CALABI–YAU COMPLETE INTERSECTIONS IN EXCEPTIONAL GRASSMANNIANS

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Abstract. We classify completely reducible equivariant vector bundles on Grassmannians of exceptional Lie groups which give Calabi–Yau 3-folds as complete intersections. We also calculate Hodge numbers for those Calabi–Yau 3-folds.

1. Introduction

A smooth projective manifold is said to be Calabi–Yau if the canonical bundle is trivial. Calabi–Yau manifolds have attracted attentions from both mathematicians and string theorists, not only because of their importance in the classification of algebraic varieties, but also because of their relation with string theory and mirror symmetry. A Calabi–Yau manifold in dimensions at most two are either an elliptic curve, an abelian surface, or a K3 surface. In dimensions greater than two, it is not known whether the number of deformation equivalence classes (or even homeomorphism types) of Calabi–Yau manifolds is finite or not.

In this paper, we study Calabi–Yau 3-folds in rational homogeneous spaces of exceptional types. The main result is the following:

Theorem 1.1. A complete intersection Calabi–Yau 3-fold of a globally generated completely reducible equivariant vector bundle $E$ on an exceptional Grassmannian $G/P$, which is not a complete intersection of line bundles on a projective space, is one of those appearing in Table 1.1.

| No. | $G/P$ | $E$ | $h^{1,1}$ | $h^{1,2}$ | deg | $c_2$ |
|-----|-------|------|----------|----------|------|------|
| 1   | $\langle 1,1 \rangle$ | $1$ | $50$ | $42$ | $84$ |
| 2   | $\langle 1,1 \rangle$ | $1$ | $50$ | $14$ | $56$ |
| 3   | $(1,0,0,0,0;0) \oplus (0,0,0,0,1;0)^{\oplus 4}$ | $1$ | $31$ | $192$ | $132$ |
| 4   | $(1,0) \oplus (2,0)$ | $1$ | $61$ | $36$ | $84$ |

Table 1.1. Complete intersection Calabi–Yau 3-folds in exceptional Grassmannians

In particular, there is no such Calabi–Yau 3-fold in exceptional Grassmannians of types $E_7$, $E_8$, and $F_4$.

In Table 1.1 we label the simple roots of $E_6$ as

\[ \alpha_1 \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_4 \Rightarrow \alpha_5 \Rightarrow \alpha_6 \]
and write the coordinates of a weight $\lambda = \lambda_1 \omega_1 + \cdots + \lambda_6 \omega_6$ with respect to the corresponding fundamental weights $\omega_1, \ldots, \omega_6$ as $(\lambda_1, \ldots, \lambda_5; \lambda_6)$. Similarly, we label the simple roots as

and write $\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2 = (\lambda_1, \lambda_2)$ for $G_2$. A weight $\lambda$ is identified with the equivariant vector bundle associated with the irreducible representation of the Levi subgroup with highest weight $\lambda$. The degree and the second Chern number are with respect to the restriction of the ample generator of the Picard group of the ambient space.

Note that the classification itself in Theorem 1.1 was already obtained in [Ben18]. The Hodge numbers are newly calculated in this work, and are also reverified in [LP, Table 2] shortly after that.

The $G_2$-Grassmannian $\mathbb{G}(1,7)$ is the zero locus of the section $s \in H^0(Q(1)) \cong \bigwedge^3 \mathbb{C}^7$ corresponding to the $G_2$-invariant 3-form, and $E(1,1)$ is the restriction of $S(1)$, where $S$ and $Q$ are the universal subbundle and the universal quotient bundle on $Gr(2,7)$. Hence No. 1 in Table 1.1 is the same as No. 16 in [LM19, Table 1], which is known by [LM19, Proposition 5.1] to be deformation-equivalent to the intersection of the image of $Gr(2,7)$ and a linear subspace of codimension 7 in $\mathbb{P}(\bigwedge^2 \mathbb{C}^7)$.

The $G_2$-Grassmannian $\mathbb{G}(1,7)$ is a smooth quadric hypersurface in $\mathbb{P}^6$. Calabi–Yau 3-folds contained in a (not necessarily smooth) quadric 5-fold are classified in [KK16, Section 5], and it is shown in [KK16, Theorem 7.1] that No. 2 in Table 1.1 is deformation-equivalent to the Pfaffian Calabi–Yau 3-fold appearing in [Rød00].

