Asymptotics of Eigenvalues and Eigenfunctions for the Laplace Operator in a Domain with Oscillating Boundary. Multiple Eigenvalue Case.

Youcef Amirat\(^\flat\), Gregory A. Chechkin\(^\natural\), Rustem R. Gadyl’shin\(^\sharp\).

\(^\flat\) Laboratoire de Mathématiques Appliquées
"CNRS UMR 6620
Université Blaise Pascal
63177 Aubière cedex, France
amirat@math.univ-bpclermont.fr

\(^\natural\) Department of Differential Equations
Faculty of Mechanics and Mathematics
Moscow State University
Moscow 119992, Russia
chechkin@mech.math.msu.su

\(^\sharp\) Department of Mathematical Analysis
Faculty of Physics and Mathematics
Bashkir State Pedagogical University
Ufa 450000, Russia
gadylshin@yandex.ru, gadylshin@bspu.ru

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Abstract

We study the asymptotic behavior of the solutions of a spectral problem for the Laplacian in a domain with rapidly oscillating boundary. We consider the case where the eigenvalue of the limit problem is multiple. We construct the leading terms of the asymptotic expansions for the eigenelements and verify the asymptotics.

Résumé

Nous étudions le comportement asymptotique des solutions d’un problème spectral associé à l’opérateur de Laplace dans un domaine à frontière oscillante. Nous considérons le cas où la valeur propre du problème limite est multiple. Nous construisons les termes principaux des développements asymptotiques des éléments propres et nous donnons une justification rigoureuse des approximations asymptotiques.

Keywords. spectral problems, oscillating boundary, asymptotic expansions

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Introduction.

Boundary-value problems involving oscillating boundaries or interfaces appear in many fields of natural sciences and engineering, such as the scattering of acoustic and electro–magnetic waves on small periodic obstacles (for instance, whispering gallery effects [36] and scanning of the surface of oceans from the outer space [6]), the vibrations of strongly inhomogeneous elastic bodies [7] [15], the friction of details in complex engineering structures [10], the flows over rough walls [1], or behavior of coupled fluid-solid periodic structures (structures having soft and hard phases [9]). Recent years many other mathematical works (purely theoretical as well as applied) were devoted to asymptotic analysis of these problems, see for instance, [4], [5], [13]–[17], [21], [25], [26], [30]–[34], [37].

In the paper [5] the authors considered spectral problems for a general $2m$-order elliptic operator in a domain of a special type with partially oscillating boundary with the Dirichlet type of boundary conditions on the oscillating part of the boundary. They proved the convergence theorem for the eigenvalues and eigenfunctions. Also it should be noted that similar convergence results were given in [35], as application of the method for the approximation of eigenvalues and eigenvectors of self-adjoint operators.

In [3] we considered a spectral problem for the Laplace operator in a bounded domain with the boundary which part, depending on a small parameter $\varepsilon$, is rapidly oscillating. The authors assumed that the frequency and the amplitude of oscillations of the boundary are of the same order $\varepsilon$. The case of simple eigenvalue of the limit problem is studied: the authors constructed the leading terms of the asymptotic expansions for the eigenelements and verified the asymptotics.

In this paper we deal with the same spectral problem in the case when the eigenvalue of the limit problem is multiple. Our aim is to construct accurate asymptotic approximations, as $\varepsilon \to 0$, of the eigenvalues and corresponding eigenfunctions. We use the method of matching of asymptotic expansions (see [22], [23] and [24]) to construct the leading terms of the asymptotic expansions for the eigenelements. Then we prove the asymptotic estimates of the difference between the solutions of the original problem and the approximate asymptotic expansions (see also papers [11] and [12]).

The case of the domain with totally oscillating boundary is considered in [33]. For such a domain the author constructed two terms asymptotics of the eigenvalues. Neumann boundary-value problems were considered in [29] and also in [31], [32].

The outline of the paper is as follows: in Section 1 we introduce the notations, set the problem, give preliminary propositions and statements
of the main results. In Section 2 we derive the formal asymptotics for the
eigenelements, in Section 3 we give a rigorous justification of the asymptotics,
and in Appendix we prove two auxiliary Propositions.

1 Setting of the problem, preliminary propositions and statements of the main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$, located in the upper half space. We
assume the boundary $\partial \Omega$ to be piecewise smooth, consisting of the parts:
$\partial \Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where $\Gamma_0$ is the segment $(-\frac{1}{2}, \frac{1}{2})$ on the abscissa axis,
$\Gamma_2$ and $\Gamma_3$ belong to the straight lines $x_1 = -\frac{1}{2}$ and $x_1 = \frac{1}{2}$, respectively. Let
$\varepsilon = \frac{1}{2N+1}$ be a small parameter, where $N$ is a large positive number. Given a
smooth negative 1-periodic in $\xi_1$ even function $F(\xi_1)$, such that $F'(\xi_1) = 0$
for $\xi_1 = \pm \frac{1}{2}$ and $\xi_1 = 0$, we set
$$\Pi_\varepsilon = \{x \in \mathbb{R}^2 : (x_1, 0) \in \Gamma_0, \varepsilon F\left(\frac{x_1}{\varepsilon}\right) < x_2 \leq 0\}$$
and then we denote
$$\Omega^\varepsilon = \Omega \cup \Pi_\varepsilon.$$
Thus, the boundary of $\Omega^\varepsilon$ consists of four parts: $\partial \Omega^\varepsilon = \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_2, \varepsilon \cup \Gamma_3, \varepsilon$,
where
$$\Gamma_\varepsilon = \{x \in \mathbb{R}^2 : (x_1, 0) \in \Gamma_0, x_2 = \varepsilon F\left(\frac{x_1}{\varepsilon}\right)\},$$
$$\Gamma_{2, \varepsilon} = \Gamma_2 \cup \{x \in \mathbb{R}^2 : x_1 = -\frac{1}{2}, \varepsilon F\left(-\frac{1}{2\varepsilon}\right) \leq x_2 \leq 0\},$$
$$\Gamma_{3, \varepsilon} = \Gamma_3 \cup \{x \in \mathbb{R}^2 : x_1 = \frac{1}{2}, \varepsilon F\left(\frac{1}{2\varepsilon}\right) \leq x_2 \leq 0\}.$$Denote by $\Gamma = \{\xi \in \mathbb{R}^2 : -\frac{1}{2} < \xi_1 < \frac{1}{2}, \xi_2 = F(\xi_1)\}$ in $\xi = \xi$ variables, and
let $\Pi = \{\xi \in \mathbb{R}^2 : -\frac{1}{2} < \xi_1 < \frac{1}{2}, \xi_2 > F(\xi_1)\}$ be a semi-infinite strip.

Denote by $\nu$ the outward unit normal vector. The following statement is
proved in [3].

**Lemma 1.1.** Assume that the multiplicity of the eigenvalue $\lambda_0$ of Problem

$$\begin{cases}
-\Delta u_0 = \lambda_0 u_0 & \text{in } \Omega, \\
u_0 = 0 & \text{on } \Gamma_0, \\
\frac{\partial u_0}{\partial \nu} = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\end{cases}$$

(1)
is equal to \( p \). Then there are \( p \) eigenvalues of Problem

\[
\begin{aligned}
\left\{ 
-\Delta u_\varepsilon &= \lambda_\varepsilon u_\varepsilon & \text{in } \Omega_\varepsilon, \\
  u_\varepsilon &= 0 & \text{on } \Gamma_\varepsilon, \\
  \frac{\partial u_\varepsilon}{\partial \nu} &= 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. 
\end{aligned}
\]  

(2)

(with multiplicities taken into account) converging to \( \lambda_0 \), as \( \varepsilon \to 0 \).

In [3] we considered the case where \( \lambda_0 \) is simple: we constructed the leading terms of the asymptotic expansions for the eigenelements and verified the asymptotics. Here we deal with the case where \( \lambda_0 \) is multiple.

