On hybrid stochastic population models with impulsive perturbations

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\begin{abstract}
This paper considers the dynamic behaviours of a hybrid stochastic population model with impulsive perturbations. The existence of the global positive solution is studied in this paper. Moreover, under some conditions on the noises and impulsive perturbations, the properties of the persistence and extinction, stochastic permanence, global attractivity and stability in distribution are presented. Our results illustrate that impulsive perturbations play a crucial role in these properties. The bounded impulse term will not affect these properties, however, when the impulse term is unbounded, some of the properties, such as the persistence and extinction may be changed significantly. As a part of this paper, a couple of examples and numerical simulations are provided to illustrate our results.
\end{abstract}

\section{Introduction}

The famous logistic growth model is often described by the ordinary differential equation

\[ \frac{dx(t)}{dt} = ax(t) - bx^2(t), \]

where \( a \) be the intrinsic growth rate. In ecosystems, the population dynamics are inevitably affected by various environmental noises. First of all, let us consider one type of environmental noise, namely white noise. When we incorporate white noise in the intrinsic growth rate \( a \) of (1), then it becomes \( a + \sigma \dot{B}(t) \), where \( \dot{B}(t) \) is white noise and \( \sigma \) is a positive number representing the intensity of the noise. As a result, (1) becomes the following Itô stochastic differential equation (see e.g. [5,27])

\[ dx(t) = [ax(t) - bx^2(t)] \, dt + \sigma x(t) \, dB(t), \]

which exhibits an approximate solution for the transient partial differential equations of \( x(t) \). The dynamic behaviours of (2) have been studied by many authors (see e.g. [2,6,10,11,15,20,21]). Now, let us take a further step by considering coloured noise, say telegraph noise (see e.g. [16,24,28]). This kind of noise can be illustrated as a switching between finite regimes of the environment. The switching is memoryless and the waiting
time for the next switch has an exponential distribution. Hence, we assume there are $N$ regimes and (2) becomes the following hybrid stochastic logistic model

$$dx(t) = [a(r(t))x(t) - b(r(t))x^2(t)] \, dt + \sigma(r(t))x(t) \, dB(t),$$

(3)

where $r(t)$ is a right-continuous Markov chain which takes values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$. The model (3) can be regarded as the results of the following $N$ stochastic differential equations

$$dx(t) = [a(i)x(t) - b(i)x^2(t)] \, dt + \sigma(i)x(t) \, dB(t) \quad i \in \mathbb{S}$$

switching from one to the other according to the movement of the Markov chain. One can understand the hybrid stochastic model (3) as: if $r(0) = j \in \mathbb{S}$, the system satisfies (3) with $r(t) = j$ until $t = \tau_1$, when $r(t)$ jumps to another state $r(t) = l$, the system will obey (3) with $r(t) = l$. Recently, the author in [17] studied the dynamics of a stochastic predator-prey model under regime-switching, modified Leslie-Gower Holling-type II and prey harvesting. Considering Lévy and Markov chain noises, the authors in [22] gave the existence of the stationary distribution, ergodicity of a stochastic hybrid competition model.

Moreover, the impulsive process is often studied in various areas of science and technology such as biological systems, population dynamics, physics, chemistry, and optimal control theories (see e.g. [3,14]). There are many important and interesting results about the behaviours of the population dynamic systems of impulsive differential equations. For example, S. Ahmad and I. Stamova in [1] investigated competitive systems with delays and impulsive perturbations, they also gave the conditions for asymptotic stability of competitive systems. The global dynamics for the periodic logistic equation and periodic multi-species predator-prey system under the impulsive perturbations were studied in [8,19], respectively. The permanence and global stability for nonautonomous $N$-species impulsive Lotka-Volterra competitive system were considered by J. Hou, Z. Teng and S. Gao in [9]. Persistence and extinction of a stochastic predator-prey model with impulsive toxicant input were considered in [23].

Motivated by the work of persistence and extinction, stochastic permanence, the existence of stationary distribution, ergodicity and stability in distribution for stochastic population model. In this paper, we will study the following hybrid stochastic logistic model with impulsive perturbations, which is defined as follows:

$$dx(t) = [a(r(t))x(t) - b(r(t))x^2(t)] \, dt + \sigma(r(t))x(t) \, dB(t), \quad t \neq t_k$$

$$x(t_k) = A_k(r(t_{k-1}))x(t_{k-}^+), \quad k \in \mathbb{N}$$

(4)

where $A_k(r(t_{k-1}))$ is the impulse gain coefficient, $N$ is the set of positive integers, and for $k \in \mathbb{N}$, $t_k$ satisfies $0 < t_1 < t_2 < \cdots$ such that $\lim_{k \to +\infty} t_k = +\infty$, $x(t_k) = A_k(r(t_{k-1}))x(t_{k-}^+)$ is the impulse at the moment and $x(t_k)$ right-continuous at $t_k$, i.e. $x(t_k) = x(t_{k-}^+)$. For any fixed $i \in \mathbb{S}$, $a(i), b(i)$ and $\sigma(i)$ are continuous bounded functions on $\mathbb{R}_+$. In the biological sense, it is reasonable to assume that

$$\inf_{i \in \mathbb{S}} b(i) > 0, \quad A_k(i) > 0, \quad k \in \mathbb{N}, \quad i \in \mathbb{S}.$$  

If $A_k(i) > 1$, we say that the species are planting, while $A_k(i) < 1$ stands for harvesting.
Remark 1.1: Our motivations for studying the properties of hybrid stochastic model (4) stem from the emerging and existing applications of the biological systems under the influence of random environments and impulse. The model (4) is different from the model given by Liu and Wang in [21]. The impulses occur at a fixed sequence of times, \( t_k \), which are dependent on the environmental state in (4). These results in the underlying problems challenging to handle due to the interaction of the impulse times, continuous processes, the tangled and hybrid information pattern.

In our paper, the coloured noises (Markov chain), the white noises (Brownian noises), and impulsive perturbations are considered in the classical logistic model. We investigate how these noises and impulses affect the dynamic behaviours of (4). As far as we know, there was no much contribution to this kind of stochastic differential equation. The authors in [30,31] studied the impulsive perturbations at the switching points of a hybrid stochastic differential equation with impulsive perturbations. This motivated us to investigate some properties of (4).

We consider the regime switching in the impulsive term. Our results show how the impulse and the environmental noises affect the properties of the logistic model, such as the existence and uniqueness of the positive solution, persistence and extinction, and global attractivity. More importantly, we give the result of asymptotic stability in distribution for this hybrid stochastic model with impulsive perturbations. Some simulation examples are given in the last part of this paper to illustrate our theoretical results.

