DG MODULE STRUCTURES AND MINIMAL FREE RESOLUTIONS
MODULO AN EXACT ZERO DIVISOR

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ABSTRACT. Let $Q$ be a local ring with maximal ideal $n$ and let $f, g \in n \setminus n^2$ with $fg = 0$. When $M$ is a finite $Q$-module with $fM = 0$, we show that a minimal free resolution of $M$ over $Q$ has a differential graded module structure over the differential graded algebra $Q\langle y, t \mid \partial(y) = f, \partial(t) = gy \rangle$. When $(f, g)$ is a pair of exact zero divisors, we use this structure to describe a minimal free resolution of $M$ over $Q/(f)$.

INTRODUCTION

Differential graded (dg) structures provide an effective and elegant tool in commutative algebra, with particularly strong applications in the study of homological behavior under a change of ring, see Avramov [1] for the basics of the theory.

Let $\varphi: Q \to R$ be a ring homomorphism and $M$ an $R$-module. One would like to be able to describe a projective resolution of $M$ over $R$, given a projective resolution $A$ of $R$ over $Q$ and a projective resolution $U$ of $M$ over $Q$. Iyengar [5] provides such a construction when the resolutions $A$ and $U$ have appropriate differential graded (dg) structures and discusses the existence of such structures. When $Q$ is local (meaning also commutative noetherian) with maximal ideal $n$ and $\varphi$ is surjective, we are particularly interested in minimal free resolutions. In this case, if $A$ is a resolution of $R$ over $Q$ admitting a dg algebra structure and $U$ is a minimal free resolution of $M$ over $Q$ admitting a semi-free dg $A$-module structure, it is known that $U \otimes_A R$ is a minimal free resolution of $M$ over $R$, see Lemma 1.3.

When $f \in n \setminus n^2$ and $R = Q/(f)$, a construction of Shamash can be used to build a semi-free dg module structure of $U$ over the Koszul complex on $f$, see [1, Proposition 2.2.2]. Consequently, this leads to a description of a minimal free resolution of $M$ over $R$ when $f$ is, in addition, a non-zero divisor, cf. [1, Theorem 2.2.3]. In this paper, we extend the Shamash construction when $f$ is a zero divisor.

Theorem. Let $(Q, n)$ be a local ring and let $f, g \in n \setminus n^2$ such that $fg = 0$. Set $R = Q/(f)$ and $A = Q\langle y, t \mid \partial(y) = f, \partial(t) = gy \rangle$. If $M$ is a finitely generated $R$-module and $U$ is a minimal free resolution of $M$ over $Q$, then:

(a) $U$ has a structure of semi-free dg module over $A$.
(b) Furthermore, if $\text{ann}(f) = (g)$ and $\text{ann}(g) = (f)$, then $U \otimes_A R$ is a minimal free resolution of $M$ over $R$.

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The dg algebra $A$ in the theorem is obtained from $Q$ by adjoining an exterior algebra variable $y$ in degree 1 and a divided powers variable $t$ in degree 2, see 1.4 for details on the notation, which is based on a construction usually referred to as a Tate construction. When the hypothesis of part (b) holds, we say that $(f, g)$ is a pair of exact zero divisors and that $f$, respectively $g$, is an exact zero divisor. In this case, $A$ is a minimal free resolution of $R$ over $Q$. Although not all rings admit exact zero divisors, there exist significant classes of rings that do. See, for example, Henriques and Šega [4] for a treatment of a class of rings that are known to admit exact zero divisors, due to a result of Conca, Rossi, and Valla. Such elements can play an important role in understanding the structure and homological properties of artinian algebras.

Let $Q \to R$ be a surjective homomorphism of local rings such that a minimal free resolution $A$ of $R$ over $Q$ admits a dg algebra structure and $M$ is a finite $R$-module. The question of whether a minimal free resolution $U$ of $M$ over $Q$ admits a dg $A$-module structure was originally asked by Buchsbaum and Eisenbud [3], in the case when $A$ is a Koszul complex. Subsequently, this question was studied in the work of several authors, including Avramov, Kustin, Iyengar, Miller, and Srinivasan, see [2], [5], [6], [7], [8], [9], with both positive and negative answers. In all previously known results in which existence is established, either $A$ is a Koszul complex, or the resolution $U$ has a finite length. Our result provides a first positive answer (to our knowledge) in a case when both $A$ and $U$ are infinite.

The minimal free resolution of $M$ over $R$ constructed in the theorem can be used to recover a known relationship [4, Theorem 1.7] between the betti numbers of $M$ over $Q$ and the betti numbers of $M$ over $R$ when $(f, g)$ is a pair of exact zero divisor, see Remark 2.3. We note, however, that part (b) of the theorem, together with the construction of the dg module structure in the proof of (a) and the more detailed description of the complex $U \otimes_A R$ in Construction 1.2, give a full description of the differential, and not just the betti numbers.

1. The Tate construction

In this section, we describe in detail the construction of the dg algebra $A$ in the main theorem, and we set up notation that will be used in the proof of the theorem.