Here and throughout we assume, without loss of generality, that the multiplicity of \( \lambda_0 \) equals two. Let then \( u_0^{(l)} \) \( (l = 1, 2) \) be the basis of the eigensubspace corresponding to \( \lambda_0 \), formed by eigenfunctions of Problem (1), orthonormalized in \( L^2(\Omega) \):

\[
\begin{aligned}
\left\{ 
-\Delta u_0^{(l)} &= \lambda_0 u_0^{(l)} & \text{in } \Omega, \\
u_0^{(l)} &= 0 & \text{on } \Gamma, \\
\frac{\partial u_0^{(l)}}{\partial \nu} &= 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\
\int_{\Omega} (u_0^{(l)})^2 dx &= 1, & \int_{\Omega} u_0^{(1)} u_0^{(2)} dx &= 0, & l = 1, 2.
\end{aligned}
\]  

(3)

It is easy to see that the eigenvalues can be chosen to satisfy an additional orthogonality condition on \( \Gamma_0 \)

\[
\int_{\Gamma_0} \frac{\partial u_0^{(1)}}{\partial \nu} \frac{\partial u_0^{(2)}}{\partial \nu} ds = 0.
\]  

(4)

Note that the similar orthogonality condition on the boundary of the type (4) was used in [8] and [12]. In addition for simplicity we assume that

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial u_0^{(1)}}{\partial x_2} \right)^2 dx_1 \neq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial u_0^{(2)}}{\partial x_2} \right)^2 dx_1.
\]  

(5)

Due to Lemma 1.1, there are two eigenvalues of Problem (2), denoted \( \lambda_\varepsilon^{(1)} \) and \( \lambda_\varepsilon^{(2)} \), converging to \( \lambda_0 \), as \( \varepsilon \to 0 \). Throughout we denote by \( u_\varepsilon^{(l)} \) \( (l = 1, 2) \) the corresponding eigenfunctions, orthonormalized in \( L^2(\Omega_\varepsilon) \). We then have

\[
\begin{aligned}
\left\{ 
-\Delta u_\varepsilon^{(l)} &= \lambda_\varepsilon^{(l)} u_\varepsilon^{(l)} & \text{in } \Omega_\varepsilon, \\
u_\varepsilon^{(l)} &= 0 & \text{on } \Gamma_\varepsilon, \\
\frac{\partial u_\varepsilon^{(l)}}{\partial \nu} &= 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\end{aligned}
\]  

(6)
\[
\int_{\Omega_{\varepsilon}} (u_{\varepsilon}^{(l)})^2 \, dx = 1, \quad \int_{\Omega_{\varepsilon}} u_{\varepsilon}^{(1)} u_{\varepsilon}^{(2)} \, dx = 0, \quad l = 1, 2. \tag{7}
\]

Our aim is the construction of accurate asymptotic approximations, as \(\varepsilon \to 0\), for the eigenvalues \(\lambda_{\varepsilon}^{(1)}\) and \(\lambda_{\varepsilon}^{(2)}\) and for the corresponding eigenfunctions.

It is proved in [3] that Problem
\[
\begin{cases}
\Delta_{\varepsilon} X = 0 \text{ in } \Pi, \\
X = 0 \text{ on } \Gamma, \\
\frac{\partial X}{\partial \xi_1} = 0 \text{ as } \xi_1 = \pm \frac{1}{2}, \\
X \sim \xi_2 \text{ as } \xi_2 \to +\infty.
\end{cases} \tag{8}
\]
has a solution with the asymptotics
\[
X(\xi) = \xi_2 + C(F) \quad \text{as } \xi_2 \to +\infty, \tag{9}
\]
up to exponentially small terms, where \(C(F)\) is a positive constant depending on the function \(F\). Note that, due to the evenness of the function \(F\), the solution \(X\) is even in \(\xi_1\) and can be extended to a 1-periodic function in \(\xi_1\). Later on we use the same notation \(X\) for the extension.

Our main goal is to prove the following statement.

**Theorem 1.1.** Assume that the multiplicity of \(\lambda_0\) of Problem (1) equals two, the associated eigenfunctions \(u_0^{(l)}\) \((l = 1, 2)\) satisfy the conditions (3)–(5). Then eigenvalues \(\lambda_{\varepsilon}^{(l)}\) of Problem (2), converging to \(\lambda_0\) as \(\varepsilon \to 0\), and the associated eigenfunctions \(u_{\varepsilon}^{(l)}\) orthonormalized in \(L_2(\Omega_{\varepsilon})\) have the following asymptotics:

\[
\lambda_{\varepsilon}^{(l)} = \lambda_0 + \varepsilon \lambda_1^{(l)} + o\left(\varepsilon^{\frac{1}{2} - \sigma}\right) \quad \text{for any } \sigma > 0, \tag{10}
\]

\[
\lambda_1^{(l)} = -C(F) \int_{\Gamma_0} \left(\frac{\partial u_0^{(l)}}{\partial \nu}\right)^2 \, ds, \tag{11}
\]

\[
\|u_{\varepsilon}^{(l)} - u_0^{(l)}\|_{H^1(\Omega)} + \|u_{\varepsilon}^{(l)}\|_{H^1(\Omega \setminus \Pi)} = o(1). \tag{12}
\]

**Remark 1.1.** In the next section we construct four terms asymptotics of the eigenelements of Problem (2). Moreover given algorithm allows to construct (see Remark 2.2) and to justify (see Remark 3.1) the complete asymptotic expansions of the eigenvalues and eigenfunctions.
2 Formal construction of the asymptotics.

In this section we formally construct the asymptotics of the eigenvalues \( \lambda_{\varepsilon}^{(l)} \) \( (l = 1, 2) \) converging to \( \lambda_0 \) as \( \varepsilon \to 0 \), and the asymptotics for corresponding eigenfunctions \( u_{\varepsilon}^{(l)} \). We use the method of matching of asymptotic expansions (see [22]–[24], [18]–[20] and also [3]). We construct the asymptotics outside a small neighborhood of \( \Gamma_0 \) (external expansion) in the form:

\[
  u_{\varepsilon}^{(l)}(x) = u_0^{(l)}(x) + \varepsilon u_1^{(l)}(x) + \varepsilon^2 u_2^{(l)}(x) + \varepsilon^3 u_3^{(l)}(x) + \sum_{i=4}^{\infty} \varepsilon^i u_i^{(l)}(x), \tag{13}
\]

the series for the eigenvalues as follows:

\[
  \lambda_{\varepsilon}^{(l)} = \lambda_0 + \varepsilon \lambda_1^{(l)} + \varepsilon^2 \lambda_2^{(l)} + \varepsilon^3 \lambda_3^{(l)} + \sum_{i=4}^{\infty} \varepsilon^i \lambda_i^{(l)} \tag{14}
\]

and the expansion in a small neighborhood of \( \Gamma_0 \) (inner expansion) in the form:

\[
  u_{\varepsilon}^{(l)}(x) = \varepsilon v_1^{(l)}(\xi; x_1) + \varepsilon^2 v_2^{(l)}(\xi; x_1) + \varepsilon^3 v_3^{(l)}(\xi; x_1) + \sum_{i=4}^{\infty} \varepsilon^i v_i^{(l)}(\xi; x_1), \tag{15}
\]

where \( \xi = \frac{x}{\varepsilon} \). Substituting (13) and (14) in (6) we deduce that the coefficients of (13) are to satisfy the following equations and boundary conditions

\[
\begin{align*}
  &\begin{cases}
    -\Delta u_1^{(l)} = \lambda_0 u_1^{(l)} + \lambda_1^{(l)} u_0^{(l)} & \text{in } \Omega, \\
    \frac{\partial u_1^{(l)}}{\partial n} = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,
  \end{cases} \tag{16} \\
  &\begin{cases}
    -\Delta u_2^{(l)} = \lambda_0 u_2^{(l)} + \lambda_1^{(l)} u_1^{(l)} + \lambda_2^{(l)} u_0^{(l)} & \text{in } \Omega, \\
    \frac{\partial u_2^{(l)}}{\partial n} = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,
  \end{cases} \tag{17} \\
  &\begin{cases}
    -\Delta u_3^{(l)} = \lambda_0 u_3^{(l)} + \lambda_1^{(l)} u_2^{(l)} + \lambda_2^{(l)} u_1^{(l)} + \lambda_3^{(l)} u_0^{(l)} & \text{in } \Omega, \\
    \frac{\partial u_3^{(l)}}{\partial n} = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
  \end{cases} \tag{18}
\end{align*}
\]

To complete the problems we add boundary conditions on \( \Gamma_0 \):

\[
  u_i^{(l)} = \alpha_i^{(l)} \quad \text{on } \Gamma_0, \quad i = 1, 2, \ldots, \tag{19}
\]

where \( \alpha_i^{(l)}(x_1) \) are unknown functions, satisfying the conditions:

\[
  \left. \frac{d^{2k+1} \alpha_i^{(l)}}{dx_1^{2k+1}} \right|_{x_1=\pm \frac{1}{2}} = 0, \quad k = 0, 1, \ldots \tag{20}
\]
We shall find these functions later. The condition (20) is necessary for solvability of recurrent system of boundary value problems (16)–(19) in $C^\infty(\Omega)$. Moreover such solutions do exist if these problems are solvable in $H^1(\Omega)$ and in addition due to boundary value problems the following formulae

$$
\left. \frac{d^{2k+1} \alpha_i^{(l)}}{dx_1^{2k+1}} \right|_{x_1=\pm \frac{1}{2}} = 0, \quad k = 0, 1, \ldots \quad (21)
$$

are true, where

$$
\alpha_{ij}^{(l)}(x_1) = \frac{1}{j!} \left. \frac{\partial^j u_i^{(l)}}{\partial x_j^2} \right|_{x_2=0}. \quad (22)
$$

Also it should be noted that due to Problem (1) the following formula holds.

Note that, if $F \in H^1(\Omega)$ and $\alpha \in H^{1/2}(\Gamma_0)$, then for solvability in $H^1(\Omega)$ of the boundary value problem

$$
\begin{align*}
\begin{cases}
-\Delta u &= \lambda_0 u + F \quad \text{in } \Omega, \\
u &= \alpha \quad \text{on } \Gamma_0, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\end{cases}
\end{align*}
$$

it is necessary and sufficient to have two identities:

$$
\int_{\Omega} F u_0^{(l)} \, dx = \int_{\Gamma_0} \alpha \frac{\partial u_0^{(l)}}{\partial \nu} \, ds, \quad l = 1, 2. \quad (25)
$$

In analogous way we obtain boundary value problems for $v_i^{(l)}$.

