A SHORT PROOF OF THE INTEGRALITY OF THE MACDONALD \((q,t)\)-KOSTKA COEFFICIENTS

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Abstract. The Macdonald polynomials can be obtained by acting on the constant 1 with creation operators. Three different expressions for these operators are derived, one from the other, in a rather succinct way. When the last of these expressions is used, the formalism is seen to imply straightforwardly the integrality of the \((q,t)\)-Kostka coefficients, that is of the expansion coefficients for the Macdonald functions in terms of Schur functions.

1. Introduction and background

Let \( \Lambda_N \) denote the ring of symmetric functions in the variables \( x_1, x_2, \ldots, x_N \) and denote by \( \mathbb{Q}(q,t) \) the field of rational functions of the parameters \( q \) and \( t \) with rational coefficients. The Macdonald polynomials \( J_\lambda(x; q,t) \) are symmetric polynomials labelled by partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \), where \( \lambda_1 \geq \lambda_2 \geq \ldots \) i.e. sequences of non-negative integers in decreasing order. These polynomials form a basis for \( \Lambda_N \otimes \mathbb{Q}(q,t) \) and can be characterized as the joint eigenfunctions of the commuting operators \( \{ M^k_N, k = 0, \ldots, N \} \) defined as follows:

\[
M^k_N = \sum_I t^{(N-k)k+k(k-1)/2} \tilde{A}_I(x; t) \prod_{i \in I} T_{q,x_i},
\]

with \( M^0_N = 1 \). Here, the sum goes over all \( k \)-element subsets \( I \) of \( \{1, \ldots, N\} \),

\[
\tilde{A}_I(x; t) = \prod_{j \in \{1, \ldots, N\} \setminus I} \frac{x_i - t^{-1}x_j}{x_i - x_j}
\]

and \( T_{q,x_i} \) stands for the \( q \)-shifted operator in the variables \( x_i \) \( (T_{q,x_i} f(x_1, \ldots, x_i, \ldots) = f(x_1, \ldots, qx_i, \ldots)) \). This notation will be used throughout the paper. The eigenvalue equations that the Macdonald polynomials satisfy are conveniently written in terms of the generating function

\[
M_N(X; q, t) = \sum_{k=0}^{N} M^k_N X^k,
\]

where \( X \) is an arbitrary parameter. With \( J \), a set of cardinality \( |J| = j \), we shall also use the notation \( M_J(X; q, t) \) to designate \( M_J(X; q, t) \) in the variables \( x_i, i \in J \). If \( \ell(\lambda) \) denotes the number of non-zero parts of \( \lambda \), for \( \ell(\lambda) \leq N \) we have

\[
M_N(X; q, t) J_\lambda(x; q,t) = a_\lambda(X; q,t) J_\lambda(x; q,t),
\]

with

\[
a_\lambda(X; q,t) = \prod_{i=1}^{N} (1 + X q^{\lambda_i} t^{N-i}).
\]
It is customary to denote by $a(s)$ and $\ell(s)$ the number of squares in the diagram of $\lambda$ that are respectively to the south and east of the square $s$ in the diagram of the partition $\lambda$. A complete characterization of the Macdonald polynomials can now be given by supplementing (4) with the following decomposition of $J_\lambda(x; q, t)$ on the monomial symmetric functions $m_\mu$:

$$J_\lambda = \sum_{\mu \leq \lambda} v_{\lambda\mu}(q, t)m_\mu;$$  \hfill (6)

with

$$v_{\lambda\mu}(q, t) \equiv c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)}t^{\ell(s)+1}).$$  \hfill (7)

The sum in (6) is over partitions that are smaller than (or equal to) $\lambda$ in the dominance order. Recall that $m_\mu = \sum_{\text{distinct permutations}} x_1^{\mu_1}x_2^{\mu_2} \cdots$.

