ON TWO-ELEMENTARY K3 SURFACES WITH FINITE AUTOMORPHISM GROUP

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Abstract. We study complex algebraic K3 surfaces of Picard ranks 11, 12, and 13 of finite automorphism group that admit a Jacobian elliptic fibration with a section of order two. We prove that the K3 surfaces admit a birational model isomorphic to a projective quartic hypersurface and construct geometrically the frames of all supported Jacobian elliptic fibrations. We determine the dual graphs of all smooth rational curves for these K3 surfaces, the polarizing divisors, and the embedding of the reducible fibers in each frame into the corresponding dual graph.

1. Introduction

Let $X$ be a smooth algebraic K3 surface defined over the field of complex numbers. Denote by $\text{NS}(X)$ the Néron-Severi lattice of $X$. Let $L$ be a choice of even indefinite lattice of rank $\rho_L$ and signature $(1, \rho_L - 1)$ where $1 \leq \rho_L \leq 19$. As in [1, Section 2B], fix $h$ to be a very irrational vector in $L \otimes \mathbb{R}$ of positive norm. Then, following [1, Definition 2.6], a lattice polarization on $X$ is, by definition, a choice of a primitive lattice embedding $i: L \to \text{NS}(X)$, such that $i(h)$ is big and nef. Two $L$-polarized K3 surfaces $(X, i)$ and $(X', i')$ are said to be isomorphic if there exists an isomorphism $\alpha: X \to X'$ and a lattice isometry $\beta \in \text{O}(L)$ such that $\alpha^* \circ i' = i \circ \beta$, where $\alpha^*$ is the induced morphism at cohomology level.

It is known [12] that $L$-polarized K3 surfaces are classified, up to isomorphism, by a coarse moduli space $\mathcal{M}_L$, which is a quasi-projective variety of dimension $20 - \rho_L$. A very general $L$-polarized K3 surface $(X, i)$ satisfies $i(L) = \text{NS}(X)$. In particular, the Picard number of $X$ is $\rho_L$.

If the standard rank-two hyperbolic lattice $H$ primitively embeds into $L$, then a $L$-polarized K3 surface $X$ has a Jacobian elliptic fibration, which is an elliptic fibration $\pi_X: X \to \mathbb{P}^1$ with a choice of section [4, Lemma 3.6]. If furthermore, the negative definite rank-eight Nikulin lattice [20, Def. 5.3] $N$ also primitively embeds into $L$, then the Mordell-Weil group $\text{MW}(X, \pi_X)$ associated to the elliptic fibration $\pi_X$ has an element of order-two [28]. This element can be seen geometrically as a second section in the elliptic fibration $\pi_X$, which determines an order-two point in each smooth fiber. Fiber-wise translation by these order-two points further determines a canonical symplectic involution known as a van Geemen-Sarti involution (see Section 2.2).

In this article, we study families of lattice polarized K3 surfaces whose polarization lattice $L$ satisfies the following conditions (see Section 2.1 for details).

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Conditions for $L$.
(a) $L \cong H \oplus K$, with $K$ a negative-definite lattice of ADE type,
(b) $L$ admits a canonical primitive lattice embedding $H \oplus N \cong L$,
(c) $L$ is of finite automorphism group type, in the sense of Nikulin [21, 22].

By standard classification results on lattices [21] and elliptic fibrations [14, 26], one can exhaust all the lattices $L$ satisfying conditions (a)-(c) up to isometry, which are displayed in column 2 in Table 1.

| $\rho_L$ ($e_L, h_L$) | $L$ | $K^\text{root}$ | $\mathcal{W}$ | construction | dual graph | embedding |
|----------------------|-----|-----------------|---------------|--------------|------------|----------|
| 10 (6, 0)            | $H(2) \oplus D_4(-1)^{\otimes 2} \cong H \oplus N$ | $8A_1$ | $\mathbb{Z}/2\mathbb{Z}$ | [28] | [25, 28] |
| 11 (7, 1)            | $H \oplus D_4(-1) \oplus A_1(-1)^{\otimes 5}$ | $D_4 + 5A_1$, $9A_1$ | $\mathbb{Z}/2\mathbb{Z}$ | Thm. 4.9 Fig. 2 Thm. 5.3(5) |
| 12 (6, 1)            | $H \oplus D_4(-1) \oplus A_1(-1)^{\otimes 2}$ | $D_4 + 4A_1$ | $\mathbb{Z}/2\mathbb{Z}$ | Thm. 4.16 Fig. 3 Thm. 5.4(5) |
| 13 (5, 1)            | $H \oplus E_7(-1) \oplus A_1(-1)^{\otimes 4}$ | $E_7 + 4A_1$, $D_4 + 3A_1$ | $\mathbb{Z}/2\mathbb{Z}$ | Thm. 4.19 Fig. 4 Thm. 5.5(5) |
| 14 (4, 0)            | $H \oplus D_4(-1) \oplus D_4(-1)$ | $D_4 + D_4$, $E_7 + 5A_1$ | $\mathbb{Z}/2\mathbb{Z}$ | [10] | [10, 17] |
| 15 (4, 1)            | $H \oplus E_8(-1) \oplus A_1(-1)^{\otimes 2}$ | $E_8 + 4A_1$, $D_10 + 2A_1$ | $\mathbb{Z}/2\mathbb{Z}$ | [10] | [10, 17] |
| 16 (2, 1)            | $H \oplus D_4(-1)$ | $E_6 + D_6$, $E_7 + D_4$ | $\mathbb{Z}/2\mathbb{Z}$ | [7, 8] | [8, 17, 21] |
| 17 (1, 1)            | $H \oplus E_7(-1) \oplus E_7(-1)$ | $E_8 + E_7$, $D_{12} + 2A_1$ | $\mathbb{Z}/2\mathbb{Z}$ | [5, 6, 19] | [5, 17, 21] |
| 18 (0, 0)            | $H \oplus E_8(-1) \oplus E_8(-1)$ | $2E_8$, $D_{16}$ | $\mathbb{Z}/2\mathbb{Z}$ | [4, 27] | [4, 17, 21] |

**Table 1**

As it turns out, there is one lattice $L$ satisfying (a)-(c) in each rank $11 \leq \rho_L \leq 18$, with the exception of rank $\rho_L = 14$, for which two inequivalent instances exist. Note that the lattice $H \oplus N$ in the first row does not satisfy (a). Nevertheless we have included the lattice in the table: each $L$-polarized K3 surface has a unique $H \oplus N$-polarization. Moreover, the $H \oplus N$-polarized family is in fact a family of Jacobian elliptic surfaces (see Section 3.1), obtained from a different family of K3 surfaces of Picard rank 10. From the latter family we will also derive birational models for the families of higher Picard rank in Table 1.
Note that the rank 19 lattice \( L = H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1) \) is the only other lattice that is not shown in the table that satisfies conditions (a) and (c), and has a \( H \oplus N \) embedding. However, the \( H \oplus N \) embedding is not unique.

The topics of interest here are the geometric construction of the frames of the Jacobian elliptic fibrations for each \( L \)-polarized family and their dual graphs of rational curves: see columns 6 and 7 on Table 1. For those families of Picard rank at least 14 and the \( H \oplus N \)-polarization family, these questions were discussed in previous works by the authors and others. The present paper fills in the gap for the remaining three families of Picard rank 11, 12 and 13. Moreover, we present an embedding of the irreducible fibers of each Jacobian elliptic fibration of these three families into the corresponding dual graphs by coloring their vertices: see column 8 on Table 1. A summary of the results of this paper is displayed on rows 2 to 4 of Table 1.

The outline of the paper is as follows: in Section 2, we give some background material important for this work. In particular, we explain the conditions satisfied by the lattice \( L \). We also study the \( H \oplus N \)-polarized family in detail. In Section 3, we introduce a family of K3 surfaces of Picard rank 10 whose associated Jacobian surfaces gives the \( H \oplus N \)-polarized family. We also derive a projective quartic hypersurface model and an associated double sextic model. We show, in Section 4, that these birational models can be specialized to models for the families of Picard rank 11, 12 and 13 in Table 1. There, we also introduce a normal form for the quartic hypersurface model as a multi-parameter generalization of the Inose quartic \([15]\), and construct the supported Jacobian elliptic fibrations from pencils of curves. In Section 5, we determine the dual graphs of all smooth rational curves as well as the polarizing divisors for the constructed projective surfaces. We color the vertices of the dual graphs to denote embeddings of the irreducible fibers of the Jacobian elliptic fibrations. Symmetries of the colored dual graphs for each alternate fibration describe the underlying van Geemen-Sarti involution.

To our knowledge, for \( L \)-polarization with \( \rho_L = 11, 12, 13 \), the quartic surfaces to be described in Section 4 have not appeared in the literature before.

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2. Background

2.1. Elliptic fibrations of \( L \)-polarized K3 surfaces. Let us first recall the definitions of a few lattice invariants. Given a lattice \( L \), we denote by \( D(L) = L^\vee/L \) the associated discriminant group with corresponding discriminant form \( q_L \). A lattice \( L \) is then called 2-elementary if \( D(L) \) is a 2-elementary abelian group, i.e., \( D(L) \cong (\mathbb{Z}/2\mathbb{Z})^\ell \). Here \( \ell = \ell_L \) is the length of \( L \), i.e., the minimal number of generators of the group \( D(L) \). As a third relevant invariant for \( L \), one also has the parity \( \delta_L \in \{0, 1\} \). By definition, \( \delta_L = 0 \) if \( q_L(x) \) takes values in \( \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z} \) for all \( x \in D(L) \).
and $\delta_L = 1$ otherwise. A result of Nikulin [21, Thm. 4.3.2] asserts that hyperbolic, even, 2-elementary lattices embedding into the K3 lattice are uniquely determined by their rank $\rho_L$, length $\ell_L$, and parity $\delta_L$.

Let $\mathcal{X}$ be a K3 surface with Neron-Severi lattice $L$ that satisfies Conditions (a)-(c) mentioned in Section 1. Condition (a) is equivalent to saying that $\mathcal{X}$ carries a Jacobian elliptic fibration: the general fiber and section classes in $L$ generate a sublattice of hyperbolic type which gives a primitive lattice embedding $j: H \hookrightarrow L$. The frame of the Jacobian elliptic fibration, which is the isometry class of the lattice $K := j(H)\perp$, is negative definite and has rank $\rho_L - 2$. Moreover, all frames of $\mathcal{X}$ share the same discriminant group and discriminant form with $L$. Hence they belong to the same lattice genus, canonically associated with the surface $\mathcal{X}$. Let us also note that, given the frame $K$, the root sub-lattice $K^{\text{root}}$ is always of ADE type and the factor group $W = K/K^{\text{root}}$ is isomorphic to the Mordell-Weil group of the Jacobian elliptic fibration. The pair $(K^{\text{root}}, W)$ gives the type of the frame $K$. We shall refer to the elliptic fibration associated the decomposition $L = H \oplus K$ with trivial Mordell-Weil group $W$ as standard.

The discussion of Condition (b), which says that $\mathcal{X}$ has a canonical Jacobian elliptic fibration with a two-torsion section, or $W = \mathbb{Z}/2\mathbb{Z}$, will be deferred to Section 2.2. The elliptic fibration associated with the $H$-embedding of this condition will be referred to as the alternate fibration.

Lastly, we explain the meaning of Condition (c): a lattice $L$ is said to be of finite automorphism type if the factor group $O(L)/W(L)$ is finite. Here $O(L)$ is the group of isometries of $L$ and $W(L)$ is the subgroup of isometries generated by the reflections with respect to roots of $L$. Following results by Pjatecki˘ı-Šapiro and Šafarevič [24] and Nikulin [21], Condition (c) is in fact the same as the condition for the group of automorphism $\text{Aut}(\mathcal{X})$ to be finite.

