Twisted Homotopy
A Group Theoretic Approach

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Abstract
After summarising the physical approach leading to twisted homotopy and after developing the
cohomological approach further with respect to our previous work we propose a third alternative
approach to twisted homotopy based on group theoretic considerations.In this approach the funda-
mental group Π(m) isomorphic to Z which describes homotopic loops on the punctured plane\(R^2/(0)\)
is enhanced in a special way to the continuous SO(2) group . This is performed by letting the pa-
rameter of the group \(m \rightarrow \lambda\) while keeping its generator unchanged .It is shown that such non-trivial
procedure has the effect of introducing well defined self-interactions among loops which are at the
basis of twisted homotopy where the angle \(\lambda\) plays the role of the self coupling constant.

KEYWORDS: HOMOTOPY, GROUP THEORY,QUANTUM MECHANICS
MSC:55Q35; PACS:02.20.Fh ; 03.65.Fd
1 Physical approach

The punctured plane $R^2/\{0\}$ is the space of interest here with its simple but not trivial topology. From the homotopy point of view, the objects which describe the punctured plane are loops encircling the hole. These loops have a structure characterized by the fundamental group we denote $\Pi(n); n \in \mathbb{Z}$ which is isomorphic to $\mathbb{Z}$ and the winding number $W$ which indicates how much times a loop encircles the hole. It will help to regard homotopic loops as intrinsic "physical" objects rather than just appropriate mathematical objects used to tag the puncture of the plane. Adopting this point of view we will therefore associate to the loop of winding number $n$ the winding number eigenstate $|n\rangle$. The group elements in turn become operators which map the state $|n\rangle$ into the state $|n+m\rangle$. We therefore summarise the homotopy information on the punctured plane by the set of relations.

$$W | n\rangle = n | n\rangle$$

$$\Pi(m) | n\rangle = | n+m\rangle$$

$$\sum_{n \in \mathbb{Z}} |n\rangle\langle n| = 1$$

(1)

$$\langle n | m\rangle = \delta_{nm}$$

The above system describe the standard homotopy of the punctured plane. The states we have are objects of definite winding number and as observable we have the winding number $W$. The group elements $\Pi(m)$ play the role of creation and annihilations operators and may serve to build up other type of observables. At this point we may conceive another alternative to probe the puncture of the plane which we call twisted homotopy which from the homotopic point of view is equivalent the above one. In previous works we have introduced twisted homotopy via essentially two different ways. The first one adopts directly the point of view that these loops are "physical" in the sense that they may undergo interactions. As a consequence of the assumed interactions new states show up and serve to probe the punctured plane in a different but equivalent way. The second one regards the twisted homotopy which in the case of the punctured plane is isomorphic to twisted homology, as the homological dual to the twisted cohomology first introduced by E.Witten in the context of topological quantum mechanics. In this paper we would like to develop to some length the second approach as it was only sketched in and to principally introduce a third procedure to derive twisted homotopy which is based on group theoretical considerations. Let us first briefly summarise the first method. In the physical approach an external field is provided to couple to loops through the winding charges with strength $\beta$. The coupling to $W$ has been shown to be linear in order for the field not to destroy the group structure of loops. Thus the Hamiltonian is

$$H_0 = \beta W$$

(2)

In addition self-interactions are introduced asking for them to reflect the group composition law in the same way as for the external field. It is important to note that the above requirement for the interactions to preserve the group structure is the guide line to find the appropriate interactions similar in this respect to gauge symmetries in quantum field theories. The Hamiltonian which satisfies the above requirements is worked out to be of the form

$$H(\beta, \lambda, \rho) = \beta W(\lambda, \rho)$$

$$W(\lambda, \rho) = W + \lambda \sum_{m \in \mathbb{Z}} \rho(m)\Pi(m)$$

(3)
Where $\lambda \rho(m)$ is m dependent self-coupling constant. The eigenstates of the above hamiltonian $|n, \lambda, \rho>$ are related to the unperturbed one through the formula:

$$|n, \lambda, \rho> = e^{-i\lambda J} |n>$$

With $J = -i \sum_{m \in \mathbb{Z}/(0)} \frac{\rho(m)}{m} \Pi(m)$ (4)

