A new construction for scalar wave equations in inhomogeneous media

S. DE TORO ARIAS* and C. VANNESTE†

Laboratoire de Physique de la Matière Condensée, CNRS UMR 6622,
Université de Nice-Sophia Antipolis,
Parc Valrose, B. P. 71, 06108 Nice Cedex 2, France

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Abstract : The paper describes a formulation of discrete scalar wave propagation in an inhomogeneous medium by the use of elementary processes obeying a discrete Huygens’ principle and satisfying fundamental symmetries such as time-reversal, reciprocity and isotropy. Its novelty is the systematic derivation of a unified equation which, properly tuned by a single parameter, leads to either the Klein-Gordon equation or the Schrödinger equation. The generality of this method enables one to consider its extension to other types of discrete wave equations on any kind of discrete lattice.

Résumé : Cet article formule un modèle discret de propagation d’ondes scalaires dans un milieu hétérogène à l’aide de processus élémentaires qui obéissent à un principe de Huygens discret et satisfont à certaines symétries fondamentales telles que le renversement temporel, la réciprocité et l’isotropie. Son originalité consiste en la dérivation systématique d’une équation unifiée qui selon la valeur d’un seul paramètre conduit soit à l’équation de Klein-Gordon, soit à l’équation de Schrödinger. La méthode est suffisamment générale pour pouvoir envisager son extension à d’autres types d’équations d’onde sur toute sorte de réseau discret.

1. INTRODUCTION

The formulation of wave propagation by the use of Huygens’ principle has stimulated extensive work in the past. Such an approach was first pioneered by P. B. Johns and coworkers who introduced the Transmission Line Matrix Modeling method (TLM) to solve the Maxwell equations in large electromagnetic structures which was later extended to the description of diffusive processes. Linear wave propagation is mediated by short voltage impulses which propagate along the bonds and are scattered on each node of a Cartesian mesh of transmission lines. The main idea behind such a formulation is a discrete equivalent of Huygens’ principle, namely a principle of action-by-proximity obeyed by the pulses when they propagate at each time step from one node to the neighbor nodes and the emission of secondary wavelets at each node as described by the scattering process which radiates the incident energy in all directions. Besides the appealing viewpoint of Huygens’ principle, the advantages of the method as a powerful tool for the numerical computation of wave propagation in complex and large structures have already been emphasized in the TLM method. A similar approach in which the pulses were just scalar quantities propagating along the bonds of a Cartesian

*E-mail : sdetoro@spec.saclay.cea.fr
†Author for correspondence. E-mail : vanneste@ondine.unice.fr
lattice, was recently advocated for studying time-dependent wave propagation in large inhomogeneous media.

Here, we focus on the nature of the wave equations which result from such a formulation, especially on the possibility of retrieving standard wave equations. This question has already been raised in previous works where the scalar wave equation, the Klein-Gordon equation and Maxwell equations have been exhibited. In the TLM method, the equivalence between the voltage and current impulses with the electric and magnetic fields of Maxwell equations was initially demonstrated on a two-dimensional mesh. The method was then extended to three dimensions and different features have been incorporated by many authors in order to describe losses, inhomogeneous media and boundary conditions. Another recent formulation lead their authors to the classical wave equation, the Klein-Gordon equation and to the Schrödinger equation. However, due to restricting assumptions, these last results were limited to scalar waves in homogeneous media, and the Schrödinger equation was ill-defined at the continuum limit.

This paper describes what is essentially a formulation from “first principles” of such an approach. The aim is to generalize the previous results to various kinds of waves in inhomogeneous media by considering the most basic properties obeyed by the scattering nodes. Such a formulation not only leads to the wave equations already mentioned but also results in a time-dependent Schrödinger equation which is well defined. Another advantage is the possible generalization of the method from the scalar waves considered in this paper to spinor or vector waves.

At each time step, the time-dependent discretized wave is defined as a linear combination of incident currents at each node of the Cartesian lattice. The nature of the wave depends on the number of distinct propagating currents on each bond of a given node. For example, one or several kinds of currents impinging on one node by the same bond will describe a scalar, a spinor or a vector field. Since they need only one type of current, scalar waves correspond to the simplest situation. For the sake of illustration, they will be the focus of this paper. In addition to the propagating currents, a current is attached to each node in order to describe an inhomogeneous medium. This current does not propagate but participates in the scattering process which involves the propagating currents. We shall demonstrate that a few elementary properties and symmetries of the scatterers like time reversal symmetry, reciprocity and isotropy lead to the classical wave, Klein-Gordon or Schrödinger equations. At that stage, we find that these discrete equations are not yet completely general because the mass, the velocity or the potential remain correlated. We show that it is possible to remove this limitation by attaching a second current to each node.

The paper is organized as follows. In section II, we introduce the currents which propagate on the mesh of a Cartesian lattice. Scatterers are described by scattering matrices located on each node of the lattice. The field is then defined as a function of incident currents. In order for the field to obey a discretized wave equation, the currents must be eliminated from its time evolution. We show in section III how this condition strongly compels the form of the scattering matrices. Time-reversal symmetry, reciprocity and isotropy of the scatterers are introduced in sections IV, V and VI respectively. The resulting discrete wave equations are discussed in section VII. This leads us to attach a second current to each node as described in section VIII. We conclude in section IX by discussing possible extensions of this work.

II. CURRENTS AND FIELDS

A. Definition of the currents

A current is a real or complex number, which propagates along a bond of a $d$-dimensional Cartesian lattice. For scalar waves, it is sufficient to consider that each lattice bond carries two currents propagating...
in opposite directions (Figure 1). At any time $t$, the system is completely defined by the values of all currents in the lattice. We assume that time and space variables are discrete. In one time step $\tau$, a current propagates between two nodes of a given bond. All currents are synchronized. In other words, the currents are simultaneously outgoing from the nodes at some time $t$ and become incident currents on neighbor nodes at time $t + \tau$.

![Figure 1. Currents propagating along the bonds of a two-dimensional Euclidean lattice.](image)

Each node of the lattice is a scatterer. The scatterers are identical in a periodic system or are distinct, and randomly chosen in a disordered quenched system. Each scatterer is described by a $2d \times 2d$ matrix $S$ which transforms the $2d$ incident currents $E_l$, at any time $t$, in $2d$ outgoing currents $S_k$ at the same time (Figure 2) according to

$$S_k = \sum_{l=1}^{2d} s_{kl} E_l \quad k = 1, \ldots, 2d,$$

where the $s_{kl}$ are the matrix elements of the matrix $S$.

![Figure 2. Scattering process.](image)

1The results described in this paper are valid for any $d$-dimensional space. However, for convenience, all figures will represent a two-dimensional Cartesian lattice.
It turns out that the model is not sufficient to describe wave propagation in an inhomogeneous medium like a material in which the index of refraction is a function of the position. As shown later, this problem is solved by attaching to each node an additional current that will be called throughout this paper the node current. For notational convenience in sums over the currents, we shall write the node currents $E_{2d+1}$ and $S_{2d+1}$. However, we emphasize that the subscript $2d+1$ is not ascribed to a spatial dimension in contrast with the other indices ranging from 1 to $2d$. The node current participates in the same scattering process as the other currents but does not propagate to neighbor nodes. Instead, the outgoing current $S_{2d+1}$ becomes the incident current $E_{2d+1}$ on the same node at the next time step. In other words, the propagation process for the node current reads

$$E_{2d+1}(t+\tau) = S_{2d+1}(t).$$

We can imagine the node current as propagating in one time step along a loop attached to the node. This loop is equivalent to the "permittivity stub" introduced in the TLM method to describe inhomogeneous electromagnetic structures [10].

In conclusion, the general scheme of the time evolution of the currents is suitably represented in Fig. 3.

We write

$$S_k = \sum_{l=1}^{2d+1} s_{kl} E_l, \quad k = 1, \ldots, 2d+1,$$

where the $s_{kl}$ are the matrix elements of the $(2d+1) \times (2d+1)$ scattering matrix $S$. This matrix is a function of the position in an inhomogeneous medium.

**B. Definition of the field**

The previous definition of the currents includes the three essential features of the model: the principle of action-by-proximity, which is described by the current propagation between neighbor nodes, the scattering process in which the nodes act like secondary sources of "spherical" wavelets and the linearity of this process. These features can be viewed as a discretized realization of Huygens' principle. However, additional ingredients and choices are needed to achieve the description of the model. For example, the form of the field propagation equation will strongly depend on the choice of the scattering matrices. Among all the questions to be answered in our approach, the first one is how to define the field.

The field is supposed to obey a discretized linear wave equation which depends on the physical problem we are interested in: the scalar wave equation for an acoustic wave, the Klein-Gordon equation for a zero-spin massive particle, etc. At time $t$, the field is defined as a real or complex quantity $\psi(i_1, i_2, \ldots, i_d, t)$ on each node of the lattice, where $i_1, i_2, \ldots, i_d$ are the discrete indices which locate the node position $x_k = i_k l, k = 1, \ldots, d$, as a function of the mesh parameter $l$. For briefness, we note $r$ the set of indices $i_1, i_2, \ldots, i_d$ of an arbitrary node. There is no reason for the field to be identified with one of the currents previously defined. Actually, we just postulate that the value of the field $\psi(r, t)$ is a function of the incident...
and outgoing currents at node $r$ and at time $t$. Therefore, by taking account of linearity, the most general definition reads

$$\psi(r, t) = \sum_{k=1}^{2d+1} [\lambda'_k(r)E_k(r, t) + \lambda''_k(r)S_k(r, t)],$$

where $\lambda'_k$ and $\lambda''_k$ are complex coefficients to be determined.

