GAUSS-BONNET THEOREM AND CROFTON TYPE FORMULAS IN COMPLEX SPACE FORMS

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Abstract. We give an expression, in terms of the so-called Hermitian intrinsic volumes, for the measure of the set of complex $r$-planes intersecting a regular domain in any complex space form. Moreover, we obtain two different expressions for the Gauss-Bonnet-Chern formula in complex space forms. One of them expresses the Gauss curvature integral in terms of the Euler characteristic and some Hermitian intrinsic volumes. The other one, which is shorter, involves the measure of complex hyperplanes meeting the domain. As a tool, we obtain variation formulas in integral geometry of complex space forms. Finally, we express the average over the complex $r$-planes of the total Gauss curvature of the intersection with a domain.

1. Introduction

In the space of constant sectional curvature $k$, $\mathbb{M}^n(k)$, Santaló [San04, p. 310] found the expression of the measure of the set of $r$-planes meeting a regular domain in terms of the mean curvature integrals. Let $\mathcal{L}_r$ denote the space of $r$-dimensional geodesic planes in $\mathbb{M}^n(k)$ and let $d\mathcal{L}_r$ be a measure on $\mathcal{L}_r$ invariant under the isometry group of $\mathbb{M}^n(k)$. If $\Omega \subset \mathbb{M}^n(k)$ is a compact domain with smooth boundary and $r = 2l$, then

$$\int_{\mathcal{L}_{2l}} \chi(\Omega \cap L_{2l})dL_{2l} = c_0 \text{vol}(\Omega) + \sum_{i=1}^{l} c_i k^{l-i} M_{2i-1}(\partial \Omega),$$

where $c_i$ are known coefficients depending only on $n$, $r$ and $i$, while $M_j(\partial \Omega)$ denotes the mean curvature integral defined as

$$M_j(\partial \Omega) = \left(\frac{n-1}{j}\right)^{-1} \int_{\partial \Omega} \sigma_j(\mathbb{II})dx$$

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where $\sigma_j(\Pi)$ is the $j$-th symmetric elementary function of the eigenvalues of the second fundamental form. An analogous formula holds in the case of odd-dimensional planes.

In the proof of formula (1), Santaló used the Gauss-Bonnet-Chern theorem in $\mathbb{M}^n(k)$, that for $n$ even states

$$\frac{1}{2} \chi(\Omega) = M_{n-1}(\partial \Omega) + kc_{n-3}M_{n-3}(\partial \Omega) + \cdots + k^{n-2}c_1M_1(\partial \Omega) + k^{n-2}c_1 \text{vol}(\Omega).$$

He also used the reproductive property of the mean curvature integrals in $\mathbb{M}^n(k)$,

$$\int_{L^r} M_i^{(r)}(\partial \Omega \cap L_r) dL_r = cM_i(\partial \Omega),$$

where $M_i^{(r)}(\partial \Omega \cap L_r)$ denotes the $i$-th mean curvature integral of $\partial \Omega \cap L_r$ as a hypersurface in $L_r$ and $c$ is known and depends only on $n$, $r$ and $i$.

In this paper we generalize formula (1) and (3) to the space $\mathbb{C}K^n(\epsilon)$ of constant holomorphic curvature $4\epsilon$. The role of $L_r$ in (1) will be played by $L_r^C$, the space of complex $r$-planes (totally geodesic complex submanifolds). We restrict to regular domains (i.e. compact with smooth boundary) for simplicity, although our results could be extended to more general objects.

The method used by Santaló cannot be applied in this situation. Mean curvature integrals are defined in a complex space form as in (2), but equation (1) does not generalize to our setting. Moreover, in the complex projective space and in the complex hyperbolic space, an explicit Gauss-Bonnet formula was not known. Instead, we will use variational arguments, as well as some facts about the theory of valuations, which we briefly describe next.

**Definition 1.1.** Let $\mathcal{K}(V)$ denote the family of non-empty compact convex subsets of a finite dimensional real vector space $V$ of dimension $n$. A scalar valued functional $\phi : \mathcal{K}(V) \to \mathbb{R}$ is called a valuation if

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$.

The space of valuations that are continuous (with respect to Hausdorff distance), and invariant under the group of euclidean unit ball was studied by Hadwiger [Had57], who proved that the dimension of this space is $n + 1$. Mean curvature integrals, volume and Euler characteristic form a basis of this space.

On the standard Hermitian space $\mathbb{C}^n$ with its isometry group $IU(n) = \mathbb{C}^n \rtimes U(n)$, Alesker [Ale03] proved that there are more isometry invariant valuations than in the euclidean case: the dimension of this space is $\binom{n+2}{2}$. Both the Euler characteristic, and the measure of complex planes intersecting the domain belong to this space.
Bernig and Fu [BF08], consider several valuation bases on $\mathbb{C}^n$. Here we will use the basis of the so-called Hermitian intrinsic volumes $\{\mu_{k,q}\}_{k,q}$. These valuations were first introduced by Park in complex space forms (cf. [Par02]). In section 2 we recall their definition.

The main results of this paper can be stated as follows.

**Theorem 1.2.** Let $\Omega$ be a regular domain in $\mathbb{C}K^n(\epsilon)$. Then, for $r = 1, \ldots, n - 1$

$$\int_{\mathcal{L}_r^C} \chi(\Omega \cap L_r) dL_r = \text{vol}(G_{n-1,r}^C) \left( \begin{array}{c} n-1 \\ r \end{array} \right)^{-1} (\epsilon^r (r + 1) \text{vol}(\Omega) +$$

$$+ \sum_{k=n-r}^{n-1} \frac{1}{k} \frac{\omega_{2n-2k} \left( \begin{array}{c} n \\ k \end{array} \right)^{-1} \cdot ((k + r - n + 1) \mu_{2k,k}(\Omega) +$$

$$+ \sum_{q=\max\{0,2k-n\}}^{k-1} \frac{1}{4k-q} \left( \begin{array}{c} 2k - 2q \\ k - q \end{array} \right) \mu_{2k,q}(\Omega))\right)$$

where $dL_r$ denotes an invariant measure in the space of complex $r$-planes $\mathcal{L}_r^C$, and $\omega_i$ denotes the $i$-dimensional volume of the euclidean unit ball. Moreover,

$$O_{2n-1} \chi(\Omega) = 2n(n + 1)\epsilon^n \text{vol}(\Omega) +$$

$$+ \sum_{c=0}^{n-1} \frac{e^c O_{2n-2c-1}}{n-1} \left( \sum_{q=\max\{0,2c-n\}}^{c-1} \frac{1}{4c-q} \left( \begin{array}{c} 2c - 2q \\ c - q \end{array} \right) \mu_{2c,q}(\Omega) + (c + 1) \mu_{2c,c}(\Omega) \right)$$

where $O_i$ denotes the volume of the euclidean unit sphere.

Formula (5) is a generalization of (1) since the Hermitian intrinsic volumes $\{\mu_{k,q}\}$ are given by integrals of certain invariant polynomials of the second fundamental form. This answers a question posed by Naveira in [Nav05]. In case $r = 1$, formula (5) was already proved by different methods in [Aba09]. For $2r \geq n$, and $\epsilon = 0$, formula (5) was proved in [BF08].

Formula (6) generalizes to complex space forms the Gauss-Bonnet theorem (3). In complex dimensions $n = 2, 3$, formula (6) was obtained in [Par02].

Combining expressions (5) and (6) we obtain

$$O_{2n-1} \chi(\Omega) = M_{2n-1}(\partial \Omega) + 2n \epsilon \int_{\mathcal{L}_{n-1}^C} \chi(\Omega \cap L_{n-1}) dL_{n-1} +$$

$$+ \sum_{k=1}^{n-1} \frac{e^k O_{2n-2k-1}}{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right)^{-1} \mu_{2k,k}(\Omega) + 2n \epsilon^n \text{vol}(\Omega).$$
This expression is similar to the following one for real space forms (cf. [Sol06]). Given a regular domain \( \Omega \subset M^n(k) \), it follows

\[
O_{n-1}\chi(\Omega) = M_{n-1}(\partial \Omega) + k \frac{2(n-1)}{O_{n-2}} \int_{L_{n-2}} \chi(\Omega \cap L_{n-2}) dL_{n-2}.
\]

The main idea for the proof of Theorem 1.2 is to take variation along a vector field in \( CK^n(\epsilon) \) in both sides of equalities (5) and (6), and to compare them.

In order to obtain a first expression of the variation of \( \int_{L_r} C_r \chi(\partial \Omega \cap L_r) dL_r \) along a vector field in \( CK^n(\epsilon) \), we proceed as in [Sol06] (see Section 3.1). In \( \mathbb{C}^n \), the variation of the Hermitian intrinsic volumes was obtained by Bernig and Fu in [BF08]. Here we use the same method to find the generalization for \( \epsilon \neq 0 \) (see Section 3.2).

