Log ordinarity and log plurigenera of a proper SNCL scheme over a log point in characteristic $p > 0$

Yukiyoshi Nakkajima *

Abstract

In this article we give two applications of the spectral sequence of the log Hodge-Witt cohomology of a proper SNCL scheme over the log point of a perfect field of characteristic $p > 0$.

1 Introduction

This article consists of two parts. One is a criterion of the log ordinarity of a proper SNCL (=simple normal crossing log) scheme over a family of log points in characteristic $p > 0$ and the other is the lower semi-continuity of the log geometric plurigenera and the log Iitaka-Kodaira dimension of a proper SNCL scheme over a log point of any characteristic. Though arithmetic geometers have not yet shown their interest in log geometric plurigenera and log Iitaka-Kodaira dimensions, it seems that they are important notions as in the theory of log geometric plurigenera and log Iitaka-Kodaira dimensions for the case of the “horizontal” log structure developed in [Ii1] and [Ii2].

Let $g: Y \to T$ be a morphism of fine log schemes in the sense of Fontaine-Illusie-Kato ([K]). Let $\Omega^\bullet_{Y/T}$ be the log de Rham complex of $Y/T$ (this was denoted $\omega^\bullet_{Y/T}$ in [K] and [HK]). Let $\mathcal{O}_Y$ be the underlying scheme of $Y$. Set $\Omega^i_{Y/T} := \text{Im}(d: \Omega^{i-1}_{Y/T} \to \Omega^i_{Y/T})$ ($\Omega^i_{Y/T}$ is only an abelian sheaf on $\mathcal{O}_Y$). Assume that $T$ is of characteristic $p > 0$ and that $g$ is proper and log smooth. In this article, we say that $g$ (or simply $Y/T$) is log ordinary (in the sense of Bloch-Kato-Illusie) if $R^jg_*(\Omega^i_{Y/T}) = 0$ for any $i, j \in \mathbb{Z}_{\geq 0}$. This is the variation of the log ordinarity defined in [H1], [Ii2], [Nakk1] and [L]. (The log ordinarity in [loc. cit.] is the log version of the ordinarity defined in [BK] and [IR].)

Let $\kappa$ be a perfect field of characteristic $p > 0$. Let $B$ be the spectrum of a (complete) discrete valuation ring of mixed characteristics with perfect residue field $\kappa$. Let $\mathcal{X}/B$ be a proper strict semistable family. Let $X/s$ be the log special fiber of $\mathcal{X}/B$ with canonical log structure. Let $\mathcal{W}\Omega^\bullet_X$ be the log de Rham-Witt complex of $X/s$. Let $k$ be a nonnegative integer. Let $\mathcal{X}^{(k)}$ be the disjoint union of the $(k + 1)$-fold intersections of the different irreducible components of $\mathcal{X}$. In [H1] Hyodo has claimed the following theorem without the proof of it in the case where $B$ is of mixed characteristics:

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Theorem 1.1 ([H1], [Il2]). If \( X^{(k)}/\kappa \) is ordinary for all \( k \in \mathbb{Z}_{\geq 0} \), then \( X/s \) is log ordinary.

This theorem has an important role in several articles, e.g., in [RX] (of course in [Il2]) by using Illusie’s result in [Il2]: the blow up of an ordinary proper smooth subscheme in an ordinary proper smooth scheme is ordinary. Following [Il2], we call this theorem Hyodo’s criterion. In [Il2] Illusie has given a proof of this fact including the case where \( B \) is of equal characteristic. He has stated another very simple method to prove Hyodo’s criterion under the assumption of the existence of the “\( p \)-adic weight spectral sequence” of \( H^h(X, \mathcal{W}\Omega^i_X) \) (\( h, i \in \mathbb{Z}_{\geq 0} \)) which is compatible with the operator \( F \), which was not constructed at that time. In the special case \( i = 0 \), this is classically well-known (cf. [RZS, §2]): for a proper SNC (=simple normal crossing) scheme \( Z/\kappa \), there exists the following spectral sequence:

\[
E_1^{-k,h+k} = H^{h+k}(Z^{(-k+1)}, \mathcal{W}(\mathcal{O}_{Z^{(-k+1)}})) \Rightarrow H^h(Z, \mathcal{W}(\mathcal{O}_Z)) \quad (-k, h \in \mathbb{Z}_{\geq 0}),
\]

In the general \( i \), the spectral sequence has been already constructed in [Nakk3] (cf. [Ma]):

Theorem 1.2 ([Nakk3, (4.1.1)]). Let \( s \) be the log point of \( \kappa \), that is, \( s = (\text{Spec}(\kappa), \mathbb{N} \oplus \kappa^*) \rightarrow \kappa) \). Here \( 1 \in \mathbb{N} \) goes to \( 0 \in \kappa \) by the morphism \( \mathbb{N} \oplus \kappa^* \rightarrow \kappa \). Let \( X/s \) be a proper SNC scheme defined in [Nakk5] (1.1.15) (cf. [Nakk2] (1.1.10.2)). Let \( i \) be a nonnegative fixed integer. Then there exists the following spectral sequence:

\[
E_1^{-k,h+k} = \bigoplus_{j \geq \max(-k,0)} H^{h-i-j}(\hat{X}^{(2j+k)}, \mathcal{W}\Omega^{j-k}_{\hat{X}^{(2j+k)}})(-h-k) \Rightarrow H^{h-i}(X, \mathcal{W}\Omega^i_X) \quad (q \in \mathbb{N}).
\]

Here \((-j-k)\) means the usual Tate twist with respect to the Frobenius endomorphism of \( X \).

(In [Nakk3, (4.7)] we have proved that this spectral sequence degenerates at \( E_2 \).) Because this spectral sequence is compatible with the operator \( F \), we can prove Hyodo’s criterion in a very simple way as follows by following Illusie’s idea.

By a fundamental theorem in [IR], a proper smooth scheme \( Y/\kappa \) is ordinary if and only if the operator \( F \) on \( H^h(Y, \mathcal{W}\Omega^i_Y) \) is bijective. Hence \( F \) is bijective on \( H^{h-i}(X, \mathcal{W}\Omega^i_X) \) by (1.2.1). Furthermore, by the log version of this fundamental theorem due to Lorenzon ([IL]), this implies that \( X/s \) is log ordinary. This is the first proof of Hyodo’s criterion in this article. We also give the proof of Hyodo’s criterion by following Illusie’s original proof in [Il2] as possible. This is the second proof of Hyodo’s criterion in this article. In this article we simplify his proof and generalize it. We give the three proofs of the following theorem (the variation of Hyodo’s criterion):

Theorem 1.3. Let \( S \) be a family of log points in characteristic \( p > 0 \) defined in [Nakk5] (1.1)]. Let \( f : X \rightarrow S \) be a proper SNC scheme defined in [Nakk5] (1.1.15). Let \( T \rightarrow S \) be a morphism of fine log schemes. Set \( X_T := X \times_S T \) and let \( f_T : X_T \rightarrow T \) be the base change of \( f \). Let \( \hat{X}^{(k)}(k \in \mathbb{Z}_{\geq 0}) \) be a scheme over \( \hat{S} \) defined in [loc. cit.]. If \( \hat{X}^{(k)}/\hat{S} \) is ordinary for all \( k \), then \( f_T \) is log ordinary.

In the case \( S = s \), the \( \hat{X}^{(k)} \) in (1.3) is equal to the \( \hat{X}^{(k)} \) in (1.1). Hence (1.3) is a generalization of (1.1). In [Il2] Illusie has defined a nonstandard Poincaré residue
isomorphism for $\Omega^1_{X/S}$ in the case $S = s$ and he has used this isomorphism for the proof of $\text{(1.3)}$ (see $\text{(1.1)}$ below for the generalization of his isomorphism). In this article we prefer to use only a standard Poincaré residue isomorphism for $\Omega^1_{X/S}$ in $\text{[Nakk5]}$ (1.3.14)] when we give the second proof of Hyodo’s criterion. The key lemma for the second and the third proofs of $\text{(1.3)}$ is the following simple one:

**Lemma 1.4 (Key lemma).** Let $\theta$ be a global section of $\Omega^1_{X/S}$ defined in $\text{[Nakk5]}$ (1.3)]. Then the following sequence

$$0 \rightarrow B\Omega^{i-1}_{X/S} \xrightarrow{\theta^\wedge} B\Omega^i_{X/S} \rightarrow B\Omega^i_{X/S} \rightarrow 0 \quad (i \in \mathbb{N})$$

of $f^{-1}(\mathcal{O}_S)$-modules is exact.

The proof of this lemma is very easy. The second proof of $\text{(1.3)}$ consists of the following two steps. By using the assumption in $\text{(1.3)}$, the standard Poincaré residue isomorphism for $\Omega^1_{X/S}$ and a variant of the log Cartier isomorphism, we first prove that $R^j f_*(B\Omega^i_{X/s}) = 0$ for $\forall i, j \in \mathbb{N}$. Using induction on $i$, we next see that $R^j f_*(B\Omega^i_{X/S}) = 0$ (and $R^j f_*(B\Omega^i_{X/T/S}) = 0$) very immediately.

Let $\mathcal{W}\Omega^1_X$ be the log de Rham-Witt complex of $X/s$ in the case $S = s$ defined in $\text{[Mo]}$ and $\text{[Nakk3]}$. The third proof of $\text{(1.3)}$ uses $\text{(1.3)}$ for the case $S = s$ and uses a spectral sequence of $H^j(X, \mathcal{W}\Omega^1_X)$ instead of $\text{(1.2.1)}$. (This spectral sequence has been constructed in $\text{[Nakk5]}$ and it is compatible with the operator $F$. ) The third proof is the shortest one and makes us feel that the proof of $\text{(1.3)}$ is obvious if we notice $\text{(1.4)}$.

Set $X_T := X \times_{s} T$. By generalizing the second proof of $\text{(1.3)}$, we can generalize $\text{(1.3)}$ for a certain integrable connection

$$\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{X_T}} \Omega^1_{X_T/S}$$

where $\mathcal{E}$ is a flat quasi-coherent $\mathcal{O}_{X_T}$-module such that $\mathcal{E}$ is “locally generated by horizontal sections of $\nabla$”. $\text{(1.3)}$ is a special case of the generalized theorem for the case where $(\mathcal{E}, \nabla)$ is the usual derivative $d: \mathcal{O}_{X_T} \rightarrow \Omega^1_{X_T/S}$. See $\text{(3.1)}$ and $\text{(3.15)}$ below for this generalized statement. An example of our $\mathcal{E}$’s is given in the following way. For a log scheme $Y$ of characteristic $p > 0$, let $F_Y: Y \rightarrow Y$ be the Frobenius endomorphism of $Y$. Set $X[p]_T := X \times_{s} T_{-1/F} T$ and let $F: X_T \rightarrow X_T$ be the induced morphism by $F_{X_T}$. Let $\mathcal{E}'$ be a quasi-coherent $\mathcal{O}_{X_T}$-module. Set $\mathcal{E} := F^* (\mathcal{E}) := F^{-1}(\mathcal{E}') \otimes_{\mathcal{O}_{X_T}} \mathcal{O}_{X_T}$ and consider a morphism $\text{id}_{F^{-1}(\mathcal{E})} \otimes d: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_{X_T}} \Omega^1_{X_T/S}$ of $f^{-1}(\mathcal{O}_T)$-modules. It is easy to check that $\text{id}_{F^{-1}(\mathcal{E})} \otimes d$ is an integrable connection on $\mathcal{E}$. The connection $(\mathcal{E}, \text{id}_{F^{-1}(\mathcal{E})} \otimes d)$ is an example of the integrable connection in our mind.

In the second part of this article, we introduce new invariants of a proper SNCL scheme $X$ over the log point $s$ of $\kappa$ of pure dimension $d$: the log (geometric) plurigenera of $X/s$ and the log Iitaka-Kodaira dimension of it.

Set $p_g(X/s, n, 1) := \text{length}_{\mathcal{W}X} H^0(X, \mathcal{W}X, \Omega^d_X)$. We call this the log geometric genus of $X/s$ of level $n$. By using the finite length version of $\text{(1.2.1)}$, we prove the following:
Theorem 1.5. \( p_g(\hat{\mathcal{X}}(1)/\kappa, n, 1) \leq p_g(X/s, n, 1) \).

We call this inequality the lower semi-continuity of the log geometric genus of \( X/s \).
This inequality bounds the number of the irreducible components \( \hat{X}_\lambda \)'s of \( \hat{X} \) such that \( p_g(\hat{X}, n, 1) \geq 1 \).

More generally, set \( p_g(X/s, n, r) := \text{length}_{\mathcal{W}_n} H^0(X, (\mathcal{W}_n \Omega^d_X)^{\otimes r}) \) \((r \in \mathbb{Z}_{\geq 1})\). We call this the log (geometric) plurigenus of \( X/s \) of level \( n \). (In the following, we omit to say the adjective “geometric” in the terminology “log geometric plurigenus”.) In the case where the characteristic of the base field of \( \kappa \) is 0, we set \( p_g(X/s, r) := \dim_k H^0(X, (\Omega^d_X)^{\otimes r}) \) and we also call this the log plurigenus of \( X/s \). We would like to prove the following lower semi-continuity of the log plurigenus of \( X/s \) of level \( n \):

Conjecture 1.6.

\[
\begin{cases}
p_g(\hat{\mathcal{X}}(1)/\kappa, n, r) \leq p_g(X/s, n, r) & \text{if } \text{ch}(k) = p > 0, \\
p_g(\hat{\mathcal{X}}(1)/\kappa, r) \leq p_g(X/s, r) & \text{if } \text{ch}(k) = 0.
\end{cases}
\]

It is clear that the finite length version of (1.7.1) itself is not useful for the proof of this lower semi-continuity in the case \( r \geq 2 \). To prove this, we need a new idea: we construct the following canonical morphism of \( \mathcal{W}_n(\mathcal{O}_X) \)-modules:

(1.6.1) \( \Psi_{X,n,r} : a_*((\mathcal{W}_n \Omega^d_{\mathcal{X}(1)})^{\otimes r}) \to (\mathcal{W}_n \Omega^d_{X})^{\otimes r} \quad (n, r \in \mathbb{Z}_{\geq 1}). \)

Here \( a : \hat{\mathcal{X}}(1) \to \hat{X} \) is the natural morphism. (One may think that this morphism is a strange morphism at first glance.) We prove that \( \Psi_{X,1,r} \) is injective. However we do not know whether \( \Psi_{X,n,r} \) is injective for any \( n \in \mathbb{Z}_{\geq 2} \). In the case \( \text{ch}(k) = 0 \), we also construct the following canonical morphism of \( \mathcal{O}_X \)-modules:

(1.6.2) \( \Psi_{X,r} : a_*((\Omega^d_{\mathcal{X}(1)})^{\otimes r}) \to (\Omega^d_{X})^{\otimes r} \quad (r \in \mathbb{Z}_{\geq 1}) \)

and we prove that this morphism is injective. Consequently we obtain the following:

Theorem 1.7. Let \( \kappa \) be a field of any characteristic. Let \( s \) be the log point of \( \kappa \). Let \( X/s \) be a proper SNCL scheme of pure dimension \( d \). Let \( r \) be a positive integer. Then

(1.7.1) \( p_g(\hat{\mathcal{X}}(1)/\kappa, 1, r) \leq p_g(X/s, 1, r) \) \quad if \( \text{ch}(k) = p > 0 \)

and

(1.7.2) \( p_g(\hat{\mathcal{X}}(1)/\kappa, r) \leq p_g(X/s, r) \) \quad if \( \text{ch}(k) = 0 \).

We can also construct the obvious analogue of the morphism (1.6.2) for the analytic case and can prove the obvious analogue of the inequality (1.7.1) in this case.

This should be compared with the works of Noboru Nakayama (Nakay1), Siu (S) and Hajime Tsuji (T1) using non-algebraic methods. They have conjectured and H. Tsuji has claimed that the following holds, though he has not given the detail of the proof of the following theorem:

Theorem 1.8 (T1). Let \( \Delta \) be the unit disk. Let \( X \to \Delta \) be the analytification of a projective strict semistable family. Let \( t \) be an element of \( \Delta \setminus \{O\} \). Let \( X \) and \( X_t \) be the special fiber and the generic fiber of this family, respectively. Then

(1.8.1) \( p_g(\hat{\mathcal{X}}(1)/\mathbb{C}, r) \leq p_g(X_t/\mathbb{C}, r) \quad (r \in \mathbb{Z}_{\geq 1}). \)
An algebraic proof of (1.8) has not been known.

Because the log plurigenera is upper semi-continuous, this implies the smooth deformation invariance of the log plurigenera of a proper smooth algebraic family in the case $\text{ch}(\kappa) = 0$. On the other hand, the smooth deformation invariance of the log plurigenera of a proper smooth analytic case does not necessarily hold by [Nakam], [Ni] and [U1] and our result for the analytic case.

In the case where the proper or projective SNCL scheme is the log special fiber of a proper or projective strict semifamily over a (complete) discrete valuation ring, we would like to raise the following problem:

**Problem 1.9.** Let $\mathcal{V}$ be a (complete) discrete valuation ring of mixed characteristics or equal characteristic $p > 0$. Let $K$ be the fraction field of $\mathcal{V}$. Let $X/\mathcal{V}$ be a proper or projective strict semistable family. Let $X/s$ and $X_K$ be the log special fiber of $X/B$ and the generic fiber of $X/B$, respectively. When does the following equality hold?

$$p_g(X/s, n, r) = p_g(X_K/K, n, r) \quad (n, r \in \mathbb{Z}_{\geq 1}).$$

(1.9.1)

We think that nothing nontrivial about (1.9) has been known. In the analytic case, the analogous equality to (1.9.1) does not hold in general by [Nakam].

The contents of this article are as follows.

In §2 we prove (1.3) quickly by using (1.2.1). We also prove the analogue of (1.3) for an open log scheme.

In §3 we prove (1.3) in a similar way to that in [Il2]. However we do not use Illusie’s Poincaré residue isomorphism. We simplify his proof by using the Key lemma. In addition, we prove the Hyodo’s criterion for the integrable connection $(\mathcal{E}, \nabla)$ as already stated.

In §4 we prove (1.3) without using (1.2.1) nor Illusie’s Poincaré residue isomorphism. Instead we use the sheaf $\mathcal{W}Omega_X^1$ in [Mo] and [Nakk3] and a spectral sequence constructed in [Nakk5].

In §5 we prove the lower semi-continuity of the log genus by using the finite length version of (1.2.1).

In §6 we prove the lower semi-continuity of the log plurigenus of level 1. It is a future problem to show whether the lower semi-continuity of the log plurigenera of level $n$ holds for any $n$.

In §7 we generalize Illusie’s Poincaré residue isomorphism.

In §8 we give an easy criterion for the quasi-$F$-splitness for an SNCL scheme.

In §9 we give an easy remark about the ordinarity at 0 defined in [Nakk6].

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**Notations.** (1) For a complex $(\mathcal{F}^\bullet, d^\bullet)$ of sheaves on a topological space, we denote $\text{Ker}(d^i: \mathcal{F}^i \to \mathcal{F}^{i+1})$ (resp. $\text{Im}(d^{i-1}: \mathcal{F}^{i-1} \to \mathcal{F}^i)$) mainly by $Z^i(\mathcal{F}^\bullet)$ and sometimes by $Z\mathcal{F}^i$ (resp. mainly by $B^i(\mathcal{F}^\bullet)$ and sometimes by $B\mathcal{F}^i$) depending on the length of the letter $\mathcal{F}$.

