Partition functions for the rigid string and membrane at any temperature

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Abstract

Exact expressions for the partition functions of the rigid string and membrane at any temperature are obtained in terms of hypergeometric functions. By using zeta function regularization methods, the results are analytically continued and written as asymptotic sums of Riemann-Hurwitz zeta functions, which provide very good numerical approximations with just a few first terms. This allows to obtain systematic corrections to the results of Polchinski et al., corresponding to the limits $T \to 0$ and $T \to \infty$ of the rigid string, and to analyze the intermediate range of temperatures. In particular, a way to obtain the Hagedorn temperature for the rigid membrane is thus found.

PACS: 11.17, 03.70, 04.50.

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1 Introduction

The problem of deciding if QCD is actually (some limit of) a string theory (or, in general, of a theory of extended objects) is almost twenty years old and goes back to 't Hooft [1] (for recent reviews, see [2,3]). In very new contributions, Polchinski and Yang have tried to answer this question, both for the Nambu-Goto [4] and for the rigid string cases [5], by calculating their partition functions and seeing if they actually match with the partition function corresponding to QCD at different ranges of temperature. Due to the difficulties involved in the calculations, only a direct analysis of the two sharp limits $T \to 0$ and $T \to \infty$ has been done. Such approximations are rightfully claimed to have overlapping validity. However, in the intermediate region corrections to the first terms of the series expansion may surely become important.

We shall here fill this gap, by obtaining exact analytical continuations of the partition functions, out of which approximations valid in different ranges of temperature will be quite easy to obtain. With some additional work, we will also be able to derive the corresponding expressions for bosonic membranes, both with and without rigid term. Extending the ideas of ref. [5] to the case of the rigid membrane, we can also characterize the Hagedorn transition in this case, and even for any kind of membrane and $p$-brane in general. As is known, this has always been a very elusive subject. The high-temperature limits of the free energy per unit area for the bosonic membrane and for the rigid membrane are found not to agree with that of the QCD large-$N$ limit.

The paper is organized as follows. In sect. 2, after a very short review of the results of Polchinski et al., we use our formalism for the calculation of an analytic partition function of the rigid string which is exact to one-loop in the coupling constant. In two separate subsections, we obtain the low-temperature and the high-temperature limits of the partition function, respectively, that is, the dominant terms plus first order corrections in each case. We also show explicitly how higher order terms can be obtained —in a way consistent with the loop expansion. In sect. 3 we proceed with the calculation of the partition functions corresponding to bosonic membranes and $p$-branes (first subsection) and to rigid membranes and rigid $p$-branes (second subsection). The calculation of the extremum with respect to the parameters, in each case, is done in sect. 4. Establishing a parallelism with the results of ref. [5], we characterize the Hagedorn transition for the
case of the rigid membrane and we study the very high temperature limits corresponding to the pure bosonic and rigid membrane. Finally, sect. 5 is devoted to conclusions.

2 Analytical partition function for the rigid string

The first two terms in the loop expansion

$$S_{\text{eff}} = S_0 + S_1 + \cdots$$

of the effective action corresponding to the rigid string [6,7]

$$S = \frac{1}{2\alpha_0} \int d^2\sigma \left[ \rho^{-1} \partial^2 X^\mu \partial^2 X_\mu + \lambda^{ab} (\partial_a X^\mu \partial_b X_\mu - \rho \delta_{ab}) \right] + \mu_0 \int d^2\sigma \rho,$$

where $\alpha_0$ is the dimensionless, asymptotically free coupling constant, $\rho$ the intrinsic metric, $\mu_0$ the explicit string tension (important at low energy) and $\lambda^{ab}$, $a, b = 1, 2$, the usual Lagrange multipliers, are given —in the world sheet $0 \leq \sigma_1 \leq L$ and $0 \leq \sigma_2 \leq \beta t$ (an annulus of modulus $t$)— by

$$S_0 = \frac{L \beta t}{2\alpha_0} \left[ \lambda^{11} + \lambda^{22} t^{-2} + \rho (2\alpha_0 \mu_0 - \lambda^{aa}) \right]$$

at tree level, and by

$$S_1 = -\frac{d-2}{2} \ln \det (\partial^4 - \rho \lambda^{ab} \partial_a \partial_b)$$

$$= \frac{d-2}{2} L \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \ln \left[ k^2 + \frac{4\pi^2 n^2}{\beta^2 t^2} \right] + \rho \left( \lambda^{11} k^2 + \frac{4\pi^2 \lambda^{22} n^2}{\beta^2 t^2} \right)$$

at one-loop order, respectively. Of course, to make sense, this last expression needs to be regularized and its calculation is highly non-trivial—as explicitly quoted in refs. [4,5]. There, it has been obtained only in the very strict limits $T \to 0$ and $T \to \infty$ (just terms of highest order have been kept in both cases) and around some extremizing configuration of the parameters $\rho$, $\lambda^{11}$, $\lambda^{22}$ and $t$. A semiclassical discussion of rigid strings can be found in refs. [8,9] also.

Here, we shall make use of the zeta function procedure [10], as developed in [11] after the rigorous proof of the zeta function regularization theorem, and further extended in [12] to the more elaborate situations we need to deal with here. One can write

$$S_1 = -(d-2)L \frac{d}{ds} \zeta_A(s/2) \bigg|_{s=0}, \quad \zeta_A(s/2) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} (k^2 + y_+^2)^{-s/2} (k^2 + y_-^2)^{-s/2},$$
where

$$y_\pm = \frac{a}{t} \left[ n^2 + \frac{\rho t^2 \lambda^{11}}{2a^2} \pm \sqrt{\frac{t}{a}} \left( (\lambda^{11} - \lambda^{22})n^2 + \frac{\rho t^2 \lambda^{11}}{4a^2} \right)^{1/2} \right]^{1/2}, \quad a \equiv \frac{2\pi}{\beta}. \quad (6)$$

Following refs. [4,5], we may consider two basic approximations of overlapping validity: one for low temperature, $\beta^{-2} << \mu_0$, and the other for high temperature, $\beta^{-2} >> \mu_0$. Both these approximations (overlapping region included) can be obtained from the expression above, which on its turn can be written in the form

$$\zeta_A(s/2) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \frac{y_-}{(y+y_-)^s} F(s/2, 1/2; s; 1 - \eta), \quad \eta \equiv \frac{y_-^2 - y_+^2}{y_-^2}. \quad (7)$$

This is an exact formula. To finish this short introduction, we should point out that the above approach is somewhat different from the standard approach to the free energy of the bosonic string at non-zero temperature (for a review and a list of references see, for example, [14]).

