Exact controllability of a first-order hyperbolic equation with an intermediate point memory

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Abstract. In this paper, we discuss the exact controllability of a first-order hyperbolic equation with an intermediate point memory by boundary control. We choose a Volterra and Fredholm transformation transferring the original control system into a target control system with controllability property. Then, we prove the exact controllability of the original control system in terms of the invertibility of the transformation and the controllability of the target control system.

1. Introduction

In this paper, we mainly study the exact controllability of

\[
\begin{cases}
  u_t(x,t) = u_x(x,t) + \mu u(x_0, t), & (x, t) \in (0,1) \times (0,T), \\
  u(1, t) = U(t), & t \in (0,T), \\
  u(x, 0) = u_0(x),
\end{cases}
\]

where \(x_0 \in (0,1)\) is the mediate point, \(u(x, t)\) is the state, the initial data \(u_0(\cdot) \in H^1(0,1)\), \(\mu\) is any nonzero constant, \(U(\cdot) \in H^1(0,T)\) is a boundary control. The first-order hyperbolic equation describes quite a different set of physical problems, for example, a monolithic heat exchange process can be described by a first-order hyperbolic partial differential equation with boundary input. The term \(u(x_0, t)\) describes a specially spatial memory at point \(x_0\) in system (1). The definition of exact controllability is the normal definition, that is, for any given \(u_1(\cdot) \in H^1(0,1)\), there exists a control \(U(\cdot)\), such that \(u(x, T) = u_1(x)\). According to the semigroup theory, there exists a unique solution \(u \in C([0,T]; H^1(0,1))\). In fact, system (1) is a special case of the following system

\[
\begin{cases}
  u_t(x,t) = u_x(x,t) + \int_0^1 g(x,y)u(y,t)dy, \\
  u(1, t) = U(t), & t > 0, \\
  u(x, 0) = u_0(x),
\end{cases}
\]  

with \(g(x,y) = \mu \delta(y - x_0)\). That is to say, the system (1) can be regarded as a control system with spatial integral item.

In [1-4], the controllability of heat equation with spatial memory term was considered under some different assumptions. In [5], the controllability of first-order integro-differential hyperbolic equations
was considered, it proved that the exact controllability of (2) is equivalent to the finite time stabilization of system (2) for $H^1$ integral kernel, however, the integral kernel $g(x, y) = \mu \delta(y - x_0)$ in system (1) does not belong to $H^1$, therefore, we can not cite the Theorem in [5] to show the controllability of (1). The motivation of this paper lies in presenting the backstepping method to consider the exact controllability of (2) with memory kernel $g(x, y) = \delta(y - x_0)$.

Backstepping method was successfully used in the stabilization and observing design of various ODEs and PDEs (see [6-16] and the reference therein). The advantage of backstepping method embodies efficiency in designing the explicit stabilization feedback control. In fact, this method can also deal with the controllability of PDEs (see [17-19]). In this paper, we will use the backstepping method to consider exact controllability of the first-order hyperbolic system (1). The main idea in this paper lies in transferring the original control system into a target control system with some known controllability property. The key point lies in choosing suitable transformation to deal with the special spatial integral term in the system. We will obtain the exact controllability of the original control system in terms of the invertibility of the transformation and the controllability of the target control system.

The paper is organized as follows. In Section II, we present the controllability control design. In Section III, we prove the existence of kernels in the forward transformation. In Section IV, we give the existence of kernels in the inverse transformation. In Section V, we prove the main Theorem of this paper. Finally, we summarize this paper and present further considering problems.

2. Controllability control design

In this Section, we will present the procedure of designing the controller for the boundary controllability of the original system. We choose a target control system as

\[
\begin{cases}
  w_t(x, t) = w_x(x, t), & x \in (0,1), \ t > 0, \\
  w(1, t) = W(t), & t > 0, \\
  w(x, 0) = w_0(x),
\end{cases}
\]  

with the initial data $w_0(\cdot) \in H^1(0,1)$ and boundary control $W(\cdot) \in H^1(0,T)$. The advantage of choosing the target control system (3) lies in its controllability property which is easy to investigate. Meanwhile, we introduce an invertible transformation

\[
w(x, t) = u(x, t) - \int_0^x k(x, y)u(y, t)dy - \int_0^{x_0} r(x, y)u(y, t)dy,
\]

