Supplementary Material to: Ranking Procedures for Repeated Measures Designs with Missing Data - Estimation, Testing and Asymptotic Theory

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We present in this supplementary material selected results from the type-I error and power simulations. Furthermore, we present the proofs of all theorems and propositions from the paper.

1 Results of the Simulation Study
Within this section, we will present additional tables and figures of the type-I error and power results of the simulation study. We simulated

1. the WTS in (16) with a critical value from a χ²_j distribution,
2. the ATS (1) in (17) with the proposed $F$-approximation,
3. the ATS (2) in (18) with the proposed $F$-approximation,
4. the MCTP $T_0$ in (19) with a $t_{n-1}(0, \mathbf{R}_n)$ approximation,

and compared them with

5. the WTS and ATS statistics for testing $H_{F0}$ as proposed by Domhof et al.\textsuperscript{1} in different homo- and heteroscedastic repeated measures designs with different rates of missing values. Even though Domhof et al.\textsuperscript{1} reported a liberal behavior of the WTS (for testing $H_{F0}$), we added the method as a competing procedure for completeness. We thus also investigated their robustness to variance heteroscedasticity. Since all of the methods above use all-available data, we additionally compared them with two MCTP-based approaches: a complete case analysis and a naive imputation approach, in which we either

6. deleted the whole observation vector $X_k$ of subject $k$ if any $X_{ik}$ was missing ($\lambda_{ik} = 0$), or
7. if $X_{ik}$ was missing ($\lambda_{ik} = 0$), we calculated $\text{median}(\lambda_{i1} X_{i1}, \ldots, \lambda_{in} X_{in})$, and assigned it to $X_{ik}$ and set $\lambda_{ik} = 1$.

Data have been generated using discretized, by rounding to integers, normal and log-normal distributions with varying numbers of time points $d \in \{3, 4\}$, sample sizes $n \in \{20, 30, 50\}$, amount of missing values $r \in \{0\%, 10\%, 30\%\}$ and six different types of covariance matrices

\[
\Sigma_1 = \begin{pmatrix}
1 & 0.5 & 0.5 \\
0.5 & 1 & 0.5 \\
0.5 & 0.5 & 1
\end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix}
1 & 0.5 & 0.5 & 0.5 \\
0.5 & 1 & 0.5 & 0.5 \\
0.5 & 0.5 & 1 & 0.5 \\
0.5 & 0.5 & 0.5 & 1
\end{pmatrix}, \quad (1)
\]

\[
\Sigma_3 = \begin{pmatrix}
1 & 0.3 & 0.6 \\
0.3 & 1.2 & 0.9 \\
0.6 & 0.9 & 1.5
\end{pmatrix}, \quad \Sigma_4 = \begin{pmatrix}
1 & 0.2 & 0.4 & 0.6 \\
0.2 & 2 & 0.7 & 0.5 \\
0.4 & 0.7 & 2.5 & 0.6 \\
0.6 & 0.5 & 0.6 & 3
\end{pmatrix},
\]

\[
\Sigma_5 = \begin{pmatrix}
1 & 0.6 & 0.36 & 0.216 \\
0.6 & 1 & 0.6 & 0.36 \\
0.36 & 0.6 & 1 & 0.6 \\
0.216 & 0.36 & 0.6 & 1
\end{pmatrix}, \quad \Sigma_6 = \begin{pmatrix}
1 & 0.8 & 0.64 & 0.512 \\
0.8 & 1.5 & 0.8 & 0.64 \\
0.64 & 0.8 & 2 & 0.8 \\
0.512 & 0.64 & 0.8 & 2.5
\end{pmatrix}.
\]

The covariance matrices were chosen to model a broad selection of dependency patterns, including homoscedastic ($\Sigma_1$ and $\Sigma_2$) as well as heteroscedastic marginals. Note that $H_{F0}$ holds only under $\Sigma_1$ and $\Sigma_2$. We furthermore investigate the methods sensitivity to both MCAR and MAR data to cover realistic scenarios. In order to generate the former, we multiplied the observations with randomly chosen indicators $\lambda_{ik} \sim B(1 - r)$, with
a zero entry being interpreted as a missing observation, whereas we followed Santos et al.\(^2\) for the latter. Hereby we defined pairs of observations \(\{X_{\text{obs}}, X_{\text{miss}}\}\), where \(X_{\text{obs}}\) determines the probability that \(X_{\text{miss}}\) was actually observed. For instance, in case of \(d = 4\) we defined the pairs \(\{X_{1k}, X_{2k}\}\) and \(\{X_{3k}, X_{4k}\}\). Following the idea of Amro et al.\(^3\), we investigated two different types of MAR scenarios, MAR (1) and MAR (2). First, for the MAR (1) scenario, we divided \(X_{i,\text{obs}}\) into three groups: (1) \(\{X_{ik} = X_{i,\text{obs}} \in (\,-\infty,-\sigma_i), k = 1,\ldots,n\}\), (2) \(\{X_{ik} = X_{i,\text{obs}} \in (-\sigma_i, \sigma_i), k = 1,\ldots,n\}\) and (3) \(\{X_{ik} = X_{i,\text{obs}} \in (\sigma_i, \infty), k = 1,\ldots,n\}\), where \(\sigma_i^2\) is the variance of \(X_{i,\text{obs}}\). Then, we assigned a missing rate of 10% to the first and third group and a missing rate of 30% to the second group. Second, in the MAR (2) scenario, data was divided into two groups using the median, following the idea of Zhu et al.\(^4\). Specifically, we defined (1) \(\{X_{ik} = X_{i,\text{obs}} \in (-\infty, \text{median}(X_{i,\text{obs}})), k = 1,\ldots,n\}\) and (2) \(\{X_{ik} = X_{i,\text{obs}} \in (\text{median}(X_{i,\text{obs}}), \infty), k = 1,\ldots,n\}\). Here, we assigned a missing rate of 0% to the first group and a missing rate of 10% to the second group.

