Point Process Analysis of Vortices in a Periodic Box

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Abstract

The motion of assemblies of point vortices in a periodic parallelogram can be described by the complex position $z_j(t)$ whose time derivative is given by the sum of the complex velocities induced by other vortices and the solid rotation centered at $z_j$. A numerical simulation up to 100 vortices in a square periodic box is performed with various initial conditions, including single and double rows, uniform spacing, checkered pattern, and complete spatial randomness. Point process theory in spatial ecology is applied in order to quantify clustering of the distribution of vortices. In many cases, clustering of the distribution persists after a long time if the initial condition is clustered. In the case of positive and negative vortices with the same absolute value of strength, the $L$ function becomes positive for both types of vortices. Scattering or recoupling of pairs of vortices by a third vortex is remarkable.
1 Introduction

A motion of point vortices (PVs) is one of the simple and fundamental issues in fluid dynamics [3]. There have been a number of numerical studies of PVs, whose motion is bounded by a circular wall. Recently, stable triangular vortex lattices have been observed in rotating Bose-Einstein condensates of Na atoms [4] in which the circulation of vortices is quantized with the same strength and the lattice contains over 100 vortices. Many experimental reports of superfluid vortex lattices increased in these several years show triangular patterns [4, 5, 6], which can be compared with the stability analysis of vortex lattices by Tkachenko [7, 8] in 1966. Conformal theory of irrotational flows and complex functions are invoked and the role of the Weierstrass zeta function is crucial.

This paper supplements the Letter [1] by showing the spatial distribution of PVs for various type of initial conditions. Tkachenko’s work 40 years ago and recent strong interests by physicists on vortex lattices in Bose-Einstein condensates stimulate the author to consider the numerical study of motions of PVs with periodic boundary conditions.

In order to quantify clustering of the point distribution, we apply point process theory used in spatial ecology. The computed $K$ and $L$ functions judge the distribution to be clustered or not. This method can be applied to other systems which include information of the point distribution in two or more dimensions.

2 Velocity field by periodic point vortices

The velocity field by a single periodic PV of strength $\kappa = 2\pi$ is given by [7, 9, 1]

$$w(z) = u + iv = i\zeta(z) - i\Omega z,$$

where $z = x + iy$ and the overbar denotes the complex conjugate. The corresponding streamfunction is denoted by

$$\psi = -\text{Re} \ln \sigma(z) + \Omega|z|^2/2. \quad (2)$$

The Weierstrass zeta and sigma functions ($\zeta$ and $\sigma$) contain two complex parameters $\omega_1$ and $\omega_2$, which are half periods on the complex plane. The second term including $\Omega = \pi/|4\text{Im}(\omega_1\omega_2)|$ expresses the solid rotation in order to cancel the circulation on sides of the parallelogram induced by the PV.

The equations of motions for PVs of strength $2\pi\mu_i$, $\mu_i = \pm 1$, can be described by

$$\dot{z}_i = \sum_{j \neq i} \mu_j w(z_i - z_j). \quad (3)$$
For a single type of PVs with the total number $N$, the index $i$ is $i = 1, \cdots, N$. For positive and negative PVs with the same strength, the index is $i = 1, \cdots, N_+, N_+ + 1, \cdots, N(= N_+ + N_-)$. There are three known conserved quantities; the Hamiltonian and two components of the linear impulse \[\text{T}\].

3 K and L functions

The $K$ function used in spatial ecology \[\text{[10]}\] is defined by

$$K(r) = (\lambda N)^{-1} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \theta(r - |\vec{x}_i - \vec{x}_j|),$$

(4)

where $N$ is the total number of points, $\lambda = N/S$ is the number density, $S$ is the area, $\vec{x}_j$ is the position of the $j$th points, and $\theta(r)$ is the step function. A function for the edge correction on the right hand side of (4), which usually appears in data analysis of spatial ecology, is unnecessary in the present periodic case. The $L$ function is given by

$$L(r) = \sqrt{K(r)/\pi} - r.$$  

(5)

If the distribution of points is complete spatial randomness (CSR), then $K = \pi r^2$ and $L = 0$. If the points are clustered, $L > 0$. The uniformly spaced (US) distribution leads to $L < 0$. It is generally difficult to distinguish between CSR and US distributions only if we see raw data of the point distribution. However, the $L$ function gives judgment on the degree of clustering clearly.

