Combinatorial space from loop quantum gravity.

José A. Zapata *
†

Abstract

The canonical quantization of diffeomorphism invariant theories of connections in terms of loop variables is revisited. Such theories include general relativity described in terms of Ashtekar-Barbero variables and extension to Yang-Mills fields (with or without fermions) coupled to gravity.

It is argued that the operators induced by classical diffeomorphism invariant or covariant functions are respectably invariant or covariant under a suitable completion of the diffeomorphism group. The canonical quantization in terms of loop variables described here, yields a representation of the algebra of observables in a separable Hilbert space. Furthermore, the resulting quantum theory is equivalent to a model for diffeomorphism invariant gauge theories which replaces space with a manifestly combinatorial object.

PACS number(s): 04.60.Nc, 04.60.Ds
KEY WORDS: Diffeomorphism invariance; loop quantization, combinatorial.

*zapata@phys.psu.edu
†Department of Physics, The Pennsylvania State University, 104 Davey Laboratory, University Park, PA 16802
I. INTRODUCTION

After ten years of ‘new variables’ [1] and loop representation [2], the theory has matured significantly. This approach to quantum gravity has gained clarity, borrowed and developed powerful tools, and sharpened its picture of physical space. Specifically, after solving the spin (Mandelstam) identities by the use of spin networks [3] the formulation of the theory has become clear and it allows a deeper understanding. After this clarification was made, explicit geometric operators [4] that encode loop quantum gravity’s picture of space were written. These geometric operators predict a geometry that is polymer-like [5], non-commutative [6] and quantized [4]. Also lattice versions of the framework [7,8] ready for explicit computation, and several proposals for the Hamiltonian constraint of the theory [9,10] have been developed.

Now the early results (on the classical/macrosopic limit [11] and incorporating other fields and matter [12]) have to be “upgraded,” and using the new tools and sharper notions other problems, like the computation of transition amplitudes [13] or the statistical mechanics behind black hole thermodynamics [14], are within reach.

Loop quantization [15,16] applies to any gauge theory with compact gauge group and particularly to general relativity casted in the Ashtekar-Barbero variables [17]. Information about the connection and the triad is stored in a set of functions of the holonomies along paths and a set of functions of the surface smeared triads respectively. Loop quantization produces an operator for every element of this family of functions. This is done by completing the space of holonomy functions to make it a $C^*$ algebra; and turning it into a Hilbert space by giving it an inner product that makes the operators induced by former real valued functions be Hermitian operators.

To represent the algebra of observables one needs a space of invariant states. Since quantization involves completing the algebra of holonomy functions, the quantum gauge group is an appropriate completion of the classical gauge group. In the case of the internal gauge transformations one can solve the Gauss constraint after quantization, or give an intrinsically gauge invariant formulation. Both constructions agree if the quantum gauge group is taken to be a completion of the classical internal gauge group. For the diffeomorphism gauge group there is no intrinsically invariant construction; one can only solve the diffeomorphism constraint after quantization. In this article I argue that there is a natural candidate for the quantum gauge group, and it turns out to be a completion of the diffeomorphism group.

According to this refined treatment of diffeomorphism invariance an old expectation is realized. Namely, diffeomorphism invariance plays a double role. It forces one to consider an uncountable set of graphs to label the kinematical states of loop quantum gravity. However, it yields a representation of the algebra of observables (diffeomorphism invariant functions) in a separable Hilbert space spanned by states labeled by knot-classes of graphs. In contrast, Grot and Rovelli found that the space of invariant states of the previous formulation of loop quantization contains families of orthogonal states labeled by continuous parameters [18]. In the version of loop quantization that uses the completed diffeomorphism group, one can exhibit a countable basis of invariant states (the spin-knot basis). In fact, the completed diffeomorphism group simplifies the formalism, and the resulting quantum theory is equivalent to a model for diffeomorphism invariant gauge theories which replaces the space manifold with a manifestly combinatorial object [8]. Just as loop quantization conduces to a notion of quantum geometry with discrete areas of non-commutative nature, it also conduces
to an intrinsically combinatorial picture of physical space.

I revisit loop quantization emphasizing the issue of diffeomorphism invariance. For completeness, the kinematics of loop quantization is briefly reviewed in section II. Internal and diffeomorphism gauge invariance of the classical and quantum theories are addressed in section III—the main section of the article. In that section, a refined treatment of diffeomorphism invariance is presented, and its consequences are studied. A discussion section ends the article.

II. KINEMATICS: GENERAL RELATIVITY IN TERMS OF CONNECTIONS AND HOLONOMIES

Recall that gravity, expressed in (real) Ashtekar-Barbero variables, is a Hamiltonian theory of connections that shares the phase space with $SU(2)$ Yang-Mills theory [17]. That is, the configuration variable is a connection $A_i^a$ taking values in the Lie algebra of $SU(2)$, and the canonically conjugate momentum is a triad $\tilde{E}_i^a$ of densitized vector fields. In these variables the contravariant spatial metric is determined by $q_{ab} \det q = \tilde{E}_i^a \tilde{E}_j^b$, which makes contact with the usual geometrodynamics treatment of general relativity. In this formulation, Einstein’s equations are equivalent to a series of constraints: a set which generates diffeomorphisms on the Cauchy surface and constitutes a closed subalgebra of the constraint algebra, and a set of constraints generating motions transverse to the initial data surface.

