Non-Commutative Correction to Thin Shell Collapse in Reissner Nordstr"{o}m Geometry

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Abstract

This paper investigates the polytropic matter shell collapse in the non-commutative Reissner-Nordstr"{o}m geometry. Using the Israel criteria, equation of motion for the polytropic matter shell is derived. In order to explore the physical aspects of this equation, the most general equation of state, $p = k\rho^{(1+\frac{1}{n})}$, has been used for finite and infinite values of $n$. The effective potentials corresponding to the equation of motion have been used to explain different states of the matter shell collapse. The numerical solution of the equation of motion predicts collapse as well as expansion depending on the choice of initial data. Further, in order to include the non-commutative correction, we modify the matter components and re-formulate the equation of motion as well as the corresponding effective potentials by including non-commutative factor and charge parameter. It is concluded that charge reduces the velocity of the expanding or collapsing matter shell but does not bring the shell to static position. While the non-commutative factor with generic matter favors the formation of black hole.

Keywords: Gravitational collapse; Israel junction conditions; Non-commutative theory.

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1 Introduction

Over the past decades, an extensive development has been made on the Penrose Cosmic Censorship Hypothesis (CCH).

According to this hypothesis, the singularities arising from the gravitational collapse of regular initial data are always invisible to the far away observer as these are clothed by the event horizons of black hole. On the other hand, if a naked singularity is formed as end state of the collapse, then it is visible to the external observer. Since black hole and naked singularity have different properties, it becomes important to explore the problem of collapse in order to have a satisfactory answer of the CCH. Depending on the choice of initial data, the gravitational collapse of a realistic matter leads to the formation of black hole or naked singularity.

Virbhadra and his collaborators introduced the idea of gravitational lensing to determine the nature of singularity. In one of his papers, he presented an improved form of the CCH, which is a source of inspiration for researchers.

The study of gravitational collapse requires that one must consider the appropriate geometry of the interior and exterior regions of a star and junction conditions which allow the smooth matching of these regions. According to Israel, there are two types of the boundary surface one with thin layer of surface matter distribution and the other without such thin layer. The matching conditions over the second type of the boundary surface demands the continuity of the metric coefficients as well as the extrinsic curvature components. This leads to the classical model of the relativistic gravitational collapse formulated by Oppenheimer and Snyder. On the other hand, matching the interior and exterior regions over a boundary surface with thin layer of matter requires the discontinuity of the extrinsic curvature over the boundary surface. This approach has been used successfully for more than thirty years to analyze the nature of singularity in the relativistic gravitational collapse.

Pereira and Wang studied the gravitational collapse of cylindrical shells made of counter rotating dust particles by using the Israel thin shell formalism. Sharif and Ahmad have extended this work to plane symmetric spacetime. Recently, Sharif and his collaborators have investigated the spherically symmetric gravitational collapse for a class of spacetimes using Israel thin shell formalism. This approach was generalized to thin charged shell without pressure by de Ia Cruz and Israel. Kuchar and Chase are among the first authors who treated the charged thin shell problem with
pressure by using the polytropic equation of state. There are a number of papers devoted to handle the charged thin shell problems. Boulware\(^{19}\) studied the time evolution of the charged thin shell and showed that their collapse can form a naked singularity.

According to Doplicher et al.\(^{20}\), when the energy density becomes sufficiently large, a black hole is formed. Heisenberg uncertainty principle states that measurement of a spacetime separation causes an uncertainty in momentum, i.e., momentum is inversely proportional to the extent of the separation. When the separation is small enough (as in the Schwarzschild spacetime, \(r \to 2m\)), the system leads to the formation of black hole which prevents any information escaping from the system. Thus there is a lower bound for the measurement of length. The condition for preventing the gravitational collapse can be expressed as uncertainty relation for the coordinates. This relation can be derived from the commutation relation of the coordinates.