1.1. Dg algebras and dg modules. Let $Q$ denote a commutative ring. If $W$ is a complex of $Q$-modules, we denote by $W^\#$ the underlying graded $Q$-module and we use $|w|$ for the degree of a homogeneous element $w$ of $W$.

A dg algebra over $Q$ is a complex $A = (A, \partial)$, concentrated in non-negative homological degrees, where $A^\#$ is a $Q$-algebra that is (graded) commutative, meaning that

$$ab = (-1)^{|a||b|} ba \quad \text{for } a, b \in A \quad \text{and} \quad a^2 = 0 \text{ when } |a| \text{ is odd},$$

and such that the Leibniz formula holds:

$$\partial(ab) = (\partial a)b + (-1)^{|a|}a(\partial b) \quad \text{for } a, b \in A.$$

If $A$ and $B$ are dg $Q$-algebras, then $A \otimes_Q B$ is also a dg-algebra, with multiplication given by

$$(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|}aa' \otimes bb'$$

and the usual differential defined on a tensor product of complexes. We usually identify $A$ and $B$ with their images in $A \otimes_Q B$ and we write $ab$ instead of $a \otimes b$. 

A dg module $U$ over $A$ is a complex $(U, \partial)$ with $U^\#$ an $A^\#$-module such that

$$\partial(au) = (\partial a)u + (-1)^{|a|}a(\partial u) \quad \text{for} \ a \in A \text{ and } u \in U.$$  

A bounded below dg $A$-module $U$ is said to be semi-free if $U^\#$ is free over $A^\#$. Existence of semi-free dg module structures is particularly useful, as they give an easy construction of free resolutions over $R$ from those over $Q$. For a more general construction, that does not require the semi-free assumption, see [5]. The benefit of the setting in Lemma 1.3 below is that, in the local case, one obtains a minimal resolution when starting with a minimal one.

**Construction 1.2.** Let $Q \to R$ be a surjective homomorphism of commutative rings. Let $A$ be a dg $Q$-algebra algebra with $H_0(A) = R$ and let $U = (U, \partial)$ be a dg semi-free $A$-module. Let $\{e_\lambda\}_{\lambda \in A}$ denote a basis of $U^\#$ over $A^\#$. For each integer $n$, let $V_n$ denote the free $Q$-summand of $U_n$ with basis $\{e_\lambda \mid \lambda \in A, |e_\lambda| = n\}$ and let $\pi_{n-1}: U_{n-1} \to V_{n-1}$ denote the projection map. Then $U \otimes_A R$ can be identified with the complex $U' = (U', \partial')$, with

$$U'_n = V_n \otimes_Q R \quad \text{and} \quad \partial'_n = \pi_{n-1}\partial_n|_{V_n} \otimes_Q 1_R.$$

Indeed, assume $R = Q/I$ for an ideal $I$, and $Q \to R$ is the natural projection. Let $\varepsilon: A \to R$ denote the augmentation map and set $J = \text{Ker}(\varepsilon)$. We have then $R = A/J$, and thus $U \otimes_A R \cong U/1_R$. Let $n \geq 0$. The definition of $V_n$ gives $U_n = V_n \oplus W_n$, where $W_n = (A_{>1}U)_n$. We have thus:

$$\begin{align*}
(U \otimes_A R)_n & \cong \left(\frac{U}{JU}\right)_{n} \cong \frac{U_n}{U_n + W_n} \cong \frac{U_n}{W_n} \otimes_Q R \cong V_n \otimes_Q R.
\end{align*}$$

The differential of $U'$ is induced by the one on $U \otimes_A R$, which is in turn induced by the differential of $U$, and this yields directly the claimed expression for $\partial'$.

The following is a standard argument that appears, for instance, in the proof of [1, Theorem 2.2.3].

**Lemma 1.3.** Let $Q \to R$ be a surjective homomorphism of commutative rings. Let $A$ be a free resolution of $R$ over $Q$ that has a structure of dg algebra, and let $U = (U, \partial)$ be a free resolution of $M$ over $Q$, that has a structure of semi-free dg-module over $A$. Then $U \otimes_A R$ is a free resolution of $M$ over $R$. In particular, if $Q$ is local and $U$ is minimal, then $U \otimes_A R$ is a minimal free resolution of $M$ over $R$.

**Proof.** Since $A$ is a free resolution of $R$ over $Q$, the augmentation map $\varepsilon: A \to R$ is a quasi-isomorphism (that is, it induces an isomorphism in homology). By [1, Proposition 1.3.2], $\varepsilon: A \to R$ induces a quasi-isomorphism $U \to U \otimes_A R$. This shows that $U \otimes_A R$ is a free resolution of $M$ over $R$. The last statement, regarding minimality, follows directly from Construction 1.2. \qed

1.4. **The Tate construction.** We describe below a basic procedure of adjoining variables to create new dg algebras, originally used by Cartan and further adapted by Tate. We refer to [1] for the general construction, and we adapt it here to the case of interest.

We denote by $Q(y)$ the graded free $Q$-module with basis $1, y$, where $|y| = 1$, and define a $Q$-linear multiplication by setting

$$y^2 = 0.$$
Let \( f \in Q \). We write \( K = Q \{ y \mid \partial(y) = f \} \) for the dg \( Q \)-algebra with underlying graded algebra \( Q \{ y \} \) and differential satisfying
\[
\partial(y) = f.
\]
This is precisely a Koszul complex on \( f \).

