Blow-up rate of sign-changing solutions to nonlinear parabolic systems

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Abstract

We present a blow-up rate estimate for a solution to the parabolic Gross-Pitaevskii and related systems on entire space with Sobolev subcritical nonlinearity. We extend the results of [Y. Giga, S. Matsui and S. Sasayama, Indiana Univ. Math. J. 53 (2004), 483–514] to the parabolic systems.

1 Introduction

Let $U = (u_1, \ldots, u_M)$ be a (classical) solution to the Cauchy problem for a semilinear parabolic system

$$\begin{cases}
\partial_t U - \Delta U = (\nabla G)(U) & \text{in } \mathbb{R}^N \times (0, T), \\
U(x, 0) = U_0(x) & \text{in } \mathbb{R}^N,
\end{cases} \quad (P)$$

where $N \geq 1$, $M \geq 1$, $T > 0$ and $U_0 \in BC^2(\mathbb{R}^N)$. Here

$$G(U) = \frac{1}{2(r+1)} \sum_{i,j=1}^{M} \beta_{ij} |u_i|^{r+1} |u_j|^{r+1} \quad \text{with } r > 0, \quad (1.1)$$

where $\{\beta_{ij}\} \subset \mathbb{R}$ satisfies

$$\beta_{ij} \geq 0, \quad \beta_{ii} > 0, \quad \beta_{ij} = \beta_{ji}. \quad (1.2)$$

Then $\{u_i\}_{i=1}^{M}$ satisfies

$$\begin{cases}
\partial_t u_i - \Delta u_i = \sum_{j=1}^{M} \beta_{ij} |u_i|^{r-1} |u_j|^{r+1} u_i & \text{in } \mathbb{R}^N \times (0, T), \\
u_i(x, 0) = u_{i,0}(x) & \text{in } \mathbb{R}^N.
\end{cases} \quad (1.3)$$
Parabolic system (1.3) can be regarded as a parabolic generalization of the Gross-Pitaevskii system
\[
\frac{1}{\sqrt{-1}} \partial_t u_i - \Delta u_i = \sum_{j=1}^{M} \beta_{ij} |u_j|^2 \quad \text{in} \quad \mathbb{R}^N \times (0, T), \quad \text{for} \quad i = 1, \ldots, M,
\]
and it is called parabolic Gross-Pitaevskii and related system. Parabolic system (1.3) has nice mathematical structure such as scaling invariance and energy structure. In this paper, thanks to these nice properties of parabolic system (1.3), we study a blow-up rate estimate for a solution to problem (P).

Blow-up rate of nonlinear parabolic problems has been studied in many papers, see e.g., [2–7, 9–22, 24–36, 39–41, 45–47]. (See also the monograph [44].) Let us consider problem (P) with \( M = 1 \), that is
\[
\left\{ \begin{array}{l}
 u_t - \Delta u = |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N \times (0, T), \\
 u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N,
\end{array} \right.
\]
where \( p > 1 \). Assume that
\[
\limsup_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty.
\]
We say that the blow-up is of type I if
\[
\limsup_{t \to T} (T - t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} < \infty,
\]
and of type II otherwise. It is known that if \( 1 < p < p_S \), then the solution \( u \) to problem (1.4) exhibits type I blow-up, where
\[
p_S := \frac{N + 2}{N - 2} \quad \text{if} \quad N \geq 3, \quad p_S := \infty \quad \text{if} \quad N = 1, 2.
\]
See [17, 19, 20]. On the other hand, if \( p \geq p_S \), the solution \( u \) to problem (1.4) does not necessarily exhibit type I blow-up. See e.g., [2, 5, 7, 10, 21, 27, 35, 46, 47].

For the parabolic system, blow-up estimates are much less understood. See e.g., [31, 37, 38, 42, 43]. Among others, Phan and Souplet [38] established a parabolic Liouville-type theorem for problem (P) and proved that a solution \( U = (u_1, \ldots, u_M) \) of problem (P) satisfies the type I blow-up estimate
\[
\|U\|_{L^\infty(\mathbb{R}^N)} \leq C(T - t)^{-\frac{1}{p-1}}, \quad 0 < t < T,
\]
provided that \( 1 < p := 2r + 1 < p_B \) and \( U \geq 0 \) in \( \mathbb{R}^N \times (0, T) \), that is, \( u_i \geq 0 \) in \( \mathbb{R}^N \times (0, T) \), where \( i = 1, \ldots, M \). Here
\[
p_B := \frac{N(N + 2)}{(N - 1)^2} \quad \text{if} \quad N \geq 3, \quad p_B := \infty \quad \text{if} \quad N = 1, 2.
\]
On the other hand, it seems difficult to apply their arguments to problem (P) without the restriction of the sign of $U$, since the parabolic Liouville-type theorem acts only on nonnegative solutions. In this paper we develop the arguments in [19, 20] and obtain the type I blow-up estimate (1.5) without the restriction of the sign of $U$ for $1 < p < p_S$. Note that $p_S > p_B$ if $N \geq 3$.

We formulate a solution to problem (P).

**Definition 1.1** Let $M = 1, 2, \ldots$, $T > 0$ and $U_0 \in BC^2(\mathbb{R}^N)$. A function $U = (u_1, \ldots, u_M)$ in $\mathbb{R}^N \times [0, T)$ is said a solution to problem (P) if $u_i, \partial_t u_i, \nabla u_i, \nabla^2 u_i$ are bounded and continuous on $\mathbb{R}^N \times [0, \tau]$ for $0 < \tau < T$ and $i = 1, \ldots, M$ and $U = (u_1, \ldots, u_M)$ solves (P).

We are ready to state the main result of this paper.

**Theorem 1.1** Let $M = 1, 2, \ldots$, $T > 0$ and $U_0 \in BC^2(\mathbb{R}^N)$. Assume (1.1) and (1.2). Let $U$ be a solution to problem (P). Then there exists $C > 0$ such that

$$\|U(t)\|_{L^\infty(\mathbb{R}^N)} \leq C(T - t)^{-\frac{1}{p-1}}, \quad 0 < t < T,$$

provided that $1 < p := 2r + 1 < p_S$.

We explain the idea of the proof of Theorem 1.1. Let $U$ be a solution in $\mathbb{R}^N \times [0, T)$. We may assume $T = 1$ since problem (P) possesses a similarity transformation such as $Z(x, t) = T^{\frac{1}{p-1}}U(\sqrt{T}x, Tt)$.

