OPERATOR SEGAL ALGEBRAS IN FOURIER ALGEBRAS

BRIAN E. FORREST, NICO SPRONK AND PETER J. WOOD

Abstract. Let $G$ be a locally compact group, $A(G)$ its Fourier algebra and $L^1(G)$ the space of Haar integrable functions on $G$. We study the Segal algebra $S^1A(G) = A(G) \cap L^1(G)$ in $A(G)$. It admits an operator space structure which makes it a completely contractive Banach algebra. We compute the dual space of $S^1A(G)$. We use it show that restriction operator $u \mapsto u|_H : S^1A(G) \to A(H)$, for some non-open closed subgroups $H$, is a surjective complete quotient map. We also show that if $N$ is a non-compact closed subgroup, then the averaging operator $\tau_N : S^1A(G) \to L^1(G/N)$, $\tau_N u(sN) = \int_N u(sn)dn$ is a surjective complete quotient map. This puts an operator space perspective on the philosophy that $S^1A(G)$ is “locally $A(G)$ while globally $L^1$”. Also, using the operator space structure we can show that $S^1A(G)$ is operator amenable exactly when $G$ is compact; and we can show that it is always operator weakly amenable. To obtain the latter fact, we use E. Samei’s theory of hyper-Tauberian Banach algebras.

1. OPERATOR SEGAL ALGEBRAS

1.1. Notation. For any Banach space $\mathcal{X}$ we let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of bounded linear operators from $\mathcal{X}$ to itself, and $b_1(\mathcal{X})$ the set of all vectors of norm not exceeding 1.

For details on classical harmonic analysis, we use [15, 23]. We will always let $G$ denote a locally compact group with a fixed left invariant Haar measure $m$. For $1 \leq p \leq \infty$, $L^p(G)$ is the usual $L^p$-space with respect to $m$. If $f, g$ are Borel measurable functions and $s \in G$, we...
have for almost every $t$ in $G$ that
\[ s*f(t) = f(s^{-1}t), \quad f*g(t) = \int_G f(s)s*g(t)dt \quad \text{and} \quad \hat{f}(t) = f(t^{-1}) \]
denotes the left group action, convolution (when the integrand makes sense) and inversion. We note that $L^1(G)$ is a Banach algebra with respect to convolution.

We let $A(G)$ and $B(G)$ denote the Fourier and Fourier-Stieltjes algebras of $G$ which are Banach algebras of continuous functions on $G$ and were introduced in [7]. We recall, from that article, that $A(G)$ consists exactly of functions on $G$ of the form $u(s) = \langle \lambda(s)f|g \rangle = g\ast \hat{f}(s)$, where $\lambda : G \to B(L^2(G))$ is the left regular representation given by $\lambda(s)f = s\ast f$. The dual of $A(G)$ is the von Neumann algebra $VN(G)$, which is generated by $\lambda(G)$ in $B(L^2(G))$.

Our standard references for operator spaces are [6, 22]. An operator space is a complex Banach space $V$ equipped with an operator space structure: for each space of matrices $M_n(V)$ with entries in $V$, $n \in \mathbb{N}$, we have a norm $\|\cdot\|_{M_n(V)}$, and the norms satisfy Ruan’s axioms in addition to that $\|\cdot\|_{M_1(V)}$ is the norm on $V = M_1(V)$. A map $T$ from $V$ to another operator space $W$ is said to be completely bounded if the family of linear operators $[v_{ij}] : M_n(V) \to M_n(W)$ is uniformly bounded over $n$. If $A$ is an algebra and an operator space, for which $V$ is a left module over $A$, we say $V$ is a completely bounded $A$-module if there is $C > 0$ so for each $[a_{ij}]$ in $M_n(A)$ and each $[v_{kl}]$ in $M_m(V)$ we have
\[ \| [a_{ij}v_{kl}] \|_{M_{mn}(V)} \leq C \| [a_{ij}] \|_{M_{nm}(A)} \| [v_{kl}] \|_{M_{mn}(V)}. \]
We say $V$ is a completely contractive $A$-module if we can set $C = 1$. This is the same as asserting that the module multiplication, extends to a map on the operator projective tensor product $A \hat{\otimes} V \to V$ is bounded at all matrix levels by $C$. We say $A$ is a completely bounded (contractive) Banach algebra if it itself is a completely bounded (contractive) $A$-module. We note that any $C^*$-algebra admits a canonical operator space structure. The algebras $L^1(G)$ and $A(G)$ will always have the standard predual structures (see [3]), in their respective roles as the preduals of $L^\infty(G)$ and $VN(G)$. With these operator space structures, these are completely contractive Banach algebras.

1.2. Abstract Operator Segal Algebras. Let $A$ be a completely contractive Banach algebra. A (contractive) [left] operator Segal algebra in $A$ is a dense [left] ideal $SA$ equipped with an operator space structure $\{\|\cdot\|_{M_n(SA)} : M_n(SA) \to \mathbb{R}_{\geq 0}\}$ under which
\[ (OSA1) \quad (SA, \|\cdot\|_{SA}) \text{ is a Banach space,} \]
(OSA2) the identity map $SA \hookrightarrow A$ is completely bounded (contractive), and

(OSA3) $SA$ is a completely bounded (contractive) $A$-bimodule.

Note that (OSA2) and (OSA3) imply that $SA$ is a completely bounded Banach algebra. Moreover $SA$ is a completely contractive Banach algebra if the associated maps and module actions are completely contractive.

Let us see that operator Segal algebras are reasonably common.

(i) For any Banach space $X$, let $\text{max}^X$ denote the maximal operator space whose underlying Banach space is $X$. If $A$ is any Banach algebra with abstract Segal subalgebra $SA$, then $\text{max}^X A$ is an operator Segal algebra in $\text{max}^X A$.

(ii) Let $H$ be a Hilbert space, $K(H)$ the C*-algebra of compact operators on $H$ and for $1 \leq p < \infty$, $S^p(H)$ be the Shatten $p$-class operators. Since we have $S^1(H) \cong K(H)^*$ under the dual pairing $(s,k) \mapsto \text{trace}(sk)$, we see that $S^1(H)$ is a completely contractive $K(H)$-module in its dual operator space structure, and hence is an operator Segal algebra. For $1 < p < \infty$, we assign to $S^p(H)$ the interpolated operator space structure $S^p(H) = (K(H), S^1(H))_{1/p}$. See [22] for more on this. By the functorial properties of operator interpolation we can verify that (OSA2) and (OSA3) obtain for $S^p(H)$, and, in fact, $S^p(H)$ is a contractive Segal algebra in $K(H)$.

(iii) Similarly to (ii), above, we can see that if $I$ is any set and $c_0(I)$ denotes the $C^*$-algebra of functions on $I$ vanishing at infinity, then $\ell^1(I) \cong c_0(I)^*$ is a contractive operator Segal algebra in $c_0(I)$. With the operator interpolation structure $\ell^p(I) = (c_0(I), \ell^1(I))_{1/p}$ ($1 \leq p \leq \infty$) we have that $\ell^p(I)$ is a contractive operator Segal algebra in $c_0(I)$.