Using formula (5), we prove in Section 6 that the total Gauss curvature does not satisfy the reproductive property and we get in \( \mathbb{C}^n \) the following expression:

\[
\int_{L_r} M_{2r-1}(\partial \Omega \cap L_r) dL_r = 2r \omega_{2r}^2 \text{vol}(G_{n-1,r}^C) (n-1) \left(\begin{array}{c} n-1 \\ r \end{array}\right) \left(\begin{array}{c} n \\ r \end{array}\right)^{-1} \mu_{n-2r,2r}^2(\Omega).
\]

(7)

In [Aba09], it is proved that the reproductive property (4) is not satisfied by the mean curvature integral either.

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**2. Hermitian intrinsic volumes**

Let \( \mathbb{C}K^n(\epsilon) \) be a (simply connected) complex space form with constant holomorphic curvature \( 4\epsilon \). We denote by \( S(\mathbb{C}K^n(\epsilon)) \) the unit tangent bundle of \( \mathbb{C}K^n(\epsilon) \).

**Definition 2.1.** Let \( \Omega \) be a regular domain in \( \mathbb{C}K^n(\epsilon) \). The unit (inner) normal bundle of \( \partial \Omega \) is defined as

\[
N(\Omega) = \{ (p, v) \in S(\mathbb{C}K^n(\epsilon)) : p \in \partial \Omega, v \text{ inner normal to } S_p \partial \Omega \}.
\]

Given a \( 2n-1 \) form \( \omega \) in \( S(\mathbb{C}K^n(\epsilon)) \), and a smooth measure \( \eta \) we may consider, for every regular domain \( \Omega \)

\[
\int_{\Omega} \eta + \int_{N(\Omega)} \omega.
\]

The resulting functional is called a *smooth valuation*. 
Let \((z, e_1) \in S(\mathbb{C}K^n(\epsilon))\) and let \(\{z; e_1, \ldots, e_n\}\) be a moving frame defined on an open subset \(U \subset S(\mathbb{C}K^n(\epsilon))\). We denote by \(\{\omega_1, \omega_2, \ldots, \omega_n\}\) the 1-forms in \(S(\mathbb{C}K^n(\epsilon))\) defined as the dual basis of \(\{e_1, \ldots, e_n\}\), and by \(\{\omega_{ij}\}\) the connection forms of \(\mathbb{C}K^n(\epsilon)\). That is, if \((\cdot, \cdot)\) denotes the Hermitian product on \(\mathbb{C}K^n(\epsilon)\) and \(\nabla\) the Levi-Civita connection, then

\[
\omega_j = (dz, e_j) \quad \text{and} \quad \omega_{jk} = (\nabla e_j, e_k) \quad \text{where} \quad j, k \in \{1, \ldots, n\}.
\]

Thus, these forms are \(\mathbb{C}\)-valued. We denote
\[
\omega_j = \alpha_j + i\beta_j,
\]
\[
\omega_{jk} = \alpha_{jk} + i\beta_{jk}.
\]

**Remark 2.2.** Forms \(\alpha_1, \beta_1\) and \(\beta_{11}\) are global forms in \(S(\mathbb{C}K^n(\epsilon))\). We denote them by \(\alpha, \beta, \gamma\) respectively. Note that \(\alpha\) coincides with the contact form of the unit tangent bundle.

**Lemma 2.3.** Let \(\Omega \subset \mathbb{C}K^n(\epsilon)\) be a regular domain. Then \(\alpha\) and \(d\alpha\) vanish at \(N(\Omega) \subset S(\mathbb{C}K^n(\epsilon))\).

**Proof.** Let \(V \in T(\pi, v)N(\Omega)\). Then, \(\alpha(V)(\pi, v) = \langle d\pi(V), v \rangle = 0\) where \(\pi : S(\mathbb{C}K^n(\epsilon)) \to \mathbb{C}K^n(\epsilon)\) is the canonical projection. Clearly \((d\alpha)|_{N(\Omega)} = d(\alpha|_{N(\Omega)}) = 0\).

Consider the following invariant 2-forms in \(S(\mathbb{C}K^n(\epsilon))\)

\[
\theta_0 = -\text{Im}((\nabla e_1, \nabla e_1)) = -\text{Im}(\sum_{i=1}^{n} \omega_{1i} \otimes \overline{\omega}_{1i})
\]
\[
\theta_1 = -\text{Im}((dz, \nabla e_1) - (\nabla e_1, dz))
\]
\[
= -\text{Im}(\sum_{i=1}^{n} \omega_i \otimes \overline{\omega}_{1i} - \sum_{i=1}^{n} \omega_{1i} \otimes \overline{\omega}_i)
\]
\[
\theta_2 = -\text{Im}((dz, dz)) = -\text{Im}(\sum_{i=1}^{n} \omega_i \otimes \overline{\omega}_i)
\]
\[
\theta_s = \text{Re}((dz, \nabla e_1) - (\nabla e_1, dz))
\]
\[
= \text{Re}(\sum_{i=1}^{n} \omega_i \otimes \omega_{1i} - \sum_{i=1}^{n} \overline{\omega}_{1i} \otimes \overline{\omega}_i).
\]

**Remark 2.4.** These forms coincide with the invariant 2-forms, \(\theta_0, \theta_1, \theta_2\) and \(\theta_s\) defined in \(S(\mathbb{C}^n)\) by Bernig and Fu [BF08, p.14]. Note that \(\theta_s\) is the symplectic form of \(T\mathbb{C}K^n(\epsilon)\). Park considered in [Par02] similar 2-forms in \(S(\mathbb{C}K^n(\epsilon))\).

Next we recall the exterior derivative of \(\theta_0, \theta_1\) and \(\theta_2\), which can be found in [BF08] when \(\epsilon = 0\), or in [Par02] for general \(\epsilon\).
**Lemma 2.5** ([Par02]). In $S(\mathbb{C}K^n(\varepsilon))$ it is satisfied
\[
\begin{align*}
\delta\alpha &= -\theta_s, & \delta\theta_0 &= -\varepsilon(\alpha \wedge \theta_1 + \beta \wedge \theta_s), \\
\delta\beta &= \theta_1, & \delta\theta_1 &= 0, \\
\delta\gamma &= 2\theta_0 - 2\varepsilon\theta_2 - 2\varepsilon\alpha \wedge \beta, & \delta\theta_2 &= 0.
\end{align*}
\]

The forms $\beta_{k,q}$ and $\gamma_{k,q}$ defined in $S(\mathbb{C}^n)$ in [BF08], can be extended to $S(\mathbb{C}K^n(\varepsilon))$ from $\mathbb{C}^n$.

**Definition 2.6.** For positive integers $k, q \in \mathbb{N}$ with $\max\{0, k - n\} \leq q \leq \frac{k}{2} < n$, we define the following $(2n - 1)$-forms in $S(\mathbb{C}K^n(\varepsilon))$
\[
\beta_{k,q} := c_{n,k,q} \beta \wedge \theta_0^{n-k+q} \wedge \theta_1^{k-2q-1} \wedge \theta_2^{q} \in \Omega^{2n-1}(S(\mathbb{C}K^n(\varepsilon))), \quad \text{if } k \neq 2q
\]
\[
\gamma_{k,q} := \frac{c_{n,k,q}}{2} \gamma \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^{q} \in \Omega^{2n-1}(S(\mathbb{C}K^n(\varepsilon))), \quad \text{if } n \neq k-q
\]
where
\[
c_{n,k,q} := \frac{1}{q!(n-k+q)!(k-2q)!}\omega_{2n-k}
\]
and $\omega_{2n-k}$ denotes the volume of the $(2n - k)$-dimensional euclidean ball.

Given a regular domain $\Omega \subset \mathbb{C}K^n(\varepsilon)$, we define (for $\max\{0, k - n\} \leq q \leq \frac{k}{2} < n$)
\[
B_{k,q}(\Omega) := \int_{\partial(\Omega)}\beta_{k,q} \quad (k \neq 2q), \quad \Gamma_{k,q}(\Omega) := \int_{\partial(\Omega)}\gamma_{k,q} \quad (n \neq k-q).
\]

In $\mathbb{C}^n$, it is satisfied $B_{k,q}(\Omega) = \Gamma_{k,q}(\Omega)$ (cf. [BF08]). Next, we give the relation among \{B_{k,q}(\Omega)\} and \{\Gamma_{k,q}(\Omega)\} in $\mathbb{C}K^n(\varepsilon)$ which generalizes the relation in $\mathbb{C}^n$.

