha\beta}^{2} \langle \eta^\beta \eta^\gamma | \gamma\gamma | \eta^\gamma \rangle + \sqrt{2n_f} \langle \gamma\gamma | G \rangle \langle \gamma\gamma | G \rangle \delta_{a0} + a_{em}^{a} \alpha_{8} \pi \langle \gamma\gamma | F_{\mu\nu} \bar{F}_{\mu\nu} | 0 \rangle$$

at zero momentum, where we have used the fact that the propagators are diagonal in the $\eta^\alpha, G$ basis. The l.h.s. vanishes at zero momentum as there is no massless particle coupling to the axial current. Then, using the explicit expressions (3.10) and (3.11) for the $G$ and $\eta^\alpha$ propagators, we find the decay constant formulae:

$$f^{a\alpha} g_{\eta^\alpha \gamma\gamma} + \sqrt{2n_f} A g_{G \gamma\gamma} \delta_{a0} = a_{em}^{a} \alpha_{8} \pi$$

The novel coupling $g_{G \gamma\gamma}$ is precisely defined from the matrix element $\langle \gamma\gamma | G \rangle$ in analogy with eq.(2.3) for the conventional couplings.

Finally, notice that the mixing of states is conjugate to the mixing of fields. In particular, the mixing for the states corresponding to eqs.(3.8) and (3.9) for the fields $G$ and $\eta^\alpha$ is

$$|G\rangle = |Q\rangle$$

and

$$|\eta^\alpha\rangle = (f^{-1})^{\alpha\alpha} (\Phi_{ab} | \phi_{b}^{k} \rangle - \sqrt{2n_f} A \delta_{a0} | Q \rangle)$$

In this sense, we see that we can regard the physical $\eta'$ (and, with SU(3) breaking, the $\eta$) as an admixture of quark and gluon components, while the unphysical state $|G\rangle$ is purely gluonic. This is why we can usefully picture the unphysical coupling $g_{G \gamma\gamma}$ as the coupling of the photons to the anomaly-induced gluonic component of the $\eta'$, as already mentioned in section 2.
4. Phenomenology

In this section, we use the experimental data on the radiative decays $\eta, \eta' \to \gamma\gamma$ to deduce values for the pseudoscalar meson decay constants $f_{\eta'}$, $f_\eta$, $f_{8\eta'}$ and $f_{8\eta}$ from the set of decay formulae (2.5),(2.6) and DGMOR relations (2.14)-(2.16). We will also find the value of the unphysical coupling parameter $g_{G\gamma\gamma}$ and test the realisation of the $1/N_c^2$ expansion in real QCD.

The two-photon decay widths are given by

$$\Gamma(\eta'\to\gamma\gamma) = \frac{m_{\eta'}^3}{64\pi}|g_{\eta'\gamma\gamma}|^2$$

(4.1)

The current experimental data, quoted in the Particle Data Group tables [27], are

$$\Gamma(\eta' \rightarrow \gamma\gamma) = 4.28 \pm 0.19 \text{ KeV}$$

(4.2)

which is dominated by the 1998 L3 data [28] on the two-photon formation of the $\eta'$ in $e^+e^- \to e^+e^-\pi^+\pi^−\gamma$, and

$$\Gamma(\eta \rightarrow \gamma\gamma) = 0.510 \pm 0.026 \text{ KeV}$$

(4.3)

which arises principally from the 1988 Crystal Ball [29] and 1990 ASP [30] results on $e^+e^- \to e^+e^-\eta$. (Notice that we follow the note in the 1994 PDG compilation [31] and use only the two-photon $\eta$ production data.)

From this data, we deduce the following results for the couplings $g_{\eta'\gamma\gamma}$ and $g_{\eta\gamma\gamma}$:

$$g_{\eta'\gamma\gamma} = 0.031 \pm 0.001 \text{ GeV}^{-1}$$

(4.4)

and

$$g_{\eta\gamma\gamma} = 0.025 \pm 0.001 \text{ GeV}^{-1}$$

(4.5)

which may be compared with $g_{\pi\gamma\gamma} = 0.024 \pm 0.001 \text{ GeV}$.

We also require the pseudoscalar meson masses:

$$m_{\eta'} = 957.78 \pm 0.14 \text{ MeV} \quad m_{\eta} = 547.30 \pm 0.12 \text{ MeV}$$

$$m_K = 493.68 \pm 0.02 \text{ MeV} \quad m_{\pi} = 139.57 \pm 0.00 \text{ MeV}$$

(4.6)

and the decay constants $f_{\pi}$ and $f_K$. These are defined in the standard way, so we take the following values (in our normalisations) from the PDG [27]:

$$f_K = 113.00 \pm 1.03 \text{ MeV} \quad f_{\pi} = 92.42 \pm 0.26 \text{ MeV}$$

(4.7)

giving $f_K/f_\pi = 1.223 \pm 0.012$.

The final input, as explained in section 2, is the lattice calculation of the topological susceptibility in pure Yang-Mills theory. The most recent value, obtained in ref.[25], is

$$\chi(0)|_{YM} = -(191 \pm 5 \text{ MeV})^4 = -(1.33 \pm 0.14) \times 10^{-3} \text{ GeV}^4$$

(4.8)
This supersedes the original value $\chi(0)|_{YM} \simeq -(180 \text{ MeV})^4$ obtained some time ago [32]. Similar estimates are also obtained using QCD spectral sum rule methods [33]. Using the argument explained in section 2, we therefore adopt the value

$$A = (1.33 \pm 0.14) \times 10^{-3} \text{ GeV}^4$$  \hspace{1cm} (4.9)

for the non-perturbative parameter determining the topological susceptibility in full QCD.

The strategy is now to solve the set of five simultaneous equations (2.5),(2.6) and (2.14),(2.15), (2.16) for the five remaining unknowns $f^{0\eta'}, f^{0\eta}, f^{8\eta'}, f^\eta$ and $g_{G\gamma\gamma}$. The results are

$$f^{0\eta'} = 104.2 \pm 4.0 \text{ MeV} \quad f^{0\eta} = 22.8 \pm 5.7 \text{ MeV}$$
$$f^{8\eta'} = -36.1 \pm 1.2 \text{ MeV} \quad f^{8\eta} = 98.4 \pm 1.4 \text{ MeV}$$  \hspace{1cm} (4.10)

that is

$$\frac{f^{0\eta'}}{f_\pi} = 1.13 \pm 0.04 \quad \frac{f^{0\eta}}{f_\pi} = 0.25 \pm 0.06$$
$$\frac{f^{8\eta'}}{f_\pi} = -0.39 \pm 0.01 \quad \frac{f^{8\eta}}{f_\pi} = 1.07 \pm 0.02$$  \hspace{1cm} (4.11)

and

$$g_{G\gamma\gamma} = -0.001 \pm 0.072 \text{ GeV}^{-4}$$  \hspace{1cm} (4.12)

These are the main results of this paper.

