Intermittency in the large $N$-limit of a spherical shell model for turbulence

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Abstract

A spherical shell model for turbulence, obtained by coupling $N$ replicas of the Gledzer, Okhitani and Yamada shell model, is considered. Conservation of energy and of an helicity-like invariant is imposed in the inviscid limit. In the $N \to \infty$ limit this model is analytically soluble and is remarkably similar to the random coupling model version of shell dynamics. We have studied numerically the convergence of the scaling exponents toward the value predicted by Kolmogorov theory (K41). We have found that the rate of convergence to the K41 solution is linear in $1/N$. The restoring of Kolmogorov law has been related to the behaviour of the probability distribution functions of the instantaneous scaling exponent.

Key words: Fully developed turbulence, intermittency, Kolmogorov scaling laws.
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The small-scale statistics of a fluid in a regime of fully-developed turbulence is an interesting and still open question. Among the many problems which have to be faced there is the understanding of anomalous scaling of structure functions $S_p(r)$ defined as:

$$S_p(r) \equiv \langle |\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})|^p \rangle = \langle |\delta_{\mathbf{r}} \mathbf{v}|^p \rangle \sim r^{\zeta_p} \quad r = |\mathbf{r}|,$$

for $r$ in the inertial range, i.e., for those scales, much smaller than the integral scale and much larger than the viscous scales, where inertial non-linear terms of Navier-Stokes (N-S) equation are dominant.

The phenomenological theory of Kolmogorov (K41) is based on the hypothesis of an energy cascade from large to small spatial scales, local in scale size, in which all statistical information about large scale is lost, save for the mean energy-dissipation rate. It leads, by simple dimensional arguments, to the celebrated K41 law: $\zeta_p^{K41} = p/3$. This result is exact for $p = 3$ and in good agreement with the experimentally and numerically measured exponent for $p = 2$ - the energy spectrum scaling $E(k) \sim k^{-5/3}$ is very well fitted - but there are strong evidences of its breakdown for larger $p$.

Phenomenological theories that modify the Kolmogorov approach assigning a key role to the statistics of the spatial distribution of energy dissipation have been introduced in the last thirty years. These approaches are able to fit with good accuracy the intermittent deviations from the $p/3$ law but do not have any direct link with the dynamical evolution of N-S equations. On the other hand, the goal of analytically obtaining the corrections to K41 exponents directly from N-S equations is far from being achieved. Until now, neither in closure theories nor by a renormalization group approach have any satisfying results been obtained on this problem.

A very interesting approach was proposed by Kraichnan in 1961. In the so-called random coupling model (RCM) he replaced the N-S equation by a set of modified equations having the same basic structural properties of N-S: quadratic non-linearity, non-linear quadratic invariants, existence of truncated inviscid equipartition solutions. The velocity field is replaced by $N$ fictitious random fields $\mathbf{u}^\alpha(\mathbf{k}, t)$ satisfying the equations:

$$\left(\frac{d}{dt} + \nu k^2\right) u_i^\alpha(\mathbf{k}, t) = -\frac{i}{N} k_m P_{ij}(\mathbf{k}) \int_{\mathbf{p} + \mathbf{q} = \mathbf{k}} \Phi_{\alpha\beta\sigma} u_j^\beta(\mathbf{p}, t) u_m^\sigma(\mathbf{q}, t) d\mathbf{p}$$

