Virtual Structure Constants as Intersection Numbers of Moduli Space of Polynomial Maps with Two Marked Points

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Abstract

In this paper, we derive the virtual structure constants used in the mirror computation of the degree $k$ hypersurface in $\mathbb{C}P^{N-1}$, by using a localization computation applied to moduli space of polynomial maps from $\mathbb{C}P^1$ to $\mathbb{C}P^{N-1}$ with two marked points. We also apply this technique to the non-nef local geometry $\mathcal{O}(1) \oplus \mathcal{O}(-3) \to \mathbb{C}P^1$ and realize the mirror computation without using Birkhoff factorization.

1 Introduction

Analysis of mirror symmetry from the point of view of the Gauged Linear Sigma Model is very important both in the mathematical and physical study of mirror symmetry [17],[15]. As was suggested by Morrison and Plesser in [15], the Gauged Linear Sigma Model is directly connected to the B-model in mirror symmetry. Indeed, they constructed the moduli space of instantons of the Gauged Linear Sigma Models corresponding to Calabi-Yau 3-folds in $\mathbb{P}(1,1,1,1,1)(=\mathbb{C}P^4)$ and (the blow-up of) $\mathbb{P}(1,1,2,2,2)$, computed intersection numbers on these moduli spaces and showed that their generating functions coincide with the B-model Yukawa couplings used in the mirror computation. Further analysis of mirror symmetry of Calabi-Yau 3-folds in this direction was also pursued by Batyrev and Materov [1].

In this paper, we try to generalize this kind of analysis to the mirror computations of non-nef geometries. Mirror symmetry of non-nef geometries has been studied by various authors,[14],[3],[8],[10],[5],[6]. Since the mirror computation in this case is rather complicated, it is hard to define objects that are directly connected to the B-model, or to the Gauged Linear Sigma Model. In this paper, we compute the "virtual structure constants" $\hat{L}_{n}^{N,k,d}$ used in mirror computation of a non-nef degree $k$ hypersurface in $\mathbb{C}P^{N-1}$ ($k > N$) [10]. Originally, the virtual structure constants are defined by recursive formulas that represent $\hat{L}_{n}^{N,k,d}$ in terms of weighted homogeneous polynomials in $L_{n+1}^{N+1,k,d'}$, ($d' \leq d$)[9]. Later, we showed that $\hat{L}_{n}^{N,k,d}$ can be directly computed from the virtual Gauss-Manin system associated with Givental’s ODE:

$$\left(\left(\frac{d}{dx}\right)^{N-1} - k \cdot e^{x} \cdot (k \frac{d}{dx} + k - 1)(k \frac{d}{dx} + k - 2) \cdots (k \frac{d}{dx} + 1)\right)w(x) = 0,$$

(1.1)

[11]. In [8], Iritani showed that the virtual structure constants can be obtained after Birkhoff factorization of connection matrices of the Gauss-Manin system associated with Givental’s ODE with ”$h$” parameter:

$$\left(h^{N-1} \frac{d}{dx}\right)^{N-1} - h^{k-1}k \cdot e^{x} \cdot (k \frac{d}{dx} + (k - 1))(k \frac{d}{dx} + (k - 2)) \cdots (k \frac{d}{dx} + 1)\right) w(x,h) = 0,$$

(1.2)

generalizing the method invented by Guest et al [7],[16].
In this paper, we start from the recursive formula in [11] that determines \( \tilde{L}^{N,k,d}_n \). Next, we propose a "conjectural" residue integral representation of \( \tilde{L}^{N,k,d}_n \), which is speculated by solving the recursive formula for low degrees. Then we show that this representation can be interpreted as the result obtained by applying the localization computation to the moduli space of polynomial maps of degree 1 with two marked points. By the word "polynomial map", we mean a birational map \( p \) of degree \( d \) from \( CP^1 \) to \( CP^{N-1} \) given by,

\[
p(s : t) = \left( \sum_{j=0}^{d} a_j^1 s^j t^{d-j} : \sum_{j=0}^{d} a_j^2 s^j t^{d-j} : \cdots : \sum_{j=0}^{d} a_j^N s^j t^{d-j} \right).
\] (1.3)

The moduli space of polynomial maps of degree \( d \) can be identified with \( CP^{N(d+1)-1} \), which is the moduli space of instantons of the Gauged Linear Sigma Model. The two marked points are fixed to \( (0 : 1), (1 : 0) \in CP^1 \). Then we consider polynomial maps of degree \( d \) such that the image of two marked points are well-defined in \( CP^{N-1} \) and divide the corresponding moduli space by \( C^* \), the automorphism group of \( CP^1 \) fixing the two marked points. After resolving the singularities of the resulting space, we obtain the moduli space of polynomial maps with two marked points mentioned above. With this set up, we can derive the residue integral representation by applying the localization computation to this space.

Our geometrical derivation using localization has the following by-product. We can also apply this technique to the local geometries \( \oplus_{j=1}^m \mathcal{O}(k_j) \rightarrow CP^{N-1} \). In this paper, we consider two examples of local geometry, \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow CP^1 \) and \( \mathcal{O}(1) \oplus \mathcal{O}(-3) \rightarrow CP^1 \). By applying the localization computation to the moduli space of polynomial maps with two marked points, we can compute the "virtual structure constants" for these models. In the latter case, it was hard for us to define virtual structure constants because we don’t have an appropriate Picard-Fuchs equation like (1.1). These virtual structure constants give us the mirror map and B-model-like two point functions, which is expected from analogy with the behavior of the virtual structure constants of the quintic 3-fold in \( CP^4 \) [4]. As a result, we can perform the mirror computation of \( \mathcal{O}(1) \oplus \mathcal{O}(-3) \rightarrow CP^1 \) without using Birkhoff factorization that had been inevitable in our previous analysis [5].

