Exercises in simplest dynamical random walk,
or Quantum path integral approach to true diffusion law
and 1/f noise of classical particle interacting with ideal gas

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Abstract
Statistics of classical Hamiltonian random walk of particle colliding with atoms of ideal gas is considered from viewpoint of earlier suggested exact pseudo-quantum path integral representation of the problem, and qualitative agreement is demonstrated between results of an naturally arising simple approximation of this integral and results obtained by formally different methods, thus in a new fashion showing inevitability of scaleless 1/f-type fluctuations in rates of molecular Brownian motions and other dynamical transport and evolution processes.

Key words: Brownian motion, diffusion, molecular random walks, fundamental 1/f-noise, kinetic theory of gases, dynamical foundations of kinetics, statistics of transport and irreversible processes

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1. Introduction
In this paper we turn once again to intriguing questions about statistics of walk of a classical particle interacting with atoms of thermodynamically equilibrium classical ideal gas. This subject already was discussed by us from different points of view in [1–5] as a principal part of more general problem, - considered by us in [6]-[33], - of molecular “Brownian” motion in fluids, molecular and electronic transport processes in various many-particle dynamical systems, and related fundamental 1/f noises 1. In [1] we suggested an original exact representation of characteristic function and thus probability distribution of displacement (path) of the “Brownian particle” in terms of effective quantum transition amplitudes or equivalent holomorphic path integrals (on them see e.g. [34,35]). Now, we shall discern these mathematical objects more carefully and try some their analytical approximations which visually reveal their main physical contents. Namely, the fact that the “Brownian particle” (BP) has no definite diffusion and drift rates (diffusivity and mobility) but walks as if these quantities undergo 1/f-type low-frequency fluctuations.

Generally, all transport processes in many-particle dynamical (Hamiltonian) systems are statistical analogues of Brownian motion and all also have no certain (a priori predictable time-averaged) rate. In essence,

1 Short partial reviews of corresponding ideas and results can be found in [6,7,9,12,18,21,25,28,33] (see also introductory and discussion sections in [8,22,24,27,29,30,32]).
this was foreseen by N. Krylov as long ago as at end of the forties [36]. In the eighties, this truth was realized independently [6,9–13] along with its direct relation to the widely observed 1/f-noise (“flicker noise”). Unfortunately, today’s priests of statistical mechanics still stay constrained by prejudices which took beginning in the works by L. Boltzmann on gas kinetics and were hallowed by his authority although later disclosed in [36]. Such the situation notably damages theoreticians’ “human rights” in statistical physics and enforces applied science and engineering to search for 1/f-noise sources in particular physical objects, - like e.g. “charge traps” or some (usually mysterious) “two-level fluctuators”, - instead of general statistical properties of transport processes produced by deterministic Hamiltonian dynamics.

This explains our interest in gases and even so “simple” system as a particle in ideal gas: clearly, if these systems create 1/f-noise then its existence does not require “traps”, “slow fluctuators” and any “long memory times” at all. Of course, charge traps and two-level defect structures in solids may take part in formation of 1/f-noise but not necessarily as its sources. The true universal origin of 1/f-noise is that complex enough (“mixing”) many-particle dynamics constantly forgets its past and therefore does not control relative frequencies of elementary kinetic events (various “collisions”, “scatterings”, “tunnel transitions”, “trappings” and “de-trappings”, “reactions”, etc.). Hence, to assume for them a priori settled “probabilities per unit time” means to lose a good few of actual chances, namely, lose 1/f-noise. That is just what Boltzmann-like kinetic models always do 2.

By these reasons, unprejudiced investigations of actual gas kinetics, - based on exact evolution equations for one-, two-, three- and all other many-particle probability distribution functions, - is of key importance for kinetics as the whole including phenomenon of 1/f-noise. Since the latter by its physical nature manifests memoryless features of underlying dynamics, its formal extraction from exact equations of dynamics is very fine mathematical operation. Therefore any its correct enough intuitively transparent simplification would be very useful. Below, we shall make some steps in this direction, at that using definitions, designations and formulae from [1] (see also [20] or [21]).

2. Pseudo-quantum representations of diffusion law

1. In comparison with [1], here, - like in [4,5], - we shall consider more general case of possibly non-zero external force \( f \) applied to the “Brownian” particle (BP) after initial time moment \( t = 0 \). At \( t = 0 \) the BP is settled to start from point \( R(0) = 0 \) possessing Maxwellian velocity distribution and being surrounded by equilibrium (ideal) gas.

Now, all the correlation functions (CF) \( V_n \) acquire additional argument \( f \), and evolution operator in the exact evolution equation

\[
\frac{\partial \mathcal{V}(t)}{\partial t} = \hat{L} \mathcal{V}(t)
\]

for their generating functional \( \mathcal{V}(t) = \mathcal{V}(t, ik, P, \psi; \nu, f) \) looks as

\[
\hat{L} = i k \cdot \mathbf{V} - f \cdot \frac{\partial}{\partial P} + \int \psi(x) \left[ (\mathbf{V} - \mathbf{v}) \cdot \frac{\partial}{\partial \rho} + \Phi'(\rho) \cdot \left( \frac{\partial}{\partial \rho} - \frac{\partial}{\partial P} \right) \right] \frac{\delta}{\delta \psi(x)} - \frac{\partial}{\partial P} \int \Phi'(\rho) \frac{\delta}{\delta \psi(x)} + \nu \int G_m(p) E'(\rho) \psi(x) \cdot (\mathbf{V} + \mathbf{T} \frac{\partial}{\partial P})
\]

or, equivalently,

\[
\hat{L} = i k \cdot \mathbf{V} - f \cdot \frac{\partial}{\partial P} + \int [1 + \psi(x)] \left[ (\mathbf{V} - \mathbf{v}) \cdot \frac{\partial}{\partial \rho} + \Phi'(\rho) \cdot \left( \frac{\partial}{\partial \rho} - \frac{\partial}{\partial P} \right) \right] \frac{\delta}{\delta \psi(x)} + \nu G_m(p) E(\rho)
\]

\( \text{At the expense of logical and mathematical self-consistency of theory, as it was especially highlighted in [28] and in last section of [1].} \)
The same as before (equilibrium) initial condition to Eq. 1 is \( V\{t = 0\} = G_M(P) \) (Eq. 30 from [1]). Recall that \( k \) is wave vector conjugated with BP’s coordinate \( R \), \( \nu \) is mean gas density, \( x = \{p, p\} \) enumerate phase space points of gas atoms, - with \( \rho = r - R \) denoting distance of an atom from BP, - \( \Phi(\rho) \) is BP-atom interaction potential, \( E(\rho) = \exp[-\Phi(\rho)/T] \), and \( G_m(p) \) and \( G_M(P) \) are Maxwellian distributions for momenta of gas atoms and BP, respectively, with \( p = mv \) and \( P = MV \) (please, see [1] for details).

Introducing, as in [1], auxiliary boson creation and annihilation operators:

\[
A^\dagger \equiv -\sqrt{T/M} \frac{\partial}{\partial P} , \quad A \equiv \sqrt{\frac{M}{T}} \left( V + T \frac{\partial}{\partial P} \right) , \quad A_\alpha A^\dagger_\beta - A^\dagger_\beta A_\alpha = \delta_{\alpha\beta} ,
\]

\[
a^\dagger(x) = c(x) \psi(x) , \quad a(x) = c^{-1}(x) \frac{\delta}{\delta \psi(x)} , \quad a(x) a^\dagger(y) - a^\dagger(y) a(x) = \delta(x - y) ,
\]

with \( c(x) = \sqrt{\nu G_m(P) E(\rho)} \), and operators

\[
\hat{L}_1 = \hat{L}_1(ik, f) = u_0 ik \cdot A + u_0 \left( ik + \frac{f}{T} \right) \cdot A^\dagger ,
\]

\[
\hat{L}_2 = \int a^\dagger(x) \left[ -v \cdot \frac{\partial}{\partial \rho} + \Phi'(\rho) \cdot \frac{\partial}{\partial P} \right] a(x) + u_0 \int c(x) \frac{\Phi'(\rho)}{T} \cdot \left[ a(x) A^\dagger - A^\dagger(x) A \right] ,
\]

\[
\hat{L}_3 = u_0 \int a^\dagger(x) \left[ (A^\dagger + A) \cdot \frac{\partial}{\partial \rho} + (A^\dagger - A) \cdot \frac{\Phi'(\rho)}{2T} \right] a(x) ,
\]

\[
\hat{L}(ik, f) = \hat{L}_1(ik, f) + \hat{L}_2 + \hat{L}_3 ,
\]

with \( u_0 = \sqrt{T/M} \), we represent the characteristic function of BP’s displacement (path) in the form of “vacuum average”:

\[
V_0(t, ik; \nu, f) = \langle e^{ik \cdot [R(t) - R(0)]} \rangle = \langle 0 | e^{ \hat{L}(ik, f) } | 0 \rangle
\]

At that, the vacuum state corresponds to complete statistical equilibrium in \( \{ \rho, p, P \} \)-space (absence of any BP-gas statistical correlations except purely equilibrium ones).

2. We may allow the wave vector \( k \), as well as the external force, to be arbitrary probe function of time, \( k = k(t) \). Then instead of (9) we have to write

\[
V_0(t) = \langle \exp \left[ \int_0^t ik(t') V(t') dt' \right] \rangle = \langle 0 | \exp \left[ \int_0^t \hat{L}(ik(t'), f(t')) dt' \right] | 0 \rangle
\]

3. Path integral representations of diffusion law and its stochastic formulation

1. Reducing standardly [34,35] the “vacuum-vacuum quantum transitions amplitudes” (9) and (10) to the holomorphic, or coherent-state, path integrals, one has

\[
V_0(t) = \int \exp \left\{ \int_0^t \left[ \frac{1}{2} (A^\ast A - A^\ast A) + \hat{L}(ik, f, A^\ast, A) \right] dt \right\} \prod_{\xi} \frac{dA^\ast dA}{2\pi i} ,
\]

with \( A^\ast = A^\ast(\xi) = \{ a^\ast(x, \xi), A^\ast(\xi) \} \) and \( A = A(\xi) = \{ a(x, \xi), A(\xi) \} \) being complex variables, in place of boson operators, and edge conditions \( A^\ast(\xi = t) = 0 \), \( A(\xi = 0) = 0 \).

