Micromaser line broadening without photon exchange

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Abstract. We perform a calculation of the linewidth of a micromaser, using the master equation and the quantum regression approach. A ‘dephasing’ contribution is identified from pumping processes that conserve the photon number and do not appear in the photon statistics. We work out examples for a single-atom maser with a precisely controlled coupling and for a laser where the interaction time is broadly distributed. In the latter case, we also assess the convergence of a recently developed uniform Lindblad approximation to the master equation; it is relatively slow.

1 Introduction

The linewidth of a laser and the spectrum of a resonantly driven two-level system are paradigmatic examples that probably spawned the field of quantum optics \cite{1,2,3}. In both cases, one has to deal with an open quantum system that is driven towards a steady state by the balance between pumping and dissipation. The characteristic features are intimately linked to the quantized nature of the electromagnetic field, with spontaneous processes determining the (laser) linewidth and the triplet spectrum (resonance fluorescence).

A fertile playground to investigate these issues in great detail is the micromaser or single-atom laser \cite{4,5}. Its basic ingredients are well known: a high-$Q$ cavity sustaining an electromagnetic field mode, and a dilute atomic beam that delivers energy by entering the cavity in an electronically excited state. In this device, a delicate control of the interaction time is possible and permits the observation of non-classical behaviour of the cavity field like quantum jumps \cite{6} or the collapse into photon number states \cite{7,8}. In addition, the atoms leaving the cavity provide information that can be used, for example, to measure in a non-destructive way the cavity photon number \cite{9,10}.

In this paper, we discuss the linewidth of the micromaser or in other words, the coherence time of the cavity field. This has been studied a few years after the introduction of the basic principles of the micromaser \cite{12} by Scully and co-workers \cite{13}. The basic ideas are simple: the maser (or laser) spectrum is defined by the Fourier transform of the two-time correlation (or coherence) function of the cavity field operator \cite{14}. To calculate these correlations, one invokes the so-called ‘quantum regression formula’ \cite{15,16} that provides a link to the time evolution of the system density operator. This evolution is conveniently given in terms of a master equation whose eigenstates and eigenvalues determine the maser spectrum. Their computation typically requires numerical work, and it is difficult to find a physical interpretation of the results. In particular, more than one eigenvalue contributes to the result around the trapping states where phase transitions happen in the micromaser \cite{17}. We suggest here a formulation of the maser linewidth (i) that provides a simple physical interpretation; (ii) that can be computed
fairly easily once the steady state solution to the master equation is known; and (iii) that interpolates in a natural way between different eigenvalues of the master equation. Our methods are similar to those used by Scully [13] and McGowan and Schieve [18] where a ‘linewidth operator’ that depends on the photon number is identified, whose expectation value with respect to the steady-state density operator yields the linewidth. The main difference of our approach is its straightforward formulation and simplicity. The results are generally in good agreement with previous work. We predict a weak oscillation of the linewidth as a function of the pumping strength that is related to a dephasing process related to the non-destructive monitoring of the cavity photon number by atoms that do not deposit their excitation energy in the cavity.

The outline is as follows. We fix the notation in Sec.2.1, introduce our formula for the linewidth (Eq.(4) of Sec.2.2) and work out the details for two cases: a ‘perfect maser’ where the interaction time \( \tau \) is fixed (Sec.3.1) and a ‘conventional laser’ where \( \tau \) is exponentially distributed (Sec.3.2). We conclude by a comparison to a recently developed approximation [19] that turns out to describe the laser above threshold fairly well, but becomes increasingly complex in the strong pumping regime (Sec.3.3). The Appendix contains some technical details of the calculation of the micromaser linewidth.