This paper is organized as follows. In Section 2, we briefly introduce the virtual structure constants \( \tilde{L}^{N,k,d}_n \) and its residue integral representation. In Section 3, we define the moduli space of polynomial maps of degree \( d \) with two marked points, introduce a torus action on this space and determine the fixed point sets under the torus action. Then we derive the residue integral representation introduced in Section 2 by using a localization computation. In Section 4, we apply the method of Section 3 to \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow CP^1 \) and \( \mathcal{O}(1) \oplus \mathcal{O}(-3) \rightarrow CP^1 \). Section 5 gives concluding remarks.

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2 Virtual Structure Constants

2.1 Virtual Structure Constants and Givental’s ODE

In this subsection, we introduce the virtual structure constants \( \tilde{L}^{N,k,d}_m \) which are non-zero only if \( 0 \leq m \leq N - 1 + (k - N)d \). The original definition of \( \tilde{L}^{N,k,d}_m \) in [11] is given by the initial condition:

\[
\sum_{m=0}^{k-1} \tilde{L}^{N,k,1}_m u^m = k \cdot \prod_{j=1}^{k-1} (jw + (k - j)), \quad (N - k \geq 2),
\] (2.4)

and the recursive formulas that describe \( \tilde{L}^{N,k,d}_m \) as a weighted homogeneous polynomial in \( \tilde{L}^{N+1,k,d'}_m \) (\( d' \leq d \)) of degree \( d \). See [9] for the explicit form of the recursive formulas. In [11], we showed that the virtual structure constants are directly connected with Givental’s ODE:

\[
\left( \frac{d}{dx} \right)^{N-1} - k \cdot e^x \cdot (k \frac{d}{dx} + k - 1)(k \frac{d}{dx} + k - 2) \cdots (k \frac{d}{dx} + 1) \right) w(x) = 0,
\] (2.5)

for arbitrary \( N \) and \( k \) via the virtual Gauss-Manin system defined as follows:

**Definition 1** We call the following rank 1 ODE for the vector valued function \( \tilde{\psi}_m(x) \)

\[
\frac{d\tilde{\psi}_{N-2-m}(x)}{dx} = \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}^{N,k,d}_m \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x).
\] (2.6)
the virtual Gauss-Manin system associated with the quantum Kähler sub-ring of $M^k_N$, where $m$ runs through $0 \leq m \leq N - 2$ if $N - k \geq 1$, $0 \leq m \leq N - 1$ if $N - k = 0$, and $m \in \mathbb{Z}$ if $N - k < 0$.

Here, we restate the main result in [11].

**Theorem 1** We can derive the following relation from the virtual Gauss-Manin system (2.6).

$$\tilde{\psi}_{N-1}(x) = \left(\frac{d}{dx}\right)^{N-1} - k \cdot e^{x} \cdot (k \frac{d}{dx} + k - 1) \cdots (k \frac{d}{dx} + 2) \cdot (k \frac{d}{dx} + 1) \left(\frac{d}{dx}\right)^{\beta} \tilde{\psi}_{-}\beta(x) \quad (2.7)$$

where $\beta = 0$ if $N - k \geq 1$, $\beta = 1$ if $N - k = 0$, and $\beta = \infty$ if $N - k < 0$.

We can also compute $\tilde{L}_{m, n, k, d}$ only by using the above theorem, and this process is an analogue of the B-model computation in the Calabi-Yau case. Explicitly, the following recursive formula of the virtual structure constants holds.

**Corollary 1** The virtual structure constants $\tilde{L}_{m, n, k, d}$ can be fully reconstructed from the relation (2.7). As a result, we can compute all the virtual structure constants by using the initial condition and the recursive formula:

$$\sum_{n=0}^{k-1} L_{n, k}^{N, k, 1} w^n = k \cdot \prod_{j=1}^{k-1} (jw + (k - j)),$$

$$\sum_{m=0}^{N-1+(k-N)d} \tilde{F}_{m, n, k, d} = m =$$

$$\sum_{l=2}^{d} (-1)^l \sum_{\sigma \in \text{OP}_d} \prod_{j=0}^{N-1+(k-N)d} \prod_{j_1=0}^{j_2} \cdots \prod_{j_{n-1}=0}^{j_n} \prod_{j_{n+1}=0}^{j_{n+1}} \left(1 + \frac{1}{d} \right)^{j_n-j_{n-1}+1} \tilde{L}_{m, n, k, d}.$$

(2.8)

We can regard (2.8) as an alternate definition of the virtual structure constants.

### 2.2 Residue Integral Representation of Virtual Structure Constants

By solving the recursive formula (2.8) for low degrees explicitly, we reached a residue integral representation of the virtual structure constants. In the following, we give some definitions necessary to describe the formula we have obtained. First, we define rational functions $F_d(z, w)$ ($d \in \mathbb{N}$) in $z, w$ by,

**Definition 2**

$$F_d(z, w) := k \prod_{j=1}^{d-1} \left(\frac{d}{jz + (d-j)w}\right)^N \prod_{j=1}^{k-1} \left(\frac{jz + (kd-j)w}{d}\right).$$

(2.9)

Next, we introduce the ordered partition of a positive integer $d$, which plays a central role in this paper.

