BLACK HOLES FROM NUCLEATING STRINGS

Jaume Garriga\textsuperscript{1,2} and Alexander Vilenkin\textsuperscript{1,3}

\textsuperscript{1}Tufts Institute of Cosmology, \\
Department of Physics and Astronomy, Tufts University, Medford, MA 02155

\textsuperscript{2}Institute for Theoretical Physics, \\
University of California \\
Santa Barbara, California, 93106-4030

\textsuperscript{3}Lyman Laboratory of Physics, Harvard University, \\
Cambridge, MA 02138

Abstract

We evaluate the probability that a loop of string that has spontaneously nucleated during inflation will form a black hole upon collapse, after the end of inflation. We then use the observational bounds on the density of primordial black holes to put constraints on the parameters of the model. Other constraints from the distortions of the microwave background and emission of gravitational radiation by the loops are considered. Also, observational constraints on domain wall nucleation and monopole pair production during inflation are briefly discussed.
1 Introduction

It has been proposed [1, 2] that loops of string can spontaneously nucleate during an inflationary period of expansion in the early universe. This is a quantum tunneling process, somewhat analogous to the spontaneous nucleation of spherical bubbles of true vacuum in the problem of false vacuum decay [3]. It has been shown that loops can nucleate at considerable rates provided that their tension $\mu$ is not much larger than $H^2$ (here $H$ is the expansion rate during inflation) and, in this case, the distribution of loops emerging from inflation may have significant cosmological consequences.

In this scenario, loops nucleate with sizes of the order of $H^{-1}$, and are subsequently stretched to large sizes by the inflationary expansion. After inflation, in the radiation dominated era, the loops eventually fall within the Hubble radius and start oscillating under their tension.

Because the instanton describing the nucleation of a loop is maximally symmetric, the nucleated loops tend to be nearly circular. It is well known that an exactly circular loop would form a black hole upon collapse [4, 5]. However, the nucleated loops will not be exactly circular. The reason is that, during inflation, quantum fluctuations on the string on subhorizon scales generate shape instabilities as their wavelengths become larger than the horizon [6, 7] (see also [8]). The question then arises of how many of the nucleated loops will be circular enough to form a black hole after the end of inflation. This is the problem we address in the present paper.

The plan of the paper is the following. In Section 2 we calculate the probability of black hole formation from a nucleating string. In Section 3 we use this result and the observational bounds on the abundance of primordial black holes to put constraints on the parameters of the model (which are essentially $\mu$ and $H_0$). Additional constraints from the distortions in the microwave background and the emission of gravitational waves by the strings are considered. In Section 4, we briefly discuss observational constraints on domain wall nucleation and monopole-antimonopole pair production, which can also occur during inflation [9]. We summarize our conclusions in Section 5. Some details are left to the appendices. Appendix A deals with linearized perturbations on a circular loop of string in flat space, and in Appendix B we consider the effect of damping processes on the probability of black hole formation.
2 Probability of black hole formation

In this section we estimate the probability that a loop of string that has nucleated during inflation will form a black hole once it collapses under its tension, after the end of inflation.

The classical worldsheet of a circular loop of string that has nucleated during inflation is given by \[ R(t) = H^{-1} \sqrt{e^{2H(t-t_0)}} + 1, \] (1)

where \( R \) is the physical radius of the loop, \( H \) is the expansion rate of the inflationary universe and \( t \) is the usual cosmological time in the flat Friedman-Robertson-Walker coordinates. The parameter \( t_0 \) can be interpreted (in a loose sense) as the time at which the loop nucleates. Throughout this paper, we shall work in the approximation in which the strings are infinitely thin, although nucleation of strings whose thickness is not much smaller than \( H^{-1} \) is also possible \[2\].

The loops nucleate with a size of the order of the horizon size \( R \sim H^{-1} \), and afterwards they are stretched by the inflationary expansion. For \( R \gg H^{-1} \) they grow proportionally to the scale factor

\[ R(t) \approx H^{-1} e^{H(t-t_0)}. \]

After inflation, in the radiation dominated era, the loops continue to be stretched by the expansion of the universe, \( R(t) \propto t^{1/2} \), until they enter the cosmological horizon, at \( t \sim R \). Soon after the loop comes within the horizon, the effects of expansion become negligible and the loop starts behaving approximately as it would behave in flat space.

Let \( R_c \) be the radius of a particular loop at horizon crossing. The mass of this loop is \( 2\pi R_c \mu \), where \( \mu \) is the string tension. The Schwarzschild radius corresponding to this mass is

\[ R_s = 4\pi G \mu R_c. \] (2)

As the loop shrinks under its tension, its rest mass is converted into kinetic energy, so that the total energy of the loop remains constant. It is clear that if the loops were exactly circular, they would all eventually shrink to a size smaller than \( R_s \), thus forming black holes (of course this argument assumes
that $R_s$ is larger than the thickness of the string, which will always be true for sufficiently large loops.)

However, loops nucleated during inflation will not be exactly circular. Quantum fluctuations around the symmetric solution \[ (1) \] will cause small departures from the circular shape, and this will determine the probability of black hole formation. Consider a circular loop of radius $R(t)$ lying in the $z = 0$ plane and centered at the origin. Using cylindrical coordinates $(\rho, \theta, z)$, we can parametrize a perturbed (non-circular) loop as follows:

\[
\rho(\theta, t) = R(t) + \Delta_r(\theta, t),
\]

\[
z(\theta, t) = \Delta_t(\theta, t).
\]

That is, we decompose the perturbations into a radial part $\Delta_r$ parallel to the plane of the unperturbed loop, and a transverse part $\Delta_t$ perpendicular to that plane. In eq. (3), $\rho$ and $z$ are physical coordinates (rather than co-moving.)

To study the evolution of a loop, we shall divide its history into three epochs:

- The loop nucleates and then expands during inflation.

- After inflation, while the loop is larger than the cosmological horizon it is stretched by the expansion until it crosses the horizon.

- Once the loop comes within the horizon, it shrinks under its tension.

We will follow the evolution of the perturbations $\Delta_r, \Delta_t$ through the epochs $a$, $b$ and $c$. For this it will be useful to expand them in Fourier modes:

\[
\Delta_r = \sum_{L=2}^{\infty} \left[ \Delta^{(1)}_{r,L} \frac{\cos L\theta}{\sqrt{\pi}} + \Delta^{(2)}_{r,L} \frac{\sin L\theta}{\sqrt{\pi}} \right],
\]

and similarly for $\Delta_t$. Note that the sum does not include the modes with $L = 0$ and $L = 1$. This is because, as it was shown in Ref. [7], such modes do not correspond to true perturbations, but only to spatial rotations and spacetime translations of the unperturbed solution \[ (1) \].

