Convex Floating Bodies of Equilibrium

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Abstract

We study a long standing open problem by Ulam, which is whether the Euclidean ball is the unique body of uniform density which will float in equilibrium in any direction. We answer this problem in the class of origin symmetric n-dimensional convex bodies whose relative density to water is $\frac{1}{2}$. For $n = 3$, this result is due to Falconer.

1 Introduction and results

1.1 Ulam floating bodies

A long standing open problem asked by Ulam in [16] (see also [9], Problem 19), is whether the Euclidean ball is the unique body of uniform density which floats in a liquid in equilibrium in any direction. We call such a body Ulam floating body.

A two-dimensional counterexample was found for relative density $\rho = \frac{1}{2}$ by Auerbach [2] and for densities $\rho \neq \frac{1}{2}$ by Wegner [17]. These counterexamples are not origin symmetric. For higher dimension, Wegner obtained results for non-convex bodies (holes in the body are allowed) in [18]. The problem remains largely open in the class of convex bodies in higher dimension. In order to study Ulam floating bodies, we use the notion of the convex floating body, which was introduced independently by Bárány and Larman [3] and by Schütt and Werner [13]. Let $K$ be a convex body in $\mathbb{R}^n$ and let $\delta \in \mathbb{R}$, $0 \leq \delta \leq \frac{1}{2}$. Then the convex floating body $K_\delta$ is defined as

$$K_\delta = \bigcap_{u \in S^{n-1}} H_{\delta,u}^-.$$

Here $H_{\delta,u}^+$ is the halfspace with outer unit normal vector $u$, which “cuts off” a $\delta$ proportion of $K$, i.e. $\text{vol}_n (K \cap H_{\delta,u}^+) = \delta \text{vol}_n(K)$. The convex floating body is a

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natural variation of Dupin’s floating body $K_{[\delta]}$ (see [4]). $K_{[\delta]}$ is this convex body contained in $K$ such that every support hyperplane cuts off a set of volume $\delta \vol_n(K)$ exactly. In general $K_{[\delta]}$ need not exist. An example is e.g., the simplex in $\mathbb{R}^n$. Dupin showed that if it exists, each supporting hyperplane $H$ touches $K_{[\delta]}$ at the centroid of $K \cap H$. If the floating body $K_{[\delta]}$ exists, it is equal to the convex floating body $K_{\delta}$. It was shown in [10] that for a symmetric convex body $K$, one has $K_{[\delta]} = K_{\delta}$.

We recall the relation between the density $\rho$ and the volume $\delta \vol_n(K)$ that is cut off. If the liquid has density 1 and the body $K$ has unit volume and density $\rho$, then by Archimedes’ law the submerged volume equals the total mass of the body, i.e. $\rho$, and consequently the floating part has volume $\delta = 1 - \rho$.

In [7], the authors defined the metronoid $M_\delta(K)$ of a convex body $K$ to be the body whose boundary consists of centroids of the floating parts of $K$, i.e. $K \cap H_{\delta,u}^+$. More precisely, denoting $X_{K,\delta}(u) = \frac{1}{\delta \vol_n(K)} \int_{K \cap H_{\delta,u}^+} x \, dx$, they defined $M_\delta(K)$ by

$$\partial M_\delta(K) = \{X_{K,\delta}(u) : u \in S^{n-1}\},$$

and showed that $X_{K,\delta} : S^{n-1} \rightarrow \partial M_\delta(K)$ is the Gauss map of $M_\delta(K)$, i.e. the normal to $M_\delta(K)$ at $X_{K,\delta}(u)$ is $u$. Huang, Slomka and Werner showed that $K$ is an Ulam floating body if and only if $M_\delta(K)$ is a ball (see [8, Section 2.2] for details). We utilize this characterization in our proof of Theorem 1.1.

1.2 Main results

We present two results concerning Ulam’s problem. We first establish a relation between Ulam floating bodies and a uniform isotropicity property of sections. Our second result provides a short proof of a known answer to Ulam’s problem in the class of symmetric convex bodies with relative density $\rho = 1/2$.

In the next theorem, and elsewhere, we use the notation $g(B)$ for the centroid of a set $B$.

