Coverings of foliation algebras

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Abstract

This article is devoted to the geometric construction which states a natural correspondence between topological coverings of a foliated manifolds and noncommutative coverings of the operator algebras. However this correspondence is not one to one because there are noncommutative coverings of foliations which do not comply with discussed in this article construction.

1 Motivation. Preliminaries

Gelfand-Naïmark theorem [1] states the correspondence between locally compact Hausdorff topological spaces and commutative $C^*$-algebras.

**Theorem 1.1.** [1] (Gelfand-Naïmark). Let $A$ be a commutative $C^*$-algebra and let $X$ be the spectrum of $A$. There is the natural $*$-isomorphism $\gamma : A \cong C_0(X)$.

So any (noncommutative) $C^*$-algebra may be regarded as a generalized (noncommutative) locally compact Hausdorff topological space. Following theorem yields a pure algebraic description of finite-fold coverings of compact spaces.

**Theorem 1.2.** [11] Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff connected spaces and $p : \mathcal{Y} \to \mathcal{X}$ is a continuous surjection. If $C(\mathcal{Y})$ is a projective finitely generated Hilbert module over $C(\mathcal{X})$ with respect to the action

$$(f\xi)(y) = f(y)\xi(p(y)), \ f \in C(\mathcal{Y}), \ \xi \in C(\mathcal{X}),$$

respect to the action

\begin{align*}
(f\xi)(y) &= f(y)\xi(p(y)), \ f \in C(\mathcal{Y}), \ \xi \in C(\mathcal{X}),
\end{align*}

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then p is a finite-fold covering.

An article [7] contains pure algebraic generalizations of following topological objects:

- Coverings of noncompact spaces,
- Infinite coverings.

Here the described in [7] theory is applied to operator algebras of foliations.

Following table contains special symbols.

| Symbol | Meaning |
|--------|---------|
| \( A^+ \) | Unitisation of \( C^* \)-algebra \( A \) |
| \( A_+ \) | Cone of positive elements of \( C^* \)-algebra, i.e. \( A_+ = \{ a \in A \mid a \geq 0 \} \) |
| \( A^G \) | Algebra of \( G \) - invariants, i.e. \( A^G = \{ a \in A \mid ga = a, \forall g \in G \} \) |
| \( A'' \) | Enveloping von Neumann algebra of \( A \) |
| \( B(\mathcal{H}) \) | Algebra of bounded operators on a Hilbert space \( \mathcal{H} \) |
| \( C(\mathbb{R}) \) | Field of complex (resp. real) numbers |
| \( C(\mathcal{X}) \) | \( C^* \)-algebra of continuous complex valued functions on a compact space \( \mathcal{X} \) |
| \( C_0(\mathcal{X}) \) | \( C^* \)-algebra of continuous complex valued functions on a locally compact topological space \( \mathcal{X} \) equal to 0 at infinity |
| \( C_b(\mathcal{X}) \) | \( C^* \)-algebra of bounded continuous complex valued functions on a locally compact topological space \( \mathcal{X} \) |
| \( G(\tilde{\mathcal{X}}, \mathcal{X}) \) | Group of covering transformations of covering \( \tilde{\mathcal{X}} \to \mathcal{X} \) [13] |
| \( \delta_{jk} \) | Delta symbol. If \( j = k \) then \( \delta_{jk} = 1 \). If \( j \neq k \) then \( \delta_{jk} = 0 \) |
| \( \mathcal{H} \) | Hilbert space |
| \( K = K(\mathcal{H}) \) | \( C^* \)-algebra of compact operators on the separable Hilbert space \( \mathcal{H} \) |
| \( \ell^2(A) \) | Standard Hilbert \( A \)-module |
| \( \lim \) | Direct limit |
| \( \lim^{-} \) | Inverse limit |
| \( M(A) \) | A multiplier algebra of \( C^* \)-algebra \( A \) |
| \( \mathbb{N} \) | A set of positive integer numbers |
| \( \mathbb{N}^0 \) | A set of nonnegative integer numbers |
| \( \text{supp} \varphi \) | Support of a continuous map \( \varphi : \mathcal{X} \to \mathbb{C} \) |
| \( \mathbb{Z} \) | Ring of integers |
| \( \mathbb{Z}_n \) | Ring of integers modulo \( n \) |
| \( X\setminus A \) | Diff and only difference of sets \( X \setminus A = \{ x \in X \mid x \notin A \} \) |
| \( |X| \) | Cardinal number of a finite set \( X \) |
| \( f|_{A'} \) | Restriction of a map \( f : A \to B \) to \( A' \subset A \), i.e. \( f|_{A'} : A' \to B \) |
1.1 Hilbert modules

We refer to [2] for the definition of Hilbert C*-modules, or simply Hilbert modules. Let \( A \) be a C*-algebra, and let \( X_A \) be an A-Hilbert module. Let \( \langle \cdot , \cdot \rangle_{X_A} \) be the \( A \)-valued product on \( X_A \). For any \( \xi, \zeta \in X_A \) let us define an \( A \)-endomorphism \( \theta_{\xi, \zeta} \) given by \( \theta_{\xi, \zeta}(\eta) = \xi \langle \zeta, \eta \rangle_{X_A} \) where \( \eta \in X_A \). The operator \( \theta_{\xi, \zeta} \) shall be denoted by \( \langle \xi \rangle \langle \zeta \rangle \). The norm completion of a generated by operators \( \theta_{\xi, \zeta} \) algebra is said to be an algebra of compact operators \( K(X_A) \). We suppose that there is a left action of \( K(X_A) \) on \( X_A \) which is \( A \)-linear, i.e. action of \( K(X_A) \) commutes with action of \( A \). For any C*-algebra \( A \) denote by \( \ell^2(A) \) the standard Hilbert \( A \)-module given by

\[
\ell^2(A) = \left\{ \{a_n\}_{n \in \mathbb{N}} \in A^\mathbb{N} \mid \sum_{n=1}^{\infty} a_n^* a_n < \infty \right\},
\]

(1.1)

1.2 C*-algebras and von Neumann algebras

In this section I follow to [12].

Definition 1.3. [12] Let \( H \) be a Hilbert space. The strong topology on \( B(H) \) is the locally convex vector space topology associated with the family of seminorms of the form \( x \mapsto \|x\zeta\|, x \in B(H), \zeta \in H \).

Definition 1.4. [12] Let \( H \) be a Hilbert space. The weak topology on \( B(H) \) is the locally convex vector space topology associated with the family of seminorms of the form \( x \mapsto |(x\xi, \eta)|, x \in B(H), \zeta, \eta \in H \).

Theorem 1.5. [12] Let \( M \) be a C*-subalgebra of \( B(H) \), containing the identity operator. The following conditions are equivalent:

- \( M = M'' \) where \( M'' \) is the bicommutant of \( M \);
- \( M \) is weakly closed;
- \( M \) is strongly closed.

Definition 1.6. Any C*-algebra \( M \) is said to be a von Neumann algebra or a \( W^* \)-algebra if \( M \) satisfies to the conditions of Theorem 1.5.

Definition 1.7. [12] Let \( A \) be a C*-algebra, and let \( S \) be the state space of \( A \). For any \( s \in S \) there is an associated representation \( \pi_s : A \to B(H_s) \). The representation \( \bigoplus_{s \in S} \pi_s : A \to \bigoplus_{s \in S} B(H_s) \) is said to be the universal representation. The universal representation can be regarded as \( A \to B(\bigoplus_{s \in S} H_s) \).

Definition 1.8. [12] Let \( A \) be a C*-algebra, and let \( A \to B(H) \) be the universal representation \( A \to B(H) \). The strong closure of \( \pi(A) \) is said to be the enveloping von Neumann algebra or the enveloping \( W^* \)-algebra of \( A \). The enveloping von Neumann algebra will be denoted by \( A'' \).
2 Noncommutative finite-fold coverings

Definition 2.1. If $A$ is a $C^*$-algebra then an action of a group $G$ is said to be involutive if $ga^* = (ga)^*$ for any $a \in A$ and $g \in G$. The action is said to be non-degenerated if for any nontrivial $g \in G$ there is $a \in A$ such that $ga \neq a$.

Definition 2.2. Let $A \hookrightarrow \tilde{A}$ be an injective $*$-homomorphism of unital $C^*$-algebras. Suppose that there is a non-degenerated involutive action $G \times \tilde{A} \to \tilde{A}$ of a finite group $G$, such that $A = \tilde{A}^G \overset{\text{def}}{=} \{ a \in \tilde{A} \mid a = ga; \forall g \in G \}$. There is an $A$-valued product on $\tilde{A}$ given by

$$\langle a, b \rangle_{\tilde{A}} = \sum_{g \in G} g(a^*b)$$

and $\tilde{A}$ is an $A$-Hilbert module. We say that $(A, \tilde{A}, G)$ is an unital noncommutative finite-fold covering if $\tilde{A}$ is a finitely generated projective $A$-Hilbert module.

Remark 2.3. Above definition is motivated by the Theorem 1.2.

Definition 2.4. Let $A, \tilde{A}$ be $C^*$-algebras such that following conditions hold:

(a) There are unital $C^*$-algebras $B, \tilde{B}$ and inclusions $A \subset B, \tilde{A} \subset \tilde{B}$ such that $A$ (resp. $B$) is an essential ideal of $\tilde{A}$ (resp. $\tilde{B}$),

(b) There is an unital noncommutative finite-fold covering $(B, \tilde{B}, G)$,

(c) $$\tilde{A} = \{ a \in \tilde{B} \mid \langle \tilde{B}, a \rangle_{\tilde{B}} \in A \}.$$  

The triple $(A, \tilde{A}, G)$ is said to be a noncommutative finite-fold covering. The group $G$ is said to be the covering transformation group (of $(A, \tilde{A}, G)$) and we use the following notation

$$G \left( \tilde{A} \mid A \right) \overset{\text{def}}{=} G.$$  

Lemma 2.5. Let us consider the situation of the Definition 2.4. Following conditions hold:

(i) From (2.2) it turns out that $\tilde{A}$ is a closed two sided ideal of $\tilde{B}$,

(ii) The action of $G$ on $\tilde{B}$ is such that $G\tilde{A} = \tilde{A}$, i.e. there is the natural action of $G$ on $\tilde{A}$,

(iii) $$A \cong \tilde{A}^G = \{ a \in \tilde{A} \mid a = ga; \forall g \in G \}.$$  

Remark 2.6. The Definition 2.5 is motivated by the Theorem 4.1.
Definition 2.7. The injective $^*$-homomorphism $A \hookrightarrow \tilde{A}$, which follows from (2.4) is said to be a noncommutative finite-fold covering.

Definition 2.8. Let $(A, \tilde{A}, G)$ be a noncommutative finite-fold covering. Algebra $\tilde{A}$ is a Hilbert $A$-module with an $A$-valued product given by

$$\langle a, b \rangle_{\tilde{A}} = \sum_{g \in G} g(a^*b) ; \quad a, b \in \tilde{A}. \quad (2.5)$$

We say that this structure of Hilbert $A$-module is induced by the covering $(A, \tilde{A}, G)$. Henceforth we shall consider $\tilde{A}$ as a right $A$-module, so we will write $\tilde{A}_A$.

3 Noncommutative infinite coverings

This section contains a noncommutative generalization of infinite coverings.

Definition 3.1. Let $S = \{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \ldots \}$ be a sequence of $C^*$-algebras and noncommutative finite-fold coverings such that:

(a) Any composition $\pi_{n_1} \circ \ldots \circ \pi_{n_{l+1}} \circ \pi_{n_0} : A_{n_0} \to A_{n_1}$ corresponds to the noncommutative covering $(A_{n_0}, A_{n_1}, G(A_{n_1} \mid A_{n_0}))$;

(b) If $k < l < m$ then $G(A_m \mid A_k) A_l = A_l$ (Action of $G(A_m \mid A_k)$ on $A_l$ means that $G(A_m \mid A_k)$ acts on $A_m$, so $G(A_m \mid A_k)$ acts on $A_l$ since $A_l$ a subalgebra of $A_m$);

(c) If $k < l < m$ are nonnegative integers then there is the natural exact sequence of covering transformation groups

$$\{e\} \to G(A_m \mid A_l) \xrightarrow{\iota} G(A_m \mid A_k) \xrightarrow{\pi} G(A_l \mid A_k) \to \{e\}$$

where the existence of the homomorphism $G(A_m \mid A_k) \xrightarrow{\pi} G(A_l \mid A_k)$ follows from (b).

The sequence $S$ is said to be an (algebraical) finite covering sequence. For any finite covering sequence we will use the notation $S \in \text{FinAlg}$.

Definition 3.2. Let $\hat{A} = \lim_{\to} A_n$ be the $C^*$-inductive limit [9], and suppose that $\hat{G} = \lim_{\to} G(A_n \mid A)$ is the projective limit of groups [13]. There is the natural action of $\hat{G}$ on $\hat{A}$. $\hat{A}$ non-degenerate faithful representation $\hat{A} \to B(\mathcal{H})$ is said to be equivariant if there is an action of $\hat{G}$ on $\mathcal{H}$ such that for any $\xi \in \mathcal{H}$ and $g \in \hat{G}$ following condition holds

$$(ga)\xi = g \left( a \left( g^{-1}\xi \right) \right). \quad (3.1)$$
Definition 3.3. Let $\pi : \hat{A} \to B(\mathcal{H})$ be an equivariant representation. A positive element $\overline{a} \in B(\mathcal{H})_+$ is said to be special (with respect to $\mathcal{S}$) if following conditions hold:

(a) For any $n \in \mathbb{N}^0$ the following series

$$a_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g\overline{a}$$

is strongly convergent and the sum lies in $A_n$, i.e. $a_n \in A_n$;

(b) If $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given by

$$f_\varepsilon(x) = \begin{cases} 0 & x \leq \varepsilon \\ x - \varepsilon & x > \varepsilon \end{cases} \quad (3.2)$$

then for any $n \in \mathbb{N}^0$ and for any $z \in A$ following series

$$b_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g(z\overline{a}^*)$$
$$c_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g(z\overline{a}^*)^2$$
$$d_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g f_\varepsilon(z\overline{a}^*)$$

are strongly convergent and the sums lie in $A_n$, i.e. $b_n, c_n, d_n \in A_n$;

(c) For any $\varepsilon > 0$ there is $N \in \mathbb{N}$ (which depends on $\overline{a}$ and $z$) such that for any $n \geq N$ a following condition holds

$$\left\| b_n^2 - c_n \right\| < \varepsilon. \quad (3.3)$$

An element $\overline{a}' \in B(\mathcal{H})$ is said to be weakly special if

$$\overline{a}' = x\overline{a}y; \text{ where } x, y \in \hat{A}, \text{ and } \overline{a} \in B(\mathcal{H}) \text{ is special.}$$

Lemma 3.4. If $\overline{a} \in B(\mathcal{H})_+$ is a special element and $G_n = \ker(\hat{G} \to G(A_n | A))$ then from

$$a_n = \sum_{g \in G_n} g\overline{a},$$

it follows that $\overline{a} = \lim_{n \to \infty} a_n$ in the sense of the strong convergence. Moreover one has $\overline{a} = \inf_{n \in \mathbb{N}} a_n$.

Corollary 3.5. Any weakly special element lies in the enveloping von Neumann algebra $\hat{A}''$ of $\hat{A} = \lim_{n \to \infty} A_n$. If $A_\pi \subset B(\mathcal{H})$ is the $C^*$-norm completion of an algebra generated by weakly special elements then $A_\pi \subset \hat{A}''$. 
Lemma 3.6. [7] If \( \overline{\pi} \in B(\mathcal{H}) \) is special, (resp. \( \overline{\pi}' \in B(\mathcal{H}) \) weakly special) then for any \( g \in \hat{G} \) the element \( g\overline{\pi} \) is special, (resp. \( g\overline{\pi}' \) is weakly special).

Corollary 3.7. [7] If \( \overline{\pi}_\pi \subset B(\mathcal{H}) \) is the \( C^* \)-norm completion of algebra generated by weakly special elements, then there is a natural action of \( \hat{G} \) on \( \overline{\pi}_\pi \).

