A Wave-Packet View of Neutrino Oscillation and Pion Decay

Chikara Fuji

Department of General Education, Junior College Funabashi Campus
Nihon University, Funabashi 274-8501, Japan

Yasumasa Matsuura, Toshihiro Shibuya and S.Y.Tsai

Institute of Quantum Science and Department of Physics
College of Science and Technology, Nihon University,
Tokyo 101-8308, Japan

Abstract

Kinematical aspects of pion decay $\pi \rightarrow \mu \nu$ is studied, with neutrino mixing taken into account. An attempt is made to derive the transition probability for such a sequence of processes: a $\pi^+$ produced at $(\vec{x}_\pi, t_\pi)$ with momentum $\vec{p}_\pi$ decays into a $\mu^+$ and a $\nu_\mu$ somewhere in space-time and then the $\mu^+$ is detected at $(\vec{x}_\mu, t_\mu)$ with momentum $\vec{p}_\mu$ and a $\nu_\alpha$ (a neutrino with flavor $\alpha = e, \mu, \cdots$) is detected at $(\vec{x}_\nu, t_\nu)$ with momentum $\vec{p}_\nu$. It is shown that

1. if all the particles involved are treated as plane-waves, that is, if each particle is assumed to possess a strictly fixed momentum, the energy-momentum conservation would eliminate the neutrino oscillating terms, leaving each mass-eigenstate to contribute separately to the transition probability;

2. if one, taking into account that the momenta relevant may not be free from some uncertainty (or dispersion), treats all the particles involved as wave-packets, the neutrino oscillating terms would appear and would be multiplied by two suppression factors, which result from distinction in velocity and in energy between the two interfering neutrino mass-eigenstates;

3. as $\sigma^2 \equiv \sigma_\pi^2 + \sigma_\mu^2 + \sigma_\nu^2, \sigma_\pi, \sigma_\mu$ and $\sigma_\nu$ being uncertainty associated respectively with $\vec{p}_\pi, \vec{p}_\mu$ and $\vec{p}_\nu$, becomes larger (smaller), the feature that each of the particles involved propagates along its classical trajectory (energies and momenta of the particles involved are conserved during the decay) becomes more prominent; and

4. in the limit of $\sigma^2 \rightarrow \infty$, the oscillating terms would again be suppressed away.

An approximate treatment which takes account of the two complementary features mentioned above is proposed and similarity and difference between our approach and that of Dolgov et al. are discussed.

*E-mail address: t-fuji@phys.ge.cst.nihon-u.ac.jp
†E-mail address: tsai@phys.cst.nihon-u.ac.jp
1. Introduction

Neutrino oscillation\[1\] is now one of the most exciting topics in particle physics, in non-accelerator as well as accelerator high energy physics and in astrophysics\[2\]. There are now plenty of evidence in favor of presence of neutrino oscillations from atmospheric neutrino experiments, solar neutrino experiments, reactor neutrino experiments and accelerator neutrino experiments\[3\].

The standard formulas used to analyze the probability of neutrino oscillation between two flavor-eigenstates, e.g., $\nu_e$ and $\nu_\mu$, read

$$P_{\nu_\mu \rightarrow \nu_\mu} = P_{\nu_e \rightarrow \nu_e} = 1 - \sin^2(2\theta) \sin^2\left(\frac{m_1^2 - m_2^2}{4E_\nu} D\right),$$
$$P_{\nu_\mu \rightarrow \nu_e} = P_{\nu_e \rightarrow \nu_\mu} = \sin^2(2\theta) \sin^2\left(\frac{m_1^2 - m_2^2}{4E_\nu} D\right),$$

(1.1)

where $E_\nu$ and $D$ are the energy and traveling distance of the neutrinos, $m_1$ and $m_2$ are the masses of two interfering mass-eigenstates, and $\theta$ is the mixing angle.

$l_{1\leftrightarrow 2} \equiv \frac{4\pi E_\nu}{|m_1^2 - m_2^2|}$

(1.2)

is often referred to as the oscillation length. These formulas are usually derived on the basis of a plane-wave approach and without paying particular attention to how neutrinos are created/detected.

As is well known, the main physics involved in neutrino oscillation is quantum mechanics and this aspect of neutrino oscillation has widely been discussed in the literature\[4\] since the pioneering work by Kayser\[5\]. There are also not a few papers which treat neutrino oscillation field-theoretically\[6\]. In particular, a field-theoretical approach of neutrino oscillation with both production process and detection process taken into account were recently developed by Giunti et al. in \[7\] and by Asahara et al. in \[8\]. In these works, the neutrino is treated as an intermediate particle, while all the external particles are treated as wave-packets.

We also have, in a series of papers\[9\]∼\[12\], developed wave-packet treatments of neutrino oscillation and addressed ourselves, in particular, to such questions as "How do neutrinos propagate?", "Equal energy or equal momentum or else?" and "How and why does the factor-of-two paradox arise?". In the present paper, after extending our previous treatment given in \[11\] to a three-dimensional case, we shall go one step further to study pion decay $\pi \rightarrow \mu\nu$, with neutrino mixing taken into account and with emphasis placed on formal structure of its transition probability.

$\pi \rightarrow \mu\nu$ decay with neutrino oscillation taken into account has been investigated with emphasis placed on such a question as "Do muons oscillate?" by Dolgov et al. in \[13\]. In their treatment, $(\vec{x}_0, t_0)$, space-time point where and when a pion decay occurs, is specified, each of the particles involved is supposed to propagate along its classical trajectory, and energy-momentum conservation is imposed by hand.\[1\] They

\[1\] They, however, at the same time mention that, in quantum field theory, one includes integration over space-time point of decay in the definition of the relevant amplitude, and this integration leads to the conservation law.
first treat all the particles as plane-waves and then introduce momentum distribution for the pion alone.

The present work is more or less stimulated by the interesting work by Dolgov et al.\[13\] and is organized as follows. In Sec.2 and in Appendix A, we focus on neutrino oscillation without paying attention to how the neutrino is created/detected. The neutrinos are treated as wave-packets and a number of comments related to those questions mentioned above are given. Pion decay with neutrino mixing taken into account is investigated in Sec.3 and Sec.4. In Sec.3, we define and derive the transition amplitude and probability for a sequence of processes $\pi^+ \rightarrow \mu^+ \nu_\mu \rightarrow \mu^+ \nu_\alpha$ with all the particles involved treated as wave-packets. The space-time point of creation of $\pi^+$ and of detection of $\mu^+$ and $\nu_\alpha$ is specified, while the space-time point of decay $(\vec{x}_0, t_0)$ is integrated out. The plane-wave limit and the total transition probability are also examined. In Sec.4, we propose an approximate treatment which takes account of the two complementary features: energies and momenta are conserved on the one hand and each of the particles propagates along its classical trajectory on the other hand. Section 5 compares our approach with that of Dolgov et al. and Section 6 gives some concluding remarks. In Appendix B, some algebra relevant to Sec.4 is given.

2. Three-dimensional wave-packet treatment of neutrino oscillation

Let $|\nu_\alpha\rangle$ ($\alpha = e, \mu, \cdots$) represent neutrino states associated with the electron, muon, $\cdots$, which are superpositions of the mass-eigenstates $|\nu_k\rangle$ having mass $m_k$ ($k = 1, 2, \cdots$):

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k} |\nu_k\rangle. \tag{2.1}$$

Suppose that a $\nu_\mu$ of momentum $\vec{p}_\nu$ with uncertainty (or dispersion) $\sigma_\nu$ is created at space-time point $(\vec{x}_0, t_0)$. Then, its state vector at $(\vec{x}_\nu, t_\nu)$ may be written as

$$|\nu_\mu(\vec{x}_\nu, t_\nu; \vec{x}_0, t_0; \vec{p}_\nu, \sigma_\nu)\rangle = \sum_k U_{\mu k} |\nu_k\rangle \phi_k(\vec{x}_{\nu 0}, t_{\nu 0}; \vec{p}_\nu, \sigma_\nu), \tag{2.2}$$

$$\phi_k(\vec{x}, t; \vec{p}_\nu, \sigma_\nu) = \left(\frac{\sigma_\nu}{\sqrt{\pi}}\right)^{3/2} \exp\left\{i(\vec{p}_\nu \vec{x} - E_k(\vec{p}_\nu)t) - \frac{1}{2}\sigma_\nu^2(\vec{x} - \vec{v}_k(\vec{p}_\nu)t)^2\right\}, \tag{2.3}$$

where

$$\vec{x}_{\nu 0} = \vec{x}_\nu - \vec{x}_0, \quad t_{\nu 0} = t_\nu - t_0,$$

$$E_k(\vec{p}) = \sqrt{\vec{p}^2 + m_k^2}, \quad \vec{v}_k(\vec{p}) = \frac{dE_k(\vec{p})}{d\vec{p}} = \frac{\vec{p}}{E_k(\vec{p})}.$$

2 We shall assume $m_k \neq m_l$ for $k \neq l$. Also, since we are not interested in CP or T violation, we shall take $U = (U_{\alpha k})$ to be a real orthogonal matrix. In the two-generation case, $U_{\alpha k}$ may be expressed in terms of a single mixing angle $\theta$: $U_{e1} = U_{\mu 2} = \cos \theta$, $U_{e2} = -U_{\mu 1} = \sin \theta$. 

3
As is well known\(^\text{[14]}\), the wave function of the form given by Eq.(2.3) follows readily, if one

1. starts from a superposition of plane-wave functions\(^3\)

\[
\phi_k(\vec{x}, t; \vec{p}_\nu, \sigma_\nu) = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^{3/2}} f(\vec{p}; \vec{p}_\nu, \sigma_\nu) \exp\{i(\vec{p}\vec{x} - E_k(\vec{p})t)\}; \quad (2.4)
\]

2. takes, as the momentum distribution function \(f(\vec{p}; \vec{p}_\nu, \sigma_\nu)\),

\[
f(\vec{p}; \vec{p}_\nu, \sigma_\nu) = \frac{1}{(\sqrt{\pi}\sigma_\nu)^{3/2}} \exp\{-\frac{(\vec{p} - \vec{p}_\nu)^2}{2\sigma_\nu^2}\}; \quad (2.5)
\]

3. expands \(E_k(\vec{p})\) around \(\vec{p}_\nu\) as

\[
E_k(\vec{p}) = E_k(\vec{p}_\nu) + \vec{v}_k(\vec{p}_\nu)(\vec{p} - \vec{p}_\nu); \quad (2.6)
\]

4. and performs the \(\vec{p}\)-integration involved in Eq.(2.4) explicitly.

Note that we have normalized \(f(\vec{p}; \vec{p}_\nu, \sigma_\nu)\) and \(\phi_k(\vec{x}, t; \vec{p}_\nu, \sigma_\nu)\) as

\[
\int_{-\infty}^{\infty} d^3p |f(\vec{p}; \vec{p}_\nu, \sigma_\nu)|^2 = \int_{-\infty}^{\infty} d^3x |\phi_k(\vec{x}, t; \vec{p}_\nu, \sigma_\nu)|^2 = 1, \quad (2.7)
\]

and that, to keep this normalization condition valid, we shall define and express the plane-wave limit of \(\phi_k(\vec{x}, t; \vec{p}_\nu, \sigma_\nu)\) as

\[
\lim_{\sigma_\nu \to 0} \phi_k(\vec{x}, t; \vec{p}_\nu, \sigma_\nu) = \frac{1}{V^{1/2}} \exp\{i(\vec{p}_\nu \vec{x} - E_k(\vec{p}_\nu)t)\}, \quad (2.8)
\]

that is, letting \(\sigma_\nu \to 0\) in the exponent and, at the same time, on confining the neutrino plane-waves within a spatial cube of volume \(V\), letting the normalization constant \(N_\nu \equiv (\sigma_\nu/\sqrt{\pi})^{3/2} \to 1/V^{1/2}\)\(^4\).