We note that for general $L$, a lattice-theoretic classification of Jacobian elliptic fibrations on $L$-polarized K3 surfaces with finite automorphism group was given by the authors in [11]. This work was largely based on Nikulin’s classical lattice classification theory [21], as well as on Shimada’s work [26] on Jacobian elliptic fibrations. The present paper focuses on K3 surfaces $\mathcal{X}$ with finite automorphism group that also carry a Jacobian elliptic fibration endowed with a 2-torsion section. By the classification in [11, Tables 2 and 3], this list consists of K3 surfaces with 2-elementary Néron-Severi lattices only. Moreover, one has:

**Theorem 2.1.** Let $\mathcal{X}$ be a K3 surface satisfying Conditions (a)-(c) in Section 1. Then, up to isometry, the lattice $L$ belongs to the list given in Table 1. Table 1 also includes the frames of all possible Jacobian elliptic fibrations supported on $\mathcal{X}$.

**Remark 2.2.** Since each entry $\mathcal{X}$ in Table 1 has automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the alternate fibration is always unique up to automorphism of $\mathcal{X}$. In contrast, there could be multiple inequivalent decompositions $L = H \oplus K$ with trivial Mordell-Weil groups.

**Remark 2.3.** In the context of Condition (a), an important question that was also studied in [11] is - how many non-isomorphic Jacobian elliptic fibrations on $\mathcal{X}$ can be
associated to a given frame. This number, called the multiplicity of the frame, can be described via a lattice-theoretic procedure due to Festi and Veniani [13]. In Table 1, the multiplicity of the unique frame \((K^\text{root}, \mathcal{W})\) with \(\mathcal{W} = \mathbb{Z}/2\mathbb{Z}\) for each lattice \(L\) with \(\rho_L > 10\) is in fact 1. We direct interested readers to [11, Section 5] for more details.

Before we move on to studying explicit families of lattice polarized K3 surfaces and their elliptic fibrations, we would like to recall some facts about Jacobian surfaces of elliptic K3 surfaces [16]. Given an elliptic K3 surface \(f : \mathcal{X} \to \mathbb{P}^1\), the Jacobian surface of \(\mathcal{X}\) is the elliptic surface equipped with the relative Jacobian fibration \(\text{J}(\mathcal{X}) \to \mathbb{P}^1\) whose fiber at a point \(z \in \mathbb{P}^1\) is \(J(f_z)\), the Jacobian of the fiber of \(f\) at \(z\). In the absence of any multiple fibers, \(J(\mathcal{X})\) has the same Picard number and singular fibers of the same types as \(\mathcal{X}\). In this paper, very often we would consider both the K3 surface and its Jacobian surface for the analysis of singular fibers of an elliptic fibration.

2.2. K3 surfaces with van Geemen-Sarti involution. In this subsection, we study \(H \oplus N\)-polarized K3 surfaces. Their geometric properties, including the types of singular fibers of a Jacobian elliptic fibration and the van Geemen-Sarti involution associated to the alternate fibration, can be understood using different birational models for the K3 surface such as a suitable Weierstrass model and double quadrics model.

Let \(\mathcal{X}\) be a K3 surface with a \(H \oplus N\)-polarization, and let \(\pi : \mathcal{X} \to \mathbb{P}^1\) be the alternate fibration. A Weierstrass model for \(\pi\), which –by a slight abuse of notation– we will also denote by \(\mathcal{X}\), is induced by the polarization with fibers in \(\mathbb{P}^2 = \mathbb{P}(X, Y, Z)\) varying over \(\mathbb{P}^1 = \mathbb{P}(u, v)\). It is given by the equation
\[
(2.1) \quad \mathcal{X}: \quad Y^2Z = X\left(X^2 + a(u, v)XZ + b(u, v)Z^2\right),
\]
where \(a\) and \(b\) are homogeneous polynomials of degree four and eight, respectively. The alternate fibration admits a section \(\sigma : [X : Y : Z] = [0 : 1 : 0]\) and a 2-torsion section \([X : Y : Z] = [0 : 0 : 1]\), and it has the discriminant
\[
(2.2) \quad \Delta_{\text{alt}} = b(u, v)^2\left(a(u, v)^2 - 4b(u, v)\right).
\]
We also assume Equation \((2.1)\) satisfies a minimality condition:

**Lemma 2.4.** Assume that \(b \neq 0, b \neq a^2/4\). The minimal resolution of Equation \((2.1)\) is a K3 surface if and only if there is no polynomial \(c \in \mathbb{C}[u, v]\) of deg \(c > 0\) such that \(c^2\) divides \(a\) and \(c^4\) divides \(b\).

**Proof.** For \(b = 0\) or \(b = a^2/4\), Equation \((2.1)\) becomes \(Y^2Z = X^2(X + aZ)\) or \(Y^2Z = X(X + aZ/2)^2\) respectively. Otherwise, Equation \((2.1)\) can be brought into the Weierstrass normal form
\[
(2.3) \quad Y^2Z = X^3 + f(u, v)XZ + g(u, v)Z^3.
\]
with coefficients
\[
(2.4) \quad f(u, v) = \frac{3b(u, v) - a(u, v)^2}{3}, \quad g(u, v) = \frac{a(u, v)(2a(u, v)^2 - 9b(u, v))}{27},
\]
and discriminant \(\Delta_{\text{alt}} = 4f^3 + 27g^2 = b^2(a^2 - 4b)\). One can check that Equation \((2.3)\) has a \((4, 6, 12)\)-point at \((f, g, \Delta_{\text{alt}})\) if and only if there is a polynomial \(c\) so that \(c^2\) divides \(a\) and \(c^4\) divides \(b\). \(\square\)
From Equation (2.1), one can check that a very general $H \oplus N$-polarized K3 surface $X$ has 8 fibers of type $I_1$ over the zeroes of $a^2 - 4b = 0$ and 8 fibers of type $I_2$ over the zeroes of $b = 0$ under the alternate fibration $\pi_X$. The Mordell-Weil group $\text{MW}(X, \pi_X)$ is easily shown to be $\mathbb{Z}/2\mathbb{Z}$. Let us consider the translation by 2-torsion whose fiberwise action is given by

$$\iota_X: \begin{bmatrix} X : Y : Z \end{bmatrix} \mapsto \begin{bmatrix} b(u, v)XZ : -b(u, v)YZ : X^2 \end{bmatrix}$$

for $[X : Y : Z] \neq [0 : 1 : 0], [0 : 0 : 1]$, and by swapping $[0 : 1 : 0] \leftrightarrow [0 : 0 : 1]$. This is easily seen to be a Nikulin involution as it leaves the holomorphic 2-form invariant. The involution $\iota_X$ is called a van Geemen-Sarti involution.

**Remark 2.5.** If one factors $X$ by the action of $\iota_X$ and then resolves the eight occurring $A_1$-type singularities, then one can obtain a new $H \oplus N$-polarized K3 surface $\mathcal{Y}$ with Weierstrass model

$$\mathcal{Y}: \begin{bmatrix} Y^2Z = X \left( X^2 - 2a(u, v)XZ + (a(u, v)^2 - 4b(u, v))Z^2 \right) \end{bmatrix}. $$

In fact, if one repeats the same construction on $\mathcal{Y}$, then the original K3 surface $X$ is recovered. The two surfaces $X$ and $\mathcal{Y}$ are therefore referred as van Geemen-Sarti duals (see [10]).

**Remark 2.6.** One can construct a different birational model for $X$ as follows. For any $n \in N_0$, we consider the Hirzebruch surface $F_n$ as the GIT quotient

$$F_n \cong (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\}) / (\mathbb{C}^* \times \mathbb{C}^*),$$

where the action of $\mathbb{C}^* \times \mathbb{C}^*$ on the coordinates $(u, v)$ and $(s, t)$ is given by

$$\begin{bmatrix} \lambda \mu \cdot (u, v, s, t) = (\lambda u, \lambda v, \mu s, \lambda^{-n}\mu t) \end{bmatrix}.$$ 

In this way, the branch locus in Equation (2.1) is contained in $F_4$. Suitable blowups transform Equation (2.1) into a double cover of $F_0 = \mathbb{P}(s, t) \times \mathbb{P}(u, v)$ branched along a curve of bi-degree $(4,4)$, i.e., along a section in the line bundle $O_{F_0}(4,4)$. Such a cover is known as double quadric surface and the two rulings of the quadric $F_0$, induced by the projections $\pi_i: F_0 \rightarrow \mathbb{P}^1$ for $i = 1, 2$, give the alternate fibration and a standard fibration, respectively. For more details, see [8].

3. K3 surfaces with a different rank 10 polarization

In this section, we study two equivalent birational models, namely the double sextics model and the projective quartic model, of a certain very general K3 surface $Z$ of Picard rank 10. We will prove in Proposition 3.1 that the associated Jacobian surface $J(Z)$ of $Z$ has an $H \oplus N$-polarization. We will then specialize the constructed model to give birational models for the K3 surfaces polarized by the lattices of rank 11, 12 and 13 in Table 1. Note that some examples of equations relating the elliptic fibrations of K3 surfaces with 2-elementary Néron-Severi lattices and double sextics or quartic hypersurfaces have also been provided in [2,14].
3.1. Double Sextics model. K3 surfaces that are the minimal resolution of double covers of $\mathbb{P}^2 = \mathbb{P}(u, v, w)$ branched along plane sextic curves are called double sextic surfaces, or double sextics for short. The Picard number of a double sextic is in general at least one, and it increases if there exist curves in special position. For example, a sextic might possess tritangents or contact conics. In [23] this approach was used to construct a number of examples for sextics defining K3 surfaces with the maximal Picard number 20.

In this paper our focus is on K3 surfaces of lower Picard rank, and we consider double covers branched over the strict transform of a (reducible) sextic, given as the union of a nodal quartic $\mathcal{N}$ and a conic $\mathcal{C}$. Such a double sextic can be written in the form

$$(3.1) \quad S: \quad \tilde{y}^2 = C(u, v, w) \cdot Q(u, v, w),$$

where $C$ and $Q$ are homogeneous polynomials of degree 2 and 4 respectively, such that $\mathcal{C} = V(C)$ and $\mathcal{N} = V(Q)$ and $\tilde{y}$ has weight 3.

The double sextics to be studied in this section are of the form

$$(3.2) \quad S: \quad \tilde{y}^2 = (j_0 u^2 + v w + h_0 w^2)(c_2(u, v) w^2 + e_3(u, v) w + d_4(u, v)),$$

where $c_2, d_4, e_3$ are homogeneous polynomials of the degree indicated by the subscript and $h_0, j_0 \in \mathbb{C}$. Note that in the branch locus of $S$, the irreducible component $\mathcal{N} = V(Q)$ has the node $n: [u : v : w] = [0 : 0 : 1]$, and $c_2(u, v) = 0$ is the equation of the two tangents at $n$. The other irreducible component $\mathcal{C} = V(C)$ is obtained from a general conic $h_0 w^2 + k_1(u, v) w + j_2(u, v)$ by coordinate shifts of the form $w \mapsto w + \rho_1 u + \rho_2 v$ and $u \mapsto u + \rho_0 v$ which keep the node $n$ fixed. Moreover, for $h_0 \neq 0$ the conic is not coincident with the node.

The following shows that the minimal resolution $\mathcal{Z}$ of $S$ is a K3 surface whose Jacobian surface is $H \oplus N$-polarized.

**Proposition 3.1.** Let $c_2, e_3, d_4$ be general polynomials and $h_0, j_0 \in \mathbb{C}$ with $h_0j_0 \neq 0$ and $S$ the double sextic in Equation (3.2). The pencil of lines through the node $n \in \mathcal{N}$ induces an elliptic fibration without section. The relative Jacobian fibration is Equation (2.1) with

$$(3.3) \quad a = v e_3 - 2 j_0 u^2 c_2 - 2 h_0 d_4, \quad b = \frac{1}{4}a^2 - \frac{1}{4}(e_3^2 - 4 c_2 d_4) (v^2 - 4 h_0 j_0 u^2),$$

and has singular fibers $8I_2 + 8I_1$ and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$.