2. The exponential in Eq. 11 is quadratic in respect to the gas variables \( \{ a^\ast(x, \xi), a(x, \xi) \} \) as well as BP variables \( \{ A^\ast(x, \xi), A(x, \xi) \} \) separately, therefore integration over one of them can be performed exactly. Integration over gas yields
\[ V_0(t, ik; \nu, f) = \int \exp \left\{ \int_0^t \left[ \frac{1}{2} (\dot{A}^* \cdot A - A^* \cdot A) + ik \cdot A u_0 + (ik + f/T) \cdot A^* u_0 \right] d\xi \right\} \times \]

\[ \times \exp \left[ - \int_{t>\xi_1>\xi_2>0} A_0^*(\xi_1) G_{\alpha\beta}(\xi_1, \xi_2, A^*, A) A_0(\xi_2) d\xi_2 d\xi_1 \right] \prod_{\xi} \frac{dA^* dA}{(2\pi i)^d} = \]

\[ = \int \exp \left\{ \int_0^t \left[ \frac{1}{2} (\dot{A}^* \cdot A - A^* \cdot \dot{A}) + ik \cdot A u_0 + (ik + f/T) \cdot A^* u_0 \right] d\xi \right\} \times \]

\[ \times \exp \left\{ \int c(x) \left( \exp \int_0^t \lambda(A^*, A) d\xi \right) - 1 \right\} c(x) \right\} \prod_{\xi} \frac{dA^* dA}{(2\pi i)^d} , \]

with edge conditions \( A^*(t) = 0 \), \( A(0) = 0 \), summation over repeated indices, and \( \int ... = \int \int ... dp \, d\rho = \int ... dx \). The kernel in Eq.12 is

\[ G_{\alpha\beta}(\xi_1, \xi_2, A^*, A) = \frac{u_0^2}{T^2} \int c(x) \Phi'_\rho(\rho) \exp \left[ \int_{\xi_2}^{\xi_1} \lambda(A^*, A) d\xi \right] \Phi'_\rho(\rho) c(x) , \]

and \( \lambda(A^*, A) \) in Eqs.12-13 means Liouville-like operator

\[ \lambda(A^*, A) = -v \cdot \frac{\partial}{\partial \rho} + \Phi'(\rho) \cdot \frac{\partial}{\partial p} + u_0 \left[ (A^* + A) \cdot \frac{\partial}{\partial \rho} + (A^* - A) \cdot \Phi'(\rho) \right] \]

3. Integration in (11) firstly over BP’s variables \( A \) and \( A^* \) leads to holomorphic path integral

\[ V_0(t, ik; \nu, f) = \int \exp \left\{ \int_0^t \int_0^t \left[ \frac{1}{2} (\dot{a}^* a - a^* \dot{a}) + a^* \lambda_0 a \right] dx d\xi \right\} \times \]

\[ \times \exp \left\{ u_0^2 \int_0^t \int_0^{\xi_1} \left[ ik + K^*(a^*(\xi_1), a(\xi_1)) \right] \cdot \left[ ik + \frac{f}{T} + K(a^*(\xi_2), a(\xi_2)) \right] \right\} \prod_{x, \xi} \frac{da^* da}{2\pi i} , \]

with edge conditions \( a^*(x, \xi = t) = 0 \), \( a(x, \xi = 0) = 0 \), functionals

\[ K^*(a^*, a) = -\int c(x) \frac{\Phi'(\rho)}{T} a(x) dx + \int a^*(x) \left[ \frac{\partial}{\partial \rho} - \frac{\Phi'(\rho)}{2T} \right] a(x) dx , \]

\[ K(a^*, a) = \int c(x) \frac{\Phi'(\rho)}{T} a(x) dx + \int a^*(x) \left[ \frac{\partial}{\partial \rho} + \frac{\Phi'(\rho)}{2T} \right] a(x) dx , \]

and Liouville operator

\[ \lambda_0 = -v \cdot \frac{\partial}{\partial \rho} + \Phi'(\rho) \cdot \frac{\partial}{\partial p} \]

of an atom interacting with BP as if the latter was immovable.

4. Let us introduce, as in Sec.6 in [1], the “seed” Gaussian measure \( M_0\{A^*, A\} \) in space of holomorphic paths by

\[ dM_0\{A^*, A\} = \exp \left\{ \int \frac{1}{2} (\dot{A}^* A - A^* \dot{A}) d\xi \right\} \prod_{\xi} \frac{dA^* dA}{2\pi i} \]
In comparison with [1], we added subscript “0” to underline the “seed” character of this measure. It is completely determined by its characteristic functional

\[
\left\langle \exp \left\{ \int_0^t \left[ b(\xi) \cdot A^*(\xi) + b^*(\xi) \cdot A(\xi) \right] \, d\xi \right\} \right\rangle_0 \equiv \int_0^t \exp \left\{ \int_0^t \left[ b(\xi) \cdot A^*(\xi) + b^*(\xi) \cdot A(\xi) \right] \, d\xi \right\} \, dM_0[A^*, A] = \exp \left[ \int_0^t \int_0^{t'} b^*(t') \cdot b(t'') \right]
\]

or, equivalently, by corresponding pair correlators

\[
\langle A_\alpha(t_1) A_\beta(t_2) \rangle_0 = \langle A^*_\alpha(t_1) A^*_\beta(t_2) \rangle_0 = 0, \quad \langle A_\alpha(t_1) A^*_\beta(t_2) \rangle_0 = \delta_{\alpha,\beta} \Theta(t_1 - t_2),
\]

where \( \Theta(t) \) denotes the Heaviside function.

Thus, for arbitrary functional \( \mathbb{F}(A^*(\xi), A(\xi)) \) we may write

\[
\int \exp \left\{ \int_0^t \frac{1}{2} (A^* A - A^* \dot{A}) \, d\xi + \mathbb{F}(A^*(\xi), A(\xi)) \right\} \prod_\xi \frac{dA^* dA}{2\pi i} = \langle \exp \mathbb{F}(A^*(\xi), A(\xi)) \rangle_0
\]

In particular, consequently,

\[
V_0(t, ik; \nu, f) = \langle 0 | e^{\int_t^0 \hat{L}(A^*, A)} | 0 \rangle = \left\langle \exp \left( \int_0^t \hat{L}(A^*(\xi), A(\xi)) \, d\xi \right) \right\rangle_0
\]

with function \( \hat{L}(A^*, A) \) ("normal symbol" [34,35] of operator \( \hat{L}(A^!, A) \)) defined by (6)-(8), and

\[
V_0(t, ik; \nu, f) = \langle \exp \{ \int_0^t u_0 \left[ ik \cdot A(\xi) + (ik + f/T) \cdot A^*(\xi) \right] \, d\xi - \int_{t_1+\xi_1>0} A^*(\xi_1) \cdot \mathcal{G}(\xi_1, \xi_2, A^*, A) \cdot A(\xi_2) \, d\xi_2 \, d\xi_1 \} \rangle_0
\]

with Gaussian random processes \( A^*(t) \) and \( A(t) \) obeying (18).

5. Notice that formula (13), - and hence (12) and (20), - is in fact exactly the same as, firstly, visually different path integral expressed by Eqs.42-44 from [14], and, secondly, the “dynamical virial expansion” of BP’s diffusion law presented in [5] by Eq.7 or Eq.11 (at \( s = 0 \)) and derived directly from non-equilibrium partition function.

4. The interaction kernel and fixed relaxation rate approximation

1. Obviously, properties of BP’s characteristic function (20) and thus its diffusion law

\[
V_0(t, \Delta R; \nu, f) = \int e^{-ik \cdot \Delta R} V_0(t, ik; \nu, f) \, \frac{d^3k}{(2\pi)^3}
\]

are fully determined by properties of the kernel \( \mathcal{G} \). From \( \mathcal{G} \)’s definition (14)-(15) one can see that it describes relaxation and simultaneously thermal agitation of BP’s momentum and velocity due to BP’s interactions with atoms. Below, for certainty and simplicity, we shall think of these interactions as “collisions”, thus
assuming the potential $\Phi(\rho)$ repulsive and short-range enough (so that atoms can not form “bound states” with BP). Let us discuss the kernel in corresponding terms.

Dependence of the kernel $G\{\xi_1, \xi_2, A^*, A\}$ on BP’s boson variables reflects three physical factors. First of them is that results of any BP-atom collision are sensitive to non-inertial character of BP’s motion during this collision and thus to atom-BP mass ratio $m/M$. In Eq.20 this is described by $A(\xi') - A^*(\xi'')$ correlations “inside the kernel” (at $\xi_1 > \xi' > \xi'' > \xi_2$).

Second factor is that effect of any particular collision depend on degree of current non-equilibrium of the system and thus on pre-history of its evolution. This is described by cross-correlations between the kernel’s inner variables and outer variables what “frame” it $(A(\xi_1)$ and $A^*(\xi_2)$ in Eq.20).

Third factor is that contribution of any particular collision to summary BP’s path $\Delta R(t) = R(t) - R(0)$ interferes with all other, previous and next, collisions (moreover, previous ones determine very occurance of present one). In Eq.20 this is described by inter-correlations of $A(\xi')$ and $A^*(\xi'')$ belonging to different copies of the kernel (for instance, between $A(\xi')$ from $G\{\xi_1, \xi_2, A^*, A\}$ and $A^*(\xi'')$ from $G\{\xi_3, \xi_4, A^*, A\}$ at $\xi_2 > \xi_3$).

It is important to realize that usual (“Boltzmannian”) kinetics deals with the first and partly with the second factors only and loses the third since avoids consideration of actual variety of spatial-temporal clusters of collisions.

2. Clearly, to take into account the first of these factors, it is quite sufficient to approximate the exact functional $A^*, A$-dependent kernel $G$ by its average value, that is to make replacement

$$
G_{\alpha\beta}(\xi_1, \xi_2, A^*, A) \Rightarrow G_{\alpha\beta}(\xi_1 - \xi_2) \equiv \langle G_{\alpha\beta}(\xi_1, \xi_2, A^*, A)\rangle_0 =
$$

$$
= \frac{u_0^2}{T^2} \int c(x) \Phi'_\alpha(\rho) \left\langle \hat{\exp} \left[ \int_{\xi_2}^{\xi_1} \tilde{\Lambda}(A^*(\xi), A(\xi)) d\xi \right] \Phi'_\beta(\rho) c(x) \right\rangle_0
$$

(21)

At that, according to definitions of holomorphic path integration and our related seed averaging $\langle \ldots \rangle_0$ (17), we can turn back to the pseudo-quantum and then original description of BP’s motion and rewrite expression (21) as

$$
G_{\alpha\beta}(\tau) = \left\langle 0 \left| \frac{u_0^2}{T^2} \int c(x) \Phi'_\alpha(\rho) \exp \left[ \tau \tilde{\Lambda}(A^*, A) \right] \Phi'_\beta(\rho) c(x) dx \right| 0 \right\rangle =
$$

$$
= \int dP \tilde{G}_{\alpha\beta}(\tau, V, \nabla P) G_M(P)
$$

(22)

In the latter row we introduced operators

$$
\tilde{G}_{\alpha\beta}(\tau, V, \nabla P) \equiv \frac{\nu}{TM} \int \Phi'_\alpha(\rho) \exp \left[ \tau \tilde{\Lambda}(V, \nabla P) \right] \Phi'_\beta(\rho) g(x) dx ,
$$

$$
\tilde{\Lambda}(V, \nabla P) = (V - v) \cdot \nabla + \tilde{\Phi}'(\rho) \cdot (\nabla P - \nabla P) ,
$$

(23)

with designations $g(x) = G_m(\rho) E(\rho), \ \nabla = \partial/\partial \rho, \ \nabla P = \partial/\partial P, \ \nabla_p = \partial/\partial P$ (we also used definitions of BP’s boson operators (4) and its “vacuum” $|0\rangle = G_M(P), \ |0\rangle = \int dP$).