We are looking for creation operators $B_k$, $k = 1, \ldots, N$, such that, for any partition $\lambda$ with $\ell(\lambda) \leq k$,

$$B_k J_\lambda(x) = J_{\lambda+(1^k)}(x).$$  \hfill (8)

Given such creation operators, it is clear that the Macdonald polynomials associated to any partition can be constructed by acting repeatedly with these $B_k$ on $J_0(x) = 1$. Indeed, the following Rodrigues formula for $J_\lambda(x)$ is an immediate consequence of (8):

$$J_\lambda(x) = B_N^{1N}B_{N-1}^{\lambda_N-1-\lambda_N} \cdots B_1^{\lambda_1-\lambda_2} \cdot 1.$$  \hfill (9)

Three expressions $B_k^{(i)}$, $i = 1, 2, 3$, have been obtained for the creation operators. In fact, we also know a fourth that will not be used here; see eq. (43)-(44) in [2]. When $B_k^{(3)}$ is used, formula (9) is readily seen to imply \cite{3, 4} the integrality of the $(q, t)$-Kostka coefficients $K_{\lambda\mu}(q, t)$. The purpose of this note is to provide a simple derivation of these expressions and therefore, a short proof of the fact that the $K_{\lambda\mu}(q, t)$ are polynomials in $q, t$ with integral coefficients.

We need more notation \cite{3} to write down the expressions $B_k^{(1)}$, $B_k^{(2)}$ and $B_k^{(3)}$. For $n$ an integer, the $q$-shifted factorials are defined by

$$(a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a); \quad (a; q)_0 = 1.$$  \hfill (10)

They satisfy various identities among which:

$$a^{(a; q)_n} = (a; q)_n(aq^n; q)_n$$

and

$$a^{-k} = \frac{(a; q)_n}{(q^{1-n}/a; q)_k}\left(\frac{q}{a}\right)^k q^{k(k-1)/2-nk}.$$  \hfill (11)

The $_1\phi_1$ basic hypergeometric series is

$$1\phi_1(a; b; q, x) = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2}\frac{(a; q)_n}{(q; q)_n(b; q)_n}x^n;$$  \hfill (12)

it can be summed when $x = b/a$ and, in particular, we have

$$1\phi_1(t^{-n}; t^{-n+1}q^{-1}; t, t/q) = \frac{1}{(t^{-n+1}q^{-1}; t)_n}.$$  \hfill (13)
At some point in the proofs, we shall also need another summation formula, this one a special case of a result due to Garsia and Tesler ([6], Proposition 3.1): let 
\[ J \subseteq \{1, \ldots, N\}, J^c = \{1, \ldots, N\} \setminus J \] and \( x_J = \prod_{j \in J} x_j \), for all \( k \) and \( m \) we have

\[
\sum_{|J| = k} x_J \sum_{J' \subseteq J^c \atop |J'| = m} \tilde{A}_{J \cup J'} = t^{-m(N-k-m)} \binom{N-k}{m} \sum_{|J| = k} x_J,
\]

where \( \tilde{A}_{J \cup J'} \) is defined in (2) and where the second sum on the l.h.s. of (14) is over all subsets \( J' \) of \( J^c \) with cardinality \( |J'| = m \). The \( q \)-binomial coefficient is

\[
\begin{aligned}
\binom{n}{k}_q &= \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.
\end{aligned}
\]

We can now give the three expressions \( B^{(i)}_k \) \( i = 1, 2, 3 \), \( k = 1, \ldots, N \) of the creation operators that we will derive in the next section.

- **Expression 1**

\[
B^{(1)}_k = \frac{1}{(q^{-1}; t^{-1})^{N-k}} M_N(-t^{k+1-Nq^{-1}}; q, t) e_k.
\]

Here \( e_k \) stands for the \( k \)th elementary function

\[
e_k(x) = \sum_{|I| = k} x_I = m_{(1^k)}.
\]

- **Expression 2**

\[
B^{(2)}_k = \sum_{|I| = k} x_I \sum_{m=0}^{N-k} \sum_{J' \subseteq J^c \atop |J'| = m} \frac{q^{-m}}{(t^{k+1-Nq^{-1}}; t)_m} \tilde{A}_{J \cup J'} M_{I \cup J'}(-t^{1-m}; q, t).
\]

