LIMIT LINEAR SERIES, THE IRRATIONALITY OF $M_g$, AND OTHER APPLICATIONS

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ABSTRACT. We describe degenerations and smoothings of linear series on some reducible algebraic curves. Applications include a proof that the moduli space of curves of genus $g$ has general type for all $g \geq 24$, a proof that the monodromy action is transitive on the set of linear series of dimension $r$ and degree $d$ on a general curve of genus $g$ when $\rho := g - (r + 1)(g - d + r) = 0$, a proof that there exist Weierstrass points with every semigroup of a certain class—in particular, on curves of genus $g$, all those semigroups with weight $w \leq g/2$ occur and a proof that the monodromy group acts as the full symmetric group on the $g^3 - g$ Weierstrass points of the general curve.

Curves will here be reduced, connected, and complex algebraic.

The study of general curves (Brill-Noether theory, etc.) and of moduli of curves depends on the degeneration of smooth curves to singular ones. Originally, the singular curves used were irreducible curves with nodes ([G-H] is a recent avatar) or, more recently, cusps [E-H1], but from the work of Mumford and others on the moduli space of stable curves it is apparent that reducible curves should be considered as well.

Unfortunately the degeneration of a linear series on a curve which degenerates to a reducible curve has not been well understood except in the particularly simple case of pencils; there the "limit" of the linear series, after removing base points, corresponds to an admissible covering, in the sense of Beauville, Knudsen and Harris-Mumford [B, K, H-M], of a curve of genus 0. The potential of a general theory is indicated, for example, by work of Gieseker [G].

In this announcement we describe the limits of linear series on some reducible curves and give some applications.

We call a curve tree-like if its irreducible components meet only two at a time, in ordinary nodes, in such a way that its dual graph (a vertex for each component, an edge for each intersection between distinct components) has no loops.

We say that a curve is of compact type if its (generalized) Jacobian is compact, or, equivalently, if it is tree-like and its irreducible components are all nonsingular.

Received by the editors November 8, 1983.

1980 Mathematics Subject Classification. Primary 14H10.

The authors are grateful to the NSF, and the second author is grateful to the Alfred P. Sloan Foundation, for partial support of this work.
DEFINITION. A limit $g_d^r$ on a tree-like curve $Y$ is a collection of $g_d^r$'s, one on each irreducible component $Z$ of $Y$,

$$L_Z$$ a line bundle of degree $d$ on $Z$,
$$V_Z \subset H^0(Z, L_Z)$$ an $r+1$-dimensional subspace

such that whenever two components of $Y$ meet in a point, say $p = Z_1 \cap Z_2$, there is for each $\sigma \in V_{Z_1}$ a $\tau \in V_{Z_2}$ such that $\text{ord}_p \sigma + \text{ord}_p \tau = d$.

The following result is implicit in [E-H3]:

**Theorem 1.** Let $O$ be a discrete valuation ring, and let $X \to \text{Spec } O$ be a family of curves with irreducible geometric general fiber $X^g$ and reduced, special fiber of compact type. Given a line bundle $L$ and a $g_d^r(k(\eta))^{r+1} \equiv V \subset H^0(X, L)$ on $X$, there is a family $\pi' : X' \to \text{Spec } O'$ obtained from $X$ by base change, blow-ups of points in the central fiber, and normalizations, with reduced, special fiber $Y$ of compact type such that:

1. For each irreducible component $Z \subset Y$ there is an extension $L_Z$ of $L$ to $X$ with
$$\text{deg}(L_Z|_Z) = d,$$
$$\text{deg}(L_Z|_{Z'}) = 0$$ for irreducible components $Z' \neq Z$.

2. The images
$$V_Z = \text{im}(V \subset \pi'_*(L_Z|_Z) \to H^0(Z, L_Z|_Z))$$

form a limit $g_d^r$ on $Y$.

See [E-H2,3] for applications of this result to Brill-Noether theory.

We will say that a limit $g_d^r$ on a tree-like curve $Y$ is smoothable if it can be obtained from a family with geometrically irreducible general fiber as in Theorem 1. Every limit $g_d^r$ is smoothable, as is shown in [H-M]; an explicit analytic smoothing can actually be constructed with little effort. Unfortunately there are nonsmoothable $g_d^r$'s with $r \geq 2$. But these only occur on rather atypical curves, as our next result shows:

**Theorem 2.** Let $X \to B$ be a family of tree-like curves of arithmetic genus $g$ over an irreducible base $B$, and let $G_d^r(X/B)$ be the corresponding family of limit $g_d^r$'s. Set $\rho = g - (r+1)(g-d+r)$ ($\rho$ may be negative) if $\dim G_d^r(X/B) \leq \dim B + \rho$, then every limit $g_d^r$ on every curve of the family is smoothable.

Curves satisfying the hypothesis of Theorem 2 (with $B$ a point) may be found in [E-H2,3]. It is also satisfied (for every $r$, $d$) by the union of three general curves of genus $g_1, g_2$ with $g_1 + g_2 = g$, joined at general points of each, and by many other simple curves and families of curves.

Theorem 2 is proved by giving explicitly the “right number” of local equations for the family of $g_d^r$'s (or rather, for a certain associated frame-bundle) in the neighborhood of a given limit $g_d^r$. This approach was suggested by conversations with Ziv Ran, to whom we are grateful.

We now indicate three applications beyond those of [E-H2,3]:

First, we may complete and simplify the ideas in the second half of [H-M] and [H], where it is shown that the moduli space $M_g$ of curves of genus $g$ has general type for $g$ odd and $\geq 25$ or even and $\geq 40$: 
APPLICATION 1 [E-H5]. $M_g$ has general type for all $g \geq 24$.

