Criteria for Starlikeness Using Schwarzian Derivatives

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Dedicated to Professor Milutin Obradović

Abstract. For a normalised analytic function \( f \) defined on the open unit disk in the complex plane, we determine several sufficient conditions for starlikeness in terms of the quotients \( Q_{ST} := zf' (z)/f(z) \), \( Q_{CV} := 1 + zf''(z)/f'(z) \) and the Schwarzian derivative \( Q_{SD} := z^2( (f''(z)/f'(z))^2 - (f''(z)/f'(z))^2 / 2) \). These conditions were obtained by using the admissibility criteria of starlikeness in the theory of second order differential subordination.

1. Introduction

A function \( f : D := \{ z \in \mathbb{C} : |z| < 1 \} \rightarrow \mathbb{C} \) is starlike if \( tf(z) \in f(D) \) for all \( z \in D \) and \( t \in [0, 1] \). We shall restrict our functions to belong to the class \( A \) of all analytic functions \( f : D \rightarrow \mathbb{C} \) normalized by the condition \( f(0) = f'(0) - 1 = 0 \). Let \( S \subset A \) consists of univalent functions and \( S^* \subset A \) be the class of starlike functions. A function \( f \in A \) is convex if \( f(D) \) is convex and the class of all convex functions is denoted by \( K \). Analytically, starlike and convex functions are characterized by \( \text{Re} Q_{ST} > 0 \) and \( \text{Re} Q_{CV} > 0 \) where \( Q_{ST} := zf'(z)/f(z) \) and \( Q_{CV} := 1 + zf''(z)/f'(z) \). For \( 0 \leq \gamma < 1 \), the class \( S^*(\gamma) \) of starlike functions of order \( \gamma \) is defined by \( S^*(\gamma) := \{ f \in A : \text{Re} Q_{ST} > \gamma \} \) and the class \( K(\gamma) \) of convex functions of order \( \gamma \) is defined by \( K := \{ f \in A : \text{Re} Q_{CV} > \gamma \} \). The functions in the classes \( S^* \) and \( K \) are univalent. A well-known univalence criteria of Nehari involves the Schwarzian derivative of function \( f \in A \) defined by \( \{ f, z \} := ( (f''(z)/f'(z))^2 - (f''(z)/f'(z))^2 / 2) \) and \( Q_{SD} := z^2\{ f, z \} \). Nehari [13, 14] studied necessary and sufficient conditions relating Schwarzian derivatives to univalency of functions \( f \in A \). Schwarzian derivatives were studied by several authors (see [5, 6]). Sharma et al. [20] discussed sufficient conditions for strong starlike functions and Cho et al. [3] studied higher order Schwarzian derivatives for Janowski classes.

Obradović [16] has shown that the condition \( |f''(z)| < 1 \) implies starlikeness of \( f \in A \) and the condition \( |f''(z)| < 1/2 \) implies convexity. These simple conditions were further studied, among others, by Tuneski [24, 26, 27] and Kown and Sim [7]. Our interest is to provide such simple sufficient conditions for starlikeness using \( Q_{ST}, Q_{CV} \) and \( Q_{SD} \). Our

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main tool in getting these result is the general theory of differential subordination introduced by Miller and Mocanu \[11\]. Miller and Mocanu \[12\, pp.244\] discussed on admissibility conditions related to the starlikeness and convexity; they proved that if \( f \in A \), with \( f(z)f'(z)/z \neq 0 \), \( \Re \psi(Q_{ST},Q_{CV},Q_{SD}) > 0 \) then the function \( f \) is starlike, provided the function \( \psi : C^3 \to C \) satisfy \( \Re \psi(i\rho,i\tau,\xi+i\eta) \leq 0 \), whenever \( \rho, \tau, \xi, \eta \in \mathbb{R} \), \( \rho \tau \geq (1+3\rho^2)/2 \) and \( \rho \eta \geq 0 \). Several authors (see, for example, \[4, 8, 9, 15, 18, 22, 29\]) applied this theory to investigate criteria for the functions to be starlike or convex. Ravichandran et al. \[19\] proved that if \( f \in A \) satisfies \( \Re Q_{ST}Q_{CV} > (\gamma+1)(\gamma-1/2) \), then the function \( f \) is starlike of order \( \gamma \). Motivated essentially by these works, we have a systematic discussion on various criteria involving \( Q_{ST}, Q_{CV} \) and \( Q_{SD} \) for the starlikeness of functions in class \( A \).

For a given set \( \Omega \subset C \), the class \( \Psi(\Omega) \) of admissible functions consists of functions \( \psi : C^3 \times D \to C \) satisfying the admissibility condition
\[
\psi(i\rho,i\tau,\xi+i\eta) \notin \Omega \quad (1.1)
\]
for \( z \in D \), and for all real \( \rho, \tau, \xi, \eta \) with
\[
\rho \tau \geq \frac{1}{2}(1+3\rho^2), \quad \rho \eta \geq 0. \quad (1.2)
\]
Our theorems are proved by making use of the following extension of the criteria of Miller and Mocanu for the starlikeness of functions \( f \in A \) given in terms of the Schwarzian derivatives:

**Theorem 1.1.** \[1\, p. 9\] If \( f \in A \) with \( f(z)f'(z)/z \neq 0 \) satisfies
\[
\psi(Q_{ST},Q_{CV},Q_{SD}) \in \Omega
\]
for some \( \psi \in \Psi(\Omega) \), then the function \( f \) is starlike.

**2. Criteria for starlikeness**

Lewandowski et al. \[8\] discussed the criterion for starlikeness of a function \( f \in A \). Many authors have developed sufficient conditions for starlikeness and convexity of functions. See \[4, 9, 15, 18, 22, 29\]. In this section, we derive results relating the Schwarzian derivatives and starlikeness of functions in the class \( A \).

**Theorem 2.1.** Let \( \alpha \geq 0 \) and \( \beta \geq 0 \). If the function \( f \in A \) satisfy any of the following inequalities
\[
(i) \quad \Re (Q_{ST}(\alpha Q_{CV} + \beta Q_{SD})) > -\alpha/2,
(ii) \quad \Re (Q_{CV}(\alpha Q_{ST} + \beta Q_{SD})) > -\alpha/2,
(iii) \quad \Re (Q_{ST}(\alpha Q_{ST} + \beta Q_{SD})) > 0,
(iv) \quad \Re (Q_{CV}(\alpha Q_{CV} + \beta Q_{SD})) > 0,
\]
then the function \( f \) is starlike.

**Proof.** For \( i = 1, 2, 3, 4 \), let \( \Omega_i \) be defined by \( \Omega_1 := \{ w \in C : \Re w > -\alpha/2 \} =: \Omega_2 \) and \( \Omega_3 := \{ w \in C : \Re w > 0 \} =: \Omega_4 \) and the functions \( \psi_i : C^3 \to C \) be defined by
\[
\psi_1(u,v,w) = u(\alpha v + \beta w),
\psi_2(u,v,w) = v(\alpha u + \beta w),
\psi_3(u,v,w) = u(\alpha u + \beta w)
\]
and

\[ \psi_4(v, w) = v(\alpha u + \beta w). \]

The hypothesis of the theorem shows that the function \( \psi_i \) satisfies

\[ \psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega_i \quad \text{for} \quad i = 1, 2, 3, 4. \]

It then follows from Theorem 1.1 that the function \( f \) is starlike provided \( \psi_i \in \Psi(\Omega_i) \). We complete the proof by showing that the function \( \psi_i \in \Psi(\Omega_i) \).

