Bound states at exceptional points in the continuum

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Abstract. In this work, an example of exceptional points in the continuous spectrum of a Hamiltonian of von Neumann-Wigner type is presented and discussed. Remarkably, these exceptional points are not associated with a double pole in the scattering matrix but with a double pole in the normalization factor of the Jost eigenfunctions normalized to unit flux. At the exceptional points the two unnormalized Jost eigenfunctions are no longer linearly independent but coalesce to give rise to two Jordan cycles of generalized bound state energy eigenfunctions in the continuum and a Jordan block representation of the Hamiltonian. The regular scattering eigenfunction vanishes at the exceptional points and the irregular scattering eigenfunction has a double pole at the exceptional points. The scattering matrix is a regular analytical function of the wave number \( k \) for all \( k \) real including the exceptional points.

1. Introduction.
The readings in a measuring device are real numbers which, according to Quantum Mechanics, correspond to the points in the spectrum of an operator that represents an observable. This physical condition is stated in mathematical form by demanding that the spectrum of an operator representing an observable should be real [1]. When the observable is represented by a self-adjoint operator the condition is automatically satisfied. But the reality of the spectrum of an operator does not necessarily mean that the operator is self-adjoint [2, 3]. This non-equivalence of selfadjointness and the reality of the spectrum of an operator was made evident by the discovery and subsequent discussion of a large class of non-Hermitian PT-symmetric Schrödinger Hamiltonian operators with a complex valued potential term but with real energy eigenvalues [4, 5, 6]. Even in the case of a radial Schrödinger Hamiltonian with a real potential term, the reality of the energy spectrum does not necessarily mean that the Hamiltonian is self-adjoint. In many cases, the spectrum of a real non-self-adjoint Hamiltonian is real but differs from the spectrum of self-adjoint ones in two essential features. These are,

(i) the possible presence of exceptional points
(ii) the possible presence of spectral singularities.

Exceptional points already appear in the finite dimensional case of non-Hermitian Hamiltonian matrices depending on a set of control parameters [7, 8, 9, 10, 11, 12, 13], whereas spectral
singularities are characteristic features of Hamiltonians having a continuous energy spectrum [14, 15, 16, 17, 18, 19, 20]. Hence, they are not possible for finite dimensional operators. Exceptional points in the real continuous spectrum of a Schrödinger Hamiltonian with a real potential have received much less attention than in the finite dimensional case [21, 22]. With the purpose of clarifying the topological nature of the exceptional points in the real, continuous energy spectrum of a quantum system, in the following we will present and discuss an example of exceptional points in a Hamiltonian with a real potential of von Neumann-Wigner type. This potential is generated from the eigenfunctions of a free particle Hamiltonian by means of a four times iterated Darboux transformation when the transformation functions are degenerated in the continuum of eigenfunctions of the free particle Hamiltonian [23, 24].

This paper is organized as follows: In section 2, we give a brief description of bound states in the continuum through the mathematical formalism of Darboux transformations. In section 3, by means of a four times iterated and completely degenerated Darboux transformation, we generate an $H[4]$ Hamiltonian which has two exceptional points in its real and continuous degenerate spectrum. In section 4, we compute the regular and irregular scattering solutions of $H[4]$. In section 5, we show that at the exceptional points $k = \pm q$, the Jost solutions coalesce, giving rise to a Jordan chain of generalized eigenfunctions of bound states. In section 6, we show that the Hamiltonian has a Jordan block matrix representation and a Jordan cycle of generalized eigenfunctions. Section 7 is devoted to show that the scattering solution vanishes at the exceptional points $k = \pm q$, while the irregular scattering solution has a double pole at $k = \pm q$. In section 8, we calculate the scattering matrix $S(k)$ and we show that at the exceptional points $k = \pm q$, it is a regular function of $k$, thus has no poles. We end with a brief summary and conclusions.

2. Bound states embedded in the continuum.

A variety of exactly solvable Hamiltonians of von Neumann-Wigner type that support a bound state in the continuum may be constructed from the known energy eigenfunctions of some initial Hamiltonian $\hat{H}_0$ by means of the Darboux transformation [23].

Let us consider an initial Hamiltonian $\hat{H}_0$ and its eigenfunction $\phi_\lambda(r)$ with eigenvalue $E_\lambda$, the Darboux transformation of the solution $\phi_\lambda$ is defined by

$$\phi_\lambda(r) \rightarrow \psi_\lambda[1](r) = \frac{W(\phi_\mu, \phi_\lambda)}{\phi_\mu(r)},$$

where $W(\phi_\mu, \phi_\lambda)$ is the usual Wronskian determinant.

The potential function in $\hat{H}_0$, $V_0$ transform into $V[1]$

$$V_0 \rightarrow V[1](r) = V_0 - 2 \frac{d^2}{dr^2} \ln \phi_\mu(r).$$

Then, the function $\psi_\lambda[1](r)$ satisfies the Schrödinger radial equation

$$H[1](r)\psi_\lambda[1](r) = E_\lambda \psi_\lambda[1](r),$$

with the new radial Hamiltonian,

$$H[1] = -\frac{d^2}{dr^2} + V[1](r).$$

The fixed eigenfunction $\phi_\mu(r)$ of the initial Hamiltonian $\hat{H}_0$ is known as the transformation function.
The Darboux transformation may be iterated $N$ times and the resulting potential $V[N](r)$ and the corresponding eigenfunctions $\Psi[N]$ may still be expressed in terms of the eigenfunctions of the initial Hamiltonian $\hat{H}_0$. This generalization is known in the literature as the theorem of Crum [26] which states that the function

