Towards Synthetic Descriptive Set Theory: An instantiation with represented spaces

Arno Pauly
Clare College
University of Cambridge, United Kingdom
Arno.Pauly@cl.cam.ac.uk

Matthew de Brecht
National Institute of Information and Communications Technology
Kyoto, Japan
matthew@nict.go.jp

Using ideas from synthetic topology, a new approach to descriptive set theory is suggested. Synthetic descriptive set theory promises elegant explanations for various phenomena in both classic and effective descriptive set theory. Presently, we mainly focus on developing the ideas in the category of represented spaces.

1 Introduction

Synthetic descriptive set theory is the idea that descriptive set theory can be reinterpreted as the study of certain endofunctors and derived concepts, primarily in the category of represented spaces. It is proposed as an abstract framework explaining the many similarities between descriptive set theory (e.g. [24]), effective descriptive set theory (e.g. [28]) and facets of recursion theory. A crucial novel aspect is that classes of functions such as the \( \Sigma^0_n \)-measurable ones that are commonly seen as generalizations of the continuous functions are now considered to be a special case – thus making the observation that they share many properties with the continuous functions trivial. The change in viewpoint proceeds via the recognition that the concepts \( \Sigma^0_n \), \( \Delta^0_n \), \( \Delta^1_1 \), etc., can be considered as endofunctors acting on a suitable category.

This research programme can be seen as a continuation of Escardó’s synthetic topology [13]. While the central concepts can be formulated in a generic setting of a cartesian closed category with a Sierpiński-space-like object, we focus on the expression of these general concepts in the category of represented spaces that underlies the TTE-approach to computable analysis ([46]).

This category does seem to be a very appropriate setting for descriptive set theory, in particular it contains the structures considered in classical descriptive set theory such as separable metric spaces or Borel equivalence relations.

To some extent we can view this work as reinterpreting the classical field of descriptive set theory into a kind of type theory, where spaces are types and the endofunctors are certain kinds of modal operators. There is a long history of these kinds of type theories in theoretical computer science, such as Moggi’s work [27] which models computational semantics with monads, and also various flavors of Jean-Yves Girard’s Light Linear Logic [15] which use substructural logic and modal operators to characterize computational complexity. In synthetic descriptive set theory, the modal operators characterize topological complexity or non-constructiveness.

A development inside descriptive set theory that to some extent mirrors the introduction of notions derived from computing machines suggested here is the use of games to characterize...
function classes. Pioneered by Wadge [44, 45], a culmination can be found in [31] by Motto-Ros. Nobrega has provided a translation of these results into the language of Weihrauch degrees in his Master’s thesis [21], which makes them even more accessible for our purposes.

Various results in the literature can – in hindsight – be read as contributing to synthetic descriptive set theory, this pertains to [4, 11, 29, 35, 19, 20, 8, 39, 38] by Brattka, Moschovakis, Higuchi, Kihara, Schröder, Selivanov and the authors (and this list quite certainly is incomplete).

After recalling (extremely briefly) some of the relevant concepts from synthetic topology on the one hand, and then from descriptive set theory on the other hand, we will present the new ideas in two main sections. First, the core ideas of synthetic descriptive theory are introduced, in a primarily category-theoretic language. The reader may find it difficult to see the connections to classical descriptive set theory until the next section. There, concrete endofunctors are examined regarding their connections to well-known concepts in descriptive set theory. The proofs in Section 2 are generally extremely short; while Section 3 would see some longer (though not very involved) proofs. The latter can be seen as reflecting some less-elegant aspects of traditional definitions in descriptive set theory.

1.1 Synthetic topology

The core idea of synthetic topology is that in any cartesian closed category (i.e. a category allowing the formation of function spaces) which has a special object $S$ behaving suitably like the Sierpiński-space in topology, it is possible to introduce a variety of concepts from topology, such as the space of open (closed, compact) subsets of a given space, and properties of spaces such as being compact or being Hausdorff. In this the morphisms of the category are pretended to be the continuous functions. [13, 14, 2] introduced and developed synthetic topology; while Taylor’s abstract stone duality [42, 43] features some similar ideas.

Admissibility as a property of objects in a category subjected to synthetic topology can, following the work of Schröder [37, 36], be understood as marking those space whose behaviour in the category (as codomain of morphisms) is fully determined by their topological properties. In essence, the admissible spaces in a cartesian closed category form a cartesian closed subcategory that also is a subcategory 1 of the category $\text{Top}$ of topological spaces and continuous maps.

A self-contained treatment of synthetic topology instanced with the category of represented spaces can be found in [34]. Here, we just recall a few formal definitions relevant for the development of synthetic descriptive set theory:

A represented space is a pair $(X, \delta_X) := X$ where $\delta_X :\subseteq \{0, 1\}^N \to X$ is a partial surjection. For $f : X \to Y$ and $F :\subseteq \{0, 1\}^N \to \{0, 1\}^N$, we call $F$ a realizer of $f$ (notation $F \vdash f$), iff $\delta_Y(F(p)) = f(\delta_X(p))$ for all $p \in \text{dom}(f \delta_X)$. A map between represented spaces is called computable (continuous), iff it has a computable (continuous) realizer. A priori, the notion of a continuous map between represented spaces and a continuous map between topological spaces are distinct and should not be confused!

We consider two categories of represented spaces, one equipped with the computable maps, and one equipped with the continuous maps. We call the resulting structure a category extension (cf. [33]), as the former is a subcategory of the latter, and shares its structure (products,

1Note however that the admisibly represented spaces do not inherit their product from $\text{Top}$, but rather from the category of sequential spaces. In other words, the topology on the product of two admisibly represented spaces is the sequentialization of the product topology.
coproducts, exponentials).

The set of continuous functions from \( X \) to \( Y \) can be turned into a represented space \( C(X, Y) \) itself, with the evaluation map being computable due to the UTM-theorem. This establishes our category to be cartesian closed.

We want to make use of two special represented spaces, \( \mathbb{N} = (\mathbb{N}, \delta_\mathbb{N}) \) and \( S = (\{\emptyset, \top\}, \delta_S) \). The representation are given by \( \delta_\mathbb{N}(0^{n}1^{\mathbb{N}}) = n \), \( \delta_\mathbb{N}(0^{\mathbb{N}}) = \bot \) and \( \delta_S(p) = \top \) for \( p \neq 0^\mathbb{N} \).

Computability on \( \mathbb{N} \) coincides with the classical notion of computability. The functions \( \land, \lor : S \times S \to S \) and \( \lor : C(\mathbb{N}, S) \to S \) are computable.

Now we define the set of open subsets of a space \( X \) to be the set of functions \( C(X, S) \), where we identify a set with its characteristic function. We immediately obtain that \( (f, U) \mapsto f^{-1}(U) : C(X, Y) \times O(Y) \to O(X) \) is computable for all represented spaces \( X, Y \) – this is just composition of functions! Hence, any continuous function between represented spaces matches the definition of continuity for functions between topological spaces. An alternative formulation is that \( f \mapsto f^{-1} : C(X, Y) \to C(O(Y), O(X)) \) is computable.

Given a represented space \( X \), consider the map \( \kappa_X : X \to O(O(X)) \) mapping any point to its neighborhood filter, i.e. \( \kappa(x) = \{ U \in O(X) \mid x \in U \} \). Let \( X_\kappa \) be the image of \( \kappa_X \). Now we call \( X \) (computably) admissible, if \( \kappa \) is injective and \( \kappa^{-1} : X_\kappa \to X \) is continuous (computable).

Note that \( X_\kappa \) is always computably admissible, i.e. isomorphic to \( (X_\kappa)_\kappa \).

Now a space \( Y \) is (computable) admissible if and only if the map \( f \mapsto f^{-1} : C(X, Y) \to C(O(Y), O(X)) \) is continuously (computably) invertible. Hence, for admissible spaces, the inherent (represented space) definition of continuity coincides with the topological version.

