The news of the passing of Georges de Rham on 8 October 1990 has by now surely reached and moved all in the mathematical community. This journal would like to bid him farewell with the following communication. He was one of the important figures of mathematics in our century, his name and work belong to its enduring legacy, and the influence of his ideas on its development has by no means been exhausted yet.

He was close to Elemente der Mathematik, which is broadly aimed at both instructors and students, and indeed for many years was among its contributors. Everything that was connected with teaching lay as close to his heart as research did. He was in the habit of saying, “Teaching, the conveying of essentials, making the beautiful intelligible and evident, that is what gives me joy; and instruction is always accompanied by interpretation.”

He understood how to enlighten, in an unassuming yet memorable fashion, students at all levels about mathematics; perhaps they could unconsciously sense what a great mathematical powerhouse was at work here. The closer to him one became, whether as a novice or as a colleague, the more one was impressed by his personality: by his—there is no other way to say it—refined, yet not distant, bearing; by his charm, which came from the heart; by his unyielding straightforwardness and intensity; by his warmhearted friendship, loyalty, and readiness to help.

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Biographical Sketch

Georges de Rham came from a prominent family of the Swiss canton of Vaud. He grew up in Lausanne and studied there at the university, receiving his licence. In Paris, where he spent several years, he came into contact with the work of Henri Poincaré and Élie Cartan. Here he was led to focus his research on the mathematical area to which his 1931 doctoral thesis is devoted, a piece of work that immediately made his name world-famous. He qualified as university lecturer at Lausanne in 1932, and from 1936 until he took emeritus status he was professor at the universities of Lausanne and Geneva; he also maintained his contacts with Paris throughout. Although many centers of mathematics all over the world strove to lure him, he never left Romandy, and he succeeded in creating a world-class school of mathematics there. He encouraged young talent in a selfless, collegial, and fatherly way. With great vigor, he invested much time and effort providing services to the larger scientific world, as president of the Swiss Mathematical Society during the difficult time in 1944/45, as editor of Commentarii Mathematici Helvetici, and as a member of the research council of the Swiss National Science Foundation. From 1963 till 1966 the International Mathematical Union entrusted him with its highest office, that of president, and here, thanks to his reliable and universally trusted leadership, he succeeded in promoting worldwide scientific cooperation.

His friends knew that he was accomplished not only as a mathematician, but as an alpinist as well. In this endeavor too he felt himself drawn toward difficult challenges. Thanks to first ascents in the Alps, an excellently written book, and lectures, de Rham's name was a household word in mountainclimbing circles. After the war, when the borders were once again open, he recounted with pride that his first invited lecture abroad was not for mathematicians, but for the English “Rucksack Club.”

It is hardly astonishing that many honors, local, national, and international, were bestowed upon de Rham. This is not the place to list them all, and we mention only three here: honorary member of the Swiss Mathematical Society, 1960; honorary doctorate, ETH Zurich, 1961; and the federally awarded “Marcel Benoist Prize,” 1966.

De Rham’s Theorems

De Rham’s mathematical work lies in the areas of global analysis, differential geometry, and topology. It is striking for its astonishing synthesis of the most disparate of methods: analytical, geometric, combinatorial-topological, and algebraic. It was typical of him that he worked only within the scope of a few specific topics, but impressed upon them the stamp of his powers of penetration. They are as follows: differential forms on manifolds; combinatorial invariants of cell complexes; the definitive version of the Hodge theory of harmonic differential forms; the decomposition of Riemannian manifolds into products of indecomposable submanifolds; and “currents,” a synthesis, making use of Schwartz distributions, of differential forms and the chains of algebraic topology.

Here we restrict ourselves to summarizing his contribution to the first subject—it is the most
The famous de Rham’s theorem, announced in 1928 and worked out in his thèse (Paris, 1931), is today usually formulated, in the terminology of cohomology theory, as an isomorphism:

$$R^p(M) \cong H^p(M).$$

Let us briefly recall the meaning of these symbols.

We consider differential forms of degree $p$, $0 \leq p \leq n$, on a closed differentiable manifold $M$ of dimension $n$; they make up a real vector space $\Omega^p$. The exterior differential $d\alpha^p$ of $\alpha^p \in \Omega^p$ provides a linear mapping $d : \Omega^p \to \Omega^{p+1}$, and we have $dd = 0$; we set $\Omega^{-1} = \Omega^{n+1} = 0$. With this, the image $d\Omega^{p-1}$ lies in the kernel $\hat{\Omega}^p$ of $d$, and the “cohomology groups”

$$\hat{\Omega}^p / d\Omega^{p-1} = R^p(M)$$

are the de Rham groups of $M$ (which are real vector spaces). A form $\alpha^p \in \hat{\Omega}^p$, which thus has the differential $d\alpha^p = 0$, is said to be closed.

