Boundary flex control for the systems governed by Boussinesq equation with the nonstandard boundary conditions

Gol Kim\textsuperscript{a}, Gennady Valentinovich Alekseev\textsuperscript{b}

\textsuperscript{a} Center of Natural Sciences, University of Sciences, Kwahakdong-1, Unjong District, Pyongyang, DPR Korea
\textsuperscript{b} Computational Fluid Dynamics, Laboratory of Institute for Applied Mathematics, FEB RAS, 7, Radio St., Vladivostok, 690041, Russia (E-mail: alekseev@iam.dvo.ru)

Abstract. In this paper, the boundary flex control problem of non-stationary equation governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are dependent on temperature has been studied. The boundary condition for velocity of fluid is non-standard boundary condition: specifically the case where dynamical pressure is given on some part of the boundary and the boundary condition for temperature of fluid is mixed boundary condition has been considered. First, we have proved the existence of the optimal control. Then the optimal condition has been derived. Pontryagin's maximum principle in the special case has been derived.

Keyword: Boussinesq equation, boundary flex control, press boundary condition, mixed boundary condition

1. Introduction

The extremal problem for Navier-Stokes equation and optimal control of fluid dynamical equation are studied by various authors (for example; [6-10], [17-19]).

In [22] the existence of time optimal controls for the Boussinesq equation has been obtained and derived Pontryagin’s maximum principle of time optimal control problem governed by the Boussinesq equation. In [23] an optimal control problem governed by a system of nonlinear partial differential equations modeling viscous incompressible flows submitted to variations of temperature has been consider. A generalized Boussinesq approximation has been used. The existence of the optimal control as well as first order optimality conditions of Pontryagin type by using the Dubovitskii-Milyutin formalism has been obtained. In [24] the stationary Boussinesq equations describing the heat transfer in the viscous heat-conducting fluid under inhomogeneous Dirichlet boundary conditions for velocity and mixed boundary conditions for temperature are considered. The optimal control problems for these equations with tracking-type functionals are formulated. A local stability of the concrete control problem solutions with respect to some disturbances of both cost functionals and state equation is proved.

In [25] the boundary control problems of the model of heat and mass transfer in a viscous incompressible heat conducting fluid has been considered. The model consists of the Navier-Stokes equations and the convection-diffusion equations for the substance concentration and the temperature that are nonlinearly related via buoyancy in the Oberbeck–Boussinesq approximation and via convective mass and heat transfer.

In [25] control problems for stationary magnetohydrodynamic equations of a viscous heat-conducting fluid under mixed boundary conditions has been considered.

In [15, 16] the Karhunen-Loeve Galerkin method for the inverse problems of Boussinesq equation have been studied.

In [20] the problem of stabilization of the Boussinesq equation via internal feedback controls has been studied. In [21] the problem of local internal controllability of the Boussinesq system has been studied. Solvability of control problems for stationary equations of magnetohydrodynamics of a viscous fluid and control problems for stationary magnetohydrodynamic equations of a viscous heat-conducting fluid under mixed boundary conditions have been researched by [30], [28] respectively.

In this paper, the boundary flex optimal control for the evolution equation governing the coupled
mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are dependent on temperature has been studied. The boundary condition for velocity of fluid is non-standard boundary condition: specifically the case where dynamical pressure is given on some part of the boundary will be considered. The boundary condition for temperature of fluid is mixed boundary condition.

The existence of the optimal control has been proved. Then the optimal condition has been derived.

Let \( \Omega \subset \mathbb{R}^N \) (N=2, 3) be a bounded domain with smooth boundary \( \Gamma \). Let \( \Gamma \) be divided by into two parts \( \Gamma_1, \Gamma_2 \) such that \( \Gamma = \Gamma_1 \cup \Gamma_2 (\Gamma_1 \cap \Gamma_2 \neq \emptyset) \), \( T (0<T<\infty) \) is given number.

We denote \( Q=\Omega \times (0,T), \Sigma_i = \Gamma_i \times (0,T) \) (i=1,2), \( \Sigma = \Gamma \times (0,T) \).

We assume that the state of control systems is given by non-stationary Boussinesq equation with the dynamical pressure condition and mixed boundary condition on some part of the boundary as follows:

\[
\begin{align*}
\frac{\partial z}{\partial t} - \nu \Delta z + (z, \nabla) z &+ \beta g w = -\text{grad } \pi; & Q \\
\text{div } z & = 0; & Q \\
\frac{\partial w}{\partial t} - k \Delta w + (z, \nabla) w & = 0; & Q \\
z_i &= 0, \pi + \frac{1}{2} |z|^2 = \nu_1, w = 0; & \Sigma_i \\
z &= 0, -k \frac{\partial w}{\partial n} = \nu_2; & \Sigma_2 \\
z(0) &= z_0, w(0) = w_0; & \Omega
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N \) (N=2, 3) is a bounded domain with smooth boundary \( \Gamma \). \( \Gamma \) is divided by into two parts \( \Gamma_1, \Gamma_2 \) such that \( \Gamma = \Gamma_1 \cup \Gamma_2 (\Gamma_1 \cap \Gamma_2 \neq \emptyset) \), \( T (0<T<\infty) \) is given number.

We denote \( Q=\Omega \times (0,T) \), \( \Sigma_i = \Gamma_i \times (0,T) \) (i=1,2), \( \Sigma = \Gamma \times (0,T) \) and \( n \) note the outer normal vector to \( \Gamma \). In the Eqs. (1)-(6) \( z(x,t) \in \mathbb{R}^N \) denotes the velocity of the fluid at point \( x \in \Omega \) at time \( t \in [0,T] \); \( \pi(x,t) \in \mathbb{R} \) is the hydrostatic pressure; \( w(x,t) \in \mathbb{R} \) is temperature; \( g \) is the gravitational vector, and \( \nu > 0 \) and \( k > 0 \) are kinematic viscosity and thermal conductivity, respectively; \( \beta \) is a positive constant associated to the coefficient of volume expansion; \( \nu_1 \) and \( \nu_2 \) are the given functions on \( \Sigma_i \) and \( \Sigma_2 \) respectively. In Eqe.(1) \( \beta > 0 \) is the coefficient of volume expansion and \( \xi \) is the gravitational function.

