Selective survey on spaces of closed subgroups of topological groups

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Abstract. We survey different topologizations of the set $S(G)$ of all closed subgroups of a topological group $G$ and demonstrate some applications in Topological Groups, Model Theory, Geometric Group Theory, Topological Dynamics.

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Some words in place of introduction. For a topological group $G$, $S(G)$ denotes the set of all closed subgroup of $G$. There are many ways to endow $S(G)$ with a topology related to the topology of $G$. Among them, the most intensively studied are Chabauty topology rooted in Geometry of Numbers and the Vietories topology went from General Topology; both coincide if $G$ is compact. The spaces of closed subgroups are interesting by their own sake but also have some deep applications in Topological Groups and Model Theory, Geometric Group Theory and Dynamical Systems. The survey is my subjective look at this area.

Content: Chabauty spaces; Victories spaces; Other topologizations.

1 Chabauty spaces

1.1. From Minkowski to Chabauty. We recall that a lattice $L$ in $\mathbb{R}^n$ is a discrete subgroup of rank $n$. We denote $\min L$ the length of a shortest non-zero vector from $L$, $\text{vol}(\mathbb{R}^n/L)$ is the volume of a basic parallelepiped of $L$.

A sequence $(L_m)_{m \in \omega}$ of lattices in $\mathbb{R}^n$ converges to the lattice $L$ if, for each $m \in \omega$, one can choose a basis $a_1(m), \ldots, a_n(m)$ of $L_m$ and a basis $a_1, \ldots, a_n$ of $L$ such that the sequence $(a_i(m))_{m \in \omega}$ converges to $a_i$ for each $i \in \{1, \ldots, n\}$. This convergence of lattices was introduced by H. Minkowski [4], and its usage in Geometry of Numbers (see [2]) is based on the following theorem of K. Mahler [3].

**Theorem 1.1.** Let $\mathcal{M}$ be a set of lattices in $\mathbb{R}^n$. Every sequence in $\mathcal{M}$ has a convergent subsequence if and only if there exist two constants $C > 0$, $c > 0$ such that $\min L > c$, $\text{vol}(\mathbb{R}^n \setminus L) < C$ for each $L \in \mathcal{M}$.

What we know now as Chabauty topology was invented by C. Chabauty in [4] in order to extend Theorem 1.1 to lattices in connected Lie groups. A discrete subgroup $L$ of a connected Lie group $G$ is called a lattice if the quotient space $G/L$ is compact.
Let $X$ be a Hausdorff locally compact space and let $\exp X$ denotes the set of all closed subsets of $X$. The sets
\[
\{ F \in \exp X : F \cap K = \emptyset \}, \{ F \in \exp X : F \cap U \neq \emptyset \},
\]
where $K$ runs over all compact subsets of $X$ and $U$ runs over all open subsets of $X$, form the subbase of the Chabauty topology on $\exp X$. The space $\exp X$ is compact and Hausdorff. If $X$ is discrete then $\exp X$ is homeomorphic to the Cantor cube $\{0, 1\}^{[X]}$.

We note also that a net $(F_\alpha)_{\alpha \in \mathcal{I}}$ converges in $\exp X$ to $F$ if and only if

- for every compact $K$ of $X$ such that $K \cap F = \emptyset$, there exists $\beta \in \mathcal{I}$ such that $F_\alpha \cap K = \emptyset$ for each $\alpha > \beta$;
- for every $x \in F$ and every neighbourhood $U$ of $x$, there exists $\gamma \in \mathcal{I}$ such that $F_\alpha \cap U \neq \emptyset$ for each $\alpha > \gamma$.

If $G$ is a locally compact group then $S(G)$ is a closed subspace of $\exp G$ (so $S(G)$ is compact); $S(G)$ is called the Chabauty space of $G$.

**Theorem 1.2**[4]. Let $G$ be a connected unimodular Lie group. A set $\mathcal{M}$ of lattices in $G$ is relatively compact in $\mathcal{M}$ if and only if there exists constant $C > 0$ and a neighbourhood $U$ of the identity $e$ of $G$ such that $L \cap U = \{e\}$ and $\text{vol} (G/L) < C$ for each $L \in \mathcal{M}$.