Combining this kernel’s approximation with exact Eq.20, we come to Gaussian path integral and, - like in Sec.6 of [1], - obtain

$$
V_0(t, ik; \nu, f) \approx \exp \left[ u_0^2 \int_{t > \xi_1 > \xi_2 > 0} ik \cdot Q(\xi_1 - \xi_2) \cdot \left( ik + \frac{f}{T} \right) d\xi_2 d\xi_1 \right]
$$

(24)

with kernel $Q(\xi)$ determined by formulae
\[ Q = \Theta - \Theta \otimes \Theta + \Theta \otimes \Theta \otimes \Theta - \ldots, \]
\[ \int_0^\infty e^{-zt} Q(\tau) d\tau = \left[ z + \int_0^\infty e^{-zt} G(\tau) d\tau \right]^{-1}, \tag{25} \]

and \( \otimes \) symbolizing causal time convolution.

3. Obviously, first, both the matrix functions \( G(\tau) \) and \( Q(\tau) \) are diagonal: \( G_{\alpha\beta} = \delta_{\alpha\beta} G, \) \( Q_{\alpha\beta} = \delta_{\alpha\beta} Q. \)

Second, under our above assumptions about the interaction potential, \( G(\tau) \) is fast decaying function of \( \tau \), vanishing at \( \tau \gtrsim \tau_c \sim r_0/v_0' \), where \( \tau_c \) is typical collision duration, \( r_0 \) is characteristic radius of BP-atom interaction, and \( v_0' = \sqrt{T/m'} \) with BP-atom reduced mass \( m' = mM/(m + M) \).

Hence, if being interested in BP’s observation times much longer than mean BP’s velocity relaxation time, or BP’s mean free path time, \( \tau_0 \), we can make replacements

\[ G(\tau) \Rightarrow \Theta \delta_+(\tau), \quad \int_0^\infty e^{-zt} G(\tau) d\tau \Rightarrow \Theta, \quad \int_0^\infty e^{-zt} Q(\tau) d\tau \Rightarrow \frac{1}{z + \Theta}, \tag{26} \]

\[ \Theta = \int_0^\infty G(\tau) d\tau = \frac{2m}{M + m} \nu \int \int |\mathbf{v} - \mathbf{V}| \Sigma(|\mathbf{v} - \mathbf{V}|) G_m(p) G_M(P) dp dP \]

Here \( \Sigma \) is effective full relative velocity-dependent cross-section of BP-atom collision, \( \Sigma \sim \pi r_0^2 \). Evidently, \( \Theta \) is nothing but mean BP’s velocity relaxation rate. Therefore we may write \( \Theta = 1/\tau_0 \) and say also that \( \Theta \) is characteristic “probability of collision per unit time”.

These simplifications imply

\[ V_0(t, i\mathbf{k}; \nu, f) \approx \exp \left[ u_0^2 \int \int ik e^{-\Theta(\xi_1 - \xi_2)} \left( ik + \frac{f}{T} \right) \right] \approx e^{ikD_0(i\mathbf{k} + f/T)t}, \tag{27} \]

where \( D_0 = u_0^2/\Theta = u_0^2/\tau_0 \) is BP’s diffusivity, and the latter exponential corresponds to time-independent wave (probe) vector and time-independent force.

5. Improved consideration and time-scaleless relaxation rate fluctuations

1. The result (27) of above considered approximation (21) is standard diffusive random process (“Ornstein-Uhlenbeck process”) possessing purely Gaussian statistics with exponential velocity correlation function. Thus, we neglected possibly non-exponential behavior of this function, that is non-linearity of BP’s velocity relaxation. However, from viewpoint of diffusion law, - i.e. BP’s total path probability distribution at \( t \gg \tau_0 \) - this non-linearity is quite insignificant since does not cancel the Gaussian long-range asymptotic of \( V_0(t, \Delta R; \nu, f) \).

   Of course, the non-linearity may induce some dependence of BP’s diffusivity and mobility on the force \( f \). At that, nevertheless, diffusivity and mobility remain well certain quantities, one and the same for all possible realizations of BP’s random walk and all (long enough) fragments of any particular realization.

   It would be quite another matter if beyond the considered approximation (21) we found qualitatively different random walk which has no certain diffusivity and mobility. Below, we want to argue that just such picture arises from analysis of third of the three above mentioned factors, that is interference of BP’s collisions.

2. First, it is useful to see mere existence of “interference of collisions” or, in other words, essential statistical correlations between them.

   With this purpose, consider pair self-correlation of the kernel in Eq.20,

\[ G^{(2)}(\xi_1, \xi_2; \xi_3, \xi_4) \equiv \langle G\{\xi_1, \xi_2, A^*, A\} \otimes \{\xi_3, \xi_4, A^*, A\} \rangle_0, \tag{28} \]
omitting indices for brevity, and taking in mind most interesting variant of time-separated intervals \((\xi_1, \xi_2)\) and \((\xi_3, \xi_4)\), e.g. when \(\xi_1 > \xi_2 > \xi_3 > \xi_4\). Then, clearly, the function \(G^{(2)}(\xi_1, \xi_2; \xi_3, \xi_4)\) delegates mutual correlation of two different collisions.

If this function reduced to product of the averaged kernels, \(G(\xi_1 - \xi_2)\) and \(G(\xi_3 - \xi_4)\), at any or some finite (of order of \(\tau_0\)) \(\xi_2 - \xi_1\), then it would be substantial ground for the above “fixed relaxation rate approximation”. In fact, however, \(G^{(2)}(\xi_1, \xi_2; \xi_3, \xi_4)\) has no natural tendency to such reduction. Indeed, let us write out the average (28) in more detail in the form

\[
G^{(2)}(\xi_1, \xi_2; \xi_3, \xi_4) = \left(\frac{\alpha^2}{T^2}\right)^2 \int \int c(x_1) c(x_2) \Phi' (\rho_1) \Phi' (\rho_2) \left(\exp \left[\int \hat{\Lambda}_1 (A^* (\xi), \Lambda (\xi)) d\xi\right]\right)_0 \Phi' (\rho_1) \Phi' (\rho_2) c(x_1) c(x_2) ,
\]

(29)

where subscripts of operators \(\hat{\Lambda}_{1,2}\) indicate that they act in different spaces \(x_1\) and \(x_2\). We can unify these two operators into single but additionally time-dependent one, \(\hat{\Lambda}(\xi; A^*, A)\), which coincides with \(\hat{\Lambda}_{1,2}\) in intervals \((\xi_1, \xi_2)\) and \((\xi_3, \xi_4)\), respectively, and turns to zero otherwise. Then (29) transforms to

\[
G^{(2)}(\xi_1, \xi_2; \xi_3, \xi_4) = \left(\frac{\alpha^2}{T^2}\right)^2 \int \int c(x_1) c(x_2) \Phi' (\rho_1) \Phi' (\rho_2) \left(\exp \left[\int \hat{\Lambda}_2 (A^* (\xi), \Lambda (\xi)) d\xi\right]\right)_0 \Phi' (\rho_1) \Phi' (\rho_2) c(x_1) c(x_2)
\]

(30)

Here again, similarly to (21)-(22), we can make transition back to pseudo-quantum representation and rewrite expression (30) first as

\[
G^{(2)}(\xi_1, \xi_2; \xi_3, \xi_4) = \left(\frac{\alpha^2}{T^2}\right)^2 \int \int c(x_1) c(x_2) \Phi' (\rho_1) \Phi' (\rho_2) \left(\exp \left[\int \tilde{\Lambda}(\xi; A^*, A) d\xi\right]\right)_0 \Phi' (\rho_1) \Phi' (\rho_2) c(x_1) c(x_2)
\]

(31)

and then, obviously, as

\[
G^{(2)}(\xi_1, \xi_2; \xi_3, \xi_4) = G^{(2)}(\xi_1 - \xi_2; \xi_3 - \xi_4) \equiv \langle 0 | \tilde{G}(\xi_1 - \xi_2) \tilde{G}(\xi_3 - \xi_4) | 0 \rangle ,
\]

(32)

with \(\tilde{G}\) being quantum operator analogue of the kernel \(G\), namely,

\[
\tilde{G}_{\alpha\beta}(\tau) \equiv \tilde{G}_{\alpha\beta}(\tau, A^*, A) \equiv \frac{\alpha^2}{T^2} \int c(x) \Phi'_{\alpha}(\rho) \exp [\tau \tilde{\Lambda}(A^*, A)] \Phi'_{\beta}(\rho) c(x) dx
\]

(33)

Or, equivalently, returning to our basic notations,

\[
\tilde{G}^{(2)}(\tau; \tau') = \int dP \tilde{G}(\tau, V, \nabla_P) \tilde{G}(\tau', V, \nabla_P) G_M(P)
\]

(34)

with operator \(\tilde{G}\) already defined in (22)-(23). Clearly, according to (4),

\[
\tilde{G}(\tau, V, \nabla_P) = \tilde{G}(\tau, -\sqrt{TM} \nabla_P, \sqrt{M/T} (V + T \nabla_P))
\]

(35)

From Eq.32 it visually follows that the correlation function of interaction kernel is indifferent to time separation of the intervals \((\xi_1, \xi_2)\) and \((\xi_3, \xi_4)\) where collisions take place. Clearly, if we considered third- and higher-order statistical moments of the kernel \(G\), that is

\[
G^{(n)}(\xi_1, \xi_2; \ldots ; \xi_{2n-1}, \xi_{2n}) \equiv \langle G\{\xi_1, \xi_2, A^*, A\} \ldots G\{\xi_{2n-1}, \xi_{2n}, A^*, A\} \rangle_0 ,
\]

(36)
we quite similarly would find their independence on time distances between collision intervals:

\[ G^{(n)}(\xi_1, \xi_2; \ldots; \xi_{2n-1}, \xi_{2n}) = G^{(n)}(\xi_1 - \xi_2; \ldots; \xi_{2n-1} - \xi_{2n}) = \equiv \langle \hat{G}(\xi_1 - \xi_2) \ldots \hat{G}(\xi_{2n-1} - \xi_{2n}) \rangle_0 = \]

\[ = \int d\mathbf{P} \hat{G}(\xi_1 - \xi_2, V, \nabla P) \ldots \hat{G}(\xi_{2n-1} - \xi_{2n}, V, \nabla P) G_M(\mathbf{P}) \]

for mutually non-intersecting intervals \((\xi_1, \xi_2)\).

Thus, BP’s relaxation rate undergoes non-decaying, or “infinitely long-living”, fluctuations. Notice that this could be predicted merely as manifestation of non-decaying character of the “seed” correlators (18).

3. Generally, existence of non-decaying relaxation rate (RR) fluctuations does not prevent diffusive chaotic behavior of BP’s walk, since collisions make their work at any values of \( G \) in (25) comparable with \( \tau \) from (26)). Non-decaying, - or time-scaleless, - RR fluctuations just make BP’s walk purely chaotic: it is becomes so much random that even has no certain \((a \text{ priori})\) “probability of collisions per unit time”.

Such free behavior of actual (physical) random walks is origin of 1/f-type fluctuations in their diffusivity and mobility, which can be explained and described [9–13] also as logically inevitable property of such many-particle dynamical systems constantly forgetting their past (e.g. forgetting pre-history of collisions or other kinetic and transport events). Thus, relaxation always appears together with its scaleless fluctuations [6–8,15,18,21,22,27,30,32,33].