2. The existence of the global positive solution

Throughout this paper, let \( (\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, P) \) be a complete probability space with a filtration \( \mathcal{F}_{t \geq 0} \), \( B(t) \) be a given one-dimensional standard Brownian motion defined on this probability space and \( r(t) \) be a right-continuous Markov chain which is independent of \( B(t) \). The generator \( \Gamma = (\gamma_{ij})_{N \times N} \) of \( r(t) \) is given by

\[
\mathbb{P}\{r(t + \delta) = j \mid r(t) = i\} = \begin{cases} 
\gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\
1 + \gamma_{ii}\delta + o(\delta) & \text{if } i = j,
\end{cases}
\]

where \( \delta > 0 \) and \( \gamma_{ij} \geq 0 \) be the transition rate from \( i \) to \( j \) satisfying \( \gamma_{ij} > 0 \) if \( i \neq j \) while \( \gamma_{ii} = -\sum_{j \neq i} \gamma_{ij} \).

Since \( r(t) \) is right-continuous, almost every sample path of \( r(t) \) is a right-continuous step function with a finite number of sample jumps in the finite subinterval of \( \mathbb{R}_+ \). That is to say, there exists a sequence of stopping times, \( 0 = t_0 < t_1 < \cdots < t_k \), and \( t_k \to \infty \) almost surely such that \( r(t) \) is a constant in every interval \([t_{k-1}, t_k)\) for any \( k \geq 1 \), i.e.

\[
r(t) = r(t_{k-1}), \quad \forall \ t \in [t_{k-1}, t_k), \quad k \geq 1.
\]

Moreover, we assume that Markov chain \( r(t) \) is irreducible which can guarantee that the hybrid system controlled by \( r(t) \) will always switch from one regime to another. Under this condition, Markov chain \( r(t) \) has a unique stationary (probability) distribution.
Let $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in \mathbb{R}^{1 \times N}$, which can be determined by solving the following linear equation

$$\pi \Gamma = 0$$

subject to

$$\sum_{k=1}^{N} \pi_k = 1 \quad \text{and} \quad \pi_k > 0, \quad \text{for} \quad \forall \ k \in \mathbb{S}.$$ 

Let $x(t) = \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) y(t)$. Consider the following hybrid stochastic differential equation without impulsive perturbations

$$dy(t) = y(t)\left[a(r(t)) - \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) b(r(t)) y(t(t))\right] dt + \sigma(r(t)) y(t) dB(t).$$

We can check that if $y(t)$ is a solution of (6), then $x(t) = \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) y(t)$ is a solution of (4). Conversely, if $x(t)$ is a solution of (4), then $y(t) = \prod_{0 < t_{k-1} < t} A_k^{-1}(r(t_{k-1})) x(t)$ is a solution of (6), so we get the following result.

**Theorem 2.1:** For any initial value $x(0) = x > 0$, $r(0) = i \in \mathbb{S}$, the model (4) has a unique continuous solution $(x(t), r(t))$ on $t \geq 0$ which can be represented by

$$x(t) = \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) \exp\left\{\int_0^t [a(r(s)) - \frac{1}{2} \sigma^2(r(s))] ds + \int_0^t \sigma(r(s)) dB(s)\right\} \frac{\frac{1}{x} + \int_0^t \prod_{0 < t_{k-1} < s} A_k(r(t_{k-1})) b(r(s)) ds}{\exp\left\{\int_0^t [a(r(\tau)) - \frac{1}{2} \sigma^2(r(\tau))] d\tau + \int_0^\tau \sigma(r(\tau)) dB(\tau)\right\} ds}$$

and the solution will remain in $\mathbb{R}^+ \times \mathbb{S}$ almost surely.

Details of the proof are omitted to conserve space and given in Appendix 1.

**Remark 2.1:** From Theorem 2.1, one can find that if $\prod_{0 < t_{k-1} < s} A_k(r(t_{k-1})) b(r(s)) < 0$ the solution may explode at a finite time, and $\inf_{t \in \mathbb{S}} b(i) > 0$, $A_k(i) > 0$, $\forall i \in \mathbb{S}$ is a necessary condition for the global existence of solutions.

**Remark 2.2:** Theorem 2.1 indicates that the solution of (4) with the positive initial value will remain positive. This property provides us with an opportunity to check persistence and extinction, stochastic permanence, the existence of the stationary distribution and stability in distribution.

### 3. The dynamics behaviours of the model (4)

In this section, we will discuss the dynamic behaviours of the model (4). The properties, such as stochastically permanence, persistence and extinction, global attractivity, all have great biological meanings. These proofs and conclusions can be found in Appendix 2, and here we give some necessary explanations for our results.
In Section 2, we showed that (4) has a positive solution for any initial values. In this section, we will investigate how the solution varies in $\mathbb{R}^+$. For convenience and simplicity in the following discussion, for any constant sequence $c(i), i \in \mathbb{S}$, let

$$\hat{c} = \max_{i \in \mathbb{S}} c(i), \quad \check{c} = \min_{i \in \mathbb{S}} c(i)$$

and

$$\beta(i) = b(i) - \frac{1}{2} \sigma^2(i), \quad i \in \mathbb{S}$$

$$A(t) = \sum_{0 < t_k < t} \ln[A_k(r(t_k))] + \int_0^t \beta(r(s)) \, ds.$$ 

**Definition 3.1:** The solution $x(t)$ to (4) is said to be extinctive with probability one if

$$\lim_{t \to +\infty} x(t) = 0 \text{ a.s.}$$

**Theorem 3.2:** For any initial value $x(0) = x \in \mathbb{R}^+$, the solution $x(t)$ to (4) has the property that

$$\limsup_{t \to +\infty} \frac{\ln x(t)}{t} \leq \limsup_{t \to +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(A_k(r(t_k))) \right] + \sum_{i=1}^N \pi_i \beta(i) =: A^*, \quad \text{a.s.}$$

In particular, if $A^* < 0$, then $\lim_{t \to +\infty} x(t) = 0 \text{ a.s.}$, that is, the model (4) is extinctive.

**Remark 3.1:** If there are no impulsive perturbations, then Theorem 3.2 indicates that the model (4) is extinctive with probability one when $\pi \beta < 0$, which is a result in [17].

**Remark 3.2:** Notice that, by the definition of extinction, the population system will be extinctive with probability one if $A^* < 0$. Ma and his co-authors in [7, 25, 29] proposed the concepts of the persistence, including non-persistence in mean, weakly persistent, persistence in mean, for some deterministic models. For stochastic systems, the concept of persistence is complex, we first discuss some kinds of persistence, such as nonpersistent in mean, weakly persistent and persistent in mean, which are the analogous concepts to the deterministic population models.

