THE ESSENTIAL NUMERICAL RANGE AND A THEOREM OF SIMON ON THE ABSORPTION OF EIGENVALUES

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Abstract. Let $A(t)$ be a holomorphic family of self-adjoint operators of type (B) on a complex Hilbert space $\mathcal{H}$. Kato-Rellich perturbation theory says that isolated eigenvalues of $A(t)$ will be analytic functions of $t$ as long as they remain below the minimum of the essential spectrum of $A(t)$. At a threshold value $t_0$ where one of these eigenvalue functions hits the essential spectrum, the corresponding point in the essential spectrum might or might not be an eigenvalue of $A(t_0)$. Our results generalize a theorem of Simon to give a sufficient condition for the minimum of the essential spectrum to be an eigenvalue of $A(t_0)$ based on the rate at which eigenvalues approach the essential spectrum. We also show that the rates at which the eigenvalues of $A(t)$ can approach the essential spectrum from below correspond to eigenvalues of a bounded self-adjoint operator. The key insight behind these results is the essential numerical range which was recently extended to unbounded operators by Bögli, Marletta, and Tretter.

1. Introduction

Let $A(t)$ be a holomorphic family of self-adjoint operators on a complex Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$. For bounded operators, this means that

$$A(t) = A_0 + tA_1 + t^2A_2 + \ldots$$

where each $A_k$ is a bounded self-adjoint operator on $\mathcal{H}$, and there is an $r > 0$ such that the series converges absolutely in norm when $|t| < r$. If $\lambda_0$ is an isolated eigenvalue of $A(t_0)$ with finite multiplicity $m$, Kato-Rellich perturbation theory predicts that there is a holomorphic family of $m$ mutually orthogonal unit vectors $x_j(t) \in \mathcal{H}$ such that each $x_j(t)$ is an eigenvector of $A(t)$ for $1 \leq j \leq m$ and $t$ in a neighborhood of $t_0$, and the corresponding eigenvalues $\lambda_j(t)$ satisfy $\lambda_j(t_0) = \lambda_0$ for each $1 \leq j \leq m$.

This theory was first introduced by Rellich [10] and later refined by Kato [9] and Sz.-Nagy [13]. In addition to bounded self-adjoint families, the theory also applies to certain special families of unbounded self-adjoint operators, the so called type (A) and the more general type (B) holomorphic families [6]. For both type (A) and type (B) families, the operators $A(t)$ are self-adjoint and bounded below. The main difference between the two types is that the domain of $A(t)$ is constant for type (A) families, but this is not assumed for type (B). Kato-Rellich theory applies to isolated eigenvalues with finite multiplicity in both cases, however.

If $\lambda(t)$ is an element in the discrete spectrum of $A(t)$ that depends analytically on $t$ when $t > t_0$, but $\lambda(t)$ approaches an element of the essential spectrum of

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$A(t_0)$ as $t \to t_0^+$, then it is not possible in general to analytically continue $\lambda(t)$ beyond $t_0$, and in fact, $\lim_{t \to t_0^+} \lambda(t)$ may not even be an eigenvalue of $A(t_0)$. Still, there are situations where it might be important to know the rate at which $\lambda(t)$ approaches the essential spectrum or to determine whether or not the limit of $\lambda(t)$ is an eigenvalue of $A(t_0)$. Such problems arise in mathematical physics when one considers perturbations of a Schrödinger operator $-\Delta + tV$ [3, 7, 9] (see also [3] and the references therein). They also arise when considering the boundary curves of the numerical range of a bounded operator [2].

One result, due to Simon [11, Theorem 2.1] is the following.

**Theorem 1.1.** Let $A, B$ be self-adjoint operators on $\mathcal{H}$ with $\mathcal{D}(|A|^{1/2}) \subseteq \mathcal{D}(|B|^{1/2})$. Suppose that $A \geq 0$, $0$ is in the essential spectrum of $A$, and $B$ is relative $A$-form compact, that is, $|B|^{1/2}(A + I)^{-1}|B|^{1/2}$ is compact. Suppose also that $A + tB$ has a largest negative eigenvalue $\mu(t)$ which is non-degenerate (i.e., has multiplicity one) for all $0 < t_0 < t < \epsilon$ for some $t_0$ and $\epsilon > 0$. If $\mu(t)$ converges to $0$ as $t \to t_0^+$, and no other eigenvalue of $A + tB$ converges to $0$, then either

1. $\lim_{t \to t_0^+} \mu(t)/(t - t_0) = 0$, or
2. $0$ is an eigenvalue of $A + t_0B$.

In the later case, suppose that $0$ is not an eigenvalue of $A$. Then $0$ is a simple eigenvalue of $A + t_0B$ and

$$\lim_{t \to t_0^+} \mu(t)/(t - t_0) = \langle B\eta, \eta \rangle$$

where $\eta$ obeys $(A + t_0B)\eta = 0, \|\eta\| = 1$. (In particular, if $0$ is not an eigenvalue of $A$, then $\lim_{t \to t_0^+} \mu(t)/(t - t_0) \neq 0$ if and only if $0$ is an eigenvalue of $A + t_0B$.)

Note that the operator sum $A + tB$ may not be self-adjoint if $B$ is not bounded, so the expression $A + tB$ in the theorem above is meant to denote a self-adjoint extension of the operator sum of $A$ and $tB$. Such an extension is always possible, and in fact forms a self-adjoint holomorphic family of type (B). In this paper, we generalize Theorem [11] to any self-adjoint holomorphic family of type (B), denoted $A(t)$. Our results describe the rates at which the eigenvalues of $A(t)$ approach the minimum of the essential spectrum of $A(t_0)$ from below as $t \to t_0^+$. Specifically, we prove that there is a bounded self-adjoint operator $B_0$ such that if $\lambda(t)$ is an eigenvalue of $A(t)$ below the essential spectrum that depends continuously on $t$ when $t_0 < t < \epsilon$, if $\Sigma(t)$ is the minimum of the essential spectrum of $A(t)$ for any $t$, and if $\lim_{t \to t_0^+} \lambda(t) = \Sigma(t_0)$, then $\lim_{t \to t_0^+} (\lambda(t) - \Sigma(t_0))/(t - t_0)$ is an eigenvalue of $B_0$. We also give a sufficient condition for the minimum of the essential spectrum to be an eigenvalue of $A(t_0)$ at a threshold value $t_0$ where eigenvalues from the discrete spectrum are absorbed by the essential spectrum. Our results also generalize some other previously known results, see [7, Theorem 1.1] and [11, Theorem 3.1].

In order to prove these results, we use the essential numerical range of an auxiliary sesquilinear form determined by the type (B) family $A(t)$. The essential numerical range was introduced by Stampfli and Williams for bounded operators in [12]. Recently, Bögli, Marletta, and Tretter extended the notion of essential numerical range to unbounded operators and sesquilinear forms [1]. They also apply it to the problem of spectral pollution. Their extension of the essential numerical range to unbounded operators is what allows us to generalize Theorem [11].
The paper is organized as follows. In Section 2, we introduce the essential numerical range for operators and for sesquilinear forms. We also prove that the essential numerical range of a densely defined unbounded operator is either empty or unbounded, which appears to be a new result. In Section 3, we review type (B) holomorphic families of self-adjoint operators and their properties. Our main results are detailed in Section 4. Section 5 contains a minor observation about analytic perturbations of isolated eigenvalues with infinite multiplicity, and we conclude with examples in Section 6.

Throughout the paper, we will use the following notation. \( H \) will denote a complex Hilbert space. All operators are assumed to act on \( H \), unless stated otherwise. For an operator \( T \), the domain of \( T \) will be denoted \( \mathcal{D}(T) \) and the spectrum of \( T \) is \( \sigma(T) \). If \( x_n \in H \) converges to \( x \) in norm, we write \( x_n \rightarrow x \) and if \( x_n \) converges weakly to \( x \), then we write \( x_n \rightharpoonup x \). For a sesquilinear form \( t \) on \( H \), we write \( t[x,x] \) as shorthand for \( t[x,x] \). We denote the closure of a set \( A \) by \( \overline{A} \) and the convex hull by \( \text{conv} \ A \).

2. The Essential Numerical Range

For an operator \( T \) with domain \( \mathcal{D}(T) \subseteq H \), the numerical range of \( T \) is the set
\[
W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{D}(T) \text{ with } \|x\| = 1 \}.
\]
According to the Toeplitz-Hausdorff theorem, the numerical range of an operator is always a convex set. The essential numerical range of \( T \) is the set
\[
W_e(T) = \{ \lambda \in \mathbb{C} : \exists x_n \in \mathcal{D}(T) \text{ with } \|x_n\| = 1, x_n \rightharpoonup 0, \langle Tx_n, x_n \rangle \rightarrow \lambda \}.
\]
There are several alternative definitions of essential spectrum, but we will follow [1] and use the following:
\[
\sigma_e(T) = \{ \lambda \in \mathbb{C} : \exists x_n \in \mathcal{D}(T) \text{ with } \|x_n\| = 1, x_n \rightharpoonup 0, \| (T - \lambda)x_n \| \rightarrow 0 \}.
\]
The following lemma collects some of the basic properties of the essential numerical range. The first two assertions are [1 Proposition 2.2], while the third follows immediately from the definition of \( W_e(T) \).

