ON DIMENSION AND REGULARITY OF BUNDLE MEASURES

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Abstract. The paper contains theorems connecting antisymmetry of Fourier transforms of certain vector valued measures with their dimensional properties. Our results are expressed in terms of a geometric condition related to Uchiyama’s theorem. We also enclose a result concerning regularity of investigated class: we show that its elements disappear on 1-purely unrectifiable sets.

1. Introduction

Geometric structure and dimensional properties of distributional gradients of functions from $BV(\mathbb{R}^n)$ are well studied. It is known, for example, that their lower Hausdorff dimension is at least $n-1$ and that it is an optimal bound. Moreover, $(n-1)$-dimensional part, if exists, can be exhausted by a countably $(n-1)$-rectifiable set (see Lemma 3.76 and Theorem 3.78 in [AFP]). For the class of bundle measures, which generalizes aforementioned example (see [RW] and [SW]), we can consider analogous problems.

Definition 1.1. By $G(m, E)$ let us denote Grassmannian of $m$-dimensional subspaces of some fixed, finitely dimensional vector space $E$. We call a bundle any continuous function $\phi : \mathbb{R}^n \setminus \{0\} \to G(m, E)$. If additionally $\phi(a\xi) = \phi(\xi)$ for any positive $a$, then we refer to it as a homogenous bundle.

Above setting gives us a possibility to define bundle measures by imposing Fourier analytic rigidity conditions:

Definition 1.2. For any homogenous bundle $\phi$, by $M_\phi(\mathbb{R}^n, E)$ we denote set of vector measures taking values in $E$ such that $\hat{\mu}(\xi) \in \phi(\xi)$ for $\xi \neq 0$. (For convenience, when it is known from the context, we sometimes neglect $E$.)

In this article, unless it is explicitly stated, we limit our interest to the case of line bundles ($m = 1$). Due to the presence of Schwartz functions in proofs, in later chapters we also assume that bundles are $C^\infty$.

Our main result links antisymmetry of a bundle with dimension of vector measures.

Definition 1.3. By lower Hausdorff dimension of a (scalar or vector) measure $\mu$ we understand

$$\dim_H(\mu) = \inf\{\alpha : \exists F - \text{Borel set}, \mu(F) \neq 0 \dim F \leq \alpha\}.$$
Definition 1.4. We say that a nonconstant line bundle \( \phi \) is antisymmetric on \( k \)-dimensional subspheres or, simply, \( k \)-antisymmetric \((k = 0, 1, \ldots, n-1)\), if for each \((k+1)\)-dimensional subspace \( V \subset \mathbb{R}^n \) there exist \( \xi_1, \xi_2 \in V \cap S^{n-1} \) such that \( \phi(\xi_1) \neq \phi(\xi_2) \). Denote
\[
a(\phi) = \min\{k : \phi \text{ is } k \text{-antisymmetric}\}.
\]
Under assumption that coordinates of \( \mu \) are \( k \)-rectifiable (with common \( k \)), we prove below claim:

Conjecture 1.5. If \( \mu \) is a bundle measure subordinated to a smooth bundle \( \phi \), then
\[
\dim_H(\mu) \geq n - a(\phi).
\]

Our method works in greater generality and in fact we obtain estimate of some kind of rectifiable dimension (see Theorem 3.5 and Corollary 3.6).

The statement is inspired by Uchiyama’s theorem on multiplier characterization of Hardy spaces (see Chapter 3), which gives a proof when \( a(\phi) = 0 \). Above question originated from the paper \( \text{[RW]} \). Let us invoke some of its ingredients:

Theorem 1.6. Let \( \phi \) be a nonconstant, line bundle, Hölder with exponent \( > \frac{1}{2} \). Then \( \dim_H(\mu) \geq 1 \) for each \( \mu \in M_\phi(\mathbb{R}^n) \).

Conjecture 1.7. If a Fourier transform of a bundle measure \( \mu \) contains \( n \) linearly independent vectors and \( \mu \in M_\phi(\mathbb{R}^n) \) for some line bundle \( \phi \), then \( \dim_H(\mu) \geq n - 1 \).

