THE VARIETY OF LIE ALGEBRAS OF MAXIMAL CLASS

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Abstract. We present an explicit description of the affine variety $M_{Fil}$ of Lie algebras of the maximal class (filiform Lie algebras): the formulas of polynomial equations that determine this variety are written. The affine variety $M_{Fil}$ can be considered as the base of the nilpotent versal deformation of $\mathbb{N}$-graded Lie algebra $m_0$.

Introduction

The methods of the Lie algebras deformation theory are used in the study of nilpotent Lie algebras for many years. One knows that there are only a finite number of pairwise nonisomorphic Lie algebras in dimensions less or equal six, but already in the dimension 7 an one-parameter family of isomorphism classes of such algebras appears for the first time. In the higher dimensions the varieties of nilpotent Lie algebras are considered. More often it means the variety of Lie algebra laws, defined on the $n$-dimensional vector space $K^n$ with a fixed basis $e_1, \ldots, e_n$. As a result one gets an affine variety in $K^{n^3}$ (the coordinates in $K^{n^3}$ are the components of the tensor $c^k_{ij}$, $[e_i, e_j] = c^k_{ij} e_k$, and polynomial equations on $c^k_{ij}$ are obtained from the Jacobi identity as well as from the nilpotency condition). The generic points of the variety $M^n$ are so called filiform Lie algebras (the term that was introduced by Michelle Vergne) – nilpotent Lie algebras with the maximal nil-index $n-1$ for a given dimension $n$. Vergne have shown, that an arbitrary filiform Lie algebra is isomorphic to some deformation $[,] + \Psi$ of the graded filiform Lie algebra $m_0(n)$ with the bracket $[,]$, when the integrability condition for the cocycle $\Psi \in H^2_+(m_0(n), m_0(n))$ (i.e. the Jacobi identity for $[,] + \Psi$) is equivalent to the vanishing of its Nijenhuis-Richardson square $[\Psi, \Psi] = 0$. The algebra $m_0(n)$ is defined by its basis $e_1, \ldots, e_n$ and nontrivial commutator relations: $[e_1, e_i] = e_{i+1}, i = 2, \ldots, n-1$. Vergne have calculated the dimensions of the spaces $H^2_+(m_0(n), m_0(n))$. Later her approach has become the cornerstone of the whole number of researches (classifications of filiform Lie algebras of small dimensions, the description of the irreducible components of the variety $M^n$, and etc.) of the whole number of authors: Khakimdjanov, Goze, Ancochea-Bermudez and others. after decomposing vector $\Psi = \sum x_{i,r} \Psi_{i,r}$ on the basis $\Psi_{i,r}$ of the space $H^2_+(m_0(n), m_0(n))$ (the basis that was introduced by Khakimdjanov [8]) the integrability condition $[\Psi, \Psi] = 0$ is equivalent to some system of quadratic equations on variables $x_{i,r}$. The Affine variety, defined by this system, began to be called the variety of filiform Lie algebras. At the same time, the study of these brackets up to an isomorphism was carried out only in small dimensions(dim $\mathfrak{g} \leq 12$) and the answer, as a rule,
was given in the form of the table of the computer calculations with the aid of some package of the symbolic calculations like Maple or Mathematica. In this case the whole set of equations on unknowns $x_{i,r}$ for an arbitrary dimension $n$ have been never written out. For instance in [4] only an algorithm for finding of such equations was proposed.

Beginning from the 50ths of the last century $p$-groups of maximal class began actively to be studied in group theory: groups of order $p^n$ and of index of the nilpotency $n-1$. Later for the study of $p$-groups and pro-$p$-groups the notion of index of the nilpotency have been substituted on a new invariant, on the so-called coclass of the group. The application of methods of Lie algebra theory with the study of $p$- and pro-$p$-groups led to the fact, that Lie algebras of maximal class began to studied in finite characteristic. In some papers they are called also narrow or thin Lie algebras (see [13]). Residually nilpotent Lie algebra $g$ is called a Lie algebra of maximal class or of coclass 1, if

$$\sum_{i \geq 1} (\dim(C^i g/C^{i+1} g) - 1) = 1,$$

where $C^i g$ denotes $i$-th ideal of lower descending series of $g$. If $\dim g < \infty$, the this condition is equivalent to the fact, that $g$ is a filiform Lie algebra.

The most elementary example of infinite dimensional Lie algebra of maximal class is the algebra $m_0$ — the direct limit of algebras $m_0(n)$. It follows from Vergne’s theorem [14], that an arbitrary finitely generated Lie algebra of maximal class is isomorphic to some special deformation of $m_0$. The space of such deformations has the structure of a direct limit $M_{Fil}$ of affine varieties, imbedded in $\lim_n K^n$. $M_{Fil}$ is a base of nilpotent versal deformation of the algebra $m_0$. The main result of the present article is the Theorem 2.6 with explicit formulas of polynomial equations, that define $M_{Fil}$. The answer in finite dimensional case is also provided.

On has to remark that the study of homogeneous deformations of $m_0$ was started by Fialowski and Wagemann in [7].

The paper is organized in the following way. In the first section definitions and facts concerning Lie algebras of maximal class are presented. Section 2 is devoted to the cohomology of $N$-graded Lie algebra $m_0$. In the third section we describe the affine variety of Lie algebras of maximal class in the form of nilpotent versal deformation. Let us remark that all considerations are valid for a field of zero characteristic. In the section 4 we consider a variety of filiform Lie algebras of finite dimension $n$. The number of unknowns of the system is equal to the dimension of non negative part $\dim H^2_+(m_0((n), m_0(n))$ of the second cohomology with the coefficients in the adjoint representation [14], and the dimension $\dim H^3_+(m_0(n), m_0(n))$ answers for the number of equations.

1. Filiform Lie algebras and Lie algebras of maximal class

The sequence of ideals of a Lie algebra $g$

$$C^1 g = g \supset C^2 g = [g, g] \supset \ldots \supset C^k g = [g, C^{k-1} g] \supset \ldots$$

is called the lower descending central series of the Lie algebra $g$. 
A Lie algebra $\mathfrak{g}$ is called nilpotent, if there exists $s$ such that

$$C^{s+1}\mathfrak{g} = [\mathfrak{g}, C^s\mathfrak{g}] = 0, \quad C^s\mathfrak{g} \neq 0.$$ 

The natural number $s$ is said to be the nil-index of the nilpotent Lie algebra $\mathfrak{g}$.

**Proposition 1.1.** Let $\mathfrak{g}$ be a $n$-dimensional nilpotent Lie algebra. Then we have the following estimate for its nil-index: $s \leq n - 1$.

**Definition 1.2.** A nilpotent $n$-dimensional Lie algebra $\mathfrak{g}$ is called filiform if its nil-index is equal to $s = n - 1$.

**Example 1.3.** A Lie algebra $\mathfrak{m}_0(n)$, that is defined by its basis $e_1, e_2, \ldots, e_n$ with commutator relations:

$$[e_1, e_i] = e_{i+1}, \; \forall \; 2 \leq i \leq n-1,$$

is filiform.

Let $\mathfrak{g}$ be a Lie algebra. We will call the set $F$ of subspaces

$$\mathfrak{g} \supset \ldots \supset F^i \mathfrak{g} \supset F^{i+1} \mathfrak{g} \supset \ldots, \quad i \in \mathbb{Z},$$

a decreasing filtration $F$ of the Lie algebra $\mathfrak{g}$, if the set $F$ is compatible with the Lie algebra structure:

$$[F^k \mathfrak{g}, F^l \mathfrak{g}] \subset F^{k+l} \mathfrak{g} \quad \forall k, l \in \mathbb{Z}.$$ 

Let $\mathfrak{g}$ be a filtered Lie algebra. The graded Lie algebra

$$\text{gr}_F \mathfrak{g} = \bigoplus_{k=1}^\infty (\text{gr}_F \mathfrak{g})_k, \quad (\text{gr}_F \mathfrak{g})_k = F^k \mathfrak{g}/F^{k+1} \mathfrak{g},$$

is called the associated graded Lie algebra $\text{gr}_F \mathfrak{g}$.

The ideals $C^k \mathfrak{g}$ of lower descending series form decreasing filtration $C$ of a Lie algebra $\mathfrak{g}$

$$C^1 \mathfrak{g} = \mathfrak{g} \supset C^2 \mathfrak{g} \supset \ldots \supset C^k \mathfrak{g} \supset \ldots,$$

$$[C^k \mathfrak{g}, C^l \mathfrak{g}] \subset C^{k+l} \mathfrak{g}.$$ 

Let us consider also the associated graded Lie algebra $\text{gr}_C \mathfrak{g}$.