- **Expression 3**

\[
B^{(3)}_k = \sum_{|I| = k} x_I \tilde{A}_I M_I(-t; q, t).
\]

Our proofs of these formulas will proceed like this. The first expression for the creation operators will be seen to follow from the Pieri formula which gives the action of the elementary functions \( e_k \) on the monic Macdonald polynomials \( P_\lambda = 1/c_\lambda(q, t) J_\lambda \). This formula reads

\[
e_k P_\lambda = \sum_\mu \Psi_{\mu/\lambda} P_\mu
\]

where the sum is over all partitions \( \mu \) containing \( \lambda \) such that the set-theoretic difference \( \mu - \lambda \) is \( k \)-dimensional with the property that \( \mu_i - \lambda_i \leq 1, \forall i \geq 1 \). If \( C_{\mu/\lambda} \) and \( R_{\mu/\lambda} \) respectively denote the union of the columns and of the rows that intersect \( \mu - \lambda \), the coefficients \( \Psi_{\mu/\lambda} \) are given by

\[
\Psi_{\mu/\lambda} = \prod_{s \in C_{\mu/\lambda}} \frac{b_\mu(s)}{b_\lambda(s)}
\]

\[
\prod_{s \in C_{\mu/\lambda}} b_\mu(s) \prod_{s \notin R_{\mu/\lambda}} b_\lambda(s)
\]
where
\[ b_\lambda(s) = \begin{cases} 
\frac{1 - q^a(s) \ell(s) + 1}{1 - q^{a(s)} + 1} & \text{if } s \in \lambda \\
1 & \text{otherwise}
\end{cases} \quad (22) \]

Second, we shall show that \( B_k^{(2)} = B_k^{(1)} \) with the help of the summation formula (13) and (14). The third expression for the creation operators will then be easily arrived at using the second one by observing that \( B_k^{(2)} \) reduces to \( B_k^{(3)} \) when acting on Macdonald polynomials; in other words, we shall prove that \( B_k^{(2)} J_\lambda = B_k^{(3)} J_\lambda = J_{\lambda + (1^k)} \) for \( \ell(\lambda) \leq k \). Creation operators of type 3 were first obtained in [7] in the limit case of the Jack polynomials \( q = t^a, t \to 1 \). A realization of these operators in terms of Dunkl operators was given and the Rodrigues formula they entail implied the integrality of the expansion coefficients of the Jack polynomials over the symmetric monomial basis [8].

In the case of the Macdonald polynomials, Expression 1 was first derived in [3]; Expression 3 is also given there in the form of a conjecture. Meanwhile Kirillov and Noumi provided a realization of \( B_k^{(3)} \) in terms of Dunkl-Cherednik operators [3, 4] and also gave two proofs of the fact that they are indeed creation operators for the Macdonald polynomials. Expression 2 seems new; it plays an important role as the essential intermediate step in the concise derivation of \( B_k^{(3)} \) from \( B_k^{(1)} \) that we present next.

In a separate paper [9], we present another derivation of \( B_k^{(3)} \). The starting point is again Expression 1, i.e. \( B_k^{(1)} \). The approach followed in [9] differs from the one adopted in the present one; it uses in an essential way the specifics of the realization in terms of Dunkl-Cherednik operators and properties of affine Hecke algebras. The derivation is more involved and lengthy than the one given here. It offers however the advantage of being more constructive allowing one to basically arrive at Expression 2 from Expression 1. Similar developments in the case of the Jack polynomials deserve special attention and are the object of a forthcoming publication [10].

2. Proofs

We now proceed along the lines indicated in the introduction to show in turn that \( B_k^{(1)}, B_k^{(2)} \) and \( B_k^{(3)} \) act as creation operators for the Macdonald polynomials.