Although the families No. 1 and No. 2 were known, their relation with $G_2$-Grassmannians were new, and led to the discoveries of an L-equivalence [MOU19], a derived equivalence [Kuz18], and a 7-fold flop [Ued19].

To the best of our knowledge, no known Calabi–Yau 3-fold has the same topological invariants as No. 3 in Table 1.1. The restriction $O_X(1)$ of the ample generator of the ambient space is primitive since

$$\chi(O_X(t)) = 32t^3 + 11t$$

by the Hirzebruch–Riemann–Roch theorem (4.8).

Families of Calabi–Yau 3-folds described as complete intersections of line bundles on the $G_2$-Grassmannian $\mathbb{G}(1,7)$ are omitted in Theorem 1.1 since they are complete intersections of line bundles in $\mathbb{P}^6$.

Calabi–Yau complete intersection 3-folds of completely reducible equivariant bundles on the Cayley plane containing $E_{\omega_5}$ as a direct summand turn out to be the empty set, since the zero locus of a general section of $E_{\omega_5}$ on the Cayley plane is the empty set because $h^0(O_X) = 0$.

The proof of Theorem 1.1 also shows the following:

**Theorem 1.2.** There is no complete intersection Fano 4-fold of a globally generated completely reducible equivariant vector bundle on an exceptional Grassmannian, which is neither a complete intersection of line bundles on a projective space nor a hypersurface of the $G_2$-Grassmannian $\mathbb{G}(1,7)$.

We also classify a class of Calabi–Yau 3-folds in flag varieties of exceptional types.

**Theorem 1.3.** A globally generated completely reducible equivariant vector bundle $E$ on an exceptional flag variety $G/P$ of Picard number greater than one satisfying $\text{rank } E = \dim G/P - 3$ and $c_1(E) = c_1(G/P)$ is one of those appearing in Table 1.2.
Table 1.2. Complete intersection Calabi–Yau 3-folds in exceptional flag manifolds

| No. | $G/P$ | $\mathcal{E}$ | $h^{1,1}$ | $h^{1,2}$ |
|-----|-------|---------------|-----------|-----------|
| 5   | $\blacklozenge$ | $(1,0) \oplus (0,1) \oplus (1,1)$ | 2 | 48 |
| 6   | $\blacklozenge$ | $(1,0) \oplus (0,1)$ | 2 | 38 |
| 7   | $\blacklozenge$ | $(2,0) \oplus (0,1)^\oplus$ | 2 | 58 |

The condition $c_1(\mathcal{E}) = c_1(G/P)$ is sufficient for the zero locus $X$ to be Calabi–Yau. This condition will be necessary if the restriction $\text{Pic } G/P \to \text{Pic } X$ is injective, and it is an interesting problem to decide when this is the case.

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2. Equivariant vector bundles over $G/P$

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $r$. The corresponding simply-connected Lie group is denoted by $G$. Fix a Cartan subgroup $H \subset G$ with the associated Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and set

$$\mathfrak{g}_\alpha := \{ v \in \mathfrak{g} \mid [h, v] = \alpha(h)v \text{ for any } h \in \mathfrak{h} \} \quad (2.1)$$

for $\alpha \in \mathfrak{h}^\vee := \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C})$. One has the root decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad (2.2)$$

where

$$\Delta := \{ \alpha \in \mathfrak{h}^\vee \mid \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0 \}. \quad (2.3)$$

We choose a system of simple roots $\mathcal{S} := \{ \alpha_1, \ldots, \alpha_r \} \subset \Delta$. This choice is equivalent to the choice of the sets $\Delta^+$ and $\Delta^-$ of positive and negative roots.

