FAMILIES OF MINIMALLY NON-GOLOD COMPLEXES
AND THEIR POLYHEDRAL PRODUCTS

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Abstract. We consider families of simple polytopes $P$ and simplicial
complexes $K$, well-known in polytope theory and convex geometry, and
show that their moment-angle complexes have some remarkable ho-
motopy properties which depend on combinatorics of the underlying
complexes and algebraic properties of their Stanley–Reisner rings. We
introduce infinite series of Golod and minimally non-Golod simplicial
complexes $K$ with moment-angle complexes $Z_K$ having free integral
cohomology but not homotopy equivalent to a wedge of spheres or a
connected sum of products of spheres respectively. We then prove a
criterion for a simplicial multiwedge and composition of complexes to
be Golod and minimally non-Golod and present a class of minimally
non-Golod polytopal spheres.

1. Introduction

We denote by $K$ a simplicial complex of dimension $n - 1$ on $m$ vertices
and by $k$ a field or the ring of integers. Let $k[v_1, \ldots, v_m]$ be the graded
polynomial algebra on $m$ variables, $\deg(v_i) = 2$. The face ring (or the
Stanley–Reisner ring) of $K$ over $k$ is the quotient ring

$$k[K] = k[v_1, \ldots, v_m]/I_K$$

where $I_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_k}$
for which $\{i_1, \ldots, i_k\}$ is not a simplex in $K$. We refer to $I_K$ as the Stanley–
Reisner ideal of $K$. Note that $k[K]$ is a $k$-algebra and a module over
$k[v_1, \ldots, v_m]$ via the quotient projection.

Let $P$ be a simple $n$-dimensional convex polytope with $m$ facets (i.e faces
of codimension 1) $F_1, \ldots, F_m$. Denote by $K_P$ the boundary $\partial P^*$ of the dual
simplicial polytope. It can be viewed as a $(n - 1)$-dimensional simplicial complex on the set $[m]$, whose simplices are subsets $\{i_1, \ldots, i_k\}$ such that
$F_{i_1} \cap \ldots \cap F_{i_k} \neq \emptyset$ in $P$.

Suppose $(X, A) = \{(X_i, A_i)\}_{i=1}^m$ is a set of topological pairs. A polyhedral
product is a topological space:

$$(X, A)^K = \bigcup_{I \in K} (X, A)^I,$$
where \((X, A)^I = \prod_{i=1}^{m} Y_i\) for \(Y_i = X_i\), if \(i \in I\), and \(Y_i = A_i\), if \(i \notin I\). Particular cases of a polyhedral product \((X, A)^K\) are moment-angle complexes \(Z_K = (\mathbb{D}^2, S^1)^K\) and real moment-angle complexes \(R_K = (\mathbb{D}^1, S^0)^K\). We also call \(Z_P = Z_{KP}\) the moment-angle manifold of \(P\). By [6, Corollary 6.2.5], \(Z_P\) has a structure of a smooth manifold of dimension \(m + n\).

The Tor-groups of \(\mathbb{k}[K]\) acquire a topological interpretation by means of the following result on the cohomology of \(Z_K\).

**Theorem 1.1** ([6, Theorem 4.5.4] or [18, Theorem 4.7]). The cohomology algebra of the moment-angle complex \(Z_K\) is given by the isomorphisms

\[
H^{*,*}(Z_K; \mathbb{k}) \cong \text{Tor}^{*,*}_{\mathbb{k}[v_1, \ldots, v_m]}(\mathbb{k}[K], \mathbb{k})
\]

\[
\cong H[\Lambda[u_1, \ldots, u_m] \otimes \mathbb{k}[K] , d]
\]

\[
\cong \bigoplus_{I \subset [m]} H^*(K_I),
\]

where bigrading and differential in the cohomology of the differential bigraded algebra are defined by

\[
\text{bideg } u_i = (-1, 2), \text{ bideg } v_i = (0, 2); \quad du_i = v_i, \quad dv_i = 0.
\]

In the third row, \(H^*(K_I)\) denotes the reduced simplicial cohomology of the full subcomplex \(K_I\) of \(K\) (the restriction of \(K\) to \(I \subset [m]\)). The last isomorphism is the sum of isomorphisms

\[
H^p(Z_K) \cong \sum_{I \subset [m]} H^{p-|I|-1}(K_I),
\]

and the ring structure is given by the maps

\[
\tilde{H}^{p-|I|-1}(K_I) \otimes \tilde{H}^{q-|J|-1}(K_J) \to \tilde{H}^{p+q-|I|-|J|-1}(K_{I \cup J}),
\]

which are induced by the canonical simplicial maps \(K_{I \cup J} \hookrightarrow K_I * K_J\) (join of simplicial complexes) for \(I \cap J = \emptyset\) and zero otherwise.