(2) For a log scheme $Z$, we denote by $Z$ the underlying scheme of $Z$. 
2 The first proof of (1.3)

In this section we give the first proof of (1.3) by using the $p$-adic weight spectral (1.2.1) ([Nakk3, cf. Mo]). The idea of the proof in this section is Illusie’s one ([Il2, Appendix (2.8)]). Let $\kappa$ be a perfect field of characteristic $p > 0$. Let $s$ be a fine log smooth scheme such that $s = \text{Spec}(\kappa)$. Let $Y/s$ be a log smooth scheme of Cartier type. Let $W_n\Omega_Y^n$ $(n \in \mathbb{Z}_{\geq 1}$ or nothing) be the log de Rham-Witt complex defined in [HK, (4.1)]. Let $\mathcal{W}_n(Y)$ be the canonical lift of $Y$ over $W_n(\kappa)$. Identify $\mathcal{O}_{\mathcal{W}_n(Y)} = W_n(\mathcal{O}_Y)$ with $W_n\Omega_Y^n$ via the canonical isomorphism $s_n : W_n(\mathcal{O}_Y) \rightarrow W_n\Omega_Y^n$ ([Nakk3 (7.5)]). First let us recall the following result. This is the log version of [II] I (3.11) if one forgets the structure of $W_{n+1}(\mathcal{O}_Y)$-modules of sheaves in (2.1.1) below. A variant of the following will be used in §4.

Proposition 2.1 ([Nakk6 (3.10)]). Let $F_{W_n(Y)} : W_n(Y) \rightarrow W_n(Y)$ be the Frobenius endomorphism of $W_n(Y)$. Let $i$ be a nonnegative integer. Let $C^{-1} : \Omega_{Y/s}^i \rightarrow F_{Y,s}(\Omega_{Y/s}^i) = F_{Y,s}(\mathcal{H}^i(\mathcal{O}_Y^\dagger))$ be the log inverse Cartier isomorphism due to Kato (K). Let $R : W_{n+1}\Omega_Y^j \rightarrow W_n\Omega_Y^n$ be the projection. Then the following hold:

1. The following sequence

$W_{n+1}\Omega_Y^j \xrightarrow{F} F_{W_n(Y)}(W_n\Omega_Y^n) \xrightarrow{F_{W_n(Y)}(\text{proj})} B_nW_1\Omega_Y^{i+1} \rightarrow 0$

is an exact sequence of $W_{n+1}(\mathcal{O}_Y)$-modules. Here $B_nW_1\Omega_Y^{i+1} = C^{-1}(B_n\Omega_Y^{i+1})$, where $B_n\Omega_Y^{i+1}$ is an $F^n_{Y,s}(\mathcal{O}_Y)$-module defined inductively by the following isomorphism $C^{-1} : B_n\Omega_Y^{i+1} \rightarrow B_{n+1}\Omega_Y^{i+1}/F^n_{Y,s}(B_{n+1}\Omega_Y^{i+1})$ and $B_n\Omega_Y^{i+1} := F_{Y,s}(\text{proj})^n(\Omega_{Y/s}^i)$.

2. Let $C : B_{n+1}W_1\Omega_Y^{i+1} \rightarrow B_nW_1\Omega_Y^{i+1}$ be the following composite morphism:

$B_{n+1}W_1\Omega_Y^{i+1} \xrightarrow{C^{-1}(\text{proj})} C^{-1}(B_{n+1}\Omega_Y^{i+1}/B_1\Omega_Y^{i+1}) \xrightarrow{C^{-1}(\text{proj})} C^{-1}(B_0\Omega_Y^{i+1}) = B_nW_1\Omega_Y^{i+1}.$

Then the following diagram is commutative:

$\begin{array}{ccc}
W_{n+1}(\mathcal{O}_Y)^j & \xrightarrow{F} & F_{W_n(Y)}(W_n\Omega_Y^n) \\
\downarrow{R} & & \downarrow{C} \\
W_n(\mathcal{O}_Y)^j & \xrightarrow{F_{W_n(Y)}(\text{proj})} & B_nW_1\Omega_Y^{i+1}
\end{array}$

(2.1.2)

Proposition 2.2. Let $\kappa$ be a perfect field of characteristic $p > 0$. Then (1.3) holds in the case where $S$ is the log point $s$ of $\kappa$.

Proof. By using (2.1.1) and noting that $R : W_{n+1}\Omega_Y \rightarrow W_n\Omega_Y$ is surjective, we have the following exact sequence:

$0 \rightarrow \Omega_Y \xrightarrow{F_{\mathcal{W}(\mathcal{O}_Y)^j}} F_{\mathcal{W}(\mathcal{O}_Y)^j}(W\Omega_Y) \rightarrow \lim_{\rightarrow n} B_nW_1\Omega_Y^{i+1} \rightarrow 0.$

Hence, by the log version of [IR IV (4.11.6), (4.13)], $X/s$ is log ordinary if and only if the operator $F^i : H^j(X, W\Omega_X^j) \rightarrow H^j(X, W\Omega_X^j)$ is bijective for any $i$ and $j$. (This has also been obtained in [L (4.1)].) Let $\{E_{ij}^n\}_{i,j \in \mathbb{Z}}$ be the $E_1$-terms of the spectral
sequence (1.2.1). Here we ignore the Tate twist \((-j-k)\). Let \((\mathcal{W}_n A^n_X, P) (n \in \mathbb{Z}_{\geq 1})\) be the filtered double complex defined in [Nak3 §2] and denoted by \((\mathcal{W}_n A^n_X, P)\) (cf. [M]). By the definition of the operator \(F\) and by considering a local log smooth lift \(X/s\) over \(W_n(s)\), the following two diagrams are commutative:

\[
\begin{array}{ccc}
\mathcal{W}_{n+1} \Omega^i_X & \xrightarrow{\theta^\wedge, \sim} & \mathcal{W}_{n+1} A^n_X \\
F \downarrow & & \downarrow F \\
\mathcal{W}_n \Omega^i_X & \xrightarrow{\theta^\wedge, \sim} & \mathcal{W}_n A^n_X,
\end{array}
\]

\[
\begin{array}{ccc}
g_t^P \mathcal{W}_{n+1} A^n_X & \xrightarrow{\text{Res}, \sim} & \bigoplus_{j \geq \max\{-k,0\}} \mathcal{W}_{n+1} \Omega^{i-j-k}_{X/(2j+k)} \{−j\} \\
F \downarrow & & \downarrow F \\
g_t^P \mathcal{W}_n A^n_X & \xrightarrow{\text{Res}, \sim} & \bigoplus_{j \geq \max\{-k,0\}} \mathcal{W}_n \Omega^{i-j-k}_{X/(2j+k)} \{−j\}.
\end{array}
\]

Because the spectral sequence (1.2.1) is obtained by these commutative diagrams, it is compatible with \(F\)'s. By the assumption, the operator \(F: E_1^{i-k,h+k} \rightarrow E_1^{i,k+h+k}\) is bijective. Hence the operator \(F: H^j(X, \mathcal{W} \Omega^i_X) \rightarrow H^j(X, \mathcal{W} \Omega^i_X)\) is bijective. We complete the proof of (2.2). \(\square\)

**Proposition 2.3 ([II2 Appendix (1.2), (1.9)]).** Let \(g: Y \rightarrow T\) be a proper log smooth scheme of fine log schemes. Let \(i\) be an integer. Then the following are equivalent:

1. \(R^jg_*(B \Omega^i_Y/T) = 0\) for \(\forall j \geq 0\).
2. For any exact closed point \(t \in T\) (\(t\) is a closed point of \(T\) endowed with the inverse image of the log structure of \(T\)), \(H^j(Y_t, B \Omega^i_{Y/t}) = 0\) for \(\forall j \geq 0\). Here \(Y_t := Y \times_T t\).

**Proof.** Set \(Y' := Y \times_T F_t T\) and let \(F_{Y/T}: Y \rightarrow Y'\) be the relative Frobenius morphism of \(Y/T\). Let \(g': Y' \rightarrow T\) be the structural morphism. For an exact closed point \(t \in T'\), let \(g'_t: Y'_t \rightarrow t\) be the base change morphism of \(g'\). Set \(B_1 \Omega^i_{Y'/T} := F_{Y/T*}(B \Omega^i_{Y/T})\). Then \(B_1 \Omega^i_{Y'/T}\) is an \(O_Y\)-module.

Because \(\tilde{F}_{Y/T}\) is a homeomorphism of topological spaces ([SGA 5 XV Proposition 2 a)]), it suffices to prove that the following are equivalent:

1. \(R^jg'_*(B_1 \Omega^i_{Y'/T}) = 0\) for \(\forall j \geq 0\).
2. For any exact closed point \(t \in T\), \(H^j(Y'_t, B_1 \Omega^i_{Y'/t}) = 0\) for \(\forall j \geq 0\). By [L (1.13)] the sheaf \(B_1 \Omega^i_{Y'/T}\) \((m, i \in \mathbb{N})\) is a locally free sheaf of \(O_Y\)-modules of finite rank. By the log Küneth formula ([K (6.12)]), we have the following isomorphism

\[
Rg'_*(B_1 \Omega^i_{Y'/T}) \otimes^L_{O_Y} \kappa_t \simto Rg'_*(B_1 \Omega^i_{Y'/t}).
\]

The rest of the proof is the same as that of [II2 Appendix (1.2), (1.9)]. Indeed, the implication \((2') \implies (1')\) follows from the fact that \(Rg'_*(B_1 \Omega^i_{Y'/T})\) is a perfect complex of \(O_T\)-modules (since \(g'\) is proper). The implication \((1') \implies (2')\) follows from this perfectness and Nakayama’s lemma. \(\square\)

We prove (1.3). Let the notations be as in (1.3). Set \(S := S \times_T \mathcal{O}\) and \(X := X \times_S S\). Let \(f_T: X_T \rightarrow S_T\) be the structural morphism. Because \(B \Omega^i_{X_T/S_T}\)
commutes with the base change morphism \( T \rightarrow S \).

\[
R^j f^*_{T*}(B\Omega^i_{\mathcal{X}_T/T}) = R^j f^*_{T*}(B\Omega^j_{\mathcal{X}_T/T} \otimes_{\mathcal{O}_T} \mathcal{O}_T) = R^j f^*_{T*}(B\Omega^j_{\mathcal{X}_T/T}/S^j_T).
\]

(To obtain (2.3.1), one may use a fact that \( B\Omega^j_{\mathcal{X}_T/T} = B\Omega^j_{\mathcal{X}_T/T} \) (since \( \Omega^j_{\mathcal{X}_T/T} = \Omega^j_{\mathcal{X}_T/T} \)). Hence we may assume that \( T = S \) and in fact, \( T = S \). By (2.3) we can assume that \( S \) is the log point. Consider the perfection \( k^\text{pf} \) of \( k \) and let \( s^\text{pf} \) be the log point whose underlying scheme is \( \text{Spec}(k^\text{pf}) \) and whose log structure is the inverse image of the log structure of \( s \) by the natural morphism \( \text{Spec}(k^\text{pf}) \rightarrow \text{Spec}(k) \). Set \( X_{s^\text{pf}} := X \times_s s^\text{pf} \). By (2.2), \( H^j(X_{s^\text{pf}}, B\Omega^i_{X_{s^\text{pf}},s^\text{pf}}) = 0 \) for \( \forall i \) and \( \forall j \).

Because \( H^j(X_{s^\text{pf}}, B\Omega^i_{X_{s^\text{pf}},s^\text{pf}}) = H^j(X, B\Omega^i_{X,s}) \otimes_k k^\text{pf}, H^j(X, B\Omega^i_{X,s}) = 0 \). By (2.3), \( R^j f^*(B\Omega^i_{X/k}) = 0 \) for \( \forall i \) and \( \forall j \). We complete the proof of (1.3).

We can also obtain the Hyodo’s criterion for an open smooth scheme as follows.

**Theorem 2.4.** Let \( S \) be a scheme of characteristic \( p > 0 \). Let \( f : (X, D) \rightarrow S \) be a proper smooth scheme with relative SNCD (\([\text{NS}_1 (2.1.7)]\)). Let \( D^{(i)} (i \in \mathbb{Z}_{\geq 0}) \) be a scheme defined in [loc. cit., (2.2.13.2)]. Then if \( D^{(i)} \) is ordinary for all \( i \), then \( f \) is log ordinary.

To prove this theorem, we have only to use the following spectral sequence ([Nak5 (5.7.1)]):

\[
E_1^{k,h+k} = H^{h-i}(D^{(k)}, W\Omega^i_{D^{(k)}}(-k)) \Rightarrow H^{h-i}(X, W\Omega^i_X(\log D)).
\]

instead of the spectral sequence (1.2.1) in the case \( S = \text{Spec}(k) \).

We say that \( f \) is log ordinary with compact support if \( R^j f^*(d\Omega^i(\log D)) = 0 \) for \( \forall i, j \). We can also obtain the following theorem:

**Theorem 2.5.** Let the notations be as in (2.4). If \( D^{(i)} \) is ordinary for all \( i \), then \( f \) is log ordinary with compact support.

To prove this theorem, we have only to use the following spectral sequence ([Nak5 (5.7.2)]):

\[
E_1^{k,q-k} = H^{k-h-i}(D^{(k)}, W\Omega^i_{D^{(k)}}) \Rightarrow H^{k-h-i}(X, W\Omega^i_X(\log D)).
\]

in the case \( S = \text{Spec}(k) \).

### 3 The second proof of (1.3)

In this section we prove (1.3), (2.4) and (2.5) without using the \( p \)-adic weight spectral sequences (1.2.1), (2.4.1) and (2.5.1), respectively. The proof of (1.3) in this section includes a simplification of the proof of Hyodo’s criterion in [H2 Appendix] as already stated in the Introduction: the filtration \( P \) on \( \Omega^\bullet_{X/k} \) induced by the filtration \( P \) on \( \Omega^\bullet_{X/k} \) defined in [Nak5] nor a generalization of the residue map in [loc. cit., (2.1.3)] is unnecessary; we use only the filtration \( P \) on \( \Omega^\bullet_{X/k} \). The filtration \( P \) on \( \Omega^\bullet_{X/k} \) is more directly related with the complex \( \Omega^\bullet_{X/k}(j \in \mathbb{N}) \) than the filtration \( P \) on \( \Omega^\bullet_{X/k} \). From the earlier part of this section, we consider a certain integrable connection without
log poles including the derivative $d: \mathcal{O}_X \to \Omega^1_{X/S}$. The main result in this section is (3.1.3) below. This includes (1.3).

Let $S$ be a family of log points ([Nakk5 (1.1)]) and let $Y/S$ be a log smooth scheme. Let $g: Y \to S$ be the structural morphism. Let $e$ be a local section of $M_S$ such that the image of $e$ in $M_S/O_S^*$ is the local generator. Set $\theta_S := d\log e \in \Omega^1_{S/S}$. Then this section is independent of the choice of $e$ and it is globalized. Set $\theta := g^*(d\log e) \in \Omega^1_{Y/S}$. Let $\pi^*: \Omega^*_{Y/S} \to \Omega^*_{Y/S}$ be the natural projection. Because $Y/S$ is log smooth, the following sequence

$$0 \to g^*(\Omega^1_{S/S}) \xrightarrow{\theta \wedge} \Omega^1_{Y/S} \xrightarrow{\pi^1} \Omega^1_{Y/S} \to 0$$

is locally split ([K (3.12)]). By a very special case of [Nakk5 (1.7.20.1)], the following sequence is exact:

$$0 \to \Omega^*_{Y/S}[-1] \xrightarrow{\theta \wedge} \Omega^*_{Y/S} \xrightarrow{\pi^*} \Omega^*_{Y/S} \to 0.$$

The following lemma is a key lemma which enables us to simplify Illusie’s proof for Hyodo’s criterion in [Il2].

**Lemma 3.1.** For each $i$, the resulting sequences of (3.0.2) by the operations $B^i$, $Z^i$ and $H^i$ ($i \in \mathbb{Z}_{\geq 0}$) are exact.

**Proof.** We have only to prove that the following sequences are exact:

$$0 \to B\Omega^{i-1}_{Y/S} \xrightarrow{\theta \wedge} B\Omega^i_{Y/S} \xrightarrow{\pi^1} B\Omega^i_{Y/S} \to 0$$

and

$$0 \to Z\Omega^{i-1}_{Y/S} \xrightarrow{\theta \wedge} Z\Omega^i_{Y/S} \xrightarrow{\pi^1} Z\Omega^i_{Y/S} \to 0.$$

Because the problem is local, we may assume that there exists a basis $\{\theta, \{\omega_i\}_{i=1}^d\}$ of $\Omega^1_{Y/S}$. Then $\{\pi^1(\omega_i)\}_{i=1}^d$ is a basis of $\Omega^1_{Y/S}$. Let $\iota: \Omega^1_{Y/S} \to \Omega^1_{Y/S}$ be the splitting of the projection $\pi^1: \Omega^1_{Y/S} \to \Omega^1_{Y/S}$. Let $\Omega^i_{Y/S} \to \Omega^i_{Y/S}$ be the induced splitting by $\iota$ above and denote it by $\iota: \Omega^i_{Y/S} \to \Omega^i_{Y/S}$ again. Then $\Omega^i_{Y/S} = \iota(\Omega^i_{Y/S}) \oplus \theta \wedge (\Omega^i_{Y/S})$. Let $d: \Omega^i_{Y/S} \to \Omega^{i+1}_{Y/S}$ and $d_{Y/S}: \Omega^i_{Y/S} \to \Omega^{i+1}_{Y/S}$ be the standard differentials. By expressing a local section of $\Omega^i_{Y/S}$ by the basis $\{\theta, \{\omega_i\}_{i=1}^d\}$ of $\Omega^i_{Y/S}$ (cf. the first proof of (3.3) below), it is obvious that the following diagram is commutative:

$$\begin{array}{ccc}
\Omega^i_{Y/S} & \xrightarrow{\iota} & \theta(\Omega^i_{Y/S}) \oplus \theta \wedge (\Omega^i_{Y/S}) \\
\downarrow d & & \downarrow \iota(d_{Y/S}) \oplus (-\theta \wedge d_{Y/S}) \\
\Omega^{i+1}_{Y/S} & \rightarrow & \theta(\Omega^{i+1}_{Y/S}) \oplus \theta \wedge (\Omega^{i+1}_{Y/S})
\end{array}$$

Hence

$$B\Omega^i_{Y/S} = \iota(B\Omega^i_{Y/S}) \oplus \theta \wedge (B\Omega^{i-1}_{Y/S})$$
and
\[ Z\Omega_{Y/S}^i = (Z\Omega_{Y/S}^i) \oplus \theta \wedge (Z\Omega_{Y/S}^{i-1}). \]
This implies that the sequences (3.1.1) and (3.1.2) are exact. We complete the proof of (3.1).