The amplitude for a \(\nu_\mu\) created at \((\vec{x}_0, t_0)\) to be detected as a \(\nu_\alpha\) at \((\vec{x}_\nu, t_\nu)\) is calculated as

\[
A_{\nu_\mu \to \nu_\alpha}(\vec{x}_\nu, t_\nu | \vec{x}_0, t_0) = \langle \nu_\alpha | \nu_\mu(\vec{x}_\nu, t_\nu; \vec{x}_0, t_0; \vec{p}_\nu, \sigma_\nu) \rangle
\]

\[
= \sum_k U_{\mu k} U_{\alpha k} \phi_k(\vec{x}_\nu, t_\nu; \vec{x}_0, t_0; \vec{p}_\nu, \sigma_\nu)
\]

\[
= N_\nu \sum_k U_{\mu k} U_{\alpha k} \exp\{i(\vec{p}_\nu \vec{x}_\nu - E_k(\vec{p}_\nu)t_\nu) - \frac{1}{2} \sigma_\nu^2(\vec{x}_\nu - \vec{v}_k(\vec{p}_\nu)t_\nu)^2\}, \quad (2.9)
\]

\(^3\)Neglecting spin degree of freedom, we shall assume that the neutrinos (and the charged leptons as well) obey the Klein-Gordon equation.

\(^4\) From the uncertainty relation, one may interpret \(V_{\text{wave–packet}} \equiv (\sqrt{\pi}/\sigma_\nu)^3\) as the volume of a spatial region within which the neutrino wave-packets are appreciable and express \(N_\nu \to 1/V^{1/2}\) as \(V_{\text{wave–packet}} \to V\).
and the corresponding propability is calculated as
\[ P_{\nu \rightarrow \nu}(\vec{x}_\nu, t_\nu) = \left| A_{\nu \rightarrow \nu}(\vec{x}_\nu, t_\nu) \right|^2 \]
\[ = \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \phi_k(\vec{x}_\nu, t_\nu; \vec{p}_\nu, \sigma_\nu) \phi_l^*(\vec{x}_\nu, t_\nu; \vec{p}_\nu, \sigma_\nu) \]
\[ = N_\nu^2 \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \exp\left\{-i E_{[k]}(\vec{p}_\nu) t_\nu \right\} \]
\[ \exp\left\{-\frac{1}{2} \sigma_\nu^2 (\vec{x}_\nu - \vec{v}_k t_\nu)^2 - \frac{1}{2} \sigma_\nu^2 (\vec{x}_\nu - \vec{v}_l t_\nu)^2 \right\} \]
\[ = N_\nu^2 \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \cos(E_{[k]}(\vec{p}_\nu) t_\nu) \]
\[ \exp\left\{-\sigma_\nu^2 (\vec{x}_\nu - \vec{v}_{[k]}(\vec{p}_\nu)) t_\nu \right\} \exp\left\{-\frac{1}{4} \sigma_\nu^2 (\vec{v}_{[k]}(\vec{p}_\nu))^2 t_\nu^2 \right\}, \quad (2.10) \]
where
\[ E_{[k]}(\vec{p}_\nu) = \frac{1}{2}(E_k(\vec{p}_\nu) + E_l(\vec{p}_\nu)), \quad E_{[kl]}(\vec{p}_\nu) = E_k(\vec{p}_\nu) - E_l(\vec{p}_\nu), \]
\[ \vec{v}_{[kl]}(\vec{p}_\nu) = \frac{1}{2}(\vec{v}_k(\vec{p}_\nu) + \vec{v}_l(\vec{p}_\nu)), \quad \vec{v}_{[k]}(\vec{p}_\nu) = \vec{v}_k(\vec{p}_\nu) - \vec{v}_l(\vec{p}_\nu). \]

With Eq.(2.7), we have
\[ \sum_{\alpha} \int d^3 x_\nu P_{\nu \rightarrow \nu}(\vec{x}_\nu, t_\nu) = 1. \quad (2.11) \]

A couple of remarks are in order as regards Eq.(2.10).

1. Each term in Eq.(2.10) contains explicitly a factor which implies that the space-time point \((\vec{x}_\nu, t_\nu)\) where and when the neutrinos are detected nearly satisfies
\[ \vec{x}_\nu = \vec{x}_\nu + \vec{v}_{[k]}(\vec{p}_\nu)(t_\nu - t_0), \quad (2.12) \]
which is precisely what could be referred to as classical trajectory of the neutrinos. We may therefore, assuming that \((\vec{x}_\nu + \vec{v}_{[k]}(\vec{p}_\nu)) t_\nu \approx 0\) is satisfied for any \(k\) and \(l\), express Eq.(2.10) approximately as
\[ P_{\nu \rightarrow \nu}(\vec{x}_\nu, t_\nu) \]
\[ \approx N_\nu^2 \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \cos(E_{[k]}(\vec{p}_\nu) t_\nu) \exp\left\{-\frac{1}{4} \sigma_\nu^2 (\vec{v}_{[k]}(\vec{p}_\nu))^2 t_\nu^2 \right\} \]
\[ = N_\nu^2 \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \cos(2\pi \frac{D}{l_{k+l}}) \exp\left\{-\frac{1}{4} \sigma_\nu^2 (\vec{v}_{[k]}(\vec{p}_\nu))^2 D^2 \right\}, \quad (2.13) \]
where \(D \equiv |\vec{x}_\nu| = |\vec{v}_{[k]}(\vec{p}_\nu)| t_\nu\) is the traveling distance of the neutrinos, and
\[ l_{k \leftrightarrow l} = \frac{2\pi |\vec{v}_{[k]}(\vec{p}_\nu)|}{|E_{[k]}(\vec{p}_\nu)|} = \frac{4\pi E_{[k]}(\vec{p}_\nu)|\vec{v}_{[k]}(\vec{p}_\nu)|}{|m_k^2 - m_l^2|}. \quad (2.14) \]
2. In each of interference terms, the oscillating factor \( \cos(E_{kl}(\vec{p}_\nu)t_{\nu 0}) \) is multiplied by a factor, denoted by \( \xi_{kl} \), which implies that this term is significant only when the condition

\[
t_{\nu} - t_0 \lesssim \frac{2}{\sigma_\nu |\vec{v}_{kl}(\vec{p}_\nu)|} \quad \text{or} \quad D \lesssim \frac{2|\vec{v}_{kl}(\vec{p}_\nu)|}{\sigma_\nu |\vec{v}_{kl}(\vec{p}_\nu)|},
\]

is satisfied. This effect, often referred to as coherent condition for neutrino oscillation\[15\], manifests itself explicitly in our wave-packet treatment but not in conventional plane-wave treatments. Note also that \( \vec{v}_k(\vec{p}_\nu) \neq \vec{v}_l(\vec{p}_\nu) \) for \( k \neq l \) only results in appearance of this effect, but by no means affects the oscillating factor and hence the oscillation period \( 2\pi/E_{kl}(\vec{p}_\nu) \) or oscillation length \( l_{k \to l} \).

5. If, noting that the magnitude of the amplitude \( A_{\nu_\mu \rightarrow \nu_\alpha}(\vec{x}_0, t_0) \), Eq.(2.9), has a peak at \( \vec{x}_{\nu 0} = \vec{v}_k(\vec{p}_\nu)t_{\nu 0} \), one substitutes this into its phase factor to obtain \( \exp\{-i(m_k^2/E_k(\vec{p}_\nu))t_{\nu 0}\} \), the phase factor of the probability (2.10) would become \( \exp\{-i(m_k^2/E_k(\vec{p}_\nu))t_{\nu 0} - m_l^2/E_l(\vec{p}_\nu))t_{\nu 0}\} \), which will result in an oscillation period nearly twice as large as the standard one for relativistic neutrinos. This is where the so-called "factor-of-two paradox"\[16\] arises and is obviously an inadequate prescription.

6. We shall give a more general three-dimensional wave-packet treatment of neutrino oscillation in Appendix A. Note that, for relativistic neutrinos and in the two-generation case, Equations (2.13) and (2.14) coincide (apart from the normalization factor) with Eqs.(1.1) and (1.2), if one let \( \sigma_\nu \rightarrow 0 \) and identifies \( E_{(kl)}(\vec{p}_\nu) \) with \( E_\nu \).
propagate to its detection point \((\vec{x}_\mu, t_\mu)\) and the amplitude for a \(\pi^+\) produced at \((\vec{x}_\pi, t_\pi)\) to propagate to its decay point may be described respectively by\(^7\)

\[
A_\mu(\vec{x}_\mu, t_\mu) = \phi_\mu(\vec{x}_\mu, t_\mu; \vec{p}_\mu, \sigma_\mu) = \int \frac{d^3p}{(2\pi)^3/2} f(\vec{p}, \vec{p}_\mu, \sigma_\mu) \exp\{i(\vec{p}\vec{x}_\mu - E_\mu(\vec{p})t_\mu)\}
\]

\[
= N_\mu \exp\{i(\vec{p}_\mu\vec{x}_\mu - E_\mu t_\mu) - \frac{1}{2}\sigma_\mu^2(\vec{x}_\mu - \vec{v}_\mu t_\mu)^2\},
\]

\[
A_\pi(\vec{x}_\pi, t_0) = \phi_\pi(\vec{x}_\pi, t_0; \vec{p}_\pi, \sigma_\pi) = \phi_\pi^*(\vec{x}_\pi, t_0; \vec{p}_\pi, \sigma_\pi) = \int \frac{d^3p}{(2\pi)^3/2} f(\vec{p}, \vec{p}_\pi, \sigma_\pi) \exp\{-i(\vec{p}\vec{x}_\pi - E_\pi(\vec{p})t_\pi)\}
\]

\[
= N_\pi \exp\{-i(\vec{p}_\pi\vec{x}_\pi - E_\pi t_\pi) - \frac{1}{2}\sigma_\pi^2(\vec{x}_\pi - \vec{v}_\pi t_\pi)^2\},
\] (3.1)

where \(\sigma_\mu\) and \(\sigma_\pi\) are uncertainties associated respectively with \(\vec{p}_\mu\) and \(\vec{p}_\pi\) and

\[
N_\lambda = \left(\frac{\sigma_\lambda}{\sqrt{\pi}}\right)^{3/2}, \quad \vec{x}_\lambda = \vec{x}_\lambda - \vec{x}_0, \quad \mu_\lambda = \mu - t_0,
\]

\[
E_\lambda = \sqrt{\vec{p}_\lambda^2 + \mu_\lambda^2}, \quad \vec{v}_\lambda = \frac{\vec{p}_\lambda}{E_\lambda}, \quad \lambda = \mu, \pi.
\]