Before going on to consider their significance, we can re-express the results for the decay constants in terms of one of the two-angle parametrisations used in the literature. We reiterate that in our view there is no particular virtue in parametrising in this way, but in order to help comparison with other analyses it may be helpful to present our results in this form as well. Refs.[14, 15] define

$$\begin{pmatrix} f^{0\eta'} & f^{0\eta} \\ f^{8\eta'} & f^{8\eta} \end{pmatrix} = \begin{pmatrix} f_0 \cos \theta_0 & -f_0 \sin \theta_0 \\ f_8 \sin \theta_8 & f_8 \cos \theta_8 \end{pmatrix}$$  \hspace{1cm} (4.13)

Translating from eq.(4.11), we find

$$f_0 = 106.6 \pm 4.2 \text{ MeV} \quad f_8 = 104.8 \pm 1.3 \text{ MeV}$$
$$\theta_0 = -12.3 \pm 3.0 \text{ deg} \quad \theta_8 = -20.1 \pm 0.7 \text{ deg}$$  \hspace{1cm} (4.14)

that is

$$\frac{f_0}{f_\pi} = 1.15 \pm 0.05 \quad \frac{f_8}{f_\pi} = 1.13 \pm 0.02$$  \hspace{1cm} (4.15)

\footnote{Note that this analysis includes the errors from the experimental inputs and the lattice evaluation of $\chi(0)|_{YM}$, but \textit{not} the systematic effect of the approximation $A = -\chi(0)|_{YM}$. The errors on the singlet decay constants are dominated by the error on $A$. Isolating this, we have

$$f^{0\eta'} = 104.2 \pm 0.3 \pm 4.0 \text{ MeV} \quad f^{0\eta} = 22.8 \pm 3.5 \pm 4.5 \text{ MeV}$$

The octet decay constants are of course unaffected by the value of $A$.}
We emphasise, however, that since our definitions of the decay constants differ from those in refs.[14, 15], any comparison of the numbers above should be made with care.

The most striking feature of the results (4.10) for the decay constants is how close the diagonal ones, \( f_{0}'\eta' \) and \( f_8\eta' \), are to \( f_\pi \), even the singlet. Predictably, the off-diagonal ones are strongly suppressed, especially \( f_{0}'\eta \). This supports the approximations used in section 2 in deriving the large \( N_c \) Witten-Veneziano limit of the DGMOR relations.

To see this in more detail, it is interesting to compare numerically the magnitudes of the various terms appearing in the DGMOR relations, together with their formal orders in \( 1/N_c \). We find (all terms in units of \( 10^{-3}\text{GeV}^4 \)):

\[
(f_{0}'\eta)^2 m_{\eta'}^2 [N_c; 9.96] + (f_{0}\eta)^2 m_{\eta}^2 [N_c; 0.15] = \frac{1}{3} f_\pi^2 m_\pi^2 [N_c; 0.06] + \frac{2}{3} f_K^2 m_K^2 [N_c; 2.07] + 6A [1; 7.98] \tag{4.16}
\]

\[
f_{0}'f_{8}' m_{\eta'}^2 [N_c; -3.45] + f_{0}f_{8} m_{\eta}^2 [N_c; 0.67] = 2\sqrt{\frac{2}{3}} f_\pi^2 m_\pi^2 [N_c; 0.16] - 2\sqrt{\frac{2}{3}} f_K^2 m_K^2 [N_c; -2.94] \tag{4.17}
\]

\[
(f_{8}'\eta)^2 m_{\eta'}^2 [N_c; 1.19] + (f_{8}\eta)^2 m_{\eta}^2 [N_c; 2.90] = -\frac{1}{3} f_\pi^2 m_\pi^2 [N_c; -0.06] + \frac{4}{3} f_K^2 m_K^2 [N_c; 4.15] \tag{4.18}
\]

The interesting feature here is the explicit demonstration that the dominant term in the flavour singlet DGMOR (Witten-Veneziano) relation is the topological susceptibility factor \( 6A \), even though it is formally suppressed by a power of \( 1/N_c \) relative to the others. It is matched by the subdominant \( O(1) \) contribution to \( (f_{0}'\eta)^2 m_{\eta'}^2 \), which arises because of the numerically large but \( O(1/N_c) \) anomaly-induced part of the \( m_{\eta'}^2 \) which survives in the chiral limit. The numerical results therefore confirm the theoretical intuition expressed in section 2.

To emphasise this point further, we can summarise the numerical magnitudes in the combined singlet-octet relation (2.19), which reduces to the full Witten-Veneziano formula (2.18):

\[
(f_{0}'\eta)^2 m_{\eta'}^2 [N_c; 9.96] + (f_{0}\eta)^2 m_{\eta}^2 [N_c; 0.15] + (f_{8}'\eta)^2 m_{\eta'}^2 [N_c; 1.19] + (f_{8}\eta)^2 m_{\eta}^2 [N_c; 2.90] - 2f_K^2 m_K^2 [N_c; -6.22] = 6A [1; 7.98] \tag{4.19}
\]

The validity of the large \( N_c \) limit leading to eq.(2.18) is particularly transparent in this form. Numerically, the surprising accuracy of the approximate formula (2.18) is seen to be in part due to a cancellation between the underestimates of \( f_{8}'\eta' \) (taken to be 0) and \( f_K \) (set equal to \( f_\pi \)).
Now consider the decay formulae themselves. The numerical magnitudes and $1/N_c$ orders of the various contributions in this case are (in units of $10^{-3}$):

$$f^{0\eta'}g_{\eta'\gamma\gamma} [N_c; 3.23] + f^{0\eta}g_{\eta\gamma\gamma} [N_c; 0.57] + \sqrt{6}A g_{G\gamma\gamma} [1; -0.005 \pm 0.23] = a^0_{em} \frac{\alpha_{em}}{\pi} [N_c; 3.79]$$

and

$$f^{8\eta'}g_{\eta'\gamma\gamma} [N_c; -1.12] + f^{8\eta}g_{\eta\gamma\gamma} [N_c; 2.46] = a^8_{em} \frac{\alpha_{em}}{\pi} [N_c; 1.34]$$

The interest here is in the realisation of the $1/N_c$ approximation in the flavour singlet decay formula. As explained in section 2, the coupling $g_{G\gamma\gamma}$ is renormalisation group invariant and $O(1/N_c)$ suppressed and our conjecture is that such terms would indeed be relatively small. Remarkably, evaluated at the central value of the topological susceptibility found in ref.[25], the coupling $g_{G\gamma\gamma}$ is essentially zero. This is probably a numerical coincidence, since we can not think of a dynamical reason why this coupling should vanish identically. What is more reasonable is to consider its value across the range of error of the topological susceptibility. In this case, we see from eq.(4.20) that the suppression is numerically still under 10%, which is closer to that expected for a typical OZI correction although still remarkably small.