**Theorem 1.1.** Let $\delta \in (0, \frac{1}{2}]$ and let $K \subset \mathbb{R}^n$ be a convex body such that $K_{\delta}$ is $C^1$ or reduces to a point. Then $K$ is an Ulam floating body if and only if there exists $R > 0$ such that for all $u \in S^{n-1}$ and $v \in S^{n-1} \cap u^\perp$,

$$\int_{K \cap H_{\delta,u}^+} \langle x, v \rangle^2 - \langle g(K \cap H_{\delta,u}), v \rangle^2 \, dx = \delta \vol_n(K)R.$$ (1)

In that case, $M_\delta(K)$ is a ball of radius $R$.

**Remark.** Note that if $K_{\delta}$ reduces to a point, which without loss of generality we can assume to be 0, then the condition (1) reduces to

$$\int_{K \cap H_{\delta,u}^+} \langle x, v \rangle^2 \, dx = \delta \vol_n(K)R.$$ (2)
We use the characterization given in Theorem 1.1 to give a short proof of the following result which was proved in dimension 3 by Falconer [5]. It also follows from a result in [11].

**Theorem 1.2.** Let \( K \subset \mathbb{R}^n \) be a symmetric convex body of volume 1 and density \( \frac{1}{2} \). If \( K \) is an Ulam floating body, then \( K \) is a ball.

## 2 Background

We collect some definitions and basic results that we use throughout the paper. For further facts in convex geometry we refer the reader to the books by Gardner [6] and Schneider [12].

The radial function \( r_{K,p} : S^{n-1} \to \mathbb{R}^+ \) of a convex body \( K \) about a point \( p \in \mathbb{R}^n \) is defined by

\[
r_{K,p}(u) = \max\{\lambda \geq 0 : \lambda u \in K - p\}.
\]

If \( 0 \in \text{int}(K) \), the interior of \( K \), we simply write \( r_K \) instead of \( r_{K,0} \).

Let \( K \subset \mathbb{R}^n \) be a convex body containing a strictly convex body \( D \) in its interior, and let \( H \) be a hyperplane supporting \( D \) at a point \( p \). If \( u \) is the outer unit normal vector at \( p \), we denote the restriction of the radial function \( r_{K \cap H,p} \) to \( S^{n-1} \cap u^\perp \) by \( r_{K,D}(u, \cdot) \).

We denote by \( B^n_2 \) the Euclidean unit ball centered at 0 and by \( S^{n-1} = \partial B^n_2 \) its boundary. The spherical Radon transform \( R : C(S^{n-1}) \to C(S^{n-1}) \) is defined by

\[
Rf(u) = \int_{u^\perp \cap S^{n-1}} f(x)dx \tag{3}
\]

for every \( f \in C(S^{n-1}) \).

### 2.1 Some results on floating bodies

Since \( \delta > \frac{1}{2} \) implies \( K_\delta = \emptyset \), we restrict our attention to the range \( \delta \in [0, \frac{1}{2}] \). It was shown in [14] that there is \( \delta_c, 0 < \delta_c \leq \frac{1}{2} \) such that \( K_{\delta_c} \) reduces to a point. It can happen that \( \delta_c < \frac{1}{2} \). An example is the simplex.

In fact, Helly’s Theorem (and a simple union bound) implies that \( \delta_c > \frac{1}{n+1} \), so we have \( \delta_c \in \left( \frac{1}{n+1}, \frac{1}{2} \right] \).

As mentioned above, when Dupin’s floating body \( K_{[\delta]} \) exists, it coincides with the convex floating body \( K_\delta \). The following lemma states that existence of \( K_{[\delta]} \) is also guaranteed by smoothness of \( K_\delta \). We use this for Theorem 1.1.
Lemma 2.1. If $K_\delta$ is $C^1$, then $K_{[\delta]}$ exists and $K_\delta = K_{[\delta]}$.

Proof. Let $x \in \partial K_\delta$. By [14], there is at least one hyperplane $H$ through $x$ that cuts off exactly $\delta \text{vol}_n(K)$ from $K$ and this hyperplane touches $\partial K_\delta$ in the barycenter of $H \cap K$. As $K_\delta$ is $C^1$, the hyperplane at every boundary point $x \in K_\delta$ is unique. Thus $K_{[\delta]}$ exists and $K_\delta = K_{[\delta]}$.

We know that $K_{1/2} = \{0\}$ for every centrally symmetric convex body $K$. The Homothety Conjecture [14] (see also [15, 19]), states that the homothety $K_\delta = t(\delta)K$ only occurs for ellipsoids. We treat the following conjecture which has a similar flavor.

Conjecture 2.2. Let $K \subset \mathbb{R}^n$ be a convex body and let $\delta \in (0, \frac{1}{2})$. If $K_\delta$ is a Euclidean ball, then $K$ is a Euclidean ball.

