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SIGNED:  Philipp Strack
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Dedicated to my grandfather Georg Walter Strack (1923 –)
Abstract

In the standard approach to neutrino transport in the simulation of core-collapse supernovae, one calculates the spatial, temporal, and spectral evolution for each neutrino species in terms of the classical density in phase-space or the associated specific intensity. The neutrino radiation is coupled to matter by source and sink terms on the “right-hand-side” of the transport equation and together with the equations of hydrodynamics this set of coupled partial differential equations describes, in principle, the evolution of core collapse and explosion. However, with the possibility of neutrino oscillations between species, a purely quantum-physical effect, how to generalize this set of classical Boltzmann equations to reflect oscillation physics has not been clear. To date, the formalisms developed have retained the character of quantum operator physics and have not been suitable for easy incorporation into standard supernova codes. In this thesis, we derive generalized Boltzmann equations for quasi-classical, real-valued phase-space densities with which any numerical code for the classical transport equation can easily be generalized to include neutrino oscillations. This new, extended set of equations incorporates the full mixing Hamiltonian consisting of the vacuum oscillations, the matter-induced effective mass for the $\nu_e$, and the self-interactions from neutrino-neutrino scattering. The equations are intrinsically nonlinear for two reasons: blocking correction at high densities and the off-diagonal $\nu\nu$ Hamiltonian which maintains SU(N) flavor symmetry for an N-flavor system. Standard oscillation phenomenology including resonant flavor conversion (the MSW effect) and the interplay between decohering matter coupling and flavor oscillations are retained in a quantum-physically consistent fashion. The extension of the formalism to curved spacetime is discussed. One has to take into account two modifications. Firstly, spinors in curved spacetime must be analyzed with the spin connection in a noncoordinate basis and, secondly, for the kinetic terms on the left-hand side of the Boltzmann equation the Christoffel connection must be employed. Both changes are brought to an algebraic footing via vielbein (tetrad) fields associated with the square root of the metric. The generalized kinetic equations in curved space derived in this work describe neutrino evolution in the most general setting encountered in astrophysical contexts such as the early universe or in the vicinity of supernova cores.
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Mixed particles and their oscillations are fundamental for a wide range of interesting physics: quark-mixing by the Cabbibo-Kobayashi-Maskawa matrix, its leptonic analog for massive neutrinos (Bilenky & Petcov 1977 and 1987), hypothetical photon-axion and photon-graviton oscillations in the presence of external magnetic fields (Moertsell, Bergstroem & Goobar 2002, Raffelt & Stodolsky 1988) and $K^0 - \bar{K}^0$ oscillations (Akhmedov, Barroso & Keraenen 2001).

Physically, these quantum systems are coupled to macroscopic systems and through external interaction their quantum evolution is altered. One prominent astrophysical context in which such oscillation for macroscopic systems is important, involves the neutrinos in supernova cores that may execute flavor oscillations while simultaneously interacting with ambient supernova matter (Takahashi, Sato, Burrows & Thompson 2003, Thompson, Burrows & Horvath 2000, Burrows & Gandhi 1995, Raffelt 1996, Pantaleone 1995, Qian & Fuller 1995, and Wolfenstein 1979).

The primary motivation of this work is to provide a straightforward generalization of the Boltzmann formalism with which to analyze the kinetics of oscillating neutrinos with collisions including energy redistribution and inelastic scattering. By taking ensemble-averaged matrix elements of quantum field operators for mixed particles, following the pioneering work in the references (Raffelt & Sigl 1993, Raffelt, Sigl & Stodolsky 1992, Akhiezer & Peletminskii 1981, and de Groot, van Leeuwen & van Weert 1980), we obtain quasi-classical phase-space densities that satisfy simple Boltzmann equations with coupling terms that account for the neutrino oscillations. The formalism is clear, numerically tractable, and does not contain operators or wavefunctions.

Finally, we discuss how to extend our formalism to curved spacetime. This includes the treatment of generalized resonances similar to the MSW effect. We present kinetic equations for oscillating neutrinos interacting with a background medium in curved spacetime.
The plan of this thesis is as follows.

- In chapter 2 we introduce the Wigner phase-space density operator approach from which we derive our formalism involving classical phase-space neutrino flavor densities and their off-diagonal, overlap correlates. The latter couple the different flavor states to account for neutrino oscillations.

- In chapter 3 the different contributions to the mixing Hamiltonian originating in the vacuum oscillations, matter-induced effective mass for the $\nu_e$, and the neutrino-neutrino scattering are discussed. The full set of Boltzmann equations is derived for both the flavor phase-space densities and the associated specific intensities. The advantage of the specific intensity representation is that collision terms can be written in a simple and intuitive fashion.

- In chapter 4 we demonstrate the usefulness and applicability of our formalism by successfully testing it on its quantum-mechanical accuracy. We demonstrate with several simple examples that the set of equations reproduces what one would expect from the standard quantum mechanics wavefunction approach.

Firstly, we give analytic solutions to the standard vacuum flavor oscillations for neutrinos in a box. Secondly, we consider flavor oscillations with absorptive matter coupling. This is frequently referred to as quantum decoherence (Raffelt, Sigl & Stodolsky 1993). Thirdly, it is shown that our formalism is capable of calculating the neutrino evolution in phase-space when the resonant matter-induced flavor conversion (MSW effect) is important. We show complete analytic and numerical congruence to the existing wavefunction formalism as discussed in the papers by Wolfenstein 1978, Bethe 1986, Mikheyev & Smirnov 1986. We show that our formalism can provide a tool to analyze active-sterile neutrino conversion in a beam with absorptive matter coupling. One can observe “indirect decoherence” of the active-sterile conversion. This means that even though the sterile neutrinos do not interact with a medium their oscillatory pattern is indirectly influenced since the $\nu_e$’s execute absorptive collisions.

- In part II we sketch how to make the relevant changes if one wishes to apply our formalism in a curved spacetime. This is potentially important in astrophysical contexts such as the early universe and in the vicinity of supernova cores.

- In chapter 6 we analyze in a heuristic fashion how neutrino oscillation phenomenology is modified in curved space. Therefore, we introduce the vielbein formalism which is the necessary ingredient to do calculations involving spinors in the framework of the general theory of relativity. Furthermore the vielbein fields are useful tools to write the Liouville operator on the left-hand-side of the Boltzmann equation in a manifestly covariant fashion to account for the curvature. This is done in section 7.1.
Culminating in our efforts, we present new kinetic equations for oscillating neutrinos with collisions in curved spacetime in chapter 7. The equations of section 7.3 include the possibility of spin-flips into a sterile right-handed species by coupling of the anomalous magnetic moment of the neutrinos to an external magnetic field. In principle, eqs. (7.24) govern the evolution of the quasi-classical density in phase-space for interacting, massive neutrinos including all known effects – from the quantum-mechanical viewpoint as well as from the gravitational perspective.
Part I

Flat Spacetime
Wigner Phase-Space Density Operator

2.1 Construction

To analyze a multi-particle system and reflect its inherent statistics, one usually sets up the density matrix of the system (Akhiezer & Peletminskii 1981, de Groot, van Leeuwen & van Weert 1980). For classical systems this is the phase-space density. For quantum fields, the conceptually most similar analog is the Wigner phase-space density operator (Wigner 1932 & 1984). In what follows we shall briefly sketch how to construct this quantity.

For Dirac neutrinos, we begin with the Dirac equation

\[ i(\gamma^\mu \partial_\mu - m)\psi = 0 \tag{2.1} \]

which describes the dynamics of spin-$\frac{1}{2}$ fields, $\psi$, in Minkowski space-time with the metric

\[ \eta^{\mu\nu} = \text{diag (1, } -1, -1, -1) \tag{2.2} \]

$\gamma^\mu$ are Dirac matrices, which satisfy

\[ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2.3} \]

Curly brackets denote matrix anti-commutators. Following the standard second quantization procedure, we write for the momentum eigenstate expansion of general Dirac fermion annihilation operator in position space:

\[ \psi(\mathbf{r}, t) = \int \frac{d^3p}{(2\pi)^3} \sum_s a_p^s \psi^s(\mathbf{p}, E_p)e^{-ip\mathbf{r}}e^{-iEt} + b_p^s \psi^s(\mathbf{p}, E_p)e^{ip\mathbf{r}}e^{iEt}, \tag{2.4} \]

and for the Dirac fermion creation operator in position space:
\[ \psi^\dagger(r, t) = \int \frac{d^3 p}{(2\pi)^3} \sum s b^s_p \psi^s(p, E_p) e^{-ipr} e^{-iE_pt} + a^s_p \psi^s(p, E_p) e^{ipr} e^{iE_pt}, \]  

(2.5)

where \( u^s(p, t) \) and \( v^s(p, t) \) are four-component Dirac spinors describing fermionic and antifermionic states, respectively. The summation is over additional quantum numbers, e.g. spin states of the particle. We now tailor \( \psi(r, t) \) to the physical properties of the neutrino and define

\[ \psi(r, t) = \int \frac{d^3 p}{(2\pi)^3} [a_p(t) u(p, E_p) + b^\dagger_{-p}(t) v(p, E_p)] e^{ipr}. \]  

(2.6)

A Fourier transformation connects momentum and position space representation of the operators:

\[ a(p, t) = \int \frac{d^3 r}{(2\pi)^3} u^\dagger(r, E_p) \psi(r, t) e^{-ipr}. \]  

(2.7)

Note that here the operators are explicitly time-dependent. This corresponds to the Heisenberg picture. For mixed neutrinos, the representation in Fourier space cannot diagonalize the energy dependence and the momentum dependence simultaneously.

The Dirac spinors \( u(p, E_p) \) and \( v(p, E_p) \) refer to negative-helicity particles and positive-helicity antiparticles, respectively.

We need to incorporate the appropriate spin-statistics properties into the description. At equal times the Pauli exclusion principle is implemented by anti-commutation relations of the creation and annihilation operators in momentum space:

\[ \{a(p, t), a^\dagger(p', t)\} = \{b(p, t), b^\dagger(p', t)\} = (2\pi)^3 \delta^3(p - p'). \]  

(2.8)

One can now write the Wigner phase-space density operator for Dirac neutrinos in terms of the creation and annihilation operators in momentum space as

\[ \rho(r, p, t) = \int \frac{d^3 p'}{(2\pi)^3} e^{-i\frac{\vec{p}'r}{\hbar}} a^\dagger(p - \frac{1}{2}\vec{p}', t) a(p + \frac{1}{2}\vec{p}', t). \]  

(2.9)

For antineutrinos we have the expression

\[ \bar{\rho}(r, p, t) = \int \frac{d^3 p'}{(2\pi)^3} e^{-i\frac{\vec{p}'r}{\hbar}} b^\dagger(p - \frac{1}{2}\vec{p}', t) b(p + \frac{1}{2}\vec{p}', t). \]  

(2.10)

The same Wigner phase-space density operator can be formulated in terms of the creation and annihilation operators in position space as long as the integration variable is changed:

\[ \rho(r, p, t) = \int \frac{d^3 r'}{(2\pi \hbar)^3} e^{-i\frac{\vec{r}'r}{\hbar}} \psi^\dagger(r - \frac{1}{2}\vec{r}', t) \psi(r + \frac{1}{2}\vec{r}', t). \]  

(2.11)
and similarly for antineutrinos

\[ \tilde{\rho}(\mathbf{r}, \mathbf{p}, t) = \int \frac{d^3 \mathbf{r}'}{(2\pi\hbar)^3} e^{-i\mathbf{p}\mathbf{r}'} \tilde{\psi}^\dagger(\mathbf{r} - \frac{1}{2} \mathbf{r}', t) \tilde{\psi}(\mathbf{r} + \frac{1}{2} \mathbf{r}', t). \]  

(2.12)

The Wigner phase-space density operator possesses the sought-after properties of a quantum-mechanical generalization of the classical phase-space density. To see this, we integrate eq. (2.11) with respect to \( \mathbf{p} \) and use the representation of the delta-function in Fourier space to write

\[ \int d^3 \mathbf{p} \rho(\mathbf{r}, \mathbf{p}, t) = |\psi(\mathbf{r}, t)|^2, \]  

(2.13)

which resembles the correct observable probability. When eq. (2.11) is integrated with respect to \( \mathbf{r} \) one has

\[ \int d^3 \mathbf{r} \rho(\mathbf{r}, \mathbf{p}, t) = \left| \int d^3 \mathbf{r} \psi(\mathbf{r}, t) e^{-i\mathbf{p}\mathbf{r}} \right|^2 = |\psi(\mathbf{p}, t)|^2, \]  

(2.14)

which is again the correct expression. This can be achieved by a shift of variables:

\[ \mathbf{r} - \frac{1}{2} \mathbf{r}' = \mathbf{v} \]

\[ \mathbf{r} + \frac{1}{2} \mathbf{r}' = \mathbf{u}, \]  

(2.15)

and a subsequent relabeling of the integration variable. The above derivation was done for fermions. The procedure is identical for bosons however. Then, we impose commutation relations instead of the anti-commutation relations for the operators. Furthermore no restrictions to the mass of the mixed particle of interest are made.

In the general case of mixed particles (this includes oscillating fermions such as neutrinos but also mixing bosons such as photon-axion conversion systems), the creation and annihilation operators naturally yield mixed particles. To project these composite objects onto their “irreducible” subspaces we take the matrix elements in the appropriate basis. We simultaneously perform an ensemble average. We show that this transforms the Wigner phase-space operator into its matrix elements. The matrix elements are of great use since the diagonal elements are quasi-classical phase-space densities. In the off-diagonal of this matrix we have macroscopic overlap functions. One is now calculating numbers and has not to worry about the time evolution of operators over Fock space. Instead the value of a number changes with time.

In principle, these matrix elements are very similar to the density matrix or statistical operator of a quantum-mechanical system as for instance for the spin density matrix of particle beams.
The quantum number spin can formally be brought on the same page as the quantum number flavor (Friedland & Lunardini 2003). The description of both systems respects the SU(N) symmetry for a N-composite problem. The form of the interaction is alike and has been useful as a calculational analogy.

