Sasaki-Ricci flow equation on five-dimensional Sasaki-Einstein space $Y^{p,q}$

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Abstract

We analyze the transverse Kähler-Ricci flow equation on Sasaki-Einstein space $Y^{p,q}$. Explicit solutions are produced representing new five-dimensional Sasaki structures. Solutions which do not modify the transverse metric preserve the Sasaki-Einstein feature of the contact structure. If the transverse metric is altered, the deformed metrics remain Sasaki, but not Einstein.

Keywords: contact geometry, Sasaki-Einstein space $Y^{p,q}$, Sasaki-Ricci flow.

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1 Introduction

The concept of Ricci flow was originally introduced by Hamilton in 1982 \cite{1} representing a major tool that allows to continuously deform a Riemannian manifold.

The complex analogue of Hamilton’s Ricci flow, known as Kähler-Ricci flow was introduced by Cao \cite{2} to give a parabolic proof of the Calabi-Yau theorem. Eventually the Ricci flow, as a method to deform Riemannian metrics, was applied to Sasaki manifolds \cite{3,4}. In physics this tool has gained significant attention in the framework of general relativity, renormalization group equations, evolution of wormholes and black holes, cosmological models, etc.

In the last time Sasaki-Einstein geometry, as an odd-dimensional cousin of Kähler-Einstein geometry, has played an important role in the AdS/CFT correspondence. There has been a growing interest in contact geometry to study mechanical systems when the Hamilton function explicitly depends on

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time. Contact geometry is also used in thermodynamics, in the description of systems with dissipation, geometric optics, geometric quantization, control theory, etc (see e.g. [5] for a recent review of applications of contact Hamiltonian dynamics in various fields).

In the present paper we study the transverse Kähler-Ricci flow on Sasaki-Einstein space $Y^{p,q}$. In the framework of AdS/CFT correspondence, $Y^{p,q}$ spaces have been employed to provide an infinite class of dualities. In order to investigate the Sasaki-Ricci flow equation we introduce a set of local complex coordinate to parametrize the transverse holomorphic structure and the Sasakian analogue of the Kähler potential for the Kähler geometry.

In spite of the complexity of the Kähler-Ricci flow equation, we find some particular explicit solutions. If the transverse part of the metric is not changed, the deformed $Y^{p,q}$ remains Sasaki-Einstein. In the opposite case, a modification of the transverse part of the metric leads to a Sasakian structure, but not Einstein.

The paper is organized as follows. In the next Section we recall the definitions and main facts about Sasakian structures and Sasaki-Ricci flow. In Section 3 we investigate the transverse Kähler-Ricci flow equation on the Sasaki-Einstein space $Y^{p,q}$ and produce families of deformed metrics. The paper ends with conclusions in the last Section.

2 Background

In this section we recall the key concepts in the theory of Sasaki manifolds and transverse Kähler-Ricci flow mainly based on [6 4 7].

By a contact manifold it is understood a pair $(M, \eta)$ where $M$ is a smooth manifold of odd dimensions $(2n+1)$ together with 1-form $\eta$ such that

$$\eta \wedge (d\eta)^n \neq 0,$$

is a volume form.

Associated with $\eta$ there is a unique vector field $\xi$, called Reeb vector field, characterized by

$$\eta(\xi) = 1 \quad \text{and} \quad d\eta(\xi, \cdot) = 0.$$

The Reeb vector field $\xi$ is a generator of $\ker d\eta$ and there is a natural splitting of the tangent bundle of $M$

$$TM = D \oplus L_{\xi},$$

where $L_{\xi}$ is a vertical subspace generated by $\xi$ and $D$ is a horizontal distribution induced by $D = \ker \eta$.

A Sasakian manifold is a Riemannian manifold $(M, g)$ with the property that its metric cone $(C(M), \bar{g})$

$$C(M) = \mathbb{R}_{>0} \times M, \quad \bar{g} = dr^2 + r^2 g,$$

is Kähler. Here, $r \in (0, \infty)$ is the standard coordinate on the positive real line $\mathbb{R}_{>0}$. 

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On a contact manifold there is a tensor $\Phi$ of type $(1, 1)$ satisfying

$$\Phi^2 = -I + \eta \otimes \xi \quad \text{and} \quad g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields $X, Y$ on $M$.

