RIEMANN-ROCH THEOREMS VIA DEFORMATION QUANTIZATION

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1. Introduction

In this note we outline the proof of a formula expressing the Euler class of a perfect complex of modules over a symplectic deformation quantization of a complex manifold in terms of the Chern character of the associated symbol complex, the \( \hat{A} \)-class of the manifold and a characteristic class of the deformation quantization. As a consequence we obtain the conjecture of P. Schapira and J.-P. Schneiders (\cite{SS}, Conjecture 8.5, see Conjecture 2.2.2) and the corollaries thereof.

In order to formulate the conjecture one needs a construction of a local Chern character (cf. \cite{SS}, p.93). Such a construction is provided in Section 4 and appears to be new.

Specifically, we define the Euler class and the Chern character of a perfect complex of sheaves of modules over a sheaf of algebras by “sheafifying” the Dennis trace map and the Goodwillie-Jones map defined by R. McCarthy in the generality of exact categories in \cite{McC}. The construction, as described in Section 4, applies to sheaves of algebras on a topological space, but, in fact, can be carried out for a sheaf of algebras on a site (using, for example, the work of J. F. Jardine).

We reduce the Riemann-Roch theorem for the Euler class to the Riemann-Roch theorem for periodic cyclic cochains of deformed sheaves of algebras of functions on a symplectic manifold. We reduce the latter to the local Riemann-Roch theorem for periodic cyclic cochains of the Weyl algebra. The latter result is contained in \cite{NT1}, \cite{NT2} and draws on the ideas in \cite{FT2}.

For the sake of brevity we restrict our attention to the “absolute” case. Analogous constructions can be carried out for families of symplectic manifolds without any additional difficulties. Two techniques whose discussion is not included here are

- Chern–Weil construction of Lie algebra cocycles with coefficients “constant up to homotopy” (used to reduce the Riemann-Roch theorem to a local statement);
• noncommutative differential calculus for periodic cyclic cochains needed in the proof of the local Riemann-Roch theorem.

Admittedly, our presentation of the preliminaries is extremely sketchy. A more detailed exposition will be given in a future paper.

Here are the contents of the paper in brief.

In Section 2 we recall some of the basic definitions and facts about elliptic pairs leading up to Conjecture 2.2.2.

In Section 3 we review basic properties of quantized rings of functions on symplectic manifolds and relate them to algebras of differential and microdifferential operators. This relationship allows us to place the traditional Riemann-Roch theorem into the context of deformation quantization.

Section 4 contains the background material on Hochschild and cyclic homology, and their relationship to algebraic K-theory necessary to define the Euler class and the Chern character of a perfect complex of sheaves of modules over a sheaf of algebras.

In Section 5 we summarize the results on the Hochschild and cyclic homology of algebras of (micro)differential operators as well as quantized function algebras and compare characteristic classes and trace density maps.

Section 6 is devoted to the statement of our main technical result (Theorem 6.1.1) which may be considered as a local Riemann-Roch type theorem for periodic cyclic chains of the quantized ring of functions. We apply Theorem 6.1.1 to characteristic classes of perfect complexes of modules over the quantized rings of functions as well as the algebras of differential and microdifferential operators to obtain Conjecture 2.2.2 and analogous statements as corollaries.

Section 7 leads up to the statement of Theorem 7.6.1 which is the analog of Theorem 6.1.1 in the particular case when the symplectic manifold in question is the formal neighborhood of the origin in a symplectic vector space.

In Section 8 we review the basic facts concerning Fedosov connections and Gel’fand-Fuchs cohomology necessary to, loosely speaking, establish a map from Theorem 7.6.1 to Theorem 6.1.1. As a consequence of the existence of this (Gel’fand-Fuchs) map the former theorem implies the latter.

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2. THE INDEX THEORY FOR ELLIPTIC PAIRS

The application of our results to the index theory of elliptic pairs is of particular importance. Here we recall the results of [SS] restricting ourselves to the “absolute” case for the sake of simplicity.

2.1. Let $X$ be a complex manifold. An elliptic pair $(\mathcal{M}^\bullet, F^\bullet)$ on $X$ consists of an object $\mathcal{M}^\bullet$ of $\mathcal{D}^b_{\text{good}}(\mathcal{D}^\bullet_X)$ (the bounded derived category of complexes of right $\mathcal{D}^\bullet_X$-modules with good cohomology and an object $F^\bullet$ of $\mathcal{D}^b_{\text{R-c}}(X)$ (the bounded derived category of $\mathbb{R}$-constructible complexes of sheaves of vector spaces over $\mathbb{C}$ on $X$) which satisfy

$$\text{char}(\mathcal{M}^\bullet) \cap \text{SS}(F^\bullet) \subseteq T^*_X X.$$ 

If $\text{Supp}(\mathcal{M}^\bullet, F^\bullet) \overset{\text{def}}{=} \text{Supp} \mathcal{M}^\bullet \cap \text{Supp} F^\bullet$ (where the support of a complex of sheaves is understood to be the cohomological support) is compact one has

$$\dim H^\bullet(X; F^\bullet \otimes M^\bullet \otimes L^D_X \mathcal{O}_X) < \infty.$$ 

Thus, the Euler characteristic

$$\chi(X; F^\bullet \otimes M^\bullet \otimes L^D_X \mathcal{O}_X) \overset{\text{def}}{=} \sum (-1)^i \dim H^i(X; F^\bullet \otimes M^\bullet \otimes L^D_X \mathcal{O}_X)$$

is defined. As particular cases one obtains

- the Euler characteristic $\chi(X; L^\bullet)$ of $X$ with coefficients in a compactly supported perfect complex $L^\bullet$ of $\mathcal{O}_X$-modules (taking $F^\bullet = \mathbb{C}_X$, $M^\bullet = L^\bullet \otimes \mathcal{O}_X \mathcal{D}_X$);
- the index of an elliptic complex $L^\bullet$ on a compact real analytic manifold $X_0$ (taking $X$ to be a complexification of $X_0$ such that $L^\bullet$ extends to a complex $\mathcal{L}^\bullet$ of $\mathcal{O}_X$-modules and differential operators, $F^\bullet = \mathbb{C}_{X_0}$, $M^\bullet = \text{Diff}(\mathcal{O}_X, \mathcal{L}^\bullet)$).

The index theorem for elliptic pairs ([SS], Theorem 5.1) says that, for an elliptic pair $(\mathcal{M}^\bullet, F^\bullet)$ with compact support on $X$ of dimension $\dim_{\mathbb{C}} X = d$

$$\chi(X; F^\bullet \otimes M^\bullet \otimes L^D_X \mathcal{O}_X) = \int_{T^*X} \mu \text{eu}(M^\bullet) \sim \mu \text{eu}(F^\bullet)$$

where $\mu \text{eu}(M^\bullet) \in H^d_{\text{char}(M^\bullet)}(T^*X; \mathbb{C})$ and $\mu \text{eu}(F^\bullet) \in H^d_{\text{SS}(F^\bullet)}(T^*X; \mathbb{C})$ are defined in [SS].

The class $\mu \text{eu}(F^\bullet)$ is the characteristic cycle of the constructible complex $F^\bullet$ as defined by Kashiwara (see [KS] for more details). For

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1A $\mathcal{D}_X$-module is called good if it admits a good filtration in a neighborhood of every compact subset of $X$. 
example, if $Y \subset X$ is a closed real analytic submanifold, one has 

$\mu \text{eu}(C_Y) = |T^*_Y X|.$

With regard to $\mu \text{eu}(M^\bullet)$ P.Schapira and J.-P.Schneiders conjectured that it is related to a certain characteristic class of the symbol $\sigma(M^\bullet)$ of $M^\bullet$ ([SS], Conjecture 8.5, see Conjecture 2.2.2).

2.2. The Riemann-Roch type formula. Suppose that $M^\bullet$ is an object of $D^b_{\text{good}}(\mathcal{D}^\text{op}_X)$. Let $\pi : T^*X \to X$ denote the projection.

If $M^\bullet$ admits a global good filtration (and this is the case when $X$ is compact) then the symbol complex of $M$ is defined by

$$\sigma(M^\bullet) = \pi^{-1} \text{gr } M^\bullet \otimes_{\pi^{-1} \text{gr } D^X} \mathcal{O}_{T^*X}.$$

The assumption that the filtration is good amounts to the fact that $\sigma(M^\bullet)$ has $\mathcal{O}_{T^*X}$-coherent cohomology, and the characteristic variety is defined by

$$\text{char}(M^\bullet) \overset{\text{def}}{=} \text{Supp } \sigma(M^\bullet).$$

For $\Lambda$ a closed subvariety of $T^*X$ let $K^0_\Lambda(T^*X)$ denote the Grothendieck group of perfect complexes of $\mathcal{O}_{T^*X}$-modules supported on $\Lambda$ (i.e. acyclic on the complement of $\Lambda$ in $T^*X$).

For $\Lambda$ containing $\text{char}(M)$ let $\sigma_\Lambda(M^\bullet)$ denote the class of $\sigma(M^\bullet)$ in $K^0_\Lambda(T^*X)$.

**Remark 2.2.1.** As is easy to show, both the characteristic variety and the class of the symbol in the Grothendieck group are independent of the choice of the good filtration, thus the existence of a good filtration locally is sufficient to define $\sigma_\Lambda(M^\bullet)$.

One can define the Chern character (see (4.3))

$$ch_\Lambda : K^0_\Lambda(T^*X) \to \bigoplus_i H^{2i}_\Lambda(T^*X; \mathbb{C}).$$

For $\alpha$ an element of a graded object let $[\alpha]^p$ denote the homogeneous component of $\alpha$ of degree $p$.

In ([SS], P.Schapira and J.-P.Schneiders make the following conjecture.

**Conjecture 2.2.2.** For $M^\bullet$ in $D^b_{\text{good}}(\mathcal{D}^\text{op}_X)$, $\Lambda$ a conic subvariety of $T^*X$ containing $\text{char}(M^\bullet)$

$$\mu \text{eu}(M^\bullet) = [ch_\Lambda(\sigma(M^\bullet)) \sim \pi^*Td(TX)]^{2d}.$$

We will obtain the above conjecture as a corollary of a Riemann-Roch type formula in the context of deformation quantization of symplectic manifolds.
3. Deformation quantization

3.1. Review of deformation quantization. A deformation quantization of a manifold $M$ is a formal one parameter deformation of the structure sheaf $\mathcal{O}_M$, i.e. a sheaf of algebras $\mathbb{A}^h_M$ flat over $\mathbb{C}[[\hbar]]$ together with an isomorphism of algebras $\mathbb{A}^h_M \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \to \mathcal{O}_M$.

The formula
$$\{f, g\} = \frac{1}{\hbar} [\tilde{f}, \tilde{g}] + \hbar \cdot \mathbb{A}^h_M,$$
where $f$ and $g$ are two local sections of $\mathcal{O}_M$ and $\tilde{f}, \tilde{g}$ are their respective lifts $\mathbb{A}^h_M$, defines a Poisson structure on $M$ called the Poisson structure associated to the deformation quantization $\mathbb{A}^h_M$.

Remark 3.1.1. The definition of “deformation quantization” as given above is essentially the one given by [BFSS] in the case of $C^\infty$ manifolds. It is not at all clear whether the scope of generality of the above definition above is sufficiently broad ...