We denote by \( Q \{ t \} \) the graded free module with basis \( \{ t^{(i)} \} \), where \( |t^{(i)}| = 2i \), and define a \( Q \)-linear multiplication by
\[
t^{(0)} = 1 \quad \text{and} \quad t^{(i)} t^{(j)} = \frac{(i+j)!}{i! j!} t^{(i+j)}.
\]
We set \( t = t^{(1)} \); this element is called a divided powers variable and the basis elements \( \{ t^{(i)} \} \) are called a system of divided powers of \( t \).

Let \( g \in Q \) with \( fg = 0 \), so that \( gy \) is a cycle in \( K \). We write
\[
A = Q \{ y, t \mid \partial(y) = f, \partial(t) = gy \}
\]
for the dg \( Q \)-algebra with underlying graded algebra \( K \otimes_Q Q \{ t \} \), with differential extending the one on \( K \) and satisfying
\[
\partial(t^{(i)}) = gyt^{(i-1)}.
\]
Note that we are using product notation for tensor products of elements, e.g. we are writing \( yt \) instead of \( y \otimes t \). The multiplication on \( A \) comes from the multiplication on \( K \) and that on \( Q \{ t \} \) using formula (1.1.1). In particular, we have
\[
yt^{(n)} = t^{(n)} y \quad \text{for all } n \geq 0.
\]
It is known and easy to check that, with these definitions, \( A \) is indeed a dg algebra, that is referred to as the Tate construction on \( f, g \).

1.5. Notation. We set
\[
f_i = \begin{cases} 
  f & \text{if } i \text{ is odd} \\
  g & \text{if } i \text{ is even}.
\end{cases}
\]
For all \( i \geq 0 \) we set
\[
y_{2i} = t^{(i)} \quad \text{and} \quad y_{2i+1} = t^{(i)} y.
\]
We set \( y_i = 0 \) if \( i < 0 \). With this notation, note that \( A \) is a free \( Q \)-module with basis \( \{ y_n \}_{n \geq 0} \) such that \( |y_n| = n \) for all \( n \), and
\[
\partial_i(y_i) = f_i y_{i-1} \quad \text{for all } i.
\]
The multiplicative structure of \( A \) is given by the unit \( y_0 \) and the relations
\[
y_{2i+1} y_{2j+1} = 0
\]
\[
y_{2i} y_{2j} = \binom{i+j}{i} y_{2(i+j)}
\]
\[
y_{2i} y_{2j+1} = \binom{i+j}{i} y_{2(i+j)+1} = y_{2j+1} y_{2i}
\]
for all \( i, j \geq 0 \). Using these relations, define \( \alpha_{i,j} \in Q \) such that
\[
y_i y_j = \alpha_{i,j} y_{i+j} \quad \text{for all } i \geq 0, j \geq 0.
\]
More precisely, we have \( \alpha_{i,j} = 0 \) when both \( i \) and \( j \) are odd, \( \alpha_{i,j} = \binom{i+j}{i} \) if both \( i, j \) are even, \( \alpha_{i,j} = \binom{i+j-1}{i} \) if \( i \) is even and \( j \) is odd, and \( \alpha_{i,j} = \binom{i+j-1}{j} \) if \( i \) is odd and \( j \) is even.
For all $i,j \geq 0$, note that
\[ \alpha_{i,j} = \alpha_{j,i}. \]

Further, one can easily verify that
\[
\alpha_{i,j} \alpha_{i+j,k} = \alpha_{j,k} \alpha_{i,j+k} \quad \text{for all } i,j,k \geq 0 \tag{1.5.2}
\]
\[
\alpha_{i-1,j} f_i + (-1)^i \alpha_{i,j-1} f_j = \alpha_{i,j} f_{i+j} \quad \text{for all } i,j \geq 1. \tag{1.5.3}
\]

In fact, (1.5.2) is equivalent to the associativity of $A$, namely $(y_i y_j) y_k = y_i (y_j y_k)$, and (1.5.3) is equivalent to the Leibniz rule for $A$, namely:
\[
\partial(y_i y_j) = \partial(y_i) y_j + (-1)^i y_i \partial(y_j).
\]

Finally, observe that the Tate construction $A$ is a periodic complex of rank 1 free $Q$-modules:
\[
\cdots \to Qy_{2n+1} \xrightarrow{f} Qy_{2n} \xrightarrow{g} Qy_{2n-1} \xrightarrow{f} Qy_{2n-2} \to \cdots \to Qy_2 \xrightarrow{g} Qy_1 \xrightarrow{f} Qy_0 \to 0.
\]

**Remark 1.6.** Let $A$ be as in 1.4, and set $R = Q/(f)$. Then, with the assumptions and notation introduced in Construction 1.2, we have an isomorphism of free $Q$-modules
\[
U_n = V_n \oplus y_1 V_{n-1} \oplus y_2 V_{n-2} \oplus \cdots \oplus y_n V_0 \cong \bigoplus_{i=0}^n V_n. \tag{1.6.1}
\]

**Definition 1.7 (Exact zero divisors).** Let $f, g \in Q$. We say that $(f, g)$ is a pair of exact zero divisors if $\text{ann}(f) = (g)$ and $\text{ann}(g) = (f)$; we also refer to $f$ or $g$ as an exact zero divisor. In this case, the Tate construction on $f, g$ is a minimal free resolution of $Q/(f)$ over $Q$.

