Cosmological Perturbations via Quantum Corrections in M-Theory

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Abstract: In the early universe, it is important to take into account the quantum effect of gravity to explain the feature of inflation. In this paper, we consider the M-theory effective action which consists of 11-dimensional supergravity and (Weyl) 4 terms. The equations of motion are solved perturbatively, and the solution describes the inflation-like expansion in 4-dimensional spacetime. Equations of motion for tensor perturbations around this background are derived perturbatively. We also check that the equations of motion are obtained from the effective action up to the second order of the perturbations. Finally, we solve the equations of motion for the tensor perturbations perturbatively and obtain analytic expressions for them.

Keywords: inflation; M-theory; quantum gravity

1. Introduction

Inflationary expansion of the early universe resolves problems of the hot Big Bang scenario, such as the horizon problem or flatness problem [1–5]. It also generates perturbations to the homogeneous background, which become initial conditions for the structure formation of the present universe. Recent observations of cosmological parameters restrict the quantities of these perturbations, and hence models of the inflation [6–8]. In order to realize the inflation via the 4-dimensional effective theory of gravity, it is usual to introduce an inflation field or some higher curvature terms 1. The inflaton field causes the inflationary expansion while it slowly rolls down the potential [5,17–20]. The inflation via higher curvature term was first studied by Starobinsky [1], which is surprisingly consistent with the current observations.

Since the energy scale of the inflationary era will be near that of the grand unified theory, it is important to analyze this scenario by using ultraviolet completion of quantum gravity. Actually, there have been many attempts to explain the inflation from supergravity theory, which is a low-energy effective action of superstring theory. It is difficult, however, to obtain de Sitter (dS) vacua in the effective theory obtained by compactification of the supergravity theory, which is stated as a no-go theorem [21–23] or dS swampland conjecture [24–26]. Especially, the dS swampland conjecture says that low-energy effective theories which realize the inflation by the inflaton field cannot be consistent with quantum theory of gravity at Planck scale.

To evade the no-go theorem, we need to take into account corrections to the supergravity theory. One way is to introduce sources of branes in the superstring theory, and it was proposed in [27] that the de Sitter vacua can be constructed in the superstring theory with orientifold planes and flux compactification. Some recent consistency checks with the swampland conjecture are reported in [28]. Another way is to introduce higher curvature corrections in the superstring theory, and some earlier works in this direction can be found in [29–33].
In this paper, we pursue the possibility of the inflationary scenario in M-theory. The M-theory is defined as a strong coupling limit, or uplift to 11 dimensions, of type IIA superstring theory, and the low-energy limit is approximated by 11-dimensional supergravity. Since the type IIA superstring theory contains one-loop corrections, the uplift of these terms also gives corrections to the 11-dimensional supergravity. As for the metric, it gives products of four Weyl tensors, abbreviated as $W^4$, so it is important to understand the effect of these terms in relation to the inflationary scenario in the M-theory. Actually, it is shown in [34,35] that if the spacetime is divided into four dimensions and seven internal spatial directions, we obtain a perturbative solution where the three spatial directions are expanding and the internal ones are shrinking. Scalar perturbations around the inflationary background are discussed in [36]. The purpose of this paper is to analyze tensor perturbations around the inflation-like background. Equations of motion for tensor perturbations around this background are derived perturbatively. In addition, we also check that the equations of motion are obtained from the effective action up to the second order of the perturbations. Finally, we solve the equations of motion for the tensor perturbations perturbatively and obtain analytic expressions for them.

The organization of this paper is as follows. In Section 2, we briefly review the background geometry of [34]. In Section 3, we consider the tensor perturbations around the background and obtain equations of motion perturbatively. The effective action for the tensor perturbations are also derived. In Section 4, we solve the equations of motion for the tensor perturbations perturbatively and obtain analytic solutions. Conclusions and discussion are given in Section 5. The scalar perturbations and their solutions are summarized in Appendix A. Some discussion on the power spectrum of the cosmological perturbations are given in Appendixes B and C.

2. Effective Action and Inflationary Solution in M-Theory

Low-energy effective action of the superstring theory is obtained by analyzing scattering amplitudes [37,38] or conformal invariance of loop corrections on the string worldsheet [39,40]. The low-energy effective action of the M-theory is described by 11-dimensional supergravity, and its leading correction is obtained by uplifting the results of the type IIA superstring theory. The bosonic part of the effective action of the M-theory is given by [41,42]

$$ S_{11} = \frac{1}{2\pi^2} \int d^{11}x \, e(R + \Gamma Z), $$

where $a, b, c, \cdots = 0, 1, 2, \cdots, 10$ are local Lorentz indices and $W_{abcd}$ is a Weyl tensor. The 11-dimensional gravitational constant $2\kappa^2_{11}$ and a coefficient $\Gamma$ are dimensionful parameters and expressed in terms of the 11-dimensional Planck length $\ell_p$ as

$$ 2\kappa^2_{11} = (2\pi)^8 \ell_p^9, \quad \Gamma = \frac{\pi^2 \ell_p^9}{216}. $$

Note that the above action can be derived directly in 11 dimensions by imposing local supersymmetry [43–49]. The $Z$ terms in the action (1) are related to a topological term via local supersymmetry and the numerical values in this action are protected against higher perturbations. Of course, there exist higher derivative terms of $O(\Gamma^n)$ for $n \geq 2$, but those contributions are unknown so far. Some thoughts on the possible terms of $n = 2$ can be found in [36].
In this paper, we solve the equations of motion perturbatively up to the linear order of $\Gamma$. By varying the effective action (1), we obtain following equations of motion \cite{50}:

$$E_{ab} = R_{ab} - \frac{1}{2} \eta_{ab} R + \Gamma \left\{ - \frac{1}{2} \eta_{ab} Z + R_{cd} \eta^{cd} b - 2D_{(c} D_{d)} \eta^{cd} a \right\} = 0. \quad (4)$$

Here, $D_a$ is a covariant derivative for local Lorentz index, and tensors $X_{abcd}$ and $Y_{abcd}$ are defined as

$$X_{abcd} = W_{abc} W_{fg} W_{efgh} - 4W_{ab} W_{cdefgh} + 4W_{a} W_{b} W_{cdefgh} - 4W_{defgh} W_{abc} W_{fg} + 4W_{bc} W_{defgh} W_{abc} W_{fg} - 4W_{defgh} W_{abc} W_{fg} - 8W_{defgh} W_{abc} W_{fg} - 8W_{defgh} W_{abc} W_{fg} - 4W_{defgh} W_{abc} W_{fg} - 4W_{defgh} W_{abc} W_{fg} ,$$

$$Y_{abcd} = X_{abcd} - \frac{1}{9} (\eta_{ac} X_{bd} - \eta_{bc} X_{ad} - \eta_{ad} X_{bc} + \eta_{bd} X_{ac}) + \frac{1}{90} (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}) X , \quad (5)$$

where $Y^a_{\phantom{a}abcd} = 0$, $X_{ab} = X^c_{\phantom{c}abc}$ and $X = X^a_{\phantom{a}a}$. Let us consider the solution of effective action (1) in the early universe. We assume that the 10-dimensional space directions are divided into a 3-dimensional flat homogeneous space and a 7-dimensional internal space compactified on the 7-dimensional flat torus. Then, the ansatz of the metric is expressed as

$$ds^2 = -dt^2 + a(t)^2 dx_i^2 + b(t)^2 dy_m^2 , \quad (7)$$

where $i = 1, 2, 3$ and $m = 4, \cdots, 10$. $a(t)$ and $b(t)$ are scale factors for the 3-dimensional space and the 7-dimensional internal one, respectively. By inserting the above ansatz into Equation (4), we obtain differential equations for $H(t) = \frac{a(t)}{a(t)}$ and $G(t) = \frac{b(t)}{b(t)}$. Here, the dot represents the time derivative. The solution up to linear order of $\Gamma$ is given by \cite{34}

$$H(\tau) = \frac{H_1}{\tau} + \frac{c_h H_1^2}{\tau^2} + O \left( \frac{H_1^3}{\tau^3} \right) ,$$

$$G(\tau) = \frac{-7 + \sqrt{21}}{14} H_1 \frac{1}{\tau} + \frac{c_g H_1^2}{\tau^2} + O \left( \frac{H_1^3}{\tau^3} \right) , \quad (8)$$

with

$$c_h = \frac{13824(477087 - 97732 \sqrt{21})}{8575} \sim 47111,$$

$$c_g = \frac{-41472(532196 - 110451 \sqrt{21})}{60025} \sim -17996. \quad (9)$$

Here, $H_1$ is an integral constant and $\tau$ is defined by

$$\tau = \frac{(-1 + \sqrt{21}) H_1 t + 2}{2} . \quad (10)$$
Notice that \( \tau \) is the dimensionless parameter and takes the range of \( 1 \leq \tau \). By integrating Equation (8), the scale factors \( a(\tau) \) and \( b(\tau) \) are written as

\[
\log \left( \frac{a}{a_E} \right) = \frac{1 + \sqrt{21}}{10} \log \tau - \frac{1 + \sqrt{21}}{60} c_h \Gamma H_1^{6} \frac{1}{\tau^6} + O \left( \frac{\Gamma^2 H_1^{12}}{\tau^{12}} \right),
\]

\[
\log \left( \frac{b}{b_E} \right) = -\frac{3\sqrt{21}}{70} \log \tau - \frac{1 + \sqrt{21}}{60} c_\Gamma \Gamma H_1^{6} \frac{1}{\tau^6} + O \left( \frac{\Gamma^2 H_1^{12}}{\tau^{12}} \right),
\]

respectively. Here, \( a_E \) and \( b_E \) are integral constants and correspond to scale factors around the end of the inflation or the deflation, respectively. Since the string theory suppresses the divergence in the UV region, we expect that the scale factor \( a(\tau) \) and \( b(\tau) \) become convergent functions in the region \( 1 < \tau \). This means that coefficients of the higher-order terms will become the same order. In Figure 1, we show plots of \( \frac{1 + \sqrt{21}}{10} \log \tau, \frac{1 + \sqrt{21}}{60} c_h \Gamma H_1^{6} \frac{1}{\tau^6} \) and \( \frac{1 + \sqrt{21}}{60} c_\Gamma \Gamma H_1^{6} \frac{1}{\tau^6} \) for \( \Gamma H_1^{6} = 0.014 \) and \( c_h = \frac{c_h}{\Gamma H_1^{6}} \), respectively. From this, we see that higher-order corrections \( \frac{1}{\tau^6} (2 \leq n) \) are suppressed in the region \( 1.5 < \tau \), and the inflationary expansion or deflation is realized by each \( 1/\tau^6 \) term in Equation (11).

