A BRIEF SURVEY OF HIGGS BUNDLES

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Abstract

Considering a compact Riemann surface of genus greater than two, a Higgs bundle is a pair composed of a holomorphic bundle over the Riemann surface, joint with an auxiliary vector field, so-called Higgs field. This theory started around thirty years ago, with Hitchin’s work, when he reduced the self-duality equations from dimension four to dimension two, and so, studied those equations over Riemann surfaces. Hitchin baptized those fields as Higgs fields because in the context of physics and gauge theory, they describe similar particles to those described by the Higgs boson. Later, Simpson used the name Higgs bundle for a holomorphic bundle together with a Higgs field. Today, Higgs bundles are the subject of research in several areas such as non-abelian Hodge theory, Langlands, mirror symmetry, integrable systems, quantum field theory (QFT), among others. The main purposes here are to introduce these objects, and to present a brief but complete construction of the moduli space of Higgs bundles, and some of its stratifications.

Keywords: Higgs bundles, Hodge bundles, moduli spaces, stable triples, vector bundles.

MSC classes: Primary 14H60; Secondaries 14D07, 55Q52.

Resumen

Considerando una superficie compacta de Riemann de género mayor que dos, un fibrado de Higgs es un par compuesto por un fibrado holomorfo sobre la superficie de Riemann, junto con un campo vectorial auxiliar, llamado campo de Higgs. Esta teoría inició hace unos treinta años, con el trabajo de Hitchin, cuando él reduce las ecuaciones de autodualidad de dimensión cuatro a dimensión dos, y así, estudiar esas ecuaciones sobre superficies de Riemann. Hitchin bautizó esos campos como campos de Higgs pues en el contexto de la física y de la teoría de gauge, describen partículas similares a las descritas por el bóson de Higgs. Más tarde, Simpson usó el nombre fibrado de Higgs para un fibrado holomorfo junto con un campo de Higgs. Hoy, los fibrados de Higgs son objeto de investigación en varias áreas tales como la teoría de Hodge no abeliana, Langlands, simetría de espejo, sistemas integrables, teoría cuántica de campos (QFT), entre otros. Los propósitos principales aquí son introducir estos objetos y presentar una breve pero completa construcción del espacio móduli de los fibrados de Higgs y algunas de sus estratificaciones.

Palabras clave: Fibrados de Higgs, Fibrados de Hodge, Espacios Móduli, Triples Estables, Fibrados Vectoriales.

Introduction

Nineteen years ago, the Clay Institute of Cambridge, Massachusetts, presented seven millenium challenging problems. Yang-Mills and Mass Gap is one of those challenging unsolved problems. Newton’s classical mechanics laws stand to planets and celestial bodies as Quantum laws stand to elementary particles. Sixty five years ago, C. N. Yang and R. L. Mills [14] introduced a remarkable framework to describe particles using structures that occur in geometry. Quantum Yang-Mills theory is nowadays the foundation of elementary particle physics theory and its predictions have been tested at many laboratories; nevertheless, its

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We recall some basic facts about differential geometry, the mass gap: the quantum particles have positive masses, even though the classical waves travel at the speed of light. Establishing the existence of the Yang-Mills theory and a mass gap will require the introduction of fundamental new ideas both in physics and in mathematics. Thirty two years ago, Hitchin \cite{Hitchin} reduced Yang-Mills self-duality equations from $\mathbb{R}^4$ to $\mathbb{R}^2$ imposing invariance translation in two directions, and considering an auxiliar vector field, so-called Higgs field. Hitchin’s reduction of Yang-Mills self-duality equations in $\mathbb{R}^2$ have the important property of conformal invariance, which allows them to be defined on Riemann surfaces.

Let $\Sigma = \Sigma_g$ be a compact (closed and connected) Riemann surface of genus $g > 2$. Let $K = K_\Sigma = T^*\Sigma$ be the canonical line bundle over $\Sigma$ (its cotangent bundle).

**Remark 0.1.** Algebraically, $\Sigma_g$ is also a complete irreducible nonsingular algebraic curve over $\mathbb{C}$: 

$$\dim_{\mathbb{C}}(\Sigma_g) = 1.$$  

The paper is organized as follows: in section 1 we recall some basic facts about differential geometry, specifically about smooth vector bundles and holomorphic vector bundles; in section 2, we present the gauge group action over the set of connections and the induced action over the holomorphic structures; in section 3, we define the moduli space of Higgs bundles, and we present the two principal constructions: Hitchin construction in 3.2 and Simpson construction in 3.3; finally, in 4, we present the most recent result of the author, published in \cite{MPP} in terms of homotopy.

## 1 Preliminary definitions

We shall recall some basic definitions we will need, like bundles and some of their invariants such as rank, degree and slope. We also present the definition of stability.