2. **Proof of the main theorem**

Our main theorem is an extension of a construction of Shamash [10], along the lines of the proof in [1, Proposition 2.2.2, Theorem 2.2.3]. We recall below the statement of the main theorem from the introduction and then proceed to prove it. A corollary about Betti numbers is given at the end of the section.

**Theorem 2.1.** Let $(Q,n)$ be a local ring and let $f, g \in n \setminus n^2$ such that $fg = 0$. Set $R = Q/(f)$ and $A = Q[y,t \mid \partial(y) = f, \partial(t) = gy]$. If $M$ is a finitely generated $R$-module and $U$ is a minimal free resolution of $M$ over $R$, then:

(a) $U$ has a structure of semi-free $dg$ module over $A$.

(b) Furthermore, if $\text{ann}(f) = (g)$ and $\text{ann}(g) = (f)$, then $U \otimes_A R$ is a minimal free resolution of $M$ over $R$.

**Proof.** (a) For all integers $i \geq 0$ and $n$ we will construct homomorphisms
\[
\sigma_{i,n} : U_n \to U_{i+n}
\]
and free $Q$-modules $V_n$ with $V_n \subseteq U_n$ such that:

(1) $\sigma_{0,n} = \text{id}_{U_n}$ for all $n$.

(2) For all $n$ and $i \geq 1$ the following condition holds:
\[
\mathcal{A}(i,n) : \quad \partial_{i+n} \sigma_{i,n} + (-1)^i \sigma_{i,n-1} \partial_n = f_i \sigma_{i-1,n}. \tag{2.1.1}
\]

(3) For all $n$ and all $i, j$ with $i \geq 0$ and $j \geq 0$ the following condition holds:
\[
\mathcal{B}(i,j,n) : \quad \sigma_{i,n} \sigma_{j,n-j} = \alpha_{i,j} \sigma_{i+j,n-j}.
\]
(4) For all $n$ the following condition holds:

\[ C(n): \text{ The map } \Phi_n: V_0 \oplus V_1 \oplus \cdots \oplus V_n \rightarrow U_{n+1} \text{ given by} \]

\[ \Phi_n(x_0, x_1, \ldots, x_n) = \sum_{i=1}^{n+1} \sigma_{i,n+1-i}(x_{n+1-i}) \]

is a split injection.

(5) For all $n$ the following holds:

\[ D(n): \quad U_n = \sum_{k=0}^{n} \sigma_{k,n-k}(U_{n-k}) = \bigoplus_{k=0}^{n} \sigma_{k,n-k}(V_{n-k}) , \]

where the direct sum indicates an \textit{internal} direct sum, that is, every element of $U_n$ can be written uniquely as a sum of elements in $\sigma_{k,n-k}(V_{n-k})$.

Once these conditions are established, we define

\[ y_iu = \sigma_{i,n}(u) \quad \text{for all } n, i \geq 0 \text{ and } u \in U_n \]

and then extend linearly to a multiplication $A \times U \rightarrow U$. Conditions (1)-(3) imply that this rule defines a dg $A$-module structure on $U$. Item (5) shows that $U$ is a semi-free dg $A$-module. Indeed, condition $D(n)$ can be re-written as $U_n = \bigoplus_{k=0}^{n} y_k V_{n-k}$, implying that a basis of $U^\alpha$ over $A^\alpha$ is $\cup_{i \geq 0} B_i$, where $B_i$ denotes a $Q$-basis of $V_i$.

Before proceeding with the bulk of the proof, we show:

\textit{Claim.} Let $n$ be an integer. Assume $C(m)$, $D(m)$ and $B(i,j,m)$ hold for all $m$ with $0 \leq m \leq n-1$ and all $i, j \geq 0$. Then

\[ \sum_{k=j}^{n} \sigma_{k,n-k}(U_{n-k}) = \bigoplus_{k=j}^{n} \sigma_{k,n-k}(V_{n-k}) \subseteq U_n \quad \text{for all } j \geq 1 . \quad \text{(2.1.2)} \]

\textit{Proof of Claim.} Let $j \geq 1$. We have:

\[ \sum_{k=j}^{n} \sigma_{k,n-k}(U_{n-k}) = \sum_{k=j}^{n} \sum_{k'=0}^{n-k} \sigma_{k,n-k} \sigma_{k',n-k-k'}(V_{n-k-k'}) \]

\[ \subseteq \sum_{k=j}^{n} \sum_{k'=0}^{n-k} \sigma_{k+k',n-(k+k')}(V_{n-(k+k')}) \]

\[ \subseteq \sum_{l=j}^{n} \sigma_{l,n-l}(V_{n-l}) \]

\[ \subseteq \sum_{k=j}^{n} \sigma_{k,n-k}(U_{n-k}) . \]

We used $D(n-k)$ in the first line, $B(k, k', n - k)$ in the second, and $V_i \subseteq U_i$ in the fourth. (Note that $k \geq j \geq 1$ and thus $n-k \leq n-1$.) It follows that all inclusions above are in fact equalities. Further, the sum $\sum_{k=j}^{n} \sigma_{k,n-k}(V_{n-k})$ is an internal direct sum, as can be seen from $C(n-1)$. This finishes the proof of the claim.