The deformation quantization $\mathbb{A}^h_M$ is called symplectic if the associated Poisson structure is nondegenerate. In this case $M$ is symplectic. In what follows we will only consider symplectic deformation quantizations, so assume that $\mathbb{A}^h_M$ is symplectic from now on.

It is known (and not difficult to show) that all symplectic deformation quantizations of $M$ of dimension $\dim_{\mathbb{C}} M = 2d$ are locally isomorphic to the standard deformation quantization of $\mathbb{C}^{2d}$. That is, for any point $x \in M$ and small neighborhoods $U$ of $x$ and $U'$ of the origin in $\mathbb{C}^{2d}$ there is an isomorphism
$$\mathbb{A}^h_{\mathbb{C}^{2d}}(U') \overset{\text{def}}{=} \mathcal{O}_{\mathbb{C}^{2d}}(U')[[[\hbar]]] \cong \mathbb{A}^h_M(U) \tag{3.1}$$
of algebras over $\mathbb{C}[[\hbar]]$, continuous in the $\hbar$-adic topology, where the product on $\mathbb{A}^h_{\mathbb{C}^{2d}}(U')$ is given, in coordinates $x_1, \ldots, x_d, \xi_1, \ldots, \xi_d$ on $\mathbb{C}^{2d}$ by the standard Weyl product
$$(f * g)(\underline{x}, \underline{\xi}) = \exp \left( \frac{-1}{2} \sum_{i=1}^{d} \left( \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial y_i} - \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial x_i} \right) \right) f(\underline{x}, \underline{\xi}) g(\underline{y}, \underline{\eta})|_{\underline{x} = \underline{y}, \underline{\xi} = \underline{\eta}}$$
where $\underline{x} = (x_1, \ldots, x_d)$, $\underline{\xi} = (\xi_1, \ldots, \xi_d)$, $\underline{y} = (y_1, \ldots, y_d)$, $\underline{\eta} = (\eta_1, \ldots, \eta_d)$. Note that the reduction of (3.1) modulo $\hbar$ is an isomorphism of Poisson algebras
$$\mathcal{O}_{\mathbb{C}^{2d}}(U') \cong \mathcal{O}_M(U)$$
with the Poisson brackets associated to the standard symplectic structure on $\mathbb{C}^{2d}$ and to the deformation quantization respectively. In particular (the images of) $x_1, \ldots, x_d, \xi_1, \ldots, \xi_d$ form a Darboux coordinate system on $U \subset M$. 
To a symplectic deformation quantization $A^h_M$ one associates a characteristic class $\theta \in H^2(M; \mathbb{C}[[\hbar]])$ with the property that the coefficient of $\frac{1}{\sqrt{-1 \hbar}}$ is the class of the symplectic form associated to the deformation quantization.

3.2. Microlocalization. We start with a coherent $\mathcal{D}_X$-module $M$ equipped with a good filtration $F_*M$ and consider the graded module

$$\mathcal{R}M \overset{def}{=} \bigoplus_q F_qM \cdot \hbar^q$$

over the graded ring

$$\mathcal{R}\mathcal{D}_X \overset{def}{=} \bigoplus_q F_q\mathcal{D}_X \cdot \hbar^q \subset \mathcal{D}_X[\hbar]$$

where $F_*\mathcal{D}_X$ is the filtration by the order of the differential operator.

There exists a deformation quantization $A^h_{T^*X}$ of $T^*X$, i.e. a formal deformation of the structure sheaf $\mathcal{O}_{T^*X}$ and faithfully flat maps

$$\pi^{-1}\mathcal{R}\mathcal{D}_X \to \mathcal{R}\mathcal{E}_X \to A^h_{T^*X}$$

of algebras over $\mathbb{C}[\hbar]$ where $\mathcal{E}_X$ is the sheaf (on $T^*X$) of microdifferential operators. The characteristic class $\theta$ of the deformation $A^h_{T^*X}$ is equal to $\frac{1}{2}\pi^*c_1(TX)$ (note that the symplectic form is exact in this case).

Consider the “microlocalization”

$$\mu M \overset{def}{=} \pi^{-1}\mathcal{R}M \otimes_{\pi^{-1}\mathcal{R}\mathcal{D}_X} A^h_{T^*X}$$

of the filtered $\mathcal{D}_X$-module $M$. Note that $\mu M$ is $h$-torsion free. Then, clearly, there is an isomorphism

$$\sigma(M) \cong \mu M \otimes_{\mathcal{A}^h_{T^*X}} \mathcal{O}_{T^*X} \cong \mu M / \mu M \cdot h .$$

Defining the symbol of an $h$-torsion free $\mathcal{A}^h_{T^*X}$-module $N$ by

$$\sigma(N) = N \otimes_{\mathcal{A}^h_{T^*X}} \mathcal{O}_{T^*X}$$

we have

$$\sigma(M) \cong \sigma(\mu M) .$$

4. Characteristic classes of perfect complexes

In this section we construct the Euler class (with values in Hochschild homology) and the Chern character (with values in negative cyclic homology) for a perfect complex of sheaves of modules over a sheaf of algebras over a topological space.
4.1. Review of Hochschild and cyclic homology. Let \( k \) denote a commutative algebra over a field of characteristic zero and let \( A \) be a flat \( k \)-algebra with \( 1_A \cdot k \) contained in the center, not necessarily commutative. Let \( C_p(A) \) denote \( A^{\otimes p+1} \) and let

\[
b : C_p(A) \rightarrow C_{p-1}(A)
\]

\[
a_0 \otimes \cdots \otimes a_p \mapsto (-1)^p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p .
\]

Then \( b^2 = 0 \) and the complex \( (C_\bullet, b) \), called the standard Hochschild complex of \( A \) represents \( A \otimes_{A \otimes k A^{op}} A \) in the derived category of \( k \)-modules.

The map

\[
B : C_p(A) \rightarrow C_{p+1}(A)
\]

\[
a_0 \otimes \cdots \otimes a_p \mapsto \sum_{i=0}^{p} (-1)^{pi} 1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{i-1}
\]

satisfies \( B^2 = 0 \) and \([B, b] = 0\) and therefore defines a map of complexes

\[
B : C_\bullet(A) \rightarrow C_\bullet(A)[-1] .
\]

For \( i, j, p \in \mathbb{Z} \) let

\[
CC_p^-(A) = \prod_{i+j=p \mod 2} C_{i+j}(A)
\]

\[
CC_p^{per}(A) = \prod_{i+j=p \mod 2} C_{i+j}(A) .
\]

The complex \((CC_p^-(A), B+b)\) (respectively \((CC_p^{per}(A), B+b)\)) is called the negative (respectively periodic) cyclic complex of \( A \).

There are inclusions of complexes

\[
CC_p^-(A)[-2] \hookrightarrow CC_p^-(A) \hookrightarrow CC_p^{per}(A)
\]

and the short exact sequence

\[
0 \rightarrow CC_p^-(A)[-2] \rightarrow CC_p^-(A) \rightarrow C_\bullet(A) \rightarrow 0 .
\]

Suppose that \( X \) is a topological space and \( A \) is a flat sheaf of \( k \)-algebras on \( X \) such that there is a global section \( 1 \in \Gamma(X; A) \) which restricts to \( 1_{A_x} \) and \( 1_{A_x} \cdot k \) is contained in the center of \( A_x \) for every point \( x \in X \). Let \( C_\bullet(A) \) (respectively \( CC_p^-(A), CC_p^{per}(A) \)) denote the complex of sheaves of \( k \)-modules associated to the presheaf with value \( C_\bullet(A(U)) \) (respectively \( CC_p^-(A(U)), CC_p^{per}(A(U)) \)) on an open subset
U of X. Then \( C_\bullet(A) \) represents \( A \otimes_{A \otimes_k A^{\text{op}}} L \) in the derived category of sheaves of \( k \)-modules on \( X \).

4.2. **Perfect complexes.** We briefly recall the notion of perfection as introduced in [Ill].

A module over \( A \) is said to be **free of finite type** if it is isomorphic to \( A \oplus n \) for some \( n \in \mathbb{Z} \).

A complex \( P^\bullet \) of (sheaves of) \( A \)-modules is called **strictly perfect** if

1. \( P^p = 0 \) for almost all \( p \in \mathbb{Z} \);
2. for any \( p \in \mathbb{Z} \) and every point \( x \in X \) there exists a neighborhood \( U \) of \( x \) such that \( P^p|_U \) is a direct summand of a free \( A|_U \)-module of finite type.

A complex \( F^\bullet \) of (sheaves of) \( A \)-modules is called **perfect** if for any point \( x \in X \) there exists an open neighborhood \( U \) of \( x \), a strictly perfect complex \( P^\bullet \) of \( A|_U \)-modules and a quasiisomorphism \( P^\bullet \to F^\bullet|_U \).

4.3. **The Euler class in Hochschild homology.** For a perfect complex \( F^\bullet \) of \( A \) modules we define the **Lefschetz map** as the morphism in the derived category of sheaves of \( k \)-modules given by the composition

\[
\text{R Hom}_A^\bullet(F^\bullet, F^\bullet) \xrightarrow{\sim} \text{R Hom}_A^\bullet(F^\bullet) \otimes_k F^\bullet \otimes_{A \otimes_k A^{\text{op}}} A \xrightarrow{ev \otimes \text{id}} A \otimes_{A \otimes_k A^{\text{op}}} A
\]

and will denote it by \( L_A(F^\bullet) \) or simply by \( L_A \).

We define the **Euler map** of \( F^\bullet \) as the morphism in the derived category given by the composition

\[
k \xrightarrow{1 \otimes \text{id}} \text{R Hom}_A^\bullet(F^\bullet, F^\bullet) \xrightarrow{L_A} A \otimes_{A \otimes_k A^{\text{op}}} A
\]

and will denote it by \( \text{Eu}_A(F^\bullet) \).

Suppose that \( Z \) is a closed subset of \( X \) such that \( Z \supset \text{Supp} F^\bullet \) and \( H^p F^\bullet \). Then, clearly, the canonical morphism \( \text{R} \Gamma_Z(\text{R Hom}_A^\bullet(F^\bullet, F^\bullet)) \to \text{R Hom}_A^\bullet(F^\bullet, F^\bullet) \) is an isomorphism.

Thus, after applying the functor \( \text{R} \Gamma(X; \text{R} \Gamma_Z(\bullet)) \) and passing to cohomology, the morphism \( \text{Eu}_A(F^\bullet) \) determines a cohomology class in \( H^0_Z(X; A \otimes_{A \otimes_k A^{\text{op}}} A) \) which we will refer to as the **Euler class** and will denote by \( \text{eu}_A^Z(F^\bullet) \).

For \( Z \) a closed subset of \( X \) let \( K^0_Z(A) \) denote the Grothendieck group of perfect complexes of \( A \)-modules supported on \( Z \); set \( K^0(A) \) and \( K^0_X(A) \). The Euler class defined above determines a homomorphism of groups

\[
\text{eu}_A^Z : K^0_Z(A) \to H^0_Z(X; A \otimes_{A \otimes_k A^{\text{op}}} A) .
\]
4.4. **Review of K-theory.** Recall that, to a category $\mathcal{C}$ with cofibrations and weak equivalences one associates the simplicial category $S_*\mathcal{C}$ (the $S$-construction) whose definition can be found in [W]. The $K$-theory spectrum $K(\mathcal{C})$ of $\mathcal{C}$ is defined by $K(\mathcal{C}) = \Omega |S_*\mathcal{C}|$ which describes the zeroth $\Omega_\infty$-space whose deloopings are given by iterating the $S$-construction. The groups $K_i(\mathcal{C})$ are defined by

$$K_i(\mathcal{C}) = \pi_i K(\mathcal{C}) .$$

To a ring $A$ one associates the spectrum $K^{\text{naive}}(A)$ (respectively $K(A)$) defined as the $K$-theory spectrum of the category of strictly perfect (respectively perfect) complexes of $A$-modules.