**Lemma 2.1.** Let \( T \) be an operator (unbounded or bounded) on \( H \). Then

1. \( W_e(T) \) is closed and convex.
2. \( \sigma_e(T) \subseteq W_e(T) \)
3. \( W_e(cT) = cW_e(T) \) for all \( c \in \mathbb{C} \).

If \( A \) is a self-adjoint operator on \( H \) that is bounded below, then \( \min \sigma_e(A) = \min \sigma_e(A) \) [1 Theorem 3.8].

The following result shows that pre-images of points in the numerical range have a kind of continuity property. We need it to prove an important fact about the essential numerical range. A weaker version appeared as [2 Theorem 2] for matrices in \( \mathbb{C}^{n \times n} \) and for bounded linear operators as [8 Proposition 3.2].

**Theorem 2.2.** Let \( T \) be an operator on \( H \). Let \( z = \langle Tx, x \rangle \) for some \( x \in \mathcal{D}(T) \) with \( \|x\| = 1 \). For any \( 0 < \epsilon < 1 \), the set
\[
\{ \langle Ty, y \rangle : y \in \mathcal{D}(T), \|y\| = 1, \text{ and } |\langle x, y \rangle|^2 \geq 1 - \epsilon \}
\]
contains \( \epsilon W(T) + (1 - \epsilon)z \).
Proof. It suffices to prove the theorem for the compression of \( T \) onto any two dimensional subspace of \( D(T) \) that contains \( x \). Therefore we will assume that \( \mathcal{H} = \mathbb{C}^2 \). In that case, Davis [3] observed that the numerical range of \( T \) is the image of the set \( S = \{yy^* : y \in \mathbb{C}^2, \|y\| = 1 \} \) under the linear transformation \( F : X \mapsto \text{tr}(TX) \), and the set \( S \) is a sphere contained in the 3-dimensional real affine space of 2-by-2 Hermitian matrices with trace equal to one.

The set of 2-by-2 Hermitian matrices has inner product \( \langle X, Y \rangle = \text{tr}(XY) \). For \( x, y \in \mathbb{C}^2 \),
\[
\langle xx^*, yy^* \rangle = \text{tr}(xx^*yy^*) = x^*y \text{tr}(xy^*) = |\langle x, y \rangle|^2.
\]
Let \( C = \{yy^* : y \in \mathbb{C}^2, \|y\| = 1, \text{ and } |\langle x, y \rangle|^2 \geq 1 - \epsilon \} \). Thus \( C \) is the spherical cap formed by intersecting \( S \) with the half-space \( H \) consisting of all 2-by-2 Hermitian matrices \( Y \) such that \( \langle Y, xx^* \rangle \geq 1 - \epsilon \). The set \( \epsilon S + (1 - \epsilon)xx^* \) is contained in both the convex hull of \( S \) and in the half-space \( H \), therefore it is contained in the convex hull of \( C \). It is not hard to show that the image of the spherical cap \( C \) under the linear transformation \( F \) is a convex set [2, Lemma 3]. Therefore
\[
\epsilon W(T) + (1 - \epsilon)z = F(\epsilon S + (1 - \epsilon)xx^*) \subset F(\text{conv } C) = F(C)
\]
\[
= \{(Ty, y) : y \in \mathbb{C}^2, \|y\| = 1, \text{ and } |\langle x, y \rangle|^2 \geq 1 - \epsilon \}.
\]
\[ \square \]

Theorem 2.2 lets us prove the following generalization of [1, Proposition 2.4].

**Theorem 2.3.** Let \( T \) be an unbounded operator on \( \mathcal{H} \). Suppose \( w \in W_c(T) \) and \( z_n \) is a sequence in \( W(T) \) such that \( |z_n| \to \infty \). If \( \frac{z_n - w}{|z_n - w|} \) has a limit point \( v \), then the ray \( \{w + tv : t \geq 0\} \) is contained in \( W_c(T) \).

**Proof.** Since \( w \in W_c(T) \), there is a sequence \( x_n \in D(T) \) with \( \|x_n\| = 1, \langle Tx_n, x_n \rangle \to w \) and \( x_n \xrightarrow{w} 0 \). Let \( w_n = \langle Tx_n, x_n \rangle \). For any \( t > 0 \), let \( \epsilon_n = t/|z_n - w_n| \). Then \( \epsilon_n \to 0 \) as \( n \to \infty \). We can assume that \( \epsilon_n < 1 \) for all \( n \). Observe that
\[
w_n + t\frac{z_n - w_n}{|z_n - w_n|} = w_n + \epsilon_n(z_n - w_n) \in \epsilon W(T) + (1 - \epsilon)w_n
\]
for all \( n \). By Theorem 2.2, there is a \( y_n \in D(T) \) with \( \|y_n\| = 1 \) such that \( \langle Ty_n, y_n \rangle = w_n + t\frac{z_n - w_n}{|z_n - w_n|} \) and \( |\langle x_n, y_n \rangle|^2 \geq 1 - \epsilon_n \). Since \( \langle T(\omega y_n), \omega y_n \rangle = \langle Ty_n, y_n \rangle \) for all unimodular constants \( \omega \in \mathbb{C} \), we may assume that \( |\langle x_n, y_n \rangle| > 0 \). Then \( \|x_n - y_n\|^2 \leq 2 - 2\sqrt{1 - \epsilon_n} \). This means that \( \|x_n - y_n\| \to 0 \) and therefore \( y_n \xrightarrow{w} 0 \). Then, since \( \langle Ty_n, y_n \rangle = w_n + t\frac{z_n - w_n}{|z_n - w_n|} \) has a subsequence converging to \( w + tv \), we conclude that \( w + tv \in W_c(T) \). \[ \square \]

One consequence of Theorem 2.3 is that if \( W_c(T) \) is nonempty and \( W(T) \) is unbounded, then \( W_c(T) \) is also unbounded. It is possible, however, for \( W(T) \) to be unbounded and \( W_c(T) \) to be empty, see [1, Example 2.6]. It is well known that the numerical range of a densely defined unbounded operator is unbounded. This can be seen from the polarization identity
\[
\langle Tx, y \rangle = \frac{1}{4}((\langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle
\]
\[+ i \langle T(x + iy), x + iy \rangle - i \langle T(x - iy), x - iy \rangle) \text{ for all } x, y \in D(T).\]
Therefore Theorem 2.3 implies the following.

**Corollary 2.4.** If \( T \) is a densely defined unbounded operator on a complex Hilbert space \( \mathcal{H} \), then \( W_c(T) \) is either empty or unbounded.
The essential numerical range can also be defined for sesquilinear forms. Let $t$ be a sesquilinear form on $\mathcal{H}$ with domain $\mathcal{D}(t)$. The numerical range $W(t)$ and the essential numerical range $W_e(t)$ are defined as

$$W(t) = \{ t[x] : x \in \mathcal{D}(t), \|x\| = 1 \}$$

and

$$W_e(t) = \{ z \in \mathbb{C} : \exists x_n \in \mathcal{D}(t) \text{ with } \|x_n\| = 1, x_n \xrightarrow{w} 0, t[x_n] \to z \}.$$

The definition of the essential numerical range for forms was introduced in [1]. Note that both the numerical range and the essential numerical range of a sesquilinear form are convex, and the essential numerical range is closed. In addition, Theorem 2.3 applies to $W_e(t)$ for any sesquilinear form $t$, and the proof is the same.

For a sesquilinear form $t$, the adjoint $t^*$ is defined by

$$t^*[x,y] = \overline{t[y,x]}$$

for all $x,y \in \mathcal{D}(t)$ and $\mathcal{D}(t^*) = \mathcal{D}(t)$. A sesquilinear form $t$ is symmetric if $t = t^*$. The real and imaginary parts of a form $t$ are

$$\text{Re } t = \frac{1}{2}(t + t^*) \text{ and } \text{Im } t = \frac{1}{2i}(t - t^*).$$

Unlike with unbounded operators, it is always possible to define the real and imaginary part of a sesquilinear form, and the domains of the real and imaginary parts are the same as the domain of the original form.

A sesquilinear form $t$ is sectorial if there is a vertex $v \in \mathbb{C}$ and an angle $\theta < \pi/2$ such that $|\arg(z-v)| \leq \theta$ for all $z \in W(t)$. Likewise, an operator $T$ on $\mathcal{H}$ is sectorial if the sesquilinear form $q_T$ defined by $q_T[x,y] = \langle Tx, y \rangle$ with domain $\mathcal{D}(q_T) = \mathcal{D}(T)$ is sectorial. The following lemma is essentially [1] Theorem 3.10(ii), although there it is stated for $m$-sectorial operators, rather than sectorial forms. The proof is the same, however, and we include it because it is short.