**Definition 1.1.** For any smooth vector bundle $\mathcal{E} \rightarrow \Sigma$, we denote the rank of $\mathcal{E}$ by $\text{rk}(\mathcal{E}) = r$ and the degree of $\mathcal{E}$ by $\text{deg}(\mathcal{E}) = d$. Then, for any smooth complex bundle $\mathcal{E} \rightarrow \Sigma$ the slope is defined to be

$$\mu(\mathcal{E}) := \frac{\text{deg}(\mathcal{E})}{\text{rk}(\mathcal{E})} = \frac{d}{r}. \quad (1)$$

**Definition 1.2.** A smooth vector bundle $\mathcal{E} \rightarrow \Sigma$ is called semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for any $\mathcal{F}$ such that $0 \subseteq \mathcal{F} \subseteq \mathcal{E}$. Similarly, a vector bundle $\mathcal{E} \rightarrow \Sigma$ is called stable if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for any nonzero proper subbundle $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$. Finally, $\mathcal{E}$ is called polystable if it is the direct sum of stable subbundles, all of the same slope.

Let $\mathcal{E} \rightarrow \Sigma$ be a complex smooth vector bundle with a hermitian metric on it.

**Definition 1.3.** A connection $d_A$ on $\mathcal{E}$ is a differential operator 

$$d_A : \Omega^0(\Sigma, \mathcal{E}) \rightarrow \Omega^1(\Sigma, \mathcal{E})$$

such that 

$$d_A(fs) = df \otimes s + fd_A s$$

for any function $f \in C^\infty(\Sigma)$ and any section $s \in \Omega^0(\Sigma, \mathcal{E})$ where $\Omega^0(\Sigma, \mathcal{E})$ is the set of smooth 0-forms of $\Sigma$ with values in $\mathcal{E}$. Locally: 

$$d_A = d + A = d + Cdz + Bdz$$

where $A$ is a matrix of 1-forms: $A_{ij} \in \Omega^1(\Sigma, \mathcal{E})$, and $B, C$ are matrix valued functions depending on the hermitian metric on $\mathcal{E}$.

**Definition 1.4.** When a connection $d_A$ is compatible with the hermitian metric on $\mathcal{E}$, i.e. 

$$d \langle s, t \rangle = \langle d_A s, t \rangle + \langle s, d_A t \rangle$$

for the hermitian inner product $\langle \cdot, \cdot \rangle$ and for $s, t$ any couple of sections of $\mathcal{E}$, $d_A$ is a unitary connection. Denote $\mathcal{A}(\mathcal{E})$ as the space of unitary connections on $\mathcal{E}$, for a smooth bundle $\mathcal{E} \rightarrow \Sigma$.  

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Remark 1.5. Some authors call the matrix $A$ as a connection and call $d_A = d + A$ as its corresponding covariant derivative. We abuse notation and will not distinguish between them.

Definition 2.1. A holomorphic structure on $E$ is a differential operator:
\[
\bar{\partial}_A : \Omega^0(\Sigma, \mathcal{E}) \to \Omega^{0,1}(\Sigma, \mathcal{E})
\]
such that $\bar{\partial}_A$ satisfies Liebniz rule:
\[
\bar{\partial}_A(fs) = \bar{\partial}f \otimes s + f\bar{\partial}_A s
\]
and the integrability condition, which is precisely vanishing of the curvature
\[
F_A = 0.
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Definition 2.3. A holomorphic structure on $E$ is a differential operator:
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\[
F_A = 0.
\]

Remark 2.2. Note that conjugation by a unitary gauge transformation takes a unitary connection to a unitary connection. We denote by $\mathcal{G}^u$ the quotient of $\mathcal{G}$ by the central subgroup $U(1)$, and by $\mathcal{G}^C$ the complex gauge group. Besides, we denote by $BG^u$ and by $BG^C$ the classifying spaces of $\mathcal{G}$ and $\mathcal{G}$ respectively. There is a homotopical equivalence: $\mathcal{G} \simeq \mathcal{G}^C$.

Definition 2.4. Denote the space of holomorphic structures on $E$ by $\mathcal{A}^{0,1}(\mathcal{E})$, and consider the collection
\[
\mathcal{A}^{0,1}(r, d) = \{ \mathcal{A}^{0,1}(\mathcal{E}) : \text{rk}(\mathcal{E}) = r \quad \text{and} \quad \deg(\mathcal{E}) = d \}. Here, the subcollections
\[
\mathcal{A}^{0,1}_{\text{ss}}(r, d) \subseteq \mathcal{A}^{0,1}(r, d),
\]
\[
\mathcal{A}^{0,1}_{\text{ss}}(r, d) = \{ \mathcal{A}^{0,1}(\mathcal{E}) : \mathcal{E} \text{ is semistable} \} \subseteq \mathcal{A}^{0,1}(r, d),
\]
\[
\mathcal{A}^{0,1}_{\text{ps}}(r, d) = \{ \mathcal{A}^{0,1}(\mathcal{E}) : \mathcal{E} \text{ is polystable} \} \subseteq \mathcal{A}^{0,1}(r, d),
\]
will be of particular interest for us.