**Proposition 1.4.** Let $\mathfrak{g}$ be a filiform Lie algebra and $\text{gr}_C \mathfrak{g} = \bigoplus_i (\text{gr}_C \mathfrak{g})_i$ be the corresponding associated (with respect to the canonical filtration $C$) graded Lie algebra. Then

$$\dim(\text{gr}_C \mathfrak{g})_1 = 2, \quad \dim(\text{gr}_C \mathfrak{g})_2 = \ldots = \dim(\text{gr}_C \mathfrak{g})_{n-1} = 1.$$ 

**Corollary 1.5.** Let $\mathfrak{g}$ be a filiform Lie algebra, then

$$\sum_{i=1}^{\dim \mathfrak{g}} (\dim(C^i \mathfrak{g}/C^{i+1} \mathfrak{g}) - 1) = 1,$$

where only the first summand is non equal to zero.

**Definition 1.6.** A Lie algebra $\mathfrak{g}$ is called residually nilpotent if

$$\cap_{i=1}^{\infty} C^i \mathfrak{g} = 0.$$
Definition 1.7. The coclass of a Lie algebra $\mathfrak{g}$ (which can be equal to infinity) is the natural number $cc(\mathfrak{g})$ that is defined as $cc(\mathfrak{g}) = \sum_{i \geq 1} (\dim(C^i \mathfrak{g}/C^{i+1} \mathfrak{g}) - 1)$. Finitely generated residually nilpotent Lie algebras of coclass 1 is also called as algebras of maximal class (infinite dimensional filiform Lie algebras).

Example 1.8. Let us define $L_k$ as the Lie algebra of polynomial vector fields on the real line $\mathbb{R}^1$, having the zero $x = 0$ of order not less than $k + 1$.

The algebra $L_k$ can be defined by its infinite basis and structure relations

$$e_i = x^{i+1} \frac{d}{dx}, \quad i \in \mathbb{N}, \quad i \geq k, \quad [e_i, e_j] = (j - i)e_{i+j}, \quad \forall i, j \in \mathbb{N}.$$ 

Let consider $k = 1$, it is simple to note, that $L_1$ is a residually nilpotent Lie algebra of maximal class, generated by two elements: $e_1$ and $e_2$.

We recall that $\mathbb{Z}$-graded Lie algebra $W$ defined by its basis $e_i, \quad i \in \mathbb{Z}$ and relations

$$[e_i, e_j] = (j - i)e_{i+j}, \quad \forall i, j \in \mathbb{Z},$$

is called the Witt algebra. The Lie algebra $L_1$ sometimes is called the positive part $W_+ = \bigoplus_{i > 0} (W)_i$ of the Witt Lie algebra.

Let us give the two additional examples infinite dimensional $\mathbb{N}$-graded Lie algebras of maximal class.

Example 1.9. Let us denote by $m_0$ the direct limit of Lie algebras $m_0(n)$ when $n \to \infty$. This infinite-dimensional Lie algebra $m_0$ can be defined by a basis $e_1, e_2, \ldots, e_n, \ldots$ and structure relations:

$$[e_1, e_i] = e_{i+1}, \quad \forall i \geq 2.$$ 

Example 1.10. A Lie algebra $m_2$ is defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ and structure relations

$$[e_1, e_i] = e_{i+1}, \quad \forall i \geq 2; \quad [e_2, e_j] = e_{j+2}, \quad \forall j \geq 3.$$ 

We will call the filtration $C$ of a residually nilpotent Lie algebra $\mathfrak{g}$ the canonical filtration of $\mathfrak{g}$.

Remark. There are the following isomorphisms $\text{gr}_C L_1 \cong \text{gr}_C m_2 \cong \text{gr}_C m_0 \cong m_0$.

Theorem 1.11 (Vergne [11]). Let $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$ be a $\mathbb{N}$-graded $n$-dimensional filiform Lie algebra and

$$\dim \mathfrak{g}_1 = 2, \quad \dim \mathfrak{g}_2 = \ldots = \dim \mathfrak{g}_{n-1} = 1.$$ 

Then

1) if $n = 2k + 1$, then $\mathfrak{g}$ is isomorphic to $m_0(2k + 1)$;
2) if $n = 2k$, then $\mathfrak{g}$ is isomorphic either to $m_0(2k)$ or to the algebra $m_1(2k)$ defined by the basis $e_1, \ldots, e_{2k}$ and structure relations

$$[e_1, e_i] = e_{i+1}, \quad i = 2, \ldots, 2k-1; \quad [e_j, e_{2k+1-j}] = (-1)^{j+k} e_{2k}, \quad j = 2, \ldots, k.$$
Remark. In the formulation of the theorem 1.11 the Lie algebra gradings $\mathfrak{m}_0(n), \mathfrak{m}_1(n)$ are defined as $\mathfrak{g}_1 = \langle e_1, e_2 \rangle, \mathfrak{g}_i = \langle e_{i+1} \rangle, i = 2, \ldots, n-1$.

Both the examples of infinite dimensional Lie algebras that we have considered are nonaccidentally the direct limits of some suites of filiform Lie algebras. It follows directly from the definition, that an arbitrary Lie algebra of maximal class can be considered as the direct limit of some infinite chain of inserted in each other filiform Lie algebras, where each subsequent algebra of the chain is obtained as a one-dimensional central extension of the previous algebra. Specifically, such direct limits of filiform Lie algebras we will consider in the sequel. Constructing by recursion a suite of the added elements $e_i$ with the central extensions, one gets the following corollary:

**Corollary 1.12.** Let $\mathfrak{g}$ be finitely generated residually nilpotent Lie algebra of maximal class. Then

$$\text{gr}_{C} \mathfrak{g} \cong \mathfrak{m}_0.$$  

In $\mathfrak{g}$ one can choose an infinite basis $e_1, e_2, \ldots, e_n, \ldots$ such as

$$[e_1, e_i] = e_{i+1}, \quad \forall i \geq 2, \quad [e_i, e_j] = \sum_{k=0}^{N(i,j)} c_{ij}^j e_{i+j+k}.$$

By means of some fixed basis $e_1, e_2, \ldots, e_n, \ldots$ one can defined another one (non canonical) decreasing filtration $\tilde{F}$ of the algebra $\mathfrak{g}$:

$$\tilde{F}^k \mathfrak{g} = \langle e_k, e_{k+1}, e_{k+2}, \ldots \rangle, \quad k \geq 1.$$  

In this case the homogeneous components of the corresponding associated graded Lie algebra $\text{gr}_{\tilde{F}} \mathfrak{g}$ will be already one-dimensional.

The classification of such associated graded Lie algebras is known.

**Theorem 1.13 ([2, 12]).** Let $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ be a $\mathbb{N}$-graded Lie algebra of maximal class such that

$$\dim \mathfrak{g}_i = 1, \quad \forall i \geq 1.$$  

Then $\mathfrak{g}$ is isomorphic to one (and the only one) Lie algebra of three given ones:

$$\mathfrak{m}_0, \mathfrak{m}_2, L_1.$$  

2. Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be its linear representation (the vector space $V$ is a $\mathfrak{g}$-module). Let us denote by $C^q(\mathfrak{g}, V)$ the space of $q$-linear skew-symmetric mappings from $\mathfrak{g}$ to $V$. The one can consider the following algebraic complex:

$$V \xrightarrow{d_0} C^1(\mathfrak{g}, V) \xrightarrow{d_1} C^2(\mathfrak{g}, V) \xrightarrow{d_2} \ldots \xrightarrow{d_{q-1}} C^q(\mathfrak{g}, V) \xrightarrow{d_q} \ldots$$
where the differential $d_q$ is defined as:

$$
(d_q f)(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \rho(X_i)(f(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1})) + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}).
$$

(1)

The cohomology of the complex $(C^*(g, V), d)$ is called the cohomology of the Lie algebra $g$ with coefficients in the representation $\rho : g \to V$.

Further we will examine two following representations:
1) $V = \mathbb{K}$ and the morphism $\rho : g \to \mathbb{K}$ is trivial.
2) $V = g$ and the adjoint representation $\rho = ad : g \to g$.

The cohomology of the complex $(C^*(g, \mathbb{K}), d)$ are called the cohomology with trivial coefficients of the Lie algebra $g$ and it is denoted by $H^*(g)$. We will fix the notation $H^*(g, g)$ for the cohomology of a Lie Algebra $g$ with coefficients in the adjoint representation.