If only the constraints that generate spatial diffeomorphisms are imposed and the Hamiltonian constraint is dropped, one gets a well-defined model to study diffeomorphism invariant theories of connections. This model is called the Husain-Kuchař model and can be derived from an action principle [19]; it shares the phase space, the gauge constraint and the diffeomorphism constraint with general relativity and has local degrees of freedom. More than a toy model, the Husain-Kuchař model provides an intermediate step in the quantization of general relativity; a quantization of the model requires to set up a kinematical framework and regularize and solve the gauge and diffeomorphism constraints. After a satisfactory quantum version of the model is developed, a quantization of general relativity amounts to the difficult tasks of regularizing and solving the Hamiltonian constraint and verifying that GR is recovered in the classical limit. This article is about the treatment of diffeomorphism invariance in the loop quantization framework; therefore it pertains to any diffeomorphism invariant theory of connections, in particular, to general relativity (possibly coupled to Yang-Mills fields) and the Husain-Kuchař model. For the sake of concreteness, the problems and the results are stated in reference to the the quantization of the Husain-Kuchař model. Issues like whether the algebra of the constraints is correct or if there is a classical limit in the theory resulting from Thiemann’s Hamiltonian constraint [10] is matter of hot debate [20]. Since the study includes diffeomorphism covariant functions and the Hamiltonian constraint is diffeomorphism covariant, the results presented in this article may shine some light on the difficult problem of regularizing the Hamiltonian constraint.

The cornerstone of loop quantization is the use of holonomies along loops as “coordinates on the classical configuration space” [16,10]. For primary momentum functions one can use the triad (whose dual is a two form) smeared on surfaces [14], or, in the manifestly gauge invariant treatment, a combination of holonomies and triads called the strip functions [21,16].
In this article, the term *loop variables* is some times used as a collective name for the configuration and momentum variables described. This choice of variables is due to the symmetries of the theory; using them one can explicitly solve the gauge and diffeomorphism constraints of the quantum theory.

It was proven [22] that all the information about the connection is contained in the set of holonomies of the connection around every smooth path $e$

$$h_e(A) = \text{Pexp}(i \int e a^i ds^a)$$

(2.1)

where $\sigma_i = \frac{1}{2} \sigma_i$ are the $SU(2)$ generators [10]. The loop variables $h_e(A)$ are an overcomplete set of configuration functions that coordinatize the space of smooth connections $A$ in the sense that two connections can always be differentiated by the loop variables. If only closed loops are used, the set of traces of the holonomies coordinatizes the space of connections modulo internal gauge transformations. Also, any two smooth triads can be differentiated by smearing the triads (two forms) over some surface. This property ensures that by keeping only functions of the loop variables as primary functions, and recovering everything from them after quantization, no relevant information is omitted. Thus, at least in principle, any phase space function can be expressed in terms of functions of the loop variables. The holonomy functions are special because they form a subalgebra of the algebra of configuration functions; and this subalgebra is preserved by the primary momentum functions, the surface smeared triads. These important properties lie at the heart of loop quantization.

The classical algebra that is actually quantized is the algebra $\text{Cyl}_0$. A cylindrical function $f_\gamma(A) \in \text{Cyl}_0$ is a function of the holonomies along the edges of the graph $\gamma$. With this definition, the product of two cylindrical functions is another cylindrical function if the edges of the two original graphs are contained in the set of edges of a bigger graph. To satisfy this condition, it was first proposed to consider only graphs with piecewise analytic edges [15]. Since among the cylindrical functions one has all the loop variables, it is clear that one can use the cylindrical functions as primary functions in the space of smooth connections. After $\text{Cyl}_0$ is quantized the primary configuration functions become operators that act by multiplication, and the primary momentum functions (the surface smeared triads) become operators that act as derivative operators. Thus, loop quantization produces a regularized operator from any phase space function written in terms of the loop variables.

$\text{Cyl}_0$ is quantized by following a series of steps. First, completing it to form a $C^*$ algebra Cyl. Second, represent the cylindrical functions and linear in momenta functions in $\mathcal{H}_{\text{kin}} = L^2(\mathcal{A}, \mu)$ (by multiplicative and derivative operators respectively), where $\mathcal{A}$ is the spectrum of Cyl and $\mu$ is the Ashtekar-Lewandowski measure, which is selected by the reality conditions [15].

At a more operational level, the Hilbert space of gauge invariant states (under $SU(2)$ gauge transformations) is spanned by spin network states $|S\rangle$ [3]. A spin network $S$ is labeled by a colored graph $\vec{\gamma}$ and represents the function of the holonomies along its edges given by

---

1 I would loosely use the term smooth to mean real analytic; except in the last paragraphs of the article where I comment on the smooth ($C^\infty$) category.
where the colors on the edges $j(e)$ are irreducible representations of $SU(2)$, and the vertices are labeled by gauge invariant contractors $c(v)$ that match all the indices (in the formula denoted by ‘.’) of the holonomies of the edges. An inner product in the space of gauge invariant states $L^2(A/\mathcal{G}, \mu)$ is given, alternatively, by the Ashtekar-Lewandowski measure or by recoupling theory. According to this inner product, two spin network states are orthogonal if their coloring or labeling graphs are different. Using a convenient set of contractors one can form an orthonormal basis with spin network states $\langle S | S' \rangle = \delta_{SS'}$.