- **Evolution during inflation.** In the process of nucleation, $\Delta^{(i)}_{\lambda,L}$ ($i = 1, 2; \lambda = t, r$) have to be treated as quantum variables. The linearized theory of quantum fluctuations around the classical solution \[ (1) \] was studied in Ref. [4].
using a covariant formalism. The worldsheet has the internal geometry of a 1+1 dimensional de Sitter space, and it was shown that the perturbations $\Delta_r$ and $\Delta_t$ behave as two uncoupled scalar fields of tachyonic mass $m^2 = -2H^2$ ‘living’ in this lower dimensional de Sitter space. The symmetries of the problem suggest that when the string nucleates, the fields $\Delta_r, \Delta_t$ should be in a de Sitter invariant quantum state (see also Ref.[9]). This state was constructed in Ref.[7] using the Heisenberg picture. The corresponding Schrödinger picture wave functional $\Psi(\{\Delta_{\lambda,L}(i)\})$ can be obtained using well known manipulations (see e.g. [9]). Since we are dealing with a linearized theory, the probability distribution associated with $\Psi$ has the Gaussian form

$$P(\{\Delta_{\lambda,L}^{(i)}\}) = |\Psi|^2 = \prod_{\lambda,i,L} (2\pi \sigma_{\lambda,L}^2)^{-1/2} \exp \left[ \frac{-\langle \Delta_{\lambda,L}^{(i)} \rangle^2}{2 \sigma_{\lambda,L}^2} \right],$$

(5)

where the standard deviations $\sigma_{\lambda,L}$ are given by

$$\sigma_{\lambda,L}^2(t) \equiv \langle \psi | (\Delta_{\lambda,L}^{(i)})^2 | \psi \rangle = \frac{1}{2\mu L} \left[ \frac{H^2 R^2(t)}{L^2 - 1} + 1 \right].$$

(6)

For wavelengths much larger than the horizon, $HR >> 1$, we have $\sigma_L \propto R$, i.e., the amplitude of the perturbations is conformally stretched by the expansion (recall that $R$ grows like the scale factor during inflation). For wavelengths much smaller than the horizon the second term in (6) dominates and we have $\sigma_L \approx \text{constant}$, as expected. This second term is just the flat space contribution to the quantum fluctuations, and we shall subtract it in what follows in order to avoid the well known ultraviolet divergences in $\langle \Delta^2 \rangle$.

Soon after a given wavelenth becomes larger than the horizon, it can be treated as classical. Here we adopt the point of view that Eq.(5) gives the probability distribution for the initial amplitudes of the perturbations, which will evolve classically thereafter.

b- After inflation: while the loop is larger than the cosmological horizon.

As we mentioned before, the loops are conformally stretched until they enter the horizon. Since we do not have analytical solutions to describe this epoch, we shall use the following approximations [10, 11]: for wavelengths larger than the horizon the amplitude of the perturbations is conformally stretched by the expansion, whereas for wavelengths within the horizon the amplitude of the perturbations remains constant.
Perturbations with wave number $L$ enter the horizon at time $t_L \sim R(t_L)/L$. At this time we have $\sigma_L = [2\mu L(L^2 - 1)]^{-1/2}HR(t_L)$. The loop enters the horizon at the time

$$t_c \sim R(t_c) \sim \frac{R^2(t_L)}{t_L} \sim LR(t_L).$$

At this time we still have the same $\sigma_L$ than at $t_L$, since the amplitude of the perturbations is frozen after $t_l$, so we write

$$\sigma_L(t_c) = [2\mu L(L^2 - 1)]^{-1/2}HR(t_L) = [2\mu L^3(L^2 - 1)]^{-1/2}HR_c. \quad (7)$$

c- Loops inside the horizon. Once a loop comes within the horizon we can ignore the effects of the expansion of the Universe and consider the collapse of a loop in flat space. In this case the evolution of the perturbations can be solved analytically. The details can be found in Appendix A. We find that during this period the behaviour of radial perturbations $\Delta_r$ is different from that of transverse perturbations $\Delta_t$. The amplitude of radial perturbations shrinks by a factor of $L$ as the loop collapses, while the amplitude of transverse perturbations remains constant. For $R \ll R_c$ we find

$$\sigma_{r,L} \approx (2\mu L^7)^{-1/2}HR_c, \quad (8)$$

$$\sigma_{t,L} \approx (2\mu L^5)^{-1/2}HR_c.$$ 

Introducing $(8)$ in $(5)$, we obtain the probability distributions for the variables $\Delta^{(i)}_{\lambda,L}$ at the time when the radius of the unperturbed loop enters the Schwarzschild radius $R_s$.

The dynamics of gravitational collapse and black hole formation from a given loop configuration is, of course, a complicated issue that cannot be treated in detail analytically. However, we know that if the amplitude of the perturbations is much smaller than $R_s$ at the time $t_s$ when the unperturbed loop enters the Schwarzschild radius, then a black hole will form. On the contrary, if the perturbations are large compared with $R_s$ the loop will probably self intersect and fragment into smaller loops before it shrinks to a size $R_s$ (the resulting loops will be far from circular, so a black hole will never form in this case). To compute the probability of black hole formation, we shall count a particular loop as a black hole with gaussian weight

$$e^{-\delta^2/\alpha^2 R_s^2}. \quad (9)$$
Here $\delta$ is the r.m.s. perturbation when one averages over the circumference of the loop (at time $t_s$)

$$
\delta^2 \equiv \frac{1}{2\pi} \int_0^{2\pi} \Delta^2(\theta, t_s) = \frac{1}{2\pi} \sum_{i,\lambda,L} (\Delta^{(i)}_{\lambda,L})^2,
$$

where we have used (4). In Eq. (9), $\alpha$ is an unknown parameter of order 1, which models our lack of knowledge on the dynamical details. We should emphasize that the precise form of the “Gaussian window” (9) is not very important here; we have taken the Gaussian shape just for computational convenience. Had we taken, for instance, a step function window, our final conclusions would be the same.

