On Zero–Mass Ground States in Super–Membrane Matrix Models

Jürg Fröhlich and Jens Hoppe
Theoretical Physics
ETH-Hönggerberg
CH–8093 Zürich

Abstract We recall a formulation of super-membrane theory in terms of certain matrix models. These models are known to have a mass spectrum given by the positive half-axis. We show that, for the simplest such matrix model, a normalizable zero-mass ground state does not exist.
1 Introduction and Summary of Results

Some time ago [1], super-membranes in $D$ space-time dimensions were related to supersymmetric matrix models where, in a Hamiltonian light-cone formulation, the $D - 2$ transverse space coordinates appear as non-commuting matrices [2].

It has been proven in [3] that the mass spectrum of any one of these matrix models, which is given by the (energy) spectrum of some supersymmetric quantum-mechanical Hamilton operator [4], fills the positive half-axis of the real line. This property of the mass spectrum in super-membrane models is in contrast to the properties of mass spectra in bosonic membrane matrix models [2] which are purely discrete; see [5]. One of the important open questions concerning super-membrane matrix models is whether they have a normalizable zero-mass ground state. Such states would describe multiplets of zero-mass one-particle states, including the gravitation; (see [1]). A new interpretation of the mass spectrum of super-membrane matrix models (in terms of multi-membrane configurations) has been proposed in [6].

A first step towards answering the question of whether there are normalizable zero-mass ground states in super-membrane matrix models has been undertaken in [1]. In this note, we continue the line of thought described in [7] and show that, in the simplest matrix model, a normalizable zero-mass ground state does not exist.

Let us recall the definition of super-membrane matrix models. The configuration space of the bosonic degrees of freedom in such models consists of $D - 2$ copies of the Lie-algebra of $SU(N)$, for some $N < \infty$, where $D$ is the dimension of space-time, with $D = 4, 5, 7, 11$. A point in this configuration space is denoted by $X = (X_j)$ with

$$X_j = \sum_{A=1}^{N^2-1} X_j^A T_A, \quad j = 1, \cdots, D - 2,$$

(1.1)

where $\{T_A\}$ is a basis of $su(N)$, the Lie algebra of $SU(N)$. In order to describe the quantum-mechanical dynamics of these degrees of freedom, we make use of the Heisenberg algebra generated by the configuration space coordinates $X_j^A$ and the canonically conjugate momenta $P_j^A$ satisfying canonical commutation relations

$$[X_j^A, X_k^B] = [P_j^A, P_k^B] = 0,$$

$$[X_j^A, P_k^B] = i \delta^{AB} \delta_{jk}.$$  

(1.2)

To describe the quantum dynamics of the fermionic degrees of freedom, we make use of the Clifford algebra with generators $\Theta^A_{\alpha}$, $A = 1, \cdots, N^2 - 1$, $\alpha = 1, \cdots, 2 \left[\frac{D - 2}{2}\right]$; and commutation relations

$$\{\Theta^A_{\alpha}, \Theta^B_{\beta}\} = \delta_{\alpha\beta} \delta^{AB}.$$  

(1.3)
The generators $\Theta_A^\alpha$ can be expressed in terms of fermionic creation- and annihilation operators:

$$
\Theta_A^{2\alpha-1} = \frac{b_A^\alpha + c_A^\alpha}{\sqrt{2}},
$$

$$
\Theta_A^A = \frac{i (b_A^\alpha - c_A^\alpha)}{\sqrt{2}},
$$

(1.4)

with $(b_A^\alpha)^* = c_A^\alpha$, $\alpha = 1, \cdots, \frac{1}{2} 2^{[\frac{D}{2} - 1]}$, and

$$
\{b_A^\alpha, b_B^\beta\} = \{c_A^\alpha, c_B^\beta\} = 0,
$$

$$
\{b_A^\alpha, c_B^\beta\} = \delta_{\alpha\beta} \delta^{AB}.
$$

(1.5)

The Hilbert space, $\mathcal{H}$, of state vectors (in the Schrödinger representation) is a direct sum of subspaces $\mathcal{H}_k$, $k = 0, \cdots, K := (N^2 - 1) \frac{1}{2} 2^{[\frac{D}{2} - 1]}$. A vector $\Psi \in \mathcal{H}_k$ is given by

$$
\Psi = \sum_{k=0}^K \frac{1}{k!} b_{A_1}^{\alpha_1} \cdots b_{A_k}^{\alpha_k} \psi_{A_1 \cdots A_k}^{\alpha_1 \cdots \alpha_k} (X),
$$

(1.6)

where $X = \{X_j^A\}$, $j = 1, \cdots, D - 2$, $A = 1, \cdots, N^2 - 1$. We require that

$$
c_A^\alpha \Psi = 0, \text{ for all } \Psi \in \mathcal{H}_0.
$$

(1.7)

The scalar product of two vectors, $\Psi$ and $\Phi$, in $\mathcal{H}$ is given by

$$
\langle \Psi, \Phi \rangle = \sum_{k=0}^K \frac{1}{k!} \sum_{\alpha_1, \cdots, \alpha_k} \int \prod_{j,A} dX_j^A \psi_{A_1 \cdots A_k}^{\alpha_1 \cdots \alpha_k} (X) \times \phi_{A_1 \cdots A_k}^{\alpha_1 \cdots \alpha_k} (X).
$$

(1.8)

The Hilbert space $\mathcal{H}$ carries unitary representations of the groups $SU(N)$ and $SO(D - 2)$. Let $\mathcal{H}^{(0)}$ denote the subspace of $\mathcal{H}$ carrying the trivial representation of $SU(N)$.

