DETERMINATION OF BAUM-BOTT RESIDUES OF HIGHER CODIMENSIONAL FOLIATIONS

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ABSTRACT. Let $\mathcal{F}$ be a singular holomorphic foliation, of codimension $k$, on a complex compact manifold such that its singular set has codimension $\geq k + 1$. In this work we determinate Baum-Bott residues for $\mathcal{F}$ with respect to homogeneous symmetric polynomials of degree $k + 1$. We drop the Baum-Bott’s generic hypothesis and we show that the residues can be expressed in terms of the Grothendieck residue of an one-dimensional foliation on a $(k + 1)$-dimensional disc transversal to a $(k + 1)$-codimensional component of the singular set of $\mathcal{F}$. Also, we show that Cenkl’s algorithm for non-expected dimensional singularities holds dropping the Cenkl’s regularity assumption.

1. INTRODUCTION

In [2] P. Baum and R. Bott developed a general residue theory for singular holomorphic foliations on complex manifolds. More precisely, they proved the following result:

Theorem 1.1 (Baum-Bott). Let $\mathcal{F}$ be a holomorphic foliation of codimension $k$ on a complex manifold $M$ and $\varphi$ be a homogeneous symmetric polynomials of degree $d$ satisfying $k < d \leq n$. Let $Z$ be a compact connected component of the singular set $\text{Sing}(\mathcal{F})$. Then, there exists a homology class $\text{Res}_\varphi(\mathcal{F}, Z) \in H_{2(n-d)}(Z; \mathbb{C})$ such that:

i) $\text{Res}_\varphi(\mathcal{F}, Z)$ depends only on $\varphi$ and on the local behavior of the leaves of $\mathcal{F}$ near $Z$,

ii) Suppose that $M$ is compact and denote by $\text{Res}(\varphi, \mathcal{F}, Z) := \alpha_\ast \text{Res}_\varphi(\mathcal{F}, Z)$, where $\alpha_\ast$ is the composition of the maps

$$H_{2(n-d)}(Z; \mathbb{C}) \xrightarrow{i_\ast} H_{2(n-d)}(M; \mathbb{C})$$

and

$$H_{2(n-d)}(M; \mathbb{C}) \xrightarrow{P} H^{2d}(M; \mathbb{C})$$

with $i_\ast$ is the induced map of inclusion $i : Z \to M$ and $P$ is the Poincaré duality. Then

$$\varphi(\mathcal{N}_\mathcal{F}) = \sum_Z \text{Res}(\varphi, \mathcal{F}, Z).$$
The computation and determination of the residues is difficult in general. If the foliation $\mathcal{F}$ has dimension one with isolated singularities, Baum and Bott in [1] show that residues can be expressed in terms of a Grothendieck residue, i.e., for each $p \in \text{Sing}(\mathcal{F})$ we have

$$\text{Res}_p(\mathcal{F}, Z) = \text{Res}_p[\varphi(JX) \frac{dz_1 \wedge \cdots \wedge dz_n}{X_1 \cdots X_n}],$$

where $X$ is a germ of holomorphic vector field at $p$ tangent to $\mathcal{F}$ and $JX$ is the jacobian of $X$.

The subset of $\text{Sing}(\mathcal{F})$ composed by analytic subsets of codimension $k+1$ will be denoted by $\text{Sing}_{k+1}(\mathcal{F})$ and it is called the singular set of $\mathcal{F}$ with expected codimension. Baum and Bott in [2] exhibit the residues for generic components of $\text{Sing}_{k+1}(\mathcal{F})$. Let us recall this result:

An irreducible component $Z$ of $\text{Sing}_{k+1}(\mathcal{F})$ comes endowed with a filtration. For given point $p \in Z$ choose holomorphic vector fields $v_1, \ldots, v_s$ defined on an open neighborhood $U_p$ of $p \in M$ and such that for all $x \in U_p$, the germs at $x$ of the holomorphic vector fields $v_1, \ldots, v_s$ are in $\mathcal{F}_x$ and span $\mathcal{F}_x$ as an $\mathcal{O}_x$-module. Define a subspace $V_p(\mathcal{F}) \subset T_p M$ by letting $V_p(\mathcal{F})$ be the subspace of $T_p M$ spanned by $v_1(p), \ldots, v_s(p)$. We have

$$Z^{(i)} = \{ p \in Z : \dim(V_p(\mathcal{F})) \leq n - k - i \} \text{ for } i = 1, \ldots, n-k.$$

Then,

$$Z \supseteq Z^{(1)} \supseteq Z^{(2)} \supseteq \cdots \supseteq Z^{(n-k)}$$

is a filtration of $Z$. Now, consider a symmetric homogeneous polynomial $\varphi$ of degree $k+1$. Let $Z \subset \text{Sing}_{k+1}(\mathcal{F})$ be an irreducible component. Take a generic point $p \in Z$ such that $p$ is a point where $Z$ is smooth and disjoint from the other singular components. Now, consider $B_p$ a ball centered at $p$, of dimension $k+1$ sufficiently small and transversal to $Z$ in $p$. In [2, Theorem 3, pg 285] Baum and Bott proved under the following generic assumption

$$\text{cod}(Z) = k+1 \text{ and } \text{cod}(Z^{(2)}) < k+1$$

that we have

$$\text{Res}(\mathcal{F}, \varphi; Z) = \text{Res}_p(\mathcal{F}|_{B_p}; p)[Z],$$

where $\text{Res}_p(\mathcal{F}|_{B_p}; p)$ represents the Grothendieck residue at $p$ of the one dimensional foliation $\mathcal{F}|_{B_p}$ on $B_p$ and $[Z]$ denotes the integration current associated to $Z$.