This is a unitary transformation provided the "spectral" function $\rho(m)$ obey the requirement relation for the hamiltonian in eq 3 to be hermitian that is $\rho^*(m) = \rho(-m)$. In this new framework the homotopy of the punctured plane is described by a new basis which is the unitary transform of the old one and which define what we called twisted homotopy. The latter is specified as follows.

$$J |n, \lambda, \rho> = -i \partial_\lambda |n, \lambda, \rho>$$

$$\Pi(m) |n, \lambda, \rho> = |n + m, \lambda, \rho>$$

$$\sum_{n\in \mathbb{Z}} |n, \lambda, \rho><n, \lambda, \rho| = 1$$

$$<n, \lambda, \rho|m, \lambda, \rho> = \delta_{nm}$$ (5)

The states $|n, \lambda, \rho>$ which now figure out the "loops" around the hole are no longer of definite winding number. In twisted homotopy the winding number is no longer an observable. It is the generator $J$ built out of the creation and annihilation operator $\Pi(m)$ which is the observable and which play a role analogous to the winding number, where the canonical variables are $\theta$ and $\lambda$ rather than $n$ and $\theta$.

Before passing to the second approach let us mention the possible applications we have found so far of twisted homotopy apart from the already known applications of it in the form of twisted cohomology to Morse theory [5]. In a recent investigation of Bessel functions in the context of homotopy and their relations to the punctured plane [6] we have demonstrated the following correspondence.

$$|n> \Leftrightarrow J_n(z)_{\frac{z^n}{2}}$$ (6)

Where $J_n(z)$ is the Bessel function of integer order $n$. On the other hand the states defining twisted homotopy $|n + \lambda>$ written for the case where the spectral function $\rho(m)$ is of the form $\rho(m) = (-1)^m$ have been shown to correspond to reduced Bessel function of real orders. That is.

$$|n + \lambda> \Leftrightarrow J_{n+\lambda}(z)_{\frac{z^{n+\lambda}}{2}}$$ (7)

And as a consequence of these correspondences, reduced Bessel functions become unified through the relation [6] in a single formula independently of their orders being integers or reals.

$$\frac{J_{n+\lambda}(z)}{z^{n+\lambda}} = e^{-i\lambda J} \frac{J_n(z)}{z^n}$$ (8)

The second application concerns the observable $J$ which plays a role analogous to the winding number. In studying observables in cohomological quantum mechanics defined on the punctured plane we have investigated in details in [6] it turns out that according to the form of the prepotential one has, the observable of the theory may appear in the form of $W$ which is the winding number operator in standard homotopy or in the form of $J$ which is its analogue in twisted homotopy.
2 Cohomological approach

The second approach to twisted homotopy starts from the twisted cohomology. Let $d$ be the exterior derivative which define the De Rham cohomology of the target manifold $M$ which will correspond to the punctured plane in our case and define a "twisted" exterior derivative as

$$d_\lambda[V] = e^{-\lambda V}d e^{\lambda V}$$

(9)

Where $V : M \to \mathbb{R}$ is a Morse function called the prepotential in the framework of topological (supersymmetric) quantum mechanics. At this point we modify the above formula by taking the scaling factor $\lambda$ pure imaginary. That is we consider

$$d_\lambda[V] = e^{-i\lambda V}d e^{i\lambda V}$$

(10)

The factor $i$ is essential as it will ensure the hermiticity of the effective winding number which results from the twisting. Now we add an important input to eq (10) which is the right form to give to the prepotential $V$. In the above cited paper [4] we have shown that the most general prepotential for the punctured plane one can have consistent with the topology is

$$V = K\theta + \Phi(\theta)$$

(11)

