Sampling properties of random graphs: the degree distribution

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We discuss two sampling schemes for selecting random subnets from a network: Random sampling and connectivity dependent sampling, and investigate how the degree distribution of a node in the network is affected by the two types of sampling. Here we derive a necessary and sufficient condition that guarantees that the degree distribution of the subnet and the true network belong to the same family of probability distributions. For completely random sampling of nodes we find that this condition is fulfilled by classical random graphs; for the vast majority of networks this condition will, however, not be met. We furthermore discuss the case where the probability of sampling a node depends on the degree of a node and we find that even classical random graphs are no longer closed under this sampling regime. We conclude by relating the results to real E.coli protein interaction network data.

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I. INTRODUCTION

Most networks investigated today are parts of much larger networks. These subnets can come in two different forms: first, we can choose a region of a network and consider all nodes that are in this region and only the edges between these nodes (for example a connected component of the larger network would be one such subnet). Looking at networks defined by all servers in a country, or the interaction network of all proteins which are confined to the mitochondria would be real-world examples [1, 2, 3]. Such networks may not be representative of the network as a whole but can give valuable insights into communication or biological processes within a defined sphere. More complicated is a second type of subnet where each node of the global network is included in the subnet with a certain probability $p$ and only the connections between pairs of nodes which are both included in the subnet are studied. This type of subnet is radically different from the regional-based subnets. It is, however, a frequent scenario in the analysis of technological and biological networks: most studies of molecular networks, such as protein-protein interaction [4, 5], gene-regulation [6] and metabolic networks [7], test for connections between a subset of the known molecular entities (proteins, genes and enzymes/metabolites, respectively). The process by which these entities (or corresponding probes) are chosen may reflect the bias of the experimenter or merely chance, and this will in turn influence the extent to which the subnet reflects properties of the global network in a meaningful way. In light of the relative straightforwardness of studying the sampling properties of networks, and their obvious importance for the analysis of current network data sets it is surprising that this problem has not been addressed previously.

Here we will focus on the simplest, and perhaps most parsimonious, process of sampling nodes: the case where each node in the network is included with probability $0 < p < 1$. In the present analysis we will concentrate on the sampling properties of the degree distribution of a network. The degree distribution, henceforth denoted by $\Pr(k)$, specifies the probability for a node to have $k$ connections, $k = 0, 1, \ldots$, and is probably the most common summary statistic used in the analysis of networks [8]. In particular the potential scale-free nature of real networks...
is often identified from the empirical degree distribution, which for scale-free networks takes on a power-law form, \( \Pr(k) \propto k^{-\gamma} \). Frequently a model is considered scale-free if the tail (i.e. for \( k \) sufficiently large) of the degree distribution takes such an asymptotic power-law form \( [1,12] \). Here we will consider this case as whether the degree distribution of randomly sampled subnets has the same properties as the degree distribution of the entire network. Thus far this question has been ignored in the literature, but as we will show, is of great importance in the analysis of real networks, which in their vast majority, are only subnets of larger networks. Unless explicitly stated otherwise we shall consider the thermodynamic limit, \( N \to \infty \).

II. THE DEGREE DISTRIBUTION OF A RANDOM SUBNET

A. Sampling from networks

We use \( N \) to denote a network with \( N \) nodes (we allow \( N \to \infty \)) drawn from a statistical ensemble of random networks \( [13,14] \) defined by some (potentially vector valued) parameter \( \Omega \) and let \( \Pr(k) \) be its degree distribution; the total number of edges is given by \( M \). Here we will be especially concerned with the case of a subnet \( S \) generated from the global network \( N \) by randomly sampling each node \( i \in N \) with a certain probability \( 0 \leq p_i \leq 1 \). Thus if a node of degree \( k \) gets picked for inclusion in the subnet, its degree in the subnet will depend on the number of its neighbours which are also included in \( S \).