1.3. The $1$-Segal Fourier algebra. We define the $1$-Segal Fourier algebra to be the space

$$S^1A(G) = A(G) \cap L^1(G).$$

For $u$ in $S^1A(G)$ we let

$$\|u\|_{S^1A} = \|u\|_{A} + \|u\|_{L^1}.$$ 

In [12] this space is denoted $L^A(G)$. It is shown in Lemma 1.1 of that article that it is complete; and in Proposition 2.5 that it is a Segal algebra in $A(G)$. We will reserve the notation $L^A(G)$ for $S^1A(G)$, when it is treated as a Segal algebra in $L^1(G)$, and call it the Lebesgue-Fourier algebra. If $G$ is abelian, with dual group $\hat{G}$, then there is an isometric algebra isomorphism $S^1A(G) \cong L^A(\hat{G})$. 
Let us assign to $S^1A(G)$ a natural operator space structure. The norm on $S^1A(G)$ was gained via the embedding $S^1A(G) \hookrightarrow A(G) \oplus_1 L^1(G) : u \mapsto (u, u)$. The space $A(G) \oplus_1 L^1(G)$ is the predual of the von Neumann algebra $VN(G) \oplus_\infty L^\infty(G)$ and as such inherits a natural operator space structure $\mathfrak{3} \mathfrak{9}$. Thus we identify the matrix space $M_n(S^1A(G))$ as a subspace of $A(G) \oplus_1 L^1(G) \cong CB^\sigma(VN(G) \oplus_\infty L^\infty(G), M_n)$. Hence if $[u_{ij}] \in M_n(S^1A(G))$ we obtain

$$
\||[u_{ij}]||_{M_n(S^1A)} = \sup \left\{ \|\langle T_{pq}, u_{ij} \rangle + \langle \varphi_{pq}, u_{ij} \rangle \|_{M_{nr}} : [T_{pq}] \in b_1(M_r(VN(G))), [\varphi_{pq}] \in b_1(M_r(L^\infty(G))) \right\}
$$

where $\langle T_{pq}, u_{ij} \rangle$ indicates the $VN(G)$-$A(G)$ dual pairing and $\langle \varphi_{pq}, u_{ij} \rangle = \int_G \varphi_{pq}(s) u_{ij}(s) ds$ indicates the $L^\infty(G)$-$L^1(G)$ dual pairing, for each index quartuple $p, q, i, j$. It follows immediately that $\||[u_{ij}]||_{M_n(S^1A)} \geq \||u_{ij}||_{M_n(A)}$, so (OSA2) is satisfied. It remains to check the axiom (OSA3). Suppose $[v_{kl}] \in b_1(M_m(A(G)))$. Then $\||v_{kl}||_{M_m(L^\infty)} \leq [v_{kl}]||_{M_m(A)} \leq 1$, since the injection $A(G) \hookrightarrow L^\infty(G)$ is a contraction, hence a complete contraction, so $\||[\varphi_{pq} v_{kl}]||_{M_{mr}(L^\infty)} \leq 1$ for each $[\varphi_{pq}]$ in $b_1(M_r(L^\infty(G)))$. Hence we have that

$$
\||[v_{kl} u_{ij}]||_{M_{mm}(S^1A)} = \sup \left\{ \|\langle T_{pq}, v_{kl} u_{ij} \rangle + \langle \varphi_{pq}, v_{kl} u_{ij} \rangle \|_{M_{nr}} \right\}
$$

$$
= \sup \left\{ \|\langle T_{pq} v_{kl}, u_{ij} \rangle + \langle \varphi_{pq} v_{kl}, u_{ij} \rangle \|_{M_{nr}} \right\}
$$

$$
\leq \sup \left\{ \|\langle T_{pq}, u_{ij} \rangle + \langle \varphi_{pq}, u_{ij} \rangle \|_{M_{nr}} \right\}
$$

$$
= \||u_{ij}||_{M_n(S^1A)}
$$

where the suprema are taken over choices $[T_{pq}]$ in $b_1(M_r(VN(G)))$, $[\varphi_{pq}]$ in $b_1(M_r(L^\infty(G)))$ and varying $r$ in $\mathbb{N}$. Thus if $[v_{kl}] \in b_1(M_m(A(G)))$ and $[u_{ij}] \in M_n(S^1A(G))$ then

$$
\||v_{kl} u_{ij}||_{M_{mm}(S^1A)} \leq \||v_{kl}||_{M_m(A)} \||u_{ij}||_{M_n(S^1A)}.
$$

Collecting these facts together we obtain

**Proposition 1.1.** $S^1A(G)$ is a contractive operator Segal algebra in $A(G)$.

The convolution algebra $LA(G)$ was shown in $\mathfrak{1} \mathfrak{2}$ to be a left Segal algebra in $L^1(G)$. It is immediate that under the operator space structure developed above that the injection $LA(G) \hookrightarrow L^1(G)$ is a complete contraction and, since $L^1(G)$ has the maximal operator space structure, that $LA(G)$ is a completely contractive $L^1(G)$-module. Thus $LA(G)$ is an operator Segal algebra as well.
1.4. Other Segal algebras. If $1 < p < \infty$ we define the Segal $p$-Fourier algebra by

$$S^p A(G) = A(G) \cap L^p(G).$$

It is standard to show that if we embed $S^p A(G) \hookrightarrow A(G) \oplus_1 L^p(G)$, then we obtain a Segal algebra in $A(G)$. If $G$ is abelian with dual group $\hat{G}$, then $S^p A(G)$ is the Segal algebra in $L^1(\hat{G})$ given by

$$\{ f \in L^1(\hat{G}) : \hat{f} \in L^p(G) \};$$

see [23]. If we admit on $L^p(G)$ an operator space structure for which it is a completely contractive $L^\infty(G)$-module – we might refer to this as an “$L^\infty$-homogeneous operator space structure”, then we may assign an operator space structure on $S^p A(G)$ analogously to that which we assigned to $S^1 A(G)$, above. The result is still a contractive operator Segal algebra. If $p = 2$, there are many candidate operator space structures including row, column and the “operator Hilbert space” $OL^2(G)$. For any $p$, the interpolated structures $L^p(G) = (\min L^\infty(G), \max L^1(G))_{1/p}$ suffice. See [22] for information on interpolation and $OL^2(G)$. By [20, Theo. 3.5] or [19] there are certain row and column operator space structures ROW($L^p(G)$) and COL($L^p(G)$) which are also $L^\infty$-homogeneous operator space structures.

For $0 < q < \infty$ the Figà-Talamanca-Herz algebra $A_q(G)$ has been shown in [20] to admit an operator space structure under which it is a completely bounded Banach algebra. If $1 < p < \infty$ we can define the Segal $p,q$-Figà-Talamanca-Herz algebra by

$$S^{p,q} A_q(G) = A_q(G) \cap L^p(G).$$

If on $L^p(G)$ we admit a $L^\infty$-homogeneous operator space structure, then similarly as above we obtain an operator Segal algebra in $A_q(G)$. Note that $S^{p,q} A_q(G)$ may not be a completely contractive Banach algebra as $A_q(G)$ is not known to be.

2. Dual Spaces

In this section we develop the dual of $S^1 A(G)$ and apply this to determining some restriction and averaging theorems.

We begin with a useful lemma. We denote the group action of right translation by

$$t \cdot f(s) = f(st).$$

for $t, s$ in $G$, where $f$ is any function on $G$. If $S^1(G)$ is a left Segal algebra in $L^1(G)$, then we say that $S^1(G)$ has continuous right translations if for any $u$ in $S^1(G)$, $t \cdot u \in S^1(G)$, and $t \mapsto t \cdot u : G \to S^1(G)$ is
Proposition 2.2. Let $L$ be a quotient of $VN(G)$ equivalent:

We note that $\{\text{continuous right translations}\}$, then, for any $u$ in $S^1(G)$, $u*e_U \in S^1(G)$ for each $U$ and $\lim_{U \in \mathcal{U}} \|u*e_U - u\|_{S^1} = 0$.