Non-gauge invariant spin network states are constructed by just dropping the gauge invariant contractors and the Ashtekar-Lewandowski measure induces an inner product in $H_{\text{kin}}$.

### III. DIFFEOMORPHISM INVARIANCE IN THE CLASSICAL AND QUANTUM THEORIES

Classical observables, gauge and diffeomorphism invariant functions, induce functions in the reduced phase space; loop quantization’s objective is to produce a faithful representation of the algebra of observables. First the operators are regularized from their expressions as functions of the loop variables. The resulting operators are expected to be invariant under “quantum gauge transformations” and “quantum diffeomorphisms.” Finally, from the algebra of invariant operators one induces (by dual action) a faithful representation on the space of diffeomorphism invariant states. Here, this process is followed, but special care is paid to the character acquired by diffeomorphism invariance after loop quantization.

In the description of the classical theory in terms of smooth fields there is a harmony between the space of smooth connections and the gauge group. As far as the internal gauge transformations, the internal gauge group may be characterized as the set of $SU(2)$-matrix valued functions $g$ such that given a smooth connection $A \in \mathcal{A}$, the connection $g(A_a) = g^{-1}A_ag + g^{-1}\partial_ag$ is also smooth. Similarly, the diffeomorphism group can be characterized as the subgroup of the homeomorphism group composed by all the transformations which leave the space of smooth connections invariant

$$\text{Diff} = \{ \phi \in \text{Hom} | \phi^*(A) \in \mathcal{A} \quad \text{for all} \quad A \in \mathcal{A} \}$$

This compatibility between configuration space and gauge group acquires a different form after loop quantization. Quantization takes the space of smooth connections and, by completing it, constructs the quantum configuration space $\bar{\mathcal{A}}$. A generalized connection $A \in \bar{\mathcal{A}}$ simply assigns group elements to piecewise analytic paths; that is, it acts as a connection which does not need to be smooth. Completing the configuration space requires adapting the gauge group also. The quantum internal gauge group $\bar{\mathcal{G}}$ is formed by the transformations acting at the end points of the paths, $g(A)[e] = g^{-1}(e_0)g(A)[e]g(e_1)$. A quantum gauge transformation maps every generalized connection to another generalized connection. This group contains the classical internal gauge group, but it is not the classical gauge group.
It is the completion of the group of smooth internal gauge transformations according to the operator norm. Most of the quantum gauge transformations would transform a smooth connection into a non-smooth connection.

In the diffeomorphism part of the gauge group a similar phenomena happens. The family of piecewise analytic graphs is left invariant by a bigger group than the group of smooth diffeomorphisms, but if one transforms a smooth connection using a non-smooth map one obtains a non-smooth connection. Again, because quantization involves completing the configuration space, the generalized connections are covariant with respect to a certain completion of the diffeomorphism group; \( \phi^*(A)[e] := A[\phi(e)] \) is defined for a certain completion of the diffeomorphism group\(^2\). As a consequence, the primary configuration and momentum variables induce operators that are covariant with respect to the mentioned completion of the diffeomorphism group. Since every operator of the quantum theory is constructed from the primary configuration and momentum operators, this extended covariance becomes a feature of the quantum theory. Functions of the phase space with a geometrical label (like the holonomy functions, surface smeared triads, surface area functions, volume functions, etc) are diffeomorphism covariant, but operators coming from these functions with geometrical labels are naturally covariant under a certain completion of the diffeomorphism group. Note that the Hamiltonian constraint is diffeomorphism covariant and some of its regularizations have the mentioned extended covariance (comments on the Hamiltonian constraint are reserved for the discussion section).

More importantly, given the extended notion of covariance, it is necessary to review the notion of observable in the quantum theory. Observables (diffeomorphism invariant functions) naturally arise from covariant functions where the geometrical labels become dynamical. For example, area functions of surfaces specified by matter fields. If the fields specifying the geometrical labels also acquire the extended covariance, as they would if they are quantized using loop quantization, then the natural notion of an observable would be to be invariant under the mentioned completion of the diffeomorphism group.

To explain the details of the previous discussion let me show you how piecewise analytic diffeomorphisms come about. Consider the following situation. The Cauchy surface is \( \mathbb{R}^3 \); an example of nonsmooth map is \( \phi : \mathbb{R}^3 \to \mathbb{R}^3 \) defined to be the identity above the \( x-y \) plane and below the plane \( x-y \) plane it is defined by \( \phi(x, y, z) = (x, y + mz, z) \). This map is smooth above and below the \( x-y \) plane but at the \( x-y \) plane its derivative from above and its derivative from below do not match (in the direction normal to the \( x-y \) plane). One can see that \( \phi \) maps some smooth loops to loops with kinks. Given any smooth connection \( A \in \mathcal{A} \), one would like to say that the functions

\[
h_l(\phi^*(A)) := h_{\phi(l)}(A). \tag{3.2}
\]