**Example 2.1.** Let $g$ be a $\mathbb{N}$-graded Lie algebra defined by its infinite basis $e_1, e_2, \ldots, e_n, \ldots$ and structure relations

$$[e_i, e_j] = c_{ij}e_{i+j}.$$

Let us examine the dual basis $e^1, e^2, \ldots, e^n, \ldots$. One can consider a grading (that we will call the weight) of $\Lambda^*(g^*) = C^*(g, \mathbb{K})$:

$$\Lambda^*(g^*) = \bigoplus_{\lambda=1}^{\infty} \Lambda^*_\lambda(g^*),$$

where the subspace $\Lambda^*_\lambda(g^*)$ is spanned by $q$-forms $\{e^{i_1} \wedge \ldots \wedge e^{i_q}, i_1 + \ldots + i_q = \lambda\}$. For instance a monomial $e^{i_1} \wedge \ldots \wedge e^{i_q}$ has the degree $q$ and the weight $\lambda = i_1 + \ldots + i_q$.

In its turn the complex $(C^*(g, g), d)$ is $\mathbb{Z}$-graded:

$$C^*(g, g) = \bigoplus_{\mu \in \mathbb{Z}} C^\mu_{(g, g)},$$

where $C^\mu_{(g, g)}$ spanned by monomials $\{e_l \otimes e^{i_1} \wedge \ldots \wedge e^{i_q}, i_1 + \ldots + i_q + \mu = l\}$.

The cohomology algebra $H^*(m_0)$ was computed in [6].

In order to formulate the main result [6] we will need the following linear operators that act on the exterior algebra $\Lambda^*(e_2, e_3, \ldots)$ with generators $e_2, e_3, \ldots$:

1) $D_1 : \Lambda^*(e_2, e_3, \ldots) \to \Lambda^*(e_2, e_3, \ldots)$,

$$D_1(e^2) = 0, \quad D_1(e^i) = e^{i-1}, \forall i \geq 3,$$

(2)

$$D_1(\xi \wedge \eta) = D_1(\xi) \wedge \eta + \xi \wedge D_1(\eta), \forall \xi, \eta \in \Lambda^*(e^2, e^3, \ldots).$$

2) and its right-inverse operator $D_{-1} : \Lambda^*(e^2, e^3, \ldots) \to \Lambda^*(e^2, e^3, \ldots)$,

$$e^i = e^{i+1}, \quad D_{-1}(\xi \wedge e^i) = \sum_{l \geq 0} (-1)^l D_1^l(\xi) \wedge e^{i+l},$$

(3)
where $i \geq 2$ and $\xi$ is an arbitrary form in $\Lambda^*(e^2, \ldots, e^{i-1})$. The sum in the definition (3) of the operator $D_{-1}$ is always finite, as. $D_1^l$ decreases the second grading by $l$. For instance

$$D_{-1}(e^i \wedge e^k) = \sum_{l=0}^{i-2} (-1)^l e^{i-l} \wedge e^{k+l+1}.$$ 

**Proposition 2.2.** The operators $D_1$ and $D_{-1}$ have the following properties:

$$d\xi = e^1 \wedge D_1\xi, \ e^1 \wedge \xi = dD_{-1}\xi, \ D_1D_{-1}\xi = \xi, \ \xi \in \Lambda^*(e^2, e^3, \ldots).$$

**Remark.** The operator $D_1$ is in fact the operator $ad^* e_1$ that is the adjoint operator to $ade_1 : (e_2, e_3, \ldots) \to (e_2, e_3, \ldots)$ extended from $(e_2, e_3, \ldots)^*$ to the whole exterior algebra $\Lambda^*(e^2, e^3, \ldots)$ like a derivation of degree zero.

**Theorem 2.3 (H).** The infinite dimensional bigraded cohomology $H^*(m_0) = \bigoplus_{k,q} H^q_k(m_0)$ is spanned by the cohomology classes of the linear forms $e^1$, $e^2$ and the following homogeneous cocycles:

$$\omega(e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i_{q+1}}) = \sum_{l \geq 0} (-1)^l D_1^l(e^{i_1} \wedge \ldots \wedge e^{i_q}) \wedge e^{i_{q+1}+l},$$

where $q \geq 1$, $2 \leq i_1 < i_2 < \ldots < i_q$.

The formula (4) defines a homogeneous closed $(q+1)$-form with the second grading equal to $i_1 + \ldots + i_{q-1} + 2i_q + 1$. There is the only one monomial in its decomposition of the form $\xi \wedge e^i \wedge e^{i+1}$ and it is equal to $e^{i_1} \wedge \ldots \wedge e^{i_q} \wedge e^{i_{q+1}}$.

The total number of linear independent $q$-cocycles with the second grading $k + \frac{q(q+1)}{2}$ is equal to

$$\dim H^q_{k+\frac{q(q+1)}{2}}(m_0) = P_q(k) - P_q(k-1),$$

where $P_q(k)$ denotes the number (not ordered) partitions of a natural number $k$ into $q$ parts.

**Example 2.4.** One can choose the following basis in $H^2(m_0)$:

$$e^2 \wedge e^3, e^3 \wedge e^4 - e^2 \wedge e^5, \ldots, \omega(e^j \wedge e^{j+1}) = \sum_{l=0}^{j-2} (-1)^l e^{j-l} \wedge e^{j+1+l}, \ldots$$

**Example 2.5.** In $H^3(m_0)$ the following cocycles form a basis:

$$\omega(e^i \wedge e^j \wedge e^{i+j+1}) = \sum_{l \geq 0} (-1)^l D_1^l(e^i \wedge e^j) \wedge e^{i+j+1+l}, \ 2 \leq i < j.$$ 

In particular we have for the cocyle $\omega(e^5 \wedge e^6 \wedge e^7)$:

$$\omega(e^5 \wedge e^6 \wedge e^7) = e^5 \wedge e^6 \wedge e^7 - e^4 \wedge e^6 \wedge e^8 + (e^3 \wedge e^6 + e^4 \wedge e^5) \wedge e^9 - (e^2 \wedge e^6 + 2e^3 \wedge e^5) \wedge e^{10} + (3e^2 \wedge e^5 + 2e^3 \wedge e^4) \wedge e^{11} - 5e^2 \wedge e^4 \wedge e^{12} + 5e^2 \wedge e^3 \wedge e^{13}.$$
The cohomology $H^*(m_0, m_0)$ were calculated in [1]. We will not completely give the appropriate result. For the purposes of the present work we will need only "non-negative" cohomology $H^2_+(m_0, m_0) = \oplus_{s \geq 0} H^2_s(m_0, m_0)$ and $H^3_+(m_0, m_0) = \oplus_{s \geq 0} H^3_s(m_0, m_0)$.

**Theorem 2.6 ([1]).** The graded spaces $H^2_+(m_0, m_0)$ and $H^3_+(m_0, m_0)$ are infinite dimensional spaces of formal series $\sum x_{j,s} \Psi_{j,s}$ and $\sum x_{i,j,s} \Psi_{i,j,s}$ respectively and homogeneous cocycles $\Psi_{j,s} \in H^2_s(m_0, m_0)$ and $\Psi_{i,j,s} \in H^3_s(m_0, m_0)$ are defined by the formulas:

\[
\Psi_{j,s} = \sum_{k=0}^{\infty} e_{2j+1+s+k} \otimes D_{-1}^k \omega(e^j \wedge e^{j+1}), \quad 2 \leq j, s \geq 0;
\]

\[
\Psi_{i,j,s} = \sum_{k=0}^{\infty} e_{i+2j+1+s+k} \otimes D_{-1}^k \omega(e^i \wedge e^j \wedge e^{j+1}), \quad 2 \leq i < j, s \geq 0.
\]

2-form $\Psi_{j,s}$ is uniquely determined by the fact that it is closed and by the following condition:

\[
\Psi_{j,s}(e_j, e_{j+1}) = e_{2j+1+s}, \\
\Psi_{j,s}(e_k, e_{k+1}) = 0, \quad 2 \leq k \neq j.
\]

The cocycle $\Psi_{i,j,s}$ is also uniquely determined by the similar conditions:

\[
\Psi_{i,j,s}(e_i, e_j, e_{j+1}) = e_{i+2j+1+s}, \\
\Psi_{i,j,s}(e_l, e_k, e_{k+1}) = 0, \quad 2 \leq l < k, \quad l \neq i, k \neq j.
\]

It follows from the properties of the operator $D_{-1}$ that

\[
D_{-1}^k \omega(e^j \wedge e^{j+1}) = \sum_{l=0}^{j-2} (-1)^l \binom{k+l}{l} e^{j-l} \wedge e^{j+1+l+k},
\]

which makes it possible to rewrite the formula of $\Psi_{j,s}$ in the following way:

\[
\Psi_{j,s} = \sum_{l=0}^{+\infty} \sum_{r=0}^{j-2} (-1)^r \binom{l+r}{r} e_{s+l} \otimes e^{j-r} \wedge e^{j+1+r+l}.
\]

It is equivalent to define the cocycle $\Psi_{j,s}$ by the table of its values:

\[
\Psi_{j,s}(e_k, e_m) = (-1)^{j-k} \binom{m-j-1}{j-k} e_{m+k-2j-1+s}, \quad 2 \leq k \leq j < m, \quad k + m \geq 2j + 1,
\]

where on the remaining vector pairs $e_k, e_m$ the cocycle $\Psi_{j,s}$ vanishes.

