On the Free-Energy of Three-Dimensional CFTs and Polylogarithms

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Abstract

We study the $O(N)$ vector model and the $U(N)$ Gross-Neveu model with fixed total fermion number, in three dimensions. Using non-trivial polylogarithmic identities, we calculate the large-$N$ renormalized free-energy density of these models, at their conformal points in a “slab” geometry with one finite dimension of length $L$. We comment on the possible implications of our results.

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1 Introduction

Conformal field theories (CFTs) in dimensions $d > 2$ have recently attracted much interest \cite{1}. Generic results based solely on conformal invariance are not very restrictive in $d > 2$, (see however \cite{2, 3, 4}), and most of the information is extracted by studying explicit models. Although many CFTs have been recently discovered in $d = 4$ \cite{5}, a large amount of work has also been devoted to the study of CFT models in $2 < d < 4$, such as the $O(N)$ vector model \cite{6, 4} and the Gross-Neveu model \cite{9} at their conformal points for large-$N$.

In this work, we study the $O(N)$ vector model and the $U(N)$ Gross-Neveu model with fixed total fermion number, in $d = 3$. Using non-trivial polylogarithmic identities, we calculate the free-energy density of these models to leading order in the $1/N$ expansion, at their conformal points in a “slab” geometry with one finite dimension of length $L$. The free-energy density of the three-dimensional $O(N)$ vector model in a “slab” geometry was first calculated in \cite{10, 11}. Recently, the free-energy density of (super)conformal field theories in $d = 4$ was also calculated in terms of polylogarithms \cite{12}.

Based on earlier ideas \cite{13}, it has been recently argued in \cite{14} that the free-energy density of four dimensional quantum field theories (QFTs) encodes non-perturbative information for the massless degrees of freedom coupled to a fixed point. Therefore, it may be interpreted as a measure of their expected reduction from the UV to the IR \cite{15}. Our leading-$N$ analytic results are consistent with an extension of such an interpretation for the free-energy density to three-dimensional QFTs.

2 The Conformally Invariant $O(N)$ Vector Model in $d = 3$

We begin by reviewing the results of \cite{11}. Consider the partition function of the $O(N)$ vector model in $d = 3$, obtained after integrating out the fundamental scalar fields $\phi^\alpha(x)$, $\alpha = 1, 2, ..., N$,

$$Z_B = \int (\mathcal{D}\sigma) \exp \left[ -N S_{ef}(\sigma, g) \right] ,$$

$$S_{ef}(\sigma, g) = \frac{1}{2} \text{Tr} \ln(-\partial^2 + \sigma) - \frac{1}{2g} \int d^3x \sigma(x) ,$$

where $\sigma(x)$ is an auxiliary scalar field and $g$ is the coupling. Setting $\sigma(x) = M^2 + (1/\sqrt{N}) \sigma_1(x)$, \cite{6} can be calculated in a renormalisable $1/N$ expansion \cite{16}, provided the gap equation

$$\frac{1}{g} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + M^2} ,$$

is satisfied. To any fixed order in $1/N$, a non-trivial CFT is obtained by tuning the coupling to the critical value $1/g \equiv 1/g_\ast = (2\pi)^{-3} \int d^3p/p^2$ \cite{14}. Then, the renormalised mass (or inverse correlation

\[ \text{...} \]
length) \( M(1/\xi) = 0 \).

When the model is placed in a “slab” geometry with one finite dimension of length \( L \) and periodic boundary conditions, the gap equation reads

\[
\frac{1}{g} = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \omega_n^2 + M_L^2}, \quad \mathbf{p} = (p_1, p_2),
\]