**Definition 3** Let $\text{OP}_d$ be the set of ordered partitions of the positive integer $d$:

$$\text{OP}_d = \{\sigma_d = (d_1, d_2, \cdots, d_{l(\sigma_d)}) \mid \sum_{j=1}^{l(\sigma_d)} d_j = d, \ d_j \in \mathbb{N}\}.$$

(2.10)

From now on, we denote a ordered partition $\sigma_d$ by $(d_1, d_2, \cdots, d_{l(\sigma_d)})$. In (2.10), we denote the length of the ordered partition $\sigma_d$ by $l(\sigma_d)$.

With this set up, the residue integral representation mentioned above is given as follows:

**Conjecture 1**

$$\sum_{j=0}^{N-1-(N-k)d} \frac{L_{n, k, d}^{N, k, d}}{d} z^j w^{N-1-(N-k)d-j}$$

$$= \sum_{\sigma \in \text{OP}_d} \frac{1}{2\pi i \prod_{j=1}^{l(\sigma_d)} dj} \oint_{C_0} \cdots \oint_{C_0} dz_1 \cdots dz_{l(\sigma_d)-1} \prod_{j=1}^{l(\sigma_d)-1} \left(\frac{z_j-z_{j-1}}{d_j} + \frac{z_j-z_{j+1}}{d_{j+1}}\right) \prod_{j=1}^{l(\sigma_d)} F_d(z_{j-1}, z_j),$$

(2.11)

where $z_0 = z$, $z_{l(\sigma_d)} = w$. 

3
In (2.11), \( \frac{1}{2\pi\sqrt{-1}} \oint_{C_0} dz_j \) represents the operation of taking the residue at \( z_j = 0 \). The residue integral in (2.11) depends heavily on the order of integration, and we have to take residues of \( z_j \)'s in descending (or ascending) order of subscript \( j \). We have a proof of the above formula up to \( d = 3 \) and checked numerically its validity up to \( d = 6 \). Let us look at the formula (2.11) more closely in the \( d = 1 \) and \( d = 2 \) cases. In the \( d = 1 \) case, the ordered partition of 1 is just (1), and (2.11) reduces to,

\[
\sum_{j=0}^{k-1} \tilde{L}_n^{N,k,1} z_j w^{k-1-j} = F_1(z, w) = k \prod_{j=1}^{k-1} (jz + (k - j)w),
\]

(2.12)

which is nothing but the initial condition of the virtual structure constants in (2.8). In the \( d = 2 \) case, the ordered partitions of 2 are (2) and (1, 1), and (2.11) takes the following form:

\[
\sum_{j=0}^{N-1-2(N-k)} \frac{\tilde{L}_n^{N,k,2}}{2} z^j w^{N-1-2(N-k)-j} = \frac{1}{2} F_2(z, w) + \frac{1}{2\pi\sqrt{-1}} \oint_{C_0} \frac{u^{1-N} du}{u - z + u - w} F_1(z, u) F_1(u, w).
\]

(2.13)

### 3 Geometrical Derivation

In this section, we derive (2.11) as an integral of Chern classes on the moduli space of polynomial maps with two marked points, by using the localization computation.

### 3.1 Moduli Space of degree \( d \) Polynomial Map with two marked Points and its Fixed Point Sets

Let \( a_j \), \( (j = 0, 1, \cdots, d) \) be vectors in \( \mathbb{C}^N \) and let \( \pi_N : \mathbb{C}^N \to CP^{N-1} \) be the projection map.

In this paper, we define a degree \( d \) polynomial map \( p \) from \( \mathbb{C}^2 \) to \( \mathbb{C}^N \) as the map that consists of \( \mathbb{C}^N \)-valued degree \( d \) homogeneous polynomials in two coordinates \( s, t \) of \( \mathbb{C}^2 \):

\[
p : \mathbb{C}^2 \to \mathbb{C}^N
\]

\[
p(s, t) = a_0 s^d + a_1 s^{d-1} t + a_2 s^{d-2} t^2 + \cdots + a_d t^d.
\]

(3.14)

The map space is described by \( \mathbb{C}^{N(d+1)} = \{ (a_0, a_1, \cdots, a_d) \} \). We denote by \( M_{p_0, 2}(N, d) \) the space obtained from dividing \( \{ (a_0, \cdots, a_d) \in \mathbb{C}^{N(d+1)} \mid a_0 \neq 0, a_d \neq 0 \} \) by two \( \mathbb{C}^* \) actions induced from the following two \( \mathbb{C}^* \) actions on \( \mathbb{C}^2 \) via the map \( p \) in (3.14).

\[
(s, t) \to (\mu s, \mu t), \quad (s, t) \to (s, \nu t).
\]

(3.15)

By the above two torus actions, \( M_{p_0, 2}(N, d) \) can be regarded as a parameter space of degree \( d \) birational maps from \( CP^1 \) to \( CP^{N-1} \) with two marked points in \( CP^1 \): \( 0 (= (1 : 0)) \) and \( \infty (= (0 : 1)) \). In particular, the second torus action corresponds to the automorphism group of \( CP^1 \) that keeps 0 and \( \infty \) invariant. The condition \( a_0, a_d \neq 0 \) ensures that the images of 0 and \( \infty \) are well-defined in \( CP^{N-1} \). In the \( d = 1 \) case, \( M_{p_0, 2}(N, 1) \) is identified with \( CP^{N-1} \times CP^{N-1} \) by the two torus actions (3.15). But in the \( d \geq 2 \) cases, \( M_{p_0, 2}(N, d) \) has singularities, and we have to resolve them. We denote by \( \tilde{M}_{p_0, 2}(N, d) \) the space obtained after resolution. This \( \tilde{M}_{p_0, 2}(N, d) \) is the moduli space of degree \( d \) polynomial maps with two marked points. Let us consider the resolution of \( M_{p_0, 2}(N, 2) \) as an example. \( M_{p_0, 2}(N, 2) \) is obtained by dividing the space,

\[
\{(a_0, a_1, a_2) \mid a_j \in \mathbb{C}^N, a_0 \neq 0, a_2 \neq 0 \}
\]