**Proof.** The pencil of lines through $n$ induces an elliptic fibration given by the projection onto $\mathbb{P}(u, v)$ in Equation (3.2). We then write Equation (3.2) in the form

$$(3.4) \quad \tilde{y}^2 = q_0 w^4 + q_1(u, v) w^3 + q_2(u, v) w^2 + q_3(u, v) w + q_4(u, v),$$

with $q_0 = c_2 h_0, q_1 = c_2 k_1 + e_3 h_0$, etc. The relative Jacobian of Equation (3.2) is

$$(3.5) \quad \eta^2 = \xi^3 + f(u, v) \xi + g(u, v),$$

where $f, g$ are given by

$$(3.6) \quad f = q_1 q_3 - 4 q_0 q_4 - \frac{1}{3}q_2^2, \quad g = q_0 q_2^2 + q_1^2 q_4 - \frac{8}{3} q_0 q_2 q_4 - \frac{1}{3} q_1 q_2 q_3 + \frac{2}{27} q_2^3.$$
and \((\xi, \eta) \in \mathbb{C}^2\) are affine coordinates on the elliptic fibers. One checks that Equation (3.5) can be brought into the form of Equation (2.6) with \(a, b\) given by Equation (3.3).

3.2. Projective quartic model. Given a double sextics model \(S\) as in Equation (3.1), one can derive a projective quartic hypersurface \(K \subset \mathbb{P}^3 = \mathbb{P}(u, v, w, y)\) given by

\[
K: \quad 0 = -y^2C(u, v, w) + Q(u, v, w),
\]

where \(C\) and \(Q\) are the same polynomials as in Equation (3.1). In this way, for every quartic surface \(K\) given by Equation (3.7) we find an associated double sextic \(S\) given by Equation (3.1) and vice versa.

In this section, we specifically consider the family of quartic surfaces

\[
K: \quad 0 = -(j_0u^2 + vw + h_0w^2)y^2 + c_2(u, v) w^2 + e_3(u, v) w + d_4(u, v),
\]

where \(c_2, d_4, e_3\) are general homogeneous polynomials of the degree indicated by the subscript and \(h_0, j_0 \in \mathbb{C}\). It has the associated double sextic \(S\) in Equation (3.2). The quartic hypersurface \(K\) is another birational model of the K3 surface \(Z\):

**Proposition 3.2.** The minimal resolution of \(K\) in Equation (3.8) is a smooth K3 surface if and only if Equation (2.1) with \(a, b\) in (3.3) defines a minimal Weierstrass equation. In particular, \(K\) has exactly 2 singularities at points \([u : v : w : y]\), given by

\[
p_1 = [0 : 0 : 0 : 1], \quad p_2 = [0 : 0 : 1 : 0],
\]

which are rational double points.

**Proof.** The double sextic in Equation (3.1) is birational to the quartic projective hypersurface in Equation (3.7) in \(\mathbb{P}^3 = \mathbb{P}(u, v, w, y)\). This can be seen by setting \(\tilde{y} = C(u, v, w) y\) in Equation (3.1). It follows from Proposition 3.1 that the pencil of lines through the node of the quartic induces an elliptic fibration on this double sextic whose relative Jacobian fibration is Equation (2.1). Since the degree of Equation (3.7) is four, the minimal resolution of \(K\) is a smooth K3 surface if and only if Equation (2.1) is a minimal Weierstrass equation. Hence, the singularities of \(K\) must be rational double points. One checks that the points in Equation (3.9) are the only singularities. \(\square\)

**Remark 3.3.** Given the existence of the singular point \(p_1\) on \(K\) where the \(y\) coordinate does not vanish, we may also see how the double sextic model \(S\) naturally arises. By considering the lines through \(p_1\) in \(\mathbb{P}^3\), we have a projection

\[
\mathbb{P}^3 - \{p_1\} \to \mathbb{P}^2.
\]

Restricting to \(K - p_1\), one checks that the image is the set of lines on \(K\) tangent to \(p_1\), which is a sextic in \(\mathbb{P}^2\). Moreover, one can check that this map is a double cover.

4. Construction of elliptic fibrations of families of Picard rank 11, 12 and 13

In this section, we consider certain special configurations for the curves \(C\) and \(N\) in the branch locus of the double sextics \(S\), which in turn correspond to specializations
of Equation (3.2). Due to Proposition 3.2, this also specializes the corresponding projective quartic models. Particular normal forms for the specialized projective quartic hypersurfaces will also be established; they will be denoted by \(Q \subset \mathbb{P}^3 = \mathbb{P}(X, Y, Z, W)\). The normal forms turn out to be generalizations of the famous Inose quartic \([15]\), given by

\[
0 = Y^2ZW - 4X^3Z + 3\alpha XYZ^2 + \beta ZW^3 - \frac{1}{2}(Z^2W^2 + W^4).
\]

We will prove that the minimal resolution of each surface \(\mathcal{S}, \mathcal{K}\) or \(\mathcal{Q}\) to be discussed is a K3 surface polarized by the lattices of rank 11, 12 or 13 in Table 1. From these models, we also compute explicit equations of pencils which induce the Jacobian elliptic fibrations supported on the families. To do so we generalize the methods that were used in \([8]\) and \([10]\), where the same was done for the families of Picard rank 14, 15 and 16 in Table 1.

4.1. **Family of Picard rank 11.** We first consider the case when the conic \(C\) in Equation (3.2) splits into 2 lines. After a suitable shift in the coordinates \(v\) and \(w\), the double sextic becomes

\[
\mathcal{S}: \quad y^2 = w(v + h_0w)(c_2(u, v)w^2 + c_3(u, v)w + d_4(u, v)).
\]

The branch locus in Equation (4.2) has three irreducible components: the quartic curve \(N = V(Q)\) with the node \(n = [0 : 0 : 1]\), and the lines \(\ell_1 = V(v + h_0w), \ell_2 = V(w)\) which are not coincident with \(n\) (for \(h_0 \neq 0\)) and satisfy \(\ell_1 \cap \ell_2 \cap N = \emptyset\).

The following gives the geometric constructions of the elliptic fibrations induced by the double sextics \(\mathcal{S}\).

**Proposition 4.1.** Let \(c_2, e_3, d_4\) be general polynomials and \(h_0 \in \mathbb{C}^\times\) and \(\mathcal{S}\) the double sextic in Equation (4.2). The following holds:

(i) the pencil of lines through the node \(n \in N\) induces the Jacobian elliptic fibration (2.1) with

\[
a = ve_3 - 2h_0d_4, \quad b = \frac{1}{4}a^2 - \frac{1}{4}v^2(e_3^2 - 4c_2d_4),
\]

and has singular fibers \(9I_2 + 6I_1\) and Mordell-Weil group \(\mathbb{Z}/2\mathbb{Z}\).

(ii) a pencil of lines through a point in \((\ell_1 \cap N) \cup (\ell_2 \cap N)\) induces a Jacobian elliptic fibration with singular fibers \(I_0^* + 5I_2 + 8I_1\) and a trivial Mordell-Weil group,

(iii) the pencil of lines through \(\ell_1 \cap \ell_2\) induces an elliptic fibration without section and singular fibers \(2I_0^* + I_2 + 10I_1\).

**Proof.** For (i) we follow the proof of Proposition 3.1; it is obvious that the pencil of lines now induces an elliptic fibration with two sections. We recover the normal form for the alternate fibration with singular fibers \(9I_2 + 6I_1\). For (ii) we construct a pencil through a point in \(\ell_2 \cap N\), the case \(\ell_1 \cap N\) will then be analogous. To do so, we set \(d_4(u, v) = (v_0u - u_0v)d_3(u, v)\) where \(d_3\) is a polynomial of degree 3 with \(d_3(u_0, v_0) \neq 0\). One point in \(N \cap \ell_2\) is \([u_0 : v_0 : 0]\). A pencil of lines through this intersection point is

\[0 = s(v_0u - u_0v) + tw\]
with \([s : t] \in \mathbb{P}^1\). Upon eliminating \(u\) in Equation (4.2) we obtain a Jacobian elliptic fibration over \(\mathbb{P}(s,t)\) with singular fibers \(I_0 + 5I_2 + 8I_1\). For (iii) we use the pencil \(0 = sv - tw\) and proceed as in (ii). However, we only obtain a genus-one fibration over \(\mathbb{P}(s,t)\). We compute the relative Jacobian elliptic fibration as in (i), confirming the types of the singular fibers.

We can verify that Equation (4.2) parametrises a birational model for every very general \(H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}\)-polarized K3 surfaces:

**Proposition 4.2.** A very general \(H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}\)-polarized K3 surface is birational to a double sextic \(S\) that is branched on a unimodal quartic \(\mathcal{N}\) and 2 lines \(\ell_1, \ell_2\) not coincident with the node and \(\ell_1 \cap \ell_2 \cap \mathcal{N} = \emptyset\). Conversely, every such double sextic is birational to an \(H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}\)-polarized K3 surface.

**Proof.** Due to Remark 2.3, a very general \(H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}\)-polarized K3 surface admits a unique Jacobian elliptic fibration (2.1) with singular fiber \(9I_2 + 6I_1\). Then, after a suitable change of coordinates, one has \(a(u, v)^2/4 - b(u, v) = v^2b'(u, v)\) where \(b'\) is a homogeneous polynomial of degree 6. Given the polynomials \(a, b'\), we choose a factorization of \(b(u, v) = a(u, v)^2/4 - v^2b'(u, v) = d_4(u, v)d_4'(u, v)\) into homogeneous polynomials of degree 4 with \(d_4(1,0) \neq 0\). We can then find \(h_0 \in \mathbb{C}^\times\) so that \(e_3 = (a + 2h_0d_4)/v\) and \(c_2 = (d_4'(u, v) + h_0^2d_4(u, v) + h_0a(u, v))/v^2\) are polynomials of degree 3 and 2 respectively.

In fact, if we write \(a = \sum_{n=0}^4 a_nu^{4-n}v^n, d_4 = \sum_{n=0}^4 \delta_nu^{4-n}v^n\) (with \(\delta_0 \neq 0\)), and \(d_4' = \sum_{n=0}^4 \delta'_nu^{4-n}v^n\), and if we have

\[
(4.4) \quad \delta'_0 = \frac{\alpha_0}{4\alpha_0}, \quad \delta'_1 = \frac{\alpha_0(2\alpha_1\delta_0 - \alpha_0\delta_1)}{4\delta_0^2}, \quad h_0 = \frac{\alpha_0}{2\delta_0},
\]

then \(h_0, c_2, e_3, d_4\) are a solution of Equation (4.3) for a given pair \(a, b'\) of polynomials. This determine a double sextic \(S\) via the birational transformation \(X = h_0d_4 + d_4v/w\).

The other direction follows from Proposition 4.1. \(\square\)

Now following the same argument as in Section 3.2, we may construct a family of quartic surfaces

\[
(4.5) \quad \mathcal{K}: 0 = -(v - \rho w)wy^2 + c_2(u, v)w^2 + e_3(u, v)w + d_4(u, v)
\]

from Equation (4.2), assuming the conditions of Proposition 3.2. In order to give explicit equations for the pencils of lines in Proposition 4.1 which induce elliptic fibrations of the K3 surface, we introduce a normal form \(\mathcal{Q}\) for the projective quartic hypersurface \(\mathcal{K}\). It is a multi-parameter generalizations of the Inose quartic in Equation (4.1). For Picard number 11, \(\mathcal{Q}\) is given by the equation

\[
(4.6) \quad \mathcal{Q}: 0 = 2Y^2Z(W - \rho Z) - c_2(2X, W)Z^2 - e_3(2X, W)Z - d_4(2X, W).
\]

One checks the following:

**Lemma 4.3.** Assume that the conditions of Proposition 3.2 are satisfied. The surface \(\mathcal{K}\) in Equation (4.5) is isomorphic to the surface \(\mathcal{Q}\) in Equation (4.6), by

\[
(4.7) \quad [u : v : w : y] \mapsto [2X : W : Z : \sqrt{2}Y].
\]
Remark 4.4. By substituting $h_0 = -\rho$ and the same choices of $c_2, e_3$ and $d_4$ made in Proposition 4.2 into Equation (4.5), it follows that any $H \oplus D_4(-1) \oplus A_1(-1)^8$-polarized K3 surface is birational to a quartic surface $\mathcal{K}$. Furthermore, Lemma 4.3 implies that $\mathcal{Q}$ is a birational model: this will be shown explicitly in the proof of Theorem 4.9.