(4)

with the momentum along the finite dimension taking the values \( \omega_n = 2\pi n/L \), \( n = 0, \pm 1, \pm 2, \ldots \) We may then study the existence of a finite-temperature phase transition in the dimensionally continued version of the model, i.e. in \( d - 1 \) infinite dimensions \([16, 17, 18]\). Since UV renormalisation is insensitive to putting the system in a finite geometry, the coupling constant on the l.h.s. of (4) can be set to its renormalised value in the bulk which explicitly depends on a mass scale \( M_0 \), such that the system is in its \( O(N) \)-ordered phase for zero “temperature” \( T \sim 1/L \). Then, we can obtain an equation which gives the dependence of \( M_L \) on \( M_0 \) and \( T \). The finite-temperature phase transition corresponding to \( O(N) \) symmetry restoration, occurs when \( M_L = 0 \) for some critical temperature \( T_* \) which is related to \( M_0 \) through the above equation. It can be shown that the finite-temperature phase transition cannot take place for \( 2 < d \leq 3 \) in accordance to the Mermin-Wagner-Coleman theorem, but can only occur for \( 3 < d < 4 \). The critical temperature \( T_* \) obtained this way agrees \([19]\) with the well-known results, i.e. see \([20]\). The theory at \( T_* \) is a \( d - 1 \)-dimensional CFT, however the OPE structure of its correlation functions is less clear \([19]\).

On the other hand, when the coupling is fixed to its bulk critical value \( 1/g_* \), (4) has the solution

\[
M_L \equiv M_* = \frac{2}{L} \ln \left( \frac{1 + \sqrt{5}}{2} \right),
\]

(5)

which corresponds to the physical situation of finite-size scaling \([21]\) of the correlation length. Then, the subtracted \( ^1 \) free-energy density for this configuration reads

\[
f_\infty - f_L = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \ln p^2 - \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \ln(p^2 + \omega_n^2 + M_*^2) + \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{M_*^2}{p^2}
\]

\[
= \frac{M_*^3}{12\pi} - \frac{1}{2\pi L^3} \left[ \ln(e^{-LM_*}) Li_2(e^{-LM_*}) - Li_3(e^{-LM_*}) \right]
\]

\[
= \frac{4}{5} \frac{\zeta(3)}{2\pi L^3},
\]

(6)

\(^1\)Since the UV singularities are the same in the bulk and in the finite geometry, subtraction of the bulk free-energy density ensures a UV-finite result.
where $L_i(x)$ are the usual polylogarithms \[22\]. The third line in (3) follows from the second, by virtue of non-trivial polylogarithmic identities \[22\], and fits into the general formula \[23\]

$$f_\infty - f_L = \tilde{c} \frac{2\zeta(d)}{S_d L^d},$$

(7)

for the finite-size scaling of the free-energy density in conformal field theories, with $\tilde{c}/N = 4/5$. It is quite remarkable that the value of $\tilde{c}/N$ obtained from (3), turns out to be a rational number. This is reminiscent of central charge calculations in two-dimensional CFTs, (see \[25\] for a recent reference). There is strong evidence \[26\] that correlation functions at the above finite-size critical point can be described by operator product expansions of the bulk $O(N)$ vector model, in accordance with earlier ideas \[23\].

The result (8) is consistent with the interpretation of $\tilde{c}$ as a measure of the massless degrees of freedom coupled to a critical point and their expected reduction from the UV to the IR. To see this, we recall that the large-$N$ critical point of the $O(N)$ vector model is identical with the large-$N$ IR critical point of the $O(N)$ invariant $\phi^4$ theory \[28\]. The latter theory has as UV critical limit the free theory of $N$ massless bosons, for which it is well-known that $\tilde{c}/N = 1$ in $d = 3$. Therefore, according to the interpretation given in (14), we expect that

$$\tilde{c}_{UV} = \tilde{c}(N \text{ massless free bosons}) > \tilde{c}(O(N) \text{vector model}) = \tilde{c}_{IR},$$

(8)

which is satisfied since $1 > 4/5$. Note that (8) is a non-perturbative result from the point of view of the renormalisation group (RG) flow, since the large-$N$ IR critical point of the $\phi^4$ theory is non-perturbative in the coupling.