1.2 Descriptive Set Theory

A central part of descriptive set theory is the Borel hierarchy. Consider a separable metric space \( X \). Now let \( \Sigma^0_0(X) := O(X) \), \( \Pi^0_0(X) := \{ X \setminus U \mid U \in \Sigma^0_0(X) \} \), \( \Sigma^0_{\alpha+1}(X) = \{ \bigcup_{i \in \mathbb{N}} A_i \mid \forall i \in \mathbb{N} A_i \in \Pi^0_\alpha(X) \} \) and \( \Sigma^0_\beta(X) = \bigcup_{\alpha < \beta} \Sigma^0_\alpha(X) \) for limit ordinals \( \beta \). Moreover, let \( \Delta^0_\alpha(X) = \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X) \).

The \( \Sigma^0_\alpha \)-sets behave in some ways like the open set: They are closed under countable unions and finite intersections, and the preimages of a \( \Sigma^0_\alpha \)-set under a continuous function is a \( \Sigma^0_\alpha \)-set again. We also find that \( \Sigma^0_\alpha(X) \subseteq \Sigma^0_{\alpha'}(X) \) if \( \alpha < \alpha' \).

For non-metric topological spaces that are still countably based and \( T_0 \), Selivanov [40] suggest a modified definition of the Borel hierarchy, using \( \Sigma^0_{\alpha+1}(X) := \{ \bigcup_{i \in \mathbb{N}} (U_i \setminus U'_i) \mid \forall i \in \mathbb{N} U_i, U'_i \in \Sigma^0_\alpha(X) \} \) instead. This modification ensures that \( \Sigma^0_\alpha(X) \subseteq \Sigma^0_{\alpha'}(X) \) if \( \alpha < \alpha' \) remains true, and is equivalent to the original definitions for metric spaces.

If we start only with the effectively open sets, and demand all countable unions to be uniform, we obtain the effective Borel hierarchy instead. Formalizing the uniformity conditions for the countable unions can be slightly cumbersome, and is omitted here.

Let \( \mathcal{B} \in \{ \Sigma^0_\alpha, \Pi^0_\alpha, \Delta^0_\alpha \} \). We call a function \( f : X \to Y \) \( \mathcal{B} \)-measurable, if \( f^{-1}(U) \in \mathcal{B}(X) \) for any \( U \in O(Y) \). A common theme in descriptive set theory is to provide alternative characterizations of some class of \( \mathcal{B} \)-measurable functions.

Say that the Baire class 0 functions are the continuous functions, the Baire class 1 functions the \( \Sigma^0_2 \)-measurable functions\(^2\), the Baire class \( \alpha \) functions the point-wise limits functions of Baire

\(^2\)For some special metric spaces, the following theorem would hold without explicitly demanding truth for \( \alpha = 1 \), i.e. with the Baire class 1 functions being the point-wise limits of continuous functions. This fails for other spaces, though: If \( X \) is connected and \( Y \) discrete, then point-wise limits of continuous functions are continuous themselves (Example taken from [30]). Such exceptions marring the theory disappear when moving
class $< \alpha$. Now we can formulate the:

**Theorem 1** (Lebesgue – Hausdorff – Banach). Let $X$, $Y$ be separable metric spaces. Then a function $f : X \to Y$ is Baire class $\alpha$ iff it is $\Sigma^0_{\alpha+1}$-measurable.

Next, we shall call a function $f : X \to Y$ piecewise continuous, if there is a cover $(A_i)_{i \in \mathbb{N}}$ of $X$ of closed sets (i.e. $\Pi^0_1$-sets), such that any $f|_{A_i}$ is continuous. The corresponding characterization result is:

**Theorem 2** (Jayne & Rogers [22]). Let $X$ be Polish and $Y$ be separable. Then a function $f : X \to Y$ is piecewise continuous iff it is $\Delta^0_2$-measurable.

## 2 Core concepts of synthetic descriptive set theory

Our investigation starts with endofunctors on the category of continuous functions between represented spaces. An endofunctor is an operation $d$ from a category to itself which maps objects to objects, morphisms to morphisms, preserves identity morphisms, and is compatible with composition, i.e. $d(f \circ g) = (df) \circ (dg)$. For any two represented spaces $X$, $Y$, an endofunctor $d$ induces a map $d : C(X, Y) \to C(dX, dY)$. If this map is always computable, we call $d$ computable. In the following, $d$ shall always be some computable endofunctor.

The typical examples relevant for our development of descriptive set theory will be operators that keep the underlying set of a represented spaces the same, and modify the representation in a sufficiently uniform way to ensure the requirements for computable endofunctors. Such operators have been called *jump operators* in [3], and specific examples can be found both there and in Section 3. For computable endofunctors that do change the underlying sets in a significant way, the interpretation of many of the following definitions becomes less clear, but an example of a computable endofunctor that still produces sensible notions is given in Subsection 3.5.

The computable endofunctors we study correspond to classes of sets such as $\Sigma^0_2$, $\Sigma^0_3$, $\Delta^0_2$, etc.; with the closure properties of the set-classes depending on how the endofunctor interacts with products. We say that $d$ preserves binary products, if $d(X \times X) \cong dX \times dX$ (where $\cong$ denotes computable isomorphism) for any represented space $X$, and that $d$ preserves products if $dC(N, X) \cong C(N, dX)$ for any represented space $X$.

### 2.1 The $d$-open sets

For a represented space $X$, we shall call $C(X, dS)$ the space of $d$-open sets $O^d(X)$. If $dS$ still has the underlying set $\{\bot, \top\}$ the elements of $O^d(X)$ actually are subsets of $X$ in the usual way. The complements of $d$-open sets are $d$-closed sets, to the synthetic approach.

In the presence of exponentials and a final object, an endofunctor $d$ may have an internal characterization. For fixed objects $X$, $Y$, let $D : C(X, Y) \to C(dX, dY)$ be an internal realization of $d$, if the following holds: Let $f : X \to Y$ be a morphism, and $f' : X \times 1 \to Y$ the corresponding morphism up to equivalence. By definition of the exponential, we then have a map $\lambda f' : 1 \to C(X, Y)$. In the same way, there is a map $\lambda (df)' : 1 \to C(dX, dY)$. The criterion now is $\lambda (df)' = D \circ \lambda f'$.

With the continuity/computability distinction, this would be a special case of an enriched endofunctor, if we understand a cartesian closed category to be enriched over itself.

Predicates in fuzzy logic would be an example of an entity analogue to the characteristic function of a set that crucially has not $\{\bot, \top\}$ as codomain. A somewhat similar example is presented in Subsection 3.5.
denoted by $A_d^d(X)$. A variety of nice closure properties follows immediately, with the proofs being straight-forward modifications of those for the corresponding results for open sets in \[34\] Proposition 6:

**Proposition 3.** The following operations are computable for any represented spaces $X$, $Y$:

1. $(f, U) \mapsto f^{-1}(U) : C(X, Y) \times O^d(Y) \to O^d(X)$
2. Cut : $Y \times O^d(X \times Y) \to O^d(X)$ mapping $(y, U)$ to $\{x \mid (x, y) \in U\}$

If $d$ preserves binary products, we additionally obtain:

3. $\cap, \cup : O^d(X) \times O^d(X) \to O^d(X)$
4. $\times : O^d(X) \times O^d(Y) \to O^d(X \times Y)$

If $d$ preserves products, we additionally obtain:

5. $\bigcup : C(N, O^d(X)) \to O^d(X)$

**Proof.**

1. This is just function composition.

2. And this is partial evaluation.

3. Given that $\land, \lor : S \times S \to S$ are computable functions, and $d$ is a computable endofunctor, we find that $\land, \lor : d(S \times S) \to dS$ are computable. Now $d$ is assumed to preserve binary products, and $\land, \lor$ are obtained by composing with $\land, \lor$.