As in [17], for any $a \in \mathbb{R}^N$, we introduce the following rescaled function of $U$

$$W^a(y, s) := (T - t)^{\beta}U(x, t)$$

with

$$y = \frac{x - a}{\sqrt{T - t}}, \quad s = -\log(T - t), \quad \beta = \frac{1}{p - 1}.$$

Then $W^a(y, s) = (w_1, \ldots, w_M)$ satisfies

$$\partial_s w_i - \Delta w_i + \frac{1}{2}y \cdot \nabla w_i + \beta w_i - \sum_{j=1}^M \beta_{ij} |w_i|^{r-1} |w_j|^{r+1} w_i = 0, \quad (1.6)$$

that is

$$\rho \partial_s w_i - \nabla \cdot (\rho \nabla w_i) + \beta \rho w_i - \rho (\nabla G)(W^a) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty) \quad (1.7)$$
for $i = 1, \ldots, M$, where $\rho(y) = \exp(-|y|^2/4)$. Furthermore, Definition 1.1 implies that

$$w_i, (1 + |y|)^{-1} \partial_s w_i, \nabla w_i, \nabla^2 w_i$$

are bounded and continuous on $\mathbb{R}^N \times [0, S]$ for $i = 1, 2, \ldots, M$ and any $S < \infty$. (1.8)

We introduce the global energy of $W$

$$E[W](s) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla W|^2 + \beta |W|^2) \rho \, dy - \int_{\mathbb{R}^N} G(W) \rho \, dy$$

(1.9)

and the local energy of $W$

$$E_\varphi[W] := \frac{1}{2} \int_{\mathbb{R}^N} \varphi^2 (|\nabla W|^2 + \beta |W|^2) \rho \, dy - \int_{\mathbb{R}^N} \varphi^2 G(W) \rho \, dy,$$

(1.10)

where

$$|\nabla W|^2 = \sum_{i=1}^M |\nabla w_i|^2 \quad \text{and} \quad |W|^2 = \sum_{i=1}^M |w_i|^2.$$ 

Here $\varphi(y)$ is a cut-off function. Developing the argument in [19], we prove the boundedness of the global energy and the local energy of $W$, and obtain

$$\sup_{s \geq 0} \int_s^{s+1} \left( \int_{B(R)} |W|^{p+1} \rho \, dy \right)^q \, d\tau \leq C_{q,R}$$

(1.11)

for any $R > 0$ and $q \geq 2$. Here $C_{q,R}$ is a constant depending on $q$ and $R$. In [19], for the case of $M = 1$, the authors combined (1.11) and the interior regularity theorem for parabolic equations to obtain the $L^\infty_{loc}$ estimate of $W$. However, this argument fails for parabolic system (1.3) in the case where $M \geq 2$ and $0 < r < 1$ because of the nonlinearity of $|w_i|^{r-1}$. In this paper, in order to overcome the difficulty, we introduce a function $w$ defined by

$$w := \sum_{i=1}^M |w_i|,$$

which is a weak subsolution to

$$\partial_s w \leq \Delta w - \frac{1}{2} y \cdot \nabla w - \beta w + \sum_{i,j=1}^M \beta_{ij} w^{2r+1} \quad \text{in} \quad \mathbb{R}^N \times (0, \infty).$$

Then we apply the parabolic regularity theorems to the weak subsolution $w$ and obtain the estimate

$$\|w\|_{L^\infty(B_R)} \leq C \quad \text{for} \quad s \geq 0.$$ 

This enables us to obtain the type I estimate, and Theorem 1.1 follows.

The rest of this paper is organized as follows. In Sections 2 and 3 we obtain the estimate for the global energy and the local energy respectively. In Section 4 we prove (1.11) and Theorem 1.1.
2 Estimate of the global energy

In this section we obtain the monotonicity of the energy \( E[W](s) \) and its related inequalities. In what follows we write \( F(U) := (\nabla G)(U) \) and often use the following structure conditions on \( G \):

\[
\begin{align*}
G(\lambda U) &= \lambda^{p+1} G(U) \quad \text{for } \lambda \geq 0 \text{ and } U \in \mathbb{R}^M; \\
G(U) &> 0 \quad \text{for } U \in \mathbb{R}^M \setminus \{0\}.
\end{align*}
\]  

(2.1)

It follows from (2.1) that

\[
\begin{align*}
F(\lambda U) &= \lambda^p F(U) \quad \text{for } \lambda \geq 0 \text{ and } U \in \mathbb{R}^M; \\
|F(U)| &\leq C_F |U|^p \quad \text{for } U \in \mathbb{R}^M; \\
c_G |U|^{p+1} &\leq G(U) \leq C_F |U|^{p+1} \quad \text{for } U \in \mathbb{R}^M; \\
U \cdot F(U) &= (p + 1) G(U) \quad \text{for } U \in \mathbb{R}^M.
\end{align*}
\]  

(2.2)

Here \( c_G \) and \( C_F \) are positive constants such that

\[
c_G = \min_{|U|=1} G(U) \quad \text{and} \quad C_F = \max_{|U|=1} |F(U)|.
\]

In Propositions 2.1 and 2.2 we show the monotonicity and the nonnegativity of the global energy \( E[W](s) \), respectively.

**Proposition 2.1** Let \( W \) satisfy (1.6) with (1.8). Then

\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} |W|^2 \rho \, dy = -2E[W](s) + (p - 1) \int_{\mathbb{R}^N} G(W)\rho \, dy,
\]

(2.3)

\[
\frac{d}{ds} E[W](s) = -\int_{\mathbb{R}^N} |W_s|^2 \rho \, dy,
\]

(2.4)

for \( s \geq 0 \).

**Proof.** It follows from (1.7), (1.9) and (2.2) that

\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} |W|^2 \rho \, dy = \int_{\mathbb{R}^N} W \cdot W_s \rho \, dy
\]

\[
= -2E[W](s) + (p - 1) \int_{\mathbb{R}^N} G(W)\rho \, dy.
\]

(2.5)

This implies (2.3). Furthermore, since

\[
\frac{d}{ds} \int_{\mathbb{R}^N} G(W)\rho \, dy = \int_{\mathbb{R}^N} F(W) \cdot W_s \rho \, dy,
\]

\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} |\nabla W|^2 \rho \, dy = -\sum_{i=1}^M \int_{\mathbb{R}^N} w_i \nabla \cdot (\rho \nabla w_i) \, dy,
\]

by (2.5) we have (2.4). The proof is complete. \( \square \)
Proposition 2.2 Let $W$ satisfy (1.6) with (1.8). Then

$$0 \leq E[W](s) \leq E[W](0) \quad \text{for} \quad s \geq 0. \quad (2.6)$$

Proof. Due to (2.4) it suffices to prove the nonnegativity of $E[W](s)$. By (2.2), applying Jensen’s inequality to (2.5), we have

$$\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} |W(s)|^2 \rho \, dy \geq -2E[W](s) + c_1 \left( \int_{\mathbb{R}^N} |W(s)|^2 \rho \, dy \right)^{\frac{p+1}{2}}, \quad (2.7)$$

where $c_1$ is a positive constant depending on $c_G$. If $E[W](s_*) < 0$ for some $s_* \geq 0$, then by (2.7) and the monotonicity of $E[W](s)$ we find $s'_* \in (s_*, \infty)$ such that

$$\int_{\mathbb{R}^N} |W(s)|^2 \rho \, dy \to \infty \quad \text{as} \quad s \to s'_*.$$