## 2 The micromaser and its linewidth

### 2.1 Master equation

The master equation for the micromaser we use here is of the form [20,3]

\[
\frac{d\rho}{dt} = \kappa(1 + n_{th}) \left[ a\rho(t)a^\dagger - \frac{1}{2} \{ \rho(t)a^\dagger a + a^\dagger a\rho(t) \} \right] + \kappa n_{th} \left[ a^\dagger \rho(t)a - \frac{1}{2} \{ \rho(t)aa^\dagger + aa^\dagger \rho(t) \} \right] + \int dp(\tau) \left[ r \cos(g\tau \hat{\varphi})\rho(t) \cos(g\tau \hat{\varphi}) - r \rho(t) + \kappa \theta^2 a^\dagger \text{sinc}(g\tau \hat{\varphi}) \rho(t) \text{sinc}(g\tau \hat{\varphi}) a \right]
\]

The first two lines describe cavity damping and its thermal excitation by the surrounding in the usual way. The thermal occupation number \( n_{th} \) depends on temperature and gives the average photon number at equilibrium, without pumping. The maser is pumped by a stream of excited two-level atoms that enter the cavity one by one with Poissonian statistics (rate \( r \)). The atoms interact with the maser mode for a time \( \tau \) that is distributed according to the measure \( dp(\tau) \), and \( g \) is the one-photon Rabi frequency for this interaction. The operator \( \hat{\varphi} = (aa^\dagger)^{1/2} \) and \( \text{sinc}(x) = \sin(x)/x \). The conventional pumping parameter is given by \( \theta = (r/\kappa)^{1/2} g\tau \).

In the strong coupling regime, non-classical states of the radiation field like number states are generated whenever the photon number \( n \) satisfies \( g\tau \sqrt{n + 1} = \pi, 2\pi, \ldots \) [12]. Then, the last term in Eq. (1) is zero and the photon number in the cavity remains unchanged. Physically speaking, each incoming atom performs an integer number of full Rabi cycles and leaves the cavity in the excited state again. This perfect scenario is perturbed by the fluctuations of the interaction time \( \tau \) and a not perfectly controlled coupling strength \( g \) that depends, for example, on the atom crossing the laser mode at a node or an anti-node. A typical choice for the distribution \( dp(\tau) \) in conventional lasers is an exponential one, its mean value \( \bar{\tau} \) giving the lifetime of the excited atomic states.

### 2.2 Maser spectrum

The maser spectrum is defined by the Fourier transform of the two-time correlation function [3,14]

\[
g(t) = \langle a^\dagger(t_0 + t)a(t_0) \rangle
\]

(2)
and is usually computed in the stationary regime where the correlation function only depends on the
time difference $t$ with respect to which the Fourier transform is taken. We therefore drop the $t_0$ argument
from Eq.(2). Alternative, but equivalent definitions are based on the decay of the average electric field
operator $\langle a(t) \rangle$ (Ref.[13]) or the correlation function of $e^{i \phi(t)}$ with $\phi$ being the phase operator (Ref.[21]).

Using arguments in the spirit of the quantum regression formula, one shows that the correlation
function can be computed as the expectation value $[14,21,17]$

$$g(t) = \text{tr} \left[ a^\dagger P(t) \right]$$

(3)
of a photon creation operator with respect to a ‘skew density operator’ $P$. The latter obeys an equation
of motion identical to the master equation (1), but with the ‘initial condition’ $P(0) = a \rho_{ss}$ where $\rho_{ss}$ is
the stationary solution that turns out to be diagonal in the photon number basis. A trivial consequence
is the normalization to the average photon number $g(0) = \text{tr} (\hat{n} \rho_{ss}) \equiv \langle \hat{n} \rangle_{ss}$.

