Qualitative analysis for a variable delay system of differential equations of second order

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ABSTRACT
This paper analyzes the stability, uniformly stability, asymptotically stability, boundedness, uniformly boundedness and square integrability of solutions of a system of differential equations of second order with variable delay by applying the direct method of Lyapunov- Krasovskii. By means of a new Lyapunov-Krasovskii functional, we simplify and extend some previous work that is found in the recent literature. Finally, the validity and applicability of the proceeded results are indicated by some numerical examples applying MATLAB-Simulink. By the results of this paper, we can obtain the results of Omeike et al. [Stability and boundedness of solutions of certain vector delay differential equations. J Nigerian Math Soc. 2018;37(2):77–87], Theorem 1.1 and Theorem 1.2, under weaker conditions. In addition, we establish two new results on the uniformly boundedness and square integrability of solutions of a system of differential equations.

1. Introduction
In the past few decades, qualitative analysis of solutions of ordinary or delay differential equations of second order have attracted increasing attention due to its wide application in physics, engineering, signal processing, medicine, population dynamics and so on. Due to these facts, stability and some related concepts as important index of control systems receive considerable attention. A large number of papers are devoted to various kinds of stability, boundedness, convergence and some other properties of ordinary and delay differential equations and systems of differential equations [1–61]. To the best of our information, the results of these papers are derived by means of the Lyapunov or Lyapunov-Krasovskii direct method applying various Lyapunov functions or Lyapunov-Krasovskii functionals. In this paper, we are not interested in the details of obtained results and used methods. However, during qualitative analysis of that differential equations, we should point out that suitable candidate function(s) or functional(s) are very effective for construction of stronger and weaker conditions.

In Omeike et al. [27], Omeike et al. considered the following Lienard delay differential equation (DDE) with the variable delay $r(t) \geq 0$:

$$X'' + AX' + H(X(t - r(t))) = P(t, X, X'),$$

in which $t \in \mathbb{R}^+, \mathbb{R}^+ = [0, \infty) \times \mathbb{R}^n, r(t) \leq r(t) \leq \gamma, \gamma r'(t) \leq \xi, 0 < \xi < 1, An \times n - H: \mathbb{R}^n \to \mathbb{R}^nP: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^nH(0) = 0$.

Omeike et al. [27] proved the following two theorems on the asymptotically stability and uniformly boundedness, uniformly ultimately boundedness of solutions, respectively, when $P(t, X, X') \equiv 0$ and $P(t, X, X') \neq 0$. The results of Omeike et al. [27] are given by the following theorem.

Let

$$P(t, X, X') = 0.$$

Theorem 1.1 ([27]): Consider DDE (1), let $H(0) = 0$ and suppose that:

(A1) $0 \leq r(t) \leq \gamma, \gamma$ is a positive constant, $r'(t) \leq \xi, 0 < \xi < 1$;

(A2) the matrices $A$ and $J_h(X)$ (Jacobian matrix of $H(X)$) are symmetric and positive definite, and furthermore that the eigenvalues $\lambda_i(A)$ and $\lambda_i(J_h(X)), (i = 1, 2, \ldots, n)$, of $A$ and $J_h(X)$, respectively, satisfy

$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a, \delta_h \leq \lambda_i(J_h(X)) \leq \Delta_h$ for $X \in \mathbb{R}^n$,

where $\delta_a, \delta_h, \Delta_a$ and $\Delta_h$ are finite constants;

(A3) the matrices $A$ and $J_h(X)$ commute.