In essence, such logics was suggested already by N. Krylov [36] who visually connected phenomenon of relation to exponential instability and “mixing” of system’s phase trajectories and simultaneously pointed out that these features in no way presume appearance of some \(a \text{ priori}\) definite (phase trajectory-independent) relaxation rates (relative frequencies of kinetic events) and “probabilistic laws” 3.

In more concrete terms, a small change of initial BP’s velocity \( V(0) \) leads to exponentially growing changes of its velocity \( V(t) \) and, all the more, its path \( \Delta R(t) \) after a sequence of BP’s collisions with atoms. At that, characteristic relaxation times \( \tau_+ = \partial \Delta R(t)/\partial V(0) \) and \( \tau_- = -\partial \Delta R(t)/\partial V(t) \) are essentially random quantities, - such that \( \langle \tau_+^2/\tau_0^2 \rangle \approx \langle \tau_-^2/\tau_0^2 \rangle \rightarrow \infty \) nearly exponentially at \( t/\tau_0 \rightarrow \infty \), - and essentially correlated one with another, so that \( \langle \tau_+\tau_- \rangle \leq \tau_+^2 \) regardless of how long is the observation time 4. This is nearly what is said by Eqs.32 and 36 5.

6. Free relaxation rate approximation

1. Let us leave “philosophy” and return to the exact expression (20). We can rewrite it in the form

\[ V_0(t, \mathbf{i}k; \nu, f) = \langle \exp \{ \int_0^t u_0 [ \mathbf{i}k \cdot \mathbf{A}^*(\xi) + (\mathbf{i}k + f/T) \cdot \mathbf{A}^*(\xi) ] d\xi \rangle_0 \]

\[ - \int_{t_0 > \xi_1 > \xi_2 > 0} \mathbf{A}^*(\xi_1) \cdot G(\xi_1, \xi_2, \mathbf{B}^*, \mathbf{B}) \cdot \mathbf{A}(\xi_2) d\xi_2 d\xi_1 \) 0

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with new additional random variables \( \{B^*, B\} \) in place of \( \{A^*, A\} \) inside the kernel, if these new variables are statistically identical to the old ones. This means that

\[
\langle B_\alpha(t_1) B_\beta(t_2) \rangle_0 = \langle B^*_\alpha(t_1) B^*_\beta(t_2) \rangle_0 = 0, \quad \langle B_\alpha(t_1) B^*_\beta(t_2) \rangle_0 = \delta_{\alpha\beta} \theta(t_1 - t_2),
\]

and, clearly,

\[
\langle A_\alpha(t_1) B_\beta(t_2) \rangle_0 = \langle A^*_\alpha(t_1) B^*_\beta(t_2) \rangle_0 = 0, \\
\langle A_\alpha(t_1) B^*_\beta(t_2) \rangle_0 = \langle B_\alpha(t_1) A^*_\beta(t_2) \rangle_0 = \delta_{\alpha\beta} \theta(t_1 - t_2)
\]

Then, taking in mind our previous reasonings, we may say that (i) \( A - A^* \)'s inter-correlations in Eq.37 are responsible for relaxation of BP's velocity and resulting diffusive BP's motion instead of ballistic one, (ii) \( B - B^* \)'s inter-correlations represent randomness of the relaxation rate (RR) in itself, - that is “interference of collisions”, - while (iii) cross-correlations \( A - B^* \) and \( B - A^* \) reflect statistical influence of BP’s walk onto RR and its fluctuations.

The latter item includes the second of three above mentioned factors. It is not critical for BP’s walk, at least in qualitative sense. Therefore we can consider approximation which excludes this factor by neglecting the \( A - B^* \) and \( B - A^* \) cross-correlations, i.e. turning all right-hand sides in (39) to zero. This formal simplification again, - similar to the “fixed RR approximation” in Sec.4, - makes BP’s velocity relaxation forcibly linear. However, now relaxation rate (RR) and hence BP’s diffusivity are not fixed but random quantities, although they will be treated with more or less significant losses because of the same simplification. In particular, loss of fine temporal structure of BP’s random walk and thus excess idealization of concept of RR fluctuations, as if RR could be measured irrespectively of the walk. In this sense, the resulting model can be named “free RR approximation”.

2. Accepting such approximation, clearly, we can perform in Eq.37 averaging first in respect to \( \{A^*, A\} \) and later in respect to \( \{B^*, B\} \). The first operation yields

\[
V_0(t) \approx \left\langle \exp \left[ \frac{1}{2} \int \int \frac{i k \cdot Q\{{\xi}_1, {\xi}_2, B^*, B\} \cdot \left( i k + \frac{f}{T} \right) d {\xi}_2 d {\xi}_1} \right] \right\rangle_0
\]

with kernel \( Q\{{\xi}_1, {\xi}_2, B^*, B\} \) now given by infinite series of convolutions,

\[
Q = \Theta - \Theta \otimes G \otimes \Theta + \Theta \otimes G \otimes \Theta \otimes G \otimes \Theta - \ldots ,
\]

which generalizes series from (25) to functional \( Q = Q\{{\xi}_1, {\xi}_2, B^*, B\} \) in place of the function \( Q\{\xi_1 - \xi_2\} \).

Equivalently,

\[
Q\{t_1, t_2, B^*, B\} = \theta(t_1 - t_2) - \int_{t_1 > \xi_1 > \xi_2 > t_2} \theta(t_1 - \xi_1) G\{\xi_1, \xi_2, B^*, B\} Q\{\xi_2, t_2, B^*, B\} d {\xi}_2 d {\xi}_1
\]

The exponential in Eq.40 represents Gaussian random walk with randomly varying diffusivity and mobility (diffusion and drift rates). Importantly, that are long-living, or time-scaleless, variations, which follows from previous section and merely from non-decaying of the \( B - B^* \) correlator (38).

Unfortunately, result of averaging of the exponential in Eq.40 hardly can be written in transparent or at least generally closed form. Hence, we need in further simplifications. Next, let us consider some possibilities.

7. Free RR approximation for low-density gas

1. Naturally, main difficulties in calculation of the average (40) come from time non-locality of the kernel \( G\{\xi_1, \xi_2, B^*, B\} \). They become lighten, however, if gas is rare enough, that is \( \nu n_0^3 \ll 1 \) and hence \( \tau_c/\tau_0 \ll 1 \) (all the more in the low-density, or Boltzmann-Grad, limit). Then, apparently, we can try to
neglect contributions of mutual time intersections of factors $G$ in power series expansion of the exponential in (40).

At that, $G\{\xi_1,\xi_2,B^*,B\}$ is treated as delta-function of $\xi_1 - \xi_2$, that is, similar to (26),

$$G\{t',t'',B^*,B\} \Rightarrow \Gamma\{t',B^*,B\} \delta_+(t' - t'') ,$$

(43)

where, nevertheless, $\Gamma\{t',B^*,B\}$ is thought as definite functional of $\{B^*(\xi), B(\xi)\}$ with $\xi$ from some time interval $\sim \tau_c$ around $t'$. Moreover, when total observation time is much greater than $\tau_0$, then the kernel $Q\{\xi_1,\xi_2,B^*,B\}$ also may be treated in time-local fashion, with replacement

$$u_0^2 Q\{t',t'',B^*,B\} \Rightarrow D\{t',B^*,B\} \delta_+(t' - t'') ,$$

(44)

where $D\{t,B^*,B\}$ is functional depending mainly on $\{B^*(\xi), B(\xi)\}$ with $|\xi - t| \sim \tau_0$ and representing random (fluctuating) BP's diffusivity.

2. The dilute gas approximation (43) allows us to use relations (22),(32) and (36) and replace the averaging in (40) by equivalent operations in the original or pseudo-quantum formalisms. In the original form, we have first to write out operator analogue of Eq.43:

$$\tilde{G}(\tau, V, \nabla_P) \Rightarrow \tilde{\Gamma}(V, \nabla_P) \delta_+ \tau , \quad \tilde{\Gamma} = \int_0^\infty \tilde{G}(\tau) \, d\tau ,$$

(45)

with operator $\tilde{G}$ defined in (23), and $\tilde{\Gamma}(V, \nabla_P)$ playing role of RR operator.

Combining formulae (22),(32) and (36) with (45) and substituting them to Eq.40, we arrive to

$$V_0(t) \approx \int dP \, \exp \left[ u_0^2 \int_{\tau > \xi_1 > \xi_2 > 0} i k \cdot \tilde{Q}(\xi_1 - \xi_2, V, \nabla_P) \cdot \left( i k + \frac{f}{T} \right) \, d\xi_2 \, d\xi_1 \right] \, G_M(P)$$

(46)

The operator kernel $\tilde{Q}$ here is directly determined by Eqs.41-42 together with (36),

$$\tilde{Q}(\tau, V, \nabla_P) = e^{-\tilde{\Gamma}(V, \nabla_P) \tau}$$

(47)

At constant wave vector and force and at sufficiently large observation time, expression (46) turns to

$$V_0(t, ik; \nu, f) \approx \int dP \, e^{i k \cdot \tilde{D}(V, \nabla_P) \cdot (ik + f/T) t} \, G_M(P) ,$$

$$\tilde{D}(V, \nabla_P) = u_0^2 \tilde{\Gamma}^{-1}(V, \nabla_P) ,$$

(48)

with $\tilde{D}(V, \nabla_P)$ in the role of BP’s diffusivity operator (operator image of $D\{t',B^*,B\}$ from Eq.44).

To go to the pseudo-quantum representation, we have to use relation (35) and replacements

$$\tilde{\Gamma}(V, \nabla_P) \leftrightarrow \tilde{\Gamma}(A^\dagger, A) = \int_0^\infty \tilde{G}(\tau) \, d\tau ,$$

$$\tilde{Q}(\tau, V, \nabla_P), \tilde{D}(V, \nabla_P) \leftrightarrow \tilde{Q}(\tau, A^\dagger, A) = \exp[-\tau \tilde{\Gamma}(A^\dagger, A)] , \tilde{D}(A^\dagger, A) = u_0^2 [\tilde{\Gamma}(A^\dagger, A)]^{-1} ,$$

$$\int dP \ldots G_M(P) \leftrightarrow \langle 0 | \ldots | 0 \rangle$$

(49)

3. It is necessary to notice that, strictly speaking, just made application of formulae (22),(32) and (36) requires chronological time ordering of the operator images $\tilde{\Gamma}(V, \nabla_P)$ of functionals $\Gamma\{t, B^*, B\}$. Although these operators get no literal time dependence, such the requirement creates obstacles to “disentangling”
of operator images of time-intersecting copies of \( G\{t', t'', B^*, B\} \). The matter is that \( \tilde{\Gamma}(V, \nabla_P) \) along with \( \Gamma(t, B^*, B) \) is tensor-like (matrix-like) object whose components may be mutually non-commuting.

By these reason, transition from Eq.40 to Eq.46 is approximate, - together with (44), (48) and other related formulas, - even under the dilute gas condition.