Proof. If $u \in S^1(G)$ then for each $U$ in $\mathcal{U}$ and almost every $s$ in $G$ we have

$$u*e_U(s) = \int_G u(t)e_U(t^{-1}s)dt = \int_G u(st)e_U(t)dt = \frac{1}{m(U)} \int_U f(st)dt$$

where we used symmetry of $U$ to obtain that $\bar{e}_U = e_U$. Since right translation is continuous on $G$, and $U$ is relatively compact, we may regard

$$u*e_U = \frac{1}{m(U)} \int_U t^{-1}dt$$

as a Bochner integral, converging in $S^1(G)$. We then obtain

$$\|u*e_U - u\|_{S^1} = \left\|\frac{1}{m(U)} \int_U (t^{-1}dt - u)dt\right\|_{S^1}$$

$$\leq \frac{1}{m(U)} \int_U \|t^{-1}dt - u\|_{S^1} dt \leq \sup_{t \in U} \|t^{-1}dt - u\|_{S^1} \longrightarrow 0.$$ 

We note that $\{e_U\}_{U \in \mathcal{U}}$ is a well-known symmetric bounded approximate identity in $L^1(G)$. \hfill \Box

2.1. $L^\infty$-convolvers. Since the metrical structure on $S^1A(G)$ is determined by an embedding $S^1A(G) \hookrightarrow A(G) \oplus_1 L^1(G)$, the dual $S^1A(G)^*$ is a quotient of $VN(G) \oplus_\infty L^\infty(G)$ by the annihilator $S^1A(G)^\perp$. The $L^\infty$-convolvers allow us to describe this annihilator.

Proposition 2.2. Let $\varphi \in L^\infty(G)$. Then the following statements are equivalent:

(i) $\sup\{\|\varphi*f\|_{L^2} : f \in L^2(G) \cap L^1(G)^\perp$ and $\|f\|_{L^2} \leq 1\} < +\infty$, and

(ii) $\sup\left\{\left|\int_G \varphi(s)u(s)ds\right| : u \in S^1A(G)$ and $\|u\|_A \leq 1\right\} < +\infty$. 
In this case the operator $f \mapsto \varphi * f : L^2(G) \cap L^1(G)^\vee \to L^2(G)$ extends uniquely to a bounded linear operator $\Lambda(\varphi)$ on $L^2(G)$. Furthermore $\Lambda(\varphi) \in VN(G)$, and the quantities in (i) and (ii) are each equal to $\|\Lambda(\varphi)\|_{VN}$.

In (i), $L^1(G)^\vee = \{\check{f} : f \in L^1(G)\}$; which is $L^1(G)$ itself exactly when $G$ is unimodular. By [15, 20.16], if $\varphi \in L^\infty(G)$ and $f \in L^1(G)^\vee$, then $\varphi * f$ makes sense; though, in general, it is not square integrable, even if we further assume that $f$ is. For example, if $G$ is non-compact then $1 * f = (\int_G \check{f} \, dm) \check{1} \not\in L^2(G)$ if $\int_G \check{f} \, dm \neq 0$.

We call such a $\varphi$, which satisfies the conditions of the proposition, an $L^\infty$-convolver and $\Lambda(\varphi)$ its convolution operator. We denote

$$Conv^{\infty}(G) = \{\varphi \in L^\infty(G) : \varphi \text{ is a convolver}\}.$$  

**Proof.** (i)$\Rightarrow$(ii) Since $L^2(G) \cap L^1(G)^\vee$ defines a dense subspace of $L^2(G)$, the operator $f \mapsto \varphi * f : L^2(G) \cap L^1(G)^\vee \to L^2(G)$ is a continuous linear operator and hence extends uniquely to a bounded linear operator $\Lambda(\varphi)$ in $\mathcal{B}(L^2(G))$.

If $\rho : G \to \mathcal{B}(L^2(G))$ is the right regular representation given by $\rho(t)f = \Delta(t)^{-1/2}t \cdot f$ for $f$ in $L^2(G)$, and $t$ in $G$, then it is easy to check that $(\Lambda(\varphi)\rho(t) - \rho(t)\Lambda(\varphi))f = 0$ for each $f$ in $L^2(G) \cap L^1(G)^\vee$, and hence for each $f$ in $L^2(G)$. This implies that $\Lambda(\varphi) \in VN(G)$.

If $f \in L^2(G) \cap L^1(G)^\vee$ and $g \in L^2(G) \cap L^1(G)$, then we have that the associated coefficient function satisfies

$$\langle \lambda(\cdot)f, g \rangle = \check{g} * \check{f} \in S^1A(G).$$

Moreover, in the VN($G$)-A($G$) dual pairing we obtain

$$\langle \Lambda(\varphi), \check{g} * \check{f} \rangle = \langle \varphi * f, g \rangle = \int_G \left( \int_G \varphi(s)f(s^{-1}t) \, ds \right) \check{g}(t) \, dt$$

(2.2) $$= \int_G \varphi(s) \left( \int_G f(s^{-1}t) \check{g}(t) \, dt \right) \, ds = \int_G \varphi(s) \check{g} * \check{f}(s) \, ds$$

where the use of Fubini’s Theorem is justified by our choices for $f$ and $g$. Now if $u \in S^1A(G)$ then we have that \(\check{u} \in L^\infty(G) \cap L^1(G) \subseteq L^2(G)\), so $\check{u} \in L^2(G) \cap L^1(G)$. Moreover, if $(e_u)_{u \in U}$ is the bounded approximate identity from Lemma 2.1 then each $e_u \in L^2(G) \cap L^1(G)^\vee$ with $\check{e}_u = e_u$.

Also, $\lim_{U \in U} \|ue_u - u\|_{S^1A} = 0$. We then have that $\langle \lambda(\cdot)e_u, \check{u} \rangle = u * e_u$ and

$$\langle \Lambda(\varphi), u \rangle = \lim_{U \in U} \langle \Lambda(\varphi), u * e_u \rangle = \lim_{U \in U} \int_G \varphi(s)u * e_u(s) \, ds = \int_G \varphi(s)u(s) \, ds.$$
Hence we have that
\[
\sup \left\{ \left| \int_G \varphi(s)u(s)ds \right| : u \in S^1A(G) \text{ and } \|u\|_A \leq 1 \right\}
\leq \sup \left\{ |\langle \Lambda(\varphi), u \rangle| : u \in b_1(A(G)) \right\} = \|\Lambda(\varphi)\|_{VN} < +\infty.
\]

(ii) \Rightarrow (i) Since \(L^2(G) \cap L^1(G)\) is dense in \(L^2(G)\) we have for each \(f \in L^2(G) \cap L^1(G)^\vee\)
\[
\|\varphi^* f\|_{L^2} = \sup \left\{ |\langle \varphi^* f, g \rangle| : g \in L^2(G) \cap L^1(G) \text{ and } \|g\|_{L^2} \leq 1 \right\}.
\]
Restating (2.2) and (2.1) we obtain for \(f \in L^2(G) \cap L^1(G)^\vee\) and \(g \in L^2(G) \cap L^1(G)\) that
\[
\langle \varphi^* f, g \rangle = \int_G \varphi(s) \langle \lambda(s)f, g \rangle \text{ and } \langle \lambda(\cdot)f, g \rangle \in S^1A(G).
\]
Thus we have that
\[
\sup \left\{ \|\varphi^* f\|_{L^2} : f \in L^2(G) \cap L^1(G)^\vee \text{ and } \|f\|_{L^2} \leq 1 \right\}
\leq \sup \left\{ \left| \int_G \varphi(s)u(s)ds \right| : u \in S^1A(G) \text{ and } \|u\|_A \leq 1 \right\} < +\infty.
\]

Unfortunately, we cannot determine if the formula (2.3) obtains for any \(u \in \Lambda(G)\), since if \(u \not\in S^1A(G)\), then the limit of the integrals in cannot be expected to hold. \(\square\)

We note that if \(G\) is discrete, then it is well-known that every element of \(VN(G)\) is an \(\ell^2\)-convolver, and in particular an \(\ell^\infty\)-convolver, i.e. \(VN(G) = \Lambda(\text{Conv}^{\infty}(G))\). This corresponds to the fact that \(S^1A(G) = A(G) \cap \ell^1(G) = \ell^1(G)\).

If \(G\) is compact, then \(L^\infty(G) \subseteq L^1(G)\), and hence \(L^\infty(G) = \text{Conv}^{\infty}(G)\). This corresponds to the fact that \(S^1A(G) = A(G)\), in this case. Here we see that \(\Lambda(\text{Conv}^{\infty}(G))\) is a dense \(*\)-subalgebra of the (reduced) group \(C^*\)-algebra \(C^*_r(G)\).

