Background Field Method and Structure of Effective Action in $N = 2$ Super Yang-Mills Theories

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Abstract: This paper is a brief review of background field method and some of its applications in $N = 2$ super Yang-Mills theories with a matter within harmonic superspace approach. A general structure of effective action is discussed, an absence of two-loop quantum corrections to first non-leading term in effective action is proved and $N = 2$ non-renormalization theorem in this approach is considered.

$N = 2$ supersymmetric field theories have attracted much attention due to significant progress in understanding their quantum aspects. Modern interest to such theories was inspired by seminal papers by Seiberg and Witten [1] where exact instanton contribution to low-energy effective action has been found. This result has demonstrated once more the wonderful features of the above theories and led to forming a research directions associated with study a general structure of effective action in $N = 2$ super Yang-Mills theories.

An adequate description of quantum $N = 2$ supersymmetric field theories should be based on formulating these theories in terms of unconstrained $N = 2$ superfields defined on an appropriate $N = 2$ superspace. Such a description is achieved within harmonic superspace approach [2].

The background field method is a powerful and highly efficient tool for study structure of quantum gauge theories (see e.g. [3]). The attractive features of the background field method is that it allows to preserve the manifest classical gauge invariance in quantum theory. Due to this circumstance the background field method is very convenient both for investigation of general properties of effective action in
gauge theories and for carrying out the calculations in concrete field models with gauge symmetries.

This paper is a brief review of background field method for $N = 2$ super Yang-Mills theories in harmonic superspace and some of its applications \[4, 5\].

The harmonic superspace is defined as a supermanifold parametrized by the coordinates $x^m_A, \theta^\pm_\alpha, \bar{\theta}^\pm_\dot{\alpha}, u^\pm_i$ where $x^m_A$ and $u^\pm_i$ are the bosonic coordinates and $\theta^\pm_\alpha, \bar{\theta}^\pm_\dot{\alpha}$ are the fermionic ones. The details of denotations are given in ref.\[2\]. The remarkable property of the harmonic superspace approach is that the set of coordinates and fermionic ones. The details of denotations are given in ref.\[2\]. The remarkable property of the harmonic superspace approach is that the set of coordinates is called an analytic subspace \[2\]. The analytic subspace is just that appropriate manifold for formulating the $N = 2$ supersymmetric field theories.

The pure $N = 2$ super Yang-Mills models are described in harmonic superspace by the superfield $V^{++} = V^{++a} T^a$ where $V^{++a}$ is analytic superfield (that is it defined on analytic subspace), $T^a$ are the internal symmetry generators and the denotation $++$ means that this superfield $V^{++}$ has $U(1)$-charge +2. The action for the superfield $V^{++}$ is given as follows \[4, 5\]

$$S_{SYM}[V^{++}] = \frac{1}{g^2} \int d^{12}z \sum_{n=0}^\infty \frac{(-i)^n}{n} \int du_1 \ldots du_n \frac{\text{tr}V^{++}(z, u_1) \ldots V^{++}(z, u_n)}{(u_1^+, u_2^+ \ldots (u_n^+, u_1^+))} \tag{1}$$

Here $z \equiv (x^m, \theta^i_\alpha, \bar{\theta}^\dot{i}_\dot{\alpha}); i = 1, 2; (u^+_1, u^+_2) = u^+_1 u^+_2$ and $g$ is a coupling. This action is invariant under the gauge transformations \[2\]

$$\delta V^{++} = -D^{++} \Lambda - i[V^{++}, \Lambda] \tag{2}$$

where $\Lambda$ is analytic superfield parameter and the operator $D^{++}$ was defined in ref.\[4\].

$N = 2$ matter hypermultiplets are described by the analytic superfields $q^+(x_A, \theta^+, \bar{\theta}^+, u^±)$ or $\omega(x_A, \theta^+, \bar{\theta}^+, u^±)$. The corresponding actions have the forms

$$S_q[\tilde{q}^+, q^-] = \int d\zeta^{(-4)} du \tilde{q}^+ \nabla^{++} q^+ \tag{3}$$

and

$$S_\omega[\omega] = \int d\zeta^{(-4)} du (\nabla^{++} \omega)(\nabla^{++} \omega) \tag{4}$$

with $\nabla^{++} = D^{++} + iV^{++}$ and $d\zeta^{(-4)}$ be analytic measure \[2\]. Action $S_{SYM} + S_q + S_\omega$ describes interacting system of super Yang-Mills fields and $q^+$ and $\omega$ hypermultiplets.