With some technical improvement made in [5], the paper [4] is included in [6, Chapter 8].

**1.2. Pontryagin-Chabauty duality.** This duality was established in [7] and detailed in [8]. We use the standard abbreviation LCA for a locally compact Abelian group. Let $G$ be a LCA-group, $G^\wedge$ denotes its dual group, $G^\wedge = \text{Hom} (G, \mathbb{R}/\mathbb{Z})$ and let $\varphi$ denotes the canonical bijection $S(G) \to S(G^\wedge)$, $\varphi(X) = \{ f \in G^\wedge : X \subseteq \ker f \}$.

**Theorem 1.3.** For every LCA-group $G$, the bijection $\varphi : S(G) \to S(G^\wedge)$ is a homeomorphism.

Typically, Theorem 1.3 applies to replace $S(G)$ by $S(G^\wedge)$ in the case of a compact Abelian group $G$.

In what follows we use the notations: $\mathbb{C}_n$ is the cyclic group of order $n$, $\mathbb{C}_p^\infty$ is the quasi-cyclic (or Prüfer) $p$-group, $\mathbb{Z}$ is the discrete group of integers, $\mathbb{Z}_p$ is the group of $p$-adic integers, $\mathbb{Q}_p$ is the additive group of the field of $p$-adic numbers.

**1.3. $S(G)$ for compact $G$.** The following two lemmas from [9] are the basic technical tools in this area.

**Lemma 1.1.** If $G, H$ are compact groups and $\varphi : G \to H$ is a continuous surjective homomorphism then the mapping $S(G) \to S(H)$, $X \mapsto \varphi(X)$ is continuous and open.

The continuity is easy but to prove the openness we need

**Lemma 1.2.** Let $G$ be a compact group, $X \in S(G)$. Then the following subsets from a base of neighbourhoods of $X$ is $S(G)$:
\[ \mathcal{N}_X(U, N, x_1, \ldots, x_n) = \{ u^{-1}Yu : Y \in \mathcal{S}(G), Y \subseteq XN, Y \cap x_1U \neq \emptyset, \ldots, Y \cap x_nU \neq \emptyset, \} \]

where \( U \) is a neighbourhood of the identity of \( G \), \( N \) is closed normal subgroup such that \( G/N \) is a Lie group, \( x_1, \ldots, x_n \) are arbitrary elements of \( X \), \( n \in \mathbb{N} \).

In particular, if \( G \) is a compact Lie group then Lemma 1.2 states that there is a neighbourhood \( N \) of \( X \) such that each subgroup \( Y \in \mathcal{N} \) is conjugated to some subgroup of \( X \). The key part in the proof of Lemma 1.2 plays the Montgomery-Yang theorem on tubes [10], see also [11, Theorem 5.4 from Chapter 2].

We recall that the cellularity (or Souslin number) \( c(X) \) of a topological space \( X \) is the supremum of cardinalities of disjoint families of open subsets of \( X \). A topological space \( X \) is called dyadic if \( X \) is a continuous image of some Cantor cube \( \{0, 1\}^\kappa \).

The weight \( w(X) \) of a topological space \( X \) is the minimal cardinality of open bases of \( X \).

**Theorem 1.4** [9]. For every compact group \( G \), we have \( c(\mathcal{S}(G)) \leq \aleph_0 \). In addition, if \( w(G) \leq \aleph_1 \) then \( \mathcal{S}(G) \) is dyadic.

**Theorem 1.5** [12]. Let a group \( G \) be either profinite or compact and Abelian. If \( w(G) > \aleph_2 \) then the space \( \mathcal{S}(G) \) is not dyadic.

**Theorem 1.6** [12]. Let \( G \) be an infinite compact Abelian group such that \( w(G) \leq \aleph_1 \). Then the space \( \mathcal{S}(G) \) is homeomorphic to the Cantor cube \( \{0, 1\}^{w(G)} \) if and only if \( \mathcal{S}(G) \) has no isolated points.

An Abelian group \( G \) is called Artinian if every increasing chain of subgroups of \( G \) is finite; every such group is isomorphic to the direct sum \( \oplus_{p \in F} \mathbb{C}_{p^{\infty}} \oplus K \), where \( F \) is a finite set of primes, \( K \) is a finite subgroup. An Abelian group \( G \) is called minimax if \( G \) has a finitely generated subgroup \( N \) such that \( G/N \) is Artinian.

