Chern-Simons-Matter Theory and Mirror Symmetry

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Abstract

In this paper we study supersymmetric Chern-Simons-matter (CSM) theories with several Higgs branches. Two such theories at small Chern-Simons level are conjectured to describe the superconformal field theory at the infrared fixed point of $\mathcal{N} = 4$ QED with $N_f = 2, 3$. In particular, the mirror symmetry which exchanges the Coulomb and Higgs branches of $\mathcal{N} = 4$ QED with $N_f = 2$ is manifest in the Chern-Simons-matter description. We also study the quantum corrections to the moduli space of a class of $\mathcal{N} = 2$ CSM theories.
1 Introduction

Chern-Simons-matter (CSM) theories in 2+1 dimensions provide a large class of (super-)conformal field theories with Lagrangian descriptions [1, 2, 3, 4, 5, 6, 7, 8]. These theories have received much attention recently due to the discovery of their $AdS_4$ gravity duals [9] (see also [10, 11, 12, 13, 14, 15]). In this paper, we explore abelian CSM theories whose moduli space has different Higgs branches that meet at a point, exhibiting the quantum criticality previously well known between the Coulomb and Higgs branch of 2+1 dimensional gauge theories [16, 17, 18].
We start with a class of $\mathcal{N} = 4$ CSM theories with $U(1) \times U(1)$ gauge group at level $(k, -k)$, which are of the BF type studied in [4]. These theories have two Higgs branches, parameterized by the scalars of the hypermultiplet matter fields. An important ingredient is that the hypermultiplet moduli spaces in CSM theories can receive quantum corrections, unlike in 2+1 dimensional $\mathcal{N} = 4$ Yang-Mills theories. The quantum corrected Higgs branches of these theories turn out to be the same as the Coulomb and Higgs branches of the infrared superconformal fixed point of $\mathcal{N} = 4$ $U(1)$ gauge theory with $N_f$ charged hypermultiplets. This leads us to conjecture that the CSM theory describes the same SCFT as the IR limit of $\mathcal{N} = 4$ SQED!

In the cases $N_f = 2, 3$, we give further evidence for this equivalence using a brane construction, and find that the infrared theory describes a certain fractional M2 brane in an $\mathcal{N} = 4$ orbifold. The $\mathcal{N} = 4$ SQED with $N_f = 2$ is known [17] to be self-mirror and has enhanced global symmetry at the IR fixed point. The mirror symmetry exchanges the Coulomb and Higgs branches of the theory. In the dual CSM description, the mirror symmetry is manifest in the Lagrangian, and exchanges the two Higgs branches. The enhanced global symmetry in the CSM theory can be understood in terms of ’t Hooft operators, which allows the construction of new symmetry currents at small Chern-Simons levels.

We then move to $\mathcal{N} = 2$ CSM theories and ask whether they can have different Higgs branches meeting at a quantum critical point. This is easy to realize in the classical theory. Quantum mechanically, the moduli spaces receive nontrivial corrections. In fact, in the class of $\mathcal{N} = 2$ theories we will consider, the classical moduli space will always be lifted by quantum effects due to the shift of Chern-Simons level when massive charged chiral multiplets are integrated out. This problem can be avoided if we start with shifted bared CS levels, such that the classical moduli space is lifted by the D-term potential, but the moduli space is restored when quantum corrections are taken into account. We find that the one-loop corrected metric of the Higgs branch is the Kähler metric of a symplectic quotient space of the form $\mathbb{C}^{M+1}/U(1)$.

The paper is organized as follows. Section 2 describes the Lagrangian of a class of $\mathcal{N} = 4$ CSM theories. The moduli spaces of these theories are studied in section 3. In section 4 we describe the brane constructions, and argue that these theories describe the same SCFT as that of the IR fixed point of $\mathcal{N} = 4$ QED with $N_f$ flavors. We also discuss a simple nonabelian generalization. In section 5, we study the moduli space of a class of $\mathcal{N} = 2$ CSM theories with no superpotential. We conclude in section 6.
2 \( \mathcal{N} = 4 \) Chern-Simons-matter theories as quantum critical points

2.1 Model II

We start by considering \( \mathcal{N} = 3 \) Chern-Simons-matter theory with gauge group \( U(1) \times U(1) \), at Chern-Simons level \((k, -k)\), and matter hypermultiplets \((X_i, \tilde{X}_i)\) of charge \((+1, +1)\) and \((-1, -1)\), \(i = 1, \ldots, N_1\), and \((Y_{i'}, \tilde{Y}_{i'})\) of charge \((+1, -1)\) and \((-1, +1)\), \(i' = 1, \ldots, N_2\). In \( \mathcal{N} = 2 \) language, we have gauge multiplets \((A_\mu, \sigma, D; \chi)\) and \((\tilde{A}_\mu, \tilde{\sigma}, \tilde{D}; \tilde{\chi})\), and chiral multiplets \(X_i, \tilde{X}_i, Y_{i'}, \tilde{Y}_{i'}\). The D-term scalar potential before integrating out the auxiliary fields is

\[
V_D = \frac{k}{2\pi} D\sigma - \frac{k}{2\pi} \tilde{D}\tilde{\sigma} + \sum_i (\sigma + \tilde{\sigma})^2(|X_i|^2 + |\tilde{X}_i|^2) + \sum_{i'} (\sigma - \tilde{\sigma})^2(|Y_{i'}|^2 + |\tilde{Y}_{i'}|^2)
\]

\[
+ D(\sum_i |X_i|^2 - \sum_i |\tilde{X}_i|^2 + \sum_{i'} |Y_{i'}|^2 - \sum_{i'} |\tilde{Y}_{i'}|^2)
\]

\[
+ \tilde{D}(\sum_i |X_i|^2 - \sum_i |\tilde{X}_i|^2 - \sum_{i'} |Y_{i'}|^2 + \sum_{i'} |\tilde{Y}_{i'}|^2).
\]

Integrating out \( D \) and \( \tilde{D} \) sets

\[
\sigma = -\frac{2\pi}{k}(\sum_i |X_i|^2 - \sum_i |\tilde{X}_i|^2 + \sum_{i'} |Y_{i'}|^2 - \sum_{i'} |\tilde{Y}_{i'}|^2),
\]

\[
\tilde{\sigma} = \frac{2\pi}{k}(\sum_i |X_i|^2 - \sum_i |\tilde{X}_i|^2 - \sum_{i'} |Y_{i'}|^2 + \sum_{i'} |\tilde{Y}_{i'}|^2).
\]

So we obtain the scalar potential

\[
V_D = 16\pi^2 k^2 \left( \left( \sum_i |X_i|^2 - \sum_i |\tilde{X}_i|^2 \right)^2 \left( \sum_{i'} |Y_{i'}|^2 + \sum_{i'} |\tilde{Y}_{i'}|^2 \right) \right.
\]

\[
+ \left( \sum_{i'} |Y_{i'}|^2 - \sum_{i'} |\tilde{Y}_{i'}|^2 \right)^2 \left( \sum_i |X_i|^2 + \sum_i |\tilde{X}_i|^2 \right) \right]
\]

There is an \( \mathcal{N} = 2 \) superpotential

\[
W = \frac{8\pi}{k} \sum_i X_i \tilde{X}_i \sum_{j'} Y_{j'} \tilde{Y}_{j'}
\]
giving rise to the $F$-term scalar potential

$$V_F = \frac{64\pi^2}{k^2} \left[ \sum_i X_i \bar{X}_i \right]^2 \left( \sum \left| Y_{i'} \right|^2 + \sum \left| \bar{Y}_{i'} \right|^2 \right) + \left| \sum \bar{Y}_{i'} \right|^2 \left( \sum \left| X_i \right|^2 + \sum \left| \bar{X}_i \right|^2 \right)$$

(2.5)

The Higgs branch moduli space contains the locus where all $X_i$’s vanish and $Y_{i'}$’s arbitrary, and all $Y_{i'}$’s vanish and $X_i$’s arbitrary. However, for $N_1, N_2 > 1$, there is also the locus given by

$$\sum \left| X_i \right|^2 - \sum \left| \bar{X}_i \right|^2 = 0, \quad \sum X_i \bar{X}_i = 0,$$

$$\sum \left| Y_{i'} \right|^2 - \sum \left| \bar{Y}_{i'} \right|^2 = 0, \quad \sum \bar{Y}_{i'} Y_{i'} = 0.$$  