### 2.1 The low temperature approximation

The low temperature case is already quite involved and the refined methods of ref. [12] must be used. The term $n = 0$ in (7) is non-vanishing and must be treated separately from the rest. It gives

$$\zeta_A^{n=0}(s/2) = \frac{1}{2\pi} \frac{\Gamma((1 - s)/2)\Gamma(s - 1/2)}{\Gamma(s/2)} (\lambda^{11}\rho)^{1/2-s}, \quad \frac{1}{2} < \Re(s) < 1. \quad (8)$$

This is again an exact expression, that yields

$$\left. \frac{d}{ds} \zeta_A^{n=0}(s/2) \right|_{s=0} = -\frac{1}{2} \sqrt{\rho\lambda^{11}} \quad (9)$$

and

$$S_1^{(n=0)} = \frac{d - 2}{2} \sqrt{\rho\lambda^{11}}. \quad (10)$$

This contribution must be added to the one coming from the remaining terms in eq. (7) above

$$\zeta_A(s/2) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{y_-}{(y+y_-)^s} F(s/2, 1/2; s; 1 - \eta), \quad \Re(s) > 1 \quad (11)$$
(the prime is here no derivative, it just means that the term \( n = 0 \) is absent from the sum). Within this approximation, and working around the classical, \( T = 0 \) solution: \( \lambda^{11} = \lambda^{22} = \alpha_0 \mu_0, \rho = t^{-2} = 1 \), we obtain

\[
\zeta_A'(s/2) = \frac{1}{\sqrt{\pi}} \Gamma(s - 1/2) \left( \frac{a}{t} \right)^{1-2s} \left[ F_0(s) + F_1(s) + F_2(s) \right],
\]

where

\[
F_0(s) \equiv \sum_{n=1}^{\infty} \frac{n^{1-s}}{(n^2 + \sigma_2^2)^{s/2}} \left[ F(s/2, 1/2; s; \sigma_1^2/(n^2 + \sigma_1^2)) - 1 - \frac{\sigma_1^2}{4n^2} \right],
\]

\[
F_1(s) \equiv \sum_{n=1}^{\infty} \frac{n^{1-s}}{(n^2 + \sigma_2^2)^{s/2}}, \quad F_2(s) \equiv \frac{\sigma_1^2}{4} \sum_{n=1}^{\infty} \frac{n^{-1-s}}{(n^2 + \sigma_2^2)^{s/2}}, \quad \sigma_i^2 \equiv \frac{\lambda^{ii} \rho t^2 \beta^2}{4\pi^2}, \quad i = 1, 2.
\]

In this formula we have already taken advantage of the low energy limit and from now on in this subsection we shall compute perturbations around it.

The study of the functions (13) is done in [12] in full detail. Substituting the results back into (12), we obtain [12]

\[
\zeta_A'(s/2) = \frac{1}{\sqrt{\pi}} \Gamma(s - 1/2) \left( \frac{a}{t} \right)^{1-2s} \left[ 1 + \gamma s + O(s^2) \right]
\]

\[
\times \left[ s F_0(0) + \frac{\sigma_1^2}{8} + \frac{\sigma_1^2 \gamma s}{4} - \frac{\sigma_2^2 s}{4} - \frac{s}{12} + O(s^2) \right],
\]

therefore

\[
\zeta_A'(0) = -\frac{a\sigma_1^2}{4t}
\]

and

\[
\left. \frac{d}{ds} \zeta_A'(s/2) \right|_{s=0} = -\frac{a}{t} \left[ \psi(-1/2) \frac{\sigma_1^2}{4} - \ln(a/t) \frac{\sigma_1^2}{2} + \frac{(1 + \gamma) \sigma_1}{4} - \frac{\sigma_1^2}{2} \ln(\sigma_1/2) - \frac{\sigma_1}{2} \right.
\]

\[
- \left. \frac{\sigma_2^2}{2} - \frac{1}{12} + \frac{1}{\pi} \int_{\sigma_1}^{\infty} dr \ln \left(1 - e^{-2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma_1^2}} \right].
\]

Finally

\[
S_1^{(n \neq 0)} = (d - 2) \frac{La}{4t} \left\{ \sigma_1^2 \left[ -2 \ln(a\sigma_1/t) + 3 \right] - 2\sigma_1 \right.
\]

\[
- \left. 2\sigma_2^2 - \frac{1}{3} + \frac{4}{\pi} \int_{\sigma_1}^{\infty} dr \ln \left(1 - e^{-2\pi r} \right) \frac{r}{\sqrt{r^2 - \sigma_1^2}} \right\}.
\]
Adding now (10) and (17) we get the desired expansion of the one loop effective action near $T = 0$. It reads

$$S_1 = \frac{(d-2) L}{2} \sqrt{\rho \lambda_{11}} + \frac{(d-2) \pi L}{\beta t} \left[-\sigma_1^2 \ln \left(\frac{\sqrt{\rho \lambda_{11}}}{\mu}\right) + \frac{3\sigma_1^2}{2}\right]$$

$$- \sigma_1 - \sigma_2^2 - \frac{1}{6} + \frac{2}{\pi} \int_\sigma^\infty dr \ln \left(1 - e^{-2\pi r}\right) \frac{r}{\sqrt{r^2 - \sigma_1^2}}$$,

where $\mu$ is the ordinary mass constant that appears in the general procedure of zeta-function regularization [13].