The restriction of the Sasaki metric $g$ to $\mathcal{D}$ gives a well-defined Hermitian metric $g^T$. This Hermitian structure is in fact Kähler.

One can introduce local coordinates $(x, z^1, \ldots, z^n)$ on a small neighborhood of $U = I \times V$ with $I \in \mathbb{R}$ and $V \in \mathbb{C}^n$. In the chart $U$ we may write

$$\xi = \frac{\partial}{\partial x},$$

$$\eta = dx + i \sum_{j=1}^{n} (K_{j} dz^j) - i \sum_{j=1}^{n} (K_{j} d\bar{z}^j),$$

$$d\eta = -2i \sum_{j,k=1}^{n} K_{j\bar{k}} dz^j \wedge d\bar{z}^k,$$

$$g = \eta \otimes \eta + g^T = \eta \otimes \eta + 2 \sum_{j,\bar{k}=1}^{n} K_{j\bar{k}} dz^j d\bar{z}^\bar{k},$$

$$\Phi = -i \sum_{j=1}^{n} [(\partial_j - i K_{j}\partial_x) \otimes dz^j] + i \sum_{j=1}^{n} (\partial_j + i K_{j}\partial_x) \otimes d\bar{z}^j,$$

where $K_{j} = \frac{\partial}{\partial z^j} K$ and $K_{j\bar{k}} = \frac{\partial^2}{\partial z^j \partial \bar{z}^\bar{k}} K$. The function $K : U \rightarrow \mathbb{R}$ is the Sasakian analogue of the Kähler potential which does not depend on $x$, i.e. $\partial_x K = 0$.

In what follows we consider deformations of the Sasaki structures which preserve the Reeb vector field $\xi$. For this purpose it is necessary to introduce the basic forms. A $r$-form $\alpha$ on $M$ is called basic if

$$\iota_\xi \alpha = 0, \quad \mathcal{L}_\xi \alpha = 0,$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field $\xi$. In the system of coordinates $(x, z^1, \ldots, z^n)$ considered above, a basic $r$-form of type $(p, q)$, $r = p + q$, has the form

$$\alpha = \alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_q},$$

where $\alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q}$ does not depend on $x$. In particular a function $\varphi$ is basic if and only if $\xi(\varphi) = 0$. That is the case of the Sasaki potential $K$.

Let $\varphi$ be a basic function and consider the deformation of the contact form $\eta$:

$$\tilde{\eta} = \eta + \tilde{d}_{\xi} \varphi, \quad (1)$$
where $d'_{B} = \frac{4}{3}(\partial_{B} - \partial_{\bar{B}})$ with

$$\partial_{B} = \sum_{j=1}^{n} dz^{j} \frac{\partial}{\partial z^{j}}, \quad \partial_{\bar{B}} = \sum_{j=1}^{n} \bar{d}z^{j} \frac{\partial}{\partial \bar{z}^{j}}.$$ 

This deformation implies that other fundamental tensors are also modified:

\begin{align*}
\tilde{\Phi} &= \Phi - (\xi \otimes (d'_{B} \varphi)) \circ \Phi, \\
\tilde{g} &= d\tilde{\eta} \circ (1 \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}, \\
d\tilde{\eta} &= d\eta + d_{B}d'_{B} \varphi.
\end{align*}

To introduce the transverse Kähler-Ricci flow, also called Sasaki-Ricci flow, we consider the flow $(\xi, \eta(t), \Phi(t), g(t))$ with initial data $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g)$ generated by a basic function $\varphi(t)$. The Sasaki-Ricci flow equation is

$$\frac{\partial g_{T}}{\partial t} = -Ric_{g(t)}^{T} + (2n + 2)g_{T}(t),$$

where $Ric^{T}$ is the transverse Ricci curvature. In the case of the deformation (1) with a basic function $\varphi$, in local coordinates the Sasaki-Ricci flow can be expressed as a parabolic Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \ln \det(g_{j\bar{k}}^{T} + \varphi_{j\bar{k}}) - \ln(\det g_{j\bar{k}}^{T}) + (2n + 2)\varphi.$$  \hspace{1cm} (2)