with undetermined kernel functions $k(x, y)$ and $r(x, y)$ transferring the original system into the target control system (3), at the same time, the inverse transformation of (4) is presented as the form

\[
u(x, t) = w(x, t) + \int_0^x l(x, y)w(y, t)dy + \int_0^{x_0} h(x, y)w(y, t)dy,
\]

where the functions $l(x, y)$ and $h(x, y)$ will be determined through some detailed calculations later on. If we find the controllability control $W(t)$ in system (3), then, taking $x = 1$ in the transformation (4), we can obtain the controllability control $U(t)$ for the original control system (1),

\[
U(t) = u(1, t) = W(t) + \int_0^1 l(1, y)w(y, t)dy + \int_0^{x_0} h(1, y)w(y, t)dy
\]

After finding the forward and backward transformation, now, we can state the main Theorem in this paper.

Theorem 1. System (1) is exactly controllable in $H^1(0,1)$. 

Remark 1. The controllability time $T$ should satisfy $T \geq 1$ because the object system is null controllable only when $T \geq 1$. Meanwhile, there are some other tools that can be used to construct the controller for the exact controllability of the original system, but here, our purpose is to show that the backstepping method can also be used to deal with the controllability problem of some PDE systems with spatial memories.

Next, we will prove Theorem 1 in several steps. We firstly make some preparations in the following Sections.

3. Existence of kernels in the forward transformation

In order to obtain the existence of $k(x,y)$ and $r(x,y)$ in the forward transformation, we differentiate the transformation (4) with respect to $x$ and $t$ on both sides, subtract and we have

\[ w_t(x,t) - w_x(x,t) = \int_0^x \left( k_x(x,y) + k_y(x,y) \right) u(y,t) dy + \int_0^x \left( r_x(x,y) + r_y(x,y) \right) u(y,t) dy + (r(x,0) + k(x,0))u(0,t) + f_{k,r}(x)u(x_0,t). \]

where $f_{k,r}(x) = \mu - \mu \int_0^x k(x,y) dy - \mu \int_0^x r(x,y) dy - r(x,x_0)$.

Choosing functions $k(x,y)$ and $r(x,y)$ to satisfy the following equations

\[ \begin{cases} k_x(x,y) + k_y(x,y) = 0, \\ k(x,0) + r(x,0) = 0, \\ r_x(x,y) + r_y(x,y) = 0, \\ f_{k,r}(x) = 0, \end{cases} \tag{6} \]

then, $w(x,t)$ satisfies $w_t(x,t) - w_x(x,t) = 0$ in the target control system (3).

Theorem 2. There exist solutions $k(x,y)$ and $r(x,y)$ to equations (6).

Proof. The equation of $k(x,y)$ is written as

\[ \begin{cases} k_x(x,y) + k_y(x,y) = 0, \\ k(x,0) + r(x,0) = 0. \end{cases} \tag{7} \]

The solution is

\[ k(x,y) = -r(x-y,0). \tag{8} \]

For simplifying the calculation of finding the possible solutions $k(x,y)$ and $r(x,y)$, we assume the possible formal solution is a function with separation of variables, that is, we set

\[ r(x,y) = p(x)q(y). \tag{9} \]

Substituting (9) into (6), we obtain

\[ p(x)q'(y) + p'(x)q(y) = 0, \tag{10} \]

with the compatibility condition

\[ f_{p,q,k}(x) = \mu - \mu \int_0^x k(x,y) dy - \mu p(x) \int_0^x q(y) dy - p(x) q(x_0) = 0. \tag{11} \]

Supposing that $p(x) \neq 0$, $q(y) \neq 0$, we know $\frac{p'(x)}{p(x)} = -\frac{q'(y)}{q(y)} = a$. Then, $p(x) = be^{ax}$, $q(y) = ce^{-ay}$, where $a$, $b$ and $c$ are undetermined nonzero constants.
Next, we verify that the compatibility condition \( f_{pqk}(x) = 0 \) holds when choosing suitable constants \( a, b \) and \( c \). Taking derivative of \( f_{pqk}(x) \), according to (6), (8) and (11), we obtain

\[
f'_{pqk}(x) = \mu p(x) q(0) - \mu p'(x) \int_0^x q(y) dy - p'(x) q(x_0) = 0
\]

(12)

Then, \( \frac{pr(x)}{p(x)} = \frac{\mu q(0)}{\mu \int_0^x q(y) dy + q(x_0)} = a \). Substitute \( p(x) \) and \( q(y) \) into the right side of the above equation, we know \( (a - \mu)e^{-ax_0} = 0 \). Therefore, if we choose \( a = \mu, f_{pqk}(x) \) is constant. In terms of \( f_{pqk}(x) = 0 \), we have \( f_{pqk}(0) = \mu - \mu p(0) \int_0^x q(y) dy - p(0)q(x_0) = \mu - bc = 0 \).