We note that the value of $L(r)$ depends on $r$. For examples, the checkered pattern gives a positive value of $L(r)$ for $2r \simeq l_s$, where $l_s$ is the size of the smallest square, but $L(r) < 0$ for $4r \simeq l_s$.

4 Numerical simulation of a single type of PVs

The system (3) with $N$ up to 100 is simulated numerically by using Mathematica 5.2. The textbook by Trott (2006) \[\text{[11]}\] gives a number of numerical codes for Mathematica programming, including samples of point vortices with other boundary conditions.

We first consider the following four initial conditions: (I) an infinite row that is a discrete model of the vortex sheet, (II) PVs located randomly in checkered patterns, (III) PVs located randomly in the $10 \times 10$ subsquares, and (IV) CSR in the unit square. The typical initial and final distributions of PVs are shown in Figures 1-4. The corresponding $L$ functions can be found in \[\text{T}\].
Figure 1: (I) The positions of PVs at $t = 0.002$ and $t = 0.01$ denoting the discrete approximation of the vortex sheet.

Figure 2: (II) The positions of PVs at $t = 0$ and $t = 0.1$, denoting the checkered pattern.

Figure 3: (III) The positions of PVs at $t = 0$ and $t = 0.1$, denoting the uniformly spaced pattern.
In Case (I), we have an analytical expression of $L(r)$ for a single row. The roll-up of PVs starts immediately corresponding to the Kelvin-Helmholtz instability. $L > 0$ for the row and it persists at the final time $t = t_f$. The relative errors of the conserved quantities are confirmed to be below $10^{-5}$ at the final time in all numerical examples.

5 Numerical simulation of two types of PVs

Next, we consider the three cases of positive and negative PVs, (V) the Kármán vortex street, (VI) positive and negative PVs located alternately in checkered segments (16 subsquares), and (VII) the CSR distribution. We can define similarly the $K_{lm}$ and $L_{lm}$ functions for $(l, m) = (+, +), (-, -),$ and $(+, -)$, where $+(-)$ denotes the positive (negative) PVs.

In Case (V), the pairing of positive and negative PVs appears. This corresponds to the pairing instability of the double row of the staggered PVs [3]. The typical final time can be estimated as $10 t_e$, where $t_e$ is the average eddy turnover time [1].

Clustering of both types of PVs is remarkable again in Case (VII). The vortex pair moves linearly at a velocity proportional to $1/h$ where $h$ is the distance of two PVs. Then, the pair is scattered by a third vortex, or recoupling occurs. This type of scattering on the unbounded plane was solved analytically by Aref [12]. We made a GIF animation of motions of PVs. The motion is quite interesting. It looks like a motion of gas molecules, which does not exist in the case of the single type of PVs.

Another method to quantify the distribution of PVs in squares of the same size [2] is called the Quadrat method [10] in spatial ecology.
Figure 5: (V) The positions of positive and negative PVs at $t = 0.001$ and $t = 0.01$, denoting the model of Karman Vortex Street. Black (white) circles denote positive (negative) PVs.

Figure 6: (VI) The positions of positive and negative PVs at $t = 0.001$ and $t = 0.1$, denoting the checkered pattern.

Figure 7: (VII) The positions of positive and negative PVs at $t = 0.001$ and $t = 0.1$, denoting the CSR distribution.
6 Conclusions and Discussions

Clustering of PVs is studied numerically based on point process theory and the \( L \) functions are computed from the final distributions of PVs. Clustering persists up to 10 eddy turnover time. We also did a numerical simulation for slightly perturbed square vortex lattices, which can be regarded as a special circumstance of Case (III). It showed inactive motions of PVs compared with the other examples shown in this paper. It may imply that the instability of square lattices requires more than 100 PVs, or the rate of growth is very small. The author is grateful for Professor T. Yamagata for support through his research on fluid dynamics over these several years.

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