**Proposition 2.7.** In $\mathbb{C}K^n(\varepsilon)$, for any positive integers $k, q$ such that $\max\{0, k - n\} < q < k/2 < n$ it is satisfied
\[
\Gamma_{k,q}(\Omega) = B_{k,q}(\Omega) - \varepsilon\frac{c_{n,k,q}}{c_{n,k+2,q+1}}B_{k+2,q+1}(\Omega).
\]

**Proof.** We denote by $I$ the ideal generated by $\alpha$, $d\alpha$ and the exact forms in $\mathcal{N}(\Omega)$. If $\lambda$, $\rho$ are $(2n - 1)$-forms in $\mathcal{N}(\Omega)$ equal modulo $I$, then by Lemma 2.3
\[
\int_{\mathcal{N}(\Omega)}\lambda = \int_{\mathcal{N}(\Omega)}\rho.
\]

Thus, it is enough to prove
\[
(10) \quad \gamma_{k,q} \equiv \beta_{k,q} - \varepsilon\frac{c_{n,k,q}}{c_{n,k+2,q+1}}\beta_{k+2,q+1} \mod I.
\]

Consider the form $\eta = (\theta_s - \beta \wedge \gamma) \wedge \theta_0^{n-k+q-1} \theta_1^{k-2q-1} \theta_2^{q}$. By Lemma 2.5 it follows that modulo $I$
\[
d\eta \equiv -\theta_0^{n-k+q-1} \theta_1^{k-2q} \theta_2^{q} + 2\beta \theta_0^{n-k+q-1} \theta_1^{k-2q-1} \theta_2^{q} - 2\varepsilon\beta \theta_0^{n-k+q-1} \theta_1^{k-2q-1} \theta_2^{q+1}.
\]
Using the definition of $\gamma_{k,q}$ and $\beta_{k,q}$ we obtain the relation in (10). □
Remark 2.8. In complex dimensions $n = 2, 3$, the previous relations were found in [Par02].

In view of the previous equalities, we define (for $\max\{0, k - n\} \leq q \leq \frac{k}{2} < n$)

$$
\mu_{k,q}(\Omega) := \begin{cases} 
B_{k,q}(\Omega) & \text{if } k \neq 2q \\
\Gamma_{2q,q}(\Omega) & \text{if } k = 2q.
\end{cases}
$$

Remark 2.9. In [BF08], it is proved that valuations $\{\mu_{k,q}, \text{vol}\}$ where $k \in \{0, \ldots, n - 1\}$ and $q \in \{\max\{0, k - n\}, \ldots, \lfloor k/2 \rfloor\}$ form a basis of invariant continuous valuations on $\mathbb{C}^n$.

3. Variation Formulas

3.1. Variation of Hermitian intrinsic volumes. In order to study the variation on $\mathbb{C}K^n(\epsilon)$ of the Hermitian intrinsic volumes, we follow the method used by Bernig and Fu [BF08] in Corollary 2.6. First, we recall the definition of the Rumin operator, introduced in [Rum94], and the definition of the Reeb vector field in a contact manifold.

**Definition 3.1.** Given $\mu \in \Omega_{2n-1}^{2n-1}(S(\mathbb{C}K^n(\epsilon)))$, let $\alpha \wedge \xi \in \Omega_{2n-1}^{2n-1}(S(\mathbb{C}K^n(\epsilon)))$ be the unique form such that $d(\mu + \alpha \wedge \xi)$ is a multiple of $\alpha$ (cf. [Rum94]). Then the Rumin operator $D$ is defined as

$$
D\mu := d(\mu + \alpha \wedge \xi).
$$

**Definition 3.2.** Let $M$ be a contact manifold and let $\alpha$ be the contact form. The Reeb vector field $T$ is the unique vector field over $M$ such that

$$
\begin{cases} 
i_T \alpha = 1, \\
L_T \alpha = 0.
\end{cases}
$$

If the contact manifold is the unit tangent bundle of a Riemannian manifold, then the Reeb vector field is the geodesic flow (cf. [Bla76, p. 17]).

**Lemma 3.3.** In $S(\mathbb{C}K^n(\epsilon))$, it is satisfied

$$
\begin{align*}
i_T \alpha &= 1, \\
i_T \theta_1 &= \gamma, \\
i_T \theta_2 &= \beta, \\
i_T \beta &= i_T \gamma = i_T \theta_0 = i_T \theta_s = 0.
\end{align*}
$$

**Proof.** The first equality comes directly from the definition (cf. (12)). As $T$ is the geodesic flow, we have $\alpha_i(T) = \beta_i(T) = 0$ and $\alpha_{1i} = \beta_{1i} = 0$, $i \in \{2, \ldots, n\}$. By definition in (9), we obtain the result. Moreover, $i_T \theta_s = -i_{i\epsilon} d\alpha = di_T \alpha - L_T \alpha = 0$.□

Given a smooth valuation $\mu$, and a vector field $X$ with flow $\phi_t$, we are interested in computing

$$
\delta_X \mu(\Omega) := \frac{d}{dt} \bigg|_{t=0} \mu(\phi_t(\Omega)).
$$

This can be done by means of the following result stated in [BF08].
Lemma 3.4 (Lemma 2.5 [BF08]). Suppose $\Omega \subset \mathbb{R}^n$ is a regular domain, $N$ is the outward unit normal field to $\partial \Omega$, $X$ is a smooth vector field on $\mathbb{R}^n$ and $\mu$ is a smooth valuation given by an $(n-1)$-form $\rho$ in $S(\mathbb{R}^n)$. Then

$$\delta_X \mu(\Omega) = \int_{\partial(\Omega)} \langle X, N \rangle \text{i}_T(D\rho)$$

where $T$ is the Reeb vector field on $S(\mathbb{R}^n)$ and $D\rho$ is the Rumin operator of $\rho$.

Although this result is stated and proved in $\mathbb{R}^n$, the given proof is also valid in an arbitrary Riemannian manifold.

From Lemma 2.5, we obtain the exterior differential of the forms $\beta_{k,q}$ and $\gamma_{k,q}$.

Lemma 3.5. In $\mathbb{C}K^n(\epsilon)$

$$d\beta_{k,q} = c_{n,k,q}(\theta_0^{n-k+q} \wedge \theta^k_1 \wedge \theta_2^q - \epsilon(n-k+q)\alpha \wedge \beta \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q)$$

and

$$d\gamma_{k,q} = c_{n,k,q}(\theta_0^{n-k+q} \wedge \theta^k_1 \wedge \theta_2^q - \epsilon \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q + 1$$

$$- \epsilon \alpha \wedge \beta \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q$$

$$- \epsilon \frac{(n-k+q-1)}{2} \alpha \wedge \gamma \wedge \theta_0^{n-k+q-2} \wedge \theta_1^{k-2q+1} \wedge \theta_2^q$$

$$- \epsilon \frac{(n-k+q-1)}{2} \beta \wedge \gamma \wedge \theta_0^{n-k+q-2} \wedge \theta_1^{k-2q} \wedge \theta_2^q).$$

Notation 3.6. Let $\Omega$ be a regular domain in $\mathbb{C}K^n(\epsilon)$ and let $N$ be the outward unit normal field to $\partial \Omega$. Let $X$ be a smooth vector field on $\mathbb{C}K^n(\epsilon)$. We denote

$$\tilde{B}_{k,q} = \tilde{B}_{k,q}(\Omega) := \int_{\partial \Omega} \langle X, N \rangle \beta_{k,q}, \quad \tilde{\Gamma}_{k,q} = \tilde{\Gamma}_{k,q}(\Omega) := \int_{\partial \Omega} \langle X, N \rangle \gamma_{k,q}.$$

The variation of the valuations $\{\mu_{k,q}\}$ on $\mathbb{C}^n$ was found in [BF08, Proposition 4.6]. We extend this result to $\mathbb{C}K^n(\epsilon)$ as follows.

Proposition 3.7. Let $X$ be a smooth vector field in $\mathbb{C}K^n(\epsilon)$ and let $\Omega \subset \mathbb{C}K^n(\epsilon)$ be a regular domain. Then

$$\delta_X B_{k,q}(\Omega) = 2c_{n,k,q}(c_{n,k-1,q}^{-1}(k-2q)^2 \Gamma_{k-1,q} - c_{n,k-1,q-1}^{-1}(n+q-k)q \Gamma_{k-1,q-1}$$

$$+ c_{n,k-1,q-1}^{-1}(n+q-k+1)q \tilde{B}_{k-1,q-1} - c_{n,k-1,q-1}^{-1}(k-2q)(k-2q-1) \tilde{B}_{k-1,q}$$

$$+ \epsilon c_{n,k+1,q+1}^{-1}(k-2q)(k-2q-1) \tilde{B}_{k+1,q+1} - c_{n,k+1,q}^{-1}(n-k+q)(q+1/2) \tilde{B}_{k+1,q})$$
Proof. Lemma 3.4 provides an expression for the variation of a smooth valuation. In order to use this lemma, it is enough to find \( i_T D \beta_{k,q}, i_T D \gamma_{2q,q} \) modulo \( \alpha \), \( do \) since the latter forms vanish over \( N(\Omega) \) (by Lemma 2.3), and \( i_T d\alpha = 0 \) (by Lemma 3.3). We will use the following fact from the proof of Proposition 4.6 in [BF08]: for \( \max\{0, k-n\} \leq q < k/2 < n \), there exists an invariant form \( \xi_{k,q} \in \Omega^{2n-1}(S(\mathbb{C}^n)) \) such that

\[
\text{(13)} \quad do \wedge \xi_{k,q} \equiv -\theta_0^{n-k+q} \theta_1^{k-2q} \theta_2^q \mod(\alpha),
\]

and

\[
\text{(14)} \quad \xi_{k,q} \equiv \beta \gamma \theta_0^{n+q-k-1} \theta_1^{k-2q-2} \theta_2^q \mod(\alpha, do).
\]

In order to find \( \delta_X B_{k,q} \) for general \( \epsilon \), we take a form \( \xi^\epsilon \in \Omega^{2n-1}(S(\mathbb{C}^n(\epsilon))) \) such that \( \xi_{(p,v)}^\epsilon = \xi_{(p',v')}(\epsilon) \), where we identify \( T_{(p,v)} S(\mathbb{C}^n(\epsilon)) \) and \( T_{(p',v')} \mathbb{C}^n \), for every \( (p, v) \in S(\mathbb{C}^n(\epsilon)) \) and \( (p', v') \in S(\mathbb{C}^n) \). Then, it is clear from Lemma 3.5 and (13) that \( d(\beta_{k,q} + c_{n,k,q} \alpha \wedge \xi^\epsilon) \equiv 0 \) modulo \( \alpha \).