Then the zero solution of DDE (1) is asymptotically stable provided that

$$\gamma < \min \left( \frac{2\delta_a \delta_h}{\Delta_a \Delta_h}, \frac{\delta_a}{\mu + \Delta_h} \right).$$
Now, let
\[ P(t, X, X') \neq 0. \]

**Theorem 1.2 ([27]):** If, in addition to the conditions (A1), (A2) and (A3) of Theorem 1.1, the inequality
\[ ||p(t, X, Y)|| \leq m + \delta (||X|| + ||Y||), \]
\[(\text{positive constants}),\]
holds, then the solutions of DDE (1) are uniformly bounded and uniformly ultimately bounded provided the constant \( \gamma \) satisfies
\[ \gamma < \min \left( \frac{2\delta \theta}{\Delta a \Delta h}, \frac{2\delta a (1 - \xi)}{\Delta a \Delta h[\Delta x + 2 - 2(2 - \xi)]} \right). \]

Motivated by the results of Omeike et al. [27], that is, the above Theorem 1.1 and Theorem 1.2, and the mentioned sources, we consider the following system of differential equations of second order with the variable delay \( \tau(t) \):
\[ X'' + F(X, X')X' + H(X(t - \tau(t))) = P(t, X, X'), \]
in which \( t \in \mathbb{R}^+, \mathbb{R}^+ = [0, \infty), X \in \mathbb{R}^n, \tau(t) \leq t \leq \gamma, \gamma \tau(t) \leq \xi, 0 < \xi < 1; F_n X_n - H : \mathbb{R}^n \to \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+ \).

DDE (2) is the vector version of the below nonlinear differential equations of second order:
\[ x_i'' + \sum_{k=1}^{n} f_{ik}(x_1, \ldots, x_n; x_1', \ldots, x_n') + h_i(x_1(t - \tau(t)), \ldots, x_n(t - \tau(t))) = p_i(t, x_1, \ldots, x_n, x_1', \ldots, x_n'), (i = 1, 2, \ldots, n). \]

We can write DDE (2) as the below differential system:
\[ X' = Y, \]
\[ Y' = -F(X, Y)Y - H(X) \]
\[ + \int_{t-\tau(t)}^{t} J_b(X(s))Y(s)ds + P(t, X, Y), \]
\[ \text{where } J_b(X) \text{ is the Jacobian matrix of } H(X) \text{ defined by } \]
\[ J_b(X) = \left( \frac{\partial h_i}{\partial X_j} \right), (i, j = 1, 2, \ldots, n), \]
\[(x_1, x_2, \ldots, x_n) \text{ and } (h_1, h_2, \ldots, h_n) \text{ are the components of } X \text{ and } H, \text{ respectively.}\]

It is assumed that the Jacobian matrix \( J_b(X) \) exists and is continuous. Let \( X, Y \in \mathbb{R}^n \). Then, we define
\[ \langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i, \langle X, X \rangle = ||X||^2 \text{ and } ||A|| = \sum_{i=1}^{n} |a_i|. \]

For brevity in notation, if a function is written without its argument, we mean that the argument is always \( t \).

For example, \( X \) represents \( X(t) \).

The aim of this paper is to obtain the results of [27] under weaker conditions and give some additional new results. Besides, the validity and applicability of the results to be proceed are indicated by some numerical examples applying MATLAB-Simulink. These are contributions of the results of this paper to be given below.

**2. Basic definitions and fundamental results**

For a given number \( r \geq 0 \), let \( C^r \) denote the space of continuous functions mapping the interval \([-r, 0]\) into \( \mathbb{R}^n \) and for \( \varphi \in C^r, ||\varphi|| = \sup_{-r \leq \xi \leq 0} ||\varphi(\xi)|| \).

\( C^r \) denotes the set of \( \varphi \) in \( C^r \) for which \( ||\varphi|| < H \). For any continuous function \( x(u) \) defined on \([-r, 0] \), \( H > 0, \) any fixed \( t, 0 \leq t \leq B, \) the symbol \( x_t \) denotes the function \( x(t + \theta), -r \leq \theta \leq 0 \).

We consider the autonomous delay differential equation (DDE):
\[ x'(t) = g(x_t), \quad t \geq 0. \]

It is assumed that \( g(\varphi) \) is a function defined for every \( \varphi \in C^r \) and \( x'(t) \) is the right side derivative of \( x(t) \). We say \( x(\varphi) \) is a solution of DDE (4) with the initial condition \( \varphi \in C^r \) at \( t = 0 \) if there is a \( B > 0 \) such that \( x(\varphi) \) is a function from \([-r, B) \mathbb{R}^n x_t(\varphi) C^r| 0 \leq t < B, x_0(\varphi) = \varphi x(\varphi)(t) \leq t < B \).