![Figure 1](https://example.com/image.png)

**Figure 1.** Plots of \( \frac{1 + \sqrt{21}}{10} \log \tau \) (blue), \( \frac{1 + \sqrt{21}}{60} c_h \Gamma H_1^{6} \frac{1}{\tau^6} \) (orange) and \( \frac{1 + \sqrt{21}}{60} c_\Gamma \Gamma H_1^{6} \frac{1}{\tau^6} \) (green). Parameters are chosen as \( \Gamma H_1^{6} = 0.014 \) and \( c_h = \frac{c_h}{\Gamma H_1^{6}} \).

The motivation for the inflation is to resolve the horizon problem. This requires that the particle horizon \( \int \frac{dt}{a(t)} \) during the inflationary era is almost equal to that after the radiation-dominated era. The particle horizon during the inflationary era is given by

\[
\frac{\sqrt{21} + 1}{10 H_1} \int_{1}^{2} \frac{dt}{a(\tau)} = \frac{\sqrt{21} + 1}{10 a_E H_1} \int_{1}^{2} d\tau \frac{\tau^2}{\tau^6} e^{\frac{1 + \sqrt{21}}{60} c_h \Gamma H_1^{6} \frac{1}{\tau^6}}.
\]

On the other hand, if we approximate the scale factor as \( a = a_E \tau^{\frac{1 + \sqrt{21}}{60}} \) after \( \tau = 2 \), the particle horizon during this era is evaluated as

\[
\frac{\sqrt{21} + 1}{10 H_1} \int_{2}^{t_0} \frac{dt}{a(\tau)} = \frac{\sqrt{21} + 1}{10 a_E H_1} \int_{2}^{t_0} d\tau \frac{\tau^2}{\tau^6} \tau^{\frac{\tau^2}{\tau^6} \frac{1 + \sqrt{21}}{60} \Gamma H_1^{6}} \sim \frac{\sqrt{21} + 1}{10 a_E H_1} \frac{9 + \sqrt{21}}{6},
\]

where \( t_0 \) is the value at current time \( t_0 \). We also obtain \( a(\tau) H(\tau) = a_E H_1 \tau^{\frac{\tau^2}{\tau^6} \frac{1 + \sqrt{21}}{60}} \).

Now we define the e-folding number as \( N_e = \log \frac{a(t)}{a(\tau)} \). This means that \( t_0 = 2 e^{\frac{\tau^2}{\tau^6} N_e} \). By equating Equation (12) with Equation (13), we obtain

\[
\int_{1}^{2} d\tau \frac{\tau^2}{\tau^6} e^{\frac{1 + \sqrt{21}}{60} c_h \Gamma H_1^{6} \frac{1}{\tau^6}} \sim \frac{9 + \sqrt{21}}{6} \frac{a_E H_1}{2} \frac{1 + \sqrt{21}}{60} e^{\frac{\tau^2}{\tau^6} N_e}.
\]
This gives a relation between $\Gamma H_0^6$ and $N_0$, and we obtain $\Gamma H_1^6 \sim 0.014$ for $N_0 = 69$, for example.

We also obtain $a(\tau_0) H(\tau_0) = a_0 H_1 \tau_0^{-1} \sim e^{-55} a_0 H_1$ for $N_0 = 69$.

3. Tensor Perturbations in M-Theory

3.1. Equations of Motion for the Tensor Perturbations

In this section, we investigate the tensor perturbations around the background geometry (7) up to the linear order of $\Gamma$. Since the action (1) contains complicated $W^4$ terms, we employ a Mathematica code to obtain the results here, even though these expressions are analytic. We deal with equations of motion for perturbations order by order, in the spirit of $[51]$. We also consult the calculations of cosmological perturbations in the modified gravity $[52,53]$.

The tensor perturbations around the background metric are chosen as follows:

$$ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dx^j + b^2 dy_m^2.$$  \hspace{1cm} (15)

In addition, $h_{ij}$ can be divided into two polarization modes, $h_+$ and $h_\times$. In the vielbein formalism, it is expressed as

$$e^a_{\mu} = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & a(1+\frac{3}{2}h_+) & 0 & 0 & \cdots & 0 \\
0 & a h_\times & a(1-\frac{1}{2}h_+) & 0 & \cdots & 0 \\
0 & 0 & 0 & a & \cdots & 0 \\
0 & 0 & 0 & 0 & b & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & b
\end{pmatrix},$$ \hspace{1cm} (16)

up to the linear order of the perturbation. Linearized equations of motion for the metric perturbations are obtained by varying Equation (4). The result is given as follows:

$$\delta E_{ab} = \delta R_{ab} - \frac{1}{2} \eta_{ab} \delta R + \Gamma \left\{ \frac{1}{2} \eta_{ab} \delta R_{cdef} Y^{cdef} + \delta R_{cdea} Y^{cede} + R^{cede} \delta Y_{cdea} ight\}$$

$$- 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} \frac{\partial}{\partial y_a} - 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} d + 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} + 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} d$$

$$+ 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} + 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} d - 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - 2 \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} d + \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} - \delta \omega_{(a}^{\epsilon} D_b Y_{(b)}^{\epsilon} d$$

$$\left\{ 0 \right\}.$$ \hspace{1cm} (17)

In this paper, we will solve the above equations of motion in the momentum space for the tensor perturbations. $h_0(a = +, \times)$ are expanded as

$$h_0(t, x, y) = \int d^3 k \sum_l \left\{ h_0(t, k, l)e^{i k_i x^i + il_m y^m} + h_0(t, k, l)^*e^{-i k_i x^i - il_m y^m} \right\},$$ \hspace{1cm} (18)

where $k_i$ is a momentum in the 3 spatial directions, and $l_m$ is a discretized momentum on the internal 7-dimensional torus. The internal momentum appears in the equations of motion with a factor of $\frac{k^2}{a^2}$, so these terms decouple from the equations of motion as $b$ becomes quite small during the inflation $[36]$. Hence, we neglect the contributions of internal momentum in this paper.

By inserting the expression (16) into the equations of motion (17), we obtain following differential equation for $h_0$:

$$0 = \ddot{h}_0 + (3H + 7G)\dot{h}_0 + \frac{k^2}{a^2} h_0 + \Gamma (A_0 \dot{h}_0 + A_1 \dot{h}_0 + A_2 \dot{h}_0 + A_3 \dot{h}_0 + A_4 \dot{h}_0),$$ \hspace{1cm} (19)
where $A_i (i = 0, 1, 2, 3, 4)$ are functions of $H$ and $G$. The explicit forms of $A_i$ are derived by employing Mathematica codes, which are reported in [54], and the results are written as

$$A_0 = \frac{3584}{105357} (164G^6 - 156051G^3H + 552087G^4H^2 + 126177G^4H - 169557GG^4 - 634247G^3H^3 + 424203G^3H + 383772GG^3H - 227652G^3HH - 45450G^3\ddot{C} + 237309G^2H^4 - 3501GG^2H^2 - 145794G^2H^2H + 131070G^2H\ddot{C} - 118800G^2HH - 3030G^2\ddot{H} - 21708G^2H^2 + 39906GG^2H - 3030G^2\ddot{C} - 15018G^2C^2 - 954GH^5 + 29592GH^4 - 10710C^2G - 240306GGH^3 + 243966GH^3H - 74910GH^3\ddot{C} + 59460GH^2H - 6210H^2\ddot{C} - 30111G^2H^2 - 12588\dot{C}H^2H - 24720GG\ddot{H} - 27900\ddot{G}GH + 21540\ddot{G}HH - 32520H^2\ddot{C} - 29340\ddot{G}HH - 6360\ddot{G}H + 9240GH\ddot{C} - 9240\ddot{G}HH - 3180\ddot{G}H - 3180\ddot{G}H + 18504GH\ddot{G}H + 115749GH\ddot{H} - 307158\ddot{G}GH\ddot{H} - 10737\ddot{G}H^2 - 6753G^2H^2 + 31080GG\ddot{G}H^2 + 3180G^2 + 3180GG\ddot{C} + 8081G^3 + 1692H^6 + 3258H^4H + 16920H^3H + 6210H^2\ddot{H} + 4587H^2H^2 + 3180H^2 + 35700H^2HH + 3180H\ddot{H} + 9409H^3)$$

$$- \frac{1433614}{6754^a} (92G^4 - 322G^3H + 401G^2H^2 - 46G^2H + 46G^2C^2 - 204G^3H + 72C^2H - 118G\ddot{G}H + 118GGH + 26G - 13C^2 + 33H^4 - 72H^2H - 13H^2),$$

