Yukawa Quasi-Unification

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Abstract

We construct concrete supersymmetric grand unified theories based on the Pati-Salam gauge group $SU(4)_{c} \times SU(2)_{L} \times SU(2)_{R}$ which naturally lead to a moderate violation of ‘asymptotic’ Yukawa unification and thus can allow an acceptable $b$-quark mass even with universal boundary conditions. We consider the constrained minimal supersymmetric standard model which emerges from one of these theories with a deviation from Yukawa unification which is adequate for $\mu > 0$. We show that this model possesses a wide and natural range of parameters which is consistent with the data on $b \to s\gamma$, the muon anomalous magnetic moment, the cold dark matter abundance in the universe, and the Higgs boson masses. The lightest supersymmetric particle can be as light as about 107 GeV.

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I. INTRODUCTION

The most restrictive version of the minimal supersymmetric standard model (MSSM) with gauge coupling unification is based on radiative electroweak breaking with universal boundary conditions from gravity-mediated soft supersymmetry (SUSY) breaking and is known as constrained MSSM (CMSSM). It is desirable to further restrict this model by assuming that the \( t \)-quark, \( b \)-quark and \( \tau \)-lepton Yukawa couplings unify ‘asymptotically’, i.e., at the SUSY grand unified theory (GUT) scale \( M_{\text{GUT}} \approx 2 \times 10^{16} \, \text{GeV} \). This assumption (Yukawa unification) naturally restricts \[ \] the \( t \)-quark mass to large values compatible with the data. Also, the emerging model is highly predictive.

Yukawa unification can be achieved by embedding the MSSM in a SUSY GUT with a gauge group containing \( SU(4)_c \) and \( SU(2)_R \). Indeed, assuming that the electroweak Higgs superfields \( h_{1w}^w, h_{2w}^w \) and the third family right handed quark superfields \( t^c, b^c \) form \( SU(2)_R \) doublets, we obtain \[ \] the ‘asymptotic’ Yukawa coupling relation \( h_t = h_b \) and, hence, large \( \tan \beta \approx m_t/m_b \). Moreover, if the third generation quark and lepton \( SU(2)_L \) doublets (singlets) \( q_3 \) and \( l_3 \) \( (b^c \text{ and } \tau^c) \) form a \( SU(4)_c \) 4-plet \( (\mathbf{4}) \)-plet \) and the Higgs doublet \( h_{1w}^w \) which couples to them is a \( SU(4)_c \) singlet, we get \( h_b = h_\tau \) and the ‘asymptotic’ relation \( m_b = m_\tau \) follows. The simplest GUT gauge group which contains both \( SU(4)_c \) and \( SU(2)_R \) is the Pati-Salam (PS) group \( G_{\text{PS}} = SU(4)_c \times SU(2)_L \times SU(2)_R \). Higher groups with the same property are \( SO(10) \) or \( E_6 \).

One problem, which is faced in trying to incorporate Yukawa unification into the CMSSM, is due to the generation of sizeable SUSY corrections to \( m_b \) \[ \] which have the same sign as the MSSM parameter \( \mu \). Consequently, for \( \mu > 0 \), the tree-level \( m_b(M_Z) \), which is predicted \[ \] (the sign of \( \mu \) in this reference is opposite to the one adopted here) from Yukawa unification already above the 95% confidence level (c.l.) experimental range \( 2.684 - 3.092 \, \text{GeV} \), receives large positive corrections. This range is derived from the 95% c.l. range for \( m_b(m_b) \) in the \( \overline{\text{MS}} \) renormalization scheme which is found \[ \] (see also Ref. \[ \]) to be \( 3.95 - 4.55 \, \text{GeV} \). This is evolved \[ \uparrow \] up to \( M_Z \) via the three-loop \( \overline{\text{MS}} \) renormalization group (RG) equations \[ \] with \( \alpha_s(M_Z) = 0.1185 \) and then converted to the \( \overline{\text{DR}} \) scheme using the appropriate one-loop factor \[ \]. For \( \mu < 0 \), the tree-level \( m_b(M_Z) \) is \[ \] smaller and close to the upper edge of the above range. However,
the radiative corrections are now negative and drive $m_b(M_Z)$ below this range. The discrepancy is, though, considerably smaller than in the $\mu > 0$ case.

We see that, for both signs of $\mu$, the hypothesis of exact Yukawa unification leads to an unacceptable $b$-quark mass. However, we are not obliged to abandon Yukawa unification altogether. We can rather modestly correct it by including an extra $SU(4)_c$ non-singlet Higgs superfield with Yukawa couplings to the quarks and leptons. The Higgs $SU(2)_L$ doublets contained in this superfield can naturally develop subdominant vacuum expectation values (vevs) and mix with the main electroweak doublets which are assumed to be $SU(4)_c$ singlets and form a $SU(2)_R$ doublet. This mixing can, in general, violate $SU(2)_R$. Consequently, the resulting electroweak Higgs doublets $h_1^{ew}, h_2^{ew}$ do not form a $SU(2)_R$ doublet and also break $SU(4)_c$. The required deviation from exact Yukawa unification is expected to be more pronounced in the $\mu > 0$ case. Despite this, we choose to study here this case since, for $\mu < 0$, the present experimental data on the inclusive decay $b \to s\gamma$ restrict the sparticle masses to considerably higher values and, thus, this case is phenomenologically less interesting. Moreover, the recent results on the muon anomalous magnetic moment also imply heavy sparticles for $\mu < 0$ (see below).

We will construct here a concrete SUSY GUT model which naturally leads to a modest deviation from Yukawa unification allowing an acceptable $m_b(M_Z)$ even with universal boundary conditions. (For models which violate universality rather than Yukawa unification see Ref. [14].) We will then show that this model possesses a wide range of parameters which is consistent with all the phenomenological and cosmological constraints. We consider only the $\mu > 0$ case, which is experimentally more attractive.

In Sec. I, we construct a SUSY GUT model which is based on the PS gauge group and provides, in a natural way, a suppressed violation of Yukawa unification. Variants of this model which can yield bigger deviations from Yukawa unification are also presented. In Sec. II, we concentrate on one of these variants which can violate Yukawa unification by an amount that is adequate for $\mu > 0$. We then describe the resulting MSSM under the assumption of universal boundary conditions and introduce the various phenomenological and cosmological requirements which restrict its parameter space. In Sec. III, we study the range of parameters which is compatible with all these requirements in this particular CMSSM. Finally, in Sec. IV, we summarize our conclusions.
II. THE SUSY GUT MODEL

We take the SUSY GUT model of Ref. [15] (see also Ref. [16]) as our starting point. This is based on the PS gauge group \( G_{PS} \), which is the simplest gauge group that can lead to Yukawa unification. The ‘matter’ superfields are \( F_i = (4, 2, 1) \) and \( F_i^c = (\bar{4}, 1, 2) \) \((i = 1, 2, 3)\), while the electroweak Higgs doublets belong to the superfield \( h = (1, 2, 2) \). So, all the requirements for exact Yukawa unification are fulfilled. The breaking of \( G_{PS} \) down to the standard model (SM) gauge group \( G_s \) is achieved by the superheavy vevs \((\sim M_{GUT})\) of the right handed neutrino type components of a conjugate pair of Higgs superfields \( \bar{H}^c = (4, 1, 2), H^c = (4, 1, 2) \). The model also contains a gauge singlet \( S \) which triggers the breaking of \( G_{PS} \), a \( SU(4)_c \) 6-plet \( G = (6, 1, 1) \) which gives \( [17] \) masses to the right handed down quark type components of \( \bar{H}^c, H^c \), and a pair of gauge singlets \( \bar{N}, N \) for solving \( [18] \) the \( \mu \) problem of the MSSM via a Peccei-Quinn (PQ) symmetry. In addition to \( G_{PS} \), the model possesses two global \( U(1) \) symmetries, namely a PQ and a R symmetry, as well as a \( Z_{2}^{mp} \) symmetry (‘matter parity’) under which \( F, F^c \) change sign. For details on the charge assignments, the full superpotential and the phenomenological and cosmological properties of this model, the reader is referred to Ref. [15].

