NEVANLINNA PAIR AND ALGEBRAIC HYPERBOLICITY

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Abstract. We introduce the notion of the Nevanlinna pair for a pair $(X, D)$, where $X$ is a projective variety and $D$ is an effective Cartier divisor on $X$. This notion links and unifies the Nevanlinna theory, the complex hyperbolicity (Brody and Kobayashi hyperbolicity), the big Picard type extension theorem (more generally the Borel hyperbolicity), as well as the algebraic hyperbolicity. The key is to use the Nevanlinna theory on parabolic Riemann surfaces recently developed by Păun and Sibony [21].

Dedicated to Professor Shiing-Shen Chern on his 110th birth anniversary

1. Introduction

The classical big Picard theorem says that every holomorphic map from the punctured unit disk $\Delta^*$ into $\mathbb{P}^1(\mathbb{C})$ whose image omits three points can be extended to a holomorphic map from $\Delta$ into $\mathbb{P}^1(\mathbb{C})$. On the other hand, it is well known that $\mathbb{P}^1(\mathbb{C})$ minus three points is hyperbolic. This suggests that the big-Picard-type results are strongly related to the complex hyperbolicity. In fact, a theorem of Kwack and Kobayashi states that every holomorphic map $f : \Delta^* \to U$ extends to a holomorphic map $f : \Delta \to \overline{U}$ if $U$ is a quasi-projective variety and is hyperbolically embedded in some compactification $\overline{U}$ (cf. [17], Theorem 6.3.7 or [18], II §2), a notion invented by Kobayashi expressly for this purpose.

The big-Picard-type extension problem is important and is often linked to the study of the algebraicity of holomorphic maps into a fixed variety. It recently attracted our interest because of the work by Javanpeykar-Kucharczyk [10] on the algebraicity of analytic maps. According to [10], let $X$ and $Y$ be two finite type schemes over $\mathbb{C}$, their associated analytic spaces are denoted by $X^{an}$, $Y^{an}$. Let $\phi : X^{an} \to Y^{an}$ be a holomorphic map. The map $\phi$ is said to be algebraic if there

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is a morphism of \( \mathbb{C} \)-schemes \( f : X \to Y \) such that \( f^{an} = \phi \). To “algebraize” (in the language of \([16]\)), one usually needs the extension property. For this reason, in \([16]\), they introduced a new notion of hyperbolicity: a quasi-projective variety \( U \) is said to be \textit{Borel hyperbolic} if any holomorphic map from a quasi-projective variety \( V \) to \( U \) is necessarily algebraic. They proved that if the big Picard property for a quasi-projective variety \( U \) holds, then \( U \) is Borel hyperbolic. We refer the reader to \([16]\), §1, for their motivation on the Borel hyperbolicity. Subsequently, Deng-Lu-Sun-Zuo \([6]\), and Deng \([5]\) also obtained results about the big Picard property. For the same reason, the big-Picard-type results are also related to the algebraic hyperbolicity, a notion introduced by J. P. Demailly \([4]\) and generalized to the logarithmic pairs by Xi Chen \([3]\). It is known, due to the result of Pacienza and Rousseau \([20]\), if \( X \setminus D \) is hyperbolically imbedded in \( X \) then \( (X, D) \) is algebraically hyperbolic.

The original proof of Kwack and Kobayashi’s extension theorem is rather conceptual and involved. It seems to us that the Nevanlinna theory (in particular, the application of the logarithmic derivative lemma) is a more natural approach to attack the Picard extension problems, as shown by M. Green \([13]\). Recently, Y.T. Siu \([25]\) revived this method. To study the \( \triangle^* \)-extension property, we make a change of variable \( z := 1/\zeta \). Then the problem is reduced to the existence of essential singularity at \( \infty \). It is known that, for a holomorphic map \( \phi : \mathbb{C} - \overline{\Delta(r_0)} \to \mathbb{P}^n(\mathbb{C}) \), if \( T_\phi(r, r_0) \leq \text{exc} \log r \) (see below for the notation) then \( \phi \) can be extended to a holomorphic map from \( \mathbb{C} \cup \{\infty\} - \overline{\Delta(r_0)} \) into \( \mathbb{P}^n(\mathbb{C}) \). For this reason, we introduce the concept of \textit{quantitatively big Picard} (see Definition \( \ref{def-qbp} \) below), and prove the following result.

\textbf{Theorem 1.1} (See Theorem \( \ref{thm-qbp} \)). \textit{If \( X \setminus D \) is hyperbolically imbedded in \( X \), then \( X \setminus D \) is quantitatively big Picard.}

More generally, we introduce the concept of \textit{Nevanlinna pair} through holomorphic maps on an open parabolic Riemann surface. We briefly recall some notations (For details, see Section 2 below and \([21]\)). A non-compact Riemann surface \( Y \) is \textit{parabolic} if it admits a parabolic exhaustion function, i.e. a continuous proper function \( \sigma : Y \to [0, \infty) \) such that \( \log \sigma \) is harmonic outside a compact subset of \( Y \). However, in this paper, we restrict it to a special case for simplicity which is sufficient for our purpose, i.e., by a \textit{parabolic Riemann surface} we mean an open Riemann surface \( Y \), together with a proper continuous function \( \sigma : Y \to [0, \infty) \) (called a parabolic exhaustion function) such that
• $\log \sigma$ is harmonic outside possibly a finite set $\Sigma := \{P_1, \ldots, P_k\}$ on $Y$.
• At each $P_i \in \Sigma$, in a coordinate chart $(U, z)$ centered at $P_i$ that does not contain other points in $\Sigma$, $\log \sigma(z) = k_i \log |z| + h_{P_i}(z)$, where $h_{P_i}$ is a harmonic function on $U$.

Note this notion of parabolic Riemann surface, first appeared in [12], is slightly stronger than the standard one given in [21]. We also note that all the inequalities we established in the paper still hold on a standard parabolic Riemann surface (in the sense of [21]) with the term $2\varsigma \log r$ (see below) replaced by a constant multiple $O(\log r)$ of $\log r$ depending on $Y$. Denote by $B(r)$ the parabolic disk $\{y \in Y : \sigma(y) < r\}$ and by $S(r)$ the parabolic circle $\{y \in Y : \sigma(y) = r\}$. By Sard’s theorem, $S(r)$ is smooth for almost all $r > 0$, in that case we denote by $\mu_r$ the measure induced by the differential $d\log \sigma|_{S(r)}$ and write $d\mu_r = d\log \sigma|_{S(r)}$. Let

\begin{equation}
\varsigma := \int_{S(r)} d\mu_r,
\end{equation}

which is independent of $r$ for $r$ large enough because $\log \sigma$ is harmonic. Let $\chi_{\sigma}(r)$ be the Euler characteristic of $B(r)$, and define

\begin{equation}
\mathcal{X}_{\sigma}(r) := \int_1^r \chi_{\sigma}(t) \frac{dt}{t}.
\end{equation}

Throughout the paper, we fix a nowhere vanishing global holomorphic vector field $\xi \in \Gamma(Y, T_Y)$ on $Y$. Such a vector field exists because $Y$ is non-compact and consequently its holomorphic tangent bundle $T_Y$ is holomorphically trivial (see Theorem 5.3.1, [10]). We also define

\begin{equation}
\mathcal{E}_{\sigma}(r) := \int_{S(r)} \log^{-} |d\sigma(\xi)|^2 d\mu_r,
\end{equation}

where, for a positive real number $x$, $\log^+ x = \max\{0, \log x\}$ and $\log^- x = -\min\{0, \log x\}$. Note that $\mathcal{E}_{\sigma}(r)$ is closely related to $\mathcal{X}_{\sigma}(r)$. When $\mathcal{X}_{\sigma}(r) = O(\log r)$, we also have $\mathcal{E}_{\sigma}(r) = O(\log r)$. More precisely, according to Lemma 2.10 we have

\begin{equation}
\int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r = -\mathcal{X}_{\sigma}(r) + 2\varsigma \log r + O(1).
\end{equation}

**Definition 1.2.** Let $X$ be a projective variety and $D$ be an effective Cartier divisor on $X$. We say that $(X, D)$ is a **Nevanlinna pair** if there is a positive $(1, 1)$-form $\eta$ on $X$ such that for any parabolic Riemann surface $Y$ and every holomorphic map $f : Y \to X$ with $f(Y) \not\subset D$ and for $\delta > 0$, one has

\begin{equation}
T_{f,0}(r) \leq_{\text{exc}} \overline{\nabla} f(r, D) - \mathcal{X}_{\sigma}(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_{\sigma}(r) + O(1),
\end{equation}
where $O(1)$ is a constant which may depend on $f$ and $Y$, $ς$ is the constant given by (1), and $\leq_{exc}$ means that the inequality holds for all $r \in (0, \infty)$ except a set of finite measure depending on $δ$.

**Remark:** Note that the coefficient $(δ + 2ς)$ appeared before $\log r$ is crucial in dealing with the algebraic hyperbolicity, hence one of the main focuses is to keep the constants appeared in the paper to be independent of $f$ and $Y$. For this reason, throughout the paper, when we mention a constant $C > 0$, we always mean that $C$ is independent of $f$ and $Y$ unless otherwise specified. On the other hand, when we write $O(\log r)$ or $O(1)$, we mean that the involved constants may depend on $f$ and $Y$.

Note that the complex plane $\mathbb{C}$ together with exhaustion function $ς(z) = |z|$ is a parabolic Riemann surface with $ς = \frac{1}{2}$. In this case, $X_ς(r) = \log r, E_ς(r) = O(1)$, so if $(X, D)$ is a Nevanlinna pair then there exists a positive $(1, 1)$-form $η$ on $X$ such that for every holomorphic map $f : \mathbb{C} \to X \setminus D$ and $δ > 0$,

$$T_{f,η}(r) \leq_{exc} δ \log r + O(1).$$

This implies that $f$ is constant. Hence we have the following statement: *If $(X, D)$ is a Nevanlinna pair, then $X \setminus D$ is Brody hyperbolic.*

Using the recent result of Brotbek and Brunebarbe (II Theorem 6.2), we prove the following result.

**Theorem 1.3** (See Theorem 2.7). *If $X \setminus D$ is hyperbolically imbedded in $X$, then $(X, D)$ is a Nevanlinna pair.*

According to Demailly [4] and Chen [3], the pair $(X, D)$ is said to be *algebraically hyperbolic* if there exists a positive $(1, 1)$-form $ω$ on $X$ such that for any compact Riemann surface $R$ and every holomorphic map $f : R \to X$ with $f(R) \not\subset D$, the following inequality holds

$$\int_R f^*ω \leq \tilde{n}_f(D) + \max\{0, 2g - 2\},$$

where $\tilde{n}_f(D)$ is the number of points of $f^{-1}(D)$ on $R$ and $g$ is the genus of $R$. We show that Nevanlinna pair implies algebraically hyperbolic.

**Theorem 1.4** (See Theorem 5.1). *If $(X, D)$ is a Nevanlinna pair, then $(X, D)$ is algebraically hyperbolic.*
We summarize our results in the following diagram:

\[ X \setminus D \text{ is hyperbolically embedded} \quad \Rightarrow \quad (X, D) \text{ is a Nevanlinna Pair} \quad \Rightarrow \quad X \setminus D \text{ is Brody hyperbolic} \quad \Rightarrow \quad (X, D) \text{ is algebraically hyperbolic} \quad \Rightarrow \quad X \setminus D \text{ is Picard hyperbolic} \]

We conjecture that if \((X, D)\) is a Nevanlinna pair, then \(X \setminus D\) is Kobayashi hyperbolic. Therefore, in our opinion, Nevanlinna pair is a suitable notion to unify the Nevanlinna theory, the complex hyperbolicity (Brody and Kobayashi hyperbolicity), the big Picard type extension property (more generally the Borel hyperbolicity), as well as the algebraic hyperbolicity.

When \(X\) is \(\mathbb{P}^n(\mathbb{C})\) and \(D\) consists of hyperplanes, we prove that Nevanlinna pair, Brody hyperbolicity and the big Picard type extension theorem (more generally the Borel hyperbolicity) are indeed equivalent.

**Theorem 1.5** (See Theorem 4.2). Let \(\mathcal{H}\) be a finite set of hyperplanes in \(\mathbb{P}^n(\mathbb{C})\). Let \(|\mathcal{H}| := \sum_{H \in \mathcal{H}} H\). Then the following statements are equivalent.

(a) \((\mathbb{P}^n(\mathbb{C}), |\mathcal{H}|)\) is a Nevanlinna pair.
(b) \(\mathbb{P}^n(\mathbb{C})\setminus|\mathcal{H}|\) is Brody hyperbolic.
(c) \(\mathbb{P}^n(\mathbb{C})\setminus|\mathcal{H}|\) is Picard hyperbolic.

In this paper, we also provide examples of the Nevanlinna pair \((X, D)\) where, in some cases, \(X \setminus D\) may not be hyperbolically imbedded in \(X\). Indeed, in each case, we obtain a more precise Second Main Theorem type inequality for holomorphic maps on open parabolic Riemann surfaces.

**Theorem 1.6** (See Theorem 3.4 and Theorem 5.1). Let \(X\) be a projective manifold of dimension \(n \geq 2\) and let \(A\) be a very ample line bundle over \(X\). Let \(D \in |A^m|\) be a general smooth hypersurface with

\[ m \geq (n + 2)^{n+3}(n + 1)^{n+3}. \]

Then there exists a constant \(C > 0\) such that for every holomorphic map \(f : Y \to X\) with \(f(Y) \not\subset D\) where \(Y\) is a parabolic Riemann surface, we have, for \(\delta > 0\),

\[ T_{f, A}(r) \leq \text{exc } \frac{\overline{N}_f(r, D)}{C} + C(\log T_{f, A}(r) - \chi_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1). \]
In particular \((X, D)\) is a Nevanlinna pair. The pair \((X, D)\) is also algebraically hyperbolic.

**Theorem 1.7** (See Theorem 3.6 and Theorem 5.1). Let \(A\) be an abelian variety and \(D\) be an ample divisor on \(A\). Then there exists a constant \(C > 0\) such that for every holomorphic map \(f : Y \to A\) with \(f(Y) \not\subset D\) where \(Y\) is a parabolic Riemann surface, we have, for some \(k_0 > 0\),

\[
T_{f,D}(r) \leq \text{exc} \left[ k_0 \right] f(r, D) + C\left(\log T_{f,A}(r) - X_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)\right) + O(1).
\]

In particular, \((A, D)\) is a Nevanlinna pair. The pair \((A, D)\) is also algebraically hyperbolic.