In order to investigate the power of the procedures, a simulation study was conducted using four-dimensional normal and log-normal distributions with \(\mu = (\mu_1, \mu_2, \mu_3, \mu_4)'\) and covariance matrices \(\Sigma_2, \Sigma_4, \Sigma_5, \Sigma_6\). In particular, three different types of shift-alternatives were considered

\[
\begin{align*}
\text{Alternative 1} & \quad \mu = (0, 0, 0, \delta)' \\
\text{Alternative 2} & \quad \mu = (0, 0, \delta, \delta)' \\
\text{Alternative 3} & \quad \mu = (0, 1\delta, 2\delta, 3\delta)',
\end{align*}
\]

with ranging \(\delta = (0.2, 0.4, 0.6, 0.8, 1, 1.5)\) and different amount of missing values. As the WTS turned out to be inappropriate for small sample sizes, it was not included into the power analysis. Moreover, since the second version of the ATS for testing \(H_0^P\) showed a more accurate behaviour than the first version, we only present results for the second version.

For each design, 10,000 simulation runs were performed using the R software package of statistical computing, version R 3.6.4\(^5\). The complete simulation code is available on https://github.com/KerstinRubarth/RM_Miss.

### 1.1 Type-I error

The type-I error rates under the MCAR mechanism for different sample sizes \(n\), covariances matrices, missing rates \(r\) and discretized distributions can be found in Tables 1 – 3. A graphical comparisons of rather small and large sample sizes and no, medium and high missing rates can be found in Figure 1.
Table 1. Type-I-error ($\alpha = 5\%$) simulation results of the ATS and WTS for testing $H_{F0}$ and $H_{P0}$ and MCTP in different designs with covariance matrices $\Sigma_1$ and $\Sigma_2$ as given in (23). Here, $r$ denotes the rate of missing data. Data is MCAR. Note that $H_{F0}$ holds.

| n  | r(%) | Dist | $\Sigma_1$ | $\Sigma_2$ |
|----|------|------|------------|------------|
|    |      |      | $H_{F0}$   | $H_{P0}$   | $H_{F0}$   | $H_{P0}$   |
|    |      |      | WTS ATTS  | WTS ATTS   | WTS ATTS  | WTS ATTS  |
| 20 | 0    | N    | 0.102 0.066 | 0.075 0.051 | 0.141 0.062 | 0.098 0.047 |
| 20 | 10   | N    | 0.097 0.066 | 0.078 0.053 | 0.138 0.060 | 0.104 0.050 |
| 20 | 30   | N    | 0.099 0.061 | 0.090 0.058 | 0.137 0.058 | 0.118 0.056 |
| 30 | 0    | N    | 0.086 0.062 | 0.070 0.054 | 0.102 0.056 | 0.082 0.047 |
| 30 | 10   | N    | 0.081 0.059 | 0.070 0.051 | 0.100 0.056 | 0.083 0.049 |
| 30 | 30   | N    | 0.080 0.059 | 0.077 0.057 | 0.104 0.057 | 0.095 0.054 |
| 50 | 0    | N    | 0.068 0.056 | 0.061 0.052 | 0.075 0.051 | 0.066 0.047 |
| 50 | 10   | N    | 0.066 0.056 | 0.061 0.052 | 0.075 0.052 | 0.067 0.048 |
| 50 | 30   | N    | 0.068 0.055 | 0.065 0.053 | 0.079 0.056 | 0.075 0.054 |
| 20 | 0    | LN   | 0.100 0.059 | 0.074 0.046 | 0.139 0.060 | 0.097 0.045 |
| 20 | 10   | LN   | 0.096 0.059 | 0.076 0.049 | 0.142 0.060 | 0.109 0.050 |
| 20 | 30   | LN   | 0.096 0.060 | 0.087 0.057 | 0.140 0.058 | 0.117 0.054 |
| 30 | 0    | LN   | 0.082 0.060 | 0.068 0.051 | 0.100 0.054 | 0.081 0.047 |
| 30 | 10   | LN   | 0.085 0.062 | 0.073 0.056 | 0.097 0.055 | 0.082 0.047 |
| 30 | 30   | LN   | 0.077 0.057 | 0.074 0.056 | 0.095 0.050 | 0.088 0.047 |
| 50 | 0    | LN   | 0.063 0.052 | 0.058 0.048 | 0.075 0.052 | 0.066 0.047 |
| 50 | 10   | LN   | 0.066 0.053 | 0.061 0.049 | 0.079 0.054 | 0.072 0.049 |
| 50 | 30   | LN   | 0.069 0.056 | 0.067 0.054 | 0.079 0.053 | 0.076 0.052 |
Table 2. Type-I-error ($\alpha = 5\%$) simulation results of the ATS and WTS for testing $H_F^0$ and $H_P^0$ and MCTP in different designs with covariance matrices $\Sigma_3$ and $\Sigma_4$ as given in (23). Here, $r$ denotes the rate of missing data. Data is MCAR. Note that $H_P^0$ holds.