Remark 4.5. In [6], the authors obtained a configuration of rational curves on a very general $H \oplus E_8(-1) \oplus E_8(-1)$-polarized K3 surface by resolving the singularities on its projective quartic model $\mathcal{Q}$. Here, one checks by explicit computation that on a general surface $\mathcal{Q}$, there are two singularities at points $[W : X : Y : Z]$ given by

\begin{equation}
(4.8) \quad P_1 = [0 : 0 : 1 : 0], \quad P_2 = [0 : 0 : 1].
\end{equation}

Moreover, $P_1$ and $P_2$ are rational double points of types $A_3$ and $A_1$, respectively. In Section 5.1, the curves appearing from resolving $P_1$ and $P_2$ will be denoted by $a_1, a_2, a_3$ and $b_1$ respectively.

Next, we introduce a set of complex coefficients $\{\gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, \omega\}$ such that the polynomials $c_2$ and $d_4$ in both Equations (4.5) and (4.6) are given by

\begin{equation}
(4.9) \quad c_2(u, v) = (\gamma u - \delta v)(\varepsilon u - \zeta v), \quad d_4(u, v) = (\eta u - \nu v)(\kappa u - \lambda v)(\mu u - \nu v)(\xi u - \omega v).
\end{equation}

With the set of coefficients, we can give explicit expressions for the vertices of the pencils of lines in Proposition 4.1. The intersection points of $\ell_2 = V(w)$ and the nodal quartic $\mathcal{N}$ are given by the roots of the polynomial $d_4$. As for the vertices in $l_1 \cap \mathcal{N}$, we need to introduce more coefficients. First note that for $\rho \neq 0$, the line $\ell_1 = V(v - \rho w)$ is not coincident with the node $n \in \mathcal{N}$. Moreover, we can assume $\gamma \epsilon \eta \kappa \mu \xi \neq 0$ in the general case. To make the points $\ell_1 \cap \mathcal{N}$ explicit, we set $w = (v + \bar{w})/\rho$ in Equation (4.5) and obtain an equivalent equation of the form

\begin{equation}
(4.10) \quad \mathcal{K}: \quad 0 = -\rho(v + \bar{w})\bar{w}y^2 + c_2(u, v)\bar{w}^2 + e_3'(u, v)\bar{w} + d_4'(u, v),
\end{equation}

where $\ell_1 = V(\bar{w})$ and

\begin{equation}
(4.11) \quad e_3'(u, v) = \rho e_3(u, v) + 2c_2(u, v)v, \quad d_4'(u, v) = \rho^2 d_4(u, v) + \rho e_3(u, v)v + c_2(u, v)v^2.
\end{equation}

The roots of $d_4'$ determine the intersection points of $\ell_1 \cap \mathcal{N}$. So, by writing

\begin{equation}
(4.12) \quad d_4'(u, v) = (\eta' u - \nu' v)(\kappa' u - \lambda' v)(\mu' u - \nu' v)(\xi' u - \omega' v),
\end{equation}

we have the coordinates for the points in $\ell_1 \cap \mathcal{N}$.

Note that the expanded coefficient set $(\gamma, \delta, \varepsilon, \zeta, \eta, \ldots, \nu, \omega')$ also determines the polynomials $\rho, e_3, e_3'$ in Equations (4.5), (4.6) and (4.10), using the equations

\begin{equation}
(4.13) \quad \rho^2 = \frac{\eta' \kappa' \mu' \xi'}{\eta \kappa \mu \xi}, \quad e_3(u, v) = \frac{d_4(u, v) - \rho^2 d_4(u, v) - c_2(u, v)v^2}{\rho v}, \quad e_3'(u, v) = \frac{d_4'(u, v) - \rho^2 d_4(u, v) + c_2(u, v)v^2}{v}.
\end{equation}

Remark 4.6. We may also express the Weierstrass model $X$ (2.1) in terms of these complex coefficients in Equation (4.2). By substitution of Equation (4.3), the polynomials $a$ and $b$ in the Jacobian elliptic fibration (2.1) are given by

\begin{equation}
(4.14) \quad a(u, v) = \frac{d_4(u, v) + \rho^2 d_4(u, v) - c_2(u, v)v^2}{\rho}, \quad b(u, v) = d_4(u, v)d_4'(u, v).
\end{equation}
Conversely, the proof of Proposition 4.2 shows how the Weierstrass model $X$ determines the quartic surface $K$.

**Remark 4.7.** Different sign choices $\pm \rho$ in Equation (4.13) (resulting in sign changes $\pm \alpha$ and $\pm \alpha$ in Equation (4.14)) yield isomorphic surfaces in Equation (4.5) (resp. in Equation (2.1)) related by $[u : v : w : y] \mapsto [u : v : -w : iy]$ (resp. $[X : Y : Z] \mapsto [-X : iY : Z]$).

**Remark 4.8.** Assume that the conditions of Proposition 3.2 are satisfied. The van Geemen-Sarti involution (2.5) in Theorem 5.1, after composition with the hyperelliptic involution, induces a Nikulin involution on the minimal resolution of $K$. The Nikulin involution is induced by the projective automorphism $\mathbb{P}^3 \to \mathbb{P}^3$ given by

$$
(4.15) \quad [u : v : w : y] \mapsto [d(u, v, w)u : d(u, v, w)v : -d(u, v)(v - \rho w) : d(u, v, w)y]
$$

with $d(u, v, w) = \rho d_4(u, v) + \rho w d_3(u, v) + w v d_2(u, v)$.

Let us resume the computation of explicit expressions for the pencils of lines in Proposition 4.1. For $Q$ in Equation (4.6), we now introduce the lines $L_4, L_5, L_6, L_7$ on $Q$ such that each line passes through a different intersection point in $l_2 \cap N$, as well as the lines $L_4', L_5', L_6', L_7'$ on $Q$ such that each line passes through a different intersection point in $l_2 \cap N$. To be specific, we define the lines $L_4$ and $L_4'$ by the equations

$$
(4.16) \quad L_4: \quad Z = 2\eta X - iW = 0, \quad L_4': \quad W - \rho Z = 2\eta X - iW = 0.
$$

From the equation for $L_4$, we define equations for $L_5, L_6, L_7$ by replacing the parameters $(\eta, i)$ by $(\kappa, \lambda), (\mu, \nu),$ and $(\xi, o)$ respectively. Similarly, from the equation of $L_4'$, equations for $L_5', L_6', L_7'$ can be defined by replacing the parameters $(\eta', i')$ by $(\kappa', \lambda'), (\mu', \nu'),$ and $(\xi', o')$ respectively. For general parameters, the lines are distinct and concurrent, meeting at $P_1$. Each line induces a pencil; they are denoted by $L_i(u,v)$ and $L_i'(u,v)$ for $i = 4, 5, 6, 7$. For example, we set $L_4(u,v) = uZ - v(2\eta X - iZ)$. Combined with Proposition 4.1, we then have the following geometric constructions of Jacobian elliptic fibrations:

**Theorem 4.9.** Assume that the conditions of Proposition 3.2 are satisfied and let $L = H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$. The minimal resolution of $Q$ in Equation (4.6) is a $K3$ surface endowed with a canonical $L$-polarization. Conversely, an $L$-polarized $K3$ surface has a birational projective model (4.6). In particular, Jacobian elliptic fibrations of the type determined in Theorem 2.1 are attained as follows:

| # | singular fibers | MW | reducible fibers | pencil |
|---|---|---|---|---|
| 1 | $9I_2 + 6I_1$ | $\mathbb{Z}/2\mathbb{Z}$ | $\tilde{A}_1^{\oplus 9}$ | family of quartic curves through $P_1$ and $P_2$, intersection of $2vX + uW = 0$ and $Q$ |
| 2 | $6I_2 + 5I_1$ | $\{1\}$ | $\tilde{D}_4 + \tilde{A}_1^{\oplus 5}$ | residual surface intersection of $L_i(u,v) = 0$ or $L_i'(u,v) = 0$ ($i = 4, 5, 6, 7$) and $Q$ |

**Proof.** To obtain the fibration (1) we make the substitution

$$
(4.17) \quad W = 2v^2X(X + \rho d_4(u,v)Z), \quad X = uvX(X + \rho d_4(u,v)Z), \quad Y = \sqrt{2}Y(X + \rho d_4(u,v)Z), \quad Z = 2v^2d_4(u,v)XZ,
$$

$$
\tilde{d}_4(u, v, w) = \rho d_4(u, v) + \rho w d_3(u, v) + w v d_2(u, v).
$$
in Equation (4.6). This determines the Jacobian elliptic fibration (2.1) where $a, b$ are
given by Equation (4.3) with $h_0 = -\rho$. Fibration (1) is induced by a family of quartic
curves through $P_1$ and $P_2$, which is obtained as the intersection of then pencil of
hyperplanes $2vX + uW = 0$ and $Q$. In general, $X = W = 0$ does not define a line on $Q$
if $\rho \neq 0$.

The fibrations induced by the pencils $L_n(u, v) = 0$ and $L_n^0(u, v) = 0$ for $n = 4, 5, 6, 7$
are all found in the same way. For example, if one substitutes $L_4(u, v) = 0$ into
Equation (4.3), one obtains a genus-one fibration. The fibration has a section since
$2\eta X - \nu W = 0$ is a root of the polynomial $d_4(2X, W)$. By bringing the equation into
Weierstrass normal form one obtains fibration (2). The fiber of type $I_0^*$ is located
over $u = 0$, one fiber of type $I_2$ over $v = 0$, and the remaining fibers of type $I_2$
over
\begin{equation}
\frac{u}{v} = \frac{\rho(\eta' - \nu'\mu')}{\eta'}, \frac{\rho(\eta' - \nu'\kappa')}{\kappa'}, \frac{\rho(\eta' - \nu'\xi')}{\xi'}.
\end{equation}

Similar results can be obtained for the pencils $L_i(u, v)$ for $i = 5, 6, 7$. Using Lemma 4.3
and Equation (4.10) as an equivalent way of writing Equation (4.5), it follows that
the same results hold if we interchange the lines $L_n$ and $L_n^0$ for $n = 4, 5, 6, 7$.

The converse is given by Proposition 4.2 and Remark 4.4. Alternatively, upon solving Equations (4.17) for $u, v, X, Y$ and plugging the result into Equation (2.1), the
proper transform is a quartic surface $Q$ which given by an equation of the form (4.6).
\hfill \Box

4.2. Family of Picard rank 12. We continue to specialize the double sextic model
(4.2) to obtain a geometric construction of elliptic fibrations of a very general $H \oplus
D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surfaces. We will follow the structure and organiza-
tion of Section 4.1, skipping over similar proofs and computations.

We consider the case when one of the lines in Equation (4.2) becomes coincident with
the node by setting $h_0 = 0$, i.e.,
\begin{equation}
\mathcal{S} : \tilde{y}^2 = vw(c_2(u, v) w^2 + c_3(u, v) w + d_4(u, v)).
\end{equation}
The branch locus in Equation (4.19) has three irreducible components: the quartic
curve $N = V(Q)$ with the node $n = [0:0:1]$ and the lines $\ell_1 = V(v), \ell_2 = V(w)$ such
that $n \in \ell_1 \cap N$ and $\ell_1 \cap \ell_2 \cap N = \emptyset$. The following proposition describes the elliptic
fibrations induced by $\mathcal{S}$.

Proposition 4.10. Let $c_2, c_3, d_4$ general polynomials and $\mathcal{S}$ the double sextic in Equation
(4.19). The following holds:

(i) the pencil of lines through the node $n \in N$ induces the Jacobian elliptic fibra-
tion (2.1) with
\begin{equation}
a = ve_3, \quad b = v^2 e_4,
\end{equation}
and has singular fibers $I_0^* + 6I_2 + 6I_1$ and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$,

(ii) a pencil of lines through a point in $\ell_2 \cap N$ induces a Jacobian elliptic fibra-
tion with singular fibers $2I_2^* + 2I_2 + 8I_1$ and a trivial Mordell-Weil group,

(iii) a pencil of lines through a point in $\ell_1 \cap N - \{n\}$ induces a Jacobian elliptic
fibration with singular fibers $I_2^* + 4I_2 + 8I_1$ and a trivial Mordell-Weil group,
(iv) the pencil of lines through $\ell_1 \cap \ell_2$ induces an elliptic fibration without section and singular fibers $I_2^* + I_0^* + 10I_1$.

