Uniform Long-Time and Propagation of Chaos Estimates for Mean Field Kinetic Particles in Non-convex Landscapes

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Abstract
Combining the results of Guillin (Uniform Poincaré and logarithmic Sobolev inequalities for mean field particles systems, 2019) and Monmarché (Stoch Process Appl 127(6):1721–1737, 2017), the trend to equilibrium in large time is studied for a large particle system associated to a Vlasov–Fokker–Planck equation. Under some conditions (that allow non-convex confining potentials) the convergence rate is proven to be independent from the number of particles. From this are derived uniform in time propagation of chaos estimates and an exponentially fast convergence for the nonlinear equation itself.

Keywords Mean field interaction · Hypocoercivity · Propagation of chaos · Vlasov–Fokker–Planck equation

Mathematics Subject Classification 82B40 · 60J60 · 35K58

1 Introduction
This work is devoted to the study of the long-time convergence of the solutions of the Vlasov-Fokker-Planck equation, governing the evolution of the density of interacting and diffusive matter in the space of positions and velocities, and of the associated system of interacting particles, as well as the convergence of the latter to the former as the number of particles increases. More precisely, following the notations of [21], the Vlasov–Fokker–Planck equation is
\[ \partial_t m_t + y \cdot \nabla_x m_t = \nabla_y \cdot \left( \frac{\sigma^2}{2} \nabla_y m_t + \left( \int_{\mathbb{R}^d} \nabla_x U(x, x') m_t(x', y') dx' dy' + \gamma y \right) m_t \right) \]

(1)

where \( m_t(x, y) \) is a density at time \( t \) of particles at \( x \in \mathbb{R}^d \) with velocity \( y \in \mathbb{R}^d \), \( d \in \mathbb{N}_{\ast} \), \( \sigma, \gamma > 0 \), \( \nabla \) and \( \nabla \cdot \) stand for the gradient and divergence operators and the potential \( U \) is a \( C^1 \) function from \( \mathbb{R}^{2d} \) to \( \mathbb{R} \) with \( U(x, x') = U(x', x) \) for all \( x, x' \in \mathbb{R}^d \). For \( N \in \mathbb{N}_{\ast} \), the system of \( N \) interacting particles which is standardly associated to (1) is the Markov process \( Z_N = (X_i, Y_i)_{i \in [1, N]} \) on \( \mathbb{R}^{2dN} \) that solves the stochastic differential equation

\[
\forall i \in [1, N], \begin{cases}
dX_i = Y_i dt \\
dY_i = -\gamma Y_i dt - \left( \frac{1}{N} \sum_{j=1}^{N} \nabla_x U(X_i, X_j) \right) dt + \sigma dB_i
\end{cases}
\]

(2)

with the initial conditions \( (X_i(0), Y_i(0)) \) being i.i.d. random variables of law \( m_0 \), independent from the standard Brownian motion \( B = (B_1, \ldots, B_N) \) on \( \mathbb{R}^{dN} \). As \( N \to \infty \), one expects that the particles are approximately independent so that a Law of Large Number holds and the empirical law

\[
M_t^N = \frac{1}{N} \sum_{i \in [1,N]} \delta_{(X_i, Y_i)},
\]

which is a random probability measure on \( \mathbb{R}^{2d} \), is close to the common law of the \( (X_i, Y_i) \)'s, whose evolution in time should thus approximately follow Equation (1). This is the so-called propagation of chaos phenomenon, as introduced by [18] and further developed by [24]. Rigorous statements are provided below.

The long-time behavior of \( m_t \) has been studied in various settings. Convergence to equilibrium without quantitative speed is addressed in [10]. Decomposing the potential \( U(x, x') = V(x) + V(x') + W(x, x') \) where \( V \) and \( W \) are respectively called the confinement and interaction potentials, exponentially fast long-time convergence is established by perturbation of the linear case in [6,17] when the interaction is sufficiently small. Such a quantitative result is also proven in [4] when the potential is close to a quadratic function, and in [21] when \( x \mapsto U(x, x') \) is strictly convex for all \( x' \). Similarly to [21], in the present work, we will obtain the long-time convergence of \( m_t \) from the long-time convergence of \( m_t^{(N)} \) the law of \( Z_N(t) \).

Indeed, notice that \( Z_N \) is a classical Langevin diffusion, for which relaxation toward equilibrium has been addressed, under various assumptions on the potential, in a broad number of works and with various techniques like Meyn-Tweedie or coupling probabilistic approaches [12,25] or hypocoercive modified entropy methods [7,9,25,26], see also [3] and within for more recent references. With respect to all this literature, the specificities of [21] which are relevant in the present mean-field framework are twofold: first, the long-time convergence has to be quantified in relative entropy (total variation distance or \( L^2 \) or \( H^1 \) norms would not be suitable for the limit \( N \to \infty \)) and, second, the convergence rate should be independent from \( N \) (which is not the case for example in [3]). From this, combined with crude propagation of chaos estimates, long-time convergence is obtained in [21] for the non-linear limit equation (1), together with uniform in time propagation of chaos estimates. It turns out that there is mainly one step in [21] where the convexity of the potential is crucially used, which is the proof that \( m_{\infty}^{(N)} \), the invariant measure of \( Z_N \) satisfies a log-Sobolev inequality with constant independent from \( N \). However, in the recent [15], such a uniform inequality is proven for the invariant measure of the overdamped version of the
system (2), which is exactly the $x$-marginal of $m^{(N)}_\infty$, under assumptions that allows non-convex potentials but with superquadratic confinement. Since log-Sobolev inequalities are stable under tensorization and since such an inequality is clearly satisfied by the $y$-marginal of $m^{(N)}_\infty$, which is a Gaussian law, we are in position to extend the results of [21] to a much broader class of potentials.

The plan of the paper is quite simple. Section 2 will present the results and comparisons with existing results, while proofs are provided in Sect. 3. We now detail these results.

2 Results

For $N \in \mathbb{N}$, denoting $\beta := 2\gamma/\sigma^2$, we consider the Gibbs measure with Hamiltonian

$$H_N(x, y) = \beta \left( \frac{|y|^2}{2} + U_N(x) \right), \quad \text{where} \quad U_N(x) = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} U(x_i, x_j),$$

namely the measure on $\mathbb{R}^{2dN}$ with Lebesgue density

$$m^{(N)}_\infty(x, y) = Z_N^{-1} \exp \left( -H_N(x, y) \right), \quad Z_N := \int_{\mathbb{R}^{2dN}} \exp \left( -H_N(x, y) \right) \, dx \, dy. \quad (3)$$

In all the paper we use the same symbol for a probability measure and for its Lebesgue density.

Assumption 1 The potential $U$ is given by $U(x, x') = V(x) + V'(x') + W(x, x')$ where $V \in C^\infty(\mathbb{R}^d)$ and $W \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with all their derivatives of order larger than 2 bounded. There exist $c_U > 0$, $c'_U, c'_W \geq 0$ and $c_W \in \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}^d$,

$$\langle \nabla V(x) - \nabla V(y) \rangle \cdot (x - y) \geq c_V |x - y|^2 - c'_V |x - y| \mathbb{1}_{||x-y|| \leq R} \quad (4)$$

$$\langle \nabla_x W(x, z) - \nabla_y W(y, z) \rangle \cdot (x - y) \geq c_W |x - y|^2 - c'_W |x - y| \mathbb{1}_{||x-y|| \leq R} \quad (5)$$

Moreover, $U$ is the sum of a strictly convex function and of a bounded function, $W$ is lower bounded, $c_V + c_W > ||\nabla_{x,x'}^2 W||_\infty$ and $\beta < \beta_0$ where

$$\beta_0 := \frac{4}{(c'_U + c'_W)R} \ln \left( \frac{c_V + c_W}{||\nabla_{x,x'}^2 W||_\infty} \right) \quad (:= +\infty \text{ if } (c'_U + c'_W)R = 0).$$

Note that Assumption 1 discards singular potentials such as considered in [3]. Indeed, we focus here on the question of having uniform estimates (in $t$ when $N \to \infty$ or in $N$ when $t \to \infty$) in non-convex cases, which is already interesting and new in cases where $U$ is smooth with bounded derivatives.

Example. For $d = 1$, consider a double well potential $V(x) = x^2/2 + e^{-x^2}$ and an harmonic attractive interaction potential $W(x, x') = \alpha(x - x')^2/2$ for some $\alpha > 0$. Then (4) holds for some $c_V, c'_V > 0$, and (5) holds with $c_W = \alpha = ||\nabla_{x,x'}^2 W||_\infty$ and $c'_W = 0$. In particular the condition $c_V + c_W > ||\nabla_{x,x'}^2 W||_\infty$ is met whatever the value of $\alpha > 0$, which is not required to be small. As a consequence, Assumption 1 holds for $\beta$ small enough, i.e. at high temperature. Besides, it is well known (see e.g. [10] and references within) that, with this choice of $V$ and $W$, the results stated below (in particular uniqueness for the equilibrium of the non-linear equation in Theorem 3) are false at low temperature (i.e. high values of $\beta$).

