Rosenthal-type inequalities for the maximum of partial sums of stationary processes and examples

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Summary

The aim of this paper is to propose new Rosenthal-type inequalities for higher moments of the maximum of partial sums of stationary sequences including martingales and their generalizations. As in the recent results by Peligrad et al. (2007) and Rio (2009), the estimates of the moments are expressed in terms of the norms of projections of partial sums. The proofs of the results are essentially based on a new maximal inequality generalizing the Doob’s maximal inequality for martingales and dyadic induction. Various applications are also provided.

1 Introduction

For independent random variables, the Rosenthal inequalities relate higher moments of partial sums of random variables with the variance of partial sums. One variant of this inequality is the following (see Rosenthal (1970), p. 279): let $(X_k)_k$ be independent and centered real valued random variables with finite moments of order $p$, $p \geq 2$. Then for every positive integer $n$,

$$
E(\max_{1 \leq j \leq n} |S_j|^p) \ll \sum_{k=1}^{n} E(|X_k|^p) + \left( \sum_{k=1}^{n} E(X_k^2) \right)^{p/2},
$$

(1)

where $S_j = \sum_{k=1}^{j} X_k$; the notation $a_n \ll b_n$ will often replace the Vinogradov symbol $O$ and means that there is a positive constant $C$ (that may depend on $p$ in this paper) such that $a_n \leq C b_n$ for all positive integers $n$.

Besides of being useful to compare the norms $L^p$ and $L^2$ of partial sums, these inequalities are important tools to obtain a variety of results, including convergence rates with respect to the strong law of large numbers (see for instance Wittmann (1985)) or almost sure invariance principles (see Wu (2007) and Gouëzel (2010) for recent results). Since the 70’s, there has been a great amount of works which extended the inequality (1) to dependent sequences. See, for instance among many others: Peligrad (1985) and Shao (1995) for the case of $\rho$-mixing sequences; Shao (1988), Peligrad (1989) and Utev (1991) for the case of $\phi$-mixing sequences; Peligrad and Gut (1999) and Utev and Peligrad (2003) for interlaced mixing; Theorem 2.2 in Viennet (1997) for $\beta$-mixing processes; Theorem 6.3 in Rio (2000) for the strongly mixing case; Rio (2009) for projective criteria.

1 Key words: Moment inequality, Maximal inequality, Rosenthal inequality, Stationary sequences, Martingale, Projective conditions.

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The main goal of the paper is to generalize the Rosenthal inequality from sequences of independent variables to stationary dependent sequences including martingales, allowing then to consider examples that are not necessarily dependent in the sense of the dependence structures mentioned above.

In order to present our results, let us first introduce some notations and definitions used all along the paper.

**Notation 1** Let \((\Omega, \mathcal{A}, \mathbf{P})\) be a probability space and let \(T : \Omega \to \Omega\) be a bijective bi-measurable transformation preserving the probability \(\mathbf{P}\). Let \(\mathcal{F}_0\) be a \(\sigma\)-algebra of \(\mathcal{A}\) satisfying \(\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)\). We then define the nondecreasing filtration \((\mathcal{F}_i)_{i \in \mathbb{Z}}\) by \(\mathcal{F}_i = T^{-i}(\mathcal{F}_0)\) and the stationary sequence \((X_i)_{i \in \mathbb{Z}}\) by \(X_i = X_0 \circ T^i\) where \(X_0\) is a real-valued random variable. The sequence will be called adapted to the filtration \((\mathcal{F}_i)_{i \in \mathbb{Z}}\) if \(X_0\) is \(\mathcal{F}_0\)-measurable. The following notations will also be used: \(\mathbf{E}_k(X) = \mathbf{E}(X|\mathcal{F}_k)\) and the norm in \(L^p\) of \(X\) is denoted by \(||X||_p\). Let \(S_n = \sum_{j=1}^n X_j\).

In the rest of this section the sequence \((X_i)_{i \in \mathbb{Z}}\) is assumed to be stationary and adapted to \((\mathcal{F}_i)_{i \in \mathbb{Z}}\) and the variables are in \(L^p\).

If \((X_k)_k\) are stationary martingale differences, the martingale form of the inequality (1) is

\[
\max_{1 \leq j \leq n} |S_j|_p \ll n^{1/p} \|X_1\|_p + \left\| \sum_{k=1}^n \mathbf{E}_{k-1}(X_k^2) \right\|_{p/2}^{1/2} \quad \text{for any } p \geq 2,
\]

(see Burkholder (1973)). One of our goals is to replace the last term in this inequality with a new one containing terms of the form \(||\mathbf{E}_0(S_n^2)||_p/2||\). The reason for introducing this term comes from the fact that for many stationary sequences \(||\mathbf{E}_0(S_n^2)||_p/2\) is closer to the variance of partial sums. In addition, we are interested to point out a Rosenthal-type inequality for a larger class of stationary adapted sequences that includes the martingales.

Two recent results by Peligrad and Utev (2005) and Wu and Zhao (2008) show that

\[
\max_{1 \leq j \leq n} |S_j|_p \ll n^{1/p} \|X_1\|_p + \sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k)\|_p \quad \text{for any } 1 \leq p \leq 2.
\]

To find a suitable extension of this inequality for \(p > 2\), the first step in our approach is to establish the following maximal inequality that has interest in itself:

\[
\max_{1 \leq j \leq n} |S_j|_p \ll n^{1/p} \left( \max_{1 \leq j \leq n} \|S_j\|_p/n^{1/p} + \sum_{k=1}^n \frac{1}{k^{1+1/p}} \|\mathbf{E}_0(S_k)\|_p \right) \quad \text{for any } p > 1.
\]

This inequality can be viewed as a generalization of the well-known Doob’s maximal inequality for martingales.

Then, we combine the inequality (3) with several inequalities for \(||S_n||_p\) that will further be established in this paper.

As we shall see in Section 3.1 in Rio (2009). When \(p \geq 4\), we shall see that the last term in the right hand side dominates the second term that can be then omitted in this case. Inequality (3) shows that in order to relate \(||\max_{1 \leq j \leq n} |S_j|_p||\) to the vector \(||S_j||_2\)\(_{1 \leq j \leq n}||\) we have to control \(\sum_{k=1}^n k^{-\delta/p} \|\mathbf{E}_0(S_k)\|_p\) and \(\sum_{k=1}^n k^{-(p+2\delta)/p} \|\mathbf{E}_0(S_k^2) - E(S_k^2)\|_{p/2}^-\).

\[\text{2}\]
In Section 3.3.1, we study the case of stationary martingale difference sequences showing that for all even powers \( p \geq 4 \), the inequality (4) holds with \( \delta = 2/(p-2) \). This result is possible for stationary martingale differences with the help of a special symmetrization for martingales initiated by Kwapień and Woyczynski (1991). In addition, by using martingale approximation techniques, we obtain, for any even integer \( p \), another form of Rosenthal-type inequality than (4) for stationary adapted processes (see the section 3.3.2), that gives, for instance, better results for functionals of linear processes with independent innovations.

We also investigate the situation when the conditional expectation with respect to both the past and the future of the process is used. For instance, when \( p \geq 4 \) is an even integer, and the process is reversible, then the inequality (4) holds (see Theorem 9 and Corollary 30) with \( \delta = 1 \).

In Section 3.2 we show that our inequalities imply the Burkholder-type inequality as stated in Theorem 1 of Peligrad, Utev and Wu (1997). For the sake of applications in Section 3.4 we express the terms that appear in our Rosenthal inequalities in terms of individual summands.

Our paper is organized as follows. In Section 2, we prove a new maximal inequality allowing to relate the moments of the maximum of partial sums of an adapted sequence, that is not necessarily stationary, to the moments of its partial sums. The maximal inequality (3) combined with moment estimates allows us to obtain the Rosenthal-type inequalities stated in Theorems 6 and 9 of Section 3.1.

Section 3.3 is devoted to Rosenthal-type inequalities for even powers for the special case of stationary martingale differences and to an application to stationary processes via a martingale approximation technique. In Section 4, we give others applications of the maximal inequalities stated in Section 2 and provide examples for which we compute the quantities involved in the Rosenthal-type inequalities of Section 3. One of the applications presented in this section is a Bernstein inequality for the maximum of partial sums for strongly mixing sequences, that extends the inequality in Merlevède et al. (2009).

The applications are given to Arch models, to functions of linear processes and reversible Markov chains. In Section 5, we apply the inequality (4) to estimate the random term of the \( L^p \)-integrated risk of kernel estimators of the unknown marginal density of a stationary sequence that is assumed to be \( \beta \)-mixing in the weak sense (see the definition 33). Some technical results are postponed to the appendix.

2 Maximal inequalities for adapted sequences

The next proposition is a generalization of the well-known Doob’s maximal inequality for martingales to adapted sequences. It states that the moment of order \( p \) of the maximum of the partial sums of an adapted process can be compared to the corresponding moment of the partial sum plus a correction term which is zero for martingale differences sequences. The proof is based on convexity and chaining arguments.

Proposition 2 Let \( p > 1 \) and let \( q \) be its conjugate. Let \( Y_i, 1 \leq i \leq 2^r \) be real random variables in \( L^p \), where \( r \) is a positive integer. Assume that the random variables are adapted to an increasing filtration \( (F_i)_i \). Let \( S_r = Y_1 + \cdots + Y_r \) and \( S_r^* = \max_{1 \leq i \leq r} |S_i| \). We have that

\[
\| \max_{1 \leq m \leq 2^r} |S_m| \|_p \leq q \|S_r^*\|_p + \frac{q}{2} \sum_{l=0}^{r-1} \sum_{k=1}^{2^{r-l}-1} \|E(S_{(k+1)2^l} - S_{k2^l}|F_{k2^l})\|_p^{1/p} \quad .
\]

(5)

Corollary 3 In the stationary case, we get that for any integer \( r \geq 1 \),

\[
\| \max_{1 \leq m \leq 2^r} |S_m| \|_p \leq q \|S_r^*\|_p + q2^r \sum_{l=0}^{r-1} 2^{-l/p} \|E(S_{2^l}|F_0)\|_p \quad .
\]
Remark 4. The inequality in Corollary 3 easily implies that
\[
\| \max_{1 \leq m \leq n} |S_m| \|_p \leq 2q \max_{1 \leq m \leq n} \|S_m\|_p + (2^{1/p} q)n^{1/p} \sum_{l=0}^{r-1} 2^{-l/p} \|E(S_{2^l} | F_0)\|_p ,
\]
for any integer \( n \in [2^{r-1}, 2^r] \), where \( r \) is a positive integer. Moreover, due to the subadditivity of the sequence \( \|E(S_n | F_0)\|_p \) \( n \geq 1 \), according to Lemma 38 we also have that for any positive integer \( n \),
\[
\| \max_{1 \leq m \leq n} |S_m| \|_p \leq 2q \max_{1 \leq m \leq n} \|S_m\|_p + (q^{2+2/p} 2^{1/p - 1})n^{1/p} \sum_{j=1}^{n} j^{-1-1/p} \|E(S_j | F_0)\|_p .
\]

For several applications involving exponential bounds we point out the following proposition.

Proposition 5. Let \( p > 1 \) and \( q \) be its conjugate. Let \( (Y_i)_{i \geq 1} \), be real random variables in \( L^p \). Assume that the random variables are adapted to an increasing filtration \( (F_i) \). Let \( S_n = Y_1 + \cdots + Y_n \) and \( S_n^* = \max_{1 \leq i \leq n} |S_i| \). Let \( \varphi \) be a nondecreasing, non negative, convex and even function. Then for any positive real \( x \), and any positive integer \( r \), the following inequality holds
\[
P(S_{2^r} \geq 2x) \leq \frac{1}{\varphi(x)} E(\varphi(S_{2^r})) + q^p x^{-p} \left( \sum_{l=0}^{r-1} \left( \sum_{k=1}^{2^r - 1} \|E(S_{k+1} | F_{2^r}) - S_{k+1} | F_{2^r} \|_p \right) \right)^{1/p} .
\]
Assume in addition that there exists a positive real \( M \) such that \( \sup_k \|Y_i\|_\infty \leq M \). Then for any positive real \( x \), and any positive integer \( r \), the following inequality holds
\[
P(S_{2^r}^* \geq 4x) \leq \frac{1}{\varphi(x)} E(\varphi(S_{2^r})) + q^p x^{-p} \left( \sum_{l=0}^{r-1} \left( \sum_{k=1}^{2^r - 1} \|E(S_{v+k+1} | F_{2^r}) - S_{v+k+1} | F_{2^r} \|_p \right) \right)^{1/p} ,
\]
with \( v = \lfloor x/M \rfloor \) (where \( \lfloor . \rfloor \) denotes the integer part).

Proof of Propositions 2 and 5. For any \( m \in [0, 2^r - 1] \), we have that
\[
S_{2^r-m} = E(S_{2^r} | F_{2^r-m}) - E(S_{2^r} - S_{2^r-m} | F_{2^r-m}) .
\]
So,
\[
S_{2^r}^* \leq \max_{0 \leq m \leq 2^r - 1} |E(S_{2^r} | F_{2^r-m})| + \max_{0 \leq m \leq 2^r - 1} |E(S_{2^r} - S_{2^r-m} | F_{2^r-m})| .
\]
Since \( E(S_{2^r} | F_k) \) \( k \geq 1 \) is a martingale, we shall use Doob’s maximal inequality to deal with the first term in the right-hand side of (10). Hence, since \( \varphi \) is a nondecreasing, non negative, convex and even function, we get that
\[
P(\max_{0 \leq m \leq 2^r - 1} |E(S_{2^r} | F_{2^r-m})| \geq x) \leq \frac{1}{\varphi(x)} E(\varphi(S_{2^r})) ,
\]
and also that
\[
\| \max_{0 \leq m \leq 2^r - 1} |E(S_{2^r} | F_{2^r-m})| \|_p \leq q \|S_{2^r}\|_p .
\]
Write now \( m \) in basis 2 as follows:
\[
m = \sum_{i=0}^{r-1} b_i(m) 2^i, \quad \text{where } b_i(m) = 0 \text{ or } b_i(m) = 1 .
\]
Set \( m_t = \sum_{i=0}^{r-1} b_i(m)2^i \) and notice that for any \( p \geq 1 \), we have

\[
|E(S_{2r} - S_{2r-m}|\mathcal{F}_{2r-m})|^p \leq \left( \sum_{l=0}^{r-1} |E(S_{2r-m_{l+1}} - S_{2r-m_{l}}|\mathcal{F}_{2r-m_{l}})|^p \right)^{1/p}.
\]

Hence setting

\[
\alpha_l = \left( \sum_{k=1}^{2^{r-l}-1} ||E(S_{(k+1)2^l} - S_{k2^l}|\mathcal{F}_{k2^l})||^p \right)^{1/p} \text{ and } \lambda_l = \frac{\alpha_l}{\sum_{l=0}^{r-1} \alpha_l},
\]

we get by convexity

\[
|E(S_{2r} - S_{2r-m}|\mathcal{F}_{2r-m})|^p \leq \sum_{l=0}^{r-1} \lambda_l^{1-p} |E(S_{2r-m_{l+1}} - S_{2r-m_{l}}|\mathcal{F}_{2r-m_{l}})|^p
\]

\[
\leq \sum_{l=0}^{r-1} \lambda_l^{1-p} |E(|E(S_{2r-m_{l+1}} - S_{2r-m_{l}}|\mathcal{F}_{2r-m_{l}})||\mathcal{F}_{2r-m})|^p.
\]

Now \( m_t \neq m_{t+1} \) only if \( b_t(m) = 1 \), and in that case \( m_t = k_m2^l \) with \( k_m \) odd. It follows that

\[
|E(S_{2r-m_{l+1}} - S_{2r-m_{l}}|\mathcal{F}_{2r-m_{l}})| \leq \max_{1 \leq k \leq 2^{r-l}, k \text{ odd}} |E(S_{2r-(k-1)2^l} - S_{2r-k2^l}|\mathcal{F}_{2r-k2^l})|
\]

\[
:= A_{r,t}.
\]

Hence, using the fact that if \( |X| \leq |Y| \) then \( E(|X|\mathcal{F}) \leq E(|Y|\mathcal{F}) \), we get that

\[
||\mathbb{E}(S_{2r} - S_{2r-m}|\mathcal{F}_{2r-m})||_p \leq \sum_{l=0}^{r-1} \lambda_l^{1-p} E(\max_{0 \leq m \leq 2^{r-1}} |\mathbb{E}(A_{r,t}|\mathcal{F}_{2r-m})|)^p.
\]

Notice that \( (\mathbb{E}(A_{r,t}|\mathcal{F}_k))_{k \geq 1} \) is a martingale and by Doob’s maximal inequality, we get that

\[
E(\max_{0 \leq m \leq 2^{r-1}} |\mathbb{E}(A_{r,t}|\mathcal{F}_{2r-m})|)^p \leq q^p \|A_{r,t}\|_p^p \leq q^p \alpha_l^p.
\]

Using the definition of \( \lambda_l \), we then get that

\[
||\max_{0 \leq m \leq 2^{r-1}} |\mathbb{E}(S_{2r} - S_{2r-m}|\mathcal{F}_{2r-m})||_p \leq q^p \left( \sum_{l=0}^{r-1} \alpha_l \right)^p
\]

\[
\leq q^p \left( \sum_{l=0}^{r-1} \sum_{k=1}^{2^{r-l}-1} ||E(S_{(k+1)2^l} - S_{k2^l}|\mathcal{F}_{k2^l})||_p^{1/p} \right)^p,
\]

Starting from (10) and using (11) and (13) combined with Markov’s inequality, Inequality (8) of Proposition 5 follows. To end the proof of Proposition 2, we start from (10) and consider the bounds (12) and (13).