Further suppose that $M$ is subdivided into cells, and let $C_p$ be the vector space of $p$-chains, that is, the vector space of linear combinations, with real coefficients, of $p$-dimensional cells. The boundary $\partial \sigma_p$ of a $p$-cell $\sigma_p$ is a $(p-1)$-chain (with coefficients $\pm 1$), and through linearity we obtain a mapping

$$\partial : C_p \to C_{p-1}$$

such that $\partial \partial = 0$; we set $C_{-1} = C_{n+1} = 0$. The vector space dual to $C_p$, that is, the space of the real linear forms defined on $p$-chains, will be denoted by $\hat{C}^p$. For these, we have the “coboundary” operator

$$\delta : \hat{C}^p \to \hat{C}^{p+1},$$

which is dual to $\partial$ and defined for $f^p \in \hat{C}^p$ by $\delta f^p(c_{p+1}) = f^p(\partial c_{p+1})$, where $c_{p+1} \in C_{p+1}$; we have $\delta \delta = 0$. Hence $\delta\hat{C}^{p-1}$ lies in the kernel $\hat{\hat{C}}^p$ of $\delta$, and one obtains the cohomology groups

$$H^p(M) = \hat{\hat{C}}^p / \delta\hat{C}^{p-1}$$

(which again are real vector spaces). These are topological invariants of $M$, and even homotopy invariants, even though the left-hand side of the isomorphism in (1) is obviously defined in terms of the differentiable structure on $M$.

However, de Rham’s theorem asserts still more, namely that the isomorphism $R^p(M) \to H^p(M)$ is realized through the integration of differential forms. We assume that the cells $\sigma_p$ are differentiable and use them as domains of integration; thus, for example, $\int_{\sigma_1} \alpha^1$, $\alpha^1 \in \Omega^1$, is an ordinary line integral. For $\alpha^p \in \Omega^p$, $\int_{\sigma_p} \alpha^p$ extends, through linearity, to all chains $c_p \in C_p$; $f^p(c_p) = \int_{\sigma_p} \alpha^p$ defines a linear form $f^p \in \hat{C}^p$, and we obtain linear mappings

$$\Phi : \Omega^p \to \hat{C}^p, \quad p = 0, 1, ..., n.$$
For these, the equation
\[ \Phi(d\alpha^p) = \delta\Phi(\alpha^p) \] (2)
holds. This is nothing other than Stokes’s theorem:
\[ \int_{c_{p+1}} d\alpha^p = \int_{\partial c_{p+1}} \alpha^p = f^p(\partial c_{p+1}) = \delta f^p(c_{p+1}) \]
for all \( c_{p+1} \in C_{p+1} \).

From (2), we see that \( \Phi \) maps the groups \( \Omega^p \) and \( d\Omega^{p-1} \) into \( \hat{\Omega}^p \) and \( \delta \Omega^{p-1} \), respectively, and hence gives rise to a linear mapping of the quotient groups \( R^p(M) \) into \( H^p(M) \)—and de Rham’s theorem asserts that this is the isomorphism in (1).

This is not exactly the original version given in the thèse. In order to arrive at the original form, one has to consider the homology groups \( H_p(M) \), that is, the quotient groups \( \hat{C}_p/\partial C_{p+1} \), where \( \hat{C}_p \) is the kernel of \( \partial : C_p \rightarrow C_{p-1} \). Chains \( c_p \) such that \( \partial c_p = 0 \) are called “cycles,” and those such that \( c_p = \partial c_{p+1} \), “boundaries”; they have a clear geometric significance. Cycles are closed domains of integration, and an element of \( H_p(M) \) is a class of \( p \)-cycles differing from one another “only” by boundaries.