A moderate violation of Yukawa unification can be accommodated in this model by adding a new Higgs superfield \( h' = (15, 2, 2) \) with Yukawa couplings \( FF^ch' \). Actually, this is the only representation, besides \((1, 2, 2)\), which possesses such couplings to the fermions. The existence of these couplings requires that the quantum numbers of \( h' \) coincide with the ones of \( h \). So, its PQ and R charges are \( PQ = 1 \) and \( R = 0 \) respectively. In order to give superheavy masses to the color non-singlet components of \( h' \), we need to include one more Higgs superfield \( \bar{h}' = (15, 2, 2) \) with the superpotential coupling \( \bar{h}'h' \), whose coefficient is of the order of \( M_{GUT} \). The field \( \bar{h}' \) then has \( PQ = -1 \) and \( R = 1 \). The full superpotential which is consistent with all the symmetries contains, in addition to the couplings mentioned above and in Ref. [15], the following extra terms too:

\[
F \bar{H}^c H^c hh', \quad F \bar{H}^c H^c h'h', \quad FFH^c H^c hh', \quad FFH^c H^c h'h', \quad (\bar{H}^c)^4 \bar{h}'h, \quad (H^c)^4 h'h,
\]

\[
\bar{H}^c H^c \bar{h}'h, \quad N^2(\bar{H}^c)^4 hh', \quad N^2(H^c)^4 hh', \quad N^2 \bar{H}^c H^c hh', \quad N^2 h'h'. \quad (1)
\]

Note that all the superpotential terms can be multiplied by arbitrary powers of the
combinations \((\bar{H}^c)^4, (H^c)^4, \bar{H}^c H^c\). The first two couplings in Eq.(1) (as well as all the ‘new’ couplings containing \((\bar{H}^c)^4, (H^c)^4\)) give rise to additional baryon and lepton number violation. However, their contribution to proton decay is subdominant to the one discussed in Ref. [15] and proton remains practically stable.

Let us now discuss how the introduction of \(\bar{h}'\), \(h'\) can lead to a moderate correction of exact Yukawa unification. The Higgs field \(h'\) contains two color singlet \(SU(2)_L\) doublets \(h'_1, h'_2\) which, after the breaking of \(G_{PS}\) to \(G_S\), can mix with the corresponding doublets \(h_1, h_2\) in \(h\). This is mainly due to the terms \(\bar{h}'h', \bar{H}^c H^c \bar{h}'h\). The latter, being non-renormalizable, is suppressed by the string scale \(M_S \approx 5 \times 10^{17}\) GeV. Actually, \(\bar{H}^c H^c \bar{h}'h\) corresponds to two independent couplings. One of them is between the \(SU(2)_R\) singlets in \(\bar{h}'\) and \(h\). So, we finally obtain two bilinear terms \(\bar{h}'_1 h_1\) and \(\bar{h}'_2 h_2\) with different coefficients, which are suppressed by \(M_{GUT}/M_S\). These terms together with the terms \(\bar{h}'_1 h'_1\) and \(\bar{h}'_2 h'_2\) from \(\bar{h}'h'\), which have equal coefficients, generate different mixings between \(h_1, h'_1\) and \(h_2, h'_2\). Consequently, the resulting electroweak doublets \(h_{ew}^1, h_{ew}^2\) contain \(SU(4)_c\) violating components suppressed by \(M_{GUT}/M_S\) and fail to form a \(SU(2)_R\) doublet by an equally suppressed amount. So, Yukawa unification is moderately violated. Unfortunately, as it turns out, this violation is not adequate for correcting the \(b\)-quark mass for \(\mu > 0\).

In order to allow for a more sizable violation of Yukawa unification, we further extend the model by including a superfield \(\phi = (15, 1, 3)\) with the coupling \(\bar{\phi} \bar{h}' h\). The field \(\phi\) is then neutral under both the PQ and R symmetries. To give superheavy masses to the color non-singlets in \(\phi\), we introduce one more superfield \(\bar{\phi} = (15, 1, 3)\) with the coupling \(\bar{\phi} \phi\), whose coefficient is of order \(M_{GUT}\). The charges of \(\phi\) are \(PQ = 0, R = 1\). Additional superpotential couplings which are allowed by the symmetries of the model are:

\[
\bar{\phi}(\bar{H}^c)^4, \bar{\phi}(H^c)^4, \bar{\phi} \bar{H}^c H^c, \phi N^2 h h'.
\]  

(2)

Multiplications of the superpotential terms by \(\phi, (\bar{H}^c)^4, (H^c)^4, \bar{H}^c H^c\) are generally allowed with the exception of the multiplication of the \(G_{PS}\) singlets \(S, \bar{N}^2 N^2, N^2 h^2\) by a single \(\phi\), which is a \(SU(4)_c\) 15-plet.
The terms \( \bar{\phi}\phi \) and \( \bar{\phi}H^cH^c \) imply that, after the breaking of \( G_{PS} \) to \( G_S \), the field \( \phi \) acquires a superheavy vev of order \( M_{GUT} \). The coupling \( \phi h' h \) then generates \( SU(2)_R \) violating unsuppressed bilinear terms between the doublets in \( \bar{h}' \) and \( h \). These terms can certainly overshadow the corresponding ones from the non-renormalizable term \( H^cH^c h'h'. \) The resulting \( SU(2)_R \) violating mixing of the doublets in \( h \) and \( h' \) is then unsuppressed and we can obtain stronger violation of Yukawa unification.

Instead of the pair \( \bar{\phi}, \phi \) one can use a pair of \( SU(2)_R \) singlet superfields \( \bar{\phi}' = (15, 1, 1), \phi' = (15, 1, 1) \) with the same PQ and R charges and the couplings \( \phi h'h \) and \( \bar{\phi}'\phi' \). In this case, the mixing is \( SU(2)_R \) invariant and the ‘asymptotic’ relation \( h_t = h_b \) is not violated. The additional superpotential couplings allowed by the symmetries are:

\[
\bar{\phi}'(H^c)^8, \quad \phi'(H^c)^8, \quad \bar{\phi}'H^cH^c, \quad \phi' N^2 hh'.
\] (3)

Note the absence of the couplings \( \bar{\phi}'(H^c)^4, \phi'(H^c)^4 \) which are \( SU(2)_R \) triplets. Multiplications of the superpotential terms by \( \phi', (H^c)^4, (H^c)^4, \bar{H}^cH^c \) are generally allowed with the exception of the multiplication of \( S, \bar{N}^2N^2, N^2h^2 \) by a single \( \phi' \), which is a \( SU(4)_c \) 15-plet, or a single \( \phi'(H^c)^4, \phi'(H^c)^4 \), which are \( SU(2)_R \) triplets.

Finally, one could introduce both the pairs \( \bar{\phi}, \phi \) and \( \bar{\phi}', \phi' \) at the same time. The allowed superpotential terms include all the above mentioned terms and

\[
\bar{\phi}'\phi(\bar{H}^c)^4, \quad \bar{\phi}'\phi(\bar{H}^c)^4, \quad \bar{\phi}'\phi^2.
\] (4)

Multiplications of the superpotential terms by \( \phi, \phi', (\bar{H}^c)^4, (H^c)^4, \bar{H}^cH^c \) are generally allowed with the exception of the multiplication of \( S, \bar{N}^2N^2, N^2h^2 \) by a single factor of \( \phi, \phi', \phi\phi' \), which are \( SU(4)_c \) 15-plets, or a single factor of \( \phi'(\bar{H}^c)^4, \phi'(H^c)^4 \). Also, multiplication of \( \bar{\phi}'\phi' \) by a single \( \phi \) is not allowed.