We turn now to the proof of Inequality (9). We start from (10) and we write that

\[
S_{2r}^* \leq \max_{0 \leq m \leq 2^{r-1}} |\mathbb{E}(S_{2r}|\mathcal{F}_{2r-m})| + \max_{0 \leq m \leq 2^{r-1}} |\mathbb{E}(S_{2r+v} - S_{2r+v-m}|\mathcal{F}_{2r-m})|
\]

\[
+ \max_{0 \leq m \leq 2^{r-1}} |\mathbb{E}(S_{2r+v} - S_{2r}|\mathcal{F}_{2r-m})| + \max_{0 \leq m \leq 2^{r-1}} |\mathbb{E}(S_{2r+v-m} - S_{2r-m}|\mathcal{F}_{2r-m})|.
\]
By the fact that the variables are uniformly bounded by $M$, we then derive that
\[
S_2^r \leq \max_{0 \leq m \leq 2^{r-1}} |E(S_2^r | F_{2^r-m})| + \max_{0 \leq m \leq 2^{r-1}} |E(S_2^{r+v} - S_2^{r+v-m} | F_{2^r-m})| + 2vM.
\]
Since $vM \leq x$, it follows that
\[
P(S_2^r \geq 4x) \leq P(\max_{0 \leq m \leq 2^{r-1}} |E(S_2^r | F_{2^r-m})| \geq x) + P(\max_{0 \leq m \leq 2^{r-1}} |E(S_2^{r+v} - S_2^{r+v-m} | F_{2^r-m})| \geq x).
\]
Using chaining arguments, convexity and the Doob’s maximal inequality, as above, we infer that for any $p > 1$,
\[
\| \max_{0 \leq m \leq 2^{r-1}} |E(S_2^{r+v} - S_2^{r+v-m} | F_{2^r-m})| \|_p^p \\
\leq q^p \left( \sum_{l=0}^{r-1} \left( \sum_{k=1}^{2^{r-l-1}} \|E(S_{v+(k+1)2^l} - S_{v+k2^l} | F_{k2^l})\|_p \right)^{1/p} \right)^p.
\]
Starting from (14) and using (11) and (15) combined with Markov’s inequality, Inequality (9) of Proposition 5 follows. \(\diamond\)

3 Moment inequalities for the maximum of partial sums under projective conditions

3.1 Rosenthal-type inequalities for stationary processes

Using a direct approach that combines the maximal inequality (7) and the Lemma (8), we obtain the following Rosenthal inequality for the maximum of the partial sums of a stationary process for all powers $p > 2$.

**Theorem 6** Let $p > 2$ be a real number and let $(X_i)_{i \in \mathbb{Z}}$ be an adapted stationary sequence in the sense of Notation (7). Then, for any positive integer $n$, the following inequality holds:

\[
E\left( \max_{1 \leq j \leq n} |S_j|^p \right) \ll nE(|X_1|^p) + cn\left( \sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \|E_0(S_k)\|_p \right)^p + n\left( \sum_{k=1}^{n} \frac{1}{k^{1+2\delta/p}} \|E_0(S_k^\delta)\|_{p/2} \right)^{p/(2\delta)},
\]

for any positive integer $r$ with $\delta = \min(1, 1/(p-2))$ and $c = 1$. When $p \geq 4$ we can take $c = 0$ by enlarging the constant involved.

**Comment 7** 1. Notice that for $2 < p \leq 3$ the inequality holds with $\delta = 1$ and therefore it provides a maximal form for Theorem 3.1 in Rio (2009).

2. It is interesting to indicate the monotonicity of the right-hand side of the inequality in $\delta$. To be more precise, for any $0 < \delta \leq \gamma \leq 1$, the following inequality holds:

\[
\left( \sum_{k=1}^{n} k^{-1-2\gamma/p} \|E_0(S_k^\gamma)\|_{p/2}^{\gamma} \right)^{1/\gamma} \leq 2^{(1+\gamma)(\gamma-\delta)/((\delta) \gamma)} \left( \sum_{k=1}^{n} k^{-1-2\delta/p} \|E_0(S_k^\delta)\|_{p/2}^{\delta} \right)^{1/\delta}.
\]

To see this, we notice the subadditivity property $\|E_0(S_{k+1}^\delta)\|_{p/2} \leq 2\|E_0(S_k^\delta)\|_{p/2} + 2\|E_0(S_k^\delta)\|_{p/2}$ and apply then the item 3 of Lemma (8) with $C = 2.$
3. On the other hand for any $0 < \delta < 1$ and any $\gamma > 1/\delta - 1$, by Hölder’s inequality, there exists a positive constant $C$ depending on $p$, $\gamma$ and $\delta$, such that

$$
\left( \sum_{k=1}^{n} k^{-1-2\delta/p} \|E_0(S_k^2)\|_p^{\delta/p} \right)^{p/(2\delta)} \leq C \left( \sum_{k=1}^{n} k^{-1-2/p} (\log k)^\gamma \|E_0(S_k^2)\|_p \right)^{p/2}.
$$

(16)

4. As a matter of fact we shall prove first the inequality from Theorem 6 in a slightly different form which is equivalent up to multiplicative constants: for any positive integer $r$ and any integer $n$ such that $2^{r-1} \leq n < 2^r$, ($\delta$ and $c$ as above)

$$
E\left( \max_{1\leq j\leq n} |S_j|^p \right) \ll nE(|X_1|^p) + cn \left( \sum_{k=0}^{r-1} 2^{-k/p} \|E_0(S_{2^k})\|_p \right)^p + n \left( \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|E_0(S_{2^k})\|_p^{\delta/p} \right)^{p/(2\delta)}.
$$

(17)

With applications to Markov processes in mind, by conditioning with respect to both the future and to the past of the process, our next result gives an alternative inequality than the one given in Theorem 6 when $p$ is an even integer. For this case, the power $\delta$ appearing in Theorem 6 is always equal to one. Before stating the result, we first introduce the following notation to define the additional nonincreasing filtration that we consider.

**Notation 8** Let $\mathcal{F}_0$ be a $\sigma$-algebra of $\mathcal{F}$ satisfying $T^{-1}(\mathcal{F}_0) \subseteq \mathcal{F}_0$. We then define the nonincreasing filtration $(\mathcal{F}_r)_{r\in\mathbb{Z}}$ by $\mathcal{F}_r = T^{-r}(\mathcal{F}_0)$. In what follows, we use the notation $\bar{E}_k(Y) = E(Y|\mathcal{F}_k)$.

**Theorem 9** Let $p \geq 4$ be an even integer and let $X_0$ be a real valued random variable such that $\|X_0\|_p < \infty$ and measurable with respect to $\mathcal{F}_0$ and to $\mathcal{F}_0$. We construct the stationary sequence $(X_k)_{k\in\mathbb{Z}}$ as in Notation 7. Then for any integer $n$,

$$
E\left( \max_{1\leq j\leq n} |S_j|^p \right) \ll nE(|X_1|^p) + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \left( \|E_0(S_k)\|_p + \|\bar{E}_{k+1}(S_k)\|_p \right) \right)^p
$$

$$
+ n \left( \sum_{k=1}^{n} \frac{1}{k^{1+2/p}} \left( \|E_0(S_k^2)\|_p^{1/2} + \|\bar{E}_{k+1}(S_k^2)\|_p^{1/2} \right) \right)^{p/2}.
$$

As a corollary to the proof of Theorem 7 we obtain:

**Theorem 10** Let $p \geq 4$ be a real number and let $X_0$ be a real valued random variables such that $\|X_0\|_p < \infty$ and measurable with respect to $\mathcal{F}_0$ and to $\mathcal{F}_0$. We construct the stationary sequence $(X_k)_{k\in\mathbb{Z}}$ as in Notation 7. Then for any integer $n$,

$$
E\left( \max_{1\leq j\leq n} |S_j|^p \right) \ll nE(|X_1|^p) + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \left( \|E_0(S_k^2)\|_p^{1/2} + \|\bar{E}_{k+1}(S_k^2)\|_p^{1/2} \right) \right)^p.
$$

This theorem is also valid for $2 < p < 4$. In this range however, according to the comment of Theorem 6, Theorem 7 gives better bounds.

**Proof of Theorem 6.** The proof of this theorem is based on dyadic induction and involves several steps. With the notation $a_n = ||S_n||_p$, we shall establish a recurrence formula: for any positive integer $r$,

$$
a_r^p \leq 2^r \left( a_0^p + c_1 \sum_{k=0}^{r-1} 2^{-k} a_k^{p-1} \|E_0(S_{2^k})\|_p + c_2 \sum_{k=0}^{r-1} 2^{-k} a_k^{p-2\delta} \|E_0(S_{2^k})\|_p^{\delta/p} \right),
$$

(18)

where $c_1$ and $c_2$ are positive constants depending only on $p$. Before proving it, let us show that (18) implies our result that we state as a lemma.
Lemma 11  Assume that for some $0 < \delta \leq 1$ the recurrence formula (18) holds. Then the inequalities (17) and (14) hold with the same $\delta$.

Let us prove the lemma. We shall establish first the inequality (17). Due to the maximal inequality (9), it suffices to prove that the inequality is satisfied for $\max_{1 \leq j \leq n} E(|S_j|^p)$ instead of $E(\max_{1 \leq j \leq n} E(|S_j|^p))$. Denote $S_n = X_{n+1} + \cdots + X_{2n}$.

The proof is divided in several steps. The goal is to establish that for any positive integer $r$ and any integer $n$ such that $2^{r-1} \leq n < 2^r$,

$$\max_{1 \leq j \leq n} E(|S_j|^p) \leq n E(|X_1|^p) + cn \left( \sum_{k=0}^{r-1} 2^{-k/p} \|E_0(S_{2^k})\|_{p} \right)^p + n \left( \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|E_0(S_{2^k})\|_{p/2}^\delta \right)^{p/2\delta}. \quad (19)$$

With the notation $B_r = \max_{0 \leq k \leq r} (a^p_{2^k}/2^k)$, starting from (18), we get

$$B_r \leq a^p_{2^0} + c_1 B_1^{1-1/p} \sum_{k=0}^{r-1} 2^{-k/p} \|E_0(S_{2^k})\|_{p} + c_2 B_1^{1-2\delta/p} \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|E_0(S_{2^k})\|_{p/2}^\delta. \quad (20)$$

Therefore, taking into account that either $B_r \leq 3a^p_{2^0}$ or $B_1^{1/p} \leq 3c_1 \sum_{k=0}^{r-1} 2^{-k/p} \|E_0(S_{2^k})\|_{p}$ or $B_2^{2\delta/p} \leq 3c_2 \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|E_0(S_{2^k})\|_{p/2}^\delta$, we derive that

$$a^p_{2^r} \leq 2^r \left( 3a^p_{2^0} + \left( 3c_1 \sum_{k=0}^{r-1} 2^{-k/p} \|E_0(S_{2^k})\|_{p} \right)^p + \left( 3c_2 \sum_{k=0}^{r-1} 2^{-2k\delta/p} \|E_0(S_{2^k})\|_{p/2}^\delta \right)^{p/2\delta} \right). \quad (21)$$

Let now $2^{r-1} \leq n < 2^r$ and write its binary expansion:

$$n = \sum_{k=0}^{r-1} 2^k b_k \text{ where } b_{r-1} = 1 \text{ and } b_k \in \{0, 1\} \text{ for } k = 0, \ldots, r-2. \quad (21)$$

Notice that by stationarity

$$\|S_n\|_p \leq \sum_{k=0}^{r-1} b_k \|S_{2^k}\|_p.$$

Then, by using (20) and the fact that $\sum_{k=0}^{r-1} b^k 2^{k/p} \leq 2^{r/p}/(1 - 2^{-1/p})$, we derive the inequality (19) for $E(|S_n|^p)$ and also for $\max_{1 \leq j \leq n} E(|S_j|^p)$. The inequality (17) follows now by the maximal inequality (9).

We indicate now how to derive from (17), the inequality stated in Theorem 6. Notice that, by stationarity, for any integers $i$ and $j$,

$$\|E_0(S_{i+j})\|_p \leq \|E_0(S_i)\|_p + \|E_0(S_j)\|_p,$$

and also that for any $0 < \delta \leq 1$,

$$\|E_0(S_{i+j}^2)\|_{p/2}^{\delta} \leq 2^\delta \|E_0(S_i^2)\|_{p/2}^\delta + 2^\delta \|E_0(S_j^2)\|_{p/2}^\delta.$$

Using Item 1 of Lemma 38 it follows that

$$\sum_{k=0}^{r-1} 2^{-k/p} \|E_0(S_{2^k})\|_p \leq \sum_{k=1}^{n} k^{-1-1/p} \|E_0(S_k)\|_p.$$

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and
\[ \sum_{k=0}^{r-1} 2^{-2k \delta/p} \| E_0(S_{2k}^2) \|_{p/2} \leq \sum_{k=1}^{n} k^{-1-2 \delta/p} \| E_0(S_k^2) \|_{p/2} . \]  \hspace{1cm} (22)

The results follows by the above considerations via the inequality (17). \( \Box \)

**End of the proof of Theorem 6** It remains to establish the recurrence formula (18). We divide the proof in three cases according to the values of \( p \).