(3.16)

by the two torus actions:

\[
(a_0, a_1, a_2) \to (\mu^2 a_0, \mu^2 a_1, \mu^2 a_2),
\]

\[
(a_0, a_1, a_2) \to (a_0, \nu a_1, \nu^2 a_2).
\]

(3.17)

Since \( a_0 \neq 0 \), we can use the first torus action to reduce \( (a_0, a_1, a_2) \) to \( (\pi_N(a_0), a_1, a_2) \). By the second torus action, the locus \( (\pi_N(a_0), 0, a_2) \) becomes singular and we have to blow it up. Then the exceptional locus can be identified with \( (\pi_N(a_0), \pi_N(a_1), a_2) \) with \( a_1 \neq 0 \). At this locus, the second torus action acts in the following way:

\[
(\pi_N(a_0), \pi_N(a_1), a_2) \to (\pi_N(a_0), \pi_N(a_1), \nu^2 a_2).
\]

(3.18)
Since \( a_2 \neq 0 \), the exceptional locus can be identified with \( \{(\pi_N(a_0), \pi_N(a_1), \pi_N(a_2))\} = (CP^{N-1})^3 \). This result suggests that we have to consider a chain of two degree 1 polynomial maps:
\[
(a_0 s_1 + a_1 t_1) \cup (a_1 s_2 + a_2 t_2),
\]
(3.19)
in addition to the usual degree 2 polynomial maps. In (3.19), the two torus actions (3.15) are extended to each \((s_j, t_j)\), \(j = 1, 2\). In the general \( d \) case, the resolution of \( Mp_{0,2}(N, d) \) forces us to consider a chain of polynomial maps labeled by ordered partition \( \sigma_d = (d_1, d_2, \cdots, d_l(\sigma_d)) \):
\[
\cup_{j=1}^{l(\sigma_d)} \left( \sum_{m_j=0}^{d_j} a_j \sum_{i=1}^{m_j} (s_j)^{m_j} (t_j)^{d_j-m_j} \right) \quad \text{where} \quad (a_j) \neq 0, \quad j = 0, 1, \cdots, l(\sigma_d),
\]
(3.20)
where the two torus actions (3.15) are extended to each \((s_j, t_j)\), \(j = 1, 2, \cdots, l(\sigma_d)\).

From now on, we introduce the following \( C^n \) action on \( C^{N(d+1)} \) and determine the fixed point set of \( Mp_{0,2}(N, d) \):
\[
(a_0, a_1, \cdots, a_d) \rightarrow (e^{\lambda u} a_0, e^{\lambda t_1} a_1, \cdots, e^{\lambda t_d} a_d)
\]
(3.21)
First, we look at the simplest map in \( \tilde{M}_{P_{0,2}}(N, d) \) given by \(^1\)
\[
a_0 s^d + a dt^d.
\]
(3.22)
We can easily see that this map is invariant under the torus action (3.21) because of the second torus action of (3.15), hence we call the map (3.22) the type I fixed point. Due to the two torus actions in (3.15), the type I fixed point set is identified with \( (\pi_N(a_0), \pi_N(a_d)) \in (CP^{N-1})^2 \). We also have to note that \( Z_d = \{ \exp(\frac{2\sqrt{-1}j}{d}) | j = 0, 1, \cdots, d - 1 \} \) naturally acts on \( CP^1 \) in the following way:
\[
(s : t) \rightarrow (s : \exp(\frac{2\sqrt{-1}j}{d})) t
\]
(3.23)
but leaves \( (\pi_N(a_0), \pi_N(a_d)) \) invariant. This means that when integrating over this fixed point set, we have to divide the result by \( |Z_d| = d \).

Let us consider the normal bundle of this fixed point set in \( \tilde{M}_{P_{0,2}}(N, d) \). Obviously, this bundle is spanned by the degrees of freedom of deformation of the map (3.22) by using \( a_j \), \( j = 1, 2, \cdots, d - 1 \). Therefore, normal vector space is given by the following \( N(d - 1) \) dimensional vector space:
\[
C^N s^{d-1}t + C^N s^{d-2}t^2 + C^N s^{d-3}t^3 + \cdots + C^N s^d t^{d-1}.
\]
(3.24)
We can easily see that the fixed points coming from usual degree \( d \) polynomial maps are exhausted by type I fixed points. The remaining fixed points can be found from the exceptional locus given by the chain of polynomial maps (3.20). In (3.20), we have extended the two torus actions (3.15) to each \((s_j, t_j)\). By using this fact, we can construct a chain of type I-like maps which remains fixed under the torus action (3.21):
\[
\cup_{j=1}^{l(\sigma_d)} \left( \sum_{m_j=0}^{d_j} a_j \sum_{i=1}^{m_j} (s_j)^{m_j} (t_j)^{d_j-m_j} \right).
\]
(3.25)
Note that image of the chain of maps (3.25) in \( CP^{N-1} \) is a nodal rational curve with \( l(\sigma_d) - 1 \) nodal singularities given by \( \pi_N(a_j) \), \( j = 1, 2, \cdots, l(\sigma_d) - 1 \).

