Discrete index transformations with Bessel and Lommel functions

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Abstract
Discrete analogs of the index transforms, involving Bessel and Lommel functions are introduced and investigated. The corresponding inversion theorems for suitable classes of functions and sequences are established.

Keywords Bessel functions · Modified Bessel functions · Lommel functions · Fourier series · Index transforms

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1 Introduction and preliminary results

Our goal in this paper is to investigate the mapping properties and prove inversion formulas for the following transformations between suitable sequences \( \{a_n\}_{n \geq 1} \) and functions \( f \) in terms of the series and integrals, which are associated with Bessel and Lommel functions (cf. [2], Ch. 10, 11), namely,

\[
f(x) = \sum_{n=0}^{\infty} \frac{a_n}{\cosh(\pi n/2)} \text{Re}[J_n(x)], \quad x > 0, \tag{1}
\]

\[
a_n = \frac{1}{\cosh(\pi n/2)} \int_0^\infty \text{Re}[J_n(x)] f(x) \, dx, \quad n \in \mathbb{N}_0, \tag{2}
\]

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\[ f(x) = \sum_{n=1}^{\infty} \frac{a_n}{\sinh(\pi n/2)} \text{Im}[J_n(x)], \quad x > 0, \tag{3} \]

\[ a_n = \frac{1}{\sinh(\pi n/2)} \int_{0}^{\infty} \text{Im}[J_n(x)] f(x) \, dx, \quad n \in \mathbb{N}, \tag{4} \]

\[ f(x) = \sum_{n=1}^{\infty} a_n \Gamma\left(\frac{1-\mu-in}{2}\right) \Gamma\left(\frac{1-\mu+in}{2}\right) S_{\mu,n}(x), \quad x > 0, \tag{5} \]

\[ a_n = \Gamma\left(\frac{1-\mu-in}{2}\right) \Gamma\left(\frac{1-\mu+in}{2}\right) \int_{0}^{\infty} S_{\mu,n}(x) f(x) \, dx, \quad n \in \mathbb{N}. \tag{6} \]

Here \(i\) is the imaginary unit and \(\text{Re}, \text{Im}\) denote the real and imaginary parts of a complex-valued function. We call transformations (1)–(6) the discrete index transforms, comparing them with continuum analogs (cf. [4]). Bessel functions \(J_{\mu}(z), Y_{\nu}(z), z, \nu \in \mathbb{C}\) of the first and second kind, respectively, are solutions of the Bessel differential equation

\[ z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - \nu^2) u = 0. \tag{7} \]

These functions have the following asymptotic behavior at infinity and near the origin

\[ J_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} (2\nu + 1)\right) \left[1 + O(1/z)\right], \quad z \to \infty, \quad |\arg z| < \pi, \tag{8} \]

\[ J_{\nu}(z) = O(z^{\nu}), \quad z \to 0, \tag{9} \]

\[ Y_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} (2\nu + 1)\right) \left[1 + O(1/z)\right], \quad z \to \infty, \quad |\arg z| < \pi, \tag{10} \]

\[ Y_{\nu}(z) = O\left(|z|^{-1}\right), \quad z \to 0, \quad \nu \neq 0, \tag{11} \]

\[ Y_{0}(z) = O(\log(|z|)), \quad z \to 0. \tag{12} \]

On the other hand, the modified Bessel functions \(I_{\nu}(z), K_{\nu}(z), z, \nu \in \mathbb{C}\) are solutions of the modified Bessel differential equation

\[ z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2) u = 0, \tag{13} \]

having the corresponding asymptotic behavior

\[ I_{\nu}(z) = O\left(|z|^\nu\right), \quad z \to 0, \tag{14} \]
\[ I_\nu(z) = O\left(\frac{e^z}{\sqrt{2\pi z}}\right), \quad z \to \infty, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \quad (15) \]

\[ K_\nu(z) = O\left(|z|^{-\left\lfloor \Re \nu \right\rfloor}\right), \quad z \to 0, \quad \nu \neq 0, \quad K_0(z) = O\left(|\log(|z|)|\right), \quad z \to 0, \quad (16) \]

\[ K_\nu(z) = O\left(\sqrt{\frac{\pi}{2z}} e^{-z}\right), \quad z \to \infty, \quad |\arg z| < \frac{3\pi}{2}. \quad (17) \]

Bessel functions are related by the equalities
\[ Y_\nu(z) = \frac{1}{\sin(\pi \nu)} [J_\nu(z) \cos(\pi \nu) - J_{-\nu}(z)], \quad (18) \]
\[ K_\nu(z) = \frac{\pi}{2\sin(\pi \nu)} [I_{-\nu}(z) - I_\nu(z)]. \quad (19) \]

Meanwhile, considering the inhomogeneous Bessel equation
\[ z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - \nu^2) u = z^\mu, \quad (20) \]
we find its solution as the Lommel function \( s_{\mu,\nu}(z) \) [2], Ch. 11. Its companion \( S_{\mu,\nu}(z) \) is defined by the equality (see [2], Entry 11.9.5)
\[ S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{1}{2}(\mu + \nu + 1)\right) \Gamma\left(\frac{1}{2}(\mu - \nu + 1)\right) \]
\[ \times \left(\sin\left(\frac{1}{2}(\mu - v)\pi\right) J_\nu(z) - \cos\left(\frac{1}{2}(\mu - v)\pi\right) Y_\nu(z)\right). \quad (21) \]
where \( \Gamma(z) \) is Euler’s gamma function [2], Ch. 5 and \( \mu \pm \nu \neq -1, -2, \ldots \). It behaves at infinity by virtue of [2], Entry 11.9.9 as follows
\[ S_{\mu,\nu}(z) = O(z^{\mu-1}), \quad z \to \infty. \quad (22) \]

In the sequel we will provide existence conditions for discrete transformations (1)-(6) and establish their inversion formulas for suitable sequences and functions. To do this, we will employ integral representations of the Bessel and Lommel functions in the kernels of these operators and classical Fourier series for Lipschitz functions.

### 2 Inversion theorems

We begin with

**Theorem 1** Let a sequence \( \{a_n\}_{n \in \mathbb{N}} \) satisfy the condition
Then the discrete transformation (1) can be inverted by the formula

\[ a_n = \frac{2}{\pi} \int_0^\infty \Phi_n(x)f(x)dx, \quad n \in \mathbb{N}_0, \]  

(24)

where the kernel \( \Phi_n(x) \) is defined by

\[ \Phi_n(x) = \int_0^\pi \sin(x \cosh(u)) \sinh(u) \cos(nu)du, \quad x > 0, \quad n \in \mathbb{N}_0, \]

(25)

and integral (24) converges in the improper sense.