Additively the following theorem due to Hochster holds.

**Theorem 1.2** ([11]). For any simplicial complex \(K\) on \(n\) vertices we have:

\[
\text{Tor}^{-i,2j}_{\mathbb{k}[v_1, \ldots, v_m]}(\mathbb{k}[K], \mathbb{k}) \cong \bigoplus_{J \subset [m], |J|=j} \tilde{H}^{j-i-1}(K_J).
\]

The ranks of the bigraded components of the Tor-algebra

\[
\beta^{-i,2j}(\mathbb{k}[K]) = \text{rk}_\mathbb{k} \text{Tor}^{-i,2j}_{\mathbb{k}[v_1, \ldots, v_m]}(\mathbb{k}[K], \mathbb{k})
\]

are called the bigraded Betti numbers of \(\mathbb{k}[K]\) or \(K\), when the base field \(\mathbb{k}\) is fixed.

A face ring \(\mathbb{k}[K]\) is called Golod if the multiplication and all higher Massey operations in \(\text{Tor}_{\mathbb{k}[v_1, \ldots, v_m]}(\mathbb{k}[K], \mathbb{k})\) are trivial. This property was first considered by Golod [8] in his study of local rings with rational Poincaré series, see also Gulliksen and Levin [10]. Due to the result of Berglund and Jöllenbeck [5, Theorem 5.1] \(\mathbb{k}[K]\) is Golod when the product in the Tor-algebra is trivial over a field \(\mathbb{k}\). We say that \(K\) is a Golod complex when \(\mathbb{k}[K]\) is a Golod ring over any \(\mathbb{k}\).
If $K$ itself is not Golod but deleting any vertex $v$ from $K$ turns the restricted complex $K - v$ into a Golod one, then $k[K]$ and $K$ itself are called minimally non-Golod.

We need the next result due to Bahri, Bendersky, Cohen and Gitler which is true in a much more general situation of polyhedral products.

**Theorem 1.3** ([3]). For any moment-angle complex its suspension $\Sigma Z_K$ is homotopy equivalent to the wedge of suspensions over all non-simplex induced subcomplexes: $\bigvee_{J \subseteq K} \Sigma^{2+|J|}[K_J]$.

The structure of this paper is as follows. We introduce several triangulations $K$ of classical manifolds with nice combinatorial properties for which we compute the homotopy types of $Z_K$ and work out the Golod and minimally non-Golod properties of the complexes in Section 2. In particular, we prove Theorem 2.5 and Proposition 3.4 which give infinite series of moment-angle complexes $Z_K$ with free integral cohomology and $K$ being Golod or minimally non-Golod over any field, such that it is not homotopy equivalent to a wedge of spheres or a connected sum of products of spheres respectively (cf. Theorem 2.6). In section 3 we prove a criterion when the simplicial multiwedge and composition of complexes are Golod and minimally non-Golod complexes (Theorem 3.3 and Theorem 3.8). We use simplicial multiwedge construction to prove a criterion for minimal non-Golodness of the nerve complexes $K_P$ for simple polytopes $P$ with few facets (Theorem 3.5).

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2. Some Golod complexes and their moment-angle complexes

In this section we consider several well-known minimal triangulations of classical surfaces, as simplicial complexes $K$ for which we discuss the homotopy types and cohomology of $Z_K$. We denote by $X^{\vee k}$ the $k$-fold wedge of $X$.

**Example 2.1.** Suppose $K$ is a 6-vertex minimal triangulation of $\mathbb{R}P^2$. Due to the result of Grbić, Panov, Theriault and Wu [9, Example 3.3] $K$ is Golod and $Z_K$ has a homotopy type of a wedge:

$Z_K \simeq (S^5)^\vee_{10} \vee (S^6)^\vee_{15} \vee (S^7)^\vee_{6} \vee \Sigma^7 \mathbb{R}P^2$.