One can define supercharges, $Q_\alpha$ and $Q_\alpha^\dagger$, $\alpha = 1, \cdots, \frac{1}{2} 2^{[\frac{D}{2} - 1]}$, with the properties that, on the subspace $\mathcal{H}^{(0)}$,

$$
\{Q_\alpha, Q_\beta\} \big|_{\mathcal{H}^{(0)}} = \{Q_\alpha^\dagger, Q_\beta^\dagger\} \big|_{\mathcal{H}^{(0)}} = 0,
$$

(1.9)

and

$$
\{Q_\alpha, Q_\beta^\dagger\} \big|_{\mathcal{H}^{(0)}} = \delta_{\alpha\beta} H \big|_{\mathcal{H}^{(0)}},
$$

(1.10)

where $H = M^2$, and $M$ is the mass operator of the super-membrane matrix model.

Precise definitions of the supercharges and of the operator $H$ can be found in [1] (formulas (4.7) through (4.12)). In [3] it is shown that the spectrum of $H \big|_{\mathcal{H}^{(0)}}$ consists of the positive half-axis $[0, \infty)$. The problem addressed in this note is to determine whether
$O$ is an eigenvalue of $H$ corresponding to a normalizable eigenvector $\Psi_0 \in \mathcal{H}^{(0)}$. Using eqs. (1.9) and (1.10), one can show that $\Psi_0$ must be a solution of the equations

$$Q_\alpha \Psi = Q^\dagger_\alpha \Psi = 0, \quad \text{for some } \alpha, \Psi \in \mathcal{H}^{(0)}.$$  

If eqs. (1.11) have a solution, $\Psi_0 = \Psi_{\alpha_0}$, for $\alpha = \alpha_0$, they have a solution for all values of $\alpha$, (by $SO(D - 2)$ covariance). The problem to determine whether eqs. (1.11) have a solution, or not, can be understood as a problem about the cohomology groups determined by the supercharges $Q_\alpha$. We define

$$\mathcal{H}_+ := \bigoplus_{l \geq 0} \mathcal{H}_{2l}^{(0)}, \quad \mathcal{H}_- := \bigoplus_{l \geq 0} \mathcal{H}_{2l+1}^{(0)}.$$  

We define the cohomology groups

$$H_{\sigma,\alpha} := \{ \Psi \in \mathcal{H}_{\sigma}^{(0)} | Q_\alpha \Psi = 0 \} / \{ \Psi | \Psi = Q_\alpha \Phi, \Phi \in \mathcal{H}_{-\sigma}^{(0)} \},$$

$\sigma = \pm 1$. If $H_{\sigma,\alpha}$ is non-trivial, for some $\sigma$ and some $\alpha$, then eqs. (1.11) have a solution.

## 2 The $(D = 4, N = 2)$ model

The goal of this note is a very modest one: We show that, for $D = 4$ and $N = 2$, eqs. (1.11) do not have any normalizable solutions. Our proof is not conceptual; it relies on explicit calculations and estimates and does therefore not extend to the general case in any straightforward way.

When $D = 4$ and $N = 2$ we use the following notations:

$$\vec{q}_j := (X_j^1, X_j^2, X_j^3), \quad j = 1, 2,$$

$$\vec{\lambda} = (\lambda^1, \lambda^2, \lambda^3) := (b_1^\alpha, b_2^\alpha, b_3^\alpha),$$

and

$$\frac{\partial}{\partial \lambda^3} = \left( \frac{\partial}{\partial \lambda^1}, \frac{\partial}{\partial \lambda^2}, \frac{\partial}{\partial \lambda^3} \right) := (c_1^\alpha, c_2^\alpha, c_3^\alpha),$$

$\alpha = 1$. The operators representing the generators of $su(2)$ on $\mathcal{H}$ are given by

$$\vec{L} = -i \left( \vec{q}_1 \wedge \vec{\nabla}_1 + \vec{q}_2 \wedge \vec{\nabla}_2 + \vec{\lambda} \wedge \frac{\partial}{\partial \lambda^3} \right).$$

The supercharges are given by (see [1], eq. (4.20))

$$Q = \left( \vec{\nabla}_1 - i \vec{\nabla}_2 \right) \cdot \frac{\partial}{\partial \lambda} + i \vec{q} \cdot \vec{\lambda},$$

and

$$Q^\dagger = - \left( \vec{\nabla}_1 + i \vec{\nabla}_2 \right) \cdot \vec{\lambda} - i \vec{q} \cdot \frac{\partial}{\partial \lambda},$$

3
where
\[ \vec{q} = \vec{q}_1 \land \vec{q}_2 , \] (2.4)
and \( \land \) denotes the vector product. We then have that
\[ Q^2 = (\vec{q}_1 - i \vec{q}_2) \cdot \vec{L} , \quad (Q^\dagger)^2 = (\vec{q}_1 + i \vec{q}_2) \cdot \vec{L} , \]
and
\[ H = \{ Q, Q^\dagger \} . \] (2.5)

A vector \( \Psi \in \mathcal{H}_+ \) can be written as
\[ \Psi = \psi + \frac{1}{2} \left( \vec{\lambda} \land \vec{\lambda} \right) \cdot \vec{\psi} \]
\[ = \psi + \frac{1}{2} \epsilon_{ABC} \lambda^A \lambda^B \psi^C . \] (2.6)