To construct effective action $\Gamma[V^{++}]$ depending on $V^{++}$ we split the superfield $V^{++}$ into background $V^{++}$ and quantum $v^{++}$ superfields, $V^{++} \rightarrow V^{++} + g v^{++}$. The gauge transformations \[2\] can be realized as background gauge transformations $\delta V^{++} = -\nabla^{++} \Lambda, \delta v^{++} = +i[\Lambda, v^{++}]$ and as quantum gauge transformations

$$\delta V^{++} = 0 \quad \delta v^{++} = -\frac{1}{g} \nabla^{++} \Lambda - i[v^{++}, \Lambda] \tag{5}$$

where $\nabla^{++} \Lambda = D^{++} \Lambda + i[V^{++}, \Lambda]$. It is worth to point out here that the form of background - quantum splitting and corresponding background and quantum gauge
transformations are absolutely analogous to the conventional Yang-Mills theory but not to $N = 1$ super Yang-Mills theory (see f.e. [7]).

To quantize a gauge theory within background field method one should fix only quantum gauge transformations (5). We introduce the gauge fixing functions in the form

$$F^{(4)} = \nabla^{++} v^{++}$$

and apply Faddeev-Popov procedure. As a result we obtain effective action $\Gamma[V^{++}]$ in the form (see the details in ref. [4]).

$$e^{i\Gamma[V^{++}]} = e^{i S_{\text{SYM}}[V^{++}]} \int Dv^{++} Dc Db D\phi D\tilde{q}^+ Dq^+ D\omega \text{Det}^{1/2}(-\Box) e^{i S_{\text{total}}}[v^{++}, b, c, \phi, q^+, \omega, V^{++}]$$

where

$$S_{\text{total}}[v^{++}, b, c, \phi, q^+, \omega, V^{++}] = S_2[v^{++}, b, c, \phi, q^+, \omega, V^{++}] + S_{\text{int}}[v^{++}, b, c, q^+, \omega, V^{++}]$$

Here $S_2$ plays a role of action of free theory

$$S_2[v^{++}, b, c, \phi, q^+, \omega, V^{++}] = -\frac{1}{2} \int d\zeta (-4) du \text{tr} v^{++} \Box v^{++} -$$

$$-\int d\zeta (-4) du \text{tr} (\nabla^{++} b)(\nabla^{++} c) - \frac{1}{2} \int d\zeta (-4) du \text{tr} (\nabla^{++} \phi)(\nabla^{++} \phi) +$$

$$+ \int d\zeta (-4) du \tilde{q}^+ \nabla^{++} q^+ + \int d\zeta (-4) du (\nabla^{++} \omega)(\nabla^{++} \omega)$$

The action $S_{\text{int}}$ describes the interactions

$$S_{\text{int}}[v^{++}, b, c, \phi, q^+, \omega, V^{++}] = -\int d^2 z \sum_{n=2}^{\infty} \frac{(-ig)^{n-2}}{n} \int du_1 \ldots du_n \times$$

$$\times \frac{\text{tr} v^{++}_{\tau}(z, u_1) \ldots v^{++}_{\tau}(z, u_n)}{(u_1^+, u_2^+) \ldots (u_n^+ u_1^+)} + \int d\zeta (-4) du \tilde{q}^+ V^{++} q^+ +$$

$$+ \int d\zeta (-4) du (\nabla^{++} \omega v^{++} + v^{++} \omega \nabla^{++} + (v^{++} \omega))(v^{++} \omega)$$

Here $\Omega$ is a background bridge superfield [2]. The operator $\Box = \Box \pm$ terms depending on $V^{++}$ is defined in ref. [4]. The analytic superfields $b$ and $c$ are Faddeev-Popov ghosts, the real analytic superfield $\phi$ is third (or Nilsen-Kallosh) ghost.

The path integral (1) for effective action $\Gamma[V^{++}]$ has the form standard for quantum field theory. The free action $S_2$ (1) defines the propagators of pure super Yang-Mills field, matter fields and ghosts fields. The interaction $S_{\text{int}}$ (1) defines the vertices. Eqs. (7-11) completely determine the structure of perturbation expansion for calculating the effective action $\Gamma[V^{++}]$ in a manifestly $N = 2$ supersymmetric and gauge invariant form.

As in conventional field theory one can suggest that the effective action $\Gamma[V^{++}]$ is described in terms of effective Lagrangians

$$\Gamma[V^{++}] = \int d^4 x d^4 \bar{\theta} d^4 \theta \mathcal{L}_{\text{eff}} + (\int d^4 x d^4 \theta \mathcal{L}_{\text{eff}}^c) + c.c.$$
where the $\mathcal{L}_{\text{eff}}$ and $\mathcal{L}_{\text{eff}}^{(c)}$ can be called a general effective Lagrangian and chiral effective Lagrangian respectively.