To further analyze the mixing of the doublets in \( h \) and \( h' \), we must first define properly the relevant couplings \( \bar{h}'h', \phi h'h \) and \( \phi h'h' \). The superfield \( h = (1, 2, 2) \) is written as

\[
h = \begin{pmatrix}
h^+_2, \quad h^0_1 \\
h^0_2, \quad h^-_1
\end{pmatrix} \equiv \begin{pmatrix} h_2, \quad h_1 \end{pmatrix}.
\] (5)

The color singlet components of the fields \( \bar{h}', h' \) can be similarly represented. Under \( U_c \in SU(4)_c, U_L \in SU(2)_L \) and \( U_R \in SU(2)_R \), the relevant fields transforms as:
where the transpose of a matrix is denoted by tilde. The Yukawa couplings are $FhF^c$, $Fh'F^c$. From Eq.(3) and the identities $U_L\epsilon U_L = U_R\epsilon U_R = \epsilon$ ($\epsilon$ is the $2 \times 2$ antisymmetric matrix with $\epsilon_{12} = 1$) which follow from $\det U_L = \det U_R = 1$, one finds that $\bar{h}\epsilon \to U_R\epsilon U_L^\dagger$, $\bar{h}'\epsilon \to U_L\bar{h}'\epsilon U_R^\dagger$, $\bar{h}'\epsilon \to U_L\bar{h}'\epsilon U_R^\dagger U_L^\dagger$. We see that $\text{tr}(\bar{h}'\epsilon\bar{h}'\epsilon)$, $\text{tr}(\bar{h}'\epsilon\phi\bar{h}')$, $\text{tr}(\bar{h}'\epsilon\phi'\bar{h}')$ (the traces are taken with respect to the $SU(4)_c$ and $SU(2)_L$ indices) are invariant under $G_{PS}$. They correspond to the ‘symbolic’ couplings $\bar{h}'\epsilon$, $\phi\bar{h}'$, $\phi'\bar{h}'$ respectively.

After the breaking of $G_{PS}$ to $G_S$, the fields $\phi$, $\phi'$ acquire vevs $\langle \phi \rangle$, $\langle \phi' \rangle \sim M_{GUT}$. Substituting them by these vevs in the above couplings and using Eq.(5), we obtain

\begin{equation}
\text{tr}(\bar{h}'\epsilon\bar{h}'\epsilon) = \bar{h}_1\epsilon h_2' + \bar{h}_1'\epsilon h_2' + \cdots,
\end{equation}

\begin{equation}
\text{tr}(\bar{h}'\epsilon\phi\bar{h}') = \langle \phi \rangle \frac{\sqrt{2}}{\sqrt{2}} \text{tr}(\bar{h}'\epsilon\phi\bar{h}') = \langle \phi \rangle \frac{\sqrt{2}}{\sqrt{2}} \bar{h}_1\epsilon h_2 - \bar{h}_1\epsilon h_2',
\end{equation}

\begin{equation}
\text{tr}(\bar{h}'\epsilon\phi'\bar{h}') = \langle \phi' \rangle \frac{\sqrt{2}}{\sqrt{2}} \text{tr}(\bar{h}'\epsilon\phi'\bar{h}') = \langle \phi' \rangle \frac{\sqrt{2}}{\sqrt{2}} \bar{h}_1\epsilon h_2 + \bar{h}_1\epsilon h_2',
\end{equation}

where only the colorless components of $\bar{h}'$ and $h'$ are shown in the right hand side of Eq.(7) and $\sigma_3 = \text{diag}(1, -1)$. The bilinear terms between $h_1$, $h_1'$, $h_1'$ and $h_2$, $h_2'$, $h_2'$ which appear in Eqs.(7), (8) and (9) turn out to be the dominant bilinear terms between these doublets. Collecting them together, we obtain

\begin{equation}
m\bar{h}_1\epsilon(h_2' + \alpha_2 h_2) + m(\bar{h}_1' + \alpha_1 h_1)\epsilon h_2',
\end{equation}

where $m$ is the superheavy mass parameter which multiplies the term in Eq.(7) and $\alpha_1 = (-p(\phi) + q(\phi'))/\sqrt{2}m$, $\alpha_2 = (p(\phi) + q(\phi'))/\sqrt{2}m$ with $p$ and $q$ being the dimensionless coupling constants which correspond to the $SU(2)_R$ triplet and singlet terms in Eqs.(8) and (9) respectively. Note that $\alpha_1$, $\alpha_2$ are in general complex. So, we get two pairs of superheavy doublets with mass $m$. They are predominantly given by

\begin{equation}
\bar{h}_1', \frac{h_2' + \alpha_2 h_2}{(1 + |\alpha_2|^2)^{1/2}} \text{ and } \frac{h_1' + \alpha_1 h_1}{(1 + |\alpha_1|^2)^{1/2}}, \bar{h}_2'.
\end{equation}

The orthogonal combinations of $h_1$, $h_1'$ and $h_2$, $h_2'$ constitute the electroweak doublets:

\begin{equation}
h_1'^w = \frac{h_1 - \alpha_1^* h_1'}{(1 + |\alpha_1|^2)^{1/2}} \text{ and } h_2'^w = \frac{h_2 - \alpha_2^* h_2'}{(1 + |\alpha_2|^2)^{1/2}}.
\end{equation}
We see that, although $h_1$, $h_2$ and $h'_1$, $h'_2$ form SU(2)$_R$ doublets, this is, in general, not true for $h'^w_1$, $h'^w_2$ since $\alpha_1$, $\alpha_2$ can be different. However, if we remove from the model the SU(2)$_R$ triplets $\phi^*$, $\phi$ (and, thus, the SU(2)$_R$ triplet coupling in Eq.(8)), we obtain $\alpha_1 = \alpha_2$ and $h'^w_1$, $h'^w_2$ do form a SU(2)$_R$ doublet. The minimal model which can provide us with an adequate violation of $t - b$ Yukawa unification for $\mu > 0$ is the one including $\phi^*$, $\phi$ but not the SU(2)$_R$ singlets $\phi^*, \phi^*$. This model, which we choose to study in detail here, yields $\alpha_1 = -\alpha_2$. Inclusion of both $\phi^*$, $\phi$ and $\phi^*$, $\phi^*$ can lead to an arbitrary relation between $\alpha_1$ and $\alpha_2$ and, thus, introduces an extra complex parameter. Finally, due to the presence of $h'_1$, $h'_2$, the electroweak doublets are generally non-singlets under SU(4)$_c$ and $b - \tau$ Yukawa unification can also be violated.

The doublets in Eq.(11) must have zero vevs, which implies that $\langle h'_1 \rangle = -\alpha_1 \langle h_1 \rangle$, $\langle h'_2 \rangle = -\alpha_2 \langle h_2 \rangle$. Eq.(12) then gives $\langle h'^w \rangle = (1 + |\alpha_1|^2)^{1/2} \langle h_1 \rangle$, $\langle h'^w \rangle = (1 + |\alpha_2|^2)^{1/2} \langle h_2 \rangle$. From the third generation Yukawa couplings $y_{33}^3 F_3 hF_3^c$, $2y_{33}^3 h'F_3^c$, we obtain

$$m_t = |y_{33}^3 \langle h_2 \rangle + y'_{33}^3 \langle h'_2 \rangle| = |(y_{33}^3 - y'_{33}^3 \alpha_2) \langle h_2 \rangle| = \left| \frac{1 - \rho \alpha_2/\sqrt{3}}{(1 + |\alpha_2|^2)^{3/2}} y_{33}^3 \langle h'^w \rangle \right|,$$  \hspace{1cm} (13)

where $\rho = y'_{33}/y_{33}$ and can be taken positive by appropriately readjusting the phases of $h$, $h'$. Note that, in the SU(4)$_c$ space, the doublets in $h'$ are proportional to diag$(1/2\sqrt{3}, 1/2\sqrt{3}, 1/2\sqrt{3}, -\sqrt{3}/2)$, which is normalized so that the trace of its square equals unity. Thus, to make $y'_{33}$ directly comparable to $y_{33}$, we included a factor of two in defining the corresponding Yukawa coupling. The masses $m_b$, $m_\tau$ are, similarly, found:

$$m_b = \left| \frac{1 - \rho \alpha_1/\sqrt{3}}{(1 + |\alpha_1|^2)^{3/2}} y_{33}^3 \langle h'^w \rangle \right|,$$ \hspace{1cm} (14)

$$m_\tau = \left| \frac{1 + \sqrt{3} \rho \alpha_1}{(1 + |\alpha_1|^2)^{3/2}} y_{33}^3 \langle h'^w \rangle \right|.$$

From Eqs.(13) and (14), we see that the exact equality of the ‘asymptotic’ Yukawa couplings $h_t$, $h_b$, $h_\tau$ is now replaced by the quasi-unification condition:

$$h_t : h_b : h_\tau = \left| \frac{1 - \rho \alpha_2/\sqrt{3}}{(1 + |\alpha_2|^2)^{3/2}} \right| : \left| \frac{1 - \rho \alpha_1/\sqrt{3}}{(1 + |\alpha_1|^2)^{3/2}} \right| : \left| \frac{1 + \sqrt{3} \rho \alpha_1}{(1 + |\alpha_1|^2)^{3/2}} \right|.$$ \hspace{1cm} (15)

This condition depends on two complex ($\alpha_1$, $\alpha_2$) and one real ($\rho > 0$) parameter.