The expressions \( \nabla, \Delta \) and \( \text{div} \) denote the gradient, Laplacian and divergence operators, respectively (sometimes, we will also denote the gradient operator by \( \text{grad} \)); \( i \)th component in Cartesian coordinates of \( (z, \nabla) z \) is given by

\[
(z, \nabla) z_i = \sum_{j=1}^N z_j \frac{\partial z_i}{\partial x_j}, \text{ also } (z, \nabla) w = \sum_{j=1}^N z_j \frac{\partial w}{\partial x_j}
\]

In the boundary condition (4) \( z_\nu = z_n, z_\nu = (z \cdot n)n \).

There are the results of research of Boussinesq equation and the generalized Boussinesq system with nonlinear thermal diffusion in [1-4, 23]. But boundary conditions of those papers are homogenous.

We assume that the cost functional \( J[v] \) is given as following:

\[
J[v] = J[v, y] = N_1 \int_0^T \int_{r_1} r_1(s,t)z_n(s,t)dsdt + N_2 \int_{r_2} r_2(s,t) \frac{\partial w}{\partial n}dsdt
\]

where \( N_1, N_2 > 0 \) are given real numbers and \( r_1(x,t) \in [L^2(\Sigma_i)]^N \), \( r_2(x,t) \in L^2(\Sigma_2) \) are given functions.
\[
\int_{\Gamma_1} r_i(s,t)z_n(s,t)ds = \int_{\Gamma_1} (r_i(s,t), z_n(s,t))ds \quad \text{and} \quad (r_i(s,t), z_n(s,t)) \text{denote the scalar product in} \ [L^2(\Sigma_i)]^N. \text{We denote} \ v = \{v_1, v_2\} \text{ and} \ y = \{z, w\}.
\]

Then, the problem that we are going to consider is to find the \(v^* \in U_a \) satisfying:

\[
\inf_{v \in U_a} J[v] = J[v^*] \quad (1.2)
\]

Here, we assume that the admissible control sets \(U_a = U_{1a} \times U_{2a}\) are defined such as:

\[
U_{1a} = \{v_1 | v_1 \in L^2(0,T; L^2(\Gamma_1))^N, 0 < \alpha_1(x,t) \leq v_1(x,t) \leq \beta_1 (\text{almost everywhere})\} \quad (1.3)
\]

\[
U_{2a} = \{v_2 | v_2 \in L^2(0,T; L^2(\Gamma_2))^N, 0 < \alpha_2 \leq v_2(x,t) \leq \beta_2 (\text{almost everywhere})\} \quad (1.4)
\]

\(\alpha_i(x,t), \beta_i(x,t)(i=1,2)\) are given functions in the function space \(L^2(\Sigma_i)\).

\(v_1(x,t) = (v_{11}(x,t), v_{12}(x,t), \ldots, v_{1N}(x,t))\) and expression \(0 < \alpha_i(x,t) \leq v_i(x,t) \leq \beta_i\) means that \(0 < \alpha_i(x,t) \leq v_{ii}(x,t) \leq \beta_i (\forall i \in N)\).

The established optimization problem (1.1)-(1.4) is an optimal boundary flex control problem.

For the convenience, we have assumed that control parameters are the flex pressure \(v_1\) on the boundary \(\Sigma_1\) and the heat flex \(v_2\) on the boundary \(\Sigma_2\).

To illustrate the example of extremal condition (1.2), we can take functions \(r_i(x,t)\) and \(r_2(x,t)\) as following;

\[
r_i(x,t) = \begin{cases} 1 & : x \in \Gamma_\epsilon \subset \Gamma_1 \\ 0 & : x \notin \Gamma_\epsilon \subset \Gamma_1 \end{cases}, \quad r_2(x,t) = \begin{cases} 1 & : x \in \Gamma_\epsilon \subset \Gamma_2 \\ 0 & : x \notin \Gamma_\epsilon \subset \Gamma_2 \end{cases}
\]

Where, \(I\) is a unit vector. Then, the optimization problem (1.2) is described as the problem that fluid flex passed the boundary \(\Gamma_\epsilon\) under the restriction for the flex pressure and heat flex passed the boundary \(\Gamma_\epsilon\) under the restriction for the heat flex must minimize.

2. Preliminaries

In this article the functions are either \(R\) or \(R^N (N = 2 \text{ or } N = 3)\) and as usual simplification, sometimes we will not distinguish them in our notations; the difference will be clear from the context. The \(L^2(\Omega)\)-product and norm are denoted by \((\cdot, \cdot)\) and \(\|\cdot\|\) respectively: the \(H^m(\Omega)\) norm is denoted by \(\|\cdot\|_m\). Here \(H^m(\Omega) = W^{m,2}(\Omega)\) is the usual Sobolev spaces (see [1] for their properties); \(H^{-1}(\Omega)\) denotes the dual spaces of \(H^1_0(\Omega)\). \(H^0(\Omega)\) is the same as \(L^2(\Omega)\) and \(\|\cdot\|_0\) is the same as \(L^2\)-norm \(\|\cdot\|\). \(D(0,T)\) is the class of \(C^\infty\) -functions with compact support in \((0,T)\). \(D(0,T)'\) are its associated spaces of distribution.

If \(B\) is a Banach space, we denoted by \(L^a(0,T;B)\) the Banach space of the B-valued functions defined in the interval \((0,T)\) that are \(L^a\) -integrable.

Now we introduce some spaces such as:

\[
\begin{align*}
D &= \{\psi | \psi \in (C^\infty(\Omega))^N, \text{div} \psi(x) = 0(x \in \Omega), \psi_\epsilon(x) = 0(x \in \Gamma_2)\} \\
H &= \text{completion of} \ D \text{ under the} \ [L^2(\Omega)]^N \text{-norm} \\
V &= \text{completion of} \ D \text{ under the} \ [H^1(\Omega)]^N \text{-norm} \\
D_{\Gamma_1} &= \{\phi : \phi \in C^\infty(\Omega), \phi(x) = 0(x \in \Gamma_1)\} \\
\tilde{H} &= \text{closure of} \ D_{\Gamma_1} \text{ in} \ L^2(\Omega) \\
W &= \text{closure of} \ D_{\Gamma_1} \text{ in} \ H^1(\Omega)
\end{align*}
\]

(2.1)

Naturally, the norm of \(H\) or \(\tilde{H}\) is also denoted by \(\|\cdot\|\), and the norm of \(V\) or \(W\) is denoted by \(\|\|\).
as well. The dual product between $V^*$ and $V$ or $W^*$ and $W$ (also the inner product in $H^{-1}(\Omega)$ and $H^1_0(\Omega)$) are denoted by $\langle \cdot , \cdot \rangle$.