**Lemma 2.1 ([8]):** Suppose \( g(0) = 0. \) Let \( V \) be a continuous functional defined on \( C^r \) with \( V(0) = 0 \) and let \( u(s) \) be a function, non-negative and continuous for \( 0 \leq s < \infty, u(s) \rightarrow \infty \) as \( s \rightarrow \infty \) with \( u(0) = 0. \) If for all \( \varphi \in C^r \), \( \varphi(0) \neq 0, \) then the zero solution of DDE (4) is stable.

Let \( R \subset C^r \) be a set of all functions \( \varphi \in C^r \) where \( V(\varphi) \neq 0. \) If \( [0] \) is the largest invariant set in \( R \), then the solution \( x = 0 \) of DDE (4) is asymptotically stable.

Let us consider the following non-autonomous delay differential equation (DDE):
\[ x'(t) = f(t, x_t), x_0 = x(t + \theta), -r \leq \theta \leq 0, t \geq 0, \]
where \( f : \mathbb{R}^+ \times C^r \to \mathbb{R}^n \) is a continuous mapping, \( f(t, 0) = 0, \) and we suppose that \( F \) takes closed bounded sets into bounded sets of \( \mathbb{R}^n \). Here \( (C, ||.||) \) is the Banach space of continuous function \( \psi : [-r, 0] \to \mathbb{R}^n \) with supremum norm, \( r \geq 0; C^r \) is the open \( H \)-ball in \( C; C^r := \{ \psi \in (C[-r, 0], \mathbb{R}^n) : ||\psi|| < H \}. \)

Let \( S \) be the set of \( \psi \in C \) such that \( ||\psi|| \geq H. \) We denote by \( S^* \) the set of all functions \( \phi \in C \) such that \( \phi(0) \geq H, \) where \( H \) is large enough.

**Definition 2.1 ([4]):** A continuous function \( W : \mathbb{R}^n \to \mathbb{R}^n \) is continuous, \( f(t, 0) = 0, \) and we suppose that \( F \) takes closed bounded sets into bounded sets of \( \mathbb{R}^n \). Here \( (C, ||.||) \) is the Banach space of continuous function \( \psi : [-r, 0] \to \mathbb{R}^n \) with supremum norm, \( r > 0; C^r \) is the open \( H \)-ball in \( C; C^r := \{ \psi \in (C[-r, 0], \mathbb{R}^n) : ||\psi|| < H \}. \)

Let \( S \) be the set of \( \psi \in C \) such that \( ||\psi|| \geq H. \) We denote by \( S^* \) the set of all functions \( \phi \in C \) such that \( \phi(0) \geq H, \) where \( H \) is large enough.

**Definition 2.2 ([4]):** Let \( D \) be an open set in \( \mathbb{R}^n \) with \( 0 = \partial D. \) A function \( V : [0, \infty) \times D \to [0, \infty) \) is defined as \( V(t, x) = 0 \) if \( t > 0, \) and \( W(t, x) \) strictly increasing is a wedge. (We denote wedges by \( W \) or \( W_t \) where \( i \) is an integer.)

**Theorem 2.1 ([4]):** Suppose that there is a continuous Lyapunov-Krasovskii for DDE (5) and wedges satisfying the following:

\[ \text{1.} \]
\[ \text{2.} \]
\[ \text{3.} \]
1) \( W_1(\phi(0)) \leq V(t, \phi) \leq W_2(\phi(t)) \), (where \( W_1(r) \) and \( W_2(r) \) are wedges);
2) \( V(t, \phi) \leq 0 \).

Then the zero solution of DDE (5) is uniformly stable.

