ON A CLASS OF SYMPLECTIC SIMILARITY TRANSFORMATION MATRICES †

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Abstract. We present a class of symplectic matrices which transform by similarity given $2n \times 2n$ -dimensional matrix into Bunse-Gerstner form. If the given matrix is skew-Hamiltonian, the transformation gives a solution of an antisymmetric Riccati matrix equation, resulting from optimal control of linear systems.

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1. Introduction.

The paper is concerned with the class of $2n \times 2n$ dimensional matrices:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

(1.1)

where $S_{ij}$ ($i, j \in \{1, 2\}$) are arbitrary real $n \times n$ matrices. Let the matrix $J$ be defined by:

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

A matrix $S$ is Hamiltonian if $J^T SJ = -S^T$. The Hamiltonian matrices come out from optimal control of linear systems [1],[2],[4],[5],[7]-[11],[17]. A matrix $S$ is skew-Hamiltonian if $J^T SJ = S^T$. For example, if $S$ is a Hamiltonian matrix, then the matrix $S^2$ is skew-Hamiltonian.

We represent some further definitions and basic results. A matrix $U$ is symplectic if $U^T J U = J$. The symplectic similar transformations keep the Hamiltonian structure of matrices, i.e. if $S$ is a Hamiltonian matrix and $U$ is a symplectic matrix, then $U^{-1} S U$ is also a Hamiltonian matrix. Analogously, it is easy to verify that the symplectic similar transformations also keep the structure of the skew-Hamiltonian matrices. The Hamiltonian matrices form Lie algebra and the symplectic matrices form the corresponding Lie group.

Bunse-Gerstner proved [6] that by application of an orthogonal symplectic transformation one can reduce an arbitrary real matrix $S$ (1.1) in a form

$$\begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}$$

(1.2)

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such that its (2,1) block $S'_{21}$ is upper-triangular and (1,1) block $S'_{11}$ is upper-Hessenberg. For the transformation matrix a composition of finite number of Givens-like [16] and Householder-like [16] symplectic orthogonal transformations is used.

Van Loan proved [15] that if $S$ is a skew-Hamiltonian (squared Hamiltonian) matrix, then $S$ is a symplectic-orthogonal similar to a matrix of the form:

$$
\begin{bmatrix}
S'_{11} & S'_{12} \\
0 & S'_{11}^T
\end{bmatrix}
$$

where $S'_{11}$ is a Hessenberg matrix and $S'_{12}$ is a skew-symmetric matrix. For the transformation matrix the same composition of finite number of Givens-like and Householder-like symplectic orthogonal transformations is used. Besides Van-Loan’s method [15],[3],[6], for this purpose, the methods from [4] or [13] may be applied.

In [14] are applied Arnoldi-like matrices for similarity transformation of $2n \times 2n$ -dimensional matrix $S$ into a form (1.2), such that the submatrix $S'_{21} = 0$, under the condition that a minimal annihilator of the first column of the transformation matrix, with respect to $S$, is of degree $n$. Although for general matrices $S$ the last condition is connected with solving a nonlinear eigenvalue-eigenvector problem, for the skew-Hamiltonian matrices $S$ these transformations result in $S'_{21} = 0$ also, as do in Van Loan’s work [15], because the minimal annihilator of skew-Hamiltonian matrices is of degree $n$.

In this paper we present a new class of symplectic matrices which transform $2n \times 2n$ -dimensional matrices into Bunse-Gerstner form (1.2), instead of the Givens-like, Householder -like or Arnoldi-like symplectic transformation matrices. We use the matrices of the form:

$$
U = \begin{bmatrix}
Q & 0 \\
YQ & Q
\end{bmatrix},
$$

where $Q$ is an orthogonal matrix and $Y$ is a symmetric matrix. This form arises naturally in solving of the antisymmetric Riccati matrix equation (ARME) [13]:

$$
-Y S_{12} Y + S_{22} Y - Y S_{11} + S_{21} = 0
$$

where $S$ is a skew-Hamiltonian matrix and $Y$ is an unknown symmetric matrix, because the matrix $Y$ in (1.4) is already a solution of ARME, in case the matrix (1.4) brings $S$ in block-triangular form. To compare our result with [15], if a solution of ARME is required after the Van Loan’s transformation [15] is performed, one has to compute the matrix $U_{21} U_{11}^{-1}$, where $U_{11}$ and $U_{21}$ are blocks of the transformation matrix. If the matrix $U_{11}$ is singular, it can not be done.