Let \( \alpha \geq 0 \) and \( \beta \geq 0 \) and \( \rho, \tau, \xi, \eta \in \mathbb{R} \) satisfy the conditions \( \rho \tau \geq (1 + 3\rho^2)/2 \) and \( \rho \eta \geq 0 \). Then, we have

\[ \text{Re} \psi_1(i\rho, i\tau, \xi + i\eta) = -\alpha \rho \tau - \beta \rho \eta \leq -\alpha (1 + 3\rho^2)/2 \leq -\alpha/2, \]

and this proves that the function \( \psi_1 \in \Psi(\Omega_1) \). For the function \( \psi_2 \), we have

\[ \text{Re} \psi_2(i\rho, i\tau, \xi + i\eta) = -\alpha \rho \tau - \beta \tau \eta \leq -\alpha (1 + 3\rho^2)/2 \leq -\alpha/2, \]

and so the function \( \psi_2 \in \Psi(\Omega_2) \). Similarly, we have

\[ \text{Re} \psi_3(i\rho, i\tau, \xi + i\eta) = -\alpha \rho^2 - \beta \rho \eta \leq 0, \]

and

\[ \text{Re} \psi_4(i\rho, i\tau, \xi + i\eta) = -\alpha \tau^2 - \beta \tau \eta \leq 0, \]

so that the functions \( \psi_3 \in \Psi(\Omega_3) \) and \( \psi_4 \in \Psi(\Omega_4) \).

**Remark 2.2.** Theorem 2.1 (i) with \( \alpha = 1 \), \( \beta = 0 \) reduces to a sufficient condition for starlikeness obtained by Ravichandran et al. [19].

Various authors [2, 10, 11] have investigated on expressions involving the product of the terms \( Q_{ST} \) and \( Q_{CV} \) for the study of starlikeness of functions. The following theorems discuss the influence of Schwarzian derivatives in many such cases.

**Theorem 2.3.** Let \( \alpha \geq 0 \) and \( \beta \geq 0 \). If the function \( f \in A \) satisfy any of the following inequalities

(i) \( \text{Re} \left( Q_{CV}(\alpha Q_{ST} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD}) \right) > -\alpha/2, \)
(ii) \( \text{Re} \left( Q_{ST}(\alpha Q_{CV} + (1 - \alpha)Q_{CV}^2 + \beta Q_{SD}) \right) > -\alpha/2, \)
(iii) \( \text{Re} \left( Q_{ST}(\alpha Q_{CV} + (1 - \alpha)Q_{CV}^2 + \beta Q_{SD}) \right) > -\alpha/2, \)
(iv) \( \text{Re} \left( Q_{CV}(\alpha Q_{ST} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD}) \right) > -\alpha/2, \)

then the function \( f \) is starlike.

**Proof.** Let \( \Omega \) be defined by \( \Omega := \{ w \in \mathbb{C} : \text{Re} w > -\alpha/2 \} \) and for \( i = 1, 2, 3, 4 \), let the functions \( \psi_i : \mathbb{C}^3 \rightarrow \mathbb{C} \) be defined by

\[ \psi_1(u, v, w) = v(\alpha u + (1 - \alpha)u^2 + \beta w), \]
\[ \psi_2(u, v, w) = u(\alpha v + (1 - \alpha)v^2 + \beta w), \]
\[ \psi_3(u, v, w) = u(\alpha v + (1 - \alpha)u^2 + \beta w) \]

and

\[ \psi_4(v, w) = v(\alpha u + (1 - \alpha)v^2 + \beta w). \]
The hypothesis of the theorem shows that the function \( \psi_i \) satisfies

\[ \psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \quad i = 1, 2, 3, 4. \]

It then follows from Theorem 1.1 that the function \( f \) is starlike provided \( \psi_i \in \Psi(\Omega) \). We complete the proof by showing that the function \( \psi_i \in \Psi(\Omega) \).

Let \( \alpha \geq 0 \) and \( \beta \geq 0 \) and \( \rho, \tau, \xi, \eta \in \mathbb{R} \) satisfy the conditions \( \rho \tau \geq (1 + 3\rho^2)/2 \) and \( \rho \eta \geq 0 \). Then, we have

\[ \Re \psi_1(i\rho, i\tau, \xi + i\eta) = -\alpha \rho \tau - \beta \tau \xi \leq -\frac{\alpha}{2}(1 + 3\rho^2) \leq -\frac{\alpha}{2}, \]

and this proves that the function \( \psi_1 \in \Psi(\Omega) \). For the function \( \psi_2 \), we have

\[ \Re \psi_2(i\rho, i\tau, \xi + i\eta) = -\alpha \rho \tau - \beta \rho \eta \leq -\frac{\alpha}{2}(1 + 3\rho^2) \leq -\frac{\alpha}{2}, \]

and so the function \( \psi_2 \in \Psi(\Omega_2) \). Similarly, we have

\[ \Re \psi_3(i\rho, i\tau, \xi + i\eta) = -\alpha \rho \tau - \beta \tau \xi \leq -\frac{\alpha}{2}(1 + 3\rho^2) \leq -\frac{\alpha}{2}, \]

and

\[ \Re \psi_4(i\rho, i\tau, \xi + i\eta) = -\alpha \rho \tau - \beta \tau \xi \leq -\frac{\alpha}{2}(1 + 3\rho^2) \leq -\frac{\alpha}{2}, \]

so that the functions \( \psi_3 \in \Psi(\Omega) \) and \( \psi_4 \in \Psi(\Omega) \).

**Theorem 2.4.** Let \( \alpha \geq 0 \) and \( \beta \geq 0 \). If the function \( f \in A \) satisfy any of the following inequalities

\[
(i) \quad \Re \left( Q_{ST}(\alpha Q_{ST} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD}) \right) > 0 \\
(ii) \quad \Re \left( Q_{CV}(\alpha Q_{CV} + (1 - \alpha)Q_{CV}^2 + \beta Q_{SD}) \right) > 0, \\
(iii) \quad \Re \left( Q_{ST}(\alpha Q_{ST} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD}) \right) > 0, \\
(iv) \quad \Re \left( Q_{CV}(\alpha Q_{CV} + (1 - \alpha)Q_{CV}^2 + \beta Q_{SD}) \right) > 0,
\]

then the function \( f \) is starlike.