We now prove the two dimensional case of the conjecture.

Theorem 2.3. Let $K \subset \mathbb{R}^2$ and suppose there is $\delta \in (0, \frac{1}{2})$ such that $K_\delta = rB_2^2$. Then $K = R B_2^2$, for some $R > 0$.

Proof. We shall prove that the radial function $r_K : S^1 \to \mathbb{R}$ is constant. If the continuous function $r_K$ is not constant, it must attain some value $R > r$ such that the angle $\theta = \arccos \left( \frac{r}{R} \right) \in \left( 0, \frac{\pi}{2} \right)$ is not a rational multiple of $\pi$. Let $u_1 \in S^1$ be the point with $r_K(u_1) = R$, and let $\{u_i\}_{i=1}^\infty$ be the arithmetic progression on $S^1$ with difference $2\theta$. We claim that $r_K$ is constant on $\{u_i\}$. Indeed, assuming $Ru_i \in \partial K$, we consider the triangle with vertices $O$, $Ru_i$, $Ru_{i+1}$ (see the figure below). The edge $[Ru_i, Ru_{i+1}]$ is tangent to $K_\delta = rB_2^2$ at its midpoint $m_i$, and since $K_\delta$ is smooth, the chord on $\partial K$ containing $Ru_i$ and $m_i$ is bisected by $m_i$, which implies $Ru_{i+1} \in \partial K$, i.e., $r_K(u_{i+1}) = R$. Since $\theta$ is not a rational multiple of $\pi$, the sequence $\{u_i\}$ is dense in $S^1$. Since $r_K$ is constant on a dense set and continuous, it is constant on $S^1$, as required.
3 Proof of the main theorems

3.1 Proof of Theorem 1.1

Proof. We first treat the case $n = 2$. Also, we first consider when $K_\delta$ reduces to a point. Without loss of generality we can assume that this point is 0. Then we have for all $u \in S^1$ that $g(K \cap H_\delta,u) = 0$ by Dupin and thus $\langle g(K \cap H_\delta,u), v \rangle = 0$, for all $v \in u^\perp \cap S^1$ and the condition reduces to $\int_{K \cap H_\delta,u} \langle x, v \rangle^2 dx = C$. This observation is true in all dimensions.

Let $u \in S^1$. Let $v \in u^\perp \cap S^1 = H_\delta,u \cap S^1$. Then, as $H_\delta,u = \text{span}\{v\}$, we get for all $v \in S^1$,

$$\int_{K \cap H_\delta,u} \langle x, v \rangle^2 dx = \int_{-r_K(v)}^{r_K(v)} x^2 dx = \frac{2}{3} r_K(v)^3,$$

as $r_K(v) = r_K(-v)$. Hence if $\int_{K \cap H_\delta,u} \langle x, v \rangle^2 dx = C$, then we get that for all $v \in S^1$ that $r_K(v) = C_1$, and hence $K$ is a Euclidean ball and therefore also $M_\delta(K)$ is a Euclidean ball.

For the other direction, we fix $u \in S^1$. We may assume that $u = (1,0)$, i.e., in polar coordinates $u_0$ corresponds to $\theta = 0$ and $r(\theta) = 1$. For $\phi$ small, let $w = (\cos \phi, \sin \phi)$ and define the sets

$$E_1 = H_{\delta,u}^+ \cap H_{\delta,w}^+ \cap K, \quad E_2 = H_{\delta,u}^+ \cap H_{\delta,w}^- \cap K, \quad E_3 = H_{\delta,u}^- \cap H_{\delta,w}^+ \cap K.$$
In order to compute the derivative of the boundary curve of \( M_\delta(K) \) we write

\[
\delta \text{vol}_2(K) \cdot [X_{K,\delta}(w) - X_{K,\delta}(u)] = \int_{E_1 \cup E_3} x \, dx - \int_{E_2} x \, dx = \int_{E_3} x \, dx - \int_{E_2} x \, dx = \int_{E_1 \cup E_2} x \, dx
\]