We note that recently Yamanoi [28] proved that \(A \setminus D\) is Kobayashi hyperbolic under the conditions in Theorem 1.7.

**Remark.** During the time we were working on the manuscript, we were not aware whether \(X \setminus D\) is hyperbolically imbedded in \(X\) implies that \((X, D)\) is a Nevanlinna pair, so we put this as a conjecture in the original version. When the manuscript was nearly finished, we discovered the preprint [1] and found that Theorem 6.2 in [1] is exactly, thanks to Brothbek and Brunebarbe, what we are looking for. Therefore, we decided to include their result as Theorem 1.8 (as well as Theorem 2.7) in the manuscript.

## 2. Quantitative big Picard and the Nevanlinna pair

### 2.1. Relating different analytic notions of hyperbolicity (Picard hyperbolicity in [5] and Borel hyperbolicity in [16]).

We first gather known results relating the different notions of hyperbolicity. We start with an extension property for holomorphic maps. Denote by \(\triangle(r)\) the disk centered at the origin with radius \(r\), and by \(\triangle^*(r) := \triangle(r) - \{0\}\) the puncture disk. We use \(\triangle\) for the unit disk and \(\triangle^*\) the unit punctured disk.

**Definition 2.1** ([16], [5]). A finite type separated scheme \(X\) over \(\mathbb{C}\) has the \(\triangle^*\)-extension property (see [16]), or is Picard hyperbolic (see [5]), if there is an open immersion \(X \subset \overline{X}\) with \(\overline{X}\) proper over \(\mathbb{C}\) such that, for every morphism \(f : \triangle^* \to X^{an}\), there is a morphism \(\triangle \to \overline{X}^{an}\) which extends \(f\).
The classical big Picard theorem can thus be stated as that $\mathbb{P}^1\{0, 1, \infty\}$ has the $\Delta^*$-extension property, or $\mathbb{P}^1\{0, 1, \infty\}$ is Picard hyperbolic.

Let $X$ be a complex analytic space. Recall that $X$ is called Brody hyperbolic provided that every holomorphic map $f : \mathbb{C} \to X$ is constant, and $X$ is said to be Kobayashi hyperbolic if the Kobayashi pseudo-distance $d_X$ is a distance. It is clear that if $X \setminus D$ is Picard hyperbolic then $X \setminus D$ is Brody hyperbolic. Conversely, according to Kwack and Kobayashi, if $U$ is a quasi-projective variety and is hyperbolically imbedded in some compactification $\overline{U}$, then $U$ has the $\Delta^*$-extension property, i.e. $U$ is Picard hyperbolic.

It is important to indicate that if the $\Delta^*$-extension property holds, then one can also obtain the higher-dimensional extension property. This indeed is a consequence of the deep extension theorem of meromorphic maps due to Siu [24]. The following is the precise statement.

**Proposition 2.2** (See [5], Proposition 3.4). Let $Y^0$ be a Zariski open set of a compact Kähler manifold $Y$. Assume that $Y^0$ is Picard hyperbolic. Then any holomorphic map $f : \Delta^p \times (\Delta^*)^q \to Y^0$ extends to a meromorphic map $f : \Delta^p+q \to Y$. In particular, any holomorphic map $g$ from a Zariski open set $X^0$ of a compact complex manifold $X^0$ to $Y^0$ extends to a meromorphic map from $X$ to $Y$.

2.2. **Quantitative big Picard and Nevanlinna pair.** To study the $\Delta^*$-extension property, we make a change of variable $z := 1/\zeta$. Then the problem is reduced to the statement that there is no essential singularity at $\infty$. In general, corresponding to the big Picard Theorem, one studies the extendability across $\infty$ of a holomorphic map $\mathbb{C} - \overline{\Delta(r_0)} \to X$ to $\mathbb{C} \cup \{\infty\} - \overline{\Delta(r_0)} \to X$ for some fixed $r_0 \geq 1$, where $X$ is a projective variety. In this paper, we use Nevanlinna theory to deal with this problem. Let $\phi : \mathbb{C} - \overline{\Delta(r_0)} \to X$ be a holomorphic map. Let $\eta$ be a positive $(1, 1)$-form on $X$. We define, for any fixed $r_1 > r_0$,

$$T_{\phi, \eta}(r, r_1) = \int_{r_1}^{r} \left( \int_{r_1 < |t| < r} f^* \eta \right) \frac{dt}{t}.$$  

Sometimes we just write it as $T_{\phi, \eta}(r)$ when $r_1$ is fixed. For the $\phi : \mathbb{C} - \overline{\Delta(r_0)} \to \mathbb{P}^n(\mathbb{C})$, we write, for $r_1 > r_0$, $T_{\phi, r_1} := T_{\phi, \eta}(r, r_1)$ when $\eta$ is taken as the Fubini-Study form of $\mathbb{P}^n(\mathbb{C})$.

The starting point is the following lemma.
Lemma 2.3 (See [25], Proposition 6.2). Let $\phi : \mathbb{C} - \Delta(r_0) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. If $T_{\phi}(r,r_1) \leq \text{exc } O(\log r)$, then $\phi$ can be extended to a holomorphic map from $\mathbb{C} \cup \{\infty\} - \overline{\Delta(r_0)}$ to $\mathbb{P}^n(\mathbb{C})$.

Motivated by the above lemma, we introduce the following definition.

Definition 2.4. Let $X$ be a projective variety and $D$ be an effective Cartier divisor on $X$. We say a holomorphic map $f : \mathbb{C} - \Delta(r_0) \to X \setminus D$ has quantitative big Picard property if there exists a positive $(1,1)$-form $\eta$ on $X$ such that, for a fixed $r_1$ with $r_1 > r_0$,

$$T_{f,\eta}(r,r_1) \leq \text{exc } O(\log r).$$

We say that $X \setminus D$ is quantitatively big Picard if every holomorphic map $f : \mathbb{C} - \Delta(r_0) \to X \setminus D$ has quantitative big Picard property.

From Lemma 2.3 we see that the quantitative big Picard implies Picard hyperbolic.

Theorem 2.5. If $X \setminus D$ is hyperbolically imbedded in $X$, then $X \setminus D$ is quantitatively big Picard.

Proof. Let $f : \mathbb{C} - \Delta(r_0) \to X \setminus D$ be a holomorphic map. It suffices to consider the case $r_0 = 1$. Let $\omega$ be a positive $(1,1)$-form on $X$ and denote by $\| \cdot \|_\omega$ the associated norm. Let $k_{X \setminus D}$ be the Kobayashi-Royden infinitesimal pseudo-norm on $X \setminus D$. Since $X \setminus D$ is hyperbolically imbedded in $X$, there exists a positive real number $c > 0$ such that

$$\| \cdot \|_\omega \leq ck_{X \setminus D}. \tag{4}$$

On the other hand, by the distance decreasing property of the Kobayashi-Royden pseudo-norm,

$$f^*k_{X \setminus D} \leq k_{\mathbb{C} - \Delta(1)}. \tag{5}$$

Therefore

$$\| \cdot \|_{f^*\omega} \leq ck_{\mathbb{C} - \Delta(1)}. \tag{6}$$

At the level of forms, this yields

$$f^*\omega \leq c\sqrt{-1} \frac{dz \wedge d\bar{z}}{|z|^2 \log^2 |z|^2}.$$
Hence
\[
\int_{r_1 \leq |z| \leq \rho} f^* \omega \leq c \left( \int_{r_1}^\rho \frac{1}{t^2 \log t} \frac{dt}{t} \right) = c \left( \frac{1}{\log r_1} - \frac{1}{\log \rho} \right).
\]
Thus,
\[
T_{f,\omega}(r, r_1) = \int_{r_1}^r \left( \int_{r_1 \leq |z| \leq \rho} f^* \omega \right) \frac{d\rho}{\rho} \leq C \log r,
\]
where \(C\) is a constant depending on \(r_1\).

Note that the above theorem gives a new and simpler proof of the result of Kwack and Kobayashi.

**Theorem 2.6** (see [6], Theorem A). Let \(X\) be a projective manifold and \(\omega\) be a Kähler metric on \(X\). Let \(D\) be a simple normal crossing divisor on \(X\). Let \(f : \Delta^* \to X \setminus D\) be a holomorphic map. Assume that there is a Finsler pseudo-metric \(h\) of \(T_X(-\log D)\) such that \(\|f'(z)\|_h^2 \neq 0\), \(\log \|f'(z)\|_h^2\) is locally integrable and that the following inequality holds in the sense of currents
\[
\partial\bar{\partial} \log \|f'(z)\|_h^2 \geq f^* \omega.
\]
Then \(f\) has quantitative big Picard property.

**Proof.** We make a change variable of \(z := 1/\zeta\) and consider \(f : \mathbb{C} - \overline{\Delta(1)} \to X\).

From the assumption we get
\[
T_{f,\omega}(r, r_1) = \int_{r_1}^r \left( \int_{r_1 \leq |z| \leq t} f^* \omega \right) \frac{dt}{t} \leq \int_{r_1}^r \left( \int_{r_1 \leq |z| \leq t} \partial\bar{\partial} \log \|f'(z)\|_h^2 \right) \frac{dt}{t} \leq \int_0^{2\pi} \log \|f'(re^{i\theta})\|_h^2 \frac{d\theta}{2\pi} - \int_0^{2\pi} \log \|f'(r_1 e^{i\theta})\|_h^2 \frac{d\theta}{2\pi} + O(1),
\]
where the third equality follows from the Green-Jensen formula. We now estimate the integral above. According to ([14], I.3), a function \(f\) is regular on \(X \subset \mathbb{P}^N\) if, for every \(P \in X\), there is a Zariski open neighborhood \(U\) with \(P \in U\), and homogeneous polynomials \(g, h\) in \((N + 1)\)-variables of the same degree with \(h\) is nowhere zero on \(U\), and \(f = g/h\) on \(U\). Since \(D\) is a simple normal crossing divisor, we can choose, at each point \(P \in \text{supp} D\), a local coordinate chart \((U, z_1, \ldots, z_n)\) near \(P\) such that \(D|_U = (z_1 \cdots z_s = 0)\). From the remark above, we see that \(z_1, \ldots, z_n\) are (global) rational functions on \(X\). Since \(X\) is compact, we can cover \(X\) by finitely many
open subsets (in the complex topology) \( \{ U_\lambda \}_{\lambda \in \Lambda} \) with coordinates \( (z_{\lambda,1}, \ldots, z_{\lambda,n}) \) such that \( D|_{U_\lambda} = (z_{\lambda,1} \cdots z_{\lambda,s(\lambda)} = 0) \) for some \( 1 \leq s(\lambda) \leq n \) (note: \( s(\lambda) \) may be empty, i.e. \( D|_{U_\lambda} \) may be an empty set). Take a relative compact subcovering \( \{ V_\lambda \}_{\lambda \in \Lambda} \) with \( \overline{V_\lambda} \subset U_\lambda \) (note that all closure are taken with respect to the complex topology). Denote by \( f_{\lambda,i} = z_{\lambda,i} \circ f \). Then, for every \( \lambda \in \Lambda \), there is a constant \( C_\lambda > 0 \) such that for \( t \in f^{-1}(V_\lambda) \),

\[
\| f'(t) \|^2_\lambda \leq C_\lambda \left( \sum_{i=1}^{s(\lambda)} \left| \frac{f'_{\lambda,i}(t)}{f_{\lambda,i}(t)} \right|^2 \right) \leq C_\lambda \left( \sum_{i=1}^{s(\lambda)} \left| \frac{f'_{\lambda,i}(t)}{f_{\lambda,i}(t)} \right|^2 + \sum_{j=s(\lambda)+1}^{n} \left| f'_{\lambda,j}(t) \right|^2 \right),
\]

where \( \hat{C}_\lambda := \max_{1 \leq i \leq n} \sup_{x \in V_\lambda} |z_{\lambda,i}(x)| \), which exists because \( z_{\lambda,i} \) are holomorphic on \( U_\lambda \) and \( \overline{V_\lambda} \) is compact. Notice that \( \Lambda \) is finite, by the logarithmic derivative lemma,

\[
\int_0^{2\pi} \log^+ \| f'(re^{i\theta}) \| r \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \log^+ \left( \sum_{\lambda \in \Lambda} \sum_{j=1}^{\lambda} \left| \frac{f'_{\lambda,j}(re^{i\theta})}{f_{\lambda,j}(re^{i\theta})} \right|^2 \right) \frac{d\theta}{2\pi} + O(1) \leq \sum_{\lambda \in \Lambda} \sum_{j=1}^{\lambda} \int_0^{2\pi} \log^+ \left| \frac{f'_{\lambda,j}(re^{i\theta})}{f_{\lambda,j}(re^{i\theta})} \right|^2 \frac{d\theta}{2\pi} + O(1) \leq O(\log^+ T_{x_{\lambda,j} \circ f}(r,r_1) + \log r) \leq O(\log^+ T_{f,\omega}(r,r_1) + \log r),
\]

where we used the fact that, for rational function \( g \), \( T_{g \circ f}(r,r_1) \leq T_{f,\omega}(r,r_1) + O(1) \) (see [19], Theorem 2.13). So the theorem is proved by combining (5) and (7).

\[ \square \]

Using the above result, Deng-Lu-Sun-Zuo [6] proved that the big Picard theorem holds for the moduli stack \( M_h \) of polarized complex projective manifolds with semiample canonical bundle and Hilbert polynomial \( h \), i.e., for an algebraic variety \( U \), a compactification \( Y \) and a quasi-finite morphism \( U \to M_h \) induced by an algebraic family over \( U \) of such manifolds, \( U \) is quantitatively big Picard.