**Proof.** Let \( \Omega \) be defined by \( \Omega := \{ w \in \mathbb{C} : \Re w > 0 \} \) and for \( i = 1, 2, 3, 4 \), let the functions \( \psi_i : \mathbb{C}^3 \rightarrow \mathbb{C} \) be defined by

\[ \psi_1(u, v, w) = u(\alpha u + (1 - \alpha)u^2 + \beta w), \]
\[ \psi_2(u, v, w) = v(\alpha v + (1 - \alpha)v^2 + \beta w), \]
\[ \psi_3(u, v, w) = u(\alpha u + (1 - \alpha)v^2 + \beta w) \]

and

\[ \psi_4(u, v) = v(\alpha v + (1 - \alpha)u^2 + \beta w). \]

The hypothesis of the theorem shows that the function \( \psi_i \) satisfies

\[ \psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \quad i = 1, 2, 3, 4. \]

It then follows from Theorem 1.1 that the function \( f \) is starlike provided \( \psi_i \in \Psi(\Omega) \). We complete the proof by showing that the function \( \psi_i \in \Psi(\Omega) \).

Let \( \alpha \geq 0 \) and \( \beta \geq 0 \) and \( \rho, \tau, \xi, \eta \in \mathbb{R} \) satisfy the conditions \( \rho \tau \geq (1 + 3\rho^2)/2 \) and \( \rho \eta \geq 0 \). Then, we have

\[ \Re \psi_1(i\rho, i\tau, \xi + i\eta) = -\alpha \rho^2 - \beta \rho \eta \leq 0, \]
and this proves that the function \( \psi_1 \in \Psi(\Omega) \). For the function \( \psi_2 \), we have
\[
\Re \psi_2(i\rho, i\tau, \xi + i\eta) = -\alpha \tau^2 - \beta \tau \eta \leq 0,
\]
and so the function \( \psi_2 \in \Psi(\Omega_2) \). Similarly, we have
\[
\Re \psi_3(i\rho, i\tau, \xi + i\eta) = -\alpha \rho^2 - \beta \rho \eta \leq 0,
\]
and
\[
\Re \psi_4(i\rho, i\tau, \xi + i\eta) = -\alpha \tau^2 - \beta \tau \eta \leq 0,
\]
so that the functions \( \psi_3 \in \Psi(\Omega) \) and \( \psi_4 \in \Psi(\Omega) \).

**Remark 2.5.** For \( \alpha = 1 \) and \( \beta = 1 \) in Part (iv) of Theorem 2.4, the obtained result is same as the one discussed by Miller and Mocanu [12], pp.247 for the expression \( \Re \left( Q_{CV}^2 + Q_{SD} \right) > 0 \).

Using the theory of differential subordination, Owa and Obradović [17] proved that the function \( f \in A \) is starlike, if \( \Re \left( (Q_{ST}^2/2) + Q_{CV} \right) > 0 \). In a generalised manner, we prove for some other cases as well.

**Theorem 2.6.** Let \( \alpha > 0 \) and \( \beta \geq 0 \). If the function \( f \in A \) satisfy any of the following inequalities
\[
(i) \ Re \left( \beta Q_{SD} Q_{ST} + \alpha (1 + Q_{CV})^2 \right) > \alpha,
(ii) \ Re \left( \beta Q_{SD} Q_{ST} + \alpha (1 + Q_{ST})^2 \right) > \alpha,
(iii) \ Re \left( \beta Q_{SD} Q_{CV} + \alpha (1 + Q_{CV})^2 \right) > \alpha,
(iv) \ Re \left( \beta Q_{SD} Q_{CV} + \alpha (1 + Q_{ST})^2 \right) > \alpha,
\]
then the function \( f \) is starlike.

**Proof.** Let \( \Omega \) be defined by \( \Omega := \{ w \in \mathbb{C} : \Re w > \alpha \} \) and for \( i = 1, 2, 3, 4 \), let the functions \( \psi_i : \mathbb{C}^3 \to \mathbb{C} \) be defined by
\[
\psi_1(u, v, w) = \beta uv + \alpha (1 + v)^2, \quad \psi_2(u, v, w) = \beta uw + \alpha (1 + u)^2,
\]
\[
\psi_3(u, v, w) = \beta vw + \alpha (1 + v)^2 \quad \text{and} \quad \psi_4(v, w) = \beta vw + \alpha (1 + u)^2.
\]
The hypothesis of the theorem shows that the function \( \psi_i \) satisfies
\[
\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for } i = 1, 2, 3, 4.
\]
It then follows from Theorem 1.1 that the function \( f \) is starlike provided \( \psi_i \in \Psi(\Omega) \). We complete the proof by showing that the function \( \psi_i \in \Psi(\Omega) \).

Let \( \alpha \geq 0 \) and \( \beta \geq 0 \) and \( \rho, \tau, \xi, \eta \in \mathbb{R} \) satisfy the conditions \( \rho \tau \geq (1 + 3\rho^2)/2 \) and \( \rho \eta \geq 0 \). Then, we have
\[
\Re \psi_1(i\rho, i\tau, \xi + i\eta) = -\beta \rho \eta - \alpha \tau^2 + \alpha \leq \alpha,
\]
and this proves that the function \( \psi_1 \in \Psi(\Omega) \). For the function \( \psi_2 \), we have
\[
\Re \psi_2(i\rho, i\tau, \xi + i\eta) = -\alpha \rho^2 - \beta \rho \eta + \alpha \leq \alpha,
\]
and so the function \( \psi_2 \in \Psi(\Omega_2) \). Similarly, we have
\[
\Re \psi_3(i\rho, i\tau, \xi + i\eta) = -\alpha \tau^2 - \beta \tau \eta + \alpha \leq \alpha,
\]
and
\[
\Re \psi_4(i\rho, i\tau, \xi + i\eta) = -\alpha \rho^2 - \beta \tau \eta + \alpha \leq \alpha,
\]
so that the functions \( \psi_3 \in \Psi(\Omega) \) and \( \psi_4 \in \Psi(\Omega) \).

Miller and Mocanu \( \text{[12]} \) determined sufficient conditions relating the starlikeness of functions in class \( \mathcal{A} \) and Schwarzian derivatives. As an application to the discussion, they obtained that for parameters \( \alpha, \beta \), the sufficient conditions \( \text{Re} \left( \alpha Q_{ST} + \beta Q_{CV} + Q_{ST} Q_{SD} \right) > 0 \) and \( \text{Re} \left( Q_{ST} Q_{CV} + Q_{ST} Q_{SD} \right) > -1/2 \) imply starlikeness. The forthcoming theorems follow as a generalisation of above observations.

**Theorem 2.7.** Let \( \alpha > 0 \) and \( \beta \geq 0 \). If the function \( f \in \mathcal{A} \) satisfy any of the following inequalities

(i) \( \text{Re} \left( \beta Q_{SD} Q_{ST} + \alpha Q_{CV} (1 + Q_{ST}) \right) > -\alpha/2 \),

(ii) \( \text{Re} \left( \beta Q_{SD} Q_{CV} + \alpha Q_{ST} (1 + Q_{CV}) \right) > -\alpha/2 \),

(iii) \( \text{Re} \left( \beta Q_{SD} Q_{CV} + \alpha Q_{CV} (1 + Q_{ST}) \right) > -\alpha/2 \),

(iv) \( \text{Re} \left( \beta Q_{SD} Q_{ST} + \alpha Q_{ST} (1 + Q_{CV}) \right) > -\alpha/2 \),

then the function \( f \) is starlike.