We can now obtain the extremum of $S_0 + S_1$:

$$\rho \simeq 1 + \frac{(d-2) \sqrt{\alpha_0}}{2\beta \sqrt{\mu_0}},$$

$$\frac{1}{t^2} \simeq 1 + \frac{(d-2) \sqrt{\alpha_0}}{2\beta \sqrt{\mu_0}} + \frac{(d-2) \alpha_0}{4\pi} \left[\ln \left(\frac{\alpha_0 \mu_0}{\mu}\right) - 3\right],$$

$$\lambda_{11} \simeq \alpha_0 \mu_0 + \frac{(d-2) \alpha_0^{3/2} \sqrt{\mu_0}}{4\beta},$$

$$\lambda_{22} \simeq \alpha_0 \mu_0 - \frac{(d-2) \alpha_0^{3/2} \sqrt{\mu_0}}{4\beta}.$$

These are the first order corrections to the classical result, which corresponds to the $T \to 0$ case. In a calculation to two loop order it would have sense to go further with the calculation of second order corrections. As is clear from the above expressions, this can be done easily.

### 2.2 The high temperature approximation

For high temperature, the ordinary expansion of the confluent hypergeometric function $F$ of eq. (7) is in order

$$F(s/2, 1/2; s; 1 - \eta) = \sum_{k=0}^\infty \frac{(s/2)_k (1/2)_k}{k! (s)_k} (1 - \eta)^k,$$

$(s)_k = s(s+1) \cdots (s+k-1)$ being Pochhamer’s symbol (the rising factorial). This series expansion is here completely consistent with the loop approximation ($\alpha_0$ is kept always small) and, actually, only the first three terms of this expansion need to be taken into account at the approximation in which we are working. However, the fact that there is
no mathematical hindrance against going further with these formulas must be properly remarked. It is useful for the future calculation to notice that

\[ \lim_{s \to 0} F(s/2, 1/2; s; 1 - \eta) = 1 + \eta^{-1/2}. \]  

(21)

We can approximate \( y_{\pm} \) by

\[ y_{\pm} \approx \frac{a}{t} \sum_{n=-\infty}^{\infty} \left[ (n \pm b)^2 + c^2 \right]^{1/2}, \]

\[ b = \frac{\beta t}{4\pi} \sqrt{\rho (\lambda^{11} - \lambda^{22})}, \quad c^2 = \frac{\rho \beta^2 t^2 (\lambda^{11} + \lambda^{22})}{16\pi^2} \approx \frac{\alpha_0^2}{16\pi^2}. \]  

(22)

Before taking the derivative of the zeta function (7) one should observe that, apart from the usual contribution coming from the derivative of \( \Gamma(s) \) at \( s = 0 \), there will be here an additional term due to the pole of the Hurwitz zeta function \( \zeta(1, \pm b) \) which emerges from the \( y_{\pm} \) in (7). After appropriate analytic continuation, the derivative of the zeta function yields

\[ \frac{d}{ds} \zeta_A(s/2) \bigg|_{s=0} = -\frac{1}{4} \sqrt{b^2 + c^2} + \frac{2\pi}{\beta t} \left\{ b^2 + \frac{1}{6} - \left[ \frac{1}{2} \ln(-b^2) + 2 - 2 \ln 2 \right] c^2 \right\}. \]  

(23)

In order to obtain this result, which comes through elementary Hurwitz zeta functions, \( \zeta(\pm 1, \pm b) \), we have used the binomial expansion in (22). This is completely consistent with the approximation employed (notice, once more, the extra terms coming from the contribution of the pole in (22)) to the derivative of \( \zeta_A(s/2) \).

It is immediate to see that the terms of highest order coincide with the result of Polchinski and Yang

\[ S_1 = -\frac{d - 2}{2} L \left[ \frac{2\pi}{3\beta t} \sqrt{\rho \lambda^{11}} + O(\beta) \right]. \]  

(24)

The outcome of the extremization of \( S_0 + S_1 \) with respect to the parameters \( \rho, \lambda^{11}, \lambda^{22} \) and \( t \) is

\[ \rho^{-1} = t^2 = 1 - \frac{(d - 2)\alpha_0}{2\beta} \sqrt{\lambda^{11}}, \]

\[ \lambda^{22} = -\frac{(d - 2)\alpha_0}{4\beta} \sqrt{\lambda^{11}} + \alpha_0 \mu_0 + \frac{(d - 2)\pi\alpha_0}{3\beta^2}, \]

(25)

\[ \lambda^{11} = \frac{3(d - 2)\alpha_0}{4\beta} \sqrt{\lambda^{11}} + \alpha_0 \mu_0 - \frac{(d - 2)\pi\alpha_0}{3\beta^2}, \]