1. Random sampling

We start by considering the case where the probability of picking a node is identical for all nodes, \( p_i = p \) for all \( i \). Here \( p = 0 \) and \( p = 1 \) are the trivial cases for which \( S = \emptyset \) and \( S = N \), respectively. Formally, the probability that a node with connectivity \( l \) in \( S \) given it has connectivity \( k \) in \( N \) is

\[
\Pr(l|k) = {k \choose l} p^l (1-p)^{k-l},
\]

where \( \Pr(x|y) \) denotes the conditional probability of \( x \) given \( y \). The degree distribution in the subnet is thus given by

\[
\Pr_S(l) = \sum_{k \geq l} \Pr(l|k) \Pr(k) = \sum_{k \geq l} {k \choose l} p^l (1-p)^{k-l} \Pr(k).
\]

This is probably the simplest and most parsimonious sampling scheme and may also be a reasonably realistic approximation, e.g. in the study of protein interaction networks where experimenters choose a set of proteins in a more or less haphazard fashion.

From Eqn. (2) we can show that

\[
\mathbb{E}_S[l] = \mathbb{E}[E[l|k]] = p \mathbb{E}[k] = pr,
\]

where \( \tau := \mathbb{E}[k] \) is the average degree in the network. Similarly we can show that the \( m \)-th moment of the descending factorial (defined by \( x[m] = x(x-1)(x-2)\ldots(x-m+1) \)) for the degree distribution of a network obeys

\[
\mathbb{E}_S[l[m]] = p^m \mathbb{E}[k[m]].
\]

Eqns. (3) and (4) are fulfilled for all networks, as long as the moments exist; for scale-free networks with exponent \( \gamma \), for example, moments of order greater or equal than \( [\gamma] \) do not exist.

2. Random sampling dependent on degree

A further sampling scheme will be considered here where the number of connections directly influences the probability, \( \pi(k) \), of sampling a node of degree \( k \); In the previous sampling scheme all nodes had the same chance of being sampled, \( \pi(k) = p \). We will focus on the particular case of an uncorrelated network.

The connectivity of a node in the subnet thus depends on the degrees of its neighbours. The probability that a node connected to a randomly chosen edge has degree \( k \) is given by

\[
\Pr^*(k) = \frac{k \Pr(k)}{\tau}
\]

where \( \tau \) is the average degree in the network; the average degree of the neighbours of a randomly chosen node is thus \( \mathbb{E}[k^2] / \mathbb{E}[k] \), if the two first moments of the degree distribution exist; below we will limit ourselves to such situations (for finite networks the moments will, of course, exist). Assuming a node is retained in the subnet then the probability of sampling a neighbouring node depends also on its connectivity and, in a mean-field approximation, the probability of retaining an edge originating from a node, \( \bar{p} \) is thus given by

\[
\bar{p} = \frac{1}{\tau} \sum_k k \Pr(k) \pi(k)
\]

The degree distribution of the subnet \( S \) is again given by binomial sampling:

\[
\Pr_S(l) = \left[ \sum_{k \geq l} {k \choose l} \bar{p}^l (1-\bar{p})^{k-l} \pi(k) \Pr(k) \right] / \sum_{k=0}^\infty \pi(k) \Pr(k).
\]

Defining

\[
\Pr_0(k) = \pi(k) \Pr(k) / \sum_{k=0}^\infty \pi(k) \Pr(k)
\]

From Eqn. (2) we can show that

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\mathbb{E}_S[l] = \mathbb{E}[E[l|k]] = p \mathbb{E}[k] = pr,
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\Pr_S(l) = \left[ \sum_{k \geq l} {k \choose l} \bar{p}^l (1-\bar{p})^{k-l} \pi(k) \Pr(k) \right] / \sum_{k=0}^\infty \pi(k) \Pr(k).
\]