The field is a linear superposition of the currents, which vanishes when all currents vanish. By noticing that, due to the scattering process, the outgoing currents $S_k(r, t)$ can be expressed as linear combinations of the incident currents $E_k(r, t)$ at the same time $t$, it is sufficient to define the field solely as a function of the incident currents

$$\psi(r, t) = \sum_{k=1}^{2d+1} \lambda_k(r)E_k(r, t).$$

(2)

It is necessary to point out that the $\lambda_k$ are functions of the node position. This property is required to describe an inhomogeneous system.

### III. CLOSURE OF THE WAVE EQUATION

The general form of the discretized wave equations we are looking for can be written:

$$\psi(r, t + \tau) = f(\psi(r', t'), \psi(r'', t''), \psi(r''', t''', \ldots)),$$

(3)

where the field at node $r$ and time $t + \tau$ is a function of the field on the same node and/or on nodes at previous times $t', t''$, etc... Such an equation is straightforwardly derived from the discretization of the corresponding continuous equation. Equation (3) is a closed equation, which means that $f$ is a function where no currents appear explicitly (so that (3) involves fields exclusively), and that the number of terms in the r.h.s. of (3) must be finite. Usually, and at the lowest order of the discretization procedure, the involved fields are fields on the same node $r = \{i_1, i_2, \ldots, i_d\}$ and on the neighbor nodes $i_1 \pm 1, i_2 \pm 1, \ldots, i_d \pm 1$, at times $t$ and $t - \tau$. In the remainder of this section we derive, by use of our model, a closed equation of the form of equation (3).

Also, we show that imposing the closure conditions strongly constrains the form of the scattering matrix.

For this purpose, we start to derive an equation of the type of Eq. (3) from the evolution rules summarized in Fig. III of the model. First of all, let us introduce some notations. In a $d$-dimensional Euclidean space, node $r$ is linked to the neighbor nodes by $2d$ bonds. We note $r_k$ the set of indices of the neighbor node which is linked to node $r$ by the bond $k$, along which the currents $E_k(r)$ or $S_k(r)$ propagate (Figure III). Moreover, bond $k$ of node $r$ is named bond $\overline{k}$ for node $r_k$, which simply means, for example, that bond 2 of the central node in Figure III corresponds to bond 1 of the right side node ($k = 2, \overline{k} = 1$). This notation immediately leads to $E_\overline{r}(r_k,t+\tau) = S_k(r,t)$ and $E_k(r,t+\tau) = S_\overline{r}(r_k,t)$. Note that for the node current ($k = 2d+1$), we have $r_{2d+1} = r$ and $\overline{k} = k$. 

![Diagram](attachment:image.png)
FIG. 4. Notation for the bond linking the central node \( r \) to one of its neighbor nodes \( r_k \) and the propagating currents defined on this bond.

The first step in attempting to derive an equation like (3), is to write down the definition of the field \( \psi(r, t + \tau) \) on a node \( r \) at time \( t + \tau \)

\[
\psi(r, t + \tau) = \sum_{k=1}^{2d+1} \lambda_k(r) E_k(r, t + \tau),
\]

\[
= \sum_{k=1}^{2d+1} \lambda_k(r) S_k(r_k, t),
\]

\[
= \sum_{k=1}^{2d+1} \lambda_k(r) \left[ \sum_{l=1}^{2d+1} s_{kl}(r_k) E_l(r_k, t) \right],
\]

(4)

where the propagation and the scattering rules have been used.

Thus, the field \( \psi(r, t + \tau) \) at time \( t + \tau \) is expressed in terms of the incident currents at time \( t \) on the same node \( r \) and on its neighbor \( r_k \). One realizes immediately that using the same rules to express the incident currents at time \( t \) as a function of the incident currents at time \( t - \tau \) and iterating the procedure, will give incident currents at times \( t - \tau, t - 2\tau, \ldots \), on nodes as distant from node \( r \) as desired. Such a process is not bounded and seems to prevent us from writing a closed equation for the field \( \psi \). This statement is true except for some particular scattering matrices of the type we describe below. For this purpose, we shall rewrite the scattering equation (1) in order to exhibit the field \( \psi \) :

\[
S_k = \sum_{k=1}^{2d+1} s_{kl} E_l = \frac{s_{km}}{\lambda_m} \psi + \sum_{l=1}^{2d+1} \left( s_{kl} - \frac{s_{km}}{\lambda_m} \right) E_l,
\]

(5)

where \( m \) is any of the \( 2d+1 \) current indices. Suppose that the following condition is satisfied :

\[
\frac{s_{kl}}{\lambda_l} = \text{constant } \equiv \rho_k, \quad \forall l,
\]

(6)

then, equation (5) becomes :

\[
S_k = \rho_k \psi,
\]

and substitution in equation (3) leads directly to :

\[
\psi(r, t + \tau) = \sum_{k=1}^{2d+1} \lambda_k(r) \rho_k(r_k) \psi(r_k, t),
\]

(7)

which is a closed equation of the type we are looking for. It turns out that condition (6) is too restrictive and can be replaced by

\[
\frac{s_{kl}}{\lambda_l} = \text{constant } \equiv \rho_k, \quad \forall l \neq k,
\]

(8)

Then, equation (5) becomes

\[
S_k = \rho_k \psi - \mu_k E_k,
\]

(9)

where \( \mu_k = \rho_k \lambda_k - s_{kk} \).

According to (9), the outgoing current on one bond is not only a function of the field \( \psi \) but also depends on the incident current on the same bond as sketched Figure 5.
It seems at first sight that we went backward in writing condition (8) instead of (6), since a current appears in equation (9) in addition to the field $\psi$. Actually, substitution of equation (9) in equation (4) gives

$$
\psi(r, t + \tau) = \sum_{k=1}^{2d+1} \lambda_k(r) \rho_k(r_k) \psi(r_k, t) - \sum_{k=1}^{2d+1} \lambda_k(r) \mu_k(r_k) E_k(r_k, t),
$$

which is represented schematically in Figure 6.

It is obvious from Figure 6 that the $2d + 1$ incident currents $E_k(r_k, t)$ at time $t$ (bold arrows) are deduced, according to the propagation rules, from the currents $S_k(r, t - \tau)$ leaving node $r$ at time $t - \tau$, as depicted in Figure 7.
Thus, the additional currents due to the looser condition \( \text{(9)} \) eventually involve only the initial node \( r \) instead of propagating new terms to remote nodes. Using again equation \( \text{(8)} \), equation \( \text{(10)} \) becomes

\[
\psi(r, t + \tau) = \sum_{k=1}^{2d+1} \lambda_k(r) \mu^2(r_k) \psi(r_k, t) \\
- \sum_{k=1}^{2d+1} \mu^2(r_k) \mu_k(r_k) \psi(r, t - \tau) + \sum_{k=1}^{2d+1} \lambda_k(r) E_k(r, t - \tau),
\]

(11)

The last term is a linear superposition of \( 2d + 1 \) incident currents upon node \( r \) at time \( t - \tau \). In order for equation \( \text{(11)} \) to be a closed field equation without current terms, it is sufficient for this superposition of currents to be proportional to \( \psi(r, t - \tau) = \sum_{k=1}^{2d+1} \lambda_k(r) E_k(r, t - \tau) \). Therefore, we must set

\[
\mu^2(r_k) \mu_k(r_k) = \mu^2(r), \quad \forall k = 1, \ldots, 2d + 1,
\]

(12)

where we denote \( \mu^2(r) \) a constant which is a priori a function of \( r \). In fact, it is simple to prove that \( \mu^2(r) \) does not depend on \( r \). To prove this we begin by recognizing that equation \( \text{(12)} \) can be written for one of the neighbor node \( r_k \)

\[
\mu^2(r_k) \mu_k(r_k) = \mu^2(r_k), \quad \forall l = 1, \ldots, 2d + 1.
\]

Now \( r_k \) plays the role of the central node \( r \) of equation \( \text{(12)} \) and \( r_{kl} \) denotes its neighbor nodes. Among the neighbor nodes \( r_{kl} \), the initial node \( r \) (\( l = k \)) is of particular interest and the last equation reads

\[
\mu_k(r) \mu^2(r_k) = \mu^2(r_k).
\]

(13)

Comparison with equation \( \text{(12)} \) leads to the required identity

\[
\mu^2(r_k) = \mu^2(r).
\]

Since the node \( r \) is arbitrary, equation \( \text{(12)} \) can be recast in a simpler form valid for any node

\[
\mu^2(r_k) \mu_k(r) = \mu^2, \quad \forall k,
\]

(14)

where now the constant \( \mu^2 \) is independent of the node’s position.

One notices that for \( k = 2d + 1, r_k = r, l = k \), and \( \mu^2(r_k) = \mu_k(r) \). Therefore

\[
\mu_{2d+1}(r) = \epsilon_{2d+1}(r) \mu,
\]

(15)

where \( \epsilon_{2d+1}(r) = \pm 1 \).