**Proof** To proceed the proof, we will appeal to the relatively convergent integral (see [3], Vol. II, Entry 2.12.15.3)

\[ \int_0^\infty \sin(t \cosh(u)) \text{Re}[J_{in}(t)]dt = \frac{\cos(nu) \cosh(\pi n/2)}{\sinh(u)}, \quad u \in (0, \pi], \quad n \in \mathbb{N}_0. \]

(26)

In fact, taking some \( T > 0 \), we have from (1)

\[
\begin{align*}
\int_0^T \sin(t \cosh(u)) & \sum_{m=0}^{\infty} \frac{a_m}{\cosh(\pi m/2)} \text{Re}[J_{im}(t)]dt \\
& = \sum_{m=0}^{\infty} \frac{a_m}{\cosh(\pi m/2)} \int_0^T \sin(t \cosh(u)) \text{Re}[J_{im}(t)]dt, 
\end{align*}
\]

(27)

where the interchange of the order of integration and summation can be justified via the uniform convergence with respect to \( t \in [0, T] \) of the series (1). Indeed, this comes immediately from the definition of the Bessel function and the condition (23), because

\[ |J_{im}(t)| = \sum_{k=0}^{\infty} \frac{|a_m|}{k! \Gamma(k + in + 1)} \leq \sum_{k=0}^{\infty} \frac{(T/2)^{2k}}{k! \Gamma(k + in + 1)} \leq e^T \sqrt{\frac{\sinh(\pi m)}{\pi n}}, \]

and therefore

\[ \sum_{m=0}^{\infty} \frac{|a_m|}{\cosh(\pi m/2)} |\text{Re}[J_{im}(t)]| \leq e^T \sum_{m=0}^{\infty} \frac{|a_m|}{\cosh(\pi m/2)} \sqrt{\frac{2 \tanh(\pi m/2)}{\pi m}} \leq e^T ||a||_1 < \infty. \]

(28)

Moreover, taking (25), the right-hand side of (24) can be written in the form
\[
\int_0^\infty \Phi_n(x)f(x)dx = \lim_{T \to \infty} \int_0^T \int_0^\pi \sin(x \cosh(u)) \sinh(u) \cos(nu)du \\
\times \sum_{m=0}^\infty \frac{a_m}{\cosh(\pi m/2)} \Re[J_{im}(x)]dx \\
= \lim_{T \to \infty} \sum_{m=0}^\infty \frac{a_m}{\cosh(\pi m/2)} \int_0^\pi \int_1^T \sin(x \cosh(u)) \sinh(u) \cos(nu) \Re[J_{im}(x)]dxdudx,
\]

where the latter interchange of the order of integration and summation is via (28).

Now we employ the integral representation of the kernel in (1) (cf. Entry 2.5.54.7 in [3], Vol. I)

\[
\Re[J_{in}(x)] = \cos(\pi n/2) = \frac{2}{\pi} \int_0^\infty \cos(nt) \sin(x \cosh(t))dt
\]

to substitute in (29), getting the equality

\[
\int_0^\infty \Phi_n(x)f(x)dx = \lim_{T \to \infty} \frac{2}{\pi} \sum_{m=0}^\infty a_m \int_0^\pi \int_1^T \sin(x \cosh(u)) \sinh(u) \cos(nu) \\
\times \int_0^\infty \cos(mt) \sin(x \cosh(t))dtdxdu.
\]

Since for some fixed \(T > 1, 1/T \leq x \leq T\) and sufficiently big \(M > T\) we have via integration by parts

\[
\left| \int_M^\infty \cos(nt) \sin(x \cosh(t))dt \right| = \left| \int_{\cosh(M)}^\infty \cos(n \log(t + \sqrt{t^2 - 1})) \frac{\sin(xt)}{\sqrt{t^2 - 1}} dt \right| \\
= \frac{1}{x} \left| \frac{\cos(nM) \cos(x \cosh(M))}{\sinh(M)} - \int_{\cosh(M)}^\infty \cos(n \log(t + \sqrt{t^2 - 1})) \frac{\cos(xt)}{(t^2 - 1)^{3/2}} dt \right| \\
= n \int_{\cosh(M)}^\infty \sin(n \log(t + \sqrt{t^2 - 1})) \frac{\cos(xt)}{t^2 - 1} dt \\
+ \int_{\cosh(M)}^\infty \frac{t}{(t^2 - 1)^{3/2}} dt + n \int_{\cosh(M)}^\infty \frac{1}{t^2 - 1} dt \\
= T \left[ \frac{2}{\sinh(M)} + \frac{n}{2} \log \left( \frac{\cosh(M) + 1}{\cosh(M) - 1} \right) \right] \to 0, \quad M \to \infty.
\]

Therefore integral (30) converges uniformly with respect to \(x \in [1/T, T]\). Consequently, we change the order of integration and calculating an elementary integral, we obtain
\[
\int_{0}^{\infty} \Phi_n(x)f(x)dx = \lim_{T \to \infty} \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_{0}^{\pi} \sinh(u) \cos(nu) dx \\
\times \int_{0}^{\infty} \cos(mt) \\
\times \left[ \frac{\sin(T(\cosh(t) - \cosh(u)))}{\cosh(t) - \cosh(u)} - \frac{\sin(T(\cosh(t) + \cosh(u)))}{\cosh(t) + \cosh(u)} \right. \\
\left. - \frac{\sin(1/T(\cosh(t) - \cosh(u)))}{\cosh(t) - \cosh(u)} + \frac{\sin(1/T(\cosh(t) + \cosh(u)))}{\cosh(t) + \cosh(u)} \right] dtdu.
\] (32)

In the meantime, since via (23)

\[
\left| \sum_{m=0}^{\infty} a_m \int_{0}^{\pi} \sinh(u) \cos(nu) dx \\
\times \int_{0}^{\infty} \cos(mt) \left[ \frac{\sin(1/T(\cosh(t) - \cosh(u)))}{\cosh(t) - \cosh(u)} - \frac{\sin(1/T(\cosh(t) + \cosh(u)))}{\cosh(t) + \cosh(u)} \right] dtdu \\
= \sum_{m=0}^{\infty} a_m \int_{0}^{\pi} \sinh(u) \cos(nu) dx \\
\times \int_{0}^{\sqrt{T}t} \cos(mt) \left[ \frac{\sin(1/T(\cosh(t) - \cosh(u)))}{\cosh(t) - \cosh(u)} - \frac{\sin(1/T(\cosh(t) + \cosh(u)))}{\cosh(t) + \cosh(u)} \right] dtdu \\
+ \sum_{m=0}^{\infty} a_m \int_{0}^{\pi} \sinh(u) \cos(nu) dx \\
\times \int_{0}^{\infty} \cos(mt) \left[ \frac{\sin(1/T(\cosh(t) - \cosh(u)))}{\cosh(t) - \cosh(u)} - \frac{\sin(1/T(\cosh(t) + \cosh(u)))}{\cosh(t) + \cosh(u)} \right] dtdu \\
\leq [\cosh(\pi) - 1] \sum_{m=0}^{\infty} |a_m| \\
\times \left[ \frac{2}{\sqrt{T}} + \int_{\sqrt{T}}^{\infty} \frac{1}{\cosh(t) - \cosh(\pi)} + \frac{1}{\cosh(t)} \right] dt \to 0, \quad T \to \infty,
\]

our goal will be to justify the existence of the limit and pass to it under series sign in the equality

\[
\int_{0}^{\infty} \Phi_n(x)f(x)dx = \lim_{T \to \infty} \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_{0}^{\pi} \sinh(u) \cos(nu) dx \\
\times \int_{0}^{\infty} \cos(mt) \\
\times \left[ \frac{\sin(T(\cosh(t) - \cosh(u)))}{\cosh(t) - \cosh(u)} - \frac{\sin(T(\cosh(t) + \cosh(u)))}{\cosh(t) + \cosh(u)} \right] dtdu.
\] (33)