The following conditions for persistence and weakly persistent of (4) are obtained. For the corresponding definitions to deterministic population models, the definitions of persistence and weakly persistent for the stochastic system is defined as the following.

**Definition 3.3:** Suppose that $x(t)$ is a solution of the model (4), then

(i) $x(t)$ is said to be nonpersistent in mean if $\lim_{t \to +\infty} (1/t) \int_0^t x(s) \, ds = 0 \text{ a.s.}$;
(ii) $x(t)$ is said to be weakly persistent if $\limsup_{t \to +\infty} x(s) > 0 \text{ a.s.}$;
(iii) $x(t)$ is said to be persistent in mean if $\liminf_{t \to +\infty} (1/t) \int_0^t x(s) \, ds > 0 \text{ a.s.}$.
Theorem 3.4: The solution $x(t)$ of the model (4) satisfies
\[
\limsup_{t \to +\infty} \frac{1}{t} \int_0^t x(s) \, ds \leq \frac{A^*}{b} \quad \text{a.s.}
\]
Moreover, if $A^* = 0$, then $\lim_{t \to +\infty} (1/t) \int_0^t x(s) \, ds = 0$ a.s., that is, the population represented by the model (4) is nonpersistent in mean.

Theorem 3.5: If $A^*$ defined in Theorem 3.2 satisfies $A^* > 0$, then the population represented by the model (4) is weakly persistent a.s., i.e.
\[
\lim_{t \to +\infty} x(t) > 0 \quad \text{a.s.} \tag{7}
\]

Theorem 3.6: Denote $A_* = \liminf_{t \to +\infty} (1/t) [\sum_{0 < t_k < t} \ln(A_k(r(t_k))) + \sum_{i=1}^N \pi_i \beta(i)]$, then the population represented by the model (4) satisfies
\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t x(s) \, ds \geq \frac{A_*}{b} \quad \text{a.s.} \tag{8}
\]
In particular, if $A_* > 0$, then $\liminf_{t \to +\infty} (1/t) \int_0^t x(s) \, ds > 0$, it is alternative to to say that the model (4) is persistent in mean almost surely.

Remark 3.3: Theorem 3.4–3.6 indicate that the intensity of impulsive perturbations and the Markov chain will affect the persistence of (4) to some extent.

In the following, we provide the result for stochastic permanence, which shows that the population system will survive in the future with large probability. The proof is omitted here as it is only a direct promotion for stochastic permanence of (4) without Markovian switching, the results for the model without impulsive perturbations can be found in [17].

Definition 3.7 (See [17]): The model (4) is said to be stochastically permanent if for any $\nu \in (0, 1)$, there exist positive constants $\chi = \chi(\nu), \delta = \delta(\mu)$ such that
\[
\liminf_{t \to +\infty} \mathbb{P}\{x(t) \leq \chi\} \geq 1 - \nu, \quad \liminf_{t \to +\infty} \mathbb{P}\{x(t) \geq \chi\} \geq 1 - \nu,
\]

Assumption 3.1: Suppose that there are two positive constants $m$ and $M$ such that $m \leq \prod_{0 < t_k < t} A_k(r(t_k)) \leq M$ for all $t > 0$ and $r(t_k) \in \mathbb{S}$.

Theorem 3.8: Under Assumption 3.1. If $\pi b =: \sum_{i=1}^N \pi_i b(i) > 0$ and $\pi \beta =: \sum_{i=1}^N \pi_i \beta(i) > 0$, then the model (4) is stochastically permanent.

Remark 3.4: Theorem 3.8 implies that the impulse does not affect stochastic permanence if the impulsive perturbations are bounded (Assumption 3.1). However, the persistence and extinction may be changed significantly if the impulsive perturbations are unbounded. We will present this result in the following examples in Section 5.
**Definition 3.9:** The model (4) is said to be globally attractive if any two solutions $x_1(t)$ and $x_2(t)$ with the initial values $x_1(0)$ and $x_2(0)$ satisfies

$$\lim_{t \to +\infty} |x_1(t) - x_2(t)| = 0 \quad \text{a.s.}$$

**Lemma 3.10 ([12]):** Suppose that stochastic process $x(t)$ satisfies

$$E|x(t) - x(s)| \leq c|t - s|^{1+\beta}, \quad 0 \leq s, t < \infty,$$

for some positive constants $\alpha, \beta$ and $c$. Then there exists a continuous modification $\tilde{x}(t)$ of $x(t)$, and almost every sample path of $\tilde{x}(t)$ is locally but uniformly Hölder continuous with exponent $\theta \in (0, \alpha/\beta)$.

**Lemma 3.11:** Let $y(t)$ be the solution of (6) for any initial values $y(0) = y_0 > 0$. Under Assumption 3.1, almost every sample path of $y(t)$ is uniformly continuous for $t \geq 0$.

**Lemma 3.12 ([4]):** Let $f$ be a non-negative function defined on $\mathbb{R}^+$ such that $f$ is integrable on $\mathbb{R}^+$ and is uniformly continuous on $\mathbb{R}^+$. Then $\lim_{t \to +\infty} f(t) = 0$.

**Theorem 3.13:** Under Assumption 3.1, the model (4) is globally attractive.

**Remark 3.5:** Theorem 3.13 indicates that every two solutions of the model (4) starting from different initial values will converge to each other if the impulsive perturbations are bounded.

### 4. Asymptotic stability in distribution

The concept of stability in distribution means that the solution of stochastic differential equation does not converge to 0 or the equilibrium state, but to some a stationary distribution. Please see [26] for the accurate definition. In this section, we will prove that the model (4) is stability in distribution under some conditions.

Let $\mathcal{P}((\mathbb{R}^+ \times \mathbb{S}))$ be the space of all probability measures on $\mathbb{R}^+ \times \mathbb{S}$, we define the metric $d_L$ in $\mathcal{P}((\mathbb{R}^+ \times \mathbb{S}))$ as

$$d_L(p_i^{(1)}, p_j^{(2)}) = \sup_{f \in \mathbb{L}} \left| \sum_{i=1}^{N} \int_{\mathbb{R}^+} f_i(x)p_i^{(1)}(dx) - \sum_{j=1}^{N} \int_{\mathbb{R}^+} f_j(x)p_j^{(2)}(dx) \right| \quad i, j \in \mathbb{S}$$

where $\mathbb{L} = \{ f_i : \mathbb{R}^+ \to \mathbb{R}^+; i \in \mathbb{S} | |f(x) - f(y)| \leq |x - y|, |f(\cdot)| \leq 1 \}$.