where it is this last equation the one which determines the possible values of \( \lambda^{11} \). Thus, the values of the parameters at the two transition points that appear at high temperature
—namely
\[ \beta_c^2 \mu_0 = \frac{(d-2)\pi}{3} - \frac{(d-2)^2 \alpha_0}{8}, \] (26)
corresponding to the Hagedorn transition, and
\[ \beta_c^2 \mu_0 = \frac{(d-2)\pi}{3} - \frac{9(d-2)^2 \alpha_0}{64}, \] (27)
beyond which the variables acquire an imaginary part— and the value of the winding-soliton mass squared
\[ M^2(\beta)^2 = \beta^2 t^2 (\lambda^{11})^2 \alpha_0^{-2}, \] (28)
are only modified by higher order terms in \( \alpha_0 \) (that is, of order \( \sqrt{\alpha_0} \) in the last case). In fact, the result for \( S_1 \) to next order is
\[ S_1^{(2)} = (d-2)L \left\{ \frac{\pi}{3\beta t} + \frac{1}{2\sqrt{\rho}} \beta \lambda^{11} - \frac{1}{8\pi} \beta t \rho (\lambda^{11} - \lambda^{22}) \right. \]
\[ + \frac{1}{8\pi} \beta t \rho (\lambda^{11} + \lambda^{22}) \left[ \frac{1}{2} \ln \left( -\frac{\beta^2 t^2 \rho (\lambda^{11} - \lambda^{22})}{16\pi^2 \mu} \right) + 2 - 2 \ln 2 \right] \}. \] (29)
From the expression of the total action, \( S \), to this order, new equations for the extremizing configuration follow. In particular, from the derivatives of \( S \) with respect to \( \lambda^{22} \) and \( \lambda^{11} \) we obtain, respectively
\[ \frac{1}{t^2} \simeq \rho - \frac{d-2}{4\pi} \alpha_0 \rho \left[ \frac{1}{2} \ln \left( \frac{(d-2)\alpha_0}{24\pi^2 \mu} \right) + \psi(-1/2) + \gamma + 1 - \frac{3}{8} \beta \sqrt{\lambda^{11}} \right], \]
\[ \frac{1}{\rho} \simeq 1 - \frac{(d-2)\alpha_0}{\beta \sqrt{\lambda^{11}}} - \frac{(d-2)\alpha_0}{4\pi} \left[ \frac{1}{2} \ln \left( \frac{(d-2)\alpha_0}{24\pi^2 \mu} \right) + 1 - 2 \ln 2 \right] \]
\[ - \frac{3}{8} \beta \sqrt{\lambda^{11}} + \frac{(d-2)\alpha_0}{\beta \sqrt{\lambda^{11}}} [3 - 2 \ln 2]. \] (30)
One can perform a consistent evaluation in terms of the dominant contribution (lowest power in \( \alpha_0 \)) obtained in ref. [5]
\[ \frac{1}{t^2} = 1 + \frac{(d-2)\alpha_0}{2\beta \sqrt{\lambda^{11}}} - \frac{(d-2)\alpha_0}{2\pi} + O(\alpha_0^{3/2}, \alpha_0^{1/2} \mu_0 \beta^2), \]
\[ \frac{1}{\rho} = 1 - \frac{(d-2)\alpha_0}{2\beta \sqrt{\lambda^{11}}} - \frac{(d-2)\alpha_0}{4\pi} \left[ \frac{1}{2} \ln \left( \frac{(d-2)\alpha_0}{24\pi^2 \mu} \right) + 1 - 2 \ln 2 \right] \]
\[ + O(\alpha_0^{3/2}, \alpha_0^{1/2} \mu_0 \beta^2). \] (31)
On the other hand, from the derivatives with respect to $\rho$ and $t$ (and within the same approximation) we get, respectively

\[
\lambda^{22} = 2\alpha_0\mu_0 - \lambda^{11} + \frac{d - 2}{2\beta} \alpha_0\sqrt{\lambda^{11}} + \frac{(d - 2)^2\alpha_0^2}{6\pi\beta^2} \left(1 - \frac{3}{2}\beta\sqrt{\lambda^{11}}\right) \\
+ \frac{(d - 2)^2\alpha_0^2}{4\pi\beta^2} \left(\beta\sqrt{\lambda^{11}} + \frac{\mu_0\beta^2}{d - 2}\right) \left[\frac{1}{2} \ln \left(\frac{(d - 2)\alpha_0}{24\pi^2\mu}\right) + \frac{3}{2} - 2\ln 2\right], \\
\lambda^{11} = -\frac{(d - 2)\pi\alpha_0}{3\beta^2} + \alpha_0\mu_0 + \frac{3(d - 2)\alpha_0}{4\beta} \sqrt{\lambda^{11}} \\
- \frac{(d - 2)\alpha_0^2\mu_0}{4\beta\sqrt{\lambda^{11}}} - \frac{3(d - 2)^2\pi\alpha_0^2}{24\beta^3\sqrt{\lambda^{11}}} + \mathcal{O}(\alpha_0^2, \alpha_0\mu_0\beta^2). \tag{32}
\]

Of course, the terms of order $\alpha_0^2$ and $\alpha_0\mu_0\beta^2$ are not to be taken seriously, because they are of the same order as the first ones that would come from the two-loop contribution to the total action.

More than the actual expressions themselves, the thing to be remarked here is the fact that we have constructed a simple procedure to obtain all higher-order contributions to the loop action and, correspondingly, to the extremizing configuration, the Hagedorn temperature, etc. in a systematic, rigorous, analytical way, both for low and for high temperature and, taking the necessary terms, for any temperature in the intermediate, overlapping region which connects both limits. In the next section we shall extend our method to membranes and $p$-branes.

### 3 Partition function for the bosonic membrane and $p$-brane

We shall consider the cases of the pure bosonic membrane and of the bosonic membrane plus rigid term, and also of their generalizations in the form of $p$-branes. The classical membrane theory has been formulated in refs. [15]. Semiclassical approaches to membrane theory can be found in refs. [16-19], and general reviews on membranes are [18,23]. One should also notice that string theories can be obtained as some compactification of membrane theories [24].
3.1 Pure bosonic membrane and corresponding p-brane

The tree level action similar to (3) is

$$S_0^{(m)} = \kappa L^2 \beta t \left[ (1 + \sigma_0)^{1/2} (1 + \sigma_1) - \left( \frac{1}{2} \lambda_0 \sigma_0 + \lambda_1 \sigma_1 \right) \right]$$  \hspace{1cm} (33)

where $\lambda_0$ and $\lambda_1$ are Lagrange multipliers and $\sigma_0$ and $\sigma_1$ are composite fields, defined in [16,17]. The one-loop contribution to the action can be written formally as follows (cf. [16,17])

$$S_1^{(bm)} = \frac{(d-3)L^2}{2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_1 dk_2}{(2\pi)^2} \left[ \lambda_1 (k_1^2 + k_2^2) + \frac{4\pi^2 \lambda_0 n^2}{\beta^2 t^2} \right].$$  \hspace{1cm} (34)

