Resonances of Multichannel Systems*

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We describe structure of the $T$-matrices, scattering matrices, and Green functions on unphysical energy sheets in multichannel scattering problems with binary channels and in the three-body problem. Based on the explicit representations obtained for the values of $T$- and $S$-matrices on the unphysical sheets, we prove that the resonances belonging to an unphysical sheet are just those energies where the correspondingly truncated scattering matrix, taken in the physical sheet, has eigenvalue zero. We show, in addition, that eigenvectors of the truncated scattering matrix associated with its zero eigenvalue are formed of the breakup amplitudes for the respective resonant states.

I. INTRODUCTION

Resonances of multichannel systems play a crucial role in various problems of nuclear, atomic, and molecular physics. In a wider sense, resonances represent one of the most interesting and intriguing phenomena observed in scattering processes, and not only in quantum physics but also in optics, acoustics, radiophysics, mechanics of continua etc. Literature on resonances is enormous and in this short introduction we have a chance to mention only several key points in the history of the subject and to refer only to a few key approaches to quantum-mechanical resonances, necessarily leaving many others a part.

With a resonance of a quantum system one usually associates an unstable state that only exists during a certain time. The original idea of interpreting resonances in quantum mechanics as complex poles of the scattering amplitude (and hence, as those of the scattering matrix) goes back to G. Gamov [8]. For radially symmetric potentials, the interpretation of two-body resonances as poles of the analytic continuation of the scattering matrix has been entirely elaborated in terms of the Jost functions [12]. Beginning with E. C. Titchmarsh [34] it was also realized that the $S$-matrix resonances may show up as poles of the analytically continued Green functions.

Another, somewhat distinct approach to resonances is known as the complex scaling (or complex rotation) method. The complex scaling makes it possible to rotate the continuous spectrum of the $N$-body Hamiltonian in such a way that resonances in certain sectors of the complex energy plane turn into usual eigenvalues of the scaled Hamiltonian. In physics literature the origins of such an approach are traced back at least to C. Lovelace [19]. A rigorous approval of the complex scaling method has been done by E. Balslev and J. M. Combes [2]. A link between the $S$-matrix interpretation of resonances and its complex rotation counterpart was established by G. A. Hagedorn [9] who has proven that for a reasonable class of quickly decreasing potentials at least a part of the scaling resonances for an $N$-body system ($N \leq 4$) turns to be also the scattering matrix resonances. We remark that the complex scaling seems to be the most popular approach to practical calculation.

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of resonances, particularly in atomic and molecular systems (see, e.g., Refs. \[5,7,11,16,21\] and references cited therein).

If support of the interaction is compact, the resonances of a two-body system can be treated within the approach created by P. Lax and R. Phillips \[18\]. An advantage of the Lax-Phillips approach is in the opportunity of giving an elegant operator interpretation of resonances. The two-body resonances show up as the discrete spectrum of a dissipative operator which is the generator of the compressed evolution semigroup. An operator interpretation of resonances in multichannel systems, based on a \(2 \times 2\) operator matrix representation of a rather generic Hamiltonian, can be found in \[20\].

For more details on the history of the subject and other approaches to resonances, as well as for the bibliography we refer to books \[1,3,4,28,31,32\] (it might also be useful to look through the review parts of papers \[22\] and \[25\]). Here we only notice that, in contrast to the “normal” bound and scattering states, the resonant ones still remain a quite mysterious object and many questions related to resonances are still unanswered. This is partly related to the fact that, unlike the “normal” spectrum, resonances are not a unitary invariant of a self-adjoint (Hermitian) operator. Moreover, following to J. S. Howland \[10\] and B. Simon \[33\], one should conclude that no satisfactory definition of resonance can rely on a single operator on an abstract Hilbert space and always an extra structure is necessary. Say, an unperturbed dynamics (in quantum scattering theory) or geometric setup (in acoustical or optical problems). Resonances are as relative as the scattering matrix is itself.

In our approach present in this report we follow the typical setup where the resonances arising due to an interaction \(V\) are considered relative to the unperturbed dynamics described by the kinetic energy operator \(H_0\). The resolvent \(G(z) = (H - z)^{-1}\) of the total Hamiltonian \(H = H_0 + V\) is an analytic operator-valued function of \(z \in \mathbb{C} \setminus \sigma(H)\). The spectrum \(\sigma(H)\) of \(H\) is a natural boundary for holomorphy domain of \(G(z)\) considered as an operator-valued function. However the kernel \(G(\cdot, \cdot, z)\) may admit analytic continuation through the continuous spectrum of \(H\). Or the form \(\langle G(z) \phi, \psi \rangle\) may do this for any \(\phi, \psi\) of a dense subset of the Hilbert space \(\mathcal{H}\). Or the “augmented” resolvent \(PG(z)P\) admits such a continuation for \(P\) the orthogonal projection onto a subspace of \(\mathcal{H}\). In any of these cases one deals with the Riemann surface of an analytical function.

In the simplest example with \(H = H_0 = -\Delta\), the two-body kinetic energy operator in coordinate representation, we have

\[
G(x, x', z) = \frac{1}{4\pi} \frac{e^{i\frac{z}{2}|x-x'|}}{|x-x'|},
\]

where \(x, x'\) are three-dimensional vectors. Clearly, \(G(x, x', z)\) as a function of the energy \(z\) has a two-sheeted Riemann surface which simply coincides with that of the function \(z^{1/2}\).

In this way one arrives at the concept of the unphysical energy sheet(s). The copy of the complex energy plane where the resolvent \(G(z)\) is considered initially as an operator-valued function is called the physical sheet. The remainder of the Riemann surface is assumed to consist of the unphysical sheets (in general, an unphysical sheet may only be a small part of the complex plane).

Meanwhile, any analytic function is uniquely defined by its values given for an infinite set of points belonging to its initial domain and having at list one limiting point. Usually one knows the \(T\)-matrix or Green function on the whole physical sheet which means that, at least in principle, it should be possible to express their values on unphysical sheets through the ones on the physical sheet.

In \[25,26\] (see also \[24,27\]) we have found just such expressions. More precisely, we have derived explicit representations for the values of the two- and three-body \(G(z), T(z),\) and \(S(z)\) on unphysical energy sheets in terms of these quantities themselves only taken on the physical sheet.
The same has been also done for analogous objects in multichannel scattering problems with binary channels [26]. The representations obtained not only disclose the structure $T$-matrices, scattering matrices, and Green functions on unphysical energy sheets but they also show which blocks of the scattering matrix taken in the physical sheet are “responsible” for resonances on a certain unphysical sheet. This result paves the way to developing new methods for practical calculation of resonances in concrete multichannel systems and, in particular, in the three-body ones (see, e.g. [13, 14, 15]). As a matter of fact we reduce all the study of resonances to a work completely on the physical sheet.

The present report essentially extends the presentation given recently in [23].