For \( \Psi \in \mathcal{H}_+^{(0)} \) (i.e., \( \Psi \in \mathcal{H}_+ \) with \( \vec{L} \Psi = 0 \)), eqs. (1.11) imply the following system (*) of first-order differential equations:
\[ i \vec{q} \psi = \left( \vec{\nabla}_1 - i \vec{\nabla}_2 \right) \land \vec{\psi} , \] (2.7)
\[ \vec{q} \cdot \vec{\psi} = 0 , \] (2.8)
and
\[ \left( \vec{\nabla}_1 + i \vec{\nabla}_2 \right) \psi = i \vec{q} \land \vec{\psi} , \] (2.9)
\[ \left( \vec{\nabla}_1 + i \vec{\nabla}_2 \right) \cdot \vec{\psi} = 0 . \] (2.10)

Moreover, the equation \( \vec{L} \Psi = 0 \) yields
\[ \left( \vec{q}_1 \land \vec{\nabla}_1 + \vec{q}_2 \land \vec{\nabla}_2 \right) \psi = 0 , \] (2.11)
\[ \left( \vec{q}_1 \land \vec{\nabla}_1 + \vec{q}_2 \land \vec{\nabla}_2 \right) \psi_A + \sum_C \epsilon_{ABC} \psi_c = 0 , \quad \forall A, B . \] (2.12)

It is straightforward to verify that, for \( \Psi \in \mathcal{H}_+^{(0)} \), eqs. (1.11) imply a system of equation equivalent to (2.7) through (2.12). This can be interpreted as a consequence of Poincaré duality.

The formal expression for the Hamiltonian \( H = \{ Q, Q^\dagger \} \) is given by
\[ H = H_B + H_F , \] (2.13)
where
\[ H_B = - \vec{\nabla}_1^2 - \vec{\nabla}_2^2 + \vec{q}^2 \]
and
\[ H_F = (\vec{q}_1 + i \vec{q}_2) \cdot (\vec{\lambda} \wedge \vec{\lambda}) - (\vec{q}_1 - i \vec{q}_2) \cdot \left( \frac{\partial}{\partial \vec{\lambda}} \wedge \frac{\partial}{\partial \vec{\lambda}} \right). \] (2.14)

As shown in [5], the spectrum of \( H_B \) is discrete, with
\[ \inf \text{ spec } H_B = E_0 > 0. \] (2.15)

The representation of the group \( SO(D - 2) \cong U(1), (D = 4) \) on \( \mathcal{H} \) is generated by the operator
\[ J = -i \left( \vec{q}_1 \cdot \vec{\nabla}_2 - \vec{q}_2 \cdot \vec{\nabla}_1 \right) - \frac{1}{2} \vec{\lambda} \cdot \frac{\partial}{\partial \vec{\lambda}}. \] (2.16)

While \( J \) does not commute with \( Q \) or \( Q^\dagger \), it does commute with \( QQ^\dagger \) and \( Q^\dagger Q \) and hence with \( H \). It is therefore sufficient to look for solutions of eqs. (2.7) through (2.12) transforming under an irreducible representation of \( U(1) \), i.e. solutions that are eigenvectors of \( J \) corresponding to eigenvalues \( j \in \frac{1}{2} \mathbb{Z} \). The spectrum of the restriction of \( J \) to the subspace \( \mathcal{H}_+ \) is the integers, while \( \text{spec } \left( J \mid_{\mathcal{H}_-} \right) \) consists of half-integers.

3 Analysis of equations (*)

In this section, we assume that \( Q\Psi = Q^\dagger \Psi = 0 \) has a solution \( \Psi \in \mathcal{H}_+^{(0)} \) and then show that \( \Psi = 0 \).

The assumption that \( Q\Psi = Q^\dagger \Psi = 0 \) implies that
\[ \langle Q\Psi, Q\Psi \rangle + \langle Q^\dagger \Psi, Q^\dagger \Psi \rangle = 0. \] (3.1)

Let \( \xi := (\vec{q}_1, \vec{q}_2) \in \mathbb{R}^6 \). Let \( g_n(\xi) \equiv g_n(|\xi|), n = 1, 2, 3, \ldots \), be a function on \( \mathbb{R}^6 \) only depending on \( |\xi| := \sqrt{\vec{q}_1^2 + \vec{q}_2^2} \) with the properties that \( g_n \) is smooth, monotonic decreasing, \( g_n(|\xi|) = 1 \), for \( |\xi| \leq n \), \( g_n(|\xi|) = 0 \) for \( |\xi| \geq 3n \), and \( \left| \left( \frac{d}{dt} g_n \right) (t) \right| \leq \frac{1}{n} \). Let \( h_k(\xi), k = 1, 2, 3, \ldots \), be an approximate \( \delta \)-function at \( \xi = 0 \) with the properties that \( h_k \) is smooth, \( h_k \geq 0 \), \( \int h_k(\xi) d^6\xi = 1 \), and
\[ \text{supp } h_k \subseteq \left\{ \xi \mid |\xi| \leq \frac{1}{k^2} \right\}. \] (3.2)

