Partition regularity of a system of De and Hindman

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Abstract

We prove that a certain matrix, which is not image partition regular over $\mathbb{R}$ near zero, is image partition regular over $\mathbb{N}$. This answers a question of De and Hindman.

1 Introduction

Let $A$ be an integer matrix with only finitely many non-zero entries in each row. We call $A$ kernel partition regular (over $\mathbb{N}$) if, whenever $\mathbb{N}$ is finitely coloured, the system of linear equations $Ax = 0$ has a monochromatic solution; that is, there is a vector $x$ with entries in $\mathbb{N}$ such that $Ax = 0$ and each entry of $x$ has the same colour. We call $A$ image partition regular (over $\mathbb{N}$) if, whenever $\mathbb{N}$ is finitely coloured, there is a vector $x$ with entries in $\mathbb{N}$ such that each entry of $Ax$ is in $\mathbb{N}$ and has the same colour. We also say that the system of equations $Ax = 0$ or the system of expressions $Ax$ is partition regular.

The finite partition regular systems of equations were characterised by Rado. Let $A$ be an $m \times n$ matrix and let $c^{(1)}, \ldots, c^{(n)}$ be the columns of $A$. Then $A$ has the columns property if there is a partition $[n] = I_1 \cup I_2 \cup \cdots \cup I_t$ of the columns of $A$ such that $\sum_{i \in I_s} c^{(i)} = 0$, and, for each $s$,

$$\sum_{i \in I_s} c^{(i)} \in \langle c^{(j)} : j \in I_1 \cup \cdots \cup I_{s-1} \rangle,$$

where $\langle \cdot \rangle$ denotes (rational) linear span.

**Theorem 1** ([Rad33]). A finite matrix $A$ with integer coefficients is kernel partition regular if and only if it has the columns property.

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The finite image partition regular systems were characterised by Hindman and Leader [HL93].

In the infinite case even examples of partition regular systems are hard to come by: see [BHL13] for an overview of what is known. De and Hindman [DH09, Q3.12] asked whether the following matrix was image partition regular over \( N \).

\[
\begin{pmatrix}
1 & 1 & \cdots \\
1 & 1 & \cdots \\
2 & 1 & \cdots \\
2 & 1 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
\]

where we have omitted zeroes to make the block structure of the matrix more apparent. This matrix corresponds to the following system of linear expressions.

\[
x_{21} + x_{22} \quad x_{21} + 2y \quad y,
\]

\[
x_{22} + 2y
\]

\[
x_{41} + x_{42} + x_{43} + x_{44} \quad x_{41} + 4y
\]

\[
x_{42} + 4y
\]

\[
x_{43} + 4y
\]

\[
x_{44} + 4y
\]

\[
\vdots
\]

\[
x_{2^{n-1} + \cdots + 2^n} \quad x_{2^{n-1} + 2^n} y
\]

\[
x_{2^{n-1} + 2^n} + 2^n y
\]

\[
\vdots
\]

\[
\vdots
\]

A matrix \( A \) is called \textit{image partition regular over} \( \mathbb{R} \) near zero if, for every \( \delta > 0 \), whenever \( (-\delta, \delta) \) is finitely coloured, there is a vector \( x \) with entries in \( \mathbb{R} \setminus \{0\} \) such that each entry of \( Ax \) is in \( (-\delta, \delta) \) and has the same colour. De and Hindman sought a matrix that was image partition regular over \( \mathbb{N} \) but not image partition regular over \( \mathbb{R} \) near zero. It is easy to show that
the above matrix is not image partition regular over \( \mathbb{R} \) near zero, so showing that it is image partition regular over \( \mathbb{N} \) would provide an example.

The main result of this paper is that De and Hindman’s matrix is image partition regular over \( \mathbb{N} \).

**Theorem 2.** For any sequence \((a_n)\) of integer coefficients, the system of expressions

\[
\begin{align*}
  x_{11} & & x_{11} + a_1 y & & y, \\
  x_{21} + x_{22} & & x_{21} + a_2 y & & x_{22} + a_2 y \\
  x_{31} + x_{32} + x_{33} & & x_{31} + a_3 y & & x_{32} + a_3 y & & x_{33} + a_3 y \\
  & & & & \vdots
\end{align*}
\]

is partition regular.

Taking \(a_n = n\) implies that De and Hindman’s matrix is image partition regular.

Barber, Hindman and Leader [BHL13] recently found a different matrix that is image partition regular but not image partition regular over \( \mathbb{R} \) near zero. Their argument proceeded via the following result on kernel partition regularity.