**Remark 2.1.** Further we construct the coefficients of the internal expansion (15) in the form of 1-periodic functions in $\xi_1$.

In $(\xi, x_1)$ variables the Laplacian and the normal derivative operator have the form

$$
\begin{align*}
\Delta &= \varepsilon^{-2} \Delta_\xi + 2\varepsilon^{-1} \frac{\partial^2}{\partial x_1 \partial \xi_1} + \frac{\partial^2}{\partial x_1^2}, \\
\frac{\partial}{\partial \nu} &= \varepsilon^{-1} \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial x_1} \quad \text{on } \Gamma_3, \quad \frac{\partial}{\partial \nu} = -\varepsilon^{-1} \frac{\partial}{\partial \xi_1} - \frac{\partial}{\partial x_1} \quad \text{on } \Gamma_2.
\end{align*}
$$

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Substituting (14), (15) and keeping in mind (26), (27) and Remark 2.1, we get the following equations and boundary conditions for $v_i^{(l)}$:

\[
\begin{cases}
\Delta_{\xi} v_1^{(l)} = 0 \quad \text{in } \Pi, \\
v_1^{(l)} = 0 \quad \text{on } \Gamma, \\
\frac{\partial v_1^{(l)}}{\partial \xi_1} = 0 \quad \text{as } \xi_1 = \pm \frac{1}{2}, \ x_1 = \pm \frac{1}{2}.
\end{cases}
\] (28)

\[
\begin{cases}
-\Delta_{\xi} v_2^{(l)} = 2 \frac{\partial^2 v_1^{(l)}}{\partial x_1 \partial \xi_1} \quad \text{in } \Pi, \\
v_2^{(l)} = 0 \quad \text{on } \Gamma, \\
\frac{\partial v_2^{(l)}}{\partial \xi_1} = -\frac{\partial v_1^{(l)}}{\partial x_1} \quad \text{as } \xi_1 = \pm \frac{1}{2}, \ x_1 = \pm \frac{1}{2}.
\end{cases}
\] (29)

\[
\begin{cases}
-\Delta_{\xi} v_3^{(l)} = 2 \frac{\partial^2 v_1^{(l)}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_3^{(l)}}{\partial x_2^2} + \lambda_0 v_1^{(l)} \quad \text{in } \Pi, \\
v_3^{(l)} = 0 \quad \text{on } \Gamma, \\
\frac{\partial v_3^{(l)}}{\partial \xi_1} = -\frac{\partial v_2^{(l)}}{\partial x_1} \quad \text{as } \xi_1 = \pm \frac{1}{2}, \ x_1 = \pm \frac{1}{2}.
\end{cases}
\] (30)

To complete the problems we need to add the conditions at infinity (as $\xi_2 \to +\infty$). These conditions we shall get from matching of asymptotic expansions. Rewriting the asymptotics of leading terms of (13) as $x_2 \to 0$ in the variables $\xi = \frac{x}{\varepsilon}$, bearing in mind (23), we deduce

\[
\sum_{i=0}^{3} \varepsilon^i u_i^{(l)}(x) = \sum_{i=1}^{3} \varepsilon^i v_i^{(l)}(\xi; x_1) + O\left(\varepsilon^4(\xi_2^4 + \xi_2)\right) \quad \text{as } x_2 = \varepsilon \xi_2 \to 0, \quad (31)
\]

where

\[
V_1^{(l)} = \alpha_{01}^{(l)}(x_1)\xi_2 + \alpha_{10}^{(l)}(x_1), \quad (32)
\]

\[
V_2^{(l)} = \alpha_{11}^{(l)}(x_1)\xi_2 + \alpha_{20}^{(l)}(x_1), \quad (33)
\]

\[
V_3^{(l)} = \alpha_{03}^{(l)}(x_1)\xi_2^3 + \alpha_{12}^{(l)}(x_1)\xi_2^2 + \alpha_{21}^{(l)}(x_1)\xi_2 + \alpha_{30}^{(l)}(x_1). \quad (34)
\]

We must find such $\lambda_i^{(l)}$ and $\alpha_{0i}^{(l)}(x_1)$ that:

- Problems (16) – (19) are to be soluble,
- Problems (28) – (30) are to be soluble with solutions having the asymptotics

\[
v_i^{(l)} \sim V_i^{(l)} \quad \text{as } \xi_2 \to +\infty \quad (35)
\]

up to exponentially small terms.

Let us define $\alpha_{10}^{(l)}(x_1)$ and $v_1^{(l)}(\xi; x_1)$. Obviously the function

\[
v_i^{(l)}(\xi; x_1) = \alpha_{0i}^{(l)}(x_1)X(\xi) \quad (36)
\]
due to (33) is the 1-periodic solution of Problem (28) and due to (33) has the asymptotics
\[ v^{(l)}(\xi; x_1) = \alpha_{01}^{(l)}(x_1)(\xi_2 + C(F)) \quad \text{as } \xi_2 \to +\infty, \]  
(37)
up to exponentially small terms. Thus, letting
\[ \alpha_{10}^{(l)}(x_1) = C(F)\alpha_{01}^{(l)}(x_1), \]
(38)
we obtain that \( v^{(l)}_1 \) defined by (36) satisfies (35), (32). Finally, we constructed \( \alpha_{10}^{(l)} \) and \( v^{(l)}_1 \).

Note that due to (21), (36)
\[ \frac{\partial v^{(l)}_1}{\partial x_1}(\xi; x_1) = 0 \quad \text{as } x_1 = \pm \frac{1}{2}, \]
(39)
Hence Problem (29) has the form
\[
\begin{cases}
-\Delta_\xi v^{(l)}_2 = 2\frac{\partial^2 v^{(l)}_1}{\partial x_1 \partial \xi_1} & \text{in } \Pi, \\
v^{(l)}_2 = 0 & \text{on } \Gamma, \\
\frac{\partial v^{(l)}_2}{\partial \xi_1} = 0 & \text{as } \xi_1 = \pm \frac{1}{2}, \ x_1 = \pm \frac{1}{2},
\end{cases}
\]
(40)
and by (27), (39) and the boundary condition from (33), we have
\[ \frac{\partial v^{(l)}_1}{\partial \nu}(\frac{x}{\xi}; x_1) = 0 \quad \text{on } \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}. \]
(41)

Let us define \( \lambda_1^{(l)} \) and \( u_1^{(l)}(x) \). The constant \( \lambda_1^{(l)} \) can be defined from the solvability condition of Problem (16), (19), which has the same form as Problem (21). From (23), (38), (3) and (4) we deduce that the sufficient solvability condition of Problem (16), (19) is
\[ \lambda_1^{(l)} = -C(F) \int_{-\frac{1}{2}}^\frac{1}{2} (\alpha_{01}^{(l)})^2(x_1) \, dx_1 \]
or (11) (taking in account (22)). We choose \( u_1^{(l)} \) in the form:
\[ u_1^{(l)} = \tilde{u}_1^{(l)} + \kappa_1^{(l)} u_0^{(r)}, \]
(42)
where
\[ \int_{\Omega} \tilde{u}_1^{(l)}(x)u_0^{(k)}(x) \, dx = 0, \quad l, k = 1, 2 \]
and the constants \( \kappa_1^{(l)} \) are arbitrary. We shall define these constants from the solvability conditions for \( u_2^{(l)} \). Here and throughout \( l^* = 1 \) if \( l = 2 \) and \( l^* = 2 \) if \( l = 1 \).

Thus,

\[
\alpha_{11}^{(l)} = \tilde{\alpha}_{11}^{(l)} + \kappa_1^{(l)} \alpha_{01}^{(l^*)},
\]

where

\[
\tilde{\alpha}_{11}^{(l)} = \left. \frac{\partial \tilde{u}_1^{(l)}}{\partial x_2} \right|_{x_2=0}, \quad \frac{d^{2k+1} \tilde{\alpha}_{11}^{(l)}}{dx_1^{2k+1}} \bigg|_{x_1=\pm \frac{1}{2}} = 0, \quad k = 0, 1, \ldots
\]

Let us define \( \alpha_{20}^{(l)}(x_1) \) and \( v_2(\xi; x_2) \). Consider an auxiliary problem:

\[
\begin{cases}
\Delta_\xi \tilde{X} = \frac{\partial X}{\partial \xi_1} & \text{in } \Pi, \\
\tilde{X} = 0 & \text{on } \partial \Pi.
\end{cases}
\]

It is proved in [3] that Problem (43) has a solution with the asymptotics

\[
\tilde{X}(\xi) = 0 \quad \text{as } \xi_2 \to +\infty,
\]

up to exponentially small terms. Note that, due to the evenness of the functions \( F \), the solution \( \tilde{X} \) of Problem (43) is odd in \( \xi_1 \), and thus has a 1-periodic extension in \( \xi_1 \) for which we keep the same notation \( \tilde{X} \).