This contradicts the global existence of $W$. Thus Proposition 2.2 follows. □

Remark 2.1 $E[W](0)$ is uniformly bounded with respect to a since $U_0 \in BC^2(\mathbb{R}^N)$. Indeed, we have

$$M_0 := \sup_{a \in \mathbb{R}^N} E[W](0) \leq C \sup_{x \in \mathbb{R}^N} (|U_0(x)|^2 + |\nabla U_0(x)|^2) < \infty.$$

For any Banach space $X$ and $f \in X$, we denote by $\|f; X\|$ the norm of $f$ in $X$. For any domain $\Omega \subset \mathbb{R}^N$ and $1 \leq q \leq \infty$, we write $L^q_\rho(\Omega) = L^q(\Omega, \rho \, dy)$. In particular,

$$\|f; L^2_\rho(\Omega)\| := \left( \int_{\Omega} |f|^2 \rho \, dy \right)^{\frac{1}{2}}, \quad f \in L^2_\rho(\Omega).$$

Furthermore, we write $W^{1,2}_\rho(\Omega) = W^{1,2}(\Omega, \rho \, dy)$ and set

$$\|f; W^{1,2}_\rho(\Omega)\| := \left( \int_{\Omega} (|\nabla f|^2 + \beta |f|^2) \rho \, dy \right)^{\frac{1}{2}}, \quad f \in W^{1,2}_\rho(\Omega).$$

Proposition 2.3 Let $W$ satisfy (1.6) with (1.8). Then

$$\int_0^\infty \|W_s; L^2_\rho(\mathbb{R}^N)\|^2 \, d\tau \leq E[W](0). \quad (2.8)$$

Moreover, there exist positive constants $K_1$, $K_2$ and $K_3$ depending only on $N$, $p$, $c_G$ and $M_0$ such that

$$\sup_{s \geq 0} \|W_s; L^2_\rho(\mathbb{R}^N)\| \leq K_1, \quad (2.9)$$

$$\sup_{s \geq 0} \int_0^{s+1} \|W_s; L^{p+1}_\rho(\mathbb{R}^N)\|^{2(p+1)} \, d\tau \leq K_2, \quad (2.10)$$

$$\|W(s); W^{1,2}_\rho(\mathbb{R}^N)\|^2 \leq K_3 (1 + \|W_s; L^2_\rho(\mathbb{R}^N)\|) \quad \text{for} \quad s \geq 0. \quad (2.11)$$
Proof. Inequality (2.8) follows from (2.4) and (2.6). We prove inequality (2.9). Let 

\[ f(s) := \|W_s; L^2_\rho(\mathbb{R}^N)\| \]

By the Sobolev inequality we find \( C > 0 \) such that

\[ \sup_{[s, s+1]} f \leq C \left( \|f'; L^2(s, s+1)\| + \|f; L^2(s, s+1)\| \right)^\theta \|f; L^{2p}(s, s+1)\|^{1-\theta} \] (2.12)

for \( s \geq 0 \), where \( \theta = 1/(p+1) \). By the definition of \( f \) and Schwartz inequality we have

\[ |f'(s)| \leq \|W_s; L^2_\rho(\mathbb{R}^N)\|. \]

This together with (2.8) implies

\[ \|f'; L^2(s, s+1)\| \leq E[W](0)^{\frac{1}{2}}. \] (2.13)

We see by (2.6) and (2.7) that

\[ 2E[W](0) + f(s)f'(s) \geq c_1 f(s)^{p+1}, \]

where \( c_1 \) is as in (2.7). If \( f(s) > 1 \), then by Schwartz inequality we have

\[ f(s)^{2p} \leq \frac{1}{c_1^2} \left( 2 |f'(s)|^2 + 8E[W](0)^2 \right). \]

Together with the case \( f(s) \leq 1 \) we get

\[ \int_s^{s+1} f(\tau)^{2p} d\tau \leq 1 + 8c_1^{-2}E[W](0)^2 + 2c_1^{-2} \int_s^{s+1} |f'(\tau)|^2 d\tau. \]

Thus by (2.13) we have

\[ \|f; L^{2p}(s, s+1)\| \leq \left( 1 + 8c_1^{-2}E[W](0)^2 + 2c_1^{-2}E[W](0) \right)^{\frac{1}{2p}}. \] (2.14)

Since \( \|f; L^2(s, s+1)\| \leq \|f; L^{2p}(s, s+1)\| \), inequality (2.9) follows from (2.12), (2.13) and (2.14) with \( K_1 \) depending on \( N, p, c_G \) and \( M_0 \).

By (2.2) and (2.5) we obtain

\[ c_G^2(p-1)^2 \left( \int_{\mathbb{R}^N} |W|^{p+1} \rho dy \right)^2 \leq \left( f(s)\|W_s; L^2_\rho(\mathbb{R}^N)\| + 2E[W](0) \right)^2 \]

\[ \leq 2f(s)^2\|W_s; L^2_\rho(\mathbb{R}^N)\|^2 + 8E[W](0)^2. \]

Now (2.10) follows from (2.9) and (2.11).

Finally we show (2.11). By the definition of \( E[W] \), (2.3), (2.6) and (2.9) we get

\[ \|W(s); W^{1,2}_\rho(\mathbb{R}^N)\|^2 = 2E[W](s) + 2 \int_{\mathbb{R}^N} G(W) \rho dy \]

\[ = \frac{2}{p-1} \left( (p+1)E[W](s) + \int_{\mathbb{R}^N} W \cdot W_{s\rho} dy \right) \]

\[ \leq \frac{2}{p-1} \left( (p+1)E[W](0) + K_1\|W_s; L^2_\rho(\mathbb{R}^N)\| \right). \]

Therefore inequality (2.11) follows. \( \square \)
3 Estimates of local energy

The aim of this section is to prove the following two propositions.

Proposition 3.1 Let $W$ satisfy (1.6) with (1.8) and $\psi \in BC^1(\mathbb{R}^N)$. Then there exists $L_1 > 0$ such that

$$E_\psi[W](s) \leq L_1, \quad \text{for } s \geq 0. \quad (3.1)$$

The constant $L_1$ depends only on $N$, $p$, $c_G$, $\|\psi\|_{BC^1(\mathbb{R}^N)}$ and $M_0$.