II. TWO-BODY PROBLEM

In general, we assume that the interaction potential $v$ falls off in coordinate space not slower than exponentially. When studying resonances of a two–bodysystem with such an interaction one can employ equally well both coordinate and momentum representations. However in the three-body case it is much easier for us to work in the momentum space (for an explanation see [25], p. 149). This is one of the reasons why we proceed in the same way in the two-body case. Thus, for the two-body kinetic energy operator $h_0$ we set

$$t(k, k') = \frac{k^2 - z}{(k^2 - z)^2},$$

where $g(z) = (h - z)^{-1}$ denotes the resolvent of the perturbed Hamiltonian $h = h_0 + v$. The operator $t$ is the solution of the Lippmann-Schwinger equation

$$t(z) = v - v g_0(t(z), z),$$

that is, in terms of its kernel we have

$$t(k, k'; z) = v(k, k') - \int_{\mathbb{R}^3} d\mathbf{q} \frac{v(k, k') t(\mathbf{q}, k', z)}{\mathbf{q}^2 - z},$$

taking into account that the free Green function $g_0$ reads

$$g_0(k, k', z) = \frac{\delta(k - k')}{(k^2 - z)}.$$  

Clearly, all dependence of $t$ on $z$ is determined by the integral term on the right-hand side of (2.3) that looks like a particular case of the Cauchy type integral

$$\Phi(z) = \int_{\mathbb{R}^N} d\mathbf{q} \frac{f(q)}{\lambda + \mathbf{q}^2 - z}$$

(2.4)
for $N = 3$. Cauchy type integrals of the same form but for various $N$ we will also have below when considering a multichannel problem with binary channels in Sec. [III] and the three-body problem in Sec. [IV]

Let $\mathcal{R}_\lambda, \lambda \in \mathbb{C}$, be the Riemann surface of the function

$$
\zeta(z) = \begin{cases} 
(z - \lambda)^{1/2} & \text{if } N \text{ is odd,} \\
\log(z - \lambda) & \text{if } N \text{ is even.}
\end{cases} 
$$

(2.5)

If $N$ is odd, $\mathcal{R}_\lambda$ is formed of two sheets of the complex plane. One of them, where $(z - \lambda)^{1/2}$ coincides with the arithmetic square root $\sqrt{z - \lambda}$, we denote by $\Pi_0$. The other one, where $(z - \lambda)^{1/2} = -\sqrt{z - \lambda}$, is denoted by $\Pi_1$.

If $N$ is even, the number of sheets of $\mathcal{R}_\lambda$ is infinite. In this case as the index $\ell$ of a sheet $\Pi_\ell$ we take the branch number of the function $\log(z - \lambda)$ picked up from the representation $\log(z - \lambda) = \log|z - \lambda| + i2\pi\ell + i\phi$ with $\phi \in [0, 2\pi)$.

Usually the point $\lambda$ is called the branching point of the Riemann surface $\mathcal{R}_\lambda$.

The following statement can be easily proven by applying the residue theorem (if necessary, consult [26] for a proof).

Lemma 1. For a holomorphic $f(q), q \in \mathbb{C}^N$, the function $\Phi(z)$ given by (2.4) is holomorphic on $\mathbb{C} \setminus [\lambda, +\infty)$ and admits the analytic continuation onto the unphysical sheets $\Pi_\ell$ of the Riemann surface $\mathcal{R}_\lambda$ as follows

$$
\Phi(z|\Pi_\ell) = \Phi(z) - \ell \pi i(\sqrt{z - \lambda})^{N-2} \int_{S^{N-1}} \hat{d}\hat{q} f(\sqrt{z - \lambda}\hat{q}),
$$

(2.6)

where $S^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$ centered at the origin.

Notice that in (2.6) and further on the writing $z|\Pi_\ell$ means that position of $z$ is taken on the unphysical sheet $\Pi_\ell$. If the reference to $\Pi_\ell$ is not present and we write simply $z$ than one understands that we deal with exactly the same energy point but lying on (dropped onto) the physical sheet $\Pi_0$.

Now return to the two-body problem and set

$$
(g_0(z)f_1, f_2) \equiv \int_{\mathbb{R}^3} d\hat{q} \frac{f_1(q)f_2(q)}{q^2 - z},
$$

where $f_1$ and $f_2$ are holomorphic. Then by Lemma 1

$$
(g_0(z|\Pi_1)f_1, f_2) = (g_0(z|\Pi_0)f_1, f_2) - \pi i \sqrt{z} \int_{S^2} d\hat{q} f_1(\sqrt{z}\hat{q})f_2(\sqrt{z}\hat{q}),
$$

which means that the continuation of the free Green function $g_0(z)$ onto the unphysical sheet $\Pi_1$ can be written in short form as

$$
g_0(z|\Pi_1) = g_0(z) + a_0(z)j^*(z)j(z),
$$

(2.7)

where $a_0(z) = -\pi i \sqrt{z}$ and $j(z)$ is the operator forcing a (holomorphic) function $f$ to set onto the energy shell, i.e. $(j(z)f)(k) = f(\sqrt{z}k)$.

Taking into account (2.7), on the unphysical sheet $\Pi_1$ the Lippmann-Schwinger equation (2.2) turns into

$$
t' = v - v(g_0 + a_0j^*)t', \quad t' = t|\Pi_1.
$$
Hence \((I + v_0) t' = v - a_0 j^\dagger j t'\). Invert \(I + v_0\) by using the fact that \(t(z) = v - v_0 t\) and, hence, \((I + v_0)^{-1} v = t\):

\[
t' = t - a_0 j^\dagger j t'.
\]  
(2.8)

Apply \(j(z)\) to both sides of (2.8) and obtain \(jt' = jt - a_0 j^\dagger j t',\) which means

\[
(I + a_0 j^\dagger j) jt' = jt,
\]  
(2.9)

where \(I\) stands for the identity operator in \(L_2(S^2)\). Then observe that \(I + a_0 j^\dagger j\) is nothing but the two-body scattering matrix \(s(z)\) since the kernel of the latter for \(z \in \Pi_0\) is known to read

\[
s(\hat{k}, \hat{k}', z) = \delta(\hat{k}, \hat{k}') - \pi i \sqrt{z} t(\sqrt{z} \hat{k}, \sqrt{z} \hat{k}', z).
\]

Hence

\[
jt' = [s(z)]^{-1}jt.
\]  
(2.10)

Now go back to (2.8) and by using (2.10) get \(t' = t - a_0 t j^\dagger[s(z)]^{-1}jt\), that is,

\[
t(z|\Pi_1) = t(z) - a_0(z) t(z) j^\dagger(z)[s(z)]^{-1} j(z)t(z).
\]  
(2.11)

All entries on the right-hand side of (2.11) are on the physical sheet. This is just the representation for the two-body \(T\)-matrix on the unphysical sheet we looked for.