**Theorem 3 ([BHL13]).** For any sequence \((a_n)\) of integer coefficients, the system of equations

\[
\begin{align*}
  x_{11} + a_1 y &= z_1 \\
  x_{21} + x_{22} + a_2 y &= z_2 \\
  & \vdots \\
  x_{n1} + \cdots + x_{nn} + a_n y &= z_n \\
  & \vdots
\end{align*}
\]

is partition regular.

In Section 2 we show that Theorem 2 can almost be deduced directly from Theorem 3. The problem we encounter motivates the proof of Theorem 2 that appears in Section 3.
2 A near miss

In this section we show that Theorem 2 can almost be deduced directly from Theorem 3.

Let \( \mathbb{N} \) be finitely coloured. By Theorem 3 there is a monochromatic solution to the system of equations

\[
\tilde{x}_{11} - a_1 y = z_1 \\
\tilde{x}_{21} + \tilde{x}_{22} - 2a_2 y = z_2 \\
\vdots \\
\tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - na_n y = z_n \\
\vdots
\]

Set \( x_{ni} = \tilde{x}_{ni} - a_n y \). Then, for each \( n \) and \( i \),

\[
x_{n1} + \cdots + x_{nn} = \tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - na_n y = z_n,
\]

and

\[
x_{ni} + a_n y = \tilde{x}_{ni},
\]

so we have found a monochromatic image for System 1. The problem is that we have not ensured that the variables \( x_{ni} = \tilde{x}_{ni} - a_n y \) are positive. In Section 3 we look inside the proof of Theorem 3 to show that we can take (most of) the \( x_{ni} \) to be as large as we please.

3 Proof of Theorem 2

The proof of Theorem 3 used a density argument. The (upper) density of a set \( S \subseteq \mathbb{N} \) is

\[
d(S) = \limsup_{n \to \infty} \frac{|S \cap [n]|}{n},
\]

where \([n] = \{1, 2, \ldots, n\}\). The density of a set \( S \subseteq \mathbb{Z} \) is \( d(S \cap \mathbb{N}) \). We call \( S \) dense if \( d(S) > 0 \). We shall use three properties of density.

1. If \( A \subseteq B \), then \( d(A) \leq d(B) \).

2. Density is unaffected by translation and the addition or removal of finitely many elements.

3. Whenever \( \mathbb{N} \) is finitely coloured, at least one of the colour classes is dense.
We will also use the standard notation for sumsets and difference sets

\[ A + B = \{ a + b : a \in A, b \in B \} \]
\[ A - B = \{ a - b : a \in A, b \in B \} \]
\[ kA = \underbrace{A + \cdots + A}_{k \text{ times}} \]

and write \( m \cdot S = \{ ms : s \in S \} \) for the set obtained from \( S \) under pointwise multiplication by \( m \).

We start with two lemmas from [BHL13].

**Lemma 4 (BHL13).** Let \( A \subseteq \mathbb{N} \) be dense. Then there is an \( m \) such that, for \( n \geq 2/d(A) \), \( nA - nA = m \cdot \mathbb{Z} \).

**Lemma 5 (BHL13).** Let \( S \subseteq \mathbb{Z} \) be dense with \( 0 \in S \). Then there is an \( X \subseteq \mathbb{Z} \) such that, for \( n \geq 2/d(S) \), we have \( S - nS = X \).

The following consequence of Lemmas 4 and 5 is mostly implicit in [BHL13]. The main new observation is that the result still holds if we insist that we only use large elements of \( A \). Write \( A_{>t} = \{ a \in A : a > t \} \).

**Lemma 6.** Let \( A \) be a dense subset of \( \mathbb{N} \) that meets every subgroup of \( \mathbb{Z} \), and let \( m \) be the least common multiple of \( 1, 2, \ldots, \lfloor 1/d(A) \rfloor \). Then, for \( n \geq 2/d(A) \) and any \( t \),

\[ A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z} \]

**Proof.** First observe that, for any \( t \), \( d(A_{>t}) = d(A) \). Let \( n \geq 2/d(A) \), and let \( X = A_{>t} - nA_{>t} \). For any \( a \in A_{>t} \), we have by Lemma 5 that

\[ (A_{>t} - a) - n(A_{>t} - a) = (A_{>t} - a) - (n + 1)(A_{>t} - a), \]

and so

\[ X = X - A_{>t} + a. \]

Since \( a \in A_{>t} \) was arbitrary it follows that \( X = X + A_{>t} - A_{>t} \), whence \( X = X + l(A_{>t} - A_{>t}) \) for all \( l \). By Lemma 4 there is an \( m_l \in \mathbb{Z} \) such that, for \( l \geq 2/d(A) \), \( l(A_{>t} - A_{>t}) = m_l \cdot \mathbb{Z} \). Hence \( X = X + m_l \cdot \mathbb{Z} \), and \( X \) is a union of cosets of \( m_l \cdot \mathbb{Z} \). Since \( A \) contains arbitrarily large multiples of \( m_l \), one of these cosets is \( m_l \cdot \mathbb{Z} \) itself.