Proposition 3.2 Let $W$ satisfy (1.6) with (1.8) and $\psi \in BC^2(\mathbb{R}^N)$ with supp $\psi \subset B_R$. Then there exists $L_2 > 0$ such that

$$E_\psi[W](s) \geq -L_2, \quad \text{for } s \geq 0. \quad (3.2)$$

The constant $L_2$ depends only on $N$, $p$, $c_G$, $R$, $\|\psi\|_{BC^2(\mathbb{R}^N)}$ and $M_0$.

For this aim we prepare the following three lemmas.

Lemma 3.1 Let $W$ satisfy (1.6) with (1.8) and $\psi \in BC^1(\mathbb{R}^N)$. Then

$$\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} \psi^2 |W|^2 \rho \, dy = \int_{\mathbb{R}^N} \psi^2 W \cdot W_s \rho \, dy$$

$$= -2E_\psi[W](s) + (p - 1) \int_{\mathbb{R}^N} \psi^2 G(W) \rho \, dy - 2 \sum_{i=1}^M \int_{\mathbb{R}^N} \psi w_i \nabla \psi \cdot \nabla w_i \rho \, dy,$$  

$$\frac{d}{ds} E_\psi[W](s) = - \int_{\mathbb{R}^N} \psi^2 |W_s|^2 \rho \, dy - 2 \sum_{i=1}^M \int_{\mathbb{R}^N} \psi w_i \rho (\nabla w_i \cdot \nabla \psi) \, dy, \quad (3.4)$$

for $s \geq 0$.

Proof. As in Proposition 2.1, we have

$$\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} \psi^2 |W|^2 \rho \, dy = \int_{\mathbb{R}^N} \psi^2 W \cdot W_s \rho \, dy$$

$$= \sum_{i=1}^M \int_{\mathbb{R}^N} \psi^2 w_i (\nabla \cdot (\rho \nabla w_i) - \beta \rho w_i + F_i(W) \rho) \, dy,$$  

$$\frac{d}{ds} \int_{\mathbb{R}^N} \left[ \frac{1}{2} \beta \psi^2 |W|^2 + \psi^2 G(W) \rho \right] \, dy$$

$$= \int_{\mathbb{R}^N} \left[ \beta \psi^2 W \cdot W_s \rho - \psi^2 F(W) \cdot W_s \rho \right] \, dy, \quad (3.5)$$

$$= \sum_{i=1}^M \int_{\mathbb{R}^N} \psi^2 w_i \rho (\nabla w_i \cdot \nabla \psi) \, dy, \quad (3.6)$$
\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} \psi^2 |\nabla W|^2 \rho \, dy = \sum_{i=1}^{M} \int_{\mathbb{R}^N} \rho \psi^2 \nabla w_i \cdot \nabla w_{is} \, dy
\]
\[
= -\sum_{i=1}^{M} \int_{\mathbb{R}^N} \psi^2 w_{is} \nabla \cdot (\rho \nabla w_i) \, dy - 2 \sum_{i=1}^{M} \int_{\mathbb{R}^N} \psi w_{is} (\nabla w_i \cdot \nabla \psi) \rho \, dy.
\]
\[
(3.7)
\]

Identity (3.3) follows from (3.5) and integration by parts. Furthermore, by (3.6), (3.7) and (1.7) we obtain (3.4). The proof is complete.

Lemma 3.2 Let \( W \) satisfy (1.6) with (1.8) and \( \psi \in BC^1(\mathbb{R}^N) \). Then there exist \( L_3 > 0 \) and \( L_4 > 0 \) such that
\[
\frac{d}{ds} E_{\psi}[W](s) \leq L_3 (1 + \|W_s(s); L^2_\rho(\mathbb{R}^N)\|),
\]
\[
\int_s^{s+1} E_{\psi}[W](\tau) \, d\tau \leq L_4,
\]
for \( s \geq 0 \). The constants \( L_3 \) and \( L_4 \) depend only on \( N, p, c_G, \|\psi\|_{BC^1(\mathbb{R}^N)} \) and \( M_0 \).

Proof. By (3.4), Cauchy’s inequality and (2.11) we see that
\[
\frac{d}{ds} E_{\psi}[W](s) \leq -\int_{\mathbb{R}^N} \psi^2 |W_s|^2 \rho \, dy + 2 \sum_{i=1}^{M} \int_{\mathbb{R}^N} |\psi||\nabla \psi||w_{is}||\nabla w_i| \rho \, dy
\]
\[
\leq -\frac{1}{2} \int_{\mathbb{R}^N} \psi^2 |W_s|^2 \rho \, dy + 2 \int_{\mathbb{R}^N} |\nabla \psi|^2 |\nabla W|^2 \rho \, dy
\]
\[
\leq 2K_3 \|\nabla \psi\|^2_\infty (1 + \|W_s(s); L^2_\rho(\mathbb{R}^N)\|).
\]
This implies inequality (3.8). By (2.8), (2.11) and Jensen’s inequality we have
\[
\int_s^{s+1} E_{\psi}[W](\tau) \, d\tau \leq \frac{\|\psi\|^2_\infty}{2} \int_s^{s+1} \int_{\mathbb{R}^N} (|\nabla W|^2 + \beta |W|^2) \rho \, dyd\tau
\]
\[
\leq K_3 \|\psi\|^2_\infty \left( 1 + \int_s^{s+1} \|W_s(\tau); L^2_\rho(\mathbb{R}^N)\| \, d\tau \right)
\]
\[
\leq K_3 \|\psi\|^2_\infty (1 + E[W](0)^{1/2}).
\]
Therefore (3.9) follows. The proof is complete.

Lemma 3.3 Let \( W \) satisfy (1.6) with (1.8) and \( \psi \in BC^2(\mathbb{R}^N) \) with \( \text{supp } \psi \subset B_R \). Then there exists \( L_5 > 0 \) such that
\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} \psi^2 |W|^2 \rho \, dy \geq -2E_{\psi}[W](s) + (p - 1) \int_{\mathbb{R}^N} \psi^2 G(W) \rho \, dy - L_5.
\]
\[
(3.10)
\]

The constant \( L_5 \) depends only on \( N, p, c_G, R, M_0, \|\psi\|_{BC^2(\mathbb{R}^N)} \).
Proof. By (3.3) it suffices to prove
\[ \sum_{i=1}^{M} \int_{\mathbb{R}^N} \psi w_i \rho (\nabla w_i \cdot \nabla \psi) \, dy \leq L_5. \]

We get from integration by parts
\[ \sum_{i=1}^{M} \int_{\mathbb{R}^N} \psi w_i \rho \nabla w_i \cdot \nabla \psi \, dy = - \sum_{i=1}^{M} \int_{\mathbb{R}^N} \nabla \psi \cdot (\psi w_i \rho \nabla \psi) \, dy \]
\[ = - \int_{\mathbb{R}^N} |\nabla \psi|^2 |W|^2 \rho \, dy - \sum_{i=1}^{M} \int_{\mathbb{R}^N} \psi w_i \rho \nabla w_i \cdot \nabla \psi \, dy \]
\[ - \int_{\mathbb{R}^N} \psi |W|^2 \Delta \rho \, dy + \frac{1}{2} \int_{\mathbb{R}^N} \psi |W|^2 \rho (y \cdot \nabla \psi) \, dy. \]