From (5) and (9), the probability of black hole formation from a loop that has nucleated during inflation is given by

$$
P_{bh} = \int |\Psi|^2 e^{-\delta^2/(\alpha R_s)^2} \prod_{\lambda,i,L} d\Delta^{(i)}_{\lambda,L} = \prod_{\lambda=r,t} \prod_{L=2}^\infty \left(1 + \frac{\sigma^2_{\lambda,L}}{\pi \alpha^2 R_s^2}\right)^{-1}.
$$

(10)

Using (8), we can rewrite it as

$$
P_{bh}(L_s) = \prod_{L=2}^\infty \left(1 + 2 \frac{L^5}{L^5}\right)^{-1} \left(1 + 2 \frac{L^5}{L^7}\right)^{-1},
$$

(11)

where we have introduced the notation

$$
L_s \equiv [16\pi^2 B(\alpha G \mu)^2]^{-1/5}.
$$

Here $B \equiv 4\pi \mu/H^2$ is the Euclidean action of the instanton describing the nucleation of the loop. Note that $P_{bh}$ does not depend on $R_c$, the radius of the loop at horizon crossing. Note, also, that the dependence on the parameters $H, \mu$ and $\alpha$ is only through the combination $L_s$.

The physical meaning of $L_s$ is the following. Modes with $L >> L_s$ have $\sigma_L << \alpha R_s$. These modes will always have amplitudes much smaller than the Schwarzschild radius and they will not contribute to $P_{bh}$. This is also clear from (11). Therefore $L_s$ is basically the number of modes whose amplitude can be large enough to prevent the formation of a black hole.

The function $P_{bh}(L_s)$ cannot be given in closed form. However, one can compute it numerically to arbitrary precision by including a sufficient number.
of terms in the product (11). A plot of \( P_{bh} \) versus \( L \) is given in Fig. 1. From the graph we see that the probability decays (essentially) as an exponential function of \( L \).

In deriving Eq. (11) we have neglected damping processes, such as gravitational radiation and friction. In principle these could reduce the amplitude of the perturbations and therefore increase the probability of black hole formation. In Appendix B we briefly discuss the damping mechanisms and the limits in which they can be neglected. We find that damping can be safely ignored from the calculation of \( P_{bh} \) provided that \( G\mu \) is in the range

\[
(\alpha^2 B)^{-2/9} \left( \frac{m_p}{M} \right)^{5/9} \lesssim G\mu < 5 \cdot 10^{-2} (\alpha^2 B)^{1/3},
\]

where \( m_p \) is the Planck mass and \( M \) is the mass of the black hole. The upper limit is due to gravitational radiation, which is important for heavy strings, while the lower limit corresponds to friction due to the surrounding matter, which is important for light strings. In the next Section we will be interested in \( P_{bh} \) for black holes of mass \( M \sim 10^{19} m_p \), so Eq. (12) leaves us with a wide range of values of \( G\mu \) for which (11) is valid.

3 Observational constraints on loop nucleation

During inflation, loops of size \( \sim H^{-1} \) are produced at a constant rate per unit physical volume. After nucleation these loops grow like the scale factor, so we expect that the number of loops with size \( \sim R \) contained in a volume \( \sim R^3 \) will be independent of \( R \). That is, we expect a scale invariant distribution of loops. The number density distribution of loops will be given by

\[
\frac{dN}{dV} = \nu \frac{dR}{R^4}.
\]

(Actually, the distribution (13) has a lower cut-off at \( R \sim H^{-1} \), since we do not have loops smaller than that. It also has an upper cut-off at \( R \sim EH^{-1} \), where \( E \) is the total e-folding factor since the onset of inflation. See Ref. [1] for details.)

The coefficient \( \nu \) is the number of loops produced during an expansion time \( H^{-1} \) in a volume \( H^{-3} \). This has been estimated in the semiclassical
approximation in Ref. [1], using the instanton methods [3]

\[ \nu = Ae^{-B}. \]  

(14)

Here, as in Section 2, \( B = 4\pi\mu/H^2 \) is the euclidean action of the instanton describing the nucleation of the string (which is just a spherical worldsheets of radius \( H^{-1} \)). The coefficient \( A \) has not been calculated and we shall leave it as a free parameter here, presumably of order one. Let us now consider various observational constraints that one can place on the parameters of the model.

**a- Constraint from primordial black hole abundance.**

After inflation, the distribution of loops with \( R >> t \) is still given by (13), since these loops continue to be stretched by the expansion. Loops with size smaller than the cosmological horizon, \( R << t \), are not stretched and are simply diluted by the expansion,

\[ \frac{dN}{dV} = \nu \left[ \frac{a(R)}{a(t)} \right]^3 \frac{dR}{R^4}, \]  

(15)

where \( a \) is the cosmological scale factor. For loops that enter the horizon during the radiation dominated era, we have

\[ \frac{dN}{dV} = \nu \left( \frac{1}{tR} \right)^{3/2} \frac{dR}{R}. \]  

(16)

Multiplying by the mass of the loops, \( M \approx 2\pi\mu R \), we obtain the mass density distribution

\[ d\rho(M) = \nu \left( \frac{2\pi G\mu}{tGM} \right)^{3/2} dM. \]  

(17)

Dividing by the critical density \( \rho_c = 3/32\pi Gt^2 \) we have, at the present time,

\[ d\Omega_s(M) \equiv \frac{d\rho(M)}{\rho_c} \approx \frac{32\pi^2\sqrt{2\pi}}{3} (G\mu)^{3/2} \nu \left( \frac{t_{eq}}{GM} \right)^{1/2} \frac{dM}{M}, \]  

(18)

where \( t_{eq} \approx 4 \cdot 10^{10} \text{ sec.} \) is the time of equal matter and radiation densities.

As we described in Section 3, some of the loops in the network will form black holes with probability given by eq. (11). To obtain the spectrum of primordial black holes that are produced through this mechanism we multiply
The fraction of the density parameter $\Omega$ in black holes of mass $\sim M$ is then given by

$$\Omega_{bh}(M) \approx P_{bh} \frac{32\pi^2\sqrt{2\pi}}{3}(G\mu)^{3/2}\nu \left(\frac{t_{eq}}{GM}\right)^{1/2}.$$  \(19\)

The strongest observational constraint on the abundance of primordial black holes comes from the emission of $\gamma$-rays by holes that are evaporating at the present time \[12\]. These black holes have a mass $M \approx 5 \cdot 10^{14} g$, and the constraint is given by

$$\Omega_{bh}(5 \cdot 10^{14} g) \lesssim 10^{-8}.$$  \(20\)

From \(19\) we have

$$P_{bh} \cdot (G\mu)^{3/2}\nu \lesssim 10^{-28}.$$  \(21\)

We can rewrite this inequality as

$$- \log_{10} P_{bh}(L_\star) - \frac{3}{2} \log_{10}(\alpha G\mu) + (\log_{10} e) B \lesssim N,$$  \(22\)

where $N = 28 + \log_{10}(A\alpha^{-3/2})$. Note that the left hand side only depends on $B$ and the combination $\alpha G\mu$. Therefore, for given $N$, eq. \(22\) will exclude a certain region in the plane $(B, \alpha G\mu)$. Since the parameters $\alpha$ and $A$ are expected to be of order 1, their contribution to $N$ will be small. Note also that even if the observational constraint \(20\) was improved by one order of magnitude, this would only increase $N$ by one. At any rate, we expect $N \approx 28$ plus or minus a few units.