From (2.11) one immediately derives representations for the continued resolvent,

\[
g(z|\Pi_1) = g + a_0(I - gv) j^\dagger[s(z)]^{-1}j(I - vg),
\]  
(2.12)

and continued scattering matrix,

\[
s(z|\Pi_1) = \mathcal{E}[s(z)]^{-1}\mathcal{E},
\]  
(2.13)

where \(\mathcal{E}\) is the inversion, \((\mathcal{E}f)(\hat{k}) = f(-\hat{k})\). Hence, the resonances are nothing but zeros of the scattering matrix \(s(z)\) in the physical sheet. That is, the energy \(z\) on the unphysical sheet \(\Pi_1\) is a resonance if and only if there is a non-zero vector \(\mathcal{A}\) of \(L_2(S^2)\) such that

\[
s(z)\mathcal{A} = 0
\]  
(2.14)

for the same \(z\) on the physical sheet.

We remark that this fact is rather well known for the partial-wave Schrödinger equations in case of centrally-symmetric potentials. For this case the statement that the resonances correspond to zeros of the partial-wave scattering matrix \(s_l\) on the physical sheet of the complex energy plane follows from its representation (see, e.g., [1])

\[
s_l(p) = (-1)^l \frac{f_l(p)}{f_l(-p)}
\]

in terms of the Jost function \(f_l(p)\) where \(l\) stands for the angular momentum and \(p\) for the (scalar) complex momentum. This property of \(s_l(p)\) was explicitly noticed in the review article [22, p. 1357]. Generalizations of the statement to the case of multichannel problems with binary channels and to the three-body problem have been given in [26] and [25], respectively. We will discuss them below in Sec. III and IV.
The eigenfunction $\mathcal{A}$ in (2.14) represents the breakup amplitude of an unstable state associated with the resonance $z$. This means that in coordinate space the corresponding “Gamov vector”, i.e. the resonance solution to the Schrödinger equation, has the following asymptotics

$$
\psi_{\text{res}}(x) \sim \mathcal{A}(-\hat{x}) \frac{\exp(i z^{1/2} |\Pi_{1}|)}{|x|} (2.15)
$$

This claim is a particular case of the statement of Lemma 2 below.

It should be stressed that the asymptotics (2.15) contains no term with the incoming spherical wave

$$
\exp(-iz^{1/2} |\Pi_{1}|) |x|^{-1/2}.
$$

We conclude the section with a remark that in [29] (see also [30, Section 2] and [22, Section 3]) Yu. V. Orlov was very close to obtaining a representation that would be a version of the representation (2.11) for partial-wave two-body $T$-matrices in the case of centrally-symmetric potentials. As a matter of fact, only the last step has not been done in [29, 30], the one analogous to the transition from equation (2.8) to equation (2.11) by using relation (2.10).

### III. MULTICHANNEL PROBLEM WITH BINARY CHANNELS

From now on assume that $h$ is an $m \times m$ matrix Schrödinger operator of the form

$$
h = \begin{pmatrix}
\lambda_1 + h_0^{(1)} & v_{12} & \cdots & v_{1m} \\
v_{21} & \lambda_2 + h_0^{(2)} & \cdots & v_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
v_{m1} & v_{m2} & \cdots & \lambda_m + h_0^{(m)} + v_{mm}
\end{pmatrix},
$$

written in the momentum representation. Thus, we assume that

$$(h_0^{(\alpha)} f_{\alpha})(k_{\alpha}) = k_{\alpha}^2 f_{\alpha}(k_{\alpha}), \quad k_{\alpha} \in \mathbb{R}^{n_\alpha}, \quad f_{\alpha} \in L_2(\mathbb{R}^{n_\alpha}), \quad \alpha = 1,2,\ldots,m.$$ 

We restrict ourselves to the case where the channel dimensions $n_\alpha$ satisfy inequalities $n_\alpha \geq 3$, $\alpha = 1,2,\ldots,m$, and $1 \leq m < \infty$. For simplicity we assume that the potential/coupling terms $v_{\alpha\beta}(k_{\alpha},k_{\beta}')$ are holomorphic functions of their variables $k_{\alpha} \in \mathbb{C}^{n_\alpha}$ and $k_{\beta}' \in \mathbb{C}^{n_\beta}$, sufficiently rapidly decreasing as $\text{Re} k_{\alpha} \to \infty$ or $\text{Re} k_{\beta}' \to \infty$ (see [26]). The thresholds $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ are supposed to be distinct and arranged in ascending order: $\lambda_1 < \lambda_2 < \ldots < \lambda_m$.

We also introduce the notations

$$
\begin{pmatrix}
\lambda_1 + h_0^{(1)} & 0 & \cdots & 0 \\
0 & \lambda_2 + h_0^{(2)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_m + h_0^{(m)}
\end{pmatrix} \quad \text{and} \quad
\begin{pmatrix}
v_{11} & v_{12} & \cdots & v_{1m} \\
v_{21} & v_{22} & \cdots & v_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
v_{m1} & v_{m2} & \cdots & v_{mm}
\end{pmatrix}
$$
for the unperturbed Hamiltonian and the total interaction, respectively. By \( g_0(z) \) and \( g(z) \) we denote the corresponding resolvents,

\[
g_0(z) = (h_0 - z)^{-1} \quad \text{and} \quad g(z) = (h - z)^{-1}.
\]

Like in the one-channel (i.e. two-body) case of Sec. III we again begin with the study of the \( T \)-matrix

\[
t(z) = v - v g(z) v
\]

that is the solution to the Lippman-Schwinger equation

\[
t(z) = v - v g_0(z) t(z). \tag{3.2}
\]

Kernels \( t_{\alpha\beta}(k_\alpha, k'_\beta, z) \) of the block entries \( t_{\alpha\beta}(z) \) of the operator matrix \( t(z) \) solve the equation system

\[
t_{\alpha\beta}(k, k', z) = v_{\alpha\beta}(k, k') - \sum_{\gamma=1}^{m} \int_{\mathbb{R}_{\alpha\gamma}} d\mathbf{q} \frac{v_{\alpha\gamma}(k, \mathbf{q}) t_{\gamma\beta}(\mathbf{q}, k', z)}{\lambda_{\gamma} + \mathbf{q}^2 - z}. \tag{3.3}
\]

Similarly to the two-body \( T \)-matrix in equation (2.3), all dependence of the kernels \( t_{\alpha\beta}(k, k', z) \) on \( z \) is determined by the integrals on the right-hand-side of (3.3), which are the Cauchy type integrals just of the form (2.4).