**Proof.** The proof is analogous to the proof of Proposition 4.1. □

It can be proved that Equation (4.19) parametrises a birational model for every very general $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface

**Proposition 4.11.** A very general $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface is birational to a double sextic $S$ that is branched on a uninodal quartic $N$ (with node $n$) and 2 lines $\ell_1, \ell_2$ so that $u \in \ell_1 \cap N$ and $\ell_1 \cap \ell_2 \cap N = \emptyset$. Conversely, every such double sextic is birational to an $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface.

**Proof.** Due to Remark 2.3, a very general $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface admits a unique Jacobian elliptic fibration (2.1) with a fiber of type $I_0^*$. We have $a(u, v) = ve_3(u, v)$, and we may group the base points of the six fibers of type $I_2$ over $b = 0$ into sets of 2 and 4 elements by writing $b(u, v) = v^2c_2(u, v)d_4(u, v)$ where $c_2, e_3, d_4$ are homogenous polynomials of degree 2, 3, and 4, respectively. We obtain the double sextic given by Equation (4.19) by blowing up in Equation (2.1) via $[x : y] = [vc_2(u, v)\tilde{x} : \tilde{y}]$ with $[(u, v, x, z)] \in \mathbb{F}_4$ and $[(u, v, \tilde{x}, \tilde{z})] \in \mathbb{F}_1$ (using the notation of Equation (2.7)) and identifying $\mathbb{F}_1 \cong \mathbb{P}(u, v, w)$ via $w = \tilde{x}/\tilde{z}$ for $\tilde{z} \neq 0$. One checks that $\ell_1 = V(v)$ is coincident with the node whereas $\ell_2 = V(w)$ is not. The other direction follows from Proposition 4.10. □

From the family $S$, we construct a family of quartic surfaces

$$K: \quad 0 = -vwz^2 + c_2(u, v)w^2 + e_3(u, v)w + d_4(u, v).$$

Again, we assume the conditions of Proposition 3.2. It can be brought to the form of a generalized Inose quartic $Q$ as in Equation (4.1) with $\rho = 0$. By Lemma 4.3, $K$ and $Q$ are isomorphic quartic surfaces. Combined with Propositions 3.2 and 4.11, we can see that both $K$ and $Q$ are birational models of a very general $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface.

By introducing a set of complex coefficients for Equation 4.21, we may obtain explicit expressions for the pencils which induce the Jacobian elliptic fibrations for a very general $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$-polarized K3 surface. By assumption, $K$ is very general so we may assume that the leading coefficients of $c_2, e_3$ and $d_4$ in Equation (4.21) do not vanish. In particular, we can write

$$c_2(u, v) = (\gamma u - \delta v)(\epsilon u - \zeta v), \quad e_3(u, v) = u^3 - 3\alpha uv^2 - 2\beta v^3;$$

$$d_4(u, v) = (\eta u - \nu v)(\kappa u - \lambda v)(\mu u - \nu v)(\xi u - \omega v),$$

where the coefficients $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, \omega) \in \mathbb{C}^{14}$ satisfies $\gamma \epsilon \eta \kappa \mu \xi \neq 0$.

**Remark 4.12.** Similar to Remark 4.6, one can obtain the polynomials $a$ and $b$ in the Weierstrass model $X$ (2.1) from $K$ (4.21) using Equation (4.20).

The converse also holds true, as in the proof of Proposition 4.11. Explicitly, if one substitutes Equation (4.22) into

$$a(u, v) := \sum_{i=0}^{4} a_iv^i = ve_3(u, v), \quad b(u, v) := \sum_{j=0}^{8} b_jv^j = v^2c_2(u, v)d_4(u, v),$$

respectively.
then the coefficients $a_i$ and $b_j$ are as follows:

\[
\begin{align*}
& a_0 = 0, \quad a_1 = 1, \quad a_2 = 0, \quad a_3 = -3\alpha, \quad a_4 = -2\beta, \\
& b_0 = b_1 = 0, \quad b_2 = \gamma\varepsilon\kappa\mu\xi, \quad b_j = (1 - 1)^{2}\gamma\varepsilon\kappa\mu\xi \cdot \sigma^{(6)}_{j-2} \left( \delta, \ldots, \frac{\alpha}{\xi} \right), \quad b_8 = \delta\zeta\iota\lambda\nuo.
\end{align*}
\]

Here, $\sigma^{(6)}_{k-2}$ for $k = 2, \ldots, 8$ are the elementary symmetric polynomials in 6 variables of degree $k - 2$. Therefore, polynomials $c_2, e_3$ and $d_4$ in Equation (4.22) can be expressed in terms of the $a_i$ and $b_j$.

**Remark 4.13.** Suppose $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, \omega) \in \mathbb{C}^{14}$ is the coefficient set considered in Remark 4.12. Since

\[
b(u, v) = v^2(\gamma u - \delta v)(\varepsilon u - \zeta v)(\eta u - \iota v)(\kappa u - \lambda v)(\mu u - \nu v)(\xi u - \omega v),
\]

the vanishing order $N_{1-r}(b) \in \{0, 1, \ldots, 6\}$ of $b(u, v)$ at $[u : v] = [1 : -r]$ for $r \neq 0$ equals the number of parameters $(u, v) \neq 0$ with

\[
[u : v] \in \left\{ \left[ \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, \omega \right] \right\},
\]

such that $[u : v] = [1 : -r]$. If we have $b(u, v) \neq 0, \frac{1}{4}a(u, v)^2$, and if $r \in \mathbb{C}$ with $\alpha = r^2, \beta = r^3, N_{1-r}(b) \geq 4$, then it can be checked by explicit computation that the rational double points $P_1$ and $P_2$ in Remark 4.5 are both of types $A_3$.

**Remark 4.14.** Let $K$ and $\widetilde{K}$ be two quartic surfaces satisfying the conditions of Remark 4.13. By Proposition 4.10, their associated double sextics admit a Jacobian elliptic fibration (2.1) with polynomials $a, b$ and $\bar{a}, \bar{b}$ respectively. Upon bringing these polynomials into the standard form, it follows that the $K3$ surfaces obtained as minimal resolutions of $K$ and $\widetilde{K}$ are isomorphic if and only if

\[
\Lambda^2a(u, v) = \bar{a}(u, \Lambda^2v), \quad \Lambda^4b(u, v) = \bar{b}(u, \Lambda^2v),
\]

for some $\Lambda \in \mathbb{C}^\times$. In other words, the coefficients of $\bar{a}, \bar{b}$ are obtained from those of $a, b$ by the following:

(a) pairwise interchanges of the elements in

\[
\left\{ \left( \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, \omega \right) \right\},
\]

(b) rescalings of the coefficients according to

\[
(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota, \kappa, \lambda, \mu, \nu, \xi, \omega) \mapsto (\Lambda^4\alpha, \Lambda^6\beta, \Lambda^{10}\gamma, \Lambda^{12}\delta, \Lambda^{-2}\varepsilon, \Lambda^{-2}\zeta, \Lambda^{-2}\eta, \Lambda^{-2}\iota, \Lambda^{-2}\kappa, \Lambda^{-2}\mu, \Lambda^{-2}\nu, \Lambda^{-2}\xi, \omega),
\]

for some $\Lambda \in \mathbb{C}^\times$.

**Remark 4.15.** By setting $\rho = 0$ in Equation (4.15), we find a projective automorphism $\mathbb{P}^3 \to \mathbb{P}^3$, given by

\[
[u : v : w : y] \mapsto \left[ c_2(u, v) uw : c_2(u, v) vw : -d_4(u, v) : c_2(u, v) wy \right],
\]

which induces a Nikulin involution coming from the van Geemen Sarti involution (2.5) on the minimal resolution of $K$. 

Recall that the points \( l_2 \cap N \) mentioned in Proposition 4.10 have already been found in Section 4.1. Again, they give rise to the pencils of lines \( L_i \) for \( i = 4, \ldots, 7 \) given in Equation (4.16). For \( Q \) now given by
\[
\begin{align*}
\mathcal{Q}: \quad 0 &= 2Y^2ZW - 8X^3Z + 6\alpha XZW^2 + 2\beta ZW^3 - (2\gamma X - \delta W)(2\epsilon X - \zeta W)Z^2 \\
&\quad - (2\eta X - \iota W)(2\kappa X - \lambda W)(2\mu X - \nu W)(2\xi X - \omega W),
\end{align*}
\]
we find additional lines denoted by \( L_1, L_2^* \):
\[
L_1: \quad W = X = 0, \quad L_2^*: \quad W = (1 \pm \chi)X + \gamma\epsilon Z = 0,
\]
where the new parameter \( \chi \) satisfies \( 4\gamma\epsilon\eta\kappa\mu\xi = 1 - \chi^2 \). The lines \( L_1, L_2^*, L_4, L_5, L_6, L_7 \) lie on the quartic surface \( Q \) in Equation (4.28). For general parameters, the lines are distinct and meet at \( P_1 \). The line \( L_1 \) is also coincident with \( P_2 \). Again, we denote by \( L_1(u, v) \) and \( L_2^*(u, v) \) the pencils of lines associated to the lines \( L_1 \) and \( L_2^* \). Note that the above pencils correspond to the pencils mentioned in Proposition 4.10 with vertices being \( n \in N \) and the points in \( l_1 \cap N - \{n\} \) respectively. We then have the following geometric construction of the Jacobian elliptic fibrations:

**Theorem 4.16.** Assume that the conditions of Remark 4.13 are satisfied, and let \( L = H \oplus D_0(-1) \oplus A_1(-1)^6 \). The minimal resolution of \( Q \) in Equation (4.28) is a K3 surface endowed with a canonical L-polarization. Conversely, a very general L-polarized K3 surface has a birational projective model (4.28). In particular, Jacobian elliptic fibrations of the type determined in Theorem 2.1 are attained as follows:

| # | singular fibers | MW | reducible fibers | pencil |
|---|----------------|-----|-----------------|--------|
| 1 | \( I^*_0 + 6I_2 + 6I_1 \) | \( \mathbb{Z}/2\mathbb{Z} \) | \( \overline{D}_4 + A_1^{\oplus 6} \) | residual surface intersection of \( L_1(u, v) = 0 \) and \( Q \) |
| 2 | \( 2I^*_0 + 2I_2 + 8I_1 \) | \( \{1\} \) | \( \overline{D}_4 + \overline{D}_4 + A_1^{\oplus 2} \) | residual surface intersection of \( L_1(u, v) = 0 \) \( (i = 4, 5, 6, 7) \) and \( Q \) |
| 3 | \( I^*_2 + 4I_2 + 8I_1 \) | \( \{1\} \) | \( \overline{D}_6 + A_1^{\oplus 4} \) | residual surface intersection of \( L_2^*(u, v) = 0 \) and \( Q \) |

**Proof.** The proof is analogous to the proof of Theorem 4.9. To obtain the fibration (1) we make the substitution
\[
W = 2v^2x, \quad X = uwx, \quad Y = \sqrt{2}y, \quad Z = 2v^2(\eta u - \nu v)(\kappa u - \lambda v)(\mu u - \nu v)(\xi u - \omega v)z,
\]
in Equation (4.28), compatible with the pencil \( L_1(u, v) = 0 \). This determines the Jacobian elliptic fibration (2.1) where the polynomials \( a, b \) are as in Remark 4.12. The other two fibrations are found in a similar way as in the proof of Theorem 4.9. If one substitutes \( L_n(u, v) = 0 \) for \( n = 4, 5, 6, 7 \) (resp. \( L_2^*(u, v) \)) into Equation (4.20), then one obtains a genus-one fibration with section. By bringing the equation into Weierstrass normal form one obtains Jacobian elliptic fibration (2) (resp. (3)). The corresponding equations for the other pencils are then generated through the symmetries in Remark 4.14.