The expansion of $P(t)$ in eigenvectors of the master equation yields a sum of exponentials in time
and hence of Lorentzian spectra. Usually, the spectra are computed in the frequency domain and their
characteristic width computed numerically. We adopt here a different approach that gives straightforward analytical results, but restricted to the regime where at most two eigenvalues are dominating. We emphasize this happens over a relatively large part of the parameter space (see Ref.[17]). Our definition
of the maser linewidth $D$ is

$$- \frac{D}{2} = \frac{1}{\langle \hat{n} \rangle_{ss}} \left. \frac{dg}{dt} \right|_0$$

(4)

where the time derivative is computed directly from the master equation, using the quantum regression
formula (3). To justify this choice, note that the eigenvector expansion for the correlation function is of the form

$$g(t) = \sum_j g_j e^{-\mu_j t/2}$$

(5)

where the prefactors satisfy the sum rule $\sum_j g_j = g(0) = \langle \hat{n} \rangle_{ss}$. This leads to

$$D = \sum_j g_j \mu_j \sum_j g_j$$

(6)
hence an arithmetic mean of the exact eigenvalues. Now, it appears from Ref.[17] that the relative weights $g_j/\langle \hat{n} \rangle_{ss}$ are often close to either zero or one. Only when the two lowest eigenvalues are crossing, both are contributing significantly. If a single eigenvalue is relevant, our linewidth $D$ coincides with it; if two contribute, Eq.(6) is just another convenient way to combine the two into a single parameter.

We note that the conventional full width at half maximum (FWHM) for a sum of two Lorentzians of equal weights, but of different widths, leads to the geometric mean of the widths. This significantly differs from the arithmetic mean only when the widths are very different. But, as mentioned above, equal weights occur typically when the eigenvalues cross.

2.3 Explicit formula and comparison

The time derivative in Eq.(4) leads to, using the regression formula (3)

$$D = - \frac{2}{\langle \hat{n} \rangle_{ss}} \text{tr} \left[ a^\dagger \left. \frac{dP}{dt} \right|_0 \right]$$

(7)

This expression involves the master equation for the skew operator $P$ which can be calculated easily by adopting a Lindblad form. One gets the familiar result

$$D\langle \hat{n} \rangle_{ss} = - \sum_\lambda \text{tr} \left\{ L_\lambda [a^\dagger, L_\lambda] a \rho_{ss} + [L_\lambda a^\dagger, L_\lambda] a \rho_{ss} \right\}$$

(8)
where the $L_\lambda$ are the so-called jump or Lindblad operators. For the micromaser, they can be written as (the index is running over $\lambda = 0, 1, (c, k), (s, k)$)

\begin{align}
L_0 &= \sqrt{\kappa(1 + n_{th})} a \\
L_1 &= \sqrt{\kappa n_{th}} a^\dagger \\
L_{c,k} &= \sqrt{r \Delta p(\tau_k) \cos(g\tau_k \hat{\phi}) + 1} f_k \\
L_{s,k} &= \sqrt{r \Delta p(\tau_k) \sin(g\tau_k \hat{\phi})}
\end{align}

where we have broken the integral over $\tau$ into a Riemann sum with weights $\Delta p(\tau_k)$. The term proportional to the unit operator in $L_{c,k}$ actually does not contribute to the Lindblad master equation; an expression for the real number $f_k$ can be found in Ref.\[19\], it is not needed here. (Note that the term $-r\rho$ in Eq.(1) is generated by the anticommutators in the Lindblad form, after performing the Riemann integral.)

We immediately see that one gets the linewidth as the average value of some operators with respect to the steady-state photon statistics, similar to what has been found previously \[13,18\]. In Appendix 4, these operators are computed and simplified using the detailed balance condition for the steady state. The result is

\begin{equation}
D \langle \hat{n} \rangle_{ss} = \kappa n_{th} + \sum_k \Delta p(\tau_k) \text{tr} \left\{ r \left( \cos(g\tau_k \hat{\phi}) - \cos(g\tau_k \hat{n}^{1/2}) \right)^2 \hat{n} \rho_{ss} \\
+ r \left( \hat{\phi} \sin(g\tau_k \hat{\phi}) - \hat{n}^{1/2} \sin(g\tau_k \hat{n}^{1/2}) \right)^2 \rho_{ss} \right\}
\end{equation}

where $\hat{n} = a^\dagger a$ is the photon number operator and we have kept for the ease of comparison the discrete summation.