In the following section 2.2, we demonstrate how the Wigner phase-space operator yields quasi-classical phase-space densities.
2.2 Matrix elements

To decompose the Wigner phase-space density operator into quasi-classical phase-space densities, scalar functions with seven-dimensional domain, we need to calculate its matrix elements. For clarity, we set the dimensions of flavor space to two. We average over an ensemble consisting of two mixing particle species and take the matrix elements in the number-density basis of Fock space. This results in the expression

\[ \mathcal{F} = \langle n_i | \rho | n_j \rangle = \left( f_{\nu e} \right)_{ij} \left( f_{\nu e}^{*} \right)_{ji}, \] (2.16)

where the indices \( i, j \) run over \( e, \mu \), and \( * \) means complex conjugation. For the off-diagonal macroscopic overlap functions we write

\[ f_{\nu e} = \langle n_{\nu e} | \rho(r, p, t) | n_{\nu e} \rangle = \int \frac{d^3p'}{(2\pi)^3} e^{ip'r} \langle n_{\nu e} | a^\dagger(p - \frac{1}{2}p', t)a(p + \frac{1}{2}p', t) | n_{\nu e} \rangle, \] (2.20)

and for the \( \nu_e \) neutrino phase-space density we have the expression:

\[ f_{\nu e} = \langle n_{\nu e} | \rho(r, p, t) | n_{\nu e} \rangle = \int \frac{d^3p'}{(2\pi)^3} e^{ip'r} \langle n_{\nu e} | a^\dagger(p - \frac{1}{2}p', t)a(p + \frac{1}{2}p', t) | n_{\nu e} \rangle, \] (2.21)

and identically for the \( \nu'_\mu \)'s.

Following (de Groot, van Leeuwen & van Weert 1980), one can define the covariant energy-momentum densities

\[ T^\mu\nu(x) = c \int \frac{d^3p}{p^0} p^\mu p^\nu \mathcal{F} \] (2.17)

and the conserved neutrino current densities

\[ J^\mu(x) = c \int \frac{d^3p}{p^0} p^\mu \left[ \mathcal{F} - \mathcal{F}' \right], \] (2.18)

wherein, by conservation of matter,

\[ \partial_\mu J^\mu = 0. \] (2.19)

\( x^\mu = (ct, \mathbf{x}) \) denotes a position four-vector and \( p^\mu = (E/c, \mathbf{p}) \) the four-momentum respectively. With the aid of these entities, eq. (2.18) and eq. (2.17), which describe a quasi-classical field, one can apply the framework of classical field theory to investigate for instance the symmetries and conserved quantities of the neutrino ensemble.
Heuristically, we can put this process of simultaneous projection and averaging as follows. The integration process is symmetric with respect to $p'$. Thus, each of the $a^\dagger(p - \frac{1}{2}p', t) a(p + \frac{1}{2}p', t)$ hits every momentum value of the $n_{\nu}$. Regarded this way, we can anti-commute $a^\dagger(p - \frac{1}{2}p', t)$ and $a(p + \frac{1}{2}p', t)$ in momentum space until they are adjacent and possess identical arguments. At that instant they act on their eigenstates and yield numbers as their eigenvalues. Subsequently, we are left with real-valued functions possessing seven-dimensional domain; quasi-classical phase-space densities $f_{\nu}$.

For completeness, we list the matrix elements for the associated antiparticles:

$$\tilde{\mathcal{F}} = \langle \tilde{n}_i | \tilde{\rho} | \tilde{n}_j \rangle = \begin{pmatrix} f_{\nu e} & f_{e \mu} \\ f_{\nu e}^* & f_{e \mu}^* \end{pmatrix}.$$

Note that it is the hermiticity of $\rho$ and $\tilde{\rho}$ that implies the off-diagonal terms of its matrix elements be complex-conjugates of one another.

A generalization to more particle species is straightforward. The diagonal terms are real-valued and denote quasi-classical phase-space densities. The off-diagonal entries are complex-valued macroscopic overlap functions. They represent the degree of coherence or macroscopic overlap of the ensemble under consideration. For completely decohered ensembles and for non-mixing ensembles the off-diagonal entries vanish. More precisely, $f_{e\mu}(r, p, t)$ for instance gives the number of $\nu$’s that result from incoherently adding up the quantum mechanical amplitudes from each otherwise $\nu$’s. This requires that flavor-entangled many-body states can be neglected or are small in number.

### 2.3 Conclusions

- The salient results of chapter 2 are eqs. (2.16) and (2.22).
- We have abandoned the operator character of the mathematical entities describing the neutrinos, eq. (2.9), and are now in position to work with real-valued phase-space densities and their complex-number-valued overlap correlates, eq. (2.16).
- In the following, we construct dynamical equations for the quasi-classical phase-space densities which are very similar to the equations of classical transport theory (see for instance eq. (3.34)).
3

Generalized Boltzmann Equations

3.1 2 × 2 Mixing Hamiltonian

Naively, we can derive a dynamical equation for the matrix elements of the Wigner phase-space density operator by merging the Boltzmann equation with the Heisenberg equation. A derivation from first-principle quantum mechanics has been performed in many different contexts (Landau 1957, Akhiezer & Peletminskii 1981, Rudzky 1990, Raffelt, Sigl & Stodolsky 1993). We need to account for the matrix character of the phase-space densities and, following the nice papers of Raffelt & Sigl 1993, generalize the “Liouville terms” on the left-hand side by introducing anti-commutators resulting in the expression

$$\frac{\partial F}{\partial t} + \frac{1}{2} \left\{ v, \frac{\partial F}{\partial r} \right\} + \frac{1}{2} \left\{ p, \frac{\partial F}{\partial p} \right\} = -i [\Omega, F] + C .$$  

(3.1)

Here, the collision matrix is given by

$$C(r, p, t) = \left( \begin{array}{cc} C_{\nu \nu} & C_{\nu \rightarrow \nu} \\ C_{\nu \rightarrow \nu} & C_{\nu \mu} \end{array} \right) ,$$  

(3.2)

where the off-diagonal elements describe flavor-changing collisions. In the standard model, flavor number is conserved at every vertex. So by restricting to these interactions one can in general set the off-diagonal elements to zero.

Oscillation phenomenology is incorporated via the commutator on the right-hand-side, the term that originated in the Heisenberg equation, with the mixing Hamiltonian given by,

$$\Omega(r, p) = \Omega_{\nu\nu}(\epsilon) + \Omega_{\nu\mu}(r) + \Omega_{\nu\nu}(r, p, t) - \Omega_{\nu\nu}(r, p, t) .$$  

(3.3)

In this cumulative expression, $\Omega_{\nu\nu}(\epsilon)$ is the usual vacuum contribution where mixing comes about through the non-degeneracy of masses and the entailed distinction between flavor or weak
interaction eigenstates and the mass eigenstates. $\Omega_{\text{mat}}(r)$ is the ordinary matter contribution that most notably results the effective mass for the $\nu_e$ neutrino in matter. $\Omega_{\nu\nu}$ is the non-diagonal Hamiltonian that describes the neutrino self-interactions or neutrino-neutrino scattering and $\tilde{\Omega}_{\tilde{\nu}\tilde{\nu}}$ is the non-diagonal neutrino-antineutrino interaction Hamiltonian.

Making use of the antineutrino phase-space densities as given in eq. (2.22) the dynamical equation for the anti-particles looks completely analogous,

$$\frac{\partial \tilde{f}}{\partial t} + \frac{1}{2} \left( \mathbf{v}, \frac{\partial \tilde{f}}{\partial \mathbf{r}} \right) + \frac{1}{2} \left( \mathbf{p}, \frac{\partial \tilde{f}}{\partial \mathbf{p}} \right) = -i \left[ \tilde{\Omega}, \tilde{f} \right] + \tilde{C} .$$

(3.4)

where apart from the obvious change in the collision terms and the different phase-space densities (denoted by a tilde), we must reverse sign for the coupling coefficients in $\Omega_{\text{mat}}, \Omega_{\nu\nu}$ and $\tilde{\Omega}_{\tilde{\nu}\tilde{\nu}},$

$$\tilde{\Omega}(r, p) = \Omega_{\text{vac}}(\varepsilon) - \Omega_{\text{mat}}(r) - \Omega_{\nu\nu}(p) + \tilde{\Omega}_{\tilde{\nu}\tilde{\nu}}(p) .$$

(3.5)

The vacuum mixing contribution is alike for neutrinos and its antiparticle,

$$\Omega_{\text{vac}}(\varepsilon) = \frac{\pi c}{L} \left( \begin{array}{c} -\cos 2\theta \\ \sin 2\theta \end{array} \right) ,$$

(3.6)

where $\theta_{\text{vac}}$ is the mixing angle for vacuum neutrino oscillations between $\nu_e$’s and $\nu_\mu$’s and $L$ is the vacuum neutrino oscillation length:

$$L = \frac{4\pi \hbar c \varepsilon}{\Delta m^2 c^4} ,$$

(3.7)

where $\varepsilon$ is the neutrino energy, $m_1$ and $m_2$ are the masses of the neutrino mass eigenstates, and $\Delta m^2 = m_2^2 - m_1^2$. The other variables have their standard meanings.

The electron neutrino finds the most leptonic interaction partners in the ambient matter. Therefore, we have the matrix:

$$\Omega_{\text{mat}}(r) = \left( \begin{array}{cc} 2A & 0 \\ 0 & 0 \end{array} \right) ,$$

(3.8)

where $A$ is the dimensionless neutrino-matter interaction amplitude (Mohapatra & Pal 2003, Giunti 2004):

$$A(r) = \frac{2 \sqrt{2} G_F}{\hbar} n_e(r) ,$$

(3.9)

where $G_F$ denotes Fermi’s constant, and $n_e$ the electron number density. The interaction amplitude $A$, through $n_e$, is a function of spatial position. For antineutrinos the sign of $A$ is reversed as
3.2 Neutrino self-interactions

The neutrino evolution in a neutrino background is nontrivial and continues to raise considerable discussions in the literature (Pantaleone 1992 A, Pantaleone 1995, Qian & Fuller 1995, Friedland & Lunardini 2003, Bell, Rawlinson & Sawyer 2003, Boyanovsky & Ho 2004, Sawyer 2004, and Sawyer 2005). Here we sketch the considerations that led to $\Omega_{\nu\nu}$ and work with the most established version that can be found in the literature.

New is, that we match the wavefunctions to the quasi-classical phase-space densities employed in our ensemble-averaged approach. In this work, we do not engage in efforts about a possible speed-up of flavor equilibration as proposed by Sawyer, et al.. We retain the one-particle description neglecting any effects arising from coherent many-body state formation or flavor entanglement (Friedland & Lunardini 2003). The exact expression for the neutral-current neutrino-neutrino interaction Hamiltonian for a test neutrino with initial momentum $p$ and the background neutrino with initial momentum $q$ is given by (Raffelt & Sigl 1993):

\[
\text{Hamiltonian} = \frac{G_F}{\sqrt{2}} \left( \bar{\nu}_\alpha \gamma^\mu \nu_\beta \eta^{\mu\nu} p^\nu q^\alpha + \text{H.c.} \right)
\]
\[ H^{\text{pq}}_{\nu
u} = \frac{1}{2} \left( \frac{G_F}{2 \cos \theta_W} \right)^2 \int d\mathbf{p}' \int d\mathbf{q}' (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \bar{\psi}_q \gamma^\mu \psi_q D_{\mu\nu} \bar{\psi}_{p'} \gamma^\nu \psi_{p'} , \]

(3.10)

where the \( \psi \)'s are the Dirac neutrino spinors as specified in eq. (2.6), \( \gamma^\mu \) are the standard Dirac matrices as given in eq. (2.3), \( \cos \theta_W \) is the Weinberg angle, \( g \) is the weak coupling constant given by

\[ g = \left( 4 \sqrt{2} G_F \right)^{1/2} m_Z , \]

(3.11)

where \( G_F \) is Fermi’s constant. \( D^{\mu\nu}_Z(p) \) is the Z-boson propagator

\[ D^{\mu\nu}_Z(p) = \left( \eta_{\mu\nu} - \frac{p^\mu p^\nu}{m^2_Z} \right) \frac{1}{m^2_Z - p^2} . \]

(3.12)

Force mediating gauge bosons that propagate from one neutrino at a given spacetime point to another neutrino at another spacetime point is a non-local effect. This can be readily discarded since in supernova cores and in most other situations found in nature one is typically interested in neutrinos with energies of the order of MeV. This is well below the mass or energy scale of the Z-boson of order 100 GeV. Therefore, we neglect the non-local gauge boson effects and work with the low-energy local four-fermion coupling. This effective Hamiltonian must satisfy SU(N) flavor symmetry for a system of N flavors (Pantaleone 1992 A) and we write the following terms:

\[ \Omega^{\text{pq}}_{\nu
u} = \frac{G_F}{\sqrt{2}} \left( \sum_i \bar{\psi}_q \gamma^\mu \psi_q \right) \left( \sum_j \bar{\psi}_q \gamma^\nu \psi_q \right) , \]

(3.13)

where the sum is over all neutrino flavors. In this form the SU(N) flavor symmetry is manifest. For a two-flavor system consisting of \( \nu_e \)'s and \( \nu_\mu \)'s one can rewrite this Hamiltonian to accentuate its off-diagonal character (Pantaleone 1992 B, Friedland & Lunardini 2003):

\[ \Omega^{\text{pq}}_{\nu
u} = \beta \left( 1 - \cos \theta_W \right) \left( |\psi_q|^2 + |\psi_{q'}|^2 + \left( \left| \psi_{\nu_e} \right|^2 \left| \psi_{\nu_e} \right|^2 + \left| \psi_{\nu_\mu} \right|^2 \left| \psi_{\nu_\mu} \right|^2 \right) \right) , \]

(3.14)

where the coupling coefficient prefers diametral scattering by inclusion of the angle \( \theta \) between the test neutrino with momentum \( \mathbf{p} \) and the background neutrino with momentum \( \mathbf{q} \).
3.2 Neutrino self-interactions

\[ \nu_e - \nu_e \text{ and } \nu_\mu - \nu_\mu \text{ scattering.} \]

\[ \int dV |\psi_{\nu_e}|^2 + |\psi_{\nu_\mu}|^2 = 1 \]

\[ \beta = \frac{\sqrt{2} G_F}{\hbar} . \quad (3.15) \]
In Fig. 3.2, the two diagonal contributions to the $\nu - \nu$ interaction are shown. The time axis is vertical. In Fig. 3.3, the off-diagonal contributions to the $\nu - \nu$ interaction are depicted. Here the point is that in forward direction the momenta of the two scattering partners are exchanged. Since the momentum eigenstates are the propagation eigenstates in the ambient matter, this is considered as a flavor-exchanging reaction.

