The emergence of background geometry 
from quantum fluctuations

J. Ambjørn\textsuperscript{a,c} R. Janik\textsuperscript{b}, W. Westra\textsuperscript{c} and S. Zohren\textsuperscript{d}

\textsuperscript{a} The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.
email: ambjorn@nbi.dk

\textsuperscript{b} Institute of Physics, Jagellonian University,
ul. Reymonta 4, 30-059 Krakow, Poland.
email: ufrjanik@if.uj.edu.pl

\textsuperscript{c} Institute for Theoretical Physics, Utrecht University,
Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands.
email: w.westra@phys.uu.nl, ambjorn@phys.uu.nl

\textsuperscript{d} Blackett Laboratory, Imperial College,
London SW7 2AZ, United Kingdom.
email: stefan.zohren@imperial.ac.uk

Abstract

We show how the quantization of two-dimensional gravity leads to an (Euclidean) quantum space-time where the average geometry is that of constant negative curvature and where the Hartle-Hawking boundary condition arises naturally.
Introduction

Two-dimensional quantum gravity is not much of a gravity theory in the sense that there are no propagating gravitons. It has nevertheless been a fertile playground when it comes to testing various aspects of diffeomorphism-invariant theories, and it is potentially important for string theory which can be viewed as two-dimensional quantum gravity coupled to specific, conformal invariant matter fields. The 2d quantum gravity aspect has been particularly important in the study of non-critical string theories.

Most of the studies where the quantum gravity aspect has been emphasized have considered two-dimensional Euclidean quantum gravity with compact space-time. The study of 2d Euclidean quantum gravity with non-compact space-time was initiated by the Zamolodchikovs (ZZ) [1] when they showed how to use conformal bootstrap and the cluster-decomposition properties to quantize Liouville theory on the pseudo-sphere (the Poincare disk).

Martinec [2] and Seiberg et al. [3] showed how the work of ZZ fitted into framework of non-critical string theory, where the ZZ-theory could be reinterpreted as special branes, now called ZZ-branes. Let $\tilde{W}_\Lambda(\tilde{X})$ be the ordinary, so-called disk amplitude for 2d Euclidean gravity on a compact space-time. $\tilde{X}$ denotes the boundary cosmological constant of the disk and $\Lambda$ the cosmological constant. It was found that the ZZ-brane of 2d Euclidean gravity was associated with the zero of

$$W_\Lambda(\tilde{X}) = (\tilde{X} - \frac{1}{2} \sqrt{\Lambda}) \sqrt{\tilde{X} + \sqrt{\Lambda}}. \quad (1)$$

At first sight this is somewhat surprising since from a world-sheet point of view the disk is compact while the Poincare disk is non-compact. In [4] it was shown how it could be understood in terms of world sheet geometry, i.e. from a 2d quantum gravity point of view: When the boundary cosmological constant $\tilde{X}$ reaches the value $\tilde{X} = \sqrt{\Lambda}/2$ where the disk amplitude $W_\Lambda(\tilde{X}) = 0$, the geodesic distance from a generic point on the disk to the boundary diverges, in this way effectively creating a non-compact space-time.

In this article we show that the same phenomenon occurs for a different two-dimensional theory of quantum gravity called quantum gravity from causal dynamical triangulations (short: CDT) [5]. This theory has been generalized to higher dimensions where potentially interesting results have been obtained [6] using computer simulations. However, here we will concentrate on the two-dimensional theory which can be solved analytically.
CDT

The idea of CDT, i.e. quantum gravity defined via causal dynamical triangulations, is two-fold: firstly, inspired by Teitelboim [7], we insist, starting in a space-time with a Lorentzian signature, that only causal histories contribute to the quantum gravity path integral, and secondly, we assume a global time-foliation.

“Dynamical triangulation” (DT) provides a simple regularization of the sum over geometries by providing a grid of piecewise linear geometries constructed from building blocks (d-dimensional simplices if we want to construct a d-dimensional geometry, see [8, 9] for reviews). The ultraviolet cut-off is the length of the side of the building blocks. CDT uses DT as the regularization of the path integral (see [5, 6] for detailed descriptions of which causal geometries are included in the grid).