The transition amplitude for \(\pi^+ \to \mu^+\nu_\mu \to \mu^+\nu_\alpha\) with the decay space-time point \((\vec{x}_0, t_0)\) specified is then given by

\[
A_{\pi \to \mu \nu_\mu \to \mu \nu_\alpha}(\vec{x}_0, t_0) = A_{\nu_\mu \to \nu_\alpha}(\vec{x}_0, t_0) A_{\mu}(\vec{x}_0, t_\mu) A_{\pi}(\vec{x}_0, t_\pi) = \sum_k U_{\mu k} U_{\alpha k} A_k(\vec{x}_0, t_0), \quad \text{(3.2)}
\]

where

\[
A_k(\vec{x}_0, t_0) = \phi_k(\vec{x}_\mu, t_\mu; \vec{p}_\nu, \sigma_\nu) \phi_\mu(\vec{x}_\mu, t_\mu; \vec{p}_\mu, \sigma_\mu) \phi_\pi^*(\vec{x}_\pi, t_\pi; \vec{p}_\pi, \sigma_\pi)
\]

\[
= N_\nu \exp\{i(\vec{p}_\nu\vec{x}_\nu - E_k t_\nu) - \frac{1}{2}\sigma_\nu^2(\vec{x}_\nu - \vec{v}_k t_\nu)^2\},
\]

\[
N_\mu \exp\{i(\vec{p}_\mu\vec{x}_\mu - E_\mu t_\mu) - \frac{1}{2}\sigma_\mu^2(\vec{x}_\mu - \vec{v}_\mu t_\mu)^2\}
\]

\[
N_\pi \exp\{-i(\vec{p}_\pi\vec{x}_\pi - E_\pi t_\pi) - \frac{1}{2}\sigma_\pi^2(\vec{x}_\pi - \vec{v}_\pi t_\pi)^2\}. \quad \text{(3.3)}
\]

The transition amplitude for the whole process we have specified in the beginning of this subsection may be obtained by integrating Eq.(3.2) with respect to \(\vec{x}_0\) and \(t_0\)\(^8\)

\[
A_{\pi \to \mu \nu_\mu \to \mu \nu_\alpha} = \int d^3\vec{x}_0 \int dt_0 A_{\pi \to \mu \nu_\mu \to \mu \nu_\alpha}(\vec{x}_0, t_0) = \sum_k U_{\mu k} U_{\alpha k} A_k, \quad \text{(3.4)}
\]

\(^7\) From now on, momentum dependence of energies and velocities will not be indicated explicitly, if not particularly necessary.

\(^8\) It is understood that the \(t_0\)-integration extends over a infinitely large time interval \(T\).
and the corresponding transition probability per unit time interval is given by
\[
P_{\pi \to \mu \nu \to \mu \nu} = \frac{1}{T} |A_{\pi \to \mu \nu \to \mu \nu}|^2 = \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} G_{kl},
\]
where
\[
A_k = \int d^3x_0 \int dt_0 A_k(\vec{x}_0, t_0),
\]
\[
G_{kl} = \frac{1}{T} A_k A^*_l = \frac{1}{T} \int d\vec{x}_0 \int dt_0 \int d\vec{x}'_0 \int dt'_0 A_k(\vec{x}_0, t_0) A^*_l(\vec{x}'_0, t'_0).
\]

### 3.2. Transition amplitude and probability

To calculate \(A_k\), Eq.(3.6), we write Eq.(3.3) as
\[
A_k(\vec{x}_0, t_0) = N \exp\{i(\theta_k - \Delta \vec{p}_k + \Delta E_k t_0) - \frac{1}{2} \sigma^2 y_k(\vec{x}_0, t_0)\},
\]
where
\[
N = N_\nu N_\mu N_\pi = \left(\frac{\sigma_\nu \sigma_\mu \sigma_\pi}{\sqrt{\pi}}\right)^{3/2}, \quad \sigma^2 = \sigma_\nu^2 + \sigma_\mu^2 + \sigma_\pi^2,
\]
\[
\Delta \vec{p} = \vec{p}_\nu + \vec{p}_\mu - \vec{p}_\pi, \quad \Delta E_k = E_k + E_\mu - E_\pi,
\]
\[
\theta_k = \vec{p}_\nu \vec{x}_\nu - E_k t_\nu + \vec{p}_\mu \vec{x}_\mu - E_\mu t_\mu - \vec{p}_\pi \vec{x}_\pi + E_\pi t_\pi,
\]
\[
y_k(\vec{x}_0, t_0) = \frac{1}{\sigma^2} \left\{ \sigma_\nu^2(\vec{x}_0 - \vec{v}_k t_0)^2 + \sigma_\mu^2(\vec{x}_0 - \vec{v}_\mu t_0)^2 + \sigma_\pi^2(\vec{x}_0 - \vec{v}_\pi t_\pi)^2 \right\}.
\]

Introducing
\[
\vec{X}_k = \vec{x}_\nu - \vec{v}_k t_\nu, \quad \vec{X}_\mu = \vec{x}_\mu - \vec{v}_\mu t_\mu, \quad \vec{X}_\pi = \vec{x}_\pi - \vec{v}_\pi t_\pi,
\]
\[
\langle \vec{u} \rangle_k = \left(\sigma_\nu^2 \vec{u}_k + \sigma_\mu^2 \vec{u}_\mu + \sigma_\pi^2 \vec{u}_\pi\right)/\sigma^2,
\]
\[
\langle \vec{u} \vec{w} \rangle_k = \left(\sigma_\nu^2 \vec{u}_k \vec{w}_k + \sigma_\mu^2 \vec{u}_\mu \vec{w}_\mu + \sigma_\pi^2 \vec{u}_\pi \vec{w}_\pi\right)/\sigma^2,
\]
\[
\langle \vec{u}, \vec{w} \rangle_k = \langle \vec{u} \vec{w} \rangle_k - \langle \vec{u} \rangle_k \langle \vec{w} \rangle_k, \quad \vec{u}, \vec{w} = \vec{v}, \vec{X},
\]
\[
\langle \vec{u}, \vec{w} \rangle_k \text{ may be expressed also as}
\]
\[
\langle \vec{u}, \vec{w} \rangle_k = \frac{1}{\sigma^4} \left\{ \sigma_\nu^2 \sigma_\mu^2 \vec{u}_{[k]} \vec{w}_{[k]} + \sigma_\mu^2 \sigma_\pi^2 \vec{u}_{[\mu]} \vec{w}_{[\mu]} + \sigma_\nu^2 \sigma_\pi^2 \vec{u}_{[\nu]} \vec{w}_{[\pi]} + \sigma_\mu^2 \sigma_\nu^2 \vec{u}_{[\mu]} \vec{w}_{[\nu]} \right\},
\]
where
\[
\vec{u}_{[\kappa, \lambda]} = \vec{u}_\kappa - \vec{u}_\lambda, \quad \kappa, \lambda = k, \mu, \pi.
\]
we further express \( y_k(\vec{x}_0, t_0) \) as

\[
y_k(\vec{x}_0, t_0) = (\vec{x}_0 - \langle \vec{v} \rangle_k t_0 - \langle \vec{X} \rangle_k)^2 + a_k(t_0 - t_k)^2 + c_k - a_k t_k^2,
\]
and \( i(\theta_k - \Delta \vec{p} \vec{x}_0 + \Delta E_k t_0) - \frac{1}{2} \sigma^2 y_k(\vec{x}_0, t_0) \) as

\[
i(\theta_k - \Delta \vec{p} \vec{x}_0 + \Delta E_k t_0) - \frac{1}{2} \sigma^2 y_k(\vec{x}_0, t_0) = i(\theta_k - \Delta \vec{p} (\vec{X}_k) + \Delta \vec{E}_k t_k)
\]

\[
- \frac{1}{2} \sigma^2 (c_k - a_k t_k^2) - \frac{1}{2 \sigma^2} (\Delta \vec{p})^2 - \frac{1}{2 \sigma^2 a_k} (\Delta \vec{E}_k)^2
\]

\[
- \frac{1}{2} \sigma^2 (\vec{x}_0 - \langle \vec{v} \rangle_k t_0 - \langle \vec{X} \rangle_k + i \frac{1}{\sigma^2} \Delta \vec{p})^2 - \frac{1}{2} \sigma^2 a_k(t_0 - t_k - i \frac{1}{\sigma^2 a_k} \Delta \vec{E}_k)^2,
\]

where

\[
a_k = \langle \vec{v}, \vec{v} \rangle_k, \quad b_k = \langle \vec{v}, \vec{X} \rangle_k, \quad c_k = \langle \vec{X}, \vec{X} \rangle_k, \quad t_k = -b_k/a_k, \quad \Delta \vec{E}_k = \Delta E_k - \langle \vec{v} \rangle_k \Delta \vec{p}.
\]

Substituting Eqs. (3.8) and (3.13) into Eq. (3.6) and performing the \( \vec{x}_0 \)- and \( t_0 \)-integrations over the whole space-time, we find

\[
A_k = N \left( \frac{2 \pi}{\sigma^2} \right)^{3/2} \left( \frac{2 \pi}{\sigma^2 a_k} \right)^{1/2} \exp \left\{ i(\theta_k - \Delta \vec{p} (\vec{X}_k) + \Delta \vec{E}_k t_k) \right\}
\]

\[
\exp \left\{ -\frac{1}{2 \sigma^2} (\Delta \vec{p})^2 - \frac{1}{2 \sigma^2 a_k} (\Delta \vec{E}_k)^2 - \frac{1}{2} \sigma^2 (c_k - a_k t_k^2) \right\}. \tag{3.15}
\]

Substituting Eq. (3.15) into Eq. (3.7), we obtain\(^{10} \)

\[
G_{kl} = \frac{1}{T} A_k A_l^* = \frac{1}{T} N \left( \frac{2 \pi}{\sigma^2} \right)^{3/2} \left( \frac{2 \pi}{\sigma^2 a_l} \right)^{1/2} \exp \left\{ -i(\theta_l - \Delta \vec{p} (\vec{X}_l) + \Delta \vec{E}_l t_l) \right\}
\]

\[
\exp \left\{ -\frac{1}{2 \sigma^2} (\Delta \vec{p})^2 - \frac{1}{2 \sigma^2 a_l} (\Delta \vec{E}_l)^2 - \frac{1}{2} \sigma^2 (c_l - a_l t_l^2) \right\}
\]

\[
N_{kl} \exp \left\{ -\sigma^2 Z_{kl} - \frac{1}{\sigma^2} H_{kl} - i \Theta_{kl} \right\}, \tag{3.16}
\]

\(^{10} \) One may substitute Eqs. (3.8) and (3.13) directly into Eq. (3.7) and perform the \( \vec{x}_0 \)-, \( t_0 \)-, \( \vec{x}_0' \)- and \( t_0' \)-integrations, by changing the integration variables from \( \vec{x}_0 \) and \( \vec{x}_0' \) to \( \vec{x}_- \equiv \vec{x}_0 - \vec{x}_0 \) and \( \vec{x}_+ \equiv (\vec{x}_0 + \vec{x}_0')/2 \), and from \( t_0 \) and \( t_0' \) to \( t_- \equiv t_0 - t_0' \) and \( t_+ \equiv (a_k t_0 + a_l t_0')/(a_k + a_l) \), to obtain the same result.
where

\[ N_{kl} = \frac{1}{T} N^2 \left( \frac{2\pi}{\sigma^2} \right)^4 \left( \frac{1}{a_k a_l} \right)^{1/2}, \]  
(3.17)

\[ Z_{kl} = \frac{1}{2} (c_k - a_k t_k^2 + c_l - a_l t_l^2), \]  
(3.18)

\[ H_{kl} = (\Delta \vec{p})^2 + \frac{1}{2a_k} (\Delta \tilde{E}_k)^2 + \frac{1}{2a_l} (\Delta \tilde{E}_l)^2, \]  
(3.19)

\[ \Theta_{kl} = -\left( \theta_k - \Delta \vec{p} (X) + \Delta \tilde{E}_t + \Delta \tilde{E}_l \right) + \left( \theta_l - \Delta \vec{p} (X) + \Delta \tilde{E}_l \right), \]  
(3.20)

A couple of remarks as regards ”trajectories of particles” are in order.