We introduce the following more general definition of Nevanlinna pair, motivated by the above discussion, as well as the Second Main Theorems in Nevanlinna theory. We use the Nevanlinna theory on open parabolic Riemann surfaces recently developed by Paun and Sibony [21]. Note that Stoll [27] developed the Nevanlinna theory on a more general parabolic complex manifold through Ahlfors’ approach.
Let $Y$ be an open parabolic Riemann surface defined in the introduction. Denote by $B(r) := (\sigma < r)$ the open parabolic disk of radius $r$ and by $S(r) := (\sigma = r)$ the parabolic circle of radius $r$. When $r$ is a regular value of $\sigma$ (which is the case for almost all $r$), $S(r)$ is smooth and one considers on it the measure $d\mu_r := d^c \log \sigma|_{S(r)}$. Fix $r_0 > 0$ such that $\log \sigma$ is harmonic outside $\overline{B(r_0)}$. Stokes theorem then implies that, for $r, r_1 > r_0$,

$$
\int_{S(r)} d\mu_r - \int_{S(r_1)} d\mu_r = \int_{B(r) \setminus B(r_1)} dd^c \log \sigma = 0.
$$

Thus $\int_{S(r)} d\mu_r$ is independent of $r$ when $r$ is large enough. We denote this constant by $\varsigma$. Let $X$ be a projective variety and $D$ be an effective Cartier divisor on $X$. Let $f : Y \to X$ be a holomorphic map, and let $\eta$ be a positive $(1, 1)$ form on $X$. We define

$$
T_{f, \eta}(r) := \int_{S(r)} \left( \int_{B(t)} f^* \eta \right) \frac{dt}{t}.
$$

Let $L$ be an ample line bundle on $X$, $h$ be a Hermitian metric on $L$, we write $T_{f,L}(r) := T_{f,c_1(L,h)}(r)$. It is independent, up to a bounded term, the choices of the metric $h$. For an effective Cartier divisor $D$, let $[D]$ denote the corresponding line bundle. Then $[D] = L_1 \otimes L_2^*$ for ample line bundles $L_1, L_2$, and we define $T_{f,D}(r) := T_{f,L_1}(r) - T_{f,L_2}(r)$. Define

$$
m_{f}(r,D) = \int_{S(r)} \log \frac{1}{\|s_D \circ f\|^2_h} d\mu_r,
$$

where $s_D$ is the canonical section of $[D]$ and $h$ is a Hermitian metric on $[D]$. For the divisor $f^*D$ on $Y$, we can write $f^*D = \sum_{a \in Y} \nu(a) \cdot a$, where $\nu(a)$ is the order of $f^*D$ at $a$. For an integer $k$, define the $k$-th truncated divisor $f^*[k] := \sum_{a \in Y} \min\{\nu(a), k\} a$. Let $n_{f,D}(r) := \sum_{a \in B(r)} \nu_{f^*D}(a)$ be degree of $f^*D$ counted inside $B(r)$, and $n^*[k]_{f,D}(r)$ denote its truncated version. In the case $k = 1$, $n^*[1]_{f,D}(r)$ coincides with the number of set theoretic preimage of $D$ in $B(r)$, which we also denote it by $\bar{n}_{f,D}(r)$. Let

$$
N_f(r,D) := \int_{1}^{r} n_{f,D}(t) \frac{dt}{t},
$$

and similarly

$$
N_{f,[k]}(r,D) := \int_{1}^{r} n^*[k]_{f,D}(t) \frac{dt}{t},
$$

$$
\mathcal{N}_f(r,D) := \int_{1}^{r} \bar{n}_{f,D}(t) \frac{dt}{t}.
$$
By identifying the divisor $f^*D$ with its current of integration and by the Poincare-Lelong formula, the following equality holds in the sense of currents:

$$-dd^c [\log \| s_D \circ f \|^2_h] = f^* c_1([-D], h) - f^* D.$$ 

Then the Green-Jensen formula for parabolic Riemann surfaces (see (11) below) implies the First Main Theorem

(8) $$m_f (r, D) + N_f (r, D) = T_{f,D} (r) + O(1).$$

With these notions, we define the notion Nevanlinna pair for $(X, D)$ as in Definition 1.2 in the introduction.

We have shown that if $(X, D)$ is a Nevanlinna pair then $X \setminus D$ is Brody hyperbolic. Conversely, similar to the Theorem 6.2 in [1], we have the following result.

**Theorem 2.7.** Assume that $X \setminus D$ is hyperbolically imbedded in $X$. Then there is a positive $(1, 1)$-form $\eta$ on $X$ such that for any parabolic Riemann surface $Y$ and every holomorphic map $f : Y \to X$ with $f(Y) \not\subset D$ and for $\delta > 0$, one has

$$T_{f,\eta} (r) \leq \text{exc} \nabla f (r, D) - \mathcal{X}_\sigma (r) + (\delta + 2\varsigma) \log r + O(1).$$

In particular, $(X, D)$ is a Nevanlinna pair.

The above theorem, with the term involving $\log r$ explicitly given by $(\delta + 2\varsigma) \log r$, is slightly stronger than the Theorem 6.2 of Brotkbe and Brunebarbe [1]. Nevertheless we emphasize that we claim no originality here.

To prove the theorem, we first recall some notations. Let $M$ be a Riemann surface with a local coordinate $z = x + \sqrt{-1} y$. Denote by

$$d = \partial + \bar{\partial}, \quad d^c = \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial)$$

so that $dd^c = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial}$.

A conformal pseudo-metric (resp. metric) on $M$ is given by $h = 2\lambda (dx^2 + dy^2) = 2\lambda dz d\bar{\xi}$ in any local coordinate $(U, z)$ with $z = x + iy$, where $\lambda$ is a nonnegative (resp. positive) smooth function. It induces a pseudo-norm on the holomorphic tangent bundle $T_M$ of $M$ defined by $\| \xi \|^2_h = h(\xi, \bar{\xi}) = 2\lambda |\gamma|^2$, where $\xi = \gamma \frac{\partial}{\partial z} \in \Gamma(U, T_M)$.

Note that, in particular,

$$\lambda (z) = \frac{1}{2} \left\| \frac{\partial}{\partial z} \right\|^2_h.$$ 

The Gaussian curvature of $h$ is given by

(9) $$K = -\frac{1}{4\lambda} \Delta \log \lambda,$$
where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$ is the usual Laplacian. Let

$$\omega = \lambda(z) \sqrt{-1} \frac{dz \wedge d\bar{z}}{2\pi}$$

be its associated Kähler pseudo form (or pseudo metric form). To $\omega$ we associate the Ricci form $\text{Ric}(\omega) := \frac{dd^c}{\log \lambda}$. Then we have

$$\text{Ric}(\omega) = \frac{dd^c}{\log \lambda} = -K \omega.$$

If $\omega$ is such that in any local coordinate chart $\log \lambda$ is locally integrable (in which case, we say abusively that $\log \omega$ is locally integrable), one can define on $M$ the current

$$\text{Ric}[\omega] = \frac{dd^c}{\log \lambda},$$

which, as in standard, this means that for any smooth function $\phi$ with compact support, one has $\text{Ric}[\omega](\phi) := \int_M (\log \lambda) dd^c \phi$.

We recall the following lemma which is standard in the potential theory.

**Lemma 2.8** (See Lemma 2.1 in [1]). Let $\psi$ be a subharmonic and $C^\infty$ function on $\Delta^*$. Suppose that $\psi$ is bounded above. Then $\psi$ extends to a subharmonic function on $\Delta$. Moreover, the $(1, 1)$-form $dd^c \psi$ is locally integrable on $\Delta$ and the following inequality holds in the sense of currents:

$$[dd^c \psi] \leq dd^c[\psi].$$

**Lemma 2.9** (Green-Jensen formula, see Proposition 3.1 in [21]). Let $Y$ be a non-compact parabolic Riemann surface equipped with a parabolic exhaustion function $\sigma$. Let $g : Y \to [-\infty, \infty]$ be a function such that $dd^c[g]$ is a current with order zero, i.e. $g$ can locally near every point of $Y$ be written as the difference of two subharmonic functions. Then, for $r \geq 1$ large enough,

$$\int_1^r \frac{dt}{t} \int_{B(t)} dd^c[g] = \int_{S(r)} g d\mu_r + O(1).$$

We need a precise integral formula which relates the exhaustion and Euler characteristic of $B(r)$, which is due to [21] with a more detailed proof available in ([1], Proposition 2.5). However here we require the explicit constant involving $\log r$, so we attach a proof here following the proof of Proposition 2.5 [1].

**Lemma 2.10** (Compare with [21], Proposition 3.3). Same notations as above. we have

$$-\int_1^r \chi_\sigma(t) \frac{dt}{t} = \int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r - 2\varsigma \log r + O(1).$$
Proof: The proof is similar to the proof of Proposition 2.5 in [1]. Under our assumption, for the parabolic exhaustion function $\sigma$ of $Y$, $\log \sigma$ is harmonic on $Y$ except at the points $\{P_1, \ldots, P_k\}$. Let $r > 1$ such that $P_i \in B(r)$ for $i = 1, \ldots, k$. Since $\xi$ is of type $(1, 0)$, one has $d\sigma(\xi) = \partial_\xi \sigma$, where $\partial_\xi \sigma$ is the holomorphic directional directive of $\sigma$ by the vector field $\xi$. Since $\xi$ is holomorphic and $\log \sigma$ is harmonic outside $\{P_1, \ldots, P_k\}$, it follows that $\partial_\xi \log \sigma$ is a holomorphic function outside $\{P_1, \ldots, P_k\}$.

A direct computation shows that $\sigma \cdot \partial_\xi \log \sigma = \partial_\xi \sigma$. Therefore, outside $\{P_1, \ldots, P_k\}$ where $\sigma$ has zero, $\partial_\xi \sigma$ vanishes only when $\partial_\xi \log \sigma$ does. Consider the vector field $v = \overline{\partial_\xi \sigma} \cdot \xi$. Let $(T_Y)_\mathbb{R}$ be the real tangent bundle of $Y$. Let $v_\mathbb{R}$ be the corresponding real vector field associated to $v$ via the canonical isomorphism $T_Y \simeq \left(T_Y \right)_\mathbb{R}$ given by $\frac{1}{2}(\frac{\partial}{\partial z} - i \frac{\partial}{\partial \overline{z}}) \mapsto \frac{\partial}{\partial z}$, it follows that $v_\mathbb{R}$ has singularities on $\{P_1, \ldots, P_k\}$ and the zeros of $\partial_\xi \log \sigma$. Take a holomorphic coordinate $z$ such that $\xi = \frac{\partial}{\partial z}$. In this coordinate, one has $\frac{\partial_\xi \sigma \cdot \xi}{\partial \xi} = \frac{\partial \sigma}{\partial z} \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial \sigma}{\partial x} + i \frac{\partial \sigma}{\partial y} \right) \frac{\partial}{\partial z}$. Observe that $d\sigma(v_\mathbb{R}) = \left( \frac{\partial \sigma}{\partial x} \right)^2 + \left( \frac{\partial \sigma}{\partial y} \right)^2$, which is strictly positive on the boundary $S(r)$ with our choice of $r$, hence $v_\mathbb{R}$ points outwards normal to $B(r)$.

Since the real vector field $v_\mathbb{R}$ has only isolated singularities on $\overline{B(r)}$ and points outwards normal on the boundary of $B(r)$, the Poincaré-Hopf theorem implies

$$\chi(\sigma)(r) = \sum_{p \in B(r)} \text{index}_p(v_\mathbb{R}) = \sum_{i=1}^{k} \text{index}_p(v_\mathbb{R}) + \sum_{p \in B(r) \setminus \{P_1, \ldots, P_k\}} \text{index}_p(v_\mathbb{R}). \quad (11)$$

Now we need to calculate the indices. We claim that for every $p \in B(r) \setminus \{P_1, \ldots, P_k\}$, $\text{index}_p(v_\mathbb{R}) = -\text{ord}_p(\partial_\xi \log \sigma)$. To see this, recall that if $g$ is a holomorphic function, then $\text{index}_0 \left( g \frac{\partial}{\partial z} \right) = \text{ord}_0 g$, where $(g \frac{\partial}{\partial z})_\mathbb{R} = \text{Re}(g) \frac{\partial}{\partial z} + \text{Im}(g) \frac{\partial}{\partial \overline{z}}$ is the real vector field associated to $g \frac{\partial}{\partial z}$. Therefore, in our situation, $\text{index}_p(\partial_\xi \log \sigma \cdot \xi)_\mathbb{R} = \text{ord}_p(\partial_\xi \log \sigma \cdot \xi)$. It follows that for $p \in B(r) \setminus \{P_1, \ldots, P_k\}$,

$$\text{ord}_p(\partial_\xi \log \sigma) = \text{index}_p(\partial_\xi \log \sigma \cdot \xi)_\mathbb{R} = \text{index}_p \left( \frac{1}{\sigma} \partial_\xi \sigma \cdot \xi \right)_\mathbb{R} \quad (12)$$

This proves our claim. Consequently, for $p \in B(r) \setminus \{P_1, \ldots, P_k\}$, $\text{index}_p(v_\mathbb{R})$ is equal to the mass of the current $-dd^c \log |\partial_\xi \log \sigma|^2$ at $p$ by the Poincaré-Lelong formula.

Next we consider the first term on the right hand side of (11). At each point of $P_i$, $1 \leq i \leq k$, recall our condition on $\sigma$ that $\log \sigma = k_i \log |w| + h_i(w)$ in a local
coordinate $w$ centered at $P_i$. Take a better coordinate $z$ centered at $P_i$ such that
\[ \log \sigma = k_i \log |z|, \]
and write $\xi = \psi(z) \frac{\partial}{\partial z}$ for a holomorphic function $\psi$ without zero, then
\[ \nu = \frac{d\xi}{\sigma} \cdot \xi = \left( \psi(z) \frac{\partial}{\partial z}(|z|^{k_i}) \right) \psi \frac{\partial}{\partial z} = \frac{k_i}{2} |\psi(z)|^2 |z|^{k_i} - 2z \frac{\partial}{\partial z}. \]

Hence
\[ (13) \quad \text{index}_{P_i} (\nu_R) = \text{index}_0 \left( z \frac{\partial}{\partial z} \right) = \text{index}_0 \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) = 1. \]

Note that here we used the fact that multiplying a vector field with a real function that is continuous and has no zero in a punctured neighbourhood of the singularity of the vector field does not influence its index at the singularity (see Exercise 7.4, [11]). On the other hand, $\frac{\partial}{\partial z} \log \sigma = \frac{\partial}{\partial z} (k_i \log |z|) = \frac{k_i}{2z}$, hence the current $dd^c [\log |\partial \xi \log \sigma|^2]$ has mass $-1$ at $P_i$. Therefore, by combining (11), (12), (13),

\[ \chi_\sigma (r) = k - \int_{B(r) \setminus \{P_1, \ldots, P_k\}} dd^c [\log |\partial \xi \log \sigma|^2] \]
\[ = k - \left( \int_{B(r)} dd^c [\log |\partial \xi \log \sigma|^2] + k \right) \]
\[ = - \int_{B(r)} dd^c [\log |\partial \xi \log \sigma|^2]. \]

Thus, by the Green-Jensen formula,
\[ - \int_1^r \chi_\sigma (t) \frac{dt}{t} = \int_1^r \frac{dt}{t} \int_{B(t)} dd^c [\log |\partial \xi \log \sigma|^2] \]
\[ = \int_{S(r)} \log |\partial \xi \log \sigma|^2 d\mu_r + O(1) = \int_{S(r)} \log \frac{|d\sigma(\xi)|^2}{\sigma} d\mu_r + O(1) \]
\[ = \int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r - \int_{S(r)} \log |\sigma|^2 d\mu_r + O(1) \]
\[ = \int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r - 2 \log r \int_{S(r)} d\mu_r + O(1) \]
\[ = \int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r - 2 \varsigma \log r + O(1). \]

This proves our proposition. \qed

We also need the following calculus lemma.

