ON SOME BUILDING BLOCKS OF HYPERGRAPHS: UNITS, TWIN-UNITS, REGULAR, CO-REGULAR, AND SYMMETRIC SETS

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Abstract. Here, we introduce and investigate different building blocks, named units, twin units, regular sets, symmetric sets, and co-regular sets in a hypergraph. Our work shows that the presence of these building blocks leaves certain traces in the spectrum and the corresponding eigenspaces of the connectivity operators associated with the hypergraph. We also show that, conversely, some specific footprints in the spectrum and in the corresponding eigenvectors retrace the presence of some of these building blocks in the hypergraph. The hypergraph remains invariant under the permutations among the vertices in some building blocks. These vertices behave similarly, in random walks on the hypergraph and play an important role in hypergraph automorphisms. Identifying similar vertices in certain building blocks results in a smaller hypergraph that contains some spectral information of the original hypergraph. The number of specific building blocks provides an upper bound of the chromatic number of the hypergraph. A pseudo metric is introduced to measure distances between vertices in the hypergraph by using one of the building blocks. Here, we use the concept of general connectivity operators of a hypergraph for our spectral study.

1. Introduction

Hypergraphs can be decomposed into a disjoint union of some subsets of vertices where each vertex has a similar incidence relationship with the hyperedges of the hypergraph. Additionally, these subsets of vertices behave identically in dynamical networks, random walks, etc. on hypergraphs. Here we study the subsets of similar vertices as building blocks of a hypergraph. We refer to them, as units, twin-units, regular sets, co-regular sets, and symmetric sets. The presence of the building blocks in a hypergraph is reflected in its eigenvalues and the corresponding eigenspaces. Here we develop a spectral method to decompose a hypergraph into different building blocks from its spectrum. We also show that the building blocks induce a smaller hypergraph, named contraction of the hypergraph, such that some eigenvalues of the original hypergraph can be derived from the smaller one. We find that an upper bound of the chromatic number of a hypergraph depends on the number of certain building blocks in it. We demonstrate that specific building blocks induce some hypergraph automorphisms. Using one of the building blocks, we introduce a pseudo metric that can be used to measure distances between the vertices in a hypergraph. We use the general operators associated with hypergraphs introduced in [2] for our spectral investigation. The study of a general operator covers the same for a wide range of different variations of the operator introduced in various literature.

A hypergraph \( H(V, E) \) is an ordered pair of sets, a vertex set \( V(\neq \emptyset) \) and an (hyper)edge set \( E \), such that \( E \subseteq P(V) \setminus \{\emptyset\} \), where \( P(V) \) is the power set of \( V \). Each \( v \in V \) and \( e \in E \) are called a vertex and hyperedge, respectively, of \( H \). The vertex set and the hyperedge set of a hypergraph \( H \) are also denoted by \( V(H) \) and \( E(H) \), respectively. If \( E(H) = \emptyset \) then \( H \) is called a null hypergraph. A hyperedge \( e \) is called a loop if \( |e| = 1 \). A vertex \( v \) is called an isolated vertex if there exists no hyperedge containing \( v \). In our study, we consider only non-null, finite
hypergraphs without any loop and isolated vertex. A hypergraph $H$ is called an $m$-uniform hypergraph if $|e| = m (\geq 2)$ for all $e \in E(H)$. The star of a vertex $v$ in $H$ is the set of edges $E_v(H) = \{ e \in E(H) : v \in e \}$. The degree of $v$ is $d(v) = |E_v(H)|$. For $u_1, \ldots, u_n \in V(H)$, we denote $E_{u_1}(H) \cap \ldots \cap E_{u_n}(H)$ as $E_{u_1 \ldots u_n}(H)$.

Here, in section 2 we introduced different building blocks of hypergraphs and demonstrate their importance in the hypergraph architecture. In Section 3, we study the relationships between the spectrum of a hypergraph and its building blocks. Using our results, we demonstrate how to compute the complete spectrum of a hypergraph from its building blocks in Section 3.3. With the help of the building blocks for some hypergraphs, we can reconstruct the hypergraph from its spectrum. This also has been shown with an example in the same section. The Section 4 is devoted to some applications of building blocks. In Section 4.1, we show that in random walks on a hypergraph, all the vertices in the same unit or in the twin units have the same hitting times. We introduce and study a pseudo metric on the set of vertices using the units in Section 4.2. In Section 4.3, we show that any upper bound of the chromatic number of the contraction of a hypergraph is also an upper bound for the same of the original hypergraph. Here we construct some independent sets using the building blocks. We study some hypergraph invariants in terms of building blocks in Section 5. In the same section, we also show that units and twin units lead to hypergraph automorphisms.

2. SOME BUILDING BLOCKS OF HYPERGRAPHS

In this section, we introduce some building blocks of a hypergraph.

2.1. Units in Hypergraph.

**Definition 2.1.** Let $H$ be a hypergraph. For $E_0 \subseteq E(H)$ if

$$W_{E_0} = \{ v : v \in \bigcap_{e \in E_0} e \text{ and } v \notin f \text{ for all } f \in E \setminus E_0 \} \neq \emptyset$$

then $W_{E_0}$ is called a unit of the hypergraph $H$. The collection of all the units in a hypergraph $H$ is denoted by $\mathcal{U}(H)$. The set $E_0$ is called the generating set of $W_{E_0}$.

In the Definition 2.1, we see that a unit is generated by a generating set. Now we figure out how to get back the generating set from a unit. The Definition 2.2 and the Proposition 2.3 help us for the same.

**Definition 2.2.** Suppose that $H$ is a hypergraph. For any $(\emptyset \neq) U \subseteq V(H)$, the star of $U$ is denoted by $E_U$ and is defined by

$$E_U = \{ e \in E : U \subseteq e \}.$$

For instance, in Figure 1, $E_{\{6\}} = E_{\{15\}} = E_{\{5,6\}} = \{e_3\}$, $E_{\{1,2\}} = \{e_5, e_6\}$.

**Proposition 2.3.** Suppose that $H$ is a hypergraph. For all $W_{E_0} \in \mathcal{U}(H)$, we have $E_{W_{E_0}} = E_0$.

**Proof.** For any $e \in E_0$, we have $W_{E_0} \subseteq e$. Therefore, $e \in E_{W_{E_0}}$ and thus, $E_0 \subseteq E_{W_{E_0}}$. If $e \in E_{W_{E_0}}$ then $W_{E_0} \subseteq e$. Thus, $e \in E_0$, since otherwise $W_{E_0} \cap e = \emptyset$. Hence $E_{W_{E_0}} \subseteq E_0$. Therefore, $E_{W_{E_0}} = E_0$. ■

**Proposition 2.4.** If $H$ is a hypergraph and $W_{E_1}, W_{E_2} \in \mathcal{U}(H)$ with generating sets $E_1$ and $E_2$, respectively, then $W_{E_1} = W_{E_2}$ if and only if $E_1 = E_2$.

**Proof.** Since $W_{E_1} = W_{E_2}$ and $E_i = \{ e \in E(H) : W_{E_i} \subseteq e \}$ for $i = 1, 2$, then $E_1 = E_2$. The converse part directly follows from the Definition 2.1. ■

In Figure 1, we see that all the units are pairwise disjoint. Now we have the following proposition.

**Proposition 2.5.** For any two units $W_{E_1}, W_{E_2} \in \mathcal{U}(H)$, either $W_{E_1} = W_{E_2}$ or $W_{E_1} \cap W_{E_2} = \emptyset$.
Suppose that $W_{E_1} \neq W_{E_2}$. Thus, $E_1 \neq E_2$. Without loss of generality we assume that $E_1 \setminus E_2 \neq \emptyset$. Take any $e \in E_1 \setminus E_2$, we have $W_{E_1} \subseteq e$ and $W_{E_2} \cap e = \emptyset$. Therefore, $W_{E_1} \cap W_{E_2} = \emptyset$.

**Definition 2.6.** A pair of distinct units $W_{E_1}, W_{E_2} \in \mathcal{U}(H)$ are called unit-neighbours if $W_{E_1} \cup W_{E_2} \subseteq e$ for some $e \in E(H)$.

In Figure 1, $W_{E_6}$ and $W_{E_1}$ are unit-neighbours, since they are contained in $e_6$.

**Definition 2.7.** A pair of units $W_{E_1}, W_{E_2} \in \mathcal{U}(H)$ are called twin units if for all $e \in E_i$, there exists a unique $f \in E_j$ such that $e \setminus W_{E_i} = f \setminus W_{E_j}$ and vice versa.

In Figure 1, any two of the units, $W_{E_3}, W_{E_4}, W_{E_5}$ are twin units and $W_{E_6}, W_{E_7}$ are twin units. Here no pair of twin units are unit-neighbours to each other. In the next proposition we see that this is true in general.

**Proposition 2.8.** Twin units can not be unit-neighbours to each other.

**Proof.** Suppose that $W_{E_1}, W_{E_2} \neq W_{E_1} \in \mathcal{U}(H)$ are twin units. If possible let us assume that $W_{E_1}, W_{E_2}$ are unit-neighbours to each other. Then, there exists $e \in E(H)$ such that $W_{E_1} \cup W_{E_2} \subseteq e$. Now, there exists $f \in E_2$ such that $e \setminus W_{E_1} = f \setminus W_{E_2}$. Since $W_{E_1} \cup W_{E_2} \subseteq e$, we have $W_{E_2} \subseteq e \setminus W_{E_1}$. This contradicts $e \setminus W_{E_1} = f \setminus W_{E_2}$. Therefore, our assumption is wrong. Hence the proof follows.

In the above proof, we see that for any two twin units $W_{E_i}, W_{E_j} \in \mathcal{U}(H)$ there exists a bijection $f_{E_i,E_j}: E_i \rightarrow E_j$ such that for any $e \in E_i$ we have $e \setminus W_{E_i} = f_{E_i,E_j}(e) \setminus W_{E_j}$. We refer this bijection as the canonical bijection from $E_i$ to $E_j$. 

**Proposition 2.9.** For any hypergraph $H$, the set of vertices $V(H)$ can be expressed as the union of a unique disjoint collection of units.

**Proof.** Since $v \in W_{E_v(H)}$ for any $v \in V(H)$, the result follows from Proposition 2.5.
Clearly $\Omega(H) = \{W_{E_i}(H) : v \in V(H)\}$.

**Proposition 2.10.** In a hypergraph $H$, any hyperedge $e(\in E(H))$ either is itself an unit or can be expressed as disjoint union of units.

**Proof.** Suppose $e = \{v_1, v_2, \ldots, v_k\} \in E(H)$. Now there exists $W_{E_{v_i}}(H) \in \Omega(H)$ for $i = 1, 2, \ldots, k$. Since either $W_{E_{v_i}}(H) = W_{E_{v_j}}(H)$ or $W_{E_{v_i}}(H) \cap W_{E_{v_j}}(H) = \emptyset$ for all $i, j = 1, 2, \ldots, k$; the result follows. $\blacksquare$

We denote the number of units in an hyperedge $e \in E(H)$ as $n_e$. Let us consider a relation $\mathcal{R}$ on $\Omega(H)$, defined by

$$\mathcal{R} = \{(W_{E_1}, W_{E_2}) \in \Omega(H)^2 : W_{E_1}, W_{E_2} \text{ are twin units}\}.$$ 

Clearly $\mathcal{R}$ is an equivalence relation and the equivalence classes of $\mathcal{R}$ forms a disjoint partition of $\Omega(H)$. We denote the $\mathcal{R}$-equivalence class of a unit $W_{E_i}$ as $\mathcal{C}(W_{E_i})$. We denote the collection of all $\mathcal{R}$-equivalence classes in $\Omega(H)$ by $\mathcal{C}(\Omega(H))$. In Figure 1, $\mathcal{C}(\Omega(H)) = \{a_1, a_2, a_3, a_4, a_5\}$, where $a_1 = \{W_{E_1}, W_{E_2}, W_{E_3}\}$, $a_2 = \{W_{E_4}\}$, $a_3 = \{W_{E_5}\}$, $a_4 = \{W_{E_6}, W_{E_7}\}$ and $a_5 = \{W_{E_8}\}$.