We now proceed to construct the maps $\sigma_{i,n}$ and the modules $V_n$. We start by setting $\sigma_{i,n} = 0$ and $V_n = 0$ whenever $n < 0$. With this definition, note that (1)-(5) hold when $n < 0$. We proceed by induction.

Consider the following statement, depending on an integer $k$. 

Induction statement I. The maps \( \sigma_{i,k} : U_k \to U_{i+k} \) are defined for all \( i \geq 0 \) and the free \( Q \)-module \( V_k \) with \( V_k \subseteq U_k \) are defined, and satisfy the properties:

- \( \sigma_{0,k} = \text{id}_{U_k} \)
- \( \mathcal{A}(i,k) \) holds for all \( i \geq 1 \)
- \( \mathcal{B}(i,j,k) \) holds for all \( i \geq 0, j \geq 0 \)
- \( \mathcal{C}(k) \) and \( \mathcal{D}(k) \) hold.

As noted above, this statement holds when \( k < 0 \). Let \( n \geq 0 \). Assume that Induction statement I holds for all \( k \leq n-1 \). To complete the induction, we define next \( \sigma_{s,n} \) and \( V_n \), and we show that all four items in Induction statement I hold when \( k = n \).

We start by defining \( \sigma_{0,n} \) to be the identity map on \( U_n \). Then, we define the free module \( V_n \) as follows: Condition \( \mathcal{C}(n-1) \) gives that \( \text{Im} \Phi_{n-1} \) is a direct summand of \( U_n \). We define \( V_n \) to be the complementary direct summand, so that

\[
U_n = \text{Im} \Phi_{n-1} \oplus V_n = \text{Im} \Phi_{n-1} \oplus \sigma_{0,n}(V_n).
\]

Observe that \( \mathcal{D}(n) \) must then hold, as it is a direct consequence of this definition and of \( \mathcal{C}(n-1) \). We need to define now \( \sigma_{i,n} \) for \( i \geq 1 \). We proceed by induction.

With \( n \) fixed as above, consider the following statement, depending on an integer \( l \geq 0 \):

Induction statement II. The map \( \sigma_{l,n} : U_n \to U_{l+n} \) is defined such that

- \( \mathcal{A}(l,n) \) holds
- \( \mathcal{B}(l,j,n) \) holds for all \( j \geq 0 \).

Note that these two conditions hold trivially when \( l = 0 \). Let \( i \geq 1 \) and assume that Induction statement II holds for all \( l \) with \( l \leq i-1 \). We now define \( \sigma_{i,n} \) and we show that the two items in Induction Statement II also hold with \( l = i \).

In view of the direct sum decomposition of \( U_n \) given by condition \( \mathcal{D}(n) \) (which we know holds, see above), in order to define \( \sigma_{i,n} \) it suffices to define the restriction functions \( \sigma_{i,n}|_{\mathcal{A}_{i,n}(V_{n-k})} \) for all \( k \) with \( 0 \leq k \leq n \).

Assume first \( k > 0 \). Note that \( \mathcal{C}(n-k) \) implies that the restricted function

\[
\sigma_{i,n}|_{\mathcal{A}_{i,n}(V_{n-k})} : V_{n-k} \to \mathcal{A}_{i,n}(V_{n-k})
\]

is bijective. We define then

\[
\sigma_{i,n}|_{\mathcal{A}_{i,n}(V_{n-k})} := \alpha_{i,k} \sigma_{i+k,n-k}|_{V_{n-k}} (\sigma_{i+k,n-k}|_{V_{n-k}})^{-1}.
\]

In other words, this definition ensures that

\[
\sigma_{i,n}|_{\mathcal{A}_{i,n}(V_{n-k})} = \alpha_{i,k} \sigma_{i+k,n-k}|_{V_{n-k}} \quad \text{when} \quad k > 0.
\]

We need to define now \( \sigma_{i,n}|_{V_n} \). To do so, we first claim that

\[
(f_i \sigma_{i-1,n} - (-1)^{i+1} \sigma_{i,n-1} \partial_n) (U_n) \subseteq \partial_{i+n}(U_{i+n}).
\]

Assuming (2.1.4) holds and recalling that \( V_n \) is a free \( Q \)-module, we can define \( \sigma_{i,n}|_{V_n} : V_n \to U_{i+n} \) so that

\[
\partial_{i+n} \sigma_{i,n}|_{V_n} = f_i \sigma_{i-1,n} - (-1)^{i+1} \sigma_{i,n-1} \partial_n|_{V_n}.
\]