We now match the the $\nu - \nu$ Hamiltonian from the wavefunction formalism to our ensemble-averaged description. To this end, we make use of our quasi-classical phase-space densities and the macroscopic overlap functions of eq. (2.16). This involves integration over all the momenta $q$ of the ensemble (note that the neutrino ensemble also constitutes the background).

The $\nu - \nu$ mixing Hamiltonian for test neutrinos with momentum $p$ in ensemble-averaged form is denoted by

$$\Omega_{\nu\nu}(p, r, t) = \begin{pmatrix} B_\nu & B_{\nu\mu} \\ B_{\mu\nu}^* & B_{\mu\mu} \end{pmatrix}, \quad (3.16)$$

where, while throwing away the overall phase term in eq. (3.14), which is proportional to the identity matrix, we write for the diagonal elements:

$$B_\nu (p, r, t) = \beta \int d^3 q \left( 1 - \cos \theta_{pq} \right) f_\nu(q, r, t)$$

$$B_\mu (p, r, t) = \beta \int d^3 q \left( 1 - \cos \theta_{pq} \right) f_\mu(q, r, t), \quad (3.17)$$

where the momentum integration goes over all momenta in the ensemble. For the off-diagonal elements we write in our notation, in terms of the macroscopic overlap functions,

$$B_{\nu\mu} (p, r, t) = \beta \int d^3 q \left( 1 - \cos \theta_{pq} \right) f_{\nu\mu}(q, r, t)$$

$$B_{\mu\nu}^* (p, r, t) = \beta \int d^3 q \left( 1 - \cos \theta_{pq} \right) f_{\mu\nu}^*(q, r, t). \quad (3.18)$$

For the antineutrinos in the ensemble and the ambient matter,

$$\tilde{\Omega}_{\nu\nu}(p, r, t) = \begin{pmatrix} \tilde{B}_\nu & \tilde{B}_{\nu\mu} \\ \tilde{B}_{\mu\nu}^* & \tilde{B}_{\mu\mu} \end{pmatrix}.$$ \quad (3.19)

And in complete analogy, we write for the interactions of the test neutrino with momentum $p$ with the antineutrinos,
\[ \tilde{B}_{\nu}(p, r, t) = \beta \int d^3q (1 - \cos \theta_{pq}) \tilde{f}_{\nu}(q, r, t) \]
\[ \tilde{B}_{\nu}(p, r, t) = \beta \int d^3q (1 - \cos \theta_{pq}) \tilde{f}_{\nu}(q, r, t) \]
\[ \tilde{B}_{e\mu}(p, r, t) = \beta \int d^3q (1 - \cos \theta_{pq}) \tilde{f}_{e\mu}(q, r, t) \]
\[ \tilde{B}_{e\mu}^*(p, r, t) = \beta \int d^3q (1 - \cos \theta_{pq}) \tilde{f}_{e\mu}^*(q, r, t) \].
\[
(3.20)
\]

The coupling coefficient in front of the integrals has to be implemented in the equations for neutrinos with reversed sign. For a similar way of parameterizing the neutrino–neutrino Hamiltonian see the nice paper by Fuller & Qian 1995.

### 3.3 Kinetic equations

We are interested in an analysis of the macroscopic, long-range transport behavior of eq. (3.1) and execute the following simplifying procedure.

- We do not take into account the effects of small-scale external homogeneities on the mixing in the ensemble (Sawyer 1990).
- We average out any small-scale correlations and do not include any other off-diagonal contributions from the complete Hamiltonian.
- We neglect off-diagonal contributions from the left-hand-side of eq. (3.1).
- We componentize eq. (3.1) and include all the contributions from the mixing Hamiltonian in eq. (3.3).

To this end it turns out to be useful to define the real part of the off-diagonal macroscopic overlap in eq. (2.16) as

\[ f_r = \frac{1}{2} \left( f_{e\mu} + f_{e\mu}^* \right) , \]
\[
(3.21)
\]
and the corresponding imaginary part as

\[ f_i = \frac{1}{2i} \left( f_{e\mu} - f_{e\mu}^* \right) , \]
\[
(3.22)
\]
respectively. For antineutrinos one proceeds analogously. For the real part one has
\[ \tilde{f}_e = \frac{1}{2} \left( \tilde{f}_\mu + \tilde{f}_\mu^* \right), \quad (3.23) \]

and for the corresponding imaginary part there is

\[ \tilde{f}_i = \frac{1}{2i} \left( \tilde{f}_\mu - \tilde{f}_\mu^* \right). \quad (3.24) \]

With this choice of parameters, everything becomes real-valued and the \( i' \)'s are eliminated from the equations. In the standard weak interaction basis or flavor basis the collision terms are diagonal and we will not consider flavor-changing reactions here. Henceforth we apply the diagonal collision matrix,

\[ C = \begin{pmatrix} C_{ee} & 0 \\ 0 & C_{\mu\mu} \end{pmatrix}. \quad (3.25) \]

For antineutrinos the collision terms differ and are denoted by

\[ \tilde{C} = \begin{pmatrix} \tilde{C}_{ee} & 0 \\ 0 & \tilde{C}_{\mu\mu} \end{pmatrix}. \quad (3.26) \]

Having made these definitions, we are in a position to componentize eq. (3.1) and eq. (3.4). We work with the general mixing Hamiltonian eq. (3.3) and eq. (3.5). For two mixing neutrino species, here denoted by \( \nu_e \) and \( \nu_\mu \), interacting with a background medium the generalized Boltzmann equations in their most generic form are:
\[
\frac{\partial f_{\nu}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{r}} + \mathbf{p} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{p}} = -f_i \left( \frac{2\pi c}{L} \sin 2\theta + 2\beta \int (1 - \cos \theta \mathbf{p}) (f_i + \tilde{f}_i) d^3q \right) + 2\beta f_i \int (1 - \cos \theta \mathbf{p}) (f_i + \tilde{f}_i) d^3q + C_{\nu},
\]

\[
\frac{\partial f_{\nu}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{r}} + \mathbf{p} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{p}} = f_i \left( \frac{2\pi c}{L} \sin 2\theta + 2\beta \int (1 - \cos \theta \mathbf{p}) (f_i + \tilde{f}_i) d^3q \right) - 2\beta f_i \int (1 - \cos \theta \mathbf{p}) (f_i + \tilde{f}_i) d^3q + C_{\nu},
\]

\[
\frac{\partial f_{\nu}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{r}} + \mathbf{p} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{p}} = f_i \left[ \frac{2\pi c}{L} (A - \cos 2\theta) + \beta \int (1 - \cos \theta \mathbf{p}) (f_i - \tilde{f}_i) d^3q \right] + (f_i - \tilde{f}_i) \beta \int (1 - \cos \theta \mathbf{p}) (f_i - \tilde{f}_i) d^3q
\]

\[
\frac{\partial f_{\nu}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{r}} + \mathbf{p} \cdot \frac{\partial f_{\nu}}{\partial \mathbf{p}} = \left( f_{\nu} - f_{\tilde{\nu}} \right) \left[ \frac{\pi c}{L} \sin 2\theta + \beta \int (1 - \cos \theta \mathbf{p}) (f_{\nu} - \tilde{f}_{\nu}) d^3q \right] - f_i \left[ \frac{2\pi c}{L} (A - \cos 2\theta) + \beta \int (1 - \cos \theta \mathbf{p}) (f_i - \tilde{f}_i) d^3q \right].
\]

(3.27)

In a very similar fashion we can write the corresponding antiparticle analog. We need to interchange and add tildes, sign-reverse the matter interaction amplitude \(A\), and complete our set of equations:
Neutrino and antineutrino evolution are nonlinearly coupled. The collision terms differ for neutrinos and antineutrinos. Our new generalized eight Boltzmann equations are entirely real-valued, yet they contain all the quantum-mechanical oscillation phenomenology.

This is the complete set of kinetic equations for the real-valued neutrino phase-space densities \( f_\nu \) and \( f_{\bar{\nu}} \) and the corresponding antineutrino phase-space densities \( \bar{f}_\nu \) and \( \bar{f}_{\bar{\nu}} \). In principle, one needs to solve the above set of eight real-valued equations to simulate and understand neutrino evolution in phase-space as for instance in numerical calculations for the supernova explosion mechanism and the early universe. Our generalized Boltzmann equations are intrinsically nonlinear due to the neutrino self-interactions from \( \Omega_{\nu \nu} \) and \( \bar{\Omega}_{\nu \bar{\nu}} \).

Note that in the absence of collisions, Liouville’s theorem for the total neutrino flavor specific intensity,

\[
\frac{d}{dt} (f_\nu + f_{\bar{\nu}}) = 0 ,
\]

where \( \frac{d}{dt} \) denotes the total time derivative, is recovered.

\( \text{3.28} \)
Moreover, at sufficiently high densities, one must include blocking corrections for the collision terms as well as for the oscillation terms which would further increase the nonlinearity of the equations. We feel that this is straightforward; there is no need to include these statistical properties in the formal framework.

For blocking corrections to be important for neutrinos, however, the densities must be sufficiently high ($\rho > 10^8$ g cm$^{-3}$). At these densities observable neutrino oscillations are suppressed due to the very small effective mixing angle in matter which can be written as (Wolfenstein 1979, Pantaleone 1989):

$$\sin 2\theta = \frac{\sin 2\theta_{\text{vac}}}{\sqrt{\sin^2 2\theta_{\text{vac}} + (\cos 2\theta_{\text{vac}} - \frac{A\epsilon}{\Delta m^2})^2}}.$$  \hfill (3.30)

In this expression for the effective mixing angle in matter, $\theta_{\text{vac}}$ is the mixing angle for vacuum neutrino oscillations and in numbers we have for the matter-induced effective mass term:

$$\frac{A\epsilon}{\Delta m^2} = 2 \sqrt{2} G_F \left(\frac{Y_e}{m_n}\right) \rho \epsilon \left[\frac{10^{-5}\text{eV}^2}{\Delta m^2}\right],$$  \hfill (3.31)

where $\rho$ denotes the mass density, $Y_e$ the number of electrons per nucleon, and $m_n$ the nucleon mass.

One must include the other constituents of the ambient matter fields such as neutrinos, nucleons and other leptons for an exact effective mixing angle (Qian & Fuller 1995). The density dependence is alike and if anything, this additional contribution amplifies the effect of matter suppression.

### 3.3.1 Transport equations for the neutrino radiation field

It is instructive to rewrite eq. (3.27) in terms of the specific intensities of the neutrino radiation field. This way, the collision integrals can be cast in a simple form. We neglect the neutrino self-interactions in the following, having shown how to include them in eq. (3.27). Also we focus on the neutrino degrees of freedom since without the self-interactions neutrino and antineutrino evolution are coupled only in the normal fashion through pair processes. We use the one-to-one relation between the invariant phase-space densities, $f_{\nu_e}, f_{\nu_x}$ and the specific intensities, $I_{\nu_e}, I_{\nu_x}$ (Burrows, et al. 2000, Mihalas 1999) and define

$$I_{\nu_e} = \frac{e^3 f_{\nu_e}}{(2\pi\hbar)^2 c^2}$$

$$I_{\nu_x} = \frac{e^3 f_{\nu_x}}{(2\pi\hbar)^2 c^2}.$$  \hfill (3.32)
The off-diagonal macroscopic overlap terms have the same units as the phase-space densities and thus we may define their associated specific intensities as

\[ R_{\nu_i} = \frac{e^3 f_{\nu_i}}{(2\pi\hbar)^3c^2} \]

\[ I_{\nu_i} = \frac{e^3 f_{\nu_i}}{(2\pi\hbar)^3c^2}. \]  

(3.33)

These physically observable variables are commonly used in hydrodynamic supernova simulations and we can write the extended set of transport equations including neutrino oscillation phenomenology in terms of the specific intensities.