This is a very encouraging result. First, it increases our confidence that we are able to identify quantities where the OZI, or leading $1/N_c$, approximation is likely to be numerically good. It also shows that $g_{G\gamma\gamma}$ gives a contribution to the decay formula which is entirely consistent with its picturesque interpretation as the coupling of the photons to the anomaly-induced gluonic component of the $\eta'$. A posteriori, the fact that its contribution is at most 10% explains the general success of previous theoretically inconsistent phenomenological parametrisations of $\eta'$ decays in which the naive current algebra formulae omitting the gluonic term are used.

5. Further discussion

We now summarise our results so far and discuss a number of aspects of their validity and applicability to other processes, both in low-energy pseudoscalar meson physics and in high-energy processes such as deep inelastic scattering or photonproduction.

In this paper, we have seen how the uncontroversial radiative decay formula

$$f^{8\eta'}g_{\eta'\gamma\gamma} + f^{8\eta}g_{\eta\gamma\gamma} = \frac{1}{\sqrt{3}} \frac{\alpha_{em}}{\pi}$$

(5.1)

together with the three DGMOR relations

$$(f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_{\eta}^2 = \frac{1}{4}(f^2_{\pi}m_{\pi}^2 + 2f^2_{K}m_{K}^2) + 6A$$

(5.2)

$$f^{0\eta'}f^{8\eta'}m_{\eta'}^2 + f^{0\eta}f^{8\eta}m_{\eta}^2 = \frac{2\sqrt{2}}{3}(f^2_{\pi}m_{\pi}^2 - f^2_{K}m_{K}^2)$$

(5.3)

$$(f^{8\eta'})^2 m_{\eta'}^2 + (f^{8\eta})^2 m_{\eta}^2 = -\frac{1}{3}(f^2_{\pi}m_{\pi}^2 - 4f^2_{K}m_{K}^2)$$

(5.4)

The topological susceptibility parameter $A$ is also renormalisation group invariant.
lead to the following identification of the four pseudoscalar meson decay constants in the mixed \( \eta', \eta \) sector:

\[
\begin{pmatrix}
  f^{0\eta'} \\
  f^{0\eta}
\end{pmatrix}
\begin{pmatrix}
  f^{8\eta'} \\
  f^{8\eta}
\end{pmatrix}
= 
\begin{pmatrix}
  104.2 \pm 4.0 & 22.8 \pm 5.7 \\
  -36.1 \pm 1.2 & 98.4 \pm 1.4
\end{pmatrix}
\text{MeV}
\]  

(5.5)

The novel feature is the use of the flavour singlet DGMOR relation. We emphasise again that this is, within the framework of PCAC or chiral Lagrangians, an exact result in the sense of being entirely independent of the \( 1/N_c \) expansion. Its large-\( N_c \) limit is the well-established Witten-Veneziano formula. This determination of the decay constants is therefore on firm theoretical ground and provides a sound basis for phenomenology.\(^6\)

The use of the flavour singlet DGMOR formula relies on the input of the non-perturbative parameter \( A \) which controls the topological susceptibility in QCD. In time, lattice calculations should be able to determine this number accurately from simulations in full QCD with massive, dynamical quarks. For the moment, we have to rely on the result obtained from the pure Yang-Mills calculation, which we have argued using \( 1/N_c \) ideas should be a good approximation. This temporary approximation is the only place where \( 1/N_c \) enters the determination of the decay constants.

To illustrate the dependence of the decay constants on the topological susceptibility, we have plotted the singlet decay constants \( f^{0\eta'} \) and \( f^{0\eta} \) against \( A \) in Fig. 1. It is clear from the DGMOR and radiative decay relations that the octet decay constants \( f^{8\eta'} \) and \( f^{8\eta} \) are themselves independent of \( A \).

![Figure 1](image)

**Figure 1:** The decay constants \( f^{0\eta'} \) and \( f^{0\eta} \) as functions of the non-perturbative parameter \( A = (x \text{ MeV})^4 \) which determines the topological susceptibility in QCD.

The remaining part of our analysis is the flavour singlet decay formula

\[
f^{0\eta'} g_{\eta'\gamma\gamma} + f^{0\eta} g_{\eta\gamma\gamma} f^{0\eta} g_{\eta\gamma\gamma} + \sqrt{6} A g_{G\gamma\gamma} \stackrel{\text{acquire}}{=} 2 \left(\frac{\sqrt{2}}{\sqrt{3}} \frac{\alpha_{em}}{\pi}\right)
\]

(5.6)

Here, because the \( \eta' \) is not a NG boson even in the chiral limit, the naive PCAC formula acquires an extra term. In our formulation, this is the parameter \( g_{G\gamma\gamma} \) which we have

\(^6\)Note that the PDG tables currently do not quote values for the flavour singlet decay constants because of the subtleties in their definition. A good case can therefore be made for adopting the definitions and experimental numbers presented here as a simple and theoretically well-motivated parametrisation of the data.
argued may reasonably, though certainly non-rigorously, be interpreted as the coupling of the photons to the anomaly-induced gluonic component of the $\eta'$, i.e. the component which removes its NG boson status. The picture is simple and attractive. The experimental value of the unphysical coupling, $g_{G\gamma\gamma} = -0.001 \pm 0.072 \text{GeV}^{-4}$, means that its contribution to the decay formula is under 10%. This is consistent with our expectations for a renormalisation group invariant, $1/N_c$ (or OZI) suppressed quantity.

This is illustrated in Fig. 2, where we have shown the size of the contributions of the various terms in eq.(5.6), showing explicitly their dependence on the topological susceptibility.

![Figure 2](image)

**Figure 2:** This shows the relative sizes of the contributions to the flavour singlet radiative decay formula (5.6) expressed as functions of the topological susceptibility parameter $A = (x \text{ MeV})^4$. The dotted (black) line denotes $2\sqrt{2} \frac{a_{\text{em}}}{\pi} \frac{a_{\text{em}}}{\pi}$. The dominant contribution comes from the term $f_{0}^{\eta'} g_{0}^{\eta'}$, denoted by the long-dashed (green) line, while the short-dashed (blue) line denotes $f_{0}^{\eta'} g_{0}^{\eta'}$. The contribution from the gluonic coupling, $\sqrt{6} A g_{G\gamma\gamma}$, is shown by the solid (red) line.