**Remark.** The last formula shows, that in the finite dimensional case our cocycle $\Psi_{j,s}$ coincides with the cocycle $\Psi_{j,2j+1+s}$ from the basis of the subspace $H^2_+(m_0(n), m_0(n))$ [8].
3. The Variety of non-negative deformations of the algebra \( \mathfrak{m}_0 \) and nilpotent versal deformation

**Definition 3.1 ([II]).** Let \( \mathfrak{g} \) be a Lie algebra with the bracket \([\cdot, \cdot]\) and \( \Psi : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \) be a skew-symmetric bilinear map. \( \Psi \) is called a deformation of the bracket \([\cdot, \cdot] \) if \([\cdot, \cdot]' = [\cdot, \cdot] + \Psi \) defines a structure of a Lie algebra on the vector space \( \mathfrak{g} \).

The Jacobi identity for the bracket \([\cdot, \cdot]' = [\cdot, \cdot] + \Psi \)

\[
[[x, y]', z] + [[y, z]', x]' + [[z, x]', y]' = 0
\]

is equivalent to the so-called deformation equation

\[
d\Psi + \frac{1}{2} [\Psi, \Psi] = 0,
\]

where the bracket \([\cdot, \cdot]\) denotes this time the Nijenhuis-Richardson bracket \( C^2(\mathfrak{g}, \mathfrak{g}) \times C^2(\mathfrak{g}, \mathfrak{g}) \rightarrow C^3(\mathfrak{g}, \mathfrak{g}) \):

\[
[\Psi, \tilde{\Psi}](x, y, z) = \Psi(\tilde{\Psi}(x, y), z) + \Psi(\tilde{\Psi}(y, z), x) + \Psi(\tilde{\Psi}(z, x), y) +
\]

\[
+ \tilde{\Psi}(\Psi(x, y), z) + \tilde{\Psi}(\Psi(y, z), x) + \tilde{\Psi}(\Psi(z, x), y).
\]

The last operation is extended to the bracket on the total space \( C^*(\mathfrak{g}, \mathfrak{g}) \):

\[
[\cdot, \cdot'] : C^n(\mathfrak{g}, \mathfrak{g}) \times C^n(\mathfrak{g}, \mathfrak{g}) \rightarrow C^{n+q-1}(\mathfrak{g}, \mathfrak{g}).
\]

Specifically for \( \alpha \in C^p(\mathfrak{g}, \mathfrak{g}) \) and \( \beta \in C^q(\mathfrak{g}, \mathfrak{g}) \) one can define \([\alpha, \beta] \in C^{p+q-1}(\mathfrak{g}, \mathfrak{g}) \):

\[
[\alpha, \beta](\xi_1, \ldots, \xi_{p+q-1}) = \sum_{1 \leq i_1 < \ldots < i_q \leq p+q-1} \alpha(\beta(\xi_{i_1}, \ldots, \xi_{i_q})\xi_1, \ldots, \xi_{i_q}, \ldots, \xi_{p+q-1}) +
\]

\[
+ (-1)^{pq+p+q} \sum_{1 \leq j_1 < \ldots < j_p \leq p+q-1} \beta(\alpha(\xi_{j_1}, \ldots, \xi_{j_p})\xi_1, \ldots, \xi_{j_p}, \ldots, \xi_{p+q-1}).
\]

The Nijenhuis-Richardson bracket determines a Lie superalgebra structure on \( C^*(\mathfrak{g}, \mathfrak{g}) \), i.e. if \( \alpha \in C^p(\mathfrak{g}, \mathfrak{g}) \), \( \beta \in C^q(\mathfrak{g}, \mathfrak{g}) \) and \( \gamma \in C^r(\mathfrak{g}, \mathfrak{g}) \) then

\[
[\alpha, \beta] = -(-1)^{(p-1)(q-1)}[\beta, \alpha];
\]

\[
(-1)^{(p-1)(q-1)}[[\alpha, \beta], \gamma] + (-1)^{(q-1)(r-1)} [[\beta, \gamma], \alpha] + (-1)^{(r-1)(p-1)} [[\gamma, \alpha], \beta] = 0.
\]

In addition to this the Nijenhuis-Richardson bracket is compatible with \( d \) of the cochain complex \( C^*(\mathfrak{g}, \mathfrak{g}) \):

\[
d[\alpha, \beta] = [d\alpha, \beta] + (-1)^{p}[\alpha, d\beta].
\]

Hence the Nijenhuis-Richardson bracket determines a Lie superalgebra structure in the cohomology \( H^*(\mathfrak{g}, \mathfrak{g}) \) also:

\[
[\cdot, \cdot] : H^p(\mathfrak{g}, \mathfrak{g}) \times H^q(\mathfrak{g}, \mathfrak{g}) \rightarrow H^{p+q-1}(\mathfrak{g}, \mathfrak{g})
\]

with properties(14).
Proposition 3.2. Let $\mathfrak{g} = \bigoplus \alpha \mathfrak{g}_\alpha$ be $\mathbb{N}$-graded Lie algebra. In this case the $\mathbb{Z}$-gradings of $C^*(\mathfrak{g}, \mathfrak{g})$ and $H^*(\mathfrak{g}, \mathfrak{g})$ are compatible with the Nijenhuis-Richardson bracket:

\[
[\cdot, \cdot] : C^p_{(\mu)}(\mathfrak{g}, \mathfrak{g}) \times C^q_{(\nu)}(\mathfrak{g}, \mathfrak{g}) \longrightarrow C^{p+q-1}_{(\mu+\nu)}(\mathfrak{g}, \mathfrak{g})
\]

Further we will examine only deformations $\Psi$ of $\mathfrak{g}$ -graded Lie algebra $\mathfrak{g} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ of the form

\[
\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \ldots + \Psi_i + \ldots, \quad \Psi_i \in C^2_{(i)}(\mathfrak{g}, \mathfrak{g}), \quad i = 1, 2, \ldots.
\]

Decomposing $\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \ldots$ in the deformation equation (11) to uniform terms and comparing the terms with same grading we come to the following system of equations on homogeneous components $\Psi_i$:

\[
d\Psi_0 + \frac{1}{2}[\Psi_0, \Psi_0] = 0, \quad d\Psi_1 + [\Psi_0, \Psi_1] = 0, \quad d\Psi_2 + [\Psi_0, \Psi_2] + \frac{1}{2}[\Psi_1, \Psi_1] = 0, \ldots
\]

\[
d\Psi_i + \frac{1}{2} \sum_{m+l=i} [\Psi_m, \Psi_l] = 0, \ldots
\]

Corollary 3.3. The set $\Psi_0, \Psi_1 = 0, \Psi_2 = 0, \ldots$ is a solution of the system (16) and hence $[\cdot, \cdot] + \Psi_0$ determines a new Lie bracket.

Proposition 3.4. An arbitrary infinite dimensional Lie algebra of maximal class $\mathfrak{g}$ is isomorphic to some non-deformable deformation $\Psi$ of the graded Lie algebra $\mathfrak{m}_0$

\[
\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \ldots + \Psi_i + \ldots, \quad \Psi_i \in C^2_{(i)}(\mathfrak{m}_0, \mathfrak{m}_0)
\]

such that

\[
\Psi(e_1, e_k) = 0, \forall k \in \mathbb{N}.
\]

Proof. The statement is just a reformulation of the Theorem 1.13 in the terms of deformation theory. \qed

We will call deformations $\Psi$ of the form (17) adapted deformations.