This result should be compared to previous calculations. In their seminal paper \[13\], Scully and co-workers make the approximation of a sufficiently narrow distribution of photon numbers and get a linewidth

\begin{equation}
D \approx 1 + \frac{2n_{th}}{4\langle \hat{n} \rangle_{ss}} + 4r \sin^2 \left( \frac{g\tau}{4\langle \hat{n} \rangle_{ss}} \right).
\end{equation}

McGowan and Schieve improve this calculation by allowing for multiple eigenvalues of the Liouville operator \[18\]. They find an expression that gives a good approximation (to a few percent) to the exact eigenvalue calculation. The method is using a numerical evaluation of the FWHM of the (non-Lorentzian) spectrum which we want to avoid here. But their approximation can also be formulated in terms of the correlation function $g(t)$, and applying our definition \[4\], one has

\begin{equation}
D \langle \hat{n} \rangle_{ss} \approx 2 \sum_k \Delta p(\tau_k) \text{tr} \left[ (r - r \sin(g\tau_k \hat{\phi}) \sin(g\tau_k \hat{n}^{1/2}) - r \cos(g\tau_k \hat{\phi}) \cos(g\tau_k \hat{n}^{1/2}) \\
+ \kappa(1 + n_{th})(\hat{n} - \frac{1}{2} - \hat{n}^{1/2}) \right] \langle \hat{n} \rangle_{ss} \right)
\end{equation}

where we have added the average over $\tau$ as before.

\section{Discussion}

\subsection{Perfect maser}

To get a better understanding of the physics behind the maser linewidth, we first fix the interaction time (no fluctuations) and plot in Fig. 1 the contribution of the Fock state $|n\rangle$ to the linewidth, that can be read off from the operators under the trace in Eqs.\[13,18\]. In the same plot, we also give the photon statistics
Fig. 1. Contribution to the maser linewidth for a given photon number. We plot the $n$-dependent term under the trace of Eq. (13) (undulating curve) and Eq. (15) (‘McGowan’). The normalization is arbitrary, and the interaction time is fixed (no average). The peaked curves give the steady-state photon statistics. Solid and dashed curves correspond to two pumping parameters: $\theta = 2.1\pi$ ($g\tau \approx 0.47$, left peak, solid line) and $\theta = 2.2\pi$ ($g\tau \approx 0.49$, right peak, dashed line). We fix the pumping rate to $r = 200 \kappa$ and $n_{th} = 0.1$.

(peak curves). The chosen parameters correspond to the first trapping transition just above $\theta \approx 2\pi$ where the maximum of the photon statistics jumps from lower to higher values. We see that Eqs. (13) predicts a weak oscillation of the linewidth as the mean photon number increases, while Eq. (15) is essentially flat in this regime. These oscillations arise from the different prefactors $\hat{\phi}$ and $\hat{n}_{1/2}$ of the sine terms in Eq. (13). Indeed, performing an expansion in $1/\hat{n}$, the trigonometric functions can be simplified and give an expression identical in leading order to Eq. (15). The remainder is of relative order $1/\hat{n}^2$ and contains the sinusoidal oscillation seen in Fig 1.

In Fig 2, we plot the linewidth as a function of the pump parameter $\theta$ and see that all three expressions give very similar values for $D$. We also note (see inset) that the product $D\langle \hat{n} \rangle_{ss}/\kappa$ that measures the deviation from the Schawlow-Townes limit of a conventional laser is essentially a quadratic function of $\theta$, similar to the formula found in Ref. [13]:

$$D\langle \hat{n} \rangle_{ss} \approx \frac{\kappa \theta^2 + \kappa(1 + 2n_{th})}{4}$$

(16)

It is interesting to note that the sharp features of $D$ near the trapping states essentially arise from the corresponding oscillations in the average photon number.