The converse direction is given by Proposition 4.11 and Remark 4.4. Alternatively, upon solving Equations (4.30) for \( u, v, X, Y \) and plugging the result into Equation (2.1), the proper transform is a quartic surface \( Q \) with an equation of the form (4.28). \( \Box \)
4.3. **Family of Picard rank 13.** By restricting the configuration of the lines \(l_1, l_2\) and the uninodal quartic \(\mathcal{N}\), we can further specialize the family of \(H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}\)-polarized K3 surfaces to a family of Picard rank 13:

**Proposition 4.17.** An \(H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}\)-polarization extends to an \(H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 3}\)-polarization if and only if for the corresponding double sextic \(S\) in Equation (4.19) either \(\ell_1\) is tangent to \(n\) or \(\ell_1 \cap \ell_2 \cap \mathcal{N} \neq \emptyset\).

**Proof.** As in the proof of Proposition 4.11, we plug Equation (4.20) into Equation (2.1). The fiber of type \(I_{\ast}^{\ast} 0\) extends to a fiber of type \(I_{\ast}^{\ast} 2\) if and only if \(b(u, v) = v^3b'(u, v)\). For the double sextic in Equation (4.19) we then have either \(c_2(u, v) = vc_1(u, v)\) or \(d_4(u, v) = vd_3(u, v)\) where \(c_1\) and \(d_3\) have degree 1 and 3, respectively. In the first case, the line \(V(v)\) in the branch locus is tangent to the node of the quartic. In the second case, the two lines in the branch locus, i.e., \(V(v)\) and \(V(w)\), which intersect at \([1:0:0]\), now intersect on the quartic. \(\square\)

Note that the pencils of lines described in Proposition 4.10 (i)-(iii) still induce Jacobian elliptic fibrations with the same types of singular fibers and the same Mordell-Weil group. The fibration in Proposition 4.10(iv) extends in the situation of Proposition 4.17 as follows: when \(\ell_1\) is tangent to \(n\), the fibration without section has singular fibers \(III^{\ast} + I_{\ast}^{\ast} 9 + I_{\ast}^{\ast} 1\). For \(\ell_1 \cap \ell_2 \cap \mathcal{N} \neq \emptyset\) the fibration has a section and is a standard fibration.

In terms of the generalized Inose quartic model (4.28), the specialization corresponds to the situation when \((\xi, o) = (0, 1)\), or equivalently when \(\chi^2 = 1\). In this case, the lines \(L_2^{\pm}\) become the lines

\[
(4.31) \quad L_2: \quad W = Z = 0, \quad \bar{L}_2: \quad W = 2X + \gamma Z = 0.
\]

Note that \(L_7\) also coincides with the line \(L_2\). Again, we denote the corresponding pencils on the quartic surface \(Q\) by \(L_2(u, v)\) and \(\bar{L}_2(u, v)\).

**Remark 4.18.** When \((\xi, o) = (0, 1)\), the singularities \(P_1\) and \(P_2\) in Remark 4.13 are of types \(A_5\) and \(A_3\) respectively.

We have the following:

**Theorem 4.19.** Assume that the conditions of Remark 4.13 are satisfied with \((\xi, o) = (0, 1)\), and let \(L = H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 3}\). The minimal resolution of \(Q\) in Equation (4.28) is a K3 surface endowed with a canonical \(L\)-polarization. Conversely, an \(L\)-polarized K3 surface has a birational projective model (4.28) with \((\xi, o) = (0, 1)\). In particular, Jacobian elliptic fibrations of the type determined in Theorem 2.1 are attained as follows:
\begin{tabular}{|c|c|c|c|c|}
\hline
\# & singular fibers & MW & reducible fibers & pencil \\
\hline
1 & $I_2^* + 5I_2 + 6I_1$ & $\mathbb{Z}/2\mathbb{Z}$ & $\bar{D}_6 + \bar{A}_1^{65}$ & residual surface intersection of $L_1(u,v) = 0$ and $Q$ \\
\hline
2 & $I_0^* + I_2^* + I_2 + 8I_1$ & $\{1\}$ & $\bar{D}_4 + \bar{D}_6 + \bar{A}_1$ & residual surface intersection of $L_i(u,v) = 0$ ($i = 2, 4, 5, 6$) and $Q$ \\
\hline
3 & $I_4^* + 3I_2 + 8I_1$ & $\{1\}$ & $\bar{D}_8 + \bar{A}_1^{83}$ & residual surface intersection of $L_2(u,v) = 0$ and $Q$ \\
\hline
4 & $III^* + 4I_2 + 7I_1$ & $\{1\}$ & $\bar{E}_7 + \bar{A}_1^{94}$ & intersection of quadric surfaces $C_i(u,v) = 0$ ($i = 4, 5, 6$) and $Q$ \\
\hline
\end{tabular}

Proof. The statements concerning fibrations (1)-(3) can be proved in a similar way as in Theorem 4.16 after setting $(\xi, o) = (0, 1)$.

Pencils of quadratic surfaces, denoted by $C_n(u,v) = 0$ with $[u:v] \in \mathbb{P}^1$ for $n = 4, 5, 6$, are constructed as follows. We set

\begin{equation}
C_4(u,v) = \mu(\kappa u - \lambda v)\left(2\mu X - \nu W\right)\left(2\kappa X - \lambda W + \gamma \varepsilon \kappa Z\right)
- \kappa(\mu u - \nu v)\left(2\kappa X - \lambda W\right)(2\mu X - \nu W + \gamma \varepsilon \mu Z),
\end{equation}

which is invariant under the permutation of parameters $(\gamma, \delta)$ with $(\varepsilon, \zeta)$. The pencils $C_5$ and $C_6$ can be obtained by replacing the parameters $((\kappa, \lambda), (\mu, \nu))$ by $((\eta, \iota), (\mu, \nu))$ or of $((\eta, \iota), (\kappa, \lambda))$. By making the substitutions

\begin{align}
W &= 2\gamma^2 \varepsilon^2 \kappa^2 \mu^2 v^2(\gamma u - \delta v)(\varepsilon u - \zeta v)xz, \\
X &= \gamma^2 \varepsilon^2 \kappa^2 \mu^2 v(\gamma u - \delta v)(\varepsilon u - \zeta v)q_1(x,z,u,v)z, \\
Y &= \sqrt{2} \gamma \kappa \mu(\gamma u - \delta v)(\varepsilon u - \zeta v)yz, \\
Z &= 2q_2(x,z,u,v)q_3(x,z,u,v),
\end{align}

in Equation (4.28) which is compatible with $C_4(u,v) = 0$, and by using the polynomials

\begin{align}
q_1(x,z,u,v) &= ux - \gamma \varepsilon \kappa \mu v(\gamma u - \delta v)(\varepsilon u - \zeta v)(\kappa u - \lambda v)(\mu u - \nu v)z, \\
q_2(x,z,u,v) &= x - \gamma \varepsilon \kappa^2 \mu^2 v(\gamma u - \delta v)(\varepsilon u - \zeta v)(\mu u - \nu v)z, \\
q_3(x,z,u,v) &= x - \gamma \varepsilon \kappa \mu^2 v(\gamma u - \delta v)(\varepsilon u - \zeta v)(\kappa u - \lambda v)z,
\end{align}

one obtains the Jacobian elliptic fibration (4). Similarly, Jacobian elliptic fibrations with the same singular fibers are obtained from $C_5(u,v)$ and $C_6(u,v)$.

5. Dual graphs of rational curves

Following Kondo [18], we define the dual graph of smooth rational curves to be the multigraph whose set of vertices is the set of all smooth rational curves on a K3 surface such that the vertices $\Sigma, \Sigma'$ are joint by an $m$-fold edge if and only if their intersection product satisfies $\Sigma \circ \Sigma' = m$. The following fact is well known; see [25, 28]:

Theorem 5.1. A very general $H \oplus N$-polarized K3 surface $X$ satisfies the following:

(1) the automorphism group of $X$ is finite, i.e., $|\text{Aut}(X)| < \infty$.  

(2) $\mathcal{X}$ has exactly 18 smooth rational curves and the dual graph of rational curves is given by Figure 1, and the van Geemen-Sarti involution (2.5) acts as horizontal flip.

We will now construct the dual graphs of all smooth rational curves for the K3 surfaces of Theorems 4.9, 4.16, and 4.19. We will also provide the embeddings of the reducible fibers appearing in the tables of the theorems into the dual graphs. With each dual graphs, we will also visualise the Nikulin involution that is induced by the van Geemen-Sarti involution in Theorem 5.1. In these figures, the components of the reducible fibers, given by extended Dynkin diagrams, are distinguished by the same colors that were used in the aforementioned theorems, and the classes of the section (and the 2-torsions section if applicable) are represented by yellow nodes.

By the adjunction formula, an irreducible curve on a K3 surface is a smooth rational curve if and only if its self-intersection number is $-2$. As we will show, the Inose-type quartic normal form derived in Section 4 allows for the simple construction of rational curves as complete intersection curves. These curves turn out to give all rational curves on the corresponding K3 surfaces. The last statement will follow from a comparison with work by Roulleau in [25]: there, the number of all rational curves and their divisor classes were determined by a lattice theoretic method implementing an algorithm due to Vinberg.

Remark 5.2. Note that in the forthcoming figures in this section which feature the rational curves on a K3 surface in this section, some intersections of the curves are not shown. However, the intersection numbers left out can be easily computed using Equations (5.6), (5.9) and (5.12).

5.1. Family of Picard rank 11. We determine the dual graph of smooth rational curves for the K3 surfaces $\mathcal{X}$ with Néron-Severi lattice $H \oplus D_4(-1) \oplus A_1(-1)^{\oplus 5}$ in Theorem 4.9. In this case, one has the following 90 rational curves:

\begin{align*}
(5.1) \quad & a_1, a_2, a_3, b_1, R_0, R_{13}, L_n, R_n, L'_n, R'_{n'}, \\
(5.2) \quad & S_{n,n'}, \tilde{S}_{n,n'}, T_{mn,m'n'}, \text{ for } m, n, m', n' \in \{4, 5, 6, 7\} \text{ with } m < n, m' < n'.
\end{align*}

The lines $L_4, \ldots, L_7$ were already determined in Section 4.1. The two sets $\{a_1, a_2, a_3\}$ and $\{b_1\}$ were mentioned in Remark 4.5.
Lastly, the curves $R_\bullet, R'_\bullet, S_\bullet, \tilde{S}_\bullet, T_\bullet$ are obtained from singular complete intersection curves of higher (arithmetic) genus on the quartic surface $Q$. When resolving the quartic surface (4.6), these curves lift to smooth rational curves on the K3 surface, which by a slight abuse of notation we shall denote by the same symbol. In the following we give these singular complete intersections curves.

\[
R_0: \begin{cases}
0 &= W \\
0 &= 2\rho Y^2Z^2 + 4X^2(c_2(1, 0)Z^2 + 2c_3(1, 0)XZ + 4d_4(1, 0)X^2),
\end{cases}
\]
\[
R_{13}: \begin{cases}
0 &= WZc_2(2X, W) + \rho Zc_3(2X, W) + \rho d_4(2X, W) \\
0 &= 2\rho Y^2 + c_2(2X, W),
\end{cases}
\]
\[
(5.3)
\]
\[
R_4: \begin{cases}
0 &= 2\eta X - \iota W \\
0 &= 2\eta^3(W - \rho Z)Y^2 - \eta c_2(\iota, \eta)W^2Z - e_3(\iota, \eta)W^3,
\end{cases}
\]
\[
R'_4: \begin{cases}
0 &= 2\eta^4\rho ZY^2 + (\eta')^2(\iota, \eta')^2\rho Y + \rho d_4(\iota', \eta')W^3.
\end{cases}
\]

Equations for $R_5, R_6, \text{and } R_7$ are obtained from $R_4$ by replacing $(\eta, \iota)$ by $(\kappa, \lambda), (\mu, \nu),$ and $(\xi, \omega)$, respectively. Similarly, the equations for $R'_5, R'_6, \text{and } R'_7$ are obtained from $R'_4$ by replacing $(\eta', \iota')$ by $(\kappa', \lambda'), (\mu', \nu'), \text{and } (\xi', \omega')$, respectively.