**Theorem 2.2 ([50]):** Suppose that there exists a continuous Lyapunov-Krasovskii functional \( V(t, \phi) \) defined for all \( t \in \mathbb{R}^n, \mathbb{R}^+ = [0, \infty) \), \( \phi \in S^* \).

3) \( a(\phi(0)) \leq V(t, \phi) \leq b_1(\phi(0)) + b_2(||\phi||) \), where \( a(r), b_1(r), b_2(r) \in C_1 \) (\( C_1 \) denotes the families of continuous increasing functions), and are positive for \( r > H \) and \( a(r) - b_2(r) \rightarrow \infty \) as \( r \rightarrow \infty \);

4) \( \gamma V(t, \phi) \leq 0 \).

Then, the solutions of DDE (5) are uniformly bounded.

### 3. Qualitative results for solutions

In this section, we ensure the main problem of this paper.

#### 3.1. Hypotheses

Suppose the following hypotheses hold:

(A1) There are positive constants \( \delta_f \) and \( \Delta_f \) such that the symmetric matrix \( F \) satisfies

\[
\delta_f \leq \lambda_1(F(X, Y)) \leq \Delta_f \quad \text{for all } X, Y \in \mathbb{R}^n.
\]

(A2) There are some positive constants \( \delta_h \) and \( \Delta_h \) such that

\[
H(0) = 0, H(X) \neq 0, (X \neq 0), J_h(X) \text{ is symmetric and }\]

\[
\delta_h \leq \lambda_1(J_h(X)) \leq \Delta_h
\]

for all \( X \in \mathbb{R}^n \); \( 0 < \tau(t) \leq \gamma, \gamma \) is a positive constant, \( \tau(s) = \xi, 0 < \xi < 1 \).

(A3) There is a continuous and non-negative function \( \alpha(t) \) such that

\[
||P(t, X, Y)|| \leq \alpha(t)||Y|| \text{ for all } t \geq t_0 \text{ and } X, Y \in \mathbb{R}^n,
\]

where \( \alpha(t) \in L^1(0, \infty) \), \( L^1(0, \infty) \) is space of integrable Lebesgue functions.

**Lemma 3.1 ([27]):** If \( A \) is a real symmetric \( n \times n \)-matrix and 

\[
\sigma_2 \geq \lambda_1(\Lambda) \geq \sigma_1 > 0, \quad (i = 1, 2, \ldots, n),
\]

then for any \( X \in \mathbb{R}^n \), we have 

\[
\sigma_1 ||X||^2 \leq \langle AX, X \rangle \leq \sigma_2 ||X||^2,
\]

where \( \sigma_1 \) and \( \sigma_2 \) are the least and the greatest eigenvalues of the matrix \( \Lambda \).

**Lemma 3.2 ([27]):** Let \( H(X) \) be a continuously differentiable vector function with \( H(0) = 0 \). Then

\[
\frac{d}{dt} \int_0^1 (H(\sigma X), Y) d\sigma = \langle H(X), Y \rangle.
\]

#### 3.3. Main results

Let 

\[
P(t, X, Y) \equiv 0.
\]

Our first result is the following theorem.

**Theorem 3.1:** Let hypotheses (A1) and (A2) hold. If \( Y < ((1 - \xi)\delta_f) / \Delta_h \) holds, then the zero solution of DDE (2) is uniformly stable and asymptotically stable.

**Proof:** We define a continuously differentiable Lyapunov-Krasovskii functional \( V(\cdot) = V(X_t, Y_t) \) by

\[
V(\cdot) = \int_0^1 (H(\sigma X), Y) d\sigma + \lambda \int_{-\tau(t)}^{t} \langle Y(\theta), Y(\theta) \rangle d\theta d\sigma,
\]

where \( \lambda \) is a positive constant, and it is chosen later.