The effect of the amplitude of the interaction on the temperature constraint in Assumption 1...
is the following: $\beta_0$ is a decreasing function of $\alpha$ and, as $\alpha \to +\infty$ (resp. 0), $\beta_0 \to 0$ (resp. $+\infty$). For this non-convex example, all the results stated below are new.

We say that a probability measure $\mu$ satisfies a log-Sobolev inequality with constant $\eta > 0$ if
\[
\forall f > 0 \text{ s.t. } \int f \, d\mu = 1, \quad \int f \ln f \, d\mu \leq \eta \int \frac{|\nabla f|^2}{f} \, d\mu. \tag{6}
\]

**Proposition 1** Under Assumption 1, there exists $\eta > 0$ such that for all $N \in \mathbb{N}$, $Z_N < +\infty$ and $m^{(N)}_\infty$ satisfies a log-Sobolev inequality with constant $\eta$.

Note that logarithmic Sobolev inequalities have direct consequences that may be useful beyond the convergence to equilibrium we look at in this paper. We refer to [2] for a general reference. For example, it yields Gaussian concentration inequalities for the measure $m^{(N)}_\infty$, uniformly in the number of particles. Another important consequence of a logarithmic Sobolev inequality is that it implies a Talagrand inequality. It will enable us to pass from entropic convergence to equilibrium to Wasserstein convergence to equilibrium. Let us detail this.

For $\mu$ and $\nu$ two probability laws on some Polish space $E$, we write
\[
\mathcal{H}(\nu \mid \mu) = \left\{ \int_E \ln \left( \frac{d\nu}{d\mu} \right) \, d\nu \right\}
\]
the relative entropy of $\nu$ with respect to $\mu$ and
\[
\mathcal{W}_2(\mu, \nu) = \inf_{\pi \in \Gamma(\mu, \nu)} \left\{ \sqrt{\mathbb{E}(|A_1 - A_2|^2)}, \, \text{Law}(A_1, A_2) = \pi \right\}
\]
their $\mathcal{W}_2$-Wasserstein distance, where the infimum is taken over the set $\Gamma(\mu, \nu)$ of transference plan between $\mu$ and $\nu$, namely the set of probability laws on $E \times E$ with marginals $\mu$ and $\nu$. Recall that the set $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ that have a finite second moment, endowed with the distance $\mathcal{W}_2$, is complete (see [27, Chap. 7], to which we also refer for a general introduction to Wasserstein distances). Similarly, denote by
\[
\| \mu - \nu \|_{TV} = \inf_{\pi \in \Gamma(\mu, \nu)} \{ \mathbb{P}(A_1 \neq A_2), \, \text{Law}(A_1, A_2) = \pi \}
\]
the total variation norm of $\mu - \nu$. Recall Pinsker’s Inequality (e.g. [27, Remark 22.12])
\[
\| \mu - \nu \|_{TV}^2 \leq 2\mathcal{H}(\mu \mid \nu)
\]
for all $\mu, \nu \in \mathcal{P}(E)$, and Talagrand’s $T_2$ Inequality
\[
\mathcal{W}_2^2(\mu, \nu) \leq \eta \mathcal{H}(\mu \mid \nu)
\]
that holds for all $\mu \in \mathcal{P}(E)$ if $\nu$ satisfies a log-Sobolev inequality with constant $\eta$, see [23].

Under Assumption 1, (2) admits a strong solution $Z_N = ((X_i, Y_i))_{i \in [1,N]}$ for any initial condition (see [20]). Denote by $m^{(N)}_t$ the law of $Z_N(t)$.

**Theorem 2** Under Assumption 1, there exist $C > 1$, $\chi > 0$ which depend only on $U, \gamma, \sigma$ such that for all $N \in \mathbb{N}$, $t \geq 0$ and all initial condition $m^{(N)}_0 \in \mathcal{P}(\mathbb{R}^{2dN})$
\[
\mathcal{H}\left( m^{(N)}_t \biggm| m^{(N)}_\infty \right) \leq C e^{-\chi t} \mathcal{H}\left( m^{(N)}_0 \biggm| m^{(N)}_\infty \right) \tag{7}
\]
\[
\mathcal{H}\left( m^{(N)}_t \biggm| m^{(N)}_\infty \right) \leq \frac{C}{(1 \wedge t)^3} \mathcal{W}_2^2\left( m^{(N)}_0, m^{(N)}_\infty \right) \tag{8}
\]
\[ \mathcal{W}_2 \left( m_t^{(N)}, m_\infty^{(N)} \right) \leq C e^{-\chi t} \mathcal{W}_2 \left( m_0^{(N)}, m_\infty^{(N)} \right). \] (9)

Notice that the fact \( C > 1 \) is of course necessary here. If not then (7) would imply back a logarithmic Sobolev inequality for \( m_\infty^{(N)} \) with the Dirichlet form given by the dynamic (2) (see the equivalence in [2, Theorem 5.2.1]), which is false since this Dirichlet form is degenerate. Thus, this result states a so-called hypocoercive convergence, by contrast with the coercive case for which \( C = 1 \). Note that, under weaker conditions, a similar result was given in \( L^2 \) and \( H^1 \) norms in [14], with a rate also independent of the number of particles. However, as mentioned in the introduction, the \( L^2 \) and \( H^1 \) norms are not suitable to obtain a result on the non-linear system (such as the next theorem).

To study the mean-field equation (1), following the notations of [15], we consider \( \alpha \) the probability measure with Lebesgue density proportional to \( \exp \left( -V(x) - \frac{|y|^2}{2} \right) \) and denote by

\[ E_f(\nu) = \mathcal{H}(\nu | \alpha) + \frac{1}{2} \int_{\mathbb{R}^{4d}} W(x, x') \nu(dx, dy) \nu(dx', dy') \]

the so-called free energy of any \( \nu \in \mathcal{P}(\mathbb{R}^{2d}) \) and

\[ \mathcal{H}_W(\nu) = E_f(\nu) - \min_{\mu \in \mathcal{P}(\mathbb{R}^{2d})} E_f(\mu) \]

the corresponding mean-field entropy.

**Theorem 3** Under Assumption 1, \( E_f \) admits a unique minimizer \( m_\infty \in \mathcal{P}(\mathbb{R}^{2d}) \). Moreover, there exist \( C, \chi > 0 \) that depend only on \( U, \gamma, \sigma \) such that for all \( t \geq 0 \) and all initial condition \( m_0 \in \mathcal{P}(\mathbb{R}^{2d}) \),

\[ \mathcal{H}_W(m_t) \leq C e^{-\chi t} \mathcal{H}_W(m_0) \] (10)

\[ \mathcal{H}_W(m_t) \leq \frac{C}{(1 \wedge t)^3} \mathcal{W}_2^2 \left( m_t, m_\infty \right) \] (11)

\[ \mathcal{W}_2 \left( m_t, m_\infty \right) \leq C e^{-\chi t} \mathcal{W}_2 \left( m_0, m_\infty \right). \] (12)

Note that \( m_\infty \) is necessarily an equilibrium of (1), and thus it solves

\[ m_\infty(x, y) \propto \exp \left( -\beta \left( V(x) + \frac{1}{2} |y|^2 + \int_{\mathbb{R}^d} W(x, x') m_\infty(x', y') \right) \right). \]

In particular, the mean-field entropy \( \mathcal{H}_W(\nu) \) differs from \( \mathcal{H}(\nu | m_\infty) \) since, up to an additive constant, writing \( H(x, y) = U(x) + |y|^2/2 \), the first one is

\[ \int_{\mathbb{R}^{2d}} v \ln v + \int_{\mathbb{R}^{2d}} H v + \frac{1}{2} \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} W v \otimes v \]

while, up to an additive constant, the second one is

\[ \int_{\mathbb{R}^{2d}} v \ln v + \int_{\mathbb{R}^{2d}} H v + \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} W v \otimes m_\infty, \]

i.e. is the linearization of the first one at \( v = m_\infty \).

What is available in practice is the empirical distribution \( M_t^N \) for finite \( t \geq 0 \) and \( N \in \mathbb{N}_* \), which motivates the next statement.
Corollary 4 Under Assumption 1, there exist $\chi > 0$ that depends only on $U$, $\gamma$, $\sigma$ such that for all initial condition $m_0^{(N)} = m_0^{\otimes N}$ with $m_0 \in \mathcal{P}_2(\mathbb{R}^{2d})$, there exists $K > 0$ such that for all $N \in \mathbb{N}_*$ and $t \geq 0$,\[
abla \mathbb{E} \left( \mathcal{W}_2^2 \left( M^N_t, m_\infty \right) \right) \leq K \left( e^{-\chi t} + a(N) \right)\]where\[a(N) = \begin{cases} N^{-1/2} & \text{if } d = 1 \\
 \ln(1 + N) N^{-1/2} & \text{if } d = 2 \\
 N^{-2/d} & \text{if } d \geq 3.\end{cases}\]

As shown in [5, Proposition 2.1], such a result yields confidence intervals with respect to the uniform metric for a numerical approximation of $m_\infty$ by $M^N_t \neq \xi$ where $\xi$ is a smooth kernel. It would also be interesting in order to get concentration inequalities independent of the number of particles for additive functionals of the trajectories of the particles.