**Proof.** Let \( \Omega \) be defined by \( \Omega := \{ w \in \mathbb{C} : \text{Re} w > -\alpha/2 \} \) and for \( i = 1, 2, 3, 4 \), let the functions \( \psi_i : \mathbb{C}^3 \to \mathbb{C} \) be defined by

\[
\psi_1(u, v, w) = \beta uw + \alpha v(1 + u), \quad \psi_2(u, v, w) = \beta vw + \alpha u(1 + v),
\]

\[
\psi_3(u, v, w) = \beta vw + \alpha v(1 + u) \quad \text{and} \quad \psi_4(v, w) = \beta uw + \alpha u(1 + v).
\]

The hypothesis of the theorem shows that the function \( \psi_i \) satisfies

\[
\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \quad i = 1, 2, 3, 4.
\]

It then follows from Theorem \( \text{[13]} \) that the function \( f \) is starlike provided \( \psi_i \in \Psi(\Omega) \). We complete the proof by showing that the function \( \psi_i \in \Psi(\Omega) \).

Let \( \alpha \geq 0 \) and \( \beta \geq 0 \) and \( \rho, \tau, \xi, \eta \in \mathbb{R} \) satisfy the conditions \( \rho \tau \geq (1 + 3\rho^2)/2 \) and \( \rho \eta \geq 0 \). Then, we have

\[
\text{Re} \psi_1(i \rho, i \tau, \xi + i \eta) = -\beta \rho \eta - \alpha \rho \tau \leq -\alpha/2 (1 + 3\rho^2) \leq -\alpha/2,
\]

and this proves that the function \( \psi_1 \in \Psi(\Omega) \). For the function \( \psi_2 \), we have

\[
\text{Re} \psi_2(i \rho, i \tau, \xi + i \eta) = -\alpha \rho \tau - \beta \tau \eta \leq -\alpha/2 (1 + 3\rho^2) \leq -\alpha/2,
\]

and so the function \( \psi_2 \in \Psi(\Omega_2) \). Similarly, we have

\[
\text{Re} \psi_3(i \rho, i \tau, \xi + i \eta) = -\alpha \rho \tau - \beta \tau \eta \leq -\alpha/2 (1 + 3\rho^2) \leq -\alpha/2,
\]

and

\[
\text{Re} \psi_4(i \rho, i \tau, \xi + i \eta) = -\alpha \rho \tau - \beta \rho \eta \leq -\alpha/2 (1 + 3\rho^2) \leq -\alpha/2,
\]

so that the functions \( \psi_3 \in \Psi(\Omega) \) and \( \psi_4 \in \Psi(\Omega) \).

**Theorem 2.8.** Let \( \alpha > 0 \) and \( \beta \geq 0 \). If the function \( f \in \mathcal{A} \) satisfy any of the following inequalities

(i) \( \text{Re} \left( \beta Q_{SD} Q_{ST} + \alpha Q_{ST} (1 + Q_{ST}) \right) > 0 \),

(ii) \( \text{Re} \left( \beta Q_{SD} Q_{CV} + \alpha Q_{ST} (1 + Q_{ST}) \right) > 0 \),

(iii) \( \text{Re} \left( \beta Q_{SD} Q_{ST} + \alpha Q_{CV} (1 + Q_{CV}) \right) > 0 \),
(iv) \( \Re (\beta Q_{SD} Q_{CV} + \alpha Q_{CV} (1 + Q_{CV})) > 0 \),
then the function \( f \) is starlike.

**Proof.** Let \( \Omega \) be defined by \( \Omega := \{w \in \mathbb{C} : \Re w > 0\} \) and for \( i = 1, 2, 3, 4 \), let the functions \( \psi_i : \mathbb{C}^3 \rightarrow \mathbb{C} \) be defined by
\[
\psi_1(u,v,w) = \beta uv + \alpha u(1 + u), \quad \psi_2(u,v,w) = \beta vw + \alpha u(1 + u), \\
\psi_3(u,v,w) = \beta uv + \alpha v(1 + v) \quad \text{and} \quad \psi_4(v,w) = \beta vw + \alpha v(1 + v).
\]
The hypothesis of the theorem shows that the function \( \psi_i \) satisfies
\[
\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \quad i = 1, 2, 3, 4.
\]
It then follows from Theorem 1.1 that the function \( f \) is starlike provided \( \psi_i \in \Psi(\Omega) \). We complete the proof by showing that the function \( \psi_i \in \Psi(\Omega) \).

Let \( \alpha > 0 \) and \( \beta \geq 0 \) and \( \rho, \tau, \xi, \eta \in \mathbb{R} \) satisfy the conditions \( \rho \tau \geq (1 + 3 \rho^2)/2 \) and \( \rho \eta \geq 0 \). Then, we have
\[
\Re \psi_1(i \rho, i \tau, \xi + i \eta) = -\alpha \rho^2 - \beta \rho \eta \leq 0,
\]
and this proves that the function \( \psi_1 \in \Psi(\Omega) \). For the function \( \psi_2 \), we have
\[
\Re \psi_2(i \rho, i \tau, \xi + i \eta) = -\alpha \rho^2 - \beta \tau \eta \leq 0,
\]
and so the function \( \psi_2 \in \Psi(\Omega_2) \). Similarly, we have
\[
\Re \psi_3(i \rho, i \tau, \xi + i \eta) = -\alpha \tau^2 - \beta \rho \eta \leq 0,
\]
and
\[
\Re \psi_4(i \rho, i \tau, \xi + i \eta) = -\alpha \tau^2 - \beta \tau \eta \leq 0,
\]
so that the functions \( \psi_3 \in \Psi(\Omega) \) and \( \psi_4 \in \Psi(\Omega) \).

Some authors \[2, 17\] considered the powers of the expressions \( Q_{ST}, Q_{CV} \) and analysed their significance in the starlikeness of a function. The following theorems with Schwarzian derivatives are examined in a similar manner.

**Theorem 2.9.** Let \( 0 \leq \alpha \leq 1 \) and \( \beta \geq 0 \). If the function \( f \in \mathcal{A} \) satisfy any of the following inequalities
\[
(i) \Re (\alpha Q_{ST} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD} Q_{CV}) > 0, \\
(ii) \Re (\alpha Q_{ST} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD} Q_{ST}) > 0, \\
(iii) \Re (\alpha Q_{ST} + (1 - \alpha)Q_{CV}^2 + \beta Q_{SD} Q_{ST}) > 0, \\
(iv) \Re (\alpha Q_{CV} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD} Q_{CV}) > 0, \\
(v) \Re (\alpha Q_{CV} + (1 - \alpha)Q_{CV}^2 + \beta Q_{SD} Q_{CV}) > 0, \\
(vi) \Re (\alpha Q_{CV} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD} Q_{ST}) > 0, \\
(vii) \Re (\alpha Q_{ST} + (1 - \alpha)Q_{CV}^2 + \beta Q_{SD} Q_{CV}) > 0, \\
(viii) \Re (\alpha Q_{CV} + (1 - \alpha)Q_{ST}^2 + \beta Q_{SD} Q_{ST}) > 0,
\]
then the function \( f \) is starlike.