In two dimensions it is natural to study the proper-time “propagator”, i.e. the amplitude for two space-like boundaries to be separated a proper time (or geodesic distance) $T$. While this is a somewhat special amplitude, it has the virtue that other amplitudes, like the disk amplitude or the cylinder amplitude, can be calculated if we know the proper-time propagator [10, 11, 12, 5]. When the path integral representation of this propagator is defined using CDT we can further, for each causal piecewise linear Lorentzian geometry, make an explicit rotation to a related Euclidean geometry. After this rotation we perform the sum over geometries in the this Euclidean regime. This sum is now different from the full Euclidean sum over geometries, leading to an alternative quantization of 2d quantum gravity (CDT). Eventually we can perform a rotation back from Euclidean proper time to Lorentzian proper time in the propagator if needed.

In the following we will use continuum notation. A derivation of the continuum expressions from the regularized (lattice) expressions can be found in [5]. We assume space-time has the topology $S^1 \times [0, 1]$, The action (rotated to Euclidean space-time) is:

$$S[g] = \Lambda \int \int dx dt \sqrt{g(x, t)} + X \oint dl_1 + Y \oint dl_2,$$

where $\Lambda$ is the cosmological constant, $X, Y$ are two boundary cosmological constants, $g$ is a metric describing a geometry of the kind mentioned above, and the line integrals refer to the length of the boundaries, induced by $g$. The propagator $G_{\Lambda}(X, Y; T)$ is defined by

$$G_{\Lambda}(X, Y; T) = \int D[g] e^{-S[g]},$$

where the functional integration is over all “causal” geometries $[g]$ such that the “exit” boundary with boundary cosmological constant $Y$ is separated a geodesic
distance $T$ from the “entry” boundary with boundary cosmological constant $X$. As shown in [5], calculating the path integral (3) using the CDT regularization and taking the continuum limit where the side-length $a$ of the simplices goes to zero leads to the following expression:\(^{1}\):

$$G_{\Lambda}(X, Y; T) = \frac{\bar{X}^{2}(T, X) - \Lambda}{\bar{X}^{2} - \Lambda} \frac{1}{\bar{X}(T, X) + Y}, \quad (4)$$

where $\bar{X}(T, X)$ is the solution of

$$\frac{d\bar{X}}{dT} = -(\bar{X}^{2} - \Lambda), \quad \bar{X}(0, X) = X, \quad (5)$$

or

$$\bar{X}(t, X) = \sqrt{\Lambda} \coth \sqrt{\Lambda}(t + t_{0}), \quad X = \sqrt{\Lambda} \coth \sqrt{\Lambda} t_{0}. \quad (6)$$

Viewing $G_{\Lambda}(X, Y; T)$ as a propagator, $\bar{X}(T)$ can be viewed as a “running” boundary cosmological constant, $T$ being the scale. If $X > -\sqrt{\Lambda}$ then $\bar{X}(T) \to \sqrt{\Lambda}$ for $T \to \infty$, $\sqrt{\Lambda}$ being a “fixed point” (a zero of the “$\beta$-function” $-(\bar{X}^{2} - \Lambda)$ in eq. (5)).

Let $L_{1}$ denote the length of the entry boundary and $L_{2}$ the length of the exit boundary. Rather than consider a situation where the boundary cosmological constant $X$ is fixed we can consider $L_{1}$ as fixed. We denote the corresponding propagator $G_{\Lambda}(L_{1}, Y; T)$. Similarly we can define $G_{\Lambda}(X, L_{2}; T)$ and $G_{\Lambda}(L_{1}, L_{2}; T)$. They are related by Laplace transformations. For instance:

$$G_{\Lambda}(X, Y; T) = \int_{0}^{\infty} dL_{2} \int_{0}^{\infty} dL_{1} \ G(L_{1}, L_{2}; T) \ e^{-XL_{1} - YL_{2}}. \quad (7)$$

and one has the following composition rule for the propagator:

$$G_{\Lambda}(X, Y; T_{1} + T_{2}) = \int_{0}^{\infty} dL \ G_{\Lambda}(X, L; T_{1}) \ G(L, Y, T_{2}). \quad (8)$$

We can now calculate the expectation value of the length of the spatial slice at proper time $t \in [0, T]$:

$$\langle L(t) \rangle_{X, Y, T} = \frac{1}{G_{\Lambda}(X, Y; T)} \int_{0}^{\infty} dL \ G_{\Lambda}(X, L; t) \ L \ G_{\Lambda}(L, Y; T - t). \quad (9)$$

In general there is no reason to expect $\langle L(t) \rangle$ to have a classical limit. Consider for instance the situation where $X$ and $Y$ are larger than $\sqrt{\Lambda}$ and where