(2.6)

so that the entire Higgs moduli space is connected. A special case, however, is when $N_1 = N_2 = 1$, where the scalar potential becomes

$$V = \frac{16\pi^2}{k^2} \left[ (|X|^2 + |\bar{X}|^2)(|Y|^2 + |\bar{Y}|^2) + (|Y|^2 + |\bar{Y}|^2)(|X|^2 + |\bar{X}|^2) \right]$$

(2.7)

In manifestly $SU(2)$ R-symmetry invariant notation, we can write $X_a = (X, \tilde{X})$, $Y_a = (Y, \tilde{Y})$, $\xi^a = (\xi, \tilde{\xi})$, $\eta^a = (\eta, \tilde{\eta})$. The fermion-boson coupling in the $\mathcal{N} = 3$ CSM with $N_1 = N_2 = 1$ is given by

$$L_F = \frac{4\pi}{k} \left[ (|X|^2 - |\bar{X}|^2)(i\eta - \tilde{\eta}) + (|Y|^2 - |\bar{Y}|^2)(i\xi - \tilde{\xi}) \right] + \frac{8\pi}{k} \left[ (X\tilde{\xi} - \tilde{X}\xi)(Y\eta - \tilde{Y}\eta) + \text{c.c.} \right] + \frac{8\pi}{k} \left( X\tilde{X}\eta\eta + Y\tilde{Y}\xi\xi + XY\tilde{\xi}\eta + X\tilde{Y}\xi\eta + \text{c.c.} \right) = \frac{8\pi}{k} \left( \tilde{X}_{(a} X_{b)} \eta^a \eta^b + \tilde{Y}_{(A} Y_{B)} \xi^A \xi^B + \tilde{X}_{a} \eta^a Y_{b} \xi^b + X_{a} \xi^a \tilde{Y}_{b} \eta^b + \tilde{Y}_{a} \xi^a X_{b} \eta^b \right)$$

(2.8)

Now we can see that the theory in fact has $SU(2)_L \times SU(2)_R$ symmetry, under which $X$ and $\eta$ transform as $(2, 1)$ whereas $Y$ and $\xi$ transform as $(1, 2)$. The $\mathcal{N} = 3$ supersymmetry is in fact enhanced to $\mathcal{N} = 4$. $X$ and $Y$ then become $\mathcal{N} = 4$ hypermultiplet and twisted hypermultiplet, respectively. We shall refer to this theory at level $(k, -k)$ as “Model II(k).” It is in fact the same as the $\mathcal{N} = 4$ BF theory studied in [4]. In the $SU(2)_L \times SU(2)_R$ invariant notation, we can write the fermion-boson coupling as

$$L_F = \frac{8\pi}{k} \left[ \tilde{X}_{(a} X_{b)} \eta^a \eta^b + \tilde{Y}_{(A} Y_{B)} \xi^A \xi^B + \tilde{X}_{a} \eta^a Y_{A} \xi^A + \tilde{Y}_{A} \xi^A X_{a} \eta^a + X_{a} \xi^a Y_{A} \tilde{\xi}^A + \tilde{Y}_{A} \tilde{\xi}^A \tilde{X}_{a} \eta^a \right]$$

(2.9)
A slightly more general case is the $\mathcal{N} = 4$ $U(1)_k \times U(1)_{-k}$ CSM theory with $N_1 = N_f - 1$, $N_2 = 1$. We shall refer to this theory as model II($N_f)_k$. It has scalar potential

$$V = \frac{16\pi^2}{k} \left( |Y|^2 + |\tilde{Y}|^2 \right)^2 \left( \sum_i |X_i|^2 + \sum_i |\tilde{X}_i|^2 \right)$$

$$+ \frac{16\pi^2}{k} \left[ \left( \sum_i |X_i|^2 - \sum_i |\tilde{X}_i|^2 \right)^2 + 4 \sum_i X_i \tilde{X}_i \right] \left( |Y|^2 + |\tilde{Y}|^2 \right)$$

(2.10)

There are two branches of Higgs moduli spaces, $\mathcal{M}_X$ of complex dimension $2(N_f - 1)$ parameterized by arbitrary $X_i, \tilde{X}_i$ and vanishing $Y, \tilde{Y}$, and $\mathcal{M}_Y$ of complex dimension 2 parameterized by arbitrary $Y, \tilde{Y}$ and vanishing $X_i, \tilde{X}_i$. They meet at the origin. Due to the Mukhi effect [19, 9], the moduli space is modded out by a discrete group of constant gauge transformations. We will discuss this as well as the quantum corrections to the moduli space in section 3. For now, we note that the $X$-branch of the moduli space, $\mathcal{M}_X$, should be singular along the locus

$$S_X : \sum_i |X_i|^2 - \sum_i |\tilde{X}_i|^2 = \sum_i X_i \tilde{X}_i = 0,$$

(2.11)

where $Y$ and $\tilde{Y}$ becomes massless.

### 2.2 Model III

The next model we shall consider is the $\mathcal{N} = 3$ $U(1)_k \times U(1)_{-k} \times U(1)_k$ CSM theory, with hypermultiplet $(X, \tilde{X})$ of charges $(+1, -1, 0)$ and $(-1, +1, 0)$ and $(Y, \tilde{Y})$ of charge $(0, -1, +1)$ and $(0, +1, -1)$. We will refer to this theory as “model III”. The overall $U(1)$ in this theory decouples, and can be integrated out. Denote the gauge fields of the three $U(1)$’s by $A_1, A_2, A_3$, and define $a = A_1 - A_2$, $b = A_3 - A_2$, then the CS term is

$$\frac{k}{4\pi} \int (A_1 \wedge dA_1 - A_2 \wedge dA_2 + A_3 \wedge dA_3)$$

$$= \frac{k}{4\pi} \int [(A_2 + a) \wedge d(A_2 + a) - A_2 \wedge dA_2 + (A_2 + b) \wedge d(A_2 + b)]$$

(2.12)

Now $A_2$ decouples from the matter fields, and up to a gauge transformation its equation of motion sets $A_2 = -a - b$, and the CS term becomes

$$-\frac{2k}{4\pi} \int a \wedge db$$

(2.13)

Upon redefining $a_\mu = A_\mu + \tilde{A}_\mu$, $b_\mu = A_\mu - \tilde{A}_\mu$, we recover model II at level $2k$. Therefore, we see that $\text{III}_k$ is the same theory as $\text{II}_{2k}$.
2.3 Model IV

Now consider the $\mathcal{N} = 3 U(1)_- \times U(1)_+ \times U(1)_- \times U(1)_+$ CSM theory, with hypermultiplets $(X, \tilde{X}), (Y, \tilde{Y}), (Z, \tilde{Z})$, of charges $(+1, -1, 0, 0), (0, +1, -1, 0)$ and $(0, 0, +1, -1)$. This theory will be denoted model IV$_k$. It has in fact $\mathcal{N} = 4$ supersymmetry as well. Writing the CS action as

$$S_{CS} = \frac{k}{4\pi} \int (-A_1 \wedge dA_1 + A_2 \wedge dA_2 - A_3 \wedge dA_3 + A_4 \wedge dA_4),$$

(2.14)

the overall $U(1)$ decouples from the matter fields. Writing $a = A_1 - A_2, b = A_2 - A_3, c = A_3 - A_4$, then integrating out $A_4$ sets $a + c = 0$. The CS action now reduces to

$$\frac{k}{2\pi} \int a \wedge db$$

(2.15)

and the hypermultiplets $X, Y, Z$ have charges $(1, 0), (0, 1)$ and $(-1, 0)$ under $(a, b)$. Redefining $a = A + \tilde{A}$ and $b = A - \tilde{A}$, the fields $X, Y, Z$ now have charges $(+1, +1), (+1, -1)$ and $(-1, -1)$ under $(A, \tilde{A})$. We can interchange $Z$ with $\tilde{Z}$, and hence model IV$_k$ is the same as model II$(N_f = 3)_{2k}$ introduced earlier.