**Theorem 1.7** [12]. For a compact Abelian group \( G \), the space \( \mathcal{S}(G) \) has an isolated point if and only if the dual group \( G^\wedge \) is minimax.

**1.4. \( \mathcal{S}(G) \) for LCA \( G \).** The space \( \mathcal{S}(\mathbb{R}) \) is homeomorphic to the segment \( [0, 1] \). By [13], \( \mathcal{S}(\mathbb{R}^2) \) is homeomorphic to the sphere \( S^4 \). For \( n \geq 3 \), \( \mathcal{S}(\mathbb{R}^n) \) is not a topological manifold and its structure is far from understanding, see [14].

**Theorem 1.8** [15]. The space \( \mathcal{S}(G) \) of a LCA-group \( G \) is connected if and only if \( G \) has a subgroup topologically isomorphic to \( \mathbb{R} \).

If \( F \) is a non-solvable finite group then \( \mathcal{S}(\mathbb{R} \times F) \) is not connected [8, Proposition 8.6].

**Theorem 1.9** [8]. The space \( \mathcal{S}(G) \) of a LCA-group \( G \) is totally disconnected if and only if \( G \) is either totally disconnected or each elements of \( G \) belongs to a compact subgroup.

Some more information on \( \mathcal{S}(G) \) for LCA \( G \) can be find in [8] and references there, in particular, on topological dimension of \( \mathcal{S}(G) \).
By Theorems 1.4 and 1.3, $c(S(G)) \leq \aleph_0$ for every discrete Abelian group. We say that a topological space $X$ has Shanin number $\omega$ if any uncountable family $\mathcal{F}$ of non-empty open subsets of $X$ has an uncountable subfamily $\mathcal{F}'$ such that $\mathcal{F}' \neq \emptyset$. Evidently, if a space $X$ has Shanin number $\omega$ then $c(X) \leq \aleph_0$. By [16, Theorem 1], for every discrete Abelian group $G$, the space $S(G)$ has Shanin number $\omega$. By [16, Theorem 3], for every infinite cardinal $\tau$, there exists a solvable discrete group $G$ such that $c(S(G)) = |G| = \tau$.

1.5. $S(G)$ as a lattice. The set $S(G)$ has the natural structure of a lattice with the operations $\lor$ and $\land$, where $A \land B = A \cap B$ and $A \lor B$ is the smallest closed subgroup of $G$ containing $A$ and $B$. In this subsection, we formulate some results from [17] on interrelations between the topological and lattice structures on $S(G)$.

For $g \in G$, $\langle g \rangle$ denotes the subgroup of $G$ topologically generated by $g$. A totally disconnected locally compact group $G$ is called periodic if $\langle g \rangle$ is compact for each $g \in G$. In this case, $\pi(G)$ denotes the set of all prime numbers such that $p \in \pi(G)$ if and only if there is $g \in G$ such that $\langle g \rangle$ is topologically isomorphic either to $\mathbb{C}_{p^n}$ or to $\mathbb{Z}_p$; this $g$ is called a topological $p$-element.

Theorem 1.10. For a compact group $G$, the following statements are equivalent:

(i) $\land$ is continuous;

(ii) $\land$ and $\lor$ are continuous;

(iii) $G$ is the semidirect product $K \rtimes P$, where $K$ is profinite with finite Sylow $p$-subgroups, $P$ is Abelian profinite and each Sylow $p$-subgroup of $G$ is $\mathbb{Z}_p$, $\pi(K) \cap \pi(P) = \emptyset$ and the centralizer of each Sylow $p$-subgroup of $G$ has finite index in $G$.

Theorem 1.11. For a locally compact group $G$, the operation $\land$ is continuous if and only if the followings conditions are satisfied:

(i) $G$ is either discrete or periodic;

(ii) $\land$ is continuous in $S(H)$ for each compact subgroup $H$ of $G$;

(iii) the centralizer of each topological $p$ element of $G$ is open.