In fact, since
\[
\sum_{m=0}^{\infty} |a_m| \int_0^\pi \sinh(u) |\cos(nu)| \int_0^\infty \left| \cos(mt) \frac{\sin(T \cosh(t) + \cosh(u))}{\cosh(t) + \cosh(u)} \right| dt du
\leq \sum_{m=0}^{\infty} |a_m| \int_0^\pi \sinh(u) du \int_0^\infty \frac{dt}{\cosh(t)} = \frac{\pi}{2} |\cosh(\pi) - 1| \sum_{m=0}^{\infty} |a_m| < \infty,
\]
we have
\[
\lim_{T \to \infty} \sum_{m=0}^{\infty} a_m \int_0^\pi \sinh(u) \cos(nu) \int_0^\infty \cos(mt) \frac{\sin(T \cosh(t) + \cosh(u))}{\cosh(t) + \cosh(u)} dt du
\]
\[
= \int_0^\pi \sinh(u) \cos(nu) \lim_{T \to \infty} \sum_{m=0}^{\infty} a_m \cos(m \log(t - \cosh(u)) + ((t - \cosh(u))^2 - 1)^{1/2}) \frac{\sin(Tt)}{t((t - \cosh(u))^2 - 1)^{1/2}} dt du = 0,
\]
owing to the Riemann-Lebesgue lemma. Further, we write
\[
\frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^\pi \sinh(u) \cos(nu) \int_0^\infty \cos(mt) \times \frac{\sin(T \cosh(t) - \cosh(u))}{\cosh(t) - \cosh(u)} dt du
\]
\[
= \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi) - 1} \cos \left( n \log(u + 1 + ((u + 1)^2 - 1)^{1/2}) \right)
\times \int_{-u}^{\infty} \cos \left( m \log \left( t + u + 1 + ((t + u + 1)^2 - 1)^{1/2} \right) \right) \times \frac{\sin(Tt)}{t((t + u + 1)^2 - 1)^{1/2}} dt du
\]
\[
= \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi) - 1} \cos \left( n \log(u + 1 + ((u + 1)^2 - 1)^{1/2}) \right)
\times \int_{-u}^{\infty} \cos \left( m \log \left( t + u + 1 + ((t + u + 1)^2 - 1)^{1/2} \right) \right)
\left( (t + u + 1)^2 - 1 \right)^{1/2}
\frac{\sin(Tt)}{t((u + 1)^2 - 1)^{1/2}} dt du
\]
\[
+ \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi) - 1} \cos \left( n \log(u + 1 + ((u + 1)^2 - 1)^{1/2}) \right)
\left( (u + 1)^2 - 1 \right)^{1/2}
\frac{\sin(Tt)}{t((u + 1)^2 - 1)^{1/2}} dt du.
\]
\[
\frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi) - 1} \cos(n \log(u + 1 + ((u + 1)^2 - 1)^{1/2})) \, du
\]
\[
\times \frac{\cos(m \log(u + 1 + ((u + 1)^2 - 1)^{1/2}))}{((u + 1)^2 - 1)^{1/2}} \int_{-u}^{\infty} \frac{\sin(Tt)}{t} \, dt \, du
\]
\[
= \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \left( \int_0^{1/\sqrt{T}} + \int_{1/\sqrt{T}}^{\cosh(\pi) - 1} \right) \cos(n \log(u + 1 + ((u + 1)^2 - 1)^{1/2})) \, du
\]
\[
\times \frac{\cos(m \log(u + 1 + ((u + 1)^2 - 1)^{1/2}))}{((u + 1)^2 - 1)^{1/2}} \int_{-u}^{\infty} \frac{\sin(Tt)}{t} \, dt \, du
\]

and, employing the second mean value theorem,

\[
\frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_{1/\sqrt{T}}^{\cosh(\pi) - 1} \cos(n \log(u + 1 + ((u + 1)^2 - 1)^{1/2})) \, du
\]
\[
\times \frac{\cos(m \log(u + 1 + ((u + 1)^2 - 1)^{1/2}))}{((u + 1)^2 - 1)^{1/2}} \int_{-u}^{\infty} \frac{\sin(Tt)}{t} \, dt \, du
\]
\[
= \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_{1/\sqrt{T}}^{\cosh(\pi) - 1} \cos(n \log(u + 1 + ((u + 1)^2 - 1)^{1/2})) \, du
\]
\[
\times \frac{\cos(m \log(u + 1 + ((u + 1)^2 - 1)^{1/2}))}{((u + 1)^2 - 1)^{1/2}} \left( \pi - \int_{uT}^{\infty} \frac{\sin(t)}{t} \, dt \right) \, du
\]

\[
= \sum_{m=0}^{\infty} a_m \int_{1/\sqrt{T}}^{\cosh(\pi) - 1} \cos(n \log(u + 1 + ((u + 1)^2 - 1)^{1/2})) \, du
\]
\[
\times \frac{\cos(m \log(u + 1 + ((u + 1)^2 - 1)^{1/2}))}{((u + 1)^2 - 1)^{1/2}} \, du
\]
\[
- \frac{1}{T \pi} \sum_{m=0}^{\infty} a_m \int_{1/\sqrt{T}}^{\cosh(\pi) - 1} \cos(n \log(u + 1 + ((u + 1)^2 - 1)^{1/2})) \, du
\]
\[
\times \frac{\cos(m \log(u + 1 + ((u + 1)^2 - 1)^{1/2}))}{((u + 1)^2 - 1)^{1/2}} \int_{uT}^{\infty} \sin(t) \, dt \, du
\]
\[
\rightarrow \sum_{m=0}^{\infty} a_m \int_0^\pi \cos(n u) \cos(mu) \, du = \frac{\pi}{2} a_n, \quad T \to \infty
\]

because
Hence,

\[
\frac{1}{T\pi} \sum_{m=0}^{\infty} a_m \int_{1/\sqrt{T}}^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + \left( (u + 1)^2 - 1 \right)^{1/2} \right) \right) \frac{\cos \left( m \log \left( u + 1 + \left( (u + 1)^2 - 1 \right)^{1/2} \right) \right)}{u^2 - 1} du \left( \frac{T}{T(1 + 2\sqrt{T})} \right)^{1/2} \sum_{m=0}^{\infty} |a_m| \to 0, \ T \to \infty,
\]

where \( A > 0 \) is an absolute constant. Hence, returning to (35) we need to establish the equality

\[
\frac{1}{\pi} \lim_{\delta \to \infty} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + \left( (u + 1)^2 - 1 \right)^{1/2} \right) \right) \frac{\cos \left( m \log \left( t + u + 1 + \left( (t + u + 1)^2 - 1 \right)^{1/2} \right) \right)}{\left( (t + u + 1)^2 - 1 \right)^{1/2}} \left( (u + 1)^2 - 1 \right)^{1/2} \sin(Tt) \frac{T}{t} dt \]