**Proposition 2.11.** If $a \in \mathcal{C}(\Omega(H))$ then for all $W_{E_i} \in a$, $|E_i| = s_a$, a constant.

**Proof.** For any two $W_{E_i}, W_{E_j} \in a$, there exists canonical bijection $f_{E_i, E_j} : E_i \to E_j$. Thus, $E_i$ and $E_j$ are equivalent. This completes the proof. $\blacksquare$

Thus, each $a \in \mathcal{C}(\Omega(H))$ corresponds an $s_a (\in \mathbb{N})$. We refer $s_a$ as the generator size of $a$.

**Proposition 2.12.** If $W_{E_1}, W_{E_2}(\in \Omega(H))$ are unit-neighbours then for every pair of units, $W_{E_i} \in \mathcal{C}(W_{E_1})$ and $W_{E_j} \in \mathcal{C}(W_{E_2})$ are unit-neighbours.

**Proof.** Now there exists $e \in E(H)$ such that $W_{E_1} \cup W_{E_2} \subseteq e$. Take any $W_{E_i} \in \mathcal{C}(W_{E_1})$, and $W_{E_j} \in \mathcal{C}(W_{E_2})$. Then $W_{E_2} \subseteq e \setminus W_{E_1} = f \setminus W_{E_1}$, for some $f \in E_i$. Therefore, $W_{E_i} \cup W_{E_2} \subseteq f$ and thus, $W_{E_i}, W_{E_2}$ are unit-neighbours. Similarly, $W_{E_i}, W_{E_j}$ are unit neighbours. $\blacksquare$

### 2.2. Contractions of a hypergraph.

For any hypergraph $H$, the collection of all the units $\Omega(H)$ forms a partition of $V(H)$ and $\mathcal{C}(\Omega(H))$ forms an partition of $\Omega(H)$. These two partition naturally induces an onto map.

We define $\pi : V(H) \to \mathcal{C}(\Omega(H))$ as $\pi(v) = \mathcal{C}(W_{E_v}(H))$. Evidently, $\pi$ is an onto map.

**Proposition 2.13.** Let $H$ be a hypergraph. If $e = \bigcup_{i=1}^{n} W_{E_i} \in E(H)$ then $e' = \bigcup_{i=1}^{n} W_{E'_i} \in E(H)$ for all $W_{E'_i} \in \mathcal{C}(W_{E_i})$, $i = 1, 2, \ldots, n$.

**Proof.** Since $W_{E_i} \in \mathcal{C}(W_{E_i})$, $W_{E_i}(H)$ and $W_{E'_i}(H)$ are twin units and there exists a canonical bijection $f_{E_i, E'_i}$. Therefore, $e' = (f_{E_1, E'_1} \circ \ldots \circ f_{E_n, E'_n})(e) \in E(H)$. $\blacksquare$

**Definition 2.14.** Let $H$ be a hypergraph. For all $e \in E(H)$, $\hat{\pi}(e) = \{\pi(v) : v \in e\}$. The contraction of $H$ is a hypergraph, denoted by $\hat{H}$, is defined as $V(\hat{H}) = \mathcal{C}(W_{E_0})$ and $E(\hat{H}) = \{\hat{\pi}(e) : e \in E(H)\}$.

For any set $V$, we denote the set of all the real-valued functions on $V$ as $\mathbb{R}^V$. For any $x \in \mathbb{R}^V(\hat{H})$, we define $\hat{x} \in \mathbb{R}^V(H)$ as $x(\pi(v)) = \hat{x}(v)$. For any $y \in \mathbb{R}^V(H)$, we define $\hat{y} \in \mathbb{R}^V(H)$ as $\hat{y}(v) = \frac{1}{W_{E_0}(H)}y(\mathcal{C}(W_{E_0}(H)))$ for all $v \in V(H)$. For any $\alpha \in \mathbb{R}^E(H)$, we define $\hat{\alpha} \in \mathbb{R}^E(\hat{H})$ as $\hat{\alpha}(\hat{\pi}(e)) = \sum_{e \in \hat{\pi}^{-1}(\hat{\pi}(e))} \alpha(e)$.

### 2.3. Regular and co-regular sets in hypergraphs.

**Definition 2.15.** Let $H$ be a hypergraph and $\alpha : E(H) \to \mathbb{R}^+$ then $U(\subseteq (V(H)))$ is said to be $\alpha$-regular if $\sum_{e \in E_u(H)} \alpha(e) = w_U^\alpha$, a constant, for all $u \in U$ and $w_U^\alpha$ is referred as the $\alpha$-regularity of $U$. 

If $\alpha$ is a constant map then instead of $\alpha$-regular we refer it as regular. A hypergraph $H$ is regular if $V(H)$ is regular.

**Definition 2.16.** Let $H$ be a hypergraph and $\alpha : E(H) \to \mathbb{R}^+$ then $U(\subseteq (V(H)))$ is said to be $\alpha$-symmetric set if for all $u \in U$, the sum $\sum_{e \in E_{uv}(H)} \alpha(e) = c^*_U$, a constant, for each $v(\neq u) \in V(H)$.

If $\alpha$ is a constant map then instead of $\alpha$-symmetric we refer it as symmetric.

**Definition 2.17.** Let $H$ be a hypergraph. $U \subseteq V(H)$ is called unit saturated if for any $W_{E_i} \in \mathcal{U}(H)$, either $W_{E_i} \subseteq U$ or $W_{E_i} \cap U = \emptyset$.

For example, for any $a(\in \mathcal{C}(\mathcal{U}(H)))$, the set $\pi^{-1}(a)$ is unit saturated.

**Definition 2.18.** Let $H$ be a hypergraph and $\alpha : E(H) \to \mathbb{R}^+$ then $U(\subseteq (V(H)))$. If $U$ is a unit saturated, $\alpha$-symmetric set and $\sum_{e \in E_{uv}(H)} \alpha(e) = \sum_{e \in E_{uv}(H)} \alpha(e)$ for all $u, v \in U$ then we refer $U$ as an $\alpha$-co-regular in $H$.

If $\alpha$ is a constant map then instead of $\alpha$-co-regular we refer it as co-regular. Evidently, each $\alpha$-co-regular set is $\alpha$-regular and $\alpha$-symmetric.

**Proposition 2.19.** Each $\alpha$-co-regular set is $\alpha$-regular and $\alpha$-symmetric and conversely every $\alpha$-regular, $\alpha$-symmetric, unit saturated set is $\alpha$-co-regular.

**Proof.** By definition, each $\alpha$-co-regular set is $\alpha$-symmetric. If $U \subseteq V(H)$ is an $\alpha$-co-regular set then for $u, v \in V(H)$, we have

$$\sum_{e \in E_{uv}(H)} \alpha(e) = \sum_{e \in E_{uv}(H)} \alpha(e) + \sum_{e \in E_{uv}(H)} \alpha(e) = \sum_{e \in E_{uv}(H)} \alpha(e) + \sum_{e \in E_{uv}(H)} \alpha(e) = \sum_{e \in E_{uv}(H)} \alpha(e),$$

and thus, $U$ is $\alpha$-regular.

Conversely, If $U$ is $\alpha$-regular, $\alpha$-symmetric, and unit saturated set then for any $u, v \in U$, we have

$$\sum_{e \in E_{uv}(H)} \alpha(e) = \sum_{e \in E_{uv}(H)} \alpha(e) - \sum_{e \in E_{uv}(H)} \alpha(e) = \sum_{e \in E_{uv}(H)} \alpha(e) - \sum_{e \in E_{uv}(H)} \alpha(e) = \sum_{e \in E_{uv}(H)} \alpha(e).$$

Thus, $U$ is $\alpha$-co-regular. $\blacksquare$

**Proposition 2.20.** Each unit in a hypergraph is an $\alpha$-co-regular set for any $\alpha : E(H) \to \mathbb{R}^+$.

**Proof.** Since for all $v \in W_{E_0}$, $E_v(H) = E_0$ for any unit $W_{E_0}$, the result follows. $\blacksquare$

Let $E_i \subseteq E(H)$, for $i = 1, 2$. For some $\alpha \in \mathbb{R}^{E(H)}$, a function $f : E_i \to E_j$ is said to be $\alpha$-preserving if $\alpha(f(e)) = \alpha(e)$.

**Proposition 2.21.** Let $H$ be a hypergraph. For any $a \in \mathcal{C}((\mathcal{U}(H)))$, the set $S_a = \bigcup_{W_{E_i} \in a} W_{E_i}$ is regular. For any $\alpha : E(H) \to \mathbb{R}^+$, if the canonical bijection $f_{E_i,E_j}$ is $\alpha$-preserving for all $W_{E_i}, W_{E_j} \in a$ then $S_a$ is $\alpha$-regular.

**Proof.** The result follows from the fact that for any $u, v \in S_a$, with $u \neq v$, we have $E_u(H) = E_i$ and $E_v(H) = E_j$ such that $W_{E_i}, W_{E_j} \in a$ and thus, there exists a canonical bijection $f_{E_i,E_j} : E_u(H) \to E_v(H)$. $\blacksquare$

All units and union of twin units are regular set, but the converse is not true. The following example shows a the same.

**Example 2.22.** Consider the hypergraph $H$ with $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{\{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}\}$. The set $U = \{1, 2, 3\}$ is a regular set, which is neither a unit nor a union of twin units.
Proposition 2.23. Let $H$ be a hypergraph. If $U$ is $\alpha$-co-regular set in $H$ for some $\alpha : E(H) \to \mathbb{R}^+$ then either $U$ is itself an unit or can be expressed as union of singleton units. Moreover if $U = \bigcup_{i=1}^{n_U} W_{E_i}$ with $\{W_{E_i} : i = 1, \ldots, n_U\} \subseteq \mathcal{U}(H)$ then for all $i, j = 1, \ldots, n_U$ \[
\sum_{e \in E_i \cap E_j} \alpha(e) = s^\alpha_{U}, \text{ a constant that depends only on } \alpha \text{ and } U.\]

Proof. Since $U$ is unit-saturated, $U = \bigcup_{i=1}^{n_U} W_{E_i}$. If possible let $|W_{E_i}| > 1$ for some $i$. In that case $u, v \in W_{E_i}$. Consider some $w \in W_{E_j}$, for some $j \neq i$. $\alpha$-co-regularity leads us to \[
\sum_{e \in E_i \cap E_j} \alpha(e) = c^\alpha_{U} = \sum_{e \in E_i \cap E_j} \alpha(e). \text{ That is } \sum_{e \in E_i} \alpha(e) = \sum_{e \in E_j} \alpha(e). \text{ Since } \alpha \text{ is positive-valued, we have } E_i \setminus E_j = \emptyset. \text{ Again by using the } \alpha \text{-co-regularity, we have } \sum_{e \in E_i \setminus E_j} \alpha(e) = 0. \text{ Thus, } E_i = E_j \text{ and } W_{E_i} = W_{E_j} \text{ for all } j \text{ and } U = W_{E_i}. \text{ Therefore, either } U \text{ is itself an unit or can be expressed as union of singleton units.}

It is enough to prove if $U = \bigcup_{i=1}^{n_U} W_{E_i}$ then $\sum_{e \in E_i \cap E_j} \alpha(e) = \sum_{e \in E_j} \alpha(e)$. Consider $v_i \in W_{E_i}$, for all $i = 1, 2, 3$. Since $U$ is $\alpha$-co-regular and therefore, $\sum_{e \in E_i \cap E_j} \alpha(e) = \sum_{e \in E_j} \alpha(e)$ and thus, the result follows. 