We recall that a torsion group $G$ is layerly finite if the set $\{g \in G : g^n = e\}$ is finite for each $n \in \mathbb{N}$. A layerly finite group $G$ is called thin if each Sylow $p$-subgroup of $G$ is finite (equivalently, $G$ has no subgroup isomorphic to $\mathbb{C}_{p}\infty$).

Theorem 1.12. Let $G$ be a locally compact group. The operations $\land$ and $\lor$ are continuous if and only if $G$ is periodic and topologically isomorphic to $A \times B \times (C \times D)$, where $C$ has a dense thin layerly finite subgroup, $A, B, D$ are Abelian with Sylow $p$-subgroups $\mathbb{C}_{p}\infty$, $\mathbb{Q}_p$ or $\mathbb{Z}_p$, the sets $\pi(A)$, $\pi(B)$, $\pi(G)$, $\pi(D)$ are pairwise disjoint and the centralizer of each Sylow $p$-subgroup of $G$ is open.

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1.6. From Chabauty to local method. A topological group $G$ is called *topologically simple* if each closed normal subgroup of $G$ is either $G$ or $\{e\}$. Every topologically simple LCA-group is discrete and either $G = \{e\}$ or $G$ is isomorphic to $\mathbb{C}_p$.

Following the algebraic tradition, we say that a group $G$ is *locally nilpotent* (solvable) if every finitely generated subgroup is nilpotent (solvable).

In [18, Problem 1.76], V. Platonov asked whether there exists a non-Abelian topologically simple locally compact locally nilpotent group. Now we sketch the negative answer to this question for locally solvable group obtained in [19].

Let $G$ be a locally compact locally solvable group. We take $g \in G \setminus \{e\}$, choose a compact neighbourhood $U$ of $G$ and denote by $F$ the family of all topologically finitely generated subgroups of $G$ containing $g$. We may assume that $G$ is not topologically finitely generated so $F$ is directed by the inclusion $\subset$. For each $F \in F$, we choose $A_F, B_F \in S(F)$ such that $B_F \subset A_F$, $A_F$ and $B_F$ are normal in $F$, $A_F \cap U \neq \emptyset$, $B_F \cap U = \emptyset$ and $A_F/B_F$ is Abelian. Since $S(G)$ is compact, we can choose two subnets $(A_\alpha)_{\alpha \in I}$, $(B_\alpha)_{\alpha \in I}$ of the nets $(A_F)_{F \in F}$, $(B_F)_{F \in F}$ which converges to $A, B \in S(G)$. Then $A, B$ are normal in $G$ and $A/B$ is Abelian. Moreover, $x \not\in B$ and $A \cap U \neq \emptyset$. If $A \neq \{G\}$ then $A$ is a proper normal subgroup of $G$; otherwise $G/B$ is Abelian.

In [20], the Chabauty topology was defined on some systems of closed subgroups of locally compact group $G$. A system $\mathfrak{A}$ of closed subgroups of $G$ is called *subnormal* if

- $\mathfrak{A}$ contains $\{e\}$ and $G$;
- $\mathfrak{A}$ is linearly ordered by the inclusion $\subset$;
- for any subset $\mathfrak{M}$ of $\mathfrak{A}$, the closure of $\bigcup_{F \in \mathfrak{M}} F \in \mathfrak{A}$ and $\bigcap_{F \in \mathfrak{M}} F \in \mathfrak{A}$;
- whenever $A$ and $B$ comprise a jump in $\mathfrak{A}$ (i.e. $B \subset A$ and no members of $\mathfrak{A}$ lie between $B$ and $A$), $B$ is a normal subgroup of $A$.

If the subgroup $A, B$ form a jump then $A/B$ is called a factor of $G$. The system is called *normal* if each $A \in \mathfrak{A}$ is normal in $G$.

A group $G$ is called an RN-group if $G$ has a normal system with Abelian factors. Among the local theorems from [20], one can find the following: if every topologically finitely generated subgroup of a locally compact group $G$ is an RN-group then $G$ is an RN-group. In particular, every locally compact locally solvable group is an RN-group.

In 1941, see [21, pp. 78-83], A.I. Mal’tsev obtained local theorems for discrete groups as applications of the following general local theorem: if every finitely generated subsystem of an algebraic system $A$ satisfies some property $\mathcal{P}$, which can be defined by some quasi universal second order formula, then $A$ satisfies $\mathcal{P}$.