In contrast to the two-body case, for \( m \geq 2 \) we arrive at a multi-sheeted Riemann surface with number of sheets greater than two. The reason is simple: every threshold \( \lambda_{\alpha}, \alpha = 1, 2, \ldots, m \), turns into a branching point. If all the channel dimensions \( n_{\alpha} \) are odd, the number of sheets should be equal to \( 2^m \), that is, in addition to the physical sheet the Riemann surface will contain \( 2^m - 1 \) unphysical ones. If at least one of \( n_{\alpha}'s \) is even, we will have a logarithmic branching point and the number of unphysical sheets will be necessarily infinite. In fact, this Riemann surface simply coincides with the Riemann surface \( \mathcal{R} \) of the vector-valued function

\[
\zeta(z) = (\zeta_1(z), \zeta_2(z), \ldots, \zeta_m(z)),
\]

where (cf. formula (2.5))

\[
\zeta_\alpha(z) = \begin{cases} 
(z - \lambda_\alpha)^{1/2} & \text{if } n_\alpha \text{ is odd,} \\
\log(z - \lambda_\alpha) & \text{if } n_\alpha \text{ is even,}
\end{cases} \quad \alpha = 1, 2, \ldots, m.
\]

To enumerate the sheets of \( \mathcal{R} \) it is natural to use a multi-index

\[
\ell = (\ell_1, \ell_2, \ldots, \ell_m),
\]

where each \( \ell_\alpha \) coincides with the branch number for the corresponding function \( \zeta_\alpha, \alpha = 1, 2, \ldots, m \). In particular, if \( n_\alpha \) is odd then \( \ell_\alpha \) may get only two values: either 0 or 1. For even \( n_\alpha \) the value of \( \ell_\alpha \) is allowed to be any integer. The sheets of \( \mathcal{R} \) are denoted by \( \Pi_\ell \). The physical sheet corresponds to the case where all components of \( \ell \) are equal to zero and thus it is denoted simply by \( \Pi_0 \).

Each sheet \( \Pi_\ell \) is a copy of the complex plane \( \mathbb{C}' \) cut along the ray \([\lambda_1, +\infty)\). The sheets are pasted to each other in a suitable way along edges of the cut segments between neighboring points in the set of the thresholds \( \lambda_\alpha, \alpha = 1, 2, \ldots, m \). In particular, if coming from the sheet \( \Pi_1(\ell_1, \ell_2, \ldots, \ell_m) \) the energy \( z \) crosses the interval \((\lambda_\alpha, \lambda_{\alpha+1}], \alpha = 1, 2, \ldots, m, \lambda_{m+1} \equiv +\infty, \) in the upward direction (i.e. passes from the region \( \text{Im} z < 0 \) to the region \( \text{Im} z > 0 \)), then it arrives at the sheet
\( \Pi(\ell_1, \ell_2, \ldots, \ell_n) \) with all indices beginning from \( \ell_{\alpha+1} \) remaining the same while the first \( \alpha \) indices \( \ell_j, 1 \leq j \leq \alpha \), change by unity. If \( n_j \) is odd then \( \ell'_j = 1 \) for \( \ell_j = 0 \) and \( \ell'_j = 0 \) for \( \ell_j = 1 \); if \( n_j \) is even then \( \ell'_j = \ell_j + 1 \). In the case where the energy \( z \) passes the same interval \( (\lambda_{\alpha}, \lambda_{\alpha+1}) \) downward, it arrives at the sheet \( \Pi(\ell_1', \ell_2', \ldots, \ell_{\alpha+1}', \ldots, \ell_m) \) where for odd \( n_j \) the indices \( \ell'_j \) are the same as in the previous case and for even \( n_j \) they change according to the rule \( \ell'_j = \ell_j - 1 \). The indices \( \ell_j \) with numbers \( j \geq \alpha + 1 \) remain unchanged.

Under the assumption that the kernels \( t_{\alpha\bar{\beta}}(\sqrt{z - \lambda_{\alpha}}, k, k', z) \) admit the analytic continuation in \( z \) through the cuts (the existence of such a continuation may be rigorously approved, see [26]) one can perform analytic continuation of the Lippman-Schwinger equation \( (3.3) \) from the physical sheet \( \Pi_0 \) onto any unphysical sheet \( \Pi_\ell \) of the surface \( \Re. \). Of course, the trajectory along which we pull \( z \) should avoid the branching points \( \lambda_{\alpha} \). Applying after each crossing the interval \( (\lambda_1, +\infty) \) the corresponding variant of formula \( (2.6) \) we arrive at the following result

\[
\begin{align*}
t_{\alpha\bar{\beta}}(k, k', z_{|\Pi_\ell}) = & v_{\alpha\bar{\beta}}(k, k') - \sum_{\gamma=1}^{m} \int_{\mathbb{R}_{\gamma}} dq \frac{v_{\alpha\gamma}(k, q) t_{\gamma\bar{\beta}}(q, k', z_{|\Pi_\ell})}{\lambda_{\gamma} + q^2 - z} \\
- & \sum_{\gamma=1}^{m} \ell_{\gamma} A_{\gamma}(z) \int_{\mathbb{R}_{\gamma}} d\bar{q} \, v_{\alpha\gamma}(k_{\alpha}, \sqrt{z - \lambda_{\gamma}\bar{q}}) t_{\gamma\bar{\beta}}(\sqrt{z - \lambda_{\gamma}\bar{q}}, k'_{\beta}, z_{|\Pi_\ell}),
\end{align*}
\]  

(3.4)

where

\[
A_{\gamma}(z) = -\pi i (\sqrt{z - \lambda_{\gamma}})^{n_{\gamma} - 2}
\]

(3.5)

Notice that the second integral term on the right-hand side of \( (3.4) \) includes the half-on-shell values \( t_{\gamma\beta}(k, k', z_{|\Pi_\ell}) \) of the \( T \)-matrix kernels \( t_{\gamma\beta}(q, k', z_{|\Pi_\ell}) \) taken on the unphysical sheet \( \Pi_\ell \). Thus, like in the two-body case of Sec. II it is convenient to introduce operators \( j_{\gamma}(z) \) forcing a holomorphic function \( f(q), q \in \mathbb{C}^n \), to set onto the corresponding energy shell, i.e.

\[
(j_{\gamma}(z) f)(\bar{q}) = f(\sqrt{z - \lambda_{\gamma}\bar{q}}), \quad \gamma = 1, 2, \ldots, m.
\]

From these operators we construct a block diagonal matrix

\[
J(z) = \begin{pmatrix}
j_1(z) & 0 & \cdots & 0 \\
0 & j_2(z) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & j_m(z)
\end{pmatrix}.
\]

Using this notation one easily rewrites equation \( (3.4) \) in the matrix form

\[
t(z_{|\Pi_\ell}) = v - v g_0(z) t(z_{|\Pi_\ell}) - v J_{\dagger}(z) L A(z) J(z) t(z_{|\Pi_\ell}),
\]

(3.6)

where \( L \) and \( A(z) \) are diagonal \( m \times m \) matrices with scalar entries,

\[
L = \begin{pmatrix}
\ell_1 & 0 & \cdots & 0 \\
0 & \ell_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \ell_m
\end{pmatrix} \quad \text{and} \quad A(z) = \begin{pmatrix}
A_1(z) & 0 & \cdots & 0 \\
0 & A_2(z) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_m(z)
\end{pmatrix},
\]

