GEOMETRIC SCHUR DUALITY OF TWO PARAMETER QUANTUM GROUP OF TYPE A

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Abstract. In this paper, we give an geometric description of the Schur-Weyl duality for two-parameter quantum algebras $U_{v,t}(gl_n)$, where $U_{v,t}(gl_n)$ is the deformation of $U_v(I,\cdot)$, the classic Shur-Weyl duality $(U_{r,s}(gl_n),V^\otimes d,H_d(r,s))$ can be seen as a corollary of the Shur-Weyl duality $(U_{v,t}(gl_n),V^\otimes d,H_d(v,t))$ by using the galois descend approach. We also establish the Shur-Weyl duality between the algebras $\widehat{U_{v,t}(gl_N)}^m$, $\widehat{U_{v,t}(gl_N)}^m$ and Hecke algebra $H_k(v,t)$.

1. Introduction

Schur-Weyl duality is a classical method to construct irreducible modules of simple Lie groups out of the fundamental representations [W46]. The quantum version for the quantum enveloping algebra $U_q(sl_n)$ and the Hecke algebra $H_q(S_m)$ has been one of the pioneering examples [13] in the fervent development of quantum groups. Two-parameter general linear and special linear quantum groups [21, 8, 4] are certain generalization of the one-parameter Drinfeld-Jimbo quantum groups [7, 12]. The two-parameter quantum groups also had their origin in the quantum inverse scattering method [20] as well as other approaches [14, 6]. So far, lots of mathematicians had studied the quantum groups and two parameter quantum group. For example, geometric Shur-Jimbo duality of type A was studied by Beilinson, Lusztig and Mcpherson [BLM90]. And the Shur-like duality of type B/C and D were discovered by Bao-Wang [BKLW14] and Fan-Li [FL14].

Especially, Fan and Li had found another version of two parameter quantum group by the way of perverse sheaves [FL13]. But the question how the two parameter quantum group $U_{v,t}(gl_n)$ can be seen as the deformation of $U_v(gl_n)$ didn’t solve in their work. So it is necessary for us to give the new graded structure on $U_v(gl_n)$ such that $U_{v,t}(gl_n)$ can be seen as the deformation of $U_v(gl_n)$.

Fan and Li found two new quantum group $U$ and $U^m$, and gave the Shur-Weyl duality between them and the Iwahori-hecke algebra of type $D_d$[FL14]. In our following paper, similar to the Fan and Li’s work, we will give two new two parameter quantum group $U_{v,t}$ and $U^m_{v,t}$. We can also give the Shur-Weyl duality between them and the two parameter Iwahori-hecke algebra of type $D_d$ through the geometric way. In order to give the comultiplication of the two new two parameter quantum group $U_{v,t}$, $U^m_{v,t}$ and use the comultiplication structure to give the Shur-Weyl duality algebraically. That is,

$$\Delta : U \rightarrow U_{v,t}(gl_N)^m \otimes U,$$
\[ \Delta : U^m \to U_{v,t}(gl_N)^m \otimes U^m. \]

So it is reasonable for us to give structure of the new quantum group \( U_{v,t}(gl_N)^m \) and the Shur-Weyl duality between them and \( H_{v,t}(d) \).

In this work, at first, we give a new version of two parameter quantum group \( U_{v,t}(gl_n) \), which is the deformation of \( U_v(gl_n) \) similar to the approach appear in [FL13]. Second, we would like to give the geometric realization of three quantum groups \( U_{v,t}(gl_n) \), \( \tilde{U}_{v,t}(gl_N)^m \), \( \hat{U}_{v,t}(gl_N)^m \). At the same time, we also give the Shur-Weyl duality between algebras \( U_{v,t}(gl_n) \), \( \tilde{U}_{v,t}(gl_N)^m \), \( \hat{U}_{v,t}(gl_N)^m \) and the Hecke algebra \( H_{v,t}(d) \). Since the classical two parameter quantum group \( U_{r,s}(gl_n) \) is the subalgebra of the new version \( U_{v,t}(gl_n) \), we would like to use the Galois descend approach to understand the two different versions of two parameter quantum groups. The classical Shur-Weyl duality \( (U_{r,s}(gl_n), V \otimes d, H_d(r,s)) \) can be seen as a corollary of the Shur-Weyl duality \( (U_{v,t}(gl_n), V \otimes d, H_d(v,t)) \) by using the galois descend theory. That is, there exist a Galois group \( G \) such that \( (U_{v,t}(gl_n)_G, V \otimes d_G, H_d(v,t)_G) \) is also Shur-Weyl duality, and \( U_{v,t}(gl_n)_G \cong (U_{r,s}(gl_n), H_d(v,t)_G \cong H_d(r,s)). \)

### 2. Deformation

#### 2.1. The algebra \( U_{v,t}(gl_n) \)

Let \( \Omega = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \) is associated to the cartan matrix of type \( A_n \). Let \( I = \{1, 2, \cdots, n\} \). To \( \Omega \), we associate the following three bilinear forms on \( \mathbb{Z}^I \).

\[
\begin{align*}
(1) & \quad \langle i, j \rangle = \Omega_{ij}, \quad \forall i, j \in I. \\
(2) & \quad \left[ i, j \right] = 2 \delta_{ij} \Omega_{ii} - \Omega_{ij}, \quad \forall i, j \in I. \\
(3) & \quad i \cdot j = \langle i, j \rangle + \langle j, i \rangle, \quad \forall i, j \in I.
\end{align*}
\]

**Definition 2.1.1.** The two-parameter quantum algebra \( U_{v,t}(gl_n) \) associated to \( A_{n-1} \) is an associative \( \mathbb{Q}(v, t) \)-algebra with 1 generated by symbols \( E_i, F_i, \forall i \in I, A_j^{\pm 1}, B_j^{\pm 1}, \forall i \in I' = \ldots \).
where $I \cup \{n\}$ and subject to the following relations.

(R1) \[
A_i^{\pm 1}A_j^{\pm 1} = A_j^{\pm 1}A_i^{\pm 1}, \quad B_i^{\pm 1}B_j^{\pm 1} = B_j^{\pm 1}B_i^{\pm 1},
\]
\[
A_i^{\pm 1}B_j^{\pm 1} = B_j^{\pm 1}A_i^{\pm 1}, \quad A_i^{\pm 1}A_i^{\pm 1} = 1 = B_i^{\pm 1}B_i^{\pm 1}.
\]

(R2) \[
A_iE_jA_i^{-1} = v^{(i,j)}t^{(i,j)}E_j, \quad B_iE_jB_i^{-1} = v^{-(i,j)}t^{(i,j)}E_j,
\]
\[
A_iF_jA_i^{-1} = v^{-(i,j)}t^{-(j,i)}F_j, \quad B_iF_jB_i^{-1} = v^{(i,j)}t^{-(j,i)}F_j.
\]

(R3) \[
E_iF_j - F_jE_i = \delta_{ij} \frac{A_iB_{i+1} - B_iA_{i+1}}{v - v^{-1}}.
\]

(R4) \[
\sum_{p+p' = 1 - 2\pi} (-1)^{pt - p(p' - 2\pi)}E_i^{(p)}E_j^{(p')} = 0, \quad \text{if } i \neq j,
\]
\[
\sum_{p+p' = 1 - 2\pi} (-1)^{pt - p(p' - 2\pi)}F_i^{(p)}F_j^{(p')} = 0, \quad \text{if } i \neq j,
\]
where $E_i^{(p)} = \frac{p}{p|_{v_i^{-1}i}}, \langle j, n \rangle = 0, \langle n, j \rangle = \begin{cases} -1 & \text{if } j = n - 1; \\ 0 & \text{else} \end{cases}, j \in I.$

The algebra $U_{v,t}(gl_n)$ has a Hopf algebra structure with the comultiplication $\Delta$, the counit $\varepsilon$ and the antipode $S$ given as follows.

\[
\Delta(A_i^{\pm 1}) = A_i^{\pm 1} \otimes A_i^{\pm 1}, \quad \Delta(B_i^{\pm 1}) = B_i^{\pm 1} \otimes B_i^{\pm 1},
\]
\[
\Delta(E_i) = E_i \otimes A_iB_{i+1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + B_iA_{i+1} \otimes F_i,
\]
\[
\varepsilon(A_i^{\pm 1}) = \varepsilon(B_i^{\pm 1}) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,
\]
\[
S(A_i^{\pm 1}) = A_i^{\mp 1}, \quad S(B_i^{\pm 1}) = B_i^{\mp 1},
\]
\[
S(E_i) = -E_iB_iA_{i+1}, \quad S(F_i) = -A_iB_{i+1}F_i.
\]

The algebra $U_{v,t}(gl_n)$ admits a $\mathbb{Z}' \times \mathbb{Z}'$-grading by defining the degrees of generators as follows.

\[
deg(E_i) = (i, 0), \quad \deg(F_i) = (0, i),
\]
\[
deg(A_j) = deg(B_j) = \begin{cases} \left(\sum_{k=j}^n (-1)^k k, \sum_{k=j}^n (-1)^k k\right) & \text{if } j \text{ is even}, \\ \left(\sum_{k=j}^n (-1)^{k+1} k, \sum_{k=j}^n (-1)^{k+1} k\right) & \text{if } j \text{ is odd}.
\end{cases}
\]

We can define a bilinear form on $\mathbb{Z}' \times \mathbb{Z}'$ by

\[
[\gamma, \eta]' = [\gamma_2, \eta_2] - [\gamma_1, \eta_1]
\]
for any $\gamma = (\gamma_1, \gamma_2), \eta = (\eta_1, \eta_2) \in \mathbb{Z}' \times \mathbb{Z}'$. Then on $U_{v,t}(gl_n)$, we can define a new multiplication $'' * ''$ by

\[
x \ast y = t^{-||x||, |y||'} xy,
\]
for any homogenous elements $x, y \in U_{v,t}(gl_n)$. Since $[\cdot, \cdot]'$ is a bilinear form, $(U_{v,t}(gl_n), *)$ is an associative algebra over $\mathbb{Q}(v,t)$. We define a multiplication, denoted by $'' * '',$ on
\( U_{v,t}(gl_n) \otimes U_{v,t}(gl_n) \) by
\[(x \otimes y) \cdot (x' \otimes y') = x \cdot x' \otimes y \cdot y'.\]

This gives a new algebra structure on \( U_{v,t}(gl_n) \otimes U_{v,t}(gl_n) \). \((U_{v,t}(gl_n), \ast)\) has a Hopf algebra structure with the comultiplication \( \Delta^* \), the counit \( \varepsilon^* \) and the antipode \( S^* \). The image of generators \( E_i, F_i, A_i \) and \( B_i^{-1} \) under the map \( \Delta^* \) (resp. \( \varepsilon^* \) and \( S^* \)) are the same as the ones under the map \( \Delta \) (resp. \( \varepsilon \) and \( S \)) defined above.

**Lemma 2.1.2.** Under the new multiplication \(" \ast \)\", the defining relations of \( U_{v,t}(gl_n) \) can be rewritten as follows.

(R1) \[ A^\pm_1 \ast A^\pm_1 = A^\pm_1 \ast A^\pm_1, \quad B^\pm_1 \ast B^\pm_1 = B^\pm_1 \ast B^\pm_1, \]
A^\pm_1 \ast B^\pm_1 = B^\pm_1 \ast A^\pm_1, \quad A^\pm_1 \ast A^\mp_1 = 1 = B^\pm_1 \ast B^\mp_1.

(R2) \[ A_i \ast E_j \ast A_i^{-1} = v^{(i,j)} E_j, \quad B_i \ast E_j \ast B_i^{-1} = v^{-(i,j)} E_j, \]
A_i \ast F_j \ast A_i^{-1} = v^{-(i,j)} F_j, \quad B_i \ast F_j \ast B_i^{-1} = v^{(i,j)} F_j.