$$
\Psi_E[N](r) = \frac{W(\phi_1, \phi_2, \ldots, \phi_N, \phi_E)}{W(\phi_1, \phi_2, \ldots, \phi_N)},
$$

is an eigenfunction of the radial Schrödinger equation with the potential

$$
V[N] = V_0 - 2 \frac{d^2}{dr^2} \ln W(\phi_1, \phi_2, \ldots, \phi_N),
$$

and eigenvalue $E$. In these expressions, $W(\phi_1, \phi_2, \ldots, \phi_N)$ and $W(\phi_1, \phi_2, \ldots, \phi_N, \phi_E)$ are the Wronskians of $N$ eigenfunctions $\phi_k(r)$ of the initial Hamiltonian $\hat{H}_0$ corresponding to the eigenvalues $E_k$, $k = 1, 2, \ldots, N$ and $\phi_E$ is an eigenfunction of $\hat{H}_0$ with eigenvalue $E$. Moreover, the Hamiltonians $\hat{H}_0$ and $\hat{H}$ with potentials $V_0$ and $V[N]$ have the same spectrum, except for the eigenvalues $E_1, E_2, \ldots, E_N$. This constraint on the spectrum makes itself apparent in the time evolution of the system.

In this work, we will be interested in a special kind of potential $V[N]$ obtained from the $N$-times iterated Darboux transformation, when the transformation functions are degenerated in a continuum of eigenvalues of $\hat{H}_0$.

The explicit form that the transformation takes in this case is obtained by taking all the transformation functions very close in energy $\phi_1 = \phi(r, q)$,

$$
\phi_k = \phi(r, q + \alpha_k), \quad k = 2, 3, \ldots, N
$$

with $\alpha_k << q$ and $\alpha_k < \alpha_{k+1}$ for all $k$.

Next, the transformation functions $\phi_k$ are expanded in a power series in $\alpha_k$ and the resulting expressions are substituted for the function $\phi_k(r)$ in eqs.(5) and (6). By making use of the properties of the Wronskians in the limit when each $\alpha_k$ vanishes, we finally get

$$
\Psi_E[N] = \frac{W_2(\phi, \partial_q \phi, \ldots, \partial_q^{N-1} \phi; \phi_E)}{W_1(\phi, \partial_q \phi, \ldots, \partial_q^{N-1} \phi)}
$$

and

$$
V[N] = V_0 - 2 \frac{d^2}{dr^2} \ln W(\phi, \partial_q \phi, \ldots, \partial_q^{N-1} \phi),
$$

the notation $\partial_q \phi$ is shorthand for $(\partial \phi/\partial q)$.

When the transformation function $\phi_q(r)$ is properly chosen, the iterated Darboux transformation with degeneracy may be used to construct potentials of von Neumann-Wigner type which support bound states in the continuum [27].

3. Jost solutions of the Hamiltonian $H[4]$.

In this section, by means of a four times iterated and completely degenerated Darboux transformation, we will generate a Hamiltonian, $H[4]$, which has two exceptional points in its spectrum.

The transformation function will be a generic solution of the Hamiltonian $\hat{H}_0$ of a free particle ($V_0 = 0$) with eigenvalue $E_q = q^2$,

$$
\phi(r) = \sin(qr + \delta(q)),
$$

(10)
the phase shift $\delta(q)$ that appears in this expression is a smooth odd function of the wave number $q$.

Then, the potential $V[4]$ is obtained from equation (9) as

$$V[4] = -2\frac{1}{W'_1(r)} \left( W''_1(r) W_1(r) - W'^2_1(r) \right),$$

where, $W'_1(r) = \partial W_1(r)/\partial r$ and

$$W_1(q,r) = W_1(\phi, \partial_q \phi, \partial^{(2)}_q \phi, \partial^{(3)}_q \phi) = 16 q^4 \gamma_1^4 - 12 q^2 \gamma_1^2 + 8 q^4 \gamma_2 \gamma - 12 q^4 \gamma_1^2 - 24 q^2 (q \gamma_1 \gamma + \gamma_2) \cos 2\theta + 3 \sin^2 2\theta + (16 q^3 \gamma_1 - 12 q^2 \gamma_1 - 2 q^3 \gamma_2) \sin 2\theta,$$

the functions $\theta(r), \gamma(r), \gamma_1(q)$ and $\gamma_2(q)$ are

$$\theta(r) = qr + \delta(q), \quad \gamma(r) = \partial_q \theta = r + \gamma_0$$

and

$$\gamma_1 = \partial^{(2)}_q \theta \quad \text{and} \quad \gamma_2 = \partial^{(3)}_q \theta,$$

$\gamma_0$ is a function of $q$ but it is independent of $r$.

**Figure 1.** The graph shows the von Neumann-Wigner potential $V[4]$ as function of $r$ for the values of the parameters in $V[4]$ fixed at $q = 1$, $\delta(q) = \pi/6$, $\gamma_0 = 0.6$, $\gamma_1 = 0.214$ and $\gamma_2 = 0.8$. This potential is such that there are exceptional points in the continuum of the spectrum of $H[4]$ at $k = \pm q$.

Explicit expressions for $W_1(q,r)$ and the potential $V[4](r)$, as functions of $r$, are obtained when the explicit expressions for $\theta(r), \gamma(r), \gamma_1$ and $\gamma_2$ as functions of $r$ are inserted in equations (11) and (12). A graphical representation of $V[4]$ as a function of $r$ is shown in figure 1.

