QUERY COMPLEXITY OF ADVERSARIAL ATTACKS

Grzegorz Gluch∗
EPFL

Rüdiger Urbanke
EPFL

ABSTRACT

Modern machine learning models are typically highly accurate but have been shown to be vulnerable to small, adversarially-chosen perturbations of the input. There are two main models of attacks considered in the literature: black-box and white-box. We consider these threat models as two ends of a fine-grained spectrum, indexed by the number of queries the adversary can ask. Using this point of view we investigate how many queries the adversary needs to make to design an attack that is comparable to the best possible attack in the white-box model. We analyze two classical learning algorithms on two synthetic tasks for which we prove meaningful security guarantees. The obtained bounds suggest that some learning algorithms are inherently more robust against query-bounded adversaries than others.

1 INTRODUCTION

Modern neural networks achieve high accuracy on tasks such as image classification (Krizhevsky et al., 2012) or speech recognition (Collobert and Weston, 2008). However, they are typically susceptible to small, adversarially-chosen perturbations of the inputs (Szegedy et al., 2014; Nguyen et al., 2015): more precisely, given a correctly-classified input \( x \), one can typically find a small perturbation \( \delta \) such that \( x + \delta \) is misclassified by the network while to the human eye this perturbation is not perceptible.

There are two main threat models considered in the literature: black-box and white-box. In the white-box model, on the one hand, the attacker (Biggio et al., 2013; Zheng et al., 2019) is assumed to have access to a full description of the model. For the case of neural networks that amounts to a knowledge of the architecture and the weights. In the black-box model, on the other hand, the adversary (Papernot et al., 2017; Chen et al., 2017; Liu et al., 2016; Xiao et al., 2018; Hayes and Danezis, 2017) can only observe the input-output behavior of the model. Many defenses have been proposed to date. To mention just some – adversarial learning (Goodfellow et al., 2014; Madry et al., 2018; Tramèr et al., 2018; Xiao et al., 2019), input denoising (Bhagoji et al., 2017; Xu et al., 2017), or more recently, randomized smoothing (Lécuyer et al., 2019; Cohen et al., 2019; Salman et al., 2019; Gluch and Urbanke, 2019). Unfortunately, most heuristic defenses break in the presence of suitably strong adversaries (Carlini and Wagner, 2017; Athalye et al., 2018; Uesato et al., 2018) and provable defenses are often impractical or allow only very small perturbations. Thus a full defense remains elusive. The current literature on this topic is considerable. We refer the reader to Chakraborty et al. (2018) for an overview of both attacks and defenses and to Bhambri et al. (2019) for a survey focused on black-box attacks.

We consider black-box and white-box models as the extreme points of a spectrum parameterized by the number of queries allowed for the adversary. This point of view is related to Ilyas et al. (2018) where the authors design a black-box attack with a limited number of queries. Intuitively, the more queries the adversary can make the more knowledge he gains about the classifier. When the number of queries approaches infinity then we transition from a black-box to a white-box model as in this limit the adversary knows the classifying function exactly. Using this point of view we ask:

How many queries does the adversary need to make to reliably find adversarial examples?

By “reliably” we mean comparable with the information-theoretic white-box performance. To be more formal, we assume that there is a distribution \( \mathcal{D} \) and a high-accuracy classifier \( f \) that maps

∗Correspondence to Grzegorz Gluch <grzegorz.gluch@epfl.ch>.
\( \mathbb{R}^d \) to classes \( \mathcal{Y} \). The adversary \( A \) only has black-box access to \( f \). Moreover, \( \epsilon \in \mathbb{R}_+ \) is an upper bound on the norm (usually \( \ell_p \) norm) of the allowed adversarial perturbation. Assume that \( f \) is susceptible to \( \epsilon \)-bounded adversarial perturbations for an \( \eta \)-fraction of the underlying distribution \( D \). The quantity \( \eta \) is the largest error an adversary, who has access to unbounded computational resources and fully knows \( f \), can achieve. We ask: How many queries to the classifier \( f \) does \( A \) need to make in order to be able to find adversarial examples for say an \( \eta/2 \) fraction of the distribution \( D \)? This question is similar to problems considered in Ashitiani et al. (2020). The difference is that in Ashitiani et al. (2020) the authors define the query complexity of the adversary as a function of the number of points for which the adversarial examples are to be found. Moreover, they require the adversary to be perfect, that is to find adversarial examples whenever they exist. This stands in contrast to our approach that only requires the adversary to succeed for say a 1/2 fraction of the adversarial examples. The question we ask is also similar to ideas in Cullina et al. (2018). In this paper the authors consider a generalization of PAC learning and ask how many queries an algorithm requires in order to learn robustly. Similar questions were also asked in Yin et al. (2019) and Schmidt et al. (2018). The difference is that we focus on the query complexity of the attacker and not the defender.

**Our contributions.** We introduce a new notion - the query complexity (QC) of adversarial attacks. This notion unifies the two most popular attack models and enables a systematic study of robustness of learning algorithms against query-bounded adversaries.

Our two most important findings are the following: (i) for a broad class of learning algorithms we prove a security guarantee against query-bounded adversaries that grows with accuracy; (ii) the higher the entropy of the decision boundaries that are created by the learning algorithm the more secure is the resulting system in our attack model. We present two scenarios for which we are able to show meaningful lower bounds on the QC. The first one is a simple 2-dimensional distribution and a nearest neighbor algorithm. For this setting we are able to prove a strong query lower bound of \( \Theta(m) \), where \( m \) is the number of samples on which the classifier was trained. For the second example we consider the well-known adversarial spheres distribution, introduced in the seminal paper Gilmer et al. (2018). For this learning task we argue that quadratic neural networks have a query lower bound of \( \Theta(d) \), where \( d \) is the dimensionality of the data. We discuss why certain learning algorithms like linear classifiers and also neural networks might be far less secure than KNN against query-bounded adversaries.

There exist tasks for which it is easy to find high-accuracy classifiers but finding robust models is infeasible. E.g., in Bubeck et al. (2019) the authors describe a situation where it is information-theoretically easy to learn robustly but there is no algorithm in the statistical query model that computes a robust classifier. In Bubeck et al. (2018) an even stronger result is proven. It is shown that under a standard cryptographic assumption there exist learning tasks for which no algorithm can efficiently learn a robust classifier. Finally, in Tsipras et al. (2019) it was shown that robust and accurate classifiers might not even exist. The query-bounded point of view shows a way to address these fundamental difficulties – even for tasks for which it is impossible to produce a model that is secure against a resource-unbounded adversary, it might be possible to defend against a query-bounded adversary.

**Organization of the paper.** In Section 2 we formally define the threat model and the query complexity of adversarial attacks. In Section 3 we show that the query complexity grows with accuracy for a rich class of learning algorithms. In Section 4.1 and 4.2 we analyze the query complexity of KNN and Quadratic Neural Network learning algorithms respectively. In Section 4.3 we present an universal defense against query-bounded adversaries. Finally, in Section 5 we summarize the results and discuss future directions. We defer all the proofs to the appendix.

## 2 The Query Complexity of Adversarial Attacks

We start by formally defining the threat model. For a data distribution \( D \) over \( \mathbb{R}^d \) and a set \( A \subseteq \mathbb{R}^d \) let \( \mu(A) := \mathbb{P}_X[X \in A] \). For simplicity we consider only separable binary classification tasks. Such tasks are fully specified by \( D \) as well as a ground truth \( h : \mathbb{R}^d \rightarrow \{-1, 1\} \). For a binary classification task with a ground truth \( h : \mathbb{R}^d \rightarrow \{-1, 1\} \) and a classifier \( f : \mathbb{R}^d \rightarrow \{-1, 1\} \) we define the error set as \( E(f) := \{x \in \text{supp}(D) : f(x) \neq h(x)\} \). For \( x \in \mathbb{R}^d \) and \( \epsilon > 0 \) we write
\[ B_{\varepsilon}(x) \] to denote the open ball with center \( x \) and radius \( \varepsilon \). We say that a function \( p : \mathbb{R}^d \to \mathbb{R}^d \) is an \( \varepsilon \)-perturbation if for all \( x \in \mathbb{R}^d \) we have \( \|p(x) - x\|_2 \leq \varepsilon \). For \( n \in \mathbb{N} \) we denote the set \( \{1, 2, \ldots, n\} \) by \( [n] \). For \( x, y \in \mathbb{R}^d \) we will use \( [x, y] \) to denote the closed line segment between \( x \) and \( y \). For \( A, B \subseteq \mathbb{R}^d \) we define \( A + B := \{x + y : x \in A, y \in B\} \). We use \( m \) to denote the sample size.

**Definition 1 (Risk).** Consider a separable, binary classification task with a ground truth \( h : \mathbb{R}^d \to \{-1, 1\} \). For a classifier \( f : \mathbb{R}^d \to \{-1, 1\} \) we define the Risk as \( R(f) := \mathbb{P}_x[f(x) \neq h(x)] \).

**Definition 2 (Adversarial risk).** Consider a binary classification task with separable classes with a ground truth \( h : \mathbb{R}^d \to \{-1, 1\} \). For a classifier \( f : \mathbb{R}^d \to \{-1, 1\} \) and \( \varepsilon \in \mathbb{R}_{\geq 0} \) we define the Adversarial Risk as:

\[
AR(f, \varepsilon) := \mathbb{P}_x[\exists \gamma \in B_{\varepsilon} : f(X + \gamma) \neq h(X) \land X + \gamma \in \text{supp}(D)]
\]

For an \( \varepsilon \)-perturbation \( p \) we define:

\[
AR(f, p) := \mathbb{P}_x[f(p(X)) \neq h(X) \land p(X) \in \text{supp}(D)],
\]

to be the adversarial risk of a specific perturbation function \( p \).

**Discussion.** Note that Definition 2 treats a point as an adversarial example only if it belongs to the \( \text{supp}(D) \). This can potentially decrease the adversarial risk significantly but as shown in the literature on manifold adversarial examples still exist. For instance in [Gilmer et al., 2018] authors show a synthetic dataset for which they successfully find adversarial examples on the data manifold. In [Athalye et al., 2018] and [Ilyas et al., 2018] the authors manage to circumvent Defense-GAN on real datasets. Defense-GAN uses a Generative Adversarial Network to project samples onto the manifold of the generator before classifying them. As it is reasonable to assume that GANs approximate data distributions well this is an evidence that adversarial examples still exist on the data manifold. The restriction simplifies the formal proof of our lower bounds (see Section 4.1 and 4.2). However, we believe that our results are not a consequence of this restriction and they carry over essentially unchanged (modulo constants).

In order to keep the exposition simple, we restrict our discussion to \( \ell_2 \)-bounded adversarial perturbations. Other norms can of course be considered and might be important in practice.

**Definition 3 (Query-bounded adversary).** For \( \varepsilon \in \mathbb{R}_{\geq 0} \) and \( f : \mathbb{R}^d \to \{-1, 1\} \) a \( q \)-bounded adversary with parameter \( \varepsilon \) is a deterministic algorithm \( \mathcal{A} \) that asks at most \( q \in \mathbb{N} \) (potentially adaptive) queries of the form \( f(x) \overset{?}{=} 1 \) and outputs an \( \varepsilon \)-perturbation \( \mathcal{A}(f) : \mathbb{R}^d \to \mathbb{R}^d \).

**Definition 4 (Query complexity of adversarial attacks).** Consider a binary classification task \( T \) for separable classes with a ground truth \( h : \mathbb{R}^d \to \{-1, 1\} \) and a distribution \( D \). Assume that there is a learning algorithm \( \text{ALG} \) for this task that given \( S \sim D^m \) learns a classifier \( \text{ALG}(S) : \mathbb{R}^d \to \{-1, 1\} \). For \( \varepsilon \in \mathbb{R}_{\geq 0} \) define the Query Complexity of adversarial attacks on \( \text{ALG} \) with respect to \( (T, m, \varepsilon) \) and denote it by \( QC(\text{ALG}; T, m, \varepsilon) \): It is the minimum \( q \in \mathbb{N} \) so that there exists a \( q \)-bounded adversary \( \mathcal{A} \) with parameter \( \varepsilon \) such that

\[
\mathbb{P}_{S \sim D^m} \left[ AR(\text{ALG}(S), \mathcal{A}(\text{ALG}(S))) \geq \frac{1}{2} : AR(\text{ALG}(S), \varepsilon) \right] \geq 0.99.
\]

In words, it is the minimum number of queries that is needed so that there exists an adversary who can achieve an error of half the maximum achievable error (with high probability over the data samples). Note that it follows from Definitions 3 and 4 that \( \mathcal{A} \) is computationally unbounded, knows the distribution \( D \) and the ground truth \( h \) of the learning task and also knows the learning algorithm \( \text{ALG} \). The only restriction that is imposed on \( \mathcal{A} \) is the number of allowed queries. What is important is that \( \mathcal{A} \) does not know \( S \) nor the potential randomness of \( \text{ALG} \) (in the generalization setting \( \text{ALG} \) can be randomized, see Definition 5) – this is what makes the QC non-degenerate. To see this, observe that if \( \mathcal{A} \) knew \( S \) and \( \text{ALG} \) was deterministic then \( \mathcal{A} \) could achieve \( AR(\text{ALG}(S), \mathcal{A}(\text{ALG}(S))) = AR(\text{ALG}(S), \varepsilon) \) without asking any queries. This is because \( \mathcal{A} \) can for every point \( x \) check if there exists \( \gamma \in B_{\varepsilon} \) such that \( \text{ALG}(S)(X + \gamma) \neq h(X) \) and \( X + \gamma \in \text{supp}(D) \), as \( \mathcal{A} \) can compute \( \text{ALG}(S) \) without asking any queries. This allows \( \mathcal{A} \) to achieve adversarial risk of \( AR(\text{ALG}(S), \varepsilon) \) (see Definition 3).

---

5We use algorithm here since this seems more natural. But we do not limit the attacker computationally nor are we concerned with questions of computability. Hence, function would be equally correct.
Definition 4 can be generalized to incorporate randomness in the learning algorithm. This extension is important as we will discuss in Section 4.3. Intuitively, the randomness in ALG can increase the entropy of the learning process and that in turn may lead to a higher QC. Further, both the approximation constant (which is chosen to be $1/2$ in Definition 4) as well as the success probability can also be generalized. We give the generalized definition (Definition 5) in the appendix. For the sake of clarity we will restrict ourselves for the most part to Definition 4. This eliminates two parameters from our expressions and restricts the attention to deterministic learning algorithms. Only when the distinction becomes important will we refer to Definition 5.

Summary: The query complexity of adversarial attacks is the minimum $q$ for which there exists a $q$-bounded adversary that carries out a successful attack. Such adversaries are computationally unbounded, know the learning task and the learning algorithm but don’t know the training set.

3 ROBUSTNESS AND ACCURACY – FOES NO MORE

It was argued in [Tsipras et al. (2019)] that there might be an inherent tension between accuracy and adversarial robustness. We argue that this potential tension disappears for a rich class of learning algorithms if we consider $q$-bounded adversaries. We show that if a learning algorithm satisfies a particular natural property then there is a lower bound for QC of this algorithm that grows with accuracy.

**Theorem 1.** Let $\epsilon \in \mathbb{R}_{\geq 0}$ and $T$ be a binary classification task on $\mathbb{R}^d$ with separable classes. Let ALG be a learning algorithm for $T$ with the following properties:

1. $\forall x \in \text{supp}(D), \mathbb{P}_{S \sim D^m}[\text{ALG}(S)(x) \neq h(x)] \leq C \cdot \delta$,
2. $\mathbb{P}_{S \sim D^m}[\text{AR}(\text{ALG}(S), \epsilon) \geq \eta] \geq 0.99$,
3. $\mathbb{P}_{S \sim D^m}[R(\text{ALG}(S)) \leq \delta] \geq 0.99$.

