Reconstructing the geometric structure of a Riemannian symmetric space from its Satake diagram

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Abstract. The local geometry of a Riemannian symmetric space is described completely by the Riemannian metric and the Riemannian curvature tensor of the space. In the present article I describe how to compute these tensors for any Riemannian symmetric space from its Satake diagram, in a way that is suited for the use with computer algebra systems; an example implementation for Maple Version 10 can be found on http://satake.sourceforge.net. As an example application, the totally geodesic submanifolds of the Riemannian symmetric space SU(3)/SO(3) are classified.

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1 Introduction

It is well-known that the behavior of any Riemannian manifold $M$ is influenced strongly by its Riemannian curvature tensor. To give just two examples, the “spreading” of the geodesics, as measured by the Jacobi fields, and the existence of a totally geodesic submanifold tangential to a given subspace of a tangent space are expressed in terms of the curvature tensor field on the manifold $M$.

Especially for (locally) Riemannian symmetric spaces, the control exerted on the local geometry of the manifold by the Riemannian curvature, together with the Riemannian metric, is total: If $M$ and $N$ are two such spaces, and there exists a linear isometry $T_pM \to T_qN$ which

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transports the curvature tensor of \( M \) at \( p \) into the curvature tensor of \( N \) at \( q \), then \( M \) and \( N \) are already locally isometric to each other; note that it suffices to consider the curvature in a single point of \( M \) and \( N \) because the curvature tensor field is parallel in this situation. This shows that for a Riemannian symmetric space \( M \), the local geometry of \( M \) is described completely by two tensors on a single tangent space \( T_p M \): the inner product given by the Riemannian metric at \( p \), and the curvature tensor at \( p \). Viewed in this way, the study of the local geometry of a Riemannian symmetric space reduces to a purely algebraic problem, namely to the study of these two tensors on the tangent space \( T_p M \). Thus we will call these two tensors the “fundamental geometric tensors” of \( M \).

One very important example for the control of the geometry of a Riemannian symmetric space \( M \) by its curvature tensor \( R \) is the following result, which permits the classification of the totally geodesic submanifolds of \( M \): A linear subspace \( U \subset T_p M \) is the tangent space of a totally geodesic submanifold of \( M \) if and only if \( U \) is a Lie triple system, i.e. if \( R(u,v)w \in U \) holds for all \( u, v, w \in U \).

In view of the above, it is very desirable to have representations of the fundamental geometric tensors, especially the curvature tensor, available for study for every Riemannian symmetric space \( M = G/K \). The well-known formula \( R(u,v)w = -[ [u,v], w] \) relating the curvature tensor \( R \) of \( M \) to the Lie bracket of the Lie algebra \( g \) of the transvection group \( G \) of \( M \) lets one calculate \( R \) relatively easily if \( G \) is a classical group (then \( g \) is a matrix Lie algebra, with the Lie bracket being simply the commutator of matrices), but not so easily if \( G \) is one of the exceptional Lie groups, because then the explicit description of \( g \) as a matrix algebra is too unwieldy to be useful generally.

Therefore, in the present paper, I will describe another representation of the fundamental geometric tensors of any Riemannian symmetric space of compact type, based on the root space decomposition of the Lie algebra of its transvection group. This representation is especially suited for the use with a computer algebra system. I have implemented the algorithms and equations given here as a Maple package, which can be found at http://satake.sourceforge.net. In a forthcoming paper, I will use this presentation to classify the totally geodesic submanifolds in the exceptional Riemannian symmetric spaces of rank 2 (based on similar methods as my classification in the 2-Grassmannians, see [K1] and [K2]).

As information about the symmetric space concerned, we will require only the Satake diagram of that space, i.e. the Dynkin diagram of the Lie algebra of the transvection group, “annotated” with further information describing the symmetric structure of the space, see for example [Lo], Section VII.3.3, p. 132ff.. The Satake diagrams are well-known and tabulated in the literature (for example, in [Lo], p. 147ff.) for every irreducible Riemannian symmetric space. It is a well-known fact that the Satake diagram already determines the local structure of the Riemannian symmetric space; however, it turns out that for the actual reconstruction of the fundamental geometric tensors in a sufficiently explicit way, some new work needs to be done.

Our consideration is based on the following well-known construction: Let us consider a Riemannian symmetric space \( M = G/K \) of compact type. Then the symmetric structure of \( M \) induces an involutive automorphism \( \sigma \) on the Lie algebra \( g \) of the transvection group \( G \) of \( M \), and \( \sigma \) gives rise to the decomposition \( g = \mathfrak{k} \oplus \mathfrak{m} \), where \( \mathfrak{k} = \text{Eig}(\sigma, 1) \) is the Lie algebra of the isotropy group \( K \) and \( \mathfrak{m} = \text{Eig}(\sigma, -1) \) is a linear subspace of \( g \) which is canonically isomorphic to the tangent space \( T_o M \) at the “origin point” \( o := eK \in G/K = M \), and if we identify \( T_o M \) with \( \mathfrak{m} \) in that way, then on each irreducible factor of \( M \), the Riemannian metric of \( M \) is a
constant multiple of the Killing form of \( g \), and the curvature tensor \( R \) of \( M \) at \( o \) is given by the formula \( R(u, v)w = -[[u, v], w] \) for \( u, v, w \in m \).

To reconstruct the curvature tensor and the inner product on \( T_o M \) from the Satake diagram, we proceed in the following way: It is well-known that the Dynkin diagram of \( g \) (which can be read off the Satake diagram of \( M \)) determines uniquely the root system \( \Delta \) of \( g \); an algorithm for the reconstruction of \( \Delta \) form the Dynkin diagram is given in Section 2. Because the root spaces \( g^C_\alpha \) of the complexification \( g^C \) of \( g \) are complex-1-dimensional, the action of the Lie bracket on each \( g^C_\alpha \times g^C_\beta \) is already determined up to a constant by the structure of \( \Delta \) via the relation \([g^C_\alpha, g^C_\beta] = g^C_{\alpha+\beta}\); more specifically, if we choose a non-zero \( X_\alpha \in g^C_\alpha \) for each \( \alpha \in \Delta \), then there exist constants \( c_{\alpha,\beta} \in \mathbb{C} \) so that \([X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}\) holds for every \( \alpha, \beta \in \Delta \) with \( \alpha + \beta \in \Delta \). As has been discovered by Weyl and Chevalley (see [W] and [C]), there exists a way to choose the vectors \( X_\alpha \) in such a way that the constants \( c_{\alpha,\beta} \) become real, and can up to sign be computed by a simple formula dependent only on the structure of the root system \( \Delta \) (see also Proposition 3.3(g) of the present paper). We call a system \((X_\alpha)_{\alpha \in \Delta}\) chosen in accordance with this a Chevalley basis of \( g^C \). In Section 3 we show how a Chevalley basis can further be adapted to the position of the compact Lie algebra \( g \) within \( g^C \), and then give an algorithm to compute the constants \( c_{\alpha,\beta} \) corresponding to such a further adapted basis, including their sign, thereby recovering the Lie algebra structure of \( g \) completely.

Section 4 then discusses the action of the involutive automorphism \( \sigma : g \to g \), which describes the symmetric structure of \( M \), on \( g \). The action of \( \sigma \) on the Cartan subalgebra (spanned by the root vectors) is already well-known from the works of Satake (indeed, its description was what induced Satake to introduce what is now known as the Satake diagram), so it remains for Section 4 to describe how \( \sigma \) acts on a Chevalley basis of \( g^C \), and thus on the root spaces of \( g \). By knowing \( \sigma \), we then know the splitting \( g = \mathfrak{t} \oplus \mathfrak{m} \), and by also knowing the Lie bracket structure of \( g \), we are then able to calculate the curvature tensor via the formula \( R(u, v)w = -[[u, v], w] \) for \( u, v, w \in m \). Because we moreover know how the Killing form evaluates for members of the Chevalley basis we chose, we can also express the Killing form on \( m \), and hence the Riemannian metric of \( M \) (which is a multiple of the Killing form on each irreducible factor of \( M \)) acting on \( m \). The resulting formulas, which describe the fundamental geometric tensors of \( M \), are given in Section 5.

Finally, to illustrate the usefulness of the presentation of the fundamental geometric tensors given herein, I apply it in Section 6 to classify the Lie triple systems, and thus the totally geodesic submanifolds, of a specific Riemannian symmetric space. Although the presentation was developed mainly with the exceptional Riemannian symmetric spaces in mind, as explained above, for the sake of simplicity I here investigate the classical Riemannian symmetric space \( SU(3)/SO(3) \). As mentioned above, I will use the same methods in a forthcoming paper to classify the totally geodesic submanifolds in all exceptional Riemannian symmetric spaces of rank 2.

I have produced an example implementation of the presentation of the fundamental geometric tensors described in this article as a package for Maple Version 10. This implementation can be downloaded from http://satake.sourceforge.net. The worksheet and the corresponding technical documentation also accompanies the version of the present paper posted on http://www.arxiv.org.
2 Reconstructing the root system

As described in the Introduction, the first step in the reconstruction of the fundamental geometric tensors of a Riemannian symmetric space (of compact type) is the reconstruction of the root system of the Lie algebra of its transvection group from its Dynkin diagram.

For this purpose, let $\mathfrak{g}$ be a compact real Lie algebra, i.e. the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X,Y) \mapsto \text{tr}(\text{ad}(X) \circ \text{ad}(Y))$ of $\mathfrak{g}$ is negative definite. We fix a Cartan subalgebra (i.e. a maximal abelian subalgebra) $\mathfrak{t}$ of $\mathfrak{g}$. For any $\mathbb{C}$-linear form $\alpha \in (\mathfrak{t}^*)^*$ on the complexification $\mathfrak{t}^\mathbb{C} := \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{t}$ we consider

$$\mathfrak{g}_\alpha := \{ X \in \mathfrak{g} \mid \forall H \in \mathfrak{t} : \text{ad}(H)^2 X = \alpha(H)^2 X \} ;$$

if we have $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, $\alpha$ is called a root of $\mathfrak{g}$ (with respect to $\mathfrak{t}$) and $\mathfrak{g}_\alpha$ is called the root space corresponding to $\alpha$; the set $\Delta$ of all roots of $\mathfrak{g}$ is called the root system of $\mathfrak{g}$.

Because the Lie algebra $\mathfrak{g}$ is compact, the roots of $\mathfrak{g}$ are purely imaginary on $\mathfrak{t}$, i.e. each $\alpha \in \Delta$ is of the form $\alpha = i\alpha'$, where $\alpha' \in \mathfrak{t}^\ast$ is a real linear form on $\mathfrak{t}$. It is clear that $\mathfrak{g}_{-\alpha} = \mathfrak{g}_\alpha$ holds, and therefore we have $-\alpha \in \Delta$ if and only if $\alpha \in \Delta$. Thus, if we fix $H_0 \in \mathfrak{a}$ so that $\alpha(H_0) \neq 0$ holds for all $\alpha \in \Delta$ (such a $H_0$ exists because $\Delta$ is finite), then the subset

$$\Delta_+ := \{ \alpha \in \Delta \mid \alpha(H_0) \in i \mathbb{R}_+ \}$$

of $\Delta$ (called the set of positive roots with respect to $H_0$) satisfies $\Delta_+ \cup (-\Delta_+) = \Delta$ and $\Delta_+ \cap (-\Delta_+) = \emptyset$; with respect to it we have the root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha . \quad (1)$$

A root $\alpha \in \Delta_+$ is called simple, if it is not the sum of two positive roots. We denote the set of simple roots in $\Delta_+$ by $\Pi$; it is a basis of $i\mathfrak{t}^\ast$. If $\beta \in \Delta$ is an arbitrary root, then the coefficients $k_\alpha$ in the unique representation $\beta = \sum_{\alpha \in \Pi} k_\alpha \cdot \alpha$ are all integers; moreover they are either all $\geq 0$ (this is the case if and only if $\beta \in \Delta_+$) or all $\leq 0$ (this is the case if and only if $\beta \in -\Delta_+$). (See [Kn], Proposition 2.49, p. 155.) Therefore the number

$$\ell(\beta) := \sum_{\alpha \in \Pi} k_\alpha ,$$

called the level of $\beta$, is an integer $\neq 0$; we have $\ell(\beta) > 0$ if and only if $\beta \in \Delta_+$ holds, and $\ell(\beta) = 1$ if and only if $\beta$ is simple. Clearly the level is additive, i.e. we have for any $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$

$$\ell(\alpha + \beta) = \ell(\alpha) + \ell(\beta) . \quad (2)$$

Next, we use the Killing form $\kappa$ of $\mathfrak{g}$ to induce an inner product on $i\mathfrak{t}$ resp. on $i\mathfrak{t}^\ast = \text{span}_\mathbb{R}(\Delta)$: Because the Lie algebra $\mathfrak{g}$ is compact, $\kappa$ is negative definite on $\mathfrak{g}$, and therefore its complexification, which we again denote by $\kappa : \mathfrak{g}^\mathbb{C} \times \mathfrak{g}^\mathbb{C} \to \mathbb{C}$, is non-degenerate; its restriction $\langle \cdot , \cdot \rangle := \kappa((i\mathfrak{t} \times i\mathfrak{t}))$ is a positive definite inner product on $i\mathfrak{t}$. It follows that for any $\alpha \in (i\mathfrak{t}^\ast)$ there exists one and only one vector $\alpha^\sharp \in i\mathfrak{t}$ so that $\alpha = \kappa(\alpha^\sharp, \cdot)$ holds. By pulling back the
inner product of it with the linear isomorphism \((it)^* \to it, \alpha \mapsto \alpha^t\), we get an inner product on \((it)^*\), which we will also denote by \(\langle \cdot, \cdot \rangle\). We will also use the associated norm \(\|v\| := \sqrt{\langle v, v \rangle}\) for \(v \in it\) resp. for \(v \in (it)^*\).