The Dynkin diagram of $\mathfrak{g}$ is a graph whose nodes correspond to the simple roots $\alpha_i \in \mathcal{S}$ and whose edges represent the Cartan integers $\langle \alpha_i, \alpha_j^\vee \rangle$, where $\langle -, - \rangle$ is the Killing form on $\mathfrak{h}^\vee$ and $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$. One has $\langle \alpha_i, \alpha_i^\vee \rangle = 2$ for all $\alpha_i \in \mathcal{S}$, and the correspondence between edges and
the Cartan integers is given by

\[
\begin{align*}
\alpha \beta & \iff \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0, \\
\alpha \beta & \iff \langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle = -1, \\
\alpha \beta & \iff \langle \alpha, \beta^\vee \rangle = -2, \quad \langle \beta, \alpha^\vee \rangle = -1, \\
\alpha \beta & \iff \langle \alpha, \beta^\vee \rangle = -3, \quad \langle \beta, \alpha^\vee \rangle = -1.
\end{align*}
\] (2.4)

(2.5)

(2.6)

(2.7)

A subgroup \( P \) of \( G \) is said to be parabolic if \( G/P \) is a projective variety. Conjugacy classes of parabolic subgroups are in one-to-one correspondence with subsets \( S_p \subset S \) of the set of simple roots in such a way that the corresponding subalgebra \( p \) is given by

\[ p := l \oplus n, \] (2.8)

where the Levi part \( l \) is

\[ l := h \oplus \bigoplus_{\alpha \in (\text{span } S_p) \cap \Delta} g_{\alpha}, \] (2.9)

and the nilpotent part \( n \) is

\[ n := \bigoplus_{\alpha \in \Delta^+ \setminus \text{span } S_p} g_{\alpha}. \] (2.10)

Here, \( \text{span } S_p \subset h^\vee \) is the linear subspace spanned by \( S_p \). The subset \( S_p \subset S \) can be described by a crossed Dynkin diagram, where elements not in \( S_p \) are crossed out (i.e., elements of \( S_p \) correspond to uncrossed nodes). The inclusion relation of \( S_p \) corresponds to the inclusion relation of \( P \). For example, the Borel subgroup is the minimal parabolic subgroup, so that all the nodes are crossed out in the corresponding crossed Dynkin diagram. We write the Weyl group of \( l \) as \( W_P \), which is the subgroup of \( W = W_G \) generated by simple reflections associated with elements of \( S_p \). One has

\[ \dim G/P = \#(\Delta^- \setminus \text{span } S_p) = \dim G/B - \dim G'/B', \] (2.11)

where \( G'/B' \) is the full flag variety corresponding to the full subgraph of the Dynkin diagram of \( G \) consisting of uncrossed nodes.

A finite-dimensional representation \( V \) of the parabolic subalgebra \( p \) naturally carries a filtration

\[ V = F^1V \supset \cdots \supset F^jV \supset F^{j+1}V \supset \cdots \supset F^sV \supset F^{s+1} = F^{s+2} = \cdots = 0 \] (2.12)

for some \( s \), where we set

\[ F^jV = n^{j-1} \cdot V \] (2.13)

for all \( j \). The nilpotent part \( n \) acts trivially on each quotient

\[ V_j := F^jV/F^{j+1}V, \] (2.14)

so that one may regard \( V_j \) as a representation of the Levi subalgebra \( l \). Since \( l \) is reductive, any finite-dimensional representation of \( l \) is completely reducible, i.e., the direct sum of irreducible representations. We write

\[ V = \sum_j V_j = V_1 + V_2 + \cdots + V_s, \] (2.15)
where \( F^j V = V_j + \cdots + V_s \) is a sub-representation. Let \( W = \sum_j W_j \) be another finite-dimensional representation of \( p \). It is easy to see

\[
V \oplus W = \sum_{j \geq 1} (V_j \oplus W_j),
\]

\[
V \otimes W = \sum_{l \geq 2} \bigoplus_{j+k=l} (V_j \otimes W_k),
\]

\[
\wedge V = \sum_{k \geq p} \bigoplus_{j \geq 1} \wedge^p V_j,
\]

\[
\text{Sym}^p V = \sum_{k \geq p} \bigoplus_{h(p)=k} \text{Sym}^p_j V_j,
\]

for any positive integer \( p \), where \( p = (p_1, \ldots, p_r) \) runs over partitions of \( p \) and \( h(p) = \sum_j j p_j \).