Then we have:

$$QC(\text{ALG}, T, m, \epsilon) \geq \log \left( \frac{\eta}{3 \cdot C \cdot \delta} \right).$$

The lower bound obtained in Theorem 1 is useful in situations when ALG($S$) has high accuracy but the adversarial risk is large. This is a typical situation when using neural networks – one is often able to find classifiers that have high accuracy but they are not adversarially robust.

Summary: For a rich class of learning algorithms our security guarantee against query-bounded adversaries increases with accuracy. A risk of $2^{-\Omega(k)}$ leads to robustness against $\Theta(k)$-bounded adversaries.

4 HIGH-ENTROPY DECISION BOUNDARIES LEAD TO ROBUSTNESS

The decision boundary of a learning algorithm applied to a given task can be viewed as the outcome of a random process: (i) generate a training set and, (ii) apply to it the, potentially randomized, learning algorithm. Recall, see Definitions 4 and 5, that a query-bounded adversary does not know the sample on which the model was trained nor the randomness used by the learner. This means that if the decision boundary has high entropy then the adversary needs to ask many questions to recover the boundary to a high degree of precision. This suggest that high-entropy decision boundaries are robust against query-bounded adversaries since intuitively it is clear that an approximate knowledge of the decision boundary is a prerequisite for a successful attack. Following this reasoning, we present two instances where high entropy of the decision boundary leads to security.

4.1 Entropy due to Locality – K-NN Algorithms

Let us now analyze the QC of K-Nearest Neighbor (K-NN) algorithms. Nearest neighbor algorithms are among the simplest and most studied algorithms in machine learning. They are also widely used as a benchmark. It was shown in [Shalev-Shwartz and Ben-David (2014)] that for a sufficiently large
training set, the risk of the 1-NN learning rule is upper bounded by twice the optimal risk. It is also known that these methods suffer from the "curse of dimensionality" – for $d$ dimensional distributions they typically require $m = 2^{\Theta(d \log(d))}$ many samples. That is why in practice one often uses some dimensionality reduction subroutine before applying $K$-NN. Moreover, $K$-NN techniques are one of the few learning algorithms that do not require any learning. In the naive implementation all computation is deferred until function evaluation. This is related to the most interesting fact from our perspective, namely that the classification rule of the $K$-NN algorithm depends only on the local structure of the training set.

We argue that this property makes $K$-NN algorithms secure against query-bounded adversaries. Intuitively, if the adversary $A$ wants to achieve a high adversarial risk she needs to understand the global structure of the decision boundary. But if the classification rule is only very weakly correlated between distant regions of the space then this intuition suggests that $A$ may need to ask $\Theta(m)$ many queries to guarantee a high adversarial risk. This is consistent with the entropy point of view. Moreover there are experimental results (see Wang et al. (2018); Papernot et al. (2016)) that show that it is hard to attack $K$-NN classifiers in the black-box model.

We make these intuitions formal in the following sense. We design a synthetic binary learning task in $\mathbb{R}^2$, where the data is uniformly distributed on two parallel intervals – corresponding to the two classes. We then show a $\Theta(m)$ lower bound for the QC of 1-NN algorithm for this learning task. This means that the number of queries the adversary needs to make to attack 1-NN is proportional to the number of samples on which the algorithm was "trained". We conjecture that a similar behavior occurs in higher dimensions as well.

4.1.1 $K$-NN – QC LOWER BOUNDS

Consider the following distribution. Let $m \in \mathbb{N}$ and $z \in \mathbb{R}_+$. Let $L_-, L_+ \subseteq \mathbb{R}^2$ be two parallel intervals of length $m$ placed at distance $z$ apart. More formally, $L_- := [(0, 0), (m, 0)]$, $L_+ = [(0, z), (m, z)]$. Let the binary learning task $T_{\text{Intervals}}(z)$ be as follows. We generate $\bar{x} \in \mathbb{R}^2$ uniformly at random from the union $L_- \cup L_+$. We assign the label $y = -1$ if $\bar{x} \in L_-$ and $y = +1$ otherwise. In Figure 1 we visualize a decision boundary of the 1-NN algorithm for a random $S \sim D^m$ on the $T_{\text{Intervals}}$. Horizontal lines, black and gray, represent the two classes, crosses are data points, white and gray regions depict the classification rule and the union of red intervals is equal to the error set. We also include more visualizations in the appendix. The main result of this subsection is:

**Theorem 2.** The 1-Nearest Neighbor algorithm applied to the learning task $T_{\text{Intervals}}(z)$ satisfies:

$$QC(1\text{-Nearest Neighbor}, T_{\text{Intervals}}, 2m, z) \geq \Theta(m),$$

provided that $z = \Omega(1)$.

**Summary:** The K-NN algorithm learns classification rules that depend only on the local structure of the data. This implies high-entropy decision boundaries, which in turn leads to robustness against query-bounded adversaries. The QC of 1-NN scales at least linearly with the size of the training set.
4.2 Entropy due to Symmetry - Quadratic Neural Networks

In this section we analyze the QC of Quadratic Neural Networks (QNN) applied to a learning task defined in [Gilmer et al. 2018]. Let $S_{k+1} := \{x \in \mathbb{R}^d : \|x\|_2 = r\}$. The distribution $\mathcal{D}$ is defined by the following process: generate $x \sim U[S^d_{k+1}]$ and $b \sim U\{-1, 1\}$ (where $U$ denotes the uniform distribution). If $b = -1$ return $(x, -1)$, otherwise return $(1.3x, +1)$.

The QNN is a single hidden-layer network where the activation function is the quadratic function $\sigma(x) = x^2$. There are no bias terms in the hidden layer. The output node computes the sum of the activations from the hidden layer, multiplies them by a scalar and adds a bias. If we assume that the hidden layer contains $h$ nodes then the network has $d \cdot h + 2$ parameters. It was shown in [Gilmer et al. 2018] that the function that is learned by QNN has the form $y(x) = \sum_{i=1}^d \alpha_i z_i^2 - 1$, where the $\alpha_i$'s are scalars that depend on the parameters of the network and $z = M(x)$ for some orthogonal matrix $M$. The decision boundary is thus $\sum_{i=1}^d \alpha_i z_i^2 = 1$, which means that it is an ellipse centered at the origin.

In a series of experiments performed for the Concentric Spheres (CS) dataset in [Gilmer et al. 2018] it was shown that a QNN trained with $N = 10^6$ many samples with $d = 500$ and $h = 1000$ learns a classifier with an estimated error of approximately $10^{-20}$ but the adversarial risk $\eta$ is high and is estimated to be $1/2$ when $\epsilon \approx 0.18$. On the theoretical side, it was proven in [Gluch and Urbanke 2019] (see Section 9.1) that

$$\epsilon \leq O\left(\frac{\log(\eta/\delta)}{d}\right).$$

In words, equation 1 gives an upper bound on the biggest allowed perturbation $\epsilon$ in terms of the risk $\delta$ the adversarial risk $\eta$ and the dimension $d$. In particular if we want the classifier to be adversarially robust for $\epsilon = \Theta(1)$ (that is for perturbations comparable with the separation between the two classes) then $\delta = 2^{-\Omega(d)}$. Even robustness of only $\epsilon = \Theta(1/\sqrt{d})$ requires the risk to be as small as $\delta = 2^{-\Omega(\sqrt{d})}$. These results paint a bleak picture of the adversarial robustness for CS.

The QC point-of-view is more optimistic. Using results from Section 3 we first show that if the network learns classifiers with risk $2^{-\Omega(k)}$ then it automatically leads to a lower bound on the QC of $\Theta(k)$. Moreover, for a simplified model of the network, we show that even if the risk of the learned classifier is only a small constant, say 0.01, then this results in a lower bound on the QC of $\Theta(d)$ for perturbations of $\Theta(1/\sqrt{d})$. Using equation 1 our result guarantees security against $\Theta(d)$-bounded adversaries for perturbations which are $\Theta(\sqrt{d})$ times bigger than the best possible against unbounded adversaries. This shows that restricting the power of the adversary can make a significant difference.

We argue that the obtained $\Theta(d)$ lower bound is close to the real QC for this algorithm and learning task. Observe that the decision boundary of the network is an ellipsoid which can be described by $O(d^2)$ parameters ($d^2$ for the rotation matrix and $d$ for lengths of principal axes). This suggest that it should be possible to design a $O(d^2)$-bounded adversary that succeeds on this task. Assuming that this is indeed the case, our lower bound is only a factor $O(d)$ away from the optimum.

The results of this section can be understood in the following way. The relatively simple structure of the decision boundaries allows the adversary to attack the model with only $O(d^2)$ queries. There is however enough entropy in the network to guarantee a lower bound for the QC of $\Theta(d)$. This entropy intuitively comes from the rotational invariance of the dataset and in turn of the learned decision boundary. We conjecture that algorithms like linear classifiers (e.g., SVMs) exhibit a similar behavior. That is, for natural learning tasks they are robust against $q$-bounded adversaries only for $q = O(poly(d))$. The reason is that all these algorithms generate classifiers with relatively simple decision boundaries which can be described by $O(poly(d))$ parameters.

But this is not the end of the story for CS. Our results don’t preclude the possibility that there exist a learning algorithm that is secure against $q$-bounded adversaries for $q \gg d$. In fact in Section 4.3 we present an off-the-shelf solution that can be applied to CS dataset and which, by injecting entropy, achieves security against $k$-bounded adversaries for $\epsilon = \Theta\left(\frac{1}{k^{\sqrt{d}}}\right)$.
4.2.1 Quadratic Neural Networks – QC lower bounds

Using the results from Section 3, one can show that increased accuracy leads to increased robustness. More precisely if QNN has a risk of $2^{-Ω(k)}$ then it is secure against $Θ(k)$-bounded adversaries for $ε = 0.3$. The proof of this fact is deferred to the appendix.

Now we argue that also in the case where the risk achieved by the network is as large as a constant then QNN are still robust against $Θ(d)$-bounded adversaries. We first argue that any reasonable optimization algorithm applied to QNN for the CS learning task gives rise to a distribution on error sets that is rotational invariant. This follows from the fact that $D$ itself is rotational invariant. Now observe that for QNN the error sets are of the form:

$$\{ x \in S_{d-1}^d : \sum_{i=1}^{d} α_i z_i^2 > 1 \} \cup \{ x \in S_{d-1}^d : \sum_{i=1}^{d} α_i z_i^2 < 1 \},$$

as the decision boundary learned by QNN is defined by $\sum_{i=1}^{d} α_i z_i^2 = 1$, where $z = M x$ for some orthonormal matrix $M$. This set might be quite complicated as it is basically defined as the intersection of a sphere and an ellipsoid. We will refer to the real distribution on error sets of QNN as $E_{QNN}$.

In the rest of this section we first introduce a set of distributions over error sets and then we state QC lower bounds for these distributions. Formal definitions are presented in Definition Distributions on Spherical Caps in the appendix, here we give a short description of what they are. For $y \in S_{d-1}^{d-1}$ let $\text{cap}(y, r, τ) := S_{d-1}^{d-1} \cap \{ x \in \mathbb{R}^d : \langle x, y \rangle \geq τ \}$. Let $τ : [0, 1] \rightarrow [0, 1]$ be such that for every $δ \in [0, 1]$ we have $μ(\text{cap}(\cdot, 1, τ(δ))) / μ(S_{d-1}^{d-1}) = δ$. For $δ \in (0, 1)$, $k \in \mathbb{N}_+$ they are: $\text{Cap}(δ) - $ one randomly rotated spherical cap occupying a $δ$ fraction of $S_{d-1}^{d-1}$; $\text{Caps}_i^{\text{i.i.d.}}(δ) - $ union of $k$ i.i.d. randomly rotated spherical caps, each occupying a $δ/k$ fraction of either $S_{d-1}^{d-1}$ or $S_{d-1}^{d-1}$, chosen uniformly at random; $\text{Caps}_G^G(δ) - $ $k$ randomly rotated spherical caps, each occupying a $δ/k$ fraction of either $S_{d-1}^{d-1}$ or $S_{d-1}^{d-1}$, chosen uniformly at random; the relative positions of cap’s normal vectors are determined by $G$, where $G$ is a given distribution on $(S_{d-1}^{d-1})^k$.

We conjecture that $\text{Cap}$, $\text{Caps}_i^{\text{i.i.d.}}$, $\text{Caps}_G^G$ have QCs that are no larger than the QC of $E_{QNN}$. The intuitive reason is that they contain less entropy than $E_{QNN}$ and so it should be easier to attack these distributions. In Lemma 1 we prove a $Θ(d)$ lower bound for Cap and, in the appendix, we give a matching upper bound of $Θ(d)$. Also in the appendix, we give two reductions that lower-bound QC of $\text{Caps}_i^{\text{i.i.d.}}$ and $\text{Caps}_G^G$ based on a conjecture (see Cap conjecture in the appendix). We summarize the proved and conjectured lower bounds in Table 1.

**Lemma 1 (Lower bound for Cap).** There exists $δ > 0$ such that if a $q$-bounded adversary $A$ succeeds on $\text{Cap}(0.01)$ with approximation constant $\geq 1 - δ$, error probability $2/3$ for $ε$ such that $\text{cap}(\cdot, 1, τ(0.01)) + B_ε = \text{cap}(\cdot, 1, 0)$. Then

$$q \geq Θ(d).$$

**Summary:** Quadratic neural networks have simple decision boundaries - they are of the form of ellipsoids. But due to the rotational symmetry there is sufficient entropy to guarantee robustness against $Θ(d)$-bounded adversaries.

---

1This lower bound is conditional on Cap conjecture (in the appendix).

| Error distribution | Lower bound |
|--------------------|-------------|
| Cap(0.01)          | $Θ(d)$     |
| Caps_i^{i.i.d.}(0.01) | $Θ(d)$   |
| Caps_G^G(0.01)    | $Θ(d/k)$   |

---

Table 1: QC for CS
4.3 HOW TO INCREASE THE ENTROPY OF AN EXISTING SCHEME – A UNIVERSAL DEFENSE

It was proven in Gluch and Urbanke (2019) that there exists a universal defense against adversarial attacks. The defense algorithm gets as an input access to a high accuracy classifier \( f \) and outputs a new classifier \( g \) that is adversarially robust. The idea of the defense is based on randomized smoothing (Cohen et al., 2019; Salman et al., 2019) and random partitions of metric spaces. Simple rephrasing of Theorem 5 from Gluch and Urbanke (2019) in the language of QC of adversarial attacks gives the following:

**Theorem 3.** For every \( d \in \mathbb{N}_+ \) there exists a randomized algorithm \( DEF \). It is given as input access to an initial classifier \( R^d \rightarrow \{-1,1\} \) and provides oracle access to a new classifier \( R^d \rightarrow \{-1,1\} \).

For every separable binary classification task \( T \) in \( \mathbb{R}^d \) with separation \( \epsilon \) the following conditions hold. Let \( ALG \) be a learning algorithm for \( T \) that uses \( m \) samples. Then for every \( S \sim D^m \) we have \( R(DEF(ALG(S))) \leq 2R(ALG(S)) \) and for every \( \epsilon' > 0 \):

\[
QC(DEF \circ ALG, T, m, \epsilon') \geq \Theta \left( \frac{\epsilon}{\sqrt{d} \cdot \epsilon'} \right).
\]

**Summary:** There exists a universal defense that can be applied on top of any learning algorithm to make it secure against query-bounded adversaries. Roughly speaking, it works by injecting additional randomness to increase the entropy of the final classifier.

