WEIGHTED POINCARÉ INEQUALITY AND THE POISSON EQUATION

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Abstract. We develop Green’s function estimate for manifolds satisfying a weighted Poincaré inequality together with a compatible lower bound on the Ricci curvature. The estimate is then applied to establish existence and sharp estimates of the solution to the Poisson equation on such manifolds. As an application, Liouville property for finite energy holomorphic functions is proven on a class of complete Kähler manifolds. Consequently, such Kähler manifolds must be connected at infinity.

1. Introduction

Recently, in [26], we have studied the existence and estimates of the solution \(u\) to the Poisson equation

\[ \Delta u = -\varphi \]

on a complete Riemannian manifold \((M^n, g)\), where \(\varphi\) is a given smooth function on \(M\). Among other things, we have obtained the following result.

**Theorem 1.1.** Let \((M^n, g)\) be a complete Riemannian manifold with bottom spectrum \(\lambda_1(\Delta) > 0\) and Ricci curvature \(\text{Ric} \geq -(n-1)K\) for some constant \(K\). Let \(\varphi\) be a smooth function such that

\[ |\varphi|(x) \leq c (1 + r(x))^{-k} \]

for some \(k > 1\), where \(r(x)\) is the distance function from \(x\) to a fixed point \(p \in M\). Then the Poisson equation \(\Delta u = -\varphi\) admits a bounded solution \(u\) on \(M\).

If, in addition, the volume of the unit ball \(B(x, 1)\) satisfies \(V(x, 1) \geq v_0 > 0\) for all \(x \in M\), then the solution \(u\) decays and

\[ |u|(x) \leq C (1 + r(x))^{-k+1} . \]

Recall that the bottom spectrum \(\lambda_1(\Delta)\) or the smallest spectrum of the Laplacian can be characterized as the best constant of the Poincaré inequality

\[ \lambda_1(\Delta) \int_M \phi^2 dx \leq \int_M |\nabla \phi|^2 dx . \]

It is known that \(\lambda_1(\Delta) > 0\) implies that \(M\) is non-parabolic, that is, there exists a positive symmetric Green’s function \(G(x, y)\) for the Laplacian. The preceding

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Theorem 1.2. Let \((M^n, g)\) be an \(n\)-dimensional complete manifold with \(\lambda_1(\Delta) > 0\) and \(\text{Ric} \geq -(n-1)K\). Then for any \(p, x \in M\) and \(r > 0\) we have

\[
\int_{B(p,r)} G(x, y) \, dy \leq C \, (1 + r)
\]

for some constant \(C\) depending only on \(n, K\) and \(\lambda_1(\Delta)\).

In the current paper, we continue to address similar issues for complete manifolds satisfying more generally a so-called weighted Poincaré inequality. Recall that Riemannian manifold \((M, g)\) satisfies a weighted Poincaré inequality if there exists a function \(\rho(x) > 0\) such that

\[
\int_M \rho \phi^2 \leq \int_M |\nabla \phi|^2
\]

for any compactly supported function \(\phi \in C_0^\infty(M)\).

Other than being a natural generalization of \(\lambda_1(\Delta) > 0\), there are various motivations for considering weighted Poincaré inequality. First, it is well-known (see [19]) that \(M\) being nonparabolic is equivalent to the validity of the weighted Poincaré inequality for some \(\rho\). Secondly, according to a result of Cheng [5], when the Ricci curvature of manifold \(M\) is asymptotically nonnegative at infinity, its bottom spectrum \(\lambda_1(\Delta) = 0\), and one is forced to work with weighted Poincaré inequalities. Thirdly, by considering weighted Poincaré inequality, it enables one to consider manifolds with Ricci curvature bounded below by a function. Typically, in geometric analysis, one assumes the curvature to be bounded by a constant so that various comparison theorems become available. As demonstrated in [19, 20], weighted Poincaré inequality allows one to go beyond this realm. Indeed, they were able to prove some structure theorems for manifolds with its Ricci curvature satisfying the inequality

\[
\text{Ric}(x) \geq -C \rho(x)
\]

for a suitable constant \(C\) for all \(x \in M\). Finally, weighted Poincaré inequality occurs naturally under various geometric settings. Indeed, a result of Minerbe [24] (see [12] for further development) implies that complete manifold \(M\) with nonnegative Ricci curvature satisfies weighted Poincaré inequality with \(\rho(x) = cr^{-2}(x)\), where \(r(x)\) is the distance from \(x\) to a fixed point \(p\) in \(M\), provided that the following reverse volume comparison holds for some constant \(C\) and \(\nu > 2\)

\[
\frac{\text{Vol}(B(p,t))}{\text{Vol}(B(p,s))} \geq C \left( \frac{t}{s} \right)^{\nu}
\]

for all \(0 < s < t < \infty\). Also, for minimal submanifold \(M^n\) of the Euclidean space \(\mathbb{R}^N\), weighted Poincaré inequality is valid on \(M\) with \(\rho(x) = \frac{(n-2)^2}{4} \hat{r}^{-2}(x)\), where \(\hat{r}(x)\) denotes the extrinsic distance function from \(x\) to a fixed point (see [3, 19]). On the other hand, for a stable minimal hypersurface in a manifold with nonnegative Ricci curvature, by the second variation formula, weighted Poincaré inequality holds for \(\rho(x)\) being the length square of the second fundamental form.

We also remark that the weighted Poincaré inequalities in various forms have appeared in many important issues of analysis and mathematical physics. Agmon
has used it in his study of eigenfunctions for the Schrödinger operators. In the interesting papers [8] and [9], Fefferman and Phong have considered the more general weighted Sobolev type inequalities for pseudodifferential operators. There are many interesting results concerning sharp form of the weight $\rho$. The classical Hardy inequality for the Euclidean space $\mathbb{R}^n$ implies that $\rho(x) = \frac{(n-2)^2}{4} r^{-2}(x)$ and it is optimal. In [2], it is shown that a sharp $\rho$ on the hyperbolic space $H^n$ is given by $\rho(x) = \frac{(n-1)^2}{4} + \frac{(n-2)^2}{4} r^{-2}(x)$. We also refer to [7] for a more systematic approach to finding an optimal $\rho$ for more general second order elliptic operators.

Throughout the paper, we will assume the weight $\rho$ in addition satisfies both (1.2) and (1.3), that is, the $\rho$-metric defined by

\begin{equation}
(1.2) \quad ds^2_\rho = \rho ds^2
\end{equation}

is complete; and for some constants $A > 0$ and $\delta > 0$,

\begin{equation}
(1.3) \quad \sup_{B(x, \sqrt{\rho(x)})} \rho \leq A \inf_{B(x, \sqrt{\rho(x)})} \rho
\end{equation}

for all $x \in M$.

We point out that these two conditions obviously hold true for a weight of the form $\rho(x) = cr^\alpha(x)$ with $\alpha \geq -2$. The metric $ds^2_\rho$ was first used by Agmon [1] to study decay estimates for eigenfunctions. It was later employed to establish $L^2$ decay estimates for the Green’s function in [10].

Our first result is an integral estimate for the minimal positive Green’s function $G(x, y)$ on $M$. In the following, we denote geodesic balls with respect to the background metric $ds^2$ by $B(x, r)$, and to the metric $ds^2_\rho$ by $B_\rho(x, r)$.

**Theorem 1.3.** Let $(M^n, g)$ be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight $\rho$ having properties (1.2) and (1.3). Assume that $\text{Ric} \geq -K \rho$ on $M$ for some $K \geq 0$. Then

$$\int_{B_\rho(p, r)} \rho(y) G(x, y) \, dy \leq C(r + 1)$$

for all $p$ and $x$ in $M$, and all $r > 0$, where $C$ depends only on $n$, $K$, $\delta$ and $A$.

As an application of Theorem 1.3 we obtain the following solvability result for the Poisson equation.

**Theorem 1.4.** Let $(M^n, g)$ be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight $\rho$ having properties (1.2) and (1.3). Assume that $\text{Ric} \geq -K \rho$ on $M$ for some $K \geq 0$. Then for smooth function $\varphi$ such that

$$|\varphi|(x) \leq c (1 + r_\rho(x))^{-k}$$

for some $k > 1$, where $r_\rho(x)$ is the $\rho$-distance function from $x$ to a fixed point $p \in M$, the Poisson equation $\Delta u = -\rho \varphi$ admits a bounded solution $u$ on $M$.

If, in addition, there exists $v_0 > 0$ such that

$$V_\rho(x, 1) = \int_{B_\rho(x, 1)} \rho(y) \, dy \geq v_0$$

for all $x \in M$, then the solution $u$ decays and
\[ |u|(x) \leq C (1 + r_p(x))^{-k+1}. \]

Obviously, these results are faithful generalization of the ones from \( \lambda_1(\Delta) > 0 \). We also point out that Theorem 1.3 is sharp as remarked after the proof of Theorem 1.2. In passing, we mention that recently Catino, Monticelli and Punzo \cite{Catino} have studied the solvability of the Poisson equation by only assuming the essential spectrum of \( M \) is positive. In view of this, one may speculate that some of the preceding results generalize with weighted Poincaré inequality holds only for smooth functions \( \phi \) with support avoiding a fixed geodesic ball.

Our proof of Theorem 1.3 follows in part of that of Theorem 1.2. As in the proof of Theorem 1.2, we write

\[ \hat{\int}_{B_{\rho}(p, r)} \rho(y) G(x, y) \, dy = \int_{B_{\rho}(p, r) \setminus B_{\rho}(x, 1)} \rho(y) G(x, y) \, dy + \int_{B_{\rho}(p, r) \cap B_{\rho}(x, 1)} \rho(y) G(x, y) \, dy. \]

Following \cite{[26]}, the integral over \( B_{\rho}(p, r) \setminus B(x, 1) \) is estimated by the integral

\[ \int_{L_x(\alpha, \beta)} \rho(y) G(x, y) \, dy \]

over the sublevel sets

\[ L_x(\alpha, \beta) := \{ y \in M : \alpha < G(x, y) < \beta \}, \]

where \( \alpha \) and \( \beta \) are the minimum and maximum value of the Green’s function \( G(x, y) \) over \( B_{\rho}(p, r) \setminus B_{\rho}(x, 1) \), respectively. Using the weighted Poincaré inequality instead of \( \lambda_1(\Delta) > 0 \) and arguing as in \cite{[26]}, one obtains

\[ \int_{L_x(\alpha, \beta)} \rho(y) G(x, y) \, dy \leq c \int_{L_x(\beta, 2\beta)} G^{-1}(x, y) |\nabla G|^2(x, y) \, dy. \]

Now the co-area formula together with the fact that \( G(x, y) \) is harmonic on \( M \setminus B_{\rho}(x, 1) \) yields that

\[ \int_{B_{\rho}(p, r) \setminus B_{\rho}(x, 1)} \rho(y) G(x, y) \, dy \leq C (r + 1). \]

For the integral over \( B_{\rho}(p, r) \cap B_{\rho}(x, 1) \) in (1.4), however, a different approach from \cite{[26]} is needed. In the case of of \( \lambda_1(\Delta) > 0 \), the proof relies on the following double integral estimate for the minimal positive Green’s function.

\[ \int_A \int_B G(x, y) \, dy \, dx \leq e^{\frac{\sqrt{\lambda_1(\Delta)}}{\lambda_1(\Delta)}} \sqrt{V(A)} \sqrt{V(B)} (1 + r(A, B)) e^{-\sqrt{\lambda_1(\Delta)} r(A, B)} \]

for any bounded domains \( A \) and \( B \) of \( M \), where \( r(A, B) \) denotes the distance between \( A \) and \( B \), and \( V(A), V(B) \) their volumes.

Unfortunately, it is unclear to us at this point how to formulate and derive a similar estimate under the weighted Poincaré inequality. To overcome this difficulty, we decompose \( B_{\rho}(p, r) \) into a sequence of annuli and employ a similar argument.
as (1.5) for each annulus. However, instead of the weighted Poincaré inequality, we now use Poincaré inequality by appealing to a result of Li and Schoen [15] on the estimate of the bottom spectrum of a geodesic ball in terms of the Ricci curvature lower bound and its radius. This argument has the added benefit that it completely avoids the involvement of the heat kernel and treats the two integrals of (1.4) away and near the singularity of the Green’s function in a unified manner.

The Green’s function estimate in Theorem 1.3 leads to the following volume comparison estimate for geodesic ρ-balls. Define

$$V_\rho(x,r) = \int_{B_\rho(x,r)} \rho(y) dy.$$  
(1.6)

**Theorem 1.5.** Let $\left(M^n, g\right)$ be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight $\rho$ having properties (1.2) and (1.3). Assume that $\text{Ric} \geq -K\rho$ on $M$ for some $K \geq 0$. Then there exist constants $c_1$ and $c_2$ depending only on $n$, $K$, $\delta$ and $A$ such that for all $x \in M$,

$$c_1 e^{2R}V_\rho(x,1) \leq V_\rho(x,R) \leq e^{c_2 R}V_\rho(x,1)$$

for all $1 < R < \infty$.

We point out that the lower bound of the form $V_\rho(x,R) \geq c e^{2R}$ first appeared in [19], where the constant $c$ may depend on $x$.

As an application of the solvability of the Poisson equation, we prove the following result concerning the connectivity at infinity.

**Theorem 1.6.** Let $(M^n, g)$ be a complete Kähler manifold satisfying (1.1) with weight $\rho$ having properties (1.2), (1.3) and $\rho \leq C$. Assume that there exists $\nu_0 > 0$ so that for all $x \in M$

$$V_\rho(x,1) = \int_{B_\rho(x,1)} \rho(y) dy \geq \nu_0 > 0$$

and that the Ricci curvature lower bound $\text{Ric} \geq -\zeta \rho$ holds for some function $\zeta(x) > 0$ converging to zero at infinity. Then $M$ has only one end.