\(^{1}\)The asymmetry between $X$ and $Y$ is just due to the convention that the entrance boundary contains a marked point. Symmetric expressions where the the boundaries have no marked points or both have marked points can be found in [13].
$T \gg 1/\sqrt{\Lambda}$. The average boundary lengths will be of order $1/X$ and $1/Y$. But for $0 \ll t \ll T$ the system has forgotten everything about the boundaries and the expectation value of $L(t)$ is, up to corrections of order $e^{-2\sqrt{\Lambda}t}$ or $e^{-2\sqrt{\Lambda}(T-t)}$, determined by the ground state of the effective Hamiltonian $H_{\text{eff}}$ corresponding to $G_\Lambda(X,Y;T)$ (see [5] for details and [13] for a discussion of various forms of $H_{\text{eff}}$. Here we do not need the explicit expression for $H_{\text{eff}}$). One finds for this ground state $\langle L \rangle = 1/\sqrt{\Lambda}$. This picture is confirmed by an explicit calculation using eq. (9) as long as $X,Y > \sqrt{\Lambda}$. The system is thus, except for boundary effects, entirely determined by the quantum fluctuations of the ground state of $H_{\text{eff}}$.

We will here be interested in a different and more interesting situation where a non-compact space-time is obtained as a limit of the compact space-time described by (9). Thus we want to take $T \rightarrow \infty$ and at the same time also the length of the boundary corresponding to proper time $T$ to infinity. Since $T \rightarrow \infty$ forces $\bar{X}(T,X) \rightarrow \sqrt{\Lambda}$ it follows from (4) that the only choice of boundary cosmological constant $Y$ independent of $T$ where the length $\langle L(T) \rangle_{X,Y,T}$ goes to infinity for $T \rightarrow \infty$ is $Y = -\sqrt{\Lambda}$ since we have:

$$\langle L(T) \rangle_{X,Y,T} = -\frac{1}{G_\Lambda(X,Y;T)} \frac{\partial G_\Lambda(X,Y;T)}{\partial Y} = \frac{1}{X(T,X) + Y}. \quad (10)$$

With the choice $Y = -\sqrt{\Lambda}$ one obtains from (9) in the limit $T \rightarrow \infty$:

$$\langle L(t) \rangle_{X} = \frac{1}{\sqrt{\Lambda}} \sinh(2\sqrt{\Lambda}(t + t_0(X))), \quad (11)$$

where $t_0(X)$ is define in eq. (6).

We have called $L_2$ the (spatial) length of the boundary corresponding to $T$ and $\langle L(t) \rangle_{X}$ the spatial length of a time-slice at time $t$ in order to be in accordance with earlier notation [15, 5], but starting from a lattice regularization and taking the continuum limit $L$ is only determined up to a constant of proportionality which we fix by comparing with a continuum effective action. In the next section we will show that such a comparison leads to the identification of $L$ as $L_{\text{cont}}/\pi$ and we are led to the following

$$L_{\text{cont}}(t) \equiv \pi \langle L(t) \rangle_{X} = \frac{\pi}{\sqrt{\Lambda}} \sinh(2\sqrt{\Lambda}(t + t_0(X))). \quad (12)$$

Consider the classical surface where the intrinsic geometry is defined by proper time $t$ and spatial length $L_{\text{cont}}(t)$ of the curve corresponding to constant $t$. It has the line element

$$ds^2 = dt^2 + \frac{L_{\text{cont}}^2}{4\pi^2} d\theta^2 = dt^2 + \frac{\sinh^2(2\sqrt{\Lambda}(t + t_0(X)))}{4\Lambda} d\theta^2, \quad (13)$$
where \( t \geq 0 \) and \( t_0(X) \) is a function of the boundary cosmological constant \( X \) at the boundary corresponding to \( t = 0 \) (see eq. (6)). What is remarkable about the formula (13) is that the surfaces for different boundary cosmological constants \( X \) can be viewed as part of the same surface, the Poincare disk with curvature \( R = -8\Lambda \), since \( t \) can be continued to \( t = -t_0 \). The Poincare disk itself is formally obtained in the limit \( X \to \infty \) since an infinite boundary cosmological constant will contract the boundary to a point.