4. In view of this formal difficulty, it seems reasonable to supplement the “free RR” and “dilute gas” approximations with one more simplification. Namely, replacing the originally multi-component matrix RR operator by its scalar diagonal part:

\[
\tilde{\Gamma}_{\alpha\beta} \Rightarrow \delta_{\alpha\beta} \tilde{\Gamma} , \quad \tilde{\Gamma} = \frac{1}{3} \sum_{\alpha} \int_{0}^{\infty} \tilde{G}_{\alpha\alpha}(\tau) \, d\tau
\]  

(50)

After that the difficulty disappears.

We can justify this simplification by the fact that due to the gas isotropy the RR operator is natively isotropic, that is obeys spherical symmetry. Therefore, replacement (50) itself must not cause too big qualitative losses. Then, combining (48) and (50) and performing in (48) Fourier transform over \( k \), we obtain the diffusion law in the form

\[
V_0(t, \Delta R; \nu, f) \approx \int dP \left( \frac{\tilde{\Gamma}}{4\pi u_0^2 t} \right)^{3/2} \exp \left[ -\frac{\tilde{\Gamma}}{4 u_0^2 T} \left( \Delta R - u_0^{-1} f T \right)^2 \right] G_M(P) ,
\]  

(51)

with scalar RR operator, in the sense of (50). Its properties, however, will be discussed in parallel with that of the full matrix RR operator.

8. Relaxation rate operator, relaxation rate distribution, and diffusion law

1. From viewpoint of Eqs.45-48 and 51, spectral properties of the RR operator \( \tilde{\Gamma} \) are of most importance. To consider them, notice, first, that the Liouville operator \( \tilde{\Lambda}(V, \nabla_P) \) commutes with multiplication by \( g(x) G_M(P) \), i.e. \( \tilde{\Lambda} g(x) G_M(P) = g(x) G_M(P) \tilde{\Lambda} = 0 \). Using this fact (together with trivial symmetry of the interaction, \( \Phi(-\rho) = \Phi(\rho) \)), it is easy to proof operator equality

\[
\tilde{\Gamma}(V, \nabla_P) G_M(P) = G_M(P) \tilde{\Gamma}^\dagger(V, \nabla_P) = [\tilde{\Gamma}(V, \nabla_P) G_M(P)]^\dagger ,
\]  

(52)

where symbol \( \dagger \) denotes conjugation of operator \( \tilde{\Gamma}(V, \nabla_P) \) (or any other operator composed of \( V \) and \( \nabla_P = M^{-1} \nabla_V \) ) in the Sturm-Liouville sense.

Taking this into account, one can search for eigenfunctions of the RR operator in the form

\[
\tilde{\Gamma}(V, \nabla_P) G_M(P) \Psi_s(V) = \gamma_s G_M(P) \Psi_s(V) , \quad \tilde{\Gamma}^\dagger(V, \nabla_P) \Psi_s(V) = \gamma_s \Psi_s(V) ,
\]  

(53)

where index \( s \) enumerates the eigenfunctions and eigenvalues. In their terms, Eq.51 turns to

\[
V_0(t, \Delta R; \nu, f) \approx \int_{0}^{\infty} \left( \frac{\gamma}{4\pi u_0^2 t} \right)^{3/2} \exp \left[ -\frac{\gamma}{4 u_0^2 T} \left( \Delta R - \frac{u_0^2}{\gamma} f T \right)^2 \right] W(\gamma) \, d\gamma ,
\]  

(54)

with \( W(\gamma) \) representing effective RR’s probability distribution,

\[
W(\gamma) \equiv \sum_s \left| \int \Psi_s(V) G_M(P) \, dP \right|^2 \delta(\gamma - \gamma_s)
\]  

(55)

Here \( \sum_s \) means generally “continuous summation”, and eigenfunctions \( \Psi_s(V) \) are thought properly normalized:

\[
\int \Psi_s^\dagger(V) \Psi_s(V) G_M(P) \, dP = \delta_{s's} , \quad \sum_s \Psi_s(V') \Psi_s^\dagger(V) = \delta(P' - P) G_M^{-1}(P)
\]  

(56)
Then \( W(\gamma) \) also is normalized, \( \int_0^\infty W(\gamma) \, d\gamma = 1 \).

Our presumption that \( \hat{\Gamma} \)'s spectrum is real positive is dictated by physical meaning of this operator and, of course, by its formal definition in Eqs.21-22 and 45 prompting that it is positively defined operator:

\[
\int \, dP \, \Psi^\dagger(V) \hat{\Gamma} \Psi(V) G_M(P) > 0 \tag{57}
\]
at nonzero \( \Psi(V) \).

2. Further, notice, or recall, that the (full matrix) RR operator is closely connected to the Boltzmann-Lorentz (BL) kinetic operator \( \hat{B} = \hat{B}(V, \nabla_P) \). According to formulas (57)-(59) from \([1]\),

\[
\hat{B} = \nabla_P \cdot \hat{\Gamma} \cdot (P + TM \nabla_P), \quad \hat{B}G_M = MT \nabla_P \cdot \hat{\Gamma}G_M \cdot \nabla_P \tag{58}
\]

This connection, together with fact that spectrum of BL operator is non-positive, implies very important information about spectrum of the RR operator.

Indeed, let us consider expression

\[
C(a,b) = \int \, dP \, e^{-i a \cdot V} \hat{B} e^{i b \cdot V} G_M(P) \tag{59}
\]

with real-valued vectors \( a \) and \( b \). From Eq.58 it follows that

\[
C(a,b) = -\frac{T}{M} a \cdot \int \, dP \, e^{-i a \cdot V} \hat{\Gamma} e^{i b \cdot V} G_M(P) \cdot b \tag{60}
\]

Next, firstly, take into account that the BL operator is non-positively defined, therefore \( C(a,b = a) \) is negative quantity at \( a \neq 0 \), \( C(a,a) < 0 \). Secondly, notice that for any short-range repulsive interaction this quantity is bounded below,

\[
C(a,a) > -\gamma', \tag{61}
\]

with some finite \( \gamma' \), since total collision cross-section is finite. As the consequence, Eq.60 yields

\[
0 < \int \, dP \, e^{-i a \cdot V} \hat{\Gamma} e^{i a \cdot V} G_M(P) < \frac{\gamma'}{u_0^2 a^2} \tag{62}
\]

For instance, in case of hard-sphere interaction, with radius \( r_0 \), one can represent the BL operator (in essence integral) in differential form

\[
\hat{B}G_M = 2r_0^2 \nu \int \, d\Omega \sqrt{\frac{m}{2\pi T}} \int_0^\infty \, du \, u \exp \left( \frac{m'}{2T} u^2 \right) \times
\]

\[
\times \sinh \left( qu \Omega \cdot \nabla_V \right) G_M(P) \exp \left[ -\frac{m'}{2T} (\Omega \cdot V)^2 \right] \sinh \left( qu \Omega \cdot \nabla_V \right), \tag{63}
\]

with \( \Omega \) being unit vector (\( |\Omega| = 1 \)) running all over unit sphere, and \( q \equiv m/(m + M) = m'/M \) (\( m' = mM/(m + M) \)). Hence, obviously,

\[
C(a,b) = -2r_0^2 \nu \int \, d\Omega \sqrt{\frac{m'}{2\pi T}} \int_0^\infty \, du \, u \exp \left( -\frac{m'}{2T} u^2 \right) \times
\]

\[
\times \sin \left( qu \Omega \cdot a \right) \sin \left( qu \Omega \cdot b \right) \exp \left\{ -\frac{T}{2M} \left[ |a - b|^2 - q (\Omega \cdot (a - b))^2 \right] \right\}, \tag{64}
\]

and \( \gamma' = 4\nu\pi r_0^2 \sqrt{T/2\pi m'} \).
The right-hand inequality in Eq. 62 clearly shows that lower bound of the RR spectrum, i.e. \( \tilde{\Gamma} \)'s spectrum, is zero:

\[
\inf_s \tilde{\Gamma} = \inf_s \gamma_s = 0
\]  

(65)

(otherwise, on the left here we would have some nonzero number in place of zero).

Simultaneously, \( \tilde{\Gamma} \) is an unbounded non-compact operator, if being considered in a more wide class of functions than in Eqs. 59, 60 and 64, namely, a class allowing \( |\Psi(V)| \rightarrow \infty \) (polynomially or even exponentially) at \( |V| \rightarrow \infty \). Therefore [39] we surely can expect that \( \tilde{\Gamma} \)'s spectrum is continuous on \((0, \infty)\).

3 The said means, in turn, that BP’s diffusion (and drift) law, expressed by Eq. 54, possesses essentially non-Gaussian, - moreover, non-exponential, - long-range asymptotic. The latter then is determined by asymptotic of the RR distribution \( W(\gamma) \) (55) at small \( \gamma/\tau \rightarrow 0 \).

Some important statements about this distribution and thus diffusion law can be made even without detail investigation of the RR operator. Anyway, for short-range repulsive interaction, the diffusion law determined by exact basic Eq. 54 must be really “diffusive”, that is undergo characteristic time scale dependence \( \Delta R^2(t) \propto t \), at least on average, so that mean value of the diffusivity operator in Eq. 48 is finite,

\[
\bar{D} \equiv u_0^2 \int dP \tilde{\Gamma}^{-1} G_M(P) = u_0^2 \int_0^\infty \frac{1}{\gamma} W(\gamma) d\gamma < \infty ,
\]  

(66)

and thus mean value of linear (low-field) BP’s mobility, \( \bar{D}/T \), is finite too. Using relation (58), we can reformulate this statement via the BL operator (BLO), as

\[
\bar{D} = u_0^2 \int G_M (\tilde{\Gamma} G_M)^{-1} G_M dP = u_0^2 TM \int G_M \nabla_P (\tilde{B} G_M)^{-1} \nabla_P G_M dP =
\]  

\[
=- \int G_M V (\tilde{B} G_M)^{-1} V G_M dP = \int V (-\tilde{B})^{-1} V G_M dP < \infty
\]  

(67)

(under proper treatment of the latter tensor-like expressions). Clearly, the “ground-state” zero BLO’s eigenvalue (corresponding to \( \tilde{B} G_M = 0 \)) does not contribute to this formula. At the same time, any physically meaningful BLO has discrete spectrum whose non-zero part is separated from zero [38,39]. Then the inequality in (67) and in (66) certainly is satisfied, that is the mean diffusivity \( \bar{D} \) is finite in spite of the zero RR’s lower bound (65).

Consequently, we can state that RR probability distribution vanishes at zero, \( W(\gamma \rightarrow 0) \rightarrow 0 \), and then guess that it makes this nearly by power law,

\[
W(\gamma) \propto \gamma^\eta \ (\gamma \ll \tau), \quad \int_0^\infty e^{-\gamma t} W(\gamma) d\gamma = \int dP e^{-t\tilde{\Gamma}} G_M(P) \propto \frac{1}{t^{\eta+1}} \ (t \gg 1/\gamma),
\]  

(68)

with \( \eta > 0 \).