where $\Phi_{\alpha\beta\sigma}$ are quenched random phases subject to suitable restrictions (see [4]). The closure problem can be solved in the $N \to \infty$ limit and gives the...
same equations of direct-interaction approximation (DIA), a two-point closure theory introduced earlier by Kraichnan himself [3]. Recently Mou and Weichman [6] introduced an alternative large-N model, the spherical model (SPM), by taking non-random $\Phi_{\alpha\beta\sigma}$ coefficients and imposing additional symmetries on the system. Their idea is basically that of $1/N$ expansion of critical phenomena: one generalizes the model under consideration to one with an higher symmetry $G_N$ (in standard application, for example, one considers the rotation group $O(N)$, with physical values $N = 1, 2, 3$); the $N \to \infty$ limit is analytically soluble and a systematic expansion in powers of $1/N$ may be developed [7]. Mou and Weichman, following the work of Amit and Roginsky [8], chose the group $G_N = O(3)$ for all $N$ but allowed the dimension of the representation to diverge with $N$. The remarkable point is that the SPM and the RCM are similar: in both the infinite $N$ limit leads to the DIA equations [10]. Both models, then, are plagued with the same problem: DIA equations violate the basic physical principle of galilean invariance and give, consequently, the wrong energy spectrum scaling, $E(k) \sim k^{-3/2}$. The correct $-5/3$ exponent has been recovered, in fact, in a Lagrangian modification of DIA in which galilean invariance is restored (for an overview about these theories see [9]). So it would be ill advised to try to get anomalous corrections of scaling exponents starting from the wrong zeroth order and taking successive orders in which the same problems occur. However, the basic idea of getting the intermittent corrections as asymptotic expansions in powers of $1/\sqrt{N}$ [3]-[10], i.e.

$$\delta \zeta_p \equiv \zeta_{K^{41}} - \zeta_p = \sum_{k>1} \frac{c_k^{(p)}}{Nk/2}$$

(3)

can be tested on simple models in which these inconsistencies do not show up like, for example, the shell models, as Eyink has recently proposed [10].

Shell models are dynamical systems in which, exploiting the idea that active degrees of freedom in a turbulent flow are hierarchically organized, a reduced number of variables (one or two) are considered for each shell of wavevectors with modulus $k \in (k_n, k_{n+1}]$, where $k_n = k_0 2^n$. Among shell models, the so-called GOY (from Gledzer, Ohkitani and Yamada) [11]-[12] is of considerable recent interest. The model is defined by the following ODE’s:

$$\frac{d}{dt} u_n = -\nu k_n^2 u_n + ik_n \left[ u_{n+1} u_{n+2} - \frac{\varepsilon}{2} u_{n-1} u_{n+1} - \frac{1 - \varepsilon}{4} u_{n-1} u_{n-2} \right] + f_n \delta_{n,1}$$

(4)
with \( n = 1, ..., N_{\text{shells}} \) and boundary conditions with null \( u_{-1}, u_0, u_{N_{\text{shells}}+1} \) and \( u_{N_{\text{shells}}+2} \). The coefficients of the non-linear terms have been chosen in order to keep the energy constant in the inviscid unforced case; besides, for \( \varepsilon = 1/2 \), one has an inviscid invariant \( H = \sum_n (-1)^n k_n |u_n|^2 \) that is a sort of “shell model helicity” similar to the helicity \( H = \int (k \times \mathbf{v}(k)) \cdot \mathbf{v}(k) dk \) conserved in the 3d Euler equations. Considering that \( u_n \) should describe velocity fluctuations at scale \( r = 1/k_n \) the structure functions are defined by:

\[
S^n_p \equiv \langle |u_n|^p \rangle \tag{5}
\]

and show, in the inertial range, a scaling behaviour: \( S^n_p \sim k_n^{-\zeta_p} \). This model has the remarkable property of reproducing quantitatively the exponents \( \zeta_p \) experimentally measured when \( \varepsilon = 1/2 \) \cite{13,14}.

In \cite{10} Eyink defined and studied the infinite \( N \) limit of the following complex spherical shell models (SSM):

\[
\left( \frac{d}{dt} + \nu k_n^2 \right) u_n^\alpha = \sum_{ml,\beta\gamma} A_{nml} W_N^\alpha\beta\gamma (u_m^\beta u_n^\gamma)^* + f_n^\alpha \tag{6}
\]

with \( A_{nml} \) local couplings, i.e. vanishing outside a finite range of neighbors and \( W_N^\alpha\beta\gamma \) given by \cite{11}. This kind of models reduce to real shell model when \( N = 1 \) and the DIA equations obtained in the infinite \( N \) limit have stationary solutions with K41 scaling when a fixed input of energy by the external force is imposed.

