Classes of harmonic starlike functions defined by Sălăgean-type $q$-differential operators

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Abstract
Sufficient and necessary coefficient bounds, extreme points of closed convex hulls, and distortion theorems are determined for a family of harmonic starlike functions of complex order involving Sălăgean-type $q$-differential operators.

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1. Introduction
Let $A$ denote the class of functions $h$ of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let $S$ denote the subclass of $A$ consisting of functions that are univalent in $U$.

We now recall the notion of $q$-operators or $q$-difference operators that play vital roles in the theory of hypergeometric series, quantum physics and operator theory. The application of $q$-calculus was initiated by Jackson [7] who have used the fractional $q$-calculus operators in investigations of certain classes of functions which are analytic in $U$. For more details on $q$-calculus and its applications one can refer to [1,5,7,13] and the references cited therein.

For $0 < q < 1$ the Jackson’s $q$-derivative of a function $h \in S$ is given as follows [7]

$$D_q h(z) = \begin{cases} \frac{h(z) - h(qz)}{(1-q)z} & \text{for } z \neq 0, \\ h'(z) & \text{for } z = 0, \end{cases}$$

(1.2)

From (1.2), we have $D_q h(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$ where $[n]_q = \frac{1-q^n}{1-q}$ is sometimes called the basic number $n$. If $q \to 1^-$ then $[n]_q = [n] \to n$. For $h \in A$, $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$

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and \( z \in \mathbb{U} \), Govinderaj and Sivasubramanian [5] considered the Sălăgean \( q \)-differential operators

\[
D_q^0 h(z) = h(z), \\
D_q^1 h(z) = zD_q h(z), \ldots , \\
D_q^m h(z) = zD_q(D_q^{m-1} h(z)) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n.
\]

(1.3)

We note that if \( q \to 1^- \) then

\[
D_q^m h(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n \quad (m \in \mathbb{N}_0, z \in \mathbb{U})
\]

is the familiar Sălăgean derivative[15].

Let \( \mathcal{H} \) denote the family of harmonic functions \( f = h + \overline{g} \) that are orientation preserving and univalent in \( \mathbb{U} \) with \( h \) as in (1.1) and \( g \) given by

\[
g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.
\]

(1.4)

We note that the family \( \mathcal{H} \) of orientation preserving, normalized harmonic univalent functions reduces to the well known class \( \mathcal{S} \) of normalized univalent functions if the co-analytic part of \( f \) is identically zero, i.e. \( g \equiv 0 \). We let \( \mathcal{H}_q \) be the subfamily of \( \mathcal{H} \) consisting of harmonic functions \( f = h + \overline{g} \) for which \( h \) and \( g \) are given by

\[
h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad a_n \geq 0 \text{ and } b_n \geq 0.
\]

The seminal work of Clunie and Sheil-Small [4] on harmonic mappings prompted many research articles on classes of complex-valued harmonic univalent functions. In particular, [2, 6, 8, 9, 11, 12, 14, 16] have investigated properties of various subclasses of harmonic univalent functions.

For harmonic functions \( f = h + \overline{g} \in \mathcal{H} \) where \( h \) and \( g \) are, respectively, given by (1.1) and (1.4), let \( D_q^m h(z) \) be defined by (1.3) and \( D_q^m g(z) \) be defined by

\[
D_q^0 g(z) = g(z), \\
D_q^1 g(z) = zD_q g(z), \ldots , \\
D_q^m g(z) = zD_q(D_q^{m-1} g(z)) = z + \sum_{n=2}^{\infty} [n]_q^m b_n z^n.
\]

(1.5)

Recently, Jahangiri [10] considered a generalized Sălăgean \( q \)- differential operator \( \mathcal{H}_q^m(\alpha) \) defined by

\[
\Re \left( \frac{D_q^{m+1} f(z)}{D_q^m f(z)} \right) \geq \alpha; \quad 0 \leq \alpha < 1,
\]

where, \( D_q^m h(z) \) and \( D_q^m g(z) \) are, respectively, defined by (1.3) and (1.5) and

\[
D_q^m f(z) = D_q^m h(z) + (-1)^m D_q^m g(z), \quad m > -1.
\]