There has been a growing interest to study the behavior of black hole in the non-commutative field theory. Nicolini et al.\(^{21}\) examined the behavior of the non-commutative Schwarzschild black hole while Modesto and Nicolini\(^{22}\) discussed the charged rotating non-commutative black hole. Bastos et al.\(^{23},^{24}\) explored the non-canonical phase space, singularity problem and black hole in the context of non-commutativity. Recently, Bartolami and Zarro\(^{25}\) investigated the non-commutative correction to pressure, particle numbers and energy density for fermion gas and radiations. The non-commutative correction to these quantities lead to the fact that non-commutativity affects the matter dispersion relation and equation of state. Inspired by the non-commutative correction to black hole physics, Oh and Park\(^{26}\) explored the gravitational collapse of shell with smeared gravitational source in the non-commutative Schwarzschild geometry.

This paper extends the above work to the non-commutative RN geometry. The main purpose is to investigate the effects of the charge parameter and the non-commutative factor on the gravitational collapse. The plan of the paper is as follows: In the next section, we discuss the shell collapse for the RN spacetime. The non-commutative correction to the shell collapse is presented in section 3. In the last section, we summarize and conclude the results.
2 Thin Shell Collapse

We assume that the 3D timelike boundary surface Σ splits the two 4D spherically symmetric spacetimes $M^+$ and $M^-$. The internal $M^-$ and external $M^+$ regions are described by the RN metrics given by

$$(ds)^2 = L_\pm dT^2 - \frac{1}{L_\pm} dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $L_\pm(R) = 1 - \frac{2M_\pm}{R} + \frac{Q_\pm^2}{R^2}$. $M_\pm$ and $Q_\pm$ are the mass and charge, respectively. The subscripts $+$ and $-$ represent the quantities in exterior and interior regions respectively to the boundary surface Σ. Further, we assume that charge in both the regions is same i.e., $Q_- = Q_+ = Q$. The strength of the electric field on the shell can be described by the Maxwell field tensor $F_{TR} = \frac{Q}{R^2} = -F_{RT}$. The corresponding energy-momentum tensor of the electromagnetic field is

$$T^{\nu(\text{em})}_{\mu} = \frac{1}{4\pi}(-F^{\nu\lambda}F_{\mu\lambda} + \frac{1}{4}g^{\nu\pi}F_{\mu\pi}F^{\mu\pi}).$$

By employing the intrinsic coordinates $(t, \theta, \phi)$ on the Σ at $R = R(t)$, the metrics in Eq.(1) become

$$(ds)^2_\Sigma = |L_\pm(R)(\frac{dT}{dt})^2 - \frac{1}{L_\pm(R)}(\frac{dR}{dt})^2|dt^2 - R(t)^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

here we assume that $g_{00} > 0$, so that $T$ is a timelike coordinate. Also, the induced metric on the Σ is

$$(ds)^2_\Sigma = dt^2 - a(t)^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

The continuity of the first fundamental form gives

$$[L_\pm(R_\Sigma) - \frac{1}{L_\pm(R_\Sigma)}(\frac{dR_\Sigma}{dT})^2]^2dT = (dt)_\Sigma,$$

$$R(t) = a(t)_\Sigma.$$  

The unit normal $n_\mu^\pm$ to the Σ in $M^\pm$ coordinates can be evaluated as

$$n_\mu^\pm = (-\dot{R}(t), \dot{T}, 0, 0),$$

where dot represents differentiation with respect to $t$. 

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The extrinsic curvature tensor \( K_{ij}^\pm \) on the \( \Sigma \) is defined as
\[
K_{ij}^\pm = n_\sigma^\pm \left( \frac{\partial^2 x_\sigma^-}{\partial \xi^i \partial \xi^j} + \Gamma^\sigma_{\mu \nu} \frac{\partial x_\pm^\mu}{\partial \xi^i} \frac{\partial x_\pm^\nu}{\partial \xi^j} \right), \quad (i, j = 0, 2, 3). \tag{8}
\]