We refer $s^\alpha_{U}$ as the $\alpha$-co-regularity of $U$, if $\alpha = \iota_{E(H)}$, the identity map on $E(H)$, then $s^\alpha_{U} \in \mathbb{N} \cup \{0\}$.

3. Spectra of Hypergraphs and the Building Blocks

Suppose that $V$ is a non empty finite set. The set of all real valued functions on $V$, denoted by $\mathbb{R}^V$, is a finite dimensional vector space of dimension $|V|$. For any $x \in \mathbb{R}^V$, the support of $x$ is $Supp(x) = \{v \in V : x(v) \neq 0\}$. For any none empty $U \subseteq V$ the characteristic function $\chi_U \in \mathbb{R}^V$ is defined as \[
\chi_U(v) := \begin{cases} 
1 & \text{if } v \in U, \\
0 & \text{otherwise.} 
\end{cases}
\]

Now $T_U = \{x \in \mathbb{R}^V : Supp(x) \subseteq U \text{ and } \sum_{v \in U} x(v) = 0\}$ is a subspace of $\mathbb{R}^V$ of dimension $|U| - 1$. Now we recall some operators associated with hypergraph introduced in [2]. Let $H$ be a hypergraph. Consider $\delta_{V(H)} : V(H) \to \mathbb{R}^+$ and $\delta_{E(H)} : E(H) \to \mathbb{R}^+$ to be two positive real-valued functions on vertices and hyperedges, respectively. The inner products on $\mathbb{R}^{V(H)}$ and $\mathbb{R}^{E(H)}$ are defined below. For $x, y \in \mathbb{R}^{V(H)}$, an inner product defined as $\langle x, y \rangle_{V(H)} = \sum_{v \in V(H)} \delta_{V(H)}(v)x(v)y(v)$, and for $\beta, \gamma \in \mathbb{R}^{E(H)}$ the same is defined as $\langle \beta, \gamma \rangle_{E(H)} = \sum_{e \in E(H)} \delta_{E(H)}(e)\beta(e)\gamma(e)$. The function $avg : \mathbb{R}^{V(H)} \to \mathbb{R}^{E(H)}$ is defined as $(avg(x))(e) := \sum_{v \in V(H)} \frac{x(v)}{|\delta_{V(H)}(v)|}$. The general signless Laplacian operator $Q_H : \mathbb{R}^{V(H)} \to \mathbb{R}^{V(H)}$ is defined as $Q_H = avg^* \circ avg$. The expression for the general signless operator $Q_H$, the general Laplacian operator $L_H : \mathbb{R}^{V(H)} \to \mathbb{R}^{V(H)}$, the general adjacency operator $A_H : \mathbb{R}^{V(H)} \to \mathbb{R}^{V(H)}$ is given by $(Q_Hx)(v) = \sum_{e \in E(H)} \frac{\delta_{E(H)}(e)}{|\delta_{V(H)}(v)|} \frac{1}{|e|^2} \sum_{u \in e} x(u)$, $(L_Hx)(v) = \sum_{e \in E(H)} \frac{\delta_{E(H)}(e)}{|\delta_{V(H)}(v)|} \frac{1}{|e|^2} \sum_{u \in e, v \neq u} x(u)$, and $(A_Hx)(v) := \sum_{e \in E(H)} \frac{\delta_{E(H)}(e)}{|\delta_{V(H)}(v)|} \frac{1}{|e|^2} \sum_{u \in e, v \neq u} x(u)$, respectively, where $x \in \mathbb{R}^{V(H)}$ and $v \in V(H)$.

We also consider $\sigma_H : E(H) \to \mathbb{R}^+$ and $\rho_H : E(H) \to \mathbb{R}^+$ defined as $\sigma_H(e) = \frac{\delta_{E(H)}(e)}{|e|}$ and $\rho_H(e) = \frac{\delta_{E(H)}(e)}{|\delta_{V(H)}(v)|}$ respectively.
3.1. Spectra of hypergraphs in terms of regular, co-regular, and symmetric sets.

**Proposition 3.1.** Let \( H \) be a hypergraph and \( B = A_H \) or \( L_H \) or \( Q_H \) and \( \delta_V(H)(v) = c_V \) for all \( v \in W_{E_0} \), for all \( W_{E_0} \in \mathcal{U}(H) \). If \( \lambda \) is an eigenvalue of \( B \) and \( U \) is a maximal subset of \( V(H) \) such that \( T_U \) is a subspace of the eigenspace of \( \lambda \) then \( U \) is unit-saturated.

**Proof.** If possible let \( U \) be not a unit saturated set then there exists \( W_{E_0} \in \mathcal{U}(H) \) such that \( u \in W_{E_0} \cap U \) and \( v \in W_{E_0} \setminus U \). Since \( u, v \) are in the same unit, \( E_u(H) = E_v(H) \). Therefore, if \( \chi(u) - \chi(v) \) is an eigenvector of \( \lambda \) then \( \chi(u) - \chi(v) \) is an eigenvector of \( \lambda \) and this contradicts the maximality of \( U \).

**Theorem 3.2.** Let \( H \) be a hypergraph. If \( U \) is a \( \sigma_H \)-co-regular set in \( H \) with \( \sigma_H \)-co-regularity \( s_U^{\sigma_H} \) and \( \delta_V(v) = c_V \), a constant for all \( v \in U \) then \( -\frac{s_U^{\sigma_H}}{c_V} \) is an eigenvalue of \( A_H \) with multiplicity at least \(|U| - 1\) and \( T_U \) is a subspace of the eigenspace of \( -\frac{s_U^{\sigma_H}}{c_V} \).

**Proof.** Let \( U = \{v_0, \ldots, v_k\} \) and \( y_j := \chi_{\{v_j\}} - \chi_{\{v_0\}} \) for all \( j = 1, 2, \ldots, k \). If \( v \notin \{v_0, v_j\} \), then \( \sum_{u \in e \neq v} y_j(u) = 0 \) for all \( e \in E_{v_0 \setminus v_j}(H) \), \( \sum_{u \in e \neq v} y_j(u) = 1 \) for all \( e \in E_{v_0 \setminus v_j}(H) \), and \( \sum_{u \in e \neq v} y_j(u) = -1 \) for all \( e \in E_{v_0 \setminus v_j}(H) \). Since \( U \) is \( \sigma_H \)-co-regular, \( (A_Hy_j)(v) = -\frac{s_U^{\sigma_H}}{c_V} \). Similarly, we can show that \( (A_Hy_j)(v_0) = -\frac{s_U^{\sigma_H}}{c_V} \).

It is natural to ask if conversely of theorem 3.2 is true. Next result will answer the question.

**Theorem 3.3.** Let \( H \) be a hypergraph. If there exists an eigenvalue \( \lambda \) of \( A_H \) such that \( U \) is maximal subset of \( V(H) \) with \( T_U \) is a subspace of the eigenspace of \( \lambda \) then \( U \) is a \( \sigma_H \)-symmetric, and unit saturated set in \( H \).

**Proof.** By Proposition 3.1, we have \( U \) is unit saturated. For \( u, v \in U \), \( y_{uv} := \chi_{\{u\}} - \chi_{\{v\}} \in T_U \). For any \( w \notin \{u, v\} \), we have \( 0 = \lambda y_{uw}(w) = (A_Hy_{uv})(w) = \sum_{e \in E_H(H \setminus E_{uv})} \frac{\sigma_H(e)}{\delta_V(H)(w)} \). Thus, \( \sum_{e \in E_H(H \setminus E_{uv})} \frac{\sigma_H(e)}{\delta_V(H)(w)} = \sum_{e \in E_H(H \setminus E_{uv})} \sigma_H(e) = 0 \) and therefore, \( U \) is \( \sigma_H \)-symmetric.

**Theorem 3.4.** Let \( H \) be a hypergraph. If \( U \) is a \( \sigma_H \)-co-regular as well as \( \rho_H \)-regular set in \( H \) with \( \sigma_H \)-co-regularity \( s_U^{\sigma_H} \), \( \sigma_H \)-regularity \( w_U^{\sigma_H} \), \( \rho_H \)-regularity \( w_U^{\rho_H} \), and \( \delta_V(v) = c_V \), a constant for all \( v \in U \) then \( \frac{w_U^{\rho_H} - w_U^{\sigma_H} + s_U^{\sigma_H}}{c_V} \) is an eigenvalue of \( L_H \) with multiplicity at least \(|U| - 1\) and \( T_U \) is a subspace of the eigenspace of \( \frac{w_U^{\rho_H} - w_U^{\sigma_H} + s_U^{\sigma_H}}{c_V} \).

**Proof.** Let \( U = \{v_0, \ldots, v_k\} \). If \( y_j \) is defined as \( y_j := \chi_{\{v_j\}} - \chi_{\{v_0\}} \) for all \( j = 1, 2, \ldots, k \) then for any \( v \notin \{v_0, v_j\} \), using the same arguments as in the proof of Theorem 3.2, we have \( (L_Hy_j)(v) = 0 \). For \( e \in E_{v_0 \setminus v_j}(H) \), we have \( \sum_{u \in e} (y_j(u) - y_j(u)) = |e| \) and for \( e \in E_{v_0}(H \setminus E_{v_0}(H)) \), we have \( \sum_{u \in e} (y_j(u) - y_j(u)) = |e| - 1 \). Therefore, \( (L_Hy_j)(v_j) = \).
Therefore, \( L_H(\{v\}) = (\lambda_y - \|\lambda_y\|_1) - \|\lambda_y\|_0 \) is an eigenvalue of \( L_H \). Similarly, we can show that \( L_H(\{v\}) \) is an eigenvalue of \( L_H \).

This leads us to \( (L_H(\{v\})) = \sum_{\{j\}} \frac{1}{\|\lambda_j\|_0} \). Therefore, the multiplicity of the eigenvalue is at least \( |U| - 1 \).

Theorem 3.7. Let \( H \) be a hypergraph, \( H \in \mathcal{V}(V(H)) \) is a maximal subset of \( V(H) \) such that there exists an eigenvalue of \( \lambda_y \) of \( \lambda_y \) with multiplicity at least \( |U| - 1 \) and \( T_U \) is a subspace of the eigenspace of \( \lambda_y \) with \( \lambda_y \) is an eigenvalue of \( \lambda_y \).

Proof. Let \( U = \{v_1, \ldots, v_k\} \) and \( y_i = X_i - X_{\{v_i\}} \) for all \( j = 1, \ldots, k \), then we have \( \sum_{\{j\}} y_i (x_j) \) for all \( e \in \mathcal{E}_H \). Therefore, the multiplicity of the eigenvalue is at least \( |U| - 1 \).
If $v \in U$ and therefore, $U$ is $\sigma_H$-regular. Since $U$ is maximal subset with the property that $T_U$ is a subspace of the eigenspace of $\lambda$, by Proposition 3.1, $U$ is unit saturated. Thus, by Proposition 2.19, the result follows.

