MINIMAL EXCLUDANT OVER PARTITIONS INTO DISTINCT PARTS

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Abstract. The average size of the “smallest gap” of a partition was studied by Grabner
and Knopfmacher in 2006. Recently, Andrews and Newman, motivated by the work of
Fraenkel and Peled, studied the concept of the “smallest gap” under the name “minimal
excludant” of a partition and rediscovered a result of Grabner and Knopfmacher. In the
present paper, we study the sum of the minimal excludants over partitions into distinct
parts, and interestingly we observe that it has a nice connection with Ramanujan’s function
\(\sigma(q)\). As an application, we derive a stronger version of a result of Uncu.

1. Introduction

Grabner and Knopfmacher [11] studied an interesting partition statistic under the name
‘smallest gap’. They defined the smallest gap of an integer partition as the least integer miss-
ing from the partition. Fraenkel and Peled introduced the concept of a minimal excludant
of a set \(S\) of positive integers, namely, the least positive integer missing from the set, denoted by
“mex(\(S\))”. Recently, in 2019, Andrews and Newman explored the idea of minimal excludant
and in the process, rediscovered a result of Grabner and Knopfmacher on “smallest gap”.
They very naturally generalized this concept to other arithmetic progressions in their two
papers [4, 5]. Let us define

\[
\sigma_{\text{mex}}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{mex}(\pi),
\]

where \(\mathcal{P}(n)\) denotes the collection of all integer partitions of \(n\). Interestingly, Andrews and
Newman [4 Theorem 1.1] proved that

\[
\sigma_{\text{mex}}(n) = D_2(n),
\]

where \(D_2(n)\) represents the number of two-colored partitions of \(n\) into distinct parts. The
generating function of the above identity was, in fact, obtained by Grabner and Knopfmacher

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formula.
Theorem 3] in their work on the smallest gap. They also obtained the following Hardy-Ramanujan-Rademacher type exact formula for $\sigma_{\text{mex}}(n)$:

$$
\sigma_{\text{mex}}(n) = \frac{\pi}{2 \sqrt{6} (n + \frac{1}{12})} \sum_{k=1}^{\infty} A_{2k-1}(n) I_1 \left( \frac{\sqrt{2(n + \frac{1}{12})}}{\sqrt{3}(2k - 1)} \right),
$$

where

$$
A_k(n) = \sum_{0 \leq h \leq k \quad \gcd(h,k) = 1} \exp \left( 2\pi i \left( s(h,k) - s(2h,k) - \frac{hn}{k} \right) \right),
$$

and $s(h,k)$ denotes the Dedekind sum, and $I_1$ denotes the modified Bessel function of the first kind with index 1. Moreover, $\sigma_{\text{mex}}(n)$ satisfies the following asymptotic formula:

$$
\sigma_{\text{mex}}(n) \sim \frac{1}{4 \sqrt{6n^3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) \quad \text{as} \quad n \to \infty.
$$

A combinatorial proof of (1.2) has been obtained by Ballantine and Merca [6]. Recently, Chern [10] defined the maximal excludant $\text{amex}(\pi)$ as the largest non-negative integer smaller than the largest part of $\pi$ that is not a part of $\pi$. Analogous to (1.1), Chern defined

$$
\sigma_{\text{amex}}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{maex}(\pi).
$$

He [10, Theorem 1.1] obtained the following generating function identity for $\sigma_{\text{amex}}(n)$:

$$
\sum_{n=1}^{\infty} \sigma_{\text{amex}}(n) q^n = \sum_{n=1}^{\infty} \frac{n}{(q;q)_{n-1}} \sum_{m=1}^{\infty} q^{m(n+1)} (-q; q)_{m-1}.
$$

Chern [10, Theorem 1.3] also established an asymptotic formula for $\sigma_{\text{amex}}(n)$. More precisely, he showed that

$$
\sigma_{\text{amex}}(n) \sim \sigma L(n), \quad \text{as} \quad n \to \infty,
$$

where $\sigma L(n)$ denotes the sum of the largest parts of all partitions of $n$. Kessler and Livingston showed that the following Hardy-Ramanujan type asymptotic formula holds for $\sigma L(n)$:

$$
\sigma L(n) \sim \frac{\log 6n}{4\pi \sqrt{2n}} + 2\gamma \exp \left( \pi \sqrt{\frac{2n}{3}} \right), \quad \text{as} \quad n \to \infty.
$$

Chern [10, Theorem 2.2] also derived a formula for $\sigma_{\text{amex}}(n)$, that connects the divisor function and the coefficients of Ramanujan’s $q$-series $\sigma(q)$, defined in (2.1) below.