We considered a slightly modified version that gives the original GOY model at \( N = 1 \). We generalized the dynamical equations for the real and imaginary parts of the complex variable \( u_n = x_n + iy_n \) to the following dynamical system:

\[
\left( \frac{d}{dt} + \nu k_n^2 \right) x_n^\alpha = k_n W_N^\alpha\beta\gamma [x_{n+1}^\beta y_n^\gamma + y_{n+1}^\beta x_n^\gamma - \frac{\varepsilon}{2} (x_{n-1}^\beta y_n^\gamma + y_{n-1}^\beta x_n^\gamma)] - \frac{(1-\varepsilon)}{4} (x_{n-1}^\beta y_n^\gamma + y_{n-1}^\beta x_n^\gamma)]^* + f_n^\alpha \delta_{n1} \tag{7}
\]

\[
\left( \frac{d}{dt} + \nu k_n^2 \right) y_n^\alpha = k_n W_N^\alpha\beta\gamma [x_{n+1}^\beta x_n^\gamma - y_{n+1}^\beta y_n^\gamma - \frac{\varepsilon}{2} (x_{n-1}^\beta x_n^\gamma - y_{n-1}^\beta y_n^\gamma)] - \frac{(1-\varepsilon)}{4} (x_{n-1}^\beta x_n^\gamma - y_{n-1}^\beta y_n^\gamma)]^* + f_n^\alpha \delta_{n1} \tag{8}
\]
where sums over repeated indices are implied; $\alpha = -J, \ldots, J$; $N = 2J + 1$ and $n = 1, \ldots, N_{\text{shells}}$. To get the original model in the limit $J \to 0$ we have taken $x_\alpha^n$ and $y_\alpha^n$ complex variables satisfying $x^{-\alpha}_n = (-1)^\alpha (x_\alpha^n)^*$ and $y^{-\alpha}_n = (-1)^\alpha (y_\alpha^n)^*$ and $f^{-\alpha} = (-1)^\alpha (f_\alpha)^*$. So there are $2N = 2(2J + 1)$ real degrees of freedom per each shell. Following [10] we imposed the invariance of the system under the transformation:

$$v_\alpha^n \rightarrow D_{\alpha\beta}^N(U)v_\beta^n,$$

where the $D_{\alpha\beta}^N(U)$ are the Wigner $D^J$ matrices of quantum mechanics ($N$-dimensional irreducible representations of the group $SU(2)$) and $v_\alpha^n = (x_\alpha^n, y_\alpha^n)$. So one must take:

$$W_{\alpha\beta\gamma}^N = (-1)^\alpha < JJJ - \alpha |JJ\beta\gamma >$$

which are the Clebsh-Gordan coefficients. The properties of these coefficients, together with the fact that we considered only even $J$'s, guarantee:

(i) invariance of the system under any permutation of the replica indices;
(ii) conservation of volume in phase-space; (iii) conservation, in the inviscid unforced case, of energy:

$$E = \frac{1}{2} \sum_{n=1}^{N} \sum_{\alpha = -J}^{+J} |x_\alpha^n|^2 + |y_\alpha^n|^2$$

and of shell-model helicity:

$$H = \frac{1}{2} \sum_{n=1}^{N} \sum_{\alpha = -J}^{+J} (-1)^n k_n [ |x_\alpha^n|^2 + |y_\alpha^n|^2 ].$$
precision is needed to calculate deviations from K41 solution we considered the scaling properties of the following suitable triple products:

\[
\Pi_n(N) = -\mathcal{R} \left( \Delta_{n+1}(N) + \frac{1-\varepsilon}{2} \Delta_n(N) \right)
\] (13)

where:

\[
\Delta_n(N) = \sum_{\alpha\beta\gamma} W^\alpha_{N} \left[ x^\alpha_n \left( x^\beta_{n+1} y^\gamma_{n-1} + y^\beta_{n+1} x^\gamma_{n-1} \right) + y^\alpha_n \left( x^\beta_{n+1} x^\gamma_{n-1} - y^\beta_{n+1} y^\gamma_{n-1} \right) \right]
\] (14)

The variables \(\Pi_n\) have a physical relevance, \(\Pi_n k_n\) representing the energy flux from the scale \(n\) to larger wavenumber scales, and are not plagued with periodic oscillations (see fig. 1). We make use of a relation analogous to the structure equation of Kolomogorov \([15]\) as in \([16]\) and so the following scaling law is expected:

\[
\Sigma^n_p(N) = \left\langle |\Pi_n(N)|^{p/3} \right\rangle \sim k_n^{-\zeta_p(N)}
\] (15)

In figure 2 we plotted the behaviour of \(\delta \zeta_p(N) = \zeta_p^{K41} - \zeta_p(N)\) as function of \(1/N\) for different values of \(p\). The corrections to the K41 solution tend to zero when \(N \to \infty\) and the rate of convergence is linear in \(1/N\). It should then be possible to develop an asymptotic expansion for the \(\delta \zeta_p\)'s of the form \([8]\) as Eyink suggested for the shell models in \([10]\).

