GENERALIZED GROSS–PERRY–SORKIN–LIKE SOLITONS

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Abstract

In this paper, we present a new solution for the effective theory of Maxwell–Einstein–Dilaton, Low energy string and Kaluza–Klein theories, which contains among other solutions the well known Kaluza–Klein monopole solution of Gross–Perry–Sorkin as special case. We show also the magnetic and electric dipole solutions contained in the general one.
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1. INTRODUCTION

One of the exact solutions of the vacuum Einstein field equations in 5-dimensional gravity is the Gross–Perry–Sorkin spacetime (GPS) [1], which is stationary, everywhere regular and without event horizon. Actually, it represents a monopole, although there is no reason why the charge carried out by the monopole should be labeled “magnetic”. It might equally well be deemed “electric” and $A_\mu$ treated as a potential for the dual field $*F_{\mu\nu}$.

As it is well known [1], the monopoles, in addition to charge, are characterized by the topology of their spatial solutions. They carry one unit of Euler character, and therefore, one can construct stationary dipole solutions.

5D gravity is one example of the unified theories of electromagnetism and General Relativity. Einstein–Maxwell–Dilaton, Kaluza–Klein, and Low energy string theories are also examples of this kind of unified theories. Mathemetically their effective actions in four dimensions are very similar, they differ in that the value of the scalar dilatonic field coupling constant is in each case different. Thus we can write the four dimensional effective action for all of the above mentioned theories in the form:

$$S = \int d^4x \sqrt{-g}[-R + 2(\nabla \Phi)^2 + e^{-2\alpha \Phi} F_{\mu\nu} F^{\mu\nu}]$$ (1.1)

where $R$ is the Ricci scalar, $\Phi$ is the scalar dilaton field, $F_{\mu\nu}$ is the Faraday electromagnetic tensor, and $\alpha$ is the dilaton coupling constant. For $\alpha = 0$ we have the effective action of the Einstein-Maxwell-Dilaton theory, here, the scalar dilaton field appears minimaly coupled to electromagnetic one. $\alpha = 1$ represents the Low energy string theory, where only the $U(1)$-vector gauge field has not been dropped out, and $\alpha = \sqrt{3}$ reduces the action (1.1) to that of the 5D Kaluza-Klein theory. As is well known, all of these theories unify gravity and electromagnetism. It is interesting to note that for the String and Kaluza-Klein theories, the electromagnetic field can not be decoupled from the scalar dilaton field.

The Gross–Perry–Sorkin solution has already been studied in [1], and recently harmonic maps ansatz [3] has been applied to the action (1.1) in order to find exact solutions of this kind to its corresponding field equations [3].
In this work we present two new classes of exact solutions of the (1.1) associated field equations, for arbitrary values of the \( \alpha \) coupling constant, the solutions are written in terms of a harmonic map, in such a way that for spacial values of this harmonic map, the solution represents monopoles, dipoles, quadrupoles etc. If we choose the harmonic map in order to have monopoles, for the particular case \( \alpha = \sqrt{3} \) it reduces just to the Gross–Perry–Sorkin solution. This new solitonic solution is the spacetime of a monopole for some values of the free parameter \( \delta \) with \( \alpha \) arbitrary. In general it represents a soliton spacetime with a singularity at \( r = 2m \). Nevertheless, the influence of the scalar dilaton field is important only in regions near the singularity.

The plan of the paper is as follows. In section 2 we review the harmonic map ansatz. In Sec. 3 we review very briefly the Gross–Perry–Sorkin solution. In Sec. 4 we present the new classes of solutions and its correspondig spacetimes in each of the mentioned theories. In Sec. 5 we discuss the results and present the conclusions.

2. HARMONIC MAP ANSATZ

We begin by considering the Papapetrou metric in the following parametrization:

\[
dS^2 = \frac{1}{f} \left[ e^{2k} (d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2 \right] - f dt^2
\] (2.1)

The harmonic map ansatz supposes that all terms of the metric depend on a set of functions \( \lambda_i \), \( (i = 1...p) \), such that these functions \( \lambda_i \) fulfills the Laplace equation

\[
\Delta \lambda = (\rho \lambda_{\bar{z}})_{,z} + (\rho \lambda_{z})_{,\bar{z}} = 0
\] (2.2)

where

\[
z = \rho + i\zeta.
\] (2.3)

Thus the field equations derived from the Lagrangian (1.1) reduce to equations in terms of the \( \lambda_i \) functions. In general these equations are easier to solve. The advantage of this
method is that it is possible to generate exact space times for each solution of the Laplace equation.