**Lemma 2.5.** Let $t$ be a sectorial sesquilinear form on $\mathcal{H}$. Then $\text{Re } W_e(t) = W_e(\text{Re } t)$.

**Proof.** If $z \in W_e(t)$, then there is a sequence $x_n \in \mathcal{D}(t), \|x_n\| = 1, x_n \xrightarrow{w} 0$ such that $t[x_n] \to z$. Then $(\text{Re } t)[x_n] \to \text{Re } z$ so $\text{Re } z \in W_e(\text{Re } t)$. This proves that $\text{Re } W_e(t) \subseteq W_e(\text{Re } t)$. Conversely, suppose that $a \in W_e(\text{Re } t)$. Let $x_n \in \mathcal{D}(t)$ be a sequence with $\|x_n\| = 1$ and $x_n \xrightarrow{w} 0$ such that $(\text{Re } t)[x_n] \to a$. Since $t$ is sectorial, $(\text{Im } t)[x_n]$ is bounded. We may pass to a subsequence such that $(\text{Im } t)[x_n]$ converges to some $b$, and then $t[x_n] \to a + ib \in W_e(t)$. Therefore $W_e(\text{Re } t) \subseteq \text{Re } W_e(t)$. \[ \square \]

Suppose that $t$ is a sectorial form on $\mathcal{H}$. We say that $t$ is closed if for any sequence $x_n \in \mathcal{D}(t)$ such that $x_n$ converges to $x \in \mathcal{H}$ and $t[x_n - x_m] \to 0$ when $n,m \to \infty$, we have $x \in \mathcal{D}(t)$. A sectorial form is closable if it has a closed extension. The closure of a sectorial form $s$ is the smallest closed extension $\tilde{s}$, and its domain is the set of $x,y \in \mathcal{H}$ such that there exist sequences $x_n,y_n \in \mathcal{D}(s)$ such that $x_n \to x$, $y_n \to y$, and $s[x_n - x_m] \to 0$, $s[y_n - y_m] \to 0$ as $m,n \to \infty$. For any such $x$ and $y$, $t$ is defined by

$$t[x,y] = \lim_{n \to \infty} s[x_n,y_n].$$

If $t$ is a closed sesquilinear form, then a subspace $V \subseteq \mathcal{D}(t)$ is a core of $t$ if the closure of the restriction of $t$ to $V$ is $t$. See [3] Chapter VI for details. Note that if $t$ is the closure of a sesquilinear form $s$, then $W_e(t) = W_e(s)$. This was observed in [1], and is easy to verify from the definition above.
The following technical lemma will be needed for the proof of our main result, Theorem 4.1.

**Lemma 3.1.** Let $a$ be a densely defined, closed, symmetric sesquilinear form on $\mathcal{H}$ that is bounded below. Let $x, y \in D(a)$. Suppose that $x_n \in D(a)$ is a sequence that weakly converges to $x$ and $a[x_n]$ is bounded. Then $\lim_{n \to \infty} a[x_n, y] = a[x, y]$.

**Proof.** Since $a$ is bounded below, there is a constant $c \in \mathbb{R}$ such that $a + c$ is nonnegative. Then, by the second representation theorem for sesquilinear forms [6, Theorem VI.2.23], there is a nonnegative self adjoint operator $A$ such that $a[x, y] = \langle A^{1/2}x, A^{1/2}y \rangle - c \langle x, y \rangle$ for all $x, y \in D(a)$ and $D(a) = D(A^{1/2})$. Since $\|x_n\|$ and $a[x_n]$ are bounded, it follows that $\|A^{1/2}x_n\|$ is bounded. So $A^{1/2}x_n$ must have a weak limit point $u \in \mathcal{H}$. Since the graph of $A^{1/2}$ is closed in the weak topology, the pair $(x, u)$ must be in the graph, so $u = A^{1/2}x$ and $A^{1/2}x_n \xrightarrow{w} A^{1/2}x$. This implies that $a[x_n, y] = \langle A^{1/2}x_n, A^{1/2}y \rangle - c \langle x_n, y \rangle$ converges to $\langle A^{1/2}x, A^{1/2}y \rangle - c \langle x, y \rangle = a[x, y]$. \hfill \qed

## 3. Perturbations of Self-Adjoint Operators

Let $A$ be a self-adjoint operator on $\mathcal{H}$ that is bounded below. The sesquilinear form $q_A$ defined by $q_A[x, y] = \langle Ax, y \rangle$ with domain $D(q_A) = D(A)$ is closable [6, Theorem VI.1.27]. Let $a$ denote its closure. Then $a$ is a closed, densely defined, symmetric form that is bounded below. If $\xi < \min \sigma(A)$, then $D(a) = D((A - \xi)^{1/2})$, and

$$a[x, y] = \langle (A - \xi)^{1/2}x, (A - \xi)^{1/2}y \rangle + \xi \langle x, y \rangle$$

for all $x, y \in D(a)$. This follows immediately from the second representation theorem for sesquilinear forms [6, Theorem VI.2.23 and Problem VI.2.25]. We will refer to the form $a$ as the closed sesquilinear form corresponding to $A$. The correspondence also works the other way. That is, for any densely defined, closed, symmetric, sesquilinear form $a$ there is a unique self-adjoint operator $A$ such that $D(A)$ is a core for $a$ [6, Theorems VI.2.1 and VI.2.6]. In particular the essential numerical range of $A$ and $a$ are the same, so $\min \sigma_e(A) = \min W_e(A) = \min W_e(a)$.

A family $a(t)$ of sesquilinear forms defined in a neighborhood of 0 is called holomorphic of type (a) if each $a(t)$ is closed and sectorial, the domain $D(a(t))$ is constant and dense in $\mathcal{H}$, and $a(t)[x, y]$ is a holomorphic function of $t$ in a neighborhood of 0 for all $x, y \in D(a(t))$. Since the domain is constant, we’ll write $D(a)$ in place of $D(a(t))$. If each $a(t)$ is symmetric when $t$ is real, then for every real $t$ in a neighborhood of zero there is a self-adjoint operator $A(t)$ such that $\langle A(t)x, y \rangle = a(t)[x, y]$ for all $x, y \in D(A(t))$, and the domain of each $A(t)$ is a core for $D(a)$. This family $A(t)$ is a holomorphic family of type (B) [6].

**Lemma 3.1.** Let $A(t)$ be a self-adjoint holomorphic family of type (B). Let $a(t)$ be the corresponding type (a) family of sesquilinear forms. There is a constant $R > 0$ and a sequence of symmetric sesquilinear forms $a_k$ such that

$$a(t)[x, y] = a_0[x, y] + ta_1[x, y] + t^2a_2[x, y] + \ldots$$

for all $|t| < R$. Each $a_k$ has $D(a) \subseteq D(a_k)$ and for any $0 < r < R$, there are constants $b, c \geq 0$ such that

$$|a_k[x, y]| \leq \frac{c}{r^k} \sqrt{a_0[x] + b\|x\|^2} \sqrt{a_0[y] + b\|y\|^2}$$

(3.2)
for all \( x, y \in D(\mathfrak{a}) \) and \( k \in \mathbb{N} \). In particular, the series \( (5.1) \) converges absolutely for all \( x, y \in D(\mathfrak{a}) \) when \( |t| < R \).

**Proof.** The self-adjoint operator \( A = A(0) \) is bounded below, so we can choose \( b > 0 \) such that \( -b < \min \sigma(A) \). Then \( (A + bI)^{-1/2} \) is a bounded self-adjoint operator. The family \( B(t) = (A + bI)^{-1/2}A(t)(A + bI)^{-1/2} \) is a bounded-holomorphic family of self-adjoint operators (see the proof of \( \text{[6, Theorem VII.4.2]} \)). Therefore it has a series expansion which converges absolutely in norm for all \( t \) of self-adjoint operators (see Proposition 3.2). Therefore it has a constant \( b < R \), then there are constants \( \|B_k\| < c/r^k \) for all \( k \). Observe that \( \| (A + bI)^{1/2} v \|_2^2 = a_0[v] + b\|v\|_a^2 \) for any \( v \in D(\mathfrak{a}) \). Then (3.2) follows from the Cauchy-Schwarz inequality.

As observed in the proof, the constant \( b \) in (3.2) can be any value such that \( -b < \min \sigma(A) \). It will be convenient for a type (B) family \( A(t) \) to fix such a constant \( b \), and introduce the norm \( \|x\|_a = \sqrt{a_0[x] + b\|x\|_a} \). Then (3.2) becomes

\[
|a_k[x, y]| \leq \frac{c}{
\right]
\]

Since the series in Lemma 3.1 converges absolutely when \( |t| < R \), if we fixed \( 0 < r < R \), there are constants \( M_1, M_2 > 0 \) such that

\[
|a(t)[x, y] - a_0[x, y]| \leq M_1 t \|x\|_a \|y\|_a \tag{3.3}
\]

and

\[
|a(t)[x, y] - (a_0 + ta_1)[x, y]| \leq M_2 t^2 \|x\|_a \|y\|_a \tag{3.4}
\]

for all \( x, y \in D(\mathfrak{a}) \) when \( |t| \leq r < R \). If \( t > 0 \) is sufficiently small so that \( M_1 t \) and \( M_1 t + M_2 t^2 \) are both less than \( \frac{1}{2} \), then (3.3) and (3.4) imply that

\[
\frac{1}{2} \|x\|_a \leq a(t)[x] + b\|x\|_a \leq \frac{3}{2} \|x\|_a \quad \text{and} \quad \frac{1}{2} \|x\|_a \leq (a + ta_1)[x] + b\|x\|_a \leq \frac{3}{2} \|x\|_a \tag{3.5}
\]

for any \( x \in D(\mathfrak{a}) \).