### 3 Sasaki-Einstein space $Y^{p,q}$ and Sasaki-Ricci flow equation

The metric of the Sasaki-Einstein space $Y^{p,q}$ is given by the line element [8]

$$ds^{2} = \frac{1 - y^{6}}{6}(d\theta^{2} + \sin^{2}\theta d\phi^{2}) + \frac{1}{w(y)q(y)}dy^{2} + \frac{w(y)q(y)}{36}(d\beta - \cos\theta d\phi)^{2} + \frac{1}{9}[d\psi + \cos\theta d\phi + y(d\beta - \cos\theta d\phi)]^{2},$$

where

\begin{align*}
w(y) &= \frac{2(a - y^{2})}{1 - y}, \\
q(y) &= \frac{a - 3y^{2} + 2y^{3}}{a - y^{2}}, \\
f(y) &= \frac{a - 2y + y^{2}}{6(a - y^{2})}.
\end{align*}

Note that we have taken $\phi \to -\phi$ with respect to [8] and consequently there are some differences of sign.
In the case of the space $Y^{p,q}$ the contact 1-form $\eta$ is 
\[
\eta = \frac{1}{3} d\psi + \frac{1}{3} y d\beta + \frac{1 - y}{3} \cos \theta \, d\phi,
\]
and the Reeb vector field is 
\[
K_\eta = 3 \frac{\partial}{\partial \psi}.
\]  
A detailed analysis of the metric $Y^{p,q}$ showed that it is globally well-defined and there are a countable infinite number of Sasaki-Einstein manifolds characterized by two relatively prime positive integers $p, q$ with $p < q$. If $0 < a < 1$ the cubic equation
\[
Q(y) = a - 3y^2 + 2y^3 = \frac{1 - y}{2} w(y) q(y) = 0,
\]
has three real roots, one negative ($y_1$) and two positive, the smallest being $y_2$. The coordinate $y$ ranges between the two smaller roots of the cubic equation (4), i.e. $y_1 \leq y \leq y_2$.

The angular coordinates span the ranges $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 2\pi$.

In order to specify the range of the variable $\beta$, we note that it is connected with another variable $\alpha$
\[
\beta = -(6\alpha + \psi).
\]
The range of $\alpha$ is 
\[0 \leq \alpha \leq 2\pi \ell,
\]
where 
\[
\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}.
\]
The Reeb Killing vector field (3) has compact orbits when $\ell$ is a rational number and the corresponding $Y^{p,q}$ manifold is called quasi-regular. If $\ell$ is irrational the orbits of the Reeb vector field do not close densely filling the orbits of a torus and the Sasaki-Einstein manifold is said to be irregular.

For what follows it is useful to evaluate the following integrals
\[
f_1(y) = \exp \left( \int \frac{1}{H^2(y)} \, dy \right) = \sqrt{(y - y_1)^{-\frac{1}{m}} (y_2 - y)^{-\frac{1}{n}} (y_3 - y)^{-\frac{1}{n}}},
\]
\[
f_2(y) = \exp \left( \int \frac{y}{H^2(y)} \, dy \right) = \frac{1}{\sqrt{Q(y)}},
\]
where
\[
H^2(y) = \frac{1}{6} w(y) q(y) = \frac{1}{3} \frac{Q(y)}{1 - y}.
\]

We introduce a local set of transverse complex coordinates [8, 10, 11] addressing the transverse Kähler structure of $Y^{p,q}$:
\[
z^1 = \tan \frac{\theta}{2} e^{i\phi},
\]
\[
z^2 = \frac{\sin \theta}{f_1(y)} e^{i\beta}.
\]
These complex coordinates are not globally well-defined. This problem is discussed in [12] and a set of holomorphic coordinates on $Y^{p,q}$ was constructed.

We then get:

$$dz^1 = \left( \frac{1}{2 \cos^2 \frac{\theta}{2}} d\theta + i \tan \frac{\theta}{2} d\phi \right) e^{i\phi},$$

$$dz^2 = \left( -\frac{f_1^y}{f_1} dy + \cot \theta d\theta + i d\beta \right) \frac{\sin \theta}{f_1(y)} e^{i\beta}.$$

In terms of the complex coordinates [14] the Sasaki-Kähler potential is

$$K = \frac{1}{3} \left[ \left( 1 + \frac{1}{z^1 z^2} \right) f_2(y) \right] + \frac{1}{6} \ln(z^1 z^2).$$

Note that the additional term restores the correct form of the contact form $\eta$ of the space $Y^{p,q}$ without altering the transverse part of the metric.