Definition 3.8. Let \( \mathfrak{S} = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \ldots \right\} \) be an algebraical finite covering sequence. Let \( \pi : \tilde{A} \to B(\mathcal{H}) \) be an equivariant representation. Let \( \overline{\mathcal{A}}_\pi \subset B(\mathcal{H}) \) be the \( C^* \)-norm completion of algebra generated by weakly special elements. We say that \( \overline{\mathcal{A}}_\pi \) is the disconnected inverse noncommutative limit of \( \mathfrak{S} \) with respect to \( \pi \). The triple \( (A, \overline{\mathcal{A}}_\pi, G(\overline{\mathcal{A}}_\pi | A) \) is said to be the disconnected infinite noncommutative covering of \( \mathfrak{S} \) with respect to \( \pi \). If \( \pi \) is the universal representation then "with respect to \( \pi \)" is dropped and we will write \( (A, \overline{\mathcal{A}}, G(\overline{\mathcal{A}} | A) \).

Definition 3.9. A maximal irreducible subalgebra \( \overline{\mathcal{A}}_\pi \subset \overline{\mathcal{A}}_\pi \) is said to be a connected component of \( \mathfrak{S} \) with respect to \( \pi \). The maximal subgroup \( G_\pi \subset G(\overline{\mathcal{A}}_\pi | A) \) among subgroups \( G \subset G(\overline{\mathcal{A}}_\pi | A) \) such that \( G\overline{\mathcal{A}}_\pi = \overline{\mathcal{A}}_\pi \) is said to be the \( \overline{\mathcal{A}}_\pi \)-invariant group of \( \mathfrak{S} \). If \( \pi \) is the universal representation then "with respect to \( \pi \)" is dropped.

Remark 3.10. From the Definition 3.9 it follows that \( G_\pi \subset G(\overline{\mathcal{A}}_\pi | A) \) is a normal subgroup.

Definition 3.11. Let

\[
\mathfrak{S} = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \ldots \right\} \in \mathfrak{S}_{\text{FinAlg}},
\]

and let \( (A, \overline{\mathcal{A}}_\pi, G(\overline{\mathcal{A}}_\pi | A)) \) be a disconnected infinite noncommutative covering of \( \mathfrak{S} \) with respect to an equivariant representation \( \pi : \lim_{\to n} A_n \to B(\mathcal{H}) \). Let \( \overline{\mathcal{A}}_\pi \subset \overline{\mathcal{A}}_\pi \) be a connected component of \( \mathfrak{S} \) with respect to \( \pi \), and let \( G_\pi \subset G(\overline{\mathcal{A}}_\pi | A) \) be the \( \overline{\mathcal{A}}_\pi \)-invariant group of \( \mathfrak{S} \). Let \( h_\pi : G(\overline{\mathcal{A}}_\pi | A) \to G(A_n | A) \) be the natural surjective homomorphism. The representation \( \pi : \lim_{\to n} A_n \to B(\mathcal{H}) \) is said to be good if it satisfies to following conditions:

(a) The natural *-homomorphism \( \lim_{\to n} A_n \to M\left( \overline{\mathcal{A}}_\pi \right) \) is injective,

(b) If \( J \subset G(\overline{\mathcal{A}}_\pi | A) \) is a set of representatives of \( G(\overline{\mathcal{A}}_\pi | A) / G_\pi \), then the algebraic direct sum

\[
\bigoplus_{g \in J} g\overline{A}_\pi
\]

is a dense subalgebra of \( \overline{A}_\pi \),

(c) For any \( n \in \mathbb{N} \) the restriction \( h_\pi |_{G_\pi} \) is an epimorphism, i.e. \( h_\pi (G_\pi) = G(A_n | A) \).

If \( \pi \) is the universal representation we say that \( \mathfrak{S} \) is good.
**Definition 3.12.** Let $\mathcal{S} = \{A = A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n \rightarrow \ldots\} \in \text{FinAlg}$ be an algebraical finite covering sequence. Let $\pi : \hat{A} \rightarrow B(\mathcal{H})$ be a good representation. A connected component $\hat{A}_\pi \subset \hat{A}$ is said to be the inverse noncommutative limit of $\downarrow \mathcal{S}$ (with respect to $\pi$). The $\hat{A}_\pi$-invariant group $G_\pi$ is said to be the covering transformation group of $\mathcal{S}$ (with respect to $\pi$). The triple $(\hat{A}, \hat{A}_\pi, G_\pi)$ is said to be the infinite noncommutative covering of $\mathcal{S}$ (with respect to $\pi$). We will use the following notation

$$\lim_\pi \downarrow \mathcal{S} \overset{\text{def}}{=} \hat{A}_\pi,$$

$$G\left(\hat{A}_\pi \mid A\right) \overset{\text{def}}{=} G_\pi.$$ 

If $\pi$ is the universal representation then "with respect to $\pi$" is dropped and we will write $(\hat{A}, \hat{A}_\pi, G)$, $\lim_\pi \downarrow \mathcal{S} \overset{\text{def}}{=} \hat{A}$ and $G\left(\hat{A} \mid A\right) \overset{\text{def}}{=} G$.

**4 Quantization of topological coverings**

The described in the Sections 2 and 3 theory is motivated by a quantization of topological coverings. Following theorem provides a purely algebraic construction of topological finite-fold coverings.

**Theorem 4.1.** [7] If $\mathcal{X}, \bar{\mathcal{X}}$ are locally compact spaces, and $\pi : \bar{\mathcal{X}} \rightarrow \mathcal{X}$ is a surjective continuous map, then following conditions are equivalent:

(i) The map $\pi : \bar{\mathcal{X}} \rightarrow \mathcal{X}$ is a finite-fold covering with a compactification,

(ii) There is a natural noncommutative finite-fold covering $(C_\mathcal{X}(\mathcal{X}), C_\bar{\mathcal{X}}(\bar{\mathcal{X}}), G)$. 

**Remark 4.2.** The definition of a covering with a compactification is presented in [7].

**4.3.** An algebraic construction of infinite coverings can be given by an 'infinite composition' of finite ones. Suppose that there is a sequence

$$\mathcal{S}_\mathcal{X} = \{\mathcal{X}_0 \leftarrow \ldots \leftarrow \mathcal{X}_n \leftarrow \ldots\} \tag{4.1}$$

of topological spaces and finite-fold coverings, and let us consider a following diagram

$$\begin{array}{c}
\bar{\mathcal{X}} \\
\downarrow \\
\mathcal{X}_0 \leftarrow \mathcal{X}_1 \leftarrow \mathcal{X}_2 \leftarrow \ldots
\end{array}$$

where all arrows are coverings. It is proven in [7] that there is "minimal" $\bar{\mathcal{X}}$ which satisfies to the above diagram. This minimal $\bar{\mathcal{X}}$ is said to be the topological inverse limit of $\mathcal{S}_\mathcal{X}$ = $\{\mathcal{X}_0 \leftarrow \ldots \leftarrow \mathcal{X}_n \leftarrow \ldots\}$ and it is denoted by $\lim_\pi \downarrow \mathcal{S}_\mathcal{X}$. The topological inverse limits are
fully described in [7]. Any inverse limit yields an infinite topological covering \( \lim \downarrow \mathcal{S}_X \rightarrow \mathcal{X} \). Denote by \( G \left( \lim \downarrow \mathcal{S}_X \mid \mathcal{X} \right) \) the group of covering transformations of the covering \( \lim \downarrow \mathcal{S}_X \rightarrow \mathcal{X} \). The following theorem gives an algebraic construction of the topological inverse limit.

**Theorem 4.4.** [7] If \( \mathcal{S}_X = \{ \mathcal{X} = \mathcal{X}_0 \leftarrow ... \leftarrow \mathcal{X}_n \leftarrow ... \} \) is the sequence of topological spaces and coverings and
\[
\mathcal{S}_{C_0(\mathcal{X})} = \{ C_0(\mathcal{X}) = C_0(\mathcal{X}_0) \rightarrow ... \rightarrow C_0(\mathcal{X}_n) \rightarrow ... \} \in \text{FinAlg}
\]
is an algebraical finite covering sequence then following conditions hold:

(i) \( \mathcal{S}_{C_0(\mathcal{X})} \) is good,

(ii) There are isomorphisms:
\[
\begin{align*}
&\lim \downarrow \mathcal{S}_{C_0(\mathcal{X})} \approx C_0 \left( \lim \downarrow \mathcal{S}_X \right); \\
&G \left( \lim \downarrow \mathcal{S}_{C_0(\mathcal{X})} \mid C_0(\mathcal{X}) \right) \approx G \left( \lim \downarrow \mathcal{S}_X \mid \mathcal{X} \right).
\end{align*}
\]

5 Operator algebras of foliations

**Definition 5.1.** Let \( M \) be a smooth manifold and \( TM \) its tangent bundle, so that for each \( x \in M \), \( T_xM \) is the tangent space of \( M \) at \( x \). A smooth subbundle \( \mathcal{F} \) of \( TM \) is called integrable if and only if one of the following equivalent conditions is satisfied:

(a) Every \( x \in M \) is contained in a submanifold \( W \) of \( M \) such that
\[
T_y(W) = \mathcal{F}_y \quad \forall y \in W,
\]

(b) Every \( x \in M \) is in the domain \( U \subset M \) of a submersion \( p : U \rightarrow \mathbb{R}^q \) \( (q = \text{codim} \mathcal{F}) \) with
\[
\mathcal{F}_y = \text{Ker}(p_*)_y \quad \forall y \in U,
\]

(c) \( C^\infty(\mathcal{F}) = \{ X \in C^\infty(TM), X_x \in \mathcal{F}_x \quad \forall x \in M \} \) is a Lie algebra,

(d) The ideal \( J(\mathcal{F}) \) of smooth exterior differential forms which vanish on \( \mathcal{F} \) is stable by exterior differentiation.

A foliation of \( M \) is given by an integrable subbundle \( \mathcal{F} \) of \( TM \). The leaves of the foliation \( (M, \mathcal{F}) \) are the maximal connected submanifolds \( L \) of \( M \) with \( T_x(L) = \mathcal{F}_x, \forall x \in L \), and the partition of \( M \) in leaves
\[
M = \bigcup L_\alpha, \quad \alpha \in X
\]
is characterized geometrically by its “local triviality”: every point \( x \in M \) has a neighborhood \( U \) and a system of local coordinates \((x^i)_{i=1,...,\dim V}\) called foliation charts, so that the partition of \( U \) in connected components of leaves corresponds to the partition of

\[
\mathbb{R}^{\dim M} = \mathbb{R}^{\dim F} \times \mathbb{R}^{\codim F}
\]

in the parallel affine subspaces \( \mathbb{R}^{\dim F} \times \text{pt} \). The corresponding foliation will be denoted by

\[
(\mathbb{R}^n, \mathcal{F}_p)
\]

where \( p = \dim F_p \). To each foliation \((M, \mathcal{F})\) is canonically associated a \( C^* \)-algebra \( C^*_r(M, \mathcal{F}) \) which encodes the topology of the space of leaves. To take this into account one first constructs a manifold \( G, \dim G = \dim M + \dim F \), called the graph (or holonomy groupoid) of the foliation, which refines the equivalence relation coming from the partition of \( M \) in leaves \( M = \cup L \). An element \( \gamma \) of \( G \) is given by two points \( x = s(\gamma), y = r(\gamma) \) of \( M \) together with an equivalence class of smooth paths: \( \gamma(t) \in M, t \in [0,1]; \gamma(0) = x, \gamma(1) = y \), tangent to the bundle \( F \) (i.e. \( \dot{\gamma}(t) \in F_{\gamma(t)} \) \( \forall t \in \mathbb{R} \)) up to the following equivalence: \( \gamma_1 \) and \( \gamma_2 \) are equivalent if and only if the holonomy of the path \( \gamma_2 \circ \gamma_1^{-1} \) at the point \( x \) is the identity. The graph \( G \) has an obvious composition law. For \( \gamma, \gamma' \in G \), the composition \( \gamma \circ \gamma' \) makes sense if \( s(\gamma) = r(\gamma') \). If the leaf \( L \) which contains both \( x \) and \( y \) has no holonomy, then the class in \( G \) of the path \( \gamma(t) \) only depends on the pair \((y, x)\). In general, if one fixes \( x = s(\gamma), y = r(\gamma) \), the map from \( G_x = \{ \gamma, s(\gamma) = x \} \) to the leaf \( L \) through \( x \), given by \( \gamma \in G_x \mapsto y = r(\gamma) \), is the holonomy covering of \( L \). Both maps \( r \) and \( s \) from the manifold \( G \) to \( M \) are smooth submersions and the map \((r, s)\) to \( M \times M \) is an immersion whose image in \( M \times M \) is the (often singular) subset

\[
\{(y, x) \in M \times M : y \text{ and } x \text{ are on the same leaf}\}.
\]

We assume, for notational convenience, that the manifold \( G \) is Hausdorff, but as this fails to be the case in very interesting examples I shall refer to \( \mathbb{R} \) for the removal of this hypothesis. For \( x \in M \) one lets \( \Omega^1_{1/2} \) be the one dimensional complex vector space of maps from the exterior power \( \wedge^k F_x \), \( k = \dim F \), to \( \mathbb{C} \) such that

\[
\rho(\lambda v) = |\lambda|^{1/2} \rho(v) \quad \forall v \in \wedge^k F_x, \quad \forall \lambda \in \mathbb{R}.
\]

Then, for \( \gamma \in G \), one can identify \( \Omega^1_{1/2} \) with the one dimensional complex vector space \( \Omega^1_{1/2} \otimes \Omega^1_{x/2} \), where \( \gamma : x \to y \). In other words

\[
\Omega^1_{1/2} = (\Omega^1_{1/2} M \otimes s^*(\Omega^1_{1/2})).
\]

Of course the bundle \( \Omega^1_{1/2} M \) is trivial on \( M \), and we could choose once and for all a trivialisation \( \nu \) turning elements of \( C_1^o(G, \Omega^1_{1/2}) \) into functions. Let us however stress that the use of half densities makes all the construction completely canonical. For \( f, g \in C_1^o(G, \Omega^1_{1/2}) \), the convolution product \( f \ast g \) is defined by the equality

\[
(f \ast g)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2).
\]
This makes sense because, for fixed $\gamma : x \to y$ and fixing $v_x \in \wedge^k F_x$ and $v_y \in \wedge^k F_y$, the product $f(\gamma_1)g(\gamma_1^{-1})$ defines a 1-density on $G^y = \{\gamma_1 \in G, r(\gamma_1) = y\}$, which is smooth with compact support (it vanishes if $\gamma_1 \notin \text{supp } f$), and hence can be integrated over $G^y$ to give a scalar, namely $(f * g)(\gamma)$ evaluated on $v_x, v_y$. The $*$ operation is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$, i.e. if $\gamma : x \to y$ and $v_x \in \wedge^k F_x$, $v_y \in \wedge^k F_y$ then $f^*(\gamma)$ evaluated on $v_y, v_y$ is equal to $\overline{f(\gamma^{-1})}$ evaluated on $v_y, v_x$. We thus get a $*$-algebra $C_\infty(G,\Omega^{1/2})$. For each leaf $L$ of $(M,F)$ one has a natural representation of this $*$-algebra on the $L^2$ space of the holonomy covering $\hat{L}$ of $L$. Fixing a base point $x \in L$, one identifies $\hat{L}$ with $\mathcal{G}_x = \{\gamma, s(\gamma) = x\}$ and defines

$$(\rho_x(f)\xi)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)\xi(\gamma_2) \quad \forall \xi \in L^2(\mathcal{G}_x), \quad (5.2)$$

where $\xi$ is a square integrable half density on $\mathcal{G}_x$. Given $\gamma : x \to y$ one has a natural isometry of $L^2(\mathcal{G}_x)$ on $L^2(\mathcal{G}_y)$ which transforms the representation $\rho_x$ in $\rho_y$. By definition $C_\tau^r(M,F)$ is the $C^*$-algebra completion of $C_\infty(G,\Omega^{1/2})$ with the norm

$$\|f\| = \sup_{x \in M} \|\rho_x(f)\|. \quad (5.3)$$

Denote by $\mathcal{G}(M,F)$ the foliation groupoid of $(M,F)$.

**Example 5.2. Linear foliation on torus.** Consider a vector field $\tilde{X}$ on $\mathbb{R}^2$ given by

$$\tilde{X} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$$

with constant $\alpha$ and $\beta$. Since $\tilde{X}$ is invariant under all translations, it determines a vector field $X$ on the two-dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The vector field $X$ determines a foliation $\mathcal{F}$ on $T^2$. The leaves of $\mathcal{F}$ are the images of the parallel lines $L = \{(x_0 + tx, y_0 + ty) : t \in \mathbb{R}\}$ with the slope $\theta = \beta/\alpha$ under the projection $\mathbb{R}^2 \to T^2$. In the case when $\theta$ is rational, all leaves of $\mathcal{F}$ are closed and are circles, and the foliation $\mathcal{F}$ is determined by the fibers of a fibration $T^2 \to S^1$. In the case when $\theta$ is irrational, all leaves of $\mathcal{F}$ are everywhere dense in $T^2$. Denote by $(T^2,\mathcal{F}_{\theta})$ this foliation.