In Fig. 2, the boundary of the excluded region in parameter space is depicted for $N = 27, 28$ and $30$. It is seen that the result is not very sensitive to the value of $N$ and, quite generically, the bound will be satisfied provided that

$$\alpha G\mu \lesssim 10^{-4},$$  \(23\)

independently of $B$. This can be easily understood since, for such values of $\alpha G\mu$, the parameter $L_\star$ will be very large and, correspondingly, the probability of black hole formation will be exponentially small (see Fig. 1).

Also, from Fig. 2 we see that for large values of $B$, say $B \gtrsim 50$, the constraint on $G\mu$ is practically removed due to the exponential suppression
in the number of nucleated loops. Of course this limit is not very interesting cosmologically.

**b-Constraints from the microwave background**

A string moving across the sky produces a discontinuity in the observed temperature of the Cosmic Microwave Background (CMB) between both sides of the string [13], roughly of order \( (\delta T/T) \approx 10 G\mu \). For values of \( \nu \sim 1 \), the loop sizes extend all the way to the present Hubble radius, and the bound obtained in Ref. [14]

\[ G\mu \lesssim 10^{-6}, \quad (\nu \sim 1) \]

should apply. However, for \( \nu \ll 1 \) this bound should not necessarily be satisfied, since in this case all the loop radii are much smaller than the horizon.

Let us first consider the effect of loops with \( R > t_{eq} \). The angle \( \Theta \) subtended by the string is given by

\[ y \equiv \tan(\Theta/2) = \frac{R}{d_0} \left[ 1 - \frac{d_0}{3t_0} \right]^{-2}, \quad (24) \]

where \( t_0 \) is the age of the Universe and \( d_0 \) is the present distance to the string. Loops of given radius start oscillating when \( R \sim t \), and therefore they will only have an effect on the CMB provided that they are at a sufficiently low redshift. Using the fact that the light that is now reaching us from a point at a distance \( d_0 \) was emitted at time \( t = t_0[1 - (d_0/3t_0)]^3 \), loops that were oscillating when the observed light passed through them must satisfy

\[ R < t_0 \left[ 1 - \frac{d_0}{3t_0} \right]^3. \quad (25) \]

The number of loops with radius in the interval \([R, R + \Delta R]\) at a distance in the interval \([d_0, \Delta d_0]\) is given by

\[ \Delta N(R, d_0) = \frac{\nu}{R^2 t_0^2} 4\pi d_0^2 \Delta R \Delta d_0. \quad (26) \]

Changing variables \( \{R, d_0\} \rightarrow \{y, d_0\} \) through eq. (24) and integrating over all possible distances \( d_0 \) allowed by the constraint (23) we obtain

\[ dN(y) = 36\pi\nu \left[ \frac{1}{3y^3} - \frac{1}{y^2} \ln \left( 1 + \frac{1}{3y} \right) \right] dy. \quad (27) \]
The expectation value of the number of loops subtending an angle larger than $\Theta = 2 \tan^{-1} y$ is then

$$N_{>\Theta} = \int_{y}^{\infty} dN[y] = 36 \pi \nu \left( \frac{1}{6y^2} + \frac{1}{y} - 3 \left[ 1 + \frac{1}{3y} \right] \ln \left[ 1 + \frac{1}{3y} \right] \right).$$

This function is plotted in Fig. 3. It is seen that for $\Theta > 10^\circ$ (the angular resolution of the COBE experiment) and $\nu < 10^{-3}$, for instance, the expected number of events $N_{>\Theta}$ is only of order 1. This means that for such low values of $\nu$ we can hardly expect to see any loops that large.

However, one must take into account that ‘unresolved’ loops whose angular size is smaller than the detector beam angular size can still produce a signal, although of lower magnitude. For simplicity, in what follows we shall assume that the loops are nearly circular. In that case, if the plane of the loop is perpendicular to the line of sight, the temperature distortion in the CMB has the profile of a ‘top hat’ with radius equal to the radius of the loop at the time when the light rays crossed this plane \[13\]. On the angular scale set by $R$ (the radius of the loop when it is at rest), the temperature fluctuation is $|\delta T/T| \approx 10G\mu$ (the sign depending on whether the loop is expanding or contracting at the time when the light rays cross the loop). On larger angular scales, the signal will be inversely proportional to the solid angle that we are considering.

Therefore if the detector beam has an angular size $\beta$ larger than the angular size of the loop $\Theta(R, d_0)$, the measured temperature fluctuation will be

$$\Delta \equiv \frac{1}{10G\mu} \frac{\delta T}{T} \approx \left( \frac{\Theta}{\beta} \right)^2. \quad (28)$$

For small angles, the distribution (27) can be approximated by $dN(\Theta) = 48\pi \nu \Theta^{-3} d\Theta$, and using (28) to express $\Theta$ as a function of $\Delta$ we have

$$N_{>\Delta} = \frac{24\pi \nu}{\beta^2 \Delta}. \quad (29)$$

Here $N_{>\Delta}$ is the expected number of events due to unresolved loops that give a signal larger than $\delta T/T$ on a detector with beam angular size $\beta$. This equation can be rewritten as

$$\nu G\mu \approx \frac{N_{\Delta}}{240\pi} \frac{\beta^2 \delta T}{T}, \quad (29)$$
where $\beta$ has to be expressed in radians.

The COBE experiment constrains $\delta T/T$ to be less than $10^{-5}$ on scales $\beta \approx 10^\circ \approx 0.17 \text{ rad}$. It is not clear what is the maximum number of events $N_\Delta$ allowed by the COBE data. Taking $N_\Delta \lesssim 1$ seems a little over restrictive since one small string can easily be hidden behind the galaxy. To be more conservative we shall take $N_\Delta \lesssim 10$, which from (29), yields the bound

$$\nu G\mu \lesssim 10^{-8}. \quad (\nu < 10^{-2})$$

This bound comes from the contribution of unresolved loops, and it only applies for $\nu < 10^{-2}$. For larger values of $\nu$, the number of ‘resolved’ loops is sufficiently large that the usual bound

$$G\mu \lesssim 10^{-6}. \quad (\nu > 10^{-2})$$

should be used.