The case \( 2 < p \leq 3 \) was discussed in Rio (2009). We give here a shorter alternative proof. We apply inequality (82) of Lemma 37 with \( x = S_{2r-1} \) and \( y = \bar{S}_{2r-1} \). Then by taking the expectation and using Lemma 35, we get by stationarity that for any positive integer \( r \),

\[ E(|S_{2r-1}|)^p \leq 2E(|S_{2r-1}|)^p + pE(|S_{2r-1}|^{p-1} \text{sign}(S_{2r-1})E_{2r-1}(|\bar{S}_{2r-1}|^p) + C_p^2 E(|S_{2r-1}|^{p-2}E_{2r-1}(|\bar{S}_{2r-1}|^2)) , \]

where \( C_p^2 = p(p-1)/2 \). This inequality combined with the Hölder’s inequality gives

\[ a_{2r}^p \leq 2a_{2r-1}^p + pa_{2r-1}^{p-1} \| E_0(S_{2r-1}) \|_p + C_p^2 a_{2r-1}^{1/p} \| E_0(S_{2r-1}) \|_{p/2} . \]

By recurrence according to the first term in the right hand side, it follows that for any integer \( r \),

\[ a_{2r}^p \leq 2^r \left( a_{20}^p + 2^{-1/p} \sum_{k=0}^{r-1} 2^{-k} a_{2k}^{p-1} \| E_0(S_{2k}) \|_p + 2^{-1} C_p^2 \sum_{k=0}^{r-1} 2^{-k} a_{2k}^{p(1-2/p)} \| E_0(S_{2k}) \|_{p/2} \right) , \]

and therefore (18) holds with \( \delta = 1 \), \( c_1 = 2^{-1/p} \), and \( c_2 = 2^{-1} C_p^2 \).

Assume now that \( p \in [3, 4] \). Using the inequality (83) of Lemma 37 with \( x = S_{2r-1} \) and \( y = \bar{S}_{2r-1} \), taking the expectation, and using Lemma 35 we get by stationarity that for any positive integer \( r \),

\[ a_{2r}^p \leq 2a_{2r-1}^p + pa_{2r-1}^{p-1} \| E_0(S_{2r-1}) \|_p + C_p^2 a_{2r-1}^{p(1-2/p)} \| E_0(S_{2r-1}) \|_{p/2} + 2p(p-2)^{-1} a_{2r-1}^{p-2/(p-2)} \| E_0(S_{2r-1}) \|_{p/2} . \]

Since \( \| E_0(S_{2r-1}^2) \|_{p/2} \leq \| E_0(S_{2r-1}^2) \|_{p/2} \), it follows that

\[ a_{2r}^p \leq 2a_{2r-1}^p + pa_{2r-1}^{p-1} \| E_0(S_{2r-1}) \|_p + 4pa_{2r-1}^{p-2/(p-2)} \| E_0(S_{2r-1}) \|_{p/2} . \]

By recurrence, we derive that for any integer \( r \),

\[ a_{2r}^p \leq 2^r \left( a_{20}^p + 2^{-1/p} \sum_{k=0}^{r-1} 2^{-k} a_{2k}^{p-1} \| E_0(S_{2k}) \|_p + 2p \sum_{k=0}^{r-1} 2^{-k} a_{2k}^{p-2/(p-2)} \| E_0(S_{2k}) \|_{p/2} \right) . \]

It follows that (18) holds with \( \delta = 1/(p-2) \), \( c_1 = 2^{-1/p} \), and \( c_2 = 2p \).

It remains to prove the inequality (4) for \( p \geq 4 \). Using the inequality (83) of Lemma 37 with \( x = S_{2r-1} \) and \( y = \bar{S}_{2r-1} \), and taking the expectation, we get by stationarity that

\[ a_{2r}^p \leq 2a_{2r-1}^p + 4pE(|S_{2r-1}|^{p-1} |\bar{S}_{2r-1}| + |\bar{S}_{2r-1}|^{p-1} |S_{2r-1}|) . \]

Using Lemma 35 together with stationarity, it follows that

\[ E(|S_{2r-1}|^{p-1} |\bar{S}_{2r-1}|^{p-1}) \leq a_{2r-1}^{p-2/(p-2)} \| E_0(S_{2r-1}^2) \|_{p/2} \],

and that

\[ E(|S_{2r-1}|^{p-1} |\bar{S}_{2r-1}|^{p-1}) \leq a_{2r-1}^{p-1} \| E_0(S_{2r-1}^2) \|_{p/2} \leq a_{2r-1}^{p-2/(p-2)} \| E_0(S_{2r-1}^2) \|_{p/2} . \]

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From these estimates we deduce

\[ a_{2r}^p \leq 2a_{2r-1}^p + 2(4^p)a_{2r-1}^{p-2/(p-2)}\|E_0(S_{2r-1})\|_{p/2}^{1/(p-2)}.\]

By recurrence, it follows that for any integer \( r \),

\[ a_{2r}^p \leq 2^r \left( a_{20}^p + 4^p \sum_{k=0}^{r-1} 2^{-k} a_{2k}^{p-2/(p-2)} \|E_0(S_{2k})\|_{p/2}^{1/(p-2)} \right). \]

It follows that (18) holds with \( \delta \) and that (37) applied with \( \delta \) we obtain the desired result by using Lemma 38.

Noticing in addition that, by stationarity, for any integer \( i \) and \( j \)

\[ \|E_0(S_{i+j})\|_p \leq \|E_0(S_i)\|_p + \|E_0(S_j)\|_p, \quad \|\tilde{E}_{i+j+1}(S_{i+j})\|_p \leq \|\tilde{E}_{i+1}(S_i)\|_p + \|\tilde{E}_{j+1}(S_j)\|_p, \]

\[ \|\tilde{E}_{i+j+1}(S_{i+j})\|_{p/2} \leq 2\|\tilde{E}_{i+1}(S_i^2)\|_{p/2} + 2\|\tilde{E}_{j+1}(S_j^2)\|_{p/2}, \]

and that

\[ \|E_0(S_{i+j}^2)\|_{p/2} \leq 2\|E_0(S_i^2)\|_{p/2} + 2\|E_0(S_j^2)\|_{p/2}. \]

We obtain the desired result by using Lemma 38. ⊙

**Proof of Theorem 9.** Denote \( \bar{S}_n = X_{n+1} + \cdots + X_{2n} \). Starting from the inequality (55) of Lemma 37 applied with \( x = S_n \) and \( y = \bar{S}_n \) and using the notation \( a_n = \|S_n\|_p \), by stationarity, we get that

\[ a_{2n}^p \leq 2a_n^p + p\left( E(S_n^p \bar{S}_n) + E(S_n \bar{S}_n^{p-1}) \right) + 2^p \left( E(S_n^{p-2} \bar{S}_n^2) + E(\bar{S}_n^{p-2} S_n^2) \right). \]

By using the Hölder’s inequality and recurrence, we then derive that for any positive integer \( r \),

\[ a_{2r}^p \leq 2^r a_{2r-1}^p + 2^{-1} p \sum_{k=0}^{r-1} 2^{r-k} a_{2k}^{p-2/(p-2)} \left( \|E_0(S_{2k})\|_p + \|\tilde{E}_{2k+1}(S_{2k})\|_p \right) \]

\[ + 2^{r-1} \sum_{k=0}^{r-1} 2^{-k} a_{2k}^{p-2} \left( \|E_0(S_{2k}^2)\|_{p/2} + \|\tilde{E}_{2k+1}(S_{2k}^2)\|_{p/2} \right). \]

By using the arguments of the proof of Lemma 11 we get for \( 2^{r-1} \leq n < 2^r \),

\[ E\left( \max_{1 \leq j \leq n} |S_j|^p \right) \ll n E(|X_1|^p) + n \left( \sum_{k=1}^{r-1} 2^{-k/p} \left( \|E_0(S_{2k})\|_p + \|\tilde{E}_{k+1}(S_{2k})\|_p \right) \right)^p \]

\[ + n \left( \sum_{k=1}^{n} 2^{-2k/p} \left( \|E_0(S_{2k}^2)\|_{p/2} + \|\tilde{E}_{k+1}(S_{2k}^2)\|_{p/2} \right) \right)^{p/2}. \]

Noticing in addition that, by stationarity, for any integer \( i \) and \( j \)

\[ \|E_0(S_{i+j})\|_p \leq \|E_0(S_i)\|_p + \|E_0(S_j)\|_p, \quad \|\tilde{E}_{i+j+1}(S_{i+j})\|_p \leq \|\tilde{E}_{i+1}(S_i)\|_p + \|\tilde{E}_{j+1}(S_j)\|_p, \]

\[ \|\tilde{E}_{i+j+1}(S_{i+j})\|_{p/2} \leq 2\|\tilde{E}_{i+1}(S_i^2)\|_{p/2} + 2\|\tilde{E}_{j+1}(S_j^2)\|_{p/2}, \]

and that

\[ \|E_0(S_{i+j}^2)\|_{p/2} \leq 2\|E_0(S_i^2)\|_{p/2} + 2\|E_0(S_j^2)\|_{p/2}. \]

We obtain the desired result by using Lemma 38. ⊙
Proof of Theorem 10. To proof this theorem we apply the inequality (84) of Lemma 37 with $x = S_n$ and $y = \tilde{S}_n$, where $S_n = X_{n+1} + \cdots + X_{2n}$. With the notation $a_n = ||S_n||_p$, we then have by stationarity that

$$a_{2n}^p \leq 2a_n^p + 4^p\left(E(||S_n|^{p-1}||\tilde{S}_n||) + E(|S_n||\tilde{S}_n|^{p-1})\right).$$

By conditioning and then applying the Jensen’s inequality followed by the Hölder’s inequality we obtain

$$a_{2n}^p \leq 2a_n^p + 4^p\left(E(||S_n|^{p-1}E_{n+1}^{1/2}(\tilde{S}_n^2)) + E(||S_n||^{p-1}\tilde{E}_{n+1}^{1/2}(S_n^2)\right)$$

$$\leq 2a_n^p + 4^p a_n^{p-1}\left(||E_0(S_n^2)||_{p/2}^2 + ||\tilde{E}_{n+1}(S_n^2)||_{p/2}^2\right).$$

By recurrence, we then derive that for any positive integer $r$,

$$a_{2^r}^p \leq 2^r\left(a_n^p + 2^{2p-1}\sum_{k=0}^{r-1}2^{-k}a_{2^k}^{p-1}\left(||E_0(S_{2^k}^2)||_{p/2} + ||\tilde{E}_{2^k+1}(S_{2^k}^2)||_{p/2}\right)\right).$$

The proof is completed by the arguments developed in the proof of Lemma 11 and by using Lemma 38 by taking into account the inequalities (24) and (25).

3.2 Relation with the Burkholder-type Inequality.

Next lemma shows how to compare $||E_0(S_n^2)||_{p/2}$ with quantities involving only $||E_0(S_n)||_p$.

Lemma 12 Let $p \geq 2$ be a real number and let $(X_n)$ be an adapted stationary sequence in the sense of Notation 7. Then, for any positive integer $n$,

$$||E_0(S_n^2)||_{p/2} \leq n||E_0(X_1^2)||_{p/2} + n\left(\sum_{j=1}^{n}||E_0(S_j)||_{p}\right)^2.$$  \hspace{1cm} (26)

As a consequence of the above lemma, we get that for any $0 < \delta \leq 1$ and any real $p > 2$,

$$n\left(\sum_{j=1}^{n}||E_0(S_j)||_{p}\right)^{p/(2\delta)} \leq n^{p/2}||E_0(X_1^2)||_{p/2} + n^{p/2}\left(\sum_{j=1}^{n}||E_0(S_j)||_{p}\right)^p.$$  \hspace{1cm} (27)

Theorem 6 then implies the following Burkholder-type inequality that was established by Peligrad, Utev, and Wu (2007, Theorem 1):

Corollary 13 Let $p > 2$ be a real number and let $(X_n)$ be an adapted stationary sequence in the sense of Notation 7. Then, for any integer $n$,

$$E(\max_{1 \leq j \leq n}|S_j|^p) \leq n^{p/2}E(|X_1|^p) + n^{p/2}\left(\sum_{j=1}^{n}||E_0(S_j)||_{p}\right)^p.$$  \hspace{1cm} (28)

Proof of Lemma 12 We shall first prove that for any positive integer $k$,

$$||E_0(S_{2^k}^2)||_{p/2} \leq 2^{k+1}||E_0(X_1^2)||_{p/2} + 2^{k+2}\left(\sum_{j=0}^{k-1}||E_0(S_{2^j})||_{p}\right)^2.$$  \hspace{1cm} (29)
By using the notation $S_{2k-1} = X_{2^{k-1}+1} + \cdots + X_{2^k}$ and the fact that $S^2_{2k} = S^2_{2^k-1} + S^2_{2^k-1} + 2S_{2^k-1}S_{2^k-1}$, we get, by stationarity, that
\[
\|E_0(S^2_{2k})\|_{p/2} \leq 2\|E_0(S^2_{2^k-1})\|_{p/2} + 2\|E_0(S_{2^k-1}E_{2^k-1}(S_{2^k-1}))\|_{p/2}.
\]
Now Cauchy-Schwartz inequality applied first to the conditional expectation gives
\[
\|E_0(S_{2^k-1}E_{2^k-1}(S_{2^k-1}))\|_{p/2} \leq \|E_{0}^{1/2}(S^2_{2^k-1})\|_{p/2} \|E_{0}^{1/2}(E_{2^k-1}S_{2^k-1})\|_{p/2}
\leq \|E_0(S^2_{2^k-1})\|_{1/2} \|E_0(S_{2^k-1})\|_p.
\]
Hence setting $b_{2k} = \|E_0(S^2_{2k})\|_{p/2}$, it follows that
\[
b_{2k} \leq 2b_{2k-1} + 2^{1/2}b_{2^k-1}\|E_0(S_{2^k-1})\|_p.
\]
By induction, this gives that
\[
b_{2k} \leq 2^k b_0 + \sum_{j=0}^{k-1} 2^{k-j}b_{2^j}\|E_0(S_{2^j})\|_p.
\]
With the notation $B_k = \max_{0 \leq j \leq k} 2^{-j}b_{2^j}$, we derive that
\[
B_k \leq 2\max\left(b_0, B_k^{1/2} \sum_{j=0}^{k-1} 2^{-j/2}\|E_0(S_{2^j})\|_p\),
\]
implying that
\[
2^{-k}b_{2^k} \leq B_k \leq 2b_0 + 2^{k} \left(\sum_{j=0}^{k-1} 2^{-j/2}\|E_0(S_{2^j})\|_p\right)^2.
\]
This ends the proof of the inequality (27).

We turn now to the proof of (26). Let $r$ be the positive integer such that $2^{-1} \leq n < 2^r$. Starting with the binary expansion (21), and using Minkowski’s inequality twice, first with respect to the conditional expectation, and second with respect to the norm in $L^p$, we get by stationarity that
\[
\|E_0(S^2_n)\|_{p/2} \leq \left(\sum_{k=0}^{r-1} b_k \|E_0(S^2_{2^k})\|_{p/2}\right)^2 \leq \left(\sum_{k=0}^{r-1} \|E_0(S^2_{2^k})\|_{1/2}\right)^2.
\]
Using then, the inequality (27), we derive that
\[
\|E_0(S^2_n)\|_{p/2} \ll n\|E_0(X^2_1)\|_{p/2} + n\left(\sum_{j=0}^{r-1} \|E_0(S_{2^j})\|_p / 2^{j/2}\right)^2. \tag{28}
\]
Since $(\|E_0(S_n)\|_p)_{n \geq 1}$ is subadditive, using Item 1 of Lemma 38, Inequality (26) follows from (28).

### 3.3 Rosenthal inequalities for martingales and the case of even powers.

#### 3.3.1 The martingale case

For any real $p > 2$, Theorem 4 applied to stationary martingale differences gives the following inequality:
\[
E\left(\max_{1 \leq j \leq n} |S_j|^p\right) \ll nE(|X_1|^p) + n\left(\sum_{k=1}^{n} \frac{1}{k^{1+2/p}}\|E_0(S^2_k)\|_{p/2}\right)^{p/(26)}.
\]
where \( \delta = \min(1, 1/(p-2)) \).