## 6 Strong Morita equivalence

The notion of the strong Morita equivalence was introduced by Rieffel.

**Definition 6.1.** Let $A$ and $B$ be $C^*$-algebras. An $A$-$B$-equivalence bimodule is an $A$-$B$-bimodule $X$, endowed with $A$-valued and $B$-valued inner products $\langle \cdot , \cdot \rangle_A$ and $\langle \cdot , \cdot \rangle_B$ accordingly, such that $X$ is a right Hilbert $B$-module and a left Hilbert $A$-module with respect to these inner products, and, moreover,

1. $\langle x,y\rangle_A z = x\langle y,z\rangle_B$ for any $x,y,z \in X$;
2. The set \( (X, X)_A \) generates a dense subset in \( A \), and the set \( (X, X)_B \) generates a dense subset in \( B \).

We call algebras \( A \) and \( B \) strongly Morita equivalent, if there is an \( A \)-\( B \)-equivalence bimodule.

**Example 6.2.** [8] If \( \mathcal{F} \) is a simple foliation given by a submersion \( M \to B \), then the \( C^* \)-algebra \( C^r(M, \mathcal{F}) \) is strongly Morita equivalent to the \( C^* \)-algebra \( C_0(B) \).

The following theorem yields a relation between the strong Morita equivalence and the stable equivalence.

**Theorem 6.3.** [3] Let \( A \) and \( B \) are \( C^* \)-algebras with countable approximate units. Then these algebras are strongly Morita equivalent if and only if they are stably equivalent, i.e. \( A \otimes K \cong B \otimes K \), where \( K \) denotes the algebra of compact operators in a separable Hilbert space.

**Remark 6.4.** The \( C^* \)-algebra \( A \otimes K \) means the \( C^* \)-norm completion of the algebraic tensor product of \( A \) and \( K \). In general the \( C^* \)-norm completion of algebraic tensor product \( A \) and \( B \) is not unique, because it depends on the \( C^* \)-norm on \( A \otimes B \). However in [2] it is proven that \( K \) is a nuclear \( C^* \)-algebra, so there is the unique \( C^* \)-norm completion \( A \otimes K \).

**Example 6.5.** [8] If \( \mathcal{F} \) is a simple foliation given by a bundle \( M \to B \), then following condition holds
\[
C^r_\theta(M, \mathcal{F}) \cong C_0(B) \otimes K
\] (6.1)
(cf. Example 6.2).

**Example 6.6.** [8] Consider a compact foliated manifold \( (M, \mathcal{F}) \). As usual, let \( \mathcal{G} \) denote the holonomy groupoid of \( \mathcal{F} \). For any subsets \( A, B \subset M \), denote
\[
\mathcal{G}_B^A = \{ \gamma \in \mathcal{G} : r(\gamma) \in A, s(\gamma) \in B \}.
\]
In particular,
\[
\mathcal{G}_T^M = \{ \gamma \in \mathcal{G} : s(\gamma) \in T \}.
\]
If \( T \) is a transversal, then \( \mathcal{G}_T^M \) is a submanifold and a subgroupoid in \( \mathcal{G} \). Let \( C^r_\theta(\mathcal{G}_T^M) \) be the reduced \( C^* \)-algebra of this groupoid. As shown in [6], if \( T \) is a complete transversal, then the algebras \( C^r_\theta(\mathcal{G}) \) and \( C^r_\theta(\mathcal{G}_T^M) \) are strongly Morita equivalent. In particular, this implies that
\[
C^r_\theta(\mathcal{G}) \otimes K \cong C^r_\theta(\mathcal{G}_T^M) \otimes K.
\]
In [6] it is proven that \( C^r_\theta(\mathcal{G}) \) is stable, i.e. \( C^r_\theta(\mathcal{G}) \cong C^r_\theta(\mathcal{G}) \otimes K \), so what implies that
\[
C^r_\theta(\mathcal{G}) \cong C^r_\theta(\mathcal{G}_T^M) \otimes K.
\]

**Example 6.7.** Consider the linear foliation \( \mathcal{F}_\theta \) on the two-dimensional torus \( \mathbb{T}^2 \) (cf. Example 5.2), where \( \theta \in \mathbb{R} \) is a fixed irrational number. If we choose the transversal \( T \) given by
the equation $y = 0$, then the leaf space of the foliation $\mathcal{F}_\theta$ is identified with the orbit space of the $\mathbb{Z}$-action on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ generated by the rotation
$$R_\theta(x) = x + \theta \mod 1, \quad x \in S^1.$$ Elements of the algebra $C^\infty_c(\mathbb{G}^T)$ are determined by matrices $a(i,j)$, where the indices $(i,j)$ are arbitrary pairs of elements $i$ and $j$ of $T$, lying on the same leaf of $\mathcal{F}$, that is, on the same orbit of the $\mathbb{Z}$-action $R_\theta$. Since in this case the leafwise equivalence relation on the transversal is given by a free group action, the algebra $C^*_r(T^2, \mathcal{F}_\theta) = C(T^2)$ coincides with the crossed product $C(S^1) \rtimes \mathbb{Z}$ of the algebra $C(S^1)$ by the group $\mathbb{Z}$ with respect to the $\mathbb{Z}$-action $R_\theta$ on $C(S^1)$. Therefore every element of $C(T^2)$ is given by a power series
$$a = \sum_{n \in \mathbb{Z}} a_n U^n, \quad a_n \in C(S^1),$$
the multiplication is given by
$$(aU^n)(bU^m) = a(b \circ R^{-1}_n)U^{n+m}$$
and the involution by
$$(aU^n)^* = \bar{a} U^{-n}.$$ The algebra $C(S^1)$ is generated by the function $V$ on $S^1$ defined as
$$V(x) = e^{2\pi ix}, \quad x \in S^1.$$ Hence, the algebra $C(T^2)$ is generated by two elements $U$ and $V$, satisfying the relation
$$VU = \lambda UV, \quad \lambda = e^{2\pi i \theta}.$$ Thus, for example, a general element of $C^\infty_c(\mathbb{G}^T)$ can be represented as a power series
$$a = \sum_{(n,m) \in \mathbb{Z}^2} a_{nm} U^n V^m,$$ where $a_{nm} \in \mathcal{S}(\mathbb{Z}^2)$ is a rapidly decreasing sequence (that is, for any natural $k$ we have $\sup_{(n,m) \in \mathbb{Z}^2} (|n| + |m|)^k |a_{nm}| < \infty$). Since, in the commutative case ($\theta = 0$), the above description defines the algebra $C^\infty(T^2)$ of smooth functions on the two-dimensional torus, the $C^*$-norm completion of $C^\infty(T^2)$ is called the algebra of continuous functions on a noncommutative torus and denoted by $\mathcal{C}(T^2)$. Otherwise from $C^\infty_c(\mathbb{G}^T) = C^\infty(T^2)$ and from the Example 6.6 it follows that the $C^*$-algebra $C^*_r(T^2, \mathcal{F}_\theta)$ is strongly Morita equivalent to $C(T^2)$. From the Theorem 6.3 it follows that
$$C^*_r(T^2, \mathcal{F}_\theta) \approx C(T^2) \otimes \mathcal{K}. \quad (6.2)$$

7 Coverings of stable algebras

Here we find the relation between noncommutative coverings of $C^*$-algebras and noncommutative coverings of their stabilizations.
7.1 Finite-fold coverings

If $A$ is a $C^*$-algebra and $\mathcal{K} = \mathcal{K} (\mathcal{H})$ is an algebra of compact operators then the $C^*$-norm completion of $A \otimes \mathcal{K}$ is said to be the stable algebra of $A$ (cf. Remark 6.4). This completion we denote by $A \otimes \mathcal{K}$. Any non-degenerated involutive action of $G$ on $A$ uniquely induces the non-degenerated involutive action of $G$ on $A \otimes \mathcal{K}$.

**Theorem 7.1.** If $\left( A, \tilde{A}, G \right)$ is a noncommutative finite-fold covering then a triple

$$\left( A \otimes \mathcal{K}, \tilde{A} \otimes \mathcal{K}, G \right)$$

is a noncommutative finite-fold covering.

**Proof.** From the Definition 2.4 it follows that following conditions hold:

(a) There are unital $C^*$-algebras $B$, $\tilde{B}$ and inclusions $A \subset B$, $\tilde{A} \subset \tilde{B}$ such that $A$ (resp. $B$) is an essential ideal of $\tilde{A}$ (resp. $\tilde{B}$),

(b) There is an unital noncommutative finite-fold covering $\left( B, \tilde{B}, G \right)$,

(c) $\tilde{A} = \left\{ a \in \tilde{B} \mid \left\langle \tilde{B}, a \right\rangle_{\tilde{B}} \in A \right\}.$

Let $\mathcal{K}^+$ be the unitisation of $\mathcal{K}$, i.e. underlying vector space of $\mathcal{K}^+$ is the direct sum $\mathcal{K} \oplus \mathbb{C}$. Algebra $A \otimes \mathcal{K}$ (resp. $\tilde{A} \otimes \mathcal{K}$) is an essential ideal of $B \otimes \mathcal{K}^+$ (resp. $\tilde{B} \otimes \mathcal{K}^+$), i.e. condition (a) of the Definition 2.4 holds. Since $\left( B, \tilde{B}, G \right)$ is an unital noncommutative finite-fold covering the algebra $\tilde{B}$ is a finitely generated $B$ module, i.e. there are $\tilde{b}_1, \ldots, \tilde{b}_n \in \tilde{B}$ such that any $\tilde{b} \in \tilde{B}$ is given by

$$\tilde{b} = \sum_{j=1}^{n} \tilde{b}_j b; \text{ where } b_j \in B.$$

From the above equation it turns out that if

$$\tilde{b}^\mathcal{K} = \sum_{k=1}^{m} \tilde{b}_k \otimes x_k \in \tilde{B} \otimes \mathcal{K}^+; \quad \tilde{b}_k \in \tilde{B}, \; x_k \in \mathcal{K}^+$$

and

$$\tilde{b}_k = \sum_{j=1}^{n} \tilde{b}_j b_{kj}$$

then

$$\tilde{b}^\mathcal{K} = \sum_{j=1}^{n} \left( \tilde{b}_j \otimes 1_{\mathcal{K}^+} \right) b_j^\mathcal{K},$$

where $b_j^\mathcal{K} = \sum_{k=1}^{m} b_{kj} \otimes x_k \in B \otimes \mathcal{K}.$ (7.1)
From (7.1) and taking into account that the algebraic tensor product of $\bar{B}$ and $\mathcal{K}^+$ is dense in its $C^*$-norm completion, it turns out that any $\tilde{b}^\mathcal{K} \in B \otimes \mathcal{K}^+$ can be represented as

$$\tilde{b}^\mathcal{K} = \sum_{j=1}^{n} \left( b_j \otimes 1_{\mathcal{K}^+} \right) b_j^\mathcal{K}; \text{ where } b_j^\mathcal{K} \in B \otimes \mathcal{K}^+.$$ 

It follows that $\bar{B} \otimes \mathcal{K}^+$ is a finitely generated $B \otimes \mathcal{K}^+$ module. From the Kasparov Stabilization Theorem [2] it turns out that $\bar{B} \otimes \mathcal{K}^+$ is a projective $B \otimes \mathcal{K}^+$ module. So $(B \otimes \mathcal{K}^+ , \bar{B} \otimes \mathcal{K}^+ , G)$ is an unital finite-fold noncommutative covering, i.e. the condition (b) of the Definition 2.4 holds. Denote by $\tilde{A}^\mathcal{K}$ a subalgebra given by (2.2), i.e.

$$\tilde{A}^\mathcal{K} = \left\{ \tilde{a} \in \bar{B} \otimes \mathcal{K}^+ \mid \langle \tilde{B} \otimes \mathcal{K}^+ , \tilde{a} \rangle_{\bar{B} \otimes \mathcal{K}^+} \in \tilde{A} \otimes \mathcal{K} \right\}.$$ 

If $\tilde{a} \in \tilde{A} \otimes \mathcal{K}$ then for any $\tilde{b} \in \bar{B} \otimes \mathcal{K}^+$ following condition holds

$$\langle \tilde{b}, \tilde{a} \rangle_{\bar{B} \otimes \mathcal{K}^+} = \sum_{g \in G} g \langle \tilde{b}, \tilde{a} \rangle \in \tilde{A} \otimes \mathcal{K}.$$ 

Since $\langle \tilde{b}, \tilde{a} \rangle_{\bar{B} \otimes \mathcal{K}^+}$ is $G$-invariant one has $\langle \tilde{b}, \tilde{a} \rangle_{\bar{B} \otimes \mathcal{K}^+} \in \tilde{A} \otimes \mathcal{K}$, i.e. $\tilde{a} \in \tilde{A}^\mathcal{K}$. It follows that $\tilde{A} \otimes \mathcal{K} \subset \tilde{A}^\mathcal{K}$. Any $\tilde{a} \in \tilde{A}^\mathcal{K}$ satisfies to $\tilde{a} \in \tilde{A}^\mathcal{K} \setminus \tilde{A} \otimes \mathcal{K}$ if and only if one or both of two following conditions hold:

(a) If $p_\mathcal{K} : \bar{B} \otimes \mathcal{K}^+ \to \bar{B} \otimes \mathcal{K}^+ / \bar{B} \otimes \mathcal{K}$ the natural projection onto the quotient algebra then $p_\mathcal{K}(\tilde{a}) \neq 0$.

(b) If $p_\mathcal{K} : \bar{B} \otimes \mathcal{K}^+ \to \bar{B} \otimes \mathcal{K}^+ / \tilde{A} \otimes \mathcal{K}^+$ the natural projection onto the quotient algebra then $p_\mathcal{K}(\tilde{a}) \neq 0$

If

$$a = \langle \tilde{a}, \tilde{a} \rangle_{\bar{B} \otimes \mathcal{K}^+} = \sum_{g \in G} g (\tilde{a}^* \tilde{a})$$

then

$$p_\mathcal{K}(a) = \sum_{g \in G} g (p_\mathcal{K}(\tilde{a}) p_\mathcal{K}(\tilde{a})^*) = p_\mathcal{K}(\tilde{a}) p_\mathcal{K}(\tilde{a})^* + \sum_{g \in G \setminus \{e\}} g (p_\mathcal{K}(\tilde{a}) p_\mathcal{K}(\tilde{a})^*) \quad (7.2)$$

where $e \in G$ is the unity of $G$. If $\tilde{a} \notin \bar{B} \otimes \mathcal{K}$ then taking into account that all terms of (7.2) are positive one has $p_\mathcal{K}(a) \neq 0$. Hence from $\tilde{a} \notin \bar{B} \otimes \mathcal{K}$ it turns out $\tilde{a} \notin \tilde{A}^\mathcal{K}$. Similarly if $\tilde{a} \notin \tilde{A} \otimes \mathcal{K}^+$ then $p_\mathcal{K}(\tilde{a}) \neq 0$ and

$$p_B(a) = \sum_{g \in G} g (p_B(\tilde{a}) p_B(\tilde{a})^*) = p_B(\tilde{a}) p_B(\tilde{a})^* + \sum_{g \in G \setminus \{e\}} g (p_B(\tilde{a}) p_B(\tilde{a})^*). \quad (7.3)$$
Taking into account that all terms of (7.3) are positive one has \( p_B(a) \neq 0 \). Hence from \( a \notin A \otimes K^+ \) it turns out \( \tilde{a} \notin \tilde{A}^K \). In result one has

\[
\tilde{A}^K \setminus \tilde{A} \otimes K = \emptyset,
\]
\[
\tilde{A}^K = \tilde{A} \otimes K,
\]
i.e. the condition (c) of the Definition 2.4 holds.

\[\square\]

### 7.2 Infinite coverings

Suppose that

\[
S = \left\{ A = A_0 \xrightarrow{\pi_1} A_1 \xrightarrow{\pi_2} \ldots \xrightarrow{\pi_n} A_n \xrightarrow{\pi_{n+1}} \ldots \right\} \subset \text{FinAlg}
\]  

(7.4)
is an (algebraical) finite covering sequence. From the Theorem 7.1 it follows that

\[
S_K = \left\{ \tilde{A} \otimes K = A_0 \otimes K \xrightarrow{\pi_1 \otimes \text{Id}_K} A_1 \otimes K \xrightarrow{\pi_2 \otimes \text{Id}_K} \ldots \xrightarrow{\pi_n \otimes \text{Id}_K} A_n \otimes K \xrightarrow{\pi_{n+1} \otimes \text{Id}_K} \ldots \right\}
\]  

(7.5)
is an (algebraical) finite covering sequence, i.e. \( S_K \subset \text{FinAlg} \). Denote by \( \hat{A} = \lim_{\rightarrow} A_n \), \( G_n = G (A_n | A) \), \( \hat{G} = \lim_{\rightarrow} G_n \). Clearly \( \lim_{\rightarrow} (A_n \otimes K) = \hat{A} \otimes K \). If \( \pi : \hat{A} \to B \left( \hat{\mathcal{H}} \right) \) is an equivariant representation and \( K = K (\mathcal{H}) \) then \( \pi_K = \pi \otimes \text{Id}_K : \hat{A} \otimes K \to B \left( \mathcal{H}_K = \hat{\mathcal{H}} \otimes \mathcal{H} \right) \) is an equivariant representation. Let us consider a \( \hat{A}'' \)-Hilbert module \( \ell^2 \left( \hat{A}'' \right) \) then

\[
\hat{A}'' \otimes K \approx K \left( \ell^2 \left( \hat{A}'' \right) \right)
\]

**Lemma 7.2.** If \( \bar{a} \in \hat{A}'' \) is a special element with respect to \( S \), and \( p \in K \) is a rank-one projector then \( \bar{a} \otimes p \in \left( \tilde{A} \otimes K \right)'' \) is a special element with respect to \( S_K \).