**Proposition 2.2.** Suppose $K$ is a 7-vertex minimal triangulation of $\mathbb{T}^2$. Then $K$ is Golod and $Z_K$ has a homotopy type of a wedge of spheres:

$Z_K \simeq (S^5)^\vee_{21} \vee (S^6)^\vee_{49} \vee (S^7)^\vee_{42} \vee (S^8)^\vee_{14} \vee (S^9)^\vee_{2} \vee S^{10}$.

**Proof.** Let us prove Golodness of $K$. By Theorem 1.1, the map

$\tilde{H}^i(K_I) \otimes \tilde{H}^j(K_J) \to \tilde{H}^{i+j+1}(K_{I\cup J})$

is trivial. Indeed, if $i = j = 1$ then $H^{i+j+1}(K_{I\cup J}) = 0$ as $K$ has dimension $n - 1 = 2$; if $i = 0$ then $\tilde{H}^i(K_I) = 0$ as $K$ is 2-neighbourly.
Using Macaulay2 software [16], we compute the bigraded Betti numbers of $K$ (over $\mathbb{Z}$). The tables of $\beta^{-i,2j}(K)$ in what follows have $n$ rows and $m$ columns. The number in the $k$th row and $l$th column equals $\beta^{-l,2(i+k)}(K)$, where $1 \leq k \leq n$ and $2 \leq l + k \leq m$. Other bigraded Betti numbers are zero, except for $\beta^{0,0}(K) = 1$, see [6, Corollary 4.6.7]. The table below has $m = 7$ columns and $n = 3$ rows.

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
21 & 49 & 42 & 14 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
$$

Table 1. Bigraded Betti numbers of $T_7^2$.

To find the homotopy type of $Z_K$ one can see that the stable homotopy decomposition in Theorem 1.3 can be desuspended, since all the attaching maps of $k$-cells in the CW-complex $Z_K$ in dimensions $6 \leq k \leq 10$ are in the stable range. Then they are all null homotopic; the desuspension in Theorem 1.3 gives the homotopy type as in the statement required due to Theorem 1.2, see Table 1. □

The two above examples as well as all previously computed ones make it possible to ask the following question: is it true that if $K$ is a Golod complex and all its induced subcomplexes have free integral homology groups then $Z_K$ is homotopy equivalent to a wedge of spheres?

The answer is negative as the next result shows.

**Example 2.3.** Let $K$ be the minimal 9-vertex triangulation of $\mathbb{C}P^2$ which was described by Kühnel and Banchoff [13]. Kühnel and Lassmann [14] computed the symmetry group and proved combinatorial uniqueness and 3-neighbourness of $K$.

Here $K$ is on the vertex set $\{0, \ldots, 8\}$, and there are 36 maximal 4-dimensional faces which are given in the following table, see [14, p. 178]:

$$
\begin{array}{cccccc}
01234 & 70485 & 17562 \\
01237 & 70481 & 17560 \\
01267 & 70431 & 17580 \\
02345 & 74852 & 15624 \\
02367 & 74831 & 15680 \\
03467 & 78531 & 16280 \\
03456 & 78523 & 16248 \\
04567 & 75231 & 12480 \\
02358 & 74826 & 15643 \\
02368 & 74836 & 15683 \\
03568 & 78236 & 16483 \\
02458 & 74526 & 15243 \\
\end{array}
$$

Table 2. Maximal simplices of $\mathbb{C}P^2_9$.

The group $G$ of symmetries of $K$ has order 54, it acts transitively on the vertex set of $K$ and is generated by the 3 permutations $R, S, T$ (see [14, p.
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\( R = (107)(245)(863), \ S = (128)(357), \ T = (28)(46)(53). \)

We note the following combinatorial property of \( K \):

**Proposition 2.4.** \( K \) is isomorphic to its Alexander dual complex \( K^\vee \), that is the minimal non-faces of \( K \) are exactly the complements to its maximal faces.