In [22], Mal’tsev’s local theorem was generalized on topological algebraic system. The part of the model-theoretical Compactness Theorem in Mal’tsev arguments plays some convergents of closed subsets. A net $(F_\alpha)_{\alpha \in I}$ of closed subsets of a topological space $X$ $S$-converges to a closed subset $F$ if
• for every $x \in F$ and every neighbourhood $U$ of $x$, there exists $\beta \in \mathcal{I}$ such that $F_\alpha \cap U \neq \emptyset$ for each $\alpha > \beta$;

• for every $y \in X \setminus F$, there exist a neighbourhood $V$ of $y$ and $\gamma \in \mathcal{I}$ such that $F_\alpha \cap V = \emptyset$ for each $\alpha > \gamma$.

Every net of closed subsets of an arbitrary (!) topological space has a convergent subnet. If $X$ is a Hausdorff locally compact space then $S$-convergence coincides with convergence in the Chabauty topology.

1.7 Spaces of marked groups. Let $F_k$ be the free group of rank $k$ with the free generators $x_1, \ldots, x_k$ and let $G_k$ denotes the set of all normal subgroups of $F_k$. In the metric form, the Chabauty topology on $G_k$ was introduced in [23] as a reply on the Gromov’s idea of topologization of some sets of groups [24].

Let $G$ be a group generated by $g_1, \ldots, g_n$. The bijection $x_i \mapsto g_i, g_1, \ldots, g_n$ can be extended to the homomorphism $f : F_k \to G$. With the correspondence $G \mapsto \ker f$, $G_k$ is called the space marked $k$-generated groups.

A couple of papers in development of [23] is directed to understand how large in topological sense are well-known classes of finitely generated groups, or how a given $k$-generated group is placed in $G_k$, see [25]. Among applications of $G_k$, we mention the construction of topologizable Tarski Monsters in [26].

1.8 Dynamical development. Every locally compact group $G$ acts on the Chabauty space $S(G)$ by the rule: $(g, H) \mapsto g^{-1}Hg$. Under this action, every minimal closed invariant subset of $S(G)$ is called a uniformly recurrent subgroup, URS for short. The study of URSs was initiated by Glasner and Weiss [27] with the following observation.

Let a locally compact group $G$ acts on a compact $X$ so that is $G$ minimal, i.e. the orbit of each point $x \in X$ is dense. We consider the mapping $\text{Stab} : X \to S(G)$ defined by $\text{Stab}(x) = \{g \in G : gx = x\}$. Then there is the unique URS contained in the closure of $\text{Stab}(X)$. This URS is called the stabilizer URS. Glasner and Weiss asked whether every URS of a locally compact group $G$ arises as the stabilizer URS of a minimal action of $G$ on a compact space. This question was answered in the affirmative in [28].

2 Vietoris spaces

For a topological space $X$, the Vietoris topology on the set $\exp X$ of all closed subsets of $X$ is defined by the subbase of open sets

$$\{F \in \exp X : F \subseteq U\}, \{F \in \exp X : F \cap V \neq \emptyset\},$$

where $U, V$ run over all open subsets of $X$.

A net $(F_\alpha)_{\alpha \in \mathcal{I}}$ converges to $F$ in $\exp X$ if and only if
• for each open subset $U$ of $X$ such that $F \subseteq U$, there exists $\beta \in I$ such that $F_\alpha \subseteq U$ for each $\alpha > \beta$;

• for each $x \in F$ and each neighbourhood $V$ of $x$, there exists $\gamma \in I$ such that $F_\alpha \cap V \neq \emptyset$ for each $\alpha > \gamma$.

If $X$ is regular then $S(G)$ is closed in $\exp G$. As to my knowledge, the spaces $S(G)$, where $G$ needs not to be compact, endowed with the Vietoris topologies appeared in [29] with characterization of LCA-groups $G$ such that the canonical mapping $\varphi : S(G) \longrightarrow S(G^\land)$ is a homeomorphism.

2.1. Compactness. It is naively to ask a constructive description of arbitrary topological groups $G$ with compact space $S(G)$ because we know nothing even about $G$ with $S(G) = 2$.

