HYPER-COMPLEX STRUCTURES ON COURANT ALGEBROIDS

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Abstract. Hypercomplex structures on Courant algebroids unify holomorphic symplectic structures and usual hypercomplex structures. In this note, we prove the equivalence of two characterizations of hypercomplex structures on Courant algebroids, one in terms of Nijenhuis concomitants and the other in terms of (almost) torsionfree connections for which each of the three complex structures is parallel.

A Courant algebroid \([4]\) consists of a vector bundle \(\pi : E \to M\), a nondegenerate symmetric pairing \(\langle , \rangle\) on the fibers of \(\pi\), a bundle map \(\rho : E \to TM\) called anchor and an \(\mathbb{R}\)-bilinear operation \(\circ\) on \(\Gamma(E)\) called Dorfman bracket, which, for all \(f \in C^{\infty}(M)\) and \(x, y, z \in \Gamma(E)\) satisfy the relations

\[
\begin{align*}
    x \circ (y \circ z) &= (x \circ y) \circ z + y \circ (x \circ z); \\
    \rho(x \circ y) &= [\rho(x), \rho(y)]; \\
    x \circ fy &= (\rho(x)f)y + f(x \circ y); \\
    x \circ y + y \circ x &= 2D(x, y); \\
    \mathcal{D}f \circ x &= 0; \\
    \rho(x)\langle y, z \rangle &= \langle x \circ y, z \rangle + \langle y, x \circ z \rangle,
\end{align*}
\]

where \(\mathcal{D} : C^{\infty}(M) \to \Gamma(E)\) is the \(\mathbb{R}\)-linear map defined by \(\langle \mathcal{D}f, x \rangle = \frac{1}{2}\rho(x)f\).

The symmetric part of the Dorfman bracket is given by \((4)\). The Courant bracket is defined as the skew-symmetric part \([x, y] = \frac{1}{2}(x \circ y - y \circ x)\) of the Dorfman bracket. Thus we have the relation \(x \circ y = [x, y] + D(x, y)\).

A standard example is due to T. Courant \([2]\). Given a smooth manifold \(M\), the vector bundle \(TM \oplus T^*M \to M\) carries a natural Courant algebroid structure: the anchor is the projection onto the tangent component while the pairing and Dorfman bracket are given, respectively, by

\[
\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)) \quad \text{and} \quad (X + \xi) \circ (Y + \eta) = [X, Y] + (L_X \eta - \iota_Y d\xi),
\]

for all \(X, Y \in \mathfrak{X}(M)\) and \(\xi, \eta \in \Omega^1(M)\).

Definition 1. An almost hypercomplex structure on a Courant algebroid \((E, \rho, \langle , \rangle, \circ)\) is a triple \((I, J, K)\) of endomorphisms of the vector bundle \(E\), i.e. vector bundle maps over \(\text{id}_M : M \to M\), which are orthogonal transformations w.r.t. the pairing \(\langle , \rangle\) and satisfy the quaternionic relations

\[
I^2 = J^2 = K^2 = IJK = -1.
\]

Obviously, if \((I, J, K)\) is an almost hypercomplex structure, then so are \((K, I, J)\) and \((J, K, I)\).
Let \((E \to M, \rho, \langle \cdot, \cdot \rangle, \circ)\) be a Courant algebroid. Given two endomorphisms \(F\) and \(G\) of the vector bundle \(E\), the relation
\[
\mathcal{N}(F,G)(X,Y) = FX \circ GY - F(X \circ GY) - G(FX \circ Y) + FG(X \circ Y)
\]
\[
+ GX \circ FY - G(X \circ FY) - F(GX \circ Y) + GF(X \circ Y),
\]
where \(X,Y \in \Gamma(E)\), defines a \((2,1)\)-tensor \(\mathcal{N}(F,G) : E \otimes E \to E\) called Nijenhuis concomitant. Obviously, \(\mathcal{N}(F,G) = \mathcal{N}(G,F)\).

**Lemma 2.** If \((I, J, K)\) is an almost hypercomplex structure on a Courant algebroid \(E\), then \(\mathcal{N}(I,J)(X,Y) + \mathcal{N}(I,J)(Y,X) = 0\) for all \(X,Y \in \Gamma(E)\).

**Definition 3.** A hypercomplex structure on a Courant algebroid \(E\) is an almost hypercomplex structure \((I, J, K)\) such that the six Nijenhuis concomitants \(\mathcal{N}(I,I), \mathcal{N}(J,J), \mathcal{N}(K,K), \mathcal{N}(I,J), \mathcal{N}(J,K)\) and \(\mathcal{N}(K,I)\) vanish.

**Remark 4.** Let \((E \to M, \rho, \langle \cdot, \cdot \rangle, \circ)\) be a Courant algebroid and let \(I\) and \(J\) be two endomorphisms of \(E\) such that: \(I^2 = J^2 = -1; I\) and \(J\) anticommute; both \(I\) and \(J\) are orthogonal w.r.t. the pairing \(\langle \cdot, \cdot \rangle\); and the three Nijenhuis concomitants \(\mathcal{N}(I,I), \mathcal{N}(J,J)\) and \(\mathcal{N}(I,J)\) vanish. Then it is easy to check that the triple \((I,J,IJ)\) is a hypercomplex structure on the Courant algebroid. This is the way Bredthauer originally defined hypercomplex structures in [1]. See also [3].