**Proof.** Firstly, we prove that the complement to a maximal 4-face is a minimal non-face of \( K \) and all minimal non-faces on 4 vertices appear in this way. The first part follows from the fact that the symmetry group \( G \) sends (maximal) simplicies of \( K \) to (maximal) simplicies of \( K \) and acts transitively on this set, and for one of them, say for \((01234)\) its complement \((5678)\) is indeed a minimal non-face. The second part follows from the fact that the \( f \)-vector of \( K \) is \((9,36,84,90,36)\), see [14], and \( \binom{9}{4} = 126 = f_3 + f_4 \), therefore all the minimal non-faces on 4 vertices have the form above.

Suppose we have a minimal non-face of \( K \) on 5 vertices: \( I = (v_1, \ldots, v_5) \). Due to what is proved above, its complement \( J \) is a 3-simplex of \( K \) (otherwise, it should be a minimal non-face of \( K \) by 3-neighbourness of \( K \)). By transitivity of the symmetry group action, take one such 3-face, say \((0123)\) (see Table 2). But its complement is \((45678)\) and contains a non-face \((5678)\) (see above). We get a contradiction, so there are no minimal non-faces on 5 vertices in \( K \).

Finally, suppose \( I \) is a minimal non-face of \( K \) on some 6 vertices, as there are no simplices in \( K \) more than on 5 vertices, \( I = (v_1, \ldots, v_6) \). Then \((v_i, v_7, v_8, v_9)\) for \( i = 1, \ldots, 6 \) are minimal non-faces of \( K \) on 4 vertices (as complements to maximal faces). But \( K \) is 3-neighbourly and pure, so \((v_7, v_8, v_9)\) should be in one of the maximal 4-simplices of \( K \). We get a contradiction and thus we found all minimal non-faces of \( K \). \( \square \)

**Remark.** Note, that the same is true for \( K = \mathbb{R}P^2_6 \): \( K \) is combinatorially equivalent to its Alexander dual complex \( K^\vee \), but is not true for \( T^2_7 \).

**Theorem 2.5.** The 9-vertex minimal triangulation \( K \) of \( \mathbb{C}P^2 \) is a Golod complex, all its induced subcomplexes have free integral homology groups and \( Z_K \) has a homotopy type of the following suspension:

\[ Z_K \simeq (S^7)^{\vee 36} \vee (S^8)^{\vee 90} \vee (S^9)^{\vee 84} \vee (S^{10})^{\vee 36} \vee (S^{11})^{\vee 9} \vee \Sigma^{10} \mathbb{C}P^2. \]

**Proof.** We first prove Golodness of \( K \). Consider the cup-product in the Tor-algebra of \( K \) induced by a simplicial embedding of full subcomplexes on some vertex sets \( I \) and \( J \):

\[ \tilde{H}^i(K_I) \otimes \tilde{H}^j(K_J) \rightarrow \tilde{H}^{i+j+1}(K_{I \cup J}) \]

One has the following cases.

1. \( I \) and \( J \) are 4-vertex minimal non-faces and \( i = j = 2 \);
2. \( |I| = 4, |J| = 5, i = 1 \) (then \( I \cup J = [9] \)).

The first is impossible as dimension of \( K \) equals \( n - 1 = 4 \). The second is impossible by Proposition 2.4. Therefore, \( K = \mathbb{C}P^2_9 \) is a Golod complex.
As in Proposition 2.2 we compute bigraded Betti numbers of $K$ using Macaulay2 program. The following table has $m = 9$ columns and $n = 5$ rows.

|   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 36| 90| 84| 36| 9 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

Table 3. Bigraded Betti numbers of $CP^2_9$.

**Remark.** The desuspension in Theorem 1.3 for the cases of Proposition 2.2 and Theorem 2.5 follows also from the triviality of the fat wedge filtration of $RZ_K$ regarding neighbourness of these triangulations, due to the result of Iriye and Kishimoto [12, Theorem 10.9]. Another argument for this is true and a different homotopy theoretical approach to the Golod property for $K$ related to the co-H-space case for moment-angle complexes $Z_K$ can be found in the work of Beben and Grbić [4].

Now we turn to the class of minimally non-Golod complexes $K$ and their moment-angle complexes $Z_K$. We denote by $sK$ the stellar subdivision of $K$ at a maximal simplex $\sigma$. Geometrically $sK$ is obtained from $K$ by replacing $\sigma$ with a cone over its boundary; we denote the cone vertex by $v$.