The asymptotic behaviour of $V[4]$ for $r$ large is readily determined from the asymptotic behaviour of $W_1(q,r)$ for $r$ large. Since $\gamma$ is a linear function of $r$, the dominant term in the expression (12) for $W_1(q,r)$ is the term with the highest power of $\gamma$. Hence,

$$V[4] = 8q \frac{\sin 2(qr + \delta(q))}{r} + O(r^{-2}), \quad \text{for } r \text{ large.}$$
It follows from here that $V[4]$ is a potential of von Neumann-Wigner type with parameters $a = 8q$ and $b = 2q$. Hence, $V[4]$ is able to support a bound state in the continuum that belongs to the energy eigenvalue $b^2/4 = q^2$ independently of the numerical values of the parameters $\gamma_0, \gamma_1$ and $\gamma_2$.

The two independent unnormalized Jost functions of the Hamiltonian $H[4]$, that belong to the energy eigenvalue $E = k^2$ and behave as outgoing and incoming spherical waves for large values of $r$, are obtained from equation (8),

$$\psi_{k, 4} = \frac{W_2(\phi, \partial_q \phi, ..., \partial_{q(4)} \phi; e^{\pm ikr})}{W_1(\phi, \partial_q \phi, ..., \partial_{q(4)} \phi)} := f_{k, r},$$

and take the form

$$f_{k, r} = \frac{1}{W_1(q, r)} w_{k, r} e^{\pm ikr},$$

here, and in the following, $f_{k, r}$ denotes the unnormalized Jost solutions of $H[4]$.

The reduced Wronskian $w_{k, r}$ is a complex function of the arguments $k, q$ and $r$,

$$w_{k, r} = u(k, r) \pm iv(k, r).$$

A straightforward computation of $W_1(\phi, \partial_q \phi, \partial_{q(3)} \phi, \partial_{q(4)} \phi)$ and $w_{k, r}$ allows us to find the following explicit expressions for the functions $u(k, r)$ and $v(k, r)$

$$u(k, r) = 16q^4 (k^2 - q^2)^2 \gamma^4 - 12q^2 (k^4 + 6q^2k^2 + q^4) \gamma^2$$

$$+ 8q^2(k^4 + 2q^2k^2 - q^4) \gamma^2 - 12q(k^4 + 4q^2k^2 - q^4) \gamma$$

$$+ q \gamma_1(k^4 - q^2) \gamma - 12 \gamma_0 \gamma_1(k^2 + q^2 - q^4) \gamma \cos 2\theta$$

$$+ \sin^2 2\theta (k^3 + 6q^2k + q^3)(2 - 12q^2(k^4 + 4q^2k^2 - q^4)) \sin 2\theta$$

$$+ 3(k^3 + 6q^8k + q^3) \sin 2\theta \gamma$$

$$u(k, r) = 64q^4(k^2 - q^2) \gamma^3 - 24q^2(k^4 + q^2) \gamma + 8q^2q^4 \gamma^3 - 24q^2(k^4 + q^2) \gamma$$

$$- 4q^2q^4 \gamma^3 + 8q^2q^4 \gamma^3 + 24q^2(k^4 + q^2) \gamma$$

$$- 8q^2q^4 \gamma^3 + 8q^2q^4 \gamma^3 + 24q^2(k^4 + q^2) \gamma$$

$$- 8q^2q^4 \gamma^3 + 8q^2q^4 \gamma^3 + 24q^2(k^4 + q^2) \gamma$$

$$+ 6q(k^2 + q^2) \sin 4\theta.$$  

The asymptotic behaviour of the eigenfunction $f_{k, r}$ for large values of $r$ is dominated by the highest power of $\gamma$.

From equations (12), (19) and (20), we get

$$W_1(q, r) \approx 16q^4r^4[1 + O(r^{-1})],$$

$$u(k, r) \approx 16q^4(k^2 - q^2)^2r^4[1 + O(r^{-1})]$$

and

$$v(k, r) \approx 32q^4(k^2 - q^2)(2 + \cos 2\theta)r^3[1 + O(r^{-1})].$$

Hence, for large values of $r$, we obtain

$$f_{k, r} = (k^2 - q^2)^2[1 + O(r^{-1})]e^{\pm ikr},$$
the factor $(k^2 - q^2)^2$ is the flux of probability current at infinity of the unnormalized Jost solutions $f^\pm(k, r)$.

Therefore, the two Jost solutions of $H[4]$, normalized to unit flux at infinity, are

$$F^\pm(k, r) = \frac{f^\pm(k, r)}{(k^2 - q^2)^2} = \frac{1}{(k^2 - q^2)^2 W_1(q, r)} u(k, r) \pm i v(k, r) \right] e^{\pm ikr}, \quad k \neq q.$$  \hspace{1cm} (25)

4. Scattering eigenfunctions and scattering matrix.

The scattering solution $\psi_s(k, r)$ expressed in terms of the Jost solutions is

$$\psi_s(k, r) = i \left[ F^-(k, r) - S(k) F^+(k, r) \right], \quad k \neq q,$$  \hspace{1cm} (26)

where the scattering matrix $S(k)$ is

$$S(k) = \frac{f^-(k, 0)}{f^+(k, 0)} = \frac{u(k, 0) - i v(k, 0)}{u(k, 0) + i v(k, 0)},$$  \hspace{1cm} (27)

where the functions $u(k, 0)$ and $v(k, 0)$ are given by eqs.(19) and (20) with $r = 0$.

This expression may also be written in the following form:

$$\psi_s(k, r) = \frac{1}{(k^2 - q^2)^2 W_1(q, r)} e^{i \Delta(k)} \left\{ u(k, r) \sin(kr + \Delta(k)) + v(k, r) \cos(kr + \Delta(k)) \right\},$$  \hspace{1cm} (28)

where the phase $\Delta(k)$ is given by

$$\Delta(k) = -\arctan \left( \frac{v(k, 0)}{u(k, 0)} \right).$$  \hspace{1cm} (29)

The regular scattering solution $\psi_s(k, r)$ goes to zero at the origen of coordinates and behaves as $\sin(kr + \Delta(k))$ for large values of $r$. A graphical representation of the regular Jost solution is shown in the figure 2.