Fortunately, the harmonic map determines the gravitational and the electromagnetic potentials in such a way, that we can choose them to have electromagnetic monopoles, dipoles, quadripoles, etc [3].

In the Papapetrou parametrization, the field equations reduce to one equation for $f$

$$\Delta \ln f = e^{-2\alpha\Phi} \frac{1}{\rho} f A_{\varphi,z} A_{\varphi,\bar{z}}$$

and to one for the function $k$

$$2k_{,z} = 4\rho (\Phi_{,z})^2 - e^{-2\alpha\Phi} \frac{f}{\rho} (A_{\varphi,z})^2 + \rho (\ln f_{,z})^2.$$  \hspace{1cm} (2.5)

with one equation for $k_{,\bar{z}}$ with $\bar{z}$ in place of $z$. Let us suppose that the components of the Papapetrou metric depend only on one harmonic map $\lambda$. Here we present two solutions of these field equations (2.4) and (2.5). By solving the general field equations coming from the metric (2.1) in terms of one harmonic map $\lambda$ with no electromagnetic field at all, we arrive at a solution given by [3]

$$f = e^{\lambda} ; \quad k_{,z} = \frac{\rho}{2} (4\alpha^2 a^2 + 1) (\lambda_{,z})^2$$

with the following form for the scalar dilaton field:

$$e^{2\alpha\Phi} = k_0^2 e^{2\alpha^2 a \lambda} ; \quad a = \text{const.}$$  \hspace{1cm} (2.7)

with $\lambda$ a harmonic map, i.e. a solution of the equation (2.2).

The field equation for the function $k$ is always integrable if $\lambda$ is a solution of the Laplace equation. (For more details of the method see [2] and [3]).

The second solution we want to deal with here, contains electromagnetic field. It is given by

$$f = \frac{1}{(1 - \lambda)^{2+\alpha^2}}, \quad k = 0, \quad A_{\varphi,z} = Q \rho \lambda_{,z}, \quad A_{\varphi,\bar{z}} = -Q \rho \lambda_{,\bar{z}}$$  \hspace{1cm} (2.8)
and the corresponding form for the scalar dilaton field given by

$$e^{-2\alpha\Phi} = \frac{e^{-2\alpha\Phi_0}}{(1 - \lambda)^{\frac{2\alpha^2}{1 + \alpha^2}}}.$$  \hspace{1cm} (2.9)

where the magnetic charge is related with the scalar one by $Q^2 = \frac{4e^{2\alpha\Phi_0}}{1 + \alpha^2}$. This solution, contains among others, the GPS one as spacial case.

In the next section we briefly review the Kaluza–Klein monopole.

### 3. GROSS–PERRY–SORKIN MONOPOLE

The Kaluza–Klein monopole, known as Gross–Perry–Sorkin solution, represents the simplest and basic soliton, it is a generalization of the self–dual euclidean Taub–Nut solution \[4\], and is described by the following metric:

$$ds^2 = -dt^2 + (1 + 4m r)^{-1}(dx^5 + 4m(1 - \cos\theta)d\phi)^2 + (1 + 4m r)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)$$  \hspace{1cm} (3.1)

where $(r, \theta, \phi)$ are polar coordinates. For $dt = 0$ the Taub–Nut instanton is obtained. As it is well known, the coordinate singularity at $r = 0$ is absent if $x^5$ is periodic with period $16\pi m = 2\pi R$ \[5\], with $R$ the radius of the fifth dimension. Thus

$$m = \frac{\sqrt{\pi G}}{2e}$$  \hspace{1cm} (3.2)

and the electromagnetic potential $A_\mu$ is that of a monopole:

$$A_\phi = 4m(1 - \cos\theta)$$  \hspace{1cm} (3.3)

and

$$B = \frac{4mr}{r}$$  \hspace{1cm} (3.4)

The magnetic charge of the monopole is fixed by the radius of the Kaluza–Klein circle

$$g = \frac{4m}{\sqrt{16\pi G}} = \frac{R}{2\sqrt{16\pi G}} = \frac{1}{2e}.$$  \hspace{1cm} (3.5)
Moreover, the mass of the soliton is given by

\[ M = \frac{m}{G} \]  