Proposition 3.5. A form $\Psi \in C^2_\ast(\mathfrak{m}_0, \mathfrak{m}_0)$ which satisfies (27) is closed: $d\Psi = 0$. The deformation equation on the cocycle $\Psi$ will be written down as $[\Psi, \Psi] = 0$:

\[
\frac{1}{2}[\Psi_0, \Psi_0] = 0, \quad [\Psi_0, \Psi_1] = 0, \quad [\Psi_0, \Psi_2] + \frac{1}{2}[\Psi_1, \Psi_1] = 0, \ldots
\]

\[
\frac{1}{2} \sum_{m+l=i} [\Psi_m, \Psi_l] = 0, \ldots
\]

Proof. In order to verify the closeness of the form $\Psi$ it it suffices to examine the values $d\Psi$ on the triples of basic vectors. From the other side if among triples $e_i, e_j, e_k$ there are no $e_1$ then $d\Psi(e_i, e_j, e_k) = 0$ in view of the commutation relationships $\mathfrak{m}_0$ but if, for example, $e_i = e_1$, then in view of $\Psi(e_1, e_k) = 0, \forall k$ it holds

\[
d\Psi(e_1, e_j, e_k) = -[e_1, \Psi(e_j, e_k)].
\]
The kernel of the operator \( ade_1 : m_0 \to m_0 \) is one-dimensional and spanned by \( e_1 \) and hence \( d \Psi = 0 \) if and only if \( \Psi = 0 \) (\( \Psi(e_j, e_k) \neq \alpha e_1 \) because \( \Psi \in C^2_+ (m_0, m_0) \)).

as it was already noted earlier the cocycle \( \Psi \) can be decomposed in a formal series \( \Psi = \sum x_{j,s} \Psi_{j,s} \) with respect to basic cocycles \( \Psi_{j,s} \). But it is easy to see that not every formal series answers a Lie algebra in a standard meaning.

**Proposition 3.6.** A formal series \( \Psi = \sum x_{j,s} \Psi_{j,s} \) determines a Lie algebra structure \([ , ] + \Psi \) if and only if \([ \Psi, \Psi \] = 0 \) and for an arbitrary \( j \in \mathbb{N} \) exists \( N(j) \in \mathbb{N} \) such that \( x_{j,s} = 0 \) when \( s > N(j) \).

**Proof.** Let us consider a commutator \([ e_j, e_{j+1} ] = \Psi(e_j, e_{j+1}) = \sum_{s=0}^{+\infty} x_{j,s} e_{2j+1+s} \) when \( j > 1 \). Our condition is equivalent to the fact that the linear combination is finite and it is necessary condition. From the other side, if it satisfied, for an arbitrary commutator \([ e_k, e_m ] = \Psi, m > k > 1 \), one has according to the formula (10)
\[
[e_k, e_m] = \sum_{j,s} x_{j,s} \Psi_{j,s}(e_k, e_m) = \sum_{j=k}^{m+k+1} (-1)^{j-k} \binom{m-j-1}{j-k} \sum_{s=0}^{N(j)} x_{j,s} e_{m+k-2j-1+s}.
\]

\( \square \)

**Theorem 3.7.** Let us consider a formal series in the completed space \( \oplus_{s \geq 0} H^2 (m_0, m_0) \)
\[
\Psi = \sum_{s=0}^{+\infty} \sum_{j=2}^{+\infty} x_{j,s} \Psi_{j,s},
\]
where cocycles \( \{ \Psi_{j,s} \} \) are defined by the formulas (10). A cocycle \( \Psi \) satisfies the deformation equation \([ \Psi, \Psi ] = 0 \) if and only if the coefficients \( x_{j,s} \) of the series \( \Psi \) satisfy the following system of quadratic equations \( F_{j,q,r} = 0, 2 \leq j < q, r \geq 0 \):
\[
F_{j,q,r} = \sum_{t=0}^{r} \sum_{l=j}^{\left[ \frac{q-1}{2} \right]} \sum_{m=q+1}^{\left[ \frac{q+l-1}{2} \right]} (-1)^{l-j+m-q} \binom{q-l-1}{l-j} \binom{j+q-m+t-1}{m-q-1} x_{l,t} x_{m,r-t}\]
\[
+ \sum_{t=0}^{r} \sum_{l=j}^{\left[ \frac{q-1}{2} \right]} \sum_{m=q}^{\left[ \frac{q+l-1}{2} \right]} (-1)^{l-j+m-q} \binom{q-l}{l-j} \binom{j+q-m+t}{m-q} x_{l,t} x_{m,r-t}\]
\[
+ \sum_{t=0}^{r} \sum_{m=j}^{\left[ \frac{q-1}{2} \right]} (-1)^{m-j+1} \binom{2q-m+t}{m-j} x_{q,t} x_{m,r-t} = 0.
\]

**Proof.** earlier it was noted, that the Nijenhuis-Richardson bracket is compatible with the differential of the Lie algebra cochain complex with coefficients in the adjoint representation. Thus the square \([ \Psi, \Psi ] \) of an arbitrary cocycle \( \Psi \) is a closed form. In addition to this it follows from \( \Psi(e_1, e_k) = 0, \forall k \) that \([ \Psi, \Psi ](e_1, e_k, e_l) = 0, \forall k, l \). But a cocycle that satisfies this condition
uniquely determines some cohomology in $\oplus_{s \geq 0} H^3_s(m_0, m_0)$ (see the proof of the Proposition 3.4) and hence $\frac{1}{2} [\Psi, \Psi]$ can be decomposed in a formal series with respect to the basis $\Psi_{j,q,r}$:

$$\frac{1}{2} [\Psi, \Psi] = \sum_{2 \leq j < q, r \geq 0} F_{j,q,r} \Psi_{j,q,r}.$$ 

Thus $[\Psi, \Psi] = 0$ if and only when all the coefficients $F_{j,q,r}$ of this decomposition vanish. It remains to calculate them explicitly as polynomials on variables $x_{j,s}$. Let us remark for this that according to the property (8)

$$\frac{1}{2} [\Psi, \Psi](e_j, e_q, e_{q+1}) = \sum_{r=0}^{+\infty} F_{j,q,r} e_{j+2q+1+r}.$$ 

On the other side

$$\frac{1}{2} [\Psi, \Psi] = \sum_{r=0}^{+\infty} \sum_{s+l=r} \sum_{m \geq l} x_{m,s} x_{l,t} [\Psi_{m,s}, \Psi_{l,t}].$$

Let us calculate the value $[\Psi_{m,s}, \Psi_{l,t}](e_j, e_q, e_{q+1})$ when $j < q$ directly from the definition (12) of the Nijenhuis-Richardson bracket under the assumption that $m > l$:

(20)

$$[\Psi_{m,s}, \Psi_{l,t}](e_j, e_q, e_{q+1}) = \Psi_{m,s}(\Psi_{l,t}(e_j, e_q), e_{q+1}) + \Psi_{m,s}(\Psi_{l,t}(e_q, e_{q+1}), e_j) + \Psi_{m,s}(\Psi_{l,t}(e_{q+1}, e_j), e_q) + \Psi_{l,t}(\Psi_{m,s}(e_j, e_q), e_{q+1}) + \Psi_{l,t}(\Psi_{m,s}(e_q, e_{q+1}), e_j) + \Psi_{l,t}(\Psi_{m,s}(e_{q+1}, e_j), e_q).$$

Using the properties of the cocycles $\Psi_{i,r}$ we have when $t \geq 0, s \geq 0$:

$$\Psi_{m,s}(\Psi_{l,t}(e_j, e_q), e_{q+1}) = (-1)^{l-j+m-q} \binom{q-l-1}{l-j} \binom{j+q-m+t-1}{m-q-1} e_{2q+j+1+t+s},$$

assuming that $0 \leq l-j \leq q-l-1$ and $0 \leq m-q-1 \leq j+q-m+t-1$, otherwise the value is trivial. We fin analogously the third summand:

$$\Psi_{m,s}(\Psi_{l,t}(e_{q+1}, e_j), e_q) = (-1)^{l-j+m-q} \binom{q-l}{l-j} \binom{j+q-m+t}{m-q} e_{2q+j+1+t+s},$$

under $0 \leq l-j \leq q-l$ and $0 \leq m-q \leq j+q-m+t$. It is is simple to be remark that the forth and the sixth summands in (20) vanish when $m > l$. Concerning the second and the fifth summands, they are non-trivial only when $l = q$ and $m = q$ respectively. In this case

$$\Psi_{m,s}(\Psi_{q,t}(e_q, e_{q+1}), e_j) = -(-1)^{m-j} \binom{2q-m+t}{m-j} e_{2q+j+1+t+s},$$

(21)

$$\Psi_{l,t}(\Psi_{q,s}(e_q, e_{q+1}), e_j) = -(-1)^{l-j} \binom{2q-l+s}{l-j} e_{2q+j+1+t+s},$$

where if $l=q$ then it holds on that $0 \leq m-j \leq 2q-m+t$ and $m > l=q$ with $m=q$ the inequalities $0 \leq l-j \leq 2q-l+s$ and $l \leq m=q$ must hold on. The contribution of these terms with $m > l$ to
the total sum (19) is expressed in the following way

\[- \sum_{t=0}^{r} \sum_{l=j}^{q-1} (-1)^{l-j} \binom{2q-l+r-t}{l-j} x_{l,t}x_{q,r-t} - \sum_{t=0}^{r} \sum_{m=q+1}^{q+[\frac{q+t}{2}]} (-1)^{m-j} \binom{2q-m+r-t}{m-j} x_{q,t}x_{m,r-t} = 0.\]

after redesignating in the second sum the indices of the summing up from \(m, t\) to \(l, r-t\) respectively and after adding the terms with \(m = l = q\) we will get the sum from the third line of the formula (19).