\(^2\) In the previous paragraph I defined \( \bar{G} \) algebraically. The algebraic relation came from the classical theory, but the definition of \( \bar{G} \) only involved quantum objects. I will show that this construction in the case of the diffeomorphism group yields \( \bar{D} \). However, \( \bar{G} \) is the completion of \( G \) in the operator norm, and \( \bar{D} \supset D \), but according to the operator norm \( \bar{D} \) is a discrete group. Strictly speaking, \( \bar{D} \) is an algebraic extension of the diffeomorphism group rather than a completion of it.
are “covariantly” related to the loop coordinates of \( A \in \mathcal{A} \), but the connection \( A' = \phi^*(A) \) is not in the configuration space of the classical theory. However, in the quantum theory, the functions \( h_{\phi(t)}(A) \) induce an operator that is as valid as the ones induced by the functions \( h_{\pi(t)}(A) \) defined using any smooth map \( \pi \). Hence the map \( \phi \) is an object that will play a role in the quantum theory even though it did not define a canonical transformation in the classical theory. Classically, we cannot ask if the connections \( A \in \mathcal{A} \) and \( A' = \phi^*(A) \) are gauge related, but the quantum configuration space is the space of generalized connections, and \( A \in \bar{\mathcal{A}} \) if and only if \( \phi^*(A) \in \bar{\mathcal{A}} \).

Following the above example, a map \( \phi : \Sigma \rightarrow \Sigma \), that maps any piecewise analytic graph to another, would map any generalized connection to another, and define a new loop operator from a given one.

A map \( \phi : \Sigma \rightarrow \Sigma \) belongs to \( \bar{\mathcal{D}} \) iff for any piecewise analytic graph \( \gamma \) the new graph \( \phi(\gamma) \) is also piecewise analytic.

Above I gave a description of \( \bar{\mathcal{D}} \) designed to show the natural role that it will play in the quantum theory, and to emphasize the parallelism between its definition and the definition of \( \bar{\mathcal{G}} \). Alternatively, one can describe \( \bar{\mathcal{D}} \) as the group of piecewise analytic diffeomorphisms. In close analogy with the definition of a piecewise linear manifold (Regge lattice), a piecewise analytic manifold \( \Sigma \) is a topological manifold formed as a union of finitely many closed cells, each of which is an analytic manifold with boundary (these correspond to the higher dimensional simplices of the Regge lattice). Two of these cells may intersect only at their boundaries. A map \( \phi : \Sigma_1 \rightarrow \Sigma_2 \) is piecewise analytic if and only if there is a refinement of the cell decomposition of \( \Sigma_1 \) such that the restriction of \( \phi \) to every cell is an analytic map. Clear examples of piecewise analytic manifolds (maps) are real-analytic manifolds (maps) and piecewise linear manifolds (maps).

Guidance from the classical theory tells us that the operators induced by \( h_t(A) \) and \( h_{\pi(t)}(A) \) for any smooth map \( \pi \) are gauge related. However, classically one can not say that the functions \( h_t(A) \) and \( h_t(\phi^*(A)) := h_{\phi(t)}(A) \) are gauge related since the non-smooth map \( \phi \) does not define a canonical transformation because the connection \( A' = \phi^*(A) \) is not in the configuration space of the classical theory, but the quantum states are functions of generalized connections \( \text{Cyl}(\bar{\mathcal{A}}) \) and \( A \in \bar{\mathcal{A}} \) if and only if \( \phi^*(A) \in \bar{\mathcal{A}} \). Quantization involves completing the space of cylindrical functions to make it the \( C^* \) algebra \( \text{Cyl}(\bar{\mathcal{A}}) \); to account for this enlargement of the configuration space, the internal gauge group is \( \bar{\mathcal{G}} \) instead of \( \mathcal{G} \). Smooth connections and generalized connections differ in more than their “internal degrees of freedom.” Recall that in the smooth case \( \phi^*(A) \) is defined only for smooth (analytic) maps, whereas in the case of generalized connections it is defined for any piecewise analytic map.

Because of these considerations, and since any piecewise analytic map \( \phi \) can be obtained as a limit of smooth maps I will assume that the operators induced by \( h_t(A) \) and \( h_{\phi(t)}(A) \) are gauge related.

A quantum ‘diffeomorphism’ \( \phi \in \bar{\mathcal{D}} \) acts by shifting the labels of the spin networks by a diffeomorphism

\[
U_\phi |S_{\bar{\gamma},j(e'),c(e')}\rangle := |S_{\phi(\bar{\gamma}),j(e'),c(e')}\rangle . \tag{3.3}
\]

Since the measure that defines the inner product is \( \bar{\mathcal{D}} \) invariant, the operator \( U_\phi \) is unitary.
Before the significance of $D$ was understood, it was noticed that the original regularization of the area and volume operators, and some versions of the Hamiltonian constraint, were not diffeomorphism covariant, but they were covariant under a bigger group. Later a version of the volume operator that was only covariant under smooth diffeomorphisms was developed and this version of the volume operator entered in the definition of Thieman’s Hamiltonian constraint. Initially, it was believed that replacing the volume operator used by Thieman with the $D$ covariant version would change the algebra of the constraints, but now it has been proven that it produces no changes \cite{21}.