The classical effective action

Consider the non-local “induced” action of 2d quantum gravity, first introduced by Polyakov [14]

\[
S[g] = \int \! dt dx \sqrt{g} \left( \frac{1}{16} R_g - \frac{1}{\Delta_g} R_g + \Lambda \right),
\]

(14)

where \( R \) is the scalar curvature corresponding to the metric \( g \), \( t \) denotes “time” and \( x \) the “spatial” coordinate.

Nakayama [15] analyzed the action (14) in proper time gauge assuming the manifold had the topology of the cylinder with a foliation in proper time \( t \), i.e. the metric was assumed to be of the form:

\[
g = \begin{pmatrix} 1 & 0 \\ 0 & \gamma(t, x) \end{pmatrix}.
\]

(15)

It was shown that in this gauge the classical dynamics is described entirely by the following one-dimensional action:

\[
S_\kappa = \int_0^T \! dt \left( \frac{\dot{l}^2(t)}{4l(t)} + \Lambda l(t) + \frac{\kappa}{l} \right),
\]

(16)

where

\[
l(t) = \frac{1}{\pi} \int \! dx \sqrt{\gamma},
\]

(17)

and where \( \kappa \) is an integration constant coming from solving for the energy-momentum tensor component \( T_{01} = 0 \) and inserting the solution in (14).

Thus \( \pi l(t) \) is precisely the length of the spatial curve corresponding to a constant value of \( t \), calculated in the metric (15). The classical solutions corre-
sponding to action (16) are

\[ l(t) = \sqrt{\kappa} \sinh 2\sqrt{\Lambda} t, \quad \kappa > 0 \text{ elliptic case}, \quad (18) \]

\[ l(t) = \sqrt{-\kappa} \cosh 2\sqrt{\Lambda} t, \quad \kappa < 0 \text{ hyperbolic case}, \quad (19) \]

\[ l(t) = e^{2\sqrt{\Lambda} t}, \quad \kappa = 0 \text{ parabolic case}, \quad (20) \]

all corresponding to cylinders with constant negative curvature \(-8\Lambda\).

In the elliptic case, where \( t \) must be larger than zero, there is a conical singularity at \( t = 0 \) unless \( \kappa = 1 \). For \( \kappa = 1 \) the geometry is regular at \( t = 0 \) and this value of \( \kappa \) corresponds precisely to the Poincare disk, \( t = 0 \) being the “center” of the disk.

Nakayama quantized the actions \( S_\kappa \) for \( \kappa = (m+1)^2, m \) a non-negative integer, and for \( m = 0 \) he obtained precisely the propagator obtained by the CDT path integral approach.

**Quantum fluctuations**

In many ways it is more natural to fix the boundary cosmological constant than to fix the length of the boundary. However, one pays the price that the fluctuations of the boundary size are large, in fact of the order of the average length of the boundary itself \(^2\): from (10) we have

\[ \langle L^2(T) \rangle_{X,Y:T} - \langle L(T) \rangle_{X,Y:T}^2 = -\frac{\partial \langle L(T) \rangle_{X,Y:T}}{\partial Y} = \langle L(T) \rangle_{X,Y:T}^2. \quad (21) \]

Such large fluctuations are also present around \( \langle L(t) \rangle_{X,Y:T} \) for \( t < T \). From this point of view it is even more remarkable \( \langle L(t) \rangle_{X,Y=-\sqrt{\lambda},T=\infty} \) has such a nice semi-classical interpretation. Let us now by hand fix the boundary lengths \( L_1 \) and \( L_2 \). This is done in the Hartle-Hawking Euclidean path integral when the geometries \( [g] \) are fixed at the boundaries \([16]\). For our one-dimensional boundaries the geometries at the boundaries are uniquely fixed by specifying the lengths of the boundaries, and the relation between the propagator with fixed boundary cosmological constants and with fixed boundary lengths is given by a Laplace transformation as shown in eq. (7). Let us for simplicity analyze the situation where we take the length \( L_1 \) of the entrance loop to zero by taking the boundary cosmological constant \( X \to \infty \). Using the decomposition property (8) one can calculate the connected “loop-loop” correlator for fixed \( L_2 \) and \( 0 < t \leq t + \Delta < T \)

\[ \langle L(t)L(t+\Delta) \rangle_{L_2,T}^{(c)} \equiv \langle L(t+\Delta)L(t) \rangle_{L_2,T} - \langle L(t) \rangle \langle L(t+\Delta) \rangle_{L_2,T}. \quad (22) \]