Defining

\[
\Pr_0(k) = \pi(k) \Pr(k) / \sum_{k=0}^\infty \pi(k) \Pr(k)
\]
we can rewrite Eqn. (7) in the same form as Eqn. (2). With these probabilities the degree distribution in the
subnet is given analogously to Eqn. (2) as
\[ \Pr_S(l) = \sum_{k=0}^{\infty} \binom{k}{l} \hat{p}^l (1 - \hat{p})^{k-l} \Pr_0(k) \]
(9)
Obviously, when setting \( \pi(k) = p \) Eqn. (9) simplifies to Eqn. (4).
We still have to specify the functional form of \( \pi(k) \); a priori
the only constraint is that \( \pi(k) \) has to be a prob-
ability for all \( k \), i.e. \( 0 \leq \pi(k) \leq 1 \). One
possible and obvious choice is to let \( \pi(k) \propto k \); in order
to ensure that \( \pi(k) \leq 1 \) for large \( k \) we set
\[ \pi(k) = Ck \]
(10)
with \( C \) sufficiently small such that \( \pi < 1 \) for large \( k \)
we can always trivially set \( C = 2\mathbb{E}[M] \) with \( \mathbb{E}[M] \) the
expected number of edges in the network). In this case
\[ \hat{p} = \frac{C}{r} \sum_{k} k^2 \Pr(k) = C\mathbb{E}[k^2], \]
(11)
i.e. \( \hat{p} \) depends on the degree distribution solely via the
first and second moments of \( \Pr(k) \). We will refer to this
sampling scheme as preferential sampling of nodes.

B. Probability generating functions of random
subnets
We represent the degree distribution of a network \( \mathcal{N} \)
through its probability generating function (PGF) [17],
\[ G(s) = \sum_{k=0}^{\infty} \Pr(k)s^k. \]
(12)
The probability \( \Pr(k) \) follows from the PGF via the
relationship
\[ \Pr(k) = \left. \frac{1}{k!} \frac{d^k G(s)}{ds^k} \right|_{s=0} \]
(13)
With Eqns. (12) and (1) we can straightforwardly de-
rive the PGF for the subnet
\[ G_S(s) = \sum_{l=0}^{\infty} \Pr(l)s^l \]
\[ = \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \Pr(l|k)\Pr(k) \]
\[ = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} p^l (1-p)^{k-l} s^l \Pr(k) \]
\[ = \sum_{k=0}^{\infty} \Pr(k)(1-p+ps)^k \]
\[ = G(1-p+ps). \]
(14)
If nodes with degree \( l = 0 \) are ignored (as is frequently
the case in high throughput protein interaction data) then
delting all nodes with \( l = 0 \) the PGF in the subnet becomes
\[ G^*_S(s) = \frac{G(1-p+ps) - G(1-p)}{1-G(1-p)}. \]
(15)
Eqns. (14) and (15), respectively, hold generally for the
degree distributions of subnets randomly sampled from
networks, depending on whether orphaned nodes (i.e.\)
those with connectivity \( l = 0 \) are allowed or not [17].
Interestingly, if Eqn. (14) holds then also Eqn. (15)
holds with \( G(s) \) replaced by \( G^*(s) = (G(s) - \Pr(0))/(1-
Pr(0)) \); i.e. networks with orphaned nodes removed are
closed under random sampling if the networks with the
orphaned nodes retained are.

III. CLOSURE UNDER RANDOM SAMPLING
FROM NETWORKS
A. Conditions for closure: generating function

From Eqns. (14) and (15) it is apparent that degree
distributions of a subnet \( \mathcal{S} \) cannot generally be expected
to be of the same type (e.g. a Poisson distribution) as the
degree distribution of the global network \( \mathcal{N} \). For some
important types of networks, however, it can be shown
that random sampling of nodes gives rise to networks
with degree distributions of the same type as the global
network, but with a different parameter depending on \( p \),
i.e. \( \mathcal{O}' = fn(\mathcal{O}, p) \). In this case we say that a network (or
its degree distribution) is closed under random sampling
of nodes. For a network ensemble to be closed under
random sampling the following condition is necessary and
sufficient [17],
\[ G_S(s; \Omega) = G(s; \Omega') = G(1-p+ps; \Omega), \]
(16)
and
\[ G^*_S(s; \Omega) = G(s; \Omega') = \frac{G(1-p+ps; \Omega) - G(1-p; \Omega)}{1-G(1-p; \Omega)}, \]
(17)
when the subnet is not allowed to have orphaned nodes.
Necessity and sufficiency follow from Eqns. (14) and (15)
and the definition of the properties of a closed subnet.