Using the technique developed in \cite{15}, one solves the quantum diffeomorphism constraint by constructing the space of $D$ invariant states $\mathcal{H}_{\text{diff}}$. It is spanned by $s$-knot states $\langle s|$, labeled by knot-classes of colored graphs, and defined by

$$\langle s[γ,j(e),c(v)]S_{γ,j(e'),c(v')}^γ := a(γ)\delta_{[γ][η]} \sum_{[φ]\in \text{GS}(γ)} \langle S_{γ,j(e),c(v)}|U_{f-φ_0}S_{γ,j(e'),c(v')}^φ \rangle$$ (3.4)$$

where $a(γ)$ is an undetermined normalization parameter, $δ_{[γ][η]}$ is non vanishing only if there is a piecewise analytic diffeomorphism $φ_0 \in D$ that maps $η$ to a graph $γ$ that defines the knot-class $[γ]$, and $φ \in D$ is any element in the class of $[φ] \in \text{GS}(γ)$. The finite group $\text{GS}(γ)$ is the group of symmetries of $γ$; in other words, the elements of $\text{GS}(γ)$ are maps between the edges of $γ$ (for a detailed explanation see \cite{13}).

The $s$-knot states are solutions of the diffeomorphism constraint because its action is invariant under quantum diffeomorphisms by construction. An inner product for $\mathcal{H}_{\text{diff}}$ is given simply by\cite{15}

$$\langle s[γ,j(e),c(v)]|S_{γ,j(e'),c(v')}^γ : (s[γ,j(e),c(v)]|S_{γ,j(e'),c(v')}^γ) : = \langle s[γ,j(e),c(v)]|S_{γ,j(e'),c(v')}^γ \rangle$$ (3.5)$$

The observables of the Husain-Kuchař model are naturally represented on $\mathcal{H}_{\text{diff}}$. If $\hat{O}$ is a “diffeomorphism” invariant Hermitian operator on the kinematical Hilbert space, $\hat{O} : \mathcal{H}_{\text{diff}} \to \mathcal{H}_{\text{diff}}$ is defined by its dual action

$$\langle (s[γ,j(e),c(v)]|\hat{O})|S_{γ,j(e),c(v)}^γ : (s[γ,j(e),c(v)]|\hat{O})|S_{γ,j(e),c(v)}^γ \rangle : = \langle s[γ,j(e),c(v)]|\hat{O}|S_{γ,j(e),c(v)}^γ \rangle \rangle$$ . (3.6)$$

These are the foundations of the theory following from considering the extended notion of diffeomorphism covariance/invariance in loop quantization. In particular, they constitute a quantization of the Husain-Kuchař model \cite{13}, that has local degrees of freedom.

Here I will describe the properties of the quantum theory that are not shared by previous treatments of loop quantization. First, one should notice that $\mathcal{H}_{\text{diff}}$ is separable. The $s$-knot states are labeled by knot-classes of graphs $[γ]$ with respect to $D$. Since the diffeomorphism group was replaced by a bigger group, the resulting knot-classes are much bigger and therefore there are very few of them; this is why separability arises. In contrast, states in the original treatment are labeled by continuous parameters parameterizing the knot-classes of graphs with higher valence vertices \cite{13}.

\footnote{Note that this inner product is determined only up to the unknown parameters $a(γ)$.}
I sketch the proof of separability in the next few paragraphs. A mathematically rigorous proof can be found in the appendix of [8].

Consider a three dimensional triangulated manifold |K|, which can be thought of as a three dimensional Regge lattice. Since the interior of the tetrahedrons of the lattice are flat, one can define the baricenter of any simplex (tetrahedron, face or link); by adding these points to the original lattice, and also adding new links and faces (see fig. 1), one constructs the finer lattice |Sd(K)| called the baricentric subdivision of the original lattice |K|. One can do this subdivision again and again to get a sequence of lattices { |K|, |Sd(K)|, . . . , |Sd^n(K)|, . . . }. All these lattices are not disconnected, they are all subdivisions of |K|; using them, one defines a combinatorial graph γc to be a graph in |K| all whose edges are links of some of the refined lattices |Sd^n(K)|. Also consider a fixed map h : |K| → Σ that maps every combinatorial graph γc to a piecewise analytic graph h(γc) on Σ.

![Fig. 1](image.png)

**Fig. 1** A triangular face and its baricentric subdivision. Every link of |K| is divided into two links of |Sd(K)|, every face into six faces and every cell into twenty four cells of |Sd(K)|.

The sense in which the knot-classes of graphs [γ] are big is that every class contains a combinatorial graph, h(γc) ∈ [γ]. Given an arbitrary graph γ, the following series of steps generates a combinatorial graph γc and a piecewise analytic map φ : Σ → Σ such that φ(h(γc)) = γ.

1. Find n such that |Sd^n(K)| separates the vertices of h⁻¹(γ) to lie in different simplices. (The conventions are such that every point of the manifold belongs to the interior of one and only one simplex of a given triangulation).

2. Let h₁ : |K| → |K| be the piecewise linear map that fixes the vertices of |Sd^n(K)| and sends the new vertices v ∈ |Sd^{n+1}(K)| (the baricenters of the simplices of |Sd^n(K)|) to:

   (a) themselves (h(v) = v), if there is no vertex of h⁻¹(γ) in the simplex of |Sd^n(K)| which has v as baricenter.