Since for stationary martingale differences we have \( \mathbf{E}(S_n^2) = n \mathbf{E}(X_1^2) \), we can express the inequality in the following form useful for applications:

\[
\mathbf{E}\left( \max_{1 \leq j \leq n} |S_j|^p \right) \leq n^{p/2} \mathbf{E}(X_1^2)^{p/2} + n \mathbf{E}(|X_1|^p) + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+4/p(p-2)}} \mathbf{E}_0(S_k^2)^{2/(p-2)} \right)^{(p-2)/4}.
\]

As we shall see in the next result, for stationary sequence \((d_i)_{i \in \mathbb{N}}\) of martingale differences in \( \mathbf{L}^p \) for \( p \geq 4 \) an even integer, this inequality can be sharpened since it holds with \( \delta = 2/(p-2) \) (see Comment 7). As a consequence, we recover, in case \( p = 4 \), the inequality (1.6) stated in Rio (2009) that was obtained using the classical Burkholder's inequality combined with the one given in Theorem 3 in Wu and Zhao (2008) for variables in \( \mathbf{L}^q \) with \( q = p/2 \). Notice that the inequality (1.6) stated in Rio (2009) cannot be generalized for \( p > 4 \) since Theorem 3 in Wu and Zhao (2008) is only valid for variables in \( \mathbf{L}^3 \) with \( 1 < q \leq 2 \).

**Theorem 14** Let \( p \geq 4 \) be an even integer and let \( d_0 \) be a real random variable in \( \mathbf{L}^p \), measurable with respect to \( \mathcal{F}_0 \) and such that \( \mathbf{E}(d_0|\mathcal{F}_{-1}) = 0 \). Let \( d_i = d_0 \circ T^i \) and \( S_n = \sum_{i=1}^{n} d_i \). Then for any integer \( n \),

\[
\mathbf{E}\left( \max_{1 \leq j \leq n} |S_j|^p \right) \leq n \mathbf{E}(|d_1|^p) + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+4/p(p-2)}} \mathbf{E}_0(S_k^2)^{2/(p-2)} \right)^{(p-2)/4}.
\]

The technique that makes this result possible is a special symmetrization for martingales initiated by Kwapień and Woyczynski (1991).

**Proposition 15** Assume that \( e_k \) are stationary martingale differences adapted to an increasing filtration \((\mathcal{F}_k)_k\) that are conditionally symmetric (the distribution of \( e_k \) given \( \mathcal{F}_{k-1} \) is equal to the distribution of \(-e_k \) given \( \mathcal{F}_{k-1} \)). Assume, in addition, that the \( e_k \)'s are conditionally independent given a sigma algebra \( \mathcal{G} \) and such that the law of \( e_k \) given \( \mathcal{G} \) is the same as the law of \( e_k \) given \( \mathcal{F}_{k-1} \). Let \( S_n = \sum_{i=1}^{n} e_i \). Then for any even integer \( p \geq 4 \) and any integer \( n \geq 1 \),

\[
\mathbf{E}\left( \max_{1 \leq j \leq n} |S_j|^p \right) \leq n \mathbf{E}(|e_1|^p) + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+4/p(p-2)}} \mathbf{E}_0(S_k^2)^{2/(p-2)} \right)^{(p-2)/4}.
\]

**Proof of Proposition 15** Due to the Doob's maximal inequality, \( \| \max_{1 \leq j \leq n} |S_j|_p \| \leq q \| S_n \|_p \) where \( q = p(p-1)^{-1} \). Then, it suffices to show that the inequality (29) holds for \( \mathbf{E}(|S_n|^p) \). We shall base this proof again on dyadic induction. Denote \( S_n = e_{n+1} + \ldots + e_{2n} \) and \( a_n = \| S_n \|_p \).

We start from the inequality (23). Since the sequence of martingale differences \( (e_k) \) is conditionally symmetric and conditionally independent given a master sigma algebra \( \mathcal{G} \), we have \( \mathbf{E}(S_{n-1}^p S_n^p) + \mathbf{E}(S_n^p S_{n-1}^p) = 0 \) and therefore

\[
a_{2n}^p \leq 2a_n^p + 2^p(\mathbf{E}(S_{n-1}^p S_n^p) + \mathbf{E}(S_n^p S_{n-1}^p)).
\]

Using Lemma 35 we have that

\[
\mathbf{E}(S_{n-1}^p S_n^p) \leq a_{n-1}^p \mathbf{E}_0(S_n^2)^{p/2} \quad \text{and} \quad \mathbf{E}(S_n^p S_{n-1}^p) \leq a_n^p \mathbf{E}_0(S_n^2)^{2/(p-2)}.
\]

Therefore, by combining all these bounds, we obtain for every even integer \( p \geq 4 \),

\[
a_{2n}^p \leq 2a_n^p + 2^p(a_{n-2}^p \mathbf{E}_0(S_n^2)^{p/2} + a_n^p \mathbf{E}_0(S_n^2)^{2/(p-2)}) \leq 2a_n^p + 2^{p+1} a_{n-2}^p \mathbf{E}_0(S_n^2)^{2/(p-2)}.
\]
By induction we easily get that for any integer \( r \),
\[
a^{p}_{2r} \leq 2^{r} \left( a^{p}_{2r_0} + 2^{p} \sum_{k=0}^{r-1} 2^{-k} a^{p-4/(p-2)}_{2^k} \right).
\]

We end the proof by Lemma 11.

**Proof of Theorem 14.**

We consider our general martingale differences sequence \((d_k)_k\) and we construct two decoupled tangent versions \((e_k)_k\) and \((\tilde{e}_k)_k\) that are \(\mathcal{G}\)–conditionally independent between them. These are martingale differences as in Proposition 14 with the additional property that the conditional distribution of \(d_k\) given \(\mathcal{F}_{k-1}\) is equal to the distribution of \(e_k\) given \(\mathcal{F}_{k-1}\) and also to the distribution of \(\tilde{e}_k\) given \(\mathcal{F}_{k-1}\) (see Proposition 6.1.5. in de la Peña and Giné (1999)). Therefore, for any even integer \( p \),
\[
\mathbb{E} \left( \sum_{i=1}^{n} d_i \right)^{p} = \mathbb{E} \left( \sum_{i=1}^{n} (d_i - \mathbb{E}(e_i|\mathcal{G})) \right)^{p} \leq \mathbb{E} \left( \sum_{i=1}^{n} (d_i - e_i) \right)^{p}.
\]

Now we use corollary 6.6.8. in de la Peña and Giné (1999) (see also Zinn (1985)). Since \((e_i - \tilde{e}_i)_i\) is a decoupled tangent sequence of \((d_i - e_i)_i\), it follows that
\[
\mathbb{E} \left( \sum_{i=1}^{n} d_i \right)^{p} \ll \mathbb{E} \left( \sum_{i=1}^{n} (e_i - \tilde{e}_i) \right)^{p}.
\]

Notice that the distribution of \(e_i - \tilde{e}_i\) is conditionally symmetric given \(\mathcal{G}\). Therefore, using the Doob’s maximal inequality and applying Proposition 15, we obtain that for every even integer \( p \geq 4 \) and any integer \( n \),
\[
\mathbb{E} \left( \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} d_i \right)^{p} \right) \ll \mathbb{E} \left( \sum_{i=1}^{n} (e_i - \tilde{e}_i) \right)^{p} \ll n \mathbb{E}(\|e_1 - \tilde{e}_1\|^p) + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+4/p(p-2)}} \right) \mathbb{E}_0 \left( \mathbb{E} \left( \left( \sum_{i=1}^{k} (e_i - \tilde{e}_i) \right)^2 \right) \right)^{2/(p-2)} \right)^{p(p-2)/4}.
\]

Notice now that \( \sum_{i=1}^{k} \mathbb{E}(e_i^2|\mathcal{F}_{i-1}) = \sum_{i=1}^{k} \mathbb{E}(d_i^2|\mathcal{F}_{i-1}) \) since both quantities are obtained using only the conditional distributions of the \(d_i\)'s and \(e_i\)'s respectively, and these two sequences are tangent. Tangency also implies that \(d_i\) and \(e_i\) have the same distributions. Hence \(\|d_1\|_p = \|e_1\|_p\). For the same reasons, we also have \( \sum_{i=1}^{k} \mathbb{E}(\tilde{e}_i^2|\mathcal{F}_{i-1}) = \sum_{i=1}^{k} \mathbb{E}(d_i^2|\mathcal{F}_{i-1}) \) and \(\|d_1\|_p = \|\tilde{e}_1\|_p\). Therefore, \(\|e_1 - \tilde{e}_1\|_p \leq 2\|d_1\|_p\) and
\[
\mathbb{E}_0 \left( \left( \sum_{i=1}^{k} (e_i - \tilde{e}_i) \right)^2 \right)^{p/2} \leq 2 \mathbb{E}_0 \left( \sum_{i=1}^{k} \mathbb{E}(e_i^2|\mathcal{F}_{i-1}) \right)^{p/2} + 2 \mathbb{E}_0 \left( \sum_{i=1}^{k} \mathbb{E}(\tilde{e}_i^2|\mathcal{F}_{i-1}) \right)^{p/2} = 4 \mathbb{E}_0 \left( \sum_{i=1}^{k} \mathbb{E}(d_i^2|\mathcal{F}_{i-1}) \right)^{p/2} = 4 \mathbb{E}_0 \left( \sum_{i=1}^{k} \mathbb{E}(d_i^2) \right)^{p/2}.
\]

Theorem 14 follows by introducing these bounds in the inequality (31).
Next lemma is a slight reformulation of the martingale approximation result that can be found in the paper by Wu and Woodroofe (Theorem 1, 2004). See also Zhao and Woodroofe (2008) and Gordin and Peligrad (2010).

**Lemma 16** Let \( p \geq 1 \) and let \((X_n)\) be an adapted stationary sequence in the sense of Notation 1. Then there is a triangular array of row-wise stationary martingale differences satisfying

\[
D^n_0 = \frac{1}{n} \sum_{i=1}^{n} (E_1(S_i) - E_0(S_i)) \quad ; \quad D^n_k = D^n_0 \circ T^k
\]

such that

\[
S_k = M^n_k + R^n_k \quad \text{where} \quad M^n_k = \sum_{i=1}^{k} D^n_i,
\]

and

\[
\max_{1 \leq k \leq n} \| R^n_k \|_p \leq 2 \| X_0 \|_p + \frac{3}{n} \sum_{i=1}^{n} \| E_0(S_i) \|_p.
\]

We state now the Rosenthal-type inequality that can be provided with the help of the approximation result above.

**Theorem 17** Let \( p \geq 4 \) be an even integer and let \((X_i)_{i \in \mathbb{Z}}\) be as in Theorem 6. Then the following inequality is valid: for any integer \( n \),

\[
E(\max_{1 \leq k \leq n} |S_k|^p) \ll n \| D^n_0 \|_p^p + n \| X_0 \|_p^p + n^{1-p} \left( \sum_{i=1}^{n} \| E_0(S_i) \|_p \right)^p
\]

\[
+ n \left( \sum_{k=1}^{n-1} \frac{1}{k^{1+4/p(p-2)}} \| E_0(S_k^2) \|_{p/2}^{2/(p-2)} \right)^{p(p-2)/4},
\]

where \( D^n_0 \) is defined by (32).

**Remark 18** Theorems 7 and 17 are in general not comparable. Indeed, for \( p \geq 4 \), Theorem 7 applies with \( \delta = 1/(p-2) \) so the last term of the inequality stated in Theorem 17 can be bounded by the last term in the inequality from Theorem 7 (see Comment 7). However the term involving the quantity \( \| E_0(S_n) \|_p \) in Theorem 17 gives additional contribution. When, for instance, \( \| E_0(S_n) \|_p = O(1) \), (which is the case under the assumptions of Corollary 27; see also the remark 28), Theorem 17 might provide a sharper bound.

**Remark 19** According to Remark 3.3 in Dedecker, Merlevède and Peligrad (2009), since \( \| D^n_0 \|_p \leq \sum_{k=1}^{n} \| P_0(X_k) \|_p \), where \( P_0(X_i) = E_0(X_i) - E_0(S_i) \), we notice that the following bound is valid:

\[
\| D^n_0 \|_p \ll \sum_{k=1}^{n} \frac{1}{k^{1/p}} \| E_0(X_k) \|_p.
\]

**Proof of Theorem 17** By the martingale approximation of Lemma 16 combined with Theorem 14 we get that

\[
E(\max_{1 \leq k \leq n} |S_k|^p) \ll E(\max_{1 \leq k \leq n} |M^n_k|^p) + E(\max_{1 \leq k \leq n} |R^n_k|^p)
\]

\[
\ll n \ E(|D^n_0|^p) + \| X_0 \|_p^p + \left( \frac{1}{n} \sum_{i=1}^{n} \| E_0(S_i) \|_p \right)^p + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+4/p(p-2)}} \| E_0(S_k^2) \|_{p/2}^{2/(p-2)} \right)^{p(p-2)/4}.
\]
Since 
\[ E_0((M_n^n)^2) \leq 2E_0(S_k^2) + 2E_0((R_k^n)^2), \]
by using Lemma 16
\[ \|E_0((M_n^n)^2)\|_{p/2} \ll \|E_0(S_k^2)\|_{p/2} + \|X_0\|_p^2 + n^{-2} \left( \sum_{i=1}^{n} \|E_0(S_i)\|_p \right)^2, \]
and Theorem 17 follows. ◊

### 3.4 Rosenthal inequality in terms of individual summands.

For the sake of applications in this section we indicate how to estimate the terms that appear in our Rosenthal inequalities in terms of individual summands and formulate some specific inequalities. By substracting \( E(S_k^2) \) and applying the triangle inequality we can reformulate all the inequalities in terms of the quantities \( E(S_k^2), \|E_0(S_k)\|_p \) and \( \|E_0(S_k^2) - E(S_k^2)\|_{p/2} \). Next lemma proposes a simple way to estimate these quantities in terms of coefficients in the spirit of Gordin (1969).

**Lemma 20** Under the stationary setting assumptions in Notation 7, we have the following estimates:

\[ E(S_k^2) \leq 2k \sum_{j=0}^{k-1} |E(X_0X_j)|, \]  
(33)

\[ \|E_0(S_k)\|_p \leq \sum_{\ell=1}^{n} \|E_0(X_\ell)\|_p, \]  
(34)

and

\[ \|E_0(S_k^2) - E(S_k^2)\|_{p/2} \leq 2 \sum_{i=1}^{k} \sum_{j=0}^{k-i} \|E_0(X_iX_{i+j}) - E(X_iX_{i+j})\|_{p/2} \]  
(35)

\[ \leq 2 \sum_{i=1}^{k} \sum_{j=0}^{k-i} \sup_{\ell \geq 0} \|E_0(X_iX_{i+\ell}) - E(X_iX_{i+\ell})\|_{p/2} \wedge (2\|X_0E_0(X_j)\|_{p/2}) \]

\[ \leq 4 \sum_{j=1}^{k} j\|X_0E_0(X_j)\|_{p/2} + 2 \sum_{i=1}^{k} \sup_{j \geq i} \|E_0(X_iX_j) - E(X_iX_j)\|_{p/2}. \]

Mixing coefficients are useful to continue the estimates from Lemma 20. We refer to the books by Bradley (2007, Theorem 4.13 via Remark 4.7, VI), Rio (2000, Theorem 2.5 and Appendix, Section C) and Dedecker et al. (2007, Remark 2.5 and Ch 3) for various estimates of the coefficients involved in Lemma 20 and examples. We shall also provide applications and explicit computations of the quantities involved.