$$A_1 = \frac{3584}{3575} (56G^7 + 82155H^6C^6 - 286878H^2C^5 + 88465GG^5C^5 - 63454HG^5C^5 - 243124H^3C^4 + 83314H^4\ddot{G}C^4 - 26141G^4C^4 + 3967H\ddot{H}HG^4 - 24251HG^4C + 33174H^4HG^3 + 119992G^3C^3 + 80084H^2G^3C^3 - 369471G^2H\ddot{G}C^3 + 52736H\ddot{G}HG^3 + 36900H^2HG^3C - 199896G\ddot{G}HG^3C + 45356H\ddot{H}HG^3C + 1930G^2C^3 - 1930HG^3 - 61587H^2G^2C^2 - 213335HG^3C^2 - 37713H\ddot{H}H^2C^2 + 303948H^2G^2C^2 - 25977H\ddot{H}G^2C^2 + 39778G\ddot{G}HG^2C^2 - 180903H^2G^2C^2 + 241268HG\ddot{H}G^2C^2 - 37578\ddot{G}HG^2C^2 + 32157H^2\ddot{H}G^2C^2 - 38678\ddot{G}HG^2C^2 + 36478HG\ddot{H}G^2C^2 - 4690G\ddot{C}G^2C^2 + 4690H\ddot{H}G^2C^2 - 7776HHG^2C^2 + 19057HG^3C - 25346H\ddot{H}G^2C^2 - 124134H^2H^2G^2C^2 + 2020HC^2G^2 - 192414H^2H^2G + 75909\ddot{C}H^2G + 2020H^2G + 55728H^4GG + 43998H\ddot{G}C^2G^2 - 86454H\ddot{G}HG^2G + 110772HG^2G^2 + 69620C^2\ddot{H}G + 335568H^2\ddot{C}HG + 69734\ddot{H}HG^2G - 41778H\ddot{H}HG^2G + 78094H\ddot{H}GG^2G - 4040\ddot{G}HG^2G - 61374HH\ddot{H}G^2G + 510H\ddot{C}G^2G + 2020G\ddot{G}G^2 - 2020H\ddot{H}GG - 510H^2\ddot{H}G - 2020\ddot{H}G + 2020\ddot{H}HG - 2268H^7 - 24056H^3C^3 + 25095H^3H^3C^2 + 5100HC^2G^2 - 50157H^3H^2 + 19974HG\ddot{H}G^2 + 7841H\ddot{H}H^2G^2 - 5100HH^2C^2 + 4644HG^5C + 8574HG^4C + 1681G^2C^2 - 21084G^2\ddot{C}G^2 - 18738HG^3H + 29177HG^2H^2 + 23460H^2G^2 + 35604HG\ddot{C}^2 - 9522HG\ddot{C} - 11484H^4HH - 3221G^2H^2 - 9381H^2H^3 + 28344H^2\ddot{C}H + 10200H\ddot{H}C\ddot{H} + 42864H^2HH + 12602GHH + 2250\ddot{C}G^2 - 5100HG\ddot{G}G + 5100H\ddot{C}G - 2250H^3\ddot{H}H + 5100H\ddot{H}H\ddot{H} + \frac{71681}{6754} (259G^5 + 863G^4H - 3235G^3H^2 - 1744G5H^3 + 2010G^2C^2 + 2713G^2H^2 - 266G^2H + 1746GG^2H + 1142G^2H\ddot{H} + 266G\ddot{C}G - 468G^4H + 528GH^3 + 72H^2C + 792GGH^2 + 1202GHH^2 - 336GHG + 338GHH - 194HG - 194G + 1017G\ddot{H} + 222\ddot{C}GH + 241\ddot{H}H + 288\ddot{G}HH + 194\ddot{C}G + 1211G^2C - 132H^5 - 600H^3H - 72H^2H + 194\ddot{H}HH - 47H^2H^2),$$
\[
A_2 = -\frac{3584}{3375} (1952G^6 + 27187G^5H + 30271G^4H^2 - 23579G^4H + 34969G^4C^4 - 112291G^3H^3 \\
- 10190G^3H + 163296G^2G^2H - 138316G^3H^2 + 11110G^3C + 21957G^2H^4 \\
- 5062G^2G^2H^2 - 11037G^2H^2 + 29850G^2H\dot{C} - 29530G^2H\dot{H} - 920G^2\ddot{H} \\
+ 44556G^2H^2 - 139692G^2H + 920G^2\ddot{C} + 98216G^2C^2 + 24228G\dot{H}^2 - 19524G^4 \\
- 15690H^3\dot{C} - 128118G^2G^2H + 130938G^3H - 25270G^2H^2 + 21870G\ddot{H}^2 \dot{H} \\
- 2160H\dddot{G} - 7367G^2H^2 - 68874GH\dddot{H} - 40460G\dot{H}\dot{C} - 43540G^2\dot{G} + 37380G\ddot{H} \\
+ 27300G\dot{H}\dot{C} - 33460GH\dddot{H} - 30380\dot{G}\dddot{H} - 6160\dot{G}\dot{H} + 1240G\dddot{H} - 1240G\dddot{H} \\
- 3080\dot{G}\dddot{H} - 3080G\dddot{H} + 130037G^2G^2H + 126477G\dddot{H}H^2 - 262674G\dddot{H}H - 8001G^2H^2 \\
- 15099G^2H + 46620G\dddot{C} + 3080G^2 + 3080G\dddot{C} + 12733G^3 + 6696H^6 + 41994H^4 \dot{H} \\
+ 17850H\dddot{H} + 2160H^2\dddot{H} + 64587H^2H^2 + 3080H^2 + 36540H\dddot{H} - 3080H\dddot{H} \\
+ 10367H^3) + \frac{7468k^2}{67.95^4} (37G^4 + 118C^3H - 479G^2H^2 - 266G^2H + 266G^2 + 456G^3H \\
+ 72G^2 - 338G^2G + 338GH\dddot{H} - 194G\dddot{H} + 97G^2 - 132H^4 - 72H^2H + 97H^2), \tag{22}
\]

\[
A_3 = -\frac{2667}{675} (14G^5 + 272G^4H - 229C^3H^2 - 284G^3H + 330G^3C - 315C^2H^3 \\
- 46G^2H + 686G^2G^2H - 670G^2H\dddot{H} + 46C^2\dddot{C} + 119G\dddot{H}^4 - 348C\dddot{H}^3 \\
- 108H\dddot{G} - 668G\dddot{H}G + 498H^2C\dddot{H} + 62G\dddot{H}C - 62GH\dddot{H} - 154C\dddot{H} \\
- 154\dddot{C}\dddot{H} + 477GH\dddot{H} - 1108\dddot{G}GH + 293C^2H - 740\dddot{C}GH + 154\dddot{C}G \\
+ 631C^2G + 99H^5 + 456H^3\dddot{H} + 108H^2C\dddot{H} + 154H\dddot{H} + 447H^2), \tag{23}
\]

and

\[
A_4 = -\frac{14336}{675} (2G^4 + 38C^3H - 49G^2H^2 - 46C^2\dddot{C} + 46C\dddot{G}^2 - 24G^3H - 108C\dddot{H}^2 \\
+ 62G\dddot{G} + 62G\dddot{H} - 154\dddot{G}GH + 77C^2 + 33H^4 + 108H^2\dddot{H} + 77H^2). \tag{24}
\]

Since we solve the equations of motion up to the linear order of \( \Gamma \), we expand \( H, G \) and \( h_a \) as

\[
H = H_0 + \Gamma H_1, \quad G = G_0 + \Gamma G_1, \quad h_a = h_0 + \Gamma h_1. \tag{25}
\]

Notice that \( H_0 = \frac{H}{\Gamma} \) and \( G_0 = \frac{7 + \sqrt{21}}{14} H_0 \) and \( H_0 = \frac{1 - \sqrt{21}}{14} H_0^2 \). We substitute these into Equation (19) and expand it up to the linear order of \( \Gamma \). Finally, with the aid of Mathematica code [54], we obtain the explicit forms of the equations of motion. The equation of motion for \( h_0 \) is written as

\[
0 = \dot{h}_0 + (7G_0 + 3H_0)h_0 + \frac{k^2}{a^2}h_0 = \dot{h}_0 + \frac{1}{2} (\sqrt{21} - 1) H_0 h_0 + \frac{k^2}{a^2} h_0. \tag{26}
\]
In addition, the equation of motion for \( h_1 \) is given by

\[
0 = h_1 + \frac{1}{2}(\sqrt{21} - 1) H_0 h_1 + \frac{k^2}{a_0^2} h_1 \\
+ \frac{512(16940714\sqrt{21} - 85692179)}{8575/5} H_0^2 h_0 + \frac{251(613929\sqrt{21} - 2861099)}{245} H_0^6 h_0 \\
+ \frac{6144(121\sqrt{21} - 521)}{\gamma} H_0^3 h_0 + \frac{2048(20\sqrt{21} - 101)}{\gamma} H_0^4 h_0 \\
+ \frac{k^2}{a_0^2} \left\{ \left( -2a_1 - \frac{768(64897\sqrt{21} - 270367)}{245} H_0^6 h_0 \right) + \frac{1024(26\sqrt{21} - 383)}{\gamma} H_0^6 h_0 \\
+ \frac{512(25\sqrt{21} - 43)}{\gamma} H_0^7 h_0 \right\} - \frac{k^4}{a_0^4} \frac{2048(7\sqrt{21} - 16)}{\gamma} H_0^7 h_0.
\]

(27)

Here, \( a_1 = -1 + \frac{\sqrt{21}}{20} c_0 H_0^6 \), which comes from \( \frac{i^2 \rho}{20} h_0 \). Note that \( a_0 = a e^{-\frac{1 + \sqrt{21}}{10} \tau} \) is the leading part of the scale factor \( a \). We will solve Equation (26) first and then substitute the solution of \( h_0 \) into Equation (27). The solutions will be derived in Section 4.

3.2. Effective Action for the Tensor Perturbations

In this subsection, we show an effective action for the tensor perturbations up to the second order of \( h_\kappa \). This is important to fix the normalization of the tensor perturbations as well as to perform a consistency check of the equations of motion.