**Proof.** Let \( \Omega \) be defined by \( \Omega := \{w \in \mathbb{C} : \Re w > 0\} \) and for \( i = 1, 2, \cdots, 8 \), let the functions \( \psi_i : \mathbb{C}^3 \rightarrow \mathbb{C} \) be defined by
\[
\psi_1(u,v,w) = \alpha u + (1 - \alpha)u^2 + \beta v w,
\]
\[
\psi_2(u,v,w) = \alpha u + (1 - \alpha)u^2 + \beta uw,
\]
\[
\psi_3(u,v,w) = \alpha u + (1 - \alpha)v^2 + \beta uw,
\]
\[
\psi_4(u,v,w) = \alpha v + (1 - \alpha)u^2 + \beta uw,
\]
\[
\psi_5(u,v,w) = \alpha v + (1 - \alpha)v^2 + \beta vw,
\]
\[
\psi_6(u,v,w) = \alpha v + (1 - \alpha)u^2 + \beta vw,
\]
\[
\psi_7(u,v,w) = \alpha u + (1 - \alpha)v^2 + \beta vw
\]

and
\[
\psi_8(v,w) = \alpha v + (1 - \alpha)v^2 + \beta uw.
\]

The hypothesis of the theorem shows that the function \(\psi_i\) satisfies
\[
\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \quad i = 1, 2, \cdots, 8.
\]

It then follows from Theorem 1.1 that the function \(f\) is starlike provided \(\psi_i \in \Psi(\Omega)\). We complete the proof by showing that the function \(\psi_i \in \Psi(\Omega)\).

Let \(0 \leq \alpha \leq 1\) and \(\beta \geq 0\) and \(\rho, \tau, \xi, \eta \in \mathbb{R}\) satisfy the conditions \(\rho \tau \geq (1 + 3\rho^2)/2\) and \(\rho \eta \geq 0\). Then, we have
\[
\text{Re} \psi_1(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\rho^2 - \beta \tau \eta \leq 0,
\]
and this proves that the function \(\psi_1 \in \Psi(\Omega)\). For the function \(\psi_2\), we have
\[
\text{Re} \psi_2(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\rho^2 - \beta \rho \eta \leq 0,
\]
and so the function \(\psi_2 \in \Psi(\Omega_2)\). Similarly, we have
\[
\text{Re} \psi_3(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\tau^2 - \beta \rho \eta \leq 0,
\]
and
\[
\text{Re} \psi_4(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\rho^2 - \beta \rho \eta \leq 0,
\]
so that the functions \(\psi_3 \in \Psi(\Omega)\) and \(\psi_4 \in \Psi(\Omega)\). Proceeding in a similar way, we have
\[
\text{Re} \psi_5(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\tau^2 - \beta \tau \eta \leq 0,
\]
and this proves that the function \(\psi_5 \in \Psi(\Omega)\). For the function \(\psi_6\), we have
\[
\text{Re} \psi_6(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\rho^2 - \beta \tau \eta \leq 0,
\]
and so the function \(\psi_6 \in \Psi(\Omega_2)\). Similarly, we have
\[
\text{Re} \psi_7(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\tau^2 - \beta \tau \eta \leq 0,
\]
and
\[
\text{Re} \psi_8(i\rho, i\tau, \xi + i\eta) = -(1 - \alpha)\tau^2 - \beta \rho \eta \leq 0,
\]
so that the functions \(\psi_7 \in \Psi(\Omega)\) and \(\psi_8 \in \Psi(\Omega)\).

**Theorem 2.10.** Let \(0 \leq \alpha \leq 1\) and \(\beta \geq 0\). If any of the following two inequalities hold for the function \(f \in \mathcal{A}\),

(i) \(\text{Re}(Q_{CV}(\alpha Q_{ST} + (1 - \alpha)Q_{CV} + \beta Q_{SD})) > -\alpha/2,\)

(ii) \(\text{Re}(Q_{ST}(\alpha Q_{CV} + (1 - \alpha)Q_{ST} + \beta Q_{SD})) > -\alpha/2,\)

then the function \(f\) is starlike.
Proof. Let \(\Omega\) be defined by \(\Omega := \{ w \in \mathbb{C} : \text{Re}\,w > -\alpha/2 \}\) and for \(i = 1, 2\), let the functions \(\psi_i : \mathbb{C}^3 \to \mathbb{C}\) be defined by

\[
\psi_1(u, v, w) = u(\alpha u + (1 - \alpha)v + \beta w)
\]

and

\[
\psi_2(u, v, w) = u(\alpha v + (1 - \alpha)u + \beta w).
\]

The hypothesis of the theorem shows that the function \(\psi_i\) satisfies

\[
\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \quad i = 1, 2.
\]

It then follows from Theorem 1.1 that the function \(f\) is starlike provided \(\psi_i \in \Psi(\Omega)\). We complete the proof by showing that the function \(\psi_i \in \Psi(\Omega)\).

Let \(0 \leq \alpha \leq 1\) and \(\beta \geq 0\) and \(\rho, \tau, \xi, \eta \in \mathbb{R}\) satisfy the conditions \(\rho \tau \geq (1 + 3\rho^2)/2\) and \(\rho \eta \geq 0\). Then, we have

\[
\text{Re} \psi_1(i \rho, i \tau, \xi + i \eta) = -\alpha \rho \tau - (1 - \alpha)\tau^2 - \beta \tau \eta \leq -\frac{\alpha}{2}(1 + 3\rho^2) \leq -\frac{\alpha}{2},
\]

and this proves that the function \(\psi_1 \in \Psi(\Omega)\). For the function \(\psi_2\), we have

\[
\text{Re} \psi_2(i \rho, i \tau, \xi + i \eta) = -(1 - \alpha)\rho^2 - \alpha \rho \tau - \beta \tau \eta \leq -\frac{\alpha}{2}(1 + 3\rho^2) \leq -\frac{\alpha}{2},
\]

and so the function \(\psi_2 \in \Psi(\Omega)\).

Remark 2.11. Let \(\alpha = 1\) and \(\beta = 0\), then the results obtained from both Part(i) and Part(ii) of Theorem 2.10 is same as the sufficient condition for starlikeness obtained by Ramesha and Padmanabhan [18].

Theorem 2.12. Let \(\alpha > 0\) and \(\beta \geq 0\). If the function \(f \in A\) satisfy any of the following inequalities

(i) \(\text{Re} \left(\alpha (Q_{CV}/Q_{ST}) + \beta (Q_{SD}/Q_{ST})\right) < 3\alpha/2\)

(ii) \(\text{Re} \left(\alpha (Q_{CV}/Q_{ST}) + \beta (Q_{SD}/Q_{CV})\right) < 3\alpha/2\)

then the function \(f\) is starlike.