**Theorem 2.1.** [30]. Let $G$ be a locally compact group. The space $S(G)$ is compact if and only if $G$ is one of the following groups

(i) $G$ is compact;

(ii) $\mathbb{C}_{p_1^\infty} \times \ldots \times \mathbb{C}_{p_n^\infty} \times K$, where $p_1, \ldots, p_n$ are distinct prime numbers, $K$ is finite and each $p_i$ is not a divisor of $|K|$;

(iii) $\mathbb{Q}_p \times K$, where $K$ is finite and $p$ does not divide $|K|$.

Similar characterization of groups with compact $S(G)$ is given in [31] provided that $G$ has a base at the identity consisting of subgroups.

**Theorem 2.2.** [32]. Let $G$ be a locally compact group. A closed subset $F$ of $S(G)$ is compact if and only if the following conditions are satisfied

(i) every descending chain of non-compact subgroups from $F$ is finite;

(ii) every closed subset $F'$ of $F$ has only finite number of non-compact subgroups maximal in $F$;

(iii) if a closed subset $F'$ of $F$ has no non-compact subgroups then $\bigcup F'$ is compact.

Two corollaries: every compact in $\mathcal{L}(G)$ consisting of non-compact subgroups is scattered; a subset $F$ is compact if and only if $F$ is countably compact.

For locally compact groups with $\sigma$-compact space $S(G)$ see [33], a description of LCA-groups with locally compact space $S(G)$ is obtained in [34].
A topological group $G$ is called \textit{inductively compact} if every finite subset of $G$ is contained in compact subgroup. For a group $G$, $K(G)$ and $IK(G)$ denote the sets of all compact and closed inductively compact subgroups.

**Theorem 2.3.** [35]. \textit{For every locally compact group $G$, $IK(G)$ is the closure of $K(G)$.}

Two corollaries: if $G$ is a connected Lie group then $K(G)$ is closed; $S(G)$ is a $k$-space for each locally compact group $G$ of countable weight, i.e. the topology of $S(G)$ is uniquely determined by the family of all compact subsets of $S(G)$.

### 2.2. Metrizability and normality.

LCA-groups $G$ with metrizable and normal space $S(G)$ are characterized by S. Panasyuk in the candidate thesis \textit{Normality and metrizability of the space of closed subgroups}, Kyiv University, 1989. These lists are completely constructive but too cumbersome so we formulate only

**Theorem 2.4.** \textit{For a discrete Abelian group $G$, the following statements are equivalent}

(i) $S(G)$ is metrizable;

(ii) $S(G)$ is normal;

(iii) $G$ has a finitely generated subgroup $H$ such that $G/H = \mathbb{C}_{p_1^\infty} \times \ldots \times \mathbb{C}_{p_n^\infty}$, where $p_1, \ldots, p_n$ are distinct primes.

In general case, metrizability and normality of $S(G)$ are not equivalent but if $G$ a connected semisimple Lie group then $S(G)$ is metrizable iff $S(G)$ is normal iff $G$ is compact, see [36], [37]. The space $S(G)$ of every connected solvable Lie group is metrizable [36].

### 2.3. Some cardinal invariants.

We remind that $c(X)$ denotes the cellularity of $X$.

**Theorem 2.5.** [9] \textit{For every infinite locally compact group $G$, we have $c(S(G)) \leq c(G)$.}

**Theorem 2.6.** [38]. \textit{For every locally compact group $G$, the following conditions are equivalent}

(i) $S(G)$ is of countable pseudocharacter;

(ii) $S(G)$ is of countable tightness;

(iii) $S(G)$ is sequential;
(iv) $w(G) \leq \aleph_0$.

3 Other topologizations

3.1. Bourbaki uniformities. Let $(X, \mathcal{U})$ be a uniform space. The uniformity $\mathcal{U}$ induces the uniformity $\mathcal{U}$ on the set $\mathcal{F}(X)$ all non-empty closed subsets of $X$ which has as a base the family of sets of the form

$$\{(A, B) \in \mathcal{F}(X) \times \mathcal{F}(X) : B \subseteq U(A), A \subseteq U(B)\},$$

whenever $U \in \mathcal{U}$. The uniformity $\tilde{\mathcal{U}}$ was introduced in [39, Chapter 2, § 1] and $\tilde{\mathcal{U}}$ is called the Bourbaki (sometimes, Hausdorff-Bourbaki) uniformity.