1. From Eq.(3.8), one sees that \(|A_k(x_0, t_0)|\) has a peak at \(y_k(x_0, t_0) = 0\), which, from Eqs.(3.9) and (3.12), implies on the one hand that

\[ x_{\nu 0} - \vec{v}_k x_{\nu 0} = 0, \quad x_{\mu 0} - \vec{v}_\mu x_{\mu 0} = 0, \quad x_{\pi 0} - \vec{v}_\pi x_{\pi 0} = 0, \]  
(3.21)

and on the other hand that

\[ x_0 - \langle \vec{v} \rangle k t_0 - \langle \vec{X} \rangle k = 0, \quad t_0 - t_k = 0, \quad c_k - a_k t_k^2 = 0. \]  
(3.22)

The three equations in Eq.(3.22) may be derived somehow directly as some weighted averages of the three equations in Eq.(3.21)\(^{[11]}\).

2. From Eqs.(3.15) and (3.16), one sees that \(|A_k|\), as well as \(G_{kk}\), has a peak at \(c_k - a_k t_k^2 = 0\). Thus, even after the integrations with respect to \(x_0\) and \(t_0\) have been carried out, the third equation in Eq.(3.22) remains meaningful and may be regarded as something which reflects that each of the particles involved in the decay propagates along its classical trajectory, Eq.(3.21).

3. For \(k \neq l\), \(c_k - a_k t_k^2 = 0\) and \(c_l - a_l t_l^2 = 0\) are not compatible with each other and \(Z_{kl} \equiv (c_k - a_k t_k^2 + c_l - a_l t_l^2)/2 = 0\) never holds. Thus, for the interference terms, \(G_{kl}\) with \(k \neq l\), it is actually not quite clear as to if and how trajectories may be defined for the particles involved in the decay. We shall come back to this question in the next section.

\(^{[11]}\)Note that the three equations in Eq.(3.21) may be rewritten as

\[ x_0 - \vec{v}_k x_{\nu 0} = X_k, \quad x_0 - \vec{v}_\mu x_{\mu 0} = X_\mu, \quad x_0 - \vec{v}_\pi x_{\pi 0} = X_\pi, \]  
(3.23)

from which it follows that

\[ \vec{v}_{[k \mu]} t_0 = -\vec{X}_{[k \mu]}, \quad \vec{v}_{[\mu \pi]} t_0 = -\vec{X}_{[\mu \pi]}, \quad \vec{v}_{[\pi k]} t_0 = -\vec{X}_{[\pi k]}. \]  
(3.24)

The first equation in Eq.(3.22) follows from Eq.(3.23), while the second and third equations in Eq.(3.22) follow from Eq.(3.24).
3.3. Plane-wave limit

Let us examine the plane-wave limit: $\sigma \to 0$ or, equivalently, $\sigma_\nu, \sigma_\mu, \sigma_\pi \to 0$. By going back to Eqs.(3.3) and (3.6) and trivially performing integrations with respect to $x_0$ and $t_0$, one finds

$$\lim_{\sigma_\pi, \sigma_\mu, \sigma_\nu \to 0} A_k = \frac{1}{V^{3/2}} (2\pi)^4 \exp\{i\theta_k\} \delta^3(\vec{p}) \delta(\Delta E_k),$$

from which it follows that\footnote{To derive Eq.(3.25), it is taken into account that, for $k \neq l$, $\Delta E_k = 0$ and $\Delta E_l = 0$ are not compatible with each other and use is made of the familiar technique to handle with the square of a $\delta$-function\cite{17}.}

$$\lim_{\sigma_\pi, \sigma_\mu, \sigma_\nu \to 0} G_{kl} = \frac{1}{T} \frac{1}{V^3} (2\pi)^8 \exp\{-iE_{|kl|}t_\nu\} \left[\delta^3(\vec{p})\right]^2 \delta(\Delta E_k) \delta(\Delta E_l) = \delta_{kl} \frac{1}{V^2} (2\pi)^4 \delta^3(\vec{p}) \delta(\Delta E_k). \tag{3.25}$$

Substituting Eq.(3.25) into Eq.(3.5), one obtains

$$P_{\pi \to \mu \nu \mu \to \mu \nu,} = \frac{1}{V^2} (2\pi)^4 \sum_k U_{\mu k}^2 U_{\nu k}^2 \delta^3(\vec{p}_\nu + \vec{p}_\mu - \vec{p}_\pi) \delta(E_k + E_\mu - E_\pi), \tag{3.26}$$

or, in the two-generation case,

$$P_{\pi \to \mu \nu \mu \to \mu \nu,} = \frac{1}{V^2} (2\pi)^4 \delta^3(\vec{p}_\nu + \vec{p}_\mu - \vec{p}_\pi) \{\sin^4 \theta \delta(E_1 + E_\mu - E_\pi) + \cos^4 \theta \delta(E_2 + E_\mu - E_\pi)\},$$

$$P_{\pi \to \mu \nu \mu \to \mu \nu,} = \frac{1}{V^2} (2\pi)^4 \delta^3(\vec{p}_\nu + \vec{p}_\mu - \vec{p}_\pi) \sin^2 \theta \cos^2 \theta \{\delta(E_1 + E_\mu - E_\pi) + \delta(E_2 + E_\mu - E_\pi)\}.$$

It is seen that energy-momentum conservation prevents different mass-eigenstates to interfere with one another and, as a result, each of mass-eigenstates appears to contribute separately to the transition probabilities\footnote{An intuitive argument on connection between neutrino oscillation and energy-momentum conservation has been given by Kayser\cite{2} and by Lipkin\cite{18}.}

It is also interesting to examine the cases in which one of $\sigma_\nu, \sigma_\mu$ and $\sigma_\pi$ is kept finite. From Eq.(3.15), with $\sigma_\pi$ kept finite, one finds\footnote{Note that, with $\sigma_\nu, \sigma_\mu \to 0$, one has $\sigma \to \sigma_\pi$, $\Delta E_k \to \Delta E_k - \vec{v}_\pi \Delta \vec{p}$, $(\vec{X})_k \to \vec{X}_\pi$ and $a_k, b_k, c_k \to 0$, while $t_k$ remains finite.}

$$\lim_{\sigma_\mu, \sigma_\nu \to 0} A_k = \frac{1}{V} N_{\pi} \left(\frac{2\pi}{\sigma_\pi^2}\right)^{3/2} (2\pi) \delta(\Delta E_k - \vec{v}_\pi \Delta \vec{p}) \exp\{i(\vec{p}_\nu \vec{x}_\nu - E_k t_\nu + \vec{p}_\mu \vec{x}_\mu - E_\mu t_\mu)\} - \frac{1}{2\sigma_\pi^2} (\Delta \vec{p})^2\},$$
from which one derives

\[
\lim_{\sigma_\pi, \sigma_\nu \to 0} G_{kl} = \frac{1}{V^2} \frac{1}{\sigma_\nu^2} \left( \frac{2\pi}{\sigma_\pi^2} \right)^3 (2\pi)^2 \delta(\Delta E_k - \vec{v}_k \Delta \vec{p}) \delta(\Delta E_l - \vec{v}_l \Delta \vec{p})
\]

\[
\exp\{-iE_{[kl]}t_{\nu\pi} - \frac{1}{\sigma_\pi^2}(\Delta \vec{p})^2\}
\]

\[
= \delta_{kl} \frac{1}{V^2} (2\pi)^4 \left( \frac{1}{\pi \sigma_\pi^2} \right)^{3/2} \delta(\Delta E_k - \vec{v}_k \Delta \vec{p}) \exp\{-\frac{1}{2\sigma_\nu^2}(\Delta \vec{p})^2\}.
\]

(3.27)

It is interesting to see that there appears in the amplitude a single \(\delta\)-function which implies that energies are conserved in the rest frame of the pion and this \(\delta\)-function eliminates interference terms from the transition probability. In contrast, with \(\sigma_\nu\) kept finite, one has

\[
\lim_{\sigma_\pi, \sigma_\nu \to 0} A_k = \frac{1}{V} \frac{2\pi}{\sigma_\nu^2} \left( \frac{2\pi}{\sigma_\pi^2} \right)^{3/2} (2\pi) \delta(\Delta E_k - \vec{v}_k \Delta \vec{p})
\]

\[
\exp\{i(\vec{p}_\mu \vec{x}_{\mu\nu} - E_\mu t_{\mu\nu} - \vec{p}_\pi \vec{x}_{\pi\nu} + E_\pi t_{\pi\nu}) - \frac{1}{2\sigma_\nu^2}(\Delta \vec{p})^2\},
\]

\[
\lim_{\sigma_\pi, \sigma_\nu \to 0} G_{kl} = \frac{1}{V^2} (2\pi)^4 \left( \frac{1}{\pi \sigma_\pi^2} \right)^{3/2} \frac{2\pi}{T} \delta(\Delta E_k - \vec{v}_k \Delta \vec{p}) \delta(\Delta E_l - \vec{v}_l \Delta \vec{p})
\]

\[
\exp\{-\frac{1}{2\sigma_\nu^2}(\Delta \vec{p})^2\}.
\]

(3.28)

Here, remarkably, interference terms remain nonvanishing, but appear not to contain an oscillating factor\(^{[13]}\)

### 3.4. Total transition probability

From Eqs.(2.4) and (2.8), noting that the function \(f(\vec{p}; \vec{p}_\nu, \sigma_\nu)\), defined by Eq.(2.5), satisfies

\[
\int d^3p_\nu |f(\vec{p}; \vec{p}_\nu, \sigma_\nu)|^2 = 1,
\]

one may derive such a relation as\(^{[16]}\)

\[
\int d^3p_\nu \int d^3x_\nu \phi_k(\vec{x}_{\nu0}, t_{\nu0}; \vec{p}_\nu, \sigma_\nu) \phi_k^*(\vec{x}_{\nu0}, t'_{\nu0}; \vec{p}_\nu, \sigma_\nu)
\]

\[
= \int d^3p_\nu \int d^3x_\nu \lim_{\sigma_\nu \to 0} \left[ \phi_k(\vec{x}_{\nu0}, t_{\nu0}; \vec{p}_\nu, \sigma_\nu) \phi_k^*(\vec{x}_{\nu0}, t'_{\nu0}; \vec{p}_\nu, \sigma_\nu) \right].
\]

(3.29)

\(\Delta E_k - \vec{v}_k \Delta \vec{p} = 0\) and \(\Delta E_l - \vec{v}_l \Delta \vec{p} = 0\) are not incompatible with each other and the product of the two \(\delta\)-functions, \(\delta(\Delta E_k - \vec{v}_k \Delta \vec{p})\delta(\Delta E_l - \vec{v}_l \Delta \vec{p})\), may be rewritten as \(\delta(\Delta E_k - \vec{v}_k \Delta \vec{p})\delta((E_k - E_l)/(E_k + E_l + E_\mu + E_\pi)/E_k))\). Note also that, if one lets \(\sigma_\pi \to 0\) in Eq.(3.27) or lets \(\sigma_\nu \to 0\) in Eq.(3.28), these equations reduce to Eq.(3.25).