3.2. The complete spectra of hypergraphs from unit, twin units, and contraction.

Now we compute the complete spectra of the operators associated with hypergraphs in terms of $\mathcal{U}(H)$, $\mathcal{E}(\mathcal{U}(H))$, and $\tilde{H}$.

**Theorem 3.8.** If $H$ is a hypergraph and $W_{E_0}$ is an unit in the hypergraph $H$ and $\delta_{\mathcal{U}(H)}(v) = c$ for all $v \in W_{E_0}$ then $\sum_{e \in E_0} \frac{\delta_{\mathcal{U}(H)}(e)}{|e|} \frac{1}{|e|}$ and $-\sum_{e \in E_0} \frac{\delta_{\mathcal{U}(H)}(e)}{|e|} \frac{1}{|e|^2}$ are eigenvalues of $L_H$ and $A_H$ respectively, with multiplicity at least $|W_{E_0}| - 1$ and $T_{W_{E_0}}$ is a subspace of the eigenspace of the both eigenvalues.

**Proof.** The result follows from Theorem 3.4, and Theorem 3.2.

**Theorem 3.9.** Let $H$ be a hypergraph. The signless Laplacian $Q_H$ has an eigenvalue 0 with $T_{W_{E_0}}$ is a subspace of the eigenspace of the eigenvalue 0.

In the next result we show that the converse of Theorem 3.9 is also true.

**Theorem 3.10.** Let $H$ be a hypergraph and $U \subseteq V(H)$ with $|U| > 1$. If $U$ is a maximal subset such that $T_U$ is a subspace of the eigenspace of the eigenvalue 0 of $Q_H$ then $U$ is an unit.

**Proof.** If $u, v \in U$ then $y_{uv} = x_{\{u\}} - x_{\{v\}} \in T_U$. Thus, $0 = (Q_H y_j)(u) = \sum_{e \in E_a \setminus E_{\mathcal{U}(H)}} \sigma_H(e)$. Therefore, $E_a(H) \subseteq E_a(H)$ and similarly, $E_a(H) \subseteq E_a(H)$. Therefore, $E_a(H) = E_a(H)$ for all $u, v \in U$. Since $U$ is maximal subset with $E_a(H) = E_a(H)$ for all $u, v \in U$, thus, $U$ is an unit.

Therefore, the number of units with cardinality at least two in $H$ is at most the the multiplicity of the eigenvalue 0 of $Q_H$.

**Theorem 3.11.** Let $H$ be a hypergraph such that $a = \{W_{E_1}, W_{E_2}, \ldots, W_{E_{n_a}}\} \in \mathcal{E}(\mathcal{U}(H))$, $n_a > 1$. If $\delta_{\mathcal{U}(H)}(v) = c_a$ for all $v \in \bigcup_{i=1}^{n_a} W_{E_i}$, $\delta_{\mathcal{U}(H)}(e) = w_a$ for all $e \in E_a = \bigcup_{i=1}^{n_a} E_i$ then

$$\frac{w_a}{c_a} \sum_{e \in E_{E_{n_a}}} |e \setminus W_{E_{n_a}}|$$

is an eigenvalue of $L_H$ with multiplicity at least $n_a - 1$.

**Proof.** Consider $y_j = |W_{E_{n_a}}| \chi_{W_{E_j}} - |W_{E_j}| \chi_{W_{E_{n_a}}}$, for $j = 1, 2, \ldots, n_a - 1$. Since, if $e \in E(H) \setminus (E_j \cup E_{n_a})$ then $y_j(u) = 0$ for all $u \in e$. Therefore,

$$(L_H y_j)(v) = \sum_{e \in E_a(H)} \frac{\delta_{\mathcal{U}(H)}(e)}{|e|} \frac{1}{|e|} \sum_{u \in e} (y_j(v) - y_j(u)) + \sum_{e \in E_{E_{n_a}}(H)} \frac{\delta_{\mathcal{U}(H)}(e)}{|e|} \frac{1}{|e|} \sum_{u \in e} (y_j(v) - y_j(u)),$$

where $E_a(H) = E_a(H) \cap E_i$ for all $i = 1, 2, \ldots, n_a$.

For any $v \in V(H) \setminus (W_{E_j} \cup W_{E_{n_a}})$, considering the restriction of the canonical map $\mathfrak{f} = \mathfrak{f}_{E_{n_a}}(E_{n_a}(H) : E_{n_a}(H) \to E_{n_a}(H)$, we have

$$(L_H y_j)(v) = -\frac{w_a}{\delta_{\mathcal{U}(H)}(v)} \left( \sum_{e \in E_a(H), v} |W_{E_j}| |W_{E_{n_a}}| - \sum_{f(e) \in E_{E_{n_a}}(H)} |W_{E_j}| |W_{E_{n_a}}| \right) = 0.$$

For any $v \in W_{E_j}$, we have, $E_a(H) = E_j(H) = E_j, E_{E_{n_a}} = \emptyset$. Since $e \setminus W_{E_j} = f(e) \setminus W_{E_{n_a}}$, we have $(L_H y_j)(v) = \frac{w_a}{c_a} \sum_{e \in E_j} |e \setminus W_{E_j}| y_j(v) = \frac{w_a}{c_a} \sum_{e \in E_{n_a}} |e \setminus W_{E_{n_a}}| y_j(v)$. Similarly, for any $v \in W_{E_{n_a}}$.
$E_v(H) = E_v^{\text{na}}(H) = E_{v_n}, E_v^j = \emptyset$. Thus, for all $v \in W_{E_{v_n}}$, we have $(L_H y_j)(v) = \frac{w_a}{c_a} \sum_{e \in E_{v_n}} |e \setminus W_{E_{v_n}}| y_j(v)$. Therefore, $\frac{w_a}{c_a} \sum_{e \in E_{v_n}(H)} |e \setminus W_{E_{v_n}}|$ is an eigenvalue of $L_H$.

Since $\{y_j\}_{j=2}^{n_a}$ are linearly independent, the multiplicity of the eigenvalue is at least $n_a - 1$. \(\blacksquare\)

We provide similar results for $A_H$ and $Q_H$ in the next result.

**Theorem 3.12.** Let $H$ be a hypergraph such that $a = \{W_{E_1}, W_{E_2}, \ldots, W_{E_{n_a}}\} \in \mathfrak{C}(\U(H))$, $n_a(\in \mathbb{N}) > 1$, and $|W_{E_i}| = m_a$ for all $W_{E_i} \in a$. If $\delta_V(v) = c_a$ for all $v \in \bigcup_{i=1}^{n_a} W_{E_i}$, $\frac{\delta(e)}{|e|} = w_a$ for all $e \in E_a = \bigcup_{i=1}^{n_a} E_i$ then $\frac{w_a}{c_a} s_a(m_a - 1)$ and $\frac{w_a}{c_a} m_a s_a$ are eigenvalues of $A_H$ and $Q_H$ respectively with multiplicity at least $n_a - 1$, where $s_a$ is the generator size of $a$.

**Proof.** We construct $y_j \in \mathbb{R}^{V(H)}$ as in the previous proof. Now, $$(A_H y_j)(v) = \sum_{e \in E_i^j(H)} \frac{\delta(e)}{|e|} \frac{1}{|e|} \sum_{u \in e \setminus a} y_j(u) + \sum_{e \in E_v^j(H)} \frac{\delta(e)}{|e|} \frac{1}{|e|} \sum_{u \in e \setminus a} y_j(u),$$ where $E_v^j(H) = E_v(H) \cap E_i$ for all $i = 1, \ldots, n_a$.

Thus, similarly, in the previous proof, we have $(A_H y_j)(v) = 0$. Proceeding in the same way, we have $(Q_H y_j)(v) = 0$.

For any $v \in W_{E_j}$, we have, $E_v(H) = E_v^j(H) = E_j, E_v^{\text{na}}(H) = \emptyset$, and $(A_H y_j)(v) = \frac{w_a}{c_a} |E_j| (|W_{E_j}|-1) y_j(v) = \frac{w_a}{c_a} s_a(m_a - 1) y_j(v)$. Similarly, for any $v \in W_{E_{v_n}}, E_v(H) = E_v^{\text{na}}(H) = E_{v_n}, E_v^j = \emptyset$. Since $e \setminus W_{E_j} = f(e) \setminus W_{E_{v_n}}$ for all $e \in E_j$, we have $A_H y_j(v) = \frac{w_a}{c_a} s_a(m_a - 1) y_j(v)$. Therefore, $\frac{w_a}{c_a} s_a(m_a - 1)$ is an eigenvalue of $A_H$.

Since $E_v(H) = E_j$ for each $v \in W_{E_j}$ and $\sum_{u \in e} y(u) = m_a$ for all $e \in E_j$, we have $(Q_H y_j)(v) = m_a |E_j| \frac{w_a}{c_a} y_j(v) = m_a s_a \frac{w_a}{c_a} y_j(v)$. Similarly, we can prove the same for $v \in W_{E_{v_n}}$. Therefore, $(Q_H y_j)(v) = m_a s_a \frac{w_a}{c_a} y_j$. Similarly, the multiplicity of the eigenvalue is at least $n_a - 1$. \(\blacksquare\)

**Lemma 3.13.** Let $H$ be a hypergraph. If $\delta_V(H) = c \tilde{\delta}_V(H)$, $\tilde{\delta}_H = c_E \sigma_H$, and $|W_{E_0}| = c$ for all $W_{E_0} \in \U(H)$ then $(L_H \tilde{y})(v) = \frac{c_E}{c_V} (L_H y)(\pi(v))$ and $(Q_H \tilde{y})(v) = \frac{c_E}{c_V} (Q_H y)(\pi(v))$ for any $y \in \mathbb{R}^{V(H)}$, and $v \in V(H)$.

**Proof.** For all $y \in \mathbb{R}^{V(H)}$, and $v \in V(H)$ we have $$(L_H \tilde{y})(v) = \sum_{e \in \mathcal{E}_v(H)} \frac{\delta_H(e)}{\tilde{\delta}_V(H)} \sum_{u \in e} \tilde{y}(u) - \tilde{y}(u))$$

$$= \sum_{\hat{e}(u) \in \mathcal{E}_v(H) \setminus \mathcal{E}_v(H)} \frac{\delta_H(\hat{e}(u))}{c_V \delta_V(H)} \sum_{\pi(u) \in \hat{e}(u)} (y(\pi(u)) - y(\pi(u)))$$

$$= \sum_{\hat{e}(u) \in \mathcal{E}_v(H) \setminus \mathcal{E}_v(H)} \frac{c_E \sigma_H(\hat{e}(u))}{c_V \delta_V(H)} \sum_{\pi(u) \in \hat{e}(u)} (y(\pi(u)) - y(\pi(u)))$$

$$= \frac{c_E}{c_V} (L_H y)(\pi(v))$$

Similarly, we have $Q_H \tilde{y}(v) = \frac{c_E}{c_V} (Q_H y)(\pi(v))$. \(\blacksquare\)

Therefore, by Lemma 3.13, we have the following result.