More generally, let \( \mathcal{E} \) be a quasi-coherent \( \mathcal{O}_Y \)-module and let
\[ \nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^1 \]
be an integrable connection. Let
\[ \nabla_{Y/S}: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^1 \]
be the induced integrable connection by \( \nabla \). Then we have the following morphism of complexes of \( g^{-1}(\mathcal{O}_S) \)-modules:
\[ \theta \wedge: \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^*[-1] \to \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^* \]
which induces the following morphism of complexes of \( g^{-1}(\mathcal{O}_S) \)-modules:
\[ \theta \wedge: \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^*[-1] \to \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^* \]
Indeed, because we have the following equalities
\[ (\nabla \theta \wedge - \theta \wedge (-\nabla))(e \otimes \omega) = (\nabla \theta \wedge + \theta \wedge \nabla)(e \otimes \omega) \]
\[ = \{ \nabla(e) \wedge \theta \wedge \omega + e \otimes (\theta \wedge \omega) \} + \{ \theta \wedge \nabla(e) \wedge \omega + e \otimes (\theta \wedge d\omega) \} \]
\[ = \nabla(e) \wedge \theta \wedge \omega + \theta \wedge \nabla(e) \wedge \omega = 0 \]
\[ (e \in \mathcal{E}, \omega \in \Omega_{Y/S}^i (i \in \mathbb{N})), \]
\( \nabla \theta \wedge = \theta \wedge (-\nabla) \). Because the exact sequence (3.0.2) is locally split, the following sequence of complexes of \( g^{-1}(\mathcal{O}_S) \)-modules is exact:
\[ 0 \to \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^{i-1} \xrightarrow{\theta \wedge} \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^i \to \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^i \to 0. \]

**Definition 3.2.** We say that \( \mathcal{E} \) is **locally generated by horizontal sections** of \( \nabla \) if there exist local sections \( e_0, \ldots, e_m (m \in \mathbb{N}) \) which generate \( \mathcal{E} \) locally as an \( \mathcal{O}_Y \)-module such that \( \nabla(e_j) = 0 \) for \( 0 \leq j \leq m \). We say that the set \( \{ e_j \}_{j=0}^m \) a set of local horizontal generators of \( \mathcal{E} \).

We can generalize (3.1) as follows:

**Lemma 3.3.** Assume that \( \mathcal{E} \) is locally generated by horizontal sections. Then, for each \( i \), the resulting sequences of (3.1.3) by the operations \( B^i, Z^i \) and \( H^i \) \( (i \in \mathbb{Z}_{\geq 0}) \) are exact.

**Proof.** First we give the elementary and direct two proof of this lemma. Later we give another proof it when \( Y/S \) is of Cartier type and \( (\mathcal{E}, \nabla) \) is a certain connection (see (3.6) below).
We claim that the following sequences are exact:

\[
\begin{align*}
(3.3.1) \quad 0 \to & B^{i-1}(E \otimes_{O_Y} \Omega^{i-1}_{Y/S}) \xrightarrow{\theta} B^i(E \otimes_{O_Y} \Omega^i_{Y/S}) \to B^i(E \otimes_{O_Y} \Omega^i_{Y/S}) \to 0, \\
(3.3.2) \quad 0 \to & Z^{i-1}(E \otimes_{O_Y} \Omega^{i-1}_{Y/S}) \xrightarrow{\theta} Z^i(E \otimes_{O_Y} \Omega^i_{Y/S}) \to Z^i(E \otimes_{O_Y} \Omega^i_{Y/S}) \to 0.
\end{align*}
\]

The problem is local. We give the two proofs of the exactness of \((3.3.1)\).

The first proof:

Let \(M_Y\) be the log structure of \(Y\). Because the problem, we can fix a set \(\{e_j\}_{j=0}^m\) of horizontal generators of \(E\) and fix a base \(\{\theta, \omega_1, \ldots, \omega_d\}\) of \(\Omega^1_{Y/S}\) (\(\omega_i = d \log m_i\) for some \(m_i \in M_Y\)). For a nonnegative integer \(k\), set \(P_k := \{I \subset \{1, \ldots, d\} \mid |I| = k\}\). For a nonnegative integer \(k\) and an element \(I := \{i_1, \ldots, i_k\} \in P_k\), set \(E_{\omega_i} := \omega_i \wedge \cdots \wedge \omega_{i_k}\), where \(i_1 < \cdots < i_k\). For an empty set \(\emptyset\), set \(E_{\emptyset} = 1 \in O_Y\). Any local section \(e\) of \(E \otimes_{O_Y} \Omega^1_{Y/S}\) can be expressed as follows:

\[
e = \sum_{j=0}^m \{e_j \otimes \left( \sum_{I \in P_{i-1}} f_I \omega_I + \sum_{J \in P_{i-2}} g_J \theta \wedge \omega_J \right)\} \quad (f_I, g_J \in O_Y).
\]

Then

\[
\nabla(e) = \sum_{j=0}^m \{e_j \otimes \left( \sum_{I \in P_{i-1}} df_I \wedge \omega_I + \sum_{J \in P_{i-2}} dg_J \wedge \theta \wedge \omega_J \right)\}.
\]

Hence the following diagram is commutative:

\[
\begin{array}{ccc}
E \otimes_{O_Y} \Omega^i_{Y/S} & \xrightarrow{\nabla} & \iota(\Omega^i_{Y/S}) \oplus \{\theta \wedge (E \otimes_{O_Y} \Omega^1_{Y/S})\} \\
\downarrow & & \downarrow \iota(\nabla)_{Y/S} \oplus (\theta \wedge \nabla_{Y/S}) \\
E \otimes_{O_Y} \Omega^{i+1}_{Y/S} & = & \{\iota(E \otimes_{O_Y} \Omega^{i+1}_{Y/S})\} \oplus \{\theta \wedge (E \otimes_{O_Y} \Omega^{i+1}_{Y/S})\}.
\end{array}
\]

Consequently

\[
B^i(E \otimes_{O_Y} \Omega^i_{Y/S}) = \iota(B^i(E \otimes_{O_Y} \Omega^i_{Y/S})) \oplus \theta \wedge (B^{i-1}(E \otimes_{O_Y} \Omega^i_{Y/S})),
\]

\[
Z^i(E \otimes_{O_Y} \Omega^i_{Y/S}) = \iota(Z^i(E \otimes_{O_Y} \Omega^i_{Y/S})) \oplus \theta \wedge (Z^{i-1}(E \otimes_{O_Y} \Omega^i_{Y/S})).
\]

This implies that the sequences \((3.3.1)\) and \((3.3.2)\) are exact.

The second proof in the case where \(E\) is a flat quasi-coherent \(O_Y\)-module:

Here we give the exactness of only \((3.3.1)\). We can prove the exactness of \((3.3.2)\) similarly. It suffices to prove that

\[
\text{Ker}(B^i(E \otimes_{O_Y} \Omega^i_{Y/S}) \to B^i(E \otimes_{O_Y} \Omega^i_{Y/S})) \subset (\theta \wedge (B^{i-1}(E \otimes_{O_Y} \Omega^i_{Y/S}))).
\]

This is equivalent to the following inclusion

\[
(3.3.4) \quad \nabla(E \otimes_{O_Y} \Omega^{i-1}_{Y/S}) \cap (\theta \wedge (E \otimes_{O_Y} \Omega^{i-1}_{Y/S})) \subset (\theta \wedge \nabla(E \otimes_{O_Y} \Omega^{i-2}_{Y/S})).
\]
To prove the inclusion \((3.3.4)\), by the assumption we have only to prove that the following inclusion holds:

\[
(3.3.5) \quad (\mathcal{E} \otimes_{\mathcal{O}_Y} d\Omega^{i-1}) \cap (\mathcal{E} \otimes_{\mathcal{O}_Y} (\theta \wedge (\Omega^{i-1}))) \subset \mathcal{E} \otimes_{\mathcal{O}_Y} (\theta \wedge (d\Omega^{i-2})).
\]

Because \(\mathcal{E}\) is a flat \(\mathcal{O}_Y\)-module, it suffices to prove that

\[
(3.3.6) \quad d\Omega^{i-1} \cap (\theta \wedge (\Omega^{i-1})) \subset \theta \wedge (d\Omega^{i-2}).
\]

By taking a local basis \(\{\omega_1, \ldots, \omega_d, \theta\}\) of \(\Omega^1_{Y/S}\), by describing a local section of \(\Omega^{i-1}_{Y/S}\) with the use of this basis and by noting that \(d\theta = 0\), \((3.3.6)\) is an obvious inclusion. We complete the first proof of \((3.3)\) \(\square\)

**Definition 3.4.** Let \(h: Z \to T\) be a proper log smooth morphism of fine log schemes. Let \(\nabla: \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \Omega^1_{Z/T}\) be an integrable connection. We say that the connection \(\nabla\) is **log ordinary** if \(R^i h_* (B^i (\mathcal{M} \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/T})) = 0\) for \(\forall i, j \in \mathbb{N}\). If \(\nabla\) is the derivative \(d: \mathcal{O}_Z \to \Omega^1_{Z/T}\) and if \(d\) is log ordinary in the sense above, then we say that \(h\) or \(Z/T\) is **log ordinary**.

The following is a first step of the proof of Hyodo's criterion for the case of an integrable connection.

**Corollary 3.5.** Assume that \(\mathcal{E}\) is locally generated by horizontal sections. The connection \((3.3.5)\) is log ordinary if and only if \(R^i h_* (B^i (\mathcal{E} \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})) = 0\) for \(\forall i, j \in \mathbb{N}\) and \(\forall j\).

**Proof.** The implication \(\Rightarrow\) follows from \((3.3)\). The converse implication also follows from the exact sequence \((3.3.1)\) and induction on \(i\) \(\square\)

**Remark 3.6.** (1) If \(\mathcal{E}\) is not locally generated by horizontal sections of \(\nabla\), \((3.3.1)\) is not necessarily exact. Indeed, consider the case \(i = 0\) and consider a log scheme \(Y\) whose underlying scheme is \(\text{Spec}(\kappa[x, y]/(xy))\) and whose log structure is associated to a morphism \(\mathbb{N}^2 \to \kappa[x, y]/(xy)\) defined by \((1, 0) \mapsto x\) and \((0, 1) \mapsto y\). The diagonal immersion \(\mathbb{N} \to \mathbb{N}^2\) induces a morphism \(\mathcal{Y} \to s\) of log schemes. Let \(\mathcal{E}\) be a free \(\mathcal{O}_Y\)-module of rank 1 with basis \(e\). Let \(\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}\) be an integrable connection defined by the following equalities: \(\nabla(e) = e \otimes \theta\), \(\nabla(fe) = f \nabla(e) + e \otimes df\) \((f \in \mathcal{O}_Y)\). Then \(\nabla_{Y/S}(e) = 0\). Hence the sequence \((3.3.1)\) for \(i = 1\) is not exact.

(2) We give the third proof of \((3.3)\) in the following case.

Assume that \(Y/S\) is of Cartier type. Set \(Y' := Y \times_{S, F_2} S\). Let \(F: Y \to Y'\) be the relative Frobenius morphism over \(S\). Let \(\mathcal{E}'\) be a quasi-coherent flat \(\mathcal{O}_{Y'}\)-module. Set \((\mathcal{E}, \nabla) := (F^*(\mathcal{E}'), \text{id}_{\mathcal{E}'} \otimes d)\). Here \(d: \mathcal{O}_Y \to \Omega^1_{Y/S}\) is the usual derivative. It is obvious that \(\mathcal{E}\) is locally generated by horizontal sections of \(\nabla\). By a special case of \([\mathcal{O}_2]\) \((4.1.1)\) and by \([\mathcal{O}_1]\) \((1.2.5)\), there exists a morphism

\[
(3.6.1) \quad G^{-1}: \mathcal{E} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S} \to F_\ast \mathcal{H}^i (\mathcal{E} \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S}) \quad (i \in \mathbb{N})
\]
of $\mathcal{O}_Y$-modules fitting into the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{E}' \otimes_{\mathcal{O}_Y} \Omega^{-1}_Y \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_Y \longrightarrow & \mathcal{E}' \otimes_{\mathcal{O}_Y} \Omega^i_Y \otimes_{Y/S} \longrightarrow & 0 \\
\downarrow C^{-1} & \cong & \downarrow C^{-1} & \cong & \downarrow C^{-1} & \cong \\
0 & \longrightarrow & F_* \mathcal{H}^{-i-1}(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow & F_* \mathcal{H}^i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow & F_* \mathcal{H}_i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow & 0.
\end{array}
\]

Note that $C^{-1}(\theta) = \theta$. Hence the morphism (3.6.1) is an isomorphism. By [SGA 5, XV Proposition 2 a)], $\hat{\varphi}$ is a homeomorphism. Hence the following sequence

\[
0 \longrightarrow \mathcal{H}^{-i-1}(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow \mathcal{H}^i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow \mathcal{H}^i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow 0
\]

is exact. Using induction on $i$, we see that the following sequences are exact:

\[
0 \longrightarrow Z^{-i-1}(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow Z^i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow Z^i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow 0
\]

and

\[
0 \longrightarrow B^{-i-1}(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow B^i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow B^i(E \otimes_{\mathcal{O}_Y} \Omega^i_Y) \longrightarrow 0.
\]

We complete the second proof of (3.3) in the case above.

Let $Z$ be a fine log (formal) scheme over a fine log (formal) scheme $T$. Let $g: Z \longrightarrow \tilde{T}$ be the structural morphism. Let us recall the increasing filtration $P$ on the sheaf $\Omega^i_Z$ ($i \in \mathbb{N}$) of log differential forms on $Z_{zar}$ ([Nakk5 (1.3.0.1)], cf. [Nakk4 (4.0.2)]):

\[
P^i_Z := \begin{cases} 
0 & (k < 0), \\
\text{Im}(\Omega^k_Z \otimes g_{\mathcal{O}_Z} \Omega^{k-i}_Z) & (0 \leq k \leq i), \\
\Omega^k_Z & (k > i).
\end{cases}
\]

In [Nakk5] we have called this filtration $P$ the preweight filtration on $\Omega^i_Z$. For a flat quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$, set

\[
P_k(\mathcal{F} \otimes_{\mathcal{O}_Z} \Omega^i_Z) := \mathcal{F} \otimes_{\mathcal{O}_Z} P^i_Z \qquad (i \in \mathbb{N}, k \in \mathbb{Z}).
\]

Let $\mathcal{F}$ be a flat quasi-coherent $\mathcal{O}_Z$-module and let

\[
\nabla: \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Z} P_0 \Omega^1_Z
\]

be an integrable connection. That is, $\nabla$ is a morphism of $g^{-1}(\mathcal{O}_T)$-modules, $\nabla(a\omega) = \omega \otimes da + a \nabla(\omega)$ ($a \in \mathcal{O}_Z, \omega \in \mathcal{F}$) and the iteration of $\nabla$: $\nabla^2: \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Z} P_0 \Omega^2_Z$ is zero. By abuse of notation, we denote the induced morphism $\mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Z} \Omega^1_Z$ by (3.6.4) also by

\[
\nabla: \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Z} \Omega^1_Z.
\]
Let
(3.6.6) \[ \nabla_{Z/T} : F \rightarrow F \otimes_{\mathcal{O}_Z} \Omega^1_{Z/T}. \]
be the induced integrable connection by (3.6.5). Then we have complexes \( F \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/T} \) and \( F \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/T} \). Set
\[ P_k(F \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/T}) := F \otimes_{\mathcal{O}_Z} P_k \Omega^\bullet_{Z/T} \quad (k \in \mathbb{Z}). \]

By (3.6.4) we indeed the complex \( P_k(F \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/T}) \). Consequently we have a filtered complex \( (F \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/T}, P) \) of \( g^{-1}(\mathcal{O}_T) \)-modules.

**Remark 3.7.** Though \( \mathcal{O}_{Z/T} \) is not necessarily log smooth, let us define a sheaf \( R_{\mathcal{O}_{Z/T}} \) on \( Z \) as \( \text{Coker}(\Omega^1_{\mathcal{O}_{Z/T}} \rightarrow \Omega^1_{\mathcal{O}_{Z/T}}) \) following [O1, (1.3.0.1)]. Let \( F' \) be a quasi-coherent \( \mathcal{O}_Z \)-module. Then it is obvious that an integrable connection \( \nabla : F' \rightarrow F' \otimes_{\mathcal{O}_Z} \Omega^1_{Z/T} \) (3.7.1) induces a connection (3.6.4) if and only if the induced morphism
(3.7.2) \[ \nabla : F' \rightarrow F' \otimes_{\mathcal{O}_Z} R_{Z/T} \]
by (3.7.1) is zero.

Let \( h : Z \rightarrow W \) be a morphism of log schemes over \( T \). Let \( \mathcal{G} \) be a flat quasi-coherent \( \mathcal{O}_W \)-module and let
\[ \nabla : \mathcal{G} \rightarrow \mathcal{G} \otimes_{\mathcal{O}_W} P_0 \Omega^1_{W/T} \]
be an integrable connection fitting into the following commutative diagram
\[
\begin{array}{ccc}
h_*(F) & \xrightarrow{\nabla} & h_*(F \otimes_{\mathcal{O}_Z} P_0 \Omega^1_{Z/T}) \\
\uparrow & & \uparrow \\
\mathcal{G} & \xrightarrow{\nabla} & \mathcal{G} \otimes_{\mathcal{O}_W} P_0 \Omega^1_{W/T}.
\end{array}
\]
Then we have the following morphism of filtered complexes:
(3.7.3) \[ h^* : (\mathcal{G} \otimes_{\mathcal{O}_W} \Omega^\bullet_{W/T}, P) \rightarrow h^*((F \otimes_{\mathcal{O}_Z} \Omega^\bullet_{Z/T}, P)). \]

Now let \( X \) be an SNCL scheme over \( S \). Let \( \{ \tilde{X}_\lambda \}_{\lambda \in \Lambda} \) be a decomposition of \( \tilde{X} \) by smooth components of \( \tilde{X} \) over \( \tilde{S} \) ([Nakk5, (1.1.8)]). Then the set \( \{ \tilde{X}_\lambda \}_{\lambda \in \Lambda} \) gives the orientation sheaf \( \omega_{\text{zar}}(k) : (\tilde{X} / \tilde{S}) (k \in \mathbb{N}) \) as in [NS, p. 81] ([Nakk5, (1.1)]). Let \( f : X \rightarrow S \) and \( f^{(k)} : \tilde{X}^{(k)} \rightarrow \tilde{S} \) be the structural morphisms. For a nonnegative integer \( k \), set
(3.7.4) \[ \tilde{X}_{\{\lambda_0, \lambda_1, \ldots, \lambda_k\}} := \tilde{X}_{\lambda_0} \cap \cdots \cap \tilde{X}_{\lambda_k} \quad (\lambda_i \neq \lambda_j \text{ if } i \neq j) \]
For a negative integer $k$, set $\hat{X}^{(k)} = \emptyset$. In [Nakkk5 (1.11)] we have proved that $\hat{X}^{(k)}$ is independent of the choice of the set $\{X_\lambda\}_{\lambda \in \Lambda}$. Denote the natural local closed immersion $\tilde{X}_{\lambda_0 \cdots \lambda_k} \hookrightarrow \tilde{X}$ by $a_{\lambda_0 \cdots \lambda_k}$. Let $a(k) : \hat{X}^{(k)} \to \hat{X}(k \in \mathbb{N})$ be the morphism induced by the $a_{\lambda_0 \cdots \lambda_k}$'s.

The following Poincaré residue isomorphism is a special case of [Nakkk5 (1.3.14)]:

**Proposition 3.8.** Let $x$ be an exact closed point of $X$. Let $r$ be a nonnegative integer such that $M_{X,x}/\mathcal{O}_{X,x}^r \simeq \mathbb{N}^r$. Let $m_{1,x}, \ldots, m_{r,x}$ be local sections of $M_X$ around $x$ whose images in $M_{X,x}/\mathcal{O}_{X,x}^r$ are generators of $M_{X,x}/\mathcal{O}_{X,x}^r$. Then, for a positive integer $k$, the following morphism

$$\text{Res} : P_k \Omega^*_{X/S} \to a^{(k-1)}(\Omega^{*-k}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}})(X/S)$$

(cf. [D2 (3.5)]) induces the following “Poincaré residue isomorphism”

$$\text{gr}_k^P(\Omega^*_{X/S}) \sim a^{(k-1)}(\Omega^{*-k}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}})(X/S)).$$

Let

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

be an integrable connection on $\mathcal{E}_{X/S}$ locally generated horizontal sections of $\nabla$. This connection induces the following integrable connections:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_S} P_0 \Omega^1_{X/S}$$

and

$$\nabla_{X/S} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}.$$

In the following we always assume that $\mathcal{E}$ is a flat $\mathcal{O}_X$-module.