4. This uses again computable $\land : dS \times dS \to dS$, together with type-conversion.

5. If $d$ is a computable endofunctor preserving products, we can obtain computable $\bigcup : C(N, dS) \to dS$ from computable $\bigcup : C(N, S) \to S$. The rest is function composition.

\[\square\]

### 2.2 $d$-continuity and $d$-measurability

Now we can introduce the notion of $d$-measurability: We call a function $f : X \to Y$ $d$-measurable, if $f^{-1} : O(Y) \to O^d(X)$ is well-defined and continuous, i.e. if the preimages of open sets under $f$ are uniformly $d$-open. The $d$-measurable functions from $X$ to $Y$ thus form a represented space $C^d(X, Y)$, which is by construction homeomorphic to a subspace of $C(O(Y), O^d(X))$.

A $d$-continuous function from $X$ to $Y$ shall just be a continuous function $f : X \to dY$. Note again, that if $d$ alters the underlying sets, then a $d$-continuous function between represented spaces will not necessarily induce a function on the underlying sets. The notion of $d$-continuity is a generalization of the Kleisli-morphisms w.r.t. a monad – if $d$ can be turned into a monad, then the $d$-continuous functions are precisely the Kleisli-morphisms. Some, but not all, of our examples of computable endofunctors will actually be monads in a natural way.

Reminiscent of the Banach-Lebesgue-Hausdorff theorem and the Jayne Rogers theorem, a $d$-measurable function is characterized by how preimages of open sets behave (similar to the $\mathfrak{B}$-measurable function), while $d$-continuous functions are characterized by what information is available on their function values (similar to Baire class $\alpha$ functions or piecewise continuous functions).

Both the $d$-measurable and the $d$-continuous functions have some of the closure properties expected from classes of $\mathfrak{B}$-measurable functions. For their formulation, note that the composition of two computable endofunctors is a computable endofunctor again.
Proposition 4. Both $d$-continuous and $d$-measurable maps are closed under composition with continuous maps from both sides, i.e. the following maps are computable for any represented spaces $X, Y, Z$:

1. $\circ : C(X, Y) \times C(Y, dZ) \to C(X, dZ)$
2. $\circ : C(X, Y) \times C^d(Y, Z) \to C^d(X, Z)$
3. $\circ : C(X, dY) \times C(Y, Z) \to C(X, dZ)$
4. $\circ : C^d(X, Y) \times C(Y, Z) \to C^d(X, Z)$

More generally, we can consider a second computable endofunctor $e$ and obtain:

5. $\circ : C(X, eY) \times C(Y, dZ) \to C(X, edZ)$

Taking into consideration the definition of $O^d(X)$ as $C(X, dS)$, we get the special case:

6. $(f, U) : C(X, eY) \times O^d(Y) \to O^{ed}(X)$
7. $\circ : C(X, eY) \times C^d(Y, Z) \to C^{ed}(X, Z)$

Finally, we find that $e$-continuity uniformly implies $e$-measurability:

8. $\text{id} : C(X, eY) \to C^e(X, Y)$

Proof. 1. Just regular function composition.

2. Consider $C^d(Y, Z)$ as (homeomorphic to) a subspace of $C(O(Z), O^d(Y))$, likewise for $C^d(X, Z)$. Recall that $O^d(Y)$ is essentially $C(Y, dS)$. Now $(f, g) \mapsto (h \mapsto (f \circ g(h)))$ realizes the desired functional.

3. As $d$ is a computable endofunctor, we can move from $C(Y, Z)$ as second argument to $C(dY, dZ)$, and then use regular function composition.

4. Let us view $C^d(X, Y)$ and $C^d(X, Z)$ as (homeomorphic to) subspaces of $C(O(Y), O^d(Z))$ and $C(O(Z), O^d(X))$ respectively. Now $(f, g) \mapsto (h \mapsto f(g \circ h))$ realizes the desired functional.

5. As $e$ is a computable endofunctor, we can move from $C(Y, dZ)$ as second argument to $C(eY, edZ)$, and then use regular function composition.

6. Choose $Z := S$ in (5.).

7. Type conversion together with (6.).

8. By currying and considering $d := \text{id}$ in (6.).

\(\square\)

2.3 $d$-admissibility

Having seen that $d$-continuity always implies $d$-measurability, we now strive for conditions that make the converse implication true, as well. Noting that id-continuity is continuity of maps between represented spaces, and id-measurability (uniform) topological continuity, we see that we need a notion of $d$-admissibility.

As a special case of Proposition 4 (8) with $X = 1$ and using trivial isomorphisms, we obtain the computability of a canonic mapping $\kappa^d : dY \to C(O(Y), dS)$. The image of $dY$ under $\kappa^d$ shall be denoted by $\kappa^dY$ (not by $\kappa^dY!!$).
Proposition 5. The following are equivalent:

1. $\text{id} : C(X, dY) \to C^d(X, Y)$ is computably invertible for any represented space $X$.
2. $\kappa^d : dY \to C(O(Y), dS)$ is computably invertible.
3. $\kappa^d Y \cong dY$.

Proof. 1. $\Rightarrow$ 2. Choose $X := 1$, and use the canonic isomorphism $C(1, Z) \cong Z$ twice.

2. $\Rightarrow$ 3. The map $\kappa^d : dY \to C(O(Y), dS)$ is always computable: Given $y \in dY$ and $U \in O(Y) = C(Y, S)$, we start by moving to $dU \in C(dY, dS)$ using that $d$ is a computable endofunctor. Then we apply $dU$ to $y$; what remains is currying. That $\kappa^d$ has a computable inverse is asserted as (2.), the claim then follows directly.

3. $\Rightarrow$ 1. Under the assumption $\kappa^d Y \cong dY$, we may show instead that $\text{id} : C(X, \kappa^d Y) \to C^d(X, Y)$ is computably invertible. Given $f^{-1} \in C^d(X, Y) \subseteq C(O(Y), C(X, dS))$ and $x \in X$, we may obtain $(U \mapsto f^{-1}(U)(x)) : O(Y) \to dS$. This in turn is $\kappa^d(f(x)) \in \kappa^d Y$, so again just currying remains to be done.

A space $Y$ satisfying these equivalent conditions shall be called $d$-admissible. We observe the following:

Proposition 6. $S$ is $d$-admissible.

Proof. We need to show that $\kappa^d : dS \to C(O(S), dS)$ is computably invertible. To do this, simply substitute $\{ \top \} \in O(S)$ at the corresponding position.

Now, consider $\kappa^d$ as an operation on the whole category of continuous functions between represented spaces. It is not hard to verify that $\kappa^d$ itself is a computable endofunctor. Even more, we can consider $d \mapsto \kappa^d (=: \kappa)$ as an operation on computable endofunctors! As a consequence of Proposition 6, we obtain:

Corollary 7. $\kappa(\kappa^d) \cong \kappa^d$.

Proof. Note that the right hand side of $\kappa^d : dY \to C(O(Y), dS)$ depends on $d$ only via $dS$. So any $\kappa(\kappa^d) : \kappa^d X \to \kappa(\kappa^d) X$ essentially is the identity. Hence, $dS = \kappa^d S$ from Proposition 6 yields the claim.

Corollary 8. Every represented space is $\kappa^d$-admissible.

Corollary 9. $O^d(X) = O^{\kappa^d}(X)$.

Corollary 10. $d$-measurability and $\kappa^d$-continuity coincide.

Corollary 11. If $Y$ is $d$-admissible, then $C^d(X, Y)$ and $C(X, dY)$ are homeomorphic.

Corollary 12. If $Y$ is $e$-admissible, then $\circ : C^e(X, Y) \times C^d(Y, Z) \to C^{ed}(X, Z)$ is computable.

For a large class of spaces and computable endofunctors, we can provide admissibility results without having to resort to modifying the endofunctor. We start with the seemingly innocuous:

Proposition 13. Let $d$ preserve products. Then $O(N)$ is $d$-admissible.
Proof. By assumption, \( d\mathcal{O}(\mathbb{N}) \cong \mathcal{C}(\mathbb{N}, d\mathcal{S}) \cong \mathcal{O}^d(\mathbb{N}) \). Now consider the right hand side of Proposition 3 (2). As \( \mathbb{N} \) is admissible, we find that we may go from the induced subspace of \( \mathcal{C}(\mathcal{O}(\mathcal{O}(\mathbb{N})), d\mathcal{S}) \) back to \( \mathcal{C}(\mathbb{N}, d\mathcal{S}) \), thus obtaining the desired equivalence. \( \Box \)

Corollary 14. Let \( d \) preserve products, and let \( X \) be countably based and admissible. Then \( X \) is \( d \)-admissible.