It is a well-known fact that for any two roots \(\alpha, \beta \in \Delta\), the quantity \(n_{\alpha, \beta} := \frac{2\langle \alpha, \beta \rangle}{\|\alpha\|^2}\) relating the lengths and the angles of the roots can only attain the discrete values 0, ±1, ±2, ±3 (see, for example, [Kn], Proposition 2.48(c), p. 153); if \(\alpha, \beta\) are simple with \(\alpha \neq \beta\), we moreover have \(n_{\alpha, \beta} \cdot n_{\beta, \alpha} \in \{0, 1, 2, 3\}\). The matrix \(N := (n_{\alpha, \beta})_{\alpha, \beta \in \Pi}\) is called the Cartan matrix of \(\Delta_+\) (or of \(g\)); it is known that up to conjugation with a permutation matrix, it does not depend on the choices involved in obtaining \(\Delta_+\). Moreover, \(N\) is regular, and if \(g\) is simple, \(N\) does not have a non-trivial block diagonal form. Finally the Dynkin diagram of \(g\) is obtained by drawing a node for each simple root of \(g\); the nodes corresponding to \(\alpha, \beta \in \Pi\) are connected by the number of lines indicated by \(n_{\alpha, \beta} \cdot n_{\beta, \alpha} \in \{0, 1, 2, 3\}\); if \(n_{\alpha, \beta} \cdot n_{\beta, \alpha} \in \{2, 3\}\) holds, then \(\alpha\) and \(\beta\) are of unequal length, and we indicate the relation by drawing an arrow tip pointing from the longer to the shorter root.

We are now able to describe how the root system \(\Delta\) and the Killing form of \(g\) (up to a constant factor on each simple ideal of \(g\)) are reconstructed from the Dynkin diagram of \(g\): First, from the Dynkin diagram, we can reconstruct the Cartan matrix \(N = (n_{\alpha, \beta})_{\alpha, \beta \in \Pi}\) (where \(\Pi\) denotes the set of simple roots, i.e. of nodes of the Dynkin diagram): Let \(\alpha, \beta \in \Pi\) be given, then \(n_{\alpha, \beta}\) is obtained in the following way: If \(\alpha = \beta\) holds, we obviously have \(n_{\alpha, \beta} = 2\). Otherwise, let \(k\) denote the number of lines connecting \(\alpha\) and \(\beta\) in the Dynkin diagram, and for \(k \in \{2, 3\}\) suppose that \(\alpha, \beta\) are arranged in such a way that \(\alpha\) is the longer of the two roots (as indicated in the Dynkin diagram). Then for \(k = 0\) we have \(n_{\alpha, \beta} = n_{\beta, \alpha} = 0\), whereas for \(k \in \{1, 2, 3\}\) we have \(n_{\alpha, \beta} = 1\) and \(n_{\beta, \alpha} = k\).

From the Cartan matrix we can reconstruct the relative lengths of the simple roots within each simple ideal of the semisimple Lie algebra \(g\): If \(\alpha, \beta \in \Pi\) are simple roots with \(\langle \alpha, \beta \rangle \neq 0\), then we have \(\|\alpha\| = \sqrt{\frac{n_{\alpha, \beta}}{n_{\beta, \alpha}}}\); because any two simple roots \(\alpha, \beta\) in the same simple ideal of \(g\) are connected by a chain of simple roots \(\alpha = \gamma_1, \gamma_2, \ldots, \gamma_{k-1}, \gamma_k = \beta\) with \(\langle \gamma_j, \gamma_{j+1} \rangle \neq 0\) (the Dynkin diagram of each simple ideal is connected), we thereby know the relative lengths of any two such simple roots. By arbitrarily fixing the length of one simple root in each simple ideal of \(g\) (this corresponds to the choice of a factor for the metric on each simple ideal of \(g\)), we then know the length of each simple root of \(g\). Thereby we also know the inner product between any two simple roots \(\alpha, \beta \in \Pi\): \(\langle \alpha, \beta \rangle = \frac{1}{2} \|\alpha\|^2 n_{\alpha, \beta}\). Because \(\Pi\) is a basis of \(it^*\), this relationship permits us to reconstruct the inner product \(\langle \cdot, \cdot \rangle\) on all of \(it^*\).

We now state the algorithm for the reconstruction of \(\Delta_+\). The algorithm in fact reconstructs the roots ordered by level, i.e. it constructs the sets \(\Delta_j := \{ \alpha \in \Delta_+ \mid \ell(\alpha) = j \}\) for all \(j \in \{1, \ldots, L\}\), where \(L\) is the maximal level occurring in \(\Delta_+\).

(R1) [Initialization.] Let \(\Delta_1\) be the set of simple roots. Let \(\Delta_\ell := \emptyset\) for all \(\ell \geq 2\) which are needed below.

(R2) [Iterate on level.] Let \(\ell := 1\). Iterate steps (R3)–(R8) until the condition given in (R8) is satisfied.

(R3) [Iterate on roots.] Iterate steps (R4)–(R7) for all \(\beta \in \Delta_\ell\) and \(\alpha \in \Delta_1\).

(R4) [Skip, if the \(\alpha\)-string through \(\beta\) has already been generated, or if \(\beta = \alpha\).] If we have \(\beta = \alpha\), go to step (R7). If we have \(\ell \geq 2\) and \(\beta - \alpha \in \Delta_{\ell-1}\), also go to step (R7).

(R5) [Determine the length of the \(\alpha\)-string through \(\beta\).] Put \(q := -\frac{2\langle \beta, \alpha \rangle}{\|\alpha\|^2}\), where the inner product of \(it\) with the linear isomorphism \((it)^* \to it, \alpha \mapsto \alpha^t\), we get an inner product on \((it)^*\), which we will also denote by \(\langle \cdot, \cdot \rangle\). We will also use the associated norm \(\|v\| := \sqrt{\langle v, v \rangle}\) for \(v \in it\) resp. for \(v \in (it)^*\).
we consider the complexification $g$ was obtained from the Dynkin diagram of $g$. Our next task is to reconstruct the action of the Lie bracket of $\beta$, the root of minimal level in this string, $\alpha$ being a root. Consider the $\alpha$-string through $\beta$. The root of minimal level in this string, $\alpha$ being a root, is of level $1$, and by the induction hypothesis we know that at this stage $\Delta_{k\alpha}$ already contains all roots of that level. Therefore the condition in step (R4) ensures that steps (R5)/(R6) are reached only if $\beta - \alpha$ is not a root. In that case, the $\alpha$-string through $\beta$ is of the form $\{\beta + k\alpha \mid 0 \leq k \leq q\}$ with some $q \in \mathbb{N}_0$, and Equation (3) shows that this $q$ is the number calculated in step (R5). Therefore the elements $\beta + k\alpha$ inserted into $\Delta_{k\alpha}$ in step (R6) constitute exactly the $\alpha$-string through $\beta$ and are thus in particular roots; moreover the level of $\beta + k\alpha$ equals $\ell + k$, and therefore this root is inserted into the correct set $\Delta_{\ell + k}$.

We now show that indeed all roots of level $\ell + 1$ have been inserted into $\Delta_{\ell + 1}$ by the end of the iteration $\ell$. For this, let $\beta$ be any root of level $\ell + 1$. It is known that there exists a simple root $\alpha \in \Delta$ so that $\beta - \alpha$ is also a root. Consider the $\alpha$-string through $\beta$, $\{\beta + k\alpha \mid 0 \leq k \leq q\}$; because $\beta - \alpha$ is a root, we have $p \geq 1$. The root of minimal level in this string, $\beta - p\alpha$, is of level $\ell + 1 - p \leq \ell$; by induction it has therefore already been generated as a member of $\Delta_{(\ell + 1 - p)\alpha}$ at the beginning of the loop iteration $\ell$. As shown in the first part of the proof, in the loop iteration $\ell - p$, with the generation of $\beta - p\alpha$, all members of the $\alpha$-string through $\beta - p\alpha$, have been added to the appropriate $\Delta_{(\ell + 1 - p)\alpha}$; in particular $\beta$ itself has been added to $\Delta_{\ell + 1}$.

3 Determining the Lie bracket

Our next task is to reconstruct the action of the Lie bracket of $g$ from its root system $\Delta$ (which was obtained from the Dynkin diagram of $g$ in Section 2). To do so, we need to take a detour into the complex setting.

We again let $g$ be a compact Lie algebra, and use the notations of the preceding section. We consider the complexification $g^C$ of $g$; via the complexification of the Lie bracket of $g$, $g^C$ becomes a complex semisimple Lie algebra. It should be noted that the Killing form of $g^C$ equals the complexification $\kappa$ of the Killing form of $g$. The complexification $t^C$ of the Cartan subalgebra $t$ of $g$ is a Cartan subalgebra of $g^C$, and we put for any $\alpha \in (t^C)^*$

$$g^C_\alpha := \{ X \in g^C \mid \forall H \in t^C : \text{ad}(H)X = \alpha(H)X \}.$$ 

Then the root system $\{ \alpha \in (t^C)^* \setminus \{0\} \mid g^C_\alpha \neq \{0\} \}$ of $g^C$ equals the root system $\Delta$ of $g$, and we have the root space decomposition

$$g^C = t^C \oplus \bigoplus_{\alpha \in \Delta} g^C_\alpha.$$
3 Determining the Lie bracket

It is a well-known fact that all these root spaces \( g_\alpha \), \( \alpha \in \Delta \), are complex-1-dimensional.

It was famously described by Weyl and Chevalley (see [W] and [C]) how to choose bases of the root spaces \( g_\alpha \) (they consist of only one vector each, because the root spaces are complex-1-dimensional) which are adapted to the Lie bracket of \( g^F \) in the best possible way. We here cite the result in the form given in [Kn], §VI.1.

**Definition 3.1** A family of vectors \( (X_\alpha)_{\alpha \in \Delta} \) is called a Chevalley basis of \( g^F \), if we have \( X_\alpha \in g_\alpha \) for every \( \alpha \in \Delta \) and if there exists a family of real numbers \( (c_{\alpha,\beta})_{\alpha,\beta \in \Delta} \), called the Chevalley constants corresponding to \( (X_\alpha) \), so that for all \( \alpha, \beta \in \Delta \) we have

\[
[X_\alpha, X_\beta] = \begin{cases} c_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\ \alpha^2 & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases},
\]

and

\[
c_{-\alpha,-\beta} = -c_{\alpha,\beta}.
\]

For formal reasons we put \( c_{\alpha,\beta} := 0 \) wherever \( \alpha, \beta \in \Delta \) with \( \alpha + \beta \notin \Delta \) and \( \beta \neq -\alpha \).

It should be noted that the Chevalley constants do depend on the choice of the Chevalley basis. However, their squares are uniquely determined by the structure of the Lie algebra (as Proposition 3.3(g) below shows), therefore the transition from one Chevalley basis to another can change the corresponding Chevalley constants only in sign. The specific transformation behavior of the Chevalley constants is described in Proposition 3.4 below.

**Proposition 3.2** \( g^F \) has a Chevalley basis.

*Proof.* See [Kn], Theorem 6.6, p. 351. \( \square \)

**Proposition 3.3** Let \( (X_\alpha) \) be a Chevalley basis of \( g^F \) and \( (c_{\alpha,\beta}) \) be the corresponding Chevalley constants. Suppose \( \alpha, \beta, \gamma, \delta \in \Delta \). Then we have

(a) \( c_{\beta,\alpha} = -c_{\alpha,\beta} \).

(b) \( \varkappa(X_\alpha, X_{-\alpha}) = 1 \), where \( \varkappa \) is the Killing form of \( g^F \).

(c) \( \overline{X_\alpha} = a \cdot X_{-\alpha} \), with \( a := \varkappa(X_\alpha, X_{-\alpha}) < 0 \).

(d) We have \( g_\alpha = \{ V_\alpha(c) \mid c \in \mathbb{C} \} \), where we put \( V_\alpha(c) := C \frac{1}{\sqrt{2}} (cX_\alpha + \bar{c}X_{\bar{\alpha}}) \) for \( c \in \mathbb{C} \).

(e) Suppose \( \alpha + \beta + \gamma = 0 \). Then we have \( c_{\alpha,\beta} = c_{\beta,\gamma} = c_{\gamma,\alpha} \).

(f) Suppose \( \alpha + \beta + \gamma + \delta = 0 \) and that none of the roots \( \alpha, \beta, \gamma, \delta \) is the negative of one of the others. Then we have \( c_{\alpha,\beta} c_{\beta,\gamma} c_{\alpha,\delta} + c_{\beta,\gamma} c_{\gamma,\alpha} c_{\delta,\beta} = 0 \).

(g) We have

\[
c_{2\alpha,\beta} = \frac{q \cdot (1 + p)}{2} \cdot ||\alpha||^2,
\]

where \( \{ \beta + k\alpha \mid -p \leq k \leq q \} \) is the \( \alpha \)-string through \( \beta \); note that this implies that we have \( c_{\alpha,\beta} \neq 0 \) if \( \alpha + \beta \in \Delta \) holds.

*Proof.* (a) is obvious. For (b), we have \( \alpha(\alpha^2) \cdot \varkappa(X_\alpha, X_{-\alpha}) = \varkappa(\text{ad}(\alpha^2)X_\alpha, X_{-\alpha}) = \varkappa(\alpha^2, [X_\alpha, X_{-\alpha}]) \geq \varkappa(\alpha^2, \alpha^2) = \alpha(\alpha^2) \); because of \( \alpha(\alpha^2) = ||\alpha||^2 \neq 0 \), the statement follows. For (c), we note that because the complex conjugation \( X \mapsto \overline{X} \) is an involutive Lie algebra automorphism of \( g^F \) which leaves \( t \) invariant and satisfies \( \overline{\alpha^2} = -\alpha^2 \), we have \( \overline{X_\alpha} \in g_{\overline{\alpha}}^F \) and therefore there exists \( a \in \mathbb{C}^x \) so that \( \overline{X_\alpha} = a \cdot X_{-\alpha} \) holds. We
have $1 \Leftrightarrow \not= (X_\alpha, X_{-\alpha}) = 4 \not= (X_\alpha, X_{-\alpha})$ and therefore $a = \not= (X_\alpha, X_{-\alpha})$. We have $0 \not= X_\alpha + X_{-\alpha} \in g$ and therefore, because $g$ is compact and hence $\not=$ is negative definite on $g$, $0 > \not= \not= (X_\alpha + X_{-\alpha}, X_{-\alpha} + \not= X_\alpha) = 2 \not= \not= (X_\alpha, X_{-\alpha}) = 2a$, thus $a < 0$. For (d), we have by (c) $V_\alpha(c) \in (g^f \otimes g^f) \cap g = g_\alpha$, and therefore $\{ V_\alpha(c) | c \in C \} \subset g_\alpha$; because $g_\alpha$ is real-2-dimensional, equality follows. For (e) and (f). These follow essentially from the Jacobi identity, see [Be]: Lemma III.5.1, p. 171 and Lemma III.5.3, p. 172. For (g). See [Kn], Theorem 6.6, p. 351. \hfill \square

We next describe the transformation behavior of the Chevalley bases and the corresponding Chevalley constants:

**Proposition 3.4** Let $(X_\alpha)$ be a Chevalley basis of $g^f$ with the corresponding Chevalley constants $(c_{\alpha,\beta})$.