The subfamily \( \mathcal{H}_q^m(\alpha) \subset \mathcal{H}_q^m(\alpha) \) consists of harmonic functions \( f_m = h + \overline{g}_m \) for which

\[
h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g_m(z) = (-1)^m \sum_{n=1}^{\infty} b_n z^n, \quad a_n \geq 0 \text{ and } b_n \geq 0.
\]

(1.6)
For non-zero complex number \( b \) with \( |b| \leq 1 \), real number \( \gamma \) and \( 0 \leq \alpha < 1 \) we let \( \mathcal{HS}^m_q(b, \gamma, \alpha) \) be the subclass of \( \mathcal{H} \) consisting of harmonic functions \( f = h + \overline{g} \) satisfying
\[
\Re \left( 1 + \frac{1}{b} \left( (1 + e^{i\gamma}) \frac{D_{q}^{m+1}f(z)}{D_{q}^{m}f(z)} - e^{i\gamma} - 1 \right) \right) > \alpha. \tag{1.7}
\]
We also let \( \mathcal{HS}^m_q(b, \gamma, \alpha) \equiv \mathcal{HS}^m_q(b, \gamma, \alpha) \cap \mathcal{H} \).
We note that \( \mathcal{HS}^m_q(1, \gamma, \alpha) \equiv \mathcal{HR}^m_q(\gamma, \alpha) \) is generalized class of Goodman-Ronne-type harmonic starlike functions (see [14], Inequality (2), p. 46) satisfying
\[
\Re \left( 1 + e^{i\gamma} \frac{D^{m+1}_q f(z)}{D^{m}_q f(z)} - e^{i\gamma} - 1 \right) > \alpha
\]
and \( \mathcal{HS}^m_q(b, 0, \alpha) \equiv \mathcal{HR}^m_q(b, \alpha) \) is the harmonic version of generalized starlike functions of complex order (see [3], Definition 1) satisfying
\[
\Re \left( 1 + \frac{2}{b} \left( \frac{D^{m+1}_q f(z)}{D^{m}_q f(z)} - 1 \right) \right) > \alpha.
\]
It is the aim of this paper to obtain sufficient coefficient conditions, extreme points, growth theorem, and distortion bounds for harmonic functions \( f = h + \overline{g} \) in \( \mathcal{HS}^m_q(b, \gamma, \alpha) \). Moreover, we show that those sufficient coefficient conditions for \( f \in \mathcal{HS}^m_q(b, \gamma, \alpha) \) are also necessary for \( f \in \mathcal{HS}^m_q(b, \gamma, \alpha) \).

2. Main results

The sufficient coefficient condition for \( \mathcal{HS}^m_q(b, \gamma, \alpha) \) is given in the following theorem.

**Theorem 2.1.** Let \( f = h + \overline{g} \in \mathcal{H} \) where \( b \) is a non-zero complex number with \( |b| \leq 1 \), \( \gamma \) is a real number and \( 0 \leq \alpha < 1 \). If
\[
\sum_{n=1}^{\infty} \left( \frac{[n]_q^{m} [2[n]_q - 2 + (1 - \alpha)|b|]}{(1 - \alpha)|b|} |a_n| + \frac{[n]_q^{m} [2[n]_q + 2 - (1 - \alpha)|b|]}{(1 - \alpha)|b|} |b_n| \right) \leq 2, \tag{2.1}
\]
then \( f \) is harmonic univalent and orientation-preserving in \( U \) and \( f \in \mathcal{HS}^m_q(b, \gamma, \alpha) \).