By Lemma 2.5, the exterior differential of \( \xi^\epsilon \) is

\[
d\xi^\epsilon \equiv \theta_0^{n+q-k-1} \theta_1^{k-2q-2} \theta_2^q ((n-k+q) \theta_1^2 - (k-2q) (k-2q-1) \theta_0 \theta_2)
\]

\[
\wedge (\theta_1^2 - 2\beta \theta_0 + 2\epsilon \beta^2 \theta_2) \mod(\alpha, do)
\]

and the contraction of \( d\beta_{k,q} \) with respect to the field \( T \), by Lemma 3.3, is

\[
i_T d\beta_{k,q} \equiv c_{n,k,q} \theta_0^{n+q-k-1} \theta_1^{k-2q-1} \theta_2^q \equiv -\epsilon(n-k+q) \beta \theta_0 \theta_2 + q \beta \theta_0 \theta_1 - \epsilon(n-k+q) \beta \theta_1 \theta_2 \mod(\alpha).
\]

By substituting the last expressions in \( i_T D \beta_{k,q} \equiv i_T d\beta_{k,q} - c_{n,k,q} d\xi^\epsilon \), we get the result.

To compute \( \delta_X \Gamma_{2q,q} \), note that \( d\gamma_{2q,q} \) has 3 terms which are not multiple of \( \alpha \) (cf. Lemma 3.5). As before, we consider \( \xi_1^\epsilon, \xi_2^\epsilon \in \Omega^{2n-1}(S(\mathbb{C}^n(\epsilon))) \) corresponding to \( \xi_{2q,q} \), and \( \xi_{2q+2,q+1} \) respectively. Let us consider also

\[
\text{(15)} \quad \xi_3^\epsilon = \frac{n-q-1}{2} \beta \gamma \theta_0^{n-q-2} \theta_2^q.
\]
Then the Rumin differential of $\gamma_{2q,q}$ is given by $D\gamma_{2q,q} = d(\gamma_{2q,q} + c_{n,2q,q}\alpha \wedge (\xi_1^q - \xi_2^q - \xi_3^q))$. Indeed, $d\alpha \wedge \xi_1^q$ cancels the first term of $D\gamma_{2q,q}$ modulo $\alpha$, and $d\alpha \wedge \xi_2^q$ cancels the second one. The third term is canceled exactly by $d\alpha \wedge \xi_3^q$.

Now, using Lemmas 3.5 and 3.3

$$i_T d\gamma_{2q,q} = q \beta \theta_0^{n-q} - \theta_1^q - q + 2) \beta \theta_0^{n-q-1} \theta_2^q \frac{n - q - 1}{2} \gamma_0^{n-q-2} \theta_1 \theta_2^q \mod(\alpha, d\alpha).$$

From (14) and (15)

$$d\xi_1^q \equiv (n - q) q \beta \theta_0^{n-q-1} \theta_2^q \beta \theta_0 - 2 \beta \theta_2 \mod(\alpha, d\alpha).$$

$$d\xi_2^q \equiv (n - q - 1) (q + 1) \theta_0^{n-q-2} \theta_2^q \beta \theta_0 - 2 \beta \theta_2 \mod(\alpha, d\alpha).$$

$$d\xi_3^q \equiv (n - q - 1) \frac{2}{2} \theta_0^{n-q-2} \theta_2^q \beta \theta_0 - 2 \beta \theta_2 \mod(\alpha, d\alpha).$$

Plugging this into $i_T D\gamma_{2q,q} = i_T d\gamma_{2q,q} - c_{n,2q,q}(d\xi_1^q - ed\xi_2^q - ed\xi_3^q) \mod(\alpha, d\alpha)$ gives the result. \qed

### 3.2. Variation of the measure of the set of complex $r$-planes intersecting a regular domain.

We denote by $\mathcal{L}^C_r$, $r \in \{1, \ldots, n-1\}$ the space of complex $r$-planes in $\mathbb{C}K^n(\epsilon)$. Complex $r$-planes are totally geodesic submanifolds of complex dimension $r$ isometric to $\mathbb{C}K^r(\epsilon)$ (cf. [Gol99, Lemma 2.2.4]). Moreover, Santaló proved the following properties of this space (as usual, $J$ denotes the complex structure).

**Lemma 3.8 (San52).** $\mathcal{L}^C_r$ is a homogeneous space and $\mathcal{L}^C_r \cong U_\epsilon(n)/U_\epsilon(r) \times U(n-r)$

where

$$U_\epsilon(n) = \left\{ \begin{array}{ll} \mathbb{C}^n \times U(n), & \text{if } \epsilon = 0, \\ U(1+n), & \text{if } \epsilon > 0, \\ U(1,n), & \text{if } \epsilon < 0. \end{array} \right.$$ 

Let $\{g; g_1, Jg_1, \ldots, g_n, Jg_n\}$ be a local orthonormal frame associated to the elements of an open set $V \subset \mathcal{L}^C_r$ such that $\{g_{n-r+1}, Jg_{n-r+1}, \ldots, g_n, Jg_n\}$ generate $T_xL$ for each $L \in V$. The invariant density of $\mathcal{L}^C_r$ is given by

$$dL_r = \left| \bigwedge_{i=1}^{n-r} \omega_i \wedge \overline{w_i} \right| \left| \bigwedge_{i=n-r+1}^{n} \omega_{ij} \wedge \overline{w_{ij}} \right|$$

where $\{\omega_i, \omega_{ij}\}_{i,j}$ are defined as in (8).

On $\partial \Omega$ there is a canonical vector field given by $JN$, and a distribution $\mathcal{D} = \langle N, JN \rangle$. At every point $x \in \partial \Omega$, $D_x$ is the maximal complex linear subspace of $T_x\mathbb{C}K(\epsilon)$ contained in $T_x\partial \Omega$. We shall consider the bundle $G^C_{n,r}(T\partial \Omega)$ whose fiber at every point $x \in \partial \Omega$ is the Grassmanian $G^C_{n,r}(T_x\partial \Omega)$ of $r$-dimensional complex subspaces of $D_x$; i.e., $G^C_{n,r}(T\partial \Omega) = \{(x, l) | x \in \partial \Omega, l \text{ is a } J\text{-invariant } r\text{-dimensional linear subspace of } T_x\partial \Omega\}$.
Proposition 3.9. Suppose $\Omega \subset \mathbb{C}^n(\epsilon)$ is a regular domain, $X$ is a smooth vector field on $\mathbb{C}^n(\epsilon)$, $\phi_t$ is the flow associated to $X$ and $\Omega_t = \phi_t(\Omega)$, then

\[
\frac{d}{dt} \left|_{t=0} \int_{L^C_r} \chi(\Omega_t \cap L_r) dL_r \right| = \int_{\partial \Omega} \langle \partial \phi_t / \partial t, N \rangle \left( \int_{G^C_{n,r}(T_x \partial \Omega)} \sigma_{2r}(II|_V) dV \right) dx
\]

where $N$ is the outward normal field and $\sigma_{2r}(II|_V)$ denotes the 2r-th symmetric elementary function of II restricted to $V \in G^C_{n,r}(T_x \partial \Omega)$.

Proof. We follow the same procedure as in [Sol06, Theorem 4].