**Corollary 3.9.** (1) The morphism (3.8.1) induces the following morphism

$$\text{Res} : P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}) \to a^{(k-1)}(\mathcal{E} \otimes_{\mathcal{O}_S} \Omega^{*-k}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}})(X/S))$$

of complexes which induces the following Poincaré residue isomorphism

$$\text{gr}_k^P(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}) \sim a^{(k-1)}(\mathcal{E} \otimes_{\mathcal{O}_S} \Omega^{*-k}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}})(X/S)).$$

of complexes.

(2)

(3.9.3)

$$B^{(k-1)}(\mathcal{E} \otimes_{\mathcal{O}_S} \Omega^{*-k}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{X(k-1)/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}})(X/S)).$$
Proof. (1): By noting the connection \(3.8.3\) induces the connection \(3.8.4\), it is easy to check that the morphism \(3.9.1\) is a morphism of complexes. It is also easy to check that this morphism induces the isomorphism \(3.9.2\).

(2): (2) follows from (1). \(\square\)

Next we recall the description of \(P_0\Omega^*_{X/S} (k \in \mathbb{Z})\) in [Nakk5]. The following has been proved in [Nakk5 (1.3.21)] as a special case:

**Proposition 3.10** (cf. [Mo, Lemma 3.15.1], [Nakk3, (6.29)]). The natural morphism

\[
\Omega^*_{X/S} \rightarrow a_*^0(\Omega^*_{X^{(0)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S))
\]

induces a morphism

\[
P_0\Omega^*_{X/S} \rightarrow a_*^0(\Omega^*_{X^{(0)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S))
\]

and the following sequence

\[
0 \rightarrow P_0\Omega^*_{X/S} \rightarrow a_*^0(\Omega^*_{X^{(0)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S)) \rightarrow a_*^1(\Omega^*_{X^{(1)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S)) \rightarrow \cdots
\]

of complexes of \(f^{-1}(\mathcal{O}_S)\)-modules is exact. Here

\[
\iota^{(k)}_* : a_*^k(\Omega^*_{X^{(k)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S))) \rightarrow a_*^{k+1}(\Omega^*_{X^{(k+1)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S)) \quad (k \in \mathbb{N})
\]

is the morphism defined in [Nakk5 (1.3.20.5)].

The following is not included in [Nakk5 (1.3.21)]:

**Corollary 3.11.** The following sequence

\[
0 \rightarrow P_0(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}) \rightarrow a_*^0(\mathcal{E} \otimes_{\mathcal{O}_{X^{(0)}}} \Omega^*_{X^{(0)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S)))
\]

\[
\iota_*^{(0)} \rightarrow a_*^1(\mathcal{E} \otimes_{\mathcal{O}_{X^{(1)}}} \Omega^*_{X^{(1)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S))) \rightarrow \cdots
\]

of complexes of \(f^{-1}(\mathcal{O}_S)\)-modules is exact.

**Proof.** This immediately follows from (3.10). \(\square\)

Set

\[
\Omega^*_k(\mathcal{E}) := a_*^k(\mathcal{E} \otimes_{\mathcal{O}_{X^{(k)}}} \Omega^*_{X^{(k)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{zar}}(X/S))) \quad (k \in \mathbb{N}).
\]

(Our numbering \(k\) is different from the numbering for \(L_k\) in [Il2, p. 394].) Let \(F^{(k)} : \tilde{X}^{(k)} \rightarrow (\tilde{X}^{(k)})'\) be the relative Frobenius morphism of \(\tilde{X}^{(k)}/\tilde{S}\). Let \(F_{\tilde{S}} : \tilde{S} \rightarrow \tilde{S}\) be the Frobenius endomorphism of \(\tilde{S}\) and set \(X[^p] := X \times_{F_{\tilde{S}}} \tilde{S}\). In particular, \(S[^p] := S \times_{S,F_{\tilde{S}}} \tilde{S}\). Note that we do not consider the base change \(X' := X \times_{S,F_{\tilde{S}}} S\) by the Frobenius endomorphism \(F_{\tilde{S}} : S \rightarrow S\) of \(S\) since \(X'\) is not an SNCL scheme.

16
Then \((X^{\bar{p}})^{(k)} = (\tilde{X}^{(k)})')\). Let \(a^{(k)'}: (\tilde{X}^{(k)})' \rightarrow (X^{\bar{p}})^\circ\) be the analogous morphism to \(a^{(k)}\). Let \(F: X \rightarrow X^{\bar{p}}\) be the abrelative Frobenius morphism of \(X\) (1.5.14) induced by the Frobenius endomorphism of \(X\). Let \(\mathcal{F}\) be a quasi-coherent flat \(\mathcal{O}_{X^{\bar{p}}}\)-module. Set

\[ \Omega^i_k(\mathcal{F}) := a^{(k)'*}(\mathcal{F}) \otimes_{(X^{\bar{p}})^{\circ}} \Omega^i_{(\tilde{X}^{(k)})'/\tilde{S}} \otimes_{\mathbb{Z}} \varpi^{(k)}((X^{\bar{p}})^\circ) \quad (k \in \mathbb{Z}_{\geq 0}) \]

for each \(i\). Then the following sequence is exact:

\[ 0 \rightarrow P_0(\mathcal{F} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S}) \rightarrow \Omega^i_0(\mathcal{F}) \xrightarrow{\iota^{(0)}(\mathcal{F})} \Omega^i_1(\mathcal{F}) \xrightarrow{\iota^{(1)}(\mathcal{F})} \cdots \]

Here \(\iota^{(k)}\) is the analogous morphism to \(\iota^{(k)}\) for \(X^{\bar{p}}/S\). Consider the following four conditions (I) \sim (IV):

(I) There exists an isomorphism

\[ C^{-1}_k: \Omega^i_k(\mathcal{F}) \xrightarrow{\sim} F_*\mathcal{H}^i(\Omega^i_k(\mathcal{E})) \]

of \(\mathcal{O}_{(X^{\bar{p}})^{\circ}}\)-modules for any \(i \in \mathbb{Z}_{\geq 0}\) and any \(k \in \mathbb{Z}_{\geq 0}\) fitting into the following commutative diagram

\[
\begin{array}{ccc}
F_*\mathcal{H}^i(\Omega^i_k(\mathcal{E})) & \longrightarrow & F_*\mathcal{H}^i(\Omega^i_{k+1}(\mathcal{E})) \\
C^{-1}_k \uparrow \sim & & \sim \uparrow C^{-1}_{k+1} \\
\Omega^i_k(\mathcal{F}) & \longrightarrow & \Omega^i_{k+1}(\mathcal{F}).
\end{array}
\]

(II) There exists the following morphism

\[ C^{-1}: \mathcal{F} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S} \longrightarrow F_*\mathcal{H}^i(\mathcal{E} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S}) \]

of \(\mathcal{O}_{X^{\bar{p}}}\)-modules which induces the following morphism

\[ C^{-1}: P_k(\mathcal{F} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S}) \longrightarrow F_*\mathcal{H}^i(P_k(\mathcal{E} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S})) \quad (k \in \mathbb{Z}). \]

(III) The following diagram

\[
\begin{array}{ccc}
P_0(\mathcal{F} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S}) & \longrightarrow & \Omega^i_0(\mathcal{F}) \\
C^{-1} \downarrow & & \sim \downarrow C^{-1}_0 \\
F_*\mathcal{H}^i(P_0(\mathcal{E} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S})) & \longrightarrow & F_*\mathcal{H}^i(\Omega^i_0(\mathcal{E}))
\end{array}
\]

is commutative.

(IV) The following diagram is commutative for \(k \in \mathbb{Z}_{\geq 1}\):

\[ F_*\mathcal{H}^i(\text{gr}_k^{\mathcal{P}}(\mathcal{E} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S})) \xrightarrow{\sim} F_*\mathcal{H}^{i-k}(\Omega^i_k(\mathcal{E})) \]

\[ C^{-1} \uparrow \sim \uparrow C^{-1}_k \]

\[ \text{gr}_k^{\mathcal{P}}(\mathcal{F} \otimes \mathcal{O}_{X^{\bar{p}}} \Omega^i_{X^{\bar{p}}/S}) \xrightarrow{\sim} \Omega^i_k(\mathcal{F}). \]
If the conditions (I) and (III) are satisfied, then the following morphism is an isomorphism

\[(3.11.9)\quad C^{-1} : P_0(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}) \cong \mathcal{H}^i(\mathcal{P}_0(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S})).\]

We easily see that the morphism \((3.11.5)\) is an isomorphism by \((3.11.9)\) and \((3.11.8)\).

The following is a generalization of \([\text{Il}2, \text{ Appendice (1.6)}]\):

**Proposition 3.12.** Assume that the condition (I) holds. Then the resulting sequences of \((3.11.1)\) by the operations \(\mathcal{H}^i, B^i\) and \(Z^i\) \((i \in \mathbb{Z}_{\geq 0})\) are exact.

**Proof.** The proof of \((3.12)\) is the same as that of \([\text{Il}2, \text{ Appendice (1.6)}]\). Indeed, the problem is local. Hence we may assume that \(\pi\) is quasi-compact. For \(\mathcal{G} := \mathcal{E} \otimes \mathcal{F}\), set \(K_k^i(\mathcal{G}) := \ker(\Omega^i_{\mathcal{G}}(\mathcal{G}) \to \Omega^i_{k+1}(\mathcal{G}))\). Then the following sequences

\[0 \to K^i_k(\mathcal{E}) \to \Omega^i_k(\mathcal{G}) \to K^i_{k+1}(\mathcal{E}) \to 0\]

and

\[0 \to K^i_k(\mathcal{F}) \to \Omega^i_k(\mathcal{F}) \to K^i_{k+1}(\mathcal{F}) \to 0\]

are exact by \((3.12.1)\) and \((3.12.2)\), respectively. Hence we have the following exact sequence

\[(3.12.1)\quad \cdots \to \mathcal{H}^i(K^i_k(\mathcal{E})) \to \mathcal{H}^i(\Omega^i_k(\mathcal{G})) \to \mathcal{H}^i(K^i_{k+1}(\mathcal{E})) \to \cdots\]

and the following commutative diagram

\[(3.12.2)\]

\[
\begin{array}{cccccc}
0 & \to & F_*\mathcal{H}^i(K^i_k(\mathcal{E})) & \to & F_*\mathcal{H}^i(\Omega^i_k(\mathcal{G})) & \to & F_*\mathcal{H}^i(K^i_{k+1}(\mathcal{E})) & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & K^i_k(\mathcal{F}) & \to & \Omega^i_k(\mathcal{F}) & \to & K^i_{k+1}(\mathcal{F}) & \to & 0
\end{array}
\]

of sequences. Let \(u^i_k\) be the left vertical morphism. If \(k \gg 0\), then \(K^i_k(\mathcal{G}) = 0\) since \(\pi^{(k)}(\emptyset) = \emptyset\). Assume that \(u^i_k\) is an isomorphism for \(\forall k \geq k_0\) for some \(k_0\) and \(\forall i \in \mathbb{Z}_{\geq 0}\). Because \(F_*\mathcal{H}^i(\Omega^i_k(\mathcal{E})) \to F_*\mathcal{H}^i(K^i_k(\mathcal{E}))\) is surjective for \(k = k_0 - 1\), the morphism \(F_*\mathcal{H}^{i+1}(K^i_{k_0-1}(\mathcal{E})) \to F_*\mathcal{H}^{i+1}(\Omega^i_{k_0-1}(\mathcal{E}))\) is injective. Obviously the morphism \(F_*\mathcal{H}^0(K^i_{k_0-1}(\mathcal{E})) \to F_*\mathcal{H}^0(\Omega^i_{k_0-1}(\mathcal{E}))\) is also injective. Hence \(u^i_{k_0-1}\) \((i \in \mathbb{Z}_{\geq 0})\) is an isomorphism. Descending induction on \(k\) shows that the upper sequence of \((3.12.2)\) is exact. Because \(\bar{F}\) is a homeomorphism of topological spaces \((\text{SGA 5 XV Proposition 2 a)}\)), the following sequence

\[0 \to \mathcal{H}^i(K^i_k(\mathcal{E})) \to \mathcal{H}^i(\Omega^i_k(\mathcal{E})) \to \mathcal{H}^i(K^i_{k+1}(\mathcal{E})) \to 0\]

is exact. Now ascending induction on \(i\) shows that the resulting sequences of \((3.10.1)\) by the operations \(Z^i\) and \(B^i\) \((i \in \mathbb{Z}_{\geq 0})\) are exact. \(\square\)

**Proposition 3.13 (cf. \([\text{Il}2, \text{ Corollaire (2.5)}]\)).** Assume that the conditions (I) \sim (IV) hold. Then the resulting sequences of the following sequence by the operations \(\mathcal{H}^i, B^i\) and \(Z^i\) \((i \in \mathbb{Z}_{\geq 0})\) are exact:

\[(3.13.1)\quad 0 \to P_{k-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}) \to P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}) \to \mathcal{H}^i_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/S}) \to 0.\]
Proof. We use the argument of [3.12] again. We may assume that $k \geq 1$. We have the following exact sequence

$$\cdots \rightarrow \mathcal{H}^i(P_{k-1}(E \otimes \mathcal{O}_X \Omega^\bullet_{X/S})) \rightarrow \mathcal{H}^i(P_k(E \otimes \mathcal{O}_X \Omega^\bullet_{X/S})) \rightarrow \mathcal{H}^i(gr_k^p(E \otimes \mathcal{O}_X \Omega^\bullet_{X/S})) \rightarrow \cdots$$

First consider the case $k = 1$. By [5.3.1.4] and the conditions (II) and (IV), we obtain the following commutative diagram

$$\begin{array}{c}
0 \longrightarrow F_*(\mathcal{H}^i(P_0(E \otimes \mathcal{O}_X \Omega^i_{X/S}))) \longrightarrow F_*(\mathcal{H}^i(P_1(E \otimes \mathcal{O}_X \Omega^i_{X/S}))) \longrightarrow \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
$U \hookrightarrow R^if_{U*}(B^i(E \otimes \mathcal{O}_X \Omega^{•}_{X \times_{\mathbb{S}} U}))$, where $f_{U*} : X \times_{\mathbb{S}} U \rightarrow U$ is the structural morphism. Hence, to prove that $R^if_{U*}(B^i(E \otimes \mathcal{O}_X \Omega^{•}_{X \times_{\mathbb{S}} U})) = 0$, we can assume that $B^i(E \otimes \mathcal{O}_X \Omega^{•}_{X \times_{\mathbb{S}} U}) = P_kB^i(E \otimes \mathcal{O}_X \Omega^{•}_{X \times_{\mathbb{S}} U})$ for some $k \in \mathbb{N}$. We can also assume that $\mathbb{X}$ is quasi-compact. Now we obtain the implication (a)$\Rightarrow$(b) by (3.12), (3.9.3) and (4.10). The converse implication follows from (3.13.3) and (3.13).

(2): Obvious.

(3): (3) follows from (3.12).

In the following, we give examples of the integrable connections $(E, \nabla)$’s which satisfy the conditions (I) $\sim$ (IV).

**Proposition 3.15.** (1) $\Omega^i_{X'/\mathbb{S}} = \Omega^i_{X[S]/\mathbb{S}} (i \in \mathbb{N})$.

(2) There exists a morphism

$$C^{-1} : \Omega^i_{X[S]/\mathbb{S}} \rightarrow F_*\mathcal{H}^i(\Omega^{•}_{X/\mathbb{S}}) (i \in \mathbb{N})$$

of $\mathcal{O}_{X[S]}$-modules fitting into the following commutative diagram for $i \in \mathbb{N}$:

$$
\begin{array}{cccccc}
0 & \rightarrow & \Omega^{i-1}_{X[S]/\mathbb{S}} & \xrightarrow{\theta^\wedge} & \Omega^i_{X[S]/\mathbb{S}} & \rightarrow & 0 \\
& & C^{-1} \downarrow \simeq & & C^{-1} \downarrow \simeq & & \\
0 & \rightarrow & F_*\mathcal{H}^{i-1}(\Omega^{•}_{X/\mathbb{S}}) & \xrightarrow{\theta^\wedge} & F_*\mathcal{H}^i(\Omega^{•}_{X/\mathbb{S}}) & \rightarrow & 0.
\end{array}
$$

Here

$$C^{-1} : \Omega^i_{X[S]/\mathbb{S}} \xrightarrow{\sim} F_*\mathcal{H}^i(\Omega^{•}_{X/\mathbb{S}})$$

is the log inverse Cartier isomorphism due to Kato ([K (4.12)]). Consequently the morphism (3.15.1) is an isomorphism.

(3) The isomorphism (3.15.1) induces the following isomorphism

$$C^{-1} : P_k\Omega^i_{X[S]/\mathbb{S}} \xrightarrow{\sim} F_*\mathcal{H}^i(P_k\Omega^{•}_{X/\mathbb{S}}) (i, k \in \mathbb{N}).$$

(4) The following diagram

$$
\begin{array}{ccc}
\text{gr}^P_i(\Omega^i_{X[S]/\mathbb{S}}) & \xrightarrow{\text{Res.} \sim} & a_*^{(k-1)}(\Omega^{i-k}_{X[S]/\mathbb{S}} \otimes_{\mathbb{Z}} \mathbb{W}^{(k-1)}_{\text{zar}}(X'/\mathbb{S})) \\
\downarrow \simeq & & \downarrow \simeq \\
F_*\mathcal{H}^i(\text{gr}^P_k(\Omega^{•}_{X/\mathbb{S}})) & \xrightarrow{\text{Res.} \sim} & F_*a_*^{(k-1)}(\Omega^{i-k}_{X[S]/\mathbb{S}} \otimes_{\mathbb{Z}} \mathbb{W}^{(k-1)}_{\text{zar}}(X'/\mathbb{S}))
\end{array}
$$

is commutative for $k \in \mathbb{Z}_{\geq 1}$.

(5) The following diagram

$$
\begin{array}{ccc}
P_k\Omega^i_{X[S]/\mathbb{S}} & \xrightarrow{\sim} & a_*^{(0)}(\Omega^i_{X[S]/\mathbb{S}} \otimes_{\mathbb{Z}} \mathbb{W}^{(k-1)}_{\text{zar}}(X'/\mathbb{S})) \\
\downarrow \simeq & & \downarrow \simeq \\
F_*\mathcal{H}^i(P_k\Omega^{•}_{X/\mathbb{S}}) & \rightarrow & F_*a_*^{(0)}(\Omega^{i}_{X[S]/\mathbb{S}} \otimes_{\mathbb{Z}} \mathbb{W}^{(k-1)}_{\text{zar}}(X'/\mathbb{S}))
\end{array}
$$

20
is commutative.

Consequently the isomorphism \(3.15.1\) satisfies the conditions (I) ~ (IV) for the case \(\mathcal{E} = \mathcal{O}_X\) and \(\mathcal{F} = \mathcal{O}_{X[p]}\).

**Proof.** (1): Because the morphism \(X \to S\) is integral, \((X')^o = (X[p])^o\). (1) follows from [K] (1.7).

(2): The existence of \(3.15.1\) is a special case of [O2] V (4.1.1)]. The commutativity of \(3.15.2\) follows from [loc. cit.] and [K] (4.12)]. (Note that \(C^{-1}(\theta) = \theta\).