Other references are mentioned in the text.

2. Main results.

Let be given the following $2n \times 2n$ block matrix

$$
S = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix},
$$

(2.1)
where $S_{ij}$ ($i, j \in \{1, 2\}$) are arbitrary $n \times n$ matrices. In this section we present a finite algorithm for similar transformation by a matrix $U$ of the matrix $S$, i.e.

$$U^{-1}SU = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} = S', $$

where $S'_{11}$ is an upper Hessenberg matrix and $S'_{21}$ is an upper triangular matrix.

Let us consider the following set of regular $2n \times 2n$ matrices:

$$G = \{ U = \begin{bmatrix} L & 0 \\ YL & L^{-T} \end{bmatrix} : Y^T = Y \}. $$

It is easy to verify that the set $G$ with the matrix multiplication is a group, a subgroup of the group of the symplectic matrices.

Applying the similar transformation $U^{-1}SU$, we obtain

$$U^{-1}SU = \begin{bmatrix} L^{-1} & 0 \\ -L^T Y & L^T \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} L & 0 \\ YL & L^{-T} \end{bmatrix} = \begin{bmatrix} L^{-1}(S_{11} + S_{12}Y)L \\ L^T(S_{21} + S_{22}Y - YS_{11} - YS_{12}Y)L \\ L^T(S_{22} - YS_{12}L^{-T}) \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} = S'. $$

The matrix $S$ will be transformed $n - 1$ times with respect to the matrices

$$U_i = \begin{bmatrix} L_i & 0 \\ Y_iL_i & L_i^{-T} \end{bmatrix}, \quad Y_i^T = Y_i, \quad (1 \leq i \leq n - 1). $$

By the first transformation, the first column of the new matrix $S_{21}$ will become upper triangular and the first column of the new matrix $S_{11}$ will become Hessenberg. Continuing this process, by the $i$-th transformation, the $i$-th column of the new matrix $S_{21}$ will become upper triangular such that the previous $i - 1$ columns of that matrix will remain upper triangular, and the $i$-th column of the new matrix $S_{11}$ will become Hessenberg, such that previous $i - 1$ columns of that matrix will remain Hessenberg.

From

$$U = \begin{bmatrix} L_1 & 0 \\ Y_1L_1 & L_1^{-T} \end{bmatrix} \begin{bmatrix} L_2 & 0 \\ Y_2L_2 & L_2^{-T} \end{bmatrix} \cdots \begin{bmatrix} L_{n-1} & 0 \\ Y_{n-1}L_{n-1} & L_{n-1}^{-T} \end{bmatrix} = \begin{bmatrix} L & 0 \\ YL & L^{-T} \end{bmatrix}, $$

we obtain

$$L = L_1L_2 \cdots L_{n-1}, $$

$$Y = Y_1 + L_1^{-T}Y_2L_1^{-1} + L_1^{-T}L_2^{-T}Y_3L_2^{-1}L_1^{-1} + \cdots + L_1^{-T}L_2^{-T} \cdots L_{n-2}^{-T}Y_{n-1}L_{n-1}^{-1}L_1^{-1} \cdots L_1^{-T}L_2^{-T} \cdots L_{n-2}^{-T}Y_{n-1}L_{n-1}^{-1}L_1^{-1} $$

and hence $Y$ is a symmetric matrix. Further we will show explicitly the structures of $L_i$ and $Y_i$ which are needed for defining the $i$-th step of the algorithm. For the sake of simplicity, the indices of the matrices $Y_i$ will be omitted.
Let us put

\[ Y = \alpha vv^T = \alpha \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & \bar{v}v^T & \ldots & 0 \end{bmatrix} , \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{v} \end{bmatrix} , \quad \bar{v} = \begin{bmatrix} v_{i+1} \\ \vdots \\ v_n \end{bmatrix} , \]