The effective action for \( h_\kappa \) is obtained by substituting the perturbation (16) into the action (1). The results are derived by using the Mathematica code [54], and the effective action for \( h_\kappa \) is obtained as

\[
S_{pt}^{(2)} = \frac{1}{2a^2 \kappa} \int d^3 x a^3 \bar{h}^7 \left[ \frac{1}{2} \bar{h}_\kappa^2 - \frac{k^2}{2a^2} \bar{h}_\kappa^2 + \Gamma \left( B_0 \bar{h}_\kappa^2 + B_1 \bar{h}_\kappa^2 + B_2 \bar{h}_\kappa^2 \right) \right].
\]

(28)

In addition, explicit forms of \( B_i \) \((i = 0, 1, 2)\) are given by

\[
B_0 = -\frac{1792k^2}{10125a^2} (164G^4 - 156051G^5 H + 552087G^4 H^2 + 126177G^4 H - 169557G^6) \\
- 634247G^4 H^3 + 42420G^5 H + 38372G^4 H - 227652G^3 H H - 45450G^6 \bar{C} \\
+ 237309G^2 H^4 - 3501G^2 H^2 - 14574G^2 H^2 H + 13107G^2 H H - 118800G^2 H H \\
+ 3030G^4 H + 21708G^3 H + 39906G^4 H^2 - 3030G^3 H^2 - 15018G^2 H^2 - 954G^5 H \\
+ 29592G^4 H^2 - 10710G^4 H^3 + 24300G^4 H^4 - 7491O^4 H^4 \bar{G} \\
+ 59460G^4 H^2 - 621G^4 H^2 - 30111G^2 H^2 - 12588G^2 H H - 24720G^2 H \\
- 2790G^4 H + 21540G H H - 32520G H H - 29340G H H \\
- 6360G H + 9240G H H - 3180G H - 3180G H + 18504G^2 G H \\
+ 11574G^2 H H - 307158G H H H - 10737G^2 H^2 - 6753G^2 H - 31080G \bar{G} \\
+ 3180G^2 + 3180G \bar{G} + 8081G^3 + 1692H^4 + 3258H H H + 1692H H H \\
+ 6210H^4 H + 45879H^2 H^2 + 3180H^2 + 35700H H \bar{H} + 3180H \bar{H} + 9409H^3 \bar{G} \\
+ \frac{716814}{675a^2} (94G^4 - 322G^3 H + 401G^2 H^2 - 46G^2 H + 46G \bar{C}^2 - 204G H^3 \\
+ 72G H^2 - 118G H + 118G H H + 26G H - 13G^2 + 33H^4 - 72H^2 H - 13H^2). \]

(29)
\[ B_1 = \frac{1792}{3375} (8G^6 + 11733G^5H - 46011G^4H^2 - 10741G^4H + 12631G^3H + 54451G^3H^3 \]
\[ - 1930G^3H - 25696G^3H^2 + 18316G^3HH + 1930G^3HH - 18597G^2H^4 \]
\[ - 15477G^2H^2 + 21657G^2H^2 - 4690G^2HH + 4690G^2HH + 8824G^2H^2 \]
\[ - 18748G^2H^2 + 9924G^2G^2 - 828GH^5 + 1824GH^4 + 2250H^3G + 26718G^2G^3 \]
\[ - 24498GH^3H + 510GH^3H - 510GH^3H - 3147G^2H^2 + 13554G^2H^2 \]
\[ - 2020GH^2H + 2020GH^2H + 2020GH^2H + 5100GH^2H + 5100GH^2H \]
\[ + 5100GH^2H - 23717G^2GH - 15357G^2G^2 + 39074G^2GHH + 2741G^2H^2 \]
\[ - 1201G^2H + 2020G^2G - 113G^3 - 756H^6 - 4734H^4 - 2250H^3H \]
\[ - 10407H^2H^2 - 5100HHH - 1427H^3 \]
\[ + \frac{5384}{6750} (37G^4 + 118G^3H - 479C^2H^2 + 266G^2H^2 + 166G^2G^2 + 456GH^3 \]
\[ + 72G^2H^2 - 338G^2GH + 338G^2HH - 194G^2H + 97G^2H^2 - 132H^4 - 72H^2H + 97H^2) \]
\[ \text{and} \]
\[ B_2 = \frac{7169}{675} (2G^4 + 38G^3H - 49G^2H^2 - 46G^2H^2 + 46G^2G^2 - 24GH^3 - 108GH^2 \]
\[ + 62G^2GH - 62GH^2H - 154G^2H + 77G^2H + 33H^4 + 128G^2H^2 + 77H^2) \]

While not shown explicitly, it is also possible to write down the effective action for \( h_+ \), which is slightly different from the above expression. We remark, however, that the effective action for \( h_+ \) is the same as that of \( h_\times \) after substituting the background metric.

By substituting Equation (25) into the action (28), we obtain the effective action for the tensor perturbation up to the second order of \( h_0 \) and \( \theta_1 \). The result is

\[ S^{(2)}_{\text{pt}} = \frac{1}{2\kappa_1^2} \int d^4x a_0^3 b_0^2 \left[ \frac{1}{2} \dot{h}_0^2 - \frac{k^2}{2a_0^2} \dot{h}_0^2 + \Gamma \left\{ \frac{k^4}{4a_0^4} \frac{1024(7\sqrt{37} - 16)}{7} H_0 h_0^2 \right\} \right. \]
\[ + \frac{k^2}{a_0^2} \left\{ \left( - \frac{1}{2} \dot{a_1} - \frac{7}{2} \dot{b}_1 + \frac{384(64897\sqrt{37} - 270367)}{245} H_0^6 \right) h_0^2 + \frac{256(25\sqrt{37} - 43)}{7} H_0^4 h_0^2 - h_0^2 \right\} \]
\[ + \left( \frac{3}{2} \dot{a_1} + \frac{7}{2} \dot{b}_1 + \frac{128(74649\sqrt{37} - 289019)}{245} H_0^6 \right) h_0^2 - \frac{1024(20\sqrt{37} - 101)}{7} H_0^4 h_0^2 + h_0^2 \right] \].

Here, \( \dot{a_1} = -\frac{1+\sqrt{37}}{4\sqrt{37}} c_6 H_0^6 \) and \( \dot{b}_1 = -\frac{1+\sqrt{37}}{4\sqrt{37}} c_6 H_0^6 \).

It is easy to see that the equation of motion (26) is obtained from the variation of \( h_1 \),

\[ \delta h_1 S^{(2)}_{\text{pt}} = -\frac{1}{2\kappa_1^2} \int d^4x a_0^3 b_0^2 \Gamma \left[ \dot{h}_0 + \frac{\sqrt{21}}{2} - \frac{1}{2} H_0 h_0 + \frac{k^2}{a_0^2} \right] \delta h_1. \]

Here, we used the relation of the background \( G_0 = -\frac{7+\sqrt{37}}{4\sqrt{37}} H_0 \). Note also the relations of \( H_0 = \frac{1-\sqrt{21}}{4} H_0, \dot{a_1} = c_6 H_0^6 \) and \( \dot{b}_1 = c_6 H_0^6 \). By using these relations, it is possible to evaluate the variation of the effective action with respect to \( h_0 \) as
\[ \delta \rho_0 S_{\text{pt}}^{(2)} = -\frac{1}{2 \kappa^2_{11}} \int d^4x a_0^3 b_0^2 \left[ \dot{\rho}_0 + \frac{\sqrt{21}}{2} H_0 \dot{h}_0 + \frac{k^2}{a_0^2} h_0 + \Gamma \left\{ -\frac{k^4}{a_0^4} \frac{2048 \sqrt{21} - 16}{7} H_0^2 h_0 \right. \right. \\
+ \left. \left. \frac{k^2}{a_0^2} \left( a_0 + 7 b_1 - 768(649897 \sqrt{21} - 270367) H_0^6 \right) h_0 + \frac{512(25 \sqrt{21} - 43)}{7} \left( H_0^4 h_0 - \frac{1}{2} \frac{\sqrt{21}}{7} H_0^2 h_0 \right) \right) \right) + \left( \frac{\sqrt{21}}{2} H_0 (3a_1 + 7b_1 + 256(74649 \sqrt{21} - 289019) H_0^6) \right) \\
+ \left( 3c_h + 7c_s + 256(74649 \sqrt{21} - 289019) \right) \left( 1 - \frac{\sqrt{21}}{7} \right) H_0^2 \dot{h}_0 \right) \right] \delta h_0 \right]
\\
= -\frac{1}{2 \kappa^2_{11}} \int d^4x a_0^3 b_0^2 \left[ \dot{\rho}_0 + \frac{\sqrt{21}}{2} H_0 \dot{h}_0 + \frac{k^2}{a_0^2} h_0 + \Gamma \left\{ -\frac{k^4}{a_0^4} \frac{2048 \sqrt{21} - 16}{7} H_0^2 h_0 \right. \right. \\
+ \left. \left. \frac{k^2}{a_0^2} \left( -2a_1 - 768(649897 \sqrt{21} - 270367) H_0^6 \right) h_0 + \frac{1024(26 \sqrt{21} - 383)}{7} \right. \right. H_0^2 h_0 + \frac{512(25 \sqrt{21} - 43)}{7} \left( H_0^4 h_0 \right) \right) \right] \delta h_0 \right]
\\
+ \frac{512(169407498 \sqrt{21} - 85692179)}{8575} H_0^2 \dot{h}_0 + \frac{256(613929 \sqrt{21} - 2861099)}{245} H_0^2 \dot{h}_0 \\
+ \frac{6144(121 \sqrt{21} - 521)}{7} H_0^2 \dot{h}_0 + \frac{256(20 \sqrt{21} - 101)}{7} H_0^2 \dot{h}_0 \\
+ \dot{h}_1 + \frac{\sqrt{21}}{2} H_0 h_1 + \frac{k^2}{a_0^2} h_1 + \left( \dot{h}_0 + \frac{\sqrt{21}}{2} H_0 h_0 + \frac{k^2}{a_0^2} h_0 \right) (3a_1 + 7b_1) \right] \right] \delta h_0 \right].}

From this, we see that the equation of motion (27) is correctly derived if we use Equation (26). Although we rely on the Mathematica codes for the calculations, this gives a non-trivial check between Equations (26) and (27) and the effective action (32).