Theorem 1.6 covers the case \( a(\phi) = n - 1 \). Conjecture 1.7 was inspired by the example of measures from \( \text{BV} \); they are subordinated to the bundle \( \phi(\xi) = \text{span}\{\frac{\xi}{|\xi|}\} \). Let us mention that a particular case of the main result from \( \text{[SW]} \) is a proof of above conjecture for measures from homogenous Sobolev spaces \( \phi(\xi) = \text{span}\{\frac{\xi}{|\xi|}\}^m \) for any natural \( m \). For such bundles we have \( a(\phi) = 1 \).

Another novelty is certain improvement of the bound from Theorem 1.6 to \( \frac{n}{2} \) for general bundles. However, unlike above, we strongly rely on a global assumption on rectifiability.

The paper contains also a small step towards geometric characterization of bundle measures: we present some description of possible directions of their values on singularities (Theorem 1.2). Also, we show that such measures disappear on 1-purely unrectifiable sets (Theorem 5.3).

1.1. Conventions. By \( M(\mathbb{R}^n) \) we understand the set of finite Radon measures. For \( f \in L^2(\mathbb{R}^n) \) and \( \mu \in M(\mathbb{R}^n) \) we choose the following normalization of the Fourier transform:
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} f(x) dx,
\]
ON DIMENSION AND REGULARITY OF BUNDLE MEASURES

\[ \hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x \rangle} d\mu(x). \]

In the paper we use below definition of rectifiability

**Definition 1.8.** A set \( E \subset \mathbb{R}^n \) is called \( k \)-rectifiable, if there exist Lipschitz functions \( f_i : \mathbb{R}^k \to \mathbb{R}^n, i = 1, 2, ..., \) such that

\[ \mathcal{H}^k(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)) = 0. \]

A set \( F \subset \mathbb{R}^n \) is called purely \( k \)-unrectifiable if \( \mathcal{H}^k(F \cap E) = 0 \) for every \( k \)-rectifiable \( E \). We call a (scalar or vector) measure \( \mu \) \( k \)-rectifiable if there exist a \( k \)-rectifiable set \( E \) and a Borel function (scalar or vector) \( f \) such that \( \mu = f\mathcal{H}^k\mid_E \).

For a vector space \( V \) and a vector \( u \) we denote \( p_V, p_u \) orthogonal projections on \( V \) and on \( \text{span}\{u\} \) respectively. A symbol \( \mathcal{D}(\mathbb{R}^n) \) means for us space of smooth functions with compact support. By spectrum of a tempered Radon measure we understand support of its distributional Fourier transform.

2. Preliminary facts

### 2.1. Tangent measures and rectifiability.

**Definition 2.1.** For a given \( r > 0 \) and a Radon measure \( \mu \) we define a blow-up measure by formula \( \mu_{r,x}(A) = \mu(x + rA) \). Any measure \( \nu \) which is a weak-* limit in \( M(\mathbb{R}^n) \) of a sequence of the type \( r_n^{-\alpha} \mu_{x,r_n} \) for some positive \( \alpha \) and \( r_n \downarrow 0 \) we call tangent measure to \( \mu \) at point \( x \). We denote the set of those measures by \( \text{Tan}(\mu, x) \).

A straightforward generalization of Theorem 4.8 from [D] or Theorem 2.83 from [AFP] is

**Theorem 2.2.** Let \( \mu = f\mathcal{H}^k\mid_E \) be a \( k \)-rectifiable vector measure. Then for \( \mathcal{H}^k \)-a.e. \( x \in E \) there exists a \( k \)-dimensional vector space \( V_x \) such that

\[ r^{-k} \mu_{x,r} \to f(x)\mathcal{H}^k\mid_{V_x}, \]

as \( r \downarrow 0 \). Convergence is understood as a weak-* convergence of coordinates in \( M(\mathbb{R}^n) \).

**Corollary 2.3.** If \( \mu \in M(\mathbb{R}^n) \) is a \( k \)-rectifiable measure, then for \( \mathcal{H}^k \)-a.e. \( x \) there exists \( C_x > 0 \) such that \( |\mu|(B(x,r)) \leq C_x r^k \).