**Lemma 2.11.** Let $H$ be a positive, strictly increasing function defined on $(0, \infty)$. Then the set of $s \in (0, \infty)$ satisfying the following inequality
\[ H'(s) > H^{1+\delta}(s) \]
is of finite Lebesgue measure.
Let $\eta$ be a non-negative $(1,1)$-form on $Y$. Note that the relationship between $\eta$ and the pseudo-norm $h_\eta$ on $T_Y$ induced from $\eta$ is given by

$$\eta = \lambda(z) \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} \longleftrightarrow h_\eta = 2\lambda dz d\bar{z} \longleftrightarrow \|\xi\|_{h_\eta}^2 = 2\lambda|\gamma|^2,$$  

if $\xi = \gamma \frac{\partial}{\partial z}$.

For simplicity, we just write $\|\cdot\|_{\eta}$ instead of $\|\cdot\|_{h_\eta}^2$. Write

$$T_\eta(r) := \int_1^r \frac{dt}{t} \int_{B(t)} \eta.$$  

When $Y = \mathbb{C}$, by applying Proposition 2.11 one gets the following consequence of the calculus lemma,

$$\log \int_0^{2\pi} \lambda(re^{i\theta}) \frac{d\theta}{2\pi} \leq \text{exc} (1 + \delta)^2 \log T_\eta(r) + \delta \log r + O(1).$$

Similarly, in the parabolic setting, we have the following corollary of the calculus lemma.

**Lemma 2.12.** Let $\eta$ be a non-negative $(1,1)$-form on $Y$. Then, for any $\delta > 0$,

$$\int_{S(r)} \log \|\xi\|_{\eta}^2 d\mu_r \leq \text{exc} (1 + \delta)^2 \log T_\eta(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + O(1).$$

**Proof.** We first observe (by the Fubini’s theorem) that for any smooth 1-form $\psi$ on $Y$ and any $r > 1$,

$$\int_{B(r)} d\sigma \wedge \psi = \int_0^r \left( \int_{S(t)} \psi \right) dt.$$  

Taking derivative on $r$ we get

$$\frac{d}{dr} \int_{B(r)} d\sigma \wedge \psi = \int_{S(r)} \psi.$$  

Applying (18) with $\psi = \frac{\|\xi\|_{\eta}^2}{|d\sigma(\xi)|^2} d\sigma \wedge \psi$, we obtain,

$$\int_{S(r)} \frac{\|\xi\|_{\eta}^2}{|d\sigma(\xi)|^2} d\mu_r = \frac{1}{r} \frac{d}{dr} \int_{B(r)} \frac{\|\xi\|_{\eta}^2}{|d\sigma(\xi)|^2} d\sigma \wedge d^c \sigma.$$  

Now we claim the following identity

$$\frac{\|\xi\|_{\eta}^2}{|d\sigma(\xi)|^2} d\sigma \wedge d^c \sigma = 2\eta.$$  

Indeed, taking a local coordinate $z$ where $\xi = \frac{\partial}{\partial z}$, by direct computation $d\sigma \wedge d^c \sigma = \frac{\sqrt{-1}}{2\pi} \left| \frac{\partial}{\partial z} \right|^2 dz \wedge d\bar{z}$, and hence

$$\frac{\|\xi\|_{\eta}^2}{|d\sigma(\xi)|^2} d\sigma \wedge d^c \sigma = \frac{\|\frac{\partial}{\partial z}\|_{\eta}^2 \sqrt{-1}}{2\pi} \left| \frac{\partial}{\partial z} \right|^2 dz \wedge d\bar{z} = \left| \frac{\partial}{\partial z} \right|^2 \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = 2\eta.$$
which proves the claim. It follows that

$$\int_{S(r)} \frac{\|\xi\|^2}{|d\sigma(\xi)|^2} d\mu_r = \frac{2}{d} \int_{B(r)} \eta = \frac{2}{d} \int_{B(r)} \eta \left( \frac{dT\eta(r)}{dr} \right).$$

Therefore, by applying the Calculus lemma (Lemma 2.11) twice, we get, for $\delta > 0$,

$$\int_{S(r)} \frac{\|\xi\|^2}{|d\sigma(\xi)|^2} d\mu_r \leq 2r^2 T^{(1+\delta)}(r).$$

This gives

$$\log \int_{S(r)} \frac{\|\xi\|^2}{|d\sigma(\xi)|^2} d\mu_r \leq \text{exc}(1 + \delta) T^{(1+\delta)}(r).$$

Recall from Lemma 2.10 that

$$-X_\sigma(r) = \int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r - 2\varsigma \log r + O(1).$$

Therefore, we get, using the concavity of the logarithm,

$$\int_{S(r)} \log \frac{\|\xi\|^2}{|d\sigma(\xi)|^2} d\mu_r = \int_{S(r)} \log \left( \frac{\|\xi\|^2}{|d\sigma(\xi)|^2} \right) d\mu_r + \int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r$$

$$\leq \log \int_{S(r)} \frac{\|\xi\|^2}{|d\sigma(\xi)|^2} d\mu_r - X_\sigma(r) + 2\varsigma \log r + O(1)$$

$$\leq \text{exc}(1 + \delta)^2 T^{(1+\delta)}(r) - X_\sigma(r) + (\delta + 2\varsigma) \log r + O(1).$$

This proves the lemma.

**Proof of Theorem 2.7** The idea is similar to the proof of Theorem 2.5. Let $\eta$ be a positive (1, 1)-form on $X$. Let $\Sigma := (f^* D)_{\text{red}}$ be the set theoretic preimage of the divisor $D$. Let $Y^* := Y \setminus \Sigma$. Note that $Y^*$ is hyperbolic, and denote by $\omega_{Y^*}$ the (1, 1)-form associated to the Kobayashi metric on $Y^*$. By the same argument in obtaining (1), there exists a constant $c > 0$ such that

$$cf^*\eta \leq \omega_{Y^*}.$$ 

This implies, by taking integration,

$$cT_{f,\eta}(r) \leq T_{\omega_{Y^*}}(r) := \int_1^r \frac{dt}{t} \int_{B(t)} \omega_{Y^*}.$$

We now analyze $\omega_{Y^*}$. Using the fact that $Y^*$ is hyperbolic, we know that the universal cover of $Y^*$ is given by the unit disk $\Delta$, and the standard (normalized) Poincaré metric on unit disk $\Delta$ descends to the (1, 1)-form $\omega_{Y^*}$. Its Gaussian curvature is $K = -1$.

We make the following claims.

(a) Both of the currents $[\text{Ric} \omega_{Y^*}]$ and $\text{Ric} [\omega_{Y^*}]$ are well-defined on $Y$. 

(b) $[\text{Ric}_{\omega_Y^*}] \leq [\Sigma] + \text{Ric}_{[\omega_Y^*]}$ holds on $Y$.

The statement is local, so we restrict ourselves to a contractible neighborhood $U$ of a point $p \in Y$ (note that $p$ is an isolated point). Let $z$ be a local coordinate on $U$ centered at $y$, and we assume that $U^* := U \setminus \{p\} = \Delta^*$. Write $\omega_Y^* = a(z) \sqrt{1-dz \wedge d\bar{z}}$. By the distance decreasing property of the Kobayashi metric, we have, for some constant $\delta > 0$,

$$a(z) \leq \frac{1}{|z|^2 \log^2(2\delta)}.$$  

This implies that $a$ is locally integrable so $\text{Ric}_{[\omega_Y^*]}$ is well-defined on $Y$. Let $\psi(z) = a(z)|z|^2$, then $\log \psi$ is subharmonic in $U^*$ due to the curvature property of $\omega_Y^*$, and is bounded above due to the inequality above. By Lemma 2.8, $\psi(z)$ extends to a subharmonic function on $U$ and the $(1,1)$-form $dd^c \log \psi$ is locally integrable on $U$. This shows that the current $[\text{Ric}_{\omega_Y^*}]$ is well-defined on $Y$ since $dd^c \log \psi = dd^c \log a$ outside $p$. Moreover, from Lemma 2.8, $[dd^c \log \psi] \leq dd^c \log |z|$. Hence,

$$[\text{Ric}_{\omega_Y^*}] = [dd^c \log \psi] \leq dd^c \log |z| = dd^c \log |z|^2 + dd^c \log a = [\Sigma] + \text{Ric}_{[\omega_Y^*]}.$$  

This proves the claim. On the other hand, using the condition that the Gauss curvature is $-1$, we have $\omega_Y^* = \text{Ric}_{\omega_Y^*}$ (in terms of the differential forms). Hence by the claim we get

$$T_{\omega_Y^*}(r) = \int_1^r \frac{dt}{t} \int_{B(t)} \omega_Y^* = \int_1^r \frac{dt}{t} \int_{B(t)} \text{Ric}_{\omega_Y^*}$$

$$\leq \int_1^r \frac{dt}{t} \int_{B(t)} [\text{Ric}_{\omega_Y^*}]$$

$$\leq N_f(r,D) + \int_1^r \frac{dt}{t} \int_{B(t)} \text{Ric}_{[\omega_Y^*]}.$$  

It remains to estimate the last term. As we have noted above, $\text{Ric}_{[\omega_Y^*]}$, regarded as a current on $Y$, is of order zero. Hence the Green-Jensen formula (see Lemma 2.9) can be applied. Using the fact that $\text{Ric}_{[\omega_Y^*]} = dd^c \log(\|\xi\|_{\omega_Y^*})$, by the Green-Jensen formula (see Lemma 2.8) and by (17) we get

$$\int_1^r \frac{dt}{t} \int_{B(t)} \text{Ric}_{[\omega_Y^*]} = \int_1^r \frac{dt}{t} \int_{S(r)} \log \|\xi\|_{\omega_Y^*}^2 \, d\mu_r + O(1)$$

$$\leq_{exc} \frac{(1+\delta)^2 \log T_{\omega_Y^*}(r) - \mathcal{X}_\sigma(r)}{r} + (\delta + 2\varsigma) \log r + O(1).$$  

Thus

$$T_{\omega_Y^*}(r) \leq_{exc} N_f(r,D) + (1+\delta)^2 \log T_{\omega_Y^*}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + O(1),$$  

which completes the proof.
which gives
\[ \frac{1}{2} T_{w_Y}(r) \leq \text{exc} \overline{N}_Y(r, D) - X_\sigma(r) + (\delta + 2\varsigma) \log r + O(1). \]

Combing the above with (20), we get, for some positive constant \( c > 0 \),
\[ c T_{f, \eta}(r) \leq \text{exc} \overline{N}_Y(r, D) - X_\sigma(r) + (\delta + 2\varsigma) \log r + O(1). \]

This finishes the proof. \( \square \)

3. Examples of the Nevanlinna pair \((X, D)\)

We provide examples of the Nevanlinna pair \((X, D)\) such that \( X \setminus D \) may not be hyperbolically imbedded in \( X \). In each case, we obtain a more precise Second Main Theorem type result. To simplify the notation, for a meromorphic function \( g \) on \( Y \), we use \( g' \) to denote the directional derivative \( dg(\xi) \).

3.1. Parabolic version of the logarithmic derivative lemma. The main tool is the logarithmic derivative lemma on parabolic Riemann surfaces due to Paun-Sibony [21], slightly modified to serve our purpose. The only change is replacing Proposition 3.3 in their original paper by our Lemma 2.10, so we omit the detail.

**Proposition 3.1** (Compare with [21], Theorem 3.7). Let \( f \) be a non-constant meromorphic function on a parabolic Riemann surface \( Y \). Then, for \( \delta > 0 \),
\[
m_{f'/f}(r, \infty) \leq \text{exc} \frac{1}{2} \left( (1 + \delta)^2 \log^+ T_f(r) - X_\sigma(r) + (\delta + 2\varsigma) \log r + E_\sigma(r) \right) + O(1),
\]
and
\[
m_{f^{(k)}}(r, \infty) \leq \text{exc} \frac{k}{2} \left( (1 + \delta)^2 \log^+ T_f(r) - X_\sigma(r) + (\delta + 2\varsigma) \log r + E_\sigma(r) \right) + O(\log^+ \log T_f(r) + \log \log r).
\]

3.2. Parabolic version of the logarithmic derivative lemma for jet differentials. The parabolic version of the logarithmic derivative lemma can be extended to jet differentials. We first recall some notions. Let \( X \) be an \( n \)-dimensional complex manifold. We denote by \( J_k X := \bigcup_{x \in X} J_k(X)_x \) the (fiber) bundle of \( k \)-jets, where \( J_k(X)_x \) consists of equivalence classes of germs of holomorphic curves \( f : (\mathbb{C}, 0) \to (X, x) \) with the equivalence relation \( f \sim_k g \) if and only if all derivatives \( f^{(j)}(0) = g^{(j)}(0) \) coincide for \( j = 0, \ldots, k \). The equivalent class of a holomorphic germ \( f : (\mathbb{C}, 0) \to (X, x) \) is called the \( k \)-jet of \( f \), denote by \( j_k f \). Note that \( J_k(X)_x \)
is isomorphic to $\mathbb{C}^{kn}$ via the identification $f \mapsto (f'(0), \ldots, f^{(k)}(0))$. Observe also that there is a natural $\mathbb{C}^*$-action on the fibers of $J_kX$ given by

$$\lambda \cdot j_k f := j_k (t \mapsto f(\lambda t)), \ \forall \lambda \in \mathbb{C}^*.$$ 

For an open subset $U \subset X$, a jet differential of order $k$ on $U$ is an element $P \in \mathcal{O}(p_k^{-1}(U))$, where $p_k : J_kX \to X$ is the projection map. The sheaf of jet differentials is defined to be $\mathcal{E}_{k}^{GG} \Omega_X := (p_k)_* \mathcal{O}_{J_kX}$. A $k$-jet differential $P \in \mathcal{E}_{k,m}^{GG} \Omega_X(U) = \mathcal{O}(p_k^{-1}(U))$ is said to be of weighted degree $m$ if for any $j_k f \in p_k^{-1}(U)$ one has

$$P(\lambda \cdot j_k f) = \lambda^m P(j_k f), \ \forall \lambda \in \mathbb{C}^*.$$ 

The Green-Griffiths sheaf $\mathcal{E}_{k,m}^{GG} \Omega_X$ is defined to be the subsheaf of $\mathcal{E}_{k}^{GG} \Omega_X$ of order $k$ and weighted degree $m$. In local coordinates, any element $P \in \mathcal{E}_{k,m}^{GG} \Omega_X(U)$ can be written as

$$P(z, dz, \ldots, d^k z) = \sum_{|\alpha|=m} c_\alpha(z)(dz)^{\alpha_1} \cdots (d^k z)^{\alpha_k},$$ 

where $c_\alpha \in \mathcal{O}(U)$ for any $\alpha := (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}^m)^k$, and where $|\alpha| := |\alpha_1| + 2|\alpha_2| + \cdots + k|\alpha_k|$ is the usual multi-index notation for the weighted degree. The sheaf $\mathcal{E}_{k,m}^{GG} \Omega_X$ is locally free, and we denote its associated vector bundle by $E_{k,m}^{GG} \Omega_X$.