4. Solution for the Tensor Perturbations

Let us analytically solve Equations (26) and (27) in this section. In order to solve these equations, we introduce a new time coordinate \( \eta \) instead of \( t \), which is defined by
\[
\frac{dt}{a_0} = \frac{3 + \sqrt{21} \sqrt{\frac{2}{3} a_E H_1 \eta}}{6 a_E H_1} \left( 1 - \frac{1}{b_0^{\frac{3}{2}}} \right) \frac{d\tau}{\eta}.
\]

and \( a_0 \) and a Hubble parameter \( H_0 \) with respect to \( \eta \) are given by
\[
a_0 = a_E \left( \frac{\sqrt{21} - 3}{2} a_E H_1 \right)^{\frac{3 + \sqrt{21}}{\sqrt{6}}} \quad \text{and} \quad H_0 = \frac{a_0'}{a_0} = \frac{3 + \sqrt{21}}{6} \frac{1}{\eta},
\]

Here, the prime \( ' \) represent \( \frac{d}{d\eta} \). Then, by defining \( h_0 = a_0' \frac{1}{2} u_0 \) and multiplying \( a_0^2 \) to Equation (26), we obtain
\[
0 = h_0'' + \frac{\sqrt{21} - 3}{2} H_0 h_0' + k^2 h_0 = a_0^2 \left[ u_0'' + \left( k^2 + \frac{1}{6 \eta^2} \right) u_0 \right].
\]

In addition, the above differential equation for \( u_0 \) can be solved as
\[
u_0 = c_1 \sqrt{21} h_0 \eta (k \eta) + c_2 \sqrt{21} \eta (k \eta).
\]
Here, \( f_0 \) and \( Y_0 \) are Bessel functions of the first and second kind, respectively. \( c_1 \) and \( c_2 \) are integral constants which have the dimension of mass and depend on \( k \). In order to fix the ratio of \( c_2 \), we demand that \( u_0 \) behaves like \( e^{-ik\eta} \) as \( \eta \) goes to infinity. This means that the tensor perturbation is approximated by the free field as \( \eta \) goes to infinity. Since 
\[
\sqrt{x}f_0(x) \sim \sqrt{\frac{2}{\pi}} \cos(x - \frac{\pi}{4}) \quad \text{and} \quad \sqrt{x}Y_0(x) \sim \sqrt{\frac{2}{\pi}} \sin(x - \frac{\pi}{4}) \quad \text{as} \quad x \to \infty,
\]
we choose \( c_2 = -i \) and \( u_0 \) is given by
\[
0 = c_1 \sqrt{k\eta} J_0^2(k\eta).
\]

(39)

\( J_0^2(x) \) is the Hankel function of the second kind. \( c_1 \) is proportional to \( \frac{1}{\sqrt{2\pi}} \) and the coefficient is determined by the normalization of the action (32). From this, \( h_0 \) is expressed as
\[
h_0 = c_1 a_0 \sqrt{\frac{3 - \sqrt{2\pi}}{8\sqrt{3}}} \sqrt{k\eta} J_0^2(k\eta) = c_1 J_0^2(k\eta),
\]
\[
c_1 = c_1 a_E \left( \frac{\sqrt{2\pi} + 3}{6} \frac{k}{a_E H_1} \right)^{\frac{1}{2}}. \tag{40}
\]

If we take \( k\eta \to \infty \), \( h_0 \) approaches to \( c_1 \sqrt{\frac{3 - \sqrt{2\pi}}{8\sqrt{3}}} \sqrt{k\eta} e^{-ik\eta} \).

Next, we investigate the equation of motion (27) for \( h_1 \). By multiplying \( a_0^2 \) to Equation (27) and using \( \eta \), we obtain
\[
0 = h_1'' + \frac{1}{2} \left( \sqrt{2\pi} - 3 \right) h_0 h_1' + k^2 h_1
+ \frac{1}{a_0^2} \left[ \frac{256(4950813 \sqrt{2\pi} - 33216993)}{245} \eta h_0 H_0'' - 768(484793 - 93483 \sqrt{2\pi}) H_0^2 h_0''
+ 6144(48 \sqrt{2\pi} - 319) \eta H_0^2 h_0''' - 512(404 - 80 \sqrt{2\pi}) \eta^2 H_0^2 h_0''''
+ k^2 \left( - \frac{2768(64897 \sqrt{2\pi} - 270367)}{245} \eta H_0^2 - 2a_0^2 \eta H_0 \right) h_0 + 1536(9 \sqrt{2\pi} - 241) \eta H_0^2 h_0'' - 512(43 - 28 \sqrt{2\pi}) \eta H_0^2 h_0'' \right], \tag{41}
\]

In order to solve the above equation, we redefine \( h_1 \) as \( h_1 = a_0^3 \sqrt{\frac{3 - \sqrt{2\pi}}{8\sqrt{3}}} u_1 \). Then, a differential equation for \( u_1 \) is expressed as
\[
0 = u_1'' + \left( k^2 - \frac{3(\sqrt{2\pi} - 5)}{8} \right) \frac{u_1''}{\eta^2} + \frac{1}{a_0^2} \left[ \frac{576(2799893 \sqrt{2\pi} - 128441773)}{245} \eta^2 u_1 + 4608(11177351 \sqrt{2\pi} - 51407611) \eta^2 u_1''
- 768(1252723 - 338133 \sqrt{2\pi}) \eta^2 u_1''' + 8392(101 \sqrt{2\pi} - 420) \eta^2 u_1'''
+ k^2 \left( - \frac{2768(55797 \sqrt{2\pi} - 234982)}{245} \eta^2 - 2a_0^2 \eta \right) u_1 + 1024(43 \sqrt{2\pi} - 525) \eta^2 u_1''
- 512(43 - 28 \sqrt{2\pi}) \eta^2 u_1'' \right]
\]
\[
= u_1'' + \left( k^2 + \frac{1}{4\eta^2} \right) u_1 + \frac{28}{3} \frac{\sqrt{2\pi} + 3}{6} \left( \frac{3 - \sqrt{2\pi}}{8\sqrt{3}} \right)^{3 + \sqrt{2\pi}} \eta^2 u_1
\]
\[
\left[ (73500(47 \sqrt{2\pi} + 217) k^6 \eta^3 + 6(1032913 \sqrt{2\pi} + 4661457) k^3 \eta^2) (c_1 f_1(k\eta) + c_2 Y_1(k\eta))
+ (-44100(8 \sqrt{2\pi} + 37) k^4 \eta^4 + (6318421 \sqrt{2\pi} + 29265657) k^2 \eta^2) (c_1 f_0(k\eta) + c_2 Y_0(k\eta)) \right]. \tag{42}
\]
In the second line, we substituted Equations (36) and (38). A particular solution for the above equation is given by

\[
    u_1 = \frac{576(999 - 218\sqrt{21})}{60025} \frac{H_{\ell}^k \sqrt{\eta}}{(2\sqrt{21} - 3a_1 H_{\ell} \eta)} (c_{1u_1} + \sqrt{\pi} c_{2u_1}).
\]

Here, the explicit forms of \(u_1\) and \(u_2\) are given by

\[
    u_1 = 7\sqrt{\frac{\pi}{\eta}} \lambda_0(k\eta) \left\{ -14700(146\sqrt{21} + 669)k^2\eta^3G_{3,5}^{2,2}(k\eta, \frac{1}{2}, \frac{5}{4}, \frac{2\sqrt{21} + 17}{4}, \frac{1}{4}, \frac{2\sqrt{21} + 13}{4}) \\
    + 24500(857\sqrt{21} + 3927)k^2\eta^2G_{3,5}^{2,2}(k\eta, \frac{1}{2}, \frac{3}{4}, \frac{2\sqrt{21} + 19}{4}, \frac{3}{4}, \frac{2\sqrt{21} + 15}{4}) \\
    + (38465701\sqrt{21} + 176254865)k\eta G_{3,5}^{2,2}(k\eta, \frac{1}{2}, \frac{5}{4}, \frac{2\sqrt{21} + 21}{4}, \frac{1}{4}, \frac{2\sqrt{21} + 17}{4}) \\
    + 6(6206377\sqrt{21} + 28445153)G_{3,5}^{2,2}(k\eta, \frac{1}{2}, \frac{5}{4}, \frac{2\sqrt{21} + 23}{4}, \frac{3}{4}, \frac{2\sqrt{21} + 19}{4}) \right\} (44)
\]