   (b) the vertex of the graph (h(v) = w), in the case when the simplex of |Sd^n(K)| which has v as baricenter contains a vertex of the graph (w ∈ h⁻¹(γ)) in its interior.

3. Find m such that h₁(|Sd^{n+m}(K)|) separates the edges of h⁻¹(γ) in the interiors of different simplices.

---

4 In the case of a graph γ with two or more edges meeting at a vertex this step needs to be refined. One needs to find an integer m and a piecewise analytic map ψ : Σ → Σ (with analycity domains given by h ◦ h₁(|Sd^{n+m}(K)|) see next footnote) such that ψ ◦ h ◦ h₁(|Sd^{n+m}(K)|) separates the
4. Let a cell be a (closed) image (by \( h : |K| \to \Sigma \)) of a simplex of \( h_1(|Sd^{m+m}(K)|) \). Let \( \phi = \phi_1 \circ h \circ h_1 \circ h^{-1} : \Sigma \to \Sigma \), where \( \phi_1 \) is a piecewise analytic map that is equal to the identity when restricted to cells which do not intersect \( \gamma \), and sends the cells which intersect \( \gamma \) to themselves, but has nontrivial analyticity domains. The analyticity domains divide the cell into the subcells given by the image (by \( h : |K| \to \Sigma \)) of the simplices of \( h_1(|Sd^{m+m+1}(K)|) \). 

\( \phi_1 \) must be such that the intersection of \( \gamma \) and the cell lies in the image (by \( \phi_1 \)) of the boundaries of the subcells; since only one (analytic) edge of \( \gamma \) intersects the interior of the original cell, a map \( \phi_1 \) with the requested property always exists.

From the construction of \( \phi : \Sigma \to \Sigma \) it is immediate that \( \phi(h(\gamma_c)) = \gamma \).

The sense in which there are very few knot-classes of graphs is that the set of combinatorial graphs \( \{\gamma_c\} \) is countable. One can easily convince oneself that this is the case because every \( \gamma_c \) belongs to \( |Sd^n(K)| \) for some \( n \), and there are countably many of these triangulations, each of which has finitely many links. This property implies that the set of labels of the s-knot states is countable; that is, the Hilbert space of ‘diffeomorphism’ invariant states \( \mathcal{H}_{\text{diff}} \) is separable.

I used the combinatorial graphs to prove the separability of the Hilbert space, but there is a deeper consequence of the existence of such a subspace of \( \mathcal{H}_{\text{kin}} \). It has a manifestly combinatorial origin and is capable of generating all the states in the space of solutions to the diffeomorphism constraint. As far as observables are concerned, the combinatorial states are sufficient; meaning that the manifestly combinatorial framework yields a unitarily equivalent representation of the algebra of observables (see the appendix of [8] for a rigorous proof).

Equivalence with a manifestly combinatorial model is not so surprising if one remembers that observables in generally covariant theories are supposed to measure only relative ‘positions’ of the dynamical fields. One may object that in pure gravity there are not enough explicitly known observables as to serve as a basis of any argument. But, physically meaningful observables will arise if other fields are coupled to pure gravity (or to the Husain-Kuchar model). In these systems one can study observables that measure the gravitational field; for example, any covariant operator of pure gravity, say an area operator, whose labeling surface becomes dynamical after coupling other fields becomes an observable. They are generally covariant systems with plenty of observables measuring the gravitational field. Proving equivalence with a manifestly combinatorial model explicitly exhibits the relational nature of loop quantization.

In contrast with the treatment of diffeomorphism invariance presented in this article, the original study of the quantization of the Husain-Kuchar model considered the diffeomorphism group as the quantum gauge group. By using the same kinematical Hilbert space, but averaging over edges of \( \gamma \). Using this refinement, the rest of the construction has a clear extension.

---

5 A piecewise analytic map is a continuous map whose restriction to the interior of any of its analyticity domains is analytic.

6 One can triangulate a compact manifold with finitely many simplices and a paracompact manifold with countably many simplices. I sketch in the argument for the compact case, but it is immediate to extend it to the paracompact case, which includes all the Cauchy surfaces of asymptotically flat spacetimes.
the diffeomorphism group instead of $\hat{D}$ to generate the solutions of the diffeomorphism constraint \[3.4\], they constructed the space of “physical” states $\mathcal{H}_{\text{Diff}^*}$. This difference implies, in particular, that $\mathcal{H}_{\text{Diff}^*}$ is not separable \[8\] and that the nature of the theory is not combinatorial.

It was argued that classical functions which are diffeomorphism invariant/covariant induce, after loop regularization, $\hat{D}$ invariant/covariant operators on $\mathcal{H}_{\text{kin}}$. Because the operators are invariant under a larger group, the algebra of observables can be represented in $\mathcal{H}_{\text{Diff}^*}$; however, the representation of such operators yields a continuum of superselected sectors \[25\]. This superselection is not surprising after one knows that the same operators are naturally represented in the separable Hilbert space $\mathcal{H}_{\text{diff}}$.

IV. SUMMARY AND DISCUSSION

In this article I studied the loop quantization of diffeomorphism invariant theories of connections. Such theories include general relativity described in terms of Ashtekar-Barbero variables and extension to Yang-Mills fields (with or without fermions \[12,26\]) coupled to gravity. For the sake of concreteness the results were stated for the Husain-Kuchar model \[19\], which shares the phase space with general relativity, but it does not have a Hamiltonian constraint.

