Cosmological backreaction and spatially averaged spatial curvature

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Abstract

It has been suggested that the accelerated expansion of the Universe is due to backreaction of small scale density perturbations on the large scale spacetime geometry. While evidence against this suggestion has accumulated, it has not yet been definitively ruled out. Many investigations of this issue have focused on the Buchert formalism, which computes spatial averages of quantities in synchronous comoving gauge. We argue that, for the deceleration parameter of this formalism to agree with observations, the spatial average of the three dimensional Ricci scalar (spatial curvature) must be large today, with an $\Omega_k$ in the range of $1 \leq \Omega_k \leq 1.3$. We argue that this constraint is difficult to reconcile with observations of the location of the first Doppler peak of the CMBR. We illustrate the argument with a simple toy model for the effect of backreaction, which we show is generically incompatible with observations.
I. INTRODUCTION AND SUMMARY

Measurements of luminosity distance as function of redshift for type Ia supernovae, as well as measurements of inhomogeneities in the cosmic microwave background radiation, indicate that the expansion of the Universe is accelerating today [1, 2, 3]. Explanations of this phenomenon usually involve either an introduction of "dark energy" – a form of matter with negative pressure, or a modification of general relativity. Recently, a different explanation has been put forward [4, 5, 6, 7, 8, 10, 11, 12, 13], where the acceleration is a consequence of subhorizon density perturbations. According to this idea, small scale cosmological density perturbations evolve in a nonlinear manner to produce backreaction that affects the large scale spacetime geometry and modifies the expansion of the Universe. This explanation is controversial and many authors have argued that backreaction can not explain the current acceleration of the Universe [14, 15, 16]. Our viewpoint is that backreaction is likely to be too small to produce a significant modification to the large scale expansion of the universe. However, it deserves to be investigated in detail since the backreaction explanation has not yet been definitively refuted.

To quantify the rate of expansion of an inhomogeneous Universe, Buchert [13, 17] introduced a particular method of taking a spatial average of the Einstein equations. He specialized to comoving synchronous gauge, and on each surface of constant time, denoted by \( t \), he considers a spatial domain \( D(t) \) such that the boundary of \( D(t) \) is comoving [i.e. \( D(t) \) is independent of time in the synchronous comoving coordinates]. Defining \( V_D(t) \) to be the proper volume of this domain, the effective scale factor \( a_D(t) \) is given by

\[
\frac{4\pi}{3} a_D^3(t) = V_D(t).
\]

By averaging the Einstein equations for an irrotational dust Universe, Buchert derived the following evolution equations for \( a_D(t) \)

\[
\left( \frac{a_D'}{a_D} \right)^2 = \frac{8\pi}{3} \rho_{eff}, \tag{1.1}
\]

\[-\frac{a_D''}{a_D} = \frac{4\pi}{3} (\rho_{eff} + 3p_{eff}), \tag{1.2}\]

where prime denotes differentiation with respect to cosmic time \( t \), and throughout this paper we use geometrized units where \( G = c = 1 \). Equations (1.1,1.2) have the same form as the Friedmann equations, except that their sources are an effective density and an effective
pressure, $\rho_{\text{eff}}$ and $p_{\text{eff}}$, respectively, which are defined by

$$\rho_{\text{eff}} \equiv \langle \rho \rangle_D - \frac{1}{16\pi}(\langle R_3 \rangle_D + \langle Q \rangle_D), \quad (1.3)$$

$$p_{\text{eff}} \equiv -\frac{1}{16\pi}(\langle Q \rangle_D - \frac{3}{4}\langle R_3 \rangle_D). \quad (1.4)$$

Here $\rho$ denotes the matter density, and $R_3$ denotes the spatial three-dimensional Ricci scalar. The brackets $\langle \ldots \rangle_D$ denote an average over the domain $D(t)$, for example

$$\langle R_3 \rangle_D \equiv \int_{D} R_3 \sqrt{\det(g_{ij})}dV / \int_{D} \sqrt{\det(g_{ij})}dV = V^{-1}_D \int_{D} R_3 \sqrt{\det(g_{ij})}dV,$$

where $\det(g_{ij})$ denotes the determinant of the spatial 3-dimensional induced metric, and $dV$ denotes the three-dimensional coordinate volume-element. The quantity denoted $\langle Q \rangle_D$ is defined by

$$\langle Q \rangle_D \equiv \frac{2}{3}(\langle \theta - \langle \theta \rangle_D \rangle)^2_D - \langle \sigma_{\alpha\beta} \sigma^{\alpha\beta} \rangle_D. \quad (1.5)$$

Here $\theta$ denotes the dust expansion parameter and $\sigma_{\alpha\beta}$ denotes the shear tensor. Notice that the quantity $\langle Q \rangle_D$ vanishes for a homogeneous and isotropic Universe, but becomes nonzero if one includes density perturbations. Furthermore, Eq. (1.2) implies that a sufficiently large value of $\langle Q \rangle_D$ could produce a negative value for $\rho_{\text{eff}} + 3p_{\text{eff}} = \langle \rho \rangle_D - (1/4\pi)\langle Q \rangle_D$, and by virtue of Eq. (1.2) give rise to an accelerated expansion $a''_D > 0$.