The natural inclusion of the former category into the latter induces the natural morphism of spectra

$$K^{\text{naive}}(A) \to K(A) .$$

The inclusion of the category $\mathcal{P}_A$ of finitely generated projective $A$-modules into the category of strictly perfect complexes induces the natural weak equivalence

$$K(\mathcal{P}_A) \to K^{\text{naive}}(A) .$$

4.5. **Hochschild and cyclic homology revisited.** We summarize some of the results of [McC] concerning the Hochschild and cyclic homology of exact categories.

To a $k$-linear additive category $\mathcal{C}$ ($k$ a ring) one associates the cyclic nerve $CN_*\mathcal{C}$ which is a cyclic $k$-module.

In the case when $\mathcal{C}$ is a $k$-linear additive category with one object, i.e. a $k$-algebra $A$, one has $CN_*\mathcal{C} \cong C_\bullet(A)$.

In what follows we will not make notational distinctions between complexes and corresponding simplicial Abelian groups.

To a category $\mathcal{C}$ with cofibrations and weak equivalences one associates the Hochschild complex $C_\bullet(\mathcal{C})$ by

$$C_\bullet(\mathcal{C}) = \Omega |CN_*S_*\mathcal{C}| = \text{Tot}(CN_*S_*\mathcal{C})[-1] .$$

Using the cyclic structure on $C_\bullet(\mathcal{C})$ one defines the cyclic (respectively the negative cyclic, respectively the periodic cyclic) complex of $\mathcal{C}$ which we denote by $CC_\bullet(\mathcal{C})$ (respectively $CC^{-}_\bullet(\mathcal{C})$, respectively $CC^{\text{per}}_\bullet(\mathcal{C})$).

There is natural commutative diagram

$$
\begin{array}{c}
CC^{-}_\bullet(\mathcal{C}) \longrightarrow CC^{\text{per}}_\bullet(\mathcal{C}) \longrightarrow CC_\bullet(\mathcal{C})[2] \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \text{id} \\
C_\bullet(\mathcal{C}) \longrightarrow CC_\bullet(\mathcal{C}) \longrightarrow CC_\bullet(\mathcal{C})[2]
\end{array}
$$

with rows exact triangles.
Let \( \mathcal{P}_A \) denote the category of finitely generated projective \( A \)-modules. There is a natural quasiisomorphism of cyclic \( k \)-modules
\[
C_\bullet(A) \to C_\bullet(\mathcal{P}_A)
\]
which induces quasiisomorphisms of respective cyclic (negative cyclic, periodic cyclic) complexes.

The natural inclusion
\[
S_\bullet \mathcal{C} \to CN_\bullet S_\bullet \mathcal{C}
\]
induces the natural morphism of spectra
\[
\text{Eu}_\mathcal{C} : K(\mathcal{C}) \to C_\bullet(\mathcal{C})
\]
which is the Dennis trace map in the case \( \mathcal{C} = \mathcal{P}_A \). The morphism \( \text{Eu} \) has a natural lifting
\[
ch_\mathcal{C} : K(\mathcal{C}) \to CC_\bullet(\mathcal{C})
\]
which is the Chern character (the Goodwillie-Jones map) in the case \( \mathcal{C} = \mathcal{P}_A \).

One can show that the (exact) inclusion of \( \mathcal{P}_A \) into the category of strictly perfect complexes of \( A \)-modules induces quasiisomorphisms of respective Hochschild, cyclic, negative cyclic, and periodic cyclic complexes. In particular one has the Euler class and the Chern character
\[
\text{Eu}_A : K(\mathcal{C}) \to C_\bullet(A), \quad ch_A : K(\mathcal{C}) \to CC_\bullet(A).
\]

4.6. Presheaves of spectra. The following is a summary of the definitions and constructions of Section 3 of \cite{M}. We refer the reader to \cite{M} (and references therein, particularly \cite{M1}) for further details.

To a presheaf \( \mathcal{S} \) of spectra on \( X \) (and, more generally, a presheaf with values in a category with filtered colimits, products and coproducts) one can associate the functorial cosimplicial Godement resolution in the usual way. The stalk \( \mathcal{S}_x \) of \( \mathcal{S} \) at \( x \in X \) is defined by
\[
\mathcal{S}_x = \text{colim} \mathcal{S}(U)
\]
where the colimit is taken over all open neighborhoods \( U \) of \( x \). For \( U \) an open subset of \( X \) let
\[
T(S)(U) = \prod_{x \in U} S_x.
\]
Then \( T(S) \) is a presheaf on \( X \), \( T \) is a functor from presheaves to presheaves and in fact a monad. Thus, \( T \) gives rise to the functorial (in \( S \)) cosimplicial object \( T^\bullet(S) \) in presheaves on \( X \) called the Godement resolution of \( S \).
For $S$ a presheaf of spectra on $X$ and an open subset $U$ of $X$ let
\[ \mathbb{H}(U; S) = \holim_{\Delta} T^\bullet(S)(U). \]
The assignment $U \mapsto \mathbb{H}(U; S)$ determines a presheaf of spectra on $X$. Note that there is a natural morphism of presheaves of spectra $S \rightarrow \mathbb{H}(\bullet; S)$.

For $Z \subset X$ a closed subset let $\Gamma_Z S$ denote the presheaf of spectra whose value on an open set $U \subset X$ is defined to be the homotopy fiber of the restriction map $S(U) \rightarrow S(U \setminus Z)$. In particular there is a canonical morphism $\Gamma_Z S \rightarrow S$. Set
\[ \mathbb{H}_Z(U; S) \overset{\text{def}}{=} \holim_{\Delta} \Gamma_Z T^\bullet(S)(U). \]

**4.7. The Euler class and the Chern character revisited.** Let $K_Z(A)$ (respectively $K^{\text{naive}}_Z(A)$) denote the $K$-theory spectrum of the category of perfect complexes (respectively strictly perfect complexes) of sheaves of $A$-modules on $X$ supported on $Z$ (i.e. acyclic on the complement of $Z$ in $X$). The inclusion of the category of strictly perfect complexes into the category of all perfect complexes induces the natural morphism of spectra $K^{\text{naive}}_Z(A) \rightarrow K_Z(A)$.

The assignment $U \mapsto K^{\text{naive}}(A(U))$ (respectively $U \mapsto K^{\text{naive}}(A|_U)$, $U \mapsto K(A|_U)$) determines a presheaf of spectra on $X$. The functor $M \mapsto M \otimes_{A(U)} A|_U$ and the inclusion of strictly perfect complexes into perfect complexes induce morphism of presheaves of spectra
\[ K^{\text{naive}}(A(\bullet)) \rightarrow K^{\text{naive}}(A|_\bullet) \rightarrow K(A|_\bullet). \] (4.1)

**Lemma 4.7.1.** The morphisms (4.1) induce equivalences on stalks.

**Corollary 4.7.2.** The morphism (4.1) induce equivalences
\[ \mathbb{H}_Z(X; K^{\text{naive}}(A(\bullet))) \rightarrow \mathbb{H}_Z(X; K^{\text{naive}}(A|_\bullet)) \rightarrow \mathbb{H}_Z(X; K(A|_\bullet)). \] (4.2)

Combing the equivalences (4.2) with the the Euler class (Dennis trace map) and the canonical morphism $K_Z(A) \rightarrow \mathbb{H}_Z(X; K(A|_\bullet))$ we obtain the morphism of spectra
\[ \text{Eu} : K_Z(A) \rightarrow \mathbb{H}_Z(X; C(\bullet)) = R\Gamma_Z(X; C(\bullet)). \]

Using the Chern character instead of the Euler class we obtain the morphism
\[ ch : K_Z(A) \rightarrow \mathbb{H}_Z(X; CC(\bullet)) = R\Gamma_Z(X; CC(\bullet)). \] (4.3)
All in all, we have constructed Euler class and Chern character for a perfect complex.

5. Characteristic classes on complex manifolds

5.1. Notations and conventions. In what follows we will be considering, for a complex manifold $X$, the sheaves of algebras $\mathcal{O}_X$, $\mathcal{D}_X$, $\mathcal{E}_X$, and $\mathbb{A}^h_{T^*X}$. All of these are sheaves of topological vector spaces. In what follows all tensor products are understood to be projective tensor products. In particular, let

$$
\mathcal{O}_X^e = \mathcal{O}_X \hat{\otimes} \mathcal{O}_X,
\mathcal{D}_X^e = \mathcal{D}_X \hat{\otimes} \mathcal{D}_X^{op},
\mathcal{E}_X^e = \mathcal{E}_X \hat{\otimes} \mathcal{E}_X,
(\mathbb{A}^h_M)^e = \mathbb{A}^h_M \hat{\otimes}_\mathbb{C}[[\hbar]](\mathbb{A}^h_M)^{op}.
$$

If $\mathcal{A}$ is one of $\mathcal{O}_X$, $\mathcal{D}_X$, $\mathcal{E}_X$, $\mathbb{A}^h_M$ then $\mathcal{A}$ has a natural structure of an $\mathcal{A}^e$-module and there is a natural map $\mathcal{A} \otimes \mathcal{A}^{op} \to \mathcal{A}^e$ which induces a map of complexes

$$
\mathcal{A} \otimes_{\mathcal{A} \hat{\otimes}_\mathbb{C} \mathcal{A}}^{L} \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}^e}^{L} \mathcal{A}.
$$

In what follows we will only consider the composition

$$
\mathbf{R} \text{Hom}^*_{\mathcal{A}}(\mathcal{F}^*, \mathcal{F}^*) \xrightarrow{\mathcal{L}_X} \mathcal{A} \otimes_{\mathcal{A} \hat{\otimes}_\mathbb{C} \mathcal{A}}^{L} \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{A}^e}^{L} \mathcal{A},
$$

refer to it as the Lefschetz map and denote it by $\mathcal{L}_X$. Similarly, the Euler map $\text{Eu}_{\mathcal{A}}(\mathcal{F}^*)$ will denote the composition

$$
k \xrightarrow{1 \mapsto \text{id}} \mathbf{R} \text{Hom}^*_{\mathcal{A}}(\mathcal{F}^*, \mathcal{F}^*) \xrightarrow{\mathcal{L}_X} \mathcal{A} \otimes_{\mathcal{A}^e}^{L} \mathcal{A}.
$$

5.2. Some examples of Hochschild and cyclic homology. Here we recall some facts about the Hochschild and cyclic homology of $\mathcal{O}_X$, $\mathcal{D}_X$, $\mathcal{E}_X$ and $\mathbb{A}^h_M$ for a complex manifold $X$ of dimension $\dim \mathbb{C} X = d$ from [13].