\(^2\)This is true also in Liouville quantum theory, the derivation essentially the same as that given in (21), as is clear from \([4]\).
One finds
\[
\langle L(t)L(t+\Delta) \rangle^{(c)}_{L_2,T} = \frac{2 \sinh^2 \sqrt{\Lambda} t \sinh^2 \sqrt{\Lambda}(T-(t+\Delta))}{\Lambda} + \frac{2L_2 \sinh^2 \sqrt{\Lambda} t \sinh \sqrt{\Lambda}(t+\Delta) \sinh \sqrt{\Lambda}(T-(t+\Delta))}{\sqrt{\Lambda} \sinh^3 \sqrt{\Lambda} T}.
\]  

We also note that
\[
\langle L(t) \rangle_{L_2,T} = \frac{2 \sinh \sqrt{\Lambda} t \sinh \sqrt{\Lambda}(T-t)}{\sqrt{\Lambda} \sinh \sqrt{\Lambda} T} + L_2 \frac{\sinh \sqrt{\Lambda} t}{\sinh^2 \sqrt{\Lambda} T}.
\]  

For fixed $L_2$ and $T \to \infty$ we obtain
\[
\langle L(t)L(t+\Delta) \rangle^{(c)}_{L_2} = \frac{1}{2\Lambda} e^{-2\sqrt{\Lambda} \Delta} \left(1 - e^{-2\sqrt{\Lambda} t} \right)^2
\]  

and
\[
\langle L(t) \rangle_{L_2} = \frac{1}{\sqrt{\Lambda}} \left(1 - e^{-2\sqrt{\Lambda} t} \right).
\]

Eqs. (25) and (26) tell us that except for small $t$ we have $\langle L(t) \rangle_{L_2} = 1/\sqrt{\Lambda}$. The quantum fluctuations $\Delta L(t)$ of $L(t)$ are defined by $(\Delta L(t))^2 = \langle L(t)L(t) \rangle^{(c)}$. Thus the spatial extension of the universe is just quantum size (i.e. $1/\sqrt{\Lambda}$, $\Lambda$ being the only coupling constant) with fluctuations $\Delta L(t)$ of the same size. The time correlation between $L(t)$ and $L(t+\Delta)$ is also dictated by the scale $1/\sqrt{\Lambda}$, telling us that the correlation between spatial elements of size $1/\sqrt{\Lambda}$, separated in time by $\Delta$ falls off exponentially as $e^{-2\sqrt{\Lambda} \Delta}$.

The above picture is precisely what one would expect from the action (2): if we force $T$ to be large and choose a $Y$ such that $\langle L_2(T) \rangle$ is not large, the universe will be a thin tube, “classically” of zero width, but due to quantum fluctuations of average width $1/\sqrt{\Lambda}$.

A more interesting situation is obtained if we choose $Y = -\sqrt{\Lambda}$, the special value needed to obtain a non-compact geometry in the limit $T \to \infty$. To implement this in a setting where $L_2$ is not allowed to fluctuate we fix $L_2(T)$ to the average value (10) for $Y = -\sqrt{\Lambda}$:
\[
L_2(T) = \langle L(T) \rangle_{X,Y=-\sqrt{\Lambda},T} = \frac{1}{\sqrt{\Lambda}} \frac{1}{\coth \sqrt{\Lambda} T - 1}.
\]

From (23) and (24) we have in the limit $T \to \infty$:
\[
\langle L(t) \rangle = \frac{1}{\sqrt{\Lambda}} \sinh 2\sqrt{\Lambda} t
\]  

in accordance with (11), and for the “loop-loop”-correlator
\[
\langle L(t+\Delta)L(t) \rangle^{(c)} = \frac{2}{\Lambda} \sinh^2 \sqrt{\Lambda} t = \frac{1}{\sqrt{\Lambda}} \left( \langle L(t) \rangle - \frac{1}{\sqrt{\Lambda}} \left(1 - e^{-2\sqrt{\Lambda} t} \right) \right).
\]
It is seen that the “loop-loop”-correlator is independent of ∆. In particular we have for ∆=0:

\[(\Delta L(t))^2 \equiv \langle L^2(t) \rangle - \langle L(t) \rangle^2 \sim \frac{1}{\sqrt{\Lambda}} \langle L(t) \rangle\]  