Note that the mixing between $h_1$, $h'_1$ (i.e., $\alpha_1 \neq 0$) and the fact that the Higgs superfield $h'$ possesses Yukawa couplings to the ‘matter’ superfields (i.e., $\rho \neq 0$) are crucial for violating $b - \tau$ Yukawa unification, which is though not affected by the mixing
between $h_t$, $h_b$ (i.e., the value of $\alpha_2$). On the contrary, violation of $t-b$ Yukawa unification requires that $\alpha_1 \neq \alpha_2$, which can be achieved only in the presence of a $SU(2)_R$ triplet bilinear term between $\bar{h}'$ and $h$. In summary, the minimal requirements for full violation of Yukawa unification are $\rho$, $\alpha_1 \neq 0$, $\alpha_1 \neq \alpha_2$. In the minimal model (with $\bar{\phi}$, $\phi$ but not $\bar{\phi}'$, $\phi'$) which we will study here, $\alpha_1 = -\alpha_2$ and, thus, Eq.(15) takes the simple form

$$h_t : h_b : h_\tau = |1 + c| : |1 - c| : |1 + 3c|,$$

where $c = \rho \alpha_1 / \sqrt{3}$. This ‘asymptotic’ relation depends on a single complex parameter. For simplicity, we will restrict our analysis to real values of $c$ only.

For completeness, we also give the ‘asymptotic’ relation between the Yukawa couplings in the model without $\bar{\phi}$, $\phi$, $\bar{\phi}'$, $\phi'$, which, although not suitable for $\mu > 0$, may be adequate for $\mu < 0$. Under $G_{PS}$, the superfields $\bar{H}^c$, $H^c$ transform as: $\bar{H}^c \rightarrow \bar{H}^c U_R \bar{U}_c$, $H^c \rightarrow U_c \bar{U}_R^\dagger H^c$. Thus, $\bar{H}^c \bar{H}^c \rightarrow U_c \bar{U}_R \bar{H}^c U_R^\dagger \bar{U}_c^\dagger$, and tr($\bar{h}' \bar{\phi} \bar{H} \bar{e} \bar{h} \bar{e}$) is invariant under $G_{PS}$ containing both the $SU(2)_R$ singlet and triplet ‘symbolic’ couplings $\bar{H}^c \bar{H}^c \bar{h} \bar{h}$. In the $SU(2)_R$ space, $\langle \bar{H}^c \rangle \langle \bar{H}^c \rangle = \text{diag}(v_0^2, 0) = v_0^2 \sigma_0 / 2 + v_0^2 \sigma_3 / 2$, where $v_0 = M_{GUT} / g_{GUT}$, with $g_{GUT}$ being the GUT gauge coupling constant, and $\sigma_0$ is the unit $2 \times 2$ matrix. Moreover, in the $SU(4)_c$ space, $\langle \bar{H}^c \rangle \langle \bar{H}^c \rangle = \text{diag}(0, 0, v_0^2)$. So, replacing $H^c$, $\bar{H}^c$ by their vevs in the singlet and triplet couplings in tr($\bar{h}' \bar{\phi} \bar{H} \bar{e} \bar{h} \bar{e}$) (with dimensionless coefficients equal to $q'$ and $p'$), we obtain $(-\sqrt{3} v_0^2 / 4) \text{tr}(\bar{h}' \bar{\phi} \bar{H} \bar{e} \bar{h} \bar{e})$ and $(-\sqrt{3} v_0^2 / 4) \text{tr}(\bar{h}' \sigma_3 \bar{H} \bar{e} \bar{h} \bar{e})$. We, thus, again end up with Eq.(15) but with $\alpha_1 = -\sqrt{3} v_0^2 (q' - p') / 4mM_S$, $\alpha_2 = -\sqrt{3} v_0^2 (q' + p') / 4mM_S$, which are suppressed by $M_{GUT} / M_S$.

III. THE RESULTING MSSM

We will now concentrate on the minimal model which includes $\bar{\phi}$, $\phi$ but not $\bar{\phi}'$, $\phi'$. This model, below $M_{GUT}$, reduces to the MSSM supplemented by the ‘asymptotic’ Yukawa coupling quasi-unification condition in Eq.(19), where $c$ is taken real for simplicity. We will assume universal soft SUSY breaking terms at $M_{GUT}$, i.e., a common mass for all scalar fields $m_0$, a common gaugino mass $M_{1/2}$ and a common trilinear scalar coupling $A_0$. So the resulting MSSM is actually the CMSSM. Furthermore, we will concentrate on the $\mu > 0$ case for reasons which we already explained.
We will closely follow the notation as well as the RG and radiative electroweak breaking analysis of Ref. [19] for the CMSSM with the improvements of Ref. [5] (recall that the sign of $\mu$ in these references is opposite to the one adopted here). These improvements include the employment of the full one-loop corrections to the effective potential for the electroweak breaking and to certain particle masses taken from Ref. [20]. They also include the incorporation of the two-loop corrections to the CP-even neutral Higgs boson mass matrix by using FeynHiggsFast [21] and the introduction [22] of a variable common SUSY threshold $M_{\text{SUSY}} = (\tilde{m}_1 \tilde{m}_2)^{1/2}$ ($\tilde{t}_{1,2}$ are the stop mass eigenstates) where the effective potential is minimized, $m_A$ (the CP-odd Higgs boson mass) and $\mu$ are evaluated, and the MSSM RG equations are replaced by the SM ones. Note that, at $M_{\text{SUSY}}$, the size of the one-loop corrections to $m_A$ and $\mu$ is [22] minimal and, thus, the accuracy in the determination of these quantities is maximal.

For any given $m_b(M_Z)$ in its 95% c.l. range ($2.684 - 3.092$ GeV), we can determine the parameters $c$ and $\tan \beta$ at $M_{\text{SUSY}}$ so that the ‘asymptotic’ condition in Eq.(16) is satisfied. We use fixed values for the running top quark mass $m_t(m_t) = 166$ GeV and the running tau lepton mass $m_\tau(M_Z) = 1.746$ GeV and incorporate not only the SUSY correction to the bottom quark mass but also the SUSY threshold correction to $m_\tau(M_{\text{SUSY}})$ from the approximate formula of Ref. [20]. This correction arises mainly from chargino/tau sneutrino ($\tilde{\nu}_\tau$) loops and, for $\mu > 0$, leads [5] to a small reduction of $\tan \beta$.

After imposing the conditions of gauge coupling unification, successful electroweak breaking and Yukawa quasi-unification in Eq.(14), we are left with three free input parameters $m_0$, $M_{1/2}$ and $A_0$ (see Refs. [3,19]). In order to make the notation physically more transparent, we replace $m_0$ and $M_{1/2}$ equivalently by the mass $m_{\text{LSP}}$ (or $m_{\tilde{\chi}}$) of the lightest supersymmetric particle (LSP), which turns out to be the lightest neutralino ($\tilde{\chi}$), and the relative mass splitting $\Delta_{\tilde{\tau}_2} = (m_{\tilde{\tau}_2} - m_{\tilde{\chi}})/m_{\tilde{\chi}}$ between the lightest stau mass eigenstate ($\tilde{\tau}_2$) and the LSP. We will study the parameter space of this model which is compatible with all the available phenomenological and cosmological constraints.

An important constraint is obtained by considering the inclusive branching ratio $\text{BR}(b \rightarrow s\gamma)$ of the decay process $b \rightarrow s\gamma$. The present best average of this branching ratio, which is found from the available experimental data [12], is $3.24 \times 10^{-4}$ with an experimental error of $\pm 0.38 \times 10^{-4}$ and an asymmetric ‘theoretical’ error due to model
dependence of $[+0.26, -0.23] \times 10^{-4}$ (assuming no correlation between the experimental systematics, which should not be too far from reality).