However, while the flavour singlet decay formula is sensible and theoretically consistent, it is necessarily non-predictive. To be genuinely useful, we would need to find another process in which the same coupling enters. The problem here is that, unlike the decay constants which are universal, the coupling $g_{G\gamma\gamma}$ is process-specific just like $g_{\eta\gamma\gamma}$ or $g_{\eta\gamma\gamma}$. There are of course many other processes to which our methods may be applied such as $\eta'(\eta) \rightarrow V\gamma$, where $V$ is a flavour singlet vector meson $\rho, \omega, \phi$, or $\eta'(\eta) \rightarrow \pi^+\pi^-\gamma$. The required flavour singlet formulae may readily be written down, generalising the naive PCAC formulae. However, each will introduce its own gluonic coupling, such as $g_{GV\gamma}$. Although strict predictivity is lost, our experience with the two-photon decays suggests that these extra couplings will give relatively small, at most $O(10\% - 20\%)$, contributions if like $g_{G\gamma\gamma}$ they can be identified as RG invariant and $1/N_c$ suppressed.

Interestingly, these novel gluonic couplings may also arise in high-energy processes. For example, the standard analysis of the two-photon deep-inelastic scattering process $e^+e^- \rightarrow$
\[ e^+e^-X \] reduces the problem of finding the first moment of the polarised photon structure function \( g_1^\gamma \) to a non-perturbative evaluation of the off-shell matrix element \( \langle \gamma | J_{\mu 5}^0 | \gamma \rangle \). The difference with \( \eta'(\eta) \to \gamma\gamma \) is that in the DIS scenario, the photons are off-shell and the interest in the first moment sum rule for \( g_1^\gamma \) is precisely how it depends on the target photon momentum [7, 8, 9]. Nevertheless, the problem may be formulated in terms of form factors \( g_{\eta'(\eta)}(k^2) \) and \( g_{\gamma\gamma}(k^2) \) of which the couplings discussed here are simply the \( k^2 = 0 \) limit [9]. It has also been suggested that the gluonic couplings \( g_{G\phi\gamma} \) and \( g_{G\eta\eta} \) could play a dominant role in the photoproduction process \( \gamma N \to \phi N \) [34]. This interpretation is less clear, but it is an interesting subject for future work to look at a variety of electro or photoproduction experiments in the light of the PCAC methods developed here.

6. Pseudoscalar meson couplings of the nucleon and the \( U(1)_A \) Goldberger-Treiman relation

A further particularly interesting application of these ideas is to the pseudoscalar couplings of the nucleon. For the pion, the relation between the axial-vector form factor of the nucleon and the pion-nucleon coupling \( g_{\pi NN} \) is the well-known Goldberger-Treiman relation. Here, we are concerned with its generalisation to the flavour-singlet sector, which involves the anomaly and gluon topology. This \( U(1)_A \) Goldberger-Treiman relation was first developed in refs.[11, 12, 13]. In this case, the corresponding high-energy process involves the measurement of the first moment of the polarised structure function of the nucleon \( g_1^N \) in deep-inelastic scattering. In the flavour-singlet sector, this is the so-called ‘proton spin’ problem. (For reviews, see e.g. refs[35, 36].)

The axial-vector form factors are defined from

\[
\left\langle N | J_{\mu 5}^a | N \right\rangle = 2m_N \left( G_A^a(k^2) s_{\mu} + G_P^a(k^2) k. s k_{\mu} \right)
\]

where \( s_{\mu} = \bar{u} \gamma_{\mu} \gamma_5 u / 2m_N \) is the covariant spin vector. In the absence of a massless pseudoscalar, only the form factors \( G_A(0) \) contribute at zero momentum. Using the ‘\( U(1)_A \) PCAC’ substitution (3.13) for \( \partial_{\mu} J_{\mu 5}^a \) and repeating the steps explained in section 3 (particularly at eq.(3.15)), we straightforwardly find the following generalisation of the Goldberger-Treiman relation:

\[
2m_NG_A^a(0) = f^{a\alpha}g_{\eta^\alpha NN} + \sqrt{2} m_f A g_{GNN} \delta_{a0}
\]

with the obvious definition of the gluonic coupling \( g_{GNN} \) in analogy to \( g_{\eta^\alpha NN} \).

For the individual flavour components, this reads (abbreviating \( G_A(0) = G_A^a \)):

\[
2m_NG_A^a = f_\pi g_{\pi NN}
\]

\[
2m_NG_A^8 = f_8 g_{\gamma NN} + f_8^\eta g_{\eta NN}
\]

\[
2m_NG_A^0 = f_0 g_{\gamma NN} + f_0^\eta g_{\eta NN} + \sqrt{6} A g_{GNN}
\]
Eq. (6.5) is the $U(1)_A$ Goldberger-Treiman relation. Notice that the flavour-singlet coupling $G^0_A$ is not renormalisation group invariant and so depends on the RG scale. This is reflected in the RG non-invariance of the gluonic coupling $g_{GNN}$ \[13\].

In the notation that has become standard in the DIS literature, the axial couplings are

\[G^3_A = \frac{1}{2} a^3, \quad G^8_A = \frac{1}{2\sqrt{3}} a^8, \quad G^0_A = \frac{1}{\sqrt{6}} a^0\] (6.6)

and have the following interpretation in terms of parton distribution functions:

\[
a^3 = \Delta u - \Delta d, \quad a^8 = \Delta u + \Delta d - 2\Delta s, \quad a^0 = \Delta u + \Delta d + \Delta s - \frac{3\alpha_s}{2\pi}\Delta g\] (6.7)

Experimentally,

\[
a^3 = 1.267 \pm 0.004, \quad a^8 = 0.585 \pm 0.025\] (6.8)

from low-energy data, while the latest result for $a^0$ quoted by the COMPASS collaboration \textit{[37, 38]} is

\[
a^0|_{Q^2=4\text{GeV}^2} = 0.237^{+0.024}_{-0.029}\] (6.9)

It is the fact that $a^0$ is much less than $a^8$, as would be predicted on the basis of the simple quark model (the Ellis-Jaffe sum rule \textit{[39]}), that is known as the ‘proton spin’ problem. For a careful analysis of the distinction between the angular momentum (spin) of the proton and the axial coupling $a^0$, see however refs.\textit{[40, 41]}.

From the standard Goldberger-Treiman relation (6.3), we immediately find the following result for the (dimensionless) pion-nucleon coupling,

\[g_{\pi NN} = 12.86 \pm 0.06\] (6.10)

consistent to within $\sim 5\%$ with the experimental value $13.65(13.80) \pm 0.12$ (depending on the dataset used) \textit{[42]}.