The case \(m = l\) is investigated analogously. gathering together the results of the calculations we get the expression for the coefficient \(F_{j,q,r}\) with \(e^{2q+j+1+r}\) standing in the decomposition \([\Psi, \Psi](e_j, e_q, e_{q+1})\).

A natural analogy with the singularity theory says that the affine variety \(M_{Fil}\) is nothing else that a variety of parameters of nilpotent versal deformation of the algebra \(m_0\); an arbitrary residually nilpotent Lie algebra \(g\) of maximal class is isomorphic to some algebra from our variety \(M_{Fil}\). The last property is called versality (in contrast to the universality) because a point of the variety \(M_{Fil}\) is determined for an algebra \(g\) not uniquely.

**Proposition 3.8.** Two deformations \(\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \cdots + \Psi_i + \cdots\) and \(\tilde{\Psi} = \beta \Psi_0 + \beta \alpha \Psi_1 + \beta \alpha^2 \Psi_2 + \cdots + \beta \alpha^i \Psi_i + \cdots\) of the Lie algebra \(m_0\) determine isomorphic Lie algebras with \(\alpha \neq 0, \beta \neq 0\).

**Proof.** it suffices to examine an automorphism \(\varphi\) of the algebra \(m_0\) of the following form:

\[\varphi(e_i) = \beta \alpha^i e_i, \quad i \in \mathbb{N}.\]

In connection with nilpotent algebras our definition of nilpotent versal deformation appears more convenient than more general abstract definition [5]. But if we nevertheless follow the abstract approach, let us consider the quotient of an associative commutative polynomial algebra

\[A = \mathbb{K}[[\{x_{j,s}\}]]/(\{F_{j,q,r}\})\]

from the infinite collection of the variables \(\{x_{j,s}, j \geq 2, s \geq 0\}\) with the unit 1 over ideal generated by the infinite system of quadratic polynomials\(\{F_{j,q,s}, 2 \leq j < q, s \geq 0\}\) of this algebra? defined by formulas (19). The augmentation \(\varepsilon : A \to \mathbb{K}\) of the algebra \(A\) is defined in a standard way: \(\varepsilon(P) = P(0)\), where a polynomial \(P \in A\). The completed tensor product \(A \hat{\otimes} m_0\) is not only a linear space over \(\mathbb{K}\) but and a \(A\)-module: for all \(x_{j,s} \in A\) and \(x \in m_0\) we suppose that \(x_{j,s} \cdot 1 \otimes x = x_{j,s} \otimes x\).

The theorem 3.7 can be formulated in the new terms.

**Proposition 3.9.** An \(A\)-linear Lie bracket \([\cdot, \cdot]_A\) is defined on the completed tensor product \(A \hat{\otimes} m_0\) which is assigned as follows

\[[1 \otimes x, 1 \otimes y]_A = 1 \otimes [x, y]_{m_0} + \sum_{j,s} x_{j,s} \otimes \Psi_j, s(x, y),\]

\(1 \otimes [x, y]_{m_0} = 1 \otimes \varepsilon(x) - \varepsilon(y) = 1 \otimes (x - y).\)
where the values $\psi_{j,s}(x,y) \in m_0$ are defined by the formula (10). It is evident that an application $\varepsilon \otimes id : A \otimes m_0 \to K \otimes m_0 = m_0$
is a Lie $K$-algebra homomorphism.

The completed tensor product $A \hat{\otimes} m_0$ with bracket $[,]_A$ is called a deformation of the algebra $m_0$ with the base $A$. It is possible to call the corresponding affine variety $M_{Fil}[5]$ the base of such deformation. It is evident that $A \hat{\otimes} m_0$ is a residually nilpotent Lie $A$-algebra of maximal class. We will not formulate here the versality condition in terms of formal algebra referring for details to [5]. We will call also the Lie $A$-algebra $A \otimes m_0$ as nilpotent versal deformation of the algebra $m_0$.

Remark. let us extract three first equations of the system (19):

\[ F_{2,3,0} = -3x_3^2 + x_3x_2 + 2x_2x_1 = 0, \quad F_{3,4,0} = -4x_4^2 + 3x_4x_5 + 3x_3x_5 = 0. \]

\[ F_{2,4,0} = 6x_2^2 - 4x_3x_4 - x_4x_5 + 2x_2x_5 - x_3x_5 = 0, \]

Khakimdjanov in [8] remarked that an affine variety, that is determined by them in four-dimensional space with the coordinates $(x_2, x_3, x_4, x_5)$ is the union of three straight lines

\[ l_1 = \{(t, 0, 0, 0)\}, l_2 = \left\{ \left( t, \frac{t}{10}, \frac{t}{70}, \frac{t}{420} \right) \right\}, \quad l_3 = \{(0, 0, 0, t)\}. \]

this observation allowed it to classify graded filiform Lie algebras with one-dimensional homogeneous components in dimensions $n \geq 12$. although its initial list [8] was not complete, in its later version [9] was also an algebra was missed, that was finally added in [10], where the alternative classification of such algebras was carried out. Let us note that non-zero points only of two first straight lines continue to a solution $\psi_0$ of entire system. Points of the first straight line correspond to deformations with $\Psi_0 = t\Psi_2,0$ (they are isomorphic to $m_0$) and the points of the second one correspond to the cocycle $\Psi_0 = t \sum_{k=2}^{+\infty} \frac{(k-2)!(k-1)!}{(2k-1)!} \Psi_{k,0}$ (they are isomorphic to $L_1$) and finally to the trivial solution corresponds $\Psi_0 = 0$ and hence the algebra $m_0$. this reasoning was repeated recently by Wagemann and Fialowski [7].

Let $\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \cdots + \Psi_i + \cdots$ be a deformation of the algebra $m_0$ moreover $\Psi_0 = \Psi_1 = \cdots = \Psi_{s-1} = 0$ and $\Psi_1 \neq 0$ for some integer $s \geq 0$.

Definition 3.10 (8). A cocycle $\Psi$ is called sill cocycle of deformation $\Psi$.

A homogeneous cocycle $\Psi_s \in H^2_+(m_0, m_0)$ is sill cocycle for some deformation $\Psi$ then and only then $[\Psi_s, \Psi_s] = 0$.

Proposition 3.11. Let $\Psi_s \in H^2_+(m_0, m_0)$ be a sill cocycle i.e. $[\Psi_s, \Psi_s] = 0$. A Lie algebra $\mathfrak{g}$ with the deformed bracket $[,]_{m_0} + \Psi_s$ has the following structure of $\mathbb{N}$-graded Lie algebra with trivial or one-dimensional homogeneous components:

\[ \mathfrak{g} = \bigoplus_{i=1}^{+\infty} \mathfrak{g}_i, \quad \text{dim} \mathfrak{g}_i = \begin{cases} 1, & i = 1, s+2, s+3, \ldots; \\ 0, & 2 \leq i \leq s+1. \end{cases} \]
Proof. The grading is defined in an obvious way:

\[ g_1 = \langle e_1 \rangle, \quad g_i = \langle e_{i-s} \rangle, \quad i \geq s+2, \]

where \( e_1, e_2, \ldots, e_i, \ldots \) is an infinite basis of the algebra \( m_0 \). \qed

The reverse is true also

Proposition 3.12. Let \( g \) be an \( \mathbb{N} \)-graded Lie algebra such that

\[ g = \bigoplus_{i=1}^{\infty} g_i, \quad [g_1, g_i] = g_{i+1}, \quad i \geq s+2, \quad \dim g_i = \begin{cases} 1, & i = 1, s+2, s+3, \ldots; \\ 0, & 2 \leq i \leq s+1. \end{cases}, \]

for some integer \( s \geq 0 \). Then the Lie algebra \( g \) is isomorphic to some deformation of \( m_0 \) defined by a homogeneous cocycle \( \Psi_s \in H^2_s(m_0, m_0) \) such that \( [\Psi_s, \Psi_s] = 0 \).