Convergence from Theorem 2.2 is tested on functions from \( C_c(\mathbb{R}^n) \). However, for our applications we need convergence in \( \mathcal{S}'(\mathbb{R}^n) \). This can be achieved with the following lemma:
Lemma 2.4. Suppose that \( \mu \in M(\mathbb{R}^n) \) satisfies \( |\mu|(B(x,r)) \leq Cr^m \). If \( r^{-m}\mu_{x,r} \to \nu \) in \( M(\mathbb{R}^n) \), then also \( r^{-m}\mu_{x,r} \to \nu \) in \( S'(\mathbb{R}^n) \).

Proof. Choose any \( \varphi \in S(\mathbb{R}^n) \). We can write \( \varphi = \sum \varphi_k \), where \( \varphi_k \in C_\infty \), \( \text{supp}(\varphi_k) \subset B(0,k) \) and \( \|\varphi_k\|_\infty \leq \|\varphi|_{B(0,k)}\|_\infty \). Then
\[
\frac{1}{r^m} \int \varphi d\mu_{x,r} - \int \varphi d\nu \leq \frac{1}{r^m} \int \sum_{k=1}^n \varphi_k d\mu_{x,r} - \int \sum_{k=1}^n \varphi_k d\nu + \frac{1}{r^m} \int \sum_{k>n} \varphi_k d\mu_{x,r} + \int \sum_{k>n} \varphi_k d\nu.
\]
Second term we can majorize by
\[
\sum_{k>n} \|\varphi_k\|_\infty |\mu|(B(x,kr)) \leq C \sum_{k>n} k^m \|\varphi_k\|_\infty,
\]
while the third one is bounded by \( \|\nu\| \sup_{k>n} \|\varphi_k\|_\infty \). After taking sufficiently big \( n \) and then choosing suitable \( r_0 \), we see that for \( r < r_0 \) the starting expression is smaller than any a priori given positive number. \( \square \)

2.2. Distributional definition of bundle measures. We say that the bundle \( \phi : \mathbb{R}^n \setminus \{0\} \to G(k,E) \) is \( C_\infty \) if (locally) \( \phi(x) = \text{span}\{e_1(x),...,e_k(x)\} \), where \( (e_1(x),...,e_k(x)) \) is an orthonormal system and \( e_l(x) \) are \( C_\infty \) functions. For a bundle \( \phi \) we can define pointwise its orthogonal complement by \( \phi^\perp(x) := \phi(x)^\perp \). Of course, if \( \phi \) is \( C_\infty \), then so is \( \phi^\perp \) (one can apply Gram-Schmidt orthogonalization).

Definition 2.5. For a bundle \( \phi \) of class \( C_\infty \), by \( S_{\phi}(\mathbb{R}^n) \) we denote the set of vector valued Schwartz functions \( f \) such that \( f(x) \in \phi(x) \) for \( x \in \mathbb{R}^n \setminus \{0\} \).

Definition 2.6. By \( S'_{\phi}(\mathbb{R}^n) \) we understand a class of vectors of tempered distributions \( (\Lambda_1,...,\Lambda_m) \) satisfying
\[
\sum_{i=1}^m \langle \Lambda_i, f_i \rangle = 0
\]
for an arbitrary \( (f_1,...,f_m) \in S_{\phi^\perp}(\mathbb{R}^n) \). It is equivalent to
\[
\sum_{i=1}^m \langle \Lambda_i, \hat{f}_i \rangle = 0.
\]
Further we prove that this class contains bundle measures and that it is preserved by taking limits of blow-up process. We use Parseval identity (see [K]):

Theorem 2.7. If \( \mu \in M(\mathbb{R}^n) \) and \( f \in S(\mathbb{R}^n) \), then
\[
\langle f, \mu \rangle = \int \hat{f}(\xi) \hat{\mu}(-\xi) d\xi.
\]
Lemma 2.8. Let $\mu \in M_\phi(\mathbb{R}^n)$. If in some point $x$ there exists a distributionally tangent (vector) measure $\nu$, then it belongs to $S_\phi'(\mathbb{R}^n)$.

