Phases in noncommutative quantum mechanics on (pseudo)sphere

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Abstract

We compare the non-commutative quantum mechanics (NCQM) on sphere and the discrete part of the spectrum of NCQM on pseudosphere (Lobachevsky plane, or AdS\textsuperscript{2}) in the presence of a constant magnetic field $B$ with planar NCQM. We show, that (pseudo)spherical NCQM has a critical point, where the system becomes effectively one-dimensional and two different phases, which the phases of the planar system originate from, specified by the sign of the parameter $\gamma = \frac{1}{2} B$. The critical point of (pseudo)spherical NCQM corresponds to the $\gamma = 0$ point of conventional planar NCQM, and to the $\gamma = 1$ point of the so-called exotic\textsuperscript{2} planar NCQM, with a symplectic coupling of the (commutative) magnetic field.

Introduction

Noncommutative quantum mechanics theories have been studied extensively during the last several years owing to their relationship with M-theory compactifications\textsuperscript{1}, string theory in nontrivial backgrounds\textsuperscript{2}, and quantum Halle\textsuperscript{3}ect\textsuperscript{4} (see e.g.\textsuperscript{5} for a recent review). At low energies the one-particle sectors become relevant, which prompted an interest in the study of noncommutative quantum mechanics (NCQM)\textsuperscript{6,7}. In these studies some attention was paid to two-dimensional NCQM in the presence of a constant magnetic field: such systems were considered on a plane\textsuperscript{8,9}, torus\textsuperscript{10}, sphere\textsuperscript{11}, pseudosphere (Lobachevsky plane, or AdS\textsuperscript{2})\textsuperscript{12,13}. NCQM on a plane has a critical point, specified by the zero value of the dimensionless parameter

$$\gamma = \frac{1}{2} B$$

where the system becomes effectively one-dimensional\textsuperscript{14,15}. Out of the critical point, the rotational properties of the model become qualitatively dependent on the sign of $\gamma$: for $\gamma > 0$ the system could have an infinite number of states with a given value of the angular momentum, while for $\gamma < 0$ the number of such states is finite\textsuperscript{16} (see also\textsuperscript{17}). In appropriate limits the NCQM on a (pseudo)sphere should be reduced to the planar one. Hence, out of the (pseudo)spherical system originate, in some sense, the phases of planar NCQM. Although this issue was touched upon in Refs.\textsuperscript{18,19,20}, the complete understanding of this question has yet to be achieved.

In the present paper, we study the relationship between NCQM on (pseudo)sphere and plane. Considering the planar limit of NCQM on (pseudo)sphere we are lead to the conclusion that NCQM on a sphere and the discrete part of the spectrum of NCQM on pseudosphere possess the phases, which yield the phases of the planar system. The critical point of (pseudo)spherical NCQM results in the point $\gamma = 0$ of the planar system suggested in Refs.\textsuperscript{14,15} and in the critical point $\gamma = 1$ of the so-called exotic\textsuperscript{2} NCQM\textsuperscript{12}, where the magnetic field is introduced via `minimal', or symplectic coupling.

NCQM on plane, sphere and pseudosphere

The `conventional' two-dimensional noncommutative quantum mechanical system with arbitrary central potential in the presence of a constant magnetic field $B$, suggested by Nair and Polychronakos, is given by the Hamiltonian \textsuperscript{10}

$$H_{\text{plane}} = \frac{p^2}{2} + V(q^2);$$

and the operators $p_1; q_1; q_2$ which obey the commutation relations

$$[q_1, p_1] = i; \quad [q_1, p_2] = i; \quad [p_1, p_2] = iB; \quad \gamma = 1/2.$$
where the noncommutativity parameter $> 0$ has the dimension of length.  