(a) Let constants $z_\alpha \in \mathbb{R}^X$ for every $\alpha \in \Delta$ be given, so that the following properties are satisfied:

(i) For every $\alpha \in \Delta$ we have $z_\alpha \cdot z_{-\alpha} = 1$.

(ii) For every $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ we have $\varepsilon_{\alpha,\beta} := \frac{z_\alpha \cdot z_{-\beta}}{z_{\alpha+\beta}} \in \{ \pm 1 \}$.

Then $(z_\alpha \cdot X_\alpha)_{\alpha \in \Delta}$ is another Chevalley basis of $g^f$, the corresponding Chevalley constants are $(\varepsilon_{\alpha,\beta}, c_{\alpha,\beta})_{\alpha,\beta \in \Delta}$.

(b) Every Chevalley basis of $g^f$ is obtained by the construction of (a).

**Proof.** For (a). Put $\tilde{X}_\alpha := z_\alpha \cdot X_\alpha$ and $\tilde{c}_{\alpha,\beta} := z_{\alpha,\beta} \cdot c_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. Obviously $\tilde{X}_\alpha \in g^f$ holds for all $\alpha \in \Delta$, the numbers $\tilde{c}_{\alpha,\beta}$ are real by property (ii), and it easily follows from (i) and (ii) that $((\tilde{X}_\alpha), (\tilde{c}_{\alpha,\beta}))$ satisfies Equation (2). Moreover, for any $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ we have

$$\varepsilon_{-\alpha,-\beta} = \frac{z_{-\alpha} \cdot z_{-\beta}}{z_{-(\alpha+\beta)}} = \frac{z_{\alpha} \cdot z_{-\beta}}{z_{\alpha+\beta}} = \varepsilon_{\alpha,\beta}$$

Therefrom it follows that $((\tilde{X}_\alpha), (\tilde{c}_{\alpha,\beta}))$ also satisfies Equation (2).

For (b). Let two Chevalley bases $(X_\alpha)$ and $(\tilde{X}_\alpha)$ of $g^f$ (with corresponding Chevalley constants $(c_{\alpha,\beta})$ resp. $(\tilde{c}_{\alpha,\beta})$) be given. For each $\alpha \in \Delta$, the non-zero vectors $X_\alpha$ and $\tilde{X}_\alpha$ lie in the complex-1-dimensional root space $g_\alpha^\delta$, so there exists $z_\alpha \in \mathbb{C}^X$ so that $\tilde{X}_\alpha = z_\alpha \cdot X_\alpha$ holds. It remains to show that the constants $(z_\alpha)$ satisfy the conditions (i) and (ii) of (a). For (i): For any $\alpha \in \Delta$ we have

$$\alpha^2 \Leftrightarrow [\tilde{X}_\alpha, \tilde{X}_\alpha] = z_\alpha \cdot z_{-\alpha} [X_\alpha, X_{-\alpha}] \Leftrightarrow z_\alpha \cdot z_{-\alpha} = 1$$

and therefore $z_{\alpha} \cdot z_{-\alpha} = 1$. For (ii): For any $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ we have

$$X_{\alpha+\beta} = \frac{1}{z_{\alpha+\beta}} \tilde{X}_{\alpha+\beta} \Leftrightarrow \frac{1}{z_{\alpha+\beta}} \tilde{X}_{\alpha+\beta} = \frac{z_{\alpha} \cdot z_{-\beta}}{z_{\alpha+\beta}} [X_\alpha, X_\beta] \Leftrightarrow \frac{z_{\alpha} \cdot z_{-\beta}}{z_{\alpha+\beta}} = \frac{z_{\alpha} \cdot z_{-\beta}}{z_{\alpha+\beta}} \cdot X_{\alpha+\beta}$$

and therefore $\frac{z_{\alpha} \cdot z_{-\beta}}{z_{\alpha+\beta}} = \frac{\tilde{c}_{\alpha,\beta}}{\tilde{c}_{\alpha,\beta}}$. It is a consequence of Proposition 3.4 that $\tilde{c}_{\alpha,\beta} = \pm c_{\alpha,\beta}$ holds, and therefore we have $\frac{z_{\alpha} \cdot z_{-\beta}}{z_{\alpha+\beta}} \in \{ \pm 1 \}$. \hfill \square

The following proposition describes a way to choose a Chevalley basis in such a way that it is adapted to the position of the compact Lie algebra $g$ within $g^f$, see property (i) in the proposition.

**Proposition 3.5** For every non-simple, positive root $\alpha \in \Delta_+ \setminus \Pi$, fix a decomposition $\alpha = \zeta_\alpha + \eta_\alpha$ with $\zeta_\alpha, \eta_\alpha \in \Delta_+$. Then there exists a Chevalley basis $(X_\alpha)$ (with corresponding Chevalley constants $(c_{\alpha,\beta})$) with the following properties:

(i) For every $\alpha \in \Delta_+$ we have $X_{-\alpha} = -X_\alpha$. (Compare Proposition 3.4(c).)

(ii) For every $\alpha \in \Delta_+ \setminus \Pi$ we have $c_{\zeta_\alpha,\eta_\alpha} > 0$.

Any two such Chevalley bases have the same Chevalley constants $(c_{\alpha,\beta})$.

**Remark 3.6** For each root $\alpha \in \Delta_+ \setminus \Pi$ a decomposition $\alpha = \zeta_\alpha + \eta_\alpha$ as required in Proposition 3.5 indeed exists: For example, it is well-known that for each $\alpha$ there exists a simple root $\eta_\alpha$ such that $\zeta_\alpha := \alpha - \eta_\alpha$ is again a positive root.
Proof of Proposition 3.3. We will show by induction on \( \ell \geq 1 \) that there exists a Chevalley basis \((X_\alpha)\) of \( g^F \), with corresponding Chevalley constants \((c_{\alpha,\beta})\), which has the property (i), and also satisfies \(c_{\alpha,\eta_\alpha} > 0\) for all \(\alpha \in \Delta_+\) with \(2 \leq \ell(\alpha) \leq \ell\).

First suppose \( \ell = 1 \). Then we are to show that there exists a Chevalley basis of \( g^F \) which satisfies (i). For this we let an arbitrary Chevalley basis \((X_\alpha)\) of \( g^F \) be given, and denote the corresponding Chevalley constants by \((c_{\alpha,\beta})\). By Proposition 3.3(c) there exists for every \(\alpha \in \Delta\) some \(a_\alpha < 0\) with \(X_\alpha = a_\alpha \cdot X_{-\alpha}\). We have

\[ a_\alpha \cdot a_{-\alpha} = 1. \]  

Next we will show that for any \(\alpha, \beta \in \Delta\) with \(\alpha + \beta \in \Delta\) we have

\[ a_{\alpha + \beta} = -a_\alpha \cdot a_\beta. \]  

For this we calculate \([X_\alpha, [X_\alpha, X_\beta]]\) in two different ways: On one hand we have

\[ [X_\alpha, [X_\alpha, X_\beta]] = [a_\alpha X_{-\alpha}, [X_\alpha, X_\beta]] = a_\alpha c_{\alpha,\beta} [X_{-\alpha}, X_{-\alpha + \beta}] = a_\alpha c_{\alpha,\beta} [a_{-\alpha} (a_{-\alpha + \beta}) X_\beta = a_\alpha c_{\alpha,\beta}^2 X_\beta, \]

for the equals sign marked (\(\ast\)) notice that we have \(-\alpha + (\alpha + \beta) - \beta = 0\) and therefore by Proposition 3.3(e),(a) and Equation (5): \(c_{-\alpha, (\alpha + \beta)} = c_{-\alpha, -\alpha} = -c_{-\alpha, -\alpha} = c_{\alpha, \beta}\). On the other hand we also have

\[ [X_\alpha, [X_\alpha, X_\beta]] = [X_\alpha, [X_\alpha, X_\beta]] = c_{\alpha,\beta} [X_\alpha, X_{-\alpha + \beta}] = c_{\alpha,\beta} a_{\alpha + \beta} [X_{-\alpha}, X_{-\alpha} - \beta] = c_{\alpha,\beta} a_{\alpha + \beta} c_{\alpha, (-\alpha - \beta)} X_\beta, \]

for the equals sign marked (i) notice that because of \(\alpha + (-\alpha - \beta) + \beta = 0\) we have by Proposition 3.3(e),(a) \(c_{\alpha, (-\alpha - \beta)} = c_{\alpha, \beta} = -c_{\alpha, \beta}\). By comparing the preceding two calculations we obtain Equation (7).

Now put \(z_\alpha := 1/\sqrt{-a_\alpha}\) for every \(\alpha \in \Delta\). Then it follows from Equation (7) that we have \(z_\alpha \cdot z_{-\alpha} = 1\), and it follows from Equation (7) that we have \(\bar{z}_\alpha := z_\alpha \cdot X_\alpha = \frac{\alpha + \beta}{\alpha - \beta} \in \{\pm 1\}\). By Proposition 3.3(a), \((\tilde{X}_\alpha)\) with \(\tilde{X}_\alpha := z_\alpha \cdot X_\alpha\) therefore is another Chevalley basis of \( g^F \), and we have

\[ \tilde{X}_\alpha = \frac{\alpha + \beta}{\alpha - \beta} \cdot X_\alpha = \alpha X_{-\alpha} = -\frac{1}{\alpha} X_{-\alpha} = -z_{-\alpha} X_{-\alpha} = -\tilde{X}_{-\alpha}, \]

hence the new Chevalley basis \((\tilde{X}_\alpha)\) enjoys property (i) of the proposition.

We now suppose that \((X_\alpha)\) is a Chevalley basis of \( g^F \), with Chevalley constants \((c_{\alpha,\beta})\), which satisfies property (i), and further satisfies \(c_{\alpha,\eta_\alpha} > 0\) for every \(\alpha \in \Delta_+\) with \(2 \leq \ell(\alpha) \leq \ell - 1\). Then put

\[ z_\alpha := \begin{cases} \frac{\alpha + \beta}{\alpha - \beta} & \text{for } \alpha \in \Delta_+ \text{ with } \ell(\alpha) \neq \ell \\ \text{sign}(c_{\alpha,\eta_\alpha}) & \text{for } \alpha \in \Delta_+ \text{ with } \ell(\alpha) = \ell. \end{cases} \]

In this way we have \(z_\alpha \in \{\pm 1\}\) in any case, and therefore \(z_\alpha \cdot z_{-\alpha} = 1\) and \(\varepsilon_{\alpha,\beta} := \frac{z_\alpha \cdot z_{-\beta}}{z_{-\alpha} \cdot z_{-\beta}} \in \{\pm 1\}\). Therefore Proposition 3.3(a) shows that with \(\tilde{X}_\alpha := z_\alpha \cdot X_\alpha\) and \(\tilde{c}_{\alpha,\beta} := \varepsilon_{\alpha,\beta} \cdot c_{\alpha,\beta}\), \((\tilde{X}_\alpha)\) is another Chevalley basis of \( g^F \), its Chevalley constants being \((\tilde{c}_{\alpha,\beta})\). For any \(\alpha \in \Delta\) we have \(\tilde{X}_\alpha = z_{\alpha} \cdot X_{-\alpha} = -z_{-\alpha} \cdot X_{-\alpha} = -\tilde{X}_{-\alpha}\), hence the Chevalley basis \((\tilde{X}_\alpha)\) satisfies property (i). Moreover, for any \(\alpha \in \Delta\) with \(2 \leq \ell(\alpha) \leq \ell - 1\) we have \(z_{\alpha} = z_{\alpha} = z_{\eta_\alpha} = 1\), hence \(c_{\alpha,\eta_\alpha} = c_{\alpha,\eta_\alpha} = c_{\alpha,\eta_\alpha} > 0\), and for any \(\alpha \in \Delta\) with \(\ell(\alpha) = \ell\) we have \(z_{\alpha} = z_{\alpha} = 1\) and \(z_{\alpha} = \text{sign}(c_{\alpha,\eta_\alpha})\), hence \(c_{\alpha,\eta_\alpha} = \text{sign}(c_{\alpha,\eta_\alpha})\) and therefore \(c_{\alpha,\eta_\alpha} = c_{\alpha,\eta_\alpha} > 0\). Thus we have shown \(c_{\alpha,\eta_\alpha} > 0\) for all \(\alpha \in \Delta_+\) with \(2 \leq \ell(\alpha) \leq \ell\).

We postpone the proof of the uniqueness statement for the Chevalley constants until after we have given the algorithm for calculating these constants. \(\square\)

To completely describe the Lie bracket of \( g^F \), it suffices to fix a Chevalley basis \((X_\alpha)\) of \( g^F \), and calculate the corresponding Chevalley constants \((c_{\alpha,\beta})\). Up to sign, they are determined by Proposition 3.3(g). However, to determine the sign correctly, we have to invest a bit more work. For this we now how to do it, if \((X_\alpha)\) is a Chevalley basis of the kind of Proposition 3.3 corresponding to a family of decompositions \((\alpha = \zeta_\alpha + \eta_\alpha)_{\alpha \in \Delta_+ \setminus \Pi}\).

By Proposition 3.3(e),(a) and Equation (5) we have for any \(\alpha, \beta \in \Delta_+\) with \(\beta \neq \alpha\)

\[ c_{\alpha,\beta} = \begin{cases} c_{(\beta - \alpha),\alpha} & \text{if } \beta - \alpha \in \Delta_+ \\ c_{(\alpha - \beta),\beta} & \text{if } \beta - \alpha \in (-\Delta_+) \text{ and } c_{-\alpha,\beta} = -c_{\alpha,\beta} \end{cases} \]  

\[ \text{otherwise.} \]
Therefore it suffices to calculate $c_{\alpha,\beta}$ for $\alpha, \beta \in \Delta_+$. This is achieved by the following algorithm.