| n  | r(%) | Dist | $\Sigma_3$ | $H_F^0$ | $H_P^0$ | $\Sigma_4$ | $H_F^0$ | $H_P^0$ |
|----|------|------|------------|---------|---------|------------|---------|---------|
|    |      |      | WTS  | ATS   | WTS  | ATS1 | ATS2 | MCT | WTS  | ATS   | WTS  | ATS1 | ATS2 | MCT   |
| 20 |  0   | N    | 0.111| 0.066 | 0.078| 0.053| 0.043| 0.050 | 0.130| 0.058 | 0.108| 0.047| 0.038| 0.054 |
| 20 | 10   | N    | 0.102| 0.065 | 0.077| 0.054| 0.044| 0.049 | 0.134| 0.060 | 0.118| 0.055| 0.045| 0.060 |
| 20 | 30   | N    | 0.100| 0.064 | 0.088| 0.059| 0.048| 0.059 | 0.140| 0.062 | 0.133| 0.062| 0.051| 0.070 |
| 30 |  0   | N    | 0.087| 0.063 | 0.065| 0.052| 0.044| 0.050 | 0.100| 0.053 | 0.087| 0.048| 0.041| 0.051 |
| 30 | 10   | N    | 0.082| 0.061 | 0.069| 0.053| 0.045| 0.051 | 0.099| 0.054 | 0.090| 0.050| 0.043| 0.053 |
| 30 | 30   | N    | 0.084| 0.061 | 0.078| 0.057| 0.050| 0.058 | 0.105| 0.059 | 0.099| 0.057| 0.051| 0.066 |
| 50 |  0   | N    | 0.071| 0.057 | 0.061| 0.052| 0.048| 0.049 | 0.082| 0.056 | 0.071| 0.051| 0.047| 0.051 |
| 50 | 10   | N    | 0.070| 0.056 | 0.062| 0.051| 0.047| 0.050 | 0.080| 0.055 | 0.072| 0.051| 0.048| 0.057 |
| 50 | 30   | N    | 0.069| 0.055 | 0.066| 0.053| 0.048| 0.054 | 0.083| 0.057 | 0.080| 0.055| 0.051| 0.060 |
| 20 |  0   | LN   | 0.105| 0.064 | 0.074| 0.051| 0.041| 0.046 | 0.136| 0.058 | 0.113| 0.052| 0.042| 0.056 |
| 20 | 10   | LN   | 0.107| 0.070 | 0.083| 0.058| 0.047| 0.055 | 0.130| 0.056 | 0.115| 0.051| 0.040| 0.061 |
| 20 | 30   | LN   | 0.100| 0.063 | 0.088| 0.060| 0.047| 0.058 | 0.141| 0.060 | 0.130| 0.060| 0.049| 0.073 |
| 30 |  0   | LN   | 0.089| 0.061 | 0.068| 0.053| 0.044| 0.049 | 0.105| 0.059 | 0.093| 0.053| 0.046| 0.056 |
| 30 | 10   | LN   | 0.084| 0.063 | 0.073| 0.056| 0.047| 0.052 | 0.110| 0.064 | 0.102| 0.060| 0.052| 0.065 |
| 30 | 30   | LN   | 0.080| 0.058 | 0.075| 0.055| 0.047| 0.056 | 0.108| 0.058 | 0.109| 0.058| 0.049| 0.067 |
| 50 |  0   | LN   | 0.070| 0.054 | 0.060| 0.048| 0.044| 0.049 | 0.087| 0.058 | 0.080| 0.053| 0.050| 0.058 |
| 50 | 10   | LN   | 0.066| 0.056 | 0.059| 0.052| 0.047| 0.049 | 0.087| 0.059 | 0.082| 0.055| 0.050| 0.059 |
| 50 | 30   | LN   | 0.069| 0.060 | 0.066| 0.058| 0.054| 0.054 | 0.087| 0.062 | 0.085| 0.060| 0.055| 0.063 |
Table 3. Type-I-error ($\alpha = 5\%$) simulation results of the ATS and WTS for testing $H^F_0$ and $H^P_0$ and MCTP in different designs with covariance matrices $\Sigma_5$ and $\Sigma_6$ as given in (23). Here, $r$ denotes the rate of missing data. Data is MCAR. Note that $H^P_0$ holds.

| n  | r(%) | Dist | $\Sigma_5$ | $H^F_0$ | $H^P_0$ | MCT | $\Sigma_6$ | $H^F_0$ | $H^P_0$ | MCT |
|----|------|------|------------|--------|--------|-----|------------|--------|--------|-----|
| 20 | 0    | N    | 0.141      | 0.058  | 0.097  | 0.051  | 0.042 | 0.049 | 0.147 | 0.060 | 0.050  | 0.041 | 0.052 |
| 20 | 10   | N    | 0.139      | 0.057  | 0.104  | 0.051  | 0.042 | 0.052 | 0.146 | 0.062 | 0.111  | 0.053 | 0.042 | 0.058 |
| 20 | 30   | N    | 0.138      | 0.057  | 0.119  | 0.055  | 0.045 | 0.064 | 0.147 | 0.063 | 0.125  | 0.060 | 0.051 | 0.068 |
| 30 | 0    | N    | 0.100      | 0.054  | 0.080  | 0.048  | 0.042 | 0.046 | 0.114 | 0.059 | 0.087  | 0.052 | 0.044 | 0.049 |
| 30 | 10   | N    | 0.097      | 0.052  | 0.081  | 0.048  | 0.040 | 0.048 | 0.102 | 0.055 | 0.083  | 0.049 | 0.041 | 0.050 |
| 30 | 30   | N    | 0.102      | 0.056  | 0.094  | 0.055  | 0.047 | 0.058 | 0.104 | 0.060 | 0.094  | 0.057 | 0.049 | 0.058 |
| 50 | 0    | N    | 0.076      | 0.054  | 0.066  | 0.052  | 0.048 | 0.050 | 0.083 | 0.055 | 0.069  | 0.050 | 0.045 | 0.051 |
| 50 | 10   | N    | 0.074      | 0.052  | 0.067  | 0.049  | 0.044 | 0.051 | 0.080 | 0.053 | 0.070  | 0.049 | 0.045 | 0.050 |
| 50 | 30   | N    | 0.074      | 0.050  | 0.073  | 0.049  | 0.045 | 0.056 | 0.078 | 0.052 | 0.073  | 0.049 | 0.045 | 0.053 |
| 20 | 0    | LN   | 0.141      | 0.060  | 0.101  | 0.049  | 0.040 | 0.050 | 0.148 | 0.064 | 0.107  | 0.054 | 0.043 | 0.054 |
| 20 | 10   | LN   | 0.138      | 0.059  | 0.109  | 0.051  | 0.041 | 0.054 | 0.139 | 0.062 | 0.104  | 0.054 | 0.046 | 0.054 |
| 20 | 30   | LN   | 0.142      | 0.062  | 0.124  | 0.061  | 0.051 | 0.066 | 0.142 | 0.059 | 0.122  | 0.058 | 0.049 | 0.068 |
| 30 | 0    | LN   | 0.102      | 0.060  | 0.084  | 0.054  | 0.048 | 0.052 | 0.110 | 0.064 | 0.089  | 0.055 | 0.047 | 0.054 |
| 30 | 10   | LN   | 0.096      | 0.051  | 0.079  | 0.046  | 0.040 | 0.051 | 0.104 | 0.063 | 0.088  | 0.058 | 0.050 | 0.056 |
| 30 | 30   | LN   | 0.098      | 0.054  | 0.091  | 0.053  | 0.047 | 0.059 | 0.100 | 0.056 | 0.093  | 0.054 | 0.047 | 0.056 |
| 50 | 0    | LN   | 0.080      | 0.058  | 0.070  | 0.054  | 0.050 | 0.054 | 0.088 | 0.063 | 0.077  | 0.057 | 0.052 | 0.057 |
| 50 | 10   | LN   | 0.075      | 0.052  | 0.069  | 0.050  | 0.046 | 0.049 | 0.084 | 0.060 | 0.073  | 0.055 | 0.051 | 0.056 |
| 50 | 30   | LN   | 0.075      | 0.051  | 0.072  | 0.051  | 0.046 | 0.054 | 0.079 | 0.057 | 0.075  | 0.055 | 0.051 | 0.054 |
Figure 1. Type-I error rates of the newly proposed Wald- (WTS), ANOVA-type (ATS (1) and ATS (2)) and MCT procedures and the procedures of Domhof et al.\textsuperscript{1} in MCAR scenarios.