B. Conditions for closure: moments

Equations (14) and (15) can be applied to all types
of degree distributions. Inspired by Eqns. (3) and (4)
we here derive a general condition in terms of moments
for a subnet to be of the same type as the global
network. We assume the moments determine the degree
distribution uniquely (in particular, this implies that all
moment exist), which is true under mild regularity con-
ditions. Let an ensemble of random networks be given
which is parameterized by \( \Omega \). For example, the ensemble of classical or Erdős-Rényi random graphs \([13, 15]\) has \( \Pr(k) = \exp(-\lambda)\lambda^k/k! \) and \( \Omega = \lambda \) is the average connectivity. We seek a condition that, provided nodes are sampled with probability \( p \), ensures that the degree distribution of the subnet remains in the same ensemble of random networks. Without loss of generality we can assume that \( \Omega \) has the form \( \Omega = (\tau, \psi) \), where \( \tau \) is the average degree in the network and \( \psi \) is an additional (potentially vector valued) parameter.

From Eqn. \( 3 \) we know that the average connectivity in the sampled subnet, \( \tau_p \), is given by \( \tau_p = \tau p \). We can use Eqn. \( 4 \) to show that a family of degree distributions is closed under random sampling of nodes if and only if the descending factorial moments obey the relationship

\[
\mathbb{E}[k_{[m]}] = a_m(\psi)\tau^m, \tag{18}
\]

where \( a_m(\psi) \) is a constant that depends only on \( m \) and \( \psi \) but not on \( \tau \) and the sampling probability \( p \), and where \( a_1(\psi) = 1 \).

To prove that Eqn. \( 18 \) is necessary we assume that the network is closed under random sampling of nodes and write \( \tau = \mathbb{E}(k) \) and \( g_m(\tau, \psi) = \mathbb{E}(k_{[m]}) \). Because of Eqns. \( 3 \) and \( 4 \) we can immediately write

\[
g_m(p\tau, \psi) = p^m g_m(\tau, \psi) \tag{19}
\]

and

\[
g_m(p\tau, \psi)/((p\tau)^m) = g_m(\tau, \psi)/\tau^m. \tag{20}
\]

Thus \( g_m(\tau, \psi)/\tau^m = \text{const.} \) (for all \( \tau \)) or

\[
g_m(\tau, \psi) = a_m(\psi)\tau^m, \tag{21}
\]

with \( a_1(\psi) = 1 \) as required.

To prove sufficiency assume that the descending moments of \( k_{[m]} \) fulfill Eqn. \( 18 \); using Eqn. \( 4 \) the descending factorial moments of the nodal degrees in the subnet follow the relationship

\[
\mathbb{E}_S[k_{[m]}] = a_m(\psi)(p\tau)^m. \tag{22}
\]

Since the descending moments determine the moments, \( \mathbb{E}(k^m) \) of a degree distribution, which in turn determine the distribution uniquely (by assumption), then the degree distribution of the subnet is given by a distribution that is of the same type as the degree distribution but with a rescaled parameter. Thus Eqn. \( 18 \) is a necessary and sufficient condition for a network ensemble to be closed under random sampling of nodes. □

### C. Analytical Examples

We can use relationships \([10, 13]\) to determine whether a degree distribution is closed under random sampling. We will discuss this for three commonly observed degree distributions. Note that we only consider a degree distribution to be closed under (random) sampling if the degree distributions of the network and the subnet belong to the same family of probability distributions.

Classical random graphs have a Poisson degree distribution, \( \text{Po}(\lambda) \). It is straightforward to show that the descending moments of the Poisson distributed random variables are given by

\[
\mathbb{E}[k_{[m]}] = \tau^m = \lambda^m. \tag{23}
\]

Thus \( a_m = 1 \) for all \( m \geq 1 \) and the degree distribution of classical random graphs is closed under random sampling of nodes. If we therefore have a subnet \( S \) of size \( M \) drawn from a larger network \( N \) of known size \( N \) we can determine \( \lambda \) from \( \lambda_S = \lambda M/N \). The subnet is therefore informative about the global network.