Now, generally, we assume that $v_1 \in L^2(0,T;(H^{-1/2}(\Gamma_1))^N)$, $v_2 \in L^2(0,T;H^{-1/2}(\Gamma_2))$.

Then, we shall prove the existence of the weak solution for state equation (1.1) Suppose that $(z,w)$ is a classical solution of (1.1). Multiplier the first equation of (1.1) by $\psi \in V$, integrate by parts over $\Omega$ and take the boundary condition into account to get

$$\frac{d}{dt} (z,\psi) + v_1(z,\psi) + b(z,z,\psi) + (\beta \xi w,\psi) = \langle v_1,\psi \rangle - \langle \psi, n \rangle.$$  \hfill (2.2)

Multiplier the second equation of (1.1) by $\varphi \in W$, integrate by parts over $\Omega$, and take the boundary conditions into account to obtain

$$\frac{d}{dt} (w,\varphi) + k\alpha_2(w,\varphi) + c(z,w,\varphi) = \langle v_2,\varphi \rangle.$$  \hfill (2.3)

Now let $\chi \in C^1[0,T]$ be a function such that $\chi(T) = 0$. Multiplier Equations (2.2) and (2.3) by $\chi$ respectively, and integrate by parts to yield

$$\int_0^T -(z(t),\chi') dt + \int_0^T (v_1(z(t),\chi(t)) + b(z(t),z(t),\chi(t)) + (\beta \xi w(t)\xi,\chi(t))) dt$$

$$= \int_0^T (v_1(t)\chi(t)) dsdt + (z_0,\chi)(0) \hfill (2.4)$$

$$\int_0^T -(w(t),\varphi') dt + \int_0^T (k\alpha_2(w(t),\varphi(t)) + c(z(t),w(t),\varphi(t))) dt$$

$$= \int_0^T (v_2(t),\varphi(t)) dsdt + (w_0,\varphi)(0) \hfill (2.5)$$

where $z(\cdot,t) = z(t), w(\cdot,t) = w(t), v_i(\cdot,t) = v_i(t), i = 1,2$ by abuse of notation without confusion from the context, and

$$(z,\psi) = \sum_{j=1}^N \int_{\Omega} z_j(x,\cdot)\psi_j(x)dx, \quad (w,\varphi) = \int_{\Omega} w(x,\cdot)\varphi(x)dx$$

$$a_1(u,\psi) = (\text{rot} z(x,\cdot),\text{rot} \psi) = \int_{\Omega} (\nabla \times u) \cdot (\nabla \times \phi) dx, \quad b(z,z,\psi) = \int_{\Omega} (\text{rot} z(\cdot) \times (x,\cdot))\psi(x)dx,$$

$$a_2(w,\varphi) = \sum_{j=1}^N \int_{\Omega} \frac{\partial w(x,\cdot)}{\partial x_j} \frac{\partial \varphi}{\partial x_j} dx, \quad c(z,w,\varphi) = \sum_{j=1}^N \int_{\Omega} z_j(x,\cdot)\frac{\partial w(x,\cdot)}{\partial x_j} \varphi(x)dx$$

The following Lemmas 2.1-2.3 can be obtained by the Sobolev inequalities and the compactness theorem. We can also refer to theorem 1.1 of [31] on page 107 and lemmas 1.2, 1.3 of [31] on page 109 (see also chapter 2 of [13]). The similar arguments can be also found in lemmas 1, 5 of [30].

**Lemma 2.1.** The bilinear forms $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ are coercive over $V$ and $W$ respectively. That is, there exist constants $c_1,c_1' > 0$ such that

$$a_1(z,z) \geq c_1\|z\|^2, \forall z \in V \text{ and } a_2(w,w) \geq c_1'\|w\|^2, \forall w \in W$$

**Lemma 2.2.** The trilinear forms $b(\cdot,\cdot,\cdot)$ is a linear continuous functional on $[H^1(\Omega)]^N$. That is, there exist a constant $c_2 > 0$ such that

$$|b(u,v,w)| \leq c_2\|u\|\|v\|\|w\|. \quad \forall u,v,w \in [H^1(\Omega)]^N$$

Moreover, the following properties hold true

Moreover, the following properties hold true
Lemma 2.3. The tri- linear form $c(\cdot, \cdot, \cdot)$ is a linear continuous functional defined on $V \times W \times W$. That is, there exist a constant $c_3 > 0$ as following:

\[ |c(z, w, \varphi)| \leq c_3 \|z\| \|w\| \|\varphi\|, \quad \forall z \in V, \quad \forall w, \varphi \in W \]

Moreover, the following properties hold true

(i) $c(z, w, w) = 0, \forall z \in V, \forall w \in W$

(ii) $c(z, w, \varphi) = -c(z, \varphi, w), \forall z \in V, \forall w, \varphi \in W$

(iii) If $z_m \rightharpoonup z$ weakly on $V$ and $w_m \rightharpoonup w$ strongly on $H$, then

\[ c(z_m, w_m, \varphi) \rightharpoonup c(z, w, \varphi), \forall z \in V, w \in \tilde{H}, \varphi \in W. \]

Definition 1. Let $Y = Z \times W = (L^2(0, T; V) \cap L^\infty(0, T; H)) \times (L^2(0, T; W) \cap L^\infty(0, T; \tilde{H})$.

Suppose that $v_1 \in (L^2(0, T; (H^{-1/2}(\Gamma_1))^N), v_2 \in (L^2(0, T; H^{1/2}(\Gamma_2)), z_0 \in H, w_0 \in \tilde{H}, g \in L^\infty(\Omega)$.