By the theorem of Hochschild-Kostant-Rosenberg there is an isomorphism in the derived category of $\mathcal{O}_X$-modules

$$
\mu_{\mathcal{O}_X} : \mathcal{O}_X \otimes_{\mathcal{O}_X}^{L} \mathcal{O}_X \to \bigoplus_p \Omega^p_X[p]
$$

given, in terms of the standard Hochschild complex representing $\mathcal{O}_X \otimes_{\mathcal{O}_X}^{L} \mathcal{O}_X$ $\mathcal{O}_X$, by the formula

$$
a_0 \otimes \cdots \otimes a_p \mapsto \frac{1}{p!} a_0 da_1 \wedge \cdots \wedge da_p.
$$
The same formula gives the quasiisomorphism

$$\tilde{\mu}_O : CC^\bullet_{\text{per}}(\mathcal{O}_X) \to \prod_{p \in \mathbb{Z}} \Omega^\bullet_X[2p]$$

(5.1)

of the periodic cyclic complex of $\mathcal{O}_X$ and the de Rham complex of $X$, since it is easily verified that the diagram

$$
\begin{array}{ccc}
C^\bullet(\mathcal{O}_X) & \xrightarrow{B} & C^\bullet(\mathcal{O}_X)[-1] \\
\mu & & \mu \\
\oplus_p \Omega^p_X & \xrightarrow{d} & \oplus_p \Omega^p_X[p - 1]
\end{array}
$$

(where $d$ denotes exterior differentiation) is commutative.

In view of the fact that the canonical map $\mathbb{C}_X \to \Omega^\bullet_X$ is a quasiisomorphism we will view the map $\tilde{\mu}_O$ as the isomorphism (in the derived category)

$$\tilde{\mu}_O : CC^\bullet_{\text{per}}(\mathcal{O}_X) \to \prod_{p \in \mathbb{Z}} \mathbb{C}_X[2p].$$

The inverse to $\tilde{\mu}_O$ is provided by the map of periodic cyclic complexes induced by the inclusion $\mathbb{C}_X \hookrightarrow \mathcal{O}_X$.

According to J.-L. Brylinski ([Bry]),

$$H^p(D_X \otimes_{D^*_X} D_X) \cong \begin{cases} 0 & \text{if } p \neq -2d \\ \mathbb{C}_X & \text{if } p = -2d \end{cases}$$

Consider a point $x \in X$, an open neighborhood $U$ of $x$ in $X$ and a local coordinate system $x_1, \ldots, x_d$ centered at $x$. The class of the Hochschild cycle $\text{Alt}(1 \otimes x_1 \otimes \cdots \otimes x_d \otimes \frac{\partial}{\partial x_1} \otimes \cdots \otimes \frac{\partial}{\partial x_d})$ represents a global section $\Phi^D_U$ of $H^{-2d}(D_U \otimes_{D^*_U} D_U)$. Let

$$\mu_D : D_X \otimes_{D^*_X} D_X \to \mathbb{C}_X[2d]$$

denote the isomorphism in the derived category of sheaves which corresponds to the global section of $H^{-2d}(D_X \otimes_{D^*_X} D_X)$ determined by the condition that it restricts to $\Phi^D_U$ for every sufficiently small open set $U \subset X$. Let

$$\mu^h_D : D_X \otimes_{D^*_X} D_X \to \mathbb{C}_X[\hbar^{-1}, \hbar][2d]$$

denote the composition $\frac{1}{\hbar^d} \mu_D$.

Similar result hold for the sheaf $\mathcal{E}_X$ of microdifferential operators. Specifically,

$$H^p(\mathcal{E}_X \otimes_{\mathcal{E}^*_X} \mathcal{E}_X) \cong \begin{cases} 0 & \text{if } p \neq -2d \\ \mathbb{C}_{T^*X} & \text{if } p = -2d \end{cases}$$
Let $\xi_i$ denote the symbol of $\frac{\partial}{\partial x_i}$ where $x_1, \ldots, x_d$ are local coordinates on $X$ as before. The class of the Hochschild cycle $\text{Alt}(1 \otimes x_1 \otimes \cdots \otimes x_d \otimes \xi_i \otimes \cdots \otimes \xi_d)$ determines a global section $\Phi^E_U$ of $H^{-2d}(E_U \otimes_{E_U} E_U)$. Let

$$\mu_E : E_X \otimes_{E_X}^{L} E_X \to \mathbb{C}_{T^*X}[2d]$$

denote the isomorphism in the derived category of sheaves which corresponds to the global section of $H^{-2d}(E_X \otimes_{E_X}^{L} E_X)$ determined by the condition that it restricts to $\Phi^E_U$ for every sufficiently small open set $U \subset X$. Let

$$\mu^h_E : E_X \otimes_{E_X}^{L} E_X \to \mathbb{C}_{T^*X}[h^{-1}, h][2d]$$

denote the composition $\frac{1}{h^d} \mu_E$.

Suppose that $A^h_M$ is a symplectic deformation quantization of a complex manifold $M$ of dimension $\text{dim}_\mathbb{C}M = 2d$. The sheaf of algebras $A^h_M[h^{-1}]$ exhibits properties similar to those of $E_X$. We will show that

$$H^p(A^h_M \otimes_{(A^h_M)^e}^{L} A^h_M)[h^{-1}] \cong \begin{cases} 0 & \text{if } p \neq -2d \\ \mathbb{C}_M[h^{-1}, h] & \text{if } p = -2d \end{cases}$$

Consider a “local trivialization” of $A^h_M$ as in (3.1). It induces an isomorphism

$$H^{-2d}(A^h_{M'}, \otimes_{(A^h_{M'})^e}^{L} A^h_{M'})[h^{-1}] \xrightarrow{\text{iso}} H^{-2d}(A^h_{M'} \otimes_{(A^h_{M'})^e}^{L} A^h_{M'})[h^{-1}] .$$

(5.2)

The expression $\text{Alt}(1 \otimes x_1 \otimes \cdots \otimes x_d \otimes \frac{\xi_1}{h} \otimes \cdots \otimes \frac{\xi_d}{h})$ represents a (non-trivial) global section of $H^{2d}(A^h_{M'} \otimes_{(A^h_{M'})^e}^{L} A^h_{M'})[h^{-1}]$ whose image $\Phi^h_U$ under the isomorphism (5.2) is independent of the local trivialization (3.1).

Let

$$\mu^h_A : A^h_M \otimes_{(A^h_M)^e}^{L} A^h_M \to \mathbb{C}_M[h^{-1}, h][2d]$$

denote the isomorphism in the derived category of sheaves which corresponds to the global section of $H^{2d}(A^h_M \otimes_{(A^h_M)^e}^{L} A^h_M)[h^{-1}]$ determined by the condition that it restricts to $\Phi^h_U$ for every sufficiently small open set $U \subset M$ as above.
we now turn to the deformation quantization $A^h_{T^*X}$ as in [3.2]. The compatibility of all of the maps defined above is expressed by the commutativity of the following diagram:

\[
\begin{array}{cccc}
\pi^{-1}D_X & \longrightarrow & \mathcal{E}_X & \longrightarrow & A^h_{T^*X}[-1] \\
\downarrow & & \downarrow & & \downarrow \\
\pi^{-1}\left(D_X \otimes_{\mathcal{O}_X} D_X\right) & \longrightarrow & \mathcal{E}_X \otimes_{\mathcal{E}_X}^L \mathcal{E}_X & \longrightarrow & A^h_{\mathcal{U}} \otimes_{(A^h_{\mathcal{U}})^\wedge} A^h_{\mathcal{U}}[-1] \\
\mu^h_D \downarrow & & \mu^h \downarrow & & \mu^h \downarrow \\
\pi^{-1}C_X[h^{-1}, h][2d] & \longrightarrow & C_{T^*X}[h^{-1}, h][2d] & \longrightarrow & C_{T^*X}[h^{-1}, h][2d]
\end{array}
\]

Although we will restrict ourselves to the discussion of the cyclic homology of $A^h_M[h^{-1}]$, analogs of the statements below hold for the algebras $D_X$ and $\mathcal{E}_X$.

Since there are no nontrivial morphisms $C_\infty M[h^{-1}, h] \rightarrow C_\infty M[h^{-1}, h][-1]$ in the derived category it follows that the map $B' : C_\bullet(A^h_M)[h^{-1}] \rightarrow C_\bullet(A^h_M)[h^{-1}][-1]$ represents the trivial morphism (in the derived category) and, consequently, there are isomorphisms (in the derived category)

\[
CC^\ast_\bullet(A^h_M)[h^{-1}] \cong \prod_{p=0}^{\infty} C_\ast(A^h_M)[h^{-1}][-2p] \cong \prod_{p=0}^{\infty} C_\ast M[h^{-1}, h][2d - 2p]
\]

and

\[
CC^\text{per}_\bullet(A^h_M)[h^{-1}] \cong \prod_{p=-\infty}^{\infty} C_\ast(A^h_M)[h^{-1}][-2p] \cong \prod_{p=-\infty}^{\infty} C_\ast M[h^{-1}, h][2d - 2p]
\]

which are induced by $\mu^h_\wedge$ on each factor. We will denote the latter composition by $\tilde{\mu}^h_\wedge$.

It is not difficult to show that the inverse to $\tilde{\mu}^h_\wedge$ is provided by the map of periodic cyclic complexes induced by the inclusion $C_M[[h]] \hookrightarrow A^h_M$.

5.3. Euler classes of $D_X$-, $\mathcal{E}_X$- and $A^h_{T^*X}$-modules. Consider a perfect complex $\mathcal{M}^\bullet$ of $D_X$-modules and a closed subvariety $\Lambda$ of $T^*X$ containing char($\mathcal{M}^\bullet$). It is well known that char($\mathcal{M}^\bullet$) = Supp($\pi^{-1}\mathcal{M}^\bullet \otimes_{\pi^{-1}D_X} \mathcal{E}_X$) and that the microlocal Euler class $\mu \text{eu}(\mathcal{M}^\bullet) \in H^2_\wedge(T^*X; \mathbb{C})$ depends only on the microlocalization $\pi^{-1}\mathcal{M}^\bullet \otimes_{\pi^{-1}D_X} \mathcal{E}_X$. In fact, it is not difficult to establish the equality

\[
\mu \text{eu}(\mathcal{M}^\bullet) = \mu_\mathcal{E} \left( \text{eu}_\mathcal{E}^\wedge(\pi^{-1}\mathcal{M}^\bullet \otimes_{\pi^{-1}D_X} \mathcal{E}_X) \right)
\]
Consider a perfect complex $\mathcal{N}^\bullet$ of $\mathcal{E}_X$-modules and a closed subset $\Lambda$ of $T^*X$ containing $\text{Supp}(\mathcal{N}^\bullet)$. The commutativity of the diagram

\[
\begin{array}{c}
\text{R} \text{Hom}_{\mathcal{E}_X}(\mathcal{N}^\bullet, \mathcal{N}^\bullet) \longrightarrow \text{R} \text{Hom}_{\mathcal{A}_M(h^{-1})}(\mathcal{N}^\bullet \otimes_{\mathcal{E}_X} \mathcal{A}^h_{T^*X}[h^{-1}], \mathcal{N}^\bullet \otimes_{\mathcal{E}_X} \mathcal{A}^h_{T^*X}[h^{-1}]) \\
\downarrow \quad \downarrow \\
\mathcal{E} \otimes_{\mathcal{E}_X} \mathcal{E}_X \longrightarrow \mathcal{A}^h_{T^*X} \otimes_{\mathcal{A}_M(h^{-1})} \mathcal{A}^h_{T^*X}[h^{-1}]
\end{array}
\]

implies the identity $\mu^h_\mathcal{E} \circ \text{Eu}_\mathcal{E}(\mathcal{N}^\bullet) = \mu^h_\mathcal{E} \circ \text{Eu}_{\mathcal{A}_M(h^{-1})}(\mathcal{N}^\bullet \otimes_{\mathcal{E}_X} \mathcal{A}^h_{T^*X}[h^{-1}])$, and, consequently, the equality

\[
\mu^h_\mathcal{E}(\text{eu}_\mathcal{E}(\mathcal{N}^\bullet)) = \mu^h_\mathcal{E}(\text{eu}_\mathcal{E}(\mathcal{N}^\bullet \otimes_{\mathcal{E}_X} \mathcal{A}^h_{T^*X}[h^{-1}])),
\]

Thus, calculation of microlocal Euler classes reduces to calculation of Euler classes for $\mathcal{A}_M^h[h^{-1}]$-modules.