Such the asymptotic of RR spectrum, being applied in Eq. 51 or Eq. 54, results in strongly non-exponential diffusion law at large values of BP’s displacement (for “large deviations”). In particular, in absence of external force \( (f = 0) \), i.e. in case of equilibrium Brownian motion, we have

\[
V(t, \Delta \mathbf{R}; \nu, f = 0) \propto \frac{(\bar{D} t)^{\eta+1}}{(|\Delta \mathbf{R}^2|)^{\eta+5/2}} \ (|\Delta \mathbf{R}^2| \gg \bar{D} t),
\]  

(69)

that is power-law long tail for probabilities of of large deviations from typical behavior (mean-square displacement).

4. Further, let us consider the RR operator (RRO) more carefully, by concrete example of the hard-sphere interaction. In agreement with relation (58), RRO what corresponds to the hard-sphere BLO (63) naturally can be suggested in the form
\((\hat{\Gamma}G_M)_{\alpha\beta} = 2r_0^2\nu \int d\Omega \sqrt{\frac{m}{2\pi T}} \int_0^\infty du \exp \left( \frac{-m'}{2T} u^2 \right) \times \Omega \sinh \left( \frac{qu \Omega \cdot \nabla V}{u_0 (\Omega \cdot \nabla V)} \right) G_M(P) \exp \left[ -\frac{m}{2T} (\Omega \cdot V)^2 \right] \frac{\sinh \left( \frac{qu \Omega \cdot \nabla V}{u_0 (\Omega \cdot \nabla V)} \right)}{\Omega \beta} \right) \) \tag{70}

It should be noticed that in general so simple formal connection between BLO and RRO has no place, moreover, RRO can not be unambiguously extracted from BLO and Eq.58. However, in special hard-sphere case the expression (70) seems well consistent with our basic RRO \(\hat{\Gamma}(V, \nabla \rho)\)’s definition in (45).

For more visuality, following the scalar RRO’s approximation, let us concentrate on scalar (mean diagonal) \(\hat{\Gamma}\)’s component and represent it as evidently integral operator, in such way that

\[
\frac{1}{3} \sum_\alpha G^{-\frac{1}{2}}_M(\mathbf{p}) \hat{\Gamma}_{\alpha \alpha} G^{\frac{1}{2}}_M(\mathbf{p}) \Xi(\mathbf{V}) = \int \Gamma(\mathbf{V}, \mathbf{V'}) \Xi(\mathbf{V'}) d\mathbf{V'}, \tag{71}
\]

with definitely positive and symmetric integral kernel \(\Gamma(\mathbf{V}, \mathbf{V'}) = \Gamma(\mathbf{V}', \mathbf{V})\). Making in Eq.70 transition to dimensionless velocities, \(V/u_0 \Rightarrow V\) and \(u/u_0 \Rightarrow u\), then introducing vectorial integration variable \(qu\Omega\), and using equality \(\sinh(x)/x = \frac{1}{2} \cosh(cx) dc/2\), one can transform Eq.70 to

\[
\Gamma(\mathbf{V}_1, \mathbf{V}_2) = \frac{2r_0^2\nu u_0}{3q^2} \int_\frac{1}{2} \frac{1}{4} \int \frac{dc_1dc_2}{(c_1+c_2)^4} |\mathbf{U}| \times \nonumber \]

\[
\exp \left\{ -\frac{\mathbf{U}^2}{8q} \frac{4-\frac{c_1-c_2}{c_1+c_2}^2}{(c_1+c_2)^2} + \frac{\mathbf{U}^2}{8} - \frac{r}{2} \left[ \frac{1}{|\mathbf{U}|} \cdot \frac{|\mathbf{U}|c_1+c_2}{c_1-c_2} \right]^2 \right\}, \tag{72}
\]

where \(\mathbf{U} \equiv \mathbf{V}_1 - \mathbf{V}_2\), \(V \equiv \frac{1}{2} (\mathbf{V}_1 + \mathbf{V}_2)\), and \(r \equiv m/M = q/(1-q)\). Changing of the integration variables,

\[
x = \frac{4 - \frac{c_1-c_2}{c_1+c_2}^2}{(c_1+c_2)^2} - 1, \quad y = \frac{c_1-c_2}{c_1+c_2},
\]

and then \(y \Rightarrow 2qy/|\mathbf{U}|\), simplifies this integral to

\[
\Gamma(\mathbf{V}_1, \mathbf{V}_2) = \frac{\gamma'}{6\pi q\sqrt{1-q}} \frac{|\mathbf{U}|}{32} \exp \left( -\frac{\mathbf{U}^2}{8r} \right) \times \nonumber \]

\[
\int \frac{dy}{2|y|} \int\frac{dx}{2|y|} \exp \left\{ -x \frac{\mathbf{U}^2}{8q} - \frac{r}{2} \left[ \frac{1}{|\mathbf{U}|} \cdot y \frac{|\mathbf{U}|c_1+c_2}{c_1-c_2} \right]^2 \right\} = \frac{\gamma'q}{12\pi \sqrt{1-q}} \frac{1}{|\mathbf{U}|^2} \exp \left( -\frac{\mathbf{U}^2}{8r} \right) I(\mathbf{U}, \Delta E), \tag{73}
\]

with \(\Delta E \equiv (\mathbf{V} \cdot \mathbf{U}) = \frac{\mathbf{V}_1^2}{2} - \frac{\mathbf{V}_2^2}{2}\) and function

\[
I(\mathbf{U}, \Delta E) = \int \exp \left\{ -|y| \frac{|\mathbf{U}|}{2} - \frac{r}{2} \left[ \frac{\Delta E}{|\mathbf{U}|} - y \right]^2 \right\} dy = \sqrt{\frac{2}{\pi r}} \int_{-\infty}^{\infty} \exp \left( \frac{i\zeta}{2} \Delta E - \frac{\zeta^2 |\mathbf{U}|^2}{8r} \right) \frac{d\zeta}{1 + \zeta^2} \tag{74}
\]

It can be exactly expressed via the error function (recall that \(\mathbf{U} = \mathbf{V}_1 - \mathbf{V}_2\)).

These formulas show that at \((\mathbf{V}_1 - \mathbf{V}_2)^2 \ll 1\), i.e. at small velocity changes,

\[
\Gamma(\mathbf{V}_1, \mathbf{V}_2) \approx \frac{\gamma'}{3} \frac{q}{2\pi} \frac{1}{(\mathbf{V}_1 - \mathbf{V}_2)^2} \exp \left( -\frac{\Delta E}{2} \right), \tag{75}
\]
while in the opposite case $(V_1 - V_2)^2 \gg 1$ one has

$$\Gamma(V_1, V_2) \approx \frac{2\gamma q}{3\pi \sqrt{1 - q}} \cdot \frac{1}{|V_1 - V_2|^3} \exp \left[ -\frac{|V_1 - V_2|^2}{8r} - \frac{r}{2} \left( \frac{\Delta E}{|V_1 - V_2|} \right)^2 \right]$$

(recall that $\Delta E = (V_1^2 - V_2^2)/2$).

Thus, of course, relaxation rates (RR) for BP’s transitions between states with nearly equal energies (when $V_1^2 \approx V_2^2$ and approximately $U \perp V$) always are greater or much greater than RR for transitions between significantly different energies (when approximately $U \parallel V$).

5. The latter statement means that lower part of the RR spectrum (55) (small $\Gamma$’s eigenvalues), responsible for long-range asymptotics of diffusion law, is essentially determined by $\Gamma(V_1, V_2)$’s and hence $I(U, \Delta E)$’s dependence on BP’s energy changes $\Delta E = (U \cdot V) = (V_1^2 - V_2^2)/2$. Especially as, due to obvious spherical symmetry of the operator $\tilde{\Gamma}$, the sum (integral) in Eq.55 in fact consists of its spherically symmetric eigen-states only.

Therefore, the index $s$ in Eq.55 can be treated as one-dimensional continuous parameter, let $\gamma$, varying in $0 < \gamma < \infty$ and uniquely enumerating non-degenerated spherically symmetric solutions of eigenvalue problem

$$\int \Gamma(V_1, V_2) \Xi_\gamma(V_2^2) dV_2 = \gamma \Xi_\gamma(V_1^2) ,$$

(77)

At that, firstly, if the eigenfunctions are normalized to delta-function,

$$\int \Xi_\gamma^*(V_2) \Xi_\gamma(V_2^2) dV = 4\pi \int_0^\infty \Xi_\gamma^*(v^2) \Xi_\gamma(v^2) v^2 dv = \delta(\gamma' - \gamma) ,$$

(78)

then, according to Eq.56 and obvious relation $\Psi_\gamma(V) = \Xi_\gamma(V^2) G_M^{-1/2}(P)$, instead of Eq.55 we have to write

$$W(\gamma) = \left| \int \Xi_\gamma(V^2) G_M^{1/2}(P) dP \right|^2 = 4\sqrt{2\pi} \left| \int_0^\infty v^2 \Xi_\gamma(v^2) e^{-v^2/4} dv \right|^2$$

(79)

Secondly, from all the aforesaid it follows that lower eigenvalues in Eq.77 are contributed by more frequently oscillating eigenfunctions.

Thirdly, we beforehand can perform averaging of the kernel $\Gamma(V_1, V_2)$ over directions of the velocities. This manipulation, with the help of Eq.74, reduces the problem (77) to

$$\int_0^\infty \bar{\Gamma}(v_1, v_2) v_2 \Xi_\gamma(v_2^2) dv_2 = \int_{-\infty}^\infty \bar{\Gamma}(v_1, v_2) v_2 \Xi_\gamma(v_2^2) dv_2 = \gamma v_1 \Xi_\gamma(v_1^2) ,$$

(80)

with $v_1 \equiv |V_1|, v_2 \equiv |V_2|$, and functions

$$\bar{\Gamma}(v_1, v_2) = \gamma_0 \int_{(v_1 + v_2)^2/r}^{(v_1 - v_2)^2/r} \int_x^\infty dy \sqrt{y} \exp \left[ -\frac{y}{8} - \frac{(v_1^2 - v_2^2)^2}{8y} \right] =$$

$$= \bar{\Gamma}(v_1, v_2) - \bar{\Gamma}(v_1, -v_2) ,$$

$$\bar{\Gamma}(v_1, v_2) = \gamma_0 \int_{(v_1 - v_2)^2/r}^{(v_1 + v_2)^2/r} \ln \frac{xr}{(v_1 + v_2)^2} \exp \left[ -\frac{x}{8} - \frac{(v_1^2 - v_2^2)^2}{8x} \right] dx ,$$

(81)

where $\gamma_0 \equiv \nu v_0^2 u_0 \sqrt{\pi/2}/3$. 

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Notice that the kernel $\overline{\Gamma}(v_1, v_2)$ has characteristic form

$$\overline{\Gamma}(v_1, v_2) = \gamma_0 F\left(\frac{v_1 - v_2}{\sqrt{F}}, \frac{\sqrt{v_1 + v_2}}{2}\right) = \gamma_0 F'\left(\frac{v_1 - v_2}{\sqrt{F}}, \frac{v_1^2 - v_2^2}{2}\right), \quad (82)$$

with functions $F(a,b)$ and $F'(a,ab)$ independent on the only RRO’s dimensionless free parameter, i.e. mass ratio $r = m/M$. Namely, according to Eq.81,

$$F'(a,ab) \equiv \int_a^\infty \ln \frac{x}{a^2} \exp \left[-\frac{x - a^2 b^2}{2x}\right] \frac{dx}{\sqrt{x}} = |a| \int_1^\infty \ln x \exp \left[-\frac{x a^2}{8} - \frac{b^2}{2x}\right] \frac{dx}{\sqrt{x}} \equiv F(a,b) \quad (83)$$

In other terms, property (82) means that

$$\overline{\Gamma}(v_1, v_2; r) = \overline{\Gamma}(v_1 \cosh \theta + v_2 \sinh \theta, v_2 \cosh \theta + v_1 \sinh \theta; r_0) \quad (84)$$

with $\exp \theta \equiv \sqrt{r/r_0}$, where the kernel’s parameter is evidently included into list of its arguments.