(3), (4), (5): By the characterization of \(C^{-1}\) in [O2] V (4.1.1)], the morphism \(3.15.1\) induces the following morphism

\[ C^{-1}: P_s \Omega^i_{X[p]/S} \to F_s \mathcal{H}^i(P_s \Omega^*_{X/S}) \quad (k \in \mathbb{Z}). \]

We also have the commutative diagram \(3.15.3\). Hence descending induction on \(k\) shows that the morphism above is an isomorphism. We also have the commutative diagram \(3.15.4\). \(\square\)

**Example 3.16.** Let \(\mathcal{E}'\) be a quasi-coherent flat \(\mathcal{O}_{X[p]}\)-module. Let \(F: X \to X[p]\) be the induced morphism of the Frobenius endomorphism of \(X\). Set \((\mathcal{E}, \nabla) \coloneqq (F^*\mathcal{E}', \text{id}_{\mathcal{E}'} \otimes d)\). By (3.15) (1) and [O1] (1.2.5) we have the following isomorphism

\[ C^{-1}: \mathcal{E}' \otimes_{\mathcal{O}_{X[p]}} \Omega^i_{X[p]/S} \to F_s \mathcal{H}^i(\mathcal{E} \otimes_{\mathcal{O}_{X[p]}} \Omega^i_{X[p]/S}) \quad (i \in \mathbb{N}). \]

Hence, by the same proof as that of (3.15), we have the following isomorphism

\[ 3.16.1: \mathcal{E}' \otimes_{\mathcal{O}_{X[p]}} \Omega^i_{X[p]/S} \to F_s \mathcal{H}^i(\mathcal{E} \otimes_{\mathcal{O}_{X[p]}} \Omega^i_{X[p]/S}) \quad (i \in \mathbb{N}) \]

of \(\mathcal{O}_{X[p]}\)-modules fitting into the following commutative diagram for \(i \in \mathbb{N}\):

\[
\begin{array}{cccccc}
0 & \to & \mathcal{E}' \otimes_{\mathcal{O}_{X[p]}} \Omega^i_{X[p]/S[p]} & \to & \mathcal{E}' \otimes_{\mathcal{O}_{X[p]}} \Omega^i_{X[p]/S} & \to & 0 \\
\downarrow C^{-1} & & \downarrow \theta^\wedge & & \downarrow C^{-1} & & \downarrow C^{-1} & & \approx & & \approx \\
0 & \to & F_s \mathcal{H}^{-i}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}) & \to & F_s \mathcal{H}^i(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}) & \to & 0.
\end{array}
\]

One can check that the isomorphism \(3.16.1\) satisfies the conditions (I) ~ (IV) without difficulty.

Note that the log \(p\)-curvature of \((\mathcal{E}, \nabla)\) in [O1] is zero.

**Remark 3.17.** (1) The inverse Cartier morphism \(C^{-1}: \Omega^i_{X/S} \to F_s(\mathcal{H}^i(\Omega^i_{X/S}))\) fits into the following commutative diagram

\[ 3.18.1 \]

\[
\begin{array}{cccccc}
\Omega^i_{X/S} & \to & P_0 \Omega^i_{X/S} & \to & F_s(\mathcal{H}^i(P_0 \Omega^i_{X/S})).
\end{array}
\]

(2) By using the log inverse Cartier isomorphism \(3.15.1\), the commutative diagram \(3.15.2\) and the argument of the proof of (3.12), we can also obtain \(3.3\) for \(Y = X\) and \((\mathcal{E}, \nabla)\) in (3.16).
Now we can give the second proof of \[ (3.13) \].

Replace \( S \) by \( \tilde{T} := S \times_S \tilde{T} \). Set \( X_{\tilde{T}} := X \times_S S_{\tilde{T}} \). Let \( f_{\tilde{T}} : X_{\tilde{T}} \rightarrow S_{\tilde{T}} \) be the structural morphism. By the assumption, the condition (a) in (3.13) for \( \mathcal{E} = \mathcal{O}_X \) is satisfied. Hence \( R^j f_{\tilde{T}}^*(B^1 H_{E/T}^1(K_{X_{\tilde{T}}/S_{\tilde{T}}})) = 0 \) for \( \forall i \) and \( \forall j \). By (2.3.1), \( R^j f_{\tilde{T}}^*(B^1 H_{E/T}^1) = 0 \). More generally, we obtain the following:

**Theorem 3.18.** Let \( (\mathcal{E}, \nabla) \) be the integrable connection \( (3.8.3) \) satisfying the conditions (I) \( \sim \) (IV). Let \( p_T : X_T \rightarrow X \) be the projection. If \( a(k)^*(\mathcal{E}, \nabla) \) for \( \forall k \in \mathbb{N} \) is ordinary, then the induced connection

\[
p_T^*(\mathcal{E}) \rightarrow p_T^*(\mathcal{E}) \otimes_{\mathcal{O}_{X_T}} \Omega^1_{X_T/T}
\]

by \( (\mathcal{E}, \nabla_{X/S}) \) is log ordinary.

**Proof.** One has to use only (\ref{13}) and the argument before (3.18).

In the following we give another proof of the existence of the log inverse Cartier isomorphism satisfying the conditions (I) \( \sim \) (IV) for a unit root \( F \)-crystal on \( \tilde{X}/\mathcal{W}(\tilde{S}) \) when \( \tilde{S} \) is a perfect scheme of characteristic \( p > 0 \).

Assume that \( \tilde{S} \) is a perfect scheme of characteristic \( p > 0 \).

Let \( E \) be a unit root \( F \)-crystal of \( \mathcal{O}_{\tilde{X}/\mathcal{W}(\tilde{S})} \)-module. Set \( \mathcal{E} := E_{\tilde{X}} \) and \( \mathcal{F} := q^*(\mathcal{E}) \).

Then, by (\ref{3.1.1}), we have the isomorphism \( (3.11.3) \). In the following we review a log version of Etesse’s results (\ref{3.2.3}, (3.3.1), (3.4.1), (4.2.1)) proved in (\ref{Nakk5}).

Let \( Y/S \) be a log smooth scheme. Let \( * \) be ‘ or nothing. Let \( E \) be a unit root \( F \)-crystal of \( \mathcal{O}_{\tilde{Y}/\mathcal{W}(\tilde{S})} \)-module. Set \( E_n := E_{(\mathcal{W}_n(Y), \log_{\mathcal{W}_n(Y)})} \).

\( (\mathcal{W}_n \mathcal{O}_Y^* \otimes \mathcal{O}_Y)^* \) be the complex defined in (\ref{Nakk5}, (2.2.11.10)). Let \( F : \mathcal{W}_n(Y) \rightarrow \mathcal{W}_n(Y) \) be the Frobenius endomorphism. Though we have assumed that \( \tilde{S} = \text{Spec}(\mathcal{O}) \) in [\cite{loc. cit.}], the same argument as that in [\cite{loc. cit.}] works for the case where \( \tilde{S} \) is a perfect scheme of characteristic \( p > 0 \).

**Proposition 3.19 (\cite{Nakk5}, (2.2.15)).** The following formula holds:

\[
F^r(\text{Fil}^n(E_{n+r} \otimes_{\mathcal{W}_{n+r}(\mathcal{O}_Y)} (\mathcal{W}_{n+r}\mathcal{O}_Y^*))) = F^r \nabla V^n(E_r \otimes_{\mathcal{W}_n(\mathcal{O}_Y)} (\mathcal{W}_r\mathcal{O}_Y^{-1})^*).
\]

Consequently the morphism \( F^r : E_{n+r} \otimes_{\mathcal{W}_{n+r}(\mathcal{O}_Y)} (\mathcal{W}_{n+r}\mathcal{O}_Y^* \otimes_{\mathcal{O}_Y} (\mathcal{W}_n\mathcal{O}_Y^*)^*) \) of \( \mathcal{W}_n(\mathcal{O}_Y)^* \)-modules induces the following morphism

\[
F^r : E_n \otimes_{\mathcal{W}_n(\mathcal{O}_Y)} (\mathcal{W}_n\mathcal{O}_Y^*)^* 
\]

\[
F^r \{ E^n_{n+r} \otimes_{\mathcal{W}_{n+r}(\mathcal{O}_Y)} (\mathcal{W}_{n+r}\mathcal{O}_Y^* \otimes_{\mathcal{O}_Y} (\mathcal{W}_n\mathcal{O}_Y^*)^*) \}
\]

of \( \mathcal{W}_n(\mathcal{O}_Y)^*-\)modules. For the case \( r = n \), \( F^n \) induces the following morphism

\[
F^n : E_n \otimes_{\mathcal{W}_n(\mathcal{O}_Y)} (\mathcal{W}_n\mathcal{O}_Y^*)^* 
\]

\[
F^n \mathcal{H}^i(E_n \otimes_{\mathcal{W}_n(\mathcal{O}_Y)} (\mathcal{W}_n\mathcal{O}_Y^*)^*)
\]

of \( \mathcal{W}_n(\mathcal{O}_Y)^*\)-modules.
Theorem 3.20 ([Nakkk\textsuperscript{3} (2.2.16)]). The morphism $V^r: E_n \otimes_{W_n(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_n})^* \rightarrow E_{n+r} \otimes_{W_{n+r}(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_{n+r}})^*$ of $W_n(\mathcal{O}_Y)^*$-modules induces the following morphism

\begin{equation}
V^r: F^*_r \{E_n \otimes_{W_n(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_n})^*/F^r \nabla V^n(E_r \otimes_{W_r(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_r})^*)\} \rightarrow E_{n+r} \otimes_{W_{n+r}(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_{n+r}})^*.
\end{equation}

There exists a generalized Cartier isomorphism

\begin{equation}
\tilde{C}^r: F^*_r \{E_n \otimes_{W_n(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_n})^*/F^r \nabla V^n(E_r \otimes_{W_r(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_r})^*)\}
\end{equation}

of $W_n(\mathcal{O}_Y)^*$-modules, which is the inverse of $\tilde{F}^r$. The morphism $\tilde{C}^r$ satisfies a relation $\tilde{C}^r \circ \tilde{F}^r = V^r$. In particular, there exist the following isomorphisms

\begin{equation}
\tilde{F}^n: E_n \otimes_{W_n(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_n})^* \rightarrow F^*_n(\mathcal{H}^t(E_n \otimes_{W_n(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_n})^*))
\end{equation}

and

\begin{equation}
\tilde{C}^n: F^*_n(\mathcal{H}^t(E_n \otimes_{W_n(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_n})^*)) \rightarrow E_n \otimes_{W_n(\mathcal{O}_Y)} (\Omega^1_{\tilde{E}_n})^*
\end{equation}

of $W_n(\mathcal{O}_Y)^*$-modules, which are the inverse of another.

Proposition 3.21. There exist morphisms \([311,3]\) and \([311,4]\) satisfying I, II, III and IV for the case $\mathcal{E} = \mathcal{F} = E_1$.

Proof. We define an isomorphism

\begin{equation}C^{-1}: \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \rightarrow F_* (\mathcal{H}^t(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X))\end{equation}

of $\mathcal{O}_Y$-modules as the following composite isomorphism:

$C^{-1}: \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{id \otimes C^{-1}} \mathcal{E} \otimes_{\mathcal{O}_X} F_* (\mathcal{H}^t(\Omega^1_X)) \xrightarrow{\tilde{F}^*} F_* (\mathcal{H}^t(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X))$

This composite morphism preserves the filtration $P$. Because $\tilde{F}$ is the induced morphism by "id", the commutativity of the diagrams \([311,7]\) and \([311,8]\) are obvious.\hfill \Box

In the rest of this section, we give the open analogue of \([311,8]\).

Lemma 3.22. Let the notations be as in \([24]\). Let $\mathcal{E}$ be a quasi-coherent flat $\mathcal{O}_X$-module. Let $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{(X,D)/S}$ be an integrable connection. (Note that $\Omega^1_{(X,D)/S} = \Omega^1_{X/S} (\log D)$). For $k \in \mathbb{Z}_{\geq 0}$, let $D^{(k)}$ be the closed subscheme of $X$ defined in \([23]\) and let $a^{(k)}: D^{(k)} \rightarrow X$ be the natural morphism. Assume that $\mathcal{E}$ is locally generated by horizontal sections of $\nabla$. Set $P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{(X,D)/S}) := \mathcal{E} \otimes_{\mathcal{O}_X} P^* k \Omega^1_{(X,D)/S}$. Set

$\Omega^*_k (\mathcal{E}) := a^{(k)} (a^{(k)*} (\mathcal{E}) \otimes_{\mathcal{O}_{D^{(k)}}} \Omega^1_{D^{(k)}/S} \otimes_{\mathbb{Z}} \mathbb{Z}^{(k)} (D/S)) \quad (k \in \mathbb{Z}_{\geq 0})$

for each $k$. Then the following hold:

1. There exists the following Poincaré residue isomorphism of complexes:

\begin{equation}
\text{gr}^P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{(X,D)/S}) \xrightarrow{\sim} \Omega^*_{-k} (\mathcal{E}).
\end{equation}
(2) The following sequence is exact:

\[(3.22.2) \quad 0 \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{(X,D)/S}(-D) \rightarrow \Omega^\bullet_0(\mathcal{E}) \xrightarrow{(0)^*} \Omega^\bullet_1(\mathcal{E}) \xrightarrow{(1)^*} \cdots\]

of complexes of \(f^{-1}(\mathcal{O}_S)\)-modules is exact.

Proof. (1): By [NS, (2.2.21.3)] we have the following Poincaré residue isomorphism:

\[\text{Res} : \text{gr}_k^p(\Omega^\bullet_{(X,D)/S}) \xrightarrow{\sim} a_k^{(k)}(\Omega_{D(k)/S}^{-k} \otimes_{\mathbb{Z}} \varpi_{\text{zar}}^k(D/S)).\]

(3.22.1) immediately follows from this isomorphism.

(2): By [DI, (4.2.2) (a), (c)] the sequence (3.22.2) for the case \(\mathcal{E} = \mathcal{O}_X\) is exact. Now we see that the sequence (3.22.2) for the general case is exact.

\[\square\]

Definition 3.23. We say that \((\mathcal{E}, \nabla)\) is log ordinary with compactly support if \(R^j f_* (B^i(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{(X,D)/S}(-D))) = 0\) for any \(i \in \mathbb{Z}_{\geq 0}\) and \(j \in \mathbb{Z}_{\geq 0}\).

The following (1) is the open analogue of (3.18).

Theorem 3.24. Let the notations be as in (2.2). Let \(F : (X, D) \rightarrow (X', D')\) be the relative Frobenius morphism over \(S\). Let \(\mathcal{E}'\) be a quasi-coherent \(\mathcal{O}_{X'}\)-module. For \(k \in \mathbb{Z}_{\geq 0}\), let \(a^{(k)} : D'(k) \rightarrow X'\) be the natural morphism. Set

\[\Omega_k^i(\mathcal{E}') := a'^{(k)}(a^{(k)^*}(\mathcal{E}') \otimes_{\mathcal{O}_{D'}(k)} \Omega_{D'(k)/S}^i \otimes_{\mathbb{Z}} \varpi_{\text{zar}}(D'/S)) \quad (k \in \mathbb{Z}_{\geq 0})\]

for each \(i\). Assume that the following two conditions (I) and (II) are satisfied:

(I): There exists an isomorphism

\[(3.24.1) \quad C_k^{-1} : \Omega_k^i(\mathcal{E}') \xrightarrow{\sim} F_* \mathcal{H}^i(\Omega_k^\bullet(\mathcal{E}))\]

of \(\mathcal{O}_{D'(k)}\)-modules for any \(i \in \mathbb{Z}_{\geq 0}\) and any \(k \in \mathbb{Z}_{\geq 0}\) fitting into the following commutative diagram

\[(3.24.2) \quad \begin{array}{ccc}
\Omega_k^i(\mathcal{E}') & \xrightarrow{\sim} & \Omega_k^i(\mathcal{E}') \\
| \Uparrow C_k^{-1} | & | \Uparrow C_k^{-1} |
\end{array} \]

(II) The following diagram is commutative for \(k \in \mathbb{Z}_{\geq 0}\):

\[(3.24.3) \quad \begin{array}{ccc}
\text{gr}_k^p(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{(X,D)/S}^\bullet) & \xrightarrow{\sim} & F_* \mathcal{H}^{-k}(\Omega_k^\bullet(\mathcal{E})) \\
| \Uparrow C_k^{-1} | & | \Uparrow C_k^{-1} |
\end{array} \]

Then the following hold:

(1) If \(a^{(k)*}(\mathcal{E}, \nabla)\) for all \(k \in \mathbb{N}\) is ordinary, then \((\mathcal{E}, \nabla)\) is log ordinary.

(2) If \(a^{(k)*}(\mathcal{E}, \nabla)\) for all \(k \in \mathbb{N}\) is ordinary, then \((\mathcal{E}, \nabla)\) is log ordinary with compact support.

Proof. (1): Because the proof of (1) by using (3.22) (1) is easier than that of (3.18), we leave the detail of the proof to the reader.

(2): Because the proof of (2) by using (3.22) (2) is easier than that of (3.18), we leave the detail of the proof to the reader.
4 The third proof of (1.3)

In this section we give the proof of (1.3) without using the $p$-adic weight spectral sequence (2.1) nor the filtration $P$ on $\Omega^*_X$. Instead we use the $p$-adic weight spectral sequence for $\mathcal{W}_X$ in the case $S = s$. To obtain this spectral sequence, we use the filtration $P$ on $\mathcal{W}_X$.

First assume that $S = s$. Then we have proved the following in Nakke5 as a very special case of Nakke5 (2.3.37).

**Theorem 4.1.** Let $i$ be a nonnegative integer. Set

$$
E^{-k,h+k}_1 = \begin{cases} 
0 & (k < 0), \\
H^h(X, P_0 \mathcal{W}_X^i) & (k = 0), \\
H^{h-k}(\tilde{X}^{(k-1)}, \mathcal{W}_X^i \otimes_{\mathcal{O}_{\text{zar}}} (X/\kappa))(-k) & (k > 0).
\end{cases}
$$

Then there exists the following spectral sequence

(4.1.1) \[ E^{-k,h+k}_1 \Rightarrow H^h(X, \mathcal{W}_X^i). \]

There also exists the following spectral sequence

(4.1.2) \[ E^{k,h-k}_1 = H^{h-k}(\tilde{X}^{(k-1)}, \mathcal{W}_X^i \otimes_{\mathcal{O}_{\text{zar}}} (X/\kappa)) \Rightarrow H^h(X, P_0 \mathcal{W}_X^i). \]

The spectral sequences (4.1.1) and (4.1.2) are compatible with the operator $F$.

**Proposition 4.2** ([Nakke6 (3.4)]). The obvious analogue of (2.1) holds for $\mathcal{W}_X$.

Now we give the third proof of (1.3) as follows.

First assume that $S = s$. By (3.4) the operator $F : H^i(X, \mathcal{W}_X^i) \rightarrow H^i(X, \mathcal{W}_X^i)$ is bijective for any $i$ and $j$. By (3.2) the operator $F : H^i(X, \mathcal{W}_X^i) \rightarrow H^i(X, \mathcal{W}_X^i)$ is bijective for any $i$ and $j$ if and only if $H^j(X, B \mathcal{W}_X^i) = 0$ (cf. IR IV (4.13)). Let $R$ be the Cartier-Dieudonné-Raynaud algebra of $\kappa$ (IR). Set $R_n := R/(V^n R + dV^n R)$. Then the following three facts hold by Nakke3 (6.21.1), (6.16.1), (6.27):

(i) $\text{Im}(F^n : W_n \mathcal{W}_X \rightarrow W_n \mathcal{W}_X) = \text{Ker}(d : W_n \mathcal{W}_X \rightarrow W_n \mathcal{W}_X)$,

(ii) $d^{-1}(p^n W_n \mathcal{W}_X^{i+1}) = F^n W_n \mathcal{W}_X^i$,

(iii) $R_n \otimes_R \mathcal{W}_X^i = W_n \mathcal{W}_X^i$.