Then it is easy to see that, due to (3), (11), (21), (43) and (44), the function

\[
v_2^{(l)}(\xi; x_1) = \alpha_{11}^{(l)}(x_1)X(\xi) - 2(\alpha_{01}^{(l)})(x_1)\tilde{X}(\xi)
\]

is the 1-periodic solution to Problem (40), which has the asymptotics

\[
v_2^{(l)}(\xi; x_1) = \alpha_{11}^{(l)}(x_1)(\xi_2 + C(F)) \quad \text{as } \xi_2 \to +\infty
\]

up to exponentially small terms, and also which satisfies the conditions (35), (33) for

\[
\alpha_{20}^{(l)}(x_1) = C(F) \left( \tilde{\alpha}_{11}^{(l)} + \kappa_1^{(l)} \alpha_{01}^{(l^*)} \right).
\]

Thus we defined \( v_2^{(l)} \) and \( \alpha_{20}^{(l)} \) up to \( \kappa_1^{(l)} \), which is unknown yet.

It is easy to verify that, due to (21), (45) and the boundary condition from (43),

\[
\frac{\partial v_2^{(l)}}{\partial x_1}(\xi; x_1) = 0 \quad \text{as } \xi = \frac{\pi}{2} \text{ and } x_1 = \pm \frac{1}{2}.
\]
Hence Problem (30) takes the form

$$
\begin{cases}
-\Delta \xi v_3^{(l)} = 2 \frac{\partial^2 v_3^{(l)}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_3^{(l)}}{\partial x_1^2} + \lambda_0 v_1^{(l)} & \text{in } \Pi, \\
v_3^{(l)} = 0 & \text{on } \Gamma, \\
\frac{\partial v_3^{(l)}}{\partial \xi_1} = 0 & \text{as } \xi_1 = \pm \frac{1}{2}, \ x_1 = \pm \frac{1}{2}.
\end{cases}
$$

and by (27), (47) and the boundary condition from (40), we have

$$
\frac{\partial v_2^{(l)}}{\partial \nu}(\frac{x}{\varepsilon}; x_1) = 0 \text{ on } \Gamma_{2, \varepsilon} \cup \Gamma_{3, \varepsilon}.
$$

Let us define $\lambda_2^{(l)}$, $u_2^{(l)}$ and $\kappa_1^{(l)}$. From (25), (46), (3) and (4) we deduce that the sufficient solvability conditions of Problem (17), (19) are

$$
\lambda_2^{(l)} = -C(F) \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{\alpha}_{11}^{(l)}(x_1) e_{01}^{(l)}(x_1) \, dx_1
$$

and

$$
\kappa_1^{(l)} = \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{\alpha}_{11}^{(l)}(x_1) \alpha^{(l^*)}_{01}(x_1) \, dx_1}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \left( \alpha_{01}^{(l)} \right)^2 - \alpha_{01}^{(l^*)} \right)^2 \, dx_1}.
$$

Thus, we defined constants $\lambda_2^{(l)}$, $\kappa_1^{(l)}$ and in particular because of (12) the function $u_1^{(l)}(x)$. We choose $u_1^{(l)}$ in the form:

$$
u_2^{(l)}(x) = \tilde{u}_2^{(l)} + \kappa_2^{(l)} u_0^{(l^*)},
$$

where

$$
\int_{\Omega} \tilde{u}_2^{(l)}(x) u_0^{(k)}(x) \, dx = 0, \quad l, k = 1, 2
$$

and the constants $\kappa_2^{(l)}$ are arbitrary. We shall define these constants from the solvability conditions for $u_3^{(l)}$.

Thus,

$$
\alpha_{21}^{(l)} = \tilde{\alpha}_{21}^{(l)} + \kappa_2^{(l)} \alpha_{01}^{(l^*)},
$$

12
where
\[ \tilde{\alpha}_2^{(l)} = \frac{\partial \tilde{u}_2^{(l)}}{\partial x_2} \bigg|_{x_2=0}, \quad \frac{d^{2k+1} \tilde{\alpha}_2^{(l)}}{dx_1^{2k+1}} \bigg|_{x_1=\pm \frac{1}{2}} = 0, \quad k = 0, 1, \ldots. \]

Let us define \( \alpha_{30}^{(l)}(x_1) \) and \( v_3^l(\xi; x_2) \). Consider auxiliary problems:

\[
\begin{cases}
\Delta \xi \tilde{X}_{(I)} = \frac{\partial \xi}{\partial \xi_1} \quad \text{in } \Pi, \\
\tilde{X}_{(I)} = 0 \quad \text{on } \Gamma, \quad \frac{\partial \tilde{X}_{(I)}}{\partial \xi_1} = 0 \quad \text{as } \xi_1 = \pm \frac{1}{2}.
\end{cases} \tag{51}
\]

\[
\begin{cases}
\Delta \xi \tilde{X}_{(II)} = X \quad \text{in } \Pi, \\
\tilde{X}_{(II)} = 0 \quad \text{on } \Gamma, \quad \frac{\partial \tilde{X}_{(II)}}{\partial \xi_1} = 0 \quad \text{as } \xi_1 = \pm \frac{1}{2}.
\end{cases} \tag{52}
\]

In Section 4 we shall prove the following two statements.

**Proposition 2.1.** Problem (51) has a solution with the asymptotics
\[ \tilde{X}_{(I)}(\xi) = C(I)(F) \quad \text{as } \xi_2 \to +\infty, \] up to exponentially small terms, where \( C(I)(F) \) is a constant depending on the function \( F \).

**Proposition 2.2.** Problem (52) has a solution with the asymptotics
\[ \tilde{X}_{(II)}(\xi) = \frac{1}{6} \xi_2^3 + \frac{1}{2} C(F) \xi_2^2 + C(II)(F) \quad \text{as } \xi_2 \to +\infty, \] up to exponentially small terms, where \( C(II)(F) \) is a constant depending on the function \( F \).

Note that, due to the evenness of the function \( F \) and the evenness of the right-hand sides of the equations in (51) and (52) the solutions \( \tilde{X}_{(I)} \) and \( \tilde{X}_{(II)} \) of Problems (51) and (52) respectively are even in \( \xi_1 \), and thus have 1-periodic extensions in \( \xi_1 \) for which we keep the same notation \( \tilde{X}_{(I)} \) and \( \tilde{X}_{(II)} \).

Then it is easy to see that, due to (53), (54), (51), (52), (53), (54), (51)–(54) the function
\[ v_3^l(\xi; x_1) = \alpha_{21}^{(l)}(x_1)X(\xi) - 2(\alpha_{11}^{(l)})'(x_1)\tilde{X}(\xi) + 4(\alpha_{01}^{(l)})''(x_1)\tilde{X}_{(I)}(\xi) - (\alpha_{01}^{(l)})''(x_1)\tilde{X}_{(II)}(\xi) - \lambda_0 \alpha_{01}^{(l)}(x_1)\tilde{X}(\xi), \]
is the 1-periodic solution to Problem (48), which has the asymptotics

\[ v_3^{(l)}(\xi; x_1) = \alpha_{21}^{(l)}(x_1) (\xi_2 + C(F)) + 4C(l)(F)(\alpha_{01}^{(l)})''(x_1) - \left( (\alpha_{01}^{(l)})''(x_1) + \lambda_0 \alpha_{01}^{(l)}(x_1) \right) \left( \frac{1}{6} \xi_2^3 + \frac{1}{2} C(F) \xi_2^2 + C(l)(F) \right) \]  

(56)

as \( \xi_2 \to +\infty \),

up to exponentially small terms. Note that due to equations for \( u_k^{(l)} \) from (16), (17) we have:

\[ -\frac{1}{6} \left( (\alpha_{01}^{(l)})''(x_1) + \lambda_0 \alpha_{01}^{(l)}(x_1) \right) = \alpha_{03}^{(l)}(x_1) \]

and

\[ -\frac{1}{2} C(F) \left( (\alpha_{01}^{(l)})''(x_1) + \lambda_0 \alpha_{01}^{(l)}(x_1) \right) = \alpha_{12}^{(l)}(x_1). \]

Hence from (56) we conclude that \( v_3 \) satisfies the condition (55), (54) for

\[ \alpha_{30}^{(l)}(x_1) = C(F) \left( \alpha_{21}^{(l)}(x_1) + \kappa_2^{(l)} \alpha_{12}^{(l)}(x_1) \right) + 4C(l)(F)(\alpha_{01}^{(l)})''(x_1) - \]

\[ - C(l)(F) \left( (\alpha_{01}^{(l)})''(x_1) + \lambda_0 \alpha_{01}^{(l)}(x_1) \right). \]  

(57)

Thus we defined \( v_3^{(l)} \) and \( \alpha_{30}^{(l)} \) up to \( \kappa_2^{(l)} \), which is unknown yet.