Therefore we obtain by (2.9)
\[ \sum_{i=1}^{M} \int_{\mathbb{R}^N} \psi w_i \rho \nabla w_i \cdot \nabla \psi \, dy = - \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 |W|^2 \rho \, dy - \frac{1}{2} \int_{\mathbb{R}^N} \psi \Delta |W|^2 \rho \, dy \]
\[ + \frac{1}{4} \int_{\mathbb{R}^N} \psi |W|^2 \rho (y \cdot \nabla \psi) \, dy \]
\[ \leq \frac{1}{2} \|\psi\|_{\infty} \|\Delta \psi\|_{\infty} K_1^2 + \frac{1}{4} \|\psi\|_{\infty} \|\nabla \psi\|_{\infty} RK_1^2 =: L_5. \]

This implies (3.10). \[ \square \]

We are ready to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Let \( s \geq 1 \). We set \( s_1 \in (s-1, s) \) such that
\[ E_{\psi}[W](s_1) = \int_{s-1}^{s} E_{\psi}[W](\tau) \, d\tau. \] (3.11)

Then we use (3.8), Jensen’s inequality and (2.8) to get
\[ E_{\psi}[W](s) - E_{\psi}[W](s_1) \leq L_3 \left( 1 + \int_{s-1}^{s} \|W_{\epsilon}(s); L_2^2(\mathbb{R}^N)\| \, d\tau \right) \]
\[ \leq L_3 \left( 1 + (E[W](0))^{1/2} \right). \]

Combining this inequality with (3.9) and (3.11), we get
\[ E_{\psi}[W](s) \leq L_3 \left( 1 + (E[W](0))^{1/2} \right) + L_4 \]
for \( s \geq 1 \). For \( s \leq 1 \) we have
\[ E_{\psi}[W](s) \leq L_3 \left( 1 + (E[W](0))^{1/2} \right) + E_{\psi}[W](0) \]
in the same way. Therefore inequality (3.1) follows, and the proof is complete. □

Proof of Proposition 3.2. By (3.10), (2.2) and Jensen’s inequality we have

\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^N} \psi^2 |W|^2 \rho \, dy \geq -2E_\psi[W](s) + c_G(p - 1) \int_{\mathbb{R}^N} \psi^2 |W|^{p+1} \rho \, dy - L_5
\]

\[
\geq -2E_\psi[W](s) + c_2 \left( \int_{\mathbb{R}^N} \psi^2 |W|^2 \rho \, dy \right)^{\frac{p+1}{2}} - L_5,
\]

where \( c_2 = (4\pi)^{-\frac{N(p-1)}{4}} (p - 1)c_G \| \psi \|_\infty^{(p-1)} \). Set

\[
T_1 := \int_0^\infty \frac{d\tau}{1 + 2c_2 \tau^{\frac{p+1}{2}}},
\]

and we show \( E_\psi[W](s) \geq -L_2 \) for \( L_2 = L_3(T_1 + E[W](0)^{1/2}) + L_5/2 + 1/4 \). If not, then there exists some point \( s_2 \in (0, \infty) \) such that \( E_\psi[W](s_2) < -L_2 \).

For \( s \in [0, T_1] \), as in the proof of Proposition 3.1 by (3.8) we have

\[
E_\psi[W](s_2 + s) < -L_2 + L_3 \int_{s_2}^{s_2 + s} \left( 1 + \| W(s) \| ; L_3^2(\mathbb{R}^N) \right) \, d\tau
\]

\[
\leq -L_2 + L_3T_1 + L_3E[W](0)^{1/2}
\]

\[
= -\frac{L_5}{2} - \frac{1}{4}.
\]

Therefore, for \( s \in [s_2, s_2 + T_1] \), we get

\[
\frac{d}{ds} \int_{\mathbb{R}^N} \psi^2 |W|^2 \rho \, dy \geq 1 + 2c_2 \left( \int_{\mathbb{R}^N} |W|^2 \rho \, dy \right)^{\frac{p+1}{2}}.
\]

Since the solution to \( f' = 1 + 2c_2 f^{(p+1)/2} \) and \( f(0) = 0 \) blows up at \( T_1 \), this is a contradiction. Therefore we obtain (3.2), and the proof is complete. □

4 Proof of Theorem 1.1.

In this section we obtain (1.11) by a bootstrap argument in Proposition 4.1 and prove Theorem 1.1. Fix \( \phi \in C^\infty([0, \infty)) \) such that \( \phi(t) = 1 \) for \( t \leq 1 \) and \( \phi(t) = 0 \) for \( t \geq 2 \). For \( R > 0 \), set \( \varphi \)

\[
\varphi(x) = \phi \left( \frac{|x|}{R} \right), \quad x \in \mathbb{R}^N.
\]

Let \( B_R \) be an open ball of radius \( R \) centered at the origin of \( \mathbb{R}^N \). We give the key estimate for the proof of Theorem 1.1.
Proposition 4.1 Assume that $1 < p < p_s$. Let $W$ satisfy (1.6) with (1.8). For $q \geq 2$ and $R > 0$, there exists $C_{q,R} > 0$ such that
\[
\sup_{s \geq 0} \int_s^{s+1} \|W;W^1_{\rho}(B_R)\|^{2q} d\tau \leq C_{q,R}.
\] (4.1)

First we prove Proposition 4.1 for the case $q = 2$.

**Proof of Proposition 4.1 for $q=2$.**

For any $R > 0$ and $s \geq 0$, we see by (2.8) and (2.11) that
\[
\int_s^{s+1} \|W;W^1_{\rho}(B_R)\|^4 d\tau \leq 2K_3^2 \left( 1 + \int_s^{s+1} \|W;L^2_\rho(R^N)\|^2 d\tau \right)
\]
\[
\leq 2K_3^2 (1 + E[W](0)).
\]
Therefore Proposition 4.1 follows for $q = 2$. \(\Box\)

Hereafter we assume $q \geq 2$. As in [20], we introduce several constants $p_1$, $\overline{q}$, $\lambda_q$, $\lambda$, $\theta$ and $\alpha$, satisfying
\[
\begin{align*}
\frac{1}{p_1} &= 1 + \frac{1}{p}, & q < \overline{q} < q + \frac{1}{p+1}, & \lambda_q = p + 1 - \frac{p - 1}{q + 1}, \\
2 < \lambda < \lambda_q, & \theta = \frac{(p+1)(\lambda - 2)}{(p-1)\lambda}, & \alpha = \frac{2}{1 - \theta \overline{q}}.
\end{align*}
\] (4.2)

Proposition 4.2 Assume that $q \geq 2$. Let $p_1$, $\overline{q}$, $\lambda_q$, $\lambda$, $\theta$ and $\alpha$ be as in (4.2). Then there exists $\lambda_0 \in (2, \lambda_q)$ depending only on $p$ and $q$ such that
\[
\alpha > 1,
\] (4.3)
\[
1 < \frac{\theta \overline{q} \alpha'}{p_1} < q,
\] (4.4)
where $\alpha'$ is the Hölder conjugate of $\alpha$, i.e., $1/\alpha + 1/\alpha' = 1$.

