Linear energy divergences in Coulomb gauge
QCD

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Abstract
The structure of linear energy divergences is analysed on the example of
one graph to 3-loop order. Such dangerous divergences do cancel when
all graphs are added, but next to leading divergences do not cancel out.

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1 Introduction

The Coulomb gauge in non-Abelian gauge theories is a very good example of a physical gauge. It is manifestly unitary. Although there are ghosts, their propagators have no poles. The propagators are closely related to the polarization states of real spin-1 particles. Nevertheless there are problems concerned with energy divergences [1]. In individual Feynman graphs there appear even linear energy divergences. These are divergences over the energy integration in a loop, for fixed values of the 3-momentum, of the form

$$\int dk_0 F$$

where $F$ is independent of $k_0$. They do cancel when all graphs are combined [2]. However, it makes one uneasy in manipulating divergent and unregulated integrals.

2 The graph $2B(b, 0i0)$

We have studied the renormalization in Coulomb gauge QCD to three-loop order in Hamiltonian formalism [3]. It was shown that to three loops the UV divergences cannot be consistently absorbed by the Christ-Lee term [4]. In this paper we show in detail how dangerous the linear divergences are on the example of one graph with fermion loop three-point function. The graph is shown in fig.1.

![Graph 2B(b, 0i0)](image)

Figure 1: Graph $2B(b, 0i0)$ which is an example of the graph containing linear energy divergences

We use the same notation and graphical conventions as in [5]. Using the Ward identities for high energies we have derived the expression for the quark loop three-point function with two Coulomb and one transverse line,

$$V_{001}(k_1, k_2, k_3) \approx \frac{K_{2i}}{k_{10}} [k_{20}S(k_2) + k_{30}S(k_3)] - \frac{K_{1i}}{k_{20}} [k_{30}S(k_3) + k_{10}S(k_1)],$$  (2)
where the gluon self-energy from the quark loop is
\[
\text{tr}(t^a t^b) S_{\mu_1 \mu_2}(p) = g^2 C_q \delta^{ab} (p_{\mu_1} p_{\mu_2} - p^2 \delta_{\mu_1 \mu_2}) S(p^2)
\]
(3)
with
\[
S(p^2) = 8i \pi^{\frac{5}{2}} \frac{1}{\Gamma(4 - \epsilon)} \frac{\Gamma^2(2 - \frac{\epsilon}{2})}{\Gamma(4 - \epsilon)} \left[ (-p^2 - i\eta)^{\frac{\epsilon}{2}} - (\mu^2)^{\frac{\epsilon}{2}} \right],
\]
(4)
where a renormalization subtraction at a mass \( \mu \) has been made and \( \epsilon = 4 - n \), \( n \) is the number of space-time dimensions. Applying (2), (3) and (4) to the graph in fig.1 we obtain for the \( K^2 \) part the expression
\[
2B(b, 0; 0) = -\frac{1}{4} g^6 (2\pi)^{-8} C_q^2 T(R) \delta_{ab} K^2 \int d^4p \int d^4q \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \times \left\{ \frac{1}{p_0 p_0'} [S(r') - S(q)] + \frac{1}{p_0 q_0'} [S(r') - S(p')] - \frac{1}{p_0 p_0'} [S(q) + S(p')] \right\}.
\]
(5)

The momenta are defined as \( p' = p - k \), \( q' = q - k \), \( r' = k - p - q \), \( p^2 = p_0^2 - P^2 \) and in the high energy limit we have used the approximation
\[
\frac{p_0}{p_0^2 - P^2 + i\eta} \approx \frac{1}{p_0}.
\]
(6)

The first term in (5) is explicit linear energy divergence. It is the difference of two integrals, one with \( S(r') \) and the other with \( S(q) \).

3 Linear energy divergence

Let us consider the first integral in (5).
\[
J_{ij} = \frac{1}{7} \int d^4p \int d^4q \frac{p_0}{p_0^2 - P^2 + i\eta} \cdot \frac{p_0'}{p_0^2 - P'^2 + i\eta} \times \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \left[ (p_0 + q_0 - k_0)^2 - (P + Q - K)^2 + i\eta \right]^\frac{1}{2}
\]
(7)