The pair $(y, w)$ is said to be a weak solution of (1.1) if it satisfies

\[ \begin{align*}
(y, w', \psi) + v_1 a_1 (z, \psi') + b(z, z, \psi) + (\beta \xi w, \psi) &= v_1 \psi_n \geq \Gamma_1, \quad \forall \psi \in V \\
(w', \varphi) + k a_2 (w, \varphi) + c(z, w, \varphi) &= v_2 \varphi \geq \Gamma_2, \quad \forall \varphi \in W \\
z(0) = z_0, w(0) = w_0
\end{align*} \]  \hspace{1cm} (2.7)

Next, we reformulate Equation (2.7) into the operator equation. To this purpose, it is noticed that for a fixed $\psi \in V$, the functional $\psi(\in V) \rightarrow a_1 (z, \psi)$ is linear continuous. So there exists an $A_1 z \in V^*$ such that

\[ < A_1 z, \psi > = a_1 (z, \psi), \quad \forall \psi \in V \]  \hspace{1cm} (2.8)

Similarly, for fixed $u, v \in V, w \in V \rightarrow b(u, v, w)$ is a linear continuous functional on $V$. Hence there exist a $B(u, v) \in V^*$ such that

\[ < B(u, v), w > = b(u, v, w), \quad \forall w \in V \]  \hspace{1cm} (2.9)

We denote $B(u) = B(u, u)$. Define

\[ L_1 (\psi) = (v_1, \psi_n)_{\Gamma_1} = \int_{\Omega_1} v_1 \psi_n ds, \quad \forall \psi \in V. \]  \hspace{1cm} (2.10)

Then, fixed $v_1 \in (L^2(0, T; (H^{-1/2}(\Gamma_1))^N),$ the functional $\psi(\in V) \rightarrow L_1 (\psi) = (v_1, \psi_n)_{\Gamma_1}$ is linear continuous. So that there exist constant $c_4 > 0$ such that

\[ \|L_1 \psi\| < c_4 \|\psi\|, \forall \psi \in V. \]

So there exists an $H_1 v_1 \in V^*$ such that

\[ < H_1, v_1, \psi > = (v_1, \psi_n)_{\Gamma_1}, \forall \psi \in V \]  \hspace{1cm} (2.11)

With these operators at hand, we can write the second equation of (2.7) as

\[ \frac{dz}{dt} + v A_1 z + B(z) + \beta \xi w = H_1 v_1 \]  \hspace{1cm} (2.12)

Similarly, we have

\[ < A_2 w, \varphi > = a_2 (w, \varphi), \quad < C(z, w, \varphi) > = c(z, w, \varphi), \quad A_2 w, C(z, w) \in W^* \]  \hspace{1cm} (2.13)

Define

\[ L_2 (\varphi) = (v_2, \varphi)_{\Gamma_2} = \int_{\Gamma_2} v_2 \varphi ds, v_2 \in L^2(\Gamma_2), \forall \varphi \in W \]  \hspace{1cm} (2.14)
Then, fixed \( v_2 \in (L^2(0,T;H^{-1/2}(\Gamma_2))) \) the functional \( \varphi(\in W) \to L_2(\varphi) = (v_2, \varphi)_{\Gamma_2} \) is linear continuous.

Then, the operator \( L_2 \) is a linear continuous functional defined on \( W \) and so there exists constant \( c_5 > 0 \) such that
\[
\|L_2\varphi\| \leq c_5 \|\varphi\|_W, \quad \forall \varphi \in W .
\]

So there exists an \( H_2v_2 \in W^* \) such that
\[
< H_2v_2, \varphi > = (v_2, \varphi)_{\Gamma_2}, \quad \forall \varphi \in V
\]

By these operators defined above, we can write the first equation of (2.7) as
\[
\frac{dw}{dt} + kA_2w + C(z,w) = H_2v_2 \tag{2.16}
\]

Combining (2.12) and (2.16), we can write (2.7) in the abstract evolution equation as follows:
\[
\begin{cases}
\frac{dz}{dt} + vA_1z + B(z) + \beta\xi_\nu = H_1v_1, \\
\frac{dw}{dt} + kA_2w + C(z,w) = H_2v_2,
\end{cases} \tag{2.17}
\]

\[z(0) = z_0, w(0) = w_0\]

**Lemma 2.4.** If \( z \in L^2(0,T;V) \), then \( B(z) \in L^1(0,T;V^*) \); and if \( w \in L^2(0,T;W) \), then \( C(z,w) \in L^1(0,T;W^*) \).

Proof. By applying Hölder inequality and compactness of embedding \( H^1(\Omega) \subset L_4(\Omega) \), we obtain
\[
\|B(z,\varphi)\| = |B(z,\varphi)| \leq c'_6 \|\varphi\|_V \leq c_6 \|\varphi\|_{L^1} \leq c_6 \|\varphi\| \leq c_6 \|\varphi\|, 
\]

for some constants \( c'_6, c_6 > 0 \) and hence \( B(z) \|_V \leq c_6 \|\varphi\| \) which shows that \( B(z) \in L^1(0,T;V^*) \).

The proof is complete.

Similarly, we have
\[
\|C(z,w,\varphi)\| = |C(z,\varphi,w)| \leq c_6 \|\varphi\|_V \leq c_6 \|\varphi\| \leq c_6 \|\varphi\|, 
\]

for some constants \( c_6 > 0 \) and hence \( B(z,\varphi) \|_V \leq c_6 \|\varphi\| \) which shows that \( C(z,w) \in L^1(0,T;W^*) \) for all \( \varphi \in W \). The proof is complete.

We specify the constants \( c_i, i = 1,2,\ldots \) used in this section in the remaining part of the paper. The following Lemma 2.5 comes from theorem 2.2 of [12] on page 220

**Lemma 2.5.** Let \( X_0, X, X_1 \) be Hilbert spaces with the compact embedding relations
\[
X_0 \subset X \subset X_1
\]

Then, for arbitrary bounded set \( K \subset R^1 \), \( v > 0 \), embedding \( H^v_K(R^1;X_0, X_1) \subset L^2(R^1;X) \) is compact, where
\[
H^v_K(R^1;X_0, X_1) = \{v \in H^v_K(R^1;X_0, X_1) : \text{sup } p \in K, \hat{p}^v \in L^2(R;X_1)\}
\]
\[
H^v_K(R^1;X_0, X_1) = \{v \in L^2(R^1;X_0), \hat{p}^v \in L^2(R^1;X_1)\}
\]
\[
\hat{p}^v(\tau) = (2\pi i\tau)^v \hat{v}(\tau), \quad \hat{v}(\tau) = \int_{-\infty}^{\infty} e^{-2\pi i\tau \varepsilon} v(t) dt
\]
\[
\|\hat{v}\|_{L^2(R;X_1)} \leq \|\hat{v}\|_{L^2(R;X_0)} + \|\tau^v \hat{v}\|_{L^2(R;X_1)}
\]