(R3) \[ E_i \ast F_j - F_j \ast E_i = \delta_{ij} \frac{A_i \ast B_{i+1} - B_i \ast A_{i+1}}{v - v^{-1}}, \quad \forall i,j \in I. \]

(R4) \[
\sum_{p+p' = 1 - a_{ij}} (-1)^p \left[ \frac{1 - a_{ij}}{p} \right] E_i^{p} \ast E_j \ast E_i^{p'} = 0, \quad \text{if } i \neq j, \\
\sum_{p+p' = 1 - a_{ij}} (-1)^p \left[ \frac{1 - a_{ij}}{p} \right] F_i^{p} \ast F_j \ast F_i^{p'} = 0 \quad \text{if } i \neq j,
\]

where \( a_{ij} = 2^{i+j} \) and \( E_i^{p} = E_i \ast E_i \ast \cdots \ast E_i \) for \( p \) copies. We notice that these relations are the specialization of (R1)-(R4) at \( t = 1 \).

**Proof.** The relation R3, R4 agrees with the one in [FL13 4. 2], whose proof is also the same as the one for type A case. Next we show R*2.

\[ A_i \ast E_j = t^{-||A_i||} \langle E_j \rangle A_i E_j = t^{-||A_i||} \langle E_j \rangle t^{(i,j)} E_j A_i = t^{||E_j|| - ||A_i||} \langle E_j \rangle t^{(i,j)} E_j A_i \]
and
\[ ||E_j|| - ||A_i|| = 0 = \langle j, i \rangle - \langle i, j \rangle - \langle j, i+1 \rangle - \langle i+1, j \rangle + \cdots + (-1)^{n-i} \langle j, n \rangle - \langle n, j \rangle. \]

Therefore,
\[ A_i \ast E_j = v^{(i,j)} E_i \ast A_i. \]

All other identity in R2 can be shown similarly. \(\square\)
The one-parameter quantum algebra $U_v(I, \cdot)$ associated to $(I, \cdot)$ is defined as the associative $\mathbb{Q}(v)$-algebra with 1 generated by symbols $E_i, F_i, A_i^{\pm1}, B_i^{\pm1}, \forall i \in I$ and subject to relations (R*1)-(R*4). $U_v(I, \cdot)$ has a Hopf algebra structure with the comultiplication $\Delta_1$, the counit $\varepsilon_1$ and the antipode $S_1$. The image of generators $E_i, F_i, A_i, B_i$ under the map $\Delta_1$ (resp. $\varepsilon_1$ and $S_1$) are the same as the ones under the map $\Delta$ (resp. $\varepsilon$ and $S$) defined above.

Let $U_{v,t}(I, \cdot) := U_v(I, \cdot) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v, t)$. The Hopf algebra structure on $U_v(I, \cdot)$ can be naturally extended to $U_{v,t}(I, \cdot)$. From the above analysis, we have the following theorem.

**Theorem 2.1.3.** If $(I, \cdot)$ is the Cartan datum associated to $\Omega_n$, then there is a Hopf-algebra isomorphism

$$(U_{v,t}(gl_n), *, \Delta^*, \varepsilon^*, S^*) \simeq (U_{v,t}(I, \cdot), *, \Delta_1, \varepsilon_1, S_1),$$

sending the generators in $U_{v,t}$ to the respective generators in $U_{v,t}(I, \cdot)$.

**3. A geometric setting**

**3.1. Preliminary.** Let $\mathbb{F}_q$ be a finite field of $q$ elements and of odd characteristic. $d$ is a fixed positive integer, $n$ is a positive integer. We fix a vector space $\mathbb{F}_q^d$. Consider the following sets.

- The set $\mathcal{X}$ of $n$-step flags $V = (V_i)_{0 \leq i \leq n}$ in $\mathbb{F}_q^d$ such that $V_0 = 0, V_i \subseteq V_{i+1}$.
- The set $\mathcal{Y}$ of complete flags $F = (F_i)_{0 \leq i \leq d}$ in $\mathbb{F}_q^d$ such that $F_i \subset F_{i+1}$, $|F_i| = i$.

where we write $|F_i|$ for the dimension of $F_i$.

Let $G = GL(V)$. Then $G$ acts naturally on sets $\mathcal{X}$ and $\mathcal{Y}$. Moreover, $G$ acts transitively on $\mathcal{Y}$. Let $G$ act diagonally on the product $\mathcal{X} \times \mathcal{X}$ (resp. $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{Y} \times \mathcal{Y}$). Set

$$(6) \quad \mathcal{A} = \mathbb{Z}[v^{\pm1}, t^{\pm1}].$$

Let

$$(7) \quad S_{\mathcal{X}} = \mathcal{A}_G(\mathcal{X} \times \mathcal{X})$$

be the set of all $\mathcal{A}$-valued $G$-invariant functions on $\mathcal{X} \times \mathcal{X}$. Clearly, the set $S_{\mathcal{X}}$ is a free $\mathcal{A}$-module. Moreover, $S_{\mathcal{X}}$ admits an associative $\mathcal{A}$-algebra structure ‘$*$’ under a standard convolution product as discussed in [BKLW14, 2.3]. In particular, when $v$ is specialized to $\sqrt{q}$, we have

$$(8) \quad f * g(V, V') = \sum_{V'' \in \mathcal{X}} f(V, V'')g(V'', V'), \quad \forall V, V' \in \mathcal{X}.$$ 

Similarly, we define the free $\mathcal{A}$-modules

$$(9) \quad \mathcal{V} = \mathcal{A}_G(\mathcal{X} \times \mathcal{Y}) \quad \text{and} \quad \mathcal{H}_\mathcal{Y} = \mathcal{A}_G(\mathcal{Y} \times \mathcal{Y}).$$

A similar convolution product gives an associative algebra structure on $\mathcal{H}_\mathcal{Y}$ and a left $S_{\mathcal{X}}$-action and a right $\mathcal{H}_\mathcal{Y}$-action on $\mathcal{V}$. Moreover, these two actions commute and hence we have the following $\mathcal{A}$-algebra homomorphisms.

$$S_{\mathcal{X}} \rightarrow \text{End}_{\mathcal{H}_\mathcal{Y}}(\mathcal{V}) \quad \text{and} \quad \mathcal{H}_\mathcal{Y} \rightarrow \text{End}_{S_{\mathcal{X}}}(\mathcal{V}).$$

Similar to [P09, Theorem 2.1], we have the following double centralizer property.

**Lemma 3.1.1.** $\text{End}_{\mathcal{H}_\mathcal{Y}}(\mathcal{V}) \simeq S_{\mathcal{X}}$ and $\text{End}_{S_{\mathcal{X}}}(\mathcal{V}) \simeq \mathcal{H}_\mathcal{Y}$, if $n \geq d$. 
We note that the result in [P09, Theorem 2.1] is obtained over the field \( \mathbb{C} \) of complex numbers, but the proof can be adapted to our setting over the ring \( \mathcal{A} \).

We shall give a description of the \( G \)-orbits on \( X \times X \), \( X \times Y \), and \( Y \times Y \). We start by introducing the following notations associated to a matrix \( M = (m_{ij})_{1 \leq i,j \leq c} \).

\[
\text{ro}(M) = \left( \sum_{j=1}^{c} m_{ij} \right)_{1 \leq i \leq c},
\]
\[
\text{co}(M) = \left( \sum_{i=1}^{c} m_{ij} \right)_{1 \leq j \leq c}.
\]

We also write \( \text{ro}(M)_i \) and \( \text{co}(M)_j \) for the \( i \)-th and \( j \)-th component of the row vectors of \( \text{ro}(M) \) and \( \text{co}(M) \), respectively.

For any pair \((V, V')\) of flags in \( X \), we can assign an \( n \times n \) matrix whose \((i, j)\)-entry equals

\[
\dim_{\mathbb{C}} V_i - 1 + \dim_{\mathbb{C}} V_i \cap V'_j - 1.
\]

\[
G \setminus X \times X \cong \Theta_d,
\]

where \( \Theta_d \) is the set of all matrices \( \Theta_d \) in \( \text{Mat}_{n \times n}(\mathbb{N}) \) such that \( \sum_{i,j}(\Theta_d)_{i,j} = d \).

A similar assignment yields two bijection

\[
G \setminus X \times Y \cong \Pi,
\]
\[
G \setminus Y \times Y \cong \Sigma,
\]

where the set \( \Pi \) consists of all matrices \( B = (b_{ij}) \) in \( \text{Mat}_{n \times d}(\mathbb{N}) \) subject to

\[
\text{co}(B)_j = 1, \quad \forall j \in [1, d].
\]

and \( \Sigma \) is the set of all matrices \( \sigma \equiv (\sigma_{ij}) \) in \( \text{Mat}_{d \times d}(\mathbb{N}) \) such that

\[
\text{ro}(\sigma)_i = 1, \quad \text{ro}(\sigma)_j = 1.
\]

Moreover, we have

\[
\#\Sigma = d! \quad \text{and} \quad \#\Pi = n^d.
\]

4. Calculus of the algebra \( S \) and \( H_Y \)

Recall from the previous section that \( S_X \) is the convolution algebra on \( X \times X \) defined in (7). For simplicity, we shall denote \( S \) instead of \( S_X \). In this section, we determine the generators for \( S \) and the associated multiplication formula. We also will \( H_Y \) action on \( Y \).
4.1. Defining relations of \( S \). For any \( i \in [1, n-1] \), \( a \in [1, n] \), set

\[
E_i(V, V') = \begin{cases} 
  v^{-|V'_i/V'_i|} t^{-|V_i/V_{i-1}|}, & \text{if } V_i \supset V'_i, V_j = V_j', \forall j \in [1, n] \setminus \{i\}; \\
  0, & \text{otherwise.}
\end{cases}
\]

\[
F_i(V, V') = \begin{cases} 
  v^{-|V_{i+1}/V'_i|} t^{|V_i'/V'_i|}, & \text{if } V_i \subset V'_i, V_j = V_j', \forall j \in [1, n] \setminus \{i\}; \\
  0, & \text{otherwise.}
\end{cases}
\]

\[
A^\pm_a(V, V') = \begin{cases} 
  v^\pm |V'_a/V'_a| t^\pm |V_a/V_a' - 1|, & \text{if } V = V'; \\
  0, & \text{otherwise.}
\end{cases}
\]

\[
B^\pm_a(V, V') = \begin{cases} 
  v^\pm |V'_a/V'_a| t^\pm |V_a/V_a' - 1|, & \text{if } V = V'; \\
  0, & \text{otherwise.}
\end{cases}
\]

(15)

It is clear that these functions are elements in \( S \).