Finally, utilizing equation \( \text{(13)} \), equation \( \text{(11)} \) reads

\[
\psi(r, t + \tau) = \sum_{k=1}^{2d+1} \lambda_k(r) \rho_k(r_k) \psi(r_k, t) + \mu^2 \left[ 1 - \sum_{k=1}^{2d+1} \lambda_k(r) \rho_k(r_k) \right] \psi(r, t - \tau).
\]

(16)

Equation \( \text{(13)} \) is a closed field equation in which currents do not appear. As the field \( \psi \) is exhibited at times \( t + \tau \), \( t \) and \( t - \tau \), this new equation is an improvement over equation \( \text{(7)} \) in the sense that a discrete second time derivative \( \psi(t + \tau) - 2 \psi(t) + \psi(t - \tau) \) can be obtained as required in, for example, the discrete time-dependent wave equation. Making use of condition \( \text{(8)} \), the general matrix element reads

\[
s_{kl} = \rho_k \lambda_l - \mu_k \delta_{kl},
\]

(17)

where \( \mu_k = \rho_k \lambda_k - s_{kk} \) and \( \delta_{kl} \) is the Kronecker symbol. Instead of \( (2d + 1)^2 \) elements, the scattering matrix depends now on the \( 3(2d + 1) \) complex coefficients \( \rho_k, \lambda_k \) and \( \mu_k \). These coefficients vary independently from one node to another in an inhomogeneous system except for the coefficients \( \mu_k \) which obey condition \( \text{(13)} \) relating neighbor nodes. Furthermore, \( \mu_{2d+1} = \pm \mu \), according to equation \( \text{(14)} \).
In summary, equation (15) is the general equation which describes the time evolution of the field $\psi$. This closed equation has been derived from the evolution rules of the currents, the definition of $\psi(2)$ and from conditions (8) and (13). Let us discuss the status of these conditions. Condition (8) cannot be avoided if we look for a closed equation in $\psi$, since a looser condition would make terms appear on all nodes of the system. Meanwhile, condition (13), which allows one to replace the currents in the last term of equation (11) by the field $\psi(r, t - \tau)$, is sufficient but not necessary. Indeed, suppose we consider that these incident currents defined on node $r$ at time $t - \tau$ are the result of propagation and scattering of currents at previous times $t - 2\tau$ and $t - 3\tau$ on the neighbor nodes and on node $r$. Reproducing the same calculation steps as those which lead to equation (11), the last term may be replaced on the one hand by new terms involving the field $\psi$ on node $r$ at time $t - 3\tau$ and on its neighbors $r_k$ at time $t - 2\tau$ respectively, and on the other hand by a new linear superposition of incident currents on node $r$ at time $t - 3\tau$. Therefore, it is possible to recover a new closure condition (analogous to equation (13)) by taking this new linear superposition of incident currents to be proportional to $\psi(r, t - 3\tau)$. Additionally, the same procedure can be iterated and similar superpositions appear at times $t - 5\tau$, $t - 7\tau$, etc. So that the resulting field equation can be closed at any step of the iteration procedure. The net result is to open the possibility of describing high order time derivatives of the field $\psi$. We shall not consider these possibilities further and restrict ourselves with equation (15) which is sufficient to obtain the second order time derivative of $\psi$.

The description of the model is far from being achieved. To make further progress, additional properties of the scatterers are needed. The most basic properties to consider are suitable symmetries of the scattering process which will restrict again the number of independent parameters and make precise the physical content of equation (15). We shall successively introduce some of the most natural symmetries we can think of, namely time-reversal symmetry, reciprocity and isotropy. It is worthwhile to point out that such symmetries, like the results obtained so far, will only involve the discrete geometry of the Cartesian lattice and assume neither any topological space nor any Euclidean or Riemannian structure. Nevertheless, we shall need the definition of underlying manifolds provided with such structures when taking later the continuous limit of the discrete wave equation obeyed by the field $\psi$.

IV. TIME-REVERSAL SYMMETRY

In this section, we introduce the time-reversal symmetry by formulating its natural action on both currents and fields.

A. Currents

Time-reversal is naturally completed by reversing the direction of the current arrows in such a way that the currents propagate and are scattered backward in time. They will obey time-reversal symmetry only if they describe the past states of the system in reverse order. It is then obvious that the scattering process must be reversible. Therefore, we state that the forward scattering process depicted in Figure 8 (a) implies the existence of the reverse process depicted in Figure 8 (b). Since the currents are, in general, complex quantities, complex conjugate currents are involved in the time-reversed process. If the currents are real, this will be equivalent to reversing the original currents.
Using matrix notation, time-reversal invariance reads

$$(S_k) = S(E_k) \Rightarrow (E_k^*) = S(S_k^*),$$

where $(S_k)$ and $(E_k)$ are the column vectors of outgoing and incident currents respectively.

Then, taking the complex conjugate $(E_k) = S^*(S_k)$, one obtains

$$SS^* = S^*S = I_{2d+1},$$

where $I_{2d+1}$ is the $(2d + 1) \times (2d + 1)$ unit matrix.

This condition and the general form (16) for the scattering matrix elements leads to the following conditions

$$\begin{align*}
\frac{\mu_k \rho_k^*}{\rho_k} &= \text{constant} = e^{i\gamma_1}, \\
\frac{\mu_k \lambda_k^*}{\lambda_k} &= \text{constant} = e^{i\gamma_2}, \quad \forall k = 1, \ldots, 2d + 1, \\
|\mu_k| &= 1,
\end{align*}$$

and

$$\Lambda_1 \equiv \sum_{k=1}^{2d+1} \lambda_k \rho_k^* = e^{i\gamma_1} + e^{-i\gamma_2}. \quad (18)$$

Again, let us point out that all the quantities appearing in these relations are functions of the node $r$. This statement holds in particular for the constants $C_1 = e^{i\gamma_1}$ and $C_2 = e^{i\gamma_2}$. 

FIG. 8. Processes involved in the time-reversal symmetry. (a) forward process. (b) time-reversed process.
B. Time-reversal symmetry of the field

Equation (15) which describes the time evolution of the field can be viewed as some general relation which expresses the field $\psi$ at time $t + \tau$ as a function of $\psi$ at times $t$ and $t - \tau$

$$\psi(r, t + \tau) = f[\psi(r_k, t), \psi(r, t - \tau)].$$

(19)

In general, $\psi$ is complex valued function so that equation (19) obeys time-reversal symmetry if the reverse-time equation is verified by the complex conjugate field $\psi^*$, i.e.,

$$\psi^*(r, t - \tau) = f[\psi^*(r_k, t), \psi^*(r, t + \tau)].$$

(20)

Therefore, our task is to check that the time-reversal symmetry translated on currents, which was considered in the previous section, enables us to obtain equation (20) from equation (19). For this purpose, we first substitute the definition (2) of the field in the previous section, enables us to obtain equation (20) from equation (19). For this purpose, we first substitute the definition (2) of the field $\psi$ in (19)

$$2d+1 \sum_{k=1}^{2d+1} \lambda_k(r)E_k(r, t + \tau) = \left[2d+1 \sum_{l=1}^{2d+1} \lambda_l(r_k)E_l(r_k, t), \sum_{k=1}^{2d+1} \lambda_k(r)E_k(r, t - \tau) \right].$$

(21)

Then, Figure 8 tells us that reversing the direction of current propagation and substituting the complex conjugate currents $S_k^*$ for $E_k$ gives the current values of the reverse-time system. In other words, the time-reversal symmetry applied to the currents, appearing in equation (21), leads to

$$2d+1 \sum_{k=1}^{2d+1} \lambda_k(r)S_k^*(r, t - \tau) = \left[2d+1 \sum_{l=1}^{2d+1} \lambda_l(r_k)S_l^*(r_k, t), \sum_{k=1}^{2d+1} \lambda_k(r)S_k^*(r, t + \tau) \right].$$

(22)

This equation is similar to equation (20). However, it involves the quantities $\sum_{k=1}^{2d+1} \lambda_k S_k^*$ instead of $\psi^* = \sum_{k=1}^{2d+1} \lambda_k E_k^*$. Therefore, to demonstrate the time-reversal invariance of the field $\psi$, it is necessary to show that these two quantities are identical. Actually, they just need to be proportional to each other since the function $f$ is linear. Using the scattering formula (2), we can write

$$\sum_{k=1}^{2d+1} \lambda_k S_k^* = \Lambda_1 \psi^* - \sum_{k=1}^{2d+1} \lambda_k \mu_k^* E_k^*.$$

Since $\lambda_k \mu_k^* = C_2^* \lambda_k^*$ according to the second equation of (17), one obtains

$$\sum_{k=1}^{2d+1} \lambda_k S_k^* = (\Lambda_1 - C_2^*) \psi^* = C_1 \psi^*,$$

where we have used equation (18). However, we have previously pointed out that $C_1$ is a function of the node $r$. Therefore, substituting $C_1 \psi^*$ for $\sum_{k=1}^{2d+1} \lambda_k S_k^*$ in equation (24) leads to

$$C_1(r)\psi^*(r, t - \tau) = f[C_1(r_k)\psi^*(r_k, t), C_1(r)\psi^*(r, t + \tau)].$$