So far we have considered loops with $R > t_{eq}$. A similar analysis can be carried out for $R < t_{eq}$ to show that these loops cannot produce relative temperature fluctuations in excess of $10^{-5}$ provided that the constraints (30), (31), and Eq. (23) from the previous subsection are satisfied.

**c-Constraint from gravitational radiation**

The millisecond pulsar observations place a constraint on the density parameter in gravitational radiation of period $\sim 1$ year [15],

$$\Omega_g < 4 \cdot 10^{-7}h^{-2}. \quad (32)$$

From strings, we have [14]

$$\Omega_g \sim \frac{128\pi}{9} \nu \left(\frac{G\mu}{\gamma_g}\right)^{1/2} \Omega_r$$

where $\Omega_r \sim 4 \cdot 10^{-5}h^{-2}$ is the present density parameter in radiation and $\gamma_g \sim 100$. With this, the bound (32) reads

$$\nu(G\mu)^{1/2} \lesssim 2 \cdot 10^{-3}.$$  

This is always satisfied provided that the bounds (23), (30) and (31) from the previous subsections are satisfied.

**c-Structure formation**
A question of cosmological interest is whether the constraints derived so far are compatible with a scenario in which the nucleated strings would seed the formation of large-scale structure in the Universe. Let us make a rough estimate of the values of the parameters needed for such purpose.

Assuming, for simplicity, cold dark matter, the mass accreted by a loop of radius \( R < t_{eq} \) at the present time is

\[
M = 2\pi\mu R z_{eq}, \quad (R < t_{eq})
\]  

(33)

Here \( z_{eq} \approx 2 \cdot 10^{-4} h^2 \) is the redshift at time \( t_{eq} \), with \( h \) the Hubble constant in units of \( 100 \text{km s}^{-1}\text{Mpc}^{-1} \). This is because perturbations start growing at time \( t_{eq} \) and they grow proportionally to the scale factor. From (16), the number density of loops of radius \( \sim R \) is

\[
n_R \sim \nu (t_{eq} R)^{-3/2} z_{eq}^{-3}.
\]

Using (33), the mean separation between objects of mass \( M \) is given by

\[
d_M \sim n_R^{-1/3} \sim \nu^{-1/3} \left( \frac{M_{t_{eq} z_{eq}}}{2\pi\mu} \right)^{1/2}.
\]

(34)

Rich clusters of mass \( M_{cl} \sim 10^{15} h^{-1} M_\odot \) are separated by distances of order \( d_{cl} \sim 50 h^{-1} \text{Mpc} \). Therefore, from (34), we obtain the normalization

\[
\nu^{2/3} G\mu \approx 5 \cdot 10^{-8} h^{-1}. \quad (\nu \gtrsim 10^{-2})
\]

(35)

This normalization is valid for \( G\mu \gtrsim 10^{-6} h \) (hence \( \nu \gtrsim 10^{-2} \)). For smaller values of \( G\mu \) the loops that accrete masses of order \( M_{cl} \) have \( R > t_{eq} \), and (33) does not apply.

For \( G\mu \lesssim 10^{-6} h \), loops start accreting matter when they enter the horizon, and we have

\[
M = 2\pi\mu\nu R \left( \frac{t_0}{R} \right)^{2/3}, \quad (R > t_{eq}).
\]

Using \( n_R = \nu/(R t_0^2) \) we obtain

\[
d_M \sim n_R^{-1/3} \sim \nu^{-1/3} M (2\pi\mu)^{-1}.
\]

Again, normalizing for rich clusters, we find

\[
\nu^{1/3} G\mu \approx 2 \cdot 10^{-7}. \quad (\nu \gtrsim 10^{-2})
\]

(36)
The region excluded by the constraints (23), (30) and (31) in parameter space is plotted schematically in Fig. 4 (shaded region). It is seen that the parameters needed for structure formation (thick solid line) [from (35) and (36)] lie marginally in the allowed region. Of course, the normalizations (35) and (36) should not be taken too literally, and one expects large error bars in the thick solid line of Fig. 4. However, a scenario in which the nucleated strings may seed some of the observed structure in our universe does not seem to be ruled out.

We should now mention the effects of compensation [17] which has been ignored in the above discussion. When a loop nucleates, its energy is balanced by a corresponding deficit in the local densities of matter and radiation. This compensation reduces the gravitational effect of the loop on surrounding matter and on background photons on scales greater than $R$. When a compensated loop radiates away its energy, it leaves behind an underdense region. As the density contrast grows, this region will evolve into a void with a dense central object seeded by the loop. The compensation is somewhat reduced by radiation and neutrinos streaming into the underdense regions and can be further reduced by loop fragmentation. The effect of compensation on structure formation has not been fully investigated, but it seems reasonable to assume that this effect will not be dramatic on scales crossing the horizon before $t_{eq}$. Then the relations (35),(36) should still be valid by order of magnitude.

4 Domain walls and monopoles

Just as in the case of strings, spherical domain walls can spontaneously nucleate at a constant rate per unit volume during inflation. As a result, the Universe will get filled with a scale invariant distribution of walls given by (13) and (14), where now $B = 2\pi^2\sigma H^{-3}$ is the action of the instanton describing the nucleation of the wall.

The observational constraints on domain wall nucleation are very different from the ones that one can impose on strings, since the mass of a spherical wall

$$M = 4\pi \sigma R^2$$

(37)

grows quadratically rather than linearly with the radius. An immediate con-
sequence is that walls of size
\[ R_c > (8\pi G\sigma)^{-1} \]  
will all collapse to form black holes \[ R_s > R_c. \]

To estimate the microwave background anisotropy induced by domain wall bubbles, one has to take proper account of the compensation effect discussed at the end of the previous section. This is a somewhat complicated issue and we shall not attempt to address it in detail here. However, one can make a rough estimate of what this effect should be by using the following arguments.

In a pure dust universe the compensation would be complete, and background photons would be unperturbed on scales greater than the co-moving scale of \( R_s \) at horizon crossing. Therefore, on larger scales, the whole effect should be due to the underdensity in radiation that is needed to compensate for the mass of the black hole. This underdensity causes radiation from neighboring regions to move in, partially destroying compensation. As this happens, the initial underdensity propagates away from the black hole as a sound wave at the speed of light. At time \( t_{eq} \) the mass of the underdensity, \( M_r \), is comparable to the mass of the black hole, \( M_{bh} \). At later times, we have \( M_r \approx M_{bh} z / z_{eq} \), due to cosmological redshift. The gravitational potential of \( M_r \) extends up to scales of order \( t \), and we have \( \phi \sim GM_t^{-1} \). This will cause temperature distortions of order
\[ \frac{\delta T}{T} \sim \phi \sim \frac{GM_{bh} z^{5/2}}{t_0 z_{eq}}, \]
on scales of the order of the cosmological horizon at redshift \( z \).