Proof. Step 1. For each $r$, $r^{-\alpha}\mu_{x,r} \in M_\phi(\mathbb{R}^n)$. For a fixed coordinate $\mu^{(i)}$ we have

$$r^{-\alpha}\hat{\mu}_{x,r}^{(i)}(\xi) = r^{-\alpha} \int_{\mathbb{R}^n} e^{-2\pi i(\xi, \frac{y}{r})} d\mu^{(i)}(y) =$$

$$r^{-\alpha} e^{2\pi i(\xi, \frac{y}{r})} \int_{\mathbb{R}^n} e^{-2\pi i(\frac{y}{r}, y)} d\mu^{(i)}(y) = r^{-\alpha} e^{2\pi i(\xi, \frac{y}{r})} \hat{\mu}^{(i)}(\frac{\xi}{r}),$$

hence $r^{-\alpha}\hat{\mu}_{x,r}(\xi) \parallel \hat{\mu}(\xi)$.

Step 2. If $\mu = (\mu_1, \ldots, \mu_m) \in M_\phi(\mathbb{R}^n)$, then $\nu \in S_\phi'(\mathbb{R}^n)$. Let $(f_1, \ldots, f_m) \in S_{\phi^{\perp}}(\mathbb{R}^n)$. By Parseval identity we have

$$\sum_{i=1}^m \langle \mu_i, \hat{f}_i \rangle = \int_{\mathbb{R}^n} \sum_{i=1}^m f_i(\xi) \hat{\mu}_i(\xi) d\xi = 0,$$

because $(f_1, \ldots, f_m)$ and $(\hat{\mu}_1, \ldots, \hat{\mu}_m)$ are perpendicular at each $\xi \neq 0$.

Step 3. Let $(f_1, \ldots, f_m) \in S_{\phi^{\perp}}(\mathbb{R}^n)$. Then

$$0 = \lim_{r \to 0} r^{-\alpha} \sum_{i=1}^m \langle \mu_{x,r}^{(i)}, \hat{f}_i \rangle = \sum_{i=1}^m \langle \nu^{(i)}, \hat{f}_i \rangle.$$

\[ \square \]

3. Dimension vs. Antisymmetry

3.1. Uchiyama-type theorem. Our first result is inspired by the celebrated theorem by Uchiyama (see \text{[U]}; here we enclose its dual form).

Theorem 3.1. Let $\theta_1(\xi), \ldots, \theta_m(\xi) \in C^\infty(S^{n-1})$. If

$$\text{rank} \begin{bmatrix} \theta_1(\xi) & \theta_2(\xi) & \cdots & \theta_m(\xi) \\ \theta_1(-\xi) & \theta_2(-\xi) & \cdots & \theta_m(-\xi) \end{bmatrix} = 2$$

for $\xi \in S^{n-1}$, then

$$H^1(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : \forall_i K_{\theta_i} f \in L^1(\mathbb{R}^n) \},$$

where $K_{\theta_i} f = F^{-1}(\theta_i(\frac{\xi}{|\xi|})F(f))$.

Above theorem shows that Conjecture \text{[L5]} is true for 0-antisymmetric bundles.

Corollary 3.2. If $a(\phi) = 0$, then $\mu \in M_\phi(\mathbb{R}^n)$ implies that $\mu \in L^1$.