**Proof.** One can select a basis of the Hilbert \( \mathcal{H} \) space such that \( p \) is represented by a following infinite matrix

\[
\begin{pmatrix}
1 & 0 & \ldots \\
0 & 0 & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix}.
\]

If \( z \in A \otimes K \subset K \left( \ell^2 \left( \hat{A} \right) \right) \) is represented by an infinite matrix

\[
z = \begin{pmatrix}
a_{11} & a_{12} & \ldots \\
a_{21} & a_{22} & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix}; \quad \text{where } a_{jk} \in A
\]

then

\[
z (\bar{a} \otimes p) z^* = \begin{pmatrix}
a_{11} \bar{a}_{11} & a_{11} \bar{a}_{12} & \ldots \\
a_{12} \bar{a}_{21} & a_{12} \bar{a}_{22} & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]

where \( a_{jk} \in A \).
is represented by the matrix

\[
 z (\overline{\pi} \otimes p) z^* = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \overline{\pi} \begin{pmatrix} a^*_{11} & 0 & \cdots \\ a^*_{12} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.
\]

Let us select an orthogonal basis \( \{ \xi_n \in \ell^2 (\hat{A}) \}_{n \in \mathbb{N}} \) such that

\[
 \xi_1 = \begin{pmatrix} a^*_{11} \\ a^*_{12} \\ \vdots \end{pmatrix}.
\]

If \( p_K \in \mathcal{K} (\ell^2 (A)) \subset \mathcal{K} (\ell^2 (\hat{A}')) \) a projector onto a submodule \( A\xi_1 \subset \ell^2 (A) \) there is \( \overline{z} \in A \) such that

\[
 \begin{pmatrix} a^*_{11} & 0 & \cdots \\ a^*_{12} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \overline{z} p_K.
\]

It turns out

\[
 z (\overline{\pi} \otimes p) z^* = p_K \overline{\pi} \overline{z} \overline{z}^* p_K. \tag{7.6}
\]

Let \( f_\varepsilon \) is given by (3.2). The element \( \overline{\pi} \) is special, it follows that the series

\[
 a_n = \sum_{g \in \text{ker}(\hat{G} \to \overline{G}(A_n | A))} g ( \overline{\pi} ), \\
 b_n = \sum_{g \in \text{ker}(\hat{G} \to \overline{G}(A_n | A))} g ( \overline{z} \overline{\pi} \overline{z}^* ), \\
 c_n = \sum_{g \in \text{ker}(\hat{G} \to \overline{G}(A_n | A))} g ( \overline{z} \overline{\pi} \overline{z}^* )^2, \\
 d_n = \sum_{g \in \text{ker}(\hat{G} \to \overline{G}(A_n | A))} g f_\varepsilon ( \overline{z} \overline{\pi} \overline{z}^* )
\]

are strongly convergent and the sums lie in \( A_n \), i.e. \( b_n, c_n, d_n \in A_n \); For any \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) (which depends on \( \overline{\pi} \) and \( z \)) such that for any \( n \geq N \) a following condition holds

\[
 \left\| b_n^2 - c_n \right\| < \varepsilon.
\]
It turns out
\begin{align*}
  a^K_n &= \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g(\bar{a} \otimes p) = a_n \otimes p \in A_n \otimes K, \\
  b^K_n &= \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g(z(\bar{a} \otimes p)z^*) = p_Kb_n p_K \in A_n \otimes K, \\
  c^K_n &= \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g(z(\bar{a} \otimes p)z^*)^2 = p_Kc_n p_K \in A_n \otimes K, \\
  d^K_n &= \sum_{g \in \ker(\hat{G} \to G(A_n | A))} gf(z(\bar{a} \otimes p)z^*) = p_Kd_n p_K \in A_n \otimes K,
\end{align*}
i.e. conditions (a), (b) of the Definition 3.3 hold. From
\begin{align*}
  \left\| b^K_n - c_n \right\| &= \left\| p_Kb_n p_K - p_Kc_n p_K \right\| = \left\| \left( b^K_n \right)^2 - c^K_n \right\|
\end{align*}
it follows that for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that
\begin{align*}
  \left\| \left( b^K_n \right)^2 - c^K_n \right\| < \varepsilon
\end{align*}
for any $n \geq N$. It means that the conditions (c) of the Definition 3.3 holds. \qed

**Corollary 7.3.** Let $\overline{\mathcal{A}}_\pi \subset B\left(\hat{H}\right)$ be the disconnected inverse noncommutative limit of $\downarrow \mathcal{G}$ with respect to $\pi$. If $\overline{\mathcal{A}}_{\pi \otimes Id_K}$ is be the disconnected inverse noncommutative limit of $\downarrow \mathcal{G}_K$ with respect to $\pi \otimes Id_K$ then
\begin{align*}
  \overline{\mathcal{A}}_\pi \otimes K \subset \overline{\mathcal{A}}_{\pi \otimes Id_K}.
\end{align*}

**Proof.** The linear span of given by the Lemma 7.2 special elements $\bar{a} \otimes p \in \left(\hat{A} \otimes K\right)^{''}$ is dense in $\overline{\mathcal{A}}_\pi \otimes K$, so $\overline{\mathcal{A}}_\pi \otimes K \subset \overline{\mathcal{A}}_{\pi \otimes Id_K}$. \qed

**Lemma 7.4.** Let $\mathcal{G}$ be an algebraical finite covering sequence given by (7.4). Let $\pi : \hat{A} \to B\left(\hat{H}\right)$ be an equivariant representation. Let $\overline{\mathcal{A}}_\pi \subset B\left(\hat{H}\right)$ be the disconnected inverse noncommutative limit of $\downarrow \mathcal{G}$ with respect to $\pi$. Let $\mathcal{G}_K$ be an algebraical finite covering sequence given by (7.5). If $\overline{\pi}^K \in B\left(\hat{H} \otimes \hat{H}\right)_{+}$ is a is special element (with respect to $\mathcal{G}_K$) then
\begin{align*}
  \overline{\pi}^K \in \overline{\mathcal{A}}_\pi \otimes K.
\end{align*}

**Proof.** From the Corollary 3.3 it turns out that $\overline{\pi}^K \in \left(\hat{A} \otimes K\right)^{''}$, i.e. $\overline{\pi}^K \in B\left(\ell^2\left(\hat{A}^\prime\prime\right)\right)$. If $\overline{\pi}^K \notin \overline{\mathcal{A}}_\pi \otimes K$ then there are $\xi, \eta \in \ell^2(\overline{\mathcal{A}}_\pi)$ such that
\begin{align}
  \langle \xi, \overline{\pi}^K \eta \rangle_{\ell^2(H_{\hat{A}^\prime\prime})} \notin \overline{\mathcal{A}}_\pi. \tag{7.7}
\end{align}
The element $\overline{a}^K$ is positive, so it is self-adjoint, hence (7.7) is equivalent to the existence of $\xi \in \ell^2(\overline{A}_\pi)$ such that
\[
\langle \xi, \overline{a}^K \xi \rangle_{\ell^2(\overline{A}_\pi')} \notin \overline{A}_\pi.
\] (7.8)

If we select a basis $\{\xi_n \in \ell^2(\overline{A}_\pi')\}_{n \in \mathbb{N}}$ of $\ell^2(\overline{A}_\pi')$ of such that $\xi_1 = \xi$ then $\overline{a}^K$ is represented by an infinite matrix
\[
\overline{a}^K = \begin{pmatrix}
\overline{a}_{11} & \overline{a}_{12} & \cdots \\
\overline{a}_{21} & \overline{a}_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]
and from (7.8) it turns out $a_{11} \notin \overline{A}_\pi$. Let us select $z \in A$ such that $a_{11}$ does not satisfy to the Definition 3.3, i.e. at last one of the following (a)-(c) does not hold:

(a) For any $n \in \mathbb{N}^0$ the following series
\[
a_n = \sum_{g \in \ker(\overline{G} \to G(A_n | A))} g\overline{a}_{11}
\]
is strongly convergent and the sum lies in $A_n$, i.e. $a_n \in A_n$;

(b) If $f : \mathbb{R} \to \mathbb{R}$ is given by (3.2) then any $n \in \mathbb{N}^0$ and for any $z \in A$ following series
\[
b_n = \sum_{g \in \ker(\overline{G} \to G(A_n | A))} g(z\overline{a}_{11}z^*),
\]
\[
c_n = \sum_{g \in \ker(\overline{G} \to G(A_n | A))} g(z\overline{a}_{11}z^*)^2,
\]
\[
d_n = \sum_{g \in \ker(\overline{G} \to G(A_n | A))} gf(z\overline{a}_{11}z^*)
\]
are strongly convergent and the sums lie in $A_n$, i.e. $b_n, c_n, d_n \in A_n$;

(c) For any $\epsilon > 0$ there is $N \in \mathbb{N}$ (which depends on $\overline{a}$ and $z$) such that for any $n \geq N$ a following condition holds
\[
\|b_n^2 - c_n\| < \epsilon.
\]

If $z_K$ is given by
\[
z_K = \begin{pmatrix}
z & 0 & \cdots \\
0 & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]
then

\[ a^K_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g\pi = \left( \sum_{g \in \ker(\hat{G} \to G(A_n | A))} a_n \right) \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g\pi_{12} \cdots \right),
\]

\[ b^K_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g(z_Kz_K^* K) = \begin{pmatrix} b_n & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},
\]

\[ c^K_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g(z_Kz_K^* K)^2 = \begin{pmatrix} c_n & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},
\]

\[ d^K_n = \sum_{g \in \ker(\hat{G} \to G(A_n | A))} g f(z_Kz_K^* K) = \begin{pmatrix} d_n & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.
\]

From above equation it turns out

\[ a_n \notin A_n \Rightarrow a^K_n \notin A_n \otimes K,
\]

\[ b_n \notin A_n \Rightarrow b^K_n \notin A_n \otimes K,
\]

\[ c_n \notin A_n \Rightarrow c^K_n \notin A_n \otimes K,
\]

\[ d_n \notin A_n \Rightarrow d^K_n \notin A_n \otimes K.
\]

Following condition holds

\[ \left\| \left( b^K_n \right)^2 - c^K_n \right\| = \left\| b_n^2 - c_n \right\|.
\]

Hence if there is \( \varepsilon > 0 \) such that for any \( N \in \mathbb{N} \) there is \( n \geq N \) which satisfy to the following condition

\[ \left\| b_n^2 - c_n \right\| \geq \varepsilon
\]

then for any \( N \in \mathbb{N} \) there is \( n \geq N \) such that

\[ \left\| \left( b^K_n \right)^2 - c^K_n \right\| \geq \varepsilon.
\]

We have a contradiction. From this contradiction it turns out

\( \pi^K \in \overline{A}_n \otimes K \).
Corollary 7.5. Let $\overline{A}_\pi \subset B \left( \hat{H} \right)$ be the disconnected inverse noncommutative limit of $\downarrow \mathcal{S}$ with respect to $\pi$. If $\overline{A}_\pi^{\mathcal{K}} \otimes \text{Id}_\mathcal{K}$ is the disconnected inverse noncommutative limit of $\downarrow \mathcal{S}_\mathcal{K}$ with respect to $\pi \otimes \text{Id}_\mathcal{K}$ then

$$\overline{A}_\pi^{\mathcal{K}} \otimes \text{Id}_\mathcal{K} \subset \overline{A}_\pi \otimes \mathcal{K}.$$

Theorem 7.6. Let $\mathcal{S}$ be an algebraical finite covering sequence given by (7.4). Let $\pi : \hat{A} \to B \left( \hat{H} \right)$ be good representation. Let $\overline{A}_\pi \subset B \left( \hat{H} \right)$ be the disconnected inverse noncommutative limit of $\downarrow \mathcal{S}$ with respect to $\pi$. If $\mathcal{S}_\mathcal{K}$ be an algebraical finite covering sequence given by (7.4) then $\pi \otimes \text{Id}_\mathcal{K} : \lim \leftarrow A_n \otimes \mathcal{K} \to B \left( \hat{H} \otimes \mathcal{H} \right)$ is a good representation. Moreover if

$$\lim \leftarrow \mathcal{S} = \overline{A}_\pi,$$

$$G \left( \overline{A}_\pi | A \right) = G_\pi.$$

then

$$\lim \leftarrow \mathcal{S}_\mathcal{K} = \overline{A}_\pi \otimes \mathcal{K},$$

$$G \left( \lim \leftarrow \mathcal{S}_\mathcal{K} | A \otimes \mathcal{K} \right) \cong G \left( \overline{A}_\pi \otimes \mathcal{K} | A \otimes \mathcal{K} \right) \cong G_\pi.$$

It means that if $\left( A, \overline{A}_\pi, G_\pi \right)$ is an infinite noncommutative covering of $\mathcal{S}$ (with respect to $\pi$) then $\left( A \otimes \mathcal{K}, \overline{A}_\pi \otimes \mathcal{K}, G_\pi \right)$ is an infinite noncommutative covering of $\mathcal{S}$ (with respect to $\pi \otimes \text{Id}_\mathcal{K}$).

Proof. From the Corollaries 7.3 and 7.5 it follows that if $\overline{A}_\pi^{\mathcal{K}} \otimes \text{Id}_\mathcal{K}$ is the disconnected inverse noncommutative limit of $\downarrow \mathcal{S}_\mathcal{K}$ with respect to $\pi \otimes \text{Id}_\mathcal{K}$ then

$$\overline{A}_\pi^{\mathcal{K}} \otimes \text{Id}_\mathcal{K} = \overline{A}_\pi \otimes \mathcal{K}.$$

If $J \subset \mathcal{I}$ is a set of representatives of $\lim G_n / G_\pi$ then $\overline{A}_\pi$ is the $C^*$-norm completion of the following algebraic direct sum

$$\bigoplus_{g \in J} g \overline{A}_\pi.$$

Hence $\overline{A}_\pi^{\mathcal{K}} \otimes \text{Id}_\mathcal{K}$ is the $C^*$-norm completion of the following algebraic direct sum

$$\bigoplus_{g \in J} g \left( \overline{A}_\pi \otimes \mathcal{K} \right).$$
So \( \tilde{A}_\pi \otimes K \) is a maximal irreducible component of \( \tilde{A}_{\pi \otimes \text{Id}_K} \) and \( G_\pi \) is the maximal subgroup of among subgroups \( G \subset \varprojlim G_n \) such that
\[
G \left( \tilde{A}_\pi \otimes K \right) = \tilde{A}_\pi \otimes K.
\]
Clearly (a)-(c) of the Definition 3.11 hold.

\[\square\]

8 Foliations and coverings

8.1 Lifts and restrictions of foliations

Let \( M \) be a smooth manifold and let is an \( \mathcal{F} \subset T\,M \) be an integrable subbundle. If \( \pi : \tilde{M} \to M \) is a covering and \( \tilde{\mathcal{F}} \subset T\,\tilde{M} \) is the lift of \( \mathcal{F} \) given by a following diagram
\[
\begin{array}{ccc}
\tilde{\mathcal{F}} & \hookrightarrow & T\tilde{M} \\
\downarrow & & \downarrow \\
\mathcal{F} & \hookrightarrow & TM
\end{array}
\]
then \( \tilde{\mathcal{F}} \) is integrable.