Finally, we consider the limit $N \to +\infty$. For $n \in \llbracket 1, N \rrbracket$, let $m_t^{(n,N)}$ be the law of $((X_1, Y_1), \ldots, (X_n, Y_n))$.

Corollary 5 Under Assumption 1, there exists $\kappa > 0$ that depend only on $U$, $\gamma$, $\sigma$ such that for all initial condition $m_0^{(N)} = m_0^{\otimes N}$ with $m_0 \in \mathcal{P}_2(\mathbb{R}^{2d})$, there exists $K > 0$ such that for all $N \in \mathbb{N}_*$, $n \in \llbracket 1, N \rrbracket$ and $t \geq 0$,\[
\mathcal{W}_2 \left( m_t^{(n,N)}, m_t^{\otimes n} \right) \leq \frac{K \sqrt{n}}{N^{3\kappa}},
\|
m_t^{(n,N)} - m_t^{\otimes n} \|_{TV} \leq \frac{K \sqrt{n}}{N^{\kappa}}.\]

Note that our approach for proving such uniform (in time) propagation of chaos estimates does not lead to an optimal exponent $\kappa$ for the Wasserstein distance, which can be seen to be $1/2$ in some more constrained example, as in [4]. It would be interesting to consider a direct coupling approach to prove this result with sharp speed, as in [11].

3 Proofs

3.1 Uniform Log-Sobolev Inequalities

Lemma 6 Under Assumption 1, there exist $\alpha_1, \alpha_2, \alpha_3 > 0$ such that for all $x, x' \in \mathbb{R}^d$,\[
\alpha_1 \left( |x|^2 + |x'|^2 \right) - \alpha_3 \leq U(x, x') \leq \alpha_2 \left( |x|^2 + |x'|^2 \right) + \alpha_3.
\]

$|\nabla x U(x, x')| \leq \alpha_2 \left( |x| + |x'| \right) + \alpha_3$.

Proof This is a straightforward consequence of the uniform bound on $\nabla^2 U$ and on the fact $U$ is the sum of a strictly convex and of a bounded function. \hfill $\square$

From Lemma 6, we get that $Z_N < +\infty$ for all $N \in \mathbb{N}_*$, which is the first claim of Proposition 1. As we now explain, the second claim, i.e. the uniform log-Sobolev inequalities for $m_\infty^{(N)}$, $N \in \mathbb{N}_*$, follows from [15, Theorem 8] (itself based on [28, Theorem 0.1]). Denote by\[
\pi_\infty^{(N)}(x) = \int_{\mathbb{R}^{dN}} m_\infty(x, y) dy = \tilde{Z}_N^{-1} e^{-\beta U_N(x)} , \quad \tilde{Z}_N = \int_{\mathbb{R}^{dN}} e^{-\beta U_N(x)} dx\]
the $x$-marginal of $m_{\infty}^{(N)}$. A straightforward consequence of the log-Sobolev inequality for the Gaussian law and of the tensorization property of the log-Sobolev inequalities (see for example [2]) is the following:

**Lemma 7** Suppose that $\pi_{\infty}^{(N)}$ satisfies a log-Sobolev inequality for some constant $\eta_N$. Then $m_{\infty}^{(N)}$ satisfies a log-Sobolev inequality with constant $\max(\eta_N, \beta)$.

The study is thus reduced to $\pi_{\infty}^{(N)}$, which is precisely the topic of [15]. We now introduced the framework of the latter. As a first step, without loss of generality we suppose that $\beta = 1$.

**Assumption 2** The potential $U$ is given by $U(x, y) = V(x) + V(y) + W(x, y)$ where

1. The confinement potential $V \in C^2(\mathbb{R}^d)$, its Hessian matrix is bounded from below and there are two positive constants $c_1, c_2$ such that $x \cdot \nabla V(x) \geq c_1|x|^2 - c_2$ for all $x \in \mathbb{R}^d$.
2. The interaction potential $W \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$, its Hessian matrix is bounded and

$$\int_{\mathbb{R}^{2d}} e^{-[V(x)+V(y)+\lambda W(x,y)]} \, dx \, dy \ < \ +\infty , \quad \forall \lambda > 0 .$$

**Assumption 3** (Zegarlinski’s condition) Denoting

$$b_0(r) = \sup_{x, y, z \in \mathbb{R}^d; |x - y| = r} \left( -\frac{x - y}{|x - y|} \cdot \left( \nabla_x U(x, z) - \nabla_y U(y, z) \right) \right)$$

for $r > 0$, then

$$c_L := \frac{1}{4} \int_0^\infty \exp \left( \frac{1}{4} \int_0^s b(r) \, dr \right) \, ds < +\infty .$$

Moreover,

$$\gamma_0 := c_L \sup_{x, y \in \mathbb{R}^d, |z| = 1} |\nabla^2_{x, y} U(x, y)z| \ < \ 1 . \quad (13)$$

**Assumption 4** (Uniform conditional log-Sobolev inequality) There exists $\rho > 0$ such that for all $N \in \mathbb{N}_+$ and all $x_{\neq 1} = (x_2, \ldots, x_N) \in \mathbb{R}^{d(N-1)}$, the conditional law $\pi_{\infty}^{(N)}$ on $\mathbb{R}^d$ with density proportional to $x_1 \mapsto \pi_{\infty}^{(N)}(x)$ satisfies a log-Sobolev inequality with constant $\rho$.

Note that the convention on what is called the constant of the log-Sobolev inequality is different in [15] and in the present paper, so that $\rho$ here corresponds to $1/(2\rho_{LS, m})$ in [15]. This has no impact on the result, in both cases the one-particle conditional law is required to satisfy a log-Sobolev inequality with a constant (in either sense) uniform in $N$ and in $x_{\neq 1}$. The same remark applies for the next result.

**Theorem 8** (Theorem 8 of [15]) Under Assumptions 2, 3 and 4, there exists $\eta > 0$ such that $\pi_{\infty}^{(N)}$ (with $\beta = 1$) satisfies a log-Sobolev with constant $\eta$ for all $N \in \mathbb{N}$.

In [15], the one-particle conditional log-Sobolev inequality (i.e. Assumption 4) is proven under the assumption that the confining potential is superconvex, meaning that $\nabla^2 V \rightarrow +\infty$ at infinity. This is not compatible with the boundedness condition in Assumption 1 but it is far from necessary.

In view of Lemma 7 and Theorem 8, Proposition 1 thus follows from the following result:
Lemma 9 Assumption 1 implies that Assumptions 2, 3 and 4 are satisfied by the potential \( U_\beta = \beta U \) on \( \mathbb{R}^{2d} \) and for the potential \( H_\beta = \beta H \) on \( \mathbb{R}^{4d} \) where \( H \) is given by 
\[
H(x, y, x', y') = U(x, x') + (|y|^2 + |y'|^2)/2.
\]

Proof We only detail the case of Shigekawa result [1] the assumption that
\[
Q_x \leq \beta(c_V + c_W)
\]
and thus
\[
\gamma_0 \leq \frac{1}{(c_V + c_W)} \frac{\beta(c_V' + c_W') R}{4} \frac{\|
abla_{x,y}^2 U \|_\infty}{\|
abla_{x,y} U \|_\infty} < 1
\]

where we used that \( \beta < \beta_0 \).

Finally, Assumption 1 implies that \( U = U_1 + U_2 \) where \( U_1 \) is \( \rho \)-convex for some \( \rho > 0 \) and \( \|U_2\|_\infty < \infty \). Fix any \( N \in \mathbb{N}_* \) and \( x \neq x_0 \in \mathbb{R}^{d(N-1)} \). Then \( x_0 \mapsto U_N(x) \) is the sum of a \( \rho \)-convex function and of a function bounded by \( \|U_2\|_\infty < \infty \), so that the probability law with density proportional to \( x_0 \mapsto \pi^{(N)}(x) \) satisfies a log-Sobolev with constant \( c^2\|U_2\|_\infty / \rho \), by using Bakry-Emery’s condition and Holley-Stroock perturbation argument (see e.g. [2, Propositions 5.1.6 and 5.7.1]).

Notice that we may also consider other perturbation arguments, for instance with the Aida-Shigekawa result [1] the assumption that \( U_2 \) is bounded could be replaced by the assumption that \( U_2 \) is Lipschitz continuous with Lipschitz constant less than \( \rho/2 \).

3.2 First Propagation of Chaos Estimates

First, we establish uniform in time moment estimates, both for the particles and non linear systems, using standard Lyapunov arguments.