In [5] and [8] the authors determine the residue $\text{Res}(\mathcal{F}, c_1^{k+1}; Z)$, but even in this case they do not show that we can calculate these residues in terms of the Grothendieck residue of a foliation on a transversal disc. In [15] Vishik proved the same result under the Baum-Bott’s generic hypotheses but supposing that the foliation has locally free tangent sheaf. In [3] F. Bracci and T. Suwa study the behavior
of the Baum-Bott residues under smooth deformations, providing an effective way of computing residues.

In this work we drop the Baum-Bott’s generic hypotheses and we prove the following:

**Theorem 1.2.** Let $\mathcal{F}$ be a singular holomorphic foliation of codimension $k$ on a compact complex manifold $M$ such that $\operatorname{cod}(\operatorname{Sing}(\mathcal{F})) \geq k + 1$. Then,

$$\operatorname{Res}(\mathcal{F}, \varphi; Z) = \operatorname{Res}_\varphi(\mathcal{F}|_{B_p}; p)[Z],$$

where $\operatorname{Res}_\varphi(\mathcal{F}|_{B_p}; p)$ represents the Grothendieck residue at $p$ of the one dimensional foliation $\mathcal{F}|_{B_p}$ on a $(k + 1)$-dimensional transversal ball $B_p$.

Finally, in the last section we apply Cenkl’s algorithm for non-expected dimensional singularities [7]. Moreover, we drop Cenkl’s regularity hypothesis and we conclude that it is possible to calculate the residues for foliations whenever $\operatorname{cod}(\operatorname{Sing}(\mathcal{F})) \geq k + s$, with $s \geq 1$.

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2. **Holomorphic foliations**

Denote by $\Theta_M$ the tangent sheaf of $M$. A foliation $\mathcal{F}$ of codimension $k$ on an $n$-dimensional complex manifold $M$ is given by an exact sequence of coherent sheaves

$$0 \rightarrow T\mathcal{F} \rightarrow \Theta_M \rightarrow N_\mathcal{F} \rightarrow 0,$$

such that $[T\mathcal{F}, T\mathcal{F}] \subset T\mathcal{F}$ and the normal sheaf $N_\mathcal{F}$ of $\mathcal{F}$ is a torsion free sheaf of rank $k \leq n - 1$. The sheaf $T\mathcal{F}$ is called the tangent sheaf of $\mathcal{F}$. The singular set of $\mathcal{F}$ is defined by $\operatorname{Sing}(\mathcal{F}) := \operatorname{Sing}(N_\mathcal{F})$. The dimension of $\mathcal{F}$ is $\dim(\mathcal{F}) = n - k$.

Also, a foliation $\mathcal{F}$, of codimension $k$, can be induced by a exact sequence

$$0 \rightarrow N^\vee_\mathcal{F} \rightarrow \Omega^1_M \rightarrow Q_\mathcal{F} \rightarrow 0,$$

where $Q_\mathcal{F}$ is a torsion free sheaf of rank $n - k$. Moreover, the singular set of $\mathcal{F}$ is $\operatorname{Sing}(Q_\mathcal{F})$. Now, by taking the wedge product of the map $N^\vee_\mathcal{F} \rightarrow \Omega^1_M$ we get a morphism

$$\bigwedge^k N^\vee_\mathcal{F} \rightarrow \Omega^k_M$$

and twisting by $(\bigwedge^k N^\vee_\mathcal{F})^\vee = \det(N_\mathcal{F})$ we obtain a morphism

$$\omega : \mathcal{O}_M \rightarrow \Omega^k_M \otimes \det(N_\mathcal{F}).$$
Therefore, a foliation is induced by a twisted holomorphic $k$-form

$$\omega \in H^0(X, \Omega^k_M \otimes \det(N_{\mathscr{F}}))$$

which is locally decomposable outside the singular set of $\mathscr{F}$. That is, by the classical Frobenius Theorem for each point $p \in X \setminus \text{Sing}(\mathscr{F})$ there exists a neighbourhood $U$ and holomorphics 1-forms $\omega_1, \ldots, \omega_k \in H^0(U, \Omega^1_U)$ such that

$$\omega|_U = \omega_1 \wedge \cdots \wedge \omega_k$$

and

$$d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0$$

for all $i = 1, \ldots, k$.

3. Proof of the Theorem

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \geq 0$ for $j = 1, \ldots, k$, consider the homogeneous symmetric polynomial of degree $k + 1$, $\varphi = c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_k^{\alpha_k}$ such that $1\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = k + 1$.