Hence, when choosing \( bc = \mu \), the compatibility condition \( f_{pqk}(x) = 0 \) holds. Finally, we obtain the explicit solution

\[
r(x, y) = p(x) q(y) = \mu e^{\mu(x-y)},
\]

(13)

\[
k(x, y) = -r(x - y, 0) = -\mu e^{\mu(x-y)}.
\]

(14)

4. Existence of kernels in the inverse transformation

In this Section, we will prove the existence of the kernels in the inverse transformation, which will show that the forward and inverse transformations are mutual invertible pairs. Following the similar procedure as in Section III, computing \( u_x \) and \( u_t \) from (5), we obtain that

\[
u_t(x, t) - u_x(x, t) - \mu u(x_0, t) = \begin{align*}
(h(x, x_0) - \mu)w(x_0, t) - & \left( h(x, 0) + l(x, 0) \right) w(0, t) - \int_0^x (l_x(x, y) + l_y(x, y)) w(y, t) dy \\
& - \int_0^x h_x(x, y) + h_y(x, y) + \mu l(x_0, y) + \mu h(x_0, y))w(y, t) dy.
\end{align*}
\]

Choosing the functions \( h(x, y) \) and \( l(x, y) \) to satisfy

\[
\begin{cases}
l_x(x, y) + l_y(x, y) = 0, \\
l(x, 0) + h(x, 0) = 0, \\
h_x(x, y) + h_y(x, y) + \mu l(x_0, y) + \mu h(x_0, y) = 0, \\
h(x, x_0) = \mu,
\end{cases}
\]

(15)

then, we know the function \( u(x, t) \) defined by (5) satisfies the equation in (1).

**Lemma 1.** There exist solutions \( h(x, y) \) and \( l(x, y) \) to equations (15).

**Proof.** From (15), \( l(x, y) \) needs to satisfy

\[
\begin{cases}
l_x(x, y) + l_y(x, y) = 0, \\
l(x, 0) + h(x, 0) = 0.
\end{cases}
\]

(16)

The solution is

\[
l(x, y) = -h(x - y, 0).
\]

(17)

Next, we solve

\[
\begin{cases}
h_x(x, y) + h_y(x, y) + \mu l(x_0, y) + \mu h(x_0, y) = 0, \\
h(x, x_0) = \mu.
\end{cases}
\]

(18)
Similarly, as in Section III, assuming that (18) has a formal solution with separation of variables, that is, \( h(x, y) \) can be taken as

\[
h(x, y) = m(x)n(y), \tag{19}
\]

substituting (17) and (19) into (18), we obtain \( m(x) \) and \( n(y) \) satisfying

\[
\begin{align*}
(m(x)n'(y) + m'(x)n(y) - \mu m(x_0 - y)n(0) + \mu m(x_0)n(y) &= 0, \\
(m(x)n(x_0) &= \mu.
\end{align*}
\tag{20}
\]

As can be seen from the above formula, \( m(x) = \frac{\mu}{n(x_0)} \neq 0 \), that means \( m(x) \) is a constant. Then \( n(y) \) satisfies

\[
n'(y) + \mu n(y) - \mu n(0) = 0.
\]

For the sake of generality, take \( n(0) = 1 \), we have \( n(y) = 1 \). Then we can get \( m(x) = \frac{\mu}{n(x_0)} = \mu \).

By (17) and (19), we obtain

\[
h(x, y) = m(x)n(y) = \mu \tag{21}
\]

and

\[
l(x, y) = -m(x - y)n(0) = -\mu, \tag{22}
\]

which finishes the proof of Lemma 1.