The preceding corollary relies on Weihrauch’s observation [46] that the countably-based admissible spaces are just the subspaces of \( \mathcal{O}(\mathbb{N}) \), together with \( d \)-admissibility being closed under formation of subspaces. Additionally, it may be the reason that countably-based \( T_0 \)-spaces seem to form a natural demarkation line for the extension of descriptive set theory \[9\]. Combining its statement with Proposition 3 we see that any computable endofunctor preserving products nicely characterizes a \( \Sigma \)-like class of sets and the corresponding measurable functions on all countably based admissible spaces.

2.4 Further concepts

The other concepts from synthetic topology studied for represented spaces in [34], namely Hausdorff, discreteness, compactness and overtness, can also be lifted along some endofunctor, and retain most of their nice properties. Rather than listing all of these statements and definitions, we shall only consider those used later in applications.

Definition 15. A space \( X \) is called computably \( d \)-Hausdorff, iff \( x \mapsto \{x\} : X \to A^d(X) \) is computable.

Proposition 16. The following are equivalent:

1. \( X \) is computably \( d \)-Hausdorff.
2. \( \neq : X \times X \to d\mathcal{S} \) is computable.

If \( d \) preserves binary products, then the following are also equivalent to those above:

3. \( \{(x,x) \mid x \in X\} \in A^d(X \times X) \) is computable.
4. Graph : \( \mathcal{C}(Y,X) \to A^d(Y \times X) \) is well-defined and computable for any represented space \( Y \).

Some properties related to \( d \)-Hausdorff have been studied by Schröder and Selivanov in [39, 38].

Proposition 17. If \( d \) preserves binary products and \( X \) is computably Hausdorff, then \( dX \) is computably \( d \)-Hausdorff.

Definition 18. A space \( X \) is called \( d \)-overt, iff IsNonEmpty : \( \mathcal{O}^d(X) \to d\mathcal{S} \) is computable.

Proposition 19. If \( f : X \to Y \) is a computable surjection and \( X \) is \( d \)-overt, then so is \( Y \).

Proposition 20. If \( d \) preserves products and each \( X_n \) is \( d \)-overt, then so is \( \bigcup_{n \in \mathbb{N}} X_n \).
2.5 The Markov-variant

In effective descriptive set theory, the notion of affectivity between higher-order objects being employed often is not computability, but rather Markov computability. A function \( f : X \to Y \) is called Markov-computable, if there is some computable partial function \( \phi : \subseteq \mathbb{N} \to \mathbb{N} \), such that whenever \( i \) is an index of a computable element in \( X \), then \( \phi(i) \) is an index of \( f(i) \). Any computable function is Markov-computable, while the converse fails.

Subsequently, an endofunctor \( d \) is called Markov-computable, if any \( \text{C}(dX, dY) \rightarrow \text{C}(dX, dY) \) is Markov-computable. The effective measurability notion going with Markov-computable endofunctors is (weak) non-uniform computability, i.e. if for any computable \( U \in \mathcal{O}(Y) \) we find \( f^{-1}(U) \in \mathcal{O}(X) \) to be computable, we call \( f \) to be Markov-\( d \)-measurable. The represented space \( \text{C}^{Md}(X, Y) \) of Markov-\( d \)-measurable functions essentially represents a function by some oracle \( p \) paired with a table listing indices of computably open sets and their \( p \)-computably \( d \)-open preimages.

**Proposition 21.** Let \( d \) be Markov-computable. Then \( \text{id} : \text{C}(dX, dY) \rightarrow \text{C}^{Md}(X, Y) \) is computable.

*Proof.* Let us be given a function \( f \in \text{C}(X, dY) \), i.e. we have an index \( n \) and an oracle \( p \) that realize \( f \). The oracle involved is retained. To construct the table, let us further be given an index \( i \) for a computable \( U \in \mathcal{O}(Y) \). By \( d \) being Markov-computable, we can obtain an index \( j \) for \( dU : dY \rightarrow dS \). Composing the machines of \( n \) and \( j \) yields an index for \( f^{-1}(U) \) relative to \( p \). \( \square \)

We can define the Markov-variant of \( \kappa_{d} \) via letting \( \eta_{d} : dY \rightarrow C^{Md}(1, Y) \) be the canonical map, and subsequently obtain a notion of Markov-\( d \)-admissibility with just the same properties as before.

2.6 Adjoint endofunctors

A computable endofunctor \( d \) is computably-left-adjoint to a computable endofunctor \( e \) (and \( e \) is right-adjoint to \( d \)), if \( \text{C}(dX, Y) \) and \( \text{C}(X, eY) \) are computably isomorphic, and the isomorphisms are natural in \( X \) and \( Y \).

Likewise, a Markov-computable endofunctor \( d \) is Markov-computably-left-adjoint to a Markov-computable endofunctor \( d \), if \( \text{C}(dX, Y) \) and \( \text{C}(X, eY) \) are Markov-computably isomorphic, and the isomorphisms are natural in \( X \) and \( Y \). Note that a statement that \( X \) and \( Y \) are Markov-computably isomorphic only refers to the cardinality (as there has to be a bijection) and to the computable elements. Note further that for computable endofunctors being Markov-computably-adjoint is a weaker condition than being computably adjoint (and that both concepts formally make sense).

At the current state, we do not have interesting examples of pairs of computably-adjoint computable endofunctors. We will discuss two cases of Markov-computably adjoint Markov-computable functors later.

It is quite illuminating to see the special case of the definitions above where \( Y := S \). We see that if \( d \) is (Markov)-computably-left-adjoint to \( e \), then the (computably) \( e \)-open subsets of \( X \)

\[ \text{Note that just as a computable endofunctor is linked to the topological jump operators of } [8], \text{ Markov-computable endofunctors are linked to the computable jump operators.} \]
are precisely the (computably) open subsets of \( dX \). This aspect of our two examples below has been utilized before.

It is a central fact in the study of pairs of adjoint functors in category theory that their composition induces a monad. Some consequences of this of interest for our theory are the following:

**Proposition 22.** Let the (Markov)-computable endofunctor \( d \) be (Markov)-computably-left-adjoint to the (Markov)-computable endofunctor \( e \). Then:

1. There is a canonic computable unit map \( \eta_X : X \to edX \).
2. \( eY \) and \( edeY \) are computably isomorphic.
3. \( \circ : \mathcal{C}(X, edY) \times \mathcal{C}(Y, eZ) \to \mathcal{C}(X, eZ) \) is well-defined and computable.
4. \( (de) \cong (de)(de) \).

**Proof.**

1. \( \eta_X \in \mathcal{C}(X, edX) \) is the image of the computable map \( \text{id}_{dX} \in \mathcal{C}(dX, dX) \) under the assumed (Markov)-computable isomorphism, and Markov-computable isomorphisms map computable points to computable points.

2. \( \eta_{eY} : eY \to edeY \) from (1.) provides one direction. The computable inverse is obtained by starting from computable \( \text{id}_{eY} \in \mathcal{C}(eY, eY) \), moving to the corresponding map under the computable isomorphism in \( \mathcal{C}(dY, eY) \) and then applying the endofunctor \( e \) on both sides to reach \( \text{id} : edeY \to eY \).

3. By combining (2.) with Proposition 4 (5.).

4. A direct consequence of (2.).

\[ \square \]

Items (3.) & (4.) in the preceding proposition shows that if, given some endofunctor \( e \), we can find a Markov-computably-left-adjoint \( d \) for it, then we obtain a class functions (namely the \( de \)-continuous ones) that is closed under composition, and that if composed with an \( e \)-continuous function from the right, again yield an \( e \)-continuous function.