Let $G$ be a topological group. We endow $G$ with the left uniformity $L$ and $F(G)$ with the Bourbaki uniformity $\tilde{L}$. We denote by $L(G)$ and $B(G)$ the subspaces of $F(G)$ consisting of all subgroups and all totally bounded subsets of $G$.

**Theorem 3.1.**[40] Let a group $G$ has a base at the identity consisting of subgroups. The space $L(G)$ is compact if and only if $G$ is totally bounded and $K \cap G$ is dense in $K$ for each closed subgroup $K$ from the completion of $G$.

In particular, if $L(G)$ is compact then $G$ is totally minimal.

**Theorem 3.2.**[40] If a group $G$ is complete in the left uniformity then $B(G)$ is complete.

We recall that a topological group $G$ is almost metrizable if each neighbourhood of $e$ contains a compact subgroup $K$ such that the set of all open subsets containing $K$ has a countable base. Every metrizable and every locally compact topological group are almost metrizable.

**Theorem 3.3.**[40] If an almost metrizable group $G$ is complete in the left uniformity then $F(G)$ is complete.

In [41], Theorem 3.3 is proved with the bilateral uniformity on $G$ (and so on $F(G)$) in place of the left uniformity.

3.2. Functionally balanced groups. For a topological group $G$, the set $F(G)$ has the natural structure of a semigroup with the operation $(A, B) \mapsto cl AB$.

**Theorem 3.4.**[42] For a topological group $G$, the following statements are equivalent

(i) $F(G)$ is a topological semigroup;
(ii) for every subset $X$ of $G$ and every neighbourhood $U$ of $e$, there exists a neighbourhood $V$ of $e$ such that $VX \subseteq XU$;

(iii) every bounded left uniformly continuous function on $G$ is right uniformly continuous.

A topological group $G$ is called balanced (or a SIN-group) if left and right uniformities of $G$ coincide. A group $G$ is called functionally balanced if $G$ satisfies (iii) of Theorem 3.4. The study of functionally balanced groups was initiated by G. Itzkowitz [43].

The equivalence of (ii) and (iii) in Theorem 3.4 is a criterion for a topological group to be functionally balanced. In [44], this criterion was used to show that each almost metrizable functionally balanced group is balanced.

3.3. Lattice topologies. These topologies on a complete lattice $\mathcal{L}(G)$ of closed subgroup are algebraically defined by the lattice structure of $\mathcal{L}(G)$.

For example, a net $(A_\alpha)_{\alpha \in I}$ in $\mathcal{L}(G)$ order-converges to $A \in \mathcal{L}(G)$ if there exist two nets $(B_\alpha)_{\alpha \in I}$, $(C_\alpha)_{\alpha \in I}$ in $\mathcal{L}(G)$ such that, for each $\alpha \in I$, $B_\alpha \subseteq A_\alpha \subseteq C_\alpha$ and $\lor_{\alpha \in I} B_\alpha = \land_{\alpha \in I} C_\alpha = A$. By [45], for a compact group $G$, every net in $\mathcal{L}(G)$ has an order-convergent subset if and only if $\mathcal{L}(G)$ endowed with the Shabauty topology is a topological lattice, see Theorem 1.10.

More on the lattices topologies on $\mathcal{L}(G)$ in the case of a compact $G$ can be find in [46].

3.4. Segment topologies. Let $G$ be a topological group, $\mathcal{P}_G$ is the family of all subsets of $G$, $[G]^<\omega$ is the family of all finite subsets of $G$. Each pair $A, B$ of subsets of $\mathcal{P}_G$ closed under finite unions define the segment topology on $\mathcal{L}(G)$ with a base consisting of the segments.

$$[A, G \setminus B] = \{X \in \mathcal{L}(G) : A \subseteq X \subseteq G \setminus B\}, \ A \in A, \ B \in B.$$  

These topologies are studied in [47] in the following three cases: $A = B = [G]^<\omega$; $A = \mathcal{P}_G$ and $B = [G]^<\omega$; $A = [G]^<\omega$, $B = \mathcal{P}_G$.