The next result connects the minimum of the essential spectrum of \( A(t) \) with the essential numerical range of an auxiliary sesquilinear form.

**Proposition 3.2.** Let \( A(t) \) be a self-adjoint holomorphic family of type (B) on a complex Hilbert space \( \mathcal{H} \). Let \( \Sigma(t) = \min \sigma_e(A(t)) \) and assume that \( \Sigma(0) = 0 \), let \( a(t) \) be the family of sesquilinear forms corresponding to \( A(t) \), and let \( t = a_0 + ia_1 \) where \( a_0, a_1 \) are defined as in Lemma 3.1. Then \( \lim_{t \to 0^+} \Sigma(t)/t = \omega \) where \( \omega \) is the smallest value such that \( i\omega \in W_e(t) \).
Proof. Observe that $\Sigma(t) = \min W_e(A(t)) = \min W_e(a(t))$. The bound in (3.2) implies that $t$ is sectorial, and also that $(1 - it)t$ is sectorial when $0 < t < \epsilon$ as long as $\epsilon$ is chosen sufficiently small. So

$$\text{Re} W_e((1 - it)t) = W_e(\text{Re}((1 - it)t)) = W_e(a_0 + ta_1)$$

by Lemma 2.5. Let $\Sigma(1) \in \Sigma(0) = 0$, and $W \in (1 - \epsilon) t$ implies that $W$ be the minimum value such that $z, w < W$. That means $W$ is such that both $\Sigma(1)$ and taking the limit, we see that there is a constant $C > 0$ such that $|\Sigma(t) - z| \leq C \epsilon^2$ for all $t$ sufficiently small. Similarly, if we take a sequence $y_n \in D(a)$

At the same time, $t \omega \in \text{Re} W_e((1 - it)t)$, so $\Sigma(1) \leq t \omega$ for all $0 < t < \epsilon$. Therefore all limit points of $\Sigma(1)$ lie between $\omega$ and $\omega'$. But since $\omega'$ can be chosen arbitrarily close to $\omega$, we conclude that $\lim_{t \to 0^+} \Sigma(1)/t = \omega$.

As in the comments after Lemma 3.1, choose $b > 0$ so that $-b < \sigma(A)$. Define $\|x\|_a = \sqrt{a_0|x| + b|x|^2}$. Since $\Sigma(0) = 0$, there is a sequence $x_n \in D(a)$, $x_n \to 0$, $\|x_n\| = 1$, such that $a_0|x_n| \to 0$. That means $\|x_n\|_a \to 1$, and so it follows from (3.3) and (3.4) that $a(t)[x_n]$ and $(a_0 + ta_1)[x_n]$ are bounded. We can take a subsequence of $x_n$ such that both $a(t)[x_n]$ and $(a_0 + ta_1)[x_n]$ converge to points in $w \in W_e(a(t))$ and $z \in W_e(a_0 + ta_1)$, respectively. As long as $t$ is sufficiently small, (3.5) implies that $z, w < \frac{3}{\epsilon^2} - b$ so $\Sigma(t) \leq \frac{3}{\epsilon^2} - b$ and $\Sigma(1) \leq \frac{3}{\epsilon^2} - b$, as well.

Now, take a sequence $x_n \in D(a)$, $x_n \to 0$, $\|x_n\| = 1$ such that $a(t)[x_n] \to \Sigma(t)$. Observe that $\lim_{t \to 0^+} \|x_n\|_a \leq 2(\Sigma(t) + b)$ by (3.5), so $(a_0 + ta_1)[x_n]$ is bounded and we can pass to a subsequence such that $(a_0 + ta_1)[x_n]$ converges to $z \in W_e(a_0 + ta_1)$. By definition, $\Sigma(1) \leq z$. Applying (3.4) to the terms $a(t)[x_n]$ and $(a_0 + ta_1)[x_n]$ and taking the limit, we see that there is a constant $C > 0$ such that $|\Sigma(t) - z| \leq Ct^2$ for all $t$ sufficiently small. Similarly, if we take a sequence $y_n \in D(a)$
such that $y_n \xrightarrow{w} 0$, $\|y_n\| = 1$, and $(a_0 + ta_1)y_n \rightarrow \Sigma_1(t)$, then we have a subsequence of $y_n$ such that $a(t)[y_n] \rightarrow w \in W_\epsilon(a(t))$ and $|\Sigma_1(t) - w| \leq Ct^2$. If $\Sigma_1(t) \geq \Sigma(t)$, then $\Sigma_1(t) - \Sigma(t) \leq z - \Sigma(t) \leq Ct^2$. If, on the other hand, $\Sigma(t) \geq \Sigma_1(t)$, then $\Sigma_1(t) - \Sigma(t) \leq \Sigma_1(t) \leq Ct^2$. Either way, we conclude that $|\Sigma(t) - \Sigma_1(t)| \leq Ct^2$ when $t$ is small, and therefore $\lim_{t \rightarrow 0^+} \Sigma(t)/t = \lim_{t \rightarrow 0^+} \Sigma_1(t)/t = \omega$. □

4. **Main Results**

Let $A(t)$ be a self-adjoint holomorphic family of type (B) on a complex Hilbert space $\mathcal{H}$. Let $a(t)$ be the type (a) family of sesquilinear forms corresponding to $A(t)$, and suppose that $a(t)$ has series expansion $a(t) = a_0 + ta_1 + t^2a_2 + \ldots$ given by Lemma 3.1. If the kernel of $A = A(0)$ is nontrivial, let $P$ denote the orthogonal projection onto the kernel of $A$. Since the quadratic form defined by $x \mapsto a_1[Px]$ is bounded and symmetric, there is a bounded self-adjoint operator $B_0$ defined by the sesquilinear form

$$\langle B_0 x, y \rangle = a_1[Px, Py]$$

for all $x, y \in \mathcal{H}$. [6, Section V.2.1]. With this observation, we are now ready to state our main result.

**Theorem 4.1.** Let $A(t)$ be a self-adjoint holomorphic family of type (B). Let $\Sigma(t) = \min \sigma_-(A(t))$ and assume that $\Sigma(0) = 0$. If $\lambda(t) < \Sigma(t)$ is an eigenvalue of $A(t)$ that depends continuously on $t$ when $0 < t < \epsilon$, $\lim_{t \rightarrow 0^+} \lambda(t) = 0$, and $\lim_{t \rightarrow 0^+} (\lambda(t) - \Sigma(t))/t < 0$, then $0$ is an eigenvalue of $A(0)$. Furthermore, $\lim_{t \rightarrow 0^+} \lambda(t)/t$ exists and is equal to an eigenvalue of the self-adjoint operator $B_0$ defined by (4.1).

**Proof.** Let $\omega = \lim_{t \rightarrow 0^+} \Sigma(t)/t$ and $\beta = \lim_{t \rightarrow 0^+} \lambda(t)/t$. So $\beta < \omega$ by assumption. Note that the limit $\omega$ exists by Proposition 3.2. Let $A = A(0)$ and let $a(t)$ be the type (a) family of sesquilinear forms corresponding to $A(t)$. Suppose that $a(t)$ has series expansion $a(t) = a_0 + ta_1 + t^2a_2 + \ldots$ given by Lemma 3.1. By the Kato-Rellich theory, $\lambda(t)$ is an analytic function of $t$, except possibly at isolated points $0 < t < \epsilon$ where $\lambda(t)$ crosses another eigenvalue function. For each $t$ where $\lambda(t)$ does not have a crossing, any unit eigenvector $x(t)$ of $A(t)$ corresponding to $\lambda(t)$ will have $\lambda'(t) = a_1[x(t)]$. [6, Problem VII.4.19].

If $\lambda(t)/t$ has more than one limit point as $t \rightarrow 0^+$, then $\lambda(t)/t$ will oscillate between $\beta$ and a value greater than $\beta$ infinitely many times, so it will be possible to choose a sequence $t_n \rightarrow 0^+$ such that $\lambda(t_n)/t_n \rightarrow \beta$ and $\lambda'(t_n) \rightarrow \beta'$, $\beta' < \beta$ by the mean value theorem.