\(^{[13]}\) \(\vec{x}_{\nu0} = \vec{x}_\nu - \vec{x}_0^\nu, \ t_{\nu0} = t_\nu - t_0^\nu, \ \kappa = \nu, \mu, \pi.\)

\(^{[16]}\) \(\vec{x}_{\nu0} = \vec{x}_\nu - \vec{x}_0^\nu, \ t_{\nu0} = t_\nu - t_0^\nu, \ \kappa = \nu, \mu, \pi.\)
On substituting Eq. (3.3), we integrate Eq. (3.7) with respect to \( \vec{x}_\nu \) and \( \vec{p}_\nu \) and consult Eq. (3.29), to obtain
\[
\int d^3p_\nu \int d^3x_\nu \ G_{kl} = \frac{1}{T} \int d^3x_0 \int dt_0 \int d^3x'_0 \int dt'_0 \ \phi_\pi^*(\vec{x}_\pi t_\pi; \vec{p}_\pi, \sigma_\pi) \phi_\pi(\vec{x}'_\pi t'_\pi; \vec{p}_\pi, \sigma_\pi) \\
\phi_\mu(\vec{x}_\mu t_\mu; \vec{p}_\mu, \sigma_\mu) \phi_\mu^*(\vec{x}'_\mu t'_\mu; \vec{p}_\mu, \sigma_\mu) \\
\int d^3p_\nu \int d^3x_\nu \ \lim_{\sigma_\nu \to 0} G_{kl}.
\]
Similarly, we obtain
\[
\int d^3p_\mu \int d^3x_\mu \ G_{kl} = \int d^3p_\mu \int d^3x_\mu \ \lim_{\sigma_\mu \to 0} G_{kl}, \quad (3.31)
\]
\[
\int d^3p_\mu \int d^3x_\mu \int d^3p_\nu \int d^3x_\nu \ G_{kl} = \int d^3p_\mu \int d^3x_\mu \int d^3p_\nu \int d^3x_\nu \ \lim_{\sigma_\mu, \sigma_\nu \to 0} G_{kl}. \quad (3.32)
\]
Equation (3.32) implies that, as far as the integrated transition probability is concerned, there is no difference between treating decay products as plane-waves or as wave-packets.

Integrating Eq. (3.5) with respect to \( \vec{x}_\nu \), \( \vec{p}_\nu \), \( \vec{x}_\mu \) and \( \vec{p}_\mu \), summing it up with respect to \( \alpha \) and consulting Eqs. (3.32) and (3.27), we obtain the total transition probability or total decay rate as
\[
P_{\pi \to \mu \nu} = \sum_{\alpha} \int d^3p_\mu \int d^3x_\mu \int d^3p_\nu \int d^3x_\nu \ P_{\pi \to \mu \nu, \nu \to \mu \alpha} \\
= \sum_k U_{\mu k}^2 \int d^3p_\mu \int d^3x_\mu \int d^3p_\nu \int d^3x_\nu \ \lim_{\sigma_\mu, \sigma_\nu \to 0} G_{kk} \\
= (2\pi)^4 \left( \frac{1}{\pi\sigma_\pi^2} \right)^{3/2} \sum_k U_{\mu k}^2 \int d^3p_\mu \int d^3p_\nu \\
\delta(\Delta E_k - \vec{v}_\pi \Delta \vec{p}) \ \exp\left\{-\frac{1}{\sigma_\pi^2}(\Delta \vec{p})^2\right\}, \quad (3.33)
\]
which reduces, in the limit of \( \sigma_\pi \to 0 \), to
\[
P_{\pi \to \mu \nu} = (2\pi)^4 \sum_k U_{\mu k}^2 \int d^3p_\mu \int d^3p_\nu \\
\delta^3(\vec{p}_\nu + \vec{p}_\mu - \vec{p}_\pi) \ \delta(E_k + E_\mu - E_\pi). \quad (3.34)
\]
Equation (3.34) as well as Equation (3.26) are expected results, which serves as a check of consistency of our treatment as a whole.
4. An approximate treatment

In this section, we like to propose an approximate prescription to handle with the trajectory factor \(\exp\{-\sigma^2 Z_{kl}\}\), the energy-momentum factor \(\exp\{-H_{kl}/\sigma^2\}\) and the phase factor \(\exp\{-i\Theta_{kl}\}\).

4.1. Trajectory factor

As noted before, the factor \(\exp\{-\sigma^2 Z_{kk}\} \equiv \exp\{-\sigma^2(c_k - a_k t_k^2)\}\), contained in \(G_{kk}\), may be interpreted as something which reflects that each of the particles involved propagates nearly along its classical trajectory, Eq.(3.21), while, for \(k \neq l\), \(c_k - a_k t_k^2 = 0\) and \(c_l - a_l t_l^2 = 0\) are not compatible with each other and \(Z_{kl} \equiv (c_k - a_k t_k^2 + c_l - a_l t_l^2)/2 = 0\) never holds. We have encountered a similar situation in Sec.2. There, writing

\[
\vec{v}_{k,l} = \frac{1}{2} \vec{v}_{[kl]},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]

we have reasonably regarded Eq.(2.12) as a classical trajectory of the interfering neutrinos. Here, on writing \(Z_{kl}\) as

\[
Z_{kl} = c_{(kl)} - a_{(kl)} t_{(kl)}^2 + \zeta_{kl},
\]
representing that each of the particles involved propagates nearly along its classical trajectory.

\( \zeta_{kl} \), defined by Eq.(4.2), is given explicitly by

\[
\zeta_{kl} \equiv \frac{1}{2} \left( c_k - a_k t_k^2 + c_l - a_l t_l^2 \right) - \left( c_{(kl)} - a_{(kl)} t_{(kl)}^2 \right)
\]

\[
= \frac{1}{a_{(kl)}(a_{(kl)} + a_{(kl)}')^2 - (a_{(kl)}')^2} \times \left\{ -a_{(kl)}(c_{(kl)} + a_{(kl)}') \{ a_{(kl)} c_{(kl)}'' - 2a_{(kl)} b_{(kl)} b_{(kl)}' - 2a_{(kl)}' a_{(kl)}'' + a_{(kl)} c_{(kl)}'' - (b_{(kl)}')^2 \} - (a_{(kl)} b_{(kl)}' - b_{(kl)} a_{(kl)}')^2 \right\} - \left( c_{(kl)}(a_{(kl)}')^2 - 2b_{(kl)} a_{(kl)}' b_{(kl)} + a_{(kl)} (b_{(kl)}')^2 \right). \tag{4.6}
\]

When Equation (4.5) holds, which implies

\[
\bar{X}_{[\kappa \lambda]} = -\bar{v}_{[\kappa \lambda]} \bar{t}_0, \quad \kappa, \lambda = (kl), \mu, \pi, \tag{4.7}
\]

\( \bar{t}_0 \) being a constant independent of \( \kappa \) and \( \lambda \), \( \zeta_{kl} \) reduces to \( \zeta_{kl}(t_{\nu} - \bar{t}_0)^2 \), where

\[
\bar{\zeta}_{kl} = \frac{a_{(kl)} + a_{(kl)}''}{(a_{(kl)} + a_{(kl)}'')^2 - (a_{(kl)}')^2} \tag{4.8}
\]

and the trajectory factor \( \exp\{-\sigma^2 Z_{kl}\} \) may be approximated as

\[
\exp\{-\sigma^2 Z_{kl}\} \approx z_{kl} \bar{\xi}_{kl}^{(1)}, \tag{4.9}
\]

where

\[
\bar{\xi}_{kl}^{(1)} = \exp\{-\sigma^2 \bar{\zeta}_{kl}(t_{\nu} - \bar{t}_0)^2\}. \tag{4.10}
\]

Necessary conditions for interference terms to be appreciable follow from Eqs.(4.3), (4.4), (4.9) and (4.10):

(a) \( c_{(kl)} - a_{(kl)} t_{(kl)} \lesssim 1/\sigma^2 \); and

(b) \( \zeta_{kl} \) or \( \bar{\zeta}_{kl}(t_{\nu} - \bar{t}_0)^2 \lesssim 1/\sigma^2 \).

The former condition applies to the diagonal terms \( G_{kk} \) too and implies that each of the particles involved should propagate along its classical trajectory at least approximately, or that those mass-eigenstates which do not satisfy this condition contribute little to the transition probability, while the latter is a condition which corresponds to Eq.(2.15), known as coherent condition for neutrino oscillation[15], and implies that those two mass-eigenstates, if their masses do not satisfy this condition, interfere little with each other.

From Eq.(4.7), it follows that

\[
\bar{x}_{\nu} - \bar{v}_{[kl]}(t_{\nu} - \bar{t}_0) = \bar{x}_{\mu} - \bar{v}_{[\mu]}(t_{\mu} - \bar{t}_0) = \bar{x}_{\pi} + \bar{v}_{[\pi]}(\bar{t}_0 - t_{\pi}) \equiv \bar{x}_0. \tag{4.11}
\]

Since these equations appear to be exactly similar to Eq.(3.21) which involves the decay space-time point \( \bar{x}_0, t_0 \), we shall refer to \( (\bar{x}_0, \bar{t}_0) \) as pseudo decay space-time point. Note that the pseudo decay time \( \bar{t}_0 \), and hence the pseudo traveling time of the neutrinos \( t_{\nu} - \bar{t}_0 \) as well, may in principle be deduced from knowledge of \( (\bar{x}_{\nu}, t_{\nu}) \) and \( \bar{v}(kl) \) and of \( (\bar{x}_{\mu}, t_{\mu}) \) and \( \bar{v}_{[\mu]} \) (or of \( (\bar{x}_{\pi}, t_{\pi}) \) and \( \bar{v}_{[\pi]} \)).
4.2. Energy-momentum factor

In the limit of $\sigma \to 0$, $\exp\{-H_{kk}/\sigma^2\}$ gives rise to $\delta(E_k + E_\mu - E_\pi)\delta^3(\vec{p}_\nu + \vec{p}_\mu - \vec{p}_\pi)$, while $\exp\{-H_{kl}/\sigma^2\}$ with $k \neq l$ vanishes (see also Sec.3.3)\(^{\text{18}}\). In order for interference terms to remain nonvanishing and appreciable, the two constituting factors of $\exp\{-H_{kl}/\sigma^2\}$, $\exp\{-\Delta \vec{p}/2\sigma^2 - (\Delta \vec{E}_k)^2/2\sigma^2 a_k\}$ and $\exp\{-\Delta \vec{p}/2\sigma^2 - (\Delta \vec{E}_l)^2/2\sigma^2 a_l\}$, have to overlap appreciably with each other.

To look into this condition more deeply, we rewrite $(\Delta \vec{E}_k)^2/2a_k + (\Delta \vec{E}_l)^2/2a_l$ as

$$
\frac{1}{2a_k}(\Delta \vec{E}_k)^2 + \frac{1}{2a_l}(\Delta \vec{E}_l)^2 = \frac{1}{a_{(kl)}}(\Delta \vec{E}_{(kl)})^2 + \eta_{kl},
$$

(4.12)

and, accordingly, $\exp\{-H_{kl}/\sigma^2\}$ as

$$
\exp\{-\frac{1}{\sigma^2} H_{kl}\} = h_{kl} \xi_{kl}^{(2)},
$$

(4.13)

where

$$
\begin{align*}
 h_{kl} &= \exp\{-\frac{1}{\sigma^2}((\Delta \vec{p})^2 + \frac{1}{a_{(kl)}}(\Delta \vec{E}_{(kl)})^2)\}, \\
 \xi_{kl}^{(2)} &= \exp\{-\frac{1}{\sigma^2} \eta_{kl}\}.
\end{align*}
$$

(4.14)

The factor $h_{kl}$ has a peak at

$$
\begin{align*}
\Delta \vec{p} &\equiv \vec{p}_\nu + \vec{p}_\mu - \vec{p}_\pi = 0, \\
\Delta E_{(kl)} &\equiv E_{(kl)} + E_\mu - E_\pi = 0,
\end{align*}
$$

(4.15)

which may reasonably be regarded as representing the energy-momentum conservation in the presence of neutrino mixing.