(30)

for \(t \gg \frac{1}{\sqrt{\Lambda}}\). The interpretation of eq. (30) is in accordance with the picture presented below (26): we can view the curve of length \(L(t)\) as consisting of \(N(t) \approx \sqrt{\Lambda} L(t) \approx e^{2\sqrt{\Lambda}t}\) independently fluctuating parts of size \(1/\sqrt{\Lambda}\) and each with a fluctuation of size \(1/\sqrt{\Lambda}\). Thus the total fluctuation \(\Delta L(t)\) of \(L(t)\) will be of order \(1/\sqrt{\Lambda} \times \sqrt{N(t)}\), i.e.

\[\frac{\Delta L(t)}{\langle L(t) \rangle} \sim \frac{1}{\sqrt{\Lambda} \langle L(t) \rangle} \sim e^{-\sqrt{\Lambda}t},\]

(31)

i.e. the fluctuation of \(L(t)\) around \(\langle L(t) \rangle\) is small for \(t \gg 1/\sqrt{\Lambda}\). In the same way the independence of the “loop-loop”-correlator of ∆ can be understood as the combined result of \(L(t + \Delta)\) growing exponentially in length with a factor \(e^{2\sqrt{\Lambda}\Delta}\) compared to \(L(t)\) and, according to (25), the correlation of “line-elements” of \(L(t)\) and \(L(t + \Delta)\) decreasing by a factor \(e^{-2\sqrt{\Lambda}\Delta}\).

**Discussion**

We have described how the CDT quantization of 2d gravity for a special value of the boundary cosmological constant leads to a non-compact (Euclidean) Ads-like space-time of constant negative curvature dressed with quantum fluctuations. It is possible to achieve this non-compact geometry as a limit of a compact geometry as described above. In particular the assignment (27) leads to a simple picture where the fluctuation of \(L(t)\) is small compared to the average value of \(L(t)\). In fact the geometry can be viewed as that of the Poincare disk with fluctuations correlated only over a distance \(1/\sqrt{\Lambda}\).

Our construction is similar to the analysis of ZZ-branes appearing as a limit of compact 2d geometries in Liouville quantum gravity [4]. In the CDT case the non-compactness came when the running boundary cosmological constant \(\bar{X}(T)\) went to the fixed point \(\sqrt{\Lambda}\) for \(T \to \infty\). In the case of Liouville gravity, represented by DT (or equivalently matrix models), the non-compactness arose when the running (Liouville) boundary cosmological constant \(\bar{X}_{\text{Liouville}}(T)\) went to the value where the disk-amplitude \(W_{\Lambda}(\bar{X})=0\), i.e. to \(\bar{X} = \sqrt{\Lambda}/2\) (see eq. (1)). It is the same process in the two cases since the relation between Liouville gravity
and CDT is well established and summarized by the mapping [20]:

\[
\frac{X}{\sqrt{\Lambda}} = \sqrt{\frac{2}{3}} \sqrt{1 + \frac{X}{\sqrt{\Lambda}}},
\]

between the coupling constants of the two theories. The physical interpretation of this relation is discussed in [20, 5]: one obtains the CDT model by chopping away all baby-universes from the Liouville gravity theory, i.e. universes connected to the “parent-universe” by a worm-hole of cut-off scale, and this produces the relation (32). It is seen that \(X \to \sqrt{\Lambda}\) corresponds precisely to \(\tilde{X} \to \sqrt{\tilde{\Lambda}}/2\).

While the starting point of the CDT quantization was the desire to include only Lorentzian, causal geometries in the path integral, the result (13) shows that after rotation to Euclidean signature this prescription is in a natural correspondence with the Euclidean Hartle-Hawking no-boundary condition since all of the geometries (13) have a continuation to \(t = -t_0\) where the space-time is regular. It would be interesting if this could be promoted to a general principle also in higher dimensions. The computer simulations reported in [6] seems in accordance with this possibility.

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\(3\)The relation (32) is similar to the one encountered in regularized bosonic string theory in dimensions \(d \geq 2\) [17, 18, 19]: the world sheet degenerates into so-called branches polymer. The two-point function of these branched polymers is related to the ordinary two-point function of the free relativistic particle by chopping off (i.e. integrating out) the branches, just leaving for each branched polymer connecting two points in target space one path connecting the two points. The mass-parameter of the particle is then related to the corresponding parameter in the partition function for the branched polymers as \(X/\sqrt{\Lambda}\) to \(\tilde{X}/\sqrt{\tilde{\Lambda}}\) in eq. (32).
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