We define a bounded operator, \( R_{n,k} \), on \( \mathcal{H} \) by setting
\[ (R_{n,k}\Phi)(\xi) = g_n(\xi) \int h_k(\xi - \xi') \Phi(\xi') d^6\xi', \] (3.3)
for any \( \Phi \in \mathcal{H} \). Clearly
\[ s-lim_{n \to \infty} R_{n,k} \Phi = \Phi, \] (3.4)
for any \( \Phi \in \mathcal{H} \). Next, we note that, for a vector \( \Phi \) in the domain of the operator \( Q \),

\[
([Q, R_{n,k}] \Phi) (\xi) = I_{n,k} (\xi) + II_{n,k} (\xi),
\]

where

\[
I_{n,k} (\xi) = \left( (\vec{\nabla}_1 - i\vec{\nabla}_2) g_n \right) (\xi) \cdot \int h_k (\xi - \xi') \frac{\partial}{\partial \lambda} \Phi (\xi') d^n \xi',
\]

and

\[
II_{n,k} (\xi) = i g_n (\xi) \int h_k (\xi - \xi') \left( \vec{q}(\xi) - \vec{q}(\xi') \right) \cdot \vec{\lambda} \Phi (\xi') d^n \xi'.
\]

The operator norm of the operators \( \frac{\partial}{\partial \lambda} \) and \( \lambda A \), \( A = 1, 2, 3 \), is bounded by 1. Since \( \left| \frac{d}{dt} g_n(t) \right| \leq \frac{1}{n} \), the operator norm of the multiplication operator \( \left( (\vec{\nabla}_1 - i\vec{\nabla}_2) g_n \right) (\cdot) \) is bounded above by \( \frac{1}{n} \). The operator norm of the convolution operator \( \Phi (\xi) \mapsto \int h_n (\xi - \xi') \Phi (\xi') d^n \xi' \) is equal to 1. This implies

\[
\| I_{n,k} \| \leq \frac{6}{n} \left\| \frac{\partial}{\partial \lambda} \Phi \right\| \leq \frac{18}{n} \| \Phi \|.
\]

Next, we note that, for \( \xi \) in the support of the function \( g_n \),

\[
\left| h_k (\xi - \xi') \left( \vec{q}(\xi) - \vec{q}(\xi') \right) \right| \leq \frac{7n}{k^2} h_k (\xi - \xi').
\]

Thus, for \( k \geq n \)

\[
\| II_{n,k} \| \leq \frac{21}{n} \| \Phi \|.
\]

In conclusion

\[
\| [Q, R_{n,k}] \Phi \| \leq \frac{40}{n} \| \Phi \|,
\]

for \( k \geq n \).

A similar chain of arguments shows that, for \( \Phi \) in the domain of \( Q^\dagger \),

\[
\| [Q^\dagger, R_{n,k}] \Phi \| \leq \frac{40}{n} \| \Phi \|,
\]

for \( k \geq n \).

Next, we suppose that \( \Psi \) solves (3.1). We claim that, given \( \varepsilon > 0 \), there is some finite \( n(\varepsilon) \) such that, for \( \Psi_{n,k} := R_{n,k} \Psi \),

\[
\| \Psi \| \geq \| \Psi_{n,k} \| \geq (1 - \varepsilon) \| \Psi \|,
\]

and

\[
\langle Q \Psi_{n,k}, Q \Psi_{n,k} \rangle + \langle Q^\dagger \Psi_{n,k}, Q^\dagger \Psi_{n,k} \rangle \leq \varepsilon \| \Psi \|^2,
\]

for all \( k \geq n \geq n(\varepsilon) \). Inequality (3.12) follows directly from (3.4) and the fact that the operator norm of \( R_{n,k} \) is 1. To prove (3.13), we note that, for \( k \geq n \),

\[
\langle Q^# \Psi_{n,k}, Q^# \Psi_{n,k} \rangle = \langle [Q^#, R_{n,k}] \Psi, [Q^#, R_{n,k}] \Psi \rangle \leq \left( \frac{40}{n} \right)^2 \langle \Psi, \Psi \rangle,
\]

\[\text{6}\]
where $Q^\# = Q$ or $Q^\dagger$. This follows from the equations $Q\Psi = Q^\dagger\Psi = 0$ and inequalities (3.10) and (3.11).

We now observe that, by the definition of $R_{n,k}$, $\Psi_{n,k} = R_{n,k}\Psi$ is a smooth function of compact support in $\mathbb{R}^6$, for all $n \leq k < \infty$. It therefore belongs to the domain of definition of the operators $Q Q^\dagger$ and $Q^\dagger Q$. Thus, for all $n \leq k < \infty$,

$$
\langle Q\Psi_{n,k}, Q\Psi_{n,k} \rangle + \langle Q^\dagger\Psi_{n,k}, Q^\dagger\Psi_{n,k} \rangle = \langle \Psi_{n,k}, \{Q, Q^\dagger\} \Psi_{n,k} \rangle,
$$

(3.15)

where $H_B$ and $H_F$ are given in eq. (2.14), (and it is obvious from (2.14) that $\Psi_{n,k}$ belongs to the domains of definition of $H_B$ and $H_F$).