It is clear that \( V(0, 0) = 0 \). By Lemmas 3.1–3.3, it follows that

\[
V(\cdot) \geq \frac{1}{2} \delta_h ||X||^2 + ||Y||^2 + \lambda \int_{-\tau(t)}^{t} ||Y(\theta)||^2 d\theta d\sigma
\]

\[
\geq K_1(||X||^2 + ||Y||^2),
\]

where \( K_1 = \min \{2^{-1} \delta_h, 1\} \).
By the similar procedure, we can derive that

\[ V(.) \leq \frac{1}{2} \Delta_h ||X||^2 + ||Y||^2 + \lambda \int_{-\tau(t)}^{t} \int_{t+s}^{t} ||Y(\theta)||^2 \, d\theta \, ds. \]

Then, we can find a continuous function \( u(s) \) such that

\[ u(||\varphi(0)||) \leq V(\varphi), u(||\varphi(0)||) \geq 0. \]

The time derivative of the Lyapunov-Krasovskii functional \( V(.) \) along any solution of DDS (2) is given by

\[
\frac{d}{dt} V(X_t, Y_t) = -(H(X), Y) - (F(X, Y), Y) \\
+ \int_{-\tau(t)}^{t} \langle Y(t), J_h(X(s)) Y(s) \rangle \, ds \\
+ \frac{d}{dt} \int_{0}^{t} \langle H(\sigma X), X \rangle \, d\sigma \\
+ \lambda \frac{d}{dt} \int_{-\tau(t)}^{t} \int_{t+s}^{t} \langle Y(\theta), Y(\theta) \rangle \, d\theta \, ds.
\]

It is clear that

\[
\frac{d}{dt} \int_{0}^{t} \langle H(\sigma X), X \rangle \, d\sigma = \langle H(X), Y \rangle
\]

and

\[
\frac{d}{dt} \int_{-\tau(t)}^{t} \int_{t+s}^{t} \langle Y(\theta), Y(\theta) \rangle \, d\theta \, ds = \tau(t) \langle Y(t), Y(t) \rangle - (1 - \tau'(t)) \int_{-\tau(t)}^{t} \langle Y(\theta), Y(\theta) \rangle \, d\theta.
\]

Hence, we have

\[
\frac{d}{dt} V(X_t, Y_t) = -(F(X, Y), Y) \\
+ \int_{-\tau(t)}^{t} \langle Y(t), J_h(X(s)) Y(s) \rangle \, ds \\
+ \lambda \tau(t) \langle Y(t), Y(t) \rangle - \lambda (1 - \tau'(t)) \\
\times \int_{-\tau(t)}^{t} \langle Y(\theta), Y(\theta) \rangle \, d\theta.
\]

The assumptions \( \delta_t \leq \lambda_t (F(X, Y)), \lambda_t (J_h(X)) \leq \Delta_h \) and the inequality \( 2|f| |g| \leq f^2 + g^2 \) (with \( f \) and \( g \) are real numbers) combined with the classical Cauchy-Schwartz inequality leads

\[
\frac{d}{dt} V(X_t, Y_t) \leq -\delta_t ||Y||^2 + \Delta_h \int_{-\tau(t)}^{t} \langle Y(s), Y(s) \rangle \, ds \\
+ \lambda \gamma ||Y||^2 - \lambda (1 - \xi) \int_{-\tau(t)}^{t} \langle Y(\theta), Y(\theta) \rangle \, d\theta.
\]

Then

\[
\frac{d}{dt} V(X_t, Y_t) \leq -(\delta_t - \lambda \gamma) ||Y||^2 - (\lambda (1 - \xi) - \Delta_h) \\
\times \int_{-\tau(t)}^{t} \langle Y(\theta), Y(\theta) \rangle \, ds.
\]

Let

\[
\lambda = \frac{\Delta_h}{1 - \xi}.
\]

Hence, this equality implies

\[
\frac{d}{dt} V(X_t, Y_t) \leq -\left( \delta_t - \frac{\Delta_h}{1 - \xi} \right) ||Y||^2.
\]

If \( Y < ((1 - \xi) \delta_t / \Delta_h) \), then there exists a positive constant \( \rho \) such that

\[
\frac{d}{dt} V(X_t, Y_t) \leq -\rho ||Y||^2 \leq 0.
\]

This inequality shows that the time derivative of the Lyapunov-Krasovskii functional \( V(X_t, Y_t) \) is negative semidefinite. Hence, we can conclude that the zero solution of DDE (2) is uniformly stable. On the same time, the zero solution of DDE (2) is also stable.