Let us consider the twisted $k$-form $\omega \in H^0(M, \Omega^k_M \otimes \det(N_{\mathscr{F}}))$ induced by $\mathscr{F}$. Denote by $\text{Sing}_{k+1}(\mathscr{F})$ the union of the irreducible components of $\text{Sing}(\mathscr{F})$ of pure codimension $k + 1$. Consider an open subset $U \subset M \setminus \text{Sing}(\mathscr{F})$. Thus, the form $\omega|_U$ is decomposable and integrable. That is, $\omega|_U$ is given by a product of $k$ 1-forms $\omega_1 \wedge \cdots \wedge \omega_k$. Then, it is possible to find a matrix of $(1,0)$-forms $(\theta^*_s)$ such that

$$\partial \omega_l = \sum_{s=1}^k \theta^*_s \wedge \omega_s, \quad \overline{\partial} \omega_l = 0, \quad \forall \ l = 1, \ldots, k.$$

We have that $\omega_1, \ldots, \omega_k$ is a local frame for $N^*_\mathscr{F}|_U$ and the identity above induces on $U$ the Bott partial connection

$$\nabla : C^\infty(N^*_\mathscr{F}|_U) \to C^\infty((T\mathscr{F}^* \oplus T\overline{M}) \otimes N^*_\mathscr{F}|_U)$$

defined by

$$\nabla_v(\omega_l) = i_v(\partial \omega_l), \quad \nabla_u(\omega_l) = i_u(\overline{\partial} \omega_l) = 0,$$

where $v \in C^\infty(T\mathscr{F}|_U)$ and $u \in C^\infty(T\overline{M}|_U)$ which can be extended to a connection $D^* : C^\infty(N^*_\mathscr{F}|_U) \to C^\infty((TM^* \oplus T\overline{M}) \otimes N^*_\mathscr{F}|_U)$ in the following way

$$D^*_v(\omega_l) = \sum_{s=1}^k i_v(\pi(\theta^*_s)) \omega_s, \quad D^*_u(\omega_l) = i_u(\overline{\partial} \omega_l) = 0$$

where $v \in C^\infty(TM|_U)$ and $u \in C^\infty(T\overline{M}|_U)$ and $\pi : TM^*|_U \to N^*_\mathscr{F}|_U$ is the natural projection. Let $\theta^*$ be the matrix of the connection $D^*$, then $\theta := [-\theta^*]^t$ is the matrix of the induced connection $D$ with respect to the frame $\{\omega_1, \ldots, \omega_k\}$.

Let $K$ be the curvature of the connection $D$ of $N_{\mathscr{F}}$ on $M \setminus \text{Sing}(\mathscr{F})$. It follows from Bott’s vanishing Theorem [13, Theorem 9.11, pg 76] that $\varphi(K) = 0$. Let $V$ be a small neighborhood of $\text{Sing}_{k+1}(\mathscr{F})$. We regularize $\theta$ and $K$ on $V$, i.e. we
choose a matrix of smooth forms $\hat{\theta}$ and $\hat{K}$ coinciding with $\theta$ and $K$ outside of $V$, respectively. By hypothesis $\dim(\text{Sing}(\mathcal{F})) \leq n-k-1$ we conclude by a dimensional reason that, for $\deg(\varphi) = k + 1$, only the components of dimension $n-k-1$ of $\text{Sing}(\mathcal{F})$ play a role. In fact, since $\text{Res}_{\varphi}(\mathcal{F}, Z) \in H_{2(n-k-1)}(Z, \mathbb{C})$, components of dimension smaller than $n-k-1$ contribute nothing. This means that $\varphi(\hat{K})$ localizes on $\text{Sing}_{k+1}(\mathcal{F})$. Then, $\varphi(\hat{K})$ has compact support on $V$, where $V$ is a small neighborhood of $\text{Sing}_{k+1}(\mathcal{F})$. That is,

$$\text{Supp}(\varphi(\hat{K})) \subset V.$$ 

Then

$$\varphi(\hat{K}) = \sum_{Z_i} \hat{\lambda}_i(\varphi)[Z_i],$$

where $Z_i$ is an irreducible component of $\text{Sing}_{k+1}(\mathcal{F})$ and $\hat{\lambda}_i(\varphi) \in \mathbb{C}$. On the other hand, we have that

$$\varphi(N_{\mathcal{F}}) = \sum_{Z_i} \text{Res}(\varphi, \mathcal{F}, Z_i) = \sum_{Z_i} \lambda_i(\varphi)[Z_i].$$

We will show that $\lambda_i(\varphi) = \hat{\lambda}_i(\varphi)$, for all $i$. In particular, this implies that $\varphi(\hat{K}) = \varphi(N_{\mathcal{F}})$. Thereafter, we will determinate the numbers $\hat{\lambda}_i(\varphi)$.

Consider the unique complete polarization of the polynomial $\varphi$, denoted by $\bar{\varphi}$. That is, $\bar{\varphi}$ is a symmetric $k$-linear function such that

$$\left(\frac{1}{2\pi i}\right)^{k+1} \bar{\varphi}(\hat{K}, \ldots, \hat{K}) = \left(\frac{1}{2\pi i}\right)^{k+1} \varphi(\hat{K}).$$

Take a generic point $p \in Z_i$, that is, $p$ is a point where $Z_i$ is smooth and disjoint from the other components. Let us consider $L \subset M$ a $(k+1)$-ball intersecting transversally $\text{Sing}_{k+1}(\mathcal{F})$ at a single point $p \in Z_i$ and non intersecting other component. Define

$$BB(\mathcal{F}, \varphi; Z_i) := \left(\frac{1}{2\pi i}\right)^{k+1} \int_L \varphi(\hat{K}).$$