There are natural questions concerning the extent of \(\text{Conv}^{\infty}(G)\). We note that in both of the above cases that \(\text{Conv}^{\infty}(G) \subseteq L^2(G)\). Moreover, we have that \(L^\infty(G) \cap L^1(G)\), which is necessarily always contained in \(\text{Conv}^{\infty}(G)\), is always a subset of \(L^2(G)\). Thus we ask: \textit{for which} \(G\) is \(\text{Conv}^{\infty}(G) \subseteq L^2(G)\)? It is also natural to wonder about the norm closure \(\overline{\Lambda(\text{Conv}^{\infty}(G))}\). \textit{When is }\overline{\Lambda(\text{Conv}^{\infty}(G))}\textit{ a }\ast\text{-subalgebra of }VN(G)\textit{? How big is }\overline{\Lambda(\text{Conv}^{\infty}(G))}\textit{?} We note that \(\overline{\Lambda(\text{Conv}^{\infty}(G))}\) always contains the reduced \(C^*\)-algebra \(C^*_r(G)\) of \(G\).
2.2. The dual of the 1-Segal Fourier algebra.

**Theorem 2.3.** There is an isometric isomorphism

\[ S^1A(G)^* \cong VN(G) \oplus_\infty L^\infty(G)/\{(\Lambda(\varphi), -\varphi) : \varphi \in Conv^\infty(G)\}. \]

In particular, Conv^\infty(G) is linearly isomorphic to a norm-closed subspace of VN(G) \oplus_\infty L^\infty(G).

**Proof.** We have an isometry \( S^1A(G) \hookrightarrow A(G) \oplus_1 L^1(G) \). Hence by the Hahn-Banach Theorem every continuous linear functional may be realised as one from \( VN(G) \oplus_\infty L^\infty(G) \cong (A(G) \oplus_1 L^1(G))^* \) via the form

\[ \langle (T, \varphi), u \rangle = \langle T, u \rangle + \int_G \varphi u \, dm \]

for \((T, \varphi)\) in \( VN(G) \oplus_\infty L^\infty(G) \) and \( u \) in \( S^1A(G) \), where \( \langle T, u \rangle \) denotes the VN(A)-A(dual pairing). It follows that the annihilator of \( S^1A(G) \) will consist exactly of those pairs \((T, \varphi)\) for which \( \langle T, u \rangle = -\int_G \varphi u \, dm \) for each \( u \) in \( S^1A(G) \). But then we have that

\[ \sup \left\{ \left| \int_G \varphi u \, dm \right| : u \in S^1A(G) \text{ and } \|u\|_A \leq 1 \right\} \]

\[ \leq \sup \{ | \langle T, u \rangle | : u \in b_1(A(G)) \} = \|T\|_{VN} < +\infty. \]

Hence \( \varphi \in Conv^\infty(G) \). Moreover, since \( S^1A(G) \) is dense in \( A(G) \), it follows that \( T = -\Lambda(\varphi) \).

We have found no obvious direct computation which allows us to obtain the following corollary, in general. We note that, by [11 Theo. 2.1], \( S^1A(G) \), being a Segal algebra in \( A(G) \), has spectrum \( G \). If \( G \) were assumed to have an approximate unit (maybe unbounded!), then the corollary would be a consequence of [11 Theo. 1.1]. We let \( A_c(G) \) denote the subalgebra of compactly supported elements of \( A(G) \) and recall that \( A(G) \) is Tauberian, as \( A_c(G) \) is dense in \( A(G) \).

**Corollary 2.4.** \( S^1A(G) \) is an essential \( A(G) \)-module, i.e., \( A(G) \cdot S^1A(G) \) is a dense subspace of \( S^1A(G) \). In particular \( S^1A(G) \) is a Tauberian Banach algebra.

**Proof.** Let \((T, -\varphi)\) in \( VN(G) \oplus_\infty L^\infty(G) \) be such that

\[ \langle T, uv \rangle - \int_G \varphi uv \, dm = 0 \]

for every \( u \) in \( A(G) \) and \( v \) in \( S^1A(G) \). Then we see from the theorem above that \( Tu = \Lambda(\varphi u) \) for every \( u \) in \( A(G) \). It is clear that \( A_c(G) \subset \)
Thus, for any $v \in A_c(G)$ we then see that
$$\langle T, uv \rangle = \langle Tu, v \rangle = \int_G \varphi uv \, dm = \langle \Lambda(\varphi), uv \rangle$$
for all $u$ in $A(G)$, from which it follows that $\langle T, w \rangle = \langle \Lambda(\varphi), w \rangle$ for any $w \in A_c(G)$. Since $A_c(G)$ is dense in $A(G)$, $T = \Lambda(\varphi)$. It follows, again for the theorem above, that $\langle T, w \rangle = \langle \Lambda(\varphi), w \rangle$ for any $w \in A(G)$. Since $A_c(G)$ is dense in $A(G)$, $T = \Lambda(\varphi)$. It follows, again for the theorem above, that $(T, -\varphi)$ is the zero functional on $S_1^1A(G)$. The Hahn-Banach Theorem then implies that $A(G) \cdot S_1^1A(G)$ is a dense subset of $S_1^1A(G)$. We observe that $A_c(G) = A_c(G) \cdot S_1^1A(G)$ consists of the compactly supported elements in $S_1^1A(G)$ and is dense in there too. □

2.3. The dual of the 2-Segal Fourier algebra. We note that the computations in this section work analogously for $S_2^2A(G)$. Let us briefly indicate how. Since translations are continuous on $S_2^2A(G)$ we can obtain that $\lim_{U \in U} \|u * e_U - u\|_{S_2^2A} = 0$ just as in Lemma 2.1. Then we find for a fixed $h$ in $L^2(G)$ that the following two quantities
(i) $\sup \{\|h*f\|_{L^2} : f \in L^2(G) \cap L^2(G)\} \text{ and } \|f\|_{L^2} \leq 1 \}$
(ii) $\sup \left\{ \left| \int_G hu \, dm \right| : u \in S_2^2A(G) \text{ and } \|u\|_A \leq 1 \right\}$
are both equal, in particular both finite if one of them is. We call such functions $h L^2$-convolvers and denote the set of them by $\text{Conv}^2(G)$. If $h \in \text{Conv}^2(G)$ then it defines a bounded operator $\Lambda_2(h)$ on $L^2(G)$ which is an element of $\text{VN}(G)$. We thus obtain the linear isometric identification
\[
S_2^2A(G)^* \cong \text{VN}(G) \oplus_\infty L^2(G)/\{(A_2(h), -h) : h \in \text{Conv}^2(G)\}.
\]
We note that if $G$ is an abelian group with dual group $\hat{G}$ and $U : L^2(G) \to L^2(\hat{G})$ is the Plancherel unitary, then $U(\text{Conv}^2(G)) = L^\infty(\hat{G}) \cap L^2(\hat{G})$. Is there an analogous description for $U\Lambda(\text{Conv}^\infty(G))U^*$?

3. Restriction and Averaging Operations

Let us first note that if $H$ is an open subgroup of $G$, then we have restrictions $A(G)|_H = A(H)$, and $L^1(G)|_H = L^1(H)$. Both restriction operations are quotient maps, the latter so provided the Haar measure on $H$ is the restricted Haar measure from $G$. Hence we obtain the restriction
$$S_1^1A(G)|_H = S_1^1A(H).$$
We shall see that for a non-open closed subgroup, this result can be much different.
3.1. **A distance formula.** If \( H \) is any closed subgroup of \( G \), the restriction map \( u \mapsto u|_H : A(G) \to A(H) \) is also a surjective quotient map by \[14\] [28]. The kernel of this map is the ideal \( \text{I}(H) = \{ u \in A(G) : u|_H = 0 \} \). Dual to this is the fact that the subalgebra \( VN_H(G) \), which is the weak operator closure of the span of \( \{ \lambda(s) : s \in H \} \) in \( VN(G) \), is the annihilator of \( \text{I}(H) \) in \( G \) and hence \( * \)-isomorphic to \( A(H)^* \cong VN(H) \).

