GENERALIZED FIBRE SUMS OF 4-MANIFOLDS AND THE CANONICAL CLASS

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ABSTRACT. In this article we determine the integral homology and cohomology groups of a closed 4-manifold $X$ obtained as the generalized fibre sum of two closed 4-manifolds $M$ and $N$ along embedded surfaces of genus $g$ and self-intersection zero. If the homologies of the 4-manifolds are torsion free and the surfaces represent indivisible homology classes, we derive a formula for the intersection form of $X$. If the 4-manifolds $M$ and $N$ are symplectic and the surfaces symplectically embedded we also derive a formula for the canonical class of the symplectic fibre sum.

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1. INTRODUCTION

In this article we are interested in the generalized fibre sum of closed oriented 4-manifolds $M$ and $N$ along closed embedded surfaces $\Sigma_M$ and $\Sigma_N$ of genus $g$. We always assume that both surfaces represent non-torsion homology classes and have self-intersection zero, i.e. their normal bundles are trivial, and choose embeddings

\[ i_M : \Sigma \to M \]
\[ i_N : \Sigma \to N \]

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that realize the surfaces as images of a fixed closed surface $\Sigma$ of genus $g$. The
generalized fibre sum $X = X(\phi) = M\#_{\Sigma_M=\Sigma_N}N$ is defined as
\[
X(\phi) = M' \cup_\phi N'
\]
where $M'$ and $N'$ denote the manifolds with boundary $\Sigma \times S^1$ obtained by deleting
the interior of tubular neighbourhoods $\Sigma \times D^2$ of the surfaces in $M$ and $N$ and $\phi$
is an orientation reversing diffeomorphism $\phi : \partial M' \to \partial N'$ that preserves the $S^1$
fibration and covers the diffeomorphism $i_N \circ i_M^{-1}$ between the surfaces.

In the first part of this article in Sections 4 to 6 we calculate the integral homology and
cohomology groups of the 4-manifold $X$ obtained as a generalized fibre sum of $M$ and $N$. To do so, we will first calculate in Section 3 the homology and
cohomology of the complement $M' = M \setminus \text{int} \nu \Sigma_M$ and determine the mapping
induced by the gluing diffeomorphism $\phi$ on the homology of the boundaries $\partial M'$
and $\partial N'$ in Section 2.2. The homology and cohomology groups of $X$ are then derived by Mayer-Vietoris arguments from the corresponding groups of $M'$ and $N'$.

For example, having calculated the first cohomology $H^1(X)$, we can determine the Betti numbers of $X$: Let $d$ denote the dimension of the kernel of the linear map
\[
i_M \oplus i_N : H_1(\Sigma; \mathbb{R}) \to H_1(M; \mathbb{R}) \oplus H_1(N; \mathbb{R})
\]
of $\mathbb{R}$-vector spaces. Then we have:

**Corollary 1.** The Betti numbers of a generalized fibre sum $X = M\#_{\Sigma_M=\Sigma_N}N$
along surfaces $\Sigma_M$ and $\Sigma_N$ of genus $g$ and self-intersection 0 are given by
\[
\begin{align*}
b_0(X) &= b_4(X) = 1 \\
b_1(X) &= b_3(X) = b_1(M) + b_1(N) - 2g + d \\
b_2(X) &= b_2(M) + b_2(N) - 2 + 2d \\
b_2^+(X) &= b_2^+(M) + b_2^+(N) - 1 + d \\
b_2^-(X) &= b_2^-(M) + b_2^-(N) - 1 + d.
\end{align*}
\]

The formulae for $H^1(X)$, $H_1(X)$ and $H^2(X)$ can be found in Theorems 38, 41 and 56. Similar formulae exist in several places in the literature, in particular in some special cases, for example 2, 4, 5, 6, 8, 12.

In Section 6 we derive a formula for the intersection form of $X$ in the case that the cohomologies of $M$, $N$ and $X$ are torsion free and the surfaces $\Sigma_M$ and $\Sigma_N$
represent indivisible classes. Let $B_M$ be a surface in $M$ such that $B_M \cdot \Sigma_M = 1$
and let $P(M)$ denote the orthogonal complement to the subgroup $\mathbb{Z}B_M \oplus \mathbb{Z}\Sigma_M$ in
the second cohomology of $M$. We then get a splitting
\[
H^2(M) = P(M) \oplus \mathbb{Z}B_M \oplus \mathbb{Z}\Sigma_M.
\]

An analogous splitting exists for $H^2(N)$. We want to derive such a splitting for $H^2(X)$ together with a formula for the intersection form. First, a choice of framing
for the surface $\Sigma_M$, i.e. a trivialization of the normal bundle, determines a push-off $\Sigma^M$ of the surface into the boundary $\partial M'$ and this surface determines under
inclusion a surface $\Sigma_X$ in $X$. A similar surface $\Sigma'_X$ is determined by a framing
for \( \Sigma_N \). Depending on the homology of \( X \) and the gluing diffeomorphism, the surfaces \( \Sigma_X \) and \( \Sigma'_X \) do not necessarily define the same homology class in \( X \). The surfaces \( B_M \) and \( B_N \) minus a disk sew together to define a surface \( B_X \) in \( X \) with intersection number \( B_X \Sigma_X = 1 \). We then have:

**Theorem 2.** Let \( X = M \# \Sigma_M = \Sigma_N \# N \) be a generalized fibre sum of closed oriented 4-manifolds \( M \) and \( N \) along embedded surfaces \( \Sigma_M, \Sigma_N \) of genus \( g \) which represent indivisible homology classes. Suppose that the cohomology of \( M, N \) and \( X \) is torsion free. Then there exists a splitting

\[
H^2(X; \mathbb{Z}) = P(M) \oplus P(N) \oplus (S'(X) \oplus R(X)) \oplus (\mathbb{Z}B_X \oplus \mathbb{Z} \Sigma_X),
\]

where

\[
(S'(X) \oplus R(X)) = (\mathbb{Z}S_1 \oplus \mathbb{Z}R_1) \oplus \ldots \oplus (\mathbb{Z}S_d \oplus \mathbb{Z}R_d).
\]

The direct sums are all orthogonal, except the direct sums inside the brackets. In this decomposition of \( H^2(X; \mathbb{Z}) \), the restriction of the intersection form \( Q_X \) to \( P(M) \) and \( P(N) \) is equal to the intersection form induced from \( M \) and \( N \) and has the structure

\[
\begin{pmatrix}
B^2_M + B^2_N & 1 \\
1 & 0
\end{pmatrix}
\]
on \( \mathbb{Z}B_X \oplus \mathbb{Z} \Sigma_X \) and the structure

\[
\begin{pmatrix}
S^2_i & 1 \\
1 & 0
\end{pmatrix}
\]
on each summand \( \mathbb{Z}S_i \oplus \mathbb{Z}R_i \).

In this formula the classes in \( S'(X) \) are split classes, sewed together from surfaces in \( M' \) and \( N' \) which bound curves on the boundaries \( \partial M' \) and \( \partial N' \) that get identified under the gluing diffeomorphism \( \phi \). The group \( R(X) \) is the group of rim tori in \( X \) and the integer \( d \) is the dimension of the kernel of the map \( i_M \oplus i_N \) above. The formula for the intersection form is similar to a formula for the simply-connected elliptic surfaces \( E(n) \) which can be found, for example, in \([12]\).

Given the decomposition of \( H^2(M) \) as a direct sum above, we get an embedding of \( H^2(M) \) into \( H^2(X) \) by mapping \( B_M \) to \( B_X \), \( \Sigma_M \) to \( \Sigma_X \) and taking the identity on \( P(M) \). This embedding does not in general preserve the intersection form, since \( B^2_M = B^2_M + B^2_N \). There exists a similar embedding for \( H^2(N) \) into \( H^2(X) \), mapping \( B_N \) to \( B_X \) and \( \Sigma_N \) to \( \Sigma_X \). Heuristically, we can think of \( H^2(X) \) as being formed by joining \( H^2(M) \) and \( H^2(N) \) along their "nuclei" \( \mathbb{Z}B_M \oplus \mathbb{Z} \Sigma_M \) and \( \mathbb{Z}B_N \oplus \mathbb{Z} \Sigma_N \), which form the new nucleus \( \mathbb{Z}B_X \oplus \mathbb{Z} \Sigma_X \) of \( X \) by sewing together \( B_M \) and \( B_N \) to the surface \( B_X \). There are additional summands coming from the split classes and rim tori groups that do not exist in the closed manifolds \( M \) and \( N \) separately.

Finally, we consider the symplectic case: Suppose that \( M \) and \( N \) are symplectic manifolds and \( \Sigma_M \) and \( \Sigma_N \) symplectically embedded surfaces of genus \( g \). Then the generalized fibre sum \( X \) admits a symplectic form. In Section \([7]\) we derive a formula for the canonical class of \( X \) under the same assumptions as in the theorem on the intersection form of \( X \). The formula can be written as:
Corollary 3. Under the embeddings of $H^2(M)$ and $H^2(N)$ into $H^2(X)$, the canonical class of $X = M \# \Sigma_M = \Sigma_N N$ is given by

$$K_X = K_M + K_N + \Sigma_X + \Sigma'_X - (2g - 2)B_X + \sum_{i=1}^{d} t_i R_i,$$

where $t_i = K_M D_i^M - K_N D_i^N$.

The numbers $K_M D_i^M$ and $K_N D_i^N$ in the rim tori contribution to the formula are the evaluation of the canonical classes of $M$ and $N$ on certain surfaces in $M'$ and $N'$ bounding curves on the boundary. In particular, suppose that $g = 1$, the coefficients $t_1, \ldots, t_d$ vanish and $\Sigma_X = \Sigma'_X$. Then we get the classical formula for the generalized fibre sum along tori

$$K_X = K_M + K_N + 2\Sigma_X,$$

which can be found in the literature, e.g. [23].

The final Section 8 contains some applications of the formula for the canonical class, for example to the case of fibre sums along tori when there exists a rim tori contribution. We will also compare the formula to a formula of Ionel and Parker [15]. Using the formula for the canonical class and for the intersection form, we can also directly check the equation

$$K^2_X = K_M^2 + K_N^2 + (8g - 8)$$

which can be derived in an elementary way from the identity $c_1^2(X) = 2e(X) + 3\sigma(X)$.

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2. Definition of the Generalized Fibre Sum

In the following, we use for a topological space $Y$ the abbreviations $H_*(Y)$ and $H^*(Y)$ to denote the homology and cohomology groups of $Y$ with $\mathbb{Z}$-coefficients. Other coefficients will be denoted explicitly. The homology class of a surface and the surface itself are often denoted by the same symbol. Poincaré duality is often suppressed, so that a class and its Poincaré dual are denoted by the same symbol.

Let $M$ and $N$ be closed, oriented, connected 4-manifolds. Suppose that $\Sigma_M$ and $\Sigma_N$ are closed, oriented, connected embedded surfaces in $M$ and $N$ of the same genus $g$. Let $\nu \Sigma_M$ and $\nu \Sigma_N$ denote the normal bundles of $\Sigma_M$ and $\Sigma_N$. The normal bundle of the surface $\Sigma_M$ is trivial if and only if the self-intersection number $\Sigma_M^2$ is zero. This follows because the Euler class of the normal bundle is given by $e(\nu \Sigma_M) = i^* PD[\Sigma_M]$, where $i: \Sigma_M \to M$ denotes the inclusion. Hence the evaluation of the Euler class on the fundamental class of $\Sigma_M$ is equal to the self-intersection number of $\Sigma_M$ in the 4-manifold $M$. From now on we will assume that $\Sigma_M$ and $\Sigma_N$ have zero self-intersection.
For the construction of the generalized fibre sum we choose a closed oriented surface $\Sigma$ of genus $g$ and smooth embeddings
\[ i_M : \Sigma \rightarrow M \]
\[ i_N : \Sigma \rightarrow N, \]
with images $\Sigma_M$ and $\Sigma_N$. We assume that the orientation induced by the embeddings on $\Sigma_M$ and $\Sigma_N$ is the given one.

Since the normal bundles of $\Sigma_M$ and $\Sigma_N$ are trivial, there exist trivial $D^2$-bundles $\nu \Sigma_M$ and $\nu \Sigma_N$ embedded in $M$ and $N$, forming closed tubular neighbourhoods for $\Sigma_M$ and $\Sigma_N$. We fix once and for all embeddings
\[ \tau_M : \Sigma \times S^1 \rightarrow M \]
\[ \tau_N : \Sigma \times S^1 \rightarrow N, \]
with images $\partial \nu \Sigma_M$ and $\partial \nu \Sigma_N$, commuting with the embeddings $i_M$ and $i_N$ above and the natural projections
\[ p : \Sigma \times S^1 \rightarrow \Sigma \]
\[ p_M : \partial \nu \Sigma_M \rightarrow \Sigma_M \]
\[ p_N : \partial \nu \Sigma_N \rightarrow \Sigma_N. \]

The maps $\tau_M$ and $\tau_N$ form fixed reference trivialisations for the normal bundles of the embedded surfaces $\Sigma_M$ and $\Sigma_N$ and are called framings. Taking the image of $\Sigma$ times an arbitrary point on $S^1$, the framings $\tau_M$ and $\tau_N$ determine sections of the $S^1$-bundles $\partial \nu \Sigma_M$ and $\partial \nu \Sigma_N$. These sections correspond to push-offs of the surfaces $\Sigma_M$ and $\Sigma_N$ into the boundary of the tubular neighbourhoods. Since trivializations of vector bundles are linear, the framings are completely determined by such push-offs.

**Definition 4.** We denote by $\Sigma^M$ and $\Sigma^N$ push-offs of $\Sigma_M$ and $\Sigma_N$ into the boundaries $\partial \nu \Sigma_M$ and $\partial \nu \Sigma_N$ given by the framings $\tau_M$ and $\tau_N$.

We set
\[ M' = M \setminus \text{int} \nu \Sigma_M \]
\[ N' = N \setminus \text{int} \nu \Sigma_N \]
which are compact, oriented 4-manifolds with boundary. The orientations are chosen as follows: On $\Sigma \times D^2$ choose the orientation of $\Sigma$ followed by the standard orientation of $D^2$ given by $dx \wedge dy$. We can assume that the framings $\tau_M$ and $\tau_N$ induce orientation preserving embeddings of $\Sigma \times D^2$ into $M$ and $N$ as tubular neighbourhoods. We define the orientation on $\Sigma \times S^1$ to be the orientation of $\Sigma$ followed by the counter-clockwise orientation of $S^1$. This determines orientations on $\partial M'$ and $\partial N'$. Both conventions together imply that the orientation on $\partial M'$ followed by the orientation of the normal direction pointing out of $M'$ is the orientation on $M$. Similarly for $N$.

We want to glue $M'$ and $N'$ together using diffeomorphisms between the boundaries which preserve the fibration of the $S^1$-bundles $\partial \nu \Sigma_M$ and $\partial \nu \Sigma_N$ and cover
the diffeomorphism \( i_N \circ i_M^{-1} \). Since the group \( \text{Diff}^+(S^1) \) retracts onto \( SO(2) \), every orientation and fibre preserving diffeomorphism \( \Sigma \times S^1 \to \Sigma \times S^1 \) covering the identity is isotopic to a diffeomorphism of the form
\[
F: \Sigma \times S^1 \to \Sigma \times S^1, \\
(x, \alpha) \mapsto (x, C(x) \cdot \alpha),
\]
where \( C: \Sigma \to S^1 \) is a map and multiplication is in the group \( S^1 \). Conversely, every smooth map \( C \) from \( \Sigma \) to \( S^1 \) defines such an orientation preserving bundle isomorphism. Let \( r \) denote the orientation reversing diffeomorphism
\[
r: \Sigma \times S^1 \to \Sigma \times S^1, (x, \alpha) \mapsto (x, \bar{\alpha}),
\]
where \( S^1 \subset \mathbb{C} \) is embedded in the standard way and \( \bar{\alpha} \) denotes complex conjugation. Then the diffeomorphism
\[
\rho = F \circ r: \Sigma \times S^1 \to \Sigma \times S^1, \\
(x, \alpha) \mapsto (x, C(x)\bar{\alpha})
\]
is orientation reversing. We define
\[
\phi = \phi(C) = \tau_N \circ \rho \circ \tau_M^{-1}.
\]
Then \( \phi \) is an orientation reversing diffeomorphism \( \phi: \partial \nu \Sigma_M \to \partial \nu \Sigma_N \), preserving the circle fibres. If \( C \) is a constant map then \( \phi \) is a diffeomorphism which identifies the push-offs of \( \Sigma_M \) and \( \Sigma_N \).

**Definition 5.** Let \( M \) and \( N \) be closed, oriented, connected 4-manifolds \( M \) and \( N \) with embedded oriented surfaces \( \Sigma_M \) and \( \Sigma_N \) of genus \( g \) and self-intersection 0. The **generalized fibre sum** of \( M \) and \( N \) along \( \Sigma_M \) and \( \Sigma_N \), determined by the diffeomorphism \( \phi \), is given by
\[
X(\phi) = M' \cup_{\phi} N'.
\]
\( X(\phi) \) is again a differentiable, closed, oriented, connected 4-manifold.

See the references [10] and [19] for the original construction. The generalized fibre sum is often denoted by \( M \#_{\Sigma_M=\Sigma_N} N \) or \( M \#_{\Sigma} N \) and is also called the Gompf sum or the normal connected sum. If \( (M, \omega_M) \) and \( (N, \omega_N) \) are symplectic 4-manifolds and \( \Sigma_M, \Sigma_N \) symplectically embedded surfaces, the manifold \( M \#_{\Sigma_M=\Sigma_N} N \) admits a symplectic structure, cf. Section 7.

In the general case, the differentiable structure on \( X \) is defined in the following way: We identify the interior of slightly larger tubular neighbourhoods \( \nu \Sigma'_M \) and \( \nu \Sigma'_N \) via the framings \( \tau_M \) and \( \tau_N \) with \( \Sigma \times D \) where \( D \) is an open disk of radius 1. We think of \( \partial M' \) and \( \partial N' \) as the 3-manifold \( \Sigma \times S \), where \( S \) denotes the circle of radius \( 1/\sqrt{2} \). Hence the tubular neighbourhoods \( \nu \Sigma_M \) and \( \nu \Sigma_N \) above have in this convention radius \( 1/\sqrt{2} \). We also choose polar coordinates \( r, \theta \) on \( D \). The manifolds \( M \setminus \Sigma_M \) and \( N \setminus \Sigma_N \) are glued together along \( \text{int} \nu \Sigma'_M \setminus \Sigma_M \) and \( \text{int} \nu \Sigma'_N \setminus \Sigma_N \) by the diffeomorphism
\[
\Phi: \Sigma \times (D \setminus \{0\}) \to \Sigma \times (D \setminus \{0\}) \\
(x, r, \theta) \mapsto (x, \sqrt{1 - r^2}, C(x) - \theta).
\]
This diffeomorphism is orientation *preserving* because it reverses on the disk the orientation on the boundary circle and the inside-outside direction. It preserves the fibration by punctured disks and identifies \( \partial M' \) and \( \partial N' \) via \( \phi \).

**Definition 6.** Let \( \Sigma_X \) denote the genus \( g \) surface in \( X \) given by the image of the push-off \( \Sigma^M \) under the inclusion \( M' \to X \). Similarly, let \( \Sigma'_X \) denote the genus \( g \) surface in \( X \) given by the image of the push-off \( \Sigma^N \) under the inclusion \( N' \to X \).

In general (depending on the diffeomorphism \( \phi \) and the homology of \( X \)) the surfaces \( \Sigma_X \) and \( \Sigma'_X \) do not represent the same homology class in \( X \) but differ by a rim torus, cf. Lemma 48.

### 2.1. Isotopic gluing diffeomorphisms.

Different choices of gluing diffeomorphisms \( \phi \) can result in non-diffeomorphic manifolds \( X(\phi) \). However, if \( \phi \) and \( \phi' \) are isotopic, then \( X(\phi) \) and \( X(\phi') \) are diffeomorphic. We want to determine how many different isotopy classes of gluing diffeomorphisms \( \phi \) of the form above exist. We first make the following definition:

**Definition 7.** Let \( C : \Sigma \to S^1 \) be the map used to define the gluing diffeomorphism \( \phi \) in equation (1). Then the integral cohomology class \( [C] \in H^1(\Sigma) \) is defined by pulling back the standard generator of \( H^1(S^1) \). We sometimes denote \( [C] \) by \( C \) if a confusion is not possible.

