GLOBAL BOUNDEDNESS AND DECAY FOR A
MULTI-DIMENSIONAL CHEMOTAXIS-HAPTOTAXIS SYSTEM
WITH NONLINEAR DIFFUSION

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Abstract. This paper deals with a parabolic-elliptic-ODE chemotaxis-haptotaxis system with nonlinear diffusion
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (\varphi(u) \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u (1 - u - w), \\
0 &= \Delta v - v + u, \\
\frac{\partial w}{\partial t} &= -vw,
\end{align*}
\]
under Neumann boundary conditions in a smooth bounded domain \(\Omega \subset \mathbb{R}^n\) \((n \geq 1)\), where \(\chi, \xi, \mu\) are positive parameters and \(\varphi(u)\) is a nonlinear diffusion. Under the non-degenerate diffusion and some suitable assumptions on positive parameters \(\chi, \xi, \mu\), it is shown that the corresponding initial boundary value problem possesses a unique global classical solution that is uniformly bounded in \(\Omega \times (0, \infty)\). Moreover, under the degenerate diffusion, it is proved that the corresponding problem admits at least one nonnegative global bounded-in-time weak solution. Finally, for the suitably small initial data \(w_0\), we give the decay estimate of \(w\).

1. Introduction. In this paper, we consider the following chemotaxis-haptotaxis system with nonlinear diffusion
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (\varphi(u) \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u (1 - u - w), \\
0 &= \Delta v - v + u, \\
\frac{\partial w}{\partial t} &= -vw, \\
\varphi(u) \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \\
u(x, 0) &= u_0(x), \quad w(x, 0) = w_0(x),
\end{align*}
\]
where \(\Omega \subset \mathbb{R}^n\) \((n \geq 1)\) is a bounded domain with smooth boundary \(\partial \Omega\), \(\frac{\partial}{\partial \nu}\) denotes the differentiation with respect to the outward normal derivative on \(\partial \Omega\), and \(\varphi\) is

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the nonlinear diffusion. The parameters $\chi$, $\xi$ and $\mu$ are positive and the initial data $(u_0, w_0)$ is supposed to be satisfied the following conditions

\[
\begin{align*}
  &u_0 \in C^0(\Omega) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\
  &w_0 \in C^{2+\alpha}(\Omega) \text{ with } \alpha \in (0, 1) \text{ w}_0 > 0 \text{ in } \Omega \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial \Omega .
\end{align*}
\]  \tag{2}

The oriented movement of biological cells or organisms in response to a chemical gradient is called chemotaxis. The pioneering works of chemotaxis model were introduced by Patlak [21] in 1953 and Keller and Segel [17] in 1970, and we refer the reader to the survey [11, 13, 14] where a comprehensive information of further examples illustrating the outstanding biological relevance of chemotaxis can be found.

In recent years, chemotactic mechanisms have also been detected to be crucial in the process of cancer invasion, where they usually interact with haptotaxis, the correspondingly directed cell movement in response to gradients of non-diffusible signals. The combination of these two cell migration mechanisms was initially proposed by Chaplain and Lolas in [6, 7] to describe cancer cell invasion into surrounding healthy tissue. More precisely, their model accounts for both chemotactic migration of cancer cells towards a diffusible matrix-degrading enzyme (MDE) secreted by themselves, and haptotactic migration towards a static tissue, also referred to as extracellular matrix (ECM). In this context, $u(x, t)$ represents the density of cancer cell, $v(x, t)$ denotes the concentration of MDE, and $w(x, t)$ stands for the density of ECM. In addition to random movement, cancer cells are supposed to bias their movement both towards increasing concentrations of urokinase plasminogen activator by chemotaxis (see [3]), and towards increasing densities of the non-diffusible ECM through detecting the macromolecules adhered therein by haptotaxis (see [2]). It is assumed that the cancer cells undergo birth and death in a logistic manner, competing for space with the ECM. The MDE is assumed to be produced by cancer cells, and to diffuse and decay, whereas the ECM is stiff in the sense that it does not diffuse, but it could be degraded upon contact with MDE.

In order to better understand model (1), let us mention some previous contributions in this direction. In recent years, the following initial boundary value problems have been studied by many authors

\[
\begin{align*}
  &u_t = \nabla \cdot (\varphi(u)\nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, t > 0, \\
  &\tau v_t = \Delta v - v + u, \quad x \in \Omega, t > 0, \\
  &w_t = -vw, \quad x \in \Omega, t > 0, \\
  &\varphi(u) \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
  &u(x, 0) = u_0(x), \quad \tau v(x, 0) = \tau v_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega ,
\end{align*}
\]  \tag{3}

where $\tau \in \{0, 1\}$, $\chi > 0$, $\xi > 0$, $\mu > 0$ and $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary $\partial \Omega$.

When $\tau = 1$, for the special case $\varphi(u) = 1$ in (3), Tao and Wang [25] proved that model (3) possesses a unique global-in-time classical solution for any $\chi > 0$ in one space dimension, and for small $\frac{\chi}{\mu} > 0$ in two and three space dimensions. Later, Tao [23] improved the result of [25] for any $\mu > 0$ in two space dimension. Moreover, If $\varphi(u) \in C^2([0, \infty))$, $\varphi(0) > 0$ and $\varphi(u) \geq \delta u^{m-1}$, where $\delta > 0$ and
m > \max\{1, m\} \text{ with }$

\begin{align*}
m = \begin{cases} 
\frac{2n^2+4n-4}{n(n+4)}, & \text{if } n \leq 8, \\
\frac{2n^2+3n+2-\sqrt{8n(n+1)}}{n(n+2)}, & \text{if } n \geq 9,
\end{cases}
\end{align*} \tag{4}

Tao and Winkler [28] proved that model (3) possesses at least one nonnegative global classical solution, however, their boundedness is left as an open problem. In [12], Hillen et.al studied the global boundedness and asymptotic behavior of model (3) with \( \varphi(u) = 1 \) in one space dimension and proposed an open problem about boundedness in the higher-dimensional case. For the case \( \varphi(u) = 1 \) in (3), Tao [24] showed that under appropriate regularity assumption on the initial data \((u_0, v_0, w_0)\), the corresponding initial-boundary problem possesses a unique classical solution which is global in time and bounded in two space dimensions. Moreover, Cao [4] proved the boundedness of solutions with the case \( \mu > 0 \) in two space dimensions. Furthermore, in [41], the results in [24] have been extended to the nonlinear diffusion case and the decay of \( w \) was studied. Recently, the boundedness of model (3) was derived in higher-dimensional case (see [34, 19]).