As proven in [5],

$$
\langle \Phi, H_B\Phi \rangle \geq E_0 \|\Phi\|^2,
$$

(3.16)

for some strictly positive constant $E_0 (= \inf \text{spec } H_B)$, for all vectors $\Phi$ in the domain of $H_B$. Thus, for $k \geq n \geq n(\varepsilon)$, and using (3.13), we have that

$$
\varepsilon \|\Psi\|^2 \geq \langle \Psi_{n,k}, H_B\Psi_{n,k} \rangle + \langle \Psi_{n,k}, H_F\Psi_{n,k} \rangle \geq (1 - \varepsilon)^2 E_0 \|\Psi\|^2 + \langle \Psi_{n,k}, H_F\Psi_{n,k} \rangle.
$$

(3.17)

Our next task is to analyze $\langle \Psi_{n,k}, H_F\Psi_{n,k} \rangle$. If $\Phi = (\varphi, \vec{\varphi}) \in \mathcal{H}_+$ belongs to the domain of definition of $H_F$ then

$$
\langle \Phi, H_F\Phi \rangle = 2 \int \varphi(\xi) (\vec{q}_1 - i\vec{q}_2) \cdot \vec{\varphi}(\xi) \, d^6 \xi + \text{c.c.}
$$

(3.18)

Note that $\Psi_n := \lim_{k \to \infty} \Psi_{n,k}$, where $\Psi_{n,k} = R_{n,k}\Psi$ and $\Psi$ solves (3.1), belongs to the domain of definition of $H_F$. Since $\Psi = (\psi, \vec{\psi})$ solves the equations $Q\Psi = Q^\dagger \Psi = 0$, see (3.1), we can use eqs. (2.8) and (2.9) to eliminate $\vec{\psi}$: For $\vec{q} \neq 0$, we find that

$$
\vec{\psi}(\xi) = \frac{i\vec{q}}{q^2} \wedge \left( \vec{\nabla}_1 + i\vec{\nabla}_2 \right) \psi(\xi)
$$

(3.19)

(recall that $\vec{q} = \vec{q}_1 \wedge \vec{q}_2$). Inserting (3.19) on the R.S. of (3.18), for $\Phi = \Psi_n$, we arrive at the equation

$$
\langle \Psi_n, H_F\Psi_n \rangle = 2 \int \left( g_n(\xi) (\vec{q}_1 - i\vec{q}_2) \right) \left( \frac{i\vec{q}}{q^2} \wedge \left( \vec{\nabla}_1 + i\vec{\nabla}_2 \right) \psi(\xi) \right) \, d^6 \xi + \text{c.c.}
$$

(3.20)

Let

$$
T := 2 (\vec{q}_1 - i\vec{q}_2) \cdot \left( \frac{i\vec{q}}{q^2} \wedge \left( \vec{\nabla}_1 + i\vec{\nabla}_2 \right) \right).
$$
Then
\[ \langle \Psi_n, H_F \Psi_n \rangle = \langle \Psi_n, T \Psi_n \rangle + \text{c.c.} \]
\[ - \int |\psi(\xi)|^2 g_n(\xi) [T, g_n](\xi) d^6\xi + \text{c.c.} \quad (3.21) \]

Next, we make use of the fact that \( \Psi \) must be SU(2)–invariant. This is expressed in eq. (2.11), which implies that \( \psi(\xi) \) only depends on SU(2)–invariant combinations of the variables \( \vec{q}_1 \) and \( \vec{q}_2 \), i.e., on
\[ r_1 := |\vec{q}_1|, \quad r_2 := |\vec{q}_2|, \quad x := \frac{\vec{q}_1 \cdot \vec{q}_2}{r_1 r_2}. \quad (3.22) \]

Instead, we may use variables \( q, p \) and \( \varphi \) defined by
\[ p e^{i\varphi} := \frac{1}{2} (\vec{q}_1 + i \vec{q}_2)^2 = \frac{1}{2} (r_1^2 - r_2^2) + i r_1 r_2 x, \quad q := |\vec{q}_1 \wedge \vec{q}_2| \quad (3.23) \]
with
\[ 0 \leq p < \infty, \quad 0 \leq \varphi < 4\pi, \quad 0 \leq q < \infty. \quad (3.24) \]

If \( F \) is an SU(2)–invariant function then
\[ \int d^6\xi \ F(\xi) = c \int_0^\infty dq \int_0^\infty dp \int_0^{4\pi} d\varphi \frac{q p}{\sqrt{q^2 + p^2}} F(q, p, \varphi), \quad (3.25) \]
where \( c \) is some positive constant.

If \( \varphi \) is SU(2)–invariant then
\[ T \varphi = c' \sqrt{q^2 + p^2} \frac{\partial}{\partial q} \varphi, \quad (3.26) \]
where \( c' \) is a positive constant.

Using (3.26) in (3.21), we find that
\[ \langle \Psi_n, H_F \Psi_n \rangle = c'' \int_0^\infty dp \int_0^{4\pi} d\varphi \int_0^\infty dq \ p \frac{\partial}{\partial q} |\psi_n(q, p, \varphi)|^2 \]
\[ - c'' \int_0^\infty dp \int_0^{4\pi} d\varphi \int_0^\infty dq \ p |\psi(q, p, \varphi)|^2 \frac{\partial}{\partial q} \left( g_n \left( 2 \sqrt{q^2 + p^2} \right)^2 \right), \quad (3.27) \]
with \( c'' = c c' > 0. \)

By the definition of \( g_n \),
\[ \frac{\partial}{\partial q} \left( g_n \left( 2 \sqrt{q^2 + p^2} \right)^2 \right) \leq 0, \]
pointwise. Therefore
\[ \langle \Psi_n, H F \Psi_n \rangle \geq -c'' \int_0^\infty dp \int_0^{4\pi} d\varphi \, p \, |\psi_n(q = 0, p, \varphi)|^2. \] (3.28)

In passing from (3.27) to (3.28), we have used that \( \frac{\partial}{\partial q} |\psi_n(q, p, \varphi)|^2 \) is an \( L^1 \)–function with respect to the measure \( p \, dp \, d\varphi \, dq \) and that
\[ \int_0^\infty dp \int_0^{4\pi} d\varphi \, p \, |\psi_n(q, p, \varphi)|^2 \]
is right-continuous at \( q = 0 \). These facts will be proven below.