The irregular scattering solution is

$$\psi_is(k, r) = \frac{1}{2} \left[ F^-(k, r) + e^{2i \Delta(k)} F^+(k, r) \right],$$  \hspace{1cm} (30)

therefore

$$\psi_is(k, r) = \frac{1}{(k^2 - q^2)^2 W_1(q, r)} e^{i \Delta(k)} \left\{ u(k, r) \cos(kr + \Delta(k)) - v(k, r) \sin(kr + \Delta(k)) \right\},$$  \hspace{1cm} (31)

where $u(k, r)$ and $v(k, r)$ are given in equations (17) and (18). The figure 3 shows the behaviour of the irregular scattering solution $\psi_is(k, r)$, this solution is divergent at the origen and it has an oscillatory behaviour for large values of $r$.

5. Exceptional points of $H[4]$ and singularities of the normalized Jost solutions

In the general case, the Hamiltonian $H[4]$ has a continuous spectrum extending from zero to infinity and to each point in this spectrum belong two linearly independent energy eigenfunctions. Clearly, the continuous spectrum of $H[4]$ is doubly degenerate. If the wave number $k$ is allowed to take complex values, the region of analyticity of the energy eigenfunctions as functions of $k$ is
Figure 2. The graph shows the scattering solution \( \psi_s(k, r) \) as function of \( r \) for \( k = 0.8 \) and the parameters in \( V[4] \), fixed at the values \( q = 1, \delta(q) = \pi/6, \gamma_0 = 0.6, \gamma_1 = 0.214 \) and \( \gamma_2 = 0.8 \).

Figure 3. The graph shows the irregular scattering solution \( \psi_{is}(k, r) \) as function of \( r \) for \( k = 0.8 \) and the values of the parameters in \( V[4] \), fixed at \( q = 1, \delta(q) = \pi/6, \gamma_0 = 0.6, \gamma_1 = 0.214 \) and \( \gamma_2 = 0.8 \).

extended to all points in the complex \( k \)-plane. When \( k \neq \pm q \), the energy eigenfunctions are the normalized, Jost solutions \( F^\pm(k, r) \) which behave at infinity as incoming and outgoing waves. In this section, it will be shown that at the points \( k = \pm q \), the two unnormalized Jost solutions are no longer linearly independent but coalesce to give rise to a Jordan chain of generalized bound state eigenfunctions of length 2.

From equations (17), (18) and \( k > 0 \), a straightforward computation of the Wronskian of the Jost solutions gives

\[
W(f^+(k, r), f^-(k, r)) = -2i \frac{W(u, v)}{u^2 + v^2 + k} f^+(k, r)f^-(k, r),
\]

the Wronskian of the functions \( u(k, r) \) and \( v(k, r) \) is readily computed from eqs.(19) and (20), we get

\[
W(u, v) = -q(u^2 + v^2),
\]
substitution of this result in the previous equation gives
\[ W(f^+(k, r), f^-(k, r)) = -2if^+(k, r)f^-(k, r)(k-q), \quad k > 0. \] (34)

Now, since the potential \( V[4] \) is an even function of \( q \), time reversal invariance of the Hamiltonian plus boundary conditions implies that [28],
\[ f^+(-k, r) = +f^-(k, r), \] (33)
for \( k \) real. In this case, we obtain
\[ W(f^+(k, r), f^-(k, r)) = +2if^+(k, r)f^-(k, r)(k+q), \quad k < 0. \] (36)

Since the product of the unnormalized Jost solutions is non-vanishing for all \( k \),
\[ f^+(k, r)f^-(k, r) > 0, \] (37)
the two Jost solutions are linearly independent for \( k \neq \pm q \).

The points \( k = \pm q \) are real non-vanishing zeroes of the Wronskian of the two Jost solutions of \( H[4] \). This property identifies the points \( k = \pm q \) as exceptional points in the spectrum of the Hamiltonian \( H[4] \).

The unnormalized Jost solutions of \( H[4] \), \( f^{\pm}(k, r) \), given in eqs.(17-20), are entire functions of \( k \) for all \( k \). Since the flux of probability current at infinity of these solutions is \((k^2 - q^2)^2\), the Jost solutions \( F^{\pm}(k, r) \) of \( H[4] \) normalized to unit flux at infinity have double poles as function of \( k \) at \( k = \pm q \), see eq.(25). In order to uncover the exceptional points of \( H[4] \), we separate the singular from the regular parts of the normalized Jost solutions \( F^{\pm}(k, r) \) as functions of \( k[29] \)

\[
\frac{f^{\pm}(k, r)}{(k^2 - q^2)^2} = \frac{1}{4q^2} \left[ \frac{1}{q} \left( \frac{\partial f^{\pm}(k, r)}{\partial k} \right) - \frac{1}{q} \frac{f^\pm(q, r)}{k-q} \right] + h^{\pm}_< (k, r) + \frac{1}{4q^2} \left[ \frac{1}{q} \left( \frac{\partial f^{\pm}(k, r)}{\partial k} \right) + \frac{1}{q} \frac{f^\pm(-q, r)}{k+q} \right] + h^{\pm}_> (k, r),
\] (38)

where
\[
h^{\pm}_>(k, r) = \frac{1}{4q^2} 2g^{\pm}_>(k, r) + \frac{k}{q} g^{\pm}_>(k, r) + \frac{1}{q} \left( \frac{\partial f^{\pm}(k, r)}{\partial k} \right) \pm q, \]
(39)

and
\[
g^{\pm}_>(k, r) = f^\pm(k, r) - \left( f^\pm(\pm q, r) + \left( \frac{\partial f^\pm(k, r)}{\partial k} \right) \pm q \right) \frac{1}{(k \mp q)^2}. \]
(40)