(3.6)

The Gorss–Perry–Sorkin soliton solutions are soliton solutions also of the effective four dimensional theory, for the four metric \( g_{\mu\nu} \) and a massless scalar dilaton field \( \Phi \), as well, with

\[
d s_4^2 = -\frac{d t^2}{\sqrt{1 + 4m/r}} + \sqrt{1 + \frac{4m}{r}} (d r^2 + r^2 d \theta^2 + r^2 \sin^2 \varphi^2) \]  

(3.7)

and

\[ \Phi = \frac{\sqrt{3}}{4} \ln \left(1 + \frac{4m}{r}\right) \]  

(3.8)

Although this is a singular solution of the effective four dimensional theory it is a perfectly sensible soliton. The singularity arises because the conformal factor \( e^{2\alpha \Phi} \) is singular at \( r = 0 \).

### 4. GENERALIZED GROSS–PERRY–SORKIN SOLUTION

In the Boyer-Lindquist coordinates:

\[ \rho = \sqrt{r^2 - 2mr \sin \theta}, \quad \zeta = (r - m) \cos \theta. \]  

(4.1)

the group of metrics we want to deal with can be written as follows:

\[
d s_4^2 = (1 - \lambda)^{\frac{2}{1 + \alpha^2}} \left\{ \left[ 1 - \frac{2m}{r} + \frac{m^2 \sin^2 \theta}{r^2} \right] \left[ \frac{d r^2}{1 - \frac{2m}{r}} + r^2 d \theta^2 \right] \right. \\
\left. + (1 - \frac{2m}{r}) r^2 \sin^2 \theta d \varphi^2 \right\} - \frac{d t^2}{(1 - \lambda)^{\frac{2}{1 + \alpha^2}} \lambda} \]  

(4.2)

with the scalar dilaton field given by

\[ e^{2\Phi} = \frac{e^{2\Phi_0}}{(1 - \lambda)^{\frac{2}{1 + \alpha^2}}} \]  

(4.3)
A. Case $m = 0$

This corresponds to conformally spherically symmetric spacetimes. Choosing $m = 0$ in (4.2), the metric reduces to:

$$ds^2 = (1 - \lambda)^{\frac{2}{1+\alpha^2}} \{dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\} - \frac{dt^2}{(1 - \lambda)^{\frac{2}{1+\alpha^2}}}$$

(4.4)

with $\lambda$ a solution of the Laplace equation:

$$\Delta \lambda = (r^2 \lambda_r)_r + \frac{1}{\sin\theta}(\sin\theta \lambda_\theta)_\theta = 0$$

(4.5)

and as is well known, the spherical armonics are all solutions of (4.5). Moreover, it is easy to show, that it is possible to construct arbitrary monopoles, by performing the following identification:

$$A_{3,z} = Q \rho \lambda_{\zeta}$$

(4.6)

$$A_{3,\bar{z}} = -Q \rho \lambda_{\bar{\zeta}}$$

(4.7)

with $z, \rho$ and $\zeta$ given by (2.3) and (4.1) respectively. The Gross–Perry–Sorkin is thus obtained by choosing the electromagnetic potential $A_3 = Q(1 - \cos\theta)$ and $\alpha = \sqrt{3}$, consequently

$$ds^2 = (1 + \frac{4M}{r})^4 (dr^2 + r^2d\theta^2 + r^2\sin^2\varphi^2) - (1 + \frac{4M}{r})^{-\frac{1}{2}} dt^2; \quad e^{2\Phi} = \frac{e^{2\Phi_0}}{(1 + \frac{4M}{r})^{\frac{1}{2}}} ,$$

(4.8)

being $\alpha$ is the dilaton coupling constant. As mentioned in the introduction, for $\alpha = 0$ this is a solution in the framework of the Einstein–Maxwell plus Dilaton theory, with the same $A_3$:

$$ds^2 = (1 + \frac{4M}{r})^2 (dr^2 + r^2d\theta^2 + r^2\sin^2\varphi^2) - (1 + \frac{4M}{r})^{-2} dt^2; \quad e^{2\Phi} = e^{2\Phi_0} ,$$

