Bosonization and operatorial extensions of supersymmetric Korteweg-de Vries equations

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Abstract. We discuss the bosonization of supersymmetric KdV as a first step on the construction of operatorial extensions of that system. We obtain the hamiltonian structure of those operatorial extensions using extended Miura tranformations. We also discuss their integrability properties in terms of extended Gardner tranformations. We give explicit examples of some operatorial extensions.

1. Introduction

Some years ago Mathieu [1] obtained an interesting supersymmetric extension of Korteweg-de Vries (KdV) equation with associated infinite local and non-local conserved quantities [1, 2, 3, 4]. That property was deduced following and idea of Gardner [5].

Such extension ($N = 1$ SKdV), equivalent to the Super KdV equations obtained in [6] by reduction from the super Kadomtsev-Petviashvili hierarchy, consists on a system of coupled partial differential equations for a commuting and an anticommuting field, taking values at the even and odd part respectively of a given Grassmann algebra. Basic ingredients in that extension are a superfield, a covariant derivative and a preferred generator in the algebra. Later on, in a similar way by increasing the number of preferred elements of the algebra, supersymmetric extensions of KdV were obtained [7, 8, 9, 10]. All such supersymmetric extensions have at least one associated Hamiltonian.

Recently, expanding the corresponding fields in terms of the generators of the algebra, via a bosonization approach [11], were obtained new interesting solutions for such supersymmetric extensions [12, 13].

In [11] an operatorial extension of KdV, which contains several systems as particular cases including $N = 1$ SKdV, preserving the property of having infinite local conserved quantities was given. In the following, we present a new operatorial extension of KdV with infinite local conserved quantities. We obtain its Hamiltonian structure, following the Dirac procedure for constraints systems [14, 15, 16, 17].

The hamiltonian analysis may be performed directly in terms of the odd variables but also one may give a bosonized hamiltonian formulation in terms of even variables. Both formulations are completely equivalent.
2. General algebraic framework

We consider an algebra of even and odd elements. The even ones belong to a commutative algebra \( \mathcal{P} \) with unit while the odd ones satisfy

\[
\begin{align*}
\mathcal{Q} \mathcal{P} & \subset \mathcal{Q} \\
[\mathcal{Q}, \mathcal{P}] & = 0 \\
[\mathcal{Q}, \mathcal{Q}] & \subset \mathcal{P}.
\end{align*}
\]

The odd part of the algebra \( \mathcal{Q} \) is characterized by the condition:

\[
[q, \mathcal{Q}] \neq 0,
\]

for any \( q \in \mathcal{Q} \). That is, given \( q \) there always exists \( \hat{q} \in \mathcal{Q} \) such that \([q, \hat{q}] \neq 0\).

In fact, otherwise \( q \) would be an even element.

Examples of this algebraic structure are:

(i) Operatorial Phase Space.

Given \( q \) and its conjugate momenta \( \frac{\partial}{\partial q} \), then: \([q, p] = 1, [q, q] = 0, [p, p] = 0\). We identify the even part \( \mathcal{P} \) with elements proportional to the unit, while \( q \) and \( p \) belong to the odd part of the algebra \( \mathcal{Q} \).

(ii) Grassmann algebras.

In this case the odd elements satisfy \( \mathcal{Q} \mathcal{Q} \subset \mathcal{P} \) and in addition for any \( q_1, q_2 \):

\[ q_1 q_2 + q_2 q_1 = 0. \]

In particular, the odd elements are nilpotent.

(iii) Extensions of the Grassmann algebras.

The algebra satisfies: \( \mathcal{Q} \mathcal{P} \subset \mathcal{Q}, \mathcal{Q} \mathcal{Q} \subset \mathcal{P}, [\mathcal{Q}, \mathcal{P}] = 0 \).