The angle \( \alpha \) subtended by the horizon at redshift \( z \) is given by \( \tan \alpha \approx (z^{1/2} - 1)^{-1} \), so at small angles \( \delta T/T \propto \alpha^{-5} \). In particular, for \( \alpha \approx 10^\circ \), we have
\[ \frac{\delta T}{T} \sim \frac{M_{bh}}{M_u} \quad (\alpha \approx 10^\circ), \] 
whereas for \( \alpha \approx 90^\circ \) we have a much lower effect \( \delta T/T \sim M_{bh}/(M_u z_{eq}) \). Here \( M_u \equiv G^{-1}t_0 \) is approximately the mass of our observable universe.

Let us denote by \( R_{\text{max}} \) the radius of the largest wall that may have existed in our observable universe (up to distances \( d_0 = 3t_0 \)). How massive can this
From (39), and using $\delta T/T \lesssim 10^{-5}$, we have

$$M_{\text{max}} \lesssim 10^{-5} M_u \approx 10^{52} g.$$  \hfill (40)

From (37) and (40) we can see that $R_{\text{max}}$ must be less than $t_{\text{eq}}$ even if $\sigma$ is as low as the electroweak scale. Using

$$\frac{dN}{dV} = \frac{\nu}{R^{5/2} t_{\text{eq}}^{3/2} (t_0)} (t_{\text{eq}})^2 dR,$$

the number of walls with size larger than $R$ within a volume $4\pi/3 (3 t_0)^3$ is

$$N_{\geq R} = 24\pi \nu \frac{t_{\text{eq}}^{1/2} t_0}{R^{3/2}}.$$ 

Therefore, the larger wall that we can expect to find in the observable universe has radius $R_{\text{max}} \approx \left[24\pi \nu t_{\text{eq}} q^{1/2} t_0\right]^{2/3}$, and Eq. (41) yields the bound

$$\nu^{4/3} \frac{\sigma}{m^3} \lesssim 10^{-64}. \hfill (41)$$

Let us now turn to monopole pair production during inflation. Unlike walls and strings, monopoles are not stretched to enormous sizes by the inflationary expansion. They are just diluted. An immediate consequence is that the only monopoles that will be relevant at the end of inflation are the ones that have been produced during the last expansion time. This density is given by

$$n = H^2 A e^{-2\pi m/H}, \hfill (42)$$

where $m$ is the mass of the monopole (here, as in the previous sections, the exponent $2\pi m H^{-1}$ is the action of the relevant instanton.)

A constraint on the abundance of nucleated monopoles comes from the fact that they should not recreate the monopole problem that inflation was aimed to solve. Of course this can be easily achieved by choosing the ratio $m/H$ to be sufficiently large, but as we shall see, one does not necessarily have to impose that.

Actually, the number density of monopoles at the time of reheating depends very much on the details of how inflation ended. Consider, as an example, a model in which the usual exponential expansion is followed by a
short period of power law expansion $a \propto t^p$ that starts at time $t_1$ and ends at time $t_2$ ($p$ may be larger that 1, in which case we have power law inflation, but it need not be). At time $t_2$ reheating is completed and we enter the usual radiation dominated era.

It is clear that, if no monopoles are created during the power law epoch, we will have, at time $t_2$,

$$n(t_2) = n(t_1) \cdot f^{3p},$$

where $n(t_1)$ is given by [11] and

$$f \equiv \left(\frac{t_1}{t_2}\right) \approx 1.6N^{1/2} \frac{T_r^2}{H m_p}.$$

Here $N$ is the effective number of massless degrees of freedom at reheating and we have used $t_1 = pH^{-1}$ and $t_2 = 2p m_p (10N)^{-1/2} T_r^{-2}$, with $m_p$ the Planck mass and $T_r$ the reheating temperature. Even if we assume that monopoles continue to be produced at a rate per unit volume given by

$$A H^4 e^{-2\pi m H},$$

where $H \equiv \dot{a}/a$ is the instantaneous expansion rate, it is not difficult to show (for $2\pi m H^{-1} > 3p - 4$) that the density of monopoles during the power law era is dominated by the ones that were already present at $t_1$. Therefore Eq. (43) is still valid in this case.

After reheating, the monopoles continue to be diluted as $a^{-3}$ whereas the density in radiation decays as $a^{-4}$. Therefore the contribution of the monopoles to the density parameter at the time $t_{eq} \approx 10^{10} s$ given by

$$\Omega_m = \frac{m \cdot n(t_2)}{\rho_c(t_2)} \left(\frac{t_{eq}}{t_2}\right)^{1/2} \approx 10^{29} f^{3[p-\frac{1}{2}]^3/2} \left[\frac{H}{m_p}\right]^{3/2} A e^{-2\pi m H} \frac{m}{m_p}.$$  

The Parker bound [18] requires $\Omega_m \lesssim 10^{-4} (m/m_p)$ (assuming Dirac charges for the monopoles), so we must impose

$$\left[\frac{H}{m_p}\right]^{3/2} f^{3[p-\frac{1}{2}]^3/2} A e^{-2\pi m H} \lesssim 10^{-33}.$$  

As mentioned above, this bound can be trivially satisfied by supressing the creation of monopoles with a sufficiently large $m/H$ ratio. Alternatively, if $p > 1/2$, then $\Omega_m$ will decrease with the reheating temperature through the supression factor $f$. For instance, taking $p = 3.5$ and $f \sim 10^{-4}$ the bound is automatically satisfied without any assumptions on $m/H$.  

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5 Conclusions

In this paper we calculated the probability of black hole formation by a string loop spontaneously nucleated during inflation. The calculation was based on the quantum theory of perturbations on strings in de Sitter space developed in Ref. [7]. The result is given in Eq. (11) and in Fig. 1. We then used the probability (11) to derive observational constraints on the mass parameter of the string and on the loop density parameter $\nu$. The strongest constraint comes from the emission of $\gamma$-rays by evaporating black holes. Additional constraints are due to the absence of characteristic hot and cold spots that would be produced by large oscillating loops on the microwave sky. The gravitational radiation background generated by the loops imposes no new constraints. The excluded region of the parameter space is sketched in Fig. 4.

We have also briefly discussed the cosmological implications of nucleated domain wall bubbles and monopole-antimonopole pairs. For sufficiently large bubbles, the formation of black holes is inevitable. Although these black holes can have enormous masses, their effect on structure formation and on the background radiation anisotropy is diminished due to the compensation effect [17]. The predicted density of the monopoles is given by Eq. (4). Depending on the values of the parameters, it can be too high, negligible, or it can lie in the interesting range and be potentially observable.