Indeed, fix a basis of \( V_n \) and define \( \sigma_{i,n}|_{V_n}(\epsilon) \) for a basis element \( \epsilon \) to be the preimage of \((f_i \sigma_{i-1,n} - (-1)^{i+1} \sigma_{i,n-1} \partial_n)(\epsilon)\) under \( \partial_{i+n} \), and then extend by linearity. Note that this definition depends on the choice of preimage, and thus is not unique.
To prove (2.1.4), we compute:
\[
\partial_{i+n-1}(f_i\sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n)
\]
\[
= f_i\partial_{i+n-1}\sigma_{i-1,n} + (-1)^i\partial_{i+n-1}\sigma_{i,n-1}\partial_n
\]
\[
= f_i \left( f_i - 1 \right) \sigma_{i-2,n} - (-1)^i \sigma_{i-1,n-1} \partial_n + (-1)^i \left( f_i \sigma_{i-1,n-2} \partial_n - (-1)^{i+1}\sigma_{i-1,n-2}\partial_{n-1}\partial_n \right)
\]
\[
= 0 ,
\]
where for the second equality we used the induction hypothesis that \(A(i-1,n)\) and \(A(i,n-1)\) hold, and for the third equality we used \(f_i f_{i-1} = fg = 0\) and \(\partial^2 = 0\). The computation above shows
\[
\left( f_i \sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n \right) \left( U_n \right) \subseteq \text{Ker} \left( \partial_{i+n-1} \right) .
\]
Since \(i \geq 1\) and \(n \geq 0\), we have \(i + n - 1 \geq 0\). When \(i + n - 1 > 0\), we use the fact that \(U\) is acyclic, and hence \(\text{Ker} \left( \partial_{i+n-1} \right) = \text{Im} \left( \partial_{i+n} \right)\) and thus (2.1.4) holds. When \(i = 1\) and \(n = 0\) we have
\[
f_1 \sigma_{0,0} - (-1)^2 \sigma_{1,-1} \partial_0 = f_1 \text{id}_{U_0} .
\]
Since \(U\) is a minimal free resolution of \(M\), we have \(M = U_0/\partial_1(U_1)\). Since \(f M = 0\), we have \(f_1 U_0 \subseteq \partial_1(U_1)\). This finishes the proof of (2.1.1) and the definition \(\sigma_{i,n}\).

Let \(j \geq 0\). We want to prove that \(B(j,i,n)\) holds. Since \(\sigma_{0,n}\) is the identity map, \(B(i,0,n)\) holds.

Assume now \(j > 0\). In view of the direct sum decomposition provided by \(D(n-j)\), in order to prove \(B(i,j,n)\), it suffices to show
\[
\sigma_{i,n}\sigma_{j,n-j-1}(V_{n-j-k}) = \alpha_{i,j}\sigma_{i,j,n-j}(V_{n-j-k})
\]
for all \(k\) with \(0 \leq k \leq n-j\). We need to check that
\[
\sigma_{i,n}\sigma_{j,n-j}\sigma_{k,n-j-k}(V_{n-j-k}) = \alpha_{i,j}\sigma_{i,j,n-j}\sigma_{k,n-j-k}(V_{n-j-k}) .
\]
Indeed, we have:
\[
\sigma_{i,n}\sigma_{j,n-j}\sigma_{k,n-j-k}(V_{n-j-k}) = \alpha_{i,j+k}\sigma_{i,n}\sigma_{j+k,n-j-k}(V_{n-j-k})
\]
\[
= \alpha_{i,j+k}\alpha_{i,j+k}\sigma_{i,j+k,n-j-k}(V_{n-j-k})
\]
\[
= \alpha_{i,j+k}\sigma_{i,j+k,n-j-k}(V_{n-j-k})
\]
\[
= \alpha_{i,j+k}\sigma_{i,j+k,n-j-k}(V_{n-j-k}) .
\]
Here, we used \(B(j,k,n-j)\) in the first equality, (2.1.3) in the second, (1.5.2) in the third, and \(B(i+j,k,n-j)\) in the last. This finishes the proof of \(B(i,j,n)\).

Next, we want to prove that \(A(i,n)\) holds. Given the definition of \(\sigma_{i,n}(V_n)\), we know that the relation holds when we restrict the functions to \(V_n\). We need to also check this relation when the functions are restricted to \(\sigma_{k,n-k}(V_{n-k})\) with \(k > 0\). To this extent, it suffices to show
\[
(f_i \sigma_{i-1,n} - (-1)^{i+1}\sigma_{i,n-1}\partial_n)\sigma_{k,n-k} = \partial_{i+n}\sigma_{i,n}\sigma_{k,n-k} \quad \text{for all } k > 0 .
\]
Let $k > 0$. The computation below achieves the desired conclusion:

$$(f_i \sigma_{i-1,n} - (-1)^{i+1} \sigma_{i,n-1} \partial_n) \sigma_{k,n-k}$$

$$= f_i \sigma_{i-1,n} \sigma_{k,n-k} - (-1)^{i+1} \sigma_{i,n-1} \partial_n \sigma_{k,n-k}$$

$$= \alpha_{i-1,k} f_i \sigma_{i-1,k,n-k} - (-1)^{i+1} \sigma_{i,n-1} (f_k \sigma_{i,k,n-k} - (-1)^{k+1} \sigma_{k,n-k-1} \partial_n)$$