In an ideal world where $g_{\eta NN}$ and $g_{\eta'NN}$ were both known, we would now verify the octet formula (6.4) then determine the gluonic coupling $g_{GNN}$ from the singlet Goldberger-Treiman relation (6.5). However, the experimental situation with the $\eta$ and $\eta'$-nucleon couplings is far less clear. (See refs.\textit{[43, 44]} for reviews of the relevant experimental literature and recent results.) One would hope to determine these couplings from the near threshold production of the $\eta$ and $\eta'$ in nucleon-nucleon collisions, i.e. $pp \rightarrow pp\eta$ and $pp \rightarrow pp\eta'$.

The original form as quoted in ref.\textit{[13]} applies to the chiral limit and reads, in the notation of \textit{[13]} but allowing for our different normalisation of the singlet,

\[2m_N G^0_A = Fg_{\eta'NN} + \frac{1}{\sqrt{2\pi f}} F^2 m_{\eta'}^2 g_{GNN}\]

where $F$ is a RG invariant decay constant defined from the two-point Green function of the pseudoscalar field $\phi_5$. In the chiral limit, where there is no SU(3) mixing, this is reproduced by the definition (3.12) of $f_{\eta'\eta'}$. The off-diagonal decay constant $f^{\eta\eta'}$ vanishes. The final term is reproduced by eq. (6.5) by virtue of the flavour-singlet DGMOR relation (2.7) in the chiral limit.
measured for example at COSY-II [45]. However, the $\eta$ production is dominated by the $S_{11}$ nucleon resonance $N^*(1535)$ which decays to $N\eta$, and as a result very little is known about $g_{\eta NN}$ itself. The detailed production mechanism of the $\eta'$ is not well understood. However, since there is no known baryonic resonance decaying into $N\eta'$, we may simply assume that the reaction $pp \to ppp'$ is driven by the direct coupling supplemented by heavy-meson exchange. This allows an upper bound to be placed on $g_{\eta' NN}$ and on this basis ref.[46] quotes $g_{\eta' NN} < 2.5$. This is supported by an analysis [47] of very recent data from CLAS [48] on the photoproduction reaction $\gamma p \to p\eta'$. Describing the cross-section data with a model comprising the direct coupling together with $t$-channel meson exchange and $s$ and $u$-channel resonances, it is found that equally good fits can be obtained for several values of $g_{\eta' NN}$ covering the whole region $0 < g_{\eta' NN} < 2.5$.

In view of this experimental uncertainty, we shall use the octet and singlet GT relations to plot the predictions for $g_{\eta NN}$ and $g_{GNN}$ as a function of the experimentally uncertain $\eta'$-nucleon coupling in the range $0 < g_{\eta' NN} < 2.5$. The results are shown in Fig. 3.

![Figure 3](image-url)

Figure 3: These figures show the dimensionless $\eta$-nucleon coupling $g_{\eta NN}$ and the gluonic coupling $g_{GNN}$ in units of GeV$^{-3}$ expressed as functions of the experimentally uncertain $\eta'$-nucleon coupling $g_{\eta' NN}$, as determined from the flavour octet and singlet Goldberger-Treiman relations (6.4) and (6.5).

Our main interest in the $U(1)_A$ Goldberger-Treiman relation lies of course in the gluonic coupling $g_{GNN}$. Unlike its counterpart $g_{G\gamma\gamma}$ in the radiative decay formula, $g_{GNN}$ is not a renormalisation group invariant coupling. However, like $g_{G\gamma\gamma}$ it is suppressed at large $N_c$. The various terms in eq.(6.5) have the following orders: $G_A = O(N_c)$, $f^{0\eta}, f^{0\eta'} = O(\sqrt{N_c})$, $A = O(1)$, $g_{\eta NN}, g_{\eta' NN} = O(\sqrt{N_c})$, $g_{GNN} = O(1)$. So the final term $Ag_{GNN}$ is $O(1)$, down by a power of $1/N_c$ compared to all the others, which are $O(N_c)$.

The intuition we have developed through experience with flavour singlet physics and the large-$N_c$ expansion is that while we expect $O(1/N_c)$ suppressed RG invariant quantities to be numerically small, in line with expectations from the OZI rule, we do not expect this to be necessarily true for RG non-invariant quantities such as $g_{GNN}$. So unlike $g_{G\gamma\gamma}$ in

---

9More precisely, we expect the OZI approximation to be unreliable for quantities which have a different RG behaviour in QCD itself and in the OZI limit. The complicated RG non-invariance of $g_{GNN}$ in QCD is induced by the axial anomaly, since $G_A^0$ itself is required to scale with the anomalous dimension $\gamma$ of
the flavour-singlet radiative decay formula, we would not be surprised if $g_{GNN}$ makes a sizeable numerical contribution to the $U(1)_A$ GT relation.\footnote{Of course, since $g_{GNN}$ is defined with dimension GeV$^{-3}$ whereas $g_{\eta NN}$ and $g_{\eta' NN}$ are dimensionless, we cannot make a direct comparison of the couplings themselves.}

This is quantified in Fig. 4, where we have plotted the contribution of each term in the $U(1)_A$ GT relation as a function of $g_{\eta' NN}$. We see immediately that the contribution from $f^{0\eta'} g_{\eta' NN}$ is relatively constant around 0.08, compared with $2m_N G_A^0 \sim 0.18$. This means that the variation of $f^{0\eta} g_{\eta NN}$ over the experimentally allowed range is compensated entirely by the variation of $\sqrt{6} A g_{GNN}$. For generic values of $g_{\eta' NN}$, there is no sign of a significant suppression of the contribution of the gluonic coupling $g_{GNN}$ relative to the others. This should be contrasted with the corresponding plot for $g_{G\gamma\gamma}$ in the radiative decay formula (Fig. 2).

![Figure 4: This shows the relative sizes of the contributions to the $U(1)_A$ Goldberger-Treiman relation from the individual terms in eq.(6.5), expressed as functions of the coupling $g_{\eta' NN}$. The dotted (black) line denotes $2m_N G_A^0$. The long-dashed (green) line is $f^{0\eta'} g_{\eta' NN}$ and the short-dashed (blue) line is $f^{0\eta} g_{\eta NN}$. The solid (red) line shows the contribution of the novel gluonic coupling, $\sqrt{6} A g_{GNN}$, where $A$ determines the QCD topological susceptibility. To see this in more detail, consider a representative value, $g_{\eta' NN} = 2.0$, which would correspond to the direct coupling contributing substantially to the cross sections for $pp \to pp\eta'$ and $\gamma p \to p\eta'$. In this case, $g_{\eta NN} = 3.96 \pm 0.16$ and $g_{GNN} = -14.6 \pm 4.3$ GeV$^{-3}$. The contributions to the $U(1)_A$ GT relation are then (in GeV):

$$2m_N G_A^0 [N_c; 0.18] = f^{0\eta'} g_{\eta' NN} [N_c; 0.21] + f^{0\eta} g_{\eta NN} [N_c; 0.09] + \sqrt{6} A g_{GNN} [O(1); -0.12]$$

(6.11)

The anomalously small value of $G_A^0$ compared to $G_A^8$ is due to the partial cancellation of the sum of the two meson-coupling terms, which together contribute close to the expected OZI value ($2m_N G_A^8 = 0.32$), by the gluonic coupling $g_{GNN}$. Although this is formally $O(1/N_c)$ suppressed, numerically it gives the dominant contribution to the large OZI violation in the flavour-singlet current $J_{\mu5}^a$ and all the other terms in the $U(1)_A$ GT relation are RG invariant. The anomaly, and thus the anomalous dimension $\gamma$, vanishes in the large-$N_c$ limit leaving $g_{GNN}$ RG invariant as $N_c \to \infty$.}
$G^0_A$. This is in line with our expectations and would provide further evidence that the insights developed in our body of work on both low-energy $\eta$ and $\eta'$ physics and related high-energy phenomena such as the ‘proton spin’ problem are on the right track.