For every $V \in G^C_{n,r}(T_x \partial \Omega)$, we make the parallel translation $V_t$ of $V$ along $\phi_t(x)$. Recall that parallel translation preserves the complex structure (cf. [O'N83, p. 326]). Then we project orthogonally $V_t$ onto $D_{\phi_t(x)}$, obtaining a complex $r$-plane $V'_t$ (at least for small values of $t$). We define

\[
\psi : G^C_{n,r}(T \partial \Omega) \times (-\epsilon, \epsilon) \rightarrow L^C_r \quad ((x, V), t) \mapsto \exp_{\phi_t(x)}(V'_t).
\]

From Proposition 3 in [Sol06] (whose proof works without change in our setting), we have

\[
\frac{d}{dt} \left|_{t=0} \int_{L^C_r} \chi(\Omega_t \cap L_r) dL_r \right| = \lim_{h \to 0} \frac{1}{h} \int_{L^C_r} \sum \text{sign} \left( \frac{\partial \phi_t}{\partial t}, N \right) \text{sign} \left( \sigma_{2r}(II|_V) \right) \psi^*_t dL_r
\]

where the sum runs over the tangencies of $L_r$ with the hypersurfaces $\partial \Omega_t$ with $0 < t < h$. As

\[
\psi^*_t(dL_r) = \iota_{\partial t}(\psi^*_t(dL_r)) dt = \psi^*_t(\iota_{\partial \phi_t} dL_r) dt
\]

where $\psi_t = \psi(\cdot, t)$, using the co-area formula we get

\[
\frac{d}{dt} \left|_{t=0} \int_{L^C_r} \chi(\Omega_t \cap L_r) dL_r \right| = \int_{G^C_{n,r}(T \partial \Omega)} \left( \frac{\partial \phi_t}{\partial t}, N \right) \text{sign} \left( \sigma_{2r}(II|_V) \right) \psi^*_t(\iota_{\partial \phi_t} dL_r).
\]

Let $\{g, g_1, Jg_1, \ldots, g_n, Jg_n\}$ be a local orthonormal frame defined on $G^C_{n,r}(T \partial \Omega_0) \times (-\epsilon, \epsilon)$ such that $g((x, l), t) = \phi_t(x, t), g_1((x, l), t)$ is orthogonal to $\partial \Omega_t$ (at $\phi_t(x)$) and $\psi = \langle g, g_{n-r+1}, Jg_{n-r+1}, \ldots, g_n, Jg_n \rangle \cap \mathbb{C}^n(\epsilon)$. We may assume the frame is defined in a neighborhood of $L_r$, since we are only interested in regular points of $\psi$.

Consider the curve $L_r(t)$ given by the parallel translation of $L_r$ along the geodesic given by $N$, the outward normal vector to $\partial \Omega_0$. If $P \in T_{L_r} L^C_r$ denotes the tangent vector to $L_r(t)$ at $t = 0$, then

\[
\omega_i(P) = \langle dg(P), g_i \rangle = \left( \frac{d}{dt} \right|_{t=0} g(L_r(t)), g_i \rangle = 0,
\]

\[
\omega_1(P) = \langle dg(P), N \rangle = 1,
\]

\[
\omega_{kj}(P) = \langle dg_k(P), g_j \rangle = \left( \frac{d}{dt} \right|_{t=0} g_k(L_r(t)), g_j \rangle = 0
\]
where \( i \in \{2, \ldots, n-r, \frac{n-r}{2}\}, j \in \{1, \ldots, n-r, \frac{n-r}{2}\}, \) and \( k \in \{n-r+1, \ldots, n, \bar{n}\}. \) By (16) and last equations we get the following equality between densities
\[
dL_r = |\omega_1| t_p dL_r
\]
since
\[
t_{d\psi \partial t} dL_r = |\omega_1(d\psi \partial t)| t_p dL_r + |\omega_1| t_{d\psi \partial t} t_p dL_r
\]
and
\[
\omega_1(d\psi \partial t) = \langle dg(d\psi \partial t), N \rangle = \langle \frac{\partial \phi}{\partial t}, N \rangle,
\]
\[
\psi^*_0(\omega_1)(v) = \langle dg(d\psi_0(v)), N \rangle = 0 \quad \forall v \in T_{(p, v)} G_n^c(T \partial \Omega_0).
\]
So,
\[
\psi^*_0(t_{d\psi \partial t} dL_r) = |\langle \frac{\partial \phi}{\partial t}, N \rangle| \psi^*_0(t_p dL_r).
\]
Finally, using that \( \psi^*_0(t_p dL_r) = |\sigma_{2^r}(II|_V)| dV dx \), we get the result. \( \square \)

**Remark 3.10.** The integral
\[
\int_{G_n^c} \sigma_{2^r}(II|_V) dV
\]
seems difficult to be computed directly. However, we will find it by an indirect method. Recall that the analogous integral in real space forms is a multiple of an elementary symmetric function of the principal curvatures.

### 4. Crofton type formulas

#### 4.1. In the standard Hermitian space. Now we are ready to prove formula (5) for \( \epsilon = 0. \) The following lemma will be used.

**Lemma 4.1.** Let \( \Omega \) be a regular domain in \( \mathbb{C}^n \) and \( \varphi: \partial \Omega \to N(\Omega) \) the canonical map. Fix a point \( x \) in \( \partial \Omega \) and a reference \( \{e_1 = \varphi(x), e_\mathbb{T} = Je_1, \ldots, e_n, e_\mathbb{T} = Je_n\} \) at \( x. \) Then \( \varphi^*(\gamma_{k,q}) = P_{k,q} dx \) where \( dx \) is the volume element of \( \partial \Omega \) and \( P_{k,q} \) is a polynomial of degree \( 2n-k-1 \) in the entries of the second fundamental form \( h_{ij} = II(e_i, e_j), i, j \in \{1, 2, \ldots, n\}. \) Each of the monomials of \( P_{k,q} \) containing only entries of the form \( h_{ii} \) contains the factor \( h_{\mathbb{T} \mathbb{T}} \) and exactly \( n+q-k-1 \) factors of the form \( h_{ij} h_{\mathbb{T} \mathbb{T}} \), \( i \in \{1, 2, \ldots, n\}, j \in \{2, \ldots, n\}. \)

**Proof.** From (9) we have
\[
\theta_0 = \sum_{i=2}^n \alpha_{1i} \wedge \beta_{1i}, \quad \theta_1 = \sum_{i=2}^n (\alpha_i \wedge \beta_{1i} - \beta_i \wedge \alpha_{1i}), \quad \theta_2 = \sum_{i=2}^n \alpha_i \wedge \beta_i.
\]
For convenience we write $\alpha_i^* = \beta_i$ and $\alpha_i^* = \beta_i$ for $i \in \{2, \ldots, n\}$. Using $\alpha_{ij} = \sum_{i \in I} h_{ij} \alpha_i$ for $j \in I := \{1, 2, \ldots, \pi\}$ yields

$$\varphi^*(\gamma_{k,q}) = \frac{c_{n,k,q}}{2} \left( \sum_{j \in I} h_{ij} \alpha_j \right) \wedge \left( \sum_{i=2}^{n} \sum_{j,l \in I} h_{ij} h_{il} \alpha_j \alpha_l \right)^{n+q-k-1} \wedge$$

$$\wedge \left( \sum_{i=2}^{n} \left( \sum_{j \in I} h_{ij} \alpha_j - \sum_{l \in I} h_{il} \alpha_l \right) \right) \bigg) \quad \wedge \left( \sum_{i=2}^{n} \alpha_i \alpha_i^* \right) \wedge \left( \sum_{i=2}^{n} \alpha_i \alpha_i^* \right).$$

Thus, $\varphi^*(\gamma_{k,q}) = P_{k,q} dx$ with $P_{k,q}$ a polynomial of degree $2n - k - 1$ in $h_{ij}$. The terms in the previous expression containing only entries of type $h_{ii}$ are

$$\frac{c_{n,k,q}}{2} h_{ii} \alpha_i^* \wedge \left( \sum_{i=2}^{n} h_{ii} h_{ii} \alpha_i \alpha_i^* \right)^{n+q-k-1} \wedge$$

$$\wedge \left( \sum_{i=2}^{n} h_{ii} \alpha_i \alpha_i^* - h_{ii} \alpha_i \alpha_i^* \right) \wedge \left( \sum_{i=2}^{n} \alpha_i \alpha_i^* \right),$$

and the result follows.

\[ \square \]

**Theorem 4.2.** Let $\Omega \subset \mathbb{C}^n$, let $X$ be a smooth vector field over $\mathbb{C}^n$, let $\phi_t$ be the flow associated to $X$ and let $\Omega_t = \phi_t(\Omega)$. Then

$$\frac{d}{dt} \bigg|_{t=0} \int_{L_r} \chi(\Omega_t \cap L_r) dL_r = \text{vol}(G_{n-1,r}^C) \omega_{2r+1} (r+1) \binom{n-1}{r}^{-1} \binom{n}{r}^{-1} \cdot$$

$$\cdot \left( \sum_{q=\max\{0, n-2r-1\}}^{n-r-1} \binom{2n-2r-2q-1}{n-r-q} \frac{1}{4^{n-r-q-1}} \tilde{B}_{2n-2r-1,q}(\Omega) \right)$$

and

$$\int_{\mathbb{C}^n} \chi(\Omega \cap L_r) dL_r = \text{vol}(G_{n-1,r}^C) \omega_{2r} \binom{n-1}{r}^{-1} \binom{n}{r}^{-1} \cdot$$

$$\cdot \left( \sum_{q=\max\{0, n-2r\}}^{n-r} \frac{1}{4^{n-r-q}} \binom{2n-2r-2q}{n-r-q} \mu_{2n-2r,q}(\Omega) \right).$$

**Proof.** In order to simplify the following computations, we consider

$$B_{k,q}' = B_{k,q}'(\Omega) := c_{n,k,q}^{-1} B_{k,q}(\Omega), \quad \Gamma_{k,q}' = \Gamma_{k,q}'(\Omega) := 2c_{n,k,q}^{-1} \Gamma_{k,q}(\Omega)$$

and

$$\tilde{B}_{k,q}' = c_{n,k,q}^{-1} \tilde{B}_{k,q}, \quad \tilde{\Gamma}_{k,q}' = 2c_{n,k,q}^{-1} \tilde{\Gamma}_{k,q}'.$$
The functional $\int_{L^r} \chi(\Omega \cap L_r) dL_r$ is a valuation on $\mathbb{C}^n$ with degree of homogeneity $2n - 2r$. Thus, it can be expressed as a linear combination of the elements of a basis of valuations with the same degree of homogeneity. Then, by Remark 2.9 and (11), we have

\begin{equation}
\int_{L^r} \chi(\Omega \cap L_r) dL_r = \sum_{q=\max\{0, n-2r\}}^{n-2r-1} C_q B_{2n-2r,q} + D \Gamma_{2n-2r,n-r}
\end{equation}

for certain constants $C_q, D$ which we wish to determine. This will be done by comparing the variation of both sides of this equality. From here on we assume $2r < n$. The case $2r \geq n$ can be treated in the same way (cf. Remark 4.3).