We now consider the set defined by

\[
l_s \equiv \{ (X, Y) : \frac{d}{dt} V(X_t, Y_t) = 0 \}.
\]

If we apply the LaSalle’s invariance principle, then we observe that \( (X, Y) \in l_s \) implies that \( Y = 0 \). Hence, DDS (3) and together \( Y = 0 \), necessarily, implies \( H(X) = 0 \).

Since \( Y = 0 \Rightarrow X = 0 \), then \( X = \xi, \xi (\neq 0) \), is a vector. This equality can be hold if and only if

\[
H(\xi) = 0.
\]

Hence,

\[
H(\xi) = 0 \Leftrightarrow \xi = 0
\]

so that \( H(X) = 0 \Leftrightarrow X = 0 \). Therefore, we have \( X = Y = 0 \). In fact, this result shows that the largest invariant set contained in the set \( l_s \) is \((0, 0) \in l_s \). Therefore, we can conclude that the zero solution of DDE (2) is asymptotically stable. This result completes the proof of Theorem 3.1.

**Corollary 3.1:** In the light of the assumptions of Theorem 3.1, it can be proceeded that all solutions of DDE (2) are uniformly bounded. We omit the details of the proof.

**Theorem 3.2:** If assumptions (A1) and (A2) hold, then the first derivatives of the solutions of DDE (2) are square integrable when \( P(.) \equiv 0 \).

**Proof:** We now give our attention to the Lyapunov-Krasovskii \( V(X_t, Y_t) \) which is used in the proof of Theorem 3.1.
It is known from Theorem 3.1 that
\[
\frac{d}{dt} V(X_t, Y_t) \leq -\rho ||Y||^2 \leq 0.
\]
Integrating this inequality from 0 to \( t \), we have
\[
V(X_0, Y_t) = V(X_0, Y_0) - \rho \int_0^t ||Y(s)||^2 ds \leq 0.
\]
Hence, it is clear that
\[
V(X_t, Y_t) + \rho \int_0^t ||Y(s)||^2 ds \leq V(X_0, Y_0).
\]
Since \( V(X_t, Y_t) \) is positive definite, then we can assume
\[
V(X_0, Y_0) = K_2, \quad K_2 \in \mathbb{N}, \quad K_2 > 0.
\]
Hence, we can derive
\[
\rho \int_0^t ||Y(s)||^2 ds \leq V(X_t, Y_t) + \rho \int_0^t ||Y(s)||^2 ds \leq V(X_0, Y_0) = K_2.
\]
Then
\[
\int_0^\infty ||Y(s)||^2 ds = \int_0^\infty ||\dot{X}(s)||^2 ds \leq \rho^{-1} K_2 < \infty.
\]
This result completes the proof of Theorem 3.2.

**Theorem 3.3:** If the assumptions of Theorem 3.1 hold, then all solutions of DDE (2) and their first order derivatives are bounded as \( t \to \infty \) when \( P(.) \equiv 0 \).

**Proof:** We again consider the estimates
\[
V(X_t, Y_t) \leq V(X_t, Y_t) + \rho \int_0^\infty ||Y(s)||^2 ds \leq K_2
\]
and
\[
||X||^2 + ||Y||^2 \leq K_1^{-1} V(X_t, Y_t),
\]
which can be found in the former proofs.

Then, in view of these inequalities, we can conclude that
\[
||X||^2 + ||Y||^2 \leq K_1^{-1} K_2.
\]
This inequality completes the proof of Theorem 3.3.

Let
\[
P(t, X, X') \neq 0.
\]
Our fourth and the last result is the following theorem.