**Theorem 3.14.** Let $H$ be a hypergraph. If $\delta_V(H) = c \tilde{\delta}_V(H)$, $\tilde{\delta}_H = c_E \sigma_H$, and $|W_{E_0}| = c$ for all $W_{E_0} \in \U(H)$ then for each eigenvalue $\lambda_{L_\tilde{y}}$ of $L_\tilde{y}$ with eigenvector $y$ we have $\frac{c_E}{c_V} \lambda_{L_\tilde{y}}$ is an eigenvalue of $L_H$ with eigenvector $\tilde{y}$, and for each eigenvalue $\lambda_{Q_\tilde{y}}$ of $Q_\tilde{y}$ with eigenvector $z$ we have $\frac{c_E}{c_V} \lambda_{Q_\tilde{y}}$ is an eigenvalue of $Q_H$ with eigenvector $\tilde{z}$. \(\blacksquare\)
Proof. By Lemma 3.13, we have \( (L_H \tilde{y})(v) = \frac{c_{\tilde{v}}}{c_{v}} (L_H y)(\pi(v)) \) for any \( y \in \mathbb{R}^{|H|} \), and \( v \in V(H) \). Since \( \lambda_{L_H} \) is an eigenvalue with eigenvector \( y \) and \( \tilde{y}(v) = \frac{1}{c_y} y(\mathcal{C}(W_{E_v(H)})) = \frac{1}{c_y} y(\mathcal{C}(W_{E_v(H)})) \) for all \( v \in V(H) \), we have \( (L_H \tilde{y})(v) = \frac{c_{\tilde{v}}}{c_{v}} (L_H y)(\pi(v)) = \frac{c_{\tilde{v}}}{c_{v}} \lambda_{L_H} y(\pi(v)) = \frac{c_{\tilde{v}}}{c_{v}} \lambda_{L_H} \tilde{y}(v) \) for any \( y \in \mathbb{R}^{|H|} \), and \( v \in V(H) \). The proof for the part of \( Q_H \) is similar to that of \( L_H \) and hence is omitted.

For any hypergraph \( H \) with \( \mathcal{C}(\mathcal{U}(H)) = \{a_1, a_2, \ldots, a_m\} \), \( a_i = \{W_{E_1}, W_{E_2}, \ldots, W_{E_{a_i}}\} \). Since Theorem 3.8 provides \( \sum_{W_{E_i} \in \mathcal{U}(H)} (|W_{E_i}| - 1) \) eigenvalues, Theorem 3.11 provides \( \sum_{a \in \mathcal{C}(\mathcal{U}(H))} (|a| - 1) \) eigenvalues the remaining number of eigenvalue of \( L_H \) are \(|\mathcal{C}(\mathcal{U}(H))|\). If all the units of \( H \) are of same cardenality then Theorem 3.14 provide us the remaining eigenvalues. Similarly, our results provide the complete spectra of \( Q_H \). Now we find the complete spectra of \( A_H \). Suppose that \( W_{E_i}, W_{E_j} \in \mathcal{U}(H) \). For any \( u \in W_{E_i} \) and \( v \in W_{E_j} \), if \( i \neq j \) then \( \sum_{uv \in W_{E_i} W_{E_j}} \sigma_{H}(e) = |W_{E_i}| \sum_{E_i \cap E_j} \sigma_{H}(e) \), and when \( i = j \), \( \sum_{uv \in W_{E_i} E_{uv}(H)} \sigma_{H}(e) = (|W_{E_i}| - 1) \sum_{E_i} \sigma_{H}(e) \). We further assume that \( \delta_{V(H)}(v) = c_i \) for all \( v \in W_{E_i} \) for all \( W_{E_i} \in \mathcal{U}(H) \). Let \( B_H = (b_{ij})_{W_{E_i}, W_{E_j} \in \mathcal{U}(H)} \) be a matrix defined as \( b_{ii} = \frac{1}{c_i} (|W_{E_i}| - 1) \sum_{E_i} \sigma_{H}(e) \), and \( b_{ij} = \frac{1}{c_i} |W_{E_j}| \sum_{E_i \cap E_j} \sigma_{H}(e) \) if \( i \neq j \).

**Theorem 3.15.** Let \( H \) be a hypergraph with \( \delta_{V(H)}(v) = c_i \) for all \( v \in W_{E_i} \) for all \( W_{E_i} \in \mathcal{U}(H) \).

For each eigenvalue \( \lambda \) of \( B_H \) with an eigenvector \( y \), \( \lambda \) is also an eigenvalue of \( A_H \) with the eigenvector \( \tilde{y} \) defined as \( \tilde{y}(v) = \frac{1}{c_i} y(W_{E_i}) \), if \( v \in W_{E_i} \), for all \( W_{E_i} \in \mathcal{U}(H) \).

**Proof.** For any \( v \in W_{E_i} \in \mathcal{U}(H) \),

\[
(A_H \tilde{y})(v) = \frac{1}{\delta_{V(H)}(v)} \sum_{u(\neq)v \in V(H)} \tilde{y}(u) \sum_{e \in E_{uv}(H)} \sigma_{H}(e)
= \frac{1}{\delta_{V(H)}(v)} \sum_{W_{E_j} \in \mathcal{U}(H)} y(W_{E_j}) b_{ij}
= \frac{1}{\delta_{V(H)}(v)} (B_H y)(W_{E_i})
= \frac{\lambda}{\delta_{V(H)}(v)} y(W_{E_i}) = \lambda \tilde{y}(v).
\]

Let \( \mathcal{C}(\mathcal{U}(H)) = \{a_1, \ldots, a_k\} \) for a hypergraph \( H \). Suppose that \( \delta_{V(H)}(v) = c_p \) for all \( v \in \pi^{-1}(a_p) \), all canonical bijections are \( \sigma_H \) preserving, and \( |W_{E_i}| = m_p \) for all \( W_{E_i} \in a_p \) for all \( p = 1, \ldots, k \). We define \( C_H = (c_{pq})_{a_p, a_q \in \mathcal{C}(\mathcal{U}(H))} \) as \( c_{pq} = \frac{1}{c_y} |\pi^{-1}(a_p)| \sum_{E_i \cap E_j} \sigma_{H}(e) \), where \( a_p = \mathcal{C}(W_{E_i}), a_q = \mathcal{C}(W_{E_j}) \) and \( p \neq q \) and \( c_{pp} = (m_p - 1) \sum_{v \in E_i} \sigma_{H}(e) \).

**Theorem 3.16.** Let \( \mathcal{C}(\mathcal{U}(H)) = \{a_1, \ldots, a_k\} \) for a hypergraph \( H \). If \( \delta_{V(H)}(v) = c_p \) for all \( v \in \pi^{-1}(a_p) \), all canonical bijections are \( \sigma_H \) preserving, and \( |W_{E_i}| = m_p \) for all \( W_{E_i} \in a_p \) for all \( p = 1, \ldots, k \) then for each eigenvalue \( \lambda \) of \( C_H \) with eigenvector \( y \) we have \( \lambda \) is an eigenvalue of \( A_H \) with eigenvector \( \tilde{y} \) defined by \( \tilde{y}(v) = \frac{y(a_p)}{c_p} \), for all \( v \in \pi^{-1}(a_p) \), \( p = 1, \ldots, k \).

**Proof.** For all \( v \in \pi^{-1}(a_p) \),

\[
(A_{H} \tilde{y})(v) = \frac{1}{\delta_{V(H)}(v)} \sum_{u(\neq)v \in V(H)} \tilde{y}(u) \sum_{e \in E_{uv}(H)} \sigma_{H}(e)
\]
Thus, from Theorem 3.15 and Theorem 3.8 (or from Theorem 3.8, Theorem 3.12, Theorem 3.16) we get the complete spectrum of $A_H$. The eigenvectors provided in Theorem 3.8, Theorem 3.12, corresponding to the eigenvalues of $A_H$ are associated with units and twin units in $H$. These eigenvectors have some negative components with respect to the basis \{\delta_{\mathcal{V}(H)(v)}\chi(v)\}_{v \in \mathcal{V}(H)}$. Therefore, the Perron eigenvalue (the eigenvalue with maximum absolute value) of $A_H$ is one of the eigenvalues obtained from Theorem 3.15 and Theorem 3.16. Thus, the spectral radius of $A_H$, $B_H$ and $C_H$ are the same.

3.3. Example: Spectra of hyperflowers. Here we compute spectra of a class of hypergraphs, $(l, r)$-hyperflower with $t$-twins and $m$-homogeneous center, using our previous results. A $(l, r)$-hyperflower with $t$-twins and $m$-homogeneous center is a hypergraph $H$ where $\mathcal{V}(H)$ can be expressed as the disjoint partition $\mathcal{V}(H) = U \cup W$. The set $U$ can be partitioned into disjoint $t$-element sets as $U = \bigcup_{i=1}^{l} U_i$. That is $|U_i| = t$, $U_i = \{u_{i1}, \ldots, u_{it}\}$ for all $i = 1, 2, \ldots, l$ and $U_i \cap U_j = \emptyset$ for all $i \neq j$ and $i, j = 1, \ldots, l$. We refer each $U_i$ as a peripheral component and $U$ as the periphery of the hyperflower. There exists $r$-disjoint set of vertices $e_1, \ldots, e_r \in \mathcal{P}(W)$, the power set of $W$, such that, $|e_k| = m$ for all $k = 1, 2, \ldots, r$, and $E = \{e_k \cup U_i : k = 1, 2, \ldots, r, i = 1, 2, \ldots, l\}$ and $W = \bigcup_{i=1}^{r} e_r$. Each $e_k$ is called a central component and $W$ is called the center of the hyperflower.

Hyperflower to spectra. Any $(l, r)$-hyperflower contains only two $\mathcal{R}$-equivalence class, one of them contains $l$ peripheral components and the other one contains $r$ central components. Each one among the peripheral and central components is a unit. Therefore, by using Theorem 3.9, Theorem 3.8, Theorem 3.12, and Theorem 3.11, we have the following result.

**Theorem 3.17.** Let $H$ be a $(l, r)$-hyperflower with $t$-twins and $m$-homogeneous center with periphery $U$, center $W$, peripheral components $U_i$, for $i = 1, \ldots, l$ and layers of the center $e_j$, for $j = 1, \ldots, r$.

1. For each $U_i$, one eigenvalue of $Q_H$ is 0 with multiplicity $|U_i| - 1$ and if $\delta_{\mathcal{V}(H)(v)} = c_a^i$, for all $v \in U_i$, then $-\frac{1}{c_a} \sum_{e \in E_{U_i}} \frac{\delta_{E(H)(e)}}{|e|}$ and $\frac{1}{c_a} \sum_{e \in E(H)_{U_i}} \frac{\delta_{E(H)(e)}}{|e|}$ are eigenvalues of $A_H$ and $L_H$ respectively, with multiplicity at least $|U_i| - 1$ for all $i = 1, 2, \ldots, l$.

2. For each $e_j$, one eigenvalue of $Q_H$ is 0 with multiplicity $|e_j| - 1$ and if $\delta_{\mathcal{V}(H)(v)} = c_b^j$, for each $v \in e_j$, then $-\frac{1}{c_b} \sum_{e \in E_{e_j}} \frac{\delta_{E(H)(e)}}{|e|}$ and $\frac{1}{c_b} \sum_{e \in E(H)_{e_j}} \frac{\delta_{E(H)(e)}}{|e|}$ are eigenvalues of $A_H$ and $L_H$ respectively, with multiplicity at least $|e_j| - 1$ for all $e_j, j \in [r]$.