Then $\hat{\lambda}_i(\varphi) = BB(\mathcal{F}, \varphi; Z_i)$. In fact

$$BB(\mathcal{F}, \varphi; Z_i) = \left(\frac{1}{2\pi i}\right)^{k+1} \int_L \varphi(\hat{K}) = [L] \cap [\varphi(\hat{K})] = \hat{\lambda}_i(\varphi)[L] \cap [Z_i] = \hat{\lambda}_i(\varphi)$$

since $[L] \cap [Z_i] = 1$ and $[L] \cap [Z_j] = 0$ for all $i \neq j$. For each $j = 1, \ldots, k$, define the polynomial

$$\varphi_j(\hat{\theta}, \hat{K}) := \bar{\varphi}(\underbrace{-\hat{\theta} \wedge \hat{\theta}, \ldots, -\hat{\theta} \wedge \hat{\theta}}_{j-1}, \hat{K}, \ldots, \hat{K}).$$

Now, we consider the $(2k+1)$-form

$$\varphi_{\alpha}(\hat{\theta}, \hat{K}) = \sum_{j=0}^{k+1} (-1)^j \frac{(k-1)!}{2^j (k-j-1)! (k+j)!} \varphi_j(\hat{\theta}, \hat{K}).$$
It follows from [15] Lemma 2.3, pg 5 that on $X \setminus \text{Sing}_{k+1}(\mathcal{F})$ we have
\[ d(\varphi_{\alpha}(\hat{\theta}, \hat{K})) = \varphi(\hat{K}). \]

Consider $i : B \to M$ an embedding transversal to $Z_i$ on $p$ as above, i.e., $i(B) = L$.
We have then an one-dimensional foliation $\mathcal{F}|_L = i^*\mathcal{F}$ on $B$ singular only on $i^{-1}(p) = 0$. We have that
\[ \hat{\lambda}_i(\varphi) = BB(\mathcal{F}, \varphi; Z_i) = \left( \frac{1}{2\pi i} \right)^{k+1} \int_L \varphi(\hat{K}) = \left( \frac{1}{2\pi i} \right)^{k+1} \int_B \varphi(i^*\hat{K}). \]

Now, by Stokes’s theorem we obtain
\[ \hat{\lambda}_i(\varphi) \left( \frac{1}{2\pi i} \right)^{k+1} \int_B \varphi(i^*\hat{K}) = \left( \frac{1}{2\pi i} \right)^{k+1} \int_B d(\varphi_{\alpha}(i^*\hat{\theta}, i^*\hat{K})) = \left( \frac{1}{2\pi i} \right)^{k+1} \int_{\partial B} \varphi_{\alpha}(i^*\hat{\theta}, i^*\hat{K}). \]

Firstly, it follows from [15] Lemma 4.6 that on $X \setminus \text{Sing}_{k+1}(\mathcal{F})$ we have
\[ 0 \to \mathcal{E}_r \to \mathcal{E}_{r-1} \to \cdots \to \mathcal{E}_0 \to N_{\mathcal{F}} \otimes \mathcal{A}_M \to 0. \]

Let $D_q, D_{q-1}, \ldots, D_0$ be connections for $\mathcal{E}_q, \mathcal{E}_{q-1}, \ldots, \mathcal{E}_0$, respectively. Set the curvature of $D_i$ by $K_i = K(D_i)$. By using Baum-Bott notation [2] pg 297 we have that
\[ \varphi(K_q|K_{q-1}| \cdots |K_0) = \varphi(N_{\mathcal{F}}). \]

Consider on $V$ a locally free resolution of the tangent sheaf of $\mathcal{F}$:
\[ 0 \to \mathcal{E}_q \to \mathcal{E}_{q-1} \to \cdots \to \mathcal{E}_1 \to T\mathcal{F} \otimes \mathcal{A}_V \to 0. \]

Combining this sequence with the sequence
\[ 0 \to T\mathcal{F} \otimes \mathcal{A}_V \to TV \to N_{\mathcal{F}} \otimes \mathcal{A}_V \to 0. \]

we get
\[ 0 \to \mathcal{E}_q \to \mathcal{E}_{q-1} \to \cdots \to \mathcal{E}_1 \to TV \to N_{\mathcal{F}} \otimes \mathcal{A}_V \to 0. \]

Pulling back the sequence (3) by $i : B \to V$ we obtain an exact sequence on $B$:
\[ 0 \to i^*\mathcal{E}_q \to i^*\mathcal{E}_{q-1} \to \cdots \to i^*\mathcal{E}_1 \to i^*(T\mathcal{F} \otimes \mathcal{A}_V) \to 0. \]