We briefly recall the construction of Demayl-Semple jet tower as follows, we refer the reader [3] or the original paper of Demailly [4] for details. Let $G_k$ be the group of germs of $k$-jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps $\phi : t \mapsto a_1 t + a_2 t^2 + \cdots + a_k t^k$, $a_j \in \mathbb{C}^*$, $a_j \in \mathbb{C}$ for $j > 1$, in which the composition law is taken modulo terms $t^j$ of degree $j > k$. The group $G_k$ has a natural action on $J_k(X)$ given by $\phi \cdot j_k(f) := j_k(f \circ \phi)$ for any $\phi \in G_k$. Denote by

$$J_{k}^{\text{reg}} X := \{ j_k f \in J_k X \mid f'(0) \neq 0 \},$$ 

the space of non-constant jets. Consider the pairs $(X', V')$, where $X'$ is a complex manifold, $V' \subset T_{X'}$ is a subbundle, and denote by $\pi' : V' \to X'$ the natural projection. Starting from $(X, T_X)$, define $X_1 := \mathbb{P}(T_X)$, let $\pi_{0,1} : X_1 \to X$ the natural projection. Define the bundle $V_1 \subset T_{X_1}$ fiberwise by

$$V_{1,(x,[v])} := \{ w \in T_{X_1}(x,[v]) : (d\pi_{0,1})_x(w) \in \mathbb{C}v \}.$$ 

In other words, $V_1$ is characterized by the exact sequence

$$0 \to T_{X_1} \to V_1 \xrightarrow{d\pi_{0,1}} \mathcal{O}_{X_1}(-1) \to 0$$.
where $\mathcal{O}_X(-1)$ is the tautological bundle on $X_1$, and $T_{X_1/X}$ is the relative tangent bundle corresponding to the fibration $\pi_{0,1}$. Inductively by this procedure, we get the Demaillÿ-Semple tower

$$(X_k, V_k) \xrightarrow{\pi_{k-1,k}} (X_{k-1}, V_{k-1}) \xrightarrow{\pi_{k-2,k-1}} \cdots \to (X_1, V_1) \xrightarrow{\pi_{0,1}} (X, T_X).$$

Then we have an embedding $J^*_k \mathcal{O}_X / G_k \to X_k$. Denote by $X^\text{reg}_k$ the image of this embedding in $X_k$ and denote by $X^\text{sing}_k := X_k \setminus X^\text{reg}_k$. Then $X^\text{sing}_k$ is a divisor in $X_k$. Denote by $\pi_k : X_k \to X$ the projection, then the direct image sheaf $\pi_\ast \mathcal{O}_{X_k}(m)$ is isomorphic to $\pi_\ast \mathcal{O}_{X_k}(m) \simeq E_{k,m} \Omega_X$, whose sections are precisely the invariant jet differentials $P$, i.e., for any $g \in G_k$ and any $j_k f \in J^*_k \mathcal{O}_X$, $P(j_k(f \circ g)) = g'(0)^m P(j_k f)$. We shall denote the associated vector bundle by $E_{k,m} \Omega_X$. The fiber of $\pi_k$ at a non-singular point of $X$ is denoted by $\mathcal{R}_{n,k}$, it is a rational manifold, and it is a compactification of the quotient $(\mathbb{C}^n \setminus \{0\}) / G_k$.

The above definition can be also extended to the logarithmic setting. Let $D$ be a simple normal crossing divisor on $X$ (i.e. $D = D_1 + \cdots + D_c$ where $D_1, \ldots, D_c$ are smooth irreducible divisors intersecting transversely). The logarithmic cotangent sheaf, denoted by $\Omega_X(\log D)$, is a locally-free sheaf generated by, on $U$,

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_s}{z_s}, \frac{dz_{s+1}}{z_{s+1}}, \ldots, \frac{dz_n}{z_n},$$

where $U \subset X$ is an open subset with local coordinates $(z_1, \ldots, z_n)$ such that $D|_U = (z_1 \cdots z_s = 0)$. Let $\mathcal{J}_k(X, -\log D)$ (called the logarithmic k-jet sheaf) be the sheaf of germs of local holomorphic sections $\alpha$ of $J_k X$ such that, for any $\omega \in \Omega_X(\log D)|_x$, $(d^{j-1} \omega)(\alpha)$ are all holomorphic for $j = 1, \ldots, k$. A local meromorphic k-jet differential $\omega$ on $U$ is called a logarithmic k-jet differential if $\omega(\alpha)$ is holomorphic for any logarithmic k-jet field $\alpha \in \mathcal{J}_k(X, -\log D)(U)$. The sheaf of logarithmic k-jet differential is denoted by $\mathcal{E}_{k,m} \Omega_X(\log D)$. The logarithmic Green-Griffiths sheaf $\mathcal{E}_{k,m} \Omega_X(\log D)$ is the subsheaf of $\mathcal{E}_{k,m} \Omega_X(\log D)$ with weighted degree $m$. In local coordinates $z_1, \ldots, z_n$ on $U$ with $D|_U = (z_1 \cdots z_s = 0)$, any element $P \in \mathcal{E}_{k,m} \Omega_X(\log D)(U)$ can be written as

$$P(z, dz_1, \ldots, dz_n) = \sum_{|\alpha| = m} c_\alpha \left( \frac{dz_1}{z_1} \right)^{\alpha_1,1} \cdots \left( \frac{dz^n}{z_n} \right)^{\alpha_n,k},$$

(22)

where each $c_\alpha \in \mathcal{O}(U)$, the summation is over the $kn$-tuples $\alpha := (\alpha_1, \ldots, \alpha_k) \in (\mathbb{N}^n)^k$, and where we used the usual multi-index notation for the weighted degree.
|α| := |α_1| + 2|α_2| + \cdots + k|α_k|. The associated vector bundle is denoted by $E_{k,m}^{GG}(log D)$.

We now briefly recall the construction of the logarithmic version of Demailly-Semple tower due to Dethloff-Lu [7]. A logarithmic directed manifold is a triple $(X, D, V)$ where $(X, D)$ is a log-manifold, $V$ is a subbundle of $T_X(-log D)$. A morphism between logarithmic directed manifolds $(X', D', V')$ and $(X, D, V)$ is given by a holomorphic map $f : X' \rightarrow X$ such that $f^{-1}D \subset D'$ and $f_*V' \subset V$.

The logarithmic Demailly-Semple $k$-jet tower $(X_k(D), D_k, V_k)$ is constructed inductively as follows: Starting from $V_0 = V = T_X(-log D)$, define $X_k(D) : = P(V_{k-1})$, and let $\pi_{k-1,k} : X_k(D) \rightarrow X_{k-1}(D)$ be the natural projection.

Set $D_k := (\pi_{k-1,k})^{-1}(D_{k-1})$ which is a simple normal crossing divisor. Note that $\pi_{k-1,k}$ induces a morphism

$$(\pi_{k-1,k})_* : T_{X_k(D)}(-log D_k) \rightarrow (\pi_{k-1,k})^*T_{X_{k-1}(D)}(-log D_{k-1}).$$

Define $V_k := (\pi_{k-1,k})_*^{-1}O_{X_k(D)}(1) \subset T_{X_k(D)}(-log D_k)$, where $O_{X_k(D)}(1) := O_{\mathbb{P}(V_{k-1})}(1)$ is the tautological line bundle, which by definition is also a subbundle of $\pi_{k-1,k}^*V_{k-1}$.

By Proposition 3.9 in [7], we have

$${\mathcal{E}}_{k,m}Omega_X(log D) = (\pi_k)_*O_{X_k(D)}(m),$$

where $E_{k,m}^{GG}(log D)$ is the subsheaf of $E_{k,m}^{GG}(log D)$ consisting of invariant logarithmic differential operator $P$.

Let $f : Y \rightarrow X$ be a holomorphic map. Let $P \in H^0(X, E_{k,m}^{GG}(log D))$. Write $f^*P := P(j_kf)$. The parabolic version of the logarithmic derivative lemma extends to the following jet differentials (see also [15], theorem 3.1).

**Theorem 3.2.** Let $X$ be a complex projective variety and $D$ be a simple normal crossing divisor on $X$ (possibly empty).

(a) Let $P \in H^0(X, E_{k,m}^{GG}(log D))$. Then there exists a constant $C > 0$ such that for any parabolic Riemann surface $Y$, every holomorphic map $f : Y \rightarrow X$ with $f(Y) \not\subset D$ and with $f^*P \neq 0$, for $\delta > 0$, one has

$$\int_{S(r)} \log^+ |P(j_kf)|d\mu_r \leq exc C(\log^+ T_{f,E}(r) - X_\sigma(r) + (\delta + 2\zeta) \log r + C_\sigma(r)) + O(1),$$
where $E$ is an (thus any) ample divisor on $X$.

(b) Let $A$ be an ample line bundle on $X$. For any positive integer $N$, let $P \in H^0(Y, E^{CG}_{k,m} \omega_X (\log D) \otimes A^{-N})$. Then there exists a constant $C > 0$ such that for every holomorphic map $f : Y \to X$ with $f(Y) \not\subset D$ where $Y$ is a parabolic Riemann surface, if $f^* P \neq 0$, then for $\delta > 0$,

$$T_{f,A}(r) \leq \frac{m}{N} \cdot \sup_{\lambda \in \Lambda} \int_{S(r)} \log^+ |P(j_k f)| \cdot d\mu_r + C \cdot (\log^+ T_{f,A}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\zeta) \log r + \mathcal{E}_\sigma(r) + O(1)).$$

Proof. Same as in the proof of Theorem 2.6, we can cover (with respect to the complex topology). Let $f^* \lambda, z_\lambda, n$ are (global) rational functions on $X$. Take a relative compact subcovering $\{V_\lambda\}_{\lambda \in \Lambda}$ with $\bigcup V_\lambda \subset U$ (note that all closure are taking with respect to the complex topology). Let $f_\lambda, z_\lambda, \sigma$. From (22), there exists a constant $C_\lambda > 0$ such that for $z \in Y$ with $f(z) \in V_\lambda$,

$$\log^+ |P(j_k f)| \leq C_\lambda \sum_{\lambda} \left( \sum_{i=1}^n \log^+ \left| \frac{f_{\lambda, i}(z)}{f_{\lambda, i}} \right| \right) + C_2.$$ 

For $\delta > 0$, the logarithmic derivative lemma (Proposition 3.1) implies, for each $\lambda \in \Lambda$, $1 \leq i \leq n$, there are constants $C_3, C_4 > 0$ such that

$$\int_{S(r)} \log^+ \left| \frac{f_{\lambda, i}}{\lambda, i} \right| d\mu_r \leq \sup_{\lambda \in \Lambda} C_3 (\log^+ T_{\lambda, i}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\zeta) \log r + \mathcal{E}_\sigma(r) + O(1)).$$

where the last inequality follows from the fact that $\log T_{g, \sigma}(r) \leq C \cdot (\log T_{f, \sigma}(r))$ for any rational function $g$ on $X$ (see [19], Theorem 2.13). By integrating (24) on $S(r)$ and applying (26), we get, for some $C > 0$,

$$\int_{S(r)} \log^+ |P(j_k f)| d\mu_r \leq \sup_{\lambda \in \Lambda} C_3 (\log^+ T_{\lambda, i}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\zeta) \log r + \mathcal{E}_\sigma(r) + O(1)).$$
This proves (a).

(b) Since $A$ is an ample divisor on $X$, fix a Hermitian metric $h$ on the line bundle associate to $A$, then the first Chern form $\omega := c_1(A, h) > 0$. The Poincaré-Lelong formula implies

$$dd^c \log \| P(j_k f) \|_h^2 = N f^* \omega - [P(j_k f)],$$

where $[P(j_k f)]$ is the divisor of zero associated to $P(j_k f)$. Write $D$ locally as $(z_1 \cdots z_s = 0)$, and let $f_j = z_j(f)$. Notice that the pole order of $(\log f_j)$ at $z \in Y$ is at most $\min\{\text{ord}_z(f_j), 1\}$. Hence, see that, from the local expression of $P$ in [22],

$$[P(j_k f)] \leq m(f^* D)_{[1]} ,$$

where, for a divisor $f^* D = \sum_{p \in Y} n_p p$ on $Y$, $(f^* D)_{[1]} := \sum_{p \in Y} \min\{n_p, 1\} p$. Hence

$$dd^c \log \| P(j_k f) \|_h^2 \geq N f^* \omega - m \cdot (f^* D)_{[1]}.$$}

Taking integral $\int_{B(0)} \frac{1}{r} \int B(t)$ both sides and apply Green-Jensen formula (Lemma 2.9), we get

$$NT_{f, A}(r) \leq mN_f(r, D) + \int_{S(r)} \log \| P(j_k f) \|_h^2 d\mu_r.$$}

Note that on each $V_\lambda$ in the proof of (a), $\| P(j_k f) \|_h \leq C'_{\lambda} |P(j_k f)|$ for some $C'_{\lambda} > 0$. Hence (b) follows from the above inequality and (a). \hfill \Box

### 3.3. The complement of hypersurfaces

Let $X$ be a smooth projective variety and $D$ be a Cartier divisor on $X$. Recall that the stable locus of $D$ is defined by $B(D) := \bigcap_{m \in \mathbb{N}, F \in [mD]} F$. As a consequence of Theorem 3.2 we can get the following result.