**Proposition 4.1.1.** The functions \( E_i, F_i, A^\pm_a, \) and \( B^\pm_a \) in \( S \), for any \( i \in [1, n-1] \), \( a \in [1, n] \), satisfy the following relations.

\[
\begin{align*}
(R1) \quad & A^\pm_i A^\pm_j = A^\pm_j A^\pm_i, \quad B^\pm_i B^\pm_j = B^\pm_j B^\pm_i, \\
& A^\pm_i B^\pm_j = B^\pm_j A^\pm_i, \quad A^\pm_i A^\mp_i = 1 = B^\mp_i B^\pm_i. \\
(R2) \quad & A_i E_j A^{-1}_i = v^{(i,j)} t^{(i,j)} E_j, \quad B_i E_j B^{-1}_i = v^{-(i,j)} t^{(i,j)} E_j, \\
& A_i F_j A^{-1}_i = v^{-(i,j)} t^{-(i,j)} F_j, \quad B_i F_j B^{-1}_i = v^{(i,j)} t^{-(i,j)} F_j. \\
(R3) \quad & E_i F_j - F_j E_i = \delta_{ij} \frac{A_i B_{i+1} - B_i A_{i+1}}{v - v^{-1}}. \\
(R4) \quad & \sum_{p+p'=1-2\frac{d}{d+1}} (-1)^p t^{-p(p'-2\frac{d}{d+1}) + p(p'-2\frac{d}{d+1})} E_i^{(p')} E_j^{(p)} = 0, \quad \text{if } i \neq j, \\
& \sum_{p+p'=1-2\frac{d}{d+1}} (-1)^p t^{-p(p'-2\frac{d}{d+1}) + p(p'-2\frac{d}{d+1})} F_i^{(p)} F_j^{(p')} = 0, \quad \text{if } i \neq j, \\
(R5) \quad & \prod_{i=1}^n A_i = v^d t^d, \quad \prod_{i=1}^n B_i = v^{-d} t^d, \\
(R6) \quad & \prod_{l=0}^d (A_j - v^l t^l) = 0, \quad \prod_{l=0}^d (B_j - v^{-l} t^l) = 0 \forall j \in [1, n]. \\
(R7) \quad & E_i^{d+1} = 0, F_i^{d+1} = 0.
\end{align*}
\]
Proof. The proofs of the identities of R1, R7 are straightforward. Let \( \lambda_i' = |V_i'/V_{i-1}'| \). We show the first identity in R2. we have

\[
(A_i E_j)(V, V') = \begin{cases}
  v^{\lambda_i'-\lambda_j'_{-1} t_{\lambda_i'} + \lambda_j'} & \text{if } V_j' \supset V_j' \text{ and } i = j + 1, \\
v^{\lambda_i'-\lambda_j'_{+1} + 2} & \text{if } V_j' \supset V_j' \text{ and } i = j, \\
v^{\lambda_i'-\lambda_j'_{+1} + 1} & \text{if } V_j' \supset V_j' \text{ and } i \neq j, j + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

That is, \( A_i E_j A_j^{-1}(V, V') = v^{(i,j) t(i,j)} E_j(V, V') \). All other identities can be shown similarly.

we show the identity in R3. By a direct calculation. We have

\[
(E_i F_j - F_j E_i)(V, V') = \begin{cases}
  v^{\lambda_i'-\lambda_j'_{+1} t_{\lambda_i'} + \lambda_j'} & \text{if } V = V' \text{ and } i = j, \\
0 & \text{otherwise}.
\end{cases}
\]

It is easy to check that the right hand side is equal to \( \delta_{ij} A_i B_i^{-1}(V, V') \).

At last, We now show the first identity in R4. By a direct calculation, we have

\[
E_i^2 E_{i+1}(V, V') = \begin{cases}
  (v^2 + 1)v^{-2\lambda_i'-\lambda_{i+1}'_{-1} t_{2\lambda_i'} + \lambda_{i+1}'} & \text{if } V_i' \supset V_i' \text{ and } V_{i+1} \supset V_{i+1}' \text{ and } V_i \supset V_i', \\
0 & \text{otherwise}.
\end{cases}
\]

The first identity in R4 follows. By the same way, the other three identities can be shown directly.

Let’s prove the first identity in R5, we have

\[
\prod_{i=1}^{n} A_i(V, V') = \begin{cases}
  v^{\lambda_1'+\cdots+\lambda_n'} t^{\lambda_1'+\cdots+\lambda_n'} & \text{if } V = V', \\
0, & \text{otherwise}.
\end{cases}
\]

Since \( \lambda_1' + \cdots + \lambda_n' = d \), the first identities follows. The other identities can be shown similarly.

At last, let’s prove the first identity in R6, we have

\[
\prod_{l=0}^{d} (A_j - v^l t^l)(V, V') = \begin{cases}
  (v^{\lambda_j'} t^{\lambda_j'} - 1)(v^{\lambda_j'} t^{\lambda_j'} - vt) \cdots (v^{\lambda_j'} t^{\lambda_j'} - v^d t^d) & \text{if } V = V', \\
0, & \text{otherwise}.
\end{cases}
\]

Since \( 0 \leq \lambda_j' \leq d \), the first identities follows. The other identities can be shown similarly. □
4.2. Multiplication formulas in $S$. For any $n \in \mathbb{Z}, k \in \mathbb{N}$, set

$$ (n)_v = \frac{v^{2n} - 1}{v^2 - 1}, \quad \text{and} \quad \binom{n}{k}_v = \frac{k \prod_{i=1}^{k} (n + 1 - i)_v}{(i)_v}. $$

Let $E_{ij}$ is the $n \times n$ matrix whose $(i,j)$-entry is 1 and all other entries are 0. Let $e_a$ be the characteristic function of the $G$-orbit corresponding to $a \in \theta_d$. It is clear that the set $\{e_a | a \in \theta_d\}$ forms a basis of $S$.

We assume that the ground field is an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ when we talk about the dimension of a $G$-orbit or its stabilizer. Set $d(a) = \dim \mathcal{O}_a$ and $r(a) = \dim \mathcal{O}_b$, \forall $a \in \theta_d$ or $\Pi$,

where $b = (b_{ij})$ is the diagonal matrix such that $b_{ii} = \sum k a_{ik}$. Denote by $C_G(V,V')$ the stabilizer of $(V,V')$ in $G$.

**Lemma 4.2.1.** If $a \in \Pi$, We have

$$ \dim C_G(V,V') = \sum_{i \geq k, j \geq l} a_{ij} a_{kl}, \quad \text{if} \ (V,V') \in \mathcal{O}_a, $$

$$ \dim \mathcal{O}_a = \sum_{i < k \text{ or } j < l} a_{ij} a_{kl}, $$

$$ d(a) - r(a) = \sum_{i \geq k, j < l} a_{ij} a_{kl}. $$

**Proof.** The proof is similar with [BLM90], The only difference we consider is that $a \in \Pi$ should be the $n \times d$ matrix. We can find the subspace $Z_{ij}$ of $V$ such that $V_a = \bigoplus_{i \leq a;j} Z_{ij}$ for all $a$, $V'_b = \bigoplus_{i \leq b;j} Z_{ij}$ for all $b$. $V = \bigoplus_{i \leq a;j} Z_{ij}$. Consider $T \in \text{End}(V)$, $T$ is determined by a family of linear maps $T_{ijkl}: Z_{ij} \rightarrow Z_{kl}$. If $T|_{V_a} = V_a, T|_{V'_b} = V'_b$, one can obtain that if $T_{ijkl} \neq 0$, then $i \geq k, j \geq l$. So we have $\dim C_G(V,V') = \sum a_{ij} a_{kl}$, $\dim \mathcal{O}_a = \dim GL(V) - \dim C_G(V,V') = \sum a_{ij} a_{kl}$. Since $r(a) = \dim (V,V)$, we have $d(a) - r(a) = \sum a_{ij} a_{kl} - \sum a_{ij} a_{kl} = \sum a_{ij} a_{kl}$. \hfill \blacksquare

For any $a \in \theta_d, \Pi$, let

$$ \{a\} = v^{-(d(a) - r(a))} t^{(d(a) - r(a))} e_a. $$

We define a bar involution $\overline{}$ on $\mathcal{A}$ by $\overline{\nu} = v^{-1}$.

**Proposition 4.2.2.** Suppose that $a, b, c \in \Theta_d$, $h \in [1, n - 1]$ and $r \in \mathbb{N}$. 
(a) If \( \text{co}(b) = \text{ro}(a) \), and \( b - rE_{h,h+1} \) is diagonal, then we have

\[
\{b\} \ast \{a\} = \sum_{t \sum_{u=1}^{n} t_u = r} v^{\beta(t)}_t t^{\alpha(t)}_t \prod_{u=1}^{n} \left( \frac{a_{hu} + t_u}{t_u} \right)^v \{a_t\}, \text{ where }
\]

\[
\alpha(t) = \sum_{j \geq l} a_{hj} t_l + \sum_{j > l} a_{h+1,j} t_l - \sum_{j < l} t_j t_l,
\]

\[
\beta(t) = \sum_{j \geq l} a_{hj} t_l - \sum_{j > l} a_{h+1,j} t_l + \sum_{j < l} t_j t_l,
\]

\[
a_t = A + \sum_{u=1}^{n} t_u (E_{hu} - E_{h+1,u}) \in \theta_d.
\]

(b) If \( \text{co}(c) = \text{ro}(a) \) and \( c - rE_{h+1,h} \) is diagonal, then

\[
\{c\} \ast \{a\} = \sum_{t \sum_{u=1}^{n} t_u = r} v^{\beta'(t)}_t t^{\alpha'(t)}_t \prod_{u=1}^{n} \left( \frac{a_{h+1,u} + t_u}{t_u} \right)^v \{a(h,t)\}, \text{ where }
\]

\[
\alpha'(t) = \sum_{j \leq l} a_{h+1,j} t_l + \sum_{j < l} a_{hj} t_l - \sum_{j > l} t_j t_l,
\]

\[
\beta'(t) = \sum_{j \leq l} a_{h+1,j} t_l - \sum_{j < l} a_{hj} t_l + \sum_{j > l} t_j t_l,
\]

\[
a(h,t) = A - \sum_{u=1}^{n} t_u (E_{hu} - E_{h+1,u}) \in \theta_d.
\]

**Proof.** In order to give the proof of (a), We only need to proof the formula \( a(t) \). By the direct computation,

\[
d(b) - r(b) = \sum_{j,u} a_{hj} t_u,
\]

\[
d(a) - r(a) = \sum_{i \geq k, j < l} a_{ij} a_{kl},
\]

\[
d(a_t) - r(a_t) = \sum_{i \geq k, j < l} a_{ij} a_{kl} + \sum_{j < u} a_{hj} t_u - \sum_{t > u} a_{h+1,t} t_u + \sum_{u < u'} t_u t_{u'}.
\]

Then,

\[
\alpha(t) = d(b) - r(b) + d(a) - r(a) - (d(a_t) - r(a_t)) = \sum_{j \geq l} a_{hj} t_l + \sum_{j > l} a_{h+1,j} t_l - \sum_{j < l} t_j t_l.
\]

Similarly, we can obtain the proposition of (b). \( \square \)

4.3. \( S \)-action on \( V \). A degenerate version of Proposition 4.2.2 gives us an explicit description of the \( S \)-action on \( V = A_G(\mathcal{X} \times \mathcal{Y}) \) as follows. For any \( r_j \in [1, n] \), we denote \( \tilde{r}_j = r_j + 1 \) and \( \hat{r}_j = r_j - 1 \).
Corollary 4.3.1. For any $1 \leq i \leq n-1, 1 \leq a \leq n-1$, we have

$$E_i \cdot \{e_{r_1 \ldots r_d}\} = v^{\sum_{j<p} \delta_{a,r_j} - \delta_{a+1,r_j}} t^{1+\sum_{j<p} \delta_{a,r_j} + \delta_{a+1,r_j}} \{e_{r_1 \ldots r_{p-1}r_{p+1} \ldots r_d}\},$$

$$F_i \cdot \{e_{r_1 \ldots r_d}\} = \sum_{1 \leq p \leq d : r_p = i} v^{\sum_{j<p} \delta_{a,r_j} - \delta_{a+1,r_j}} t^{\sum_{j<p} \delta_{a,r_j} + \delta_{a+1,r_j}} \{e_{r_1 \ldots r_{p-1}r_{p+1} \ldots r_d}\},$$

$$A^\pm_a \cdot \{e_{r_1 \ldots r_d}\} = v^{\sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\sum_{1 \leq j \leq d} \delta_{a,r_j}} \{e_{r_1 \ldots r_d}\} \quad \text{and}$$

$$B^\pm_a \cdot \{e_{r_1 \ldots r_d}\} = v^{+\sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\sum_{1 \leq j \leq d} \delta_{a,r_j}} \{e_{r_1 \ldots r_d}\} \quad \text{and}$$

\[\begin{align*}
A^\pm_a \cdot \{e_{r_1 \ldots r_d}\} &= v^{\sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\sum_{1 \leq j \leq d} \delta_{a,r_j}} \{e_{r_1 \ldots r_d}\} \\
B^\pm_a \cdot \{e_{r_1 \ldots r_d}\} &= v^{\sum_{1 \leq j \leq d} \delta_{a,r_j}} t^{\sum_{1 \leq j \leq d} \delta_{a,r_j}} \{e_{r_1 \ldots r_d}\}
\end{align*}\]

Proof. The first two identities follow directly from Proposition 4.2.2. The last two identities are straightforward.