**Theorem 4.1:** If assumptions (A1)–(A3) hold, then there exists a positive constant \( K_2 \) such that all solutions DDE (2) satisfy the inequalities
\[
||X(t)|| \leq K_3, \quad ||X'(t)|| \leq K_3
\]
as \( t \to +\infty \) when \( P(t, X, X') \neq 0 \).
Figure 1. (a) Orbits of $x_1(t)$ for Example 4.1. (b) Orbits of $x_2(t)$ for Example 4.1.

It is clear that

$$P(t, X, X') = 0,$$

$$F(X, X') = \begin{bmatrix} 10 + \sin t(1 + x_1^2) + x_1^2 + x_2^2 + x_2'^2)^{-1} & 0 \\ 5 + \cos t(1 + x_1^2) + x_1^2 + x_2^2 + x_2'^2)^{-1} & 0 \end{bmatrix},$$

where $X = (x_1, x_2)$,

$$H(X) = \begin{bmatrix} 3x_1 + \arctgx_1 \\ 3x_2 + \arctgx_2 \end{bmatrix}$$

and the variable delay $\tau(t) = (1/4)\sin^2 t$ satisfies

$$0 \leq \tau(t) = (1/4)\sin^2 t \leq (1/4) = \gamma,$$

$$\tau'(t) = \frac{1}{2} \sin t \cos t \leq \frac{1}{2} = \xi, \text{ that is, } 0 < \xi = \frac{1}{2} < 1.$$

Next, as eigenvalues of the matrix $F(.)$, we have

$$\lambda_1(F(.)) = 10 + \frac{\sin t}{1 + x_1^2 + x_2^2 + x_2'^2},$$

and

$$\lambda_2(F(.)) = 5 + \frac{\cos t}{1 + x_1^2 + x_2^2 + x_2'^2}.$$
We can derive
\[ \delta_f = 4 \leq \lambda_i(F) \leq \Delta_f = 11, \quad (i = 1, 2). \]
Next, the Jacobian matrix of \( H(X) \) is
\[
J_h(X) = \begin{bmatrix}
3 + \frac{1}{1 + x_1^2} & 0 \\
0 & 3 + \frac{1}{1 + x_2^2}
\end{bmatrix}.
\]
Hence
\[ \delta_h = 3 \leq \lambda_i(J_h(X)) \leq 4 = \Delta_h, \quad (i = 1, 2). \]
Thus, all hypotheses \((A1)\) and \((A2)\) of Theorem 2.1 hold.
In addition, since
\[ \gamma = \frac{1}{4}, \xi = \frac{1}{2}, \delta_f = 4, \Delta_h = 4, \]
then \( (((1 - \xi)\delta_f)/\Delta_h) = (1/2) \), that is, \( \gamma = (1/4) < (((1 - \xi)\delta_f)/\Delta_h) = (1/2) \) holds. For the particular case of DDE (2) when \( P(t) \equiv 0 \), we give DDE (6) in Example 4.1. Then, it can be seen that the zero solution of DDE (6) is uniformly stable, asymptotically stable, and all solutions of DDE (6) are uniformly bounded, their first derivatives are square integrable. Besides, all solutions of DDE (6) and their first order derivatives are bounded as \( t \to \infty \) (see Figure 1a and 1b).

Now, let
\[ P(t, X, X') \neq 0. \]

**Example 4.2:** In the particular case of DDE (2), we now consider the following DDE with the variable delay,
Further, we proceed

\[
\begin{bmatrix}
  x_1' \\
  x_2'
\end{bmatrix}
+ \begin{bmatrix}
  10 + \sin t(1 + x_1^2 + x_2^2 + x_3^2)(t - 4^{-1}\sin^2 t) \\
  5 + \cos t(1 + x_1^2 + x_2^2 + x_3^2)(t - 4^{-1}\sin^2 t)
\end{bmatrix}
\]

obtain the results of [27] under weaker conditions. At the same time, in addition, we obtain new results on the uniformly stability and integrability of solutions of the considered system. Further, two examples are given to illustrate the validity and feasibility of the main results of this paper.

**Contribution**

All the authors have equal contribution in this paper and there is no competing interest.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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