Now if \( f^p \in \hat{C}_p \), \( \delta f^p = 0 \), then \( f^p(\partial c_{p+1}) = \partial f^p(c_{p+1}) = 0 \), that is, \( f^p \) is 0 on \( \partial C_{p+1} \). If we consider \( f^p \) only on the cycles of \( \hat{C}_p \), then we thereby obtain a linear form on \( H_p(M) = \hat{C}_p/\partial C_{p+1} \); this is identically 0 if \( f^p = \delta g^{p-1} \). Conversely, we obtain all real linear forms \( H_p(M) \rightarrow \mathbb{R} \) in this way (because a linear form on \( \hat{C}_p \) can be extended to the entire vector space \( C_p \) in an arbitrary manner). Therefore
\[ H^p(M) = \text{Hom}(H_p(M), \mathbb{R}), \]
where \( \text{Hom}(V, \mathbb{R}) \) denotes the vector space of all real linear forms on \( V \), that is, the vector space dual to \( V \). The mapping \( \Phi \) can now be interpreted as \( R^p(M) \rightarrow \text{Hom}(H_p(M), \mathbb{R}) \). It associates a closed differential form \( \alpha^p \in \Omega^p \) to a linear form that is defined on the cycles and is equal to 0 on the boundaries; its values are called the \textit{periods} of \( \alpha_p \). That \( \Phi \) is an isomorphism means two things:

I. If all periods of the closed differential form \( \alpha^p \) are equal to 0, then \( \alpha^p \) is a differential \( d\beta^{p-1} \).
   In other words, \( \Phi \) is injective.

II. Given specified periods \( \varphi \in \text{Hom}(H_p(M), \mathbb{R}) \), there exists a closed differential form \( \alpha^p \) that has exactly these periods. In other words, \( \Phi \) is surjective.

This was the form in which de Rham worked out the proof: at the time, this was a monumental achievement! Upon closer analysis it reveals itself to be very similar to subsequent proofs, which were rendered simple and transparent thanks to the techniques of sheaf theory (a formalization of the passage from the local to the global).
In addition to Theorems I and II, the thèse also contains yet another, third theorem. It deals with the (exterior) product $\alpha^p \wedge \beta^q$ of differential forms, for which the equation
\[ d(\alpha^p \wedge \beta^q) = d\alpha^p \wedge \beta^q + (-1)^p \alpha^p \wedge d\beta^q \]
holds; it makes the direct sum $R^*(M) = \bigoplus R^p(M)$ into a graded ring. Likewise, $H^*(M) = \bigoplus H^p(M)$ is a graded ring by means of the Alexander multiplication. We then have that $\Phi$ is product-preserving, and thus a ring isomorphism $R^*(M) \cong H^*(M)$. Of course, de Rham did not have the Alexander product at his disposal; however, according to Poincaré duality, $H^p(M) \cong H_{n-p}(M)$, and thus to each closed form $\alpha^p$ there corresponds a cycle and hence a homology class $\Psi(\alpha^p) \in H_{n-p}(M)$. De Rham proved that

III. $\Psi(\alpha^p \wedge \beta^q) = S(\Psi(\alpha^p), \Psi(\beta^q)),$

where $S$ denotes the “intersection” of the cycles $c_{n-p} = \Psi(\alpha^p)$ and $d_{n-q} = \Psi(\beta^q)$. $S(c_{n-p}, d_{n-q})$ lies in dimension $n - (p + q)$ and is Poincaré-dual to the Alexander product of $f^p = \Phi(\alpha^p)$ and $g^q = \Phi(\beta^q)$. This theorem expresses a still closer relationship between the algebra of differential forms and the topology of the manifold.

Outlook

De Rham’s theorems, which have only been outlined here, simultaneously signified the end of a line of development and a new beginning. On one hand, the theory of differential forms growing out of analysis and differential geometry and, on the other hand, the emergence of the concept of homology in combinatorial topology had led Élie Cartan—and perhaps earlier even Poincaré—to conjecture that these theorems must be true. But the proof had to wait until de Rham succeeded in providing the necessary link from the local to the global. Not only the results, but also the methods would soon lead to a wellspring of new ideas, which would later be further developed in the works of Weil, Henri Cartan, Serre, Grothendieck, Sullivan, and others; these would be concerned not only with manifolds (real, open, complex, algebraic), but also with much more general cell complexes.

Georges de Rham’s manner of presentation, both verbally and in his writings, was crystal clear and “concrete” in the best sense of the word. Any kind of “abstract generalized nonsense,” if not rooted in some way in a concrete problem, was foreign to his thinking. Yet according to my personal recollection, he also appreciated the value and the expressive power of generalizations, especially of structural concepts that were abstracted from mathematical substance. He was thoroughly positive toward the efforts of the young Bourbaki group, to which he was linked by genuine friendships. And he would without a doubt have taken great joy in today’s mathematics, with its numerous surprising connections between seemingly disparate areas and its synthesis of concrete problems and abstract general methods.
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