There exists a so-called "exotic" NCQM suggested by Duval and Horvathy [12]. Its difference from the "conventional" planar NCQM lies in the coupling of external magnetic field. Instead of a naive, or algebraic approach, used in conventional NCQM, the minimal or symplectic, coupling is used there, in the spirit of Souriau [24]. This coupling assumes that the closed two-form describing the magnetic field is added to the symplectic structure of the underlying Hamiltonian mechanics

\[ H^{\text{plane}}; \mathbf{J} = 0 + B \mathbf{q} \wedge \mathbf{p} ; \quad H^{\text{plane}}; \mathbf{J} = B \mathbf{q} \wedge \mathbf{p} ; \quad \text{for} \quad \text{non-constant} \]  

The corresponding quantum mechanical commutators (out of the point $= 0$) read

\[ [\mathbf{q}_1; \mathbf{q}_2] = i\mathbf{J} ; \quad [\mathbf{q}_1; \mathbf{p}_2] = i\mathbf{J} ; \quad [\mathbf{p}_1; \mathbf{p}_2] = i\mathbf{J} ; \quad [\mathbf{p}_1; \mathbf{q}_2] = i\mathbf{J} ; \quad \text{conventional} \]

\[ [\mathbf{q}_1; \mathbf{q}_2] = i; \quad [\mathbf{q}_1; \mathbf{p}_2] = i\mathbf{J} ; \quad [\mathbf{p}_1; \mathbf{p}_2] = i\mathbf{J} ; \quad [\mathbf{p}_1; \mathbf{q}_2] = i; \quad \text{exotic} \]

The Hamiltonian is the same as in the "conventional" NCQM, [3].

It is convenient to represent these systems as follows:

\[ H^{\text{plane}} = \left( \frac{\mathbf{q} \cdot \mathbf{J}}{2} \right) + V(\mathbf{q}^2) ; \]

where the operators and $\mathbf{q}$ are given by the expressions

\[ 1 = \mathbf{p}_2 \mathbf{q}_2 ; \quad 2 = \mathbf{p}_1 \mathbf{q}_2 + \mathbf{q}_1 \mathbf{p}_2 ; \quad [\mathbf{q}_1; \mathbf{q}_2] = 0 ; \quad [\mathbf{q}_1; \mathbf{p}_2] = i; \quad [\mathbf{p}_1; \mathbf{q}_2] = i; \quad \text{conventional} \]

\[ [\mathbf{q}_1; \mathbf{q}_2] = i; \quad [\mathbf{q}_1; \mathbf{p}_2] = i; \quad [\mathbf{p}_1; \mathbf{p}_2] = i; \quad [\mathbf{p}_1; \mathbf{q}_2] = i; \quad \text{exotic} \]

The angular momentum of these systems is defined by the operator (out of the point $= 0$)

\[ L = \begin{cases} \mathbf{q}^2=2 ; & \text{conventional} \\ \mathbf{q}^2=2 ; & \text{exotic} \end{cases} \]

Its eigenvalues are given by the expression

\[ l = (n_1; n_2) ; \quad n_1; n_2 = 0; \cdots \]

where $(n_1; n_2)$ denote, respectively, the eigenvalues of the operators $(\mathbf{q}^2; \mathbf{J})$ for the "conventional" NCQM and of the $(\mathbf{q}^2; \mathbf{J})$ for the "exotic" one, the upper sign corresponds to the "conventional" system, and the lower sign to the "exotic" one. Hence, the rotational properties of NCQM qualitatively depend on the sign of $\mathbf{J}$.

At the "critical point", i.e. for $\mathbf{J} = 0$, these systems become effectively one-dimensional [3, 12]

\[ H^{\text{plane}} = \begin{cases} \mathbf{q}^2=2 + V(\mathbf{q}^2) ; & \text{conventional} \\ V(\mathbf{q}^2) ; & \text{exotic} \end{cases} \]

Let us remind [24], that for non-constant $B$ the Jacobi identities failed in the "conventional" model, while in the "exotic" model the Jacobi identities hold for any $B = A_{ij} \mathbf{q}^i \mathbf{q}^j$, by definition. This rectifies the different origin of magnetic fields appearing in these two models. In the "conventional" model, $B$ appears as the strength of a non-commutative magnetic field, while in the "exotic" model $B$ appears as a commutative magnetic field, obtained by the Seiberg-Witten map from the non-commutative one. In the quantum mechanical context this question was considered in [3].