**Theorem 2.6.** The following statements hold:

(a) If $K = RP^2_6$ then $sK$ is Golod over any field $k$, except for $\text{char}(k) = 2$. In the latter case $sK$ is minimally non-Golod.

(b) If $K = T^2_7$ then $sK$ is minimally non-Golod.

(c) If $K = CP^2_9$ then $sK$ is minimally non-Golod.

Moreover, in all these three cases $Z_{sK}$ is not homotopy equivalent to any connected sum of products of spheres.

**Proof.** Let us prove statement (a).

Suppose the new vertex $7$ is the vertex $v$ of a cone over the facet $(456)$. As $K = RP^2_6$ is a 2-neighbourly complex of dimension $n - 1 = 2$, the only nontrivial product in the Tor-algebra of $sK$ is of the form:

$$\widetilde{H}^0(K_I; k) \otimes \widetilde{H}^1(K_J; k) \to \widetilde{H}^2(RP^2; k),$$

where $I \sqcup J = [7]$ and $7 \in I$. We have $\widetilde{H}^2(RP^2; k) \neq 0$ if and only if $\text{char}(k) = 2$, which finishes the proof in this case.

Let us prove statement (b).

Suppose the new vertex is $8$ over the facet $(123)$. Observe that the complex
$sK$ is non-Golod as we have a nontrivial product:

\[ \tilde{H}^0(sK_I) \otimes \tilde{H}^1(sK_J) \to \tilde{H}^2(T^2), \]

where $I \cup J = [7]$ and $7 \in I$. If we delete a vertex $v$ from $sK$ then there are no more 2-dim cohomology classes, and the 2-neighbourness of $K$ implies that for $K' = sK - v$: $\tilde{H}^0(K'_I)$ and $\tilde{H}^0(K'_J)$ cannot be nonzero simultaneously when $I \cap J = \emptyset$. Therefore, the product in Tor-algebra of the induced complex $K'$ is trivial and $sK$ is minimally non-Golod.

Finally, we prove statement (c).

Suppose, $K$ is on the vertices $\{0, \ldots, 8\}$ and $9$ is the vertex of the cone over the facet $(01234)$. The complex $sK$ is non-Golod by the same reason as in the previous case. To prove that $sK$ is minimally non-Golod we consider the complex $K'$ obtained by deleting vertex $v$ from $K$. There are 3 cases:

1) $v \in \{0, 1, 2, 3, 4\}$. Then we have: $sK - v = (K - v) \cup \Delta^3 \Delta^4$.

This is a Golod complex by [15, Proposition 3.1].

2) $v \in \{5, 6, 7, 8\}$. According to the above description of the symmetry group $G$, it is enough to consider the case $v = 5$. We first compute the bigraded Betti numbers of $K' = sK - v$:

\[
\begin{array}{cccccccc}
8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 60 & 68 & 36 & 9 & 1 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

Table 4. Bigraded Betti numbers of $K'$.

Due to Theorem 1.2 this means that $H^1(K'_I) = 0$ for all full subcomplexes on $I$ vertices in $K'$. If $\tilde{H}^0(K'_I) \neq 0$ then $9 \in I$. Applying the same argument to the following table of bigraded Betti numbers of $K' - 9$, one can see that $H^2(K'_J) = 0$ for all $J$ with $9 \notin J$:

\[
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 40 & 33 & 14 & 2 & 0 & 0 & 0 \\
1 & 8 & 9 & 2 & 0 & 0 & 0 & 0
\end{array}
\]

Table 5. Bigraded Betti numbers of $K' - 9$.

Thus, the cup-product in the Tor-algebra of $K' = sK - v$ is trivial.

3) $v$ is a new vertex 9. Then $|K'| \cong CP^2 - D^4 \simeq S^2$, and the neighbourness of $K$ implies that $\tilde{H}^0(K'_I) = 0$ for all $I$, so the product in the Tor-algebra of $K' = sK - v$ is trivial.

Therefore, $sK$ is minimally non-Golod for $K = CP^2_9$. 

As for the last statement of the theorem, note that $K$ is not a Gorenstein* complex (and not even Cohen-Macaulay) in either of the three cases. By the Avramov-Golod theorem (see [6, Theorem 3.4.4]) its Tor-algebra is not a Poincaré algebra. Then by Theorem 1.1, $Z_K$ can not be homotopy equivalent to a closed oriented manifold, nor is it homotopy equivalent to a connected sum of sphere products.