Using this expression, one can find the following non-vanishing components of the extrinsic curvature
\[
K_{tt}^\pm = \frac{d}{dR} \sqrt{\dot{R}^2 + L_\pm}, \quad K_{\theta \theta}^\pm = -R \sqrt{\dot{R}^2 + L_\pm}, \quad K_{\phi \phi}^\pm = K_{\theta \theta}^\pm \sin^2 \theta. \tag{9}
\]

The surface energy-momentum tensor is defined by
\[
S_{ij} = \frac{1}{\kappa} \{ [K_{ij}] - \gamma_{ij}[K] \}, \tag{10}
\]
where \( \kappa \) is the coupling constant, \( \gamma_{ij} \) is the induced metric on the \( \Sigma \) and
\[
[K_{ij}] = K_{ij}^+ - K_{ij}^-, \quad [K] = \gamma^{ij}[K_{ij}]. \tag{11}
\]

The surface energy-momentum tensor for a fluid of density \( \rho \) and pressure \( p \) is
\[
S_{ij} = \rho \omega_i \omega_j + p(\theta_i \theta_j + \phi_i \phi_j), \quad (i, j = t, \theta, \phi), \tag{12}
\]
where \( \omega_i, \theta_i \) and \( \phi_i \) are unit vectors defined on the \( \Sigma \) by the relations
\[
\omega_i = \delta^t_i, \quad \theta_i = a(t) \delta^\theta_i, \quad \phi_i = a(t) \delta^\phi_i \sin \theta. \tag{13}
\]

Using Eqs.\( \text{(6)}, \text{(10)}, \text{(12)} \) and \( \text{(13)} \), we can find
\[
\rho = \frac{2}{\kappa R^2} [K_{\theta \theta}], \quad p = \frac{1}{\kappa} \{ [K_{tt}] - [K_{\theta \theta}] / R^2 \}. \tag{14}
\]

With the help of the non-zero extrinsic curvature components, we get
\[
(\zeta_+ - \zeta_-) + \frac{\kappa}{2} \rho R = 0, \tag{15}
\]
\[
\frac{d}{dR} (\zeta_+ - \zeta_-) + \frac{1}{R} (\zeta_+ - \zeta_-) - \kappa p = 0, \tag{16}
\]
where \( \zeta_\pm = \sqrt{\dot{R}^2 + L_\pm} \). Equations \( \text{(15)} \) and \( \text{(16)} \) can be reduced to the following single equation
\[
\frac{d\rho}{d \log R} + 2(\rho + p) = 0. \tag{17}
\]
The equation of state for the polytropic matter is

\[ p = k \rho^{(1 + \frac{1}{n})}, \quad (18) \]

where \( k \) is the equation of state parameter and \( n \) denotes the polytropic index. Notice that different values of \( n \) correspond to different types of matter, for example, for \( n \to \infty \), we have perfect fluid. The solution of Eq.(17), by using (18), for finite and infinite values of \( n \) are

\[ \rho = \{ (k + \rho_0 \frac{1}{n}) (\frac{R}{R_0})^\frac{2}{n} - k \}^{-n}, \quad (19) \]

\[ \rho = \rho_0 (\frac{R_0}{R})^{2k+2}, \quad (20) \]

respectively, where \( R_0 \) is the position of the shell at \( t = t_0 \) and \( \rho_0 \) is the density of matter on the shell at position \( R_0 \). It is mentioned here that in case of finite \( n \), energy density diverges at \( R = R_0 (\frac{k}{k+\rho_0 \frac{1}{n}})^{\frac{1}{n}} \).

Twice squaring Eq.(15), we obtain equation of motion of the shell

\[ \dot{R}^2 + V_{\text{eff}}(R) = 0, \quad (21) \]

where the effective potential \( V_{\text{eff}}(R) \) is

\[ V_{\text{eff}}(R) = \frac{1}{2} (L_+ + L_-) - \frac{(L_+ - L_-)^2}{(\kappa \rho R)^2} - \frac{1}{16} (\kappa \rho R)^2. \quad (22) \]

Using \( x = \frac{R}{R_0} \) and \( \tau = \frac{t}{R_0} \), this equation reduces to the following form

\[ \dot{x}^2 + V_{\text{eff}}(x) = 0. \quad (23) \]