(3.7)

and \( J_{\dagger}(z) \) is the “transpose” of \( J(z) \) which means that the product \( t(z) J_{\dagger}(z) \) has half-on-shell kernels of the form \( v_{\alpha\beta}(k, \sqrt{z - \lambda_{\beta}k}) \).
When rearranging (3.6) we first transfer the term \(v_g(z)t(z|\Pi_t)\) to the left-hand side of (3.6) and obtain
\[
(I + v_g(z))t(z|\Pi_t) = v - vJ^\dagger(z)L\alpha(z)J(z)t(z|\Pi_t).
\] (3.8)

Our next step is to invert the operator \((I + v_g(z))\) (of course, this is only possible for \(z\) not belonging to the discrete spectrum of \(h\)). Here, we keep in mind that the energy \(z\) in this operator is from the physical sheet where the Lippmann-Schwinger equation (3.2) holds and thus \((I + v_g(z))^{-1}v = t(z)\). Using this inversion formula we then derive from (3.8) that
\[
t(z|\Pi_t) = t(z) - t(z)J^\dagger(z)L\alpha(z)J(z)t(z|\Pi_t).
\] (3.9)

At this point it is convenient to introduce another diagonal scalar \(m \times m\) matrix
\[
\tilde{L} = \begin{pmatrix} \tilde{\ell}_1 & 0 & \ldots & 0 \\ 0 & \tilde{\ell}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{\ell}_m \end{pmatrix}
\] (3.10)
whose diagonal entries are
\[
\tilde{\ell}_\alpha = \left\{ \begin{array}{ll} 0 & \text{if } \ell_\alpha = 0, \\ \text{Sign}(\ell_\alpha) = \frac{\ell_\alpha}{|\ell_\alpha|} & \text{if } \ell_\alpha \neq 0. \end{array} \right.
\]
Clearly, the matrices \(L, \tilde{L}, \) and \(A(z)\) commute. Moreover, \(\tilde{L}L = L\). Using these facts one rewrites (3.9) in a slightly different form
\[
t(z|\Pi_t) = t(z) - t(z)J^\dagger(z)L\alpha(z)\tilde{L}J(z)t(z|\Pi_t).
\] (3.11)

which means that the value \(t(z|\Pi_t)\) of the \(T\)-matrix \(t\) at a point \(z\) on the unphysical sheet \(\Pi_t\) is expressed through the value of \(t\) itself taken at the same point \(z\) on the physical sheet as well as through the half-on-shell value \(J(z)t(z|\Pi_t)\) taken still for \(z|\Pi_t\) and, in addition, multiplied by \(\tilde{L}\) from the left. Applying the product \(\tilde{L}J(z)\) to both side of (3.11) we arrive at a closed equation for \(\tilde{L}J(z)t(z|\Pi_t)\):
\[
[\hat{I} + \tilde{L}J(z)t(z)J^\dagger(z)L\alpha(z)]\tilde{L}J(z)t(z|\Pi_t) = \tilde{L}J(z)t(z),
\] (3.12)

where \(\hat{I}\) denotes the identity operator in the sum Hilbert space
\[
\mathcal{G} = L_2(S^{m_1-1}) \oplus L_2(S^{m_2-1}) \oplus \ldots \oplus L_2(S^{m_m-1}).
\] (3.13)

Therefore, at any point \(z\) in the physical sheet where the operator
\[
s_\ell(z) = \tilde{I} + \tilde{L}J(z)t(z)J^\dagger(z)L\alpha(z)
\] (3.14)
is invertible, we will have
\[
\tilde{L}J(z)t(z|\Pi_t) = [s_\ell(z)]^{-1}\tilde{L}J(z)t(z).
\] (3.15)

Notice that \(s_\ell(z)\) commutes with \(\tilde{L}\), i.e.
\[
\tilde{L}s_\ell(z) = s_\ell(z)\tilde{L},
\]
and hence
\[ LA(z)s_\ell(z)^{-1}\tilde{L} = LA(z)s_\ell(z)^{-1}. \] (3.16)

Taking into account equalities \( (3.15) \) and \( (3.16) \) we obtain from \( (3.12) \) the following result:
\[
\begin{align*}
t(z|_{\Pi_\ell}) &= t(z) - t(z)\hat{J}^\dagger(z)LA(z)[s_\ell(z)]^{-1}\hat{L}J(z)t(z) \\
&= t(z) - t(z)\hat{J}^\dagger(z)LA(z)[s_\ell(z)]^{-1}(z)J(z)t(z).
\end{align*}
\] (3.17) (3.18)

These are just the representations for \( t(z|_{\Pi_\ell}) \) we look for: in \( (3.17) \) and \( (3.18) \) values of the multichannel \( T \)-matrix on an arbitrarily chosen unphysical energy sheet \( \Pi_\ell \) are explicitly written in terms of the entries whose values are taken from the physical sheet. Formulas \( (3.17) \) and \( (3.18) \) are just the ones that represent a generalization of the two-body representation \( (2.11) \) to the case of multichannel Schrödinger operators with binary channels. A slightly different version of the representations \( (3.17) \) and \( (3.18) \) was first published in \([26]\).

The operator matrix \( s_\ell(z) \) given by \( (3.14) \) is closely related to the total scattering matrix for the problem which reads
\[ s(z) = \tilde{I} + J(z)t(z)\hat{J}^\dagger(z)A(z), \] (3.19)

Of course, the total scattering matrix contains neither entry \( L \) nor entry \( \tilde{L} \). For the matrix \( s_\ell(z) \) these entries play an important role. Depending on the unphysical sheet \( \Pi_\ell \) under consideration, certain rows and columns of the difference matrix \( (s_\ell(z) - \tilde{I}) = LJ(z)t(z)\hat{J}^\dagger(z)L \) completely consist of zero entries. Nullification takes place for those rows and columns of the difference matrix \( (s(z) - \tilde{I}) = J(z)t(z)\hat{J}^\dagger(z) \) whose numbers \( \alpha \) are such that the corresponding indices \( \ell_\alpha \) equal zero. This is a reason why we call \( s_\ell(z) \) the truncated scattering matrix associated with the unphysical sheet \( \Pi_\ell \).