(C1) [Set $c_{\alpha,\beta}$ with $\alpha + \beta \not\in \Delta$.] Iterate the following for all $\alpha, \beta \in \Delta_+$: If $\alpha + \beta \not\in \Delta_+$, put $c_{\alpha,\beta} := 0$.

(C2) [Iterate on level.] Iterate steps (C3)–(C8) for $\ell = 2, \ldots, L$, where $L$ denotes the maximal level of roots occurring in $\Delta$.

(C3) [Iterate on roots of level $\ell$.] Iterate steps (C4)–(C8) with $\alpha$ running through all positive roots in $\Delta$ of level $\ell$.

(C4) Set $\zeta := \zeta_\alpha$ and $\eta := \eta_\alpha$.

(C5) [Calculate $c_{\zeta,\eta}$ and $c_{\eta,\zeta}$.] Let $p$ be the smallest integer so that $\eta - (p + 1) \zeta \not\in \Delta$ holds, and put $q := p - 2\frac{\langle \eta, \zeta \rangle}{\| \zeta \|^2}$.

$$c_{\zeta,\eta} := \sqrt{\frac{q \cdot (1 + p)}{2} \cdot \| \zeta \|}$$

and $c_{\eta,\zeta} := -c_{\zeta,\eta}$.

(C6) [Iterate on the decompositions of $\lambda$.] Iterate step (C7) for all pairs $(\gamma, \delta)$ of positive roots with $\gamma + \delta = \alpha$ and $\gamma, \delta \not\in \{ \zeta, \eta \}$.

(C7) [Calculate $c_{\gamma,\delta}$.] Put

$$c_{11} := \begin{cases} c_{\eta,(\gamma-\eta)} & \text{if } \gamma - \eta \in \Delta_+ \\ c_{\eta,(\gamma-\eta)} & \text{if } \gamma - \eta \in -\Delta_+ \\ 0 & \text{otherwise} \end{cases}$$

$$c_{12} := \begin{cases} c_{\zeta,(\zeta-\gamma)} & \text{if } \zeta - \gamma \in \Delta_+ \\ c_{\zeta,(\zeta-\gamma)} & \text{if } \zeta - \gamma \in -\Delta_+ \\ 0 & \text{otherwise} \end{cases}$$

$$c_{21} := \begin{cases} c_{(\zeta-\gamma),\gamma} & \text{if } \zeta - \gamma \in \Delta_+ \\ c_{(\zeta-\gamma),\gamma} & \text{if } \zeta - \gamma \in -\Delta_+ \\ 0 & \text{otherwise} \end{cases}$$

$$c_{22} := \begin{cases} c_{\zeta,(\zeta-\delta)} & \text{if } \zeta - \delta \in \Delta_+ \\ c_{\zeta,(\zeta-\delta)} & \text{if } \zeta - \delta \in -\Delta_+ \\ 0 & \text{otherwise} \end{cases}$$

Then put $c_{\gamma,\delta} := \frac{1}{c_{\zeta,\eta}} \cdot (c_{11} c_{12} + c_{21} c_{22})$.

(C8) (End of loops.)

Proof for the correctness of the algorithm. It is clear that the assignment $c_{\alpha,\beta} := 0$ for every $\alpha, \beta \in \Delta_+$ with $\alpha + \beta \not\in \Delta_+$ in step (C1) is correct, and that within the loop (C2)–(C8), every $c_{\alpha,\beta}$ with $\alpha, \beta, \alpha + \beta \in \Delta_+$ is assigned to exactly once (either in step (C5) or in step (C7)). We have to show that the latter assignments are correct.

In step (C5), the numbers $p$ and $q$ are chosen such that $\{ \eta + k \zeta \mid - p \leq k \leq q \}$ is the $\zeta$-string through $\eta$ (for the correctness of $q$, see Equation (9)). Therefore, and because we have $c_{\zeta,\eta} > 0$ by definition, Proposition 3.11(g) shows that the assignment to $c_{\zeta,\eta}$ in step (C5) is correct. The correctness of the following assignment $c_{\eta,\zeta} := -c_{\zeta,\eta}$ follows by Proposition 3.13(a).

For the correctness of the assignment in step (C7): In the situation of that step, we have $\zeta + \eta = \alpha = \gamma + \delta$ and therefore $-\zeta - \eta + \gamma + \delta = 0$, and none of the four summands in the latter equation is the negative of one of the others. Therefore Proposition 3.13(f) shows that we have

$$c_{-\zeta,-\eta} c_{\gamma,\delta} + c_{-\eta,\gamma} c_{-\zeta,\delta} + c_{\gamma,-\zeta} c_{-\eta,\delta} = 0$$

and thus (note that $c_{-\zeta,-\eta} = -c_{\zeta,\eta}$ by Equation (5))

$$c_{\gamma,\delta} = \frac{1}{c_{\zeta,\eta}} \cdot (c_{-\eta,\gamma} c_{-\zeta,\delta} + c_{\gamma,-\zeta} c_{-\eta,\delta}) \cdot$$

(9)

If $\gamma - \eta$ is not a root, then we have $c_{-\eta,\gamma} = 0$. Otherwise we have either $\gamma - \eta \in \Delta_+$, or else $\gamma - \eta \in -\Delta_+$ and then $\eta - \gamma \in \Delta_+$. In either case, by Proposition 3.13(e)(a) and Equation (9)

$$c_{-\eta,\gamma} (e) = c_{\gamma,(\gamma-\eta)} (e) = c_{\gamma,-\eta} (e) = c_{\eta,(\gamma-\eta)} (e)$$

(10)

holds. Notice that the level of $\eta + (\gamma - \eta) = \gamma$ (if $\gamma - \eta$ is positive) or of $\gamma + (\eta - \gamma) = \eta$ (if $\gamma - \eta$ is negative) is strictly less than $\ell$, and therefore the value of $c_{\eta,(\gamma-\eta)}$ resp. of $c_{\gamma,(\eta-\gamma)}$ has already been calculated by the
4 Reconstructing the symmetric involution

We now suppose that a Riemannian symmetric space $M = G/K$ of compact type is given. Then $G$ is a semisimple Lie group, and the symmetric structure of $M$ is given by an involutive Lie algebra automorphism $\sigma$ of the Lie algebra $\mathfrak{g}$ of $G$. It will now be our objective to describe how to reconstruct the action of $\sigma$ on $\mathfrak{g}$ from the information contained in the Satake diagram of $M$.

$\sigma$ induces the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ of $\mathfrak{g}$, where $\mathfrak{k} := \text{Eig}(\sigma, 1)$ is the Lie algebra of the isotropy group $K$ and $\mathfrak{m} := \text{Eig}(\sigma, -1)$ is a linear subspace of $\mathfrak{g}$ which is isomorphic to the tangent space $T_pM$ in a canonical way.

We let $\mathfrak{a}$ be a maximal flat subspace of $\mathfrak{m}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{a} \subseteq \mathfrak{t}$. Then $\mathfrak{t}$ is invariant under $\sigma$ (see [Lo], Lemma VI.3.2, p. 72).

In the sequel, we apply the results of the preceding sections in this situation. For this we again also consider the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ as a complex Lie algebra. We denote the complexification of $\sigma$ also by $\sigma$; this is an involutive Lie algebra automorphism of $\mathfrak{g}^C$.

The action of $\sigma$ on $\mathfrak{t}$ has been described by Satake in the following way (see, for example, [Lo], Section VII.3.3, p. 132ff.): There exists a partition of the set $\Pi$ of simple roots into two subsets: $\Pi = \{\alpha_1, \ldots, \alpha_r\} \cup \{\beta_1, \ldots, \beta_s\}$ (with $r, s \in \mathbb{N}_0$, $r + s = \text{rk}(\mathfrak{g})$) and an involutive permutation $\pi \in \mathfrak{S}_r$ so that for each $\alpha_k$ ($k = 1, \ldots, r$) we have

$$\sigma(\alpha^\pi_k) = -\alpha^\pi_{\pi(k)} - \sum_{j=1}^{n_{kj}} n_{kj} \beta^\pi_j$$ (11)

with some non-negative integers $n_{kj}$, whereas for each $\beta_k$ ($k = 1, \ldots, s$) we have

$$\sigma(\beta^\pi_k) = \beta^\pi_k.$$ (12)

The Satake diagram of the symmetric space $M$ is obtained by “annotating” the Dynkin diagram of $\mathfrak{g}$ in the following way: We color the node corresponding to each simple root of $\mathfrak{g}$ either white or black, according to whether it is one of the $\alpha_k$ or one of the $\beta_k$ (these simple roots are thus called white roots and black roots, respectively), and wherever two white roots are interchanged by the involutive permutation $\pi$, we indicate this fact by drawing a curved both-sided arrow between them.

We now show how to reconstruct the action of $\sigma$ on $\mathfrak{g}^C$ (or on $\mathfrak{g}$) from the Satake diagram of $M$.

First, it was already described by Satake how to reconstruct the action of $\sigma$ on $\mathfrak{t}$: For this, we note that the partition $\Pi = \{\alpha_1, \ldots, \alpha_r\} \cup \{\beta_1, \ldots, \beta_s\}$ of the set of simple roots into the sets of white resp. black roots, and the involutive permutation $\pi \in \mathfrak{S}_r$ can be read off the Satake diagram of $M$ immediately. To apply formulas (11) and (12) to obtain the action of $\sigma$ on the
simple roots, we therefore only need to determine the numbers $n_{kj}$ occurring in Equation (11), and this can be done in the following way: From Equation (11) it follows, putting $\beta_k^\alpha := \frac{2\beta_k^\alpha}{\|\beta_k^\alpha\|^2}$:

$$
\sum_{j=1}^s \beta_j(\beta_k^\alpha) n_{kj} = -\alpha_k(\sigma(\beta_k^\alpha)) - \alpha_{\pi(k)}(\beta_k^\alpha) \quad (\text{12})
- \alpha_k(\beta_k^\alpha) - \alpha_{\pi(k)}(\beta_k^\alpha).
$$

The numbers $\beta_j(\beta_k^\alpha)$, $\alpha_k(\beta_k^\alpha)$ and $\alpha_{\pi(k)}(\beta_k^\alpha)$ are known from the Dynkin diagram of $g$. Because the matrix $(\beta_j(\beta_k^\alpha))_{j=1,\ldots,s}$ is invertible (it is the Cartan matrix of the Lie algebra $\mathfrak{t}^\mathbb{C} = \{ X \in \mathfrak{t} \mid [X,a] = 0 \}$), we can solve for $n_{kj}$. Because $\Pi$ is a basis of it, from Equations (11) and (12) we therefrom know the action of $\sigma$ on all of $\mathfrak{t}^\mathbb{C}$.

But now also we want to describe the action of $\sigma$ on the root spaces of $g^\mathbb{C}$. For this, we let a Chevalley basis $(X_\alpha)$ of $g^\mathbb{C}$ of the kind of Proposition 3.5 be given, in particular we have for any $\alpha \in \Delta$

$$
X_{-\alpha} = -X_\alpha.
$$

We denote the Chevalley constants corresponding to $(X_\alpha)$ by $(c_{\alpha,\beta})$. Because $\sigma$ is an involutive automorphism of $g$, we have for every $\alpha \in \Delta$: $\sigma(X_\alpha) \in g^\mathbb{C}_{\sigma(\alpha)} = g^\mathbb{C}_{\sigma(\alpha)}$ (where we write also $\sigma(\alpha)$ for $\sigma(\alpha)\sigma^{-1}$ in the sequel; note that with this notation, $(\sigma(\alpha))^{\sharp} = \sigma(\alpha^{\sharp})$ holds) and therefore there exists $s_\alpha \in \mathbb{C}$ with

$$
\sigma(X_\alpha) = s_\alpha \cdot X_{\sigma(\alpha)}.
$$

Clearly the action of $\sigma$ on the root spaces is determined completely by the constants $s_\alpha$, and it is our objective in the following to determine these constants.

**Proposition 4.1** Suppose $\alpha, \beta \in \Delta$.

(a) $s_{-\alpha} = s_{\alpha}^{-1} = s_{\sigma(\alpha)}$.

(b) $|s_\alpha| = 1$.

(c) If $\alpha + \beta \in \Delta$ holds, we have $s_{\alpha + \beta} = \frac{c_{\alpha,\beta}}{c_{\alpha,\beta}} s_\alpha s_\beta$.

(d) If $\sigma(\beta) = \beta$ holds, we have $s_\beta = 1$.

**Proof.** For (a), By the involutivity of $\sigma$ we have $X_\alpha = \sigma^2(X_\alpha) \equiv \sigma(s_\alpha X_{\sigma(\alpha)}) \equiv s_\alpha s_{\sigma(\alpha)} X_\alpha$ and therefore $1 = s_\alpha s_{\sigma(\alpha)}$. We also have $\sigma(\alpha^{\sharp}) \equiv \sigma([X_\alpha,X_{-\alpha}]) = [\sigma(X_\alpha),\sigma(X_{-\alpha})] \equiv s_\alpha s_{-\alpha} [X_{\sigma(\alpha)},X_{-\sigma(\alpha)}] \equiv s_\alpha s_{-\alpha} \sigma(\alpha^{\sharp})$ and hence $1 = s_\alpha s_{-\alpha}$.

For (b). We have $\frac{s_{-\alpha}}{s_\alpha} X_{\sigma(\alpha)} \equiv \sigma(\alpha^{\sharp}) \equiv \sigma([X_\alpha,X_{-\alpha}]) = [\sigma(X_\alpha),\sigma(X_{-\alpha})] \equiv s_\alpha s_{-\alpha} [X_{\sigma(\alpha)},X_{-\sigma(\alpha)}] \equiv s_\alpha s_{-\alpha} \sigma(\alpha^{\sharp})$ and hence $1 = s_{-\alpha} s_\alpha$.