Lemma 10 Under Assumption 1, for all initial conditions \( m_0 \in \mathcal{P}_2(\mathbb{R}^{2d}) \), there exists \( K > 0 \) depending only on \( U, \gamma, \sigma, m_0 \) such that, if \( m_0^{(N)} = m_0^\otimes N \), then for all \( N \in \mathbb{N}_* \) and \( t \geq 0 \),
\[
\mathbb{E} \left( |X_1(t)|^2 + |Y_1(t)|^2 \right) + \int_{\mathbb{R}^{2d}} (|x|^2 + |y|^2) m_t(x, y) \, dx \, dy \leq K.
\]

Proof Under Assumption 1, for \( N \in \mathbb{N}_* \), \( U_N \) satisfies, for all \( x \in \mathbb{R}^{dN} \),
\[
x \cdot \nabla U_N(x) = \sum_{i=1}^N x_i \cdot \nabla x U_N(x_i, x_j) \geq \sum_{i=1}^N \left( (c_V + c_W) |x_i|^2 - (c_V' + c_W') |x_i| \right)
\geq \frac{(c_V + c_W)}{2} |x|^2 - N \frac{(c_V' + c_W')^2}{2(c_V + c_W)}.
\]
Moreover, from Lemma 6, for all $x \in \mathbb{R}^{dN}$,
\begin{equation}
\alpha_1 |x|^2 - \alpha_3 N \leq U_N(x) \leq \alpha_2 |x|^2 + \alpha_3 N.
\end{equation}

From these estimates, the proof is then similar to the one of [21, Lemma 11]. In the remaining of the proof, $c_i$ for $i \in \mathbb{N}$ stand for various positive constants that are independent from $N$ and $t$. The infinitesimal generator of $Z_N$ is
\begin{equation}
\mathcal{L}_N = y \cdot \nabla_x - (\nabla U_N(x) + y) \cdot \nabla_y + \frac{\sigma^2}{2} \Delta_y.
\end{equation}

For some $\varepsilon > 0$, let
\[ \tilde{H}(x, y) = U_N(x) + \frac{1}{2} |y|^2 + \varepsilon x \cdot y. \]

Then, using (14) and (15), for $\varepsilon$ small enough (and independent from $t$ and $N$),
\[ \tilde{H}(x, y) \geq \frac{\alpha_1}{2} |x|^2 + \frac{1}{4} |y|^2 - \alpha_3 N \]
and
\[ \mathcal{L}_N \tilde{H}(x, y) = -(\gamma - \varepsilon)|y|^2 + \frac{\sigma^2}{2} dN - \varepsilon \gamma x \cdot y - \varepsilon x \cdot \nabla U_N(x) \leq -c_1 H + c_2 N \]
for some $c_1, c_2 > 0$. The Grönwall Lemma yields
\[ \mathbb{E}(\tilde{H}(Z_N(t))) \leq \mathbb{E}(\tilde{H}(Z_N(0))) + \frac{c_2 N}{c_1}. \]

Using the interchangeability of particles,
\[ \mathbb{E}(|X_1(t)|^2 + |Y_1(t)|^2) \leq \frac{c_3}{N} \mathbb{E}(\tilde{H}(Z_N(t))) + c_3 \leq c_4 \mathbb{E}(|X_1(0)|^2 + |Y_1(0)|^2) + c_4 \]
for some $c_3, c_4 > 0$.

Similarly, for $\varepsilon > 0$, let
\[ R_t = \int_{\mathbb{R}^{dN}} (U(x, x') + |y|^2 + \varepsilon x \cdot y) m_t(x, y)m_t(x', y')dxdydx'dy'. \]

From Assumption 1 and Lemma 6, for $\varepsilon$ small enough,
\[ R_t \geq c_5 \int_{\mathbb{R}^{2d}} (|x|^2 + |y|^2) m_t(x, y)dxdy - c_6 \]
for some $c_5, c_6 > 0$ and
\[ \partial_t R_t = \int_{\mathbb{R}^{dN}} (-2\gamma - \varepsilon)|y|^2 + 2d - \varepsilon \gamma x \cdot y - \varepsilon x \cdot \nabla U(x, x') m_t(x, y)m_t(x', y')dxdydx'dy' \leq -c_7 R_t + c_8 \]
for some $c_7, c_8 > 0$. As a consequence,
\[ \int_{\mathbb{R}^{2d}} (|x|^2 + |y|^2) m_t(x, y)dxdy \leq \frac{1}{c_5} \left( R_0 + \frac{c_8}{c_7} \right) + \frac{c_6}{c_5} \]
\[ \leq c_9 \int_{\mathbb{R}^{2d}} (|x|^2 + |y|^2) m_0(x, y)dxdy + c_9 \]
for some $c_9 > 0$, which concludes.
The next proposition provides a first crude time dependent propagation of chaos estimate.

**Proposition 11** Under Assumption 1, there exists \( b \) (depending only on \( U, \gamma, \sigma \)) such that for all initial condition \( m_0 \in \mathcal{P}_2(\mathbb{R}^{2d}) \) (and \( m_0^{(N)} = m_0^{\otimes N} \)), there exist \( K > 0 \) (depending only on \( U, \gamma, \sigma \) and \( m_0 \)) such that for all \( N \in \mathbb{N}^* \) and \( t \geq 0 \),

\[
\mathcal{W}_2^2 \left( m_t^{\otimes N} , m_t^{(N)} \right) \leq K \left( e^{bt} - 1 \right).
\]

**Proof** This is a classical result, obtained with a parallel coupling of the system (2) with a system of independent non-linear particles. More precisely, consider a system \( \bar{Z}_N = (\bar{X}_i, \bar{Y}_i)_{i \in [1,N]} \) with \( \bar{Z}_N(0) = Z_N(0) \) and

\[
\forall i \in [1,N] \left\{ \begin{array}{l}
d\bar{X}_i = \bar{Y}_i dt \\
d\bar{Y}_i = -\gamma \bar{Y}_i dt - \int_{\mathbb{R}^{2d}} \nabla_x U(\bar{X}_i,x) m_t(x,y) dx dy dt + \sigma dB_i
\end{array} \right.
\]

driven by the same Brownian motion as (2). The forces \( \nabla_x U \) being Lipschitz, it is clear that there exist \( b' > 0 \) such that

\[
|Z_N - \bar{Z}_N|^2 \leq b' |Z_N - \bar{Z}_N|^2 dt
\]

\[
- \frac{2}{N} \sum_{i=1}^{N} (Y_i - \bar{Y}_i) \sum_{j=1}^{N} \left( \nabla_x W(X_i, X_j) - \int \nabla_x W(X_1, u) m_t(u,v) dt \right)
\]

Decomposing the last term as

\[
(\nabla_x W(X_i, X_j) - \nabla W(X_i, \bar{X}_j)) + \left( \nabla W(X_1, \bar{X}_j) - \int \nabla W(X_1, u) m_t(u,v) \right),
\]

using that \( \nabla_x W \) is Lipschitz, taking the expectation and using that particles are interchangeable, we obtain

\[
\partial_t \mathbb{E} \left( |Z_N - \bar{Z}_N|^2 \right) \
\leq b \mathbb{E} \left( |Z_N - \bar{Z}_N|^2 \right)
\]

\[
+ \frac{1}{N} \mathbb{E} \left( \left| \sum_{j=1}^{N} \left( \nabla_x W(X_1, X_j) - \int \nabla_x W(X_1, u) m_t(u,v) \right) \right|^2 \right)
\]

for some \( b > 0 \). Finally, using that the \( (\bar{X}_i, \bar{Y}_i)_{i \in [1,N]} \) are independent and distributed according to \( m_t \),

\[
\mathbb{E} \left( \left| \sum_{j=1}^{N} \left( \nabla_x W(X_1, X_j) - \int \nabla_x W(X_1, u) m_t(u,v) \right) \right|^2 \right)
\]

\[
= \mathbb{E} \left( \sum_{j=1}^{N} \left| \nabla_x W(X_1, X_j) - \int \nabla_x W(X_1, u) m_t(u,v) \right|^2 \right)
\]

\[
\leq N \| \nabla^2 W \|_\infty^2 \int_{\mathbb{R}^{2d}} |x|^2 m_t(x,y) dx dy.
\]

The moment estimates of Lemma 10 and Grönwall’s Lemma conclude. \( \square \)
We will also need some first (time dependent) propagation of chaos estimates in entropy, which are inherited here from the previous estimates for the $W_2$-Wasserstein distance.