### 6. Riemann-Roch Type Theorems

#### 6.1. The Riemann-Roch Theorem for Periodic Cyclic Cocycles

The following theorem constitutes the central result of this note.

**Theorem 6.1.1.** The diagram (in the derived category of Abelian sheaves on $M$)

\[
\begin{array}{ccc}
CC^\text{per}_\bullet(\mathcal{A}_M^h) & \xrightarrow{\sigma} & CC^\text{per}_\bullet(\mathcal{O}_M) \\
\downarrow \quad \downarrow & & \downarrow \quad \downarrow \\
CC^\text{per}_\bullet(\mathcal{A}_M^h)[h^{-1}] & \xrightarrow{\bar{\mu}^h} & \prod_{p=-\infty}^\infty \mathbb{C}[h^{-1}, h][{-2p}]
\end{array}
\]

is commutative.

The proof of Theorem 6.1.1 is postponed until the later sections. In Section 3 we introduce the methods of Gel’fand-Fuchs cohomology and reduce (see Corollary 3.2.2) Theorem 6.1.1 to an analogous statement (Theorem 6.3.1) in the case when $M$ is a formal neighborhood of the origin in a symplectic vector space over $\mathbb{C}$ which is formulated in Section 7. The rest of this section is devoted to corollaries of Theorem 6.1.1.

#### 6.2. Riemann-Roch for $\mathcal{A}_M^h$-modules

The commutativity of the diagram

\[
\begin{array}{ccc}
K^0_{\Lambda}(\mathcal{A}_M^h[h^{-1}]) & \xleftarrow{\varepsilon} & K^0_{\Lambda}(\mathcal{A}_M^h) & \xrightarrow{\sigma} & K^0_{\Lambda}(\mathcal{O}_M) \\
\downarrow \quad \downarrow & & \downarrow & & \downarrow \quad \downarrow \\
H^0_{\Lambda}(M; CC^\text{per}_\bullet(\mathcal{A}_M^h)[h^{-1}]) & \xleftarrow{\varepsilon} & H^0_{\Lambda}(M; CC^\text{per}_\bullet(\mathcal{A}_M^h)) & \xrightarrow{\sigma} & H^0_{\Lambda}(M; CC^\text{per}_\bullet(\mathcal{O}_M))
\end{array}
\]

yields the following.
Corollary 6.2.1. Suppose that $M$ is a complex manifold, $\mathbb{A}^h_M$ is a symplectic deformation quantization of $M$, $\mathcal{M}^\bullet$ is a perfect complex of $\mathbb{A}^h_M$-modules and $\Lambda$ is a closed subvariety of $M$ containing $\text{Supp}(\mathcal{M}^\bullet)$. Then

$$\mu^h(\text{ch}_{\mathbb{A}^h_M[h^{-1}]}(\mathcal{M}^\bullet[h^{-1}])) = \hat{\mu}_\mathcal{O}(\text{ch}_{\mathcal{O}}(\sigma(\mathcal{M}^\bullet))) \sim \hat{A}(TM) \sim e^\theta$$

in $H^\bullet(M; \mathbb{C}[h^{-1}, h])$, where $\theta$ is the characteristic class of the deformation quantization $\mathbb{A}^h_M$.

Note that the class $\hat{A}(E)$ is defined for any symplectic vector bundle $E$, for example by choosing a reduction of the (symplectic) structure group of $E$ to the unitary group.

Recall that, for an element $\alpha$ of a graded object, $[\alpha]^p$ denotes the homogeneous component of $\alpha$ of degree $p$.

Corollary 6.2.2. Under the assumptions of Corollary 6.2.1

$$\mu^h\left(\text{eu}_{\mathbb{A}^h_M[h^{-1}]}(\mathcal{M}^\bullet[h^{-1}])\right) = \left[\hat{\mu}_\mathcal{O}(\text{ch}_{\mathcal{O}}(\sigma(\mathcal{M}^\bullet))) \sim \hat{A}(TM) \sim e^\theta\right]^\text{dim}_C M$$

in $H^\bullet(M; \mathbb{C}[h^{-1}, h])$.

6.3. Riemann-Roch for $\mathcal{D}$- and $\mathcal{E}$-modules. If $M = T^*X$ for a complex manifold $X$, and $\mathbb{A}^h_{T^*X}$ is the deformation quantization with the characteristic class $\theta = \frac{1}{2}\pi^*c_1(X)$, then

$$\hat{A}(TM) \sim e^\theta = \pi^*Td(TX)$$

and the right hand side of (6.2) is, clearly, independent of $\hbar$; if $\mathcal{M}^\bullet[h^{-1}]$ is obtained by an extension of scalars from a complex of $\mathcal{E}_X$-modules, then, clearly, so is the left hand side.

Thus, we obtain Conjecture 2.2.2 of P.Schapira and J.-P.Schneiders.

Corollary 6.3.1. Suppose that $X$ is a complex manifold, $(\mathcal{M}^\bullet, F_\bullet)$ is a perfect complex of $\mathcal{D}_X$-modules with a good filtration and $\Lambda$ is a closed subvariety of $T^*X$ containing $\text{char} \mathcal{M}^\bullet$. Then

$$\mu_{\mathcal{E}_X}(\mathcal{M}^\bullet) = \left[\text{ch}_{\mathcal{O}_{T^*X}}(\sigma(\mathcal{M}^\bullet)) \sim \pi^*Td(TX)\right]^\text{dim}_X$$

in $H^2_{\text{dim}_C X}(T^*X; \mathbb{C})$.

7. The Riemann-Roch formula in the formal setting

The Weyl algebra of a symplectic vector space $(V, \omega)$ over $\mathbb{C}$ may be considered as a symplectic deformation quantization of the completion of $V$ at the origin. In this section we introduce the notations and the facts necessary to state the analogue of Theorem 6.1.1 in this setting.
In what follows $(V, \omega)$ is viewed as a symplectic manifold.

7.1. **The Weyl algebra.** Here we briefly recall the definition and the basic properties of the Weyl algebra $W = W(V)$ of a (finite dimensional) symplectic vector space $(V, \omega)$ over $\mathbb{C}$. Let $V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C})$.

Let $I = I(V)$ denote the kernel of the map (of $\mathbb{C}$-algebras)

$$\text{Sym}^\bullet(V^*) \otimes \mathbb{C}[h] \to \mathbb{C}.$$ 

Let

$$\widehat{\text{Sym}}^\bullet(V^*)[[h]] = \lim_{\leftarrow} \text{Sym}^\bullet(V^*) \otimes \mathbb{C}[h] / I^n$$

and let $\widehat{I} = \widehat{I}(V)$ denote the kernel of the map $\widehat{\text{Sym}}^\bullet(V^*)[[h]] \to \mathbb{C}$.

The Moyal-Weyl product on $\widehat{\text{Sym}}^\bullet(V^*)[[h]]$ is defined by the formula

$$f \ast g = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\sqrt{-1}h}{2} \right)^n \omega(d^n f, d^n g), \quad (7.1)$$

where

$$d^n : \widehat{\text{Sym}}^\bullet(V^*) \to \widehat{\text{Sym}}^\bullet(V^*) \otimes \text{Sym}^n(V)$$

assigns to a jet of a function the symmetric tensor composed of its $n$-th order partial derivatives, and $\omega$ is extended naturally to a bilinear form on $\text{Sym}^n(V)$.

The Moyal-Weyl product endows $\widehat{\text{Sym}}^\bullet(V^*)[[h]]$ with a structure of an associative algebra with unit over $\mathbb{C}[[h]]$ which contains $\widehat{I}$ as a twosided ideal. Moreover, the Moyal-Weyl product is continuous in the $\widehat{I}$-adic topology. Let $W = W(V)$ denote the topological algebra over $\mathbb{C}[[h]]$ whose underlying $\mathbb{C}[[h]]$-module is $\widehat{\text{Sym}}^\bullet(V^*)[[h]]$ and the multiplication is given by the Moyal-Weyl product.

Let $F_pW = \widehat{I}^{-p}$. Then $(W, F_\bullet)$ is a filtered ring. Note also that the center of $W$ is equal to $\mathbb{C}[[h]]$ and $[W, W] = h \cdot W$.

Clearly, the association $(V, \omega) \mapsto W(V)$ is functorial. In particular, the group $\text{Sp}(V)$ acts naturally on $W(V)$ by continuous algebra automorphisms.

7.2. **Derivations of the Weyl algebra.** Let $\mathfrak{g} = \mathfrak{g}(\mathfrak{W})$ denote the Lie algebra of continuous, $\mathbb{C}[[h]]$-linear derivations of $W$. Then there is a central extension of Lie algebras

$$0 \to \frac{1}{\sqrt{-1}h} \mathbb{C}[[h]] \to \frac{1}{\sqrt{-1}h} W \to \mathfrak{g} \to 0.$$
where the Lie algebra structure on $\sqrt{-\hbar}W$ is given by the commutator (note that $\left[\frac{1}{\sqrt{-\hbar}}W, \frac{1}{\sqrt{-\hbar}}W\right] \subseteq \frac{1}{\sqrt{-\hbar}}W$) and the second map is defined by $\frac{1}{\sqrt{-\hbar}}f \mapsto \frac{1}{\sqrt{-\hbar}}[f, \cdot]$.

Let

$$F_p g = \{ \mathcal{D} \in g \mid \mathcal{D}(\mathcal{F}_{i}W) \subseteq \mathcal{F}_{i+p}W \text{ for all } i \}$$

Then $(g, \mathfrak{f}_\bullet)$ is a filtered Lie algebra and the action of $g$ on $W$ respects the filtrations, i.e. $[F_p g, \mathfrak{f}_q g] \subseteq F_{p+q} g$ and $F_p g \mathfrak{f}_q W \subseteq \mathfrak{f}_{p+q} W$.

The following properties of the filtered Lie algebra $(g, \mathfrak{f}_\bullet)$ are easily verified:

1. $Gr^F_p g = 0$ for $p > 1$ (in particular $g = \mathfrak{f}_1 g$), hence $Gr^F_1 g$ is Abelian;
2. the composition $\frac{1}{\sqrt{-\hbar}}V \hookrightarrow \frac{1}{\sqrt{-\hbar}}W \to g \to Gr^F_1 g$ is an isomorphism;
3. the composition $\mathfrak{sp}(\mathcal{V}) \to \text{Sym}^2(\mathcal{V}^*) \to \frac{1}{\sqrt{-\hbar}}\hat{I} \to \mathfrak{f}_0 g \to Gr^F_0 g$ is an isomorphism;
4. under the above isomorphisms the action of $Gr^F_0 g$ on $Gr^F_1 g$ is identified with the natural action of $\mathfrak{sp}(\mathcal{V})$ on $V$ (in particular there is an isomorphism $g/\mathfrak{f}_{-1} g \cong \mathcal{V} \ltimes \mathfrak{sp}(\mathcal{V})$);
5. the Lie algebra $F_{-1} g$ is pro-nilpotent.