Suppose that \( C, C' : \Sigma \to S^1 \), are smooth maps which determine self-diffeomorphisms \( \rho \) and \( \rho' \) of \( \Sigma \times S^1 \) and gluing diffeomorphisms \( \phi, \phi' : \partial \nu \Sigma_M \to \partial \nu \Sigma_N \) as before.

**Proposition 8.** The diffeomorphisms \( \phi, \phi' : \partial \nu \Sigma_M \to \partial \nu \Sigma_N \) are smoothly isotopic if and only if \( [C] = [C'] \in H^1(\Sigma) \). In particular, if the maps \( C \) and \( C' \) determine the same cohomology class, then the generalized fibre sums \( X(\phi) \) and \( X(\phi') \) are diffeomorphic.

**Proof.** Suppose that \( \phi \) and \( \phi' \) are isotopic. The equation

\[
\rho = \tau_N^{-1} \circ \phi \circ \tau_M,
\]

implies that the diffeomorphisms \( \rho, \rho' \) are also isotopic, hence homotopic. The maps \( C, C' \) can be written as

\[
C = pr \circ \rho \circ \iota, \quad C' = pr \circ \rho' \circ \iota,
\]

where \( \iota : \Sigma \to \Sigma \times S^1 \) denotes the inclusion \( x \mapsto (x, 1) \) and \( pr \) denotes the projection onto the second factor in \( \Sigma \times S^1 \). This implies that \( C \) and \( C' \) are homotopic, hence the cohomology classes \( [C] \) and \( [C'] \) coincide.

Conversely, if the cohomology classes \( [C] \) and \( [C'] \) coincide, then \( C \) and \( C' \) are homotopic maps. We can choose a smooth homotopy

\[
\Delta : \Sigma \times [0, 1] \to S^1, \quad (x, t) \mapsto \Delta(x, t)
\]
with $\Delta_0 = C$ and $\Delta_1 = C'$. Define the map

$$R: (\Sigma \times S^1) \times [0, 1] \to \Sigma \times S^1,$$

$$(x, \alpha, t) \mapsto R_t(x, \alpha),$$

where

$$R_t(x, \alpha) = (x, \Delta(x, t) \cdot \overline{\alpha}).$$

Then $R$ is a homotopy between $\rho$ and $\rho'$. The maps $R_t: \Sigma \times S^1 \to \Sigma \times S^1$ are diffeomorphisms with inverse

$$(y, \beta) \mapsto (y, \Delta(y, t)^{-1} \cdot \beta),$$

where $\Delta(y, t)^{-1}$ denotes the inverse as a group element in $S^1$. Hence $R$ is an isotopy between $\rho$ and $\rho'$ that defines via the framings $\tau_M$ and $\tau_N$ an isotopy between $\phi$ and $\phi'$.

\[\Box\]

2.2. Action of the gluing diffeomorphism on homology. In this subsection we determine the action of the gluing diffeomorphism $\phi: \partial M' \to \partial N'$ on the homology of the boundaries $\partial M'$ and $\partial N'$. We first choose bases for these homology groups using the framings $\tau_M$ and $\tau_N$.

First choose a basis for $H_1(\Sigma)$ consisting of oriented embedded loops $\gamma_1, \ldots, \gamma_{2g}$ in $\Sigma$. For each index $i$, we denote the loop $\gamma_i \times \{\ast\}$ in $\Sigma \times S^1$ also by $\gamma_i$. Let $\sigma$ represent the loop $\{\ast\} \times S^1$ in $\Sigma \times S^1$. Then the loops

$$\gamma_1, \ldots, \gamma_{2g}, \sigma,$$

represent homology classes (denoted by the same symbols) which determine a basis for $H_1(\Sigma \times S^1) \cong \mathbb{Z}^{2g+1}$.

**Definition 9.** The basis for the first homology of $\partial \nu \Sigma_M$ and $\partial \nu \Sigma_N$ is chosen as:

$$\gamma_i^M = \tau_M \ast \gamma_i, \quad \sigma^M = \tau_M \ast \sigma$$

$$\gamma_i^N = \tau_N \ast \gamma_i, \quad \sigma^N = \tau_N \ast \sigma.$$

For $H_2(\Sigma \times S^1)$ we define a basis consisting of elements $\Gamma_1, \ldots, \Gamma_{2g}, \Sigma$ such that the intersection numbers satisfy the following equations:

$$\Gamma_i \cdot \gamma_i = 1, \quad i = 1, \ldots, 2g,$$

$$\Sigma \cdot \sigma = 1,$$

and all other intersection numbers are zero. By Poincaré duality, this basis can be realized by choosing the dual basis $\gamma_1^*, \ldots, \gamma_{2g}^*, \sigma^*$ in the group $H^1(\Sigma \times S^1) = \text{Hom}(H_1(\Sigma \times S^1), \mathbb{Z})$ to the basis for the first homology above and defining

$$\Gamma_i = PD(\gamma_i^*), \quad i = 1, \ldots, 2g,$$

$$\Sigma = PD(\sigma^*).$$

**Definition 10.** The basis for the second homology of $\partial \nu \Sigma_M$ and $\partial \nu \Sigma_N$ is defined as:

$$\Gamma_i^M = \tau_M \ast \Gamma_i, \quad \Sigma^M = \tau_M \ast \Sigma$$

$$\Gamma_i^N = \tau_N \ast \Gamma_i, \quad \Sigma^N = \tau_N \ast \Sigma.$$
The class $\Sigma^M$ can be represented by a push-off of $\Sigma_M$ determined by the framing $\tau_M$. The classes $\Gamma^M_i$ have the following interpretation: We can choose a basis $\pi_1, \ldots, \pi_{2g}$ of $H_1(\Sigma)$ such that

$$\pi_i \cdot \gamma_j = -\delta_{ij}$$

for all indices $i, j$. The existence of such a basis follows because the intersection form on $H_1(\Sigma)$ is symplectic. The intersection numbers of the immersed tori $\pi_i \times \sigma$ in $\Sigma \times S^1$ with the curves $\gamma_j$ are given by

$$(\pi_i \times \sigma) \cdot \gamma_j = \delta_{ij},$$

hence these tori represent the classes $\Gamma_i$.

**Definition 11.** For the basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(\Sigma)$ above define the integers

$$a_i = \deg(C \circ \gamma_i : S^1 \to S^1) = \langle [C], \gamma_i \rangle = \langle C, \gamma_i \rangle \in \mathbb{Z}.$$ 

The integers $a_i$ together determine the cohomology class $[C]$. Since the map $C$ can be chosen arbitrarily, the integers $a_i$ can (independently) take any possible value.

**Lemma 12.** The map $\phi_* : H_1(\partial \nu \Sigma_M) \to H_1(\partial \nu \Sigma_N)$ is given by

$$\phi_* \Gamma^M_i = \gamma^N_i + a_i \sigma^N, \quad i = 1, \ldots, 2g$$
$$\phi_* \sigma^M = -\sigma^N.$$

**Proof.** We have

$$\rho(\gamma_i(t), *) = (\gamma_i(t), (C \circ \gamma_i)(t) \cdot \bar{t}),$$

which implies on differentiation $\rho_* \gamma_i = \gamma_i + a_i \sigma$ for all $i = 1, \ldots, 2g$. Similarly,

$$\rho(*, t) = (*, C(*) \cdot \bar{t}),$$

which implies $\rho_* \sigma = -\sigma$. The claim follows from these equations and equation $[2]$. \qed

**Lemma 13.** The map $\phi_* : H_2(\partial \nu \Sigma_M) \to H_2(\partial \nu \Sigma_N)$ is given by

$$\phi_* \Gamma^M_i = -\Gamma^N_i, \quad i = 1, \ldots, 2g$$
$$\phi_* \Sigma^M = -\left( \sum_{i=1}^{2g} a_i \Gamma^N_i \right) + \Sigma^N.$$

**Proof.** We first compute the action of $\rho$ on the first cohomology of $\Sigma \times S^1$. By the proof of Lemma $[12]$

$$(\rho^{-1})_* \gamma_i = \gamma_i + a_i \sigma, \quad i = 1, \ldots, 2g$$
$$(\rho^{-1})_* \sigma = -\sigma.$$
We claim that
\[
(\rho^{-1})^*(\gamma_i^*) = \gamma_i^*, \quad i = 1, \ldots, 2g,
\]
\[
(\rho^{-1})^*(\sigma^*) = \left( \sum_{i=1}^{2g} a_i \gamma_i^* \right) - \sigma^*.
\]
This is easy to check by evaluating both sides on the given basis of \(H_1(\Sigma \times S^1)\) and using \(\langle (\rho^{-1})^*\mu, v \rangle = \langle \mu, (\rho^{-1})_* v \rangle\). By the formula
\[
\lambda_* (\lambda^* \alpha \cap \beta) = \alpha \cap \lambda_* \beta,
\]
for continuous maps \(\lambda\) between topological spaces, homology classes \(\beta\) and cohomology classes \(\alpha\) (see [1, Chapter VI, Theorem 5.2]), we get for all classes \(\mu \in H^*(\Sigma \times S^1)\),
\[
\rho_* PD(\rho^* \mu) = \rho_* (\rho^* \mu \cap [\Sigma \times S^1])
\]
\[
= \rho_* [\Sigma \times S^1]
\]
\[
= -\rho_* [\Sigma \times S^1]
\]
\[
= -PD(\mu).
\]
since \(\rho\) is orientation reversing. This implies \(\rho_* PD(\mu) = -PD((\rho^{-1})^* \mu)\) and hence
\[
\rho_* \Gamma_i = -\Gamma_i, \quad i = 1, \ldots, 2g,
\]
\[
\rho_* \Sigma = -\left( \sum_{i=1}^{2g} a_i \Gamma_i \right) + \Sigma.
\]
The claim follows from this. \(\square\)

**Proposition 14.** The diffeomorphism \(\phi\) is determined up to isotopy by the difference of the homology classes \(\phi_* \Sigma^M\) and \(\Sigma^N\) in \(\partial \nu \Sigma_N\).

**Proof:** This follows because by the formula in Lemma [13] above, the difference determines the coefficients \(a_i\). Hence it determines the class \([C]\) and by Proposition [8] the diffeomorphism \(\phi\) up to isotopy. \(\square\)

We fix the following notation for some inclusions of manifolds into other manifolds:
\[
\rho_M: M' \to M
\]
\[
\mu_M: \partial \nu \Sigma_M \to M'
\]
\[
j_M: \sigma^M \to M'
\]
\[
\eta_M: M' \to X,
\]
and corresponding maps for \(N\). For simplicity, the maps induced on homology and homotopy groups will often be denoted by the same symbol.
3. The homology and cohomology of $M'$

Let $M$ be a closed, oriented 4-manifold and $\Sigma_M \subset M$ a closed, oriented, connected embedded surface of genus $g$ of self-intersection zero. We denote a closed tubular neighbourhood of $\Sigma_M$ by $\nu \Sigma_M$ and let $M'$ denote the complement

$$M' = M \setminus \text{int} \nu \Sigma_M.$$ 

Then $M'$ is an oriented manifold with boundary $\partial \nu \Sigma_M$. As above we choose a fixed closed oriented surface $\Sigma$ of genus $g$ and an embedding

$$i : \Sigma \to M$$

with image $\Sigma_M$. We continue to use the same notations for the framing and the embeddings as in the previous section. However, we will often drop the index $M$ on the maps to keep the formulae notationally more simple.

We always assume in this section that the surface $\Sigma_M$ represents a non-torsion class, denoted by the same symbol $\Sigma_M \in H^2(M)$. On the closed 4-manifold $M$, the Poincaré dual of $\Sigma_M$ acts as a homomorphism on $H^2(M)$,

$$\langle PD(\Sigma_M), - \rangle : H^2(M) \to \mathbb{Z}.$$ 

Since the homology class $\Sigma_M$ is non-torsion, the image of this homomorphism is non-zero and hence a subgroup of $\mathbb{Z}$ of the form $k_M \mathbb{Z}$ with $k_M > 0$. We assume that $\Sigma_M$ is divisible by $k_M$, i.e. there exists a class $A_M \in H^2(M)$ such that $\Sigma_M = k_M A_M$. This is always true, for example, if $H^2(M) \cong H^2(M)$ is torsion free. We also define the homology class $A'_M$ to be the image of $A_M$ under the homomorphism

$$H^2(M) \to H^2(M, \Sigma_M) \cong H^2(M', \partial M'),$$

where the first map comes from the long exact homology sequence for the pair $(M, \Sigma_M)$ and the second map is by excision and the 5-Lemma.

**Remark 15.** The results in Section 3.1 and Proposition 21 hold without the assumption that the surface $\Sigma_M$ has self-intersection zero.

In the following calculations we will often use the Mayer-Vietoris sequence for the decomposition $M = M' \cup \nu \Sigma_M$:

$$\ldots \to H_k(\partial M') \to H_k(M') \oplus H_k(\Sigma) \to H_k(M) \to H_{k-1}(\partial M') \to \ldots$$

with homomorphisms

$$H_k(\partial M') \to H_k(M') \oplus H_k(\Sigma), \quad \alpha \mapsto (\mu_s \alpha, p_s \alpha)$$

$$H_k(M') \oplus H_k(\Sigma) \to H_k(M), \quad (x, y) \mapsto \rho_s x - i_s y.$$ 

We also use the Mayer-Vietoris sequence for cohomology groups.

**Lemma 16.** The following diagram commutes up to sign for every integer $p$:

$$\begin{array}{ccc}
H_p(M', \partial M') & \xrightarrow{\partial} & H_{p-1}(\partial M') \\
\cong \downarrow & & \cong \downarrow \\
H^{4-p}(M') & \xrightarrow{\mu^*} & H^{4-p}(\partial M')
\end{array}$$
where the vertical isomorphisms are Poincaré duality.

A proof for this Lemma can be found in [1, Chapter VI, Theorem 9.2].

3.1. Calculation of $H_1(M')$ and $H^1(M')$. We begin with the calculation of the first cohomology of the complement $M'$.

**Proposition 17.** There exists an isomorphism $H^1(M') \cong H^1(M)$.

**Proof.** This follows from the long exact sequence in homology for the pair $(M, \Sigma_M)$:

$$0 \to H_3(M) \to H_3(M, \Sigma_M) \to H_2(\Sigma_M) \xrightarrow{i} H_2(M) \to \ldots$$

The map

$$i : H_2(\Sigma_M) \cong \mathbb{Z} \to H_2(M)$$

$$a \mapsto a\Sigma_M,$

is injective, since the class $\Sigma_M$ is non-torsion. This implies that $H_3(M, \Sigma_M) \cong H_3(M)$. Hence by excision and Poincaré duality

$$H^1(M') \cong H_3(M', \partial M') \cong H_3(M, \Sigma_M) \cong H_3(M) \cong H^1(M).$$

□

We can now calculate the first homology of the complement.

**Proposition 18.** There exists an isomorphism $H_1(M') \cong H_1(M) \oplus \mathbb{Z}_{k_M}$.

**Proof.** By Proposition 21 we have

$$
\text{Tor}H^2(M') \cong \text{Tor}(H^2(M)/\mathbb{Z}PD(\Sigma_M))
\cong \text{Tor}(H^2(M)/\mathbb{Z}k_M PD(A_M))
\cong \text{Tor}H^2(M) \oplus \mathbb{Z}PD(A_M)/k_M PD(A_M)
\cong \text{Tor}H^2(M) \oplus \mathbb{Z}_{k_M} PD(A_M).
$$

The third step follows because the class $A_M$ is indivisible and of infinite order. The Universal Coefficient Theorem implies that

$$\text{Tor}H^2(M) = \text{Ext}(H_1(M), \mathbb{Z}) \cong \text{Tor}H_1(M),$$

and similarly for $M'$. This implies

$$\text{Tor}H_1(M') \cong \text{Tor}H_1(M) \oplus \mathbb{Z}_{k_M}.$$ 

Using again the Universal Coefficient Theorem we get

$$H_1(M') \cong H^1(M') \oplus \text{Tor}H_1(M')$$

$$\cong H^1(M) \oplus \text{Tor}H_1(M) \oplus \mathbb{Z}_{k_M}$$

$$\cong H_1(M) \oplus \mathbb{Z}_{k_M}.$$ 

□

A similar calculation has been done in [14] and [22] for the case of a 4-manifold $M$ under the assumption $H_1(M) = 0$. The fundamental group of the complement $M'$ can be calculated as follows:
Proposition 19. The fundamental groups of $M$ and $M'$ are related by
$$\pi_1(M) \cong \pi_1(M')/N(\sigma^M),$$
where $N(\sigma^M)$ denotes the normal subgroup in $\pi_1(M')$ generated by the meridian $\sigma^M$ to the surface $\Sigma_M$.

The proof, which we omit, is an application of the Seifert-van Kampen theorem. Taking the abelianization of the exact sequence
$$0 \to N(\sigma^M) \to \pi_1(M') \to \pi_1(M) \to 0$$
we get with Proposition 18:

Corollary 20. The first integral homology groups of $M'$ and $M$ are related by the exact sequence

(7) $$0 \to \mathbb{Z}_{k_M} \xrightarrow{j} H_1(M') \xrightarrow{\rho} H_1(M) \to 0,$$

which splits. The image of $j$ is generated by the meridian $\sigma^M$ to the surface $\Sigma_M$.

Proofs for the last two statements can be found for example in [13, Appendix].

3.2. Calculation of $H^2(M')$. In this subsection we determine the second cohomology of the complement $M'$.

Proposition 21. There exists a short exact sequence

$$0 \to H^2(M)/\mathbb{Z}\Sigma_M \xrightarrow{\nu^*} H^2(M') \to \ker(i: H_1(\Sigma_M) \to H_1(M)) \to 0.$$ 

This sequence splits, because $H_1(\Sigma_M)$ is torsion free. Hence there exists an isomorphism

(8) $$H^2(M') \cong \left(H^2(M)/\mathbb{Z}\Sigma_M\right) \oplus \ker i.$$

Proof. We consider the long exact sequence in cohomology associated to the pair $(M, M')$: 

$$\ldots \to H^2(M, M') \xrightarrow{\partial} H^3(M, M') \xrightarrow{\partial} H^3(M', M') \xrightarrow{\partial} H^3(M) \to \ldots$$

By excision, Poincaré duality and the deformation retraction $\nu\Sigma_M \to \Sigma_M$ we have for all indices $m$: 

$$H^m(M, M') \cong H^m(\nu\Sigma_M, \partial\nu\Sigma_M) \cong H_{4-m}(\nu\Sigma_M) \cong H_{4-m}(\Sigma_M).$$

It follows that the map $H^m(M, M') \to H^m(M)$ is equivalent under Poincaré duality to the map $i: H_{4-m}(\Sigma_M) \to H_{4-m}(M)$. With $m = 2, 3$, this proves the claim. \hfill \Box

Remark 22. An explicit splitting can be defined as follows: The images of loops representing the classes in $\ker i$ under the embedding $i: \Sigma \to M$ bound surfaces in $M$. Using the trivialization $\tau_M$ we can think of these loops to be on the push-off $\Sigma^M$ and the surfaces they bound in $M'$. In this way the elements in $\ker i$ determine classes in $H^2(M') \cong H_2(M', \partial M')$. 

Definition 23. Choose a class \( B_M \in H^2(M) \) with \( B_M \cdot A_M = 1 \). Such a class exists because \( A_M \) is indivisible. We denote the image of this class in \( H^2(M') \cong H_2(M', \partial M') \) by \( B'_M \). The surface representing \( B_M \) can be chosen such that it intersects the surface \( \Sigma_M \) in precisely \( k_M \) transverse positive points: By Proposition 20 the curve \( k_M \sigma_M \) on \( \partial M' \) is null-homologous in \( M' \), hence it bounds a surface \( B'_M \). In the manifold \( M \) we can then sew \( k_M \) disk fibres of the tubular neighbourhood of \( \Sigma_M \) to the surface \( B'_M \) to get the surface \( B_M \) that intersects \( \Sigma_M \) in exactly \( k_M \) points.