We formulate the following proposition:

**Proposition 21** Let \( p > 2 \) be a real number and let \( (X_i)_{i \in \mathbb{Z}} \) be a stationary sequence of real-valued random variables in \( L_p \) adapted to an increasing filtration \( (\mathcal{F}_i) \). For any \( j \geq 1 \), let

\[ \lambda(j) = \max \left( \|X_0E_0(X_j)\|_{p/2}, \sup_{i \geq j} \|E_0(X_iX_j) - E(X_iX_j)\|_{p/2} \right). \]  
(36)
Then for every positive integer \( n \),
\[
\left\| \max_{1 \leq j \leq n} |S_j| \right\|_p \ll n^{1/2} \left( \sum_{k=0}^{n-1} \left| \mathbb{E}(X_0 X_k) \right| \right)^{1/2} + n^{1/p} \left| X_1 \right|_p \\
+ cn^{1/p} \sum_{k=1}^{n} \frac{1}{k^{1/p}} \left| \mathbb{E}_0(X_k) \right|_p + n^{1/p} \left( \sum_{k=1}^{n} k^{1-2/p} (\log k)^\gamma \lambda(k) \right)^{1/2}.
\]
where \( \gamma \) can be taken \( \gamma = 0 \) for \( 2 < p \leq 3 \) and \( \gamma > p - 3 \) for \( p > 3 \); \( c = 1 \) for \( 2 < p < 4 \) and \( c = 0 \) for \( p \geq 4 \). The constant that is implicitly involved in the notation \( \ll \) depends on \( p \) and \( \gamma \) but it does not depend on \( n \).

**Proof of Proposition 21.** The proof of this proposition is basically a combination of Theorem 6 and Lemma 20. By the triangle inequality
\[
\left\| \mathbb{E}_0(S_k^2) \right\|_{p/2} \leq \left\| \mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2) \right\|_{p/2} + \left\| \mathbb{E}_0(S_k^2) \right\|_{p/2}.
\]
By (33), for any \( p > 2 \) and any \( \delta > 0 \), we easily obtain
\[
\left( \sum_{k=1}^{n} \frac{1}{k^{1+2/\delta}} \left( \mathbb{E}(S_k^2) \right)^{\delta} \right)^{1/(2\delta)} \ll n^{1/2-1/p} \left( \sum_{j=0}^{n-1} \left| \mathbb{E}(X_0 X_j) \right| \right)^{1/p}.
\]
Then, we use inequality (34) and changing the order of summation
\[
\sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \left| \mathbb{E}_0(S_k) \right|_p \ll \sum_{k=1}^{n} \frac{1}{k^{1/p}} \left| \mathbb{E}_0(X_k) \right|_p.
\]
Now for the situation \( 0 < \delta < 1 \), by Hölder’s inequality,
\[
\left( \sum_{k=1}^{n} \frac{1}{k^{1+2/\delta}} \left\| \mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2) \right\|_{p/2}^{\delta} \right)^{p/(2\delta)} \ll \left( \sum_{k=1}^{n} \frac{1}{k^{1+2/\delta}} \left| \mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2) \right|_{p/2} \right)^{p/2}.
\]
where \( \gamma > 1/\delta - 1 \). We continue the estimate by using (35) and get
\[
\sum_{k=1}^{n} \frac{1}{k^{1+2/\delta}} \left\| \mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2) \right\|_{p/2} \ll \sum_{k=1}^{n} \left( \log k \right)^\gamma k^{1-2/p} \lambda(k).
\]
Proposition 21 follows by using Theorem 6 combined with all the above estimates. \( \diamond \)

We give now a consequence of Theorem 6 that will be used in one of our applications. The proof is omitted since it follows the spirit of the proof Proposition 21; namely, the use of Lemma 20 combined with the Hölder’s inequality, and the fact that \( \left\| \mathbb{E}_0(S_k) \right\|_p \leq \left\| \mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2) \right\|_{p/2} + \left( \mathbb{E}(S_k^2) \right)^{1/2} \).

**Proposition 22.** Let \( p > 2 \) be a real number and let \( (X_i)_{i \in \mathbb{Z}} \) be a stationary sequence of real-valued random variables in \( L_p \) adapted to an increasing filtration \( (\mathcal{F}_t) \). Let \( (\lambda(j))_{j \geq 1} \) be defined by (36). For every positive integer \( n \), the following inequality holds: for any \( \varepsilon > 0 \),
\[
\mathbb{E} \left( \max_{1 \leq j \leq n} |S_j|^p \right) \ll n^{p/2} \left( \sum_{k=0}^{n-1} \left| \mathbb{E}(X_0 X_k) \right| \right)^{p/2} + n \mathbb{E}(|X_1|^p) + n \sum_{k=1}^{n} k^{-2+\varepsilon} \lambda^{p/2}(k).
\]
The constant that is implicitly involved in the notation \( \ll \) depends on \( p \) and \( \varepsilon \) but it does not depend on \( n \).
4 Applications and examples

As we have seen Propositions 2 and 5 give a direct approach to compare the moments of order $p$ of the maximum of the partial sums to the corresponding ones of the partial sum. We start this section by presenting two additional applications of these propositions to the convergence of maximum of partial sums and to the maximal Bernstein inequality for dependent structures. In the last three examples, we apply our results on the Rosenthal type inequalities to different classes of processes.

4.1 Convergence of the maximum of partial sums in $L^p$.

**Corollary 23** Let $p \geq 2$ and let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence of centered real-valued random variables in $L^p$ adapted to an increasing and stationary filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$. Assume that

$$
\lim_{n \to \infty} n^{-1/p} \|S_n\|_p = 0. \tag{37}
$$

Assume in addition that

$$
\sum_{n \geq 1} \|\mathbb{E}(S_n | \mathcal{F}_0)\|_p \frac{1}{n^{1+1/p}} < \infty. \tag{38}
$$

Then

$$
\lim_{n \to \infty} n^{-1/p} \max_{1 \leq k \leq n} |S_k| \|p = 0. \tag{39}
$$

**Remark 24** This corollary is particularly useful for studying the asymptotic behavior of a partial sum via a martingale approximation. Assume there exists a strictly stationary sequence $(d_i)_{i \in \mathbb{Z}}$ of martingale differences with respect to $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ that are in $L^p$, such that $\lim_{n \to \infty} n^{-1/p} \|S_n - \sum_{i=1}^n d_i\|_p = 0$. Then, if condition (38) holds for the sequence $(X_i)_{i \in \mathbb{Z}}$, then by a construction in Woodroofe and Zhao (2008) and by the uniqueness of the martingale approximation, the sequence $(X_i - d_i)_{i \in \mathbb{Z}}$ is still a strictly stationary sequence and by our theorem $\lim_{n \to \infty} n^{-1/p} \max_{1 \leq k \leq n} |S_k - \sum_{i=1}^k d_i| \|_p = 0$. As a matter of fact, for $p = 2$, our corollary leads to the functional form of the central limit theorem for $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ (see also Theorem 1.1 in Peligrad and Utev, 2005).

**Proof of Corollary 23**

Let $m$ be an integer and $k = k_{n,m} = [n/m]$ (where $[x]$ denotes the integer part of $x$).

The initial step of the proof is to divide the variables in blocks of size $m$ and to make the sums in each block. Let

$$
X_{i,m} = \sum_{j=(i-1)m+1}^{im} X_j, \ i \geq 1.
$$

Notice first that

$$
\| \sup_{t \in [0,1]} \left| \sum_{j=1}^{[nt]} X_j - \sum_{i=1}^{[kt]} X_{i,m} \right| \|_p \leq \| \sup_{t \in [0,1]} \left| \sum_{i=[nt]}^{[kt]m+1} X_i \right| \|_p \leq m \max_{1 \leq i \leq n} \|X_i\|_p.
$$

Since for every $\varepsilon > 0$,

$$
\mathbb{E}(\max_{1 \leq i \leq n} |X_i|^p) \leq \varepsilon^p + \sum_{i=1}^n \mathbb{E}(|X_i|^p \mathbb{1}_{\{|X_i| > \varepsilon\}}),
$$

and since $\|X_i\|_p < \infty$ for all $i$, we derive that for any fixed $m$, $\lim_{n \to \infty} m \max_{1 \leq i \leq n} X_i \|/n^{1/p} = 0$. Hence to prove (39) it remains to show that

$$
\lim_{m \to \infty} \limsup_{n \to \infty} n^{-1/p} \sup_{t \in [0,1]} \left| \sum_{i=1}^{[kt]} X_{i,m} \right| \|_p = 0. \tag{40}
$$
Applying Proposition 2 to the variables \((X_{i,m})_{1 \leq i \leq k}\) which are adapted with respect to \(\mathcal{F}_{im}\), and taking into account the remark 4, we get that

\[
\| \sup_{t \in [0,1]} \bigg| \sum_{i=1}^{[kt]} X_{i,m} \bigg|_p \| \ll \max_{1 \leq j \leq k} \bigg\| \sum_{\ell=1}^{jm} X_{\ell} \bigg\|_p + k^{1/p} \sum_{j=1}^{k} \frac{\| \mathbb{E}(S_{jm}|\mathcal{F}_0) \|_p}{j^{1+1/p}},
\]

where for the last term we used the fact that for any positive integer \(u\), \(\| \mathbb{E}\left(\sum_{j=um2^t+1}^{(u+1)m2^t} X_j|\mathcal{F}_{um2^t}\right) \|_p = \| \mathbb{E}\left(\sum_{j=1}^{m2^t} X_j|\mathcal{F}_0\right) \|_p\). Condition (37) implies that

\[
\max_{1 \leq j \leq k} \bigg\| \sum_{\ell=1}^{jm} X_{\ell} \bigg\|_p = o((km)^{1/p}) = o(n^{1/p}).
\]

Now, by subadditivity of the sequence \((\| \mathbb{E}(S_n|\mathcal{F}_0) \|_p)_{n \geq 1}\) and applying Lemma 38 we have

\[
n^{-1/p}k^{1/p} \sum_{j=1}^{k} \frac{\| \mathbb{E}(S_{jm}|\mathcal{F}_0) \|_p}{j^{1+1/p}} \ll \sum_{\ell=1}^{m} \frac{\| \mathbb{E}(S_{\ell}|\mathcal{F}_0) \|_p}{(\ell + m)^{1+1/p}} + \sum_{j=m}^{n} \frac{\| \mathbb{E}(S_{j}|\mathcal{F}_0) \|_p}{j^{1+1/p}}.
\]

Hence, under (38) and using Fatou’s lemma for the first term in the right-hand side of (41), we get that

\[
\lim_{m \to \infty} \limsup_{n \to \infty} n^{-1/p}k^{1/p} \sum_{j=1}^{k} \frac{\| \mathbb{E}(S_{jm}|\mathcal{F}_0) \|_p}{j^{1+1/p}} = 0,
\]

which ends the proof of the corollary. 

### 4.2 Maximal exponential inequalities for strongly mixing.

Let us first recall the definition of strongly mixing sequences, introduced by Rosenblatt (1956): For any two \(\sigma\) algebras \(\mathcal{A}\) and \(\mathcal{B}\), we define the \(\alpha\)-mixing coefficient by

\[
\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.
\]

Let \((X_k, k \geq 1)\) be a sequence of real-valued random variables defined on \((\Omega, \mathcal{A}, \mathbb{P})\). This sequence will be called strongly mixing if

\[
\alpha(n) := \sup_{k \geq 1} \alpha(\mathcal{F}_k, \mathcal{G}_{k+n}) \to 0 \text{ as } n \to \infty,
\]

where \(\mathcal{F}_j := \sigma(X_i, i \leq j)\) and \(\mathcal{G}_j := \sigma(X_i, i \geq j)\) for \(j \geq 1\).

In 2009, Merlevède, Peligrad and Rio have proved (see their Theorem 2) that for a strongly mixing sequence of centered random variables satisfying \(\sup_{i \geq 1} \|X_i\|_\infty \leq M\) and for a certain \(c > 0\)

\[
\alpha(n) \leq \exp(-cn),
\]

the following Bernstein-type inequality is valid: there is a constant \(C\) depending only on \(c\) such that for all \(n \geq 2\),

\[
\mathbb{P}(|S_n| \geq x) \leq \exp\left(- \frac{Cx^2}{n^2 + M^2 + xM(\log n)^2}\right),
\]
where

\[ v^2 = \sup_{i > 0} \left( \text{Var}(X_i) + 2 \sum_{j > i} |\text{Cov}(X_i, X_j)| \right). \]  

(45)

Proving the maximal version of the inequality (44) cannot be handled directly neither using Theorem 2.2 in Móricz, Serfling and Stout (1982), nor using Theorem 1 in Kevei and Mason (2010) since the left-hand side of (44) does not satisfy the assumptions of both these papers. However, an application of Proposition 5 leads to the maximal version of Theorem 2 in Merlevède, Peligrad and Rio (2009).

**Corollary 25** Let \((X_j)_{j \geq 1}\) be a sequence of centered real-valued random variables. Suppose that there exists a positive \(M\) such that \(\sup_{i \geq 1} \|X_i\|_\infty \leq M\) and that the strongly mixing coefficients \((\alpha(n))_{n \geq 1}\) of the sequence satisfy (43). Then there exists constants \(C = C(c)\) and \(K = K(M, c)\) such that for all integer \(n \geq 2\) and all real \(x > K \log n\),

\[ \mathbb{P}\left( \max_{1 \leq k \leq n} |S_k| \geq x \right) \leq \exp \left(-\frac{C_x^2}{v^2 n + M^2 + xM(\log n)^2}\right). \]  

(46)

**Proof of Corollary 25** We first apply Inequality (9) of Proposition 5 with \(p = 2\) and \(\varphi(x) = e^{tx}\) where \(t\) is a positive integer. According to Theorem 2 in Merlevède, Peligrad and Rio (2009), there exist positive constants \(C_1\) and \(C_2\) depending only on \(c\) such that for all \(n \geq 2\) and any positive \(t\) such that \(t < \frac{1}{C_1 M (\log n)^2}\), the following inequality holds:

\[ \log \mathbb{E}\left( \exp(tS_n) \right) \leq \frac{C_2 t^2 (nv^2 + M^2)}{1 - C_1 tM(\log n)^2}. \]

Then an optimization on \(t\) gives that there is a constant \(C_3\) depending only on \(c\) such that for all \(n \geq 2\) and any positive real \(x\),

\[ \mathbb{P}\left( \max_{1 \leq k \leq n} |S_k| \geq 4x \right) \leq \exp \left(-\frac{C_3 x^2}{v^2 n + M^2 + xM(\log n)^2}\right) 
+ 4x^{-2} \left( \sum_{l=0}^{r-1} \left( \sum_{k=1}^{2^{r-l}-1} \|\mathbb{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})\|^2 \right)^{1/2} \right)^2, \]  

(47)

where \(r\) is the positive integer satisfying \(2^{r-1} < n \leq 2^{r}\) and \(v = \lfloor x/M \rfloor\).