The Hamiltonian of the axially-symmetric NCQM on the sphere [4, 17, 18] and pseudospheres [18, 21] in the presence of a constant magnetic field, looks precisely as in the commutative case (up to the dimensionless parameter $B$)

\[ H = \frac{\mathbf{q}^2}{2\mathbf{s}^2} + V(\mathbf{q}^2) ; \]

where the rotation and position operators $J_i = (J_i; J_{ij})$, $x_i = (x_i; x_{ij})$ obey commutation relations

\[ [J_i; J_{ij}] = 0; \quad [J_i; x_{ij}] = 0; \quad [x_i; x_{ij}] = 0; \quad [x_i; x_{ij}] = 0; \quad [x_i; x_{ij}] = 0; \quad \text{in} \]

In [4, 18] the constant term $s^2=2\mathbf{s}^2$ was ignored; also in [3] the factor was chosen to be unity as well.
Here and after, for squaring the operators and for rising/lowering the indices, we use the diagonal metric diag(1;1;1) for the sphere and diag(1; 1; 1) for the pseudosphere. The upper sign corresponds to a sphere, and the lower one to a pseudosphere. The noncommutativity parameter has the dimension of length and is assumed to be positive, $>0$. The values of the Casimir operators of the algebra are fixed by the equations

$$
C_0 \ x^2 = r_0^2 > 0; \quad C_1 \ J_x \ \frac{J^2}{2} = r_0 S(s;r_0);
$$

where $r_0$ is the radius of the (pseudo)sphere and $s$ is the monopole number. In the commutative limit $r_0 \to 0$ the parameters $S$ and should have a limit:

$$
S(s;r_0) ! s = B r_0^2;
$$

where $B$ is a strength of the magnetic field.

The angular momentum of the system is defined by the operator $J_3 : [\mathcal{H}; J_3] = 0$.

The algebra (12) can be split into two independent copies of $su(2)/su(1,1)$,

$$
K_i = J_i \ \frac{x_i}{r_0} : [K_1; x_1] = 0; \quad [K_1; x_1] = 1_{ijk} K^j; \quad [k_1; x_1] = 1_{ijk} x^j:
$$

In these terms the Casimir operators read $C_0 = x^2$ and $C_1 = (x^2 - K^2)/2$. For the NCQM on a sphere, the Casimir operators $C_0, K^2$ are positive. For a pseudosphere $C_1$ is positive, whereas another Casimir operator, i.e. $K^2$, could get positive, zero or negative values. We restrict ourselves to the case of positive $K^2$ which is responsible for the description of the discrete part of the energy spectrum. Hence, the Casimir operators take the following values:

$$
x_0^2 = 2m (m \ 1); \quad 2s = \frac{1}{2}; \quad [k; 1] = m (m \ 1); \quad k^2 = 2m (m \ 1);
$$

where $m, k$ are non-negative (half) integers xing the representation of $SU(2)$, in the case of sphere, and $m, k > 1$ are real numbers, xing the representation of $SU(1,1)$, in the case of pseudosphere.

It is unclear, how the qualitatively different planar phases $> 0$ and $< 0$ originate in the (pseudo)spherical NCQM, as well as whether the limit of (pseudo)spherical NCQM results in the conventional or the exotic planar system. Some steps in relating (pseudo)spherical NCQM with conventional planar system were performed in [14, 17, 21]. In particular, the following expressions for $\ , \ , s$ parameters were found there:

$$
k = m; \quad s = k \ m;
$$

while the arising of planar phases was explained as a projection "one to two" (?). A further inconsistency of the above picture is that upon the choice of parameters [13], the (pseudo)spherical NCQM becomes effectively one-dimensional at the point $K = 0$, which yields $1$, instead of $0$. At this point, the Hamiltonian consists of a potential term only, which seems to be in agreement with the planar exotic Hamiltonian at the point $0$.

In order to clarify the above listed questions, in the next section we compare the planar limits of NCQM on sphere and of the discrete part of the NCQM on pseudosphere with both conventional and exotic versions of planar NCQM.