In fact, denoting the expression under the limit sign in (37) by \( I(T) \) and fixing a small positive \( \delta \), we get

\[
I(T) = \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + \left( (u + 1)^2 - 1 \right)^{1/2} \right) \right) \frac{\cos \left( m \log \left( t + u + 1 + \left( (t + u + 1)^2 - 1 \right)^{1/2} \right) \right)}{\left( (t + u + 1)^2 - 1 \right)^{1/2}} \left( (u + 1)^2 - 1 \right)^{1/2} \sin(Tt) \frac{T}{t} dt du + \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + \left( (u + 1)^2 - 1 \right)^{1/2} \right) \right) \frac{\cos \left( m \log \left( t + u + 1 + \left( (t + u + 1)^2 - 1 \right)^{1/2} \right) \right)}{\left( (t + u + 1)^2 - 1 \right)^{1/2}} \left( (u + 1)^2 - 1 \right)^{1/2} \sin(Tt) \frac{T}{t} dt du = I_1(T) + I_2(T).
\]

Hence,
\[
\lim_{T \to \infty} I_2(T) = \frac{1}{\pi} \lim_{T \to \infty} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + (u + 1)^2 - 1 \right) \right) \\
\times \int_{-u}^{\delta} \frac{\cos \left( m \log \left( t + u + 1 + ((t + u + 1)^2 - 1)^{1/2} \right) \right) \sin(Tt)}{t^2} dt du \\
- \frac{1}{\pi} \lim_{T \to \infty} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + (u + 1)^2 - 1 \right) \right) \\
\times \frac{\cos \left( m \log \left( u + 1 + (u + 1)^2 - 1 \right) \right) \sin(Tr)}{u^{1/2}} du \int_{\delta T}^{\infty} \frac{\sin(t)}{t} dt = 0
\]

via the Riemann-Lebesgue lemma and the remainder of the convergent integral. Concerning the expression \( I_1(T) \), we appeal to the Leibniz differentiation formula under the integral sign to write

\[
I_1(T) = \frac{a_0}{\pi} \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + (u + 1)^2 - 1 \right) \right) \\
\times \int_{-u}^{\delta} \left[ \frac{1}{((t + u + 1)^2 - 1)^{1/2}} - \frac{1}{((u + 1)^2 - 1)^{1/2}} \right] \frac{\sin(Tt)}{t} dt du \\
+ \frac{1}{\pi} \sum_{m=1}^{\infty} a_m \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + (u + 1)^2 - 1 \right) \right) \\
\times \left[ \frac{d}{du} \int_{-u}^{\delta} \sin \left( m \log \left( t + u + 1 + ((t + u + 1)^2 - 1)^{1/2} \right) \right) \right] \frac{\sin(Tt)}{t} dt du \\
+ \frac{1}{\pi} \sum_{m=0}^{\infty} a_m \int_0^{\cosh(\pi)-1} \cos \left( n \log \left( u + 1 + (u + 1)^2 - 1 \right) \right) \\
\times \sin \left( m \log \left( u + 1 + (u + 1)^2 - 1 \right) \right) \frac{\sin(Tu)}{u} du \\
= N_1(T) + N_2(T) + N_3(T).
\]

Then since
\begin{align*}
\int_0^{\cosh(\pi)-1} & \left| \cos\left(n \log\left(u + 1 + (u + 1)^2 - 1\right)^{1/2}\right) \right| \\
\times & \int_{-u}^{\delta} \left[ \frac{1}{(t + u)(t + u + 2)^{1/2}} - \frac{1}{(u + 2)^{1/2}} \right] \frac{\sin(T)}{t} dt du \\
\leq & \int_0^{\cosh(\pi)-1} \frac{1}{(u + 2)^{3/4}} \int_{-\delta}^{\delta} \frac{t + 2(u + 1)}{(t + u)(t + u + 2)^{3/4}} dt du < \infty ,
\end{align*}

it yields \( \lim_{T \to \infty} N_1(T) = 0 \) via the Riemann-Lebesgue lemma. Analogously, since

\[
\cos\left(n \log\left(u + 1 + (u + 1)^2 - 1\right)^{1/2}\right) \times \sin\left(m \log\left(u + 1 + (u + 1)^2 - 1\right)^{1/2}\right) \frac{1}{u} \in L_1(0, \cosh(\pi) - 1),
\]

we get, taking into account (23), \( \lim_{T \to \infty} N_3(T) = 0 \). Finally, the middle term in (38) can be treated via integration by parts. In fact, we find

\[
N_2(T) = \frac{(-1)^n}{\pi} \sum_{m=1}^{\infty} a_m \int_{1-\cosh(\pi)}^{\delta} \sin\left(m \log\left(t + \cosh(\pi) + (t + \cosh(\pi)^2 - 1)^{1/2}\right) \right) \frac{\sin(T)}{t} dt \\
- \frac{1}{\pi} \sum_{m=1}^{\infty} a_m \int_{0}^{\delta} \sin\left(m \log\left(t + 1 + (t + 1)^2 - 1\right)^{1/2}\right) \frac{\sin(T)}{t} dt \\
+ \frac{2n}{\pi} \sum_{m=1}^{\cosh(\pi)-1} \frac{1}{\sin\left(m \log\left(u + 1 + (u + 1)^2 - 1\right)^{1/2}\right)} \\
\times \int_{-u}^{\delta} \sin\left(\frac{m}{2} \log\left(\frac{t + u + 1 + ((t + u + 1)^2 - 1)^{1/2}}{u + 1 + (u + 1)^2 - 1}\right) \right) \times \frac{\sin(T)}{t} dt du ,
\]

and we see that \( \lim_{T \to \infty} N_2(T) = 0 \) by the same reasons. Thus, combining with (34), (35), (36), we return to (33) to establish the inversion formula (24), completing the proof of Theorem 1.

The discrete transformation (2) can be inverted by the following theorem.