**3 Existence of the weak solution for state equation**

This section discusses the existence of the weak solution defined by Definition 1 to Equation (1.1).
The main idea is to construct a Galerkin approximation scheme and make some prior estimates. Choose two orthogonal bases \( \{ u_j \}_{j=1}^m \) for \( V \) and \( \{ \mu_j \}_{j=1}^m \) for \( W \) respectively. Construct the Galerkin approximation solutions:

\[
z_m(x,t) = \sum_{j=1}^m q_{jm}(t)u_j(x), \quad w_m(x,t) = \sum_{j=1}^m h_{jm}(t)\mu_j(x)
\]  

(3.1)
such that for all \( j \in \mathbb{N}^+ \), \( \{ z_m, w_m \} \) satisfies

\[
\begin{align*}
(\z_m(t), u_j) + a_1(z_m(t), z_m(t)) + \beta(w_m(t)\xi, z_m) &= (v_1(t), z_m)_{\Gamma_t} \\
(w_m(t), \mu_j) + a_2(w_m(t), w_m(t)) &= (v_2(t), \mu_j)_{\Gamma_t}
\end{align*}
\]

(3.2)

where \( \z_m(0) = z_{m0} \to z_0 \) in \( H \), \( w_m(0) = w_{m0} \to w_0 \) in \( \tilde{H} \), \( j = 1, 2, \ldots \)

where \( z_{m0} \) is the orthogonal projection of \( z_0 \) in \( H \) on the subspace spanned by \( \{ u_j \}_{j=1}^m \) and \( w_{m0} \) is the orthogonal projection of \( w_0 \) in \( \tilde{H} \) on the subspace spanned by \( \{ \mu_j \}_{j=1}^m \).

Once again, we write \( z_m(x,t) = z_m(\cdot,t) \), \( w_m(x,t) = w_m(\cdot,t) \) by abuse of notation without the confusion from the context. It is seen that for any \( m \in \mathbb{N}^+ \), system (3.2) is a system of nonlinear differential equations with the unknown variables \( \{ q_{jm}(t), h_{jm}(t) \} \) and the initial values \( \{ q_{jm}(0) = (z_0, u_j), h_{jm}(0) = (w_0, \mu_j) \}_{j=1}^m \). By the assumption, this initial value problem admits a solution in some interval \([0, t_m] \). We need to show that \( t_m = T \equiv t \).

**Lemma 3.1.** Let \( \{ z_m, w_m \} \) be the sequence satisfying (3.2). Then there exists a subsequence of \( \{ z_m, w_m \} \), still denoted by itself without confusion, such that

\[
z_m \to z \text{ weakly in } L^2(0,T;V) \text{ and } z_m \to z \text{ weakly star in } L^\infty(0,T;H),
\]

(3.3)

where \( z \in L^2(0,T;V) \cap L^\infty(0,T;H) \), and

\[
w_m \to w \text{ weakly in } L^2(0,T;W) \text{ and } w_m \to w \text{ weakly star in } L^\infty(0,T;\tilde{H}),
\]

(3.4)

where \( w \in L^2(0,T;W) \cap L^\infty(0,T;\tilde{H}) \).

Proof. Sum for \( j \) from 1 to \( m \) in (3.2) and apply Lemmas 2.2, 2.3, to get

\[
\begin{align*}
(\z_m(t), z_m(t)) + a_1(z_m(t), z_m(t)) + \beta(w_m(t)\xi, z_m) &= (v_1(t), z_m)_{\Gamma_t} \\
w_m(t), w_m(t) &= (v_2(t), w_m)_{\Gamma_t}
\end{align*}
\]

(3.5)

By assumption (2.6), for any given \( \varepsilon > 0 \), we can get from (3.5) that

\[
\frac{d}{dt}|z_m(t)|^2 + c_1\|z_m(t)\|^2 \leq -\beta\|w_m(t)\|\xi, z_m(t)) + (v_1(t), z_m)_{\Gamma_t} \leq \frac{1}{2}\beta\|z_m\|^2 + \frac{1}{2}\beta\|w_m(t)\|^2
\]

(3.6)

Here and in what follows, we denote \( \|z_m(t)\|_{H^{1/2}(\Gamma_t)} \) and \( \|v_1(t)\|_{H^{1/2}(\Gamma_t)} \) simply by \( \|z_m(t)\|_{H^{1/2}(\Gamma_t)} \) and \( \|v_1(t)\|_{H^{1/2}(\Gamma_t)} \) respectively by abuse of the notation.

By the trace theorem from \( H^1(\Omega) \) to \( H^{1/2}(\Gamma) \), there exists a constant \( c_{10} \) such that

\[
\|z_m(t)\|_{H^{1/2}(\Gamma_t)} \leq c_{10}\|z_m(t)\|
\]

Substitute above into (3.6) to yield

\[
\frac{d}{dt}|z_m(t)|^2 + [c_1 - (\beta\frac{1}{e}\|z_m\|^2 + c_{10}e^2)]\|z_m(t)\|^2 \leq -\beta\|z_m\|^2 + \frac{1}{e^2}\|w_m(t)\|^2 + \frac{1}{e^2}\|v_1(t)\|_{H^{1/2}(\Gamma_t)}^2
\]

(3.7)

Setting \( e^2 = c_1 / (2\beta\frac{1}{e}\|z_m\|^2 + c_{10}) \) in (3.7) gives
\[
\frac{d}{dt} \left| z_m(t) \right|^2 + \frac{c_1}{2} \left\| z_m(t) \right\|^2 \\
\leq \frac{2(\beta\|\mathbf{e}\|_w + c_{10})}{c_1} \beta\|\mathbf{e}\|_w \left\| w_m(t) \right\|^2 + \frac{2(\beta\|\mathbf{e}\|_w + c_{10})}{c_1} \left\| v_1(t) \right\|^2_{H^{-1/2}(\Gamma_1)}
\]
(3.8)

By assumption (2.6) again, for any given \( \varepsilon > 0 \), we can get from (3.5) that
\[
\frac{d}{dt} \left| w_m(t) \right|^2 + c'_1 \left\| w_m(t) \right\|^2 \leq \varepsilon^2 \left\| w_m(t) \right\|^2_{H^{-1/2}(\Gamma_1)} + \frac{1}{\varepsilon^2} \left\| v_2(t) \right\|^2_{H^{-1/2}(\Gamma_1)}
\]
(3.9)