Notice that if instead of \( (3.2) \) we start with the transposed Lippmann-Schwinger equation
\[ t(z) = v - t(z)g_0(z)v, \]
then in the same way we obtain for \( t(z|_{\Pi_\ell}) \) another representation that can be considered as a transposed version of the representation \( (3.17) \):
\[
\begin{align*}
t(z|_{\Pi_\ell}) &= t(z) - t(z)\hat{J}^\dagger(z)\tilde{L}[s_\ell^\dagger(z)]^{-1}A(z)LJ(z)t(z) \\
&= t(z) - t(z)\hat{J}^\dagger(z)[s_\ell^\dagger(z)]^{-1}A(z)LJ(z)t(z),
\end{align*}
\] (3.20) (3.21)

where
\[ s_\ell^\dagger(z) = \tilde{I} + LA(z)J(z)t(z)\hat{J}^\dagger(z)\tilde{L}. \]

The operator \( s_\ell^\dagger(z) \) represents the result of truncation of the transposed \( S \)-matrix
\[ s^\dagger(z) = \tilde{I} + A(z)J(z)t(z)\hat{J}^\dagger(z). \]

From the uniqueness of the analytic continuation by \( (3.17) \) and \( (3.20) \) it immediately follows that
\[ t(z)\hat{J}^\dagger(z)LA(z)s_\ell(z)^{-1}J(z)t(z) = t(z)\hat{J}^\dagger(z)[s_\ell^\dagger(z)]^{-1}A(z)LJ(z)t(z). \]

To describe structure of the scattering matrices \( s(z) \) or \( s^\dagger(z) \) analytically continued to an unphysical sheet \( \Pi_\ell \) we need some more notations. First, introduce a block diagonal operator matrix
\( \mathcal{E}(\ell) \) of the form \( \mathcal{E} = \text{diag} (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m) \) where \( \mathcal{E}_\alpha \) is the identity operator on \( L_2(S^{n_\alpha - 1}) \) if \( \ell_\alpha \) is even and \( \mathcal{E}_\alpha \) is the involution, \( (\mathcal{E}_\alpha f)(\mathbf{k}) = f(-\mathbf{k}) \), if \( \ell_\alpha \) is odd. Second, let \( e(\ell) \) be a scalar diagonal matrix, \( e = \text{diag} (e_1, e_2, \ldots, e_m) \), with the main diagonal entries \( e_\alpha \) defined by

\[
e_\alpha = \begin{cases} +1 & \text{for any } \ell_\alpha = 0, \pm 1, \pm 2, \ldots \text{ if } n_\alpha \text{ is even}, \\ +1 & \text{if } n_\alpha \text{ is odd and } \ell_\alpha = 0, \\ -1 & \text{if } n_\alpha \text{ is odd and } \ell_\alpha = 1. \end{cases}
\]

That is, \( e_\alpha \) only depend on the corresponding \( n_\alpha \) and \( \ell_\alpha \). It is obvious that if a matrix-valued function \( A(z) \) is defined on the physical sheet of the Riemann surface \( \mathbb{R} \) by formulas (3.5) and (3.7), then after the analytic continuation to the sheet \( \Pi_\ell \) it acquires the form

\[
A(z) \big|_{\Pi_\ell} = A(z)e(\ell).
\]

Now we are ready to present our main result concerning the \( S \)-matrices. We claim that after continuation to the sheet \( \Pi_\ell \) their values are expressed by the formulas

\[
s(z \big|_{\Pi_\ell}) = \mathcal{E} \left[ \hat{I} + i A e - i LA s_\ell e^{-1} i A e \right] \mathcal{E},
\]

\[
s^\dagger(z \big|_{\Pi_\ell}) = \mathcal{E} \left[ \hat{I} + e A^\dagger e - e A e^{-1} A L e \right] \mathcal{E},
\]

where we use another shorthand notation

\[
\tilde{r}(z) = J(z) t(z) J^\dagger(z).
\]

The argument \( z \) of the operator-valued functions \( s_\ell(z) \), \( s_\ell^\dagger(z) \), \( J(z) \), \( J^\dagger(z) \), and \( A(z) \) on the right-hand sides of (3.23)–(3.24) is a point on the physical sheet \( \Pi_0 \) having just the same position on the complex plane as the point \( z \big|_{\Pi_\ell} \) on the sheet \( \Pi_\ell \) on the left-hand sides of (3.23) and (3.24), respectively.

At last, we present the representation for the continued resolvent on the sheet \( \Pi_\ell \):

\[
g(z \big|_{\Pi_\ell}) = g + (I - g v)^\dagger A L s_\ell e^{-1} J (I - v g),
\]

\[
= g + (I - g v)^\dagger [s_\ell^\dagger(z)]^{-1} A L J (I - v g).
\]

In this report we skip derivation of the representations (3.23)–(3.26). The interested reader may find it in [27, Sections 1.4 and 1.5] (see also [26]). Here we only remark that the derivation is rather straightforward being based directly on the representations (3.20) or (3.21) for the \( T \)-matrix.

The most important consequence of the representations (3.23)–(3.26) is the fact that all energy singularities of the \( T \)-matrix, scattering matrices, and resolvent on an unphysical sheet \( \Pi_\ell \), differing of those in the physical sheet, are just the singularities of the inverse truncated scattering matrix \( [s_\ell(z)]^{-1} \) (or, and this is the same, the ones of its transpose \( [s_\ell^\dagger(z)]^{-1} \)). This means that

\[
\text{resonances on sheet } \Pi_\ell \text{ correspond exactly to the points } z \text{ on the physical sheet where the operator } s_\ell(z) \text{ has eigenvalue zero,} \tag{R}
\]

i.e. the resonances on \( \Pi_\ell \) are those energies \( z \) on \( \Pi_0 \) where equation

\[
s_\ell(z) \mathcal{A} = 0
\]

(3.27)
has a non-trivial solution $\mathcal{A} \neq 0$ in the sum Hilbert space $\mathfrak{S}$ given by (3.13).

Eigenvectors of the truncated scattering matrices $s_{\ell}(z)$ associated with resonances have a quite transparent physical meaning. Assume that $z$ is a resonance on the unphysical sheet $\Pi_{\ell}$. This implies that for the same energy $z$ on the physical sheet $\Pi_0$ equation (3.27) has a solution $\mathcal{A} \neq 0$, $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$. Clearly, the components $\mathcal{A}_\alpha$ of the vector $\mathcal{A}$ are non-zero only for the channels $\alpha$ such that $l_\alpha \neq 0$. Taking into account that (3.27) can be written in the equivalent form

$$\mathcal{A} = -\mathcal{L}Jt(z)J^\dagger LA(z)\mathcal{A},$$

this conclusion follows from equality

$$(I - \mathcal{L})\mathcal{A} = 0.$$ 

Notice that the latter holds since $\mathcal{L}(I - \mathcal{L}) = 0$.

Along with the vector $\mathcal{A}$ we also consider an “extended” vector $\tilde{\mathcal{A}}$ that is obtained of $\mathcal{A}$ as a result of replacing the projection $\mathcal{L}$ on the right-hand side of (3.28) with the identity operator, i.e.

$$\tilde{\mathcal{A}} = -Jt(z)J^\dagger LA(z)\mathcal{A}.$$ 

Clearly, $\mathcal{A} = \mathcal{L}\tilde{\mathcal{A}}$.