2 Gauge Group Action

The gauge theory arises first in the context of physics, specifically in general relativity and classical electromagnetism. Here, we present the mathematical formalism.

Definition 2.2. A gauge transformation is an automorphism of $\mathcal{E}$. Locally, a gauge transformation $g \in \text{Aut}(\mathcal{E})$ is a $C^\infty$-function with values in $GL_r(\mathbb{C})$. A gauge transformation $g$ is unitary if $g$ preserves the hermitian inner product.

We denote by $\mathcal{G}$ the group of unitary gauge transformations. This gauge group $\mathcal{G}$ acts on $\mathcal{A}(\mathcal{E})$ by conjugation:
\[
g \cdot d_A = g^{-1}d_A g \forall g \in \mathcal{G} \quad \text{and for} \quad d_A \in \mathcal{A}(\mathcal{E}).
\]

Remark 1.2. Some authors call the matrix $A$ as a connection and call $d_A = d + A$ as its corresponding covariant derivative. We abuse notation and will not distinguish between them.
3 Moduli Space of Higgs Bundles

A central feature of Higgs bundles, is that they come in collections parametrized by the points of a quasi-projective variety: the Moduli Space of Higgs Bundles.

3.1 Moduli space of vector bundles

The starting point to classify these collections, is the Moduli Space of Vector bundles. Using Mumford’s Geometric Invariant Theory (GIT) from the work of Mumford Fogrty and Kirwan [10], first Narasimhan & Seshadri [11], and then Atiyah & Bott [1], build and characterize this family of bundles by Morse theory and by their stability. Both classifications are equivalent. We use the one from stability here:

**Definition 3.1.** Define the moduli space of (polystable) vector bundles \( E \to \Sigma \) as the quotient

\[
\mathcal{N}(r, d) = A^{0,1}_p(r, d)/G^C.
\]

Respectively, define the moduli space of stable vector bundles as the quotient

\[
\mathcal{N}_s(r, d) = A^{0,1}_s(r, d)/G^C \subseteq \mathcal{N}(r, d).
\]

**Remark 3.2.** The quotient \( A^{0,1}_p(r, d)/G^C \) is not Hausdorff so, we may lose a lot of interesting properties.

Narasimhan & Seshadri [11] characterize this variety, and then Atiyah & Bott [1] conclude in their work that:

**Theorem 3.3** (9, [1]). If \( \text{GCD}(r, d) = 1 \), then \( A^{0,1}_p = A^{0,1}_s \) and \( \mathcal{N}(r, d) = \mathcal{N}_s(r, d) \) becomes a (compact) differential projective algebraic variety with dimension

\[
\dim_C (\mathcal{N}(r, d)) = r^2(g-1) + 1.
\]

3.2 Hitchin Construction

Hitchin [7] works with the Yang-Mills self-duality equations (SDE)

\[
\begin{cases}
F_A + [\phi, \phi^*] = 0 \\
\bar{\partial}_A \phi = 0
\end{cases}
\]

where \( \phi \in \Omega^{1,0}(\Sigma, \text{End}(\mathcal{E})) \) is a complex auxiliary field and \( F_A \) is the curvature of a connection \( d_A \) which is compatible with the holomorphic structure of the bundle \( E = (\mathcal{E}, \bar{\partial}_A) \). Hitchin calls such a \( \phi \) as a Higgs field, because it shares a lot of the physical and gauge properties to those of the Higgs boson.

The set of solutions

\[
\beta(\mathcal{E}) := \{ (\bar{\partial}_A, \phi) \mid \text{solution of (2)} \} \subseteq A^{0,1}(\mathcal{E}) \times \Omega^{1,0}(\Sigma, \text{End}(\mathcal{E}))
\]

and the collection

\[
\beta_{ps}(r, d) := \{ \beta(\mathcal{E}) \mid \mathcal{E} \text{ polystable}, \text{rk(}\mathcal{E}\text{)} = r, \deg(\mathcal{E}) = d \}
\]

allow Hitchin to construct the Moduli space of solutions to SDE (2)

\[
\mathcal{M}^{YM}(r, d) = \beta_{ps}(r, d)/G^C,
\]

and

\[
\mathcal{M}^{YM}_s(r, d) = \beta_s(r, d)/G^C \subseteq \mathcal{M}^{YM}(r, d)
\]

the moduli space of stable solutions to SDE (2).