3 The $U(N)$ Gross-Neveu Model in $d = 3$

Following the considerations above, it is possible to study the free-energy density of other CFT models. Consider, for example, the $U(N)$ invariant Gross-Neveu model in $d = 3$ whose partition function, after integrating out the fundamental Dirac fermionic fields $\psi^\alpha(x), \bar{\psi}^\alpha(x), \alpha = 1, 2, \ldots, N$, reads

$$Z_F = \int (D \lambda) \exp \left[-N I_{eff}(\lambda, G)\right],$$

(9)

$$I_{eff}(\lambda, G) = \frac{1}{2G} \int d^3 x \lambda^2(x) - \text{Tr}[\ln(\theta + \lambda)],$$

(10)

\[2\]Recall that, in two dimensions $\tilde{c}$ coincides with the central charge \[24\].
where \( \lambda(x) \) is an auxiliary scalar field and \( G \) is the coupling. We use the notation \( \theta = \gamma_{\mu} \partial_{\mu} \) and the following two-dimensional Hermitian representation for the Euclidean gamma matrices in \( d = 3 \) \[27, 28\]

\[
\begin{align*}
\gamma_1 &= \sigma^1, \\
\gamma_2 &= \sigma^2, \\
\gamma_3 &\equiv \gamma_0 = \sigma^3,
\end{align*}
\]

(11)

where \( \sigma^i, i = 1, 2, 3 \) are the usual Pauli matrices. This model describes fermion mass generation through the breaking of space parity \[28\]. The partition function (9) can be evaluated in a renormalisable \( 1/N \) expansion \[27\] when one sets \( \lambda(x) = m + (1/\sqrt{N}) \lambda_1(x) \), provided the following gap equation is satisfied

\[
\frac{1}{G} = 2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + m^2}.
\]

(12)

At the critical coupling \( 1/G = 1/G_s = 2(2\pi)^{-d} \int d^dp/p^2 \), the theory is conformally invariant and \( m = 0 \).

When the system is placed in a “slab” geometry with one finite dimension of length \( L \), the fermions acquire antiperiodic boundary conditions and the gap equation reads

\[
\frac{1}{G} = \frac{2}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \omega_n^2 + m_L^2},
\]

(13)

with \( \omega_n = (2n+1)\pi/L, n = 0, \pm 1, \pm 2, \ldots \). Again, we may study the finite-temperature phase transition in the dimensionally continued version of the model, (i.e. in \( d - 1 \) infinite dimensions) \[27, 17\], by setting \( 1/G \) to its bulk renormalised value which explicitly depends on the mass \( m_0 \) of the fundamental fermionic fields. This means that the system is in its “broken phase” for zero “temperature” \( T \sim 1/L \). Then, we can obtain an equation which gives the dependence of \( m_L \) on \( m_0 \) and \( T \). The second order finite-temperature phase transition corresponding to space parity restoration, occurs when \( m_L = 0 \) for some critical temperature \( T_* \) which is related to \( m_0 \) through the above equation. The finite-temperature phase transition is now possible for all \( 2 < d < 4 \) \[27, 17\], due to the absence of zero modes for fermions and antiperiodic boundary conditions.

On the other hand, when the coupling stays at its bulk critical value \( 1/G_s \), (13) is satisfied for

\[
m_L \equiv m_* = 0.
\]

(14)
This essentially means that, to leading order in $1/N$, the free-energy density of the system is given by the free-field theory result. Indeed we easily find

$$\frac{f_\infty - f_L}{N} = \frac{3}{2} \frac{\zeta(3)}{2\pi L^3},$$

(15)

which implies that $\tilde{c}/N = 3/2$ in agreement with the results of [29].

Although the result (15) may seem trivial, it encodes important information. This is again seen if we recall that the large-$N$ critical point of the $U(N)$ Gross-Neveu model is identical with the large-$N$ IR critical point of the Gross-Neveu-Yukawa model [28]. The latter model has as UV critical point the free theory of $N$ massless Dirac fermions plus one massless boson for which $\tilde{c}/N = 3/2 + 1/N$ in $d = 3$. Then, according to [14] we expect

$$c_{UV} = \tilde{c}(N \text{ massless Dirac fermions} + 1 \text{ massless boson}) > \tilde{c}(\text{Gross-Neveu}) = \tilde{c}_{IR},$$

(16)

which is consistent with (15) for large and finite $N$. Again, the non-perturbative nature of (16) is a consequence of the non-perturbative nature of the large-$N$ IR critical point of the Gross-Neveu-Yukawa model.