Proof. The obvious verification. \qed

\( \mathbb{N} \)-graded Lie algebras considered in two previous propositions we will call \( \mathbb{N} \)-graded Lie algebras with lacunas in the grading. It is easy to construct examples of this type from algebras already examined.

Proposition 3.13. Let \( g = \bigoplus_{i=1}^{\infty} g_i \) be an \( \mathbb{N} \)-graded Lie algebra then its subalgebra \( g(s) = g_1 \oplus \bigoplus_{i=s+2}^{\infty} g_i \) with \( s \geq 1 \) is \( \mathbb{N} \)-graded Lie algebra with lacunas in the grading from 2 to \( s+1 \).

The subalgebra \( L_1(s) \) will be isomorphic to a semi-direct sum \( K \oplus L_{s+2} \). Obviously that \( m_0(s) \cong m_2(s) \cong m_0 \) with \( s \geq 1 \). But not all algebras with lacunas can be obtained by the deletion of the basic vectors from some other algebra.

Example 3.14. A \( \mathbb{N} \)-graded Lie algebra \( m_k \) is defined by its infinite basis \( e_1, e_k, e_{k+1}, \ldots, e_i, \ldots \) and the structure relations:

\[ [e_1, e_i] = e_{i+1}, \quad i \geq k, \quad [e_k, e_i] = e_{k+i}, \quad i \geq k+1. \]

The algebra \( m_k \) is isomorphic to the deformation of \( m_0 \) defined by the sill cocycle \( \Psi_{2,k-2} \).

The question: how much exists the sill cocycles \( \Psi_s \) of the grading equal to \( s \)? Let us try to find an answer in the case of \( s = 2 \). We will search for the solution of the system \( (19) \) such that \( x_{j,s} = 0 \) with \( s \neq 2 \). Let us write down seven equations \( (19) \) on unknowns \( x_{2,2}, x_{3,2}, \ldots, x_{8,2} \) moreover for simplification in the record we will omit the second upper script in the unknowns.
Conjecture. Let grading equal to \( 2 \) calculations by Vaughan-Lee (see calculations in \( 7 \)).

After computing with Maple we will obtain the following collection of solutions of the system \((23)\):

\[
\{(t, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, t, 4t, 14t)\},
\]

\[
\left\{ \left( \begin{array}{ccccccc} t & t & t & t & t & t & t \\ 70 & 420 & 2310 & 12012 & 60060 & 291720 & 1385670 \end{array} \right) \right\}.
\]

One can show that only nontrivial points of the first and last straight lines from this collection are extendable to a solution of entire infinite system. The corresponding deformations are isomorphic to algebras \( m_k \) and \( L_1(2) \). It follows evidently from this proposition that up to an isomorphism there are only three \( N \)-graded Lie algebras of maximal class with a lacuna in the grading equal to 2: \( L_1(1), m_3 \) and \( m_6 \). The last statement was obtained directly using computer calculations in \( 7 \). In the light of the Theorem \( 1.13 \) and non published results of computer calculations by Vaughan-Lee (see \( 13 \) page 956), the following conjecture is completely plausible.

Conjecture. Let \( g \) be a \( N \)-graded Lie algebra such that

\[
g = \bigoplus_{i=1}^{+\infty} g_i, \quad [g_0, g_i] = g_{i+1}, i \geq k, \quad \dim g_i = \begin{cases} 1, & i = 1, k, k+1, \ldots; \\ 0, & 2 \leq i \leq k-1. \end{cases}
\]

Then \( g \) is isomorphic to one from three given algebras

\[
m_0, \quad L_1(k-2), \quad m_k.
\]

4. The finite dimensional case

In this section we will consider the affine variety \( M_{Fil}(n) \) of \( n \)-dimensional filiform Lie algebras. One has immediately to note that the structure of the system equations that define \( M_{Fil}(n) \) will depend on parity of \( n \).

We will need a finite dimensional version of the Theorem \( 2.6 \)
Theorem 4.1. 1) as the basis of the graded space \( H^2_+(m_0(n), m_0(n)) = \bigoplus_{s \geq 0} H^2_s(m_0(n), m_0(n)) \) one can take the following collection of cocycles:

\[
\Psi_{j,s} = \sum_{k=0}^{n-2j-1-s} e_{2j+1+s+k} \otimes D^k_{-1} \omega(e^j \wedge e^{j+1}), \ 2 \leq j, \ s \geq 0; \ 2j+1+s \leq n.
\]

2) The following collection of cocycles

\[
\Psi_{i,j,s} = \sum_{k=0}^{n-i-2j-1-s} e_{i+2j+1+s+k} \otimes D^k_{-1} \omega(e^i \wedge e^j \wedge e^{j+1}), \ 2 \leq i < j, \ s \geq 0, \ i+2j+1+s \leq n.
\]

can serve as a basis of the graded space \( H^3_+(m_0(n), m_0(n)) = \bigoplus_{s \geq 0} H^3_s(m_0(n), m_0(n)) \).

Proof. The first part of the theorem is just a reformulation of the Vergne theorem on two-cohomology with the coefficients in the adjoint representation of the algebra \( m_0(n) \) \[14\] (see also the Remark \[2\]. Vergne have not calculated the three-cohomology \( H^3(m_0(n), m_0(n)) \). For the non-negative part \( H^3_+(m_0(n), m_0(n)) \) the computation by means of the spectral sequence \[1\] is still valid.

We recall that according the Theorem \[11,17\] the following holds on

Proposition 4.2. An arbitrary filiform Lie algebra \( g \) of odd dimension \( n = 2k+1 \) is isomorphic to some non-negative deformation \( \Psi = \Psi_0 + \Psi_1 + \Psi_2 + \cdots + \Psi_{n-5} \) of the graded Lie algebra \( m_0(n) \). In the case of even dimension \( n = 2k \) a filiform Lie algebra is non-negative deformation of: either a) the graded Lie algebra \( m_0(n) \), either b) the graded Lie algebra \( m_1(2k) \).

In each case the deformation \( \Psi \) can be chosen adapted:

\[
\Psi(e_1, e_k) = 0, \forall k \in \mathbb{N}.
\]

The Lie algebra \( m_1(2k) \) can be regarded as deformation \( m_0(2k) \):

\[
[,]_{m_1(2k)} = [,]_{m_0(2k)} + \Psi_{k,-1},
\]

\( \Psi_{k,-1} \in H^2_{-1}(m_0(2k), m_0(2k)) \) is defined by the formula

\[
\Psi_{k,-1} = e_{2k} \otimes \omega(e^k \wedge e^{k+1}).
\]

let us combine both cases into one, considering with \( n = 2k \) deformations \( \Psi \) of the algebra \( m_0(2k) \)

\[
\Psi = x \Psi_{k,-1} + \Psi_0 + \Psi_1 + \Psi_2 + \cdots + \Psi_{n-5},
\]

where the variable \( x \) can take only two values: \( x = 1 \) (and we get non-negative deformations of the algebra \( m_1(2k) \)) or \( x = 0 \) (we will deal with deformations of the algebra \( m_0(2k) \)).

Theorem 4.3. 1) Let \( n = 2k+1 \geq 9 \). Let us consider a linear combination

\[
\Psi = \sum_{s=0}^{n-5} \sum_{j=2}^{\left\lfloor \frac{n-1}{2} \right\rfloor} x_{j,s} \Psi_{j,s},
\]
of basic cocycles $Ψ_{j,s}$ from the subspace $⊕_{s≥0} H^3_{i} (m_0(n), m_0(n))$. A cocycle $Ψ$ satisfies the deformation equation if and only if its coordinates $x_{j,s}$ satisfy the finite system of quadratic equations

$$\{ F_{j,q,r} = 0, \quad 2 ≤ j < q, \quad 9 ≤ j+2q+1+r ≤ n, \quad r ≥ 0. $$

2) Let $n = 2k ≥ 10$. Let us consider a linear combination

$$Ψ = \sum_{s=0}^{n-5} \sum_{j=2}^{[n+1]} x_{j,s} Ψ_{j,s}. $$

A cocycle $xΨ_{k,-1} + Ψ$ satisfies the deformation equation if and only if its coordinates $x_{j,s}$ with respect to the basis $Ψ_{j,s}$ satisfy the following system of equations

$$\begin{align*}
F_{j,q,r} &= 0, & 2 ≤ j < q, \quad 9 ≤ j+2q+1+r < 2k, \quad r ≥ 0, \\
F_{j,q,r} &= F_{j,q,r} + (-1)^{k-j-q} xG_{j,q,r} = 0, & 2 ≤ j < q, \quad j+2q+1+r = 2k, \quad r ≥ 0, \\
F_{j,q,-1} &= xG_{j,q,-1} = 0, & 2 ≤ j < q, \quad j+2q = 2k.
\end{align*}$$

where the polynomials $F_{j,q,r}$ are defined by the formulas (19) and the polynomial $G_{j,q,r}$ with $r ≥ 1$ is defined by the following

$$G_{j,q,r} = \sum_{l=j}^{[j+q-1]} (-1)^l \binom{q-l-1}{l-j} x_{l,r+1} + \sum_{l=j}^{[j+q]} (-1)^l \binom{q-l}{l-j} x_{l,r+1} - (-1)^q x_{q,r+1}. $$