3.2 Exponentially distributed interaction time

Our formula (13) can be averaged explicitly over an exponentially distributed interaction time $\tau$. The result takes the form (for simplicity, we make the replacement $g\tilde{\tau} \Rightarrow \tilde{g}$)

$$D\langle \hat{n} \rangle_{ss} = \kappa \text{tr} \left[ \frac{1 + g^2(3\tilde{n} + 2) + g^4(\tilde{n} + 1)(4\tilde{n} + 1)}{1 + 4g^2(\tilde{n} + 1)(1 + 2\tilde{g}^2(2\tilde{n} + 1) + \tilde{g}^4)} \rho_{ss} \right]$$

(17)

The corresponding photon statistics is given by the recurrence relation

$$p_{n+1} = \frac{2\tilde{g}^2}{1 + 4\tilde{g}^2(\tilde{n} + 1)}p_n$$

(18)

where $\tilde{\theta} = g\tilde{\tau}(r/\kappa)^{1/2}$ is the pumping parameter corresponding to $\tilde{\tau}$. Here, we have neglected the thermal effects and set $n_{th} = 0$.

The linewidth (17) is plotted in Fig 3 as a function of the (average) pump parameter $\tilde{\theta}$. The sharp features at the trapping transitions are smoothed due to the averaging over $\tau$. Above the threshold,
Fig. 2. Comparing expressions for the micromaser linewidth vs. the pumping parameter $\theta = (r/\kappa)^{1/2}g\tau$. No average over the interaction time $\tau$ here. Main plot: the expressions by Scully [Eq.(14), dashed line] and by McGowan and Schieve [Eq.(15), red triangles] practically coincide, except far below threshold ($\theta/\pi \ll 1/\pi$). Eq.(13) suggested here predicts in addition weak oscillations. Inset: linewidth normalized to the Schawlow-Townes value, $D\langle \hat{n} \rangle_{ss}/\kappa$ vs. pumping parameter $\theta$ around the first trapping state. A similar smooth behaviour is found also at larger values of $\theta$.

Parameters: $r = 50\kappa$, $n_{th} = 0.01$. We scan $\theta$ via the product $g\tau$.

$\theta \geq 1/\sqrt{2}$, the linewidth is quadratic in $\theta$. This can be derived from the Scully formula (14) by assuming that the average photon number is large and expanding the sine to lowest order. This gives a term $r(g\tau)^2/4\langle \hat{n} \rangle_{ss} = \kappa\theta^2/4\langle \hat{n} \rangle_{ss}$, and taking the average over an exponential distribution for $\tau$, we get

$$D \approx 1 + 2\frac{\theta^2}{4\langle \hat{n} \rangle_{ss}}$$  \hspace{1cm} (19)

This is plotted in dashed in Fig.3 and gives good agreement way above threshold.

3.3 Uniform approximation to the master equation

We also give in Fig.3 the results from a recently developed approximation to the laser master equation that is built to be valid beyond the weak coupling regime $g\tau \ll 1$, see Ref.[19]. The basic idea is to replace the large set of Lindblad operators in Eqs.(9)–(12) by a smaller one, using an expansion adapted to the probability measure $d\rho(\tau)$. More specifically, we represent the operators $\cos(g\tau\hat{\varphi})$ and $\sin(g\tau\hat{\varphi})$ as sums of normalized orthogonal polynomials in $\tau/\bar{\tau}$, with coefficients that are rational functions of $g\tau\hat{\varphi}$. The expansion is truncated at polynomials of a given order and gives good agreement for the average photon number and its variance. The agreement persists above the laser threshold where a simple expansion in $g\tau$ fails, as illustrated in Ref.[19]. Fig.3 shows, however, that for the laser linewidth, polynomials of fairly high order (up to the seventh) have to be taken into account to describe the strong pumping regime $\theta \gg 1$. In fact, if we compare the $n$-dependent linewidth (inside the trace of Eq.(3)) with its uniform expansion, we see a good agreement only up to photon numbers where $g\tau n^{1/2}$ is comparable to the maximum retained order. Whenever the photon statistics covers photon numbers beyond that value, the (truncated) uniform approximation breaks down. This can be translated into an upper limit for the pumping parameter that scales linearly with the order—a behaviour that can be qualitatively seen in Fig.3.