**Corollary 3.3.** Let $X$ be a complex projective variety and $D$ be a divisor on $X$ with simple normal crossing (possibly empty). Let $A$ be an ample line bundle on $X$. Let $\pi_{0,k} : X_k(D) \to X$ be the log Demailly tower associated to the pair $(X, D)$. For any positive integers $k, N, N'$, assume that the stable base locus

$$B(O_{X_k(D)}(N) \otimes \pi_{0,k}^* A^{-N'}) \subset X_k(D)^{sing} \cup \pi_{0,k}^{-1}(D).$$

Then exists a constant $C > 0$ such that for every holomorphic map $f : Y \to X$ with $f(Y) \not\subset D$ where $Y$ is an open parabolic Riemann surface, if $f^* P \neq 0$, then, for $\delta > 0$,

$$T_{f, A}(r) \leq \text{exc} \frac{N}{N'} \overline{N}_f(r, D) + C(\log^+ T_{f, A}(r) - \overline{\chi}_\sigma(r) + (\delta + 2\varsigma) \log r + \overline{\chi}_\sigma(r)) + O(1).$$


We establish the following result which is an extension of Brotbek and Deng [2], Corollary 4.9.

**Theorem 3.4.** Let $X$ be a projective manifold of dimension $n \geq 2$ and let $A$ be a very ample line bundle over $X$. Let $D \in |A^m|$ be a general smooth hypersurface with

$$m \geq (n+2)^{n+3}(n+1)^{n+3}.$$  

Then there exists a constant $C > 0$ such that for any parabolic Riemann surface $Y$ and every holomorphic map $f : Y \to X$ with $f(Y) \not\subset D$, for $\delta > 0$, one has

$$T_{f,A}(r) \leq \text{exc} N_f(r,D) + C(\log^+ T_{f,A}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).$$

In particular $(X,D)$ is a Nevanlinna pair.

The proof of the above theorem relies on the following key result in the paper of Brotbek and Deng [2].

**Proposition 3.5** ([2], Corollary 4.5). We keep the notation in Corollary 4.5, [2]: Let $\epsilon$ be a positive integer. Let $k = n+1, k' = \frac{k(k+1)}{2}$ and $d = (k+1)n+k$. Let $l$ be an integer such that $l > d^{k-1}k(\epsilon + kd)$. Then there exist $\beta, \tilde{\beta} \in \mathbb{N}$ such that, for any $\alpha \geq 0$, and for a general hypersurface $D \in |A^{+\epsilon(l+k)d}|$, denoting by $X_k(D)$ the log Demailly $k$-jet tower, the stable base locus

$$B(\mathcal{O}_{X_k(D)}(\beta + \alpha d^{k-1}k') \otimes \pi_{0,k}^* A^{\tilde{\beta} + \alpha(d^{k-1}k(r+kd) - l)}) \subset X_k(D)^{\text{sing}} \cup \pi_{0,k}^{-1}(D).$$

**Proof of Theorem 3.4.** Note that $k = n+1, d = (k+1)n+k = n^2 + 3n + 1$ and set

$$l_0 = d^{k-1}k' + d^{k-1}(d+1)^2 = d^{k-1}(d+1)\left(d + \frac{3}{2}\right).$$

By the basic inequality

$$k(k + d - 1 + kd) < (d+1)^2,$$

one can show that any $m \geq (l_0 + k)d + 2d$ can be written in the form

$$m = \epsilon + (l + k)d$$

with $k \leq \epsilon \leq k + d - 1$, and $l \geq d^{k-1}k' + d^{k-1}k(\epsilon + kd)$. In particular, applying Corollary 3.3 and Proposition 3.5, we see that for such $m$ and a general hypersurface $D \in |A^m|$, there exists a constant $C > 0$ such that

$$T_{f,A}(r) \leq \text{exc} \frac{\beta + \alpha d^{k-1}k'}{-\beta - \alpha(d^{k-1}k(\epsilon + kd) - l)} N_f(r,D) + C(\log^+ T_{f,A}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).$$
However, when $\alpha \to \infty$,

$$\frac{\beta + \alpha d^{k-1}k'}{-\beta - \alpha(d^{k-1}(\epsilon + kd) - l)} \to \frac{d^{k-1}k'}{l - (d^{k-1}(\epsilon + kd))} < 1.$$ 

Thus

$$T_{f,A}(r) \leq \text{exc}\, N_f(r,D) + C(\log^+ T_{f,A}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).$$

It remains to give a bound on $(l_0 + k)d + 2d$. Indeed we have

$$(l_0 + k)d + 2d = \left(d^{k-1}(d + 1) \left(d + \frac{3}{2}\right) + k + 2\right)d$$

$$< (n + 2)^{n+3}(n + 1)^{n+3}.$$ 

This proves the theorem.

3.4. The Abelian variety case. We now consider the abelian variety case, and use Theorem 3.2 to extend the result of Siu-Yeung [26]. Let $A$ be an abelian variety of dimension $n$. Then the universal covering $\mathbb{C}^n \to A$ induces global holomorphic 1-forms $\omega_1 := dw_1, \ldots, \omega_n := dw_n$, where $(w_1, \ldots, w_n)$ are standard coordinates on $\mathbb{C}^n$. The global 1-forms give global holomorphic coordinates on $J_k A$ so that $J_k A \simeq A \times \mathbb{C}^{kn}$. We fix such a trivialization for every $k$. The induced coordinates will be called below “the jet coordinates of $J_k A$”. Let $f : Y \to A$ be a holomorphic map. Fix a global vector field $\xi$ on $Y$, then the lifting of $f$, denoted by $j_k f : Y \to A \times J_k A = A \times \mathbb{C}^{kn}$, is given by

$$j_k f = (f, f_1', \ldots, f_n', f_1^{(k)}, \ldots, f_n^{(k)})$$ 

where $f_i' = (f^* \omega_i)(\xi)$ for $i = 1, \ldots, n$, and inductively, $f_i^{(k)} = (d f_i^{(k-1)})(\xi)$.

Theorem 3.6. Let $A$ be an abelian variety. Let $D$ be an ample divisor on $A$. Then there exists a constant $C > 0$ such that for every holomorphic map $f : Y \to A$ with $f(Y) \not\subseteq D$ where $Y$ is a parabolic Riemann surface, we have, for some $k_0 > 0$,

$$T_{f,D}(r) \leq \text{exc}\, N_f^{[k_0]}(r,D) + C(\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).$$

In particular, $(A, D)$ is a Nevanlinna pair.

Proof. We first deal with the case when $f(Y)$ is Zariski dense in $A$. Let $X_k(f)$ be the Zariski closure of the image of $Y$ under $j_k f$. Let $I_k$ denote the restriction on $X_k(f)$ of the second projection $\iota_k : J_k A = A \times \mathbb{C}^{nk} \to \mathbb{C}^{nk}$. Let $J_k(D)$ be the $k$-jets of $A$ which are annihilated by $\sigma, d\sigma, \ldots, d^k \sigma$ where $\sigma$ is a local defining function of $D$ (when $D$ is a smooth subvariety of $A$ it coincides with the $k$-jet bundle of $D$).
Claim: There exists an integer \( k \geq 0 \) such that \( I_k(X_k(f)) \cap I_k(J_k(D)) \neq I_k(J_k(D)) \).

To prove the claim, fix a point \( y_0 \in Y \), it suffices to show \( I_k(j_k(f(y_0))) \notin I_k(J_k(D)) \) for some \( k \geq 0 \). Assume this is not true, i.e. \( I_k(j_k(f(y_0))) \in I_k(J_k(D)) \) for all \( k \). Then \( J_k(D) \cap I_k^{-1}(I_k(j_k(f(y_0)))) \neq \emptyset \) for all \( k \). Define
\[
V_k := p_k(J_k(D) \cap I_k^{-1}(I_k(j_k(f(y_0)))) \neq \emptyset,
\]
where \( p_k \) is the projection \( J_k(A) \to A \). Note that \( V_k \) is Zariski closed (because \( p_k : J_k(A) \to A \) has a section \( \text{id}_A \times \{ I_k(j_k(f(y_0))) \} : A \to J_k(A) \), and \( V_k \) is the pull-back of \( \text{supp} J_k(D) \) by this section), and note that \( V_{k+1} \subset V_k \), we obtain a decreasing sequence of closed subsets on \( A \)
\[
D \supset V_1 \supset V_2 \supset \cdots
\]
which eventually stabilizes to a closed set called \( V \). By assumption the elements in the decreasing sequence are non-empty hence \( V \) is non-empty. Let \( a \) be an element in \( V \), so \( a + j_k(f(y_0)) \in J_k(D) \). Define \( \tilde{f}(y) := f(y) + a - f(y_0) \), then \( \tilde{f}(y_0) = a \), so \( j_k(\tilde{f}(y_0)) \in J_k(D) \) for any \( k \geq 0 \). By a power series argument, we get \( \tilde{f}(Y) \subset D \), which contradicts to \( f \) being algebraically non-degenerate. This proves the claim.

Note that \( I_k \) is proper, therefore \( Y_k := I_k(X_k(f)) \) is an irreducible algebraic subset of \( \mathbb{C}^{nk} \). By the claim, there is \( k_0 \) for which there is a polynomial \( P \) on \( \mathbb{C}^{nk_0} \) satisfying
\[
P|_{Y_{k_0}} \neq 0, \quad P|_{J_{k_0}(D)} \equiv 0.
\]
Let \( \{ U_\lambda \}_{\lambda \in \Lambda} \) be a finite open covering of \( A \) such that \( D \cap U_\lambda = (\sigma_\lambda = 0) \), where \( \sigma_\lambda \) is regular on a Zariski open neighborhood of \( U_\lambda \), same as in the proof of Theorem 2.6, we can view \( \sigma_\lambda \) as a rational functions on \( A \). Then
\[
\sigma_\lambda = d\sigma_\lambda = \cdots = d^k\sigma_\lambda = 0
\]
give defining equations of \( J_k(D)|_{U_\lambda} \), hence on each \( U_\lambda \) one obtains the following equation:
\[
(a_{\lambda_0}\sigma_\lambda + \cdots + a_{\lambda_{k_0}}d^k\sigma_\lambda = I_{k_0}^*P|_{U_\lambda}.
\]
Here \( a_{\lambda_j} \) are polynomials in jet coordinates with coefficients of rational holomorphic functions on \( U_\lambda \) restricted on \( J_{k_0}(A)|_{U_\lambda} \).

Let \( \{ h_\lambda \} \) be a Hermitian metric on the line bundle \([D]\) associated to \( D \), i.e.
\[
\|\sigma\|^2 = h_\lambda|\sigma_\lambda|^2,
\]
Choose relatively compact open subsets \( V_\lambda \) of \( U_\lambda \) so that \( \bigcup_\lambda V_\lambda = A \). Let \( w_{ij}, 1 \leq i \leq n, 1 \leq j \leq k_0 \), be the coordinate system on \( \mathbb{C}^{nk_0} \) coming from restriction of the jet coordinates of \( J_\lambda A \). Since \( a_\lambda \) are polynomials in the jet coordinates with coefficients given by holomorphic functions on \( U_\lambda \), for every \( \lambda \), there exist a positive constant \( C_\lambda \) and an integer \( d_\lambda > 0 \) such that, for every \( z \in Y \) with \( f(z) \in V_\lambda \) and every \( 1 \leq t \leq k_0 \),

\[
 h_\lambda^{-1/2}(f(z)|a_\lambda(j_{k_0} f(z))| \leq C_\lambda \left( 1 + \sum_{1 \leq i \leq n, 1 \leq j \leq k_0} |w_{ij}(j_{k_0} f(z))| \right)^{d_\lambda}.
\]

Hence, for every \( z \in Y \) with \( f(z) \in V_\lambda \),

\[
\frac{|(I_{k_0}^* P)(j_{k_0} f(z))|}{\|\sigma(f(z))\|} \leq C_\lambda \left( 1 + \sum_{1 \leq i \leq n, 1 \leq j \leq k_0} |w_{ij}(j_{k_0} f(z))| \right)^{d_\lambda} \times \left( 1 + \sum_{k=1}^{k_0} \left| \frac{d^k \sigma_\lambda(f(z))}{\sigma_\lambda(f(z))} \right| \right).
\]

On the other hand, by Theorem 3.2 (part (a)), we get, for any \( 1 \leq i \leq n, 1 \leq j \leq k_0 \),

\[
\int_{S(r)} \log^+ |w_{ij}(j_{k_0} f)| |d\mu_r| \leq \text{exc} C_1 (\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).
\]

Hence

\[
m_f(r,D) \leq \text{exc} \ C_2 \left( \sum_{\lambda_1 \leq j \leq k_0} m \left( r, \frac{(\sigma_\lambda \circ f)^{(j)}}{\sigma_\lambda \circ f} \right) \right) + m \left( r, \frac{1}{(I_{k_0}^* P)(j_{k_0} f)} \right) + C_3 (\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).
\]

Again, by Theorem 3.2 (a) or by the logarithmic derivative lemma

\[
m \left( r, \frac{(\sigma_\lambda \circ f)^{(j)}}{\sigma_\lambda \circ f} \right) \leq \text{exc} C_4 (\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).
\]

Hence,

\[
m_f(r,D) \leq \text{exc} m \left( r, \frac{1}{(I_{k_0}^* P)(j_{k_0} f)} \right) + C_5 (\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).
\]
Note here, for a meromorphic function \( g \) on \( Y \), we also use \( m(r, g) \) to denote \( m_g(r, \infty) \) and \( N(r, g) \) to denote \( N_g(r, \infty) \). By adding \( N_f(r, D) \) both sides of the above inequality and applying the First Main Theorem \([3]\), we get

\[
T_{f,D}(r) \leq \text{exc} \quad N_f(r, D) + m \left( r, \frac{1}{(I^*_k P)(j_{k_0}f)} \right)
+ C_6(\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).
\]

From \([29]\), we see that, for any \( z \in Y \),
\[
\operatorname{ord}_z f^*D - \min\{\operatorname{ord}_z f^*D, k_0\} \leq \operatorname{ord}_z ((I^*_k P)(j_{k_0}f))_0,
\]
where \(((I^*_k P)(j_{k_0}f))_0 \) is the divisor of zeroes on \( Y \) associated to \((I^*_k P)(j_{k_0}f)\).
Hence

\[
N_f(r, D) - N^{[k_0]}_f(r, D) \leq N((I^*_k P)(j_{k_0}f))(r, 0).
\]

Therefore, by the First Main Theorem,

\[
T_{f,D}(r) \leq \text{exc} \quad N^{[k_0]}_f(r, D) + m \left( r, \frac{1}{(I^*_k P)(j_{k_0}f)} \right)
+ C_6(\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1)
\]

(29)

(30)

Applying Theorem \([32]\) (a) again we get

\[
m(r, (I^*_k P)(j_{k_0}f)) \leq \text{exc} \quad C_6(\log^+ T_{f,D}(r) - \mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r + \mathcal{E}_\sigma(r)) + O(1).
\]

Combining \([29]\), \([30]\) finishes the proof in this case.