3 Quantum corrections to hypermultiplet moduli space

In 2+1 dimensional $\mathcal{N} = 4$ Yang-Mills theories coupled to hypermultiplet matter fields, the Higgs branch moduli space, i.e. the moduli space of hypermultiplets, is not corrected by quantum effects because the Yang-Mills coupling constant can be promoted to a vector superfield, which decouples from the hypermultiplets at the level of kinetic terms. This non-renormalization argument does not apply to $\mathcal{N} = 4$ CSM theories $\mathcal{E}$. So the hypermultiplet moduli space in general can and in fact will get quantum corrections, as we shall argue.

Let us start with model II$(N_f)_{k}$. There are $N_f - 1$ hypermultiplets $X_i$ of charge $(+1, +1)$ and 1 hypermultiplet $Y$ of charge $(+1, -1)$, under the $U(1) \times U(1)$ gauge fields $A_\mu, \tilde{A}_\mu$. Write $a_\mu = A_\mu + \tilde{A}_\mu, b_\mu = A_\mu - \tilde{A}_\mu$, and let $n_a$ and $n_b$ be the magnetic flux of $a_\mu$ and $b_\mu$ on the sphere at infinity. The constant gauge transformations $e^{2\pi i n_a}$ and $e^{2\pi i n_b}$ satisfy

$$\frac{k}{2}(\eta_a n_b + \eta_b n_a) \in \mathbb{Z}$$

(3.1)

where the factor of $1/2$ is due to the normalization of the twisted CS term $\int a \wedge db$. A priori, it follows from Dirac quantization condition that $n_a$ and $n_b$ should be integer
valued. So the constant gauge transformations are given by

\[ \eta_a, \eta_b \in \frac{2}{k} \mathbb{Z}, \]  

(3.2)

For even \( k \), they generate the subgroup \( \mathbb{Z}_{k/2} \times \mathbb{Z}_{k/2} \subset U(1) \times U(1) \). In the previous section we have obtained model II(\( N_f \)) with even \( k \) by integrating out the overall \( U(1) \) in model III and IV, the latter involving only bifundamentals and can be embedded straightforwardly in string theory. So we will assume \( k \) is even in model II for now.

The classical moduli space has two Higgs branches \( \mathcal{M}_X \) and \( \mathcal{M}_Y \). After modding out by constant gauge symmetries, we have \( \mathcal{M}_X^1 \simeq \mathbb{C}^{2(N_f - 1)}/\mathbb{Z}_{k/2} \), and \( \mathcal{M}_Y^1 \simeq \mathbb{C}^2/\mathbb{Z}_{k/2} \). Let us first consider \( \mathcal{M}_Y^1 \). Along this moduli space, the fields \((X_i, \bar{X}_i)\) are massive, and can be integrated out. The hypermultiplet mass for \((R\text{-symmetry }, \pm 1)\) the origin of \( \mathcal{M}_Y \) together with the rigidity of hyperkähler metrics and the homogeneity in \((Y, \bar{Y})\) then fixes \( \mathcal{M}_Y \). After integrating out \( N_f - 1 \) hypermultiplets, we end up with \( \mathcal{M}_Y \simeq \mathbb{C}^2/\mathbb{Z}_{k/2}^{N_f - 1} \).

To justify our proposal, one may compute the correction to the metric on \( \mathcal{M}_Y \) by integrating out \((X_i, \bar{X}_i)\) at one-loop. As well known in the Coulomb branch moduli space in SQED [16, 17, 23], the one-loop contribution to the kinetic term from a hypermultiplet of mass \( \bar{m}_X \) is to shift the degree of this circle bundle by +1. The topology of \( \mathcal{M}_Y \) together with the rigidity of hyperkähler metrics and the homogeneity in \((Y, \bar{Y})\) then fixes \( \mathcal{M}_Y \). After integrating out \( N_f - 1 \) hypermultiplets, we end up with \( \mathcal{M}_Y \simeq \mathbb{C}^2/\mathbb{Z}_{k/2}^{N_f - 1} \).

\[
\frac{4\pi}{k} Y_a Y_b \equiv \frac{4\pi}{k} \left( \frac{1}{2}(|Y|^2 - |\bar{Y}|^2), Y \bar{Y}, Y \bar{Y} \right) \equiv \bar{m}_X \tag{3.3}
\]

\( \mathcal{M}_Y^1 \) is a \( S^1 \)-bundle over the \( \mathbb{R}^3 \) parameterized by \( \bar{m}_X \), such that on any \( S^2 \) around the origin of \( \mathbb{R}^3 \) the fibration is a circle bundle of degree \( k/2 \). We propose that the effect of integrating out one hypermultiplet of mass \( \bar{m}_X \) is to shift the degree of this circle bundle by +1. The topology of \( \mathcal{M}_Y \) together with the rigidity of hyperkähler metrics and the homogeneity in \((Y, \bar{Y})\) then fixes \( \mathcal{M}_Y \). After integrating out \( N_f - 1 \) hypermultiplets, we end up with \( \mathcal{M}_Y \simeq \mathbb{C}^2/\mathbb{Z}_{k/2}^{N_f - 1} \).

To justify our proposal, one may compute the correction to the metric on \( \mathcal{M}_Y \) by integrating out \((X_i, \bar{X}_i)\) at one-loop. As well known in the Coulomb branch moduli space in SQED [16, 17, 23], the one-loop contribution to the kinetic term from a hypermultiplet of mass \( \bar{m} \) coupled to gauge field \( a_\mu \) takes the form

\[
\int \frac{1}{8\pi|\bar{m}|} (\partial_\mu \bar{m} \cdot \partial^\mu \bar{m} - |da|^2) + e^{\frac{i}{2} \phi} \partial_\mu (\frac{1}{8\pi|\bar{m}|}) a_\mu \partial_\nu m_j \partial_\rho m_k \tag{3.4}
\]

\[
= \int \frac{1}{8\pi|\bar{m}|} (\partial_\mu \bar{m} \cdot \partial^\mu \bar{m} - |da|^2) + (\ast da)^\mu \omega_i \partial_\mu m_i \]

where \( \omega_i \) is the vector potential of a Dirac monopole in the \( \bar{m} \)-space. We can dualize \( a_\mu \) by replacing \( da \) with an independent two-form field \( \tilde{F}_a \) and introducing the Lagrangian multiplier field \( \varphi_Y \) of periodicity 1. The bosonic part of the action is

\[
\int |D_\mu Y|^2 + \frac{1}{8\pi|\bar{m}|} (\partial_\mu \bar{m} \cdot \partial^\mu \bar{m} - |\tilde{F}_a|^2) + \int \tilde{F}_a \wedge (d\varphi_Y + \omega_i dm_i + \frac{k}{4\pi} b) \tag{3.5}
\]

Integrating out \( \tilde{F}_a \) gives

\[
\int |D_\mu Y|^2 + \frac{1}{8\pi|\bar{m}|} \partial_\mu \bar{m} \cdot \partial^\mu \bar{m} + 2\pi |\bar{m}| (\partial_\mu \varphi_Y + \omega_i \partial_\mu m_i + \frac{k}{4\pi} b_\mu)^2 \tag{3.6}
\]
Finally, integrating out $b_\mu$ then identifies $\frac{2}{k}\varphi_Y$ with the overall phase of $Y$. The $U(1)$ gauge symmetry acts on the fields as

$$Y \rightarrow e^{iA}Y, \quad b_\mu \rightarrow b_\mu + \partial_\mu \Lambda, \quad \varphi_Y \rightarrow \varphi_Y - \frac{k}{4\pi} \Lambda$$

The constant gauge transformations $\mathbb{Z}_{k/2}$ act on $Y$ as $Y \rightarrow e^{i\pi i/k}Y$, i.e. the phase of $Y$ has periodicity $4\pi/k$. Therefore the effect of the one-loop correction is to have the $S^1$ parameterized by the phase of $Y$ fibered over $\mathbb{R}^3 = \{\vec{m}\}$ with degree shifted from $-\frac{k}{2}$ to $\frac{k}{2} + 1$. This is due to the coupling $k|\vec{m}|^{\omega_i}e^{i\mu_i}$, which effectively shifts phase rotation of $Y$ when one sends $\varphi_Y \rightarrow \varphi_Y + 1$. Similarly, when $N_f - 1$ $(X_i, \tilde{X}_i)$’s are integrated out, the degree is shifted from $\frac{k}{2}$ to $\frac{k}{2} + N_f - 1$.