**Definition 4.1:** The model (4) is said to be asymptotically stable in distribution if transition probability density function $(p_1(t, x, \cdot), p_1(t, x, \cdot), \ldots, p_N(t, x, \cdot))$ of the solution $(x(t), r(t))$ converges weakly to a probability measure $(\pi_1(\cdot), \pi_2(\cdot), \ldots, \pi_N(\cdot))$ of the probability measures space as $t \to \infty$. 
Lemma 4.2: Let $x(t)$ be the solution of the model (4) with the initial value $x(0) = x$, $r(0) = i$, then under Assumption 3.1, for any $p > 0$, $x(t)$ satisfies that

$$
\sup_{0 \leq t < \infty} E[|x(t)|^p] < \infty, \quad (x, i) \in \mathbb{R}^+ \times \mathbb{S}. \tag{9}
$$

**Proof:** For $t \in (t_{k-1}, t_k)$, $k = 0, 1, \ldots, N$, $x(t)$ satisfies that

$$
\text{dx}(t) = a(t)x(t) - b(t)x^2(t) \, dt + \sigma(t)x(t) \, dB(t).
$$

Moreover, it is easy to check that $x(t)$ satisfies (9). For $t = t_k$, because $x(t) = x(t_k) = A_k(r(t_k))x(t_k^-) = A_k(r(t_k)) \lim_{t \to t_k^-} x(t)$, so $x(t)$ also satisfies (9) immediately. \(\blacksquare\)

**Theorem 4.3:** Suppose that Assumption 3.1 holds. Then the model (4) is asymptotically stable in distribution.

**Proof:** Let $K \in \mathbb{R}^+$ be any compact set, from the global attractivity of (4) (Theorem 3.13), for any $x, y \in K$, $i, j \in \mathbb{S}$ and $f \in \mathbb{L}$, we can compute that

$$
\sup_{f \in \mathbb{L}} |Ef(x(t), r^i(t)) - Ef(y(t), r^j(t))| \leq \varepsilon \quad \forall \ t \geq T
$$

as the proof of Lemma 5.6 in [26], where $(x(t), r^i(t))$ and $(y(t), r^j(t))$ are the two solutions of (4) with the initial value $(x, i)$ and $(y, j)$, respectively. By the definition of $d_{\mathbb{L}}$, for any $x, y \in K$ and $i, j \in \mathbb{S}$, we have

$$
\lim_{t \to \infty} d_{\mathbb{L}}(p_i(t, x, \cdot), p_j(t, y, \cdot)) = 0. \tag{10}
$$

Furthermore, Lemma 4.2 implies that $\{p(t, 0, \cdot) : t \geq 0\}$ is Cauchy in the space $\mathcal{P}((0, \infty))$ with metric $d_{\mathbb{L}}$. So there is a unique probability measure $\pi_i(\cdot) \in \mathcal{P}(\mathbb{R}^+)$, $i \in \mathbb{S}$ satisfies

$$
\lim_{t \to \infty} d_{\mathbb{L}}(p_i(t, 0, 1, \cdot), \pi_i(\cdot)) = 0. \tag{11}
$$

Therefore, the Equations (10) and (11) imply that

$$
\lim_{t \to \infty} d_{\mathbb{L}}(p_i(t, x, \cdot), \pi_i(\cdot)) \\
\leq \lim_{t \to \infty} [d_{\mathbb{L}}(p_i(t, x, \cdot), p_1(t, 0, \cdot)) + d_{\mathbb{L}}(p_1(t, 0, \cdot), \pi_i(\cdot))] \\
= 0.
$$

By the definition of the weak convergence of probability measures, for any $x \in \mathbb{R}^+$, $i \in \mathbb{S}$ the transition probabilities $\{p_i(t, x, \cdot) : t \geq 0\}$ converge weakly to the probability measure $\pi(\cdot)$. So the model (4) is asymptotically stable in distribution. \(\blacksquare\)

**Remark 4.1:** Theorem 4.3 depends on the existence and uniqueness, the moment boundedness and global attractivity of the solution to the model (4). For the deterministic model (1), it always has a globally stability equilibrium solution $x(t) = a/b$. A very natural question is what does this equilibrium solution become under the noise perturbations?
According to our results, we know that this equilibrium solution becomes a stationary distribution that is stable (stability in distribution) under the perturbations of the white noise, the Markov chain and the impulse.

Stability in distribution implies that the regime-switching diffusion process \((x(t), r(t))\) defined by (4), has a unique stationary distribution \(\mu(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_N(\cdot))\).

For the unique stationary distribution \(\mu(\cdot)\) of the solution process \((x(t), r(t))\) to the model (4), we have the following result according to [13].

**Theorem 4.4:** Under the conditions of Theorem 4.3, the distribution \(\mu(\cdot)\) has a density 
\[
\pi(x) = (\pi_1(x), \pi_2(x), \ldots, \pi_N(x))
\]
with respect to Lebesgue measure in \(\mathbb{R}^N\), where
\[
\mu_i(A_i) = \mathbb{P}\{x(t) \in A_i, r(t) = i\} = \int_{A_i \subseteq \mathbb{R}} \pi_i(x) \, dx.
\]
Moreover, the density \(\pi(x)\) is the unique bounded solution of the equations
\[
\mathcal{L}_i^* \pi_i = \frac{\sigma^2(i)}{2} \frac{d^2}{dx^2} \pi_i x^2 - \frac{d}{dx} \left[ x(a(i) - b(i)x) \pi_i \right] + \sum_{j=1}^{N} \gamma_{ij} \pi_j = 0, \quad i = 1, 2, \ldots, N
\]
satisfying the additional condition
\[
\sum_{j=1}^{N} \int_{-\infty}^{\infty} \pi_j(x) \, dx = 1.
\]

5. **Concluding remarks and examples**

The hybrid system is often used to describe the sudden changes in the process that appeared in various fields of science and technology. This paper investigates the dynamic behaviours of the hybrid stochastic logistic model with impulsive perturbations. The switching may occur at the impulse points in our model. We first give the existence and uniqueness of the global positive solution to the model (4), and then obtain sufficient conditions for the persistence and extinction, stochastic permanence, global attractivity and stability in distribution. Also, we will illustrate some of our results given above in the following two examples.

**Example 5.1:** Consider
\[
dx(t) = [a(r(t))x(t) - b(r(t))x^2(t)] \, dt + \sigma(r(t))x(t) \, dB(t), \quad t \neq t_k
\]
\[
x(t_k) = A_k(r(t_k))x(t_k^-), \quad k \in \mathbb{N}
\]
where the Markov chain \(r(t)\) taking values in \(S = \{1, 2\}\). Then the model (12) can be regarded as a switching between
\[
dx(t) = [a(1)x(t) - b(1)x^2(t)] \, dt + \sigma(1)x(t) \, dB(t), \quad t \neq t_k
\]
\[
x(t_k) = A_k(1)x(t_k^-), \quad k \in \mathbb{N}
\]
and
\[\frac{dx(t)}{dt} = [a(2)x(t) - b(2)x^2(t)] dt + \sigma(2)x(t) dB(t), \quad t \neq t_k\]
\[x(t_k) = A_k(2)x(t_k^-), \quad k \in \mathbb{N}\]
according to the movement of the Markov chain \(r(t)\), where
\[a(1) = 1, \quad b(1) = 0, \quad \sigma(1) = 0.5; \quad a(2) = 2, \quad b(2) = 1, \quad \sigma(2) = \sqrt{5}.