**Proof.** First we establish that \( f \) is orientation preserving in \( U \). In other words, we need to show that \( |D^{m+1}_q h(z)| \geq |D^{m+1}_q g(z)| \). This is accomplished using the properties of absolute values and the coefficient inequality (2.1).

\[
|D^{m+1}_q h(z)| \geq 1 - \sum_{n=2}^{\infty} [n]_q^{m+1} |a_n| n^{-1} > 1 - \sum_{n=2}^{\infty} [n]_q^{m+1} |a_n|
\]
\[
\geq 1 - \sum_{n=2}^{\infty} \left[ \frac{2[n]_q - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} \right] [n]_q^{m} |a_n| \]
\[
\geq \sum_{n=1}^{\infty} \left[ \frac{2[n]_q + 2 - (1 - \alpha)|b|}{(1 - \alpha)|b|} \right] [n]_q^{m} |b_n| \]
\[
\geq \sum_{n=1}^{\infty} [n]_q^{m+1} |b_n| \geq \sum_{n=1}^{\infty} [n]_q^{m+1} |b_n| n^{-1} \geq |D^{m+1}_q g(z)|.
\]

To show \( f \) is univalent in \( U \) we use a method that was first used by Jahangiri [8]. We will show that \( f(z_1) \neq f(z_2) \) when \( z_1 \neq z_2 \). Consider \( z_1 \) and \( z_2 \) in \( U \) so that \( z_1 \neq z_2 \). Since the unit disc \( U \) is simply connected and convex, we have \( z(t) = (1 - t)z_1 + tz_2 \in U \) for \( 0 \leq t \leq 1 \). Then we may write
\[
D^{m+1}_q f(z_2) - D^{m+1}_q f(z_1) = \frac{1}{0} \int [(z_2 - z_1)(D^{m+1}_q h(z(t)) + (z_2 - z_1)(D^{m+1}_q g(z(t)))dt.
\]
Dividing the above equation by $z_2 - z_1$ and taking the real parts we obtain

$$
\Re \left( \frac{D_{q}^{m+1}f(z_2) - D_{q}^{m+1}f(z_1)}{z_2 - z_1} \right) = \int_{0}^{1} \Re[D_{q}^{m+1}h(z(t)) + \frac{(z_2 - z_1)D_{q}^{m+1}g(z(t))}{z_2 - z_1}] dt \quad (2.2)
$$

On the other hand

$$
\Re(D_{q}^{m+1}h(z(t))) - |D_{q}^{m+1}g(z(t))| \geq \Re(D_{q}^{m+1}h(z(t))) - \sum_{n=1}^{\infty} |n|^m |a_n| - \sum_{n=1}^{\infty} |n|^m b_n
$$

$$
\geq 1 - \sum_{n=2}^{\infty} |n|^m |a_n| - \sum_{n=1}^{\infty} |n|^m b_n
$$

$$
\geq 1 - \sum_{n=2}^{\infty} |n|^m \left[ \frac{2|n|q - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} \right] |a_n|
$$

$$
- \sum_{n=1}^{\infty} |n|^m \left[ \frac{2|n|q + 2 - (1 - \alpha)|b|}{(1 - \alpha)|b|} \right] b_n
$$

$$
\geq 0 \text{ by } (2.1).
$$

This together with inequality (2.2) implies the univalence of $f$.

Next we show that if the condition (2.1) holds then $f \in \mathcal{S}_q^m(b, \gamma, \alpha)$. In other words, we need to show that the condition (1.7) is satisfied if (2.1) holds.

Using the fact that $\Re(w(z)) \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$ for $0 \leq \alpha < 1$ it suffices to show that

$$
|(2b - \alpha b - e^{i\gamma} - 1)(D_{q}^{m}h(z) + (-1)^m D_{q}^{m}g(z)) + (1 + e^{i\gamma})(D_{q}^{m+1}h(z) - (-1)^m D_{q}^{m+1}g(z))|
$$

$$
- |(1 + \alpha b + e^{i\gamma})(D_{q}^{m}h(z) + (-1)^m D_{q}^{m}g(z))| - (1 + e^{i\gamma})(D_{q}^{m+1}h(z) - (-1)^m D_{q}^{m+1}g(z))| \geq 0.
$$