If \( H \) is a closed subgroup of \( G \) we say \( H \) admits a **bounded approximate indicator** in \( G \) (after [1]) if there is a bounded net \((u_\alpha)\) from \( B(G) \) such that

(i) \( \lim_\alpha u_\alpha|_H v = v \) for each \( v \) in \( A(H) \), and
(ii) \( \lim_\alpha u_\alpha u = 0 \) for each \( u \in \text{I}(H) \).

By [1, Theo. 3.7] we can suppose that \( \|u_\alpha\|_B \leq 1 \) for each \( \alpha \). We will have a bounded approximate indicator for \( H \) in \( G \) provided:

- \( H \) is neutral in \( G \), i.e. there is a neighbourhood basis \( V \) of \( e \) for which \( VH = HV \) for each \( V \) in \( V \) (see [18, Prop. 2.2]) – this is always true if \( G \) is a small invariant neighbourhood group; or
- \( G \) is amenable (see [11, Theo. 1.3] and [1, Prop. 4.1]).

Since there is a completely contractive injection \( S^1A(G) \hookrightarrow A(G) \) with dense range, there is a completely contractive injection \( VN(G) \hookrightarrow S^1A(G)^* \). Using Theorem [23] we can obtain a lower bound on the norms of the range of this map: if \( T \in VN(G) \) then

\[
\|T\|_{S^1A^*} = \inf \{ \max \{ \|T + \Lambda(\varphi)\|_{VN} : \|\varphi\|_{L^\infty} \} : \varphi \in \text{Conv}^\infty(G) \}
\geq \inf \{ \|T + \Lambda(\varphi)\|_{M_n(VN)} : \varphi \in \text{Conv}^\infty(G) \}
= \text{dist}_{VN}(T, \Lambda(\text{Conv}^\infty(G))).
\]

We obtain a similar bound for matricial norms: if \( [T_{ij}] \in M_n(VN(G)) \), then

\[
\|[T_{ij}]\|_{M_n(S^1A^*)} \geq \text{dist}_{M_n(VN)}([T_{ij}], M_n(\Lambda(\text{Conv}^\infty(G)))).
\]

Under certain assumptions, the quantity on the right is as big as it can be.

**Theorem 3.1.** If \( H \) is a non-open closed subgroup of \( G \) which admits a contractive approximate indicator \((u_\alpha)\), then for any \([T_{ij}]\) in \( M_n(VN_H(G)) \)

\[
\text{dist}_{M_n(VN)}([T_{ij}], M_n(\Lambda(\text{Conv}^\infty(G)))) = \|[T_{ij}]\|_{M_n(VN)}.
\]

**Proof.** Let us first suppose that \( n = 1 \). Given any \( \varepsilon > 0 \) find \( u \in b_1(A(G)) \) for which \( \text{supp}(u) \) is compact and

\[
|\langle T, u \rangle| > \|T\|_{VN} - \varepsilon.
\]
Now let $\varphi \in \text{Conv}^\infty(G)$. We then have for any $\alpha$ that

$$\langle T + \Lambda(\varphi), uu_\alpha \rangle = \langle T, uu_\alpha \rangle + \int_G \varphi uu_\alpha \, dm.$$  

(3.2)

If we can find $\alpha_0$ for which

$$\left| \langle T, u \rangle - \langle T, uu_\alpha \rangle \right| < \varepsilon \quad \text{and} \quad \left| \int_G \varphi uu_\alpha \, dm \right| < \varepsilon$$  

(3.3)

then it follows that

$$\left| \langle T + \Lambda(\varphi), uu_\alpha_0 \rangle \right| > \|T\|_{VN} - 3\varepsilon.$$  

(3.4)

But then $\text{dist}_{VN}(T, \Lambda(\text{Conv}^\infty(G))) \geq \|T\|_{VN}$, whence equality must hold.

Let us verify the first inequality from (3.3). Since $T \in VN_H(G)$ there is $T_H$ in $VN(H)$ such that $\langle T, v \rangle = \langle T_H, v \rangle_H$ for each $v$ in $\Lambda(G)$. Hence by condition (i) in the definition of $(u_\alpha)$ we have

$$\langle T, uu_\alpha \rangle = \langle T_H, (uu_\alpha)|_H \rangle \xrightarrow{\alpha} \langle T_H, u|_H \rangle = \langle T, u \rangle$$

and we can find $\alpha_1$ for which the desired inequality is satisfied for any $\alpha_0 \geq \alpha_1$. Now let us verify the second inequality from (3.3). Since $H$ is not open it is locally null [15, 20.17], so $m(S \cap H) = 0$ where $S = \text{supp}(u)$. Let $C$ be any compact neighbourhood of $S \cap H$ with

$$m(C) < \frac{\varepsilon}{2 \|\varphi\|_{L^\infty} + 1}.$$  

Let $U$ be a symmetric neighbourhood of $e$ for which $(S \cap H)U^2 \subseteq C$. Then if for $s$ in $G$ we let

$$v(s) = \frac{1}{m(U)} \langle \lambda(s)1_U, 1_{(S \cap H)U} \rangle = \frac{m(sU \cap (S \cap H)U)}{m(U)}$$

we obtain that

$$\|v\|_{L^\infty} = 1, \quad v|_{S \cap H} = 1 \quad \text{and} \quad \text{supp}(v) \subseteq (S \cap H)U^2 \subseteq C.$$  

Now we have that

$$\int_G \varphi uu_\alpha \, dm = \int_G \varphi uu_\alpha v \, dm + \int_G \varphi uu_\alpha(1 - v) \, dm$$

where $\|uu_\alpha v\|_{L^\infty} \leq 1$ so

$$\left| \int_G \varphi uu_\alpha v \, dm \right| \leq \int_G |\varphi| \, dm \leq \|\varphi\|_{L^\infty} m(C) < \frac{\varepsilon}{2}$$

while $\|u_\alpha u(1 - v)\|_{L^\infty} \leq \|u_\alpha u(1 - v)\|_A \xrightarrow{\alpha} 0$ as $u(1 - v) \in I(H)$, so

$$\left| \int_G \varphi uu_\alpha(1 - v) \, dm \right| \leq \int_S \|\varphi\|_{L^\infty} \|uu_\alpha(1 - v)\|_{L^\infty} \, dm \xrightarrow{\alpha} 0.$$
Hence we can find $\alpha_2$ so that the desired inequality holds for $\alpha_0 \geq \alpha_2$.

To obtain (3.3), we now choose $\alpha_0 \geq \alpha_1, \alpha_2$.

We can now obtain the theorem for $n > 1$. First, given $\varepsilon > 0$ we can $m$ and $[u_{kl}]$ in $b_1(M_n(A(G)))$ for which each $\text{supp}(u_{ij})$ is compact and

$$\|\langle T_{ij}, u_{kl}\rangle\|_{M_m} > \|\langle T_{ij}\rangle\|_{M_n(VN)} - \varepsilon.$$ 

Then, as in (3.2), if $[\varphi_{ij}] \in M_n(\text{Conv}^\infty(G))$ we have

$$\langle T_{ij} + \Lambda(\varphi_{ij}), u_{kl}u_{\alpha}\rangle = \langle T_{ij}, u_{kl}u_{\alpha}\rangle + \int_G \varphi_{ij} u_{kl} u_{\alpha} dm.$$ 

As above we can arrange to find an $\alpha_0$ for which

$$|\langle T_{ij}, u_{kl}u_{\alpha_0}\rangle - \langle T_{ij}, u_{kl}\rangle| < \frac{\varepsilon}{n^2} \quad \text{and} \quad \left|\int_G \varphi_{ij} u_{kl} u_{\alpha_0} dm\right| < \frac{\varepsilon}{n^2}$$

then that the matricial analogue of (3.4) is satisfied. \hfill \Box

### 3.2. A restriction theorem.

We conjecture that the following theorem holds for general closed non-open subgroups of $G$.