We also have the following complete intersection curve:

\[
S_{4,4}: \begin{cases}
0 &= \eta'\big(W - \rho Z\big) - \eta\big(2\eta'X - \rho'Z\big) \\
0 &= 2\rho\eta'Y^2 + \frac{(\eta')^2(2\eta X - \iota W)c_2(2X, W)}{2\eta X - \iota W} + \frac{\rho'((\eta' - \iota')\eta)c_3(2X, W)}{2\eta X - \iota W} + \frac{\rho^2(\eta' - \iota')^2d_4(2X, W)}{(2\eta X - \iota W)(2\eta X - \iota' W)},
\end{cases}
\]

where the second equation is a homogeneous polynomial of degree three. Equations for $S_{5,4}, S_{6,4}, \text{and } S_{7,4}$ are obtained from $S_{4,4}$ by replacing $(\eta, \iota)$ by $(\kappa, \lambda), (\mu, \nu),$ and $(\xi, \omega)$, respectively. The equations for $S_{n,5}, S_{n,6}, \text{and } S_{n,7}$ are obtained from $S_{n,4}$ for $n = 4, 5, 6, 7$ by replacing $(\eta', \iota')$ by $(\kappa', \lambda'), (\mu', \nu'), \text{and } (\xi', \omega')$, respectively. This procedure generates the 16 rational curves $S_{4,4}, S_{5,4}, \ldots, S_{7,4}, S_{7,7}$. Applying the Nikulin involution in Equation (4.8), we obtain from $S_{4,4}$ the complete intersection curve:

\[
\tilde{S}_{4,4}: \begin{cases}
0 &= \frac{1}{2\eta'X - \iota' W}\big(\eta'WZc_2(2X, W) + \rho'Zc_3(2X, W) + \rho(\eta'W - \rho Z)J_{\iota'W}(2X, W) + \rho\rho'ZJ_{\iota'W}(2X, W) + \rho^2J_{\iota'W}(2X, W)\big) \\
0 &= 2\rho\eta'Y^2 + \frac{(\eta')^2(2\eta X - \iota W)c_2(2X, W)}{2\eta X - \iota W} + \frac{\rho'((\eta' - \iota')\eta)c_3(2X, W)}{2\eta X - \iota W} + \frac{\rho^2(\eta' - \iota')^2d_4(2X, W)}{(2\eta X - \iota W)(2\eta X - \iota' W)},
\end{cases}
\]

where the first and second equation are again homogeneous polynomials. Equations for $\tilde{S}_{5,4}, \ldots, \tilde{S}_{7,7}$ are then obtained from $\tilde{S}_{4,4}$ in the same way as before.

For $m, n, m', n' \in \{4, 5, 6, 7\}$ with $m < n, m' < n'$, we also introduce the following complete intersection curves:

\[
T_{mn, m'n'}: \begin{cases}
0 &= Q_{mn, m'n'}(2X, W) + \rho Q_{mn, m'n'}(2X, W)W - \rho Z \\
0 &= 2\rho Y^2 + \frac{\rho^2 P_{mn, m'n'}(2X, W)c_2(2X, W)}{P_{mn, m'n'}(2X, W)} + \frac{(\rho^2 P_{mn, m'n'}(2X, W) - Q_{mn, m'n'}(2X, W))c_3(2X, W)}{Q_{mn, m'n'}(2X, W)W} + \frac{(\rho^2 P_{mn, m'n'}(2X, W) - Q_{mn, m'n'}(2X, W))d_4(2X, W)}{P_{mn, m'n'}(2X, W)Q_{mn, m'n'}(2X, W)W^2},
\end{cases}
\]
where the first and second equation are homogeneous polynomials. For \( m = 4, n = 5 \) and \( m' = 4, n' = 5 \), we set

\[
P_{45}(2\mathbf{X}, \mathbf{W}) = \mu \xi (2\eta \mathbf{X} - \iota \mathbf{W})(2\kappa \mathbf{X} - \lambda \mathbf{W}),
\]

\[
Q_{45}(2\mathbf{X}, \mathbf{W}) = \mu' \xi' (2\eta' \mathbf{X} - \iota' \mathbf{W})(2\kappa' \mathbf{X} - \lambda' \mathbf{W}),
\]

and equations for \( T_{46,45}, \ldots, T_{67,45}, T_{67,46}, \ldots, T_{67,67} \) are obtained from \( T_{45,45} \) by interchanging parameters in an analogous manner as above. This procedure generates 36 rational curves. Applying the Nikulin involution in Equation (4.8), we obtain from the curve \( T_{mn,m'n'} \) the curve \( T_{kl,k' l'} \) with \( \{k,l,m,n\} = \{k',l',m',n'\} = \{4,5,6,7\} \) and \( m < n, k < l, m' < n', k' < l' \).

We have the following:

**Theorem 5.3.**

1. The rational curves \( a_1, a_2, a_3, L_4, L_5, L_6, L_7, R_0, R_4, R_5, R_6 \) generate the Néron-Severi lattice \( H \oplus D_4(-1) \oplus A_1(-1)^\oplus 5 \).
2. For the K3 surfaces in Theorem 4.9 the dual graph of all smooth rational curves is shown in Figure 2.
3. On Figure 2 the Nikulin involution acts as a horizontal flip, exchanging the nodes \( a_1 \) and \( b_3 \).
4. As divisor classes in the Néron-Severi lattice, one has the following relations:

\[
b_1 \sim 2a_1 + L_4 + L_5 + L_6 + L_7 - R_0, \quad R_7' \sim a_2 - L_7 + R_0,
\]

\[
S_{n,n'} \sim 2a_1 + L_k + L_l + L_m - R_0 + R_0',
\]

\[
T_{mn,m'n'} \sim 2a_1 + L_k + L_l - R_0 + R_0' + R_0',
\]

\[
R_{13} \sim 2D_{11} - b_1, \quad S_{n,n'} \sim 2D_{11} - S_{n,n'}, \quad T_{mn,m'n'} \sim 2D_{11} - T_{kl,k'l'},
\]

for \( \{k,l,m,n\} = \{k',l',m',n'\} = \{4,5,6,7\} \) and \( m < n, k < l, m' < n', k' < l' \), and the divisor \( D_{11} = a_1 + 2a_2 + a_3 + R_0 \), which is invariant under the action of the Nikulin involution.

5. Figures 5-6 determine the embeddings of the reducible fibers from Theorem 4.9.

**Proof.** Items (1), (3) and (4) can be shown by explicit computation. For (2), the comparison of the divisor classes for the constructed rational curves with [25, Sec. 11.4] shows that they constitute all rational curves. Finally for (5), there is only one way of embedding the reducible fibers of fibration (1) in Theorem 4.9 into the graph in Figure 2; see Figure 5. For all other Jacobian fibrations, there is a second embedding of the reducible fibers, related by the action of the Nikulin involution. The embeddings are depicted in Figure 6.

5.2. **Family of Picard rank 12.** We also determine the dual graph of all rational curves for the K3 surfaces in Theorem 4.16 with Néron-Severi lattice \( H \oplus D_6(-1) \oplus A_1(-1)^\oplus 4 \). In this situation one has the following 59 rational curves:

\[
a_1, a_2, a_3, b_1, b_2, b_3, L_1, L_2^\pm, R_1, R_3, L_n, R_n,
\]

\[
S_n^\pm, \tilde{S}_n^\pm, T_{mn}^\pm, \tilde{T}_{mn}^\pm \text{ for } m,n \in \{4,5,6,7\} \text{ with } m < n.
\]
Figure 2. Dual graph of all smooth rational curves with 1-fold (thin) and some 2-fold (thick) edges for $\text{NS}(\mathcal{X}) = H \oplus D_4(-1) \oplus A_1(-1)^{\otimes 5}$
Again, the lines $L_1, L_2^*, L_n$ for $n = 4, 5, 6, 7$ were mentioned in Section 4.2. The two sets of lines $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ comes from resolving the singularities at $P_1$ and $P_2$ in Remark 4.13, respectively. The remaining curves can be given as complete intersections. We set

\[
R_i: \begin{cases}
0 &= 2\gamma X - \delta W \\
0 &= 2\gamma Y^2 Z - \gamma e_3(\delta, \gamma) W^2 Z - d_4(\delta, \gamma) W^3,
\end{cases}
\]

where the polynomials $c_2, e_3$ and $d_4$ are determined in Equation 4.22. An equation for $R_3$ is obtained from $R_1$ by replacing $(\gamma, \delta)$ by $(\epsilon, \zeta)$, and $R_5, R_6,$ and $R_7$ are obtained from $R_4$ by replacing $(\eta, \iota)$ by $(\kappa, \lambda), (\mu, \nu)$ and $(\xi, \omicron)$ respectively. That is, they are related by the symmetries in Remark 4.14. We also have

\[
S^+_4: \begin{cases}
0 &= -\iota(1 + \chi)W + 2\eta(1 + \chi)X + 2\gamma e\eta Z \\
0 &= 4\gamma e\eta(1 + \chi) Y^2 + \frac{1}{W}(\frac{1}{2} + \iota(1 + \chi)^2(2\eta X - \epsilon W) c_2(2X, W) W - 2\gamma e\eta(1 + \chi) e_3(2X, W) + 4\gamma^2 e^2 \eta^2 \frac{d_4(2X, W) W}{2}\),
\end{cases}
\]

\[
\tilde{S}^+_4: \begin{cases}
0 &= c_2(2X, W) Z + \frac{2\eta(1 + \chi) Y^2}{(2\eta X - \epsilon W) W} W \frac{1}{(1 + \chi)^2(2\eta X - \epsilon W) c_2(2X, W)} \\
0 &= 4\gamma e\eta(1 + \chi) Y^2 + \frac{1}{W}(\frac{1}{2} + \iota(1 + \chi)^2(2\eta X - \epsilon W) c_2(2X, W) W - 2\gamma e\eta(1 + \chi) e_3(2X, W) + 4\gamma^2 e^2 \eta^2 \frac{d_4(2X, W) W}{2}\),
\end{cases}
\]

where the second equation for $S^+_4$ is a homogeneous polynomial of degree two, and $\chi$ is as in Equation 4.29 with $\chi^2 \neq 0, 1$. Equations for $S^+_5, S^+_6,$ and $S^+_7$ are obtained from $S^+_4$ by replacing $(\eta, \iota) \leftrightarrow (\kappa, \lambda), (\eta, \iota) \leftrightarrow (\mu, \nu)$, and $(\eta, \iota) \leftrightarrow (\xi, \omicron)$, respectively; in the same way, $\tilde{S}^+_5, \tilde{S}^+_6,$ and $\tilde{S}^+_7$ are obtained from $\tilde{S}^+_4$. Divisors $S^+_n$ and $S^-_n$ for $n = 4, 5, 6, 7$ are interchanged by the sign change $\chi \mapsto -\chi$. Note that $\tilde{S}^+_4$ is obtained from $S^+_4$ by applying the Nikulin involution in Remark 4.15. Divisors $S^+_n$ and $\tilde{S}^-_n$ as well as $L_n$ and $R_n$ for $n = 4, 5, 6, 7$ are also related by the action of the Nikulin involution. Finally, we set

\[
T^+_{45}: \begin{cases}
0 &= (1 + \chi)(2\gamma X - \delta W)Z + 2\gamma \mu \xi (1 + \chi)(2\eta X - \epsilon W) (2\eta X - \lambda W) \\
0 &= 4\gamma e\kappa \mu \xi (1 + \chi) Y^2 + \frac{1}{W}(\frac{1}{2} + \iota(1 + \chi)(2\eta X - \epsilon W) (2\eta X - \lambda W) c_2(2X, W) W + 4\gamma^2 e^2 \kappa^2 \mu^2 \xi^2(2\eta X - \epsilon W)(2\eta X - \lambda W) c_2(2X, W) W - 2\gamma e\kappa \mu \xi (1 + \chi) e_3(2X, W) + 4\gamma^2 e^2 \kappa^2 \mu^2 \xi^2(2\eta X - \epsilon W)(2\eta X - \lambda W) c_2(2X, W) W )
\end{cases}
\]

Equations for $T^+_{46}, T^+_{47}, \ldots, T^+_{67}$ are obtained from $T^+_{45}$ by interchanging parameters in an analogous manner as above. Applying the Nikulin involution in Remark 4.15, we also obtain $\tilde{T}^+_{45}, \ldots, \tilde{T}^+_{67}$. For the latter, applying the Nikulin involution essentially interchanges $(\gamma, \delta)$ and $(\epsilon, \zeta)$.