This equation will be equivalent to equation (20) if we impose the condition

$$C_1(r_k) = C_1(r), \quad \forall k.$$

Thus, the constant $C_1$ must be the same on the node $r$ and on its neighbors $r_k$, for any node $r$ in the system. One must conclude that $C_1$ is a constant over the whole lattice. Therefore, the first equation of (17) becomes

$$\frac{\mu_k(r)\rho_k^*(r)}{\rho_k(r)} = e^{i\gamma_1}, \quad \forall r, k.$$

(23)
C. Form of the time-dependent equation

We now proceed to show that the results obtained so far have already strong consequences on the form of the evolution equation of $\psi$. First, the coefficients of $\psi(r, t - \tau)$ and $\psi(r_k, t)$ in equation (15) can be recast in the following form, with the help of identities (17), (18) and (23)

$$
\mu^2 \left[ 1 - \sum_{k=1}^{2d+1} \frac{\lambda_k(r)\rho_k(r)}{\mu_k(r)} \right] = -\frac{\mu^2}{C_1C_2(r)},
$$

and

$$
\lambda_k(r)\rho_\tau(r_k) = \epsilon_k(r)|\lambda_k(r)||\rho_\tau(r_k)|\left[ \frac{\mu^2}{C_1C_2(r)} \right]^{1/2},
$$

where the modulus and phases of $\lambda$ and $\rho$ have been separated in the above expression. The sign $\epsilon_k(r) = \pm 1$ depends on the choice made in taking the square root. Plugging these two results into (15) leads to the time-reversal invariant equation of the field

$$
\psi(r, t + \tau) + \frac{\mu^2}{C_1C_2(r)}\psi(r, t - \tau) = \sum_{k=1}^{2d+1} \lambda_k(r)\rho_\tau(r_k)\psi(r_k, t),
$$

where $|\mu^2/(C_1C_2(r))| = 1$ since $|\mu| = |C_1| = |C_2(r)| = 1$.

It is now easy to convince ourselves that the form of (24) is either $\partial^2\psi(r, t)/\partial t^2 = L\psi(r, t)$ or $i\partial\psi(r, t)/\partial t = L\psi(r, t)$, where $L$ is a real operator. Let us notice that the left hand side of equation (24) contains the terms which are needed to build the time derivatives of the field. There are only two possibilities: $\mu^2/(C_1C_2(r)) = 1$ or $\mu^2/(C_1C_2(r)) = -1$. If $\mu^2/(C_1C_2(r)) = 1$, the left hand side of equation (24) becomes $\psi(r, t + \tau) + \psi(r, t - \tau)$ which allows us to construct a second order time derivative: $[\psi(r, t + \tau) + \psi(r, t - \tau) - 2\psi(r, t)]/\tau^2$. If $\mu^2/(C_1C_2(r)) = -1$, the difference $\psi(r, t + \tau) - \psi(r, t - \tau)$ is directly related to the first order time derivative: $[\psi(r, t + \tau) - \psi(r, t - \tau)]/2\tau$. The first or second order time derivatives of the field must be the same on the whole lattice, which is achieved by setting $C_2(r) = C_2$.

Lastly, we conclude that according to the two values we assign to $\mu^2/(C_1C_2)$, equation (24) becomes

$$
\frac{\mu^2}{(C_1C_2)} = 1 \rightarrow \psi(r, t + \tau) + \psi(r, t - \tau) = \sum_{k=1}^{2d+1} \epsilon_k(r)|\lambda_k(r)||\rho_\tau(r_k)|\psi(r_k, t),
$$

$$
\frac{\mu^2}{(C_1C_2)} = -1 \rightarrow i[\psi(r, t + \tau) - \psi(r, t - \tau)] = \sum_{k=1}^{2d+1} \epsilon_k(r)|\lambda_k(r)||\rho_\tau(r_k)|\psi(r_k, t),
$$

as was claimed above. In particular, it is worthwhile to point out that, due to the square root of $\mu^2/(C_1C_2)$ in the right hand side of (24), the factor $i$ appears when the equation is of first order in time.

In the remainder of the paper, we shall write

$$
\frac{\mu^2}{(C_1C_2)} = \mu^2 e^{-i(\gamma_1 + \gamma_2)} = \epsilon_e,
$$

with $\epsilon_e = \pm 1$. 

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V. RECIPROCITY

In this section, we go further by imposing a new symmetry, reciprocity, on the evolution rules of the currents. The definition is sketched in Figure 9 where \( E \) and \( S \) are respectively an input and an outgoing current defined on two arbitrary bonds of the same node. We assume that the direct process in Fig. 9 (a) implies that the reciprocal process in Fig. 9 (b) also exists. In the reciprocal process, the currents \( E \) and \( S \) have the same values as in Figure 9 (a) but the two arbitrary bonds have been exchanged.

Formulated in this way, this symmetry is the counterpart for the scattering process, of the reciprocity theorem which is well known for propagation in inhomogeneous media (see for instance [11–14]). This is our main motivation for introducing it. Let us discuss its consequences.

Using matrix notation, the process described in Fig. 9 (a) can be written as \( S_l = s_{lk} E_k \) where \( E_k \) is the input current and \( S_l \) is any of the outgoing currents. We can also imagine a second process for which the only non-vanishing input current is incident on bond \( l \) while we are looking at the outgoing current on bond \( k \), i.e. \( S_k = s_{kl} E_l \). If both processes are chosen so that \( E_k = E_l \), reciprocity tells us that \( S_k = S_l \), leading to \( s_{kl} = s_{lk} \). Since the two bonds \( k \) and \( l \) are arbitrary, one concludes that the scattering matrix is symmetrical \( S = S^T \).

Additionaly, the scattering matrix is time-reversal invariant : \( S^{-1} = S^* \). Hence, the scattering matrix is unitary. As a direct consequence of this result, the sum of current intensities is conserved in any scattering process on the lattice

\[
\sum_{k=1}^{2d+1} |S_k|^2 = \sum_{k=1}^{2d+1} |E_k|^2.
\]

(26)
Since the current intensities are also conserved in the propagation stage, their total sum over the whole lattice keeps the same value at each time step. This property guarantees the stability of the current dynamics. To understand its importance, let us consider a system of linear size \( L \). The linear evolution of the whole set of currents can be described by a \((2d+1)L^d \times (2d+1)L^d\) matrix which transforms the \((2d+1)L^d\) input currents at time \( t \) into the \((2d+1)L^d\) input currents at the next time step \( t + \tau \). Such a linear system is characterized by its eigenvectors and the corresponding eigenvalues. Any arbitrary initial state of the currents can be expanded on the basis of these eigenvectors. Let us choose one arbitrary eigenvector, characterized by its eigenvalue \( \zeta \), as the initial state. The current intensities will diverge to infinity if \( |\zeta| > 1 \) or decay to zero if \( |\zeta| < 1 \). The above result, equation (26), prevents us from such a scenario since the total sum of current intensities is constant regardless of the initial state. We conclude that all the eigenvalues are bound to be of modulus 1 and that any initial linear superposition of eigenvectors is preserved by the time evolution of the currents. This result is pleasing because it is equivalent to what is expected from the acceptable solutions of the usual linear wave equations such as the classical wave or Schrödinger equations. In order to break the stability of the current dynamics, losses or interactions of the system with the "outside world" must be introduced in some way.

Note that instead of introducing reciprocity, we could have required first the conservation law (26), and then combining this law with time-reversal symmetry would have led to reciprocity.

From \( s_{kl} = s_{lk} \) and (11), we find

\[
\rho_k \lambda_l = \rho_l \lambda_k, \quad \forall k, l,
\]

or

\[
\frac{\rho_k}{\lambda_k} = Re^{i\gamma},
\]

where \( R \) and \( \gamma \) are independent of \( k \).

Combining this result with the two first equations of (17) leads directly to

\[
2\gamma = \gamma_2 - \gamma_1. \tag{27}
\]

The matrix element becomes

\[
s_{kl} = \rho_k \lambda_l - \mu_k \delta_{kl} = \mu_k \left[ \frac{(1 + \epsilon_e/\mu^2) \lambda_k^* \lambda_l}{\sum_{k=1}^{2d+1} |\lambda_k|^2} - \delta_{kl} \right].
\]

Remembering that, according to the second equation of (17), \( \mu_k = e^{i\gamma_2} \lambda_k/\lambda_k^* \), the scattering matrix is parametrized by two constant phases \( \gamma_1 \) and \( \gamma_2 \) and by \( 2d + 1 \) complex coefficients \( \lambda_k \) which are functions of the location \( r \).

VI. ISOTROPY

Isotropy is not required in order for the model to be solvable. However, we impose isotropy to simplify the model since anisotropic scatterers could be considered as well. Let us consider a process where only one of the propagating input currents \( E_i \) is different from zero (Figure 10(a)). This process is by definition isotropic if all outgoing currents have the same value except for two outgoing currents : \( S_i \) on the input bond and the node current \( S_{2d+1} \)

\[
S_j = S_k, \quad \forall j, k / j, k \neq i, j, k \neq 2d + 1.
\]
In other words, when viewed from one particular bond, all other bonds are equivalent except the node bond because of its special status. The scatterer will be isotropic if the scattering process is isotropic for any chosen input bond.