By the trace theorem from \( H^1(\Omega) \) to \( H^{1/2}(\Gamma) \), there exists a constant \( c_{11} \) such that
\[
\left\| w_m(t) \right\|^2_{H^{1/2}(\Gamma_1)} \leq c_{11} \left\| w_m(t) \right\|
\]

Substitute above into (3.9) to yield
\[
\frac{d}{dt} \left| w_m(t) \right|^2 + \frac{c'_1}{2} \left\| w_m(t) \right\|^2 \leq \frac{1}{\varepsilon^2} \left\| v_2(t) \right\|^2_{H^{-1/2}(\Gamma_1)}
\]
(3.10)

Setting \( \varepsilon^2 = c'_1/(2c_{11}) \) in (3.10) gives
\[
\frac{d}{dt} \left| w_m(t) \right|^2 + \frac{c'_1}{2} \left\| w_m(t) \right\|^2 \leq \frac{2c_{11}}{c_1} \left\| v_2(t) \right\|^2_{H^{-1/2}(\Gamma_1)}
\]
(3.11)

Integrate (3.11) over \([0; T]\) with respect to \( t \) to give
\[
\left| w_m(T) \right|^2 + \frac{c'_1}{2} \int_0^T \left\| w_m(t) \right\|^2 dt \leq \frac{2c_{11}}{c_1} \int_0^T \left\| v_2(t) \right\|^2_{H^{-1/2}(\Gamma_1)} dt + \left| w_m(0) \right|^2
\]
(3.12)

Since the right-hand side of (3.12) is bounded, we have
\[
\{ w_m \} \text{ is a bounded sequence in } L^2(0,T;W)
\]
(3.13)

Replace \( T \) by \( t \in [0, T] \) in (3.12) to obtain
\[
\text{ess sup}_{t} \left| w_m(t) \right|^2 \leq \frac{2c_{11}}{c_1} \int_0^T \left\| v_2(t) \right\|^2_{H^{-1/2}(\Gamma_1)} dt + \left| w_m(0) \right|^2
\]
(3.14)

Hence
\[
\{ w_m \} \text{ is a bounded sequence in } L^\infty(0,T;\tilde{H})
\]
(3.15)

On the other hand, integrate (3.8) over \([0; T]\) with respect to \( t \) to give
\[
\left| z_m(T) \right|^2 + \frac{c'_1}{c} \int_0^T \left| z_m(t) \right|^2 dt \leq \frac{2(\beta\|\mathbf{e}\|_w + c_{10})}{c_1} \beta\|\mathbf{e}\|_w \int_0^T \left\| w_m(t) \right\|^2 dt + \frac{2(\beta\|\mathbf{e}\|_w + c_{10})}{c_1} \int_0^T \left\| v_1(t) \right\|^2_{H^{-1/2}(\Gamma_1)} dt + \left| z(0) \right|^2
\]
(3.16)

Therefore
\[
\{ z_m \} \text{ is a bounded sequence in } L^2(0,T;V)
\]
(3.17)

Replace \( T \) by \( t \in [0, T] \) in (3.16) to get
\[
\text{ess sup}_{t} \left| w_m(t) \right|^2 \leq \frac{2(\beta\|\mathbf{e}\|_w + c_{10})}{c_1} \beta\|\mathbf{e}\|_w \int_0^T \left\| w_m(t) \right\|^2 dt + \frac{2(\beta\|\mathbf{e}\|_w + c_{10})}{c_1} \int_0^T \left\| v_1(t) \right\|^2_{H^{-1/2}(\Gamma_1)} dt + \left| z(0) \right|^2
\]
(3.18)

Therefore,
\[
\{ z_m \} \text{ is a bounded sequence in } L^\infty(0,T;H)
\]
(3.19)
(3.3) and (3.4) then follow from (3.13), (3.17), (3.15) and (3.19).

**Lemma 3.2.** Let \( \{z_m, w_m\} \) be the sequence determined by Lemma 3.1. Then there exists a sequence of \( \{z, w\} \), still denoted by itself without confusion, such that

\[
z_m \to z \text{ strongly in } L^2(0; T; H), \quad w_m \to w \text{ strongly in } L^2(0; T; \tilde{H}).
\]

Proof. By virtue of Lemmas 2.1-2.3, we can write (3.2) as follows:

\[
\begin{aligned}
\frac{dz_m(t)}{dt}, u_j &= (H_1 - \beta \tilde{w}_m(t) - \nu A_1 z_m(t) - B(z_m(t)), u_j), \forall j = 1, 2, \ldots, m, \\
\frac{dw_m(t)}{dt}, \mu_j &= (H_2 - C(z_m(t), w_m(t)) - kA_2 w_m(t), \mu_j), \forall j = 1, 2, \ldots, m,
\end{aligned}
\]

Denote by \( \{\tilde{z}_m, \tilde{w}_m\} \) the \( \{z_m, w_m\} \) with zero values outside of \([0; T]\) and \( \{\hat{z}_m, \hat{w}_m\} \) the Fourier transformations of \( \{\tilde{z}_m, \tilde{w}_m\} \). We claim that there exists a \( \nu > 0 \) such that

\[
\int_{-\infty}^{+\infty} \|\tilde{z}_m(\tau)\|^2 d\tau < \infty.
\]

To this end, we write the \( \_\_\_r \)st equation of (3.21) as

\[
\frac{d}{dt} (z_m, u_j) = <\tilde{f}_m, u_j > + (z_{0m}, u_j) \delta_0 - (z_m(T), u_j) \delta_T
\]

where \( \delta_0, \delta_T \) are Dirac functions, and

\[
\tilde{f}_m(t) = f_m(t) \text{ for } t \in [0; T] \text{ and } \tilde{f}_m(t) = 0 \text{ for } t > T
\]

Take Fourier transform for Equation (3.23) to get

\[
2\pi \int \tilde{z}_m(\tau) <\tilde{f}_m, u_j > + (z_{0m}, u_j) \delta_0 - (z_m(T), u_j) e^{-2\pi i \tau T} \]

where \( \hat{f}_m \) is the Fourier Transform of \( \tilde{f}_m \).