$$= \alpha_{i-1,k} f_i \sigma_{i-1,k,n-k} + (-1)^i \alpha_{i,k-1} f_k \sigma_{i,k-1,n-k} + (-1)^i \alpha_{i,k} \sigma_{i,k,n-k-1} \partial_n$$

$$= \alpha_{i,k} \partial_{i+n} \sigma_{i,k,n-k}$$

$$= \partial_{i+n} \sigma_{i,n} \sigma_{k,n-k}$$

We used $\mathcal{B}(i-1, k, n)$ and $\mathcal{A}(k, n-k)$ for the second equality, $\mathcal{B}(i, k-1, n-1)$ and $\mathcal{B}(i, k, n-1)$ for the third, $(1.5.3)$ for the fourth, $\mathcal{A}(i+k, n-k)$ for the fifth and $\mathcal{B}(i, k, n)$ (which was proved earlier) for the last equality.

At this point, we verified Induction statement II for $l = i$, hence we have a definition of $\sigma_i$ for all $i$, such that $\mathcal{A}(i, n)$ holds and $\mathcal{B}(i, j, n)$ holds for all $i, j \geq 0$. We need to finalize the proof of the Induction statement I for $k = n$. The only remaining item is to show $C(n)$, namely that the map $\Phi_n$ is a split injective. We will show that if $\Phi_n(x) \in nU_{n+1}$ for

$$x = (x_0, x_1, \ldots, x_n) \in V_0 \oplus V_1 \oplus \cdots \oplus V_n,$$

then $x_i \in nV_i$ for all $i$.

Assume $\Phi_n(x) \in nU_{n+1}$. Then $\partial_{n+1} \Phi_n(x) \in n^2 U_n$. For the purposes of the next computation, we set $x_{n+1} = 0$. We have then:

$$\partial_{n+1} \Phi_n(x) = \sum_{i=1}^{n+1} \partial_{n+1} \sigma_{i,n+1-i} (x_{n+1-i})$$

$$= \sum_{i=1}^{n+1} f_i \sigma_{i-1,n+1-i} (x_{n+1-i}) - (-1)^{i+1} \sigma_{i,n-i} \partial_{n+1-i} (x_{n+1-i})$$

$$= \sum_{j=0}^{n} f_j \sigma_{j,n-j} (x_{n-j}) + \sum_{j=0}^{n} (-1)^j \sigma_{j,n-j} \partial_{n+1-j} (x_{n+1-j})$$

$$= \sum_{j=0}^{n} \sigma_{j,n-j} (f_j + x_{n-j} - \partial_{n-j+1} (x_{n-j+1}))$$

We used $\mathcal{A}(i, n+1-i)$ for the second equality. Next, we set

$$w_j = \sigma_{j,n-j} (f_j + x_{n-j} - \partial_{n-j+1} (x_{n-j+1}))$$

so that (2.1.5) becomes

$$\partial_{n+1} \Phi_n(x) = \sum_{j=0}^{n} w_j.$$

We prove by induction on $i$ with $0 \leq i \leq n+1$ that $x_{n+1-i} \in nV_{n+1-i}$. This is true when $i = 0$, as we defined $x_{n+1} = 0$. Let $k$ be so that $0 \leq k \leq n$ and assume that $x_{n+1-i} \in nV_{n+1-i}$ for all $i$ with $0 \leq i \leq k$. The fact that that $f_i \in n$ for all $i$ and the minimality of $U$ imply

$$w_j \in n^2 V_{n-j}$$

for all $j \leq k-1$.  

To complete the induction argument, we show that \( x_{n-k} \in nV_{n-k} \). We have
\[
f_{k+1}\sigma_{k,n-k}(x_{n-k}) + \sum_{j=k+1}^{n} w_j = w_k + \sigma_{k,n-k}\partial_{n-k+1}(x_{n-k+1}) + \sum_{j=k+1}^{n} w_j
\]
\[
= \sigma_{k,n-k}\partial_{n-k+1}(x_{n-k+1}) + \partial_{n+2}\Phi_n(x) - \sum_{j=0}^{k-1} w_j
\]
\[
\in n^2U_n.
\]
Recall the direct sum decomposition of \( U_n \) given by property \( D(n) \), and notice that \( f_{k+1}\sigma_{k,n-k}(x_{n-k}) + \sum_{j=k+1}^{n} w_j \) belong to distinct summands in this decomposition. Indeed, we have \( f_{k+1}\sigma_{k,n-k}(x_{n-k}) \in \sigma_{k,n-k}(V_{n-k}) \), while \( (2.1.2) \) implies
\[
\sum_{j=k+1}^{n} w_j \in \sum_{j=k+1}^{n} \sigma_{j,n-j}(U_{n-j}) = \bigoplus_{j=k+1}^{n} \sigma_{j,n-j}(V_{n-j}).
\]
We conclude \( f_{k+1}\sigma_{k,n-k}(x_{n-k}) \in n^2U_n \), and hence \( \sigma_{k,n-k}(x_{n-k}) \in nU_n \), since \( f_{k+1} \notin n^2 \). Since \( \Phi_n \) is a split injection by \( C(n-1) \), it follows \( x_{n-k} \in nV_{n-k} \).