Proof. Let \(\Omega\) be defined by \(\Omega := \{ w \in \mathbb{C} : \text{Re}\,w < 3\alpha/2 \}\) and for \(i = 1, 2\), let the functions \(\psi_i : \mathbb{C}^3 \to \mathbb{C}\) be defined by

\[
\psi_1(u, v, w) = \alpha(v/u) + \beta(w/u) \quad \text{and} \quad \psi_2(u, v, w) = \alpha(v/u) + \beta(w/v).
\]

The hypothesis of the theorem shows that the function \(\psi_i\) satisfies

\[
\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \quad i = 1, 2.
\]

It then follows from Theorem 1.1 that the function \(f\) is starlike provided \(\psi_i \in \Psi(\Omega)\). We complete the proof by showing that the function \(\psi_i \in \Psi(\Omega)\).

Let \(\alpha > 0\) and \(\beta \geq 0\) and \(\rho, \tau, \xi, \eta \in \mathbb{R}\) satisfy the conditions \(\rho \tau \geq (1 + 3\rho^2)/2\) and \(\rho \eta \geq 0\). Then, we have

\[
\text{Re} \psi_1(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \tau}{\rho} + \frac{\beta \eta}{\rho} \geq \frac{\alpha (1 + 3\rho^2)}{2\rho^2} \geq \frac{3\alpha}{2},
\]

and this proves that the function \(\psi_1 \in \Psi(\Omega)\). For the function \(\psi_2\), we have

\[
\text{Re} \psi_2(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \tau}{\tau} + \frac{\beta \eta}{\tau} \geq \frac{\alpha (1 + 3\rho^2)}{2\rho^2} \geq \frac{3\alpha}{2}.
\]
and so the function $\psi_2 \in \Psi(\Omega_2)$.

**Remark 2.13.** The sufficient conditions for starlikeness obtained by Tuneski [25, pp.523] follows from Part(i) and Part(ii) of Theorem 2.12 when $\alpha = 1$ and $\beta = 0$.

**Theorem 2.14.** Let $\alpha > 0$ and $\beta \leq 0$. If the function $f \in A$ satisfy any of the following inequalities

(i) $\Re\left(\alpha(Q_{ST}/Q_{CV}) + \beta(Q_{SD}/Q_{CV})\right) > 2\alpha/3$,

(ii) $\Re\left(\alpha(Q_{ST}/Q_{CV}) + \beta(Q_{SD}/Q_{ST})\right) > 2\alpha/3$,

then the function $f$ is starlike.

**Proof.** Let $\Omega$ be defined by $\Omega := \{w \in \mathbb{C} : \Re w > 2\alpha/3\}$ and for $i = 1, 2$, let the functions $\psi_i : \mathbb{C}^3 \to \mathbb{C}$ be defined by

$\psi_1(u, v, w) = \alpha(u/v) + \beta(w/v)$ and $\psi_2(u, v, w) = \alpha(u/v) + \beta(w/u)$.

The hypothesis of the theorem shows that the function $\psi_i$ satisfies

$\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega$ for $i = 1, 2$.

It then follows from Theorem 1.1 that the function $f$ is starlike provided $\psi_i \in \Psi(\Omega)$. We complete the proof by showing that the function $\psi_1 \in \Psi(\Omega)$.

Let $\alpha > 0$ and $\beta \leq 0$ and $\rho, \tau, \xi, \eta \in \mathbb{R}$ satisfy the conditions $\rho \tau \geq (1 + 3\rho^2)/2$ and $\rho \eta \geq 0$. Then, we have

$$\Re\psi_1(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \rho}{\tau} + \frac{\beta \eta}{\tau} \leq \frac{2\alpha \rho^2}{(1 + 3\rho^2)} \leq \frac{2\alpha}{3},$$

and this proves that the function $\psi_1 \in \Psi(\Omega)$. For the function $\psi_2$, we have

$$\Re\psi_2(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \rho}{\tau} + \frac{\beta \eta}{\rho} \leq \frac{2\alpha \rho^2}{(1 + 3\rho^2)} \leq \frac{2\alpha}{3},$$

and so the function $\psi_2 \in \Psi(\Omega_2)$.

**Theorem 2.15.** Let $\alpha < 0$ and $\beta \in \mathbb{R}$. If the function $f \in A$ satisfy any of the following inequalities

(i) $\Re\left(\alpha(Q_{SD}/Q_{ST}) + \beta Q_{ST}\right) > 0$,

(ii) $\Re\left(\alpha(Q_{SD}/Q_{CV}) + \beta Q_{ST}\right) > 0$,

(iii) $\Re\left(\alpha(Q_{SD}/Q_{ST}) + \beta Q_{CV}\right) > 0$,

(iv) $\Re\left(\alpha(Q_{SD}/Q_{CV}) + \beta Q_{CV}\right) > 0$,

then the function $f$ is starlike.

**Proof.** Let $\Omega$ be defined by $\Omega := \{w \in \mathbb{C} : \Re w > 0\}$ and for $i = 1, 2, 3, 4$, let the functions $\psi_i : \mathbb{C}^3 \to \mathbb{C}$ be defined by

$\psi_1(u, v, w) = \alpha(w/u) + \beta u$, \quad $\psi_2(u, v, w) = \alpha(w/v) + \beta u$,

$\psi_3(u, v, w) = \alpha(w/u) + \beta v$ and $\psi_4(v, w) = \alpha(w/v) + \beta v$.

The hypothesis of the theorem shows that the function $\psi_i$ satisfies

$\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega$ for $i = 1, 2, 3, 4$. 

It then follows from Theorem 1.1 that the function $f$ is starlike provided $\psi_i \in \Psi(\Omega)$. We complete the proof by showing that the function $\psi_i \in \Psi(\Omega)$.

Let $\alpha < 0$ and $\beta \in \mathbb{R}$ and $\rho, \tau, \xi, \eta \in \mathbb{R}$ satisfy the conditions $\rho \tau \geq (1 + 3\rho^2)/2$ and $\rho \eta \geq 0$. Then, we have

$$\text{Re} \psi_1(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\rho} \leq 0,$$

and this proves that the function $\psi_1 \in \Psi(\Omega)$. For the function $\psi_2$, we have

$$\text{Re} \psi_2(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\tau} \leq 0,$$

and so the function $\psi_2 \in \Psi(\Omega_2)$. Similarly, we have

$$\text{Re} \psi_3(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\rho} \leq 0,$$

and

$$\text{Re} \psi_4(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\tau} \leq 0,$$

so that the functions $\psi_3 \in \Psi(\Omega)$ and $\psi_4 \in \Psi(\Omega)$.

Many authors have dedicated significant part of their works on developing conditions for the functions to be starlike. In such a way, the quotients $Q_{CV}/Q_{ST}, (Q_{CV} - \gamma)/Q_{ST}$ were introduced and examined. See [21, 23, 25, 28]. Along with the quotients, we consider Schwarzian derivatives and discuss its consequences in the study of starlikeness of functions.