When \( \tau = 0 \), i.e. the diffusion rate of the MDE is much greater than that of cancer cells [7]. Moreover, similar quasi-steady-approximations for corresponding chemotactarctant equations were frequently used to study classical chemotaxis systems (for instance [16, 22]). For the special case \( \varphi(u) = 1 \) in (3), Tao and Wang [26] proved that model (3) possesses a unique global bounded classical solution for any \( \mu > 0 \) in two space dimension, and for large \( \mu > 0 \) in three space dimensions. Furthermore, in [30], Tao and Winkler studied global boundedness for model (3) with \( \varphi(u) = 1 \) under the condition \( \mu > \frac{(n-2)\chi}{n} \), furthermore, in additional explicit smallness on \( w_0 \), they gave the exponential decay of \( w \) in the large time limit. When \( w \equiv 0 \), model (3) is reduced to the following chemotaxis-only system

\begin{align*}
\begin{cases}
\begin{aligned}
\partial_t u &= \nabla \cdot (\varphi(u)\nabla u) - \chi \nabla \cdot (u\nabla v) + \mu u(1-u), & x \in \Omega, t > 0, \\
0 &= \Delta v - v + u, & x \in \Omega, t > 0,
\end{aligned}
\end{cases}
\end{align*} \tag{5}

This system has been widely studied by many authors in these years. In the case \( \varphi(u) = 1 \) in (5), Tello and Winkler [31] proved that the solutions of semilinear parabolic-elliptic problem are global and bounded provided that either \( n \leq 2 \), or \( n \geq 3 \) and \( \mu > \frac{(n-2)\chi}{n} \) with \( \chi > 0 \). Moreover, for any \( n \geq 1 \) and \( \mu > 0 \), the existence of global weak solution was shown under some additional conditions. Furthermore, if the logistic source \( f(u) \leq a - bu^k \), \( k > 2 - \frac{1}{n} \), some global very weak solutions of semilinear parabolic-elliptic model were constructed by Winkler [36]. When \( \varphi(u) \geq c(u + 1)^p \) with \( p \in \mathbb{R} \) and \( \mu > \left(1 - \frac{2}{n(1-p^r)}\right) \chi \) with \( \chi > 0 \), Cao and Zheng [5] proved that the simplified parabolic-elliptic model (5) has a unique global classic solution, which is uniformly bounded. Recently, Wang et.al in [33] investigated the boundedness and asymptotic behavior for model (5) with the special case \( \varphi(u) \geq C_D u^{m-1} \) \( (m \geq 1) \) under other additional technique conditions. In the recent paper [37], for the case of \( \varphi(u) = 1 \), \( f(u) = ru - \mu u^2 \) with \( r \geq 0 \) and \( \mu > 0 \), in one-dimensional case, Winkler proved that going beyond carrying capacities actually is a genuinely dynamical feature of (5) provided that \( \mu < 1 \) and diffusion is sufficiently weak, moreover, he investigated global boundedness and finite-time blow-up for a corresponding hyperbolic-elliptic limit problem. Furthermore, Lankeit [18] extended the results of [37] to the higher dimensional radially symmetric case.
Motivated by the above works, the present paper deals with global boundedness for model (1) under some suitable conditions. The crucial assumption in our result is related to the diffusion function $\varphi(u)$. Our main results in this paper are stated as follows.

Firstly, we consider global boundedness for model (1) in the case of non-degenerate diffusion. To do this, we assume that $\varphi \in C^2([0, \infty))$ satisfies

$$\varphi(s) \geq c_0(s + 1)^{-p} \quad \text{for all } s \geq 0, p \in \mathbb{R} \text{ and some } c_0 > 0.$$  \hfill (6)

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with smooth boundary. Assume that $\varphi \in C^2([0, \infty))$ satisfies (6). Suppose that $\chi > 0$, $\xi > 0$ and $\mu > \mu^*$, where

$$\mu^* = \begin{cases} 0, & \text{if } p < \frac{2}{n} - 1, \\ \frac{(p+1)n-2}{(p+1)n} \chi, & \text{if } p \geq \frac{2}{n} - 1. \end{cases}$$  \hfill (7)

Then for any $(u_0, v_0)$ fulfilling (2), model (1) possesses a unique global classical solution $(u, v, w)$ which is uniformly bounded in $\Omega \times (0, \infty)$ in the sense that there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0.$$  

**Remark 1.** Compared with the chemotaxis-only system in [5, 33], it is easy to see that the non-diffusive interaction with the ECM variable $w$ in (1) does not affect the global boundedness under some suitable assumptions. Moreover, when $\varphi(u) = 1$, the results in Theorem 1.1 are consistent with those in [30].

Next, we consider weak solutions in degenerate case of (1), because in general there is no classical solution. Suppose that $\varphi \in C^2([0, \infty))$ satisfies

$$\varphi(s) = c_1 s^{-p} \quad \text{for all } s \geq 0, p \in \mathbb{R} \text{ and some } c_1 > 0.$$  \hfill (8)

Before stating our second result, we first give the definition of weak solutions of (1).

**Definition 1.2.** Let $T \in (0, \infty)$. A pair of nonnegative function $(u, v, w)$ defined in $\Omega \times (0, T)$ is called a weak solution of model (1), if (i) $(u, v, w)$ satisfies the following

\begin{align*}
&u \in L^2((0, T); L^2(\Omega)), \varphi(u) \nabla u \in L^2((0, T); L^2(\Omega)), v \in L^2((0, T); W^{1,2}(\Omega)), \\
&u \nabla v \in L^2((0, T); L^2(\Omega)), w \in L^2((0, T); W^{1,2}(\Omega)), u \nabla w \in L^2((0, T); L^2(\Omega)), \quad (9) \\
&uv \in L^2((0, T); L^2(\Omega)) \text{ and } uw \in L^2((0, T); L^2(\Omega)),
\end{align*}

(ii) the following integral equalities

\begin{align*}
- \int_0^T \int_\Omega u \varphi \xi dx dt - \int_\Omega u_0 \xi(x, 0) dx dt \\
= - \int_0^T \int_\Omega \varphi(u) \nabla u \cdot \nabla \xi dx dt + \chi \int_0^T \int_\Omega u \nabla v \cdot \nabla \xi dx dt \\
+ \xi \int_0^T \int_\Omega u \nabla w \cdot \nabla \xi dx dt + \mu \int_0^T \int_\Omega u \zeta(1 - u - w) \quad (10)
\end{align*}

and

\begin{align*}
- \int_0^T \int_\Omega \nabla v \cdot \nabla \eta dx dt - \int_0^T \int_\Omega \eta dx dt + \int_0^T \int_\Omega \eta dx dt = 0 \quad (11)
\end{align*}
as well as
\[
\int_0^T \int_\Omega w \theta dxdt = \int_0^T \int_\Omega w_0 e^{\int_0^t v(x,s)ds} dxdt
\]
(12)
hold for all \((\zeta, \eta, \theta) \in (C_0^\infty(\Omega \times [0, T]))^3\). Furthermore, if \((u, v, w)\) is a weak solution of model (1) in \(\Omega \times (0, T)\) for all \(T \in (0, \infty)\), it is said that \((u, v, w)\) is a global weak solution.

**Theorem 1.3.** Let \(\Omega \subset \mathbb{R}^n\), \(n \geq 1\) be a bounded domain with smooth boundary. Assume that \(\varphi \in C^2([0, \infty))\) satisfies (8). Suppose that \(\chi > 0\), \(\xi > 0\) and \(\mu > \mu^*\), where \(\mu^*\) is defined by (7). Then for any \((u_0, w_0)\) fulfilling (2), model (1) possesses at least one nonnegative global bounded weak solution \((u, v, w)\).