**Theorem 1.** For any partition \( \lambda \) with \( l(\lambda) \leq k \), the operators \( B_k^{(1)} \) act as follows on the Macdonald polynomials \( J_\lambda(x; q, t) \):
\[ B_k^{(1)} J_\lambda(x) = J_{\lambda + (1^k)}(x). \quad (23) \]

**Proof.** The following lemma is an immediate consequence of the Pieri formula.

**Lemma 2.** For \( \lambda \) a partition such that \( l(\lambda) \leq k \), the action of \( e_k \) on \( P_\lambda \) is given by
\[ e_k P_\lambda = P_{\lambda + (1^k)} + \sum_{\mu \neq \lambda + (1^k)} \Psi_{\mu/\lambda} P_\mu, \quad (24) \]
where all the \( \mu \)'s in the sum are such that \( \mu_{k+1} = 1 \).
Indeed, the only way to construct a $\mu$ with $\mu_{k+1} \neq 1$ is to add a 1 in each of the first $k$ entries of $\lambda$. From Lemma 2 and (4) and (5), we have

$$M_N(-t^{k+1-N}q^{-1}; q, t)e_k P_\lambda = \prod_{i=1}^{k} (1 - t^{k+1-i}q^{-1}) (q^{-1}; t^{-1})_{N-k} P_{\lambda + (1^k)}$$

(25)

since the eigenvalues

$$a_\mu(-t^{k+1-N}q^{-1}; q, t) = \prod_{i=1}^{N} (1 - t^{k+1-i}q^{-1}),$$

(26)

of $M_N(-t^{k+1-N}q^{-1}; q, t)$ on the $P_\mu$’s in (24) vanish if $\mu_{k+1} = 1$.

From the definition given in (7), it is easy to check that

$$\prod_{i=1}^{k} (1 - t^{k+1-i}q^{-1}).$$

(27)

Using this result and passing from $P_\lambda$ to $J_\lambda$ we see that

$$B_k^{(1)} J_\lambda = \frac{1}{(q^{-1}; t^{-1})_{N-k}} M_N(-t^{k+1-N}q^{-1}; q, t)e_k J_\lambda = J_{\lambda + (1^k)}$$

(28)

when $\ell(\lambda) \leq k$. This proves Theorem 1.

**Theorem 3.**

$$B_k^{(2)} = B_k^{(1)},$$

(29)

hence $B_k^{(2)}$ is also such that $B_k^{(2)} J_\lambda = J_{\lambda + (1^k)}$ for $\ell(\lambda) \leq k$.

Proof. We first prove the following lemma.

**Lemma 4.** Let $J$ and $J'$ be complementary subsets of \{1, \ldots, $N$\} \setminus \{1, \ldots, $\ell$\}, $\ell \leq N$. For all $n, k$ such that $N - \ell \geq k - n \geq 0$,

$$\sum_{m'=0}^{N-k-\ell+n} \sum_{|J|=k-n} x_{J'} \times \sum_{J' \subseteq J \vdash} A_{J' \times 1 \ldots 1, \ldots, \ell} = \frac{1}{(q^{-1}; t^{-1})_{N-k}} \sum_{|J|=k-n} x_{J},$$

(30)

where

$$\tilde{A}_{J'} = \prod_{\substack{i \in J' \atop j \in J \setminus J'}} \frac{x_i - x_j^{-1}}{x_i - x_j}.$$  

(31)

To derive this relation, one first uses formula (14) in the l.h.s. of (30). After simplifying the factors $\sum_{|J|=k-n} x_{J}$, one thus finds that (30) amounts to

$$\sum_{m'=0}^{N-k-\ell+n} \frac{q^{-m'} t^{-m'} (N-k-\ell+n-m')}{(q^{-1}; t^{-1})_{m'+\ell-n} m'} \left[ N - k - \ell + n \right]_{m'} = \frac{1}{(q^{-1}; t^{-1})_{N-k}}.$$  

(32)

With the help of the identities (11), one easily shows that

$$\text{l.h.s. of (32)} = \frac{1}{(q^{-1}; t^{-1})_{\ell-n}} \phi_1(t^{k-N+\ell-n}, t^{k-N+\ell-n+1} q^{-1}; t, t/q).$$  

(33)
At this point, we recall the $\phi_1$ sum in (13) to find that
\[
\text{l.h.s. of (32)} = \frac{1}{(t^{k-N+\ell-n+1} q^{-1}; t)_n (t^{k-N+\ell-n+1} q^{-1}; t)_{N-k-\ell+n}}
\]
and to see using (11) that (30) is indeed verified.