Networks which grow by random attachment of new nodes give rise to exponential degree distributions such that asymptotically (large \( N \)) \( \Pr(k) = (1 - e^{-\alpha})e^{-\alpha k} \). For such a distribution it is easily shown that

\[
\mathbb{E}[k_{[m]}] = m!e^{-m\alpha}/(1 - e^{-\alpha})^m = m!\tau^m, \tag{24}
\]

since \( \mathbb{E}[k] = e^{-\alpha}/(1 - e^{-\alpha}) \). This means that \( \mathbb{E}[k_{[m]}] \) can be written in the form specified by Eqn. \( 18 \) and therefore exponential degree distributions are closed under random sampling. Binomial (as for classical finite-sized random graphs) and negative binomial distributions are also closed under random sampling as is easily verified. An explicit construction of probability distributions which are closed is discussed in appendix A.

If the probability of attaching to a node is proportional to its degree the resulting network will asymptotically have a power-law degree distribution with exponent \( 3 \). For models where an existing node is duplicated and each of its connections is kept with certain probability \( \gamma < 1 \), all moments greater than \( \gamma \) diverge and we therefore have to use the PGF formalism. The PGF for the global network is given by

\[
G(s; \gamma) = \frac{1}{\zeta(\gamma)} \sum_{k=1}^\infty s^k k^{-\gamma} \tag{25}
\]

and since \( k = 0 \) is explicitly forbidden in a scale-free network, we use Eqn. \( 17 \) to construct the PGF in the subnet, whence

\[
G^*_S(s; \gamma) = \frac{\sum_{k=1}^\infty [(1-p+ps)^k - (1-p)^k] k^{-\gamma}}{\zeta(\gamma) - \sum_{k=1}^\infty (1-p)^k k^{-\gamma}}. \tag{26}
\]

Clearly for \( p \to 1 \) we obtain the original PGF, \( G(s; \gamma) \). For \( 0 < p < 1 \), however, it is impossible to determine...
an exponent $\gamma'$ such that $G_S$ could be written in terms of the PGF of a power law. Therefore random subnets drawn from exact scale-free networks are not themselves scale-free. This can also be shown explicitly using a series expansion \[17\]. We note, however, that the tail of the degree distribution of the subnet still takes on a powerlaw form for $k$ sufficiently large. The same analysis applied to other fat-tailed probability distributions also shows that other fat-tailed degree distributions such as the log-normal and the stretched exponential families \[20\] are not closed under random sampling.

D. Numerical Examples

The effect of random sampling on the degree distribution is most straightforwardly illustrated using numerical solutions of Eqns. \(2\) and \(6-9\). Here we do this for networks of infinite size and for simplicity focus on the canonical models of the classical random graph and the exact scale-free network, respectively.

In part (a) of figure 2 we show the Poisson distribution with $\lambda = 5$ and the distributions of random subnet with $p = 0.8$ and $p = 0.2$, respectively. The subnet distributions are identical with the Poisson distributions with parameters $\lambda = 4$ and $\lambda = 2$. This also means that as $p\lambda$ becomes smaller than one the subnet will move through the phase-transition where the giant connected component dissolves and the size distribution of connected parts of the subnet becomes exponential.

In part (b) of the same figure we show the power-law distribution with $\gamma = 3$ and again the respective subnet degree distributions (renormalized such that $Pr_S(0) = 0$ in the subnet). We find that the subnet degree distributions are no longer straight lines but that as $k$ becomes large they run parallel to the original distributions. That is, as already described above, the tails of degree distributions of subnets sampled randomly from scale-free networks also fall off in the same power-law fashion as the original network. But at low connectivities the departure from the scale-free network is quite pronounced: probability mass moves from the tail towards the lowly connected nodes with $k = 1$, which become more abundant than would be expected for a true scale-free network. This will have quite considerable effects for finite size networks. The deviation of the subnet degree distribution from a pure power-law at small to intermediate connectivities increases with $\gamma$ (as well as, naturally, with decreasing sampling probability $p$). We note however, that the tail of the degree distribution will retain a powerlaw form; thus for an alternative definition of scale-free behaviour which only requires $Pr_S(k) \propto k^{-\gamma}$ for $k \to \infty$ random subnets will retain scale-free behaviour in the sense that the tail is still described by a powerlaw $Pr_S(k) \propto k^{-\gamma'}$ for $k \to \infty$. In general, however, when the whole degree distribution is considered scale-free networks are not closed under random sampling.