(2.5)

\[ L' = \begin{bmatrix} I_i & 0 \\ 0 & \bar{L} \end{bmatrix} , \quad \bar{L} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ w_{i+2} & 1 & 0 & \ldots & 0 \\ w_{i+3} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_n & 0 & 0 & \ldots & 1 \end{bmatrix} , \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_{i+1} \\ w_{i+2} \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w_{i+2} \\ w_n \end{bmatrix} , \]

where \( \alpha \) is a scalar, \( v \) and \( w \) are vector-columns. If the \( i \)-th column of \( S_{21} \) is already upper triangular, we put \( Y = 0 \). The matrix \( Y \) is symmetric.

Let us denote

\[ S_{11} = [r_1, \ldots, r_n] = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix} , \quad S_{21} = [t_1, \ldots, t_n] = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{bmatrix} . \]

The \( i \)-th column of the matrix \( S'_{21} \) is upper triangular, if and only if

\[ t_i - \alpha vv^T r_i = \begin{bmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} , \]

i.e. the last \( n - i \) elements of this vector should be equal to zero. If

\[ \sum_{j=i+1}^{n} t_{ji}r_{ji} \neq 0 , \]

(2.6)

the solution of this system of \( \alpha \) and \( v \) yields to

\[ v_j = t_{ji} , \quad (i + 1 \leq j \leq n) \quad \text{and} \quad \alpha = \frac{1}{v_i^T r_i} = \frac{1}{\sum_{j=i+1}^{n} t_{ji}r_{ji}} . \]

The vector \( w \) which takes part in the construction of the matrix \( L' \), will be determined from the condition that the \( i \)-th column of the matrix \( L^{-1} S_{11} L \) be Hessenberg. It will be done as follows. If that column is already Hessenberg, we put \( L = I \). If it is not Hessenberg, then among the elements \( r_{i+1,i}, r_{i+2,i}, \ldots, r_{ni} \) we choose that one of maximal module. Let
Let $r_{ki}$ be such an element. For such $k$ we do a similar transformation of the matrix $S_{11}$ which consists of interchanging the $k$-th with the $(i+1)$-th row and also the $k$-th with the $(i+1)$-th column of $S_{11}$. Thus the maximal module element will come on position $(i+1,i)$. For the sake of simplicity, the obtained matrix will also be denoted by $S_{11}$, and the matrix $L'$ which will be obtained further should be multiplied from left with the permutation matrix in order to obtain the matrix $L$. The permutation matrix on $S$ belongs to the group $G$.

The $i$-th column of $L^{-1}S_{11}L$ and hence of $S'_{11}$ is Hessenberg if and only if:

$$
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-w_{i+2} & 1 & 0 & \cdots & 0 \\
-w_{i+3} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-w_n & 0 & 0 & \cdots & 1
\end{bmatrix} \begin{bmatrix} r_{i+1,i} \\ r_{i+2,i} \\ s \\ r_{ni} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
$$

Hence the solution for the vector $w$ is given by

$$w_{i+2} = \frac{r_{i+2,i}}{r_{i+1,i}}, \ldots, w_n = \frac{r_{ni}}{r_{i+1,i}}.$$
it follows that the first row and the first column of the matrix $Y$ are zero.