We call the chain of maps given in (3.25) a type II fixed point. The type II fixed point set labeled by \( \sigma_d \) is identified with
\[
(\pi_N(a_0), \pi_N(a_{d_1}), \pi_N(a_{d_1+d_2}), \cdots, \pi_N(a_{d-d_l(\sigma_d)}), \pi_N(a_d)) \in (CP^{N-1})^{l(\sigma_d)+1}.
\]
Let \( p_j : CP^1 \rightarrow CP^{N-1} \) be \( \pi_N(a \sum_{i=1}^{d_j} (s_j)^d_j + a \sum_{j=1}^{d_j} (t_j)^d_j) \) and \( C_j \) be the image of \( p_j \) in \( CP^{N-1} \). In the same way as in the type I case, \( Z_d \) acting on \((s_j, t_j)\) keeps \( p_j \) invariant. Therefore, in integrating over the type II fixed point set labeled by \( \sigma_d \), we have to divide the result by \( \prod_{j=1}^{l(\sigma_d)} |Z_d_j| = \prod_{j=1}^{l(\sigma_d)} d_j \).

\(^1\)If \( a_0 = a_d \), the map (3.22) becomes constant map from \( CP^1 \) to \( CP^{N-1} \), but we don’t eliminate this locus in this paper.
The normal vector space of the type II fixed point set labeled by \(\sigma_d\) is spanned by the degrees of freedom coming from deforming \(p_j\) individually and by the degrees of freedom associated to resolving nodal singularities of the image curve:

\[
\bigoplus_{j=1}^{l(\sigma_d)} \left( C N_j^{s_j, t_j} \oplus C N_j^{s_j, t_j-2} \oplus C N_j^{s_j, t_j-3} \oplus \cdots \right) \bigoplus C N_j^{s_j, t_j-1}.
\]

\[
\bigoplus_{j=1}^{l(\sigma_d)-1} \left( T \cdot C_j \ominus T_0 C_{j+1} \right).
\]  

### 3.2 Localization Computation of Virtual Structure Constants

In this subsection, we compute the \(\frac{i_{N,k,d}}{d}\), which may be regarded as the B-model analogue of 2-pointed Gromov-Witten invariants:

\[
\frac{1}{d} \left( \mathcal{O}_{h^{N-2-n}} \mathcal{O}_{h^{N-1+(N-k)d}} \right)_{0,d} = \frac{1}{k} \left( \mathcal{O}_{h^{N-2-n}} \mathcal{O}_{h^{N-1+(N-k)d}} \right)_{0,d}
\]

\[
= \frac{1}{k} \int_{M_{0,2}^{N,d}(CP^{N-1},d)} c_{top} \left( R^0(\pi_*ev_1^*(\mathcal{O}(k))) \wedge ev_1^*(h^{N-2-n}) \wedge ev_2^*(h^{N-1+(N-k)d}) \right),
\]

by using the moduli space introduced in the previous subsection. In (3.27), \(h\) is the hyperplane class of \(CP^{N-1}\), \(M_{0,n}(CP^{N-1},d)\) represents moduli space of degree \(d\) stable maps from genus 0 stable curve to \(CP^{N-1}\) with \(n\) marked points, \(ev_i : M_{0,n}(CP^{N-1},d) \rightarrow CP^{N-1}\) is the evaluation map of the \(i\)-th marked point and \(\pi : M_{0,3}(CP^{N-1},d) \rightarrow M_{0,2}(CP^{N-1},d)\) is the forgetful map. We also use the localization technique (Bott residue formula) associated with the torus action (3.21). However, in the following computation, we only consider the case \(\lambda_0 = \lambda_1 = \cdots = \lambda_d = 0\), for simplicity. To compensate for this choice, we have to treat the order of integration carefully. We will discuss these subtleties in the last part of this subsection.

First, we determine the contribution from the type I fixed point set \((CP^{N-1})^2\). From now on, we denote by \((CP^{N-1})_0\) (resp. \((CP^{N-1})_1\)) the first (resp. the second) \(CP^{N-1}\) of \((CP^{N-1})^2\). We define \(p_i : CP^1 \rightarrow CP^{N-1}\) be the map defined by,

\[
p_i(s : t) := \pi_N(a_0 s^d + a_d t^d).
\]

In the construction of \(M_{0,2}(N, d)\), the two marked points of \(CP^1\) are fixed to 0 = (1 : 0) and \(\infty = (0 : 1)\). Obviously, we have

\[
p_1(1 : 0) = \pi_N(a_0), \quad p_1(0 : 1) = \pi_N(a_d).
\]

Therefore, the classes that correspond to \(ev_i^*(h^{N-2-n})\) and \(ev_2^*(h^{N-1+(N-k)d})\) are given by \(h_0^{N-2-n}\) and \(h_1^{N-1+(N-k)d}\) respectively. Then we consider the vector bundle corresponding to \(R^0\pi_*ev^*_i\mathcal{O}(k)\). The corresponding vector space is spanned by \(H^0(CP^1, p_1^\top \mathcal{O}(k))\), and can written as

\[
\bigoplus_{j=0}^{kd} C N_j^{s_j, t^{kd-j}}.
\]