Let \( \tilde{q}_m(t) \) be the function of \( q_m(t) \) in (3.1), which is zero outside of \([0; T]\) and let \( \hat{q}_m(t) \) be its Fourier transform. Multiplier Equation (3.24) by \( \hat{q}_m(t) \) and sum for \( j \) from 1 to \( m \) to obtain

\[
2\pi \int \|z_m(\tau)\|^2 <\tilde{f}_m, \hat{z}_m(\tau) > + (z_{0m}, \hat{z}_m(\tau)) \delta_0 - (z_m(T), \hat{z}_m(\tau)) e^{-2\pi i \tau T} \]

We thus conclude that

\[
\int_0^T \|\hat{f}_m(t)\|^2 dt \leq \sum_0^T \|f_1(t)\|^2 + c_1 \|z_m(t)\|^2 + c_1 \|z_m(t)\|^2 dt
\]

where we used the fact \( \|B(z_m(t))\| \leq c_1 \|z_m(t)\|^2 \). By (3.13), (3.15) and (3.17), it follows from (3.26) that

\[
\sup_{t \in R} \|\hat{f}_m(t)\|^2, c < \infty
\]

Apply (3.27) and the fact \( \sup_{m \in Z^+} \left( \|z_m(0)\| + \|z_m(T)\| \right) < \infty \) to (3.25) to yield

\[
\tau \|\hat{z}_m(\tau)\|^2 \leq c_5 \|\hat{z}_m(\tau)\|^2 + c_4 \|\hat{z}_m(\tau)\|^2 + c_5 \|\hat{z}_m(\tau)\|^2 + c_5 = c_5 + c_4
\]

For fixed \( 0 < \nu < 1/4 \), observe that

\[
|\tau|^{2\nu} \leq c'(\nu) \frac{1+|\tau|}{1+|\tau|^{1-2\nu}}, \forall \tau \in R
\]

for some constant \( c'(\nu) \). From this inequality, we obtain

\[
\int_{-\infty}^{+\infty} |\tau|^{2\nu} |\hat{z}_m(\tau)|^2 d\tau \leq c'(\nu) \int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\nu}} |\hat{z}_m(\tau)|^2 d\tau
\]
By (3.28), there are constants $c_0' > 0$ and $c_1' > 0$ such that

$$\int_{-\infty}^{+\infty} |\tau|^{2\nu} |\hat{z}_m(\tau)|^2 \, d\tau \leq c'(\nu) \int_{-\infty}^{+\infty} \frac{|\tau|}{1 + |\tau|^{1-2\nu}} |\hat{z}_m(\tau)|^2 \, d\tau$$

$$\leq c_0' \int_{-1}^{+\infty} \frac{|\hat{z}_m(\tau)|}{1 + |\tau|^{1-2\nu}} \, d\tau + c_1' \int_{-\infty}^{+\infty} |\hat{z}_m(\tau)|^2 \, d\tau$$

By (3.20) and the Parseval identity, the second term on the right-hand side of above inequality is bounded as $m \to \infty$. Therefore, (3.22) is proved if we can show that

$$\int_{-1}^{+\infty} \frac{|\hat{z}_m(\tau)|}{1 + |\tau|^{1-2\nu}} \, d\tau < \infty$$

However, this is a consequence of the Schwarz inequality and the Parseval identity that

$$\int_{-1}^{+\infty} \frac{|\hat{z}_m(\tau)|}{1 + |\tau|^{1-2\nu}} \, d\tau < (\int_{-1}^{+\infty} \frac{d\tau}{1 + |\tau|^{1-2\nu}})^{1/2} \left( \int_{-\infty}^{+\infty} |\hat{z}_m(\tau)|^2 \, d\tau \right)^{1/2},$$

where we used the facts $0 < \nu < 1/4$ and the boundedness of $\{z_m\}$ in $L^\infty(0,T;H)$ as $m \to \infty$ claimed by (3.19). So (3.22) is valid.

By (3.22), (3.17) and (3.19), we conclude that

$$\{z_m\} \text{ is bounded in } H^1(R;V) \cap H^\nu(R;H)$$

(3.30)

By Lemma 2.5, there exists a subsequence of $\{z_m, w_m\}$ that is still denoted by itself without confusion such that

$$z_m \to z \text{ strongly in } L^2(0,T;H), w_m \to w \text{ strongly in } L^2(0,T;\tilde{H}).$$

This is (3.20).

**Theorem 3.1.** There exists a weak solution to (1.1).

Proof. Let $\Psi$ and $\theta$ be continuous differentiable vector functions defined on $[0; T]$ with $\Psi(T) = \theta(T) = 0$. Multiply the first equation of (3.2) by $\Psi$ and integrate over $[0; T]$ with respect to $t$ to give

$$-\int_0^T (z_m(t), \Psi'(t)u_j) \, dt + \int_0^T [w_z(z_m(t), \Psi(t)u_j) + b(z_m(t), z_m(t), \Psi(t)u_j) + \beta(w_m(t)\xi, \Psi(t)u_j)] \, dt$$

$$= (z_{0m}, u_j) \Psi(0) + \int_0^T <v_1(t), \Psi(t)u_m >_{\Gamma_1} \, dt$$

(3.31)

Multiply the second equation of (3.2) by $\theta$ and integrate over $[0; T]$ with respect to $t$ to give

$$-\int_0^T (w_m(t), \theta'(t)\mu_j) \, dt + \int_0^T [ka_z(w_m(t), \theta'(t)\mu_j) + c(z_m(t), w_m(t), \theta(t)\mu_j)] \, dt$$

$$= (w_{0m}, \mu_j) \theta(0) + \int_0^T <v_2(t), \theta(t)\mu_j >_{\Gamma_2} \, dt$$

(3.32)

Passing to the limit as $m \to \infty$ in (3.31) and (3.32) by applying (3.3), (3.20), the properties (iii) in Lemmas 2.2 and 2.3 for $b$ and $c$, we obtain

$$-\int_0^T (z(t), \Psi'(t)u_j) \, dt + \int_0^T [w_z(z(t), \Psi(t)u_j) + b(z(t), z(t), \Psi(t)u_j) + \beta(w(t)\xi, \Psi(t)u_j)] \, dt$$

$$= (z_0, u_j) \Psi(0) + \int_0^T <v_1(t), \Psi(t)u_m >_{\Gamma_1} \, dt$$

(3.33)
\[- \int_0^T (w(t), \theta'(t) \mu_j) dt + \int_0^T [ka_2(w(t), \theta'(t) \mu_j) + c(z(t), w(t), \theta(t) \mu_j)] dt \]

\[= (w_0, \mu_j) \theta(0) + \int_0^T \langle v_2(t), \theta(t) \mu_j \rangle_{T_2} dt \]