Irreducible representations of the parabolic subalgebra \( p \) correspond bijectively to those of the Levi subalgebra \( l \), which are highest weight representations. Since \( g \) and \( l \) share the same Cartan subalgebra, weights of \( l \) can be regarded naturally as weights of \( g \). Since \( G \) and \( L \) share the same Cartan subgroup, the notion of integrality of weights is the same for both \( g \) and \( l \). The fundamental weight associated with the simple root \( \alpha_i \in \mathfrak{S} \) is denoted by \( \omega_i \); \n
\[
\langle \omega_i, \alpha_j^* \rangle = \delta_{ij}, \quad i, j = 1, \ldots, r.
\]

A weight \( \lambda = \sum_{i=1}^r \lambda_i \omega_i \) is
- \textit{integral} if \( \lambda_i \in \mathbb{Z} \) for any \( i = 1, \ldots, r \),
- \textit{\( p \)-dominant} if \( \lambda_i \in \mathbb{N} \) for any \( i \) such that \( \alpha_i \in \mathfrak{S}_p \), and
- \textit{\( g \)-dominant} if \( \lambda_i \in \mathbb{N} \) for any \( i = 1, \ldots, r \).

A highest weight representation of \( I \) integrates to a representation of \( P \) if and only if the highest weight is integral and \( p \)-dominant (see e.g. [BE89, Remark 3.1.6]). The irreducible representations of \( P \) and \( G \) with highest weight \( \lambda \) are denoted by \( V^P_\lambda \) and \( V^G_\lambda \) respectively.

The category of equivariant vector bundles on \( G/P \) is equivalent to that of representations of \( P \). For a representation \( V \) of \( P \), the corresponding equivariant vector bundle on \( G/P \) is denoted by

\[
\mathcal{E}_V := G \times_P V.
\]

We write the filtration of an equivariant bundle corresponding to the filtration \((2.15)\) as

\[
\mathcal{E} = \sum_{j \geq 1} \mathcal{E}_j = \mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_s.
\]

For an irreducible equivariant vector bundle, we set

\[
\mathcal{E}_\lambda := \mathcal{E}_\lambda^\vee.
\]

which is globally generated if and only if \( \lambda \) is \( g \)-dominant. The Borel–Weil–Bott theorem gives an isomorphism

\[
H^\ell(w)(G/P, \mathcal{E}_\lambda) \cong (V^G_{w, \lambda})^\vee
\]

of \( G \)-vector spaces, where \( \rho := 1/2 \sum_{\alpha \in \Delta^+} \alpha \) is the Weyl vector,

\[
w.\lambda := w(\lambda + \rho) - \rho
\]
is the affine Weyl action, $w \in W_P$ is the unique element such that $w, \lambda$ is $g$-dominant (the left hand side of (2.24) is zero if there is no such $w$), and $\ell(w)$ is the length of (the minimal representative in $W$ of) $w \in W_P$. The filtration (2.22) gives a spectral sequence
\[ E_1^{pq} = H^{p+q}(G/P, \mathcal{E}_p) \Rightarrow H^{p+q}(G/P, \mathcal{E}), \tag{2.26} \]
which allows us to compute the cohomology of $\mathcal{E}$ using the Borel–Weil–Bott theorem.

The Picard group $\text{Pic} G/P$ is isomorphic to the group $\text{Hom}(P, \mathbb{C}^\times)$ of characters of $P$. The set of weights of a representation $V$ of $P$ is denoted by $\Delta(V)$. One has
\[ \text{rank} \mathcal{E}_V = \dim V = |\Delta(V)|, \quad \text{det} \mathcal{E}_V \cong \mathcal{E}_{\text{det} V}, \quad \text{and} \quad \Delta(\text{det} V) = \left\{ \sum_{\lambda \in \Delta(V)} \lambda \right\}. \tag{2.27} \]

The tangent bundle $T_{G/P}$ corresponds to the $P$-vector space $g/p$ with respect to the adjoint action;
\[ T_{G/P} \cong \mathcal{E}_{g/p}. \tag{2.28} \]
Since the weights of the adjoint action are roots, one has
\[ \Delta(g/p) = \Delta(g) \setminus \Delta(p) = \Delta^- \setminus \text{span} S_p. \tag{2.29} \]
Although the tangent bundle $T_{G/P}$ and hence the cotangent bundle $\Omega^1_{G/P}$ are indecomposable if $G$ is simple, they are not irreducible unless $G/P$ is a Hermitian symmetric space [Ise60, Theorem 6]. Instead, it carries the following filtration. Recall that the height of a positive root is the sum of the coefficients of the simple roots.