The novelty of the result is that the assumption on the Ricci curvature is essentially imposed only at infinity, yet we are able to conclude that the manifold is connected at infinity. This is of course not true in the Riemannian setting. Indeed, the connected sum of copies of $\mathbb{R}^n$ for $n \geq 3$ has non-negative Ricci curvature outside a compact set and satisfies a weighted Poincaré inequality of the form $\rho(x) = cr^{-2}(x)$. Obviously, it can have as many ends as one wishes.

We remark that our assumption is vacuous when $\rho = \lambda_1(\Delta)$ is constant according to the aforementioned result of Cheng [5]. However, in the case $\lambda_1(\Delta) > 0$, there are various results concerning the number of ends for both Riemannian and Kähler manifolds. We refer to the papers [17, 18, 21, 25] for more information and further references. It should also be noted, although not explicitly stated there, that the argument in [19] already implies that $M$ necessarily has finitely many ends, without assuming $M$ is Kähler.

To prove Theorem 1.6 we first observe the assumption that

$$V_\rho(x,1) = \int_{B_\rho(x,1)} \rho(y) dy \geq \nu_0 > 0$$
ensures all ends of $M$ must be nonparabolic. Therefore, by the result of Li and Tam [16], $M$ admits a nonconstant bounded harmonic function $u$ with finite energy if it is not connected at infinity. According to [13], such $u$ must be pluriharmonic as $M$ is Kähler. One may view $u$ as a holomorphic map from $M$ into the hyperbolic disk. The proof is then completed by establishing a Liouville type result for such maps. It is well-known from Yau’s Schwarz lemma [29] that such map $u$ must be constant if the Ricci curvature of the domain manifold $M$ is nonnegative. The result was generalized by Li and Yau [23] to address the case that the negative part of the Ricci curvature of $M$ is integrable. They concluded that $u$ is necessarily a constant map if $M$ is in addition nonparabolic. Our next result may be viewed as further development along this line.

**Theorem 1.7.** Let $(M, g)$ be a complete Kähler manifold satisfying the assumptions of Theorem 1.6. Assume that $F : M \to N$ is a finite energy holomorphic map into a complex Hermitian manifold $N$ of non-positive bisectional curvature. Then $F$ must be a constant map.

The paper is organized as follows. In Section 2 after making some preliminary observations relating $\rho$-balls to the background metric balls, we translate Poincaré inequality, Sobolev inequality and gradient estimate from the background metric balls to the $\rho$-balls. With these preparations, we prove Theorem 1.3 in Section 3. Section 4 is devoted to the Poisson equation and the proof of Theorem 1.4. In Section 5, we discuss applications of the Poisson equation and prove the Liouville property for finite energy holomorphic maps. Section 6 contains a new treatment of Theorem 1.2. Comparing to the original proof in [26], we believe the new one is more streamlined. The proof relies on estimates of heat kernel and avoids level set consideration. It remains to be seen if this new approach can be adapted to handle Theorem 1.3 as well.

2. Properties of the $\rho$-distance

In this section, we make preparations for proving Theorem 1.3 by relating both the geometry and analysis of the $\rho$-balls to the background metric balls. Consider the $\rho$-distance function, defined to be

$$r_\rho(x, y) = \inf_\gamma l_\rho(\gamma),$$

the infimum of the length with respect to metric $ds^2_\rho$ of all smooth curves joining $x$ and $y$. For a fixed point $x \in M$, one checks readily that $|\nabla r_\rho|^2(x, y) = \rho(y)$. When there is no confusion, the $\rho$-distance from $x$ to a fixed point $p$ is simply denoted by $r_\rho(x)$. More generally, for any function $v \in C^1(M)$, denote by $\nabla_\rho v$ the gradient of $v$ with respect to $ds^2_\rho$. Then its length with respect to $ds^2_\rho$ is given by $|\nabla_\rho v|^2 = \frac{1}{\rho} |\nabla v|^2$.

We denote geodesic balls with center $x$ and radius $r$ with respect to $ds^2$ by $B(x, r)$ and those with respect to $ds^2_\rho$ by $B_\rho(x, r)$. Our first result shows that $B \left( x, \frac{r}{\sqrt{\rho(x)}} \right)$ and $B_\rho(x, r)$ are comparable when $r \leq 1$. Without loss of generality, we may assume the constants $A$ and $\delta$ specified in 1.3 satisfy $A > 16$ and $\delta < 1$. Throughout this section, we use $c$ and $C$ to denote constants depending only on dimension $n$, the
constant $K$ from the Ricci curvature lower bound, and the constants $A$ and $\delta$ in (1.3). Any other dependencies will be explicitly stated.

**Proposition 2.1.** Let $M$ be a complete Riemannian manifold satisfying weighted Poincaré inequality (1.1) with weight $\rho$ having properties (1.2) and (1.3). Then there exists $C > 0$ depending only on $A$ and $\delta$ such that for any $x \in M$,

$$
\sup_{B_{\rho}(x,1)} \rho \leq C \inf_{B_{\rho}(x,1)} \rho.
$$

Furthermore, there exist $c_0 > 0$ and $C_0 > 0$ depending only on $A$ and $\delta$ such that

$$
B \left( x, \frac{c_0}{\sqrt{\rho(x)}} r \right) \subset B_{\rho}(x,r) \subset B \left( x, \frac{C_0}{\sqrt{\rho(x)}} r \right)
$$

for all $x \in M$ and $0 < r \leq 1$.

**Proof.** Let $x \in M$ and $0 < r \leq 1$. Let $\tau(t)$, $0 \leq t \leq T$, be a minimizing $\rho$-geodesic starting from $x$. We claim that either

$$
\tau([0,T]) \subset B \left( x, \frac{\delta}{\sqrt{\rho(x)}} r \right) \quad \text{or} \quad l_{\rho}(\tau) > \frac{\delta}{A} r.
$$

Indeed, if $\tau$ is not entirely contained in $B \left( x, \frac{\delta}{\sqrt{\rho(x)}} r \right)$, then there exists $0 < t_1 < T$ so that $\tau(t) \in B \left( x, \frac{\delta}{\sqrt{\rho(x)}} r \right)$ for all $0 \leq t \leq t_1$ and $\tau(t_1) \in \partial B \left( x, \frac{\delta}{\sqrt{\rho(x)}} r \right)$. Let $\bar{\tau}$ be the restriction of $\tau$ to $[0,t_1]$. Then

$$
l_{\rho}(\bar{\tau}) = \int_{\bar{\tau}} |\bar{\tau}'|_{\rho}(t) \, dt
$$

$$
= \int_{\bar{\tau}} \sqrt{\rho(\bar{\tau}(t))} |\bar{\tau}'(t)| \, dt
$$

$$
\geq \frac{1}{\sqrt{A}} \sqrt{\rho(x)} \int_{\bar{\tau}} |\bar{\tau}'(t)| \, dt
$$

$$
= \frac{1}{\sqrt{A}} \sqrt{\rho(x)} l(\bar{\tau}),
$$

where in the third line we have used (1.3) and that $\bar{\tau}(t) \in B \left( x, \frac{\delta}{\sqrt{\rho(x)}} r \right)$ for all $t \leq t_1$. Since $\tau(t_1) \in \partial B \left( x, \frac{\delta}{\sqrt{\rho(x)}} r \right)$, we have $l(\bar{\tau}) \geq \frac{\delta}{\sqrt{\rho(x)}} r$. Consequently,

$$
l_{\rho}(\bar{\tau}) \geq \frac{\delta}{A} r.
$$

This proves (2.1).

We infer from the claim that $r(x,y) < \frac{\delta}{\sqrt{\rho(x)}} r$ when $r_{\rho}(x,y) < \frac{\delta}{A} r$. In other words,

$$
B_{\rho} \left( x, \frac{\delta}{A} r \right) \subset B \left( x, \frac{\delta}{\sqrt{\rho(x)}} r \right)
$$

(2.2)
for all \( x \in M \) and all \( 0 < r \leq 1 \). By (1.3), this implies

\[
\sup_{B_{\rho}(x, \frac{r}{A})} \rho \leq A \inf_{B_{\rho}(x, \frac{r}{A})} \rho.
\]

Now for \( x, y \in M \) with \( r_{\rho}(x, y) \leq 1 \), let \( \tau \) be a minimizing \( \rho \)-geodesic from \( x \) to \( y \). Applying (2.3) successively on each interval of \( \rho \)-length \( \frac{\delta}{A} \) along \( \tau \), we conclude that

\[
\frac{1}{C} \rho(x) \leq \rho(y) \leq C \rho(x),
\]

where \( C = A^\frac{2\delta}{A} \). Therefore,

\[
\sup_{B_{\rho}(x, 1)} \rho \leq C \inf_{B_{\rho}(x, 1)} \rho
\]

for all \( x \in M \). This proves the first part of the proposition.

Note that by (2.2), for any \( z_1, z_2 \in M \) and \( 0 < r \leq 1 \),

\[
r(z_1, z_2) < \frac{\delta}{\sqrt{\rho(z_1)}} r \text{ whenever } r_{\rho}(z_1, z_2) < \frac{\delta}{A} r.
\]

So for \( x, y \in M \) with \( r_{\rho}(x, y) \leq r \), applying (2.5) successively on intervals of \( \rho \)-length \( \frac{\delta}{A} r \) along a minimizing \( \rho \)-geodesic \( \tau \) from \( x \) to \( y \) and using (2.4), one concludes that

\[
r(x, y) \leq \frac{C_0}{\sqrt{\rho(x)}} r
\]

for some \( C_0 > 0 \) depending on \( A \) and \( \delta \). Hence,

\[
B_{\rho}(x, r) \subset B \left( x, \frac{C_0}{\sqrt{\rho(x)}} r \right)
\]

for all \( x \in M \) and \( r \leq 1 \).

We now show that

\[
B \left( x, \frac{c_0}{\sqrt{\rho(x)}} r \right) \subset B_{\rho}(x, r)
\]

for all \( x \in M \) and \( r \leq 1 \) with \( c_0 = \frac{\delta}{A} \).

Indeed, for \( y \in B \left( x, \frac{c_0}{\sqrt{\rho(x)}} r \right) \) and \( \gamma(t), 0 \leq t \leq T < \frac{c_0}{\sqrt{\rho(x)}} r \), a minimizing geodesic joining \( x \) and \( y \), we have
\[ l_ρ(γ) = \int_γ |γ'(t)| \, dt \]
\[ = \int_γ \sqrt{ρ(γ(t))} |γ'(t)| \, dt \]
\[ \leq \sqrt{A} \sqrt{ρ(x)} \int_γ |γ'(t)| \, dt \]
\[ = \sqrt{A} \sqrt{ρ(x)} l(γ) \]
\[ \leq c_0 \sqrt{A} r < r, \]

where in the third line we have used (1.3) together with \( γ(t) \in B \left( x, \frac{δ}{\sqrt{ρ(x)}} r \right) \) for all \( 0 \leq t \leq T \). This proves (2.7).

From (2.7) and (2.6) we conclude that

\[ B \left( x, \frac{c_0}{\sqrt{ρ(x)}} r \right) \subset B_{ρ}(x, r) \subset B \left( x, \frac{C_0}{\sqrt{ρ(x)}} r \right) \]

for all \( x \in M \) and \( r \leq 1 \). This proves the proposition. \( \square \)

The previous result enables us to translate some properties on geodesic balls of metric \( ds^2 \) to those of \( ds_ρ^2 \). Denote by \( λ_1(B_{ρ}(x, r)) \) the first Dirichlet eigenvalue of \( B_{ρ}(x, r) \) with respect to metric \( ds^2 \). Then

\[ λ_1(B_{ρ}(x, r)) \int_{B_{ρ}(x, r)} φ^2 \leq \int_{B_{ρ}(x, r)} |∇φ|^2 \]

for any \( φ \in C_0^{∞}(B_{ρ}(x, r)) \). Here and in the following, all integrals are with respect to the Riemannian measure induced by the metric \( ds^2 \). Similarly, we use \( CS(B_{ρ}(x, r)) \) to denote the optimal constant for the following Dirichlet Sobolev inequality on \( B_{ρ}(x, r) \).

\[ CS(B_{ρ}(x, r)) \left( \int_{B_{ρ}(x, r)} φ^2 \right)^{\frac{α_2}{α_2}} \leq \int_{B_{ρ}(x, r)} |∇φ|^2 + \frac{ρ(x)}{r^2} \int_{B_{ρ}(x, r)} φ^2 \]

for \( φ \in C_0^{∞}(B_{ρ}(x, r)) \), where \( \int_{B_{ρ}(x, r)} u \) is the average value of function \( u \) over the set \( B_{ρ}(x, r) \), namely,

\[ \int_{B_{ρ}(x, r)} u = \frac{1}{V(B_{ρ}(x, r))} \int_{B_{ρ}(x, r)} u \]

with \( V(B_{ρ}(x, r)) \) being the volume of \( B_{ρ}(x, r) \) with respect to metric \( ds^2 \). We refer to \( CS(B_{ρ}(x, r)) \) as the Dirichlet Sobolev constant for \( B_{ρ}(x, r) \).