Finally, for the suitably small initial data \(w_0\), we consider the decay estimate of \(w\).

**Theorem 1.4.** Under the same conditions of Theorem 1.1, assume that the initial data \((u_0, w_0)\) satisfies (2) and
\[
||w_0||_{L^\infty(\Omega)} + \frac{\xi^2}{\mu K} \cdot \left[ \frac{||w_0||_{L^\infty(\Omega)}}{4e} + \frac{||\nabla w_0||_{L^2(\Omega)}}{||\Omega||} \right] < 1,
\]
(13)
where \(K := \min_{0 \leq s \leq ||u(\cdot, t)||_{L^\infty(\Omega)}} c_0(s+1)^{-p} > 0\). Then there exist positive constants \(\kappa\) and \(C\) such that the third solution component \(w\) satisfies the following decay estimate
\[
||w(\cdot, t)||_{W^{1, \infty}(\Omega)} \leq Ce^{-\kappa t} \text{ for all } t > 0.
\]
(14)

This paper is organized as follows. In Section 2, we show the local-in-time existence of a classical solution to model (1) and give some preliminary inequalities which are important for our proofs. In Section 3, we consider the global existence and boundedness of solutions for model (1) under some suitable conditions and prove Theorem 1.1. In Section 4, we concern with global weak solutions of (1) and prove Theorem 1.3. Finally, we prove Theorem 1.4 in Section 5.

2. Preliminaries. We first state one result concerning local-in-time existence of classical solution to model (1).

**Lemma 2.1.** Let \(\chi > 0\), \(\xi > 0\) and \(\mu \geq 0\), and assume that the function \(\varphi \in C^2([0, \infty))\) satisfies (6). Then for any initial data \((u_0, w_0)\) fulfilling (2), there exists a maximal existence time \(T_{\text{max}} \in (0, \infty)\) such that model (1) possesses a unique classical solution
\[
\begin{align*}
&u \in C^0(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})), \\
v \in C^0((0, T_{\text{max}}); C^2(\Omega))
\end{align*}
\]
and
\[
w \in C^{2,1}(\overline{\Omega} \times [0, T_{\text{max}})).
\]
Moreover, the solution \((u, v, w)\) satisfies
\[
\begin{align*}
u &\geq 0, \ v \geq 0 \text{ and } 0 < w \leq ||w_0||_{L^\infty(\Omega)} \text{ for all } (x, t) \in \Omega \times [0, T_{\text{max}}). \quad (15)
\end{align*}
\]
Finally, if \(T_{\text{max}} < +\infty\), then
\[
\lim_{t \to T_{\text{max}}} ||u(\cdot, t)||_{L^\infty(\Omega)} = \infty.
\]
(16)
Proof. The claims concerning local-in-time existence of classical solution to model (1) are well-established by a fixed-point argument in the context of chemotaxis-haptotaxis systems. By the maximum principle, it is easy to obtain that $u \geq 0$ and $v \geq 0$ for all $(x, t) \in \Omega \times [0, T_{\text{max}})$. Integrating the third equation in (1), it follows from (2) and $v \geq 0$ that $0 < w \leq \|w_0\|_{L^\infty(\Omega)}$ for all $(x, t) \in \Omega \times [0, T_{\text{max}})$. The proof is quite standard, for details, we refer the readers to [8, 28, 31, 38, 15, 33, 39]. 

Next, based on the ideas developed in [24, 30], we give the following one-sided pointwise estimate for $-\Delta w$, which will be served as a cornerstone for our subsequent analysis.

**Lemma 2.2.** Assume that $(u, v, w)$ is a classical solution for model (1). Then we have

$$-\Delta w(x, t) \leq M \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{\text{max}}),$$

where

$$M := \|\Delta w_0\|_{L^\infty(\Omega)} + 4\|\nabla \sqrt{w_0}\|_{L^\infty(\Omega)}^2 + \frac{\|w_0\|_{L^\infty(\Omega)}}{e}. \quad (18)$$

Proof. The main idea of the proof is very similar to those of Lemma 2.2 in [24] and Lemma 2.2 in [30], thus we refrain us from repeating it here.

Finally, let us collect some basic statements about the Gagliardo-Nirenberg inequality which will be used in the forthcoming proof of $L^r$-boundedness for model (1). For details, we refer the readers to [9, 20, 38] (see also [27]).

**Lemma 2.3.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with smooth boundary and assume that $r \in (0, p)$ and $\phi \in W^{1,2}(\Omega) \cap L^r(\Omega)$. Then there exists a positive constant $C_{\text{GN}}$ such that

$$\|\phi\|_{L^p(\Omega)} \leq C_{\text{GN}}(\|\nabla \phi\|_{L^2(\Omega)}^\lambda \|\phi\|_{L^r(\Omega)}^{1-\lambda} + \|\phi\|_{L^r(\Omega)}), \quad (19)$$

holds with $\lambda \in (0, 1)$ satisfying

$$\frac{1}{p} = \lambda \left(\frac{1}{2} - \frac{1}{n}\right) + (1-\lambda) \frac{1}{r}. \quad (20)$$

3. **Global boundedness.** In this section, we consider the global existence and boundedness of solutions for model (1) under some suitable conditions. To do this, we first give the following a priori estimates for model (1).

**Lemma 3.1.** Assume that $(u, v, w)$ is a classical solution for model (1). Then there exists a constant $m^* > 0$ such that the first component $u$ of the solution to (1)

satisfies the following estimate

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq m^* := \max\{\|\Omega\|, \|u_0\|_{L^1(\Omega)}\} \quad \text{for all } t \in (0, T_{\text{max}}). \quad (21)$$

Proof. Integrating the first equation in (1), we deduce from $w > 0$ that

$$\frac{d}{dt} \int_\Omega u(x, t)dx = \int_\Omega \mu u(1-u-w)dx \leq \mu \int_\Omega udx - \mu \int_\Omega u^2dx. \quad (22)$$

According to Hölder’s inequality, we have

$$\frac{d}{dt} \int_\Omega u(x, t)dx \leq \mu \int_\Omega udx - \mu \frac{1}{\|\Omega\|} \left(\int_\Omega udx\right)^2. \quad (23)$$

By the comparison argument of ODE, we derive

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq \max\{\|u_0(x)\|_{L^1(\Omega)}, \|\Omega\|\} := m^*. \quad (24)$$
The proof of Lemma 3.1 is complete. 