4.4. $\mathcal{H}_Y$-action on $\mathcal{V}$.

Definition 4.4.1. The two parameter Iwahori-Hecke algebra $\mathcal{H}_d(v,t)$ of type $A_d$ is a unital associative algebra over $\mathbb{Q}(v,t)$ generated by $T_i$ for $i \in [1, d-1]$ and subject to the following relations.

$$T_i^2 = (vt - v^{-1}t)T_i + t^2, \quad 1 \leq i \leq d-1,$$

$$T_i T_{j+1} T_j = T_{j+1} T_j T_{j+1}, \quad 1 \leq j \leq d-2,$$

$$T_i T_j = T_j T_i, \quad |i - j| > 1.$$

We shall provide an explicit description of the action of $\mathcal{H}_Y$ on $\mathcal{V}$. For any $1 \leq j \leq d-1$, we define a function $T_j$ in $\mathcal{H}_Y$ by

$$T_j(F, F') = \begin{cases} v^{-1}t, & \text{if } F_i = F'_i \forall i \in [1, d] \setminus \{j\}, F_j \neq F'_j; \\
0, & \text{otherwise}. \end{cases}$$

Lemma 4.4.2. The assignment of sending the functions $T_j$, for $1 \leq j \leq d-1$, in the algebra $\mathcal{H}_Y$ to the generators of $\mathcal{H}_d$ in the same notations is an isomorphism.

Given $B = (b_{ij}) \in \Pi$, let $r_c$ be the unique number in $[1, n]$ such that $b_{rc} = 1$ for each $c \in [1, d]$. The correspondence $B \mapsto \tilde{B} = (r_1, \ldots, r_d)$ defines a bijection between $\Pi$ and the set of all sequences $(r_1, \ldots, r_d)$. Denote by $e_{r_1 \ldots r_d}$ the characteristic function of the $G$-orbit corresponding to the matrix $B$ in $\mathcal{V}$. It is clear that the collection of these characteristic functions provides a basis for $\mathcal{V}$.

Lemma 4.4.3. The action of $\mathcal{H}_Y$ on $\mathcal{V}$ is described as follows. For $1 \leq j \leq d-1$, we have

\[\{e_{r_1 \ldots r_d}\} T_j = \begin{cases} \{e_{r_1 \ldots r_j-1 r_{j+1} \ldots r_d}\}, & r_j < r_{j+1}; \\
vt \{e_{r_1 \ldots r_d}\}, & r_j = r_{j+1}; \\
(vt - v^{-1}t) \{e_{r_1 \ldots r_d}\} + t^2 \{e_{r_1 \ldots r_j-1 r_{j+1} \ldots r_d}\}, & r_j > r_{j+1}. \end{cases}\]

Proof. Formula (18) similar with the one in [GL92, 1. 12], whose proof is also almost the same as one parameter of type-A case.
4.5. Generators of $S$. Define a partial order "$\preceq$" on $\Theta_d$ by $a \preceq b$ if $\mathcal{O}_a \subset \mathcal{O}_b$. For any $a = (a_{ij})$ and $b = (b_{ij})$ in $\Xi_d$, we say that $a \preceq b$ if and only if the following two conditions hold.

\begin{align}
\sum_{r \leq i, s \geq j} a_{rs} &\leq \sum_{r \leq i, s \geq j} b_{rs}, \quad \forall i < j. \\
\sum_{r \geq i, s \leq j} a_{rs} &\leq \sum_{r \geq i, s \leq j} b_{rs}, \quad \forall i > j.
\end{align}

The relation "$\preceq$" defines a second partial order on $\Theta_d$. We say that $a < b$ if $a \preceq b$ and at least one of the inequalities in (19) is strict. We shall denote by "$\{m\} + \text{lower terms}$" an element in $S$ which is equal to $\{m\}$ plus a linear combination of $\{m'\}$ with $m' < m$. By Proposition (4.2.2), we have

**Corollary 4.5.1.** Assume that $1 \leq h < n$, $1 \leq h \leq n$, $M = (m_{ij}) \in \Theta_d$.

(a) Assume that $m_{h,j} = 0, \forall j > k, m_{h+1,j} = 0, \forall j \geq k$. Let $r = m_{h,k}, a = (a_{ij}) \in \Xi_d$ satisfies the following two conditions: $a_{h,k} = 0, a_{i,k} = r, a_{i,j} = m_{i,j}$ for all other $i, j$. If $b$ is subject to $b - re_{h,h+1}$ is diagonal, $\text{co}(b) = \text{ro}(a)$, then

\[ \{b\} \ast \{a\} = \{M\} + \text{lower terms}. \]

(b) Assume that $m_{h,j} = 0, \forall j \leq k, m_{h+1,j} = 0, \forall j < k$. Let $r = m_{h+1,k}, a = (a_{ij}) \in \Theta_d$ satisfies the following two conditions: $a_{h,k} = r, a_{i,k} = 0, a_{i,j} = m_{i,j}$ for all other $i, j$. If $c$ is subject to $c - re_{h,h+1}$ is diagonal, $\text{co}(c) = \text{ro}(a)$, then

\[ \{c\} \ast \{a\} = \{M\} + \text{lower terms}. \]

**Proof.** In case (a), from the proof of the [BLM90] 3.8, we have that $\{M\}$ is correspondence to $t = (0, \cdots, 0, R, 0, \cdots, 0)$, where $R$ is in the $k$ place. Therefore, $\alpha(t) = \sum_{j \geq k} a_{h,j}t_k + \sum_{j > k} a_{h+1,j}t_k - \sum_{j < l} t_j t_l = 0$. Then (a) follows.

In case (b), we have that $\{M\}$ is correspondence to $t = (0, \cdots, 0, R, 0, \cdots, 0)$, where $R$ is in the $k$ place. Therefore, $\alpha'(t) = \sum_{j \leq l} a_{h+1,j}t_l + \sum_{j < l} a_{h,j}t_l - \sum_{j < l} t_j t_l = 0$. Then (b) follows. \[ \square \]

**Theorem 4.5.2.** For any $a = (a_{ij}) \in \Theta_d$. The following identity holds in $S$

\[ \prod_{1 \leq i \leq h < j \leq n} \{D_{i,j} + a_{ij}E_{h,h+1}\} \ast \prod_{1 \leq j \leq h < i \leq n} \{D_{i,j} + a_{ij}E_{h+1,h}\} = \{a\} + \text{lower terms}, \]

where the product is taken in the following order. The factors in the first product are taken in the following order: $(i, h, j)$ comes before $(i', h', j')$ if either $j > j'$ or $j = j'$, $h - i < h' - i'$, or $j = j', h - i = h' - i', i' > i$. The factors in the second product are taken in the following order: $(i, h, j)$ comes before $(i', h', j')$ if either $i < i'$ or $i = i', h - j > h' - j'$, or $i = i', h - j = h' - j', j' < j$. The matrices $D_{i,j}$ are diagonal with entries in $\mathbb{N}$. Which are uniquely determined.

**Proof.** The proof of this theorem is similar to the [BLM90] 3.9. \[ \square \]

We have immediately

**Corollary 4.5.3.** The products $m_a = \prod_{1 \leq i \leq h < j \leq n} \{D_{i,j} + a_{ij}E_{h,h+1}\} \ast \prod_{1 \leq j \leq h < i \leq n} \{D_{i,j} + a_{ij}E_{h+1,h}\}$ for any $a \in \Theta_d$ in Theorem 4.5.2 form a basis for $S$.\[ \square \]
Corollary 4.5.4. The algebra $\mathcal{S}$ (resp. $\mathcal{Q}(v) \otimes_{A} \mathcal{S}$) is generated by the elements $[\epsilon]$ such that $\epsilon - RE_{i,i+1}$ (resp. either $\epsilon$ or $\epsilon - RE_{i,i+1}$) is diagonal for some $R \in \mathbb{N}$ and $i \in [1,n-1]$.

Observe that $E_{i} = \sum t(\{b\}, F_{i} = \sum \{a\}, A_{a}^{\pm 1} = \sum v^{\pm da_{t} \pm da} \{d\}, B_{a}^{\pm 1} = \sum v^{\pm da_{t} \pm da} \{d\}, \forall i \in [1,n-1], a \in [1,n]$, where $b, c$ and $d$ run over all matrices in $\Theta_{d}$ such that $b - E_{i,i+1}$, $c - E_{i+1,i}$ and $d$ are diagonal, respectively, and $d_{a}$ is the $(a,a)$-entry of the matrix in $d$. We have the following corollary by Corollary 4.5.4.

Corollary 4.5.5. The algebra $\mathcal{Q}(v, t) \otimes_{A} \mathcal{S}$ is generated by the functions $E_{i}, F_{i}, A_{a}^{\pm 1}, B_{a}^{\pm 1}$ for any $i \in [1,n-1], a \in [1,n]$.

5. The limit algebra $\mathcal{K}$

5.1. Stabilization. Let $I$ be the identity matrix. We set $pA = A + pI$. Let $\bar{\Theta}$ be the set of all $n \times n$ matrices with integer entries such that the entries off diagonal are $\geq 0$.

Let

$$\mathcal{K} = \text{span}_{A}\{\{a\} \mid a \in \bar{\Theta}\},$$

where the notation $\{a\}$ is a formal symbol. Let $v', t'$ be a independent indeterminates, and we denote by $\mathfrak{K}$ the ring $\mathcal{Q}(v, t)[v', t']$.

Proposition 5.1.1. Suppose that $a_{1}, a_{2}, \ldots, a_{r}$ ($r \geq 2$) are matrices in $\bar{\Theta}$ such that $\text{co}(a_{i}) = \text{ro}(a_{i+1})$ for $1 \leq i \leq r - 1$. There exist $\tilde{\beta}_{1}, \ldots, \tilde{\beta}_{m} \in \bar{\Theta}$, $G_{j}(v, v', t, t') \in \mathfrak{K}$ and $p_{0} \in \mathbb{N}$ such that in $\mathcal{S}_{d}$ for some $d$, we have

$$[p_{0}a_{1}] * [p_{0}a_{2}] * \cdots * [p_{0}a_{r}] = \sum_{j=1}^{m} G_{j}(v, v^{-p}, t, t') [p_{0}\tilde{\beta}_{j}], \quad \forall p \geq p_{0}.$$

Proof. The proof is essentially the same as the one for Proposition 4.2 in [BLM90] by using Corollary 4.2.2 and Theorem 4.5.5. The main difference is that we should give how the twists $\alpha(t)$ and $\alpha'(t)$ change when $a$ is replaced by $p_{0}a$.