This is a particular case of (1) but different from (i) and includes (ii). It is different from (i) because in the phase space \( qq \) is not proportional to the unit and hence it does not satisfy \( \mathcal{Q} \mathcal{Q} \subset \mathcal{P} \). It is more general than (ii) because we do not require \( q_1 q_2 + q_2 q_1 = 0 \). Particular violations of this condition occur if we consider, for example, the matrix realization

\[
q_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad q_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & a & 1 & 0
\end{pmatrix}, \quad a \in \mathbb{R}.
\]

We get

\[
q_1 q_1 = 0, \quad q_2 q_2 = 0, \quad q_1 q_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad q_2 q_1 = a q_1 q_2.
\]

If \( a = -1 \), we obtain an example of a Grassmann algebra. If \( a \neq -1 \) we have a deformation of the Grassmann algebra.

3. The Lagrangian of an extension of Super Miura Korteweg-de Vries (Super MKdV) equations

We introduce a Lagrangian formulated in terms of fields \( w \) and \( \eta \) valued on \( \mathcal{P} \) and \( \mathcal{Q} \) respectively, which satisfy (1):

\[
\mathcal{L}(w, \eta) = \frac{1}{2} \dot{w} w' + \frac{1}{2} [\dot{\eta}, \eta] - \frac{1}{2} (w'')^2 - \frac{1}{2} (w')^4 - \frac{1}{2} [\eta, \eta']^2 - \frac{1}{2} [\eta'', \eta'] - \frac{3}{2} (w')^2 [\eta', \eta].
\]
The fields equations are obtained by taking variations with respect to $w$ and $\eta$. By taking variations with respect to $w$ we obtain
\[ \dot{v} + v''' - 6v^2v' - 3\left(v \left[ \eta', \eta \right] \right)' = 0 \] (4)
where $v \equiv w'$.

By taking variations with respect to $\eta$ we obtain
\[ \langle \left[ \delta \eta, F \right] \rangle_{x,t} = 0 \] (5)
where
\[ F = \dot{\eta} + \eta''' - 3v^2\eta' - 3vv'\eta + \eta' \left[ \eta, \eta' \right] + \frac{1}{2} \eta \left[ \eta, \eta'' \right] \] (6)
and $\langle \rangle_{x,t}$ denotes integration on $x$ and $t$.

(5) implies $[Q, F] = 0$ for any odd element and from (2) we get
\[ F = 0 \] (7)
as the field equation associated to general variations of the odd field $\eta$. If $P, Q$ generates a Grassmann algebra then the terms $\eta' \left[ \eta, \eta' \right]$ and $\frac{1}{2} \eta \left[ \eta, \eta'' \right]$ in (6) are zero and the field equations are exactly the $N = 1$ Super MKdV equations obtained from the $N = 1$ Super KdV equations by performing a Miura transformation.

We now proceed to construct the Hamiltonian structure of the system (4) and (7).

4. Hamiltonian structure

The conjugate momenta to $w$ and $\eta$ will be denoted by $p$ and $\mu$ respectively. We obtain directly from their definitions
\[ \phi \equiv p - \frac{1}{2} v = 0 \]
\[ \psi \equiv \mu - \frac{1}{2} \eta = 0 \] (8)
These are primary constraints of the phase space formulation. It turns out, following the Dirac approach to determine the constrained structure of the phase space, that they are the only constraints of the theory.

They satisfy the following Poisson bracket relations:
\[ \{ \phi(x), \phi(y) \}_{PB} = -\partial_x \delta(x, y) \]
\[ \{ \phi(x), \psi(y) \}_{PB} = 0 \]
\[ \{ \psi(x), \psi(y) \}_{PB} = -\delta(x, y), \]
consequently (8) are second class constraints.

The Poisson structure of the constrained phase space is then given by the Dirac brackets defined by
\[ \{ F, G \}_{DB} = \{ F, G \}_{PB} - \left\langle \left\langle \{ F, \varphi_i(x) \}_{PB} \{ \varphi_i(x), \varphi_j(x) \}_{PB} \{ \varphi_j(x), G \}_{PB} \right\rangle_{\tilde{x}} \right\rangle_{\tilde{x}} \]
where $i = 1, 2$ and $\varphi_1 \equiv \phi, \varphi_2 \equiv \psi$ (summation on $i$ and $j$ is understood).