**Theorem 3.2.** If $H$ is a closed non-open subgroup of $G$ which admits a bounded approximate indicator in $G$, then the restriction map

$$u \mapsto u|_H : S^1A(G) \to A(H)$$

is a surjective complete quotient map.

**Proof.** The adjoint of the restriction map is the composition of the injective $*$-homomorphism $J : VN(H) \to VN_H(G) \subset VN(G)$ with the completely contractive injection $VN(G) \hookrightarrow S^1A(G)^*$. It follows from (3.3) and Theorem 3.1 that for $[T_{ij}]$ in $M_n(VN(H))$, we have

$$\|JT_{ij}\|_{M_n(S^1A^*)} \geq \text{dist}_{M_n(VN)}([JT_{ij}], M_n(A(\text{Conv}^\infty(G))))$$

$$= \|JT_{ij}\|_{M_n(VN)} = \|T_{ij}\|_{M_n(VN_H)}.$$ 

Hence $J : VN_H(G) \to S^1A(G)^*$ is a complete isometry, so the restriction map must be a surjective complete quotient map. \hfill \Box

We recall that for the operator projective tensor product $\hat{\otimes}$ we have formulas

$$L^1(G) \hat{\otimes} L^1(G) \cong L^1(G \times G) \quad \text{and} \quad A(G) \hat{\otimes} A(G) \cong A(G \times G).$$

It is thus a little surprising that the $1$-Segal Fourier algebra $S^1A(G)$ admits no such formula.

**Corollary 3.3.** If $G$ is a non-compact, non-discrete group which is either amenable or admits small invariant neighbourhoods, then

$$S^1A(G) \hat{\otimes} S^1A(G) \neq S^1A(G \times G).$$
Proof. If the two spaces were isomorphic, then the multiplication map
$S^1A(G) \otimes S^1A(G) \to S^1A(G) \subset A(G)$ would be isomorphic to the restriction map $w \mapsto w|_D : S^1A(G \times G) \to A(D) \cong A(G)$, where $D = \{(s, s) : s \in G\}$ is the diagonal subgroup. Since our assumptions on $G$ allow $D$ a bounded approximate indicator in $G \times G$ by [11, Theo. 2.4], the restriction map is surjective. But this contradicts that $S^1A(G) \not\subset A(G)$.

\[\square\]

3.3. An averaging theorem. We note an analogous result to Theorem 3.2 which applies to $LA(G)$, which we recall is $L^1(G) \cap A(G)$ qua Segal algebra of $L^1(G)$.

Let $N$ be a normal subgroup of $G$, admitting left Haar integral $\int_N \cdots \, dn$ (normalised if $N$ is compact). It is well-known that the $N$-averaging operator $\tau_N : L^1(G) \to L^1(G/N)$ given by

$$\tau_N f(tN) = \int_N f(tn) \, dn$$

for almost every $tN$ in $G/N$, is a contractive surjective algebra homomorphism. See [23, III.4], for example.

Let us note that $\tau_N$ is a quotient map, but offer an alternative proof to the standard one. We let

$$L_\infty(G:N) = \left\{ \varphi \in L_\infty(G) : \varphi(tn) = \varphi(t) \text{ for every } n \in N \right\}$$

which is clearly a closed subspace of $L_\infty(G)$. We note that if $\varphi \in L_\infty(G:N)$, then for locally almost every $t$ in $G$ and every $n$ in $N$ we have $\varphi(nt) = \varphi(t \cdot t^{-1}nt) = \varphi(t)$. If $q : G \to G/N$ is the quotient map then $\varphi \mapsto \varphi \circ q$ is a linear isometry from $L_\infty(G/N)$ onto $L_\infty(G:N)$. Now let us compute the adjoint of $\tau_N$: if $\varphi \in L_\infty(G/N)$ and $f \in L^1(G)$, then

$$\langle \tau_N^* \varphi, f \rangle = \langle \varphi, \tau_N f \rangle = \int_{G/N} \varphi(tN) \int_N f(tn) \, dn \, dtN$$

$$= \int_{G/N} \int_N \varphi \circ q(tn)f(tn) \, dn \, dtN = \int_G \varphi \circ q(t)f(t) \, dt = \langle \varphi \circ q, f \rangle$$

by Weil’s integral formula, so $\tau_N^* \varphi = \varphi \circ q$. Thus $\tau_N^* : L_\infty(G/N) \to L_\infty(G)$ is an isometry, so $\tau_N$ is a quotient map.

If $N$ is a compact group, then we may let $L^1(G:N)$ be defined similarly as $L_\infty(G:N)$, and we note it is a subalgebra of $L^1(G)$ which is isometrically algebraically isomorphic to $L^1(G/N)$. We may consider $\tau_N$ to have its range as $L^1(G:N)$. It is well known that $\tau_N A(G) = A(G)$:
\[ N = \{ u \in A(G) : u \text{ is constant on cosets of } N \} \cong A(G/N), \text{ and that } \tau_N u = u \text{ for } u \text{ in } A(G). \]
Hence it follows that
\[ \tau_N (LA(G)) = LA(G:N) \cong LA(G/N) \]
where \( LA(G:N) = \{ u \in LA(G) : u \text{ is constant on cosets of } N \} \). This result does not hold if \( N \) is not compact. In the case that \( G \) is abelian, the following due to Krogstad, whose unpublished result is announced in [8]. Our proof is for general locally compact \( G \).

Theorem 3.4. If \( N \) is a non-compact closed normal subgroup of \( G \), then the \( N \)-averaging operator \( \tau_N : LA(G) \to L^1(G/N) \) is a surjective complete quotient map.

As with Theorem 3.2, the proof relies on a distance formula.

Lemma 3.5. If \( N \) is a non-compact closed normal subgroup of \( G \), and \( [\varphi_{ij}] \in M_n(L^\infty(G:N)) \), then
\[ \text{dist}_{M_n(L^\infty)}([\varphi_{ij}], M_n(\text{Conv}^\infty(G))) = ||[\varphi_{ij}]||_{M_n(L^\infty)}. \]

Proof. Let us begin with the “scalar” case. Let \( \varphi \in L^\infty(G:N) \) and \( \varepsilon > 0 \). Let \( f \in (L^1(G) \cap L^2(G))^\circ \) be so \( ||f||_{L^1} = 1 \) and \( |\langle \varphi, f \rangle| > ||\varphi||_{L^\infty} - \varepsilon \). Then for any \( \psi \in \text{Conv}^\infty(G) \) and any \( n \in N \) we have
\[ \langle \varphi + \psi, n*f \rangle = \langle n^{-1} \varphi, f \rangle + \langle \psi, n*f \rangle = \langle \varphi, f \rangle + \int_G \psi(t) \tilde{f}(t^{-1}n)dt = \psi \ast \tilde{f}(n). \]
By [15, 20.16] we find that \( h = \psi \ast \tilde{f} \) is left uniformly continuous. Since \( \psi \in \text{Conv}^\infty(G) \), \( h \in L^2(G) \). It then follows that \( h \) is a continuous function vanishing at \( \infty \). Indeed, if not, then there is an \( \delta > 0 \) and a net \((t_\alpha)\) in \( G \) such that \( \lim_\alpha t_\alpha = \infty \) and for which \( |h(t_\alpha)| > \delta \) for each \( \alpha \). Then, by uniform continuity, there is a compact neighbourhood \( U \) of the identity in \( G \) such that \( |h(t_\alpha s)| > \delta/2 \) for \( s \in U \). By dropping to a subnet, we may assume that \( t_\alpha U \cap t_\beta U = \emptyset \) if \( \alpha \neq \beta \). But then, selecting any finite collection \( F \) of indices we find
\[ \int_G |h|^2dm \geq \sum_{\alpha \in F} \int_{t_\alpha U} |h|^2dm \geq \frac{|F|\delta^2}{4} \]
which, since \( |F| \) can be chosen arbitrarily large, contradicts that \( h \in L^2(G) \).