It is easy to verify that, due to (21), (51) and the boundary condition from (51) and (52),

\[ \frac{\partial v_3^{(l)}}{\partial x_1}(\xi; x_1) = 0 \quad \text{as} \quad \xi = \frac{x}{\varepsilon} \text{ and } x_1 = \pm \frac{1}{2}, \]  

(58)

and hence, by (27), (58) and the boundary condition from (51), we have

\[ \frac{\partial v_3^{(l)}}{\partial \nu}(\xi; x_1) = 0 \quad \text{on} \quad \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}. \]  

(59)

Let us define \( \lambda_3^{(l)} \), \( u_3^{(l)} \) and \( \kappa_2^{(l)} \). From (25), (57), (3) and (4) we deduce that the sufficient solvability conditions of Problem (18), (19) are

\[ \lambda_3^{(l)} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_{30}^{(l)}(x_1) \alpha_{01}^{(l)}(x_1) \, dx_1 = -C(F) \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_{21}^{(l)}(x_1) \alpha_{01}^{(l)}(x_1) \, dx_1 + \]

\[ \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \alpha_{01}^{(l)}(x_1) \, dx_1. \]
\[
+ \left( C_{II}(F) - 4C_{I}(F) \right) \int_{-1}^{1} \left( \alpha_{01}^{(l)}(x_1) \alpha_{01}^{(l)}(x_1) dx_1 \right) dx_1 + \\
+ \lambda_0 C_{II}(F) \int_{-1}^{1} \left( \alpha_{01}^{(l)} \right)^2 (x_1) dx_1
\]

and

\[
\lambda_1^{(l)} \kappa_2^{(l)} + \lambda_2^{(l)} \kappa_1^{(l)} = -C(F) \int_{-1}^{1} \left( \tilde{\alpha}_{21}^{(l)}(x_1) \alpha_{01}^{(l)}(x_1) + \kappa_2^{(l)} \left( \alpha_{01}^{(l)} \right)^2 (x_1) \right) dx_1 + \\
+ \left( C_{II}(F) - 4C_{I}(F) \right) \int_{-1}^{1} \left( \alpha_{01}^{(l)} \right)^2 (x_1) dx_1.
\]

From the last formula we deduce the expression for \( \kappa_2^{(l)} \). Thus, we defined constants \( \lambda_3^{(l)} \), \( \kappa_2^{(l)} \) and in particular because of (50) the function \( u_2^{(l)}(x) \). We fix the arbitrariness in choosing of \( u_3^{(l)} \) by the following

\[
u_3^{(l)} = \tilde{\nu}_3^{(l)} + \kappa_3^{(l)} u_0^{(l)}
\]

where

\[
\int_{\Omega} \tilde{\nu}_3^{(l)}(x) u_0^{(k)}(x) dx = 0, \quad l, k = 1, 2.
\]

Here constants \( \kappa_3^{(l)} \) are arbitrary and we shall define them in a unique way from the solvability conditions for \( u_4^{(l)} \).

**Remark 2.2.** In the same way we can construct the complete asymptotic expansion of eigenelements of Problem (6) in the form (13)–(15). Substituting (13) and (14) in (6), we deduce the boundary value problems for coefficients of (13):

\[
\begin{align*}
-\Delta u_i^{(l)} &= \lambda_0 u_i^{(l)} + \sum_{k=1}^{i-1} \lambda_k^{(l)} u_{i-k}^{(l)} + \lambda_i^{(l)} u_0^{(l)} & \text{in } \Omega, \\
\frac{\partial u_i^{(l)}}{\partial n} &= 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\
u_i^{(l)} &= \alpha_{00}^{(l)} & \text{on } \Gamma_0.
\end{align*}
\]
We construct the solution of Problem (60) in the form
\[ u_i^{(l)} = \tilde{u}_i^{(l)} + \kappa_i^{(l)} u_i^{(r)}, \]  \hspace{1cm} (61)
where
\[ \int_{\Omega} \tilde{u}_i^{(l)}(x)u_0^{(k)}(x) \, dx = 0, \quad l, k = 1, 2. \]  \hspace{1cm} (62)

Rewriting the asymptotics of (13) as \( x_2 \to 0 \) in the fast variables \( \xi = \frac{x_2}{\varepsilon} \), we obtain that
\[ \sum_{i=0}^{\infty} \varepsilon^i u_i^{(l)}(x) = \sum_{i=1}^{\infty} \varepsilon^i V_i^{(l)}(\xi; x_1), \quad \text{as} \quad x_2 = \varepsilon x_1 \to 0, \]  \hspace{1cm} (63)
where
\[ V_i^{(l)}(\xi; x_1) = \tilde{V}_i^{(l)}(\xi; x_1) + \left( \frac{\tilde{\alpha}^{(l)}_{i-1,1}(x_1) + \kappa_i^{(l)} u_i^{(r)}}{\alpha_{i-1,1}(x_1) + \kappa_i^{(l)} u_i^{(r)}} \right) \xi_2 + \alpha_{i0}^{(l)}(x_1) \]
and \( \tilde{V}_i^{(l)}(\xi; x_1) \) is independent of \( u_j \) for \( j \geq i - 1 \), \( \tilde{\alpha}_{i-1,1}(x_1) \) depends only on \( \tilde{u}_{i-1} \). Substituting (14), (15) in (6) keeping in mind (63), we get the equations and the boundary conditions for the terms \( v_i^{(l)} \) of (13):
\[
\begin{align*}
-\Delta \xi v_i^{(l)} &= 2\frac{\partial^2 v_i^{(l)}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_i^{(l)}}{\partial x_1^2} + \sum_{k=0}^{i-3} \lambda_k v_i^{(l)-k} \quad \text{in} \, \Pi, \\
v_i^{(l)} &= 0 \quad \text{on} \, \Gamma, \\
\partial_\nu v_i^{(l)} &= -\frac{\partial v_i^{(l)}}{\partial x_1} \quad \text{as} \, \xi_1 = \pm \frac{1}{2}, \ x_1 = \pm \frac{1}{2}, \\
v_i^{(l)} &\sim V_i^{(l)} \quad \text{as} \, \xi_2 \to +\infty.
\end{align*}
\]  \hspace{1cm} (64)

Before the \( i \)-th step we have defined \( \lambda_i^{(l)} , \ v_i^{(l)} , \ k \leq i - 1 \), \( u_j^{(l)} , \ j \leq i - 2 \), \( \tilde{u}_{i-1}^{(l)} \) (and consequently \( \tilde{V}_i^{(l)} \) and \( \tilde{\alpha}_{i-1,1}(x_1) \)).

On the \( i \)-th step solving Problem (64), we find the boundary condition \( \alpha_{i0}^{(l)} \) in the form
\[ \alpha_{i0}^{(l)} = \tilde{\alpha}_{i0}^{(l)} + \kappa_i^{(l)} C(F)\alpha_{i1}^{(r)}, \]  \hspace{1cm} (65)
where \( \tilde{\alpha}_{i0}^{(l)} \) depends only on \( \tilde{V}_i^{(l)} \) and \( \tilde{\alpha}_{i-1,1}(x_1) \). Then from the solvability condition for Problem (60) with \( \alpha_{i0}^{(l)} \) defined in (65), we derive \( \lambda_i^{(l)} \) and \( \kappa_i^{(l)} \) (hence exactly define \( u_i^{(l-1)} \)). We choose \( u_i^{(l)} \) in the form (61), (62), where the constants \( \kappa_i^{(l)} \) are arbitrary. We shall define these constants in the next step. Thus we completed the \( i \)-th step.

Acting in the same way, we construct complete formal asymptotic expansions \( 13 \)–\( 15 \) of eigenelements.

On the justification of these asymptotics see Remark 3.1.
3 Verification of the asymptotics.

Denote
\[ \tilde{\chi}_\varepsilon^{(l)} = \lambda_0 + \varepsilon \lambda_1^{(l)} + \varepsilon^2 \lambda_2^{(l)} + \varepsilon^3 \lambda_3^{(l)}, \]
\[ \tilde{w}_\varepsilon^{(l)}(x) = \left( u_0^{(l)}(x) + \sum_{i=1}^3 \varepsilon^i u_1^{(l)}(x) \right) \chi \left( \frac{x_2}{\varepsilon^\beta} \right) \]
\[ + \left( \sum_{i=1}^3 \varepsilon^i v_1^{(l)} \left( \frac{x}{\varepsilon}, x_1 \right) \right) \left( 1 - \chi \left( \frac{x_2}{\varepsilon^\beta} \right) \right), \]
where \( \chi(s) \) is a smooth cut-off function, equals to zero as \( s < 1 \) and equals to one as \( s > 2 \), and \( \beta \) is a fixed number \((0 < \beta < 1)\). Obviously, \( \tilde{w}_\varepsilon^{(l)} \in C^\infty(\bar{\Omega}) \)

**Lemma 3.1.** The function \( \tilde{w}_\varepsilon^{(l)} \) is the solution of Problem
\[
\left\{ \begin{array}{l}
-\Delta \tilde{w}_\varepsilon^{(l)} = \tilde{\chi}_\varepsilon \tilde{w}_\varepsilon^{(l)} + f_\varepsilon^{(l)} \text{ in } \Omega^c, \\
\tilde{w}_\varepsilon^{(l)} = 0 \text{ on } \Gamma_\varepsilon, \quad \frac{\partial \tilde{w}_\varepsilon^{(l)}}{\partial \nu} = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \varepsilon \cup \Gamma_3, \varepsilon,
\end{array} \right.
\]
where
\[ \| f_\varepsilon^{(l)} \|_{L^2(\Omega^c)} = O(\varepsilon^{5\beta/2}). \]

**Proof.** Due to boundary conditions from (1), (16), (17) and (18) for \( u_0^{(l)}, u_1^{(l)}, u_2^{(l)}, u_3^{(l)} \) and the boundary conditions (11), (19), (59) for \( v_1^{(l)}, v_2^{(l)}, v_3^{(l)} \), we get that the function \( \tilde{w}_\varepsilon^{(l)} \) satisfies the boundary conditions of Problem (68).