Now, let us define a more convenient notation
\[
\psi_B(q, r) = \frac{f^\pm(q, r)}{4q^2} e^{\pm i\delta(q)}, \]
(41)

and
\[
\chi^\pm_B(q, r) = \frac{1}{4q^2} \left[ \left( \frac{\partial f^\pm(k, r)}{\partial k} \right) \right]_{k=q} - \frac{1}{q} f^\pm(q, r) e^{\pm i\delta(q)}. \]
(42)

Then, substitution of expressions (41-42) in equation (25) allows us to write the normalized Jost solutions of \( H[4] \) as the sum of singular and regular parts
\[
F^\pm(k, r) = \frac{\psi_B(q, r) e^{\mp i\delta(q)}}{(k-q)^2} + \frac{\chi^\pm_B(q, r) e^{\mp i\delta(q)}}{(k-q)^2} \pm \frac{\psi_B(-q, r) e^{\mp i\delta(q)}}{(k+q)^2}.
\]
Figure 4. The graph shows the bound state eigenfunction $\psi_B(q,r)$ as a function of $r$ computed for $q = 1$ and the values of the parameters in $V[4]$ fixed at $\theta = \pi/6$, $\gamma_0 = 0.6$, $\gamma_1 = 0.214$ and $\gamma_2 = 0.8$

$$\psi_B(q,r) = \frac{24q^2}{W_1(q,r)} \left[ -2q^2 \gamma^3 \sin \theta + 3q^2 \gamma^2 \cos \theta + 3q^2 \gamma \sin \theta \right]$$

(45)

where

$$\psi_B(q,r) = \frac{\chi_B^\pm(q,r) e^{i \Theta(q)}}{h^\pm(k,r)} + \frac{\psi_B(q,r)}{h^\pm(k,r)} + \frac{\chi_B^\pm(-q,r) e^{-i \Theta(q)}}{h^\pm(k,r)}.$$  

(43)

Explicit expressions for the functions $\psi_B(q,r)$ and $\chi_B^\pm(q,r)$ are readily obtained from the definitions (41), (42) and equations (12) and (17-20)

$$\psi_B(q,r) = \frac{24q^2}{W_1(q,r)} \left[ -2q^2 \gamma^3 \sin \theta + 3q^2 \gamma^2 \cos \theta + 3q^2 \gamma \sin \theta \right]$$

(45)

and

$$\chi_B^\pm(q,r) = \frac{\chi_B^\pm(q,r) e^{i \Theta(q)}}{h^\pm(k,r)} + \frac{\psi_B(q,r)}{h^\pm(k,r)} + \frac{\chi_B^\pm(-q,r) e^{-i \Theta(q)}}{h^\pm(k,r)}.$$  

(43)

Since $W_1(q,r) \sim 16(qr)^4[1 + O(r^{-1})]$, the generalized eigenfunctions $\psi_B(q,r)$ and $\chi_B^\pm(q,r)$ vanish as $r^{-3}$ for $r$ large, and, hence they are quadratically integrable. Furthermore, $\psi_B(q,r)$ is an even function of $q$ and $\chi_B(q,r)$ is an odd function of $q$.

Now, it is straightforward to verify that the functions $\psi_B(q,r)$ and $\chi_B(q,r)$ are the elements of a Jordan cycle of length two of generalized eigenfunctions of $H[4]$.

From the radial Schrödinger equation satisfied by $F^\pm(k,r)$

$$H[4]F^\pm(k,r) = k^2 F^\pm(k,r),$$

(48)

and the expansion of $F^\pm(k,r)$ in singularities, equation (43), we get

$$\left(1 + \left(\frac{k-q}{k+q}\right)^2\right)H[4]\psi_B(q,r) + (k-q)H[4]\left(\chi_B^\pm(q,r) + \frac{k-q}{k+q}\chi_B^\pm(-q,r)\right)$$

\[+ \psi_B(q,r) + \frac{\chi_B^\pm(q,r) e^{i \Theta(q)}}{h^\pm(k,r)} + \frac{\psi_B(q,r)}{h^\pm(k,r)} + \frac{\chi_B^\pm(-q,r) e^{-i \Theta(q)}}{h^\pm(k,r)}\]
Figure 5. The generalized eigenfunction $\chi_B(q,r)$, as a function of $r$ computed $q = 1$ and the values of the parameters in $V[4]$ fixed at $\delta = \pi/6$, $\gamma_0 = 0.6$, $\gamma_1 = 0.214$ and $\gamma_2 = 0.8$

\[
+(k - q)^2 e^{\pm i\delta(q)} H[4] h^\pm(k,r) = k^2 \left(1 + \left(\frac{k - q}{k + q}\right)^2\right) \psi_B(q,r)
+k^2(k - q) \left(\chi^+_B(q,r) + \frac{k - q}{k + q} \chi^+_B(q-r,r)\right) + k^2(k - q)^2 h^\pm(k,r) e^{\pm i\delta(q)}.
\]  

(49)

Taking the limit when $k \to q$, we obtain

\[
H[4] \psi_B(q,r) = q^2 \psi_B(q,r).
\]  

(50)

Substitution of this result in (49) gives

\[
H[4]\left(\chi^+_B(q,r) + \frac{k - q}{k + q} \chi^+_B(q-r,r)\right) + (k - q)e^{\pm i\delta(q)} H[4] h^\pm(k,r) =
2\frac{k^2 + q^2}{k + q} \psi_B(q,r) + k^2 \left(\chi^+_B(q,r) + \frac{k - q}{k + q} \chi^+_B(q-r,r)\right) +
k^2(k - q)e^{\pm i\delta(q)} h^\pm(k,r),
\]  

(51)

again, taking the limit when $k \to q$, and using (47) and (50) we get

\[
H[4] \chi_B(q,r) = q^2 \chi_B(q,r) + 2q \psi_B(q,r).
\]  

(52)

Graphical representations of the generalized eigenfunctions $\psi_B(q,r)$ and $\chi_B(q,r)$ as functions of $r$ are shown in Figs. 4 and 5.