Proof. If $\mu \in M_\phi$, then there exist $\theta_1, \ldots, \theta_m$ as above and a bounded function $g(\xi)$ such that

$$\hat{\mu}_i(\xi) = \hat{\theta}_i(\frac{\xi}{|\xi|})g(\xi)$$
for \( \xi \neq 0 \). Take any approximate identity \( \{F_n\} \) (from Schwartz class). Then \( \mu \ast F_n = K_\theta(\tilde{g} \ast F_n) \in L^1 \), so by previous theorem \( \tilde{g} \ast F_n \in H^1 \). By weak-* closedness of \( H^1 \), a tempered distribution \( \tilde{g} \ast F_n \in H^1 \). By weak-* closedness of \( H^1 \), a tempered distribution \( \tilde{g} \ast F_n \in H^1 \). By weak-* closedness of \( H^1 \), a tempered distribution \( \tilde{g} \ast F_n \in H^1 \). By weak-* closedness of \( H^1 \), a tempered distribution \( \tilde{g} \ast F_n \in H^1 \). By weak-* closedness of \( H^1 \), a tempered distribution \( \tilde{g} \ast F_n \in H^1 \).

As already mentioned in the introduction, Theorem 1.6 gives a proof for the other endpoint of the scale. Now let us focus on the intermediate dimensions. We begin from invoking a well-known fact, whose proof can be found in [H] (Theorem 7.1.25).

**Lemma 3.3.** If \( V \subset \mathbb{R}^n \) is a \( k \)-dimensional linear subspace, then \( \mathcal{H}^k_{\perp V} = \mathcal{H}^{n-k}_{\perp V} \).

**Lemma 3.4.** Suppose that a measure \( \mu \in M_\phi(\mathbb{R}^n) \) has a tangent measure of the form

\[
\nu = v\mathcal{H}^k_{\perp V},
\]

where \( \dim V = k \), and \( v \) is some nonzero vector. Then \( \phi \equiv \text{span}\{v\} \) on \( V_{\perp} \setminus \{0\} \).

**Proof.** Let us take any vector-valued function \( F \in S_\phi(\mathbb{R}^n) \). Then, by preceding lemma and the definition of \( S_\phi(\mathbb{R}^n) \) we obtain:

\[
\int_{\mathbb{R}^n} \langle F(x), v \rangle d\mathcal{H}^{n-k}_{\perp V}(x) = 0
\]

(brackets under integral sign mean scalar product in \( \mathbb{R}^n \)). Let us assume that in some \( x_0 \in V_{\perp} \setminus \{0\} \) we have \( \phi(x_0) \neq \text{span}\{v\} \). It implies existence of \( w \in \phi^{-1}(x_0) \) such that \( \langle w, v \rangle \neq 0 \), say \( \langle w, v \rangle > 0 \). Take any function \( g \in S_\phi(\mathbb{R}^n) \) such that \( g(x_0) = w \). Obviously, \( \langle g(x), v \rangle > 0 \) in some neighbourhood of \( x_0 \). Multiplying \( g \) coordinatewise by suitable mollifier and substituting it in place of \( F \) we get a contradiction. \( \square \)

**Theorem 3.5.** If \( \mu \) is a bundle measure subordinated to \( \phi \) satisfies following conditions:

1. there exists \( x \) such that for some \( \alpha > 0 \) we have \( |\mu|(B(x, r)) \leq Cr^\alpha \),
2. \( \mu \) has at \( x \) nonzero tangent measure of the form \( \nu = v\mathcal{H}^k_{\perp V} \), \( \dim V = k \), then we have

\[
k \geq n - a(\phi).
\]

**Corollary 3.6.** If \( \mu \) is a \( k \)-rectifiable, bundle measure subordinated to \( \phi \), then

\[
k \geq n - a(\phi).
\]

**Proof.** Tangent measure at generic point \( x \) is of the form \( f(x)\mathcal{H}^k_{\perp V_x} \) \((f(x) \) is the density and \( V_x \) is the tangent plane at \( x \)), so we have \( \phi \equiv \text{span}\{f(x)\} \) on the plane orthogonal to the support at \( x \). \( \square \)
Remark 3.7. If we repeat above reasoning for measures of the form \( \mu = \sum g_i \mathcal{H}^{k_i} \), where each \( M_i \) is a \( k_i \)-dimensional \( C^1 \) submanifold and \( g_i \) are continuous functions, then we see that for them Conjecture 1.5 is still true (higher dimensional parts disappear after blow-up process).

4. More on singular directions

4.1. Localization of exceptional directions. As we have seen in the proof of Corollary 3.6, the homogenity condition gives us a possibility to relate geometry of sets with values of bundles measures. Here we present „affine” version of this fact, working without any geometric assumptions.