**Lemma 3.2.** Let $\chi > 0$, $\xi > 0$ and $\mu > 0$. Assume that the initial data $(u_0, w_0)$ satisfies (2) and $\varphi$ fulfills (6). Then for all $\gamma \in \left(1, \frac{\chi}{(\chi - \mu +)}\right)$, there exists $C > 0$ such that

$$||u(\cdot, t)||_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T_{\mathrm{max}}).$$

(25)

**Proof.** Multiplying the first equation in (1) by $u^{\gamma-1}$ ($\gamma > 1$) and integrating by parts, we have

$$\frac{1}{\gamma} \frac{d}{dt} \int_\Omega u^\gamma dx = -(\gamma - 1) \int_\Omega u^{\gamma-2} \varphi(u)|\nabla u|^2 dx + \chi(\gamma - 1) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla v dx$$

$$+ \xi(\gamma - 1) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla w dx + \mu \int_\Omega u^{\gamma}(1-u-w) dx$$

$$\leq \chi(\gamma - 1) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla v dx + \xi(\gamma - 1) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla w dx$$

$$+ \mu \int_\Omega u^{\gamma} dx - \mu \int_\Omega u^{\gamma+1} dx \text{ for all } t \in (0, T_{\mathrm{max}}).$$

(26)

Integrating by parts once more, it follows from the second equation in (1) and $v \geq 0$ that

$$\chi(\gamma - 1) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla v dx = \frac{\chi(\gamma - 1)}{\gamma} \int_\Omega \nabla u^\gamma \cdot \nabla v dx$$

$$= -\frac{\chi(\gamma - 1)}{\gamma} \int_\Omega u^{\gamma} \Delta v dx$$

$$= -\frac{\chi(\gamma - 1)}{\gamma} \int_\Omega u^{\gamma} (v - u) dx$$

$$\leq \frac{\chi(\gamma - 1)}{\gamma} \int_\Omega u^{\gamma+1} dx \text{ for all } t \in (0, T_{\mathrm{max}}).$$

(27)

Similarly, we deduce from Lemma 2.2 that

$$\xi(\gamma - 1) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla w dx = \frac{\xi(\gamma - 1)}{\gamma} \int_\Omega \nabla u^\gamma \cdot \nabla w dx$$

$$= -\frac{\xi(\gamma - 1)}{\gamma} \int_\Omega u^{\gamma} \Delta w dx$$

$$\leq M\xi(\gamma - 1) \int_\Omega u^{\gamma} dx \text{ for all } t \in (0, T_{\mathrm{max}}).$$

(28)

Combining (26)-(28), we derive

$$\frac{d}{dt} \int_\Omega u^\gamma dx \leq (M\xi(\gamma - 1) + \mu\gamma) \int_\Omega u^\gamma dx + (\chi(\gamma - 1) - \mu\gamma) \int_\Omega u^{\gamma+1} dx$$

(29)

Due to the fact that $\gamma < \frac{\chi}{(\chi - \mu +)}$, it is easy to see that

$$\chi(\gamma - 1) - \mu\gamma < 0.$$  

(30)

By Hölder’s inequality, we obtain

$$\int_\Omega u^\gamma dx \leq \left( \int_\Omega u^{\gamma+1} dx \right)^{\gamma+1} |\Omega|^{\frac{\gamma}{\gamma+1}}.$$  

(31)
Collecting (29)-(31), we have
\[
\frac{d}{dt} \int_{\Omega} u^\gamma dx \leq (M\xi(\gamma - 1) + \mu\gamma) \int_{\Omega} u^\gamma dx - (\mu\gamma - \chi(\gamma - 1))|\Omega|^{\frac{1}{2}} \left( \int_{\Omega} u^\gamma dx \right)^{\frac{\gamma+1}{\gamma}}.
\]
(32)

By the comparison argument of ODE, we derive
\[
\int_{\Omega} u^\gamma dx \leq \max \left\{ \int_{\Omega} u_0^\gamma dx, \left( \frac{M\xi(\gamma - 1) + \mu\gamma}{\mu\gamma - \chi(\gamma - 1)} \right)^{\gamma} |\Omega| \right\}
\]
for all \( t \in (0, T_{\max}) \). The proof of Lemma 3.2 is complete. \( \square \)

**Lemma 3.3.** Let \((u, v, w)\) be a classical solution of model (1) and assume that the conditions in Theorem 1.1 hold. Then for all \( \gamma \in (1, \infty) \), there exists a positive constant \( C_\gamma > 0 \) such that
\[
||u(\cdot, t)||_{L^\gamma(\Omega)} \leq C_\gamma \text{ for all } t \in (0, T_{\max}).
\]
(34)

**Proof.** Multiplying the first equation in model (1) by \((1+u)^{\gamma-1}\) with \( \gamma > \max\{1, p\} \) and integrating by parts, we have
\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} (1+u)^\gamma dx
\]
\[
= -(\gamma - 1) \int_{\Omega} (1+u)^{\gamma-2} \varphi(u) |\nabla u|^2 dx + \chi(\gamma - 1) \int_{\Omega} (1+u)^{\gamma-2} u \nabla u \cdot \nabla v dx
\]
\[
+ \xi(\gamma - 1) \int_{\Omega} (1+u)^{\gamma-2} u \nabla u \cdot \nabla w dx + \mu \int_{\Omega} u(1+u)^{\gamma-1}(1-u-w) dx
\]
\[
= -(\gamma - 1) \int_{\Omega} (1+u)^{\gamma-2} \varphi(u) |\nabla u|^2 dx + \chi(\gamma - 1) \int_{\Omega} \nabla \Psi(u) \cdot \nabla v dx
\]
\[
+ \xi(\gamma - 1) \int_{\Omega} \nabla \Psi(u) \cdot \nabla w dx + \mu \int_{\Omega} u(1+u)^{\gamma-1}(1-u-w) dx
\]
:= \( I + II + III + IV \),
(35)

where
\[
\Psi(u) = \int_0^u (1+\sigma)^{\gamma-2} \sigma d\sigma \leq \int_0^u (1+\sigma)^{\gamma-1} d\sigma \leq \frac{1}{\gamma} (1+u)^\gamma.
\]
(36)

According to the condition (6), we obtain
\[
I = -(\gamma - 1) \int_{\Omega} (1+u)^{\gamma-2} \varphi(u) |\nabla u|^2 dx
\]
\[
\leq -c_0(\gamma - 1) \int_{\Omega} (1+u)^{\gamma-p-2} |\nabla u|^2 dx
\]
\[
= -\frac{4c_0(\gamma - 1)}{(\gamma-p)^2} \int_{\Omega} |\nabla (1+u)^{\gamma-p} |^2 dx.
\]
(37)

By the second equation in (1) and (36), we have
\[
II = -\chi(\gamma - 1) \int_{\Omega} \Psi(u) \Delta v dx
\]
\[
= -\chi(\gamma - 1) \int_{\Omega} \Psi(u)(v-u) dx
\]
\begin{align}
&\leq \chi(\gamma - 1) \int_{\Omega} \Psi(u) ud\sigma x \\
&\leq \frac{\chi(\gamma - 1)}{\gamma} \int_{\Omega} (1 + u)^{\gamma+1} dx.
\end{align}

Similarly, by integrating by parts, it follows from Lemma 2.2 that

\begin{align}
III &= -\xi(\gamma - 1) \int_{\Omega} \Psi(u) \Delta w dx \\
&\leq M\chi(\gamma - 1) \int_{\Omega} \Psi(u) dx \\
&\leq \frac{M\chi(\gamma - 1)}{\gamma} \int_{\Omega} (1 + u)^{\gamma} dx.
\end{align}