We now return to the proof of (29), that is of the identity $B_k^{(2)} = B_k^{(1)}$. Let $\ell \leq N$. Since $B_k^{(1)}$ and $B_k^{(2)}$ are symmetric, it will suffice to show that the coefficient to the left of the operators $T_{q,x_1} \cdots T_{q,x_\ell}$ in $B_k^{(1)}$ and $B_k^{(2)}$ are identical.

To that end, let
\[
I = L \cup J, \quad I' = \bar{L} \cup J',
\]
\[
L \subseteq \{1, \ldots, \ell\}, \quad \bar{L} = \{1, \ldots, \ell\} \setminus L,
\]
\[
J, J' \subseteq \{1, \ldots, N\} \setminus \{1, \ldots, \ell\} = J \cup J',
\]
\[
J \cap \bar{J} = \emptyset, J' \subseteq \bar{J},
\]
and define
\[
[\ell, k] = \begin{cases} 
\ell & \text{if } \ell \leq k \\
 k & \text{if } \ell > k 
\end{cases}
\]
Collecting the factors to the left of $T_{q,x_1} \cdots T_{q,x_\ell}$ in $B_k^{(1)}$ and $B_k^{(2)}$ (see (16) and (18)) gives respectively:
\[
B_k^{(1)}|_{T_{q,x_1} \cdots T_{q,x_\ell}} = \frac{1}{(q^{-1}; t^{-1})_{N-k}} (-t^{k+1-N} q^{-1})^\ell
\times \tilde{A}_{1, \ldots, \ell} t^{(\ell-1)/2} \ell^\ell (N-\ell) \sum_{n=0}^{[\ell,k]} q^n x_L \sum_{|J|=k-n} x_J
\]
and
\[
B_k^{(2)}|_{T_{q,x_1} \cdots T_{q,x_\ell}} = \sum_{n=0}^{[\ell,k]} \sum_{|L|=n} x_L \sum_{|J|=k-n} x_J \sum_{m=0}^{N-k} \sum_{|L \cup J'|=m} \frac{q^{-m}}{(t^{k+1-N} q^{-1}; t)^m} (-t^{1-m})^\ell \ell^\ell (m+k-\ell) \tilde{A}_{L \cup J} \tilde{A}_J
\]
where we have used in (38) the fact that $\tilde{A}_{L \cup J'} \tilde{A}_{J' \setminus \ell} = \tilde{A}_{L \cup J'} \tilde{A}_J \tilde{A}_{J' \setminus \ell}$. It is then immediate to see that the equality
\[
B_k^{(1)}|_{T_{q,x_1} \cdots T_{q,x_\ell}} = B_k^{(2)}|_{T_{q,x_1} \cdots T_{q,x_\ell}}
\]
holds, since after trivial simplifications it is seen to amount to
\[
\sum_{n=0}^{[\ell,k]} \sum_{|L|=n} q^{n-\ell} x_L \left( \frac{1}{(t^{-1}; t^{-1})_{N-k}} \sum_{|J|=k-n} x_J \right)
= \sum_{n=0}^{[\ell,k]} \sum_{|L|=n} q^{n-\ell} x_L \left( \sum_{m=0}^{N-k-\ell+n} \frac{q^{-m'}}{(t^{k-N+\ell-n+1} q^{-1}; t)^{m'+\ell-n}} \sum_{|J'|=m'} x_J \sum_{|J'|=m'} \tilde{A}_{J \cup J'} \right)
\]
and hence to follow from Lemma 4.