**Theorem 2**  Let \( f \) be a complex-valued function on \( \mathbb{R}_+ \) which is represented by the integral

\[
f(x) = \int_0^\pi \sin(x \cosh(u)) \varphi(u) du , \quad x > 0,
\]

where \( \varphi(u) = \psi(u) \sinh(u) , \quad \psi(0) = 0 \) and \( \psi \) is a continuously differentiable even \( 2\pi \)-periodic function. Then for all \( x > 0 \) the following inversion formula for transformation (2) holds
\[ f(x) = \frac{2}{x\pi} \sin \left( \frac{x}{2} (\cosh(\pi) - 1) \right) \sin \left( \frac{x}{2} (\cosh(\pi) + 1) \right) + \frac{2}{\pi} \sum_{n=0}^{\infty} \Phi_n(x)a_n, \quad (40) \]

where \( \Phi_n \) is defined by (25).

**Proof** Plugging the right-hand side of the representation (39) in (2), we change the order of integration, employ (26) and the definition of \( \varphi \) to obtain

\[ a_n = \int_0^{\pi} \psi(u) \cos(nu) du. \quad (41) \]

The interchange of the order of integration can be justified in the following way. We have

\[
\int_0^\infty \Re[J_n(x)] f(x) dx = \lim_{T \to \infty} \int_0^T \Re[J_n(x)] f(x) dx \\
= \lim_{T \to \infty} \int_0^\pi \varphi(u) \int_0^T \Re[J_n(x)] \sin(x \cosh(u)) dx du \\
\times \cosh \left( \frac{\pi n}{2} \right) \int_0^\pi \psi(u) \cos(nu) du \\
- \lim_{T \to \infty} \int_0^\pi \varphi(u) \int_T^\infty \Re[J_n(x)] \sin(x \cosh(u)) dx du,
\]

where the interchange is allowed owing to the continuity of the integrand on the rectangle \([0, \pi] \times [0, T]\). To show that the latter limit is zero we appeal to the asymptotic behavior of the Bessel function (8) to find for each \( n \in \mathbb{N} \)

\[ \Re[J_n(x)] = \cosh \left( \frac{\pi n}{2} \right) \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{4} \right) \left[ 1 + O(1/x) \right], \quad x \to \infty. \]

Hence for sufficiently big \( T > 0 \) and by virtue of second mean value theorem we get

\[
\int_0^\pi \varphi(u) \int_T^\infty \Re[J_n(x)] \sin(x \cosh(u)) dx du \\
= \cosh \left( \frac{\pi n}{2} \right) \sqrt{\frac{1}{2\pi}} \int_0^\pi \varphi(u) \int_T^\infty \sin \left( x(\cosh(u) - 1) - \frac{\pi}{4} \right) \frac{dx du}{\sqrt{x}} + O \left( \frac{1}{\sqrt{T}} \right) \\
= O \left( \frac{1}{\sqrt{T}} \int_0^\pi \frac{\varphi(u)}{\cosh(u) - 1} du \right) \to 0, \quad T \to \infty
\]

since the latter integral is convergent under conditions of the theorem. Thus, returning to (41), we substitute \( a_n \) and \( \Phi_n \) by (25) into the partial sum of the series (40) \( S_N(x) \), and it becomes

\[ S_N(x) = \frac{2}{\pi} \sum_{n=0}^{N} \int_0^\pi \sin(x \cosh(t)) \sinh(t) \cos(nt) dt \int_0^\pi \psi(u) \cos(nu) du. \quad (42) \]

Hence, calculating the sum via the known identity
\[
\sum_{n=0}^{N} \cos(nt) \cos(nu) = \frac{1}{4} \left[ 2 + \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} + \frac{\sin((2N+1)(u+t)/2)}{\sin((u+t)/2)} \right],
\]

we obtain from (42)

\[
S_N(x) = \frac{2}{x\pi} \sin \left( \frac{x}{2} (1 - \cosh(\pi)) \right) \sin \left( \frac{x}{2} (\cosh(\pi) + 1) \right) + \frac{1}{2\pi} \int_{0}^{\pi} \sin(x \cosh(t)) \sin(t) \times \int_{-\pi}^{\pi} \psi(u) \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} du dt.
\]

(43)

Since \( \psi \) is 2\( \pi \)-periodic, we treat the latter integral with respect to \( u \) as follows

\[
\int_{-\pi}^{\pi} \psi(u) \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} du
\]

\[
= \int_{t-\pi}^{t+\pi} \psi(u) \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} du
\]

\[
= \int_{-\pi}^{\pi} \psi(u+t) \frac{\sin((2N+1)u/2)}{\sin(u/2)} du.
\]

Moreover,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(u+t) \frac{\sin((2N+1)u/2)}{\sin(u/2)} du - \psi(t)
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(u+t) - \psi(t)] \frac{\sin((2N+1)u/2)}{\sin(u/2)} du.
\]

When \( u + t > \pi \) or \( u + t < -\pi \) then we interpret the value \( \psi(u+t) \) by formulas

\[
\psi(u+t) - \psi(t) = \psi(u+t - 2\pi) - \psi(t - 2\pi),
\]

\[
\psi(u+t) - \psi(t) = \psi(u+t + 2\pi) - \psi(t + 2\pi),
\]

respectively. Then since \( \psi \in C^1[-\pi, \pi] \), it satisfies the Lipschitz condition on \([-\pi, \pi] \)

\[
|\psi(u) - \psi(v)| \leq C|u - v|, \quad \forall \ u, v \in [-\pi, \pi],
\]

(44)

where \( C > 0 \) is an absolute constant. Hence we have the uniform estimate for any \( t \in [-\pi, \pi] \)

\[
\frac{|\psi(u+t) - \psi(t)|}{|\sin(u/2)|} \leq C \left| \frac{u}{\sin(u/2)} \right|.
\]

Therefore, owing to the Riemann-Lebesgue lemma
\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du = 0
\] 

(45)

for all \( t \in [-\pi, \pi] \). Besides, returning to (43), we estimate the iterated integral

\[
\int_{0}^{\pi} |\sin(x \cosh(t))| \sin(t) \int_{-\pi}^{\pi} |[\psi(u + t) - \psi(t)] 
\times \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, dudt \leq C|\cosh(\pi) - 1| \int_{-\pi}^{\pi} \left| \frac{u}{\sin(u/2)} \right| \, du < \infty.
\]

Consequently, via the dominated convergence theorem it is possible to pass to the limit when \( N \to \infty \) under the integral sign, and recalling (45), we derive

\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{0}^{\pi} \sin(x \cosh(t)) \sin(t) \int_{-\pi}^{\pi} [\psi(u + t) - \psi(t)] 
\times \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, dudt = \frac{1}{2\pi} \int_{0}^{\pi} \sin(x \cosh(t)) \sin(t) 
\times \lim_{N \to \infty} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, dudt = 0.
\]

Hence, combining with (43), we obtain by virtue of the definition of \( \varphi \) and \( f \)

\[
\lim_{N \to \infty} S_N(x) = \frac{2}{x\pi} \sin \left( \frac{x}{2} (1 - \cosh(\pi)) \right) \sin \left( \frac{x}{2} (\cosh(\pi) + 1) \right) + \int_{0}^{\pi} \sin(x \cosh(t)) \varphi(t) \, dt 
= f(x) + \frac{2}{x\pi} \sin \left( \frac{x}{2} (1 - \cosh(\pi)) \right) \sin \left( \frac{x}{2} (\cosh(\pi) + 1) \right),
\]

where the integral (39) converges since \( \varphi \in C[0, \pi] \). Thus we established (40), completing the proof of Theorem 2.