In [4], Andrews and Newman studied another arithmetic function, namely,

$$
a(n) = \sum_{\pi \in \mathcal{P}(n) \quad \text{max}(\pi) \text{ odd}} 1.
$$

They observed that $\sigma_{\text{mex}}(n) \equiv a(n) \pmod{2}$ and that $a(n)$ is almost always even and is odd exactly when $n$ is of the form $j(3j \pm 1)$.

1Note that there is a typo in [4, p. 250, Theorem 1.2], in which the words odd and even are exchanged.
We study the questions raised by Andrews and Newman for the function $\sigma_{\text{mex}}(n)$ in [4], but restricted to partitions into distinct parts. We define the function $\sigma_{d\text{mex}}(n)$ by

$$\sigma_{d\text{mex}}(n) := \sum_{\pi \in \mathcal{D}(n)} \text{mex}(\pi),$$

(1.5)

where $\mathcal{D}(n)$ denotes the collection of partitions of $n$ into distinct parts. We also define

$$a_d(n) := \sum_{\pi \in \mathcal{D}(n), \text{mex}(\pi) \text{ odd}} 1.$$

In fact, the generating function for $a_d(n)$ was considered by Uncu [14] in a different combinatorial context.

In the next section we state the main results.

2. Main Results

Before stating the main results, let us begin with one of the most important $q$-series of Ramanujan, which has been a constant source of study from the point of view of both algebraic and analytic number theory. It is given by

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q)_n}.$$  

(2.1)

Readers are encouraged to see [1], [2] and [3], for deeper results related to $\sigma(q)$. Surprisingly, the generating function for $\sigma_{d\text{mex}}(n)$ is directly connected to $\sigma(q)$ as stated below.

**Theorem 2.1.** We have

$$\sum_{n=0}^{\infty} \sigma_{d\text{mex}}(n) q^n = (-q; q)_{\infty} \sigma(q).$$

The next result gives us an asymptotic formula for $\sigma_{d\text{mex}}(n)$.

**Theorem 2.2.** We have

$$\sigma_{d\text{mex}}(n) \sim \frac{\exp \left( \pi \sqrt{\frac{2}{3n^3}} \right)}{2(3n^3)^{1/4}}, \quad \text{as } n \to \infty.$$  

(2.2)

Now we will state a result due to Uncu [14, Theorem 3].

**Theorem 2.3.** Let $U(n)$ be the sequence of numbers defined by

$$\sum_{n=0}^{\infty} U(n) q^n = (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q; q)_n}.$$  

Then $U(n) \geq 0$ for all $n \geq 0$.

Interestingly, we observe that the generating function for $U(n)$ and $a_d(n)$ are indeed the same.
Theorem 2.4. We have
\[ \sum_{n=0}^{\infty} U(n)q^n = \sum_{n=0}^{\infty} a_d(n)q^n = (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)}_{n}}{(-q; q)_n}. \] \hspace{1cm} (2.3)

An immediate consequence of this result is

Corollary 2.5. For any \( n \geq 0 \), we have
\[ U(n) > 0 \quad \text{except for} \quad n = 1. \]

Andrews and Newman also defined \( \text{moex}(\pi) \) to be the smallest odd integer missing from \( \pi \). This naturally led them to define the function \( \sigma_{\text{moex}}(n) \), given by
\[ \sigma_{\text{moex}}(n) := \sum_{\pi \in \mathcal{P}(n)} \text{moex}(\pi). \]

We analogously define a quantity for partitions into distinct parts and study its generating function. Define
\[ \sigma_{d\text{moex}}(n) := \sum_{\pi \in \mathcal{D}(n)} \text{moex}(\pi). \]

Theorem 2.6. The generating function for \( \sigma_{d\text{moex}}(n) \) is
\[ \sum_{n=0}^{\infty} \sigma_{d\text{moex}}(n)q^n = (-q; q)_{\infty} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q; q^{2})_n} \right) \]
\[ = (-q; q)_{\infty} \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^{n-1} q^n (q^2; q^{2})_{n-1} \right). \]

In other words,
\[ \sum_{n=0}^{\infty} \sigma_{d\text{moex}}(n)q^n = (-q; q)_{\infty} (1 + \sigma^{*}(-q)), \]
where
\[ \sigma^{*}(q) := 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q; q^2)_n}. \]

In literature, the \( q \)-series \( \sigma(q) \) and \( \sigma^{*}(q) \) are found to appear simultaneously at many places. Andrews, Dyson and Hickerson [3] proved that the coefficients of these two \( q \)-series are very small and related to the arithmetic of the quadratic real field \( \mathbb{Q}(\sqrt{6}) \).