By a simple computation, we deduce from $u \geq 0$ and $w > 0$ that

\begin{align}
IV &= \mu \int_{\Omega} u(1 + u)^{\gamma-1}(1 - u - w) dx \\
&\leq \mu \int_{\Omega} (1 + u)^{\gamma-1} ud\sigma x \\
&\leq \mu \int_{\Omega} (1 + u)^{\gamma} dx.
\end{align}

Hence, it follows from (35), (37)-(40) that

\begin{align}
\frac{d}{dt} \int_{\Omega} (1 + u)^{\gamma} dx \\
&\leq -\frac{4c_0 \gamma(\gamma - 1)}{(\gamma - p)^2} \int_{\Omega} |\nabla (1 + u)^{\frac{\gamma-p}{2}}|^2 dx + \chi(\gamma - 1) \int_{\Omega} (1 + u)^{\gamma+1} dx \\
&\quad + (M\chi(\gamma - 1) + \mu\gamma) \int_{\Omega} (1 + u)^{\gamma} dx.
\end{align}

By Young’s inequality, we derive

\begin{align}
(M\chi(\gamma - 1) + \mu\gamma) \int_{\Omega} (1 + u)^{\gamma} dx &\leq \frac{\gamma (M\chi(\gamma - 1) + \mu\gamma)}{\gamma + 1} \int_{\Omega} (1 + u)^{\gamma+1} dx \\
&\quad + \frac{(M\chi(\gamma - 1) + \mu\gamma)|\Omega|}{\gamma + 1}.
\end{align}

Thus, combining (41) with (42), we obtain

\begin{align}
\frac{d}{dt} \int_{\Omega} (1 + u)^{\gamma} dx &\leq -\frac{4c_0 \gamma(\gamma - 1)}{(\gamma - p)^2} \int_{\Omega} |\nabla (1 + u)^{\frac{\gamma-p}{2}}|^2 dx + C_1 \int_{\Omega} (1 + u)^{\gamma+1} dx + C_2,
\end{align}

where $C_1 = \chi(\gamma - 1) + \frac{\gamma (M\chi(\gamma - 1) + \mu\gamma)}{\gamma + 1} > 0$ and $C_2 = \frac{(M\chi(\gamma - 1) + \mu\gamma)|\Omega|}{\gamma + 1} > 0$. We now divide the proof of Lemma 3.3 into the following two steps.

**Step 1.** For the case $p < \frac{2}{n-1}$, we proceed in a similar way in [8]. By Gagliardo-Nirenberg’s inequality in Lemma 2.3, it follows from Lemma 3.1 that there exist positive constants $C_3$ and $C_4$ such that

\begin{align}
&\int_{\Omega} (1 + u)^{\gamma+1} dx \\
&= \|((1 + u)^{\frac{2}{2-p}} \frac{2(\gamma+1)}{2-p}) (\Omega)
\end{align}
By the comparison argument of ODE, we obtain

\[ \lambda = \frac{(\gamma - p)n}{\gamma - p} + \frac{(\gamma - p)n}{2(\gamma + 1)} = \frac{\gamma(\gamma - p)n}{(\gamma + 1)(2 + (\gamma - p)n - n)}. \]  

Due to the conditions \( p < \frac{2}{n} - 1 \) and \( \gamma > 1 \), we know

\[
(\gamma + 1)(2 + (\gamma - p)n - n) = \gamma(\gamma - p)n + 2\gamma + 2 - pm - n > \gamma(\gamma - p)n + 2 - pm - n > \gamma(\gamma - p)n > 0,
\]

which implies \( \lambda \in (0, 1) \).

Since \( p < \frac{2}{n} - 1 \), we obtain

\[
\frac{2\lambda(\gamma + 1)}{\gamma - p} = \frac{2\gamma n}{2 + n(\gamma - p - 1)} < 2. 
\]

Thus, using Young’s inequality, it follows from (44) and (47) that there exists \( C_5 > 0 \) such that

\[
(C_1 + 1) \int_{\Omega} (1 + u)^{\gamma + 1} dx \leq \frac{4c_0\gamma(\gamma - 1)}{(\gamma - p)^2} \int_{\Omega} |\nabla(1 + u)|^{\frac{\gamma - p}{2}} dx + C_5. 
\]

Combining (43) with (48), we derive

\[
\frac{d}{dt} \int_{\Omega} (1 + u)^{\gamma} dx \leq - \int_{\Omega} (1 + u)^{\gamma + 1} dx + C_6
\]

with a positive constant \( C_6 = C_2 + C_5 \).

According to Hölder’s inequality, we have

\[
\int_{\Omega} (1 + u)^{\gamma + 1} dx \geq |\Omega|^{-\frac{1}{\gamma}} \left( \int_{\Omega} (1 + u)^{\gamma} dx \right)^{\frac{\gamma + 1}{\gamma}}. 
\]

It follows from (49) and (50) that

\[
\frac{d}{dt} \int_{\Omega} (1 + u)^{\gamma} dx \leq -|\Omega|^{-\frac{1}{\gamma}} \left( \int_{\Omega} (1 + u)^{\gamma} dx \right)^{\frac{\gamma + 1}{\gamma}} + C_6. 
\]

By the comparison argument of ODE, we obtain

\[
\int_{\Omega} (1 + u)^{\gamma} dx \leq \max \left\{ \int_{\Omega} (1 + u_0)^{\gamma} dx, (C_6|\Omega|^{-\frac{1}{\gamma}})^{\frac{\gamma}{\gamma + 1}} \right\},
\]

where

\[
\lambda = \frac{(\gamma - p)n}{\gamma - p} + \frac{(\gamma - p)n}{2(\gamma + 1)}.
\]
which implies that (34) holds.

**Step 2.** For the case \( p \geq \frac{2}{n} - 1 \) and \( \mu > \frac{(p+1)n-2}{(p+1)n} \chi \), we proceed in a similar way in [40] and fix \( \gamma' \in \left( \frac{p+1}{2}, \frac{\chi}{\chi-\mu} \right) \), then it follows from Lemma 3.2 that there exists a positive constant \( C_7 \) such that

\[
||u(\cdot, t)||_{L^{\gamma'}(\Omega)} \leq C_7 \quad \text{for all } t \in (0, T_{\text{max}}).
\] (53)

Therefore, selecting \( \gamma > \gamma' \) and using Gagliardo-Nirenberg’s inequality in Lemma 2.3, it follows from (33) that there exist positive constants \( C_8 \) and \( C_9 \) such that

\[
\int_{\Omega} (1 + u)^{\gamma+1} dx
\]

\[
= ||(1 + u)^{\frac{2\gamma}{\gamma'}}\frac{L^{2+\gamma}}{L^{\frac{2\gamma}{\gamma'}}(\Omega)}||
\]

\[
\leq C_8 \left( ||\nabla (1 + u)^{\frac{2\gamma}{\gamma'}} ||_{L^2(\Omega)} \cdot ||(1 + u)^{\frac{2\gamma}{\gamma'}} ||_{L^{\frac{2\gamma}{\gamma'}}(\Omega)} + ||(1 + u)^{\frac{2\gamma}{\gamma'}} ||_{L^{\frac{2\gamma}{\gamma'}}(\Omega)} \right)^{\frac{2(\gamma+1)}{2\gamma'-p}}
\]

\[
= C_8 \left( ||\nabla (1 + u)^{\frac{2\gamma}{\gamma'}} ||_{L^2(\Omega)} \cdot ||1 + u||_{L^{\frac{2\gamma}{\gamma'}}(\Omega)} + ||1 + u||_{L^{\frac{2\gamma}{\gamma'}}(\Omega)} \right)^{\frac{2(\gamma+1)}{2\gamma'-p}}
\]

\[
\leq C_9 \left( ||\nabla (1 + u)^{\frac{2\gamma}{\gamma'}} ||_{L^2(\Omega)} + 1 \right),
\]