By Proposition 3.7, the variation of the right hand side of (21) is a linear combination of the following type

\begin{equation}
\sum_{q=\max\{0, n-2r\}}^{n-2r-1} c_q B_{2n-2r-1,q} + \sum_{q=n-2r}^{n-2r-1} d_q \Gamma_{2n-2r-1,q}
\end{equation}

where the coefficients $c_q$ and $d_q$ can be expressed in terms of a linear combination with known coefficients of the variables $C_q$ and $D$, that still remain unknown.

The variation of the left hand side of (21), by Proposition 3.9 is

\begin{equation}
\frac{d}{dt} \bigg|_{t=0} \int_{L^r} \chi(\Omega_t \cap L_r) dL_r = \int_{\partial \Omega} \langle \partial \phi / \partial t, N \rangle \int_{G_{n-1,r}} \sigma_2(II|_V) dV dx.
\end{equation}

From Lemma 4.1 when pulling-back the form $\gamma_{k,q}$ from $N(\Omega)$ to $\partial \Omega$, one gets a polynomial expression $P_{k,q}$ of degree $2n-k-1$ in the coefficients $h_{ij}$ of II with $i, j \in \{1, 2, \ldots, n\}$. Moreover, for each $q$ the monomials in $P_{k,q}$ containing only entries of the form $h_{ii}$ contain the factor $b_{TT} = II(JN, JN)$ and do not appear in any other $P_{k,q'}$ with $q' \neq q$. Therefore, every non-trivial linear combination of $\{P_{k,q}\}_q$ must contain the variable $b_{TT}$. On the other hand, the integral $\int_{G_{n-1,r}} \sigma_2(II|_V) dV$ is a polynomial of the second fundamental form II restricted to the distribution $D = \langle N, JN \rangle^\perp$, hence a polynomial not involving $b_{TT}$.

Comparing the expressions of (22) and (23), it follows that $d_q = 0$ for all $q \in \{n - 2r, \ldots, n - r - 1\}$.

As $c_q$ and $d_q$ depend on $C_q$ and $D$, we will obtain the value of $c_q$ once we know the value of $C_q$ and $D$. We will get their value from the equalities $\{d_q = 0\}$. Note that this gives $r$ equations, since $q$ runs from $n - 2r$ to $n - r - 1$ in (22). As for the unknowns, we need to find $r$ constants $C_q$ plus the constant $D$ in (21).

We will get an extra equation by taking $II|_D = Id$ and equating (23) to (22). Then, for any pair $(n, r)$ we have a compatible linear system since constants in (21) exist. Next we find the solution, and we show it is unique.
Let us relate explicitly the coefficients \( \{c_q\} \) and \( \{d_q\} \) in (22) with \( C_q \) and \( D \) in (21). To simplify the range of the subscripts, we denote \( d_{n-r-a} \) with \( a = 1, \ldots, r \) and \( c_{n-r-a} \) with \( a = 1, \ldots, r + 1 \).

**Coefficient** \( d_{n-r-1} \). From the variation of \( B'_{k,q} \) in \( \mathbb{C}^n \) (Proposition 3.7), the coefficient of \( \Gamma'_{2n-2r-1,n-r-1} \) comes from the variation of \( B'_{2n-2r,n-r-1} \) and \( \Gamma'_{2n-2r,n-r} \). Then,

\[
d_{n-r-1} = 4C_{n-r-1} - 2r(n - r)D.
\]

**Coefficient** \( d_{n-r-a} \), \( a = 2, \ldots, r \). The coefficient of \( \Gamma'_{2n-2r-1,n-r-a} \) comes from the variation of \( B'_{2n-2r,n-r-a} \) and \( \Gamma'_{2n-2r,n-r-a+1} \). Then,

\[
d_{n-r-a} = 4a^2C_{n-r-a} - (r - a + 1)(n - r - a + 1)C_{n-r-a+1}.
\]

**Coefficient** \( c_{n-r-1} \). The coefficient of \( B'_{2n-2r-1,n-r-1} \) comes from the variation of \( B'_{2n-2r,n-r-1} \) and \( \Gamma'_{2n-2r,n-r} \). Then,

\[
c_{n-r-1} = 2(2r + 1)(n - r)D - 4C_{n-r-1}.
\]

**Coefficient** \( c_{n-r-a} \), \( a = 2, \ldots, r - 2 \). The coefficient of \( \tilde{B}'_{2n-2r-1,n-r-a} \) comes from the variation of \( B'_{2n-2r,n-r-a} \) and \( \tilde{B}'_{2n-2r,n-r-a+1} \). Then,

\[
c_{n-r-a} = -4a(2a - 1)C_{n-r-a} + (2r - 2a + 3)(n - r - a + 1)C_{n-r-a+1}.
\]

**Coefficient** \( c_{n-2r-1} \). The coefficient of \( \tilde{B}'_{2n-2r-1,n-2r-1} \) comes from the variation of \( B'_{2n-2r,n-2r} \). Then,

\[
c_{n-2r-1} = (n - 2r)C_{n-2r}.
\]

Now, we solve the linear system given by \( \{d_{n-r-a} = 0\} \) where \( a \in \{1, \ldots, r\} \). From equations (24) and (25) the system is given by:

\[
\begin{align*}
    r(n - r)D &= 2C_{n-r-1} \\
    4a^2C_{n-r-a} &= (n - r - a + 1)(r - a + 1)C_{n-r-a+1}.
\end{align*}
\]

Thus,

\[
C_{n-r-a} = \frac{(n - r - a + 1) \cdots (n - r)(r - a + 1) \cdots r}{2 \cdot 4^a a^2(a - 1)^2 \cdots 1^2} D
\]

\[
= \frac{(n - r)! r!}{2^{2a - 1} (n - r - a)!(r - a)! a! a!} D
\]

(29)

(29)

To obtain the value of \( D \), we calculate \( \int_{G^C_{n-1,r}} \sigma_{2r}(p) dV \) and \( \beta_{2n-2r-1,n-r-a} \) in case \( \Gamma(p) = \lambda Id \) for \( \lambda > 0 \), which occurs when \( \Omega \) is a metric ball. On the one hand, we have

\[
\int_{G^C_{n-1,r}} \sigma_{2r}(p)(\lambda Id|_V) dV = \lambda^{2r} \text{vol}(G^C_{n-1,r}).
\]
On the other hand, if $\Pi|_\mathcal{D} = \lambda \text{Id}$, then the connection forms satisfy $\alpha_{1i} = \lambda \omega_i$ and $\beta_{1i} = \lambda \omega_i$. Thus, $\theta_1 = 2\lambda \theta_2$ and $\theta_0 = \lambda^2 \theta_2$ and we obtain

$$\beta^r_{2n - 2r - 1, n - r - a}(p) = \lambda^{2r}(\beta \wedge \theta_0^{r - a + 1} \wedge \theta_1^{2a - 2} \wedge \theta_2^{n - r - a})(p) = 2^{2a - 2} \lambda^{2r}(\beta \wedge \theta_2^{n - 1})(p) = 2^{2a - 2} \lambda^{2r}(n - 1)!.$$

So, the equation

$$\text{vol}(G^C_{n-1,r}) = \sum_{a=1}^{r+1} c_{n-r-a} 2^{2a-2}(n-1)!$$

must be satisfied.