6. Jordan cycle of generalized eigenfunctions and Jordan block representation of $H[4]$

The generalized eigenfunctions $\psi_B(q,r)$ and $\chi_B(q,r)$ belong to the same point $k = q$ in the spectrum of $H[4]$ and satisfy the coupled equations (50 and 52), which identify $\psi_B(q,r)$ and $\chi_B(q,r)$ as the elements of a Jordan cycle of length two of generalized eigenfunctions of $H[4]$ associated with a Jordan block representation of $H[4]$ at the exceptional point.

In order to make this property evident, it will be convenient to write equations (50 and 52) in matrix form,

\[
H[4]_{12x2} \Psi_B(q,r) = \mathcal{H}_B(q) \Psi_B(q,r),
\]  

(53)
where
\[ \Psi_B(q,r) = \begin{pmatrix} \psi_B(q,r) \\ \chi_B(q,r) \end{pmatrix} \] (54)
and
\[ \mathcal{H}_B(q) = \begin{pmatrix} q^2 & 0 \\ 2q & q^2 \end{pmatrix}. \] (55)

From (53), it is evident that the matrix \( \mathcal{H}_B(q) \) is a matrix representation of the Hamiltonian \( H[4] \) in the two dimensional functional space spanned by the generalized eigenfunctions \( \{\psi_B(q,r), \chi_B(q,r)\} \).

(i) Let us recall that the domain \( \mathcal{D}(H[4]) \) of the Hamiltonian \( H[4] \) is a rigged Hilbert space of continuous complex functions of the variables \((k,r)\) with continuous first and second derivatives with respect to \( r \), with \( r \) in the semi-infinite straight line \( 0 \leq r < \infty \).

The two dimensional subspace of functions spanned by the generalized eigenfunctions \( \{\psi_B(q,r), \chi_B(q,r)\} \) is in the domain \( \mathcal{D}(H[4]) \) of \( H[4] \).

(ii) The matrix \( \mathcal{H}_B(q) \) is a Jordan block\([30]\) of \( 2 \times 2 \) and can not be brought to diagonal form by means of a similarity transformation with a unitary operator \([31]\). Hence, also the Hamiltonian \( H[4] \) can not be diagonalized by means of a similarity transformation with a unitary operator.

(iii) The real non-symmetric matrix \( \mathcal{H}_B(q) \) is not Hermitian \( \mathcal{H}_B(q) \neq \mathcal{H}_B(q)^\dagger \). Therefore, the system of coupled equations (50) and (52) plus boundary conditions is not self-adjoint. This lack of self-adjointness of \( \mathcal{H}_B(q) \) implies that also the operator \( H[4] \) on the left hand side of eq.(53), when acting in the functional space spanned by the generalized eigenfunctions, is also not self-adjoint, that is, non-Hermitian.

7. Scattering solutions.

In this section it will be shown that the regular scattering solution \( \psi_s(k,r) \) vanishes at \( k = \pm q \), and the irregular Jost solution \( \psi_{is}(k,r) \) has a double pole at the exceptional points \( k = \pm q \).

The regular scattering solution of \( H[4] \) is obtained from the Jost solutions, eqs. (25-27), as
\[ \psi_s(k,r) = i \frac{1}{2(k^2 - q^2)^2} W_1(q,r) w^+(k,0) w^+(k,0) w^+(k,0) w^+(k,0) \left[ w^+(k,0) w^-(k,r) e^{ikr} - w^-(k,0) w^+(k,r) e^{ikr} \right], \] (57)
where
\[ \frac{i}{2} w^+(k,0) w^+(k,0) w^+(k,0) w^+(k,0) \left[ w^+(k,0) w^-(k,r) e^{ikr} - w^-(k,0) w^+(k,r) e^{ikr} \right] = e^{i\Delta} u(k,r) \sin(kr + \Delta) + v(k,r) \cos(kr + \Delta), \] (58)
explicit expressions for \( u(k,r) \) and \( v(k,r) \) as functions of \((k,r)\) are given in equations (19-20).

In order to make explicit the properties of the scattering solutions as functions of \( k \), for \( k \) close to the exceptional points at \( k = \pm q \), we will write the reduced Wronskian \( w^\pm(k,r) \) as an expansion in powers of \((k - q)\)
\[ w^\pm(k,r) = u(k,r) \pm iv(k,r) = e^{i\theta(q,r)} \sum_{\ell=0}^4 w^\pm_\ell(q,r)(k-q)^\ell, \] (59)
explicit expressions for the coefficients $w^\pm_\ell(q,r)$, as functions of their arguments are given in the
appendix.