Where $\theta$ is the polar angle, $K$ an arbitrary constant and $\Phi(\theta)$ is an arbitrary but periodic function of the argument. More importantly the function should be an even function of $\theta$. Note that $\theta$ is not a periodic function. Now on the subspace of periodic functions defined on the punctured plane $u(\theta)$, the exterior derivative takes the form $d = d\theta \partial_\theta$. Inserting the form of the prepotential into the twisted exterior derivative which we may rewrite as $d_\lambda[V] = \partial_\theta^\lambda[V]d\theta$ and dividing by $d\theta$ we obtain

$$\partial_\theta^\lambda[V] = e^{-i\lambda\phi}\partial_\theta e^{i\lambda\phi} + i\lambda K$$

(12)

To make the transition with the hamiltonian form in eq (3) we define the states $| \theta >$ which are the Fourier transform of the states $| n >$.

$$u(\theta) = \langle \theta | u \rangle$$

$$| \theta > = \sum_{n \in \mathbb{Z}} e^{in\theta} | n >$$

(13)

The winding number operator $W$ acts on the wavefunctions $u(\theta)$ or the states $| \theta >$ as $W = -i\partial_\theta$ While the group elements $\Pi(m)$ act as $\Pi(m) = e^{-im\theta}$. The function $\Phi(\theta)$ being periodic by definition, we decompose it as a Fourier series.

$$\Phi(\theta) = \sum_{m \in \mathbb{Z}/(0)} \sigma(m) e^{-im\theta}$$

(14)

Where $\sigma(m)$ is odd function of $m$ as a consequence of the evenness of $\Phi$. We may formally rewrite $\Phi(\theta)$ as the mean value on the theta states of an operator $J$ as follows.

$$\Phi(\theta) = \langle \theta | J | \theta \rangle$$

$$J = \sum_{m \in \mathbb{Z}/(0)} \sigma(m) \frac{\Pi(m)}{m}$$

(15)
Multiply both sides of the equation eq 12 by the factor -i, we may arrange the resulting equation in operatorial forms where the operators now act on the states $|n\rangle$.

$$\langle \theta | W(\lambda, \rho) | u \rangle = \partial_\theta [V] u(\theta)$$

$$W(\lambda, \rho) = e^{-i\lambda J} W e^{i\lambda J} + \lambda K$$

$$= W + i\lambda \sum_{m \in \mathbb{Z}} \rho(m) \Pi(m)$$

(16)

Where the function $\rho(m)$ extends the function $\sigma(m)$ which vanishes at $m=0$, as.

$$\rho(m) = \begin{cases} 
\sigma(m) & m \neq 0 \\
-iK & m = 0 
\end{cases}$$

(17)

Note that to get the second equation in eq 16 from the first one we use the commutation relation $[W, \Pi(m)] = m\Pi(m)$ one may extract from the defining equations of the involved operators. We thus recover the formula of the twisted winding number $W(\lambda, \rho)$ we found in the first approach.

3 Group theoretic approach

In this section we would like to show how enhancing the fundamental group $\Pi(m)$ from discrete to continuous in a specific but not trivial way has the effect of introducing interactions among loops and precisely those worked out in the hamiltonian above. The only limitation of the approach is that it corresponds to the spectral function in its simplest form of a phase $\rho(m) = (-)^m$. The reason of this limitation is obvious as only when loops $|m\rangle$ are assigned equal weights (couplings) $\rho(m)$ up to a phase does one expect them to still form a group structure and therefore to extract some group properties. The starting point is to rewrite the fundamental group elements in a form familiar to continuous groups.

$$\Pi(m) = e^{-iJm}$$

(18)

$J$ is an operator which will become the generator of the resulting continuous group. The action of the generator $J$ on the $|\theta\rangle$ states defined earlier can be inferred from eq 18

$$J | \theta \rangle = \theta | \theta \rangle$$

(19)