We derive the local expressions of the derivatives of the Sasaki-Kähler potential. We can simplify calculation by defining [10]

$$f_1^y(y) = \sigma = \frac{\sin^2 \theta}{z^1 z^2}.$$

First we evaluate

$$\frac{dy}{d\sigma} = \frac{w(y)q(y)}{12 f_1^y(y)},$$

$$\frac{\partial \sigma}{\partial z^1} = \frac{\sigma}{z^1} \cos \theta,$$

$$\frac{\partial \sigma}{\partial z^2} = -\frac{\sigma}{z^2},$$

$$\frac{\partial y}{\partial z^1} = \frac{w(y)q(y)}{12 z^1} \cos \theta,$$

$$\frac{\partial y}{\partial z^2} = -\frac{w(y)q(y)}{12 z^2}.$$

Derivatives of the Sasaki-Kähler potential are

$$K_{,1} = \left[ -\frac{1}{3} \cos^2 \frac{\theta}{2} + \frac{y}{6} \cos \theta \right] \cot \frac{\theta}{2} e^{-i\phi},$$

$$K_{,2} = \left[ -\frac{1}{3} \frac{1}{\cos \frac{\theta}{2}} + \frac{y}{6} \frac{1}{\cos \theta} \right] \cot \frac{\theta}{2} e^{-i\phi}.$$
\[ K_{11} = \frac{1}{3}(1-y) \cos^4 \theta + \frac{w(y)q(y) \cos^2 \theta}{72 \tan^2 \frac{\theta}{2}}, \]

\[ K_{22} = \frac{w(y)q(y) f_1^2(y)}{72 \sin^2 \theta}, \]

\[ K_{21} = -\frac{1}{72} w(y)q(y) \cos \theta \cot \theta e^{-i\phi - i\beta}, \]

Consequently we get

\[ g^T = 2 \left( K_{11} dz_1 d\bar{z}_1 + K_{12} dz_1 d\bar{z}_2 + K_{21} dz_2 d\bar{z}_1 + K_{22} dz_2 d\bar{z}_2 \right) \]

\[ = \frac{1-y}{6} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \frac{1}{w(y)q(y) dy^2} + \frac{w(y)q(y)}{36} (d\beta - \cos \theta d\phi)^2, \quad (7) \]

\[ \det g^T = 4(K_{11} K_{22} - K_{12} K_{21}) = \frac{w(y)q(y)}{216} f_1^2(y) (1-y) \cot^2 \frac{\theta}{2}. \]

For the purpose of studying the Sasaki-Ricci flow equation we evaluate the derivatives \( \frac{\partial}{\partial z}. \) We obtain for the derivatives involving the complex coordinate \( z_1: \)

\[ \frac{\partial}{\partial z_1} = \cos^2 \theta e^{-i\phi} \left( \frac{\partial}{\partial \rho} - i \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right), \]

\[ \frac{\partial}{\partial \bar{z}_1} = \cos^2 \theta e^{i\phi} \left( \frac{\partial}{\partial \rho} + i \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right), \]

\[ \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} = \cos^4 \theta \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \cot \theta \frac{\partial}{\partial \rho} \frac{\partial}{\partial \phi} \right). \]

To have similar expressions for the partial derivatives involving \( z_2 \) we write

\[ z_2 = \rho e^{i\beta}, \]

with

\[ \rho = \frac{\sin \theta}{f_1(y)}. \]