In what follows $\mathfrak{h}$ will denote the Lie subalgebra of $g$ which is the image of the embedding $\mathfrak{sp}(\mathcal{V}) \hookrightarrow g$.

### 7.3. The Weyl algebra as a deformation quantization.

The (commutative) algebra $W/\hbar \cdot W$ is naturally isomorphic to the completion $\hat{\mathcal{O}} = \hat{\mathcal{O}}_{V}$ of the ring $\mathcal{O}_V$ of regular functions on $V$ at the origin, i.e. with respect to the powers of the maximal ideal $m$ of functions which vanish at $0 \in V$. The natural surjective map

$$\sigma : W(V) \to \hat{\mathcal{O}}_V$$

is strictly compatible with the $\hat{I}$-adic filtration on $W$ and the $m$-adic filtration on $\hat{\mathcal{O}}_V$.

The Lie algebra $g$ acts by derivations on $\hat{\mathcal{O}}$ by the formula

$$D(f) = \sigma(D(\tilde{f})),$$

where $D \in g$ and $\tilde{f} \in W$ is such that $\sigma(\tilde{f}) = f$. Thus, $\sigma$ is a map of $g$-modules.

### 7.4. The Hochschild homology of the Weyl algebra.

We recall the calculation of the Hochschild homology of the Weyl algebra ([FT1], [Bry]).
The Hochschild homology of $W$ may be computed using the Koszul resolution of $W$ as a $W$ module.

The Koszul complex $(K^\bullet, \partial)$ is defined by

$$K^{-q} = W \otimes \bigwedge^q V^* \otimes W$$

with the differential acting by

$$\partial(f \otimes v_1 \wedge \ldots \wedge v_q \otimes g) = \sum_i (-1)^i f v_i \otimes v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_q$$

$$+ \sum_i (-1)^i f \otimes v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_q \otimes v_i g.$$

Here we consider $V^*$ embedded in $W$ and $\bigwedge^q V^*$ embedded in $(V^*)^\otimes q$.

The map $K^\bullet \rightarrow W$ of complexes of $W_e$-modules with the only non-trivial component (in degree zero) given by multiplication is easily seen to be a quasiisomorphism. The map

$$K^\bullet(W) \overset{def}{=} K^\bullet \otimes_{W^*} W \rightarrow C^\bullet(W)$$

defined by

$$f \otimes v_1 \wedge \ldots \wedge v_q \otimes g \otimes h \mapsto fhg \otimes \text{Alt}(v_1 \otimes \cdots \otimes v_q)$$

is easily seen to be a quasiisomorphism. Hence it induces a quasiisomorphism

$$K^\bullet(W) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{-1}, \hbar] \rightarrow C^\bullet(W) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{-1}, \hbar].$$

Below we will use $(\bullet)[\hbar^{-1}]$ to denote $(\bullet) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{-1}, \hbar]$.

Under the natural isomorphism

$$K^{-q}(W) \rightarrow W \otimes \bigwedge^q V^*$$

defined by

$$f \otimes v_1 \wedge \ldots \wedge v_q \otimes g \otimes h \mapsto fhg \otimes v_1 \wedge \ldots \wedge v_q$$

the induced differential acts on the latter by

$$\partial(f \otimes v_1 \wedge \ldots \wedge v_q) = \sum_i [f, v_i] \otimes v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_q.$$

Suppose that $\dim V = 2d$. Let $\widehat{\Omega}^\bullet = \widehat{\Omega}^\bullet_V$ denote the de Rham complex of $V$ with formal coefficients (i.e. $\widehat{\Omega}^q_V = \Omega^q_V \otimes_{\mathcal{O}_V} \widehat{\mathcal{O}}_V$). The map

$$W \otimes \bigwedge^q V^* \rightarrow \widehat{\Omega}^{2d-q}[[\hbar]]$$

given by

$$f \otimes v_1 \wedge \ldots \wedge v_q \mapsto f \cdot \iota_{v_1} \cdots \iota_{v_q} (\omega^d)$$
(where \( f \in \hat{O} \)) is easily seen to determine an isomorphism of complexes

\[
(K^\bullet(W), \partial) \to (\hat{\Omega}^\bullet[[h]], h \cdot d)[2d].
\]

The formal Poincaré Lemma implies that

\[
H^q(K^\bullet \otimes_W W[h^{-1}]) \cong HH^q(W; W)[h^{-1}] \cong \begin{cases} 0 & \text{if } q \neq 2d \\ \mathbb{C}[h^{-1}, h] & \text{if } q = 2d \end{cases}
\]

The canonical generator of \( HH_{2d}(W; W)[h^{-1}] \) is represented by the cycle \( 1 \otimes \frac{1}{h!} \omega^d \in W \otimes \Lambda^{2d} V^* \). If \( x_1, \ldots, x_d, \xi_1, \ldots, \xi_d \) is (dual to) a symplectic basis of \( V \) (so that \( \omega = \sum_i x_i \wedge \xi_i \)), then the canonical generator is represented by the cycle \( \text{Alt}(1 \otimes x_1 \otimes \cdots \otimes x_d \otimes \xi_1 \otimes \cdots \otimes \xi_d) \in C_{2d}(W) \). We will denote this cycle (and its class) by \( \Phi = \Phi_V \). Note also that \( \Phi \) corresponds under the above (quasi)isomorphisms to the cocycle \( 1 \in \hat{\Omega}^0 \).

Observe that the Lie algebra \( g \) acts on all of the complexes introduced above (by Lie derivative) and all maps defined above are, in fact, \( g \)-equivariant. The cycle \( \Phi \) is not invariant under the action of \( g \). It is, however, invariant under the action of the subalgebra \( h \).

### 7.5. Characteristic classes in Lie algebra cohomology

We will presently construct the classes in relative Lie algebra cohomology of the pair \((g, h)\) which enter the Riemann-Roch formula in the present setting.

#### 7.5.1. The trace density

Since \( h \) acts semi-simply on \( C^\bullet(W)[h^{-1}] \) and \( \hat{\Omega}^\bullet[h^{-1}, h] \), the quasiisomorphism

\[
\hat{\Omega}^\bullet[h^{-1}, h]][2d] \to C^\bullet(W)[h^{-1}]
\]

constructed above admits an \( h \)-equivariant splitting

\[
\mu^h_{(0)} : C^\bullet(W)[h^{-1}] \to \hat{\Omega}^\bullet[h^{-1}, h]][2d]
\]

which is a quasiisomorphism.

We will consider the map \( \mu^h_{(0)} \) as a relative Lie algebra cochain

\[
\mu^h_{(0)} \in C^0(g, h; \text{Hom}^0(C^\bullet(\mathfrak{g})[h^{-1}], \hat{\Omega}^\bullet[h^{-1}, h]][2d])).
\]

**Lemma 7.5.1.** \( \mu^h_{(0)} \) extends to a cocycle \( \mu^h = \sum_p \mu^h_{(p)} \) with

\[
\mu^h_{(p)} \in C^p(g, h; \text{Hom}^{-p}(C^\bullet(\mathfrak{g})[h^{-1}], \hat{\Omega}^\bullet[h^{-1}, h]][2d])).
\]

Moreover, any two such extensions are cohomologous.
Since the complex $C^\bullet(\mathfrak{g}, \mathfrak{h}; \text{Hom}^\bullet_{\mathfrak{c}[u]}(\mathfrak{c}(2\mathfrak{W})[h^{-1}], \widehat{\Omega}^\bullet[h^{-1}, h]][2\mathfrak{d}])$ represents (in the derived category) the object $R\text{Hom}^\bullet_{\mathfrak{g}, \mathfrak{h}}(C^\bullet(\mathfrak{W})[h^{-1}], \widehat{\Omega}^\bullet[h^{-1}, h]][2d])$, $\mu^h$ represents a well defined isomorphism

$$\mu^h : C^\bullet(\mathfrak{W})[h^{-1}] \to \widehat{\Omega}^\bullet[h^{-1}, h]][2d]$$

in the derived category of $(\mathfrak{g}, \mathfrak{h})$-modules. The image of $\mu^h$ under the functor of forgetting the module structure is $\mu^h_{(0)}$.

Cup product with $\mu^h$ induces the quasiisomorphism of complexes

$$\mu^h : C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathfrak{c}(\mathfrak{W})[h^{-1}]) \to \mathfrak{c}^\bullet(\mathfrak{g}, \mathfrak{h}; \widehat{\Omega}^\bullet[h^{-1}, h]][2d])$$

unique up to homotopy.

**Lemma 7.5.2.** $\mu^h_{(0)}$ extends to an $\mathfrak{h}$-equivariant quasiisomorphism of complexes

$$\tilde{\mu}^h_{(0)} : C_{\text{per}}^\bullet(W)[h^{-1}] \to \prod_{p \in \mathbb{Z}} \widehat{\Omega}^\bullet[h^{-1}, h][2p] .$$

We will consider the map $\tilde{\mu}^h_{(0)}$ as a relative Lie algebra cochain

$$\tilde{\mu}^h_{(0)} \in C^0(\mathfrak{g}, \mathfrak{h}; \text{Hom}^0(\mathfrak{c}C_{\text{per}}^\bullet(\mathfrak{W})[h^{-1}], \prod_{p \in \mathbb{Z}} \widehat{\Omega}^\bullet[h^{-1}, h][2p])) .$$

**Lemma 7.5.3.** $\tilde{\mu}^h_{(0)}$ extends to a cocycle $\tilde{\mu}^h = \sum_p \tilde{\mu}^h_{(p)}$ with

$$\mu^h_{(p)} \in C^p(\mathfrak{g}, \mathfrak{h}; \text{Hom}^{-p}(\mathfrak{c}C_{\text{per}}^\bullet(\mathfrak{W})[h^{-1}], \prod_{q \in \mathbb{Z}} \widehat{\Omega}^\bullet[h^{-1}, h][2q])) .$$

Moreover, any two such extensions are cohomologous.

Since the complex $C^\bullet(\mathfrak{g}, \mathfrak{h}; \text{Hom}^\bullet(\mathfrak{c}C_{\text{per}}^\bullet(\mathfrak{W})[h^{-1}], \prod_{p \in \mathbb{Z}} \widehat{\Omega}^\bullet[h^{-1}, h]][2p]))$ represents the object $R\text{Hom}^\bullet_{\mathfrak{g}, \mathfrak{h}}(C_{\text{per}}^\bullet(W)[h^{-1}], \prod_{p \in \mathbb{Z}} \widehat{\Omega}^\bullet[h^{-1}, h]][2d])$, $\tilde{\mu}^h$ represents a well defined isomorphism

$$\tilde{\mu}^h : C_{\text{per}}^\bullet(W)[h^{-1}] \to \prod_{p \in \mathbb{Z}} \widehat{\Omega}^\bullet[h^{-1}, h][2d]$$

in the derived category of $(\mathfrak{g}, \mathfrak{h})$-modules. The image of $\tilde{\mu}^h$ under the functor of forgetting the module structure is $\tilde{\mu}^h_{(0)}$.