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Appendix A

In this Appendix we study linear perturbations to a circular loop of string in flat space. Before going to the particular case of a circular loop, we shall consider perturbations to an arbitrary string configuration whose worldsheet is given by $x^\mu(\xi^a)$ ($\xi^a$ are arbitrary coordinates on the worldsheet). This will
be a trivial extension of the general results given in Refs. [6, 7] for the case of domain walls.

Following Ref. [6], we parametrize the perturbed worldsheet $\tilde{x}^{\mu}$ as

$$\tilde{x}^{\mu}(\xi) = x^{\mu}(\xi) + \sum_{A=1}^{2} n^{A\mu} \phi^{A},$$

(A1)

where $n^{A\mu}$ are the two vectors normal to the unperturbed worldsheet. They satisfy

$$n^{A\mu} \partial_{a} x_{\mu} = 0, \quad n^{A\mu} n^{B}_{\mu} = \delta^{AB}.$$  (A2)

The perturbation fields $\phi^{A}$ have the meaning of normal displacements to the worldsheet, as measured by an observer that is moving with the unperturbed string.

The dynamics of the string is governed by the Nambu action

$$S[x^{\mu}(\xi)] = -\mu \int \sqrt{-g} \, d^{2}\xi,$$  (A3)

where $g$ is the determinant of the metric induced on the worldsheet $g_{ab} = \partial_{a} x^{\mu} \partial_{b} x_{\mu}$. The equations of motion that result from (A3) are well known

$$\Box x^{\mu}(\xi^{a}) = 0,$$  (A4)

where $\Box$ stands for the covariant d’Alembertian on the worldsheet. Multiplying (A4) by $n^{A}_{\mu}$ and integrating by parts one obtains

$$g^{ab} K_{ab}^{A} = 0,$$  (A5)

where $K_{ab}^{A} \equiv -\partial_{a} n^{A\mu} \partial_{b} x_{\mu}$ is the extrinsic curvature corresponding to the normal $n^{A}$.

The effective action for the perturbation fields $\phi^{A}$ on a given background $x^{\mu}(\xi)$ can be obtained by introducing (A1) into the action (A3) and then expanding to quadratic order in $\phi^{a}$. After some lengthy algebra, the result can be written as

$$S[\tilde{x}^{\mu}] = S[x^{\mu}] + S_{\phi}.$$  

The first term is just the action for the unperturbed solution, while the second is given by

$$S_{\phi} = -\mu \int \sqrt{-g} \left[ \frac{1}{2} \phi_{,a}^{A} \phi^{A},^{a} - \frac{1}{2} K_{ab}^{A} K^{Bab} \phi^{A} \phi^{B} + S \right] d^{2}\xi.$$  (A6)
with
\[ S = \frac{1}{2} \phi^A \phi^B n^C \partial_{\mu} n^A \partial^\mu n^B + \phi^A \phi^B n^A \partial_{\mu} n^B. \]

In deriving this effective action we have used the equations of motion (A5) to eliminate the terms linear in \( \phi^A \). From (A6) we see that, in general, the two fields \( \phi^A \) are coupled to each other in a complicated way.

Particularizing to the case of a circular loop, matters will simplify considerably. The unperturbed worldsheet in cartesian coordinates \((t, x, y, z)\) is given by
\[ x^\mu = (t, R_c \cos \theta \cos t, R_c \sin \theta \cos t, 0). \] (A7)

Here \( R_c \) is just the radius of the loop at \( t = 0 \) (when the loop is at rest). The two vectors normal to the worldsheet, can be chosen as
\[ n^{(1)}_\mu = \frac{1}{\cos \frac{t}{R_c}} (\sin \frac{t}{R_c}, \cos \theta, \sin \theta, 0), \]
\[ n^{(2)}_\mu = (0, 0, 0, 1). \] (A8)

The first one is a radial normal vector, while the second one is transverse to the plane of the loop. It is easy to see that in this case \( K_{ab}^{(2)} = 0 \) and \( S = 0 \), so the fields \( \phi^{(1)} \) and \( \phi^{(2)} \) decouple from each other. They simply behave like free scalar fields in the curved geometry of the unperturbed worldsheet.

In particular, the transverse perturbation behaves like a massless minimally coupled field. The corresponding equation of motion
\[ \Box \phi^{(2)} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{ab} \partial_{\nu} \phi^{(2)}) = 0 \] (A9)
can be solved trivially. The metric on the worldsheet is given by
\[ ds^2 = g_{ab} d\xi^a d\xi^b = \cos^2(t/R_c) [-dt^2 + R_c^2 d\theta^2]. \]

Introducing this metric in (A9) we obtain a flat-space wave equation
\[ \ddot{\phi}^{(2)} - \frac{1}{R_c^2} \phi^{(2)} = 0, \] (A10)
where a prime denotes derivative with respect to \( \theta \). Eq.(A10) is not surprising, since in 1+1 dimensions the conformal coupling is the same as the
minimal coupling (see e.g. [19]). The mode solutions of (A10) are just standing-waves that oscillate with constant amplitude. Note that $\phi^{(2)}$ coincides with $\Delta_r$ of Section 2, so we have the following result: *transverse perturbations to a circular loop oscillate with constant amplitude as the loop collapses.*

The equation of motion for the radial perturbations $\phi^{(1)}$ can be found from (A6) (with $K_{ab}^{(2)} = S = 0$). We have

$$\Box \phi^{(1)} + K_{ab}^{(1)} K^{(1)ab} \phi^{(1)} = 0. \tag{A11}$$

The extrinsic curvature can be obtained from (A7) and (A8):

$$K^{(1)}_{tt} = \frac{1}{R_0}, \quad K^{(1)}_{\theta\theta} = R_0, \quad K^{(1)}_{t\theta} = 0.$$ 

The proper perturbation $\phi^{(1)}$, as measured by a local observer that is moving with the string, is related to the radial perturbation $\Delta_r$ defined in Section 2 through a Lorentz contraction factor

$$\Delta_r = \phi^{(1)} \sqrt{1 - \dot{R}^2}.$$ 

Writing (A11) in terms of $\Delta_r$ and decomposing in Fourier modes [as in (4)] we have, after some algebra,

$$\ddot{\Delta}_r + \frac{2}{R_c} \tan \left( \frac{t}{R_c} \right) \dot{\Delta}_r + \frac{(L^2 - 1)}{R_c^2} \Delta_r = 0.$$ 