Let us now consider the moduli space $\mathcal{M}_X$. The hypermultiplet mass of $(Y, \tilde{Y})$ along $\mathcal{M}_X^l$ is given by

$$\frac{4\pi}{k} \sum_{i=1}^{N_f-1} \tilde{X}_i X_{ib} = \frac{4\pi}{k} \left( \frac{1}{2} \sum_i (|X_i|^2 - |	ilde{X}_i|^2), \sum_i X_i \tilde{X}_i, \sum_i \tilde{X}_i \tilde{X}_i \right) \equiv \vec{m}_Y$$

We can think of (3.8) as a fibration of $\mathcal{M}_X^l$ over $\mathbb{R}^3 = \{\vec{m}_Y\}$, whose fiber is an $S^1$-bundle $L_{k/2}$ over $T^*\mathbb{C}P^{N_f-2}$. Here $L_{k/2}$ is fibered over the $\mathbb{C}P^{N_f-2}$ with degree $k/2$, due to quotienting by constant gauge symmetries $\mathbb{Z}_{k/2}$. Following our discussion above, the effect of integrating out $(Y, \tilde{Y})$ is to tensor this $S^1$-bundle (which is also fibered over $\mathbb{R}^3$) with the degree +1 circle bundle over $\mathbb{R}^3 - \{0\}$. Spelling this out explicitly, the corrected moduli space $\mathcal{M}_Y$ can be expressed as

$$\left( \frac{1}{2} \sum_i (|X_i|^2 - |	ilde{X}_i|^2), \sum_i X_i \tilde{X}_i, \sum_i \tilde{X}_i \tilde{X}_i \right) = \frac{k}{4\pi} \vec{m}_Y,$$

$$\left( \frac{1}{2} (|Q|^2 - |	ilde{Q}|^2), Q \tilde{Q}, Q \tilde{Q} \right) = \frac{k}{4\pi} \vec{m}_Y,$$

modulo $U(1) : (X_i \rightarrow e^{\frac{2i}{k} \theta} X_i, \tilde{X}_i \rightarrow e^{-\frac{2i}{k} \theta} \tilde{X}_i, Q \rightarrow e^{-i\theta} Q, \tilde{Q} \rightarrow e^{i\theta} \tilde{Q})$,

where we have introduce the variables $(Q, \tilde{Q})$ to parameterize an $S^1$ fibered over $\mathbb{R}^3 - \{0\}$ of degree 1, so that after quotienting by the $U(1)$, the degree of $L_{k/2}$ over $\mathbb{R}^3 - \{0\}$ is shifted by +1. In other words, $\mathcal{M}_Y$ is the hyperkähler quotient of $\mathbb{C}^{2N_f} / \mathbb{Z}_{k/2} / / / U(1)$, where the coordinates $(X_i, \tilde{X}_i, Q, \tilde{Q})$ are assigned charges $(1, -1, -\frac{k}{2}, \frac{k}{2})$ under the $U(1)$. For $N_f = 2$, $\mathcal{M}_Y$ reduces to $\mathbb{C}^2 / \mathbb{Z}_{2N_f}$.
4 Brane construction, enhanced global symmetry, and $\mathcal{N} = 4$ QED

4.1 Model III and $\mathcal{N} = 4$ QED with $N_f = 2$

Our model $\text{III}_k$ can be engineered in type IIB string theory \cite{20, 21, 22, 27} as a D3-brane suspended from one NS5-brane, across a $(1, k)$ 5-brane and a NS5-brane, to another $(1, k)$ 5-brane, arranged to preserve $\mathcal{N} = 3$ supersymmetry. After putting this system on a circle, taking the small radius limit, performing T-duality and lifting to M-theory, we end up the toric hyperkähler manifold $X_8$:

$$
\text{ds}_8^2 = U_{ab} d\vec{x}^a \cdot d\vec{x}^b + U^{ab}(d\phi_a + A_a)(d\phi_b + A_b),
$$

$$
U_{ab} = \frac{2}{|\vec{x}_1|} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{2}{|\vec{x}_1 + k\vec{x}_2|} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}
$$

The suspended D3 brane dualizes to a fractional M2 brane at the singularity at the origin in this space, since the D3 did not entirely wrap the circle. After the change of variables

$$
\vec{x}'_1 = \vec{x}_1, \quad \vec{x}'_2 = \vec{x}_1 + k\vec{x}_2, \quad \phi'_1 = \phi_1 - \frac{1}{k}\phi_2, \quad \phi'_2 = \frac{1}{k}\phi_2.
$$

we see that $X_8$ is $(\mathbb{C}^2/\mathbb{Z}_2)^2/\mathbb{Z}_k$. For general $k > 2$, $X_8$ has symmetry $SU(2) \times SU(2)$. This is the R-symmetry of the $\mathcal{N} = 4$ CSM theory. For $k = 1, 2$, however, the symmetry of $X_8$ is enhanced to $SO(4) \times SO(4) \simeq SU(2)_F \times SU(2)'_F \times SU(2)_L \times SU(2)_R$. The D3-brane has turned into a fractional M2-brane sitting at the singularity of $X_8$.

Note that model III at level $k = 1$ has the same moduli space and global symmetry as the IR SCFT of $\mathcal{N} = 4 U(1)$ gauge theory with $N_f = 2$ hypermultiplet matter fields. We shall refer to this SCFT as SQED-2. The quantum moduli space of both theories are two branches of $\mathbb{C}^2/\mathbb{Z}_2$, meeting at the origin. We shall argue that III$_1$ and SQED-2 are in fact the same SCFT.

We can start with the brane configuration of a pair of NS5-branes and a pair of D5-branes in between, and a D3-brane stretching from one NS5 to the other, crossing the two D5-branes. Explicitly, the configuration is (with the axion $\chi$ set to zero)

$$
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{NS5} & \times & \times & \times & 0 & \times & \times & \times \\
\text{NS5}' & \times & \times & \times & L & \times & \times & \times \\
\text{D5} & \times & \times & \times & y_1 & \times & \times & \times \\
\text{D5}' & \times & \times & \times & y_2 & \times & \times & \times \\
\text{D3} & \times & \times & \times & & & & & & \\
\end{array}
$$
with $0 < y_1 < y_2 < L$. The low energy world volume theory on the D3-brane is 2+1 dimensional $\mathcal{N} = 4$ QED with $N_f = 2$. Now we can move D5' to the right, crossing NS5', and end up with $0 < y_1 < L < y_2$. A single D3 brane is created between the NS5' and the D5' by the Hanany-Witten effect [20], so the D3-brane is now stretched from NS5 to D5'. We can perform the $SL(2, \mathbb{Z})$ duality $\tau \rightarrow \tau/(1 - \tau)$, so that the D5-branes are turned into $(1,1)$ 5-branes, while the NS5-branes stay the same. The low energy world volume theory is now described by model III at level $k = 1$. This leads us to the conjecture that model IIII is the same as SQED-2.

The SQED-2 has been conjectured to have enhanced $SU(2)_F \times SU(2)'_F \times SU(2)_L \times SU(2)'_L$ global symmetry, where the $SU(2)'_F$ is not manifest in the UV description, and emerges at the IR fixed point [17]. The $SU(2)_F$ and $SU(2)'_F$ are exchanged by mirror symmetry. In model III$k=1$, neither $SU(2)_F$ nor $SU(2)'_F$ is manifest in the Lagrangian of the theory, but the mirror symmetry is a manifest $\mathbb{Z}_2$ symmetry of the Lagrangian, and so is the symmetry between the two branches of the Higgs moduli space.

To see how the $SU(2)_F \times SU(2)'_F$ emerges in model III at $k = 1$, or model II at $k = 2$, note firstly that for $II_k=1,2$, by combining with a 't Hooft operator, we can shift the $U(1)$ charge of $\tilde{X}_a$ to that of $X_a$, etc. We shall denote the field $\tilde{X}_a$ dressed by the 't Hooft operator as $C\tilde{X}_a$. This makes it possible for $X_a$ and $C\tilde{X}_a$ to fit into a multiplet of $SU(2)_F \times SU(2)_L$, for some $SU(2)_F$ symmetry.