Then, the term in square brackets in the right hand side of eq.(57) may also be expressed as
an expansion in powers of $(k-q)$

$$w^+(k,0)w^-(k,r)e^{-ikr} - w^-(k,0)w^+(k,r)e^{ikr} =$$

$$\sum_{\ell=0}^{4} \sum_{m=0}^{4} w^+_{\ell m}(q,r)e^{-i(k-q)r} - w^-_{\ell m}(q,r)e^{i(k-q)r}.$$

Now, from (A1) and (A2) in the appendix, the first three terms in the summation in the right
hand side of eq.(60) are proportional to $w^+_0(q,0)$ and / or $w^-_0(q,0)$, but

$$w^+_0(q,r) = w^-_0(q,r) = 4q^2W_1(q,r)\psi_B(q,r),$$

since $\psi_B(q,0)$, vanishes the first three terms in the summation in the right hand side of (60)
also vanish.

It follows that

$$\psi_s(k,r) = \frac{i}{2} \frac{(k-q)}{(k+q)^2 W_1(q,r) w^+(q,0)} \sum_{\ell=3}^{8} \sum_{m=0}^{\ell} w^+_{\ell m}(q,r)e^{-i(k-q)r}$$

$$- w^-_{\ell m}(q,r)e^{i(k-q)r} \frac{e^{i\Delta(k)}}{(k-q)\ell-3},$$

with the restriction $w^+_0(q,r) = 0$, for $\ell \geq 3$.

Therefore, at the exceptional point, the regular scattering solution also vanishes,

$$\psi_s(q,r) = 0.$$

The energy eigenfunctions at the exceptional point are the generalized bound state eigenfunctions
$\psi_B(q,r)$ and $\chi_B(q,r)$.

Let us now turn our attention to the irregular scattering solution, equation (31),

$$\psi_{is}(k,r) = \frac{1}{(k^2 - q^2)^2 W_1(q,r)} \frac{1}{2} e^{i\Delta(k)} \{ u(k,r) \cos(kr + \Delta(k)) - v(k,r) \sin(kr + \Delta(k)) \}$$

In this case, we have

$$\frac{1}{2} w^+(k,0)w^-(k,r)e^{-ikr} + w^-(k,0)w^+(k,r)e^{ikr}.$$

Notice that, when the expression (59) is substituted for $w^\pm(k,r)$ in (65) we obtain an expansion
of the right hand side of (65) in powers of $(k-q)$ similar to (60), but now the terms proportional
to $w^+_0(q,0)w^+_0(q,r)$ and $w^-_0(q,0)w^-_0(q,r)$ and $w^+_0(q,0)w^+_0(q,r)$ are added. Hence, the singular
solutions in the expressions (43) for the Jost solutions add instead of cancelling which makes $\psi_{is}(k,r)$
singular at the exceptional points. Hence, the irregular scattering solution $\psi_{is}(k,r)$, as a function
of $k$, has a pole of second order at the exceptional points.

Substitution of the expressions (59-60) for $w^\pm(k,r)$ in (65) and, multiplying the result times
$(k-q)^2$ and taking the limit, $k \to q$ gives

$$\lim_{k \to q} (k-q)^2 \psi_{is}(k,r) = \psi_B(q,r).$$

Therefore, the bound state eigenfunction $\psi_B(q,r)$ is the residue of second order of the irregular
scattering solution $\psi_{is}(k,r)$ at $k = q$. 

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Figure 6. The phase shift $\Delta(k)$ as a function of the wave number $k$ computed with the parameters in $V[4]$ fixed at the value $\gamma_0 = 0.61$, $\gamma_1 = 0.214$ and $\gamma_2 = 0.8$.

8. The scattering matrix $S(k)$ in the limit $k \to q$ and the differential cross section.

The scattering matrix $S(k)$ is, by definition,

$$S(k) = \frac{F^-(k,0)}{F^+(k,0)} = \frac{u(k,0) - iv(k,0)}{u(k,0) + iv(k,0)},$$

in this expression $F^{\pm}(k,0)$ are the normalized Jost solutions for incoming and outgoing waves and $u(k,0) \pm iv(k,0)$ is the reduced Wronskian defined in eq. (18), and given in explicit form as a function of the arguments $(k,r)$ in eqs. (19) and (20).

The reduced Wronskian, evaluated at the origin of coordinates, may be written as

$$u(k,0) + iv(k,0) = N(k)e^{-i\delta(q)},$$

where

$$N(k) = \sum_{\ell=0}^{4} w_{+}^{\ell}(q,0)(k-q)^{\ell}.$$  

Thus, the scattering matrix may be written as

$$S(k) = \frac{N^*(k)}{N(k)}e^{i2\delta(q)}.$$  

Now, at $k = q$, only the first term in the expansion (69) contributes

$$N(q) = w_{0}(q,0).$$

Since $w_{0}(q,0)$ is real,

$$\frac{N^*(k)}{N(k)}|_{k=q} = 1.$$  

Therefore, the scattering matrix evaluated at the exceptional point is finite and takes the form

$$\lim_{k \to q} S(k) = e^{i2\delta(q)}.$$
The cross section $\sigma(k)$ as a function of the wave number $k$ computed with the parameter in $V[4]$ fixed at the values $\gamma_0 = 0.6$, $\gamma_1 = 0.214$ and $\gamma_2 = 0.8$

We have shown that $S(k)$ as function of $k$, is finite at the exceptional points, that is, it does not have a singularity, pole or otherwise, at $k = \pm q$. Hence, the bound state generalized eigenfunction are not associated with a singularity of the $S(k)$ matrix.

The cross section is

$$\sigma(k) = \frac{4\pi}{k^2} \sin^2 \Delta(k).$$

(74)

The graphical representation of phase $\Delta(k)$ and the cross section are given in the figures 6 and 7.