While the effective scale factor $a_D$ is a mathematically well defined quantity, its relation to cosmological observations is not completely clear. The physical interpretation of $a_D$ faces three main difficulties. First, $a_D$ is a quantity defined on a spacelike hypersurface, and so, in general, it can not be directly related to cosmological observations which are determined by quantities on the past lightcone of the observer. Second, the time evolution of $a_D$ does not provide sufficient information to allow calculation of cosmological observables such as luminosity distance as function of redshift, which requires a metric for its calculation. Third, defining a quantity related to a constant time hypersurface is somewhat arbitrary, since one is free to choose a different time coordinate that defines a different spacetime foliation. Despite these difficulties, it has been argued that if the Buchert formalism predicts an effective declaration parameter $q_D$ which is approximately $-1/2$, then it is likely that the predicted value of the actual declaration will also be large and negative. In this paper we will adopt this point of view, and ignore the above mentioned difficulties with the interpretation of $a_D$. 

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In Ref. [8], Buchert’s formalism is used to calculate the evolution of $a_D$ in a perturbed Friedmann Robertson Walker (FRW) irrotational dust Universe. In particular the term $\langle Q \rangle_D$ which presumably drives the accelerated expansion of the Universe is calculated to second order in perturbation theory, and is found to be a boundary term, depending only on the metric perturbations on the boundary of the domain $D(t)$. This result generalizes other previous computation using Newtonian cosmological perturbation theory, which calculates a quantity related to $\langle Q \rangle_D$ which is also found to be a boundary term [9]. However, there is a difficulty in reconciling this property of $\langle Q \rangle_D$ at second-order with the interpretation of $\langle Q \rangle_D$ as the source of backreaction. To see this, suppose that the Universe is spatially compact, and that the domain $D$ is chosen to be the complete space. In this case any boundary term must vanish identically, and can have no affect on the time evolution of the Universe. While there is no observational evidence for a compact Universe, the fact that an FRW Universe has a particle horizon implies that a noncompact Universe is observationally indistinguishable from a spatially compact one as long as the scale of compactness is larger then the observer’s particle horizon at decoupling. This bound translates into a lower bound on the scale of compactness today of about twice the size of the horizon. We are therefore free to choose the scale of compactness to be roughly twice the horizon size today, without changing anything we can measure. This implies that $\langle Q \rangle_D = 0$ for $D \approx 2 \times \text{horizon}$. Now Buchert’s formalism is valid for all choices of $D$ and gives no guidelines as to what choice of $D$ to make. This ambiguity is part of the overall problem of relating the Buchert formalism to observations. Yet, it seem plausible that the correct answer (if it exist) should be roughly $D \approx \text{horizon size today}$. It seems reasonable that the value of $\langle Q \rangle_D$ should not change much if we reduce $D$ from $2 \times \text{horizon}$ to roughly the horizon size, and if so, for this choice of $D$, the Buchert formalism predicts that expansion of the Universe is unaffected by the vanishing $\langle Q \rangle_D$ at second-order. Nevertheless, there remains the possibility that third order and higher order perturbations could produce a large backreaction effects [5] which can not be represented by a boundary term so the backreaction issue is not settled.

In this paper we argue that general considerations suggest that it is hard to reconcile a large cosmological backreaction described by the Buchert formalism with observational constraints coming from measurements of luminosity distance and angular-diameter distance as functions of redshift. Observations of the the first Doppler peak of the CMBR together with baryon acoustic oscillation and supernovae data has been used to severely constrain
the spatial curvature of a ΛCDM Universe. By combining these observations with the assumptions that the dynamics of Universe is governed by the FRW metric and that the effect of density perturbations is negligible it has been found that spatial curvature satisfies \( \Omega_K = -0.0052 \pm 0.0064 \) (68% CL) \cite{24}, where \( \Omega_K = -R_3(t_0)/(6H_0^2) \), \( t_0 \) denotes the current time and \( H_0 \) denotes the current Hubble rate. In this paper we shall consider a more general theoretical framework that include perturbations and possibly large backreaction. One might expect that these observations should also place constraints on the average curvature \( \langle R_3(t_0) \rangle_D \).

In order for the Buchert formalism to reproduce the desired accelerated expansion from backreaction alone, it must have a significant averaged curvature with \( 0.975 \leq \Omega_K \leq 1.294 \) (see Sec. II B), where we have defined \( \Omega_K \equiv \langle R_3(t_0) \rangle_D/[6H_D^2(t_0)] \), and denoted the effective Hubble rate by \( H_D = a'_D/a_D \). This large averaged curvature seems to be hard to reconcile with the flat Universe implied by observations.