Definition 4.1. For \( A \subset \mathbb{R}^n \), by \( N(A) \) we denote set of its exceptional directions, that is \( \{ v \in \mathbb{R}^n : \| v \| = 1, \lambda (p_v(A)) = 0 \} \). Here, for \( v \in \mathbb{R}^n \setminus \{0\} \), by \( p_v : \mathbb{R}^n \rightarrow \text{span}\{v\} \) we understand orthogonal projection on \( \text{span}\{v\} \) and \( \lambda \) is a suitable 1-dimensional Lebesgue measure.

Theorem 4.2. If \( \phi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathcal{G}(k, E) \) is a homogenous bundle, hölder with exponent \( > \frac{1}{2} \), then for each \( \mu \in M_\phi(\mathbb{R}^n, E) \) and arbitrary Borel set \( A \subset \mathbb{R}^n \) we have
\[
N(A) \subset \phi^{-1}(\mu(A)) := \{ u \in \mathbb{R}^n : \| u \| = 1, \mu(A) \in \phi(u) \}.
\]

Above result is essentially a qualitative reformulation of Theorem 1.6 from [RW]. The proof relies on a fact which could be called degenerated theorem of Forelli (though it cannot be derived from it in a direct way).

Theorem 4.3. (Forelli on \( \mathbb{R}^2 \)) Suppose that a measure \( \mu \) has its spectrum outside some acute angle \( \alpha \). Then, if a Borel set \( F \) can be covered with countably many (unbounded) strips, orthogonal to directions from \( \alpha \), with arbitrary small sum of widths, then \( \mu(F) = 0 \).

General formulation and proof can be found in [R]. Aforementioned relative can be stated as follows:

Definition 4.4. Let \( \ell \) be a line in \( \mathbb{R}^n \). We say that \( A \subset \mathbb{R}^n \) is a parabolic set (with asymptote \( \ell \)) if for each vector \( a \in \mathbb{R}^n \) and arbitrary bounded set \( B \), set \( (\ell + a) \cap (A + B) \) is contained in some halfline. (In particular, for \( B = \{0\}, (\ell + a) \cap A \forall a \in \mathbb{R}^n \) is contained in some halfline.)

For example, concave subsets of the plane which are bounded by graphs of power functions are parabolic sets. Their complements also satisfy the definition, but examples of this type are useless for further applications and in such case F. and M. Riesz’s theorem gives even absolute continuity with respect to the Lebesgue measure.

Theorem 4.5. If a measure \( \mu \in M(\mathbb{R}^n) \) has its spectrum inside some parabolic set with asymptote \( \ell \), then \( \mu(A) = 0 \) for each Borel set \( A \) which satisfies \( \mathcal{H}^1_\ell(\mu(A)) = 0 \).

We present an elementary proof from [RW] (compare with [G]).
Definition 4.6. For linear subspace $V \subset \mathbb{R}^n$ and measure $\nu \in M(\mathbb{R}^n)$ we denote $p(V, \nu) \in M(V)$ orthogonal projection of $\nu$ on $V$, i.e. a measure defined by $p(V, \nu)(A) = \nu(A \times V^\perp)$ for each $A \subset V$.

An easy consequence of Fubini’s theorem is the following lemma:

Lemma 4.7. Let $\nu$ and $V$ be as in above definition and let us choose coordinates $t = (t', t'') \in V \times V^\perp = \mathbb{R}^n$. Then, for arbitrary function $f \in C_b(V)$ we have

$$
\int_V f(t') dp(V, \nu)(t') = \int_{\mathbb{R}^n} f(t') d\nu(t).
$$

Proof. (of Theorem 4.5) Of course, we can assume that $\ell$ is a linear subspace of $\mathbb{R}^n$; let $N$ be its orthogonal complement and $\lambda$ be Lebesgue measure on $\ell$. As previously, let us choose coordinates $t = (t', t'') \in \ell \times N$. For an arbitrary vector $a \in \mathbb{R}^n$ denote $\mu_a$ a measure satisfying $d\mu_a = e^{-2\pi i \langle a, \cdot \rangle} d\mu$.