4.4 DISCUSSION

For a given task the QC can vary considerably depending on the learning algorithm. Consider e.g. the CS data set and compare the QCs of K-NN and QNN. As we mentioned in Section 4.1 for CS K-NN might need as many as \( m = 2^{\Theta(d \log d)} \) samples. This means that the lower bound \( \Theta(m) \) becomes \( 2^{\Theta(d \log d)} \). Comparing that with the conjectured QC of \( O(d^2) \) for QNN we see that there can be an exponential (in \( d \)) gap between QCs of different learning algorithms for the same task. A unifying point of view is that the QC of \( \Theta(d^2) \) and \( \Theta(m) \) correspond in both cases to the number of “degrees of freedom” of decision boundary of the respective model. For the case of QNN it was \( O(d^2) \) and for the case of 1-NN it is \( \Theta(m) \). We conjecture that neural networks are susceptible to black-box attacks because their decision boundaries are low-entropy.

In this section, we demonstrated that the entropy of learned decision boundaries is closely related to security against query-bounded adversaries. Moreover we observe that sources of the entropy can be varied. For K-NN the entropy is high due to the locality of the learning algorithm whereas for QNN it comes from the rotational symmetry of the data. The entropy can also be increased by introducing randomness in the learning algorithm itself.

5 CONCLUSIONS AND TAKEAWAYS

We investigate robustness of learning algorithms against query-bounded adversaries. We start by introducing a definition of QC of adversarial attacks and then proceed to study its properties. We show a series of lower bounds for classical learning algorithms. Specifically, we show that improvements in accuracy of a model lead to an improved security against query-bounded adversaries. We give a lower bound of \( \Theta(d) \) for QNNs and a lower bound of \( \Theta(m) \) for 1-NN algorithm. Our analysis identifies properties of learning algorithms that make them (non-)robust. These results give a rule-of-thumb: ”The higher the entropy of decision boundary the better” for assessing the QC of a given algorithm.

We believe that a systematic investigation of learning algorithms from the point of view of QC will lead to more adversarially-robust systems. Specifically, it should be possible to design generic defenses that can be applied on top of any learning algorithm. One example of such a defense was given in Section 4.3. Significantly more work is needed in order to fulfill the potential of this approach. But imagine that this type of defense could be applied efficiently with only a black-box access to the underlying classifier. And imagine further, that it could guaranteed a QC of, say \( q = 2^{\Theta(d)} \). This would arguably solve the adversarial robustness problem.
REFERENCES

Ashtiani, H., Pathak, V., and Urner, R. (2020). Black-box certification and learning under adversarial perturbations. ArXiv; abs/2006.16520.

Athalye, A., Carlini, N., and Wagner, D. A. (2018). Obfuscated gradients give a false sense of security: Circumventing defenses to adversarial examples. In Proceedings of the 35th International Conference on Machine Learning, ICML 2018, Stockholmsmässan, Stockholm, Sweden, July 10-15, 2018, pages 274–283.

Bhagoji, A., Cullina, D., and Mittal, P. (2017). Dimensionality reduction as a defense against evasion attacks on machine learning classifiers.

Bhambri, S., Muku, S., Tulasi, A. S., and Buduru, A. B. (2019). A study of black box adversarial attacks in computer vision. ArXiv; abs/1912.01667.

Biggio, B., Corona, I., Maiorca, D., Nelson, B., Šrndić, N., Laskov, P., Giacinto, G., and Roli, F. (2013). Evasion attacks against machine learning at test time. In Blockeel, H., Kersting, K., Nijssen, S., and Železný, F., editors, Machine Learning and Knowledge Discovery in Databases, pages 387–402, Berlin, Heidelberg. Springer Berlin Heidelberg.

Bubeck, S., Lee, Y. T., Price, E., and Razenshteyn, I. (2019). Adversarial examples from computational constraints. In Chaudhuri, K. and Salakhutdinov, R., editors, Proceedings of the 36th International Conference on Machine Learning, volume 97 of Proceedings of Machine Learning Research, pages 831–840, Long Beach, California, USA. PMLR.

Bubeck, S., Lee, Y. T., Price, E., and Razenshteyn, I. P. (2018). Adversarial examples from cryptographic pseudo-random generators. CoRR; abs/1811.06418.

Carlini, N. and Wagner, D. A. (2017). Adversarial examples are not easily detected: Bypassing ten detection methods. In Proceedings of the 10th ACM Workshop on Artificial Intelligence and Security, AISec@CCS 2017, Dallas, TX, USA, November 3, 2017, pages 3–14.

Chakraborty, A., Alam, M., Dey, V., Chattopadhyay, A., and Mukhopadhyay, D. (2018). Adversarial attacks and defences: A survey. ArXiv; abs/1810.00069.

Chen, P.-Y., Zhang, H., Sharma, Y., Yi, J., and Hsieh, C.-J. (2017). Zoo: Zeroth order optimization based black-box attacks to deep neural networks without training substitute models. In Proceedings of the 10th ACM Workshop on Artificial Intelligence and Security, AISec ’17, page 15–26, New York, NY, USA. Association for Computing Machinery.

Cohen, J., Rosenfeld, E., and Kolter, Z. (2019). Certified adversarial robustness via randomized smoothing. In Chaudhuri, K. and Salakhutdinov, R., editors, Proceedings of the 36th International Conference on Machine Learning, volume 97 of Proceedings of Machine Learning Research, pages 1310–1320, Long Beach, California, USA. PMLR.

Collobert, R. and Weston, J. (2008). A unified architecture for natural language processing: Deep neural networks with multitask learning. In Proceedings of the 25th International Conference on Machine Learning, ICML ’08, pages 160–167, New York, NY, USA. ACM.

Cullina, D., Bhagoji, A. N., and Mittal, P. (2018). Pac-learning in the presence of evasion adversaries. In Proceedings of the 32nd International Conference on Neural Information Processing Systems, NIPS ’18, page 228–239, Red Hook, NY, USA. Curran Associates Inc.

Gilmer, J., Metz, L., Faghri, F., Schoenholz, S. S., Raghu, M., Wattenberg, M., and Goodfellow, I. J. (2018). Adversarial spheres. In 6th International Conference on Learning Representations, ICLR 2018, Vancouver, BC, Canada, April 30 - May 3, 2018, Workshop Track Proceedings.

Gluch, G. and Urbanke, R. L. (2019). Constructing a provably adversarially-robust classifier from a high accuracy one. CoRR; abs/1912.07561.

Goodfellow, I. J., Shlens, J., and Szegedy, C. (2014). Explaining and harnessing adversarial examples. cite arxiv:1412.6572.
Hayes, J. and Danezis, G. (2017). Machine learning as an adversarial service: Learning black-box adversarial examples.

Ilyas, A., Engstrom, L., Athalye, A., and Lin, J. (2018). Black-box adversarial attacks with limited queries and information. In Dy, J. and Krause, A., editors, Proceedings of the 35th International Conference on Machine Learning, volume 80 of Proceedings of Machine Learning Research, pages 2137–2146, Stockholmsmässan, Stockholm Sweden. PMLR.

Ilyas, A., Jalal, A., Asteri, E., Daskalakis, C., and Dimakis, A. G. (2017). The robust manifold defense: Adversarial training using generative models. CoRR, abs/1712.09196.

Krizhevsky, A., Sutskever, I., and Hinton, G. E. (2012). Imagenet classification with deep convolutional neural networks. In Pereira, F., Burges, C. J. C., Bottou, L., and Weinberger, K. Q., editors, Advances in Neural Information Processing Systems 25, pages 1097–1105. Curran Associates, Inc.

Lecuyer, M., Atlidakis, V., Geambasu, R., Hsu, D., and Jana, S. (2019). Certified robustness to adversarial examples with differential privacy. In 2019 IEEE Symposium on Security and Privacy, SP 2019, San Francisco, CA, USA, May 19-23, 2019, pages 656–672.

Liu, Y., Chen, X., Liu, C., and Song, D. (2016). Delving into transferable adversarial examples and black-box attacks. CoRR, abs/1611.02770.

Madry, A., Makelov, A., Schmidt, L., Tsipras, D., and Vladu, A. (2018). Towards deep learning models resistant to adversarial attacks. In International Conference on Learning Representations.

Nguyen, A. M., Yosinski, J., and Clune, J. (2015). Deep neural networks are easily fooled: High confidence predictions for unrecognizable images. In CVPR, pages 427–436. IEEE Computer Society.

Papernot, N., McDaniel, P., Goodfellow, I., Jha, S., Celik, Z. B., and Swami, A. (2017). Practical black-box attacks against machine learning. pages 506–519.

Papernot, N., McDaniel, P., and Goodfellow, I. J. (2016). Transferability in machine learning: from phenomena to black-box attacks using adversarial samples. ArXiv, abs/1605.07277.

Salman, H., Yang, G., Li, J., Zhang, P., Zhang, H., Razenshteyn, I. P., and Bubeck, S. (2019). Provably robust deep learning via adversarially trained smoothed classifiers. ArXiv, abs/1906.04584.

Schmidt, L., Santurkar, S., Tsipras, D., Talwar, K., and Madry, A. (2018). Adversarially robust generalization requires more data. In Proceedings of the 32nd International Conference on Neural Information Processing Systems, NIPS’18, page 5019–5031, Red Hook, NY, USA. Curran Associates Inc.

Shalev-Shwartz, S. and Ben-David, S. (2014). Understanding Machine Learning: From Theory to Algorithms. Cambridge University Press, USA.

Szegedy, C., Zaremba, W., Sutskever, I., Bruna, J., Erhan, D., Goodfellow, I., and Fergus, R. (2014). Intriguing properties of neural networks. In International Conference on Learning Representations.

Tramèr, F., Kurakin, A., Papernot, N., Goodfellow, I., Boneh, D., and McDaniel, P. (2018). Ensemble adversarial training: Attacks and defenses. In International Conference on Learning Representations.

Tsipras, D., Santurkar, S., Engstrom, L., Turner, A., and Madry, A. (2019). Robustness may be at odds with accuracy. In International Conference on Learning Representations.

Uesato, J., O’Donoghue, B., Kohli, P., and van den Oord, A. (2018). Adversarial risk and the dangers of evaluating against weak attacks. In Proceedings of the 35th International Conference on Machine Learning, ICML 2018, Stockholmsmässan, Stockholm, Sweden, July 10-15, 2018, pages 5032–5041.
Wang, Y., Jha, S., and Chaudhuri, K. (2018). Analyzing the robustness of nearest neighbors to adversarial examples. Volume 80 of *Proceedings of Machine Learning Research*, pages 5133–5142, Stockholmsmässan, Stockholm Sweden. PMLR.

Xiao, C., Li, B., Zhu, J.-Y., He, W., Liu, M., and Song, D. (2018). Generating adversarial examples with adversarial networks.

Xiao, K. Y., Tjeng, V., Shafiullah, N. M. M., and Madry, A. (2019). Training for faster adversarial robustness verification via inducing reLU stability. In *International Conference on Learning Representations*.

Xu, W., Evans, D., and Qi, Y. (2017). Feature squeezing: Detecting adversarial examples in deep neural networks.

Yin, D., Kannan, R., and Bartlett, P. (2019). Rademacher complexity for adversarially robust generalization. Volume 97 of *Proceedings of Machine Learning Research*, pages 7085–7094, Long Beach, California, USA. PMLR.

Zheng, T., Chen, C., and Ren, K. (2019). Distributionally adversarial attack. *Proceedings of the AAAI Conference on Artificial Intelligence*, 33:2253–2260.
A Generalized definitions

We give here a generalization of Definition 4 to randomized learning algorithms and general approximation constants and error probabilities.

Definition 5 (Query complexity of adversarial attacks - generalized). Consider a binary classification task $T$ for separable classes with a ground truth $h : \mathbb{R}^d \to \{-1, 1\}$ and a distribution $D$. Assume that there is a randomized learning algorithm ALG for this task that given $S \sim D^m$ and a sequence of random bits $B \sim B$ learns a classifier $ALG(S, B) : \mathbb{R}^d \to \{-1, 1\}$. For $\epsilon \in \mathbb{R}_{\geq 0}, \kappa, \alpha \in [0, 1]$ define the Query Complexity of the adversarial attacks on ALG with respect to $(T, m, \epsilon, \alpha, \kappa)$ and denote it by $QC(ALG, T, m, \epsilon, \alpha, \kappa)$: It is the minimum $q \in \mathbb{N}$ such that there exists a $q$-bounded adversary $A$ with parameter $\epsilon$ such that

$$
P_{S \sim D^m, B \sim B} [AR(ALG(S, B), A(ALG(S, B))) \geq \alpha \cdot AR(ALG(S, B), \epsilon)] \geq 1 - \kappa.
$$

If the above holds for $A$ we will refer to $\alpha$ as the approximation constant of $A$ and to $\kappa$ as the error probability of $A$. 
B Omitted proofs - Reduction and a lower bound

We now present the reduction that we will use for our QC lower bounds later on. Before delving into the details let us explain the intuition behind this approach. Let us recall the set-up. The classifier is trained on a sample $S$ that is unknown to the adversary. This classifier has a particular error set. We say that an adversary succeeds if, after asking some queries, it manages to produce an $\epsilon$-perturbation with the property that this perturbation moves “sufficient” mass into the error set of the classifier. Here, sufficient means at least half of what is possible if the adversary had known the classifier exactly. Let us say in this case that an $\epsilon$-perturbation is consistent with an error set.

The following theorem states that if for every $\epsilon$-perturbation the probability that an error set of a classifier is consistent with that perturbation is small then the QC is high. This is true since if for every $\epsilon$-perturbation only a small fraction of probability space (i.e., the possible classifiers) is consistent with this perturbation then $\mathcal{A}$’s protocol has to return many distinct $\epsilon$-perturbations depending on the outcome of its queries. And to distinguish which perturbation it should return it has to ask many queries.

**Theorem 4.** [Reduction.] Let $\epsilon \in \mathbb{R}_{\geq 0}$ and let $T$ be a binary classification task on $\mathbb{R}^d$ with separable classes. Let $\text{ALG}$ be a randomized learning algorithm for $T$ that uses $m$ samples. Then for every $\kappa \in [0, 1]$ the following holds:

$$
\text{QC} (\text{ALG}, T, m, \epsilon, 1/2, \kappa) \geq \log \left( \frac{1 - \kappa}{\sup_{p : \epsilon\text{-perturbation}} \mathbb{P}_{S \sim D^m, B \sim B} \left[ \mu(p^{-1}(E(\text{ALG}(S, B)))) \geq \frac{\text{AR}(\text{ALG}(S, B), \epsilon)}{2} \right]} \right).
$$

**Proof.** We first prove the Theorem when $\text{ALG}$ is deterministic. Let $\mathcal{A}$ be a $q$-bounded adversary that performs a successful attack on $\text{ALG}$ with respect to $(T, m, \epsilon, 1/2, \kappa)$ (as per Definition 5). We will show that $q$ is lower-bounded by the value from the statement of the Theorem.

The behavior of $\mathcal{A}$ can be represented as a binary tree $T$ where each non-leaf vertex $v \in T$ contains a query point $x_v \in \mathbb{R}^d$ and each leaf $l \in T$ contains an $\epsilon$-perturbation $p_l : \mathbb{R}^d \to \mathbb{R}^d$. Then $\mathcal{A}$ works as follows: it starts in the root $r \in T$ and queries the vertex $x_r$. Depending on $f(x_r) \equiv 1$ it proceeds left or right. It continues in this manner, querying the points stored in the visited vertices until it reaches a leaf $l$. At the leaf it outputs the perturbation function $p_l$.