The corresponding effective potential from Eqs.(19) and (22) for finite \( n \) is

\[ V_{\text{eff}}(x) = 1 - \frac{\varepsilon_+}{x} + \frac{\tilde{Q}^2}{x^2} - \frac{\varepsilon_+^2 \eta^2 (x^\frac{2}{n} - d)^2 n}{\eta^2 (x^\frac{2}{n} - d)^{2n}} - \frac{x^2}{\eta^2 (x^\frac{2}{n} - d)^{2n}}, \quad (24) \]

where \( \varepsilon_\pm = \frac{(M_\pm + M_\cd)}{R_0}, \quad \tilde{Q} = \frac{Q}{R_0}, \quad d = \frac{k}{k+\rho_0 \frac{1}{n}}, \quad \eta = 4 (\frac{k+\rho_0 \frac{1}{n}}{\kappa R_0})^n. \) For infinite \( n \) (perfect fluid), it turns out to be

\[ V_{\text{eff}}(x) = 1 - \frac{\varepsilon_+}{x} + \frac{\tilde{Q}^2}{x^2} - \frac{\eta^2}{x^{2+4k}} - \frac{\varepsilon_+^2 x^{4k}}{4 \eta^2}, \quad (25) \]
Figure 1: (Color online) Both graphs represent the effective potential for the polytropic matter shell (24). The left graph corresponds to $n = 30$ and $k = 2$ while the right graph corresponds to $n = -30$ and $k = 2$. For both graphs the values of the parameters are $M_- = 0, M_+ = R_0 = \rho_0 = Q = 1, \kappa = 8\pi$. These values of the parameters will remain the same for each graph while the extra parameters will be mentioned.

Figure 2: (Color online) The left graph shows the effective potential for the perfect fluid shell (25) with $k = 2$. The right graph represents the increase and decrease in the shell radius with the increase of time for $x(0) = 25$. 


where $\bar{\eta} = \frac{1}{4} \kappa \eta R_0^{(1-4k)}$.

Now we discuss Eqs. (23)-(25) graphically following the recent papers.\textsuperscript{27, 28)\textsuperscript{27}} Fig. 1 describes the behavior of effective potential (24) for the collapsing polytropic matter with finite $n$ and shell initial data. The left graph in Fig. 2 indicates the behavior of effective potential (25) for the perfect fluid shell depending on the EoS parameter and shell initial data. All these graphs show that $V_{\text{eff}} \leq 0$, thus Eq.(23) implies that motion is possible as $x^2 \geq 0$. The left graphs in both figures show that the effective potential increases from $-\infty$ to 0 and then decreases from 0 to finite negative value. This implies that expanding or collapsing shell of matter comes to static state and then expands or collapse. The right graph in Fig. 1 represents the following three phases:

1. Initially $V_{\text{eff}} \to -\infty$ as $x \to 0$, in this case, polytropic matter shell will expand to infinity for large initial radius or collapses to zero radius, forming a black hole or naked singularity. In both cases bouncing would occur if initial shell velocity is negative or positive, respectively.

2. When $V_{\text{eff}} \to 0$ for $x > 0$. The matter shell attains the non-static equilibrium state as $x$ increases.

3. $V_{\text{eff}} \to -\infty$ for some values of $x$. This implies that static shell comes to the state of expansion or collapse. In case of no crossing to $x$-axis, the shell collapses to zero or expands to infinity depending on the choice of the shell initial data.

The effect of charge parameter on the dynamics of the shell can be determined by the shell equation of motion given by

$$
\dot{x} = \pm \sqrt{-1 + \frac{\varepsilon}{x} - \frac{\bar{Q}^2}{x^2} + \frac{\varepsilon^2 \eta^2}{4x^4} (x^2 - d)^{2n} + \frac{x^2}{\eta^2 (x^2 - d)^{2n}}}.
$$

Here $\pm$ correspond to expansion and collapse of the shell, respectively. It follows from this equation that the presence of the charge term (Coulomb repulsive force) decreases the velocity of the expanding or collapsing shell. Also, we would like to mention here that this behavior of charge parameter is the same for the cases of finite and infinite $n$. In this case, the shell motion is slow as compared to the Schwarzschild spacetime.