If the theory under consideration is quantized with background field method the effective action $\Gamma[V^{++}]$ will be gauge invariant under initial classical gauge transformations (background gauge transformations). In this case this effective action should be constructed only from strengths $W$ and $\bar{W}$ and their covariant derivatives. Therefore the effective Lagrangians must have the following general structure

$$\mathcal{L}_{\text{eff}} = \mathcal{H}(W, \bar{W}) + \text{terms depending on covariant derivatives of } W \text{ and } \bar{W}$$

$$\mathcal{L}_{\text{eff}}^{(c)} = \mathcal{F}(W) + \text{terms depending on covariant derivatives of strengths} \quad \text{(12)}$$

and preserving chirality

The term $\mathcal{F}(W)$ in chiral effective Lagrangian depending only on $W$ is called a holomorphic effective Lagrangian. This term is leading in low-energy limit and describes vacuum structure theory. Namely holomorphic effective Lagrangian was a main object of Seiberg-Witten theory \cite{1}. The term

$$\int d^4x d^4\theta d^4\bar{\theta}\mathcal{H}(W, \bar{W}) \quad \text{(13)}$$

defines a first non-leading correction to low-energy effective action and describes an effective low-energy dynamics.

The structure of effective action \cite{11,13} is turned out to be analogous to structure of effective action depending on chiral and antichiral superfields in $N = 1$ case. To be more precise, the chiral effective potential \cite{8,5} in $N = 1$ case is analogous to holomorphic effective Lagrangian $\mathcal{F}(W)$. The first non-leading correction $\mathcal{H}(W, \bar{W})$ in $N = 2$ case is analogous to Kählerian effective potential in $N = 1$ case \cite{8,6}. The explicit calculations of $\mathcal{F}(W)$ and $\mathcal{H}(W, \bar{W})$ in one-loop approximation for hypermultiplets coupled to abelian gauge superfield have been given within harmonic superspace formulation in \cite{10}. It has been shown that $\mathcal{F}(W)$ is obtained in the form analogous to Seiberg one for pure $N = 2$, $SU(2)$ super Yang-Mills model \cite{11}. The $\mathcal{H}(W, \bar{W})$ was given in a form of a series in a power of $W\bar{W}$ where a first term proportional to $(W\bar{W})^2$ is $N = 2$ generalization of known Heisenberg-Euler effective Lagrangian.

A simple consequence of the background field formulation is that there are no quantum corrections to $\mathcal{H}(W, \bar{W})$ at two loops in the pure $N = 2$ super Yang-Mills theory. All two-loop supergraphs contributing to the effective action within background field method are given in Fig.1

![Figure 1: Fig.1](image)

Here the wavy line corresponds to the super Yang-Mills propagator and the dotted line to the ghost propagator. These propagators are defined by the action $S_2$ \cite{9} and
have the form

\[ \langle v^+_r(1)v^+_r(2) \rangle = -\frac{i}{\hbar} (D^+_r)^4 \left\{ \delta^{12}(z_1 - z_2)\delta^{(-2,2)}(u_1, u_2) \right\} \]

\[ \langle c_r(1)b_r(2) \rangle = -\frac{i}{\hbar} (D^+_r)^4 \left\{ \delta^{12}(z_1 - z_2)\frac{(u_1^t u_2^t)}{(u_1^t u_2^t)^3} \right\} (D^+_r)^4 \quad (14) \]

Here \( v^+_r, c_r, b_r \) and the derivatives \( D^+_r \) are given in so called \( \tau \)-frame \([2, 3]\) and the distributions \( \delta^{(-2,2)}(u_1, u_2), (u_1^t u_2^t)^{-3} \) were introduced in refs. [12].