**Proposition 12** Under Assumption 1, there exists $K$ (depending only on $U, \gamma, \sigma$ and $m_0$) such that for all $t \geq 0$ and $N \in \mathbb{N}$,

$$\mathcal{H}\left( m_i^{(N)} \mid m_i^{\otimes N} \right) \leq K \left( t + \sqrt{N} \int_0^t \mathcal{W}_2\left( m_s^{\otimes N}, m_s^{(N)} \right) \, ds \right).$$

**Proof** We follow the idea of [19, Lemma 3.15] (see also [21, Lemma 14]), namely we compute the derivative of

$$F(t) = \mathcal{H}\left( m_i^{(N)} \mid m_i^{\otimes N} \right).$$

To do so, let $u_1 = m_i^{(N)}$, $u_2 = m_i^{\otimes N}$,

$$b_1(x, y) = \left( -\gamma y - \nabla_x U_N(x) \right), \quad b_2(x, y) = \left( -\gamma y - \nabla_x U_N(x) \right)$$

with

$$\overline{U}_N(x) = \sum_{i=1}^N \int U(x_i, v)m_i(v, w) \, dv \, dw,$$

and $L_i f = -\nabla \cdot (b_i f) + \frac{\sigma^2}{2} \Delta_y f$ for $i = 1, 2$. With these notations, $\partial_t (u_i) = L_i u_i$, and the dual in the Lebesgue sense of $L_i$ is $L'_i = b_i \cdot \nabla + \frac{\sigma^2}{2} \Delta_y$. From the conservation of the mass of $u_1$, we get

$$0 = \partial_t \left( \int \frac{u_1}{u_2} u_2 \right) = \int \left( L_1 u_1 - \frac{u_1}{u_2} L_2 u_2 + L'_2 \left( \frac{u_1}{u_2} \right) u_2 \right).$$

Since $L'_1$ is a diffusion operator with carré du chap operator $\Gamma f = \frac{\sigma^2}{2} |\nabla_y f|^2$ (see [2, p. 20 & 42] for the definitions),

$$u_1 L'_1 \ln \left( \frac{u_1}{u_2} \right) = u_1 \frac{L'_1 \left( \frac{u_1}{u_2} \right)}{\frac{u_1}{u_2}} - u_1 \Gamma \left( \frac{u_1}{u_2} \right)^2 = u_2 L'_1 \left( \frac{u_1}{u_2} \right) - u_1 \Gamma \left( \ln \frac{u_1}{u_2} \right).$$

Using both these relations,

$$\partial_t \left( \int \ln \left( \frac{u_1}{u_2} \right) u_1 \right) = \int \left( \frac{L_1 u_1}{u_1} - \frac{L_2 u_2}{u_2} + L'_1 \ln \left( \frac{u_1}{u_2} \right) \right) u_1$$

$$= \int -\Gamma \left( \ln \frac{u_1}{u_2} \right) u_1 + u_2 L'_1 \left( \frac{u_1}{u_2} \right) - u_2 L'_2 \left( \frac{u_1}{u_2} \right)$$

$$= \int -\Gamma \left( \ln \frac{u_1}{u_2} \right) u_1 + (b_1 - b_2) \cdot \nabla \ln \left( \frac{u_1}{u_2} \right) u_1.$$

Applying Young’s Inequality, we get

$$F'(t) \leq \frac{1}{2\sigma^2} \int \left| \nabla U_N(x) - \nabla \overline{U}_N(x) \right|^2 m_i^{(N)}$$
by interchangeability. Developing the square of the sum, the $N$ diagonal terms are bounded by

$$\frac{1}{N^2} \left\| \nabla^2 W \right\|^2 \left( \mathbb{E} (|X_j|^2) + \int |v|^2 m_t(v,w) \right) \leq \frac{K}{N^2}$$

for some $K > 0$, where we used Lemma 10. For the extra-diagonal terms, we consider an optimal coupling $(\bar{Z}_N, Z_N)$ of $m_t^N$ and $m_t^{(N)}$ in the sense that

$$\mathbb{E} \left( (\bar{Z}_N(t) - Z_N(t))^2 \right) = \mathcal{W}_2^2 \left( m_t^N, m_t^{(N)} \right)$$

(see [27, Corollary 5.22] for the existence of such an optimal coupling) and write, for $j \neq k$,

$$\left( \nabla W(X_1, X_j) - \int \nabla W(X_1, v) m_t \right) \left( \nabla W(X_1, X_k) - \int \nabla W(X_1, v) m_t \right)$$

$$= (\nabla W(X_1, X_j) - \nabla W(X_1, X_j)) \left( \nabla W(X_1, X_k) - \int \nabla W(X_1, v) m_t \right)$$

$$+ (\nabla W(X_1, X_j) - \int \nabla W(X_1, v) m_t) (\nabla W(X_1, X_k) - \nabla W(X_1, X_k))$$

$$+ (\nabla W(X_1, X_j) - \int \nabla W(X_1, v) m_t) (\nabla W(X_1, X_k) - \int \nabla W(X_1, v) m_t) .$$

The $X_i$’s being independent with law the first marginal of $m_t$, the expectation of the third term vanishes, while the expectations of the two other terms is bounded by the Cauchy-Schwarz inequality and interchangeability by

$$\left\| \nabla^2 W \right\|^2 \sqrt{\mathbb{E} (|X_1 - \bar{X}_1|^2) (\mathbb{E} (|X_1|^2) + \mathbb{E} (|\bar{X}_1|^2))} \leq \frac{K}{\sqrt{N}} \mathcal{W}_2 \left( m_t^N, m_t^{(N)} \right)$$

for some $K > 0$, where we used again interchangeability and Lemma 10 for the second inequality. As a conclusion, we have obtained that for all $t \geq 0$

$$F'(t) \leq K + K \sqrt{N} \mathcal{W}_2 \left( m_t^N, m_t^{(N)} \right)$$

for some $K > 0$ independent from $t$ and $N$, and the claim follows from the fact $F(0) = 0$. □

**Lemma 13** For all $t \geq 0$, $N \in \mathbb{N}_0$ and $n \in [1, N]$,

$$\mathcal{W}_2^2 \left( m_t^N, m_t^{(n,N)} \right) \leq \frac{n}{N} \mathcal{W}_2^2 \left( m_t^N, m_t^{(N)} \right)$$

$$\mathcal{H} \left( m_t^{(n,N)} \mid m_t^N \right) \leq \frac{1}{[N/n]} \mathcal{H} \left( m_t^{(N)} \mid m_t^N \right) .$$

**Proof** Let $Z_N = ((X_1, Y_1), \ldots, (X_N, Y_N))$ and $\bar{Z}_N = ((\bar{X}_1, \bar{Y}_1), \ldots, (\bar{X}_N, \bar{Y}_N))$ be a $\mathcal{W}_2$-optimal coupling of $m_t^N$ and $m_t^{\otimes N}$, i.e. be such that $Z_N \sim m_t^{(N)}$, $\bar{Z}_N \sim m_t^{\otimes N}$ and

$$\mathcal{W}_2^2 \left( m_t^{(N)}, m_t^{\otimes N} \right) = \mathbb{E} (|Z_N - \bar{Z}_N|^2)$$

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(again, see [27, Corollary 5.22]). Then \(((X_1, Y_1), \ldots, (X_n, Y_n))\) and \((\overline{X}_1, \overline{Y}_1), \ldots, (\overline{X}_n, \overline{Y}_n))\) form a coupling of \(m_t^{(n,N)}\) and \(m_t^{\otimes N}\) and, by exchangeability,

\[
\mathcal{W}_2^2 \left( m_t^{(n,N)}, m_t^{\otimes n} \right) \leq \sum_{i=1}^n \mathbb{E} \left( |X_i - \overline{X}_i|^2 + |Y_i - \overline{Y}_i|^2 \right)
\]

\[
= \frac{n}{N} \mathcal{W}_2^2 \left( m_t^{(N)}, m_t^{\otimes N} \right).
\]

The second claim follows from the Csiszár’s inequality which, for \(n = 1\), is [8, Inequality (2.10)]. Let us establish it for any \(n \in \llbracket 1, N \rrbracket\). Set \(k = \lfloor N/n \rfloor\) and \(s = N - kn\). Then

\[
\int_{\mathbb{R}^{2dN}} m_t^{(N)} \ln \left( \frac{m_t^{(N)}}{m_t^{\otimes N}} \right) \, dz
\]

\[
= \int_{\mathbb{R}^{2dN}} m_t^{(N)} \ln \left( \frac{m_t^{(N)}}{m_t^{(n,N)} \otimes m_t^{\otimes s}} \right) \, dz
\]

\[
+ \int_{\mathbb{R}^{2dN}} m_t^{(N)} \ln \left( \frac{m_t^{(n,N)} \otimes k \otimes m_t^{\otimes s}}{m_t^{\otimes N}} \right) \, dz
\]

\[
\geq k \int_{\mathbb{R}^{2dN}} m_t^{(n,N)} \ln \left( \frac{m_t^{(n,N)}}{m_t^{\otimes n}} \right) \, dz
\]

where we used that the first term is positive (as a relative entropy) and the interchangeability of \(m_t^{(N)}\).

\[\square\]

### 3.3 Long-Time Convergence

The proof of Theorem 2 is based on the following quantitative results of hypocoercivity for diffusion processes.