It remains to bound up the second term in the right-hand side of the above inequality. Notice first that for any centered variable \(Z\) such that \(\|Z\|_\infty \leq B\),

\[ \|\mathbb{E}(Z|\mathcal{F})\|_2^2 \leq B \|\mathbb{E}(Z|\mathcal{F})\|_1 = BCov(\text{sign}(\mathbb{E}(Z|\mathcal{F})), Z). \]

Hence by the Ibragimov’s covariance inequality (see Theorem 1.11 in Bradley, 2007),

\[ \|\mathbb{E}(Z|\mathcal{F})\|_2^2 \leq 4B^2 \alpha(\mathcal{F}, \alpha(Z)). \]

Therefore, applying this last estimate with \(Z = S_{v+(k+1)2^l} - S_{v+k2^l}\) and \(\mathcal{F} = \mathcal{F}_{k2^l}\), we get that

\[ \|\mathbb{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})\|^2 \leq 4M^22^{2l}\alpha(v). \]

implying that

\[ \left( \sum_{l=0}^{r-1} \left( \sum_{k=1}^{2^{r-l}-1} \|\mathbb{E}(S_{v+(k+1)2^l} - S_{v+k2^l}|\mathcal{F}_{k2^l})\|^2 \right)^{1/2} \right)^2 \leq 4M^22^{2r}(\sqrt{2} + 1)^2\alpha(v). \]
Since $2^{2r} \leq 4n^2$ and $[x/M] \geq x/(2M)$, for $x \geq 2M$, by using (45), we get that, for any $x \geq 2M$,

$$
\left( \sum_{i=0}^{r-1} \left( \sum_{k=1}^{2^{r-1}-1} \| \mathbb{E}(S_{v+(k+1)2^i} - S_{v+k2^i}) \|_2^2 \right)^{1/2} \right)^2 \leq 3 \times 2^5 M^2 n^2 \exp(-cx/(2M)).
$$

(48)

Starting from (47) and using (48), we then derive that for any $x \geq 2M \max(1, 4c^{-1} \log n)$,

$$
P(S_{2^r}^2 \geq 4x) \leq \exp \left( - \frac{C_3 x^2}{n^2 + M^2 + xM(\log n)^2} \right) + 96 \exp \left( - \frac{xc}{4M} \right),
$$

proving the inequality (46). \(\diamondsuit\)

### 4.3 Application to Arch models.

Theorem 6 applies to the case where $(X_i)_{i \in \mathbb{Z}}$ has an ARCH($\infty$) structure as described by Giraitis et al. (2000), that is

$$X_n = \sigma_n \eta_n, \quad \text{with } \sigma_n^2 = c + \sum_{j=1}^{\infty} c_j X_{n-j}^2,$$

(49)

where $(\eta_n)_{n \in \mathbb{Z}}$ is a sequence of i.i.d. centered random variables such that $\mathbb{E}(\eta_0^2) = 1$, and where $c \geq 0$, $c_j \geq 0$, and $\sum_{j \geq 1} c_j < 1$. Notice that $(X_i)_{i \in \mathbb{Z}}$ is a stationary sequence of martingale differences adapted to the filtration $(\mathcal{F}_i)$ where $\mathcal{F}_i = \sigma(\eta_k, k \leq i)$.

Let $p > 2$ and assume that $||\eta||_p < \infty$. Notice first that

$$||\mathbb{E}(X_j^2 | \mathcal{F}_0) - \mathbb{E}(X_0^2) ||_{p/2} = ||\mathbb{E}(\sigma_j^2 | \mathcal{F}_0) - \mathbb{E}(\sigma_0^2) ||_{p/2}. \quad (50)$$

In addition, since $\mathbb{E}(\eta_0^2) = 1$ and $\sum_{j \geq 1} c_j < 1$, the unique stationary solution to (49) is given by Giraitis et al. (2000):

$$\sigma_n^2 = c + c \sum_{\ell=1}^\infty \sum_{j_1, \ldots, j_\ell = 1}^\infty c_{j_1} \cdots c_{j_\ell} \eta_{n-j_1}^2 \cdots \eta_{n-j_\ell}^2. \quad (51)$$

Starting from (50) and using (51), one can prove that

$$||\mathbb{E}(X_j^2 | \mathcal{F}_0) - \mathbb{E}(X_0^2) ||_{p/2} \leq 2c||\eta||_p^{\infty} \sum_{\ell=1}^\infty \sum_{i=\ell/\ell}^\infty c_i,$$

where $\kappa = ||\eta||_p^2 \sum_{j \geq 1} c_j$ (see Section 6.6 in Dedecker and Merlevède (2010) for more detailed computations). Consequently if

$$||\eta||_p^{\infty} \sum_{j \geq 1} c_j < 1 \quad \text{and} \quad \sum_{j \geq n} c_j = O(n^{-b}) \quad \text{for } b > 1 - 2/p, \quad (52)$$

we get that for any $\delta \in [0, 1]$,

$$\sum_{k=1}^\infty \frac{1}{k^{1+2\delta/p}} ||\mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2)||_{p/2}^\delta < \infty.$$

Applying Theorem 6 for the martingale case, we then get the following corollary:

**Corollary 26** Let $(X_i)_{i \in \mathbb{Z}}$ be defined by (43) and $S_n = \sum_{i=1}^n X_i$. Let $p > 2$ and assume that (53) is satisfied. Then for any integer $n$,

$$\mathbb{E} \left( \max_{1 \leq k \leq n} |S_k|^p \right) \ll \left( \mathbb{E}(X_0^2) \right)^{p/2} \mathbb{E}(|X_0|^p) + n \left( 1 + \mathbb{E}(|X_0|^p) \right).$$

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4.4 Application to functions of linear processes.

Let

\[ X_k = h \left( \sum_{i \in \mathbb{Z}} a_i \varepsilon_{k-i} \right) - \mathbb{E} \left( h \left( \sum_{i \in \mathbb{Z}} a_i \varepsilon_{k-i} \right) \right), \tag{53} \]

where \((\varepsilon_i)_{i \in \mathbb{Z}}\) is a sequence of i.i.d. random variables. Denote by \(w_h(\cdot, M)\) the modulus of continuity of the function \(h\) on the interval \([-M, M]\), that is

\[ w_h(t, M) = \sup \{|h(x) - h(y)|, |x - y| \leq t, |x|, |y| \leq M\}. \]

Applying Theorem 17, the following result holds:

**Corollary 27** Let \((a_i)_{i \in \mathbb{Z}}\) be a sequence of real numbers in \(\ell^2\) and \((\varepsilon_i)_{i \in \mathbb{Z}}\) be a sequence of i.i.d. random variables in \(L^2\). Let \(X_k\) be defined as in (53). Assume that \(h\) is \(\gamma\)-Hölder on any compact set, with \(w_h(t, M) \leq C t^\gamma M^\alpha\), for some \(C > 0\), \(\gamma \in ]0, 1]\) and \(\alpha \geq 0\). Let \(p \geq 4\) be an even integer. Assume that for \(\lambda > p/2 - 2\),

\[ \mathbb{E}(\bar{\varepsilon}_0^{2\nu(\alpha + \gamma)p}) < \infty \quad \text{and} \quad \sum_{i \geq 1} i^{1-2/p}(\log i)^\lambda \left( \sum_{j \geq 1} a_j^2 \right)^{\gamma/2} < \infty, \tag{54} \]

and that

\[ \text{there exists } \beta > 2/p \text{ such that } n^{-\beta} \mathbb{E}(S_n^2) \text{ is increasing.} \tag{55} \]

Then for any integer \(n\),

\[ \mathbb{E}(\max_{1 \leq k \leq n} |S_k|^p) \ll n \left( 1 + \mathbb{E}(|X_0|^p) \right) + \left( \mathbb{E}(S_n^2) \right)^{p/2}. \]

**Remark 28** The proof of this result is based on Theorem 17. Our proof reveals that an application of Theorem 6 would involve a more restrictive condition on \(\lambda\), namely \(\lambda > p - 3\).

**Proof of Corollary 27.**

Applying Theorem 17, for any positive integer \(n\), we get that

\[ \mathbb{E}(\max_{1 \leq j \leq n} |S_j|^p) \ll n \left( \sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} (\mathbb{E}(S_k^2)^{2/(p-2)}) \right)^{p(p-2)/4} + n \left( \sum_{k=1}^n \|\mathbb{E}_0(X_k)\|_p \right)^p + n \|X_0\|_p^p 
\]

\[ + n \left( \sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} \|\mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2}^{2/(p-2)} \right)^{p(p-2)/4}. \]

Using the condition (55), it follows that

\[ n \left( \sum_{k=1}^n \frac{1}{k^{1+4/p(p-2)}} (\mathbb{E}(S_k^2)^{2/(p-2)}) \right)^{p(p-2)/4} \ll (\mathbb{E}(S_n^2))^{p/2}. \]

Therefore the theorem follows if we prove that (55) implies that

\[ \sum_{k \geq 1} \|\mathbb{E}_0(X_k)\|_p < \infty \quad \text{and} \quad \sum_{k \geq 1} \frac{1}{k^{1+4/p(p-2)}} \|\mathbb{E}_0(S_k^2) - \mathbb{E}(S_k^2)\|_{p/2}^{2/(p-2)} < \infty. \tag{56} \]

Notice now that by Hölder’s inequality and (55) to prove the second part of (56), it suffices to prove that for \(\lambda > p/2 - 2\),

\[ \sum_{k \geq 1} \frac{(\log k)^\lambda}{k^{1+2/p}} \sum_{i=1}^{k-i} \sum_{j=0}^i \|\mathbb{E}_0(X_iX_{j+i}) - \mathbb{E}(X_iX_{j+i})\|_{p/2} < \infty. \]
Corollary 27 is then a consequence of the following proposition applied with $b = \lambda$ and $c = 2/p$ (see the proof of Theorem 4.2 in Dedecker, Merlevède and Rio 2009, page 988).

**Proposition 29** Let $(a_i)_{i \in \mathbb{Z}}$, $(\varepsilon_i)_{i \in \mathbb{Z}}$ and $(X_i)_{i \in \mathbb{Z}}$ be as in Corollary 27. Let $(\varepsilon'_i)_{i \in \mathbb{Z}}$ be an independent copy of $(\varepsilon_i)_{i \in \mathbb{Z}}$. Let $V_0 = \sum_{i \geq 0} a_i \varepsilon_{-i}$ and

$$M_{i,i} = |V_0| \vee \left| \sum_{0 \leq j < i} a_j \varepsilon_{-j} + \sum_{j \geq i} a_j \varepsilon'_{-j} \right|.$$ 

Let $p \geq 2$. If for $b \geq 0$ and $0 < c \leq 1$,

$$\sum_{i \geq 1} i^{1-c} (\log i)^b \| w_h \left( \left| \sum_{j \geq i} a_j \varepsilon_{-j} \right|, M_{i,i} \right) \|_p < \infty,$$  

then $\sum_{k \geq 1} \| \mathbb{E}_0(X_k) \|_p < \infty$ and

$$\sum_{k \geq 1} \frac{(\log k)^b}{k^{1+c}} \sum_{i=1}^{k} \sum_{j=0}^{k-i} \| \mathbb{E}_0(X_iX_{j+i}) - \mathbb{E}(X_iX_{j+i}) \|_{p/2} < \infty.$$ 

The proof of the above proposition is direct following the lines of the proof of Proposition 4.2 Dedecker, Merlevède and Rio (2009).

### 4.5 Application to a stationary reversible Markov chain.

First we want to mention that all our results can be formulated in the Markov chain setting. We assume that $(\zeta_n)_{n \in \mathbb{Z}}$ denotes a stationary Markov chain defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a measurable space $(E, \mathcal{E})$. The marginal distribution and the transition kernel are denoted by $\pi(A) = \mathbb{P}(\zeta_0 \in A)$ and $Q(\zeta_0, A) = \mathbb{P}(\zeta_1 \in A | \zeta_0)$. In addition $Q$ denotes the operator acting via $(Qf)(\zeta) = \int_E f(s)Q(\zeta, ds)$. Next, let $f$ be a function on $E$ such that $\int_E |f|^p d\pi < \infty$ and $\int_E f d\pi = 0$.

Denote by $\mathcal{F}_k$ the $\sigma$–field generated by $\zeta_i$ with $i \leq k$, $X_i = f(\zeta_i)$, and $S_n(f) = \sum_{i=1}^{n} X_i$. Notice that any stationary sequence $(Y_k)_{k \in \mathbb{Z}}$ can be viewed as a function of a Markov process $\zeta_k = (Y_i; i \leq k)$, for the function $g(\zeta_k) = Y_k$.

The Markov chain is called reversible if $Q = Q^*$, where $Q^*$ is the adjoint operator of $Q$. In this setting, an application of Theorem 10 gives the following estimate:

**Corollary 30** Let $(\zeta_n)$ be a reversible Markov chain. For any even integer $p \geq 4$ and any positive integer $n$,

$$\mathbb{E}(\max_{1 \leq k \leq n} |S_k(f)|^p) \ll n \mathbb{E}(|f(\zeta_1)|^p) + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \| \mathbb{E}_0(S_k(f)) \|_p \right)^p$$

$$+ n \left( \sum_{k=1}^{n} \frac{1}{k^{1+2/p}} \| \mathbb{E}_0(S_k^2(f)) - \mathbb{E}(S_k^2(f)) \|_{p/2} \right)^{p/2} + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+2/p}} \| \mathbb{E}(S_k^2(f)) \|_{p/2} \right)^{p/2}.$$ 

Moreover using Theorem 10 we obtain:

**Corollary 31** Let $(\zeta_n)$ be a reversible Markov chain. For any real number $p > 4$ and any positive integer $n$,

$$\mathbb{E}(\max_{1 \leq k \leq n} |S_k(f)|^p) \ll n \mathbb{E}(|f(\zeta_1)|^p)$$

$$+ n \left( \sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \| \mathbb{E}_0(S_k^2(f)) - \mathbb{E}(S_k^2(f)) \|_{p/2} \right)^{p/2} + n \left( \sum_{k=1}^{n} \frac{1}{k^{1+2/p}} \| S_k(f) \|_2 \right)^p.$$
Therefore, for any real $2 < p \leq 4$. For this range however, according to the comment \[7\] Theorem \[8\] (respectively Corollary \[20\]) gives a better bound for $p \in ]2, 4[$ (respectively for $p = 4$).

For a particular example let $E = [-1, 1]$ and let $v$ be a symmetric atomless law on $E$. The transition probabilities are defined by

$$Q(x, A) = (1 - |x|)\delta_x(A) + |x|v(A),$$

where $\delta_x$ denotes the Dirac measure. Assume that $\theta = \int_E |x|^{-1}v(dx) < \infty$. Then there is a unique invariant measure $\pi(dx) = \theta^{-1}|x|^{-1}v(dx)$

and the stationary Markov chain $(\zeta_i)_i$ is reversible and positively recurrent.

Assume the following assumption on the measure $v$: there exists a positive constant $c$ such that for any $x \in [0, 1]$,

$$\frac{dv}{dx}(x) \leq cx^{p/2 - 1}(\log(1 + 1/x))^{-\lambda} \text{ for some } \lambda > 0. \quad (58)$$

As an application of Corollary \[31\] we shall establish:

**Corollary 32** Let $p > 2$ be a real number and let $f(-x) = -f(x)$ for any $x \in E$. Assume that $|f(x)| < C|x|^{1/2}$ for any $x$ in $E$ and a positive constant $C$. Assume in addition that \[58\] is satisfied for $\lambda > p$. Then for any integer $n$,

$$E \left( \max_{1 \leq k \leq n} |S_k(f)|^p \right) \ll n + n^{p/2} \left( \int_0^1 f^2(x)x^{-2}v(dx) \right)^{p/2}. \quad (59)$$

Notice that this example of reversible Markov chain has been considered by Rio (2009, Section 4) under a slightly more stringent condition on the measure than \[58\]. Corollary \[32\] then extends Proposition 4.1 (b) in Rio (2009) to all real $p > 2$.