Thus

\[
\frac{d}{d\phi} [X_{K,\delta}(w) - X_{K,\delta}(u)] = \frac{2}{3 \delta \text{vol}_2(K)} r_K \left( \phi + \frac{\pi}{2} \right)^3 (- \sin \phi, \cos \phi)
\]

and hence

\[
\left| \frac{d}{d\phi} [X_{K,\delta}(w) - X_{K,\delta}(u)] \right| = \frac{2}{3 \delta \text{vol}_2(K)} r_K \left( \phi + \frac{\pi}{2} \right)^3,
\]

where \( |\cdot| \) denotes the Euclidean norm. With \( z = w^+ \), we get from (4) and (5) that

\[
\left| \frac{d}{d\phi} [X_{K,\delta}(w) - X_{K,\delta}(u)] \right| = \frac{1}{\delta \text{vol}_2(K)} \int_{K \cap H_{\delta,w}} \langle x, z \rangle^2 \, dx.
\]

If \( M_\delta(K) \) is a Euclidean ball with radius \( R \), we write \( X_{K,\delta} (\cos \phi, \sin \phi) = R (\cos \phi, \sin \phi) \) in polar coordinates and get from (6)

\[
\frac{1}{\delta \text{vol}_2(K)} \int_{K \cap H_{\delta,w}} \langle x, z \rangle^2 \, dx = \left| \frac{d}{d\phi} [X_{K,\delta}(w) - X_{K,\delta}(u)] \right| = \left| \frac{d}{d\phi} [X_{K,\delta}(w)] \right| = R.
\]

To treat the case when \( K_\delta \) does not consist of just one point, we introduce the following coordinate system for the complement of an open, strictly convex body \( D \subset \mathbb{R}^2 \) with smooth boundary (see also e.g., [20]). Let \( \gamma : [0, 2\pi] \to \partial D \) be the inverse Gauss map, and \( T : [0, 2\pi] \to S^1 \) be the unit tangent vector to the curve at \( \gamma(\theta) \), oriented counterclockwise, i.e.,

\[
n(\theta) := n_D(\gamma(\theta)) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad T(\theta) = \frac{\gamma'(\theta)}{|\gamma'(\theta)|} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}.
\]

The coordinate system \( F : \mathbb{R} \times [0, 2\pi] \to \mathbb{R}^2 \setminus D \) is defined by

\[
F(r, \theta) = \gamma(\theta) + r T(\theta).
\]
Since $\frac{\partial F}{\partial r} = T$ and $\frac{\partial F}{\partial \theta} = \gamma' - rn$, the Jacobian of $F$ is given by $|r|$. Now we fix $0 < \delta < 1/2$ and assume that $K_\delta$ does not just consist of one point. We then set $D = \text{int}(K_\delta)$, which has smooth boundary by assumption. Without loss of generality, we can assume that $0 \in \text{int}(K_\delta)$. It was shown in [14] that $K_\delta$ is strictly convex. Let $u \in S^1$, and assume without loss of generality that $u = n(0) = (1, 0)$. Let $w = n(\phi)$ for an angle $\phi > 0$ small enough, such that the lines $H_{\delta,u}$ and $H_{\delta,w}$ intersect in the interior of $K$. Define the sets

$$E_1 = H_{\delta,u}^+ \cap H_{\delta,w}^+ \cap K, \quad E_2 = H_{\delta,u}^+ \cap H_{\delta,w}^- \cap K, \quad E_3 = H_{\delta,u}^- \cap H_{\delta,w}^+ \cap K,$$

and let $E_4$ be the bounded connected component of $(H_{\delta,u}^- \cap H_{\delta,w}^-) \setminus K_\delta$, see Figure.

![Figure](image_url)

Again, in order to compute the derivative of the boundary curve of $M_\delta(K)$ we write

$$\delta \text{ vol}_2(K) \cdot [X_{K,\delta}(w) - X_{K,\delta}(u)] = \int_{E_1 \cup E_3} x \, dx - \int_{E_1 \cup E_2} x \, dx$$

$$= \int_{E_3 \cup E_4} x \, dx - \int_{E_2 \cup E_4} x \, dx.$$
Now we use the definition of $F$. Thus

$$X_{K,\delta}(w) - X_{K,\delta}(u) = \frac{1}{\delta \, \text{vol}_2(K)} \int_0^\phi \int_{-r_{K,\delta}(\theta)}^{r_{K,\delta}(\theta)} r\gamma(\theta) + r^2T(\theta) \, dr \, d\theta = \frac{1}{\delta \, \text{vol}_2(K)} \int_0^\phi \frac{2}{3} r^3_{K,\delta}(\theta) \, T(\theta) \, d\theta.$$

As $M_\delta(K)$ is strictly convex and $C^1$ by [8], we can differentiate with respect to $\phi$ and get,

$$\frac{dX_{K,\delta}(n(\phi))}{d\phi} = \frac{2}{3} \frac{r^3_{K,\delta}(\phi)}{\delta \, \text{vol}_2(K)} \, T(\phi). \quad (8)$$

