Conditional generation of $N$-photon entangled states of light

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We propose a scheme for conditional generation of two-mode $N$-photon path-entangled states of traveling light field. These states may find applications in quantum optical lithography and they may be used to improve the sensitivity of interferometric measurements. Our method requires only single-photon sources, linear optics (beam splitters and phase shifters), and photodetectors with single photon sensitivity.

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Quantum entanglement represents one of the most remarkable and intriguing features of the quantum mechanics. Recently, entanglement has been identified as a fundamental resource necessary for quantum information processing such as quantum teleportation and quantum computing. The entangled states may also help to improve the sensitivity of interferometric measurements and they form a key ingredient of quantum optical lithography, which employs $N$-photon entangled states to fabricate patterns on lithographic substrate with resolution $\lambda/(2N)$, where $\lambda$ is the optical wavelength.

In view of these potential applications, it is highly desirable to build a source of $N$-photon path-entangled states of traveling light field,

$$|\psi_N\rangle = \sum_{k=0}^{N} c_k |k, N-k\rangle.$$  \hfill (1)

Here $|k, N-k\rangle$ denotes the usual Fock state with $k$ photons in mode $a$ and $N-k$ photons in mode $b$. Of particular interest could be the entangled state

$$|\psi_0^N\rangle = \frac{1}{\sqrt{2}} (|N, 0\rangle + |0, N\rangle).$$  \hfill (2)

In Schrödinger picture, $|\psi_0^N\rangle$ evolves in time according to $|\psi_0^N(t)\rangle = \text{exp}(-i\omega t)|\psi_0^N(0)\rangle$, where $\omega = 2\pi c/\lambda$. We can interprete (2) as a state of a quasiparticle with energy $N\hbar\omega$ and effective de-Broglie wavelength $\lambda_{\text{eff}} = \lambda/N$. This specific feature of $|\psi_0^N\rangle$ is the origin of the improvement of the resolution in quantum optical lithography.

For $N = 2$, the state (2) can be generated by feeding two-ports of a balanced beam splitter with single-photon Fock states, e.g., signal and idler photons generated by spontaneous parametric down-conversion. For $N > 2$, however, single-photon sources and linear optics are not sufficient for deterministic preparation of state (2), and Kerr or other nonlinear media are required. Unfortunately, sufficiently strong nonlinear interactions between single traveling photons are not currently available.

Nevertheless, one may avoid the necessity of nonlinear interactions. In a recent paper, Lee et al. showed that the states (2) can be prepared probabilistically using only Fock-state sources, linear optical elements and single-photon counting detectors. Lee et al. provided schemes for $N = 3$ and $N = 4$ but were not able to extend them to higher $N$. In the present paper we design scheme for conditional generation of arbitrary entangled $N$-photon states (3) for any $N$. We first present a generic scheme and then, as an application, we shall consider preparation of the state (3).

The quantum-state preparation schemes whose success is conditioned on the results of quantum measurements have attracted considerable amount of attention recently. Schemes for probabilistic preparation of Fock states, arbitrary superpositions of Fock states of single-mode field, and Schrödinger cat states, have been found. Experimental conditional preparation of the single-photon Fock state with negative Wigner function has been reported. In cavity QED, the state (3) can be generated by injection of a sequence of $N$ suitably prepared three-level A-type atoms into a two-mode resonator. In that scheme, one detects whether the atom leaving the resonator is in excited or ground state and the desired state (3) is prepared only if all atoms are in a ground state. Here, we design scheme for generation of two-mode entangled states of traveling light field.

For our purposes it is convenient to express the target state (3) in terms of bosonic creation operators $a^\dagger$ and $b^\dagger$ acting on two-mode vacuum state,

$$|\psi_N\rangle = \sum_{k=0}^{N} d_k a^{k} b^{N-k} |0, 0\rangle,$$  \hfill (3)

where $d_k = c_k/\sqrt{k!(N-k)!}$. The polynomial on the right-hand side of Eq. (3) can be factorized into a product of $N$ terms linear in creation operators,

$$|\psi_N\rangle = \frac{1}{\sqrt{N!}} \prod_{k=1}^{N} (\cos \theta_k a^\dagger - e^{i\phi_k} \sin \theta_k b^\dagger) |0, 0\rangle,$$  \hfill (4)