Upon substituting for $D_{q}^{m}h(z)$ and $D_{q}^{m}g(z)$ we obtain

$$
|(2b - \alpha b - (1 + e^{i\gamma})) \left[ z + \sum_{n=2}^{\infty} |n|^m a_n z^n + (-1)^m \sum_{n=1}^{\infty} |n|^m b_n z^n \right]
$$

$$
+ (1 + e^{i\gamma}) \left[ z + \sum_{n=2}^{\infty} |n|^m a_n z^n - (-1)^m \sum_{n=1}^{\infty} |n|^m b_n z^n \right] |
$$

$$
- |(1 + \alpha b + e^{i\gamma}) \left[ z + \sum_{n=2}^{\infty} |n|^m a_n z^n + (-1)^m \sum_{n=1}^{\infty} |n|^m b_n z^n \right]
$$

$$
- (1 + e^{i\gamma}) \left[ z + \sum_{n=2}^{\infty} |n|^m a_n z^n - (-1)^m \sum_{n=1}^{\infty} |n|^m b_n z^n \right] |
$$
2.1
follows from
To prove the

Let $g$
Since

Theorem

Proof.

The functions

$$f(z) = z + \sum_{n=2}^{\infty} \left( \frac{(2 - \alpha)|b|}{2[n]_q - 2 + (1 - \alpha)|b|} \right) x_n z^n + \sum_{n=1}^{\infty} \left( \frac{(1 - \alpha)|b|}{2[n]_q + 2 - (1 - \alpha)|b|} \right) y_n z^n,$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, shows that the coefficient bound given by (2.1) is sharp.

The next theorem shows that condition (2.1) is also necessary for $f \in \mathcal{HS}_q^m(b, \gamma, \alpha)$.

**Theorem 2.2.** Let $f_m = h + g_m$ be given by (1.6) where $b$ is a non-zero complex number with $|b| \leq 1$, $\gamma$ is a real number and $0 \leq \alpha < 1$. Then $f_m$ is harmonic univalent and orientation-preserving in $U$ and $f_m \in \mathcal{HS}_q^m(b, \gamma, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left( \frac{[n]_q |2[n]_q - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} a_n + \frac{[n]_q |2[n]_q + 2 - (1 - \alpha)|b|}{(1 - \alpha)|b|} b_n \right) \leq 2. \quad (2.3)$$

**Proof.** Since $\mathcal{HS}_q^m(b, \gamma, \alpha) \subset \mathcal{HS}_q^m(b, \gamma, \alpha)$, the if part of the Theorem 2.2 follows from Theorem 2.1. To prove the only if part, we will show that if (2.3) does not hold then $f_m$ is not in $\mathcal{HS}_q^m(b, \gamma, \alpha)$.

For $f_m \in \mathcal{HS}_q^m(b, \gamma, \alpha)$ we must have

$$\Re \left( 1 + \frac{1}{b} \left( (1 + e^{i\gamma}) \frac{D_q^{m+1}h(z) - (-1)^m D_q^{m+1}g_m(z)}{D_q^m h(z) + (-1)^m D_q^m g_m(z) - (e^{i\gamma} + 1)} \right) \right) \geq \alpha.$$
Or equivalently
\[
\Re \left( (1 - \alpha) bz - \sum_{n=2}^{\infty} [(1 - \alpha)b + ([n]_q - 1)(1 + e^{i\gamma})][n]_q^m |a_n| z^n \right)
\]
\[
= \Re \left( (1 - \alpha)b^2 - \sum_{n=2}^{\infty} [(1 - \alpha)b + ([n]_q - 1)(1 + e^{i\gamma})][n]_q^m |a_n| z^n \right)
\]
\[
= \Re \left( \frac{(1 - \alpha)b^2 - \sum_{n=2}^{\infty} [(1 - \alpha)b + ([n]_q - 1)(1 + e^{i\gamma})][n]_q^m |a_n| z^n}{|b|^2 \left( 1 - \sum_{n=1}^\infty |n|_q^m |a_n| z^{n-1} + \sum_{n=1}^\infty |n|_q^m |b_n| z^{(n)_q-1} \right)} \right)
\geq 0.
\]