We calculate $\text{BR}(b \to s\gamma)$ using the formalism of Ref. [23], where the SM contribution is factorized out. This contribution includes the next-to-leading order (NLO) QCD and the leading order (LO) QED corrections. The charged Higgs boson contribution to $\text{BR}(b \to s\gamma)$ is evaluated by including the NLO QCD corrections from Ref. [24]. The dominant SUSY contribution includes the NLO QCD corrections from Ref. [25], which hold for large $\tan\beta$ (see also Ref. [26]). The error in the SM contribution to $\text{BR}(b \to s\gamma)$ is made of two components. The first is due to the propagation of the experimental errors in the input parameters and is about $\pm 0.23 \times 10^{-4}$. The second originates from the dependence on the renormalization and matching scales and is about $\pm 6\%$ if the NLO QCD corrections are included. These errors remain basically the same even after including the charged Higgs and SUSY contributions corrected at NLO.

In order to construct the 95% c.l. range of $\text{BR}(b \to s\gamma)$, we first add in quadrature the experimental error ($\pm 0.38 \times 10^{-4}$) and the error originating from the input parameters ($\pm 0.23 \times 10^{-4}$). This yields the overall standard deviation ($\sigma$). The asymmetric error coming from model dependence ($[+0.26, -0.23] \times 10^{-4}$) and the error from scale dependence (6%) are then added linearly on both ends of the $\pm 2 - \sigma$ range. For simplicity, we take a constant error from scale dependence, which we evaluate at the central experimental value of $\text{BR}(b \to s\gamma)$. The 95% c.l. range of this branching ratio then turns out to be about $(1.9 - 4.6) \times 10^{-4}$.

The recent measurement [13] of the anomalous magnetic moment of the muon $a_\mu \equiv (g_\mu - 2)/2$ provides an additional significant constraint. The deviation of $a_\mu$ from its predicted value in the SM, $\delta a_\mu$, is found to lie, at 95% c.l., in the range from $-6 \times 10^{-10}$ to $58 \times 10^{-10}$. This range is derived using the calculations (see e.g., Ref. [30]) of $a_\mu$ in the SM which are based on the evaluation of the hadronic vacuum polarization contribution of Ref. [31]. However, we take here the recently corrected [32] sign of the pseudoscalar pole contribution to the light-by-light scattering correction to $a_\mu$. This corrected sign reduces considerably the discrepancy between the SM and the measured value of $a_\mu$. It also relaxes the restrictions on the parameter space of the CMSSM. In particular, the $\mu < 0$ case, which leads to negative $\delta a_\mu$, is no longer disfavored. However, the sparticles,
in this case, cannot be as light as in the \( \mu > 0 \) case, where \( \delta a_\mu > 0 \). The calculation of \( \delta a_\mu \) in the CMSSM is performed here by using the analysis of Ref. [33].

Another constraint results from the requirement that the relic abundance \( \Omega_{LSP} h^2 \) of the LSP in the universe does not exceed the upper limit on the cold dark matter (CDM) abundance which is derived from observations (\( \Omega_{LSP} \) is the present energy density of the LSP over the critical energy density of the universe and \( h \) is the present value of the Hubble parameter in units of 100 km sec\(^{-1}\) Mpc\(^{-1}\)). From the recent results of DASI [34], one finds that the 95\% c.l. range of \( \Omega_{CDM} h^2 \) is 0.06 – 0.22. Therefore, we require that \( \Omega_{LSP} h^2 \) does not exceed 0.22.

Here, the LSP (\( \tilde{\chi} \)) is an almost pure bino. Its relic abundance will be calculated by micrOMEGAs [35], which is the most complete code available. (A similar calculation has appeared in Ref. [36].) It includes all the coannihilations [37] of neutralinos, charginos, sleptons, squarks and gluinos. The exact tree-level cross sections are used and are accurately thermally averaged. Also, poles and thresholds are properly handled and one-loop QCD corrected Higgs decay widths [38] are used, which is the main improvement provided by Ref. [35]. The SUSY corrections [39] to these widths are, however, not included. Fortunately, in our case, their effect is much smaller than that of the QCD corrections.

In order to have an independent check of micrOMEGAs, we also use the following alternative method for calculating \( \Omega_{LSP} h^2 \) in our model. In most of the parameter space where coannihilations are unimportant, \( \Omega_{LSP} h^2 \) can be calculated by using DarkSUSY [40]. This code employs the complete tree-level (and in some cases one-loop) cross sections of the relevant processes with properly treating all resonances and thresholds and performs accurate numerical integration of the Boltzmann equation. Its neutralino annihilation part is in excellent numerical agreement with the recent exact analytic calculation of Ref. [41]. Moreover, it includes all coannihilations between neutralinos and charginos which are, however, insignificant for our model. Its main defect is that it uses the tree-level Higgs decay widths. This can be approximately corrected if, in evaluating the Higgs decay widths, we replace \( m_\phi(m_\phi) \) by \( m_\phi \) at the mass of the appropriate Higgs boson in the couplings of the \( b \)-quark to the Higgs bosons (see Ref. [35]).

In the region of the parameter space where coannihilations come into play, the next-to-lightest supersymmetric particle (NLSP) turns out to be the \( \tilde{\tau}_2 \) and the only relevant
coannihilations are the bino-stau ones [19,42]. In this region, which is given by \( \Delta_{\tau_2} < 0.25 \), we calculate \( \Omega_{LSP} h^2 \) by using an improved version of the analysis of Ref. [19] (the sign of \( \mu \) in this reference is opposite to the one adopted here). This analysis, which has been applied in Refs. [7,13], includes bino-stau coannihilations for all \( \tan \beta \)'s (see also Refs. [7,44]) and is based on a series expansion of the thermally averaged cross sections in \( x_F = T_F/m_\chi \), with \( T_F \) being the freeze-out temperature. This expansion is, however, inadequate when we have to treat resonances, and these are crucial for bino annihilation.

So, we need to improve the bino annihilation part in Ref. [19]. This can be achieved by evaluating the corresponding expansion coefficients \( a_{\chi\chi} \) and \( b_{\chi\chi} \) not as in this reference but as follows. From DarkSUSY, we find the values of \( \Omega_\chi h^2 \) and \( x_F \) which correspond to essentially just bino annihilation. Using the appropriate formulas of Ref. [19], we then extract the ‘improved’ expansion coefficients \( a_{\chi\chi} \) and \( b_{\chi\chi} \).

There is one more improvement which we need to do in Ref. [19] (and was already used in Ref. [15]) to make it applicable to the present case. The cross sections of the processes \( \tilde{\tau}_2 \tilde{\tau}_2^* \rightarrow W^+ W^- \), \( H^+ H^- \) were calculated without including the tree-graphs with \( \tilde{\nu}_\tau \) exchange in the t-channel [10]. Thus, the contribution to \( a_{\tilde{\tau}_2 \tilde{\tau}_2^*} \) from the process \( \tilde{\tau}_2 \tilde{\tau}_2^* \rightarrow W^+ W^- \) appearing in Table II of Ref. [19] should be corrected by adding to it

\[
\frac{g_{\tilde{\tau}_2 \tilde{\nu}_\tau, W^\pm}^2}{4\pi m_W^4 (1 - m_W^2 + m_{\tilde{\tau}_2})} \left[ 2g_{\tilde{\tau}_2 \tilde{\nu}_\tau, W^\pm}^2 \left( \frac{1 - m_W^2)^2 m_{\tilde{\tau}_2}^2}{1 - m_W^2 + m_{\tilde{\nu}_\tau}^2} \right) \right] - (2 - 3m_W^2 + m_{\tilde{W}}^2) \left( \frac{g_{\tilde{\tau}_2 \tilde{\nu}_\tau, W^\pm}^2}{m_W^4} \left( \frac{g_{\tilde{\tau}_2 \tilde{\nu}_\tau, W^\pm}^2}{m_W^2} - \frac{g_{\tilde{\tau}_2 \tilde{\nu}_\tau, W^\pm}^2}{m_W^2} \right) \right),
\]

(17)

with \( g_{\tilde{\tau}_2 \tilde{\nu}_\tau, W^\pm} = g_{\tilde{\tau}_2 \tilde{\nu}_\tau, W^\pm} / \sqrt{2} \). Also, the contribution to \( a_{\tilde{\tau}_2 \tilde{\tau}_2^*} \) from \( \tilde{\tau}_2 \tilde{\tau}_2^* \rightarrow H^+ H^- \) is now given by Eq. (26) of Ref. [19] with the expression in the last parenthesis corrected by adding to it the quantity \( g_{\tilde{\tau}_2 \tilde{\nu}_\tau, H^\pm}^2 / (m_W^2 - m_{\tilde{\nu}_\tau}^2 + 1) \), where

\[
g_{\tilde{\tau}_2 \tilde{\nu}_\tau, H^\pm} = g \left[ (m_W^2 \sin 2\beta - m_{\tilde{\nu}_\tau}^2 \tan \beta) s_\theta + (A_\tau \tan \beta + \mu) m_{\tilde{\nu}_\tau} c_\theta \right] / \sqrt{2} m_W
\]

(with the present sign convention for \( \mu \)). The resulting correction to \( \Omega_\chi h^2 \), in the case of the process \( \tilde{\tau}_2 \tilde{\tau}_2^* \rightarrow W^+ W^- \), varies from about 8% to about 1% as \( \Delta_{\tau_2} \) increases from 0 to 0.25. On the contrary, the correction from \( \tilde{\tau}_2 \tilde{\tau}_2^* \rightarrow H^+ H^- \) is negligible.