On the other hand, for any $\theta \in [0, 2\pi]$,

$$\int_{K \cap H_{\delta,n(\theta)}} \langle x, T(\theta) \rangle^2 \, dx = \int_{-r_{K,\delta}(\theta)}^{r_{K,\delta}(\theta)} \langle \gamma(\theta) + rT(\theta), T(\theta) \rangle^2 \, dr = 2r_{K,\delta}(\theta) \, \langle \gamma(\theta), T(\theta) \rangle^2 + 2\langle \gamma(\theta), T(\theta) \rangle \int_{-r_{K,\delta}(\theta)}^{r_{K,\delta}(\theta)} r \, dr + \int_{-r_{K,\delta}(\theta)}^{r_{K,\delta}(\theta)} r^2 \, dr$$

$$= \langle g(K \cap H_{\delta,n(\theta)}), T(\theta) \rangle^2 \, \text{vol}_1(K \cap H_{\delta,n(\theta)}) + \frac{2}{3} \frac{r^3_{K,\delta}(\theta)}{\delta}, \quad (9)$$

since $\gamma(\theta)$ is the centroid $g(K \cap H_{\delta,n(\theta)})$ of $K \cap H_{\delta,n(\theta)}$. Combining (8) and (9), we get that for $\theta = \phi$,

$$\left| \frac{dX_{K,\delta}(n(\phi))}{d\phi} \right| = \frac{1}{\delta \, \text{vol}_2(K)} \int_{K \cap H_{\delta,n(\phi)}} \langle x, T(\phi) \rangle^2 - \langle g(K \cap H_{\delta,n(\phi)}), T(\phi) \rangle^2 \, dx.$$

By [7, 8], the normal to $\partial M_\delta(K)$ at $X_{K,\delta}(n(\phi))$ is $n(\phi) = (\cos \phi, \sin \phi)$ and as $M_\delta(K)$ is strictly convex and $C^1$ by [8], $\xi(\phi) = X_{K,\delta}(n(\phi))$ is a parametrization of $\partial M_\delta(K)$ with respect to the angle of the normal. The curvature is given by $\frac{d\xi}{ds}$ where $s$ is the arclength along the curve. Since $\frac{d\xi}{ds}$ is a unit vector, we get by the chain rule $\frac{d}{ds} = \frac{d}{d\phi} \frac{d\phi}{ds}$ that the radius of curvature is given by

$$R(\phi) = \left| \frac{dX_{K,\delta}(n(\phi))}{d\phi} \right| = \frac{1}{\delta \, \text{vol}_2(K)} \int_{K \cap H_{\delta,n(\phi)}} \langle x, T(\phi) \rangle^2 - \langle g(K \cap H_{\delta,n(\phi)}), T(\phi) \rangle^2 \, dx.$$

Since $M_\delta(K)$ is a disk if and only if its radius of curvature is constant, the theorem follows.

Let now $n \geq 3$.

Let $u \in S^{n-1}$ be arbitrary, but fixed, and let $v \in S^{n-1} \cap u^\perp$. We denote by $W = \text{span}\{u, v\}$ the span of $u$ and $v$ and by $W^\perp$ the $(n-2)$-dimensional subspace that is the orthogonal complement of $W$. $\bar{K} = K|W$ is the orthogonal projection of the convex
body $K$ onto the 2-dimensional subspace $W$. For a small $\phi$, let $w = \cos \phi \ u + \sin \phi \ v$. We define $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ as follows,

$$
\bar{E}_1 = (H_{\delta,u}^+ \cap H_{\delta,w}^+)|W, \quad \bar{E}_2 = (H_{\delta,u}^+ \cap H_{\delta,\eta}^-)|W, \quad \bar{E}_3 = (H_{\delta,u}^- \cap H_{\delta,w}^+)|W
$$

and $\bar{E}_4$ is the curvilinear triangle enclosed by $H_{\delta,u}|W$, $H_{\delta,w}|W$, and the boundary of $\bar{K}_{\delta} = K_{\delta}|W$. Then the picture is identical to the previous Figure. We let

$$
E_i = \bar{E}_i \times W^\perp, \quad \text{for } i = 1, 2, 3, 4.
$$

When $K_{\delta}$ reduces to a point, we can assume without loss of generality that $K_{\delta} = \{0\}$. As noted before, the condition then reduces to $\int_{K \cap H_{\delta,u}} (x, v)^2 \, dx = C$. In this case $\bar{E}_4 = \emptyset$ and the proof continues along the same lines as below. Alternatively, one can replace the coordinate system (7) by the usual polar coordinates in $W$ as it was done in the case $n = 2$.