3.5. $(\Sigma, \Theta)$-topologies. This general construction for topologizations of the set $\mathcal{L}(G)$ of closed subgroups of a topological group $G$ from [48] produces Chabauty, Vietoris, Bourbaki topologies and a plenty of other topologies.

We assume that, for each $H \in \mathcal{L}(G)$, $\Sigma(H)$ is some family of open subsets of $G$, $\Sigma = \bigcup_{H \in \mathcal{L}(G)} \Sigma(H)$ and the following conditions are satisfied

- if $U, V \in \Sigma(H)$ then $U \cap V$ contains some $W \in \Sigma(H)$;
- for every $U \in \Sigma(H)$, there exists $V \in \Sigma(H)$ such that $U \in \Sigma(K)$ for each $K \in \mathcal{L}(G)$, $K \subseteq V$;
- $\bigcap_{U \in \Sigma(H)} U = H$ for each $H \in \mathcal{L}(G)$.  

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Then the family \( \{ X \in L(G) : X \subseteq U \} \), \( U \in \Sigma \) is a base for the \( \Sigma \)-topology on \( L(G) \).

Let \( \tau \) denotes the topology of \( G \), \( \mathcal{P}_\tau \) is the family of all subsets of \( \tau \). We assume that, for each \( H \in L(G) \), \( \Theta(H) \) is some subset of \( \mathcal{P}_\tau \) such that the following conditions are satisfied

- for every \( \alpha, \beta \in \Theta(H) \), there is \( \gamma \in \Theta(H) \) such that \( \alpha < \gamma, \beta < \gamma \) (\( \alpha < \beta \) means that, for every \( U \in \alpha \), there exists \( V \in \beta \) such that \( V \subseteq U \));
- for every \( \alpha \in \Theta(H) \), there exists \( \beta \in \Theta(H) \) such that if \( K \in L(G) \) and \( K \cap V \neq \emptyset \) for each \( V \in \beta \), then \( \alpha < \gamma \) for some \( \gamma \in \Theta(K) \);
- for each \( H \in L(G) \) and every neighbourhood \( V \) of \( x \), there exists \( \alpha \in \Theta(H) \) such that \( x \in U, U \subseteq V \) for some \( U \in \alpha \).

Then the family \( \{ X \in L(G) : X \cap U \neq \emptyset \text{ for each } U \in \alpha \} \), where \( \alpha \in \Theta(H), H \in L(G) \), is a base for the \( \Theta \)-topology on \( L(G) \).

The upper bound of \( \Sigma \)- and \( \Theta \)-topologies is called the \( (\Sigma, \Theta) \)-topology.

A net \( (H_\alpha)_{\alpha \in \mathcal{I}} \) converges in \( (\Sigma, \Theta) \)-topology to \( H \in L(G) \) if and only if

- for any \( U \in \Sigma(H) \), there exists \( \beta \in \mathcal{I} \) such that \( H_\alpha \subseteq U \) for each \( \alpha > \beta \);
- for any \( \alpha \in \Theta(H) \), there exists \( \gamma \in \mathcal{I} \) such that \( H_\alpha \cap V \neq \emptyset \) for each \( \alpha > \gamma \).

In [48], one can find characterizations of \( G \) with compact and discrete \( L(G) \) in some concrete \( (\Sigma, \Theta) \)-topologies.

3.6. Hyperballeans of groups. Let \( G \) be a discrete group. The set \( \{ Fg : g \in G, F \in [G]^{<\omega} \} \) is a family of balls in the finitary coarse structure on \( G \). For coarse structures and balleans see [49] and [50]. The finitary coarse structure on \( G \) induces the coarse structure on \( L(G) \) in which \( \{ X \in L(G) : X \subseteq FA, A \in FA \} \), \( F \in [G]^{<\omega} \) is the family of balls centered at \( A \in L(G) \). The set \( L(G) \) endowed with structure is called a hyperballean of \( G \). Hyperballeans of groups carefully studied in [51] can be considered as asymptotic counterparts of Bourbaki uniformities.

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