However, it may still be that $g_{\eta'NN}$ is significantly smaller, implying a relatively small contribution to the production reactions $pp \rightarrow ppp\eta$ and $\gamma p \rightarrow p\eta'$ from the direct coupling. In particular, there is a range of $g_{\eta'NN}$ around $0.7 - 1.3$ where the gluonic coupling $g_{GNN}$ only contributes to the $U(1)_A$ GT relation at the level expected of a typical OZI-suppressed quantity, despite its RG non-invariance. In the extreme case where $g_{\eta'NN} = 1.0$, we have $g_{\eta NN} = 3.59 \pm 0.15$, $g_{GNN} = 0.5 \pm 3.8\text{GeV}^{-3}$, and the contributions to the $U(1)_A$ GT formula become (in GeV):

$$2m_N G^0_A[N_c; 0.18] = f^{0\eta'} g_{\eta'NN}[N_c; 0.10] + f^{0\eta} g_{\eta NN}[N_c; 0.08] + \sqrt{6} A g_{GNN}[O(1); -0.004]$$

(6.12)

This scenario would be similar to the radiative decays, where we found that the corresponding coupling $g_{G\gamma\gamma} \simeq 0$ using the central value of the lattice determination of the topological susceptibility and contributes only at $O(10\%)$ within the error bounds on $A$. This would then suggest that RG non-invariance is not critical after all and the $O(1/N_c)$ suppressed gluonic couplings are indeed numerically small. It would also leave open the possibility that all couplings of type $g_{GXX}$ are close to zero, which in the picturesque interpretation discussed earlier would imply that the gluonic component of the $\eta'$ wave function is small. The suppression of $G^0_A$ relative to $G^8_A$ would then not be due to gluonic, anomaly-induced OZI violations but rather to the particular nature of the flavour octet-singlet mixing in the $\eta - \eta'$ sector. In the parton picture of the ‘proton spin’ problem, this would be a hint that the suppression in $a^0$ is not primarily due to the polarised gluon distribution $\Delta g$ but also involves a strong contribution from the polarised strange quark distribution $\Delta s$.

Although we would consider this alternative scenario rather surprising and prefer the more theoretically motivated interpretation of the flavour singlet sector in which $g_{\eta'NN} \simeq 2$ and gluon topology plays an important role, ultimately the decision rests with experiment. Clearly, a reliable determination of $g_{\eta'NN}$, or equivalently $g_{\eta NN}$, would shed considerable light on the $U(1)_A$ dynamics of QCD.

### Acknowledgments

I would like to thank G. Veneziano for interesting comments and collaboration on the original investigations of $U(1)_A$ physics on which this paper is based. This work is supported in part by PPARC grants PPA/G/O/2002/00470 and PP/D507407/1.
A. Appendix: Comparison with chiral Lagrangians

It is useful to compare the results presented in this paper with those arising in extensions of the chiral Lagrangian formalism to include the $\eta'$ and low-energy flavour singlet physics. The large-$N_c$ expansion plays a crucial role in this approach, since it is only in the limit $N_c \to \infty$ that the anomaly disappears, the chiral symmetry is enlarged to $U(3)_L \times U(3)_R$ and the $\eta'$ appears as a light NG degree of freedom to be included in the fundamental fields $\varphi^a$ ($a = 0, 1 \ldots 8$) of the chiral Lagrangian. Large-$N_c$ chiral Lagrangians have been developed by a number of authors, notably ref.[18], though here we shall focus on the results obtained by Kaiser and Leutwyler [14, 15, 16].

The elegance of chiral Lagrangians should not obscure the fact that all the dynamical approximations we have made in deriving our results, such as pole dominance by NG bosons, weak momentum dependence of pole-free quantities like decay constants and couplings and a judicious use of the $1/N_c$ expansion, are necessarily also made in the chiral Lagrangian formalism, where they are built in to the structure of the initial Lagrangian. Indeed, our results can also be systematised into an effective Lagrangian (see ref.[1] for details). The principal merit of the chiral/effective Lagrangian approach in general is in providing a systematic way of calculating higher-order corrections.

The physical results obtained using the two methods should therefore be equivalent. However, a number of our definitions, notably of the flavour singlet decay constants, differ from those made by Kaiser and Leutwyler so the comparison is not straightforward. However, we would argue that in many ways the formalism developed in this paper provides a better starting point for the description of $U(1)_A$ phenomenology, especially in the use of RG invariant decay constants and our natural generalisation of the Witten-Veneziano formula as the flavour singlet DGMOR relation.

The fundamental fields in the KL chiral Lagrangian are assembled into matrices $U = \exp[i\varphi^aT^a]$ so that, up to mixing, the fields are in one-to-one correspondence with the NG bosons in the chiral and large-$N_c$ limits.\textsuperscript{11} The Lagrangian is a simultaneous expansion in three parameters - momentum ($p$), quark mass ($m$) and $1/N_c$. For bookkeeping purposes, KL consider these to be related as follows: $p^2 = O(\delta)$, $m = O(\delta)$, $1/N_c = O(\delta)$, and expand consistently in the small parameter $\delta$. It will be clear, however, that this is mere bookkeeping and should be treated with considerable caution. As we have seen, the realisation of the $1/N_c$ expansion in the singlet sector is extremely delicate and it cannot simply be assumed that quantities, especially RG non-invariant ones, that are $1/N_c$ suppressed are necessarily numerically small or indeed of $O(p^2, m)$.

Nevertheless, arranging the allowed terms in the chiral Lagrangian according to their order in $\delta$, KL find:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \ldots \quad (A.1)$$

\textsuperscript{11}We use the same normalisation for the generators as in the rest of this paper, so our singlet field differs from the $\psi$ of refs.[14, 15, 16] by $\varphi^0 = \sqrt{2/3} \psi$. The normalisation of the singlet currents and decay constants is, however, the same. We assume isospin symmetry and represent the quark mass matrix by $M = \text{diag}(m_u, m_d, m_s)$ with $m_u = m_d = m$. 