**Proof.** The condition $\alpha > 1$ is equivalent to
\[
\overline{q} < \frac{2}{1 - \theta} = \frac{(p-1)\lambda}{(p+1) - \lambda}.
\]
Since the right hand side is monotone increasing for $\lambda \in (2, \lambda_q)$ and
\[
\frac{(p - 1)\lambda_q}{(p + 1) - \lambda_q} = (p + 1) \left( q + \frac{2}{p + 1} \right) > q + \frac{1}{p + 1},
\]
we can choose $\lambda' = \lambda'(p, q)$ such that (4.3) holds true for $\lambda > \lambda'$.
We prove (4.4). Since 
\[ \theta \overline{q} \alpha' = \frac{2 \theta \overline{q}}{2 - (1 - \theta) \overline{q}}. \]
the condition \( p_1 < \theta \overline{q} \alpha' \) is equivalent to 
\[ \overline{q} > \frac{2p_1}{(2 - p_1) \theta + p_1}. \]
This together with 
\[ \frac{2p_1}{(2 - p_1) \theta + p_1} < 2 < \overline{q} \]
implies that \( p_1 < \theta \overline{q} \alpha' \) for any \( \lambda > \lambda' \). On the other hand, the condition \( \theta \overline{q} \alpha' < p_1 q \) is equivalent to 
\[ \overline{q} < \frac{2p_1 q}{(2 - p_1 q) \theta + p_1 q}. \]
Since the right hand side is monotone increasing for \( \lambda \in (2, \lambda_q) \) and
\[ \frac{2p_1 q}{(2 - p_1 q) \theta + p_1 q} = q + \frac{2}{p + 1} > q + \frac{1}{p + 1}, \]
we can choose \( \lambda'' = \lambda''(p, q) \) so that (4.3) holds for \( \lambda > \lambda'' \). Therefore the proof is complete. \( \square \)

Hereafter we fix \( \lambda = \lambda_0 \) so that relations (4.2), (4.3) and (4.4) hold. To prove Proposition 4.1, we prepare the following four lemmas.

**Lemma 4.1** Assume that \( 1 < p < p_s \). Let \( W \) satisfy (1.6) with (1.8). Assume that inequality (4.1) holds with \( B \) replaced by \( B_{2R} \) for some \( q \geq 2 \). Let \( p_1, \overline{q}, \lambda_q, \lambda, \theta \) and \( \alpha \) be as in (4.2). Then there exists \( J_1 > 0 \) such that
\[
\int_{s}^{s+1} \| \varphi W_s(\tau); L_{p_1}^q(B_{2R}) \|^{\theta \overline{q} \alpha'} d\tau \leq J_1 \left( 1 + \int_{s}^{s+1} \| W \|^{p}; L_{p_1}^q(B_{2R}) \|^{\theta \overline{q} \alpha'} d\tau \right) \tag{4.5}
\]
for all \( s \geq 1 \) with some \( \sigma \in [s - 1/4, s] \).

**Proof.** Let \( \sigma \in [s - 1/4, s] \). We apply the maximal regularity theorem for the heat equation [20, Lemma 6.5] to
\[ (\varphi w_i) - \Delta(\varphi w_i) = -2 \nabla \varphi \cdot \nabla w_i - \Delta \varphi w_i - \frac{\varphi}{2} y \cdot \nabla w_i - \beta \varphi w_i + \varphi F_i(W) =: f_i \]
in \( B_{2R} \times (\sigma, s + 1) \) to get
\[
\int_{\sigma}^{s+1} \| \varphi W_s(\tau); L_{p_1}^q(B_{2R}) \|^{\theta \overline{q} \alpha'} d\tau \leq C\left( \int_{\sigma}^{s+1} \| f; L_{p_1}^q(B_{2R}) \|^{\theta \overline{q} \alpha'} d\tau + \| W(\sigma); C^2(B_{2R}) \|^{\theta \overline{q} \alpha'} \right), \tag{4.6}
\]
where \( f = (f_1, \ldots, f_M) \) and \( C_* \) is a positive constant. Let \( c_3 = (4\pi)^{N(p-1)/(2p+1)} \). We see by Hölder’s inequality and (2.9) that

\[
\| \Delta \varphi W; L^p_\rho (B_{2R}) \| \leq \left( \int_{B_{2R}} |W|^2 \rho \, dy \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |\Delta \varphi|^{2(p+1)/(p-1)} \rho \, dy \right)^{\frac{p-1}{2(p+1)}} \leq c_3 K_1 \| \Delta \varphi \|_{\infty}. \tag{4.7}
\]

In the same way, we obtain

\[
\| \beta \varphi W; L^p_\rho (B_{2R}) \| \leq \beta c_3 K_1, \tag{4.8}
\]

\[
\| |\nabla \varphi| |\nabla W|; L^p_\rho (B_{2R}) \| \leq c_3 \| \nabla \varphi \|_{\infty} \| \nabla W; L^2_\rho (B_{2R}) \|, \tag{4.9}
\]

\[
\left( \frac{\varphi}{2} |y| \| \nabla W|; L^p_\rho (B_{2R}) \| \leq c_3 R \| \nabla W; L^2_\rho (B_{2R}) \|. \tag{4.10}
\]

Since \( \theta q \alpha' < 2q \), by (4.9) we see that

\[
\left( \frac{\varphi}{2} |y| \| \nabla W|; L^p_\rho (B_{2R}) \| \| \nabla W; L^2_\rho (B_{2R}) \| \right)^{1/q} \leq 4 c_3 \| \nabla \varphi \|_{\infty} \left( \int_{B_{2R}} \| \nabla W; L^2_\rho (B_{2R}) \|^{2q} \, d\tau \right)^{1/2q} \leq 8 c_3 \| \nabla \varphi \|_{\infty} C^{1/2q}_{q,2R}. \tag{4.11}
\]

In the same way, by (4.10) we obtain

\[
\left( \int_{\sigma}^{s+1} \| |\nabla \varphi| |\nabla W|; L^p_\rho (B_{2R}) \| \| \nabla W; L^2_\rho (B_{2R}) \| \, d\tau \right)^{1/\theta q \alpha'} \leq 4 c_3 R C^{1/2q}_{q,2R}. \tag{4.12}
\]