Then we get

\[ \frac{\partial}{\partial z_2} = \frac{e^{-i\beta}}{2} \left( \frac{\partial}{\partial \rho} - i \frac{1}{\rho} \frac{\partial}{\partial \beta} \right), \]

\[ \frac{\partial}{\partial \bar{z}_2} = \frac{e^{i\beta}}{2} \left( \frac{\partial}{\partial \rho} + i \frac{1}{\rho} \frac{\partial}{\partial \beta} \right), \]

\[ \frac{\partial^2}{\partial z_2 \partial \bar{z}_2} = \frac{1}{4} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \beta^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right), \]
where $c$ becomes simpler:

$$
\frac{\partial^2}{\partial z^2} - \cos \frac{\partial}{2} e^{i \beta - i \phi} \left( \frac{\partial^2}{\partial \rho \partial \theta} + i \frac{\partial^2}{\partial \rho \partial \phi} - \frac{i}{\rho} \frac{\partial^2}{\partial \beta \partial \theta} + \frac{1}{\rho} \frac{\partial^2}{\partial \beta \partial \phi} \right),
$$

$$
\frac{\partial^2}{\partial z_2 \partial z_1} = \cos \frac{\partial}{2} e^{i \beta + i \phi} \left( \frac{\partial^2}{\partial \rho \partial \theta} - i \frac{\partial^2}{\partial \rho \partial \phi} + i \frac{\partial^2}{\partial \beta \partial \theta} + \frac{1}{\rho} \frac{\partial^2}{\partial \beta \partial \phi} \right).
$$

Sasaki-Ricci flow equation (2) on $Y^p$ becomes:

$$
d\varphi = \ln \left\{ \varphi,_{11} \varphi,_{22} - \varphi,_{12} \varphi,_{21} + \left[ \frac{1}{3} (1 - y) \cos^4 \frac{\theta}{2} + \frac{w(y)q(y) \cos^2 \theta}{\tan^2 \frac{\theta}{2}} \right] \varphi,_{12} 
+ \frac{w(y)q(y)}{72} \cos \theta \cot \frac{\theta}{2} \frac{f_1(y)}{\sin \theta} e^{-i \phi + i \beta} \varphi,_{12} 
+ \frac{w(y)q(y)}{72} \cos \theta \cot \frac{\theta}{2} \frac{f_1(y)}{\sin \theta} e^{-i \phi + i \beta} \varphi,_{12} 
+ \frac{w(y)q(y)}{121} \left( \frac{f_1(y)}{1 - y} \cot^2 \frac{\theta}{2} \right) \right\} 
- \ln \left\{ \frac{w(y)q(y)}{216} f_1(y)(1 - y) \cot^2 \frac{\theta}{2} \right\} + 6\varphi.
$$

(8)

Let us assume that the dependence of $\varphi$ on $z^1$ and $z^2$ separates

$$
\varphi = f(t) \left[ g_1(z^1, z^2) + g_2(z^2, z^2) \right],
$$

where the functions $\varphi, g_1, g_2$ are to be determined. In this case the mixed derivatives $\varphi,_{12}$ and $\varphi,_{21}$ vanish and the evaluation of the derivatives $\varphi,_{11}, \varphi,_{22}$ becomes simpler:

$$
\varphi,_{11} = \cos^4 \frac{\theta}{2} \left( \frac{\partial^2 g_1}{\partial \theta^2} + \frac{1}{\sin \theta} \frac{\partial^2 g_1}{\partial \phi^2} + \cot \theta \frac{\partial g_1}{\partial \theta} \right) f(t),
$$

(9)

$$
\varphi,_{22} = \frac{1}{4} \left( \frac{\partial^2 g_2}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 g_2}{\partial \beta^2} + \frac{1}{\rho} \frac{\partial g_2}{\partial \rho} \right) f(t).
$$

(10)

Explicit solutions of the Sasaki-Ricci flow equation can be obtained assuming

$$
\varphi,_{11} = \cos \frac{\theta}{2} c_1 f(t),
$$

(11)

$$
\varphi,_{22} = c_2 f(t),
$$

(12)

where $c_1, c_2$ are arbitrary constants.

Analogous assumptions were implied in the study of the Sasaki-Ricci flow on $T^{1,1}$ [13]. Using (11) from (9) we obtain for $g_1$ an explicit expression:

$$
g_1(\theta, \phi) = \frac{d_1}{2} \phi^2 + e_1 \ln \tan \frac{\theta}{2} - \frac{d_1}{2} \left( \ln \tan \frac{\theta}{2} \right)^2 - c_1 \ln \sin \theta,
$$

involving the arbitrary constants $c_1, d_1, e_1$. 