We find that the alternative method for calculating the neutralino relic abundance in our model, which we have just described, yields results which are in excellent agreement
with micrOMEGAs. In practice, however, we use the code micrOMEGAs since it has the extra advantage of being much faster.

We will also impose the 95% c.l. LEP bound on the lightest CP-even neutral Higgs boson mass $m_h > 114.1 \text{ GeV}$ $^{47}$. In the CMSSM, this bound holds almost always for all tan $\beta$'s, at least as long as CP is conserved. Finally, for the values of tan $\beta$ which appear here (about 60), the CDF results yield the 95% c.l. bound $m_A > 110 \text{ GeV}$ $^{48}$.

IV. THE ALLOWED PARAMETER SPACE

We now proceed to the derivation of the restrictions which are imposed by the various phenomenological and cosmological constraints presented in Sec.III on the parameters of the CMSSM (with $\mu > 0$) supplemented with the Yukawa quasi-unification condition in Eq.(16) (with $0 < c < 1$). The restrictions on the $m_{LSP} - \Delta \tilde{\tau}_2$ plane, for $A_0 = 0$ and with the central value of $\alpha_s(M_Z) = 0.1185$, are shown in Fig.1. The dashed (dotted) lines correspond to the 95% c.l. lower (upper) experimental bound on $m_b(M_Z)$ which is 2.684 GeV (3.092 GeV) (see Sec.II), while the solid lines correspond to the central experimental value of $m_b(M_Z) = 2.888 \text{ GeV}$. We will follow this convention in the subsequent four figures too. From left to right, the dashed (dotted) lines depict the 95% c.l. lower bounds on $m_{LSP}$ from the constraints $m_A > 110 \text{ GeV}$, $\text{BR}(b \rightarrow s\gamma) > 1.9 \times 10^{-4}$ and $\delta a_\mu < 58 \times 10^{-10}$, and the 95% c.l. upper bound on $m_{LSP}$ from $\Omega_{LSP} \, h^2 < 0.22$. The constraints $\text{BR}(b \rightarrow s\gamma) < 4.6 \times 10^{-4}$ and $\delta a_\mu > -6 \times 10^{-10}$ do not restrict the parameters since they are always satisfied for $\mu > 0$. The left solid line depicts the lower bound on $m_{LSP}$ from $m_h > 114.1 \text{ GeV}$ which does not depend much on $m_b(M_Z)$, while the right solid line corresponds to $\Omega_{LSP} \, h^2 = 0.22$ for the central value of $m_b(M_Z)$.

We observe that the lower bounds on $m_{LSP}$ are generally not so sensitive to the variations of $m_b(M_Z)$ within its 95% c.l. range. The most sensitive of these bounds is the one from $m_A > 110 \text{ GeV}$, while the bound from $m_h > 114.1 \text{ GeV}$ is practically $m_b(M_Z)$-independent. Actually, this bound overshadows all the other lower bounds for all $\Delta \tilde{\tau}_2$'s and, being essentially $\Delta \tilde{\tau}_2$-independent too, sets, for $A_0 = 0$ and $\alpha_s(M_Z) = 0.1185$, an overall constant 95% c.l. lower bound on $m_{LSP}$ of about 138 GeV. Contrary to the lower bounds, the line from the LSP relic abundance is extremely sensitive to the value
of $m_b(M_Z)$. In particular, its almost vertical part is considerably displaced to higher $m_{LSP}$'s as $m_b(M_Z)$ decreases. We will explain this behavior later.

In Fig.2, we depict $m_A$ and $M_{SUSY}$ versus $m_{LSP}$ for various $\Delta_{\tilde{\tau}_2}$'s, $A_0 = 0$ and the central values of $m_b(M_Z)$ and $\alpha_s(M_Z)$. We see that $m_A$ is always smaller than $2m_{LSP}$ but close to it. Thus, generally, the neutralino annihilation via the s-channel exchange of an $A$-boson is by far the dominant (co)annihilation process. We also observe that, as $m_{LSP}$ or $\Delta_{\tilde{\tau}_2}$ increase, we move away from the $A$-pole, which thus becomes less efficient. As a consequence, $\Omega_{LSP} h^2$ increases with $m_{LSP}$ or $\Delta_{\tilde{\tau}_2}$ (see Fig.3).

In Fig.3, we show $\Omega_{LSP} h^2$ as a function of $m_{LSP}$ for various $\Delta_{\tilde{\tau}_2}$'s, $A_0 = 0$, $m_b(M_Z) = 2.888$ GeV and $\alpha_s(M_Z) = 0.1185$. The solid lines are obtained by using micrOMEGAs. The crosses are from the alternative method described in Sec. [11] which combines DarkSUSY with the bino-stau coannihilation calculation of Ref. [19]. In order to mimic, in DarkSUSY, the effect of the one-loop QCD corrections to the Higgs decay widths, we can generally use the $b$-quark mass (including the SUSY corrections) evaluated at $m_A$ since the $A$-pole is by far the most important. As it turns out, $m_b(m_A)$ depends very weakly on the values of the input parameters $m_{LSP}$ and $\Delta_{\tilde{\tau}_2}$, and thus the uniform use of a constant mean value of $m_b(m_A)$ should be adequate. We find that excellent agreement between micrOMEGAs and our alternative method is achieve if, in DarkSUSY, we use a constant default value for the $b$-quark mass which is equal to about 2.5 GeV (see Fig.3).

The importance for our calculation of the one-loop QCD corrections to the Higgs decay widths can be easily concluded from Fig.4, where we show $\Omega_{LSP} h^2$ versus $m_{LSP}$ for $\Delta_{\tilde{\tau}_2} = 1$, $A_0 = 0$ and the central values of $m_b(M_Z)$, $\alpha_s(M_Z)$. The LSP relic abundance is calculated by micrOMEGAs with (thick solid line) or without (faint solid line) the inclusion of the one-loop QCD corrections to the Higgs widths. We see that the results differ appreciably. For comparison, we also draw $\Omega_{LSP} h^2$ calculated by our alternative method with (thick crosses) or without (faint crosses) the above ‘correction’ of the $b$-quark mass. The agreement with micrOMEGAs is really impressive in both cases.

In Fig.5, we present $m_A$ and $M_{SUSY}$ as functions of $m_{LSP}$ for $\Delta_{\tilde{\tau}_2} = 1$, $A_0 = 0$ and with the lower, upper and central values of $m_b(M_Z)$ for $\alpha_s(M_Z) = 0.1185$. We see that $m_A$ increases and approaches $2m_{LSP}$ as $m_b(M_Z)$ decreases. As a consequence, the effect of the $A$-pole is considerably enhanced and $\Omega_{LSP} h^2$ is reduced. This explains the large
displacement of the vertical part of the $\Omega_{LSP} \ h^2 = 0.22$ line in Fig.1 at low $m_b(M_Z)$'s.

As $\Delta_{\tilde{\tau}_2}$ becomes small and approaches zero, bino-stau coannihilations become very important and strongly dominate over the pole annihilation of neutralinos. This leads to a very pronounced reduction of the neutralino relic abundance and, consequently, to a drastic increase of the upper bound on $m_{LSP}$. We, thus, obtain the almost horizontal parts of the $\Omega_{LSP} \ h^2 = 0.22$ lines in Fig.1.