For the proof of this we make use of the ideas and methods of the first 3 sections of [H-M] as described in the introduction to [H]; these methods require the choice and computation of a divisor in $M_g$ with certain properties.

We distinguish 2 (overlapping) cases:

(i) If $g + 1$ is not prime, then for suitable $r$ and $d$ we have

$$\rho = g - (r + 1)(g - d + r) = -1,$$

and the closure of the set of smooth curves possessing a $g_d$ forms a suitable divisor in $M_g$ if $g \geq 24$. This covers in particular the cases $g$ odd and $g = 24, 26$.

(ii) If $g$ is even, say $g = 2k - 2$, and $g \geq 28$, we use the closure of the ramification divisor of the map from the moduli space of curves $C$ of genus $g$ with chosen pencil $C^2 \cong V \subset \mathbb{H}^0(C, \mathcal{L})$ of degree $k$ to $M_g$, in accordance with the program expressed in the introduction to [H-M]. To circumvent the problem mentioned in the introduction [H] we interpret ramification as being signalled by the presence of a nonzero section of $K_C \otimes \mathcal{L}^{-2}$, where $K_C$ is the canonical class of $C$.

As a second application, we can complete, in a certain sense, the result of Fulton and Lazarsfeld [F-L] who prove (using the result of Gieseker proved in [G] and [E-H3]) that if $C$ is a general curve, then the variety $G_d(C)$ of $g_d$'s on $C$ is irreducible as long as $\rho := g - (r + 1)(g - d + r) > 0$. For $\rho = 0$ and $C$ general, $G_d(C)$ is a reduced set of points. We prove:

APPLICATION 2 [E-H4]. Assume $\rho = g - (r + 1)(g - d + r) = 0$. The fundamental group of the moduli space of curves $C$ with $G_d(C)$ reduced and finite acts transitively by monodromy on each such $G_d(C)$. Equivalently, there is a family of such curves $X \to B$ such that the associated family $G_d(X/B)$ is irreducible.

The key to the proof of this is the fact that on the curve used in [E-H3] the different $g_d$'s can be labelled, in the $\rho = 0$ case, by certain chains of Schubert cycles in a Grassmann variety. Further, if two of these chains differ in only one element, then a family of curves can be constructed (by allowing two "elliptic tails" to hang at varying points from one rational component of a curve as in [E-H3]) whose monodromy interchanges the corresponding $g_d$'s. Since the simplicial complex of chains of Schubert cycles is connected in codimension 1 (even Cohen-Macaulay—see for example [D-E-P]), this suffices to prove transitivity.

APPLICATION 3. Certain semigroups occur as the Weierstrass semigroups of smooth curves. In particular, if $\Gamma = \{0, a_1, a_2, \ldots \} \subset \mathbb{N}$ is a subsemigroup without common divisor of the natural numbers, then $\Gamma$ occurs as the Weierstrass semigroup of a curve of genus $g = |N-G|$ if $a_1 > w$ or, more particularly, $w \leq g/2$, where $w = \sum_{i=1}^{g+1} (g + i - a_i)$ is the weight of $\Gamma$. Moreover, there is at least one component of the subvariety of Weierstrass points with semigroup $\Gamma$, in $M_g^1$, with codimension $w$.

This is proved inductively by smoothing "limit canonical series" on curves of the form
where $C$ is a curve of genus $g - 1$ with a suitable Weierstrass point $p$ of a certain type, moving in a family whose dimension is the weight of $p$, $E$ is an elliptic curve, $q - p$ is torsion of a suitable order, and the limit series is chosen to have ramification at $q$ corresponding to a Weierstrass point of the desired type.

**APPLICATION 4.** The monodromy group acts on the $g^3 - g$ Weierstrass points of a general curve as the symmetric group on $g^3 - g$ letters.

This is proved by specializing to a reducible curve with a positive dimensional family of "limit canonical series", and examining the monodromy of this family.

**REMARK.** It seems possible to give a related, but substantially more complicated, description of "limit $g^r$'s" on arbitrary stable curves. It may be possible to use this fact to study other types of Weierstrass points of low weight.

**References**

[A] E. Arbarello, *Weierstrass points and moduli of curves*, Compositio Math. 29 (1974), 325–342.

[B] A. Beauville, *Prym varieties and the Schottky problem*, Invent. Math. 41 (1977), 149–196.

[D-E-P] C. DeConcini, D. Eisenbud and C. Procesi, *Hodge algebras*, Astérisque 91 (1982).

[D] S. Dias, *Exceptional Weierstrass points and the divisor on moduli space that they define*, Ph.D. thesis, Brown Univ. Providence, R.I., 1982.

[E-H1] D. Eisenbud and J. Harris, *Linear series on general curves and cuspidal rational curves*, Invent. Math. (1983).

[E-H2] ______, *A short proof of the Brill-Noether theorem*, Proc. Ravello Conf. Algebraic Geometry.

[E-H3] ______, *A simpler proof of the Gieseker-Petri theorem*, Invent. Math. (1983).

[E-H4] ______, *Linear series on reducible curves, and applications to linear series with $p = 0$* (in preparation).

[E-H5] ______, $M_g$ is of general type for $g \geq 24$ (in preparation).

[F-L] W. Fulton and R. Lazarsfeld, *On connectedness of degeneracy loci and special divisors*, Acta Math. 146 (1981), 271–283.

[G] D. Gieseker, *Stable curves and special divisors*, Invent. Math. 66 (1982), 251–275.

[G-H] P. A. Griffiths and J. Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. 47 (1980), 233–272.

[H] J. Harris, *On the Kodaira dimension of the moduli space of curves. II: The even genus case*, Invent. Math. (to appear).

[H-M] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. 67 (1982), 23–86.

[K] F. Knudsen, *The projectivity of the moduli space of stable curves*, Math. Scand. 52 (1983).