It suffices to show that $\mu_{\perp B \times N}(\cdot) = 0$ for any $B \subset \ell$ satisfying $\lambda_\ell(B) = 0$. For $\xi = (\xi', 0)$, by using Lemma 4.7 we obtain:

$$
p(\ell, \mu_a)\hat{}(\xi') = \int_\ell e^{-2\pi i \langle \xi', t' \rangle} dp(\ell, \mu_a)(t') = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi + a, t \rangle} d\mu(t) = \hat{\mu}(\xi + a),
$$

hence, by the definition of parabolic set, spectrum $p(\ell, \mu_a)$ is inside some halfline. F. and M. Riesz’s theorem guarantees absolute continuity of this measure with respect to $\lambda_\ell$. From this conclusion we derive $\mu_a(B \times N) = p(\ell, \mu_a)(B) = 0$. Finally, for each $a \in \mathbb{R}^n$:

$$
\mu_{\perp B \times N}(\cdot)(a) = \int_{B \times N} e^{-2\pi i \langle a, t \rangle} d\mu(t) = \mu_a(B \times N) = 0,
$$

which completes the proof.

Next we will generalize above theorem on tempered measures.

Theorem 4.8. If $\mu$ is a tempered Radon measure and if support of its (distributional) Fourier transform is contained inside some parabolic set with asymptote $\ell$, then $\mu(A) = 0$ for each Borel set $A$ such that $H^1_\ell(p_\mu(A)) = 0$.

Proof. Suppose that there exists a bounded set $F$ contradicting our thesis. For arbitrary $\delta > 0$ we can find a function $f \in S(\mathbb{R}^n)$ such that $\hat{f} \in D(\mathbb{R}^n)$ and $|f(x) - 1| < \delta$ for $x \in F$.

Construction: Take $g \in D(\mathbb{R}^n)$ such that $\int g = 1$ and denote $f = \hat{g}$. Then $f(0) = 1$ and there exists a neighbourhood $U \ni 0$ such that $|f(x) - 1| < \delta$ for $x \in U$. Of course $\forall r > 0 f(\xi) \hat{} \in D(\mathbb{R}^n)$. Taking $r$ such that $F \subset r U$ we get a suitable function.

Denote $\nu = f d\mu$. For sufficiently small $\delta$, $\nu(F) \neq 0$, $\nu$ is a finite measure and $\text{spec}(\nu) \subset \text{spec}(\mu) + \text{spec}(f)$ ($\nu$ is a product of tempered distribution $\mu$ and Schwartz function $f$) and $\text{spec}(f)$ is a bounded set. Hence, spectrum $\nu$ is contained in parabolic set with asymptote $\ell$. Previous theorem gives $\nu(F) = 0$, which leads to a contradiction.
Further we will use an „$L^2$-perturbation” of the previous fact.

**Corollary 4.9.** If $\Omega$ is a parabolic set with asymptote $\ell$ and $\mu$ is a tempered Radon measure such that restriction of $\hat{\mu}$ to $\mathbb{R}^n \setminus \Omega$ is a function from $L^2(\mathbb{R}^n)$, then the conclusion of previous theorem still holds.

**Proof.** Let $f$ be a function satisfying assumptions. By Theorem 4.8 we have $(\mu - \hat{f})(F) = 0$ for each bounded $F$, whose projection on $\ell$ has zero Lebesgue measure. Because $\hat{f}$ is in $L^1_{\text{loc}}$, we get $\mu(F) = 0$. □

Now let us prove main result of the section.

**Proof.** (of Theorem 4.2) Let $A \subset \mathbb{R}^n$, $\mu(A) = e$, $\lambda(p_v(A)) = 0$ and assume that thesis does not hold, i.e. $v \notin \phi^{-1}(e)$ for some $v$. It can happen only if $e \neq 0$ and henceforth we assume it.