Definition 8.1. In the above situation we say that a foliation \((\tilde{M}, \tilde{\mathcal{F}})\) is the induced by \( \pi \) covering of \((M, \mathcal{F})\) or the \( \pi \)-lift of \((M, \mathcal{F})\).

Remark 8.2. The \( \pi \)-lift of a foliation is described in [10].

Definition 8.3. If \((M, \mathcal{F})\) is a foliation and \( U \subset M \) be an open subset then there is a restriction \((U, \mathcal{F}|_U)\) of \((M, \mathcal{F})\).

Remark 8.4. It is proven in [4] that any restriction of foliation induces an injective *-homomorphism
\[
C^*_r(U, \mathcal{F}|_U) \hookrightarrow C^*_r(M, \mathcal{F}). \tag{8.1}
\]

8.2 Extended algebra of foliation

Let \((M, \mathcal{F})\) be a foliation \( x \in M \), and let us consider a representation, \( \rho_x \) given by (5.2), i.e.
\[
(\rho_x(f) \xi)(\gamma) = \int_{\gamma \gamma_1 = \gamma} f(\gamma_1) \xi(\gamma_2) \quad \forall \xi \in L^2(\mathcal{G}_x).
\]

Denote by \( \mathcal{H} = \bigoplus_{x \in M} L^2(\mathcal{G}_x) \). Representations \( \rho_x \) yield an inclusion
\[
C^*_r(M, \mathcal{F}) \hookrightarrow B(\mathcal{H}) \tag{8.2}
\]
If \( C_b (M) \) is a \( C^* \)-algebra of bounded continuous functions then for any \( x \in M \) there is the natural representation \( C_b (M) \to B \left( L^2 (\mathcal{G}_x) \right) \) given by

\[
a \mapsto (\xi \mapsto a (x) \xi) \text{ where } a \in C_b (M), \, \xi \in L^2 (\mathcal{G}_x).
\]

From this fact it turns out that there is the natural faithful representation

\[
C_b (M) \to B (\mathcal{H}).
\]  \hspace{1cm} (8.3)

Let us consider both \( C^*_r (M, \mathcal{F}) \) and \( C_b (M) \) as subalgebras of \( B (\mathcal{H}) = B \left( \bigoplus_{x \in M} L^2 (\mathcal{G}_x) \right) \).

**Definition 8.5.** The \( C^* \)-subalgebra of \( B (\mathcal{H}) \) generated by images of \( C^*_r (M, \mathcal{F}) \) and \( C_b (M) \) given by \( (8.2), (8.3) \) is said to be the extended algebra of the foliation \((M, \mathcal{F})\). The extended algebra will be denoted by \( E^* (M, \mathcal{F}) \).

**Lemma 8.6.** An algebra of a foliation \( C^*_r (M, \mathcal{F}) \) is an essential ideal of an extended algebra \( E^* (M, \mathcal{F}) \) of the foliation.

**Proof.** If \( a \in E^* (M, \mathcal{F}) \setminus C^*_r (M, \mathcal{F}) \) then there are \( x \in M, z \in \mathbb{C} \) and \( k \in K \left( L^2 (\mathcal{G}_x) \right) \) such that

\[
z \neq 0, \quad \rho_x (a) \xi = (z + k) \xi, \, \forall \xi \in L^2 (\mathcal{G}_x).
\]  \hspace{1cm} (8.4)

From \( (8.4) \) it follows that \( C^*_r (M, \mathcal{F}) \) is an ideal. If \( a \in C_b (M) \) is not trivial then there is \( x \in M \) such that \( a (x) \neq 0 \). On the other hand there is \( b \in C^*_r (M, \mathcal{F}) \) such that \( \rho_x (b) \neq 0 \). However \( \rho_x (ab) = a (x) \rho_x (b) \neq 0 \). It turns out that \( C^*_r (M, \mathcal{F}) \) is an essential ideal. \(\square\)

### 8.3 Finite-fold coverings

**8.7.** If \( \gamma : [0, 1] \to M \) is a path which corresponds to an element of the holonomy groupoid then we denote by \([\gamma]\) its equivalence class, i.e. element of groupoid. There is the space of half densities \( \Omega^{1/2} \) on \( M \) which is a lift of the space of half densities \( \Omega^{1/2}_M \) on \( M \). If \( L \) is a leaf of \((M, \mathcal{F})\), \( L' = \pi^{-1} (L) \) then a space \( \tilde{L} \) of holonomy covering of \( L \) coincides with the space of the holonomy covering of \( L' \). It turns out that \( L^2 (\tilde{G}_{\tilde{x}}) \approx L^2 (\tilde{G}_{\pi (\tilde{x})}) \) for any \( \tilde{x} \in \tilde{M} \). If \( \mathcal{G} \) (resp. \( \tilde{\mathcal{G}} \)) is a holonomy groupoid of \((M, \mathcal{F})\) (resp. \((\tilde{M}, \tilde{\mathcal{F}})\)) then there is a map \( \pi_{\mathcal{G}} : \tilde{\mathcal{G}} \to \mathcal{G} \) given by

\[
[\tilde{\gamma}] \mapsto [\pi \circ \tilde{\gamma}]
\]

The map \( \pi_{\mathcal{G}} : \tilde{\mathcal{G}} \to \mathcal{G} \) induces a natural involutive homomorphism

\[
C^\infty_c (\mathcal{G}, \Omega^{1/2}_M) \hookrightarrow C^\infty_c (\tilde{\mathcal{G}}, \Omega^{1/2}_\tilde{M})
\]

Completions of \( C^\infty_c (\mathcal{G}, \Omega^{1/2}_M) \) and \( C^\infty_c (\tilde{\mathcal{G}}, \Omega^{1/2}_\tilde{M}) \) with respect to given by \( (5.3) \) norms gives an injective *- homomorphism \( C^*_r (M, \mathcal{F}) \hookrightarrow C^*_r (\tilde{M}, \tilde{\mathcal{F}}) \) of \( C^* \)-algebras. The action
of the group $G \left( \tilde{M} \mid M \right)$ of covering transformations on $\tilde{M}$ naturally induces an action of $G \left( \tilde{M} \mid M \right)$ on $C^{\ast}_{r} \left( \tilde{M}, \tilde{F} \right)$ such that $C^{\ast}_{r} \left( M, F \right) = C^{\ast}_{r} \left( \tilde{M}, \tilde{F} \right) \bigotimes G \left( \tilde{M} \mid M \right)$.

8.8. If $\pi : \tilde{X} \to X$ is a finite-fold covering of compact Hausdorff spaces then there is a finite set $\{ \mathcal{U}_{i} \subset X \}_{i \in I}$ of connected open subsets of $X$ evenly covered by $\pi$ such that $X = \bigcup_{i \in I} \mathcal{U}_{i}$. There is a partition of unity, i.e.

$$1_{C(X)} = \sum_{i \in I} a_{i}$$

where $a_{i} \in C(X)_{+}$ is such that $\text{supp } a_{i} \subset \mathcal{U}_{i}$. Denote by $e_{i} = \sqrt{a_{i}} \in C(X)_{+}$. For any $i \in I$ we select $\tilde{U}_{i} \subset \tilde{X}$ such that $\tilde{U}_{i}$ is homomorphically mapped onto $\mathcal{U}_{i}$. If $\tilde{e}_{i} \in C(\tilde{X})$ is given by

$$\tilde{e}_{i}(\tilde{x}) = \begin{cases} e_{i}(\pi(\tilde{x})) & \tilde{x} \in \tilde{U}_{i} \\ 0 & \tilde{x} \notin \tilde{U}_{i} \end{cases}$$

and $G = G \left( \tilde{X} \mid X \right)$ then

$$1_{C(\tilde{X})} = \sum_{(g, i) \in G \times I} g e_{i}^{2},$$

$$\tilde{e}_{i}(g \tilde{e}_{i}) = 0; \text{ for any nontrivial } g \in G.$$ 

If $\tilde{I} = G \times I$ and $\tilde{e}_{(g, i)} = g \tilde{e}_{i}$ the from the above equation it turns out

$$1_{C(\tilde{X})} = \sum_{i \in I} \tilde{e}_{i} \langle \tilde{e}_{i} \rangle.$$  \hfill (8.5)

**Theorem 8.9.** Let $(M, F)$ be a foliation, and let $\pi : \tilde{M} \to M$ is a finite-fold covering. Let $(\tilde{M}, \tilde{F})$ be a covering of $(M, F)$ induced by $\pi$. If $\tilde{M}$ is compact then the triple

$$\left( C^{\ast}_{r} (M, F), C^{\ast}_{r} (\tilde{M}, \tilde{F}), G = G \left( \tilde{M} \mid M \right) \right)$$

is a noncommutative finite-fold covering.

**Proof.** Let $E^{\ast}(M, F)$ (resp. $E^{\ast}(\tilde{M}, \tilde{F})$) be an extended algebra of the foliation $(M, F)$ (resp. $(\tilde{M}, \tilde{F})$). Denote by $A \overset{\text{def}}{=} C^{\ast}_{r} (M, F)$, $B \overset{\text{def}}{=} E^{\ast}(M, F)$, $\tilde{B} \overset{\text{def}}{=} E^{\ast}(\tilde{M}, \tilde{F})$. Both $B$ and $\tilde{B}$ are unital. From the Lemma 8.6 it turns out that $A$, (resp. $C^{\ast}_{r}(\tilde{M}, \tilde{F})$) is an essential ideal of $B$, (resp. $\tilde{B}$), i.e. these algebras satisfy to the condition (a) of the Definition 2.4. From the Theorem 1.2 it turns out that $C(\tilde{M}) = C_{b}(\tilde{M})$ is a finitely generated $C(M) = C_{b}(M)$ module. Moreover from (8.5) it turns out that there is a finite set $\{ e_{i} \}_{i \in I}$ such that

$$1_{C(\tilde{M})} = \sum_{i \in I} \tilde{e}_{i} \langle \tilde{e}_{i} \rangle.$$

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Clearly \(1_{C(\hat{M})} = 1_{E^*(\hat{M}, \bar{F})} = 1_{\bar{B}}\). It turns out that any \(\tilde{b} \in \bar{B}\) is given by

\[
\tilde{b} = \sum_{i \in I} \tilde{e}_ib_i,
\]

\[
b_i = \langle \tilde{b}, \tilde{e}_i \rangle _{\bar{B}} \in B,
\]

i.e. \(\bar{B}\) is a finitely generated right \(B\) module. From the Kasparov Stabilization Theorem \([2]\) it turns out that \(\bar{B}\) is a projective \(B\) module. From \(\bar{B}^G = B\) it follows that the triple \((B, \bar{B}, G)\) is an unital noncommutative finite-fold covering, i.e. the condition (b) of the Definition \([2,4]\) is satisfied. Let \(\hat{A}\) be a \(C^*\)-algebra which satisfies to the condition (c) of the Definition \([2,4]\), i.e.

\[
\hat{A} = \{ a \in \bar{B} \mid \langle \bar{b}, a \rangle _{\bar{B}} \subset A \}.
\]

If \(\tilde{a} \in C^*_r(\hat{M}, \bar{F})\) then from the Lemma \([8.6]\) it follows that \(\tilde{b}\tilde{a} \in C^*_r(\hat{M}, \bar{F})\) for any \(\tilde{b} \in \bar{B} = E^*(\hat{M}, \bar{F})\), hence

\[
\langle \tilde{b}, \tilde{a} \rangle _{\bar{B}} = \sum_{g \in G} g(\bar{b}a) \in C^*_r(\hat{M}, \bar{F}).
\]

Otherwise \(\langle \tilde{b}, \tilde{a} \rangle _{\bar{B}}\) is \(G\)-invariant, so \(\langle \tilde{b}, \tilde{a} \rangle _{\bar{B}} \in A\). It turns out \(\tilde{a} \in \hat{A}\), hence \(C^*_r(\hat{M}, \bar{F}) \subset \hat{A}\). If \(\tilde{a} \in \hat{A}\backslash C^*_r(\hat{M}, \bar{F})\) then there is \(x \in \hat{M}, z \in \mathbb{C}\) and \(k \in K \left( L^2 \left( \hat{G}_x \right) \right) \) such that

\[
z \neq 0,
\]

\[
\rho_x (\tilde{a}) = z1_{B(L^2(\hat{G}_x))} + k.
\]

It follows that

\[
\rho_{\pi(x)} (\langle \tilde{a}, \tilde{a} \rangle _{\bar{B}}) = |z|^2 1_{B(L^2(\hat{G}_x))} + k'
\]

where \(k' \in K \left( L^2 \left( \hat{G}_x \right) \right) \). From the above equation it turns out that \(\langle \tilde{a}, \tilde{a} \rangle _{\bar{B}} \notin A\), hence \(\tilde{a} \notin \hat{A}\). From this contradiction it turns out

\[
\hat{A}\backslash C^*_r(\hat{M}, \bar{F}) = \emptyset,
\]

\[
\hat{A} \subset C^*_r(\hat{M}, \bar{F}).
\]

In result one has and \(\hat{A} = C^*_r(\hat{M}, \bar{F})\).

\[\square\]

Example 8.10. Let us consider a foliation \((T^2, \mathcal{F}_\theta)\) given by the Example \([5,2]\). From the Example \([6,7]\) and the Theorem \([6,3]\) it turns out

\[
C^*_r(\mathbb{T}^2, \mathcal{F}_\theta) = C \left( \mathbb{T}^2_\theta \right) \otimes K.
\]
There is a homeomorphism $T^2 = S^1 \times S^1$. Let $m \in \mathbb{N}$ be such that $m > 1$, and let $\tilde{T}^2 \approx T^2$ and let us consider an $m^2$-fold covering $\pi : \tilde{T}^2 \to T$ given by
\[
\tilde{T}^2 \approx S^1 \times S^1 \xrightarrow{(\times m, \times m)} S^1 \times S^1 \approx T^2
\]
where $\times m$ is an $m$-listed covering of the circle $S^1$. If $\tilde{\nu} = \nu$ is the $\pi$-lift of $(T^2, \mathcal{F}_\theta)$ then
\[
(\tilde{\nu}, \tilde{\mathcal{F}}) = \left(\tilde{T}^2, \mathcal{F}_{\theta/m^2}\right)
\]
From the Theorem 8.9 it follows that the triple
\[
(C^* \left(T^2, \mathcal{F}_\theta\right), C^* \left(\tilde{T}^2, \mathcal{F}_{\theta/m^2}\right), C \left(\tilde{T}^2 \mid T^2\right) = \mathbb{Z}_{m^2})
\]
is a noncommutative finite-fold covering. From the Theorem 7.1 it turns out that the triple
\[
\left(C^* \left(T^2, \mathcal{F}_\theta\right) \otimes \mathcal{K}, C^* \left(\tilde{T}^2, \mathcal{F}_{\theta/m^2}\right) \otimes \mathcal{K}, \mathbb{Z}_{m^2}\right)
\]
is a noncommutative finite-fold covering. Otherwise in [7] it is proven that the triple
\[
\left(C \left(T^2_\theta \right), C \left(T^2_{\theta/m^2}\right), \mathbb{Z}_{m^2}\right)
\]
is a noncommutative finite-fold covering, and from the Theorem 7.1 it turns out that the triple
\[
\left(C \left(T^2_\theta \right) \otimes \mathcal{K}, C \left(T^2_{\theta/m^2}\right) \otimes \mathcal{K}, \mathbb{Z}_{m^2}\right)
\]
is a noncommutative finite-fold covering. There are natural *-isomorphisms
\[
\begin{align*}
C \left(T^2_\theta \right) \otimes \mathcal{K} & \approx C^* \left(T^2, \mathcal{F}_\theta\right), \\
C \left(T^2_{\theta/m^2}\right) \otimes \mathcal{K} & \approx C^* \left(\tilde{T}^2, \mathcal{F}_{\theta/m^2}\right).
\end{align*}
\]
From the above isomorphism it turns out that a noncommutative finite fold covering
\[
\left(C^* \left(T^2, \mathcal{F}_\theta\right), C^* \left(\tilde{T}^2, \mathcal{F}_{\theta/m^2}\right), \mathbb{Z}_{m^2}\right)
\]
is equivalent to
\[
\left(C \left(T^2_\theta \right) \otimes \mathcal{K}, C \left(T^2_{\theta/m^2}\right) \otimes \mathcal{K}, \mathbb{Z}_{m^2}\right).
\]