One possible avenue for evading this observational constraint in spatial curvature, suggested in Ref. \cite{13}, is the fact that in the Buchert formalism the effective energy density in the spatial curvature need not have the standard scaling \( \propto a^{-2} \). It is not clear whether the strong constraints coming from CMBR are more sensitive to the low redshift curvature or high redshift curvature. If the constraint principally applies to high redshift curvature, then an evolving curvature that is negligible at high redshift could evade the CMBR constraints. In this paper we shall argue that this avenue for evading the constraint is unlikely. Any nonstandard time evolution of the spatial curvature is quite constrained, since at high redshifts the density perturbations evolve linearly and the Universe is accurately described by a weakly perturbed CDM Universe. Non-standard time evolution must therefore be confined to the low redshifts, where nonlinear effects are presumably important. However, in this regime the spatial curvature is constrained by the requirement that the backreaction formalism reproduces the correct luminosity distance as function of redshift that agrees with supernovae data. These observations constrain the time evolution of the metric. Therefore, a nonstandard time evolution of the curvature in this regime would require a nonstandard evolution of the metric such that supernovae data observations are reproduced despite the large spatial curvature. In this scenario the full time evolution of the metric has two regimes. In the first low redshift regime, the metric evolves in a highly non-standard manner, and in the second high redshift regime, it evolves according to a standard weakly perturbed CDM Universe. The difficulty that it is not guaranteed that this time evolution reproduces the
correct angular power spectrum as measured by WMAP.

In this paper we construct a simple toy model that illustrates the observational difficulties that arise in models with a large value of averaged spatial curvature today, even allowing for nonstandard evolution of that curvature. For this purpose, we adopt the point of view of the Buchert backreaction formalism, and assume that we can replace the actual spacetime geometry by a set of averaged quantities. To be able to make predictions, we assemble these quantities and construct an averaged metric that allows us to calculate observables. Here we should make the following remarks.

First, the spatial averaged curvature in the Buchert formalism need not be dominated by low spatial frequency components, it may be mostly high spatial frequency components. Nevertheless, CMBR photons traveling along our past lightcone experience some sort of average curvature along their way. While this average is different from that of the Buchert formalism, it is plausible that they are not too much different. We will not address this issue in this paper. Second, in this paper we shall calculate an average spatial curvature using an expression for an averaged metric. However, this calculation is in general different from an average of the curvature of the true metric. We shall ignore this discrepancy in this paper.

We construct the following averaged metric toy model

\[
ds^2 = a^2(\eta) \left[ -d\eta^2 + \frac{dr^2}{1 - k(\eta)r^2} + r^2 d\Omega^2 \right].
\]  

(1.6)

The time coordinate \( t \) is related to \( \eta \) by \( dt^2 = a^2 d\eta^2 \). It should be emphasized that this metric should be thought of as an averaged metric and so it does not have to satisfy the Einstein field equations. Here, \( a(\eta) \) and \( k(\eta) \) are certain functions of the conformal time \( \eta \), where we set the present value of the scale factor to unity \( a(\eta_0) = 1 \). This averaged metric is designed to allow for a time evolution of the averaged spatial curvature to mimic what is presumably produced by backreaction. Notice that for every constant time hypersurface the induced three dimensional metric obtained from \( (1.6) \) coincides with a corresponding induced three-metric of an FRW constant time slice, and so this three-metric is isotropic and homogeneous about every point. However, for a generic function \( k(\eta) \), the overall four-dimensional spacetime is not maximally symmetric. Finally, we should mention here that after completing this work we learned that the form \( (1.6) \) of an averaged metric has been suggested before in Refs. \[13\], see also Refs. \[19, 20\].

Recently the toy model \( (1.6) \) was studied in detail by Larena et. al. \[18\]. This study
claims that there is a good agreement between this toy model and data from WMAP and supernovae observations, while we reach the opposite conclusion. We believe that the reason for this discrepancy originates from the fact that Larena et al. use a different expression for the redshift in terms of the scale factor and the function $k(\eta)$. In their study it is argued that under some approximation the relation between redshift and scale factor is the standard $1 + z \propto a^{-1}$ relation [see their Eq. (31)]. Using the standard definition of redshift (2.11) we show that the nonstandard time evolution of the spatial curvature significantly changes this relation, and the correct relation is given by Eq. (2.15). As we show, this nonstandard expression for the redshift has a significant effect on the calculation of observables in this model.