The cases 2) and 3) in Theorem 2.6 provide counterexamples to a question raised in [15], and we therefore give its following modified version:

**Question 2.7.** Assume $K$ is a triangulated sphere. Then $Z_K$ is topologically equivalent to a connected sum of sphere products with two spheres in each product if and only if $K$ is minimally non-Golod and torsion free (that is $H^*(Z_K)$ is a free group).

### 3. Minimally non-Golod complexes, simple polytopes and polyhedral products

We begin with the definition of the simplicial multiwedge construction due to Bahri, Bendersky, Cohen and Gitler [2].

**Definition.** Let $K$ be an $(n-1)$-dimensional simplicial complex on $m$ vertices on the vertex set $\{v_1, \ldots, v_m\}$, and let $J = (j_1, \ldots, j_m)$ be a sequence of positive integers. Then the simplicial multiwedge $K(J)$ is the complex on $j_1 + \ldots + j_m$ vertices whose minimal non-faces have the following form:

$$\{v_{i_1}, \ldots, v_{i_{j_1}}, \ldots, v_{i_k}, \ldots, v_{i_{j_k}}\},$$

where $\{v_{i_1}, \ldots, v_{i_k}\}$ is a minimal non-face of $K$.

The definition provides an explicit description of the Stanley–Reisner ideal of $K(J)$. Obviously, $K(1, \ldots, 1) = K$.

**Remark.** If $K = K_P$ is the nerve complex of a simple $n$-polytope, then there is a simple polytope $P(J)$ satisfying $K_P(J) = K_P(J)$, see [2]. Obviously, $P(J)$ has $j_1 + \ldots + j_m$ facets and is of dimension $(j_1 - 1) + \ldots + (j_m - 1) + n$, so the number $m - n$ is preserved by simplicial multiwedge operation.

**Example 3.1.** Let $P$ be a 6-gon. Then $P(2, 1, 1, 1, 1, 1)$ is a simple 3-polytope with 7 facets. It is easy to see that it is a truncation polytope: $P(J) = ve^3(\Delta^3)$.

In what follows we need the following result about polyhedral products of simplicial multiwedges.

**Theorem 3.2** ([2, Theorem 7.5, Corollary 7.6]). There is an action of $\mathbb{T}^m$ on $(D^2, S^1)^K$ and on $(D^2, S^1)^{K(J)}$ with respect to which they are equivariantly homeomorphic. This yields the spaces $(D^2, S^1)^K$ and $(D^2, S^1)^{K(J)}$ have isomorphic ungraded cohomology rings.

The simplicial multiwedge construction preserves the Golod and minimal non-Golod properties of simplicial complexes:

**Proposition 3.3.** Let $J = (j_1, \ldots, j_m)$ be a sequence of positive integers:

(a) $K$ is a Golod complex if and only if $K(J)$ is Golod.
(b) \( K \) is a minimally non-Golod complex if and only if \( K(J) \) is minimally non-Golod.

Proof. Statement (a) follows directly from Theorem 3.2. We prove statement (b). Suppose \( K \) is minimally non-Golod. Then \( K(J) \) is non-Golod by statement (a). Suppose \( K \) is minimally non-Golod. Then \( K(J) \) is non-Golod by statement (a). Consider the complex obtained by removing a vertex \( v \) from \( K(J) \). Let \( v = v_{i_j} \), for \( 1 \leq i \leq m, 1 \leq l \leq i \) in the notation from the definition of the simplicial multiwedge. Then

\[
K(J) - v = (K - i)(j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_m)
\]

is a Golod complex by statement (a).

Therefore, we proved that \( K(J) \) is minimally non-Golod if \( K \) is minimally non-Golod. The opposite statement is proved in the same way. □

Proposition 3.4. Let \( K = \mathbb{C}P^2(J) \) for a sequence of positive integers \( J \). Then \( Z_K \) is not homotopy equivalent to a wedge of spheres.

Proof. In the case \( J = (1, \ldots, 1) \) we know the homotopy type of \( Z_K \) (see Theorem 2.5). The Steenrod square \( Sq^2 \) is nonzero in cohomology of any suspension over \( \mathbb{C}P^2 \) and all cohomology operations are trivial on wedges of spheres.