**Proof of Corollary 32.**

To get this result we shall apply Corollary \[31\] along with Lemma \[20\]. We start by noticing that $f$ being an odd function we have

$$E(f(\zeta_k)|\zeta_0) = (1 - |\zeta_0|)^k f(\zeta_0) \text{ a.s.} \quad (60)$$

Therefore, for any $j \geq 0$,

$$E(X_0X_j) = E(f(\zeta_0)E(f(\zeta_j)|\zeta_0)) = \theta^{-1} \int_E f^2(x)(1 - |x|)^j|x|^{-1}v(dx).$$

Then

$$E(S_k^2(f)) \leq 2k\theta^{-1} \left( \int_0^1 f^2(x)x^{-1}v(dx) + 2 \sum_{j=1}^{k-1} \int_0^1 f^2(x)(1 - x)^2x^{-1}v(dx) \right) \quad (61)$$

$$\leq 2k\theta^{-1} \left( \int_0^1 f^2(x)x^{-1}v(dx) + 2 \int_0^1 f^2(x)x^{-2}v(dx) \right).$$

We estimate next the quantity $\|E_0(S_n^2(f)) - E(S_n^2(f))\|_{p/2}$. Notice first that

$$\|E_0(S_n^2(f)) - E(S_n^2(f))\|_{p/2} \leq \sum_{k=1}^{n} \left( \int_E \left| \delta_x Q^k - \pi \right| (f^2 + 2f \sum_{k=1}^{n-k} Q^k f)^{p/2} \pi(dx) \right)^{2/p},$$

$$\|E_0(S_n^2(f)) - E(S_n^2(f))\|_{p/2} \leq \sum_{k=1}^{n} \left( \int_E \left| \delta_x Q^k - \pi \right| (f^2 + 2f \sum_{k=1}^{n-k} Q^k f)^{p/2} \pi(dx) \right)^{2/p},$$

$$= \sum_{k=1}^{n} \left( \int_E \left| \delta_x Q^k - \pi \right| (f^2 + 2f \sum_{k=1}^{n-k} Q^k f)^{p/2} \pi(dx) \right)^{2/p}.$$

$$\leq \sum_{k=1}^{n} \left( \int_E \left| \delta_x Q^k - \pi \right| (f^2 + 2f \sum_{k=1}^{n-k} Q^k f)^{p/2} \pi(dx) \right)^{2/p}.$$
by using the fact that for any positive \( k \), \( \pi Q^k = \pi \) (see also the inequality (4.12) in Rio (2009)). Now, by using the relation \( \pi Q^k \), one can prove that for any \( x \in E \),

\[
(f^2 + 2f \sum_{k=1}^{n} Q^k f)(x) = f^2(x)(1 + 2(1 - |x|)^n(|x|^{-1} - 1))
\]

(see the computations in Rio (2009) leading to his relation (4.13)). Then, since \( |f(x)| \leq C|x|^{1/2} \), it follows that

\[
\sup_{x \in E} |f^2(x) + 2f(x) \sum_{k=1}^{n} Q^k f(x)| \leq 2C^2.
\]

Therefore, for any \( p > 2 \),

\[
\|E_0(S_n^2(f)) - E(S_n^2(f))\|_{p/2} \leq 4C^2 \sum_{k=1}^{n} \left( \int_E \|Q^k(x, \cdot) - \pi(\cdot)\|\pi(dx) \right)^{2/p},
\]

(62)

where \( \|\mu(\cdot)\| \) denotes the total variation of the signed measure \( \mu \) (see also the inequality (4.15) in Rio (2009)). We estimate next the coefficients of absolute regularity \( \beta_n \) as defined in (63). Let \( a = p/2 - 1 \).

We shall prove now that under (58), there exists a positive constant \( K \) such that

\[
2\beta_n := \int_E \|Q^\alpha(x, \cdot) - \pi(\cdot)\|\pi(dx) \leq Kn^{-\alpha}(\log n)^{-\lambda}.
\]

(63)

Notice first that by Lemma 2, page 75, in Doukhan, Massart and Rio (1994), we have that

\[
\beta_n \leq 3 \int_E (1 - |x|)^{n/2} \pi(dx).
\]

(64)

Let \( k \geq 2 \) be an integer. Clearly, for any \( \alpha \in ]0, 1[ \),

\[
\int_0^1 (1 - x)^k \pi(dx) \leq c \int_0^{k^{-\alpha}} (1 - x)^k x^{a-1}(log(1 + 1/x))^{-\lambda} dx + c \int_{k^{-\alpha}}^1 (1 - x)^k x^{a-1}(log(1 + 1/x))^{-\lambda} dx.
\]

(65)

Notice now that

\[
\int_0^{k^{-\alpha}} (1 - x)^k x^{a-1}(log(1 + 1/x))^{-\lambda} dx \leq (\alpha \log k)^{-\lambda} \int_0^1 (1 - x)^k x^{a-1}dx.
\]

Hence, by the properties of the Beta and Gamma functions,

\[
\lim_{k \to \infty} k^\alpha (log k)^\lambda \int_0^{k^{-\alpha}} (1 - x)^k x^{a-1}(log(1 + 1/x))^{-\lambda} dx \leq \alpha^{-\lambda} a \Gamma(a).
\]

(66)

On the other hand, we have that

\[
\int_{k^{-\alpha}}^1 (1 - x)^k x^{a-1}(log(1 + 1/x))^{-\lambda} dx \leq (log 2)^{-\lambda}(1 - k^{-\alpha})^k \int_0^1 x^{a-1}dx,
\]

and then, since \( \alpha < 1 \), we easily obtain

\[
\lim_{k \to \infty} k^\alpha (log k)^\lambda \int_{k^{-\alpha}}^1 (1 - x)^k x^{a-1}(log(1 + 1/x))^{-\lambda} dx = 0.
\]

(67)
Starting from \(O(124)\) and taking into account \(O(55)\), \(O(90)\) and \(O(053)\) follows. Then by using the inequality \(O(52)\) combined with \(O(53)\), we derive that
\[
\|E_0(S_n^2(f)) - E(S_n^2(f))\|_{p/2} \ll n^{2/p}(\log n)^{-2\lambda/p}.
\]
Therefore, for any \(\delta \in ]0, 1[\),
\[
\sum_{k=1}^{n} \frac{1}{k^{1+1/p}} \|E_0(S_k^2(f)) - E(S_k^2(f))\|_{p/2}^{1/2} = O(1) \text{ for } \lambda > p.
\]
(68)

Considering the estimates \(O(11)\) and \(O(55)\), Corollary \(O(31)\) follows from an application of Corollary \(O(31)\) (taking also into account the comment after its statement for \(2 < p \leq 4\)).

5 Application to density estimation

In this section, we estimate the \(L^p\)-integrated risk for \(p \geq 4\), for the kernel estimator of the unknown marginal density \(f\) of a stationary sequence \((Y_i)_{i \geq 0}\).

Applying our theorem \(O(3)\) we shall show that if the coefficients of dependence ((\(\beta_{2,Y}(k)\))\(i \geq 1\) (see the definition \(O(53)\) below) of the sequence \((Y_i)_{i \in \mathbb{Z}}\) satisfy \(\beta_{2,Y}(k) = O(n^{-a})\) for \(a > p - 1\), then the bound of the \(L^p\)-norm of the random term of the risk is of the same order of magnitude as the one obtained in Bretagnolle and Huber (1979) in the independence setting (see their corollary 2), provided that the density is bounded and the kernel \(K\) satisfies the assumption \(A_p\) below.

**Assumption \(A_p\).** \(K\) is a BV (bounded variation) function such that
\[
\int_{\mathbb{R}} |K(u)| du < \infty \text{ and } \int_{\mathbb{R}} |K(u)|^p du < \infty.
\]

**Definition 33** Let \((Y_i)_{i \in \mathbb{Z}}\) be a stationary sequence of real valued random variables, and let \(F_0 = \sigma(Y_i, i \leq 0)\). For any positive \(i\) and \(j\), define the random variables
\[
b(F_0, i, j) = \sup_{(s,t) \in \mathbb{R}^2} |P(Y_i \leq t, Y_j \leq s| F_0) - P(Y_i \leq t, Y_j \leq s)|.
\]
Define now the coefficient
\[
\beta_{2,Y}(k) = \sup_{i \geq j \geq k} E(b(F_0, i, j)).
\]

**Proposition 34** Let \(p \geq 4\) and \(K\) be any real function satisfying assumption \(A_p\). Let \((Y_i)_{i \geq 0}\) be a stationary sequence with unknown marginal density \(f\) such that \(\|f\|_{\infty} < \infty\). Define
\[
X_{k,n}(x) = K(h_n^{-1}(x - Y_k)) \text{ and } f_n(x) = \frac{1}{nh_n} \sum_{k=1}^{n} X_{k,n}(x),
\]
where \((h_n)_{n \geq 1}\) is a sequence of positive real numbers. Assume that for some \(\eta > 0\),
\[
\beta_{2,Y}(n) = O(n^{-p-1+\eta}).
\]
(69)

Then there exists positive constants \(C_1\) and \(C_2\) depending on \(\eta\) and \(p\) such that for any positive integer \(n\),
\[
E \int_{\mathbb{R}} |f_n(x) - E(f_n(x))|^p dx \leq C_1(nh_n)^{-p/2} \|dK\|^{p/2} \|f\|^{p/2-1} \left( \int_{\mathbb{R}} |K(u)| du \right)^{p/2}
\]
\[
+ C_2(nh_n)^{1-p} \left( \int_{\mathbb{R}} |K(u)|^p du + \|dK\| \right) \left( \int_{\mathbb{R}} |K(u)|^{p-1} du + \|dK\|^2 \right) \left( \int_{\mathbb{R}} |K(u)|^{p-2} du \right),
\]
where \(\|dK\|\) is the total variation norm of the measure \(dK\).
The bound obtained in Proposition 34 can be also compared to the one obtained in Theorem 3.3 in Viennet (1997) under the assumption that the strong $\beta$-mixing coefficients in the sense of Rozanov and Volkonskii (1959) of the sequence $(Y_i)_{i \in \mathbb{Z}}$, denoted by $\beta_\infty(k)$, satisfy: $\sum_{k \geq 1} k^{p-2}\beta_\infty(k) < \infty$. Our condition is then comparable to the one imposed by Viennet (1997) but less restrictive in the sense that many processes are such that the sequence $\beta_{2,Y}(n)$ tends to zero as $n \to \infty$ which is not the case for $\beta_\infty(n)$ (see the examples given in Dedecker and Prieur (2007)).

If we assume that $f$ has a derivative of order $s$, where $s \geq 1$ is an integer and that the following bound holds for the bias term:

$$\int_\mathbb{R} |f(x) - \mathbb{E}(f_n(x))|^p dx \leq Mh_n^s ||f||_p^{s+1} , \quad (71)$$

where $M$ is a constant depending on the Kernel $K$, then the choice of $(nh_n)^{p/2}h_n^{sp} = O(1)$ leads to the following estimate:

$$\mathbb{E} \int_\mathbb{R} |f_n(x) - f(x)|^p dx = O(n^{-sp/(2s+1)}) . \quad (72)$$

We mention that (71) holds for any Parzen Kernel of order $s$ (see Section 4 in Bretagnolle and Huber (1979)). We also mention that if we only assume that \( \sum_{k \geq 1} k^{p-2}\beta_{2,Y}(k) < \infty \) instead of (69) in Proposition 34 then the inequality (70) is valid with $(nh_n)^{1-p/2}n^s$ (for any $\varepsilon > 0$) replacing $(nh_n)^{1-p}$ in the second term of the right-hand side. In this situation, the bound (71) combined with a choice of $h_n$ of order $n^{-1/(1+2\varepsilon)}$ still leads to the estimate (72).

**Proof of Proposition 34**

Setting $X_{i,n}(x) = K((x - Y_i)/h_n) - \mathbb{E}(K((x - Y_i)/h_n))$, we have that

$$\mathbb{E} \int_\mathbb{R} |f_n(x) - \mathbb{E}(f_n(x))|^p dx \leq (nh_n)^{-p} \int_\mathbb{R} \mathbb{E} \left| \sum_{i=1}^n X_{i,n}(x) \right|^p dx . \quad (73)$$

Starting from (73) and applying Proposition 22 to the stationary sequence $(X_{i,n}(x))_{i \in \mathbb{Z}}$, Proposition 34 follows provided we establish the following bounds (in what follows $C$ is a positive constant not depending on $n$ which may vary from line to line):

$$\int_\mathbb{R} \mathbb{E} |X_{1,n}(x)|^p dx \leq 2^{p+1}h_n \int_\mathbb{R} |K(u)|^p du , \quad (74)$$

$$\int_\mathbb{R} \left( \sum_{j=0}^{n-1} \mathbb{E}(X_{0,n}(x)X_{j,n}(x)) \right)^{p/2} dx \leq Ch_n^p ||dK||^{p/2} ||f||_\infty^{p/2-1} \left( \int_\mathbb{R} |K(u)| du \right)^{p/2} , \quad (75)$$

and that for $\varepsilon > 0$ small enough,

$$\sum_{j=1}^n j^{p-2+\varepsilon} \int \|X_{0,n}(x)\mathbb{E}_0(X_{j,n}(x))\|_{p/2}^{p/2} dx \leq Ch_n ||dK|| \int_\mathbb{R} |K(u)|^{p-1} du , \quad (76)$$

and

$$\sum_{j=1}^n j^{p-2+\varepsilon} \sup_{i \geq j} \int \|\mathbb{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbb{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} dx$$

$$\leq Ch_n ||dK||^2 \int_\mathbb{R} |K(u)|^{p-2} du . \quad (77)$$

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In what follows, we shall prove these bounds. Notice first that
\[
\int_{\mathbb{R}} E |X_{1,n}(x)|^p dx \leq 2^{p+1} \int_{\mathbb{R}} \int_{\mathbb{R}} |K((x-y)h_n^{-1})|^p f(y) dx dy ,
\]
proving (74) by the change of variables \( u = (x-y)h_n^{-1} \). To prove (75), we first apply Item 1 of Lemma 36 implying that
\[
\sum_{j=0}^{n-1} |E(X_{0,n}(x)X_{j,n}(x))| \leq ||dK|| E(b(\mathcal{F}_0,n) |K((x-Y_0)/h_n)) ,
\]
where \( b(\mathcal{F}_0,n) = \sum_{j=0}^{n-1} b(\mathcal{F}_0,j,j) \). An application of Hölder’s inequality as done in Viennet (1997) at the bottom of page 474, then gives
\[
\int_{\mathbb{R}} \left( \sum_{j=0}^{n-1} |E(X_{0,n}(x)X_{j,n}(x))| \right)^{p/2} dx \leq h_n^{p/2} ||f||_2^{1/2} E(b(\mathcal{F}_0,n))^{p/2} \left( \int_{\mathbb{R}} |K(u)| du \right)^{p/2} .
\]
This proves (75) since \( E(b(\mathcal{F}_0,n))^{p/2} \leq C \sum_{k=1}^{n} k^{p-2} \beta_2(Y(k) \leq \sum_{k=1}^{n} k^{p-2} \beta_2(Y(k) = O(1) \) by condition (69).

We turn now to the proof of (76). With this aim we notice that
\[
||X_{0,n}(x)E_0(X_{j,n}(x))||_{p/2} = E(Z_0(x)E_0(X_{j,n}(x))) = E(Z_0(x)X_{j,n}(x)) ,
\]
where \( Z_0(x) = |X_{0,n}(x)|^{p/2} |E_0(X_{j,n}(x))|^{p/2-1} \text{sign}(E_0(X_{j,n}(x))) \). Consequently, by using Item 1 of Lemma 36, we derive that
\[
||X_{0,n}(x)E_0(X_{j,n}(x))||_{p/2} = \text{Cov} \left( Z_0(x), K((x-Y_j)/h_n) \right) \leq ||dK|| E(b(\mathcal{F}_0,j,j) |Z_0(x)) .
\]
Notice now that by using the elementary inequality: \( x^\alpha y^{1-\alpha} \leq x+y \) valid for \( \alpha \in [0, 1] \) and nonnegative \( x \) and \( y \), we get that \( |Z_0(x)| \leq (|X_{0,n}(x)| + |E_0(X_{j,n}(x))|)^{p-1} \). Therefore, some computations involving the Jensen’s inequality lead to
\[
\int_{\mathbb{R}} |Z_0(x)| dx \leq 4^p h_n \int_{\mathbb{R}} |K(u)|^{p-1} du .
\]
Starting from (78), we end the proof of (76) by taking into account (79) and the fact that
\[
\sum_{j=1}^{n} j^{p-2+\varepsilon} E(b(\mathcal{F}_0,j,j)) \leq \sum_{j=1}^{n} j^{p-2+\varepsilon} \beta_{2,Y}(j)
\]
is convergent by condition (69) for any \( \varepsilon < \eta \).