**Lemma 2.2.** Let \((M^n, g)\) be a complete manifold satisfying (1.1), (1.2) and (1.3). Assume that \( \text{Ric} \geq -Kρ \) on \( M \) for some \( K \geq 0 \). Then for some \( C > 0 \),
\[ \lambda_1 (B_p(x, r)) \geq \frac{1}{Cr^2} \rho(x) \]

\[ C_S (B_p(x, r)) \geq \frac{1}{Cr^2} \rho(x) \]

for any \( x \in M \) and \( 0 < r \leq \frac{\delta}{2C_0} \). Here \( C_0 \) is the constant specified in Proposition 2.1.

**Proof.** According to Li-Schoen [15], if \( \text{Ric} \geq -H \) on \( B(x, 2R) \), then

\[ \lambda_1 (B(x, R)) \geq \frac{1}{R^2} e^{-C(1+R\sqrt{H})} \]

with \( C \) depending only on dimension. For \( r \leq \frac{\delta}{2C_0} \) we have

\[ B \left( x, \frac{2C_0}{\sqrt{\rho(x)}} r \right) \subset B \left( x, \frac{\delta}{\sqrt{\rho(x)}} \right). \]

The Ricci curvature lower bound assumption together with (1.3) implies that

\[ \text{Ric} \geq -c \rho(x) \text{ on } B \left( x, \frac{2C_0}{\sqrt{\rho(x)}} r \right). \]

Using (2.8) and (2.9) we get

\[ \lambda_1 \left( B \left( x, \frac{C_0}{\sqrt{\rho(x)}} r \right) \right) \geq \frac{1}{Cr^2} \rho(x) \]

for any \( x \in M \) and \( 0 < r \leq \frac{\delta}{2C_0} \). Since Proposition 2.1 asserts

\[ B_p(x, r) \subset B \left( x, \frac{C_0}{\sqrt{\rho(x)}} r \right), \]

it follows that

\[ \lambda_1 (B_p(x, r)) \geq \frac{1}{Cr^2} \rho(x) \]

for any \( x \in M \) and \( 0 < r \leq \frac{\delta}{2C_0} \). This proves the eigenvalue lower bound.

To prove the Sobolev constant bound, we use a result of Saloff-Coste [28] that the following Sobolev inequality holds on \( B(x, R) \) if \( \text{Ric} \geq -H \) on \( B(x, 2R) \).

\[ \left( \int_{B(x, R)} |\nabla \phi|^2 \right)^{\frac{2}{n}} \leq \frac{1}{R^2} e^{-C(1+\sqrt{n}H)} \left( \int_{B(x, R)} \phi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \]

for any \( \phi \in C_0^\infty (B(x, R)) \).

Now for \( R = \frac{C_0}{\sqrt{\rho(x)}} r \), in view of (2.9), applying (2.11), we get...
\[
\frac{1}{C} \frac{\rho(x)}{r^2} V(B(x,R))^\frac{n}{2} \left( \int_{B(x,R)} \phi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \int_{B(x,R)} |\nabla \phi|^2 + \frac{\rho(x)}{r^2} \int_{B(x,R)} \phi^2.
\]
for any \( \phi \in C_0^\infty(B(x,R)) \).

However, by (2.10), we have
\( B_\rho(x,r) \subset B(x,R) \).

It follows for \( \phi \in C_0^\infty(B_\rho(x,r)) \) that
\[
\frac{1}{C} \frac{\rho(x)}{r^2} V(B_\rho(x,r))^\frac{n}{2} \left( \int_{B_\rho(x,r)} \phi^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq \int_{B_\rho(x,r)} |\nabla \phi|^2 + \frac{\rho(x)}{r^2} \int_{B_\rho(x,r)} \phi^2.
\]
This completes the proof of the lemma. \( \square \)

A well known result of Cheng and Yau [6] says that for \( u > 0 \) a harmonic function on \( B(x,R) \),
\[
(2.12) \quad \sup_{B(x,\frac{R}{2})} |\nabla \ln u| \leq c \left( \sqrt{H} + \frac{1}{R} \right)
\]
for some constant \( c > 0 \) depending only on dimension \( n \) provided that the Ricci curvature \( \text{Ric} \geq -H \) on \( B(x,R) \) for some nonnegative constant \( H \). We now use Proposition 2.1 to translate this estimate to \( \rho \)-balls.

**Lemma 2.3.** Let \((M^n, g)\) be a complete manifold satisfying (1.1), (1.2) and (1.3). Assume that \( \text{Ric} \geq -K \rho \) on \( M \) for some \( K \geq 0 \). Then there exists \( c > 0 \) such that for \( u > 0 \) a harmonic function on \( B_\rho(x,r) \) with \( 0 < r \leq 1 \),
\[
\sup_{B_r(x,\frac{r}{2})} |\nabla \rho \ln u|_\rho \leq \frac{c}{r}.
\]
Consequently,
\[
u(y) \leq c \nu(z)
\]
for \( y, z \in B_\rho(x,\frac{r}{2}) \).

**Proof.** For \( y \in B_\rho(x,\frac{r}{2}) \), the triangle inequality implies that \( B_\rho(y,\frac{r}{2}) \subset B_\rho(x,r) \).

On the other hand, by Proposition 2.1 we have
\[
B\left(y, \frac{c_0}{2\sqrt{\rho(y)}}r \right) \subset B_\rho\left(y, \frac{r}{2} \right).
\]
Therefore, \( u \) is harmonic on \( B\left(y, \frac{c_0}{2\sqrt{\rho(y)}}r \right) \). Using (1.3) one sees that \( \text{Ric} \geq -c_0 \rho(y) \) on \( B\left(y, \frac{c_0}{2\sqrt{\rho(y)}}r \right) \). In conclusion, by (2.12),
\[
|\nabla \ln u|_\rho(y) \leq \frac{c}{r} \sqrt{\rho(y)}.
\]
This can be rewritten into
\[
(2.13) \quad |\nabla \rho \ln u|_\rho(y) \leq \frac{c}{r}.
\]

Integrating (2.13) along a minimizing \( \rho \)-geodesic joining \( x \) and \( y \) yields
\[ \frac{1}{c} u(y) \leq u(x) \leq cu(y) \]
for \( y \in B_\rho \left( x, \frac{r}{2} \right) \). This obviously implies
\[ u(y) \leq cu(z) \]
for \( y, z \in B_\rho \left( x, \frac{r}{2} \right) \). The lemma is proved. \( \square \)

3. Green’s function estimates

With the preparations in the previous section, we now prove Theorem 1.3. Throughout this section, unless otherwise specified, we continue to use \( c \) and \( C \) to denote constants depending only on \( n, K, \delta \) and \( A \).

Let us first note the following simple consequence of Lemma 2.3 which will be used repeatedly below. For any \( 0 < r \leq 1 \) and \( y \in M \setminus B_\rho (x, r) \), apply the local gradient estimate Lemma 2.3 to the harmonic function \( u(q) = G(x, q) \) on \( B_\rho (y, r) \).

Then
\[
\sup_{z \in B_\rho (y, \frac{r}{2})} |\nabla_\rho \ln G|_\rho (x, z) \leq \frac{c}{r},
\]
where the gradient is computed with respect to variable \( z \). Consequently, we have
\[
G(x, z_1) \leq CG(x, z_2)
\]
for all \( z_1, z_2 \in B_\rho (y, \frac{r}{2}) \).

We first establish a local Harnack estimate.

Lemma 3.1. Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \( \rho \) having properties (1.2) and (1.3). Assume that \( \text{Ric} \geq -K\rho \) on \( M \) for some \( K \geq 0 \). Then
\[
G(x, y) \leq C \left( \frac{r_2}{r_1} \right)^{C_1} G(x, z)
\]
for any \( y \in \partial B_\rho (x, r) \) and \( z \in \partial B_\rho (x, s) \), where
\[ 0 < r_1 \leq r, s \leq r_2 \leq 1. \]

Proof. Suppose first that both \( y, z \in \partial B_\rho (x, r) \) for some \( 0 < r \leq 1 \). Since the estimate \((3.2)\) implies
\[ G(x, y) \leq CG(x, z) \]
for \( z \in B_\rho (y, \frac{r}{2}) \), it suffices to prove \((3.3)\) for \( y \) and \( z \) satisfying
\[ r_\rho (y, z) \geq \frac{1}{2} r. \]
Let \( \tau(t) \) and \( \eta(t) \), \( 0 \leq t \leq r \), be minimizing \( \rho \)-geodesics from \( x \) to \( y \) and \( z \), respectively. We claim that \( r_\rho (y, \eta) \geq \frac{1}{4} r \) and \( r_\rho (z, \tau) \geq \frac{1}{4} r \). Indeed, suppose \( r_\rho (y, \eta(t_0)) < \frac{1}{4} r \) for some \( t_0 \in (0, r) \). Since \( r_\rho (x, y) = r_\rho (x, z) = r \) and \( r_\rho (y, z) \geq \frac{1}{2} r \), the triangle inequality implies
\[ r_\rho(z, \eta(t_0)) \geq r_\rho(y, z) - r_\rho(y, \eta(t_0)) \]
\[ > \frac{1}{4}r \]

and
\[ r_\rho(x, \eta(t_0)) \geq r_\rho(x, y) - r_\rho(y, \eta(t_0)) \]
\[ > \frac{3}{4}r. \]

Adding up these two inequalities we get
\[ r_\rho(x, z) = r_\rho(x, \eta(t_0)) + r_\rho(\eta(t_0), z) > r. \]

This contradiction shows that \( r_\rho(y, \eta) \geq \frac{1}{4}r \) as claimed. The proof of \( r_\rho(z, \tau) \geq \frac{1}{4}r \) is similar.

Consequently, \( u(q) = G(y, q) \) is harmonic on \( B_\rho(\eta(t), \frac{1}{4}r) \) for all \( t \in [0, r] \). It follows from (3.1) that
\[ G(y, x) \leq C G(y, z). \]

Similarly, as \( r_\rho(z, \tau) \geq \frac{1}{4}r \), the function \( u(q) = G(z, q) \) is harmonic on \( B_\rho(\tau(t), \frac{1}{4}r) \) for all \( t \in [0, r] \). By (3.1) we get
\[ G(z, y) \leq C G(z, x). \]

Combining (3.4) with (3.5) we conclude that
\[ G(x, y) \leq C G(x, z) \]
as claimed in (3.3). This proves (3.3) when both \( y, z \in \partial B_\rho(x, r) \).

Now let \( y \in \partial B_\rho(x, r) \) and \( z \in \partial B_\rho(x, s) \) with
\[ 0 < r_1 \leq r, s \leq r_2 \leq 1. \]

Let us assume first that \( r < s \). Let \( \eta(t), 0 \leq t \leq s \), be a minimizing \( \rho \)-geodesic from \( x \) to \( z \). Applying (3.6) to \( y \in \partial B_\rho(x, r) \) and \( \eta(r) \in \partial B_\rho(x, r) \), we get that
\[ G(x, y) \leq C G(x, \eta(r)). \]

Note that the function \( u(q) = G(x, q) \) is harmonic on \( B_\rho(\eta(t), t) \) for all \( r \leq t \leq s \). Hence, according to (3.1),
\[ |\nabla_\rho \ln G|_{\rho}(x, \eta(t)) \leq \frac{c}{t}. \]

Integrating (3.8) in \( t \) from \( r \) to \( s \) implies that
\[ G(x, \eta(r)) \leq c \left( \frac{s}{r} \right)^c G(x, z). \]

Together with (3.7) and the fact \( \frac{s}{r} \leq \frac{s}{r_1} \), one concludes
\[ G(x, y) \leq c \left( \frac{r_2}{r_1} \right)^c G(x, z) \]
for any \( y \in \partial B_\rho(x, r) \) and \( z \in \partial B_\rho(x, s) \). This proves the result in the case \( r \leq s \).

The remaining case of \( s < r \) is similar, using (3.8) along a minimizing \( \rho \)-geodesic \( \tau(t) \) joining \( x \) and \( y \) instead. \( \square \)
We now establish a similar result for any radius.

**Lemma 3.2.** Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality \(\text{(1.1)}\) with weight \(\rho\) having properties \(\text{(1.2)}\) and \(\text{(1.3)}\). Assume that \(\text{Ric} \geq -K\rho\) on \(M\) for some \(K \geq 0\). Then

\[
G(x, y) \leq e^{Cr} G(x, z)
\]

for any \(p \in M\), \(x \in B_\rho(p, r)\), and any \(y, z \in B_\rho(p, r) \setminus B_\rho(x, 1)\).

**Proof.** For \(y, z \in B_\rho(p, r) \setminus B_\rho(x, 1)\), let \(\tau(t), 0 \leq t \leq T_1\), and \(\eta(\bar{t}), 0 \leq \bar{t} \leq T_2\), be minimizing \(\rho\)-geodesics from \(x\) to \(y\) and from \(x\) to \(z\), respectively. Since \(y, z \in B_\rho(p, r)\) and \(r_\rho(x) < r\), the triangle inequality implies that \(T_1, T_2 < 2r\).

Let \(y_1 = \tau(1) \in \partial B_\rho(x, 1)\) and \(z_1 = \eta(1) \in \partial B_\rho(x, 1)\) be the intersection points of \(\tau\) and \(\eta\) with \(\partial B_\rho(x, 1)\). By Lemma 3.1 we have

\[
G(x, y_1) \leq c G(x, z_1).
\]

On the other hand, by (3.1),

\[
|\nabla_\rho \ln G|_\rho(x, \tau(t)) \leq c
\]

for all \(1 \leq t\). Integrating (3.10) in \(t\) from \(1\) to \(T_1\) yields that

\[
G(x, y) \leq e^{cr} G(x, y_1).
\]

Similarly, we have

\[
G(x, z_1) \leq e^{cr} G(x, z).
\]

In view of (3.9) we conclude that

\[
G(x, y) \leq e^{cr} G(x, z).
\]

This proves the lemma. \(\square\)

To prove Theorem 1.3 we will need to consider the level sets of the Green’s function. Denote by

\[
l_x(t) = \{ y \in M : G(x, y) = t \}
\]

and

\[
L_x(\alpha, \beta) = \{ y \in M : \alpha < G(x, y) < \beta \}.
\]

We will make extensive use of the following lemma. For a proof, see lemma 3.3 in [26].