3. $\frac{m}{c_a}(t - 1)$, $\frac{m}{c_a}r$, $\frac{m}{c_a}r_m$, and $\frac{m}{c_a}r_t$ are eigenvalues of $A_H$, $L_H$, and $Q_H$ respectively, with multiplicity at least $t - 1$.

4. $\frac{m}{c_b}(l - 1)$, $\frac{m}{c_b}l_t$, and $\frac{m}{c_b}l_m$ are eigenvalues of $A_H$, $L_H$, and $Q_H$ respectively, with multiplicity at least $l - 1$.

It is easy to verify that $H$ is a uniform hypergraph with $\hat{H} = K_2$, complete graph with 2 vertices if and only if $H$ is a $(l, r)$-hyperflower with $t$-twins and $m$-homogeneous center for some $l, r, t, m \in \mathbb{N}$. The theorem 3.17, provide all the eigenvalues from the units and twin units, except two. The next Theorem gives us these remaining two eigenvalues for $A_H$, $L_H$, and $Q_H$. 

Theorem 3.18. Suppose that $H$ is a $(l,r)$-hyperflower with $t$-twins and an $m$-homogeneous center. If i) $\delta_{e}(e) = \frac{c}{w}$ for all $e \in E(H)$ for some constant $w$ and ii) $\delta_{v}(v) = c$ for all $v \in V(H)$ for some constant $c$ then

(1) $\frac{w}{c}r(t-1 + m\gamma)$ is an eigenvalue of $A_{H}$ with eigenvector $\gamma \chi_{W} + \chi_{U}$, where $\gamma$ is a root of the equation

$$rm\gamma^{2} + [r(t-1) - l(m-1)]x - lt = 0.$$  (3.1)

(2) $\frac{w}{c}|V(H)|$ is an eigenvalue of $L_{H}$ with eigenvector $\gamma \chi_{U} - |U| \chi_{W}$.

(3) $\frac{w}{c}(lm + rt)$ and 0 are two eigenvalues of $Q_{H}$ with eigenvectors $y = l\chi_{W} + r\chi_{U}$ and $z = \sum_{i=1}^{l} \chi(u) - \sum_{j=1}^{r} \chi(v)$ respectively, for some fixed $u_{i} \in U$, $v_{j} \in E_{i}$, $i = 1, \ldots, l$, and $j = 1, \ldots, r$.

Proof. Suppose that $y_{\gamma} = \gamma \chi_{W} + \chi_{U}$. Therefore,

$$(A_{H}y_{\gamma})(v) = \begin{cases} \frac{w}{c}(t-1 + m\gamma) & \text{if } v \in U, \\ l\gamma + \gamma(m-1) & \text{otherwise}. \end{cases}$$

Then, $y_{\gamma}$ is an eigenvector of $A_{H}$ then $\gamma$ is a root of the equation

$$rm\gamma^{2} + [r(t-1) - l(m-1)]x - lt = 0,$$

and in that case $r\frac{w}{c}(t-1 + m\gamma)$ is an eigenvalue of $A_{H}$.

Consider $y = |W|\chi_{U} - |U|\chi_{W} \in \mathbb{R}^{V}$. Note that for any $v \in U_{i}$, one has $E_{v}(H) = E_{U_{i}}$ for all $i \in [l]$. Since, the center of the hyperflower is $m$-uniform and $|E_{U_{i}}| = r$ for all $i \in [l]$, one has $\sum_{u \in e}(y(v) - y(u)) = m(|W| + |U|)$ for any $e \in E_{U_{i}}$. Therefore, by using the facts $\frac{\delta_{e}(e)}{|e|^{2}} = w$ for all $e \in E(H)$ for some constant $w$ and $\delta_{v}(v) = c$ for all $v \in V(H)$ for some constant $c$, for any $v \in U_{i}$, we have if $v \in U$ then $(Ly)(v) = \frac{w}{c}(|U| + |W|)mr = \frac{w}{c}(|U| + |W|)y(v)$.

Similarly, using the facts, $|U_{i}| = t$ for all $i \in [l]$ and $|E_{e_{i}}| = l$ for all $j \in [r]$, we have if $v \in W$ then $(Ly)(v) = \frac{w}{c}(|U| + |W|)lt = \frac{w}{c}(|U| + |W|)y(v)$. The result on $Q_{H}$ follows from the fact that $\sum_{u \in e} y(u) = (lm + rt)$ and $\sum_{u \in e} z(u) = 0$ for all $e \in E(H)$. This completes the proof. \[ \Box \]

Spectra to hyperflower. We can reconstruct the hyperflower from its spectra( of $Q_{H}$, $A_{H}$ and $L_{H}$). Since we know $H$ is a hyperflower, we need to either figure out the peripheral and central components or compute the value of $l, r, t, m$.

If we know the spectra of $Q_{H}$ of a hyperflower, consider the eigenvector $y$ corresponding to the largest eigenvalue. The range of $y$ consists of only two real number $a, b$. One of the $\{y^{-1}(a), y^{-1}(b)\}$ is the center $W$ and the another is the periphery $U$ of the hyperflower. For each maximal subset $U_{i}$ of $U$ with $T_{U_{i}}$ is a subspace of the eigenspace of 0 we have $U_{i}$ is a peripheral component. Similarly, for each maximal subset $e_{k}$ of $W$ with $T_{e_{k}}$ is a subspace of the eigenvalue 0, we have $e_{k}$ is a central component. Similarly, considering the eigenvector corresponding to the largest eigenvalue of $A_{H}$, we can separate the periphery and center of $H$. Each maximal subset $U_{i}$ of the periphery with $T_{U_{i}}$ is a subspace of the eigenspace of some eigenvalue of $A_{H}$ and similarly, we can point out the central components. In the same way, We can reconstruct a hyperflower from the spectra of $L_{H}$.

4. Applications

4.1. Random-walk on hypergraphs using units. Random walk on graphs ([3]) is an active area of study for a long time([5]). In the last decade random walks on hypergraphs has attracted considerable attentions [1, 6, 7]. A random walk on hypergraph $H$ is a map $\tau_{H} : T \rightarrow V(H)$, such that $T = \mathbb{N} \cup \{0\}$ and $\tau_{H}(t)$ depends only on its previous state $\tau_{H}(t - 1)$ for all $t \neq 0$. Thus, $\tau_{H}$ depends on a matrix $P_{H}$ of order $|V(H)|$, referred as the probability transition marix, defined as $P_{H}(u,v) = Prob(\tau_{H}(i + 1) = v | \tau_{H}(i) = u)$, and $\sum_{u \in V(H)} P_{H}(u,v) = 1$. Suppose that
$x_i(\in [0,1]^{V(H)})$ is the probability distribution at time $i$, that is, $x_i(v) = \text{Prob}(r_H(i) = v)$ and the iteration $x_{i+1} = P_Hx_i$ defines the random walk.

Here, we consider the transition probability matrix $P_H$ for random walk studied in [2] and is defined as

$$P_H(u,v) = \begin{cases} \frac{r(u)}{e} \sum_{e \in E_H \cap C_v} \frac{\delta_{E_H}(e)}{\delta_{V_H}(u)} |e|^{-1} & \text{if } u \neq v, \\ 0 & \text{otherwise,} \end{cases}$$

for all $u, v \in V(H)$, where $r(u) := \sum_{e \in E_H} \frac{\delta_{E_H}(e)}{\delta_{V_H}(u)} |e|^{-1}$.

Let $\Delta_H = I - P_H$. For any $x \in \mathbb{R}^{V(H)}$, and $v \in V(H)$ we have

$$(\Delta_H x)(u) = x(u) - (P_H x)(u)$$

$$= \sum_{e \in E_H} \frac{\delta_{E_H}(e)}{\delta_{V_H}(u)} |e|^{-1} \sum_{u \in e} (x(u) - x(v)).$$

If we chose $\delta_{H}(v) = r(v)\delta_{V_H}(v)$ for all $v \in V(H)$ and $\delta_{E_H} = \delta_{E_H}$ then $\Delta_H$ becomes the general Laplacian operator of $H$ and using Theorem 3.8, Theorem 3.11, Theorem 3.14 we can compute all the eigenvalues of $\Delta_H$. For any eigenvalue $\lambda$ of $\Delta_H$, $1 - \lambda$ is an eigenvalue of $P_H$. Therefore, we have the following results.

**Theorem 4.1.** Let $H$ be a hypergraph and $W_{E_0} = \{v_0, \ldots, v_n\} \in \Upsilon(H)$. If $\delta_{V_H}(v) = c$ for all $v \in W_{E_0}$ then $1 - \sum_{e \in E_0} \frac{\delta_{E_H}(e)}{c} \times c$ is an eigenvalue of $P_H$ with $y_j = \chi_{\{v_0\}} - \chi_{\{v_j\}}$ for some $j = 1, \ldots, n$.

**Theorem 4.2.** Let $H$ be a hypergraph such that $a = \{W_{E_1}, \ldots, W_{E_n}\} \in \mathcal{E}(\Upsilon(H))$, $n_a > 1$. If $r(v)\delta_{V_H}(v) = c_a$ for all $v \in \bigcup_{i=1}^{n_a} W_{E_i}$ and $\delta_{E_H}(e) \in |e|^\alpha = w_a$ for all $e \in \bigcup_{i=1}^{n_a} E_i$ then $1 - \frac{w_a}{c_a} \sum_{e \in E_{E_0}} |e| \in W_{E_0}$ is an eigenvalue of $P_H$ with multiplicity $n_a - 1$ and the eigenvectors are $|W_{E_j}|\chi_{W_{E_j}} - |W_{E_0}|\chi_{W_{E_0}}$, for $j = 1, \ldots, n_a - 1$.

**Theorem 4.3.** Let $H$ be a hypergraph. If $\delta_{V_H} = c \delta_{V_H}$, $\hat{\delta}_{E_H} = c \delta_{E_H}$, and $|W_{E_0}| = c$ for all $W_{E_0} \in \Upsilon(H)$ then for each eigenvalue $\lambda_{E_H}$ with eigenvector $y$, we have $1 - \frac{c \delta_{E_H}}{c} \lambda_{E_H}$ is an eigenvalue of $P_H$ with eigenvector $\hat{y}$.

Let $r_H$ be a random walk on a hypergraph $H$. For any $v \in V(H)$ and $t \in \mathbb{N}$, we denote the event $\{r_H(t) = v, r_H(i) \neq v \text{ for } 0 \leq i < t\}$ of first hitting the vertex $v$ at time $t$ as $(T_v = t)$. The event $\{r_H(t) = v, r_H(i) \neq v \text{ for } 0 \leq i < k \mid r_H(0) = u\}$ is referred as the first hitting of $v$ at time $t$ where $r_H$ starts at $u$ and is denoted by $(T^u_v = t)$. The expected time taken by the random walk to reach from $u$ to $v$ is referred as expected hitting time from $u$ to $v$ and is defined as $ET(u,v) = \sum_{k \in \mathbb{N}} k \text{Prob}(T^u_v = k)$.

**Theorem 4.4.** Let $r_H$ be a random walk on a hypergraph $H$. If $W_{E_0} \in \Upsilon(H)$ then for any $t \in \mathbb{N}$, we have $\text{Prob}(T^u_v = t) = \text{Prob}(T^u_ {v_2} = t)$ for any $u_1, u_2 \in W_{E_0}$ and $u \in V(H) \setminus \{u_1, u_2\}$.