The shaded area depicted in Fig.1, which lies between the $m_h = 114.1$ GeV line and the $\Omega_{LSP} \ h^2 = 0.22$ line corresponding to the central value of $m_b(M_Z)$, constitutes the allowed range in the $m_{LSP} - \Delta_{\tilde{\tau}_2}$ plane for $A_0 = 0$, $m_b(M_Z) = 2.888$ GeV and $\alpha_s(M_Z) = 0.1185$. For practical reasons, we limited our investigation to $\Delta_{\tilde{\tau}_2} \leq 2$ and $m_{LSP} \leq 500$ GeV. We see that allowing $m_b(M_Z)$ to vary within its 95% c.l. experimental range considerably enlarges the allowed area in Fig.1. On the other hand, fixing $m_b(M_Z)$ to its upper bound yields the smallest possible allowed area for $A_0 = 0$ and $\alpha_s(M_Z) = 0.1185$, which is, however, still quite wide. In any case, we conclude that, for $A_0 = 0$, there exists a natural and wide parameter space which is consistent with all the available phenomenological and cosmological requirements.

In the allowed (shaded) area of Fig.1 which corresponds to the central value of $m_b(M_Z)$, the parameter $c (\tan \beta)$ varies between about 0.15 and 0.20 (58 and 59). For $m_b(M_Z)$ fixed to its lower or upper bound, we find that, in the corresponding allowed area, the parameter $c (\tan \beta)$ ranges between about 0.17 and 0.23 (59 and 61) or 0.13 and 0.17 (56 and 58). We observe that, as we increase $m_b(M_Z)$, the parameter $c$ decreases and we get closer to exact Yukawa unification. This behavior is certainly consistent with the fact that the value of $m_b(M_Z)$ which corresponds to exact Yukawa unification lies well above its 95% c.l. range. The LSP mass is restricted to be higher than about 138 GeV for $A_0 = 0$ and $\alpha_s(M_Z) = 0.1185$, with the minimum being practically $\Delta_{\tilde{\tau}_2}$-independent.

At this minimum, $c \approx 0.16 - 0.20$ ($c \approx 0.13 - 0.23$) and $\tan \beta \approx 59$ ($\tan \beta \approx 58 - 61$) for $m_b(M_Z) = 2.888$ GeV ($m_b(M_Z) = 2.684 - 3.092$ GeV).

The ‘asymptotic’ Yukawa quasi-unification condition in Eq.(16) with $0 < c < 1$ yields

$$\delta h \equiv - \frac{h_b - h_t}{h_t} = \frac{h_{\tau} - h_t}{h_t} = \frac{2c}{1 + c}. \quad (18)$$

This means that the bottom and tau Yukawa couplings split from the top Yukawa cou-
pling by the same amount but in opposite directions, with \( h_b \) becoming smaller than \( h_t \). For \( A_0 = 0 \) and with the central values of \( m_b(M_Z) \) and \( \alpha_s(M_Z) \), the splitting \( \delta h \) ranges from about 0.26 to 0.33. Allowing \( m_b(M_Z) \) to vary in its 95\% c.l. range, however, we obtain a larger range for \( \delta h \) which is about 0.22 - 0.38. At the minimum of \( m_{LSP} \approx 138 \text{ GeV} \), \( \delta h \approx 0.28 - 0.33 \) (\( \delta h \approx 0.23 - 0.38 \)) for \( m_b(M_Z) = 2.888 \text{ GeV} \) (\( m_b(M_Z) = 2.684 - 3.092 \text{ GeV} \)).

For simplicity, \( \alpha_s(M_Z) \) was fixed to its central experimental value (equal to 0.1185) throughout our calculation. We could, however, let it vary in its 95\% c.l. experimental range 0.1145 - 0.1225, which would significantly widen the 95\% c.l. range of \( m_b(M_Z) \) to 2.616 - 3.157 GeV, with the lower (upper) bound corresponding to the upper (lower) bound on \( \alpha_s(M_Z) \). This would lead to a sizable enlargement of the allowed area due to the sensitivity of the LSP relic abundance to the \( b \)-quark mass.

In Figs. 6, 7 and 8, we present the restrictions on the \( m_{LSP} - A_0/M_{1/2} \) plane for \( m_b(M_Z) = 2.888 \text{ GeV} \), \( \alpha_s(M_Z) = 0.1185 \), and \( \Delta_{\tilde{\tau}_2} = 0, 1 \) and 2 respectively. We use solid, dashed, dot-dashed and dotted lines to depict the lower bounds on \( m_{LSP} \) from \( m_A > 110 \text{ GeV} \), \( \text{BR}(b \rightarrow s\gamma) > 1.9 \times 10^{-4} \), \( \delta a_\mu < 58 \times 10^{-10} \) and \( m_h > 114.1 \text{ GeV} \) respectively. The upper bounds on \( m_{LSP} \) from \( \Omega_{LSP} h^2 < 0.22 \) are represented by double dot-dashed lines. Note that, in Fig.6, the upper bound on \( m_{LSP} \) does not appear since it lies at \( m_{LSP} > 500 \text{ GeV} \). This is due to the fact that, since the bino and stau masses are degenerate in this case, the bino-stau coannihilation is considerably enhanced leading to a strong reduction of the LSP relic abundance. In order to illustrate the quick reduction of \( \Omega_{LSP} h^2 \) as \( \Delta_{\tilde{\tau}_2} \) approaches zero, we display in Fig.6 the bounds from \( \Omega_{LSP} h^2 < 0.22 \) (double dot-dashed lines) for \( \Delta_{\tilde{\tau}_2} = 0.1 \) and 0.03 too. The reduction of \( \Omega_{LSP} h^2 \) by bino-stau coannihilation is also the reason for the fact that the allowed (shaded) areas in Figs. 6, 7 and 8 are more sizable for smaller \( \Delta_{\tilde{\tau}_2} \)’s. The lower bound on \( m_{LSP} \) overshadows all the other lower bounds in all cases and is only very mildly dependent on \( \Delta_{\tilde{\tau}_2} \).

In Figs. 9 and 10, we present the restrictions on the \( m_{LSP} - \Delta_{\tilde{\tau}_2} \) plane for \( m_b(M_Z) = 2.888 \text{ GeV} \), \( \alpha_s(M_Z) = 0.1185 \), and \( A_0/M_{1/2} = 3 \) and \( -3 \) respectively. We follow the same notation for the various bounds on \( m_{LSP} \) and the allowed areas as in Figs. 6, 7 and 8. We observe that, for \( A_0/M_{1/2} = 3 \) \( (A_0/M_{1/2} = -3) \), the lower bound on \( m_{LSP} \) from the constraint \( \text{BR}(b \rightarrow s\gamma) > 1.9 \times 10^{-4} \) is overshadowed by (overshadows) the bounds
from the requirements $m_A > 110$ GeV and $\delta a_\mu < 58 \times 10^{-10}$. In all cases, however, the overall lower bound on $m_{LSP}$ is set by $m_h > 114.1$ GeV.

For presentation purposes, we restricted our analysis to $m_{LSP} \leq 500$ GeV, $\Delta \tilde{\tau}_2 \leq 2$, $-3 \leq A_0/M_{1/2} \leq 3$. The allowed ranges in Figs.1, 6, 7, 8, 9 and 10 constitute sections or boundaries of the part of the overall allowed parameter space for $m_b(M_Z) = 2.888$ GeV and $\alpha_s(M_Z) = 0.1185$ which is contained in the investigated range of parameters. We found that, within this part of the overall allowed parameter space, $\tan \beta$ ranges between about 58 and 61. Also, the parameter $c$ ranges between about 0.15 and 0.21 and, thus, the ‘asymptotic’ splitting between the bottom (or tau) and the top Yukawa couplings varies in the range $26 - 35\%$. Moreover, the smallest value of $m_{LSP}$ is about 107 GeV and is achieved at $A_0/M_{1/2} = -3$ and practically any $\Delta \tilde{\tau}_2$. The corresponding $\delta h \approx 0.30 - 0.35$ and $\tan \beta \approx 60 - 61$. Allowing $m_b(M_Z)$ and $\alpha_s(M_Z)$ to vary within their 95\% c.l. ranges will further widen the overall allowed parameter space within the investigated range of parameters as well as the allowed ranges of the parameters $c$ (or $\delta h$) and $\tan \beta$. The minimal value of $m_{LSP}$ is, though, not much affected by varying $m_b(M_Z)$. Variations of $\alpha_s(M_Z)$, however, lead to small ($\sim 10$ GeV) fluctuations of the minimal $m_{LSP}$.