Thus, \( N \) is non-compact, we may find \( n \) in \( N \) for which
\[ |\langle \varphi + \psi, n*f \rangle| > ||\varphi||_{L^\infty} - \varepsilon. \]
Hence \( \text{dist}_{L^\infty}(\varphi, \text{Conv}^\infty(G)) = ||\varphi||_{L^\infty}. \)
The general matricial case can be deduced from the scalar case, exactly as in the proof of Theorem 3.1.

**Proof of Theorem 3.4** The adjoint of \( \tau_N : \text{LA}(G) \to L^1(G/N) \) is the composition of the injective \(*\)-homomorphism \( \varphi \mapsto \varphi q : L^\infty(G/N) \to L^\infty(G : N) \subset L^\infty(G) \) with the contractive inclusion \( L^\infty(G) \hookrightarrow \text{LA}(G)^\ast \).

We recall that \( \text{LA}(G)^\ast = S^1A(G)^\ast \), which is described in Theorem 2.3.

Now if \( [\varphi_{ij}] \in M_n(L^\infty(G/N)) \), then we obtain, just as in (3.4), and using the lemma above

\[
\| [\varphi_{ij}] \|_{M_n(L^\infty)} \geq \| [\tau_N^\ast \varphi_{ij}] \|_{M_n(\text{LA}^\ast)} \\
\geq \text{dist}_{M_n(L^\infty)}([\varphi_{ij} \circ q], M_n(\text{Conv}^\infty(G))) \\
= \| [\varphi_{ij} \circ q] \|_{M_n(L^\infty)} = \| [\varphi_{ij}] \|_{M_n(L^\infty)}.
\]

Hence \( \tau_N^\ast : L^\infty(G/N) \to \text{LA}(G)^\ast \) is a complete isometry, which implies that \( \tau_N : \text{LA}(G) \to L^1(G/N) \) is a surjective complete quotient map.

It was shown in [13] that if \( G \) is a unimodular group – so \( \text{LA}(G) \) is a symmetric Segal algebra – then \( \text{LA}(G) \) is Arens regular if and only if \( G \) is compact. Let us briefly note that this result holds without assuming \textit{a priori} that \( G \) is unimodular. We refer reader to the survey article [5] for details on, and functorial properties of, Arens regularity.

**Corollary 3.6.** If \( G \) is not unimodular, then \( \text{LA}(G) \) is not Arens regular.

**Proof.** Let us first note that if \( G \) is admits a continuous homomorphism \( \delta : G \to \mathbb{R} \) for which \( \ker \delta \) is compact, then for and \( a > 0 \), \( \delta^{-1}([-a,a]) \) is a compact neighbourhood of the identity, invariant under inner automorphisms, making \( G \) an \([\text{IN}]\)-group. Then it is well known (see [21 12.1.9], for example) that \( G \) is unimodular.

Hence, if \( G \) is not unimodular, then \( N = \ker \Delta \) is necessarily a non-compact closed subgroup of \( G \), with \( G/N \) isomorphic to an infinite subgroup of \( \mathbb{R} \). Hence \( L^1(G/N) \) is not Arens regular and \( \tau_N : \text{LA}(G) \to L^1(G/N) \) is a quotient homomorphism.

4. Cohomological Properties

In this section we discuss amenability and operator amenability of the 1-Segal Fourier algebra \( S^1A(G) \). We have opted to keep our presentation simple by focusing on \( S^1A(G) \). With suitable modifications to the proofs, we believe that all of our results hold for \( S^pA_q(G) \), which was defined and given an operator space structure in Section 1.4 where \( 1 \leq p < \infty, 1 < q < \infty \).
4.1. Amenability. Ruan’s result, that \(A(G)\) is \textit{operator amenable} if and only if \(G\) is amenable \cite{24}, is one of the most important results which justifies treating \(A(G)\) as an operator space. We recall that a completely contractive Banach algebra \(\mathcal{A}\) is \textit{operator amenable} if every completely bounded derivation \(D : \mathcal{A} \to \mathcal{V}^*\), where \(\mathcal{V}^*\) is the operator dual space to completely bounded \(\mathcal{A}\)-bimodule, is inner. We note that there exist compact groups, for example \(G = \text{SO}(3)\), such that \(A(G)\) is not amenable \cite{17}.

**Proposition 4.1.** \(S^1A(G) = A(G)\) completely isomorphically if and only if \(G\) is compact.

**Proof.** By \cite[Prop. 2.6]{12}, \(S^1A(G) = A(G)\) if and only if \(G\) is compact. Thus it remains to show that if \(G\) is compact then the identity map \(j : A(G) \to S^1A(G)\) is completely bounded. We have for \([u_{ij}]\) in \(M_n(A(G))\) that

\[
\|[u_{ij}]\|_{M_n(S^1A)} = \|[u_{ij}]1\|_{M_n(S^1A)} \leq \|[u_{ij}]\|_{M_n(A)} \|1\|_{S^1A} = 2 \|[u_{ij}]\|_{M_n(A)}.
\]

Thus \(j\) is completely bounded with \(\|j\|_{cb} \leq 2\). In fact, since \(2 = \|j(1)\|\) we have that \(2 \leq \|j\| \leq \|j\|_{cb}\), as well. Since \(j^{-1} : S^1A(G) \to A(G)\) is completely contractive we obtain that \(S^1A(G) = A(G)\) completely isomorphically. \(\square\)

Thus we obtain the main result of this section.

**Theorem 4.2.** \(S^1A(G)\) is operator amenable if and only if \(G\) is compact.

**Proof.** \((\Rightarrow)\) \(S^1A(G)\) is operator amenable only if it has a bounded approximate identity \cite{24, 16}. This happens if and only if \(G\) is compact by \cite[Prop. 2.6]{12}.

\((\Leftarrow)\) By Proposition \ref{prop:cb} above, any completely bounded \(S^1A(G)\)-module \(\mathcal{V}\) is a completely bounded \(A(G)\)-module. Moreover, every completely bounded derivation \(D : S^1A(G) \to \mathcal{V}^*\) induces a completely bounded derivation \(D \circ j : A(G) \to \mathcal{V}^*\). By \cite{24} \(D \circ j\) is inner, whence so too is \(D\). \(\square\)

The main result of \cite{9} states that \(A(G)\) is amenable if and only if \(G\) admits an abelian subgroup of finite index. Combining this with \cite[Cor. 2.7]{12} we obtain the following.

**Corollary 4.3.** \(S^1A(G)\) is amenable if and only if \(G\) is compact and admits an abelian subgroup of finite index.
4.2. Weak Amenability. The theory of hyper-Tauberian Banach algebras, developed by Samei [26], extends very easily to Segal algebras. We are grateful to E. Samei for pointing out an error we made, applying his work, in an earlier draft of this article.

Let us first recall some basic definitions. Let \( A \) be a semi-simple abelian (completely contractive) Banach algebra with Gelfand spectrum \( X \). Hence we identify \( A \) as a subspace of \( C_0(X) \). We define for \( a \) in \( A \) its support by \( \text{supp}(a) = \{ x \in X : a(x) \neq 0 \} \). We say that \( A \) is Tauberian if the subalgebra of compactly supported elements, \( A_c \), is dense in \( A \). If \( V \) is a symmetric Banach \( A \)-module and \( v \in V \), we define the support of \( v \) over \( A \) by
\[
\text{supp}_A(v) = \{ x \in X : a(x) = 0 \text{ whenever } a \cdot v = 0 \}.
\]

If \( V = A^* \), with the usual dual module action: \( a \cdot f(b) = f(ba) \), this agrees with the usual notion of support of a linear functional. If \( V = A \), then \( \text{supp}_A(a) = \text{supp}(a) \) for any \( a \) in \( A \). If \( V \) and \( W \) are (completely bounded) Banach \( A \)-modules a linear operator \( T : V \to W \) is called \( A \)-local if
\[
\text{supp}_A(Tv) \subseteq \text{supp}_A(v) \text{ for any } v \in V.
\]

We say that \( A \) is (operator) hyper-Tauberian if every (completely) bounded local operator \( T : A \to A^* \) is an \( A \)-module map.