IV. CONNECTIVITY DEPENDENT SAMPLING

There is no unique and obvious way in which the probability of sampling a node may depend on the connectivity. Here we briefly outline the behaviour of the degree distribution under the simple schemes outlined above where the probability of sampling a node is no longer uniform but linearly proportional to its connectivity, i.e. if $\tau(k) \propto k$; we assume that $\tilde{p}(k)$ is given by Eqn. \[11\].

For a Poisson degree distribution with parameter $\lambda$ we have $\mathbb{E}[k^2] = \tau^2 + \tau = \lambda^2 + \lambda$ and $\mathbb{E}[M] = N\lambda/2$, (assuming the network is large and finite) whence $\tilde{p} =$

![Figure 2: Degree distributions of full network and subnets obtained by sampling each node with probability $p = 0.8$ and $p = 0.2$, respectively, for classical random graphs (a) and scale-free networks (b).](image)
(\lambda + 1)/(N\lambda) and
\[
\Pr_0(k) = \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!},
\]
if we set \(C = 2\mathbb{E}[M]\) in Eqn. (10). In this case Eqn. (9)
becomes
\[
\Pr_S(l) = \sum_{k \geq l} \binom{k}{l} \tilde{p}^l (1 - \tilde{p})^{k-l} \Pr_0(k)
= \frac{(\lambda \tilde{p})^l}{l!} e^{-\lambda \tilde{p}} \left(1 - \tilde{p} + \frac{l}{\lambda}\right)
\]
for \(l = 0, 1, \ldots\). The distribution in the subnet is thus
not a pure Poisson distribution but one multiplied by
a factor \(1 - \tilde{p} + l/\lambda\). Under this connectivity dependent
sampling classical random graphs are therefore not closed
and subnets \(S\) are qualitatively (if perhaps only rather
slightly) different from the overall network \(N\).

For scale-free networks with \(\gamma \leq 3\) the second
moment diverges, \(\mathbb{E}[k^2] \to \infty\), and we therefore focus on fi-
nite (though potentially very large networks). Networks
with a powerlaw degree distribution can, for example, be
constructed using standard methods \[21, 22, 23\]. For
such a scale-free graph with \(N\) nodes we have to numer-
ically evaluate the expected number of edges \(\mathbb{E}[M] = \sum_{k=1}^{N} k^{-\gamma+1}/\zeta(\gamma)\) and \(\tilde{p}\), given by Eqn. (11). For
\(\Pr_0(k)\) we obtain for scale-free networks
\[
\Pr_0(k) = \frac{k^{1-\gamma}}{\zeta(\gamma - 1)}
\]
Proportional sampling from a scale-free network defined
by a powerlaw exponent \(\gamma\) is thus identical to sampling
from a network with powerlaw exponent \(\gamma - 1\) and sam-
pling probability \(\tilde{p}\). Therefore we can use the results ob-
tained above and conclude that the scale-free network (in
the strict sense outlined above) is not closed under propor-
tional sampling of nodes; for sufficiently large degrees,
however, the tail of the degree distribution will still have
a powerlaw form.

V. PROTEIN INTERACTION NETWORK DATA

In figure \(3\) we show three degree distributions cor-
responding to the protein interaction network (PIN) data
from \(E.coli\) which was available in April 2003, 2004 and 2005 in the database of interaction proteins
(DIP: \url{dip.doe-mbi.ucla.edu}); the resulting networks
are made up of the interactions among 228, 373 and 480
proteins and have 293, 515 and 760 interactions, respec-
tively. Figure \(3\) confirms the results of the theoretical
analysis presented above: as the fraction of sampled net-
work nodes decreases statistical weight shifted from the
tail towards lower degrees; the degree of the single highly
connected node, \(k = 54\), was already known in the 2003
dataset (no further interactions have been added to this
node since). The statistical weight of sparsely connected
nodes, \(k = 1\), increases as the fraction of sampled nodes
decreases. We note that the present data samples only a
small subnet of the \(E.coli\) PIN which consists of interactions
among approximately 4000 proteins. Moreover (i) it is
well established that PIN data is highly unreliable
and very noisy, and (ii) the true sampling scheme under-
lying the sampling scheme will generally be more com-
plicated than the first order model employed here. The
behaviour appears, however, to be qualitatively similar
to our theoretical analysis.