4 The $U(N)$ Gross-Neveu Model in $d = 3$ with fixed total fermion number

The Gross-Neveu model can also be studied [30, 31] for fixed total fermion number $B$. To this effect, we introduce a delta-function constraint $\delta(\hat{N} - B)$ into the functional integral (9) at finite temperature, where

$$\hat{N} = \int d^2x \psi^\dagger(x)\psi(x), \quad x = (x_1, x_2)$$

(17)

is the fermion number operator. Using an auxiliary scalar field $\theta(x_3)$, the above delta-function constraint is exponentiated and after integrating out the fermions we obtain the partition function

$$Z_f = \int (D\lambda)(D\theta) \exp \left[-N \mathcal{I}_{eff}(\lambda, G; \theta, \tilde{B})\right],$$

(18)

$$\mathcal{I}_{eff}(\lambda, G; \theta, \tilde{B}) = i\tilde{B} \int L \theta(x_3) d x_3 + \frac{1}{2G} \int L d^3x \lambda^2(x) - \text{Tr}\ln(\theta + i\gamma_3\theta + \lambda)|_L,$$

(19)

where $\tilde{B} = B/N$ is assumed to be finite for large-$N$ and $1/G$ is the new coupling. The subscript $L$ denotes $x_3$-integration up to $L$ and the latter quantity plays here the rôle of inverse temperature $1/T$.

There exist solutions of (13) with imaginary $m_L$ which will be discussed elsewhere.
For large-$N$, the functional integral (18) can be calculated by the steepest descent method, since it is dominated by the uniform stationary points $\langle \lambda \rangle$ and $\langle \theta \rangle$ of $I_{\text{eff}}$. These stationary points are obtained as the solutions of the following set of saddle-point equations

$$
\frac{1}{G} = \frac{2}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + (\omega_n + \langle \theta \rangle)^2 + \langle \lambda \rangle^2},
$$

$$
i\tilde{b} = \lim_{\tau \to 0} \frac{2}{L} \int \frac{d^2p}{(2\pi)^2} \frac{e^{i\omega_n \tau} (\omega_n + \langle \theta \rangle)}{p^2 + (\omega_n + \langle \theta \rangle)^2 + \langle \lambda \rangle^2},
$$

where $\tilde{b} = (\tilde{B}L/\Omega)$, with $\Omega$ the total volume. The regulating term $e^{i\omega_n \tau}$ on the r.h.s of (21) has been discussed in [31] and ensures a finite result in the limit $\tau \to 0$, after the Matsubara sum has been performed. For $\langle \theta \rangle$ purely imaginary, which corresponds to having a real chemical potential $\mu = -i\langle \theta \rangle$ [30, 31], (20) coincides with a similar saddle-point equation obtained in [17, 27]. We can renormalise (20) by substituting for $1/G$ the bulk renormalised coupling $1/G^*$ from (12), since the presence of $\langle \theta \rangle$ does not alter its UV behavior. In this way, we can study the finite-temperature phase transition of the model in terms of the renormalised mass of the bulk fermionic fields and the chemical potential.