The generalized Boltzmann equations in the laboratory (Eulerian frame) for the radiation field of two oscillating neutrino species with collisions are then:

\[
\frac{1}{c} \frac{\partial I_{\nu_e}}{\partial t} + \frac{v}{c} \cdot \frac{\partial I_{\nu_e}}{\partial r} + \frac{e^3 \dot{p}}{c} \cdot \frac{\partial (I_{\nu_e} \epsilon^{-1})}{\partial p} = -\frac{2\pi}{L} I_{\nu_e} \sin 2\theta + C'_{\nu_e}
\]

\[
\frac{1}{c} \frac{\partial I_{\nu_\mu}}{\partial t} + \frac{v}{c} \cdot \frac{\partial I_{\nu_\mu}}{\partial r} + \frac{e^3 \dot{p}}{c} \cdot \frac{\partial (I_{\nu_\mu} \epsilon^{-1})}{\partial p} = -\frac{2\pi}{L} I_{\nu_\mu} \sin 2\theta + C'_{\nu_\mu}
\]

\[
\frac{1}{c} \frac{\partial R_{\nu_i}}{\partial t} + \frac{v}{c} \cdot \frac{\partial R_{\nu_i}}{\partial r} + \frac{e^3 \dot{p}}{c} \cdot \frac{\partial (R_{\nu_i} \epsilon^{-1})}{\partial p} = -\frac{2\pi}{L} (\cos 2\theta - A) R_{\nu_i}
\]

\[
\frac{1}{c} \frac{\partial I_{\nu_i}}{\partial t} + \frac{v}{c} \cdot \frac{\partial I_{\nu_i}}{\partial r} + \frac{e^3 \dot{p}}{c} \cdot \frac{\partial (I_{\nu_i} \epsilon^{-1})}{\partial p} = \frac{2\pi}{L} \left( \frac{I_{\nu_e} - I_{\nu_\mu}}{2} \right) \sin 2\theta + (\cos 2\theta - A) R_{\nu_i}
\]  

(3.34)

These equations and eqs. (3.27) are our major results. The collision terms can be conveniently written as (Burrows, et al. 2000)

\[ C'_{\nu_i} = -\kappa^3_{\nu_i} \frac{B_{\nu_i} - I_{\nu_i}}{1 - F_{\nu_i}^{eq}} + \kappa^a_{\nu_i} \int \frac{\Phi_{\nu_i}(\Omega, \Omega') I_{\nu_i}(\Omega')d\Omega'}{4\pi}, \]  

(3.35)

where \( \Phi_{\nu_i} \) is a phase function for scattering into the beam integrated over the solid angle \( d\Omega' \). Furthermore, \( \kappa^a_{\nu_i} \) is the sum of all absorption processes \( \sum_i n_i \sigma^a_i \), where \( n_i \) is the number density of matter species \( i \) and \( \sigma^a_i \) denotes the absorption cross sections (for scattering processes the superscript \( a \) is replaced with \( s \)).

\( F_{\nu_i}^{eq} \) is the equilibrium Fermi-Dirac occupation probability

\[ F_{\nu_i}^{eq} = \frac{1}{\exp[(\epsilon_{\nu_i} - (\mu_\nu - \hat{\mu}))\beta] + 1}, \]  

(3.36)

where \( \mu_\nu \) and \( \hat{\mu} \) denote the electron chemical potential and the difference between neutron and proton chemical potential, respectively, and \( B_{\nu_i} \) is the corresponding black body specific intensity.
Changing the subscript from $\nu_e$ to $\nu_\mu$ yields the corresponding parameters for the $\nu_\mu$'s. For sterile neutrinos, one substitutes $\nu_s$ for $\nu_\mu$ and sets the scattering and absorption terms to zero. Similarly, one can write a set of equations for antineutrinos with different collision terms and with the sign of $A$ reversed. Here, neutrino and antineutrino evolution are implicitly coupled through pair processes.

### 3.4 Conclusions

- We have formulated the most generic form of the generalized Boltzmann equations including the nonlinear evolution in a neutrino background in eq. (3.27).

- We have succeeded to incorporate the purely quantum-physical phenomenon of flavor oscillations into the classical transport equation (3.34).

- In the following, we employ the simple version of our new kinetic equations in terms of the specific intensities, eq. (3.34), and perform explicit analytic and numerical tests of this formalism in chapter 4.

- We show that our formalism is completely consistent with the existing single-particle wavefunction approach in subsection 4.3.1.
In practice, it is interesting how the neutrino ensemble behaves with time when there are flavor oscillations and collisions simultaneously. Since the interaction rates for the differently flavored neutrinos can differ dramatically—depending on the surrounding matter constituents—, the possibility of flavor oscillations must be taken into account for.

Eqs. (3.27) and (3.34) serve this purpose. For the illustrative examples in the present chapter we neglect the neutrino self-interactions, work with eq. (3.34), and demonstrate the correct limiting behavior of the formalism.

4.1 Flavor oscillations with absorptive matter coupling

As a first example, we solve eqs. (3.34) for $\nu_e - \nu_\mu$ oscillations in a box of isotropic neutrinos that can also experience decohering absorption on matter. This is realized by including the neutrino absorption reaction on nucleons as given in Appendix A. Every absorptive collision decoheres the oscillation cycle. In reality, however, due to the small cross-sections of neutrinos, frequent collisions with the ambient matter fields require high densities. As indicated above in eq. (3.30), observable flavor oscillations are suppressed at the high densities found in stellar collapse (Wolfenstein 1979). To remedy this problem artificially, we “turn off” matter suppression by setting the matter interaction amplitude $A$ of $\Omega_{\text{mat}}$ in eqs. (3.30) and (3.34) to zero. This is not the situation found in nature.

The goal of the following definitions is to simplify eq. (3.34) for the present purpose and write it in terms of dimensionless variables. Therefore, we define the approximate oscillation time,

$$t_{osc} = \frac{L}{2\pi c} = \frac{2\hbar c}{\Delta m^2 e^4}, \quad (4.1)$$
and artificially set the interaction rate of the $\nu_e$’s equal to the interaction rate of the $\nu_\mu$’s $^1$. And we define the characteristic absorption time,

$$t_{\nu_e} = \frac{1}{\kappa_{\nu_e}^c} = \frac{(1 - \tau_{\nu_e}^c)}{cN_A\rho Y_n\sigma_{\nu_e n}},$$

(4.2)

where $N_A$ denotes Avogadro’s number, $Y_n$ is the neutron fraction per nucleon, and $\sigma_{\nu_e n}^a$ is the cross section for absorption on neutrons as given in Appendix A. The absorption time is flavor-independent in our toy calculation. As a “critical parameter” for the characteristic time evolution of the system we define the ratio of the oscillation to the absorption timescales:

$$\alpha = \frac{t_{\nu_e}}{t_{\nu_e}^c},$$

(4.3)

and finally we define the dimensionless time coordinate

$$\tau = \frac{t}{t_{\nu_e}},$$

(4.4)

Furthermore, we set $B_{\nu_e} = B_{\nu_\mu} = B_\nu$ and denote the dimensionless specific intensities by

$$\tilde{I}_{\nu_e}(\tau) = \frac{I_{\nu_e}}{B_\nu},$$

$$\tilde{I}_{\nu_\mu}(\tau) = \frac{I_{\nu_\mu}}{B_\nu},$$

(4.5)

and for the off-diagonal macroscopic overlap functions we write

$$\tilde{R}_{\nu_e \nu_\mu}(\tau) = \frac{R_{\nu_e \nu_\mu}}{B_\nu},$$

$$\tilde{I}_{\nu_e \nu_\mu}(\tau) = \frac{I_{\nu_e \nu_\mu}}{B_\nu}.$$  

(4.6)

Note that the black-body function is flavor-independent, since we take both neutrino flavors to be of the same energy (chemical potentials have also been set to zero; they do not affect the solutions significantly). The resulting dimensionless version of eq. (3.34) reads:

---

$^1$ This definition of flavor-independent interaction rates is again not so realistic. Cross sections depend on the mass of the associated leptons which differ by a factor of $\approx 200$. Furthermore the abundance of the potential interaction partners in the ambient matter is normally much greater for the $\nu_e$’s.
4.1 Flavor oscillations with absorptive matter coupling

\[\frac{\partial \hat{I}_{\nu e}}{\partial \tau} = -\hat{I}_{e\mu} \sin 2\theta + \alpha \left(1 - \hat{I}_{\nu e}\right)\]
\[\frac{\partial \hat{I}_{\nu\mu}}{\partial \tau} = \hat{I}_{e\mu} \sin 2\theta + \alpha \left(1 - \hat{I}_{\nu\mu}\right)\]
\[\frac{\partial \hat{R}_{e\mu}}{\partial \tau} = -\hat{I}_{e\mu} \cos 2\theta\]
\[\frac{\partial \hat{I}_{e\mu}}{\partial \tau} = \frac{\hat{I}_{\nu e} - \hat{I}_{\nu\mu}}{2} \sin 2\theta + \hat{R}_{e\mu} \cos 2\theta.\]

(4.7)

Since here we consider isotropic neutrinos, it is easy to show that the scattering in and out of the beam – represented by the integral over the phase-function – and the scattering part of the extinction coefficient cancel \(^2\).

In matter and for the initial conditions \(\hat{I}_{\nu e} = \hat{R}_{e\mu} = \hat{I}_{e\mu} = 0\) and \(\hat{I}_{\nu\mu} \neq 0\), one can derive an harmonic oscillator equation for the early rate of evolution of \(\hat{I}_{\nu e}\):

\[\frac{\partial^2 \hat{I}_{\nu e}}{\partial \tau^2} + \left(\frac{1}{2} - \alpha^2\right) \hat{I}_{\nu e} = \text{const.}\]  

(4.9)

As expected, for \(\alpha \ll 1\), the early time dependence of the solution is predominantly sinusoidal: neutrino oscillations dominate. For \(\alpha \gg 1\), an exponential decay/increase dominates with a timescale of \(1/\alpha\). Further algebraic manipulations convert eq. (4.7) into an inhomogeneous ordinary differential equation of fourth order for the dimensionless electron-neutrino specific intensity, valid for all \(\tau\):

\(^2\)In a vacuum (\(\alpha = 0\)) and with the initial conditions \(\hat{I}_{\nu e} = 1, \hat{I}_{\nu\mu} = 0\) (and consequently no off-diagonal overlap terms at \(\tau = 0\)), we obtain

\[\hat{I}_{\nu e}(\tau) = 1 - \sin^2 2\theta \sin^2 \frac{\tau}{2}\]
\[\hat{I}_{\nu\mu}(\tau) = \sin^2 2\theta \sin^2 \frac{\tau}{2}\]
\[\hat{R}_{e\mu}(\tau) = \frac{1}{2} \sin 2\theta \cos 2\theta (\cos \tau - 1)\]
\[\hat{I}_{e\mu}(\tau) = \frac{1}{2} \sin 2\theta \sin \tau.\]

(4.8)

This behavior of the radiation field is unambiguously identical to the probability density obtained by squaring the amplitude of a single-neutrino wavefunction in a beam. The off-diagonal terms representing the macroscopic overlap peak when mixing of \(\nu_e\) and \(\nu_\mu\) neutrinos is maximal and vanish when the ensemble is single-flavored.
\[ \frac{\partial^4 \hat{I}_{\nu}}{\partial \tau^4} + 2\alpha \frac{\partial^3 \hat{I}_{\nu}}{\partial \tau^3} + \left( \alpha^2 + \cot^2 2\theta + 1 \right) \frac{\partial^2 \hat{I}_{\nu}}{\partial \tau^2} + \alpha \left( 2 \cot^2 2\theta + 1 \right) \frac{\partial \hat{I}_{\nu}}{\partial \tau} + (\alpha \cot 2\theta)^2 \left( \hat{I}_{\nu} - 1 \right) = 0 . \] (4.10)

In the following, we evaluate eqs. (4.7) and (4.10) for three different regimes (without matter suppression since the matter interaction amplitude for the $\nu_e$, $A$, has been set to zero artificially). Initial conditions are $\hat{I}_{\nu_e} = \hat{R}_{e\mu} = \hat{I}_{e\mu} = 0$ and $\hat{I}_{\nu_e} = 0.8$ and from the large-mixing-angle solution (LMA) (Fukuda, et al. 2002) we apply $\sin 2\theta = 0.9$ and $\Delta m^2 = 6.9 \times 10^{-5} \text{ eV}^2$.

4.1.1 Oscillation-dominated regime: $\alpha \ll 1$

We tailor eq. (4.10) according to the smallness of $\alpha$ and write

\[ \frac{\partial^4 \hat{I}_{\nu_e}}{\partial \tau^4} + \left( \cot^2 2\theta + 1 \right) \frac{\partial^2 \hat{I}_{\nu_e}}{\partial \tau^2} = 0 , \] (4.11)

where we have dropped all terms with $\alpha$. This is an approximation – even in this regime.

The time dependence of the solution to this equation is predominantly sinusoidal with oscillation periods of the order of unity.

As depicted in Fig. 4.1, flavor oscillations dominate the temporal evolution of the neutrinos. Asymptotically, the diagonal specific intensities for oscillating $\nu_e$ and $\nu_\mu$ neutrinos equilibrate at 1.

Absorption on neutrons, and by detailed balance, the resulting emissivity, drive the system to radiative equilibrium at the blackbody intensity. The real part of the off-diagonal term, $\hat{R}_{e\mu}$, takes negative values, whereas the imaginary part, $\hat{I}_{e\mu}$, oscillates symmetrically around zero. The oscillation amplitude of these terms is comparable to the change in magnitude due to absorption. Both, $\hat{R}_{e\mu}$ and $\hat{I}_{e\mu}$ converge toward zero as the ensemble becomes increasingly decohered; the system eventually approaches flavor equilibrium and no oscillations persist.

In other words, the density matrix of the system becomes diagonal with time due to absorptive coupling with matter.
4.1 Flavor oscillations with absorptive matter coupling

Fig. 4.1. Oscillation-dominated regime. (a): dimensionless specific intensities. (b): macroscopic overlap functions. Parameters: $\epsilon_{\nu_e} = \epsilon_{\nu_\mu} = 10$ MeV, $\rho = 4 \times 10^{11}$ g cm$^{-3}$, and $T = 1$ MeV.
4.1.2 Oscillating and absorptive regime: \( \alpha \approx 1 \)

In this case, all terms of eq. (4.10) have to be retained. The solution to the homogenous equation consists of a sum of exponentials with the four roots of the following equation in Fourier space as exponents:

\[
\omega^4 - 2i\omega^3 - \left( \cot^2 2\theta + 2 \right) \omega^2 + i \left( 1 + 2 \cot^2 2\theta \right) \omega + \cot^2 2\theta = 0 ,
\]  

where we have set \( \alpha = 1 \). This equation has two roots of real value and two roots of complex value with magnitudes of the order of unity. Thus, decaying exponentials and oscillating exponentials are represented equally.