In the localization computation, we can identify \(s\) with \(\mathcal{O}_{(CP^{N-1})_0}(\frac{1}{d})\) and \(t\) with \(\mathcal{O}_{(CP^{N-1})_1}(\frac{1}{d})\) through (3.28). Therefore, the first Chern class of \(C N_j^{s_j, t^j}\) is given by,

\[
i h_0 + j h_1.
\]

In this way, the class that corresponds to \(c_{top}(R^0\pi_*ev^*_i\mathcal{O}(k))\) turns out to be

\[
\prod_{j=0}^{kd} \left( \frac{i h_0 + (kd - j) h_1}{d} \right)
\]

If we look back at (3.24), we can also determine the top Chern class of the normal bundle of the type I fixed point set as follows:

\[
\prod_{j=1}^{d-1} \left( \frac{j h_0 + (d - j) h_1}{d} \right)^N.
\]
Putting these pieces together, we can write down the contribution from the type I fixed point set by the localization theorem:

\[
\frac{1}{d} \int (\mathbb{C}P_{N-1})^2 h_0^{N-2-n} h_1^{n-1+(N-k)d} \prod_{j=0}^{kd} \left( \frac{h_0 + (kd-j)h_1}{d} \right) \prod_{j=1}^{d-1} \left( \frac{h_0 + (d-j)h_1}{d} \right)^N,
\]

(3.34)

where the factor \( \frac{1}{d} \) comes from the \( \mathbb{Z}_d \) action mentioned in the previous subsection.

Next, we determine the contribution from the type II fixed point set labeled by \( \sigma_d \). Let \( (\mathbb{C}P^{N-1})_i \) be the \( (i+1) \)-th \( \mathbb{C}P^{N-1} \) of \( (\mathbb{C}P^{N-1})^{(\sigma_d)+1} \) considered as a type II fixed point set, and let \( h_i \) be its hyperplane class. The computation goes in the same way as in the type I case, except for the effect of nodal singularities. Therefore, we consider here the contributions of these singularities. As for the vector bundle corresponding to \( R^0\pi_*ev^*_3\mathcal{O}(k) \), we have to consider the exact sequence:

\[
0 \to H^0((\mathbb{C}P^{N-1})_i C_j, (\mathbb{C}P^{N-1})_j (\mathbb{C}P^{N-1})) \to \oplus_{j=1}^{l(\sigma_d)} H^0(C_j, D^*_j \mathcal{O}(k)) \to \oplus_{j=1}^{l(\sigma_d)-1} \mathcal{O}_{\pi}(\sum_{i=1} a_i)(k) \to 0.
\]

(3.35)

From this exact sequence, we can easily see that we have to insert an additional \( \prod_{j=1}^{l(\sigma_d)-1} \frac{1}{h_{d_j}} \). We next turn to the effect of \( T'_\epsilon C_j \otimes T_0 C_{j+1} \) in (3.26). Obviously this can be written as,

\[
\mathbb{C}O \frac{d}{d(\frac{d}{s_j+1})} \frac{d}{d(\frac{d_{j+1}}{s_{j+1}})}
\]

(3.36)

and the \( s_j, t_j \) are identified with \( \mathcal{O}_{\mathbb{C}P^{N-1}}(\frac{1}{s_j}), \mathcal{O}_{\mathbb{C}P^{N-1}}(\frac{1}{s_{j+1}}) \) respectively. Hence its first Chern class is given by,

\[
h_j - h_{j-1} + \frac{h_j - h_{j+1}}{d_j}
\]

(3.37)

By the localization theorem, we also have to insert \( \prod_{j=1}^{l(\sigma_d)-1} \left( \frac{h_i - h_{j-1}}{d_j} + \frac{h_j - h_{j+1}}{d_{j+1}} \right)^{-1} \). Combining these considerations, we can write down the contributions from the type II fixed point set labeled by \( \sigma_d \):

\[
\frac{1}{l(\sigma_d)l_d} \int (\mathbb{C}P^{N-1})^{(\sigma_d)+1} h_0^{N-2-n} h_1^{n-1+(N-k)d} \prod_{j=1}^{l(\sigma_d)} \left( \prod_{i=0}^{kd} \left( \frac{h_i + (kd-j)h_1}{d} \right) \prod_{i=1}^{d-1} \left( \frac{h_i + (d-j)h_1}{d} \right)^N \right) \prod_{j=1}^{l(\sigma_d)-1} \frac{1}{h_j} \left( \frac{h_i - h_{j-1}}{d_j} + \frac{h_j - h_{j+1}}{d_{j+1}} \right)^{l(\sigma_d)-1},
\]

(3.38)

where the factor \( \frac{1}{l(\sigma_d)l_d} \) comes from the \( \mathbb{Z}_d \) action on \( C_j \) \( (j = 1, 2, \cdots, l(\sigma_d)) \). Then we integrate out (3.34) and (3.38) and divide the result by \( k \). After all this, we arrive at the corresponding summands in the formula (2.11). Lastly, we have to mention order of integration. In practice, we have to order the integrations of all the summands in descending (or ascending) order of subscript \( j \) of \( (\mathbb{C}P^{N-1})_j \subset (\mathbb{C}P^{N-1})^{(\sigma_d)+1} \).

4 Applications to Local Mirror Symmetry of Vector Bundles over \( \mathbb{C}P^1 \)

Our geometrical computation is also applicable to complete intersections in \( \mathbb{C}P^{N-1} \) and to the local geometries \( \oplus_{i=1}^n \mathcal{O}(k_i) \to \mathbb{C}P^{N-1} \). In this section, we consider two examples of local mirror symmetry of \( \mathbb{C}P^1 \), \( \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1 \) and \( \mathcal{O}(1) \oplus \mathcal{O}(3) \to \mathbb{C}P^1 \). The first one is the simplest example of local mirror symmetry, and the second one is a typical example of non-nef local mirror symmetry, which was analyzed extensively in [5],[6].