**Remark 3.4.** Recall that \( \text{GCD}(r, d) = 1 \) implies \( A^{0,1}_p = A^{0,1}_s \) and so \( \mathcal{M}^{YM}(r, d) = \mathcal{M}^{YM}_s(r, d) \).

Nitsure [12] computes the dimension of this space in his work:

**Theorem 3.5** (Nitsure (1991)). The space \( \mathcal{M}(r, d) \) is a quasi–projective variety of complex dimension

\[
\dim_C (\mathcal{M}(r, d)) = (r^2 - 1)(2g - 2).
\]
3.3 Simpson Construction

An alternative concept arises in the work of Simpson [13]:

**Definition 3.6.** A Higgs bundle over $\Sigma$ is a pair $(E, \phi)$ where $E \to \Sigma$ is a holomorphic vector bundle together with $\phi$, an endomorphism of $E$ twisted by $K = T^*(\Sigma) \to \Sigma$, the canonical line bundle of the surface $\Sigma$: $\phi : E \to E \otimes K$. The field $\phi$ is what Simpson calls Higgs field.

There is a condition of stability analogous to the one for vector bundles, but with reference just to subbundles preserved by the endomorphism $\phi$:

**Definition 3.7.** A subbundle $F \subseteq E$ is said to be $\phi$-invariant if $\phi(F) \subseteq F \otimes K$. A Higgs bundle is said to be semistable [respectively, stable] if $\mu(F) \leq \mu(E)$ [respectively, $\mu(F) < \mu(E)$] for any nonzero $\phi$-invariant subbundle $F \subseteq E$ [resp., $F \subseteq E$]. Finally, $(E, \phi)$ is called polystable if it is the direct sum of stable $\phi$-invariant subbundles, all of the same slope.

With this notion of stability in mind, Simpson [13] constructs the Moduli space of Higgs bundles as the quotient

$$\mathcal{M}^H(r, d) = \{ (E, \phi) | E \text{ polystable } \} / G^C$$

and the subspace

$$\mathcal{M}^H_s(r, d) = \{ (E, \phi) | E \text{ stable } \} / G^C \subseteq \mathcal{M}^H(r, d)$$

of stable Higgs bundles.

**Remark 3.8.** Again, $\text{GCD}(r, d) = 1$ implies $\mathcal{M}^H(r, d) = \mathcal{M}^H_s(r, d)$. See Simpson [13] for details.

Finally, Simpson [13] concludes:

**Proposition 3.9 (Prop. 1.5. [13]).** There is a homeomorphism of topological spaces

$$\mathcal{M}^H(r, d) \cong \mathcal{Y}M(r, d). \quad \square$$

4 Recent Results

If we consider a fixed point $p \in X$ as a divisor $p \in \text{Sym}^1(X) = X$, and $L_p$ the line bundle that corresponds to that divisor $p$, we get a complex of the form

$$E \xrightarrow{\Phi^k} E \otimes K \otimes L_p^\otimes k$$

where $\Phi^k \in H^0(X, \text{End}(E) \otimes K \otimes L_p^\otimes k)$ is a Higgs field with poles of order $k$. So, we call such a complex as a $k$-Higgs bundle and $\Phi^k$ as its $k$-Higgs field. As well as for a stable complex, a $k$-Higgs bundle $(E, \Phi^k)$ is stable (respectively semistable) if the slope of any $\Phi^k$-invariant subbundle of $E$ is strictly less (respectively less or equal) than the slope of $E : \mu(E)$. Finally, $(E, \Phi^k)$ is called polystable if $E$ is the direct sum of stable $\Phi^k$-invariant subbundles, all of the same slope.

Motivated by the results of Bradlow, García–Prada, Gothen [3] and the work of Hausel [5], in terms of the homotopy groups of the moduli space of Higgs bundles and $k$-Higgs bundles respectively, the author has tried to improve the result of Hausel to higher rank. So far, here is one of the main results. Recall that here, only announce the result, no proof is given. The reader can find the proof in [17].

**Theorem 4.1 (Cor. 4.14. [17]).** Suppose the rank is either $r = 2$ or $r = 3$, and $\text{GCD}(r, d) = 1$. Then, for all $n$ exists $k_0$, depending on $n$, such that

$$\pi_j \left( \mathcal{M}^H_k(r, d) \right) \xrightarrow{\cong} \pi_j \left( \mathcal{M}^\infty_k(r, d) \right)$$

for all $k \geq k_0$ and for all $j \leq n - 1$. \quad \square

There is a lot of interesting recent results related with the homotopy of the moduli space of Higgs bundles, in terms of its stratifications (see [4]), and in terms of Morse Theory and Variations of Hodge Structures (see [17]). Nevertheless, the introduction of a wide bunch of notation would be needed. The author encourages the reader to explore those results in the references above mentioned.
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