Next, similar to the definition (1.5), we define \( \sigma_{d\text{maex}}(n) \) as
\[ \sigma_{d\text{maex}}(n) := \sum_{\pi \in \mathcal{D}(n)} \text{maex}(\pi). \]

The next result provides us a generating function identity for \( \sigma_{d\text{maex}}(n) \).

Theorem 2.7. We have
\[ \sum_{n=0}^{\infty} \sigma_{d\text{maex}}(n)q^n = \sum_{k=1}^{\infty} k(-q; q)_{k-1} \sum_{m=1}^{\infty} q^{\frac{m(n+1)}{2} + km}. \]
One can easily observe that Theorem 2.7 is an analogue of Chern’s identity \[10\] (2). Before going on to the proofs of our main results, we establish an auxiliary result in the next section.

3. Preliminaries

For the proof of the asymptotic formula for \(\sigma_d \text{mex}(n)\), namely, Theorem 2.2, we derive a monotonic property of \(\sigma_d \text{mex}(n)\), which is itself of independent interest.

**Proposition 3.1.** For \(n \geq 7\), we have that

\[
\sigma_d \text{mex}(n + 1) > \sigma_d \text{mex}(n).
\]

**Proof.** We consider two cases depending on whether \(n\) is a triangular number or not.

**Case 1:** Suppose \(n \geq 7\) is not a triangular number, that is, \(n \neq 1 + 2 + \cdots + k\) for any positive integer \(k\). We define a map \(f\) from \(\mathcal{D}(n)\) to \(\mathcal{D}(n + 1)\) as follows: Consider a distinct parts partition \(\pi\) of \(n\). It will be of the form

\[
\pi = (a_1, a_2, \ldots, a_k) \quad \text{with} \quad a_1 > a_2 > \cdots > a_k \geq 1, \quad a_1 + a_2 + \cdots + a_k = n. \quad (3.1)
\]

Then let \(f(\pi) := \pi' = (a_1 + 1, a_2, \ldots, a_k)\). Clearly, this is a distinct parts partition of \(n + 1\). Also see that \((a_1 + 1) - a_2 \geq 2\), i.e., a partition of the form \(f(\pi)\) has a gap of at least two between its largest part and the next largest part. Moreover, \(f\) preserves minimal excludants, that is \(\text{mex}(\pi) = \text{mex}(\pi')\). This is because as \(n\) is not a triangular number, \(\pi = (a_1, \ldots, a_k)\) cannot be the partition \((k, k-1, \ldots, 1)\) and consequently, there exists an \(i\), \(1 \leq i \leq k\) such that \(a_i - a_{i+1} > 1\) with the interpretation that \(a_{k+1} = 0\). Choosing \(i_0\) to be the largest integer \(i\) for which \(a_i - a_{i+1} > 1\) implies that \(\text{mex}(\pi) = \text{mex}(\pi') = a_{i_0+1} + 1\).

For example, if \(n = 9\), then the partition \(\pi_1 = (4, 3, 2)\) satisfies \(k = i_0 = 3\) with \(\text{mex}(\pi_1) = \text{mex}(f(\pi_1)) = 1\) and the partition \(\pi_2 = (8, 1)\) satisfies \(k = 2, i_0 = 1\) with \(\text{mex}(\pi_2) = \text{mex}(f(\pi_2)) = 2\). Thus, we have matched up the minimal excludants of partitions in \(\mathcal{D}(n)\) with those of a certain collection of partitions in \(\mathcal{D}(n + 1)\) and hence \(\sigma_d \text{mex}(n) \leq \sigma_d \text{mex}(n + 1)\). We next exhibit a partition in \(\mathcal{D}(n + 1)\) that does not lie in the image of \(f\), thus proving that the inequality \(\sigma_d \text{mex}(n + 1) > \sigma_d \text{mex}(n)\) holds. The idea is to get hold of a partition \(\pi\) in which both \(\ell(\pi)\) and \(\ell(\pi) - 1\) occur as parts so that it cannot be in the image of \(f\).

**If \(n\) is odd:** Consider the partition \(\lambda_1 = \left( \frac{n+1}{2}, \frac{n-1}{2}, 1 \right)\) of \(n + 1\). Note that \(\frac{n+1}{2} - \frac{n-1}{2} = 1\) and that \(\lambda_1\) has distinct parts as \((n-1)/2 > 1\).