Since $B$ is a small ball we have the splitting $i^*TV = TB \oplus N_{B|V}$, where $N_{B|V}$ denotes its normal bundle. We consider the projection $\xi : i^*TV \to TB$ and we map $i^*TV$ to $N_{i^*\mathcal{F}}$ via
\[ i^*TV \overset{\xi}{\to} TB \to N_{i^*\mathcal{F}} \]

which give us an exact sequence
\[ 0 \to i^*(T\mathcal{F} \otimes \mathcal{A}_V) \to i^*TV \to N_{i^*\mathcal{F}} \otimes \mathcal{A}_B \to 0. \]
Now, combining the exact sequences \( \text{(3)} \) and \( \text{(6)} \) we obtain an exact sequence
\[
0 \to i^* E_q \to i^* E_{q-1} \to \cdots \to i^* E_1 \to i^* TV \to N_{r, \mathcal{F}} \otimes A_B \to 0.
\]
Let \( D_q, D_{q-1}, \ldots, D_0 \) be connections for \( E_q, E_{q-1}, \ldots, E_1, TV \), respectively. Observe that
\[
i^* \varphi(K_q|K_{q-1}| \cdots |K_0) = \varphi(i^* K_q|i^* K_{q-1}| \cdots |i^* K_0) = \varphi(N_{r, \mathcal{F}}).
\]
Finally, it follows from \( \text{[2, Lemma 7.16]} \) that
\[
\text{Res}_\varphi(i^* \mathcal{F}; 0) = \left( \frac{1}{2 \pi i} \right)^{k+1} \int_B \varphi(i^* K_q|i^* K_{q-1}| \cdots |i^* K_0) = \left( \frac{1}{2 \pi i} \right)^{k+1} \int_B i^* \varphi(K_q|K_{q-1}| \cdots |K_0)
\]
and \( \text{[2, 9.12, pg 326]} \) that
\[
\text{Res}_\varphi(i^* \mathcal{F}; 0) = \left( \frac{1}{2 \pi i} \right)^{k+1} \int_B i^* \varphi(K_q|K_{q-1}| \cdots |K_0) = \lambda_i(\varphi).
\]

Thus, we conclude from \( \text{[2]} \) and \( \text{(7)} \) that \( \lambda_i(\varphi) = \hat{\lambda}_i(\varphi) \), for all \( i \). This implies that \( \varphi(\hat{K}) = \varphi(N_{r, \mathcal{F}}) \).

Now, we will determinate the numbers \( \hat{\lambda}_i(\varphi) \). Let \( X = \sum_{i=1}^{k+1} X_i \partial/\partial z_i \) be a vector field inducing \( i^* \mathcal{F} \) on \( B \) and \( J(X) \) denotes the Jacobian of \( X \). Let \( \omega \) be the 1-form on \( B \setminus \{0\} \) such that \( i_X(\omega) = 1 \). It follows from \( \text{[15, Corollary 4.7]} \) that
\[
\hat{\lambda}_i(\varphi) = \left( \frac{1}{2 \pi i} \right)^{k+1} \int_{\partial B} \varphi \left( i^* \hat{\theta}, i^* \hat{K} \right) = \left( \frac{1}{2 \pi i} \right)^{k+1} \int_{\partial B} \omega \wedge (\partial \omega)^k \varphi \left( -J(X) \right).
\]

Thus,
\[
\hat{\lambda}_i(\varphi) = \left( \frac{1}{2 \pi i} \right)^{k+1} \int_{\partial B} (-1)^{k+1} \omega \wedge (\partial \omega)^k \varphi \left( J(X) \right).
\]

By using Martinelli’s formula \( \text{[9, pg. 655]} \) we have
\[
\hat{\lambda}_i(\varphi) = \left( \frac{1}{2 \pi i} \right)^{k+1} \int_{\partial B} (-1)^{k+1} \omega \wedge (\partial \omega)^k \varphi \left( J(X) \right) = \text{Res}_\varphi \left[ \varphi(JX) \frac{dz_1 \wedge \cdots \wedge dz_{k+1}}{X_1 \cdots X_{k+1}} \right],
\]

Therefore,
\[
\hat{\lambda}_i(\varphi) = \text{Res}_\varphi(i^* \mathcal{F}; 0) = \text{Res}_\varphi(\mathcal{F}|_L; p),
\]

where \( \text{Res}_\varphi(\mathcal{F}|_L; p) \) represents the Grothendieck residue at \( p \) of the one dimensional foliation \( \mathcal{F}|_L \) on a \((k+1)\)-dimensional transversal ball \( L \).

4. Examples

In the next examples, with a slight abuse of notation, we write \( \text{Res}(\mathcal{F}, \varphi; Z_i) = \lambda_i(\varphi) \).

**Example 4.1.** Let \( \mathcal{F} \) be the logarithmic foliation on \( \mathbb{P}^3 \) induced, locally in \((\mathbb{C}^3, (x, y, z))\) by the polynomial 1-form
\[
\omega = yzdx + xzdy + xydz.
\]

In this chart, the singular set of \( \omega \) is the union of the lines \( Z_1 = \{ x = y = 0 \} \); \( Z_2 = \{ x = z = 0 \} \) and \( Z_3 = \{ y = z = 0 \} \). We have \( \text{Res}(\mathcal{F}, c_i^2; Z_i) = \text{Res}_{c_i^2}(\mathcal{F}; p_i), \)
where $\mathcal{F}$ is a foliation on $D_i$ with $D_i$ a 2-disc cutting transversally $Z_i$. Consider $D_1 = \{(x,y), ||(x,y)|| \leq 1, z = 1\}$ then, we have

$$\omega|_{D_1} = \omega_1 = ydx + xdy$$

with dual vector field $X_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

Then, $D_1 \cap Z_1 = \{p_1 = (0, 0, 1)\}$. Now, a straightforward calculation shows that

$$JX_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Thus,

$$\text{Res}_{c_2}(\mathcal{F}; p_1) = \frac{c_2^2(JX_1(p_1))}{\det(JX_1(p_1))} = 0.$$

The same holds for $Z_2$ and $Z_3$. The foliation $\mathcal{F}$ is induced, in homogeneous coordinates $[X, Y, Z, T]$, by the form

$$\tilde{\omega} = YZTdX + XZTdY + XYTdZ - 3XYZdT.$$

The singular set of $\mathcal{F}$ is the union of the lines $Z_1, Z_2, Z_3$, and $Z_4 = \{T = X = 0\}$, $Z_5 = \{T = Y = 0\}$ and $Z_6 = \{T = X = 0\}$.