$\eta_{kl}$, defined by Eq.(4.12), is given explicitly by

$$
\eta_{kl} = \frac{1}{2a_k}(\Delta \vec{E}_k)^2 + \frac{1}{2a_l}(\Delta \vec{E}_l)^2 - \frac{1}{a_{(kl)}}(\Delta \vec{E}_{(kl)})^2
\begin{align*}
&= \frac{1}{4a_{(kl)}}((a_{(kl)} + a''_{kl})^2 - (a'_{kl})^2) \\
&\quad \times [ (a_{(kl)}(a_{(kl)} + a''_{kl})((\Delta \vec{E}_{(kl)})^2 - 4a_{(kl)}a'_{kl}\Delta \vec{E}_{(kl)}\Delta \vec{E}_{(kl)}' \\
&\quad + 4((a'_{kl})^2 - a''_{kl}(a_{(kl)} + a''_{kl}))(\Delta \vec{E}_{(kl)})^2].
\end{align*}
$$

(4.16)

\(^{\text{18}}\) More precisely, one has

$$
\lim_{\sigma \to 0} \exp\{-\frac{1}{\sigma^2} H_{kl}\} = \delta_{kl} (\pi \sigma^2)^{3/2} (\pi \sigma^2 a_k)^{1/2} \delta^3(\Delta \vec{p}) \delta(\Delta E_k),
$$

and, accordingly,

$$
\lim_{\sigma, \sigma_\mu, \sigma_\pi \to 0} G_{kl} = \frac{\delta_{kl}}{V^2} \frac{1}{T} \frac{1}{(\frac{\pi}{\sigma^2 a_k})^{1/2} (2\pi)^4} \delta^3(\Delta \vec{p}) \delta(\Delta E_k).
$$

In order for this expression to coincide with Eq.(3.25), one needs to identify $T$ with $(\pi/\sigma^2 a_k)^{1/2}$.
When Equation (4.15) holds, \( \eta_{kl} \) reduces to

\[
\hat{\eta}_{kl} = \left( a_{kl} + a''_{kl} \right)
\]

\[
\frac{4((a_{kl} + a''_{kl})^2 - (a'_{kl})^2)}{(E_{[kl]}^2)}, \tag{4.17}
\]

and \( \exp\{-H_{kl}/\sigma^2\} \) may be approximated as

\[
\exp\left\{-\frac{1}{\sigma^2}H_{kl}\right\} \approx h_{kl} \zeta_{kl}^{(2)}, \tag{4.18}
\]

where

\[
\zeta_{kl}^{(2)} = \exp\left\{-\frac{1}{\sigma^2}h_{kl}\right\}. \tag{4.19}
\]

Necessary conditions for interference terms to be appreciable now follow from Eqs. (4.13), (4.14), (4.18) and (4.19):

\begin{itemize}
  \item[(c)] \((\Delta p)^2 + (\Delta \tilde{E}_{(kl)})^2/a_{(kl)} \lesssim \sigma^2; \) and \\
  \item[(d)] \( \eta_{kl} \) or \( \hat{\eta}_{kl} \lesssim \sigma^2. \)
\end{itemize}

The former condition applies to the diagonal terms \( G_{kk} \) too and implies that energies and momenta involved should conserved at least approximately, or that those mass-eigenstates which do not satisfy this condition contribute little to the transition probability, while the latter condition implies that those two mass-eigenstates, if their masses do not satisfy this condition, interfere little with each other. It is to be noted also that \( \xi_{kl}^{(2)} \) or \( \zeta_{kl}^{(2)} \) seems to have something to do with the suppression factor, given by \( \exp\left\{-\left(\tilde{E}_{[kl]}^2/2\sigma^2(\tilde{v}_k^2 + \tilde{v}_l^2)\right)\right\} \), which appears in a more general wave-packet treatment of neutrino oscillation (see Appendix A).

### 4.3. Phase factor

\( \Theta_{kl} \), Eq.(3.20), may be rewritten as

\[
\Theta_{kl} = \Delta \tilde{E}_{kl}'\left(t_{\nu} - \frac{1}{2}(t_k + t_l)\right) - \Delta \tilde{E}_{(kl)}(t_k - t_l)
\]

\[
= \Delta \tilde{E}_{kl}'\{t_{\nu} + \left\{\frac{a_{(kl)} + a''_{(kl)}(b_{(kl)} + b''_{(kl)}) - a'_{kl}b'_{kl}}{(a_{(kl)} + a''_{(kl)})^2 - (a'_{kl})^2}\right\}
\]

\[
+ 2\Delta \tilde{E}_{(kl)}\left(\frac{b_{kl}(a_{(kl)} + a''_{(kl)}) - a'_{kl}(b_{(kl)} + b''_{(kl)})}{(a_{(kl)} + a''_{(kl)})^2 - (a'_{kl})^2}\right). \tag{4.20}
\]

When Equation (4.7) holds, \( \Theta_{kl} \) reduces to \( \bar{\Theta}_{kl}(t_{\nu} - \bar{t}_0) \), where

\[
\bar{\Theta}_{kl} = \Delta \tilde{E}_{kl}'\\frac{a_{(kl)}(a_{(kl)} + a''_{(kl)}) - (a'_{kl})^2/2}{(a_{(kl)} + a''_{(kl)})^2 - (a'_{kl})^2} - \Delta \tilde{E}_{(kl)}\\frac{a'_{kl}(a_{(kl)} - a''_{(kl)})}{(a_{(kl)} + a''_{(kl)})^2 - (a'_{kl})^2}. \tag{4.21}
\]

If Equations (4.15) are further applied, \( \Theta_{kl} \) reduces to \( \hat{\Theta}_{kl}(t_{\nu} - \bar{t}_0) \), where

\[
\hat{\Theta}_{kl} = \frac{a_{(kl)}(a_{(kl)} + a''_{(kl)}) - (a'_{kl})^2/2}{(a_{(kl)} + a''_{(kl)})^2 - (a'_{kl})^2} \tilde{E}_{[kl]}^2. \tag{4.22}
\]
4.4. Summary

We have rewritten Eq.(3.16),

\[ G_{kl} \equiv N_{kl} \exp\{-\sigma^2 Z_{kl} - \frac{1}{\sigma^2} H_{kl} - i\Theta_{kl}\}, \]

as

\[ G_{kl} = N_{kl} z_{kl} \xi_{kl}^{(1)} \xi_{kl}^{(2)} \exp\{-i\Theta_{kl}\}, \quad (4.23) \]

where \( z_{kl}, h_{kl}, \xi_{kl}^{(1)} \) and \( \xi_{kl}^{(2)} \) are given by Eqs.(4.4) and (4.14). Substituting Eqs.(3.16) and (4.23) into Eq.(3.5), one obtains

\[ P_{\pi \rightarrow \mu_\nu \rightarrow \mu_\alpha} = \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} N_{kl} \exp\{-\sigma^2 Z_{kl} - \frac{1}{\sigma^2} H_{kl} - i\Theta_{kl}\} \]

\[ = \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} N_{kl} z_{kl} h_{kl} \xi_{kl}^{(1)} \xi_{kl}^{(2)} \cos(\Theta_{kl}), \quad (4.24) \]

We have then shown that Equation (4.24) may be approximated as

\[ P_{\pi \rightarrow \mu_\nu \rightarrow \mu_\alpha} \approx \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \bar{\xi}_{kl}^{(1)} \xi_{kl}^{(2)} \cos(\hat{\Theta}_{kl}(t_\nu - \bar{t}_0)), \quad (4.25) \]

where \( \bar{\xi}_{kl}^{(1)}, \xi_{kl}^{(2)} \), and \( \hat{\Theta}_{kl} \) are given respectively by Eqs.(4.10), (4.19) and (4.22).

We have furthermore discussed implications of each factor in Eqs.(4.24) and (4.25) and derived necessary conditions for oscillating terms to be appreciable. It is to be emphasized here that the two features implied respectively by \( z_{kl} \) and \( h_{kl} \), that is, each of the particles involved propagates nearly along its classical trajectory on the one hand and energies and momenta of the particles involved are nearly conserved on the other hand, are complementary to each other in the sense that, as \( \sigma \) becomes larger (smaller), the former (latter) becomes more prominent, and that the two suppression factors \( \xi_{kl}^{(1)} \) or \( \bar{\xi}_{kl}^{(1)} \) and \( \xi_{kl}^{(2)} \) or \( \xi_{kl}^{(2)} \) are also complementary to each other in the sense that, as \( \sigma^2 \) becomes larger (smaller), the former (latter) becomes more effective.

If \( c_{(kl)} - a_{(kl)} t_{(kl)}^2 \ll 1/\sigma^2 \) is assumed to hold for any \( k \) and \( l \), Equations (4.24) and (4.25) may be approximated as

\[ P_{\pi \rightarrow \mu_\nu \rightarrow \mu_\alpha} \approx \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \bar{N}_{kl} h_{kl} \bar{\xi}_{kl}^{(1)} \xi_{kl}^{(2)} \cos(\bar{\Theta}_{kl}(t_\nu - \bar{t}_0))) \]

\[ \approx \sum_{k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \bar{N}_{kl} h_{kl} \xi_{kl}^{(1)} \xi_{kl}^{(2)} \cos(\hat{\Theta}_{kl}(t_\nu - \bar{t}_0)), \quad (4.26) \]

where \( \bar{\Theta}_{kl} \) is given by (4.21). Equations (4.25) and (4.26) are to be contrasted with Eqs.(2.10) and (2.13) and also with Eq.(3.26).
5. Comparison with the approach by Dolgov et al.

In the wave-packet treatment of $\pi \to \mu \nu$ decay given in [13], assuming that a pion has been created with some momentum distribution $f(\vec{p}_\pi)$, the authors define a transition amplitude and the corresponding transition probability by

$$A^D_{\pi \to \mu \nu} (\vec{x}_0, t_0) = \sum_k U_{\mu k} U_{\nu k} \int d^3 p_\pi \ f(\vec{p}_\pi) \ \exp \{i \varphi^D_k (\vec{p}_\pi)\},$$

and

$$P^D_{\pi \to \mu \nu} (\vec{x}_0, t_0) = |A^D_{\pi \to \mu \nu} (\vec{x}_0, t_0)|^2 = \sum_{k,l} U_{\mu k} U_{\nu k} U_{\mu l} U_{\nu l} G^D_{kl}(\vec{x}_0, t_0), \quad (5.1)$$

where

$$G^D_{kl}(\vec{x}_0, t_0) = \int d^3 p_\pi \int d^3 p'_\pi f(\vec{p}_\pi) f(\vec{p}'_\pi) \exp \{i (\varphi^D_k (\vec{p}_\pi) - \varphi^D_l (\vec{p}'_\pi))\},$$

with

$$\varphi^D_k (\vec{p}_\pi) = \vec{p}_{\nu k} \vec{x}_\nu 0 - E_k (\vec{p}_{\nu k}) t_\nu 0 + \vec{p}_{\mu k} \vec{x}_\mu 0 - E_\mu (\vec{p}_{\mu k}) t_\mu 0 - \vec{p}_\pi \vec{x}_\pi 0 + E_\pi (\vec{p}_\pi) t_\pi 0,$$

$$\varphi^D_l (\vec{p}'_\pi) = \vec{p}'_{\nu l} \vec{x}_\nu 0 - E_l (\vec{p}'_{\nu l}) t_\nu 0 + \vec{p}'_{\mu l} \vec{x}_\mu 0 - E_\mu (\vec{p}'_{\mu l}) t_\mu 0 - \vec{p}_\pi \vec{x}_\pi 0 + E_\pi (\vec{p}'_\pi) t_\pi 0.$$