**Proof.** For any $u, v(\neq u) \in V(H)$, and $t \in \mathbb{N}$, we have $(T^u_v = t) = A_t \cap A_{t-1} \cap \ldots \cap A_0$, where $A_t = (r_H(t) = v)$, $A_i = (r_H(i) \neq v)$ for all $i = 1, \ldots, t-1$, $A_0 = (r_H(0) = u)$. Therefore,

$$\text{Prob}(T^u_v = t) = \left( \sum_{r_H(t)(v) \in V(H)} P_H(r_H(t-1), v) \right) \prod_{i=1}^{t-1} \left( \sum_{r_H(t-i)(v) \in V(H)} P_H(r_H(t-i-1), r_H(t-i)) \right).$$

For any $u_1, u_2 \in W_{E_0}$, since $P_H(u, u_1) = P_H(u, u_2)$ for all $u \in V(H) \setminus \{u_1, u_2\}$ and there exists a bijection between $(T^u_{v_1} = t), (T^u_{v_2} = t)$ defined by $(v_0 = u, v_1, \ldots, v_l = u_1) \leftrightarrow (v_0 = u, v_1, \ldots, v_{l-1}, v_l = u_2)$, we have $\text{Prob}(T^u_{v_1} = t) = \text{Prob}(T^u_{v_2} = t)$ for any $u_1, u_2 \in W_{E_0}$ and $u \in V(H) \setminus \{u_1, u_2\}$. 


Similarly, we can show $\text{Prob}(T_{u_1}^u = t) = \text{Prob}(T_{u_2}^u = t)$ for any $u_1, u_2 \in W_{E_0}$ and $u \in V(H) \setminus \{u_1, u_2\}$.

**Corollary 4.5.** Let $r_H$ be a random walk on a hypergraph $H$. If $W_{E_0} \subseteq \Omega(H)$ and $u_1, u_2 \in W_{E_0}$ then $ET(u, u_1) = ET(u, u_2)$ and $ET(u_1, u) = ET(u_2, u)$ for all $u \in V(H) \setminus \{u_1, u_2\}$.

We can show the similar result for symmetric sets.

**Theorem 4.6.** Let $t_H$ be a random walk on a hypergraph $H$. If $U$ is a $\sigma_H$-symmetric set with $\delta_{V(H)}(v) = c$, a constant for all $v \in U$ then for any $t \in \mathbb{N}$, we have $\text{Prob}(T_{u_1}^u = t) = \text{Prob}(T_{u_2}^u = t)$ for any $u_1, u_2 \in U$ and $u \in V(H) \setminus \{u_1, u_2\}$.

**Proof.** Since $U$ is a $\sigma_H$-symmetric set with $\delta_{V(H)}(v) = c$, a constant for all $v \in U$ we have $P_H(u_1, u) = P_H(u_2, u)$ for all $u \in V(H) \setminus \{u_1, u_2\}$. Thus, by proceeding like the previous proof, the result follows. $\blacksquare$

Suppose that $W_{E_1}$ and $W_{E_2}$ are twin units. If there exists a bijection $h_0 : W_{E_1} \to W_{E_2}$ and $x \in \mathbb{R}^{V(H)}$ such that $x(h_0(v)) = x(v)$ for all $v \in W_{E_1}$ then $h_0 : W_{E_1} \to W_{E_2}$ is called $x$-preserving. Similarly, the canonical bijection $f_{E,E_j} : E_i \to E_j$ is called $\alpha$-preserving for some $\alpha(\in \mathbb{R}^{E(H)})$ if $\alpha(f_{E,E_j}(e)) = \alpha(e)$ for all $e \in E_i$.

**Theorem 4.7.** Let $t_H$ be a random walk on a hypergraph $H$. If $W_{E_1}$ and $W_{E_2}$ are twin unit and $h_0 : W_{E_1} \to W_{E_2}$ is a $\delta_V(V(H))$-preserving bijection with $h_0(u_1) = u_2$ and the canonical bijection $f_{E,E_j} : E_i \to E_j$ is $\delta(E(H))$-preserving then for any $t \in \mathbb{N}$, we have $\text{Prob}(T_{u_1}^u = t) = \text{Prob}(T_{u_2}^u = t)$ for any $u_1 \in W_{E_1}$ and $u_2 \in W_{E_2}$ and $u \in V(H) \setminus (W_{E_1} \cup W_{E_2})$.

**Proof.** For each event $F_t = \{t_H(t) = u_1|t_H(i) = v_i, 1 \leq i < t, t_H(0) = u\}$, there exist an event $F_t' = \{t'_H(t) = u_2|t'_H(i) = v'_i, 1 \leq i < t, t'_H(0) = u\}$ such that

$$v'_k = \begin{cases} v_k & \text{if } v_k \in V(H) \setminus (W_{E_1} \cup W_{E_2}) \\ h^{-1}_{ij}(v_k) & \text{if } v_k \in W_{E_1} \\ h_{ij}(v_k) & \text{if } v_k \in W_{E_2}. \end{cases}$$

Since $h_{ij}$ is a $\delta_V(V(H))$-preserving bijection and the canonical bijection $f_{E,E_j}$ is $\delta(E(H))$-preserving, we have $\text{Prob}(F_t) = \text{Prob}(F_t')$. Therefore, $\text{Prob}(T_{u_1}^u = t) = \text{Prob}(T_{u_2}^u = t)$ for any $u_1 \in W_{E_1}$ and $u_2 \in W_{E_2}$ and $u \in V(H) \setminus (W_{E_1} \cup W_{E_2})$. $\blacksquare$

**4.2. Unit-distance: A pseudo metric.** In a hypergraph $H$, a path between $v_0, v_l(\in V(H))$ of length $l$ is an alternating sequence of distinct vertices and edges $v_0 e_1 v_1 \ldots v_{l-1} e_l v_l$ with $v_{i-1}, v_i \in e_i$ for all $i = 1, \ldots, l$ (see [1]). In case of a walk between $v_0, v_l$, the vertices and hyperedges may not be distinct. This notion of path and walk naturally induces a metric defined by $d(u, v) = \text{length of the smallest path between } u, v(\in V(H))$. These notion of paths and distance have some limitations.

![Figure 2. A path of length 3 between 1 and 6.](image)

Consider the path of length 3 between 1 and 6 in Figure 2 composed of the hyperedges $e, f, g$. This path have multiple representations and they are $1e2f4g6, 1e3f4g6, 1e2f5g6, 1e3f5g6$. Thus, this notion of path leads us to an unavoidable redundancy that may counts the same path multiple number of times. This redundancy going to effect any notion that depends on the number of path(or walk) between two vertices. For example, for a graph $G$, the $(u, v)$-th
element of the $k$-th power of adjacency is the number of walk of length $k$ between $u, v \in V(G)$. This result does not hold for hypergraph.

A vertex $u$ should be more close to a vertex that belongs the same unit than two another vertex outside the unit. This is not taken care of by the usual distance. For example, in the path given in Figure 2, we have $d(2,3) = 1 = d(2,4)$ though $3, 2$ belongs to the same unit and $2, 4$ belong to different units. Here we construct a pseudo metric space on the set of vertices using the units. Suppose that $H$ is a hypergraph. Consider the graph $G_H$, defined by $V(G_H) = \mathcal{U}(H)$ and

$$E(G_H) = \{(W_{E_i}, W_{E_j}) \in \mathcal{U}(H)^2 : W_{E_i} \text{ and } W_{E_j} \text{ are neighbours}\}.$$

For any hypergraph $H$, an unit-walk, and unit-path between $u, v$ of length $l$ are respectively the walk and path of length $l$ between $W_{E_i}(H), W_{E_j}(H)$ in $G_H$ and the unit-distance $d_H^l : V(H) \times V(H) \rightarrow [0, \infty)$ is defined by $d_H^l(u, v) =$length of the smallest path between $W_{E_i}(H), W_{E_j}(H)$ in $G_H$. Here $d_H^l(u, v) = 0$ if $u, v$ belongs to the same unit.

For any graph $G$, the distance $d_G : V_G \times V(G) \rightarrow [0, \infty)$ is defined as $d_G(u, v) =$length of smallest path between $u, v \in V(G)$. Evidently, for any hypergraph $H$, we have $d_H^l(u, v) = d_G(W_{E_i}(H), W_{E_j}(H))$ for all $u, v \in V(H)$. We now generalize some notions of distance based graph-centrality for the hypergraph using the notion of unit-distance. For any connected hypergraph $H$, and $v \in V(H)$ the unit-eccentricity $\xi_H(v) \in \mathbb{R}^V(H)$ defined by $\xi_H(v) = \max_{u \in V(H)} d_H^\infty(u, v)$.

The unit-diameter of $H$ is $\max_{v \in V(H)} \xi_H(v)$ and the unit-radius is $\min_{v \in V(H)} \xi_H(v)$. Thus, clearly, for any hypergraph $H$, the number of hyperedges is greater than or equal to the unit-diameter of $H$. For a hypergraph $H$, we define the unit-girth of $H$ as the girth of $G_H$. For each $e \in E(H)$, there exists $n_e \in \mathbb{N}$ such that $e = \bigcup_{i=1}^{n_e} W_{E_i}$ where $W_{E_i} \in \mathcal{U}(H)$ for all $i = 1, \ldots, n_e$. The maximum partition number of $H$ is $\max_{e \in E(H)} n_e$ and minimum partition number of $H$ is $\min_{e \in E(H)} n_e$. Thus, if the minimum partition number of a hypergraph $H$ is at least $3$ then the minimum partition number of $H$ is at least equal to the unit-girth of $H$, which is equal to $3$. The unit-clique number of a hypergraph is the clique number of $G_H$.

**Proposition 4.8.** The maximum partition number of a hypergraph $H$ can be at most the unit-clique number of $H$.

**Proof.** Let $H$ be a hypergraph and $c_H$ be the unit-clique number of $H$. It is enough to prove $n_e \leq c_H$ for all $e \in E(H)$. If possible let there exists $e \in E(H)$ such that $n_e > c_H$. Thus, $e = \bigcup_{i=1}^{n_e} W_{E_i}$ and this leads us to a $n_e$-clique in $G_H$ with $n_e > c_H$, a contradiction. Therefore, $n_e \leq c_H$ for all $e \in E(H)$.

For any graph $G$, consider the usual adjacency matrix $A_G = (a_{uv})_{u,v \in V(G)}$ defined by

$$a_{uv} = \begin{cases} 1 & \text{if } \{u, v\} \in E(G) \\ 0 & \text{otherwise}. \end{cases}$$

Note that if $\delta_{E(G)}(e) = \frac{|e|^2}{|e|-1}$ for all $e \in E(G)$ and $\delta_{V(G)}(v) = 1$ for all $v \in V(G)$ then the matrix of the general adjacency operator with respect to usual basis becomes this usual adjacency matrix, that is for above mentioned choice of $\delta_{E(H)}, \delta_{V(H)}$, we have $A_Gx = \hat{A}_Gx$, for all $x \in \mathbb{R}^V(G)$. Since in a graph $G$, the $u, v$-th entry of $A_G^k$ is the number of walk of length $k$ between $u, v \in V(G)$. For any hypergraph $H$, and $u, v \in V(H)$, the $(u, v)$-th entry of $\hat{A}_G^k$ is the number of unit-walk of length $k$ between $u, v$. We can incorporate some distance-based graph-centralities into the hypergraph theory using the concept of unit-distance. For instance, in a connected hypergraph $H$, the unit-closeness centrality of each $v \in V(H)$ is defined by $c_H(v) = \frac{\sum_{u \in V(H)} 1}{d_H(u, v)}$, the unit-eccentricity based centrality of $v \in V(H)$ is defined as $c_E(v) = \frac{1}{\xi_H(v)}$. 