**If \(n\) is even:** Look at the partition \(\lambda_2 = \left( \frac{n}{2} + 1, \frac{n}{2} \right)\) of \(n + 1\), which is again in \(\mathcal{D}(n + 1) \setminus f(\mathcal{D}(n))\).

**Case 2:** Assume that \(n \geq 7\) is a triangular number, so that \(n = k + (k-1) + \cdots + 1\) for a unique positive integer \(k\). As in Case 1, for any \(\pi\) in \(\mathcal{D}(n)\) other than the partition \(\mu = (k, k-1, \ldots, 1)\), the partition \(f(\pi)\) lying in \(\mathcal{D}(n + 1)\) will have the same minimal excludant as \(\pi\). But for \(\mu\), whose minimal excludant is \(k+1\), we see that \(\text{mex}(f(\mu)) = k\). This means that
\[ \sum_{\pi \in \mathcal{D}(n)} \text{mex}(\pi) = 1 + \sum_{\pi \in f(\mathcal{D}(n))} \text{mex}(\pi). \]

In other words, to prove the required inequality we have to show that the contribution of the partitions from \( \mathcal{D}(n+1) \setminus f(\mathcal{D}(n)) \) to the sum of minimal excludants is at least 2. We once again proceed according to the parity of \( n \).

If \( n \) is odd: Consider the partition \( \nu_1 = (\frac{n+1}{2}, \frac{n-1}{2}, 1) \) of \( n+1 \). Observe that \( (n-1)/2 > 2 \) and so \( \nu_1 \in \mathcal{D}(n+1) \setminus f(\mathcal{D}(n)) \) with its minimal excludant being 2.

If \( n \) is even: In this case, we look at two partitions given by \( \nu_2 = (\frac{n}{2} + 1, \frac{n}{2}) \) and \( \nu_3 = (\frac{n}{2}, \frac{n}{2} - 1, 2) \). Clearly, \( \nu_2 \in \mathcal{D}(n+1) \setminus f(\mathcal{D}(n)) \) with minimal excludant 1. Again, note that \( \nu_3 \) has distinct parts since \( \frac{n}{2} - 1 > 2 \) and hence \( \text{mex}(\nu_3) = 1 \).

With this, we complete the proof of the proposition. \( \square \)

In the upcoming section, we provide the proofs of our results.

4. PROOFS OF THE MAIN RESULTS

4.1. Proof of Theorem 2.1

4.1.1. First proof of Theorem 2.1

Let \( p_{d \text{ex}}(m, n) \) be the number of partitions of \( n \) into distinct parts whose minimal excludant is \( m \). Then, we have

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} p_{d \text{ex}}(m, n) z^m q^n = \sum_{m=1}^{\infty} z^m q^1 \cdot q^2 \cdots q^{m-1} \prod_{k=m+1}^{\infty} (1 + q^k) \\
= \sum_{m=1}^{\infty} z^m q^{m(m-1)/2} \prod_{k=m+1}^{\infty} (1 + q^k) \\
= (-q; q)_\infty \sum_{m=1}^{\infty} \frac{z^m q^{m(m-1)/2}}{(-q; q)_m}. \tag{4.1}
\]

Differentiating both sides of (4.1) with respect to \( z \) and putting \( z = 1 \), we get

\[
\sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} m p_{d \text{ex}}(m, n) \right) q^n = (-q; q)_\infty \sum_{m=1}^{\infty} \frac{mq^{m(m-1)/2}}{(-q; q)_m}.
\]

But \( \sum_{m=1}^{\infty} m p_{d \text{ex}}(m, n) = \sigma_d \text{mex}(n) \), the sum of minimal excludants in all the partitions of \( n \) into distinct parts. Thus,

\[
\sum_{n=0}^{\infty} \sigma_d \text{mex}(n) q^n = (-q; q)_\infty \sum_{m=1}^{\infty} \frac{mq^{m(m-1)/2}}{(-q; q)_m}. \tag{4.2}
\]

We now show that the sum on the right hand side of (4.2) is nothing but Ramanujan’s series \( \sigma(q) \). Start with

\[
\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q; q)_n} = \sum_{n=0}^{\infty} \frac{(n+1) - n}{(-q; q)_n} q^{n(n+1)/2}
\]
\[
\sum_{n=0}^{\infty} \frac{(n+1)q^{n+1/2}}{(-q;q)_n} - \sum_{n=0}^{\infty} \frac{nq^{n+1/2}}{(-q;q)_n} \\
= \sum_{n=1}^{\infty} \frac{nq^{n-1/2}}{(-q;q)_{n-1}} - \sum_{n=1}^{\infty} \frac{nq^{n+1/2}}{(-q;q)_n} \\
= \sum_{n=1}^{\infty} \frac{nq^{n-1/2}}{(-q;q)_{n-1}} \left(1 - \frac{q^n}{1+q^n}\right) \\
= \sum_{n=1}^{\infty} \frac{nq^{n-1/2}}{(-q;q)_n}.
\]