As noted in the proof of (L.4.1) (for the case $W_n \mathcal{W}_X^i$ for a log smooth scheme over a fine log scheme $Y$ whose underlying scheme is Spec($\kappa$)), these imply that the following sequence

$$
0 \rightarrow H^i(X, \mathcal{W}_X^i)/(F^n + V^n)H^i(X, \mathcal{W}_X^i) \xrightarrow{d} H^i(X, BW_n \mathcal{W}_X^i) \rightarrow (V^n)^{-1}F^n H^{i+1}(X, \mathcal{W}_X^i)/F^n H^{i+1}(X, \mathcal{W}_X^i) \rightarrow 0.
$$

is exact by the argument of the log version of IR IV (4.13). Hence $H^i(X, BW_n \mathcal{W}_X^i) = 0$ for any $i$, $j$ and $n$. In particular, $H^i(X, BW_1 \mathcal{W}_X^i) = 0$ for any $i$ and $j$. This is equivalent to the vanishing of $H^j(X, B \mathcal{W}_X^i) = 0$. By (3.3), $H^i(X, B \mathcal{W}_X^i) = 0$ for any $i$ and $j$.

In the case of the general $S$, the rest of the proof is the same as the proof after (2.3).
Let $n$ be a positive integer. Let $Y/s$ be a log smooth scheme. By [Nakk5] (2.2.3.1) we have the following exact sequence:

$$(4.2.1) \quad 0 \to \mathcal{W}_n \Omega^\bullet_Y[-1] \to \mathcal{W}_n \Omega^\bullet_Y \to 0.$$

Because this exact sequence is compatible with projections, we have the following exact sequence:

$$(4.2.2) \quad 0 \to \mathcal{W} \Omega^\bullet_Y[-1] \to \mathcal{W} \Omega^\bullet_Y \to 0.$$

One can generalize (3.1) in the case $s \to 0$ of $\mathcal{W}$ $\Omega^\bullet_Y$. By the operations $B^i$, $Z^i$ and $\mathcal{H}^i$, $(i \in \mathbb{Z}_{\geq 0})$ are exact. Consequently the resulting sequences of $\mathcal{W} \Omega^\bullet_Y$ by the operations $B^i$, $Z^i$ and $\mathcal{H}^i$, $(i \in \mathbb{Z}_{\geq 0})$ are exact.

**Proposition 4.3.** For each $i$, the resulting sequences of $\mathcal{W} \Omega^\bullet_Y$ by the operations $B^i$, $Z^i$ and $\mathcal{H}^i$, $(i \in \mathbb{Z}_{\geq 0})$ are exact. Consequently the resulting sequences of $\mathcal{W} \Omega^\bullet_Y$ by the operations $B^i$, $Z^i$ and $\mathcal{H}^i$, $(i \in \mathbb{Z}_{\geq 0})$ are exact.

**Proof.** This is a local problem. We may assume that there exists a log smooth lift $\mathcal{Y}$ of $Y$ over $\mathcal{W}_2n(s)$. Because we have the following commutative diagram

$$
\begin{array}{cccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & \mathcal{W}_n \Omega^\bullet_Y[-1] & \mathcal{W}_n \Omega^\bullet_Y & 0 \\
\quad \downarrow{p^n} & \quad \downarrow{p^n} & \quad \downarrow{p^n} \\
0 & \mathcal{W}_n \Omega^\bullet_Y[-1] & \mathcal{W}_n \Omega^\bullet_Y & 0 \\
\quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & \mathcal{W}_n \Omega^\bullet_Y[-1] & \mathcal{W}_n \Omega^\bullet_Y & 0 \\
\quad \downarrow & \quad \downarrow & \quad \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
$$

we have the following commutative diagram of exact sequences:

$$
\begin{array}{cccc}
\cdots & \mathcal{H}^{i-1}(\Omega^\bullet_{}/W_2n(s)) & \mathcal{H}^i(\Omega^\bullet_{}/W_n) & \mathcal{H}^i(\Omega^\bullet_{}/W_2n(s)) \\
\downarrow{p^n} & \downarrow{p^n} & \downarrow{p^n} \\
\cdots & \mathcal{H}^{i+1}(\Omega^\bullet_{}/W_2n(s)) & \mathcal{H}^{i+1}(\Omega^\bullet_{}/W_n) & \mathcal{H}^{i+1}(\Omega^\bullet_{}/W_2n(s)) \\
\end{array}
$$

We may also assume that there exists a splitting $i: \Omega^\bullet_{}/W_2n(s) \to \Omega^\bullet_{}/W_2n(s)$ of the projection $\mathcal{W}_n \Omega^\bullet_{}/W_2n(s) \to \mathcal{W}_n \Omega^\bullet_{}/W_2n(s)$ as in the proof of (3.3). Hence the following sequence

$$
0 \to \mathcal{H}^{i-1}(\Omega^\bullet_{}/W_2n(s)) \to \mathcal{H}^i(\Omega^\bullet_{}/W_n) \to \mathcal{H}^i(\Omega^\bullet_{}/W_2n(s)) \to 0 \quad (m = n, 2n)
$$

is split. Hence the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{H}^i(\Omega^\bullet_{X}/W_n) & \mathcal{H}^{i+1}(\Omega^\bullet_{X}/W_n) \\
\downarrow{p^n} & \downarrow{p^n} \\
\mathcal{H}^{i+1}(\Omega^\bullet_{X}/W_n) & \mathcal{H}^{i+1}(\Omega^\bullet_{X}/W_n) \\
\end{array}
$$

(4.3.1)
Consequently
\[ Z^{ie}(\Omega^i_X/W_n) = \iota(Z^{ie}(\Omega^i_X/W_n(s))) \oplus \theta \wedge (Z^{ie-1}(\Omega^i_X/W_n(s))), \]
\[ B^{ie}(\Omega^i_X/W_n) = \iota(B^{ie}(\Omega^i_X/W_n(s))) \oplus \theta \wedge (B^{ie-1}(\Omega^i_X/W_n(s))) \]
and
\[ H^i(H^*(\Omega^i_X/W_n)) = \iota(H^i(H^*(\Omega^i_X/W_n(s)))) \oplus \theta \wedge (H^{i-1}(\Omega^i_X/W_n(s))). \]
\[ \tag{\Box} \]

**Remark 4.4.** We can give another proof of [3] by using the following inverse log Cartier isomorphisms (cf. [2, (11.1)]) and by using the argument in the proof of [4.13].

\[ C^{-n}: W_n\Omega^i_X \rightsquigarrow H^i(W_n\Omega^i_X) \]

and

\[ C^{-n}: W_n\Omega^i_X \rightsquigarrow H^i(W_n\Omega^i_X) \]

fitting into the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & W_n\Omega^{i-1}_Y & \longrightarrow & W_n\Omega^i_Y & \longrightarrow & 0 \\
& & \downarrow{\theta} & & \downarrow{\theta} & & \\
& & W_n\Omega^i_Y & \longrightarrow & W_n\Omega^i_Y & \longrightarrow & 0 \\
& & \downarrow{c^{-n}} & & \downarrow{c^{-n}} & & \\
0 & \longrightarrow & H^{i-1}(W_n\Omega^i_X) & \longrightarrow & H^i(W_n\Omega^i_X) & \longrightarrow & 0.
\end{array}
\]

5 Lower semi-continuity of log genera

In this section we give applications of the finite length version of the spectral sequence (5.1.1). This has been proved in [3].

**Theorem 5.1** ([3, (11.1)n]). Let \( i \) be a nonnegative fixed integer. Then there exists the following spectral sequence:

\[ E_1^{k,h+k} = \bigoplus_{j \geq \max(-k,0)} H^{h-i-j}(X^{(2j+k)}, W_n\Omega^{i-j-k}_X(\mathcal{O}_X)) \]  
\[ \Rightarrow H^{h-i}(X, W_n\Omega^i_X) \]  
\[ (h \in \mathbb{N}). \]

Let \( X \) be a proper log smooth scheme over \( s \) of pure dimension \( d \). Let \( K_0 \) be the fraction field of \( W \). Let \( (\mathcal{T}, \mathcal{A}) \) be a ringed topos. For an \( \mathcal{A} \)-module \( \mathcal{F} \) of \( \mathcal{T} \) and for a positive integer \( r \), denote \( \mathcal{F} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{F} \) by \( \mathcal{F}^r \).

**Definition 5.2.** (1) (cf. [21 §11.2]) We call

\[ p_g(X/s, n, r) := \text{length}_{W_n}(H^0(X, W_n\Omega^i_X)) \]
the log plurigenus of \( X/s \) of level \( n \). When \( n = 1 \), we call \( p_g(X/s, n, r) \) the log plurigenus of \( X/s \). When \( r = 1 \), we call \( p_g(X/s, n, r) \) the log genus of \( X/s \). When \( r = 1 \) and \( n = 1 \), we call \( p_g(X/s, n, r) \) the log genus of \( X/s \).
(2) (the “vertical” log version of the Iitaka-Kodaira dimension defined in \[11\] and \[12\] §10.5, §11.2) Set
\[
\kappa(X/s, n) := \lim_{r \to \infty} \frac{\log p_g(X, n, r)}{\log r}.
\]
We call \(\kappa(X/s, n)\) the log Iitaka-Kodaira dimension of \(X/s\) of level \(n\). We call \(\kappa(X/s, 1)\) the log Iitaka-Kodaira dimension of \(X/s\) and we denote it by \(\kappa(X/s)\).

(3) We call
\[
p_g(X/W(s), r) := \text{rank}_{W}(H^0(X, \otimes_{W(C_X)}^W \Omega^1_X))
\]
the log Witt plurigenus of \(X/s\). When \(r = 1\), we call \(p_g(X/W(s), r)\) the log Witt genus of \(X/s\).

(4) Set
\[
\kappa(X/W(s)) := \lim_{r \to \infty} \frac{\log p_g(X, \infty, r)}{\log r}.
\]
We call \(\kappa(X/W(s))\) the log Witt-Iitaka-Kodaira dimension of \(X/W(s)\).

First we check that \(p_g(X/W(s), r) \neq \infty\). To prove this, we recall the following:

**Proposition 5.3** ([Nakk6 (3.10)]). Let \(t\) be a fine log scheme whose underlying scheme is \(\text{Spec}(\kappa)\). Let \(Z/t\) be a log smooth scheme of Cartier type. Let \(F_{W_n(Z)} : W_n(Z) \to W_n(Z)\) be the Frobenius endomorphism of \(W_n(Z)\). Then the following hold:

(1) The following sequence
\[
(F_{W_n(Z)})(\Omega^1_{Z}) \xrightarrow{\nu} \Omega^1_{Z} \xrightarrow{F_n} Z_0\Omega^1_{Z} \to 0
\]
is an exact sequence of \(W_{n+1}(O_Z)\)-modules.

(2) The following sequence
\[
W_{n+1}(O^1_Z) \xrightarrow{F} (F_{W_n(Z)})(\Omega^1_{Z}) \xrightarrow{F_{W_n(Z)}(\nu)} B_n\Omega^1_{Z} \to 0
\]
is an exact sequence of \(W_{n+1}(O_Z)\)-modules.

The following is a log version of \[11\] II (2.16):

**Corollary 5.4.** Let \(i, j\) be nonnegative integers. If \(\dim_{\kappa} H^j(Y, Z_0\Omega^1_{Y/i}) (n \in \mathbb{Z}_{\geq 0})\) is bounded, then \(\dim_{\kappa} H^j(Y, (W_O^1)'/V H^j(Y, W_O^1)) < \infty\). Furthermore, \(\dim_{\kappa} H^j(Y, B_n\Omega^1_{Y/i}) (n \in \mathbb{Z}_{\geq 0})\) is bounded, then \(H^j(Y, (W_O^1)'/V H^j(Y, W_O^1))\) is a finitely generated \(W\)-module.

**Proof.** The proof is the same as that of \[11\] II (2.16) by using (5.3). \(\square\)

**Corollary 5.5.** Let \(i\) be a nonnegative integer. Then \(H^0(Y, (W_O^1)'\nu)\) is a free \(W\)-module of finite type.

**Proof.** The proof is the same as that of \[11\] II (2.17) by using (5.4). \(\square\)

Consider the spectral sequence \[5.1.1\] for the case \(i = d:\)
\[
E_1^{i-k, h+k} = \bigoplus_{j \geq \text{max}(-k, 0)} H^{h-d-j}(X^{(2j+k)}, W_n\Omega^d_{X^{(2j+k)}}, \nu)(-j-k) \Rightarrow H^{h-d}(X, W_n\Omega^d_X) (h \in \mathbb{N}).
\]

Let \(\Gamma(X)\) be the dual graph of the simple normal crossing variety \(X/\kappa\).
Theorem 5.6. (1) The following inequalities hold:

\begin{equation}
(5.6.1)
\end{equation}

\[
p_g(\mathcal{O}(0)/\kappa, n, 1) \leq p_g(X/s, n, 1) \leq p_g(\mathcal{O}(0)/\kappa, n, 1) + \sum_{k=1}^{d} \text{length}_{\mathcal{O}_n} \ker(H^0(\mathcal{O}(k), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d-k}(X/k)) \to H^1(\mathcal{O}(k-1), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d-k+1}(X/k)) \leq \sum_{s=0}^{d} p_g(\mathcal{O}(s)/\kappa, n, 1).
\]

(2) Let \( n \) be a positive integer. If \( \dim \mathcal{O}(k) = 1 \), then \( p_g(X/s, n, 1) = p_g(\mathcal{O}(0)/\kappa, n, 1) + \text{length}_{\mathcal{O}_n}(H^1(\mathcal{O}(X), \mathcal{O}_n)). \)

(3) Let \( n \) be a positive integer. Set

\[ \Psi^1(\mathcal{O}, n) := \text{length}_{\mathcal{O}_n} \text{Coker}(H^1(\mathcal{O}(0), \mathcal{O}_n(\mathcal{O}(0))) \to H^1(\mathcal{O}(1), \mathcal{O}_n(\mathcal{O}(0))))). \]

If \( \dim \mathcal{O} = 2 \) and if the boundary map \( d_2^{2,4} : E_2^{2,4} \to E_2^{0,3} \) of \( (5.5.1) \) is zero, then the following formula holds:

\[
p_g(X/s, n, 1) = p_g(\mathcal{O}(0)/\kappa, n, 1) + \Psi^1(\mathcal{O}, n) + \text{length}_{\mathcal{O}_n}(H^2(\mathcal{O}(X), \mathcal{O}_n)).
\]

Proof. (1): Let \( \{E^{k,k+d}_\infty\}_{k \in \mathbb{Z}} \) be the set of the \( E_\infty \)-terms of the spectral sequence \( (5.5.1) \). It is clear that \( p_g(X/s, n, 1) = \sum_{k=0}^{d} \text{length}_{\mathcal{O}_n}(E^{k,d+k}_\infty) \). Because \( \dim \mathcal{O}(k) = d-(2j+k) \), the non-vanishing term \( H^{h-d-j}(\mathcal{O}(2j+k), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d-j-k}(X/k)) \) can arise only in the case \( d-(2j+k) \geq d-j-k \). This inequality implies that \( j = 0 \). Hence the spectral sequence \( (5.5.1) \) is equal to the following spectral sequence

\begin{equation}
(5.6.2)
\end{equation}

\[
E_1^{h,k} = H^{h-d}(\mathcal{O}(X/k), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d-k}(X/k))(k) \Longrightarrow H^{h-d}(X, \mathcal{O}_n \Omega_{\mathcal{O}(X)}^{d}) \quad (h \in \mathbb{N}).
\]

If \( k < 0 \) or \( h < d \), then \( E_1^{h,k} = 0 \). Hence \( E_\infty^{k,d+k} (k \geq 0) \) is a submodule of

\begin{equation}
(5.6.3)
\end{equation}

\[
E_2^{k,d+k} = \ker(H^0(\mathcal{O}(k), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d-k}(X/k))(k) \to H^1(\mathcal{O}(k-1), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d-k+1}(X/k))(k))
\]

Furthermore, \( E^{0,d}_\infty = E_1^{0,d} = H^0(\mathcal{O}(0), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d}(X/k)) \). Hence the following natural morphism

\begin{equation}
(5.6.4)
\end{equation}

\[
H^0(X, \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d}(X/k)) \to H^0(\mathcal{O}(0), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{d}(X/k))
\]

is surjective. In conclusion, we obtain the inequalities in \( (5.6.1) \).

(2): By \( (5.6.2) \) we have the following exact sequence

\[
0 \to H^0(\mathcal{O}(0), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{1}(X/k)) \to H^0(X, \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{1}(X/k)) \to \ker(H^0(X, \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{1}(X/k)) \to H^1(\mathcal{O}(0), \mathcal{O}_n \Omega_{\mathcal{O}(0)}^{1}(X/k)) \to 0.
\]

29
The boundary morphism $H^0(X^{(1)}, W_n\mathcal{O}_{X^{(1)}})(-1) \rightarrow H^1(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^1_{\hat{X}^{(0)}})$ is the Čech Gysin morphism ([Mg 4.9]). Note that $H^0(X^{(1)}, W_n\mathcal{O}_{X^{(1)}}) = H^0_{\text{cryst}}(X^{(1)}/W_n)$ and $H^1(\hat{X}^{(0)}, W_n\Omega^1_{\hat{X}^{(0)}}) = H^2_{\text{cryst}}(\hat{X}^{(0)}/W_n)$. Hence

$$\ker(H^0(X^{(1)}, W_n\mathcal{O}_{X^{(1)}})(-1) \rightarrow H^1(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^1_{\hat{X}^{(0)}})) = H^1(\Gamma(\hat{X}), W_n)(-1)$$

(cf. [loc. cit. 5.3]). Thus we can complete the proof of (1).

(3): Because $\dim \hat{X} = 2$, we obtain the following equalities:

$$E_{1}^{0,2} = H^0(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^2_{\hat{X}^{(0)}}), \quad E_{1}^{1,3} = H^0(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^1_{\hat{X}^{(0)}})(-1),$$

$$E_{1}^{0,3} = H^1(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^2_{\hat{X}^{(0)}}), \quad E_{1}^{1,4} = H^0(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^1_{\hat{X}^{(0)}})(-2),$$

$$E_{1}^{2,4} = H^1(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^2_{\hat{X}^{(0)}})(-1), \quad E_{1}^{3,4} = H^2(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^2_{\hat{X}^{(0)}}).$$

The other $E_1$-terms of (5.6.2) are zero. Hence $E_{1}^{-1,3} = E_{2}^{-1,3}$. By the assumption, $E_{2}^{-2,4} = E_{2}^{-2,4} = H^2(\Gamma(\hat{\mathcal{O}}_{X}), W_n)(-2)$. By the duality of Ekedahl ([Ek (2.2.23)]), $E_{2}^{-1,3}$ is the dual of $\text{Coker}(H^1(\hat{\mathcal{O}}_{X^{(0)}}, W_n(\mathcal{O}_{X^{(0)}})) \rightarrow H^1(X^{(1)}, W_n(\mathcal{O}_{X^{(1)}})))$. Hence (3) follows.