Another process to consider is the one where the node current is different from zero (Figure 10 (b)). This process is isotropic if all outgoing currents have the same value except the node current \( S_{2d+1} \), i.e.

\[
S_j = S_k, \quad \forall j, k / j, k \neq 2d + 1
\]

Actually, both kinds of processes lead to the same conclusions because of the form of the scattering elements discussed in the previous section.

\[
\rho_j = \rho_k, \lambda_j = \lambda_k, \mu_j = \mu_k, \quad \forall j, k / j, k \neq 2d + 1. \tag{28}
\]

![FIG. 10. The two scattering processes considered to define isotropy.](image)

The scattering matrix now depends only on two complex parameters \( \lambda_1 \) and \( \lambda_{2d+1} \), and on the two phases \( \gamma_1 \) and \( \gamma_2 \) (since \( \mu_k \) and \( \rho_k \) can be expressed as functions of \( \lambda_k \) according to the second equations of (17) and (27)).

Conditions (28) and (13) provide an additional constraint for the value of \( \mu_1 \). First, suppose the value of \( \mu_1 \) is known at node \( \mathbf{r} \). Then, according to (13), the value of \( \mu_{d} \) on the nearest-neighbors of node \( \mathbf{r} \) reads

\[
\mu_{d}(\mathbf{r}) = \frac{\mu^2}{\mu_k(\mathbf{r})} = \frac{\mu^2}{\mu_1(\mathbf{r})}, \quad \forall k \neq 2d + 1.
\]

Since the scatterers \( \mathbf{r}_k \) are also isotropic, \( \mu_{d}(\mathbf{r}_k) = \mu_1(\mathbf{r}_k) \) for \( k \neq 2d = 1 \), we obtain

\[
\mu_1(\mathbf{r}_k) = \frac{\mu^2}{\mu_1(\mathbf{r})}, \quad \forall k \neq 2d + 1.
\]
Thus, \( \mu_1(\mathbf{r}_k) \) has a constant value on the nearest neighbors \( \mathbf{r}_k \) of node \( \mathbf{r} \). Next, starting from nodes \( \mathbf{r}_k \), we obtain for the second nearest-neighbors \( \mathbf{r}_{kl} \) of node \( \mathbf{r} \):

\[
\mu_1(\mathbf{r}_{kl}) = \frac{\mu^2}{\mu_1(\mathbf{r}_k)} = \mu_1(\mathbf{r}).
\]

Finally, proceeding in the same way from node to node, we find that the whole system is made of two intermingled Cartesian lattices characterized by their respective values \( \mu_1 \) and \( \mu_2/\mu_1 \). It is important to note that this result is peculiar to a Cartesian lattice, which has been considered from the beginning. Actually, the model we have discussed so far does not depend on the type of the discrete lattice. Indeed, if we had considered, for instance, a triangular lattice, a quasiperiodic lattice or even a random lattice (where the node connectivity changes from node to node, see Appendix), the time dependent equation of the field or the form of the scattering matrices would not have changed. However, the conclusion leading to the two values of \( \mu_1 \) on a Cartesian lattice would need to be modified on an arbitrary lattice. Consider a triangular lattice in which the elementary cell contains three nodes connected by three bonds. Following an elementary closed loop of three bonds starting at node \( \mathbf{r} \), we would find after a complete cycle

\[
\mu_1(\mathbf{r}) = \frac{\mu^2}{\mu_1(\mathbf{r})},
\]

i.e.

\[
\mu_1(\mathbf{r}) = \epsilon_1 \mu,
\]

where \( \epsilon_1 = \pm 1 \). Moreover, \( \epsilon_1 \) should have the same value at the three nodes and as a result over the whole system. As this conclusion is true for almost any arbitrary lattice, we are led to adopt (29) for the sake of generality. Remembering that \( \mu \) is defined by its square \( \mu^2 \) in equation (13), it is possible to choose once and for all \( \epsilon_1 = 1 \), i.e. \( \mu_1 = \mu \).

Let us combine now these results with time reversal symmetry and reciprocity. For this purpose, we introduce the ratio \( \nu = \lambda_{2d+1}/\lambda_1 \). From the second equation of (17) and (14), we write

\[
\mu_1 = \mu_{2d+1} \frac{\nu^*}{\nu} = \epsilon_{2d+1} \mu \frac{\nu^*}{\nu}.
\]

Since \( \mu_1 = \mu \), we get

\[
\frac{\nu^*}{\nu} = \epsilon_{2d+1} = \pm 1.
\]

That is, the ratio \( \nu = \lambda_{2d+1}/\lambda_1 \) is either real or imaginary.

On the other hand, reciprocity leads to \( \rho_{2d+1} = \rho_1 \nu \). Hence, \( \rho_{2d+1} \lambda_1 = \rho_1 \lambda_{2d+1} = \rho_1 \lambda_1 \nu \) and \( \rho_{2d+1} \lambda_{2d+1} = \rho_1 \lambda_1 \nu^2 \). Therefore, the scattering matrix elements become

\[
\begin{align*}
\begin{cases}
s_{kl} = \rho_1 \lambda_1 - \mu \delta_{kl}, & \forall k, l \neq 2d + 1, \\
s_{kl} = \rho_1 \lambda_1 \nu, & \forall k \neq 2d + 1, l = 2d + 1, \\
s_{kl} = \rho_1 \lambda_1 \nu^2 - \mu \epsilon_{2d+1}, & k = l = 2d + 1,
\end{cases}
\end{align*}
\]

where, using (27)

\[
\rho_1 \lambda_1 = \frac{\mu + \epsilon_2/\mu}{2d + \nu \nu^*}.
\]

Let us make a few remarks about these last results. First, the coefficients \( \lambda_k \) which appear in the expression of the field \( \psi(\mathbf{r}, t) = \sum_{k=1}^{2d+1} \lambda_k(r) E_k(\mathbf{r}, t) \) are defined modulo a multiplicative constant. Obviously, the scattering matrix cannot depend on such an arbitrary constant, and indeed, one can see that it is only
function of the ratio \( \nu = \lambda_{2d+1}/\lambda_1 \), according to (31) and (31). Next, \( \mu \) and \( \epsilon_c \) are constants defined over the whole system. Hence, \( \nu \) is the sole quantity which depends upon the location \( r \) in the system (with the restriction \( \nu^* / \nu = \epsilon_{2d+1} = \pm 1 \)). Given the definition \( \nu = \lambda_{2d+1}/\lambda_1 \), we see then that this is the node current which enables us to describe an inhomogeneous system as asserted before. A model without node currents, which corresponds to \( \nu = 0 \), would have restrained us to an homogeneous system. We can interpret the node current as trapping a fraction of the wave energy before releasing it to neighboring nodes. The parameter \( \nu(r) \) measures the local strength of this trapping effect. We shall see in the next section that the velocity of the wave is directly related to \( \nu(r) \).

We can now write the final form of the wave equation (24) as

\[
\psi(r, t + \tau) + \epsilon_c \psi(r, t - \tau) = \lambda_1(r) \left[ \sum_{k=1}^{2d} \rho_1(r_k) \psi(r_k, t) + \nu^2(r) \rho_1(r) \psi(r, t) \right].
\]

Given this equation, we are led to introduce the new field

\[
\phi(r, t) = \rho_1(r) \psi(r, t) = \rho_1(r) \lambda_1(r) \left[ \sum_{k=1}^{2d} E_k(r, t) + \nu(r) E_{2d+1}(r, t) \right],
\]

which obeys the following equation

\[
\phi(r, t + \tau) + \epsilon_c \phi(r, t - \tau) = \frac{\mu + \epsilon_c / \mu}{2d + \nu(r) \nu^*(r)} \left[ \sum_{k=1}^{2d} \phi(r_k, t) + \nu^2(r) \phi(r, t) \right].
\]

Like the scattering matrix elements (30), this equation depends only on the constants \( \mu \) and \( \epsilon_c \) and on the local parameter \( \nu(r) \). We note that the coefficient \( \lambda_1(r) \) comes into the expressions of the field, of the scattering matrix and of the wave equation as the product \( \lambda_1(r) \rho_1(r) \). Therefore, it is possible and convenient to rename the quantities \( \lambda_1(r) \rho_1(r) \) and \( \lambda_{2d+1}(r) \rho_1(r) \) as \( \lambda_1(r) \) and \( \lambda_{2d+1}(r) \) respectively. With this new notation, we can reformulate the above equations as

\[
\phi(r, t) = \lambda_1(r) \left[ \sum_{k=1}^{2d} E_k(r, t) + \nu(r) E_{2d+1}(r, t) \right], \quad \text{(32)}
\]

\[
\lambda_1(r) = \frac{\mu + \epsilon_c / \mu}{2d + \nu(r) \nu^*(r)}, \quad \text{(33)}
\]

\[
\phi(r, t + \tau) + \epsilon_c \phi(r, t - \tau) = \lambda_1(r) \left[ \sum_{k=1}^{2d} (\phi(r_k, t) - \phi(r, t)) + (2d + \nu^2(r)) \phi(r, t) \right], \quad \text{(34)}
\]

where we have shown explicitly the term \( \sum_{k=1}^{2d} (\phi(r_k, t) - \phi(r, t)) \), which corresponds to the discrete Laplacian on an Euclidean lattice. Similarly, the scattering matrix is given by

\[
S(r) = \lambda_1(r) \left( \begin{array}{cccc} 1 & \cdots & 1 & \nu(r) \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & \nu(r) & \nu^2(r) \end{array} \right) - \mu \left( \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 1 & \vdots \\ 0 & \cdots & 1 & \epsilon_{2d+1}(r) \end{array} \right),
\]

where \( \epsilon_{2d+1}(r) = \nu^*(r)/\nu(r) = \pm 1 \).
VII. DISCUSSION OF THE WAVE EQUATION