where $\mathcal{N}$ is a normalization factor and $z_k = e^{i\phi_k} \tan \theta_k$ are complex roots of the polynomial $\sum_{k=0}^{N} d_k z^k$. The factorization (4) suggests that we can prepare the state $|\psi_N\rangle$ from the vacuum state $|0, 0\rangle$ by applying $N$-times a non-
The setup under consideration is shown in Fig. 1. In addition to modes $a$ and $b$, which contain the state $|\psi\rangle$, we also need two auxiliary modes $c$ and $d$, initially in a single-photon Fock state $|1,0\rangle_{cd}$. With the help of the beam splitter $BS_1$ with transmittance $\cos \theta_k$ and a suitable phase shifter we transform this state to

$$|\varphi\rangle_{cd} = \cos \theta_k |1,0\rangle_{cd} - e^{i\phi_k} \sin \theta_k |0,1\rangle_{cd}. \tag{7}$$

The next step consists of mixing the mode $a$ with $c$ at a beam splitter $BS_2$ while the mode $b$ is mixed with $d$ at $BS_3$. The beam splitters $BS_2$ and $BS_3$ are identical. The corresponding unitary transformation describing the operation of $BS_2$ and $BS_3$ can be thus parametrized by a single real number $\kappa$.

$$U = \exp(\kappa a^\dagger c - \kappa c^\dagger a) \exp(\kappa b^\dagger d - \kappa d^\dagger b). \tag{8}$$

The photodetectors $PD_1$ and $PD_2$ measure number of photons in the output modes $c_{\text{out}}$ and $d_{\text{out}}$. In our scheme the operation (8) is successfully applied if and only if both detectors do not detect any photons. This means that $PD_1$ need not resolve between single and two-photon states, they should only distinguish vacuum state from any Fock state with nonzero number of photons. Efficient avalanche photodiodes are suitable for this purpose.

If no photons are detected in output modes $c_{\text{out}}$ and $d_{\text{out}}$, then the photon contained in the input state $|\varphi\rangle_{cd}$ has been added to modes $a$ or $b$. This intuitively explains the principle of operation of the scheme shown in Fig. 1. In order to provide a rigorous mathematical treatment, we rewrite the unitary transformation (8) in a disentangled form.

$$U = e^{-K a^\dagger c} e^{-K b^\dagger d} (\cos \kappa)^{n_{ab} - n_{cd}} e^{K a^\dagger c} e^{K b^\dagger d}, \tag{9}$$

where $K = \tan \kappa$ and $n_{cd} = c^\dagger c + d^\dagger d$. The conditionally generated state $|\psi_k\rangle$ in the output modes $c_{\text{out}}$ and $d_{\text{out}}$ can be obtained from the transformed input state $U|\psi_{k-1}\rangle_{ab} |\varphi_k\rangle_{cd}$ by applying a projection operator

$$\Pi = \mathbb{1}_{ab} \otimes |0,0\rangle_{cd} \langle 0,0|, \tag{10}$$

which describes the conditioning on no photons present in the modes $c_{\text{out}}$ and $d_{\text{out}}$. Here $\mathbb{1}_{ab}$ is an identity operator acting on Hilbert space of modes $a$ and $b$. Thus we can write

$$|\psi_k\rangle_{ab} |0,0\rangle_{cd} = \Pi U |\psi_{k-1}\rangle_{ab} |\varphi_k\rangle_{cd}. \tag{11}$$

We insert the factorized form (9) of the operator $U$ into Eq. (11) and make use of the vacuum stability condition $cd|0,0\rangle_{cd} = c_{\text{out}}|0,0\rangle_{cd}$ to simplify Eq. (11) as follows,

$$|\psi_k\rangle_{ab} |0,0\rangle_{cd} = \Pi \exp(K a^\dagger a + K b^\dagger b) |\psi_{k-1}\rangle_{ab} |\varphi_k\rangle_{cd}. \tag{12}$$

Now we expand the two exponentials in Taylor series. Since the state $|\varphi_k\rangle_{cd}$ contains only a single photon, we have to keep only terms up to linear in annihilation operators $c$ and $d$.

$$\exp(K a^\dagger a + K b^\dagger b) \to 1 + K a^\dagger a + K b^\dagger b. \tag{13}$$

Inserting (13) back into Eq. (12) and taking into account that $cd|0,0\rangle_{cd} = 0$ we obtain

$$|\psi_k\rangle_{ab} = q_k (\cos \theta_k a^\dagger - e^{i\phi_k} \sin \theta_k b^\dagger) |\psi_{k-1}\rangle_{ab}, \tag{14}$$

where

$$q_k = (\cos \kappa)^{k-1} \sin \kappa \tag{15}$$

and we have used that the state $|\psi_{k-1}\rangle$ is an eigenstate of the total photon number operator $n_{ab}$.