**Theorem 14** (from Theorem 10 of [22]) Consider a diffusion generator \(L\) on Hörmander form

\[L = B_0 + \sum_{i=1}^d B_i^2\]

where the \(B_j\)’s are derivation operators. Suppose there exist \(N_c \in \mathbb{N}\) and \(\lambda, \Lambda, m, \rho, K > 0\) such that for \(i \in \llbracket 0, N_c + 1 \rrbracket\) there exist smooth derivation operators \(C_i\) and \(R_i\) and a scalar field \(Z_i\) satisfying:

1. \(C_{N_c+1} = 0\), and \([B_0, C_i] = Z_{i+1} C_{i+1} + R_{i+1}\) for all \(i \in \llbracket 0, N_c \rrbracket\), where \([A, B] = AB - BA\) stands for the Poisson bracket of two operators,
2. \([B_j, C_i] = 0\) for all \(i \in \llbracket 0, N_c \rrbracket\), \(j \in \llbracket 1, d \rrbracket\),
3. \(\lambda \leq Z_i \leq \Lambda\) for all \(i \in \llbracket 0, N_c \rrbracket\),
4. \(|C_0 f| < m \sum_{j \geq 1} |B_j f|^2\) and \(|R_i f|^2 < m \sum_{j < i} |C_j f|^2\) for all \(i \in \llbracket 0, N_c + 1 \rrbracket\) and smooth Lipschitz \(f\).
5. \(\sum_{i \geq 0} |C_i f|^2 \geq \rho |\nabla f|^2\).
Suppose moreover that there exists a probability measure $\mu$ which is invariant for $e^{tL}$ and satisfies a log-Sobolev inequality with constant $\eta$.

Then for all $t > 0$ and for all $f > 0$ with $\int f \, d\mu = 1$,

$$\int \left(e^{tL} f \right) \ln \left(e^{tL} f \right) \, d\mu \leq e^{-\eta \kappa t} (1 - e^{-t})^{2N_c} \int f \ln f \, d\mu$$

with

$$\kappa = \frac{\rho}{\eta} \left( \frac{100}{\lambda} \left( N_c^2 + \frac{\Lambda^2}{\lambda} + m \right) \right)^{-20N_c^2}.$$  

**Proof (Proof of Theorem 2)** From Theorem 14, the uniform log-Sobolev inequality given by Proposition 1 and the bound on $\|\nabla^2 UN\|_\infty$ that is uniform in $N$, the proof of (7) is similar to the proof of [21, Theorem 1]. The generator (16) is on Hörmander form

$$B_0 + \sum_{i=1}^N \sum_{j=1}^d B_{i,j}$$

with, writing $y_i = \left( y_i^{(1)}, \ldots, y_i^{(d)} \right) \in \mathbb{R}^d$,

$$B_0 = -y \cdot \nabla_x + (\nabla U_N(x) - \gamma y) \cdot \nabla_y$$

$$B_{i,j} = \frac{\sigma}{\sqrt{2}} \partial_{y_i^{(j)}}.$$

Since

$$[B_0, \nabla_y] = [L_N, \nabla_y] = \nabla_x + \gamma \nabla_y, \quad [B_0, \nabla_x] = [L_N, \nabla_x] = -\nabla^2 U_N \nabla_y,$$

Theorem 14 applies with

$$C_0 = \nabla_y, \quad C_1 = \nabla_x, \quad R_1 = \gamma \nabla_y, \quad R_2 = -\nabla^2 U_N \nabla_y,$$

$$Z_1 = Z_2 = N_c = \lambda = \Lambda = \rho = 1, \quad m = \frac{2}{\sigma^2} + \gamma^2 + (\|\nabla^2 V\|_\infty + 2\|\nabla^2 W\|_\infty)^2$$

and $\eta$ given by Proposition 1. This gives (7).

The second part of Theorem 2, namely (8), follows from [16] whose results which are useful in our work are gathered in the following proposition.

**Proposition 15** (From Corollary 4.7 in [16]) Consider the stochastic differential equation on $\mathbb{R}^m \times \mathbb{R}^d$

$$dX_t = AY_t \, dt, \quad dY_t = dB_t + Z_t(X_t, Y_t) \, dt$$

with initial conditions $(X_0, Y_0) = (x, y)$, and associated semigroup $P_t$. Assume that

$$|\nabla^x Z(x, y)| \leq K_1, \quad |\nabla^y Z(x, y)| \leq K_2, \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^d.$$  

Suppose also that $P_t$ has an invariant probability measure $\mu$ and let $P_t^*$ be the adjoint of $P_t$ in $L^2(\mu)$, then for all $t > 0$ and any function $f \geq 0$ with $\mu(f) = 1$,

$$\mu(P_t^* f \log P_t^* f) \leq \frac{C}{(1 + t)^2} W_2^2(f \mu, \mu)$$

where $C$ only depends on $K_1$ and $K_2$.

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It is a regularization result, namely a control in small time of the entropy along the flow of the particle system by the initial Wasserstein distance. A crucial point for us is that this quantitative regularization has to be independent from the number of particles. Let us check that, indeed, the constant \( C \) obtained does not depend on \( N \). Applied to our case, the notations read \( A = I_{dN} \) and \( Z(x, y) = -\nabla U_N(x) - \gamma y \). Under Assumption 1, the Jacobian matrix of this \( Z \) is bounded uniformly in \( N \), which means that \( K_1 \) and \( K_2 \) do not depend on \( N \), and thus neither does \( C \), which concludes.

Finally, at least for \( t \geq 1 \), (9) is a straightforward consequence of the previous claims of Theorem 2 and of the Talagrand \( T_2 \) inequality implied (see [23]) by the log-Sobolev inequality given by Proposition 1. Indeed, for \( t \geq 1 \),

\[
\mathcal{W}_2^2\left( m_{t}^{(N)}, m_{\infty}^{(N)} \right) \leq \eta \mathcal{H} \left( m_{t}^{(N)} \mid m_{\infty}^{(N)} \right)
\]

\[
\leq \eta C e^{-x(1-\gamma)t} \mathcal{H} \left( m_{1}^{(N)} \mid m_{\infty}^{(N)} \right)
\]

\[
\leq \eta C^2 e^{-\gamma t} W_2^2 \left( m_{0}^{(N)}, m_{\infty}^{(N)} \right).
\]

For \( t \in [0, 1] \), we simply consider two solutions \( Z_N, \tilde{Z}_N \) of (2) driven by the same Brownian motion but with two different initial condition. More precisely, we suppose that \((Z_N(0), \tilde{Z}_N(0)) \) is an \( \mathcal{W}_2 \)-optimal coupling of \( m_{0}^{(N)} \) and \( m_{\infty}^{(N)} \), so that

\[
\mathbb{E} \left( |Z_N(0) - \tilde{Z}_N(0)|^2 \right) = \mathcal{W}_2^2 \left( m_{0}^{(N)}, m_{\infty}^{(N)} \right).
\]

Since \( \|\nabla^2 U_N\|_{\infty} \) is bounded uniformly in \( N \), we immediately get that

\[
d|Z_N(t) - \tilde{Z}_N(t)|^2 \leq b|Z_N(t) - \tilde{Z}_N(t)|^2 dt\]

for some \( b > 0 \) that does not depend on \( N \). Conclusion follows from

\[
\mathcal{W}_2^2 \left( m_{t}^{(N)}, m_{\infty}^{(N)} \right) \leq \mathbb{E} \left( |Z_N(t) - \tilde{Z}_N(t)|^2 \right) \leq e^{bt} \mathbb{E} \left( |Z_N(0) - \tilde{Z}_N(0)|^2 \right).
\]

Let us now transfer the results obtained on the particles system to the nonlinear equation.

**Proof (Proof of Theorem 3)** In this proof, we use repeatedly results from [15] but applied to the potential \( H_{\beta} \) defined in Lemma 9. It is possible to do so since, according to Lemma 9, this potential satisfies the assumptions of [15] (in particular the condition \( c_L \|\nabla^2 \phi \|_{\infty} < 1 \)).

The fact that \( E_f \) admits a unique minimizer \( m_{\infty} \) over \( \mathcal{P}(\mathbb{R}^{2d}) \) is proven in [15, Lemma 21]. Moreover, as established in the proof of [15, Theorem 10], \( \mu_{1, N} \) weakly converges to \( m_{\infty} \) and for all \( v \in \mathcal{P}_2(\mathbb{R}^{2d}) \),

\[
\mathcal{W}_2^2(v, m_{\infty}) \leq \liminf_{N \to +\infty} \frac{1}{N} \mathcal{W}_2^2 \left( v^{\otimes N}, m_{\infty}^{(N)} \right).
\]

Moreover, according to [15, Lemma 17], for all \( v \in \mathcal{P}(\mathbb{R}^{2d}) \) such that \( \mathcal{H}(v \mid \alpha) < +\infty \),

\[
\frac{1}{N} \mathcal{H} \left( v^{\otimes N} \mid m_{\infty}^{(N)} \right) \underset{N \to +\infty}{\longrightarrow} \mathcal{H}_W (v).
\]