If $r = 2$ and $a_{1}$ is chosen such that $a_{1} - RE_{h,h+1}$ is a diagonal with $R \in \mathbb{N}$, the structure constant $G_{t}(v, v', t, t')$ is defined by

$$G_{t}(v, v', t, t') = v^{\beta(t)} \prod_{1 \leq h \leq n} \left( a_{h,u} + t_{u} \right) \prod_{1 \leq h \leq t} \left( v^{2(a_{h,u} + t_{u})} - 1 \right) t^{\sum_{h<ht} u_{h}}.$$

Similarly, if $r = 2$ and $a_{1}$ is chosen such that $a_{1} - RE_{h+1,h}$ is diagonal with $R \in \mathbb{N}$, the structure constant $G_{t}(v, v', t, t')$ is defined by

$$G_{t}(v, v', t, t') = v^{\beta'(t)} \prod_{1 \leq u \leq n, u \neq h+1} \left( a_{h+1,u} + t_{u} \right) \prod_{1 \leq h \leq t} \left( v^{2(a_{h+1,u} + t_{u})} - 1 \right) t^{\sum_{h<ht} u_{h}},$$

Keep in mind the above modifications, the rest of the proof for Proposition 4.2 in [BLM90] can be repeated here. \qed
By specialization $v', t'$ at $v' = 1, t' = 1$, there is a unique associative $A$-algebra structure on $K$, without unit, where the product is given by

$$\{a_1\} \cdot \{a_2\} \cdot \cdots \cdot \{a_r\} = \sum_{j=1}^{m} G_j(v, 1, 1) [\hat{a}_j]$$

if $a_1, \ldots, a_r$ are as in Proposition 5.1.1.

Let $a$ and $b$ be elements of $\hat{\Theta}$ so that $b - rE_{h,h+1}$ is diagonal for some $1 \leq h < n, r \in \mathbb{N}$ satisfying $co(b) = ro(a)$. Then we have

$$\{b\} \cdot \{a\} = \sum_t v^{\beta(t)} t^{\alpha(t)} \prod_{u=1}^{N} \left( \frac{a_{hu} + t_u}{t_u} \right) \{a_t\},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^N$ such that $\sum_{u=1}^{n} t_u = r$ and $t_u \leq a_{h,u}u$ for all $u \neq h + 1, \alpha(t), \beta(t), a_u \in \hat{\Theta}$ are defined in (16).

Similarly, if $a, c \in \hat{\Theta}$ are chosen such that $c - rE_{h+1,h}$ is diagonal for some $1 \leq h < n, r \in \mathbb{N}$ satisfying $co(c) = ro(a)$, then we have

$$\{c\} \cdot \{a\} = \sum_t v^{\beta(t)} t^{\alpha(t)} \prod_{u=1}^{N} \left( \frac{a_{hu+1} + t_u}{t_u} \right) \{a(h, t)\},$$

where the sum is taken over all $t = (t_u) \in \mathbb{N}^N$ such that $\sum_{u=1}^{n} t_u = r$ and $t_u \leq a_{h,u}$ for all $u \neq h, \alpha(t), \beta(t), (a, h, t) \in \hat{\Theta}$ are defined in (17).

5.2. The algebra $U$. In this section, we shall define a new algebra $U$ in the completion of $K$ similar to [BLM90] Section 5.

Let $\hat{K}$ be the $\mathbb{Q}(v, t)$-vector space of all formal sum $\sum_{a \in \hat{\Theta}} \xi_a \{a\}$ with $\xi_a \in \mathbb{Q}(v, t)$ and a locally finite property, i.e., for any $t \in \mathbb{Z}^n$, the sets $\{a \in \hat{\Theta} | ro(a) = t, \xi_a \neq 0\}$ and $\{a \in \hat{\Theta} | co(a) = t, \xi_a \neq 0\}$ are finite. The space $\hat{K}$ becomes an associative algebra over $\mathbb{Q}(v, t)$ when equipped with the following multiplication:

$$\sum_{a \in \Xi_D} \xi_a \{a\} \cdot \sum_{b \in \Xi_D} \xi_b \{b\} = \sum_{a,b} \xi_a \xi_b \{a \cdot b\},$$

where the product $\{a\} \cdot \{b\}$ is taken in $K$.

Observe that the algebra $\hat{K}$ has a unit element $\sum \{\emptyset\}$, the summation of all diagonal matrices.

We define the following elements in $\hat{K}$. For any nonzero matrix $a \in \hat{\Theta}$, let $\hat{a}$ be the matrix obtained by replacing diagonal entries of $a$ by zeroes. We set

$$\Theta^0 = \{\hat{a} | a \in \hat{\Theta}\}.$$ 

For any $\hat{a}$ in $\Theta^0$ and $j = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, we define

$$\hat{a}(j) = \sum_{\lambda} v^{\lambda_1 j_1 + \cdots + \lambda_n j_n} t^{\lambda_1 |j_1| + \cdots + \lambda_n |j_n|} \{\hat{a} + D_\lambda\}$$

where the sum runs through all $\lambda = (\lambda_i) \in \mathbb{Z}^n$ such that $\hat{a} + D_\lambda \in \hat{\Theta}$, where $D_\lambda$ is the diagonal matrices with diagonal entries($\lambda_i$).
For $i \in [1, n-1]$, let
\[
E_i = E_{i,i+1}(0) \quad \text{and} \quad F_i = E_{i+1,i}(0).
\]

Let $\mathcal{U}$ be the subalgebra of $\hat{\mathcal{K}}$ generated by $E_i, F_i, 0(j)$ for all $i \in [1, n-1]$ and $j \in \mathbb{Z}^n$.

**Proposition 5.2.1.** The following relations hold in $\mathcal{U}$.

\[
\begin{align*}
(24) & \quad 0(j)0(j') = 0(j')0(j), \\
(25) & \quad 0(j)E_h = v^{j_h - j_{h+1}} t^{j_h} E_h 0(j), \quad 0(j)F_h = v^{-j_h + j_{h+1}} t^{-j_h} E_h 0(j), \\
(26) & \quad t(E_h F_h - F_h E_h) = (v - v^{-1})^{-1}(0(h - h + 1) - 0(h + 1 - h)), \\
(27) & \quad E_i^2 E_{i+1} - (vt + v^{-1}) E_i E_{i+1} + t^2 E_{i+1} E_i^2 = 0, \\
(28) & \quad t^2 E_i^2 E_{i+1} - (vt + v^{-1}) E_i E_{i+1} + E_i E_{i+1} E_i = 0, \\
(29) & \quad F_i^2 F_{i+1} - (vt - v^{-1}) F_i F_{i+1} + t F_{i+1} F_i^2 = 0, \\
(30) & \quad t^{-2} F_i^2 F_{i+1} - (vt - v^{-1}) F_{i+1} F_i + F_i F_{i+1}^2 = 0.
\end{align*}
\]

where $j, j' \in \mathbb{Z}^n$, $h, i, j \in [1, n]$ and $\mathbf{1} \in \mathbb{N}^N$ is the vector whose $i$-th entry is 1 and 0 elsewhere.

**Proof.** We show (25). 
\[
0(j)E_h = \sum_{\lambda} v^{\sum \lambda_k j_k} t^{\sum \lambda_k j_k ^{}} \{ D_\lambda \} \sum_{\lambda'} \{ E_{h,h+1} + D_{\lambda'} \} = \sum_{\lambda} v^{\sum \lambda_k j_k + j_h} t^{\sum \lambda_k j_k ^{}} \{ E_{h,h+1} + D_{\lambda'} \},
\]

where the sums run through in an obvious range by the definition in (23).
\[
E_h 0(j) = \sum_{\lambda, \lambda'} v^{\sum \lambda_k j_k} t^{\sum \lambda_k j_k ^{}} \{ E_{h,h+1} + D_{\lambda'} \} \{ D_\lambda \} = \sum_{\lambda} v^{\sum \lambda_k j_k + j_h} t^{\sum \lambda_k j_k ^{}} \{ E_{h,h+1} + D_{\lambda'} \}.
\]

So we have the first identity in (25). All other identities in (24) and (25) can be shown similarly.

We show (26). We have
\[
E_h F_h = \sum_{\lambda} \{ E_{h,h+1} + D_{\lambda} \} \{ E_{h+1,h} + D_{\lambda} \} = \sum_{\lambda} \{ E_{h,h+1} + D_{\lambda} \} \{ E_{h+1,h} + D_{\lambda} \}
\]

\[
= \sum_{\lambda} (v^{\lambda_h - \lambda_{h+1}} t^{\lambda_h + \lambda_{h+1}} (\frac{\lambda_{h+1} + 1}{1}) v^{\lambda_{h+1}}) \{ D_\lambda + E_{h,h} \}
\]

\[
+ \{ E_{h+1,h} + E_{h,h+1} + D_\lambda - E_{h+1,h+1} \}).
\]

Similarly,
\[
F_h E_h = \sum_{\lambda} \{ E_{h+1,h} + D_{\lambda} \} \{ E_{h,h+1} + D_{\lambda} \}
\]

\[
= \sum_{\lambda} (v^{\lambda_{h+1} - \lambda_h} t^{\lambda_h + \lambda_{h+1}} (\frac{\lambda_{h+1} + 1}{1}) v^{\lambda_{h+1}}) \{ D_\lambda + E_{h+1,h+1} \}
\]

\[
+ \{ E_{h+1,h} + E_{h,h+1} + D_\lambda - E_{h,h} \}).
\]

Therefore,
\[
t(E_h F_h - F_h E_h) = \sum_{\lambda} \frac{v^{\lambda_h - \lambda_{h+1}} t^{\lambda_h + \lambda_{h+1} + \lambda_h + \lambda_{h+1}} v^{\lambda_{h+1}}}{v - v^{-1}} \{ D_\lambda \}
\]

\[
= (v - v^{-1})^{-1}(0(h - h + 1) - 0(h + 1 - h)).
\]
At last, we show (27).  
\[ E_h^2 E_{h+1} = \sum_{\lambda} vt(v^{-2} + 1)\{D_\lambda + E_{h,h+1} + E_{h,h+2}\} + \sum_{\lambda} v^{-1} t^3(v^{-2} + 1)\{D_\lambda + E_{h+1,h+2} + 2E_{h,h+1}\}; \]
\[ E_h E_{h+1} E_h = \sum_{\lambda} t^2(v^{-2} + 1)\{D_\lambda + 2E_{h,h+1} + E_{h,h+2}\} + \sum_{\lambda} \{D_\lambda + E_{h,h+1} + E_{h,h+2}\}; \]
\[ E_{h+1} E_h^2 = \sum_{\lambda} vt(v^{-2} + 1)\{D_\lambda + 2E_{h,h+1} + E_{h+1,h+2}\}. \]

Then the first identity of (27) follows. All other identities can be shown similarly.

The Corollary directly follows.

**Corollary 5.2.2.** The assignment \( E_i \mapsto tE_i, F_i \mapsto F_i, A_a \mapsto 0(a) \) and \( B_a \mapsto 0(-a) \), for any \( i \in [1, n-1], a \in [1, n] \), defines an algebra isomorphism \( \Upsilon : U_{v,t}(gl_n) \rightarrow \mathcal{U} \).

### 6. Schur dualities for two parameter case of type \( A_d \)

In this section, we shall formulate algebraically the dualities between algebras \( U_{v,t}(gl_n) \) and the two parameter Iwahori-Hecke algebras \( H_d(v, t) \) of type \( A_d \).

Let \( V \) be a vector space over \( \mathbb{Q}(v, t) \) of dimension \( n \). We fix a basis \( (v_i)_{1 \leq i \leq n} \) for \( V \). Let \( V^{\otimes d} \) be the \( d \)-th tensor space of \( V \). Thus we have a basis \( (v_{r_1} \otimes \cdots \otimes v_{r_d}) \), where \( r_1, \cdots, r_d \in [1, n] \), for the tensor space \( V^{\otimes d} \).

For a sequence \( r = (r_1, \cdots, r_d) \), we write \( v_r \) for \( v_{r_1} \otimes \cdots \otimes v_{r_d} \).