The general solution to this equation can be written as

$$\Delta_L(t) = \frac{\Delta_L(0)}{L} \left[ \sqrt{1 - \frac{R^2}{R_c^2}} \sin \frac{Lt}{R_c} + L \frac{R}{R_c} \cos \frac{Lt}{R_c} \right] +$$

$$\dot{\Delta}_L(0) \frac{R_c}{1 - L^2} \left[ \sqrt{1 - \frac{R^2}{R_c^2}} \cos \frac{Lt}{R_c} - L \frac{R}{R_c} \sin \frac{Lt}{R_c} \right], \tag{A12}$$

where $\Delta_L(0)$ and $\dot{\Delta}_L(0)$ are the values of the perturbation and its derivative at $t = 0$ (we have dropped the subindex $r$), and $R = R_c \cos(t/R_c)$ is the radius of the unperturbed loop.
In the cosmological problem that we are interested in, $t = 0$ corresponds to the moment when the loop enters the horizon and starts collapsing. The probability distribution for the initial conditions $\Delta L(0)$ will be given by (5) and (7). We also need the initial conditions $\dot{\Delta} L(0)$. When the wavelength of a perturbation is within the horizon we expect $\langle \dot{\Delta}^2 \rangle = \left( L/R_c \right)^2 < \Delta^2 L$, where the brackets indicate average over one oscillation period. Therefore, by the time the loop enters the horizon, the perturbations will have developed velocities of order $\dot{\Delta}(t = 0) \sim (L/R_c)\Delta L(t = 0)$. Using this in (A12) we have

$$\Delta L(R << R_c) \approx \frac{\Delta L(R_c)}{L}.$$ 

That is to say, the radial perturbations shrink by a factor of $L$ as the loop collapses.

**Appendix B**

Damping mechanisms, such as gravitational radiation and friction, may decrease the amplitude of the perturbations on a circular loop of string and therefore increase the probability of black hole formation. In this appendix we briefly discuss the limits in which the effects of gravitational radiation and friction can be ignored.

Gravitational radiation can only smooth out perturbations on wavelengths smaller than $\lambda g \equiv \gamma g G \mu t$, where $\gamma g \sim 100$ (see e.g. [20]). Taking $t \sim R_c$, this corresponds to wave numbers larger than $L g \equiv 2\pi(\gamma g G \mu)^{-1}$. It is clear, from the physical interpretation of $L_*$ discussed in Section 2, that gravitational damping can be neglected in the calculation of $P_{bh}$ providing that $L_g > L_*$. This inequality gives

$$G \mu < (2\pi \gamma g^{-1})^{5/3}(16\pi^2 \alpha^2 B)^{1/3} \approx (17\gamma g^{-1})^{5/3}(\alpha^2 B)^{1/3}. \quad (B1)$$

Typically, the right hand side of (B1) will be of order $10^{-2}$ or larger, so this is not a very strong condition.

Similarly we can consider the effects of friction. The dominant contribution to friction comes from Aharonov-Bohm scattering of ambient particles off the string in the radiation dominated era (see Ref. [21]). Assuming a situation in which the wavelength of the perturbations is conformally stretched by the expansion of the universe (as it is in the present case), it is shown
in Ref.[22] that friction can only be important for wavelengths smaller than \(\lambda_f \equiv \gamma_f (G\mu T)^{-1}\). Here \(T\) is the temperature and \(\gamma_f\) is a numerical coefficient of order one. The physical reason is that for \(\lambda > \lambda_f\), the friction term in the equations of motion will “switch off” before the perturbations cross the cosmological horizon and start to oscillate.

The wave number corresponding to \(\lambda_f\) is

\[
L_f \equiv 2\pi \gamma_f^{-1}RTG\mu = \gamma_f^{-1}GMT_c,
\]

where \(T_c\) is the temperature of the Universe at the time \(t_c\) when the loop crosses the horizon, \(M\) is the mass of the loop at \(t_c\) (which is also the mass of the resulting black hole) and we have used that the product \(RT\) is independent of time before the loop crosses the horizon. Using \(T_c \approx (10G\mathcal{N})^{-1/4}t_c^{-1/2}\) and \(t_c \approx R_c = M(2\pi\mu)^{-1}\) (where \(\mathcal{N} \sim 10^2\) is the effective number of massless degrees of freedom) we have

\[
L_f \equiv (10\mathcal{N})^{-1/4}\gamma_f^{-1}(2\pi G\mu)^{1/2}\left(\frac{M}{m_p}\right)^{1/2}.
\]

Similarly to the case of gravitational radiation, friction can be neglected in the calculation of \(P_{bh}\) providing that \(L_f > L_\star\). This condition is equivalent to

\[
G\mu \gtrsim \left(\frac{\mathcal{N}\gamma_f}{230}\right)^{5/18}(\alpha^2 B)^{-2/9}\left(\frac{m_p}{M}\right)^{5/9}. \tag{B2}
\]

Therefore, the range of values of \(G\mu\) for which friction can be ignored depends on the mass of the black hole that one wishes to consider.

Putting (B1) and (B2) together and taking \(\gamma_f \sim 1, \gamma_g \sim 10^2\), we conclude that damping is unimportant for values of \(G\mu\) in the range

\[
(\alpha^2 B)^{-2/9}\left(\frac{m_p}{M}\right)^{5/9} \lesssim G\mu < 5 \cdot 10^{-2}(\alpha^2 B)^{1/3}.
\]

**Figure captions**

- **Fig. 1** The function \(P_{bh}(L_\star)\) can be computed numerically from Eq.(11) to arbitrary precision. We see that, essentially, the probability of black hole formation decays as an exponential function of \(L_\star\).
• **Fig. 2** We represent the boundary of the region excluded by the constraint (22) in parameter space ($\alpha G\mu$ versus $B$) for $N = 27$ (dashed line), $N = 28$ (solid line) and $N = 30$ (dotted line). The result is not very sensitive to the value of $N$ and, quite generically, the bound is satisfied provided that $\alpha G\mu \lesssim 10^{-4}$.

• **Fig. 3** A plot of the number of loops $N_{>\Theta}$ that would be seen in the sky at an angular size $\Theta$.

• **Fig. 4** Schematic plot of the region excluded in parameter space by the observational constraints on loop nucleation. The horizontal boundary at $G\mu \approx 10^{-6}$ is due to distortions in the CMB caused by loops with angular size larger than $10^\circ$. The slanted part of the boundary is due to ‘unresolved’ loops, with angular size less than $10^\circ$. The horizontal boundary at $G\mu \approx 10^{-4}$ comes from limits on the abundance of primordial black holes. The thick line is meant to represent values of the parameters that roughly satisfy the normalizations (35) and (36), and which may be adequate for structure formation.

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