4.1 \( \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1 \)

In this example, we compute the virtual structure constants \( a_n^d \) that correspond to local Gromov-Witten invariants:

\[
\langle \mathcal{O}_{h_n^1} \mathcal{O}_{h_2^{1-n}} \rangle_{0,d} = \int_{\mathcal{M}_{0,2}(\mathbb{C}P^1,d)} \left( c_{\text{top}}(R^1\pi_*ev^*_3\mathcal{O}(1)) \right)^2 \wedge ev_1^*(h^n) \wedge ev_2^*(h^{2-n}), \quad (n = 0, 1, 2).
\]

(4.39)
Following the computation in the previous section, we obtain a closed formula that computes $\alpha_n^d$:

$$
\alpha_n^d = \sum_{\sigma_d \in OP_d} \frac{1}{\prod_{j=1}^{l(\sigma_d)} d_j} \int_{(CP^1)^{|l(\sigma_d)|+1}} h_n^0 h_{(\sigma_d)}^{2-n} \prod_{j=1}^{l(\sigma_d)-1} G_{d_j}(h_{j-1}, h_j) \prod_{j=1}^{l(\sigma_d)-1} \frac{(-h_j)^2}{h_j-h_{j-1} + h_j-h_{j+1}}, \tag{4.40}
$$

where

$$
G_d(x, y) := \frac{\prod_{j=1}^{d-1} \left(1 - \frac{jx-(d-j)y}{d}\right)^2}{\prod_{j=1}^{d-1} \left(1 + \frac{jx+(d-j)y}{d}\right)^2} = 1. \tag{4.41}
$$

In deriving (4.40), the exact sequence (3.35) is replaced by the following exact sequence:

$$
0 \to \oplus_{j=1}^{l(\sigma_d)-1} \mathcal{O}_{\pi_N(m_j)}(\sum_{i=1}^{d_i}) (1) \to H^1(U_j^{l(\sigma_d)} C_j, (\cup_{j=1}^{l(\sigma_d)} p_j)^* \mathcal{O}(-1)) \to \oplus_{j=1}^{l(\sigma_d)} H^1(C_j, p_j^* \mathcal{O}(-1)) \to 0. \tag{4.42}
$$

Since (4.41) holds, (4.40) can be further simplified to,

$$
\alpha_n^d = \sum_{\sigma_d \in OP_d} \frac{1}{\prod_{j=1}^{l(\sigma_d)} d_j} \int_{(CP^1)^{|l(\sigma_d)|+1}} h_n^0 h_{(\sigma_d)}^{2-n} \prod_{j=1}^{l(\sigma_d)-1} \frac{(h_j)^2}{h_j-h_{j-1} + h_j-h_{j+1}}. \tag{4.43}
$$

Due to the fact that $h^2_j = 0$ in $H^*((CP^1)_j, \mathbb{C})$, the summand in (4.43) vanishes if $l(\sigma_d) > 1$. Hence we obtain

$$
\alpha_n^d = \frac{1}{d} \delta_n^1. \tag{4.44}
$$

If we consider this in analogy with the Calabi-Yau hypersurface case [4], we expect that $t = x + \sum_{d=1}^{\infty} \alpha_n^d e^{dx}$ gives us the mirror map of this model. (4.44) says that it is trivial in this case. Therefore, $\alpha_n^d$ should coincide with the corresponding local Gromov-Witten invariant $\langle \mathcal{O}_h \mathcal{O}_{h^d} \rangle$. (4.44) agrees with this expectation.

### 4.2 $\mathcal{O}(1) \oplus \mathcal{O}(-3) \to CP^1$

In this case, we compute equivariant virtual structure constants $\beta_{mn}^d(z)$ that correspond to the equivariant Gromov-Witten invariants:

$$
\langle \mathcal{O}_{h^m} \mathcal{O}_{h^n}(z) \rangle_{0,d} = \int_{M_{0,2}(CP^{1},d)} \left( \frac{\sum_{j=0}^{3d-1} c_j(R^{\ast} \pi_{\ast} e_{\ast} v_{\ast} \mathcal{O}(-3))}{\sum_{j=0}^{d} x^j c_j(R^{\ast} \pi_{\ast} e_{\ast} v_{\ast} \mathcal{O}(1))} \right) \wedge ev_{1}^{\ast}(h^m) \wedge ev_{2}^{\ast}(h^n), \ (0 \leq m, n \leq 1). \tag{4.45}
$$

In the same way as in the previous subsection, $\beta_{mn}^d(z)$ is given as follows:

$$
\beta_{mn}^d(z) = \sum_{\sigma_d \in OP_d} \frac{1}{\prod_{j=1}^{l(\sigma_d)} d_j} \int_{(CP^1)^{|l(\sigma_d)|+1}} h_n^0 h_{(\sigma_d)}^{2-n} \prod_{j=1}^{l(\sigma_d)-1} H_{d_j}(h_{j-1}, h_j, z) \prod_{j=1}^{l(\sigma_d)-1} \frac{(1 + z h_j)(1 - 3 h_{j+1})}{h_j-h_{j-1} + h_j-h_{j+1}}, \tag{4.46}
$$

where

$$
H_d(x, y, z) := \frac{\prod_{j=0}^{3d-1} \left(1 - \frac{jx+(3d-j)y}{d}\right)^2}{\prod_{j=0}^{d} \left(1 + \frac{jx+(d-j)y}{d}\right)^2}. \tag{4.47}
$$