Using the Schwinger representation for the propagators \( J_{ij} \) becomes
\[
J_{ij} = (-i)^{2 + \frac{5}{2}} \int_{-\infty}^{\infty} d\rho_0 d\rho_0 (p - k) \int d^{3 - \epsilon} P \int_{-\infty}^{\infty} d\rho_0 \int d^{3 - \epsilon} Q \times \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\gamma \gamma^{\frac{5}{2} - 1} \frac{P_i Q'_j}{P^2 P'^2 Q^2 Q'^2} \times e^{i\alpha (\rho_0^2 - P^2 + i\eta)} e^{i\beta (\rho_0^2 - P'^2 + i\eta)} e^{i\gamma (r_0^2 - R^2 + i\eta)}
\]
(8)
Performing the $q_0$ and $p_0$ integrations with Gaussian integrals followed by integration over the parameter $\gamma$, we obtain

$$J_{ij} = (-i)^{3+\frac{3}{2}} \pi \Gamma\left(\frac{\epsilon - 1}{2}\right) \int d^3\epsilon P \int d^3\epsilon Q \frac{P_i Q_j}{P^2 P'Q^2 Q'Q'^2} \int_0^\infty d\alpha \int_0^\infty d\beta e^{-i\alpha P^2 - i\beta P'^2 - \eta(\alpha + \beta)}$$

$$\times \frac{1}{(\alpha + \beta)^{3/2}} e^{ik_0^2 \alpha \beta} \cdot \left[ \frac{i}{2} - \frac{\alpha \beta}{\alpha + \beta} k_0^2 \right] \cdot (\eta + iR^2)^{3/2}. \quad (9)$$

Changing the variables of integration $\alpha$ and $\beta$ as

$$\alpha = \lambda v, \quad \beta = \lambda(1 - v), \quad \frac{\partial \alpha, \partial \beta}{\partial \lambda, \partial v} = \lambda,$$

$$0 < v < 1, \quad 0 < \lambda < \infty, \quad (10)$$

makes $\lambda$-integration easy, leading to

$$J_{ij} = \Gamma\left(\frac{\epsilon - 1}{2}\right) \frac{1}{2} \pi^{3/4} e^{-\pi^2 v^2} \int d^3\epsilon P \int d^3\epsilon Q \int_0^1 dv \frac{P^2 v + P'^2(1 - v)}{[P^2 v + P'^2(1 - v) - k_0^2 v(1 - v) - i\eta]^3/2}$$

$$\times \frac{P_i Q_j}{P^2 P'Q^2 Q'^2} \cdot \frac{1}{(R^2)^{3/2}}. \quad (11)$$

Repeating the same operations with the other integral containing $S(q)$, we obtain for the linear energy divergence term in (5) the expression

$$L_{ij} = \int d^4p \int d^4q \frac{P_i Q_j}{P^2 P'Q^2 Q'^2} \cdot \frac{1}{p_0 p'_0} [S(q') - S(q)]$$

$$= \Gamma\left(\frac{\epsilon - 1}{2}\right) \frac{1}{2} \pi^{3/2} e^{-\pi^2 v^2} \int d^3\epsilon P \int d^3\epsilon Q \int_0^1 dv \frac{P^2 v + P'^2(1 - v)}{[P^2 v + P'^2(1 - v) - k_0^2 v(1 - v) - i\eta]^3/2}$$

$$\times \frac{P_i Q_j}{P^2 P'Q^2 Q'^2} \left[ \frac{1}{(R^2)^{3/2}} - \frac{1}{(Q'^2)^{3/2}} \right]. \quad (12)$$

The linear energy divergence reflects in the factor $\Gamma\left(\frac{\epsilon - 1}{2}\right)$. The $Q$-integral is

$$X_j = \int d^3\epsilon Q \frac{Q_j}{Q^2 Q'^2} \left[ \frac{1}{(R^2)^{3/2}} - \frac{1}{(Q'^2)^{3/2}} \right]. \quad (13)$$

The second integral is easy.

$$B_j = \int d^3\epsilon Q \frac{Q_j}{Q'^2 Q'^2} \left[ \frac{1}{(R^2)^{3/2}} - \frac{1}{(Q'^2)^{3/2}} \right]$$

$$= -K_j (K^2)^{\epsilon - \frac{3}{2}} \frac{\Gamma(\epsilon)}{\Gamma\left(\frac{1 + \epsilon}{2}\right)} \frac{\Gamma(2 - \epsilon)\Gamma\left(\frac{1 - \epsilon}{2}\right)}{\Gamma\left(\frac{5 - \epsilon}{2}\right) \Gamma\left(\frac{3 + \epsilon}{2}\right)} \quad (14)$$
Let us study the first integral in (13).

\[
A_j = \int d^{3-\epsilon} Q Q_j' Q'^2 Q \Gamma(\frac{3-\epsilon}{2}) \frac{1}{(R^2)^{\frac{\epsilon}{2}}} \tag{15}
\]

We combine the denominators \(Q^2\) and \(Q'^2\) with the Feynman parameter \(x\) and then \((R^2)^{\frac{\epsilon}{2}}\) with the parameter \(y\) (remembering that \(R' = -P' - Q\), \(Q' = Q - K\)), then integrate in \(d^{3-\epsilon} Q\).

\[
A_j = \frac{\pi^{\frac{3-\epsilon}{2}}}{\Gamma(\frac{3-\epsilon}{2})} \int_0^1 dx \int_0^1 dy y^{\frac{3-\epsilon}{2}} (1-y) \cdot [K_j x(1-y) - K_j - P_j' y] \\
\times [P^2 + K^2 x(1-y) - (K x(1-y) - P^\prime y)^2]^{-\epsilon} \tag{16}
\]