After performing some calculations we obtain the Dirac bracket algebra for the fields $w$ and $\eta$. It turns out that one may directly evaluate the algebra of the fields $v \equiv w'$ and $\eta$. We obtain
\[ \{ v(x), v(y) \}_{DB} = \partial_x \delta(x, y) \]
\[ \{ v(x), \eta(y) \}_{DB} = 0 \]
\[ \{ \eta(x), \eta(y) \}_{DB} = \delta(x, y). \]
The Hamiltonian associated to the Lagrangian (3) is obtained via a Legendre transformation. We get
\[ H \equiv p\dot{w} + \frac{1}{2} [\dot{\eta}, \eta] - \mathcal{L} = \frac{1}{2} (v')^2 + \frac{1}{2} v^4 + \frac{1}{2} [\eta, \eta']^2 + \frac{1}{2} [\eta'', \eta'] + \frac{3}{2} v^2 [\eta', \eta] . \]
The canonical field equations
\[ \dot{v} = \{ v, H \}_DB \]
\[ \dot{\eta} = \{ \eta, H \}_DB \]
(9)
where \( H = \langle H \rangle_x \), exactly agree with the lagrangian field equations (4),(7), as it should be.

We now consider the most general Miura transformation. A dimensional analysis shows that it has the form
\[ u = v' + v^2 - [\eta, \eta'] \]
\[ \xi = \eta' + v\eta. \]
(10)
It then follows the field equations for \( u \) and \( \xi \). We obtain
\[ \dot{u} = \{ u, H \}_DB = -u'''' + 6 uu' - 3[\xi \xi''] \]
\[ \dot{\xi} = \{ \xi, H \}_DB = -\xi'' + 3(\xi u)' + \theta' + v\theta \]
(11)
where
\[ \theta \equiv [\eta, \eta'] \eta' + v [\eta, \eta''] \eta. \]
Under the restricted algebra satisfying:
\[QP \subset Q, QQ \subset P, [Q, P] = 0 \]
(12)
we have
\[ \theta = 0, \]
even when the odd elements are not necessarily nilpotent. The resulting equations (11) under the restricted algebra (12) were analyzed in [11]. It was proven, via a Gardner transformation, that they have an infinite sequence of local conserved quantities. In fact, in that case we have
\[ u_t = -u'''' + 6 uu' + 3[\xi''', \xi] \]
\[ \xi_t = -\xi''' + 3(u\xi)', \]
(13)
\[ z_t = (z'' + 3z^2 + 3[\sigma', \sigma])' + e^2(2z^3 + 3z[\sigma', \sigma]'), \]
\[ \sigma_t = (\sigma'' + 3z\sigma)' + e^2(3z^2[\sigma'] + 2z\sigma + \sigma'[\sigma', \sigma]), \]
\[ u = z + e\sigma' + e^2(z^2 + [\sigma', \sigma]) \]
\[ \xi = \sigma + e\sigma' + e^2z\sigma. \]
(14)
\[ \theta = \sigma' + e^2z. \]
(15)
(14) and (15) are the Gardner equations and associated Gardner transformations respectively.

After simplifying by crossing out derivatives in \( x \) and using the inverse Gardner transformation, the first four nontrivial conserved quantities for the operator-extended KdV system (13) are:
\[ H_0 = \int u dx \]
\[ H_2 = \int (u^2 + [\xi', \xi]) dx \]
\[ H_4 = \int (2u^3 + (u')^2 + 4u[\xi', \xi] + [\xi'', \xi']) dx \]
\[ H_6 = \int \left( 5u^4 + 10u(u')^2 + 15u^2[\xi', \xi] - 2u[\xi'', \xi'] + 3[\xi', \xi]^2 + [\xi'''', \xi'] \right) dx. \]
(16)
5. Conclusions
We have then introduced an extension of the Super KdV equations, obtained its Hamiltonian and its Poisson structure on the constrained phase space, and shown it has an infinite sequence of conserved quantities. We notice that the case of the operatorial phase space analyzed in [11] does not satisfy (12) hence the Hamiltonian structure discussed here does not apply to that case.

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