Here, the energy and momentum of the neutrino mass-eigenstate and of the muon are all regarded as depending, through the energy-momentum conservation law, on the mass of the neutrino and hence carrying the suffix $k$ or $l$. Assuming that the dispersion of $f(\vec{p}_\pi)$ is small and that each of the particles involved should propagate along its classical trajectory (cf. Eq.(3.21)), they claim that Equation (5.1) reduces to

$$P^D_{\pi \to \mu \nu} (\vec{x}_0, t_0) = \sum_{k,l} U_{\mu k} U_{\nu k} U_{\mu l} U_{\nu l} \cos (m^2_k - m^2_l 2E_\nu (t_\nu 0 - t_0)). \quad (5.2)$$

Although we may also define an $(\vec{x}_0, t_0)$-dependent transition probability from Eqs.(3.2) and (3.3), we like to point out here that, in spite that the approach we have developed in Sec.3 and Sec.4 is characterized among other things by the space-time point of decay, $(\vec{x}_0, t_0)$, being integrated out, the transition probability we have derived, Eq.(4.26), seems to describe a situation very much close to what Dolgov et al. claim to describe with Eq.(5.2): Equation (4.26) is derived with each of the particles involved assumed to propagate along its classical trajectory, and, as a result, becomes depending on the pseudo traveling time $t_\nu - t_0$, which just corresponds to the traveling time $t_\nu - t_0$ in the $(\vec{x}_0, t_0)$-dependent probability (5.2) of Dolgov et al.. It is to be noted at the same time, however, that Equation (4.26) appears to
be distinct from Eq.(5.2) in the following three points: (a) each term in Eq.(4.26) contains a factor which implies that the energy-momentum conservation holds only approximately; (b) each of the oscillating factors in Eq.(4.26) is multiplied by the two suppression factors $\xi_{kl}^{(1)}$ and $\xi_{kl}^{(2)}$ which are absent in Eq.(5.2); and (c) the oscillation period in Eq.(4.26) deviates in general from the oscillation period $4\pi E_\nu/|m_\nu^2 - m^2|$ in Eq.(5.2).

In addition to the case in which both neutrinos and muons are detected (Case A), Dolgov et al.[13] consider also the case in which only muons are detected (Case B) and the case in which only neutrinos are detected (Case C). In our approach, the transition probability corresponding to Case B may be obtained by integrating Eq.(3.5) with respect to $\vec{x}_\nu$ and $\vec{p}_\nu$ and summing it up with respect to $\alpha$:

$$P_{\pi \rightarrow \mu\nu \rightarrow \mu} \equiv \sum_\alpha \int d^3 p_\nu \int d^3 x_\nu P_{\pi \rightarrow \mu\nu \rightarrow \mu\alpha}$$

$$= \sum_{\alpha,k,l} U_{\mu k} U_{\alpha k} U_{\mu l} U_{\alpha l} \int d^3 p_\nu \int d^3 x_\nu G_{kl}$$

$$= \sum_k U_{\mu k}^2 \int d^3 p_\nu \int d^3 x_\nu G_{kk}. \quad (5.3)$$

As seen, interference terms drop out, implying that muons do not oscillate. Although our approach gives, not only for Case A but also for Case B and Case C, results more or less different from what Dolgov et al. claim, we share with them the conclusion that muons do not oscillate.[19]

### 6. Conclusions

Our wave-packet approach to pion decay is characterized by treating all particles involved as wave-packets and by integrating out the space-time point of decay $(\vec{x}_0, t_0)$. We have seen that, in such an approach, (1) energy-momentum conservation appears to hold only approximately, (2) exact energy-momentum conservation, which holds in the plane-wave limit, would eliminate neutrino oscillating terms from the transition probability, and (3) treating only one of the particles involved as a wave-packet is insufficient to allow for neutrino oscillating terms to appear. We have furthermore developed an approximate treatment, which allows one to define a pseudo decay space-time point $(\vec{r}_0, \bar{t}_0)$ and to express the transition probability, originally given by Eq.(4.24), approximately as Eqs.(4.25) and (4.26).

To conclude, we like to mention that phenomenological implications of the outcomes of the present study need to be examined separately and that our approach may readily be applied to various decay as well as production processes.

---

[19] Note however that there have been some debates as regards whether or not muons can oscillate.[19].
Acknowledgments
The authors are indebted to Professor S.Kamefuchi, Professor M.Obu and the members of particle physics group at Nihon University for discussions and encouragement.

Appendix

Appendix A. A more general three-dimensional wave-packet treatment of neutrino oscillation

More generally, one may conceive a situation in which the peak momentum of the neutrino mass-eigenstates, $\vec{p}_\nu$ in Eqs.(2.4) and (2.5), depends on the suffix $k$ [11, 20]. If $\vec{p}_\nu$ in Eqs.(2.2) and (2.3) is replaced by $\vec{p}_k$, one would be led to

$$P_{\nu_\mu \to \nu_\alpha}(\vec{x}_{\nu_0}, t_{\nu_0}) = N_\nu^2 \sum_{k,l} U_{\mu k} U_{\alpha l} U_{\mu l} \exp\{i(\vec{p}_k \cdot \vec{x}_{\nu_0} - E_{[kl]} t_{\nu_0})\} \exp\{-\frac{1}{2}\sigma^2_\nu (\vec{x}_{\nu_0} - \vec{v}_k t_{\nu_0})^2 \} \exp\{-\frac{1}{2}\sigma^2_\nu (\vec{v}_0 t_{\nu_0})^2 \} \exp\{-\frac{1}{4}\sigma^2_\nu (\vec{v}_0)^2 t_{\nu_0}^2 \} \cos\{\langle \vec{p}_k \vec{v}_0 \rangle - E_{[kl]} t_{\nu_0} \} \exp\{-\frac{1}{4}\sigma^2_\nu (\vec{v}_0)^2 t_{\nu_0}^2 \}, \quad (A.1)$$

which may be approximated as

$$P_{\nu_\mu \to \nu_\alpha}(\vec{x}_{\nu_0}, t_{\nu_0}) \simeq N_\nu^2 \sum_{k,l} U_{\mu k} U_{\alpha l} U_{\mu l} \cos\{\langle \vec{p}_k \vec{v}_0 \rangle - E_{[kl]} t_{\nu_0} \} \exp\{-\frac{1}{4}\sigma^2_\nu (\vec{v}_0)^2 t_{\nu_0}^2 \}, \quad (A.2)$$

where $\vec{p}_{[kl]} = \vec{p}_k - \vec{p}_l$. With a little algebra, one may verify that

$$\vec{p}_{[kl]} \vec{v}_{[kl]} - E_{[kl]} = -\frac{m_k^2 - m_l^2}{2E_{[kl]}} - \frac{\vec{p}_{[kl]} \vec{v}_{[kl]} - E_{[kl]} \vec{x}_{\nu_0}}{4E_{[kl]}}. \quad (A.3)$$

In practical experiments, $D \equiv |\vec{x}_{\nu_0}|$ and $t_{\nu_0}$ are not measured in an independent way; instead, the former is measured, while the latter is inferred from the former. Such a situation corresponds to $t_\nu$ or $t_0$ to be integrated out in Eq.(A.1). One then obtains

$$P_{\nu_\mu \to \nu_\alpha}(\vec{x}_{\nu_0}) \equiv \int dt_{\nu} P_{\nu_\mu \to \nu_\alpha}(\vec{x}_{\nu_0}, t_{\nu_0})$$

$$= N_\nu^2 \sum_{k,l} U_{\mu k} U_{\alpha l} U_{\mu l} \left(\frac{2\pi}{\sigma^2_\nu (\vec{v}_k^2 + \vec{v}_l^2)}\right)^{1/2} \cos\{\langle \vec{p}_k \vec{v}_0 \rangle - \frac{2E_{[kl]} \vec{v}_{[kl]} \vec{x}_{\nu_0}}{\vec{v}_k^2 + \vec{v}_l^2} \} \exp\{-\frac{(E_{[kl]})^2}{2\sigma^2_\nu (\vec{v}_k^2 + \vec{v}_l^2)} \} \exp\{-\frac{(E_{[kl]})^2}{2\sigma^2_\nu (\vec{v}_k^2 + \vec{v}_l^2)} \}. \quad (A.4)$$
Equations (A.2) ~ (A.4) coincide in the one-dimensional case with the corresponding equations derived in [11] and [20].

It is interesting to observe the following.

1. In Eq.(A.2), there appears a correction term to the "standard oscillation period", $4\pi E_{kl}/|m_k^2 - m_l^2|$, which appears in Eq.(2.10) or Eq.(2.13). Although this correction term vanishes in either of the equal-energy, equal-momentum and equal-velocity cases, it is rather artificial, in the framework of the wave-packet treatment presented in this paper, to assume $E_k = E_l$ or $\vec{v}_k = \vec{v}_l$.

2. In Eq.(A.4), each of oscillating factors is multiplied by two suppression factors: one which corresponds to that already present in Eq.(A.2), and in Eq.(2.10) or Eq.(2.13) as well, and the other which gives rise to another necessary condition for neutrino oscillation to be significant: $|E_{kl}|/\sqrt{2(\vec{v}_k^2 + \vec{v}_l^2)} \lesssim \sigma_\nu$.

Appendix B. Some algebra relevant to Sec.4

If Equation (4.1) is substituted, $\langle \vec{v} \rangle_{k,l}$ and $\langle \vec{X} \rangle_{k,l}$ (defined by Eq.(3.11)) and $\langle \vec{u}, \vec{w} \rangle_{k,l}$ (given by Eq.(3.10)) may be expressed as

$$
\langle \vec{v} \rangle_{k,l} = \langle \vec{v} \rangle_{(kl)} \pm \frac{\sigma_\nu^2}{2\sigma_\nu^2} \vec{v}_{[kl]}, \quad \langle \vec{X} \rangle_{k,l} = \langle \vec{X} \rangle_{(kl)} \pm \frac{\sigma_\nu^2}{2\sigma_\nu^2} \vec{v}_{[kl]} t_{\nu},
$$

$$
\langle \vec{u}, \vec{w} \rangle_{k,l} = \langle \vec{u}, \vec{w} \rangle_{(kl)} \pm (\vec{u}, \vec{w})'_{kl} + (\vec{u}, \vec{w})''_{kl}, \quad (B.1)
$$

where

$$
\langle \vec{u}, \vec{w} \rangle'_{kl} = \frac{1}{\sigma_\nu^4} \left\{ \sigma_\nu^2 \sigma_\mu^2 \vec{u}_{[(kl)]} \vec{w}_{[(kl)]} + \sigma_\nu^2 \sigma_\pi^2 \vec{u}_{[(kl)] \pi} \vec{w}_{[(kl)] \pi} + \sigma_\mu^2 \sigma_\pi^2 \vec{u}_{[\mu \pi]} \vec{w}_{[\mu \pi]} \right\},
$$

$$
\langle \vec{u}, \vec{w} \rangle''_{kl} = \frac{1}{2\sigma_\nu^4} \left\{ \sigma_\nu^2 (\vec{u}_{[(kl)] \pi} \vec{w}_{[(kl)] \pi} + \vec{u}_{[(kl)] \pi} \vec{w}_{[(kl)] \pi}) + \sigma_\pi^2 (\vec{u}_{[(kl)]} \vec{w}_{[(kl)]} + \vec{u}_{[(kl)]} \vec{w}_{[(kl)]}) \right\},
$$

$$
\langle \vec{u}, \vec{w} \rangle''_{kl} = \frac{1}{4\sigma_\nu^4} \sigma_\nu^2 \sigma_\mu^2 \vec{u}_{[\mu \pi]} \vec{w}_{[\mu \pi]}.(B.2)
$$