For $Z_4 = \{X = T = 0\}$ we can consider the local chart $U_y = \{Y = 1\}$. Then, we have,

$$\omega_y := \tilde{\omega}|_{U_y} = ztdx + xtdz - 3xzdtt.$$

Take a 2-disc transversal $D_2 = \{(x, t), ||(x, t)|| \leq 1, z = 1\}$.

$$\omega_2 := \omega_y|_{D_2} = tdx - 3xdt$$

with dual vector field $X_2 = -3x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t}$.

Thus, $Z_4 \cap D_2 = \{(0, 1, 0) = : p_4\}$ and

$$JX_2(p_4) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore,

$$\text{Res}_{c_2}(\mathcal{F}; p_4) = \frac{c_2^2(JX_2)(p_4)}{\det(JX_2)(p_4)} = \frac{16}{3}.$$ 

An analogous calculation shows that

$$\text{Res}_{c_2}(\mathcal{F}; p_5) = \text{Res}_{c_2}(\mathcal{F}; p_6) = \frac{16}{3}.$$ 

Now, we will verify the formula

$$c_2^2(N_\mathcal{F}) = \sum_{i=1}^{6} \text{Res}(\mathcal{F}, c_2^2; Z_i)[Z_i].$$ 

On the one hand, Since $\det(N_\mathcal{F}) = O_{P^3}(4)$, then

$$c_2^2(N_\mathcal{F}) = c_2^2(\det(N_\mathcal{F})) = 16h^2,$$

where $h$ represents the hyperplane class. On the other hand, by the above calculations and since $[Z_i] = h^2$, for all $i$, we have

$$\sum_{i=1}^{6} \text{Res}(\mathcal{F}, c_2^2; Z_i)[Z_i] = 0[Z_1] + 0[Z_2] + 0[Z_3] + \frac{16}{3}[Z_4] + \frac{16}{3}[Z_5] + \frac{16}{3}[Z_6] = 16h^2.$$
The following example is due to D. Cerveau and A. Lins Neto, see [6]. It originates from the so-called exceptional component of the space of codimension one holomorphic foliations of degree 2 of \( \mathbb{P}^n \). We can simplify the computation as done by M. Soares in [12].

**Example 4.2.** Consider \( \mathcal{F} \) be a holomorphic foliation of codimension one on \( \mathbb{P}^3 \), given locally by the 1-form

\[
\omega = z(2y^2 - 3x)dx + z(3z - xy)dy - (xy^2 - 2x^2 + yz)dz.
\]

The singular set of this foliation has one connected component, denoted by \( Z \), with 3 irreducible components, given by:

1) the twisted cubic \( \Gamma : y \mapsto (2/3y^2, y, 2/9y^3) \),
2) the quadric \( Q : y \mapsto (y^2/2, y, 0) \),
3) the line \( L : y \mapsto (0, y, 0) \).

We consider a transversal 2-disc \( D \subset \{ y = 1 \} \) and we take the restriction of \( \mathcal{F} \) on the affine open \( \{ y = 1 \} \). We have an one-dimensional holomorphic foliation, denoted by \( \mathcal{G} \), given by the 1-form on \( H \)

\[
\tilde{\omega} = (2z - 3xz)dx + (2x^2 - x - z)dz
\]

with dual vector field

\[
X = (2x^2 - x - z)\frac{\partial}{\partial x} + (-2z + 3xz)\frac{\partial}{\partial z}.
\]

The singular set of \( \mathcal{G} \) is given by

\[
\text{Sing}(X) = \{ p_1 = (2/3, 1, 2/9); p_2 = (1/2, 1, 0); p_3 = (0, 1, 0) \}.
\]

We know how to calculate the Grothendieck residue of the foliation \( \mathcal{G} \):

\[
\text{Res}_{c_2}(\mathcal{G}; p_1) = \frac{c_2(JX(p_1))}{\text{det}(JX(p_1))} = \frac{25}{6},
\]

\[
\text{Res}_{c_2}(\mathcal{G}; p_2) = \frac{c_2(JX(p_2))}{\text{det}(JX(p_2))} = -\frac{1}{2},
\]

\[
\text{Res}_{c_2}(\mathcal{G}; p_3) = \frac{c_2(JX(p_3))}{\text{det}(JX(p_3))} = \frac{9}{7}.
\]

Now, we will verify the formula

\[
c_1^2(N_{\mathcal{G}}) = \text{Res}(\mathcal{F}, c_1^2; \Gamma)[\Gamma] + \text{Res}(\mathcal{F}, c_1^2; Q)[Q] + \text{Res}(\mathcal{F}, c_1^2; L)[L]
\]