For (c). We have $s_{\alpha + \beta} X_{\sigma(\alpha + \beta)} \equiv \sigma(X_{\alpha + \beta}) \equiv \frac{1}{c_{\alpha,\beta}} \sigma([X_\alpha,X_\beta]) = \frac{1}{c_{\alpha,\beta}} [\sigma(X_\alpha),\sigma(X_\beta)]$.

For (d). We now suppose that $\sigma(\beta) = \beta$ holds. Then we have $s_\beta \in \{ \pm 1 \}$ by (a). Assume that $s_\beta = -1$ holds. Then with $v : = X_\beta - X_{-\beta} \in g$ we have $\sigma(v) = -v$ and therefore $v \in \mathfrak{m}$. By hypothesis we have $\beta \circ \sigma = \beta$, and therefore $\beta$ vanishes on $\mathfrak{a} \subset \text{Eig}(\sigma, -1)$. For every $H \in \mathfrak{a}$ we therefore have $[H,v] = [H,X_\beta] - [H,X_{-\beta}] = \beta(H) \cdot (X_\beta + X_{-\beta}) = 0$, and thus $a \oplus \mathbb{R}v$ is a flat subspace of $\mathfrak{m}$, in contradiction to the maximality of $\mathfrak{a}$. Therefore we have $s_\beta = 1$.

**Proposition 4.2** Suppose that $M$ is irreducible.

(a) (i) If $M$ is of one of the types $AIII(n,q) = SU(n)/S(U(q) \times U(n-q))$ with $2q < n - 1$, $DIII(n) = SO(2n)/U(n)$ with $n$ odd, or $EIII = E_6/(U(1) \cdot SO(10))$, we label simple roots as in the following Satake diagrams:
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\[ AIII(n, q) \]
with \( 2q < n - 1 \)

\[ DIII(n) \]
with \( n \) odd

\[ EIII \]

Then we have

\[ s_{\alpha_\pi(1)} = \frac{c_{\beta, \alpha_1}}{c_{\alpha_\pi(1), \beta}} s_{\alpha_1} \quad \text{and} \quad s_{\alpha_\pi(k)} = s_{\alpha_k} \quad \text{for} \ k \notin \{1, \pi(1)\}. \]

(ii) If \( M \) is of a type not mentioned in (i), we have \( s_{\alpha_\pi(k)} = s_{\alpha_k} \) for all \( k \).

(b) \( \sigma \) is congruent to another involutive automorphism \( \tilde{\sigma} \) of \( \mathfrak{g} \) so that (with \( \tilde{s}_\alpha \) being defined for \( \tilde{\sigma} \) analogous to Equation \( 1.2 \)) we have:

(i) If \( M \) is of one of the types \( AIII(n, q) \) with \( 2q < n - 1 \), \( DIII(n) \) with \( n \) odd, and \( EIII \),

then \( \tilde{s}_{\alpha_\pi(1)} = \frac{c_{\beta, \alpha_1}}{c_{\alpha_\pi(1), \beta}} \) and \( \tilde{s}_{\alpha_k} = 1 \) for all \( k \neq \pi(1) \) (where \( \pi \), \( \alpha_k \) and \( \beta \) have the same meaning as in (a)(i)).

(ii) If \( M \) is of any other type, then \( \tilde{s}_{\alpha_k} = 1 \) for all \( k \).

In either case, we have \( \tilde{s}_\alpha \in \{ \pm 1 \} \) for all \( \alpha \in \Delta \).

Remarks 4.3 (a) If \( M \) is not irreducible, then the statement of Proposition 4.2 holds within each irreducible factor of \( M \).

(b) The quotient \( \frac{c_{\beta, \alpha_1}}{c_{\alpha_\pi(1), \beta}} \) occurring in Proposition 4.2 is always \( \in \{ \pm 1 \} \), and the Chevalley basis of \( \mathfrak{g} \) can be chosen so that it equals 1; such a Chevalley basis can, for example, be obtained by applying Proposition 3.5 resp. Algorithm (C) with \( \zeta_{\beta + \alpha_1} = \beta \), \( \eta_{\beta + \alpha_1} = \alpha_1 \), \( \zeta_{\alpha_\pi(1) + \beta} = \alpha_{\pi(1)} \) and \( \eta_{\alpha_\pi(1) + \beta} = \beta \). If we use such a Chevalley basis, then the involutive automorphism \( \tilde{\sigma} \) of Proposition 4.2(b) satisfies \( \tilde{s}_{\alpha_k} = 1 \) for all \( k \) in any case.

(c) Even if the Chevalley basis is set up so that we have \( \tilde{s}_\alpha = 1 \) for every simple root \( \alpha \), we do not generally have \( \tilde{s}_\alpha = 1 \) for all \( \alpha \in \Delta \).

Proof. For (a). If \( \pi = \text{id} \) holds (i.e. the Satake diagram of \( M \) has no arrows), then there is nothing to show. If \( M \) has no black roots (this is also the case if \( M \) is a simple Lie group seen as symmetric space), then it follows from Equation 1.1 that for any \( k \in \{1, \ldots, r\} \), \( \sigma(\alpha_k) = -\alpha_{\pi(k)} \) holds, and therefore we have by Proposition 4.1(a):

\[ s_{\alpha_k} = s_{-\sigma(\alpha_k)} = s_{\alpha_{\pi(k)}}. \]

Thus it only remains to consider the spaces whose Satake diagrams contain both arrows and black roots, and an inspection of the Satake diagrams of all the irreducible Riemannian symmetric spaces (see for example [14], p. 147f.) shows that they are only the spaces of type \( AIII(n, q) \) with \( 2q < n - 1 \), \( DIII(n) \) with \( n \) odd, and \( EIII \).
Let us therefore now suppose that \( M \) is of one of these three types. Then we have \( \sigma(\beta) = \beta \), \( -\sigma(\alpha_1) = \alpha_1 + \beta \) and \( -\sigma(\alpha_{1(1)}) = \alpha_1 + \beta \) (with the roots \( \alpha_1 \), \( \alpha_{1(1)} \) and \( \beta \) being defined as in the relevant part of the proposition) and therefore

\[
\sigma_\alpha \equiv s_{-\sigma(\alpha_1)} = s_{\alpha_{1(1)}} + \beta = \frac{c_{\sigma(\alpha_{1(1)})}, \sigma(\beta)}{c_{\alpha_{1}}(1)} \cdot s_\beta \cdot s_\alpha = \frac{c_{-\alpha_1(1), \beta}}{c_{\alpha_{1}}(1)} \cdot s_{\alpha_{1(1)}} \cdot s_\alpha.
\]

(Herein, the letters \( (a) \), \( (c) \) and \( (d) \) refer to the respective parts of Proposition 4.1 and the equals sign marked \( (\ast) \) follows from Proposition 3.3(e).)

Moreover, if \( M \) is of the type \( A_{2l}(n, q) \) with \( 2q < n - 1 \), then we have \( \sigma(\alpha_k) = -\alpha_{1(k)} \) for any \( k \not\in \{1, \pi(1)\} \) and therefore for such \( k \) by Proposition 4.1(a): \( s_{\alpha_k} = s_{-\sigma(\alpha_k)} = s_{\alpha_{1(k)}} \). On the other hand, if \( M \) is of type \( D_{2l}(2n + 1) \) or of type \( E_{III} \), then we have \( \pi(k) = k \) for all \( k \not\in \{1, \pi(1)\} \), and therefore \( s_{\alpha_{1(k)}} = s_{\alpha_k} \) then trivially holds.

For \( (b) \). For arbitrary \( H \in \mathfrak{t} \), the map \( \text{ad}(H) \) is a derivation of \( \mathfrak{g} \) with \( \text{ad}(H)|_{\mathfrak{t}} = 0 \), therefore \( B := \exp(\text{ad}(H)) \) is a Lie algebra automorphism of \( \mathfrak{g} \) with \( B|_{\mathfrak{t}} = \text{id}_{\mathfrak{t}} \). Hence \( \tilde{\sigma} := B \circ \sigma \circ B^{-1} \) then is another involutive automorphism of \( \mathfrak{g} \) with \( \tilde{\sigma}|_{\mathfrak{t}} = \sigma|_{\mathfrak{t}} \). Thus \( \tilde{\sigma} \) describes the same symmetric structure as \( \sigma \) does, and we can define \( \tilde{s}_\alpha \) with respect to \( \tilde{\sigma} \) analogous to Equation 14. Doing so, the results of the present section, especially part (a) of the present proposition, are true mutatis mutandis with \( \tilde{s}_\alpha \) in the place of \( s_\alpha \).

Because of (a) it therefore suffices to show that \( H \) can be chosen in such a way that for every \( k \in \{1, \ldots, r\} \) at least one of the equations \( \tilde{s}_{\alpha_k} = 1 \) and \( \tilde{s}_{\sigma(\alpha_k)} = 1 \) holds. For this purpose, for every \( k \) we let \( \tau(k) = \tau(\pi(k)) \) be an arbitrarily chosen element of \( \{k, \pi(k)\} \) and let \( t_k \in \mathbb{R} \) be such that

\[
s_{\alpha_k} = e^{it_k}
\]

holds. Then we let \( H \in \mathfrak{t} \) be the element characterized by

\[
\langle H, \alpha_k^\sharp \rangle = \langle H, \alpha_k^\sharp \rangle = \frac{1}{2} it_k \tau(k) \quad \text{for every } k
\]

and

\[
\langle H, \beta_k^\sharp \rangle = 0 \quad \text{for every black root } \beta_k.
\]

Then we have for every \( k \in \{1, \ldots, r\} \)

\[
\text{ad}(H) X_{\alpha_k} = [H, X_{\alpha_k}] = \alpha_k(H) X_{\alpha_k} = \langle H, \alpha_k^\sharp \rangle X_{\alpha_k} = \frac{1}{2} it_k \tau(k) X_{\alpha_k}
\]

and for every \( k \in \{1, \ldots, s\} \)

\[
\text{ad}(H) X_{\beta_k} = [H, X_{\beta_k}] = \beta_k(H) X_{\beta_k} = \langle H, \beta_k^\sharp \rangle X_{\beta_k} = 0.
\]

Therefore we obtain

\[
B(X_{\alpha_k}) = e^{\frac{1}{2} it_k \tau(k)} X_{\alpha_k}
\]

and – with the notation from Equation 11 –

\[
B(X_{\sigma(\alpha_k)}) = \langle H, \sigma(\alpha_k^\sharp) \rangle X_{\sigma(\alpha_k)} = \langle H, -\alpha_k^\sharp \rangle - \sum_{j=1}^{s} n_{kj} \beta_j^\sharp X_{\sigma(\alpha_k)} = e^{-\frac{1}{2} it_k \tau(k)} X_{\sigma(\alpha_k)}.
\]

Therefrom it follows that

\[
\tilde{\sigma}(X_{\sigma(\alpha_k)}) = (B \circ \sigma \circ B^{-1})(X_{\sigma(\alpha_k)}) = (B \circ \sigma)(e^{-\frac{1}{2} it_k \tau(k)} X_{\sigma(\alpha_k)}) = B(s_{\alpha_{1(k)}} e^{-\frac{1}{2} it_k \tau(k)} X_{\sigma(\alpha_{1(k)})})
\]

and therefore \( \tilde{s}_{\alpha_{1(k)}} = 1 \) holds.

We are now able to reconstruct the action of \( \sigma \) on the root spaces of \( \mathfrak{g}^{\mathbb{F}} \). For this we may suppose without loss of generality that the Chevalley basis \( (X_\alpha) \) and the involutive automorphism \( \sigma \) are adapted to each other in such a way that \( \sigma \) equals the automorphism \( \tilde{\sigma} \) from Proposition 4.1(b). The following algorithm then computes the \( s_\alpha \) for \( \alpha \in \Delta_+ \); by Proposition 4.1(a) we then know \( s_\alpha \) for all \( \alpha \in \Delta \).

(S1) [Compute \( s_\alpha \) for \( \alpha \in \Pi \).]
5 Formulas for the fundamental geometric tensors

- If $M$ is of one of the types $A_{II}(n,q)$ with $2q < n-1$, $D_{III}(n)$ with $n$ odd, and $E_{III}$, then put
  \[ s_{\alpha_{\pi(1)}} := \frac{c_{\beta,\alpha}}{c_{\alpha_{\pi(1)},\beta}} \quad \text{and} \quad s_{\alpha} := 1 \quad \text{for all} \quad \alpha \in \Pi \setminus \{\alpha_{\pi(1)}\}, \]
  where $\alpha_k$ and $\beta$ have the same meaning as in Proposition 4.2(a)(i).

- If $M$ is of any other type, then put $s_{\alpha} := 1$ for all $\alpha \in \Pi$.

(S2) [Iterate on level.] Iterate steps (S3)–(S6) for $\ell = 2, \ldots, L$, where $L$ denotes the maximal level of roots occurring in $\Delta$.

(S3) [Iterate on roots of level $\ell$.] Iterate steps (S4)–(S6) with $\alpha$ running through all the roots in $\Delta$ of level $\ell$.

(S4) [Find a decomposition of $\alpha$.] Let $\zeta, \eta$ be positive roots so that $\alpha = \zeta + \eta$ holds.

(S5) [Compute $s_{\alpha}$.] Put $s_{\alpha} := \frac{c_{\sigma\zeta,\sigma\eta}}{c_{\zeta,\eta}} s_{\zeta} s_{\eta}$.

(S6) [End of loops.]

Remark 4.4 For the decomposition $\alpha = \zeta + \eta$ of the positive root $\alpha$ requested in step (S4) of the algorithm, we can again use the decomposition $\alpha = \zeta + \eta_\alpha$ already used in the computation of the Chevalley constants by algorithm (C), compare Remark 3.6.

Proof for the correctness of the algorithm. Notice that each of the $s_{\alpha}$ with $\alpha \in \Delta_+$ is assigned to exactly once in the course of the algorithm, either in step (S1) (if $\alpha$ is simple), or in step (S5) (if $\alpha$ is not simple). The assignments in step (S1) are correct by Proposition 4.2(b) (for the white roots) and Proposition 4.1(d) (for the black roots). The correctness of the assignments in step (S5) follow by induction on the level $\ell$ via Proposition 4.1(c); note that $\ell(\zeta), \ell(\eta) < \ell$ holds in the situation of that step. □

5 Formulas for the fundamental geometric tensors

We are now ready to describe explicitly first the Lie bracket of $\mathfrak{g}$, the inner product induced by its Killing form, and the involution $\sigma$, thereby also the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ induced by $\sigma$, and then the fundamental geometric tensors of $M$, namely the inner product on $\mathfrak{m}$ and the curvature tensor $R$ of $M$.