Let us partition all possible data sets $S \in (\mathbb{R}^d)^m$ depending on which leaf is reached by $\mathcal{A}$ when interacting with $\text{ALG}(S)$. Let $l_1, \ldots, l_n$ be the leaves of $T$ and $C_1, \ldots, C_n \subseteq (\mathbb{R}^d)^m$ be the respective families of data sets that end up in the corresponding leaves. Let $Z := \{S \in (\mathbb{R}^d)^m : \mathcal{A} \text{ succeeds on } S\}$. By assumption $\mathcal{A}$ is guaranteed to succeed with probability $1 - \kappa$, so

$$
\mathbb{P}_{S \sim D^m}[S \in Z] \geq 1 - \kappa.
$$

(2) Now observe that for every $i \in [n]$ and $S \in C_i \cap Z$

$$
\mu(p_i^{-1}(E(\text{ALG}(S)))) \geq \frac{\text{AR}(\text{ALG}(S), \epsilon)}{2}.
$$

In words, for every $S \in C_i \cap Z$ the adversary $\mathcal{A}$ succeeds if at least $\frac{\text{AR}(\text{ALG}(S), \epsilon)}{2}$ of the probability mass of $D$ is moved by $p_i$, into the error set of $\text{ALG}(S)$. Thus we get that for every $i \in [n]$:

$$
\mathbb{P}_{S \sim D^m}[S \in C_i \cap Z] \leq \sup_{p : \epsilon\text{-perturbation}} \mathbb{P}_{S \sim D^m} \left[ \mu(p^{-1}(E(\text{ALG}(S)))) \geq \frac{\text{AR}(\text{ALG}(S), \epsilon)}{2} \right].
$$

(3) By standard properties of entropy we know that for a discrete random variable $W$ any protocol asking yes-no questions that finds the value of $W$ must on average ask at least $H(W)$ many questions. Let $W$ be a random variable that takes values in $\{1, 2, \ldots, n\}$, where for every $i \in [n]$ we have $\mathbb{P}[W = i] := \mathbb{P}_{S \sim D^m}[S \in C_i \cap Z]/\mathbb{P}_{S \sim D^m}[S \in Z]$. Note that $\mathcal{A}$’s protocol can be directly used to find a protocol that asks yes-no questions and finds the value of $W$ with at most $q$ queries. It is enough to prove a lower-bound on $H(W)$ as the expected number of questions can only be lower than the maximum number.
Note that by equation \[2\] and equation \[3\] we get that for every \(i \in [n]\):
\[
\mathbb{P}[W = i] \leq \frac{1}{1 - \kappa} \sup_{p: \varepsilon \text{-perturbation}} \mathbb{P}_{S \sim D^{m}} \left[ \mu(p^{-1}(E(\text{ALG}(S)))) \geq \frac{\text{AR}(\text{ALG}(S), \epsilon)}{2} \right].
\]

By properties of entropy we know that \(H(W) \geq \log(1/ \max_{i \in [n]} \mathbb{P}[W = i])\), so in the end we get that:
\[
H(W) \geq \log \left( \frac{1 - \kappa}{\sup_{p: \varepsilon \text{-perturbation}} \mathbb{P}_{S \sim D^{m}} \left[ \mu(p^{-1}(E(\text{ALG}(S)))) \geq \frac{\text{AR}(\text{ALG}(S), \epsilon)}{2} \right]} \right).
\]

The proof for the case when ALG is randomized in analogous. The only difference is that instead of partitioning the space \((\mathbb{R}^{d})^{m}\) we partition the space \((\mathbb{R}^{d})^{m} \times \text{supp}(B)\).

\[\Box\]

**Remark 1.** For the sake of clarity and consistency with the standard setup we fixed the approximation constant to be equal 1/2 and the data generation process to be \(S \sim D^{m}\). We note however, that Theorem \[2\] (and its proof with minor changes) is also true for all approximation constants and for general data generation processes. By different generation process we mean anything different from \(S \sim D^{m}\), for instance a case where samples are dependent or where the number of samples \(m\) is itself a random variable. This distinction will become important in the proof of Theorem \[4\].

The following theorem states that if an algorithm ALG applied to a learning task satisfies the following: ALG learns low-risk classifier with constant probability, the adversarial risk is high with constant probability and every point from the support of the distribution is misclassified with small probability then the QC of ALG is high. The core of the proof is the reduction from Theorem \[4\].

**Theorem 1.** Let \(\epsilon \in \mathbb{R}_{>0}\) and \(T\) be a binary classification task on \(\mathbb{R}^{d}\) with separable classes. Let ALG be a learning algorithm for \(T\) with the following properties:

1. \(\forall x \in \text{supp}(D)\), \(\mathbb{P}_{S \sim D^{m}}[\text{ALG}(S)(x) \neq h(x)] \leq C \cdot \delta\),
2. \(\mathbb{P}_{S \sim D^{m}}[\text{AR}(\text{ALG}(S), \epsilon) \geq \eta] \geq 0.99\),
3. \(\mathbb{P}_{S \sim D^{m}}[\text{R}(\text{ALG}(S)) \leq \delta] \geq 0.99\).

Then we have:
\[
\text{QC}(\text{ALG}, T, m, \epsilon) \geq \log \left( \frac{\eta}{3 \cdot C \cdot \delta} \right).
\]

**Proof.** Let \(\rho : \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}\) be any function. For simplicity we introduce the notation \(\rho := \mathbb{P}_{S \sim D^{m}}[\text{AR}(\text{ALG}(S), \epsilon) \geq \eta \land \text{R}(\text{ALG}(S), \epsilon) \leq \delta]\). We define two new data distributions:
\[
\mathcal{D}_{1} := D^{m} \{ (\text{AR}(\text{ALG}(S), \epsilon) \geq \eta \land \text{R}(\text{ALG}(S), \epsilon) \leq \delta) \},
\]
\[
\mathcal{D}_{2} := D^{m} \{ (\text{AR}(\text{ALG}(S), \epsilon) < \eta \lor \text{R}(\text{ALG}(S), \epsilon) > \delta) \}.
\]

Observe that \(\text{supp}(\mathcal{D}_{1}) \cap \text{supp}(\mathcal{D}_{2}) = \emptyset\) and:
\[
D^{m} = \rho \cdot \mathcal{D}_{1} + (1 - \rho) \cdot \mathcal{D}_{2}.
\]

Let \(A\) be an adversary that succeeds on \(D^{m}\) with probability 0.99. By equation \[4\] and the union bound \(A\) has to succeed on \(\mathcal{D}_{1}\) with probability of success \(s\) that satisfies:
\[
\rho \cdot s + (1 - \rho) \geq 0.99,
\]
or, equivalently,
\[
s \geq \frac{1}{\rho} (0.99 - (1 - \rho)).
\]

By Assumption \[2\] and \[3\] this implies
\[
s \geq 0.97.\quad (5)
\]
Now observe:

\[
\begin{align*}
\mathbb{E}_{S \sim D_1}[\mu(p^{-1}(E(ALG(S))))] & = \int_{\mathbb{R}^d} \mathbb{P}_{S \sim D_1}[p(x) \in E(ALG(S))] \, d\mu \\
& = \int_{\mathbb{R}^d} \mathbb{P}_{S \sim D} [p(x) \in E(ALG(S)) \cap AR(ALG(S), \epsilon) \geq \eta \land R(ALG(S), \epsilon) \leq \delta] \, d\mu \\
& \leq \int_{\mathbb{R}^d} P_{S \sim D}[p(x) \in E(ALG(S))] \, d\mu \\
& \leq \frac{1}{0.98 \cdot C \cdot \delta}, \quad (6)
\end{align*}
\]

where the second equality follows from the definition of \(D_1\), third equality follows from the definition of conditioning, first inequality follows from the fact that intersection decreases probability, second inequality is a result of Assumption 1 and the last inequality is obtained by Assumptions 2, 3 and the union bound. Using equation 6 we get:

\[
\begin{align*}
\mathbb{P}_{S \sim D_1} \left[ \mu(p^{-1}(E(ALG(S)))) \geq \frac{AR(ALG(S), \epsilon)}{2} \right] & \leq \frac{2 \cdot \mathbb{E}_{S \sim D_1}[\mu(p^{-1}(E(ALG(S)))]}{AR(ALG(S), \epsilon)} \quad \text{by Markov inequality} \\
& \leq \frac{2 \cdot \frac{1}{0.98} \cdot C \cdot \delta}{\eta} \quad \text{by equation 6 and definition of } D_1 \quad (7)
\end{align*}
\]

Applying Theorem 4 to equation 5 and equation 7 we get that:

\[
QC(ALG, T, m, \epsilon) \geq \log \left( \frac{0.97 \cdot 0.98 \cdot \eta}{2 \cdot C \cdot \delta} \right) \geq \log \left( \frac{\eta}{3 \cdot C \cdot \delta} \right).
\]

\[\square\]
C Omitted Proofs - K-NN

Theorem 2. The 1-Nearest Neighbor algorithm applied to the learning task $T_{\text{intervals}}(z)$ satisfies:

$$QC(1\text{-Nearest Neighbor}, T_{\text{intervals}}, 2m, z) \geq \Theta(m).$$

provided that $z = \Omega(1)$.

Before we start with the formal proof let us give a quick overview. Rather than picking the points on the two line segments uniformly at random we will think of them as the result of a spatial Poisson process. This way the “interpoint distance” between two consecutive points follows an exponential distribution and the lengths of two consecutive such intervals are independent. We can hence concentrate on one such interval. Ignoring boundary effects, the length of this interval is an exponential distribution and the lengths of two consecutive such intervals are independent. We can hence construct in higher dimensions where this difference can likely be made significantly higher. The final ingredient is then to find a lower bound on the QC. Here the crucial realization is that the decision boundaries are very local due to the memoryless property of the interarrival process. As a result, the knowledge of the decision boundaries up to a given point in space quickly looses its value when we move further down the line. This is why the QC scales linearly in the number of sample points.

Proof. For $x \in L_- \cup L_+$ and $\rho \in \mathbb{R}$ we will use $x + \rho$ to denote $x + (\rho, 0)$. Finally, for $x \in L_- \cup L_+$ we will use $g(x)$ to denote the closest point to $x$ in the other interval.

Data generation process. Instead of letting $S \sim D^{2m}$ we will use a standard trick and employ a Poisson sampling scheme. This will simplify our proof considerably. Specifically, we think of the samples as being generated by two Poisson processes: Let $N_-$ be a homogeneous Poisson process on the line defined by the extension of $L_-$ and $N_+$ be an independent of $N_-$ homogeneous Poisson process on the line defined by the extension of $L_+$, both of rate $\lambda = 1$. Then we define $A_- := ([0, m] \times \{0\}) \cap N_-$, $A_+ := ([0, m] \times \{z\}) \cap N_+$ and finally:

$$S := \{(x, -1) : x \in A_-\} \cup \{(x, +1) : x \in A_+\} \quad \text{and} \quad \tilde{S} := \{(x, -1) : x \in N_-\} \cup \{(x, +1) : x \in N_+\}.$$ 

By design we have $\mathbb{E}[|S|] = 2m$ as $|S|$ is distributed according to Pois$(2m)$. Moreover, using a standard tail bound for a Poisson random variable, we get that for every $t > 0$:

$$\mathbb{P}[|S| - 2m \geq t] \leq 2e^{-t^2/(2m + 1)}.$$ 

(8)

This means that the size of the dataset generated with the new process is concentrated around $2m$ (with likely deviations of order $\sqrt{m}$). Let $\{x_1^-, x_2^-, \ldots\}$ be the points from $N_-$ with non-negative first coordinate ordered in the increasing order and similarly let $\{x_1^+, x_2^+, \ldots\}$. Then note that $A_- = \{x_1^-, x_2^-, \ldots, x_{|A_-|}^-\}$ and $A_+ = \{x_1^+, x_2^+, \ldots, x_{|A_+|}^+\}$. To simplify notation we let $E(S) := E(1\text{-Nearest Neighbor}(S)), E(\tilde{S}) := E(1\text{-Nearest Neighbor}(\tilde{S}))$, where we recall that $E$ denotes the error set. Moreover let:

$$x_0^- := \max_{x \in N_-} x, \quad x_0^+ := \min_{x \in N_+} x.$$

We also define the corresponding random variables $\{X_0^-, X_1^-, \ldots\}$ and $\{X_0^+, X_1^+, \ldots\}$, where for every $i$ we have $x_i^- \sim X_i^-$ and $x_i^+ \sim X_i^+$. 

16
Lower-bounding $AR(1$-Nearest Neighbor$(S), z$). We will focus on $L_-$ as the argument for $L_+$ is analogous. Let $a, b \in A_-$ be two consecutive points from $A_-$ such that $b - a > 2z + 2\rho$ for a parameter $\rho$ to be chosen later. By construction, $E(S) \cap [a, b] \subseteq [a + z, b - z]$. Our aim is to lower bound 

$$
P([a + z + \rho, b - z - \rho] \subseteq E(S)).$$

Observe that if there exist points $c, d \in L_+$ such that $c \in [g(a + z + \rho) - \sqrt{\rho^2 + 2\rho z}, g(a + z + \rho) + \sqrt{\rho^2 + 2\rho z}], d \in [g(b - z - \rho) - \sqrt{\rho^2 + 2\rho z}, g(b - z - \rho) + \sqrt{\rho^2 + 2\rho z}]$ then $[a + z + \rho, b - z - \rho] \subseteq E(S)$. As $N_+$ is a Poisson process we get that:

$$|N_+ \cap [g(a + z + \rho) - \sqrt{\rho^2 + 2\rho z}, g(a + z + \rho) + \sqrt{\rho^2 + 2\rho z}]| \sim \text{Pois}(2\sqrt{\rho^2 + 2\rho z}),$$

so $\mathbb{P}[N_+ \cap [g(a + z + \rho) - \sqrt{\rho^2 + 2\rho z}, g(a + z + \rho) + \sqrt{\rho^2 + 2\rho z}] \neq \emptyset] = 1 - e^{-2\sqrt{\rho^2 + 2\rho z}},$ and by the union bound over the two intervals we get that:

$$
\mathbb{P}([a + z + \rho, b - z - \rho] \subseteq E(S)) \geq 1 - 2e^{-2\sqrt{\rho^2 + 2\rho z}}. \tag{9}
$$

For $i \in \mathbb{N}_+$ let $\tilde{Y}_i$ be the random variable defined as:

$$
\tilde{Y}_i := \nu((E(S) \cap [\tilde{x}_i, \tilde{x}_{i+1}]) + B_2),
$$

where $\nu$ is one dimensional Lebesgue measure. In words, $\tilde{Y}_i$ is the random variable that is equal to how much the interval $[\tilde{x}_i, \tilde{x}_{i+1}]$ contributes to $AR(1$-Nearest Neighbor$(S), z$). Observe that $\tilde{Y}_i$ is primarily determined by the length of $[\tilde{x}_i, \tilde{x}_{i+1}]$ as well as where the points of $N_+$ are located with respect to $[\tilde{x}_i, \tilde{x}_{i+1}]$. Using the fact that $\tilde{x}_{i+1} - \tilde{x}_i$ has an exponential distribution with mean $1$ and equation 9 we get the following properties of $\tilde{Y}_i$:

1. $\tilde{Y}_i$ is non-negative,
2. $\forall t \geq 2z, \mathbb{P}[\tilde{Y}_i > t] \geq e^{-t - 2\rho} \left(1 - 2e^{-2\sqrt{\rho^2 + 2\rho z}}\right),$
3. $\tilde{Y}_i$’s are i.i.d.

The first property (non-negativity) is true by definition. To see the second, note that in order for the interval to contribute to the adversarial risk it needs to contribute to the standard risk, i.e., it needs to contain points of the error set $E(S)$. The first factor bounds the probability that the interval has length at least $t + 2\rho$, while the second factor (by equation 9) bounds the probability that this interval contributes at least $t$ to the adversarial risk. The last property is in turn a consequence of the fact that the inter-arrival times of a Poisson process are iid and that the points on the “other” line are Poisson as well and independent of the first line.