The wave equation (34) is a function of two discrete parameters, $\epsilon_e$ and $\epsilon_{2d+1}(r)$. The constant $\epsilon_e$ is defined over the whole system. We have already seen that its value $\pm 1$ controls the order of the time dependent wave equation, which will later be discussed separately. The parameter $\epsilon_{2d+1}(r)$ has a different status. Being defined by $\epsilon_{2d+1}(r) = \nu^*(r)/\nu(r)$ with $\nu(r)$ controlling the disorder of the system, its value $\pm 1$ depends on the node $r$. However, we would like to describe should depend continuously on the location in the system, such as a potential, a wave velocity or an index of refraction which varies smoothly with the position. Therefore, we shall assume that the value of $\epsilon_{2d+1}$ is fixed over the whole system whereas $\nu(r)$ is a smooth function of $r$. If $\epsilon_{2d+1} = 1$ or $-1$, $\nu(r)$ is real or imaginary, respectively. Both cases will also be considered separately. Hence, four possibilities will be discussed according to the values of $\epsilon_e$ and $\epsilon_{2d+1}$.

A. Second order time equation ($\epsilon_e = 1$)

1. $\epsilon_{2d+1} = 1$

Since in this case $\nu(r)$ is real, $\nu(r)\nu^*(r) = \nu^2(r)$. The wave equation becomes

$$\phi(r,t+\tau) + \phi(r,t-\tau) - 2\phi(r,t) = \frac{2\cos\theta}{2d + \nu^2(r)} \sum_{k=1}^{2d} (\phi(r_k,t) - \phi(r,t)) + 2(\cos\theta - 1)\phi(r,t),$$

where we have written explicitly the second order time derivative and $\mu = e^{i\theta}$.

We have already pointed out that our results were not peculiar to a Cartesian lattice. A similar wave equation would have been obtained in any discrete lattice. However, in order to consider this equation as the discretized version of the corresponding continuous wave equation, we need now to limit ourselves to a Cartesian lattice so that we can introduce the lattice constant $l$ and the current velocity $c_0 = l/\tau$. Hence, one writes

$$\left(\frac{2\cos\theta}{2d + \nu^2(r)} c_0^2\right) \sum_{k=1}^{2d} \frac{1}{l^2} (\phi(r_k,t) - \phi(r,t))$$

$$- \frac{1}{\tau^2} (\phi(r,t+\tau) + \phi(r,t-\tau) - 2\phi(r,t)) = \frac{2(1 - \cos\theta)}{\tau^2} \phi(r,t),$$

which is the discretized version of the Klein-Gordon equation

$$c^2(r)\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = a^2 \phi.$$  \hspace{1cm} (37)

In the particular case where the Klein-Gordon equation describes a scalar particle of mass $m$, the coefficient $a$ would be given by $a^2 = m^2c^4/\hbar^2$. Comparison between (36) and (37) provides the velocity $c$ of the wave and the coefficient $a$

$$\begin{cases} c^2(r) = c_0^2 \left(\frac{2\cos\theta}{2d + |\nu(r)|^2}\right), \\ a^2 = \frac{2(1 - \cos\theta)}{\tau^2}. \end{cases}$$  \hspace{1cm} (38)

The special choice $\theta = 0$, i.e. $\mu = 1$, leads to the scalar wave equation $c^2(r)\nabla^2 \phi - \partial^2 \phi/\partial t^2 = 0$. The first equation of (38) shows clearly that the velocity of the wave is directly connected to the parameter $\nu(r)$. Its maximum value $c_{\text{max}} = c_0(\cos\theta/d)^{1/2}$ is reached when $\nu(r) = 0$. As stated before, the role of the node current is to slow down the wave. If $d > 1$, we also note that $c_{\text{max}}$ is always strictly smaller than the velocity $c_0$ of the currents.
In this case $\nu(r)$ is imaginary, thus $\nu^2(r) = -|\nu(r)|^2$. The wave equation becomes

$$\left(\frac{2\cos \theta}{2d + |\nu(r)|^2}\right) \sum_{k=1}^{2d} \frac{1}{l^2} \left(\phi(r_k, t) - \phi(r, t)\right) - \frac{1}{\tau^2} \left(\phi(r, t + \tau) + \phi(r, t - \tau) - 2\phi(r, t)\right) = \frac{2}{\tau^2} \left[ \cos \theta \left(\frac{2d - |\nu(r)|^2}{2d + |\nu(r)|^2}\right) + 1 \right] \phi(r, t),$$

(39)

This equation is also a discretized version of (37) since the factor of $\phi(r, t)$ in the right hand side of (39) is positive. The velocity of the wave is given by

$$c^2(r) = c_0^2 \left(\frac{2\cos \theta}{2d + |\nu(r)|^2}\right).$$

However, (39) is not as versatile as (36). First, in order to obtain the scalar wave equation, we must set the double condition $\theta = 0$ and $\nu(r) = 0$. Hence, the wave velocity is pinned to its maximum value $c = c_0/\sqrt{d}$. This describes a uniform system without node current. Next, it is easy to show that the same double condition holds if we take the continuous limit (39) in fixing the value of $a^2$. Under these circumstances, we conclude that an imaginary node current is not compatible with equation (37).

Hence, from the analysis of these two cases ($\epsilon_{2d+1} = \pm 1$), in order to correctly describe the Klein-Gordon equation and the scalar wave equation in an inhomogeneous system by our current model, we must set $\epsilon_e = 1$ and $\epsilon_{2d+1} = 1$. The resulting equation is given by (36). This result is not completely satisfactory since the necessity of setting $\epsilon_{2d+1} = 1$ is not derived from the general kind of reasoning which has been used in our approach up to this stage. We will see in section VIII that adding a second node current to each scatterer will remove this arbitrary condition.

B. First order time equation ($\epsilon_e = -1$)

1. $\epsilon_{2d+1} = 1$

With $\nu(r)$ being real, the wave equation becomes

$$\frac{i}{2\tau} \left(\phi(r, t + \tau) - \phi(r, t - \tau)\right) = -\frac{l^2}{\tau} \left(\frac{\sin \theta}{2d + \nu^2(r)}\right) \sum_{k=1}^{2d} \left(\phi(r_k, t) - \phi(r, t)\right) - \frac{\sin \theta}{\tau} \phi(r, t).$$

We recognize the discretized version of the Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi + V(r)\phi,$$

(40)

with

$$\frac{V(r)}{\hbar} = -\frac{\sin \theta}{\tau},$$

and

$$\frac{\hbar}{2m} = \frac{l^2}{\tau} \left(\frac{\sin \theta}{2d + |\nu(r)|^2}\right) = -\frac{l^2 V(r)}{\hbar(2d + |\nu(r)|^2)}.$$

The expression for $V(r)$ shows that we would obtain a Schrödinger equation in which the potential is bound to stay constant over the whole system. Therefore, we shall reject the choice $\epsilon_e = -\epsilon_{2d+1} = -1$. 

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Again, $\nu^2(r) = -|\nu(r)|^2$. We obtain

$$
\frac{i}{2\tau} (\phi(r, t + \tau) - \phi(r, t - \tau)) = -\frac{l^2}{\tau} \sum_{k=1}^{2d} \left( \phi(r_k, t) - \frac{2d - |\nu(r)|^2}{l^2} \phi(r, t) \right) - \frac{\sin \theta}{\tau} \left( \frac{2d - |\nu(r)|^2}{2d + |\nu(r)|^2} \right) \phi(r, t).
$$

In a similar way, identification with the discrete Schrödinger equation (40) gives

$$
\frac{V(r)}{\hbar} = -\frac{\sin \theta}{\tau} \left( \frac{2d - |\nu(r)|^2}{2d + |\nu(r)|^2} \right),
$$

and

$$
\frac{\hbar}{2m} = \frac{l^2}{\tau} \left( \frac{\sin \theta}{2d + |\nu(r)|^2} \right) = -\frac{l^2V(r)}{\hbar(2d - |\nu(r)|^2)}.
$$

In contrast with the previous section, the potential $V(r)$ is a function of the position $r$. The last equality shows that the mass $m$ also depends on $r$. This is not a surprise in the context of quantum waves on a lattice where one usually introduces an effective mass which may be a function of position (see [17] for instance). However, the Schrödinger equation (41) is not completely general. We note that the variations of $m$ and $V$ with $r$ are correlated. In particular, the mass cannot be kept fixed at a constant value if the potential is a function of $r$. In the context of the localization of quantum waves in an inhomogeneous medium, equation (41) would correspond to the Anderson model with correlated “diagonal” and “off-diagonal” disorder [13]. Hence, one may wonder which conditions must be imposed in our approach in order to derive a Schrödinger equation where the two types of disorder can be described separately. The most natural idea which comes to mind, is that one needs an additional degree of freedom which would allow to separate the mass and the potential spatial variations. Introducing a second node current not only achieves this task but also removes the arbitrary condition discussed in section VII A.