### 8.4 Infinite coverings

Let $(M, \mathcal{F})$ be a Hausdorff foliation, on a compact $M$ and let
\[
\mathcal{G}_M = \{M = M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_1 \leftarrow \cdots\} \quad (8.6)
\]
be a sequence of regular finite fold coverings. From the Theorem 8.9 it follows that there is an (algebraical) finite covering sequence
\[
\mathcal{G}_{C^*_r(M, \mathcal{F})} = \{C^* \left(M, \mathcal{F}\right) = C^* \left(M_0, \mathcal{F}_0\right) \rightarrow C^* \left(M_1, \mathcal{F}_1\right) \rightarrow \cdots \rightarrow C^* \left(M_n, \mathcal{F}_n\right) \rightarrow \cdots\} \in \mathbb{P}_{\text{filAlg}}
\]
and for any $n \in \mathbb{N}$ the there is an isomorphism of covering transformation groups

$$G \left( C^*_r (M_n, F_n) \mid C^*_r (M, F) \right) \cong C \left( M_n \mid M \right).$$

Denote by $G_n \overset{\text{def}}{=} C \left( M_n \mid M \right)$ and  $\hat{G} \overset{\text{def}}{=} \lim \leftarrow G_n$ an inverse limit of groups \cite{13}. Denote by $\hat{M} \overset{\text{def}}{=} \lim \leftarrow M_n$ the inverse limit of topological spaces \cite{13} and $C^*_r (\hat{M}, \hat{F}) \overset{\text{def}}{=} \lim \leftarrow C^*_r (M_n, F_n)$ $C^*$-inductive limit \cite{9}. In \cite{7} it is proven that there is the disconnected inverse limit of $\hat{G}_M$ i.e. a topological space $\hat{M}$ and a bicontinuous map $\hat{M} \rightarrow \hat{M}$ such that for any $n \in \mathbb{N}$ the composition map $\pi_n : \hat{M} \rightarrow M \rightarrow M_n$ of the following diagram.

\[
\begin{array}{ccc}
\lim \leftarrow \downarrow \hat{G}_M \\
\downarrow \downarrow \\
\hat{M} \\
\downarrow \\
M_0 & \leftarrow M_1 & \leftarrow M_2 & \ldots
\end{array}
\]

is a covering. In the above diagram $\lim \leftarrow \downarrow \hat{G}_M$ is the topological inverse noncommutative limit (cf. \ref{4.3}) which is a connected component of $\hat{M}$. The group $\hat{G} = \lim \leftarrow G \left( M_n \mid M \right)$ naturally acts on $\hat{M}$ and $\hat{M}$. In \cite{7} it is proven that

$$\hat{M} = \bigsqcup_{\pi \in \hat{G}} \left( \lim \leftarrow \downarrow \hat{G}_M \right) \quad (8.8)$$

where $\pi \in \hat{G}$ is a set of representatives of $\hat{G} / G \left( \lim \leftarrow \downarrow \hat{G}_M \mid M \right)$. The group $G \left( \lim \leftarrow \downarrow \hat{G}_M \mid M \right)$ is a maximal among subgroups $G \subset \hat{G}$ such that

$$G \left( \lim \leftarrow \downarrow \hat{G}_M \right) = \lim \leftarrow \downarrow \hat{G}_M.$$

**8.4.1 Equivariant geometric representation**

Denote by

$$\mathcal{H} = \bigoplus_{\pi \in \hat{M}} L^2 (\pi)$$

the Hilbert norm completion of an algebraic direct sum. There is a natural isomorphism $L^2 (\pi) \cong L^2 \left( \hat{G}_{\pi_n (\pi)} \right)$. It turns out that for any $a_n \in C^*_r (M_n, F_n)$ and $\pi \in \hat{M}$ one can define $\rho_{\pi} (a_n)$ as isomorphic image of $\rho_{\pi_n (\pi)} (a_n)$, so there is a representation
\( \rho_\pi : C^*_\pi (M_n, \mathcal{F}_n) \to L^2 (\mathbb{C}_\pi) \). The direct sum of representations \( \rho_\pi \) yields a representation \( \rho_n : C^*_\pi (M_n, \mathcal{F}_n) \to B (\mathcal{H}) \). There is a following commutative diagram.

\[
\begin{array}{ccc}
C^*_\pi (M_n, \mathcal{F}_n) & \xrightarrow{\rho_n} & C^*_\pi (M_{n+1}, \mathcal{F}_{n+1}) \\
\downarrow & & \downarrow \\
B (\mathcal{H}) & & B (\mathcal{H})
\end{array}
\]

From the above diagram it follows that there is an equivariant representation

\[
\pi_{geom} : \widehat{C}^*_\pi (M, \mathcal{F}) = \lim_{\rightarrow} C^*_\pi (M_n, \mathcal{F}) \to B (\mathcal{H}) \tag{8.9}
\]

Otherwise if \( C^*_\pi (\overline{M}, \overline{\mathcal{F}}) \) is the \( \overline{\pi} \)-lift of \( C^*_\pi (M_n, \mathcal{F}_n) \) then there is the natural representation

\[
C^*_\pi (\overline{M}, \overline{\mathcal{F}}) \hookrightarrow B (\overline{\mathcal{H}}) \tag{8.10}
\]
given by (5.2).

**Definition 8.11.** We say that representation (8.9) is a geometric representation of the sequence (8.7).

### 8.4.2 Inverse noncommutative limit

From (8.10) it follows that any \( \pi \in C^*_\pi (\overline{M}, \overline{\mathcal{F}}) \) can be regarded as element in \( B (\overline{\mathcal{H}}) \), i.e. \( \pi \in B (\overline{\mathcal{H}}) \).

**Lemma 8.12.** Let \( \overline{U} \subset \overline{M} \) be a connected open subset homeomorphically mapped onto \( U = \pi (\overline{U}) \). Let \( \overline{b} : C^*_\pi (\overline{U}, \overline{\mathcal{F}}_{\overline{U}}) \hookrightarrow C^*_\pi (\overline{M}, \overline{\mathcal{F}}) \) be given by (8.1) \(*\)-homomorphism. If \( \overline{b} \in C^*_\pi (\overline{U}, \overline{\mathcal{F}}_{\overline{U}}) \) and \( \pi = \overline{b} (\overline{b}) \) then \( \pi \in B (\overline{\mathcal{H}}) \) is a special element of the sequence (8.7).

**Proof.** Denote by \( U_n = \pi (\overline{U}) \). For any \( n \in \mathbb{N}^0 \) there is the natural isomorphism

\[
\varphi_n : C^*_\pi (\overline{U}, \overline{\mathcal{F}}_{\overline{U}}) \cong C^*_\pi (U_n, \mathcal{F}_n | U_n)
\]

such that

\[
\sum_{g \in \ker (G \to G_n)} g \overline{\vartheta} (\overline{b}) = \theta_n \circ \varphi_n (\overline{b}) \tag{8.11}
\]

where \( \theta_n : C^*_\pi (U_n, \mathcal{F}_n | U_n) \hookrightarrow C^*_\pi (M_n, \mathcal{F}_n) \) is given by (8.1). It follows that

\[
a_n = \sum_{g \in \ker (G \to G_n)} g \overline{a} = \sum_{g \in \ker (G \to G_n)} g \overline{\vartheta} (\overline{b}) = \theta_n \circ \varphi_n (\overline{b}) \in C^*_\pi (M_n, \mathcal{F}_n)
\]

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i.e. \( \mathcal{F} \) satisfies to the condition (a) of the Definition 3.3. If \( f : \mathbb{R} \to \mathbb{R} \) is given by (3.2) then for any \( n \in \mathbb{N}_0 \) and for any \( z \in C^*_\tau (M, \mathcal{F}) \) following conditions hold

\[
    b_n = \sum_{g \in \ker(G \to G_n)} g(z\bar{z}^*) = z(\theta_n \circ \varphi_n(\overline{b}))z^* \in C^*_\tau (M, \mathcal{F}),
\]

\[
    c_n = \sum_{g \in \ker(G \to G_n)} g(z\bar{z}^*)^2 = (z(\theta_n \circ \varphi_n(\overline{b}))z^*)^2 = b^2_n \in C^*_\tau (M, \mathcal{F}),
\]

\[
    d_n = \sum_{g \in \ker(G \to G_n)} gf(z\bar{z}^*) = f(z(\theta_n \circ \varphi_n(\overline{b}))z^*) \in C^*_\tau (M, \mathcal{F}),
\]

i.e. the condition (b) of the Definition 3.3 holds. From \( c_n = b^2_n \) it turns out the condition (c) of the Definition 3.3 holds. \( \Box \)

Denote by \( \overline{C^*_\tau (M, \mathcal{F})} \) the disconnected inverse limit of the sequence \( \mathfrak{S}_{C^*_\tau (M, \mathcal{F})} \in \text{FinAlg} \) given by (8.7), with respect to the geometric representation \( \pi_{\text{geom}} \) (cf. (8.9)).

**Corollary 8.13.** Following condition holds

\[ C^*_\tau (\overline{M}, \mathcal{F}) \subset C^*_\tau (M, \mathcal{F}). \]

**Proof.** The \( C^* \)-norm completion of the algebra generated by special elements given by the Lemma 8.12 coincides with \( C^*_\tau (\overline{M}, \mathcal{F}) \). \( \Box \)

8.14. Denote by \( p = \text{dim } \mathcal{F}, q = \text{codim } \mathcal{F} \). Let us consider a foliation chart \( \phi : \mathbb{R}^{\text{dim } M} \to M \) such that leafs of the foliation \( \left( \mathbb{R}^{\text{dim } M}, \mathcal{F}_p \right) \) given by (5.1) are mapped into leafs of \( (M, \mathcal{F}) \). The set

\[ \mathcal{V} = \phi(\{0\} \times \mathbb{R}^q) \]  

(8.12)

is a submanifold of \( M \) which is traversal to \( \mathcal{F} \). If \( b \in C_0(\mathbb{R}^p \times \mathbb{R}^q)_+ = C_0\left( \mathbb{R}^{\text{dim } M}_+ \right) \) is a positive function such that \( b \left( \mathbb{R}^{\text{dim } M} \right) \subset [0, 1] \) then \( b \) can be represented by a following way

\[ b = \left( 1_{C_b(\mathbb{R}^p)} \otimes b' \right) b'' \]

(8.13)

where \( b' \in C_0(\mathbb{R}^q)_+ \) and \( b'' \in C_0\left( \mathbb{R}^{\text{dim } M} \right) \) and following conditions hold:

\[ b' \left( \mathbb{R}^q \right) \subset [0, 1], \]

\[ b'' \left( \mathbb{R}^{\text{dim } M} \right) \subset [0, 1]. \]

Clearly \( b', b'' \) correspond to contractible operators. If \( \mathcal{U} = \phi\left( \mathbb{R}^{\text{dim } M} \right) \) then any \( a \in C_0(\mathcal{U}) \) is an image of \( b \in C_0(\mathbb{R}^p \times \mathbb{R}^q) \) and the factorization (8.13) corresponds to the factorization

\[ a = a'a'' \]

(8.14)
where \( a' \) (resp. \( a'' \)) corresponds to \( 1_{C^\infty_0(\mathbb{R}^p)} \otimes b' \) (resp. \( b'' \)). The element \( a' \) can be regarded as an element of \( C^\infty_0(Y) \). The manifold \( M \) is compact, so there is a finite set 

\[
\left\{ \phi_i : \mathbb{R}^{\dim M} \to U_i \subset M \right\}_{i \in I}
\]

of foliation charts such that \( M = \bigcup_{i \in I} U_i \). There is a partition of unity 

\[
1_{C^\infty_0(M)} = \sum_{i \in I} a_i = \sum_{i \in I} a'_i a''_i \quad (8.15)
\]

where \( a_i \in C^\infty_0(M) \), \( \text{supp} \ a_i \subset U_i \), and \( a_i = a'_i a''_i \) is a factorization given by (8.14). From the Example 6.5, it follows that \( C^\infty_0(\mathbb{R}^{\dim M}, F) \cong C^\infty_0(\mathbb{R}^p) \otimes K \) similarly one has \( C^\infty_0(U_i, F_{U_i}) \cong C^\infty_0(V_i) \otimes K \) where \( V_i \) is a transversal manifold given by (8.12). The formula \( C^\infty_0(U_i, F_{U_i}) \cong C^\infty_0(V_i) \otimes K \) implies that there is a separable Hilbert space \( H \) with an orthogonal basis \( \xi_1, \ldots, \xi_j, \ldots \) such that elements of \( C^\infty_0(U_i, F_{U_i}) \) are operators 

\[
C_b(V_i) \otimes H \to C^\infty_0(V_i) \otimes H.
\]

Above operators can be regarded as compact operators in \( K(\ell^2(C^\infty_0(V_i))) \). If \( p_j \in K = K(H) \) is a projector along \( \xi_j \) then from 

\[
1_{M(K)} = \sum_{j=1}^{\infty} p_j
\]

it follows that 

\[
1_{M(C^\infty_0(V_i) \otimes K)} = 1_{C^\infty_0(V_i)} \otimes \sum_{j=1}^{\infty} p_j
\]

where the sum of the above series implies the strict convergence \[2\]. From above formulas it follows that the series 

\[
1_{C(M)} = 1_{M(C^\infty_0(M,F))} = \sum_{i \in I} a'_i a''_i \otimes \sum_{j=1}^{\infty} p_j \quad (8.16)
\]

is strictly convergent.

**Remark 8.15.** Clearly \( a'_i a''_i \otimes p_j, a''_i \otimes p_j \in C^\infty_0(M, F) \).

**8.16.** Let us consider the geometric representation 

\[
\pi_{\text{geom}} : C^\infty_0(M,F) = \lim_{\mathbb{N}} C^\infty_n(M_n, F_n) \to B(H)
\]

given by (8.9). Suppose that \( \overline{\pi} \in B(\overline{H}) \) is a special element of the sequence (8.7). We would like to proof that 

\[
\overline{\pi} \in C^\infty_0(M,F).
\]  

The equation (8.17) follows from three facts:
1. There is a decomposition
\[ \pi = \sum_{\ell' \in I} \sum_{\ell'' \in \ell} \sum_{j=1}^{\infty} (a_{\ell'} a_{\ell''} \otimes p_j) \pi (a_{\ell'} a_{\ell''} \otimes p_k), \] (8.18)

2. One has \((a_{\ell'} a_{\ell''} \otimes p_j) \pi (a_{\ell'} a_{\ell''} \otimes p_k) \in C^* (\mathcal{M}, \mathcal{F})\) for any \(i', i'' \in I, j, k \in \mathbb{N}\).

3. The decomposition (8.18) is norm convergent.

The decomposition (8.18) follows from (8.16). Facts 2 and 3 are proven below.