The same scheme can be applied to invert discrete \( Im \)-transformations (3), (4). It involves the following analogs of the integrals (26), (30) (cf. [3], Vol. II, Entry 2.12.15.3, Vol. I, Entry 2.5.54.7)

\[
\int_{0}^{\infty} \cos(t \cosh(u)) \text{Im}[J_n(t)] \, dt = -\frac{\cos(nu) \sinh(\pi n/2)}{\sinh(u)}, \quad u \in (0, \pi], \quad n \in \mathbb{N},
\]

(46)

\[
\frac{\text{Im}[J_n(x)]}{\sinh(\pi n/2)} = -\frac{2}{\pi} \int_{0}^{\infty} \cos(nt) \cos(x \cosh(t)) \, dt.
\]

(47)

We will formulate the corresponding theorems, leaving the proofs to the interested reader.

**Theorem 3** Let a sequence \( \{a_n\}_{n \in \mathbb{N}} \) satisfy the condition (23). Then the discrete transformation (3) can be inverted by the formula

\[
a_n = -\frac{2}{\pi} \int_{0}^{\infty} \Psi_n(x)f(x) \, dx, \quad n \in \mathbb{N}_0,
\]

(48)

where the kernel \( \Psi_n(x) \) is defined by
\[
\Psi_n(x) = \int_0^\pi \cos(x \cosh(u)) \sinh(u) \cos(nu) du, \quad x > 0, \quad n \in \mathbb{N}_0, \quad (49)
\]
and integral (48) converges in the improper sense.

**Theorem 4** Let \( f \) be a complex-valued function on \( \mathbb{R}_+ \) which is represented by the integral

\[
f(x) = \int_0^\pi \cos(x \cosh(u)) \varphi(u) du, \quad x > 0, \quad (50)
\]
where \( \varphi(u) = \psi(u) \sinh(u) \), \( \psi(0) = 0 \) and \( \psi \) is a continuously differentiable even \( 2\pi \)-periodic function. Then for all \( x > 0 \) the following inversion formula for transformation (4) holds

\[
f(x) = \frac{2}{x\pi} \sin\left(\frac{x}{2} (\cosh(\pi) - 1)\right) \cos\left(\frac{x}{2} (\cosh(\pi) + 1)\right) + \frac{2}{\pi} \sum_{n=0}^{\infty} \Psi_n(x) a_n, \quad (51)
\]
where \( \Psi_n \) is defined by (49).

In order to establish the inversion formula for the transformation (5), involving the Lommel function (21) in the kernel, we will set up the following lemma.

**Lemma 1** Let \( \Re \mu < 0, u \in (0, \pi] \), \( n \in \mathbb{N} \). Then the following formula takes place

\[
\frac{1}{\sinh(u) \sinh(\pi n) \Gamma((1 - \mu - in)/2) \Gamma((1 - \mu + in)/2)} \int_0^\infty \sin\left(x \cosh(u) - \frac{\pi i u}{2}\right) S_{\mu, in}(x) dx = \frac{2^\mu \pi^2 \sin(nu)}{2^\mu \pi^2 \sin(nu)} \quad (52)
\]
and integral (52) converges absolutely.

**Proof** Taking the Mellin-Barnes representation for the Lommel function \( S_{\mu, in}(x) \) (see [3], Vol. III, Entry 8.4.27.3) with the use of the reflection formula for the gamma function, we find

\[
S_{\mu, in}(2x) = \frac{2^{\mu-1}}{4i \Gamma((1 - \mu - in)/2) \Gamma((1 - \mu + in)/2)} \times \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma\left(\frac{s + in}{2}\right) \Gamma\left(\frac{s - in}{2}\right) \frac{x^{-s}}{\cos(\pi(s + \mu)/2)} ds, \quad (53)
\]
where \( x > 0, \quad -1 - \Re \mu, 0 < \gamma < 1 - \Re \mu \). Then, shifting the contour in the integral (53) to the left and to the right within this vertical strip and appealing to the Stirling asymptotic formula for the gamma function (see [2], Entry 5.11.9), one can guarantee the convergence of the following iterated integral
\[
\left( \int_0^1 + \int_1^\infty \right) \left[ \left| \cos \left( \frac{\pi \mu}{2} \right) \sin(2x \cosh(u)) \right| + \left| \sin \left( \frac{\pi \mu}{2} \right) \cos(2x \cosh(u)) \right| \right] \\
\times \int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{\Gamma \left( \frac{s + in}{2} \right) \Gamma \left( \frac{s - in}{2} \right)}{\cos(\pi(s + \mu)/2)} \right| ds \right| dx
\leq \int_0^1 \left[ \left| \cos \left( \frac{\pi \mu}{2} \right) \right| + \left| \sin \left( \frac{\pi \mu}{2} \right) \right| \right] \frac{dx}{x^\gamma}
\times \int_{\gamma-i\infty}^{\gamma+i\infty} \left[ \left| \frac{\Gamma \left( \frac{s + in}{2} \right) \Gamma \left( \frac{s - in}{2} \right)}{\cos(\pi(s + \mu)/2)} \right| \right] ds < \infty,
\]

where we choose \( \gamma \in (\max(-1 - \text{Re}\mu, 0), 1) \) for the integration over \((0, 1)\) and \( \gamma \in (1, 1 - \text{Re}\mu) \) for the integration over \((1, \infty)\). Therefore, making use the integral (53), we deduce from (52) after the interchange of the order of integration by Fubini’s theorem and involving Entries 8.4.5.1, 8.4.5.2 in [3], Vol. III