and

\[
    u_2 = \sqrt{\frac{\pi}{\eta}} \lambda_0(k\eta) \left\{ 51450(61\sqrt{21} + 279)\sqrt{\pi}(k\eta)^{5/2}F_3 \left( \frac{1}{2}, -\frac{\sqrt{21}}{2}, \frac{21}{2}, 1, 1, -\frac{\sqrt{21}}{2}; -k^2\eta^2 \right) \\
    - 85750(179\sqrt{21} + 819)\sqrt{\pi}(k\eta)^{3/2}F_3 \left( \frac{1}{2}, -\frac{2\sqrt{21}}{3}, -\frac{1}{2}, 2, 2, -\frac{2\sqrt{21}}{3}; -k^2\eta^2 \right) \\
    - (46502521\sqrt{21} + 213002167)\sqrt{\pi}(k\eta)^{3/2}F_3 \left( \frac{1}{2}, -\frac{k^2\eta^2}{2}, 1, 2, -\frac{k^2\eta^2}{2}; -\frac{k^2\eta^2}{2} \right) \\
    - 3(7499743\sqrt{21} + 34391077)\sqrt{\pi}(k\eta)^{3/2}F_3 \left( \frac{1}{2}, -\frac{2\sqrt{21}}{3}, 2, 2, -\frac{2\sqrt{21}}{3}; -k^2\eta^2 \right) \\
    - 205800(146\sqrt{21} + 669)k^3\eta^3G_{3,5}^{3,1}(k\eta, \frac{1}{2}, \frac{3}{4}, \frac{2\sqrt{21} + 17}{4}, \frac{3}{4}, \frac{2\sqrt{21} + 19}{4}) \\
    + 343000(857\sqrt{21} + 3927)k^2\eta^2G_{3,5}^{3,1}(k\eta, \frac{1}{2}, \frac{1}{4}, \frac{2\sqrt{21} + 19}{4}, \frac{1}{4}, \frac{2\sqrt{21} + 17}{4}) \\
    + 14(38465701\sqrt{21} + 176254865)k\eta G_{3,5}^{3,1}(k\eta, \frac{1}{2}, \frac{1}{4}, \frac{2\sqrt{21} + 21}{4}, \frac{1}{4}, \frac{2\sqrt{21} + 15}{4}) \\
    + 84(6206377\sqrt{21} + 28445153)G_{3,5}^{3,1}(k\eta, \frac{1}{2}, \frac{1}{4}, \frac{2\sqrt{21} + 23}{4}, \frac{1}{4}, \frac{2\sqrt{21} + 19}{4}) \right\} (45)
\]

Here, the function \(C_{F\ell}^{m,n}(z,r)\) is the generalized Meijer G-function, and the function \(\mu_\ell F_\eta(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\) is the generalized hypergeometric
function. The ratio of $\frac{\omega}{\epsilon}$ should be fixed by $\frac{\omega}{\epsilon} = -i$ as explained before. Thus, we have solved Equation (41) as

$$h_1 = -\frac{576(999 - 218\sqrt{21})}{60025} \frac{\epsilon_1 \eta_0 \tau^{\frac{3}{2} + \pi}}{(\sqrt{21} - \frac{\epsilon_1}{\sqrt{21}})^9 + \sqrt{21}} \left(\mu_{11} - i\sqrt{\pi} \mu_{12}\right)$$

$$= -\frac{576(999 - 218\sqrt{21})}{60025} \frac{H_1^6}{\epsilon_1} \left(\mu_{11} - i\sqrt{\pi} \mu_{12}\right).$$

Here, we used $\tau^6 = (\sqrt{21} - \frac{\epsilon_1}{\sqrt{21}} H_{11})^{9 + \sqrt{21}}$.

In conclusion, we derived analytic form of the tensor perturbations which are given by Equations (40) and (46). If we know the effective action of the M-theory beyond the linear order of $\Gamma$, we could derive the solution up to the same order by using the method developed in this paper. Some discussions on the power spectrum of the perturbations are given in Appendix B.

5. Conclusions and Discussion

In this paper, we examined the inflationary solution via higher derivative corrections in the M-theory and examined tensor perturbations around such a background. As a result, we have obtained the solutions of tensor perturbations analytically up to the linear order of $\Gamma$. Although our calculations are limited up to the linear order of $\Gamma$, our method developed in this paper is applied to higher-order effective action once its form is revealed.

The effective action of the M-theory contains (Weyl)$^4$ terms. If we assume that the 10-dimensional space is divided into a 3-dimensional homogeneous space and a 7-dimensional internal one, it is possible to solve the equations of motion perturbatively with respect to $\Gamma$ and obtain a inflationary solution due to the presence of (Weyl)$^4$ terms. The tensor perturbations are analyzed around this inflationary background. These are expanded up to the linear order of $\Gamma$ such as $h_4 = h_0 + \Gamma h_1$, and the equation of motion for $h_0$ is written by Equation (26) and that for $h_1$ is given by Equation (27). On the other hand, the effective action for the tensor perturbation for the cross mode is evaluated up to the linear order of $\Gamma$. In addition, the effective action with respect to $h_0$ and $h_1$ is given by Equation (32). We could derive the same equations of motion from this action, and this gives a self-consistency check of our calculations, which are mainly done by using Mathematica codes. Although we omitted the calculations, the effective action for the plus mode is the same as that of the cross mode once we insert the background metric.

The equations of motion for $h_4$ are solved perturbatively. The solution of $h_0$ is given by the linear combinations of Bessel functions. Then, we put the solution of $h_0$ into the equation of motion for $h_1$. The analytic form of $h_1$ is given by Equation (46). In this way, we developed a method to calculate tensor perturbations by using Mathematica codes. If we know the effective action of the M-theory beyond the linear order of $\Gamma$, we could derive the solution up to the same order by using the method developed in this paper [55].

Since we know the analytic solution, at least perturbatively, it is straightforward to evaluate the power spectrum of the tensor perturbations. However, the vacuum of our inflationary solution is highly interacting, it is different from usual Bunch–Davies vacuum. So, in order to evaluate the power spectrum, we need to put an assumption on the amplitude of the power spectrum. Some discussions on the evaluation of the power spectrum are given in Appendix B.

As a future work, it is interesting to apply the method developed here to more a complicated internal geometry, such as the $G_2$ manifold [56]. It is also interesting to apply the analyses of this paper to the heterotic superstring theory with nontrivial internal space, which contains $\mathcal{R}^2$ corrections [57], and reveal several problems in string cosmology [58]. Unification of the inflationary expansion and late time acceleration in modified gravity, such as $f(R)$ gravity or mimetic gravity, is an interesting direction to be explored [59–62].
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Appendix A. The Scalar Perturbations

In this appendix, we briefly summarize the results of scalar perturbations obtained in [36]. The metric for the scalar perturbations is written as

$$ds^2 = -(1 + 2\alpha)dt^2 - 2ad\beta dt dx^i + a^2(\delta_{ij} + 2\delta_i\gamma + 2\psi\delta_j)dx^idx^j + b^2dy_m^2.$$  (A1)

By inserting the above into Equation (17), we obtain four independent equations with respect to \(\alpha\), \(\chi = a(\beta + a\gamma)\) and \(P = H^{-1}\psi\). By using two equations out of four, it is possible to solve \(\alpha\) and \(\chi\) up to the linear order of \(\Gamma\), and one equation becomes redundant. Then, we obtain a differential equation for \(P = P_0 + \Gamma P_1\) up to the linear order of \(\Gamma\).

$$0 = P_0'' - \frac{\sqrt{21}}{2} - 1 \mathcal{H}_0 P_0' + \left(\frac{k^2}{a_0^2} - \frac{\sqrt{21}}{2} - 11 \mathcal{H}_0^2\right) P_0 + \Gamma\left[\frac{P_1'' - \sqrt{21} - 1}{2} \mathcal{H}_0 P_1' + \left(\frac{k^2}{a_0^2} - \frac{\sqrt{21}}{2} - 11 \mathcal{H}_0^2\right) P_1\right]$$

$$+ \frac{1}{a_0^2} \left\{\frac{1536(49692383\sqrt{21} - 7093438)}{8575} \mathcal{H}_0^6 P_0 - 124991079 \mathcal{H}_0^2 P_0\right\} + \frac{768(36412229\sqrt{21} - 124991079)}{1715} \mathcal{H}_0^6 P_0 + \frac{768(3567079\sqrt{21} - 17688)}{245} \mathcal{H}_0^6 P_0' + \frac{3072(2261\sqrt{21} - 23271)}{1225} \mathcal{H}_0^4 P_0 + \frac{3072(2261\sqrt{21} - 23271)}{1225} \mathcal{H}_0^4 P_0' + \frac{3072(8533\sqrt{21} - 265416)}{1225} \mathcal{H}_0^4 P_0 + \frac{1536(9479\sqrt{21} - 66369)}{1225} \mathcal{H}_0^4 P_0 + \frac{k^4}{a_0^4} \left\{6144(1633\sqrt{21} - 9288) \mathcal{H}_0^4 P_0\right\}. $$  (A2)

As in the case of the tensor perturbations, in order to solve Equation (A2), we use the variable \(\eta\) which is defined in Equation (35). By multiplying \(a_0^2\) to Equation (A2), we obtain

$$0 = P_0'' - \frac{\sqrt{21}}{2} + 1 \mathcal{H}_0 P_0' + \left(k^2 - \frac{\sqrt{21}}{2} - 11 \mathcal{H}_0^2\right) P_0 + \Gamma\left[\frac{P_1'' - \sqrt{21} + 1}{2} \mathcal{H}_0 P_1' + \left(k^2 - \frac{\sqrt{21}}{2} - 11 \mathcal{H}_0^2\right) P_1\right]$$

$$+ \frac{1}{a_0^2} \left\{\frac{1536(49692383\sqrt{21} - 7093438)}{8575} \mathcal{H}_0^6 P_0 - 1536(15053494\sqrt{21} - 40585737) \mathcal{H}_0^2 P_0\right\} + \frac{6912(1812421\sqrt{21} - 16802761)}{8575} \mathcal{H}_0^6 P_0' + \frac{6144(57047\sqrt{21} - 107067)}{1225} \mathcal{H}_0^5 P_0''$$

$$+ \frac{3072(2261\sqrt{21} - 23271)}{1225} \mathcal{H}_0^4 P_0' + \frac{k^2}{a_0^2} \left\{\frac{768(3567079\sqrt{21} - 29260239)}{8575} \mathcal{H}_0^5 P_0'' - 2a_0^2 \bar{a}_1\right\} P_0 + \frac{1536(161183\sqrt{21} - 464463)}{1225} \mathcal{H}_0^5 P_0' + \frac{1536(9479\sqrt{21} - 66369)}{1225} \mathcal{H}_0^5 P_0' + \frac{k^4}{a_0^4} \left\{6144(1633\sqrt{21} - 9288) \mathcal{H}_0^4 P_0\right\}. $$  (A3)
Let us solve the above equation perturbatively. First, by setting $P_0 = a_0^{\frac{\sqrt{21}+1}{2}} U_0$, a part of $O(\Gamma^0)$ becomes

\[
0 = P'_0 - \frac{\sqrt{21} + 1}{2} \mathcal{H}_0 P'_0 + \left( k^2 - \frac{\sqrt{21} - 11}{2} \mathcal{H}_0^2 \right) P_0 \\
= a_0 \left[ \sqrt{21+1} \left( k^2 + \frac{1}{4\eta^2} \right) U_0 \right].
\]  