Consider the subgroup in \( H^2(M) \) generated by the classes \( B_M \) and \( A_M \).

Definition 24. Let \( P(M) = (\mathbb{Z}B_M \oplus \mathbb{Z}A_M)^\perp \) denote the orthogonal complement in \( H^2(M) \) with respect to the intersection form. The elements in \( P(M) \) are called perpendicular classes.

Since \( A_M^2 = 0 \) and \( A_M \cdot B_M = 1 \), the intersection form on these (indivisible) elements looks like

\[
\begin{pmatrix}
B_M^2 & 1 \\
1 & 0
\end{pmatrix}
\]

Since this form is unimodular (it is equivalent to \( H \) if \( B_M^2 \) is even and to \((-1) \oplus (+1)\) if \( B_M^2 \) is odd) it follows that there exists a direct sum decomposition

\[ H^2(M) = \mathbb{Z}B_M \oplus \mathbb{Z}A_M \oplus P(M). \] (9)

The restriction of the intersection form to \( P(M) \) modulo torsion is again unimodular (see [12, Lemma 1.2.12]) and \( P(M) \) has rank equal to \( b_2(M) - 2 \). The following lemma determines the decomposition of an arbitrary element in \( H^2(M) \) under the direct sum (9):

Lemma 25. For every element \( \alpha \in H^2(M) \) define a class \( \overline{\alpha} \) by the equation

\[ \alpha = (\alpha \cdot A_M)B_M + (\alpha \cdot B_M - B_M^2(\alpha \cdot A_M))A_M + \overline{\alpha}. \]

Then \( \overline{\alpha} \) is an element in \( P(M) \), hence orthogonal to both \( A_M \) and \( B_M \).

Proof. Writing \( \alpha = aA_M + bB_M + \overline{\alpha} \), the equations \( \overline{\alpha} \cdot A_M = 0 \) and \( \overline{\alpha} \cdot B_M \) determine the coefficients \( a \) and \( b \). \( \square \)

Since \( \Sigma_M = k_M A_M \), we can now write

\[ H^2(M)/\mathbb{Z}\Sigma_M \cong \mathbb{Z}k_M A_M \oplus \mathbb{Z}B_M \oplus P(M). \]

Definition 26. We define \( P(M)_{A_M} = H^2(M)/ (\mathbb{Z}\Sigma_M \oplus \mathbb{Z}B_M) = \mathbb{Z}k_M A_M \oplus P(M). \)

Proposition 27. Suppose that the surface \( \Sigma_M \) has zero self-intersection. Then there exists an isomorphism

\[ H^2(M') \cong P(M)_{A_M} \oplus \mathbb{Z}B_M \oplus \ker i. \]

\(^{1}\)This subgroup corresponds to the Gompf nucleus in elliptic surfaces defined as a regular neighbourhood of a cusp fibre and a section, cf. [11][12].
This proposition shows that the cohomology group $H^2(M')$ decomposes into elements in the interior of $M'$ coming from $M$ (i.e. elements in $P(M)_{A_M}$), classes bounding multiples of the meridian to $\Sigma_M$ (multiples of $B_M$) and classes bounding curves on the push-off $\Sigma^M$ (elements of ker $i$).

Before we continue with the calculation of $H^2(M')$, we derive explicit expressions for the maps

$$
\mu^* : H^k(M') \to H^k(\partial M'), \quad k = 1, 2
$$

and choose certain framings such that $\mu^* : H_1(\partial M') \to H_1(M')$ has a normal form.

### 3.3. Adapted framings

In this subsection, we define a particular class of framings $\tau_M$ which are adapted to the splitting of $H_1(M')$ into $H_1(M)$ and the torsion group determined by the meridian of $\Sigma_M$ as in Proposition [18]. This is a slightly "technical" issue which will make the calculations much easier. If the homology class representing $\Sigma_M$ is indivisible ($k_M = 1$), then there is no restriction and every framing is adapted.

By Corollary [20] there exists an isomorphism of the form

$$
s : H_1(M') \to H_1(M) \oplus \mathbb{Z}_{k_M},
\alpha \mapsto (\rho_* \alpha, A(\alpha)).
$$

The framing $\tau_M$ should be compatible with this isomorphism in the following way: The composition

$$
(11) \quad H_1(\partial M') \xrightarrow{\mu} H_1(M') \xrightarrow{s} H_1(M) \oplus \mathbb{Z}_{k_M}
$$

should be given on generators by

$$
\gamma_i^M \mapsto (i_* \gamma_i, 0)
$$

$$
\sigma^M \mapsto (0, 1).
$$

Consider the exact sequence

$$
(12) \quad H_1(\partial M') \to H_1(M') \oplus H_1(\Sigma) \to H_1(M),
$$

coming from the Mayer-Vietoris sequence for $M$. It maps

$$
\gamma_i^M \mapsto (\mu_* \gamma_i^M, \gamma_i) \mapsto \rho_* \mu_* \gamma_i^M - i_* \gamma_i
$$

$$
\sigma^M \mapsto (\mu_* \sigma^M, 0) \mapsto \rho_* \mu_* \sigma^M.
$$

By exactness of the Mayer-Vietoris sequence, we have $\rho_* \mu_* \gamma_i^M = i_* \gamma_i$ and $\rho_* \mu_* \sigma^M = 0$, where as before $\gamma_i^M$ is determined by $\gamma_i$ via the trivialization $\tau_M$. The isomorphism $s$ above maps

$$
\mu_* \gamma_i^M \mapsto (\rho_* \mu_* \gamma_i^M, A(\mu_* \gamma_i^M)) = (i_* \gamma_i, A(\mu_* \gamma_i^M))
$$

$$
\mu_* \sigma^M \mapsto (0, 1).$$
Let \([c_i^M]\) denote the numbers \(A(\mu_*\gamma_i^M) \in \mathbb{Z}_{k_M}\). It follows that the composition in equation (11) is given on generators by
\[
\gamma_i^M \mapsto (i_*\gamma_i, [c_i^M]) \\
\sigma^M \mapsto (0, 1).
\]

We can change the reference framing \(\tau_M\) to a new framing \(\tau'_M\) such that \(\gamma_i^M\) changes to
\[
\gamma_i^{M'} = \gamma_i^M - c_i^M \sigma^M,
\]
for all \(i = 1, \ldots, 2g\) and \(\sigma^M\) stays the same. This change can be realized by a suitable self-diffeomorphism of \(\partial\nu\Sigma_M\) according to the proof of Lemma 12. The composition in equation (11) now has the form
\[
\gamma_i^{M'} \mapsto (i_*\gamma_i, 0) \\
\sigma^M \mapsto (0, 1).
\]

**Lemma 28.** There exists a trivialization \(\tau_M\) of the normal bundle of \(\Sigma_M\) in \(M\), such that the composition
\[
H_1(\partial M') \xrightarrow{\mu} H_1(M') \xrightarrow{s} H_1(M) \oplus \mathbb{Z}_{k_M}
\]

is given by
\[
\gamma_i^M \mapsto (i_*\gamma_i, 0), \quad i = 1, \ldots, 2g \\
\sigma^M \mapsto (0, 1).
\]

A framing with this property is called adapted.

Every framing is adapted if \(k_M = 1\), since in this case \(H_1(M')\) and \(H_1(M)\) are isomorphic and \(\sigma^M\) is null-homologous.

3.4. **Calculation of the map** \(\mu^* \circ \rho^* : H^1(M) \rightarrow H^1(\partial M')\). We consider the map
\[
\mu^* : H^1(M') \rightarrow H^1(\partial M').
\]

By the proof of Proposition 17 the map \(\rho^* : H^1(M) \rightarrow H^1(M')\) is an isomorphism. The framing \(\tau_M\) defines an identification
\[
H^1(\partial M') \cong H^1(\Sigma) \oplus \mathbb{Z} PD(\Sigma^M),
\]
where \(\Sigma^M\) denotes the push-off of the surface \(\Sigma_M\) and \(PD(\Sigma^M) = \sigma^{M*}\). We want to derive a formula for the composition
\[
H^1(M) \cong H^1(M') \xrightarrow{\mu^*} H^1(\partial M') \cong H^1(\Sigma) \oplus \mathbb{Z} PD(\Sigma^M).
\]
Let \(\alpha \in H^1(M)\). Then by the exactness of the Mayer-Vietoris sequence (12)
\[
\langle \mu^* \rho^* \alpha, \gamma_i^M \rangle = \langle \alpha, \rho_* \mu_* \gamma_i^M \rangle \\
= \langle \alpha, i_* \gamma_i \rangle \\
= \langle i^* \alpha, \gamma_i \rangle,
\]

and

\[ \langle \mu^* \rho^* \alpha, \sigma^i \rangle = \langle \alpha, \rho_* \mu_* \sigma^M \rangle = 0. \]

Hence we have:

**Lemma 29.** The composition

\[ \mu^* \circ \rho^* : H^1(M) \to H^1(\Sigma) \oplus \mathbb{Z} PD(\Sigma^M) \]

is equal to \( i^* \oplus 0 \).

**3.5. Calculation of the map** \( \mu^* : H^2(M') \to H^2(\partial M') \).

Using the framing \( \tau_M \) of the surface \( \Sigma_M \) we can identify

\[ H^2(\partial M') \cong H_1(\Sigma \times S^1) \cong \mathbb{Z} \oplus H_1(\Sigma), \]

where the \( \mathbb{Z} \) summand is spanned by \( PD(\sigma^M) \). We can then consider the composition

\[ (H^2(M)/\Sigma_M) \oplus \ker i \cong H^2(M') \overset{\mu^*}{\twoheadrightarrow} H^2(\partial M') \cong \mathbb{Z} \oplus H_1(\Sigma). \]

**Proposition 30.** The composition

(13) \[ \mu^* : (H^2(M)/\Sigma_M) \oplus \ker i \to \mathbb{Z} \oplus H_1(\Sigma) \]

is given by

\[ ([V], \alpha) \mapsto (V \cdot \Sigma_M, \alpha). \]

The map is well-defined in the first variable since \( \Sigma_M^2 = 0 \) and takes image in \( k \mathbb{Z} \), since \( \Sigma_M \) is divisible by \( k_M \). The map in the second variable is inclusion.

**Proof.** On the second summand, the map \( \mu^* \) is the identity by Lemma [16] and the choice of splitting in Remark [22]. It remains to prove that

\[ \mu^* \rho^*[V] = (V \cdot \Sigma_M) PD(\sigma^M) \]

Exactness of the Mayer-Vietoris sequence for \( M = M' \cup \nu \Sigma_M \) implies the equality \( \mu^* \rho^* V = p^* i^* V. \) Since

\[ \langle i^* V, \Sigma \rangle = \langle V, \Sigma_M \rangle = V \cdot \Sigma_M \]

the class \( i^* V \) is equal to the class \( (V \cdot \Sigma_M)1 \), where 1 denotes the generator of \( H^2(\Sigma) \), Poincaré dual to a point. Since \( p^*(1) \) is the Poincaré dual of a fibre in \( \partial M' = \partial \nu \Sigma_M \), where \( p \) denotes the projection \( \partial \nu \Sigma_M \to \Sigma_M \), the claim follows.

\( \square \)

**Corollary 31.** The composition

\[ \mu^* : P(M)_A M \oplus \mathbb{Z} B_M \oplus \ker i \to \mathbb{Z} \oplus H_1(\Sigma) \]

is given by

\[ (c, xB_M, \alpha) \mapsto (xk_M, \alpha). \]
3.6. **Calculation of** $H_2(M')$. Consider the following sequence coming from the long exact sequence for the pair $(M', \partial M')$:

$$(14) \quad H_3(M', \partial M') \xrightarrow{\partial} H_2(\partial M') \xrightarrow{\mu} H_2(M') \rightarrow H_2(M', \partial M') \xrightarrow{\partial} H_1(\partial M').$$

This induces a short exact sequence

$$(15) \quad 0 \rightarrow H_2(\partial M')/\ker \mu \rightarrow H_2(M') \rightarrow \ker \partial \rightarrow 0.$$  

We will give a more explicit description of the terms on the left and right hand side of this sequence.

Under Poincaré duality, the boundary homomorphism $\partial$ on the right hand side of the exact sequence (14) can be replaced by $\mu^*$:

$$\mu^*: H^2(M') \rightarrow H^2(\partial M'),$$

as in equation (16). Hence the kernel of $\partial$ can be replaced by the kernel of $\mu^*$, which can be written according to Corollary 31 as

$$\ker \mu^* \cong P(M)_{\Lambda M} = H^2(M)/(\mathbb{Z} \Sigma M \oplus \mathbb{Z} B M).$$

This determines the term on the right hand side of sequence (15).

To determine the term on the left side, consider the following part of the exact sequence (14):

$$H_3(M', \partial M') \xrightarrow{\partial} H_2(\partial M') \xrightarrow{\mu} H_2(M').$$

Under Poincaré duality and the isomorphism $\rho^*: H^1(M) \rightarrow H^1(M')$ this sequence becomes

$$(17) \quad H^1(M) \xrightarrow{\mu^* \circ \rho^*} H^1(\partial M') \xrightarrow{\mu^* \circ PD} H_2(M').$$

By Lemma 29 this sequence can be written as

$$(18) \quad H^1(M) \xrightarrow{\tau \oplus 0} H^1(\Sigma) \oplus \mathbb{Z} PD(\Sigma M) \xrightarrow{\tau \oplus \mu} H_2(M'),$$

where $\tau_M$ denotes the homomorphism

$$\tau_M = \mu_M \circ PD \circ p^*_M: H^1(\Sigma) \rightarrow H_2(M'),$$

and $\mu$ is the restriction of this map to $\mathbb{Z} PD(\Sigma M)$. Exactness of sequence (18) implies that the kernel of the homomorphism $\tau_M$ is equal to the image of $i^*$. Hence:

**Proposition 32.** The framing induces an isomorphism

$$H_2(\partial M')/\ker \mu \cong H^1(\Sigma)/\ker \tau_M \oplus \mathbb{Z} PD(\Sigma M)$$

and $H^1(\Sigma)/\ker \tau_M = \text{coker} \ i^*.$

The term $H^1(\Sigma)/\ker \tau_M$ has the following interpretation: Consider the homomorphism $\tau_M$ above. The map $PD \circ p^*_M$ determines an isomorphism of $H^1(\Sigma)$ onto $\ker pf_M$. In our standard basis, this isomorphism is given by

$$H^1(\Sigma) \rightarrow \ker pf_M$$

$$(19) \quad \sum c_i \gamma_i \mapsto \sum c_i \Gamma_i^M.$$
Lemma 33. Every element in the image of $r_M$ can be represented by a smoothly embedded torus in the interior of $M'$.

Proof. Note that the classes $\Gamma_i^M \subset H_2(\partial M')$ are of the form $\chi_i^M \times \sigma^M$ where $\chi_i^M$ is a curve on $\Sigma_M$. Hence every element $T \in \ker p_M$ is represented by a surface of the form $c^M \times \sigma^M$, where $c^M$ is a closed, oriented curve on $\Sigma_M$ with transverse self-intersections. A collar of $\partial M' = \nu \Sigma_M$ in $M'$ is of the form $\Sigma_M \times S^1 \times I$. We can eliminate the self-intersection points of the curve $c^M$ in $\Sigma_M \times I$, without changing the homology class. If we then take $c^M$ times $\sigma^M$, we see that $\mu_M(T) = c^M \times \sigma^M$ can be represented by a smoothly embedded torus in $M'$. □

We make the following definition, see e.g. [3, 7, 15].

Definition 34. The map $r_M$ given by

\[ r_M = \mu_M \circ PD \circ p_M : H^1(\Sigma) \to H_2(M') \]

is called the rim tori homomorphism and the image of $r_M$, denoted by $R(M')$, the group of rim tori in $M'$.

Rim tori are already “virtually” in the manifold $M$ as embedded null-homologous tori. Some of them can become non-zero homology classes if the tubular neighbourhood $\nu \Sigma_M$ is deleted. The set of elements in $H^1(\Sigma)$ whose associated rim tori are null-homologous in $M'$ is given by the kernel of the rim tori map $r_M$, which is isomorphic to the image of $i^*$ by equation (18).

Proposition 35. The short exact sequence (15) for the calculation of $H_2(M')$ can be written as

\[ 0 \to R(M') \oplus \mathbb{Z}PD(\Sigma^M) \to H_2(M') \to P(M)_{A_M} \to 0. \]

4. Calculation of $H_1(X)$ and $H^1(X)$

Let $X = M \#_{\Sigma_M=\Sigma_N} N$ denote the generalized fibre sum of two closed 4-manifolds $M$ and $N$ along embedded surfaces $\Sigma_M$ and $\Sigma_N$ of genus $g$ and self-intersection zero. We assume as in Section 3 that $\Sigma_M$ and $\Sigma_N$ represent non-torsion classes of maximal divisibility $k_M$ and $k_N$ and choose adapted framings for both surfaces as defined in Section 3.3.

Consider the homomorphisms

\[ i_M \oplus i_N : H_1(\Sigma; \mathbb{Z}) \to H_1(M; \mathbb{Z}) \oplus H_1(N; \mathbb{Z}) \]

\[ \lambda \mapsto (i_M(\lambda), i_N(\lambda)), \]

and

\[ i_M^* + i_N^* : H^1(M; \mathbb{Z}) \oplus H^1(N; \mathbb{Z}) \to H^1(\Sigma; \mathbb{Z}) \]

\[ (\alpha, \beta) \mapsto i_M^* \alpha + i_N^* \beta. \]

The kernels of $i_M \oplus i_N$ and $i_M^* + i_N^*$ are free abelian groups, but the cokernels may have torsion terms. Both homomorphisms can also be considered for homology and cohomology with $\mathbb{R}$-coefficients.
Definition 36. Let \( d \) denote the integer \( d = \dim \ker (i_M \oplus i_N) \) for the linear map
\[
i_M \oplus i_N: H_1(\Sigma; \mathbb{R}) \rightarrow H_1(M; \mathbb{R}) \oplus H_1(N; \mathbb{R})
\]
of \( \mathbb{R} \)-vector spaces.

Lemma 37. Consider the homomorphisms \( i_M \oplus i_N \) and \( i^*_M + i^*_N \) for homology and cohomology with \( \mathbb{R} \)-coefficients. Then
\[
dim \ker (i^*_M + i^*_N) = b_1(M) + b_1(N) - 2g + d = \dim \coker (i_M \oplus i_N)
\]
\[
dim \coker (i^*_M + i^*_N) = d = \dim \ker (i_M \oplus i_N),
\]
where \( g \) denotes the genus of the surface \( \Sigma \).

Proof. By linear algebra, \( i^*_M + i^*_N \) is the dual homomorphism to \( i_M \oplus i_N \) under the identification of cohomology with the dual vector space of homology with \( \mathbb{R} \)-coefficients. Moreover,
\[
\dim \coker (i_M \oplus i_N) = b_1(M) + b_1(N) - \dim \im (i_M \oplus i_N)
\]
\[
= b_1(M) + b_1(N) - (2g - \dim \ker (i_M \oplus i_N))
\]
\[
= b_1(M) + b_1(N) - 2g + d.
\]
This implies
\[
\dim \ker (i^*_M + i^*_N) = \dim \coker (i_M \oplus i_N) = b_1(M) + b_1(N) - 2g + d
\]
\[
\dim \coker (i^*_M + i^*_N) = \dim \ker (i_M \oplus i_N) = d.
\]

In the following calculations we will often use the Mayer-Vietoris sequence associated to the decomposition \( X = M' \cup N' \), given by
\[
\ldots \rightarrow H_k(\partial M') \xrightarrow{\psi_k} H_k(M') \oplus H_k(N') \rightarrow H_k(X) \rightarrow H_{k-1}(\partial M') \rightarrow \ldots
\]
with homomorphisms
\[
\psi_k: H_k(\partial M') \rightarrow H_k(M') \oplus H_k(N'), \quad \alpha \mapsto (\mu_M \alpha, \mu_N \phi_\ast \alpha)
\]
\[
H_k(M') \oplus H_k(N') \rightarrow H_k(X), \quad (x, y) \mapsto \eta_M x - \eta_N y.
\]
We also use the Mayer-Vietoris sequence for cohomology groups.