Suppose, on the other hand, that $\lambda(t)/t$ converges to $\beta$. Observe that

$$\frac{d}{dt} \lambda(t) \bigg|_{t = t_0} = \frac{1}{t} \left( \lambda'(t) - \frac{\lambda(t)}{t} \right),$$

and therefore for any $0 < t < t_0$, we have

$$\frac{\lambda(t_0)}{t_0} - \frac{\lambda(t)}{t} = \int_t^{t_0} \frac{1}{\tau} \left( \lambda'(\tau) - \frac{\lambda(\tau)}{\tau} \right) d\tau.$$

Fix $\delta > 0$ and suppose that $t_0$ is small enough so that $|\lambda(t)/t \beta| < \delta$ for every $0 < t \leq t_0$. If $\lambda'(t) > \beta + 2\delta$ for almost every $0 < t < t_0$, then

$$2\delta > \frac{\lambda(t_0)}{t_0} - \frac{\lambda(t)}{t} = \int_t^{t_0} \frac{1}{\tau} \left( \lambda'(\tau) - \frac{\lambda(\tau)}{\tau} \right) d\tau \geq \delta \ln \left( \frac{t_0}{t} \right).$$
This is a contradiction, since \( \ln(t_0/t) \to \infty \) as \( t \to 0^+ \). Therefore, for every \( t_0, \delta > 0 \), it is possible to choose \( t \) such that \( 0 < t < t_0 \) and \( \lambda'(t) < \beta + 2\delta \). So we can choose a sequence \( t_n \to 0^+ \) such that \( \lambda(t_n)/t_n \to \beta \) and \( \lim_{n \to \infty} \lambda(t_n) = \beta' \) where \( \beta' \leq \beta + 2\delta < \omega \).

Regardless of whether or not \( \lambda(t)/t \) converges, we have shown that we can construct a sequence \( t_n \to 0^+ \) such that \( \lambda(t_n)/t_n \to \beta \) and \( \lambda'(t_n) \to \beta' < \omega \). For each \( t_n \), let \( \lambda_n = \lambda(t_n) \) and let \( x_n \) be a unit eigenvector of \( A(t_n) \) corresponding to \( \lambda_n \).

By (3.3) and (3.5), \( \lim_{n \to \infty} a_0[x_n] = 0 \). Then \( \lim_{n \to \infty} (a_0 + i\alpha_1)[x_n] = i\beta' \). Since \( \omega \) is the smallest value such that \( i\omega \in W_e(a_0 + i\alpha_1) \) by Proposition 3.2, we see that \( i\beta' \) is outside \( W_e(a_0 + i\alpha_1) \). By passing to a subsequence, we can assume that \( x_n \) converges weakly to some \( x \in H \). Since \( i\beta' \not\in W_e(a_0 + i\alpha_1) \), we conclude that \( x \neq 0 \).

The domain of \( A \) is a core for \( a_0 \) and \( a_0[y] = \langle Ay, y \rangle \) for all \( y \in D(A) \). Therefore there is a sequence \( y_n \in D(A) \) such that \( \|y_n\| = 1 \), \( \|y_n - x_n\| < \frac{1}{n} \), and \( \langle Ay_n, y_n \rangle - a_0[x_n] \to \frac{1}{n} \) for all \( n \). Then

\[
\lim_{n \to \infty} \langle Ay_n, y_n \rangle = \lim_{n \to \infty} a_0[x_n] = 0.
\]

Let \( A_{\pm} \) denote the positive and negative parts of \( A \). For each negative eigenvalue \( \nu \) of \( A \) with multiplicity \( m \), there is a corresponding analytic family of \( m \) mutually orthogonal unit eigenvectors \( v_j(t), \ j = 1, \ldots, m, \) of \( A(t) \) defined in a neighborhood of \( t = 0 \) such that \( v_j(0) \) is an eigenvector of \( A \) with eigenvalue \( \nu \). Since the eigenvectors \( x_n \) correspond to eigenvalues \( \lambda_n \) that converge to 0, \( x_n \) will be orthogonal to each of the \( v_j(t_n) \) when \( n \) is large enough. Therefore \( \lim_{n \to \infty} \langle y_n, v_j(t_n) \rangle = \lim_{n \to \infty} \langle x_n, v_j(t_n) \rangle = 0 \). Since \( A_- \) is compact and self-adjoint, this implies that \( A_- x_n \to 0 \). We also know that \( \langle Ay_n, y_n \rangle \to 0 \), so we must also have \( \langle A_+ y_n, y_n \rangle = \|A_1 y_n\|^2 \to 0 \). Since both \( A_- \) and \( A_+ \) are positive operators, both have graphs that are weakly closed. Then since \( y_n \to x \), we conclude that \( A_- x = 0 \) and \( A_1 x = 0 \). This implies that \( x \in D(A) \) and \( Ax = 0 \), so 0 is an eigenvalue of \( A \).

Since \( P \) is the orthogonal projection onto the kernel of \( A \), we have \( Px = x \). For any \( y \in H \) we calculate

\[
\langle x, y \rangle = \langle Px, y \rangle = \langle x, Py \rangle = \lim_{n \to \infty} \langle x_n, Py \rangle = \lim_{n \to \infty} \langle A(t_n)x_n, Py \rangle / \lambda_n \]

Note that \( \langle A(t_n)x_n, Py \rangle = a(t_n)[x_n, Py] \). Since \( a_0[x_n] \to 0 \), (3.4) implies that there is a constant \( C > 0 \) such that

\[
|a(t_n)[x_n, Py] - (a_0 + t_n a_1)[x_n, Py]| \leq Ct_n^2
\]

for all \( n \). Therefore,

\[
\langle x, y \rangle = \lim_{n \to \infty} \langle A(t_n)x_n, Py \rangle / \lambda_n = \lim_{n \to \infty} (a_0 + t_n a_1)[x_n, Py]/\lambda_n = \lim_{n \to \infty} (a_0[x_n, Py] + t_n a_1[x_n, Py])/\lambda_n.
\]

The sesquilinear form \( a_0 + t a_1 \) is closed when \( t \) is small by [6, Theorem VI.1.33]. By Lemma 2.6, \( \lim_{n \to \infty} (a_0 + t a_1)[x_n, Py] = (a_0 + t a_1)[x, Py] \) for all \( 0 \leq t < \epsilon \). Therefore \( \lim_{n \to \infty} a_1[x_n, Py] \to a_1[x, Py] \). Since \( D(A) \) is a core for \( a_0 \) and
\(a_0[v, Py] = (Av, Py) = (PAv, y) = 0\) for all \(v \in \mathcal{D}(A)\), we can use a limiting argument to show that \(a_0[x_n, Py] = 0\) for all \(n \in \mathbb{N}\). Then, continuing the expansion of \((x, y)\) from above, we have,

\[
\langle x, y \rangle = \lim_{n \to \infty} \langle a_0[x_n, Py] + t_n a_1[x_n, Py] \rangle / \lambda_n
\]

which implies that \(<B_0x, y> = 0\) for all \(y \in \mathcal{D}(A)\). This also implies that \(<B_0x, x> = \beta \|x\|^2\), so \(x\) has eigenvalue \(\beta\).

Remark 4.2. In many examples, there is a series expansion for the self-adjoint holomorphic family in Theorem 4.1, that is,

\[
A(t) = A + tA_1 + t^2A_2 + \ldots
\]

which converges for all \(x \in \mathcal{D}(A)\) when \(|t|\) is sufficiently small. This is true for bounded families and also when \(A(t)\) is type (A) or type (B) (see [6]). In these cases \(B_0 = PA_1P\).

Corollary 4.3. Let \(A(t)\) be a self-adjoint holomorphic family of type (B). Let \(\Sigma(t) = \min \sigma_e(A(t))\). If \(\lambda(t) = \Sigma(t)\) is an eigenvalue of \(A(t)\) that depends continuously on \(t\) when \(t_0 < t < \epsilon\), then \(\lambda(t) = \Sigma(t)\), and \(\lim_{t \to t_0} (\lambda(t) - \Sigma(t))/(t - t_0) < 0\), then \(\Sigma(t_0)\) is an eigenvalue of \(A(t_0)\). Furthermore, \(\lim_{t \to t_0} (\lambda(t) - \Sigma(t))/(t - t_0)\) exists and is equal to an eigenvalue of the self-adjoint operator \(B_0\) defined by (4.1) when \(P\) is the orthogonal projection onto the kernel of \(A(t_0) - \Sigma(t_0)\).

Proof. Apply Theorem 4.1 to the type (B) self-adjoint family \(A(t)\) defined by (4.4).

If the operator \(A = A(0)\) in Theorem 4.4 has only a finite number of negative eigenvalues, then we can say more about how the minimal eigenvalues of \(A(t)\) behave as \(t\) approaches zero from above.