8.17. Let us fix \(i', i'' \in I\). The subset
\[ \mathcal{G}_{i,i'} = \{ \gamma \in \mathcal{G} (M, F) \mid s (\gamma) = U_{i'}, r (\gamma) = U_{i''} \} \]
can be decomposed into connected components, i.e.
\[ \mathcal{G}_{i,i'} = \bigcup_{\lambda \in \Lambda_{i,i''}} \mathcal{G}_{\lambda}. \] (8.19)

If \(\mathcal{W}_{i'} \subset \mathcal{V}_{i'}\) and \(\mathcal{W}_{i''} \subset \mathcal{V}_{i''}\) are given by
\[
\mathcal{W}_{i'} = \{ x \in \mathcal{V}_{i'} \mid \exists \gamma \in \mathcal{G}_{\lambda}; x \in s (\gamma) \}, \\
\mathcal{W}_{i''} = \{ x \in \mathcal{V}_{i''} \mid \exists \gamma \in \mathcal{G}_{\lambda}; x \in r (\gamma) \}
\]
then \(\mathcal{G}_{\lambda}\) corresponds to a diffeomorphism \(\varphi_{\lambda} : \mathcal{W}_{i'} \rightarrow \mathcal{W}_{i''}\). Below we drop the index \(\lambda\) and \(\mathcal{W}', \mathcal{W}''\) instead of \(\mathcal{W}_{i'}, \mathcal{W}_{i''}\). For any subset \(\mathcal{U}\) subset of a topological set \(X\) denote by \(\partial \mathcal{U} = \overline{\mathcal{U}} \setminus \mathcal{U} \subset X\) where \(\overline{\mathcal{U}}\) is a closure of \(\mathcal{U}\). Any \(a_{\mathcal{W}'} \in C_0 (\mathcal{W}')\) can be regarded as element of \(C_b (\mathcal{W}')\) such that \(a_{\mathcal{W}'} (\partial \mathcal{W}') = \{0\}\). Clearly \(\partial \mathcal{W}' \subset \partial \mathcal{W}'' \cup \partial \varphi_{\lambda}^{-1} (\mathcal{W}'').\) If \(b' \in C_0 (\mathcal{W}')\) and \(b'' \in C_0 (\mathcal{W}'')\) then the isomorphism \(\varphi_{\lambda}\) gives a product \(b'b'' \in C_b (\mathcal{W})\) such that \((b'b'') (\partial \mathcal{W}'') = \{0\}\). It follows that \(b'b'' \in C_0 (\mathcal{W}')\). Let us define an element \(y \in C^* (M, F)\) such that \(\text{supp} \ y \in \mathcal{G}_{\lambda}\) and \(y\) is given by
\[ y : C_0 (\mathcal{W}') \otimes \mathcal{H} \rightarrow C_0 (\mathcal{W}'') \otimes \mathcal{H}, \]
\[ 1_{C_0 (\mathcal{W}')} \otimes \tilde{e}_{i'} \mapsto \delta_{ij} \sqrt{a_{i'}}, \sqrt{a_{i'}} \otimes \tilde{e}_k; \ (\delta_{ij} = 1, \ \delta_{jk} = 0 \text{ if } j \neq k), \] (8.20)
and note that \(\sqrt{a_{i'}}, \sqrt{a_{i'}} \in C_0 (\mathcal{W}'').\) Moreover \(y\) can be regarded as a compact operator in \(K (L^2 (C_0 (\mathcal{W}')), L^2 (C_0 (\mathcal{W}'')))\). Denote by
\[ e_i = \sqrt{a_{i'}} \otimes p_j \in C^*_c (U_{i'}, F_{i'}) \subset C^*_c (M, F) \] (8.21)
where \(p_j \in K = K (\mathcal{H})\) is a projector along \(\tilde{e}_{i'}\), and let
\[ z = y^* + e \] (8.22)
Let \( \mathbf{a} \in B(\mathcal{H})_+ \) be a special element of the sequence \((8.7)\). From the (a) of the Definition \(3.3\) it turns out

\[
a = \sum_{g \in G} g \mathbf{a} \in C^*_r(\mathcal{M}, \mathcal{F})
\]

If \( \mathcal{G}_\varepsilon \subset \mathcal{G}(M, \mathcal{F}) \) is a set of path which a homotopic to a trivial path \( \gamma ([0, 1]) \in U_\varepsilon \) and \( s(\gamma), r(\gamma) \in U_\varepsilon \) then the restriction of \( z\mathbf{a}^* \) on \( \mathcal{G}_\varepsilon \) is a "rank-one" positive operator, given by

\[
zaz^*|_{\mathcal{G}_\varepsilon} = \theta_{\varepsilon^\mu} \mathbf{a} \otimes p_j,
\]

where \( \theta_{\varepsilon^\mu} \in C_0(\mathcal{W}_\varepsilon)^+ \). If \( f_\varepsilon : \mathbb{R} \rightarrow \mathbb{R} \) is given by \((3.2)\) then from (b) of the Definition \(3.3\) it turns out that for any \( n \in \mathbb{N}^0 \) following conditions hold

\[
b_n = \sum_{g \in \ker(\tilde{G} \rightarrow G(A_n | A))} g \mathbf{b} \in C^*_r(M_n, \mathcal{F}_n),
\]

\[
c_n = \sum_{g \in \ker(\tilde{G} \rightarrow G(A_n | A))} g \mathbf{b}^2 \in C^*_r(M_n, \mathcal{F}_n),
\]

\[
d_n = \sum_{g \in \ker(\tilde{G} \rightarrow G(A_n | A))} g f_\varepsilon (\mathbf{b}) \in C^*_r(M_n, \mathcal{F}_n).
\]

From the condition (c) of the Definition \(3.3\) it follows that for any \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) such that

\[
\|b^2_n - c_n\| < \varepsilon
\]

for any \( n \geq N \). If \( \mathcal{G}_\varepsilon |_{\mathcal{G}_\varepsilon} = \pi^{-1}(\mathcal{G}_\varepsilon) \) then similarly to \((8.23)\) one has

\[
b_n |_{\mathcal{G}_\varepsilon} = b'_n \otimes p_j,
\]

\[
c_n |_{\mathcal{G}_\varepsilon} = c'_n \otimes p_j,
\]

\[
d_n |_{\mathcal{G}_\varepsilon} = d'_n \otimes p_j,
\]

where \( b'_n, c'_n, d'_n \in C_0(\pi^{-1}(\mathcal{W}_\varepsilon))^+ \). If \( \mathbf{b} \in C_0(\pi^{-1}(\mathcal{W}_\varepsilon))^+ \) is a strong limit

\[
\mathbf{b} = \lim_{n \rightarrow \infty} b'_n
\]

then following condition holds:

\[
b'_n = \sum_{g \in \ker(\tilde{G} \rightarrow G_n)} g \mathbf{b},
\]

\[
c'_n = \sum_{g \in \ker(\tilde{G} \rightarrow G_n)} g \mathbf{b}^2,
\]

\[
d'_n = \sum_{g \in \ker(\tilde{G} \rightarrow G_n)} g f_\varepsilon (\mathbf{b}).
\]

From \( \|b^2_n - c_n\| < \varepsilon \) it follows that \( \|b^2_n - c'_n\| < \varepsilon \). Now we need a following lemma.
Lemma 8.18. Suppose that \( X \) is a locally compact Hausdorff space. Let \( \overline{\pi} \in C_0 (\overline{X})' \) be such that following conditions hold:

(a) If \( f_\varepsilon \) is given by (3.2) then following series

\[
    a_n = \sum_{g \in \ker (\hat{G} \to G_n)} g \pi,
    \quad b_n = \sum_{g \in \ker (\hat{G} \to G_n)} g \pi^2,
    \quad c_n = \sum_{g \in \ker (\hat{G} \to G_n)} g f_\varepsilon (\pi),
\]

are strongly convergent and \( a_n, b_n, c_n \in C_0 (X_n) \).

(b) For any \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) such that

\[
    \| a_n - b_n \| < \varepsilon, \quad \forall n \geq N.
\]

Then \( \pi \in C_0 (\overline{X})' \).

Corollary 8.19. If \( \overline{\pi} \in B (\overline{H})' \) is a special element of the sequence (8.7) and \( z \in C_0^+ (M, F) \) is given by (8.22) then

\[
    \overline{b} = z \pi^+ \in C_0^+ (\overline{M}, F).
\]

Proof. Let \( \varepsilon > 0 \). If \( f_\varepsilon \) is given by (3.2) and \( \overline{b}_{\varepsilon/2} = f_{\varepsilon/2} (\overline{b}) \) then

\[
    \| \overline{b} - \overline{b}_{\varepsilon/2} \| < \frac{\varepsilon}{2}.
\]

If \( \overline{b} = \overline{b} \otimes p_j \) then \( \overline{b}_{\varepsilon/2} = \overline{b}_{\varepsilon/2} \otimes p_j \) where \( \overline{b}_{\varepsilon/2} = f_{\varepsilon/2} (\overline{b}) \). From the construction 8.17 and the Lemma 8.18 it follows that if \( \overline{b} \) is given by (8.24) then \( \overline{b} \in C_0 (\pi^{-1} (W_{\varepsilon/2})) \). It follows that \( \supp \overline{b}_{\varepsilon/2} \subset \overline{\pi}^{-1} (W_{\varepsilon/2}) \) is compact. There is \( N \in \mathbb{N} \) such that for any \( n \geq N \) a restriction

\[
    \overline{\pi}_n |_{\supp \overline{b}_{\varepsilon/2}} : \supp \overline{b}_{\varepsilon/2} \simeq \overline{\pi}_n \left( \supp \overline{b}_{\varepsilon/2} \right) \subset M_n
\]

is a homeomorphism. For any \( \overline{x} \in \supp \overline{b}_{\varepsilon/2} \) there is an open neighborhood \( \overline{U} \) which is homeomorphically mapped onto \( U_\varepsilon \). Since \( \supp \overline{b}_{\varepsilon/2} \) is compact the set

\[
    F \subset \overline{G} = \left\{ g \in \overline{G} \mid g \overline{U} \bigcap \supp \overline{b}_{\varepsilon/2} \neq \emptyset \right\}
\]

is finite. If \( \overline{U}_n/2 = F \overline{U} \subset \overline{M} \) then for any \( n > N \) the set \( \overline{U}_n/2 \) is mapped homeomorphically onto \( \overline{\pi}_n \left( \overline{U}_n/2 \right) = \overline{U}_n/2 \). Let \( b_{n/2}^j \in C_0^+ (M_n, F_N) \) is given by

\[
    b_{n/2}^j = \sum_{g \in \ker (\hat{G} \to G_n)} g \overline{b}_{\varepsilon/2}.
\]
There is $b_N^\infty \in C_c^\infty (M_N, F_N)_+$ such that following conditions hold:

$$\|b_N^{1/2} - b_N^\infty\| < \frac{\varepsilon}{2}.$$

$$U_N^\infty = \{ x \in M_N \mid \exists y \in \text{supp } b_N^\infty; x = s(y) \} \subset U_N^{1/2}.$$

For any path $\gamma \in \text{supp } b_N^\infty$ there is the unique $\pi_\gamma \in \mathcal{G} (\bar{M}, \bar{F})$ such that $s(\gamma) \in \bar{U}^{1/2}$. We say this lift special. There is an element $\bar{b}^\infty \in C_c^\infty (\bar{M}, \bar{F})$ such that any "value" (half density) of $b_N^\infty$ on a path $\gamma \in \mathcal{G} (M_N, F_N)$ coincides with "value" of $\bar{b}^\infty$ on the special lift $\pi_\gamma \subset \mathcal{G} (\bar{M}, \bar{F})$ of $\gamma$. We also require that $\bar{b}^\infty$ is trivial on paths which are not special lifts. From $\|b_N^{1/2} - b_N^\infty\| < \frac{\varepsilon}{2}$ it follows that $\|\bar{b}_{\gamma/2} - \bar{b}^\infty\| < \frac{\varepsilon}{2}$ and from $\|\bar{b} - \bar{b}_{\gamma/2}\| < \frac{\varepsilon}{2}$ it follows that $\|\bar{b} - \bar{b}^\infty\| < \varepsilon$. An algebra $C^*_r (\bar{M}, \bar{F})$ is the $C^*$-norm completion of $C_c^\infty (\bar{M}, \bar{F})$ it turns out

$$\bar{b} \in C^*_r (\bar{M}, \bar{F}).$$

8.20. The described in 8.17 construction depends on indexes $'/', 'p', 'k'$ in $I$, a connected component $\mathcal{G}_\lambda \subset \mathcal{G}_{'/', 'p'}$ ($\lambda \in \Lambda_{'/', 'p'}$) in the decomposition (8.19), and $j, k \in \mathbb{N}$. Now for clarity we use $y_{'/', 'p', 'k'}$ (resp. $z_{'/', 'p', 'k'}$) instead $y$ given by (8.20) (resp. $z$ given by (8.22). Thus from the Corollary 8.19 it turns out

$$z_{'p', 'k'} \in C^*_r (\bar{M}, \bar{F}).$$

From $C^*_r (M, F) \subset M (C^*_r (\bar{M}, \bar{F}))$ it follows that $y_{'p', 'p', 'k'} \in M (C^*_r (\bar{M}, \bar{F}))$, hence

$$y_{'p', 'p', 'k'} \in C^*_r (\bar{M}, \bar{F}).$$

From (8.20) and (8.21) on has a following "formal decomposition"

$$(a_{'p'} \otimes p_{'p'}) \pi (a_{'p'} \otimes p_k) = \sum_{\lambda \in \Lambda_{'/', 'p'}} y_{'p', 'p', 'k'} z_{'p', 'p', 'k'} a_{'p'}^{*} z_{'p', 'p', 'k'} a_{'p'}^{*},$$  

(8.25)

(The "formal decomposition" word means that one should prove that the series (8.25) is norm convergent). Denote by $\Xi$ the set of quintuplets $(', 'p', 'p', 'k')$ where $'p', 'p', 'k' \in I, \lambda \in \Lambda_{'/', 'p'}$ and $j, k \in \mathbb{N}$. From (8.18) and (8.25) it follows a "formal decomposition"

$$\pi = \sum_{(', 'p', 'p', 'k') \in \Xi} a_{'p'}^{*} y_{'p', 'p', 'k'} z_{'p', 'p', 'k'} a_{'p'}^{*}.$$  

(8.26)

All terms in the series (8.26) lie in $C^*_r (\bar{M}, \bar{F})$. Any operator

$$y_{'p', 'p', 'k'} z_{'p', 'p', 'k'} a_{'p'}^{*}$$

is given by

$$1_{C_{b}(W_{'p'})} \otimes \hat{\xi}_j \mapsto \delta_{ij} \theta_{'p', 'p', 'k'} \otimes p_k$$

is given by
where $\theta_{\ell'\ell''\lambda jk} \in C_0(W_{\ell''})$. For any $\ell', \ell'' \in I$ and $\lambda \in \Lambda_{\ell'\ell''}$ the sum

$$\pi_{\ell'\ell''\lambda} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} y_{\ell'\ell''\lambda jk} z_{\ell'\ell''\lambda jk} \pi^{*}_{\ell'\ell''\lambda jk}$$  \hspace{1cm} (8.27)

can be regarded as a compact operator from $\ell^2(C_0(W_{\ell''}))$ to $\ell^2(C_0(W_{\ell''}))$, i.e. $\pi_{\ell'\ell''\lambda} \in K(\ell^2(C_0(W_{\ell''})), \ell^2(C_0(W_{\ell''})))$. If follows that for any $\delta > 0$ there is $N_\delta$ such that for any $n \geq N_\delta$ following condition holds

$$\left\| \pi_{\ell'\ell''\lambda} - \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} y_{\ell'\ell''\lambda jk} z_{\ell'\ell''\lambda jk} \pi^{*}_{\ell'\ell''\lambda jk} \right\| < \delta.$$  \hspace{1cm} (8.28)

**Lemma 8.21.** If $\pi \in B(\mathcal{H})_+$ is a special element of the sequence \((8.7)\) then $\pi \in C^*_r(\mathcal{M}, \mathcal{F})$.

**Proof.** Let $\epsilon > 0$ be a small number. Let $a = \sum_{g \in \mathcal{G}} g \pi \in C^*_r(M, \mathcal{F})$ and let $a' \in C^*_c(M, \mathcal{F})$ be a positive element such that

$$\|a - a'\| < \frac{\epsilon}{2}. \hspace{1cm} (8.29)$$

From \((8.19)\) it follows the decomposition

$$\mathcal{G}_{\ell'\ell''} = \bigcup_{\lambda \in \Lambda_{\ell'\ell''}} \mathcal{G}_\lambda.$$  \hspace{1cm} (8.30)

From $a' \in C^*_c(M, \mathcal{F})$ it turns out that for any $\ell', \ell'' \in I$ a set

$$\Lambda_{\ell'\ell''} = \{ \gamma \in \mathcal{G}_{\ell'\ell''} \mid a' \text{ is not trivial on } \gamma \}$$

is finite. Since the set $I$ from the decomposition \((8.15)\) is finite the set $\Theta = \bigcup_{\ell', \ell'' \in I} \Lambda_{\ell'\ell''}$ is also finite. Let $C = |\Theta| \in \mathbb{N}$ be the cardinal number of $\Theta$ and let $\delta = \frac{\epsilon}{C}$. From \((8.29)\) it follows that

$$\left\| \pi - \sum_{\ell', \ell'' \in I} \sum_{\lambda \in \Lambda_{\ell'\ell''}} \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} a'_{\ell'\ell''\lambda jk} \pi^{*}_{\ell'\ell''\lambda jk} \right\| \leq \frac{\epsilon}{2}.$$  \hspace{1cm} (8.31)

From \((8.28)\) it turns out that for any $\lambda \in \Lambda_{\ell'\ell''}$ there is $N_\lambda$ such that for any $n \geq N_\lambda$ following condition holds

$$\left\| \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} a'_{\ell'\ell''\lambda jk} \pi^{*}_{\ell'\ell''\lambda jk} - \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} a'_{\ell'\ell''\lambda jk} \pi^{*}_{\ell'\ell''\lambda jk} \right\| < \delta.$$  \hspace{1cm} (8.32)
If \( \bar{a}_f \) is given by
\[
\bar{a}_f = \sum_{i' \in I} \sum_{\lambda \in \Lambda} \sum_{j=1}^{N_{i'}} a_{i'} y_{i'}^\text{e}_j z_{i'j}^{\text{e}} \bar{a}_{i'}^{\text{e}_j} a^{\text{e}_j}_{i'}
\]
then \( \bar{a}_f \in C^*_r(M, \mathcal{F}) \) because \( \bar{a}_f \) is finite sum of elements each of which lies in \( C^*_r(M, \mathcal{F}) \).