Meanwhile, the transformation defines two bounded linear operators \( P: H^1(0,1) \to H^1(0,1) \),

\[
(Pu)(x) = u(x) - \int_0^x k(x,y)u(y)dy - \int_0^{x_0} r(x,y)u(y)dy, \ u \in H^1(0,1). \tag{23}
\]

and \( Q: H^1(0,1) \to H^1(0,1) \),

\[
(Qw)(x) = w(x) + \int_0^x l(x,y)w(y)dy + \int_0^{x_0} h(x,y)w(y)dy, \ w \in H^1(0,1). \tag{24}
\]

Since the functions \( k(x,y), r(x,y), h(x,y) \) and \( l(x,y) \) respectively defined by (13), (14), (21), and (22) are smooth on \([0,1] \times [0,1] \), we know that the linear operators \( P \) and \( Q \) are bounded, at the same time, it satisfies \((QPu)(x) = u(x)\), which shows that \( P \) and \( Q \) are bounded invertible linear operators.

5. Exact controllability of the original system

In this Section, we will prove the exact controllability of the original system (1) by several steps.

Proof. We will give the proof of Theorem 1 by four steps.

Step 1. Exact controllability of (1) is equivalent to the null controllability of system (1). We need to prove that the exact controllability of (1) can be deduced from the null controllability of system (1).

In fact, for any given \( u_1(\cdot) \in H^1(0,1) \), for the equation

\[
\begin{align*}
\bar{u}_1(x, t) &= \bar{u}_x(x, t) + \mu \bar{u}(x_0, t), (x, t) \in (0,1) \times (0,T), \\
\bar{u}(1, t) &= \bar{U}(t), t \in (0,T), \\
\bar{u}(x, 0) &= \bar{u}(x), 0,
\end{align*}
\tag{25}
\]

there exists an initial data \( \bar{u}(\cdot, 0) \in H^1(0,1) \) and \( \bar{U}(\cdot) \in H^1(0,T) \) such that \( \bar{u}(x, T) = u_1(x) \) and \( \bar{u} \in C([0,T]; H^1(0,1)) \). Meanwhile, according to the assumption that the system (1) is null controllable, there also exists \( \bar{U}(\cdot) \in H^1(0,T) \) for the equation
\[
\begin{aligned}
\begin{cases}
\ddot{u}(x,t) = \ddot{u}_x(x,t) + \mu \dot{u}(x_0,t), (x,t) \in (0,1) \times (0,T), \\
\dot{u}(1,t) = \overline{U}(t), t \in (0,T), \\
\ddot{u}(x,0) = u_0(x) - \overline{u}(x,0),
\end{cases}
\end{aligned}
\]  

such that \(\overline{u}(x,T) = 0\) for \(T \geq 1\).

Setting \(u(x,t) = \overline{u}(x,t) + \ddot{u}(x,t), U(t) = \overline{U}(t) + \ddot{U}(t)\), then, we know that \(u(x,t)\) satisfies the equation

\[
\begin{aligned}
\begin{cases}
\dot{u}(x,t) = u_x(x,t) + \mu u(x_0,t), (x,t) \in (0,1) \times (0,T), \\
\dot{u}(1,t) = U(t), t \in (0,T), \\
\ddot{u}(x,0) = u_0(x)
\end{cases}
\end{aligned}
\]

and

\[
\dot{u}(x,T) = u_1(x).
\]

which shows that the system (1) is exact controllable in \(H^1(0,1)\).

**Step 2.** The forward transformation is invertible, which was checked in Section IV.

**Step 3.** The target control system is null controllable. In fact, as shown in [4], for \(T \geq 1\), there exists a control \(w(\cdot) \in H^1(0,T)\) such that \(w(x,T) = 0\).

**Step 4.** Exact controllability of the original system.

According to **Step 3**, we know the target control system (3) is null controllable, then, according to the inverse transformation (5), we have \(u(x,T) = 0\) for \(T \geq 1\), which shows the original system is null controllable. In terms of **Step 1**, we know that system (1) is exact controllable in \(H^1(0,1)\), which finishes the proof of **Theorem 1**.

### 6. Conclusions

In this paper, we have studied the exact controllability in \(H^1(0,1)\) for the first-order hyperbolic system with intermediate point \(x_0 \in (0,1)\). We have presented the forward transformation transferring the original control system into the target control system, the advantage of choosing the target control system lies in its controllability property which is easy to investigate. We have proved the existence of kernels in the forward and backward transformations. Meanwhile, we proved the exact controllability of the original control system based on the invertibility of the forward transformation and null controllability of the target control system. Furthermore, the procedure of this paper also may be extended to similar problems for some evolution questions, which will be further considered in the future.

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