However, we are interested here in possible real values of $\langle \theta \rangle$ which satisfy (20) and (21). The reason is that, if $\langle \theta \rangle$ is a real number we can set in (20) the coupling $1/G$ to its bulk critical value $1/G^*$ and obtain the following equation for $\langle \lambda \rangle$

$$
L \langle \lambda \rangle + \ln \left( 1 + e^{-L\langle \lambda \rangle - iL\langle \theta \rangle} \right) + \ln \left( 1 + e^{-L\langle \lambda \rangle + iL\langle \theta \rangle} \right) = 0.
$$

This has a real solution for $\langle \lambda \rangle$ in terms of $L$, whenever we have $-1 \leq \cos (L\langle \theta \rangle) \leq -1/2$, or simply $2\pi/3 \leq L\langle \theta \rangle \leq \pi$. Note now that $\langle \lambda \rangle$, which is the renormalised inverse correlation length, is non-zero for $L\langle \theta \rangle \neq 2\pi/3$ corresponding to a finite-size scaling regime for our fermionic model. Such a regime is absent for the usual Gross-Neveu model studied in Section 3 and it is a consequence of keeping the total fermion number fixed. The dimensionless quantity $L\langle \lambda \rangle$ is plotted in Fig. 1 for the allowed values of $\langle \theta \rangle$.

The second saddle-point equation (21) ensures a fixed mean fermion number $\langle \hat{N} \rangle = B$, as imposed by the constraint. It turns out that its r.h.s. is real and has a quadratic divergence which should be subtracted. Again, if $\langle \theta \rangle$ is purely imaginary (21) assumes the usual form [32] of the expression for the conserved charge in a system exchanging particles with a reservoir. It also vanishes for $\mu \equiv -i\langle \theta \rangle = 0$, as it should, since this would correspond to the absence of conserved charges. On the other hand, for
Figure 1: \( L\langle \lambda \rangle \) for the allowed region of \( \langle \theta \rangle \).

\[ 2\pi/3 \leq L\langle \theta \rangle \leq \pi \] we obtain, after subtraction of the divergence,

\[
\tilde{b} = \frac{i}{2\pi L^2} \left[ Cl_2(2\phi) - Cl_2(2\phi - 2L\langle \theta \rangle) - Cl_2(2L\langle \theta \rangle) \right],
\]

(23)

\[
\phi = \arctan \left[ \frac{e^{-L\langle \lambda \rangle} \sin (L\langle \theta \rangle)}{1 + e^{-L\langle \lambda \rangle} \cos (L\langle \theta \rangle)} \right],
\]

(24)

where \( Cl_2(\omega) = \text{Im} \left[ Li_2(e^{i\omega}) \right] \) is Clausen’s function \([22]\). Now, \( \tilde{b} \) is related to the total fermion number of the system and in principle it should be real and positive. Therefore, from (23), the only allowed real value for \( \langle \theta \rangle \) which satisfies both saddle-point equations (20) and (21) is \( \langle \theta \rangle = \pi/L \). In this case, \( \tilde{b} = 0 \) and it seems that there are no fermions left in the system. This is consistent with the apparent bosonization of the theory for \( \langle \theta \rangle = \pi/L \) which will be discussed shortly. However, it is conceivable that the imaginary solutions for \( \tilde{b} \) may also have physical meaning, as they give rise to a real value for the free-energy density of the theory. The latter result is rather surprising and is obtained by virtue of non-trivial polylogarithmic identities.

To demonstrate the above points, we calculate the free-energy density of the model and we obtain

\[
\frac{f_\infty - f_L}{N} = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \ln(p^2 + (\omega_n + \langle \theta \rangle)^2 + \langle \lambda \rangle^2) - \int \frac{d^3p}{(2\pi)^3} \ln p^2 - \frac{\langle \lambda \rangle^2}{2G} - i\langle \theta \rangle \tilde{b}
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \left[ \ln \left( \frac{p^2 + \langle \lambda \rangle^2}{p^2} \right) - \frac{\langle \lambda \rangle^2}{p^2} \right]
\]

\[
+ \frac{1}{L} \int \frac{d^2p}{(2\pi)^2} \left\{ \ln \left( 1 + e^{-L\sqrt{p^2 + (\langle \lambda \rangle)^2} - iL\langle \theta \rangle} \right) + \ln \left( 1 + e^{-L\sqrt{p^2 + (\langle \lambda \rangle)^2} + iL\langle \theta \rangle} \right) \right\}
\]