On the one hand, Since \( \text{det}(N_{\mathcal{G}}) = O_{\mathbb{P}^3}(4) \), then

\[
c_1^2(N_{\mathcal{G}}) = c_1^2(\text{det}(N_{\mathcal{G}})) = 16h^2.
\]
where \( h \) represents the hyperplane class. On the other hand, by the above calculations and using that \([\Gamma] = 3h^2\), \([Q] = 2h^2\) and \([L] = h\) we have

\[
\sum_{i=1}^{3} \text{Res}(\mathcal{F}, c_i^\mathcal{F}; Z_i)[Z_i] = \frac{25}{6}[\Gamma] - \frac{1}{2}[Q] + \frac{9}{2}[L] = \frac{25}{6}[3h^2] - \frac{1}{2}[2h^2] + \frac{9}{2}[L] = 16h^2.
\]

**Example 4.3.** Let \( f : M \rightarrow N \) be a dominant meromorphic map such that \( \dim(N) = k + 1 \) and \( \mathcal{G} \) is an one-dimensional foliation on \( N \) with isolated singular set \( \text{Sing}(\mathcal{G}) \). Suppose that \( f : M \rightarrow N \) is a submersion outside its indeterminacy locus \( \text{Ind}(f) \). Then, the induced foliation \( \mathcal{F} = f^*\mathcal{G} \) on \( M \) has codimension \( k \) and \( \text{Sing}(\mathcal{F}) = f^{-1}(\text{Sing}(\mathcal{G})) \cup \text{Ind}(f) \). If \( \text{Ind}(f) \) has codimension \( \geq k + 1 \), we conclude that \( \text{cod}(\text{Sing}(\mathcal{F})) \geq k + 1 \). If \( q \in f^{-1}(p) \subset M \) is a regular point of the map \( f : M \rightarrow N \), then

\[
\text{Res}(f^*\mathcal{G}, \varphi; f^{-1}(p)) = \text{Res}_{\varphi}(\mathcal{G}; p)[f^{-1}(p)],
\]

where \( \text{Res}_{\varphi}(\mathcal{G}; p) \) represents the Grothendieck residue at \( p \in \text{Sing}(\mathcal{G}) \). In fact, there exist open sets \( U \subset M \) and \( V \subset N \), with \( q \in f^{-1}(p) \subset U \) and \( p \in V \), such that \( U \simeq f^{-1}(p) \times V \). Now, if we take a \((k+1)\)-ball \( B \) in \( V \) then by theorem 1.2 we have

\[
\text{Res}(f^*\mathcal{G}, \varphi; f^{-1}(p)) = \text{Res}_{\varphi}(\mathcal{G}|_B; p)[f^{-1}(p)] = \text{Res}_{\varphi}(\mathcal{G}; p)[f^{-1}(p)].
\]

For instance, if \( f : \mathbb{P}^n \rightarrow (\mathbb{P}^{k+1}, \mathcal{G}) \) is a rational linear projection and \( \mathcal{G} \) is an one-dimensional foliation with isolated singularities. Since \( \text{Ind}(f) = \mathbb{P}^{k+1} \), then \( \text{cod}(\text{Sing}(f^*\mathcal{G})) = k + 1 \). Therefore

\[
\text{Res}(f^*\mathcal{G}, \varphi; f^{-1}(p)) = \text{Res}(f^*\mathcal{G}, \varphi; \mathbb{P}^{k+1}) = \text{Res}_{\varphi}(\mathcal{G}; p)[\mathbb{P}^{k+1}].
\]

5. Cenkl algorithm for singularities with non-expected dimension

In [7] Cenkl provided an algorithm to determinate residues for non-expected dimensional singularities, under a certain regularity condition on the singular set of the foliation. We observe that this condition is not necessary. In fact, Cenkl’s conditions are the following:

Suppose that the singular set \( S := \text{Sing}(\mathcal{F}) \) of \( \mathcal{F} \) has pure codimension \( k + s \), with \( s \geq 1 \), and

(i) \( \text{cod}(S) \geq 4 \),

(ii) there exists a closed subset \( W \subset M \) such that \( S \subset W \) with the property

\[
H^j(W, \mathbb{Z}) \simeq H^j(W \setminus S, \mathbb{Z}), \quad j = 1, 2.
\]

Denote by \( M' = M \setminus S \), Cenkl show that under the above condition the line bundle \( \wedge^k(N_{\mathcal{F}})|_{M'} \) on \( M' \) can be extended a line bundle on \( M \). We observe that there always exists a line bundle \( \det(N_{\mathcal{F}})^\vee = [\wedge^k(N_{\mathcal{F}})^\vee]^{\wedge^k} \) on \( M \) which extends
\[ E_\mathcal{F} = \text{det}(N_\mathcal{F})^\vee \oplus \text{det}(N_\mathcal{F})^{\vee'} \].