4.1. Calculation of \( H^1(X) \). We begin with the calculation of the first cohomology of \( X \). Consider the following part of the Mayer-Vietoris sequence in cohomology:
\[
0 \rightarrow H^1(X) \xrightarrow{\eta_M \oplus \eta_N} H^1(M') \oplus H^1(N') \xrightarrow{\psi_1^*} H^1(\partial M').
\]
Since \( \eta^*_M - \eta^*_N \) is injective, \( H^1(X) \) is isomorphic to the kernel of \( \psi_1^* = \mu^*_M + \phi^* \mu^*_N \). Composing with isomorphisms,
\[
H^1(M) \oplus H^1(N) \cong H^1(M') \oplus H^1(N') \xrightarrow{\psi_1^*} H^1(\partial M') \cong H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^M),
\]
the map $\psi_i^* \circ \phi_i$ can be replaced by the map $\mu_M^* \rho_M^* + \phi_i^* \mu_N^* \rho_N^*$. Since the equality $\phi_i^* \gamma_i'^N = \gamma_i^M$ holds for all $i = 1, \ldots, 2g$, it follows with Lemma 29 that this composition can be replaced by the map

$$(i_M^* + i_N^*) \circ 0: H^1(M) \oplus H^1(N) \to H^1(\Sigma) \oplus \mathbb{Z}PD(\Sigma^M).$$

This implies:

**Theorem 38.** The first cohomology $H^1(X; \mathbb{Z})$ is isomorphic to the kernel of

$$i_M^* + i_N^*: H^1(M; \mathbb{Z}) \oplus H^1(N; \mathbb{Z}) \to H^1(\Sigma; \mathbb{Z}).$$

As a corollary we can compute the Betti numbers of $X$.

**Corollary 39.** The Betti numbers of a generalized fibre sum $X = M \#_{\Sigma_M = \Sigma_N} N$ along surfaces $\Sigma_M$ and $\Sigma_N$ of genus $g$ and self-intersection 0 are given by

$$b_0(X) = b_4(X) = 1$$
$$b_1(X) = b_3(X) = b_1(M) + b_1(N) - 2g + d$$
$$b_2(X) = b_2(M) + b_2(N) - 2 + 2d$$
$$b_2^+(X) = b_2^+(M) + b_2^+(N) - 1 + d$$
$$b_2^-(X) = b_2^-(M) + b_2^-(N) - 1 + d,$$

where $d$ is the integer from Definition 36.

**Proof.** The formula for $b_1(X)$ follows from Theorem 38 and Lemma 37. To derive the formula for $b_2(X)$, we use the formula for the Euler characteristic of a space decomposed into two parts $A, B$:

$$e(A \cup B) = e(A) + e(B) - e(A \cap B),$$

For $M = M' \cup \nu \Sigma_M$, with $M' \cap \nu \Sigma_M \cong \Sigma \times S^1$, we get

$$e(M) = e(M') + e(\nu \Sigma_M) - e(\Sigma \times S^1) = e(M') + 2 - 2g,$$

since $\nu \Sigma_M$ is homotopy equivalent to $\Sigma_M$ and $\Sigma \times S^1$ is a 3-manifold, hence has zero Euler characteristic. This implies

$$e(M') = e(M) + 2g - 2, \quad \text{and similarly} \quad e(N') = e(N) + 2g - 2.$$

For $X = M' \cup N'$, with $M' \cap N' \cong \Sigma \times S^1$, we then get

$$e(X) = e(M') + e(N') = e(M) + e(N) + 4g - 4.$$

Substituting the formula for $b_1(X) = b_3(X)$ above, this implies

$$b_2(X) = -2 + 2(b_1(M) + b_1(N) - 2g + d) + 2 - 2b_1(M) + b_2(M) + 2 - 2b_1(N) + b_2(N) + 4g - 4$$
$$= b_2(M) + b_2(N) - 2 + 2d.$$
It remains to prove the formula for $b_2^+(X)$. By Novikov additivity for the signature \cite[Remark 9.1.7]{12},
\[ \sigma(X) = \sigma(M) + \sigma(N), \]
we get by adding $b_2(X)$ on both sides,
\[ 2b_2^+(X) = 2b_2^+(M) + 2b_2^+(N) - 2 + 2d, \]
hence $b_2^+(X) = b_2^+(M) + b_2^+(N) - 1 + d$. This also implies the formula for $b_2^-(X)$. \hfill \Box

A direct computation of $b_2^+(X)$ as the rank of $H_2(X)$ will be given in Section 5.3.

4.2. Calculation of $H_1(X)$. In this subsection we prove a formula for the first integral homology of $X$. We make the following definition:

**Definition 40.** Let $n_{MN}$ denote the greatest common divisor of $k_M$ and $k_N$.

If $n_{MN}$ is not equal to 1, the formula for $H_1(X)$ involves an additional torsion term. Let $r$ denote the homomorphism defined by
\[ r : H_1(\Sigma; \mathbb{Z}) \to \mathbb{Z}_{n_{MN}}, \]
\[ \lambda \mapsto \langle C, \lambda \rangle \mod n_{MN}. \]

We then have the following theorem:

**Theorem 41.** Consider the homomorphism
\[ H_1(\Sigma; \mathbb{Z}) \xrightarrow{i_M \oplus i_N \oplus r} H_1(M; \mathbb{Z}) \oplus H_1(N; \mathbb{Z}) \oplus \mathbb{Z}_{n_{MN}}, \]
\[ \lambda \mapsto (i_M \lambda, i_N \lambda, r(\lambda)). \]

Then $H_1(X; \mathbb{Z})$ is isomorphic to the cokernel of $i_M \oplus i_N \oplus r$.

**Proof.** Since $\partial M'$, $M'$ and $N'$ are connected, the Mayer-Vietoris sequence for $X$ shows that
\[ H_1(X) \cong \text{coker}(\psi : H_1(\partial M') \to H_1(M') \oplus H_1(N')). \]
The homomorphism $\psi_1$ is given on the standard basis by
\[ \gamma_i^M \mapsto (\mu_M \gamma_i^M, \mu_N \gamma_i^N + a_i \mu_N \sigma^N) \]
\[ \sigma^M \mapsto (\mu_M \sigma^M, -\mu_N \sigma^N). \]

We want to replace $H_1(M')$ by $H_1(M) \oplus \mathbb{Z}_{k_M}$ and $H_1(N')$ by $H_1(N) \oplus \mathbb{Z}_{k_N}$, as in Proposition \ref{18}. We choose isomorphisms $s$ as in Section 3.3. Since we are working with adapted framings, the composition
\[ H_1(\partial M') \xrightarrow{\mu_M} H_1(M') \xrightarrow{\delta} H_1(M) \oplus \mathbb{Z}_{k_M} \]
is given on generators by
\[ \gamma_i^M \mapsto (i_M \gamma_i, 0) \]
\[ \sigma^M \mapsto (0, 1). \]
as before. An analogous map exists for \( N \). If we add these maps together, the homomorphism \( \psi_1 \) can be replaced by

\[
H_1(\partial M') \to H_1(M) \oplus k_{M} \oplus H_1(N) \oplus k_{N},
\]

\[
\gamma_i^M \mapsto (i_M \gamma_i, 0, i_N \gamma_i, a_i)
\]

\[
\sigma^M \mapsto (0, 1, 0, -1).
\]

Using the isomorphism \( H_1(\Sigma \times S^1) \cong H_1(\Sigma) \oplus \mathbb{Z} \to H_1(\partial M') \) given by the framing \( \tau_M \), we get the map

\[
H_1(\Sigma) \oplus \mathbb{Z} \to H_1(M) \oplus \mathbb{Z} \oplus M \oplus H_1(N) \oplus \mathbb{Z}.
\]

\[
(\lambda, \alpha) \mapsto (i_M \lambda, i_N \lambda, \langle C, \lambda \rangle - \alpha \mod k_N).
\]

To finish the proof, we have to show that this map has the same cokernel as the map

\[
i_M \oplus i_N \oplus r: H_1(\Sigma) \to H_1(M) \oplus H_1(N) \oplus \mathbb{Z},
\]

\[
\lambda \mapsto (i_M \lambda, i_N \lambda, \langle C, \lambda \rangle \mod n_{MN}).
\]

This follows from Lemma 42 below.

In the proof we used a small algebraic lemma which can be formulated as follows: Let \( H \) and \( G \) be abelian groups and \( f: H \to G \) and \( h: H \to \mathbb{Z} \) homomorphisms. Let \( k_M, k_N \) be positive integers with greatest common divisor \( n_{MN} \). Consider the (well-defined) map

\[
p: \mathbb{Z}_{k_M} \oplus \mathbb{Z}_{k_N} \to \mathbb{Z}_{n_{MN}},
\]

\[
([x], [y]) \mapsto [x + y].
\]

**Lemma 42.** The homomorphisms

\[
\psi: H \oplus \mathbb{Z} \to G \oplus \mathbb{Z}_{k_M} \oplus \mathbb{Z}_{k_N}
\]

\[
(x, a) \mapsto (f(x), a \mod k_M, h(x) - a \mod k_N),
\]

and

\[
\psi': H \to G \oplus \mathbb{Z}_{n_{MN}}
\]

\[
x \mapsto (f(x), h(x) \mod n_{MN})
\]

have isomorphic cokernels. The isomorphism is induced by \( \text{Id} \oplus p \).

**Proof.** The map \( \text{Id} \oplus p \) is a surjection, hence it induces a surjection

\[
P: G \oplus \mathbb{Z}_{k_M} \oplus \mathbb{Z}_{k_N} \to \text{coker} \, \psi'.
\]

We compute the kernel of \( P \) and show that it is equal to the image of \( \psi \). This will prove the lemma. Suppose an element is in the image of \( \psi \). Then it is of the form \((f(x), a \mod k_M, h(x) - a \mod k_N)\). The image under \( \text{Id} \oplus p \) of this element is \((f(x), h(x) \mod n_{MN})\), hence in the image of \( \psi' \). Conversely, let \((g, u \mod k_M, v \mod k_N)\) be an element in the kernel of \( P \). The element maps under \( \text{Id} \oplus p \) to \((g, u + v \mod n_{MN})\), hence there exists an element \( x \in H \) such
that \( g = f(x) \) and \( u + v \equiv h(x) \mod n_{MN} \). We can choose integers \( c, d, e \) such that the following equations hold:
\[
\begin{align*}
u + v - h(x) &= cn_{MN} = dk_M + ek_N.
\end{align*}
\]

Define an integer \( a = u - dk_M \). Then:
\[
\begin{align*}
u &\equiv h(x) - a + ek_N = h(x) - a \mod k_N.
\end{align*}
\]

Hence \( (g, u \mod k_M, v \mod k_N) = \psi(x, a) \) and the element is in the image of \( \psi \).

An immediate corollary of Theorem 41 is the following.

**Corollary 43.** If the greatest common divisor of \( k_M \) and \( k_N \) is equal to 1, then \( H_1(X; \mathbb{Z}) \cong \ker(i_M \oplus i_N) \). In particular, \( H_1(X; \mathbb{Z}) \) does not depend on the cohomology class \([C]\) in this case (up to isomorphism).

5. **Calculation of \( H^2(X) \)**

The computation of \( H^2(X) \) is based on the following proposition:

**Proposition 44.** The following part of the Mayer-Vietoris sequence
\[
\begin{align*}
H^1(M') \oplus H^1(N') &\xrightarrow{\psi_1^*} H^1(\partial M') \to H^2(X) \xrightarrow{\eta_M \oplus \eta_N^*} H^2(M') \oplus H^2(N') \xrightarrow{\psi_2^*} H^2(\partial M').
\end{align*}
\]

induces a short exact sequence
\[
\begin{align*}
0 \to \ker \psi_1^* \to H^2(X) \to \ker \psi_2^* \to 0.
\end{align*}
\]

By equation (21) there exists an isomorphism
\[
\begin{align*}
\ker \psi_1^* \cong \ker (i_M^* + i_N^* \oplus \mathbb{Z}PD(\Sigma^M)).
\end{align*}
\]

We calculate \( \ker (i_M^* + i_N^*) \) in the following subsection and \( \ker \psi_2^* \) in the next subsection.

**5.1. Rim tori in \( X \).** We can map every rim torus in \( M' \) under the inclusion \( \eta_M : M' \to X \) to a homology class in \( X \).

**Definition 45.** We call \( \eta_M \circ \mu_M(\alpha) \) the rim torus in \( X \) associated to the element \( \alpha \in H^1(\Sigma) \) via \( M' \). The group \( R(X) \) of rim tori in \( X \) is defined as the image of the homomorphism
\[
\begin{align*}
r_X = \eta_M \circ \mu_M \circ PD \circ p_M^* : H^1(\Sigma) \to H_2(X).
\end{align*}
\]

Similarly, we can map every rim torus in \( N' \) under the inclusion \( \eta_N : N' \to X \) to a homology class in \( X \). This defines a homomorphism
\[
\begin{align*}
r_X = \eta_N \circ \mu_N \circ PD \circ p_N^* : H^1(\Sigma) \to H_2(X).
\end{align*}
\]

The rim tori in \( X \) coming via \( M' \) and \( N' \) are related in the following way:

**Lemma 46.** Let \( \alpha \) be a class in \( H^1(\Sigma) \). Then \( r_X(\alpha) = -r_X'(\alpha) \). Hence for the same element \( \alpha \in H^1(\Sigma) \) the rim torus in \( X \) coming via \( N' \) is minus the rim torus coming via \( M' \).
Proof. The action of the gluing diffeomorphism $\phi$ on second homology is given by $\phi_\ast \Gamma_i^M = -\Gamma_i^N$. Let $\alpha \in H^1(\Sigma)$ be a fixed class,

$$\alpha = \sum_{i=1}^{2g} c_i \gamma_i^*.$$

The rim tori in $M'$ and $N'$ associated to $\alpha$ are given by

$$a_M = \sum_{i=1}^{2g} c_i \mu_M \Gamma_i^M, \quad a_N = \sum_{i=1}^{2g} c_i \mu_N \Gamma_i^N = -\sum_{i=1}^{2g} c_i \mu_N \phi_\ast \Gamma_i^M.$$

In $X$ we get

$$\eta_M a_M + \eta_N a_N = \sum_{i=1}^{2g} c_i (\eta_M \mu_M - \eta_N \mu_N \phi_\ast) \Gamma_i^M = 0,$$

by the Mayer-Vietoris sequence for $X$. This proves the claim. \hfill $\square$

Definition 47. Let $R_C$ denote the rim torus in $X$ determined by the class

$$-\sum_{i=1}^{2g} a_i \Gamma_i^M \in H_2(\partial M')$$

under the inclusion of $\partial M'$ in $X$ as in Definition 45. This class is equal to the image of the class $\sum_{i=1}^{2g} a_i \Gamma_i^N \in H_2(\partial N')$ under the inclusion of $\partial N'$ in $X$.

Recall that $\Sigma_X$ is the class in $X$ which is the image of the push-off $\Sigma^M$ under the inclusion $M' \to X$. Similarly, $\Sigma_X'$ is the image of the push-off $\Sigma^N$ under the inclusion $N' \to X$.

Lemma 48. The classes $\Sigma_X'$ and $\Sigma_X$ in $X$ differ by

$$\Sigma_X' - \Sigma_X = R_C.$$

Proof. This follows, since by Lemma 13

$$\phi_\ast \Sigma^M = -\left( \sum_{i=1}^{2g} a_i \Gamma_i^N \right) + \Sigma^N.$$

The difference is due to the fact that the diffeomorphism $\phi$ does not necessarily match the classes $\Sigma^M$ and $\Sigma^N$. We now prove the main theorem in this subsection.

Theorem 49. The kernel of $r_X$ is equal to the image of $i_M^\ast + i_N^\ast$. Hence the map $r_X$ induces an isomorphism

$$\text{coker} \left( i_M^\ast + i_N^\ast \right) \xrightarrow{\cong} R(X).$$

Lemma 50. The kernel of the map $\eta_M$ is equal to the image of $r_M \circ i_N^\ast$. 
\textbf{Proof.} Consider the following diagram:

\[
\begin{array}{c}
H_3(X, M') & \xrightarrow{\partial} & H_2(M') & \xrightarrow{\eta_M} & H_2(X) \\
\cong & & \mu_M \phi^{-1} & & \eta_N \\
H_3(N', \partial N') & \xrightarrow{\partial} & H_2(\partial N') & \longrightarrow & H_2(N')
\end{array}
\]

The horizontal parts come from the long exact sequences of pairs, the vertical parts come from inclusion. The isomorphism on the left is by excision. Hence the kernel of \(\eta_M\) is given by the image of \(\mu_M \circ \phi^{-1} \circ \partial\): \(H_3(N', \partial N') \to H_2(M')\).

Given the definition of the rim tori map \(r_M\) in equation (20), we have to show that this is equal to the image of \(\mu_M \circ PD \circ i^*_N \circ \rho^*_N\): \(H^1(\Sigma) \to H_2(M')\).

This follows in three steps: First, by Lemma 16, the image of \(\partial\) is equal to the image of \(PD \circ \mu_N \circ \rho_N\). By the Mayer-Vietoris sequence for \(N\)

\[
H^1(N) \xrightarrow{\phi^{-1}_N \oplus i^*_N} H^1(N') \oplus H^1(\Sigma) \xrightarrow{\mu_N + p^*_N} H^1(\partial N')
\]

we have \(\mu_N \circ \rho_N = p^*_N \circ i^*_N\). Finally, we use the identity

\[\phi^{-1}_s \circ PD \circ p^*_N = -PD \circ p^*_M,\]

which is equivalent to the identity \(\phi_s \Gamma^M_i = -\Gamma^N_i\), for all \(i = 1, \ldots, 2g\), from Lemma 13. \(\square\)

We can now prove Theorem 49.

\textbf{Proof.} Suppose that \(\alpha \in H^1(\Sigma)\) is in the kernel of \(r_X = \eta_M \circ r_M\). This happens if and only if \(r_M(\alpha)\) is in the kernel of \(\eta_M\). By Lemma 50, this is equivalent to the existence of a class \(\beta_N \in H^1(\Sigma)\) with

\[r_M(\alpha) = (r_M \circ i^*_N)(\beta_N)\]

By the sentence before Proposition 32, this is equivalent to the existence of a class \(\beta_M \in H^1(M)\) with

\[\alpha = i^*_M \beta_M + i^*_N \beta_N\]

This shows that the kernel of \(r_X\) is equal to the image of \(i^*_M + i^*_N\) and proves the claim. \(\square\)

\textbf{Corollary 51.} The rank of the abelian subgroup \(R(X)\) of rim tori in \(X\) is equal to the integer \(d\) from Definition 36.
5.2. Split classes. For the calculation of $H^2(X)$ it remains to calculate the kernel of

$$\psi^*_{2}: H^2(M') \oplus H^2(N') \rightarrow H^2(\partial M'),$$

where $\psi^*_{2} = \mu^*_M + \phi^* \mu^*_N$, as in equation (24). We will first replace this map by an equivalent map. By Corollary 31, the map $\mu^*_M$ can be replaced by

$$P(M)_A \oplus \mathbb{Z}B_M \oplus \ker i_M \rightarrow k_M \mathbb{Z}PD(\sigma^M) \oplus H_1(\Sigma)$$

$$(c_M, x MB_M, \alpha_M) \mapsto (x_Mk_M, \alpha_M).$$

We can replace $\mu^*_N$ by a similar map

$$P(N)_A \oplus \mathbb{Z}B_N \oplus \ker i_N \rightarrow k_N \mathbb{Z}PD(\sigma^N) \oplus H_1(\Sigma)$$

$$(c_N, x NB_N, \alpha_N) \mapsto (x_Nk_N, \alpha_N).$$

Under the identifications

$$H^2(\partial M') \cong H_1(\partial M') \cong \mathbb{Z} \oplus H_1(\Sigma),$$

and

$$H^2(\partial N') \cong H_1(\partial N') \cong \mathbb{Z} \oplus H_1(\Sigma)$$

given by the framings $\tau_M, \tau_N$, we can replace the map

$$\phi^*: H^2(\partial N') \rightarrow H^2(\partial M')$$

according to equation (5) by

$$\mathbb{Z} \oplus H_1(\Sigma) \rightarrow \mathbb{Z} \oplus H_1(\Sigma)$$

$$(x, y) \mapsto (x - \langle C, y \rangle, -y).$$

Lemma 52. The map $\psi^*_{2} = \mu^*_M + \phi^* \mu^*_N$ can be replaced by the homomorphism

$$P(M)_A \oplus P(N)_A \oplus \mathbb{Z}B_M \oplus \mathbb{Z}B_N \oplus \ker i_M \oplus \ker i_N \rightarrow \mathbb{Z} \oplus H_1(\Sigma)$$

given by

$$(c_M, c_N, x_M, x_N, \alpha_M, \alpha_N) \mapsto (x_Mk_M + x_Nk_N - \langle C, \alpha_N \rangle, \alpha_M - \alpha_N).$$

Definition 53. We consider the map

$$f: \mathbb{Z}B_M \oplus \mathbb{Z}B_N \oplus \ker (i_M \oplus i_N) \rightarrow \mathbb{Z}$$

$$(x MB_M, x NB_N, \alpha) \mapsto x_Mk_M + x_Nk_N - \langle C, \alpha \rangle.$$ 

Let $S(X)$ denote the kernel of the map $f$. We call $S(X)$ the group of split classes of $X$. It is a free abelian group of rank $d + 1$ by Lemma 32.