1.2 Power

The power simulation results for $n = 30$ and $\Sigma_4$ are given in Tables 4 and 5 for discretized normal and log-normal distributions respectively. An exploration of the power under different MAR scenarios, covariance matrices, alternatives and discretized distributions can be found in Figures 2 - 5.
Table 4. Power comparisons of the ATS for testing either $H_0^F$ (ATS (F)) or $H_0^P$ (ATS (2)) and the MCTP to detect the three alternatives listed in (24) in normal distributions with $\Sigma_4$ and $n = 30$. Data is MCAR.

| delta | r(%) | Alternative 1 | | Alternative 2 | | Alternative 3 | |
|-------|------|---------------|-----------------|---------------|-----------------|---------------|
|       |      | ATS(F) | ATS(2) | MCTP | ATS(F) | ATS(2) | MCTP | ATS(F) | ATS(2) | MCTP |
| 0.0   | 0    | 0.053  | 0.041  | 0.051 | 0.059  | 0.047  | 0.056 | 0.055  | 0.042  | 0.050 |
| 0.0   | 10   | 0.054  | 0.043  | 0.053 | 0.058  | 0.046  | 0.058 | 0.057  | 0.043  | 0.058 |
| 0.0   | 30   | 0.059  | 0.051  | 0.066 | 0.056  | 0.047  | 0.061 | 0.057  | 0.049  | 0.061 |
| 0.2   | 0    | 0.085  | 0.070  | 0.078 | 0.099  | 0.079  | 0.095 | 0.309  | 0.267  | 0.301 |
| 0.2   | 10   | 0.080  | 0.067  | 0.077 | 0.088  | 0.072  | 0.087 | 0.269  | 0.234  | 0.269 |
| 0.2   | 30   | 0.074  | 0.064  | 0.081 | 0.082  | 0.071  | 0.093 | 0.208  | 0.189  | 0.227 |
| 0.4   | 0    | 0.176  | 0.150  | 0.152 | 0.248  | 0.212  | 0.220 | 0.903  | 0.880  | 0.872 |
| 0.4   | 10   | 0.160  | 0.138  | 0.149 | 0.219  | 0.186  | 0.201 | 0.851  | 0.821  | 0.823 |
| 0.4   | 30   | 0.132  | 0.118  | 0.129 | 0.177  | 0.157  | 0.175 | 0.694  | 0.666  | 0.688 |
| 0.6   | 0    | 0.355  | 0.313  | 0.317 | 0.526  | 0.481  | 0.430 | 0.999  | 0.999  | 0.998 |
| 0.6   | 10   | 0.319  | 0.284  | 0.286 | 0.454  | 0.411  | 0.377 | 0.997  | 0.995  | 0.993 |
| 0.6   | 30   | 0.235  | 0.212  | 0.227 | 0.341  | 0.314  | 0.311 | 0.974  | 0.968  | 0.966 |
| 0.8   | 0    | 0.574  | 0.532  | 0.527 | 0.801  | 0.766  | 0.682 | 1.000  | 1.000  | 1.000 |
| 0.8   | 10   | 0.521  | 0.484  | 0.474 | 0.732  | 0.694  | 0.617 | 1.000  | 1.000  | 1.000 |
| 0.8   | 30   | 0.391  | 0.367  | 0.376 | 0.572  | 0.540  | 0.493 | 1.000  | 0.999  | 0.999 |
| 1.0   | 0    | 0.789  | 0.755  | 0.742 | 0.949  | 0.936  | 0.862 | 1.000  | 1.000  | 1.000 |
| 1.0   | 10   | 0.720  | 0.685  | 0.676 | 0.912  | 0.896  | 0.821 | 1.000  | 1.000  | 1.000 |
| 1.0   | 30   | 0.566  | 0.540  | 0.546 | 0.784  | 0.758  | 0.693 | 1.000  | 1.000  | 1.000 |
| 1.5   | 0    | 0.987  | 0.983  | 0.982 | 1.000  | 1.000  | 0.998 | 1.000  | 1.000  | 1.000 |
| 1.5   | 10   | 0.974  | 0.967  | 0.964 | 1.000  | 0.999  | 0.994 | 1.000  | 1.000  | 1.000 |
| 1.5   | 30   | 0.904  | 0.891  | 0.886 | 0.988  | 0.986  | 0.965 | 1.000  | 1.000  | 1.000 |
Table 5. Power comparisons of the ATS for testing either $H_F^0$ (ATS (F)) or $H_p^0$ (ATS (2)) and the MCTP to detect the three alternatives listed in (24) in log-normal distributions with $\Sigma_4$ and $n = 30$. Data is MCAR.