Since the equation of motion (23) is nonlinear, its exact solution is impossible but can be solved numerically. The numerical solution of this equation
for suitable choice of initial data and for some initial value of shell radius gives the behavior of the shell radius with respect to time shown in right Fig. 2. This shows that radius decreases with the increase of time which is the strong argument for a shell to collapse. Also, radius is an increasing function of time which shows expansion.

3 Non-commutative Correction to Thin Shell Collapse

In this section, we study gravitational collapse in the non-commutative RN geometry. First, we discuss the effects of the non-commutative factor $\Theta$ on junction conditions which provide the equations of motion. The spacetime non-commutativity can be encoded by the following relation

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu},\quad (27)$$

where $\Theta^{\mu\nu}$ is an anti-symmetric matrix that determines the spacetime cell discretization as $\hbar$ (Planck’s constant) discretizes the phase space. There are different approaches to the non-commutative field theory out of which one is based on the $\star$-product and another on the coordinate coherent state formalism. Recently, following the second approach, Smailagic and Spallucci have shown that the problems of Lorentz invariance and unitary (arising in the $\star$-product approach) can be solved by considering $\Theta^{\mu\nu} = \Theta diag(\epsilon_{ij}, \epsilon_{ij}, \ldots)$, where $\Theta$ is constant with dimensions of length squared. Also, the coordinate coherent state modifies the Feynman propagators. Thus it is believed that non-commutativity removes the singularities (divergences) appearing in GR. In GR, the metric field is a geometrical structure and curvature (presence of matter) measures its strength. Since non-commutativity is the fundamental property of the metric, so it affects gravity via curvature. This implies that non-commutativity influences the matter energy-momentum tensor. Thus the geometric part of the field equations is left unchanged and modification is made in the energy-momentum tensor. Following this philosophy, we shall make only modification in matter part of the junction conditions and leave the geometric part unchanged.

The line element for this metric is

$$\left(ds\right)_{\pm}^2 = L_{\pm}dT^2 - \frac{1}{L_{\pm}}dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2),\quad (28)$$
where

\[
L_{\pm}(R) = 1 - 4M_{\pm} \frac{R}{R\sqrt{\pi}} \gamma \left( \frac{3}{2}; \frac{R^2}{4\Theta} \right) + \frac{Q^2}{\pi R^2} \gamma \left( \frac{1}{2}; \frac{R^2}{4\Theta} \right) - \frac{Q^2}{\pi R\sqrt{2\Theta}} \gamma \left( \frac{1}{2}; \frac{R^2}{2\Theta} \right) + \frac{Q^2}{\pi R\sqrt{2\Theta}} \gamma \left( \frac{3}{2}; \frac{R^2}{4\Theta} \right)
\]

and lower incomplete gamma function is defined by

\[
\gamma \left( \frac{a}{b}; x \right) = \int_0^x \frac{dt}{t^{a/b}} e^{-t}.
\]

In the commutative limit \( \frac{R}{\sqrt{\Theta}} \to \infty \), i.e., \( \Theta \to 0 \), Eq. (28) reduces to conventional RN metric (1) (for detail, see the properties of gamma function in the appendix 22).

Here we assume the smeared gravitating source in the non-commutative geometry and use the modified energy density and pressure 26) \( \rho_m = \rho_s + \rho_{\Theta} \) and \( p_m = p_s + p_{\perp} \) respectively. The quantities \( \rho_s \) and \( p_s \) are the energy density and pressure of the shell used in the previous section while \( \rho_{\Theta} \) and \( p_{\perp} \) are the energy density and pressure of the smeared source in the non-commutative theory. Since pressure and density of matter are affected by the non-commutative factor \( \Theta \), so equation of state is also affected by this factor. Recently, Bertolami and Zarro 25) have pointed out this effect by studying the astrophysical objects in the context of non-commutativity. For non-commutative energy density, \( \rho_{\Theta} \) and pressure, \( p_{\perp} \), Eq. (17) can be written as

\[
\frac{d\rho_{\Theta}}{d\log R} + 2(\rho_{\Theta} + p_{\perp}) = 0.
\]