Substituting equations (26), (27) and (28) in the last equation gives

$$\frac{\text{vol}(G^C_{n-1,r})}{(n-1)!} = (2(2r + 1)(n - r)D - 4C_{n-r-1}) + \sum_{a=2}^{r} 2^{2a-2}((2r - 2a + 3)(n - r + a + 1)C_{n-r-a+1} - 4a(2a - 1)C_{n-r-a}) + 2^{2r}(n - 2r)C_{n-2r} = 2(2r + 1)(n - r)D + 4C_{n-r-1}((2r - 1)(n - r - 1) - 1) + \sum_{a=2}^{r-1} (-2^{2a-2}4a(2a - 1) + 2^{2a} (2r - 2a + 1)(n - r - a))C_{n-r-a} + C_{n-2r}(2^{2r}(n - 2r) - 2^{2r-2}4r(2r - 1)) = 2(2r + 1)(n - r)D + \sum_{a=1}^{r} 2^{2a} C_{n-r-a}((2r - 2a + 1)(n - r - a) - a(2a - 1)) = D \left( 2(n - r)! r! \sum_{a=0}^{r} \frac{(2r - 2a + 1)(n - r - a) - a(2a - 1)}{(n - r - a)!(r - a)!a!} \right) = D \frac{2n!}{r!(n - r - 1)!}.$$

Thus,

$$D = \frac{\text{vol}(G^C_{n-1,r})}{2n!} \binom{n - 1}{r}^{-1},$$

$$C_{n-r-a} = \frac{\text{vol}(G^C_{n-1,r})}{4^n n!} \binom{n - 1}{r}^{-1} \binom{n - r}{a} \binom{r}{a}$$

and, for $2r < n$, we have

$$\int_{\mathcal{L}'_\mathcal{C}} \chi(\Omega \cap L_r) dL_r = \sum_{a=1}^{r} C_{n-r-a} B'_{2n - 2r, n-r-a} + D \Gamma'_{2n-2r, n-r} = \frac{\text{vol}(G^C_{n-1,r})}{2n!} \binom{n - 1}{r}^{-1} \sum_{a=1}^{r} \binom{n - r}{a} \binom{r}{a} 2^{-2a+1} B'_{2n - 2r, n-r-a} + \Gamma'_{2n-2r, n-r}.$$
and
\[
\frac{d}{dt}\Bigg|_{t=0} \int_{\mathcal{L}_r^C} \chi(\Omega_t \cap L_r) dL_r = (2(2r+1)(n-r)D - 4C_{n-r-1}) \tilde{B}'_{2n-2r-1,n-r-1} \\
+ \sum_{a=2}^{r} ((2r-2a+3)(n-r+a+1)C_{n-r-a+1} - 4a(2a-1)C_{n-r-a}) \tilde{B}'_{2n-2r-1,n-r-a} \\
+ (n-2r)C_{n-2r} \tilde{B}'_{2n-2r-1,n-2r-1} \\
= \frac{\text{vol}(G_{n-1,r}^C)}{n!} \binom{n-1}{r}^{-1} \left( \sum_{a=1}^{r+1} \binom{n-r}{a} \binom{r+1}{a} a \frac{r}{4^{r-1}} \tilde{B}'_{2n-2r-1,n-r-a} \right).
\]

Finally, we use the relation in [19] and (11) to obtain the result. □

**Remark 4.3.** If \(2r \geq n\), then formula (17) follows directly from the relations among the different bases of valuations on \(\mathbb{C}^n\) given in [BF08] and the following relation in [Ale03]
\[
\int_{\mathcal{L}_r^C} \chi(\Omega \cap L_r) dL_r = \frac{1}{O_{2r-1}} \int_{\mathcal{L}_r^C} M_{2r-1}(\partial \Omega \cap L_r) dL_r = c U_{2(n-r),n-r}
\]
for a certain constant \(c\) coming from the different normalizations in \(dL_r\).

4.2. In Hermitian space forms.

**Corollary 4.4.** Let \(\Omega \subset \mathbb{C} \mathbb{K}^n(\epsilon)\) be a regular domain, let \(X\) be a smooth vector field over \(\mathbb{C} \mathbb{K}^n(\epsilon)\), let \(\phi_t\) be the flow associated to \(X\) and let \(\Omega_t = \phi_t(\Omega)\). Then
\[
\frac{d}{dt}\Bigg|_{t=0} \int_{\mathcal{L}_r^C} \chi(\Omega_t \cap L_r) dL_r = \text{vol}(G_{n-1,r}^C) \omega_{2r+1}(r+1) \binom{n-1}{r}^{-1} \binom{n}{r}^{-1} \\
\cdot \left( \sum_{q=\max\{0,n-2r-1\}}^{n-r-1} \binom{2n-2r-2q-1}{n-r-q} \frac{1}{4^{n-r-q-1}} \tilde{B}'_{2n-2r-1,q}(\Omega) \right).
\]

**Proof.** Comparing equation (17) and Proposition 3.9 in case \(\epsilon = 0\) shows that
\[
\int_{\partial \Omega} \langle X, N \rangle \left( \int_{G_{n,r}^C} \sigma_{2r}(II|V) dV \right) dx
\]
equals the right hand side of equation above. By taking a field \(X\) that vanishes outside an arbitrarily small neighborhood of a fixed \(x \in \partial \Omega\), we deduce the following equality between forms
\[
\left( \int_{G_{n,r}^C} \sigma_{2r}(II|V) dV \right) dx = \frac{\omega_{2r+1}}{\binom{n-1}{r} \binom{n}{r}} \\
\cdot \sum_{q=\max\{0,n-2r-1\}}^{n-r-1} \binom{2q-1}{q} \frac{c_{n,2n-2r-1,n-r-q}}{4^{q-1}} \beta \wedge \theta_{0}^{r-q+1} \wedge \theta_{1}^{2q-2} \wedge \theta_{2}^{n-r-q}.
\]
This equation extends obviously to $\mathbb{C}^n(\epsilon)$ without change. Then, using Proposition 3.9 gives the result.

**Theorem 4.5.** Let $\Omega$ be a regular domain in $\mathbb{C}^n(\epsilon)$. Then
\[
\int_{C^c_r} \chi(\Omega \cap L_r)dL_r = \text{vol}(G^C_{n-1,r}) \left( \frac{n-1}{r} \right)^{-1} (\epsilon^r (r+1) \text{vol}(\Omega) + \sum_{j=n-r}^{n-1} \epsilon^{j-n+r} \omega_{2n-2j} \left( \frac{n}{j} \right)^{-1} ((j+r-n+1)\mu_{2j,j} + \sum_{q=\max(0,2j-n)}^{j-1} \frac{1}{4j-q} \left( \frac{2j-2q}{j-q} \right) \mu_{2j,q})).
\]

(31)

\[
\delta_X \frac{\text{vol}(G^C_{n-1,r})}{n!} \left( \frac{n-1}{r} \right)^{-1} \{ \epsilon^r (r+1)n!V + \sum_{j=n-r}^{n-1} \epsilon^{j-n+r} \left( \frac{j-n+r+1}{2} \right) \tilde{\Gamma}_{2j,j} + \sum_{q=\max(0,2j-n)}^{j-1} \frac{1}{4j-q} \left( \frac{n-j}{j-q} \right) \frac{j}{q} B_{2j,q} \}.
\]

By Proposition 3.8
\[
\delta_X C_r(\Omega) = \frac{\text{vol}(G^C_{n-1,r})}{n!} \left( \frac{n-1}{r} \right)^{-1} [\epsilon^r n(r+1)\delta_2 \tilde{B}_{2n-1,n-1} + \sum_{j=n-r}^{n-1} \epsilon^{j-n+r} \frac{j-n+r+1}{2} \{ -2(n-j)j \tilde{\Gamma}_{2j-1,j-1} + 2\epsilon(n-j-1)(j+1) \tilde{\Gamma}_{2j+1,j} + 4(n-j) \tilde{B}_{2j+1,j} + 4\epsilon^2(n-j-1)(j+\frac{3}{2}) \tilde{B}_{2j+3,j+1} \}]
\]

(32)

\[
\frac{\text{vol}(G^C_{n-1,r})}{n!} \left( \frac{n-1}{r} \right)^{-1} \{ \epsilon^r n(r+1)\delta_2 \tilde{B}_{2n-1,n-1} + \sum_{j=n-r}^{n-1} \epsilon^{j-n+r} \frac{j-n+r+1}{2} \{ -2(n-j)j \tilde{\Gamma}_{2j-1,j-1} + 2\epsilon(n-j-1)(j+1) \tilde{\Gamma}_{2j+1,j} + 4(n-j) \tilde{B}_{2j+1,j} + 4\epsilon^2(n-j-1)(j+\frac{3}{2}) \tilde{B}_{2j+3,j+1} \}]
\]

We will show the previous expression is independent of $\epsilon$; i.e. all the terms containing $\epsilon$ cancel out. Hence, $\delta_X C_r(\Omega)$ coincides with (30) since we know this happens for $\epsilon = 0$. This will finish the proof.
We concentrate first on the terms with \( \tilde{B}'_{k,q} \). By putting together similar terms, the third line of (32) is

\[
\sum_{h=n-r+1}^{n-1} e^{h-n+r} 2\{ (h-n+r+1)(n-h+\frac{1}{2})h+(h-n+r)(\frac{h}{2}-(n-h+1)(2h-\frac{1}{2})) \\
+(h-n+r+1)(n-h+1)(h-\frac{1}{2}) \} \tilde{B}'_{2h-1,h-1} \\
-\epsilon^r \{(r+2)n-1 \} \tilde{B}'_{2n-1,n-1} + (2r+1)(n-r) \tilde{B}'_{2n-2r-1,n-r-1}.
\]