\section*{NCQM: (pseudo)sphere plane}

In order to obtain the planar limit of the NCQM on the (pseudo)sphere out of the point $=$ $0$, we should take the limits [14]

$$
k ! 1; \quad m ! 1;
$$

and consider all neighborhoods of the pole of coordinate and monopole sphere.

$$
x_0 = \frac{x^2}{2r_0} = \frac{x^2}{2m} ; \quad x_0 = \frac{K^2}{2K}; \quad 1; 2 = 1:
$$
In these neighborhoods the commutation relations

\[ [x_1; x_2] = i_1 \ell_1 m; \quad [K_1; K_2] = i_2 K \]  \hspace{1cm} (20)

hold, while the Hamiltonian looks as follows:

\[ H = \frac{K^2}{2 r_0^2} - 2 K x = \frac{m^2}{s^2} + V(\ell^2) \quad E_0 = \frac{(K x)^2}{2 r_0^2} + V(\ell^2); \]  \hspace{1cm} (21)

Here we introduced the notation

\[ m = p_1 (m + 1); \quad K = \frac{p_1}{k (k + 1)} = \frac{q}{m - k}; \quad m = k \]

and

\[ E_0 = \frac{(K + m)^2}{2 r_0^2}; \]  \hspace{1cm} (22)

In order to get the planar Hamiltonian with a positively defined kinetic term, we should put

\[ \text{sgn} = \frac{s}{r_0^2}; \]  \hspace{1cm} (23)

For a correspondence with the planar Hamiltonian \[ \text{[3]} \], we redefine the coordinates and momenta of the resulting system as follows:

\[ \frac{p_1}{r_0}; \quad \frac{q}{r_0}; \]  \hspace{1cm} (24)

Then, comparing their commutators with \[ \text{[3]} \], we get the following expressions for the parameters:

\[ \frac{2m^2}{r_0^2} = K; \quad \text{conventional} \]

\[ \frac{2m}{r_0^2} = i; \quad \text{exotic} \]  \hspace{1cm} (25)

and the same value of \[ \text{[3]} \] for both systems:

\[ \frac{m}{r_0}; \]  \hspace{1cm} (26)

Naively, it seems that the planar NCQM with \[ \ell < 0 \] and positive kinetic term corresponds to a (pseudo)spherical system with negative kinetic term. Fortunately, thanks to the additional term \[ s^2 = 2r_0^2 \] the kinetic term of the Hamiltonian \[ \text{[3]} \] remains positively defined! Indeed, one can identify them monopole numbers as follows:

\[ s = \frac{m + k}{m + K}; \quad \text{conventional} \]

\[ (m + K); \quad \text{exotic} \]  \hspace{1cm} (27)

which yields the vanishing of the vacuum energy \[ \text{[2]} \], and the following expressions for the magnetic field, which are in agreement with \[ \text{[4]} \]:

\[ B = \frac{B}{1/b} = \frac{1}{s}; \quad \text{conventional} \]

\[ s = r_0^2; \quad \text{exotic} \]  \hspace{1cm} (28)

One can redefine the parameters \[ s \], as follows:

\[ m \frac{1 = 2}{k \ 1 = 2}; \quad s = k \frac{1 = 2 + (m \ 1 = 2)}{k \ 1 = 2}; \quad \text{conventional} \]

\[ (m \ 1 = 2); \quad \text{exotic} \]  \hspace{1cm} (29)

In this case the monopole number is quantized on a sphere, and it remains not quantized on a pseudosphere, as in the commutative case. The constant energy term \[ E_0 \] vanishes upon this choice too.

Taking into account that the maximal value of \[ \ell^2 \] is \( (k + m) (k + m + 1) \), and the minimal one is \( k \cdot m \cdot (k + m + 1) \) \[ \text{[2]} \], we obtain

\[ \frac{\ell^2}{2 r_0^2}; \quad 0; \]  \hspace{1cm} (30)
Hence, the kinetic part of the (pseudo)spherical Hamiltonian is positively defined for any $\theta$. Expanding (pseudo)spherical NCQM near the upper/lower bound of $J^2$, we shall get the planar NCQM with $B > 0/ < 0$.