In general, the isomorphism of ungraded cohomology rings from Theorem 3.2 with coefficients in \( \mathbb{Z}/p \) (\( p \) is prime) is an isomorphism of \( \mathbb{Z}/p \)-modules commuting with the action of the Steenrod algebra (see [2, Corollary 7.7]), thus \( Z_K \) cannot be homotopy equivalent to a wedge of spheres. □

Remark. Note that, in general, \( K(J) \) for a neighbourly complex \( K \) is no longer a neighbourly simplicial complex.

Next we study minimally non-Golodness for \((n-1)\)-dimensional spheres with few vertices. Note that any \((n-1)\)-dimensional sphere with \( m \leq n + 3 \) vertices is polytopal.

Theorem 3.5. Suppose \( K = K_P \) is a nerve complex of a polytopal \((n-1)\)-sphere with \( m \leq n + 3 \) vertices. The following statements hold:

(a) If \( m = n + 1 \) then \( P \) is a simplex, \( Z_P \) is a sphere and \( K_P \) is Golod.

(b) If \( m = n + 2 \) then \( P \) is combinatorially a product of two simplices, \( Z_P \) is a product of two odd-dimensional spheres and \( K_P \) is minimally non-Golod.

(c) If \( m = n + 3 \) then \( K_P \) is minimally non-Golod if and only if \( P \) is not a product of 3 simplices.

Proof. The statements (a) and (b) are obvious since \( n \)-polytopes with \( m = n + 1 \) or \( m = n + 2 \) are determined uniquely up to affine or projective equivalence, respectively.

Let us prove the statement (c).

According to a result of Erokhovets [7, Theorem 2.3.48], if \( P \) is a simple \( n \)-polytope with \( m = n + 3 \) facets, then \( P = C^{2k-4}(2k-1)^*(j_1, \ldots, j_{2k-1}) \), where \( k \geq 3 \) and \( C^n(m) \) denotes a \( n \)-dimensional cyclic polytope with \( m \) vertices.

A cyclic polytope is neighbourly (see [19]). By [15, Proposition 3.6] the nerve complex \( K_P \) of an even dimensional dual neighbourly polytope \( P \) is minimally non-Golod. The proof is finished applying Proposition 3.3. □
The topological types of the moment-angle manifolds \( \mathcal{Z}_P \) corresponding to the statement (c) of Theorem 3.5 are described as follows.

**Proposition 3.6** ([17],[7]). Let \( P \) be a simple \( n \)-polytope with \( m = n + 3 \) vertices, so that \( P = C_{2k-4}^{2k-4}(2k-1)^{s}(j_1,\ldots,j_{2k-1}) \), for some \( k \geq 3 \). Then
\[
\mathcal{Z}_P \cong \bigotimes_{i=1}^{2k-1} S^2 \varphi_i^{-1} \times S^2 \psi_i^{-1} \cdot 2,\]
where \( \varphi_r = j_r + \ldots + j_{r+k-2} \), \( \psi_r = j_r + \ldots + j_{r+k-1} \), and all the indices are considered modulo \( 2k - 1 \).

We therefore obtain infinite families of triangulated spheres \( K \) which are minimally non-Golod and torsion free, whose corresponding \( \mathcal{Z}_K \) are connected sums of products of two spheres.

**Remark.** Minimally non-Golodness in statement (c) of Theorem 3.5 can be also deduced from the explicit description of the multiplication in \( H^*(\mathcal{Z}_P) \) in the case \( m = n + 3 \), see [7, Theorem 2.5.8]

We can also extend our results by considering the operation of composition of simplicial complexes, originally defined by Ayzenberg [1].

**Definition.** Suppose \( K \) is an \( (n-1) \)-dimensional simplicial complex on \( m \) vertices, \( K_1,\ldots,K_m \) are simplicial complexes (may be empty or with ghost vertices) on the sets \( \{l_1\},\ldots,\{l_m\} \) respectively. Then the composition of \( K \) with \( K_i,1 \leq i \leq m \) is the simplicial complex \( K \) on the set \( \bigcup \{l_j\} \) defined as follows: a set \( I = I_1 \cup \ldots \cup I_m \), with \( I_j \subset \{l_j\} \) is a simplex of \( K \) if and only if \( \{j \in [m] | I_j \neq K_j \} \in K \). The composition complex has therefore \( l_1 + \ldots + l_m \) vertices and dimension \( (n-1) + (l_1 + \ldots + l_m - m) \).