For a sequence \( r \) and a fixed integer \( p \in [1, d] \), we define the sequence \( r'_p \) and \( r''_p \) by
\[
(r'_p)_j = \begin{cases} 
  r_j, & j \neq p, \\
  r_p - 1, & j = p 
\end{cases} \quad \text{and} \quad (r''_p)_j = \begin{cases} 
  r_j, & j \neq p, \\
  r_p + 1, & j = p 
\end{cases}
\]

**Lemma 6.0.3.** There has a left \( U_{v,t}(gl_n) \)-action on \( V^{\otimes d} \) defined by, for any \( i \in [1, n-1], a \in [1, n] \),
\[
E_i \cdot v_r = \sum_{1 \leq p \leq d; r_p = i+1} v^{\sum_{j \leq p} \delta_{i,j} - \delta_{i+1,j} + \sum_{j > p} \delta_{i,j} + \delta_{i+1,j}} v_{r'_p},
\]
\[
F_i \cdot v_r = \sum_{1 \leq p \leq d; r_p = i} v^{\sum_{j < p} \delta_{i+1,j} - \delta_{i,j} + \sum_{j > p} \delta_{i,j} + \delta_{i+1,j}} v_{r''_p},
\]
\[
A_a^{\pm 1} \cdot v_r = v^{\pm \sum_{1 \leq j \leq d} \delta_{a,j} r_j} v^{\pm \sum_{1 \leq j \leq d} \delta_{a,j}} v_r,
\]
\[
B_a^{\pm 1} \cdot v_r = v^{\pm \sum_{1 \leq j \leq d} \delta_{a,j} r_j} v^{\pm \sum_{1 \leq j \leq d} \delta_{a,j}} v_r.
\]

The lemma follows Proposition 4.1.1 and Corollary 4.3.1.
Lemma 6.0.4. There has a right $H_d$-action on $V^\otimes d$ given by, for $1 \leq j \leq d - 1$,

\[
 v_{r_1 \ldots r_d}^{\sigma} \cdot T_j = \begin{cases} 
 v_{r_1 \ldots r_j - 1 r_{j+1} r_{j+2} \ldots r_d}, & r_j < r_{j+1}; \\
 v^{\sigma} v_{r_1 \ldots r_d}, & r_j = r_{j+1}; \\
 (vt - v^{-1}t)v_{r_1 \ldots r_d} + t^2 v_{r_1 \ldots r_j - 1 r_{j+1} r_{j+2} \ldots r_d}, & r_j > r_{j+1}.
\end{cases}
\]

This lemma follows Lemmas 4.4.2 and 4.4.3.

We now can state the duality.

Proposition 6.0.5. The left $U_{v,t}(gl_n)$-action in Lemma 6.0.3 and the right $H_d$-action in Lemma 4.4.3 on $V^\otimes d$ are commuting. They form a double centralizer for $n \geq d$, i.e.,

$$H_d \simeq \text{End}_U(V^\otimes d) \text{ and } U_{v,t}(gl_n) \to \text{End}_{H_d}(V^\otimes d) \text{ is surjective.}$$

The proposition follows from the previous two lemmas, Lemma 3.1.1 Proposition 4.1.1 and Corollary 4.5.5.

6.1. Galois descend approach. Let $G = \text{Gal}(\mathbb{Q}(v,t)/\mathbb{Q}(r,s))$, $r = vt, s = v^{-1}t$. It is easy to know $G \cong S_2$ which is generated by $\sigma$. $G$ act on $U_{v,t}(gl_n)$ given by a $Q$ algebra homomorphism $\sigma : U_{v,t}(gl_n) \to U_{v,t}(gl_n)$; \(E_i \mapsto -E_i, F_i \mapsto F_i, K_i \mapsto K_i, K'_i \mapsto K'_i, v \mapsto -v, t \mapsto -t\). $G$ can be also act on $V^\otimes k$ which is given by $\sigma : V^\otimes k \to V^\otimes k; v_i \otimes \cdots \otimes v_k \mapsto -v_i \otimes \cdots \otimes v_k, v \mapsto -v, t \mapsto -t$. By the directly compute we have the following lemma.

Lemma 6.1.1. The $G$-actions on $(U_{v,t}(gl_n), V^\otimes k)$ is compatible. That is $\sigma(av) = \sigma(a)\sigma(v)$, $\forall a \in U_{v,t}(gl_n), v \in V$.

Proof. We only need to check the identities $\sigma(av) = \sigma(a)\sigma(v)$ on the generators. By the lemma 6.0.3, the result is obvious.

Though the above lemma we know there is a $G$-action on $H_k(v,t)$ which is given by $\sigma : H_k(v,t) \mapsto H_k(v,t); T_i \mapsto T_i, v \mapsto -v, t \mapsto -t$.

Theorem 6.1.2. $(U_{v,t}(gl_n)^G, V^\otimes k^G, H_k(v,t)^G)$ is a shur-weyl tripple. and $U_{v,t}(gl_n)^G \cong U_{r,s}(gl_n), V^\otimes k^G$ is a $n^k$ dimension vector space over $\mathbb{Q}(r,s)$, $H_k(v,t)^G \cong H_k(r,s)$.

Proof.

Remark 6.1.3. $H_k(r,s)$ is a unital associate algebra over $\mathbb{Q}(r,s)$ with generators $\widetilde{T}_i$, $1 \leq i < k$ subject to the following relations:

1. $\widetilde{T}_i \widetilde{T}_{i+1} \widetilde{T}_i = \widetilde{T}_{i+1} \widetilde{T}_i \widetilde{T}_{i+1}, 1 \leq i < k$.
2. $\widetilde{T}_i \widetilde{T}_j = \widetilde{T}_j \widetilde{T}_i, \text{if} |i - j| \geq 2$.
3. $(\widetilde{T}_i - r)(\widetilde{T}_i + s) = 0, \forall i$.

$U_{r,s}(gl_n)$ is a $\mathbb{Q}(r,s)$ algebra generated by $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i, \widetilde{K}'_i$. 

GEOMETRIC SCHUR DUALITY OF TWO PARAMETER QUANTUM GROUP OF TYPE A 17
7. Two New Quantum Group $U_{v,t}(gl_n)^m$ and $U_{v,t}(gl_n)^m$

In order to give the comultiplication in the two parameter case of two new quantum group appeared in [FL14], we give two new quantum group $U_{v,t}(gl_n)^m$ and $U_{v,t}(gl_n)^m$ in this section. For any $i \in [1, n-1]$, $a \in [1, n]$, $m \in [1, n-1]$ , we define the function $E_i, F_i, A_i^{\pm 1}, B_i^{\pm 1}$ to be the same function in $S$ . we further define

$$J_{\pm}(V, V') = \begin{cases} 1, & \text{if } V = V' \text{ and } |V_m| = d \mod 2; \\ 0, & \text{otherwise.} \end{cases}$$

(32)

$$J_{-}(V, V') = \begin{cases} 1, & \text{if } V = V' \text{ and } |V_m| = d - 1 \mod 2; \\ 0, & \text{otherwise.} \end{cases}$$

All these functions are elements in $S$.

Proposition 7.0.4. The functions $E_i, F_i, A_i^{\pm 1}, B_i^{\pm 1}$ and $J_{\pm}$ in $S$, for any $i \in [1, n-1]$, $a \in [1, n]$ , satisfy the relations in 7.1 together with the following relations.

(R1) $J_{\pm} + J_{-} = 1$, $J_\alpha J_\beta = \delta_{\alpha\beta} J_\alpha$, $J_\pm A_a = A_a J_\pm$, $J_\pm A_a = A_a J_\pm$, $J_\pm E_i = E_i J_{\pm}$, $J_\pm F_i = F_i J_{\pm}, i \neq m$; $J_\pm E_m = E_m J_{\mp}, J_\pm F_m = F_m J_{\mp};$

Corollary 7.0.5. The algebra $Q(v, t) \otimes_\A S$ is generated by the functions $E_i, F_i, A_i^{\pm 1}, B_i^{\pm 1}$, and $J_{\pm}$ in $S$, for any $i \in [1, n-1]$, $a \in [1, n]$.

7.1. Another limit algebra $\K'$. We set $pA = A + 2pI$. Let

$$\K' = \text{span}_A \{a| a \in \Theta\},$$

where the notation $\{a\}$ is a formal symbol. Let $v', t'$ be a independent indeterminates, and we denote by $\mathfrak{R}$ the ring $Q(v, t)[v', t']$.

Proposition 7.1.1. Suppose that $a_1, a_2, \ldots, a_r (r \geq 2)$ are matrices in $\Theta$ such that $\text{co}(a_i) = \text{ro}(a_{i+1})$ for $1 \leq i \leq r - 1$. There exist $\bar{a}_1, \ldots, \bar{a}_m \in \Theta$, $G_j^r(v, v', t', t') \in \mathfrak{R}$ and $p_0 \in \mathbb{N}$ such that in $S_d$ for some $d$, we have

$$\{p a_1\} \ast \{p a_2\} \ast \cdots \ast \{p a_r\} = \sum_{j=1}^m G_j^r(v, v^{-p}, t, t')\{p \bar{a}_j\}, \quad \forall p \geq p_0.$$

By specialization $v', t'$ at $v' = 1, t' = 1$, there is a unique associative $\A$-algebra structure on $\K$, without unit, where the product is given by

$$\{a_1\} \cdot \{a_2\} \cdot \cdots \cdot \{a_r\} = \sum_{j=1}^m G_j^r(v, 1, t, 1)\{a_j\}$$

if $a_1, \ldots , a_r$ are as in Proposition 7.1.1.

Let $a$ and $b \in \Theta$ be chosen such that $b - rE_{m, m+1}$ is diagonal for some $r \in \mathbb{N}$ satisfying $\text{co}(b) = \text{ro}(a)$. Then we have

$$\{b\} \cdot \{a\} = \sum_t v^\beta(t) t^\alpha(t) \prod_{u=1}^N \frac{(a_{hu} + t_u)}{t_u} \{a_t\},$$

(33)

$$\{b\} \cdot \{a\} = \sum_t v^\beta(t) t^\alpha(t) \prod_{u=1}^N \frac{(a_{hu} + t_u)}{t_u} \{a_t\},$$
where the sum is taken over all \( t = (t_u) \in \mathbb{N}^n \) such that \( \sum_{u=1}^n t_u = r \) and \( t_u \leq a_{m+1,u} \), \( \alpha(t), \beta(t), a_t \in \tilde{\Theta} \) are defined in (19).

Similarly, if \( a, c \in \tilde{\Theta} \) are chosen such that \( c - rE_{m+2,m+1} \) is diagonal for some \( 1 \leq h < n, r \in \mathbb{N} \) satisfying \( \text{co}(c) = \text{ro}(a) \), then we have

\[
(35) \quad \{ c \} \cdot \{ a \} = \sum_t \nu^{\alpha'(t)} t^{\alpha(t)} \prod_{u=1}^N \left( \frac{a_{h+1,u} + t_u}{t_u} \right) \{ a(h,t) \},
\]

where the sum is taken over all \( t = (t_u) \in \mathbb{N}^n \) such that \( \sum_{u=1}^n t_u = r \) and \( t_u \leq a_{m+1,u} \), \( \alpha'(t), \beta'(t) \) \( a(t), a(t) \in \tilde{\Theta} \) are defined in (17).

### 7.2. The algebra \( \mathcal{U}' \)

In this section, we shall define a new algebra \( \mathcal{U} \) in the completion of \( \mathcal{K} \) similar to [BLM90] Section 5.