We insert (16) and (14) into (12).

\[
L_{ij} = \frac{1}{2} \pi^{\frac{3-\epsilon}{2}} e^{-\epsilon} \int d^{3-\epsilon} P \int_0^1 dv \frac{P^2 + P^2(1-v)}{[P^{2 v} + P^2(1-v) - k^2_0 v(1-v) - i\eta]^2} \cdot \frac{P_i}{P^2 P^2} \\
\times \{ \Gamma(\epsilon) \int_0^1 dx \int_0^1 dy y^{\frac{3-\epsilon}{2}} (1-y) [K_j x(1-y) - K_j - P_j' y] \cdot [P^2 + K^2 x(1-y) - (K x(1-y) - P^\prime y)^2]^{-\epsilon} \\
+ \Gamma\left(\frac{\epsilon}{2}\right) \Gamma(\epsilon) \cdot \frac{\Gamma(2 - \epsilon) \Gamma(\frac{1-\epsilon}{2})}{\Gamma\left(\frac{3-\epsilon}{2}\right) \Gamma\left(\frac{3}{2} - \frac{\epsilon}{2}\right)} \cdot K_j (K^2)^{-\epsilon} \} \tag{17}
\]

We notice that the \(d^{3-\epsilon} P\) integration is IR safe and also UV finite by power counting for \(K_j\) terms, while it is UV divergent for \(P_j' y\) term. Hence, for the leading divergence we can set

\[
[P^2 + K^2 x(1-y) - (K x(1-y) - P^\prime y)^2]^{-\epsilon} \approx 1, \tag{18}
\]

\[
L_{ij} = \Gamma\left(\frac{\epsilon - 1}{2}\right) \Gamma(\epsilon) \frac{1}{2} \pi^3 K_j \int d^{3-\epsilon} P \int_0^1 dv \frac{P_i}{P^2 P^2} \cdot \frac{P^2 + P^2(1-v)}{[P^{2 v} + P^2(1-v) - k^2_0 v(1-v) - i\eta]^2} \\
\times \left\{ \frac{\Gamma(3)}{2 \Gamma\left(\frac{3-\epsilon}{2}\right)} - \frac{\Gamma(2) \Gamma(\frac{1-\epsilon}{2})}{\Gamma\left(\frac{3-\epsilon}{2}\right) \Gamma\left(\frac{3}{2} - \frac{\epsilon}{2}\right)} \right\} \\
- \frac{1}{2} \pi^3 \Gamma(\epsilon) \int d^{3-\epsilon} P \int_0^1 dv \frac{P^2 + P^2(1-v)}{[P^{2 v} + P^2(1-v) - k^2_0 v(1-v) - i\eta]^{3/2}} \cdot \frac{P_i P_j'}{P^2 P^2} \int_0^1 dx \int_0^1 dy y^{\frac{3-\epsilon}{2}} (1-y) \tag{19}
\]

We can easily isolate the UV divergence from the last integral in (19). It behaves like \(\Gamma\left(\frac{\epsilon}{2}\right)\).

## 4 Conclusion

Individual Feynman graphs in Coulomb gauge QCD to three loop order contain even linear energy divergences. Our analysis shows their behaviour. It is
\[ L_{ij} = \Gamma\left(\frac{\epsilon - 1}{2}\right)\Gamma(\epsilon)K_iK_j f(k_0^2, K^2) + \Gamma(\epsilon)\Gamma\left(\frac{\epsilon}{2}\right)K_iK_j g(k_0^2, K^2). \quad \text{(20)} \]

The first term is the product of poles in \( \frac{1}{\epsilon - 1} \cdot \frac{1}{\epsilon} \) while the second is a double pole in \( \frac{1}{\epsilon} \). Such dangerous divergences do cancel in the sum of graphs with three-point and four-point fermion loop insertions, but next to leading divergences coming from \( \frac{1}{p_0 q_0} \), \( \frac{1}{p_0 r_0} \), and \( \frac{1}{q_0 r_0} \) terms do not. Hence, UV divergences from higher order graphs cannot be consistently absorbed by renormalization of the Christ-Lee term [3].

References

[1] J. C. Taylor, in Physical and Nonstandard Gauges, Proceedings, Vienna, Austria 1989, edited by P. Gaigg, W. Kummer, M. Schweda,

[2] R. N. Mohapatra, Phys. Rev. D 4(1971) 1007,

[3] A. Andraši, J. C. Taylor, Ann. Phys. 326 (2011) 1053,

[4] N. Christ, T. D. Lee, Phys. Rev. D 22 (1980) 939,

[5] A. Andraši, J. C. Taylor, Ann. Phys. 324 (2009) 2179.