(A4)

This is the same as Equation (37). Then, $U_0$, $P_0$ and $\psi_0 = H_0 P_0$ are solved as follows:

\[
U_0 = c_1 \sqrt{k\eta} H_0^{(2)}(k\eta), \\
P_0 = c_1 a_0 \frac{\sqrt{21+1}}{\sqrt{k\eta}} H_0^{(2)}(k\eta), \\
\psi_0 = \tilde{c}_1 H_0^{(2)}(k\eta), \\
\tilde{c}_1 = c_1 a_E \frac{\frac{\sqrt{21+1}}{6}}{a_E H_1}.
\]  

(A5)

In the above, we set $\frac{\xi}{\tilde{c}_1} = -i$ as in the tensor perturbations. $c_1$ is proportional to $\frac{1}{\sqrt{2\eta}}$ and the coefficient is determined by the normalization of the action in [36]. In the limit of $k\eta \to \infty$, $\psi_0(\eta, k)$ approaches to $c_1 \sqrt{2 \pi} e^{\frac{1}{2}i\eta}$.

Next, let us solve a part of the differential Equation (A3) which linearly depends on $\Gamma$. By setting $P_1 = a_0^{\frac{\sqrt{21}+1}{2}} U_1$ and using the solution (A5), the equation which linearly depends on $\Gamma$ becomes

\[
0 = U'_1 + \left( k^2 + \frac{1}{4\eta^2} \right) U_1 + \frac{28}{3 \cdot 5^2 \pi} \left( \frac{\sqrt{21} + 3}{6} \right)^3 \mathcal{H}_0 \left[ \frac{\sqrt{21}}{a_E H_1} \right]^{3 + \sqrt{21}} \\
\left[ (420(127267 + 27753\sqrt{21})k^3 \eta^3 - 78(27149229 + 5923661\sqrt{21})k(k_0 H_0(k) + c_2 Y_0(k\eta)) \\
+ ( -44100(37 + 8\sqrt{21})k\eta^4 + (6025616417 + 13142261\sqrt{21})k^2 \eta^2, \\
+ 36(494288949 + 10780291\sqrt{21})(c_1 H_0(k\eta) + c_2 Y_0(k\eta)) \right].
\]  

(A6)

A particular solution of the above is obtained by using Mathematica code [36]. The solutions of $U_1$, $P_1$ and $\psi_1 = H_1 P_1 + H_0 P_0 = H_0(c_1 H_0 P_0 + P_1)$ are expressed as follows.

\[
U_1 = \frac{-288(20727 - 4523\sqrt{21})}{300125} \frac{c_1 H_0^{(2)}(k)}{ \left( \frac{\sqrt{21} - 3}{2} a_E H_1 \eta \right)^{9 + \sqrt{21}} \left( U_1 + i \left( \frac{41 + 9\sqrt{21}}{10} \right) U_1 \right)} \\
P_1 = \frac{-288(20727 - 4523\sqrt{21})}{300125} \frac{c_1 a_0 H_0^{(2)}(k)}{ \left( \frac{\sqrt{21} - 3}{2} a_E H_1 \eta \right)^{9 + \sqrt{21}} \left( U_1 + i \left( \frac{41 + 9\sqrt{21}}{10} \right) U_1 \right)} \\
\psi_1 = \frac{\tilde{c}_1 H_0^{(2)}(k)}{a_0} \left( c_1 H_0^{(2)}(k) - \frac{288(20727 - 4523\sqrt{21})}{300125} \left( U_1 + i \left( \frac{41 + 9\sqrt{21}}{10} \right) U_1 \right) \right).
\]  

(A7)

The explicit expressions of $U_{11}$ and $U_{12}$ can be found in [36]. Here, the ratio of integration constants is fixed as $\frac{\xi}{\tilde{c}_1} = -i$. We also used $\tau^6 = \left( \frac{\sqrt{21} - 3}{2} a_E H_1 \eta \right)^{9 + \sqrt{21}}$ and $H_1 = c_1 H_0^{(2)}$ from Equation (8). Note that as $\tau$ approaches to the infinity, $\psi_1$ decreases faster than $\psi_0$.

Appendix B. Numerical Analyses for Scalar and Tensor Perturbations

In this appendix, we examine spectral indices of the scalar and tensor perturbations. First of all, we change the definition of $\tau$ by rescaling. Without loss of generality, it is possible to rescale $\tau$ like

\[
\tau \rightarrow (c_i \Gamma_0^{\frac{1}{2}})^{\frac{1}{2}} \tau.
\]  

(A8)
In addition, after this prescription, we shift the integral constants as

\[
H_I \to (c_h \Gamma H_I^6)^{-1} H_I, \quad a_E \to (c_h \Gamma H_I^6)^{-1 + \sqrt{21} \tau^{6 \sigma}} a_E, \quad b_E \to (c_h \Gamma H_I^6)^{3 \sqrt{21} \tau^{-7}} b_E. \tag{A9}
\]

Then, \(\eta, a_0, b_0, H_0\) and \(G_0\) are invariant, and \(H(\tau), G(\tau), a(\tau)\) and \(b(\tau)\) behave as

\[
\frac{H}{H_I} = \frac{1}{\tau} + \frac{1}{\tau^7} + \mathcal{O}\left(\frac{1}{\tau^{13}}\right),
\]

\[
\frac{G}{H_I} = \frac{-7 + \sqrt{21}}{14} \frac{1}{\tau} + \frac{c_\chi}{c_h} \frac{1}{\tau^6} + \mathcal{O}\left(\frac{1}{\tau^{13}}\right), \tag{A10}
\]

\[
\log \left(\frac{a}{a_E} H_I\right) = \frac{1 + \sqrt{21}}{10} \log \tau - \frac{1 + \sqrt{21}}{60} \frac{1}{\tau^6} + \mathcal{O}\left(\frac{1}{\tau^{13}}\right),
\]

\[
\log \left(\frac{b}{b_E} H_I\right) = -\frac{3 + \sqrt{21}}{70} \log \tau - \frac{1 + \sqrt{21}}{60} \frac{c_\chi}{c_h} \frac{1}{\tau^6} + \mathcal{O}\left(\frac{1}{\tau^{13}}\right).
\]

Note that the range of \(\tau\) becomes \((c_h \Gamma H_I^6)^{-1} \leq \tau\). Thus, the parameter \(\Gamma H_I^6\) disappears from the background and is absorbed into the lower bound of \(\tau\), which is determined by requiring that the e-folding number \(N_e\) is within the range of \(60 < N_e < 70\). Below we use Equation (A10) as the background and the range of \(\tau\) is given by \(\tau_1 < \tau\). The explicit value of \(\tau_1\) is irrelevant in this paper but should be determined by the e-folding number\(^4\). From Equation (35), \(\eta\) is also bounded as

\[
\frac{3 + \sqrt{21}}{6} \frac{1}{\tau^{6 \sigma}} \leq a_E H_I \eta \equiv \bar{\eta}. \tag{A11}
\]

Here, we introduced the dimensionless parameter \(\bar{\eta}\), and \(\tau = 1.0\) corresponds to \(\bar{\eta} = \frac{3 + \sqrt{21}}{6} \sim 1.3\). Below, we assume that the expansions in Equation (A10) are convergent around \(\tau \sim 0, \bar{\eta} \sim 0\) and coefficients of higher-order terms in Equation (A10) are small and negligible around \(\tau \sim 0.50, \bar{\eta} \sim 0.93\)\(^5\). A plot of the comoving Hubble radius \(\frac{a_E H_I}{a H}\) as a function of \(\bar{\eta}\) is shown in Figure A1. From this figure, we see that the inflation ends around \(\tau \sim 0.7, \bar{\eta} \sim 1.1\).

![Figure A1. Plot of the comoving Hubble radius \(\frac{a_E H_I}{a H}\) as a function of \(\bar{\eta}\).](image)
First, let us evaluate the power spectrum of the tensor perturbation. In Section 4, we normalized the tensor perturbation around the vacuum in the future infinity, $\eta \to \infty$. In addition, the quantization of the tensor perturbation becomes

$$u(\eta, x) = \int \frac{d^3k}{(2\pi)^3} (u(\eta, k)a_k e^{i k \cdot x} + u(\eta, k)^* a_k^* e^{-i k \cdot x}),$$

(A12)

$$u(\eta, k) = u_0 + \Gamma u_1.$$ 

$u_0$ is given by Equation (39) and $u_1$ is written by Equation (43). However, the power spectrum should be evaluated by using the vacuum in the past infinity, so we need the Fourier component, which we denote $u_{-\infty}(\eta, k)$, solved in the past infinity. Although the quantization of the tensor perturbation around the vacuum in the past infinity is difficult due to the complicated interaction, $u_{-\infty}(\eta, k)$ and $u(\eta, k)$ are related by the Bogoliubov transformation as

$$u_{-\infty}(\eta, k) = a(k)u(\eta, k) + \beta(k)u(\eta, k)^*,$$

(A13)

with some unknown functions of $a(k)$ and $\beta(k)$. Then, an ingredient of the power spectrum is evaluated as

$$|u_{-\infty}(\eta, k)|^2 = (|a|^2 + |\beta|^2)|u(\eta, k)|^2\left\{1 + \frac{2|a||\beta|}{|a|^2 + |\beta|^2} \cos(\theta_a - \theta_\beta + 2\theta_u)\right\},$$

(A14)

where $\theta_a, \theta_\beta$ and $\theta_u$ are arguments of $a, \beta$ and $u_{-\infty}$, respectively. The second term in the big parentheses is less than 1. In the above expression, the $\eta$ dependence appears through $u(\eta, k)$ and $\theta_u$. So, if $\theta_u$ is almost time independent, the ingredient of the power spectrum is approximated as

$$|u_{-\infty}(\eta, k)|^2 \sim C(k)|u(\eta, k)|^2,$$

(A15)

with some unknown function $C(k)$. We show a plot of $\theta_u(\eta, k)$ for small $k$ and justify the above approximation in Appendix C.