The classes in $S(X)$ are also called vanishing classes, for example in [6]. They have the following interpretation:

Lemma 54. The elements $(x_MB_M, x_NB_N, \alpha)$ in $S(X)$ are precisely those elements in $\mathbb{Z}B_M \oplus \mathbb{Z}B_N \oplus \ker (i_M \oplus i_N)$ such that $\alpha^M + x_Mk_M \sigma^M$ bounds in $M'$, $\alpha^N + x_Nk_N \sigma^N$ bounds in $N'$, and both elements get identified under the gluing diffeomorphism $\phi$.
Proof. Suppose that an element
\[ \alpha^M + r\sigma^M = \tau^M \alpha + r\sigma^M \in H_1(\partial M'), \]
with \( \alpha \in H_1(\Sigma) \), is null-homologous in \( M' \). Since we are using adapted framings, this happens if and only if \( i_M \alpha = 0 \in H_1(M) \) and \( r \) is divisible by \( k_M \), hence \( r = x_M k_M \) for some \( x_M \in \mathbb{Z} \). In this case the curve bounds a surface in \( M' \). The class \( \alpha^M + r\sigma^M \) maps under \( \phi \) to the class
\[ \alpha^N + \langle C, \alpha \rangle \sigma^N - r\sigma^N. \]
This class is null-homologous in \( N' \) if and only if \( i_N \alpha = 0 \) and \( \langle C, \alpha \rangle - r = \langle C, \alpha \rangle - x_M k_M \) is divisible by \( k_N \), hence
\[ \langle C, \alpha \rangle - x_M k_M = x_N k_N. \]
\[ \square \]

We can now prove:

**Theorem 55.** The kernel of the homomorphism
\[ \psi^*_2 : H^2(M') \oplus H^2(N') \to H^2(\partial M') \]
is isomorphic to \( S(X) \oplus P(M)_{A_M} \oplus P(N)_{A_N} \).

**Proof.** Elements in the kernel must satisfy \( \alpha^M = \alpha^N \). In particular, both elements are in \( \ker i_M \cap \ker i_N = \ker (i_M \oplus i_N) \). Hence the kernel of the replaced \( \psi^*_2 \) is given by
\[ S(X) \oplus P(M)_{A_M} \oplus P(N)_{A_N}. \]
\[ \square \]

5.3. **Calculation of** \( H^2(X) \). Using Theorem 55 and Theorem 49 we can now replace the terms in Proposition 44 to get the following theorem:

**Theorem 56.** There exists a short exact sequence
\[ 0 \to R(X) \oplus \mathbb{Z}\Sigma_X \to H^2(X; \mathbb{Z}) \to S(X) \oplus P(M)_{A_M} \oplus P(N)_{A_N} \to 0. \]

Since we already calculated the rank of each group occurring in this short exact sequence, we can calculate the second Betti number of \( X \):
\[ b_2(X) = d + 1 + (d + 1) + (b_2(M) - 2) + (b_2(N) - 2) = b_2(M) + b_2(N) - 2 + 2d. \]
This is the same number as in Corollary 49.

In the following sections we will restrict to the case that \( \Sigma_M, \Sigma_N \) represent indivisible classes, hence \( k_M = k_N = 1 \), and the cohomologies of \( M, N \) and \( X \) are torsion-free, which is equivalent to \( H^2 \) or \( H_1 \) being torsion free. For the manifold \( X \) this can be checked using the formula for \( H_1(X) \) in Theorem 41 or Corollary 43. Then the exact sequence in Theorem 56 reduces to
\[ 0 \to R(X) \oplus \mathbb{Z}\Sigma_X \to H^2(X) \to S(X) \oplus P(M) \oplus P(N) \to 0. \]
Since all groups are free abelian, the sequence splits and we get an isomorphism
\[ H^2(X) \cong P(M) \oplus P(N) \oplus S(X) \oplus R(X) \oplus \mathbb{Z}\Sigma_X. \]
6. The Intersection Form of $X$

From now on until the end of this article we will assume that $\Sigma_M$ and $\Sigma_N$ represent indivisible classes and the cohomologies of $M$, $N$ and $X$ are torsion free. The group of split classes $S(X)$ contains the element

$$B_X = B_M - B_N.$$ 

We want to prove that we can choose $d$ elements $S_1, \ldots, S_d$ in $S(X)$, forming a basis for a subgroup $S'(X)$ such that $S(X) = \mathbb{Z}B_X \oplus S'(X)$, and a basis $R_1, \ldots, R_d$ for the group of rim tori $R(X)$ such that the following holds:

**Theorem 57.** Let $X = M \# \Sigma_M = \Sigma_N N$ be a generalized fibre sum of closed oriented 4-manifolds $M$ and $N$ along embedded surfaces $\Sigma_M, \Sigma_N$ of genus $g$ which represent indivisible homology classes. Suppose that the cohomology of $M$, $N$ and $X$ is torsion free. Then there exists a splitting

$$H^2(X; \mathbb{Z}) = P(M) \oplus P(N) \oplus (S'(X) \oplus R(X)) \oplus (\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X),$$

where

$$(S'(X) \oplus R(X)) = (\mathbb{Z}S_1 \oplus \mathbb{Z}R_1) \oplus \ldots \oplus (\mathbb{Z}S_d \oplus \mathbb{Z}R_d).$$

The direct sums are all orthogonal, except the direct sums inside the brackets. In this decomposition of $H^2(X; \mathbb{Z})$, the restriction of the intersection form $Q_X$ to $P(M)$ and $P(N)$ is equal to the intersection form induced from $M$ and $N$ and has the structure

$$\begin{pmatrix}
    B^2_M + B^2_N & 1 \\
    1 & 0
\end{pmatrix}$$

on $\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X$ and the structure

$$\begin{pmatrix}
    S_i^2 & 1 \\
    1 & 0
\end{pmatrix}$$

on each summand $\mathbb{Z}S_i \oplus \mathbb{Z}R_i$.

The outline of the proof is the following: We first construct a basis for the subgroup

$$S(X) \oplus P(M) \oplus P(N) \text{ of } H_2(M', \partial M') \oplus H_2(N', \partial N'),$$

where the basis elements $S_1, \ldots, S_d$ for the subgroup $S'(X)$ consist of surfaces sewed together from surfaces $S_i^M$ and $S_i^N$ in $M'$ and $N'$ with boundaries on $\partial M'$ and $\partial N'$, such that the boundaries get identified under the gluing diffeomorphism $\phi$. The surfaces $S_i^M$ on the $M'$ side are chosen such that they have zero algebraic intersection with the basis elements in $P(M)$, the surface $B_i^M$ and the surface $\Sigma_i^M$, and similarly on the $N'$ side. Lifting the basis to $H^2(X)$ and choosing a basis for the rim tori group $R(X)$ we get a basis for

$$H^2(X) \cong P(M) \oplus P(N) \oplus (S'(X) \oplus R(X)) \oplus (\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X),$$

which is orthogonal except inside the brackets. The basis elements for the split classes $S'(X)$ are then modified using the rim tori in $R(X)$ such that the intersection form on $(S'(X) \oplus R(X))$ takes the form in Theorem 57.
The construction of the basis for $S(X)$ is rather lengthy and will be done step by step. First choose a basis $\alpha_1, \ldots, \alpha_d$ for the free abelian group $\ker(i_M \oplus i_N)$. We then get a basis of $S(X)$ consisting of the element

$$B_X = B_M - B_N$$

and $d$ further elements of the form

$$S_i = x_N(\alpha_i)B_N + \alpha_i, \quad 1 \leq i \leq d,$$

where $x_N(\alpha_i)$ are coefficients given by

$$x_N(\alpha_i) = \langle C, \alpha_i \rangle.$$

The class $B_X$ is sewed together from surfaces in $M'$ and $N'$ representing the classes $B'_M$ and $B'_N$ that bound the meridians $\sigma^M$ and $\sigma^N$ in $\partial M'$ and $\partial N'$. We also want to realize the homology classes $S_i$ by embedded surfaces in $X$.

Surfaces that bound curves on the boundary $\partial M'$ are always oriented as follows:

**Definition 58.** Suppose that $L_M$ is a connected, orientable surface in $M'$ which is transverse to the boundary $\partial M'$ and bounds an oriented curve $c_M$ on $\partial M'$. In a collar of the boundary of the form $\Sigma \times S^1 \times I$, we can assume that the surface $L_M$ has the form $c_M \times I$. The orientation of $L_M$ is then chosen such that it induces on $c_M \times I$ the orientation of $c_M$ followed by the orientation of the interval $I$ pointing out of $M'$.

This definition applies in particular to the surfaces $B'_M$ and $B'_N$ bounding the meridians in $M'$ and $N'$: The surfaces $\Sigma_M$ and $\Sigma_N$ are oriented by the embeddings $i_M, i_N$ from a fixed oriented surface $\Sigma$. The surfaces $B_M$ and $B_N$ are oriented such that $\Sigma_M B_M = +1$ and $\Sigma_N B_N = +1$.

**Lemma 59.** The restriction of the orientation of $B_M$ to the punctured surface $B'_M$ is equal to the orientation determined by Definition 58.

**Proof.** On the 2-disk $D$ in $B_M$ obtained by the intersection with the tubular neighborhood $\nu \Sigma_M$ the restriction of the orientation is given by the direction pointing into $M'$ followed by the orientation of the meridian $\sigma^M$. This is equal to the orientation of $\sigma^M$ followed by the direction pointing out of $M'$.

The extension $\Phi$ of the gluing diffeomorphism $\phi$ (see equation (3)) inverts on the 2-disk $D$ the inside-outside direction and the direction along the boundary $\partial D$. Hence the oriented surfaces $B'_M$ and $B'_N$ sew together to define an oriented surface $B_X$ in $X$.

6.1. **Construction of the surfaces $D^M_i$.** The images of the loops $\alpha_i$ on $\Sigma$ under the embedding $i_M$ are null-homologous curves in $M$. We can consider the images of the $\alpha_i$ as curves $\alpha^M_i$ on the push-off $\Sigma^M$ on the boundary of a tubular neighborhood $\nu \Sigma_M$. We also choose a basis for $P(M)$ consisting of pairwise transverse, oriented surfaces $P_1, \ldots, P_n$ embedded in $M$. The surfaces $P_1, \ldots, P_n$ can be chosen disjoint from $\Sigma_M$ and hence be considered as surfaces in $M'$.
Lemma 60. For all $i$, there exist oriented embedded surfaces $D_i^M$ in $M'$ transverse to the boundary $\partial M'$ and bounding the curves $\alpha_i^M$. We can assume that each surface $D_i^M$ intersects the surfaces $P_1, \ldots, P_n$ transversely such that the algebraic count of intersections is zero.

Proof: Since the images of the curves $\alpha_i$ under the embedding $i_M$ into $M$ are null-homologous, we can assume that the curves $\alpha_i^M$ bound connected, orientable surfaces $D_i^M$ in the complement $M'$, transverse to the boundary $\partial M'$.

We choose on $D_i^M$ the orientation determined by Definition 58. The surfaces $D_i^M$ are simplified as follows: We can connect the surface $D_i^M$ to any other closed surface in $M$ in the complement of $\nu \Sigma M$ to get a new surface which still bounds the same loop $\alpha_i^M$. We can consider the surface $D_i^M = D' M$ to be transverse to the surfaces $P_1, \ldots, P_n$ and disjoint from their intersections. Let $\delta_j$ be the algebraic intersection number of the surface $D_i^M$ with the surface $P_j$. We want to add closed surfaces to $D_i^M$ to make the intersection numbers $\delta_j$ for all $j = 1, \ldots, n$ zero. The new surface $D'$ then does not intersect algebraically the surfaces giving a basis for the free part of $P(M)$.

Let $\beta$ denote the matrix with entries $\beta_{kj} = P_k \cdot P_j$ for $k, j = 1, \ldots, n$, determined by the intersection form of $M$. This matrix is invertible over $\mathbb{Z}$ since the restriction of the intersection form to $P(M)$ is unimodular. Hence there exists a unique vector $r \in \mathbb{Z}^n$ such that

$$\sum_{k=1}^n r_k \beta_{kj} = -\delta_j.$$

Let $D' = D + \sum_{k=1}^n r_k P_k$. Then

$$D' \cdot P_j = \delta_j + \sum_{k=1}^n r_k \beta_{kj} = 0.$$

□

Lemma 61. We can assume that the surfaces $D_i^M$ are disjoint from the surface $\Sigma^M$. We can also assume that the intersections of the surfaces $D_i^M$ with the surface $B'_M$ are transverse, contained in the interior of $M'$ and have zero algebraic intersection count.

Proof: We can assume that $\Sigma^M$ is given by the push-off $\Sigma_M \times \{p\}$ in the boundary $\Sigma_M \times S^1$ and that the curves $\alpha_i^M$ are embedded in a different push-off $\Sigma_M \times \{q\}$ with $p \neq q$. This implies the first claim.

To prove the second claim, we can assume that the meridian $\sigma^M$ is given by $\{x\} \times S^1$, where $x$ is a point on $\Sigma$ disjoint from all curves $\alpha_i$. The intersections of $D_i^M$ and $B'_M$ are hence contained in the interior of $M'$ and can be assumed transverse. The push-off $\Sigma^M$ moved into the interior of $M'$ has transverse intersection +1 with $B'_M$. Tubing several copies of $\Sigma^M$ to the surface $D_i^M$ we can achieve that the intersections of $D_i^M$ with $B'_M$ add to zero. □
We want to determine the intersection numbers of the surfaces $D^M_i$ with the rim tori in $M'$. Let $\gamma_1, \ldots, \gamma_{2g}$ denote the basis of $H_1(\Sigma)$. We defined a basis for $H_2(\Sigma \times S^1)$ given by the elements $\Gamma_i = PD(\gamma_i^*)$. On the 3-manifold $\Sigma \times S^1$ we have

$$\Gamma_i \cdot \gamma_j = \delta_{ij}.$$ 

More generally, suppose that a class $T$ represents the element $\sum_{i=1}^{2g} c_i \gamma_i^*$. Then $PD(T) = \sum_{i=1}^{2g} c_i \Gamma_i$ and

$$PD(T) \cdot \gamma_j = \langle T, \gamma_j \rangle = c_j.$$ 

These relations also hold on $\partial M'$ and $\partial N'$. Consider the closed, oriented curve $\gamma_j$ on $\Sigma$. We view $\gamma_j$ as a curve on the push-off $\Sigma^M$ in $M'$. It defines a small annulus $\gamma_j \times I$ in a collar of the form $\Sigma \times S^1 \times I$ of the boundary $\partial M'$. On the annulus we choose the orientation given by Definition 58. Then the intersection number of $\Gamma_i^M$ and $\gamma_j \times I$ in the manifold $M'$ is given by

$$\Gamma_i^M \cdot (\gamma_j \times I) = \delta_{ij}$$

according to the orientation convention for $M'$, cf. Section 2.

More generally, suppose that

$$e = \sum_{i=1}^{2g} c_i \gamma_i$$

is an oriented curve on $\Sigma$ and $E_M$ the annulus $E_M = e \times I$ defined by $e$. Let $R^M_T$ be a rim torus in $M'$ induced from an element $T \in H^1(\Sigma)$. Then $R^M_T$ is the image of

$$\sum_{j=1}^{2g} \langle T, \gamma_j \rangle \Gamma_j^M$$

under the inclusion of $\partial M'$ in $M'$. We then have

$$R^M_T \cdot E_M = \sum_{j=1}^{2g} \langle T, \gamma_j \rangle e_j = \langle T, e \rangle.$$

**Lemma 62.** With our orientation conventions, the algebraic intersection number of a rim torus $R^M_T$ and an annulus $E_M$ as above is given by $R^M_T \cdot E_M = \langle T, e \rangle$.

In particular we get:

**Lemma 63.** Let $R^M_T$ be the rim torus associated to an element $T$ in $H^1(\Sigma)$. Then $R^M_T \cdot D^M_i = \langle T, \alpha_i \rangle$.

We also have:

**Lemma 64.** Every rim torus $R^M_T$ in $R(M')$ has zero intersection with the surfaces $\Sigma^M$, $B'_M$, all surfaces in $P(M)$ and all rim tori in $R(M')$.

This follows because the rim torus can be moved away from each surface mentioned in this lemma.
6.2. Construction of the surfaces $U_i^N$. Let $V_i^N$ denote the embedded surface in the closed manifold $N$, obtained by smoothing the intersection points of $x_N(\alpha_i)$ copies of $B_N$ and $x_N(\alpha_i)B_N^2$ disjoint copies of the push-off $\Sigma^N$ with the opposite orientation. Here
\[ x_N(\alpha_i) = \langle C, \alpha_i \rangle \]
as before. The surface $V_i^N$ represents the class $x_N(\alpha_i)B_N - (x_N(\alpha_i)B_N^2)\Sigma_N$ and has zero intersection with the class $B_N$ and the surfaces in $P(N)$. We can assume that the copies of $\Sigma_N$ are contained in the complement of the tubular neighbourhood $\nu\Sigma_N$. Deleting the interior of the tubular neighbourhood we obtain a surface $U_i^N$ in $N'$ bounding the disjoint union of $x_N(\alpha_i)$ parallel copies of the meridian $\sigma^N$ on the boundary $\partial N'$. The surface has zero intersection with the surface $B_N'$, the rim tori in $R(N')$ and the surfaces in $P(N)$. In addition, the intersection number with the surface $\Sigma^N$ is given by $U_i^N \cdot \Sigma^N = x_N(\alpha_i)$.

6.3. Definition of split surfaces $S_i$. Let $\alpha_1, \ldots, \alpha_d$ be a basis for the free abelian group $\ker(i_M \oplus i_N)$ and $\alpha_i^M, \alpha_i^N$ for each index $i$ the image of these loops on the boundaries $\partial\nu\Sigma_M$ and $\partial\nu\Sigma_N$ of the 4-manifolds $M'$ and $N'$. Choose for each index $i$ embedded surfaces $D_i^M$ and $D_i^N$ bounding the curves $\alpha_i^M$ and $\alpha_i^N$ with the properties of Lemma 60 and Lemma 61.

Suppose that the gluing diffeomorphism $\phi$ is isotopic to the trivial diffeomorphism that identifies the push-offs, or equivalently the cohomology class $C$ equals zero. Then we can assume without loss of generality that $\phi$ is equal to the trivial diffeomorphism, hence the curves $\alpha_i^M$ and $\alpha_i^N$ on the boundaries $\partial M'$ and $\partial N'$ get identified by the gluing map. We set
\[ S_i^M = D_i^M \]
\[ S_i^N = D_i^N \]
and define the split class $S_i$ by sewing together the surfaces $S_i^M$ and $S_i^N$.