Now suppose that the semi-simple abelian (completely contractive) Banach algebra \( A \) has an abstract (operator) Segal algebra \( S_A \). By [4, Theo. 2.1], \( S_A \) is also semi-simple with Gelfand spectrum \( X \).

**Theorem 4.4.** If \( A \) is (operator) hyper-Tauberian and \( S_A \) is an essential \( A \)-module, i.e., \( A \cdot S_A \) is dense in \( S_A \), then \( S_A \) is (operator) hyper-Tauberian.

Note that the converse follows from [26 Theo. 4.6], and (OSA2).

**Proof.** It is immediate that for \( v \) in \( S_A \)
\[
\text{(4.1)} \quad \text{supp}_A(v) = \text{supp}(v) = \text{supp}_{S_A}(v).
\]
Also, since \( S_A \) is a subalgebra of \( A \), we have that if \( f \in S_A^* \) then
\[
\text{(4.2)} \quad \text{supp}_A(f) \subseteq \text{supp}_{S_A}(f).
\]
Now suppose \( T : S_A \to S_A^* \) is a (completely) bounded \( S_A \)-local operator. Then we have for \( u \) in \( S_A \), combining (4.2) and (1.1),
\[
\text{supp}_A(Tu) \subseteq \text{supp}_{S_A}(Tu) \subseteq \text{supp}(u) = \text{supp}_A(u).
\]
Hence \( T \) is also an \( A \)-local operator. Hence by [26 Prop. 2.3] \( T \) is an \( A \)-module map. Thus it is an \( S_A \)-module map. \( \square \)
One of the main motivations for studying hyper-Tauberian algebras is the result \cite[Theo. 3.2]{26}: if $\mathcal{A}$ is (operator) hyper-Tauberian, then it is (operator) weakly amenable. We recall that $\mathcal{A}$ is (operator) weakly amenable if every (completely) bounded derivation $D : \mathcal{A} \to \mathcal{A}^*$ is inner \cite{2}. We also recall that the subalgebra $A_c(G)$ of the Fourier algebra of compactly supported elements lies within $S^1 A(G)$, and is exactly the subalgebra of compactly supported elements there.

**Corollary 4.5.** (i) $S^1 A(G)$ is always operator weakly amenable.

(ii) $S^1 A(G)$ is weakly amenable if the connected component $G_e$ of $G$ is abelian.

**Proof.** In \cite[Theo. 7.4]{26} it is shown that $A(G)$ is always operator hyper-Tauberian. Thus (i) follows from \cite[Theo. 3.2]{26}, whose statement was mentioned above, the preceding theorem, and Corollary \cite[Theo. 6.5]{24} in \cite{26} it is shown that $A(G)$ is hyper-Tauberian if $G_e$ is abelian. Hence (ii) follows, similarly to (i) above. □

We note that weak amenability of $A(G)$ is discussed in \cite{17} and \cite{9}, and operator weak amenability in \cite{10, 27, 25}.

**References**

[1] O. Yu. Aristov, V. Runde, and N. Spronk. Operator biflatness of the Fourier algebra and bounded approximate indicators for subgroups. *J. Funct. Anal.*, 209:367–387, 2004.

[2] W. G. Bade, P. C. Curtis, and H. G. Dales. Amenability and weak amenability for Beurling and Lipschitz algebras. *Proc. London Math. Soc.*, 55(3):359–377, 1987.

[3] D. P. Blecher. The standard dual of an operator space. *Pacific Math. J.*, 153(1):15–30, 1992.

[4] J. T. Burnham. Closed ideals in subalgebras of Banach algebras, I. *Proc. Amer. Math. Soc.*, 32:551–555, 1972.

[5] J. Duncan and A. R. Hosseinium. The second dual of a banach algebra. *Proc. Roy. Soc. Edinburgh Sect. A*, 84:309–325, 1979.

[6] E. G. Effros and Z.-J. Ruan. *Operator Spaces*, volume 23 of *London Math. Soc., New Series*. Claredon Press, Oxford Univ. Press, New York, 2000.

[7] P. Eymard. L’algebre de Fourier d’un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.

[8] H. G. Feichtinger. On a new Segal algebra. *Montash. Math.*, 92:269–289, 1981.

[9] B. E. Forrest and V. Runde. Amenability and weak amenability of the Fourier algebra. *Math. Z.*, 250:731–744, 2005.

[10] B. E. Forrest and P. J. Wood. Cohomology and the operator space structure of the Fourier algebra and its second dual. *Indiana Univ. Math. J.*, 50:1217–1240, 2001.

[11] B.E. Forrest, E. Kamnith, A.T. Lau, and N. Spronk. Ideals with bounded approximate identities in the fourier algebra. *J. Funct. Anal.*, 203:286–304, 2003.
[12] F. Ghahramani and A. T. M. Lau. Weak amenability of certain classes of Banach algebras without bounded approximate units. *Math. Proc. Camb. Phil. Soc.*, 133:357–371, 2002.

[13] F. Ghahramani and A. T. M. Lau. Approximate weak amenability, derivations and Arens regularity of Segal algebras. *Stud. Math*, 169:189–205, 2005.

[14] C. S. Herz. Harmonic synthesis for subgroups. *Ann. Inst. Fourier (Grenoble)*, 25(3):91–123, 1973.

[15] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis I*, volume 115 of *Grundlehren der mathematischen Wissenschaften*. Springer, New York, second edition, 1979.

[16] B. E. Johnson. *Cohomology in Banach Algebras*, volume 127 of *Mem. Amer. Math. Soc.* Amer. Math. Soc., Providence, RI, 1972.

[17] B. E. Johnson. Non-amenability of the Fourier algebra of a compact group. *J. London Math. Soc.*, 50(2):361–374, 1994.

[18] E. Kaniuth and A.T. Lau. A separation property of positive definite functions on locally compact groups and applications to fourier algebras. *J. Funct. Anal.*, 175:89–110, 2000.

[19] A. Lambert. *Operatorfolgenräume, Eine Kategorien auf dem Weg von den Banach-Räumen zu den Operatorenräumen*. PhD thesis, Universität des Saarlandes, Germany, 2002.

[20] A. Lambert, M. Neufang, and V. Runde. Operator space structure and amenability for Figà-Talamanca-Herz algebras. *J. Funct. Anal.*, 211:245–269, 2004.

[21] T. W Palmer. *Banach Algebras and the General Theory of *-algebras. Vol. II *-algebras*, volume 79 of *Encyclopedia of Mathematics and its Applications*. Cambridge Univ. Press, Cambridge, 2001.

[22] G. Pisier. *Introduction to Operator Space Theory*, volume 294 of *London Math. Soc. Lec. Note Series*. Cambridge Univ. Press, Cambridge, UK, 2003.

[23] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*. Oxford Math. Monograhs. Claredon, Oxford, 1968.

[24] Z.-J. Ruan. The operator amenability of $A(G)$. *Amer. J. Math.*, 117:1449–1474, 1995.

[25] E. Samei. Approximately local derivations. *J. London Math. Soc.*, 71:759–778, 2005.

[26] E. Samei. Hyper-Tauberian algebras and weak amenability of Figà-Talamanca-Herz algebras. Preprint, 2005.

[27] N. Spronk. Operator weak amenability of the Fourier algebra. *Proc. Amer. Math. Soc.*, 130:3609–3617, 2002.

[28] M. Takesaki and N. Tatsuuma. Duality and subgroups, II. *J. Funct. Anal.*, 11:184–190, 1972.

Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada

E-mail addresses: beforres@uwaterloo.ca, nspronk@uwaterloo.ca, pwood@uwaterloo.ca