We continue to use the notations of the preceding sections. In particular, we fix a Chevalley basis $(X_\alpha)$ of $\mathfrak{g}^k$ of the kind of Proposition 3.5 denote the corresponding Chevalley constants by $(c_{\alpha,\beta})$ – they can be calculated by algorithm (C) – and consider the quantities $n_{kj}$ and $s_{\alpha}$ describing the involution $\sigma$ as defined by Equations (11) and (14). We suppose that $\sigma$ is of the kind described in Proposition 4.2(b), so that we have $s_{\alpha} \in \{\pm 1\}$ for all $\alpha \in \Delta$, then the $s_{\alpha}$ can be calculated by algorithm (S).

To describe the relevant tensors on $\mathfrak{g}$ (the Lie bracket, the inner product, and $\sigma$), it suffices to describe the behavior of $\mathfrak{k}$ and of the root spaces $\mathfrak{g}_\alpha$ with respect to these maps, because of the root space decomposition (1).

Proposition 5.1 For any $\alpha \in \Delta$ and $z \in \mathfrak{C}$ we put $V_\alpha(z) := \frac{1}{\sqrt{2}}(z X_\alpha - \bar{z} X_{-\alpha})$. For formal reasons we put $V_\alpha(z) := 0$, whenever $\alpha \in \mathfrak{k}^*$ is a linear form which is not a root and $z \in \mathfrak{C}$.

Then we have for any $\alpha, \beta \in \Delta$:
(a) $\mathfrak{g}_\alpha = \{V_\alpha(z) \mid z \in \mathfrak{C}\}$.
(b) For $H \in \mathfrak{k}$ and $z, z' \in \mathfrak{C}$ we have

\[ [H, V_\alpha(z)] = V_\alpha(\alpha(H) z) \]
and

\[
[V_\alpha(z), V_\beta(z')] = \begin{cases} \\
\frac{1}{\sqrt{2}} (c_{\alpha,\beta} V_{\alpha+\beta}(z z') - c_{\alpha,-\beta} V_{\alpha-\beta}(z z')) & \text{for } \beta \notin \{\pm \alpha\} \\
\Im (\tau z') i\alpha^2 & \text{for } \beta = \alpha \\
\Im (z z') i\alpha^2 & \text{for } \beta = -\alpha \\
\end{cases}.
\]

(c) \( t \) is orthogonal to \( g_\alpha \) for every \( \alpha \in \Delta \); \( g_\alpha \) and \( g_\beta \) are orthogonal for every \( \alpha, \beta \in \Delta \) with \( \alpha \notin \{\pm \beta\} \); and besides the description of \( \langle \cdot, \cdot \rangle \) on \( t \times t \) in Section 4 we have for every \( \alpha \in \Delta \), \( z, z' \in \mathfrak{c} \)

\[
\langle V_\alpha(z), V_\alpha(z') \rangle = \Re (z \cdot z') .
\]

(d) Besides Equations (11) and (12), which describe the action of \( \sigma \) on \( t \), we have for any \( \alpha \in \Delta \) and \( z \in \mathfrak{c} \)

\[
\sigma(V_\alpha(z)) = s_\alpha \cdot V_{\sigma(\alpha)}(z) .
\]

Proof. For (a). Because of the property \( \overline{X}_\alpha = -X_{-\alpha} \) of the Chevalley basis (see property (i) in Proposition 5.2), we have \( \{ V_\alpha(z) \mid z \in \mathfrak{c} \} = (\mathfrak{g}^* \oplus \mathfrak{g}^* \alpha) \cap \mathfrak{g} = g_\alpha \).

For (b). This is a straightforward computation, involving the definition of \( V_\alpha(z) \) and Equations (5) and (6).

For (c). The stated pairwise orthogonality of \( t \), \( g_\alpha \) and \( g_\beta \) for \( \beta \notin \{\pm \alpha\} \) is well-known, and for \( \alpha \in \Delta \), \( z, z' \in \mathfrak{c} \) we have

\[
\langle V_\alpha(z), V_\alpha(z') \rangle = -\frac{1}{2} \langle X_\alpha - \tau X_{-\alpha}, z' X_\alpha - \overline{\tau} X_{-\alpha} \rangle \overset{(1)}{=} \frac{1}{2} \langle \tau z' + \overline{\tau} z' \rangle \Re (X_\alpha, X_{-\alpha}) \overset{(1)}{=} \Re(z z') ,
\]

where the equals sign marked (\( \ast \)) follows from the fact that \( \Re (X_\alpha, X_{-\alpha}) = 0 \) holds, and the equals sign marked (\( \dagger \)) follows from Proposition 5.2(b).

For (d). This follows immediately from the definition of \( V_\alpha(z) \) and Equation (13). \( \square \)

Let us now consider the decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \) induced by \( \sigma \). We have the root space decompositions

\[
\mathfrak{t} = (t \cap \mathfrak{t}) \oplus \bigoplus_{\alpha \in \Delta_+^\sigma} \mathfrak{t}_\alpha \quad \text{and} \quad \mathfrak{m} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta_+^\sigma} \mathfrak{m}_\alpha ,
\]

(18)

where \( \Delta_+^\sigma \subset \Delta_+ \) is a subset such that for every \( \alpha \in \Delta_+ \), exactly one of the roots \( \alpha, \sigma(\alpha), -\sigma(\alpha) \) (not necessarily pairwise unequal) is a member of \( \Delta_+^\sigma \), and where for any \( \alpha \in (\mathfrak{t}^*)^* \) we put \( \mathfrak{t}_\alpha \) := \( (\mathfrak{g}_\alpha + \mathfrak{g}_{\sigma(\alpha)}) \cap \mathfrak{t} \) and \( \mathfrak{m}_\alpha := (\mathfrak{g}_\alpha + \mathfrak{g}_{\sigma(\alpha)}) \cap \mathfrak{m} \).

To describe the fundamental geometric tensors of \( M \) on \( \mathfrak{m} \), it again suffices to describe the behavior of \( \mathfrak{a} \) and the root spaces \( \mathfrak{m}_\alpha \) with respect to them. We also describe the behavior of the Lie bracket and the inner product with regard to \( t \cap \mathfrak{t} \) and the root spaces \( \mathfrak{t}_\alpha \).

**Proposition 5.2** (a) Let \( \alpha \in \Delta \) be given.

(i) If \( \sigma(\alpha) \notin \{\pm \alpha\} \) holds, we put for any \( z \in \mathfrak{c} \)

\[
K_\alpha(z) := \frac{1}{\sqrt{2}} (V_\alpha(z) + s_\alpha V_{\sigma(\alpha)}(z)) \quad \text{and} \quad M_\alpha(z) := \frac{1}{\sqrt{2}} (V_\alpha(z) - s_\alpha V_{\sigma(\alpha)}(z)) .
\]

Then we have \( \mathfrak{t}_\alpha = \{ K_\alpha(z) \mid z \in \mathfrak{c} \} \) and \( \mathfrak{m}_\alpha = \{ M_\alpha(z) \mid z \in \mathfrak{c} \} \).

(ii) If \( \sigma(\alpha) = \alpha \) holds, we have \( \mathfrak{t}_\alpha = \{ V_\alpha(z) \mid z \in \mathfrak{c} \} \) and \( \mathfrak{m}_\alpha = \{ 0 \} \).

(iii) If \( \sigma(\alpha) = -\alpha \) holds, we put for any \( t \in \mathfrak{r} \)

\[
\widetilde{K}_\alpha(t) := \begin{cases} V_\alpha(it) & \text{if } s_\alpha = 1 \\
V_\alpha(t) & \text{if } s_\alpha = -1 \end{cases} \quad \text{and} \quad \widetilde{M}_\alpha(t) := \begin{cases} V_\alpha(t) & \text{if } s_\alpha = 1 \\
V_\alpha(it) & \text{if } s_\alpha = -1 \end{cases} .
\]

Then we have \( \mathfrak{t}_\alpha = \{ \widetilde{K}_\alpha(t) \mid t \in \mathfrak{r} \} \) and \( \mathfrak{m}_\alpha = \{ \widetilde{M}_\alpha(t) \mid t \in \mathfrak{r} \} \).


(b) For $\alpha, \beta \in \Delta$ with $\beta \not\in \{\pm \alpha, \pm \sigma(\alpha)\}$, the spaces $(t \cap t), a, f_\alpha, m_\alpha, f_\beta, m_\beta$ are pairwise orthogonal. Moreover, for any $\alpha \in \Delta$ we have

(i) If $\sigma(\alpha) \not\in \{\pm \alpha\}$: For any $z, z' \in \mathbb{C}$ we have $\langle K_\alpha(z), K_\alpha(z') \rangle = \langle M_\alpha(z), M_\alpha(z') \rangle = \text{Re}(z \cdot \overline{z'})$.

(ii) If $\sigma(\alpha) = \alpha$: For any $z, z' \in \mathbb{C}$ we have $\langle V_\alpha(z), V_\alpha(z') \rangle = \text{Re}(z \cdot \overline{z'})$.

(iii) If $\sigma(\alpha) = -\alpha$: For any $t, t' \in \mathbb{R}$ we have $\langle \tilde{K}_\alpha(t), \tilde{K}_\alpha(t') \rangle = \langle \tilde{M}_\alpha(t), \tilde{M}_\alpha(t') \rangle = t \cdot t'$.

(c) $\alpha, \beta \in \Delta$. To calculate the Lie bracket between elements of $\xi_\alpha \cup m_\alpha$ and $\xi_\beta \cup m_\beta$, we need to distinguish which of the three cases for $\alpha$, namely $\sigma(\alpha) = \alpha$, $\sigma(\alpha) = -\alpha$ or $\sigma(\alpha) \not\in \{\pm \alpha\}$, and similarly which case for $\beta$ holds. By the combination of these cases, and use of the anti-symmetry of the Lie bracket, there are in total six cases, which are handled separately in the following parts of the statement:

| $\sigma(\beta) \not\in \{\pm \beta\}$ | $\sigma(\alpha) \not\in \{\pm \alpha\}$ | $\sigma(\alpha) = \alpha$ | $\sigma(\alpha) = -\alpha$ |
|---------------------------------|---------------------------------|-------------------------|-------------------------|
| $\sigma(\beta) = \beta$        | (i)                             | (ii)                    | (iii)                   |
| $\sigma(\beta) - \beta$        | (iv)                            | (v)                     | (vi)                    |

In the formulas, we have $z, z' \in \mathbb{C}$ and $t \in \mathbb{R}$.

(i) If $\sigma(\alpha) \not\in \{\pm \alpha\}$ and $\sigma(\beta) \not\in \{\pm \beta\}$ holds, we may suppose that either $\beta \not\in \{\pm \alpha, \pm \sigma(\alpha)\}$ or $\beta = \alpha$ holds; indeed we can reduce the case $\beta \in \{\pm \alpha, \pm \sigma(\alpha)\}$ to $\beta = \alpha$ by application of the formulas

$$K_{\sigma(\beta)}(z') = s_{\beta} K_{\beta}(z'), \quad M_{\sigma(\beta)}(z') = -s_{\beta} M_{\beta}(z'), \quad \text{and} \quad K_{-\beta}(z') = -K_{\beta}(\overline{z'}), \quad M_{-\beta}(z') = -M_{\beta}(\overline{z'}).$$

Then we have:

$$[K_\alpha(z), K_\beta(z')] = \left\{ \begin{array}{l}
\mp \left( c_{\alpha, \beta} K_{\alpha+\beta}(z z') - s_{\alpha, -\beta} K_{\alpha-\beta}(\overline{z z'}) + s_{\alpha, \sigma(\alpha), \beta} K_{\alpha+\beta}(z z') - s_{\alpha, \sigma(\alpha), -\beta} K_{\alpha-\beta}(\overline{z z'}) \right)
\pm \text{Im}(\overline{z z'}) (\alpha^2 + \sigma(\beta))^2 + \sqrt{2} s_{\alpha, \sigma(\alpha), \beta} K_{\alpha+\beta}(\text{Im}(\overline{z z'}))
\end{array} \right.$$ for $\beta \not\in \{\pm \alpha, \pm \sigma(\alpha)\}$

for $\beta = \alpha$.

(ii) If $\sigma(\alpha) = \alpha$ and $\sigma(\beta) \not\in \{\pm \beta\}$ holds, we have:

$$[V_\alpha(z), K_\beta(z')] = c_{\alpha, \beta} K_{\alpha+\beta}(z z') - c_{\alpha, -\beta} K_{\alpha-\beta}(z \overline{z'})$$

$$[V_\alpha(z), M_\beta(z')] = c_{\alpha, \beta} M_{\alpha+\beta}(z z') - c_{\alpha, -\beta} M_{\alpha-\beta}(z \overline{z'}).$$

(iii) If $\sigma(\alpha) = -\alpha$ and $\sigma(\beta) \not\in \{\pm \beta\}$ holds, we put $\zeta := i$ if $s_\alpha = 1$, $\zeta := 1$ if $s_\alpha = -1$, then we have

$$[\tilde{K}_\alpha(t), K_\beta(z)] = c_{\alpha, \beta} K_{\alpha+\beta}(\zeta t z) - c_{\alpha, -\beta} K_{\alpha-\beta}(\zeta t \overline{z})$$

$$[\tilde{K}_\alpha(t), M_\beta(z)] = c_{\alpha, \beta} M_{\alpha+\beta}(\zeta t z) - c_{\alpha, -\beta} M_{\alpha-\beta}(\zeta t \overline{z})$$

$$[\tilde{M}_\alpha(t), K_\beta(z)] = c_{\alpha, \beta} M_{\alpha+\beta}(1 + i - \zeta) t z - c_{\alpha, -\beta} M_{\alpha-\beta}(1 + i - \zeta) t \overline{z})$$

$$[\tilde{M}_\alpha(t), M_\beta(z)] = c_{\alpha, \beta} K_{\alpha+\beta}(1 + i - \zeta) t z - c_{\alpha, -\beta} K_{\alpha-\beta}(1 + i - \zeta) t \overline{z})$$

(iv) If $\sigma(\alpha) = \alpha$ and $\sigma(\beta) = \beta$ holds, then $[V_\alpha(z), V_\beta(z')]$ is given by Proposition 5.7(b).