Our goal now is to show that $\sum_{i=1}^{m/2} \tilde{Y}_i \geq \frac{2}{\rho} e^{-2z - 2m}$ with high probability. Similarly to the standard proof of the Chernoff bound, for every $s, \eta > 0$:

$$
\mathbb{P} \left[ \sum_{i=1}^{m/2} \tilde{Y}_i \leq \eta m \right] = \mathbb{P} \left[ e^{-s \sum \tilde{Y}_i} \geq e^{-s \eta m} \right] 
\leq \mathbb{E} \left[ e^{-s \sum \tilde{Y}_i} \right] \cdot e^{s \eta m} \quad \text{by Markov inequality} 
= \left( \mathbb{E} \left[ e^{-s \tilde{Y}_i} \right] \cdot e^{2s \eta} \right)^{m/2} \quad \text{as } \tilde{Y}_i \text{'s are i.i.d.} \tag{10}
$$

Now using Properties 1 and integration by parts we get for every $i \in \mathbb{N}_+$:

$$
\mathbb{E} \left[ e^{-s \tilde{Y}_i} \right] = 1 - \int_0^\infty e^{-st} \cdot \mathbb{P}[\tilde{Y}_i > t] \, dt \tag{11}
$$
Now we lower bound the following:
\[
\int_0^\infty se^{-st} \cdot P[\bar{Y}_1 > t] \, dt \\
\geq \int_0^{z} se^{-st} \cdot e^{-2z-2\rho} \left(1 - 2e^{-\rho^2/2\rho^2} \right) \, dt + \\
+ \int_{z}^\infty se^{-st} \cdot e^{-t-2\rho} \left(1 - 2e^{-\rho^2/2\rho^2} \right) \, dt \\
= e^{-2\rho} \left(1 - 2e^{-\rho^2/2\rho^2} \right) \left(e^{-2z} \left(1 - e^{-2sz}\right) + \frac{s}{s+1} \cdot e^{-2(s+1)z}\right)
\]
(12)

Now we fix the parameters \( \rho := 1, \eta := \frac{z}{2}e^{-2z-2\rho} = \frac{z}{2}e^{-2z-2} \). Then using equation (11) and equation (12) we get:

\[
E \left[ e^{-s\bar{Y}_1} \right] \cdot e^{2s\eta} \\
\leq \left[ 1 - e^{-2} \left(1 - 2e^{-2\sqrt{2z+1}}\right) \left(e^{-2z} \left(1 - e^{-2sz}\right) + \frac{s}{s+1} \cdot e^{-2(s+1)z}\right) \right] \exp(sze^{-2z-2}) \\
\]  

Taking the derivative of expression from equation (13) with respect to \( s \) and evaluating it at \( s = 0 \) we get:

\[
e^{-2(z+1+\sqrt{2z+1})} \left(2 + 4z - e^{2\sqrt{2z+1}}(z+1) \right),
\]

which as long as \( z > 1 \) is negative. Note that expression from equation (13) evaluated at \( s = 0 \) is equal to 1. Thus, using the fact that the derivative at 0 is negative, we get that there exists a function \( \kappa'(1,\infty) \rightarrow \mathbb{R}_+ \) such that:

\[
\min_{s > 0} E \left[ e^{-s\bar{Y}_1} \right] \cdot e^{s\eta} e^{-2s-2} \leq 1 - \kappa'(z),
\]
(14)

which using equation (10) means that:

\[
P \left[ \sum_{i=1}^{m/2} \bar{Y}_i \leq \frac{z}{2} e^{-2z-2} \right] \leq (1 - \kappa'(z))^{m/2}.
\]
(15)

Now for \( i \in |A_-| - 1 \) let \( Y_i \) be the random variable defined as:

\[
Y_i := \nu((E(S) \cap [x_i, x_{i+1})) + B_z) \cap L_-
\]

Notice that for all \( i \in |A_-| - 1 \) we have \( Y_i = \bar{Y}_i \). Note that by Poisson tail bound we have:

\[
P \left[ |A_-| \leq \frac{m}{2} \right] \leq 2e^{-\frac{(m/2)^2}{2(m/2)+2}} = 2e^{-m/12}.
\]
(16)

Combining equation (10) and equation (15) and the union bound we get that there exists a function \( \kappa : (1,\infty) \rightarrow \mathbb{R}_+ \) such that:

\[
P \left[ \sum_{i \in |A_-| - 1} Y_i \leq \frac{z}{2} e^{-2z-2} \cdot m \right] \leq P \left( |A_-| \leq m/2 \right) \vee \left[ \sum_{i=1}^{m/2} \bar{Y}_i \leq \frac{z}{2} e^{-2z-2} \right]
\]
\[
\leq P \left[ |A_-| \leq m/2 \right] + \left[ \sum_{i=1}^{m/2} \bar{Y}_i \leq \frac{z}{2} e^{-2z-2} \right]
\]
\[
\leq 2e^{-m/12} + (1 - \kappa'(z))^{m/2}
\]
\[
\leq (1 - \kappa(z))^m.
\]

Note that we omitted the first and the last interval \( \left[ 0, x_1 \right) \) and \( \left[ x_{|A_-| - 1}, m \right) \). Omitting these intervals is valid as we are deriving a lower bound for \( AR(1\text{-Nearest Neighbor}(S), z) \). We conclude using the union bound over two intervals \( L_- \) and \( L_+ \) to obtain:

\[
P \left[ AR(1\text{-Nearest Neighbor}(S), z) \leq \frac{z}{2} e^{-2z-2} \right] \leq 2 \cdot (1 - \kappa(z))^m.
\]
(17)
Lower-bounding QC. To prove a lower-bound on the QC of 1-NN applied to this task we will use Theorem 3. This means that we need to upper-bound:

$$\sup_{p: z\text{-perturbation}} \mathbb{P}_S \left[ \mu(p^{-1}(E(1\text{-Nearest Neighbor}(S)))) \geq \frac{AR(1\text{-Nearest Neighbor}(S), z)}{2} \right].$$

where $S$ is generated from the two independent Poisson processes as described at the beginning of the proof. By Remark 1 we can use Theorem 4 in this case. Let $p$ be a $z$-perturbation. We analyze only one of the intervals, namely $L_-$, as the situation for $L_+$ is symmetric. For $i \in \mathbb{N}_+ \cup \{0\}$ let $\tilde{Z}_i$ be a non-negative random variable defined as:

$$\tilde{Z}_i := \nu(p^{-1}([x_i^- + z, x^+_{i+1} - z])).$$

Note that by construction:

$$\sum_{i=0}^{|A_-|} \tilde{Z}_i \geq \nu(p^{-1}(E(S) \cap L_-)), \quad (18)$$

as for every two consecutive points $a, b \in A_-$ we have $E(S) \cap [a, b] \subseteq [a + z, b - z]$. We divide $Z_i$'s into $k$ groups, where $k$ will be chosen later. For $i \in \mathbb{N}_+ \cup \{0\}$ we define:

$$\tilde{Z}_i^{\text{mod } k} := \tilde{Z}_i.$$

Let $g \in \{0, \ldots, k - 1\}$. We will upper-bound the probability:

$$\mathbb{P} \left[ \sum_{i=0}^{[2m/k]} \tilde{Z}_i^g \geq \frac{z}{2k} \cdot e^{-2z - 2} \cdot m \right].$$

Let $i \in [2m/k]$ and $x_0^-, x_1^-, \ldots, x_i^{- (i-1)k + g + 1} \in \mathbb{R}$ be an increasing sequence such that $x_0^- < 0 < x_1^-$. Then we have:

$$\mathbb{E} \left[ \tilde{Z}_i^g \right] X_0^- = x_0^-, X_1^- = x_1^-, \ldots, X_i^{- (i-1)k + g + 1} = x_i^{- (i-1)k + g + 1}$$

$$\mathbb{E} \left[ \tilde{Z}_i^g \right] \left[ X_0^- = x_0^-, \ldots, X_i^{- (i-1)k + g + 1} = x_i^{- (i-1)k + g + 1} \right] dt$$

$$\mathbb{E} \left[ \tilde{Z}_i^g \right] \left[ X_0^- = x_0^-, \ldots, X_i^{- (i-1)k + g + 1} = x_i^{- (i-1)k + g + 1} \right] dt$$

$$\mathbb{E} \left[ \tilde{Z}_i^g \right] \left[ X_0^- = x_0^-, \ldots, X_i^{- (i-1)k + g + 1} = x_i^{- (i-1)k + g + 1} \right] dt$$

$$\mathbb{E} \left[ \tilde{Z}_i^g \right] \left[ X_0^- = x_0^-, \ldots, X_i^{- (i-1)k + g + 1} = x_i^{- (i-1)k + g + 1} \right] dt$$

where in the first equality it is enough to integrate starting from $x_i^{- (i-1)k + g + 1}$ as $p$ is a $z$-perturbation and the leftmost point that can potentially belong to $[x_i^{- k + g} + z, x_i^{- k + g + 1} - z]$ is $x_i^{- (i-1)k + g + 1}$.

The second equality comes from conditioning on the value of $X_i^{- k + g}$ and independence of $X_i^{- k + g} - X_i^{- (i-1)k + g + 1}$ of $X_0^-, \ldots, X_i^{- (i-1)k + g + 1}$. The third equality follows from three facts:

- $X_i^{- k + g} - X_i^{- (i-1)k + g + 1}$ is distributed according to Erlang distribution with parameters $k - 1, 1$, because, by construction, it is a sum of $k - 1$ i.i.d. exponential random variables with parameter 1,
• for \( p(t) \in [x_{i,k+g}^-, x_{i,k+g+1}^- + z] \) one needs \( p(t) - z \geq x_{i,k+g}^- \) and \( x_{i,k+g+1}^- \geq p(t) + z, \)

• \( X_{i,k+g+1}^- - X_{i,k+g}^- \) is distributed according to the exponential distribution and is independent of \( X_{i,k+g}^- - X_{(i-1)k+g}^- \).

Now we bound the expression from equation 19. Function \( e^{-x} \cdot \frac{x^k}{k!} \) is increasing on \([-\infty, k]\) and decreasing on \([k, \infty]\) thus a \( p^* \) that maximizes \( \max_{p': \text{perturbation}} \int_0^\infty e^{-(p'(t)-z)} \cdot \frac{(p'(t)-z)^k}{k!} \, dt \) can be set to:

\[
p^*(t) := \begin{cases} 
  t + z, & \text{if } t \in [0, k), \\
  k + z, & \text{if } t \in [k, k + 2z), \\
  t - z, & \text{otherwise}.
\end{cases}
\]

Function \( e^{-t} \cdot \frac{t^k}{k!} \) is the density function of the Erlang distribution with parameters \((k, 1)\) thus:

\[
\int_0^\infty e^{-(p^*(t)-z)} \cdot \frac{(p^*(t)-z)^k}{k!} \, dt \\
\leq \int_0^\infty e^{-t} \cdot \frac{t^k}{k!} \, dt + 2z \cdot e^{-k} \cdot \frac{k^k}{k!} \\
\leq 1 + \frac{2z}{\sqrt{2\pi k}} \quad \text{By Stirling factorial bounds (20)}
\]

Combining equation 19 and equation 20 we get that for \( k \geq 2z^2/\pi \):

\[
\mathbb{E} \left[ Z_i^g \mid X_0^- = x_0^-, X_1^- = x_1^-, \ldots, X_{(i-1)k+g+1}^- = x_{(i-1)k+g+1}^- \right] \leq 2 \cdot e^{-2z}.
\]

Moreover as the lengths of intervals are independent we get that for every \( i \in \mathbb{N}_+ \cup \{0\} \):

\[
P \left[ Z_i^g = 0 \mid X_0^- = x_0^-, X_1^- = x_1^-, \ldots, X_{(i-1)k+g+1}^- = x_{(i-1)k+g+1}^- \right] \geq 1 - e^{-2z} \quad (22)
\]

and \( \forall t \geq 0 \):

\[
P \left[ Z_i^g \geq t + 2z \mid X_0^- = x_0^-, X_1^- = x_1^-, \ldots, X_{(i-1)k+g+1}^- = x_{(i-1)k+g+1}^- \right] \leq e^{-(t-2z)} \quad (23)
\]

equation 22 follows from the fact that if \( x_i^+ - x_i^- < 2z \) then \( Z_i = 0 \) and equation 23 follows from the fact that if \( x_i^+ - x_i^- \leq t \) then \( Z_i \leq t \) as \( p \) can move at most \( t \) of mass into an interval of length \( t - 2z \). Now we bound the probability that sum of variables from the \( g \)-th group deviates considerably from its expectation. The idea is to use a method similar to the proof of the Chernoff bound. For every \( s > 0 \):

\[
P \left[ \sum_{i=0}^{[2m/k]} Z_i^g \leq \frac{z}{2k} e^{-2s-2} \cdot m \right] \\
\leq P \left[ Z_0^g \leq \frac{z}{4k} e^{-2s-2} \cdot m \right] + P \left[ \sum_{i=1}^{[2m/k]} Z_i^g \geq \frac{z}{4k} e^{-2s-2} \cdot m \right] \quad \text{By the union bound (24)}
\]

We bound the two terms from equation 24 separately. Using equation 23 we get that for \( m \geq 4e^{2s+2}k \):

\[
P \left[ Z_0^g \geq \frac{z}{4k} e^{-2s-2} \cdot m \right] \leq \exp \left( -\frac{z}{4k} e^{-2s-2} \cdot m \right) \quad (25)
\]
Now we bound the second term from equation 24:

\[ \mathbb{P} \left[ \sum_{i=1}^{\lceil 2m/k \rceil} \bar{Z}_i^g \geq \frac{z}{4k} e^{-2z-2} \cdot m \right] \]

\[ \leq \mathbb{P} \left[ \exp \left( s \sum_{i=1}^{\lceil 2m/k \rceil} \bar{Z}_i^g \right) \geq \exp \left( s \cdot \frac{z}{4k} e^{-2z-2} \cdot m \right) \right] \]

\[ \leq \mathbb{E} \left[ \exp \left( s \sum_{i=1}^{\lceil 2m/k \rceil} \bar{Z}_i^g \right) \right] \cdot \exp \left( -s \cdot \frac{z}{4k} e^{-2z-2} \cdot m \right) \quad \text{By Markov inequality} \]

\[ \leq \mathbb{E} \left[ \prod_{i=1}^{\lceil 2m/k \rceil} \exp \left( s \cdot \left( \bar{Z}_i^g - \frac{z}{16} e^{-2z-2} \right) \right) \right] \quad (26) \]

Using the chain rule we obtain:

\[ \mathbb{E} \left[ \prod_{i=1}^{\lceil 2m/k \rceil-1} \left[ \exp \left( s \cdot \left( \bar{Z}_i^g - \frac{z}{16} e^{-2z-2} \right) \right) \right] \cdot \mathbb{E} \left[ \exp \left( s \cdot \left( \bar{Z}_{\lceil 2m/k \rceil}^g - \frac{z}{16} e^{-2z-2} \right) \right) \right] \right] \quad (27) \]

Using the fact that variables \( X_0^-, \ldots, X_{(i-1)k+g+1}^- \) determine values of \( \bar{Z}_0^g, \ldots, \bar{Z}_{i-1}^g \) and the bounds from equation 21, equation 22 and equation 23 hold for all possible realizations of \( X_0^-, \ldots, X_{(i-1)k+g+1}^- \), if we maximize the inner conditional expectation of equation 27 over variables \( \bar{Z}_i^g \) satisfying properties 21, 22 and 23 we can get an upper bound on \( \mathbb{P} \left[ \sum_{i=0}^{\lceil 2m/k \rceil} \bar{Z}_i^g \geq \frac{z}{16} e^{-2z-2} \cdot m \right] \) via equation 24, 25 and 26. More formally if we consider a class of random variables \( Z \) satisfying the following properties:

1. \( Z \geq 0 \)
2. \( \mathbb{E}[Z] \leq 2 \cdot e^{-2z} \),
3. \( \mathbb{P}[Z = 0] \geq 1 - e^{-2z} \),
4. \( \forall t \geq 0 \mathbb{P}[Z \geq t + 2z] \leq e^{-2z} \).

and we show that for some \( C \in \mathbb{R} \) there exists a function \( \xi' : (C, \infty) \to \mathbb{R}_+ \) such that:

\[ \inf_{s \in \mathbb{R}_+} \sup_{Z: Z \text{ satisfies 1, 2, 3 and 4}} \mathbb{E} \left[ \exp \left( s \cdot \left( Z - \frac{z}{16} e^{-2z-2} \right) \right) \right] \leq 1 - \xi'(z). \quad (28) \]

then, by setting \( k := 2z^2/\pi \), there exists a function \( \xi : (C, \infty) \to \mathbb{R}_+ \):

\[ \mathbb{P} \left[ \sum_{i=0}^{\lceil 2m/k \rceil} \bar{Z}_i^g \geq \frac{z}{2k} e^{-2z-2} \cdot m \right] \leq \exp \left( - \frac{z}{4k} e^{-2z-2} \cdot m \right) + \left( 1 - \xi'(z) \right)^{m/k} \leq (1 - \xi(z))^m. \quad (29) \]

Now we will prove equation 28. For \( \zeta \in [e^{-2z}, \infty) \) let a non-negative random variable \( Z_\zeta \) be defined as:

1. \( \mathbb{P}[Z_\zeta = 0] = 1 - e^{-\zeta} \)
2. \( f_{Z_\zeta}(t) = \begin{cases} 0 & \text{if } t \in (0, \zeta) \\ e^{-t} & \text{otherwise} \end{cases} \)
We argue that a variable \( Z \) that maximizes equation \( 28 \) has to be belong to \( \{ Z_{\zeta} \}_{\zeta \in [e^{-2z}, \infty)} \). Assume for the sake of contradiction that \( Z \) is a random variable that satisfies \( Z \notin \{ Z_{\zeta} \}_{\zeta \in [e^{-2z}, \infty)} \) and \( Z \) maximizes equation \( 28 \). Then this means that there exists \( t \in (2z, \infty) \) such that \( P[Z \in (0, t)] > 0 \) and \( P[Z \geq t] < e^{-t} \). Then one can distribute the mass from \((0, t)\) without changing \( E[Z] \) and violating Property 4 to increase \( E[\exp \left( s \cdot \left( Z - \frac{z}{16} e^{-2z-2} \right) \right)] \). This gives us a contradiction.

Now let \( \zeta \in [e^{-2z}, \infty] \). Note that:

\[
E[Z_{\zeta}] = \int_{\zeta}^{\infty} te^{-t} dt = (1 + \zeta)e^{-\zeta}. \tag{30}
\]

Now we bound:

\[
E \left[ \exp \left( s \cdot \left( Z_{\zeta} - \frac{z}{16} e^{-2z-2} \right) \right) \right] \\
\leq E \left[ \exp \left( s \cdot \left( Z_{\zeta} - \frac{z}{32e^2} E[Z_{\zeta}] \right) \right) \right] \quad \text{By Property 2} \\
= \exp \left( -s \frac{z}{32e^2} (1 + \zeta) e^{-\zeta} \right) \cdot E \left[ \exp \left( s \cdot (Z_{\zeta}) \right) \right] \quad \text{By equation 30} \\
= \exp \left( -s \frac{z}{32e^2} (1 + \zeta) e^{-\zeta} \right) \cdot \left[ 1 - e^{-\zeta} + \int_{\zeta}^{\infty} e^{st} e^{-t} dt \right] \\
= \exp \left( -s \frac{z}{32e^2} (1 + \zeta) e^{-\zeta} \right) \cdot \left[ 1 - e^{-\zeta} + \frac{1}{1 - s} e^{(s-1)\zeta} \right] \quad \text{Provided that} \ s < 1 \tag{31}
\]

Taking derivative of equation \( 31 \) with respect to \( s \) and evaluating at \( s = 0 \) gives:

\[
-\frac{1}{32e^2} e^{-\zeta} (z - 32e^2)(1 + \zeta), \tag{32}
\]

which as long as \( z > 32e^2 \) is negative. Note that expression from equation \( 31 \) evaluated at \( s = 0 \) is equal to 1. Combining these facts we get that there exists a function \( \xi' : (32e^2, \infty) \to \mathbb{R}^+ \) such that:

\[
\inf_{s \in \mathbb{R}^+} \sup_{Z : Z \text{ satisfies } \xi' : (32e^2, \infty) \to \mathbb{R}^+} E \left[ \exp \left( s \cdot \left( Z - \frac{z}{16} e^{-2z-2} \right) \right) \right] \leq 1 - \xi'(z).
\]

But this means that we have proven equation \( 29 \). Thus we get that there exists a function \( \psi : (32e^2, \infty) \to \mathbb{R}^+ \) such that:

\[
P \left[ \nu(p^{-1}(E(S)) \cap L_-) \geq \frac{z}{2} e^{-2z-2} \cdot m \right] \\
\leq P \left[ \sum_{i=0}^{2m} Z_i \geq \frac{z}{2} e^{-2z-2} \cdot m \right] \quad \text{By equation 18} \\
\leq P \left[ \sum_{i=0}^{2m} Z_i \geq \frac{z}{2} e^{-2z-2} \cdot m \right] \lor \left( |A_-| > 2m \right) \quad \text{Union bound and equation 8} \\
\leq P \left[ \sum_{i=0}^{2m} Z_i \geq \frac{z}{2} e^{-2z-2} \cdot m \right] + 2e^{-\frac{m^2}{m^2+1}} \quad \text{Union bound and equation 8} \\
\leq k \cdot (1 - \xi(z))^m + 2e^{-m/4} \quad \text{By equation 29 and union bound over groups} \\
\leq \frac{2z^2}{\pi} \cdot (1 - \xi(z))^m + 2e^{-m/4} \quad \text{By setting of} \ k \\
\leq 4z^2 \cdot (1 - \psi(z))^m
\]

The above fact and the union bound over the intervals \( L_- \) and \( L_+ \) together with equation \( 17 \) shows that if \( z > 32e^2 \) then:

\[
\sup_{P : \text{z-perturbation}} P_S \left[ \mu(p^{-1}(E(S))) \geq \frac{AR(1-\text{Nearest Neighbor}(S), z)}{2} \right] \\
\leq 2 \cdot (1 - \kappa(z))^m + 8z^2 \cdot (1 - \psi(z))^m \\
\leq 8z^2 \cdot (1 - \min(\kappa(z), \psi(z)))^m.
\]
This, by Theorem 4 means that if $z > 32e^2$ then:

$$QC'(1\text{-Nearest Neighbor, } T_{\text{intervals}}, 2m, z) \geq \Theta \left( \log \left( \frac{1}{8z^2 \cdot (1 - \min(\kappa(z), \psi(z)))^m} \right) \right)$$

$$\geq \Theta \left( m \cdot \log \left( \frac{1}{1 - \min(\kappa(z), \psi(z))} \right) \right)$$

$$\geq \Theta(m).$$
D  OMITTED PROOFS - QUADRATIC NEURAL NETWORK

We now present proofs of claims from Section 3.2. Recall that that section deals with quadratic neural nets applied to the concentric spheres dataset.

D.1 QC LOWER BOUNDS FOR EXPONENTIALLY SMALL RISK

We first use the results from Section 3 to argue that increased accuracy leads to increased robustness. It was experimentally shown in Gilmer et al. (2018) that increasing the sample size for QNN leads to a higher accuracy on the CS dataset. Thus we assume that for some \( m \in \mathbb{N} \) the following holds:

\[
\mathbb{P}_{S \sim \mathcal{D}}[R(QNN(S)) \leq \delta] \geq 1 - \delta.
\]

(33)

Moreover note that \( \mathcal{D} \) is symmetric and thus it is natural to assume that every point is misclassified with the same probability. Formally:

\[
\mathbb{P}_{S \sim \mathcal{D}}[\text{ALG}(S)(x) \neq h(x)] \text{ is constant for } x \in S_{1}^{d-1} \cup S_{1,3}^{d-1}.
\]

(34)

Using equation 33 we bound:

\[
\int_{\mathbb{R}^{d}} \mathbb{P}_{S \sim \mathcal{D}}[\text{ALG}(S)(x) \neq h(x)] \, d\mu = \mathbb{E}_{S \sim \mathcal{D}}[R(QNN)]
\]

\[
\leq \delta \cdot \mathbb{P}_{S \sim \mathcal{D}}[R(QNN(S)) \leq \delta] + 1 \cdot (1 - \mathbb{P}_{S \sim \mathcal{D}}[R(QNN(S)) \leq \delta]) \leq 2\delta,
\]

which by equation 34 gives that \( \forall x \in \text{supp}(\mathcal{D}) \), \( \mathbb{P}_{S \sim \mathcal{D}}[\text{ALG}(S)(x) \neq h(x)] \leq 2\delta \). Finally for \( \epsilon = 0.3 \) we have \( \forall S \in \text{supp}(\mathcal{D}^{m}) \) \( AR(QNN(S), \epsilon) \geq 1/2 \), as the distance between the classes is 0.3.

Combining the properties and applying Theorem 1 we get: \( QC(QNN, \mathcal{C}, m, 0.3) \geq \log \left( \frac{1}{2\delta} \right) \). In words, if QNN has a risk of \( 2^{-\Omega(k)} \) then it is secure against \( \Theta(k) \)-bounded adversaries for \( \epsilon = 0.3 \).

D.2 QC LOWER BOUNDS FOR CONSTANT RISK

Now present QC lower bounds for the case where the risk achieved by the network is as large as a constant. To get started, let us formally define the distributions and error sets that we will be concerned with. Recall that for \( y \in S_{1}^{d-1} \) we define \( \text{cap}(y, r, \tau) := S_{r}^{d-1} \cap \{ x \in \mathbb{R}^{d} : \langle x, y \rangle \geq \tau \} \).

Let \( \tau : [0, 1] \to [0, 1] \) be such that for every \( \delta \in [0, 1] \) we have \( \mu(\text{cap}(., 1, \tau(\delta))) / \mu(S_{1}^{d-1}) = \delta \).

**Definition 6. (Distributions on Spherical Caps)**

- **Cap.** Let \( \delta \in (0, 1) \). We define \( \text{Cap}(\delta) \) as a distribution on subsets of \( S_{1}^{d-1} \) defined by the following process: generate \( y \sim U[S_{1}^{d-1}] \). Return: \( \text{cap}(y, 1, \tau(\delta)) \).

- **Caps\(_{k}\)^{i.i.d.}**. Let \( k \in \mathbb{N}, \delta \in (0, 1) \). We define \( \text{Caps}_{k}^{i.i.d.}(\delta) \) as a distribution on subsets of \( S_{1}^{d-1} \cup S_{1,3}^{d-1} \) defined by the following process: generate a sequence of random bits \( b_{1}, \ldots, b_{k} \sim U\{-1, 1\} \), generate a sequence of random vectors \( y_{1}, \ldots, y_{k} \sim U[S_{1}^{d-1}] \). Return:

\[
\bigcup_{i : b_{i} = -1} \text{cap}(y_{i}, 1, \tau(\delta/k)) \cup \bigcup_{i : b_{i} = +1} \text{cap}(y_{i}, 1.3, 1.3\tau(\delta/k))
\]

In words \( \text{Caps}_{k}(\delta) \) generates \( k \) i.i.d. uniformly distributed spherical caps occupying \( \delta/k \mu\)-volume of a random sphere from \( S_{1}^{d-1}, S_{1,3}^{d-1} \).

- **Caps\(_{k}\)^{G}**. Let \( k \in \mathbb{N}, \delta \in (0, 1), \mathcal{G} \) be a distribution on \( (S_{1}^{d-1})^{k} \). We define \( \text{Caps}_{k}^{G}(\delta) \) as a distribution on subsets of \( S_{1}^{d-1} \cup S_{1,3}^{d-1} \) defined by the following process: generate a sequence of random bits \( b_{1}, \ldots, b_{k} \sim U\{-1, 1\} \), generate \( y_{1}, \ldots, y_{k} \sim \mathcal{G} \), generate an orthonormal matrix \( M \sim O(d) \). Return:

\[
\bigcup_{i : b_{i} = -1} M(\text{cap}(y_{i}, 1, \tau(\delta/k))) \cup \bigcup_{i : b_{i} = +1} M(\text{cap}(y_{i}, 1.3, 1.3\tau(\delta/k)))
\]

In words \( \text{Caps}_{k}^{G}(\delta) \) generates \( k \) randomly rotated spherical caps occupying \( \delta/k \mu\)-volume of a random sphere from \( S_{1}^{d-1}, S_{1,3}^{d-1} \), where relative positions of normal vectors of the caps are defined by \( \mathcal{G} \).
Note that definitions of $\text{Caps}_k^{i.i.d.}$ and $\text{Caps}_k^G$ are compatible in the following sense:

**Observation 1.** For every $k \in \mathbb{N}_+$, $\delta \in (0, 1)$, $\text{Caps}_k^{i.i.d.}(\delta) = \text{Caps}_k^{U[(S_1^{d-1})^k]}(\delta)$.

In the following lemma we show a reduction from $\text{Cap}_k^{i.i.d.}$ to $\text{Cap}$. This means that we show that if there is an adversary that uses $q$ queries and succeeds on $\text{Cap}_k^{i.i.d.}$ then there exists an adversary that succeeds on $\text{Cap}$ and also asks at most $q$ queries. The takeaway from this lemma is that the QC of $\text{Cap}_k^{i.i.d.}$ is no smaller than the QC of $\text{Cap}$. Formally:

**Lemma 2 (Reduction from $\text{Caps}_k^{i.i.d.}$ to $\text{Cap}$).** Let $k \in \mathbb{N}_+$. If there exists a $q$-bounded adversary $A$ that succeeds on $\text{Caps}_k^{i.i.d.}(0.01)$ with approximation constant $1/2$, error probability $0.01$ and $\epsilon$ such that $\text{cap}(\cdot, 1, \tau(0.01/k)) + B_\varepsilon = \text{cap}(\cdot, 1, 0)$ then there exists a $q$-bounded adversary $A'$ that succeeds on $\text{Cap}(0.01/k)$ with approximation constant $\frac{1}{2k}$, error probability of $1 - \frac{1}{3k}$ and the same $\epsilon$.

**Proof.** Algorithm 1 invoked with $\delta = 0.01$ defines the protocol for $A'$. We will show that this protocol satisfies the statement of the Lemma.