Therefore, from (4.2), we deduce that
\[
\sum_{n=0}^{\infty} \sigma_d \text{mex}(n) q^n = (-q;q)_{\infty} \sigma(q). \tag{4.3}
\]

We now give an alternate proof of (4.3), on the lines of Andrews and Newman’s second proof of the generating function for \(\sigma_{\text{mex}}(n)\). (see [4, p. 251])

4.1.2. Second proof of Theorem 2.1. Let \(D_i(n)\) denote the number of partitions of \(n\) into distinct parts for which \(\text{mex}(\pi) > i\). Then we claim that
\[
D_i(n) = p_d \left(n - \frac{i(i+1)}{2}, i\right), \tag{4.4}
\]

where \(p_d(m, i)\) denotes the number of partitions of \(m\) into distinct parts with smallest part greater than \(i\). To see this, start with a distinct parts partition \(\pi\) of \(n\) with \(\text{mex}(\pi) > i\). By the definition of minimal excludant, the integers 1 through \(i\) must all occur as parts in \(\pi\). Moreover, since \(\pi\) is a distinct parts partition, each of the numbers 1 to \(i\) appears exactly once in \(\pi\). Subtract the quantity \(1 + 2 + \cdots + i\) from \(\pi\). This gives a distinct parts partition \(\pi'\) of \(n - (1 + 2 + \cdots + i)\) (since we began with a distinct parts partition \(\pi\), removing some parts from it doesn’t affect its distinct nature). Now, since \(\pi\) has only one copy of each of 1 to \(i\), \(\pi'\) will not have any parts less than or equal to \(i\). Therefore \(\pi'\) is a distinct parts partition of \(n - \frac{i(i+1)}{2}\) with smallest part greater than \(i\).

Conversely, starting with a distinct parts partition \(\lambda\) of \(n - \frac{i(i+1)}{2}\) with \(s(\pi) > i\), we add the quantity \(1 + 2 + \cdots + i\) to \(\lambda\) to get a distinct parts partition \(\lambda'\) (since \(\lambda\) had no parts less than or equal to \(i\)) with the integers 1 to \(i\) all occurring as parts. This means that \(\text{mex}(\lambda') > i\). Hence, this bijection proves the claim in (4.4).

From the definition of \(D_i(n)\), \(\sigma_d \text{mex}(n)\) can be expressed as
\[
\sigma_d \text{mex}(n) = \sum_{i=0}^{\infty} D_i(n), \tag{4.5}
\]
since each distinct parts partition $\pi$ with $\text{mex}(\pi) = i$ is counted $i$ times on the right hand side of (4.5), once in each of $D_0(n), D_1(n), \ldots, D_{i-1}(n)$. On the left hand side of (4.5), we add together the minimal excludants over all the distinct parts partitions, thus each distinct parts partition $\pi$ contributes a weight $\text{mex}(\pi)$ to it. Thus, on both sides of equation (4.5), each distinct parts partition contributes the same number and hence the identity holds.

Now, the generating function of distinct parts partitions with $s(\pi) > i$ is simply

$$\sum_{n=0}^{\infty} p_d(n, i)q^n = (-q^{i+1}; q)_\infty.$$

Therefore, $D_i(n)$, which is the number of distinct parts partitions of $n - \frac{i(i+1)}{2}$ with smallest part greater than $i$, will be the coefficient of $q^n - \frac{i(i+1)}{2}$ in $(-q^{i+1}; q)_\infty$. Equivalently, this is the coefficient of $q^n$ in $q^{\frac{i(i+1)}{2}}(-q^{i+1}; q)_\infty$.