Theorem 4.4. With the same conditions and notation as Theorem 4.4, suppose in addition that \(A = A(0)\) has exactly \(m < \infty\) negative eigenvalues counting multiplicity. Let \(\omega = \lim_{t \to 0^+} \Sigma(t)/t\), let \(P\) denote the orthogonal projection onto the kernel of \(A\), and let \(B_0\) be defined as in (4.4). Suppose that the smallest \(k + 1\) eigenvalues of \(B_0\) (counting multiplicity) are all less than \(\omega\). Label these eigenvalues

\[
\mu_0 \leq \mu_1 \leq \ldots \leq \mu_k,
\]

and label the smallest \(m + k + 1\) eigenvalues of \(A(t)\) as

\[
\lambda_{-m}(t) \leq \ldots \leq \lambda_0(t) \leq \lambda_1(t) \leq \ldots \leq \lambda_k(t).
\]

Then \(\lim_{t \to 0^+} \lambda_k(t)/t = \mu_k\). Moreover, if \(x_k(t)\) is a family of unit eigenvectors of \(A(t)\) corresponding to \(\lambda_k(t)\) for \(0 < t < \epsilon\), then for any sequence \(t_n \to 0^+, x_k(t_n)\) has a limit point that is a unit eigenvector of \(B_0\) corresponding to \(\mu_k\).
Proof. Let \( a(t) \) be the type (a) family of sesquilinear forms corresponding to \( A(t) \). Then \( a(t) \) has a series expansion \([3.1]\) by Lemma \([3.1]\). Fix \( b > 0 \) large enough so that \(-b < \min a(A) \) and let \( \|x\|_a = \|a_0[x] + \|x\| \) for all \( x \in D(a) \). Let \( v_j \) denote a unit eigenvector of \( A \) corresponding to the eigenvalue \( \lambda_j(0) \) for \(-m \leq j < 0 \), and let \( v_j \) denote a unit eigenvector of \( B_0 \) corresponding to \( \mu_j \) for \( 0 \leq j < k \). Then \( \lim_{t \to 0^+} a(t)[v_j] = a_0[v_j] = \lambda_j(0) \) when \(-m \leq j < 0 \) and \( (a_0 + ta_1)[v_j] = t\mu_j \) for \( 0 \leq j \leq k \) and all \( 0 < t < \epsilon \). By \([3.4]\) there is a constant \( M_2 > 0 \) such that \( a(t)[v_j] \leq t\mu_j + M_2t^2\|v_j\|_a = \mu_j t + M_2bt^2 \). Therefore the Courant-Fischer-Weyl min-max principal implies that \( \lambda_k(t) \leq t\mu_k + M_2bt^2 \) for all \( t > 0 \) sufficiently small. We also know that \( \lim_{t \to 0^+} \lambda_k(t)/t \) converges by Theorem \([3.1]\). Therefore \( \lim_{t \to 0^+} \lambda_k(t)/t \leq \mu_k \). We just need to prove that \( \lim_{t \to 0^+} \lambda_k(t)/t = \mu_k \).

Let \( 0 < t_0 < \epsilon \). The Kato-Rellich theory says that for any eigenvalue \( \lambda < \Sigma(t_0) \) of \( A(t_0) \), it is possible to find analytic functions \( x(t) \) and \( \lambda(t) \) defined for \( t \) in a neighborhood of \( t_0 \) such that \( x(t) \) is a unit eigenvector of \( A(t) \), \( \lambda(t) \) is the corresponding eigenvalue, and \( \lambda(t_0) = \lambda \). As long as \( \lambda(t) < \Sigma(t) \), the functions \( x(t) \) and \( \lambda(t) \) can be analytically continued. Observe that \( \lambda(t) = (A(t)x(t), x(t)) = a(t)[x(t)] \). We also know that \( \lambda(t) = a_1[x(t)] \) for all \( t \) [6, Section VII.4.6]. By \([3.4]\) \[
|a(t)[x(t)] - (a_0 + ta_1)[x(t)]| < M_2t^2\|x(t)\|_a^2.
\]

Therefore \[
|\lambda(t) - t\lambda'(t) - a_0[x(t)]| < M_2t^2\|x(t)\|_a^2.
\]

This implies that \( \frac{d}{dt}\lambda(t) = \frac{t\lambda'(t) - \lambda(t)}{t^2} \leq -\frac{a_0[x(t)]}{t^2} + M_2\|x(t)\|_a^2. \tag{4.2} \)

For all \( x \in D(A) \), we have \( a_0[x] = \langle Ax, x \rangle = \langle A_+x, x \rangle - \langle A_-x, x \rangle \) where \( A_\pm \) are the positive and negative parts of the self-adjoint operator \( A \), that is, \( A_+ = \frac{1}{2}(|A| + A) \) and \( A_- = \frac{1}{2}(|A| - A). \tag{4.3} \)

Since \( A \) has only a finite number of negative eigenvalues, \( A_- \) is a compact operator. There is also an analytic projection operator \( P(t) \) such that \( P(0) \) is the spectral projection corresponding to the negative eigenvalues of \( A \), and \( P(t) \) is the spectral projection corresponding to the eigenvalues of \( A(t) \) that are in a neighborhood of the negative eigenvalues of \( A(0) \) when \( t > 0 \) is sufficiently small. Let \( Q(t) = I - P(t) \). Then \( Q(t) \) has a power series expansion \( Q(t) = Q_0 + tQ_1 + t^2Q_2 + \ldots \)

that is absolutely convergent in norm when \( |t| < r \) for some \( r > 0 \). This follows from the fact that \( Q(t) \) is bounded-holomorphic using Cauchy’s Inequality, see [6] Chapter VII] for details.

The series expansion for \( Q(t)A_-Q(t) \) is \( Q(t)A_-Q(t) = Q_0A_-Q_0 + (Q_0A_-Q_1 + Q_1A_-Q_0)t + o(t^2). \)

This series is also absolutely converging in norm, and since \( Q_0A_-Q_0 = A_-Q_0 = 0 \), we see that \( \|Q(t)A_-Q(t)\|/t^2 \) is bounded by some constant \( M_0 > 0 \) in a neighborhood of \( t = 0 \).

Let \( \alpha_- \) be the sesquilinear form corresponding to \( A_- \), and define \( \alpha_+ \) to be \( a_0 + \alpha_- \). Then \( D(\alpha_+) = D(a) \) since \( D(\alpha_-) = \mathcal{H} \). Also, \( D(A) \) is a core for \( a_0 \), so it is also a core for \( a_+ \). This implies that \( a_+[x] \geq 0 \) for all \( x \in D(a) \) since \( A_+ \geq 0 \).
Suppose that \( x \in \mathcal{D}(A(t)) \) has \( \|x\| = 1 \), and \( Q(t)x = x \). Then
\[
a_0[x] = a_+[x] - a_-[x] \geq -\langle A_-x, x \rangle = -\langle Q(t)A_-Q(t)x, x \rangle \geq -M_0t^2.
\]
Combined with (4.2) and the fact that \( a_0[x(t)] \) is bounded, this implies that for any eigenvalue \( \lambda(t) < \Sigma(t) \) of \( A(t) \) corresponding to an eigenvector \( x(t) \) such that \( Q(t)x(t) = x(t) \), there is a constant \( M > 0 \) such that
\[
\frac{d}{dt} \lambda(t) \leq M \text{ and } \lambda'(t) \leq \frac{\lambda(t)}{t} + M t
\]when \( t > 0 \) is sufficiently small.

Recall that \( \lambda_k'(t) \) is defined and analytic, except at isolated points where the analytic curves corresponding to the eigenvalues of \( A(t) \) near \( \lambda_k(t) \) cross. Then (4.4) implies that \( \lambda_k'(t) \leq \mu_k + Mt \) wherever it is defined. In particular, if \( \delta > 0 \) is small enough so that \( \mu_k + \delta < \omega \), then for all \( t < \delta/M \), we have \( \lambda_k'(t) \leq \mu_k + \delta < \omega \).

Even at points where \( \lambda_k'(t) \) is not defined, each of the analytic eigenvalue curves that cross \( \lambda_k(t) \) at that point will have a derivative at most \( \mu_k + \delta \). The same argument applies to each \( \lambda_j(t) \) for \( 0 < j < k \). For each \( 0 < t < \delta/M \), we can choose a mutually orthogonal collection of unit eigenvectors \( x_j(t) \) of \( A(t) \) corresponding to \( \lambda_j(t) \) for \( 0 < j < k \). For any sequence \( t_n \to 0^+ \), we can take a subsequence such that each \( x_j(t_n) \) converges weakly to some \( x_j \) for each \( 0 < j < k \). Then the argument of Theorem 4.1 implies that each \( x_j \) is a nonzero eigenvector of \( B_0 \) with an eigenvalue equal to \( \lim_{t \to 0^+} \lambda_j(t)/t \). In addition, the eigenvectors \( x_j \) are mutually orthogonal.

If \( \lim_{t \to 0^+} \lambda_k(t)/t < \mu_k \), then \( B_0 \) has \( k + 1 \) mutually orthogonal eigenvectors with eigenvalues strictly less than \( \mu_k \), but that contradicts the Courant-Fischer-Weyl min-max principal. Therefore we conclude that \( \lim_{t \to 0^+} \lambda_k(t)/t = \mu_k \).

Now consider any sequence \( t_n \to 0^+ \) such that \( x_k(t_n) \) converges weakly to \( x_k \).

We know that \( x_k \) is an eigenvector of \( B_0 \) with eigenvalue equal to \( \mu_k \). Now consider
\[
\mathbf{a}_1[x_k(t_n) - x_k] = \mathbf{a}_1[x_k(t_n)] - \mathbf{a}_1[x_k(t_n), x_k] - \mathbf{a}_1[x_k, x_k(t_n)] + \mathbf{a}_1[x_k].
\]
It was observed in the proof of Theorem 4.1 that \( \lim_{n \to \infty} \mathbf{a}_1[x_k(t_n), x] = \mathbf{a}_1[x_k] = \mu_k \|x_k\|^2 \).