It is well-known that \( \rho_\Theta \) and \( p_{\perp} \) satisfy the following relation 26)

\[
p_{\perp} = -(1 - \frac{R^2}{4\Theta})\rho_{\Theta}.
\]

Thus the non-commutative energy density is \( \rho_{\Theta} = \bar{\rho} e^{-\left(\frac{R^2 - R_0^2}{4\Theta}\right)} \), where \( \bar{\rho} \) is the value of \( \rho_{\Theta} \) at position \( R_0 \). It is interesting to note that when \( R \to 0 \) or \( \Theta \to \infty \), this matter acts as a matter of constant density. Thus the modified energy density for finite \( n \) is

\[
\rho_m = \{(k + \rho_0 \frac{R}{R_0}) \frac{R}{R_0} - k\}^{-n} + \bar{\rho} e^{-\left(\frac{R^2 - R_0^2}{4\Theta}\right)}.
\]
Also, for perfect fluid, we have
\[ \rho_m = \rho_0 \left( \frac{R_0}{R} \right)^{2k+2} + \bar{\rho} e^{-\left( \frac{k^2-\rho_0^2}{4\Theta} \right)}. \]  
(33)

Equation of motion of the shell is
\[ \dot{R}^2 + V_{eff}(R) = 0, \]  
(34)

where
\[ V_{eff}(R) = \frac{1}{2} (L_+ + L_-) - \frac{(L_+ - L_-)^2}{(\kappa\rho_m R)^2} - \frac{1}{16} (\kappa\rho_m)^2. \]  
(35)

Using \( x = \frac{R}{R_0} \), \( \tau = \frac{t}{R_0} \) as in the previous case, we get the modified energy density and effective potential for finite \( n \) as follows
\[ \rho_m = \left\{ (k + \rho_0 \frac{1}{n}) x^2 - k \right\}^{-n} + \bar{\rho} e^{-R_0^2 \left( \frac{x^2}{4\Theta} \right)}, \]  
(36)
\[ V_{eff}(x) = 1 - \frac{\varepsilon_+}{x} + \frac{\bar{\rho}^2}{x^2} \frac{4\varepsilon_+^2}{\kappa^2 R_0^2 x^4} \left\{ \left( k + \rho_0 \frac{1}{n} \right) x^2 - k \right\}^{-n} + \bar{\rho} e^{-R_0^2 \left( \frac{x^2}{4\Theta} \right)} \right)^{-2} - \frac{R_0^2 \kappa^2 x^2}{16} \left\{ \left( k + \rho_0 \frac{1}{n} \right) x^2 - k \right\}^{-n} + \bar{\rho} e^{-R_0^2 \left( \frac{x^2}{4\Theta} \right)} \right)^2. \]  
(37)

Also, using Eqs. (33) and (35), the effective potential for the perfect fluid is
\[ V_{eff}(x) = 1 - \frac{\varepsilon_+}{x} + \frac{\bar{\rho}^2}{x^2} - \frac{4\varepsilon_+^2}{\kappa^2 R_0^2 x^4} \left\{ \rho_0 \left( \frac{1}{x} \right)^{(2k+2)} + \bar{\rho} e^{-R_0^2 \left( \frac{x^2}{4\Theta} \right)} \right\}^{-2} - \frac{R_0^2 \kappa^2 x^2}{16} \left\{ \rho_0 \left( \frac{1}{x} \right)^{(2k+2)} + \bar{\rho} e^{-R_0^2 \left( \frac{x^2}{4\Theta} \right)} \right\}^2. \]  
(38)