Our goal in this paper is to confront the model (1.6) with observations. For this purpose we calculate the luminosity distance $D_L(z)$ and angular diameter distance $D_A(z)$ as functions of redshift, using the metric (1.6) and compare the results with observational constraints1. We start by choosing a set of functions $k(\eta)$ parametrized by two parameters [see Eq. (2.1) below]. For each set of values of the parameters we then choose $a(\eta)$ to enforce the equation $D_L(z) = D_L^{\Lambda CDM}(z)$ at low redshifts, where $D_L^{\Lambda CDM}(z)$ is the luminosity distance derived from a $\Lambda CDM$ FRW model with parameter values agreeing with supernovae observations. This equation is enforced up to a maximum redshift. Using this requirement we calculate $a(\eta)$ for this low redshift part of the spacetime. We focus attention only on those metrics in which the the spatial curvature vanishes at a large redshift. Once the averaged spatial curvature vanishes the backreaction effect should vanish as well, and so in this high redshift regime we assume that $a(\eta)$ follows the standard evolution of a $CDM$ cosmology without a cosmological constant. Using this law of evolution we calculate the function $a(\eta)$ for the remaining part of spacetime. Once we have calculated $a(\eta)$, we use the metric (1.6) to compare the characteristic angular scale of the CMBR power spectrum as derived from our model with observation of WMAP. We find that generically the characteristic angular scale of our model is at odds with WMAP observations.

This paper is organized as follows. In Sec. II we explain in detail how we determine the the functions $a(\eta)$ and $k(\eta)$ of our model. In Sec. III we calculate the sound horizon that

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1 In practice, it is sufficient to calculate $D_A(z)$, since $D_L(z)$ can be obtained from the relation $D_L(z) = (1 + z)^2 D_A(z)$ which is valid in any spacetime, see Refs. [21,22].
determines the location of first CMBR peak in our model. In Sec. IV we explore various values of the parameters which determines the function $k(\eta)$ and describe the results.

II. CONSTRUCTION OF THE BACKREACTION MODEL

To be able to interpolate between an initially vanishing averaged spatial curvature that corresponds to a weakly perturbed FRW Universe, and a current large value of averaged spatial curvature needed for the backreaction picture, we assume that the function $k(\eta)$ in the metric (1.6) takes the following form

$$k(\eta) = \begin{cases} H_0^2 \frac{f^2}{f+1}, & \eta \geq \bar{\eta} \\ 0, & \eta \leq \bar{\eta} \end{cases}$$

(2.1)

where

$$f \equiv \frac{H_0(\eta - \bar{\eta})}{w}.$$ 

Here $H_0 = (a^{-1} \frac{da}{dt})_{t_0}$ is the value of the Hubble constant today, and $\bar{k}$, $w$ and $H_0 \bar{\eta}$ are dimensionless parameters. The parameter $\bar{\eta}$ marks the conformal time of the transition between a conformally flat spacetime and a conformally curved spacetime, and $w$ governs the rapidity of this transition. Using the definition $\Omega_k \equiv -\langle R_3(t_0) \rangle_D /[6H_D^2(t_0)]$ together with Eq. (2.1) and identifying $H_0$ with $H_D(t_0)$ we find that

$$\Omega_k = -\bar{k} \left(1 + \frac{w^2}{(\eta_0 - \bar{\eta})^2}\right)^{-1},$$

(2.2)

where $\eta_0$ is the value of the conformal time today. Below we explore various values for the parameters $w$ and $\bar{\eta}$, while $\bar{k}$ is determined from the requirement that the Buchert formalism gives an equation of state parameter of dark energy near $-1$ [see Sec. II B].

To calculate the scale factor $a(\eta)$, we demand that the luminosity distance as function of redshift, $D_L^{\text{model}}(z)$, in our model matches observational data. Since the luminosity distance of a $\Lambda CDM$ cosmology matches observational data we impose

$$D_L^{\text{model}}(z) = D_L^{\Lambda CDM}(z),$$

(2.3)

where throughout the superscripts 'model' and ' $\Lambda CDM$' refer to our model and to a flat $\Lambda CDM$ cosmology, respectively. We impose the condition (2.3) only at low redshifts, in the range of values of $\eta$ given by $\eta \geq \bar{\eta}$. At high redshifts, we switch to imposing the Friedmann
equation, since backreaction should be negligible at high redshifts. The evolution of $a(\eta)$ for $\eta \leq \bar{\eta}$ is determined from an Einstein- de Sitter model for which the scale factor satisfies

$$\dot{a} = h\sqrt{a},$$

where an overdot denotes differentiation with respect to $\eta$. We determine the constant $h$ by demanding continuity of $\dot{a}$ at the transition time $\bar{\eta}$. The solution of this equation is given by

$$a(\eta) = \left[\frac{h}{2}(\eta - \bar{\eta}) - \sqrt{a(\bar{\eta})}\right]^2, \quad \eta \leq \bar{\eta}. \quad (2.4)$$

where we used the continuity of $a(\eta)$ at $\eta = \bar{\eta}$.