As we have noted in ref. [5], in order to get a non-zero result in two-loop supergraphs we should use twice the identity \( \delta^8(\theta_1 - \theta_2)(D^+_{1})^4(D^+_{2})^4\delta^8(\theta_1 - \theta_2) = (u_1^t u_2^t)\delta^8(\theta_1 - \theta_2) \quad [12] \). This implies that we should have 16 spinor covariant derivatives to reduce the \( \theta \)-integrals over the full \( N = 2 \) superspace to a single one. All these spinor derivatives come or from \( (D^+_r)^4 \) in the propagators \([4]\) or from expansion the operator \( \Box^{-1} \) in a power series of the \( W \) and \( \bar{W} \). After we use one \( (D^+_r)^4 \)-factor from the ghost propagator to restore the full superspace measure, we see the propagators of both gauge and ghost superfields have at most a single factor \( (D^+_r)^4 \). It is evident that the number of these \( (D^+_r)^4 \)-factors is not sufficient to form all 16 \( (D^+_r)^4 \)-factors we need in two-loop supergraphs. As to a possible way to get extra \( (D^+_r)^4 \)-factors from \( \Box^{-1} \) we observe that the spinor covariant derivatives enter the \( \Box \) always multiplied by the derivatives of \( W \) and \( \bar{W} \) (see explicit form in of \( \Box \) in ref. [4]). If we omit these derivatives the operator \( \Box \) takes the form \( \Box = \mathcal{D}^m\mathcal{D}_m + \frac{1}{2}\{W, \bar{W}\} \) and does not contain the spinor covariant derivatives. Therefore, the two-loop supergraphs given in Fig.1 do not contribute to the function \( \Gamma(W, \bar{W}) \) in effective Lagrangian \([13]\). It is worth to point out that this result is a simple consequence of the \( N = 2 \) background field method and does not demand any direct calculations of the supergraphs. Moreover, above result will be true even if we take into account the two-loop matter contributions to \( \Gamma[V^{++}] \). This is almost obvious since, after restoring the full superspace measure, the matter superfield propagator following from action \( S_2 \) \([4]\) have effectively the same structure as the gauge and ghost superfield propagators.

The \( N = 2 \) background field method leads to a simple and clear proof of the \( N = 2 \) non-renormalization theorem. See for comparison a consideratation of problem of divergences in conventional \( N = 2 \) superspace in ref. [13]. First of all, acting the same way as in the case of \( N = 1 \) non-renormalization theorem (see f.e. [4]) we can use the \( (D^+_r)^4 \)-factors in the propagators \([4]\) and in the matter superfield propagators and restore the full superspace measure \( d^2xd^4\theta d^4\bar{\theta} \) in all vertices of all supergraphs. Then, using the identity \( \delta^8(\theta_1 - \theta_2)(D^+_{1})^4(D^+_{2})^4\delta^8(\theta_1 - \theta_2) = (u_1^t u_2^t)\delta^8(\theta_1 - \theta_2) \), and making integration by part we can transform any supergraph contributing to the effective action to the form containing only a single integral over \( d^8\theta \).

Let us estimate a superficial degree of divergence for the theory under consideration. We consider an arbitrary \( L \)-loop supergraph \( G \) with \( P \) propagators, \( N_{\text{MAT}} \) external matter legs and an any number of gauge superfield external legs. We denote by \( N_D \) the number of spinor covariant derivatives acting on the external legs as a result of integration by parts in the process of transforming the contributions to a single integral over \( d^8\theta \). Taking into account the dimensions of the factors \( \Box, D^+ \) and the loop integrals over momenta we immediately obtain

\[ \omega(G) = 4L - 2P + (2P - N_{\text{MAT}} - 4L) - \frac{1}{2}N_D = -N_{\text{MAT}} - \frac{1}{2}N_D \quad (15) \]
See the details of deriving eq.(15) in ref.[5]. The eq.(15) shows that all supergraphs with external matter legs are automatically finite. As to supergraphs with pure gauge superfield legs, they will be finite only if some non-zero number of spinor covariant derivatives acts on the external legs. We will show that this is always the case beyond one loop.

Let us consider the supergraph contributions after restoring the full superspace measure at all vertices. Then we transform these contributions to \( \tau \)-frame [2, 5]. The propagators of gauge superfield, ghost superfields and matter superfields contain the background field \( V^{++} \) only via the \( \Box \) and \( D^+ \)-factors, that is, only via \( u \)-independent connections \( A_M \) [2]. But all connections \( A_M \) contain at least one spinor covariant derivative acting on background superfield \( V^{++} \) [3]. Therefore, each external leg must contain at least one spinor covariant derivative. Thus, the number \( N_D \) in eq.(13) must be greater than or equal to one. It means that \( \omega(G) < 0 \) and, hence, all supergraphs are ultravioletly finite beyond the one-loop level. As to one-loop contributions to effective action they are given in terms of functional determinants [1, 3] and demand a special and independent investigation.

Acknowledgements We are grateful to our co-authors E.I.Buchbinder and S.M. Kuzenko for collaboration and valuable discussions. We would like to thank E.A. Ivanov for critical remarks and discussions. The work of ILB was partially supported by the grants of RFBR, project 96-02-16017, by grant of RFBR-DFG, project 96-02-00180 and by grant of INTAS, INTAS 96-0308. BAO acknowledges the POE Contract OE-AC02-76-ER-03072 and the Alexander von Humboldt Foundation for partial support.

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