We consider no impulsive perturbations and constant coefficient impulsive perturbations respectively in the model (12).

1. For \(A_k(1) = A_k(2) = 0\), i.e. there are no impulsive perturbations, then the Equations (13) and (14) become
\[\frac{dx(t)}{dt} = [a(1)x(t) - b(1)x^2(t)] dt + \sigma(1)x(t) dB(t)\]
and
\[\frac{dx(t)}{dt} = [a(2)x(t) - b(2)x^2(t)] dt + \sigma(2)x(t) dB(t),\]
respectively. Figure 1(a) shows that as \(b(1) = 0\) and \(a(1) - \frac{1}{2}\sigma^2(1) = 0.875 > 0\), the Equation (15) increase to infinity. On the other hand, Figure 1(b) shows that the Equation (16) is extinctive as \(a(2) - \frac{1}{2}\sigma^2(2) = -0.5 < 0\) which is the result of Theorem 3.2. However, an interesting fact is that under the regime switching of the Markov chain \(r(t)\), the result will be different in the following two cases.
Case 1. Let the generator of the Markov chain $r(t)$ be
\[
\Gamma = \begin{pmatrix}
-1 & 1 \\
2 & -2
\end{pmatrix}.
\]

By solving (5), we obtain the unique solution
\[
\pi = (\pi_1, \pi_2) = \left(\frac{2}{3}, \frac{1}{3}\right)
\]
which is the stationary distribution of Markov chain $r(t)$. Computing
\[
\pi b = \pi_1 b(1) + \pi_2 b(2) = \frac{1}{3} > 0, \quad \pi \beta = \pi_1 \beta(1) + \pi_2 \beta(2) = \frac{5}{12} > 0.
\]
Therefore, Theorem 3.8 implies that the model (12) is stochastic permanent; see Figure 2(a).

Case 2. Let the generator of the Markov chain $r(t)$ be
\[
\Gamma = \begin{pmatrix}
-3 & 3 \\
1 & -1
\end{pmatrix}.
\]

By the same method as shown in Case 1, we get the unique solution of (5) as
\[
\pi = (\pi_1, \pi_2) = \left(\frac{1}{4}, \frac{3}{4}\right).
\]
Computing
\[
\pi b = \pi_1 b(1) + \pi_2 b(2) = \frac{1}{2} > 0, \quad \pi \beta = \pi_1 \beta(1) + \pi_2 \beta(2) = -\frac{0.625}{4} < 0.
\]
By Theorem 3.2, the model (12) is extincive; see Figure 2(b).
(2) For $A_k(1) = A_k(2) = \exp\{0.2\}$, i.e. impulse gain coefficient is a constant, then the Equations (13) and (14) become

$$dx(t) = [a(1)x(t) - b(1)x^2(t)] \, dt + \sigma(1)x(t) \, dB(t), \quad t \neq t_k$$
$$x(t_k) = \exp\{0.2\}x(t^-_k), \quad k \in \mathbb{N}$$

and

$$dx(t) = [a(2)x(t) - b(2)x^2(t)] \, dt + \sigma(2)x(t) \, dB(t), \quad t \neq t_k$$
$$x(t_k) = \exp\{0.2\}x(t^-_k), \quad k \in \mathbb{N}$$

As $b(1) = 0$ and $\ln(A_k(1)) + a(1) - \frac{1}{2}\sigma^2(1) = 1.075 > 0$, the the model (17) increases to infinity, while the model (18) is extinctive since $\ln(A_k(2)) + a(2) - \frac{1}{2}\sigma^2(2) = -0.3 < 0$ from Theorem 3.2, see Figure 3(a) and Figure 3(b), respectively.

In the following, we will consider two cases of the Markov chain $r(t)$ as defined in Case 1 and Case 2 of (1). A more interesting fact is that under the regime switching of the Markov chain $r(t)$ and constant coefficient impulsive perturbations, the population is persistent in mean although it is stochastic permanent and extinctive under the regime switching of Markov chain $r(t)$ without impulsive perturbations.

**Case 1.** Let the Markov chain $r(t)$ as in Case 1 of (1), we can get that

$$A^* = A_* = \frac{74}{12} > 0.$$

By Theorem 3.6, the model (12) is persistent in mean; see Figure 4(a).
Figure 4. The sample paths of the model (12) with Markov chain \( r(t) \) as in (1) and constant impulsive perturbations coefficient \( A_k(1) = A_k(2) = \exp(0.2) \).

Case 2. Let the Markov chain \( r(t) \) as in Case 2 of (1), we obtain that

\[ A^* = A_* = \frac{0.175}{4} > 0. \]

Therefore, Theorem 3.6 implies that the model (12) is persistent in mean; Figure 4(b).

Example 5.2: Consider the the model (12) again, take \( A_k(1) = A_k(2) = \exp((-1)^{k+1}/k^2) \), that is, the impulse gain coefficient is a function of \( k \). Then the Equations (13) and (14) become

\[
dx(t) = [a(1)x(t) - b(1)x^2(t)] \, dt + \sigma(1)x(t) \, dB(t), \quad t \neq t_k
\]

\[
x(t_k) = \exp((-1)^{k+1}/k^2)x(t_k^-), \quad k \in \mathbb{N}
\]

and

\[
dx(t) = [a(2)x(t) - b(2)x^2(t)] \, dt + \sigma(2)x(t) \, dB(t), \quad t \neq t_k
\]

\[
x(t_k) = \exp((-1)^{k+1}/k^2)x(t_k^-), \quad k \in \mathbb{N}
\]

where

\[
a(1) = 3, \quad b(1) = \frac{1}{2}, \quad \sigma(1) = \sqrt{3} \quad a(2) = 1.5, \quad b(2) = \frac{1}{3}, \quad \sigma(2) = \sqrt{2}.
\]

We can calculate that \( \beta(1) = a(1) - \frac{1}{2}\sigma^2(1) = \frac{3}{2} \) and \( \beta(2) = a(2) - \frac{1}{2}\sigma^2(2) = \frac{1}{2} \). Take \( t_k = k \), then \( \exp(0.75) < \prod_{k=1}^{\infty} A_k(\cdot) < e \). Let the generator of the Markov chain \( r(t) \) as in
Figure 5. The sample paths of the model (12) with Markov chain $r(t)$ as in (1) and impulsive perturbations coefficient function $A_k(1) = A_k(2) = \exp\{(-1)^{k+1}/k^2\}$.