Thus,

$$\sum_{n=0}^{\infty} D_i(n)q^n = q^{\frac{i(i+1)}{2}}(-q^{i+1}; q)_\infty = (-q; q)_\infty \frac{q^{\frac{i(i+1)}{2}}}{(-q; q)_i}.$$

We are ready to obtain the generating function of $\sigma_d\text{mex}(n)$. Starting with (4.5), we get

$$\sum_{n=0}^{\infty} \sigma_d\text{mex}(n)q^n = \sum_{n=0}^{\infty} q^n \sum_{i=0}^{\infty} D_i(n)q^n = \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} D_i(n)q^n$$

$$= \sum_{i=0}^{\infty} (-q; q)_\infty \frac{q^{\frac{i(i+1)}{2}}}{(-q; q)_i}$$

$$= (-q; q)_\infty \sum_{i=0}^{\infty} \frac{q^{\frac{i(i+1)}{2}}}{(-q; q)_i} = (-q; q)_\infty \sigma(q).$$

□

4.2. Proof of Theorem 2.2. We first state an important asymptotic result for coefficients of a power series, due to Ingham [12].

**Proposition 4.1.** Let $A(q) = \sum_{n=0}^{\infty} a(n)q^n$ be a power series with radius of convergence 1. Assume that $\{a(n)\}$ is a weakly increasing sequence of non-negative real numbers. If there are constants $\alpha, \beta \in \mathbb{R}$, and $C > 0$ such that

$$A(e^{-t}) \sim \alpha t^\beta \exp \left(\frac{C}{t}\right), \quad \text{as } t \to 0^+,$$

then we have

$$a(n) \sim \frac{\alpha}{2\sqrt{\pi}} \frac{C^{\frac{3}{4}}}{n^{\frac{3}{4} + \frac{1}{2}}} \exp \left(2\sqrt{Cn}\right), \quad \text{as } n \to \infty. \quad (4.6)$$
Proof of Theorem 2.2. Recall that the generating function for $\sigma_{d \text{mex}}(n)$ is given by Theorem 2.1, namely,

$$\sum_{n=0}^{\infty} \sigma_{d \text{mex}}(n)q^n = (-q; q)_{\infty} \sigma(q) := B(q).$$

In a famous work on quantum modular forms, Zagier [15, p. 7] pointed out that, for $t \to 0^+$,

$$\sigma(e^{-t}) = 2 - 2t + 5t^2 - \frac{55}{3}t^3 + \frac{1073}{12}t^4 - \frac{32671}{60}t^5 + \ldots. \quad (4.7)$$

Now, using the transformation formula for the Dedekind’s eta-function, one can show that, for $t \to 0^+$,

$$\frac{1}{(e^{-t}; e^{-t})_{\infty}} \sim \sqrt{\frac{t}{2\pi}} \exp \left( \frac{\pi^2}{6t} \right). \quad (4.8)$$

Euler's identity suggests that

$$(-q; q)_{\infty} = \frac{1}{(q; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}}{(q; q)_{\infty}}. \quad (4.9)$$

Therefore, in view of (4.8) and (4.9), we can derive, for $t \to 0^+$,

$$(-e^{-t}; e^{-t})_{\infty} \sim \frac{1}{\sqrt{2}} \exp \left( \frac{\pi^2}{12t} \right). \quad (4.10)$$

Finally, combining (4.7) and (4.10), we arrive at

$$B(e^{-t}) \sim \sqrt{2} \exp \left( \frac{\pi^2}{12t} \right), \quad \text{as } t \to 0^+. \quad (4.11)$$

Again, by Proposition 3.1 we know that the sequence $\{\sigma_{d \text{mex}}(n)\}$, for $n \geq 7$, is an increasing sequence of positive integers. Now we are ready to invoke Proposition 4.1 with $\alpha = \sqrt{2}, \beta = 0$ and $C = \frac{\pi^2}{12}$. Substituting these constants in (4.6), we obtain

$$\sigma_{d \text{mex}}(n) \sim \frac{1}{2(3n^3)^{1/4}} \exp \left( \frac{\pi}{\sqrt{3}} \sqrt{n} \right), \quad \text{as } n \to \infty. \quad (4.12)$$

This finishes the proof. \hfill \Box

In the next subsection, we provide proofs of other results.

4.3. Proofs of other results.

Proof of Theorem 2.4. Recall that $a_d(n)$ counts the number of distinct parts partitions of $n$ with an odd minimal excludant. So the least integer missing from such a partition can only be of the form $2n + 1$ for some $n \geq 0$. And all the integers from 1 through $2n$ should occur exactly once and for the integers greater than $2n + 1$, they may occur at most once. Putting this together, we may write