We have proved the following theorem:

**Theorem 2.1.** Let be given a matrix $S$ by (2.1). Applying the similar transformations on $S$ by the matrices $U_i$, $i = 1, \ldots, n - 1$ (2.3), under the conditions (2.6), we get the global transformation matrix $U$ (2.4) with matrix $Y$ in which the first row and column are equal to zero, and the transformed matrix $S'$ be such that $S'_{11}$ is an upper Hessenberg matrix and $S'_{21}$ is an triangular matrix. ■

**Remark.** One feature of the presented algorithm is that it contains only linear algebraic operations: adding, subtracting, multiplying and dividing. Thus the theorem is true also if the elements of the matrix $S$ are elements of larger set. For example it can be the field of complex numbers or the ring of the analytical functions of more variables. However, if the elements of the matrix $S$ are real numbers, from numerical viewpoint it is more convenient if the matrix $L$ is orthogonal. Therefore, we define the following algorithm.

For arbitrary given matrix $S$, we shall construct the matrix:

$$U = \begin{bmatrix} Q & 0 \\ YQ & Q \end{bmatrix}, \quad Q^TQ = I, \quad Y^T = Y$$

such that

$$\begin{bmatrix} Q^T & 0 \\ -Q^TY & Q^T \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} Q & 0 \\ YQ & Q \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix},$$

where $S'_{11}$ is an upper Hessenberg matrix and $S'_{21}$ is an upper triangular matrix, and the first row and column of $Y$ are zeros.

It verifies that the set of the matrices $U$ with respect to the matrix multiplication is a group. We apply $n - 1$ such similar transformations to $S$ determined by

$$U_i = \begin{bmatrix} Q_i & 0 \\ Y_i Q_i & Q_i \end{bmatrix}, \quad Q_i^T Q_i = I, \quad Y_i^T = Y_i, \quad (1 \leq i \leq n - 1).$$

By the first transformation, the first column of the obtained matrix $S_{21}$ will become upper triangular, and the first column of the obtained matrix $S_{11}$ will become Hessenberg. Generally, applying the $i$-th transformation, the $i$-th column of the obtained matrix $S_{21}$ will become upper triangular such that the first $i - 1$ columns will remain upper triangular and the $i$-th column of the obtained matrix $S_{11}$ will become Hessenberg such that the first $i - 1$ columns of that matrix will remain Hessenberg.
The structures of the matrices $Y_i$ are the same as in the proof of Theorem 2.1, and the matrices $Q_i$ without indices, for simplicity, are given by

$$Q = I_n - 2ww^T = \begin{bmatrix} I_i & 0 \\ 0 & I - 2\bar{w}\bar{w}^T \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_i \\ w_{i+1} \\ \vdots \\ w_n \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} w_{i+1} \\ \vdots \\ w_n \end{bmatrix},$$

$$w^T w = \bar{w}^T \bar{w} = 1,$$

where $w$ is a vector-column whose first $i$ elements are zeros. The matrix $Q$ is Householder’s 

The vector $w$ which takes part in the construction of the orthogonal matrix $Q$, will be determined such that the $i$-th column of the matrix $Q^T S_{11} Q$ to be Hessenberg, i.e.

$$\begin{bmatrix} r_{i+1,i} \\ r_{i+2,i} \\ \vdots \\ r_{ni} \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (2.10)$$

The system (2.10) together with the orthogonality condition (2.9) has the following solution for $w$:

$$w_{i+1} = \beta(r_{i+1,i} + s)$$

$$w_{i+2} = \beta r_{i+2,i}$$

$$\vdots$$

$$w_n = \beta r_{ni}$$

where

$$\beta = [2(s^2 + r_{i+1,i}s)]^{-1/2}, \quad s = [r_{i+1,i}^2 + r_{i+2,i}^2 + s + r_{ni}^2]^{1/2}.$$ 

We should prove that the first $i-1$ columns of the new obtained matrix $S'_{21}$ remain upper triangular after the $i$-th transformation, assuming that the first $i-1$ columns of the matrix $S_{21}$ are upper triangular. We have

$$S'_{21} = Q^T S_{21} Q + \alpha Q^T S_{22} v v^T Q - \alpha Q^T v v^T Q Q^T S_{11} Q - \alpha^2 v^T S_{12} v Q^T v v^T Q.$$