We claim that up to scalar factors the components $\tilde{\mathcal{A}}_1(\hat{k}_1), \tilde{\mathcal{A}}_2(\hat{k}_2), \ldots, \tilde{\mathcal{A}}_m(\hat{k}_m)$ of the eigenvector $\tilde{\mathcal{A}}$ make sense of the breakup amplitudes of the corresponding resonance state in channels 1, 2, ..., and $m$, respectively. In particular, these amplitudes determine angular dependence of coefficients at the spherical waves in the asymptotics of the channel components of the resonant solution to the Schrödinger equation in coordinate representation.

To give some details, let us denote by $h_0^\#$ and $v^\#$ the coordinate-space version (Fourier transform) of the operators $h_0$ and $v$, respectively. Namely, let

$$h_0^\# = \text{diag}(\lambda_1 - \Delta_{x_1}, \lambda_2 - \Delta_{x_2}, \ldots, \lambda_m - \Delta_{x_m}),$$

where $\Delta_{x_\alpha}, \alpha = 1, 2, \ldots, m$, stands for the Laplacian in variable $x_{\alpha} \in \mathbb{R}^{n_\alpha}$.

In the statement below we restrict ourselves to the case where absolute values of the unphysical-sheet indices corresponding to the even-dimensional channels are less than or equal unity, i.e. we assume that if $n_\alpha$ is even then $|l_\alpha| \leq 1$. Recall that if $n_\alpha$ is odd then automatically $l_\alpha = 0$ or $l_\alpha = 1$.

**Lemma 2.** Assume that $z$ is a resonance on an unphysical sheet $\Pi_{\ell}$ with multi-index $\ell = (\ell_1, \ell_2, \ldots, \ell_m)$ such that $|\ell_\alpha| \leq 1$ for all $\alpha = 1, 2, \ldots, m$. Let $\mathcal{A} \in \mathfrak{S}$ be a non-zero solution to equation (3.27) for the same energy $z$ but belonging to the physical sheet. Then for this $z$ the Schrödinger equation

$$(h_0^\# + v^\#) \psi^\# = z\psi^\#$$

has a non-zero (resonant) solution $\psi^\#_{\text{res}, \ell} = (\psi^\#_{\text{res}, 1}, \psi^\#_{\text{res}, 2}, \ldots, \psi^\#_{\text{res}, n})^\dagger$ whose components $\psi^\#_{\text{res}, \alpha}(x_\alpha)$ for $\ell_\alpha \neq 0$ possess exponentially increasing asymptotics,

$$\psi^\#_{\text{res}, \alpha}(x_\alpha) \xrightarrow{x_\alpha \to \infty} C_{\alpha}(z, \ell_\alpha)(\mathcal{A}(z_\alpha) + o(1))\frac{e^{-\sqrt{z - \lambda_\alpha}|x_\alpha|}}{|x_\alpha|^{(n_\alpha - 1)/2}},$$

while for $\ell_\alpha = 0$ their asymptotics is exponentially decreasing,

$$\psi^\#_{\text{res}, \alpha}(x_\alpha) \xrightarrow{x_\alpha \to \infty} C_{\alpha}(z, \ell_\alpha)(\mathcal{A}(z_\alpha) + o(1))\frac{e^{\sqrt{z - \lambda_\alpha}|x_\alpha|}}{|x_\alpha|^{(n_\alpha - 1)/2}}.$$
where \( \tilde{\alpha}(k_\alpha) \) stand for the corresponding components of the extended vector (3.29) and

\[
C_\alpha(z, \ell_\alpha) = \sqrt{\frac{\pi}{2}} e^{i\frac{na-(\alpha-1)}{4}}\pi(z - \lambda_\alpha)^{\frac{na-3}{4}}
\] (3.33)

For the function \( (z - \lambda_\alpha)^{\frac{na-3}{4}} \) on the right-hand side of (3.33) the main branch is chosen.

Complete proof of this statement may be found in [27, Section 1.6].

The functions \( \psi^{\text{res}}_{\alpha}(x_\alpha) \) taken altogether form the Gamov vector corresponding to the resonance energy \( z \) (see, e.g. [4, 28]). Just asymptotic formulas (3.31) and (3.32) prove that the functions \( \tilde{\alpha}(k_\alpha), \ell_\alpha \neq 0 \), and \( \tilde{\alpha}(k_\alpha), \ell_\alpha = 0 \), make sense of the breakup amplitudes describing decay of the resonant state along open and closed channels, respectively.

IV. THREE-BODY PROBLEM

In this section we give a sketch of our results on the structure of the \( T \)-matrix, scattering matrices, and Green function on unphysical energy sheets in the three-body problem. For detail exposition of this material see Refs. [25] or [27].

Let \( H_0 \) be the three-body kinetic energy operator in the center-of-mass system. Assume that there are no three-body forces and thus the total interaction reads \( V = v_1 + v_2 + v_3 \) where \( v_{\alpha}, \alpha = 1, 2, 3 \), are the corresponding two-body potentials having just the same properties as in Sec. II.

The best way to proceed in the three-body case is to work with the Faddeev components \[6\]

\[
M_{\alpha\beta} = \delta_{\alpha\beta}v_\alpha - v_\alpha G(z)v_\beta \quad (\alpha, \beta = 1, 2, 3)
\]
of the \( T \)-operator \( T(z) = V - VG(z)V \) where \( G(z) \) denotes the resolvent of the total Hamiltonian \( H = H_0 + V \). The components \( M_{\alpha\beta} \) satisfy the Faddeev equations

\[
M_{\alpha\beta}(z) = \delta_{\alpha\beta}t_\alpha(z) - t_\alpha(z)G_0(z)\sum_{\gamma \neq \alpha} M_{\gamma\beta}(z)
\] (4.1)

with \( G_0(z) = (H_0 - z)^{-1} \) and

\[
t_\alpha(P, P', z) = t_\alpha(k_\alpha, k'_\alpha, z - p_\alpha^2)\delta(p_\alpha - p'_\alpha)
\]

where \( k_\alpha, p_\alpha \) denote the corresponding reduced Jacobi momenta (see [25] for the precise definition we use) and \( P = (k_\alpha, p_\alpha) \in \mathbb{R}^6 \) is the total momentum.