Now we discuss the behavior of Eqs. (34)-(38) graphically. It follows from Figs. 4-6 that the effective potential (36) increases from negative to zero for polytropic matter with varying \( \Theta \) and fixed \( k \). The similar behavior of the effective potential (37) for the perfect fluid shell is shown in the left Fig. 7. It is mentioned here that we have only considered the case for which \( k > 0 \) in order to exclude the possibility of the exotic matter (dark energy) shell for which \( k < 0 \). This implies that depending on the choice of initial data, the shell continuously expands or collapses to a finite size then comes to rest position. After a shell attains a last stage of rest position, it has no capability to re-expand or re-collapse to zero size. All the graphs in non-commutative
Figure 3: (Color online) Both graphs show the effective potential for the polytropic matter shell in non-commutative background (37). The left graph corresponds to \( n = 30 \) and \( \Theta = 4 \) while the right graph to \( n = 30 \) and \( \Theta = 8 \). For both graphs \( k = 2, \bar{\rho} = 1 \).

Figure 4: (Color online) These indicate the effective potential for the polytropic matter shell in non-commutative background (37). The left graph corresponds to \( n = 30 \) and \( \Theta = 12 \) while the right graph to \( n = -30 \) and \( \Theta = 4 \).

Figure 5: (Color online) The left and right graphs represent the effective potential (37) corresponding to \( n = -30, \Theta = 8 \) and \( n = -30, \Theta = 12 \).
Figure 6: (Color online) The effective potential \( V_{\text{eff}} \) corresponding to \( \Theta = 4 \) and \( \Theta = 8 \) respectively.

Figure 7: (Color online) The effective potential of the perfect fluid shell corresponding to \( \Theta = 12 \) is shown in the left graph. While the right graph shows that for expanding and collapsing shell radius is increasing and decreasing function of \( t \), respectively.
background have $V_{\text{eff}} \to -\infty$ at $x > 0$, neither of them is divergent at $x = 0$. This confirms that non-commutativity can measure the short distances up to the order of Plank’s length scale. This can be seen by investigating the horizon radius and point of singularity where density diverges. The exact solution of nonlinear equation (34) is impossible. As before, we solve this equation by using numerical technique with initial condition. The behavior of the shell radius in this case is shown in right graph of Fig. 7.

The black hole horizon can be found by solving

$$
1 - \frac{4M_{\pm}}{xR_0 \sqrt{\pi \gamma}} \left( \frac{3}{2}; \frac{(xR_0)^2}{4 \Theta} \right) + \frac{Q^2}{\pi (xR_0)^2 \gamma^2} \left( \frac{1}{2}; \frac{(xR_0)^2}{4 \Theta} \right) - \frac{Q^2}{\pi (xR_0) \sqrt{2 \Theta}} \times \gamma \left( \frac{1}{2}; \frac{(xR_0)^2}{2 \Theta} \right) + \frac{Q^2}{\pi (xR_0)} \sqrt{\frac{2}{\Theta} \gamma} \left( \frac{3}{2}; \frac{(xR_0)^2}{4 \Theta} \right) = 0. \tag{39}
$$

For $R_0 = 1, Q = 1, M_+ = 1, M_- = 0$ and taking initially $x_h = 0.1$, the position of the horizon by iterative method is

$$
\begin{align*}
  x_h &= 1.35862; \quad \Theta = 4, \\
  x_h &= 1.70161; \quad \Theta = 8, \\
  x_h &= 1.94338; \quad \Theta = 12. \tag{40}
\end{align*}
$$

For finite $n$, the modified energy density (35) as well as the effective potential (37) for polytropic matter shell in non-commutative case are singular at $x_s = \frac{k^{\frac{1}{2}}}{(k + \rho_0)^{\frac{1}{2}}}$. Although this is independent of $\Theta$ but it is the only value of $x$ at which modified energy density and effective potential diverge. Further, all graphs of the effective potential for polytropic matter in Fig. 3-6 for non-commutative case imply that $V_{\text{eff}}$ diverges negatively at $x > 0$, while the right graph of Fig. 1 in commutative case for polytropic matter imply that $V_{\text{eff}} \to -\infty$ at $x = 0$. This means that the non-commutative parameter $\Theta$ has shifted the singularity from $x = 0$ to $x > 0$. Hence, for the values of the parameters, $k$, $n$ and $\rho_0$ used for the solutions previously, we get $x_s = 0.00228365$ at which polytropic matter shell in non-commutative case becomes singular. From the values of $x_h$ and $x_s$, we conclude that ”shell radii are greater than the singular point (where density diverges)” i.e., horizon covers the singularity at point $x_s = 0.00228365$ which leads to the formation of black hole as the final fate of the collapse as shown in Fig. 8.