We see that the required deviation from Yukawa unification, for $\mu > 0$, is not so small. In spite of this, the restrictions from Yukawa unification are not completely lost but only somewhat weakened. In particular, $\tan \beta$ remains large and close to 60. Actually, our model is much closer to Yukawa unification than generic models where the Yukawa couplings can differ even by orders of magnitude. Also, the deviation from Yukawa unification is generated here in a natural, systematic, controlled and well-motivated way. Finally, recall that the required size of the violation of Yukawa unification forced us to extend the initial model of Sec. II which could only provide a suppressed violation.

V. CONCLUSIONS

We constructed a class of concrete SUSY GUTs based on the PS gauge group which naturally lead to a moderate violation of ‘asymptotic’ Yukawa unification so that the $b$-quark mass can take acceptable values even with universal boundary conditions. For $\mu < 0$, a suppressed deviation from Yukawa unification may be adequate, while, for
$\mu > 0$, a more sizable deviation is required. In the $\mu < 0$ case, however, the sparticles are considerably heavier due to the constraints from $b \to s\gamma$ and the muon anomalous magnetic moment. So, the $\mu > 0$ case is more attractive for experimenters.

We considered a particular SUSY GUT from the above class with a deviation from Yukawa unification which is adequate for $\mu > 0$. We then discussed the resulting MSSM under the assumption of universal boundary conditions and the various phenomenological and cosmological requirements which restrict its parameter space. They originate from the data on the inclusive branching ratio of $b \to s\gamma$, the muon anomalous magnetic moment, the CDM abundance in the universe, and the masses $m_h$ and $m_A$.

The calculation of $\text{BR}(b \to s\gamma)$ incorporates all the LO QED and NLO QCD corrections which hold for large values of $\tan \beta$, while the LSP contribution to $\Omega_{CDM} h^2$ is evaluated by using the code micrOMEGAs which includes all possible coannihilation processes, treats poles properly and uses the one-loop QCD corrected Higgs decay widths. We also employed an alternative method for estimating the LSP relic abundance and found excellent agreement with micrOMEGAs.

We showed that, in the particular model with Yukawa quasi-unification considered, there exists a wide and natural range of CMSSM parameters which is consistent with all the above constraints. We found that, within the investigated part of the overall allowed parameter space, the parameter $\tan \beta$ ranges between about 58 and 61 and the ‘asymptotic’ splitting between the bottom (or tau) and the top Yukawa couplings varies in the range $26 - 35\%$ for central values of $m_b(M_Z)$ and $\alpha_s(M_Z)$. Also, the LSP mass can be as low as about 107 GeV.

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FIG. 1. Restrictions on the $m_{LSP} - \Delta \tau_2$ plane for $A_0 = 0, \alpha_s(M_Z) = 0.1185$. From left to right, the dashed (dotted) lines depict the lower bounds on $m_{LSP}$ from $m_A > 110$ GeV, $\text{BR}(b \to s\gamma) > 1.9 \times 10^{-4}$ and $\delta a_\mu < 58 \times 10^{-10}$, and the upper bound on $m_{LSP}$ from $\Omega_{LSP} h^2 < 0.22$ for $m_b(M_Z) = 2.684$ GeV (3.092 GeV). The left (right) solid line depicts the lower (upper) bound on $m_{LSP}$ from $m_h > 114.1$ GeV ($\Omega_{LSP} h^2 < 0.22$) for $m_b(M_Z) = 2.888$ GeV. The allowed area for $m_b(M_Z) = 2.888$ GeV is shaded.

FIG. 2. The mass parameters $m_A$ and $M_{SUSY}$ as functions of $m_{LSP}$ for various values of $\Delta \tau_2$, which are indicated on the curves, and with $A_0 = 0, m_b(M_Z) = 2.888$ GeV, $\alpha_s(M_Z) = 0.1185$. 
FIG. 3. The LSP relic abundance $\Omega_{LSP} h^2$ versus $m_{LSP}$ for various $\Delta\tilde{\tau}_2$'s (indicated on the curves) and with $A_0 = 0$, $m_b(M_Z) = 2.888$ GeV, $\alpha_s(M_Z) = 0.1185$. The solid lines (crosses) are obtained by micrOMEGAs (our alternative method). The upper bound on $\Omega_{LSP} h^2 (=0.22)$ is also depicted.

FIG. 4. The LSP relic abundance $\Omega_{LSP} h^2$ versus $m_{LSP}$ for $\Delta\tilde{\tau}_2 = 1$, $A_0 = 0$, $m_b(M_Z) = 2.888$ GeV, $\alpha_s(M_Z) = 0.1185$. The thick (faint) solid line is obtained by micrOMEGAs with (without) the one-loop QCD corrections to the Higgs widths, while the thick (faint) crosses by our alternative method with (without) the ‘correction’ of the $b$-quark mass. The upper bound on $\Omega_{LSP} h^2 (=0.22)$ is also depicted.
FIG. 5. The mass parameters $m_A$ and $M_{\text{SUSY}}$ versus $m_{\text{LSP}}$ for $\Delta \tilde{\chi}_2 = 1$, $A_0 = 0$, $\alpha_s(M_Z) = 0.1185$ and with $m_b(M_Z) = 2.684$ GeV (dashed lines), 3.092 GeV (dotted lines) or 2.888 GeV (solid lines).

FIG. 6. Restrictions on the $m_{\text{LSP}} - A_0/M_{1/2}$ plane for $\Delta \tilde{\chi}_2 = 0$, $m_b(M_Z) = 2.888$ GeV, $\alpha_s(M_Z) = 0.1185$. The solid, dashed, dot-dashed and dotted lines correspond to the lower bounds on $m_{\text{LSP}}$ from $m_A > 110$ GeV, $\text{BR}(b \to s\gamma) > 1.9 \times 10^{-4}$, $\delta a_\mu < 58 \times 10^{-10}$ and $m_h > 114.1$ GeV respectively. The upper bound on $m_{\text{LSP}}$ from $\Omega_{\text{LSP}} \, h^2 < 0.22$ does not appear in the figure since it lies at $m_{\text{LSP}} > 500$ GeV. The allowed area is shaded. For comparison, we also display the bounds from $\Omega_{\text{LSP}} \, h^2 < 0.22$ (double dot-dashed lines) for $\Delta \tilde{\chi}_2 = 0.1$ and 0.03, as indicated.
FIG. 7. Restrictions on the $m_{\text{LSP}} - A_0/M_{1/2}$ plane for $\Delta \tau_2 = 1$, $m_b(M_Z) = 2.888$ GeV and $\alpha_s(M_Z) = 0.1185$. We use the same notation for the lines which correspond to the various bounds on $m_{\text{LSP}}$ and for the allowed area as in Fig.6. The upper bound on $m_{\text{LSP}}$ from the cosmological constraint $\Omega_{\text{LSP}} h^2 < 0.22$ (double dot-dashed line) now appears inside the figure.

FIG. 8. Restrictions on the $m_{\text{LSP}} - A_0/M_{1/2}$ plane for $\Delta \tau_2 = 2$, $m_b(M_Z) = 2.888$ GeV and $\alpha_s(M_Z) = 0.1185$. We use the same notation for the lines which correspond to the various bounds on $m_{\text{LSP}}$ and for the allowed area as in Fig.7.
FIG. 9. Restrictions on the $m_{\text{LSP}} - \Delta \tilde{\tau}_2$ plane for $A_0/M_{1/2} = 3$, $m_b(M_Z) = 2.888$ GeV and $\alpha_s(M_Z) = 0.1185$. We use the same notation for the lines which correspond to the various bounds on $m_{\text{LSP}}$ and for the allowed area as in Fig.7.

FIG. 10. Restrictions on the $m_{\text{LSP}} - \Delta \tilde{\tau}_2$ plane for $A_0/M_{1/2} = -3$, $m_b(M_Z) = 2.888$ GeV and $\alpha_s(M_Z) = 0.1185$. We use the same notation for the lines which correspond to the various bounds on $m_{\text{LSP}}$ and for the allowed area as in Fig.7. The minimal value of $m_{\text{LSP}}$ in the investigated overall allowed parameter space is about 107 GeV for central values of $m_b(M_Z)$ and $\alpha_s(M_Z)$. As seen from this figure, this value is achieved at $A_0/M_{1/2} = -3$ and practically any $\Delta \tilde{\tau}_2$ (see also Figs.6, 7 and 8).