The importance of adjointness had already been noticed in [8].

### 3 Examples

To substantiate our claim that the framework of \( d \)-admissible spaces actually pertains to descriptive set theory, a few computable endofunctors are investigated. These endofunctors are not freshly introduced here, but have been studied for a while, in particular in work by Ziegler [47, 48]. As we cannot (yet?) give generic characterizations of these endofunctors in terms not specific for the category of represented spaces, we do leave behind our proto-synthetic framework at this stage.

---

\( \eta_X \) is indeed always computable, even if the endofunctors involved are only Markov-computable. The same pattern applies in the following.
3.1 $\Sigma^0_\alpha$-measurability

Consider the partial function $\lim : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ defined via $\lim(p)(n) = \lim_{i \rightarrow \infty} p(\langle n, i \rangle)$. This induces a computable endofunctor $'$ via $(X, \delta_X)' = (X, \delta_X \circ \lim)$ and $(f : X \rightarrow Y)' = f : X' \rightarrow Y'$. We iterate this endofunctor, so let $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$ and $X^{(\beta)} = \pi_2(\prod_{\gamma < \beta} X^{(\gamma)})$ for limit ordinals $\beta$. We claim that the $(\alpha)$-open subsets of a represented space are a suitable generalization of the $\Sigma^0_\alpha$-subsets of a metric space.

**Proposition 23.** $(\alpha)$ is a computable endofunctor preserving binary products. Moreover, $(\alpha+1)$ even preserves products.

**Proof.** That $(\alpha)$ is an endofunctor follows directly from its definition, which leaves the underlying sets and set-theoretical functions unchanged. It being computable for limit ordinal $\alpha$ is straightforward by slice-wise application, so it suffices to prove that $'$ is computable. Let $F \vdash f : X \rightarrow Y$. A naive attempt to obtain a realizer $F'$ for $f' : X' \rightarrow Y'$ would be to define $F'((p_0, p_1, \ldots)) = (F(p_0), F(p_1), \ldots)$. However, any $F(p_i)$ may fail to be well-defined as an element of $\mathbb{N}^\mathbb{N}$. The algorithm will, however, have to produce initial segments of the output of increasing length if $\lim_{n \in \mathbb{N}} p_i \in \text{dom}(F)$. So instead, let $F^n$ be the modification of the algorithm for $F$ that runs only for time $n$, and then stops. Now let $\lambda_{n,i} = \max\{j \leq n \mid F^n(p_j)(i) \text{ exists}\}$, and $F'(\langle p_0, p_1, \ldots \rangle)(\langle n, i \rangle) = F(p_{\lambda_{n,i}})(i)$.

That $(\alpha+1)$ preserves products is a consequence of the position-wise definition of convergence for sequences. To see that $(\alpha)$ preserves binary products for limit ordinals $\alpha$, we just need that $\sup(\beta_1, \beta_2) < \alpha$ for $\beta_1, \beta_2 < \alpha$ (the failure of this to generalize to countably many $\beta$’s is the reason why for limit ordinal $\alpha$, $(\alpha)$ will not preserve countable products). \qed

**Corollary 24** (Synthetic Lebesgue – Hausdorff – Banach Theorem). Every countably based admissible space is $(\alpha+1)$-admissible.

Most of the closure properties of the $(\alpha)$-open subsets follow directly from the general case. Additionally, as $\text{id} : X \rightarrow X'$ and $\neg : S \rightarrow S'$ are computable, we obtain that $\text{id}, C : O^{(\alpha)}(X) \rightarrow O^{(\alpha+1)}(X)$ are computable. By combining these results, also $\bigcup(C) : C(\mathbb{N}, O^{(\alpha)}(X)) \rightarrow O^{(\alpha+1)}(X)$ becomes computable.

**Proposition 25.** Let $X$ be a computable metric space. Then $\bigcup(C) : C(\mathbb{N}, O(X)) \rightarrow O'(X)$ admits a computable multi-valued inverse.

**Proof.** First, we show the claim for $X = \{0, 1\}^\mathbb{N}$, then we transfer the result to general computable metric spaces using the fact that those have effectively fiber-compact representations.

Assume we have some realizer $\chi$ of some $U \in O'(\{0, 1\}^\mathbb{N})$. As $\delta_S \circ \lim(q) = \top$ iff $\exists n \lim_{i \rightarrow \infty} q(\langle n, i \rangle) = 1$ iff $\exists n \forall i \geq k q(\langle n, i \rangle) = 1$, we have that $p \in U$ iff $\exists n \forall i \geq k \chi(p)(\langle n, i \rangle) = 1$. Now consider the closed sets $A_{n,k} = \{p \mid \forall i \geq k \chi(p)(\langle n, i \rangle) = 1\}$ and notice $U = \bigcup_{n,k \in \mathbb{N}} A_{n,k}$ and that the $A_{n,k}$ can by construction be computed from $\chi$.

Before proceeding to general computable metric spaces, we point out that the preceding proof carries over rather directly to subspaces of $\{0, 1\}^\mathbb{N}$, totality of the maps involved is not a concern.

Now, given some $U \in O'(X)$, we use Proposition 4 (6) to compute $\delta_X^{-1}(U) \in O'(\text{dom}(\delta_X))$ and use the established result to obtain some $(A_i)_{i \in \mathbb{N}}$ with $A_i \in \mathcal{A}(\{0, 1\}^\mathbb{N})$ and $\delta_X \left( \bigcup_{i \in \mathbb{N}} A_i \right) = U$.

---

10The definition for limit ordinals was suggested by Bauer at CCA 2009.

11Generally, $\text{id} : O^{(\alpha)}(X) \rightarrow O^{(\beta)}(X)$ is continuous for $\beta \geq \alpha$, and computable, if $\beta$ is computable relative to $\alpha$. 重要作用。
\[ \bigcup_{i \in \mathbb{N}} \delta_X(A_i) = U. \] Now notice that for \( A \in \mathcal{A}(\{0,1\}^\mathbb{N}) \), we find that \( x \notin \delta_X(A) \) iff \( \delta^{-1}(\{x\}) \subseteq A^c \) and choose \( \delta_X \) to be effectively fiber-compact to see that \( \delta_X(A_i) \in \mathcal{A}(X) \) holds uniformly, which concludes the proof.

\[ \square \]

**Proposition 26 (\[11\]).** For a quasi-Polish space \( X \), the elements of \( \mathcal{O}^{(\alpha)}(X) \) are precisely the \( \Sigma_0^{\alpha+1} \)-sets in Selivanov’s definition.

To substantiate the claim that Corollary \[24\] matches the Lebesgue–Hausdorff–Banach Theorem \[4\] one further observation is required. The proof of the second part of the statement relies on \[4\] Section 4.

**Proposition 27.** Let \( Y \) be computable metric spaces. Then \( \text{pw-lim} : \subseteq \mathcal{C}(\mathbb{N}, \mathcal{C}(X, Y^{(\alpha)})) \to \mathcal{C}(X, Y^{(\alpha+1)}) \) is computable. If \( \alpha > 1 \) and \( X \) is a computable metric space, too, then it admits a computable multi-valued inverse.

### 3.2 \( \Delta_2 \)-measurability

Define \( \Delta : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) via \( \Delta(p)(n) = p(n) + 1 + \max\{i \mid p(i) = 0\} - 1 \). Let the finite mindchange endofunctor be defined via \( (X, \delta_X) = (X, \delta_X \circ \Delta) \) and \( (f : X \to Y)^V = f : X^V \to Y^V \). Note that \( S^V = 2^V \), hence the map \( ^C : \mathcal{O}^V(X) \to \mathcal{O}^V(X) \) is computable (thus \( \mathcal{O}^V(X) \cong \mathcal{A}^V(X) \)). Moreover, \( x \mapsto (x, \neg x) : S^V \to S \times S' \) is computable and computably invertible, which implies that \( \mathcal{O}^V(X) \) contains exactly those sets that are both themselves and their complements members of \( \mathcal{O}'(X) \), i.e. corresponds to the \( \Delta_2 \)-sets via Proposition \[26\]. Note that \( \nabla^V \cong \nabla \), hence iteration of this endofunctor makes little sense. For computable metric spaces, \( \nabla \)-continuity is piecewise continuity as shown in \[8\].