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} p_{d \text{mex}}(m, n)z^m q^n = \sum_{k=0}^{\infty} z^{2k+1} q^1 \cdot q^2 \cdots q^{2k} \prod_{\ell=2k+2}^{\infty} (1 + q^\ell), \quad (4.11)$$

where \( p_{d}^{\text{moex}}(m, n) \) denotes the number of distinct parts partitions of \( n \) with an odd minimal excludant \( m \). Putting \( z = 1 \) in (4.11), we get

\[
\sum_{n=0}^{\infty} a_{d}(n)q^{n} = \sum_{k=0}^{\infty} q^{(2k+1)\frac{2n}{2}} \prod_{\ell=2k+2}^{\infty} (1 + q^{\ell}) = (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\frac{2n+1}{2}}}{(-q; q)_{2n+1}}. \tag{4.12}
\]

Consider the rightmost sum in (4.12). Rewriting it, we get

\[
\sum_{n=0}^{\infty} \frac{q^{\frac{2n+1}{2}}}{(-q; q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{q^{1+\cdots+2n}}{(1 + q) \cdots (1 + q^{2n+1})} = \sum_{n=0}^{\infty} \frac{q^{1+\cdots+2n}}{(1 + q) \cdots (1 + q^{2n})} \left\{ 1 - \frac{q^{2n+1}}{1 + q^{2n+1}} \right\} \\
= \sum_{n=0}^{\infty} \frac{q^{1+\cdots+2n}}{(1 + q) \cdots (1 + q^{2n})} - \sum_{n=0}^{\infty} \frac{q^{1+\cdots+(2n+1)}}{(1 + q) \cdots (1 + q^{2n+1})} \\
= \sum_{n=0}^{\infty} \frac{q^{\frac{2n+1}{2}}}{(-q; q)_{2n+1}} - \sum_{n=0}^{\infty} \frac{q^{\frac{2n+2}{2}}}{(-q; q)_{2n}} \\
= \sum_{n=0}^{\infty} \frac{(-1)^{n}q^{\frac{n+1}{2}}}{(-q; q)_{n}}.
\]

Putting this in (4.12), we see that

\[
\sum_{n=0}^{\infty} a_{d}(n)q^{n} = (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n}q^{\frac{n+1}{2}}}{(-q; q)_{n}}.
\]

This completes the proof.

\[\square\]

**Proof of Corollary 2.5.** We note that \( a_{d}(n) \) is non-negative for all \( n \geq 0 \), since it counts certain kind of partitions. Moreover, for \( n > 1 \), we always have a partition of \( n \) into distinct parts with an odd minimal excludant, namely, the partition \( n \), where the minimal excludant is 1. So \( a_{d}(n) > 0 \) for all \( n > 1 \), and hence by (2.3), we conclude that \( U(n) > 0 \) for all \( n > 1 \).

Uncu [14, Theorem 3.2], remarks that the infinite series in (2.3) is a false theta function studied by Rogers. Further, in the same paper, he gives a combinatorial explanation of the fact that the coefficients on the right hand side of (2.3) are non-negative. But, by Corollary 2.5, via our interpretation in terms of minimal excludant, we have shown that all but one of the coefficients, namely \( a_{d}(1) \), are in fact positive.

**Proof of Theorem 2.6.** Let \( p_{d}^{\text{moex}}(m, n) \) denote the number of distinct parts partitions \( \pi \) of \( n \) with \( \text{moex}(\pi) = m \). Consider the following double sum

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} p_{d}^{\text{moex}}(m, n)z^{m}q^{n} = (-q^{2}; q^{2})_{\infty} \sum_{k=0}^{\infty} z^{2k+1}q^{1+3+\cdots+(2k-1)}(-q^{2k+3}; q^{2})_{\infty}
\]
\[
\begin{align*}
&= (-q^2; q^2)_{\infty} \sum_{k=0}^{\infty} z^{2k+1} q^{k^2} (-q^{2k+3}; q^2)_{\infty} \\
&= (-q; q^2)_{\infty} (-q^2; q^2)_{\infty} \sum_{k=0}^{\infty} z^{2k+1} q^{k^2} (-q^{2k+3}; q^2)_{\infty} \\
&= (-q; q^2)_{\infty} \sum_{k=0}^{\infty} z^{2k+1} q^{k^2} (-q; q^2)_{\infty}^{k+1}.
\end{align*}
\]

Differentiate with respect to \(z\) and then put \(z = 1\) to get
\[
\sum_{n=0}^{\infty} \sigma_{d, \text{moex}}(n) q^n = (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(2n + 1) q^{n^2}}{(-q; q^2)^{n+1}}
\]
\[
= (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(2n + 1) q^{n^2}}{(-q; q^2)^n} \left\{ 1 - \frac{q^{2n+1}}{1 + q^{2n+1}} \right\}
\]
\[
= (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(2n + 1) q^{n^2}}{(-q; q^2)^n} - (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(2n + 1) q^{(n+1)^2}}{(-q; q^2)^{n+1}}
\]
\[
= (-q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(2n + 1) q^{n^2}}{(-q; q^2)^n} - (-q; q)_{\infty} \sum_{n=1}^{\infty} \frac{(2n - 1) q^{n^2}}{(-q; q^2)^n}
\]
\[
= (-q; q)_{\infty} + 2(-q; q)_{\infty} \sum_{n=1}^{\infty} \frac{q^{n^2}}{(-q; q^2)^n}.
\]