It remains to prove (77). We first write that
\[
||E_0(X_{i,n}(x)X_{j,n}(x)) - E(X_{i,n}(x)X_{j,n}(x))||_{p/2} = E(Z_0^{(0)}(x)X_{i,n}(x)X_{j,n}(x)) ,
\]
where the notation \( X^{(0)} \) stands for \( X^{(0)} = X - E(X) \) and
\[
Z_0(x) = |E_0(B_{i,j}(x))|^{p/2-1} \text{sign}(E_0(B_{i,j}(x))) ,
\]

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with $B_{i,j}(x) = X_{i,n}(x)X_{j,n}(x) - \mathbb{E}(X_{i,n}(x)X_{j,n}(x))$. Since the variables $X_{i,n}(x)$ and $X_{j,n}(x)$ are centered, an application of Item 2 of Lemma 36 then gives

$$
\|\mathbb{E}_0(X_{i,n}(x)X_{j,n}(x)) - \mathbb{E}(X_{i,n}(x)X_{j,n}(x))\|_{p/2}^{p/2} \leq \|dK\|^2 \mathbb{E}(|Z_0(x)|\langle b(F_0, i, i) + b(F_0, j, j) + b(F_0, i, j) \rangle).
$$

Notice now that since $p/2 - 1 \geq 1$, we can easily get

$$
\int_{\mathbb{R}} |Z_0(x)| dx \leq c_p h_n \int_{\mathbb{R}} |K(u)|^{p-2} du,
$$

where $c_p$ is a positive constant depending on $p$. In addition

$$
\sum_{j=1}^{n} j^{p-2+\varepsilon} \sup_{i \geq j} \mathbb{E}(b(F_0, i, i) + b(F_0, j, j) + b(F_0, i, j)) \leq 3 \sum_{j=1}^{n} j^{p-2+\varepsilon} \beta_{2,Y}(j),
$$

which is convergent by condition (69) for any $\varepsilon < \eta$. Then (77) holds and so does the proposition. \(\diamondsuit\)

6 Appendix

This section is devoted to some technical lemmas. Next lemma gives estimates for terms of the type $\mathbb{E}(X_0^uX_1^{p-u})$.

**Lemma 35** Let $p$ and $u$ be real numbers such that $0 \leq u \leq p - 2$. Let $X_0$ and $X_1$ be two positive identically distributed random variables, With the notation $a^p = \mathbb{E}(X_0^p)$, $\mathbb{E}_0(X_1) = \mathbb{E}(X_1|X_0)$ the following estimates hold

$$
\mathbb{E}(X_0^uX_1^{p-u}) \leq a^{p-2u/(p-2)}\|\mathbb{E}_0(X_1^2)\|_{p/2}^{u/(p-2)}, \quad (80)
$$

and

$$
\mathbb{E}(X_0^{p-1}X_1) \leq a^{p-1}\|\mathbb{E}_0(X_1^2)\|_{p/2}^{1/2}. \quad (81)
$$

**Proof of Lemma 35** The inequality (80) is trivial for $u = 0$. To prove it for $u = p - 2$, it suffices to write that $\mathbb{E}(X_0^{p-2}X_1^2) = \mathbb{E}(X_0^{p-2}\mathbb{E}_0(X_1^2))$, and then to use the H"older's inequality.

We prove now the inequality (80) for $0 < u < p - 2$. Select $x = (p/2 - 1)/u = (p/2 - 2)/u$. Notice that $2x > 1$ and $p - u - 1/x > 0$ since $u < p - 2$. Then, since the variables are identically distributed,

$$
\mathbb{E}(X_0^uX_1^{p-u}) = \mathbb{E}(X_0^uX_1^{1/x}X_1^{p-u-1/x}) \leq \|X_0^uX_1^{1/x}\|_{2x}\|X_1^{p-u-1/x}\|_{2x/(2x-1)}
\leq (\mathbb{E}(X_0^pX_1^2))^{u/(p-2)}(a^p)^{1-u/(p-2)}.
$$

Now, again by H"older's inequality applied with $x = p/(p - 2)$ and $1 - 1/x = 2/p$,

$$
\mathbb{E}(X_0^{p-2}X_1^2) = \mathbb{E}(X_0^{p-2}\mathbb{E}_0(X_1^2)) \leq (\mathbb{E}(X_0^p))^{(p-2)/p}(\mathbb{E}(\mathbb{E}_0(X_1^2))^{p/2})^{2/p} = (a^p)^{(p-2)/p}\|\mathbb{E}_0(X_1^2)\|_{p/2}^{1/2}.
$$

Overall

$$
\mathbb{E}(X_0^uX_1^{p-u}) \leq a^u\|\mathbb{E}_0(X_1^2)\|_{p/2}^{u/(p-2)}(a^p)^{1-u/(p-2)}
= a^{p-2u/(p-2)}\|\mathbb{E}_0(X_1^2)\|_{p/2}^{u/(p-2)},
$$

ending the proof of the inequality (80).
To prove the inequality (81), we use the Hölder’s inequality which entails that
\[
E \left( X_0^{p-1}X_1 \right) \leq E \left( X_0^{p-1}E_0^{1/2}(X_1^2) \right) \leq a^{p-1}||E_0(X_1^2)||_{p/2}^{1/2}.
\]

Next lemma gives covariance-type inequalities in terms of beta coefficients as defined in Definition 33.

**Lemma 36** Let $Z$ be a $\mathcal{F}_0$-measurable real valued random variable and let $h$ and $g$ be two $BV$ functions (denote by $||dh||$ (resp. $||dg||$) the total variation norm of the measure $dh$ (resp. $dg$)). Denote $Z^{(0)} = Z - E(Z)$, $h^{(0)}(Y_i) = h(Y_i) - E(h(Y_i))$ and $g^{(0)}(Y_j) = g(Y_j) - E(g(Y_j))$. Define the random variables $b(\mathcal{F}_0, i, j)$ as in Definition 33. Then

1. $|E(Z^{(0)}h^{(0)}(Y_i))| = |\text{Cov}(Z, h(Y_i))| \leq ||dh|| E(|Z|b(\mathcal{F}_0, i, i))$.
2. $|E(Z^{(0)}h^{(0)}(Y_i)g^{(0)}(Y_j))| \leq ||dh|| \times ||dg|| E(|Z|b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j))$.

**Proof of Lemma 36** Item 1 has been proven by Dedecker and Prieur (2005) (see Item 2 of their proposition 1). Item 2 needs a proof. We first notice that
\[
h^{(0)}(X)g^{(0)}(Y) = \iint (1_{X \leq t} - F_X(t))(1_{Y \leq s} - F_Y(s))dh(t)dg(s).
\]
Therefore
\[
E(Z^{(0)}h^{(0)}(Y_i)g^{(0)}(Y_j)) = E \left( Z \iint (1_{Y_i \leq t}1_{Y_j \leq s} - E(1_{Y_i \leq t}1_{Y_j \leq s}))dh(t)dg(s) \right)
\]
\[
= E \left( Z \iint E(1_{Y_i \leq t}1_{Y_j \leq s} - E(1_{Y_i \leq t}1_{Y_j \leq s}))|\mathcal{F}_0)dh(t)dg(s) \right),
\]
which proves Item 2 by noticing that
\[
|E(1_{Y_i \leq t}1_{Y_j \leq s} - E(1_{Y_i \leq t}1_{Y_j \leq s})|\mathcal{F}_0)| \leq b(\mathcal{F}_0, i, i) + b(\mathcal{F}_0, j, j) + b(\mathcal{F}_0, i, j).
\]

Next lemma gives different ranges of inequalities for $|x + y|^p$ where $p \geq 2$ is a real number.

**Lemma 37**

1. Let $x$ and $y$ be two real numbers and $2 \leq p \leq 3$. Then
\[
|x + y|^p \leq |x|^p + |y|^p + p|x|^{p-1}\text{sign}(x)y + \frac{p(p-1)}{2} |x|^{p-2}y^2. \tag{82}
\]
2. Let $x$ and $y$ be two real numbers and $3 < p \leq 4$. Then
\[
|x + y|^p \leq |x|^p + |y|^p + p|x|^{p-1}\text{sign}(x)y + \frac{p(p-1)}{2} |x|^{p-2}y^2 + \frac{2p}{(p-2)} |x||y|^{p-1}. \tag{83}
\]
3. Let $x$ and $y$ be two positive real numbers and $p \geq 1$ any real number. Then
\[
(x + y)^p \leq x^p + y^p + 4^p(x^{p-1}y + xy^{p-1}). \tag{84}
\]
4. Let $x$ and $y$ be two real numbers and $p$ an even positive integer. Then
\[(x + y)^p \leq x^p + y^p + p(x^{p-1}y + xy^{p-1}) + 2p(x^2y^{p-2} + x^{p-2}y^2)\tag{85}\]

**Proof of Lemma 37.** The inequality (82) was established in Rio (2007, Relation (3.3)) by using Taylor expansion with integral rest for evaluating the difference $|x + y|^p - |x|^p$. To prove the inequality (83), we also use the Taylor integral formula of order 2 that gives
\[|x + y|^p - |x|^p = p|x|^{p-1}\text{sign}(x)y + C^2_p|x|^{p-2}y^2 + 2C^2_py^2\int_0^1 (1-t)(|x + ty|^{p-2} - |x|^{p-2})dt,\tag{86}\]
where $C^2_p = p(p - 1)/2$. Notice now that, for $3 < p \leq 4$,
\[|x + ty|^{p-2} \leq \frac{x^2 + 2|x||ty| + y^2}{(|x| + |ty|)^{4-p}} \leq |x|^{p-2} + 2|x||ty|^{p-3} + |ty|^{p-2}\]

Hence
\[
2C^2_py^2\int_0^1 (1-t)(|x + ty|^{p-2} - |x|^{p-2})dt \\
\leq 2C^2_py^2\int_0^1 (1-t)t^{p-2}dt + 4C^2_p|x||y|^{p-1}\int_0^1 (1-t)t^{p-3}dt \\
= 2|x|^{p-2}C^2_p\frac{\Gamma(p-1)}{\Gamma(p+1)} + 4|x||y|^{p-1}C^2_p\frac{\Gamma(p-2)}{\Gamma(p)} \\
= |y|^p + \frac{2p}{(p-2)}|x||y|^{p-1}.\tag{87}\]

Starting from (86) and using (87), the inequality (83) follows.

The inequality (84) was observed by Shao (1995, page 957). We shall establish now (85). We start by noticing that for any $a, b$ two positive real numbers and $2 \leq k \leq p - 2$ we have
\[a^{p-k}b^k \leq a^2b^{p-2} + a^{p-2}b^2\tag{88}\]

Now for $p$ an even positive integer and $x$ and $y$ two real numbers, the Newton binomial formula gives
\[\begin{aligned}
(x + y)^p &= x^p + y^p + p(x^{p-1}y + xy^{p-1}) + \sum_{k=2}^{p-2} C_p^k x^{p-k}y^k \\
&\leq x^p + y^p + p(x^{p-1}y + xy^{p-1}) + \sum_{k=2}^{p-2} C_p^k |x|^{p-k}|y|^k
\end{aligned}\]

Whence, by (88) and the fact that $\sum_{k=0}^{p} C_p^k = 2^p$ inequality (85) follows. $\diamond$

**Lemma 38** Let $(V_i)_{i \geq 0}$ be a sequence of non negative numbers such that $V_0 = 0$ and for all $i, j \geq 0$,
\[V_{i+j} \leq C(V_i + V_j),\tag{89}\]
where $C \geq 1$ is a constant not depending on $i$ and $j$. Then

1. For any integer $r \geq 1$, any integer $n$ satisfying $2^{r-1} \leq n < 2^r$ and any real $q \geq 0$
\[\sum_{i=0}^{r-1} \frac{1}{2^q}V_{2^i} \leq C2^{q+2}(2^{r+1} - 1)^{-1}\sum_{k=1}^{n} \frac{V_k}{k^{1+q}}.\]
2. For any positive integers \(k\) and \(m\) and any real \(q > 0\),

\[
\sum_{j=1}^{k} \frac{1}{j^q} V_{jm} \leq 2^{q+1} C q^{-1} m^{-1} \sum_{\ell=1}^{m} \frac{1}{(\ell + m)^q} V_{\ell} + 2 C q^{-1} m^{-1} \sum_{\ell=m+1}^{km} \frac{1}{\ell^q} V_{\ell}.
\]

3. Let \(0 < \delta \leq \gamma \leq 1\). Then for any real \(q \geq 0\),

\[
\left( \sum_{k=1}^{n} \frac{1}{k^{1+q\gamma} V_k^\gamma} \right)^{1/\gamma} \leq 2^{1/\delta-1/\gamma} C^{(\gamma-\delta)/\delta} \left( \sum_{k=1}^{n} \frac{1}{k^{1+q\delta} V_k^\delta} \right)^{1/\delta}.
\]

**Remark 39** If \((V_i)_{i \geq 0}\) satisfies (89) with \(C = 1\), then the sequence is said to be subadditive.

**Proof of Lemma 38.**

The condition (89) implies that for any integer \(k\) and any integer \(0 \leq j \leq k\),

\[
V_k \leq C (V_j + V_{k-j}) \quad \text{and then that} \quad (k+1)V_k \leq 2C \sum_{j=1}^{k} V_j.
\]

(90)

Therefore for \(2^{r-1} \leq n < 2^r\),

\[
\sum_{i=0}^{n-1} \frac{1}{2^{iq}} V_{2^i} \leq 2C \sum_{j=1}^{2^{r-1}} V_j \sum_{i:2^i \geq j} \frac{1}{2^{iq+1}},
\]

proving Item 1. To prove Item 2, using again (90), it suffices to notice that

\[
\sum_{j=1}^{k} \frac{1}{j^q} V_{jm} \leq 2C \sum_{j=1}^{k} \frac{1}{(1+jm)^q} \sum_{\ell=1}^{jm} V_{\ell} \leq 2C m^{-1}(2m)^q \sum_{j=1}^{k} j^{-q-1} \sum_{\ell=1}^{m} \frac{1}{(\ell+m)^q} V_{\ell} + 2C q^{-1} m^{q-1} \sum_{\ell=m+1}^{km} \frac{1}{\ell^q} V_{\ell}.
\]

To prove Item 3, we first notice that (89) entails that

\[
V_{i+j}^\gamma \leq C^\gamma (V_i^\gamma + V_j^\gamma) \quad \text{and then that} \quad (k+1)V_k^\gamma \leq 2C^\gamma \sum_{j=1}^{k} V_j^\gamma.
\]

Then for any real \(q \geq 0\),

\[
k^{-q(\gamma-\delta)} V_k^{\gamma-\delta} \leq 2^{1-\delta/\gamma} C^{\gamma-\delta} \left( \sum_{j=1}^{k} j^{-(1+q\gamma)} V_j^\gamma \right)^{1-\delta/\gamma}.
\]

(91)

Writing that \(k^{-(1+q\gamma)} V_k^\gamma = k^{-(1+q\delta)} V_k^\delta \times k^{-q(\gamma-\delta)} V_k^{\gamma-\delta}\) and using (91), the following inequality holds

\[
\sum_{k=1}^{n} \frac{1}{k^{1+q\gamma}} V_k^\gamma \leq 2^{1-\delta/\gamma} C^{\gamma-\delta} \left( \sum_{k=1}^{n} k^{-(1+q\delta)} V_k^\delta \right) \left( \sum_{j=1}^{n} j^{-(1+q\gamma)} V_j^\gamma \right)^{1-\delta/\gamma},
\]

proving Item 3. ⋄
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