From Fig. 4.2, it is clear that the equilibration due to absorption happens on the same timescale as oscillations. The ensemble is guided to flavor and radiative equilibrium. Coherent flavor oscillations are disrupted by absorptive collisions. Asymptotically, the diagonal specific intensities for the \( \nu_e \)'s and \( \nu_\mu \)'s equilibrate.

Absorption on neutrons, and by detailed balance, the resulting emissivity, drive the \( \nu_e \)'s to the blackbody intensity. The oscillation amplitude decreases with time; the quantum evolution of the system is decohered through absorptive coupling with matter. This is sometimes referred to as quantum decoherence (Raffelt, Sigl & Stodolsky 1993). The real part of the off-diagonal overlap, \( \hat{R}_{eq} \), takes predominantly negative values whereas the imaginary part, \( \hat{I}_{eq} \), oscillates symmetrically around zero. Both vanish asymptotically and no oscillations persist.
4.1 Flavor oscillations with absorptive matter coupling

Fig. 4.2. $\nu_e - \nu_\mu$ oscillations of isotropic neutrinos in a box with absorptive matter coupling. (a) Specific intensities. (b) Off-diagonal macroscopic overlap functions. Parameters: $\epsilon_{\nu_e} = \epsilon_{\nu_\mu} = 10$ MeV, $\rho = 8 \times 10^{12}$ g cm$^{-3}$, $T = 5$ MeV.
4.1.3 Absorption-dominated regime: $\alpha \gg 1$

We extract the terms of eq. (4.10) consisting of $\alpha$ and write the approximate equation for this regime as

$$\frac{\partial^3 \hat{I}_{\nu_e}}{\partial \tau^3} + \frac{\alpha}{2} \frac{\partial^2 \hat{I}_{\nu_e}}{\partial \tau^2} + \frac{\alpha \cot^2 2\theta}{2} \hat{I}_{\nu_e} - \frac{\alpha \cot^2 2\theta}{2} I_{\nu_e} = 0 .$$

(4.13)

The solution is predominantly exponentially waxing or waning (depending on the initial conditions) with timescale $1/\alpha$.

The equilibration due to absorption is the dominant feature of this regime as shown in Fig. 4.3. Nonetheless, the specific intensities execute small-amplitude oscillations around the blackbody value (a small effect due to neutrino mixing). The diagonal specific intensities for $\nu_e$ and $\nu_\mu$ neutrinos equilibrate at the blackbody intensity in fractions of one reduced oscillation time $\tau$. The amplitude of the corresponding off-diagonal macroscopic overlap functions is very small (note the values on the ordinate). In a nutshell, oscillations do not influence the solution significantly.
Fig. 4.3. Absorption-dominated regime. (a): dimensionless specific intensities. (b): off-diagonal macroscopic overlap functions. Note the small values on the ordinate of (b). Parameters: $\varepsilon_{\nu_e} = \varepsilon_{\nu_\mu} = 10$ MeV, $\rho = 4 \times 10^{13}$ g cm$^{-3}$, and $T = 10$ MeV.
4.2 Active-sterile oscillation in a beam with absorptive matter coupling

The mass of hypothetical sterile neutrinos can only be constrained by observation (Abazajian, Fuller & Patel 2001). It is commonly thought that it could be much larger than the mass of the active neutrino species resulting in a comparatively large mass-squared-difference: $\Delta m^2 \sim O(eV^2)$ (Abazajian, Bell, Fuller & Wong 2004). This shifts the relevance of flavor oscillations to relatively high densities.

As in the preceding section, we artificially “turn off” matter suppression by setting the matter interaction amplitude $A$ to zero. We determine the specific intensities of a $\nu_e$ neutrino beam simultaneously oscillating into a sterile species $\nu_s$ and being absorbed by nucleons including only the reaction in Appendix A.

To this end, we define the approximate “reduced” oscillation length

$$l_{osc} \approx \frac{L}{2\pi \sin 2\theta} = \frac{2\hbar \varepsilon}{\Delta m^2 c^3 \sin 2\theta}, \quad (4.14)$$

and the characteristic absorption length

$$f_{\nu_e \text{col}} = \frac{1}{k_{\nu_e}} \left(1 - f_{\nu_e}^{eq}\right) \frac{N_A \rho Y_n \sigma_{\nu_e n}}{N_{eq} \rho Y_n \sigma_{\nu_e n}}, \quad (4.15)$$

where $N_A$ denotes Avogadro’s number, and $Y_n$ is the neutron fraction per nucleon, and $\sigma_{\nu_e n}$ is the cross section for absorption on neutrons. We define the ratio of oscillation to absorption length scales

$$\alpha = \frac{l_{osc}}{f_{\nu_e \text{col}}}, \quad (4.16)$$

and the dimensionless distance coordinate

$$\xi = \frac{x}{l_{osc}}. \quad (4.17)$$

We denote the dimensionless specific intensities and the off-diagonal macroscopic overlap functions normalized to the blackbody intensity with a hat.
4.2 Active-sterile oscillation in a beam with absorptive matter coupling

Fig. 4.4. Active-sterile conversion with absorptive matter coupling. (a) Specific intensities. (b) Off-diagonal macroscopic overlap functions. Parameters: $\epsilon_{\nu_e} = \epsilon_{\nu_s} = 40$ MeV, $\rho = 5 \times 10^{13}$ g cm$^{-3}$, $\sin 2\theta = 0.9$, $T = 10$ MeV, and $\Delta m^2 = 0.1$ eV$^2$ (Abazajian, Bell, Fuller & Wong 2004).
The resulting dimensionless version of eq. (3.34) reads:

\[
\begin{align*}
\frac{\partial \hat{I}_{\nu_e}}{\partial \xi} &= -\hat{I}_{es} + \alpha \left(1 - \hat{I}_{\nu_e}\right) \\
\frac{\partial \hat{I}_{\nu_s}}{\partial \xi} &= \hat{I}_{es} \\
\frac{\partial \hat{R}_{es}}{\partial \xi} &= -\hat{I}_{es} \cot 2\theta \\
\frac{\partial \hat{I}_{es}}{\partial \xi} &= \frac{\hat{I}_{\nu_e} - \hat{I}_{\nu_s}}{2} + \hat{R}_{es} \cot 2\theta.
\end{align*}
\]

(4.18)

Note that there are no collision terms for sterile neutrinos.

In Fig. 4.4, we depict the solutions to eq. (4.7) for a \(\nu_e\) neutrino beam simultaneously oscillating into a sterile species \(\nu_s\) and being absorbed by nucleons. In this toy example, \(\alpha = 0.2\). Initial conditions are \(\hat{I}_{\nu_e} = \hat{R}_{es} = \hat{I}_{es} = 0\) and \(\hat{I}_{\nu_s} = 0.5\). Flavor oscillations and collisions happen on the same length scale. While passing through matter the beam is guided to flavor equilibrium by the radiative equilibration. The coherence of flavor oscillations is decreased with time. Similar to the neutrinos in a box, the \(\nu_e\)’s asymptotically equilibrate at the blackbody intensity. The sterile neutrinos in the beam converge toward the value 0.475.

The interacting \(\nu_e\)’s indirectly govern the oscillatory pattern of the \(\nu_s\)’s even though the sterile neutrinos do not interact. The oscillation amplitude decreases with time; the quantum evolution of the system is damped through absorptive coupling with matter. The real part of the off-diagonal overlap, \(\hat{R}_{es}\), converges toward the finite negative value \(-0.35\). The imaginary part, \(\hat{I}_{es}\), oscillates symmetrically around zero and asymptotically vanishes.
4.3 Matter-enhanced resonant flavor conversion

When a mixed neutrino traverses matter at certain densities, its flavor composition may be resonantly altered. This is due to the, compared to the vacuum, large difference between the effective neutrino masses in matter. The $\nu_e$ receives an additional effective mass contribution due to its (compared with the other flavors) super-allowed interaction rates. This has been incorporated into the equations in this thesis; see eq. (3.8).

To demonstrate that our formalism contains the MSW effect (Wolfenstein 1979, Mikheyev & Smirnov 1986, Bethe 1986), we solve eq. (3.34) for a monoenergetic one-dimensional neutrino beam propagating down a density profile for which resonant matter-enhanced flavor conversion takes place.

We define the dimensionless distance coordinate in terms of the oscillation length

$$\hat{x} = \frac{x}{2\pi L},$$

and the dimensionless matter-induced mass term in terms of its resonance value:

$$\hat{A} = \frac{A}{A_{\text{res}}} = \frac{A}{\cos 2\theta}.$$  \hspace{1cm} (4.19)

The beam passes the resonance density for $\hat{A} = 1$. We analyze the following dimensionless version of eq. (3.34):

$$\frac{\partial I_{\nu_e}}{\partial \hat{x}} = -I_{\nu_{\mu}} \sin 2\theta$$
$$\frac{\partial I_{\nu_{\mu}}}{\partial \hat{x}} = I_{\nu_{\mu}} \sin 2\theta$$
$$\frac{\partial R_{\nu_{\mu}}}{\partial \hat{x}} = -\cos 2\theta \left(1 - \hat{A}\right) I_{\nu_{\mu}}$$
$$\frac{\partial I_{\nu_{\mu}}}{\partial \hat{x}} = \frac{I_{\nu_e} - I_{\nu_{\mu}}}{2} \sin 2\theta + \cos 2\theta \left(1 - \hat{A}\right) R_{\nu_{\mu}},$$

where $C'_{\nu_e}$ and $C'_{\nu_{\mu}}$ have been set to zero.

In Fig. 4.5 we depict the solutions to eq. (4.21) for a density profile of $\hat{A} = \beta$. We set $\beta = 900$ and thus ensure that the scale of spatial inhomogeneities is large compared to the microscopic length scales such as the neutrino de Broglie wavelength and the oscillation length. Initially, the beam contains only $\nu_e$ neutrinos. The mixing angle is arbitrarily taken to be $\sin^2 2\theta = 0.18$ (at present the LMA is favored (Fukuda, et al. 2002). From Fig. 4.5 it is clear that at the resonance density, $\hat{x} = \sqrt{\beta} = 30$, the flavor composition of the beam is radically altered. For higher values
Fig. 4.5. MSW effect. (a) Specific intensities. (b) Off-diagonal macroscopic overlap functions. An initially $\nu_e$-beam propagates down the density profile $A = \frac{4}{\beta^2}$, where $\beta = 900$. 
of $\bar{x}$, the beam executes vacuum oscillations. In this illustrative problem the density at production is much greater than the resonance density. Then, the spatially averaged survival probability of a $\nu_e$ neutrino going from matter to free space should be (Parke 1986)

$$\langle P(\nu_e \to \nu_e) \rangle \approx (1 - P_x) \sin^2 \theta + P_x \cos^2 \theta ,$$ (4.22)

where $P_x$ is the Landau-Zener probability for non-adiabatic transitions. For the chosen density profile, propagation is adiabatic and $P_x = 0$. The averaged $\nu_e$ survival probability in Fig. 4.5 converges toward $\sim 0.05$, which is congruent with the value predicted in eq. (4.22): $\sin^2 \theta \sim 0.05$. For high densities ($\bar{x} \leq 20$), matter suppression is severe. No flavor oscillations happen and the off-diagonal overlap functions $R_{\nu\mu}$ and $I_{\nu\mu}$ are close to zero. In free space for $\bar{x} > 60$, the imaginary part $I_{\nu\mu}$ oscillates symmetrically around zero whereas the real part $R_{\nu\mu}$ is positive.

### 4.3.1 Equivalence with wavefunction quantum mechanics

The solution given in Fig. 4.5 is numerically equivalent to the solution obtained using the standard wavefunction formalism (Giunti 2004, Wolfenstein 1978, Bethe 1986, Mikheyev & Smirnov 1986):

$$i \frac{\partial}{\partial \bar{x}} \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_{\mu}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} - \cos 2\theta \left(1 - 2\hat{A} \right) & \sin 2\theta \cos 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \psi_{\nu_e} \\ \psi_{\nu_{\mu}} \end{pmatrix} ,$$

$$-i \frac{\partial}{\partial \bar{x}} \begin{pmatrix} \psi_{\nu_e}^* \\ \psi_{\nu_{\mu}}^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} - \cos 2\theta \left(1 - 2\hat{A} \right) & \sin 2\theta \cos 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \psi_{\nu_e}^* \\ \psi_{\nu_{\mu}}^* \end{pmatrix} ,$$ (4.23)

when we identify the specific intensities with the probability densities

$$I_{\nu_e} \leftrightarrow |\psi_{\nu_e}|^2$$

$$I_{\nu_{\mu}} \leftrightarrow |\psi_{\nu_{\mu}}|^2 ,$$ (4.24)

and the macroscopic overlap functions with

$$R_{\nu\mu} \leftrightarrow \frac{1}{2} \left( \psi_{\nu_e} \psi_{\nu_{\mu}}^* + \psi_{\nu_e}^* \psi_{\nu_{\mu}} \right)$$

$$I_{\nu\mu} \leftrightarrow \frac{1}{2i} \left( \psi_{\nu_e} \psi_{\nu_{\mu}}^* - \psi_{\nu_e}^* \psi_{\nu_{\mu}} \right) .$$ (4.25)

Thus, our Boltzmann formalism is completely consistent with the existing description.
4.4 Conclusions

In this chapter, we have demonstrated that our generalized Boltzmann equations for the neutrino specific intensities, eq. (3.34), reproduce the expected behavior. We have considered three different settings, for which we numerically solved dimensionless versions of eq. (3.34).

- The vacuum beam solution was calculated in analytic form for a non-interacting neutrino ensemble in a box, eq. (4.8).

- We have shown how the interplay between flavor oscillation and decoherent matter coupling can be analyzed simply by solving eq. (4.7) for neutrinos undergoing absorptions on nucleons. The graphical results for three different regimes are given in Figs. 4.1, 4.2, and 4.3.