We now deal with the case when \( f(Y) \) is not Zariski dense. Let \( X \) be the Zariski closure of \( f(Y) \). We can assume that \( X \) is not a translate of an Abelian sub-variety of \( A \), since if it is, then it follows from the condition \( f(Y) \not\subset D \) that \( m_f(r, D) = m_f(r, D \cap X) \). Hence the above argument can be applied to the case \( A = X \) and \( D \) is \( D \cap X \) to get our conclusion. Furthermore, let \( A_0 \) be the quotient of the subgroup of all elements whose translates leave \( X \) invariant, i.e., if \( B = \{ a \in A \mid a + X = X \} \), then \( A_0 = A/B \). By replacing \( f \) by its composite with the quotient map \( A \to A_0 \), we can assume without loss of generality that \( X \) is not invariant by the translate of any subgroup of \( A \) with positive dimension.

Starting with \( A_0 = A \) and \( V_0 := T_X \), we consider the Demaily-Semple jet tower of directed manifolds \((A_k, V_k)_{k \geq 0}\), whose construction was recalled in the beginning
of subsection 3.2 above. Note that $A_k = A \times \mathcal{R}_{n,k}$ where $\mathcal{R}_{n,k}$ is the “universal” rational homogeneous variety $\mathbb{C}^{n,k}/G_k$. The curve $f : Y \to A$ lifts to $A_k$, and we denote by $f_k : Y \to A_k$ the lift of $f$. Let $X_k$ be the Zariski closure of the image of $f_k$, and let $\tau_k : X_k \to \mathcal{R}_{n,k}$ be the composition of the injection $X_k \to A_k$ with the projection on the second factor $A_k \to \mathcal{R}_{n,k}$. According to Proposition 5.3 in [21], if for each $k \geq 1$ the fibers of $\tau_k$ are positive dimensional, then the dimension of the subgroup $A_X$ of $A$ defined by

$$A_X := \{a \in X \mid a + X = X\}$$

is strictly positive. On the other hand, we have assumed that $X$ is not invariant by the translate of any subgroup of $A$ with positive dimension. Hence we get, for some $k \geq 1$, the map $\tau_k : X_k \to \mathcal{R}_{n,k}$ has finite generic fibers. Thus, by Proposition 5.4 in [21], there exists a jet differential $\mathcal{P}$ of order $k$ with values in the dual of an ample line bundle, and whose restriction to $X_k$ is non-identically zero. This implies, from Theorem 4(2b) (where the normal crossing divisor is taken to be empty) that, for some $\delta > 0$,

$$T_f,D(r) \leq \text{exc}(\log^+ T_f,D(r) - \chi(r) + (\delta + 2\varsigma) \log r + \mathcal{E}(r)) + O(1).$$

□

4. The hyperplane case

In this section we consider the case when $X = \mathbb{P}^n(\mathbb{C})$ and $D$ is given by a collection of hyperplanes. Let $\mathcal{H}$ be a finite set of hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Let $|\mathcal{H}| := \sum_{H \in \mathcal{H}} H$. Let $\mathcal{L}$ be the set of defining linear forms of the hyperplanes in $\mathcal{H}$.

**Definition 4.1** ([22]). $\mathcal{H}$ is said to be non-degenerate if

1. $\dim(\mathcal{L}) = n + 1$;
2. For any proper non-empty subset $\mathcal{L}'$ of $\mathcal{L}$,

$$(\mathcal{L}') \cap (\mathcal{L} \setminus \mathcal{L}') \cap \mathcal{L} \neq \emptyset,$$

where $(\mathcal{L})$ means the vector space generated by the linear forms in $\mathcal{L}$.

In [22] the second named author showed that $\mathbb{P}^n(\mathbb{C}) \setminus |\mathcal{H}|$ is Brody hyperbolic if and only if $\mathcal{H}$ is non-degenerate. Our main purpose of this section is to show that if $\mathcal{H}$ is non-degenerate then $(\mathbb{P}^n(\mathbb{C}), |\mathcal{H}|)$ is a Nevanlinna pair. Hence, since Nevanlinna pair implies Brody hyperbolic, we get
Theorem 4.2. \( (\mathbb{P}^n(\mathbb{C}), |\mathcal{H}|) \) is a Nevanlinna pair if and only if \( \mathbb{P}^n(\mathbb{C}) \backslash |\mathcal{H}| \) is Brody hyperbolic.

We first establish a parabolic version of the Ahlfors-Wyel Second Main Theorem.

Theorem 4.3. Let \( Y \) be a parabolic Riemann surface and \( f : Y \to \mathbb{P}^n(\mathbb{C}) \) be a linearly nondegenerate holomorphic map, i.e., its image is not contained in any proper linear subspaces of \( \mathbb{P}^n(\mathbb{C}) \). Let \( H_1, \ldots, H_q \) be hyperplanes on \( \mathbb{P}^n(\mathbb{C}) \) in general position, then, for \( \delta', \delta > 0 \),

\[
\sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0) \leq \text{exc}(n + 1 + \delta') T_f(r) + \left( \frac{n(n+1)}{2} + \delta' \right) [-X_\sigma(r) + (\delta + 2\varsigma) \log r + O(1)],
\]

where \( W \) is the Wronskian of \( f \).

Proof. We modify the geometric proof of the second main theorem for holomorphic curves (see A3.5 [23]) to the parabolic setting. For a local coordinate chart \( (U, z) \), let \( f = (f_0, \ldots, f_n) : U \to \mathbb{C}^{n+1} \) be a local reduced representation of \( f \), where \( f_0, \ldots, f_n \) are holomorphic functions on \( U \) with no common zeroes. For \( 0 \leq k \leq 1 \), consider the map \( F_k \) defined by

\[
F_k(z) := f(z) \wedge f'(z) \wedge \cdots \wedge f^{(k)}(z) : U \to \bigwedge^{k+1} \mathbb{C}^{n+1}.
\]

Identify \( \bigwedge^{k+1} \mathbb{C}^{n+1} \) with \( \mathbb{C}^{N_{k+1}} \), where \( N_k = \frac{(n+1)!}{(k+1)!(n-k)!} - 1 \), and let \( P : \mathbb{C}^{N_{k+1}} \to \mathbb{P}^{N_k}(\mathbb{C}) \) be the natural projection. Then the \( k \)-th associate map \( F_k := P(F_k) \) is a well-defined holomorphic map from \( Y \) to \( \mathbb{P}^{N_k}(\mathbb{C}) \).

Let \( \omega_k \) be the Fubini-Study form on \( \mathbb{P}^{N_k}(\mathbb{C}) \) and let \( \Omega_k := F_k^* \omega_k \) be its pull-back on \( Y \). Define the \( k \)-th characteristic function

\[
T_{F_k}(r) := \int_1^r \frac{dt}{t} \int_{B(t)} \Omega_k = T_{\Omega_k}(r).
\]

Fix \( \delta > 0 \), let \( T(r) := T_{F_0}(r) + \cdots + T_{F_{n-1}}(r) \). We claim that, for \( \delta' > 0 \),

\[
T(r) \leq \text{exc}(n(n+1)^2 + \delta') T_f(r) + (n(n+1)^2 + \delta') [(\delta + 2\varsigma) \log r - X_\sigma(r)] + O(1).
\]

Now we prove the claim. Following the notation in [15] for \( \|\xi\|_{\Omega_k} \), write

\[
S_k(r) := \int_{S(r)} \log \|\xi\|_{\Omega_k}^2 d\mu_r.
\]
Then (17) implies
\[ S_k(r) \leq \text{exc} (1 + \delta)^2 \log T_{F_k}(r) + (\delta + 2\zeta) \log r - \mathcal{X}_\sigma(r) + O(1) \]
\[ \leq \text{exc} (1 + \delta)^2 \log T(r) + (\delta + 2\zeta) \log r - \mathcal{X}_\sigma(r) + O(1). \] (32)

On the other hand, by the Plücker formula (see [9], Lemma 4.1 or [23], Lemma A3.5.1), \( \Omega_k = \frac{\|F_{k+1}(z)\|^2}{\|F_k(z)\|^4} \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{z} \) in any local coordinate \( z \). Choose \( z \) such that \( \xi = \frac{\partial}{\partial z} \), following the notation of (15), we have, for \( 0 \leq k \leq n \),
\[ \|\xi\|^2 \Omega_k = 2 \frac{\|F_{k-1}\|^2\|F_k\|^2}{\|F_k\|^4}, \] (33)
note that by convention we have set \( \|F_{-1}\| \equiv 1 \). Let \( \nu_k \) be the divisor of zeroes of \( \Omega_k \).

Applying \( \int_1^t \frac{ddx}{x} \log \cdot \) on (33) and use the Green-Jensen formula (Proposition 2.9), one has
\[ N_{\nu_k}(r) + T_{F_{k-1}}(r) - 2T_{F_k}(r) + T_{F_{k+1}}(r) = S_k(r). \]

From here, by an induction argument (see the proof of Theorem A3.5.3 in [23]), for \( 0 \leq q \leq p \), we can obtain
\[ T_{F_k}(r) + (p - q)T_{F_{q-1}}(r) \leq (p - q + 1)T_{F_k}(r) + \sum_{j=q}^{p-1} (p - j)S_j(r) + O(1). \]

In particular, by taking \( q = 0, p = k \) and notice that \( T_{F_{-1}}(r) \equiv 0 \), it gives
\[ T_{F_k}(r) \leq (k + 1)T_f(r) + \sum_{j=0}^{k-1} (k - j)S_j(r) + O(1) \]
\[ \leq (k + 1)T_f(r) + \frac{k(k + 1)}{2} [(1 + \delta)^2 \log T(r) + (\delta + 2\zeta) \log r - \mathcal{X}_\sigma(r)] + O(1). \] (34)

Hence by enlarging the exceptional set in \( \leq \text{exc} \) if necessary we get for \( \delta' > 0 \)
\[ T(r) \leq \text{exc} (n(n + 1)^2 + \delta')T_f(r) + (n(n + 1)^2 + \delta')[(\delta + 2\zeta) \log r - \mathcal{X}_\sigma(r)] + O(1). \]

The claim is proved.

Let \( 0 < \epsilon < \frac{\delta'}{n(n+1)^2+\delta'} \) so that by (31)
\[ \epsilon T(r) < \delta' T_f(r) + \delta' [(\delta + 2\zeta) \log r - \mathcal{X}_\sigma(r)] + O(1). \] (35)

Let \( \mu > 0 \) and define
\[ \lambda := \frac{\prod_{k=0}^{n-1} \|F_k\|^2 e}{\prod_{1 \leq j \leq q, 0 \leq k \leq n-1} \log^2(\mu/\phi_k(H_j))}. \]
Therefore Theorem 4.3 and the First Main Theorem imply the following inequality: 

$$C$$ is bounded from above for any

$$H$$

Hence

$$\int \frac{1}{x} \, dx$$

rearrange the index of

$$L$$

On the other hand, by (17) and the Green-Jensen formula (Proposition 2.9),

$$C > 0$$

for some (new) constant

$$C > 0$$. Hence, by the definition and the Green-Jensen formula (Proposition 2.9),

$$n(n + 1) \int S(r) \log \|\xi\|^2_{dd^c \log \lambda} \, d\mu_r \geq \sum_{n=1}^{q} m_j(r, H_j) - (n + 1)T_f(r) + N_W(r, 0) - \epsilon T(r) + \int S(r) \log \lambda d\mu_r.$$

On the other hand, by (17) and the Green-Jensen formula (Proposition 2.9),

$$\int S(r) \log \|\xi\|^2_{dd^c \log \lambda} \, d\mu_r \leq \text{exc} (1 + \delta)^2 \log T_{dd^c \log \lambda}(r) + (\delta + 2\zeta) \log r - \mathcal{X}_\sigma(r) + O(1)$$

$$\leq \text{exc} (1 + \delta)^2 \log \left( \int S(r) \log \lambda d\mu_r \right) + (\delta + 2\zeta) \log r - \mathcal{X}_\sigma(r) + O(1).$$

The proof is finished by observing that

$$C_0 \log \left( \int S(r) \log \lambda d\mu_r \right) - \int S(r) \log \lambda d\mu_r,$$

is bounded from above for any $$C_0 > 0$$, and combining (37), (38) with (35). \(\square\)

Let $$L_1, \ldots, L_q$$ be the linear forms defining $$H_1, \ldots, H_q$$. For each fixed $$y \in Y$$, we rearrange the index of $$L_1, \ldots, L_q$$ such that ord$$_{y}(L_1 \circ f) \geq \cdots \geq$$ ord$$_{y}(L_q \circ f) \geq n$$ ($$q_0$$ may not exist). Then ord$$_{y} W = \sum_{j=1}^{q_0} \text{ord}(f^*H_j) - n = \sum_{j=1}^{q} \text{ord}(f^*H_j) - n$$. Hence

$$\sum_{j=1}^{q} N_j(r, H_j) - N_W(r, 0) \leq \sum_{j=1}^{q} N_j^{[n]}(r, H_j).$$

Therefore Theorem 4.3 and the First Main Theorem imply the following inequality:

$$(q - (n + 1) - \delta')T_f(r) \leq \text{exc} \sum_{j=1}^{q} N_j^{[n]}(r, H_j) + \left( \frac{n(n + 1)}{2} + \delta' \right) [\mathcal{X}_\sigma(r) + (\delta + 2\zeta) \log r] + O(1).$$
A linear relation $c_1L_1 + \cdots + c_mL_m = 0$ is called *minimal* provided that all proper subsets of $\{L_1, \ldots, L_m\}$ are linearly independent. Note that any linear relation can be reduced to a linear combination of minimal linear relations. We also note that the $L_1, \ldots, L_q$, are always pairwisely linearly independent because the hyperplanes are distinct, hence necessarily $3 \leq m \leq n + 2$. Following [22], the idea to prove Theorem 4.2 is to observe that a minimal linear relation yields a Second-Main-Theorem type inequality (see Lemma 4.3) and that, if $H$ is non-degenerate, any two hyperplanes in $H$ are connected by a chain of minimal linear relations.

Let $f = (f_0, \ldots, f_n)$ be a local reduced representation of $f$. Since different reduced representations differ by a holomorphic function with no zero, $L_{\alpha \beta}$ is a well-defined meromorphic function on $Y$ for any $L_i, L_j \in \mathcal{L}$.