\[
\int_0^\infty \sin \left( 2x \cosh(u) - \frac{\pi \mu}{2} \right) S_{\mu, in}(2x) dx = 4i\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)
\times \int_{\gamma-i\infty}^{\gamma+i\infty} \left[ \frac{\Gamma \left( \frac{s + in}{2} \right) \Gamma \left( \frac{s - in}{2} \right)}{\cos(\pi(s + \mu)/2)} \right] \frac{1}{\cosh(u)} \frac{ds}{x^s}
\times \int_0^\infty \sin \left( 2x \cosh(u) - \frac{\pi \mu}{2} \right) x^{-s} dx ds
\leq \frac{2^{\mu-1}}{4i\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)}
\times \int_{\gamma-i\infty}^{\gamma+i\infty} \left[ \cos \left( \frac{\pi \mu}{2} \right) \Gamma \left( \frac{1 - s}{2} \right) - \sin \left( \frac{\pi \mu}{2} \right) \Gamma \left( \frac{1 + s}{2} \right) \right] \frac{1}{\cosh(u)} \frac{ds}{x^s}
\times \int_0^\infty \sin \left( 2x \cosh(u) - \frac{\pi \mu}{2} \right) x^{-s} dx ds
\leq \frac{2^{\mu-2}}{4i\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)}
\times \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(1 - s) \Gamma \left( \frac{s - in}{2} \right) \left( \frac{s - in}{2} \right)^{s-1} \frac{\cosh(u)}{\cos(\pi(s + \mu)/2)} ds
\times \int_0^\infty \sin \left( 2x \cosh(u) - \frac{\pi \mu}{2} \right) x^{-s} dx ds
\]

But the latter integral can be treated via the Parseval equality for the Mellin transform [1, 4] and Entries 8.4.3.1, 8.4.23.1 in [2], Vol. III, 2.16.6.1 in [2], Vol. II. Thus we obtain
2^{\mu-2}
\frac{4i\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)}{
\times \int_{j-\infty}^{j+\infty} \Gamma(1 - s)\Gamma\left(\frac{s + in}{2}\right)\Gamma\left(\frac{s - in}{2}\right)2^{s}(\cosh(u))^{s-1}ds
= \frac{2^{\mu-1}\pi}{\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)} \int_{0}^{\infty} e^{-x\cosh(u)}K_{in}(x)dx
= \frac{2^{\mu-1}\pi}{\sinh(u)\sinh(\pi n)\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)}.}

Thus, combining with (54), we arrive at (52), completing the proof of Lemma 1. □

Corollary 1 Let \(x > 0, -5/4 < \text{Re} \mu < 3/4, n \in \mathbb{N}\). The following inequality holds valid

\[ |S_{\mu, in}(x)| \leq \frac{C x^{-1/4}}{[\sinh(\pi n)]^{1/2}\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)}, \tag{55} \]

where \(C > 0\) is an absolute constant.

Proof To prove (55), we appeal to the Lebedev inequality for the modified Bessel function (cf. [4], p.219)

\[ |K_{in}(x)| \leq A \frac{x^{-1/4}}{[\sinh(\pi n)]^{1/2}}, \quad x > 0, n \in \mathbb{N}, \tag{56} \]

where \(A > 0\) is an absolute constant. Then from (53) and the Parseval equality for the Mellin transform we derive (cf. [2], Vol. II, Entry 2.16.3.15)

\[ S_{\mu, in}(x) = \frac{(2x)^{\mu+1}}{\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)} \int_{0}^{\infty} t^{-\mu}K_{in}(t)dt. \tag{57} \]

Hence with (56) we find

\[ |S_{\mu, in}(x)| \leq \frac{A(2x)^{\text{Re} \mu+1}}{[\sinh(\pi n)]^{1/2}\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)} \int_{0}^{\infty} t^{-\text{Re} \mu - 1/4}dt
= \frac{A 2^{\text{Re} \mu}x^{-1/4}\Gamma((3/4 - \text{Re} \mu)/2)\Gamma((5/4 + \text{Re} \mu)/2)}{[\sinh(\pi n)]^{1/2}\Gamma((1 - \mu - in)/2)\Gamma((1 - \mu + in)/2)}. \]

Theorem 5 Let \(-5/4 < \text{Re} \mu < 0\) and a sequence \(a = \{a_n\}_{n \in \mathbb{N}} \in l_1\), i.e.

\[ ||a||_l = \sum_{n=1}^{\infty} |a_n| < \infty. \tag{58} \]

Then the discrete transformation (5) can be inverted by the formula
\[ a_n = \frac{2^{1-\mu}}{\pi^3} \sinh(\pi n) \int_0^\infty \Omega_n(x)f(x)dx, \quad n \in \mathbb{N}, \]  

(59)

where the kernel \( \Omega_n(x) \) is defined by
\[ \Omega_n(x) = \int_0^\pi \sin \left( x \cosh(u) - \frac{\pi \mu}{2} \right) \sin(u) \sin(nu)du, \]  

(60)

and integral (59) converges in the improper sense.

**Proof**  Substituting the series (5) on the right-hand side of (59), we have
\[ \frac{2^{1-\mu}}{\pi^3} \sinh(\pi n) \int_0^\infty \Omega_n(x)f(x)dx = \frac{2^{1-\mu}}{\pi^3} \sinh(\pi n) \lim_{T \to \infty} \int_0^T \Omega_n(x) \]
\[ \times \sum_{m=1}^{\infty} a_m \Gamma \left( \frac{1-\mu - im}{2} \right) \Gamma \left( \frac{1-\mu + im}{2} \right) S_{\mu,im}(x)dx \]  

(61)

where the kernel \( X_n(x) \) is defined by
\[ X_n(x) = \int_0^\pi \sin(x \cosh(u) - \frac{\pi \mu}{2}) \cosh(u) \sinh(u) \sin(nu)du; \]  

(59)

and integral (59) converges in the improper sense.

\[ \text{Proof} \quad \text{Substituting the series (5) on the right-hand side of (59), we have} \]
\[ \frac{2^{1-\mu}}{\pi^3} \sinh(\pi n) \int_0^\infty \Omega_n(x)f(x)dx = \frac{2^{1-\mu}}{\pi^3} \sinh(\pi n) \lim_{T \to \infty} \int_0^T \Omega_n(x) \]
\[ \times \sum_{m=1}^{\infty} a_m \Gamma \left( \frac{1-\mu - im}{2} \right) \Gamma \left( \frac{1-\mu + im}{2} \right) S_{\mu,im}(x)dx \]  

(61)

where the interchange of the order of integration and summation is due to condition (58) and inequality (55). Then in order to pass to the limit under the series sign in (61), we estimate the remainder of the corresponding integral. Indeed, recalling (57), (60) and the inequality for the modified Bessel function [4] \( |K_{\mu}(x)| \leq K_0(x), \) we deduce
\[ \left| \Gamma \left( \frac{1-\mu - im}{2} \right) \Gamma \left( \frac{1-\mu + im}{2} \right) \int_0^\infty \Omega_n(x)S_{\mu,im}(x)dx \right| \]
\[ = \left| \int_T^\infty (2x)^{\mu+1} \int_0^\pi \sin \left( x \cosh(u) - \frac{\pi \mu}{2} \right) \right| \]
\[ \times \sinh(u) \sin(nu)du \int_0^\infty \frac{t^{-\mu}}{t^2 + x^2}K_{\mu}(t)dt \]  