This expression must be regularized. As in the string case, we choose the zeta function method. Calling $\zeta_2$ the corresponding zeta function, we have

$$S_1^{(bm)} = -\frac{(d-3)L^2}{2} \frac{d}{ds} \zeta_2(s) \bigg|_{s=0},$$

$$\zeta_2(s) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_1 dk_2}{(2\pi)^2} \ln \left[ \lambda_1 (k_1^2 + k_2^2) + \frac{4\pi^2 \lambda_0 n^2}{\beta^2 t^2} \right]^{-s}. \hspace{1cm} (35)$$

After some calculations, we get

$$\zeta_2(s) = \frac{1}{4\pi(s-1)\lambda_1} \left( \frac{4\pi^2 \lambda_0}{\beta^2 t^2} \right)^{1-s} \zeta_R(2s-2),$$  \hspace{1cm} (36)

where $\zeta_R$ is Riemann’s zeta function. We thus obtain

$$S_1^{(bm)} = -\frac{2(d-3)\pi\lambda_0 L^2}{\lambda_1 \beta^2 t^2} \zeta_R'(-2).$$  \hspace{1cm} (37)

A review of membrane theory at non-zero temperature can be found in ref. [21].

In the case of the bosonic $p$-brane, the corresponding expressions are

$$S_0^{(p)} = \kappa L^p \beta t \left[ (1 + \sigma_0)^{1/2} (1 + \sigma_1)^{p/2} - \frac{1}{2} \left( \lambda_0 \sigma_0 + p \lambda_1 \sigma_1 \right) \right]$$  \hspace{1cm} (38)

and

$$S_1^{(bp)} = -\frac{(d-p-1)L^p}{2} \frac{d}{ds} \zeta_p(s) \bigg|_{s=0},$$

$$\zeta_p(s) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_1 \cdots dk_p}{(2\pi)^p} \ln \left[ \lambda_1 (k_1^2 + \cdots + k_p^2) + \frac{4\pi^2 \lambda_0 n^2}{\beta^2 t^2} \right]^{-s}$$

$$= \frac{V_p \Gamma(p/2) \Gamma(s - p/2)}{(2\pi)^p \lambda_1^{p/2} \Gamma(s)} \left( \frac{4\pi^2 \lambda_0}{\beta^2 t^2} \right)^{p/2-s} \zeta_R(2s - p).$$  \hspace{1cm} (39)
where $V_p$ is the ‘volume’ of the $p-1$-dimensional unit sphere. Observe that the membrane is the particular case $p = 2$ of this formula (as it should). We get

$$S_1^{(bp)} = \frac{d - p - 1}{2} V_p L^p \Gamma(p/2) \left( \frac{\lambda_0}{\lambda_1} \right)^{p/2} \left\{ \begin{array}{ll} \frac{(-1)^{p/2}}{(p/2)!} \zeta_{\nu}(-p), & p = \text{even}, \\ \Gamma(-p/2) \zeta_{\nu}(-p), & p = \text{odd}. \end{array} \right. \quad (40)$$

We ought to remark here that very nice expressions for the derivatives of the Riemann and Hurwitz zeta functions at any negative integer value of the argument have been obtained in [22].

### 3.2 Bosonic membrane with rigid term and the corresponding p-brane

The tree level action is the same as before, eq. (33). The one-loop order contribution for the bosonic membrane is (for a semiclassical approach to the rigid $p$-brane see [20])

$$S_1^{(rm)} = \frac{d - 3}{2} L^2 \left. \frac{d}{ds} \zeta_2^r \right|_{s=0}, \quad (41)$$

$$\zeta_2^r(s) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{dk_1 dk_2}{(2\pi)^2} \left[ \frac{1}{\rho^2} \left( k_1^2 + k_2^2 + \frac{4\pi^2 n^2}{\beta^2 t^2} \right)^2 + \kappa \left( \lambda_1 (k_1^2 + k_2^2) + \frac{4\pi^2 \lambda_0 n^2}{\beta^2 t^2} \right) \right]^{-s},$$

where the label $r$ means rigid. We can write

$$\zeta_2^r(s) = \frac{\rho^{2s}}{4\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dk \, (k + c_+)^{-s}(k + c_-)^{-s}, \quad (42)$$

with

$$c_{\pm} = \frac{4\pi^2 n^2}{\beta^2 t^2} + \frac{\kappa \rho^2 \lambda_1}{2} \pm \sqrt{\kappa \rho} \left[ (\lambda_1 - \lambda_0) \frac{4\pi^2 n^2}{\beta^2 t^2} + \frac{\kappa \rho^2 \lambda_1^2}{4} \right]^{1/2}. \quad (43)$$

Here, in analogy with the rigid string case, the term corresponding to $n = 0$ must be treated separately. It yields a beta function. Also as in the rigid string case, the remaining series can be written in terms of a confluent hypergeometric function. The complete result, obtained after some work, is

$$\zeta_2^r(s) = \frac{\rho^{2s}(\kappa \lambda_1)^{1-2s} \Gamma(1-s) \Gamma(2s-1)}{4\pi \Gamma(s)} + \frac{\rho^{2s} \Gamma(2s-1)}{2\pi \Gamma(2s)} \sum_{n=1}^{\infty} c_{\pm}^{1-2s} F(s, 2s-1; 2s; 1 - c_+/c_-). \quad (44)$$
Again, it is remarkable that one can in fact obtain the exact expressions corresponding to bosonic membranes and \( p \)-branes, both for the ordinary and for the rigid case. The last one is really intricate, specially for dealing with membranes and \( p \)-branes with \( p \) even. In fact, when taking the derivative of the zeta function at \( s = 0 \) two alternative procedures can be followed. This stems from the observation that, in the end, we shall be interested in the high temperature approximation. Thus, we can either start by making the corresponding expansion in the confluent hypergeometric function, do then the analytical continuation and finally take the derivative, or else, we can keep the hypergeometric function exact during the process of analytical continuation, then take the derivative, and make the high-temperature expansion at the end. Both procedures have been seen to yield exactly the same result. The second one is much more involved but has the advantage of keeping exact expression till the end. It can be considered as zeta function regularization at its best. No other procedure can compete with it in rigor and elegance. However, let us repeat that the price to be paid is not worth in many cases—as the present, where for the high-temperature analysis in which we are interested the first alternative is much easier. We shall follow it here.