**Lemma 2.1** (cf. e.g. [BE89, Section 9.9]). For a complex semisimple simply-connected Lie group $G$ and a parabolic subgroup $P \subset G$, the cotangent bundle has the filtration
\[ \Omega^1_{G/P} = \sum_{j \geq 0} \bigoplus_{|n|=j} \mathcal{E}_{-\alpha(n)}, \tag{2.30} \]
where $n = (n_\alpha)_{\alpha}$ runs over the image of the map
\[ \pi : \Delta ((g/p)^\vee) = \Delta^+ \setminus \text{span} S_p \to \mathbb{Z}^{S \setminus S_p} \tag{2.31} \]
taking the coefficients of the roots in $S \setminus S_p$, $|n| := \sum_{\alpha \in S \setminus S_p} n_\alpha$, and $\alpha(n) \in \Delta^+ \setminus \text{span} S_p$ is the unique element of minimal height such that $\pi(\alpha) = n$.

**Sketch of proof.** Since the coadjoint action of $n$ on $(g/p)^\vee \subset g^\vee$ increases $|n|$, it suffices to show the decomposition for each graded component. For each $n \in \text{Im} \pi$, the root $\alpha(n)$ is uniquely determined, since the existence of distinct positive roots $\beta_1, \beta_2$ with the same height contradicts the fact that the difference $\beta_1 - \beta_2$ must be a (positive or negative) root [Hum72, Lemma 9.4]. For any $\beta \in \pi^{-1}(n)$, the difference $\alpha(n) - \beta$ is a root in $\text{span} S_p$, and the ladder operator of the corresponding $sl_2$-triple sends $\mathbb{C}(-\alpha(n)) \subset (g/p)^\vee \subset g^\vee$ onto $\mathbb{C}(-\beta)$ by the coadjoint action [Hum72, Proposition 8.4]. This implies the existence of the filtration (2.30).

3. **Complete Intersections of Equivariant Vector Bundles**

Let $\mathcal{E} := \mathcal{E}_V$ be the equivariant vector bundle on $F := G/P$ associated with a representation $V$ of $P$. Assume that $\mathcal{E}$ is globally generated. For a general section $s$ of $\mathcal{E}$, the zero locus $X := s^{-1}(0)$ is a smooth complete intersection by a generalization of the theorem of Bertini [Muk92, Theorem 1.10].
Since $X$ is a complete intersection, the differential $ds$ of the section $s$ induces an isomorphism

$$N_{X/F} \cong \mathcal{E}|_X.$$  \hspace{1cm} (3.1)

By taking the determinant of the exact sequence

$$0 \to T_X \to T_F|_X \to N_{X/F} \to 0,$$  \hspace{1cm} (3.2)

one obtains an isomorphism

$$\text{det } T_X \cong \text{det } T_F|_X \otimes \text{det}^{-1} \mathcal{E}|_X.$$  \hspace{1cm} (3.3)

Hence $\text{det } V \cong \text{det } (\mathfrak{g}/\mathfrak{p})$ is a sufficient condition for $\text{det } T_X \cong \mathcal{O}_X$, which is necessary if the restriction map $\text{Pic } F \to \text{Pic } X$ is injective.

The exact sequence

$$0 \to \text{Sym}^j \mathcal{E}^\vee|_X \to \cdots \to \text{Sym}^{j-k} \mathcal{E}^\vee \otimes \Omega^k_F|_X \to \cdots \to \Omega^j_F|_X \to 0$$  \hspace{1cm} (3.4)

obtained as the $j$-th exterior power of the exact sequence

$$0 \to \mathcal{E}^\vee|_X \to \Omega^1_F|_X \to \Omega^1_X \to 0$$  \hspace{1cm} (3.5)

dual to (3.2) gives the spectral sequence

$$E_{1}^{-q,p} = H^p \left( \text{Sym}^q \mathcal{E}^\vee \otimes \Omega^{j-q}_F|_X \right) \Rightarrow H^{p-q} \left( \Omega^j_X \right).$$  \hspace{1cm} (3.6)

The Koszul resolution

$$0 \to \wedge^{\text{rank } \mathcal{E}} \mathcal{E}^\vee \to \cdots \to \mathcal{E}^\vee \to \mathcal{O}_F \to \mathcal{O}_X \to 0$$  \hspace{1cm} (3.7)

gives the spectral sequence

$$E_{1}^{-q,p} = H^p \left( \wedge^q \mathcal{E}^\vee \otimes \mathcal{G} \right) \Rightarrow H^{p-q} \left( \mathcal{G}|_X \right)$$  \hspace{1cm} (3.8)

for any coherent sheaf $\mathcal{G}$ on $F$.