### A. Matching luminosity distances as function of redshift

In this section we describe how we calculate $a(\eta)$ in practice from the matching requirement (2.3). In a general spacetime the luminosity distance $D_L(z)$ is related to the angular diameter distance $D_A(z)$ by \cite{21, 22}

$$D_L(z) = D_A(z)(1 + z)^2. \quad (2.5)$$

Using Eq. (2.5), the matching requirement (2.3) takes the form of

$$D_A^{model}(z) = D_A^{\Lambda CDM}(z). \quad (2.6)$$

In a flat $\Lambda CDM$ cosmology the right hand side of Eq. (2.6) is given by

$$D_A^{\Lambda CDM}(z) = \frac{1}{(1 + z)H_0} \int_0^z [\Omega_m(1 + z')^3 + \Omega_{\Lambda}]^{-1/2} dz'. \quad (2.7)$$

Here the parameters $\Omega_m$ and $\Omega_{\Lambda}$ satisfy $\Omega_m + \Omega_{\Lambda} = 1$, and the contribution from radiation energy-density has been neglected since we confine the discussion to the epoch after recombination.

We now consider the left hand side of Eq. (2.6) and derive an expression for $D_A^{model}$. Suppose that an observer views a sizeable distant object (e.g. a distant galaxy or a structure of the CMB anisotropy) that has a transverse proper cross sectional area $\delta A$, and subtends a small solid angle $\delta \Omega$. From these quantities the observer can determine the angular diameter distance

$$D_A = \sqrt{\frac{\delta A}{\delta \Omega}}.$$
Since the wavelength of the electromagnetic radiation is typically much smaller than the spacetime curvature, we can safely use the geometric optics approximation and describe the electromagnetic radiation as a bundle of light rays that trace a congruence of null geodesics. We assume that the light rays converge at an event \( p \) at the location of the observer, which is chosen to be at the origin, so that \( r(p) = 0 \) and \( \eta(p) = \eta_0 \). Since the metric (1.6) is isotropic about the origin, the light rays trace radial null geodesics from the source at \( r(\eta) \), where \( \eta < \eta_0 \), to the observer; and the angular diameter distance is given by

\[
D_{\text{model}}^a = a(\eta)r(\eta).
\]  
(2.8)

Substituting Eqs. (2.7) and (2.8) into Eq. (2.6) and differentiating with respect to \( \eta \) gives

\[
H_0 \frac{d}{d\eta}[(1 + z)ar] = \frac{dz}{d\eta}[(\Omega_m(1 + z)^3 + \Omega_\Lambda)^{-1/2}].
\]  
(2.9)

Our goal is to solve Eq. (2.9) for \( a(\eta) \). As a preliminary step, we first calculate \( r(\eta) \) and \( z(\eta) \) and then substitute these functions into Eq. (2.9).

The calculation of \( r(\eta) \) for radial null geodesics follows directly from the metric (1.6), which gives \( \dot{r}^2 = 1 - k(\eta)r^2 \), where dot denotes differentiation with respect to \( \eta \). Later we shall assume that \( k(\eta) < 0 \) and so \( \dot{r}^2 > 0 \) implies that the null geodesics have no turning points. Since \( \eta \) is a monotonically decreasing function of \( r \) we have

\[
\dot{r} = -\sqrt{1 - k(\eta)r^2}.
\]  
(2.10)

Below we use Eq. (2.1) to specify \( k(\eta) \) and solve Eq. (2.10) numerically together with the initial condition \( r(\eta_0) = 0 \).

We now consider the calculation of the redshift \( z(\eta) \). By definition the redshift is given by

\[
1 + z = \frac{(k^\alpha u_\beta g_{\alpha\beta})_{\text{source}}}{(k^\alpha u_\beta g_{\alpha\beta})_{\text{observer}}},
\]  
(2.11)

where \( k^\alpha \) is the 4-momentum of the photon, and \( u^\alpha \) is the 4-velocity of the cosmological fluid. For simplicity we assume that both the observer and the source have four-velocities of the form \( u^\alpha = a^{-1} \delta^\alpha_\eta \) meaning that their peculiar velocities vanish. We normalize \( k^\alpha \) by demanding that \( k^\alpha u_\beta g_{\alpha\beta} = -1 \) at the observer. Some simplification is gained by considering a conformal transformation of the form

\[
g_{\alpha\beta} = a^2 \hat{g}_{\alpha\beta}, \quad k^\alpha = a^{-2} \hat{k}^\alpha.
\]  
(2.12)
Combining our choices for normalization and velocities with Eqs. (2.11) and (2.12) gives

\[ z = a^{-1} \hat{k}^\eta - 1. \]  

(2.13)

The conformal null vector field \( \hat{k}^\eta \) satisfies a geodesic equation in the conformal spacetime where the metric is \( \hat{g}_{\mu\nu} \), and so it is independent of \( a(\eta) \). Using this geodesic equation together with Eq. (2.10) we obtain

\[ (\hat{k}^\eta)^{-1} \frac{d}{d\eta} \hat{k}^\eta = -\frac{r^2 \dot{k}}{2(1 - kr^2)}. \]  

(2.14)

Integrating Eq. (2.14) and using Eq. (2.13) we find that the red shift is given by

\[ z = a^{-1} e^{1/2 \int_\eta^0 r^2 k(1 - kr^2)^{-1} d\eta'} - 1. \]  

(2.15)