Case 1 of (1). Computing $\pi b$ and $\pi \beta$ as

$$\pi b = \pi_1 b(1) + \pi_2 b(2) = \frac{4}{5} > 0, \quad \pi \beta = \pi_1 \beta (1) + \pi_2 \beta (2) = \frac{7}{6} > 0.$$ 

For the Markov chain $r(t)$ as in Case 2 of (1), we get that

$$\pi b = \pi_1 b(1) + \pi_2 b(2) = \frac{3}{5} > 0, \quad \pi \beta = \pi_1 \beta (1) + \pi_2 \beta (2) = \frac{3}{4} > 0.$$ 

Then Theorem 3.8 implies that the model (12) is stochastically permanent; see Figure 5(a) and Figure 5(b).

This paper devotes to studying the dynamic behaviours of a hybrid stochastic single population logistic model, the methods developed here may also applicable to other models, such as stochastic Gompertz model, Lotka-Volterra systems with more species, and we leave these for our future work. The results and some methods given in this paper provide a fundamental work for the further research.

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Appendices

Appendix 1

Proof of Theorem 2.1: Note that (6) is a Itô’s SDE without impulse. From a modification proof of the existence and uniqueness for the positive and global solution to autonomous stochastic logistic equation, we can get that for the initial value \( y_0 = y(0) = x \), the Equation (6) has a positive solution \( y(t) \), which can be expressed as

\[
y(t) = \frac{\exp\left[\int_0^t [a(s) - \frac{1}{2}\sigma^2(s)] \, ds + \int_0^t \sigma(s) \, dB(s)\right]}{\frac{1}{x} + \int_0^t \prod_{0 < \tau < s} A_k(r(\tau)) \, ds + \int_0^t \sigma(r) \, dB(t)}
\]

Let \( x(t) = \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) y(t) \), we can check that \( x(t) \) is a solution to (4) and the proof is similar to Theorem 1 of [32]. In fact, \( x(t) \) is continuous on \( (0, t_1) \) and each interval \( (t_k, t_{k+1}) \subset [0, \infty), \ k \in \mathbb{N}, \) for \( t \neq t_k \)

\[
dx(t) = d \left[ \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) y(t) \right] = \prod_{0 < t_{k-1} \leq t} A_k(r(t_{k-1})) dy(t)
\]

\[
= \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) y(t) \left[ a(r(t)) - b(r(t)) \right] \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) dt
\]

\[
+ \sigma(r(t)) \prod_{0 < t_{k-1} < t} A_k(r(t_{k-1})) y(t) dB(t)
\]

\[
= x(t) [a(r(t)) - b(r(t)) x(t)] dt + \sigma(r(t)) x(t) dB(t)
\]

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For every $t_k \in [0, +\infty)$, we have

$$x(t_k^+) = \lim_{t \uparrow t_k} \prod_{0 < t_{j-1} < t} A_j(r(t_{j-1}))y(t) = \prod_{0 < t_{j-1} < t} A_j(r(t_{j-1}))y(t_k^+)$$

$$= \prod_{0 < t_{j-1} < t_k} A_j(r(t_{j-1}))y(t_k) = x(t_k)$$

Similarly,

$$A_k(r(t_{k-1}))x(t_k^-) = A_k(r(t_{k-1})) \lim_{t \uparrow t_k} \prod_{0 < t_{j-1} < t} A_j(r(t_{j-1}))y(t)$$

$$= A_k(r(t_{k-1})) \prod_{0 < t_{j-1} < t_k} A_j(r(t_{j-1}))y(t_k^-)$$

$$= \prod_{0 < t_{j-1} < t_k} A_j(r(t_{j-1}))A_k(r(t_{k-1}))y(t_k^-)$$

$$= \prod_{0 < t_{j-1} < t_k} A_j(r(t_{j-1}))y(t_k) = x(t_k)$$

So from the definition of the solution for ISDE, we have proved that $(x(t), r(t))$ is a solution to (4) with the initial value $(x, i)$. Following we will prove the uniqueness of the solution to (4). For $t \in (0, t_1)$ and $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}$, the Equation (4) becomes the following classical stochastic differential equation

$$dx(t) = [a(r(t))x(t) - b(r(t))x^2(t)] \, dt + \sigma(r(t))x(t) \, dB(t). \quad (A1)$$

Note that the coefficients of (A1) are local Lipschitz continuous, by uniqueness theorem for SDE (see [26]), the solution to (A1) is unique. For $t = t_k$, $k \in \mathbb{N}$,

$$x(t) = x(t_k) = x(t_k^+) = \lim_{t \to t_k^+} x(t), \quad t \in (t_k, t_{k+1}), \quad k \in \mathbb{N}$$

exists and unique. So the uniqueness of the solution to (4) is obviously.

Moreover, if $x(t)$ is a solution to (4), then $y(t) = \prod_{0 < t_{k-1} < t} A_k^{-1}(r(t_{k-1}))x(t)$ is a solution to (6). In fact, since $x(t)$ is right-continuous on each interval $[t_k, t_{k+1})$, so for any $k = 1, 2, \ldots$,

$$y(t_k^+) = \lim_{t \to t_k^+} \prod_{0 < t_{j-1} < t} A_j^{-1}(r(t_{j-1}))x(t_k^+) = \prod_{0 < t_{j-1} < t_k} A_j^{-1}(r(t_{j-1}))x(t_k) = y(t_k).$$

Since $A_k(r(t_{k-1}))x(t_k^-) = x(t_k)$, so

$$y(t_k^-) = \lim_{t \to t_k^-} \prod_{0 < t_{j-1} < t} A_j^{-1}(r(t_{j-1}))x(t_k^-)$$

$$= \prod_{0 < t_{j-1} < t_{k-1}} A_j^{-1}(r(t_{j-1}))A_k^{-1}(r(t_{k-1}))x(t_k)$$

$$= \prod_{0 < t_{j-1} < t_k} A_j^{-1}(r(t_{j-1}))x(t_k)$$

$$= y(t_k), \quad k = 1, 2, \ldots,$$

which implies that $y(t)$ is continuous on $\mathbb{R}^+$, and then we can get that $y(t)$ is a solution to (6). The proof is complete.
Appendix 2