Combining eqs. (3.15), (3.17) and (3.28), we conclude that
\[ \varepsilon \|\Psi\|^2 \geq \lim_{k \to \infty} \left\{ \langle Q \Psi_{n,k}, Q \Psi_{n,k} \rangle + \langle Q^\dagger \Psi_{n,k}, Q^\dagger \Psi_{n,k} \rangle \right\} \]
\[ \geq (1 - \varepsilon) E_0 \|\Psi\|^2 - c'' \int_0^\infty dp \int_0^{4\pi} d\varphi \, p \, |\psi_n(q = 0, p, \varphi)|^2, \] (3.29)
for all \( n \geq n(\varepsilon) \). Choosing \( \varepsilon \) sufficiently small, we conclude that either \( \Psi = 0 \), or there is a constant \( \beta > 0 \) such that
\[ \int_0^\infty dp \int_0^{4\pi} d\varphi \, p \, |\psi_n(q = 0, p, \varphi)|^2 \geq \beta, \] (3.30)
for all sufficiently large \( n \).

Next, we explore the consequences of (3.30). Since \( \Psi \) solves (3.1), we can use (3.19) to conclude that
\[ \infty > \|\Psi\|^2 = \|\psi\|^2 + \|\vec{\psi}\|^2 \]
\[ = \int d^6 \xi \left\{ |\psi(\xi)|^2 + \frac{|(\vec{\nabla}_1 + i \vec{\nabla}_2) \psi(\xi)|^2}{|q|^2} \right\}. \]
Using that \( \Psi \) is SU(2)–invariant and passing to the variables \( q, p \) and \( \varphi \), one finds that
\[ 2 \int_0^\infty dp \int_0^\infty dq \int_0^{4\pi} d\varphi \left( |\psi_{,p} + \frac{i\psi_{,q}}{p}|^2 + |\psi_{,q}|^2 \right)(q, p, \varphi) < \tilde{K}, \] (3.31)
where \( \psi_{,x} := \frac{\partial \psi}{\partial x} \), and
\[ \int_0^\infty dp \int_0^\infty dq \int_0^{4\pi} d\varphi \frac{pq}{\sqrt{p^2 + q^2}} |\psi(q, p, \varphi)|^2 < \tilde{K}, \] (3.32)
with $\tilde{K} = \|\Psi\|_c^2 < \infty$ (with the constant $c$ appearing in (3.25)).

Inequalities (3.31) and (3.32) also hold for $\psi_n$, instead of $\psi$, with a constant $K$ that is uniform in $n \to \infty$. These inequalities prove that $\frac{\partial}{\partial q} |\psi_n(q,p,\varphi)|^2$ is an $L^1$–function with respect to the measure $p \, dp \, d\varphi \, dq$ and that

$$f_n(q) := \int_0^\infty dp \int_0^{4\pi} d\varphi \, p \, |\psi_n(q,p,\varphi)|^2$$

is right-continuous at $q = 0$, properties that were used in our derivation of (3.28).

By the Schwarz inequality and (3.31),

$$\left( \int_0^\infty dp \int_0^{4\pi} d\varphi \, \int_0^\infty dq \, p \, \left| \frac{\partial}{\partial q} |\psi_n(q,p,\varphi)|^2 \right| \right)^{1/2} \leq 2 \left( \int_0^\infty dp \int_0^{4\pi} d\varphi \, \int_0^\infty dq \, |\psi_n(q,p,\varphi)|^2 \right)^{1/2} \leq K'n^4,$$

for some finite constant $K'!$

To prove continuity of $f_n(q)$ in $q$, we note that, for $q_1 > q_2$,

$$|f_n(q_1) - f_n(q_2)| \leq \int_0^\infty dp \int_0^{4\pi} d\varphi \int_0^\infty dq \, p \, \left| \frac{\partial}{\partial q} |\psi_n(q,p,\varphi)|^2 \right|$$

with tends to 0, as $(q_1 - q_2) \to 0$, because $\frac{\partial}{\partial q} |\psi_n(q,p,\varphi)|^2$ is an $L^1$–function.