The above condition must hold for all values of \( \gamma, |z| = r < 1 \) and \( 0 < |b| < 1 \). For \( \gamma = 0 \) and \( |b| = b \) let \( z = r < 1 \) be on the positive real axis. Then the above condition becomes
\[
\frac{(1 - \alpha)b^2 - \sum_{n=2}^{\infty} [(2[n]_q + 2) - (1 - \alpha)b][n]_q^m |a_n| z^{n-1}}{|b|^2 \left( 1 - \sum_{n=2}^\infty |n|_q^m |a_n| z^{n-1} + \sum_{n=2}^\infty |n|_q^m |b_n| z^{(n)_q-1} \right)} \geq 0.
\]

Now we observe that the numerator in the above required inequality (2.4) is negative if condition (2.3) does not hold. Thus, there exists a point \( z_0 = r_0 \) in \((0,1)\) for which the quotient in the above inequalities are negative. This contradicts the required condition (1.7) for \( f_m \in \overline{\mathcal{F}S}_{q}^m(b, \gamma, \alpha) \). Hence the proof is complete.

The following theorem is a consequence of the above Theorem 2.2.

**Theorem 2.3.** Let \( f_m = h + \mathfrak{g}_m \) be given by (1.6). Then \( f_m \in \overline{\mathcal{F}S}_{q}^m(\gamma, \alpha) \) if and only if
\[
\sum_{n=1}^{\infty} \left( \frac{[n]_q^m [2[n]_q - 1 - \alpha]}{1 - \alpha} a_n + \frac{[n]_q^m [2[n]_q + 1 + \alpha]}{1 - \alpha} b_n \right) \leq 2.
\]

The extreme points of closed convex hull of \( \overline{\mathcal{F}S}_{q}^m(b, \gamma, \alpha) \), denoted by \( cldc \overline{\mathcal{F}S}_{q}^m(b, \gamma, \alpha) \), are determined in the following theorem.

**Theorem 2.4.** Let \( f_m \in cldc \overline{\mathcal{F}S}_{q}^m(b, \gamma, \alpha) \) if and only if
\[
f_m(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_m)
\]
where
\[
h_1(z) = z, h_n(z) = z - \frac{(1 - \alpha)|b|}{[n]_q^m [2[n]_q - 2 + (1 - \alpha)|b|]} z^n, \ n = 2, 3, \ldots;
\]
\[ g_{m_n}(z) = z + (-1)^m \frac{(1 - \alpha)|b|}{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]} z^n, \quad n = 1, 2, \ldots; \]

\[ \sum_{n=1}^{\infty} (X_n + Y_n) = 1, \quad X_n \geq 0 \text{ and } Y_n \geq 0. \]

In particular, the extreme points of \( \text{clco} \Re \mathcal{D}^m_q(b, \gamma, \alpha) \) are \( \{h_n\} \) and \( \{g_{m_n}\} \).

**Proof.** For functions of the form (2.5), we have

\[
\begin{align*}
    f_m(z) &= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_{m_n}) \\
    &= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{(1 - \alpha)|b|}{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]} X_n z^n \\
    &\quad + (-1)^m \sum_{n=1}^{\infty} \frac{(1 - \alpha)|b|}{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]} Y_n z^n.
\end{align*}
\]

Therefore

\[
\begin{align*}
    \sum_{n=2}^{\infty} \frac{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]}{(1 - \alpha)|b|} \left( \frac{(1 - \alpha)|b|}{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]} \right) X_n \\
    + \sum_{n=1}^{\infty} \frac{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]}{(1 - \alpha)|b|} \left( \frac{(1 - \alpha)|b|}{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]} \right) Y_n \\
    = \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1.
\end{align*}
\]