| delta | r(%) | ATS(F) | ATS(2) | MCTP | ATS(F) | ATS(2) | MCTP | ATS(F) | ATS(2) | MCTP |
|-------|------|--------|--------|------|--------|--------|------|--------|--------|------|
| 0.0   | 0    | 0.059  | 0.046  | 0.056| 0.059  | 0.047  | 0.060| 0.059  | 0.048  | 0.057|
| 0.0   | 10   | 0.064  | 0.052  | 0.065| 0.064  | 0.052  | 0.066| 0.059  | 0.047  | 0.061|
| 0.0   | 30   | 0.058  | 0.049  | 0.067| 0.059  | 0.050  | 0.065| 0.061  | 0.053  | 0.066|
| 0.2   | 0    | 0.100  | 0.083  | 0.094| 0.124  | 0.102  | 0.121| 0.366  | 0.329  | 0.366|
| 0.2   | 10   | 0.099  | 0.082  | 0.093| 0.110  | 0.091  | 0.108| 0.337  | 0.300  | 0.335|
| 0.2   | 30   | 0.085  | 0.074  | 0.088| 0.098  | 0.086  | 0.107| 0.238  | 0.214  | 0.260|
| 0.4   | 0    | 0.205  | 0.175  | 0.184| 0.303  | 0.263  | 0.261| 0.926  | 0.906  | 0.904|
| 0.4   | 10   | 0.188  | 0.162  | 0.166| 0.260  | 0.229  | 0.237| 0.880  | 0.857  | 0.859|
| 0.4   | 30   | 0.143  | 0.127  | 0.145| 0.207  | 0.186  | 0.199| 0.731  | 0.704  | 0.732|
| 0.6   | 0    | 0.397  | 0.357  | 0.350| 0.581  | 0.536  | 0.479| 0.999  | 0.999  | 0.999|
| 0.6   | 10   | 0.345  | 0.312  | 0.316| 0.505  | 0.465  | 0.428| 0.998  | 0.997  | 0.996|
| 0.6   | 30   | 0.269  | 0.246  | 0.262| 0.387  | 0.358  | 0.354| 0.977  | 0.973  | 0.973|
| 0.8   | 0    | 0.619  | 0.579  | 0.569| 0.830  | 0.798  | 0.715| 1.000  | 1.000  | 1.000|
| 0.8   | 10   | 0.548  | 0.511  | 0.504| 0.764  | 0.732  | 0.658| 1.000  | 1.000  | 1.000|
| 0.8   | 30   | 0.423  | 0.397  | 0.408| 0.608  | 0.579  | 0.535| 0.999  | 0.999  | 0.999|
| 1.0   | 0    | 0.809  | 0.777  | 0.769| 0.959  | 0.945  | 0.887| 1.000  | 1.000  | 1.000|
| 1.0   | 10   | 0.746  | 0.714  | 0.703| 0.932  | 0.917  | 0.854| 1.000  | 1.000  | 1.000|
| 1.0   | 30   | 0.596  | 0.573  | 0.574| 0.813  | 0.792  | 0.729| 1.000  | 1.000  | 1.000|
| 1.5   | 0    | 0.991  | 0.988  | 0.986| 1.000  | 1.000  | 0.999| 1.000  | 1.000  | 1.000|
| 1.5   | 10   | 0.977  | 0.971  | 0.965| 1.000  | 0.999  | 0.996| 1.000  | 1.000  | 1.000|
| 1.5   | 30   | 0.916  | 0.906  | 0.898| 0.993  | 0.992  | 0.975| 1.000  | 1.000  | 1.000|
Normal Distribution, MAR (1), n = 20

**Figure 2.** Power simulation of the ATS (2) and the MCTP for testing $H_{0\delta}$, the ATS for testing $H_{0\delta}^C$ and the MCTP for testing $H_{0\delta}^P$ using only complete cases, data is MAR (1).
Figure 3. Power simulation of the ATS (2) and the MCTP for testing $H^F_0$, the ATS for testing $H^C_0$, and the MCTP for testing $H^P_0$ using only complete cases, data is MAR (1).
Figure 4. Power simulation of the ATS (2) and the MCTP for testing $H_{0}^{p}$, the ATS for testing $H_{0}^{F}$ and the MCTP for testing $H_{0}^{p}$ using only complete cases, data is MAR (2).
LogNormal Distribution, MAR (2), n = 20

![Diagram showing power simulation for ATS (2) and MCTP]

**Figure 5.** Power simulation of the ATS (2) and the MCTP for testing $H_0^p$, the ATS for testing $H_0^F$ and the MCTP for testing $H_0^p$ using only complete cases, data is MAR (2).

### 2 Proofs

In this section we will provide the proofs of the theoretical results achieved in the paper. The strong consistency of the point and variance estimators follows from the following generalization of the Glivenko-Cantelli Theorem:

**Lemma 1:** Let $F^{(-)}_i(x) = P(X_{i1} < x)$ and $F^{(+)}_i(x) = P(X_{i1} \leq x)$ denote the left-continuous and right-continuous versions of the distribution function of $X_{i1}$ and let

\[
\hat{F}^{(-)}_i = \frac{1}{\lambda_i} \sum_{k=1}^{n} \lambda_{ik} c^{(-)}(x - X_{ik}) \quad \text{and} \quad \hat{F}^{(+)}_i = \frac{1}{\lambda_i} \sum_{k=1}^{n} \lambda_{ik} c^{(+)}(x - X_{ik})
\]

denote their empirical counterparts, where

\[
c^{(-)}(u) = \begin{cases} 0 & u \leq 0 \\ 1 & u > 0 \end{cases} \quad \text{and} \quad c^{(+)}(u) = \begin{cases} 0 & u < 0 \\ 1 & u \geq 0 \end{cases}
\]
Furthermore, let \( F_i(x) = \frac{1}{2}(F_i^-(x) + F_i^+(x)) \), \( G(x) \), \( \hat{F}_i(x) = \frac{1}{2}(\hat{F}_i^- + \hat{F}_i^+) \) and \( \hat{G}(x) = \frac{1}{d} \sum_{i=1}^{d} \hat{F}_i(x) \) denote the normalized versions of the distribution functions as used in the manuscript. Then,

\[
\left\| \hat{F}_i - F_i \right\|_{\infty} \xrightarrow{a.s.} 0, \; \lambda_i \to \infty \quad \text{and} \quad \left\| \hat{G} - G \right\|_{\infty} \xrightarrow{a.s.} 0, \; \min\{\lambda_1, \ldots, \lambda_d\} \to \infty.
\]