- An important feature of our new approach is that the MSW effect for a streaming neutrino beam can be analyzed in a very simple fashion for observable quantities, see eq. (4.21). Numerical results for an exemplary density profile are depicted in Fig. 4.5.

- For realistic physical settings in a supernova environment, one must take into account the energy spectrum of the neutrinos, energy redistribution scattering, etc., and calculate the solutions to eq. (3.34) in their full generality. This has not been performed in this work and it would also be interesting to understand better the effects of the self-interaction contributions on the right-hand-side of eq. (3.27).
We briefly highlight the main features of our formalism in flat spacetime. The analytic expressions and conclusions reached in Part 1 are published in Physical Review D (Strack & Burrows 2005).

5.1 Quantum effects

- Neutrinos are mathematically represented by spinor fields. One consequence of their spin-\(\frac{1}{2}\) nature is the Fermi-Dirac state occupation probability. We include this in the stimulated absorption terms of our formalism in eq. (3.30). Furthermore, we argue that it is straightforward to include blocking corrections for the sink and source terms originating in the flavor oscillations in eqs. (3.27) and (3.34) as well as in the corresponding collision integrals (Burrows, et al. 2004).

- In the simultaneous presence of collision and oscillations, two time scales govern the evolution of the ensemble. When the collision or interaction time is of the same order as the flavor oscillation periods, it is believed that the collisions decohere the oscillation cycles and reset their clock. Then the temporal evolution becomes non-trivial since both processes must be taken into account for. We have shown in chapter 4 and in particular the in series of graphs, Figs. 4.1, 4.2, and 4.3, that our formalism encapsulates this interplay of flavor oscillations and matter coupling in a straightforward way.

- The possibility of oscillation into a sterile neutrino species is explored in section 4.2. We can observe indirect affection of the sterile neutrinos, since their oscillating flavor partner, the \(\nu_s\)'s do interact. The plots are shown in Fig. 4.2. This is potentially important for kinetic calculations in the early universe (Dolgov 1990) and could also have an effect on the flavor composition of the neutrino spectra emerging from supernovae (Abazajian, et al. 2004).
• Abstractly speaking, the neutrino vector is resonantly rotated in flavor space through matter interaction with electrons (MSW effect) at certain densities. This has been suggested to explain the solar neutrino puzzle (Bethe 1986). We have embedded this effect in a simple quasi-classical framework and have shown analytically that our formalism correctly displays this important quantum-physical prediction in subsection 4.3.1. The corresponding plot is depicted in Fig. 4.5.

5.2 Classical framework

• The classical framework and the classical equations of radiation hydrodynamics (Mihalas 1999) are commonly employed to simulate core-collapse supernovae. The neutrinos are believed to play a crucial role in the explosion process as they carry away almost all the gravitational energy. Future detection of neutrino spectra from supernova explosions will unravel the supernova mystery a little more and buttress or falsify the theoretical models on the market. Therefore it is very important to consistently include neutrino oscillations into the propagation from the supernova to the detector but also to include it in the kinetic description close to the core.

• Our generalized set of kinetic equations, eqs. (3.27) and (3.34), will allow straightforward extension of any numerical code that simulates core-collapse, to account for the purely quantum-mechanical effects of flavor oscillations and self-interactions. The generalized Boltzmann equations without neutrino self-interactions, eq. (3.34), are very simple and strongly resemble the classical equations of transport theory.

• Conceptually, we have reduced the time evolution of neutrino field operators over Fock space, eq. (2.9), to the time evolution of real numbers by ensemble-averaging, eq. (2.16).

• We have connected three approaches: We started with quantum field operators, eq. (2.9), made the connection the equations for the quasi-classical phase-space densities and the associated specific intensities in section 3.3.1 and finally we showed that our approach is also consistent with the one-particle wavefunction quantum mechanics in subsection 4.3.1.
Part II

Curved Spacetime
6

Neutrino Phenomenology

In general astrophysical contexts such as the early universe and in the vicinity of supernova cores, one needs to consider the effects of gravitational fields on the physics under consideration. Here we seek to understand the kinetics of oscillating neutrinos interacting with a background medium in the presence of a strong, non-negligible gravitational field.

Two changes to the eq. (3.27) for flat spacetime need be implemented. Firstly, we must analyze a gravitational contribution to the mixing Hamiltonian (Cardall & Fuller 1997), which has observational consequences only if other effects such as spin precession in the presence of a magnetic field take place (Piriz, Roy & Wudka 1996, Brüggen 1998). This gravitational contribution stems from the spin connection, which we need to consider when treating spinor fields in a curved spacetime (Brill & Wheeler 1957, Weinberg 1972).

In addition to the modification of the mixing Hamiltonian we must write the left-hand-side of the Boltzmann equations, eq. (3.27), in a manifestly covariant manner so as to account for curvature. To this end we follow (Mihalas 1999 & references therein, and Stewart 1971). This will be done in section 7.1.

6.1 Neutrino oscillations (i)

We discuss how mixed neutrinos can be analyzed in a covariant manner following (Cardall & Fuller 1997). We set $G = \hbar = c = 1$ throughout this section. In a standard treatment, the neutrino flavor state is written (Kayser 1981)

$$|\nu_\alpha(x, t)\rangle = \sum_j U_{\alpha j} e^{-i(E_t - P_j x)m_j} |m_j\rangle,$$

(6.1)

where flavor (mass) indices are in Greek (Latin letters), and the transformation coefficients are given by
The mass eigenstates are taken to be energy eigenstates with common energy $E$. The three-momenta of the mass eigenstates are then

$$P_j = \sqrt{E^2 - m_j^2} \approx E - \frac{m_j^2}{2E},$$

(6.3)

where $m_j$ is the rest mass corresponding to the mass eigenstate $|m_j\rangle$. To observe oscillation it is crucial that one assumes null trajectory ($x = t$) of the mixed neutrino yielding the expression

$$|\nu_\alpha(x)\rangle = \sum_j U_{\alpha j} \exp\left(-i\frac{m_j}{2E} E x\right) |m_j\rangle.$$  

(6.4)

If the mass eigenstates could be measured at different positions (or times), the interference pattern would be destroyed. How can eq. (6.4) be generalized to curved spacetimes and be cast in a manifestly covariant form?

The transformation coefficients from eq. (6.2) (or equivalently the vacuum mixing angle $\theta$ between the two flavor species of interest), the phase and the mass eigenstates are frame independent quantities. The trajectory of a free neutrino is null geodesic, which leads to

$$|\nu_\alpha(\lambda)\rangle = \sum_j U_{\alpha j} \exp\left(i \int_{\lambda_0}^\lambda P^\mu p_\mu d\lambda\right) |m_j\rangle.$$  

(6.5)

In this expression, $P^\mu$ is the four-momentum operator that generates spacetime translation of the mass eigenstates. The quantity $p_\mu$ is the (null) tangent vector to the neutrino’s world line,

$$x^\mu(\lambda) = \begin{pmatrix} x_0(\lambda) \\ x_1(\lambda) \\ x_2(\lambda) \\ x_3(\lambda) \end{pmatrix},$$

(6.6)

and $\lambda$ is an affine parameter of the world line. In the following, we prove that eq. (6.5) is indeed the proper expression for mixed neutrinos in a curved spacetime.

Therefore, we consider motion along the $x_1$ axis. Then the four-momentum operator reads

$$P^\mu = (E, P, 0, 0),$$

(6.7)

and with the Lorentzian signature metric

$$\eta_{\mu\nu} = \text{diag} [-1, 1, 1, 1],$$

(6.8)

we have for the argument of the exponential in eq. (6.5)


\[-i \int_{\lambda_0}^{\lambda} \left( E \frac{dt}{d\lambda} - P_1 \frac{dx}{d\lambda} \right) d\lambda . \quad (6.9)\]

In this expression the \(x_1\)-component of the four-momentum operator is

\[P_1 = \eta_{\mu\nu} P^\mu \equiv E - \frac{M^2}{2E} . \quad (6.10)\]

\(M\) is the mass operator which we replace by its eigenvalue and write for the corresponding phase \(\omega_j\):

\[
\omega_j = -i \int_{\lambda_0}^{\lambda} \left( E \frac{dt}{d\lambda} d\lambda - \left( E - \frac{m_j^2}{2E} \right) \frac{dx}{d\lambda} d\lambda \right.
\]

\[= -i \int_{\lambda_0}^{\lambda} \frac{m_j^2}{2E} dx d\lambda
\]

\[= -i \frac{m_j^2}{2E} (x - x_0) . \quad (6.11)\]

Here we have used the null trajectory condition

\[
\frac{dt/d\lambda}{dx/d\lambda} = \frac{p^\prime_{\text{null}}}{p^\prime_{\text{null}}} = 1 . \quad (6.12)
\]

The important result is that eq. (6.5) is the covariant generalization of eq. (6.4) to spaces with curvature.

### 6.1.1 Matter-induced effective mass for the \(\nu_e\)

We consider Dirac neutrinos and the equation of motion is the Dirac equation. We can include matter effects into this equation. For this purpose the Dirac equation in the flavor basis can be cast in the form,

\[
\left[ i \gamma^\mu \left( \partial_\mu + A_\mu P_L \right) + M \right] \psi = 0 . \quad (6.13)
\]

Here \(\psi\) is a column spinor with four components for each flavor. \(\gamma^\mu\) are defined by eq. (6.5) and our choice shall be the chiral basis (Peskin & Schroeder 1995):

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad (6.14)
\]

where \(\sigma^j\) are the Pauli-matrices. \(P_L\) is the chirality projection on the left-handed components.
\[ P_L = \frac{1 - \gamma^5}{2}, \]  

(6.15)

where

\[ \gamma^5 = -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]  

(6.16)

Here one makes the assumption that only the left-handed components of the flavor spinor $\psi$ interact, whereas the right-handed components are taken to be sterile. The mass matrix in flavor basis $M$ satisfies,

\[ M^2 = U \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} U^\dagger, \]  

(6.17)

where $U$ has been specified in eq. (6.2) and involves the vacuum mixing angle $\theta$.  

$A^\mu$ is the flavor-basis effective potential matrix for an interaction with the electron background:

\[ A^\mu = -\sqrt{2} G_F \begin{pmatrix} N^{\mu} e \\ 0 \end{pmatrix}. \]  

(6.18)

In this expression $N^{\mu}$ is the number current of the electron fluid

\[ N^{\mu} = n_e u^\mu, \]  

(6.19)

where $n_e$ is the electron density in the fluid rest frame, and $u^\mu$ is the fluid’s four-velocity. This is the covariant extension of matrix given in eq. (3.30).  

By iteration of the Dirac equation eq. (6.13) in Fourier space one has for the mass shell relation the expression,

\[ (P^\mu + A^\mu P_L) (P_\mu + A_\mu P_L) = -M. \]  

(6.20)

Keeping terms first order in $G_F$ and employing eq. (6.7) we find

\[ P_1 \approx E - \frac{1}{2E} \left( M^2 - V \right), \]  

(6.21)

with

\[ V = -2 \sqrt{2} G_F \begin{pmatrix} E n_e P_L \\ 0 \end{pmatrix}. \]  

(6.22)

The three-momentum operator eq. (6.21) now includes effective mass contributions from background matter, and can be used in eq. (6.5).

The most important physical consequence of this effective mass is well known (MSW effect) and we have performed explicit calculations in this thesis in section 4.3.
6.1 Neutrino oscillations (i)

### 6.1.2 Spinor fields

Spinor fields and their transformation properties are investigated with help of the vielbein formalism which separates local Lorentz transformations and general coordinate transformations. This is convenient because for spinor fields one can then apply, locally, the spinor representation of the Lorentz group for which the well known $\gamma$’s of the Dirac algebra in flat spacetime eq. (2.3) can be employed. In Appendix B, we sketch how to construct locally inertial coordinate systems and “hide” the curvature in the components of the vielbeins so as to apply Lorentz transformations on spinors in curved spacetime (Carroll 2004, Weinberg 1972, and Brill & Wheeler 1957).

The Dirac equation in curved spacetime involves gravitational effects on the spin through the spin connection $\Gamma_\mu$:

$$
(i\gamma^a e_a^\mu \left[ \partial_\mu + \Gamma_\mu \right] + M) \psi = 0 ,
$$  

where Greek indices refer to the general coordinates, while the Latin indices $a, b, c, d$ refer to the locally inertial noncoordinate basis as described in Appendix B. Using the explicit expression for $\Gamma_\mu$ (Weinberg 1972):

$$
\Gamma_\mu = \frac{i}{8} \left[ \gamma^b, \gamma^c \right] e_b^\nu e^e_\nu e^\nu_\mu ,
$$  

where the semi-colon denotes the covariant derivative, one can incorporate the gravitational effects on spin into the momentum operator $P_\mu$ of eq. (6.5). The Dirac equation (6.23) features the Dirac matrix product

$$
\gamma^a \left[ \gamma^b, \gamma^c \right] = 2\eta^{ab} \gamma^c - 2\eta^{ac} \gamma^b - 2i\epsilon^{dabc} \gamma_5 \gamma_d ,
$$  

where the right-hand-side can be calculated by using the identities for Dirac matrices as for instance given in (Peskin & Schroeder 1995). Here $\epsilon^{dabc}$ is the totally antisymmetric tensor with $\epsilon^{0123} = 1$. After some manipulation we can group the terms arising from the spin connection as follows:

$$
i\gamma^a e_a^\mu \Gamma_\mu = i\gamma^a e_a^\mu \left( \delta_\mu - \frac{\gamma^a}{2} \right) ,
$$  

with

$$
\delta_\mu = \frac{i}{2} \epsilon^{dabc} (\gamma^a)^{-1} \gamma_d e^b_\mu \gamma^e e^e_\mu .
$$  

Note that in the reference (Cardall & Fuller 1997), a different convention of the $\gamma$’s is employed and the corresponding expression for $\delta_\mu$ is different (I could not see how the authors derived their expression). It is now apparent that, by adding a term proportional to the identity matrix (this has no observable consequence), we can choose either chirality to be prone to the gravitational effects and write
\[ i \gamma^0 e_a^\alpha \Gamma_\mu = i \gamma^\alpha e_a^\mu \left( \mathbf{6}_\mu \mathcal{P}_L \right), \]  
where

\[ \mathcal{P}_L = \frac{1 - \gamma^5}{2} \]  
is the projector on the left-handed states.