**Theorem 2.16.** Let $\alpha \leq 0$ and $\beta \geq 0$. If the function $f \in A$ satisfy any of the following inequalities

(i) $\text{Re} \left( \alpha \left( Q_{SD}/Q_{ST} \right) + Q_{ST}(1 + \beta Q_{ST}) \right) > 0$,
(ii) $\text{Re} \left( \alpha \left( Q_{SD}/Q_{CV} \right) + Q_{ST}(1 + \beta Q_{ST}) \right) > 0$,
(iii) $\text{Re} \left( \alpha \left( Q_{SD}/Q_{CV} \right) + Q_{CV}(1 + \beta Q_{CV}) \right) > 0$,
(iv) $\text{Re} \left( \alpha \left( Q_{SD}/Q_{ST} \right) + Q_{CV}(1 + \beta Q_{CV}) \right) > 0$,

then the function $f$ is starlike.

**Proof.** Let $\Omega$ be defined by $\Omega := \{ w \in \mathbb{C} : \text{Re} w > 0 \}$ and for $i = 1, 2, 3, 4$, let the functions $\psi_i : \mathbb{C}^3 \to \mathbb{C}$ be defined by

$$\psi_1(u, v, w) = \alpha(w/u) + u(1 + \beta u), \quad \psi_2(u, v, w) = \alpha(w/v) + u(1 + \beta u),$$

$$\psi_3(u, v, w) = \alpha(w/v) + v(1 + \beta v) \quad \text{and} \quad \psi_4(v, w) = \alpha(w/u) + v(1 + \beta v).$$

The hypothesis of the theorem shows that the function $\psi_i$ satisfies

$$\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for} \ i = 1, 2, 3, 4.$$ 

It then follows from Theorem 1.1 that the function $f$ is starlike provided $\psi_i \in \Psi(\Omega)$. We complete the proof by showing that the function $\psi_i \in \Psi(\Omega)$.

Let $\alpha \leq 0$ and $\beta \geq 0$ and $\rho, \tau, \xi, \eta \in \mathbb{R}$ satisfy the conditions $\rho \tau \geq (1 + 3\rho^2)/2$ and $\rho \eta \geq 0$. Then, we have

$$\text{Re} \psi_1(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\rho} - \beta \rho^2 \leq 0,$$
and this proves that the function $\psi_1 \in \Psi(\Omega)$. For the function $\psi_2$, we have

$$\text{Re } \psi_2(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\tau} - \beta \rho^2 \leq 0,$$

and so the function $\psi_2 \in \Psi(\Omega_2)$. Similarly, we have

$$\text{Re } \psi_3(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\tau} - \beta \tau^2 \leq 0,$$

and

$$\text{Re } \psi_4(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\rho} - \beta \tau^2 \leq 0,$$

so that the functions $\psi_3 \in \Psi(\Omega)$ and $\psi_4 \in \Psi(\Omega)$.

\[ \blacksquare \]

**Theorem 2.17.** Let $\alpha \leq 0$ and $\beta \geq 0$. If the function $f \in \mathcal{A}$ satisfy any of the following inequalities

(i) $\text{Re } (\alpha(Q_{SD}/Q_{ST}) + Q_{ST}(1 + \beta Q_{CV})) > -\beta/2$,

(ii) $\text{Re } (\alpha(Q_{SD}/Q_{ST}) + Q_{CV}(1 + \beta Q_{ST})) > -\beta/2$,

(iii) $\text{Re } (\alpha(Q_{SD}/Q_{CV}) + Q_{CV}(1 + \beta Q_{ST})) > -\beta/2$,

(iv) $\text{Re } (\alpha(Q_{SD}/Q_{CV}) + Q_{ST}(1 + \beta Q_{CV})) > -\beta/2$,

then the function $f$ is starlike.

**Proof.** Let $\Omega$ be defined by $\Omega := \{w \in \mathbb{C} : \text{Re } w > -\beta/2\}$ and for $i = 1, 2, 3, 4$, let the functions $\psi_i : \mathbb{C}^3 \to \mathbb{C}$ be defined by

$$\psi_1(u, v, w) = \alpha(w/u) + u(1 + \beta v), \quad \psi_2(u, v, w) = \alpha(w/u) + v(1 + \beta u),$$

$$\psi_3(u, v, w) = \alpha(w/v) + v(1 + \beta u) \quad \text{and} \quad \psi_4(v, w) = \alpha(w/v) + u(1 + \beta v).$$

The hypothesis of the theorem shows that the function $\psi_i$ satisfies

$$\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega \quad \text{for } i = 1, 2, 3, 4.$$

It then follows from Theorem 1.1 that the function $f$ is starlike provided $\psi_i \in \Psi(\Omega)$. We complete the proof by showing that the function $\psi_i \in \Psi(\Omega)$.

Let $\alpha \geq 0$ and $\beta \geq 0$ and $\rho, \tau, \xi, \eta \in \mathbb{R}$ satisfy the conditions $\rho \tau \geq (1 + 3\rho^2)/2$ and $\rho \eta \geq 0$. Then, we have

$$\text{Re } \psi_1(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\rho} - \beta \rho \tau \leq \frac{\beta(1 + 3\rho^2)}{2} \leq -\frac{\beta}{2},$$

and this proves that the function $\psi_1 \in \Psi(\Omega)$. For the function $\psi_2$, we have

$$\text{Re } \psi_2(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\tau} - \beta \rho \tau \leq \frac{\beta(1 + 3\rho^2)}{2} \leq -\frac{\beta}{2},$$

and so the function $\psi_2 \in \Psi(\Omega_2)$. Similarly, we have

$$\text{Re } \psi_3(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\tau} - \beta \rho \tau \leq \frac{\beta(1 + 3\rho^2)}{2} \leq -\frac{\beta}{2},$$

and

$$\text{Re } \psi_4(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \eta}{\rho} - \beta \tau \leq \frac{\beta(1 + 3\rho^2)}{2} \leq -\frac{\beta}{2},$$

so that the functions $\psi_3 \in \Psi(\Omega)$ and $\psi_4 \in \Psi(\Omega)$. 

\[ \blacksquare \]
Theorem 2.18. Let $\alpha \leq 0$ and $\beta > 0$. If the function $f \in A$ satisfy any of the following inequalities

(i) $\text{Re} \left( \alpha(Q_{SD}/Q_{ST}) + \beta Q_{ST}Q_{CV} \right) > -\beta/2$,
(ii) $\text{Re} \left( \alpha(Q_{SD}/Q_{CV}) + \beta Q_{ST}Q_{CV} \right) > -\beta/2$,
(iii) $\text{Re} \left( \alpha(Q_{SD}/Q_{CV}) + \beta Q_{ST}Q_{ST} \right) > 0$,
(iv) $\text{Re} \left( \alpha(Q_{SD}/Q_{ST}) + \beta Q_{CV}Q_{CV} \right) > 0$,
(v) $\text{Re} \left( \alpha(Q_{SD}/Q_{ST}) + \beta Q_{ST}Q_{ST} \right) > 0$,
(vi) $\text{Re} \left( \alpha(Q_{SD}/Q_{CV}) + \beta Q_{CV}Q_{CV} \right) > 0$,

then the function $f$ is starlike.