Assume that any of the three two-body subsystems has only one bound state with the corresponding energy \( \varepsilon_\alpha < 0 \), \( \alpha = 1, 2, 3 \). Assume in addition that all of these three binding energies are different. It is easy to see that the thresholds \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \), and 0 are associated with particular Cauchy type integrals in the integral equations (4.1). By Lemma 1 the two-body thresholds \( \varepsilon_\alpha \) appear to be square-root branching points while the three-body threshold 0 is the logarithmic one. In order to enumerate the unphysical sheets we introduce the multi-index \( \ell = (\ell_0, \ell_1, \ell_2, \ell_3) \) with \( \ell_0 = \ldots, -1, 0, 1, \ldots \) and \( \ell_\alpha = 0, 1 \) if \( \alpha = 1, 2, 3 \). Clearly, only encircling the two-body thresholds one arrives at seven unphysical sheets. The three-body threshold generates infinitely many unphysical sheets. (There might also be additional branching points on the unphysical sheets, in particular due to two-body resonances.)
It turns out that the analytically continued Faddeev equations (4.1) can be explicitly solved in terms of the matrix $M = \{M_{\alpha \beta}\}$ itself taken only on the physical sheet, just like in the case of the two-body $T$-matrix in Sec. II and multichannel $T$-matrix in Sec. III. The result strongly depends, of course, on the unphysical sheet $\Pi_\ell$ concerned. More precisely, the resulting representation reads as follows

$$M|_{\Pi_\ell} = M + Q_M L S_\ell^{-1} \tilde{L} \tilde{Q}_M. \quad (4.2)$$

In the particular case we deal with, $L$ and $\tilde{L}$ are $4 \times 4$ diagonal scalar matrices of the form $L = \text{diag}(\ell_0, \ell_1, \ell_2, \ell_3)$ and $\tilde{L} = \text{diag}(|\ell_0|, \ell_1, \ell_2, \ell_3)$, respectively; $S_\ell(z) = I + \tilde{L}(S(z) - I)L$ is a truncation of the total scattering matrix $\mathcal{S}(z)$ and the entries $Q_M, \tilde{Q}_M$ are explicitly written in terms of the half-on-shell kernels of $M$ (see formula (7.34) of [25]). From (4.2) one also derives explicit representations for $G(z|_{\Pi_\ell})$ and $S(z|_{\Pi_\ell})$ similar to those of (3.25) and (3.23), respectively.

Thus, to find resonances on the sheet $\Pi_\ell$ one should simply look for the zeros of the truncated scattering matrix $S_\ell(z)$, that is, for the points $z$ on the physical sheet where equation $S_\ell(z)\mathcal{A} = 0$ has a non-trivial solution $\mathcal{A}$. The vector $\mathcal{A}$ will consist of amplitudes of the resonance state to breakup into the various possible channels. Within such an approach one can also find the three-body virtual states.

In order to find the amplitudes involved in $S_\ell$, one may employ any suitable method, for example the one of Refs. [13, 14, 15] based on the Faddeev differential equations. In these works the approach we discuss has been successfully applied to several three-body systems. In particular, the mechanism of emerging the Efimov states in the $^4\text{He}$ trimer has been studied [13, 15].

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[1] V. de Alfaro and T. Regge, Potential scattering (North-Holland, Amsterdam, 1965).
[2] E. Balslev and J. M. Combes, Commun. Math. Phys. 22 (1971), 280.
[3] A. Baz, Ya. Zeldovich, and A. Perelomov, Scattering, Reactions and Decays in Nonrelativistic Quantum Mechanics (Israel Program for Scientific Translations, Jerusalem, 1969).
[4] A. Böhm, Quantum Mechanics: Foundations and Applications (Springer-Verlag, 1986).
[5] E. Brändas and N. Elander (Eds.), Resonances: The Unifying Route Towards the Formulation of Dynamical Processes — Foundations and Applications in Nuclear, Atomic, and Molecular Physics, Lect. Notes Phys. 325 (Springer-Verlag, Berlin, 1989).
[6] L. D. Faddeev, Mathematical Aspects of the Three-Body Problem in Quantum Mechanics (Israel Program for Scientific Translations, Jerusalem, 1965).
[7] D. V. Fedorov, E. Garrido, and A. S. Jensen, Few-Body Syst. 33:2-3 (2003), 153.
[8] G. Gamow, Z. Phys. 51 (1928), 204.
[9] G. A. Hagedorn, Commun. Math. Phys. 65 (1979), 181.
[10] J. S. Howland, Pacific J. Math. 55:1 (1974), 157.
[11] C.-Y. Hu and A. K. Bhatia, Muon Catalyzed Fusion 5/6 (1990/91), 439.
[12] R. Jost, Helv. Phys. Acta. 20 (1947), 250.
[13] E. A. Kolganova and A. K. Motovilov, Comp. Phys. Comm. 126 (2000), 88; arXiv: physics/9810005.
[14] E. A. Kolganova and A. K. Motovilov, Phys. Atom. Nucl. 60 (1997), 177; arXiv: nucl-th/9602001.
[15] E. A. Kolganova and A. K. Motovilov, Phys. Atom. Nucl. 62 (1999), 1179; arXiv: physics/9808027.
[16] V. I. Korobov, Phys. Rev. A 67 (2003), 062501.
[17] V. I. Kukulin, V. M. Krasnopol’sky, and J. Horáček, Theory of Resonances: Principles and Applications (Academia, Praha, 1989).
[18] P. D. Lax and R. S. Phillips, Scattering Theory (Academic Press, N.Y.–London, 1967).
[19] C. Lovelace, Phys. Rev. 135:5B (1964), 1225.
[20] R. Mennicken and A. K. Motovilov, Math. Nachr. 201 (1999), 117; arXiv: funct-an/9708001.
[21] N. Moiseyev, Phys. Rep. 302 (1998), 211.
[22] K. Möller and Yu. V. Orlov, Fiz. Elem. Chast. At. Yadra 20 (1989), 1341 (Russian).
[23] A. K. Motovilov, Few-Body Syst. 38 (2006), 115; arXiv: physics/0511238.
[24] A. K. Motovilov, Fiz. Elem. Chast. At. Yadra 32:7 (2001), 144 (Russian).
[25] A. K. Motovilov, Math. Nachr. 187 (1997), 147; arXiv: funct-an/9509003.
[26] A. K. Motovilov, Theor. Math. Phys. 97 (1993), 692; DOI: 10.1007/BF01017515.
[27] A. K. Motovilov, Theory of Resonances in Multichannel Systems, D.Sc. Thesis (JINR, Dubna, 2006; Russian); available at http://theor.jinr.ru/˜motovilv/DSc-Thesis.pdf (2007).
[28] R. G. Newton, Scattering Theory of Waves and Particles, 2nd ed. (McGraw Hill, N.Y., 1982).
[29] Yu. V. Orlov, Pis’ma v ZhETF 33:7 (1981), 380 (Russian).
[30] Yu. V. Orlov and V. V. Turovtsev, ZhETF 86 (1984), 1600 (Russian).
[31] M. Reed and B. Simon, Methods of Modern Mathematical Physics, III: Scattering Theory (Academic Press, N.Y., 1979).
[32] M. Reed and B. Simon, Methods of Modern Mathematical Physics, IV: Analysis of Operators (Academic Press, N.Y., 1978).
[33] B. Simon, Int. J. Quant. Chem. 14 (1978), 529.
[34] E. C. Titchmarsh, Eigenfunction Expansions Associated with Second Order Differential Equations, Vol. II (Oxford U. P., London, 1946).