Thus, \( f_m \in \text{clco} \Re \mathcal{D}^m_q(b, \gamma, \alpha) \). Conversely, suppose that \( f_m \in \text{clco} \Re \mathcal{D}^m_q(b, \gamma, \alpha) \). Set

\[ X_n = \frac{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]}{(1 - \alpha)|b|} |a_n|, \quad n = 2, 3, \ldots, \]

and

\[ Y_n = \frac{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]}{(1 - \alpha)|b|} |b_n|, \quad n = 1, 2, \ldots, \]

where \( \sum_{n=1}^{\infty} (X_n + Y_n) = 1 \). Then

\[
\begin{align*}
    f_m(z) &= z - \sum_{n=2}^{\infty} a_n z^n + (-1)^m \sum_{n=1}^{\infty} b_n z^n \\
    &= z - \sum_{n=2}^{\infty} \frac{(1 - \alpha)|b|}{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]} X_n z^n + (-1)^m \sum_{n=1}^{\infty} \frac{(1 - \alpha)|b|}{|n|^m_2[|n|_q + 2 - (1 - \alpha)|b|]} Y_n z^n \\
    &= z - \sum_{n=2}^{\infty} [X_n (h_n(z) - z)] + \sum_{n=1}^{\infty} [Y_n (g_{m_n}(z) - z)] \\
    &= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_{m_n}).
\end{align*}
\]

Now from Theorem 2.2, we can deduce that \( 0 \leq X_n \leq 1, \ (n \geq 2) \) and \( 0 \leq Y_n \leq 1, \ (n \geq 1) \).

Therefore \( X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n \geq 0 \). Thus \( \sum_{n=1}^{\infty} (X_n h_n + Y_n g_{m_n}) = f_m(z) \) as required in the theorem.

Finally, we determine the distortion theorem for the family \( \Re \mathcal{D}^m_q(b, \gamma, \alpha) \).
Theorem 2.5. Let $f_m \in \mathcal{F}_{q_d}^{m}(b, \gamma, \alpha)$ where $|z| = r < 1$. Then

$$|f_m(z)| \leq (1 + b_1)r + \left(\frac{(1 - \alpha)|b|}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} - \frac{4 - (1 - \alpha)|b|}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|}\right)r^2$$

and

$$|f_m(z)| \geq (1 - b_1)r - \left(\frac{(1 - \alpha)|b|}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} - \frac{4 - (1 - \alpha)|b|}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|}\right)r^2.$$  

\textbf{Proof.} We will prove the right hand inequality. The proof for the left hand inequality will be similar and is omitted. Let $f_m(z) \in \mathcal{F}_{q_d}^{m}(b, \gamma, \alpha)$. Upon taking the absolute value of $f_m$, we obtain

$$|f_m(z)| \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} n[a_n] + |b_n| |n_q^m| r^n$$

$$= (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|n_q^m| r^n$$

$$= (1 + |b_1|)r + \frac{(1 - \alpha)|b|r^2}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} \times \sum_{n=2}^{\infty} [2]_q^m \left(\frac{2[2]_q - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} |a_n| + \frac{2[2]_q - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} |b_n|\right)$$

$$\leq (1 + |b_1|)r + \frac{(1 - \alpha)|b|r^2}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} \times \sum_{n=2}^{\infty} [n_q^m] \left(\frac{2[n_q] - 2 + (1 - \alpha)|b|}{(1 - \alpha)|b|} |a_n| + \frac{2[n_q] - 2 - (1 - \alpha)|b|}{(1 - \alpha)|b|} |b_n|\right)$$

$$\leq (1 + |b_1|)r + \frac{(1 - \alpha)|b|r^2}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} \left(1 + \frac{|4 - (1 - \alpha)|b|}{(1 - \alpha)|b|} |b_1|\right)r^2$$

$$\leq (1 + |b_1|)r + \frac{(1 - \alpha)|b|r^2}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} \left(1 + \frac{4 - (1 - \alpha)|b|}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} |b_1|\right)r^2.$$

The result is sharp for

$$f(z) = z + |b_1|z + \left(\frac{(1 - \alpha)|b|}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|} - \frac{4 - (1 - \alpha)|b|}{[2]_q^m[2][2]_q - 2 + (1 - \alpha)|b|}\right)z^2,$$

where $|b_1| \leq \frac{(1 - \alpha)|b|}{4 - (1 - \alpha)|b|}$.

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