Proof. For $i = 1, 2, \cdots, 6$, let $\Omega_i$ be defined by $\Omega_1 := \{ w \in \mathbb{C} : \text{Re} w > -\beta/2 \} =: \Omega_2$ and $\Omega_3 := \{ w \in \mathbb{C} : \text{Re} w > 0 \} =: \Omega_4 := \Omega_5 := \Omega_6$ and the functions $\psi_i : \mathbb{C}^2 \to \mathbb{C}$ be defined by

$$
\psi_1(u, v, w) = \alpha(w/u) + \beta u v, \quad \psi_2(u, v, w) = \alpha(w/v) + \beta u v,
$$

$$
\psi_3(u, v, w) = \alpha(w/v) + \beta u^2, \quad \psi_4(u, v, w) = \alpha(w/u) + \beta v^2,
$$

$$
\psi_5(u, v, w) = \alpha(w/u) + \beta u^2 \quad \text{and} \quad \psi_6(v, w) = \alpha(w/v) + \beta v^2.
$$

The hypothesis of the theorem shows that the function $\psi_i$ satisfies

$$
\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega_i \quad \text{for} \quad i = 1, 2, \cdots, 6.
$$

It then follows from Theorem 1.1 that the function $f$ is starlike provided $\psi_i \in \Psi(\Omega_i)$. We complete the proof by showing that the function $\psi_i \in \Psi(\Omega_i)$.

Let $\alpha \leq 0$ and $\beta > 0$ and $\rho, \tau, \xi, \eta \in \mathbb{R}$ satisfy the conditions $\rho \tau \geq (1 + 3 \rho^2)/2$ and $\rho \eta \geq 0$. Then, we have

$$
\text{Re} \psi_1(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \eta}{\rho} - \beta \rho \tau \leq \frac{-\beta(1 + 3 \rho^2)}{2} \leq \frac{-\beta}{2},
$$

and this proves that the function $\psi_1 \in \Psi(\Omega_1)$. For the function $\psi_2$, we have

$$
\text{Re} \psi_2(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \eta}{\tau} - \beta \rho \tau \leq \frac{-\beta(1 + 3 \rho^2)}{2} \leq \frac{-\beta}{2},
$$

and so the function $\psi_2 \in \Psi(\Omega_2)$. Similarly, we have

$$
\text{Re} \psi_3(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \eta}{\tau} - \beta \rho^2 \leq 0,
$$

and

$$
\text{Re} \psi_4(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \eta}{\rho} - \beta \tau^2 \leq 0,
$$

so that the functions $\psi_3 \in \Psi(\Omega_3)$ and $\psi_4 \in \Psi(\Omega_4)$. Proceeding in a similar way, we have

$$
\text{Re} \psi_5(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \eta}{\rho} - \beta \rho^2 \leq 0
$$

and this proves that the function $\psi_5 \in \Psi(\Omega_5)$. For the function $\psi_6$, we have

$$
\text{Re} \psi_6(i \rho, i \tau, \xi + i \eta) = \frac{\alpha \eta}{\tau} - \beta \tau^2 \leq 0,
$$

and so the function $\psi_6 \in \Psi(\Omega_6)$. 

**Theorem 2.19.** Let $\alpha > 0$ and $\beta \geq 0$. If the function $f \in A$ satisfy any of the following inequalities

(i) $\Re \left( \alpha \left( \frac{Q_{CV}}{Q_{ST}} \right) - \beta Q_{SD}Q_{CV} \right) < \frac{3\alpha}{2}$,
(ii) $\Re \left( \alpha \left( \frac{Q_{CV}}{Q_{ST}} \right) - \beta SD \right) < \frac{3\alpha}{2}$,
(iii) $\Re \left( \alpha \left( \frac{Q_{ST}}{Q_{CV}} \right) + \beta Q_{SD}Q_{CV} \right) < \frac{2\alpha}{3}$,
(iv) $\Re \left( \alpha \left( \frac{Q_{ST}}{Q_{CV}} \right) + \beta Q_{SD}Q_{ST} \right) < \frac{2\alpha}{3}$,

then the function $f$ is starlike.

**Proof.** For $i = 1, 2, 3, 4$, let $\Omega_i$ be defined by $\Omega_1 := \{ w \in \mathbb{C} : \Re w < \frac{3\alpha}{2} \} =: \Omega_2$ and $\Omega_3 := \{ w \in \mathbb{C} : \Re w > \frac{2\alpha}{3} \} =: \Omega_4$ and the functions $\psi_i : \mathbb{C}^3 \to \mathbb{C}$ be defined by

$\psi_1(u, v, w) = \alpha (v/u) - \beta vw$, $\psi_2(u, v, w) = \alpha (v/u) - \beta uw$,
$\psi_3(u, v, w) = \alpha (u/v) + \beta vw$ and $\psi_4(v, w) = \alpha (u/v) + \beta uw$.

The hypothesis of the theorem shows that the function $\psi_i$ satisfies $\psi_i(Q_{ST}, Q_{CV}, Q_{SD}) \in \Omega_i$ for $i = 1, 2, 3, 4$.

It then follows from Theorem 1.1 that the function $f$ is starlike provided $\psi_i \in \Psi(\Omega_i)$. We complete the proof by showing that the function $\psi_i \in \Psi(\Omega_i)$.

Let $\alpha > 0$ and $\beta \geq 0$ and $\rho, \tau, \xi, \eta \in \mathbb{R}$ satisfy the conditions $\rho \tau \geq (1 + 3\rho^2)/2$ and $\rho \eta \geq 0$. Then, we have

$\Re \psi_1(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \tau}{\rho} + \beta \tau \eta \geq \frac{\alpha (1 + 3\rho^2)}{2\rho^2} \geq \frac{3\alpha}{2}$

and this proves that the function $\psi_1 \in \Psi(\Omega_1)$. For the function $\psi_2$, we have

$\Re \psi_2(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \tau}{\rho} + \beta \rho \eta \geq \frac{\alpha (1 + 3\rho^2)}{2\rho^2} \geq \frac{3\alpha}{2}$

and so the function $\psi_2 \in \Psi(\Omega_2)$. The real valued function $2\rho^2/(1 + 3\rho^2)$ is an increasing function and the maximum value of the function is $2/3$. Then, we have

$\Re \psi_3(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \rho}{\tau} - \beta \tau \eta \leq \frac{2\alpha \rho^2}{1 + 3\rho^2} \leq \frac{2\alpha}{3},$

and

$\Re \psi_4(i\rho, i\tau, \xi + i\eta) = \frac{\alpha \rho}{\tau} - \beta \rho \eta \leq \frac{2\alpha \rho^2}{1 + 3\rho^2} \leq \frac{2\alpha}{3},$

so that the functions $\psi_3 \in \Psi(\Omega_3)$ and $\psi_4 \in \Psi(\Omega_4)$.

**Remark 2.20.** Note that substitution of $\alpha = 1$ and $\beta = 0$ in Part(i) and Part(ii) of Theorem 2.19 provide the same result as the one obtained in [25] pp.523].
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