Together with the Hodge symmetry $h^{p,q}(X) = h^q,p(X)$ and the obvious fact that $h^{p,q}(X) = 0$ unless $0 \leq p \leq \dim X$, the Hodge numbers are often determined only from dimensions of the cohomology groups on the $E_1$-page, although there are cases where one should look at morphisms more carefully.

The topological Euler number $\chi(X)$ can be computed by

$$\chi(X) = \int_X \frac{c(T_X)}{c(T_F)} = \int_F \frac{c(T_F)}{c(\mathcal{E})} c_{\text{top}}(\mathcal{E}),$$  \hspace{1cm} (3.9)

where $c(\mathcal{G})$ and $c_{\text{top}}(\mathcal{G})$ denote the total and the top Chern classes. The first equality in (3.9) is the Chern–Gauss–Bonnet theorem, and the second equality comes from (3.1) and (3.2). From the splitting principle, it follows

$$\chi(X) = \int \prod_{\mu \in \Delta(\mathfrak{g}/\mathfrak{p}^\vee)} \left( 1 + c_1(\mathcal{L}_\mu) \right) \prod_{\nu \in \Delta(V^\vee)} \left( 1 + c_1(\mathcal{L}_\nu) \right) c_1(\mathcal{L}_\lambda),$$  \hspace{1cm} (3.10)

where $\mathcal{L}_{\lambda} = \mathcal{E}^\vee_{V^\lambda}$ is a line bundle on $G/B$ for any weight $\lambda$, and the integrand is an element of $H^*(F, \mathbb{Z})$ considered as a subgroup of $H^*(G/B, \mathbb{Z})$ by the pull-back along the natural projection $G/B \to F$; an element in $H^*(F, \mathbb{Z})$ is described as a polynomial in $x_i := c_1(\mathcal{L}_{\omega_i}) \in H^2(G/B, \mathbb{Z})$ for $i = 1, \ldots, r$. Note that $c_1(\mathcal{L}_\lambda) = \sum_{i=1}^r \lambda_i x_i$. One can perform the integral in terms of representation theory by using the following Lemma 3.1.
Lemma 3.1 (cf. e.g. [BE89, Lemma 6.3.2]). For a monomial \( x_1x_2 \ldots x_k \in H^2(G/B, \mathbb{Z}) \) of degree \( l \) and a Schubert cycle \([X_w] := [BwB/B] \in H_2(G/B, \mathbb{Z})\) associated with \( w \in W \) of length \( l \), one has

\[
x_1x_2 \ldots x_k[X_w] = \sum \langle \lambda_i, \beta_1 \rangle \langle \lambda_j, \beta_2 \rangle \cdots \langle \lambda_k, \beta_l \rangle,
\]

where the sum runs over all collections \( \beta_1, \ldots, \beta_l \in \Delta^+ \) such that \( w = \sigma_{\beta_1} \sigma_{\beta_2} \cdots \sigma_{\beta_l} \) is a reduced expression, and the parentheses on the right-hand side denote the symmetrized product.

4. Rational homogeneous spaces of exceptional types

Let \( G \) be the complex simple Lie group of exceptional type. We call a rational homogeneous space \( G/P \) an exceptional flag variety, or an exceptional Grassmannian if \( P \) is maximal.