In order to avoid the rescaling of the potential in the planar limit, we should take

$$ = = \text{conventional} \quad \text{exotic} :$$

Upon this choice, the expression reads

$$ \frac{s}{r_0^5} = \begin{cases} B & \text{conventional} \\ B & \text{exotic} : \end{cases}$$

In the conventional picture $B$ plays the role of the strength of a (commutative) magnetic field obtained by the Seiberg-Witten map from the non-commutative one [14]. In the exotic picture the same role is played by $B$. Hence, in both pictures we get the standard expression for the strength of the constant commutative magnetic field on (pseudo)sphere, and the quantization of the flux of the commutative magnetic field on the sphere, as well.

We did not consider yet the planar limit of the critical point of (pseudo)spherical NCQM, and did not establish yet, whether the latter results in the conventional or in the exotic planar NCQM, in this limit. For this purpose let us notice, that our specification of the monopole number $s$ and of the parameter yields the following values of the first Casimir operator:

$$C_0 = r_0^2 = 2m^2 \quad \text{conventional} \quad \text{exotic} :$$

Thus, in the conventional picture the (pseudo)spherical NCQM becomes one-dimensional for $R = 0$, i.e. for $s = 1$; in the exotic picture we have, instead, $m = 0$, i.e. $s = 0$.

In the exotic picture the (pseudo)spherical NCQM in the $s = 0$ limit results in the system

$$H_0 = V(x^2); \quad [x_1; x_2] = i \quad x^2 = r_0^2;$$

which reduces, immediately, to the exotic planar NCQM at the critical point.

Hence, the conventional and phases of (pseudo)spherical NCQM reduce, in the planar limit, to the respective conventional and phases of exotic NCQM, with the symplectic coupling of the commutative magnetic field.

The eigenvalues of the angular momentum of (pseudo)spherical NCQM are given by the expression

$$j_0 = k_3 + m_3; \quad k_3 = 0; \quad \text{sphere}$$

$$k_3 = k; \quad k + 1); \quad m_3 = m; \quad (m + 1); \quad \text{pseudosphere} :$$

Introducing $m_3 = 1 \quad (m + 1); \quad k_3 = 1 \quad (k + n_2)$, we get

$$j_0 = 1 (m + n_1) + 2 (k + n_2) = 1 (m + k) (n_1 + m_2);$$

which is in agreement with the angular momentum of planar NCQM [3].

Conclusion

We considered non-commutative quantum mechanics on sphere and on pseudosphere in the presence of constant magnetic field (with the strength $B$), and compared these systems with conventional and exotic models of noncommutative quantum mechanics on plane, specified by a different coupling of the magnetic field.

We have shown that the quantum mechanics on sphere and the discrete part of the spectrum of quantum mechanics on pseudosphere, defined by the Hamiltonian [11] and the commutation relations [12] essentially depend on the values of the Casimir operators,

$$C_0 = x^2 = r_0^2 > 0; \quad C_1 = Jx = \frac{\gamma^2}{2}; \quad \text{conventional} \quad \text{exotic} :$$

$$C_0 \quad C_1;$$
where \( r_0, s \) are the noncommutativity parameter, the radius of sphere and the monopole number, respectively.

When \( C_0 = C_1 = 2 \) (pseudo)spherical NCQM becomes effectively one-dimensional. It yields the conventional planar NCQM for \( \lambda = 1 \) and the exotic one at the point \( \lambda = 0 \), with the symplectic (or minimal) coupling of the commutative magnetic field \( B \).

When \( C_0 > C_1 = 2 \), the monopole number is connected with \( r_t \) by the quadratic equation, which has two solutions, corresponding to the positive and negative values of the specic parameter. In the planar limit, the \( 'phases' \) of (pseudo)spherical NCQM lead to the respective \( 'phases' \) of conventional and exotic planar NCQM.

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