**Lemma 3.3.** Let \((M^n, g)\) be a Riemannian manifold satisfying \(\text{(1.1)}\) with weight \(\rho\) having property \(\text{(1.2)}\). For any \(t > 0\) we have

\[
\int_{l_x(t)} |\nabla G| (x, \xi) dA(\xi) = 1,
\]

where \(dA\) is the Riemannian area form of \(l_x(t)\). Furthermore, for any \(0 < \alpha < \beta\) we have

\[
\int_{L_x(\alpha, \beta)} G^{-1}(x, y) |\nabla G|^2(x, y) dy = \ln \frac{\beta}{\alpha}.
\]
A useful consequence of Lemma 3.3 is that

\[(3.11) \int_{L_x(\alpha, \beta)} \rho(y) G(x, y) dy \leq c \left( 1 + \ln \frac{\beta}{\alpha} \right) \]

if the set $L_x (\frac{1}{\epsilon} \alpha, e\beta)$ is compact in $M$. In fact, one only requires the weighted Poincaré inequality (1.1) to hold for smooth functions $\phi$ with support contained in $L_x (\frac{1}{\epsilon} \alpha, e\beta)$.

Indeed, let $\phi$ be the cut-off function defined by

\[
\phi(y) = \begin{cases} 
\ln (e\beta) - \ln G(x, y) & \text{on } L_x (\beta, e\beta) \\
1 & \text{on } L_x (\alpha, \beta) \\
\ln G(x, y) - \ln (\frac{1}{\epsilon} \alpha) & \text{on } L_x (\frac{1}{\epsilon} \alpha, \alpha) \\
0 & \text{otherwise}
\end{cases}
\]

Then the weighted Poincaré inequality (1.1) implies that

\[(3.12) \int_M \rho(y) \phi^2(y) G(x, y) dy \leq \int_M \left| \nabla \left( \phi G^{\frac{1}{2}} \right) \right|^2 (x, y) dy \]

\[\leq \frac{1}{2} \int_M \phi^2(y) |\nabla G|^2(x, y) G^{-1}(x, y) dy + 2 \int_M G(x, y) |\nabla \phi|^2(y) dy \]

Using the co-area formula and Lemma 3.3, we have

\[
\int_M \phi^2(y) |\nabla G|^2(x, y) G^{-1}(x, y) dy \\
\leq \int_{L_x(\frac{1}{\epsilon} \alpha, e\beta)} |\nabla G|^2(x, y) G^{-1}(x, y) dy \\
= 2 + \ln \left( \frac{\beta}{\alpha} \right).
\]

The second term of the right hand side of (3.12) can be estimated as

\[
\int_M G(x, y) |\nabla \phi|^2(y) dy \\
\leq \int_{L_x(\beta, e\beta)} |\nabla G|^2(x, y) G^{-1}(x, y) dy + \int_{L_x(\frac{1}{\epsilon} \alpha, \alpha)} |\nabla G|^2(x, y) G^{-1}(x, y) dy \\
= 2.
\]

Combining these estimates we obtain

\[
\int_M \rho(y) \phi^2(y) G(x, y) dy \leq c \left( 1 + \ln \frac{\beta}{\alpha} \right)
\]

as claimed in (3.11).

With a further cut-off, the assumption that the set $L_x (\frac{1}{\epsilon} \alpha, e\beta)$ is compact in $M$ is in fact not needed.
Lemma 3.4. Let $(M^n, g)$ be a Riemannian manifold satisfying (1.1) with weight $\rho$ having property (1.2). Then for all $0 < \alpha < \beta$,
\[
\int_{L_x(\alpha, \beta)} \rho(y) G(x, y) \, dy \leq c \left( 1 + \ln \frac{\beta}{\alpha} \right).
\]

Proof. The argument is similar to that of (3.11), the main difference being that we use an additional cut-off in distance
\[
\psi(y) = \begin{cases} 
1 & \text{on } B_\rho(x, R) \\
R + 1 - r_\rho(x, y) & \text{on } B_\rho(x, R + 1) \setminus B_\rho(x, R) \\
0 & \text{on } M \setminus B_\rho(x, R + 1)
\end{cases}
\]
Now define $\phi = \chi \psi$, where $\chi$ is given by
\[
\chi(y) = \begin{cases} 
\ln (e^\beta - \ln G(x, y)) & \text{on } L_x(\beta, e^\beta) \\
1 & \text{on } L_x(\alpha, \beta) \\
\ln G(x, y) - \ln \left( \frac{1}{e^\alpha} \right) & \text{on } L_x\left( \frac{1}{e^\alpha}, \alpha \right) \\
0 & \text{otherwise}
\end{cases}
\]
We have
\[
\int_M \rho(y) \phi(y)^2 G(x, y) \, dy \leq \int_M \left| \nabla \left( \phi G^{\frac{1}{2}} \right) \right|^2 (x, y) \, dy
\]
\[
\leq \frac{1}{2} \int_M |\nabla G|^2 (x, y) G^{-1}(x, y) \, dy + 4 \int_M G(x, y) |\nabla \chi|^2 (y) \, dy + 4 \int_M G(x, y) |\nabla \psi|^2 (y) \chi^2 (y) \, dy.
\]
Using Lemma 3.3 as in the proof of (3.11), we get
\[
\int_M \rho(y) \phi(y)^2 G(x, y) \, dy \leq c \left( 1 + \ln \frac{\beta}{\alpha} \right) + 4 \int_M G(x, y) |\nabla \psi|^2 (y) \chi^2 (y) \, dy.
\]
Note that $|\nabla \psi|^2 (y) = \rho(y)$ on its support. Since $G > e^{-1} \alpha$ on the support of $\chi$, we get that
\[
\int_M G(x, y) |\nabla \psi|^2 (y) \chi^2 (y) \, dy \leq e \int_{B_\rho(x, R + 1) \setminus B_\rho(x, R)} \rho(y) G^2 (x, y) \, dy.
\]
However, it follows from Corollary 2.2 in [19] (cf. Theorem 2.5 in [20]) that
\[
\int_{B_\rho(x, R + 1) \setminus B_\rho(x, R)} \rho(y) G^2 (x, y) \, dy \leq C e^{-2R} \int_{B_\rho(x, R + 1) \setminus B_\rho(x, R)} \rho(y) G^2 (x, y) \, dy.
\]
In conclusion, this implies
\[
\int_M G(x, y) |\nabla \psi|^2 (y) \chi^2 (y) \, dy \leq \frac{C}{\alpha} e^{-2R} \int_{B_\rho(x, R + 1) \setminus B_\rho(x, R)} \rho(y) G^2 (x, y) \, dy.
\]
Combining (3.13) and (3.15) we obtain
\[
\hat{L}_x(\alpha,\beta) \cap B_{\rho}(x, R) \rho(y) G(x, y) dy \leq \hat{L}_x(\alpha,\beta) \rho(y) G(x, y) dy \leq c \left( 1 + \ln \frac{\beta}{\alpha} \right) + \frac{C}{\alpha} e^{-2R} \int_{B_{\rho}(x,2) \setminus B_{\rho}(x,1)} \rho(y) G^2(x, y) dy.
\]
The result follows by taking \( R \to \infty \) above. \hfill \Box

With the preceding lemmas, we now conclude the following.

**Proposition 3.5.** Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \(\rho\) having properties (1.2) and (1.3). Assume that \(\text{Ric} \geq -K\rho\) on \(M\) for some \(K \geq 0\). Then
\[
\hat{B}_{\rho}(x, r) \rho(y) G(x, y) dy \leq C (r + 1)
\]
for any \(p \in M\) and \(x \in B_{\rho}(p, r)\).

**Proof.** Let
\[
\alpha := \inf_{y \in B_{\rho}(p, r) \setminus B_{\rho}(x,1)} G(x, y) \quad \text{and} \quad \beta := \sup_{y \in B_{\rho}(p, r) \setminus B_{\rho}(x,1)} G(x, y).
\]
It follows from Lemma 3.4 that
\[
\int_{B_{\rho}(p, r) \setminus B_{\rho}(x,1)} \rho(y) G(x, y) dy \leq \int_{L_x(\alpha,\beta)} \rho(y) G(x, y) dy \leq c \left( \ln \frac{\beta}{\alpha} + 1 \right).
\]
However, Lemma 3.2 implies that
\[
\beta \leq e^{cr} \alpha.
\]
The proposition follows. \hfill \Box

We now turn to the region around the pole and establish an integral estimate for the Green’s function.

**Proposition 3.6.** Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \(\rho\) having properties (1.2) and (1.3). Assume that \(\text{Ric} \geq -K\rho\) on \(M\) for some \(K \geq 0\). Then
\[
\int_{B_{\rho}(x,1)} \rho(y) G(x, y) dy \leq C
\]
for all \(x \in M\).

**Proof.** Let
\[
(3.16) \quad \sigma(x) := \inf_{y \in \partial B_{\rho}(x,1)} G(x, y).
\]
Then by the maximum principle,
(3.17) \[ B_\rho(x,1) \subset L_x(\sigma(x),\infty). \]

Hence, it suffices to prove that
\[
\int_{L_x(\sigma(x),\infty)} \rho(y) G(x,y) \, dy \leq C.
\]

First, observe that
\[
(3.18) \sup_{y \in \partial B_\rho(x,r)} G(x,y) = \sup_{y \in \partial B_\rho(x,r)} G_i(x,y).
\]

Indeed, being the minimal positive Green’s function, \( G(x,y) \) is the limit of \( G_i(x,y) \), the Dirichlet Green’s function of compact exhaustion \( \Omega_i \subset M \). Obviously,
\[
\sup_{y \in \Omega_i \setminus B_\rho(x,r)} G_i(x,y) = \sup_{y \in \partial B_\rho(x,r)} G_i(x,y).
\]

After letting \( i \to \infty \), one sees that (3.18) holds true for \( G(x,y) \). In particular,
\[
(3.19) \sup_{y \in \partial B_\rho(x,r)} G(x,y) \text{ is decreasing in } r > 0.
\]

By Lemma 3.1 there exists \( C_1 > 0 \) so that
\[
(3.20) G(x,y) \leq C_1 \left( \frac{r_2}{r_1} \right)^{C_1} G(x,z)
\]
for any \( y \in \partial B_\rho(x,r) \) and \( z \in \partial B_\rho(x,s) \) for
\[
0 < r_1 \leq r, s \leq r_2 \leq 1.
\]

Hence,
\[
\sup_{y \in \partial B_\rho(x,1)} G(x,y) \leq C_1 \sigma(x).
\]

For \( C_1 \) in (3.20), let
\[
(3.21) \omega := C_1 4^{C_1}.
\]

Setting \( s = 1 \) and \( r = \frac{1}{2} \) in (3.20) we see that
\[
\sup_{y \in B_\rho(x,1) \setminus B_\rho(x,\frac{1}{2})} G(x,y) \leq C_1 2^{C_1} \sigma(x) < \omega \sigma(x).
\]

Together with (3.18), this proves that
\[
(3.22) l_x(\omega \sigma(x)) \subset B_\rho(x,\frac{1}{2}).
\]

We now prove by induction that
\[
(3.23) l_x(\omega^k \sigma(x)) \subset B_\rho(x,\frac{1}{2^k})
\]
for all \( k \geq 1 \).
Assume (3.23) holds for some $k \geq 1$. If it does not hold for $k + 1$, then there exists

$$y \in l_x (\omega^{k+1} \sigma (x)) \cap \left( B_\rho \left( x, \frac{1}{2^k} \right) \setminus B_\rho \left( x, \frac{1}{2^{k+1}} \right) \right),$$

that is, $y \in \partial B_\rho (x,r)$ for $\frac{1}{2^{k+1}} < r \leq \frac{1}{2^k}$ and $G (x,y) = \omega^{k+1} \sigma (x)$. Now (3.20) and (3.21) imply that

$$G (x,y) \leq C_1 2^{C_1} G (x,z) < \omega G (x,z)$$

or

$$G (x,z) > \omega^k \sigma (x)$$

for all $z \in B_\rho \left( x, \frac{1}{2^k} \right) \setminus B_\rho \left( x, \frac{1}{2^{k+1}} \right)$. Therefore, by the maximum principle,

$$\min_{y \in B_\rho (x,\frac{1}{2^k})} G (x,y) = \min_{y \in B_\rho (x,\frac{1}{2^k}) \setminus B_\rho (x,\frac{1}{2^{k+1}})} G (x,y) > \omega^k \sigma (x).$$

This violates the induction hypothesis that $l_x (\omega^k \sigma (x)) \subset B_\rho (x,\frac{1}{2^k})$. So (3.23) is true for any $k \geq 1$. In particular, we conclude that

$$L_x \left( \frac{1}{e} \omega^k \sigma (x), e \omega^{k+1} \sigma (x) \right) \subset B_\rho \left( x, \frac{1}{2^k-1} \right)$$

and the set $L_x \left( \frac{1}{e} \omega^k \sigma (x), e \omega^{k+1} \sigma (x) \right)$ is compact in $M$ for all $k \geq 2$.