For the general case of the \( p \)-brane with rigid term, the one-loop contribution to the action is

\[
S_1^{(rp)} = -\left. \frac{(d - p - 1)L^p}{2} \frac{d}{d s} \zeta_p^r(s) \right|_{s=0},
\]

\[
\zeta_p^r(s) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} \frac{d k_1 \cdots d k_p}{(2 \pi)^p} \left[ \frac{1}{\rho^2} \left( k_1^2 + \cdots + k_p^2 + \frac{4 \pi^2 n^2}{\beta^2 p^2} \right)^2 \right]^{-s}
\]

\[
+ \kappa \left( \lambda_1 (k_1^2 + \cdots + k_p^2) + \frac{4 \pi^2 \lambda_0 n^2}{\beta^2 p^2} \right)^{-s}
\]

\[
= \frac{V_p \rho^{2s}}{2(2 \pi)^p} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} d k \, k^{p/2-1} (k + c_+)^{-s} (k + c_-)^{-s},
\]

where \( V_p \) is again the ‘volume’ of the \( p - 1 \)-dimensional unit sphere and the \( c_\pm \) are again given by (43). A point about dimensions: the reader may verify (see also (38)) that \([\kappa] = M^{p+1}\) and \([\rho] = M^{(1-p)/2}\).

Proceeding as above, namely considering the \( n = 0 \) term separately, we obtain the following generalization of the formula corresponding to the rigid membrane:

\[
\zeta_p^r(s) = \frac{V_p \rho^{p-2s} (\kappa \lambda_1)^{p/2-2s}}{2(2 \pi)^p} \frac{\Gamma(p/2 - s) \Gamma(2s - p/2)}{\Gamma(s)} + \frac{V_p \rho^{2s}}{(2 \pi)^p}
\]
\[ \times \frac{\Gamma(2s - p/2)\Gamma(p/2)}{\Gamma(2s)} \sum_{n=1}^{\infty} c_{p/2-2s} F(s, 2s - p/2; 2s; 1 - c_+/c_-). \] (46)

When computing its derivative at \( s = 0 \) we must distinguish the two cases: \( p \) even and \( p \) odd. For odd \( p \) (which is the simplest case):

\[
\frac{d}{ds} \zeta_p(s) \bigg|_{s=0} = V_p \frac{\Gamma(p/2)\Gamma(-p/2)}{2(2\pi)^p} \left\{ (\kappa\rho^2\lambda_1)^{p/2} + 4\zeta(-p) \left( \frac{2\pi}{\beta t} \right)^p \right. \\
+ \left. p \zeta(2 - p) \kappa\rho^2 \left[ p/2 \lambda_1 + \left( 1 - \frac{p}{2} \right) \lambda_0 \right] \left( \frac{2\pi}{\beta t} \right)^{p-2} \right\}, \quad p = \text{odd}, \quad p \geq 3.
\] (47)

However, for even \( p \) we must take into account that a divergence in \( \Gamma(s - p/2) \) can compensate the divergence of \( \Gamma(s) \) in the denominator, when \( s \to 0 \). One should be careful with this fact, that renders the calculation more involved. The result is

\[
\frac{d}{ds} \zeta_p(s) \bigg|_{s=0} = V_p \frac{(-1)^{p/2}}{(2\pi)^p} \left\{ \frac{1}{p} (\kappa\rho^2\lambda_1)^{p/2} \left[ -\ln \left( \frac{\kappa\rho\lambda_1}{\mu} \right) - \gamma + \psi(p/2) + S(p/2) \right] \\
+ \frac{8}{p} \zeta'(-p) \left( \frac{2\pi}{\beta t} \right)^p - \frac{8}{p} \zeta(-p) \left( \frac{2\pi}{\beta t} \right)^p \ln \left( \frac{2\pi}{\beta t\sqrt{\rho}} \right) \\
- 2\zeta(2 - p)\kappa\rho^2 \left[ p/2 \lambda_1 + \left( 1 - \frac{p}{2} \right) \lambda_0 \right] \left( \frac{2\pi}{\beta t} \right)^{p-2} \ln \left( \frac{2\pi}{\beta t\sqrt{\rho}} \right) \\
+ 2\zeta'(2 - p)\kappa\rho^2 \left[ p/2 \lambda_1 + \left( 1 - \frac{p}{2} \right) \lambda_0 \right] \left( \frac{2\pi}{\beta t} \right)^{p-2} \\
+ \frac{4}{p} \zeta(-p) S(p/2) \left( \frac{2\pi}{\beta t} \right)^p + \zeta(2 - p) S(p/2 - 1) \kappa\rho^2\lambda_1 \left( \frac{2\pi}{\beta t} \right)^{p-2} \\
+ \left( \frac{p}{2} - 1 \right) \zeta(2 - p) \left[ S(p/2 - 2) - 1 \right] \kappa\rho^2(\lambda_1 - \lambda_0) \left( \frac{2\pi}{\beta t} \right)^{p-2} \right\}, \\
\] (48)

being \( S(k) \equiv \sum_{l=1}^{k} l^{-1} \). Finally, these expressions multiplied by \(-1/2(d - p - 1)L^p \) yield the one-loop contribution to the action of the rigid \( p \)-brane. In the case of the rigid membrane we get the rather simpler result

\[
\frac{d}{ds} \zeta_0(s) \bigg|_{s=0} = \frac{1}{4\pi} \kappa\rho^2\lambda_1 \left[ \ln \left( \kappa\rho^2\lambda_1(\beta t)^2 \right) - 1 \right] - \frac{8\pi \zeta'(-2)}{(\beta t)^2} \frac{1}{4\pi} \kappa\rho^2(\lambda_1 - \lambda_0). \] (49)

It is easy to check that the action corresponding to the rigid membrane is just the particular case \( p = 2 \) of the rather non-trivial formula corresponding to the rigid \( p \)-brane, for \( p \) even. The full power of zeta function regularization has been used in working out such expression.
Again, it is remarkable that one can in fact obtain the \textit{exact} expressions corresponding to bosonic membranes and \( p \)-branes, both for the ordinary and for the rigid case. The last one is really intricate, specially for dealing with membranes and \( p \)-branes with \( p \) even.