From the other hand, due to Formula (26), and equations (1), (16), (17), (18), (28), (30) and (40) for \( u_0^{(l)}, u_1^{(l)}, u_2^{(l)}, u_3^{(l)}, v_1^{(l)}, v_2^{(l)} \) and \( v_3^{(l)} \), the function \( \tilde{w}_\varepsilon^{(l)} \) satisfies the equation of Problem (68) where
\[ -f_\varepsilon^{(l)}(x) = f_\varepsilon^{(l)}(x; \varepsilon) + f_\varepsilon^{(l)}(x; \varepsilon) + f_\varepsilon^{(l)}(x; \varepsilon) + f_\varepsilon^{(l)}(x; \varepsilon), \]
\[ I_1^{(l)} = \varepsilon^2 \chi \left( \frac{x_2}{\varepsilon^\beta} \right) \left( \lambda_1^{(l)} u_3^{(l)} + \lambda_2^{(l)} v_3^{(l)} + \varepsilon \lambda_2^{(l)} u_3^{(l)} + \lambda_3^{(l)} u_1^{(l)} + \varepsilon \lambda_3^{(l)} u_1^{(l)} + \varepsilon^2 \lambda_3^{(l)} u_3^{(l)} \right), \]
\[ I_2^{(l)} = \varepsilon^2 \left( 1 - \chi \left( \frac{x_2}{\varepsilon^\beta} \right) \right) \left( \lambda_0 v_2^{(l)} + \varepsilon \lambda_0 v_2^{(l)} + \lambda_1 v_1^{(l)} + \varepsilon \lambda_1 v_1^{(l)} + \varepsilon^2 \lambda_2 v_3^{(l)} + \varepsilon^2 \lambda_3 v_3^{(l)} \right) \]
\[ + \varepsilon^2 \left( \varepsilon^2 \lambda_2 v_3^{(l)} + \varepsilon^2 \lambda_3 v_3^{(l)} + \varepsilon \lambda_3 v_3^{(l)} + \varepsilon^2 \lambda_3 v_3^{(l)} + \varepsilon^2 \lambda_3 v_3^{(l)} \right) \]
\[ + \varepsilon^2 \left( \frac{\partial^2 v_3^{(l)}}{\partial x_1^2} + \varepsilon \frac{\partial^2 v_3^{(l)}}{\partial x_1^2} + 2 \frac{\partial^2 v_3^{(l)}}{\partial x_1^2} \right), \]
\[ I_3^{(l)} = 2 \varepsilon^{1-\beta} \chi^2 \left( \frac{x_2}{\varepsilon^\beta} \right) \left( \sum_{i=1}^3 \varepsilon^i \frac{\partial u_1^{(l)}}{\partial x_2} - \sum_{i=1}^3 \varepsilon^{i-1} \frac{\partial v_1^{(l)}}{\partial x_2} \right), \]
\[ I_4^{(l)} = \varepsilon^{2-2\beta} \chi^2 \left( \frac{x_2}{\varepsilon^\beta} \right) \left( u_0^{(l)} + \varepsilon u_1^{(l)} + \varepsilon^2 u_2^{(l)} + \varepsilon^3 u_3^{(l)} - (\varepsilon v_1^{(l)} + \varepsilon^2 v_2^{(l)} + \varepsilon^3 v_3^{(l)}) \right). \]
Since the functions \( u_i^{(l)} \) are smooth, then it is obvious that
\[
\| I_1^{(l)} \|_{L_2(\Omega^\varepsilon)} = O(\varepsilon^4). \tag{70}
\]
Due to (36), (37) and (45), we obtain that
\[
\| I_2^{(l)} \|_{L_2(\Omega^\varepsilon)} = O\left( \varepsilon^{\frac{5\beta}{2}} \right). \tag{71}
\]
Bearing in mind the matching conditions (31), (35) and that the derivatives of \( \chi \left( \frac{x_2}{\varepsilon} \right) \) are not equal to zero only in the strip \( \varepsilon^{\beta} < x_2 < 2\varepsilon^{\beta} \), it is easy to see that
\[
\| I_3^{(l)} \|_{L_2(\Omega^\varepsilon)} + \| I_4^{(l)} \|_{L_2(\Omega^\varepsilon)} = O(\varepsilon^{\frac{5\beta}{2}}). \tag{72}
\]
From (70)–(72) it follows (69).

The following statement is proved in [3].

**Lemma 3.2.** Assume that the multiplicity of the eigenvalue \( \lambda_0 \) of Problem (1) is equal to \( p \). Then for any \( \lambda \) close to \( \lambda_0 \)

(i) the solution \( U_\varepsilon \) to Problem
\[
\begin{align*}
  -\Delta U_\varepsilon &= \lambda U_\varepsilon + F_\varepsilon \quad \text{in } \Omega^\varepsilon, \\
  U_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon, \quad \frac{\partial U_\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2^\varepsilon \cup \Gamma_3^\varepsilon
\end{align*}
\]  \tag{73}

admits the estimate
\[
\| U_\varepsilon \|_{H^1(\Omega^\varepsilon)} \leq C \frac{\| F_\varepsilon \|_{L_2(\Omega^\varepsilon)}}{\prod_{l=1}^p |\lambda_\varepsilon^{(l)} - \lambda|} \tag{74}
\]
where \( \lambda_\varepsilon^{(1)}, \ldots, \lambda_\varepsilon^{(p)} \) are the eigenvalues of Problem (2), which converge to \( \lambda_0 \);

(ii) if a solution \( U_\varepsilon \) to Problem (73) is orthogonal in \( L_2(\Omega^\varepsilon) \) to the eigenfunction \( u_\varepsilon^{(k)} \) of Problem (2) corresponding to \( \lambda_\varepsilon^{(k)} \), then it satisfies the estimate
\[
\| U_\varepsilon \|_{H^1(\Omega^\varepsilon)} \leq C \frac{\| F_\varepsilon \|_{L_2(\Omega^\varepsilon)}}{\prod_{l=1; l \neq k}^p |\lambda_\varepsilon^{(l)} - \lambda|}. \tag{75}
\]

For our case \( p = 2 \) from this lemma we deduce the statement.

**Corollary 1.** For any \( \lambda \) close to \( \lambda_0 \)
(i) The solution $U_\varepsilon$ to Problem (73) admits the estimate

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C \frac{\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}}{\left(\max_l |\lambda^{(l)}_\varepsilon - \lambda|\right)^2}, \quad (76)$$

where $\lambda^{(1)}_\varepsilon, \lambda^{(2)}_\varepsilon$ are the eigenvalues of Problem (2), which converge to $\lambda_0$.

(ii) If a solution $U_\varepsilon$ to Problem (73) is orthogonal in $L_2(\Omega^\varepsilon)$ to the eigenfunction $u^{(l)}_\varepsilon$ of Problem (2) corresponding to $\lambda^{(l)}_\varepsilon$, then it satisfies the estimate

$$\|U_\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C \frac{\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)}}{|\lambda^{(k)}_\varepsilon - \lambda|} \quad \text{for } k \neq l. \quad (77)$$

**Theorem 3.1.** Assume that the multiplicity of $\lambda_0$ of Problem (1) equals two, the associated eigenfunctions $u^{(l)}_0 (l = 1, 2)$ satisfy the conditions (3)–(5). Then eigenvalues $\lambda^{(l)}_\varepsilon$ of Problem (2), converging to $\lambda_0$ as $\varepsilon \to 0$, and the associated eigenfunctions $u^{(l)}_\varepsilon$ orthonormalized in $L_2(\Omega^\varepsilon)$ have the following asymptotics:

$$\lambda^{(l)}_\varepsilon = \tilde{\lambda}^{(l)}_\varepsilon + o(\varepsilon^{\frac{5\beta}{4}}) \quad \text{for any } \beta < 1, \quad (78)$$

$$\|u^{(l)}_\varepsilon - \tilde{u}^{(l)}_\varepsilon\|_{H^1(\Omega^\varepsilon)} = o(1). \quad (79)$$

**Proof.** Since $u_j^{(l)} \in C^\infty(\overline{\Omega})$, then due to (67) and (32)–(35) we derive

$$\|\tilde{u}^{(l)}_\varepsilon\|_{L_2(\Omega^\varepsilon)} = 1 + o(1) \quad \text{as } \varepsilon \to 0. \quad (80)$$

Applying item (i) of Corollary 1 for $\lambda = \tilde{\lambda}^{(l)}_\varepsilon$, $F_\varepsilon = f^{(l)}_\varepsilon$ and $U_\varepsilon = \tilde{u}^{(l)}_\varepsilon$, due to Lemma 3.1 we obtain (78).