Let

$$k_0 = \left\lceil \frac{\ln (C_0 / \delta)}{\ln 2} \right\rceil + 3.$$

Then, for all $k \geq k_0$, $\frac{1}{2^{k+1}} \leq \frac{\delta}{2^{k_0}}$ and Lemma 2.2 implies that

$$\lambda_1 \left( B_\rho \left( x, \frac{1}{2^{k-1}} \right) \right) \geq \frac{1}{C} 2^{2k} \rho (x).$$

From this and (3.25) we infer that the Poincaré inequality

$$\frac{1}{C} 2^{2k} \rho (x) \int_M \phi^2 \leq \int_M |\nabla \phi|^2$$

holds for any compactly supported function $\phi \in C^\infty_0 (L_x \left( \frac{1}{e} \omega^k \sigma (x), e \omega^{k+1} \sigma (x) \right))$ and any $k \geq k_0$. Thus, applying (3.11), we get

$$\rho (x) \int_{L_x (\omega^k \sigma (x), \omega^{k+1} \sigma (x))} G (x,y) dy \leq \frac{C}{2^{2k}} \ln \omega.$$
\[
\int_{L_x(\omega^{k_0}\sigma(x),\infty)} \rho(y) G(x,y) dy \\
= \sum_{k=k_0}^{\infty} \int_{L_x(\omega^{k}\sigma(x),\omega^{k+1}\sigma(x))} \rho(y) G(x,y) dy \\
\leq C.
\]

Note that Lemma 3.4 implies

\[
\int_{L_x(\sigma(x),\omega^{k_0}\sigma(x))} \rho(y) G(x,y) dy \leq C.
\]

Combining (3.28) and (3.29) we conclude that

\[
\int_{L_x(\sigma(x),\omega^{k_0}\sigma(x))} \rho(y) G(x,y) dy = \int_{L_x(\sigma(x),\omega^{k_0}\sigma(x))} \rho(y) G(x,y) dy \\
+ \int_{L_x(\omega^{k_0}\sigma(x),\infty)} \rho(y) G(x,y) dy \\
\leq C.
\]

This completes the proof. \(\square\)

We are now able to prove the main result of this section.

**Theorem 3.7.** Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \(\rho\) having properties (1.2) and (1.3). Assume that \(\text{Ric} \geq -K\rho\) on \(M\) for some \(K \geq 0\). Then, for any \(p, x \in M\), and \(r > 0\),

\[
\int_{B_p(p,r)} \rho(y) G(x,y) dy \leq C(r+1).
\]

**Proof.** We first remark that it suffices to prove the result for \(x \in B_p(p,r)\). Indeed, consider the function

\[
\Phi(x) = \int_{B_p(p,r)} \rho(y) G(x,y) dy.
\]

We claim that the maximum value of \(\Phi\) on \(M \setminus B_p(p,r)\) must occur on \(\partial B_p(p,r)\). This is because \(G(x,y)\) is the limit of \(G_i(x,y)\), the Dirichlet Green’s function of compact exhaustion \(\Omega_i\) of \(M\). If we let

\[
\Phi_i(x) = \int_{B_p(p,r)} \rho(y) G_i(x,y) dy,
\]

then \(\Phi_i \rightarrow \Phi\) as \(i \rightarrow \infty\). However, by the maximum principle, the maximum value of \(\Phi_i(x)\) on \(\Omega_i \setminus B_p(p,r)\) is achieved on \(\partial B_p(p,r)\). Therefore, the same is true for \(\Phi(x)\).

From now on, we assume that \(x \in B_p(p,r)\). By Proposition 3.5 and Proposition 3.6,

\[
\int_{B_p(p,r) \setminus B_p(x,1)} \rho(y) G(x,y) dy \leq C(r+1)
\]
and
\[ \int_{B_{\rho}(x,1)} \rho(y) G(x,y) \, dy \leq C. \]
Obviously, the theorem follows by combining these two estimates. \( \square \)

Let us point out that Theorem 3.7 is sharp. Indeed, for any \( \epsilon > 0 \) small enough so that \( B(x,\epsilon) \subset B_{\rho}(x,t) \), we have
\[
0 = \int_{B_{\rho}(x,t) \setminus B(x,\epsilon)} \Delta y G(x,y) \, dy \\
= \int_{\partial B_{\rho}(x,t)} \frac{\partial G}{\partial \nu}(x,\xi) \, dA(\xi) \\
- \int_{\partial B(x,\epsilon)} \frac{\partial G}{\partial r}(x,\xi) \, dA(\xi),
\]
where \( \nu \) is the unit normal of \( \partial B_{\rho}(x,t) \) with respect to \( ds^2 \).

Using the asymptotics of \( G \) near its pole, we obtain
\[
\int_{\partial B(x,\epsilon)} \frac{\partial G}{\partial r}(x,\xi) \, dA(\xi) = -1
\]
for any \( \epsilon > 0 \). So
\[
1 = - \int_{\partial B_{\rho}(x,t)} \frac{\partial G}{\partial \nu}(x,\xi) \, dA(\xi) \\
\leq \int_{\partial B_{\rho}(x,t)} |\nabla G|(x,\xi) \, dA(\xi)
\]
for any \( t > 0 \). Combining with the gradient estimate in (3.1) that
\[ |\nabla G|(x,y) \leq C \sqrt{\rho(y)} G(x,y) \]
for \( y \in M \setminus B_{\rho}(x,1) \), where the gradient is taken in variable \( y \), we conclude
\[
\int_{\partial B_{\rho}(x,t)} \sqrt{\rho(\xi)} G(x,\xi) \, dA(\xi) \geq \frac{1}{C}
\]
for all \( t \geq 1 \).

Now the co-area formula yields
\[
\int_{B_{\rho}(x,r) \setminus B_{\rho}(x,1)} \rho(y) G(x,y) \, dy \\
= \int_{1}^{r} \int_{\partial B_{\rho}(x,t)} \frac{1}{|\nabla r_{\rho}|(x,\xi)} \rho(\xi) G(x,\xi) \, dA(\xi) \, dt \\
= \int_{1}^{r} \int_{\partial B_{\rho}(x,t)} \sqrt{\rho(\xi)} G(x,\xi) \, dA(\xi) \, dt \\
\geq \frac{1}{C} (r - 1). 
\]
This shows that
\[
\int_{B_{\rho}(x,r)} \rho(y) G(x,y) \, dy \geq \frac{1}{C} (r - 1)
\]
for all \( r > 1 \), confirming the sharpness of Theorem 3.7.

The above estimate of the Green’s function leads to a volume comparison result for geodesic \( \rho \)-balls. Define

\[
V_\rho (x, r) = \int_{B_\rho(x, r)} \rho (y) \, dy.
\]

**Theorem 3.8.** Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \( \rho \) having properties (1.2) and (1.3). Assume that \( \text{Ric} \geq -K \rho \) on \( M \) for some \( K \geq 0 \). Then for all \( x \in M \),

\[
Ce^{2R} V_\rho (x, 1) \leq V_\rho (x, R) \leq e^{C(R+1)} V_\rho (x, r) \quad \text{for all} \quad 0 < r \leq 1 \leq R.
\]

**Proof.** We first prove the upper bound. Theorem 3.7 implies that

\[
(3.32) \quad \int_{B_\rho(x, t)} \rho (y) G (x, y) \, dy \leq C (t + 1)
\]

for all \( x \in M \) and \( t > 0 \). Set

\[
(3.33) \quad \sigma (x) = \inf_{y \in B_\rho(x, 1)} G (x, y) = \inf_{y \in \partial B_\rho(x, 1)} G (x, y).
\]

By Lemma 3.2,

\[
G (x, y) \geq e^{-cr_\rho(x,y)} \sigma (x)
\]

for \( y \in M \setminus B_\rho(x, 1) \). From (3.32) and (3.33) we conclude that

\[
C (t + 1) \geq \int_{B_\rho(x, t) \setminus B_\rho(x, t-1)} \rho (y) G (x, y) \, dy
\]

\[
\geq e^{-ct}\sigma (x) \int_{B_\rho(x, t) \setminus B_\rho(x, t-1)} \rho (y) \, dy
\]

for all \( t \geq 1 \). Summing over \( t \) from 1 to \( R \), we get

\[
(3.34) \quad V_\rho (x, R) \leq \frac{e^{CR}}{\sigma (x)}
\]

for all \( x \in M \) and all \( R \geq 1 \).

On the other hand, according to (3.30),

\[
\int_{\partial B_\rho(x, t)} |\nabla G| (x, \xi) \, dA (\xi) \geq 1.
\]

In view of (3.4) we obtain for all \( 0 < t \leq 1 \),

\[
(3.35) \quad 1 \leq \frac{C}{t} \int_{\partial B_\rho(x, t)} \sqrt{\rho (\xi)} G (x, \xi) \, dA (\xi).
\]
Note Lemma 3.1 implies for $0 < t \leq 1$,

$$
\sup_{y \in \partial B_\rho(x,t)} G(x,y) \leq C \left(\frac{1}{t}\right)^C \inf_{z \in \partial B_\rho(x,1)} G(x,z)
= C \left(\frac{1}{t}\right)^C \sigma(x).
$$

Plugging into (3.35) yields

$$
(3.36) \quad \int_{\partial B_\rho(x,t)} \rho(\xi) dA(\xi) \geq \frac{1}{C} \frac{\rho}{\sigma(x)},
$$

for all $0 < t \leq 1$. So for any $0 < r \leq 1$, by the co-area formula,

$$
\begin{align*}
\int_{B_\rho(x,r) \setminus B_\rho(x,\frac{r}{2})} \rho(y) dy &= \int_{\frac{r}{2}}^r \int_{\partial B_\rho(x,t)} \frac{1}{\rho(x,\xi)} \rho(\xi) dA(\xi) dt \\
&= \int_{\frac{r}{2}}^r \int_{\partial B_\rho(x,t)} \sqrt{\rho(\xi)} dA(\xi) dt \\
&\geq \frac{1}{C} \frac{r}{C^{\frac{1}{2}}} \frac{1}{\sigma(x)},
\end{align*}
$$

where in the last line we have used (3.36). Thus,

$$
(3.37) \quad \frac{1}{\sigma(x)} \leq \frac{r}{C} \mathcal{V}_\rho(x,r)
$$

for all $r \leq 1$.

Combining (3.34) and (3.37) we conclude

$$
\mathcal{V}_\rho(x,R) \leq \frac{e^{CR}}{Cr} \mathcal{V}_\rho(x,r)
$$

for any $x \in M$ and $0 < r \leq 1 \leq R$. This proves the upper bound.

We now turn to the lower bound. The same argument as in (3.34) implies that

$$
\frac{1}{C} \leq \int_{B_\rho(x,R) \setminus B_\rho(x,R-1)} \rho(y) G(x,y) dy
$$

for $R > 2$. By the Cauchy-Schwarz inequality it follows that

$$
\frac{1}{C} \leq \mathcal{V}_\rho(x,R) \int_{B_\rho(x,R) \setminus B_\rho(x,R-1)} \rho(y) G^2(x,y) dy.
$$

Therefore, combining with (3.14), we obtain

$$
(3.38) \quad \frac{1}{C} e^{2R} \leq \mathcal{V}_\rho(x,R) \int_{B_\rho(x,2) \setminus B_\rho(x,1)} \rho(y) G^2(x,y) dy.
$$

As in the proof of the upper bound, set

$$
\sigma(x) = \inf_{y \in \partial B_\rho(x,1)} G(x,y).
$$

Then Lemma 3.2 implies that
\[
\sup_{y \in B_\rho(x,2) \setminus B_\rho(x,1)} G(x,y) \leq c\sigma(x).
\]

Hence, we obtain from (3.38) that

\[
(3.39) \quad \frac{1}{C} e^{2R} \leq \sigma^2(x) V_\rho(x,2) V_\rho(x,R).
\]

Applying (3.34) for \( R = 1 \) and using the upper bound we have

\[
(3.40) \quad \sigma^2(x) V_\rho(x,2) \leq C V_\rho(x,1).
\]

Clearly, (3.40) and (3.39) imply the lower bound. \( \square \)

4. The Poisson equation

In this section, we focus on the Poisson equation and prove Theorem 1.4. We adopt the same convention that \( c \) and \( C \) denote positive constants depending on \( n, K, \delta, \) and \( A. \) We continue to denote

\[
r_\rho(x) = r_\rho(p,x).
\]

**Theorem 4.1.** Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \( \rho \) having properties (1.2) and (1.3). Assume that \( \text{Ric} \geq -K\rho \) on \( M \) for some \( K \geq 0. \) Then for any smooth function \( \varphi \) satisfying

\[
|\varphi|(x) \leq \omega(r_\rho(x)),
\]

where \( \omega(t) \) is a non-increasing function such that \( \int_0^\infty \omega(t) \, dt < \infty, \) the Poisson equation \( \Delta u = -\rho \varphi \) admits a bounded solution \( u \) on \( M \) with

\[
\sup_M |u| \leq c \left( \omega(0) + \int_0^\infty \omega(t) \, dt \right).
\]

**Proof.** We first prove that

\[
(4.1) \quad \int_M \rho(y) G(x,y) |\varphi|(y) \, dy \leq c \left( \omega(0) + \int_0^\infty \omega(t) \, dt \right)
\]

for all \( x \in M. \) Note that by Theorem 3.7 we have

\[
\int_{B_\rho(p,1)} \rho(y) G(x,y) |\varphi|(y) \, dy \leq c \sup_{B_\rho(p,1)} |\varphi| \leq c \omega(0)
\]

as \( \omega \) is non-increasing. Therefore,
\begin{equation}
\int_M \rho(y) G(x, y) |\varphi|(y) \, dy = \sum_{j=0}^{\infty} \int_{B_p(p, 2^{j+1}) \setminus B_p(p, 2^j)} \rho(y) G(x, y) |\varphi|(y) \, dy + \int_{B_p(p, 1)} \rho(y) G(x, y) |\varphi|(y) \, dy
\leq \sum_{j=0}^{\infty} \left( \int_{B_p(p, 2^{j+1}) \setminus B_p(p, 2^j)} \rho(y) G(x, y) \, dy \right) \sup_{B_p(p, 2^{j+1}) \setminus B_p(p, 2^j)} |\varphi| + c \omega(0).
\end{equation}