We showed thus that \( C(n) \) holds. This finishes the proof of Induction Statement I and concludes the proof of (a).

(b) The fact that \((f, g)\) is a pair of exact zero divisors implies that \( A \) is a minimal free resolution of \( R \) over \( Q \), as noted in Definition 1.7. The conclusion follows then from Lemma 1.3, making use of (a).

\[\square\]

**Remark 2.2.** The proof of the Main Theorem is inspired by the construction of Shamash described in [1, Proposition 2.2]. It is proved there that if \( f \in n \setminus n^2 \), then, with notation as in our theorem, \( U \) has a structure of semi-free dg module over the Koszul complex \( K = Q(y | \partial(y) = f) \). As noted in [1, Remark 2.2.1], since \( f^{id}U \) and \( 0^U \) both induce the zero map on \( M \), they are homotopic, say \( f^{id}U = \partial \sigma + \sigma 0 \), and the existence of the desired dg module structure is equivalent with the existence of a homotopy \( \sigma \) with \( \sigma^2 = 0 \). The map \( \sigma_{i,*} \) constructed in our proof above is exactly the homotopy \( \sigma \) constructed in the proof of [1, Proposition 2.2]. In fact, all maps \( \sigma_{i,*} \) with \( i \geq 1 \) are homotopies. Indeed, since \( f_{i-1}f_i = 0 \), we see from condition \( A(i,n) \) in (2.1.1) that \( f_{i-1}\sigma_{i,*} \) is a chain map of degree \( i \) on \( U \), for all \( i \geq 1 \). Consequently, \( A(i,n) \) shows that \( \sigma_{i,*} \) is a homotopy between \( f_{i}\sigma_{i-1,*} \) and the zero map on \( U \).

**Remark 2.3.** Assume that the hypothesis of the Main Theorem holds. In the proof of the theorem, we saw that a basis for \( U^\# \) over \( A^\# \) can be taken to consist of the union of the bases of the free \( Q \)-modules \( V_0, V_1, V_2, \ldots \). With this choice of basis, the modules \( V_n \) defined in the proof of the theorem coincide with the modules \( V_n \) introduced in Construction 1.2, and thus the differential of the complex \( U \otimes_A R \) can be understood as described there. Further, Construction 1.2 and (1.6.1) give
\[
U_n \cong \bigoplus_{i=0}^{n} V_i \quad \text{and} \quad (U \otimes_A R)_n \cong V_n \otimes_Q R.
\]

When \( R \) is a local ring and \( M \) is a finitely generated \( R \)-module, we let \( \beta^n(R)(M) \) denote the \( n \)th Betti number of \( M \) over \( R \), that is, the rank of the \( n \)th free \( R \)-module
in a minimal free resolution of $M$, and we let
\[
P^R_M(t) = \sum_{i=0}^{\infty} \beta_i^R(M)t^i
\]
 denote the Poincaré series of $M$ over $R$.

Corollary 2.4 below recovers the Poincaré series formula in [4, Theorem 1.7], and shows that the hypothesis on $f$, $g$ in part (b) of Theorem 2.1 is necessary.

**Corollary 2.4.** Let $(Q, n)$ be a local ring and $f, g \in n \smallsetminus n^2$ with $fg = 0$. Set $R = Q/(f)$ and $A = Q\langle y, t \mid \delta(y) = f, \delta(t) = gy \rangle$. The following are equivalent:

1. $(f, g)$ is a pair of exact zero divisors;
2. For all finitely generated $R$-modules $M$, if $U$ is a minimal free resolution of $M$ over $Q$, then $U \otimes_A R$ is a minimal free resolution of $M$ over $R$;
3. For all finitely generated $R$-modules $M$, $P^R_M(t) = (1 - t)P^Q_M(t)$.

**Proof.** The implication $(1) \implies (2)$ is established in Theorem 2.1(b).

$(2) \implies (3)$: Let $n \geq 0$. Under the hypothesis of (2), we have $\beta_n^R(M) = \text{rank}_R(U \otimes_A R)_n$ and $\beta_n^Q(M) = \text{rank}_Q(U_n)$. Then (2.3.1) gives
\[
\beta_n^Q(M) = \sum_{i=0}^{n} \beta_i^R(M)
\]
and the Poincaré series formula follows from here.

$(3) \implies (1)$: If (3) holds, take $M = R$ in the Poincaré series formula to conclude $P^Q_R(t) = (1 - t)^{-1}$, and hence $\beta_n^Q(R) = 1$ for all $n \geq 0$. Since $\text{ann}(f)$ is a second syzygy in a minimal free resolution of $R = Q/(f)$ over $Q$, we conclude that $\text{ann}(f)$ is principal. If $\text{ann}(f) = (h)$ for some $h \in n$, then, since $g \in (h)$ and $g \notin n^2$, we must have $g = uh$ for some unit $u$, and hence $\text{ann}(f) = (h) = (g)$. Then ann$(g)$ is a third syzygy in a minimal free resolution of $R$ over $Q$, and we conclude it must be principal as well. Similarly, we obtain $\text{ann}(g) = (f)$.

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