Next, we make use of the $SO(D - 2) \simeq U(1)$ symmetry with generator $J$ given in eq. (2.16). We have noted below (2.16) that $J$ commutes with $QQ^\dagger$ and $Q^\dagger Q$, and hence that $\Psi \in \mathcal{H}_+$ can be chosen to be an eigenvector of $J$ corresponding to some eigenvalue $m \in \mathbb{Z}$. In the variables $q,p,\varphi$,

$$J = -2i \frac{\partial}{\partial \varphi}.$$  

Hence we may write

$$\psi(q,p,\varphi) = e^{\frac{im}{2} \varphi} p^m \phi(q,p), \quad (3.33)$$

for some function $\phi$ independent of $\varphi$. Eqs. (3.31) and (3.32) then simply

$$\int_0^\infty dp \, p^\alpha \int_0^\infty dq \left( |\phi_p|^2 + |\phi_q|^2 \right) (q,p) < \infty \quad (3.34)$$
and
\[
\int_0^\infty dp \int_0^\infty dq \frac{p^\alpha q}{\sqrt{p^2 + q^2}} |\phi(q, p)|^2 < \infty, \tag{3.35}
\]
where \(\alpha = m + 1\). Furthermore, inequality (3.30), in the limit as \(n \to \infty\), yields
\[
\int_0^\infty dp |\phi(q = 0, p)|^2 \geq \frac{\beta}{4\pi}. \tag{3.36}
\]

Let \(I_N := \left[\frac{1}{N}, N\right]\). Then inequality (3.34) implies that there exists a set \(\Omega \subseteq [0, \infty)\) with the property that \(\Omega \cap [0, \delta]\) has Lebesgue measures \(\frac{\delta}{2}\), for any \(\delta > 0\), and such that
\[
\left(\frac{1}{N}\right)^{\alpha} \int_{I_N} dp |\phi_p(q, p)|^2 \leq K_\delta \tag{3.37}
\]
for some constant \(K_\delta\) independent of \(N\) and all \(q \in \Omega \cap [0, \delta]\). Moreover,
\[
\lim_{q \to 0} \int_{I_N} dp |\phi_p(q, p)|^2 = 0, \tag{3.38}
\]
for all \(N\). It follows that, for \(q \in \Omega \cap [0, \delta]\), \(p_1, p_2 \in I_N\), \(N < \infty\),
\[
|\phi(q, p_1) - \phi(q, p_2)| = |p_1 - p_2| \int_{p_1}^{p_2} dp \frac{\phi_p(q, p)}{|p_1 - p_2|} \leq |p_1 - p_2|^{1/2} (N^{\alpha} K_\delta)^{1/2}. \tag{3.39}
\]

Thus, for \(q \in \Omega \cap [0, \delta]\) and \(p_1, p_2 \in I_N\), \(\phi(q, p)\) is uniformly Hölder–continuous with exponent \(\frac{1}{2}\). Thus \(\phi_0(p) := \lim_{q \to 0} \phi(q, p)\) is uniformly continuous in \(p \in I_N\), for all \(N < \infty\), and it then follows from (3.38) that
\[
\phi_0(p) = \phi_0 = \text{const.} \tag{3.40}
\]
Inequality (3.36) then implies that \(|\phi_0|\) must be positive! Without loss of generality, we may then assume that \(\phi_0 > 0\).

Thus the function \(\phi\) introduced in (3.33) has the following properties

\(A\) \(\lim_{q \to 0} \phi(q, p) = \phi_0 > 0\)

\(B\) \(\int_0^\infty dp \int_0^\infty dq \frac{qq^\alpha}{\sqrt{p^2 + q^2}} |\phi(q, p)|^2 < \infty\)

\(C\) \(\int_0^\infty dp \int_0^\infty dq \frac{p^\alpha}{q} \left(|\phi_p(q, p)|^2 + |\phi_q(q, p)|^2\right) < \infty\).
We now show that such a function $\phi(q, p)$ does not exist.

Let us first consider the case $\alpha \geq 0$. We choose an arbitrary, but fixed $p \in (0, \infty)$. Using the Schwarz inequality, we find that, for $0 < q_0 < \infty$,

$$
\int_0^{q_0} \frac{dq}{q} |\phi,_{q} (q, p)|^2 \geq \frac{1}{q_0} \int_0^{q_0} dq |\phi,_{q} (q, p)|^2
$$

$$
\geq \left( \frac{1}{q_0} \int_0^{q_0} dq |\phi,_{q} (q, p)| \right)^2 \geq \frac{1}{q_0} \int_0^{q_0} dq \phi,_{q} (q, p) ,
$$

(3.41)

where $q^*(p) \in [0, q_0]$ is the point at which $|\phi(q, p)|$ takes its minimum in the interval $[0, q_0]$. Note that $\phi(q, p)$ is continuous in $q \in [0, q_0]$, for almost every $p \in [0, \infty)$. The R.S. of (3.41) is equal to

$$
\frac{1}{q_0^2} |\phi,_{q} (p) - \phi_0|^2 .
$$

Thus

$$
\left( \frac{1}{q_0} |\chi (p) - \phi_0| \right)^2 \leq \int_0^{q_0} \frac{dq}{q} |\phi,_{q} (q, p)|^2 ,
$$

(3.42)

where $\chi(p) = \phi(q^*(p), p)$. By property (C),

$$
\int_0^{\infty} dp \ p^\alpha \int_0^{q_0} \frac{dq}{q} |\phi,_{q} (q, p)|^2 \leq \varepsilon(q_0),
$$

for some finite $\varepsilon(q_0)$, will $\varepsilon(q_0) \to 0$, as $q_0 \to 0$. Hence

$$
\int_0^{\infty} dp \ p^\alpha |\chi(p) - \phi_0|^2 < q_0^2 \varepsilon(q_0).
$$

(3.43)

We define a subset $M_\delta \subseteq [0, \infty)$ by

$$
M_\delta := \left\{ p \mid |\chi(p)| \leq \phi_0 - \delta \right\} .
$$

Then

$$
\int_{M_\delta} dp \ p^\alpha \leq \frac{1}{\delta^2} \int_0^{\infty} dp \ p^\alpha |\chi(p) - \phi_0|^2 < \frac{q_0^2 \varepsilon(q_0)}{\delta^2} .
$$

