Systematics of geometric scaling

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Using all available data on the deep-inelastic cross-sections at HERA at $x < 10^{-2}$, we look for geometric scaling of the form $\sigma^{\gamma_p}(Y, Q)$ where the scaling variable $\gamma$ behaves alternatively like $\log Q^2 - \lambda Y$, as in the original definition, or $\log Q^2 - \lambda \sqrt{Y}$, which is suggested by the asymptotic properties of the Balitsky-Kovchegov (BK) equation with running QCD coupling constant. A “Quality Factor” (QF) is defined, quantifying the phenomenological validity of the scaling and the uncertainty on the intercept $\lambda$. Both choices have a good QF, showing that the second choice is as valid as the first one, predicted for fixed coupling constant. A comparison between the QCD asymptotic predictions and data is made and the QF analysis shows that the agreement can be reached, provided going beyond leading logarithmic accuracy for the BK equation.

1. Geometric scaling [1] is a remarkable empirical property verified by the data on the high-energy deep-inelastic scattering (DIS) cross-sections $\sigma^{\gamma_p}$. It has been realized that one can represent with reasonable accuracy the cross-section by the formula

$$\sigma^{\gamma_p}(Y, Q) = \sigma^{\gamma_p}\left(\frac{Q^2}{Q^2(Y)}\right), \quad (1)$$

where $Q$ is the virtuality of the photon, $Y$ the total rapidity in the $\gamma^*\text{proton}$ system and $Q^2 \propto e^{\gamma Y}$ an increasing function of $Y$. The value found for $\lambda \sim 0.3$ has been confirmed by the well-known Golec-Biernat and Wüsthoff model [2] where geometric scaling has been explicitly assumed in the parametrisation.

The property [1], also observed in DIS on nucleus [3] and diffractive processes [4], has been intimately related [5] to the concept of saturation [6], i.e. the behaviour of perturbative QCD amplitudes when the density of partons becomes high enough to exercise the unitarity bound. Indeed, there has been many theoretical arguments to infer that in a $Y, Q^2$ domain where saturation effects set in, geometric scaling is expected to occur. Within this framework, the function $Q_s(Y)$ can be called the saturation scale, since it delineates the approximate upper bound of the saturation domain.

Following a theoretical approach, it has been possible to derive the property [1] from the nonlinear Balitsky-Kovchegov (BK) equation which represents the “mean-field” approximation of the evolution equation for high energy (high density) QCD amplitudes. This equation is supposed to capture essential features of saturation effects. In the 1-dimensional approximation it reads for the density of gluons with transverse momenta $k$ in some target [7]

$$\partial_Y N(L, Y) = \ddot{\alpha} \chi(-\partial_L) N(L, Y) - \alpha N^2(L, Y), \quad (2)$$

where $L = \log(k^2/k_0^2)$, with $k_0^2$ being an arbitrary constant. In Eq. (2), the coupling constant will be considered alternatively in the following as fixed (we shall use $\ddot{\alpha} = 0.15$), or running such that $\ddot{\alpha}(L) = 1/bL$, $b = (11N_c - 2N_f)/12N_c$. The kernel will also be either taken with leading logarithm (LL) accuracy [8], as $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$, or at next-to-leading logarithm (NLL) accuracy (see e.g. [9, 10, 11]).

The key ingredient to theoretically prove geometric scaling of the asymptotic solutions of the nonlinear equation (2) is the travelling wave method [12]. Indeed, the BK equation admits solutions in the form of travelling waves $N(L - v_y \alpha Y)$. $L$ has the interpretation of a space variable while $t = \alpha Y$, interpreted as time, is an increasing function of rapidity $Y$. $v_y$ is the critical velocity of the wave, defined in this case as the minimum of the phase velocity. The travelling-wave solution for the quantity $N$ can be easily translated to the property (2) (at least assuming negligible quark masses) since it yields $N(k^2/Q_s^2(Y))$. Hence the asymptotic travelling-wave solutions of the BK equation satisfy geometric scaling (up to subdominant scaling violations which we will not analyse in the present work). Similar results can be obtained [13, 14] from an approximation of the linear equation with absorptive boundary conditions. The main theoretical results for the asymptotic saturation scale are the following:

$$\log Q_s^2 = \ddot{\alpha} \frac{\chi(\gamma_c)}{\gamma_c} (Y - Y_0) - \frac{3}{2\gamma_c} \log(Y - Y_0)$$

$$- \frac{3}{\gamma_c} \sqrt{\frac{2\pi}{\ddot{\alpha} \chi''(\gamma_c)}} \frac{1}{\sqrt{Y + Y_0}} + \ldots, \quad (3)$$

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totic predictions for the saturation scale \((3,4)\). They may
interact between the empirical formula \((2)\) with the asymp-
totic contributions, depending on the initial
conditions.