(54)

where

\[
\lambda' = \frac{(\gamma - p)n}{1 - \frac{n}{2} + \frac{(\gamma - p)n}{2\gamma'}} - \frac{(\gamma + 1)(\gamma - p)n - \gamma'(\gamma - p)n}{\gamma'(\gamma + 1)(2 - n) + (\gamma + 1)(\gamma - p)n}.
\]

(55)

According to the fact that \( \gamma > \gamma' > \frac{(p+1)n}{2} \geq 1 \), we derive

\[
\gamma'(\gamma + 1)(2 - n) + (\gamma + 1)(\gamma - p)n
\]

\[
= (\gamma + 1)(\gamma - p)n - \gamma'(\gamma - p)n + \gamma'[2\gamma - (p + 1)n + 2]
\]

(56)

\[
> (\gamma + 1)(\gamma - p)n - \gamma'(\gamma - p)n > 0,
\]

which implies \( \lambda' \in (0, 1) \).

Since \( \gamma' > \frac{(p+1)n}{2} \), we have

\[
\frac{2\lambda'(\gamma + 1)}{\gamma - p} = \frac{(\gamma + 1)n - \gamma'n}{\gamma'(2 - n) + (\gamma - p)n} < 2.
\]

(57)

Hence, using Young’s inequality, it follows from (54) and (57) that there exists \( C_{10} > 0 \) such that

\[
(C_1 + 1) \int_{\Omega} (1 + u)^{\gamma+1} dx \leq \frac{4c_0\gamma(\gamma - 1)}{(\gamma - p)^2} \int_{\Omega} |\nabla (1 + u)^{\frac{2\gamma}{\gamma'}}|^2 dx + C_{10}.
\]

(58)

The later proof is the same as step 1, thus we can obtain the desired result. The proof of Lemma 3.3 is complete. \( \square \)

**Lemma 3.4.** Let \((u, v, w)\) be a solution of model (1) and assume that the conditions in Theorem 1.1 hold. Then there exists \( C > 0 \) such that

\[
||u(\cdot, t)||_{L^{\infty}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}).
\]

(59)
Before proceeding with the proof, let us establish some auxiliary results. For later use, we introduce the following approximating equation of (1):

\[
\begin{aligned}
\begin{cases}
u_{\varepsilon t} &= \nabla \cdot (\varphi_{\varepsilon}(u_{\varepsilon}) \nabla u_{\varepsilon}) - \chi \nabla \cdot (u_{\varepsilon} \nabla v_{\varepsilon}) - \xi \nabla \cdot (u_{\varepsilon} \nabla w_{\varepsilon}) \\
&\quad + \mu u_{\varepsilon}(1 - u_{\varepsilon} - w_{\varepsilon}), &x \in \Omega, t > 0, \\
\frac{\partial u_{\varepsilon}}{\partial \nu} - \chi u_{\varepsilon} \frac{\partial v_{\varepsilon}}{\partial \nu} - \xi u_{\varepsilon} \frac{\partial w_{\varepsilon}}{\partial \nu} &= 0, &x \in \partial \Omega, t > 0, \\
u_{\varepsilon}(x, 0) &= u_{0\varepsilon}(x), w_{\varepsilon}(x, 0) = w_{0\varepsilon}(x), &x \in \Omega,
\end{cases}
\end{aligned}
\]

where \(\varphi_{\varepsilon}(s) = \varphi(s + \varepsilon)\) for all \(s \geq 0\), and the initial data \((u_{0\varepsilon}(x), w_{0\varepsilon}(x))\) \((W^{1,\infty}(\Omega))^2\) satisfies

\[
\begin{aligned}
\begin{cases}
u_{\varepsilon} \rightharpoonup u_0 \text{ weakly star in } L^\infty(\Omega), &\text{and} \\
w_{0\varepsilon} \rightharpoonup w_0 \text{ weakly star in } L^\infty(\Omega).
\end{cases}
\end{aligned}
\]

Indeed, for each \(\varepsilon \in (0, 1)\), problem (62) possesses a nonnegative classical solution \((u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon})\), which is global and bounded by Theorem 1.1.

\[
\int_0^T ||u_{\varepsilon}(\cdot, t) + \varepsilon||_{L^\gamma(\Omega)} dt \leq C(T) \quad \text{for all } \gamma > 1, T \in (0, \infty),
\]

Proof of Theorem 1.3. We first give some estimates for \(u_{\varepsilon}, v_{\varepsilon}\) and \(w_{\varepsilon}\). Similar to Lemma 2.1 and Lemma 3.2-Lemma 3.4, it is not difficult to see that for each \(T \in (0, \infty)\), there exists a constant \(C(T) > 0\) such that

\[
\int_0^T ||u_{\varepsilon}(\cdot, t) + \varepsilon||_{L^\gamma(\Omega)} dt \leq C(T) \quad \text{for all } \gamma > 1, T \in (0, \infty),
\]

Proof of Theorem 1.1. With the aid of the blow up criterion (16) and Lemma 3.4, there exists a positive constant \(C = C(||u_0||_{L^\infty(\Omega)}) > 0\) such that

\[
||u(\cdot, t)||_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, \infty).
\]

According to the maximum principle, we have \(||v(\cdot, t)||_{L^\infty(\Omega)} \leq ||u(\cdot, t)||_{L^\infty(\Omega)}\). Therefore, we obtain the desired result due to Lemma 2.1. The proof of Theorem 1.1 is complete.
\[
    \int_0^T \int_\Omega (u_\varepsilon + \varepsilon)^{\gamma - p - 2} |\nabla u_\varepsilon|^2 \, dx \, dt \leq C(T) \quad \text{for all } \gamma > 1, T \in (0, \infty),
\]
\[
    \int_0^T \|\nabla v_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \, dt \leq C(T) \quad \text{for all } T \in (0, \infty) \quad \text{(64)}
\]
\[
    \int_0^T \|w_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \, dt \leq C(T) \quad \text{for all } t \in (0, T).
\]