**Proof:** Since \( \hat{F}_i(x) \) is the mean of the left- and right continuous version, it follows

\[
\left\| \hat{F}_i - F_i \right\|_{\infty} = \left\| \frac{1}{2}(\hat{F}_i^- - F_i^-) + \frac{1}{2}(\hat{F}_i^+ - F_i^+) \right\|_{\infty} \leq \frac{1}{2} \left\| \hat{F}_i^- - F_i^- \right\|_{\infty} + \frac{1}{2} \left\| \hat{F}_i^+ - F_i^+ \right\|_{\infty} \xrightarrow{a.s.} 0, \; \lambda_i \to \infty.
\]

The proof of the almost sure convergence of the last remaining terms is given by \(^6\), page 111. Furthermore, by triangle inequality, the convergence \( \left\| \hat{G} - G \right\|_{\infty} \xrightarrow{a.s.} 0 \) follows, which completes the proof.

### 2.1 Proof of Proposition 1

1. The estimator \( \hat{p} = (\hat{p}_1, \ldots, \hat{p}_d)' \) is asymptotically unbiased, because

\[
\left| E(\hat{p}_i) - p_i \right| = \frac{1}{d} \sum_{h=1}^{d} \left| E \left( \int \hat{F}_h d\hat{F}_i - \int F_h dF_i \right) \right| \\
= \frac{1}{d} \sum_{h=1}^{d} \frac{1}{\lambda_h \lambda_i} \sum_{k=1}^{n} \sum_{\ell=1}^{n} \lambda_{ik} \lambda_{h\ell} \left( E \left( c(X_{ik} - X_{h\ell}) - \int F_h dF_i \right) \right) \\
= \frac{1}{d} \sum_{h=1}^{d} \frac{1}{\lambda_h \lambda_i} \sum_{k=1}^{n} \lambda_{ik} \lambda_{hk} \left( E \left( c(X_{ik} - X_{hk}) - \int F_h dF_i \right) \right) \\
\leq \frac{1}{d} \sum_{h=1}^{d} \frac{n}{\lambda_h \lambda_i} \sum_{k=1}^{n} \lambda_{ik} \lambda_{hk} \left( E \left( c(X_{ik} - X_{hk}) - \int F_h dF_i \right) \right) \\
\leq \frac{1}{d} \sum_{h=1}^{d} \frac{n}{\lambda_h \lambda_i} \leq \frac{1}{d} \sum_{h=1}^{d} \frac{N_0}{\lambda_i} = \frac{N_0}{\lambda_i} \to 0, \; \lambda_i \to \infty.
\]
2. The strong consistency of $\hat{p}$ follows from
\[
|\hat{p}_i - p_i| \leq \frac{1}{d} \sum_{s=1}^{d} \left| \int \hat{F}_s d\hat{F}_i - \int F_s dF_i \right|
\]
\[
= \frac{1}{d} \sum_{s=1}^{d} \left| \int \hat{F}_s d\hat{F}_i - \int F_s dF_i - \int F_s d\hat{F}_i + \int F_s d\hat{F}_i \right|
\]
\[
\leq \frac{1}{d} \sum_{s=1}^{d} \left( \left| \int (\hat{F}_s - F_s) d\hat{F}_i \right| + \left| \int F_s d(\hat{F}_i - F_i) \right| \right)
\]
\[
= \frac{1}{d} \sum_{s=1}^{d} \left( \left| \int (\hat{F}_s - F_s) d\hat{F}_i \right| + \left| \int (F_i - \hat{F}_i) dF_s \right| \right)
\]
\[
\leq \frac{1}{d} \sum_{s=1}^{d} \left( ||\hat{F}_s - F_s||_\infty + ||\hat{F}_i - F_i||_\infty \right) \xrightarrow{a.s.} 0, \lambda_i, \lambda_s \to \infty.
\]

2.2 Proof of Theorem 1
It is sufficient to prove the theorem only for the $i$-th component. By adding and subtracting $\int G d\hat{F}_i$ and $\int G dF_i$, it holds for all $i = 1, \ldots, d$ that
\[
\sqrt{n} \int \hat{G} d(\hat{F}_i - F_i) = \sqrt{n} \int G d(\hat{F}_i - F_i) + \sqrt{n} \int (\hat{G} - G) d(\hat{F}_i - F_i).
\]

If it can be shown, that
\[
\sqrt{n} Z_i = \sqrt{n} \int (\hat{G} - G) d(\hat{F}_i - F_i) \xrightarrow{p} 0, i = 1, \ldots, d,
\]
the proof will be complete. It is technically easier to prove the stronger result $E(\sqrt{n} Z_i)^2 \to 0$. Therefore, consider
\[
\sqrt{n} Z_i = \sqrt{n} \int (\hat{G} - G) d(\hat{F}_i - F_i) = \frac{1}{d} \sum_{s=1}^{d} \sqrt{n} \int (\hat{F}_s - F_s) d(\hat{F}_i - F_i)
\]
and thus
\[
E(\sqrt{n} Z_i)^2 = E \left\{ \frac{n}{d^2} \sum_{s=1}^{d} \sum_{t=1}^{d} \int (\hat{F}_s - F_s) d(\hat{F}_i - F_t) \int (\hat{F}_t - F_t) d(\hat{F}_i - F_i) \right\}.
\]