Applied with \( v = m_{\infty} \) and combined with the Talagrand’s Inequality satisfied by \( m_{\infty}^{(N)} \),

\[
\frac{1}{N} \mathcal{W}_2^2 \left( m_{\infty}^{\otimes N}, m_{\infty}^{(N)} \right) \leq \frac{\eta}{N} \mathcal{H} \left( m_{\infty}^{\otimes N} \mid m_{\infty}^{(N)} \right) \underset{N \to +\infty}{\longrightarrow} \mathcal{H}_W (m_{\infty}) = 0.
\]
In particular, dividing
\[ \mathcal{W}_2 \left( m_0^{\otimes N}, m_\infty^{(N)} \right) \leq \sqrt{N} \mathcal{W}_2 (m_0, m_\infty) + \mathcal{W}_2 \left( m_\infty^{\otimes N}, m_\infty^{(N)} \right) \]
by \( \sqrt{N} \) and letting \( N \to +\infty \), we get that
\[ \limsup_{N \to +\infty} \frac{1}{\sqrt{N}} \mathcal{W}_2 \left( m_0^{\otimes N}, m_\infty^{(N)} \right) \leq \mathcal{W}_2 (m_0, m_\infty). \] (19)

Together with Theorem 2 and Proposition 11, for all \( t \geq 0 \),
\[ \mathcal{W}_2 (m_t, m_\infty) \leq \limsup_{N \to +\infty} \mathcal{W}_2 \left( m_t^{(N)}, m_\infty^{(N)} \right) + \mathcal{W}_2 \left( m_\infty^{(N)}, m_\infty^{(N+1)} \right) \]
\[ \leq C e^{-\chi t} \mathcal{W}_2 (m_0, m_\infty). \]

Similarly, following the proof of [15, Theorem 10] we see that
\[ \mathcal{H}_W (m_t) \leq \liminf_{N \to +\infty} \frac{1}{N} \mathcal{H} \left( m_t^{(N)} \mid m_\infty^{(N)} \right). \] (20)

The proof of (10) and (11) is then concluded by dividing (7) and (8) by \( N \) and letting \( N \to +\infty \) thanks to (18), (19) and (20).

3.4 Proofs of the Corollaries

We first need some preliminary lemmas. The first one gives a control of the propagation of chaos at the level of the invariant measure (hence, at infinite time, by contrast to Proposition 11).

Lemma 16 Under Assumption 1, there exists \( K > 0 \) such that, for all \( N \in \mathbb{N} \),
\[ \mathcal{W}_2 \left( m_\infty^{\otimes N}, m_\infty^{(N)} \right) \leq K. \]

Proof For all \( N \in \mathbb{N} \) and \( t \geq 0 \),
\[ \mathcal{W}_2 \left( m_\infty^{\otimes N}, m_\infty^{(N)} \right) \leq \mathcal{W}_2 \left( m_\infty^{\otimes N}, m_t^{(N)} \right) + \mathcal{W}_2 \left( m_t^{(N)}, m_\infty^{(N)} \right). \]

Applied in the case \( m_0^{(N)} = m_\infty^{\otimes N} \) together with Proposition 11 and Theorem 2, this yields
\[ \mathcal{W}_2 \left( m_\infty^{\otimes N}, m_\infty^{(N)} \right) \leq K e^{bt} + C e^{-\chi t} \mathcal{W}_2 \left( m_\infty^{\otimes N}, m_\infty^{(N)} \right). \]

In particular, for \( t = \ln(2C)/\chi \), we get
\[ \mathcal{W}_2 \left( m_\infty^{\otimes N}, m_\infty^{(N)} \right) \leq 2K \left( 2C \right)^{b/\chi}. \]

Lemma 17 Let \( v_1 \) and \( v_2 \) be probability laws on \( \mathbb{R}^{dN} = (\mathbb{R}^d)^N \) which are fixed by any permutation of the \( d \)-dimensional coordinates (in other words, if \( (A_i)_{i \in [1,N]} \) is of law \( v \), the \( A_i \)'s are interchangeable). Let \( (A, B) = (A_i, B_i)_{i \in [1,N]} \) be a coupling of \( v_1 \) and \( v_2 \) such that
\[ \mathbb{E} (|A - B|^2) = \mathcal{W}_2^2 (v_1, v_2). \]

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Then
\[ \mathbb{E}\left( \mathcal{W}_2^2 \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{A_i}, \frac{1}{N} \sum_{i=1}^{N} \delta_{B_i} \right) \right) \leq \frac{1}{N} \mathcal{W}_2^2(v_1, v_2). \]

**Proof** Let \( I \) be uniformly distributed on \([1, N]\). Then \((A_I, B_I)\) is a coupling of \( \frac{1}{N} \sum \delta_{A_i} \) and \( \frac{1}{N} \sum \delta_{B_i} \), hence
\[ \mathbb{E}\left( \mathcal{W}_2^2 \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{A_i}, \frac{1}{N} \sum_{i=1}^{N} \delta_{B_i} \right) \right) \leq \mathbb{E} \left( |A_I - B_I|^2 \right) = \frac{1}{N} \mathbb{E} \left( |A - B|^2 \right). \]

**Proof (Proof of Corollary 5)** Let \((Z_N, \tilde{Z}_N)\) be a \(\mathcal{W}_2\)-optimal coupling of \(m_i^{(N)}\) and \(m_{\infty}^{\otimes N}\) and \(\tilde{M}_i^N\) be the empirical distribution of \(\tilde{Z}_N\). Then, using Lemma 17, we bound
\[ \mathbb{E}\left( \mathcal{W}_2^2 \left( M_i^N, m_{\infty} \right) \right) \leq 2 \mathbb{E} \left( \mathcal{W}_2^2 \left( M_i^N, \tilde{M}_i^N \right) \right) + 2 \mathbb{E} \left( \mathcal{W}_2^2 \left( \tilde{M}_i^N, m_{\infty} \right) \right) \leq \frac{2}{N} \mathcal{W}_2^2 \left( m_i^{(N)}, m_{\infty}^{\otimes N} \right) + 2 \mathbb{E} \left( \mathcal{W}_2^2 \left( \tilde{M}_i^N, m_{\infty} \right) \right). \]

From [13, Theorem 1], the second term is bounded by \(Ra(N)\) for some \(R\) independent from \(N\), and we bound the first one using Lemma 16 and Theorem 2 as
\[
\mathcal{W}_2 \left( m_i^{(N)}, m_{\infty}^{\otimes N} \right) \leq \mathcal{W}_2 \left( m_i^{(N)}, m_{\infty}^{(N)} \right) + \mathcal{W}_2 \left( m_{\infty}^{(N)}, m_{\infty}^{\otimes N} \right) \\
\leq C e^{-\chi t} \mathcal{W}_2 \left( m_0^{(N)}, m_{\infty}^{(N)} \right) + K \\
\leq C e^{-\chi t} \mathcal{W}_2 \left( m_0^{\otimes N}, m_{\infty}^{\otimes N} \right) + \mathcal{W}_2 \left( m_i^{(N)}, m_{\infty}^{(N)} \right) + K \\
\leq C e^{-\chi t} \sqrt{N} \mathcal{W}_2 \left( m_0^{(N)}, m_{\infty} \right) + K(1 + C) \\
\leq K'(\sqrt{N} e^{-\chi t} + 1)
\]
for some \(K'\) independent from \(N\) and \(t\). We have thus obtained
\[ \mathbb{E}\left( \mathcal{W}_2^2 \left( M_i^N, m_{\infty} \right) \right) \leq 4(K')^2 \left( e^{-2\chi t} + \frac{1}{N} \right) + Ra(N), \]
and conclusion follows from the fact \(1/N\) is negligible with respect to \(a(N)\) as \(N \to +\infty\).

**Proof (Proof of Corollary 5)** Combining Proposition 11 and Lemma 13,
\[ \mathcal{W}_2^2 \left( m_i^{\otimes N}, m_i^{(N)} \right) \leq \frac{K ne^{bt}}{N}. \]
Besides, combining Theorems 2 and 3 and Lemma 16,
\[
\mathcal{W}_2 \left( m_i^{\otimes N}, m_i^{(N)} \right) \leq \mathcal{W}_2 \left( m_i^{\otimes N}, m_{\infty}^{\otimes N} \right) + \mathcal{W}_2 \left( m_{\infty}^{\otimes N}, m_i^{(N)} \right) + \mathcal{W}_2 \left( m_{\infty}^{(N)}, m_i^{(N)} \right) \\
\leq C e^{-\chi t} \left( \sqrt{N} \mathcal{W}_2 \left( m_0^{(N)}, m_{\infty} \right) + \mathcal{W}_2 \left( m_i^{(N)}, m_0^{(N)} \right) \right) + K \\
\leq C e^{-\chi t} \left( 2\sqrt{N} \mathcal{W}_2 \left( m_0^{(N)}, m_{\infty} \right) + \mathcal{W}_2 \left( m_i^{(N)}, m_{\infty}^{(N)} \right) \right) + K
\]
\[
\leq K\left(\sqrt{N}e^{-\lambda t} + 1\right),
\]
for some \(K'\) independent from \(N\) and from \(t \geq 0\). Using again Lemma 13, we have thus obtained that there exists \(K''\) independent from \(N\) and \(t\) such that
\[
\mathcal{W}_2^2\left(m_t^{\otimes n}, m_t''(n,N)\right) \leq K'' n \left(\frac{e^{bt}}{N} \wedge \frac{N}{e^{2\lambda t}}\right).
\]
Distinguishing the cases \(t \leq \ln(N)/(2b)\) and \(t \geq \ln(N)/(2b)\) concludes the proof for the \(\mathcal{W}_2\) distance.