**Corollary 5.7.** The natural morphism

$$H^0(X, W\Omega^d_X) \rightarrow H^0(\hat{\mathcal{O}}_{X^{(0)}}, W\Omega^d_{\hat{X}^{(0)}})$$

is surjective. In particular,

$$p_g(X/W(s), 1) \geq p_g(\hat{\mathcal{O}}_{X^{(0)}}/W, 1).$$

**Proof.** Set $K_n := \ker(H^0(X, W_n\Omega^d_{X}) \rightarrow H^0(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^d_{\hat{X}^{(0)}}))$. By the surjectivity of the morphism (5.6.4), we obtain the following exact sequence

$$0 \rightarrow K_n \rightarrow H^0(X, W_n\Omega^d_{X}) \rightarrow H^0(\hat{\mathcal{O}}_{X^{(0)}}, W_n\Omega^d_{\hat{X}^{(0)}}) \rightarrow 0$$

of $W_n$-modules. Because $H^0(X, W_n\Omega^d_{X})$ is an artinian $W_n$-modules, so is $K_n$, and hence the projective system $\{K_n\}_{n=1}^\infty$ satisfies the Mittag-Leffler condition. Hence the following sequence $0 \rightarrow \lim_{\leftarrow n} K_n \rightarrow H^0(X, W\Omega^d_X) \rightarrow H^0(\hat{\mathcal{O}}_{X^{(0)}}, W\Omega^d_{\hat{X}^{(0)}}) \rightarrow 0$ is exact.

**Remark 5.8.** (1) As in [5.6], even if $\kappa$ is any field of any characteristic, there is the following spectral sequence of abelian groups

$$E_{1}^{-k,h+k} = H^{h-k}(X^{(k+1)}, \Omega^{d-k}_{X^{(k+1)}}/\kappa) \Rightarrow H^{h-d}(X, \Omega^{d}_{X/\kappa}).$$

Thus $p_g(X/s, 1, 1) = \sum_{k=0}^{d} \dim_\kappa E_{1}^{-k,h+k}$. In particular, the inequalities (5.6.1) for the case $n = 1$ holds.
Assume that $\tilde{s} = \text{Spec}(\mathbb{C})$ and that $X/s$ is a proper log analytic SNCL space. Let $i$ be a nonnegative fixed integer. Then we have the following spectral sequence by using the Poincaré residue isomorphism (cf. (3.8.1)):

\[
E_1^{-k,h+k} = \bigoplus_{j \geq \max\{-k,0\}} H^{h-i-j}(\tilde{X}(2j+k), \Omega^{i-j-k}_{\tilde{X}(2j+k)})(-j-k) \Rightarrow H^h(X, \Omega^i_{X/s})(h \in \mathbb{N}).
\]

Here $(-j-k)$ is the usual Tate twist. Assume that each irreducible components of $\tilde{X}$ is Kähler or the analytification of a proper scheme over $\mathbb{C}$. By using theory of mixed Hodge structures in ([1], [2]) and by [3], this spectral sequence degenerates at $E_2$. In particular, the following equality holds:

\[
p_g(X/s, 1) = p_g(\tilde{X}^{(0)}/\mathbb{C}, 1) + \sum_{k=1}^d \dim_{\mathbb{C}} \ker(H^0(\tilde{X}^{(k)}, \Omega^{d-k}_{\tilde{X}(k)})(-k) \rightarrow H^1(\tilde{X}^{(k-1)}, \Omega^{d-k+1}_{\tilde{X}(k-1)})(-k+1)).
\]

Using the Lefschetz' principle, we obtain the following:

**Corollary 5.9.** Assume that the characteristic of the base field $\kappa$ is 0. Let $X/s$ be a proper SNCL scheme. Then

\[
p_g(X/s, 1) = p_g(\tilde{X}^{(0)}/\kappa, 1) + \sum_{k=1}^d \dim_{\kappa} \ker(H^0(\tilde{X}^{(k)}, \Omega^{d-k}_{\tilde{X}(k)})(-k) \rightarrow H^1(\tilde{X}^{(k-1)}, \Omega^{d-k+1}_{\tilde{X}(k-1)})(-k+1)).
\]

**Corollary 5.10.** Let the notations be as in (5.6). Set $d := \dim \tilde{X}$. Let $\tilde{X}_\lambda$ be an irreducible component of $\tilde{X}$. Assume that the spectral sequence (5.6.2) for the case $n = 1$ degenerates at $E_2$. Then, if $p_g(X/s, 1, 1) = 1$ and if $H^d(\Gamma(\tilde{X}), \kappa) \neq 0$, then $p_g(\tilde{X}_\lambda, 1, 1) = 0$ and $H^d(\Gamma(\tilde{X}), \kappa) \simeq \kappa$.

**Proof.** Consider the spectral sequence (5.6.1). By (5.6.3),

\[
E_\infty^{-d,2d} = E_2^{-d,2d} = \ker(H^0(\tilde{X}(d), \mathcal{O}_{\tilde{X}(d)})(-d) \rightarrow H^1(\tilde{X}^{(d-1)}, \Omega^{1}_{\tilde{X}(d-1)})(-d+1)) = H^d(\Gamma(X), \kappa)(-d).
\]

**6 Lower semi-continuity of log plurigenera**

Let $\{\tilde{X}_i\}_{i \in I}$ be a finite set of the irreducible component of $\tilde{X}$. Next we would like to prove that the following inequality holds:

\[
\sum_{i \in I} p_g(\tilde{X}_i/\kappa, n, r) \leq p_g(X/s, n, r) \quad (r \in \mathbb{Z}_{\geq 1})
\]
holds. However, for general $n$, I have not been able to prove this inequality. What I have been able to prove is (6.0.1) only in the case $n = 1$ for the case $r \geq 2$ (see (6.3) below). Even in the case $n = 1$, because we do not know whether the following induced morphism by (6.6.4)

$$H^0(X, \frac{r}{\mathcal{O}_X} (\Omega^d_X)^{\otimes r}) \rightarrow H^0(X, \frac{r}{\mathcal{O}_X(0)} \Omega^d_X(0))$$

is surjective for general $r \geq 2$, we need a new argument to prove this inequality.

Let $r$ be a positive integer. Set $a := a^{(0)}: \hat{X}(0) \rightarrow \bar{X}$. Let $n$ be a positive integer or nothing. The key for the proof of this inequality is to construct the following morphism and to prove that it is injective:

**Theorem 6.1.** There exists a morphism

$$(6.1.1) \quad \Psi_{X,n,r}: a_* \left( \frac{r}{\mathcal{O}_X(0)} \mathcal{W}_n \Omega^d_X(0) \right) \rightarrow \frac{r}{\mathcal{O}_X} \mathcal{W}_n \Omega^d_X$$

of $\mathcal{W}_n(\mathcal{O}_X)$-modules, which shall be shown to be injective in (6.3) below for the case $n = 1$.

**Proof.** By [Mo, 3.15] and [Nakk3, (6.28) (9)] (cf. [Nakk3, (6.29) (1)]), the morphism $\theta_n \wedge: \mathcal{W}_n \Omega^d_X \rightarrow \mathcal{W}_n \tilde{\Omega}^{d+1}_X/P_0 \mathcal{W}_n \tilde{\Omega}^{d+1}_X$ of $\mathcal{W}_n(\mathcal{O}_X)$-modules is an isomorphism of $\mathcal{W}_n(\mathcal{O}_X)$-modules:

$$(6.1.2) \quad \theta_n \wedge: \mathcal{W}_n \Omega^d_X \sim \rightarrow \mathcal{W}_n \tilde{\Omega}^{d+1}_X/P_0 \mathcal{W}_n \tilde{\Omega}^{d+1}_X.$$ 

Consider the following exact sequence of $\mathcal{W}_n(\mathcal{O}_X)$-modules:

$$(6.1.3) \quad 0 \rightarrow \text{gr}^P \mathcal{W}_n \tilde{\Omega}^{d+1}_X \rightarrow \mathcal{W}_n \tilde{\Omega}^{d+1}_X/P_0 \mathcal{W}_n \tilde{\Omega}^{d+1}_X \rightarrow \mathcal{W}_n \tilde{\Omega}^{d+1}_X/P_1 \mathcal{W}_n \tilde{\Omega}^{d+1}_X \rightarrow 0.$$ 

By [Mo, (3.7)] we have the following isomorphism

$$(6.1.4) \quad \text{Res} : \text{gr}^P \mathcal{W}_n \tilde{\Omega}^{d+1}_X \sim \rightarrow a_*(\mathcal{W}_n \Omega^d_{\hat{X}(0)})$$

of $\mathcal{W}_n(\mathcal{O}_X)$-modules. Hence we have the following composite morphism of $\mathcal{W}_n(\mathcal{O}_X)$-modules:

$$(6.1.5) \quad a_* (\mathcal{W}_n \Omega^d_{\hat{X}(0)}) \xleftarrow{\text{Res}} \text{gr}^P \mathcal{W}_n \tilde{\Omega}^{d+1}_X \xleftarrow{\subseteq} \mathcal{W}_n \mathcal{W}_n \tilde{\Omega}^{d+1}_X/P_0 \mathcal{W}_n \tilde{\Omega}^{d+1}_X \xleftarrow{\theta_n} \mathcal{W}_n \Omega^d_X.$$ 

Consequently we have the following morphism of $\mathcal{W}_n(\mathcal{O}_X)$-modules:

$$(6.1.6) \quad \frac{r}{\mathcal{O}_X} a_* (\mathcal{W}_n \Omega^d_{\hat{X}(0)}) \rightarrow \frac{r}{\mathcal{O}_X} \mathcal{W}_n \Omega^d_X.$$ 

Let $a_i: \hat{X}_i \rightarrow \bar{X}$ be the following composite morphism $\hat{X}_i \rightarrow \hat{X}(0) \rightarrow \bar{X}$. Because

$$\frac{r}{\mathcal{O}_X} a_* (\mathcal{W}_n \Omega^d_{\hat{X}(0)}) = \frac{r}{\mathcal{O}_X} \left( \bigoplus_{i} a_{i*} (\mathcal{W}_n \Omega^d_{\hat{X}_i(\bar{X})}) \right)$$

$$= \bigoplus_{i_1, \ldots, i_r} a_{i_1*} (\mathcal{W}_n \Omega^d_{\hat{X}_{i_1}}) \otimes \mathcal{W}_n(\mathcal{O}_X) \cdots \otimes \mathcal{W}_n(\mathcal{O}_X) a_{i_r*} (\mathcal{W}_n \Omega^d_{\hat{X}_{i_r}(\bar{X})})$$

32
and the last sheaf contains $\bigoplus_{i} a_{\ast}((W_{n}\Omega^{d}_{X_{i}/\kappa})^{\otimes r})$, there exists the following morphism of $W_{n}(\mathcal{O}_X)$-modules:

\[(6.1.7) \quad \bigoplus_{i \in I} a_{\ast}((r \otimes W_{n}(\mathcal{O}_{X_{i}}))_{X_i} \rightarrow r \otimes W_{n}(\mathcal{O}_X)).\]

Since $a_{\ast}(r \otimes W_{n}(\mathcal{O}_{X_{i}}))_{X_i} = \bigoplus_{i \in I} a_{\ast}((r \otimes W_{n}(\mathcal{O}_{X_{i}}))_{X_i})$, we obtain the desired morphism \[(6.1.1).\]

\[\Box\]

**Remark 6.2.** Because the morphism \[(6.1.5)\] is injective, we have the following injective morphism

\[(6.2.1) \quad \Psi_{X,n,1} : H^{0}(\tilde{X}^{(0)}, W_{n}(\otimes_{X_{i}(0)}) \rightarrow H^{0}(X, W_{n}(\otimes_{X_{i}})).\]

Hence we obtain the inequality again

\[p_{g}(X^{(0)}/\kappa, n, 1) \leq p_{g}(X/s, n, 1),\]

which has been proved in \[(5.6.1).\]

Now we can prove the following. This result is similar to those of N. Nakayama (Nakay1 Theorem 11, Nakay2 (6.3)) (cf. [CI] (4.2), [Mori] [6]).

**Theorem 6.3.** The morphism \[(6.1.1)\] is injective. Consequently

\[(6.3.1) \quad p_{g}(X/s, 1, r) \geq p_{g}(\tilde{X}^{(0)}/\kappa, 1, r)\]

and

\[(6.3.2) \quad \kappa(X, 1) \geq \max\{\kappa(X_{i}, 1)|X_{i} : \text{an irreducible component of } X\}\]

(cf. [UL], [NI]).

**Proof.** For the time being, we do not assume that $n = 1$. Let $\tilde{X}_{i}$ be an irreducible component of $\tilde{X}$. Let $X_{sm}$ be the smooth locus of $\tilde{X}/\kappa$. We claim that the morphism \[(6.1.1)\] is an isomorphism on the nonsingular points $X_{sm}$ of $\tilde{X}$. Indeed, it suffices to show that the morphism $a_{\ast}(W_{n}\Omega^{d}_{\tilde{X}^{(0)}}) \rightarrow W_{n}\Omega^{d}_{X}$ is an isomorphism on $X_{sm}$ since $\tilde{X}^{(0)}|_{X_{sm}} = X_{sm}$. Locally on $X_{sm}$, we have a system of local parameters $x_1, \ldots, x_d$. We may assume that there is a local isomorphism $X_{sm} \simeq \text{Spec } k[t, x_1, \ldots, x_d]/(t)$ and we have a local admissible lift $\tilde{\mathcal{X}} := \text{Spec } \mathcal{W}[t, x_1, \ldots, x_d]$ of $X_{sm}$ over $\mathcal{W}[t]$. Set $X_{n} := \tilde{\mathcal{X}} \times_{\mathcal{W}[t]} W_{n}$. Then $\theta \in W_{n}\tilde{X}^{(0)} = H^{0}(\Omega^{d}_{X_{sm}})$ is the class $[d \log t]$ of $d \log t$. The inverse of the Poincaré residue isomorphism $a_{\ast}(W_{n}\Omega^{d}_{\tilde{X}^{(0)}}) \rightarrow \text{gr}^{d}W_{n}\tilde{X}^{d+1}$ is defined by $\omega \mapsto \omega [d \log t]$. Because $a_{\ast}(W_{n}\Omega^{d}_{\tilde{X}^{(0)}})|_{X_{sm}} = W_{n}\Omega^{d}_{X_{sm}} = W_{n}\Omega^{d}_{X_{sm}}|_{X_{sm}}$, the restriction of \[(6.1.1)\] to $X_{sm}$ is equal to the identity $\text{id}_{W_{n}\Omega^{d}_{X_{sm}}}$. Thus we have proved that our claim holds.

By [UL] (1.17]) the sheaf $r \otimes_{W_{n}(\mathcal{O}_X)} W_{n}\Omega^{d}_{X}$ is a coherent $W_{n}(\mathcal{O}_X)$-module. In particular, $a_{\ast}(r \otimes_{W_{n}(\mathcal{O}_{X}(0))} W_{n}\Omega^{d}_{X^{(0)}})$ is also a coherent $W_{n}(\mathcal{O}_X)$-module.
Now consider the case $n = 1$ (but $r$ is general). Let $F_{X_i} : \tilde{X}_i \to \tilde{X}_i$ be the Frobenius endomorphism of $\tilde{X}_i$. Since $(C^{-1})^{-1} : W_1 \Omega^d_{\tilde{X}_i} = F_* \mathcal{H}^d(\Omega^*_{\tilde{X}_i, /\kappa}) \to \Omega^d_{\tilde{X}_i /\kappa}$, $W_1 \Omega^d_{\tilde{X}_i}$ is an invertible $W_1(\mathcal{O}_{\tilde{X}_i})/(\simeq \mathcal{O}_{\tilde{X}_i})$-module. Hence $a_* (\otimes W_1 \Omega^d_{\tilde{X}_i, /\kappa})$ is an invertible $W_1(\mathcal{O}_{\tilde{X}_i})$-module. Because the morphism (6.1.1) is an isomorphism on a dense open subset and because an étale algebra over $\kappa[x_1, \ldots, x_d]$ has no non-zero divisor, the morphism (6.1.1) is injective.

In the rest of this section, we give the compatibility with fundamental operators. (This may be useful for checking whether $\Psi_{X,n,r}$ is injective.)

**Proposition 6.4.** Let $R : W_{n+1} \Omega^d_{\tilde{X}(0)} \to W_{n+1} \Omega^d_{\tilde{X}(0)}$ and $R : W_{n+1} \Omega^d_{\tilde{X}} \to W_{n+1} \Omega^d_{\tilde{X}}$ be the projections. (These are morphisms of $W_{n+1}(\mathcal{O}_{\tilde{X}(0)})$-modules and $W_{n+1}(\mathcal{O}_{\tilde{X}})$-modules, respectively.) Then the following diagram is commutative:

$$
\begin{array}{ccc}
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\Psi_{X,n+1,r}} & W_{n+1} \Omega^d_{\tilde{X}} \\
\downarrow_{\otimes R} & & \downarrow_{\otimes R} \\
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\Psi_{X,n,r}} & W_{n+1} \Omega^d_{\tilde{X}}.
\end{array}
$$

**(6.4.1)**

**Proof.** By Nakajima (8.4.3), (11.1) we have the following commutative diagram

$$
\begin{array}{ccc}
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\text{Res.} \simeq} & \text{gr}^P W_{n+1} \tilde{\Omega}^d_{\tilde{X}} \\
\downarrow_{a_*(R)} & & \downarrow_{a_*(R)} \\
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\text{Res.} \simeq} & \text{gr}^P W_{n+1} \tilde{\Omega}^d_{\tilde{X}}.
\end{array}
$$

**(6.4.2)**

Hence we have the following commutative diagram

$$
\begin{array}{ccc}
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\Psi_{X,n+1,r}} & W_{n+1} \Omega^d_{\tilde{X}} \\
\downarrow_{\otimes a_*(R)} & & \downarrow_{\otimes a_*(R)} \\
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\Psi_{X,n,r}} & W_{n+1} \Omega^d_{\tilde{X}}.
\end{array}
$$

**(6.4.3)**

Since the following diagram

$$
\begin{array}{ccc}
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\text{C}} & W_{n+1} \Omega^d_{\tilde{X}(0)} \\
\downarrow_{a_*(\otimes R)} & & \downarrow_{a_*(\otimes R)} \\
W_{n+1} \Omega^d_{\tilde{X}(0)} & \xrightarrow{\text{C}} & W_{n+1} \Omega^d_{\tilde{X}(0)}.
\end{array}
$$

**(6.4.4)**

is commutative, we obtain the commutative diagram (6.4.1).

**Definition 6.5.** Set $\Psi_{X,n,r} := \lim_{\rightarrow n} \Psi_{X,n,r}$. 

34
The rest of the proof is the same as that of (6.4).

(2): The proof of (2) is similar to that of (1).