**VIII. INTRODUCING A SECOND NODE CURRENT**

Let us attach to each node additional currents $E_{2d+2}$ and $S_{2d+2}$ which behave like the node currents $E_{2d+1}$ and $S_{2d+1}$. They participate in the same scattering process as the other currents but do not propagate to neighbor nodes. Instead, the outgoing current $S_{2d+2}$ becomes the incident current $E_{2d+2}$ on the same node at the next time step. The scattering process and the definition of the field become

$$
S_k = \sum_{l=1}^{2d+2} s_{kl} E_l, \quad k = 1, \ldots, 2d + 2,
$$

$$
\psi(r, t) = \sum_{k=1}^{2d+2} \lambda_k(r) E_k(r).
$$

It is easy to check that the derivation of a wave equation similar to (34) remains quite unchanged. Therefore, we will only point out the most noticeable differences and the final result.

First, equation (41) is supplemented by

$$
\mu_{2d+2}(r) = \epsilon_{2d+2}(r) \mu,
$$

where $\epsilon_{2d+2}(r) = \pm 1$. 

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Second, instead of the single ratio \( \nu = \lambda_{2d+1}/\lambda_1 \), we introduce \( \nu_{2d+1} = \lambda_{2d+1}/\lambda_1 \) and \( \nu_{2d+2} = \lambda_{2d+2}/\lambda_1 \) which obey the conditions \( \nu_{2d+1}/\nu_{2d+1} = \epsilon_{2d+1} = \pm 1 \) and \( \nu_{2d+2}/\nu_{2d+2} = \epsilon_{2d+2} = \pm 1 \). In other words, we find again that \( \nu_{2d+1} \) and \( \nu_{2d+2} \) are either real or imaginary.

Finally, equations (32), (33) and (34) become

\[
\phi(r, t) = \lambda_1(r) \left[ \sum_{k=1}^{2d} E_k(r, t) + \nu_{2d+1}(r)E_{2d+1}(r, t) + \nu_{2d+2}(r)E_{2d+2}(r, t) \right], \tag{42}
\]

\[
\lambda_1(r) = \frac{\mu + \epsilon_c / \mu}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2}, \tag{43}
\]

\[
\phi(r, t + \tau) + \epsilon_c \phi(r, t - \tau) = \lambda_1(r) \left[ \sum_{k=1}^{2d} (\phi(r_k, t) - \phi(r, t)) + (2d + \nu_{2d+1}^2(r) + \nu_{2d+2}^2(r)) \phi(r, t) \right]. \tag{44}
\]

At this stage, the discussion of section VII can be reproduced. However, as shown below, we need only to discuss the two cases \( \epsilon_c = \pm 1 \) separately. Indeed, we have been previously led to discuss the two choices \( \nu = \lambda_{2d+1}/\lambda_1 \) being real or imaginary. By adding a second node current, it happens now that we have gained additional possibilities since \( \nu_{2d+1} = \lambda_{2d+1}/\lambda_1 \) and \( \nu_{2d+2} = \lambda_{2d+2}/\lambda_1 \) can be real or imaginary independently of each other. However, one can easily convince oneself that choosing \( \nu_{2d+1} \) and \( \nu_{2d+2} \) being simultaneously real or imaginary is equivalent to having a single node current such that \( \nu^2 = \nu_{2d+1}^2 + \nu_{2d+2}^2 \). The only new possibility opened up by introducing two nodes currents corresponds to \( \nu_{2d+1} \) real and \( \nu_{2d+2} \) imaginary, or \textit{vice-versa}. Since both node currents have been equivalent until now, it is sufficient to consider one of both cases, i.e \( \nu_{2d+1} \) real \( (\nu_{2d+1}(r) = 1) \) and \( \nu_{2d+2} \) imaginary \( (\epsilon_{2d+2}(r) = -1) \).

\section*{A. Second order time equation \( (\epsilon_c = 1) \)}

By introducing the lattice constant \( l \) and the current velocity \( c_0 = l/\tau \), the wave equation (44) becomes

\[
\left( \frac{2 \cos \theta}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right)^{2d} \sum_{k=1}^{2d} \frac{1}{l^2} (\phi(r_k, t) - \phi(r, t)) \nonumber
\]

\[
- \frac{1}{\tau^2} (\phi(r, t + \tau) + \phi(r, t - \tau) - 2\phi(r, t)) = \frac{2}{\tau^2} \left[ \cos \theta \left( \frac{2d + |\nu_{2d+1}(r)|^2 - |\nu_{2d+2}(r)|^2}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right) + 1 \right] \phi(r, t), \tag{45}
\]

where we have written \( \mu = e^{i\theta} \).

Comparison between (45) and the Klein-Gordon equation (47) provides

\[
c^2(r) = \frac{c_0^2}{\left( \frac{2 \cos \theta}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right)^{2d}},
\]

\[
a^2(r) = \frac{2}{\tau^2} \left[ \cos \theta \left( \frac{2d + |\nu_{2d+1}(r)|^2 - |\nu_{2d+2}(r)|^2}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right) + 1 \right].
\]

These results are an improvement over those obtained in section VII A 1 since \( a^2 \) is a function of the position \( r \). Moreover, \( c^2(r) \) and \( a^2(r) \) can vary independently from each other by properly adjusting the variations of \( \nu_{2d+1}(r) \) and \( \nu_{2d+2}(r) \). Finally, the necessity of setting \( \epsilon_{2d+1} = 1 \) which was found quite arbitrary in section VII A is no longer pertinent. The above results show that one really needs to keep both \( \epsilon_{2d+1} = 1 \) and \( \epsilon_{2d+2} = -1 \) simultaneously in order to obtain a complete version of the Klein-Gordon equation.
B. First order time equation \( \epsilon_c = -1 \)

The wave equation \( \mathbf{14} \) becomes

\[
\frac{i}{2\tau} \left( \phi(r, t + \tau) - \phi(r, t - \tau) \right) =

-\frac{l^2}{\tau} \left( \frac{\sin \theta}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right) \sum_{k=1}^{2d} \left( \phi(k, t) - \phi(r, t) \right)

- \frac{\sin \theta}{\tau} \left( \frac{2d + |\nu_{2d+1}(r)|^2 - |\nu_{2d+2}(r)|^2}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right) \phi(r, t).
\]

Comparison between \( \mathbf{15} \) and the Schrödinger equation \( \mathbf{10} \) gives

\[
\frac{V(r)}{\hbar} = -\frac{\sin \theta}{\tau} \left( \frac{2d + |\nu_{2d+1}(r)|^2 - |\nu_{2d+2}(r)|^2}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right),
\]

and

\[
\frac{\hbar}{2m} = \frac{l^2}{\tau} \left( \frac{\sin \theta}{2d + |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2} \right).
\]

In contrast with the results discussed in section \( \mathbf{VII} \), this new version of the discrete Schrödinger equation is now well defined. The potential \( V(r) \) and the mass \( m(r) \) can vary independently from each other by properly adjusting the variations of \( \nu_{2d+1}(r) \) and \( \nu_{2d+2}(r) \). For instance, it is possible to formulate the Anderson model with diagonal and off-diagonal disorders separately. Of particular interest is the choice \( |\nu_{2d+1}(r)|^2 + |\nu_{2d+2}(r)|^2 = \text{constant} \) which corresponds to keeping \( m(r) = \text{constant} \) while \( V(r) \) is varying with \( \nu_{2d+1}(r) \) and \( \nu_{2d+2}(r) \).

Introducing two node currents could seem at first glance somehow arbitrary but is justified by the resulting well-defined Klein-Gordon and Schrödinger equations, equations \( \mathbf{12} \) and \( \mathbf{13} \) respectively. These equations do not suffer from the drawbacks of the \( 2d + 1 \) current model discussed in section \( \mathbf{VII} \).

IX. CONCLUSION

It is at first glance a curiosity that starting from a discrete formulation of Huygens’ principle and taking account of basic properties and symmetries, we have derived two equations that can be directly identified with the Klein-Gordon equation and the Schrödinger equation. On the one hand, this is not completely surprising considering that this simple model incorporates the basic features of wave propagation. On the other hand, the temporal and spatial symmetries which have been selected in building the model are simply those underlying the resulting discrete wave equations. The novelty of this approach is the systematic derivation resulting in a versatile unified equation, which properly tuned by a single parameter, yields two of the most fundamental wave equations in physics. So this derivation can be viewed as a general framework into which is merged the related approach of the TLM network model. In particular, it is worthwhile to point out that this unified equation can be derived on any discrete lattice or graph, thus underlining the generality of the construction (see Appendix).

The most unusual feature of this formulation is the introduction of currents without any initial physical meaning. This can be compared with the cellular automata approach which is now routinely used for solving hydrodynamic problems. By starting from a minimal microscopic model of non physical particles which obey the most fundamental conservation laws of physics such as energy and momentum conservation, the macroscopic behavior of real fluids is recovered. The current model described in this paper has a similar status. A physical interpretation of the currents was given in the TLM network model by identifying them with electric field impulses traveling on a mesh of transmission lines \( \mathbf{1} \). Since in that particular case it
can be shown that the voltages and currents obey the same equations as the electric and magnetic fields of Maxwell’s equations, it is not surprising to obtain Maxwell’s equations from that model. However, such an interpretation fails in the abstract context we have considered in this paper, especially when we obtain the Schrödinger equation without any explicit analytic continuation from the diffusion equation (e.g. $t \mapsto it$). Although they locally propagate as particles, and then, are scattered as waves, the currents cannot be interpreted either as fields or as particles. This is also in contrast with other approaches where the Schrödinger and Maxwell equations have been obtained from the dynamics of ensembles of Brownian particles [17–19], thus providing a particle picture to standard wave equations.