Denote

$$\tilde{u}^{(l)}_\varepsilon = \tilde{u}^{(l)}_\varepsilon - \left(\tilde{u}^{(l)}_\varepsilon, u^{(l)}_\varepsilon\right)_{L_2(\Omega^\varepsilon)} u^{(l)}_\varepsilon. \quad (81)$$

By the definition

$$\left(\tilde{u}^{(l)}_\varepsilon, u^{(l)}_\varepsilon\right)_{L_2(\Omega^\varepsilon)} = 0, \quad (82)$$

and the function $U_\varepsilon = \tilde{u}^{(l)}_\varepsilon$ is a solution of Problem (73) for $\lambda = \tilde{\lambda}^{(l)}_\varepsilon$ and

$$F_\varepsilon = f^{(l)}_\varepsilon + \left(\tilde{\lambda}^{(l)}_\varepsilon - \lambda^{(l)}_\varepsilon\right)\tilde{u}^{(l)}_\varepsilon. \quad (83)$$

From (83), (78), (80) and (69) it follows that

$$\|F_\varepsilon\|_{L_2(\Omega^\varepsilon)} = O(\varepsilon^{\frac{5\beta}{4}}). \quad (84)$$
Since due to (10) and (5) we have \(|\lambda^{(1)}_{\varepsilon} - \lambda^{(2)}_{\varepsilon}| > c \varepsilon\), where \(c > 0\), then by (84) and item (ii) of Corollary 1 it follows that
\[
\|\hat{u}^{(l)}_{\varepsilon}\|_{H^1(\Omega)} = o(1).
\] (85)
Due to (81), (85) and (80) we deduce (79).

Proof of Theorem 1.1. Since \(u^j_l \in C^\infty(\Omega)\), then due to (67) and (32)–(35) we derive
\[
\|\hat{u}^{(l)}_{\varepsilon} - u^j_l\|_{H^1(\Omega)} + \|\hat{u}^{(l)}_{\varepsilon}\|_{H^1(\Omega^\varepsilon, \mathbb{R})} = o(1).
\] (86)
Then Theorem 1.1 follows from Theorem 3.1.

Remark 3.1. In an analogous way we can justify the complete asymptotic expansion of the eigenelements of Problem (6) constructed in Remark 2.2.

Denoting
\[
\hat{\lambda}^{(l)}_{\varepsilon} = \lambda_0 + \sum_{i=1}^n \varepsilon^i \lambda^i_{l},
\]
\[
\hat{u}^{(l)}_{\varepsilon}(x) = \left( u^j_0(x) + \sum_{i=1}^n \varepsilon^i u^i_0(x) \chi \left( \frac{x_2}{\varepsilon^2} \right) \right)
+ \left( \sum_{i=1}^n \varepsilon^i v^i_l \left( \frac{x}{\varepsilon}; x_1 \right) \right) \left( 1 - \chi \left( \frac{x_2}{\varepsilon^2} \right) \right)
\] (87)
and repeating the proof of Lemma 3.1, we conclude that Lemma 3.1 holds true for
\[
\|f^{(l)}_{\varepsilon}\|_{L^2(\Omega^\varepsilon)} = O(\varepsilon^N) \quad \text{where } N \to \infty.
\] (88)
Obviously that (86) also holds true for \(\hat{u}^{(l)}_{\varepsilon}(x)\) defined in (87). Then taking into account (88) and (86) and repeating the proof of Theorem 3.1 we obtain that
\[
\lambda^{(l)}_{\varepsilon} = \hat{\lambda}^{(l)}_{\varepsilon} + O(\varepsilon^{N}) \quad \text{where } N \to \infty.
\]

4 Appendix

The proofs of Propositions 2.1 and 2.2 are similar to that of L.Tartar 40 (Lemma V.9) for a problem in a semi-infinite strip with a flat bottom (see also 28).
Lemma 4.1. Let $E$ and $E_0$ be Hilbert spaces, let $a$ be a continuous bilinear form from $E \times E_0$, and let $M$ be a continuous linear mapping from $E$ onto $E_0$. Assume that there exists $\gamma > 0$ such that

$$a(u, Mu) \geq \gamma \|u\|^2$$

for every $u \in E$, where $\| \cdot \|$ denotes the norm in $E$. Then, for every continuous linear form $L$ into $E_0$, there exists a unique $u \in E$ satisfying

$$a(u, v) = L(v)$$

for every $v \in E_0$.

Proof of Proposition 2.1. In view of [3] (proof of Proposition 1) there exists a positive constant $\varsigma > 0$ such that,

$$\left| \partial^\alpha \left( \frac{\partial \tilde{X}}{\partial \xi_1} \right) \right| \leq C_{\delta, \alpha} e^{-\varsigma \xi_2} \quad (89)$$

for any $(\xi_1, \xi_2) \in \Pi$ with $\xi_2 \geq \delta$, where $C_{\delta, \alpha}$ is a constant depending only on $\delta$ and $\alpha$. Let us introduce the Hilbert spaces

$$E_{\varsigma} = \{ v : e^{\varsigma \xi_2} v \in L_2(\Pi), e^{\varsigma \xi_2} \frac{\partial v}{\partial \xi_j} \in L_2(\Pi) \text{ for } j = 1, 2, v = 0 \text{ on } \Gamma \},
$$

$$E_{\varsigma}^0 = \{ v : v \in E_{\varsigma}, e^{\varsigma \xi_2} v \in L_2(\Pi) \},$$

equipped, respectively, with the scalar products (and associated norms)

$$(v, w)_{\varsigma} = \int_\Pi e^{2\varsigma \xi_2} \nabla v \cdot \nabla w \, d\xi,$$

$$(v, w)_{\varsigma}^0 = \int_\Pi e^{2\varsigma \xi_2} v \, d\xi + \int_\Pi e^{2\varsigma \xi_2} \nabla v \cdot \nabla w \, d\xi.$$

We consider the bilinear form $a_{\varsigma}$, continuous on $E_{\varsigma} \times E_{\varsigma}^0$,

$$a_{\varsigma}(v, w) = \int_\Pi \nabla v \cdot \nabla (e^{2\varsigma \xi_2} w) \, d\xi$$

for $v \in E_{\varsigma}$, $w \in E_{\varsigma}^0$,

and the linear form $L_{\varsigma}$, continuous on $E_{\varsigma}^0$,

$$L_{\varsigma}(v) = - \int_\Pi \frac{\partial \tilde{X}}{\partial \xi_1} e^{2\varsigma \xi_2} v \, d\xi$$

for $v \in E_{\varsigma}^0$. 21
Note here that, due to (89), \( \frac{\partial}{\partial \xi_1} \in E_0^\varsigma \) and then \( L_\varsigma \) is well-defined. We extend any \( v \in V_\varsigma \) by 0 on \( \{ \xi \in \mathbb{R}^2 : -\frac{1}{2} < \xi_1 < \frac{1}{2}, \xi_2 < F(\xi_1) \} \), and we use the same notation \( v \) for the extension. For \( v \in E_\varsigma \), we denote

\[
\overline{v}(\xi_2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\xi_1, \xi_2) \, d\xi_1, \quad \text{for } \xi_2 > 0.
\]

We have the Friedrichs–Poincaré inequality

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} |v(\xi_1, \xi_2)|^2 \, d\xi_1 \leq \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial v}{\partial \xi_1}(\xi_1, \xi_2) \right|^2 \, d\xi_1 \quad \text{for } \xi_2 < 0, \tag{90}
\]

and the Poincaré–Wirtinger inequality

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} |v(\xi_1, \xi_2) - \overline{v}(\xi_2)|^2 \, d\xi_1 \leq \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial v}{\partial \xi_1}(\xi_1, \xi_2) \right|^2 \, d\xi_1 \quad \text{for } \xi_2 > 0. \tag{91}
\]

Let \( M : E_\varsigma \to E_0^\varsigma \) defined by

\[
Mv = \begin{cases} 
  v - \tilde{v}(\xi_2) & \text{in } \Pi^+, \\
  v & \text{in } \Pi^-,
\end{cases}
\]

where

\[
\Pi^+ = \{ \xi \in \Pi : \xi = (\xi_1, \xi_2), \xi_2 > 0 \}, \quad \Pi^- = \{ \xi \in \Pi : \xi = (\xi_1, \xi_2), \xi_2 < 0 \},
\]

and \( \tilde{v} \) is the solution of the differential equation

\[
\begin{cases} 
  \frac{d\tilde{v}}{d\xi_2} + 2\varsigma \tilde{v} = 2\varsigma \overline{v} & \text{for } \xi_2 > 0, \\
  \tilde{v}(0) = 0.
\end{cases} \tag{92}
\]

We easily verify that \( e^{\xi_2}(\tilde{v} - \overline{v}) \) solves the differential equation

\[
\frac{d}{d\xi_2} \left( e^{\xi_2}(\tilde{v} - \overline{v}) \right) + \varsigma e^{\xi_2} (\tilde{v} - \overline{v}) = -e^{\xi_2} \frac{d\overline{v}}{d\xi_2} \quad \text{for } \xi_2 > 0. \tag{93}
\]

Since \( e^{\xi_2} \frac{d\overline{v}}{d\xi_2} \in L_2(\Pi) \) it follows that \( e^{\xi_2} \frac{d\overline{v}}{d\xi_2} \in L_2(0, \infty) \). Multiplying (93) by \( e^{\xi_2}(\tilde{v} - \overline{v}) \) and integrating on \( (0, \infty) \) and using the trace theorem, it follows that

\[
\| e^{\xi_2}(\tilde{v} - \overline{v}) \|_{L_2(0, \infty)} \leq C_\varsigma \| v \|_{E_\varsigma} \quad \forall v \in E_\varsigma, \tag{94}
\]