In the case that $C$ is different from zero the curves $\phi \circ \alpha_i^M$ and $\alpha_i^N$ do not coincide. By Lemma 62 we know that the curve $\phi \circ \alpha_i^M$ is homologous to
\[ \alpha_i^N + \langle C, \alpha_i \rangle \sigma^N = \alpha_i^N + x_N(\alpha_i)\sigma^N. \]

Let $\nu\Sigma_N'$ be a tubular neighbourhood of slightly larger radius than $\nu\Sigma_N$ (we indicate curves and surfaces bounding curves on $\nu\Sigma_N'$ by a prime). On the boundary of $\nu\Sigma_N'$ we consider the curve $\alpha_i^{N'}$ and $x_N(\alpha_i)$ parallel copies of the meridian $\sigma^{N'}$, disjoint from $\alpha_i^{N'}$. The curve $\alpha_i^{N'}$ bounds a surface $D_i^{N'}$ as before and the parallel copies of the meridian bound the surface $U_i^{N'}$. We can connect the curve $\phi \circ \alpha_i^M$ on $\partial\nu\Sigma_N$ to the disjoint union of the meridians and the curve $\alpha_i^{N'}$ on $\partial\nu\Sigma_N'$ by an embedded, connected, oriented surface $Q_i^N$ realizing the homology.

**Lemma 65.** We can assume that the homology $Q_i^N$ is disjoint from an annulus of the form $\{*\} \times S^1 \times I$ in the submanifold $\Sigma_N \times S^1 \times I$ bounded by $\partial\nu\Sigma_N$ and $\partial\nu\Sigma_N'$. 
Proof. This follows because the isomorphism $H_1(\Sigma^*) \to H_1(\Sigma)$ induces an isomorphism $H_1(\Sigma^* \times S^1) \to H_1(\Sigma \times S^1)$, where $\Sigma^*$ denotes the punctured surface $\Sigma \setminus \{\ast\}$. \mark\\

We set

$$S_i^M = D_i^M$$

$$S_i^N = Q_i^N \cup U_i^{N'} \cup D_i^{N'}.$$  

The connected surfaces $S_i^M$ and $S_i^N$ sew together to define the split surfaces $S_i$. The orientation of $S_i$ is defined as follows: The surfaces $S_i^M$ and $S_i^N$ are oriented according to Definition 58. The orientation of $I$ is reversed by the extension $\Phi$ of the gluing diffeomorphism, while the orientations of the curves $\alpha_i^M$ and $\phi \circ \alpha_i^M$ are identified. This implies that the surface $S_i^M$ with its given orientation and the surface $S_i^N$ with the opposite orientation sew together to define an oriented surface $S_i$ in $X$.

**Proposition 66.** For arbitrary gluing diffeomorphism $\phi$, the split classes $S_i$ satisfy

$$S_i \cdot \Sigma_X = 0$$

$$S_i \cdot B_X = 0.$$  

We also have $B_X \cdot \Sigma_X = 1$ and $B_X^2 = B_M^2 + B_N^2$.

**Proof.** We only have to check the first equation. This can be done on either the $M$ or the $N$ side: On the $M$ side we have

$$S_i \cdot \Sigma_X = D_i^M \cdot \Sigma_M = 0.$$  

On the $N$ side we have

$$S_i \cdot \Sigma_X = S_i \cdot (\Sigma'_X - R_C) = -S_i^N \cdot (\Sigma_N - R_C^N)$$

$$= -U_i^N \cdot \Sigma_N + D_i^N \cdot R_C^N = -x_N(\alpha_i) + \langle C, \alpha_i \rangle = 0.$$  

\mark

The rim torus $R_i^M$ associated to an element $T$ in $H^1(\Sigma)$ induces under the inclusion $M' \to X$ a rim torus in $X$, denoted by $R_T$.

**Proposition 67.** We have

$$R_T \cdot \Sigma_X = 0, \quad R_T \cdot B_X = 0, \quad R_T \cdot R_T = 0 \quad \text{for all } T' \in H^1(\Sigma).$$  

The intersection with a split class $S_i$ is given by $R_T \cdot S_i = \langle T, \alpha_i \rangle$.

The first three statements follow because the rim torus can be moved away from the other surfaces. The statement about intersections with split classes follows from Lemma 63.
6.4. **Proof of the formula for the intersection form.** By our assumption $k_M = k_N = 1$, the sequence in Theorem 56 simplifies to

$$0 \rightarrow \mathbb{Z}\Sigma_X \oplus R(X) \rightarrow H^2(X) \rightarrow S(X) \oplus P(M) \oplus P(N) \rightarrow 0.$$ 

Since $S(X)$ is free abelian, we can lift this group to a direct summand of $H^2(X; \mathbb{Z})$. Since we also assumed that the cohomology of $M$, $N$ and $X$ is torsion free, the whole sequence splits and we can write

$$H^2(X) = P(M) \oplus P(N) \oplus S(X) \oplus R(X) \oplus \mathbb{Z}\Sigma_X.$$

So far we have proved:

**Lemma 68.** There exists a splitting

$$H^2(X) = P(M) \oplus P(N) \oplus (S'(X) \oplus R(X)) \oplus (\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X),$$

where the direct sums are all orthogonal, except the two direct sums inside the brackets. In this direct sum, the restriction of the intersection form $Q_X$ to $P(M)$ and $P(N)$ is the intersection form induced from $M$ and $N$, it vanishes on $R(X)$ and has the structure

$$
\begin{pmatrix}
B^2_M + B^2_N & 1 \\
1 & 0
\end{pmatrix}
$$

on $\mathbb{Z}B_X \oplus \mathbb{Z}\Sigma_X$.

In the construction of the split classes $S_i$ from the surfaces $S^M_i$ and $S^N_i$ we can assume that the curves $\alpha^M_i$ are contained in pairwise disjoint parallel copies of the push-off $\Sigma^M_1$ in $\partial \nu \Sigma^M$ of the form $\Sigma_M \times \{p_i\}$. Hence the split classes $S_i$ in $X$ do not intersect on $\partial M' = \partial N'$ and we can assume that they have only transverse intersections.

We now simplify the intersection form on $S'(X) \oplus R(X)$. This will complete the proof of Theorem 57.

**Lemma 69.** The subgroup $\ker(i_M \oplus i_N)$ is a direct summand of $H_1(\Sigma)$.

**Proof.** Suppose that $\alpha \in \ker(i_M \oplus i_N)$ is divisible by an integer $c > 1$ so that $\alpha = c\alpha'$ with $\alpha' \in H_1(\Sigma)$. Then $ci_M\alpha' = 0 = ci_N\alpha'$. Since $H_1(M)$ and $H_1(N)$ are torsion free this implies that $\alpha' \in \ker(i_M \oplus i_N)$. Hence $\ker(i_M \oplus i_N)$ is a direct summand. □

According to this lemma we can complete the basis $\alpha_1, \ldots, \alpha_d$ for $\ker(i_M \oplus i_N)$ by elements $\beta_{d+1}, \ldots, \beta_{2g} \in H_1(\Sigma)$ to a basis of $H_1(\Sigma)$. Let

$$\alpha_1^*, \ldots, \alpha_d^*, \beta_{d+1}^*, \ldots, \beta_{2g}^*$$

denote the dual basis of $H^1(\Sigma)$ and $R_1, \ldots, R_{2g}$ the corresponding rim tori in $H^2(X)$. Then

$$S_i \cdot R_j = \delta_{ij}, \quad \text{for } 1 \leq j \leq d$$

time

$$S_i \cdot R_j = 0, \quad \text{for } d + 1 \leq j \leq 2g.$$

This implies that the elements $R_1, \ldots, R_d$ are a basis of $R(X)$ and $R_{d+1}, \ldots, R_{2g}$ are null-homologous, since the cohomology of $X$ is torsion free.
The surfaces $S_i$ are simplified as follows: Let $r_{ij} = S_i \cdot S_j$ for $i, j = 1, \ldots, d$ denote the matrix of intersection numbers and let

$$S'_i = S_i - \sum_{k > i} r_{ik} R_k.$$ 

The surfaces $S'_i$ are tubed together from the surfaces $S_i$ and certain rim tori. They can still be considered as split classes sewed together from surfaces in $M'$ and $N'$ bounding the loops $\alpha_i^M$ and $\phi \circ \alpha_i^N$ and still have intersection $S'_i \cdot R_j = \delta_{ij}$. However, the intersection numbers $S'_i \cdot S'_j$ for $i \neq j$ simplify to (where without loss of generality $j > i$)

$$S'_i \cdot S'_j = (S_i - \sum_{k > i} r_{ik} R_k) \cdot (S_j - \sum_{l > j} r_{jl} R_l) = S_i \cdot S_j - r_{ij} = 0.$$ 

Denote these new split classes again by $S_1, \ldots, S_d$ and the subgroup spanned by them in $S'(X) \oplus R(X)$ now has the form as in Theorem [57].

**Remark 70.** We can choose the basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(\Sigma)$ we started with in Section 2.2 as

$$\gamma_i = \alpha_i, \quad \text{for } 1 \leq i \leq d$$

$$\gamma_i = \beta_i, \quad \text{for } d + 1 \leq i \leq 2g.$$ 

This choice does not depend on the choice of $C$ since $\alpha_1, \ldots, \alpha_d$ are merely a basis for $\ker(i_M \oplus i_N)$. Then the rim tori $R_1, \ldots, R_d$ are given by the image of the classes $\Gamma_1^M, \ldots, \Gamma_d^M$ under the inclusion $\partial M' \to M' \to X$ and the rim tori determined by $\Gamma_{d+1}^M, \ldots, \Gamma_{2g}^M$ are null-homologous in $X$. In this basis the rim torus $R_C$ in $X$ is given by

$$R_C = - \sum_{i=1}^{d} a_i R_i,$$

where $a_i = \langle C, \alpha_i \rangle$.

The splitting given by Theorem [57]

$$H^2(X) = P(M) \oplus P(N) \oplus (S'(X) \oplus R(X)) \oplus (ZB_X \oplus Z\Sigma_X)$$

is similar to the splitting

$$H^2(M) = P(M) \oplus ZB_M \oplus Z\Sigma_M.$$ 

from equation (9). In particular, we can write

$$P(X) = P(M) \oplus P(N) \oplus (S'(X) \oplus R(X)).$$ 

According to equation (10), we can decompose an element $\alpha \in H^2(M)$ in the direct sum (26) as

$$\alpha = (\alpha \Sigma_M) B_M + (\alpha B_M - B_M^2(\alpha \Sigma_M)) \Sigma_M.$$

Replacing $B_M$ by $B_X$ and $\Sigma_M$ by $\Sigma_X$ we get:
Corollary 71. Under the assumptions of Theorem 57 there exists a group monomorphism \( H^2(M) \to H^2(X) \) given by
\[
\alpha \mapsto \alpha + (\alpha \Sigma_M) B_X + (\alpha B_M - B_M^2 (\alpha \Sigma_M)) \Sigma_X.
\]
There exists a similar monomorphism \( H^2(N) \to H^2(X) \) given by
\[
\alpha \mapsto \alpha + (\alpha \Sigma_M) B_X + (\alpha B_M - B_M^2 (\alpha \Sigma_M)) \Sigma_X',
\]
where \( \Sigma_X' = \Sigma_X + R_C \).

Hence \( H^2(M) \) and \( H^2(N) \) can be realized as direct summands of \( H^2(X) \). In general, the embeddings do not preserve the intersection form, because \( B_X^2 = B_M^2 + B_N^2 \). The images of both embeddings have non-trivial intersection and in general do not span \( H^2(X) \).

7. A FORMULA FOR THE CANONICAL CLASS

In this section we derive a formula for the canonical class of the symplectic generalized fibre sum \( X \) of two symplectic 4-manifolds \( M \) and \( N \) along embedded symplectic surfaces \( \Sigma_M \) and \( \Sigma_N \) of genus \( g \). We make the same assumptions as in the previous section, i.e. that \( \Sigma_M \) and \( \Sigma_N \) represent indivisible classes and the cohomologies of \( M \), \( N \) and \( X \) are torsion free.

The idea of the proof is to decompose the canonical class \( K_X \) according to the formula for \( H^2(X) \) in Theorem 57 as
\[
K_X = p_M + p_N + \sum_{i=1}^d s_i S_i + \sum_{i=1}^d r_i R_i + b_X B_X + \sigma_X \Sigma_X,
\]
where \( p_M \in P(M) \) and \( p_N \in P(N) \). Since the cohomology of \( X \) is assumed to be torsion free, the coefficients in equation (27) can be determined from the intersection numbers of the canonical class \( K_X \) and the basis elements of \( H^2(X) \).

7.1. Definition of the symplectic form on \( X \). We first recall the definition of the symplectic generalized fibre sum by the construction of Gompf [10]. Let \((M, \omega_M)\) and \((N, \omega_N)\) be closed, symplectic 4-manifolds and \( \Sigma_M, \Sigma_N \) embedded symplectic surfaces of genus \( g \). Denote the symplectic generalized fibre sum by \( X = M \#_{\Sigma_M = \Sigma_N} N \). We want to determine a formula for the canonical class \( K_X \) in terms of \( M \) and \( N \).

The symplectic generalized fibre sum is constructed using the following lemma which follows from Lemma 1.5 and Lemma 2.1 in [10]. Recall that we have fixed trivializations of tubular neighbourhoods \( \nu \Sigma_M \) and \( \nu \Sigma_N \) determined by the framings \( \tau_M \) and \( \tau_N \). Hence we can identify the interior of the tubular neighbourhoods with \( D \times \Sigma \), where \( D \) denotes the open disk of radius 1 in \( \mathbb{R}^2 \).

\footnote{In the proof of [5, Theorem 3.2] a similar formula is used to calculate the SW-basic classes for a certain generalized fibre sum of two 4-manifolds.}
Lemma 72. The symplectic structures $\omega_M$ and $\omega_N$ can be deformed by rescaling and isotopies such that both restrict on the tubular neighbourhoods $\nu \Sigma M$ and $\nu \Sigma N$ to the same symplectic form

$$\omega = \omega_\Sigma + \omega_D,$$

where $\omega_D$ is the standard symplectic structure $\omega_D = dx \wedge dy$ on the open unit disk $D$ and $\omega_\Sigma$ is a symplectic form on $\Sigma$.

It is useful to choose polar coordinates $(r, \theta)$ on $D$ such that $x = r \cos \theta$ and $y = r \sin \theta$, hence

$$dx = dr \cos \theta - r \sin \theta d\theta$$
$$dy = dr \sin \theta + r \cos \theta d\theta.$$

Then $\omega_D = r dr \wedge d\theta$. To form the generalized fibre sum, the manifolds $M \setminus \Sigma M$ and $N \setminus \Sigma N$ are glued together along the collars $\text{int} \nu \Sigma M \setminus \Sigma M$ and $\text{int} \nu \Sigma N \setminus \Sigma N$ by the orientation and fibre preserving diffeomorphism

$$\Phi : (D \setminus \{0\}) \times \Sigma \to (D \setminus \{0\}) \times \Sigma$$

$$(r, \theta, z) \mapsto (\sqrt{1 - r^2}, C(z) - \theta, z).$$

(28)

The action of $\Phi$ on the 1-forms $dr$ and $d\theta$ is given by

$$\Phi^* dr = d(r \circ \Phi) = d\sqrt{1 - r^2} = \frac{-r}{\sqrt{1 - r^2}} dr$$
$$\Phi^* d\theta = d(\theta \circ \Phi) = dC - d\theta.$$

This implies that $\Phi^* \omega_D = \omega_D - rdr \wedge dC$, while $\Phi^* \omega_\Sigma = \omega_\Sigma$.

We can think of the boundaries of $M'$ and $N'$ as $S \times \Sigma$, where $S$ denotes the circle of radius $1/\sqrt{2}$. Let $A$ denote the annulus in the unit disk $D$ between radius $1/\sqrt{2}$ and 1. To define the symplectic structure on $X$, we consider on the $N$ side the standard symplectic structure $\omega_D + \omega_\Sigma$ on $A \times \Sigma$ which extends over the rest of $N$ by the symplectic form $\omega_N$. On the boundary $\partial N'$ given by $S \times \Sigma$ this form pulls back under the gluing diffeomorphism to the form

$$\Phi^* (\omega_D + \omega_\Sigma) = \omega_D - rdr \wedge dC + \omega_\Sigma$$

on the boundary $\partial M'$. Let $\rho$ be a smooth cut-off function on the annulus $A$ which is identical to 1 near $r = 1/\sqrt{2}$, identical to 0 near $r = 1$ and depends only on the radius $r$. Consider on the $M$ side the following 2-form on $A \times \Sigma$:

$$\omega_D - \rho rdr \wedge dC + \omega_\Sigma.$$

(29)

This 2-form is closed, since the function $\rho$ depends only on $r$. The square of this 2-form is equal to $\omega_D^2 + \omega_\Sigma^2$, which is a volume form on $A \times \Sigma$, hence the form is symplectic. It follows that we can deform the symplectic structure at radius $1/\sqrt{2}$ through the symplectic form (29) on $A \times \Sigma$ such that it coincides with the standard form $\omega_D$ at $r = 1$. From there it can be extended over the rest of $M$ by the given symplectic form $\omega_M$. In this way a symplectic structure $\omega_X$ is defined on $X$ for every gluing diffeomorphism $\phi$. 

Remark 73. The Gompf construction for the symplectic generalized fibre sum can only be done if (after a rescaling) the symplectic structures $\omega_M$ and $\omega_N$ have the same volume on $\Sigma_M$ and $\Sigma_N$:

\[
\int_{\Sigma_M} \omega_M = \int_{\Sigma_N} \omega_N.
\]

To calculate this number, both $\Sigma_M$ and $\Sigma_N$ have to be oriented which we have assumed a priori. It is not necessary that this number is positive, the construction can also be done for negative volume. In the standard case the orientation induced by the symplectic forms coincides with the given orientation on $\Sigma_M$ and $\Sigma_N$ and is the opposite orientation in the second case.

We split the canonical class $K_X$ as in equation (27)

\[
K_X = p_M + p_N + \sum_{i=1}^{d} s_i S_i + \sum_{i=1}^{d} r_i R_i + b_X B_X + \sigma_X \Sigma_X.
\]

The coefficients in this formula can be determined using intersection numbers:

\[
\begin{align*}
K_X \cdot S_j &= s_j S_j^2 + r_j \\
K_X \cdot R_j &= s_j \\
K_X \cdot B_X &= b_X (B_M^2 + B_N^2) + \sigma_X \\
K_X \cdot \Sigma_X &= b_X.
\end{align*}
\]

Similarly, the coefficients $p_M$ and $p_N$ can be determined by intersecting $K_X$ with classes in $P(M)$ and $P(N)$. We assume that $\Sigma_M$ and $\Sigma_N$ are oriented by the symplectic forms $\omega_M$ and $\omega_N$. Then $\Sigma_X$ is a symplectic surface in $X$ of genus $g$ and self-intersection 0, oriented by the symplectic form $\omega_X$. This implies by the adjunction formula

\[
b_X = K_X \cdot \Sigma_X = 2g - 2,
\]

hence

\[
\sigma_X = K_X \cdot B_X - (2g - 2)(B_M^2 + B_N^2).
\]

Similarly, every rim torus $R_j$ is of the form $c_j \times \sigma^M$ in $\partial M' \subset X$ for some closed oriented curve $c_j$ on $\Sigma_M$. Writing $c_j$ as a linear combination of closed curves on $\Sigma_M$ without self-intersections and placing the corresponding rim tori into different layers $\Sigma_M \times S^1 \times t_i$ in a collar $\Sigma_M \times S^1 \times I$ of $\partial M'$, it follows that the rim torus $R_j$ is a linear combination of embedded Lagrangian tori of self-intersection 0 in $X$. Since the adjunction formula holds for each one of them,

\[
s_j = 0, \quad \text{for all } j = 1, \ldots, d
\]

hence also

\[
r_j = K_X \cdot S_j.
\]

To determine the coefficient $p_M$ we know that $\eta_M^* K_X = K_{M'} = \rho_M^* K_M$. This implies that the intersection of a class in $P(M)$ with $K_X$ is equal to its intersection with $K_M$. Recall that by equation (9) we have a decomposition

\[(30) \quad H^2(M) = P(M) \oplus \mathbb{Z} \Sigma_M \oplus \mathbb{Z} B_M.
\]
By our choice of orientation for $\Sigma_M$, the adjunction formula holds and we have $K_M \Sigma_M = 2g - 2$. By equation (10) we can decompose $K_M$ in the direct sum (30) as

$$K_M = \overline{K_M} + (K_M B_M - (2g - 2)B^2_M)\Sigma_M + (2g - 2)B_M,$$

where the element $\overline{K_M}$, defined by this equation, is in $P(M)$. It is then clear that $p_M = \overline{K_M}$.