The hypothesis on \( \varphi \) implies

\[ \sup_{B_p(p, 2^{j+1}) \setminus B_p(p, 2^j)} |\varphi| \leq \omega(2^j) \]

and Theorem 3.7 says that

\[ \int_{B_p(p, 2^{j+1}) \setminus B_p(p, 2^j)} \rho(y) G(x, y) \, dy \leq c 2^{j-1}. \]

Using these estimates in (4.2) we obtain

\[ \int_M \rho(y) G(x, y) |\varphi|(y) \, dy \leq c \omega(0) + c \sum_{j=0}^{\infty} 2^{j-1} \omega(2^j) \leq c \omega(0) + c \sum_{j=0}^{\infty} \int_{2^{j-1}}^{2^j} \omega(t) \, dt \leq c \left( \omega(0) + \int_{0}^{\infty} \omega(t) \, dt \right). \]

This proves (4.1). As \[ \int_{0}^{\infty} \omega(t) \, dt < \infty, \] it follows that the function

\[ u(x) := \int_M \rho(y) G(x, y) \varphi(y) \, dy \]

is well defined, bounded on \( M \), and verifies

\[ \Delta u = -\rho \varphi. \]

Furthermore, we have the estimate

\[ \sup_M |u| \leq c \left( \omega(0) + \int_{0}^{\infty} \omega(t) \, dt \right). \]

This proves the theorem. \( \square \)

Our next step is to prove that the solution \( u \) in Theorem 4.1 decays to zero at infinity by assuming a uniform lower bound on \( \mathcal{V}_p(x, 1) \), that is,

\begin{equation}
\mathcal{V}_p(x, 1) = \int_{B_p(x, 1)} \rho(y) \, dy \geq \nu_0 > 0
\end{equation}

for all \( x \in M \).
We first establish a pointwise decay estimate for the Green’s function. For the rest of the section, constants $c$ and $C$ may in addition depend on $v_0$.

**Theorem 4.2.** Let $(M^n, g)$ be a complete manifold satisfying the weighted Poincaré inequality \((1.1)\) with weight $\rho$ having properties \((1.2), (1.3),\) and \((4.3)\). Assume that $\text{Ric} \geq -K\rho$ on $M$ for some $K \geq 0$. Then we have

$$G(x, z) \leq Ce^{-r_\rho(x, z)}$$

for $z \in M$ with $r_\rho(x, z) \geq 1$.

**Proof.** By \((3.14)\),

\[
\int_{B_\rho(x, r+1) \setminus B_\rho(x, r-1)} \rho(y) G^2(x, y) \, dy \
\leq Ce^{-2r} \int_{B_\rho(x,3) \setminus B_\rho(x,1)} \rho(y) G^2(x, y) \, dy
\]

for any $r \geq 4$. To estimate the right hand side of \((4.4)\), by Lemma \(3.2\) we have

\[
\sup_{y \in B_\rho(x,3) \setminus B_\rho(x,1)} G(x, y) \leq c \inf_{y \in B_\rho(x,3) \setminus B_\rho(x,1)} G(x, y).
\]

Together with Theorem \(3.7\) it implies that

\[
C \geq \int_{B_\rho(x,3) \setminus B_\rho(x,1)} \rho(y) G(x, y) \, dy
\]

\[
\geq \frac{1}{c} \left( \sup_{y \in B_\rho(x,3) \setminus B_\rho(x,1)} G(x, y) \right) \int_{B_\rho(x,3) \setminus B_\rho(x,1)} \rho(y) \, dy.
\]

Consequently,

\[
\sup_{y \in B_\rho(x,3) \setminus B_\rho(x,1)} G(x, y) \leq C \left( \int_{B_\rho(x,3) \setminus B_\rho(x,1)} \rho(y) \, dy \right)^{-1}.
\]

By \((4.6)\) and \((4.3)\) we get

\[
\int_{B_\rho(x, r+1) \setminus B_\rho(x, r-1)} \rho(y) G^2(x, y) \, dy \leq Ce^{-2r} \left( \int_{B_\rho(x,3) \setminus B_\rho(x,1)} \rho(y) \, dy \right)^{-1}.
\]

But the hypothesis \((1.3)\) implies

\[
\left( \int_{B_\rho(x,3) \setminus B_\rho(x,1)} \rho(y) \, dy \right)^{-1} \leq \frac{1}{v_0}.
\]

Therefore, we conclude

\[
\int_{B_\rho(x, r+1) \setminus B_\rho(x, r-1)} \rho(y) G^2(x, y) \, dy \leq Ce^{-2r}
\]

for any $r \geq 4$.

For $z \in \partial B_\rho(x, r)$ with $r \geq 4$, since

$B_\rho(z, 1) \subset B_\rho(x, r + 1) \setminus B_\rho(x, r - 1),$
it follows that

\[
\int_{B_\rho(z,1)} \rho(y) G^2(x, y) \, dy \leq C e^{-2r_\rho(x,z)}.
\]

Using (3.1) that

\[
|\nabla_\rho G(x,y)| \leq c
\]

for all \(y \in B_\rho(z,1)\), we have

\[
G(x,z) \leq c \inf_{y \in B_\rho(z,1)} G(x,y).
\]

Plugging into (4.8), together with the hypothesis that

\[
\mathcal{V}_\rho(z,1) \geq v_0 > 0,
\]

one concludes

\[
G(x,z) \leq C e^{-r_\rho(x,z)}
\]

for \(z \in M\) with \(r_\rho(x,z) \geq 4\). This proves the result.

We now establish the decay estimate of the solution \(u\) to the Poisson equation.

**Theorem 4.3.** Let \((M^n, g)\) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \(\rho\) having properties (1.2), (1.3), and (4.3). Assume that \(\text{Ric} \geq -K\rho\) on \(M\) for some \(K \geq 0\). Then for any function \(\varphi\) satisfying

\[
|\varphi|(x) \leq \omega(r_\rho(x)),
\]

where \(\omega(t)\) is a non-increasing function such that \(\int_0^\infty \omega(t) \, dt < \infty\), the Poisson equation \(\Delta u = -\rho \varphi\) admits a bounded solution \(u\) on \(M\) such that

\[
|u|(x) \leq C \left( \int_{r_\rho(x)}^\infty \omega(t) \, dt + \mathcal{V}_\rho(p,1) \omega(0) e^{-\frac{4}{3}r_\rho(x)} \right)
\]

for all \(x \in M\), where \(\alpha\) is a constant depending only on \(n, K\), and \(\delta, A\).

**Proof.** According to Theorem [1.3], there exists a constant \(c_1 > 0\) so that

\[
\mathcal{V}_\rho(p,t) \leq e^{c_1 t} \mathcal{V}_\rho(p,1)
\]

for all \(t \geq 1\). For \(c_1\) specified in [1.10], set

\[
\alpha = \frac{1}{2(c_1 + 1)}.
\]

For \(x \in M\), let

\[
R = r_\rho(x).
\]

We may assume \(R \geq 6\) as the theorem obviously is true for \(R \leq 6\) by adjusting the constant \(C\).
Similar to Theorem 4.1 we have
\[
\int_{M \setminus B_{p}(p, \alpha R)} \rho(y) G(x, y) |\varphi|(y) \, dy \\
= \sum_{j=0}^{\infty} \int_{B_{p}(p, 2^{j+1} \alpha R) \setminus B_{p}(p, 2^j \alpha R)} \rho(y) G(x, y) |\varphi|(y) \, dy \\
\leq \sum_{j=0}^{\infty} \left( \int_{B_{p}(p, 2^{j+1} \alpha R) \setminus B_{p}(p, 2^j \alpha R)} \rho(y) G(x, y) \, dy \right) \sup_{B_{p}(p, 2^{j+1} \alpha R) \setminus B_{p}(p, 2^j \alpha R)} |\varphi| \\
\leq C \sum_{j=0}^{\infty} (2^{j-1} \alpha R) \omega(2^j \alpha R),
\]
where in the last line we have used the decay hypothesis on $\varphi$ and Theorem 3.7.

Since $\omega(t)$ is nonincreasing, it is easy to see that
\[
\sum_{j=0}^{\infty} (2^{j-1} \alpha R) \omega(2^j \alpha R) \leq \sum_{j=0}^{\infty} \int_{2^{j-1} \alpha R}^{2^j \alpha R} \omega(t) \, dt \\
\leq \int_{2^0 \alpha R}^{\infty} \omega(t) \, dt.
\]
It follows that
\[
(4.12) \quad \int_{M \setminus B_{p}(p, \alpha R)} \rho(y) G(x, y) |\varphi|(y) \, dy \leq c \int_{2^0 \alpha R}^{\infty} \omega(t) \, dt.
\]

We now proceed to obtain an estimate on $B_{p}(p, \alpha R)$. For $y \in B_{p}(p, j+1)$, where $0 < j + 1 \leq R - 2$, we get by triangle inequality that
\[
r_{\rho}(x, y) \geq r_{\rho}(p, x) - r_{\rho}(p, y) \\
\geq R - (j + 1).
\]
Hence, by Theorem 4.2,
\[
G(x, y) \leq c e^{-(R-j)}
\]
for all $y \in B_{p}(p, j+1)$, where $0 < j + 1 \leq R - 2$.

Furthermore, by (1.10),
\[
\mathcal{V}_{\rho}(p, j + 1) \leq e^{c_1(j+1)} \mathcal{V}_{\rho}(p, 1)
\]
for any $j \geq 0$. Combining these estimates together, we get
\[
(4.13) \quad \int_{B_{p}(p, j+1) \setminus B_{p}(p, j)} \rho(y) G(x, y) \, dy \leq c e^{-(R-(c_1+1)j)} \mathcal{V}_{\rho}(p, 1)
\]
for all $0 \leq j \leq R - 3$. 

Since $\alpha R \leq R - 3$, by (4.13) it follows that
\[
\int_{B_{\rho}(p,\alpha R)} \rho (y) G(x, y) \, |\varphi| (y) \, dy \\
\leq \sum_{j=0}^{[\alpha R]} \int_{B_{\rho}(p,j+1) \setminus B_{\rho}(p,j)} \rho (y) G(x, y) \, |\varphi| (y) \, dy \\
\leq c \mathcal{V}_\rho (p, 1) \sum_{j=0}^{[\alpha R]} e^{(c_1+1)j} \sup_{B_{\rho}(p,j+1) \setminus B_{\rho}(p,j)} |\varphi| \\
\leq c \mathcal{V}_\rho (p, 1) \omega (0) e^{-R(1-(c_1+1)\alpha)} \\
= c \mathcal{V}_\rho (p, 1) \omega (0) e^{-\frac{1}{2} R},
\]
where in the last line we have used that $\alpha = \frac{1}{2 (c_1+1)}$. Combining with (4.12) we arrive at
\[
\int_M \rho (y) G(x, y) \, |\varphi| (y) \, dy \leq c \int_0^\infty \omega (t) \, dt + c \mathcal{V}_\rho (p, 1) \omega (0) e^{-\frac{1}{2} R}.
\]
This proves the theorem. \(\square\)

Let us note that Theorem 1.4 follows from Theorems 4.1 and 4.3. Indeed, in the case that the function $\varphi$ decays as
\[
|\varphi| (x) \leq c (1 + r_\rho (x))^{-k}
\]
for some $k > 1$ and
\[
\mathcal{V}_\rho (x, 1) \geq v_0 > 0
\]
holds for all $x \in M$, Theorem 1.3 readily implies that the solution $u$ satisfies
\[
|u| (x) \leq C (k) (1 + r (x))^{-k+1}
\]
as claimed in Theorem 1.4.

5. Applications

In this section, we discuss some applications of the Poisson equation and prove Theorem 1.6. We continue to assume that $(M, g)$ is a complete manifold satisfying the weighted Poincaré inequality (1.1), together with (1.2) and (1.3). Furthermore, we assume that there exists $v_0 > 0$ such that the weighted volume
\[
(5.1) \quad \mathcal{V}_\rho (x, 1) = \int_{B_{\rho}(x,1)} \rho (y) \, dy \geq v_0 > 0
\]
for all $x \in M$. In the following, unless otherwise specified, the constants $c$ and $C$ depend only on $n, K, \delta, A$ and $v_0$.

We begin with a Liouville type result.

**Theorem 5.1.** Let $(M^n, g)$ be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight $\rho$ having properties (1.2), (1.3), and (5.1), and $\text{Ric} \geq -K \rho$ for some constant $K \geq 0$. Let $\eta \geq 0$ be a $C^1$ function satisfying
\[
\eta \Delta \eta \geq -\zeta \rho \eta^2 + |\nabla \eta|^2
\]
for some positive continuous function $\zeta(x)$ which converges to zero at infinity. If there exist $\varepsilon > 0$ and $\Lambda > 0$ such that

$$(5.2) \quad \eta(x) \leq \Lambda e^{-\varepsilon r_{\rho}(x)}$$
on M, then $\eta = 0$ on $M$.