For the case of infinite $n$, the energy density as well as the effective potential diverge at $x = 0$. Thus for each value of $\Theta$, the corresponding values
Figure 8: Note that shell before collapse is composed of matter, after collapse in GR matter is contracted to a point, while in non-commutative approach matter is contracted within an inner small circle. According to GR, the shell collapses to zero radius leaving behind event horizon while in non-commutativity, it collapses to non-zero radius interior to the event horizon.

of horizon radius are greater than zero, hence a singular shell of zero radius seems to be covered by the horizon radius. Consequently, we can say that perfect fluid shell collapse in non-commutative geometry always ends as a black hole. Hence in non-commutativity, polytropic matter shell collapses to a circle of small non-zero radius, while perfect fluid shell collapses to zero radius. The clear effects of $\Theta$ appear in the presence of generic polytropic matter.

4 Concluding Remarks

This paper is devoted to study the gravitational collapse of polytropic matter shell collapse in non-commutative RN geometry. Using the Israel junction conditions between the interior and exterior regions of a star, we have formulated equation of motion of the shell. It has been found that solution of this equation represents collapse and expansion depending on the choice of initial data. Further, by taking the polytropic index $n$ finite and infinite (perfect fluid), the effective potential of the system has been discussed in detail. The behavior of the effective potential for the polytropic matter with finite $n$
shows that the matter can undergo an expansion, collapse or bouncing depending on the choice of initial data. Similarly, the effective potential of the system for infinite \( n \) (perfect fluid) predicts different stages of dynamics of the shell depending on the choice of initial data.

In order to understand the effects of non-commutative factor \( \Theta \) on the gravitational collapse, a smeared source in the non-commutative RN spacetime has been taken. We have adopted the non-commutative approach\(^2\) to see the effects of the non-commutative factor \( \Theta \) on the collapse. Using the modified matter components, the shell equations of motion for polytropic matter with finite and infinite \( n \) have been derived. In this approach, we have found that in the presence of \( \Theta \) an initially an expanding or collapsing matter shell comes to static equilibrium. It cannot further, re-expand to infinity or re-collapse to zero size shown in Figs. 3-7. In other words in non-commutative the end state is a BH with some non-zero horizon radius.

It has been found that the charge parameter \( Q \) effects the velocity of the expanding or collapsing shell but does not play a dominant role to bring a shell to static equilibrium. For charged to be dominant it is necessary that effective potential must vanish in general, so that we get velocity \( \dot{x} = 0 \). Since it is not possible in general, so charge only effects the dynamics of the shell but does not prevent the collapse. Further, the solution of \( g_{00} = 0 \) for the non-commutative RN spacetime gives the horizon radius. For the larger value of \( \Theta \) factor, it has been found that the horizon radius covers the singularity where density is undefined for the case of polytropic matter with finite \( n \). For any non-zero value of \( \Theta \), perfect fluid always collapse to form a black hole. Thus in the non-commutative approach, there is a validity of the CCH depending on the choice of initial data.

After the work of Oh and Park\(^2\), the present work is the second step towards the thin shell collapse in non-commutative theory. Although both papers involve the numerical solutions however, it is expected that this kind of study would lead to find an exactly solvable collapsing model in non-commutative theory of gravity as Oppenheimer-Synder\(^1\) classical collapsing model in GR. The thin shell formalism was used for a wide class of cosmological and astrophysical problems\(^3\) which could be extended by using non-commutative approach.

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