Notice that for \( k = \text{const} \) Eq. (2.15) reduces to the FRW relation \( z + 1 \propto 1/a \). Eq. (2.15) is at odds with Eq. (30) of Ref. [18] which seems to be inconsistent with the standard definition of redshift (2.11). Below we solve for \( r(\eta) \) by specifying \( k(\eta) \) and solving Eq. (2.10). Using \( r(\eta) \) we evaluate the integral in Eq. (2.15) and obtain \( z(\eta) \). Both \( r(\eta) \) and \( z(\eta) \) are then substituted into Eq. (2.9) which is solved to give \( a(\eta) \). Finally \( a(\eta) \) and \( r(\eta) \) are inserted into Eq. (2.8) to obtain \( D_A^{\text{model}}(\eta) \), and this is combined with \( z(\eta) \) to obtain the angular diameter distance \( D_A^{\text{model}}(z) \) as function of redshift \( z \). All these calculations are done numerically.

**B. Constraining the toy model parameters**

In this section we use observational constraints on the cosmological parameters to place constraints on the parameters of our toy model. The calculation is based on Buchert’s formalism [17] which was summarized in Sec. [11]. In this formalism the equation of state parameter of dark energy is given by

\[ w_{de}(\eta) = \frac{p_{de}(\eta)}{\rho_{de}(\eta)}, \]  

(2.16)

where \( \rho_{de} \) denotes the dark energy density, and \( p_{de} \) denotes the dark energy pressure. These quantities are related to the effective density \( \rho_{eff} \) and effective pressure \( p_{eff} \), which are given by Eqs. (1.3) and (1.4), through the relations \( \rho_{eff} = \langle \rho \rangle_D + \rho_{de} \) and \( p_{eff} = p_{de} \). Eq. (2.16) together with Eqs. (1.3, 1.4) give

\[ w_{de} = \frac{-(1/3)\langle R_3 \rangle_D + \langle Q \rangle_D}{\langle Q \rangle_D + \langle R_3 \rangle_D}. \]  

(2.17)
The current value of this parameter is in the range $-1.1 \leq w_{de}(\eta_0) \leq -0.9$ \[23\]. Using this constraint together with Eq. (2.17) gives

$$-\frac{57}{17} \langle Q(\eta_0) \rangle_D \leq \langle R_3(\eta_0) \rangle_D \leq -\frac{63}{23} \langle Q(\eta_0) \rangle_D.$$  \hspace{1cm} (2.18)

We define the effective deceleration parameter by

$$q_D = -\frac{a''_D}{a_D H_D^2},$$  \hspace{1cm} (2.19)

and substitute Eq. (1.2) into Eq. (2.19) and use Eqs. (1.3) and (1.4). This gives

$$q_D(\eta_0) = \frac{1}{2} \Omega_m(D) - \frac{\langle Q(\eta_0) \rangle_D}{3H_D^2},$$  \hspace{1cm} (2.20)

where $\Omega_m(D) = 8\pi \langle \rho \rangle_D / 3H_D^2$. We demand that this expression be equal to the current deceleration of a standard flat $\Lambda CDM$ cosmology, where the deceleration parameter is given by

$$q_{\Lambda CDM}(\eta_0) = \frac{1}{2} \Omega_m - \Omega_\Lambda,$$  \hspace{1cm} (2.21)

where $\Omega_m$ and $\Omega_\Lambda$ are the $\Lambda CDM$ densities of matter and dark energy, respectively. Assuming that we can substitute $\Omega_m$ in place of $\Omega_m(D)$ we find from Eq. (2.20) and Eq. (2.21) that

$$\frac{\langle Q(\eta_0) \rangle_D}{3H_D^2} = \Omega_\Lambda.$$  \hspace{1cm} (2.22)

The 5-year WMAP data \[24\] reveals that the dark energy density is in the range $\Omega_\Lambda = 0.742 \pm 0.030$. We use the WMAP constraint on $\Omega_\Lambda$ and Eqs. (2.22), (2.18) together with the definition $\Omega_k \equiv -(R_3(t_0))_D/[6H_D^2(t_0)]$, to obtain

$$0.975 \leq \Omega_k \leq 1.294.$$  \hspace{1cm} (2.23)

By combining $\Omega_k \approx 1.1$ together with Eq. (2.22) and Eq. (2.23) we determine the parameter $\bar{k}$ once the parameters $w$ and $\eta_0$ have been specified.