In our numerical simulations we used \(N_{\text{shells}} = 15\) and 19, with \(\nu = 10^{-5}\) and \(10^{-6}\), \(f^a = 5 \times 10^{-3}(1+i)\), \(k_0 = 6.25 \times 10^{-2}\) and a number of replicas up to 21. Equations have been integrated by a slaved Adams-Bashforth algorithm and temporal averages are over several thousands of eddy turn-over times of the first shell.

Beyond these numerical results an interesting physical problem is to understand the dynamical mechanism by which K41 is restored. As Kraichnan pointed out in \([17]\), the K41 law is not ruled out just because energy-dissipation fluctuates: it could be \textit{a priori} possible if there were spacewise diffusion effects of sufficient strength to suppress fluctuations of energy transfer in the inertial range. We suppose that a similar mechanism is present in the SSM, although it is not completely clear to us to what extent the similarity can be pushed. In the GOY model the anomalous scaling law is accompanied by an intermittent behaviour of the energy transfer in the inertial range with very large fluctuations. A quantity that measure the local
(in time) singularity is the instantaneous scaling exponent for fluxes, defined as:

\[ h(t) = \frac{1}{n_{\text{max}} - n_{\text{min}} + 1} \sum_{n_{\text{min}}}^{n_{\text{max}}} \ln_2 \left| \frac{\Pi_n}{\Pi_{n+1}} \right| \]

where \( n_{\text{min}} \) and \( n_{\text{max}} \) delimit the inertial range. In a laminar regime one has \( h = 3 \), \( h = 1 \) corresponds to a K41 scaling while a value of \( h < 1 \) represents a more singular velocity field. Such a value of \( h \) is realized during the fast energy bursts \(^8\). We measured \( h(t) \) and calculated its PDF for different values of \( N \). We observed that increasing the number of coupled replicas the peak of the distribution gets sharper and the minimum value of \( h \) increases, indicating that the field is getting less singular in the inertial range (see fig. 3). This result could be due to possible exchanges of energy between the increasing number of degrees of freedom at the same scale.

To conclude we would like to stress that the SSM is the only model we know in which it has been shown that there is approach to the K41 solution by tuning a parameter (the number of replicas) starting from an intermittent solution. Let us remark, anyway, that in a different contest, that of passive scalar advection, two groups have been succeeded in calculate anomalous scaling exponents by expansion around limiting gaussian cases \(^{13,20}\). At now nothing similar has been done with the N-S equation also if it is believed that an infinite-dimension limit should converge to K41. Beyond the possibility of getting an asymptotic expansion for the anomalous corrections of the form (3) we believe that further work in this direction should deepen the comprehension of intermittency in real turbulence.

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FIGURE CAPTIONS

• FIGURE 1:
  Structure function $\Sigma_{p=6}^{n}(N)$ vs $k_n$ in a log-log plot for $N = 5$ and $N_{shells} = 19$ (crosses with line). We plotted also the scaling behaviour of the 6-th moment $\langle |u_n^{\alpha}\rangle|^{p=6}\rangle$ for $\alpha = 1$ (diamonds).

• FIGURE 2:
  Anomalous corrections $\delta\zeta_p(N)$ versus $1/N$ for $p = 5, ..., 11$. The error bars have been obtained by a least square fit. Note that the origin is an analytically calculated point.

• FIGURE 3:
  Probability distribution function of the instantaneous scaling exponents $h(t)$ of equation (16) for different values of $N$. 
\[ \ln(\Sigma_p^n(N)), \ln(|u_n^p|^p) \]
$$\delta \zeta_p(N)$$

FIG. 2
FIG. 3