**Lemma 4.4.** Let $R$ be a minimal linear relation given by $c_1L_1 + \cdots + c_uL_{u+1} = 0$. Then, for any $\alpha, \beta \in \{1, \ldots, u + 1\}$ with $\alpha \neq \beta$,

$$T_{\frac{L_{\alpha \beta}}{L_{\alpha \beta}}} (r) \leq \text{exc} \sum_{j=1}^{q} 2N_{H_j}^{|n|} (r, H_j) + (n^2 + n + 2)[-X_{\sigma} (r) + (\delta + 2\varsigma) \log r] + O(1).$$

**Proof.** By rearranging the index if necessary we can assume $\alpha = 1$ and $2 \leq \beta \leq u$. Let $g_R : Y \to \mathbb{P}^{u-1}(\mathbb{C})$ be the holomorphic map defined by

$$z \mapsto [c_1(L_1 \circ f)(z), \ldots, c_u(L_u \circ f)(z)].$$

Let $H'_1 := \{w_1 = 0\}, \ldots, H'_u = \{w_u = 0\}, H'_{u+1} = \{c_1w_1 + \cdots + c_uw_u = 0\}$ be hyperplanes on $\mathbb{P}^{u-1}(\mathbb{C})$. Since $R$ is minimal, $g_R$ is linearly non-degenerate. Note that $H'_1, \ldots, H'_{u+1}$ are in general position. Let $\delta > 0$, by applying Theorem 4.3 (more precisely, (39)) to $g_R$ and $H'_1, \ldots, H'_{u+1}$ with $\delta' = \frac{1}{2}$, we get

$$T_{\frac{L_{\alpha \beta}}{L_{\alpha \beta}}} (r) \leq T_{g_R} (r)$$

(39)

$$\leq \text{exc} \sum_{t=1}^{u} N_{L_t}^{|u-2|} (r, 0) + \frac{(u-1)u}{2} \log r + O(1)$$

$$\leq \text{exc} \sum_{t=1}^{u} N_{L_t}^{|u-2|} (r, 0) + (u^2 - u + 1)[-X_{\sigma} (r) + (\delta + 2\varsigma) \log r] + O(1)$$

$$\leq \text{exc} \sum_{t=1}^{q} N_{L_t}^{|n|} (r, 0) + (n^2 + n + 1)[-X_{\sigma} (r) + (\delta + 2\varsigma) \log r] + O(1),$$

where the last equality holds because $u \leq n + 1$. □
Lemma 4.5. If $\mathcal{H} := \{H_1, \ldots, H_q\}$ is non-degenerate, then there exist $n+1$ linearly independent linear forms $L_{i_1}, \ldots, L_{i_{n+1}}$ in $\mathcal{L}$ such that, for $\delta > 0$,

$$T_{L_{i_1}}(f) \leq \text{exc} (n-1) \left( 2 \sum_{j=1}^{q} N_f^n(r, H_j) + (n^2 + n + 2)[-\mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r] \right) + O(1),$$

for $2 \leq \alpha \leq n+1$.

Proof. Choose an arbitrary $L_1 \in \mathcal{L}$ and let $R_1$ be a minimal linear relation containing $L_1$. Let $\mathcal{L}_1$ be the set of linear forms appeared in $R_1$. Then for any $L_{i_1} \in \mathcal{L}_1$, Lemma 4.4 implies

$$T_{L_{i_1}}(f) \leq \text{exc} \sum_{j=1}^{q} 2 N_f^n(r, H_j) + (n^2 + n + 2)[-\mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r] + O(1).$$

If $\# \mathcal{L}_1 = n + 2$, we are done. Otherwise let $\mathcal{L}_2 := \mathcal{L} \setminus (\mathcal{L}_1)$. Since $\mathcal{H}$ is non-degenerate, at least one element of $\mathcal{L}_2$ belong to $(\mathcal{L} \setminus \mathcal{L}_1)$. Then for any element $L_{i_2}$ in $\mathcal{L}_2$, either $L_{i_2}$ is in $\mathcal{L}_1$, or $L_{i_2}$ satisfies a minimal linear relation involving an element $L_{i_1}$ in $\mathcal{L}_1$. It sufficies to consider the latter case. By Lemma 4.4

$$T_{L_{i_2}}(f) \leq \text{exc} \sum_{j=1}^{q} 2 N_f^n(r, H_j) + (n^2 + n + 2)[-\mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r] + O(1).$$

Observe that $\frac{T_{L_{i_2}}(f)}{L_{i_1} \circ f} = \frac{T_{L_{i_2}}(f)}{L_{i_2} \circ f} \cdot \frac{T_{L_{i_2}}(f)}{L_{i_1} \circ f}$, hence

$$T_{L_{i_2}}(f) \leq \text{exc} \left( \sum_{j=1}^{q} 2 N_f^n(r, H_j) + (n^2 + n + 2)[-\mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r] \right) + O(1).$$

If $\# \mathcal{L}_2 = n + 2$, then we are done. Otherwise, note that $\mathcal{L}_2 \supseteq \mathcal{L}_1$. Inductively, since $\dim(\mathcal{L}) = n + 1$, we obtain a finite sequence $\mathcal{L}_s \supseteq \mathcal{L}_{s-1} \supseteq \cdots \supseteq \mathcal{L}_1$ with $\# \mathcal{L}_s = n + 2$, and for any $L_{i_s} \in \mathcal{L}_s$,

$$T_{L_{i_s}}(f) \leq \text{exc} s \left( \sum_{j=1}^{q} 2 N_f^n(r, H_j) + (n^2 + n + 2)[-\mathcal{X}_\sigma(r) + (\delta + 2\varsigma) \log r] \right) + O(1).$$

The proof is finished by noting that $\# \mathcal{L}_1 \geq 3$ and hence $s \leq n - 1$. □

Proof of Theorem 4.2
By Lemma 4.5, there are $n+1$ linearly independent linear forms $L_{i_1}, \ldots, L_{i_{n+1}} \in \mathcal{L}$ such that (40) holds. Hence

\[(41)\]

\[
T_f(r) \leq \sum_{j=2}^{n+1} T_{L_{i_j}(f)}(r) \leq \text{exc} n(n-1) \left( 2 \sum_{j=1}^{q} N^{[n]}_f(r, H_j) + (n^2 + n + 2)[(\delta + 2\zeta) \log r - \mathcal{X}_f(r)] \right) + O(1)
\]

\[
\leq \text{exc} 2n^2(n-1)N_f(r, |\mathcal{H}|) + n(n-1)(n^2 + n + 2)[(\delta + 2\zeta) \log r - \mathcal{X}_f(r)] + O(1).
\]

This shows that if $\mathcal{H}$ is non-degenerate, then $(\mathbb{P}^n(\mathbb{C}), |\mathcal{H}|)$ is a Nevanlinna pair.

5. NEVANLINNA PAIR AND ALGEBRAIC HYPERBOLICITY

The concept of algebraic hyperbolicity for a compact complex manifold $X$ was introduced by Demailly in [4], Definition 2.2, and he proved (see [4], Theorem 2.1) that $X$ is algebraically hyperbolic if it is Kobayashi hyperbolic. The notion of algebraic hyperbolicity was generalized to the case of log-pairs $(X, D)$ by Chen [3]. According to Chen, $(X, D)$ is said to be algebraically hyperbolic if there exists a positive $(1,1)$-form $\omega$ on $X$ such that for any compact Riemann surface $R$ and every holomorphic map $f : R \to X$ with $f(R) \not\subset D$, one has

\[
\int_{R} f^* \omega \leq \tilde{n}_f(D) + \max\{0, 2g - 2\},
\]

where $\tilde{n}_f(D)$ is the number of points of $f^{-1}(D)$ on $R$ and $g$ is the genus of $R$.

Unlike Demailly’s theorem (as well as the theorem of Pacienza-Rousseau [20] for log-pairs $(X, D)$), it is unclear whether Kobayashi hyperbolicity or Picard hyperbolicity of $X \setminus D$ will imply the algebraic hyperbolicity of $(X, D)$. However, we prove that $(X, D)$ is algebraically hyperbolic if $(X, D)$ is a Nevanlinna pair, which is one of the main points and motivations of this paper.

**Theorem 5.1.** If $(X, D)$ is a Nevanlinna pair, then $(X, D)$ is algebraically hyperbolic.

**Proof.** Let $R$ be a compact Riemann surface with genus $g$ and $f : R \to X$ be holomorphic map with $f(R) \not\subset D$. We need to show that

\[(42)\]

\[
\int_{R} f^* \omega \leq \tilde{n}_f(D) + \max\{0, 2g - 2\}
\]
for a positive $(1,1)$-form $\omega$ on $X$ that is independent of $R$ and $f$.

Fix a point $Q \in R$ such that $f(Q) \notin \text{supp}(D)$. We view $R$ as a divisor on $R$ with degree 1. Let $L(kQ)$ be the vector space of meromorphic functions $\psi$ on $R$ such that either $\psi$ is a constant or $(\psi) + kQ \geq 0$, i.e., $\psi$ has only a pole at $Q$ with order less than or equal to $k$. By the Riemann-Roch Theorem,

$$\dim L(kQ) - \dim L(kQ - K) = k - g + 1,$$

where $K$ is the canonical divisor on $R$. This implies that $\dim L((g+1)Q) \geq 2$, so we can choose a non-constant meromorphic function $\psi$ on $R$ with a single pole at $Q$ of order less than or equal to $g + 1$. Then $\sigma := |\psi|$ is a parabolic exhaustion function for the open parabolic Riemann surface $R \setminus \{Q\}$. By the Poincaré-Lelong and Stokes’ formula, for $r > 1$ such that all zeros of $\psi$ are inside $B(r)$,

$$g + 1 \geq \sum_{p \in B(r)} \text{ord}_p \psi = \int_{B(r)} d\tilde{c} \log |\psi|^2 = 2 \int_{S(r)} \tilde{c} \log |\psi| = 2 \int_{S(r)} \tilde{c} \log \sigma = 2\varsigma.$$

Hence $\varsigma \leq \frac{g+1}{2}$. Since $(X, D)$ is a Nevanlinna pair, there exists a positive $(1,1)$-form $\eta$ on $X$ such that

$$T_{f, \eta}(r) \leq \text{exc} \, N_f(r, D) - \mathcal{X}_\sigma(r) + (\delta + g + 1) \log r + \mathcal{E}_\sigma(r) + O(1),$$

where we used the fact $\varsigma \leq \frac{g+1}{2}$. Also, since $\psi$ has a pole only at $Q$, $\psi' := d\psi(\xi)$ also has a pole only at $Q$ (otherwise $\psi'$ would be a constant). Hence $\mathcal{E}_\sigma(r) := \int_{S(r)} \log |d\sigma(\xi)|^2 d\mu_r = 0$ for $r$ big enough. Therefore, from (43) we can take a sequence $r_n \to +\infty$ such that

$$T_{f, \eta}(r_n) \leq N_f(r_n, D) - \mathcal{X}_\sigma(r_n) + (\delta + g + 1) \log r_n + O(1).$$

Now recall that

$$\mathcal{X}_\sigma(r) = \int_{1}^{r} \chi_\sigma(t) \frac{dt}{t},$$

where $\chi_\sigma(t)$ is the Euler characteristic of the domain $B(t)$. Hence,

$$\lim_{r \to \infty} \frac{\mathcal{X}_\sigma(r)}{\log r} = \chi(R - \{p\}) = \chi(R) - 1 = 1 - 2g.$$

Let

$$A(r) := \int_{B(r)} f^* \eta.$$ 

Then, for any fixed $r$, when $r_n > r$

$$A(r) \leq \frac{1}{\log r_n - \log r} \int_{r}^{r_n} A(t) \frac{dt}{t} \leq \frac{1}{\log r_n - \log r} T_{f, \eta}(r_n) \leq \frac{1}{\log r_n - \log r} (N_f(r_n, D) - \mathcal{X}_\sigma(r_n) + (\delta + g + 1) \log r_n + O(1))$$
\[
\leq \frac{1}{\log r_n - \log r} (\bar{n}_f(D) \log r_n - X_\sigma(r_n) + (\delta + g + 1) \log r_n + O(1)).
\]

By taking \( n \to \infty \) we get

\[
A(r) \leq \bar{n}_f(D) + \delta + 3g.
\]

Now let \( r \to +\infty \) and then let \( \delta \to 0 \), one gets

\[
\int_R f^*\eta \leq \bar{n}_f(D) + 3g.
\]

Observe that when \( g \geq 2, n_f(D) \geq 0 \) or \( 0 \leq g \leq 1 \) with \( \bar{n}_f(D) \geq 1 \), we have

\[
\bar{n}_f(D) + 3g \leq 6(\bar{n}_f(D) + \max\{0, 2g - 2\}).
\]

Hence, by choosing \( \omega = \frac{1}{6} \eta \),

\[
\int_R f^*\omega \leq \bar{n}_f(D) + \max\{0, 2g - 2\},
\]

which verifies (42). It remains to deal with the following two cases,

1. \( g = 0, \bar{n}_f(D) = 0 \).
2. \( g = 1, \bar{n}_f(D) = 0 \).

In these two cases, we prove that \( f \) must be a constant so (42) trivially holds. In case (1), \( R = \mathbb{P}^1(\mathbb{C}) \), so \( f \) is a holomorphic map from \( \mathbb{P}^1(\mathbb{C}) \) to \( X \setminus D \), which must be constant because \( X \setminus D \) is Brody hyperbolic. In case (2), consider the universal covering \( \pi : \mathbb{C} \to R \), let \( \tilde{f} := f \circ \pi \) be the lifting of \( f \). Then \( \tilde{f} : \mathbb{C} \to X \setminus D \) is a holomorphic curve. Since \( X \setminus D \) is Brody hyperbolic, \( \tilde{f} \) is constant, so \( f \) must be constant. \( \square \)

We have the following consequences of Theorem 5.1.

**Corollary 5.2.**  
(1) If \( \mathbb{P}^n(\mathbb{C}) \setminus |\mathcal{H}| \) is Brody hyperbolic, then \( (\mathbb{P}^n(\mathbb{C}), |\mathcal{H}|) \) is algebraically hyperbolic.

(2) If \( A \) is an abelian variety and \( D \) is an ample divisor, then \( (A, D) \) is algebraically hyperbolic.

(3) Let \( X \) be a projective manifold of dimension \( n \geq 2 \) and let \( A \) be a very ample line bundle over \( X \). If \( D \in |A^m| \) is a general smooth hypersurface with

\[
m \geq (n + 2)^{n+3} (n + 1)^{n+3},
\]

then \( (X, D) \) is algebraically hyperbolic.

**Proof.**  
(1) By combining Theorem 4.2 with Theorem 5.1.

(2) By combining Theorem 3.6 with Theorem 5.1.
(3) By combining Theorem 3.4 with Theorem 5.1.

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