Let us compute $\beta_{00}^d(z), \beta_{10}^d(z), \beta_{11}^d(z)$ by using the formula (4.46). Here, we show the results for lower degrees by using the generating function $\beta_j(e^x, z) := \sum_{d=1}^{\infty} \beta_{j}^d(z) e^{dx}$.

$$
\beta_{00}(e^x, z) := (6z + z^2 + 5)e^x + \left(\frac{645}{4} z + \frac{311}{4} z^2 + \frac{63}{4} z^3 + \frac{5}{4} z^4 + 104\right) e^{2x}
+ (6387z + 91421 z^2 + 8767 z^4 + 31473 + 83723 + \frac{121}{2} z^5 + 85 z^6) e^{3x} + \ldots
$$

$$
\beta_{10}(e^x, z) := (3z - 3)e^x + \left(-\frac{139}{4} z - \frac{33}{4} z^2 - \frac{3}{4} z^3 - \frac{177}{4}\right) e^{2x}
+ (-1131 - 5917 z - 762 z^2 - 28 z^4 - \frac{407}{2} z^5 - 19 z^6) e^{3x} + \ldots
$$

$$
\beta_{11}(e^x, z) := e^x + \left(\frac{31}{2} + \frac{9}{2} z + \frac{1}{2} z^2\right) e^{2x} + (380 + \frac{549}{2} z + 175 z^2 + 5 z^4 + \frac{27}{2} z^3) e^{3x} + \ldots \tag{4.48}
$$
we compute $z$. If we invert (4.51) regarding $\beta_{30}(e^{x}, z) + (z - 3) \beta_{10}(e^{x}, z)^2$, the result turns out to be,

$$
(14 + 6z)e^x + \left( \frac{39}{4}z^3 + \frac{271}{4}z^2 + \frac{1}{2}z^4 + \frac{885}{4}z + \frac{947}{4} \right)e^{2x}
\quad + \left( \frac{38775}{4}z^2 + \frac{211885}{36}z^5 + \frac{1493308}{9}z^4 + \frac{1693308}{108}z^6 + \frac{175334}{27} + 1947z^3 \right)e^{3x} + \cdots,
\tag{4.49}
$$

which is nothing but the other equivariant mirror map $t(q, \lambda)$ in the formula (3.23) of [5]! In this way, we have computed full equivariant mirror map without using Birkhoff factorization inevitable in the analysis in [5]. Then we set

$$
\tilde{L}_1(e^x, z) := 1 + \partial_x(\beta_{30}(e^{x}, z) + (z - 3) \beta_{10}(e^{x}, z)) = \frac{\partial t}{\partial x},
\tilde{L}_2(e^x, z) := \partial_x \beta_{11}(e^{x}, z),
\tag{4.50}
$$
in analogy with the Calabi-Yau hypersurface case [4]. The first line of (4.50) also asserts,

$$
t(x, z) = x + \beta_{30}(e^{x}, z) + (z - 3) \beta_{10}(e^{x}, z).
\tag{4.51}
$$

If we invert (4.51) regarding $z$ as a parameter, we obtain $x(t, z)$. Then again by using the same analogy as before, we compute $\frac{\tilde{L}_2(e^{x(t, z)}, z)}{\tilde{L}_1(e^{x(t, z)}, z)}$. The result turns out to be,

$$
e^t + (-3z + 3 + z^2)e^{2t} + \left( \frac{69}{4}z^2 - \frac{81}{4}z + \frac{39}{4}z^4 - \frac{27}{4}z^3 \right)e^{3t} + \cdots,
\tag{4.52}
$$

which is nothing but the equivariant mirror A-model Yukawa coupling compatible with the formula (3.23) of [5]!! We can also apply this technique to the model $\mathcal{O}(1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$ and obtain the same mirror map and A-model Yukawa coupling as the ones computed in [6].

5 Conclusion

In this paper, we derived the virtual structure constants by applying the localization computation to the moduli space of polynomial maps with two marked points. The process of computation is very similar to the well-known result of Kontsevich [13], but is much simpler because we consider a simple moduli space of polynomial maps instead of the moduli space of stable maps. As a result, we obtain a B-model analogue of standard Gromov-Witten invariants. Unlike the standard Gromov-Witten invariants, the virtual structure constants need not vanish when they have an insertion of an operator induced from the identity element of the cohomology ring. Moreover, the virtual structure constants with insertions of the identity operator give us the expansion coefficients of the mirror map in the examples we have treated. Therefore, our computation provides a geometrical construction of the mirror map.

The line of thought in this paper stems from our endeavor to interpret geometrically the computation process of the generalized mirror transformation [10]. What we are aiming at is to describe the generalized mirror transformation as a process of changing the moduli space of Gauged Linear Sigma Model into the one of stable maps (see the discussion in [2] in the nef cases) [12]. To this end, we need to characterize the expansion coefficients of mirror map geometrically, and we think that the construction given in this paper provides what we need. We also hope to generalize our construction to various weighted projective spaces, especially in the case when they have several Kähler forms. If we accomplish this task, we will obtain a concrete geometrical understanding of the mirror computation.

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2This combination is derived by introducing the metric $\eta_{ij}$ induced from classical intersection number $\eta_{ij} := \int_{\mathbb{C}P^1} \frac{h_i^* h_j}{(1 + 2h)(1 - 3h)}$. 


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