III. THE CHARACTERISTIC ANGULAR SCALE OF THE CMBR POWER SPECTRUM

The position of the peaks of the WMAP angular power spectrum is set by the characteristic angular scale $\theta_{\text{WMAP}}$ defined by

$$\theta_{\text{WMAP}} \equiv \frac{D_H(z^*)}{D_A(z^*)},$$  \hspace{1cm} (3.1)
where $D_H(z^*)$ denotes the sound horizon at the redshift of decoupling, where throughout we use the notation $*\text{ to refer to the redshift of decoupling. For a flat } \Lambda CDM \text{ cosmology the observed WMAP power spectrum is consistent with the angular scale (3.1) at a percent level of accuracy (Assuming that the sound horizon is consistent with a } \Lambda CDM \text{ cosmology the 5-year WMAP data [24] yields a comoving angular diameter distance given by } d^\text{obs}_A(z^*) = 14115^{+188}_{-191}\text{Mpc). To see if our toy model is able to reproduce the characteristic angular scale which is consistent with observations we calculate the ratio

$$
\chi = \frac{\theta^\text{model}}{\theta^\text{WMAP}} = \frac{D^\text{model}_H(z^*)D^\Lambda CDM_A(z^*)}{D^\text{model}_A(z^*)D^\Lambda CDM_H(z^*)}.
$$

(3.2)

To calculate $\chi$ we need to calculate the ratios $D^\Lambda CDM_A(z^*)/D^\text{model}_A(z^*)$ and $D^\text{model}_H(z^*)/D^\Lambda CDM_H(z^*)$. The calculation of the ratio of angular diameter distances follows from the method described in Sec. III. We now discuss calculating the ratio of the sound horizons at decoupling, for this we follow Ref. [25]. Early on our model coincides with a cold dark matter cosmology. Therefore, prior to decoupling the sound speed in the plasma of baryons and photons is given by $v_s = [3(1 + R)]^{-1/2}$, where $R \equiv 3\rho_B/4\rho_\gamma$; and $\rho_B$ and $\rho_\gamma$ denote the baryon density and the photon density, respectively. At this epoch these densities are assumed to be approximately homogeneous, and the sound horizon takes the form of

$$
D^\text{model}_H = a^* \int_0^{z^*} \frac{dt}{a\sqrt{3(1 + R)}}.
$$

(3.3)

To evaluate this integral we need to know the time evolution of the relevant densities before decoupling. In our model the baryons are assumed to be comoving, so that the evolution of the baryon density (and the matter density) at the location of the observer traces the evolution of a three dimensional volume element at $r = 0$, which gives

$$
\rho_B = \rho_{B0}a^{-3}, \ t < t^*.
$$

(3.4)

Here and throughout the subscript 0 denotes a quantity evaluated at the observer today, for example $\rho_{B0} \equiv \rho_B(\eta_0, r = 0)$. The evolution of the photon density (and the radiation density) is that of a black-body and so it is determined by the temperature. Therefore, irrespective of any spacetime symmetry we have

$$
\rho_\gamma = \rho_{\gamma0}(1 + z)^4, \ t < t^*.
$$

(3.5)

Recall that in our model there is a nonstandard relation between the redshift and the scale factor. For this reason it is instructive to introduce the quantity $\alpha \equiv a^{-1}(1 + z)$. Prior to
the growth in curvature, i.e. at a conformal time \( \eta \) where \( \eta < \bar{\eta} \), Eq. (2.15) implies that \( \alpha \) is a constant, given by

\[
\alpha = e^{1/2} \int_{\eta_0}^{\eta} r^2 k(1-kr^2)^{-1} \, d\eta.
\]

(3.6)

For a \( \Lambda CDM \) Universe we have \( \dot{k} = 0 \) and so we recover \( \alpha = 1 \). Using Eqs. (3.4,3.5,3.6) we obtain

\[
R = R_0 \alpha^{-4} a.
\]

(3.7)

Using this equation together with Eq. (3.3) we can write the sound horizon as

\[
D_{model}^H = R^* \int_0^{t^*} \frac{dt}{R \sqrt{3(1+R)}}.
\]

(3.8)

The integration is carried out by noting that \( dt = dR/H R \), \( H = \sqrt{(8\pi/3)(\rho_M + \rho_R)} \), where \( \rho_M \) and \( \rho_R \) denote the matter density and the radiation density, respectively. Introducing the notation \( R_{ACDM}^* = R_0 (1 + z^*)^{-1} \) and \( R_{EQ} = 3\rho_0 \rho_B / 4 \rho_M 0 \rho_r \) we obtain

\[
\frac{D_{model}^H(z^*)}{D_{H}^{ACDM}(z^*)} = \alpha^3 f(\alpha, R_{ACDM}^*, R_{EQ}) / f(1, R_{ACDM}^*, R_{EQ}),
\]

(3.9)

where

\[
f(\alpha, R_{ACDM}^*, R_{EQ}) = \ln \left( \frac{\sqrt{1+R_{ACDM}^* \alpha^{-3}} + \sqrt{R_{EQ} + R_{ACDM}^* \alpha^{-3}}}{1 + \sqrt{R_{EQ}}} \right).
\]

Here we assumed that the observer in our toy model Universe would measure the same densities today as an observer placed in a \( \Lambda CDM \) Universe. For a standard \( \Lambda CDM \) Universe [25] we have \( R_{ACDM}^* = 0.62 \), \( R_{EQ} = 0.21 \). Eq. (3.9) implies that at decoupling the sound horizon calculated from our toy model is different from the sound horizon in a \( \Lambda CDM \) Universe. This suggests that for generic parameters \( (\bar{\eta}, w) \) our model would give rise to a shift in the locations of the peaks of the TT power spectrum determined by the ratio \( \theta_{model}/\theta_{WMAP} \) given by Eq. (3.2). An exception to this could arise only in rare occasions where the ratio \( D_{model}^H(z^*)/D_{H}^{ACDM}(z^*) \) in Eq. (3.2) is exactly compensated by the ratio \( D_{A}^{ACDM}(z^*)/D_{A}^{model}(z^*) \) in this equation.