Cup product with $\tilde{\mu}^h$ induces the quasiisomorphism of complexes

$$\tilde{\mu}^h : C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathfrak{c}C_{\text{per}}^\bullet(\mathfrak{W})[h^{-1}]) \to \mathfrak{c}^\bullet(\mathfrak{g}, \mathfrak{h}; \prod_{p \in \mathbb{Z}} \widehat{\Omega}^\bullet[h^{-1}, h][2p])$$

unique up to homotopy.

The natural inclusion

$$\iota : C_{\text{per}}^\bullet(W) \to C_{\text{per}}^\bullet(W)[h^{-1}]$$

is a morphism of complexes of \((g, h)\)-modules, therefore determines a cocycle 
\[
\iota \in C^0(g, h; \text{Hom}^\circ(\mathcal{C}\mathcal{C}_\bullet^{\text{per}}(\mathcal{W}), \mathcal{C}\mathcal{C}_\bullet^{\text{per}}(\mathcal{W})[h^{-1}])),
\]

The cup product of \(\iota\) and \(\tilde{\mu}\) is a cocycle 
\[
\tilde{\mu}^h \cdot \iota \in C^\bullet(g, h; \text{Hom}^\bullet(\mathcal{C}\mathcal{C}_\bullet^{\text{per}}(\mathcal{W}), \prod_{p \in \mathbb{Z}} \hat{\Omega}^\bullet[h^{-1}, h][2p])
\]
of (total) degree zero which prepresents the morphism 
\[
\tilde{\mu}^h \circ \iota : CC^{\text{per}}_\bullet(W) \to \prod_{p \in \mathbb{Z}} \hat{\Omega}^\bullet[h^{-1}, h][2p]
\]
in the derived category of \((g, h)\)-modules.

7.5.2. The symbol and the Hochschild-Kostant-Rosenberg map. The maps 
\[
\sigma : CC^{\text{per}}_\bullet(W) \to CC^{\text{per}}_\bullet(\hat{O})
\]
(defined by \(f_0 \otimes \cdots \otimes f_p \mapsto \frac{1}{p!} f_0 d f_1 \wedge \cdots \wedge d f_p\)) are morphisms of complexes of \((g, h)\)-modules, therefore determine cocycles 
\[
\sigma \in C^\bullet(g, h; \text{Hom}^\bullet(\mathcal{C}\mathcal{C}_\bullet^{\text{per}}(\mathcal{W}), \mathcal{C}\mathcal{C}_\bullet^{\text{per}}(\hat{O})))
\]
and
\[
\tilde{\mu} \in C^\bullet(g, h; \text{Hom}^\bullet(\mathcal{C}\mathcal{C}_\bullet^{\text{per}}(\hat{O}), \prod_{p \in \mathbb{Z}} \hat{\Omega}^\bullet[h^{-1}, h][2p]))
\]

7.5.3. The characteristic class of the deformation. The central extension of Lie algebras 
\[
0 \to \frac{1}{\sqrt{-1}h} \mathbb{C}[[h]] \to \frac{1}{\sqrt{-1}h} W \to g \to 0
\]
restricts to a trivial extension of \(h\), therefore is classified by a class \(\theta \in H^2(g, h; \frac{1}{\sqrt{-1}h} \mathbb{C}[[h]])\) represented by the cocycle 
\[
\theta : X \wedge Y \mapsto \widetilde{[X, Y]} - \widetilde{[X, Y]}
\]
where \(\widetilde{\ )}\) is a choice of a \(\mathbb{C}[[h]]\)-linear splitting of the extension.
7.5.4. The $\hat{A}$-class. Let $\nabla : g \to h$ denote an $h$-equivariant splitting of the inclusion $h \hookrightarrow g$ and let

$$R(X, Y) = [\nabla(X), \nabla(Y)] - \nabla([X, Y])$$

for $X, Y \in g$. Then, $R \in C^2(g; h; h)$ is a cocycle (where $h$ is considered as a trivial $(g; h)$-module). The $k$-fold cup product of $R$ with itself is a cocycle $R^{\wedge k} \in C^{2k}(g; h; h^{\otimes k})$.

The splitting $\nabla$ determines the Chern-Weil map

$$CW : \widehat{\text{Sym}}^*(h) \to C^2(g; h; C)$$

de fined as the composite

$$\text{Sym}^k(h) \to C^0(h; \text{Sym}^k(h)) \to C^0(h; \text{Sym}^k(h)) \otimes C^{2k}(g; h; h^{\otimes k}) \to C^{2k}(g; h; C)$$

where the first map is the natural inclusion (of the cocycles), the second map is $P \mapsto P \otimes R^{\wedge k}$, and the last map is induced by the natural pairing on the coefficients.

Let $\hat{A}$ denote the image under the Chern-Weil map of

$$h \ni X \mapsto \det \left( \frac{\exp(ad(X)) - \exp(-ad(X))}{\exp(ad(X))} \right).$$

7.6. Riemann-Roch formula in Lie algebra cohomology. The following theorem is the analog of the Theorem 6.1.1 in the setting of this section.

**Theorem 7.6.1.** The cocycle $\tilde{\mu} \circ \iota - \tilde{\mu} \circ \hat{A} \circ e^\theta \circ \sigma$ is cohomologous to zero in $C^*(g; h; \text{Hom}^*(\hat{\Omega}^*_V \omega; \prod_{p \in Z} \hat{\Omega}^*[h^{-1}, h][2p]))$.

**Corollary 7.6.2.** The diagram

$$\begin{array}{ccc}
CC^*_{\text{per}}(W) & \xrightarrow{\sigma} & CC^*_{\text{per}}(\hat{O}) \\
\downarrow \iota & & \downarrow \tilde{\mu} \circ \hat{A} \circ e^\theta \\
CC^*_{\text{per}}(W)[h^{-1}] & \xrightarrow{\tilde{\mu}^h} & \prod_{p \in Z} \hat{\Omega}^*_V \omega[h^{-1}, h][2p]
\end{array}$$

in the derived category of $(g, h)$-modules is commutative.

**Corollary 7.6.3.** The diagram

$$\begin{array}{ccc}
C^*(g; h; \hat{\Omega}^*_V \omega) & \xrightarrow{\sigma} & C^*(g; h; \hat{\Omega}^*_V \omega) \\
\downarrow \iota & & \downarrow \tilde{\mu} \circ \hat{A} \circ e^\theta \\
C^*(g; h; \hat{\Omega}^*_V \omega[h^{-1}]) & \xrightarrow{\tilde{\mu}^h} & C^*(g; h; \prod_{p \in Z} \hat{\Omega}^*[h^{-1}, h][2p])
\end{array}$$
is homotopy commutative.

A proof of Theorem 7.6.1 may be found in [NT1], [NT2]. In Section 8 we will show that Theorem 6.1.1 reduces to Theorem 7.6.1.

8. Gel’fand-Fuchs cohomology

In this section we introduce the machinery of Fedosov connections and Gel’fand-Fuchs cohomology and reduce Theorem 6.1.1 to the analogous statement in the particular case when \( M \) is the formal neighborhood of the origin in a symplectic vector space over \( \mathbb{C} \).

Suppose given a complex manifold \( M \) and a symplectic deformation quantization \( A_M^h \) of \( M \). Let \( \omega \in H^0(M; \Omega^2_M) \) denote the associated symplectic form.

8.1. The sheaf of Weyl algebras. The sheaf of Weyl algebras \( \mathbb{W}_M \) on \( M \) is the sheaf of topological algebras over the sheaf of topological algebras \( \mathcal{O}_M[[\hbar]] \) (equipped with the \( \hbar \)-adic topology) defined as follows.

Let \( \Theta_M \) denote the sheaf of holomorphic vector fields on \( M \). Define \( \mathcal{O}_M[[\hbar]] \), the completion of \( \text{Sym}_{\mathcal{O}_M}(\Theta_M) \otimes_{\mathcal{O}_M} \mathcal{O}_M[[\hbar]] \) in the \( \mathcal{O}_M[[\hbar]] \)-adic topology, as follows.

The completion \( \mathcal{O}_M[[\hbar]] \) of \( \text{Sym}_{\mathcal{O}_M}(\Theta_M) \otimes_{\mathcal{O}_M} \mathcal{O}_M[[\hbar]] \) in the \( \mathcal{O}_M[[\hbar]] \)-adic topology is a topological \( \mathcal{O}_M[[\hbar]] \)-module. The Weyl multiplication is continuous in the \( \mathcal{O}_M[[\hbar]] \)-adic topology.

Let \( \mathbb{W}_M \) denote the kernel of the canonical map \( \mathbb{W}_M \to \mathcal{O}_M \). The Weyl multiplication is continuous in the \( \mathcal{O}_M[[\hbar]] \)-adic topology.

Let \( F_p \mathbb{W}_M = \mathcal{I}^{-p} \). Then \( (\mathbb{W}_M, F_h) \) is a filtered ring, i.e. \( F_p \mathbb{W}_M \cdot F_q \mathbb{W}_M \subseteq F_{p+q} \mathbb{W}_M \). Note that the quotients \( F_p \mathbb{W}_M / F_q \mathbb{W}_M \) are locally free \( \mathcal{O}_M \)-modules of finite rank.

8.2. Review of the Fedosov construction. We refer the reader to [F] and [NT3] for a detailed exposition of the construction of deformation quantizations via Fedosov connections.

Let \( A^{p,q}_M \) denote the sheaf of complex valued \( C^\infty \)-forms of type \( (p, q) \) on \( M \), \( A^*_M = \bigoplus_{p+q=s} A^{p,q}_M \). Let \( F_r A^*_M \) denote the Hodge filtration and \( d \) the de Rham differential. Then \( (A^*_M, d, F_h) \) is a filtered differential graded algebra (i.e. \( F_r A^*_M F_s A^*_M \subseteq F_{r+s} A^*_M \) and \( d(F_r A^*_M) \subseteq F_{r+1} A^*_M \)).
Let $\mathcal{A}_M^p(\mathbb{W}_M) = \mathcal{A}_M^p \otimes_{\mathcal{O}_M} \mathbb{W}_M$. Then
\[ \mathcal{A}_M^\bullet(\mathbb{W}_M) \overset{\text{def}}{=} \bigoplus_p \mathcal{A}_M^p(\mathbb{W}_M)[-p] \]
has a natural structure of a sheaf of graded algebras. Let
\[ F_r \mathcal{A}_M^\bullet(\mathbb{W}_M) = \sum_{p+q=r} F_p \mathcal{A}_M^\bullet(\mathbb{W}_M) \]  
Then $(\mathcal{A}_M^\bullet(\mathbb{W}_M), F_\bullet)$ is a filtered graded algebra over $(\mathcal{A}_M^\bullet, F_\bullet)$ (i.e. $F_r \mathcal{A}_M^p(\mathbb{W}_M) \subseteq F_{r+s} \mathcal{A}_M^{p+q}(\mathbb{W}_M)$ and $F_r \mathcal{A}_M^p F_s \mathcal{A}_M^q(\mathbb{W}_M) \subseteq F_{r+s} \mathcal{A}_M^{p+q}(\mathbb{W}_M)$).

Let $F_p \mathcal{A}_M = h^{-p} \mathcal{A}_M^p$.