As a consequence, the singlet decay constants $F$ couplings to the axial-vector currents:

$$L_0 = \frac{1}{4} F^2 \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{2} F^2 B \text{tr}(M^\dagger U + U^\dagger M) - \frac{3}{4} \tau(\varphi^0)^2$$

(A.2)

and

$$L_1 = 2 B L_5 \text{tr}(\partial_\mu U^\dagger \partial^\mu U(M^\dagger U + U^\dagger M)) + 4 B^2 L_8 \text{tr}((M^\dagger U)^2 + (U^\dagger M)^2)
+ \frac{1}{8} F^2 \Lambda_1 \partial^\mu \varphi^0 \partial_\mu \varphi^0 + \frac{i}{6} \sqrt{2} F^2 B \Lambda_2 \varphi^0 \text{tr}(M^\dagger U - U^\dagger M)$$

(A.3)

Most of the physics we are interested in can be derived from $L_0 + L_1$. However, we also encounter some of the terms in the next order,

$$L_2 = 2 B L_4 \text{tr}(\partial_\mu U^\dagger \partial^\mu U) \text{tr}(M^\dagger U + U^\dagger M)
+ 4 B^2 L_6 \left( \text{tr}(M^\dagger U + U^\dagger M) \right)^2 + 4 B^2 L_7 \left( \text{tr}(M^\dagger U - U^\dagger M) \right)^2
- 2 i \frac{3}{2} B L_8 \partial_\mu \varphi^0 \text{tr}(M^\dagger \partial^\mu U - \partial^\mu U^\dagger M) - 4 i \frac{3}{2} B^2 L_9 \varphi^0 \text{tr}((M^\dagger U)^2 + (U^\dagger M)^2)
+ \ldots$$

(A.4)

Here, $M$ is the quark mass matrix, $B$ sets the scale of the quark condensate, $F$ is the leading-order decay constant before $SU(3)$ breaking, and $\tau$ is a parameter which is identified at leading order in $1/N_c$ with the topological susceptibility of pure Yang-Mills. Their $1/N_c$ orders are: $F^2 = O(N_c)$ and $B, \tau = O(1)$. The coefficients entering at higher order have the following dependence: $\Lambda_1, \Lambda_2 = O(1/N_c)$, $L_5, L_8 = O(N_c)$ and $L_4, L_6, L_7, L_9, L_9, L_{25} = O(1)$.

KL define their decay constants $F^a_P$ ($P = \pi, K, \eta, \eta'$) in the conventional way as the couplings to the axial-vector currents:

$$\langle 0 | J^a_{\mu 5} | P \rangle = i k_P F^a_P$$

(A.5)

As a consequence, the singlet decay constants $F^0_{\eta'}$ and $F^0_\eta$ are not RG invariant but scale with the usual anomalous dimension $\gamma$ corresponding to the multiplicative renormalisation of the singlet current. It follows that the parameters $(1 + \Lambda_1)$ and $\tau$ are also not RG invariant, scaling with anomalous dimension $2\gamma$. This means that while $\tau$ coincides with $A$ (the non-perturbative parameter determining the QCD topological susceptibility) at $O(1)$, it differs beyond leading order in $1/N_c$.

The $SU(3)$ breaking which distinguishes the decay constants arises first from the terms in $L_1$. Beyond this order, in addition to the direct contributions from the new couplings

$$\tau_R = Z^2 \tau_B \quad 1 + \Lambda_1 R = Z^2 (1 + \Lambda_1 B)$$

where $Z$ is the usual multiplicative renormalisation factor for the axial current, $J^0_{\mu 5 R} = Z J^0_{\mu 5 B}$. In the KL chiral Lagrangian formalism, the singlet field $\varphi^0$ is itself renormalised. Allowing for a non-zero vacuum angle $\theta$, this field renormalisation is

$$\varphi_R = Z^{-1} \varphi^0 + \sqrt{\frac{2}{3} (Z^{-1} - 1)} \theta$$
in the chiral Lagrangian, there are also contributions from loop diagrams calculated using $L_0$. These give rise to the ‘chiral logarithms’ $\mu_P = \frac{m_\pi^2}{32\pi^2 F_\pi} \ln \frac{m_\pi^2}{\mu^2}$, which KL have calculated explicitly. To present their results, we again use the two-angle parametrisation (cf eq.4.13) in the octet-singlet sector:

$$
\begin{pmatrix}
F_0^{\eta'} & F_0^{\eta} \\
F_8^{\eta'} & F_8^{\eta}
\end{pmatrix} =
\begin{pmatrix}
F_0 \cos \theta_0 & -F_0 \sin \theta_0 \\
F_8 \sin \theta_8 & F_8 \cos \theta_8
\end{pmatrix}
$$

(A.6)

To this order, the KL decay constants are then [15]:

$$
F_\pi = F \left[ 1 + \frac{4B}{F^2} \left( 2 \sum m_q L_4 + 2m L_5 \right) + O(\mu_P) \right]
$$

(A.7)

$$
F_K = F \left[ 1 + \frac{4B}{F^2} \left( 2 \sum m_q L_4 + (m + m_s) L_5 \right) + O(\mu_P) \right]
$$

(A.8)

$$
F_8 = F \left[ 1 + \frac{4B}{F^2} \left( 2 \sum m_q L_4 + \frac{2}{3}(m + 2m_s) L_5 \right) + O(\mu_P) \right]
$$

(A.9)

For the singlet,

$$
F_0 = \sqrt{1 + \Lambda_1 \bar{F}_0}
$$

(A.10)

where the scale-invariant part is

$$
\bar{F}_0 = F \left[ 1 + \frac{4B}{F^2} \left( 2 \sum m_q L_4 + \frac{2}{3}(2m + m_s)(-L_5 + L_A) \right) \right]
$$

(A.11)

where $L_A = (2L_5 + 3L_{18})/\sqrt{1 + \Lambda_1} = 2L_5 + O(1)$. Notice there are no loop corrections to $F_0$. Finally, the difference in the angles $\theta_0$ and $\theta_8$ is determined from

$$
F_0^{\eta} F_8^{\eta'} + F_0^{\eta'} F_8^{\eta} = -F_0 F_8 \sin(\theta_0 - \theta_8)
= \frac{8\sqrt{2}}{3} B(m - m_s)(2L_5 + 3L_{18})
$$

(A.12)

and is proportional to the $SU(3)$ breaking $m, m_s$ mass difference. Also recall [17] that at lowest order, the pseudoscalar masses are $m_\pi^2 = 2mB$, $m_K^2 = (m + m_s)B$, $m_\eta^2 = \frac{2}{3}(m + 2m_s)B$ and $m_8^2 = \frac{2}{3}(2m + m_s)B$.