By (2.2) we see that

\[
\left( \int_{\sigma}^{s+1} \| \varphi F(W); L^p_\rho (B_{2R}) \| \| \nabla \varphi \|_{\infty} \, d\tau \right)^{1/\theta q \alpha'} \leq C_F \left( \int_{\sigma}^{s+1} \| W; L^p_\rho (B_{2R}) \| \| \nabla \varphi \|_{\infty} \, d\tau \right)^{1/\theta q \alpha'}. \tag{4.13}
\]

Thus by (4.7), (4.8), (4.11), (4.12) and (4.13) we obtain

\[
\left( \int_{\sigma}^{s+1} \| f; L^p_\rho (B_{2R}) \| \, d\tau \right)^{1/\theta q \alpha'} \leq 8 c_3 \| \nabla \varphi \|_{\infty} C^{1/2q}_{q,2R} + 4 c_3 R C^{1/2q}_{q,2R} + 2 c_3 K_1 \| \Delta \varphi \|_{\infty} + 2 \beta c_3 K_1 + C_F \left( \int_{\sigma}^{s+1} \| W; L^p_\rho (B_{2R}) \| \, d\tau \right)^{1/\theta q \alpha'}. \tag{4.14}
\]
As in \cite[Lemma 6.9 and Remark 6.10]{20}, by (2.8) and (2.11) we can choose $\sigma$ such that $\|W(\sigma); C^2(\overline{B_{2R}})\| \leq Z$ with a constant $Z$ independent of $s$ and $a$. This together with (4.6) and (4.14) implies (4.5), and the proof is complete. \hfill $\square$

**Lemma 4.2** Let $W$ satisfy (1.6) with (1.7). For any $R > 0$, then there exists $J_2 > 0$ such that

$$
\|W(s); W^{1,2}_\rho(B_R)\|^2 \leq J_2 (1 + \|\varphi^2|W(s)||W_s(s); L^1_\rho(B_{2R})\|) \quad (4.15)
$$

for $s \geq 0$.

**Proof.** It follows from (1.10) that

$$
-2E_\varphi[W](s) + (p - 1) \int_{\mathbb{R}^N} \varphi^2 G(W) \rho \, dy = \frac{p - 1}{2} \int_{\mathbb{R}^N} \varphi^2(|\nabla W|^2 + \beta|W|^2) \rho \, dy - (p + 1)E_\varphi[W](s).
$$

This together with (3.10) implies that

$$
\|W; W^{1,2}_\rho(B_R)\|^2 \leq \int_{\mathbb{R}^N} \varphi^2(|\nabla W|^2 + \beta|W|^2) \rho \, dy
$$

$$
\leq \frac{2}{p - 1} ((p + 1)L_1 + L_5 + \|\varphi^2|W||W_s; L^1_\rho(B_{2R})\|)
$$

$$
\leq J_3 (1 + \|\varphi^2|W||W_s; L^1_\rho(B_{2R})\|).
$$

Thus inequality (4.15) follows. \hfill $\square$

**Lemma 4.3** Let $W$ satisfy (1.6) with (1.7). Assume that for some $q \geq 2$, (4.1) holds with $B_R$ replaced by $B_{2R}$. Let $\lambda_q$ be as in (4.2). Then there exist $C'_{q,R} > 0$ and $C''_{q,R} > 0$ such that

$$
\sup_{s \geq 0} \int_{s}^{s+1} \|W; L^{p+1}_\rho(B_R)\|^{(p+1)q} \, d\tau \leq C'_{q,R}, \quad (4.16)
$$

$$
\sup_{s \geq 0} \|W; L^\lambda_\rho(B_R)\| \leq C''_{q,R} \quad \text{for all } \lambda < \lambda_q. \quad (4.17)
$$

**Proof.** By (2.2) and (3.2) we see that for any $R > 0$,

$$
\|W; L^{p+1}_\rho(B_R)\|^{(p+1)} \leq \frac{1}{c_G} \int_{\mathbb{R}^N} \varphi^2 G(W) \rho \, dy
$$

$$
\leq \frac{1}{c_G} (L_2 + \|W; W^{1,2}_\rho(B_{2R})\|^2). \quad (4.18)
$$
Thus we get
\[
\int_s^{s+1} \|W; L^{p+1}_\rho(B_R)\|^{(p+1)q} d\tau \leq \frac{2^{q-1}}{c^q_G} \int_s^{s+1} (L^q_2 + \|W; W_{1,2}^{1,2}(B_{2R})\|^2) d\tau \\
\leq \frac{2^{q-1}}{c^q_G} (L^q_2 + C_{q,2R}).
\]

Therefore inequality (4.16) follows. Furthermore, by (4.16) and (2.8) we obtain
\[
\sup_{s \geq 0} \int_s^{s+1} (\|W; L^{p+1}_\rho(B_R)\|^{(p+1)q} + \|W_s; L^2(B_R)\|^2) d\tau \\
\leq \exp \left( \frac{qR^2}{4} \right) \left( C'_{q,R} + E[W](0) \right).
\]

This together with the interpolation theorem [20, Lemma A.1] (See also [1]) implies (4.17). The proof is complete. \(\square\)

**Lemma 4.4** Assume that \(1 < p < p_S\). Let \(W\) satisfy (1.6) with (1.8). Assume that for some \(q \geq 2\), (4.1) holds with \(B_R\) replaced by \(B_{4R}\). Let \(p_1, q, \lambda, \theta\) and \(\alpha\) as in (4.2). Then there exists \(J_3 > 0\) such that
\[
\int_s^{s+1} \|W; W_{1,2}^{1,2}(B_R)\|^{2q} d\tau \leq J_3 \left( 1 + \int_s^{s+1} \|W_s; L^{1,2}_\rho(B_{4R})\|^{2q} d\tau \right)
\]
for all \(s \geq 1\) with some \(\sigma \in [s - 1/4, s]\).