Similarly,

$$K_N = \overline{K_N} + (K_N B_N - (2g - 2)B^2_N)\Sigma_N + (2g - 2)B_N$$

with the class $\overline{K_N}$ in $P(N)$ and we have $p_N = \overline{K_N}$.

It remains to determine the intersections of the canonical class with the split classes $B_X$ and $S_1, \ldots, S_d$. This is more difficult and will be done in the following subsections.

**7.2. Construction of a holomorphic 2-form on $X$.** We also need compatible almost complex structures: We choose the standard almost complex structure $J_D$ on $D$ which maps $J_D \partial_x = \partial_y$ and $J_D \partial_y = -\partial_x$. In polar coordinates

$$J_D \partial_r = \frac{1}{r} \partial_\theta$$

$$J_D \frac{1}{r} \partial_\theta = -\partial_r.$$

We also choose a compatible almost complex structure $J_\Sigma$ on $\Sigma$ which maps $J_\Sigma \partial_{z_1} = \partial_{z_2}$ and $J_\Sigma \partial_{z_2} = -\partial_{z_1}$ in suitable coordinates $z = (z_1, z_2)$ on the surface $\Sigma$. The almost complex structure $J_D + J_\Sigma$ on the tubular neighbourhood $D \times \Sigma$ extends to compatible almost complex structures on $M$ and $N$.

Recall that the smooth sections of the canonical bundle $K_M$ are complex valued 2-forms on $M$ which are “holomorphic”, i.e. complex linear. We choose the holomorphic 1-form $\Omega_D = dx + idy$ on $D$, which can be written in polar coordinates as

$$\Omega_D = (dr + ir \, d\theta) e^{i\theta}.$$

This form satisfies $\Omega_D \circ J_D = i \Omega_D$. We also choose a holomorphic 1-form $\Omega_\Sigma$ on $\Sigma$. This form can be chosen such that it has precisely $2g - 2$ different zeroes of index +1. We can assume that all zeroes are contained in a small disk $D_\Sigma$ around a point $q$ disjoint from the zeroes. The form $\Omega_D \wedge \Omega_\Sigma$ is then a holomorphic 2-form on $D \times \Sigma$ which has transverse zero set consisting of $2g - 2$ parallel copies of $D$. This 2-form can be extended to holomorphic 2-forms on $M$ and $N$ as sections of the canonical bundles.

Note that $J_D$ and $\Omega_D$ are not invariant under $\Phi$, even if $C = 0$: On $S \times \Sigma$, where $S$ is the circle of radius $1/\sqrt{2}$, we have

$$\Phi^* dr = -dr$$

$$\Phi^* d\theta = dC - d\theta,$$
hence
\[ \Phi^* \Omega_D = -(dr + ir(d\theta - dC))e^{-i\theta + iC}. \]

We also have on \( S \times \Sigma \)
\[ \Phi_* \partial_r = -\partial_r \]
\[ \Phi_* \partial_\theta = -\partial_\theta \]
\[ \Phi_* \partial_{z_1} = C_1 \partial_\theta + \partial_{z_1} \]
\[ \Phi_* \partial_{z_2} = C_2 \partial_\theta + \partial_{z_2}, \]
where \( C_i \) for \( i = 1, 2 \) denotes the partial derivative \( \partial C/\partial z_i \).

Setting
\[ \Phi^* J = (\Phi^*)_1 \circ J \circ \Phi^* \]
where \( J = J_D + J_\Sigma \) we obtain
\[ (\Phi^* J) \partial_r = \sqrt{2} \partial_\theta \]
\[ (\Phi^* J) \partial_\theta = -\frac{1}{\sqrt{2}} \partial_r \]
\[ (\Phi^* J) \partial_{z_1} = \frac{1}{\sqrt{2}} C_1 \partial_r + C_2 \partial_\theta + \partial_{z_2} \]
\[ (\Phi^* J) \partial_{z_2} = \frac{1}{\sqrt{2}} C_2 \partial_r - C_1 \partial_\theta - \partial_{z_1}. \]

Using the same cut-off function as above in front of the partial derivatives of \( C \), we can deform \( \Phi^* J \) through an almost complex structure on \( A \times \Sigma \) on the \( M \) side such that it coincides with the standard \( J \) at \( r = 1 \). The almost complex structure is everywhere compatible with the symplectic structure on \( A \times \Sigma \). We can also deform \( \Phi^* \Omega_D \) on \( A \times \Sigma \) with the cut-off function through a nowhere vanishing 1-form which is holomorphic for this almost complex structure such that it becomes at \( r = 1 \) equal to
\[ -(dr + ird\theta)e^{-i\theta + iC} = -\Omega_D e^{-2i\theta + iC}. \]

Hence \( \Phi^*(\Omega_D \wedge \Omega_\Sigma) \) can be deformed through a holomorphic 2-form on \( A \times \Sigma \) to the form
\[ -\Omega_D \wedge \Omega_\Sigma e^{-2i\theta + iC} \]
at \( r = 1 \).

A section \( \Omega_X \) for the canonical line bundle \( K_X \) can now be constructed in the following way: Choose a holomorphic 2-form \( \Omega_N \) on \( N \) which restricts on the tubular neighbourhood \( \nu \Sigma_N \) of radius 1 to the form \( -\Omega_D^N \wedge \Omega_\Sigma \), where
\[ \Omega_D^N = (dr + ird\theta)e^{i\theta} \]

Similarly, we choose a holomorphic 2-form \( \Omega_M \) on \( M \) which restricts on the tubular neighbourhood \( \nu \Sigma_M \) of radius 2 to the form \( \Omega_D^M \wedge \Omega_\Sigma \), where
\[ \Omega_D^M = (dr + ird\theta)e^{i\theta} \]

On the boundary \( \partial N' = S \times \Sigma \) we have the holomorphic 2-form
\[ -\Omega_D^N \wedge \Omega_\Sigma. \]
It pulls back under $\Phi$ to a holomorphic 2-form on $\partial M' = S \times \Sigma$. By the argument above this form (together with the almost complex structure) can be deformed over $A \times \Sigma$ to the holomorphic 2-form

$$(33) \quad \Omega^M_D \wedge \Omega^\Sigma e^{-2i\theta + iC}$$

at $r = 1$.

Let $A'$ denote the annulus between radius 1 and 2. We want to change the form in equation (33) over $A' \times \Sigma$ through a holomorphic 2-form to the form $\Omega^M_D \wedge \Omega^\Sigma$ at $r = 2$. This form can then be extended over the rest of $M$ using the 2-form $\Omega_M$ above. The almost complex structure will be extended by the standard one over $A' \times \Sigma$. The change will be done by changing the function $e^{-2i\theta + iC}$ at $r = 1$

---

**Figure 1.** Collars of the boundaries $\partial M'$ and $\partial N'$

---

over $A' \times \Sigma$ to the constant function with value 1 at $r = 2$. This is not possible if we consider the functions as having image in $S^1$, because they represent different cohomology classes on $S^1 \times \Sigma$. Hence we consider $S^1 \subset \mathbb{C}$ and the change will involve crossings of zero. We choose a smooth function $f : A' \times \Sigma \to \mathbb{C}$ which is transverse to 0 and satisfies

$$f_1 = e^{-2i\theta + iC} \quad \text{and} \quad f_2 \equiv 1.$$ 

The Poincaré dual of the zero set of $f$ is then the cohomology class of $S^1 \times \Sigma$ determined by the $S^1$-valued function $e^{2i\theta - iC}$.

Let $\gamma_1^M, \ldots, \gamma_{2g}^M, \sigma^M$ be a basis of $H^1(S^1 \times \Sigma_M)$ as in Section 2.2 Then the cohomology class determined by $e^{2i\theta - iC}$ is equal to

$$-\sum_{i=1}^{2g} a_i \gamma_i^M + 2\sigma^M.$$

The Poincaré dual of this class is

$$-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M.$$

This implies:
Proposition 74. There exists a 2-form $\Omega'$ on $A' \times \Sigma_M$ which is holomorphic for $J_D + J_\Sigma$ and satisfies:

- $\Omega' = \Omega_D^M \land \Omega_\Sigma e^{-2i\theta + iC}$ at $r = 1$ and $\Omega' = \Omega_D' \land \Omega_\Sigma$ at $r = 2$.
- The zeroes of the form $\Omega'$ are all transverse and the zero set represents the class $-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M$ in the interior of $A' \times \Sigma_M$ and $2g - 2$ parallel copies of $A'$.

The appearance of the zero set $-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M$ can be seen as the obstruction to extending the $S^1$-valued function on the boundary of $A' \times \Sigma$ given by $f_1$ at $r = 1$ and $f_2$ at $r = 2$ into the interior. Under inclusion in $X$, this class becomes

$$R_C + 2\Sigma_X,$$

where $R_C$ denotes the rim torus in $X$ determined by the diffeomorphism $\phi$, cf. Definition 47. We get the following corollary:

Corollary 75. There exists a symplectic form $\omega_X$ with compatible almost complex structure $J_X$ and holomorphic 2-form $\Omega_X$ on $X$ such that:

- On the boundary $\partial \Sigma_N$ of the tubular neighbourhood of $\Sigma_N$ in $N$ of radius 2 the symplectic form and the almost complex structure are $\omega_X = \omega_D + \omega_\Sigma$ and $J_X = J_D + J_\Sigma$ while $\Omega_X = - \Omega_D \land \Omega_\Sigma$.
- On the boundary $\partial \Sigma_M$ of the tubular neighbourhood of $\Sigma_M$ in $M$ of radius 2 the symplectic form and the almost complex structure are $\omega_X = \omega_D + \omega_\Sigma$ and $J_X = J_D + J_\Sigma$ while $\Omega_X = \Omega_D \land \Omega_\Sigma$.
- On the subset of $\nu \Sigma_N$ between radius $1/\sqrt{2}$ and 2, which is an annulus times $\Sigma_N$, the zero set of $\Omega_X$ consists of $2g - 2$ parallel copies of the annulus.
- On the subset of $\nu \Sigma_M$ between radius $1/\sqrt{2}$ and 2, which is an annulus times $\Sigma_M$, the zero set of $\Omega_X$ consists of $2g - 2$ parallel copies of the annulus and a surface in the interior representing $-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M$.

7.3. The formula for the canonical class of $X$. We first define the meaning of intersection numbers of the canonical class $K_M$ pulled back to $M'$ and surfaces $L^M$ in $M'$ which bound curves on $\partial M'$.

Definition 76. Let $\Omega_\Sigma$ be a given 1-form on $\Sigma$ with $2g - 2$ transverse zeroes, holomorphic with respect to a given almost complex structure $J_\Sigma$. Under the embedding $i_M$ and the trivialization $\tau_M$ of the normal bundle equip the tubular neighbourhood $\nu \Sigma_M$ of radius 2 with the almost complex structure $J_D + J_\Sigma$ and the holomorphic 2-form $\Omega_D \land \Omega_\Sigma$. Let $L^M$ be a closed oriented surface in $M' = M \setminus \text{int} \nu \Sigma_M$ which bounds a closed curve $c^M$ on $\partial \nu \Sigma_M$, disjoint from the zeroes of $\Omega_D \land \Omega_\Sigma$ on the boundary. Then $K_M L^M$ denotes the obstruction to extending the given section of $K_M$ on $c^M$ over the whole surface $L^M$. This is the number of zeroes one encounters when trying to extend the non-vanishing section of $K_M$ on $\partial L^M$ over all of $L^M$. 
There is an exactly analogous definition for $N$ with almost complex structure $J_D + J_{\Sigma}$ and holomorphic 2-form $\Omega_D \wedge \Omega_{\Sigma}$ on the tubular neighbourhood $\nu \Sigma_N$ of radius 2.

We can now calculate $K_X \cdot B_X$. Our choice of orientation for $\Sigma_M$ and $\Sigma_N$ and the fact that $\Sigma_M B_M = +1 = \Sigma_N B_N$ determines an orientation of $B_M$ and $B_N$ and hence an orientation for $B_X$.

**Lemma 77.** With this choice of orientation, we have $K_X B_X = K_M B_M + K_N B_N + 2$.

**Proof.** We extend the holomorphic 2-form $\Omega_D \wedge \Omega_{\Sigma}$ on the boundary $\partial \nu \Sigma_M$ of the tubular neighbourhood of $\Sigma_M$ in $M$ of radius 2 to the holomorphic 2-form on $\nu \Sigma_M$ given by the same formula and then to a holomorphic 2-form on $M \setminus \text{int} \nu \Sigma_M$. The zero set of the resulting holomorphic 2-form $\Omega_M$ restricted to $\nu \Sigma_M = D_M \times \Sigma_M$ consists of $2g - 2$ parallel copies of $D_M$. We can choose the surface $B_M$ such that it is parallel but disjoint from these copies of $D_M$ inside $\nu \Sigma_M$ and intersects the zero set of $\Omega_M$ outside transverse. The zero set on $B_M$ then consists of a set of points which count algebraically as $K_M B_M$. We can do a similar construction for $N$. We think of the surface $B_X$ as being glued together from the surfaces $B_M$ and $B_N$ by deleting in each a disk of radius $1/\sqrt{2}$ in $D_M$ and $D_N$ around 0. On the $M$ side we get two additional positive zeroes coming from the intersection with the class $-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M$ in Corollary 75 over the annulus in $D_M$ between radius $1/\sqrt{2}$ and 2. Adding these terms proves the claim. \hfill \Box

It remains to calculate the intersections $K_X \cdot S_i$ which determine the rim tori contribution to the canonical class. Consider the surfaces $S_i^M$ and $S_i^N$ that bound the loops $\alpha_i^M$ in $\partial M'$ and $\phi \circ \alpha_i^M$ in $\partial N'$. We choose the basis for $H_1(\Sigma)$, the rim tori $R_i$, the curves $\alpha_i^M, \alpha_i^N$ and the surfaces $S_i^M, S_i^N$ as described in Section 6.3 and Remark 70. In particular, the curve $\alpha_i^M$ is given by the basis element $\gamma_i^M$.

**Lemma 78.** With the choice of orientation as in Definition 58 we have $K_X S_i = K_M S_i^M - K_N S_i^N - a_i$.

**Proof.** The proof is the similar to the proof for Lemma 77. The minus sign in front of $K_N S_i^N$ comes in because we have to change the orientation on $S_i^N$ if we want to sew it to $S_i^M$ to get the surface $S_i$ in $X$. This time the non-zero intersections over the annulus in $D_M$ between radius $1/\sqrt{2}$ and 2 come from the intersection of the annulus $\gamma_i^M \times [1/\sqrt{2}, 2]$ and the class

$$-\sum_{i=1}^{2g} a_i \Gamma_i^M + 2\Sigma^M = -\sum_{i=1}^{d} a_i R_i + 2\Sigma^M,$$

giving $-a_i$. \hfill \Box
This term can be evaluated more explicitly, because we have

\[
S_i^M = D_i^M \\
S_i^N = Q_i^N \cup U_i^{N'} \cup D_i^{N'},
\]

where \(U_i^{N'}\) is constructed from a surface \(V_i^N\) in the closed manifold \(N\) representing \(a_i(B_N - B_N^2 \Sigma_N)\) by deleting the part in the interior of \(\nu \Sigma_N\). There are additional rim tori terms in the definition of the \(S_i\) used to separate \(S_i\) and \(S_j\) for \(i \neq j\) which we can ignore here because the canonical class evaluates to zero on them. We think of the surface \(Q_i^N\) as being constructed in the annulus between radius 2 and 3 times \(\Sigma_N\). We extend the almost complex structure and the holomorphic 2-form over this annulus without change. By Lemma 65 we can assume that \(Q_i^N\) is disjoint from the zero set of \(\Omega_D \wedge \Omega_{\Sigma}\), consisting of \(2g - 2\) parallel annuli. Hence there are no zeroes of \(\Omega_N\) on \(Q_i^N\). The surface \(D_i^N\) contributes \(K_N D_i^N\) to the number \(K_N S_i^N\) and the surface \(U_i^{N'}\) contributes

\[
K_N U_i^{N'} = a_i(K_N(B_N - (B_N^2 \Sigma_N) - a_i(K_N B_N + 1 - (2g - 2)B_N^2)).
\]

Hence we get:

**Lemma 79.** With our choice of the surfaces \(S_i^M\) and \(S_i^N\), we have

\[
K_X S_i = K_M D_i^M - K_N D_i^N - a_i(K_N b - (2g - 2)B_N^2).
\]

This formula has the advantage that the first two terms are independent of the choice of the diffeomorphism \(\phi\). By collecting our calculations it follows that we can write

\[
K_X = K_M + K_N + \sum_{i=1}^{d} r_i R_i + b_X B_X + \sigma_X \Sigma_X,
\]

where

\[
K_M = K_M - (2g - 2)B_M - (K_M B_M - (2g - 2)B_M^2) \Sigma_M \in P(M)
\]

\[
K_N = K_N - (2g - 2)B_N - (K_N B_N - (2g - 2)B_N^2) \Sigma_N \in P(N)
\]

\[
r_i = K_X S_i = K_M D_i^M - K_N D_i^N - a_i(K_N B_N + 1 - (2g - 2)B_N^2)
\]

\[
b_X = 2g - 2
\]

\[
\sigma_X = K_M B_M + K_N B_N + 2 - (2g - 2)(B_M^2 + B_N^2).
\]

In this formula \(K_X\) depends on the diffeomorphism \(\phi\) through the term

\[
-a_i(K_N B_N + 1 - (2g - 2)B_N^2)
\]

which gives the contribution

\[
(K_N B_N + 1 - (2g - 2)B_N^2) R_C = - \sum_{i=1}^{d} a_i(K_N B_N + 1 - (2g - 2)B_N^2) R_i
\]
to the canonical class. The formula for $K_X$ can be written more symmetrically by also using the class $\Sigma'_X = R_C + \Sigma_X$, induced from the push-off on the $N$ side.

We then get:

**Theorem 80.** Let $X = M \#_{\Sigma_M = \Sigma_N} N$ be a symplectic generalized fibre sum of closed oriented symplectic 4-manifolds $M$ and $N$ along embedded symplectic surfaces $\Sigma_M, \Sigma_N$ of genus $g$ which represent indivisible homology classes and are oriented by the symplectic forms. Suppose that the cohomology of $M, N$ and $X$ is torsion free. Choose a basis for $H_2(X; \mathbb{Z})$ as in Theorem 57, where the split classes are constructed from surfaces $S^M_i, S^N_i$ as in Section 6.3 and Remark 70.