**Proof.** We assume by contradiction that $\eta$ is not identically zero. We first normalize $\eta$ by defining

$$(5.3) \quad h = \frac{1}{\Lambda e \eta}.$$ Then 

$$h \leq e^{-\varepsilon r_{\rho} - 1} \text{ on } M.$$ As $h$ satisfies 

$$\Delta h \geq -\zeta \rho h + \frac{\left|\nabla h\right|^{2}}{h}$$

at all points where $h > 0$, it is easy to see that 

$$(5.4) \quad \Delta \ln h \geq -\zeta \rho$$

whenever $h > 0$. In addition, we have 

$$(5.5) \quad -\ln h \geq 1 + \varepsilon r_{\rho} \text{ on } M.$$ Denote by 

$$(5.6) \quad v = \frac{1}{(-\ln h)},$$ 

where we set $v = 0$ whenever $h = 0$. Hence, $v \in C^{0}(M)$.

Computing directly, we have 

$$\Delta v = (\Delta \ln h) v^{2} + 2 \left|\nabla \ln h\right|^{2} v^{3}.$$ Hence, by (5.4) $v$ satisfies 

$$(5.7) \quad \Delta v \geq -\zeta \rho v^{2}$$

whenever $v > 0$. Also, by (5.3), 

$$(5.8) \quad 0 \leq v \leq \frac{1}{1 + \varepsilon r_{\rho}} \text{ on } M.$$ Define continuous function 

$$(5.9) \quad \varphi = \zeta v^{2} \text{ on } M$$ and let 

$$(5.10) \quad \omega(t) = \frac{1}{(1 + \varepsilon t)^{2}} \sup_{M \setminus B_{\rho}(p,t)} \zeta.$$ Clearly, $\omega$ is non-increasing and $\int_{0}^{\infty} \omega(t) \, dt < \infty$. Furthermore, (5.8) implies that 

$$|\varphi(x)| \leq \omega(r_{\rho}(x)) \text{ on } M.$$ By Theorem 4.3 the Poisson equation 

$$(5.11) \quad \Delta u = -\rho \varphi$$
admits a bounded positive solution $u > 0$ such that

$$0 < u(x) \leq C \left( \int_{\alpha r_\rho(x)}^{\infty} \omega(t) \, dt + \mathcal{V}_\rho(p, 1, 0) e^{-\frac{1}{2} r_\rho(x)} \right)$$

for some $0 < \alpha < 1$. Since $\phi$ is continuous, we have $u \in W^{2,p}_{\text{loc}}(M)$ for any $p$.

By (5.10) we have that

$$0 < u(x) \leq \frac{C_{1}}{1 + \alpha \varepsilon r_\rho(x)} \sup_{M \setminus B(\rho, \alpha r_\rho(x))} \zeta + C \mathcal{V}_\rho(p, 1) e^{-\frac{1}{2} r_\rho(x) \sup_{M} \zeta}.$$

As $\zeta \to 0$ at infinity we conclude that for any $\sigma > 0$ there exists $R_0 > 0$ such that

$$(5.12) \quad u(x) \leq \frac{1}{\sigma r_\rho(x)}$$

for all $x \in M \setminus B(\rho, R_0)$.

We claim that

$$(5.13) \quad v \leq u \quad \text{on} \ M.$$

Suppose by contradiction that (5.13) is not true. Since by (5.8) and (5.12) both $u$ and $v$ approach 0 at infinity, the function $v - u$ must achieve its maximum at some point $x_0 \in M$, where in particular $v(x_0) > 0$. Observe that by (5.9) and (5.11) we have $\Delta u = -\zeta \rho v^2$, whereas by (5.7) we have $\Delta v \geq -\zeta \rho v^2$ at any point where $v > 0$. Then $v - u \in W^{1,2}_{\text{loc}}(M)$ is subharmonic in a neighborhood of $x_0$ and achieves its maximum at $x_0$. The strong maximum principle implies that $v - u$ is in fact constant on $M$. Obviously, the constant must be 0. This contradiction implies that (5.13) is true.

In view of (5.12) and (5.13) we have proved that for any large $\sigma > 0$, there exists $R_0 > 0$ sufficiently large such that

$$(5.14) \quad v(x) \leq \frac{1}{\sigma r_\rho(x)} \quad \text{for all} \ x \in M \setminus B(\rho, R_0).$$

We now follow the proof of Theorem 4.4 in [26] and show that $v$ decays faster than any polynomial order in the $\rho$-distance. This will be done by iterating the previous argument.

First, let us note the following fact. Define

$$|\zeta|_{\infty} := \sup_{M} \zeta.$$

Then (5.7) implies that

$$(5.15) \quad \Delta v \geq -|\zeta|_{\infty} \rho v^2$$

whenever $v > 0$. Assume that

$$v(x) \leq \theta \left( r_\rho(x) \right)$$

for some decreasing function $\theta(t)$ such that $\int_{0}^{\infty} \theta^2(t) \, dt < \infty$. Then there exists $0 < \alpha < 1$ and $\Upsilon > 0$, independent of $v$ or $\theta$, such that

$$(5.16) \quad v(x) \leq \Upsilon \left( \int_{\alpha r_\rho(x)}^{\infty} \theta^2(t) \, dt + e^{-\frac{1}{2} r_\rho(x)} \theta^2(0) \right)$$
for all $x \in M$.

Indeed, (5.16) follows in the same manner as (5.14). Define the continuous function

$$\varphi (x) = |\zeta|_{\infty} v^2$$

and note that

$$0 \leq \varphi (x) \leq \omega (r_{\rho} (x)),$$

where

$$\omega (t) = |\zeta|_{\infty} \theta^2 (t).$$

By Theorem 4.3 there exists a bounded solution $u \in W^{2,p}_{\text{loc}} (M)$ of

$$\Delta u = -\rho \varphi$$

such that

$$0 < u (x) \leq C \left( \int_{\alpha r_{\rho} (x)}^{\infty} \omega (t) \, dt + V_{\rho} (p, 1) \omega (0) e^{-\frac{1}{\alpha} r_{\rho} (x)} \right)$$

for some $0 < \alpha < 1$. Using that $\omega (t) = |\zeta|_{\infty} \theta^2 (t)$ and taking

$$\Upsilon := C |\zeta|_{\infty} \max \{1, V_{\rho} (p, 1)\},$$

we have

$$0 < u (x) \leq \Upsilon \left( \int_{\alpha r_{\rho} (x)}^{\infty} \theta^2 (t) \, dt + e^{-\frac{1}{\alpha} r_{\rho} (x)} \theta^2 (0) \right)$$

on $M$.

By (5.15) and (5.17) the function $v - u \in W^{1,2}_{\text{loc}} (M)$ is subharmonic and converges to zero at infinity. Using the maximum principle we obtain $v \leq u$ on $M$, thus proving (5.16).

Fix $b > 0$ small enough, depending only on $\alpha$ and $\Upsilon$ in (5.16), to be specified later. Note that by (5.14), there exists $B_0 > 0$ so that

$$v (x) \leq \frac{b^6}{\alpha^2 r_{\rho} (x) + 1} + B_0^2 e^{-\alpha^2}$$

on $M$.

We prove by induction on $m \geq 2$ that

$$v (x) \leq \frac{b^{2m+1} + B_0^2 e^{-\alpha^2}}{\alpha^2 r_{\rho} (x) + 1} + B^{2m-1} e^{-\alpha^2 r_{\rho} (x)}$$

on $M$, where $B$ is a large enough constant depending only on $\alpha$, $\Upsilon$ and $B_0$.

Clearly, (5.19) holds for $m = 2$ from (5.18). We now assume (5.19) holds for $m \geq 2$ and prove

$$v (x) \leq \frac{b^{2m+1} + B^{2m+1}}{\alpha^2 r_{\rho} (x) + 1} + B^{2m+1} e^{-\alpha^2}$$

on $M$.

By the induction hypothesis we have $v (x) \leq \theta (r_{\rho} (x))$, where

$$\theta (t) := \frac{b^{2m+1}}{\alpha^2 t + 1} + B^{2m-1} e^{-\alpha^2 t}.$$

By (5.16) we obtain that

$$v (x) \leq \Upsilon \left( \int_{\alpha r_{\rho} (x)}^{\infty} \theta^2 (t) \, dt + e^{-\frac{1}{\alpha} r_{\rho} (x)} \theta^2 (0) \right).$$
Obviously,

\[ (5.22) \quad \theta^2(t) \leq \frac{2b^{2m+1+2m}}{(\alpha^m t + 1)^2} + 2B^{2m+1-2m}e^{-2\alpha^m t}. \]

It follows that

\[ (5.23) \quad \int_{\alpha r_\rho(x)}^{\infty} \theta^2(t) \, dt \leq \frac{2}{\alpha^m} \frac{b^{2m+1+2m}}{\alpha^{m+1} r_\rho(x) + 1} + \frac{1}{\alpha^m} B^{2m+1-2m}e^{-\alpha^{m+1} r_\rho(x)}. \]

Furthermore, we have by (5.22) that

\[ (5.24) \quad e^{-\frac{1}{2} r_\rho(x)} \theta^2(0) \leq \frac{2}{\alpha^m} \left( b^{2m+1+2m} + B^{2m+1-2m} \right) e^{-\frac{1}{2} r_\rho(x)} \leq \frac{1}{\alpha^m} B^{2m+1-2m}e^{-\alpha^{m+1} r_\rho(x)}. \]

Plugging (5.23) and (5.24) into (5.21) yields

\[ (5.25) \quad v(x) \leq \frac{2\Upsilon}{\alpha^m} \frac{b^{2m+1+2m}}{\alpha^{m+1} r_\rho(x) + 1} + \frac{2\Upsilon}{\alpha^m} B^{2m+1-2m}e^{-\alpha^{m+1} r_\rho(x)} \]

\[ = \left( \frac{2\Upsilon}{\alpha^2 b} \right) \frac{b}{\alpha} \frac{m-2}{m} \frac{b^{2m+1+(m+1)}}{\alpha^{m+1} r_\rho(x) + 1} \]

\[ + \left( \frac{2\Upsilon}{\alpha^2 B} \right) \left( \frac{1}{\alpha B} \right)^{m-2} B^{2m+1-(m+1)}e^{-\alpha^{m+1} r_\rho(x)}. \]

Now take \( b \) sufficiently small so that \( \frac{b}{\alpha} \leq 1 \) and \( \frac{2\Upsilon}{\alpha^2 b} \leq 1 \), and \( B \) sufficiently large so that \( \frac{1}{\alpha B} \leq 1 \) and \( \frac{2\Upsilon}{\alpha^2 B} \leq 1 \). Since \( m \geq 2 \), it follows by (5.23) that

\[ v(x) \leq \frac{b^{2m+1+(m+1)}}{\alpha^{m+1} r_\rho(x) + 1} + B^{2m+1-(m+1)}e^{-\alpha^{m+1} r_\rho(x)}. \]

This proves (5.20). Hence,

\[ (5.26) \quad v(x) \leq \frac{b^{2m+m}}{\alpha^{m+1} r_\rho(x) + 1} + B^{2m-m}e^{-\alpha^m r_\rho(x)} \]

for all \( m \geq 2 \).

For \( x \in M \) with \( r_\rho(x) \) large, apply (5.20) by setting

\[ m := \left\lceil \frac{\ln r_\rho(x)}{2 \ln (2\alpha^{-1})} \right\rceil, \]

where \( \lceil \cdot \rceil \) denotes the greatest integer function. It is not difficult to conclude that there exists constant \( a > 0 \) such that

\[ (5.27) \quad v(x) \leq Ce^{-r_\rho^a(x)} \text{ on } M. \]

We now complete the proof of the theorem. By (5.6) we have that

\[ (5.28) \quad - \ln h \geq \frac{1}{C} e^{r_\rho(x)} \text{ on } M \]
and satisfies
\[ \Delta ( - \ln h ) \leq \zeta \rho . \]

Consider the function
\[ f (x) = \ln ( - \ln h ) . \]

Then it satisfies
\[ (5.29) \quad \Delta f \leq \frac{\zeta \rho}{( - \ln h )} \]
whenever \( h > 0 \). Moreover, from (5.28), \( f \) is bounded below by
\[ (5.30) \quad f (x) \geq r^a \rho (x) - C \quad \text{on } M . \]

Define
\[ \phi (x) = \frac{\zeta}{( - \ln h )}, \]
where \( \phi \) is continuously extended as \( \phi = 0 \) at points where \( h = 0 \). By Theorem 4.3 and (5.28) we can solve the Poisson equation
\[ (5.31) \quad \Delta u = - \rho \phi \]
and obtain a solution \( u \in W^{2,p}_{\text{loc}} ( M ) \) that decays to zero at infinity.

According to (5.30), the function \( f + u \) achieves its minimum at some point \( x_0 \in M \). Then \( h (x_0) > 0 \). So by (5.29) and (5.31), \( f + u \in W^{1,2}_{\text{loc}} ( M ) \) satisfies
\[ \Delta ( f + u ) \leq 0 \]
in a neighborhood of \( x_0 \). By the maximum principle, this implies that \( f + u \) is constant, which is a contradiction.