(v) If $\sigma(\alpha) = -\alpha$ and $\sigma(\beta) = \beta$ holds, we again put $\zeta := i$ if $s_\alpha = 1$, $\zeta := 1$ if $s_\alpha = -1$, then we have

$$[\tilde{K}_\alpha(t), V_\beta(z)] = \sqrt{2} c_{\alpha, \beta} K_{\alpha+\beta}(\zeta t z)$$

$$[\tilde{M}_\alpha(t), V_\beta(z)] = \sqrt{2} c_{\alpha, \beta} M_{\alpha+\beta}(\zeta t z).$$
An application: Totally geodesic submanifolds in $SU(3)/SO(3)$

As an application of the preceding construction of the curvature tensor, we show how to classify the Lie triple systems (i.e., those linear subspaces of $m$ which are invariant under the curvature tensor) in the Riemannian symmetric space $SU(3)/SO(3)$. They are exactly the tangent spaces of totally geodesic submanifolds of $SU(3)/SO(3)$ passing through the origin point.

Note that the rank of the symmetric space $M := SU(3)/SO(3)$ is 2, and thus equals the rank of its transvection group $G := SU(3)$. If we consider the splitting $g := su(3) = t \oplus m$ induced by the symmetric structure of $M$, any maximal flat subspace $a$ of $m$ therefore already is a Cartan subalgebra of $g$, and thus the root system $\Delta$ of $g$ equals the “restricted” root system of $m$.

In the present case, this root system is of type $A_2$, so if $\{\alpha_1, \alpha_2\}$ is a system of simple roots in $\Delta$, we have $\Delta = \{\pm \alpha_1, \pm \alpha_2, \pm \alpha_3\}$ with $\alpha_3 := \alpha_1 + \alpha_2$. Moreover, we have $\sigma(\alpha_k) = -\alpha_k$ and thus the root space decomposition

$$m = a \oplus 3 \bigoplus_{k=1}^3 m_{\alpha_k} \quad \text{with} \quad m_{\alpha_k} = \mathbb{R} M_{\alpha_k}(1) \quad \text{for} \quad k = 1, 2, 3$$

(see Equation (19) and Proposition 5.2(a)(iii)). For every $k \in \{1, 2, 3\}$, the linear form $\alpha_k$ is purely imaginary on $a$ because the symmetric space $M$ is of compact type, and thus we have $H_k := (\frac{1}{2} \alpha_k)^2 \in a$.

**Proposition 6.1** Let $\{0\} \neq m' \subseteq m$ be a linear subspace. Then $m'$ is a Lie triple system in $m$ if and only if there exists a maximal flat subspace $a \subset m$ and a system of simple roots $\{\alpha_1, \alpha_2\}$ in the root system $\Delta$ of $m$ (or of $g$) with respect to $a$, so that with $\alpha_3 := \alpha_1 + \alpha_2$ and $H_k$ defined in relation to these $\alpha_k$ as above, $m'$ is of one of the following types:

- **(G)** $m' = \mathbb{R} H$ with some $H \in a$
- **(T)** $m' = a$
- **(S)** $m' = \mathbb{R} H_3 \oplus \mathbb{R} M_{\alpha_3}(1)$
- **(M)** $m' = \mathbb{R} H_3 \oplus \mathbb{R} (M_{\alpha_1}(1) + M_{\alpha_2}(1))$
- **(P)** $m' = a \oplus \mathbb{R} M_{\alpha_3}(1)$
Two Lie triple systems of types other than (G) are congruent to each other under the isotropy action on m if and only if they are of the same type. A Lie triple system is maximal if and only if it is either of type (P) or of type (M).

Remark 6.2 The totally geodesic submanifolds corresponding to Lie triple systems of type (M) — they are isometric to an \( \mathbb{RP}^2 \) of sectional curvature \( \frac{1}{2} \) — are missing from the classification of maximal totally geodesic submanifolds in Riemannian symmetric spaces of rank 2 by CHEN and NAGANO in [CN] (Notice that they are not contained in the local sphere products corresponding to type (P)).

Proof of Proposition 6.1 First we note that the spaces described in the proposition are indeed Lie triple systems: For the types (G) and (T) this is true because they are flat, for the types (P) and (S) it is true because they correspond to the closed root subsystem \( \{ \pm \alpha_2 \} \) of \( \Delta \), for type (M): In the setting of that type we put \( H := H_3 \in a \) and \( v := M_{\alpha_2}(1) + M_{\alpha_3}(1) \), then we have \( \alpha_1(H) = \alpha_2(H) \), hence \( R(H, v)H \in \mathbb{R}v \subset m' \); we also see \( R(H, v)v \in \mathbb{R}H \subset m' \) by explicitly calculating this vector via the description of \( R \) developed in this paper (see Equation (22) below), and therefore \( m' = \mathbb{R}H \oplus \mathbb{R}v \) is a Lie triple system. Presuming that the list of Lie triple systems given in the proposition is complete, we see that the types (P) and (M) are maximal, whereas the inclusions \( (G) \subset (T) \subset (P) \) and \( (S) \subset (P) \) hold, showing that no other types are maximal.

It remains to show that any given Lie triple system \( m' \) of \( m \) is of one of the types given in the proposition. We will base the proof on the fact that also for \( m' \) we have a root space decomposition, and on the relations that hold between that decomposition and the root space decomposition (19) for \( m \); for a detailed description of these relations, see [K1], Section 2. We have \( \text{rk}(m') \leq \text{rk}(m) = 2 \), and therefore \( \text{rk}(m') \in \{1, 2\} \). We will handle the two possibilities for the rank of \( m' \) separately.

If \( \text{rk}(m') = 2 \), then any maximal flat subspace \( a \) of \( m' \) also is a maximal flat subspace of \( m \), and if we denote the root systems of \( m' \) resp. of \( m \) with respect to \( a \) by \( \Delta' \) resp. \( \Delta \), then \( \Delta' \subset \Delta \) holds; moreover if we consider the root space decomposition \( m' = a \oplus \bigoplus_{\alpha \in \Delta'_+} m'_\alpha \) of \( m' \), then we have \( \{0\} \neq m'_\alpha \subset m_\alpha \) and therefore \( m'_\alpha = m_\alpha \) for any \( \alpha \in \Delta' \), because \( m_\alpha \) is 1-dimensional.

There are thus three possibilities: \( \Delta' = \emptyset \), \( \Delta' = \{ \pm \alpha_1 \} \) for some \( \alpha_1 \in \Delta \), and \( \Delta' = \Delta \). If \( \Delta' = \emptyset \) holds, we have \( m' = a \), and thus \( m' \) is of type (T). If \( \Delta' = \{ \pm \alpha_1 \} \) holds, then we have \( m' = a \oplus m_\lambda_1 \), and thus \( m' \) is of type (P). Finally, \( \Delta' = \Delta \) is possible only for \( m' = m \).

If \( \text{rk}(m') = 1 \), we fix some \( H \in m' \setminus \{0\} \), then \( a' := \mathbb{R}H \) is a maximal flat subspace of \( m' \). We choose a maximal flat subspace \( a \) of \( m \) with \( a' = a \cap m' \). If \( \text{dim}(m') = 1 \) holds, we have \( m' = a' \), and therefore \( m' \) then is of type (G). We suppose \( \text{dim}(m') \geq 2 \) in the sequel. Then we have (see [K1], Proposition 2.3) either \( H \in \mathbb{R}(\frac{1}{2} \alpha_2) \) for some \( \alpha_2 \in \Delta \), or \( H \perp (\alpha_2^2 - \beta^2) \) for some \( \alpha, \beta \in \Delta \), \( \alpha \neq \beta \). It follows that there exists a system of simple roots \( \{ \alpha_1, \alpha_2 \} \) of \( \Delta \) so that with \( \alpha_3 := \alpha_1 + \alpha_2 \in \Delta \) we have (after scaling \( H \) appropriately) either \( H = H_3 \) or \( H = H_3 + \frac{1}{2}(H_1 - H_3) = H_1 + \frac{1}{2}H_2 \). We will treat these two possibilities separately below, but in either case we have \( \emptyset \neq \Delta' \subset \{ \alpha[a'] \mid \alpha \in \Delta, \alpha[a'] \neq 0 \} \) and the following root space decomposition for \( m' \):

\[
m' = a' \oplus \bigoplus_{\lambda \in \Delta'_+} m'_\lambda \quad \text{with} \quad m'_\lambda = \left( \bigoplus_{\alpha \mid \alpha[a'] = \lambda} m_\alpha \right) \cap m' \quad \text{for} \quad \lambda \in \Delta'. \tag{20}
\]

Let us now first consider the case \( H = H_3 \). Then we have \( \Delta' \subset \{ \pm \lambda, \pm 2 \lambda \} \) with \( \lambda := \alpha_1[a'] = \alpha_2[a'] \); note \( 2 \lambda = \alpha_3[a'] \). By Equation (20) we therefore have

\[
m'_\lambda \subset m_{\alpha_1} \oplus m_{\alpha_2} \quad \text{and} \quad m'_{2\lambda} \subset m_{\alpha_3}. \tag{21}
\]

Let \( v \in m'_\lambda \) be given, say \( v = M_{\alpha_1}(a) + M_{\alpha_2}(b) \) with \( a, b \in \mathbb{R} \) (see Equation (21)). Because \( m' \) is a Lie triple system, we have \( R(H, v)v \in m' \), and using the representation of the curvature tensor \( R \) given in the present paper, we can actually calculate that vector explicitly (most easily by using the Maple implementation of the algorithms mentioned in the Introduction):

\[
R(H, v)v = \frac{1}{2}(a^2H_1 + b^2H_2). \tag{22}
\]

Therefore \( R(H, v)v \) is a member of \( m' \cap a = a' = \mathbb{R}H = \mathbb{R}H_3 = \mathbb{R}(H_1 + H_2) \), and hence (22) gives \( a^2 = b^2 \) and thus \( a = \pm b \). It follows that we have either \( m'_\lambda \subset \mathbb{R}(M_{\alpha_1}(1) + M_{\alpha_2}(1)) \) or \( m'_\lambda \subset \mathbb{R}(M_{\alpha_1}(1) - M_{\alpha_2}(1)) \). In fact, we can suppose without loss of generality

\[
m'_\lambda \subset \mathbb{R}(M_{\alpha_1}(1) + M_{\alpha_2}(1)). \tag{23}
\]
6 An application: Totally geodesic submanifolds in SU(3)/SO(3)

Let us now suppose $2\lambda \in \Delta'$, then we have $m'_{\lambda} = m_{\alpha_3}$ by (21), and therefore $M_{\alpha_3}(1) \in m'$ holds. Again using our representation of $R$, we calculate the element $R(M_{\alpha_3}(1), v)v$ of $m'$:

$$R(M_{\alpha_3}(1), v)v = \frac{1}{2}ab(H_1 - H_2) + \frac{1}{4}M_{\alpha_3}(a^2 + b^2).$$  

(24)

Because the $\alpha$-component of this vector, $\frac{1}{2}ab(H_1 - H_2)$, is contained in $a' = \mathbb{R}(H_1 + H_2)$, we see that $ab = 0$ holds. Because we have already seen $b = \pm a$, $a = b = 0$ and thus $v = 0$ follows. So we have shown that $2\lambda \in \Delta$ implies $\lambda \not\in \Delta$. Thus Equations (20) and (21) imply that in this case $m' = \mathbb{R}H_3 \oplus m_{\alpha_3}$ holds, and therefore $m'$ is of type (S).

On the other hand, for $2\lambda \not\in \Delta'$ we have $\lambda \in \Delta'$ and thus by Equation (23): $m'_{\lambda} = \mathbb{R}(M_{\alpha_1}(1) + M_{\alpha_2}(1))$. It now follows from Equation (20) that $m'$ is of type (M).

Finally, we consider the case $H = H_1 + \frac{1}{2}H_2$. Then we have $\alpha_1(H) = \alpha_2(H) = 0$ and $\alpha_3(H) = 0$, and thus $\Delta' \subset \{ \pm \lambda \}$ with $\lambda := \alpha_3|a' = \alpha_3|a'$ and $m'_{\lambda} \subset m_{\alpha_2} \oplus m_{\alpha_3}$. Let $v \in m'_{\lambda}$ be given, say $v = M_{\alpha_3}(a) + M_{\alpha_2}(b)$ with $a, b \in \mathbb{R}$. We once again use our representation of $R$ to compute the element $R(H, v)v$ of $m'$ explicitly:

$$R(H, v)v = \frac{3}{2}((a^2 + b^2)H_1 + b^2H_2) - \frac{3}{4}M_{\alpha_3}(ab).$$  

(25)

Because $m'$ is orthogonal to $m_{\alpha_2}$ by Equation (20), the $m_{\alpha_2}$-component of (25) must be zero, so we have $ab = 0$. Also, the $\alpha$-component of (25) must be a member of $a' = \mathbb{R}(H_1 + \frac{1}{2}H_2)$, so we have $a^2 + b^2 = 2b^2$, hence $a = \pm b$. From these two equations, $a = b = 0$ and hence $v = 0$ follows. So we have $m'_{\lambda} = \{0\}$ and thus $m' = a'$ is 1-dimensional. This shows that the case $H = H_1 + \frac{1}{2}H_2$ in fact cannot occur (for dim$(m') \geq 2$), and this completes the classification.