In the general case we thus have that

$$
\begin{align*}
\delta \, \text{vol}_n(K) [X_{K,\delta}(w) - X_{K,\delta}(u)] &= \int_{K \cap H_{\delta,w}^+} x \, dx - \int_{K \cap H_{\delta,w}^-} x \, dx \\
&= \int_{K \cap (E_1 \cup E_3)} x \, dx - \int_{K \cap (E_1 \cup E_2)} x \, dx = \int_{K \cap E_3} x \, dx - \int_{K \cap E_2} x \, dx \\
&= \int_{K \cap (E_3 \cup E_4)} x \, dx - \int_{K \cap (E_2 \cup E_4)} x \, dx.
\end{align*}
$$

For $x \in W$, we consider the following parallel section function,

$$
A_{K,W}(x) = \text{vol}_{n-2} \left( K \cap \{W^\perp + x\} \right)
$$

and observe that by Fubini,

$$
\text{vol}_n(K) [X_{K,\delta}(w)] = \int_{K \cap H_{\delta,w}^+} z \, dz = \int_{K} \left( \int_{(x+W^\perp) \cap K \cap H_{\delta,w}^+} y \, dy \right) \, dx.
$$

We denote by $g(B) = \frac{1}{\text{vol}_n(B)} \int_B y \, dy$ the centroid of the set $B$. Then we get

$$
\begin{align*}
\delta(X_{K,\delta}(w)|W &= \left( \int_{K \cap (E_3 \cup E_4)} x \, dx \right)|W = \left( \int_{K \cap (E_3 \cup E_4)} \left( \int_{(x+W^\perp) \cap K} y \, dy \right) \, dx \right)|W \\
&= \left( \int_{K \cap (E_3 \cup E_4)} A_{K,W}(x) \, g((x+W^\perp) \cap K) \, dx \right)|W \\
&= \int_{K \cap (E_3 \cup E_4)} A_{K,W}(x) \left( g((x+W^\perp) \cap K) \right) \, dx \\
&= \int_{K \cap (E_3 \cup E_4)} A_{K,W}(x) \, x \, dx,
\end{align*}
$$
and similarly for $\delta \text{vol}_n(K)(X_{K,\delta}(u))|W$. Now we will use the coordinate system 
$F : \mathbb{R} \times [0, 2\pi] \to \mathbb{R} \setminus \text{int} \left( \tilde{K}_\delta \right)$, introduced earlier in \textcolor{red}{[7]},

$$F(r, \theta) = \gamma(\theta) + r T(\theta).$$

We can assume that $n(0) = u$. Then $T(0) = v$. We recall that the Jacobian of $F$ is given by $|r|$. We abbreviate $n = n(\theta)$ and $T = T(\theta)$. We let $r_{K,\tilde{K}_\delta}(n(\theta)), T(\theta)) = r_{K,\tilde{K}_\delta}(n, T) > 0$ be such that $\gamma(\theta) + r_{K,\tilde{K}_\delta}(n, T) T(\theta) \in \partial \tilde{K}$, and $r_{K,\tilde{K}_\delta}(n, -T) > 0$ be such that $\gamma(\theta) + r_{K,\tilde{K}_\delta}(n, T) (-T(\theta)) \in \partial \tilde{K}$. We get

$$\delta \text{vol}_n(K) (X_{K,\delta}(w) - X_{K,\delta}(u)) |W \ni \int_0^\phi \int_{r_{K,\tilde{K}_\delta}(n, T)}^{r_{K,\tilde{K}_\delta}(n, T)} F(r, \theta) A_{K,W}(F(r, \theta)) |r| dr d\theta$$

$$- \int_0^\phi \int_{-r_{K,\tilde{K}_\delta}(n, -T)}^{r_{K,\tilde{K}_\delta}(n, -T)} F(r, \theta) A_{K,W}(F(r, \theta)) |r| dr d\theta$$

$$= \int_0^\phi \int_{r_{K,\tilde{K}_\delta}(n, T)}^{r_{K,\tilde{K}_\delta}(n, T)} F(r, w) A_{K,W}(F(r, \theta)) r dr d\theta$$

$$+ \int_0^\phi \int_{-r_{K,\tilde{K}_\delta}(n, -T)}^{r_{K,\tilde{K}_\delta}(n, -T)} F(r, \theta) A_{K,W}(F(r, \theta)) r dr d\theta$$

$$= \int_0^\phi \int_{-r_{K,\tilde{K}_\delta}(n, -T)}^{r_{K,\tilde{K}_\delta}(n, T)} F(r, \theta) A_{K,W}(F(r, \theta)) r dr d\theta.$$