**Example 3.7.** Let \( K_i = \partial \Delta^{l_i-1} \) for \( 1 \leq i \leq m \). Then \( K(K_1,\ldots,K_m) = K(J) \) is the simplicial wedge.

**Theorem 3.8.** Let \( K_1,\ldots,K_m \) be simplicial complexes.

(a) \( K \) is Golod if and only if \( K_{[m]-\{s_1,\ldots,s_r\}} \) is Golod, where \( 1 \leq s_1,\ldots,s_r \leq m \) are such that \( K_{s_i} = \Delta^{l_i-1} \) for all \( 1 \leq i \leq r \) \((l_i \geq 1)\).

(b) \( K \) is minimally non-Golod if and only if \( K_{s_i} = \Delta^{l_i-1} \) for \( 1 \leq i \leq r \) and \( K_j = \partial \Delta^{l_j-1} \) for \( j \neq s_i,1 \leq i \leq r \) and \( K_{[m]-\{s_1,\ldots,s_r\}} \) is minimally non-Golod \((l_i \geq 1)\).

**Proof.** We proceed by induction on the number \( N \) of non-empty complexes in \( K_1,\ldots,K_m \). The base case \( N = 0 \) is trivial, so consider \( N = 1 \). Suppose that \( K = L \) is the only non-empty complex. Then
\[
K(L) = K(\emptyset_1,\ldots,\emptyset_m) = L \ast (K - i) \cup \Delta^{l_{i}-1} \ast \text{link}_i K,
\]
where the simplicial complexes in the union are glued along their common subcomplex \( L \ast \text{link}_i K \).

Let us prove (a).

For the “only if” part, if \( L \) is a simplex then \( K(L) = \Delta^{l_{i}-1} \ast (K - i) \) and the statement is true as \( \mathcal{Z}_{K(L)} \cong \mathcal{Z}_{K-i} \). Otherwise, take a minimal non-face \( V \).
in \( L \). Then \( K(V) \) is a full subcomplex in a Golod complex \( K(L) \) and thus \( K(V) \) is Golod, therefore \( K \) is Golod by Proposition 3.3.

For the “if” part, if \( L \) is a simplex then \( K(L) = \Delta^{i-1} \ast (K - i) \) and the statement is true. Otherwise, suppose \( K(L) \) is non-Golod and the following map is nontrivial:

\[ \tilde{H}^i(K(L)_{\{I\}}) \otimes \tilde{H}^j(K(L)_{\{J\}}) \to \tilde{H}^{i+j+1}(K(L)_{\{I, J\}}). \]

Then it is also nontrivial viewed as a cup-product in the Tor-algebra of \( K'(\partial \Delta^{(I \sqcup J) \cap L}) \), but the latter complex is Golod by Proposition 3.3.

Let us prove (b).

The “if” part was proved in Proposition 3.3. Now we prove the “only if” part. Suppose that \( K(L) \) is minimally non-Golod. If \( L \) is a simplex, then \( Z_{K(L)} \cong Z_{K - i} \); if \( L \) is the boundary of a simplex then \( K \) is minimally non-Golod by Proposition 3.3 and the statement is true. Assume that \( L \) is neither a simplex nor the boundary of a simplex. Then there is a proper subset of vertices \( V \subset L \) (a minimal non-face of \( L \)) such that \( L_V \) is the boundary of a simplex. Note that \( K(L_V) \) is a full subcomplex in \( K(L) \), so that \( K(L_V) \) is Golod, as \( K(L) \) is minimally non-Golod. Then \( K \) is Golod by Proposition 3.3 and we get a contradiction with part (a).

To make an induction step in both (a) and (b) we use the following result [1, Corollary 4.14]:

\[ K(K_1, \ldots, K_m) = K(L)(K_1, \ldots, K_{i-1}, \emptyset, \ldots, \emptyset, K_{i+1}, \ldots, K_m), \]

where there are exactly \( l_i \) empty simplicial complexes in the second substitution. This finishes the proof by induction on the number of non-empty complexes in the composition of simplicial complexes. \( \square \)

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