Approximately, in parallel to (75) and (76),

$$F(a,b) \approx \begin{cases} \sqrt{2\pi} \ln \frac{4b^2}{a^2} \exp \left(-\frac{|a||b|}{2}\right) & (|a| < 2|b|) \\ \ln \left(1 + \frac{8}{a^2}\right) |a| \int_1^\infty \exp \left(-\frac{a^2 x}{8}\right) \frac{dx}{\sqrt{x}} \exp \left(-\frac{b^2}{2}\right) & (|a| > 2|b|) \end{cases} \quad (85)$$

The upper row here comes from the saddle-point approximation (applicable at $|a| < 2|b|$). Its result becomes exact in limit case of infinitely massive atoms, $r \to \infty$, giving

$$\overline{\Gamma}(v_1, v_2; \infty) = 4\gamma_0 \sqrt{2\pi} \ln \left|\frac{v_1 + v_2}{v_1 - v_2}\right| \exp \left(-\frac{v_1^2 - v_2^2}{4}\right) \quad (86)$$

6. Unfortunately, these observations not strongly facilitate rigorous analysis of the RR distribution, i.e. calculation of quantity (55) as determined by equation (80) and condition (78). Therefore it seems reasonable to confine ourselves by semi-heuristic discussion, the more so as we are anyway restricted by our initial approximations of the basic path integral. Then, most important thing is to argue that at small RR values, $\gamma \ll \gamma$, the distribution (79), that is (55), is nearly power function of $\gamma$, as it was stated in Eq.68.

Indeed, first, according to Eqs.81,82,83,85 and 86, the kernel $\overline{\Gamma}(v_1, v_2) = \overline{\Gamma}(v_1, v_2; r)$ (even when considered on the whole axis $-\infty < v < \infty$) is not a mere difference kernel but instead a function of two differences $v_1 - v_2$ and $v_1^2 - v_2^2$, both playing significant roles. This means that solutions of the problem (80),

$$v \Xi_\gamma(v^2) = c_\gamma(v) \sin \phi_\gamma(v),$$

oscillate in a non-harmonical way, that is their phases $\phi_\gamma(v)$ are essentially non-linear functions of $v$. Therefore integral in Eq.79 is not exponential but (inverse) power functions of characteristic frequency of the oscillations (for instance, if $\phi_\gamma(v) \propto \zeta v^2$, then this integral $\propto 1/\zeta$ at large enough $\zeta$ though it would change like $\exp(-\zeta^2)$ if the phase was $\phi_\gamma(v) \propto \zeta v$).

Second, on the other hand, the characteristic frequency, $d\phi_\gamma(v)/dv$, is connected with the eigenvalues $\gamma$ also by an (inverse) power law, merely because of non-analyticity of the kernel $\overline{\Gamma}(v_1, v_2)$ at $v_1 = v_2$. Moreover, since this non-analiticality is dominated by the logarithmic factor (obvious in Eqs.81,83,85 and 86), we can state that $d\phi_\gamma(v)/dv \propto \gamma/\gamma$. At that, clearly, since $\delta(\gamma'/\gamma) = \gamma^{-2}\delta(1/\gamma'-1/\gamma)$, the normalization condition (78) requires $c_\gamma(v) \propto \gamma/\gamma$, so that $v \Xi_\gamma(v^2)$ can be rewritten more visually as

$$v \Xi_\gamma(v^2) = \frac{c_\gamma(v)}{\gamma} \sin \frac{\phi_\gamma(v)}{\gamma} \quad (87)$$
As the consequence of these two circumstances, (lower part of) RR distribution $W(\gamma)$ (79) appears approximately power function of $\gamma$, with exponent $\eta = \eta(r)$ in (68) definitely dependent on the mass ratio.

7. In the limit $r = \infty$ one may expect this exponent to turn to zero.

To ground this, first, let us represent the eigenfunctions by series over normalized Hermite functions associated with the equilibrium BP’s velocity distribution:

$$v \Xi_n(v^2) = \sum_{n=0}^{\infty} c_{\gamma n} (-1)^n \frac{H_{2n+1}(v)}{(2n+1)!} e^{-v^2/4},$$  (88)

where $H_k(x)$ are standardly defined Hermite polynomials. Formally such expansion is possible at any $r$, but most adequate it becomes at $r = \infty$ when the RRO kernel (86) directly relates to the equilibrium velocity distribution. Second, compare this series expansion with expression (87) and combine both them with equality

$$8\pi^{\gamma_0} \sqrt{2\pi} \int_{0}^{\infty} e^{-v^2/4} \Xi_n(v^2) dv = \gamma \Xi_n(0)$$  (89)

which is easy derivable from Eq.80 in the case of kernel (86). If we take into account also that

$$\left(\frac{H_{2n+1}(x)}{x}\right)_{x=0} = (-1)^n (2n+1)!!, \quad \int_{0}^{\pi} \frac{H_{2n+1}(x)}{x} e^{-x^2/2} dx = \sqrt{\pi}(-1)^n 2^n n!,$$

and use the Stirling formula, then insertions of (88) and (87) to (89) yield

$$8\pi^{\gamma_0} \sum_{n=0}^{\infty} c_{\gamma n} \left(2\sqrt{\frac{n+1}{\pi}}\right)^{-1/2} \approx \gamma \sum_{n=0}^{\infty} c_{\gamma n} \left(2\sqrt{\frac{n+1}{\pi}}\right)^{1/2} \propto \frac{1}{\gamma}$$  (90)

Here $\gamma$ was presumed so small ($\gamma \ll 8\pi\gamma_0$) that $\Xi_n(v^2)$ makes many oscillations per unit.

These relations clearly prompt that at sufficiently small $\gamma$ the coefficients $c_{\gamma n}$ can be approximately expressed through one-variable function, - let $c'(\gamma)$, - such that

$$c_{\gamma n} \propto (n+1)^{-1/4} c'(\gamma \sqrt{n}), \quad c'(0) = 1,$$

and hence $c_{\gamma n} \to \text{const} \neq 0$ at $\gamma \to 0$ for any fixed $n$ (numerically, non-zero values of limits $c_{\gamma \to 0}$ are due to unbounded growth of the amplitudes in (87)). This behavior of $c_{\gamma n}$ seems quite natural in view of the fact that $\sqrt{\pi}$ is characteristic frequency of $H_{2n+1}(x)$’s oscillations.

In particular, $c_{\gamma \to 0} = \int v^2 \Xi_{\gamma \to 0}(v^2) \exp(-v^2/4) dv \neq 0$, which means that $\eta = 0$.

8. Of course, more careful consideration of the limit case $r = \infty$ could reveal some weak, may be logarithmic, dependence of $W(\gamma \to 0)$ instead of a constant, but this do not cancel equality $\eta = 0$ in principal sense. Physically, it reflects the fact that in gas of infinitely massive (hence, immovable) atoms BP’s energy does not relax at all.

In the opposite limit of infinitely light atoms, $r \to 0$, - according to Eqs.81, Eq.83 and Eq.85, - the RRO kernel effectively reduces to delta-function,

$$\Gamma(v_1, v_2) \to 32\gamma_0 \sqrt{\pi} \delta(v_1 - v_2) \quad (r \to 0)$$  (92)

Correspondingly, the RR distribution $W(\gamma)$ also shrinks to delta-function, which formally means that $\eta \to \infty$. Thus, in this (and only this) special case one comes to conventional “Boltzmannian” kinetics with non-random RR.

9. Notice that, in principle, some properties of the RR operator can be investigated even without writing out it evidently, instead considering related more usual BL operator. Indeed, if eigen-states of the full
(matrix) RRO, - satisfying Eq.53, - can be searched in gradient form \( \Psi_s(V) = \nabla_V \Upsilon_s(V) \), then application of the divergence operation to Eq.53 leads vector to scalar problem

\[
\hat{B}_{GM}(V) \ Upsilon_s(V) = \gamma_s \nabla_V \hat{B}_{GM}(V) \nabla_V \ Upsilon_s(V)
\]

(here we applied also the RRO-BLo relation (58), but, in contrast to 53 and (58), take in mind the mentioned dimensionless units, in which \( TM = T/M = u_0^3 = 1 \) and \( P = V \).

At that, in order to construct the RR distribution (55), we can introduce resolvent of the RR operator, then transforming it as follows:

\[
h(z) \equiv \int_0^\infty \frac{W(\gamma)}{\gamma - z} d\gamma = \int dV \left[ \frac{1}{\gamma - z} \right]^{-1} G_M(V) = \int G_M \left[ -zG_M + \hat{\Gamma}G_M \right]^{-1} G_M dV = \int V \left[ z \nabla_V (V + \nabla_V) - \hat{B} \right]^{-1} V G_M dV
\]

(94)

Here again formal transformations like that in (67) are performed, - thus visually connecting RRO resolvent to the BLO, - and again BLO’s zero eigen-states do not contribute to result. Formally, if knowing (94), one can restore RR distribution with the help of

\[
W(\gamma) = \frac{1}{\pi} \Im \lim_{\epsilon \to 0} h(\gamma + i\epsilon)
\]

The last expression in Eq.94 differs from usual resolvent by operator-valued argument \( z \nabla_V (V + \nabla_V) \) in place of a (complex) \( c \)-number valued argument like \( z \). Thus, RR statistics can be determined from spectral properties of linear combination of two operators, namely, BLO and simplest Fokker-Planck operator, \( \nabla_V (V + \nabla_V) \). Each of them has (non-positive) countable spectrum. Nevertheless, as far as they do not commute one with another, their linear combination can possess continuous spectrum, as it was shown above.

9. Conclusion

"Get at the root!" (Koz’ma Prutkov)

1. We considered the original path integral which was derived in [1] from complete infinite hierarchy of the BBGKY (Bogolyubov-Born-Green-Kirkwood-Yvon) equations as exact representation for non-equilibrium probability distribution of trajectory (path) of a particle interacting with atoms of equilibrium ideal gas [1–3,5,18–21]. Since this is extremely non-Gaussian path integral, our main purpose here was, firstly, to recognize a non-trivial but calculable approximation for it and, secondly, demonstrate that corresponding results are in natural agreement with results of our earlier investigated approximate approaches to solution of the BBGKY equations [6,7,15,24,4,5].

Both these objects are achieved by means of approximation which formally separates two floors of randomness in walk of the “Brownian” particle under consideration (BP), namely, its directly observable velocity fluctuations and less visible deeper hidden fluctuations im their intensity, that is fluctuations in rate of velocity relaxation and agitation because of the BP’s collisions with gas atoms.