An important point is that the gravitational contribution $6^\mu$ is proportional to the identity matrix in flavor space, and diagonal in spin space. It cannot induce spin flips on its own. Therefore it will not have any observable effects unless there are other off-diagonal terms in spin space, e.g., from the interaction of a neutrino magnetic moment with a magnetic field (Piriz, Roy & Wudka 1996). In reality, however, the spin-gravity coupling needs a large anomalous magnetic moment of the neutrino to yield observationally relevant effects.

At present the Standard Model-like calculation for massive Dirac neutrinos predicts (Shrock & Fujikawa 1977, Mohapatra & Pal 2003)

\[ \mu \approx \frac{3 e G_F}{8 \sqrt{2} \pi^2} m_\nu, \]  
where $m_\nu$ is the mass of the neutrino. In numbers, we have

\[ \mu \approx 3.1 \times 10^{-19} \mu_B \left( \frac{m_\nu}{1 \text{eV}} \right), \]  
$\mu_B$ being the Bohr magneton, $e/(2m_e)$. This is rather puny. Even though the magnetic fields near protoneutron stars or supernova cores can be as large as $10^{12}$ Gauss, the spin-flip length scale will be nonlocal, i. e. spread out over large spatial distances when compared with the scales of other processes as collision with the ambient matter. For theoretical reasons it is important that the gravitational contribution is added to the mixing Hamiltonian eq. (3.3) even though with the present range of magnetic fields and anomalous magnetic moments this will have no physical effect.

To find effects (Lambiase 2004, Lambiase, Papini, Punzi & Scarpetta 2005), the neutrino magnetic moment must be estimated to be magnitudes larger than that following from eq. (6.31) with neutrino masses in the eV range.
6.2 Neutrino oscillations (ii)

We can now treat the effective mass contribution of eq. (6.18) and the spin connection of eq. (6.27) in the same fashion. Then the flavor state in curved space time of eq. (6.5) can be written as

\[ |\nu_\alpha(\lambda)\rangle = \sum_j U_{\alpha j} \exp\left(i \int_{\lambda_0}^\lambda p_\mu p_\mu d\lambda \right) |m_j\rangle . \] (6.32)

In this expression the argument of the exponential now encompasses spin and effective mass effects and we have

\[ P^\mu p_\mu = -\left( \frac{M^2}{2} + p^\mu \left( A_\mu + 6_\mu \right) P_L \right) . \] (6.33)

We are in position to explicitly calculate the vielbeins eq. (6.24) for interesting geometries and follow the temporal evolution of the flavor composition of the mixed neutrino spinor in curved spacetime through solving the following modified version of eq. (6.23):

\[ (i\gamma^a e^\mu_a \left[ \partial_\mu + \left( A_\mu + 6_\mu \right) P_L \right] + M) \psi = 0 . \] (6.34)

More equations of this type are derived in the references Cardall & Fuller 1997, and Piriz, Roy & Wudka 1996.

6.3 Conclusions

- In this chapter, we have discussed how to generalize the phase of a mixed neutrino to curved spaces. The result is eq. (6.33).

- We have shown the dynamical equation (6.34) for a neutrino spinor in curved spacetime. We argued that the non-vanishing contribution due to spin-gravity coupling can be written in terms of the vielbein fields, eq. (6.27). This contribution is diagonal in spin and flavor space.

- The terms in the operator of eq. (6.34), notably the gravitational contribution, \( 6_\mu \), will be used to write the mixing Hamiltonian, eq. (7.16), so as to account for spin-gravity coupling.
General-relativistic Boltzmann Formalism

The purpose of section 7.1 is to formulate the classical Boltzmann equation
\[ \frac{\partial f(r, p, t)}{\partial t} + \dot{r} \cdot \frac{\partial f(r, p, t)}{\partial r} + \dot{p} \cdot \frac{\partial f(r, p, t)}{\partial p} = \left[ \frac{\delta f}{\delta t} \right]_{\text{col}}, \]
(7.1)
in a manifestly covariant manner and generalize the Liouville operator of the left-hand-side to curved spaces. In section 7.2, we decompose the phase-space densities into their flavor degrees of freedom and into their spin degrees of freedom. We show the associated mixing Hamiltonian, eq. (7.16).

The highlight of this chapter are the new kinetic equations for oscillating neutrinos in curved spaces, eq. (7.24).

### 7.1 Classical Boltzmann equation

The phase-space for a system of relativistic particles is simply the tangent bundle over the space-time manifold, and the corresponding distribution function or phase-space density can naturally be described as a scalar field on this bundle (Stewart 1971). At every point \( x \) of the semi-riemannian manifold of our spacetime, the tangent space, \( T_x \), becomes the relativistic generalization of momentum space. Physically realizable four-momenta are timelike or null. Free particles have a geodesic trajectory and between collisions, the four-momenta of the particles,
\[ \frac{dx^\mu}{d\lambda} = p^\mu, \]
(7.2)
where \( \lambda \) is an affine parameter, satisfy the geodesic equation:
\[ \frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho} p^\nu p^\rho = 0. \]
(7.3)
In this expression $\Gamma^\mu{}_{\nu\rho}$ is the symmetric, metric-compatible and unique Christoffel connection that is linear in the derivatives of the metric:

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left( \frac{\partial g_{\alpha\nu}}{\partial x^\rho} + \frac{\partial g_{\rho\alpha}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\alpha} \right),$$

(7.4)

where $g_{\mu\nu}$ is the twice-covariant metric tensor field which multi-linearly makes two tangent vectors to real numbers.

The manifestly covariant relativistic Boltzmann equation for the invariant phase-space density $f (x^\mu, p^\mu)$ is then given by:

$$p^\mu \frac{\partial f}{\partial x^\mu} - \left( \Gamma^\mu{}_{\nu\rho} p^\nu p^\rho \right) \frac{\partial f}{\partial p^\mu} = \left[ \frac{\delta f}{\delta \lambda} \right]_{\text{coll}}.$$

(7.5)

The term on the right-hand-side is the collision term (Mihalas 1999).

There is a subtlety involved in this way of writing the Boltzmann equation. We have tacitly assumed that the invariant phase-space density $f (x^\mu, p^\mu)$ is defined for all possible four-momenta. However only the four-momenta that lie on the null-cone make sense for the description of almost massless neutrinos (as in chapter 6 we assume null geodesics for the neutrinos). Therefore, we must ensure that $p^\rho$ remains on the null cone as the neutrino propagates. This can be achieved by employing the vielbein formalism (Appendix B) because in the local vielbein frame we can fix the four-momentum to be null once and for all, since in the vielbein frame we have the Minkowski metric. All the spatial-dependence involving the curvature is “out-sourced” to the complicated structure of the vielbeins. We regard the phase-space density as a function of $(x^\mu, p^\mu)$, where the components of the spatial dependence $x^\mu$ are specified with respect to the global coordinate system. The momenta $p^\mu$, however, are given in the local vielbein frame. Then one has, in a kind of mixed representation similar to eq. (7.5),

$$p^\rho e^\mu_{\rho} \frac{\partial f}{\partial x^\mu} + \frac{dp^b}{d\lambda} \frac{\partial f}{\partial p^b} = \left[ \frac{\delta f}{\delta \lambda} \right]_{\text{coll}}.$$

(7.6)

We need to calculate $\frac{dp^b}{d\lambda}$. Recalling that the neutrino trajectories are geodesics fulfilling eq. (7.3) in the global coordinate system, there is

$$\frac{dp^b}{d\lambda} = e^b_{\mu} \frac{dp^\mu}{d\lambda} = e^b_{\mu} \left( \frac{d\phi^\mu}{d\lambda} - \frac{d\omega_{\nu}^b}{d\lambda} p^\nu \right) = e^b_{\mu} \left( -\Gamma^\mu{}_{\nu\rho} p^\nu p^\rho + \Gamma^\mu{}_{\nu\rho} e^\nu_{\mu} b^\rho \right) = e^b_{\mu} \left( e^\nu_{\mu\rho} (\phi^\rho_{\mu} + \Gamma^\rho{}_{\nu\sigma} \phi^\sigma_{\nu}) p^\mu p^\nu \right) = e^b_{\mu} e^\nu_{\mu\rho} e^\sigma_{\nu\rho} b^\sigma p^\rho = -g_{\mu\nu} p^\mu p^\nu.$$

(7.7)
In this formula, the comma denotes a partial derivative and a semi-colon stands for the covariant derivative. We can define the so-called Ricci rotation coefficient

\[ R^b_{ac} = \varepsilon^b_{\mu} e^c_{\rho} e^\mu_{cp} , \] (7.8)

and the Pfaffian derivative (this is the correct partial derivative in the vielbein frame):

\[ \partial_a = e^a_{\mu} \partial_{\mu} . \] (7.9)

In terms of these quantities the Boltzmann equation eq. (7.5) becomes

\[ p^a \left[ \partial_a - R^b_{ac} p^c \partial_{p^b} \right] f = \left[ \delta f \right]_{\text{scoll}} . \] (7.10)

This is the general form of the left-hand-side of the Boltzmann equation (sometimes referred to as the Liouville operator) in curved spacetime. In non-negligible gravitational fields this Liouville operator eq. (7.10) replaces the flat space left-hand-side of the Boltzmann equation as for example shown in eq. (7.1).

### 7.2 4 × 4 Mixing Hamiltonian

We increase the dimensionality of the mixing Hamiltonian employed in flat spacetime, eq. (3.3), to further decompose the phase-space densities and the associated intensities not only into their flavor degrees of freedom, but also into their spin degrees of freedom. The matrix elements of the Wigner phase-space density are then written as a 4 × 4 matrix:

\[ \mathcal{F} = \langle n_i | \rho | n_j \rangle = \begin{pmatrix} f_{\uparrow \uparrow} & f_{\downarrow \uparrow}^\dagger & f_{\uparrow \downarrow}^\dagger & f_{\downarrow \downarrow}^\dagger \\ f_{\uparrow \downarrow} & f_{\downarrow \downarrow} & f_{\uparrow \uparrow}^\dagger & f_{\downarrow \uparrow}^\dagger \\ f_{\downarrow \uparrow}^\dagger & f_{\uparrow \uparrow} & f_{\downarrow \downarrow}^\dagger & f_{\uparrow \downarrow}^\dagger \\ f_{\downarrow \downarrow}^\dagger & f_{\uparrow \downarrow} & f_{\downarrow \uparrow} & f_{\uparrow \uparrow} \end{pmatrix} . \] (7.11)

In this matrix \( i \) and \( j \) run over flavor degrees of freedom and the spin states. \( \uparrow \) denotes left-handed states and \( \downarrow \) means right-handed states. In addition to the previously employed macroscopic overlap for the differently flavored neutrinos in the ensemble, eq. (2.20), we must also consider mixed spin states denoted by the superscripts \( \uparrow \downarrow \) and \( \downarrow \uparrow \). Again this is only relevant if the magnetic moment of the neutrino is comparatively large and the magnetic fields are very strong (Piriz, Roy & Wudka 1996). The interaction of the neutrino spinor with the electromagnetic field stems from terms of the form

\[ \mu S^{ab} F_{ab} \psi , \] (7.12)
where $\psi$ is the neutrino spinor, as for example defined in eq. (6.23), and $\mu$ the magnetic moment of the neutrino, eq. (6.31). For $S^{ab}$ we have the expression eq. (B.9) from the Appendix B:

$$ S^{ab} = \frac{i}{4} [\gamma^a, \gamma^b] . $$

This electromagnetic interaction term can be added to the Dirac equation (6.34), yielding

$$ (i \gamma^a e_a^\mu \left[ \partial^\mu + (A^\mu + \theta^\mu) \mathcal{P}_L \right] + \mu S^{ab} F_{ab} + M) \psi = 0 . $$

In the mixing Hamiltonian, the surviving electromagnetic terms are $\mu B$ with $B$ being the component of the magnetic field perpendicular to the neutrino trajectory. Similar to as performed in the papers by Piriz, Roy & Wudka 1996, and Brüggen 1998 (these references decomposed the neutrino wavefunction into their spin and flavor states), we generalize the mixing Hamiltonian of eq. (3.3) as follows:

$$ W(x^\mu, p^\mu) = W_{vac} + W_{mat}(x^\mu) + W_{\nu\nu}(p^\mu, x^\mu) - \tilde{W}_{\nu\nu}(p^\mu, x^\mu) + W_{grav}(x^\mu) . $$

This Hamiltonian includes the weak, electromagnetic and the gravitational interactions. Adding up all the contributions, except for the self-interactions, the most general form of the mixing Hamiltonian is:

$$ W = \begin{pmatrix} V_{cc} + V_{nc} + P_\mu 6\mu^\mu / E - \frac{\Delta m^2}{4E} \cos 2\theta & \frac{\Delta m^2}{4E} \sin 2\theta & \mu_{\nu} B & \mu_{\mu} B \\ \frac{\Delta m^2}{4E} \sin 2\theta & V_{cc} + P_\mu 6\mu^\mu / E + \frac{\Delta m^2}{4E} \cos 2\theta & \mu_{\mu} B & \mu_{\nu} B \\ \mu_{\nu} B & \mu_{\mu} B & \mu_{\nu} B & \mu_{\mu} B \\ \mu_{\mu} B & \mu_{\nu} B & \mu_{\mu} B & \mu_{\nu} B \end{pmatrix} . $$

$V_{cc}$ is the charged-current contribution leading to the effective mass of the $\nu_e$ in matter multiply discussed in this work (the covariant formulation is done in subsection 6.1.1 and the standard way is treated via the matter mixing Hamiltonian in eq. (3.8):

$$ V_{cc} = 2 \sqrt{2} G_f n_e . $$

$V_{nc}$ is the neutral-current contribution (Mohapatra & Pal 2003):

$$ V_{nc} = -2 \sqrt{2} G_f n_n , $$

where $n_n$ is the neutron number density and $n_e$ is the electron density. We emphasize that the neutral-current contribution is the same for all flavors of neutrinos (which is why it was dropped in eq. (5.8), whereas the charged-current contribution affects the $\nu_e$ only (see Fig. 3.1)).
7.2.1 Generalized resonances

By virtue of the mixing Hamiltonian eq. (7.16) we realize that resonances occur whenever the diagonal elements cross. We read off the resonance conditions for spin-flips and MSW conversions.