(59)

\[ \leq 2^{\Re \mu + 1} \pi \sinh(\pi) \left[ \left| \cos \left( \frac{\pi \mu}{2} \right) \right| + \left| \sin \left( \frac{\pi \mu}{2} \right) \right| \right] \]
\[ \times \int_0^\infty r^{-\Re \mu}K_0(t) \int_T^\infty \frac{x^{\Re \mu + 1}}{t^2 + x^2}dxdt \]
\[ \leq 2^{\Re \mu + 1} \pi \sinh(\pi) \left[ \left| \cos \left( \frac{\pi \mu}{2} \right) \right| + \left| \sin \left( \frac{\pi \mu}{2} \right) \right| \right] \frac{t^{\Re \mu}}{-\Re \mu} \int_0^\infty t^{-\Re \mu}K_0(t)dt, \]

and the latter expression tends to 0, when \( T \to \infty \) by virtue of the condition \( \Re \mu < 0 \) and the convergence of the latter integral (see (16), (17)). Therefore, passing to the limit under the series sign in (61), we appeal to (52) to obtain

\[ \text{Springer} \]
Theorem 5 is proved. □

A final result is the inversion theorem for the transformation (6).

**Theorem 6** Let $\text{Re}\mu < 0$ and $f$ be a complex-valued function on $\mathbb{R}_+$ which is represented by the integral

$$f(x) = \int_{-\pi}^{\pi} \sin(x \cosh(t) - \frac{\pi \mu}{2}) \varphi(u) du, \quad x > 0,$$

where $\varphi(u) = \psi(u) \sinh(u)$ and $\psi$ is a $2\pi$-periodic function, satisfying the Lipschitz condition (44) on $[-\pi, \pi]$. Then for all $x > 0$ the following inversion formula for transformation (6) holds

$$f(x) = \frac{2^{1-\mu}}{\pi^3} \sum_{n=1}^{\infty} \sinh(\pi n) \Omega_n(x) a_n,$$

where $\Omega_n$ is defined by (60).

**Proof** Substituting $f$ by formula (62) in (6), we invoke (52) and the definition of $\varphi$ to obtain after the interchange of the order of integration

$$a_n = \frac{2^\mu \pi^2}{\sinh(\pi n)} \int_{-\pi}^{\pi} \psi(u) \sin(nu) du. \quad (64)$$

This interchange is allowed due to the absolute convergence of the corresponding iterated integral via the asymptotic behavior (22) of the Lommel function at infinity for each fixed $n \in \mathbb{N}$. Then, substituting $a_n$ by (64) and $\Omega_n$ by (60) into the partial sum of the series (63) $S_N(x)$, it becomes

$$S_N(x) = \frac{2}{\pi} \sum_{n=1}^{N} \int_{0}^{\pi} \sin(x \cosh(t) - \frac{\pi \mu}{2}) \sinh(t) \sin(nt) dt \int_{-\pi}^{\pi} \psi(u) \sin(nu) du. \quad (65)$$

Hence, calculating the sum via the known identity

$$\sum_{n=1}^{N} \sin(nt) \sin(nu) = \frac{1}{4} \left[ \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} - \frac{\sin((2N+1)(u+t)/2)}{\sin((u+t)/2)} \right],$$

we obtain from (65)
\[ S_N(x) = \frac{1}{2\pi} \int_{0}^{\pi} \sin\left( x \cosh(t) - \frac{\pi \mu}{2} \right) \sin(t) \int_{-\pi}^{\pi} \left[ \psi(u) - \psi(-u) \right] \times \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du. \] (66)

Since \( \psi \) is \( 2\pi \)-periodic, we treat the latter integral with respect to \( u \) as follows

\[
\int_{-\pi}^{\pi} \left[ \psi(u) - \psi(-u) \right] \frac{\sin((2N + 1)(u - t)/2)}{\sin((u - t)/2)} \, du
= \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du.
\]

Moreover,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(u - t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du.
\]

When \( u + t > \pi \) or \( u + t < -\pi \) then we interpret the value \( \psi(u + t) - \psi(t) \) by formulas

\[
\psi(u + t) - \psi(t) = \psi(u + t - 2\pi) - \psi(t - 2\pi),
\]

\[
\psi(u + t) - \psi(t) = \psi(u + t + 2\pi) - \psi(t + 2\pi),
\]

respectively. Analogously, the value \( \psi(-u - t) - \psi(-t) \) can be treated. Then due to the Lipschitz condition (44) we have the uniform estimate for any \( t \in [-\pi, \pi] \)

\[
\left| \frac{\psi(u + t) - \psi(t) + \psi(-t) - \psi(-u - t)}{\sin(u/2)} \right| \leq 2C \left| \frac{u}{\sin(u/2)} \right|.
\]

Therefore, owing to the Riemann-Lebesgue lemma

\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(t) + \psi(-t) - \psi(-u - t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du = 0
\] (67)

for all \( t \in [-\pi, \pi] \). Besides, returning to (66), we estimate the iterated integral

\[
\int_{0}^{\pi} \left| \sin\left( x \cosh(t) - \frac{\pi \mu}{2} \right) \right| \sin(t) \int_{-\pi}^{\pi} \left| \psi(u + t) - \psi(t) + \psi(-t) - \psi(-u - t) \right| \times \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du
\leq 2C[\cosh(\pi) - 1]
\]

\[
\times \left[ \left| \cos\left( \frac{\pi \mu}{2} \right) \right| + \left| \sin\left( \frac{\pi \mu}{2} \right) \right| \right] \int_{-\pi}^{\pi} \left| \frac{u}{\sin(u/2)} \right| \, du < \infty.
\]

Consequently, via the dominated convergence theorem it is possible to pass to the limit when \( N \to \infty \) under the integral sign, and recalling (67), we derive
\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{0}^{\pi} \sin\left( x \cosh(t) - \frac{\pi \mu}{2} \right) \sinh(t) \times \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(t) + \psi(-t) - \psi(-u - t) \right] \\
\times \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du \, dt = \frac{1}{2\pi} \int_{0}^{\pi} \sin\left( x \cosh(t) - \frac{\pi \mu}{2} \right) \sinh(t) \\
\times \lim_{N \to \infty} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(t) \right] \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \, du \, dt = 0.
\]

Hence, combining with (66), we obtain by virtue of the definition of \( \phi \) and \( f \)
\[
\lim_{N \to \infty} S_N(x) = \int_{0}^{\pi} \sin\left( x \cosh(t) - \frac{\pi \mu}{2} \right) \left[ \phi(t) + \phi(-t) \right] dt = f(x),
\]
where the integral (62) converges since \( \phi \in C[-\pi, \pi] \). Thus we established (63), completing the proof of Theorem 6. 

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**Declarations**

**Conflict of interest** The author declares that there are no competing interests.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

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