Since the first $i$ elements of the vector-row $v^T Q$ are zeros, it follows that the first $i$ columns of the second and the fourth matrices in the previous sum are zeros. The first $i-1$ columns of the matrix $Q^T S_{11} Q$ are Hessenberg according to the inductive assumption, and the $i$-th column was made Hessenberg using the vector $\bar{w}$. Thus the first $i-1$ elements of the following vector-row

$$v^T Q Q^T S_{11} Q$$
are zeros. Consequently, the first \( i - 1 \) columns of the matrix \( S'_{21} \) will be also upper triangular. The \( i \)-th column of that matrix was done upper triangular using \( \alpha \) and \( \nu \).

The first row and column of the matrix \( Y \) are zero, from the same reasons as in the proof of Theorem 2.1.

We have proved the following theorem.

**Theorem 2.2.** Let be given a matrix \( S \) with (2.1). Applying the similar transformations on \( S \) by the matrices \( U_i \), \( i = 1, \ldots, n - 1 \) (2.8), under the conditions of finiteness of \( \beta \) in each step, we get the global transformation matrix \( U \) (2.7) with matrix \( Y \) in which the first row and column are equal to zero, and the transformed matrix \( S' \) be such that \( S'_{11} \) is upper Hessenberg matrix and \( S'_{21} \) is upper triangular matrix. ■

As a consequence from these results follow two corollaries for the Hamiltonian and skew-Hamiltonian matrices.

**Corollary 2.3.** Let the matrix \( S \) be a Hamiltonian matrix. Applying \( n - 1 \) similar transformations (2.3) of the matrix \( S \) under the conditions (2.6), we get a matrix \( S' \), such that the left lower block of dimension \( n \times n \) is a diagonal matrix and the upper left block of dimension \( n \times n \) is of Hessenberg form.

**Proof.** It follows from (2.2) that the similar matrix \( S' = U^{-1}SU \) also has the Hamiltonian form, if \( U \) is any matrix of the form given in Theorems 2.1 and 2.2. The rest of the proof is obvious. ■

The second corollary is about the ARME.

**Corollary 2.4.** Let \( S \) be a skew-Hamiltonian matrix. Applying \( n - 1 \) similar transformations (2.3), under the condition (2.6), the matrix \( S \) can be reduced in a form such that lower left block of dimension \( n \times n \) is zero, the upper left block of dimension \( n \times n \) has a Hessenberg form. If the final similar transformation is given by the matrix \( U \), i.e.

\[
U = \begin{bmatrix} L & 0 \\ YL & L^{-T} \end{bmatrix} \quad \text{or} \quad U = \begin{bmatrix} Q & 0 \\ YQ & Q \end{bmatrix},
\]

then the matrix \( Y \) is a symmetric solution of the ARME

\[
-Y S_{12} Y + S_{22} Y - Y S_{11} + S_{21} = 0,
\]

such that its first row and column are zeros.

**Proof.** Using the similar transformation \( S' = U^{-1}SU \), where \( U \) is any of the matrices given in Theorems 2.1 and 2.2, it is obtained skew-Hamiltonian matrix \( S' \). The rest of the proof is obvious. ■

**Remark.** Breaking down the algorithm happens when there is no solution of ARME or there is no solution of ARME with first row and column equal to zero.