So we may group the fields into multiplets of $SU(2)_F \times SU(2)'_F \times SU(2)_L \times SU(2)_R$ for some $SU(2)_F \times SU(2)'_F$ as

$$
X_{ia} = (X_a, C\tilde{X}_a) \in (2, 1, 2, 1),
Y_{ia} = (Y_A, CY_A) \in (1, 2, 1, 2),
\xi_{ia} = (\xi_A, C\xi_A) \in (2, 1, 1, 2),
\eta_{ia} = (\eta_A, C\eta_A) \in (1, 2, 2, 1).
(4.3)
$$

Note that the scalar potential $V$ would be invariant under the $SU(2)^4$ if we naively ignore the distinct charges of $X_a, Y_a$ and $\tilde{X}_a, \tilde{Y}_a$. It is not clear to us how to see the $SU(2)^4$ symmetry in the fermion-boson coupling, due to the difficulty of describing the 't Hooft operator in the Lagrangian formalism.

Instead, let us compare the chiral primaries and $SU(2)_F \times SU(2)'_F$ in SQED-2 and model II. Let $(Q_i, \tilde{Q}^i)$, $i = 1, 2$, be the complex scalars in the hypermultiplets of SQED-2. In $\mathcal{N} = 2$ language, we have the chiral primary operators $Q_1\tilde{Q}^2$, $Q_2\tilde{Q}^1$ and $Q_1\tilde{Q}^1 - Q_2\tilde{Q}^2$, in a triplet of $SU(2)_F$. The $\mathcal{N} = 4$ R-symmetry completes them into a multiplet $(3, 3, 1)$ under $SU(2)_F \times SU(2)_L \times SU(2)_R$. In model II, there is the chiral primary $X\tilde{X}$, completely by R-symmetry into the triplet

$$
XXX, \tilde{X}\tilde{X}, |X|^2 - |\tilde{X}|^2
(4.4)
$$
in the representation \((3,1)\) of \(SU(2)_L \times SU(2)_R\). With the ’t Hooft operators that correspond to \((+1,-1)\) or \((-1,+1)\) units of magnetic flux on the \(S^2\) (denoted by \(C\) and \(C^{-1}\)), we also have

\[
C^{-1}X^2, \ C^{-1}X\bar{X}, \ C^{-1}X\tilde{X}, \ CX^2, \ CX\bar{X}, \ CX\tilde{X},
\]

(4.5)

which are protected to have dimension 1. \(C^{-1}X^2\) for instance, under the state/operator mapping, corresponds to two \(X\) particles in their ground state on the \(S^2\) with \((-1,+1)\) units of magnetic fluxes. Together with (4.4), they form a multiplet in the representation \((3,1,3,1)\) of \(SU(2)_F \times SU(2)'_F \times SU(2)_L \times SU(2)_R\).

Similarly, we can identify the \(SU(2)_F\) currents as

\[
\tilde{X}_a \overline{D}_\mu X^a + \overline{\xi}_A \sigma_\mu \xi^A, \ C^{-1}X_a \overline{D}_\mu X^a + C^{-1}\overline{\xi}_A \sigma_\mu \xi^A, \ CX_a \overline{D}_\mu \tilde{X}^a + C\overline{\xi}_A \sigma_\mu \xi^A. \]

(4.6)

They are in the same supermultiplet as the dimension 1 operators above.

### 4.2 Model IV and \(N = 4\) QED with \(N_f = 3\)

Let us consider \(N = 4\) \(U(1)\) gauge theory with \(N_f = 3\) hypermultiplet matters. It can be engineered by suspending a D3-brane between two NS5-branes, and intersecting three D5-branes in between. The 5-branes are separated along \(x^3\) direction as before, with the NS5-branes at \(x^3 = 0, L\), and the D5-branes at \(x^3 = y_1, y_2, y_3\), with \(0 < y_1 < y_2 < y_3 < L\). Now let us move the D5-branes at \(y_1\) and \(y_3\) to the left and right of the two NS5-branes, i.e. \(y_1 < 0 < y_2 < L < y_3\), again inducing the creation of stretched D3-branes. The D3-brane is then suspended from the D5-brane at \(x^3 = y_1\) to the D5-brane at \(y_3\). Further performing a \(\tau \rightarrow \tau/(1-\tau)\) turns the D5-branes into \((1,1)\) 5-branes. The low energy world volume theory on the suspended D3-brane is now the \(N = 4\) \(U(1)\)\(_{-1}\) × \(U(1)\)\(_1\) × \(U(1)\)\(_{-1}\) × \(U(1)\)\(_1\) CSM theory with three hypermultiplets \((X, \tilde{X}), (Y, \tilde{Y}), (Z, \tilde{Z})\), of charges \((+1, -1, 0, 0), (0, +1, -1, 0)\) and \((0, 0, +1, -1)\), which we called model IV at \(k = 1\).

We can consider the more general brane configuration, with the \((1,1)\) 5-branes replaced by \((1, k)\) 5-branes, at the suitable angles to preserve \(N = 3\) supersymmetry, which is enhanced to \(N = 4\) when the axio-dilaton lies on a particular curve. The infrared theory does not depend on the choice of \(\tau\), and hence also possesses this \(N = 4\) supersymmetry. After T-duality and lifting to M-theory, we obtain a fractional M2-brane at the origin of the toric hyperkähler orbifold \(X'_k = ((\mathbb{C}^2/\mathbb{Z}_2) \times (\mathbb{C}^2/\mathbb{Z}_3))/\mathbb{Z}_k\). The low energy world volume theory is model IV\(_k\). The \(N = 4\) supersymmetry is now evident from the \(SU(2)_L \times SU(2)_R\) symmetry of \(X'_k\) for general \(k\). In \(N = 2\) language, the superpotential is

\[
W = \frac{4\pi}{k} (X\tilde{X} - Z\tilde{Z})Y\tilde{Y}
\]

(4.7)
By examining the F-flatness and D-flatness conditions, we again find two branches of the Higgs moduli space, \( M_{H_1} \), parameterized by \((Y, \tilde{Y})\), with \( X = \tilde{X} = Z = \tilde{Z} = 0 \), and \( M_{H_2} \) parameterized by \((X, \tilde{X}, Z, \tilde{Z})\) unconstrained, with \( Y = \tilde{Y} = 0 \).

The SQED-3 SCFT on the other hand, has Coulomb branch moduli space \( M_{SQED}^{C} \simeq \mathbb{C}^2/\mathbb{Z}_3 \), and Higgs branch moduli space \( M_{SQED}^{H} \) being the hyperkähler quotient

\[
\left\{ \sum_{i=1}^{3} (|Q_i|^2 - |\tilde{Q}_i|^2) = 0, \quad \sum_{i=1}^{3} Q_i \tilde{Q}_i = 0 \right\} / U(1) \quad (4.8)
\]

which has \( SU(3) \) isometry.

The classical moduli space of model IV\(_{k=1}\) has two branches, isomorphic to \( \mathbb{C}^2 \) and \( \mathbb{C}^4 \), meeting at the origin. However, as we have argued in section 3, when the massive hypermultiplets \( X \) and \( Z \) are integrated out, the branch \( M_{H_1} \) is corrected into \( \mathbb{C}^2/\mathbb{Z}_3 \). This is also suggested by the fractional M2-brane picture. Furthermore, \( M_{H_2} \) is singular along the locus

\[
|X|^2 - |\tilde{X}|^2 - |Z|^2 + |\tilde{Z}|^2 = X \tilde{X} - Z \tilde{Z} = 0, \quad (4.9)
\]

where \((Y, \tilde{Y})\) become massless. The effect of integrating out \((Y, \tilde{Y})\) is to turn \( M_{H_2} \) into

\[
\left\{ |X|^2 - |\tilde{X}|^2 - |Z|^2 + |\tilde{Z}|^2 = |Q|^2 - |\tilde{Q}|^2, \quad X \tilde{X} - Z \tilde{Z} = Q \tilde{Q} \right\} / U(1) \quad (4.10)
\]

where the \( U(1) \) acts on \((X, \tilde{X}, Z, \tilde{Z}, Q, \tilde{Q})\) with charges \((1, -1, -1, 1, -1, 1)\). So we see that the quantum corrected moduli spaces \( M_{H_1} \) and \( M_{H_2} \) of the CSM theory at \( k = 1 \) are precisely the same as \( M_{SQED}^{C} \) and \( M_{SQED}^{H} \) ! We conjecture that model II\(_{3}\) (or model IV\(_1\)) is the same as the IR SCFT of \( \mathcal{N} = 4 \) QED with \( N_f = 3 \).