**Proof.** By (1.15), (4.17) and Hölder’s inequality we have
\[
\|W; W_{1,2}^{1,2}(B_R)\|^2 \leq J_2 (1 + \|\varphi W; L^{\lambda}_\rho(B_{2R})\| \|\varphi W_s; L^{\lambda'}_\rho(B_{2R})\|) \\
\leq J_2 (1 + C'_{q,2R} \|\varphi W_s; L^{p_1}_\rho(B_{2R})\|^{\theta} \|\varphi W_s; L^2_\rho(B_{2R})\|^{1-\theta}).
\]

Then we obtain
\[
\int_s^{s+1} \|W; W_{1,2}^{1,2}(B_R)\|^{2q} d\tau \\
\leq 2^{q-1} J_2^q \left( 2 + C'_{q,2R} \int_s^{s+1} \|\varphi W_s; L^{p_1}_\rho(B_{2R})\|^{\theta q} \|\varphi W_s; L^2_\rho(B_{2R})\|^{(1-\theta)q} d\tau \right).
\]

By (2.8), (4.2) and Hölder’s inequality we see that
\[
\int_s^{s+1} \|\varphi W_s; L^{p_1}_\rho(B_{2R})\|^{\theta q} \|\varphi W_s; L^2_\rho(B_{2R})\|^{(1-\theta)q} d\tau \\
\leq (E[W](0))^{1/\alpha} \left( \int_s^{s+1} \|\varphi W_s; L^{p_1}_\rho(B_{2R})\|^{\theta q \alpha'} d\tau \right)^{1/\alpha'}.
\]
Therefore we see that (4.1) holds for some \( q \) and \( R \) for \( J \) for 4.1.

**Proof of Proposition 4.1.**

We have already proved Proposition 4.1 for \( q = 2 \). Therefore inequality (4.19) follows from (4.20), (4.21) and (4.23).

By (4.18) we have

\[
\left\| W; L^{p+1}_\rho (B_R) \right\|^{p+1} \leq J_4 (1 + \left\| W; W^{1,2}_\rho (B_{2R}) \right\|^2)
\]

for \( J_4 := c^{-1}_G \max \{ L_2, 1 \} \). Furthermore, by (4.15) and (4.22) we have

\[
\left( \int_\sigma^{s+1} \| \varphi W_s; L^{p_1}_\rho (B_{2R}) \|^{\sigma q^{\alpha'}} \, d\tau \right)^{1/\alpha'} \\
\leq J_1^{1/\alpha'} \left( 1 + \int_\sigma^{s+1} \| W; L^{p_1}_\rho (B_{2R}) \|^{\sigma q^{\alpha'}} \, d\tau \right) \\
= J_1^{1/\alpha'} \left( 1 + \int_\sigma^{s+1} \| W; L^{p+1}_\rho (B_{2R}) \|^{\sigma p q^{\alpha'}} \, d\tau \right) \\
\leq J_1^{1/\alpha'} \left( 1 + 2^{\sigma q^{\alpha'}/p_1 - 1} J_4^{\sigma q^{\alpha'}/p_1} \left( 2 + \int_\sigma^{s+1} \| W; W^{1,2}_\rho (B_{4R}) \|^{2\sigma q^{\alpha'}/p_1} \, d\tau \right) \right)
\]

(4.23)

Therefore inequality (4.19) follows from (4.20), (4.21) and (4.23).

**Proof of Proposition 4.1.**

We have already proved Proposition 4.1 for \( q = 2 \). We prove Proposition 4.1 for \( q > 2 \) by a bootstrap argument starting with \( q = 2 \). If (4.11) holds for some \( q \geq 2 \) with some \( 4R > 0 \), then by (4.4) and (4.19) we have

\[
\int_\sigma^{s+1} \| W; W^{1,2}_\rho (B_R) \|^{2q} \, d\tau \leq J_3 \left( 1 + 2 \left( \int_\sigma^{s+1} \| W; W^{1,2}_\rho (B_{4R}) \|^{2q} \, d\tau \right)^{\sigma q^{\alpha'}/p_1} \right) \\
\leq J_3 \left( 1 + 2 (2C_{q,4R})^{\sigma q^{\alpha'}/p_1} \right)
\]

for \( \sigma < q + 1/(p + 1) \) and \( s \geq 1 \). On the other hand, we have

\[
\sup_{0 \leq s \leq 1} \int_\sigma^{s+1} \| W; W^{1,2}_\rho (B_R) \|^{2q} \, d\tau \leq C \sup_{0 \leq t \leq 1 - e^{-2}} \| U(t); BC^1 (\mathbb{R}^N) \|^{2q}.
\]

Therefore we see that (4.11) holds for \( \sigma < q + 1/(p + 1) \) with \( R \). Let \( q_1 > 2, R_1 > 0 \) and \( m = [(p + 1)(q_1 - 2)] + 1 \). We repeat the estimates \( m \) times starting with \( q = 2 \) and \( R = 4^m R_1 \) to obtain

\[
\sup_{s \geq 0} \int_\sigma^{s+1} \| W; W^{1,2}_\rho (B_{R_1}) \|^{2q_1} \, d\tau \leq C_{q_1, R_1}.
\]

Therefore the proof of (4.1) is complete.

**Proof of Theorem 1.1.**

We use the interior regularity theorem for linear parabolic equation. Let \( w_\pm := \max \{ \pm w_1, 0 \} \). Since \( W = (w_1, \ldots, w_M) \) satisfies

\[
\partial_t w_i = \Delta w_i - \frac{1}{2} y \cdot \nabla w_i - \beta w_i + \sum_{j=1}^M \beta_{ij} |w_1|^{r-1} |w_j|^{r+1} w_i \quad \text{in} \quad \mathbb{R}^N \times (0, \infty)
\]

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for $i = 1, \ldots, M$, $\{w_i^\pm\}_{i=1}^M$ satisfies

$$\partial_s w_i^\pm \leq \Delta w_i^\pm - \frac{1}{2} y \cdot \nabla w_i^\pm - \beta w_i^\pm + \sum_{j=1}^M \beta_{ij} |w_i|^r |w_i|^r + 1 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty)$$

in a weak sense. See e.g., [8, Chapter 1]. Therefore

$$w := \sum_{i=1}^M |w_i|$$

satisfies

$$\partial_s w \leq \Delta w - \frac{1}{2} y \cdot \nabla w - \beta w + \sum_{i,j=1}^M \beta_{ij} w^p \quad (4.24)$$

in $\mathbb{R}^N \times (0, \infty)$ in a weak sense. We choose $\alpha$, $\beta$ and $q$ so that $1/\beta + N/2\alpha < 1$, $\alpha \geq 1$ and $\alpha(p - 1) < \lambda_1(q)$. This is possible for sufficiently large $q$ provided that $1 < p < p_s$. By (2.9) and (4.17) we can apply an interior regularity theorem for a linear parabolic equation [20, Lemma A.2] (see also [23]) for (4.24) in $B_R \times (s, s+1)$ for $s \geq 0$ to find $C > 0$ such that

$$\|w; L^\infty(B_{R/2})\| \leq C \quad \text{for} \quad s > \frac{1}{2}.$$ 

This together with (1.8) implies

$$\|W; L^\infty(B_{R/2})\| \leq C \quad \text{for} \quad s \geq 0.$$ 

Since the constant $C$ is independent of $a$, we have

$$|U(x, t)| \leq C(1 - t)^{-\beta} \quad \text{for} \quad 0 < t < 1,$$

and the proof of Theorem 1.1 is complete. $\square$

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