\[
i\langle \theta \rangle \int \frac{d^2p}{(2\pi)^2} \left\{ \frac{1}{1 + e^{L\sqrt{p^2 + (\langle \lambda \rangle)^2} - iL\langle \theta \rangle}} - \frac{1}{1 + e^{L\sqrt{p^2 + (\langle \lambda \rangle)^2} + iL\langle \theta \rangle}} \right\}
\]
\[
\begin{align*}
&= -\frac{\langle \lambda \rangle^3}{6\pi} + \frac{1}{\pi L^3} \left[ \ln \left( e^{-L\langle \lambda \rangle} \right) \left( L\langle \theta \rangle \right) - L\langle \theta \rangle \right] \\
&\quad + \frac{L\langle \theta \rangle}{2\pi L^3} \left[ Cl_2(2\phi) - Cl_2(2\phi - 2L\langle \theta \rangle) - Cl_2(2L\langle \theta \rangle) \right].
\end{align*}
\]

(25)

\(Li_n(r, \theta)\) is the real part of the polylogarithm \(Li_n(re^{i\theta})\) in Lewin’s notation \([22]\). As already mentioned, the free-energy density (25) is real for \(2\pi/3 \leq L\langle \theta \rangle \leq \pi\).

For \(\langle \theta \rangle = \pi/L\), (22) has the solution \(\langle \lambda \rangle = M_L\) (the "golden mean") which corresponds to the physical situation of finite-size scaling of the correlation length in the \(O(N)\) vector model in \(d = 3\). This apparent “bosonization” of the Gross-Neveu model is effectively a transmutation between Fermi and Bose statistics, since the solution \(\langle \theta \rangle = \pi/L\) introduces a zero mode for the fermions. Also, for this particular value of \(\langle \theta \rangle\), \(\tilde{b}\) is zero which can be further interpreted as the absence of conserved charges in the bosonized version of the system. The bosonization of the model for \(\langle \theta \rangle = \pi/L\) is also discussed in \([31, 33]\) and a possible connection with anyonic physics is made in \([34]\). In this case, using the same polylogarithmic identities as in \((3)\) we obtain

\[
\frac{f_\infty - f_L}{N} = -\frac{8}{5} \frac{\zeta(3)}{2\pi L^3},
\]

(26)

which is consistent with the expected CFT result \((6)\), with \(\tilde{c}/N = -8/5\). In fact, the full expression \((25)\) is consistent with the scaling form \((7)\) for the allowed values of \(\langle \theta \rangle\), however the corresponding expressions for \(\tilde{c}/N\) involve polylogarithms and Clausen’s functions at non-exceptional arguments and are not illuminating. In Fig. 2 we plot the numerical values of \(\tilde{c}/N\) corresponding to \((25)\) for \(2\pi/3 \leq L\langle \theta \rangle \leq \pi\).

![Figure 2: \(\tilde{c}/N\) for the allowed region of \(\langle \theta \rangle\).](image)

Nevertheless, at the other end-point of the allowed \(\langle \theta \rangle\) region, \(\langle \theta \rangle = 2\pi/3L\), (22) has the solution \(\langle \lambda \rangle = 0\) which corresponds to infinite correlation length. Therefore, we may associate this point with
a temperature phase transition. Remarkably, (25) simplifies considerably for \( \langle \theta \rangle = 2\pi/3L \) giving

\[
\frac{f_\infty - f_L}{N} = \frac{1}{2\pi L^3} \left[ \frac{4\pi}{3} Cl_2 \left( \frac{\pi}{3} \right) - \frac{2}{3} \zeta(3) \right].
\]  

(27)

It may be interesting to point out that \( Cl_2(\pi/3) \) is the absolute maximum of Clausen’s function \( \ref{equation:clausen} \) which is a well-documented numerical constant.