4.3 \( N_f > 3 \)

The IR SCFT of \( \mathcal{N} = 4 \) QED with \( N_f \) flavors has Coulomb branch moduli space \( M_{SQED}^{C} = \mathbb{C}^2/\mathbb{Z}_{N_f} \) and Higgs branch moduli space \( M_{SQED}^{H} \) given by the hyperkähler quotient

\[
\left\{ \sum_{i=1}^{N_f} (|Q_i|^2 - |\tilde{Q}_i|^2) = 0, \quad \sum_{i=1}^{N_f} Q_i \tilde{Q}_i = 0 \right\} / U(1) \quad (4.11)
\]

\( M_{SQED}^{H} \) is the singular limit of \( T^* \mathbb{CP}^{N_f-1} \) where the \( \mathbb{CP}^{N_f-1} \) is shrunk to zero size. Once again, these are the same as the two branches of the quantum corrected moduli space \( M_Y \) and \( M_X \) of model II\((N_f)_{k=2}\). This leads us to conjecture that the \( \mathcal{N} = 4 \) SCFT described by model II\((N_f)_{k=2}\) is the same as SQED\(-N_f\). Although, unlike the \( N_f = 2, 3 \) cases, we do not know a brane construction that motivates this identification.
4.4 A simple nonabelian generalization

The 2+1 dimensional \( \mathcal{N} = 4 \) \( U(N) \) SQCD with \( N_f \) flavors can be engineered by suspending \( N \) D3-branes between a pair of parallel NS5-branes, intersecting \( N_f \) D5-branes, as follows

\[
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{NS5} & \times & \times & \times & 0 & \times & \times & \times \\
\text{NS5'} & \times & \times & \times & L & \times & \times & \times \\
\text{D5}_i & \times & \times & \times & y_i & \times & \times & \times \\
\text{D3} & \times & \times & \times & \times & \times & \times & \times
\end{array}
\]

where \( 0 < y_i < L, \ i = 1, \cdots, N_f \). For \( N_f \leq 3 \), we can move some of the D5-branes to the outside of the pair of NS5-branes, leaving only one D5-brane in between the NS5-branes. In doing this an additional D3-brane is created stretching between an NS5-brane and the D5-brane on the outside \[20\]. We can then perform the \( SL(2, \mathbb{Z}) \) duality \( \tau \rightarrow \tau / (1 - \tau) \) and turn the D5-branes into \((1,1)\) 5-branes as before, and obtain in the low energy limit \( \mathcal{N} = 4 \) CSM theories with gauge groups and representation of hypermultiplet matter fields

\[(1)\quad U(N)_1 \times U(N)`_1, \quad (\mathbb{N}, \overline{\mathbb{N}}); \]

\[(2)\quad U(N)_1 \times U(N)`_1 \times U(1)_1, \quad (\mathbb{N}, \overline{\mathbb{N}}, 0) \oplus (1, \mathbb{N}, -1); \]

\[(3)\quad U(1)`_1 \times U(N)_1 \times U(N)`_1 \times U(1)_1, \quad (+1, \overline{\mathbb{N}}, 1, 0) \oplus (0, \mathbb{N}, \overline{\mathbb{N}}, 0) \oplus (0, 1, \mathbb{N}, -1). \]

(4.12)

Among these, theory (1) is the simplest example of \( \mathcal{N} = 4 \) CSM theory studied by \[7\], at level \( k = 1 \). It is natural to propose that these CSM theories describe the IR SCFT of \( \mathcal{N} = 4 \) \( U(N) \) QCD with \( N_f = 1, 2, 3 \) respectively.

5 \( \mathcal{N} = 2 \) Chern-Simons-matter theories as quantum critical points

5.1 Model I: the classical theory

Let us now consider \( \mathcal{N} = 2 \) Chern-Simons-matter theory with gauge group \( U(1) \times U(1) \), at Chern-Simons level \((k, -k)\). The fields in the gauge multiplet are denoted as \((A_\mu, \sigma, D; \chi)\) and \((\tilde{A}_\mu, \tilde{\sigma}, \tilde{D}; \tilde{\chi})\) as before. The matter fields are taken to be \( M_1 \) chiral multiplets \( X_i \) \((i = 1, \cdots, M_1)\) with charge \((+1, +1)\), and \( M_2 \) chiral multiplets \( Y_{i'} \) \((i' = 1, \cdots, M_2)\) with charge \((+1, -1)\). The scalar potential before integrating out the
auxiliary fields is
\[
\frac{k}{2\pi} D\sigma - \frac{k}{2\pi} \tilde{D}\tilde{\sigma} + \sum_i |(\sigma + \tilde{\sigma})X_i|^2 + \sum_{i'} |(\sigma - \tilde{\sigma})Y_{i'}|^2 \\
+ D(\sum_i |X_i|^2 + \sum_{i'} |Y_{i'}|^2) + \tilde{D}(\sum_i |X_i|^2 - \sum_{i'} |Y_{i'}|^2)
\] (5.1)

Integrating out \(D\) and \(\tilde{D}\) sets
\[
\sigma = -\frac{2\pi}{k} \left( \sum_i |X_i|^2 + \sum_{i'} |Y_{i'}|^2 \right),
\]
\[
\tilde{\sigma} = \frac{2\pi}{k} \left( \sum_i |X_i|^2 - \sum_{i'} |Y_{i'}|^2 \right).
\] (5.2)

So we obtain the scalar potential
\[
V = \frac{16\pi^2}{k^2} \left[ \left( \sum_i |X_i|^2 \right)^2 \left( \sum_{i'} |Y_{i'}|^2 \right) + \left( \sum_{i'} |Y_{i'}|^2 \right)^2 \left( \sum_i |X_i|^2 \right) \right]
\] (5.3)

In the case \(M_1 = M_2 = 2\), this is exactly the same as the scalar potential of model II\(_k\), but the fermion couplings will be different. We see that there are two branches of the Higgs branch moduli space, parameterized by
\[
\mathcal{M}_X : \ X_i \neq 0, \ Y_{i'} = 0,
\] (5.4)
and
\[
\mathcal{M}_Y : \ Y_{i'} \neq 0, \ X_i = 0.
\] (5.5)

On each of these Higgs branches, one combination of the \(U(1) \times U(1)\) gauge symmetry acts trivially. The abelian Chern-Simons action can be written as
\[
S_{CS} = \frac{k}{4\pi} \int A \wedge dA - \tilde{A} \wedge d\tilde{A} = \frac{k}{4\pi} \int a \wedge db,
\] (5.6)

where \(a = A + \tilde{A}, \ b = A - \tilde{A}\). Dirac quantization condition implies that the fluxes of \(a\) and \(b\) are independently quantized to be integer valued. Let us assume that \(k\) is even, so it follows that the constant gauge transformations of each \(U(1)\) are the subgroup \(\mathbb{Z}_{k/2}\). The classical moduli space is then
\[
\mathcal{M}_X \simeq \mathbb{C}^{M_1} / \mathbb{Z}_{k/2}, \quad \mathcal{M}_Y \simeq \mathbb{C}^{M_2} / \mathbb{Z}_{k/2}, \quad k \text{ even}
\] (5.7)

The CSM theory has \(SU(M_1) \times SU(M_2)\) flavor symmetry. \(SU(M_2)\) acts trivially on \(\mathcal{M}_X\), and \(SU(M_1)\) acts trivially on \(\mathcal{M}_Y\). This indicates that quantum effects cannot
join the two branches of the moduli space. However, each branch may be deformed or lifted entirely. The Chern-Simons-matter SCFT lives at the singular origin where the two branches meet.