In this context, also separation principles play a rôle. Note that \( \nabla \)-Hausdorff separation is a uniform counterpart of the \( T_D \) separation principle \[11\ \[10\], it requires that \( x \mapsto \{x\} : X \to \mathcal{A}^V(X) \) is computable. We required one more concept, namely:

**Definition 28.** We call a space \( X \) completely compact, iff it has a total representation \( \delta_X : \{0,1\}^\mathbb{N} \to X \).

Note that any completely compact space is compact (inherited from \( \{0,1\}^\mathbb{N} \)), whereas there are compact but not completely compact spaces\[^{12}\]. In \[35\], the following was obtained:

**Theorem 29** (Synthetic Jayne Rogers Theorem). Any admissible completely compact \( \nabla \)-Hausdorff space is \( \nabla \)-admissible.

### 3.3 Markov \( \Delta_2^0 \)-measurability and lowness

Our next example both shows the need for the concept of a Markov-computable endofunctor, (as they are not computable endofunctors) and illuminate the rôle of adjoinness introduced in Subsection \[2.6\]. Let \( J : \{0,1\}^\mathbb{N} \to \{0,1\}^\mathbb{N} \) be the Turing-jump (i.e. \( J(p) \) is the Halting problem relative to \( p \)), and then define \( \int \) via \( \int(X, \delta_X) = (X, \delta_X \circ J^{-1}) \) with the straight-forward extension to morphisms. This yields a Markov-computable endofunctor.

As observed in more general terms in Subsection \[2.6\], the computably open subsets of \( \int X \) are just the computably \( \Sigma_2^0 \)-subsets of \( X \). Under this perspective, the space \( \int \{0,1\}^\mathbb{N} \) had already been investigated in \[26\].

\[^{12}\]A somewhat trivial example would be a space without computable points.
Note that $J^{-1}$ is computable, whereas $J$ is not, hence $\text{id} : \int X \to X$ is computable, and $\text{id} : X \to \int X$ typically not.

Now the low-endofunctor $\vee$ is defined via $X^\vee = (\int X)'$. Both $\int$ and $\vee$ were studied in \[3\]. The results there are essentially special cases of Proposition \[22\]. In particular, we see that $(\vee)'(\vee) \cong \vee$, and that if $f$ is $\ell$-continuous and $g$ is $\ell'$-continuous, then $f \circ g$ is $\ell'$-continuous again (hence the name \textit{low}).

Given that $\text{id} : \int S \to 2$ and $\text{id} : \{J(0^N), J(10^N)\} \to \int S$ are computable; and that $\{J(0^N), J(10^N)\}' = 2'$, we find $S^\vee = (\int S)' = 2' = 2^\vee = S^\vee$. Hence, the \textit{low-open} sets are just the $\Delta_2$-sets again. Thus the result (originally from \[35\]) that the Markov $\Delta_2$-measurable functions are the low-computable ones can now be phrased as:

\textbf{Theorem 30.} $\mathbb{N}^\mathbb{N}$ is Markov $\vee$-admissible.

As separating $\vee$-continuity and $\vee$ continuity on Baire space is straight-forward (it follows from the existence of a low uncomputable sequence), we also see that Markov $d$-admissibility and $d$-admissibility are clearly distinct concepts: $\mathbb{N}^\mathbb{N}$ is Markov $\vee$-admissible and $\vee$-admissible, but not Markov $\vee$ admissible (and $\vee$-admissibility is not even defined).

### 3.4 Borel equivalence relations as $^{(\alpha)}$-Hausdorff spaces

Not only are the derived spaces of our theory such as $C'(X, Y)$ not admissible, but there are well-studied examples in descriptive set theory of represented spaces that are not admissible (i.e. not understandable as topological spaces). Borel equivalence relations (see e.g. \[24\]) can be defined in our framework as follows:

\textbf{Definition 31.} We call a space $X$ a \textit{Borel equivalence relation}, if it has a total representation and is $^{(\alpha)}$-Hausdorff for some $\alpha$.

Spaces with total representations have been studied by Selivanov in \[11\], and spaces that are admissible and $^{(\alpha)}$-Hausdorff by Schröder and Selivanov in \[39\]. However, a particular Borel equivalence relation of crucial interest is $E_0$ given by $\delta_{E_0}(p) = \delta_{E_0}(q)$ iff $\exists n p_{\geq n} = q_{\geq n}$; and $E_0$ is easily seen not to be admissible.

\textbf{Theorem 32} (Harrington, Kechris & Louveau \[25\]). Let $X$ be a Borel equivalence relation. Then exactly one of the following holds:

1. $\exists \alpha \exists f : X \to \{0, 1\}^N$ such that $f$ is a continuous injection.

2. There is a continuous embedding $E_0 \hookrightarrow X$.

A curious phenomenon easily demonstrated on $E_0$ is that $O(E_0)$ is trivial (i.e. $\{0, E_0\}$), but for $\alpha > 0$, we find $O^{(\alpha)}(E_0)$ to carry a Borel-like structure. This shows that descriptive set theory can make sense on represented spaces that have the indiscrete topology as their associated topology, hence are not susceptible to any approach building $\Sigma^0_2$-sets from $\Sigma^0_1$-sets.

### 3.5 The endofunctor $\mathcal{K}$

We shall provide an example of a computable endofunctor $d$ that does change the underlying sets, and consider to what extent notions such as $b$-measurable sets or $d$-continuous functions still make sense. Our example is a very familiar one, namely the operation that takes a represented spaces $X$ to the space of compact subsets $\mathcal{K}(X)$, and a continuous function $f : X \to Y$ to its lifted version. For details, see \[31\].
To understand the $K$-open sets, we need to have a look at $\mathcal{K}(S)$. This space has three elements, $\{\emptyset, \{\top\}, S\}$, and carries the generalized Sierpiński topology, i.e. $\mathcal{O}(\mathcal{K}(S))$ has the underlying set $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\top\}\}, \{\emptyset, \{\top\}, S\}\}$. It seems sensible to interpret $\mathcal{K}(S)$ as a three-valued logic, with $\emptyset$ being unknown, $\{\top\}$ being plausible and $S$ being true. Thus, a $K$-open set actually is two open sets, with one contained in the other. Elements of the inner open set are definitely in the $K$-open, for elements of the outer but not the inner, it is plausible but unknown that they are members of the $K$-open. Operations such as preimage under continuous functions make sense for such a structure, whereas intersection could be defined in a few different ways – and as $K$ does not preserve binary products, we only make claims for the former.

Now let us consider $K$-continuous functions. A $K$-continuous function from $X$ to $Y$ maps points in $X$ to compact sets in $Y$ – and such mappings have been studied extensively as the upper hemicontinuous maps from $X$ to $Y$. Likewise, we may consider the computable endofunctor $\mathcal{V}$ that maps a space to the space of its overt subsets and lifts functions, and would obtain the lower hemicontinuous maps as the $\mathcal{V}$-continuous ones. While our framework does not have many implications for these classes (mainly closure under composition with continuous functions), their example nevertheless indicates that it is unnecessary to restrict our framework to those endofunctors leaving the underlying sets intact (which would be quite problematic for the synthetic part).

### 3.6 The analytic sets

There are various characterizations of the analytic sets in classical descriptive set theory, two of which are particularly relevant for our interests. They can be introduced either as the images of $\mathbb{N}^\mathbb{N}$ under a continuous function, or as the projections of closed subsets of $\mathbb{N}^\mathbb{N} \times X$ to the second component. In an effective setting, these two split – a situation reminiscent of (and ultimately related to) the split of the classical concept closed set into closed set and overt set in synthetic topology.