The desired $N$-photon entangled state $|\psi_N\rangle$ can be generated if we repeatedly $N$-times apply the basic transformation (8), as illustrated in Fig. 2. There are altogether $2N$ detectors. If none of them detects any photon, then the state $|\psi_N\rangle$ is generated at the output. We assume that $\kappa$ may be different for each basic building block, hence we have $3N$ parameters $\kappa_k$, $\theta_k$ and $\phi_k$ ($k = 1, \ldots, N$) characterizing the setup shown in Fig. 2. The unnormalized conditionally generated output state reads

$$|\psi_N\rangle = \prod_{k=1}^{N} q_k (\cos \theta_k a^\dagger - e^{i\phi_k} \sin \theta_k b^\dagger) |0,0\rangle. \tag{16}$$

The probability $P_N$ of generation of the state $|\psi_N\rangle$, i.e., the yield of our scheme, can be obtained as a norm of the output state (16),

$$P_N = N \prod_{k=1}^{N} q_k^2. \tag{17}$$
We can maximize the probability $P_N$ by maximizing independently each term $q_k^2$. It is convenient to introduce a transmittance $T_k = \sin^2 \theta_k$. Thus we have

$$q_k^2 = T_k (1 - T_k)^{k-1}$$

(18)

and the optimal $T_k$ maximizing $q_k^2$ reads

$$T_k = \frac{1}{k}.$$  (19)

Notice that the optimum beam-splitter transmittance does not depend on the state which we want to generate. On inserting $q_k^2 = (k-1)^{k-1}/k$ into Eq. (17) we obtain the optimum probability of generation

$$P_N = N N^{-N}.$$  (20)

The normalization factor $N$ has been introduced in Eq. (4).

As an example of application of our generic method, we shall consider generation of the entangled state $|\psi_2\rangle$. It is easy to see that this state may be written as follows,

$$|\psi_2\rangle = \frac{1}{\sqrt{2\sqrt{N}}} \prod_{k=1}^{N} (a^\dagger - e^{i\phi_k} b^\dagger) |0,0\rangle,$$  (21)

where $\phi_k = (2k+1)\pi/N$. Upon comparing Eqs. (21) and (4) we find that $\theta_k = \pi/4$. After some algebra one obtains the optimum probability of generation

$$P_N = (N - 1)! (2N)^{1-N}.$$  (22)

With the help of Stirling’s formula we find that for large $N$ we may approximate Eq. (22) as $P_N \approx 2\sqrt{2\pi N} (2e)^{-N}$. The yield decays exponentially with the number of photons $N$.

If $N$ is even, then we can simplify our scheme and reduce the number of necessary elements by a factor of two. We write the state (3) as follows,

$$|\psi_N^0\rangle = \frac{1}{\sqrt{2\sqrt{N}!}} \prod_{k=1}^{N/2} (a^{12} - e^{2i\phi_k} b^{12}) |0,0\rangle.$$  (23)

We can generate this state if we perform $N/2$-times the transformation

$$|\psi_k\rangle = (a^{12} - e^{2i\phi_k} b^{12}) |\psi_{k-1}\rangle,$$  (24)

which can be conditionally implemented with only a slight modification of the scheme shown in Fig. 1. Instead of the vacuum state $|0\rangle$, we send a single-photon Fock state $|1\rangle$ into the right input port of $BS_1$. After mixing on balanced beam-splitter $BS_1 [\theta = \pi/4]$ and passing through the phase-shifter, the state $|\varphi_k\rangle_{cd}$ of the modes $c$ and $d$ reads

$$|\varphi_k\rangle_{cd} = \frac{1}{\sqrt{2}} ((2,0)_{cd} - e^{2i\phi_k} |0,2\rangle_{cd}).$$  (25)

Similarly as before, we condition on detecting no photons in the output modes $c_{\text{out}}$ and $d_{\text{out}}$. In this way we add two photons to the modes $a$ or $b$ at each step. After $N/2$ steps we thus end up with $N$-photon entangled state.