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where $C_\varsigma$ is a constant (depending on $\varsigma$). By virtue of (91), we have
\[
\|e^{\varsigma \xi_2}(v - \overline{v})\|^2_{L^2(\Pi^+)} \leq \frac{1}{2}\|v\|^2_{E_\varsigma} \quad \forall v \in E_\varsigma.
\] (95)

Using (94) and (95) we thus obtain that $Mv \in E_\varsigma^0$ and that the mapping $M$ is continuous from $E_\varsigma$ to $E_\varsigma^0$. Let us verify that $M$ maps $E_\varsigma$ onto $E_\varsigma^0$. Given $v$ in $E_\varsigma^0$, we want to find $u \in E_\varsigma$ such that $Mu = v$. We seek $u$ in the form
\[
u = \begin{cases} v + h(\xi_2) & \text{in } \Pi^+, \\ v & \text{in } \Pi^- \end{cases}
\]
This implies $\overline{u} = \overline{v} + h$ (for $\xi_2 > 0$), then $h = \tilde{u}$ and therefore $h$ is solution of the differential equation
\[
\begin{cases} \frac{dh}{d\xi_2} + 2\varsigma h = 2\varsigma(\overline{v} + h) & \text{for } \xi_2 > 0, \\ h(0) = 0, \end{cases}
\]
that is $\frac{dh}{d\xi_2} = 2\varsigma \overline{v}$. Since $v \in E_\varsigma^0$ it follows that $e^{\varsigma \xi_2} \overline{v} \in L^2(0, \infty)$, and then $e^{\varsigma \xi_2} \frac{dh}{d\xi_2} \in L^2(0, \infty)$. Thus $u \in E_\varsigma$ and $M$ is onto. Let us now prove that there exists a number $\gamma > 0$ such that
\[
a_\varsigma(v, Mv) \geq \gamma\|v\|^2 \quad \text{for every } v \in E_\varsigma.
\]
We have
\[
a_\varsigma(v, Mv) = \int_{\Pi} \nabla v \cdot \nabla(e^{2\varsigma \xi_2} Mv) \, d\xi
\]
\[
= \int_{\Pi^+} \nabla v \cdot \nabla(e^{2\varsigma \xi_2}(v - \overline{v})) \, d\xi + \int_{\Pi^-} \nabla v \cdot \nabla(e^{2\varsigma \xi_2} v) \, d\xi
\]
\[
= \int_{\Pi^+} e^{2\varsigma \xi_2} |\nabla v|^2 \, d\xi + \int_{\Pi^-} e^{2\varsigma \xi_2} \frac{\partial v}{\partial \xi_2} \left(2\varsigma(v - \overline{v}) - \frac{d\overline{v}}{d\xi_2}\right) \, d\xi
\]
\[
+ \int_{\Pi^-} e^{2\varsigma \xi_2} |\nabla v|^2 \, d\xi + 2\varsigma \int_{\Pi^-} e^{2\varsigma \xi_2} \frac{\partial v}{\partial \xi_2} v \, d\xi.
\]
Using (92), it follows that
\[
a_\varsigma(v, Mv) = \int_{\Pi^+} e^{2\varsigma \xi_2} |\nabla v|^2 \, d\xi + 2\varsigma \int_{\Pi^+} e^{2\varsigma \xi_2} \frac{\partial v}{\partial \xi_2} (v - \overline{v}) \, d\xi
\]
\[
+ 2\varsigma \int_{\Pi^-} e^{2\varsigma \xi_2} \frac{\partial v}{\partial \xi_2} v \, d\xi.
\]
Applying the Young inequality and (90) and (91), it follows that

\[
a_\varsigma(v, Mv) \geq \int_\Pi e^{2\xi_2} |\nabla v|^2 \, d\xi - \varsigma \int_\Pi e^{2\xi_2} \left| \frac{\partial v}{\partial \xi_2} \right|^2 \, d\xi
\]

\[
\geq \frac{\varsigma}{2} \int_\Pi e^{2\xi_2} \left| \frac{\partial v}{\partial \xi_1} \right|^2 \, d\xi
\]

\[
\geq \left( 1 - \frac{3k}{2} \right) \int_\Pi e^{2\xi_2} |\nabla v|^2 \, d\xi.
\]

Thus, for \( \varsigma < \frac{2}{3} \) (that we may suppose), the bilinear form \( a_\varsigma \) satisfies

\[
a_\varsigma(v, Mv) \geq \gamma \|v\|^2 \quad \text{for every } v \in E_\varsigma,
\]

with \( \gamma > 0 \). Then, by virtue of Lemma 4.1 there is a unique solution \( \widetilde{X}(I) \) in \( E_\varsigma \) of the variational equation

\[
a_\varsigma(\widetilde{X}(I), v) = L_\varsigma(v) \quad \forall v \in E_0^\varsigma,
\]

from which follows that \( \widetilde{X}(I) \) is a weak solution of Problem (51). Let us set, for simplicity of notation, \( Y = \widetilde{X}(I) \). From \( Y \in E_\varsigma \) we deduce that \( Y \) decays exponentially fast in the Dirichlet integral, i.e., for any \( \delta > 0 \), there is a constant \( C_\delta \) such that

\[
\int_{\Pi^\delta} |\nabla Y|^2 \, d\xi \leq C_\delta e^{-2\varsigma \delta},
\]

where \( \Pi^\delta = (-\frac{1}{2}, \frac{1}{2}) \times (\delta, \infty) \). Since \( e^{\varsigma \xi_2} \frac{\partial Y}{\partial \xi_2} \in L_2(0, \infty) \) it follows that \( \overline{Y} \) admits a limit as \( \xi_2 \to +\infty \), which we denote \( C(I)(\overline{F}) \). We have \( Y - \overline{Y} \in E_0^\varsigma \) and we easily show that \( Y - C(I)(\overline{F}) \in E_0^\varsigma \). Consequently, for any \( \delta > 0 \), there is a constant \( C_\delta \) such that

\[
\int_{\Pi^\delta} |Y - C(I)(\overline{F})|^2 \, d\xi \leq C_\delta e^{-2\varsigma \delta}.
\]

Then, using the local regularizing properties of the Laplace operator and the Sobolev imbedding theorem (see, for instance, [38], [39]), we deduce, that \( \forall \delta > 0, \forall \alpha \in \mathbb{N}^2 \),

\[
|\partial^\alpha (Y - C(I)(\overline{F}))(\xi_1, \xi_2)| \leq C_{\delta, \alpha} e^{-c\xi_2}
\]
for any \((\xi_1, \xi_2) \in \Pi\) with \(\xi_2 \geq \delta\), where \(C_{\delta, \alpha}\) is another constant depending only \(\delta\) and \(\alpha\). The proposition is proved.

**Proof of Proposition 2.2** Let \(s \in C^\infty(\mathbb{R})\) be such that \(s(\xi_2) = 0\) if \(\xi_2 < 1\) and \(s(\xi_2) = 1\) if \(\xi_2 > 2\), and let \(h(\xi_2) = \left(\frac{1}{6}\xi_2^3 + \frac{1}{2}C(F)\xi_2^2\right) s(\xi_2)\). Consider the problem

\[
\begin{cases}
\Delta \xi Z = X - h''(\xi_2) & \text{in } \Pi, \\
Z = 0 & \text{on } \Gamma, \\
\frac{\partial Z}{\partial \xi_1} = 0 & \text{as } \xi_1 = \pm \frac{1}{2}.
\end{cases}
\]  

(96)

Since \(X - h''(\xi_2) = X - (\xi_2 + C(F))\) for \(\xi_2 > 2\), according to (9), \(X - h'' = 0\) as \(\xi_2 \to +\infty\), up to exponentially small terms. We then can show as for Proposition 2.1 that Problem (96) admits a solution \(Z\) which has the asymptotics

\[Z = C_{(II)}(F)\]  

as \(\xi_2 \to +\infty\),

up to exponentially small terms, where \(C_{(II)}(F)\) is a constant depending on the function \(F\). More precisely, denoting

\[C_{(II)}(F) = \lim_{\xi_2 \to +\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} Z(\xi_1, \xi_2) \, d\xi_1\]  

\((\xi_2 > 0)\)

there exists a positive constant \(\varsigma > 0\) such that, \(\forall \delta > 0\), \(\forall \alpha \in \mathbb{N}^2\),

\[|\partial^\alpha (Z(\xi_1, \xi_2) - C_{(II)}(F))| \leq C_{\delta, \alpha} e^{-\varsigma \xi_2}\]

for any \((\xi_1, \xi_2) \in \Pi\) with \(\xi_2 > \delta\), where \(C_{\delta, \alpha}\) is a constant depending only \(\delta\) and \(\alpha\). Setting \(\tilde{X}_{(II)} = Z + h\), we obtain the statement.

**Remark 4.1.** Note that the existence of periodic in \((n-1)\) variables solutions and their behavior at infinity in \(n\)-dimensional semi-space are studied, for instance, in [27]. See also [33].

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