□

We finally discuss the totally geodesic submanifolds corresponding to the various types of Lie triple systems of SU(3)/SO(3) found in Proposition 6.1. For the purpose of describing the metric of the submanifolds, we suppose that the SU(3)-invariant Riemannian metric on SU(3)/SO(3) is the one induced by the usual inner product on $\text{End}(\mathfrak{f}^3)$, namely the one given by $\langle A, B \rangle := \text{Re}(\text{tr}(B^*A))$ for $A, B \in \text{End}(\mathfrak{f}^3)$. Then the root vectors $H_k$ have length $\sqrt{2}$. With this choice of Riemannian metric on SU(3)/SO(3), the totally geodesic submanifolds corresponding to the various types of Lie triple systems have the following isometry type:

| type of Lie triple system | (G) | (T) | (S) | (M) | (P) |
|--------------------------|-----|-----|-----|-----|-----|
| isometry type            | $\mathbb{R}$ or $S^1$ | $(S^1_{\mathbb{R}} = \sqrt{\frac{1}{3}} \times S^1_{\mathbb{R}} = \sqrt{\frac{1}{3}}) / \{ \pm \text{id} \}$ | $S^2_{\mathbb{R}} = \sqrt{\frac{1}{2}}$ | $\mathbb{R}P^2_{\epsilon = 1/2}$ | $(S^4_{\mathbb{R}} = \sqrt{\frac{1}{2}} \times S^4_{\mathbb{R}} = \sqrt{\frac{1}{2}}) / \{ \pm \text{id} \}$ |

The totally geodesic submanifolds of type (G) are the traces of geodesics in SU(3)/SO(3), and the submanifolds of type (T) are the maximal flat tori in SU(3)/SO(3). A totally geodesic submanifold of SU(3)/SO(3) is reflective (i.e. is a connected component of the fixed point set of an involutive isometry, see for example Le and other papers by Leung) if and only if it is either of type (M) or of type (P). For a Lie triple system corresponding to a reflective submanifold, the orthogonal complement is again a Lie triple system, and in this way, the types (M) and (P) correspond to each other. In fact, the submanifolds of type (M) are polars (i.e. they are connected components of the fixed point set of the geodesic symmetry of SU(3)/SO(3), see [CN], §2) and the submanifolds of type (P) are the corresponding meridians.

Note that via these concepts, all totally geodesic submanifolds of SU(3)/SO(3) can be obtained in a “natural” way: The submanifolds of type (M) are the polars of SU(3)/SO(3), and the submanifolds of type (P) correspond to them as meridians. The remaining totally geodesic submanifolds, namely those of type (S), (T) and (G), are obtained as the “obvious” totally geodesic submanifolds of the meridians.

To prove that the totally geodesic submanifolds are indeed of the isometry types given in the above table, and to describe their position in SU(3)/SO(3), we now give totally geodesic embeddings for each type of totally geodesic submanifold explicitly.

**Type (S).** Consider the Lie group monomorphism

$$\Phi_0 : SU(2) \rightarrow SU(3), \ B \mapsto \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix},$$

$20$
which is isometric with regard to the Riemannian metrics induced on $SU(2)$ resp. $SU(3)$ by the usual endomorphism inner product. We have $\Phi_0^{-1}(SO(3)) = SO(2)$, and $\Phi_0$ is compatible with the Lie group involutions induced by the symmetric space structures of $SU(2)/SO(2)$ resp. $SU(3)/SO(3)$. Therefore $\Phi_0$ gives rise to a totally geodesic isometric embedding

$$\bar{\Phi}_0 : SU(2)/SO(2) \to SU(3)/SO(3),$$

its image is a totally geodesic submanifold of $SU(3)/SO(3)$, which turns out to be of type (S). Thus the totally geodesic submanifolds of type (S) are isometric to $SU(2)/SO(2)$.

$SU(2)/SO(2)$ is a simply connected, 2-dimensional, irreducible Riemannian symmetric space of compact type, and hence isometric to a 2-sphere of some specific radius $r$. The curve $\gamma : \mathbb{R} \to SU(2)/SO(2)$, $t \mapsto \left( e^{it/\sqrt{\pi}} \begin{pmatrix} 0 & 0 \\ e^{-it/\sqrt{\pi}} \\ 0 \end{pmatrix} \right) \cdot SO(2)$ is a unit speed geodesic of $SU(2)/SO(2)$ with period $\sqrt{2}\pi = 2\pi r$. Therefore $SU(2)/SO(2)$ (and hence, any totally geodesic submanifold of $SU(3)/SO(3)$ of type (S)) is isometric to $S^2_{r=1/\sqrt{\pi}}$.

**Type (P).** To describe a totally geodesic embedding of type (P), we “extend” the embedding $\Phi_0$ described above in the following way:

$$\Phi : S^1_{r=\sqrt{6}} \times SU(2) \to SU(3), \ (\lambda, B) \mapsto \left( \frac{\lambda}{\sqrt{6}} B \ 0 \ \frac{6}{\sqrt{\pi}} \right),$$

where we regard $S^1_{r=\sqrt{6}}$ as the circle $\{ z \in \mathbb{C} \mid |z|^2 = 6 \}$ in $\mathbb{C}$. Note that $\Phi(\sqrt{6}, \cdot) = \Phi_0$ holds. The differential of $\Phi$ at $(\sqrt{6}, \text{id}) \in S^1_{r=\sqrt{6}} \times SU(2)$ is given by

$$T_{\sqrt{6}}S^1_{r=\sqrt{6}} \times su(2) \to su(3), \ (it, X) \mapsto \begin{pmatrix} \frac{i}{\sqrt{6}} \text{id} + X & 0 \\ 0 & -2\frac{i}{\sqrt{6}} \end{pmatrix},$$

where we identify the tangent space of $S^1_{r=\sqrt{6}}$ at $\sqrt{6}$ with $i\mathbb{R}$. Using this presentation of the differential of $\Phi$ it is easy to see that $\Phi$ is isometric. Moreover, we have $\Phi^{-1}(SO(3)) = K \cup gK$ with $K := \{ \pm \sqrt{6} \} \times SO(2)$ and $g := (\sqrt{6} i, J)$ where $J := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SU(2)$, and again $\Phi$ is compatible with the symmetric structures of the symmetric spaces involved. Therefore $\Phi$ gives rise to a totally geodesic, isometric embedding

$$\tilde{\Phi} : (S^1_{r=\sqrt{6}} \times SU(2))/(K \cup gK) \to SU(3)/SO(3).$$

The image of $\tilde{\Phi}$ is a totally geodesic submanifold of $SU(3)/SO(3)$, which turns out to be of type (P). Therefore the totally geodesic submanifolds of $SU(3)/SO(3)$ of type (P) are isometric to $(S^1_{r=\sqrt{6}} \times SU(2))/(K \cup gK)$.

It remains to describe the isometry type of $(S^1_{r=\sqrt{6}} \times SU(2))/(K \cup gK)$ more succinctly. We have $(S^1_{r=\sqrt{6}} \times SU(2))/K = (S^1_{r=\sqrt{6}}/\{ \pm \sqrt{6} \}) \times (SU(2)/SO(2))$, where $S^1_{r=\sqrt{6}}/\{ \pm \sqrt{6} \}$ is isometric to $S^1_{r=\sqrt{6}/2}$, and $SU(2)/SO(2)$ is isometric to $S^2_{r=1/\sqrt{2}}$ as we saw in the treatment of type (S). Hence $(S^1_{r=\sqrt{6}} \times SU(2))/K$ is isometric to $S^1_{r=\sqrt{6}/2} \times S^2_{r=1/\sqrt{2}}$. Because $K, gK \in (S^1_{r=\sqrt{6}} \times SU(2))/K$ correspond to a pair of antipodal points in $S^1_{r=\sqrt{6}/2} \times S^2_{r=1/\sqrt{2}}$ under this isometry, it follows that $(S^1_{r=\sqrt{6}} \times SU(2))/(K \cup gK)$ (and hence, any totally geodesic submanifold of $SU(3)/SO(3)$ of type (P)) is isometric to $(S^1_{r=\sqrt{6}/2} \times S^2_{r=1/\sqrt{2}})/\{ \pm \text{id} \}$.

**Type (T).** The Lie triple systems of type (T) are the maximal flat subspaces of $\mathfrak{m}$, so the corresponding totally geodesic submanifolds are the maximal flat tori of $SU(3)/SO(3)$. These
Lie triple systems are contained in Lie triple systems of type (P) (as can be seen by the explicit description of the Lie triple systems in Proposition 6.1), and therefore the maximal flat tori of $\text{SU}(3)/\text{SO}(3)$ are contained in totally geodesic submanifolds of type (P). A totally geodesic, isometric embedding of type (T) can be obtained by fixing a one-parameter subgroup $C$ of $\text{SU}(2)$ which runs orthogonal to $\text{SO}(2) \subset \text{SU}(2)$ and restricting $\Phi$ to $(S^1_{r=\sqrt{6}/2} \times C)/(K \cup gK)$. Therefore the maximal flat tori of $\text{SU}(3)/\text{SO}(3)$ are isometric to $(S^1_{r=\sqrt{6}/2} \times S^1_{r=1/\sqrt{2}})/\{\pm \text{id}\}$.

To understand the geometry of the maximal tori in $\text{SU}(3)/\text{SO}(3)$ better, we consider the lattice $\tilde{\Gamma} := \mathbb{Z}(\sqrt{6}/\pi, 0) \oplus \mathbb{Z}(0, \sqrt{2}/\pi)$ in $\mathbb{R}^2$. Then $S^1_{r=\sqrt{6}/2} \times S^1_{r=1/\sqrt{2}}$ is isometric to $\mathbb{R}^2/\tilde{\Gamma}$, and therefore the maximal tori $(S^1_{r=\sqrt{6}/2} \times S^1_{r=1/\sqrt{2}})/\{\pm \text{id}\}$ of $\text{SU}(3)/\text{SO}(3)$ are isometric to $\mathbb{R}^2/\Gamma$, where $\Gamma \subset \mathbb{R}^2$ is the lattice generated by $\tilde{\Gamma}$ and the point $(\sqrt{6}/\pi, \sqrt{2}/\pi)$ which corresponds to the antipodal point of the origin in $\mathbb{R}^2/\tilde{\Gamma}$. It can be shown that $\Gamma = \mathbb{Z}(\sqrt{6}/\pi, \sqrt{2}/\pi) \oplus \mathbb{Z}(\sqrt{6}/\pi, -\sqrt{2}/\pi)$ holds. The two generators of $\Gamma$ are not orthogonal to each other (they are at an angle of $\frac{\pi}{6}$). It follows that $\mathbb{R}^2/\Gamma$, and hence any maximal flat torus of $\text{SU}(3)/\text{SO}(3)$, is diffeomorphic to $S^1 \times S^1$, but is not globally isometric to a product of circles.

**Type (G).** The totally geodesic submanifolds corresponding to the Lie triple systems of type (G) are of course the traces of the geodesics of $\text{SU}(3)/\text{SO}(3)$; each of them runs within a maximal torus, and their behavior (either they are periodic, or they are injective and then their trace is dense in the torus) depends on their starting angle in the well-known way.

**Type (M).** To construct totally geodesic embeddings of type (M), we consider the 3-dimensional complex space $V$ of symmetric complex $(2 \times 2)$-matrices. $V$ becomes a unitary space via the usual endomorphism inner product, and this inner product gives rise to the Lie group $\text{SU}(V) \cong \text{SU}(3)$. $V$ has a canonical real form $V_{\mathbb{R}} := \{ X \in V \mid \overline{X} = X \}$, which we use to define the Lie subgroup $\text{SO}(V) := \{ B \in \text{SU}(V) \mid B(V_{\mathbb{R}}) = V_{\mathbb{R}} \}$ of $\text{SU}(V)$, isomorphic to $\text{SO}(3)$. Thereby we have the realization $\text{SU}(V)/\text{SO}(V)$ of the Riemannian symmetric space $\text{SU}(3)/\text{SO}(3)$, and we will construct the totally geodesic submanifolds of type (M) in this realization.

For this, consider the Lie group homomorphism
\[
\Psi : \text{SU}(2) \to \text{SU}(V), \quad B \mapsto (X \mapsto BXB^T),
\]
where $B^T$ denotes the transpose of $B$ for any $B \in \text{SU}(2)$. Because $\text{SU}(2)$ and $\text{SU}(V)$ are simple Lie groups, $\Psi$ is a homothety with regard to the invariant Riemannian metrics induced on $\text{SU}(2)$ resp. $\text{SU}(V)$ by the endomorphism inner product, i.e. there exists $c \in \mathbb{R}_+$ so that the linearization $\Psi_L : \text{su}(2) \to \text{su}(V)$ of $\Psi$ satisfies $\|\Psi_L(H)\|^2 = c^2 \|H\|^2$ for all $H \in \text{su}(2)$. To determine the value of $c$ we note that $\Psi_L$ is given explicitly by
\[
\Psi_L : \text{su}(2) \to \text{su}(V), \quad H \mapsto (X \mapsto HX + XH^T);
\]
by explicit calculations for $H := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \text{su}(2)$ we find $\|H\|^2 = 2$ and (using the mentioned description of $\Psi_L$) $\|\Psi_L(H)\|^2 = 8$. Therefore we have $c = 2$.

Moreover, we have $\Psi^{-1}(\text{SO}(V)) = \text{SO}(2) \cup (J \cdot \text{SO}(2)) =: K$, where we again put $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SU}(2)$, and $\Psi$ is compatible with the involutions on $\text{SU}(2)$ resp. $\text{SU}(V)$ given by the symmetric space structures of $\text{SU}(2)/\text{SO}(2)$ resp. $\text{SU}(V)/\text{SO}(V)$. Therefore $\Psi$ gives rise to a totally geodesic embedding, which is homothetic with $c = 2$:
\[
\Psi : \text{SU}(2)/K \to \text{SU}(V)/\text{SO}(V).
\]
The totally geodesic submanifold of $\text{SU}(V)/\text{SO}(V)$ that is the image of $\Psi$ turns out to be of type (M).

As we saw above, $\text{SU}(2)/\text{SO}(2)$ is isometric to $S^2_{r=1/\sqrt{2}}$, and the pair of points $\text{SO}(2), J \cdot \text{SO}(2)$ of $\text{SU}(2)/\text{SO}(2)$ corresponds to a pair of antipodal points in $S^2_{r=1/\sqrt{2}}$ under that isometry. Therefore $\text{SU}(2)/K$ is isometric to $\mathbb{R}P^2_{\kappa=2}$. Because $\Psi$ is a homothety with $c = 2$, its image, and hence any totally geodesic submanifold of $\text{SU}(3)/\text{SO}(3)$ of type (M), is isometric to $\mathbb{R}P^2_{\kappa=1/2}$.

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