4 Conclusions

We have discussed in this paper an alternative calculation of the linewidth of a laser, with emphasis on the micromaser. The expression we suggest can be immediately computed from the master equation,
using the quantum regression theorem, see Eq.(3). The result emerges as a average over the photon number statistics, similar to previous work that we find good agreement with.

In our expression to the linewidth, a genuine contribution from a ‘dephasing’ mechanism is evident. It is due to the cosine terms in Eq.(13) that can be traced back to those pumping events where the pumping atom is detected again in the excited state after crossing the cavity. These processes cannot contribute to the laser gain, this is why only the sine terms appear in the photon statistics [Eq.(24)]. The combined atom+cavity system does get affected, however, by a signed probability amplitude. This is very similar to non-destructive measurements of the photon number using atoms in superposition states [10]. It is the average over these signed probability amplitudes, since they depend on the photon number, that contributes to the maser linewidth. Our formula for the linewidth nicely embodies this mechanism via the differences $\cos(g\tau \hat{n}) - \cos(g\tau \hat{\phi})$ that occur in Eq.(13). Even when the interaction time is broadly distributed, dephasing contributes a slight increase of the linewidth near the lasing threshold (Fig.3).

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**Appendix**

The contribution of the thermal absorption and emission terms to the linewidth is straightforward using the commutators $[a^\dagger, L_0] = -\sqrt{\kappa(1+n_{th})}$ and $[a, L_1] = \sqrt{\kappa n_{th}}$. This gives

$$D_{\text{th}} = \text{tr}\left[\left(\kappa(1+n_{th})\hat{n} - \kappa n_{th}\hat{n}\right)\rho_{ss}\right]$$

or $D_{\text{th}} = \kappa$. (Perfectly reasonable: only spontaneous, no stimulated processes contribute to the linewidth.) The pumping terms can be simplified with the help of the identity

$$a^\dagger F(\hat{\phi}) = F(\hat{n}^{1/2})a^\dagger,$$

valid for any smooth function $F(\hat{\phi})$. For example, commutators are transformed into differences

$$[a^\dagger, L_{c,k}] = \left[\cos(g\tau_k \hat{n}^{1/2}) - \cos(g\tau_k \hat{\phi})\right]a^\dagger$$
We thus get from the cosine terms

\[ D|_{\text{cos}} \langle \hat{n} \rangle_{ss} = r \sum_k \Delta p(\tau_k) \text{Tr} \left\{ \left( \cos(g \tau_k \hat{\varphi}) - \cos(g \tau_k \hat{n}^{1/2}) \right)^2 \hat{n}_{ss} \right\} \]  

(23)

The result from the sine terms is slightly more complicated. We now use the detailed balance for the photon statistics to bring it into a manifestly positive form. Recall that the steady state density matrix has diagonal elements \( p_n = \langle n | \rho_{ss} | n \rangle \) that satisfy the recurrence relation [12]

\[ p_{n+1} = \frac{1}{1 + n_{th}} \left( n_{th} + \theta^2 \text{sinc}^2(g \tau \sqrt{n + 1}) \right) p_n \]  

(24)

Including the average over the interaction time \( \tau \), we find

\[ 0 = -\kappa(1 + n_{th}) p_{n+1} + \kappa n_{th} p_n + \kappa \sum_k \Delta p(\tau_k) \theta_k^2 \text{sinc}^2(g \tau_k \sqrt{n + 1}) p_n \]  

(25)

with the pumping parameter \( \theta_k = (r/\kappa)^{1/2}(g \tau_k) \). We multiply this expression with \( n + 1 \), sum over \( n \) and subtract it from the linewidth. There is partial cancellation of the terms proportional to \( \kappa \) with the thermal contribution [20], and finally only the thermal occupation remains, as stated in Eq. (13).

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