For any \(f \in C^\infty(M)\) and \(X,Y \in \Gamma(E)\), let
\[
\Delta_f(X,Y) = \langle X,Y \rangle Df + \langle IX,Y \rangle IDf + \langle JX,Y \rangle JDf + \langle KX,Y \rangle KDf.
\]
It is clear that
\[
\Delta_f(X,Y) = I\Delta_f(X,Y), \quad \Delta_f(X,JY) = J\Delta_f(X,Y), \quad \Delta_f(X,KY) = K\Delta_f(X,Y)
\]
and
\[
\Delta_f(X,Y) + \Delta_f(Y,X) = 2\langle X,Y \rangle Df.
\]

**Definition 5.** Let \((I, J, K)\) be an almost hypercomplex structure on a Courant algebroid \((E \to M, \rho, \langle \cdot, \cdot \rangle, \circ)\). A hypercomplex connection is an \(\mathbb{R}\)-bilinear map
\[
\Gamma(E) \otimes \Gamma(E) : (X,Y) \mapsto \nabla_X Y
\]
such that, for all \(f \in C^\infty(M)\) and \(X,Y \in \Gamma(E)\), we have
\[
\nabla_f X Y = f \nabla_X Y
\]
and
\[
\nabla_X (fY) = (\rho(X)f)Y + f(\nabla_X Y) - \Delta_f(X,Y).
\]
Its torsion \(T^\nabla : \Gamma(E) \wedge \Gamma(E) \to \Gamma(E)\) is given by
\[
T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].
\]

**Remark 6.** If \(L\) is an isotropic subbundle of \(E\) stable under \(I, J\) and \(K\), then a hypercomplex connection on \(E\) induces a usual \(L\)-connection on \(L\).

The purpose of this note is to establish the following result.

**Theorem 7.** Let \((I, J, K)\) be an almost hypercomplex structure on a Courant algebroid \(E\). The following assertions are equivalent.
\[\begin{align*}
(a) \quad \mathcal{N}(I,J) = \mathcal{N}(J,J) &= 0
\end{align*}\]
(b) \( N(I, J) = 0 \)
(c) The triple \((I, J, K)\) is a hypercomplex structure, i.e. all six Nijenhuis concomitants vanish.
(d) There exists a hypercomplex connection \(\nabla\) satisfying
\[
\nabla I = \nabla J = \nabla K = 0
\]
and, for all \(X, Y \in \Gamma(E)\),
\[
T^\nabla(X, Y) = ID(X, Y) + JD(X, Y) + KD(X, Y).
\]
(e) There exists a hypercomplex connection satisfying (13) and (14); it is unique and given by
\[
\nabla_X Y = -\frac{1}{2}K(JY \circ IX - J(Y \circ IX) - I(JY \circ X) + JI(Y \circ X)).
\]

The remainder of this note is devoted to the proof of this theorem. Straightforward computations lead to the first two lemmas below, of which the former is a generalization of Theorem 1.1 in [7].

**Lemma 8.** Given an almost hypercomplex structure \((I, J, K)\), the relation
\[
\nabla_X Y = -\frac{1}{2}K(JY \circ IX - J(Y \circ IX) - I(JY \circ X) + JI(Y \circ X))
\]
defines a hypercomplex connection. Permuting \(I, J\) and \(K\) cyclically in (15), we obtain two other hypercomplex connections:
\[
\nabla'_X Y = -\frac{1}{2}I(KY \circ JX - K(Y \circ JX) - J(KY \circ X) + KJ(Y \circ X))
\]
\[
\nabla''_X Y = -\frac{1}{2}J(IY \circ KX - I(Y \circ KX) - K(IY \circ X) + IK(Y \circ X)).
\]

**Lemma 9.** Given an almost hypercomplex structure \((I, J, K)\), the hypercomplex connection (15) satisfies
\[
\nabla_X J = 0,
\]
\[
(\nabla_X I)Y = \frac{1}{2}KN(I, J)(X, YY) + \frac{1}{2}JN(I, J)(X, Y),
\]
and
\[
X \circ Y + \frac{1}{2}KN(I, J)(X, Y) = \nabla_X Y - \nabla_Y X + D(X, Y)
\]
\[
- (ID(X, Y) + JD(X, JY) + KD(X, KY)).
\]

**Corollary 10.** Let \((I, J, K)\) be an almost hypercomplex structure on a Courant algebroid \(E\). If \(N(I, J) = 0\), then the hypercomplex connection (15) satisfies (13) and (14).

**Proof.** We always have \(\nabla J = 0\) by (18). Since \(N(I, J) = 0\), (19) implies that \(\nabla I = 0\). And it follows from \(K = IJ\) that
\[
\nabla_X K = (\nabla_X I) \circ J + I \circ (\nabla_X J) = 0.
\]
Thus (13) is proved and (14) follows immediately from (20) and the relation \(x \circ y = [x, y] + D(x, y)\).