One can show that there exists a map
\[ \nabla : \mathcal{A}_M^\bullet(\mathbb{W}_M) \to \mathcal{A}_M^\bullet(\mathbb{W}_M)[1] \]
which has the following properties:

1. $\nabla (F_p \mathcal{A}_M^\bullet(\mathbb{W}_M)) \subseteq F_p \mathcal{A}_M^\bullet(\mathbb{W}_M)$; the induced maps
   \[ Gr_p \nabla : Gr_p F_r \mathcal{A}_M^\bullet(\mathbb{W}_M) \to Gr_p F_r \mathcal{A}_M^\bullet(\mathbb{W}_M)[1] \]
   are $\mathcal{O}_M[[h]]$-linear differential operators of order one;

2. $\nabla^2 = 0$;

3. $(\mathcal{A}_M^\bullet(\mathbb{W}_M), \nabla)$ is a sheaf of differential graded algebras over the
   $C^\infty$ de Rham complex $(\mathcal{A}_M^\bullet, d)$ (in particular $H^\bullet(\mathcal{A}_M^\bullet(\mathbb{W}_M), \nabla)$ is
   a sheaf of graded algebras over the constant sheaf $\mathcal{O}_M[[h]]$);

4. there is a filtered quasiisomorphism
   \[ (\mathcal{A}_M^h, F_\bullet) \to ((\mathcal{A}_M^\bullet(\mathbb{W}_M), \nabla), F_\bullet) \]
   of differential graded algebras over $\mathcal{O}_M[[h]]$ (in particular $H^p(F_q \mathcal{A}_M^\bullet(\mathbb{W}_M), \nabla) = 0$ for $p \neq 0$);

therefore $\nabla$ is determined by its component
\[ \nabla : \mathcal{A}_M^0(\mathbb{W}_M) \to \mathcal{A}_M^1(\mathbb{W}_M) \]
which has all the properties of a flat connection on $\mathbb{W}_M$.

Let $H = Sp(\dim M)$ and let $P \to M$ denote the $H$-principal bundle of symplectic frames in $TM$, identify $TM$ with the vector bundle associated to the standard representation of $H$.

Recall that, for an $\mathfrak{h}$-module $V$, the subcomplex $[\pi_* \mathcal{A}_P^\bullet \otimes V]^{\text{basic}} \subset \pi_* \mathcal{A}_P^\bullet$ is defined by the pull-back diagram

\[
\begin{array}{ccc}
[\pi_* \mathcal{A}_P^\bullet \otimes V]^{\text{basic}} & \longrightarrow & C^\bullet(\pi_* T_P, \mathfrak{h}; \mathcal{V}) \\
\downarrow & & \downarrow \\
\pi_* \mathcal{A}_P^\bullet \otimes V & \longrightarrow & C^\bullet(\pi_* T_P; V)
\end{array}
\]
where $T_P$ denotes the sheaf of Lie algebras of $C^\infty$ vector fields on $P$.

Let $W$ denote the Weyl algebra of the standard representation of $H$. Then $\mathbb{W}_M$ is identified with the sheaf of sections of the associated bundle $P \times_H W$ and pull-back by $\pi$

$$\mathcal{A}_M^*(\mathbb{W}_M) \to \pi_* \mathcal{A}_P^* \otimes W$$

identifies $\mathbb{W}_M$-valued forms on $M$ with the subcomplex of basic $W$-valued forms on $P$. The flat connection $\nabla$ gives rise to a basic $\mathfrak{g}(= \mathcal{D}et(\mathfrak{w}))$-valued 1-form $A \in H^0(P; \mathcal{A}_P^* \otimes \mathfrak{g})$ which satisfies the Maurer-Cartan equation $dA + \frac{1}{2}[A, A] = 0$ (so that $(d + A)^2 = 0$). Then, pull back by $\pi$ induces the isomorphism of filtered complexes

$$(\mathcal{A}_M^*(\mathbb{W}_M), \nabla) \to \left( [\pi_* \mathcal{A}_P^* \otimes \mathbb{W}]^{\text{basic}}, d + A \right).$$

Given $A$ as above and a (filtered) topological $\mathfrak{g}$-module $L$ such that the action of $\mathfrak{h} \subset \mathfrak{g}$ integrates to an action of $H$. Set

$$(\mathcal{A}_M^*(L), \nabla) \overset{\text{def}}{=} \left( [\pi_* \mathcal{A}_P^* \otimes L]^{\text{basic}}, d + A \right).$$

Note that the association $L \mapsto (\mathcal{A}_M^*(L), \nabla)$ is functorial in $L$. In particular it extends to complexes of $\mathfrak{g}$-modules.

Taking $L = \mathbb{C}$, the trivial $\mathfrak{g}$-module, we recover $(\mathcal{A}_M^*, d)$. For any complex $(L^\bullet, d_L)$ of $\mathfrak{g}$-modules as above the complex $(\mathcal{A}_M^*(L^\bullet), \nabla + d_L)$ has a natural structure of a differential graded module over $(\mathcal{A}_M^*, d)$.

The Gel'fand-Fuchs map

$$GF : C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathfrak{L}) \to \mathcal{A}_M^*(\mathfrak{L})$$

(the source understood to be the constant sheaf) is defined by the formula

$$GF(c)(X_1, \ldots, X_p) = c(A(X_1), \ldots, A(X_p)),$$

where $c \in C^p(\mathfrak{g}, \mathfrak{h}; \mathfrak{L})$ and $X_1, \ldots, X_p$ are locally defined vector fields. It is easy to verify that $GF$ takes values in basic forms and is a map of complexes. Note also, that $GF$ is natural in $L$. In particular the definition above has an obvious extension to complexes of $\mathfrak{g}$-modules.

We now proceed to apply the above constructions to particular examples of complexes $L^\bullet$ of $\mathfrak{g}$-modules. In all examples below the $\mathfrak{g}$-modules which appear have the following additional property which is easy to verify, namely,

$$H^p(\mathcal{A}_M^*(L), \nabla) = 0 \text{ for } p \neq 0.$$

If $(L^\bullet, d_L)$ is a complex of $\mathfrak{g}$-modules with the above property, then the inclusion

$$(\ker(\nabla), d_L|_{\ker(\nabla)}) \hookrightarrow (\mathcal{A}_M^*(L^\bullet), \nabla + d_L)$$

(8.1)
is a quasiisomorphism. In such a case the map (8.1) induces an isomorphism

\[ R\Gamma(M; \ker(\nabla)) \to \Gamma(M; A^*_M(L^*)) \]

in the derived category of complexes since the sheaves \( A^*_M(L^*) \) are soft and the Gel’fand-Fuchs map induces the (natural in \( L^* \)) morphism

\[ GF : C^*(g, h; L) \to R\Gamma(\mathfrak{M}; \ker(\nabla)) \] (8.2)

in the derived category.

For \( L = W \) (respectively \( g, \mathbb{C}C^\text{per}(W), \mathcal{O}, \mathcal{O}^*, \mathbb{C}C^\text{per}(\mathcal{O}) \)), \( \ker(\nabla) = A^h_M \) (respectively \( \text{Der}(A^h_M), \mathbb{C}C^\text{per}(A^h_M), \mathcal{O}_M, \mathbb{C}M, \mathcal{O}^*_M, \mathbb{C}C^\text{per}(\mathcal{O}_M) \)). We leave it to the reader to identify \( \ker(\nabla) \) for other (complexes of) \( g \)-modules which appear in Section 7 by analogy with the above examples.

The relationship between the Lie algebra cocycles defined in Section 7 and morphism defined in Section 5 established by the Gel’fand-Fuchs map (8.2) is as follows.

8.2.1. The trace density. The image of \( \tilde{\mu}^h \) (defined in Lemma 7.5.3) under (the map on cohomology in degree zero induced by)

\[ GF : C^*(g, h; \text{Hom}^*(\mathbb{C}C^\text{per}(\mathfrak{M}); h^{-1}, \prod_{p \in \mathbb{Z}} \hat{\Omega}^*[h^{-1}, h][2d - 2p]) \to \]

\[ R\Gamma(M; \text{Hom}^*(\mathbb{C}C^\text{per}(A^h_M); h^{-1}, \prod_{p \in \mathbb{Z}} \Omega^*_M[h^{-1}, h][2d - 2p])) \]

is the morphism \( \tilde{\mu}^h_A \) defined in (5.3).

Similarly, the image of \( \tilde{\mu}^h \sim \iota \) under \( GF \) is the morphism \( \tilde{\mu}^h_A \circ \iota \).

8.2.2. The symbol and the Hochschild-Kostant-Rosenberg map. The image of \( \sigma \) (defined in 7.5.2) under (the map on cohomology in degree zero induced by)

\[ GF : C^*(g, h; \text{Hom}^*(\mathbb{C}C^\text{per}(\mathfrak{M}); \mathbb{C}C^\text{per}(\hat{\mathcal{O}^*}))) \to R\Gamma(\mathfrak{M}; \text{Hom}^*(\mathbb{C}C^\text{per}(A^h_M), \mathbb{C}C^\text{per}(\mathcal{O}_M))) \]

is the morphism

\[ \sigma : \mathbb{C}C^\text{per}(A^h_M) \to \mathbb{C}C^\text{per}(\mathcal{O}_M) \]

The image of \( \tilde{\mu} \) (defined in 7.5.2) under (the map on cohomology in degree zero induced by)

\[ GF : C^*(g, h; \text{Hom}^*(\mathbb{C}C^\text{per}(\hat{\mathcal{O}^*}); \prod_{p \in \mathbb{Z}} \hat{\Omega}^*[-2p])) \to \]

\[ R\Gamma(M; \text{Hom}^*(\mathbb{C}C^\text{per}(\mathcal{O}_M); \prod_{p \in \mathbb{Z}} \Omega^*_M[-2p])) \]

is the morphism \( \tilde{\mu}_\mathcal{O} \) defined in (5.1).
8.2.3. The characteristic class of the deformation. The image of the cocycle $\theta \in C^2(\mathfrak{g}, \mathfrak{h}; \frac{1}{\sqrt{-1h}} \mathbb{C}[[h]])$ (defined in [7.5.3]) under the map

$$G: C^\bullet(\mathfrak{g}, \mathfrak{h}; \frac{1}{\sqrt{-1h}} \mathbb{C}[[h]]) \to \mathbb{A}_{\text{DR}}^\bullet[[h]]$$

is the characteristic class $\theta$ of the deformation quantization $A^h_M$ defined in [F] and [D].

8.2.4. The $\hat{A}$-class. The composition

$$\hat{\text{Sym}}^\bullet(\mathfrak{h}) \xrightarrow{\text{exp}} C^\bullet(\mathfrak{g}, \mathfrak{h}; \mathbb{C}) \xrightarrow{\phi_{\hat{A}}} \mathbb{A}_{\text{DR}}^\bullet$$

is easily seen to be the usual Chern-Weil homomorphism. In particular we have

$$GF(\hat{A}) = \hat{A}(TM),$$

where $\hat{A}$ is defined in [7.5.4].

Combining the above facts we obtain the following proposition.

**Proposition 8.2.1.** The image of the cocycle $\tilde{\mu}^h \sim i - \tilde{\mu} \sim \hat{A} \sim e^\theta \sim \sigma$ (see Theorem [7.6.1]) is the morphism $i \circ \tilde{\mu}^h - (\tilde{\mu} \sigma \hat{A} \sim e^\theta) \circ \sigma$ (see Theorem [6.1.1]).

**Corollary 8.2.2.** Theorem [7.6.1] implies Theorem [6.1.1].

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