Loop quantization regularizes operators using the expression of a phase space function in terms of “loop variables” (functions of the holonomies of the connection along the edges of graphs and functions of surface smeared triads) and the quantization of the loop variables. The loop variables are a family of covariant functions with geometric labels whose quantization is a family of operators with the same geometric labels and an extended covariance. Since the quantum theory is built over the quantization of the loop variables, the extended covariance becomes a feature of the whole quantum theory.

Guidance from the classical theory tells us that the operators induced by $h_l(A)$ and $h_{\pi(l)}(A)$ for any smooth map $\pi$ are gauge related. In the case of non-smooth maps, one can not say that the functions $h_l(A)$ and $h_l(\phi^*(A)) := h_{\phi(l)}(A)$ are gauge related since the non-smooth map $\phi$ does not define a canonical transformation because connections of the form $A' = \phi^*(A)$ are not in the configuration space of the classical theory. However, the quantum states are functions of generalized connections and $A \in \hat{A}$ if and only if $\phi^*(A) \in \hat{A}$ for any map $\phi \in \hat{D}$, where $\hat{D}$ is a completion of the diffeomorphism group.

Just as in the case of the internal gauge group, where the quantum internal gauge group is $\hat{G}$, the same equations that defined the classical gauge group in terms of smooth connections are used to define the the quantum gauge group in terms of generalized connections.

A quantum diffeomorphism belongs to $\hat{D}$, which in the analytic category is the group of piecewise analytic diffeomorphisms.

The resulting quantum theory yields a representation of the algebra of observables in a separable Hilbert space. Furthermore, the quantum theory turns out to be equivalent to a model for diffeomorphism invariant gauge theories which replaces the space manifold with a manifestly combinatorial object \[8\]. Loop quantization yields a quantum theory which is sensitive only to the combinatorial information on the space manifold. Thus, it fulfills the expectations of a framework tailored to study generally covariant theories.

Since the Hamiltonian constraint is a diffeomorphism covariant function, it is natural for its loop regularization to be $\hat{D}$ covariant (and there are versions of the Hamiltonian constraint which
are $\bar{D}$ covariant). Hence, the notion of space in loop quantum gravity is expected to remain combinatorial after the Hamiltonian constraint is imposed. It should be noticed that the original version of Thiemann’s Hamiltonian constraint uses the Ashtekar-Lewandowski volume operator which is not $\bar{D}$ covariant. However, the modification of Thiemann’s Hamiltonian constraint using the Rovelli-Smolin volume operator is $\bar{D}$ covariant, and it has been shown that it enjoys similar properties; in particular, the algebra of the constraints is not altered by using the $\bar{D}$ covariant version of the volume operator [20]. That the properties of the $\bar{D}$ covariant Hamiltonian constraint are the same as Thiemann’s is not necessarily a desirable property [20]. In spite of this feature, a combinatorial view of loop quantization does suggest new treatments of dynamics.

The combinatorial picture of space suggests a simple lattice-like regularization of the Hamiltonian constraint. As in regular lattice gauge theories one can prove that the algebra of the constraints resembles the continuum algebra, but it has corrections that vanish in the continuum limit of regular lattice gauge theories. However, in loop quantization the continuum limit (where the lattice spacing, measured in a background metric, is reduced to zero) was replaced by the projective limit, and the correction terms do not vanish in the projective limit.

I believe that there is a more promising avenue to understand the dynamics of loop quantum gravity. One can take advantage of the combinatorial formulation to make contact with the state sum models that arose borrowing ideas from topological field theories [27]. All the models that have been proposed up to today use the combinatorial setting (or the piecewise linear setting) from the outset.

Apart from the analytic category, which I have used throughout this article, there is the smooth ($C^\infty$) category [28]. The difference is that the allowed graphs have smooth edges; because of this, it is necessary to include “wild graphs,” which are graphs whose edges intersect infinitely many times between vertices. Some aspects of this framework require a more careful analysis, but the quantization strategy is essentially the same. However, in view of the results of this article, part of the motivation to develop a refined version of the smooth category is lost. The quantum gauge group constructed by loop quantization is an appropriate completion of the diffeomorphism group, not the diffeomorphism group itself. Smoothness is considered as a semiclassical/macroscopic property of space by most approaches to quantum gravity. How to reconcile this notion with the quantization of the classical theory is a puzzling problem. This is part of the motivation behind a proposal by Louko and Sorkin of considering more general groups than the diffeomorphism group as the gauge group of general relativity [29].

If smoothness is not considered as fundamental, one has to find the characteristics of the arena of the fundamental theory. By completing the diffeomorphism group, loop quantization gives a precise replacement of classical smooth space: only the combinatorial information of the manifold is relevant in the quantum theory.