The questions we want to ask deal with the confronta-
tion between the empirical formula \((2)\) with the asymp-
totic predictions for the saturation scale \((3,4)\). They may
be expressed as follows

- Is geometric scaling, demonstrated for \(\log(Q^2) \sim Y\)
in agreement with the leading term in \((3)\), remains valid for \(\log(Q^2) \sim Y^{1/2}\) as suggested by \((4)\)?
- Are the theoretical factors and subasymptotic
terms given in Eqs. \((3,4)\) visible in the data?
- Are NLL effects in the theoretical predictions phe-
nomenologically important?
- What is the role of “non-universal” terms?

2. We focus our analysis on the measurements of the
\(\gamma^p\) cross-section \((\sigma^\gamma p = (4\pi^2\alpha_c/Q^2)F_2)\) for which geo-
metric scaling is predicted. The latter being valid at
high-energy, we shall restrict ourselves to \(x \leq 0.01\) for all
values of \(Q^2\). Within that range, we shall use all avail-
able data \((404\) points from E665 \(15\), H1 \(16\), NMC \(17\)
and ZEUS \(18\)).

In order to get a quantitative answer, we shall intro-
duce an “estimator” or quality factor \((QF)\) for deter-
mining the scaling quality, whose working definition is the
following. Given a set of points \((x_i, y_i, f_i = f(x_i, y_i))\)
and a parametric scaling law \(\tau(x, y; \lambda)\), we want to deter-
mine if for some values of \(\lambda\), \(f(x, y)\) can be considered a
function of \(\tau\) only. To achieve this we define a QF which
is large when the points \((\tau_i = \tau(x_i, y_i; \lambda), f_i)\) “lies on a
unique curve”. To quantify this still ill-defined concept,
we shall first rescale the set \((\tau_i, f_i)\) into \((u_i, v_i)\) such that
\(0 \leq u_i, v_i \leq 1\) and assume that the \(u_i\)’s are ordered. We
then introduce

\[
Q(\lambda) = \left[ \sum_i \frac{(v_i - v_{i-1})^2}{(u_i - u_{i-1})^2 + \varepsilon^2} \right]^{-1}.
\]

This definition for the quality factor obviously achieves
what we want: when two successive points are close in
\(u\) and far in \(v\), we expect them “not to lie on the same
curve” and, indeed, they give a large contribution to the
sum in \((5)\), leading to a small quality factor. The con-
stant \(\varepsilon^2\) in \((5)\) is a small number (we have taken \(\varepsilon = 1/n\)
with \(n\) being the number of data points) which prevents
the sum from becoming infinite when two points have the
same value for \(u\). Finally, by studying the dependence on
\(\lambda\) of \(Q\) (as we shall see, it usually shows a Gaussian peak)
we can determine the best choice for the parameter in the
scaling function \(\tau\) and its uncertainty.

3. In order to answer the first question concerning the
compared validity of geometric scaling in \(Y\) vs. \(\sqrt{Y}\), we
performed the QF analysis in both cases. In Fig \((a)\) we
show the data and QF for the usual geometric scaling
definition with \(Y\). In Fig \((b)\), is displayed the data plot
with scale redefinition. The quality of scaling on this plot
can be read from Fig \((a)\), where the scatter plot of the
QF is displayed together with a Gaussian fit of the peak.
As obvious from Fig \((b)\), the peak value larger than 0.1

FIG. 1: Geometric Scaling in \(Y\)
show a good scaling and a Gaussian fit of the bump gives the best value and error for the lambda parameter \( \lambda = 0.321 \pm 0.056 \). Note that the results displayed on Fig. 1 can be considered as an update for the usual geometric scaling tests and a quantitative determination of the scaling parameter and its attached uncertainty. The new point is that geometric scaling is also verified, and with the same level of quality than previously, when the saturation scale is chosen as a function of \( \sqrt{Y} \). This is manifest on Fig. 2 where comparable QF heights at the peak and width attest of the quality of scaling. The peak value getting larger than 0.11 gives a good scaling. The Gaussian fit of the bump gives \( \lambda = 1.621 \pm 0.254 \).