The values of the extrema and of the Hagedorn temperature will be derived after carefully taking the high temperature approximation consistently with the loop expansion (that is, in the expansion (20) of the hypergeometric function the first three terms must be kept).

4 The minimal configuration and Hagedorn temperature for bosonic membranes

Again, the cases of the pure bosonic membrane and of the rigid membrane will be considered.

4.1 Pure bosonic membrane

We start from the formula for the effective action of the bosonic membrane to one-loop order:

\[
S_{\text{eff}}^{(bm)} = \kappa L^2 \beta t \left[ (1 + \sigma_0)^{1/2}(1 + \sigma_1) - \frac{\lambda_0}{2 \sigma_0 + \lambda_1 \sigma_1} \right] - \frac{2 \pi (d - 3) L^2 \zeta'(-2) \lambda_0}{\lambda_1 \beta^2 t^2}. \tag{50}
\]

The conditions for extremum of this action are the following

\[
\begin{align*}
\lambda_0 &= (1 + \sigma_0)^{-1/2}(1 + \sigma_1), \\
\lambda_1 &= (1 + \sigma_0)^{1/2}, \\
\lambda_1 &= -\frac{4 \pi (d - 3) \zeta'(-2)}{\kappa \sigma_0 \beta^3 t^3}, \\
\kappa \beta t \sigma_1 &= \frac{2 \pi (d - 3) \zeta'(-2) \lambda_0}{\lambda_1^2 \beta^2 t^2}, \tag{51}
\end{align*}
\]

Solving these equations for high temperature yields the following behavior for the parameters

\[
\sigma_0 \approx u^{2/3}(\beta t)^{-2} - \frac{1}{3},
\]
\[ \lambda_1 \simeq -u^{1/3}(\beta t)^{-1} - \frac{1}{3}u^{-1/3}\beta t, \]
\[ \sigma_1 \simeq -\frac{1}{3} + \frac{2}{9}u^{-2/3}(\beta t)^2, \]  
\[ \lambda_0 \simeq -\frac{2}{3}u^{-1/3}\beta t, \]  

plus terms of higher order in \( \beta \) and \( \kappa \), being

\[ u \equiv \frac{4\pi(d-3)\zeta'(-2)}{\kappa}. \]  

The asymptotic behavior of the winding soliton mass for high temperature is easily found to be

\[ M_{1}^{(bm)} \simeq \kappa^{2/3}[4\pi(d-3)|\zeta'(-2)|]^{1/3} + \frac{1}{3}\kappa^{4/3}[4\pi(d-3)|\zeta'(-2)|]^{-1/3}(\beta t)^2. \]  

The same type of behavior as for the Nambu-Goto string shows up here: the mass does not feel the increase in temperature and remains constant, while corrections are damped by a second inverse power of \( T \). One could try to identify in this last expression the Hagedorn temperature as the inverse value of \( \beta \) for which the soliton mass vanishes. However this would yield a value proportional to the coupling constant \( \kappa \), which is considered to be small, so that the whole approach would not make any sense. We shall see that the situation in this respect is completely different in the case of the rigid membrane. There, the ideas of Polchinski et al. corresponding to the rigid string can in fact be implemented.

### 4.2 Rigid membrane

The one loop action for the rigid membrane is readily obtained from

\[ S_1^{(rm)} \quad = \quad \frac{(d-3)L^2}{8\pi} \kappa \rho^2 \lambda_1 \left[ 1 - \ln \left( \kappa \rho^2 \lambda_1 (\beta t)^2 \right) \right] + \frac{4\pi(d-3)L^2}{(\beta t)^2} \zeta'(-2) \]
\[ + \quad \frac{(d-3)L^2}{8\pi} \kappa \rho^2 (\lambda_1 - \lambda_0). \]  

Here, terms up to \( k = 2 \) in the expansion (20) of the hypergeometric function of \( (\beta t) \) have been taken into account. It can be checked that we are thus absolutely consistent —to first order in the coupling constant \( \kappa \)— with the high-temperature expansion (the
suppressed contributions are at least of order $L^2 \kappa^{3/2} \rho^3 \beta$). Notice, however, that all higher-order terms would be easy to obtain from (16) and that a consistent loop expansion to any desired order can in fact be performed.

The conditions for extremum of $S^{(rm)} = S_0^{(m)} + S_1^{(rm)}$, eqs. (33) and (55), are obtained by taking the derivatives with respect to the parameters $\lambda_0$, $\lambda_1$, $\sigma_0$ and $\sigma_1$. The result is

$$
\begin{align*}
\sigma_0 &= -\frac{d-3}{4\pi} \frac{\rho^2}{\beta t}, \\
\sigma_1 &= \frac{(d-3)\rho^2}{8\pi \beta t} \left[ 1 - \ln \left( \kappa \rho \lambda_1 (\beta t)^2 \right) \right], \\
\lambda_0 &= (1 + \sigma_1)(1 + \sigma_0)^{-1/2}, \\
\lambda_1 &= (1 + \sigma_0)^{1/2}.
\end{align*}
$$

(56)

Let us now analyze these equations. We can easily identify here the transition that also takes place for the rigid string: for values of the temperature higher than the one coming from the expression

$$
\beta_c^{-1} = \frac{4\pi t}{(d-3)\rho^2},
$$

(57)

the values of the parameters, and hence of the action and of the winding soliton mass squared, acquire an imaginary part. Guided by the fact that in the rigid string case this temperature lies above the Hagedorn temperature—which, on the other hand, is supposed to be high—we conclude that in order that the whole scheme of the string case can be translated to the membrane situation we must demand that $\rho^2 \mu$ be small ($\mu$ a typical mass scale).