To state the contravariant functoriality, let us recall the following:
Proposition 6.8 ([Nakk4, (4.3)]). Let \( g : Y \to Y' \) be a morphism of fs log (formal) schemes satisfying the condition

\[
(6.8.1) \quad M_{Z,y}/\mathcal{O}_{Z,y}^* \cong \mathbb{N}^r
\]

for any point \( y \) of \( Z \) and for some \( r \in \mathbb{N} \) depending on \( y \). Let \( b^{(k)} : \hat{D}^{(k)}(M_{Y'}) \to \hat{Y}'^* \) (\( k \in \mathbb{Z} \) (\( * \) = nothing or \( ' \)) be the morphism defined before [Nakk4, (4.3)]. Assume that, for each point \( y \in \hat{Y}' \) and for each member \( m \) of the minimal generators of \( M_{Y',\hat{g}(y)}/\mathcal{O}_{Y',\hat{g}(y)}^* \), there exists a unique member \( m' \) of the minimal generators of \( M_{Y',\hat{g}(y)}/\mathcal{O}_{Y',\hat{g}(y)}^* \) such that \( g^*(m') \in m\mathbb{Z}^{>0} \). Then there exists a canonical morphism \( \hat{g}^{(k)} : \hat{D}^{(k)}(M_Y) \to \hat{D}^{(k)}(M_{Y'}) \) fitting into the following commutative diagram of schemes:

\[
\begin{array}{ccc}
D^{(k)}(M_Y) & \xrightarrow{g^{(k)}} & D^{(k)}(M_{Y'}) \\
\downarrow b^{(k)} & & \downarrow b^{(k)} \\
\hat{Y} & \xrightarrow{\hat{g}} & \hat{Y}'.
\end{array}
\]

The following has been proved in [Nakk5, (1.3.20)] (cf. [Nakk4, (4.8)]):

Proposition 6.9 (The contravariant functoriality of the Poincaré residue morphism). Let \( S \) be as above and let \( u : S \to S' \) be a morphism of family of log points. Let \( X'/S' \) be an SNCL scheme. Let \( X' \xhookrightarrow{\iota} \mathcal{P}' \) be an immersion into a log smooth scheme over \( S' \) fitting into the following commutative diagram over the morphism \( S \to S' \):

\[
\begin{array}{ccc}
X' & \xrightarrow{\iota} & \mathcal{P}' \\
\downarrow g & & \downarrow g \\
X & \xrightarrow{\iota} & \mathcal{P}.
\end{array}
\]

Let \( \mathcal{P}_{\text{ext}} \) and \( \mathcal{P}'_{\text{ext}} \) be the exactification of the immersions \( X \xhookrightarrow{\iota} \mathcal{P} \) and \( X' \xhookrightarrow{\iota} \mathcal{P}' \), respectively. Let \( g_{\text{ext}} : \mathcal{P}_{\text{ext}} \to \mathcal{P}'_{\text{ext}} \) be the induced morphism by \( g \). Let \( a^{(k)} : \hat{X}'^{(k)} \to \hat{X}' \) (\( k \in \mathbb{N} \) \( \hat{X}' \) (\( k \in \mathbb{N} \)) be the natural morphism of schemes over \( \hat{S}'_0 \). Assume that, for each point \( x \in \hat{P}_{\text{ext}} \) and for each member \( m \) of the minimal generators of \( M_{\mathcal{P}_{\text{ext}},\hat{g}(x)}/\mathcal{O}_{\mathcal{P}_{\text{ext}},\hat{g}(x)}^* \), there exists a unique member \( m' \) of the minimal generators of \( M_{\mathcal{P}_{\text{ext}},\hat{g}(x)}/\mathcal{O}_{\mathcal{P}_{\text{ext}},\hat{g}(x)}^* \) such that \( g^*(m') = m \) and such that the image of the other minimal generators of \( M_{\mathcal{P}_{\text{ext}},\hat{g}(x)}/\mathcal{O}_{\mathcal{P}_{\text{ext}},\hat{g}(x)}^* \) by \( g_{\text{ext}} \) are the trivial element of \( M_{\mathcal{P}_{\text{ext}},\hat{g}(x)}/\mathcal{O}_{\mathcal{P}_{\text{ext}},\hat{g}(x)}^* \). Let \( a^{(k)} : \hat{X}'^{(k)} \to \hat{X}' \) be an analogous morphism to \( a^{(k)} \) for \( X' \). Then the following diagram is commutative:

\[
\begin{array}{ccc}
g^{\text{ext}}_*(\text{Res}) : g^{\text{ext}}_*(\Omega^+_\mathcal{P}_{\text{ext}}/\hat{S}') & \xrightarrow{\sim} & g^{\text{ext}}_*(\Omega^+_\mathcal{P}_{\text{ext}}/\hat{S}') \\
\uparrow & & \uparrow \\
\text{Res} : \Omega^+_\mathcal{P}_{\text{ext}}/\hat{S}' & \xrightarrow{\sim} & \Omega^+_\mathcal{P}_{\text{ext}}/\hat{S}'.
\end{array}
\]
Let $s \to t$ be a morphism of the log points of perfect fields of characteristic $p > 0$. Let $Y/t$ be a proper SNCL scheme and let $b: \tilde{Y}(0) \to \tilde{Y}$ be an analogue of $a: \tilde{X}(0) \to \tilde{X}$. Let $g: X \to Y$ be a morphism of log schemes satisfying the assumption in (6.8). Then the following diagram is commutative:

$$
\begin{array}{c}
g^*(\Psi_X,n,r) \\
\uparrow \\
g^*(\Psi_Y,n,r)
\end{array}
\begin{array}{c}
g^*(r \otimes W_n(\Omega^i_{X/S})) \\
\downarrow \\
g^*(r \otimes W_n(\Omega^i_{Y/S}))
\end{array}
$$

Proposition 6.10. This immediately follows from the following functoriality of the Poincaré residue isomorphism.

### Appendix

#### 7 Generalization of Illusie’s Poincaré residue isomorphism

Let the notations be as in the beginning of §3. Especially let $E$ be a locally generated by horizontal sections of $\nabla: E \to E \otimes_{\mathcal{O}_X} \Omega^{\bullet}_{X/S}$ satisfying the conditions (I)∼(IV). Set

$$P_k(E \otimes_{\mathcal{O}_X} \Omega^{\bullet}_{X/S}) := \text{Im}(E \otimes_{\mathcal{O}_X} P_k \Omega^i_{X/S} \to E \otimes_{\mathcal{O}_X} \Omega^i_{X/S}).$$

In this section we generalize Illusie’s Poincaré residue isomorphism in [I12, Appendix (2.2)]. That is, we prove that there exists the following isomorphism:

$$g^k_!(E \otimes_{\mathcal{O}_X} \Omega^{\bullet}_{X/S}) \to K^k_!(\mathcal{E}) \quad (k \in \mathbb{N}).$$

We also generalize [I12, Appendix (2.6)]. First we begin with the following proposition:

**Proposition 7.1 (cf. [I12, p. 399]).** Let $Y/S$ be as in the beginning of §3. Let $\mathcal{G}$ be a flat quasi-coherent $\mathcal{O}_Y$-module. Then the following morphism

$$\theta \wedge: \mathcal{G} \otimes_{\mathcal{O}_Y} \Omega^{i-1}_{Y/S} \ni e \otimes \omega \mapsto e \otimes (\theta \wedge \omega) \in \mathcal{G} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S} \quad (i \in \mathbb{N})$$

of $\mathcal{O}_Y$-modules is strictly compatible with $P$’s in the following sense: for nonnegative integers $i$ and $k$,

$$\theta \wedge(P_k(\mathcal{G} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S})) = (\theta \wedge(P_{k-1}(\mathcal{G} \otimes_{\mathcal{O}_Y} \Omega^{i-1}_{Y/S}))).$$
Proof. It suffices to prove the inclusion “$\subset$” in (7.1.2). The problem is local. Let $m_1, \ldots, m_r$ be local sections of $M_Y$ such that $\{d\log m_1, \ldots, d\log m_r\}$ is a local basis of $\Omega^1_{\mathcal{O}_Y}$ over $\mathcal{O}_Y$. Let $\omega$ be a local section of $\Omega^1_{\mathcal{O}_Y}$ such that $\theta \wedge \omega \in P_k \Omega^i_{\mathcal{O}_Y}$. Express $\omega = \theta \wedge \eta + \omega'$, where $\omega' \notin (\theta \wedge \Omega^i_{\mathcal{O}_Y})$ and $\eta \in \Omega^i_{\mathcal{O}_Y}$. Then $\theta \wedge \omega = \theta \wedge \omega'$. Because $\theta \wedge \omega' \in P_k \Omega^i_{\mathcal{O}_Y}$ and $\omega' \notin (\theta \wedge \Omega^i_{\mathcal{O}_Y})$, $\omega' \in P_{k-1} \Omega^i_{\mathcal{O}_Y}$. Hence we obtain the following equality:

(7.1.3) \[(\theta \wedge \Omega^i_{\mathcal{O}_Y}) \cap P_k \Omega^i_{\mathcal{O}_Y} = (\theta \wedge) P_{k-1} \Omega^i_{\mathcal{O}_Y},\]

and we see that the following sequence

$$0 \to P_{k-1} \Omega^i_{\mathcal{O}_Y} \to \Omega^i_{\mathcal{O}_Y} \to (\theta \wedge \Omega^i_{\mathcal{O}_Y}) \cap P_k \Omega^i_{\mathcal{O}_Y} \to 0$$

of $\mathcal{O}_Y$-modules is exact. Because $\mathcal{G}$ is a flat $\mathcal{O}_Y$-module, the following sequence

(7.1.4) \[0 \to \mathcal{G} \otimes_{\mathcal{O}_Y} P_{k-1} \Omega^i_{\mathcal{O}_Y} \to \mathcal{G} \otimes_{\mathcal{O}_Y} \Omega^i_{\mathcal{O}_Y} \to (\mathcal{G} \otimes_{\mathcal{O}_Y} \Omega^i_{\mathcal{O}_Y}) \cap (\mathcal{G} \otimes_{\mathcal{O}_Y} P_k \Omega^i_{\mathcal{O}_Y}) \to 0\]

is exact. Because

$$\mathcal{G} \otimes_{\mathcal{O}_Y} ((\theta \wedge \Omega^i_{\mathcal{O}_Y}) \cap P_k \Omega^i_{\mathcal{O}_Y})) = (\theta \wedge) (\mathcal{G} \otimes_{\mathcal{O}_Y} \Omega^i_{\mathcal{O}_Y}) \cap (\mathcal{G} \otimes_{\mathcal{O}_Y} P_k \Omega^i_{\mathcal{O}_Y}),$$

the exact sequence (7.1.4) implies the equality (7.1.2). □

Corollary 7.2. Let $k$ be an integer. The following sequences

(7.2.1) \[P_{k-1}(\mathcal{G} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X})[-1] \to P_k(\mathcal{G} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) \to P_k(\mathcal{G} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) \to 0,\]

(7.2.2) \[0 \to P_{k-1}(\mathcal{G} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X})[-1] \to P_k(\mathcal{G} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) \to P_k(\mathcal{G} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) \to 0\]

of complexes of $f^{-1}(\mathcal{O}_S)$-modules are exact.

Proposition 7.3. For each $i$, the resulting sequences of (7.2.2) by the operations $Z^i$, $B^i$ and $\mathcal{H}^i$ ($i \in \mathbb{Z}_{\geq 0}$) are exact.

Proof. The analogous proof to the first proof of (3.3) works. □

Proposition 7.4 (cf. [12, Appendix (2.2)]). There exists the isomorphism (7.1.1).

Proof. First assume that $k = 0$. Then $gr_0^p(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) = P_0(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) = P_0(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) = K_0^i(\mathcal{E})$.

Next assume that $k \geq 1$. By (7.2.2) we see that the following upper sequence

(7.4.1) \[gr_{k-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X})[-1] \to gr_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) \to gr_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^i_{\mathcal{O}_X}) \to 0\]

of complexes of $f^{-1}(\mathcal{O}_S)$-modules is exact.

\[\text{Res.} \simeq \]
is exact. Because the sequence \((3.10.1)\) is exact,

\[
gr_k^P (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*) = \text{Coker}(i^{(k-1)*} : \Omega_{k-1}^{*-(k-1)}(\mathcal{E})[-1] \to \Omega_{k-1}^*(\mathcal{E})) = K_{k-k}(\mathcal{E}).
\]

\[\Box\]

**Definition 7.5.** We call the isomorphism \((7.0.1)\) the relative Poincaré residue isomorphism of \(\mathcal{E}\) for \(X/S\). This is a generalization of [12] Appendix (2.1.4). (Note that we do not need the local description of the isomorphism \((7.0.1)\) unlike in [loc. cit.] to obtain the isomorphism \((7.0.1)\).

**Proposition 7.6. (cf. [12] Appendix (2.5))** The resulting sequences of the following sequence by the operations \(\mathcal{H}^i, B^i\) and \(Z^i\) \((i \geq 0)\) are exact:

\[
0 \to P_{k-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*) \to P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*) \to gr_k^R (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*) \to 0
\]

Proof. The proof is the same as that of \((3.12)\).

\[\Box\]

**Proposition 7.7.** Let \(\mathcal{F}\) be as in \((3)\). The isomorphism \(C^{-1} : \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_{X/S[\pi]}^* \iso F_{\mathcal{H}}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*)\) induce the following isomorphism

\[
(7.7.1) \quad C^{-1} : \mathcal{F} \otimes_{\mathcal{O}_X} P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*) \iso F_{\mathcal{H}}(\mathcal{E} \otimes_{\mathcal{O}_X} P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*)) \quad (i, k \in \mathbb{N}).
\]

Proof. By \((7.2.1)\) and induction on \(k\), we obtain \((7.7)\).

\[\Box\]

**Proposition 7.8 (cf. [12] Appendix (2.6)).** Consider the following conditions:

(a) \(R^jf_*(B^iK_k^*(\mathcal{E})) = 0\) for \(\forall i, j, \forall n\).

(b) \(R^jf_*(B^i\Omega^j_k(\mathcal{E})) = 0\) for \(\forall i, j, \forall n\).

(c) \(R^jf_*(B^i(\Omega^j_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*))) = 0\) for \(\forall i, j, \forall k\).

(d) \(R^jf_*(B^i(\Omega^j_k(P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*)))) = 0\) and \(R^jf_*(B^i(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*))) = 0\) for \(\forall i, j, \forall k\).

(e) \(R^jf_*(B^i(\Omega^j_k(P_k(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*)))) = 0\) and \(R^jf_*(B^i(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/S}^*))) = 0\) for \(\forall i, j, \forall k\).

Then the following hold:

(1) (a), (b) and (c) are equivalent.

(2) (c), (d) and (e) are equivalent.

(Consequently (a), (b), (c), (d) and (e) are equivalent.)

Proof. (1) (a) follows from \((3.12), (7.3)\) and \((7.4)\).

(2): Set \(T := S\) or \(\bar{S}\). Note that the sheaf is the associated sheaf to the following presheaf for open sub-log scheme \(U \subset B\): \(U \to R^f f_U(B\Omega^j_{X/T[\pi]})\), where \(f_U : X \times_B U \to U\) is the structural morphism. Hence, to prove that \(R^f f_U(B\Omega^j_{X/T}) = 0\), it suffices to assume that \(B\Omega^j_{X/T} = P_kB\Omega^j_{X/T}\) for some \(k \in \mathbb{N}\). Note (2) follows from \((7.6)\) and \((7.3)\).

\[\Box\]

8 **An analogue of Hyodo’s criterion for quasi-F-split schemes**

In this section we give an analogue of Hyodo’s criterion for quasi-F-split schemes. This answers Illusie’s question stated in the Introduction.

Let \(Y\) be a closed subvariety of an F-split variety \(X\). Let \(\mathcal{I}\) be the ideal sheaf of \(Y\) in \(X\). Recall that \(Y\) is compatibly split if there exists a splitting \(\rho : F_*(\mathcal{O}_X) \to \)

39
\(\mathcal{O}_X\) of the operator \(F: \mathcal{O}_X \rightarrow F_*(\mathcal{O}_X)\) such that \(\rho(F_*(I)) \subset I\). It is clear that this is equivalent to the existence of a splitting \(\rho: F_*(\mathcal{O}_X) \rightarrow \mathcal{O}_X\) of the operator \(F: \mathcal{O}_X \rightarrow F_*(\mathcal{O}_X)\) which induces a splitting \(\rho: F_*(\mathcal{O}_Y) \rightarrow \mathcal{O}_Y\) of the operator \(F: \mathcal{O}_Y \rightarrow F_*(\mathcal{O}_Y)\).

**Definition 8.1.** Let \(Y\) be an SNC scheme. We say that \(Y\) is *compatibly quasi-F-split* if there exists a splitting \(\rho: F_*(\mathcal{O}_Y(0)) \rightarrow \mathcal{O}_Y(0)\) of the operator \(F: \mathcal{O}_Y(0) \rightarrow F_*(\mathcal{O}_Y(0))\) for some \(n \in \mathbb{Z}_{\geq 1}\) which induces a splitting \(\rho: F_*(\mathcal{O}_Y(n)) \rightarrow \mathcal{O}_Y(n)\) of the operator \(F: \mathcal{O}_Y(n) \rightarrow F_*(\mathcal{O}_Y(n))\).

**Proposition 8.2.** Let \(Y\) be an SNC scheme. Then \(Y\) is quasi-F-split if and only if \(Y\) is compatibly quasi-F-split.

*Proof.* By [RS] Theorem 1] the following sequence

\[
(8.2.1) \quad 0 \rightarrow \mathcal{O}_Y(n) \rightarrow \mathcal{O}_Y(n) \rightarrow \mathcal{O}_Y(n) \rightarrow \cdots \quad (n \in \mathbb{Z}_{\geq 1})
\]

is exact. Hence [8.2] follows. \(\square\)

## 9 Ordinariness at 0 of proper SNC schemes

Let \(S\) be a scheme of characteristic \(p > 0\). In this section let \(X\) be a proper SNC (=simple normal crossing) scheme over \(S\). Let \(f: X \rightarrow S\) be the structural morphism. In this section we consider only trivial log structures. Let \(F: X \rightarrow X\) be the Frobenius endomorphism. We say that \(X/S\) is *ordinary at 0* if \(F_* \mathbf{R}^h f_* (\mathcal{O}_X) \rightarrow \mathbf{R}^h f_* (\mathcal{O}_X)\) is bijective for any \(h \in \mathbb{N}\). When \(S\) is the underlying scheme of a family of log points and when \(X/S\) is the underlying morphism of a morphism of SNCL schemes \(Y/T\) with structural morphism \(g: Y \rightarrow T\), this definition is equivalent to the variation of the ordinarity at 0 defined in [Nak6] (6.1): \(\mathbf{R}^h g_* (B\Omega^1_{Y/T}) = 0\). This follows from the following tautological exact sequence

\[
0 \rightarrow \mathcal{O}_Y \rightarrow F_*(\mathcal{O}_Y) \xrightarrow{d} B_1 \Omega^1_{Y/T} \rightarrow 0.
\]

The following proposition may be of independent interest (cf. [ST] Conjecture Nn):

**Proposition 9.1.** If \(X(i)/S\) is ordinary at 0 for all \(i\), then \(X/S\) is ordinary at 0.

*Proof.* Because the following sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X(0)} \rightarrow \mathcal{O}_{X(1)} \rightarrow \cdots
\]

is exact and this exact sequence is compatible with the operator \(F\), we have the following sequence

\[
(9.1.1) \quad E_1^{-k,h+k} = \mathbf{R}^h f_* (\mathcal{O}_{X(-k+1)}) \Rightarrow \mathbf{R}^h f_* (\mathcal{O}_X) \quad (-k, q \in \mathbb{Z}_{\geq 0}),
\]

By the assumption \(F: E_1^{-k,h+k} \rightarrow E_1^{-k,h+k}\) is bijective. Hence \(F: \mathbf{R}^h f_* (\mathcal{O}_X) \rightarrow \mathbf{R}^h f_* (\mathcal{O}_X)\) is bijective. \(\square\)

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40
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Yukiyoshi Nakkajima
Department of Mathematics, Tokyo Denki University, 5 Asahi-cho Senju Adachi-ku, Tokyo 120–8551, Japan.

*E-mail address*: nakayuki@cck.dendai.ac.jp