Proof of Theorem 3.2: Let \( V(t, y) = \ln y \), then the generalised Itô’s formula on (6) implies that

\[
\begin{align*}
\frac{d \ln y(t)}{y(t)} &= \frac{dy(t)}{y(t)} - \frac{(dy(t))^2}{2y^2(t)} \\
&= \left[ a(r(t)) - \frac{1}{2}\sigma^2(r(t)) - b(r(t)) \prod_{0 < t_k < t} (A_k(t_k)) y(t) \right] dt + \sigma(r(t)) dB(t) \\
&= \left( a(r(t)) - \frac{1}{2}\sigma^2(r(t)) - b(r(t))x(t) \right) dt + \sigma(r(t)) dB(t). \tag{A2}
\end{align*}
\]

Note that

\[-b(r(t))x(t) \leq -\hat{b}|x(t)| \leq 0,
\]

and

\[a(r(t)) - \frac{1}{2}\sigma^2(r(t)) = \beta(r(t)).\]

Then substituting these two equations into (A2) yields that

\[
\frac{d \ln y(t)}{dt} \leq \beta(r(t)) dt + \sigma(r(t)) dB(t). \tag{A3}
\]

Integrating both sides of (A3) from 0 to \( t \), we obtain that

\[
\ln y(t) \leq \ln y(0) + \int_0^t \beta(r(s)) \, ds + M_1(t), \tag{A4}
\]

where \( M_1(t) = \int_0^t \sigma(r(s)) \, dB(s) \) is a local martingale, whose quadratic variation is \( \langle M_1(t), M_1(t) \rangle = \sigma^2(s) \, ds \). The strong law of large numbers of local martingales \[26\] leads to

\[
\lim_{t \to +\infty} \frac{M_1(t)}{t} = 0, \quad \text{a.s.} \tag{A5}
\]

Then from (A4), one can see that

\[
\sum_{0 < t_k < t} \ln(A_k(r(t_k))) + \ln y(t) \leq \ln y(0) + \sum_{0 < t_k < t} \ln(A_k(r(t_k))) + \int_0^t \beta(r(s)) \, ds + M_1(t).
\]

That is

\[
\ln x(t) \leq \ln x(0) + \sum_{0 < t_k < t} \ln(A_k(r(t_k))) + \int_0^t \beta(r(s)) \, ds + M_1(t). \tag{A6}
\]

It finally follows from (A6) by dividing \( t \) on both sides and then letting \( t \to \infty \) that

\[
\limsup_{t \to +\infty} \frac{\ln x(t)}{t} \leq \limsup_{t \to +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(A_k(r(t_k))) + \int_0^t \beta(r(s)) \, ds \right] \\
= \limsup_{t \to +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(A_k(r(t_k))) \right] + \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \beta(r(s)) \, ds \\
= \limsup_{t \to +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(A_k(r(t_k))) \right] + \sum_{i=1}^N \pi_i \beta(i) \\
=: A^*.
\]

Particularly, if \( A^* < 0 \), then

\[
\limsup_{t \to +\infty} \frac{\ln x(t)}{t} < 0,
\]
that is
\[ \lim_{t \to +\infty} x(t) = 0 \quad \text{a.s.} \]

This completes the proof. ■

**Proof of Theorem 3.4:** For any fixed $\varepsilon > 0$, there exists a constant $T$ such that for all $t \geq T$,
\[ \frac{\ln x(0)}{t} < \frac{\varepsilon}{3}, \quad \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln A_k(r(t_k)) \right] + \sum_{i=1}^{N} \pi_i \beta(i) \leq A^* + \frac{\varepsilon}{3}, \quad \frac{M_1(t)}{t} < \frac{\varepsilon}{3}. \]

(A7)

On the other hand,
\[ \ln x(t) = \ln x(0) + \sum_{0 < t_k < t} \ln A_k(r(t_k)) + \int_0^t \beta(r(s)) \, ds - \int_0^t b(r(s)) x(s) \, ds + M_1(t), \]

(A8)

Substituting (A7) into (A8), we can get that for all $t \geq T$
\[ \ln x(t) \leq (A^* + \varepsilon)t - \hat{b} \int_0^t x(s) \, ds, \]

holds almost surely. Let $h(t) = \int_0^t x(s) \, ds$, then
\[ \exp\{\hat{b}h(t)\} x(t) \leq \exp\{(A^* + \varepsilon)t\}, \quad t \geq T. \]

Integrating both sides of this inequality from $T$ to $t$, we have
\[ \frac{1}{\hat{b}} \left[ \exp(\hat{b}h(t)) - \exp(\hat{b}h(T)) \right] \leq \frac{1}{A^* + \varepsilon} \left[ \exp((A^* + \varepsilon)t) - \exp((A^* + \varepsilon)T) \right]. \]

(A9)

Therefore
\[ h(t) \leq \frac{1}{\hat{b}} \ln \left\{ \exp(\hat{b}(i)h(T)) + \frac{\hat{b}}{A^* + \varepsilon} [\exp((A^* + \varepsilon)t) - \exp((A^* + \varepsilon)T)] \right\}. \]

As a consequence
\[ \limsup_{t \to +\infty} \frac{1}{t} \int_0^t x(s) \, ds \leq \limsup_{t \to +\infty} \frac{1}{\hat{b}} \ln \left\{ \exp(\hat{b}(i)h(T)) + \frac{\hat{b}}{A^* + \varepsilon} [\exp((A^* + \varepsilon)t) - \exp((A^* + \varepsilon)T)] \right\} \]

Then the L'Hospital's rule implies that
\[ \limsup_{t \to +\infty} \frac{1}{t} \int_0^t x(s) \, ds \leq \frac{1}{\hat{b}} \limsup_{t \to +\infty} \frac{1}{t} \left\{ \ln \left[ \frac{\hat{b}}{A^* + \varepsilon} \exp((A^* + \varepsilon)t) \right] \right\} = \frac{A^* + \varepsilon}{\hat{b}}. \]

The required assertion follows from the arbitrariness of $\varepsilon$. ■

**Proof of Theorem 3.5:** If the assertion (7) is not true, then $P\{\omega : \limsup_{t \to +\infty} x(s, \omega) = 0\} > 0$, that is, there exist $\omega$ such that $\limsup_{t \to +\infty} x(s, \omega) = 0$. So
\[ \limsup_{t \to +\infty} \frac{\ln x(t) - \ln x(0)}{t} \leq 0. \]
It then follows from (A8) that
\[
\frac{\ln x(t) - \ln x(0)}{t} = \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(A_k(r(t_k))) + \int_{0}^{t} \beta(r(s)) \, ds \right] \\
- \frac{1}{t} \int_{0}^{t} b(r(s)) x(s) \, ds + \frac{M_1(t)}{t} \\
\leq 0. \tag{A10}
\]