VI. CONCLUSION

Both sampling schemes discussed here are necessarily
simpler than is the case in many real situations, such
as the analysis of protein interaction networks (see e.g.
\[24, 25\]. We believe, however, that between them they
retain some vestiges of reality. Crucially, however, we
wish to stress the incomplete nature of many network
data sets. For many of these data sets in fact, including
protein interaction network data, it appears that some
form of random sampling is more realistic than a process
in which the neighbourhood of a node is explored and
neighbouring sites are recruited iteratively into the ex-
perimental setup. No matter what the sampling process
is, it has to be included into the analysis from the outset;
making inferences from incomplete (in the sense that not
all nodes have been sampled) network data may give mis-
leading results. If a network is closed under random (or
connectivity dependent) sampling then it is straightforward
to infer properties of the overall network from the
subnet. For some, notably Erdős-Rényi random graphs,
this is indeed the case. In general, however, the degree distributions of the network and sampled subnets will be qualitatively different. For example, while powerlaw tails will also give rise to powerlaw tails in the subnet a network which has an exact powerlaw degree distribution is not closed under random sampling. The same is true for other broad-tailed degree distributions such as lognormal or stretched exponential distributions.

Sampling properties will also affect other network statistics, including network diameter and average path length, clustering coefficient and network motifs. These will be studied in a companion paper. We believe that sampling properties ought to be included explicitly and from the outset into any network analysis, unless there are good evidence that the whole (or the majority) of the network’s nodes have been included in the data. Quite apart from the relevance of this work in the analysis of real data we believe that a detailed analysis of sampling properties of graphs is a rich field which, surprisingly, appears to have been neglected thus far.

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APPENDIX A: CONSTRUCTION OF CLOSED DEGREE DISTRIBUTIONS

We have shown that Eqn. 15 is both a necessary and sufficient condition for a degree distribution to be closed under binomial random sampling. We can also use Eqn. 15 to construct closed distributions de novo as any series of positive numbers \(a_k\), \(k = 1, 2, \ldots\) with \(a_1 = 1\) defines a family of random variables closed under binomial sampling via the condition

\[
E[k_{[m]}] = a_m \tau^m
\]

for some \(\tau \in T = [0, t]\) and \(t \geq 0\).

First, the degenerate distribution \(\Pr(k = 0) = 1\) is defined by \(E[k_{[m]}] = 0\) for all \(m > 0\). Therefore \(0\) must be in the interval \(T\) and \(T\) is non-empty as the degenerate distribution is trivially closed under binomial sampling. Now assume that \(\tau \geq 0\) defines the distribution of \(k\) through Eqn. (A1). Any \(\tau^*\) with \(0 \leq \tau^* \leq \tau\) defines the degree distribution after binomial sampling of nodes from \(k\) with probability \(p = \tau^*/\tau\) which, by construction, has degree distribution given by \(E[k_{[m]}] = a_m (\tau^*)^m\). The distributions defined by Eqn. (A1) are therefore closed under random sampling of nodes.

Eqn. (A1) can be used to to construct arbitrary degree distributions which are closed under binomial sampling. Nontrivial examples are possible; for example

\[
ak = (k + 1)2^{-k} \quad \text{for} \quad k = 1, 2, \ldots
\]

defines a distribution closed under random sampling,

\[
\Pr(k) = \frac{(2\tau)^k}{k!} (k + 1 - 2\tau)e^{-2\tau}
\]

where \(\tau = E[k] \in [0, 0.5]\) (note that for \(\tau = 0.5\), \(\Pr(k = 1)\) defined by Eqn. (A2) is Poisson distributed).

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