I need to acknowledge the illuminating conversations, suggestions and encouragement from Abhay Ashtekar, Alejandro Corichi, Seth Major, Roberto De Pietri, Jorge Pullin, Michael Reisenberger, Carlo Rovelli, Lee Smolin and Madhavan Varadarajan. Support was provided by Universidad Nacional Autónoma de México (DGAPA), and grants NSF-PHY-9423950, NSF-PHY-9396246, research funds of the Pennsylvania State University, the Eberly Family research fund at PSU and the Alfred P. Sloan foundation.
REFERENCES

[1] Ashtekar, A. (1987) Phys. Rev. D36(6)1587
[2] Rovelli C. and Smolin L. (1990) Nuc. Phys. B331(1) 80
    Gambini R. (1991) Phys. Lett. B255 180
[3] Penrose R. (1971) in Quantum theory and beyond ed. Bassin T, Cambridge University Press;
    in Combinatorial Mathematics and its Application ed. Welsh D J A, Academic Press.
    Rovelli C. and Smolin L. (1995) Phys.Rev. D52, 5743
    Baez J. (1996) in The Interface of Knots and Physics, ed. Kauffman L, American Mathematical Society.
[4] Rovelli C. and Smolin L. (1995) Nucl. Phys. B442 593
    Ashtekar A. and Lewandowski J. (1997) Class. Quant. Grav. 1A, 55; e-Print Archive: gr-qc/9711031
[5] Ashtekar, A. (1996) To appear in Proceedings of GR-14, e-Print Archive: hep-th/9601054
[6] Ashtekar A., Corichi A. and Zapata J. A. (1998) in preparation
    Loll R. (1997) Class. Quant. Grav. 14 1725
[7] Loll R. (1995) Nucl. Phys. B444 619
    Fort H., Gambini R. and Pullin J. (1997) Phys. Rev.D56 2127
[8] Zapata J. A. (1997) Journal of Mathematical Physics 38(11): 5663.
[9] Rovelli C. and Smolin L. (1994) Phys. Rev. Lett. 72, 446
    Rovelli C. (1995) J. Math. Phys. 36, 5629
    Borissov R. (1997) Phys. Rev.D55, 6099
    Gambini R. and Pullin J. (1996) Phys. Rev.D54, 5935
    Loll R. (1996) Phys. Rev.D54, 5381
[10] Thiemann T. (1996) Phys. Lett. B 380 257
[11] Ashtekar A., Rovelli C. and Smolin L. (1992) Phys. Rev. Lett. 69, 237
    Iwasaki J. and Rovelli C. (1993) Int. J. Mod. Phys.D1 533
    Iwasaki J. and Rovelli C. (1994) Class. Quant. Grav. 11, 1653
[12] Ashtekar A. and Rovelli C. (1992) Class. Quant. Grav. 9, 1121
    Morales-Tecotl H. A. and Rovelli C. (1994) Phys. Rev. Lett. 72, 3642
    Krasnov K. (1996) Phys. Rev.D53, 1874
[13] Rovelli C. (1995) J. Math. Phys. 36, 6529
    Reisenberger M. and Rovelli C. (1997) Phys. Rev.D56 3490
[14] Rovelli C. (1996) Phys. Rev. Lett. 77 3288; (1996) Helv. Phys. Acta 69 582
    Krasnov K. (1997) Phys. Rev. D55 3505
[15] Ashtekar A., Lewandowski J., Marolf D., Mourao J. and Thiemann T. (1995) J. Math. Phys. 36 6456
[16] De Pietri R. and Rovelli C. (1996) Phys. Rev. D54, 2664
[17] Barbero F. (1995) Phys. Rev. D51 5507
[18] Grot N. and Rovelli C. (1996) J.Math.Phys. 37, 3014
[19] Husain V. and Kuchař K. V. (1990) Phys. Rev. D42, 4070
[20] Smolin L., (1996) e-Print Archive: gr-qc/9609003
    Lewandowski J. and Marolf D. (1997) e-Print Archive: gr-qc/9710001
    Gambini R., Lewandowski J., Marolf D. and Pullin J. (1997) e-Print Archive: gr-qc/9710018
[21] Smolin L. (1992) in *Quantum gravity and cosmology* eds. Pérez Mercader J. et al, World Scientific, Singapore.

[22] Giles R. (1981) *Phys. Rev.* **D24** 2160
Anandan J. (1981) *Proc. Conf. Differential Geometric Methods in Physics* (Trieste 1981) eds. Denardo G. et al, World Scientific, Singapore.
Barrett J. (1985) *Int. Jour. Theor. Phys.* **30** 1171
Lewandowski J. (1993) *Class. Quant. Grav.* **10** 879

[23] Ashtekar A. and Lewandowski J. (1993) in *Quantum Gravity and Knots*, ed. Baez J., Oxford University Press.

[24] De Pietri R. (1997) *Class. Quant. Grav.* **14**, 53
Frittelli S., Lehner L. and Rovelli C. (1996) *Class. Quant. Grav.* **13**, 2921

[25] Thiemann T. (1997) *e-Print Archive*: gr-qc/9705017

[26] Baez J., Krasnov K. (1997) *e-Print Archive*: hep-th/9703112

[27] Reisenberger M. (1997) *e-Print Archive*: gr-qc/9711052
Barrett J., Crane L. (1997) *e-Print Archive*: gr-qc/9709028
Baez J. (1997) *e-Print Archive*: gr-qc/9709052

[28] Baez J. and Sawin S. (1995) *e-Print Archive*: q-alg/9507023

[29] Louko J. and Sorkin R. D. (1997) *Class. Quant. Grav.* **14** 179