**Remark.** If the algorithm breaks down or if it is requested a solution of ARME which is non-singular matrix, or the matrix with given first row and column, we could introduce a new unknown symmetric matrix \( X \) by \( Y = M + N^{-T}XN^{-1} \), where \( M \) is a given symmetric
matrix and $N$ is a given non-singular matrix. The new equation of $X$ is also of the ARME type:

$$-XS'_{12}X - XS'_{11} + S'_{22}X + S'_{21} = 0,$$

where

$$S'_{11} = N^{-1}(S_{11} + S_{12}M)N = S'_{22}^T,$$
$$S'_{12} = N^{-1}S_{12}N^{-T} = -S'_{12}^T,$$
$$S'_{21} = N^T(S_{21} + S_{22}M - MS_{11} - MS_{12}M)N = -S'_{21}^T,$$
$$S'_{22} = N^T(S_{22} - MS_{12})N^{-T} = S'_{11}^T$$

(compare with (2.2).)

**Example.** In this example it is $n = 6$ and the matrix $S$ is skew-Hamiltonian with randomly chosen numbers from $(0,1)$ and symmetrically from $(-1,0)$. The beginning matrix $S$ is

$$S_{11} = \begin{bmatrix}
0.001251 & 0.563585 & 0.193304 & 0.808740 & 0.585009 & 0.479873 \\
0.350291 & 0.895962 & 0.822840 & 0.746605 & 0.174108 & 0.858943 \\
0.710501 & 0.513535 & 0.303995 & 0.014985 & 0.091403 & 0.364452 \\
0.147313 & 0.165899 & 0.988525 & 0.445692 & 0.119083 & 0.004669 \\
0.008911 & 0.377880 & 0.531663 & 0.571184 & 0.601764 & 0.607166 \\
0.166234 & 0.663045 & 0.450789 & 0.352123 & 0.057039 & 0.607685 \\
\end{bmatrix}, \quad S_{22} = S_{11}^T,$$

$$S_{12} = \begin{bmatrix}
0.000000 & 0.783319 & 0.802606 & 0.519883 & 0.301950 & 0.875973 \\
-0.783319 & 0.000000 & 0.726676 & 0.955901 & 0.925718 & 0.539354 \\
-0.802606 & -0.726676 & 0.000000 & 0.142338 & 0.462081 & 0.235328 \\
-0.519883 & -0.955901 & -0.142338 & 0.000000 & 0.862239 & 0.209601 \\
-0.301950 & -0.925718 & -0.462081 & -0.862239 & 0.000000 & 0.779656 \\
-0.875973 & -0.539354 & -0.235328 & -0.209601 & -0.779656 & 0.000000 \\
\end{bmatrix},$$

$$S_{21} = \begin{bmatrix}
0.000000 & 0.843654 & 0.996796 & 0.999695 & 0.611499 & 0.392438 \\
-0.843654 & 0.000000 & 0.266213 & 0.297281 & 0.840144 & 0.023743 \\
-0.996796 & -0.266213 & 0.000000 & 0.375866 & 0.092624 & 0.677206 \\
-0.999695 & -0.297281 & -0.375866 & 0.000000 & 0.056215 & 0.008789 \\
-0.611499 & -0.840144 & -0.092624 & -0.056215 & 0.000000 & 0.918790 \\
-0.392438 & -0.023743 & -0.677206 & -0.008789 & -0.918790 & 0.000000 \\
\end{bmatrix},$$

After the similar transformation (2.2), the matrix $S$ becomes

$$S_{11} = \begin{bmatrix}
0.001252 & 2.394542 & 0.628127 & -1.333712 & 1.170301 & -2.988671 \\
-0.822757 & 0.705668 & -0.064203 & -0.380534 & 0.078576 & -2.116390 \\
0.000000 & 4.483943 & 1.443984 & 3.651134 & -1.359573 & -5.058558 \\
0.000000 & -0.000000 & -0.232568 & -1.192501 & 0.923134 & -0.061005 \\
-0.000000 & -0.000000 & 0.000000 & -2.882063 & 1.947789 & -0.289124 \\
-0.000000 & -0.000000 & 0.000000 & -0.000000 & -0.112915 & -0.049842 \\
\end{bmatrix},$$

$$S_{22} = S_{11}^T.$$
with multiple application of the algorithm. For example, one could try to find the reasons for that behavior and to find an improvement of the presented algorithm. 

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