For type \( G_2 \), there are three homogeneous spaces \( G/P_1, G/P_2 \) and \( G/B \) associated with the crossed Dynkin diagrams \( \Xi \), \( \Xi \) and \( \Xi \) respectively. The sets of weights of the irreducible representations \( V_{(a,b)}^P \) with the highest weight \( (a,b) := a\omega_1 + b\omega_2 \) are given by

\[
\begin{align*}
\Delta \left( V_{(a,b)}^{P_1} \right) &= \{(a+j,b-2j) \mid j = 0,1,\ldots,b \}, \\
\Delta \left( V_{(a,b)}^{P_2} \right) &= \{(a-2j,b+3j) \mid j = 0,1,\ldots,a \}, \\
\Delta \left( V_{(a,b)}^{B} \right) &= \{(a,b)\}.
\end{align*}
\]

In particular, the dimensions and the determinants of these representations are given as follows:

| representation  | dimension | determinant               |
|-----------------|-----------|---------------------------|
| \( V_{(a,b)}^{P_1} \) | \( b+1 \) | \( (a(b+1) + b(b+1)/2,0) \) |
| \( V_{(a,b)}^{P_2} \) | \( a+1 \) | \( (0, (a+1)b + 3(a+1)/2) \) |
| \( V_{(a,b)}^{B} \)  | \( 1 \)  | \( (a,b) \)               |

By considering the action of the nilpotent parts on the roots, one can directly observe that the representations \( (\mathfrak{g}/\mathfrak{p})^\vee \) are given by

\[
\begin{align*}
(\mathfrak{g}/\mathfrak{p}_1)^\vee &\cong V_{-(1,-3)}^{P_1} + V_{(1,0)}^{P_1}, \\
(\mathfrak{g}/\mathfrak{p}_2)^\vee &\cong V_{(1,-1)}^{P_2} + V_{(0,1)}^{P_2} + V_{(1,0)}^{P_2}, \\
(\mathfrak{g}/\mathfrak{b})^\vee &\cong V_{(2,-3)}^{B} \oplus V_{(1,-2)}^{B} + V_{(1,-1)}^{B} + V_{(0,1)}^{B} + V_{(1,0)}^{B} + V_{(1,-3)}^{B} + V_{(1,0)}^{B},
\end{align*}
\]

which agree with the formula for the cotangent bundles in Lemma 2.3. The determinants are given by

\[
\begin{align*}
\det (\mathfrak{g}/\mathfrak{p}_1)^\vee &\cong V_{(3,0)}^{P_1}, \\
\det (\mathfrak{g}/\mathfrak{p}_2)^\vee &\cong V_{(0,3)}^{P_2}, \\
\det (\mathfrak{g}/\mathfrak{b})^\vee &\cong V_{(2,2)}^{B}.
\end{align*}
\]

Theorem 1.1 for \( G_2 \)-Grassmannians is an easy consequence of these facts. For example, to obtain a Calabi–Yau 3-fold from \( (G/P_1, \mathcal{E}_V^\vee) \), the completely reducible representation \( V \) must satisfy \( \dim V = 2 \) and \( \det V = V_{(3,0)}^{P_1} \) since \( \text{Pic} G/P_1 \cong \mathbb{Z} \) must inject to \( \text{Pic} X \). If \( V \) is decomposable,
then $V \cong V_{(1,0)}^{P_1} \oplus V_{(2,0)}^{P_1}$ is the only choice, and if $V$ is indecomposable, then $V \cong V_{(1,1)}^{P_1}$ is the only choice.

For the Calabi–Yau 3-fold $X$ associated with $(G/P_1, \mathcal{E}_{(1,1)})$, one can use the spectral sequences (2.26), (3.6), and (3.8) to prove $h^{0,1} = h^{0,2} = 0$, $h^{1,1} = 1$, and $h^{1,2} = 50$ by hand. The Koszul resolution (3.7) allows us to compute the cohomology of $\mathcal{O}_X(i)$, which together with the Hirzebruch–Riemann–Roch theorem

$$\chi(\mathcal{O}_X(i)) = \frac{1}{6} \deg X \cdot i^3 + \frac{1}{12} c_2(X) \cdot i$$

implies $\deg X = 42$ and $c_2(X) = 84$.

Similar calculations aided by a Mathematica package [FKS20] give Theorems 1.1, 1.2, and 1.3. The conditions rank $\mathcal{E} = \dim G/P - 3$ and $c_1(\mathcal{E}) = c_1(G/P)$ are strong, and many of 439 exceptional flag varieties are eliminated quickly. The topological invariants are calculated by using spectral sequences (2.26), (3.6), (3.8), and the Chern–Gauss–Bonnet theorem (3.10) in one case. All calculations are recorded in the ancillary files to this paper.

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