Accordingly, $a_{k,l}$, $b_{k,l}$ and $c_{k,l}$ may be expressed as

$$
a_{k,l} = a_{(kl)} \pm a'_{kl} + a''_{kl},
$$

$$
b_{k,l} = b_{(kl)} \pm b'_{kl} + b''_{kl},
$$

$$
c_{k,l} = c_{(kl)} \pm c'_{kl} + c''_{kl} \quad \text{ (B.3)}
$$

In the case in which $\vec{p}_k$ and $\vec{x}_{\nu_0}$ may be expressed as $\vec{p}_k = \vec{p}_\nu |\vec{p}_k|/|\vec{p}_\nu|$ (for any $k$) and as $\vec{x}_{\nu_0} = \vec{p}_\nu |\vec{x}_{\nu_0}|/|\vec{p}_\nu|$, one has

$$
\vec{p}_{[kl]} |\vec{v}_{[kl]}| = \frac{m_k^2 - m_l^2}{2E_{[kl]}} \left( \frac{(|\vec{p}_k| - |\vec{p}_l|)(|\vec{v}_k| - |\vec{v}_l|)}{4E_{[kl]}} \right) E_{[kl]},
$$

$$
\vec{p}_{[kl]} - \frac{2(\vec{v}_{[kl]} + E_{[kl]})}{2E_{[kl]} + \vec{v}_{[kl]} + \vec{v}_{[kl]}} = \frac{m_k^2 - m_l^2}{|\vec{p}_k| + |\vec{p}_l|} \frac{(|\vec{v}_k| - |\vec{v}_l|)(|\vec{v}_k| - |\vec{v}_l|)}{4E_{[kl]} + \vec{v}_{[kl]}^2 + \vec{v}_{[kl]}^2} E_{[kl]},
$$

$$
\vec{x}_{\nu_0}^2 - \frac{2(\vec{v}_{[kl]} |\vec{x}_{\nu_0}|^2)}{2E_{[kl]} + \vec{v}_{[kl]}^2} = \left( |\vec{x}_{\nu_0}|^2 - \frac{(|\vec{v}_k| + |\vec{v}_l|)(|\vec{v}_k| - |\vec{v}_l|)}{2(|\vec{v}_k| + |\vec{v}_l|)^2} \right) |\vec{x}_{\nu_0}|^2.
$$

22
where
\[ a_{(kl)} = \langle \vec{v}, \vec{v} \rangle_{(kl)}, \quad a'_{kl} = \langle \vec{v}, \vec{v}' \rangle_{kl}, \quad a''_{kl} = \langle \vec{v}, \vec{v}'' \rangle_{kl}, \]
\[ b_{(kl)} = \langle \vec{v}, \vec{X} \rangle_{(kl)}, \quad b'_{kl} = \langle \vec{v}, \vec{X}' \rangle_{kl}, \quad b''_{kl} = \langle \vec{v}, \vec{X}'' \rangle_{kl}, \]
\[ c_{(kl)} = \langle \vec{X}, \vec{X} \rangle_{(kl)}, \quad c'_{kl} = \langle \vec{X}, \vec{X}' \rangle_{kl}, \quad c''_{kl} = \langle \vec{X}, \vec{X}'' \rangle_{kl}. \] (B.4)

Substituting Eq.(B.3) into Eq.(4.2), which defines \( \zeta_{kl} \), one may derive Eq.(4.6). On the other hand, substituting Eqs.(B.2) and (B.4) into
\[ c_{(kl)} - a_{(kl)} t_{(kl)}^2 = \frac{1}{8\sigma_{\delta}^2 a_{(kl)} \kappa,\lambda,\lambda',\lambda''-(kl),\mu,\pi \sum_{i,j=x,y,z} \sigma_{\kappa}^2 \sigma_{\lambda}^2 \sigma_{\kappa'}^2 \sigma_{\lambda'}^2 (v_{[\kappa\lambda]i} X_{[\kappa'\lambda']j} - v_{[\kappa'\lambda']j} X_{[\kappa\lambda]i})^2. \]

Equation (4.5) therefore implies Eq.(4.7). From this, it follows that
\[ b_{(kl)} = -a_{(kl)} t_0, \quad c_{(kl)} = a_{(kl)} t_0. \] (B.5)

With the aid of Eqs.(B.5) and (B.6), one may verify that \( \zeta_{kl} \), Eq.(4.6), reduces to \( \zeta_{kl} (t_\nu - t_0)^2 \) with \( \zeta_{kl} \) given by Eq.(4.8).\(^{22}\)

Similarly, if Equation (4.1) and
\[ E_{k,l} = E_{(kl)} \pm \frac{1}{2} E_{[kl]} \] (B.7)
are substituted, \( \Delta E_{k,l} \) and \( \Delta \tilde{E}_{k,l} \) (defined respectively by Eq.(3.9) and Eq.(3.14)) may be expressed as
\[ \Delta E_{k,l} = \Delta E_{(kl)} \pm \frac{1}{2} E_{[kl]}, \quad \Delta \tilde{E}_{k,l} = \Delta \tilde{E}_{(kl)} \pm \frac{1}{2} \Delta \tilde{E}_{kl}, \] (B.8)
where
\[ \Delta \tilde{E}_{kl} = E_{[kl]} - \frac{\sigma_{\delta}^2}{\sigma^2} \tilde{v}_{[kl]} \Delta \vec{p}. \] (B.9)

Substituting Eqs.(B.3) and (B.8) into Eq.(4.12), which defines \( \eta_{kl} \), one readily verifies Eq.(4.16). Substituting Eqs.(3.9), (B.1) and (B.8) into Eq.(3.20), one may derive Eq.(4.20), which, if Equations (B.5) and (B.6) are further substituted, gives Eq.(4.21).

\(^{21}\) Note also that the following relations hold independently of Eq.(4.7):
\[ u''_{kl} = -a''_{kl} t_\nu, \quad c''_{kl} = a''_{kl} t_\nu^2. \] (B.5)

\(^{22}\) It is not difficult to verify that \( \tilde{c}_{kl} > 0 \).
References

[1] Z.Maki, M.Nakagawa and S.Sakata, "Remarks on the unified model of elementary particles", Prog.Theor.Phys. 28, 870-880 (1962); B.Pontecorvo, "Neutrino experiments and the problem of conservation of leptonic charge", Soviet Phys.JETP 26, 984-988 (1968).

[2] For a review of neutrino physics, see for example: W.C.Haxton and B.R.Holstein, "Neutrino physics", Am.J.Phys. 68, 15-32 (2000).

[3] Particle Data Group, W.-M.Yao et al., "Review of particle properties", J.Phys.G 33, 1-1232 (2006).

[4] See for example: M.M.Nieto, "Quantum interference: From kaons to neutrinos (with quantum beats in between)", hep-ph/9509370; C.Giunti and C.W.Kim, "Quantum mechanics of neutrino oscillations", hep-ph/0011074; see also E.Sassaroli, "Neutrino oscillations: a relativistic example of a two-level system", Am.J.Phys. 67, 869-875 (1999); C.Waltham, "Teaching neutrino oscillations", Am.J.Phys. 72, 742-752 (2004).

[5] B.Kayser, "On the quantum mechanics of neutrino oscillations", Phys.Rev. D24, 110-116 (1981).

[6] M.Beuthe, "Oscillations of neutrinos and mesons in quantum field theory", Phys.Rep. 375, 105-218 (2003), and references therein.

[7] C.Giunti, C.W.Kim, J.A.Lee and U.W.Lee, "Treatment of neutrino oscillations without resort to weak eigenstates", Phys.Rev. D48, 4310-4317 (1993); C.Giunti, C.W.Kim and U.W.Lee, "When do neutrinos cease to oscillate?”, Phys.Lett. B421, 237-244 (1998).

[8] A.Asahara, K.Ishikawa, T.Shimomura and T.Yabuki, "Neutrino oscillations in intermediate states II –Wave packets–", Prog.Theor.Phys. 113, 385-411 (2005).

[9] S.Y.Tsai, "Quantum interference –from neutrino oscillation to CP violation in the $B^0 - \bar{B}^0$ system–", in Proc. of 8th B Phys. Intern. Workshop held at Kawatabi, Miyagi, Japan, Oct. 29-31, 1998, ed. K.Abe et al. (Tohoku Univ. 1998), pp.95-100.

[10] Y.Takeuchi, Y.Tazaki, S.Y.Tsai and T.Yamazaki, "Wave packet approach to the equal-energy/momentum/velocity prescriptions of neutrino oscillation”, Mod.Phys.Lett. A14, 2329-2339 (1999).

[11] Y.Takeuchi, Y.Tazaki, S.Y.Tsai and T.Yamazaki, "How do neutrinos propagate? –Wave-packet treatment of neutrino oscillation–", Prog.Theor.Phys. 105, 471-482 (2001).
[12] K.Matsuda, Y.Takeuchi and S.Y.Tsai, "Quantum mechanics and kinematics of neutrino oscillation", in Proc. of Intern. Conf. on Flavor Physics (Zhang-Jia-Jie, China, May 31-June 6, 2001), ed. Y.L.Wu, (World Scientific 2002), pp.420-424.

[13] A.D.Dolgov, A.Yu.Morozov, L.B.Okun and M.G.Schepkin, "Do muons oscillate?", Nucl.Phys. B502, 3-18 (1997).

[14] See for example S.Gasiorowicz, Quantum Physics, John Wiley & Sons (1974), pp.27-43.

[15] S.Nussinov, "Solar neutrinos and neutrino mixing", Phys.Lett. 63B, 201-203 (1976).

[16] H.L.Lipkin, "Theories of non-experiments in coherent decays of neutral mesons", Phys.Lett. B348, 604-608 (1995); L.B.Okun, M.G.Schepkin and I.S.Tsukerman, "On the extra factor of two in the phase of neutrino oscillations", Nucl.Phys. B650, 443-446 (2003).

[17] See for example T.D.Lee, Particle Physics and Introduction to Field Theory, Harwood Academic Publishers (1981), p.87.

[18] H.L.Lipkin, "Neutrino oscillations as two-slit experiments in momentum space", Phys.Lett. B477, 195-200 (2000).

[19] E.Sassaroli, N.Y.Srivastava and A.Widom, "Charged lepton oscillations", hep-ph/9509261; Y.N.Srivastava, A.Widom and E.Sassaroli, "Associated lepton oscillation", in Proc. of Conf. on Results and Perspectives in Particle Physics (La Thuile, Italy, March 3-9,1996), ed. M.Greco (IFNF, Frascati, 1996), pp.125-138; S.N.Srivastava and A.Widom, "Of course muons can oscillate", hep-ph/9707268; Y.N.Srivastava, A.Widom and E.Sassaroli, "Charged lepton and neutrino oscillations", Eur.Phys.J. C2, 769-774 (1998); see also Y.N.Srivastava, A.Widom and E.Sassaroli, "Λ oscillations", Phys.Lett. B344, 436-440 (1995); J.Lowe et al., "No Λ oscillations", Phys.Lett. B384, 288-292 (1996).

[20] C.Giunti, C.W.Kim and U.W.Lee, "When do neutrinos really oscillate? Quantum mechanics of neutrino oscillations", Phys.Rev. D44, 3635-3640 (1991).