First, we shall see that the overt sets, rather than the closed set, occur as the images of $\mathbb{N}^\mathbb{N}$ under continuous functions. Unfortunately, we can only prove this for computable metric spaces for now. As shown in [6], the identity $\text{id}: \mathcal{A}(X) \rightarrow \mathcal{V}(X)$ is never computable for a non-empty space $X$, together with the following result, this implies that the first definition of analytic sets cannot yield an extension of the closed sets.

**Proposition 33** (14). The map $\text{Image}: \mathcal{C}(\mathbb{N}^\mathbb{N}, X) \rightarrow \mathcal{V}(X)$ is computable and has a computably multivalued inverse with domain $\mathcal{V}(X) \setminus \{\emptyset\}$ for any complete computable metric space $X$.

**Proof.** That Image is computable holds true for all represented spaces as a corollary of [6, Proposition 7.4 (7)].

For the computability of the inverse, let us be given a non-empty overt set $A \in \mathcal{V}(X)$. Further let $(a_n)_{n \in \mathbb{N}}$ be a computable dense sequence in $X$. We describe a function $f \in \text{Image}^{-1}(A)$ in terms of a labeled complete countably-branching tree. At the top level, test simultaneously for all $n \in \mathbb{N}$ if $A \cap B(a_n, 1) \neq \emptyset$. There is at least one such $n$. Thus, we can obtain an infinite sequence $(n_i)_{i \in \mathbb{N}}$ such that $\{n_i \mid i \in \mathbb{N}\} = \{n \mid A \cap B(a_n, 1) \neq \emptyset\}$. We then label the $i$-th child of

---

13Where we tacitly understand an overt set to be closed in order to uniquely identify it.

14This result is essentially present in [7].
the root with $B(a_n, 1)$. For all subsequent vertices, if the current vertex with depth $k$ is labeled by $B$, we proceed as above, but test $A \cap B \cap B(a_n, 2^{-k})$ instead.

From this labeled tree we can find the function $f$ by mapping a path to the unique point in the intersection of all its labels. This is a computable operation due to the properties of complete computable metric spaces, and by construction we find that $f[\mathbb{N}^\mathbb{N}] = A$.

The definition of analytic sets as projections however works nicely in our context. We will start by introducing a variant of the Sierpiński-space suitable for capturing this class.

**Definition 34.** Let the space $aS = (\{\perp, \top\}, \delta_aS)$ be defined via $\delta_aS(p) = \top$ iff $p$ codes an ill-founded tree, and $\delta_aS(p) = \perp$ otherwise.

We can extend this definition to yield an endofunctor $a$ making $aS$ understood as $a$ applied to $S$ equivalent to $aS$ defined explicitly above by understanding $aX$ to be the suitable subspace of $C(C(X, S), aS)$.

**Proposition 35.** For any represented space $X$ the map $\pi_2 : A(\mathbb{N}^\mathbb{N} \times X) \to O^a(X)$ is computable and has a computable inverse.

As shown in [17, Section 5], the continuity structure on $O^a(X)$ for Polish spaces $X$ corresponds to the structure induced by good universal system as employed in [28, 18].

Just like we introduced a Markov-computably-left adjoint Markov-computable endofunctor for $'$ in Subsection 3.3 we can introduce a Markov-computably-left adjoint Markov-computable endofunctor for $a$. Pick some standard enumeration $(A_n)_{n \in \mathbb{N}}$ of the computable $\Sigma^1_1$-subsets of Baire, and then let $j_{GH} : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be defined via $j_{GH}(p) = q$ iff $\{n \mid \exists i p(i) = n + 1\} = \{n \mid q = A_n\}$. The map $j_{GH}$ is surjective, and can thus be understood to be a representation – in fact, it is an admissible representation of the Gandy-Harrington space. A name for a point is an enumeration of all effectively analytic sets containing it.

Next, we introduce $\int_{GH}$ via $\int_{GH}(X, \delta_X) = (X, \delta_X \circ j_{GH})$ and the straight-forward extension to morphisms, and find $\int_{GH}$ to be a Markov-computable endofunctor which is Markov-computably-left-adjoint to $a$. This endofunctor turns effectively $\Sigma^1_1$-sets into effectively open sets – it seems reasonable to suspect a connection with Gregoriades’ work on turning Borel sets into clopens [16].

### 3.7 $K_{\sigma}$-property and $'$-overtness

While overtness is a classically invisible condition, its lift along the $'$-endofunctor does yield a topological property similar to the $K_{\sigma}$-property. For computable Polish spaces, it is in fact equivalent:

**Theorem 36.** A Polish space is $'$-overt iff it is $K_{\sigma}$.

**Proof.** By Corollary 38 and Lemma 39.

**Lemma 37.** Let $X$ be a computably compact computable metric space. Then $X$ is $'$-overt.

**Proof.** Using Proposition 25 we can effectively express any $'$-open set $U$ as a union $\bigcup_{n \in \mathbb{N}} A_n$ of closed sets. As $X$ is computably compact, these closed sets are uniformly compact. By definition of compactness, IsEmpty : $\mathbb{K}(X) \to S$ is computable. Then IsNonEmpty : $\mathbb{K}(X) \to S'$ is computable, too. As $'$ preserves products, we see that $\bigvee : \mathbb{C}(\mathbb{N}, S') \to S'$ is computable, and its application yields the final answer.
Corollary 38. Let $X$ be an effectively $K_\sigma$ metric space. Then $X$ is $'$-overt.

Proof. Use Lemma 37 together with Proposition 20.

Lemma 39. Let a Polish space $X$ be $'$-overt. Then $X$ is $K_\sigma$.

Proof. Assume that $X$ is not $K_\sigma$ and $'$-overt. Then by [24, Theorem 7.10] there is an embedding $\iota : \mathbb{N}^\mathbb{N} \to X$ such that $\iota[\mathbb{N}^\mathbb{N}]$ is closed in $X$. Thus, we may understand $A(\mathbb{N}^\mathbb{N}) \subseteq A(X) \subseteq O'(X)$ (maybe by employing some oracle). That $X$ is $'$-overt now yields that $\text{IsNonEmpty} : A(\mathbb{N}^\mathbb{N}) \to S'$ is continuous. Now the names of non-empty closed subsets of $\mathbb{N}^\mathbb{N}$ are essentially the ill-founded countably branching trees, which is known to be $\Pi_1^1$-complete (e.g. [12, Section 11.8]). But $\text{IsNonEmpty} : A(\mathbb{N}^\mathbb{N}) \to S'$ being continuous would imply this set to be Borel, contradiction.

4 Concluding remarks

Hopefully, our sketch of synthetic descriptive set theory convincingly outlines an exciting new paradigm – there clearly is much more to do before it can be said to rival classic or effective descriptive set theory when it comes to the breadth of the picture painted. A next step would be to obtain a better understanding of the interplay of various computable endofunctors. For example, as the $\Delta^0_2$-sets are just those $\Sigma^0_2$-sets with $\Sigma^0_2$-complement, it seems reasonable to expect a generic way of obtaining $\nabla$ from $'$. Likewise, it seems desirable to obtain the projective hierarchy from the Borel hierarchy, in particular a synthetic Suslin theorem, as GREGORIADIES pointed out.

We should not shy away from pointing out that there is an obstacle to synthetic results implying their classical counterparts, namely uniformity in function measurability: Traditionally, a function $f$ is $\mathfrak{B}$ measurable, if any preimage of an open set is in $\mathfrak{B}$ – no requirements are imposed on the associated preimage map $f^{-1}$ besides (classically) existing. In a synthetic (or a constructivist) framework, however, such conditions do not appear: The preimage map needs to be a morphism of the underlying category. As discussed already in [35], when working in the category of represented spaces it is sometimes possible to prove that the classical existence implies continuity, but these proofs can be non-trivial.