Since \( lA_{>t} - lA_{>t} \) contains a translate of \( A_{>t} \),

\[ 1/m_l = d(m_l \cdot \mathbb{Z}) \geq d(A), \]

and \( m_l \leq 1/d(A) \). So \( m_l \) divides \( m \) and

\[ A_{>t} - nA_{>t} \supseteq m \cdot \mathbb{Z}. \]

\[ \square \]
Lemma 6 will allow us to find a monochromatic image for all but a finite part of System 1. The remaining finite part can be handled using Rado’s theorem, provided we take care to ensure that it gives us a solution inside a dense colour class.

**Lemma 7 (BHL13).** Let $\mathbb{N}$ be finitely coloured. For any $l \in \mathbb{N}$, there is a $c \in \mathbb{N}$ such that $c \cdot [l]$ is disjoint from the non-dense colour classes.

We can now show that System 1 is partition regular.

**Proof of Theorem 2**. Let $\mathbb{N}$ be $r$-coloured. Suppose first that some colour class does not meet every subgroup of $\mathbb{Z}$; say some class contains no multiple of $m$. Then $m \cdot \mathbb{N}$ is $(r - 1)$-coloured by the remaining colour classes, so by induction on $r$ we can find a monochromatic image. So we may assume that every colour class meets every subgroup of $\mathbb{Z}$.

Let $d$ be the least density among the dense colour classes, and let $m$ be the least common multiple of $1, 2, \ldots, \lfloor 1/d \rfloor$. Then for any dense colour class $A$, any $t$ and $n \geq 2/d$,

$$A > t - nA > t \supseteq m \cdot \mathbb{Z}. $$

Now let $N = \lceil 2/d \rceil - 1$. We will find a monochromatic image for the the expressions containing only $y$ and $x_{ni}$ for $n \leq N$ using Rado’s theorem. Indeed, consider the following system of linear equations.

\[
\begin{align*}
  u_1 &= x_{11} & v_{11} &= x_{11} + a_1 y \\
  u_2 &= x_{21} + x_{22} & v_{21} &= x_{21} + a_2 y \\
  & & v_{22} &= x_{22} + a_2 y \\
  & & \vdots & & \vdots \\
  u_N &= x_{N1} + \cdots + x_{NN} & v_{N1} &= x_{N1} + a_N y \\
  & & \vdots & & \vdots \\
  & & v_{NN} &= x_{NN} + a_N y
\end{align*}
\]

The matrix corresponding to these equations has the form

$$\begin{pmatrix} B & -I \end{pmatrix}$$

where $B$ is a top-left corner of the matrix corresponding to System 1 and $I$ is an appropriately sized identity matrix. It is easy to check that this matrix has the columns property, so by Rado’s theorem there is an $l$ such that, whenever a progression $c \cdot [l]$ is $r$-coloured, it contains a monochromatic solution to System 2.
Apply Lemma 7 to get $c$ with $c \cdot [ml]$ disjoint from the non-dense colour classes. Then $mc \cdot [l] \subseteq c \cdot [ml]$ is also disjoint from the non-dense colour classes, and by the choice of $l$ there is a dense colour class $A$ such that $A \cap (md \cdot [l])$ contains a solution to System 2. Since the $u_n$, $v_{ni}$ and $y$ are all in $A$, $y$ and the corresponding $x_{ni}$ make the first part of System 1 monochromatic.

Now $y$ is divisible by $m$, so for $n > N$ we have that
\[-na_ny \in A_{>a_ny} - nA_{>a_ny},\]
so there are $\tilde{x}_{ni}$ and $z_n$ in $A_{>a_ny}$ such that
\[-a_ny = z_n - \tilde{x}_n1 - \cdots - \tilde{x}_{nn}.\]
Set $x_{ni} = \tilde{x}_{ni} - a_ny$. Then
\[x_{n1} + \cdots + x_{nn} = \tilde{x}_{n1} + \cdots + \tilde{x}_{nn} - na_ny = z_n,\]
and
\[x_{ni} + a_ny = \tilde{x}_{ni},\]
for each $n > N$ and $1 \leq i \leq n$. Since $\tilde{x}_{ni}$ and $z_n$ are in $A$ it follows that the whole of System 1 is monochromatic.

It remains only to check that all of the variables are positive. But for $y$ and $x_{ni}$ with $n \leq N$ this is guaranteed by Rado’s theorem; for $n > N$ it holds because $\tilde{x}_{ni} > a_ny$.

\section*{References}

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