Let us choose a functional $\theta \in E^*$ satisfying $\phi(v) \subset \ker \theta$ and $\theta(e) = 1$ (normalized coordinate of a projection on $\text{span}\{e\}$ along a subspace containing $\phi(v)$). Let $\nu \in M(\mathbb{R}^n)$ be defined by a formula $\nu = \theta(\hat{\mu})$. Then we have $\hat{\nu} = \theta(\hat{\mu}(\xi)), \nu(A) = 1$ and for some constant $C = C(\ker \theta, e)$ a following estimate holds

$$|\hat{\nu}(\xi)| \leq C|\hat{\mu}(\xi)| \cdot \sin \angle(\phi(\xi), \phi(v)).$$

It is obvious if $e$ is orthogonal to $\phi(v)$ (we can take $C = 1$). In other cases, for a fixed $\ker \theta$, each two such functionals are proportional.

In coordinates $\xi = (\xi_1, \xi_2) \in \text{span}\{v\} \times \text{span}\{v\}^\perp$ let $\Omega$ be the complement of a set

$$\{(\xi_1, \xi_2) : \xi_1 \geq 1, |\xi_2| \leq |\xi_1|^\gamma\},$$

where $\gamma < \frac{2\alpha - 1}{2n + \alpha - 1}$. Then $\Omega$ is parabolic with asymptote $\text{span}\{v\}$. By Hölder continuity and homogeneity of a bundle we obtain

$$\int_{\Omega} |\hat{\nu}(\xi)|^2 d\xi \lesssim \int_1^\infty \int_{\{|\xi_2| < |\xi_1|^{\gamma}\}} |d(\phi(\xi), \phi(v))|^2 d\xi_2 d\xi_1$$

$$\lesssim \int_1^\infty |\xi_1|^{\gamma(n-1)} \left(\frac{\xi_2^\gamma}{\xi_1}\right)^{2\alpha} d\xi_1 = \int_1^\infty t^{\gamma(2\alpha + n - 1) - 2\alpha} dt < \infty,$$

so by Corollary 4.9 we obtain $\nu(A) = 0$, which gives a contradiction. □

5. Applications

5.1. Lower Hausdorff dimension of rectifiable bundle measures.

**Theorem 5.1.** If $\mu$ is a $k$-rectifiable, bundle measure subordinated to $\phi$ and

$$k < \frac{n}{2},$$

then $\mu$ takes values in some 1-dimensional subspace (hence it can be identified with a scalar measure).
Proof. Let us observe that, by Theorem 2.2, Lemma 2.4 and Lemma 3.4 for a generic point $x$ from the set $\{y : f(y) \neq 0\}$ we have $\phi \equiv \text{span}\{f(x)\}$ on $V_x^\perp \setminus \{0\}$. But $\dim V_x^\perp > \frac{n}{2}$, which means $V_x^\perp \cap V_y^\perp \neq \{0\}$ and consequently $\text{span}\{f(x)\} = \text{span}\{f(y)\}$ for any two such points. Hence, the density $f(x)$ is $H^k$ - a.e. parallel to some fixed vector, which finishes the proof.

5.2. Regularity of bundle measures. In this section we will use a part of Besicovitch-Federer projection theorem (see Theorem 18.1 in [M]).

Theorem 5.2. Let $A \subset \mathbb{R}^n$ be a Borel set with $H^m(A) < \infty$, where $m < n$ is an integer. Then $A$ is purely $m$-unrectifiable if and only if $H^m(p_V(A)) = 0$ for almost all $V \in \mathcal{G}(m, \mathbb{R}^n)$ (with respect to the natural measure on the Grassmannian).

An immediate consequence of the above and Theorem 4.2 is

Theorem 5.3. Suppose that $\phi : \mathbb{R}^n \setminus \{0\} \to \mathcal{G}(1, E)$ is a nonconstant, homogeneous bundle, holder with exponent $> \frac{1}{2}$. Then for $\mu \in M_{\phi}(\mathbb{R}^n)$ and any 1-purely unrectifiable set $E$, $H^1(E) < \infty$, we have $\mu \ll E \equiv 0$.

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