It is remarkable that this necessary condition turns out to be sufficient in order to get sensible results. In fact, for the Hagedorn temperature, defined as the value for which the winding soliton mass

$$
M_1^{(rm)} \equiv \frac{S_{eff}^{(rm)}}{L^2}
$$

vanishes, we find

$$
\beta_H^{-1} \simeq \frac{4\pi \sqrt{-\zeta'(-2)} \ t}{\rho \sqrt{-\kappa \ln(\kappa \rho^p)}}.
$$

(59)

Before going on with the analysis, the following remark is in order. The values of the constants which determine the leading behavior of the effective action at high temperature, namely the derivative of the zeta function at the point $-2$ (in general, $-p$, respectively),
have been calculated with unchallenged precision by one of the authors in ref. [22] (in fact, asymptotic expansions for the derivatives to any order of the zeta function, which provide extremely accurate numerical results with just a few terms of the series, have been given there). In particular

\[ \zeta'(-1) = -0.16542115, \quad \zeta'(-2) = -0.03049103. \]  

(60)

We are now in the position to analyze the evolution of the winding-soliton mass for the rigid string at one loop order:

\[ M_{1}^{(rm)} \simeq \frac{4\pi(d-3)\zeta'(-2)}{\beta t^2}. \]  

(61)

Let us recall that we are assuming \( \rho^2 \) small. For the sake of precision, let us take it to be of order \( \kappa \). That is, \( \kappa \mu^{-3} \sim \rho^2 \mu \), and for the adimensional quantity \( \kappa \rho^6 \ll 1 \). When the temperature is varied from slow to higher values, the winding-soliton mass starts from its classical value near \( \kappa \beta t \) and ends at minus infinity. Similarly to what happens to the rigid string, in the meantime, as the temperature increases two different critical values can be identified. The first one occurs when the winding-soliton mass goes through zero. This corresponds to the Hagedorn transition and is obtained at the value

\[ \beta^{-1}_H \simeq \frac{t}{\rho^2 \mu^2 \kappa \sqrt{-\ln(\rho^2 \mu)}}. \]  

(62)

The second privileged value of the temperature occurs at \( T_c \) as given by expression (57). This value lies above the Hagedorn temperature and signalizes the appearance of an imaginary part in the soliton mass squared, as happened with the rigid string.

We can readily check that everything is correct in this regime: (i) the complex transition takes place for high temperature; (ii) the Hagedorn transition takes place before the complex transition; (iii) in the Hagedorn transition the behavior of the constants is completely consistent with the high-temperature expansion, in fact, \( \sigma_0 \) and \( \lambda_1 \) are of order 1, while \( \sigma_1 \) and \( \lambda_0 \) are logarithmically divergent. More precisely, assuming typical values for the constants, such as \( \rho \mu = 10^{-1} \) and \( \kappa \mu^{-3} \simeq \rho^2 \mu = 10^{-2} \), the values of the temperature at the two transition points are

\[ \beta^{-1}_H \simeq 10\pi, \quad \beta^{-1}_c \simeq \frac{4\pi \cdot 10^2}{d-3}. \]  

(63)
Substituting into (55) the values of the parameters at the extremum, we can check that we (consistently) obtain again (61), which is apparently different from the large-$N$ QCD result (although the sign is coincident).

5 Discussion and conclusions

The extension of the method of zeta function regularization developed in [12] has been used in the calculation of the partition functions for strings and membranes. These results are essential in order to decide, once and forever, if QCD can possibly be a certain limit of some string (or membrane, or $p$-brane) theory, through the analysis of the high temperature behavior of the corresponding partition functions.

We have developed a method that allows the explicit calculation of the partition functions of all these theories of extended objects at any value of the temperature and to any desired order of approximation, always consistently with the loop expansion. Although the procedure involves rather elaborate mathematical tools (mainly the so-called zeta function regularization techniques), the final results are expressed in terms of simple series, whose first few terms give normally the desired result. We can also take advantage of the high-precision calculations of the derivatives of the zeta functions that are available in the literature.

The two transitions which were shown in ref. [5] to occur in the case of the rigid string, for increasing temperature, have been also clearly characterized here in the case of the rigid membrane. In particular, assuming a reasonable behavior of the parameters of the model, we have been able to define the Hagedorn transition in general, for the rigid membrane and $p$-brane. And this has always been a very elusive concept.

The high-temperature limits of the free energy per unit area for the bosonic membrane and for the rigid membrane have been found not to agree with the high temperature behavior of large-$N$ QCD. In fact, in the case of the bosonic membrane it remains constant, suffering from the same problem as the Nambu-Goto string. On the opposite side, in the case of the rigid membrane it grows too quickly with $\beta^{-1}$ (namely, as $\beta^{-2}$, contrary to what happens with the Nambu-Goto string and bosonic membrane, which remain constant with $T$). Only the rigid string seems to yield the desired behavior $\beta^{-1}$. With
our method, a systematic analysis of all the distinct cases corresponding to the different kinds of membranes and $p$-branes has been rendered possible.

Acknowledgments

Financial support from DGICYT (Spain), research project PB90-0022, from the Generalitat de Catalunya and from the Alexander von Humboldt Foundation (Germany) is gratefully acknowledged. The paper was finished during a short visit of EE at Trento University. He thanks A.A. Bytsenko, G. Cognola, K. Kirsten, L. Vanno and S. Zerbini for discussions. SDO wishes to thank JSPS (Japan) for partial financial support of this work.
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