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Similarly, assuming equation \((10)\) has a simple solution:

\[
g_2(\rho, \beta) = \frac{d_2}{2} \beta^2 + c_2 \rho^2 + e_2 \ln \rho - \frac{d_2}{2} (\ln \rho)^2,
\]

where \(c_2, d_2, e_2\) are other arbitrary constants.

Let us remark that for \(c_1 = c_2 = 0\), the transverse metric remains unaltered and the Sasaki-Ricci flow equation reduces to

\[
\frac{d\varphi}{dt} = 6\varphi,
\]

with the obvious solution taking the initial condition \(f(0) = 0\):

\[
f(t) = e^{6t} - 1.
\]

The deformed contact form can be evaluated according to (1):

\[
\tilde{\eta} = \eta + \frac{i}{2} \sum_m \varphi, m dz^m - \frac{i}{2} \sum_m \varphi, \bar{m} d\bar{z}^m.
\]

To summarize the above analysis we have the following outcome:

**Proposition 1** The families of basic functions

\[
\varphi(t) = (e^{6t} - 1) \left[ \frac{d_1}{2} \phi^2 + c_1 \ln \tan \frac{\theta}{2} - \frac{d_1}{2} \left( \ln \tan \frac{\theta}{2} \right)^2 
+ \frac{d_2}{2} \beta^2 + e_2 \ln \rho - \frac{d_2}{2} (\ln \rho)^2 \right],
\]

with \(d_1, c_1, e_2\) arbitrary constants, stand as solutions of the transverse Kähler-Ricci flow equation on the manifold \(Y^{p,q}\).

The corresponding deformed contact structures remain Sasaki-Einstein with the contact forms

\[
\tilde{\eta} = \eta + \frac{e^{6t} - 1}{2} \left[ \frac{d_1}{\sin \theta} d\theta + \left( -c_1 + d_1 \ln \tan \frac{\theta}{2} \right) d\phi 
+ \frac{d_2}{\rho} d\rho + (-e_2 + d_2 \ln \rho) d\beta \right].
\]

When \(c_j \neq 0\), Sasaki-Ricci flow equation \((8)\) becomes more involved. In spite of the fact that we have not an explicit solution for the basic function \(\varphi(t)\), the Ricci flow produces new Sasaki structures with deformations of the transverse metric \(g^T\) \((7)\) of the source \(Y^{p,q}\) metric.

**Proposition 2** The deformed contact structures with the contact forms

\[
\tilde{\eta} = \eta + \frac{f(t)}{2} \left[ c_1 \cos \theta d\phi - c_2 \rho^2 d\beta \right] ,
\]

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with $c_j$ arbitrary constants, remain Sasaki with the deformed metrics

$$\tilde{g} = \tilde{\eta} \otimes \tilde{\eta} + g^T + \sum_{j=1}^{2} \phi_j d\bar{z}^j dz^j$$

$$= \tilde{\eta} \otimes \tilde{\eta} + g^T + f(t) \left[ \frac{c_1}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + c_2 (d\rho^2 + \rho^2 d\beta^2) \right].$$

4 Concluding remarks

To deform the Sasakian structures we exploit the transverse structure of Sasaki manifolds. The Sasaki-Ricci flow is a transverse Kähler-Ricci flow which deforms the transverse Kähler structure.

To perturb the Sasakian structure, we keep the Reeb field fixed and let vary the contact form $\eta$ by modifying it with a basic function as in (1). For small perturbing basic functions, the Sasakian structure of the manifold is preserved.

Starting with a Sasaki-Einstein manifold $Y^{p,q}$, as a seed, we generate families of new Sasakian structures. We are able to find explicit solutions of the Sasaki-Ricci flow equation depending on some arbitrary constants.

Recently we discussed the complete integrability on $T^{1,1}$ and $Y^{p,q}$ spaces [14, 15, 16, 17]. It would be interesting to investigate the contact Hamiltonian systems [18, 19], complete integrability and action angle variables on the perturbed Sasaki-Einstein spaces $Y^{p,q}$.

It is worth extending the study of Sasaki-Ricci flow on higher dimensional Sasaki-Einstein spaces as well as other contact structures as 3-Sasaki structures [20] or mixed 3-structures [21].

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