In order to achieve a strong precompactness property of \(\{u_\varepsilon\}_{\varepsilon \in (0,1)}\), let us fix \(q \geq \frac{\gamma + 2}{\gamma} \) and multiply the first equation in (62) by \(u_\varepsilon^q - \rho(x)\), where \(\rho(x) \in C^\infty_0(\Omega)\), we infer from integrating it over \(\Omega\) that
\[
    \frac{1}{q} \int_\Omega (u_\varepsilon^q) \rho(x) \, dx
\]
\[
    = \int_\Omega \nabla \cdot (\varphi_\varepsilon(u_\varepsilon) \nabla u_\varepsilon) u_\varepsilon^{q-2} \rho(x) \, dx - \chi \int_\Omega \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) u_\varepsilon^{q-2} \rho(x) \, dx
\]
\[
    - \xi \int_\Omega \nabla \cdot (u_\varepsilon \nabla w_\varepsilon) u_\varepsilon^{q-2} \rho(x) \, dx + \mu \int_\Omega u_\varepsilon^q \rho(x)(1 - u_\varepsilon - w_\varepsilon) \, dx
\]
\[
    = -(q - 1) \int_\Omega \varphi_\varepsilon(u_\varepsilon) u_\varepsilon^{q-2} \rho(x) |\nabla u_\varepsilon|^2 \, dx - \int_\Omega \varphi_\varepsilon(u_\varepsilon) u_\varepsilon^{q-2} \nabla u_\varepsilon \cdot \nabla \rho \, dx
\]
\[
    + (q - 1) \chi \int_\Omega u_\varepsilon^{q-1} \rho(x) \nabla u_\varepsilon \cdot \nabla v_\varepsilon \, dx + \chi \int_\Omega u_\varepsilon^q \nabla v_\varepsilon \cdot \nabla \rho \, dx
\]
\[
    + (q - 1) \xi \int_\Omega u_\varepsilon^{q-1} \rho(x) \nabla u_\varepsilon \cdot \nabla w_\varepsilon \, dx + \xi \int_\Omega u_\varepsilon^q \nabla w_\varepsilon \cdot \nabla \rho \, dx
\]
\[
    + \mu \int_\Omega u_\varepsilon^q \rho(x)(1 - u_\varepsilon - w_\varepsilon) \, dx.
\]

We choose \(l \in \mathbb{N}\) large enough to satisfy \(l > \frac{\gamma + 2}{\gamma}\) and hence \(W_0^{1,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)\).

Now collecting (63)-(65), according to the restriction on \(q\), there exists a constant \(C(T) > 0\) (see more details in [31, 33]) such that
\[
    \| (u_\varepsilon^q) \|_{L^1([0,T);W_0^{1,2}(\Omega)^*)} = \int_0^T \sup_{\xi \in C^\infty_0(\Omega), ||\xi||_{W^{1,2}(\Omega)} \leq 1} |\int_\Omega (u_\varepsilon^q) \xi(x) \, dx| \, dt \quad \text{(66)}
\]
\[
    \leq C(T).
\]

Now in conjunction with (64), (66) and the Aubin-Lions lemma (Theorem III.2.3 in [32]), there exists a subsequence \(\varepsilon = \varepsilon_j \searrow 0\) as \(j \to \infty\) such that
\[
    u_\varepsilon \to u \quad \text{a.e. in } \Omega \times (0, T),
\]
\[
    u_\varepsilon \rightharpoonup u \quad \text{weakly in star } L^\infty(\Omega \times (0, T)),
\]
\[
    \varphi_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \to \varphi(u) \nabla u \quad \text{in } L^2((0, T); L^2(\Omega)),
\]
\[
    v_\varepsilon \to v \quad \text{a.e. in } \Omega \times (0, T),
\]
\[
    v_\varepsilon \rightharpoonup v \quad \text{weakly in star } L^\infty(\Omega \times (0, T)),
\]
\[
    \nabla v_\varepsilon \to \nabla v \quad \text{weakly in star } L^\infty(\Omega \times (0, T)),
\]
\[
    w_\varepsilon \to w \quad \text{a.e. in } \Omega \times (0, T)
\]
\[
    w_\varepsilon \rightharpoonup w \quad \text{weakly in star } L^\infty(\Omega \times (0, T)) \quad \text{and}
\]
\[
    \nabla w_\varepsilon \to \nabla w \quad \text{weakly in star } L^\infty(\Omega \times (0, T)).
\]
Therefore, we select \( \zeta \in C^\infty_0(\Omega \times [0, T]) \), \( \eta \in C^\infty_0(\Omega \times [0, T]) \) and \( \theta \in C^\infty_0(\Omega \times [0, T]) \) for all \( T \in (0, \infty) \). Multiplying the first, second and third equations of (62) by \( \zeta \), \( \eta \) and \( \theta \), respectively, and then integrating by party, we see that \((u_\varepsilon, v_\varepsilon, w_\varepsilon)\) satisfies

\[
- \int_0^T \int_\Omega u_\varepsilon \zeta_t \, dx \, dt - \int_\Omega u_0 \zeta(x, 0) \, dx \\
= - \int_0^T \int_\Omega \varphi_\varepsilon(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \zeta \, dx \, dt + \chi \int_0^T \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla \zeta \, dx \, dt \\
+ \xi \int_0^T \int_\Omega u_\varepsilon \nabla w_\varepsilon \cdot \nabla \zeta \, dx \, dt + \mu \int_0^T \int_\Omega u_\varepsilon \zeta(1 - u_\varepsilon - w_\varepsilon)
\]

and

\[
0 = - \int_0^T \int_\Omega \nabla v_\varepsilon \cdot \nabla \eta \, dx \, dt - \int_0^T \int_\Omega v_\varepsilon \eta \, dx \, dt + \int_0^T \int_\Omega u_\varepsilon \eta \, dx \, dt
\]

as well as

\[
\int_0^T \int_\Omega w_\varepsilon \theta \, dx \, dt = \int_0^T \int_\Omega w_0 \theta e^{\int_0^t v(x, s) \, ds} \, dx \, dt.
\]

By using (63) and (67) in passing to the limit in each term of the identities (68)-(70), we obtain

\[
- \int_0^T \int_\Omega u_\varepsilon \zeta_t \, dx \, dt - \int_\Omega u_0 \zeta(x, 0) \, dx \\
= - \int_0^T \int_\Omega \varphi(u) \nabla u \cdot \nabla \zeta \, dx \, dt + \chi \int_0^T \int_\Omega u \nabla v \cdot \nabla \zeta \, dx \, dt \\
+ \xi \int_0^T \int_\Omega u \nabla w \cdot \nabla \zeta \, dx \, dt + \mu \int_0^T \int_\Omega u \zeta(1 - u - w)
\]

and

\[
0 = - \int_0^T \int_\Omega \nabla v \cdot \nabla \eta \, dx \, dt - \int_0^T \int_\Omega v \eta \, dx \, dt + \int_0^T \int_\Omega u \eta \, dx \, dt
\]

as well as

\[
\int_0^T \int_\Omega w \theta \, dx \, dt = \int_0^T \int_\Omega w_0 \theta e^{\int_0^t v(x, s) \, ds} \, dx \, dt.
\]

Hence, it is easy to see that \((u, v, w)\) is a global weak solution for (1). Finally, the boundedness statement is a straightforward consequence of the proof of Theorem 1.1. The proof of Theorem 1.3 is complete.