References

[1] C.E. Aull & W.J. Thron (1962): Separation axioms between $T_0$ and $T_1$. Proc. Nederl. Akad. Wet.
[2] Andrej Bauer & Davorin Lesnik (2012): Metric spaces in synthetic topology. Annals of Pure and Applied Logic 163(2), pp. 87 – 100.
[3] Vasco Brattka: Limit Computable Functions and Subsets. Unpublished.
[4] Vasco Brattka (2005): Effective Borel measurability and reducibility of functions. Mathematical Logic Quarterly 51(1), pp. 19–44.
[5] Vasco Brattka, Matthew de Brecht & Arno Pauly (2012): Closed Choice and a Uniform Low Basis Theorem. Annals of Pure and Applied Logic 163(8), pp. 968–1008.
[6] Vasco Brattka & Guido Gherardi (2009): Borel Complexity of Topological Operations on Computable Metric Spaces. Journal of Logic and Computation 19(1), pp. 45–76.
[7] Vasco Brattka & Gero Presser (2003): Computability on subsets of metric spaces. Theoretical Computer Science 305(1-3), pp. 43 – 76.
[8] Matthew de Brecht (2013). Levels of discontinuity, limit-computability, and jump operators. arXiv 1312.0697.
[9] Matthew de Brecht (2013): Quasi-Polish spaces. Annals of Pure and Applied Logic 164(3), pp. 354–381.
[10] Matthew de Brecht & A. Yamamoto (2010): Topological properties of concept spaces. Information and Computation 208(4), pp. 327–340.
[11] Matthew de Brecht & Akihiro Yamamoto (2009): $\Sigma_0^\alpha$ - Admissible Representations (Extended Abstract). In: Andrey Bauer, Peter Hertling & Ker-I Ko, editors: 6th Int’l Conf. on Computability and Complexity in Analysis, Schloss Dagstuhl. Available at http://drops.dagstuhl.de/opus/volltexte/2009/2264.
[12] Andrew M. Bruckner, Judith B. Bruckner & Brian S. Thomson (1997): Real Analysis. Prentice-Hall.
[13] Martin Escardó (2004): Synthetic topology of datatypes and classical spaces. Electronic Notes in Theoretical Computer Science 87.
[14] Martin Escardó (2009). Intersections of compactly many open sets are open. Available at http://www.cs.bham.ac.uk/~mhe/papers/compactness-submitted.pdf.
[15] Jean-Yves Girard (1995): Light linear logic. In: Daniel Leivant, editor: Logic and Computational Complexity, Lecture Notes in Computer Science 960, Springer Berlin Heidelberg, pp. 145–176.
[16] Vassilios Gregoriades (2012): Turning Borel sets into clopen sets effectively. Fundamenta Mathematicae 219(2), pp. 119–143.
[17] Vassilios Gregoriades, Tamás Kispéter & Arno Pauly (2014). A comparison of concepts from computable analysis and effective descriptive set theory. arXiv:1401.3325.
[18] Vassilios Gregoriades & Yiannis N. Moschovakis. Notes on effective descriptive set theory. notes in preparation.
[19] Kojiro Higuchi & Takayuki Kihara (2014): Inside the Muchnik degrees I: Discontinuity, learnability and constructivism. Annals of Pure and Applied Logic 165(5), pp. 1058 – 1114.
[20] Kojiro Higuchi & Takayuki Kihara (2014): Inside the Muchnik degrees II: The degree structures induced by the arithmetical hierarchy of countably continuous functions. Annals of Pure and Applied Logic 165(6), pp. 1201 – 1241.
[21] Hugo de Holanda Cunha Nobrega (2013): Game characterizations of function classes and Weihrauch degrees. M.Sc. thesis, University of Amsterdam.
[22] J.E. Jayne & C.A. Rogers (1982): First level Borel functions and isomorphisms. Journal de Mathematiques Pures et Appliquees 61, pp. 177–205.
[23] Miroslav Kačena, Luca Motto Ros & Brian Semmes (2012/3): Some observations on ‘A New Proof of a Theorem of Jayne and Rogers’. Real Analysis Exchange 38(1), pp. 121–132.
[24] A.S. Kechris (1995): Classical Descriptive Set Theory, Graduate Texts in Mathematics 156. Springer.
[25] A. S. Kechris L. A. Harrington & A. Louveau (1990): A Glimm-Effros dichotomy for Borel equivalence relations. Journal of the AMS 3, pp. 903–928.
[26] Joseph S. Miller (2002): $\Pi_0^1$ Classes in Computable Analysis and Topology. Ph.D. thesis, Cornell University.
[27] Eugenio Moggi (1991): Notions of computation and monads. Information and Computation 93(1), pp. 55 – 92. Selections from 1989 {IEEE} Symposium on Logic in Computer Science.
[28] Yiannis N. Moschovakis (1980): Descriptive Set Theory, Studies in Logic and the Foundations of Mathematics 100. North-Holland.
[29] Yiannis N. Moschovakis (2010): Classical descriptive set theory as a refinement of effective descriptive set theory. Annals of Pure and Applied Logic 162, pp. 243–255.
[30] Luca Motto-Ros (2008): A new characterization of the Baire class 1 functions. Real Analysis Exchange 34(1), pp. 29–48.
[31] Luca Motto Ros (2011): Game representations of classes of piecewise definable functions. Mathematical Logic Quarterly 57(1), pp. 95–112.
[32] Luca Motto Ros & Brian Semmes (2009): A New Proof of a Theorem of Jayne and Rogers. Real Analysis Exchange 35(1), pp. 195–204.
[33] Arno Pauly (2011). Many-one reductions between search problems. arXiv 1102.3151. Available at http://arxiv.org/abs/1102.3151
[34] Arno Pauly (2012). A new introduction to the theory of represented spaces. http://arxiv.org/abs/1204.3763.
[35] Arno Pauly & Matthew de Brecht (2014): Non-deterministic Computation and the Jayne Rogers Theorem. Electronic Proceedings in Theoretical Computer Science 143. DCM 2012.
[36] Matthias Schröder (2002): Admissible Representations for Continuous Computations. Ph.D. thesis, FernUniversität Hagen.
[37] Matthias Schröder (2002): Extended admissibility. Theoretical Computer Science 284(2), pp. 519–538.
[38] Matthias Schröder & Victor Selivanov (2014). Hyperprojective Hierarchy of QCB₀-spaces. arXiv 1404.0297. Available at http://arxiv.org/abs/1404.0297
[39] Matthias Schröder & Victor L. Selivanov (2013). Some hierarchies of QCB₀-spaces. arXiv 1304.1647. Available at http://arxiv.org/abs/1304.1647
[40] Victor L. Selivanov (2004): Difference hierarchy in φ-spaces. Algebra and Logic 43(4), pp. 238–248.
[41] Victor L. Selivanov (2013): Total representations. Logical Methods in Computer Science 9(2).
[42] Paul Taylor (2010): A lambda calculus for real analysis. Journal of Logic & Analysis 2(5), pp. 1–115.
[43] Paul Taylor (2011): Foundations for Computable Topology. In: Foundational Theories of Classical and Constructive Mathematics, The Western Ontario Series in Philosophy of Science 76, Springer, pp. 265–310.
[44] William W. Wadge (1972): Degrees of complexity of subsets of the Baire space. Notices of the AMS 19, pp. 714–715.
[45] William W. Wadge (1983): Reducibility and determinateness on the Baire space. Ph.D. thesis, University of California, Berkeley.
[46] Klaus Weihrauch (2000): Computable Analysis. Springer-Verlag.
[47] Martin Ziegler (2007): Real Hypercomputation and Continuity. Theory of Computing Systems 41, pp. 177 – 206.
[48] Martin Ziegler (2007): Revising Type-2 Computation and Degrees of Discontinuity. Electronic Notes in Theoretical Computer Science 167, pp. 255–274.

Acknowledgements

The first author would like to thank Vasco Brattka, Vassilis Gregoriades, Takayuki Kihara, Luca Motto-Ros, Matthias Schröder and Victor Selivanov for fruitful discussions on the subject of the present paper. The work has benefited from the Marie Curie International Research Staff Exchange Scheme Computable Analysis, PIRSES-GA-2011- 294962.