Observe that \( E_\mathcal{F}|_{M'} = \wedge^k(N_\mathcal{F}|_{M'}) \oplus \wedge^k(N_\mathcal{F}|_{M'})^{\vee} \). Thus, we conclude that Lemma 1 in [7, Proposition 5.6.10 and Proposition 5.6.12] holds in general:

**Lemma 5.1.** Consider the projective bundle \( \pi : \mathbb{P}(E_\mathcal{F}) \to M \). Then there exist a holomorphic foliation \( \mathcal{F}_\pi \) on \( \mathbb{P}(E_\mathcal{F}) \) with singular set \( \text{Sing}(\mathcal{F}_\pi) = \pi^{-1}(S) \) such that

\[
\dim(\mathcal{F}_\pi) = \dim(\mathcal{F}) \text{ and } \dim(\text{Sing}(\mathcal{F}_\pi)) = \dim(S) + 1.
\]

We succeeded in replacing the compact manifold \( M \) with a foliation \( \mathcal{F} \) and the singular set \( S \) such that \( \dim(\mathcal{F}) - \dim(\text{Sing}(\mathcal{F})) = n - s \) by another compact manifold \( \mathbb{P}(E_\mathcal{F}) \) and a foliation \( \mathcal{F}_\pi \) with singular set \( \dim(\mathcal{F}_\pi) - \dim(\text{Sing}(\mathcal{F}_\pi)) = n - s - 1 \). If this procedure is repeated \((n - s - 1)\)-times we end up with a compact complex analytic manifold with a holomorphic foliation whose singular set is a subvariety of complex dimension one less than the leaf dimension of the foliation. That is, we have a tower of foliated manifolds

\[
(P_{n-s-1}, \mathcal{F}^{n-s-1}) \xrightarrow{\pi_{n-s-1}} (P_{n-s-2}, \mathcal{F}^{n-s-2}) \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_2} (P_1, \mathcal{F}^1) \xrightarrow{\pi_1=\pi} (M, \mathcal{F})
\]

where \((P_1, \mathcal{F}^1)\) is such that \( P_1 = \mathbb{P}(E_\mathcal{F}|_{\mathcal{F}_\pi}) \) and \((P_1, \mathcal{F}^1) = (\mathbb{P}(E_\mathcal{F}), \mathcal{F}_\pi)\). Thus, by Lemma 5.1 we conclude that on \( P_{n-s-1} \) we have a foliations \( \mathcal{F}^{n-s-1} \) such that \( \text{Sing}(\mathcal{F}^{n-s-1}) = (\pi_{n-s-1} \circ \cdots \circ \pi_2 \circ \pi_1)^{-1}(S) \) and

\[
\dim(\text{Sing}(\mathcal{F}^{n-s-1})) = \dim(\mathcal{F}^{n-s-1}) - 1.
\]

That is, \( \text{cod}(\text{Sing}(\mathcal{F}^{n-s-1})) = \text{cod}(\mathcal{F}^{n-s-1}) + 1 \).

On the one hand, we can apply the Theorem 1.2 to determinate the residues of \( \mathcal{F}^{n-s-1} \). On the other hand, Cenkl show that we can calculate the residue \( \text{Res}_\varphi(\mathcal{F}^1, Z_1) \) in terms of the residue \( \text{Res}_\varphi(\mathcal{F}, Z) \) for symmetric polynomial \( \varphi \) of degree \( k + 1 \).

Let us recall the Cenkl’s construction:

Let \( \sigma_1, \ldots, \sigma_\ell \) be the elementary symmetric functions in the \( n \) variables \( x_1, \ldots, x_n \) and let \( \rho_1, \ldots, \rho_\ell \) be the elementary symmetric functions in the \( n + 1 \) variables \( x_1, \ldots, x_n, y \). It follows from [7, Corollary, pg 21] that for any polynomial \( \phi \), of degree \( \ell \), can be associated a polynomial \( \psi \) of degree \( \ell + 1 \) such that

\[
\psi(\rho_1, \ldots, \rho_\ell) = \phi(\sigma_1, \ldots, \sigma_\ell) y + \phi^0(\sigma_1, \ldots, \sigma_\ell) y^2 + \sum_{j \geq 2} \phi^j(\sigma_1, \ldots, \sigma_\ell) \cdot y^j,
\]

where \( \phi^0 \) has degree \( \ell + 1 \) and \( \phi^j \) has degree \( \ell - j + 1 \).

Let \( T_{P/M} \) be the tangent bundle associated the one-dimensional foliation induced by the \( \mathbb{P}^1 \)-fibration \((P, \mathcal{F}_\pi) \to (M, \mathcal{F})\).
Therefore, it follows from Lemma 5.1, Cenkl’s construction [7, Theorem 1] and Theorem 1.2 the following:

**Theorem 5.2.** Suppose that $\text{cod}(\text{Sing}(\mathcal{F})) \geq \text{cod}(\mathcal{F}) + 2$. If $\varphi$ is a homogeneous symmetric polynomials of degree $\text{cod}(\mathcal{F}) + 1$, then

$$\text{Res}_{\psi}(\mathcal{F} \mid |_{B_p}; p)[Z_1] = \pi^* \text{Res}_{\varphi}(\mathcal{F}, Z) \cap c_1(T_{P/M}) + \pi^*(\phi^0(N_{\mathcal{F}})) + \sum_{j \geq 2} \pi^*(\phi^j(N_{\mathcal{F}})) \cap c_1(T_{P/M})^j,$$

where $\text{Res}_{\psi}(\mathcal{F} \mid |_{B_p}; p)$ represents the Grothendieck residue at $p$ of the one dimensional foliation $\mathcal{F} \mid |_{B_p}$ on a $(k + 1)$-dimensional transversal ball $B_p$.

We believe that this algorithm can be adapted to the context of residues for flags of foliations [4].

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DETERMINATION OF BAUM-BOTT RESIDUES OF HIGHER CODIMENSIONAL FOLIATIONS

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