These decay constants satisfy a set of relations which are closely analogous, but not identical, to the DGMOR relations (2.14),(2.15) and (2.16). In fact, up to terms involving chiral logarithms, the decay constants shown above satisfy (see also ref.[19])

$$
\bar{F}_0^{\eta} F_0^{\eta'}/ F_8^{\eta'} F_8^{\eta} = \bar{(F_0^2)} = \frac{1}{3} (F_\pi^2 + 2F_K^2)
$$

(A.13)

$$
F_0^{\eta} F_8^{\eta'} + F_0^{\eta'} F_8^{\eta} = -F_0 F_8 \sin(\theta_0 - \theta_8) = \frac{2\sqrt{2}}{3}(F_\pi^2 - F_K^2)
$$

(A.14)

$$
F_8^{\eta} F_0^{\eta'} + F_8^{\eta'} F_0^{\eta} = (F_8^2) = -\frac{1}{3} (F_\pi^2 - 4F_K^2)
$$

(A.15)

The corresponding DGMOR relations, i.e. including the appropriate pseudoscalar mass terms, are broken also by the terms proportional to $L_7$ and $L_8$ (the $L_6$ contributions cancel). These are just the expected $O(m^2)$ corrections to the leading-order PCAC relations.
Our main interest, as always, is in the flavour singlet sector. Notice that the relation (A.13) is written for the scale-invariant part of the singlet decay constants only, omitting the factor involving the OZI-violating coupling $\Lambda_1$. This does not involve the topological susceptibility in any way and is not related to the Witten-Veneziano formula. This enters the formalism as follows. In the chiral limit and working to $O(\delta)$, the relevant part of the chiral Lagrangian is just

$$L \sim \frac{1}{8} F^2 (1 + \Lambda_1) \partial_\mu \phi^0 \partial^\mu \phi^0 - \frac{3}{4} \tau (\phi^0)^2$$  \tag{A.16}$$

from which it follows immediately that

$$(F_\eta^0)^2 m_{\eta^0}^2 = 6 \tau, \quad F_\eta^0 \eta' = \sqrt{1 + \Lambda_1 F}$$  \tag{A.17}$$

At leading order in $1/N_c$, this is the original Witten-Veneziano formula, since $\Lambda_1 = O(1/N_c)$ and we may interpret $\tau = -\chi(0)|_{YM} + O(1/N_c)$. However, beyond leading order and away from the chiral limit, it does not have a straightforward generalisation in terms of the QCD topological susceptibility, since $\tau$ is not identical to $A$. Fundamentally, this originates from the use of an RG non-invariant field $\phi^0$ in the formulation of the chiral Lagrangian. In contrast, the corresponding formula (2.14), i.e. the flavour singlet DGMOR relation written with our RG invariant definition of $f_{\eta^0}$ and involving the parameter $A$ determining the topological susceptibility in QCD itself, is the appropriate generalisation of the Witten-Veneziano relation and provides a more suitable basis for describing $U(1)_A$ phenomenology.

These formulae allow a numerical determination of the decay constants and mixing angles in the octet-singlet sector. The results quoted in ref.[22] are

$$F_0 = 1.25f_\pi \quad F_8 = 1.28f_\pi \quad \theta_0 = -4 \text{ deg} \quad \theta_8 = -20.5 \text{ deg}$$  \tag{A.18}$$

These should be compared with our results, eq.(4.14). Since the definitions of the octet decay constants are the same, we expect $F_8 \simeq f_8$ and $\theta_8 \simeq \theta_8$. Indeed, the angles agree while $F_8$ is a little higher than $f_8$, by around 10%. This is explicable by the fact that the KL fit incorporates the next-to-leading order corrections in the chiral expansion but not the radiative decays, whereas we have used the leading-order DGMOR relation together with the octet radiative decay formula. The definitions of the flavour singlet decay constant and mixing angle are different in the two approaches, so cannot be directly compared.

Radiative decays are described in the chiral Lagrangian approach by the Wess-Zumino-Witten term, which encodes the anomalous Ward identities. The relevant part, constructed by Kaiser and Leutwyler [16], is\(^\text{13}\)

$$\mathcal{L}_{WZW} = -\frac{\alpha N_c}{4\pi} \left[ \text{tr}(\bar{e}^2 \varphi) + \frac{1}{3} \Lambda_3 \text{tr}(\bar{e}^2) \text{tr} \varphi + 2B \Lambda_4 \text{tr}(\bar{e}^2 M \varphi) \right] F_{\mu\nu} \tilde{F}^{\mu\nu}$$  \tag{A.19}$$

\(^{13}\)In ref.[16], the constants $\Lambda_3$ and $\Lambda_4$ are called $K_1$ and $K_2$ resp. The $\Lambda_3$ notation is used in refs.[14] and [19]. The radiative decay formulae (A.20), (A.21) are identical to those quoted in ref.[19], eqs.(39) rewritten in our notation.
where \( \varphi = \varphi^a T^a \) and \( \hat{e} = \text{diag}(\frac{2}{7}, -\frac{1}{7}, -\frac{1}{7}) \) is the quark charge matrix. \( \Lambda_3 \) is \( O(1/N_c) \) and scales in the same way as \( \Lambda_1 \), while \( \Lambda_4 \) is \( O(1) \) and RG invariant. It is then straightforward to derive the following formulae (omitting the \( O(m) \) corrections arising from the \( \Lambda_4 \) term for simplicity)

\[
F^0_{\eta' \gamma \gamma} + F^0_{\gamma \eta \gamma \gamma} = (1 + \Lambda_3) a^0_{\text{em}} \frac{\alpha}{\pi} \quad (A.20)
\]

\[
F^8_{\eta' \gamma \gamma} + F^8_{\gamma \eta \gamma \gamma} = a^8_{\text{em}} \frac{\alpha}{\pi} \quad (A.21)
\]

These are the analogues of our eqs. (2.5) and (2.6). The key point is that in each case, the flavour singlet formula has to incorporate OZI breaking by the inclusion of a new, process-specific parameter, \( \Lambda_3 \) in the KL formalism and \( g_{\gamma \gamma} \) in our approach, which must be determined from the experimental data. In each case, this removes the predictivity of the formula unless, as we have discussed, we can bring theoretical arguments to bear to argue that the OZI-violating terms are small. In the KL case, this would presumably require the RG-invariant ratio \( (1 + \Lambda_3)/(1 + \Lambda_1) \) to be close to 1. However, once again, we consider that the flavour singlet decay formula presented here has the added virtue of explicitly showing the link with the topological susceptibility and giving a physical interpretation of the new OZI-violating parameter in terms of the gluonic component of the physical \( \eta' \) meson.

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