Let \( \hat{\mathcal{K}} \) be the \( \mathbb{Q}(v,t) \)-vector space of all formal sum \( \sum_{a \in \tilde{\Theta}} \xi_a \{ a \} \) with \( \xi_a \in \mathbb{Q}(v,t) \) and a locally finite property, i.e., for any \( t \in \mathbb{Z}^n \), the sets \( \{ a \in \tilde{\Theta} | \text{ro}(a) = t, \xi_a \neq 0 \} \) and \( \{ a \in \tilde{\Theta} | \text{co}(a) = t, \xi_a \neq 0 \} \) are finite. The space \( \hat{\mathcal{K}} \) becomes an associative algebra over \( \mathbb{Q}(v,t) \) when equipped with the following multiplication:

\[
\sum_{a \in \tilde{\Theta}} \xi_a \{ a \} \cdot \sum_{b \in \tilde{\Theta}} \xi_b \{ b \} = \sum_{a,b} \xi_a \xi_b \{ a \} \cdot \{ b \},
\]

where the product \( \{ a \} \cdot \{ b \} \) is taken in \( \mathcal{K} \). This is shown in exactly the same as [BLM90] Section 5.

Observe that the algebra \( \hat{\mathcal{K}} \) has a unit element \( \sum \{ a \} \), the summation of all diagonal matrices.

We define the following elements in \( \hat{\mathcal{K}} \). For any nonzero matrix \( a \in \tilde{\Theta} \), let \( \hat{a} \) be the matrix obtained by replacing diagonal entries of \( a \) by zeroes. We set \( \Theta^0 = \{ \hat{a} | a \in \tilde{\Theta} \} \).

For any \( \hat{a} \in \Theta^0 \) and \( j = (j_1, \ldots, j_n) \in \mathbb{Z}^n \), we define

\[
(36) \quad \hat{a}(j) = \sum_{\lambda} \nu^{\lambda_1 j_1 + \cdots + \lambda_n j_n} t^{\lambda_1 j_1 + \cdots + \lambda_n j_n} \{ \hat{a} + D_\lambda \}
\]

where the sum runs through all \( \lambda = (\lambda_i) \in \mathbb{Z}^n \) such that \( \hat{a} + D_\lambda, \in \tilde{\Theta} \), where \( D_\lambda \) is the diagonal matrices with diagonal entries \( (\lambda_i) \).

And we also define

\[
J_+ = \sum_{\lambda \in S_0} \{ D_\lambda \}, \quad J_- = \sum_{\lambda \in S_1} \{ D_\lambda \},
\]

Where \( S_0 = \{ \lambda | \sum_{i=1}^m \lambda_i \equiv 0 \mod 2 \}, S_1 = \{ \lambda | \sum_{i=1}^m \lambda_i \equiv \sum_{i=1}^n \lambda_i - 1 \mod 2 \} \)

For \( i \in [1, n-1] \), let

\[
E_i = E_{i, i+1}(0) \quad \text{and} \quad F_i = E_{i+1, i}(0).
\]

Let \( \mathcal{U}' \) be the subalgebra of \( \hat{\mathcal{K}} \) generated by \( E_i, F_i, 0(j) \), \( J_\pm \) for all \( i \in [1, n-1] \) and \( j \in \mathbb{Z}^n \).

### Proposition 7.2.1.

The following relations hold in \( \mathcal{U}' \).

\[
(37) \quad \begin{align*}
J_+ E_i &= E_i J_+, \quad J_+ F_i = F_i J_+, \quad i \neq m; \\
J_\pm E_m &= E_m J_\pm, \quad J_\pm F_m = F_m J_\pm;
\end{align*}
\]

\[
(38) \quad 0(j) 0(j') = 0(j') 0(j).
\]
The algebra $E$. $A_i$.

**Definition 7.3.1.** $\mathcal{U}_{v,t}(gl_n)^m.$

$U_{v,t}(gl_n)^m$ is an associative $\mathbb{Q}(v,t)$-algebra with 1 generated by symbols $E_i, F_i, A_a, B_a, J_a$ for all $i \in [1, n - 1], a \in [1, n]$ and $\alpha \in \{+, -\}$ and subject to the following relations.

$$
\begin{align*}
J_+ + J_- &= 1, \quad J_+ J_\beta = \delta_{\alpha,\beta} J_\alpha, \quad J_\pm A_a = A_a J_\pm, \quad J_\pm B_a = B_a J_\pm, \\
J_\pm E_i &= E_i J_\pm, \quad J_\pm F_i = F_i J_\pm, i \neq m, \quad J_\pm E_m = E_m J_\pm, J_\pm F_m = F_m J_\pm; \\
A_i^{+1} A_i^{-1} &= A_i^{-1} A_i^{+1}, \quad B_i^{+1} B_i^{-1} = B_i^{-1} B_i^{+1}, \\
A_i^{+1} B_i^{-1} &= B_i^{+1} A_i^{-1}, \quad A_i^{-1} A_i^{+1} = 1 = B_i^{+1} B_i^{-1}, \\
A_i E_j A_i^{-1} &= v^{(i,j)} t^{(i,j)} E_j, \quad B_i E_j B_i^{-1} = v^{-(i,j)} t^{-(i,j)} E_j, \\
A_i F_j A_i^{-1} &= v^{(i,j)} t^{-(i,j)} F_j, \quad B_i F_j B_i^{-1} = v^{(i,j)} t^{(i,j)} F_j, \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{A_i B_{i+1} - B_i A_{i+1}}{t v - t v^{-1}}.
\end{align*}
$$

$$
\begin{align*}
E_i^2 E_{i+1} - (v + v^{-1}) E_i E_{i+1} E_i + t^2 E_{i+1} E_i^2 &= 0, \\
t^2 E_{i+1}^2 E_i - (v + v^{-1}) E_{i+1} E_i E_{i+1} + E_i^2 E_{i+1}^2 &= 0, \\
F_i^2 F_{i+1} - (v + v^{-1}) F_i F_{i+1} F_i + t F_{i+1} F_i^2 &= 0, \\
t^{-2} F_{i+1}^2 F_i - (v + v^{-1}) F_{i+1} F_i F_{i+1} + F_i F_{i+1}^2 &= 0.
\end{align*}
$$

**Proposition 7.3.2.** The assignment $E_i \mapsto E_i, F_i \mapsto F_i, A_a \mapsto 0(a), B_a \mapsto 0(-a), and J_\alpha \mapsto J_\alpha$ for any $i \in [1, n - 1], a \in [1, n]$ and $\alpha \in \{+, -\}$ defines a algebra isomorphism $\Upsilon : \mathcal{U}_{v,t}(gl_n)^m \rightarrow \mathcal{U}'$.

**7.4. Defining relations of $S$.** For any $i \in [1, n - 1], a \in [1, n], m \in [1, n - 1]$ we define the function $E_i, F_i, A_a^{+1}, B_a^{+1}$ to be the same function in $S$. we further define

$$
\begin{align*}
J_+(V, V') &= \begin{cases} 1, & \text{if } V_m = V_{m+1}, |V_m| \equiv d \mod 2; \\
0, & \text{otherwise.}
\end{cases} \\
J_-(V, V') &= \begin{cases} 1, & \text{if } V_m = V_{m+1}, |V_m| \equiv d - 1 \mod 2; \\
0, & \text{otherwise.}
\end{cases}
\end{align*}
$$

$J_0 = 1 - J_+ - J_-$. 

where $j, j' \in \mathbb{Z}^n, h, i, j \in [1, n]$ and $i \in \mathbb{N}^N$ is the vector whose $i$-th entry is 1 and 0 elsewhere.
Proposition 7.4.1. The functions $E_i, F_i, A_a^{\pm 1}, B_a^{\pm 1}, \text{and} J_\alpha$ in $S$, for any $i \in [1, n - 1], \alpha \in [1, n], \alpha \in \{+,-,0\}$, satisfy the relations in Proposition 4.17 and the following relations.

(57) \[ J_+ + J_0 + J_- = 1, J_\alpha J_\beta = \delta_{\alpha,\beta} J_\alpha, J_\alpha A_a = A_a J_\alpha, J_\alpha B_a = A_a B_a; \]

(58) \[ E_i J_\pm = (1 - \delta_{i,m}) J_\pm E_i, J_\pm E_i = (1 - \delta_{i,m+1}) E_i J_\pm; \]

(59) \[ F_i J_\pm = (1 - \delta_{i,m+1}) J_\pm F_i, J_\pm F_i = (1 - \delta_{i,m}) F_i J_\pm; \]

(60) \[ J_\pm E_m E_{m+1} = E_m E_{m+1} J_\mp; \]

(61) \[ J_\pm F_{m+1} F_m = F_{m+1} F_m J_\mp; \]

(62) \[ J_\pm E_m F_m - E_m F_m J_\mp = \frac{A_m B_{m+1} - B_m A_{m+1}}{v - v^{-1}} (J_\pm - J_\mp); \]

(63) \[ J_\pm F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_\mp = \frac{B_{m+1} A_{m+2} - A_{m+1} B_{m+2}}{v - v^{-1}} (J_\pm - J_\mp). \]

Proof. The first identity in the first three rows of the relations in the proposition are straightforward. Let $\lambda'_i = |V_i'/V_{i-1}'|$. We show the identity (61) by a direct calculation. We have

$$ F_{m+1} F_m (V, V') = \begin{cases} v^{-\lambda'_{m+2}-\lambda'_m} t^{\lambda'_{m+2}+\lambda'_m}, & \text{if } V_m \subseteq V'_m \text{ and } V_{m+1} \subseteq V'_{m+1}, \\ 0, & \text{otherwise.} \end{cases} $$

$$ J_+ F_{m+1} F_m (V, V') = \begin{cases} v^{-\lambda'_{m+2}-\lambda'_m} t^{\lambda'_{m+2}+\lambda'_m}, & \text{if } V_m \subseteq V'_m, \ V_{m+1} \subseteq V'_{m+1}, \ V_m = V_{m+1} \\ 0, & \text{if } |V_m| \equiv d \mod 2, \text{ and otherwise.} \end{cases} $$

$$ F_{m+1} F_m J_- (V, V') = \begin{cases} v^{-\lambda'_{m+2}-\lambda'_m} t^{\lambda'_{m+2}+\lambda'_m}, & \text{if } V_m \subseteq V'_m, \ V_{m+1} \subseteq V'_{m+1}, \ V_m = V_{m+1} \\ 0, & \text{if } |V_m| \equiv d \mod 2, \text{ and otherwise.} \end{cases} $$

The first part of the identity (61) follows, all other identities in (61) and (60) can be shown similarly.

Then, We show the identity (63) By a direct calculation, we have

$$ F_{m+1} E_{m+1} (V, V') = \begin{cases} v^{\lambda'_{m+2}} - v^{-\lambda'_{m+2}+\lambda'_m+1} t^{\lambda'_{m+2}+\lambda'_m+1}, & \text{if } V = V' \\ 0, & \text{otherwise.} \end{cases} $$

$$ (J_+ F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_-)(V, V') = \begin{cases} v^{\lambda'_{m+2} t^{\lambda'_m+2} - \lambda'_m+1} v^{-\lambda'_{m+2} t^{\lambda'_m} + \lambda'_m+1}, & \text{if } V = V', \ V_m = V_{m+1}, \ V_m \subseteq V_{m+1} \text{ and } |V_m| \equiv d \mod 2, \\ v^{-\lambda'_{m+2} t^{\lambda'_m+2} - \lambda'_m+1} v^{\lambda'_{m+2} t^{\lambda'_m} + \lambda'_m+1}, & \text{if } V = V', \ V_m = V_{m+1}, \ V_m \subseteq d - 1 \mod 2, \text{ and otherwise.} \end{cases} $$
\[ \frac{B_{m+1}A_{m+2} - A_{m+1}B_{m+2}}{v - v^{-1}} (J_\pm - J_\mp) (V, V') = \begin{cases} \\ v^{m+2}\lambda^{m+2} t^{m+2} - v^{-}\lambda^{m+2} t^{m+2}, & \text{if } V = V', V_m = V_{m+1}, \\
& \text{and } |V_m| \equiv d \mod 2, \\
0, & \text{otherwise.} \end{cases} \]

The first part of the identity follows, all other identities in \[63\] and \[62\] can be shown similarly.

\[ \square \]

Corollary 7.4.2. The algebra \( Q(v, t) \otimes_\mathcal{A} \mathcal{S} \) is generated by the functions \( E_i, F_i, A_a^{\pm 1}, B_b^{\pm 1}, \) and \( J_\alpha \) in \( \mathcal{S} \), for any \( i \in [1, n-1], a \in [1, n], \alpha \in \{+, -, 0\} \).