Moreover,
\[
\limsup_{t \to +\infty} \frac{1}{t} \int_{0}^{t} b(r(s)) x(s) \, ds = 0
\]
since the boundedness of \( b(r(t)) \). Therefore,
\[
0 \geq \limsup_{t \to +\infty} \frac{1}{t} \left[ \ln x(t) - \ln x(0) \right] = \limsup_{t \to +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(A_k(r(t_k))) + \int_{0}^{t} \beta(r(s)) \, ds \right] \\
= \limsup_{t \to +\infty} \frac{1}{t} \left[ \sum_{0 < t_k < t} \ln(A_k(r(t_k))) \right] + \sum_{i=1}^{N} \pi_i \beta(i) \\
=: A^*
\]

This is contradict with \( A^* > 0 \). This completes the proof. \( \blacksquare \)

**Proof of Theorem 3.6:** We suppose that \( A_* > 0 \) in the following proof, then for \( \forall \varepsilon > 0 \), there exists a positive \( T \) such that for all \( t > T \)
\[
\frac{1}{t} \left[ \sum_{0 < t_k < t} \ln A_k(r(t_k)) \right] + \sum_{i=1}^{N} \pi_i \beta(i) \geq A_* - \varepsilon, \quad \frac{M_1(t)}{t} \geq \varepsilon, \quad \frac{\ln x(0)}{t} \geq \varepsilon.
\]
Substituting these into (A8) and noting that
\[
\frac{1}{t} \int_{0}^{t} b(r(s)) x(s) \, ds \leq \frac{1}{t} \int_{0}^{t} \tilde{b} x(s) \, ds.
\]
Therefore,
\[
\ln x(t) \geq (A_* - \varepsilon) t - \tilde{b} \int_{0}^{t} x(s) \, ds, \quad t > T.
\]
Let \( g(t) := \int_{0}^{t} x(s) \, ds \), then for \( t \leq T \), \( \exp[\tilde{b} \int_{0}^{t} x(s) \, ds] \geq \exp[(A_* - \varepsilon) t] \). Integrating this inequality from \( T \) to \( t \), the following proof is similar as Theorem 2.1, and then we can get the required assertion. \( \blacksquare \)

**Proof of Lemma 3.11:** We rewrite (6) as the following integral equation
\[
y(t) = y_0 + \int_{0}^{t} y(s) \left[ a(r(s)) - \prod_{0 < t_k < s} A_k(r(t_k)) b(r(s)) y(s) \right] \, ds + \int_{0}^{t} \sigma(r(s)) y(s) \, dB(s).
\]
For arbitrary $q > 0$, it is easy to see that

$$E\left| y(t)[a(r(t)) - \prod_{0<k<t} A_k(r(t))]\right|^q$$

$$= E\left[ |y(t)|^q a(r(t)) - \prod_{0<k<t} A_k(r(t)) b(r(t)) y(t) \right]^q$$

$$\leq \frac{1}{2} E|y(t)|^{2q} + \frac{1}{2} E\left| a(r(t)) - \prod_{0<k<t} A_k(r(t)) b(r(t)) y(t) \right|^{2q}$$

By the moment inequality for stochastic integral, for any $0 \leq t_1 \leq t_2$, we can get that

$$E\left| \int_{t_1}^{t_2} \sigma(r(s)) y(s) dB(s) \right|^q \leq (\tilde{E})^{2q} \left[ \frac{q(q-1)}{2} \right]^{q/2} (t_2 - t_1)^{(q-2)/2} \left| \int_{t_1}^{t_2} E|y(s)|^q ds \right|$$

For any $t \geq 0$, as Theorem 6 of [21], we have $E|y(t)|^q \leq G(q)$, $G(q)$ is related to $q$. Then for $0 < t_1 < t_2 < \infty$, one can derive that

$$E(|y(t_2) - y(t_2)|^q) = \left| \int_{t_1}^{t_2} y(s) \left[ a(r(s)) - \prod_{0<k<s} A_k(r(t)) b(r(s)) y(s) \right] ds \right|^q$$

$$+ \left| \int_{t_1}^{t_2} \sigma(r(s)) y(s) dB(s) \right|^q$$

$$\leq 2^{q-1} (t_2 - t_1)^{q/2} \left[ 1 + \left( \frac{q(q-1)}{2} \right)^{q/2} \right] K(q)$$

where $K(q)$ is related to $G(q)$. Therefore from Lemma 3.10 that almost every sample path of $y(t)$ is uniformly continuous on $t \geq 0$.

**Proof of Theorem 3.13:** Let $y_1(t)$ and $y_2(t)$ be the solutions of equation

$$dy(t) = y(t) \left[ a(r(t)) - b(r(t)) \prod_{0<k<t} A_k(r(t)) y(t) \right] dt + \sigma(r(t)) y(t) dB(t)$$

with the initial value $y_1(0)$ and $y_2(0)$ respectively. Then $x_1(t) = \prod_{0<k<t} A_k(r(t)) y_1(t)$ and $x_2(t) = \prod_{0<k<t} A_k(r(t)) y_2(t)$ are the solutions to the Equation (4) with the initial values $y_1(0)$ and $y_2(0)$ respectively. Define $V(t) = |\ln y_1(t) - \ln y_2(t)|$. Note that $V(t)$ is continuous and positive on $t \geq 0$. Calculating the right differential of $V(t)$, we get that

$$d^+ V(t) = \text{sgn}(y_1(t) - y_2(t)) d(\ln y_1(t) - \ln y_2(t))$$

$$= \text{sgn}(y_1(t) - y_2(t)) \left[ -b(r(t)) \sum_{0<k<t} A_k(r(t)) (y_1(t) - y_2(t)) \right] dt$$

$$= -b(r(t)) \sum_{0<k<t} A_k(r(t)) |y_1(t) - y_2(t)| dt$$

$$\leq -m \hat{\beta} |y_1(t) - y_2(t)| dt$$
Integrating both sides we get
\[ V(t) \leq V(0) - m\hat{b} \int_0^t |y_1(s) - y_2(s)| \, ds. \]

Rewrite this equation yields
\[ V(t) + m\hat{b} \int_0^t |y_1(s) - y_2(s)| \, ds \leq V(0) = |\ln y_1(0) - \ln y_2(0)| < \infty. \]

Therefore \(|y_1(s) - y_2(s)| \in L^2[0, \infty)\). Subsequently from Lemmas 3.11 and 3.12, we have
\[ \lim_{t \to \infty} |y_1(t) - y_2(t)| = 0. \]

By the Assumption 3.1, we can obtain
\[ \lim_{t \to \infty} |x_1(t) - x_2(t)| = \lim_{t \to \infty} \prod_{0 < t_k < t} (A_k(r(t_k))) |y_1(t) - y_2(t)| \leq M \lim_{t \to \infty} |y_1(t) - y_2(t)| = 0. \]

This completes the proof. ■