Alternatively, using Chern’s result \([10, \text{Proposition 2.1}]\) with \(x = 1, y = -1\) gives us
\[
\sum_{n=0}^{\infty} \sigma_{d, \text{moex}}(n) q^n = (-q; q)_{\infty} + 2(-q; q)_{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^n (q^2; q^2)_{n-1}.
\]

**Proof of Theorem 2.7.** Suppose \(\pi_d\) is a distinct parts partition of \(n\) with maximal excludant \(k\). Note that \(k \geq 1\), since partitions with maximal excludant 0 do not contribute to the sum
\[
\sum_{\pi \in D(n)} \max(\pi) = \sigma_d \text{max}(n).
\]
We can divide \(\pi_d\) into two components, \(\pi_d’\) and \(\pi_d''\):

The first component \(\pi_d’\) is a distinct parts partition with parts \(\leq k - 1\); and the second one \(\pi_d''\) is a gapfree distinct parts partition with \(s(\pi) = k + 1\), i.e., each integer between \(s(\pi)\) and \(\ell(\pi)\) also occurs as a part. Observe that the second component \(\pi_d''\) upon conjugation gives a gapfree partition in which the smallest part \(s(\pi) = 1\) and the largest part \(\ell(\pi)\) appears exactly \(k + 1\) times and all other parts appear exactly once. We consider a two variable generating function \(D(z, q)\) for \(p_{d,k}(n)\), the number of distinct parts partitions of \(n\) with maximal excludant \(k\). In \(D(z, q)\), the exponent of \(z\) indicates the maximal excludant of a partition \(\pi_d’\) into distinct parts, and the exponent of \(q\), as always, keeps track of the number being partitioned by \(\pi_d’\).

\[
D(z, q) := \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} p_{d,k}(n) z^k q^n = \sum_{k=1}^{\infty} (-q; q)_{k-1} z^k \sum_{m=1}^{\infty} q^{1+2+\ldots+(m-1)+(k+1)m}.
\]
Now differentiating \( D(z, q) \) with respect to \( z \) and substituting \( z = 1 \), we get the generating function for \( \sigma_d \text{maex}(n) \),

\[
\sum_{n=0}^{\infty} \sigma_d \text{maex}(n) q^n = \sum_{k=1}^{\infty} k(-q; q)_{k-1} \sum_{m=1}^{\infty} q^{\frac{m(m+1)}{2} + km}.
\]

\[\square\]

5. Concluding Remarks

Inspired by the work of Andrews and Newman, in the current paper, we studied the minimal excludant over partitions into distinct parts. We have proved that the generating function for \( \sigma_d \text{mex}(n) \) is the product of the generating function for distinct parts partition function and Ramanujan’s well-known \( q \)-series \( \sigma(q) \). We also established a Hardy-Ramanujan type asymptotic formula for \( \sigma_d \text{mex}(n) \). It would be interesting to find a Hardy-Ramanujan-Rademacher type exact formula for \( \sigma_d \text{mex}(n) \), analogous to the result \([13]\) of Grabner and Knopfmacher for \( \sigma \text{mex}(n) \).

We also examined \( a_d(n) \), which counts the number of distinct parts partitions with an odd minimal excludant. Quite surprisingly, we have observed that the generating function for \( a_d(n) \) has been studied by Uncu in a different context, which immediately improved Uncu’s result \([14] \text{ Theorem 3}\). Subsequently, we studied \( \sigma_d \text{moex}(n) \) and its generating function has been expressed as the product of the generating function for the distinct parts partition function and \( 1 + \sigma^*(q) \). It is interesting that the function \( \sigma^*(q) \) has mostly been seen to appear in the vicinity of \( \sigma(q) \). Recently, using the theory of modular forms, Barman and Singh \([7, 8]\), and Chakraborty and Ray \([9]\) found interesting congruence properties and density results for Mex-related partition functions. Readers are encouraged to see the paper of da Silva and Sellers \([13]\) for parity results and congruence properties related to Mex-related partition functions of Andrews and Newman.

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