We differentiate with respect to $\phi$,

$$\delta \text{vol}_n(K) \frac{d}{d\phi} (\langle X_{K,\delta}(w) - X_{K,\delta}(u) \rangle |W \rangle) = \delta \text{vol}_n(K) \frac{d}{d\phi} \langle (X_{K,\delta}(w)) \rangle |W \rangle$$

$$= \int_{-r_{K,\tilde{K}_\delta}(n(\phi), -T(\phi))}^{r_{K,\tilde{K}_\delta}(n(\phi), T(\phi))} F(r, \phi) A_{K,W}(F(r, \phi)) r dr.$$

Putting $\phi = 0$, results in

$$\delta \text{vol}_n(K) \frac{d}{d\phi} (\langle X_{K,\delta}(w) \rangle |W \rangle \bigg|_{\phi=0} = \int_{-r_{K,\tilde{K}_\delta}(n, v)}^{r_{K,\tilde{K}_\delta}(u, v)} F(r, 0) A_{K,W}(F(r, 0)) r dr$$

$$= \int_{-r_{K,\tilde{K}_\delta}(u, v)}^{r_{K,\tilde{K}_\delta}(u, v)} [\gamma(0) + rv] A_{K,W}(F(r, 0)) r dr$$

$$= \int_{-r_{K,\tilde{K}_\delta}(u, v)}^{r_{K,\tilde{K}_\delta}(u, v)} \gamma(0) A_{K,W}(F(r, 0)) r dr + \int_{-r_{K,\tilde{K}_\delta}(u, v)}^{r_{K,\tilde{K}_\delta}(u, v)} r^2 v A_{K,W}(F(r, 0)) dr$$

$$ v \int_{-r_{K,\tilde{K}_\delta}(u, v)}^{r_{K,\tilde{K}_\delta}(u, v)} r^2 A_{K,W}(F(r, 0)) dr, \quad (11)$$

where the last equality holds by Dupin since $H_{\delta,u} \cap K_\delta$ is the centroid of $H_{\delta,u} \cap K$, i.e.

$$\int_{-r_{K,\tilde{K}_\delta}(u, v)}^{r_{K,\tilde{K}_\delta}(u, v)} \gamma(0) A_{K,W}(F(r, 0)) r dr = 0. \quad (12)$$
Indeed, in the coordinate system (7), the centroid of $H_{\delta,u} \cap K$ is $\gamma(0)$. Thus, with coordinate system (7) we get as above

$$\text{vol}_{n-1}(K \cap H_{\delta,u}) \langle v, \gamma(0) \rangle = \int_{K \cap H_{\delta,u}} \langle v, x \rangle \, dx$$

$$= \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} \langle \gamma(0) + rv, v \rangle \, A_{K,W}(F(r,0)) \, dr$$

$$= \langle \gamma(0), v \rangle \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} A_{K,W}(F(r,0)) \, dr + \langle v, v \rangle \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} r \, A_{K,W}(F(r,0)) \, dr$$

$$= \langle \gamma(0), v \rangle \text{vol}_{n-1}(K \cap H_{\delta,u}) + \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} r \, A_{K,W}(F(r,0)) \, dr.$$  

On the other hand, again in the coordinate system (7), and also using (12),

$$\int_{K \cap H_{\delta,u}} \langle x, v \rangle^2 \, dx = \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} \langle \gamma(0) + rv, v \rangle^2 \, A_{K,W}(F(r,0)) \, dr$$

$$= \langle \gamma(0), v \rangle^2 \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} A_{K,W}(F(r,0)) \, dr + \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} r^2 \, A_{K,W}(F(r,0)) \, dr$$

$$= \langle \gamma(0), v \rangle^2 \text{vol}_{n-1}(K \cap H_{\delta,u}) + \int_{-r_{K,\delta}(u,v)}^{r_{K,\delta}(u,v)} r^2 \, A_{K,W}(F(r,0)) \, dr.$$  