5. **Decay of \( w \).** In this section, motivated by Tao and Winkler in [30], we consider the decay estimate of \( w \) under a suitable smallness condition on \( w_0 \) and prove Theorem 1.4. The proof is mainly based on a lower bound for the mass \( m(t) = \int_\Omega u(x, t) \, dx \). To do this, we need the following lemmas.

**Lemma 5.1.** Under the same conditions of Theorem 1.1, assume that the initial data \((u_0, w_0)\) satisfies (2) and

\[
||w_0||_{L^\infty(\Omega)} + \frac{\varepsilon^2}{\mu K} \cdot \left[ \frac{||w_0||_{L^\infty(\Omega)}^2}{4e} + \frac{||\nabla w_0||_{L^2(\Omega)}^2}{|\Omega|} \right] < 1,
\]

(74)
where \( K := \min_{0 \leq s \leq ||u(\cdot,t)||_{L^\infty(\Omega)}} c_0(s+1)^{-p} > 0 \). Then we can find \( \beta > 0 \) and \( \Gamma > 0 \) such that the first solution component \( u \) of (1) satisfies
\[
\int_0^t \int_\Omega u(x,s)dxds \geq \beta t - \Gamma \quad \text{for all } t > 0. \tag{75}
\]

**Proof.** The proof is mainly based on the arguments in \([30, 10]\). According to the maximum principle and the hypothesis \( u_0 \not\equiv 0 \) in (2), then we know that \( u \) is positive in \( \Omega \times (0, \infty) \). Thus, we may multiply the first equation in (1) by \( \frac{1}{u} \) and integrate by parts over \( \Omega \) to obtain
\[
\frac{d}{dt} \int_\Omega \ln u dx = \int_\Omega \frac{1}{u} \nabla \cdot (\varphi(u) \nabla u) dx - \chi \int_\Omega \frac{1}{u} \nabla \cdot (u \nabla v) dx - \xi \int_\Omega \frac{1}{u} \nabla \cdot (u \nabla w) dx \\
+ \mu |\Omega| - \mu \int_\Omega u dx - \mu \int_\Omega w dx \\
= \int_\Omega \frac{\varphi(u) |\nabla u|^2}{u^2} dx - \chi \int_\Omega \frac{1}{u} \nabla u \cdot \nabla v dx - \xi \int_\Omega \frac{1}{u} \nabla u \cdot \nabla w dx \\
+ \mu |\Omega| - \mu \int_\Omega u dx - \mu \int_\Omega w dx \\
\geq K \int_\Omega \frac{|\nabla u|^2}{u^2} dx - \chi \int_\Omega \frac{1}{u} \nabla u \cdot \nabla v dx - \xi \int_\Omega \frac{1}{u} \nabla u \cdot \nabla w dx \\
+ \mu |\Omega| - \mu \int_\Omega u dx - \mu \int_\Omega w dx \tag{76}
\]
for all \( t > 0 \), where \( K := \min_{0 \leq s \leq ||u||_{L^\infty(\Omega)}} c_0(1+s)^{-p} \), due to the condition (6) and Theorem 1.1. The later proof is the same as Lemma 4.4 in \([30]\), thus we refrain us from repeating it here. The proof of Lemma 5.1 is complete. \( \square \)

Next, we shall give the following lower bound estimate for \( v \).

**Lemma 5.2.** Let \( \chi > 0, \xi > 0 \) and \( \mu > \mu^* \), where \( \mu^* \) is defined in (7), and suppose that the initial data \( (u_0,w_0) \) satisfies (2). Then there exists \( \eta > 0 \) such that the solution \( v \) of (1) fulfills the pointwise inequality
\[
v(x,t) \geq \eta \int_\Omega u(y,t)dy \quad \text{for all } x \in \Omega \text{ and } t > 0. \tag{77}
\]

**Proof.** By using the strict positivity of the associated Green’s function for the Helmholtz operator \(-\Delta + 1\) or alternatively by positivity properties of the Neumann heat semigroup along with a representation of the corresponding resolvent of \( e^{t\Delta} \), it is not difficult to prove it. For details, please see Lemma 4.5 in \([30]\) or Lemma 2.1 in \([10]\). \( \square \)

Finally, we give the decay estimate of \( w \).

**Proof of Theorem 1.4.** By combining Lemma 5.1 and Lemma 5.2, we have
\[
\int_0^t v(x,s)ds \geq \eta \int_0^t u(y,s)dyds \geq \eta(\beta t - \Gamma) \quad \text{for all } t > 0. \tag{78}
\]
Since the third equation in (1) is an ODE, then
\[
w(x,t) = w_0(x)e^{-\int_0^t v(x,s)ds} \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{\text{max}}), \tag{79}
\]
so we obtain
\[ \nabla w(x,t) = \nabla w_0(x) e^{-\int_0^t v(s,x) ds} - w_0(x) e^{-\int_0^t v(s,x) ds} \int_0^t \nabla v(x,s) ds \] (80)
for all \( x \in \Omega \) and \( t \in (0,T_{\text{max}}) \). It follows from (78) and (79) that
\[ ||w(\cdot,t)||_{L^\infty(\Omega)} \leq ||w_0||_{L^\infty(\Omega)} e^{-\eta(\beta t - T)} = C e^{\eta t} ||w_0||_{L^\infty(\Omega)} e^{-\eta \beta t} \quad \text{for all } t > 0. \] (81)
On the other hand, by Theorem 1.1, there exists \( C_1 > 0 \) such that
\[ u(x,t) \leq ||u(\cdot,t)||_{L^\infty(\Omega)} \leq C_1 \quad \text{for all } x \in \Omega \text{ and } t > 0. \] (82)
By the elliptic regularity (see Lemma 4.3 in [10]), we can find \( C_2 > 0 \) such that
\[ ||\nabla v||_{L^\infty(\Omega)} \leq C_2 \quad \text{for all } x \in \Omega \text{ and } t > 0. \] (83)
Collecting (78), (80) and (83), we have
\[
||\nabla w(\cdot,t)||_{L^\infty(\Omega)} \leq ||\nabla w_0||_{L^\infty(\Omega)} e^{-\eta(\beta t - T)} + ||w_0||_{L^\infty(\Omega)} e^{-\eta \beta t} C_2 t \\
= e^{\eta t} ||\nabla w_0||_{L^\infty(\Omega)} e^{-\eta \beta t} + 2C_2 \frac{\eta \beta t}{2} e^{\eta t} ||w_0||_{L^\infty(\Omega)} e^{-\frac{\eta \beta t}{2}} e^{-\frac{\eta \beta t}{2}} \eta \beta t \\
\leq C_3 e^{-\frac{\eta \beta t}{2}} \quad \text{for all } t > 0, \] (84)
where \( C_3 := e^{\eta t} \left( ||\nabla w_0||_{L^\infty(\Omega)} + 2C_2 \frac{\eta \beta t}{2} ||w_0||_{L^\infty(\Omega)} \right) \) and we have used the facts that
\[ ze^{-z} \leq \frac{z}{2} \quad \text{for all } z \in \mathbb{R} \text{ and } e^{-\eta \beta t} \leq e^{-\frac{\eta \beta t}{2}}. \] Thus, combining (81) with (84), we know that (14) holds. The proof of Theorem 1.4 is complete.

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