We have the following:

**Theorem 5.4.**

(1) The rational curves $a_1, a_2, a_3, b_2, L_1, L_2^*, L_4, L_5, L_6, L_7, R_1$ generate the Néron-Severi lattice $H \oplus D_6(-1) \oplus A_1(-1)^{\oplus 4}$.

(2) For the K3 surfaces in Theorem 4.16 the dual graph of all smooth rational curves is shown in Figure 3.
(3) On Figure 3 the Nikulin involution acts as a horizontal flip, exchanging the nodes $a_1$ and $b_2$.

(4) As divisor classes in the Néron-Severi lattice, one has the following relations:

\[
S^\pm_n \sim L_k + L_l + L_m - L_2^\pm + 2a_1 + a_2, \quad \tilde{S}^\pm_n \sim 2D_{12} - S^\pm_n,
\]
\[
T^\pm_{mn} \sim L_m + L_n + R_1 - L_2^\pm + 2a_1 + a_2, \quad \tilde{T}^\pm_{mn} \sim 2D_{12} - T^\pm_{mn},
\]

for $\{k, l, m, n\} = \{4, 5, 6, 7\}$, $m < n$, and the divisor

\[
D_{12} = a_1 + 2a_2 + 3a_3 + b_2 + 2L_1 + L_2^* + L_2^-,
\]

which is invariant under the action of the Nikulin involution.

(5) Figures 7-9 determine the embeddings of the reducible fibers from Theorem 4.16.
Proof. Items (1), (3), (4) can be shown by direct computation. The comparison of the divisor classes for the constructed rational curves with [25, Sec. 12.3] gives (2). For (5), there is only one way of embedding the reducible fibers of fibration (1) in Theorem 4.16 into the graph in Figure 3. It is depicted in Figure 7. For the remaining fibrations, there are always two distinct embeddings of the reducible fibers for each Jacobian elliptic fibration, related by the action of the Nikulin involution. They are depicted in Figures 8 and 9. □

5.3. Family of Picard rank 13. Finally, we determine the dual graph of smooth rational curves for the K3 surfaces in Theorem 4.19 with Néron-Severi lattice \(H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 3}\). In this situation one has the following 39 rational curves:

\[
\begin{align*}
(5.10) & \quad a_1, \ldots, a_5, b_1, b_2, b_3, L_1, L_2, \bar{L}_2, L_n, R_1, R_3, R_n, \\
(5.11) & \quad S_2, S_n, \bar{S}_2, \bar{S}_n, \bar{T}_{i,n}, \bar{T}_{i,n} \text{ for } i \in \{1, 3\}, \ n \in \{4, 5, 6\}.
\end{align*}
\]

The lines \(L_1, L_2, \bar{L}_2, L_4, L_5, L_6\) were already determined in Section 4.2. The two sets 
\(\{a_1, a_2, a_3, a_4, a_5\}\) and \(\{b_1, b_2, b_3\}\) denote the curves appearing when resolving the singularities at \(P_1\) and \(P_2\) mentioned in Remark 4.18.

The rest of the curves can again be given as complete intersections. From the curves \(R_1\) and \(R_3\) in Section 5.2, the condition \((\xi, o) = (0, 1)\) for the equation of \(\mathcal{Q}\) restricts them to curves of the same degree which we will denote by the same symbols. Setting \(\chi = 1\) in the equations for \(S_n^+\) with \(n = 4, 5, 6\) in Picard number 12 restricts the divisors that will are denoted by \(S_n\). The equations for \(S_n^-\) become reducible combinations of \(L_4, L_5, L_6\). The same divisors are obtained if one sets \(\chi = -1\) in the equations for \(S_n^-\). Similarly, we obtain the restrictions \(\bar{S}_n\) of \(\bar{S}_n^+\). Note that \(\bar{S}_n\) is obtained from \(S_n\) by applying the Nikulin involution in Remark 4.15, and the equations for \(S_n\) (resp. \(\bar{S}_n\)) for \(n = 4, 5, 6\) are related by interchanging parameters. Moreover, we have

\[
\begin{align*}
S_2: & \quad \begin{cases} 0 = \eta \kappa \mu W - Z \\
0 = \eta \kappa^2 \mu^2 c_2(2X, W) + \gamma \mu c_3(2X, W) W + d_4(2X, W) W^2 - 2 \eta \kappa \mu Y^2, \\
0 = \eta \kappa \mu c_2(2X, W) Z - d_4(2X, W) W - \gamma \mu c_3(2X, W) W^2 + d_4(2X, W) W^2 - 2 \eta \kappa \mu Y^2, \\
0 = \gamma \kappa \mu (2 \kappa X - \zeta W) Z - (2 \kappa X - \lambda W) (2 \mu X - \nu W) W \\
0 = - (2 \kappa X - \lambda W)(2 \mu X - \nu W) c_2(2X, W) W + \gamma \mu c_3(2X, W) W^2 - (2 \kappa X - \lambda W)(2 \mu X - \nu W) W, \\
0 = \gamma \kappa \mu (2 \kappa X - \zeta W) Z - (2 \kappa X - \lambda W) (2 \mu X - \nu W) W \\
0 = \gamma \kappa \mu (2 \kappa X - \zeta W) Z - (2 \kappa X - \lambda W) (2 \mu X - \nu W) W - 2 \gamma \kappa \mu Y^2,
\end{cases}
\end{align*}
\]

where all equations are polynomial by construction. The curves \(S_n\) and \(\bar{S}_n\) (resp. \(T_{1,4}\) and \(\bar{T}_{1,4}\)) are related by the action of the Nikulin involution. Equations for \(T_{3,4}\) and \(\bar{T}_{3,4}\) are obtained from \(T_{1,4}\) and \(\bar{T}_{1,4}\) by replacing the parameters \((\gamma, \delta)\) by \((\epsilon, \zeta)\). Moreover, equations for \(T_{m,5}\) and \(\bar{T}_{m,6}\) are obtained from \(T_{m,4}\) and \(\bar{T}_{m,4}\) for \(m = 1, 3\) by replacing \((\eta, \iota)\) by \((\kappa, \lambda)\) and \((\mu, \nu)\) respectively. These divisors appeared above are in connection with the pencils \(C_n(u, v)\) for \(n = 4, 5, 6\) in the proof of Theorem 4.19.
Theorem 5.5.

(1) The rational curves \( a_1, \ldots, a_5, b_1, b_2, L_2, L_4, L_5, L_6, R_1, R_3 \) generate the Néron-Severi lattice \( H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 3} \).

(2) For the K3 surfaces in Theorem 4.19 the dual graph of all smooth rational curves is shown in Figure 4.

(3) On Figure 4 the Nikulin involution acts as a horizontal flip, exchanging the nodes \( a_1 \) and \( b_2 \).

(4) As divisor classes in the Néron-Severi lattice, one has the following relations:

\[
\begin{align*}
\hat{T}_{i,l} & \sim 2a_1 + a_2 - L_2 + L_m + L_n + R_j, & T_{i,l} & \sim D_{13} - \hat{T}_{i,l} \\
\hat{S}_n & \sim 2a_1 + a_2 - L_2 + L_n + R_1 + R_3, & S_n & \sim D_{13} - \hat{S}_n \\
S_2 & \sim 2a_1 + a_2 - L_2 + L_4 + L_5 + L_6, & \hat{S}_2 & \sim D_{13} - S_2,
\end{align*}
\]

(5.12) for \( \{i, j\} = \{1, 3\}, \{l, m, n\} = \{4, 5, 6\}, \) and the divisor

\[
D_{13} = 3a_1 + 3a_2 + 3a_3 + 2a_4 + a_5 - 2b_1 - b_2 + L_2 + L_4 + L_5 + L_6 - R_1 + R_3,
\]

which is invariant under the action of the Nikulin involution

(5) Figures 10-13 determine the embeddings of the reducible fibers from Theorem 4.19.

Proof. Items (1), (3), (4) can be shown by direct computation. The comparison of the divisor classes for the constructed rational curves with [25, Sec. 13.2] gives (2). For (5), there is only one way of embedding the reducible fibers of fibration (1), which is depicted in Figure 10. For the remaining fibrations there are two embeddings in each case, related by the action of the Nikulin involution. They are depicted in Figures 11 to 13. \( \square \)

Before we end the section, We have the following:

Theorem 5.6. The polarization divisor \( \mathcal{H} \) of \( Q \) is as follows:

\[
\mathcal{H} = L_4 + L_5 + L_6 + L_7 + 3a_1 + 2a_2 + a_3 \quad \text{in Theorem 4.9},
\]

(5.13) \[
\begin{align*}
\mathcal{H} &= L_4 + L_5 + L_6 + L_7 + 3a_1 + 2a_2 + a_3 \quad \text{in Theorem 4.16}, \\
\mathcal{H} &= L_2 + L_4 + L_5 + L_6 + 3a_1 + 3a_2 + 3a_3 + 2a_4 + a_5 \quad \text{in Theorem 4.19}.
\end{align*}
\]

One has \( \mathcal{H}^2 = 4 \) and \( \mathcal{H} \circ F = 3 \) where \( F \) is the smooth fiber class of any elliptic fibration that is obtained as the intersection with a pencil of lines.

Proof. We will prove the statement for Picard number 12. The other cases are analogous. By computing all intersection numbers of the rational curves, we may express their divisor classes as linear combinations with integer coefficients in a basis of the lattice. Let us look at fibration (2) in Theorem 4.16, i.e., the standard fibration and compute a basis of the polarization lattice. We observe that the nodes \( L_1 \) and \( a_2 \) are the extra nodes of the two extended Dynkin diagrams of \( \hat{D}_4 \). It follows that \( L_2, L_4, \ldots, L_7, a_1, a_3, b_1, b_2, b_3 \), and the fiber class \( F \) of the standard fibration form a basis in \( \text{NS}(X) \). For example, taking a fiber class in Figure 8a we have

\[
F = 2a_1 + a_2 + L_5 + L_6 + L_7.
\]

(5.14)
Figure 4. Dual graph of all smooth rational curves with 1-fold (thin) and some 2-fold (thick) edges for $\text{NS}(\mathcal{X}) = H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 3}$
Thus, the polarizing divisor can be written as a linear combination

\[
\mathcal{H} = f F + l_2 L_2^+ + \sum_{i=4}^{7} l_i L_i + \sum_{i=1}^{3} \beta_i b_i + \alpha_1 a_1 + \alpha_3 a_3.
\]

We use \(\mathcal{H} \circ a_i = \mathcal{H} \circ b_i = 0\) for \(i = 1, 2, 3\), \(\mathcal{H} \circ L_1 = \mathcal{H} \circ L_2^+ = 1\), and \(\mathcal{H} \circ L_j = 1\) for \(j = 4, \ldots, 7\). We obtain a linear system of equations for the coefficients in Equation (5.15) whose unique solution is given by Equation (5.13). \(\square\)
Figure 5. Fibration with reducible fibers $\overline{A}_1 \oplus^0$ and sections

Figure 6. Fibrations with reducible fibers $\overline{D}_4 + \overline{A}_1 \oplus^5$ and section
Figure 7. Fibration with reducible fibers $\tilde{D}_4 + \tilde{A}_1^{\oplus 6}$ and sections.

Figure 8. Fibrations with reducible fibers $\tilde{D}_4 + \tilde{D}_4 + \tilde{A}_1^{\oplus 2}$ and section.

Figure 9. Fibrations with reducible fibers $\tilde{D}_6 + \tilde{A}_1^{\oplus 4}$ and section.
Figure 10. Fibration with reducible fibers $\tilde{D}_6 + \tilde{A}_1^{\oplus 5}$ and sections

Figure 11. Fibrations with reducible fibers $\tilde{D}_4 + \tilde{D}_6 + \tilde{A}_1$ and section

Figure 12. Fibration with reducible fibers $\tilde{D}_8 + \tilde{A}_1^{\oplus 3}$ and section

Figure 13. Fibration with reducible fibers $\tilde{E}_7 + \tilde{A}_1^{\oplus 4}$ and section
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