Hence \( h \), as well as \( \eta \), must be identically zero on \( M \). \( \square \)

Let us point out that the hypothesis (5.2) on \( \eta \) is necessary and optimal. Indeed, consider
\[ \eta (x) = e^{-\ln^a (|x|^2 + e)}, \]
where \( 0 < a < 1 \) is fixed. It can be checked directly that
\[ \Delta \eta - \frac{[\nabla \eta]^2}{\eta} = \left( -\Delta \ln^a \left( |x|^2 + e \right) \right) \eta \]
\[\geq -a \frac{\Delta |x|^2}{( |x|^2 + e ) \ln^{1-a} \left( |x|^2 + e \right)} \eta \]
\[= - \frac{2na}{( |x|^2 + e ) \ln^{1-a} \left( |x|^2 + e \right)} \eta . \]

Now \( \mathbb{R}^n \) satisfies weighted Poincaré inequality with weight \( \rho(x) = \frac{(n-2)^2 \ln 4}{4 |x|^2} \). So \( \eta \) satisfies
\[ \Delta \eta \geq - \zeta \rho \eta + \frac{[\nabla \eta]^2}{\eta} \]
with
\[ \zeta (x) = \frac{c (n, a)}{\left( r^a (x) + 1 \right)^{1-a}} . \]
However, \( \eta \) violates the hypothesis \((5.2)\) as
\[
e^{-2c(n)(r_\rho(x)+1)} \leq \eta(x) \leq e^{-c(n)((r_\rho(x))+1)}.
\]

Theorem 5.1 leads to the following vanishing result for holomorphic maps.

**Theorem 5.2.** Let \((M^n, g)\) be a complete Kähler manifold satisfying the weighted Poincaré inequality \((1.1)\) with weight \(\rho\) having properties \((1.2), (1.3), (5.1)\) and \(\rho \leq C\). Assume that the Ricci curvature has lower bound \(\text{Ric} \geq -\zeta \rho\) for some function \(\zeta(x) > 0\) that converges to zero at infinity. Then any finite energy holomorphic map \(F : M \to N\), where \(N\) is a complex Hermitian manifold of non-positive bisectional curvature, is identically constant.

**Proof.** It is well known (see e.g. Theorem 1.24 in [27]) that the differential \(\eta = |dF|\) satisfies
\[
(5.32) \quad \eta \Delta \eta \geq -\zeta \rho \eta^2 + |\nabla \eta|^2.
\]
To be in the context of Theorem 5.1 we first show that \(\eta\) decays exponentially fast in the \(\rho\)-distance based on the assumption that \(\int_M \eta^2 < \infty\). Since \(\zeta\) converges to zero at infinity, by \((5.32)\) there exists \(R_0 > 0\) so that
\[
\Delta \eta \geq -\frac{1}{2} \rho \eta \quad \text{on } M \setminus B_\rho(p, R_0).
\]
Note that since \(\rho \leq C\), we have
\[
(5.33) \quad \int_M \rho \eta^2 < \infty.
\]
Hence, applying Theorem 2.1 in [19] we conclude that
\[
\int_{M \setminus B_\rho(p, r)} \rho \eta^2 \leq C e^{-r} \int_{B_\rho(p, R_0)} \rho \eta^2
\]
for \(r \geq 2R_0\).

Consequently, there exists \(\Lambda > 0\) so that
\[
(5.34) \quad \int_{B_\rho(x, 1)} \rho \eta^2 \leq \Lambda e^{-\tau_\rho(x)}
\]
for all \(x \in M\). In fact, we may take \(\Lambda = C \int_M \rho \eta^2\).

We now use DeGiorgi-Nash-Moser iteration to obtain a pointwise estimate.

For \(\delta\) in \((1.3)\) and \(C_0\) in Proposition 2.1 fix
\[
(5.35) \quad r_0 = \frac{\delta}{2C_0}.
\]
Multiply \((5.32)\) with \(\eta^{p-2} \phi^2\), where \(p \geq 2\) and \(\phi = \phi(r_\rho(x, \cdot))\) is a cut-off function with support in \(B_\rho(x, r_0)\). We have
\begin{align}
(5.36) \quad \int_M \zeta \rho^p \phi^2 & \geq - \int_M \eta^{p-1} \phi^2 \Delta \eta \\
& = (p-1) \int_M |\nabla \eta|^2 \eta^{p-2} \phi^2 - 2 \int_M \langle \nabla \phi, \nabla \eta \rangle \eta^{p-1} \phi \\
& \geq \left( p - \frac{3}{2} \right) \int_M |\nabla \eta|^2 \eta^{p-2} \phi^2 - 2 \int_M \eta^p |\nabla \phi|^2 \\
& \geq \frac{2p-3}{p^2} \int_M \left| \nabla \left( \eta^2 \phi^2 \right) \right|^2 - 3 \int_M \eta^p |\nabla \phi|^2.
\end{align}

Using the Sobolev inequality from Lemma 2.2 for $B_\rho(x, r_0)$ we get that

\begin{align}
(5.37) \quad \int_{B_\rho(x, r_0)} \left| \nabla \left( \eta^2 \phi \right) \right|^2 \\
& \geq \frac{1}{C \rho(x)} V(B_\rho(x, r_0)) \frac{\eta^p}{\eta^{n-2} \phi^{\frac{2n}{n-2}}} \frac{n-2}{n} \\
& - C \rho(x) \int_{B_\rho(x, r_0)} \eta^p \phi^2.
\end{align}

Plugging (5.37) into (5.36) and noting that

$$
|\nabla \phi|^2 = (\phi')^2 |\nabla r_\rho(x, \cdot)|^2 = \rho(x) (\phi')^2
$$

and

\begin{equation}
(5.38) \quad \sup_{B_\rho(x, r_0)} \rho \leq C \inf_{B_\rho(x, r_0)} \rho
\end{equation}

by Proposition 2.1 we obtain

$$
V(B_\rho(x, r_0))^{\frac{n}{n-2}} \left( \int_{B_\rho(x, r_0)} \eta^{\frac{n}{n-2} \phi^{\frac{2n}{n-2}}} \right)^{\frac{n-2}{n}} \leq C \int_{B_\rho(x, r_0)} \eta^p \left( \phi^2 + (\phi')^2 \right).
$$

The standard Moser iteration then gives

$$
\eta^2(x) \leq \frac{C}{V(B_\rho(x, r_0))} \int_{B_\rho(x, r_0)} \eta^2.
$$

Together with (5.38), this yields

\begin{equation}
(5.39) \quad \eta^2(x) \leq \frac{1}{V_\rho(x, r_0)} \int_{B_\rho(x, r_0)} \rho \eta^2.
\end{equation}

According to Theorem 3.8 and (5.11) we have

\begin{equation}
V_\rho(x, r_0) \geq \frac{1}{C} v_0 > 0
\end{equation}

for all $x \in M$. Then (5.39) and (5.34) imply that

\begin{equation}
(5.40) \quad \eta(x) \leq \Lambda e^{-\frac{1}{2} r_\rho(x)}
\end{equation}

for all $x \in M$, where $\Lambda$ is a constant depending on the total energy of $\eta$ on $M$.

Applying Theorem 5.1 we conclude $\eta = 0$ and $F$ is a constant map. \qed
We point out that in [23] Li and Yau proved a vanishing theorem for holomorphic maps \( F : M \to N \), where \( M \) is assumed to be non-parabolic and its Ricci curvature is bounded from below by \( \text{Ric} \geq -\bar{\rho} \) with \( \bar{\rho} \) being an integrable function. An alternative proof of this result using the Poisson equation is given as Theorem 8.6 in [27].

As a consequence of Theorem 5.2 we obtain the following structural result.

**Corollary 5.3.** Let \( (M^n, g) \) be a complete manifold satisfying the weighted Poincaré inequality (1.1) with weight \( \rho \) having properties (1.2), (1.3), (5.1) and \( \rho \leq C \). Assume that the Ricci curvature is bounded by \( \text{Ric} \geq -\zeta\rho \) for some function \( \zeta(x) > 0 \) that converges to zero at infinity. Then \( M \) has only one end.

**Proof.** Let us assume by contradiction that \( M \) has at least two ends. We denote by \( E \) a nonparabolic end and let \( F = M \setminus E \). Note that \( E \) exists because \( M \) is nonparabolic. We claim that \( F \) is nonparabolic as well. Indeed, if \( F \) were parabolic, then by [19],

\[
\int_{(M \setminus B_\delta(p, R)) \cap F} \rho(y) \, dy \leq C e^{-2R}
\]

for all \( R \). This obviously contradicts with (5.1). Hence, both \( E \) and \( F \) are non-parabolic ends. By Li-Tam [16], there exists a harmonic function \( w \) on \( M \) with the following properties.

\[
\begin{align*}
\int_M |\nabla w|^2 &< \infty \\
\limsup_{F} w &\leq 1 \\
\liminf_{E} w &\geq 0.
\end{align*}
\]

Such \( w \) is necessarily pluriharmonic according to [13]. Therefore, Theorem 5.2 is applicable to \( w \) and \( w \) must be constant. This shows that \( M \) must be connected at infinity. \( \Box \)

6. **The special case of constant weight**

In this section we specialize to the case when \( \rho = \lambda_1(\Delta) \) and present an alternative approach from [26] to Theorem 1.2. The argument relies on the heat kernel estimates and is more streamlined. Since it avoids the level set consideration, such an approach may be applicable to more general setting. In the following, \( C \) denotes a constant depending only on \( n, K \) and \( \lambda_1(\Delta) \). Denote by \( H(x, y, t) \) the minimal heat kernel of \( M \).

Let us restate Theorem 1.2 below.

**Theorem 6.1.** Let \( (M^n, g) \) be a Riemannian manifold with positive spectrum \( \lambda_1(\Delta) > 0 \) and with Ricci curvature \( \text{Ric} \geq -K \) for some constant \( K \geq 0 \). Then there exists \( C > 0 \) such that for any \( p, x \in M \) and any \( r > 0 \),

\[
\int_{B(p, r)} G(x, y) \, dy \leq C (r + 1).
\]

**Proof.** As noted in the proof of Theorem 3.7 it suffices to prove the result for \( x \in B(p, r) \).

It is well known (see e.g. Chapter 10 in [11]) that
\[ e^{\lambda_1(\Delta)t} H(x, x, t) \] is nonincreasing in \( t > 0 \).

Therefore,

\[ H(x, x, t) \leq e^{-\lambda_1(\Delta)(t-1)} H(x, x, 1) \] for all \( t \geq 1 \). Using the semi-group property and the Cauchy-Schwarz inequality, we get

\[
H(x, y, 2t) = \int_M H(x, z, t) H(y, z, t) \, dz \\
\leq \left( \int_M H(x, z, t)^2 \, dz \right)^{1/2} \left( \int_M H(y, z, t)^2 \, dz \right)^{1/2} \\
= H(x, x, 2t)^{1/2} H(y, y, 2t)^{1/2}.
\]

Together with (6.1), this proves that

\[ H(x, y, t) \leq e^{-\lambda_1(\Delta)(t-1)} H(x, x, 1)^{1/2} H(y, y, 1)^{1/2} \]
for all \( x, y \in M \) and all \( t \geq 1 \).

By Li-Yau [22] we have for all \( x \in M \)

\[ H(x, x, 1) \leq \frac{C}{V(x, 1)}, \]

where \( V(x, 1) = \text{Vol}(B(x, 1)) \). However, by the Bishop-Gromov volume comparison theorem, for any \( x \in B(p, r) \),

\[
\frac{V(p, r)}{V(x, 1)} \leq \frac{V(x, 2r)}{V(x, 1)} \leq e^{Cr}.
\]

Hence, if both \( x, y \in B(p, r) \), then we get from (6.3) that

\[ H(x, x, 1)^{1/2} H(y, y, 1)^{1/2} \leq e^{Cr} V(p, r)^{-1}. \]
Plugging this into (6.2) we conclude that

\[ H(x, y, t) \leq C e^{-\lambda_1(\Delta)t + Cr} V(p, r)^{-1} \]
for all \( x, y \in B(p, r) \) and \( t \geq 1 \). This immediately implies that for some \( C_1 > 0 \),

\[ \int_{B(p, r)} H(x, y, t) \, dy \leq C_1 e^{-\lambda_1(\Delta)t + C_1 r} \]
for any \( x \in B(p, r) \) and \( t \geq 1 \). In particular, for \( t \geq \Lambda \) with

\[ \Lambda = \max \left\{ 1, \frac{2C_1 r}{\lambda_1(\Delta)} \right\}, \]
one has

\[ \int_{B(p, r)} H(x, y, t) \, dy \leq C e^{-\frac{t}{\lambda_1(\Delta)}} \]
for all $x \in B(p,r)$. We integrate this inequality from $t = \Lambda$ to $t = \infty$ and use Fubini’s theorem to conclude that

$$
\int_{B(p,r)} \left( \int_{\Lambda}^{\infty} H(x,y,t) \, dt \right) \, dy \leq C
$$

for any $x \in B(p,r)$. On the other hand, it is well know that the minimal heat kernel satisfies

$$
\int_{M} H(x,y,t) \, dy \leq 1
$$

for all $x \in M$. It implies that

$$
\int_{B(p,r)} \left( \int_{0}^{\Lambda} H(x,y,t) \, dt \right) \, dy = \int_{0}^{\Lambda} \left( \int_{B(p,r)} H(x,y,t) \, dy \right) \, dt \leq \Lambda.
$$

In view of the choice of $\Lambda$ from (6.5) we conclude that

$$
\int_{B(p,r)} \left( \int_{0}^{\Lambda} H(x,y,t) \, dt \right) \, dy \leq C (r + 1)
$$

for all $x \in B(p,r)$. Combining (6.6) and (6.7), we obtain that

$$
\int_{B(p,r)} \left( \int_{0}^{\infty} H(x,y,t) \, dt \right) \, dy \leq C (r + 1)
$$

for all $x \in B(p,r)$. Since

$$
G(x,y) = \int_{0}^{\infty} H(x,y,t) \, dt,
$$

this shows

$$
\int_{B(p,r)} G(x,y) \, dy \leq C (r + 1)
$$

for all $x \in B(p,r)$. The theorem is proved. \qed

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