It follows with the same arguments as used by Dobler et al.\(^7\) (see equation (8)), Gao et al.\(^8\) (Lemma 2.1) or Munzel\(^9\) that
\[
E \left\{ n \int (\hat{F}_s - F_s) d(\hat{F}_i - F_t) \int (\hat{F}_t - F_t) d(\hat{F}_i - F_i) \right\} \to 0, \lambda_i \to \infty,
\]
for any fixed pair of \((s,t)\) under \((A1)\) and \((A2)\). It remains to show that

\[
\sqrt{n} \int Gd(\hat{F}_i - F_i) = \sqrt{n} \int Gd(\hat{F}_i - F_i) + o_p(1)
\]

\[
\iff \sqrt{n}(\hat{p}_i - p_i) = \sqrt{n} \left( \int Gd\hat{F}_i - \int F_idG + 1 - 2p_i \right) + o_p(1)
\]

\[
= \sqrt{n} \left( \frac{1}{\lambda_i} \sum_{k=1}^{n} \lambda_{ik} G(X_{ik}) - \int F_id \left( \frac{1}{d} \sum_{s=1}^{d} \hat{F}_s \right) + 1 - 2p_i \right) + o_p(1)
\]

\[
= \sqrt{n} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{n \lambda_{ik}}{\lambda_i} G(X_{ik}) - \frac{1}{d} \sum_{s=1}^{d} \int F_id\hat{F}_s + 1 - 2p_i \right) + o_p(1)
\]

\[
= \sqrt{n} \left( \frac{1}{n} \sum_{k=1}^{n} \frac{n \lambda_{ik}}{\lambda_i} \left( G(X_{ik}) - \frac{1}{d} F_i(X_{ik}) \right) - \frac{1}{d} \sum_{s \neq i}^{n} \frac{n \lambda_{sk}}{\lambda_s} F_i(X_{sk}) + (1 - 2p_i) \right) + o_p(1)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\Psi_{ik} - E(\Psi_{ik})) + o_P(1),
\]

which completes the proof.

### 2.3 Proof of Theorem 2

Note that \(\sqrt{n}B\) is a mean (multiplied by \(\sqrt{n}\)) of independent random variables

\[
\sqrt{n}B = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\Psi_k - E(\Psi_k)), \quad \Psi_k = (\Psi_{1k}, \ldots, \Psi_{dk})'.
\]

Furthermore, note that the random variables \(\Psi_k\) are uniformly bounded by Assumption \((A2)\) in (??). Thus, the multivariate Lindeberg Feller Theorem implies that \(\sqrt{n}B\) is asymptotically normal with covariance matrix \(\tilde{V}_n\) and the result follows from Theorem 1 and Slutsky.

### 2.4 Proof of Theorem 3

1. Let \(y = (y_1, \ldots, y_d)'\) be a vector of constants. Then it holds

\[
y'\tilde{V}_n y = y' \left\{ \frac{1}{n-1} \sum_{k=1}^{n} \left( \hat{\Psi}_k - \hat{\beta}_k \right) \left( \hat{\Psi}_k - \hat{\beta}_k \right)' \right\} y
\]

\[
= \frac{1}{n-1} \sum_{k=1}^{n} y' \left( \hat{\Psi}_k - \hat{\beta}_k \right) \left( \hat{\Psi}_k - \hat{\beta}_k \right)' y
\]

\[
= \frac{1}{n-1} \sum_{k=1}^{n} \left( \left( \hat{\Psi}_k - \hat{\beta}_k y \right)' \right)^2 \geq 0.
\]
2. For the diagonal elements, it suffices to show, that

$$\left| \frac{1}{n} \sum_{k=1}^{n} \left( (\hat{\Psi}_{ik} - \hat{\beta}_{ik})^2 - (\Psi_{ik} - E(\Psi_{ik}))^2 \right) \right| \overset{a.s.}{\longrightarrow} 0, \ n \rightarrow \infty, \forall i,$$

since

$$\frac{1}{n - 1} \sum_{k=1}^{n} (\Psi_{ik} - E(\Psi_{ik}))^2 = \frac{1}{n} \sum_{k=1}^{n} (\Psi_{ik} - E(\Psi_{ik}))^2 + O\left(\frac{1}{n}\right), \forall i.$$

First, we compute the bound of

$$|\Psi_{ik}| = \left| \frac{n \lambda_{ik}}{\lambda_i} \left\{ G(X_{ik}) - \frac{1}{d} F_i(X_{ik}) \right\} - \frac{1}{d} \sum_{s \neq i}^{n} \frac{n \lambda_{sk}}{\lambda_s} F_i(X_{sk}) \right|$$

$$= \left| \frac{1}{d} \sum_{s \neq i}^{n} \frac{n \lambda_{ik}}{\lambda_i} F_s(X_{ik}) - \frac{1}{d} \sum_{s \neq i}^{n} \frac{n \lambda_{sk}}{\lambda_s} F_i(X_{sk}) \right| \leq N_0.$$

With the same arguments, it follows $|\hat{\Psi}_{ik}| \leq N_0$, $|\beta_{ik}| \leq N_0$ and $|\hat{\beta}_{ik}| \leq N_0$.