Consistency of our main result (25) with the expected from conformal invariance scaling form (7), raises the possibility that, at the above finite-size critical point with \( 2\pi/3 \leq L \langle \theta \rangle \leq \pi \) the Gross-Neveu model with fixed total fermion number is related to conformal field theory. This is most clearly seen for \( \langle \theta \rangle = \pi/L \), when from (26) one concludes that the model is related to the \( O(N) \) vector model at its finite-size scaling critical point. Note that the free-energy density given by (26) is negative which shows that such a critical point is unstable by itself.

The interpretation of our results (26) and (27) in terms of the loss of degrees of freedom from the UV to IR is not clear, except perhaps at the point \( \langle \theta \rangle = \pi/L \) which is related to the critical point of the \( O(N) \) vector model in \( d = 3 \). In particular, the numerical value of \( \bar{c} \) in (27) is \( \sim 2.9 \) and it is larger than the sum of \( \bar{c} \)'s at the UV fixed points of the \( \phi^4 \) theory and the Gross-Neveu-Yukawa theory, when \( N > 2.5 \). The latter sum may be considered as an upper bound for \( \bar{c} \) in view of the known universality classes in three-dimensions. Therefore, the point \( \langle \theta \rangle = 2\pi/3L \) corresponds to a genuinely new phase transition for which the inequalities suggested in \( \ref{equation:inequality} \) do not seem to hold. Such a critical point may be associated with a non-unitary CFT and further with a Lee-Yang edge singularity \( \ref{equation:ly} \). In fact, an interpretation of \( i\langle \theta \rangle \) as a purely imaginary chemical potential suggests \( \ref{equation:ih} \) that, for \( 2\pi/3 \leq L \langle \theta \rangle < \pi \), we are dealing with critical points corresponding to Lee-Yang zeros \( \ref{equation:ly} \).

5 Summary and Outlook

In this work we calculated the renormalised free-energy densities of the three-dimensional \( O(N) \) vector model and the \( U(N) \) Gross-Neveu model, at their conformal points in a geometry with one finite dimension of length \( L \). Our calculations were based on non-trivial polylogarithmic identities. Our results are consistent with the recent suggestion of \( \ref{equation:inequality} \) that the free-energy density encodes non-perturbative information regarding the loss of degrees of freedom from the UV to the IR in QFTs.

Our study demonstrates the importance of three-dimensional critical models as testing grounds for the recent ideas concerning irreversibility of the RG flow and non-perturbative phenomena in QFTs \( \ref{equation:inequality} \). There are many directions in which our leading-\( N \) calculations could be extended. For example, one could study the IR limits of the \( O(N) \) vector model and the Gross-Neveu model in three-dimensions, both in the bulk and in a finite geometry. These limits would correspond to the
lower bounds of the inequalities suggested for $\tilde{c}$. The latter could be further checked by next-to-leading order calculations of $\tilde{c}$ in the above models. Based on techniques developed here, one could also study the free-energy density of other three-dimensional models exhibiting critical behavior, such as $CP^{N-1}$ models or QED$_3$. 

In an possible application of our results, we may view the critical points for $2\pi/3 \leq L\langle\theta\rangle \leq \pi$ as viable critical points in supersymmetric CFTs in $d = 3$. For example, consider the $\mathcal{N} = 1$ supersymmetric $\sigma$-model in $d = 3$ which contains Majorana fermions. This means that the large-$N$ fermionic contribution to the free-energy density of such a model with fixed total fermion number is half the r.h.s. of (25), therefore the sum of the bosonic and fermionic contributions, as seen from (6) and Fig.2, is always greater than zero. This sum vanishes for $\langle\theta\rangle = \pi/L$ corresponding to a supersymmetry restoration mechanism which deserves further study.

The relevance of non-trivial polylogarithmic identities to the calculation of free-energy densities in two-dimensional CFTs is well-known. Their appearance in studies of CFTs in higher dimensions is intriguing and requires further investigation. It would also be interesting to study the OPE structure of correlation functions in the Gross-Neveu model for fixed total fermion number, at the above critical points. Although our results simplify for $d = 3$, they can presumably be generalised for all $2 < d < 4$.

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