7.5. Limit algebra \( \mathcal{K}'' \). Let \( I' = I - E_{m+1, m+1} \) be the identity matrix. We set \( pA = A + 2pI' \)

Let \( \widetilde{\Theta}' = \{ M | M \in \widetilde{\Theta}, M_{m+1,m+1} \geq 0 \} \).

Let

\[ \mathcal{K}'' = \text{span}_\mathcal{A}\{ \{a\} | a \in \widetilde{\Theta}' \} \],

where the notation \( \{a\} \) is a formal symbol bearing no geometric meaning. Let \( v', t' \) be an independent indeterminates, and we denote by \( \mathfrak{R} \) the ring \( Q(v, t)[v', t'] \).

Proposition 7.5.1. Suppose that \( a_1, a_2, \ldots, a_r \) \( (r \geq 2) \) are matrices in \( \widetilde{\Theta}' \) such that \( \text{co}(a_i) = \text{ro}(a_{i+1}) \) for \( 1 \leq i \leq r - 1 \). There exist \( \delta_1, \ldots, \delta_m \in \widetilde{\Theta}' \), \( G_j'(v, v', t, t') \in \mathfrak{R} \) and \( p_0 \in \mathbb{N} \) such that in \( \mathcal{S}_d \) for some \( d \), we have

\[ \{p_1a_1\} \ast \{p_2a_2\} \ast \cdots \ast \{p_r a_r\} = \sum_{j=1}^{m} G_j'(v, v^{-p}, t, t') \{p_3 j\}, \quad \forall p \geq p_0. \]

By specialization \( v', t' \) at \( v' = 1, t' = 1 \), there is a unique associative \( \mathcal{A} \)-algebra structure on \( \mathcal{K} \), without unit, where the product is given by

\[ \{a_1\} \cdot \{a_2\} \cdots \cdot \{a_r\} = \sum_{j=1}^{m} G_j'(v, 1, 1, 1) \{j\} \]

if \( a_1, \ldots, a_r \) are as in Proposition 7.5.1.

Let \( a \) and \( b \in \widetilde{\Theta} \) be chosen such that \( b - rE_{m,m+1} \) is diagonal for some \( r \in \mathbb{N} \) satisfying \( \text{co}(b) = \text{ro}(a) \). Then we have

\[ \{b\} \cdot \{a\} = \sum_t v^{\beta(t)} t^\alpha(t) \prod_{u=1}^{N} \frac{a_{hu} + t_u}{t_u} \{a_t\}, \]

where the sum is taken over all \( t = (t_u) \in \mathbb{N}^n \) such that \( \sum_{u=1}^{n} t_u = r \) and \( t_u \leq a_{m+1,u} \), \( \alpha(t), \beta(t), a_t \in \widetilde{\Theta}' \) are defined in \( [16] \).

Similarly, if \( a, c \in \widetilde{\Theta} \) are chosen such that \( c - rE_{m+2,m+1} \) is diagonal for some \( 1 \leq h < n, r \in \mathbb{N} \) satisfying \( \text{co}(c) = \text{ro}(a) \), then we have

\[ \{c\} \cdot \{a\} = \sum_t v^{\beta'(t)} t^{\alpha'(t)} \prod_{u=1}^{N} \frac{a_{h+1,u} + t_u}{t_u} \{a(h, t)\}, \]

if
where the sum is taken over all \( t = (t_u) \in \mathbb{N}^n \) such that \( \sum_{u=1}^n t_u = r \) and \( t_u \leq a_{m+1,u} \), \( ,o'(t),\beta'(t) a(h, t) \in \hat{\Theta}' \) are defined in [17].

7.6. The algebra \( \mathcal{U}' \). In this section, we shall define a new algebra \( \mathcal{U}' \) in the completion of \( \mathcal{K}' \) similar to [BLM90, Section 5].

Let \( \hat{\mathcal{K}}' \) be the \( \mathbb{Q}(v,t) \)-vector space of all formal sum \( \sum_{a \in \hat{\Theta}'} \xi_a \{ a \} \) with \( \xi_a \in \mathbb{Q}(v,t) \) and a locally finite property, i.e., for any \( t \in \mathbb{Z}^n \), the sets \( \{ a \in \hat{\Theta}'|\text{ro}(a) = t, \xi_a \neq 0 \} \) and \( \{ a \in \hat{\Theta}'|\text{co}(a) = t, \xi_a \neq 0 \} \) are finite. The space \( \mathcal{K}' \) becomes an associative algebra over \( \mathbb{Q}(v,t) \) when equipped with the following multiplication:

\[
\sum_{a \in \hat{\Theta}'} \xi_a \{ a \} \cdot \sum_{b \in \hat{\Theta}'} \xi_b \{ b \} = \sum_{a, b} \xi_a \xi_b \{ a \} \cdot \{ b \},
\]

where the product \( \{ a \} \cdot \{ b \} \) is taken in \( \mathcal{K}' \). This is shown in exactly the same as [BLM90, Section 5].

Observe that the algebra \( \hat{\mathcal{K}}' \) has a unit element \( \sum \{ a \} \), the summation of all diagonal matrices.

We define the following elements in \( \hat{\mathcal{K}}' \). For any nonzero matrix \( a \in \hat{\Theta}' \), let \( \hat{a} \) be the matrix obtained by replacing diagonal entries of \( a \) by zeroes. We set

\[
\Theta^0 = \{ \hat{a} | a \in \hat{\Theta}' \}.
\]

For any \( \hat{a} \) in \( \Theta^0 \) and \( j = (j_1, \cdots, j_n) \in \mathbb{Z}^n \), we define

\[
\hat{a}(j) = \sum_{\lambda} v^{\lambda_j j_1 + \cdots + \lambda_n j_n} t^{\lambda_j j_1 + \cdots + \lambda_n j_n} \{ \hat{a} + D_{\lambda} \}
\]

where the sum runs through all \( \lambda = (\lambda_i) \in \mathbb{Z}^n \) such that \( \hat{a} + D_{\lambda}, \in \hat{\Theta}' \), where \( D_{\lambda} \) is the diagonal matrices with diagonal entries \( (\lambda_i) \).

And we also define

\[
J_+ = \sum_{\lambda \in S_0} \{ D_{\lambda} \},
\]

\[
J_- = \sum_{\lambda \in S_1} \{ D_{\lambda} \},
\]

\[
J_0 = 1 - J_+ - J_-
\]

Where \( S_0 = \{ \lambda | \lambda_{m+1} = 0, \sum_{i=1}^n \lambda_i \equiv \sum_{i=1}^n \lambda_i \mod 2 \} \), \( S_1 = \{ \lambda | \lambda_{m+1} = 0, \sum_{i=1}^n \lambda_i \equiv \sum_{i=1}^n \lambda_i - 1 \mod 2 \} \).

For \( i \in [1, n-1] \), let

\[
E_i = E_{i;i+1}(0) \quad \text{and} \quad F_i = E_{i+1;i}(0).
\]

Let \( \mathcal{U} \) be the subalgebra of \( \hat{\mathcal{K}} \) generated by \( E_i, F_i, 0(j), J_\alpha \) for all \( i \in [1, n-1], j \in \mathbb{Z}^n \) and \( \alpha \in \{ +, -, 0 \} \).

Proposition 7.6.1. The following relations hold in \( \mathcal{U}' \).

\[
J_+ + J_0 + J_- = 1, J_\alpha J_\beta = \delta_{\alpha,\beta} J_\alpha, J_\alpha A_\alpha = A_\alpha J_\alpha, J_\alpha B_\alpha = B_\alpha J_\alpha;
\]

\[
E_i J_+ = (1 - \delta_{i,m}) J_+ E_i, J_- E_i = (1 - \delta_{i,m+1}) E_i J_+;
\]

\[
F_i J_+ = (1 - \delta_{i,m}) J_+ F_i, J_- F_i = (1 - \delta_{i,m}) F_i J_+;
\]
(73) \[ J_+ E_m E_{m+1} = E_m E_{m+1} J_+; \]
(74) \[ J_+ F_{m+1} F_m = F_{m+1} F_m J_+; \]
(75) \[ J_\pm E_m F_m - E_m F_m J_\mp = \frac{A_m B_{m+1} - B_m A_{m+1}}{v - v^{-1}} (J_\pm - J_\mp); \]
(76) \[ J_\pm F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_\mp = \frac{B_{m+1} A_{m+2} - A_{m+1} B_{m+2}}{v - v^{-1}} (J_\pm - J_\mp). \]
(77) \[ 0(j)0(j') = 0(j')0(j), \]
(78) \[ 0(j)E_h = v^{j_h-j_{h+1}} t^{j_h-j_{h+1}} E_h 0(j), \]
(79) \[ t(E_i F_h - F_h E_i) = (v - v^{-1})^{-1} (0(j_h - h + 1) - 0(h + 1 - j_h)), h \neq m + 1; \]
(80) \[ E_i^2 = (v^2 - 1) E_i + t^2 E_i^2 = 0; \]
(81) \[ t^2 E_i^2 - (v^2 - 1) E_i E_i + t^2 E_i^2 E_i = 0, \]
(82) \[ F_i^2 = (v^2 - 1) F_i F_i + t^2 F_i^2 F_i = 0; \]
(83) \[ t^{-2} F_i^2 F_i - (v^{-2} - 1) F_i F_i F_i + t^{-2} F_i F_i F_i = 0; \]
where \( j, j' \in \mathbb{Z}^n, h, i, j \in [1, n] \) and \( \mathbf{1} \in \mathbb{N}^n \) is the vector whose \( i \)-th entry is 1 and 0 elsewhere.

7.7. The algebra \( \widehat{U_{v,t}(gl_n)} \).

**Definition 7.7.1.** \( \widehat{U_{v,t}(gl_n)} \) is an associative \( \mathbb{Q}(v, t) \)-algebra with 1 generated by symbols \( E_i, F_i, J_\alpha, A_\alpha, B_\alpha \) for all \( i \in [1, n - 1], \alpha \in [1, n] \) and \( \alpha \in \{+, -, 0\} \) subject to the following relations.

(84) \[ J_+ + J_0 + J_- = 1, \]
(85) \[ E_i J_+ = (1 - \delta_{i,m}) J_\pm E_i, J_\pm E_i = (1 - \delta_{i,m+1}) E_i J_\pm; \]
(86) \[ F_i J_\pm = (1 - \delta_{i,m+1}) J_\pm F_i, J_\pm F_i = (1 - \delta_{i,m}) F_i J_\pm; \]
(87) \[ J_\pm E_i E_{m+1} = E_i E_{m+1} J_\pm; \]
(88) \[ J_\pm F_{m+1} F_m = F_{m+1} F_m J_\pm; \]
(89) \[ J_\pm E_m F_m - E_m F_m J_\mp = \frac{A_m B_{m+1} - B_m A_{m+1}}{v - v^{-1}} (J_\pm - J_\mp); \]
(90) \[ J_\pm F_{m+1} E_{m+1} - F_{m+1} E_{m+1} J_\mp = \frac{B_{m+1} A_{m+2} - A_{m+1} B_{m+2}}{v - v^{-1}} (J_\pm - J_\mp). \]
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