As $n \rightarrow \infty$, it follows that

$$\left| \frac{1}{n} \sum_{k=1}^{n} \left( (\hat{\Psi}_{ik} - \hat{\beta}_{ik})^2 - (\Psi_{ik} - E(\Psi_{ik}))^2 \right) \right|$$

$$\leq \max_{k=1,\ldots,n} \left| (\hat{\Psi}_{ik} - \hat{\beta}_{ik})^2 - (\Psi_{ik} - E(\Psi_{ik}))^2 \right|$$

$$= \max_{k=1,\ldots,n} \left| (\hat{\Psi}_{ik} - \hat{\beta}_{ik} + \Psi_{ik} - E(\Psi_{ik})) (\hat{\Psi}_{ik} - \hat{\beta}_{ik} - \Psi_{ik} + E(\Psi_{ik})) \right|$$

$$\leq 4N_0 \max_{k=1,\ldots,n} \left| \hat{\Psi}_{ik} - \hat{\beta}_{ik} - \Psi_{ik} + E(\Psi_{ik}) \right|$$

$$\leq 4N_0 \max_{k=1,\ldots,n} \left| \hat{\Psi}_{ik} - \Psi_{ik} \right| + 4N_0 \max_{k=1,\ldots,n} \left| \hat{\beta}_{ik} - E(\Psi_{ik}) \right| \overset{\text{a.s.}}{\longrightarrow} 0,$$

$$\overset{(i)}{\leq} \overset{(ii)}{\leq} \left| \hat{\Psi}_{ik} - \Psi_{ik} \right| + \left| \hat{\beta}_{ik} - E(\Psi_{ik}) \right| \overset{\text{a.s.}}{\longrightarrow} 0,$$
since

\[(i) = \left| \frac{n\lambda_{ik}}{\lambda_i} \left( \hat{G}(X_{ik}) - \frac{1}{d}\hat{F}_i(X_{ik}) \right) - \frac{1}{d} \sum_{s \neq i} \frac{n\lambda_{sk}}{\lambda_s} \hat{F}_i(X_{sk}) \right| \\
- \left( \frac{n\lambda_{ik}}{\lambda_i} \left( G(X_{ik}) - \frac{1}{d}F_i(X_{ik}) \right) - \frac{1}{d} \sum_{s \neq i} \frac{n\lambda_{sk}}{\lambda_s} F_i(X_{sk}) \right) \right| \\
\leq \frac{n\lambda_{ik}}{\lambda_i} \left( |\hat{G}(X_{ik}) - G(X_{ik})| + \frac{1}{d}|\hat{F}_i(X_{ik}) - F_i(X_{ik})| \right) \\
+ \frac{1}{d} \sum_{s \neq i} \frac{n\lambda_{sk}}{\lambda_s} |\hat{F}_i(X_{sk}) - F_i(X_{sk})| \\
\leq N_0 \left( ||\hat{G} - G||_{∞} + \frac{1}{d}||\hat{F}_i - F_i||_{∞} \right) + \frac{1}{d} \sum_{s \neq i} N_0 ||\hat{F}_i - F_i||_{∞} \xrightarrow{a.s.} 0,
\]

under assumption (A2) and

\[(ii) = \left| \left( \frac{n\lambda_{ik}}{\lambda_i} \left( p_i - \frac{1}{d}p^{(ii)} \right) - \frac{1}{d} \sum_{s \neq i} \frac{n\lambda_{sk}}{\lambda_s} p^{(is)} \right) \right| \\
- \left( \frac{n\lambda_{ik}}{\lambda_i} \left( \hat{p}_i - \frac{1}{d}\hat{p}^{(ii)} \right) - \frac{1}{d} \sum_{s \neq i} \frac{n\lambda_{sk}}{\lambda_s} \hat{p}^{(is)} \right) \right| \\
\leq N_0 \left( |\hat{p}_i - p_i| + \frac{1}{d}||\hat{p}^{(ii)} - p^{(ii)}|| \right) + \frac{1}{d} \sum_{s \neq i} N_0 ||\hat{p}^{(is)} - p^{(is)}|| \xrightarrow{a.s.} 0.
\]

For the off diagonal elements, it holds that

\[
\frac{1}{n} \sum_{k=1}^{n} \left( (\hat{\Psi}_{ik} - \hat{\beta}_{ik})(\hat{\Psi}_{jk} - \hat{\beta}_{jk}) - (\Psi_{ik} - E(\Psi_{ik}))(\Psi_{jk} - E(\Psi_{jk})) \right) \Delta_{ik} \\
\leq \max_{k=1,\ldots,n} |\Delta_{ik}| \xrightarrow{a.s.} 0,
\]
since
\[ |\Delta_{ik}| = \left| (\tilde{\Psi}_{ik} - \tilde{\beta}_{ik})(\tilde{\Psi}_{jk} - \tilde{\beta}_{jk}) - (\Psi_{ik} - E(\Psi_{ik}))(\Psi_{jk} - E(\Psi_{jk})) \right| \]
\[ = \left| (\tilde{\Psi}_{ik} - \tilde{\beta}_{ik})(\tilde{\Psi}_{jk} - \tilde{\beta}_{jk}) - (\Psi_{ik} - E(\Psi_{ik}))(\Psi_{jk} - E(\Psi_{jk})) \right| + (\Psi_{jk} - E(\Psi_{jk}))\left| (\tilde{\Psi}_{ik} - \tilde{\beta}_{ik}) - (\Psi_{ik} - E(\Psi_{ik})) \right| \]
\[ \leq \left| (\tilde{\Psi}_{ik} - \tilde{\beta}_{ik})(\tilde{\Psi}_{jk} - \tilde{\beta}_{jk}) - (\Psi_{ik} - E(\Psi_{ik}))(\Psi_{jk} - E(\Psi_{jk})) \right| \]
\[ + 2N_0 \left| (\tilde{\Psi}_{jk} - \tilde{\beta}_{jk}) - (\Psi_{jk} - E(\Psi_{jk})) \right| \]
\[ \leq 2N_0 \left| (\tilde{\Psi}_{jk} - \tilde{\beta}_{jk}) - (\Psi_{jk} - E(\Psi_{jk})) \right| \]
\[ + 2N_0 \left| (\tilde{\Psi}_{ik} - \tilde{\beta}_{ik}) - (\Psi_{ik} - E(\Psi_{ik})) \right| \]
\[ \leq 2N_0 \left[ |\tilde{\Psi}_{jk} - \Psi_{jk}| + |\tilde{\beta}_{jk} - E(\Psi_{jk})| + |\tilde{\Psi}_{ik} - \Psi_{ik}| + |\tilde{\beta}_{ik} - E(\Psi_{ik})| \right] \]
\[ \xrightarrow{a.s.} 0. \]

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