(4.9)

for $\alpha = 1$, it reduces to a low energy string theory solution,

$$ds^2 = (1 + \frac{4M}{r})(dr^2 + r^2d\theta^2 + r^2\sin^2\varphi^2) - (1 + \frac{4M}{r})^{-1} dt^2; \quad e^{2\Phi} = \frac{e^{2\Phi_0}}{(1 + \frac{4M}{r})} ,$$

(4.10)
Finally, for $\alpha = \sqrt{3}$, a 5-D Kaluza-Klein solution is obtained in the sense that in this particular case

$$dS_5^2 = \frac{1}{I} dS_4^2 + I^2 (A_\mu dx^\mu + dx^5)^2$$

(4.11)

with

$$I^3 = e^{2\alpha \Phi}$$

(4.12)

This last fact suggests that in higher dimensional theories the 4-dimensional part should be multiplied by a conformal factor related to the dilaton field in order to be physically meaningful.

Further generalizations can be carried out by taking $\lambda = \frac{q \cos \theta}{r}$, $m = 0$:

$$ds^2 = (1 - \frac{q \cos \theta}{r})^{2\alpha} \{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\} - \frac{dt^2}{(1 - \frac{q \cos \theta}{r})^{1 + \alpha^2}}.$$  

(4.13)

and the scalar dilaton field takes the form

$$e^{-2\Phi} = \frac{e^{-2\Phi_0}}{(1 - \frac{q \cos \theta}{r})^{\frac{2\alpha}{1 + \alpha^2}},}$$

(4.14)

with $\alpha = 0, 1, \sqrt{3}$. This metric contains a magnetic dipole moment whose magnetical four potential is $A_3 = -\frac{1}{2\sqrt{2}} \frac{\sin^2 \theta}{r}$.

**B. Case $m \neq 0$**

The same can be done with $m \neq 0$. The monopole solutions are obtained by choosing

$\lambda = q \ln(1 - \frac{2m}{r})$ (but now with gravitational mass $\frac{2mq}{1 + \alpha^2}$), with $A_3 = -\frac{1}{\sqrt{2}} Q(1 - \cos \theta)$, thus, the corresponding metric is given by

$$ds_4^2 = (1 - q \ln(1 - \frac{2m}{r}))^{\frac{2}{1 + \alpha^2}} \{(1 - \frac{2m}{r}) + \frac{m^2 \sin^2 \theta}{r^2} \} \left\{ [1 - \frac{2m}{r} + \frac{m^2 \sin^2 \theta}{r^2}] \left[ \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 \right] \right\}

+ (1 - \frac{2m}{r}) r^2 \sin^2 \theta d\phi^2 \}

- \frac{dt^2}{(1 - q \ln(1 - \frac{2m}{r}))^{\frac{2}{1 + \alpha^2}}}$$

(4.15)

with the scalar dilaton field given by
Moreover, in order to achieve a magnetic dipole solution, we identify $\lambda = \frac{q \cos \theta}{(r-m)^2 - m^2 \cos^2 \theta}$, with the following electromagnetic potential: $A_3 = \frac{1}{2\sqrt{2}} q \theta \frac{(r-m) \sin^2 \theta}{(r-m)^2 - m^2 \cos^2 \theta}$, which corresponds to a magnetic dipole field. The metric is given by

$$ds^2_4 = (1 - \frac{q \cos \theta}{(r-m)^2 - m^2 \cos^2 \theta}) \frac{1}{1+\alpha^2} \left\{ [1 - \frac{2m}{r} + \frac{m^2 \sin^2 \theta}{r^2}] \left[ \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 \right] ight. \\
+ \left. (1 - \frac{2m}{r}) r^2 \sin^2 \theta d\varphi^2 \right\} - \frac{dt^2}{(1 - \frac{q \cos \theta}{(r-m)^2 - m^2 \cos^2 \theta}) \frac{1}{1+\alpha^2}}$$

(4.17)

with the scalar dilaton field given by

$$e^{-2\Phi} = \frac{e^{-2\Phi_0}}{(1 - \frac{q \cos \theta}{(r-m)^2 - m^2 \cos^2 \theta}) \frac{1}{1+\alpha^2}}$$

(4.18)

This metrics are all asymptotically flat and also they are flat for $q = m = 0$.

5. OUTLOOK

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