Importantly, such approximation appears next after most primitive one which neglects the relaxation rate (RR) fluctuations at all and therefore replaces actual path integral by a Gaussian integral while true BP’s random walk by Ornstein-Uhlenbeck random process obeying a Fokker-Planck kinetic equation. What is for the Boltzmann-Lorentz equation, there are no evident conditions for its appearance in some approximation. This fact once more confirms our statement, - for the first time deduced in [6] and especially grounded in [23,28], - that the Boltzmann kinetic equation is invalid even for arbitrary dilute gases and hence the Boltzmann-Lorentz (BL) equation is invalid too.
2. Dramatic defect of Boltzmannian kinetics, - explained in detail already in [6], in principle even earlier in [9–13] and additionally in other words in [2,7,8,15,17,18,21,22,26–28,31,33], - is that it always unconsciously postulates constant “probabilities of collisions per unit time”, or “collision cross-sections”, etc., in place of actually random quantities which have no tendency to “time self-averaging”.

For example, relative frequency of BP’s collisions of a given sort (e.g. with given impact parameter value) has no definite time average, merely because its random deviations from an imaginary “norm” (e.g. theoretical ensemble average) do not cause a compensating “back reaction” of the system, instead being constantly forgotten by it. Therefore, a hystogram of collisions’ distribution over sorts (impact parameters), as well as summary rate of various collisions, stay not smoothed out with time, instead undergoing time-scaleless (1/f-type) fluctuations irregularly distributed over the histogram [6,7].

In essence, all that was proved already in [36] (see Introduction) as logically inevitable consequence of the mixing property (exponential instability and chaoticity) of many-particle dynamics.

This dictates quite clear logics: if a flow of events produced by dynamical system is memoryless, - that is its past history does not influence onto its future, - then such flow has no certain (a priori predictable) “probabilities of events per unit time” 6. Besides, also logically inevitably, in such purely random flow all events are mutually statistically correlated, since all are commonly and equally responsible for resulting (a posteriori) number of events per unit time. Thus, as it was underlined in [36], in statistical mechanics physical (cause-and-consequence) independence of events does not necessarily mean their statistical independence 7.

In practice, unfortunately, scientists exploit the inverse “medieval” Bernoulli’s [37] logics, thinking that physically independent events must be statistically independent and furnished with a priori definite personal probabilities. An user of these assumptions and thus of the Bernoulli’s “law of large numbers” [37] implied by them, - e.g. in Boltzmannian kinetics, - usually is not aware that they automatically kill many potential possibilities for statistics of random events to be described 8.

Therefore, it will be not surprising if in this way of thinking the problem of “ubiquitous 1/f noise” never will be resolved 9.

3. These remarks may help to understand meaning of our path integral and its approximation corresponding to the “two-storey” division of BP’s motion into velocity fluctuations, Gaussian in themselves, and non-Gaussian fluctuations of velocity relaxation rate (RR). We named this “free RR approximation”, since in it the lower storey (RR fluctuations) keeps independent on the upper one (i.e. on resulting BP’s path).

Equivalently, one can speak about fluctuations of velocity relaxation time, or spectral intensity (power density), or BP’s diffusivity and mobility, or relative frequency and efficiency of BP’s collisions with atoms, or their effective cross-section, etc.

In the path integral approach all that is contained in any of several forms of the “interaction kernel”, particularly, in functional $\Gamma(t, A^\ast, A)$ and equivalent RR operators $\tilde{\Gamma}(A^\dagger, A)$ and $\hat{\Gamma}(V, \nabla_P)$ or, equivalently, in their integral-operator kernel $\Gamma(V_1, V_2)$, all introduced in Sections 7-8 for (most interesting for us here) the case of dilute gas and short-range BP-atom interaction.

Applying quantum terminology, these objects are analogues of “scattering matrix”, or “scattering operator”, of BP in gas. However, unlike usual kinetics, in our theory not matrix elements of this operator but its eigen-values and eigen-functions are of most importance. The eigen-values represent possible values of

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6 In other words, “probabilities per unit time” (or other characteristics with similar meaning) are different at different phase trajectories of a system in theory and in different experiments in practice.

7 To avoid confusions with folklore “ergodic hypotheses” and “ergodic theorems”, one should identify the “events” with transitions between instant system’s states rather than states themselves. On true relations between uncertainty of “probabilities” of events and 1/f-noise, on one hand, and the ergodic theory, on the other hand, see e.g. Introduction in [21].

8 Really, the law of large numbers shows that assigning personal independent probabilities to events is nothing but assigning a priori certain time limit to “number of events per unit time”. Moreover, even assigning a priori certain total number of events (since its relative uncertainty fast enough tends to zero). From physical viewpoint, this looks very strangely, as if all events were obeying some teleological external control and/or infinitely long-living common memory. Anyway, any low-frequency fluctuations a priori are excluded from such a theory.

9 On the contrary, in such way the beautiful mathematical probability theory may imperceptibly become turned into a kind of physical pseudo-science (a good example may be the factually axiomatic construction of hard-sphere gas kinetics, - see e.g. references and remarks in [1–3,21,23,28]). Though, of course, homo erratum est, and modern scientific “homo” not less than antique [40]. And, on the other hand, the firth Euclid axiom equally can be accepted or refused.
global, - that is time-averaged, - RR of BP’s velocity fluctuations. At that, spectrum of the eigenvalues never reduces to a single point (except may be limits of infinitely light atoms or infinitely hard BP). In opposite, we have shown that it occupies the whole real positive semi-axis (hence including arbitrary close vicinity of zero).

This principal mathematical fact means that BP’s random walk always possesses “kinetic non-ergodicity”, i.e. different relative frequencies and efficiencies of collisions (and, as a result, different diffusivities and mobilities) at different observations (experiments).

Thus, in the present paper for the first time a mathematical object - the RR operator - is pointed out what serves, under definite approximation, as quantitative measure of the kinetic non-ergodicity, being evidently expressed in terms of the underlying interaction potential and collision dynamics.

Inter-relation between eigen-functions of this operator and its above mentioned eigen-values says that the more irregular (sharply oscillating) is a perturbation of initial BP’s velocity probability distribution (histogram) the longer is its relaxation, in the sense that it weaker affects diversity of following BP’s trajectories and diffusivities exposed on them. In other words, in any set of BP copies (with randomly chosen initial states) all they almost surely will demonstrate substantially different diffusivities, - i.e. spectral powers of velocity fluctuations, - regardless of duration of their measurements. This, of course, is mere manifestation of exponential instability of dynamical BP’s path in respect to its perturbations.

4. At the same time, the RR operator is not something unprecedented. In fact, as was shown, at least in the case of dilute gas, it is simply connected with well known object, namely, the standard Boltzmann-Lorentz (BL) operator (on it see e.g. [38]). More concretely, the RR operator (RRO) is sealed up inside the BL operator (BLO) as its constituent part, like “genie in bottle”. There, inside the BLO, the RRO can play significant role by introducing quantitative dependence of BLO spectrum on the atom-BP mass ratio \( r = m/M \). But qualitatively, - from viewpoint of long-range statistics of BP’s random walk, - its presence there is not important. Indeed, the same (standard Gaussian) asymptotic diffusion law would take place even if we substituted the RRO inside BLO for trivial c-number constant, that is turned the BLO into Fokker-Planck operator

But the “release of the genie” in exact theory (or let approximate but correct enough in comparison with naive Boltzmannian kinetics) radically “changes things around”. Now, the mass ratio enters not only into mean BP’s diffusivity (mobility), but also into forms of RR and diffusivity (mobility) probability distributions and, consequently, into very qualitative construction and visual shape of (now non-Gaussian) diffusion law (total BP’s path probability distribution).

In particular, \( r = m/M \) determines amplitude and exponent of power-law long tails of long-time asymptotic of diffusion law, which in turn are closely connected to magnitude and spectrum exponent of low-frequency diffusivity (mobility) 1/f noise [6,15,18,21,24,5].

Unfortunately, calculation of the RR distribution was out of our possibilities here, since it requires accurate analysis of structure of the RRO eigen-functions (if not exact diagonalization of RRO). Nevertheless, for principal comparison with earlier results even our rough discussion of this distribution is sufficient (especially if remembering that the RRO itself is product of approximation of exact path integral). The discussion prompted that the RR distribution exponent \( \eta \) is nearly inversely proportional to the mass ratio, \( \eta(r) \propto 1/r \). This is in agreement with results obtained in the frameworks of two quite different approaches, - the “collisional approximation” to the BBGKY hierarchy [6,7,15,24] and the rigorous “dynamical virial relations” [17–19,21,4,5], - both yielding \( \eta(1) = 1 \) [15,18] \( \eta(r) = 1/r \) [24,5].

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10 In connection with this remark, it is interesting to recollect that the theory based on the BBGKY equations [6,7,15,24] does not lose the diffusivity 1/f noise (and related non-Gaussianity of diffusion law) if it takes into account conservation of center-of-mass velocity in every particular collision (and thus in any many-particle cluster of collisions), even though approximating momentum and energy exchange between colliding particles by artificial model Fokker-Planck collision operator. This fact highlights deep physical meaning of arbitrary small RRO’s eigen-values: they delegate conservation of total momentum of BP plus gas during collisions.

In this respect, notice that significant part of BP’s path always is collective center-of-mass flight whose contribution can be expressed by \( t[Mv + m \sum_{j=1}^{N(t)} v_j]/[M + mN(t)] \), where \( N(t) \) denotes number of past BP’s collisions with some of atoms during observation time \( t \), while \( V \) and \( v_j \) present (or initial) velocities of BP and that atoms.
Hence, we can state that now three very different approaches to the problem give essentially similar results, thus justifying one another.

5. Our consideration allows to expect that the expounded (here and in [1]) pseudo-quantum path integral representation of classical statistical dynamics can be usefully extended to “Brownian motion” in non-ideal gases (fluids) with arbitrary density. Moreover, to mutual diffusion and drift and other transport processes in mixtures of different fluids, and may be even to the hydrodynamics as the whole.

A principal base for such the extension is given by the Bogolyubov functional evolution equation [41] which, however, preliminarily should be reformulated in terms of suitably defined irreducible many-particle correlation (cumulant) functions and their generating functionals, in analogy with what was made in [1,16–19,21,4,28,5]. Then, after such preparation, one can introduce effective quantum field operators like $a^\dagger(x)$ and $a(x) = \delta/\delta a^\dagger(x)$ above, with $x$ enumerating $\mu$-phase space points (and fluid species).

Of course, on this way generally we have to deal with more complicated effective “Hamiltonians” (in essence, Liouville operators) than previously met ones. At least, with Hamiltonians containing four-particle interactions, i.e. terms with $a^\dagger(x)a^\dagger(y)a(x')a(y')$. Nevertheless, the corresponding formalism, including path integrals, undoubtedly will be able to create meaningful non-trivial approximations (and, may be, unify the pseudo-quantum and truly quantum theories \footnote{Notice that in [26,27] very close formalisms for actually quantum Brownian particles in quantum many-particle thermostats already were suggested and partly developed (up to rather general “theorem on fundamental $1/f$ noise” [30]).}). I wish good luck to followers.

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