In considering further investigations, the natural question that comes immediately to mind, concerns the possibility of describing other wave equations by the same approach. Some clues have already been indicated in this paper. For instance, we have discussed the possibility of describing higher order time derivatives of the field by releasing condition (13). More generally, as the construction relies essentially on symmetries, we may wonder what equations would be obtained if a symmetry condition is not fulfilled or is replaced by another one. For instance, it will be easy to release the time-reversal symmetry and to solve numerically the Schrödinger equation within this assumption. This would be of fundamental interest in our understanding of strong localization in random media. Finally, there is no reason for limiting ourselves to scalar equations. As discussed in the introduction, the Maxwell equations as an example of vector wave equations, have been exhibited in the TLM approach. A further possibility is the description of spinor waves by introducing two different types of currents propagating on a Cartesian lattice. This work will be described in a forthcoming paper [9].

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APPENDIX: FORMULATION OF THE CURRENT-MODEL ON A GRAPH

Although a $n$-dimensional Cartesian lattice has been considered throughout this paper, we have pointed out that most of our results did not depend on this particular lattice. This special kind of lattice, which involves an underlying manifold equipped with a metric, is required when considering the continuous limit of the discrete wave equation (44). Indeed, in that limit, the lattice spacing $l$ and the current velocity $c_0 = l/\tau$ are needed. In this appendix, we show that our model leads to the discrete wave equation (44) formulated on any arbitrary lattice. We are not going to reproduce in detail the derivation of Eq. (44), which remains quite unchanged. Instead of this, we focus on the main differences between both approaches on either a Cartesian lattice or on any arbitrary lattice.

We consider the most general lattice defined as a random lattice where each node is connected to $z(r)$ neighbor nodes, $z(r)$ denotes the coordination number. The notation $z(r)$ indicates explicitly that the number of neighbors varies from node to node, as sketched in Fig. 11. It is important to emphasize that no length is attributed to the bonds linking any two nodes. Thus, the bonds connected to one node represent only the connectivity of this arbitrary node.
As explained in the introduction, the currents propagate in one time step between a pair of nodes of a given bond. By taking into account the node currents, \( z(r) + 2 \) currents \( E_k(r,t) \) are incident on node \( r \) at each time step. Hence, a \((z(r) + 2) \times (z(r) + 2)\) scattering matrix is attached to node \( r \). The definition (2) of the field becomes

\[
\psi(r,t) = \sum_{k=1}^{z(r)+2} \lambda_k (r) E_k (r,t).
\]  

(A1)

The number of currents in the definition of the field depends now on the node \( r \). Keeping in mind these definitions, and using the previous notation for the neighbors of node \( r \), as described in Figure 4, it is easy to check that all the steps of the derivation of equation (44) are left unchanged. For instance, conditions (8) and (13) still hold to derive a closed field equation analogous to equation (15). The new closed field equation reads straightforwardly

\[
\psi(r,t + \tau) = \sum_{k=1}^{z(r)+2} \lambda_k (r) E_k (r,t) + \mu^2 \left[ 1 - \sum_{k=1}^{z(r)+2} \frac{\lambda_k (r) \rho_k (r)}{\mu_k (r)} \right] \psi(r,t - \tau).
\]  

(A2)

Then, the implementation of time-reversal symmetry and reciprocity can exactly be reproduced by substituting each summation over \( 2d + 2 \) currents by a summation over \( z(r) + 2 \) currents. Isotropy is also introduced in the same way by noticing again that the number of propagating currents involved in this local symmetry now depends on the node \( r \). As discussed in section VI, the lattice being arbitrary, equation (23) holds and the sign \( \epsilon_1 \) can be fixed equal to 1. Then, we introduce naturally the ratios \( \nu_{z(r)+1} (r) = \lambda_{z(r)+1} / \lambda_1 \) and \( \nu_{z(r)+2} (r) = \lambda_{z(r)+2} / \lambda_1 \) which satisfy the conditions \( \nu_{z(r)+1} (r) / \nu_{z(r)+1} (r) = \epsilon_{z(r)+1} = \pm 1 \) and \( \nu_{z(r)+2} (r) / \nu_{z(r)+2} (r) = \epsilon_{z(r)+2} = \pm 1 \). Finally, the equations (42), (43) and (44) are recovered in this context and read

\[
\phi(r,t) = \lambda_1 (r) \sum_{k=1}^{z(r)} E_k (r,t) + \nu_{z(r)+1} (r) E_{z(r)+1} (r,t) + \nu_{z(r)+2} (r) E_{z(r)+2} (r,t),
\]  

(A3)

\[
\lambda_1 (r) = \frac{\mu + \epsilon_\epsilon \mu}{z(r) + |\nu_{z(r)+1} (r)|^2 + |\nu_{z(r)+2} (r)|^2},
\]  

(A4)

\[
\phi(r,t + \tau) + \epsilon_\epsilon \phi(r,t - \tau) =
\]

\[
\lambda_1 (r) \sum_{k=1}^{z(r)} \left( \phi(r_k,t) - \phi(r,t) \right) + \left( z(r) + \nu_{z(r)+1}^2 (r) + \nu_{z(r)+2}^2 (r) \right) \phi(r,t),
\]  

(A5)
where $\epsilon_e = \pm 1$ and $\mu = e^{i\theta}$.

In equation (A5), one recognizes the term $\sum_{k=1}^{z(r)} (\phi(r_k, t) - \phi(r, t))$ as a topological Laplacian defined on a graph. Thus, equation (A3) is a wave propagation equation formulated for a scalar field defined on a graph. This result can be compared to the attempts of formulating discretized field theories for scalar fields [20,21].

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[1] P.B. Johns and R. L. Beurle, Numerical solution of two-dimensional scattering problems using a transmission-line matrix, Proc. Inst. Elec. Eng. 118 (1971), 1203.
[2] P.B. Johns, A new mathematical model to describe the physics of propagation, Radio Electron. Eng. 44 (1974), 657.
[3] W.R.J. Hoefer, Huygens and the computer. A powerful alliance in numerical electromagnetics, Proc. IEEE 79 (1991), 1459, and references cited therein.
[4] P.B. Johns, A simple explicit and unconditionally stable numerical routine for the solution of the diffusion equation, Int. J. Numer. Meth. Engin. 11 (1977), 1307.
[5] C. Vanneste, P. Sebbah, D. Sornette, A wave automaton for time-dependent wave propagation in random media, Europhys. Lett. 17 (1992), 715.
[6] P. Enders, Comment on “A wave automaton for time-dependent wave propagation in random media” by C. Vanneste, P. Sebbah and D. Sornette, Europhys. Lett. 21 (1992), 791.
[7] P. Sebbah, D. Sornette and C. Vanneste, Anomalous diffusion in two-dimensional Anderson-localization dynamics, Phys. Rev. B 48 (1993), 12506.
[8] D. Sornette, O. Legrand, F. Mortessagne, P. Sebbah and C. Vanneste, The wave automaton for the time-dependent Schrödinger, classical wave and Klein-Gordon equations, Phys. Lett. A 178 (1993), 292.
[9] S. De Toro Arias, D. Sornette and C. Vanneste, A new construction for spinor wave equations, to be published.
[10] P.B. Johns, The solution of inhomogeneous waveguide problems using transmission-line matrix, IEEE Trans. Microwave Theory Tech. MTT-22 (1974), 209.
[11] L.D. Landau and E.M. Lifchitz, Electrodynamics of continuous media, (Pergamon Press, 1984).
[12] L.D. Landau and E.M. Lifchitz, Fluid mechanics, (Pergamon Press, 1983).
[13] E.N. Economou, Green’s functions in quantum physics (Springer–Verlag, 1979).
[14] C. Prada, J.-L. Thomas and M. Fink, The iterative time reversal process: analysis of the convergence, J. Acoust. Soc. Am. 97 (1995), 62.
[15] P. Sheng, Introduction to wave scattering, localization, and mesoscopic phenomena, (Academic Press, 1995).
[16] P. Enders, D. De Cogan, TLM routines for the paraxial wave equation and the time-dependent Schrödinger equation, First international workshop on Transmission Line matrix (TLM) Modelling, Theory and applications.Victoria BC, (1995), 137.
[17] G.N. Ord, The Schrödinger and diffusion propagators coexisting on a lattice, J. Phys. A: Math. Gen. 29 (1996), L123.
[18] G.N. Ord, A stochastic model of Maxwell’s equations in 1 + 1 dimensions, Int. J. Theor. Phys. 35 (1996), 263.
[19] G.N. Ord, Random walks and the Schödinger’s equation in 2 + 1 dimensions, J. Phys. A: Math. Gen. 30 (1997), 819.
[20] T. Regge, General relativity without coordinates, Il Rivista del Nuovo Cimento 19 (1961), 558.
[21] C. Itzykson and J.M. Drouffe, Théorie statistique des champs, (InterEditions/Editions du CNRS, 1989). Tome 2, Chapitre XI, Section 1.2, in French.