9. Summary and Conclusions

The von Neumann-Wigner type Hamiltonian $H[4]$ discussed in this paper and the free particle Hamiltonian $H_0$ are isospectral. In the general case, to each point in this continuous spectrum, correspond two linearly independent Jost solutions which behave at infinity as incoming and outgoing waves. Clearly in both cases, this continuous spectrum is doubly degenerate. However, here we have shown that in the continuous spectrum of $H[4]$ there are two exceptional points at wave numbers $k = \pm q$, such that at these points the two unnormalized Jost eigenfunction of $H[4]$ are linearly dependent, and coalesce to give rise to Jordan cycles of length two of generalized quadratically integrable bound state eigenfunctions and a Jordan block representation of the Hamiltonian $H[4]$. At the exceptional points the regular scattering function of $H[4]$ vanishes, while the irregular scattering solution has a double pole with a second order residue equal to the bound state eigenfunction in the continuum and a Jordan block representation of the Hamiltonian. Most remarkably is the fact that the scattering matrix $S(k)$ is a regular function of $k$ at the exceptional points, that is, the Jordan cycle of generalized bound states eigenfunctions of $H[4]$ in the continuum is not associated with a pole of the scattering matrix.

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Appendix A.
The dependence of $S(k)$ on $k$ is readily made evident by expanding the reduced Wronskian, defined in eq.(18) in powers of $(k - q)$

$$w^\pm(k, r) = e^{\mp i \theta} \sum_{\ell=0}^{4} w_{\ell}^\pm(q, r)(k - q)^\ell,$$  \hspace{1cm} (A.1)

the coefficients of the powers of $(k - q)$ in this expansion are

$$w_0^\pm(q, r) = 4q^2 W_1(q, r) \psi_B(q, r)$$  \hspace{1cm} (A.2)

$$w_1^\pm(q, r) = 4qW_1(q, r) \psi_B(q, r) + q\chi_B(q, r) \mp i q\gamma \psi_B(q, r)$$  \hspace{1cm} (A.3)

$$w_2^\pm(q, r) = -2W_1(q, r) \psi_B(q, r) \cos^2 \theta - 3qW_1(q, r) \chi_B(q, r)$$
$$+ 16q \left[ 4q^4 \gamma^4 - 3q^2 \gamma^2 - 3q^4 \gamma^2 + 2q \gamma \gamma_2 - 3 \sin^2 \theta \cos^2 \theta \right] \cos \theta$$
$$\mp i 2W_1(q, r) \left\{ (2q \gamma + q^2 \gamma_1) \psi_B(q, r) + 2 \gamma \chi_B(q, r) \right\}$$  \hspace{1cm} (A.4)

$$w_3^\pm(q, r) = 8q \left\{ 3 \sin 2\theta \sin \theta - 6 (q\gamma) \sin \theta \right\} + 6 (q\gamma)^2 \cos \theta - 12 \gamma \gamma_2 \sin \theta$$
$$+ 8 (q\gamma)^4 \cos \theta - 6 (q_1 \gamma q^2) \sin \theta \cos \theta - 6 (q_1 q^2)^2 \cos \theta$$
$$- 2 (q^3 \gamma^3) \sin \theta \cos \theta - 12 \gamma \gamma q^3 \sin \theta \cos \theta$$
$$+ 4 (q_2 q^3) \sin \theta$$  \hspace{1cm} (A.5)

and

$$w_4^\pm(q, r) = 3 \sin^2 2\theta - 12 (q\gamma) \sin 2\theta + 12 (q\gamma)^2 \sin \theta$$
$$+ 16 (q\gamma)^3 \sin 2\theta + 16 (q\gamma)^4 - 12 \gamma_2 q^2 \sin \theta - 12 \gamma (q_2 q^2)^2$$
$$+ 24 (q^3 \gamma \gamma_1) \cos \theta - 4 (q^2 \gamma q^3) \sin \theta + 8 (q_2 q^4) \cos \theta$$
$$\pm i \left\{ 3 \sin^2 2\theta - 12 (q\gamma) \sin \theta + 12 (q\gamma)^2 \sin \theta \right\} \sin \theta$$
$$+ 16 (q\gamma)^3 \sin 2\theta + 16 (q\gamma)^4 - 12 \gamma_2 q^2 \sin \theta - 12 \gamma (q_2 q^2)^2$$
$$+ 24 (q^3 \gamma \gamma_1) \cos \theta - 4 (q^2 \gamma q^3) \sin \theta + 8 (q_2 q^4) \sin \theta \right\} \sin \theta.$$  \hspace{1cm} (A.6)

An expansion of the reduced wronskian, $w^\pm(k, r)$, about the point $k = -q$ is readily obtained from eqs. (A.1) and (35). From time reversal invariance of the Hamiltonian plus boundary conditions, we get

$$w^\pm(k, r) = w^\mp(-k, r).$$  \hspace{1cm} (A.7)

Hence, from (A.1)

$$w^\pm(k, r) = e^{\mp i \theta(q)} \sum_{\ell=0}^{4} (-1)^\ell w_{\ell}^\pm(q, r)(k + q)^\ell,$$  \hspace{1cm} (A.8)
the coefficients $w_\ell^\pm(q,r)$ are explicitly given in eqs. (A.2 - A.6) as entire functions of $q$. Their domain of analiticity is trivially extended to negative values of the argument $q$ just by changing $q$ by $-q$ in (A.2 - A.6). Then it may be verified that

$$(-1)^\ell w_\ell^\pm(q,r) = w_\ell^\pm(-q,r). \quad (A.9)$$

Therefore,

$$w_\ell^\pm(k,r) = e^{\pm\ell(\theta(q))} \left((-1)^\ell w_\ell^\pm(|q|,r)(k + q)\right)^\ell, \quad k < 0. \quad (A.10)$$

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