For $f^1_{\nu_e} \leftrightarrow f^1_{\nu_\mu}$:
\[ \mathcal{V}_{cc} - \frac{\Delta m^2}{2\epsilon} \cos \theta = 0 \text{ ,} \tag{7.19} \]

for $f^1_{\nu_e} \leftrightarrow f^1_{\nu_\mu}$:
\[ \mathcal{V}_{cc} + \mathcal{V}_{nc} + P_\mu \frac{6\mu}{\epsilon} = 0 \text{ ,} \tag{7.20} \]

for $f^1_{\nu_e} \leftrightarrow f^1_{\nu_\mu}$:
\[ \mathcal{V}_{cc} + \mathcal{V}_{nc} + P_\mu \frac{6\mu}{\epsilon} - \frac{\Delta m^2}{2\epsilon} \cos 2\theta = 0 \text{ ,} \tag{7.21} \]

for $f^1_{\nu_\mu} \leftrightarrow f^1_{\nu_\mu}$:
\[ \mathcal{V}_{cc} + \mathcal{V}_{nc} + P_\mu \frac{6\mu}{\epsilon} + \frac{\Delta m^2}{2\epsilon} \cos 2\theta = 0 \text{ ,} \tag{7.22} \]

and lastly for the transition $f^1_{\nu_\mu} \leftrightarrow f^1_{\nu_\mu}$:
\[ \mathcal{V}_{nc} + P_\mu \frac{6\mu}{\epsilon} = 0 \text{ .} \tag{7.23} \]

In the active-active MSW transition, eq. (7.19), the matter potential is due to the charged-current interactions only and proportional to the electron number density (see eq. (7.17)). To obtain accurate spectra in general conditions all these resonances must be taken into account. The gravitational contribution generates a shift of the resonance positions. The adiabaticity can also be altered (Brüggen 1998).
7.3 Kinetic equations

With the aid of eq. (7.10), the associated generalization to curved spacetime including the spin-states of the Heisenberg-Boltzmann equation, eq. (3.1), reads:

$$p^a \left[ \partial_a - g^{ab} \frac{\partial}{\partial p^b} \right] \bar{\sigma} = -i \{ \mathcal{W}, \bar{\sigma} \} + C.$$  \hspace{1cm} (7.24)

This equation is the main result of Part II of this thesis. Note that $\bar{\sigma}$ and $\mathcal{W}$ are now $4 \times 4$ matrices. An analogous equation can be written for antineutrinos. Particles and antiparticles couple identically to the gravitational field, so no changes need to be made for the gravitational modifications; neither for the Liouville operator in curved space, eq. (7.10), nor for the gravitational contribution $G$ in eq. (6.27) to the mixing Hamiltonian $\mathcal{W}$, eq. (7.16).

If we were to componentize them like done above in eq. (3.27), we would have to deal with sixteen coupled equations for the neutrinos without the self-interactions. The issue of making the equations real-valued can naturally be solved by defining the real and imaginary parts of the off-diagonal elements of $\bar{\sigma}$ similarly to eq. (3.22). Including the self-interactions, however, one would have to solve 32 nonlinearly coupled equations. Clearly, for real applications one would have to break up the neutrino trajectory to care for the most dominant effect one at a time.

Eq. (7.24) can also be used if one wishes to generalize eq. (3.27) to curved space without taking into account the coupling of the still speculative anomalous magnetic moment to a very large magnetic field. Then $\bar{\sigma}$ and $\mathcal{W}$ reduce to the familiar $2 \times 2$ matrices given in eqs. (2.16) and (3.3). In this case the only effect of the curvature is the standard generalization of the Liouville operator to curved space as performed in eq. (7.10).

7.4 Conclusions

- The basic fundament of eq. (7.24) is the classical, general-relativistic framework with which we generalized the Liouville operator to curved spaces in eq. (7.10).

- The quantum physics is incorporated by means of the general mixing Hamiltonian, eq. (7.16). We have included spin degrees of freedom.

- The “master equation” of Part II, eq. (7.24) describes, in principle, the evolution of the quasi-classical density in phase-space for massive neutrinos with collisions including all known effects – from the quantum-mechanical viewpoint as well as from the gravitational perspective.
8

Summary of Part II

We briefly highlight the main features of our formalism in curved spacetime and discuss physical applications for which this formalism might be helpful.

8.1 Spin-gravity coupling

Invariance under general coordinate transformations for spinors naturally leads to the vielbein fields introduced in Appendix B. In short, these considerations lead to an additional contribution for the differential operator of the Dirac equation (6.23) in terms of the spin connection. However, terms arising from the spin connection are proportional to the identity matrix in flavor space and do not lead to additional effects concerning oscillation between flavor species. Moreover, the spin connection terms are also diagonal in spin space and act on one spin orientation of the neutrino only.

The most general mixing Hamiltonian, eq. (7.16), has potentially other off-diagonal contribution stemming from spin-flips from coupling of the neutrino spinor the an external magnetic field. It is for this case only that the gravitational resonances in subsection 7.2.1 are important. It has been analyzed elsewhere that, assuming large anomalous magnetic moments, these effects can be potentially large (Piriz, Roy & Wudka 1996).

8.2 Possible applications

Combining the oscillation physics in curved spacetime from section 6 with the kinetic generalizations from section 7.1 yields the master equation of Part II, eqs. (7.24). These equations satisfy all quantum-mechanical and gravitational considerations for the kinetics of oscillating neutrinos in curved spacetime.
Potential applications include active galactic nuclei (Piriz, Roy & Wudka 1996) and the neutrino fluxes in supernova cores. In practice, however, one would break up the master equations, eq. (7.24), and solve them isolating the dominant effect. For supernova neutrinos, for instance, we do not need to care about collisions at nuclear densities because observable flavor oscillations are matter-suppressed. There, we must focus on the collision physics with the ambient matter. In the streaming region above the neutrino sphere, self-interactions become important and they must be focussed on. Spin-flips only deserve consideration for very large magnetic fields and so forth.

To close this summary of Part II, we note that it has been proposed that the nonlinear evolution in a neutrino background leads to a speed-up of flavor equilibration processes that could potentially overshadow the gravitational and all other effects (Saywer 2004 and 2005). We leave this question open for future work.
Part III

Conclusions
9

Conclusions

In this thesis, we have derived a generalized set of Boltzmann equations for real-valued phase-space densities of oscillating neutrinos interacting with a background medium. The off-diagonal functions of the Wigner phase-space density representing macroscopic overlap are explicitly included and serve to couple the flavor states to reflect neutrino oscillation physics. Conceptually, we have reduced the time evolution of creation and annihilation operators to that of real valued phase-space densities without losing quantum-physical accuracy. Important quantum effects such as matter-enhanced resonant flavor conversion and quantum decoherence through matter coupling are correctly incorporated. The generalized Boltzmann equations are simple and very similar to the equations of classical transport theory. Neutrino oscillations are incorporated by new sink and source terms that indirectly couple the expanded set of equations. Using this formalism, codes that now solve the standard Boltzmann equations for the classical neutrino phase-space density ($f_v$), or which address its angular and/or energy moments, can straightforwardly be reconfigured by the simple addition of source terms and similar transport equations for overlap densities that have the same units as $f_v$, to incorporate neutrino oscillations in a quantum-physically consistent fashion. In curved spacetime, oscillation phenomenology is altered. These effects play no observable role except for the usual redshift and the contribution to the mixing Hamiltonian from the spin connection can be isolated in the diagonal of the left-handed spin sector of the effective mixing Hamiltonian. If, however, spin flips occur through coupling of an anomalous magnetic moment of the neutrino with a large magnetic field, one must take into account gravitational resonances similar to the MSW effect in flat spacetime.
Part IV

Appendices
Cross section $\nu_e + n \rightarrow e^- + p$

A convenient reference neutrino cross section is $\sigma_0$, given by

$$
\sigma_0 = \frac{4G_F^2(m_e c^2)^2}{\pi \hbar c^2} \approx 1.705 \times 10^{-44} \text{ cm}^2, \tag{A.1}
$$

where $G_F$ is the Fermi weak coupling constant ($\approx 1.436 \times 10^{-49} \text{ ergs cm}^{-3}$). The total $\nu_e - n$ absorption cross section for the reaction $\nu_e + n \rightarrow e^- + p$ is then given by

$$
\sigma_{\nu,n}^a \sim \sigma_0 \left( \frac{1 + 3g_A^2}{4} \right) \left( \frac{\varepsilon_{\nu_e} + \Delta_{np}}{m_e c^2} \right)^2 \left[ 1 - \left( \frac{m_e c^2}{\varepsilon_{\nu_e} + \Delta_{np}} \right)^2 \right]^{1/2}, \tag{A.2}
$$

where $g_A$ is the axial–vector coupling constant ($\sim -1.23$), and $\Delta_{np} = m_n c^2 - m_p c^2 = 1.29332 \text{ MeV}$. In Fig. (A.1) we show the heuristic Feynman graph corresponding to this process.

![Feynman graph](A.1)

**Fig. A.1.** Super–allowed charged–current $\nu_e$ absorption on neutrons.
The essential quantities in the general theory of relativity such as the curvature or the energy-momentum are invariant under general coordinate transformations. This can be viewed as the construction principle of the general theory of relativity (Weinberg 1972). Luckily, these quantities behave like tensors not only under general coordinate transformation but they also behave as tensors under Lorentz transformations. This makes the transition from special relativity in flat spacetime to general relativity in curved spacetimes straightforward. For spinor fields this is not true because the tensor representations of the group \( \text{GL}(4) \) of general linear \( 4 \times 4 \) matrices behave like tensors under the subgroup of Lorentz transformations, but there are no representation of \( \text{GL}(4) \) which behave like spinors under the Lorentz subgroup. To remedy this situation, one erects at every point \( x \) a noncoordinate basis \( \xi^a(x) \) that is locally inertial. Then the metric can be written

\[
g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab} ,
\]

where for a Lorentzian spacetime \( \eta_{ab} \) represents the Minkowski metric, while in a space with positive-definite metric it would represent the Euclidean metric. The Latin indices indicate that the locally inertial coordinates are not related to any coordinate system. The vielbeins are given by

\[
e^a_\mu(x) = \left( \frac{\partial \xi^a(x')}{\partial x^\mu} \right)_{x'=x} .
\]

Coordinate tensors of arbitrary rank can be made into a set of scalars by expressing their components with respect to the locally inertial basis, that is,

\[
T^a_b = e^{\nu}_b e^\mu_c T^\mu_c .
\]
two transformations, local Lorentz transformations and general coordinate transformations, can be executed at the same time resulting in the mixed transformation law (Carroll 2004):

$$T^\mu_a{}_{\nu'b'} = \Lambda^a_{a'} \frac{\partial x^\mu}{\partial x'^a} \Lambda^b_{b'} \frac{\partial x^\nu}{\partial x'^b} T^{\nu'b'}.$$  \hfill (B.4)

In general an arbitrary field in the vielbein frame $\psi_n(x)$ transforms as

$$\psi_n(x) = \sum_m [D(\Lambda(x))]_{nm} \psi_m(x),$$  \hfill (B.5)

where $D(\Lambda)$ is a matrix representation of the Lorentz group tailored to the nature of the field. Heuristically, we have stored away the curvature in the vielbeins so as to now being able, for a spinor field, to apply the spinor representation of the Lorentz group.

We now need a derivative operator, $\mathcal{D}_a$, that is a scalar with respect to the general coordinate transformations and a Lorentz vector with respect to local position-dependent Lorentz transformations with matrix representation $A^a_{a'}(x)$. This ensures, that any action which depends on the various fields of interest, including spinor fields, will then automatically be independent of the choice of locally inertial frames if it is invariant under ordinary constant Lorentz transformations. The derivative operator must transform homogeneously:

$$\mathcal{D}_a \psi(x) \rightarrow A^a_{a'}(x) D(\Lambda(x)) \mathcal{D}_{a'} \psi(x).$$  \hfill (B.6)

It can be shown that the expression following expression is the correct derivative operator:

$$\mathcal{D}_a \psi(x) = e_a^\mu \left( \partial_\mu + \Gamma_\mu \right) \psi(x),$$  \hfill (B.7)

where the spin connection $\Gamma_\mu$ is given by (Weinberg 1972):

$$\Gamma_\mu = \frac{1}{2} S^{bc} e^b_\nu e^c_\nu \partial_\mu,$$  \hfill (B.8)

where the semicolon denotes the covariant derivative, and $S^{bc}$ is the appropriate representation of the Lorentz group. For spinor fields, employing the chiral choice of $\gamma$’s, eq. (6.14), one has

$$S^{bc} = \frac{i}{4} [\gamma^b, \gamma^c],$$  \hfill (B.9)

where square brackets denote the usual matrix commutators.

To summarize: The effects of gravitation on any physical system can be taken into account by writing down the matter action or the field equations that hold in special relativity, and then replacing all derivatives $\partial/\partial x^\mu$ with the derivative operator $\mathcal{D}_a$ of eq. (B.6). For the case of consistently describing spinor fields in curved space one must resort to this line of thought, since only in the vielbein frame can one employ the spinor representations of the Lorentz group coherently.
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