Then the canonical class of $X$ is given by

$$K_X = K_M + K_N + \sum_{i=1}^{d} t_i R_i + b_X B_X + \eta_X \Sigma_X + \eta'_X \Sigma'_X,$$

where

$$K_M = K_M - (2g - 2)B_M - (K_M B_M - (2g - 2)B^2_M)\Sigma_M \in P(M)$$

$$K_N = K_N - (2g - 2)B_N - (K_N B_N - (2g - 2)B^2_N)\Sigma_N \in P(N)$$

$$t_i = K_M D^M_i - K_N D^N_i$$

$$b_X = 2g - 2$$

$$\eta_X = K_M B_M + 1 - (2g - 2)B^2_M$$

$$\eta'_X = K_N B_N + 1 - (2g - 2)B^2_N.$$

Under the embeddings of $H^2(M)$ and $H^2(N)$ into $H^2(X)$ given by Corollary 71 the canonical classes of $M$ and $N$ map to

$$K_M \mapsto \overline{K_M} + (2g - 2)B_X + (K_M B_M - (2g - 2)B^2_M)\Sigma_X$$

$$K_N \mapsto \overline{K_N} + (2g - 2)B_X + (K_N B_N - (2g - 2)B^2_N)\Sigma'_X.$$

This implies:

**Corollary 81.** Under the assumptions in Theorem 80 and the embeddings of $H^2(M)$ and $H^2(N)$ into $H^2(X)$ given by Corollary 71 the canonical class of the symplectic generalized fibre sum $X = M \#_{\Sigma_M = \Sigma_N} N$ is given by

$$K_X = K_M + K_N + \Sigma_X + \Sigma'_X - (2g - 2)B_X + \sum_{i=1}^{d} t_i R_i,$$

where $t_i = K_M D^M_i - K_N D^N_i$.

For example, suppose that $g = 1$, the coefficients $t_1, \ldots, t_d$ vanish and $\Sigma_X = \Sigma'_X$. Then we get the classical formula for the generalized fibre sum along tori

$$K_X = K_M + K_N + 2\Sigma_X,$$

which can be found in the literature, e.g. [23].
Remark 82. The coefficients $t_i = K_M D_i^m - K_N D_i^n$ can be calculated as $K_{X_i} S_i$, where $X_0 = X(\phi_0)$ is constructed from the trivial gluing diffeomorphism that identifies the push-offs and the split surfaces $S_i$ are sewed together from $D_i^m$ and $D_i^n$.

8. Examples and Applications

To check the formula for the canonical class given by Theorem 80, we can calculate the square $K^2_X = Q_X(K_X, K_X)$ and compare it with the classical formula (34)

$$c_1(X)^2 = c_1(M)^2 + c_1^2(N) + (8g - 8),$$

which can be derived independently using the formulae for the Euler characteristic and the signature of a generalized fibre sum (see the proof of Corollary 39)

$$e(X) = e(M) + e(N) + (4g - 4)$$

$$\sigma(X) = \sigma(M) + \sigma(N)$$

and the formula $c_1^2 = 2e + 3\sigma$. We do this step by step. By Theorem 57 we have:

$$Q_X(K_M, K_M) = Q_M(K_M, K_M)$$

$$= Q_M(\overline{K_M}, K_M)$$

$$= K_M^2 - (2g - 2)K_M B_M - (2g - 2)(K_M B_M - (2g - 2)B_M^2)$$

$$= K_M^2 - (4g - 4)K_M B_M + (2g - 2)^2 B_M^2.$$

The second step in this calculation follows since by definition $\overline{K_M}$ is orthogonal to $B_M$ and $\Sigma_M$. Similarly

$$Q_X(\overline{K_N}, \overline{K_N}) = K_N^2 - (4g - 4)K_N B_N + (2g - 2)^2 B_N^2.$$

The rim torus term $\sum_{i=1}^d r_i R_i$ has zero intersection with itself and all other terms in $K_X$. We have

$$Q_X(b_X B_X, b_X B_X) = (2g - 2)^2(B_M^2 + B_N^2),$$

and

$$2Q_X(b_X B_X, \sigma_X \Sigma_X) = 2(2g - 2)(K_M B_M + K_N B_N + 2 - (2g - 2)(B_M^2 + B_N^2)).$$

The self-intersection of $\Sigma_X$ is zero. Adding these terms together, we get the expected result

$$K_X^2 = K_M^2 + K_N^2 + (8g - 8).$$

As another check we compare the formula for $K_X$ in Theorem 80 with a formula of Ionel and Parker that determines the intersection of $K_X$ with certain homology classes for symplectic generalized fibre sums in arbitrary dimension and without the assumption of trivial normal bundles of $\Sigma_M$ and $\Sigma_N$ (see [16 Lemma 2.4]).
For dimension 4 with surfaces of genus $g$ and self-intersection zero the formula can be written (in our notation for the cohomology of $X$):

$$K_X C = K_M C \text{ for } C \in P(M)$$
$$K_X C = K_N C \text{ for } C \in P(N)$$
$$K_X \Sigma_X = K_M \Sigma_M = K_N \Sigma_N = 2g - 2 \text{ (by the adjunction formula)}$$
$$K_X R = 0 \text{ for all elements in } R(X)$$
$$K_X B_X = K_M B_M + K_N B_N + 2(B_M \Sigma_M = B_N \Sigma_N)$$
$$\quad = K_M B_M + K_N B_N + 2.$$

There is no statement about the intersection with classes in $S'(X)$ that have a non-zero component in $\ker (i_M \oplus i_N)$. We calculate the corresponding intersections with the formula for $K_X$ in Theorem 80. For $C \in P(M)$ we have

$$K_X \cdot C = \overline{K_M} \cdot C = K_M \cdot C,$$

where the second line follows because the terms in the formula for $\overline{K_M}$ involving $B_M$ and $\Sigma_M$ have zero intersection with $C$, being a perpendicular element. A similar equation holds for $N$. The intersection with $\Sigma_X$ is given by

$$K_X \cdot \Sigma_X = (2g - 2)B_X \cdot \Sigma_X = 2g - 2.$$

The intersection with rim tori is zero and

$$K_X \cdot B_X = b_X B_X^2 + \sigma_X$$
$$\quad = (2g - 2)(B_M^2 + B_N^2) + K_M B_M + K_N B_N + 2 - (2g - 2)(B_M^2 + B_N^2)$$
$$\quad = K_M B_M + K_N B_N + 2,$$

which also follows by Lemma 77. Hence with the formula in Theorem 80 we get the same result as with the formula of Ionel and Parker.

The following corollary gives a criterion when the canonical class $K_X$ is divisible by $d$ as an element in $H^2(X; \mathbb{Z})$.

**Corollary 83.** Let $X$ be a symplectic generalized fibre sum $M \#_{\Sigma_M = \Sigma_N} N$ as in Theorem 80. If $K_X$ is divisible by an integer $d \geq 0$ then

- the integers $2g - 2$ and $K_M B_M + K_N B_N + 2$ are divisible by $d$, and
- the cohomology classes $K_M - (K_M B_M) \Sigma_M$ in $H^2(M; \mathbb{Z})$ and $K_N - (K_N B_N) \Sigma_N$ in $H^2(N; \mathbb{Z})$ are divisible by $d$.

Conversely, if all coefficients $r_i$ vanish, then these conditions are also sufficient for $K_X$ being divisible by $d$.

The proof is immediate by the formula for the canonical class $K_X$, since $B_X$ and $\Sigma_X$ intersect once and are orthogonal to all other classes.
8.1. **Generalized fibre sums along tori.** We consider the special case of Theorem [80] where the embedded surfaces are tori. Let $M$ and $N$ be closed symplectic 4-manifolds which contain symplectically embedded tori $T_M$ and $T_N$ of self-intersection 0, representing indivisible classes. Suppose that $M$ and $N$ have torsion free homology and both tori are contained in cusp neighbourhoods. Then each torus has two vanishing cycles coming from the cusp. We choose identifications of both $T_M$ and $T_N$ with $T^2$ such that the vanishing cycles are given by the simple closed loops $\gamma_1 = S^1 \times 1$ and $\gamma_2 = 1 \times S^1$. The loops bound embedded vanishing disks in $M$ and $N$, denoted by $(D^M_1, D^M_2)$ and $(D^N_1, D^N_2)$. The existence of the vanishing disks shows that the embeddings $T_M \to M$ and $T_N \to N$ induce the zero map on the fundamental group.

We choose for both tori trivializations of the normal bundles and corresponding push-offs $T^M$ and $T^N$. By choosing the trivializations appropriately we can assume that the vanishing disks bound the vanishing cycles on these push-offs and are contained in $M \setminus \text{int} \nu T_M$ and $N \setminus \text{int} \nu T_N$. The vanishing disks have self-intersection $-1$ if the vanishing circles on the boundary of the tubular neighbourhood are framed by the normal framing on the torus and the meridian. We consider the symplectic generalized fibre sum $X = X(\phi) = M \#_{T_M=T_N} N$ for a gluing diffeomorphism

$$\phi: \partial (M \setminus \text{int} \nu T_M) \to \partial (N \setminus \text{int} \nu T_N).$$

The vanishing cycles on both tori determine a basis for $H_1(T^2)$. If $\alpha_i = \langle C, \gamma_i \rangle$ and $\sigma$ denotes the meridians to $T_M$ in $M$ and $T_N$ in $N$, then the gluing diffeomorphism $\phi: \nu T_M \to \nu T_N$ maps in homology

$$\gamma_1 \mapsto \gamma_1 + a_1 \sigma$$
$$\gamma_2 \mapsto \gamma_2 + a_2 \sigma$$
$$\sigma \mapsto -\sigma$$

by Lemma [12]. Note that $H_1(X(\phi)) \cong H_1(M) \oplus H_1(N)$ by Theorem [41]. Hence under our assumptions the homology of $X(\phi)$ is torsion free. The group of rim tori is $R(X) = \text{coker}(i^*_M + i^*_N) \cong \mathbb{Z}^2$. Let $R_1, R_2$ denote a basis for $R(X)$.

We can calculate the canonical class of $X = X(\phi)$ by Theorem [80]. Let $B_M$ and $B_N$ denote surfaces in $M$ and $N$ which intersect $T_M$ and $T_N$ transversely once. Then the canonical class is given by

$$K_X = \overline{K}_M + \overline{K}_N + (r_1 R_1 + r_2 R_2) + b_X B_X + \eta_X T_X + \eta_X' T'_X,$$

where

$$\overline{K}_M = K_M - (K_M B_M) T_M \in P(M)$$
$$\overline{K}_N = K_N - (K_N B_N) T_N \in P(N)$$
$$r_i = K_M D^M_i - K_N D^N_i$$
$$b_X = 2g - 2 = 0$$
$$\eta_X = K_M B_M + 1$$
$$\eta'_X = K_N B_N + 1.$$
Here $T_X$ is the torus in $X$ determined by the push-off $T^M$ and $T^\prime_X$ is determined by the push-off $T^N$.

**Lemma 84.** In the situation above we have $K_M D_i^M - K_N D_i^N = 0$ for $i = 1, 2$.

**Proof.** The pairs $(D_1^M, D_1^N)$ and $(D_2^M, D_2^N)$ sew together in the generalized fibre sum $X_0 = X(\phi_0)$, where $\phi_0$ denotes the trivial gluing diffeomorphism that identifies the push-offs, and determine embedded spheres $S_1, S_2$ of self-intersection $-2$. We claim that

$$K_{X_0} S_i = K_M D_i^M - K_N D_i^N = 0, \quad i = 1, 2.$$  

This is clear by the adjunction formula if the spheres are symplectic or Lagrangian. In the general case, there exist rim tori $R_1, R_2$ in $X_0$ which are dual to the spheres $S_1, S_2$ and which can be assumed Lagrangian by the Gompf construction. Consider the pair $R_1$ and $S_1$: By the adjunction formula we have $K_{X_0} R_1 = 0$. The sphere $S_1$ and the torus $R_1$ intersect once. By smoothing the intersection point we get a smooth torus of self-intersection zero in $X_0$ representing $R_1 + S_1$. The canonical class $K_{X_0}$ is a Seiberg-Witten basic class. The adjunction inequality [20] implies that $K_{X_0}(R_1 + S_1) = 0$, which shows that $K_{X_0} S_1 = 0$. In a similar way it follows that $K_{X_0} S_2 = 0$. □

This implies:

**Proposition 85.** Let $M, N$ be closed symplectic 4-manifolds with torsion free homology. Suppose that $T_M$ and $T_N$ are embedded symplectic tori of self-intersection 0 which are contained in cusp neighbourhoods in $M$ and $N$. Then the canonical class of the symplectic generalized fibre sum $X = X(\phi) = M \#_{T_M = T_N} N$ is given by

$$K_X = \overline{K_M} + \overline{K_N} + \eta_X T_X + \eta_X^\prime T_X^\prime = K_M + K_N + T_X + T_X^\prime,$$

where

$$\overline{K_M} = K_M - (K_M B_M) T_M \in P(M)$$
$$\overline{K_N} = K_N - (K_N B_N) T_N \in P(N)$$
$$\eta_X = K_M B_M + 1$$
$$\eta_X^\prime = K_N B_N + 1.$$  

The second line in the formula for $K_X$ holds by Corollary [87] under the embeddings of $H^2(M)$ and $H^2(N)$ in $H^2(X)$.

As a special case, suppose that the tori $T_M$ and $T_N$ are contained in smoothly embedded nuclei $N(m) \subset M$ and $N(n) \subset N$, which are by definition diffeomorphic to neighbourhoods of a cusp fibre and a section in the elliptic surfaces $E(m)$ and $E(n)$, cf. [11] [12]. The surfaces $B_M$ and $B_N$ can then be chosen as the spheres $S_M, S_N$ inside the nuclei corresponding to the sections. The spheres
have self-intersection \(-m\) and \(-n\) respectively. If the sphere \(S_M\) is symplectic or Lagrangian in \(M\), we get by the adjunction formula

\[ K_M S_M = m - 2. \]

If \(m = 2\) this holds by an argument similar to the one in Lemma 84 already without the assumption that \(S_M\) is symplectic or Lagrangian. With Proposition 85 we get:

**Corollary 86.** Let \(M, N\) be closed symplectic 4-manifolds with torsion free homology. Suppose that \(T_M\) and \(T_N\) are embedded symplectic tori of self-intersection 0 which are contained in embedded nuclei \(N(m) \subset M\) and \(N(n) \subset N\). Suppose that \(m = 2\) or the sphere \(S_M\) is symplectic or Lagrangian. Similarly, suppose that \(n = 2\) or the sphere \(S_N\) is symplectic or Lagrangian. Then the canonical class of the symplectic generalized fibre sum \(X = X(\phi) = M \#_{T_M = T_N} N\) is given by

\[ K_X = K_M + K_N + (m - 1)T_X + (n - 1)T'_X, \]

where

\[ K_M = K_M - (m - 2)T_M \in P(M) \]
\[ K_N = K_N - (n - 2)T_N \in P(N). \]

As an example we consider twisted fibre sums of elliptic surfaces \(E(m)\) and \(E(n)\) which are glued together by diffeomorphisms that preserve the \(S^1\) fibration on the boundary, but not the \(T^2\) fibration:

**Example 87.** Suppose that \(M = E(m)\) and \(N = E(n)\) with general fibres \(T_M\) and \(T_N\). The framing for the tori is given by the framing induced from the elliptic fibration. Since \(K_{E(m)} = (m - 2)T_M\) and the spheres in the nuclei are symplectic, the canonical class of \(X = X(\phi) = E(m) \#_{T_M = T_N} E(n)\) is given by

\[ K_X = (m - 1)T_X + (n - 1)T'_X. \]

Using the identity \(T'_X = T_X + R_C\) this formula can be written as

\[ K_X = -(n - 1)R_C + (m + n - 2)T_X. \]

If both coefficients \(a_1\) and \(a_2\) vanish, we get the standard formula

\[ K_X = (m + n - 2)T_X \]

for the fibre sum \(E(m + n) = E(m) \#_{T_M = T_N} E(n)\). If \(n = 1\) it follows that there is no rim tori contribution to the canonical class, independent of the gluing diffeomorphism \(\phi\). This can be explained because every orientation-preserving self-diffeomorphism of \(\partial(E(1) \setminus \text{int} \nu F)\) extends over \(E(1) \setminus \text{int} \nu F\) where \(F\) denotes a general fibre. Hence all generalized fibre sums \(X(\phi)\) are diffeomorphic to the elliptic surface \(E(m + 1)\) in this case (see [12, Theorem 8.3.11]). The same argument holds if \(n \neq 1\) but \(m = 1\).

---

3This formula can also be derived from a gluing formula for the Seiberg-Witten invariants along \(T^3\), cf. [21 Corollary 22].
If both \( m \) and \( n \) are different from 1, there may exist a non-trivial rim tori contribution. For example, if we consider the generalized fibre sum \( X = X(\phi) = E(2)\#_{T_M=T_N}E(2) \) of two \( K3 \)-surfaces \( E(2) \), then

\[
K_X = -(a_1R_1 + a_2R_2) + 2T_X
\]

\[
= T_X + T_X'
\]

If the greatest common divisor of \( a_1 \) and \( a_2 \) is odd then \( K_X \) is indivisible (because there exist certain split classes in \( X \) dual to the rim tori \( R_1 \) and \( R_2 \)). In this case the manifold \( X \) is no longer spin, hence cannot be homeomorphic to the spin manifold \( E(4) \).

We consider a more general case: For integers \( p \in \mathbb{Z} \), let \( X(m,n,p) \) denote the symplectic fibre sum

\[
X(m,n,p) = E(m)\#_{T_M=T_N}E(n),
\]

of \( E(m) \) and \( E(n) \) along general fibres \( T_M, T_N \) with gluing diffeomorphism determined by \( a_1 = p, a_2 = 0 \), hence representing the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
p & 0 & -1
\end{pmatrix}.
\]

The 4-manifold \( X(m,n,p) \) is simply-connected and has the same characteristic numbers \( c_2 \) and \( \sigma \) as the elliptic surface \( E(m+n) \). In particular, \( X(m,n,0) \) is diffeomorphic to \( E(m+n) \). The canonical class of \( X = X(m,n,p) \) can be calculated by the formula in Example 87:

\[
K_X = -(n-1)pR_1 + (m+n-2)T_X.
\]

There exist homology classes \( S_1 \) and \( B_X \) in \( X \) with \( S_1R_1 = 1, B_XR_1 = 0 \) and \( S_1T_X = 0, B_XT_X = 1 \).

**Proposition 88.** If \((m+n-2)\) does not divide \((n-1)p\), then \(X(m,n,p)\) is not diffeomorphic to the elliptic surface \( E(m+n) \).

**Proof.** If \( X(m,n,p) \) is diffeomorphic to \( E(m+n) \) and \((m+n-2)\) does not divide \((n-1)p\) then we have constructed a symplectic structure on \( E(m+n) \) whose canonical class \( K_X \) is not divisible by \( m+n-2 \). Under our assumptions \( X \) has \( b_2^+ \geq 3 \). The canonical class \( K_X \) is a Seiberg-Witten basic class on \( E(m+n) \). However, it follows from the calculation of the Seiberg-Witten basic classes of \( E(m+n) \), cf. [9, 18], and a theorem of Taubes [24, 17] that every basic class on \( E(m+n) \) which can arise as the canonical class of a symplectic structure is equal to \( \pm(m+n-2)F \), where \( F \) is a general fibre of the elliptic surface \( E(m+n) \). This is a contradiction. \( \square \)

This also follows from [21 Corollary 22]. As a corollary, we get a new proof of the following known theorem (cf. [12, Theorem 8.3.11] for a more general statement):
Corollary 89. Let \( n \geq 2, p \in \mathbb{Z} \) and \( F \) a general fibre in the elliptic surface \( E(n) \) with fibred tubular neighbourhood \( \nu F \). Suppose that \( \psi \) is an orientation preserving self-diffeomorphism of \( \partial \nu F \) realizing
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
p & 0 & 1
\end{pmatrix} \in SL(3, \mathbb{Z})
\]
on \( H_1(\partial \nu F) \). Then \( \psi \) extends to an orientation preserving self-diffeomorphism of \( E(n) \setminus \text{int} \nu F \) if and only if \( p = 0 \).

Proof. Suppose that \( p \neq 0 \). If \( \psi \) extends to a self-diffeomorphism of \( E(n) \setminus \text{int} \nu F \), then \( X(m, n, p) \) is diffeomorphic to \( E(m + n) \) for all \( m \geq 1 \). Since \( n \neq 1 \) we can choose \( m \) large enough such that \( (m + n - 2) \) does not divide \( (n - 1)p \). This is a contradiction to Proposition 88.

The diffeomorphism \( \psi \) does extend in the case of \( E(1) \) for all integers \( p \in \mathbb{Z} \), as has been mentioned already in Example 87.
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