Therefore
\[
\lim_{n \to \infty} \mathbf{a}_1[x_k(t_n)] = \lim_{n \to \infty} \mathbf{a}_1[x_k(t_n) - x_k] + \mu_k \|x_k\|^2.
\]

By construction, \( x_k(t_n) - x_k \xrightarrow{w} 0 \) and a quick calculation shows that
\[
\lim_{n \to \infty} \|x_k(t_n) - x_k\|^2 = 1 - \|x_k\|^2.
\]

We also know that \( \mathbf{a}_1[x_k(t_n) - x_k] \) is bounded by (3.2) since \( \mathbf{a}_0[x_k(t_n) - x_k] \to 0 \).

Therefore we can pass to a subsequence such that \( \mathbf{a}_1[x_k(t_n) - x_k]/\|x_k(t_n) - x_k\|^2 \) converges to some \( z \in W_0(\mathbf{a}_1) \). By Proposition 3.2, \( z \geq \omega \). Then, we have
\[
\lim_{n \to \infty} \mathbf{a}_1[x_k(t_n)] = (1 - \|x_k\|^2)z + \mu_k \|x_k\|^2 \geq \mu_k.
\]
Recall from the Kato-Rellich theory that \( \lambda_k'(t) = \mathbf{a}_1[x_k(t)] \) wherever the derivative is defined. Even where the derivative is not defined because multiple analytic eigenvalue curves cross at \( \lambda_k(t) \), each of the crossing curves will have a derivative at most \( \lambda_k(t)/t + Mt \) by (4.4) and therefore \( \lim_{n \to \infty} \mathbf{a}_1[x_k(t_n)] = \mu_k \).

Then \( \|x_k\| = 1 \), so \( x_k(t_n) \) converges to \( x_k \) in norm.

If \( B \) is a self-adjoint operator on \( \mathcal{H} \) that is not bounded below, it is still possible to define a sesquilinear form corresponding to \( B \). We use the spectral theorem to
decompose $B$ into positive and negative parts, that is, $B = B_+ - B_-$ where $B_\pm$ are given by (4.3). Then define the sesquilinear form $b$ associated with $B$ to be

$$b[x, y] = \left\langle B_{1/2}^+ x, B_{1/2}^+ y \right\rangle - \left\langle B_{1/2}^- x, B_{1/2}^- y \right\rangle,$$

and $\mathcal{D}(b) = \mathcal{D}(|B|^{1/2})$. Theorem 1.1 is a special case of Corollary 4.3 and Theorem 4.4 because the assumption that $B$ is relative $A$-form compact implies that the essential numerical range of the corresponding sesquilinear form $a + ib$ is contained in the ray $[0, \infty)$.

**Lemma 4.5.** Let $A, B$ be self-adjoint operators on a Hilbert space $\mathcal{H}$ with corresponding quadratic forms $a$ and $b$, respectively. Let $t = a + ib$. Suppose that $A \geq 0$ and $B$ is relative $A$-form compact, that is, $|B|^{1/2}(A + I)^{-1}|B|^{1/2}$ is compact. Then $W_e(t) \subset [0, \infty)$ and if $A(t)$ is the type (B) family of self-adjoint operators corresponding to the sesquilinear form $a + tb$, then $\min \sigma_e(A(t)) = 0$ for all $t \in \mathbb{R}$.

**Proof.** Consider any $z \in W_e(t)$. There is a sequence $x_n$ in $\mathcal{D}(t)$ with $\|x_n\| = 1$, $x_n \xrightarrow{w} 0$, such that $t[x_n]$ converges to $z$. In particular $a[x_n]$ is bounded. Note that $\mathcal{D}(a) = \mathcal{D}(A^{1/2}) = \mathcal{D}((A + I)^{1/2})$ (see e.g., [1] Problem VI.2.25) so

$$\left\langle (A + I)^{1/2} x_n, (A + I)^{1/2} x_n \right\rangle = a[x_n] + \|x_n\|^2$$

for all $n$, and therefore $\|(A + I)^{1/2} x_n\|$ is bounded. By passing to a subsequence, we can assume that $y_n = (A + I)^{1/2} x_n$ converges weakly to some $y \in \mathcal{H}$.

Since $x_n = (A + I)^{-1/2} y_n$ and the graph of $(A + I)^{-1/2}$ is weakly closed, it follows that $x_n \xrightarrow{w} (A + I)^{-1/2} y$. We also know that $x_n \xrightarrow{w} 0$, so we conclude that $(A + I)^{-1/2} y = 0$.

For each $n$, let $\tilde{y}_n = y_n - y$. Then $\tilde{y}_n \xrightarrow{w} 0$. Observe that

$$|B|^{1/2} x_n = |B|^{1/2}(A + I)^{-1/2} y_n = |B|^{1/2}(A + I)^{-1/2} \tilde{y}_n + |B|^{1/2}(A + I)^{-1/2} y = |B|^{1/2}(A + I)^{-1/2} \tilde{y}_n$$

This converges to zero because $|B|^{1/2}(A + I)^{-1/2}$ is compact. Therefore $|B|^{1/2} x_n \rightarrow 0$ and so $b[x_n] \rightarrow 0$ which proves that $W_e(t) \subset \mathbb{R}$. Since $A \geq 0$, it follows that $W_e(t) \subset [0, \infty)$. Then we use Lemma 2.5 to observe that

$$\min \sigma_e(A(t)) = \min W_e(A(t)) = \min W_e(a + tb) = \min \text{Re } W_e((1 - it)t) = 0$$

for all $t \in \mathbb{R}$. \hfill \qed

**Remark 4.6.** Lemma 4.5 and Corollary 4.3 together imply that if

$$\lim_{t \to t_0} \mu(t)/(t - t_0) < 0,$$

in the notation of Theorem 1.1 then 0 is an eigenvalue of $A + t_0 B$. If zero is not an eigenvalue of $A$, then the converse is also true, as was observed in [11]. Let $A(\tau)$ denote the type (B) family of self-adjoint operators corresponding to the family of sesquilinear forms $(a + t_0 b) + \tau b$ (here $\tau = t - t_0$). Suppose that 0 is an eigenvalue of $A(0)$ (here $A(0) = A + t_0 B$), and $x$ is an eigenvector of $A(0)$ corresponding to 0. Then $(a + t_0 b)[x] = 0$. Since $A \geq 0$, $b[x] < 0$. Therefore the self-adjoint operator $B_0$ defined by (4.1) has a minimal eigenvalue $\beta < 0$. If $\mu(\tau)$ is the sole negative
eigenvalue of $A(\tau)$ that approaches 0 as $\tau \to 0^+$, then $\mu(\tau)/\tau \to \beta$ by Theorem 4.3.

5. ISOLATED EIGENVALUES

If $A(t)$ is a family of self-adjoint operators that depend analytically on the real parameter $t$, then the Kato-Rellich perturbation theory applies to the isolated eigenvalues of $A(t)$ with finite multiplicity. In general, it is not possible to analytically continue an eigenvalue function $\lambda(t)$ after it approaches an element of the essential spectrum, see Example 6.2. In some circumstances, the Kato-Rellich theory can be adapted to isolated eigenvalues with infinite multiplicity, as the following theorem shows.

**Theorem 5.1.** Suppose that $A(t)$ is a holomorphic family of self-adjoint bounded linear operators on $\mathcal{H}$ with power series expansion

$$A(t) = A_0 + tA_1 + t^2 A_2 + \ldots$$

defined in a neighborhood of $t = 0$. Suppose that 0 is an isolated element of the spectrum of $A_0$. Let $P$ denote the spectral projection onto the kernel of $A_0$. If $\mu$ is an element of the discrete spectrum of $PA_1 P$ with multiplicity $k$, then there is a family of $k$ analytic functions $x_j(t) \in \mathcal{H}$ defined on an open interval $I$ containing 0 such that the collection $x_j(t)$, $j = 1, \ldots, k$, is a mutually orthogonal family of unit eigenvectors of $A(t)$ with corresponding eigenvalues $\lambda_j(t)$ that satisfy $\lambda_j(0) = 0$ and $\lambda_j'(0) = \mu$ for all $1 \leq j \leq k$.