For that sake, we have considered first the LL prediction with fixed coupling \( \lambda = 0 \) with the first subleading correction, i.e. of the form \( \log(Q_s^2) \propto \lambda Y - \mu \log(Y) \) (see Fig. 3). The predicted values from \( \lambda = 0 \) are represented by a point in the \( \lambda, \mu \) plane. It happens that a QF of \( \sim 0.055 \) can be attributed to this parametrisation, as shown on the same figure by the equi-QF curves. In this case changing \( Y_0 \) or adding the third term of \( \lambda = 0 \) does not improve the QF's. Hence the asymptotic LL theoretical prediction remains only marginally verified at present energies.

Let us now turn to the analysis of the 3 different theoretical NLL schemes that we consider. As now well known, NLL corrections to the LL kernel have to be embedded in a resummation scheme in order to cancel spurious singularities. We consider asymptotic predictions \( (3) \) for the so-called S3, S4 schemes and CCS scheme which were recently derived.

On Fig. 5 we show the resulting analysis. The Quality Factor for the 3 schemes is presented as a function of \( Y_0 \), which parameterizes in \( (4) \) the non-universal contributions. There always exists a range of \( Y_0 \) for which the QF goes beyond 0.1 and thus leads to an acceptable scaling. This has to be contrasted with the situation with LL kernel. However, if \( Y_0 \) is too negative, the relevance of geometrical scaling in \( \sqrt{Y} \) is questionable. Indeed \( t = \sqrt{Y-Y_0} \sim \sqrt{|Y_0|} + Y/2\sqrt{|Y_0|} \) and thus the scaling variable is \( Y \) with a non universal \( \lambda = 1/2\sqrt{|Y_0|} \) parameter. For that reason the S4 scheme seems favoured by the analysis and the figure displays the resulting scaling curve for \( Y_0 = -5.5 \).

Using the preceding results based on the estimate of Quality Factors for scaling properties, it is now possible to answer the questions asked in the introduction.

- Geometric scaling is definitely as valid for the
choice \( \log(Q_s^2) \sim Y^{1/2} \) as it is for the original \( \log(Q_s^2) \sim Y \) suggestion.

- The theoretical predictions [3] based on the BK equation at leading log accuracy, \( i.e. \) with fixed coupling and LL kernel are only marginally verified with a QF of \( \sim 0.055 \).

- At next-to-leading accuracy, the theoretical predictions [11] on scaling may be satisfied. There exists preferable NLL schemes \( e.g. \) the S4 scheme [10].

- Parametrising “non-universal” terms, \( i.e. \) depending on initial conditions by a constant \( Y_0 \) in the rapidity evolution, we find that they do not play an essential role at LL level, while they contribute to the quality of the scaling at NLL level.

The generality of the Quality Factor method indicates its interest for a quantitative evaluation of good scaling properties beyond the specific applications we focussed on in the present work. It can be used to check empirical scaling-law proposals as well as to quantitatively evaluate the “distance” between given theoretical predictions with data. In the present application, it leads to the conclusion that geometrical scaling is a well verified empirical property of deep-inelastic cross-sections on the proton. It also shows that the theoretical predictions based on the QCD saturation mechanism are consistent with the observation, but requires to go beyond leading logarithmic accuracy and the asymptotic terms.

Finally, let us quote that recent high-energy predictions [21] of diffusive scaling, namely \( \sigma^{\tau Y} / \sqrt{Y} = \sigma(\tau) \) with \( \tau = \log(Q^2/Q_s^2) / \sqrt{Y} \), have a QF of 0.05, indicating that it might require higher energies.

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The data in the last ZEUS paper include contributions for $F_L$ and $xF_3$ but those can be neglected within the kinematical domain we consider.

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