As $w = \cos \phi \, u + \sin \phi \, v$, it follows from (11) and (13) that

$$\left| \frac{d}{d\phi} \left( X_{K,\delta}(\cos \phi \, u + \sin \phi \, v) \right) \right|_{\phi=0} = \frac{1}{\delta \text{vol}(K)} \int_{K \cap H_{\delta,u}} \langle \langle x, v \rangle^2 - \langle \gamma(0), v \rangle^2 \rangle \, dx$$

$$= \frac{1}{\delta \, \text{vol}(K)} \int_{K \cap H_{\delta,u}} \langle \langle x, v \rangle^2 - \langle g(K \cap H_{\delta,u}), v \rangle^2 \rangle \, dx.$$  

Observe that in the case when $K_{\delta} = \{0\}$, $\langle g(K \cap H_{\delta,u}), v \rangle = 0$. We have that $w = n(\phi) = \cos \phi \, u + \sin \phi \, v \in W$ is the outer unit normal to $M_{\delta}(K)$ in $X_{\delta,K}(w)$. Therefore, $w$ is the outer unit normal to $M_{\delta}(K)|W$ in $X_{\delta,K}(w)|W$. Again, as $M_{\delta}(K)$ and therefore $M_{\delta}(K)|W$ is strictly convex and $C^1$ by [8], $X_{\delta,K}(n(\phi))$ is a parametrization of the boundary of $M_{\delta}(K)|W$ with respect to the angle of the normal. Thus the curvature of $M_{\delta}(K)|W$ is constant, which implies that $M_{\delta}(K)|W$ is a disk. Since $W$ is arbitrary, we get that every two dimensional projection of $M_{\delta}$ is a disk, and it follows that $M_{\delta}(K)$ is a Euclidean ball ([9], Corollary 3.1.6).
3.2 Proof of Theorem 1.2

Proof. Since \( K \) is symmetric and has volume 1 and density \( \rho = \frac{1}{2} \), we have that \( \delta = \frac{1}{2} \), as noted above. Therefore, \( K_{\frac{1}{2}} = K_{\{\frac{1}{2}\}} = \{0\} \). Since \( K \) is an Ulam floating body, the remark after Theorem 1.1 implies that for any \( u \in S^{n-1} \) and \( v \in u^\perp \cap S^{n-1} \),

\[
\int_{u^\perp \cap K} \langle x, v \rangle^2 \, dx = C,
\]

for some constant \( C \). Let \( u \in S^{n-1} \) be arbitrary, but fixed. We pass to polar coordinates in \( u^\perp \) and get for all \( v \in u^\perp \cap S^{n-1} \),

\[
C = \int_{u^\perp \cap S^{n-1}} \int_{t=0}^{r_K(\xi)} t^n \langle \xi, v \rangle^2 \, dt \, d\sigma(\xi) = \frac{1}{n+1} \int_{u^\perp \cap S^{n-1}} r_K(\xi)^{n+1} \langle \xi, v \rangle^2 \, d\sigma(\xi).
\]

Now we integrate over all \( v \in u^\perp \cap S^{n-1} = S^{n-2} \) w.r. to the normalized Haar measure \( \mu \) on \( S^{n-2} \). We use that \( \int_{S^{n-2}} \langle \xi, v \rangle^2 \, d\mu(v) = c_n \| \xi \| = c_n \), where \( c_n = 2 \frac{\vol_{n-2}(B^{n-2})}{\vol_{n-2}(S^{n-2})} \) and get that

\[
\frac{(n+1)C}{c_n} = \int_{u^\perp \cap S^{n-1}} r_K(\xi)^{n+1} \, d\sigma(\xi) = R r_K^{n+1}(u),
\]

where \( R \) is the spherical Radon transform [3]. We rewrite this equation as

\[
\int_{u^\perp \cap S^{n-1}} d\sigma(\xi) = \int_{u^\perp \cap S^{n-1}} \frac{2 \vol_{n-2}(B^{n-2})}{(n+1)C} r_K(\xi)^{n+1} \, d\sigma(\xi),
\]

or

\[
0 = \int_{u^\perp \cap S^{n-1}} \left( \frac{2 \vol_{n-1}(B^{n-2})}{(n+1)C} r_K(\xi)^{n+1} - 1 \right) \, d\sigma(\xi).
\]

As \( u \) was arbitrary and as \( r_K \) is even, it then follows from e.g., Theorem C.2.4 of [6] that \( r_K = \text{const.} \) for \( \sigma \) almost all \( u \) and as \( r_K \) is continuous, \( r_K = \text{const.} \) on \( S^{n-1} \). Thus \( K \) is a ball. \( \square \)

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