Once Theorem 3 is proved, the main result, Theorem 5, is readily obtained.
Theorem 5. For any partition \(\lambda\), such that \(\ell(\lambda) \leq k\), the actions of \(B_k^{(2)}\) and \(B_k^{(3)}\) on the Macdonald polynomials \(J_\lambda(x)\) coincide:

\[
B_k^{(3)}J_\lambda(x) = B_k^{(2)}J_\lambda(x) = J_{\lambda+(1^k)}(x).
\]

This is shown to be true with the help of the following lemma.

Lemma 6. Let \(|I| = k\) and \(|I'| = m\), \(I' \subseteq I^c\).

\[
M_{I \cup I'}(-t^{1-m}; q, t)J_\lambda(x; q, t) = 0,
\]

if \(\ell(\lambda) \leq k\) and \(m > 0\).

Proof. Denote by \(x(I)\) the set of variables \(\{x_i, i \in J\}\). The Macdonald polynomials are known to enjoy the property according to which

\[
J_\lambda(x(I), x(I^c)) = \sum_{\mu, \nu} f_{\mu, \nu} J_\mu(x(I)) J_\nu(x(I^c))
\]

with \(f_{\mu, \nu} = 0\) unless \(\mu \subseteq \lambda\) and \(\nu \subseteq \lambda\) and in particular if \(\ell(\mu)\) or \(\ell(\nu)\) is greater than \(k\).

Since \(M_{I \cup I'}(-t^{1-m}; q, t)\) is a \(q\)-difference operator depending only on the variables \(x_i, i \in I \cup I'\), we see from (43) that

\[
M_{I \cup I'}J_\lambda(x) = \sum_{\mu, \nu} f_{\mu, \nu} J_\mu(x((I \cup I')^c)) M_{I \cup I'}J_\nu(x(I \cup I')).
\]

The proof of Lemma 6 is then completed by observing from (4)-(5) that

\[
M_{I \cup I'}(-t^{1-m}; q, t)J_\mu(x(I \cup I')) = \prod_{i=1}^{m} (1 - q^\nu_i t^{k+1-i})J_\nu(x(I \cup I')) = 0,
\]

whenever \(m > 0\), since \(\nu_{k+1} = 0\).

Theorem 5 is thus an immediate consequence of this lemma since in the expression (18) of \(B_k^{(2)}\), all the terms, except the \(m = 0\) one, are seen to act trivially on the \(J_\lambda(x)\) when \(\ell(\lambda) \leq k\).

3. Integrality of the \((q, t)\)-Kostka coefficients and conclusion

The \((q, t)\)-Kostka \(K_{\lambda \mu}(q, t)\) coefficients give the overlaps between the Macdonald and the Schur polynomials (see [1]):

\[
J_\mu(x; q, t) = \sum_\lambda K_{\lambda \mu}(q, t)S_\lambda(x; t).
\]

As mentioned in the introduction, a proof of Theorem 5 readily provides a proof of the fact that these \(K_{\lambda \mu}(q, t)\)'s are polynomials in \(q\) and \(t\) with integer coefficients as was first conjectured by Macdonald. How such a conclusion is arrived at from Theorem 5 is explained in [3, 4]. The main observation (see also [5]) is that Theorem 5 implies that the expansion coefficients \(c_{\lambda \mu}(q, t)\) in (6) are polynomials in \(q, t\) with integer coefficients. One then recalls [6] that the overlaps between the monomial symmetric functions and the Schur functions have the form \(p(t)/q(t)\), with \(p(t)\) and \(q(t)\) polynomials in \(t\) with integer coefficients and such that \(q(0) = 1\). Upon using the duality properties of the function \(J_\lambda\) and \(S_\lambda\) one then straightforwardly concludes that \(K_{\lambda \mu}(q, t) \in \mathbb{Z}[q, t]\).

Let us mention in concluding that a other proofs of the integrality of the \((q, t)\)-Kotkska coefficients have been given recently using different approaches by Garsia and Remmel [1], Garsia and Tesler [3], Knop [12, 13] and Sahi [14].
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