The fermion-boson coupling in this theory is given by

\[ L_F = -\left(\sigma + \tilde{\sigma}\right) \sum_i \bar{\xi}_i \xi_i - \left(\sigma - \tilde{\sigma}\right) \sum_{i'} \bar{\eta}_{i'} \eta_{i'} + \frac{2\pi k}{2} \left(\sum_i \bar{X}_i \xi_i + \sum_{i'} \bar{Y}_{i'} \eta_{i'}\right)^\dagger \left(\sum_i X_i^\dagger \xi_i + \sum_{i'} Y_{i'}^\dagger \eta_{i'}\right) \]

\[ = \frac{4\pi}{k} \left[ \sum_i |X_i|^2 \sum_{i'} |\eta_{i'}|^2 + \sum_{i'} |Y_{i'}|^2 \sum_i \bar{\xi}_i \xi_i + \sum_i \bar{Y}_{i'} \eta_{i'} \sum_{i'} \bar{X}_i \xi_i + \sum_{i'} \bar{Y}_{i'} \eta_{i'} \right] \tag{5.8} \]

We will refer to this CSM theory as “Model I”. A bosonic version of this model describing the critical point of a triangular lattice antiferromagnetic was studied recently in [24].

5.2 A nonabelian version of model I

Here we briefly discuss a nonabelian generalization of model I, which has various branches of Higgs moduli space. Consider \( \mathcal{N} = 2 \ U(N)_k \times U(N)_{-k} \) CSM theory with \( M_1 \) chiral multiplets in the representation \((N,N)\) and \( M_2 \) chiral multiplets in \((N,N)\), and with vanishing superpotential. The bosonic components of the fields will be denoted \( X_{iA}, Y_{iI}^A \), where \( i, i' \) are flavor indices, and \( I, J, A, B \) are the gauge indices of the two \( U(N) \)'s. We have the auxiliary fields

\[ \sigma_I^J = -\frac{2\pi}{k} \left( X_{iA}(X^\dagger)^{iJA} + Y_{i'I}^A(Y^\dagger)^{i'JA}\right), \]

\[ \tilde{\sigma}_A^B = \frac{2\pi}{k} \left( X_{iA}(X^\dagger)^{iIB} - Y_{i'I}^B(Y^\dagger)^{i'I A}\right). \tag{5.9} \]

The scalar potential is

\[ V = |\sigma_I^J X_{iJA} + \tilde{\sigma}_A^B X_{iIB}|^2 + |\sigma_I^J Y_{i'J}^A - Y_{i'I}^B \tilde{\sigma}_B^A|^2 \tag{5.10} \]

For simplicity let us consider the 1 flavor case, i.e. \( M_1 = M_2 = 1 \). In this case we have the scalar potential

\[ V = \frac{16\pi^2}{k^2} \left[ \sum_{I,A} |Y_J^B(Y^\dagger)_B X_{JA} + Y_J^B(Y^\dagger)_A X_{IB}|^2 + \sum_{I,A} |X_{IB}(X^\dagger)_B Y_J^A + X_{iB}(X^\dagger)_A Y_{iB}|^2 \right] \tag{5.11} \]
We can assume that, for instance, $X$ is diagonal by $U(N) \times U(N)$ gauge symmetry. It is then clear that the classical moduli space has $N + 1$ branches, given by

$$\mathcal{M}_n : \quad X = \text{diag}(x_1, \cdots, x_n, 0, \cdots, 0), \quad Y = \text{diag}(0, \cdots, 0, y_{n+1}, \cdots, y_N)$$

(5.12)

where $n = 0, \cdots, N$. Each branch $\mathcal{M}_n$ is isomorphic to $\text{Sym}^n(\mathbb{C}/\mathbb{Z}_k) \times \text{Sym}^{N-n}(\mathbb{C}/\mathbb{Z}_k)$. The different branches meet pairwise along

$$\mathcal{M}_n \cap \mathcal{M}_m \simeq \text{Sym}^n(\mathbb{C}/\mathbb{Z}_k) \times \text{Sym}^{N-m}(\mathbb{C}/\mathbb{Z}_k), \quad (n < m)$$

(5.13)

and all branches meet at the origin.

5.3 The quantum theory

Let us now consider quantum corrections in model I. It is well known that integrating out massive charged multiplets will shift the CS levels. Consequently, one may expect the moduli spaces to be lifted. To avoid this problem, we will define the theory with shifted “bare” CS level. The fields $X_i$’s ($i = 1, \cdots, M_1$) have charge +1 with respect to $a_\mu$, and $Y_i$’s ($i' = 1, \cdots, M_2$) have charge +1 with respect to $b_\mu$, where $a_\mu, b_\mu$ are the twisted CS gauge fields defined in (5.6). We will introduce bare CS levels $-M_1/2$ and $-M_2/2$ for $a$ and $b$ respectively, and adjust the action for the matter fields so as to continue to preserve $\mathcal{N} = 2$ supersymmetry, so that the total CS action is written as

$$\frac{k}{4\pi} \int a \wedge db - \frac{M_1}{8\pi} \int a \wedge da - \frac{M_2}{8\pi} \int b \wedge db$$

(5.14)

Classically this would lift the moduli spaces $\mathcal{M}_X$ and $\mathcal{M}_Y$. We will show that when quantum corrections are taken into account, the moduli spaces are restored. Let us assume nonzero values of $X_i$’s, and integrate out the massive $Y_i$’s. The mass of $Y_i$ is given by

$$m_Y = \frac{4\pi}{k} \sum_i |X_i|^2$$

(5.15)

Integrating out $Y_i$ at one loop generates the couplings

$$\int \frac{M_2}{8\pi m} \left[ (\partial_\mu m)^2 + (db)^2 \right] + \frac{M_2}{8\pi} \int b \wedge db$$

(5.16)

The last term cancels the bare CS level $-M_2/2$ for $b_\mu$. The kinetic term $(\partial_\mu m)^2/m$ may appear to be inconsistent with having a Kähler metric on the moduli space. In fact, it has a supersymmetric completion due to the CS coupling. We can write the $\mathcal{N} = 2$ supersymmetric effective action of the chiral fields $X_i$ and vector superfields $V_a$, and $V_b$ as

$$\int d^4\theta \left[ \sum_i X_i e^{V_a} X_i - \frac{M_1}{8\pi} V_a \Sigma_a + \frac{k}{4\pi} V_a \Sigma_b - \frac{k M_2}{32\pi^2} \sum_i X_i e^{V_a} X_i \Sigma_b^2 \right]$$

(5.17)
Here $\Sigma = iD^a \tilde{D}_a V$ is the linear multiplet field strength. In component fields, the bosonic part of the action can be written as

$$\frac{k}{4\pi} \int a \wedge db - \frac{M_1}{8\pi} \int a \wedge da + \frac{k}{4\pi} \int (\sigma_a D_b + \sigma_b D_a) - \frac{M_1}{4\pi} \sigma_a D_a$$

$$+ \int |D_\mu X_i|^2 + \sigma_a^2 |X|^2 - D_a |X|^2$$

$$+ \int \frac{kM_2}{32\pi^2 |X|^2} \left[-(db)^2 - D_b^2 + (\partial_\mu \sigma_b)^2 - 2\sigma_a \sigma_b D_b \right]$$

$$+ \int \frac{kM_2}{32\pi^2 |X|^4} \sigma_b^2 (|D_\mu X_i|^2 - \sigma_a^2 |X|^2 - D_a |X|^2) + \cdots$$

(5.18)

where $|X|^2 \equiv \sum_i |X_i|^2$. In particular, we see that the equation of motion for $D_a$ implies (up to order $O(1/k^2)$ terms)

$$\sigma_b = \frac{1}{k}(M_1 \sigma_a + 4\pi |X|^2) = m + \frac{M_1}{k} \sigma_a$$

(5.19)