Proof. The proof with $n = 2k+1$ completely repeats our reasonings in infinite dimensional case with the only difference that the Nijenhuis-Richardson bracket $[,]$ is decomposed in this case with respect to the finite basis $H^3_{i}(m_0(n), m_0(n))$. Its elements are the cocycles $Ψ_{j,q,r}$ with the upper scripts $j, q, r$, satisfying obvious conditions $2 ≤ j < q, \quad 9 ≤ j+2q+1+r ≤ n, \quad r ≥ 0.$

In the case of even dimension $n = 2k$ one has to decompose the square $[xΨ_{k,-1} + Ψ, xΨ_{k,-1} + Ψ]$ with respect to the basis $Ψ_{j,q,r}$ of the subspace $⊕_{i≥-1} H^3_{i}(m_0(2k), m_0(2k))$. Let us denote the coefficients of this decomposition by $F_{j,q,r}$. One can easily obtain the expressions for them using the formulas (19), where it is necessary to change the limits of change of the index $t$: from $−1$ to $r$ (instead of $0 ≤ t ≤ r$). As $x_{j,-1} = 0$ with $j ≠ k$, then the final formulas can be obtained by easy change of (19).

What we can say about the behavior of the number of unknowns and number of equations of the system that determines $M_{F(0)}(n)$ with an encrease in $n$? The number of unknowns (the dimension $\dim H^3_{i}(m(n), m(n))$) was computed by Vergne [13] and it is equal to $\frac{1}{2} (n-3)^2$ with odd $n$ and $\frac{1}{2} (n-2) (n-4)$ with even $n$. The number of equations in (28) is equal to the dimension $\dim H^3_{i}(m(n), m(n))$.

**Proposition 4.4.**

$$\dim H^3_{i}(m(n), m(n)) = \sum_{r=3}^{n-6} P_3(r),$$

where $P_3(r)$ denotes the partition number of a natural number $r$ exactly into three terms.
Proof. The number of basic cocycles $\Psi_{i,j,s}$ with a fixed value of upper script $s$ is equal to $\sum_{k=1}^{n-s} \dim H^3_k(m_0(n))$. Using the formula (5) for $\dim H^3_k(m_0(n))$ we get

$$\dim H^3_+(m(n), m(n)) = \sum_{s=0}^{n-5} \sum_{k=1}^{n-s} \dim H^3_k(m_0(n)) = \sum_{s=0}^{n-5} \sum_{k=1}^{n-s} (P_3(k-6)-P_3(k-7)) = \sum_{s=0}^{n-5} P_3(n-s-6).$$

Replacing in the last sum upper script $s$ by $r = n-s-6$ and taking into account that $P_3(-1) = P_3(0) = P_3(1) = P_3(2) = 0$ the required equality is obtained. $\square$

It is possible to prove in a similar way that $\dim H^2_+(m(n), m(n)) = \sum_{r=2}^{n-3} P_2(r)$.

Using the known recursion relation $P_3(m) = P_3(m-3)+P_2(m-3)+P_1(m-3)$ one can write down explicit formulas for the sum $\sum_{k=3}^{n-6} P_3(k)$ as a function on $n$ (see [3]) moreover the answer will depend on the residue class of $n$ modulo 6. We will not write them down. Let us say only that according to the formula for $P_3(m)$ from [3] the sequence $\sum_{k=3}^{n-6} P_3(k)$ has an asymptotic behavior of $n^3$ with the large numbers $n$. Usually with the classification of filiform Lie algebras of small dimensions they obtained the equations by substitution into $[\Psi, \Psi]$ of all triples of basic vectors $e_i, e_j, e_k$ with $2 \leq i < j < k \leq n$ that gave $\binom{n-1}{3}$ equations (moreover the explicit formula of equations for an arbitrary $n$ has not been written down) with the asymptotic behavior $\frac{1}{6}n^3$ with the large numbers $n$.

But the dimension $\dim H^3_+(m(n), m(n))$ will be answer only with odd values of $n$. For even ones $n = 2k$ the variable $x$ appears as well as the additional series of equations $F_{i,j,-1} = 0$. The number of equations in this series is equal to $\dim H^3_{2k+1}(m_0(2k)) = P_3(2k-5) - P_3(2k-6)$. Finally we have: with even $n = 2k$ the number of equation in the system is equal to

$$\sum_{r=3}^{n-7} P_3(r) + P_3(n-5).$$

A question about the minimization of the number of equations that determine $M_{Fil}(n)$ appeared long ago. An algorithm for finding the explicit formulas of such systems was introduced in [4]. It seems that this algorithm gives as the answer the system that was given explicitly in the present paper. At least our example $M_{Fil}(12)$ coincides completely with the analogous example from [4]. It is understood that we not say that the collection of polynomials $F_{i,j,q,r}$ the collection of polynomials forms the minimal system of generators of the corresponding ideal. It appears that it is not true beginning with certain dimension $n_0$.

Example 4.5. It is easy to find several first values $P_3(k)$:

$$P_3(3)=P_3(4)=1, P_3(5)=2, P_3(6)=3, P_3(7)=4, P_3(8)=5, P_3(9)=7, P_3(10)=8, P_3(11)=10.$$ 

with their aid it is not difficult to compose the table

| dimension $n$ | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---------------|----|----|----|----|----|----|----|----|----|----|
| number of equations | 1  | 3  | 4  | 8  | 11 | 18 | 23 | 33 | 41 | 55 |

it completely coincides with the results of the computer calculations from [4].
Example 4.6. The variety $M_{Fil}(12)$ of 12-dimensional filiform Lie algebras.

(31)

\[
\begin{align*}
\tilde{F}_{2,5,-1} &= x(-2x_{2,0} + 3x_{3,0} - x_{5,0}) = 0, \\
F_{2,3,0} &= -3x_{3,0} + x_{3,0}x_{2,0} + 2x_{2,0}x_{4,0} = 0, \\
F_{2,3,1} &= -2x_{2,0}x_{4,1} + 7x_{3,0}x_{3,1} - 3x_{4,0}x_{3,1} - 3x_{4,0}x_{2,1} - x_{3,0}x_{4,1} = 0; \\
F_{2,3,2} &= 4x_{3,1}^2 - 3x_{2,1}x_{4,1} - 3x_{3,1}x_{4,1} + x_{2,2}(2x_{5,0} - x_{4,0}) + x_{3,2}(8x_{3,0} - 6x_{4,0} + x_{5,0}) - x_{4,2}(x_{3,0} + 2x_{2,0}) = 0, \\
\tilde{F}_{2,3,3} &= x_{2,2}(2x_{5,1} - 4x_{4,1}) + x_{3,2}(9x_{3,1} - 6x_{4,1} + x_{5,1}) - x_{4,2}(3x_{3,1} + 3x_{2,1}) + x_{2,3}(5x_{5,0} - 5x_{4,0}) + x_{3,3}(9x_{3,0} - 10x_{4,0} + 4x_{5,0}) + x_{4,3}(x_{3,0} - 2x_{2,0}) - x(2x_{2,4} + x_{3,4}) = 0, \\
F_{2,4,0} &= 6x_{4,0}^2 - 4x_{3,0}x_{4,0} - 4x_{3,0}x_{5,0} + 2x_{2,0}x_{5,0} - 3x_{3,0}x_{5,0} = 0, \\
\tilde{F}_{2,4,1} &= -2x_{2,0}x_{5,1} + 5x_{3,0}x_{4,1} + x_{3,0}x_{5,1} - 10x_{4,0}x_{4,1} + 4x_{5,0}x_{4,1} - 3x_{5,0}x_{2,1} + 4x_{4,0}x_{3,1} + 2x_{5,0}x_{3,1} - 6x_{4,0}x_{4,1} + x_{4,0}x_{5,1} + x(2x_{2,2} - x_{3,2} - x_{4,2}) = 0; \\
\tilde{F}_{3,4,0} &= -4x_{4,0}^2 + 3x_{4,0}x_{5,0} + 3x_{3,0}x_{5,0} + x(2x_{3,1} + x_{4,1}) = 0.
\end{align*}
\]

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