**Algorithm 1 EMULATEIID($f, A, \delta, k$)**  
\[ f \] is the attacked classifier \[ A \] is an adversary for distribution $\text{Caps}_k^{i.i.d.}(\delta)$

1: \( \{y_1, \ldots, y_{k-1}\} \sim U[S_1^{d-1}] \)
2: \( \{b, b_1, \ldots, b_{k-1}\} \sim U\{-1, 1\} \)
3: \( T(x) := \begin{cases} 1.3 \cdot x & \text{if } b = -1 \land x \in S_1^{d-1} \\ x & \text{if } b = 1 \end{cases} \)
4: for $i = 1, \ldots, k-1$ do
5: \( C_i := \begin{cases} \text{cap}(y_i, 1, \tau(\delta/k)) & \text{if } b_i = -1 \\ \text{cap}(y_i, 1, 1.3\tau(\delta/k)) & \text{if } b_i = 1 \end{cases} \)
6: \( p := \text{Simulate } A, \text{ to query } x \text{ answer } f(T(x)) \lor (x \in C_1) \lor \cdots \lor (x \in C_{k-1}) \)
7: return $p$

At the first sight it might seem that the protocol for $A'$ uses $kq$ queries. But due to the fact that $k - 1$ caps were added artificially the answer to $(k - 1)q$ of those queries is known to $A'$ beforehand. This gives us that $A'$ is $q$-bounded as every query of $A'$ corresponds to a query of $A$. Let $C \subseteq S_1^{d-1}$ be the hidden cap that was generated from $\text{Cap}$. Observe that:

\[
T(C) \cup \bigcup_{i=1}^{k-1} C_i
\]

is distributed according to $\text{Cap}_k^{i.i.d.}$, as $C_1, \ldots, C_{k-1}$ are i.i.d. uniformly random spherical caps, $C$ is a random spherical cap of $S_1^{d-1}$ and $T$ moves the cap $C$ to $S_1^{d-1}$ with probability $1/2$. Thus by the guarantee for $A$ we know that with probability at least $0.99$:

\[
\mu \left( p^{-1} \left( T(C) \cup \bigcup_{i=1}^{k-1} C_i \right) \right) \geq \frac{1}{2} \cdot \mu \left( T(C) \cup \bigcup_{i=1}^{k-1} C_i \right) + B_\varepsilon
\]

As $T(C), C_1, \ldots, C_k$ are indistinguishable from the point of view of $A$ we get that with probability at least $0.99/k$:

\[
\mu \left( p^{-1}(T(C)) \right) \geq \frac{1}{2k} \cdot \mu \left( T(C) \cup \bigcup_{i=1}^{k-1} C_i \right) + B_\varepsilon,
\]

and finally as $T = \text{Id}$ with probability $1/2$ we get that with probability at least $0.99/2k \geq 1/3k$:

\[
\mu \left( p^{-1}(C) \right) \geq \frac{1}{2k} \cdot \mu \left( T(C) \cup \bigcup_{i=1}^{k-1} C_i \right) + B_\varepsilon,
\]

25
where \( \mu \left( \left( T(C) \cup \bigcup_{i=1}^{k} C_i \right) + B_e \right) \geq 1/4 \) as \( C \subseteq S_1^{d-1} \) and \( C + B_e \) covers 1/4 of the mass of \( \mu \). Thus we get that with probability at least 1/3k:

\[
\mu \left( p^{-1}(C) \right) \geq \frac{1}{8k}.
\]

which is equivalent to \( A' \) succeeding on Cap(0.01/k) with approximation constant of \( \frac{1}{2k} \), error probability of at most \( 1 - \frac{1}{2k} \) for the same \( \epsilon \).

\[\square\]

In the next lemma we generalize Lemma 2 to more complex distributions. More formally we show that if there is an adversary that uses \( q \) queries and succeeds on Cap(\( \mu \)) then there exists an adversary that succeeds on Cap and asks at most \( kq \) queries. Formally:

**Lemma 3 (Reduction from \( \text{Caps}_k^\mu \) to Cap).** Let \( k \in \mathbb{N}_+ \) and let \( \mathcal{G} \) be any distribution on \( (S_1^{d-1})^k \).
If there exists a \( q \)-bounded adversary \( A \) that succeeds on \( \text{Caps}_k^\mu(0.01) \) with approximation constant 1/2, error probability 0.01 and \( \epsilon \) such that \( \text{cap}(\cdot, 1, \tau(0.01/k)) + B_e = \text{cap}(\cdot, 1, 0) \) then there exists a \( kq \)-bounded adversary \( A' \) that succeeds on Cap(0.01/k) with approximation constant \( \frac{1}{2k} \), error probability 0.51 and the same \( \epsilon \).

**Proof.** Algorithm 2 defines the protocol for \( A' \). We will show that this protocol satisfies the statement of the lemma.

**Algorithm 2 EMULATEGENERAL(\( f, A, \mathcal{G}, k \))**

\[\begin{align*}
1: & \quad T(x) := \begin{cases} 1.3 \cdot x & \text{if } x \in S_1^{d-1} \\ x/1.3 & \text{if } x \in S_{1.3}^{d-1} \end{cases} \\
2: & \quad (y_1, \ldots, y_k) \sim \mathcal{G} \\
3: & \quad \text{for } i = 1, \ldots, k \text{ do} \quad \text{Any rotation satisfying the condition is valid} \\
4: & \quad R_i := \text{rotation such that } R_i(e_1) = y_i \\
5: & \quad M \sim O(d) \\
6: & \quad b_1, \ldots, b_k \sim U\{-1, 1\} \\
7: & \quad \text{for } i = 1, \ldots, k \text{ do} \quad \text{understood as a linear combination of transport maps} \\
8: & \quad T_i := \begin{cases} T & \text{if } b_i = -1 \\ \text{Id} & \text{if } b_i = +1 \end{cases} \\
9: & \quad p := \text{Simulate } A, \text{ to } x \text{ answer } f(M(R_1(T_1(x)))) \lor \cdots \lor f(M(R_k(T_k(x)))) \\
10: & \quad \text{for } i = 1, \ldots, k \text{ do} \\
11: & \quad p_i := T_i^{-1} \circ R_i^{-1} \circ M^{-1} \circ p \circ M \circ R_i \circ T_i \\
12: & \quad \text{return } p' := \frac{1}{k} \sum_{i=1}^{k} p_i \bigg|_{S_{d-1}} + \text{Id} \bigg|_{S_{d-1}^{d-1}}
\end{align*}\]

First observe that \( A' \) asks at most \( kq \) queries as every query of \( A \) is multiplied \( k \) times (see line 9 of Algorithm 2). Observe that \( p' \) is a well defined \( \epsilon \)-perturbation as all \( p_i \)'s are \( \epsilon \)-perturbations when restricted to \( S_1^{d-1} \). It follows from the fact that all \( p_i \)'s are of the form \( F^{-1} \circ p \circ F \) where \( F \) is a composition of an isometry and either \( T \) or the identity. This implies that for all \( x \in S_1^{d-1} \) we have \( \| x - F^{-1} \circ p \circ F(x) \|_2 \leq \epsilon \). Let \( C \subseteq S_1^{d-1} \) be the hidden spherical cap. Observe that:

\[
\bigcup_{i=1}^{k} M(R_i(T_i(C)))
\]

is distributed according to \( \text{Caps}_k^\mu \), as the relative positions of normal vectors of \( M(R_1(C)), M(R_2(C)), \ldots, M(R_k(C)) \) are distributed according to the process: generate \( (y_1, \ldots, y_k) \sim \mathcal{G}, M' \sim O(d) \), return \( M'((y'_1, \ldots, y'_k)) \). Thus by the fact that \( A \) succeeds with \( \alpha = 1/2 \) we know that with probability at least 0.99:
Lemma 5 (Non-adaptive query-bounded adversary). For \( \epsilon \in \mathbb{R}_{\geq 0} \) and \( f : \mathbb{R}^d \rightarrow \{-1, 1\} \), a \( q \)-bounded adversary with parameter \( \epsilon \) is a deterministic algorithm \( \mathcal{A} \) that asks at most \( q \in \mathbb{N} \) non-adaptive queries of the form \( f(x) = 1 \) and outputs an \( \epsilon \)-perturbation \( \mathcal{A}(f) : \mathbb{R}^d \rightarrow \mathbb{R}^d \).

Lemma 4. Let \( X \) be a zero-mean Gaussian with variance \( \sigma^2 \). Then for every \( t \geq 0 \):

\[
\frac{1}{\sqrt{2\pi}} \cdot \left( \frac{1}{t} - \frac{1}{t^3} \right) \cdot e^{-t^2/2} \leq \mathbb{P}_X \sim \mathcal{N}(0, \sigma^2) [X \geq \sigma \cdot t] \leq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{t} \cdot e^{-t^2/2}
\]

In Lemma 2 and Lemma 3, we showed that the QC of \( \text{Cap}^i \) and \( \text{Cap}^d \) can be lower-bounded in terms of the QC of Cap. We now show an upper bound \( \Theta(d) \) for the QC of Cap. Further, we give the sketch of the proof for a lower bound of \( \Theta(d) \). The summary of these results is presented in Table 1.

The upper-bound for Cap, that we are going to show, holds even if we restrict the adversary to be non-adaptive. I.e., the bound holds even if we require the adversary to declare the set of queries up front.

Definition 7 (Non-adaptive query-bounded adversary). For \( \epsilon \in \mathbb{R}_{\geq 0} \) and \( f : \mathbb{R}^d \rightarrow \{-1, 1\} \) a \( q \)-bounded adversary with parameter \( \epsilon \) is a deterministic algorithm \( \mathcal{A} \) that asks at most \( q \in \mathbb{N} \) non-adaptive queries of the form \( f(x) = 1 \) and outputs an \( \epsilon \)-perturbation \( \mathcal{A}(f) : \mathbb{R}^d \rightarrow \mathbb{R}^d \).

Lemma 5 (Upper bound for Cap). For every \( d \) bigger than an absolute constant there exists a non-adaptive \( \Theta(d) \)-bounded adversary \( \mathcal{A} \) that succeeds on \( \text{Cap}(0.01) \) with approximation constant 1/2, error probability 0.01 for \( \epsilon \) such that \( \text{cap}(\cdot, 1, \tau(0.01)) + B_\epsilon = \text{cap}(\cdot, 1, 0) \). Moreover \( \mathcal{A} \) can be implemented in \( O(d^2) \) time.

Proof. We will first show that there exists a randomized \( \mathcal{A} \) that satisfies the statement of the Lemma. This adversary uses Algorithm 3 invoked with \( s = \Theta(d) \) as its protocol. Later we will show how to derandomize the protocol.

\[
\mu \left( p^{-1} \left( \bigcup_{i=1}^k M(R_i(T_i(C))) \right) \right) \geq \frac{1}{2} \cdot \mu \left( \bigcup_{i=1}^k M(R_i(T_i(C))) + B_\epsilon \right).
\]

Observe that:

\[
\mu \left( \frac{1}{k} \sum_{i=1}^k p_i \right)^{-1}(C) = \frac{1}{k} \sum_{i=1}^k \mu \left( p^{-1}(M(R_i(T_i(C)))) \right) \geq \frac{1}{k} \cdot \mu \left( \bigcup_{i=1}^k M(R_i(T_i(C))) \right)
\]

Combining the two bounds we get that with probability at least 0.99:

\[
\mu \left( p'^{-1}(C) \right) \geq \frac{1}{2k} \cdot \mu \left( \bigcup_{i=1}^k M(R_i(T_i(C))) + B_\epsilon \right)
\]

We note that with probability at least 1/2 there exists \( i_0 \in [k] \) such that \( T_{i_0} = \text{Id} \). This means that with probability at least 1/2:

\[
\mu \left( \bigcup_{i=1}^k M(R_i(T_i(C))) + B_\epsilon \right) \geq 1/4,
\]

as \( \mu(M(R_{i_0}(T_{i_0}(C))) + B_\epsilon) = \mu(S_{i_0}^{d-1})/2 \). Combining equation 35 and equation 36 and using the union bound we get that with probability of at least 0.49:

\[
\mu(p'^{-1}(C)) \geq \frac{1}{8k},
\]

which is equivalent to \( \mathcal{A}' \) succeeding on \( \text{Cap}(0.01/k) \) with approximation constant of at least \( \frac{1}{2k} \), error probability of at most 0.51 for the same \( \epsilon \).

\[\square\]
Algorithm 3 \textsc{CapAdversaryRandomized}(f, s, \epsilon)
\begin{align*}
&\triangleright f \text{ is the classifier, } s \text{ is the number of sampled points per sphere} \\
&\triangleright \epsilon \text{ is the bound on allowed perturbations} \\
\end{align*}
\begin{enumerate}
\item $Q^- := \{x_1^{-}, \ldots, x_n^{-}\}$, where $x_i^{-}$'s are i.i.d. $\sim U[S_1^{d-1}]$
\item $Q^+ := \{x_1^+, \ldots, x_n^+\}$, where $x_i^+$'s are i.i.d. $\sim U[S_1^{d-1}]$
\item $R := \{x \in Q^- : f(x) = +1\} \cup \{x \in Q^+ : f(x) = -1\}$
\item $v := 1/|R| \cdot \sum_{x \in R}{x}$
\item $p(x) := \begin{cases} 
\text{argsup}_{x' \in S_1^{d-1}, ||x-x'||_2 \leq \epsilon} (x' - x, v) & \text{if } x \in S_1^{d-1} \\
\text{argsup}_{x' \in S_{1, 3}^{d-1}, ||x-x'||_2 \leq \epsilon} (x' - x, v) & \text{if } x \in S_{1, 3}^{d-1}
\end{cases}$
\item return $p$
\end{enumerate}

Randomized algorithm. First notice that $\mathcal{A}$ is non-adaptive. The queries asked by $\mathcal{A}$ are from $Q^- \cup Q^+$ which were generated (see lines 1 and 2) before any queries were asked and, hence, answered were received. Note further that $\mathcal{A}$ is $\Theta(d)$ bounded as she asks $2 \cdot s = \Theta(d)$ queries.

Run time. We first remark that $\mathcal{A}$ can be implemented in $O(d^2)$ time as the run time is dominated by asking $\Theta(d)$ queries and each vector is in $\mathbb{R}^d$. Formally, $p$ is not returned explicitly but one can imagine that $\mathcal{A}$, after preprocessing that takes $O(d^2)$ time, provides oracle access to $p$ where each evaluation takes time $O(d)$. Now we prove that $\mathcal{A}$ succeeds with probability $0.99$ with approximation constant $1/2$. Let $E$ be the hidden spherical cap that contains all errors of 1-NN and let $u \in S_1^{d-1}$ be its normal vector. First assume that $E \subseteq S_1^{d-1}$. We start by lower-bounding $|R|$. For every $i \in [s]$ let $Y_i^-$ be a random variable which is equal to $1$ if $x_i^{-} \in E$ and $0$ otherwise. Then, by the Chernoff bound, we have that for every $\delta < 1$:
\begin{equation}
\Pr \left[ \left| \sum_{i=1}^{s}{Y_i^-} - \mathbb{E} \left[ \sum_{i=1}^{s}{Y_i^-} \right] \right| > \delta \cdot \mathbb{E} \left[ \sum_{i=1}^{s}{Y_i^-} \right] \right] \leq 2e^{-\frac{\delta^2}{4} \mathbb{E}[\sum_{i=1}^{s}{Y_i^-}]} , \tag{37}
\end{equation}
Noticing that $\mathbb{E} \left[ \sum_{i=1}^{s}{Y_i^-} \right] = s \cdot 0.01$ if we set $\delta = 1/2$ we get that:
\begin{equation}
\Pr \left[ \left| \sum_{i=1}^{s}{Y_i^-} - s/100 \right| > s/200 \right] \leq 2e^{-\frac{s^2}{4} \cdot s/100} = 2e^{-s/1200} . \tag{38}
\end{equation}
So with probability at least $1 - 2e^{-s/1200}$ we have that:
\begin{equation}
|R| \geq s/200 .
\end{equation}
Now assume $R = \{z_1, \ldots, z_k\}$ and observe that for every $z \in R$ we have $\langle z, u \rangle \geq \tau(0.01)$ and note that $z_i$'s are i.i.d. uniformly distributed on $\text{cap}(u, 1, \tau(0.01))$. We will model $U[S_1^{d-1}]$ as $\mathcal{N}(0, 1/d)^d$. Then we have that:
\begin{equation}
\langle u, v \rangle = \frac{1}{k} \sum_{i=1}^{k}{\langle u, z_i \rangle} \\
= \frac{1}{k} \sum_{i=1}^{k}{\langle u, z_i \rangle} \\
\geq \tau(0.01) \quad \text{as } z_i \in R \\
\geq 2.2/\sqrt{d} \quad \text{by Lemma\textsuperscript{4}} \tag{39}
\end{equation}
Moreover if $\Pi$ is the orthogonal projection onto $u^\perp$ then $\Pi(k \cdot v) \sim \mathcal{N}(0, k/d)^{d-1}$ and $\Pi(v) \sim \mathcal{N}(0, 1/(dk))^{d-1}$ thus:
\begin{equation}
\|\Pi(v)\|_2^2 \sim \frac{1}{dk} \cdot \chi^2(d - 1)
\end{equation}
So, using standard tail bounds for $\chi^2$ distribution, we get that for all $t \in (0, 1)$:

$$\Pr \left[ \left| \frac{dk}{d-1} \cdot \|II(v)\|_2^2 - 1 \right| \geq t \right] \leq 2e^{-(d-1)t^2/8} \tag{40}$$