By putting together similar terms, the double sum in (32) (forgetting for the moment the terms with \( \tilde{\Gamma}'_{k,q} \)) becomes

\[
\sum_{h=n-r \ a = \text{max}(1,2h-n-1)}^{n-1} \sum_{a = \text{max}(0,2h-n)}^{h-1} e^{h-n+r} \frac{1}{4^{h-a}} \left( \frac{n-h}{h-a} \right)^2 \frac{1}{a} (a+1) \tilde{B}'_{2h-1,a} \\
- \sum_{h=n-r \ a = \text{max}(0,2h-n)}^{n-1} \sum_{a = \text{max}(0,2h-n)}^{h-1} e^{h-n+r} \frac{1}{4^{h-a}} \left( \frac{n-h}{h-a} \right)^2 \frac{1}{a} (a+1) \tilde{B}'_{2h-1,a} \\
+ \sum_{h=n-r+1 \ a = \text{max}(1,2h-n-1)}^{n} \sum_{a = \text{max}(0,2h-n-2)}^{h-2} e^{h-n+r} \frac{1}{4^{h-a}} \left( \frac{n-h+1}{h-a} \right)^2 \frac{1}{a} (a+1) \tilde{B}'_{2h-1,a} \\
- \sum_{h=n-r+1 \ a = \text{max}(0,2h-n-2)}^{n} \sum_{a = \text{max}(0,2h-n-2)}^{h-2} e^{h-n+r} \frac{1}{4^{h-a}} \left( \frac{n-h+1}{h-a} \right)^2 \frac{1}{a} (a+1) \tilde{B}'_{2h-1,a}.
\]

Note that the terms with \( a = -1 \) or \( a = 2h-n-2 \) vanish, if they occur. Then, one checks that all the terms in the above expression cancel out except those with \( h = n-r, n \), and those with \( a = h-1 \). Clearly the terms corresponding to \( h = n-r \) are independent of \( \epsilon \). The terms with \( h = n \) sum up \( \epsilon^r (n-1) \tilde{B}'_{2n-1,n-1} \), and together with the similar term appearing in (33) cancel out the first term in (32). Finally, the terms with \( a = h-1 \) are cancelled with the sum in (33).

With a similar but shorter analysis one checks that the multiples of \( \tilde{\Gamma}'_{k,q} \) cancel out completely. This shows that (32) is independent of \( \epsilon \), and finishes the proof.

\textbf{Remark 4.6.} The coefficients of \( \mu_{k,q} \) and vol in (31) were found by solving a linear system of equations. These equations were obtained by imposing that the variations of both sides in (31) coincide.

5. \textit{Gauss-Bonnet theorem}

\textbf{Theorem 5.1.} Let \( \Omega \) be a regular domain in \( \mathbb{C}^n(\epsilon) \). Then

\[
O_{2n-1}(\Omega) = 2n(n+1)e^n \text{vol}(\Omega) +
\sum_{c=0}^{n-1} \frac{O_{2n-2c-1}e^c}{(n-1)} \left( \sum_{q=\text{max}(0,2c-n)}^{c-1} \frac{1}{4^{c-q}} \binom{2c-2q}{c-q} \mu_{2c,q} + (c+1) \mu_{2c,c} \right).
\]
Proof. We proceed analogously to the proof of Theorem 4.5. In fact, the same computations of the previous proof show (in case \( r = n \)) that the right hand side of (34) has null variation.

For \( \varepsilon = 0 \) equation (34) is the well know Gauss-Bonnet formula in \( \mathbb{C}^n \equiv \mathbb{R}^{2n} \). For \( \varepsilon \neq 0 \), we take a smooth deformation of \( \Omega \) to get a domain contained in a ball of radius \( r \). Under this deformation, the right hand side of (34) remains constant. By taking \( r \) small enough, the difference between both sides can be made arbitrarily small. Hence, they coincide. \( \square \)

Theorem 5.2. Let \( \Omega \) be a regular domain in \( \mathbb{C}K^n(\varepsilon) \). Then

\[
O_{2n-1} \chi(\Omega) = M_{2n-1}(\partial \Omega) + 2\varepsilon \int_{\mathbb{C}^n_{n-1}} \chi(\partial \Omega \cap L_{n-1}) dL_{n-1} + \sum_{k=1}^{n-1} \varepsilon^k O_{2n-2k-1} \binom{n-1}{k}^{-1} \mu_{2k,k} + 2\varepsilon^n \nu \text{vol}(\Omega).
\]

Proof. First, we use the following relation between \( M_{2n-1}(\partial \Omega) \) and \( \mu_{0,0}(\Omega) \)

\[
M_{2n-1}(\partial \Omega) = \int_{\partial \Omega} \sigma_{2n-1}(\Pi_x) dx = \int_{N(\Omega)} \frac{\gamma \wedge \theta_0^{n-1}}{(n-1)!} = \frac{2\varepsilon_{n,0,0}^{-1} \sigma_{n,0,0}}{(n-1)!} \int_{N(\Omega)} \frac{\gamma \wedge \theta_0^{n-1}}{2} = \frac{2\varepsilon_{n,0,0}^{-1} \sigma_{n,0,0}}{(n-1)!} \nu \mu_{0,0}(\Omega)
\]

(35)

Now, from Theorems 4.5 and 5.1 it follows that

\[
\chi(\Omega) = \sum_{c=0}^{n-1} \frac{\varepsilon^c c!}{\pi^c} \left( \sum_{q=\max\{0,2c-n\}}^{n-1} \frac{1}{4^{c-q}} \binom{2c-2q}{c-q} \mu_{2c,q} + (c+1) \mu_{2k,k} \right) + \frac{\varepsilon^n (n+1)}{\pi^n} \text{vol}(\Omega)
\]

\[
= \mu_{0,0} + \sum_{c=1}^{n-1} \frac{\varepsilon^c c!}{\pi^c} \left( \sum_{q=\max\{0,2c-n\}}^{n-1} \frac{1}{4^{c-q}} \binom{2c-2q}{c-q} \mu_{2c,q} + (c+1) \mu_{2k,c} \right) + \frac{\varepsilon^n (n+1)}{\pi^n} \text{vol}(\Omega)
\]
\[
\mu_0 = \frac{\epsilon n!}{\pi^n} \sum_{c=1}^{n-1} \frac{e^{c-1}c!n^{n-c}}{n!}\left(2c - 2q\right)\mu_{2c,q} + c\mu_{2c,c} + \frac{\epsilon n!}{\pi^n} \mu_{2c,c} + \frac{\epsilon n(n+1)!}{\pi^n} \text{vol}(\Omega)
\]

\[
= \mu_0 + \frac{\epsilon n!}{\pi^n} \sum_{c=1}^{n-1} \frac{e^{c-1}c!n^{n-c}}{n!} \mu_{2c,c} + \frac{\epsilon n(n+1)!}{\pi^n} \text{vol}(\Omega)
\]

\[
= \mu_0 + \frac{\epsilon n!}{\pi^n} \int_{L_{n-1}} \chi(\partial \Omega \cap L_{n-1})dL_{n-1} + \sum_{c=1}^{n-1} \frac{\epsilon^n c!}{\pi^n} \mu_{2c,c}
\]

\[
+ \left(\frac{\epsilon^n(n+1)!}{\pi^n} - \frac{\epsilon n!n}{\pi^n}\right) \text{vol}(\Omega).
\]

\[\square\]

**Remark 5.3.** For \(n = 2\) and \(n = 3\), the Gauss-Bonnet-Chern formula in \(\mathbb{C}K^n(\epsilon)\) given in Theorem 5.1 was already stated in [Par02].

**6. Total Gauss curvature integral**

**Theorem 6.1.** Let \(\Omega\) be a regular domain in \(\mathbb{C}^n\). Then

\[
\int_{L_r} M_{2r-1}(\partial \Omega \cap L_r)dL_r = 2r\omega^2_{2r}\text{vol}(G^n_{n-1,r})\left(\frac{n-1}{r}\right)^{-1}\left(\frac{n}{r}\right)^{-1}
\]

\[
\cdot \left(\frac{1}{\text{vol}(G^n_{n-1,r})} \sum_{q=\max\{0, n-2r\}}^{n-r} \frac{2n - 2r - 2q}{n - r - q}\mu_{2n-2r,q}\right).
\]

**Proof.** On the one hand, by Gauss-Bonnet formula in \(\mathbb{C}^n\), we have

\[
\int_{L_r} \chi(\Omega \cap L_r)dL_r = \int_{L_r} \mu_{0,0}(\Omega \cap L_r)dL_r.
\]

On the other hand, by Theorem 4.2, we have

\[
\int_{L_r} \chi(\partial \Omega \cap L_r)dL_r = \text{vol}(G^n_{n-1,r})\left(\frac{n-1}{r}\right)^{-1}\left(\frac{n}{r}\right)^{-1}
\]

\[
\cdot \left(\frac{1}{\text{vol}(G^n_{n-1,r})} \sum_{q=\max\{0, n-2r\}}^{n-r} \frac{2n - 2r - 2q}{n - r - q}\mu_{2n-2r,q}\right).
\]

If we equate both expressions and we use the relation (35) between the total Gauss curvature and the valuation \(\mu_{0,0}\), we obtain the result.

\[\square\]

**Remark 6.2.** The previous theorem is not necessarily true in \(\mathbb{C}K^n(\epsilon)\) for \(\epsilon \neq 0\). Indeed, Howard’s transfer principle cannot be used here since Theorem 6.1 is not local: it does not apply to general hypersurfaces, but only to the closed embedded ones.
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