**Lemma 11.** Given an almost hypercomplex structure \((I, J, K)\), there exists at most one hypercomplex connection satisfying (13) and (14).
Proof. Assume there exist two such hypercomplex connections $\nabla^1, \nabla^2$. Let
$$\Xi(X,Y) = \nabla^2_X Y - \nabla^1_X Y.$$ 
It follows from (13) that
$$\Xi(I X, I Y) = I \Xi(X,Y), \quad \Xi(J X, J Y) = J \Xi(X,Y), \quad \Xi(K X, K Y) = K \Xi(X,Y)$$
and from (14) that $\Xi(X,Y) = \Xi(Y,X)$. Therefore
$$K \Xi(X,X) = I J \Xi(X,X) = I \Xi(X,J X) = I \Xi(J X, X) = \Xi(I X, J X) = J \Xi(I X, X) = J \Xi(X, I X) = J I \Xi(X,X) = - K \Xi(X,X).$$
Hence $\Xi(X,X) = 0$ for all $X \in \Gamma(E)$ and, consequently,
$$\Xi(X,Y) = \frac{1}{2}(\Xi(X+Y, Y) + \Xi(X, Y)) = 0$$
for all $X,Y \in \Gamma(E)$. \qed

Lemma 12. Given an almost hypercomplex structure $(I, J, K)$, if there exists a hypercomplex connection satisfying (13) and (14), then $N(I,J) = 0$.

Proof. From (14), it follows that
$$X \circ Y = \nabla_X Y - \nabla_Y X + D\langle X,Y \rangle - (ID\langle X, IY \rangle + J D\langle X, JY \rangle + K D\langle X, KY \rangle).$$
This relation can be used to evaluate each of the terms of $N(I,J)$. It follows from (13), the quaternionic relations (7), and the orthogonality of the endomorphisms $I$, $J$ and $K$ w.r.t. the pairing that $N(I,J)$ vanishes. \qed

Together, Lemma 12, Corollary 11 and Lemma 10 imply the following

Proposition 13. Given an almost hypercomplex structure $(I, J, K)$ on a Courant algebroid $E$, there exists a hypercomplex connection satisfying (13) and (14) if and only if $N(I,J) = 0$. And in that case, it coincides with all three hypercomplex connections given by (15), (16) and (17).

Proposition 14. Let $(I, J, K)$ be an almost hypercomplex structure on a Courant algebroid. The following assertions are equivalent:

(a) $N(I,I) = N(J,J) = 0$;
(b) $N(I,J) = 0$;
(c) $N(I,I) = N(J,J) = N(K,K) = N(I,J) = N(J,K) = N(K,I) = 0$.

Proof. (i)$\Rightarrow$(ii) The proof is a lengthy computation similar to that of [6, Theorem 3.1]. It is omitted. (ii)$\Rightarrow$(iii) For any pair of elements $P, Q$ in $\{I, J, K\}$, we can evaluate the Nijenhuis concomitant
$$N(P, Q)(X,Y) = PX \circ QY - P(X \circ QY) - Q(PX \circ Y) + PQ(X \circ Y) + QX \circ PY - Q(X \circ PY) - P(QX \circ Y) + QP(X \circ Y)$$
by successively making use of: primo relation (20) to get rid of all the Dorfman brackets in the r.h.s. of (21); secondo (13) and the quaternionic relations (7) to cancel all terms involving $\nabla$; terzo (7) and the orthogonality of $I$, $J$ and $K$ w.r.t. the pairing to cancel all remaining terms. (iii)$\Rightarrow$(i) This is trivial. \qed

Theorem 7 immediately follows from Propositions 13 and 14.
Example 15. Let $i, j, k$ be three almost complex structures on a smooth manifold $X$. The triple

$$I = \begin{pmatrix} -i & 0 \\ 0 & i^* \end{pmatrix}, \quad J = \begin{pmatrix} -j & 0 \\ 0 & j^* \end{pmatrix}, \quad K = \begin{pmatrix} -k & 0 \\ 0 & k^* \end{pmatrix}$$

is a hypercomplex structure on $TX \oplus T^*X$ if and only if the triple $i, j, k$ is hypercomplex in the classical sense (see [5]).

Example 16. Let $j$ be an almost complex structure on a smooth manifold $X$ and let $\omega_1$ and $\omega_2$ be two nondegenerate 2-forms on $X$. The triple

$$I = \begin{pmatrix} 0 & \omega_2^{-1} \\ -\omega_2 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -j & 0 \\ 0 & j^* \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \omega_1^{-1} \\ -\omega_1 & 0 \end{pmatrix}$$

is hypercomplex on $TX \oplus T^*X$ if and only if $\omega_1 + i\omega_2 \in \Omega^2(X)$ is a holomorphic symplectic structure on $X$. Theorem 7 has interesting consequences in this case, which we will discuss somewhere else.

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