(3.44)
By property (B),

\[
\infty > \int_0^\infty dp \int_0^\infty dq \frac{q^\alpha}{\sqrt{p^2 + q^2}} |\phi(q, p)|^2 \\
\geq \int_0^\infty dp \int_0^{q_0} dq \frac{q_0^\alpha p^\alpha}{p + q_0} |\phi(q, p)|^2 \\
\geq \int_{M_\delta}^\infty dp \int_{q_0/2}^{q_0} dq \frac{q_0^\alpha p^\alpha}{p + q_0} |\phi(q, p)|^2 \\
\geq \frac{q_0^2}{4} (\phi_0 - \delta)^2 \int_{M_\delta}^\infty dp \frac{p^\alpha}{p + q_0}.
\]

It follows that

\[
\int_{M_\delta} dp \frac{p^\alpha}{p + q_0} + \int_{M_\delta} dp \frac{p^\alpha}{p + q_0} \leq \frac{1}{q_0} \int_{M_\delta} dp p^\alpha + \int_{M_\delta} dp \frac{p^\alpha}{p + q_0} < \infty.
\]

This is a contradiction, since \( M_{\delta \cup M_\delta} = [0, \infty) \), and \( \int dp \frac{p^\alpha}{p + q_0} \) diverges.

Next, we consider the case \( \alpha \leq -1 \). We change variables, \( k := \frac{1}{p} \),

\[
dp = -\frac{1}{k^2} dk, \quad \frac{\partial}{\partial p} = -k^2 \frac{\partial}{\partial k}.
\]

Then conditions (A) – (C) take the form

\[
(A') \quad \lim_{q_0 \to 0} \phi(q, k) = \phi_0 > 0
\]

\[
(B') \quad \int_0^\infty dk \int_0^\infty dq \frac{q k^{\gamma - 2}}{\sqrt{(1/k)^2 + q^2}} |\phi(q, k)|^2 < \infty
\]

\[
(C') \quad \int_0^\infty dk \int_0^\infty dq \frac{k^{\gamma - 2}}{q} (k^4 |\phi_{,k}(q, k)|^2 + |\phi_{,q}(q, k)|^2) < \infty,
\]

where \( \gamma := -\alpha > 0 \).

Repeating the same arguments as above, we get again

\[
\left( \frac{1}{q_0} |\chi(k) - \phi_0| \right)^2 \leq \int_0^\infty dq \frac{1}{q} |\phi_{,q}(q, k)|^2,
\]

(3.45)

where \( \chi(k) \) is the value of \( \phi(q, k) \) at the minimum of \( |\phi(q, k)| \), for \( q \in [0, q_0] \). By (C'),

\[
\int_0^\infty dk \int_0^{q_0} dq \frac{k^{\gamma - 2}}{q} |\phi_{,q}(q, k)|^2 < \varepsilon'(q_0) < \infty.
\]
with $\varepsilon'(q_0) \to 0$, as $q_0 \to 0$. Hence
\[
\int_0^\infty dk \, k^{\gamma-2} |\chi(k) - \phi_0|^2 \leq q_0^2 \varepsilon'(q_0) .
\] (3.46)
Let $L_\delta \subseteq [0, \infty)$ be the set defined by
\[
L_\delta := \left\{ k \mid |\chi(k)| \leq \phi_0 - \delta \right\} .
\]
Then we have that
\[
\int_{L_\delta} dk \, k^{\gamma-2} \leq \frac{1}{\delta^2} \int_0^\infty dk \, k^{\gamma-2} |\chi(k) - \phi_0|^2 \leq \frac{q_0^2 \varepsilon'(q_0)}{\delta^2} .
\] (3.47)
Condition (B') implies that
\[
\infty > \int_0^\infty dk \, k^{\gamma-2} \int_0^\infty \frac{q}{q + k} |\phi(q, k)|^2 \\
\geq \int_0^\infty dk \, k^{\gamma-2} \int_{q_0/2}^{q_0} \frac{q_0/2}{\kappa + q_0} |\phi(q, k)|^2 \\
\geq \int_{L_\delta} dk \, k^{\gamma-1} \int_{q_0/2}^{q_0} \frac{q_0/2}{1 + k q_0} (\phi_0 - \delta)^2 \\
\geq \left( \frac{q_0}{2} \right)^2 (\phi_0 - \delta)^2 \int_{L_\delta} dk \, k^{\gamma-1} \frac{1}{1 + k q_0} .
\] (3.48)
Combining (3.47) and (3.48) we find that
\[
\int_{L_\delta} dk \, \frac{k^{\gamma-1}}{1 + k q_0} + \int_{L_\delta} dk \, \frac{k^{\gamma-1}}{1 + k q_0} \\
\leq \frac{1}{q_0} \int_{L_\delta} dk \, k^{\gamma-2} + \int_{L_\delta} dk \, \frac{k^{\gamma-1}}{1 + k q_0} < \infty .
\]
This is a contradiction, because $L_\delta \cup L_\delta = [0, \infty)$ and $\int dk \, \frac{k^{\gamma-1}}{1 + k q_0}$ diverges for $\gamma \geq 1$.
This completes the proof that functions satisfying properties (A), (B) and (C) do not exist.
We have thus proven that eq.(3.1) only has the trivial solution $\Psi = 0$. 

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