The case of the total variation distance is similar. First, from Pinsker’s and Csiszár’s inequalities, considering the initial condition \(m_0^{(N)} = m_\infty^{\otimes N}\), we get for all \(t \geq 1\)
\[
\|m_\infty^{\otimes n} - m_\infty^{(n,N)}\|_{TV}^2 \leq 2\|m_\infty^{\otimes n} - m_t^{(n,N)}\|_{TV}^2 + 2\|m_t^{(n,N)} - m_\infty^{(n,N)}\|_{TV}^2 \\
\leq \frac{8n}{N} \mathcal{H}\left(m_t^{(N)}|m_t^{\otimes N}\right) + 4\mathcal{H}\left(m_t^{(N)}|m_\infty^{(N)}\right) \\
\leq \frac{8n}{N} K' e^{bt} \sqrt{N} + 4C^2 e^{-\lambda (t-1)} \mathcal{W}_2^2\left(m_0^{(N)}, m_\infty^{(N)}\right)
\]
for some \(K'\), where we combined Propositions 11 and 12 for the first term and used Theorem 2 for the second one. Together with Lemma 16, we have obtained that for some \(K''\) independent from \(N, t, n,\)
\[
\|m_\infty^{\otimes n} - m_\infty^{(n,N)}\|_{TV}^2 \leq K'' \left(\frac{n}{\sqrt{N}} e^{bt} + e^{-\lambda t}\right) \leq nK'' \left(\frac{1}{\sqrt{N}} e^{bt} + e^{-\lambda t}\right) \leq \frac{K'''n}{N^\kappa}
\]
for some \(\kappa, K''' > 0\) when \(t = 1 + \ln(N)/(4b)\).

Now, considering any initial condition \(m_0 \in \mathcal{P}_2(\mathbb{R}^d),\)
\[
\|m_t^{\otimes n} - m_t^{(n,N)}\|_{TV}^2 \leq 3\|m_t^{\otimes n} - m_\infty^{\otimes n}\|_{TV}^2 + 3\|m_\infty^{\otimes n} - m_\infty^{(n,N)}\|_{TV}^2 + 3\|m_t^{(n,N)} - m_\infty^{(n,N)}\|_{TV}^2 \\
\leq 6\mathcal{H}\left(m_t^{\otimes n}|m_\infty^{\otimes n}\right) + 3K'' nN^{-\kappa} + 6\mathcal{H}\left(m_t^{(N)}|m_\infty^{(N)}\right) \\
\leq 6C^2 e^{-\lambda (t-1)} \left(\mathcal{W}_2^2\left(m_0^{\otimes N}, m_\infty^{\otimes N}\right) + \mathcal{W}_2^2\left(m_0^{(N)}, m_\infty^{(N)}\right)\right) + 3K''' nN^{-\kappa}
\]
for \(t \geq 1\), so that
\[
\|m_t^{\otimes n} - m_t^{(n,N)}\|_{TV}^2 \leq K \left(N e^{-\lambda t} + nN^{-\kappa}\right) \leq Kn \left(N e^{-\lambda t} + N^{-\kappa}\right)
\]
for some \(K\). Besides, from Propositions 11 and 12 and Lemma 13,
\[
\|m_t^{\otimes n} - m_t^{(n,N)}\|_{TV}^2 \leq \frac{4n}{N} \mathcal{H}\left(m_t^{(N)}|m_t^{\otimes N}\right) \leq \frac{Kn}{\sqrt{N}} e^{bt}
\]
for some \(K\), and conclusion follows again by distinguishing the cases \(t \geq 1 + \ln(N)/(4b)\) and \(t \leq 1 + \ln(N)/(4b).\)

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest. No data nor code have been used or produced in this work.
References

1. Aida, I., Shigekawa, S.: Logarithmic Sobolev inequalities and spectral gaps: perturbation theory. J. Funct. Anal. 126(2), 448–475 (1994)
2. Bakry, D., Gentil, I., Ledoux, M.: Analysis and geometry of Markov diffusion operators, volume 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Cham (2014)
3. Baudoin, F., Gordina, M., Herzog, D.P.: Gamma calculus beyond Villani and explicit convergence estimates for Langevin dynamics with singular potentials. arXiv e-prints, page arXiv:1907.03092 (2019)
4. Bolley, F., Guillin, A., Malrieu, F.: Trend to equilibrium and particle approximation for a weakly self-consistent Vlasov-Fokker-Planck equation. M2AN Math. Model. Numer. Anal. 44(5), 867–884 (2010)
5. Bolley, F., Guillin, A., Villani, C.: Quantitative concentration inequalities for empirical measures on non-compact spaces. Probab. Theory Relat. Fields 137(3–4), 541–593 (2007)
6. Carrillo, J.A., McCann, R.J., Villani, C.: Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. Rev. Mat. Iberoamericana 19(3), 971–1018 (2003)
7. Cattiaux, P., Guillin, A., Monmarché, P., Zhang, C.: Entropic multipliers method for Langevin diffusion and weighted log Sobolev inequalities. J. Funct. Anal. 277(11), 108288 (2019)
8. Csiszár, I.: Sanov property, generalized I-projection and a conditional limit theorem. Ann. Probab. 12(3), 768–793 (1984)
9. Dolbeault, J., Mouhot, C., Schmeiser, C.: Hypocoercivity for kinetic equations with linear relaxation terms. C. R. Math. Acad. Sci. Paris 347(9–10), 511–516 (2009)
10. Duong, M.H., Tugaut, J.: Stationary solutions of the Vlasov-Fokker-Planck equation: existence, characterization and phase-transition. Appl. Math. Lett. 52, 38–45 (2016)
11. Durmus, A., Eberle, A., Guillin, A., Zimmer, R.: An elementary approach for uniform in time propagation of chaos. To appear in Proc. Am. Math. Soc (2019)
12. Eberle, A., Guillin, A., Zimmer, R.: Couplings and quantitative contraction rates for Langevin dynamics. arXiv e-prints, to appear in Trans. Am. Math. Soc., page arXiv:1703.01617 (2017)
13. Fournier, N., Guillin, A.: On the rate of convergence in Wasserstein distance of the empirical measure. Probab. Theory Relat. Fields 162(3–4), 707–738 (2015)
14. Guillin, A., Liu, W., Wu, L., Zhang, C.: The Kinetic Fokker-Planck equation with mean field interaction. arXiv e-prints, page arXiv:1912.02594 (2019)
15. Guillin, A., Liu, W., Wu, L., Zhang, C.: Uniform Poincaré and logarithmic Sobolev inequalities for mean field particles systems. arXiv e-prints, page arXiv:1909.07051 (2019)
16. Guillin, A., Wang, F.-Y.: Degenerate Fokker-Planck equations: Bismut formula, gradient estimate and Harnack inequality. J. Differ. Equ. 253(1), 20–40 (2012)
17. Hérau, F., Thomann, L.: On global existence and trend to the equilibrium for the Vlasov-Poisson-Fokker-Planck system with exterior confining potential. ArXiv e-prints (2015)
18. Kac, M.: Foundations of kinetic theory. In: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 171–197. University of California Press, Berkeley and Los Angeles (1956)
19. Malrieu, F.: Logarithmic Sobolev inequalities for some nonlinear PDE’s. Stoch. Process. Appl. 95(1), 109–132 (2001)
20. Méléard, S.: Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In: Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995), vol. 1627 of Lecture Notes in Math., pp. 42–95. Springer, Berlin (1996)
21. Monmarché, P.: Long-time behaviour and propagation of chaos for mean field kinetic particles. Stoch. Process. Appl. 127(6), 1721–1737 (2017)
22. Monmarché, P.: Generalized Γ calculus and application to interacting particles on a graph. Potential Anal. 50, 459–466 (2019)
23. Otto, F., Villani, C.: Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. J. Funct. Anal. 173(2), 361–400 (2000)
24. Sznitman, A.-S.: Topics in propagation of chaos. In: École d’Été de Probabilités de Saint-Flour XIX—1989, vol. 1464 of Lecture Notes in Math., pp. 165–251. Springer, Berlin (1991)
25. Talay, D.: Stochastic Hamiltonian systems: exponential convergence to the invariant measure, and discretization by the implicit Euler scheme. Markov Process. Relat. Fields 8(2), 163–198 (2002)
26. Villani, C.: Hypocoercivity. Mem. Amer. Math. Soc. 202(950):iv+141 (2009)
27. Villani, C.: Optimal transport, old and new, vol. 338 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin (2009)
28. Zegarlinski, B.: Dobrushin uniqueness theorem and logarithmic Sobolev inequalities. J. Funct. Anal. 105(1), 77–111 (1992)

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