From the above argument and using Equations (40) and (46), the power spectrum of the tensor perturbations is approximated as

$$P(\eta, k) = k^3|h_{-\infty}|^2$$

$$\sim C(k)\left|\partial^i h_0^{(2)}(k\eta) - \frac{576(999 - 218\sqrt{21})}{60025 c_k r^6} (u_{11} - i\sqrt{\pi u_{12}})\right|^2,$$

(A16)

where $h_{-\infty} = a_0^{-1/2} u_{-\infty}$ and $\tilde{k} \equiv \frac{k}{a H}\equiv e^{-50} k \frac{1}{k^*}$ is a dimensionless momentum. Since $C(k)$ cannot be determined, we normalize the power spectrum as $\frac{P(\eta, k)}{P(0.9, k)}$. Plots of these functions with $\tilde{k} = e^{-60}, e^{-50}$ and $e^{-40}$ are shown in Figure A2. Naively, the horizon exit occurs at $k = aH$, but the corrections should modify this relation. Thus, we define the spectral index $n_t$ by evaluating the power spectrum at $\tilde{\eta} = 3.0$, where the scale factor behaves like a radiation-dominated era. A function of $\log \frac{P(3, 0, k)}{P(0.9, k)}$ is plotted in Figure A3. If we fit the curve, we obtain

$$\log \frac{P(3, 0, k)}{P(0.9, k)} = -5.0 - 0.00083 \log \frac{k}{k_*} - 0.000017 \left(\log \frac{k}{k_*}\right)^2 - 3.4 \times 10^{-7} \left(\log \frac{k}{k_*}\right)^3.$$  

(A17)

From this expression, we see that the tensor spectral index becomes $n_t = -0.00083$ and its runnings are quite small. Thus, if the power spectrum is independent of $k$ at the beginning of the inflation, it is almost scale-independent after the inflation. Note that if we evaluate $n_t$ at $\tilde{\eta} = 2.0$, it becomes closer to zero.
Figure A2. Plots of $\log \frac{P_t(\bar{\eta}, k)}{P_t(0.9, k)}$ with $\bar{k} = e^{-60}$ (blue), $e^{-50}$ (yellow) and $e^{-40}$ (green).

Figure A3. Plots of $\log \frac{P_t(3.0, \bar{k})}{P_t(0.9, k)}$ as a function of $\log \frac{k}{k_*}$.

Next, by applying the similar argument for Equation (A15), it is possible to relate the scalar perturbation $U_{-\infty}$ defined in the past infinity to $U = U_0 + \Gamma U_1$ in the future infinity. $U_0$ and $U_1$ are given by Equations (A5) and (A7), respectively.

$$|U_{-\infty}(\eta, k)|^2 \sim \tilde{C}(k)|U(\eta, k)|^2.$$  \hspace{1cm} (A18)

Then, by using Equations (A5) and (A7), the power spectrum of the scalar perturbations is expressed as

$$P_s(\bar{\eta}, k) = E^2 |\phi_{-\infty}|^2 \sim \tilde{C}(k) |\bar{c}|^2 |\left( 1 - \frac{1}{\tau^2} \right) H_0^2 (k\eta) - \frac{288(20727 - 4523\sqrt{21})}{300125\eta^9 \tau^7} \left( U_{11}^2 - \left( \frac{41 - 9\sqrt{21}\sqrt{\pi}}{10} \sqrt{\pi} U_{12} \right) \right) |^2.$$  \hspace{1cm} (A19)

where $\phi_{-\infty} = \frac{\sqrt{21}}{U_{-\infty}}$. Since the dimensional parameter $\tilde{C}(k)$ cannot be determined, we normalize the power spectrum as $\frac{P_s(\bar{\eta}, k)}{P_s(0.9, k)}$. Plots of these functions with $k = e^{-60}, e^{-50}$ and $e^{-40}$ are shown in Figure A4. In addition, a function of $\log \frac{P_s(3.0, k)}{P_s(0.9, k)}$ is plotted in Figure A5. If we fit the data, we obtain

$$\log \frac{P_s(3.0, k)}{P_s(0.9, k)} = -13.4 - 0.00080 \log \frac{k}{k_*} - 0.000016 \left( \log \frac{k}{k_*} \right)^2 - 3.3 \times 10^{-7} \left( \log \frac{k}{k_*} \right)^3.$$  \hspace{1cm} (A20)
From this expression, we see that the spectral index becomes \( n_s = 0.99920 \) and its runnings are quite small. Again, if the power spectrum is independent of \( k \) at the beginning of the inflation, it is almost scale-independent after the inflation. Note that if we evaluate \( n_s \) at \( \bar{\eta} = 2.0 \), it becomes closer to one.

The tensor-to-scalar ratio \( r \) is obtained by the power spectra of scalar and tensor perturbations. If we define \( r \) at \( \bar{k} = e^{-50} (k = k_*) \), we obtain

\[
\frac{P_t(3.0,e^{-50})}{P_s(3.0,e^{-50})} = \frac{P_t(3.0,e^{-50}) P_s(0.9,e^{-50})}{P_s(3.0,e^{-50}) P_s(0.9,e^{-50})} = e^{8.4} \frac{P_t(0.9,e^{-50})}{P_s(0.9,e^{-50})}. \tag{A21}
\]

Here, we used \( \log P_t(3.0,e^{-50}) = -5.0 \) and \( \log P_s(3.0,e^{-50}) = -13.4 \). Since \( r < 0.06 \), we need that the power spectrum of the tensor perturbation is much smaller than that of the scalar perturbation at the beginning of the inflation [8]. The mechanism of this process is not clear so far, and the knowledge of the behaviors of perturbations around \( \bar{\eta} \sim \bar{\eta} \) should be quite important. In this section, we assumed that power spectra \( P_t(0.9,\bar{k}) \) and \( P_s(0.9,\bar{k}) \) are scale-invariant, but it will be interesting if we can restrict these values from the observation.

Appendix C. Plots of \( \theta_u \) and \( \theta_U \)

\( \theta_u \) is the argument of \( u = u_0 + \Gamma u_1 \). \( u_0 \) and \( u_1 \) are given by Equations (39) and (43), respectively. We also prescribe the rescaling of (A8) and (A9). Then, the plots of \( \theta_u(\bar{\eta},\bar{k}) \) —
\[\theta_u(0.9, \bar{k})\] with \(\bar{k} = e^{-60}, e^{-50}\) and \(e^{-40}\) are shown in Figure A6. From these plots, we see that \(\theta_u\) is almost time-independent for \(e^{-60} < \bar{k} < e^{-40}\).

\[\theta_u(\eta, \bar{k}) - \theta_u(0.9, \bar{k})\]

![Figure A6](image)

\(\theta_{UJ}\) is the argument of \(U = U_0 + \Gamma U_1\). \(U_0\) and \(U_1\) are given by Equations (A5) and (A7), respectively. We also prescribe the rescaling of (A8) and (A9). Then, the plots of \(\theta_{UJ}(\eta, \bar{k}) - \theta_{UJ}(0.9, \bar{k})\) with \(\bar{k} = e^{-60}, e^{-50}\) and \(e^{-40}\) are shown in Figure A7. From these plots, we see that, except around \(\bar{k} = 1.5\), \(\theta_{UJ}\) is almost time-independent for \(e^{-60} < \bar{k} < e^{-40}\).

\[\theta_{UJ}(\eta, \bar{k}) - \theta_{UJ}(0.9, \bar{k})\]

![Figure A7](image)

**Notes**

1. There are many works and reviews in this area. For example, see [9–16].

2. It is possible to make the Riemann tensor to a Weyl tensor by using field redefinition ambiguity. A more general case is discussed in [35].

3. The action contains linear term on 3-form field \(A\) which is proportional to

\[e e_{11} f_{11} A_{f_{1}f_{2}f_{3}} (R_{abf_{4}f_{5}}R_{abf_{6}f_{7}}R_{cdf_{8}f_{9}}R_{df_{10}f_{11}} - 4R_{abf_{4}f_{5}}R_{bcf_{6}f_{7}}R_{cdf_{8}f_{9}}R_{daf_{10}f_{11}}).\]

(3)

This gives a potential source term for the 3-form field. However, if we choose the ansatz (7), nonzero components of the Riemann tensor are written by \(R_{abab}\) (without contraction) and the above source term becomes zero.

4. For example, \(N_t = 69\) corresponds to \(\tau_t \sim 0.34\) (\(\bar{\eta} \sim 0.78\)) [36]. Higher derivative terms which are not considered in this paper will also affect the explicit value of \(\tau_t\).

5. Some thoughts on this point are given in the appendix in [36].
A sum of physical degrees of freedom $a = +, \times$ are included in $\epsilon_1$. As discussed in the Section 2, we choose $\frac{\epsilon^5 a(t_0)/H(t_0)}{a^2 H} = e^{-55}$ and define a pivot scale as $k_* \equiv \epsilon^5 a(t_0)/H(t_0) = e^{-50}aH$.

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