In subsequent analysis we need however the representation of $J$ on the space of the $|n\rangle$ states. This may be obtained by inverting eq 18 which is invertible. To this end a simple way to proceed is to multiply both sides of the equation by the factor $\frac{i^{-iJm}}{m}$ then sum over all values $m \in \mathbb{Z}/(0)$.

$$\sum_{m \in \mathbb{Z}/(0)} (-)^m \frac{\Pi(m)}{m} = \sum_{m \in \mathbb{Z}/(0)} (-)^m \frac{e^{-im\theta}}{m}$$

$$= -2i \sum_{m \in \mathbb{Z}/(0)} (-)^m \frac{\sin(m\theta)}{m}$$

$$= i\theta - \pi \langle \theta | \theta \rangle \pi$$

$$= iJ$$

(20)

In the above sequence, both sides of the equations are acting on the states $|\theta\rangle$ which we do not write explicitly for clarity. Thus we end up with the expression for $J$ we look for.

$$J = -i \sum_{m \in \mathbb{Z}/(0)} (-)^m \frac{\Pi(m)}{m}$$

(21)
The second key point and the most important is to make the transition from discrete values to continuous in the parameter space of the group, while keeping the generator of the group unchanged. This leads immediately to the SO(2) group:

$$\Pi(\lambda) = e^{-iJ\lambda}$$

(22)

It is important to realise that the generator J in the above equation is not modified and keeps its defining form as a sum over the $$\Pi(m)$$s. Note that from the fact that the variable $$\theta$$ is continuous the resulting group does not have periodicity in the $$\lambda$$ variable. The next step is to see how the new operators $$\Pi(\lambda)$$ will act on the old states $$|n>$$ and write down the expression for these, then work out the operator "Hamiltonian" $$W_\lambda$$ which generates these states as eigenstates. Denoting the new states as $$|n_\lambda>$$ we have the set of defining equations, both for $$|n_\lambda>$$ and $$W_\lambda$$.

$$|n_\lambda> = e^{-iJ\lambda} |n>$$

$$W_\lambda |n_\lambda> = n_\lambda |n_\lambda>$$

(23)

The above states can be worked out as we know how J acts on the $$|n>$$ states. An immediate consequence is that the new states by their very definition "collapse" to old states of well-defined winding numbers when the angle $$\lambda$$ picks up integer values i.e.

$$|n_\lambda>_{\lambda=m} = |n+m>$$

(24)

This property is useful in that it allows the extraction of the unknown eigenvalues $$n_\lambda$$ in eq (23). Expanding $$n(\lambda)$$ in powers of $$\lambda$$ the expansion should saturates at the first order in $$\lambda$$ otherwise it will contradict equation (24). We thus obtain.

$$n_\lambda = n + \lambda$$

(25)

It remains now to work out the explicit form of the operator $$W_\lambda$$. Rewrite the eigenvalue equation (23) as

$$W_\lambda e^{-iJ\lambda} |n> = n_\lambda e^{-iJ\lambda} |n> = e^{-iJ\lambda}(W + \lambda) |n>$$

(26)

In operatorial form this equation reads.

$$W_\lambda = e^{-iJ\lambda}We^{iJ\lambda} + \lambda$$

(27)

To make loop self-interactions more explicit, we use the familiar formula.

$$e^{-iJ\lambda}We^{iJ\lambda} = W + i\lambda[W, J]$$

(28)

Together with the commutation relations we have between W and the $$\Pi(m)$$. After some algebra we get the desired result that.

$$W_\lambda = W + \lambda \sum_{m\in\mathbb{Z}} (-)^m \Pi(m)$$

(29)

This is the effective winding number operator to which the external field $$\beta$$ couples as in (8) in which the angle $$\lambda$$ plays the role of the self coupling.
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