Moreover observe:

$$
\begin{align*}
\langle u, v \rangle &= \frac{\langle u, v \rangle}{\sqrt{\langle u, v \rangle^2 + \|II(v)\|_2^2}} \\
&= \frac{1}{\sqrt{1 + \frac{d}{2} \cdot \|II(v)\|_2^2 / \langle u, v \rangle^2}} \\
&\geq \frac{1}{\sqrt{1 + \frac{d}{2} \cdot \|II(v)\|_2^2}} \quad \text{by equation} \ 39 \tag{41}
\end{align*}
$$

Observe that if $\langle u, v \rangle \leq 0$ then $\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01)))) = 1/4$, as the preimage is exactly half of the sphere $S_{d-1}^d$. Moreover $\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01))))$ is a continuous function of $\langle u, v \rangle$. Observe that in a coordinate system where the first basis vector is $v/\|v\|_2$ we have $p(\mu|_{S_{d-1}^1} \approx N(\tau(0.01), 1/d), N(0, 1/d), \ldots, N(0, 1/d))$. Assume $\langle u, v \rangle = \alpha$. We bound:

$$
\begin{align*}
\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01)))) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d}{2\pi} \cdot e^{-\frac{1}{2d}(x_1^2 + x_2^2)} \ dx_1 \ dx_2 \\
&\geq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d}{2\pi} \cdot e^{-\frac{1}{2d}(x_1^2 - 2\sqrt{d}x_2 + x_2^2)} \ dx_1 \ dx_2 \quad \text{by Lemma} \ 4 \tag{42}
\end{align*}
$$

This means that there exists $\alpha \in (0, \pi/2]$ (independent of $d$) such that for all $u$ such that $\langle u, v \rangle \leq \alpha$ we have $\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01)))) \geq 1/5$. Thus by equation 41 we get that there exists $\xi > 0$ such that if $\|\Pi(v)\|_2^2 \leq \xi/d$ then $\langle u, v \rangle \leq \alpha$ and in turn $\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01)))) \geq 1/5$.

Setting $k := \frac{2d}{\xi}, t := 1/2$ in equation 40 we get that with probability at least $1 - e^{-(d-1)/32}$ we have:

$$\|\Pi(v)\|_2^2 \leq \xi/d,$$

which in turn means that with probability at least $1 - e^{-(d-1)/32}$:

$$\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01)))) \geq 1/5. \tag{42}$$

Now combining equation 39, equation 42 and the union bound we get that if we set $s := \frac{400d}{\xi}$ then with probability at least $1 - 2e^{-s/200} - e^{-(d-1)/32} = 1 - 2e^{-2d/\xi} - e^{-(d-1)/32}$ we have:

$$\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01)))) \geq 1/5. \tag{43}$$

This probability is bigger than 0.99 if $d$ is bigger than an absolute constant that depends on $\xi$. Observing that $\mu(E + B_c) = 1/4$ we conclude that if $E \subseteq S_{d-1}^d$ then if $s = \Theta(d)$ then with probability 0.99 $A$ succeeds on Cap with approximation constant at least 1/2. To finish the proof one notices that the case $E \subseteq S_{d-1}^d$ is analogous. The final constant hidden under $\Theta(d)$ for the number of samples is a maximum of constants for $S_{d-1}^d$ and $S_{d-1}^d$.

**Deterministic algorithm.** We know how to derandomize Algorithm 3 to design an adversary $A_{det}$. We observe that in Algorithm 3 randomness was used only to generate query points $Q^-, Q^+$. Instead of generating the query points randomly we use fixed sets. We define the deterministic adversary, $A_{det}$, as:

$$A_{det} := \text{CAPADVERSARYDETERMINISTIC}(\cdot, Q^-, Q^+).$$
for fixed (for a given $d$) sets $Q^-, Q^+$ that we define next.

**Algorithm 4** CAPADVERSARYDETERMINISTIC($f, Q^-, Q^+, \epsilon$)

\[ \text{\texttt{\textbf{\texttt{CAPADVERSARYDETERMINISTIC}}}}(f, Q^-, Q^+, \epsilon) \]

\[ \begin{align*}
1: & R := \{ x \in Q^- : f(x) = +1 \} \cup \{ x \in Q^+ : f(x) = -1 \} \\
2: & v := 1/|R| \cdot \sum_{x \in R} x \\
3: & p(x) := \begin{cases} 
\text{argsup}_{x' \in S_1^{d-1}, \|x-x'\|_2 \leq \epsilon} \langle x' - x, v \rangle & \text{if } x \in S_1^{d-1} \\
\text{argsup}_{x' \in S_1^{d-1}, \|x-x'\|_2 \leq \epsilon} \langle x' - x, v \rangle & \text{if } x \in S_1^{d-1} 
\end{cases} \\
4: & \text{return } p 
\end{align*} \]

For $u \in S_1^{d-1}$ let:

\[ f_u(x) := \begin{cases} 
-1 & \text{if } x \in S_1^{d-1} \setminus \text{cap}(u, 1, \tau(0.01)), \\
+1 & \text{otherwise} 
\end{cases} \]

We say that an adversary succeeds on $f_u$ if she, run for $f_u$, returns $p$ such that $\mu(p^{-1}(\text{cap}(u, 1, \tau(0.01)))) \geq 1/8$. From equation 43 we know that for every $d \in \mathbb{N}_+$, for every $u \in S_1^{d-1}$:

\[ \mathbb{P}_{x_1, \ldots, x_{400d/\xi} \sim U[S_1^{d-1}]} [A(f_u, 400d/\xi, \epsilon) \text{ succeeds}] \geq 1 - 2e^{-2d/\xi} - e^{-(d-1)/32} \]

Thus we get that for every $d \in \mathbb{N}_+$ that:

\[ \mathbb{P}_{u, x_1, \ldots, x_{400d/\xi} \sim U[S_1^{d-1}]} [A(f_u, 400d/\xi, \epsilon) \text{ succeeds}] \geq 1 - 2e^{-2d/\xi} - e^{-(d-1)/32} \]

And finally, this means that for every $d \in \mathbb{N}_+$ there exists $Q_d^- \subseteq S_1^{d-1}, |Q_d^-| = 400d/\xi$ such that:

\[ \mathbb{P}_{u \sim U[S_1^{d-1}]} [A_{\text{det}}(f_u, Q_d^-, \emptyset, \epsilon) \text{ succeeds}] \geq 1 - 2e^{-2d/\xi} - e^{-(d-1)/32} \]

Thus, for $d$ bigger than an absolute constant we get that conditioned on $E \subseteq S_1^{d-1}$ $A_{\text{det}}$ run with $Q^- = Q_d^-$ succeeds with probability at least 0.99 and asks $|Q_d^-| = \Theta(d)$ queries. Analogous argument shows that for every $d \in \mathbb{N}_+$ there exists $Q_d^+ \subseteq S_1^{d-1}, |Q_d^+| = \Theta(d)$ such that the following holds. For every $d$ bigger than an absolute constant conditioned on $E \subseteq S_1^{d-1}$ $A_{\text{det}}$ run with $Q^+ = Q_d^+$ succeeds with probability at least 0.99. Combining these two results we get that $A_{\text{det}}$ satisfies statement of the lemma.

**Remark 2.** As we have seen in the proof of Lemma 3 it was more natural to design an adversary that was randomized. We believe that allowing the adversary to use randomness would not change the results in a fundamental way.

**Lemma 1 (Lower bound for Cap).** There exists $\delta > 0$ such that if a $q$-bounded adversary $A$ succeeds on $\text{Cap}(0.01)$ with approximation constant $\geq 1 - \delta$, error probability $2/3$ for $\epsilon$ such that $\text{cap}(\cdot, 1, \tau(0.01)) + B_\epsilon = \text{cap}(\cdot, 1, 0)$. Then

\[ q \geq \Theta(d). \]

**Proof.** To simplify computations we will sometimes approximate the uniform distribution on $S_1^{d-1}$ as a $d$ dimensional normal distributions: $\mathcal{N}(0, \frac{1}{d})$. This change is valid as the norm of $\mathcal{N}(0, \frac{1}{d})^d$ is closely concentrated around 1.

**Lower-bounding QC.** To use Theorem 4 we think that there is an algorithm $\text{ALG}$ for which the distribution of errors coincides with $\text{Cap}(0.01)$. We recall that in this case $\mu$ is a normalized Lebesgue measure on $S_1^{d-1}$. Note that by definition $\text{AR}(\text{ALG}(S), \epsilon) = 1/2$. Thus we analyze:

\[ \sup_{p: \epsilon\text{-perturbation}} \mathbb{P}_{S \sim \text{DM}} [\mu(p^{-1}(E(\text{ALG}(S))))] \geq (1 - \delta) \cdot \text{AR}(\text{ALG}(S), \epsilon) \]

\[ = \sup_{p: \epsilon\text{-perturbation}} \mathbb{P}_{E \sim \text{Cap}(0.01)} [\mu(p^{-1}(E))] \geq \frac{1 - \delta}{2}, \]

(44)

30
for a constant $\delta$ that will be fixed later. Let $p$ be an $\epsilon$-perturbation and $y \in \mathcal{S}_d^{d-1}$ be such that $\mu(p^{-1}(\text{cap}(y, 1, \tau(0.01)))) \geq \frac{1 - \delta}{2}$. We will show that for every $x \in \mathcal{S}_d^{d-1}$ if $\angle(y, x) \in \left[\frac{49\pi}{100} : \frac{51\pi}{100}\right]$ then $\mu(p^{-1}(\text{cap}(y, 1, \tau(0.01)))) < \frac{1 - \delta}{2}$. This will conclude the proof as then:

$$
\mathbb{P}_{E \sim \text{Cap}(0.01)} \left[ \mu(p^{-1}(E)) \geq \frac{1 - \delta}{2} \right] \leq 2 \cdot \mu \left( \text{cap} \left( \cdot, 1, \arccos \left( \frac{49\pi}{100} \right) \right) \right) \leq 2^{-\Omega(d)}
$$

By Lemma 4, combined with equation 44 and Theorem 4 gives the result.

Now let $x \in \mathcal{S}_d^{d-1}$ be such that $\angle(y, x) \in \left[\frac{49\pi}{100} : \frac{51\pi}{100}\right]$. To simplify notation let $C_x := \text{cap}(x, 1, \tau(0.01)), C_y := \text{cap}(y, 1, \tau(0.01))$. Now define:

$$
I := \left\{ z \in \mathcal{S}_d^{d-1} \mid d(z, C_y) \leq \epsilon \land d(z, C_x) \leq \epsilon \land d(z, C_x \cap C_y) > \epsilon \right\},
$$

where $d$ denotes the $\ell_2$ distance between sets. By Lemma 4 we have:

$$
2.2/\sqrt{d} \leq \tau(0.01) \leq 2.4/\sqrt{d}
$$

which in turn by definition of $\epsilon$ gives that for $d > 15$:

$$
\epsilon \leq \sqrt{\frac{2.4^2}{d} + \left(1 - \sqrt{1 - \frac{2.4^2}{d}}\right)^2} \leq 3/\sqrt{d}
$$

Now observe that:

$$
I \supseteq \left\{ z \in \mathcal{S}_d^{d-1} \mid \langle z, y \rangle \geq 0 \land \langle z, x \rangle \geq 0 \land \left\langle z, \frac{x + y}{\|x + y\|} \right\rangle < \frac{2.2}{\sqrt{d} \cdot \cos(\angle(y, x)/2)} - \frac{3}{\sqrt{d}} \right\}
$$

$$
\supseteq \left\{ z \in \mathcal{S}_d^{d-1} \mid \langle z, y \rangle \geq 0 \land \langle z, x \rangle \geq 0 \land \left\langle z, \frac{x + y}{\|x + y\|} \right\rangle < \frac{1}{20/\sqrt{d}} \right\} =: \hat{I}
$$

where in the first transition we used equation 45 and equation 46 and in the second transition we used that $\angle(y, x) \in \left[\frac{49\pi}{100} : \frac{51\pi}{100}\right]$. Note that $\mu(\hat{I})$ is minimized for $\angle(y, x) = \frac{51\pi}{100}$. Thus:

$$
\mu(\hat{I}) 
\geq \int_0^\infty \int_{\tan(\pi/100) \cdot x_1}^\infty d/2 \cdot e^{-\frac{1}{2}(x_1^2 + x_2^2)} \cdot 1 \left[ x_1 \cos \left( \frac{51\pi}{200} \right) + x_2 \sin \left( \frac{51\pi}{200} \right) < \frac{1}{20/\sqrt{d}} \right] dx_2 dx_1
$$

$$
= \int_0^\infty \int_{\tan(\pi/100) \cdot x_1}^\infty 1/2 \cdot e^{-\frac{1}{2}(x_1^2 + x_2^2)} \cdot 1 \left[ x_1 \cos \left( \frac{51\pi}{200} \right) + x_2 \sin \left( \frac{51\pi}{200} \right) < \frac{1}{20} \right] dx_2 dx_1,
$$

where the first equality comes from integration by substitution. The integral from equation 48 is positive, which means that there exists $\delta > 0$ such that $\mu(\hat{I}) > \delta$. Combining that with equation 47 we get that $\mu(I) > \delta$. Observe that by definition of $I$ for every $z \in I$ we have that at most one of $p(z) \in C_x, p(z) \in C_y$ can be true. Thus, using the fact that $\mu(C_x \cup B_x) = \mu(C_x \cup B_x) = 1/2$, we get that:

$$
\min(\mu(p^{-1}(C_x)), \mu(p^{-1}(C_y))) < 1/2 - \delta/2.
$$

This ends the proof as by assumption we know that $\mu(p^{-1}(C_y)) \geq 1/2 - \delta/2$, so by equation 49 we get that $\mu(p^{-1}(C_x)) < 1/2 - \delta/2$. 

Note that Lemma 1 is equivalent to the statement of Conjecture 1 for $k = 1$.

**Conjecture 1 (Cap conjecture).** For every $k \in [d]$ if a $q$-bounded adversary $A$ succeeds on $\text{Cap}(0.01/k)$ with approximation constant $\geq \frac{1}{2k}$, error probability $\leq 1 - \frac{1}{4k}$ for $\epsilon$ such that $\text{cap}(\cdot, 1, \tau(0.01/k)) + B_e = \text{cap}(\cdot, 1, 0)$. Then

$$
q \geq \Theta(d).
$$
In Figure 2, similar to Figure 1, we present visualizations of decision boundaries for 1-NN. Each subfigure represents a random decision boundary for a different sample $S \sim D^m$. The aim of these visualizations is to give an intuition for why Theorem 2 is true.

Figure 2: Random decision boundaries of 1-NN for $T_{\text{intervals}}$. 