This gives the term $M_2(\partial_\mu m)^2/m$ through $\frac{kM_2}{32\pi^2} |X|^{-2}(\partial_\mu \sigma_b)^2$. Moreover, we see that arbitrary constant $X_i$ and $\sigma_a = D_a = D_b = 0$ solves the equations of motion. So we recover the moduli space $M_X$. However, the low energy theory on this moduli space is not simply a sigma model in $X_i$’s. The effective action takes the form

$$\frac{k}{4\pi} \int a \wedge db - \frac{M_1}{8\pi} \int a \wedge da + \int |D_\mu X_i|^2 + \frac{M_2}{2k |X|^2} (\partial_\mu |X|^2)^2 - \frac{kM_2}{32\pi^2 |X|^2} (db)^2 + \text{fermions}$$

(5.20)

Note that the last term in (5.18) renormalizes the coefficient of $|D_\mu X_i|^2$ to $1 - \frac{M_1}{k}$, but this can be absorbed by a rescaling of the field $X_i$, and the Lagrangian stays in the form (5.20) up to $O(1/k^2)$ terms. Now dualizing $b_\mu$, we have

$$\int \tilde{F}_b \wedge (d\varphi + \frac{k}{4\pi} a) - \int \frac{kM_2}{32\pi^2 |X|^2} |\tilde{F}_b|^2$$

$$\rightarrow \int \frac{8\pi^2 |X|^2}{kM_2} (\partial_\mu \varphi + \frac{k}{4\pi} a_\mu)^2$$

(5.21)

We then end up with the action

$$- \frac{M_1}{8\pi} \int a \wedge da + \int |D_\mu X_i|^2 + \frac{M_2}{2k |X|^2} (\partial_\mu |X|^2)^2 + \frac{8\pi^2 |X|^2}{kM_2} (\partial_\mu \varphi + \frac{k}{4\pi} a_\mu)^2 + \text{fermions}$$

(5.22)

The $U(1)$ gauge symmetry acts as

$$X_i \rightarrow e^{i\Lambda} X_i, \quad a_\mu \rightarrow a_\mu + \partial_\mu \Lambda, \quad \varphi \rightarrow \varphi - \frac{k}{4\pi} \Lambda$$

(5.23)
The moduli space \( \mathcal{M}_X \) is parameterized by \( X_i \) and \( \varphi \), modulo the \( U(1) \) action. In other words, \( \mathcal{M}_X = \tilde{\mathcal{M}}_X/U(1) \), where the metric on \( \tilde{\mathcal{M}}_X \) is

\[
ds^2 = |dX_i|^2 + \frac{M_2}{2k|X|^2}(d|X|^2)^2 + \frac{8\pi^2|X|^2}{kM_2}d\varphi^2
\]

(5.24)

This is the induced metric on \( \mu^{-1}(0) \subset \mathbb{C}^{M_{1+1}} \), where

\[
\mathbb{C}^{M_{1+1}} = \{(X_i, z = \sqrt{\frac{2M_2}{k}\rho e^{2\pi i\varphi/M_2}})\},
\]

\[
ds^2 = \sum_{i=1}^{M_1} |dX_i|^2 + |dz|^2,
\]

(5.25)

\[
\mu = \sum_{i=1}^{M_1} |X_i|^2 - \frac{k}{2M_2}|z|^2 = \sum_{i=1}^{M_1} |X_i|^2 - \rho^2.
\]

and \( \mu \) is the moment map of the symplectic form on \( \mathbb{C}^{M_{1+1}} \)

\[
\omega = dX_i \wedge d\bar{X}_i + dz \wedge d\bar{z}
\]

\[
= dX_i \wedge d\bar{X}_i + \frac{8\pi i}{k}\rho d\rho \wedge d\varphi.
\]

(5.26)

with respect to the \( U(1) \) action (5.23). Therefore, we have found that the one-loop corrected metric on the moduli space \( \mathcal{M}_X \) is that of the (singular) symplectic quotient

\[
\mathcal{M}_X \simeq \mathbb{C}^{M_{1+1}}/U(1) = \mu^{-1}(0)/U(1).
\]

(5.27)

The \( U(1) \) acts on the \( \mathbb{C}^{M_{1+1}} \) with charges \((M_2, M_2, \cdots, M_2, -k/2)\). Similarly, the one-loop correction turns \( \mathcal{M}_Y \) into the symplectic quotient of \( \mathbb{C}^{M_{2+1}} \) by the \( U(1) \) acting with charges \((M_1, M_1, \cdots, M_1, -k/2)\). Note that the massless fields along the moduli space still couple nontrivially to a \( U(1) \) Chern-Simons gauge field, at level \(-M_1/2\) for \( \mathcal{M}_X \) and level \(-M_2/2\) for \( \mathcal{M}_Y \).

If the bare CS level for \( a_{\mu} \) is not \(-M_1/2\), the moduli space \( \mathcal{M}_X \) still exists, whereas the moduli space \( \mathcal{M}_Y \) will be lifted. Similarly, if the CS level for \( b_{\mu} \) is not \(-M_2/2\), the moduli space \( \mathcal{M}_X \) would be lifted. This is because in (5.17) there would be an additional supersymmetric CS term

\[
\frac{k_b}{4\pi} \int d^3x \int d^4\theta V_b \Sigma_b = \frac{k_b}{4\pi} \int (b \wedge db + 2\sigma_b D_b + \bar{\chi}_b \chi_b)
\]

(5.28)

As a consequence, \( \sigma_a = D_a = D_b = 0 \) would not be a solution to the equation of motion for nonzero constant \( X_i \). In fact, the moduli space of \( X_i \) will be lifted by a potential

\[
V \sim \frac{k_b^2}{k^4} (|X|^2)^3
\]

(5.29)

This would be a two-loop contribution to the effective Lagrangian, which is why it was absent from (5.16) but is needed for the supersymmetric completion of the one-loop effective Lagrangian.
6 Summary and outlook

We have presented examples of abelian CSM theories with $\mathcal{N} = 4$ supersymmetries that describe quantum critical points where different branches of the moduli space meet. A new feature of the hypermultiplet moduli spaces in CSM theories is that they can receive quantum corrections, but can still be determined exactly due to the rigidity of the hyperkähler metric. These theories can also be studied perturbatively at weak coupling in $1/k$.

We then considered $\mathcal{N} = 2$ abelian and nonabelian CSM theories. It is easy to describe classical theories with multiple Higgs branches. Quantum mechanically, the moduli space receives nontrivial corrections and may get lifted. We have analyzed the abelian case, and described $\mathcal{N} = 2$ CSM theories with two Higgs branches that are lifted in the classical theory but are restored in the quantum theory. We computed the one-loop correction to the moduli space and found that each branch is turned into a symplectic quotient of the form $\mathbb{C}^{M+1}/U(1)$. The moduli spaces of the nonabelian theories are more complicated. We hope to explore them in future works. It would be particularly interesting to find an example of CSM theory describing a quantum critical point, with a nontrivial ’t Hooft limit that also has a brane construction which allows one to identify its gravity dual.

We proposed that the model III at $k = 1$ (or model II$_{k=2}$) and model IV at $k = 1$ (or model II(3)$_{k=2}$) describe the IR SCFT of $\mathcal{N} = 4$ QED with $N_f = 2$ and $N_f = 3$, respectively. The quantum corrections to the moduli spaces are crucial for the identification, and provide highly nontrivial checks of the conjecture. Despite that such CSM theories are strong coupled, they belong to a family of CSM theories parameterized by the CS level $k$. One could hope to learn aspects of the strongly coupled theories by extrapolating from the weakly coupled ones, i.e. at large $k$. The mechanism of enhanced global symmetries at small $k$ through ’t Hooft operators clearly deserves further study. We have further proposed CSM descriptions of $\mathcal{N} = 4$ QED with $N_f > 3$ based on matching the quantum moduli spaces. It would be nice to have brane constructions for this identification.

We also generalized such constructions to $\mathcal{N} = 4$ $U(N)$ SQCD with $N_f$ flavors, for $N_f = 1, 2, 3$. Since the $U(1)$ gauge group does not decouple in SQCD, one would really like to understand the “genuinely nonabelian” $\mathcal{N} = 4$ $SU(N)$ SQCD, say for $N = 2$, and $N_f$ flavors. It is conceivable that there are CSM descriptions of the IR SCFTs of these theories as well (in particular, they should reproduce the Coulomb and Higgs branch moduli spaces of SQCD), although we do not have any proposals so far.
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