**Proof.** Since 0 is an isolated element of the spectrum of $A(0)$, we can construct an analytic spectral projection function $P(t)$ such that $P(0) = P$ [7, Theorem VII.1.7]. We will show that the expression $B(t) = t^{-1} P(t) A(t) P(t)$ has a power series expansion that converges in a neighborhood of $t = 0$. The spectral projection $P(t)$ has a power series of the form

$$P(t) = P_0 + P_1 t + P_2 t^2 + \ldots$$

Expanding the power series for $P(t) A(t) P(t)$ gives:

$$P_0 A_0 P_0 + t(P_0 A_1 P_0 + P_1 A_0 P_0 + P_0 A_0 P_1) + o(t^2)$$

Observe that $P_0 A_0 = A_0 P_0 = 0$, so the expression above simplifies to:

$$t(P_0 A_1 P_0) + o(t^2).$$

This means that $B(t) = t^{-1} P(t) A(t) P(t)$ is analytic in a neighborhood of $t = 0$. Also, $\mu$ is an isolated eigenvalue of $B(0)$ with multiplicity $k$. Therefore the Kato-Rellich perturbation theory applies, so there is a family of mutually orthogonal unit eigenvectors $x_1(t), \ldots, x_k(t)$ of $B(t)$ that are analytic functions of $t$ in an interval of 0, and such that $B(0)x_j(0) = \mu x_j(0)$ for all $j$. Each $x_j(t)$ is an eigenvector of $A(t)$ with corresponding eigenvalue $\lambda_j(t) = \langle A(t)x_j(t), x_j(t) \rangle = t \langle B(t)x_j(t), x_j(t) \rangle$.

Let $\mu_j(t)$ denote $\langle B(t)x_j(t), x_j(t) \rangle$ and observe that $\mu_j(t)$ is analytic in $t$ for all $1 \leq j \leq k$. Then $\lambda_j'(t) = \mu_j(t) + t\mu_j'(t)$ and $\lambda_j'(0) = \mu_j(0) = \mu$ for all $1 \leq j \leq k$. □

6. EXAMPLES

**Example 6.1.** Let $\mathcal{H} = L^2(0,1)$. The Volterra operator $V : \mathcal{H} \to \mathcal{H}$ is

$$(Vf)(t) := \int_0^t f(s) \, ds.$$
It is well known that the Volterra operator is a compact linear operator. The adjoint of V is \((V^*)f(t) = \int_1^t f(s) \, ds\) and therefore the real part of V is a rank one self-adjoint operator with non-zero eigenvalue equal to 1/2 and the corresponding eigenspace consists of all constant functions. Let \(V(\theta)\) denote the real part of \(e^{-i\theta}V\) and note that

\[
V(\theta) = \frac{1}{2}(e^{-i\theta}V + e^{i\theta}V^*) = \cos \theta \text{Re} V + \sin \theta \text{Im} V.
\]

Suppose that \(f\) is a unit eigenvector of \(V(\theta)\) for \(\theta \in \mathbb{R}\setminus\{0\}\). Then \(V(\theta)f = \lambda f\) for some \(\lambda \in \mathbb{R}\) and

\[
\lambda f'(x) = \frac{1}{2}e^{-i\theta}f(x) - \frac{1}{2}e^{i\theta}f(x) = -i \sin \theta f(x).
\]

This means that the eigenvectors of \(V(\theta)\) have the form \(f_n(t) = e^{-it(2\theta + 2n\pi)}\) where \(n \in \mathbb{Z}\).

Example 6.2. Let \(H\) be the operator on \(\ell_2(\mathbb{N})\) defined by

\[
H(x)_k = \begin{cases} 0 & \text{if } k = 1, \\ e^{-k\pi}x_k & \text{otherwise}. \end{cases}
\]

We also choose two elements \(a, b \in \ell_2(\mathbb{N})\) with nonnegative entries such that

\[
a_k^2 = \begin{cases} 1 & \text{if } k = 1, \\ \frac{3}{4n^2} & \text{if } k = (4n)^2 \text{ for } n \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases}
\]

and

\[
b_k^2 = \begin{cases} 1 & \text{if } k = 1, \\ \frac{1}{2} & \text{if } k = 2, \\ \frac{3}{2(4n+1)} & \text{if } k = (4n+2)^2 \text{ for } n \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}
\]

Observe that \(\|a_k\| = \|b_k\| = \sqrt{2}\). In particular, \(\sum_{k=2}^\infty a_k^2 = \sum_{k=2}^\infty b_k^2 = 1\). We also note that

\[
\sum_{k=2}^{(4n+1)^2} a_k^2 = \sum_{k=2}^{(4n+3)^2} a_k^2 = 1 - \frac{1}{4n},
\]
Since $\lambda < 0$. Then taking the inner-product of both sides of the expression above with $t$ analytically continued beyond and therefore at least one of the two eigenvalue functions $\lambda$ minimal eigenvalues of $H$ are given by $\lambda$ below. This in turn will show that the functions $\lambda(t)$ or $\lambda_b(t)$ cannot be analytically continued beyond $t = 0$.

Fix $0 < t < 1$ and suppose that $(H - tK_a)x = \lambda x$ for $x \in \ell_2$ with $\|x\| = 1$ and $\lambda < 0$. Then

$$Hx - tK_ax = \lambda x$$

$$(H - \lambda I)x = t \langle x, a \rangle a.$$ 

Since $H \geq 0$ and $\lambda < 0$, $H - \lambda I$ is invertible, so

$$x = t \langle x, a \rangle (H - \lambda I)^{-1} a$$

Taking the inner-product of both sides of the expression above with $a$ and solving for $t$, we get

$$\frac{1}{\langle (H - \lambda I)^{-1} a, a \rangle} = t$$

The expression $\langle (H - \lambda I)^{-1} a, a \rangle$ can be expanded as:

$$\frac{1}{\lambda} a_1^2 + \sum_{k=2}^{\infty} \frac{1}{e^{-k} - \lambda} a_k^2.$$ 

It is apparent that this expression is a strictly monotone function of $\lambda \in (-\infty, 0)$ and therefore so is $t$. For convenience, let $f_a(\lambda) = \langle (H - \lambda I)^{-1} a, a \rangle$, and likewise, let $f_b(\lambda) = \langle (H - \lambda I)^{-1} b, b \rangle$. The minimum eigenvalues of $H - tK_a$ and $H - tK_b$ are given by $\lambda_a(t) = f_a^{-1}(1/t)$ and $\lambda_b(t) = f_b^{-1}(1/t)$ respectively. We will show that the functions $f_a(\lambda)$ and $f_b(\lambda)$ cross infinitely many times as $\lambda$ approaches 0 from below. This in turn will show that the functions $\lambda_a(t)$ and $\lambda_b(t)$ cross infinitely many times as $t \to 0^+$.

If $k \leq (m - 1)^2$, then

$$\frac{1}{1 + e^{m^2 - k}} \leq \frac{1}{1 + e^{2m - 1}} < \frac{1}{2^m}.$$ 

and similarly if $k \geq (m + 1)^2$, then

$$\frac{1}{1 + e^{m^2 - k}} \geq \frac{1}{1 + e^{-2m - 1}} > \frac{2^m - 1}{2^m}.$$
If \( m = 4n + 1 \) and \( \lambda = -e^{-m^2} \), then
\[
fa(\lambda) = e^{m^2} \left( 1 + \sum_{k=2}^{\infty} \left( \frac{1}{1 + e^{m^2-k}} \right) a_k^2 \right)
\]
\[
< e^{m^2} \left( 1 + \sum_{k=2}^{m^2} \left( \frac{1}{1 + e^{m^2-k}} \right) a_k^2 + \sum_{k=m^2}^{\infty} a_k^2 \right)
\]
\[
< e^{m^2} \left( 1 + \sum_{k=2}^{m^2} \left( \frac{1}{2m} \right) a_k^2 + \sum_{k=m^2}^{\infty} a_k^2 \right)
\]
\[
= e^{m^2} \left( 1 + \left( \frac{1}{2m} \right) \left( 1 - \frac{1}{4^n} \right) + \frac{1}{4^n} \right)
\]
\[
= e^{m^2} \left( 1 + \frac{1}{2m} + \left( 1 - \frac{1}{2m} \right) \frac{1}{4^n} \right),
\]
while
\[
fb(\lambda) = e^{m^2} \left( 1 + \sum_{k=2}^{\infty} \left( \frac{1}{1 + e^{m^2-k}} \right) b_k^2 \right)
\]
\[
> e^{m^2} \left( 1 + \sum_{k=2}^{\infty} \left( \frac{1}{1 + e^{m^2-k}} \right) b_k^2 \right)
\]
\[
> e^{m^2} \left( 1 + \sum_{k=m^2}^{\infty} \left( \frac{2^m-1}{2^m} \right) b_k^2 \right)
\]
\[
= e^{m^2} \left( 1 + \left( \frac{2^m-1}{2^m} \right) \left( \frac{2}{4^n} \right) \right)
\]
\[
= e^{m^2} \left( 1 + \left( 1 - \frac{1}{2m} \right) \frac{2}{4^n} \right).
\]

By inspection, it is clear from the above inequalities that \( fa(\lambda) < fb(\lambda) \) when \( \lambda = -e^{-(4n+1)^2} \). Essentially the same argument shows that \( fb(\lambda) < fa(\lambda) \) when \( \lambda = -e^{-(4n+3)^2} \). Therefore the functions \( fa(\lambda) \) and \( fb(\lambda) \) cross infinitely many times as \( \lambda \to 0^- \), which proves that at least one of the eigenvalue functions \( \lambda_a(t) \) or \( \lambda_b(t) \) cannot be analytically continued past beyond \( t = 0 \).

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