The calculations of the conditionally generated output state closely follow those presented above. Since the state $|\varphi_k\rangle_{cd}$ now contains two photons, we must keep quadratic terms in the expansion (13),

$$e^{K_{ab} e^{K_{cd}} d} \rightarrow 1 + K (a^\dagger c + b^\dagger d) + K^2/2 (a^{12} e^{2i\phi_k} b^{12}) |\psi_{k-1}\rangle.$$  (27)

Assuming that the state $|\psi_k\rangle$ is an eigenstate of total number of photons, $n_{ab}|\psi_k\rangle = 2k|\psi_k\rangle$, we find that

$$|\psi_k\rangle = \frac{1}{2} (\cos \kappa)^{2k} \tan^2 \kappa (a^{12} - e^{2i\phi_k} b^{12}) |\psi_{k-1}\rangle.$$  (27)

The optimal transmittance of the k-th beam-splitter is again given by Eq. (14). The probability of generation of the state (3) (i.e., the yield) reads

$$P_N' = 2(N - 1)! N^{1-N}.$$  (28)

A comparison of the yields (25) and (22) immediately reveals that $P_N' = 2^{N} P_N$. The scheme where we add two photons in a single step is much more effective, because the number of necessary measurements is halved. To be specific, for $N = 4$ we have $P_4 = 3/256$ and $P_4' = 3/16$. Lee et al. [12] designed schemes for generation of the state $|4,0\rangle + |0,4\rangle/\sqrt{2}$ with yield 3/64 and our second method improves on this result by a factor of 4.
On the way towards experimental implementation of the scheme proposed in the present paper, two main obstacles have to be overcome. First, one needs a controlled source of single-photon Fock states. Currently available triggered single photon sources operate by means of fluorescence from a single molecule [28] or a single quantum dot [27, 29] and they exhibit very good performance. However, in our scheme, we need a synchronized arrival of $N$ single photons into $N$ input ports of $N$ beam splitters, which will be experimentally challenging.

A second obstacle stems from the less-than-unit efficiencies of the single-photon detectors. Imperfect detectors will degrade the output state which will be a mixed state described by some density matrix $\rho_{ab}$ [13]. However, in some applications, such as quantum lithography, this problem may be circumvented because the detectors are actually not necessary. If no conditioning is performed, then the mixed output state can be expressed as

$$
\rho_{ab} = P_N |\psi_N\rangle\langle\psi_N| + (1 - P_N)\rho_{ab},
$$

(29)

where the density matrix $\rho_{ab}$ represents the output state when one or more photons leak into the output auxiliary modes. This implies that the operator $\rho_{ab}$ is supported on Hilbert space of Fock states $|k, M-k\rangle$ with $M \leq N - 1$.

Now consider the quantum lithography. If the lithographic process is based on the $N$-photon absorption, then the absorption rate at the imaging surface will be proportional to the expectation value of normally ordered operator [11]

$$
\delta = \frac{e^{+N}e^{-N}}{N!},
$$

(30)

where $e = a + b$ is the effective positive-frequency field operator. It follows that the medium will respond only to the $N$-photon part of the output state $\rho_{ab}$ and we have

$$
\text{Tr} \rho_{ab} \delta = P_N \langle \psi_N | \delta | \psi_N \rangle,
$$

(31)

which is essentially the same result as for an ideal pure output state $|\psi_N\rangle\langle\psi_N|$. The rate is only reduced by factor $P_N$ representing the yield of our scheme.

In summary, we have designed a universal scheme for conditional generation of an arbitrary $N$-photon path-entangled quantum state of traveling light field. The necessary resources comprise single photon sources, beam splitters, phase shifters, and photodetectors with single-photon sensitivity. However, in certain applications, when one wishes to measure or utilize the $N$-photon coincidence rates, the conditioning is not necessary, because the desired $N$-photon part of the output state is selected automatically.

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