Let \( p_1 : V(H) \to \Omega(H) \) be a map defined as \( v \mapsto W_{E_u(H)} \). Since \( d_{GH}^H(u, v) = d_{GH}(p_1(u), p_1(v)) \), the map \( p_1 \) preserves distance. This distance preserving map \( p_1 \) leads to a topology on \( V(H) \) where \( \Omega(H) \) is a basis of that topology. Let \( p : X \to Y \) be a surjective map. If \( \tau_Y \) is a topology on \( Y \) then the quotient topology \( \tau/p \) is the smallest topology on \( X \), for which the map \( p \) is continuous. Evidently, \( \tau/p = \{ p^{-1}(U) : u \in \tau_Y \} \).

The metric \( d_{GH} \) induces the metric topology \( \tau_{d_{GH}} \) on \( V(G_H) \) generated by the open balls \( B(x, \epsilon) = \{ y \in V(G_H) : d_{GH}(x, y) < \epsilon \} \) for all \( x \in V(G_H) \), and \( \epsilon \in (0, \infty) \). Since, \( p_1 : V(H) \to \Omega(H) \) is a surjective map, we have the quotient topology \( \tau_{d_{GH}}/p_1 \) on \( H \).

**Proposition 4.9.** For any hypergraph \( H \), the collection of all the units \( \Omega(H) \) is a basis of \( \tau_{d_{GH}}/p_1 \).

**Proof.** Since \( W_{E_0} = p_1^{-1}(B(W_{E_0}, \frac{1}{2})) \) for each \( W_{E_0} \in \Omega(U) \), we have \( W_{E_0} \) is an open set in \( \tau_{d_{GH}}/p_1 \). Suppose that \( v \in V(H) \) and \( O \) is an open set in the topology \( \tau_{d_{GH}}/p_1 \) with \( v \in O \). Clearly, \( O = p^{-1}(U) \) for some \( U \in \tau_{d_{GH}}/p_1 \) and \( p(v) = W_{E_u(H)} \in U \). Therefore, \( v \in W_{E_u(H)} \subseteq O \). □

### 4.3. Proper colouring, independent set and contraction of a hypergraph.

A hypergraph \( H \) is called a simple hypergraph if for any two \( e, f \in \Omega(H) \), \( e \subseteq f \) if and only if \( e = f \). A proper colouring of a hypergraph \( H \) is a vertex colouring such that no \( e \in E(H) \) is monochromatic. If a proper colouring of a hypergraph \( H \) is possible with \( n \)-colours then \( H \) is called \( n \)-colourable hypergraph. The chromatic number of \( H \), \( \chi(H) \) is the minimum number of colours required for proper colouring of \( H \).

**Proposition 4.10.** Let \( H \) be a simple connected hypergraph \( H \) with \( |E(H)| > 1 \). If \( \hat{H} \) is \( n \)-colourable then \( H \) is \( n \)-colourable.

**Proof.** Let \( \hat{H} \) be \( n \)-colourable and \( V_i \) be the set of vertices of \( \hat{H} \), coloured by the \( i \)-th colour for all \( i = 1, \ldots, n \). Since \( e \not\subseteq V_i \) for any \( e \in E(\hat{H}) \), thus, \( e' \not\subseteq \pi^{-1}(V_i) \) for all \( e' \in E(H) \). Thus, we can colour all the vertices in \( \pi^{-1}(V_i) \) with the \( i \)-th colour, for all \( i = 1, \ldots, n \) and therefore, \( H \) is \( n \)-colourable. □

Now, we have the following result.

**Corollary 4.11.** For any simple connected hypergraph \( H \) with \( |E(H)| > 1 \), \( \chi(H) \leq \chi(\hat{H}) \leq |\mathcal{C}(\Omega(H))| \).

The equality of the above Corollary holds for many hypergraphs, for example, the chromatic number of hyperflower, blow up of any bipartite graph (in which each vertex of the bipartite graph is replaced by an unit) are equal to \( \chi(H) = |\mathcal{C}(\Omega(H))| = 2 \). For the power graph of any bipartite graph, \( |\mathcal{C}(\Omega(H))| = 3 \) but \( \chi(H) = 2 \). Any upper bound of the chromatic number of the smaller hypergraph \( \hat{H} \) is becomes an upper bound of the same for the bigger hypergraph \( H \). We refer the reader to \([3, 8]\) and references therein for different upper bounds of chromatic numbers which can be used as upper bounds for \( \chi(H) \). For any hypergraph \( H \), a set \( U \subseteq V(H) \) is said to be an independent set if \( U \) contains no hyperedge \( e \). If there exists an independent set \( U \) in \( V(H) \) then \( H \) is \( |V(H)| - |U| + 1 \)-colourable. Thus, \( \chi(H) \leq |V(\hat{H})| - \text{rank}(\hat{H}) + 2 \), where \( \text{rank}(H) = \max_{e \in E(H)} |e| \). Since \( \hat{\pi} : E(H) \to E(\hat{H}) \) is an onto map, we have the following Proposition

**Proposition 4.12.** Let \( H \) be a simple connected hypergraph \( H \) with \( |E(H)| > 1 \). For any independent set \( U \subseteq V(\hat{H}) \) in \( \hat{H} \), \( \pi^{-1}(U) \) is an independent set in \( H \).

In a simple connected hypergraph \( H \) with \( |E(H)| > 1 \), for each \( a \in \mathcal{C}(\Omega(H)) \), the set \( \pi^{-1}(a) \) is an independent set. If \( H \) is a simple hypergraph with the minimum partition number \( m \) then for any \( m - 1 \)-tuple, \( a_1, \ldots, a_{m-1} \), where \( a_i \in \mathcal{C}(\Omega(H)) \) for all \( i = 1, 2, \ldots, m - 1 \), the set \( \pi^{-1}(a_1) \cup \ldots \cup \pi^{-1}(a_{m-1}) \) is an independent set.
5. Discussion

Let $G$ and $H$ be two hypergraphs. A function $f: V(G) \to V(H)$ is called an hypergraph-isomorphism between $G$ and $H$ if $f$ is a bijection and it induced another bijection $\hat{f}: E(H) \to E(H)$, defined as, $\hat{f}(e) = \{f(v) : v \in e\}$ for all $e \in E(H)$. Thus, $E(H) = \{\hat{f}(e) : e \in E(H)\}$. A property of hypergraph $H$ is called a hypergraph-invariant if it is an invariant under any hypergraph-isomorphism. For example, the order of the hypergraph $|V(H)|$, the number of hyperedges $|E(H)|$, the rank $r_H$ and corank $c_H$. Some hypergraph-invariant of a hypergraph $H$ involving our introduced building blocks are $|\mathcal{U}(H)|$, $|V(H)|$, $|E(H)|$, $|E(G_H)|$, $\max_{W_{E_0} \in \mathcal{U}(H)} |W_{E_0}|$, $\min_{W_{E_0} \in \mathcal{U}(H)} |W_{E_0}|$, maximum partition number, minimum partition number are hypergraph invariants. Another hypergraph invariant is the monotone non-decreasing sequence of unit-cardinalities of $H \{W_{E_1}, \ldots, W_{E_k}\}$ where $\mathcal{U}(H) = \{W_{E_1}, \ldots, W_{E_k}\}$ and $W_{E_i} \leq W_{E_{i+1}}$ for all $i = 1, \ldots, k - 1$. For any hypergraph $H$, if $\mathcal{C}(\mathcal{U}(H)) = \{a_1, a_2, \ldots, a_m\}$, and $a_i = \{W_{E_1}, W_{E_2}, \ldots, W_{E_{n_i}}\}$ then $|V(H)| = \sum_{i=1}^{m} \sum_{j=1}^{n_i} |W_{E_j}|$. If $|a_i| \leq |a_{i+1}|$ then the monotone non-decreasing sequence $\{a_1, \ldots, a_k\}$ is an hypergraph invariant. The number of hyperedges $|E(H)| = \sum_{\hat{e} \in \mathcal{E}(H)} |\hat{e}|$. The number of connected component in $H$ is another hypergraph invariant, which is equal to the same of $\hat{H}$. Here we have also introduced some distance based invariant, namely, unit-radius, unit-diameter, unit-eccentricity, unit-girth of a hypergraph.

A hypergraph symmetry (or hypergraph automorphism) of a hypergraph $H$ is an bijection $\phi$ on $V(H)$ that preserves the adjacency relation. That is, $e = \{v_1, \ldots, v_m\} \in E(H)$ if and only if $\phi(e) = \{\phi(v_1), \phi(v_2), \ldots, \phi(v_m)\} \in E(H)$.

Total number of bijections on $W_{E_0}$ is $|W_{E_0}|!$. Each bijection $\phi : W_{E_0} \to W_{E_0}$, can be extended to an automorphism of $H$. Therefore, for any hypergraph $H$, each $W_{E_0} \in \mathcal{U}(H)$ corresponds to $|W_{E_0}|!$ number of hypergraph automorphism.

If $W_{E_i}, W_{E_j}$ are twin units in $H$ with $|W_{E_i}| = |W_{E_j}|$ then each bijection $\phi : W_{E_i} \to W_{E_j}$ induces a hypergraph automorphism.

The above mentioned hypergraph automorphisms alter the vertices only in a very small part of $V(H)$ (namely, a unit or a pair of twin unit). We refer these automorphisms as local automorphisms. Composition of finite number of local-automorphisms is also a hypergraph automorphism. Evidently, for each $a \in \mathcal{C}(\mathcal{U}(H))$, the set $\pi^{-1}(a)$ corresponds to a local automorphism.

Centrality of a vertex in a hypergraph $H$ is a function on $V(H)$ that determine the importance of the vertices. The notion of centrality changes with the idea of importance. If the importance of a vertex depends on the star of the vertex, then the corresponding centrality function is constant on each unit. For example, the degree centrality of $v \in V(H)$ is the $|E_v(H)|$, which is constant on each unit. In case of disease propagation, each individuals are considered as vertices and interacting communities are the hyperedges. In this network, all the vertices in a unit are equally vulnerable to the disease. Thus, one can rank the units according to their vulnerabilities and take safety measures on the whole units, instead of taking action on individuals. In the scientific collaboration network scientists are vertices and research projects are the hyperedges. Unit distance can be a good measure for the collaborative distance.

6. Conclusion

In section 2, we illustrate how the building blocks affect the structure of hypergraphs. The building blocks may be related to the hypergraph properties that are dependent on the hypergraph’s structure. The spectra of hypergraphs, for example, are affected by the building blocks. We can also see that the building blocks are important in hypergraph colouring, hypergraph automorphisms, random walks on hypergraphs, and other applications. The study of building blocks has numerous applications. In dynamical networks on hypergraphs, for example, all
dynamical systems in certain building blocks may exhibit similar behaviour. In the disease propagation model on hypergraphs, since all vertices in specific building blocks have the same expected hitting times, they form clusters of equally vulnerable people. We expect all nodes in the same unit to be informed at the same time in the information spreading model. As $\chi(H) \leq \chi(\hat{H})$, we can get sharper upper bounds of $\chi(H)$ using the existing upper bounds of $\chi(\hat{H})$. Since some building blocks are related to hypergraph automorphisms, we can use them to explore hypergraph symmetries. The concept of unit distance can be used to investigate collaborative distances in collaboration networks as well as some centralities in hypergraphs.

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