IV. RESULTS

We explored the two dimensional parameter space \( (\bar{\eta}, w) \) of the metric function \( k(\eta) \) and used the method described in Sec. 11 to determine the functions \( k(\eta) \) and \( a(\eta) \). We then
FIG. 1: Log-log plot of the angular diameter distance as function of redshift, for a ΛCDM Universe and the backreaction toy model with parameters \((H_0\bar{\eta}, w) = (-1.6, 1.2)\) which corresponds to a transition redshift of \(z(\bar{\eta}) = 1.92\). For these parameters the two graphs of \(D_{A\text{model}}^\Lambda(z)\) and \(D_{A\text{CDM}}^\Lambda(z)\) coincide. The filled box indicates the WMAP measured angular diameter distance at decoupling \(D_{A\text{obs}}^\Lambda(z^*)\) assuming that the sound horizon is consistent with a ΛCDM Universe. The empty box indicates this distance assuming that the sound horizon is consistent with the toy model.

used Eq. (2.8) and Eq. (2.15) to determine the angular diameter distance at decoupling \(D_{A\text{model}}^\Lambda(z^*)\), where \(z^* = 1090.51 \pm 0.95\) [24]. To calculate the ratio \(\chi = \theta_{A\text{model}}/\theta_{A\text{CDM}}\) we substituted the value of \(D_{A\text{model}}^\Lambda(z^*)\) and the value of \(D_{A\text{CDM}}^\Lambda(z^*)\) [given by Eq. (2.7)] into Eq. (3.2) and used Eq. (3.9). We have found that for generic values of \((\bar{\eta}, w)\) the estimator \(\chi\) is different from unity by more than a percent indicating a discrepancy between the backreaction toy-model and observations. An example of this discrepancy is demonstrated in Fig. 1 showing the angular diameter distance as function of redshift for a ΛCDM Universe and the backreaction toy model for parameters \((H_0\bar{\eta}, w) = (-1.6, 1.2)\). For these parameters the graphs of \(D_{A\text{model}}^\Lambda(z)\) and \(D_{A\text{CDM}}^\Lambda(z)\) coincide. However, the WMAP value of the angular diameter distance at decoupling is model dependent and is in agreement only with a ΛCDM Universe. There is a small region in the parameter space where \((H_0\bar{\eta}, w) \approx (-0.8, 0.8)\) and \(z(\bar{\eta}) \approx 1\) where the ratio \(D_{H\text{model}}^\Lambda(z^*)/D_{H\text{CDM}}^\Lambda(z^*)\) in Eq. (3.2) is exactly compensated by the ratio \(D_{A\text{CDM}}^\Lambda(z^*)/D_{A\text{model}}^\Lambda(z^*)\) in this equation. While this part of the parameter space of our toy model is not ruled out, we are not aware of any observation that support a sudden growth of the averaged curvature at \(z \approx 1\).

Another difficulty that showed up only in a portion of the toy-model parameter-space is that the redshift can become a non-monotonic function of the conformal time coordinate.
FIG. 2: The toy-model parameter space ($H_0\bar{\eta}, w$), showing the domain (gray) where the redshift becomes a non-monotonic function of the conformal time coordinate $\eta$.

FIG. 3: The transition redshift $\bar{z}$ as function of the model parameter $\bar{\eta}$ for different values of the parameter $w$. Showing $w = 0.05$ (thin line), $w = 0.2$ (thick line), $w = 0.8$ (dashed line).

$\eta$. The portion of the parameter space which suffers from this difficulty is shown in Fig. 2. Finally, Fig 3 shows the transition redshift $\bar{z} \equiv z(\bar{\eta})$ as function of the parameters of the toy-model. For model parameters where the redshift is a monotonic function of $\eta$ an increase in $w$ (for a fixed $\bar{\eta}$) normally implies a decrease in $\bar{z}$. Notice, however, that most of the graphs are in the portion parameter space where the redshift is not a monotonic function of $\eta$.

V. CONCLUSIONS

In this paper we studied a toy model of a backreaction mechanism. In this model the averaged spatial curvature grows at low redshifts so that the expansion of the Universe
presumably induced by backreaction could be consistent with supernovae data. In the high redshift regime, we assumed that Universe evolves according to a standard weakly perturbed CDM Universe. We showed that this model alters the predictions for the sound horizon at decoupling and that it is generically inconsistent with the power spectrum as measured by WMAP.

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