From (8.30) and (8.31) it follows that \( \| \bar{a} - \bar{a}_f \| < \epsilon \). Hence \( \bar{a} \in C^*_r(M, \mathcal{F}) \).

**Theorem 8.23.** Let us consider a sequence
\[
\mathcal{S}_C^* = \{ C^*_r(M_0, \mathcal{F}_0) \rightarrow C^*_r(M_1, \mathcal{F}_1) \rightarrow \cdots \rightarrow C^*_r(M_n, \mathcal{F}_n) \rightarrow \ldots \} \in \mathcal{FinAlg}
\]
given by (8.7). Let \( \pi : \lim_{\leftarrow} \downarrow \mathcal{S}_M \rightarrow M \) be a topological covering associated with the topological inverse limit (cf. (4.3)) of the sequence
\[
\mathcal{S}_M = \{ M = M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_1 \leftarrow \ldots \}
\]
given by (8.6). Let \( (\lim_{\leftarrow} \downarrow \mathcal{S}_M, \tilde{\mathcal{F}}) \) be the \( \tilde{\pi} \)-lift of the foliation \((M, \mathcal{F})\). If \( \pi_{\text{geom}} \) is a geometric representation
\[
\pi_{\text{geom}} : C^*_r(M, \mathcal{F}) = \mathcal{S}_C^* = \lim_{\leftarrow} C^*_r(M_n, \mathcal{F}_n) \rightarrow B(\mathcal{H})
\]
given by (8.32) then following conditions hold:

(i) The representation \( \pi_{\text{geom}} \) is good,

(ii) There are isomorphisms:
\[
\lim_{\pi_{\text{geom}}} \downarrow \mathcal{S}_C^*(M, \mathcal{F}) \cong C^*_r \left( \lim_{\leftarrow} \downarrow \mathcal{S}_M, \tilde{\mathcal{F}} \right),
\]
\[
G \left( \lim_{\pi_{\text{geom}}} \downarrow \mathcal{S}_C^*(M, \mathcal{F}) \mid C^*_r(M, \mathcal{F}) \right) \cong G \left( \lim_{\leftarrow} \downarrow \mathcal{S}_M \mid M \right).
\]

**Proof.** From the Corollaries 8.13 and 8.22 it turns out that \( C^*_r(M, \mathcal{F}) = C^*_r(M, \mathcal{F}) \). The group \( G \left( \lim_{\leftarrow} \downarrow \mathcal{S}_M \mid M \right) \) is the maximal subgroup of \( \tilde{G} \) maximal among subgroups \( G \subset \tilde{G} \) such that
\[
G \left( \lim_{\leftarrow} \downarrow \mathcal{S}_M \right) = \lim_{\pi_{\text{geom}}} \downarrow \mathcal{S}_M.
\]
From (8.8) it follows that

\[
\mathcal{M} = \bigcup_{g \in J} g \left( \lim_{\rightarrow} \downarrow \mathcal{S}_M \right)
\] (8.33)

where \( J \subset \hat{G} \) is a set of representatives of \( \hat{G}/G \left( \lim_{\rightarrow} \downarrow \mathcal{S}_M \mid M \right) \). From (8.33) it turns out that the algebraic direct sum of irreducible algebras

\[
\bigoplus_{g \in J} g C^a_r \left( \lim_{\rightarrow} \downarrow \mathcal{S}_M, \mathcal{F} \right)
\]
is dense in \( C^a_r \left( \mathcal{M}, \mathcal{F} \right) \) and \( C^a_r \left( \lim_{\rightarrow} \downarrow \mathcal{S}_M, \mathcal{F} \right) \subset C^a_r \left( \mathcal{M}, \mathcal{F} \right) \) is a maximal irreducible sub-algebra.

(i) We need check conditions (a)-(c) of the Definition 3.11. Clearly the map

\[
\lim_{\rightarrow} C^a_r \left( M_n, \mathcal{F}_n \right) \hookrightarrow M \left( C^a_r \left( \lim_{\rightarrow} \downarrow \mathcal{S}_M, \mathcal{F} \right) \right)
\]
is injective and \( \bigoplus_{g \in J} g C^a_r \left( \lim_{\rightarrow} \downarrow \mathcal{S}_M, \mathcal{F} \right) \) is dense in \( C^a_r \left( \mathcal{M}, \mathcal{F} \right) \), i.e. conditions (a), (b) of the Definition 3.11 hold. For any \( n \in \mathbb{N} \) the homomorphism \( G \left( \lim_{\rightarrow} \downarrow \mathcal{S}_M \mid M \right) \rightarrow G \left( M_n \mid M \right) \) is surjective it follows that

\[
G \left( \lim_{\rightarrow} \downarrow \mathcal{S}_C^a \mid C^a_r \left( M, \mathcal{F} \right) \right) \rightarrow G \left( C^a_r \left( M_n, \mathcal{F}_n \right) \mid C^a_r \left( M, \mathcal{F} \right) \right)
\]
is surjective, i.e. the condition (c) of the Definition 3.11 holds.

(ii) Follows from the proof of (i).

8.5 Alternative equivariant representation

Let \( \{ p_k \in \mathbb{N} \}_{k \in \mathbb{N}} \) be an infinite sequence of natural numbers such that \( p_k > 1 \) for any \( k \), and let \( m_j = \prod_{k=1}^{j} p_k \). Let us consider a foliation \((T^2, \mathcal{F}_\theta)\) given by the Example 5.2. There is a sequence of finite-fold topological coverings

\[
\mathcal{S}_{T^2} = \left\{ T^2 \left/ \left( \times p_1 \times p_1 \right) \right\uparrow T_1^2 \left/ \left( \times p_2 \times p_2 \right) \right\uparrow \cdots \left/ \left( \times p_n \times p_n \right) \right\uparrow T_n^2 \left/ \left( \times p_{n+1} \times p_{n+1} \right) \right\uparrow \cdots \right\}
\]

where \( T_n^2 \cong T^2 \) for any \( n \in \mathbb{N} \) (cf. Example 8.10). Denote by

\[
\pi_n = \left( \times p_n, \times p_n \right) \circ \cdots \circ \left( \times p_1, \times p_1 \right) : T_n^2 \rightarrow T^2.
\]

From the Theorem 8.9 it follows that there is an (algebraical) finite covering sequence

\[
\mathcal{S}_{C^*_r(T^2, \mathcal{F}_\theta)} = \left\{ C^*_r \left( T^2, \mathcal{F}_\theta \right) \rightarrow C^*_r \left( T_1^2, \mathcal{F}_\theta \mid w_1^2 \right) \rightarrow \cdots \rightarrow C^*_r \left( T_n^2, \mathcal{F}_\theta \mid w_n^2 \right) \rightarrow \cdots \right\}
\] (8.34)
where \( \left( \mathbb{T}^2_n, F_{\theta/m_2} \right) \) is the \( \pi_n \)-lift of \( \left( \mathbb{T}^2, F_\theta \right) \). The topological inverse limit of \( \downarrow \mathcal{S}_{\mathbb{T}^2} \) is \( \mathbb{R}^2 \), i.e. \( \lim \downarrow \mathcal{S}_{\mathbb{T}^2} = \mathbb{R}^2 \). From the natural infinite covering \( \pi : \mathbb{R}^2 \to \mathbb{T}^2 \) it follows that there is an induced by \( \pi \) covering \( \left( \mathbb{R}^2, \tilde{F} \right) \) of \( \left( \mathbb{T}^2, F_\theta \right) \). The foliation \( \left( \mathbb{R}^2, \tilde{F} \right) \) is simple and given by the bundle \( p : \mathbb{R}^2 \to \mathbb{R}^1 \), so from (6.1) it follows that

\[
\mathcal{C}_r^\pi \left( \mathbb{R}^2, \tilde{F} \right) \approx \mathcal{C}_0 (\mathbb{R}) \otimes \mathcal{K}.
\]

(8.35)

A following equation

\[
G \left( \lim \downarrow \mathcal{S}_{\mathbb{T}^2} = \mathbb{R}^2 \mid \mathbb{T}^2 \right) = \mathbb{Z}^2
\]

(8.36)

is well known.

**Geometric representation** If \( \pi_{\text{geom}} : \lim \nrightarrow C^\pi_\left( \mathbb{T}^2_n, F_{\theta/m_2} \right) \to B \left( \mathcal{H} \right) \) is a geometric representation given by (8.9) then from the Theorem 8.23 it follows that \( \pi_{\text{geom}} \) is good. Moreover from (8.35) and (8.36) it follows that

\[
\lim_{\pi_{\text{geom}}} \mathcal{S}_{C^\pi_\left( \mathbb{T}^2, F_\theta \right)} \cong \mathcal{C}_0 (\mathbb{R}) \otimes \mathcal{K},
\]

\[
G \left( \lim_{\pi_{\text{geom}}} \mathcal{S}_{\left( \mathbb{T}^2_\theta, F_{\theta/m_2} \right)} \mid \mathcal{C}^\pi_\left( \mathbb{T}^2, F_\theta \right) \right) \cong G \left( \mathbb{R}^2 \mid \mathbb{T}^2 \right) \cong \mathbb{Z}^2.
\]

(8.37)

**Alternative representation** A following (algebraical) finite covering sequence

\[
\mathcal{S}_{C^\left( \mathbb{T}^2_\theta \right)} = \left\{ C^\left( \mathbb{T}^2_\theta \right) \to C^\left( \mathbb{T}^2_{\theta/m_2} \right) \to \ldots \to C^\left( \mathbb{T}^2_{\theta/m_n^2} \right) \to \ldots \right\}
\]

(8.38)

and an equivariant representation

\[
\tilde{\mathcal{R}}^\oplus : \lim \to C^\left( \mathbb{T}^2_{\theta/m_2^2} \right) \to B \left( \mathcal{H} \right)
\]

are described in [7], and these objects satisfy to the following theorem.

**Theorem 8.24.** Following conditions hold:

(i) The representation \( \tilde{\mathcal{R}}^\oplus \) is good,

(ii)

\[
\lim_{\tilde{\mathcal{R}}^\oplus} \downarrow \mathcal{S}_{C^\left( \mathbb{T}^2_\theta \right)} = C_0 \left( \mathbb{R}_{\theta}^{2N} \right);
\]

\[
G \left( \lim_{\tilde{\mathcal{R}}^\oplus} \downarrow \mathcal{S}_{\mathcal{C}^\left( \mathbb{T}^2_\theta \right)} \mid \mathcal{C} \left( \mathbb{T}^2_{\theta}^{2N} \right) \right) = \mathbb{Z}^{2N}.
\]
The explanation of the $C^*$-algebra $C_0 \left( \mathbb{R}^2_\theta \right)$ is given in [7]. In [5] it is shown that $C_0 \left( \mathbb{R}^{2N}_\theta \right)$ is the $C^*$-norm completion of the Schwartz space $\mathcal{S} \left( \mathbb{R}^{2N}_\theta \right)$ with the twisted Moyal product $*_\theta$. In [5] it is proven that there are elements $\{f_{nm} \in \mathcal{S} \left( \mathbb{R}^{2N}_\theta \right)\}_{m,n \in \mathbb{N}_0}$ which satisfy to following theorems.

Theorem 8.25. [5] Let $s$ be the Fréchet space of rapidly decreasing double sequences $c = c_{nm}$ such that

$$r_k (c) \overset{\text{def}}{=} \sqrt{\sum_{m,n=0}^{\infty} |c_{mn}|^2 \left( (2m + 1)^2 (2n + 1)^2 \right)}$$

is finite for all $k \in \mathbb{N}$, topologized by the seminorms $\{r_k\}_{k \in \mathbb{N}}$. For $f \in \mathcal{S} \left( \mathbb{R}^{2N}_\theta \right)$ let $c$ be the sequence of coefficients in the expansion

$$f = \sum_{m,n=0}^{\infty} c_{mn} f_{mn}$$

Then $f \mapsto c$ an isomorphism of Fréchet space spaces from $\mathcal{S} \left( \mathbb{R}^{2N}_\theta \right)$ onto $s$.

Theorem 8.26. [5] If $a, b \in s$ correspond respectively to $f, g \in \mathcal{S} \left( \mathbb{R}^{2N}_\theta \right)$ as coefficient sequences in the twisted Hermite basis, then the sequence corresponding to the twisted product $f *_\theta g$ is the matrix product $ab$, where

$$(ab)_{mn} = \sum_{k=0}^{\infty} a_{mk} b_{kn}. \quad (8.39)$$

From the Theorems 8.25 and 8.26 it follows that there is a $*$-isomorphism of $C^*$-algebras $C_0 \left( \mathbb{R}^{2N}_\theta \right) \cong \mathcal{K}$. From the Theorem 7.1 it follows that

$$\mathcal{G} \left( C \left( \mathbb{T}^2_\theta \right) \otimes \mathcal{K} \right) = \left\{ C \left( \mathbb{T}^2_\theta \right) \otimes \mathcal{K} \rightarrow C \left( \mathbb{T}^2_\theta / m_n^2 \right) \otimes \mathcal{K} \rightarrow ... \rightarrow C \left( \mathbb{T}^2_\theta / m_n^2 \right) \otimes \mathcal{K} \rightarrow ... \right\}. \quad (8.40)$$

is an algebraical finite covering sequence. There is a representation

$$\hat{\pi} \otimes \text{Id}_\mathcal{K} : \lim_\leftarrow C \left( \mathbb{T}^2_\theta / m_n^2 \right) \otimes \mathcal{K} \rightarrow B \left( \mathcal{H} \otimes \mathcal{H} \right).$$

From the Theorems 7.6 and 8.24 it follows that

- The representation $\hat{\pi} \otimes \text{Id}_\mathcal{K}$ is good,

$$\lim_\leftarrow G \left( \hat{\pi} \otimes \text{Id}_\mathcal{K} \right) \downarrow \mathcal{G} \left( C \left( \mathbb{T}^2_\theta \right) \otimes \mathcal{K} \right) \cong \mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}.$$

$$G \left( \lim_\leftarrow \mathcal{G} \left( C \left( \mathbb{T}^2_\theta \right) \otimes \mathcal{K} \right) \mid C \left( \mathbb{T}^2_\theta \right) \otimes \mathcal{K} \right) = \mathbb{Z}^2.$$
From (6.2) it follows that sequence (8.34) is isomorphic to the sequence (8.40). From this fact it follows that there is a good equivariant representation

$$\pi_{\text{alg}} : \lim_{\rightarrow} C^+_{\mathbb{T}_n^2, F_{\theta}/m_n^2} \rightarrow B(\mathcal{H})$$

such that

$$\lim_{\rightarrow} \pi_{\text{alg}} \downarrow \mathcal{G}(\mathbb{T}_2, F_\theta) \cong \lim_{\rightarrow} \pi_{\text{alg}} \downarrow \mathcal{G}(\mathbb{T}_2^\theta) \cong \mathcal{K},$$

$$G \left( \lim_{\rightarrow} \mathcal{G}(\mathbb{T}_2, F_\theta) \mid C^+_{\mathbb{T}_2, F_\theta} \right) \cong \lim_{\rightarrow} \mathcal{G}(\mathbb{T}_2^\theta) \cong \mathbb{Z}^2. \quad (8.41)$$

A comparison of (8.37) and (8.41) gives a following result

$$\lim_{\rightarrow} \pi_{\text{alg}} \downarrow \mathcal{G}(\mathbb{T}_2, F_\theta) \not\approx \lim_{\rightarrow} \pi_{\text{geom}} \downarrow \mathcal{G}(\mathbb{T}_2, F_\theta).$$

From the above equation it follows that the construction given by the Theorem 8.23 does not yield a one to one correspondence between geometric and algebraic infinite coverings of foliations.

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