(3.34)

where in obtaining (3.33) and (3.34), we used the following facts:
- The convergence of the nonlinear terms in $b(\cdot, \cdot, \cdot)$ and $c(\cdot, \cdot, \cdot)$ can be obtained in the same way as that in Chapter 3 of [12].
- \[
\int_0^T a_1(z_m(t), \Psi(t) u_j) dt = \int_0^T (\text{rot} z_m(t), \Psi(t) \text{rot} u_j) dt = \]
  \[\rightarrow \int_0^T (\text{rot} z(t), \Psi(t) \text{rot} u_j) dt = \int_0^T a_1(z(t), \Psi(t) u_j) dt \]

where we used the facts that $\text{rot} z_m(t) \rightarrow \text{rot} z(t)$ weakly in $L^2(0, T; V)$.
- Similarly
  \[
\int_0^T a_2(w_m(t), \theta(t) \mu_j) dt \rightarrow \int_0^T a_2(w(t), \theta(t) \mu_j) dt \]

by the facts again $\nabla w_m(t) \rightarrow \nabla w(t)$ weakly in $L^2(0, T; W)$.

By the density arguments, we have that (3.33) and (3.34) hold true for any $\psi \in V$ instead of $u_j$ and $\varphi \in W$ instead on $\mu_j$, respectively. That is,

\[- \int_0^T (z(t), \Psi'(t) \psi) dt + \int_0^T [\text{rot} z_m(t), \Psi(t) \psi] dt =
\]

\[= (z_0, \psi) \Psi(0) + \int_0^T \langle v_1(t), \Psi(t) \psi \rangle_{T_1} dt, \forall \psi \in V \]

(3.35)

\[- \int_0^T (w(t), \theta'(t) \varphi) dt + \int_0^T [ka_2(w(t), \theta'(t) \varphi) + c(z(t), w(t), \theta(t) \varphi)] dt =
\]

\[= (w_0, \varphi) \theta(0) + \int_0^T \langle v_2(t), \theta(t) \varphi \rangle_{T_2} dt \]

(3.36)

Now take $\Psi \in (D(0, T))^N$ in (3.35) and $\theta \in D(0, T)$ in (3.36). Then \{z, w\} satisfies

\[
\begin{cases}
(z', \psi) + \nu a_1(z, \psi) + b(z, z, \psi) + (\beta \xi w, \psi) = & \langle v_1, \psi \rangle_{T_1}, \forall \psi \in V \\
(w', \varphi) + ka_2(w, \varphi) + c(z, w, \varphi) = & \langle v_2, \varphi \rangle_{T_2}, \forall \varphi \in W
\end{cases}

(3.37)

This is the equations in (2.1). Finally, we determine the initial value of \{z, w\}. Actually, multiply the first equation of (3.37) and integrate over $[0, T]$ with respect to $t$ to get

\[- \int_0^T (z(t), \Psi'(t) \psi) dt + \int_0^T [\text{rot} z_m(t), \Psi(t) \psi] dt =
\]

\[= (z(0), \psi) \Psi(0) + \int_0^T \langle v_1(t), \Psi(t) \psi \rangle_{T_1} dt, \forall \psi \in V \]

Subtract (3.38) from (3.35) to get $(z(0) - z_0, \psi) \Psi(0) = 0$. Take $\Psi$ so that $\Psi(0) = 1$ to get $(z(0) - z_0, \psi) = 0$ for all $\psi \in V$. So $z(0) = z_0$. The similar arguments lead to $w(0) = w_0$. The proof is complete. \(\square\)
4. Existence of the optimal pair

**Definition 2.** Suppose that \( z_0 \in H, \ w_0 \in \tilde{H}, \ g \in L^\infty(\Omega) \). The pair \( \{y, v\} = \{(z, w), (v_1, v_2)\} \) is said to be admissible pair; “state-control” of extreme value problem (1.2) if it satisfies

\[
\begin{align*}
&y = \{z, w\} \in Y, z' \in L^1(0, T, V'), \ w' \in L^1(0, T : W') \\
&\frac{dz}{dt} + wA(z) + B(z) + \beta \xi w = H_1v_1, \\
&\frac{dw}{dt} + kA_2w + C(z, w) = H_2v_2,
\end{align*}
\]

(4.1)

We denote admissible pair sets by \( M \) that is;

\[
M \equiv \{(y, v) \mid y \in Y = \mathbb{Z} \times \mathbb{W}, v = \{v_1, v_2\} \in U_a, (y, v) \text{ satisfy (4.1)}\}.
\]

**Definition 3.** The admissible pair \( \{y, v_s\} = \{(z_s, w_s), \{v_{s, 1}, v_{s, 2}\}\} \in M \) is called an optimal pair if \( v_s \) is satisfied (1.2) and it is called by optimal control and \( y_s \) is called by optimal state.

The optimal control problem (1.2) is an optimal control of the singular distributed system ([12]). We can easily see that energy inequalities

\[
\begin{align*}
&\frac{1}{2} |z(t)|^2 + \varepsilon_0 \int_0^T \|z(\tau)|^2 d\tau + \int_0^T (\beta \xi _{\mathbb{Z}} w(\tau), z(\tau)) d\tau \leq \frac{1}{2} |z_0|^2 + \int_0^T <H_1v_1, z(\tau)> d\tau \\
&\frac{1}{2} |w(t)|^2 + \kappa_0 \int_0^T \|w(\tau)|^2 d\tau \leq \frac{1}{2} |w_0|^2 + \int_0^T <H_2v_2, w(\tau)> d\tau
\end{align*}
\]

are holed.

From the inequalities (4.2), (4.3), we can obtain as following inequalities;

\[
\begin{align*}
&\|z\|_{L^2(0,T;H)}^2 + \|z\|_{L^2(0,T;V')}^2 + \left\|\frac{dz}{dt}\right\|_{L^1(0,T;V')}^2 \leq \bar{c}_1(\|z_0\|^2 + \|v_1\|_{L^2(0,T;\Gamma_1)}^2) \\
&\|w\|_{L^2(0,T;H)}^2 + \|w\|_{L^2(0,T;W')}^2 + \left\|\frac{dw}{dt}\right\|_{L^1(0,T;W')}^2 \leq \bar{c}_2(\|w_0\|^2 + \|v_2\|_{L^2(0,T;\Gamma_2)}^2)
\end{align*}
\]

(4.4)

(4.5)

where and throughout in the sequel of the proof of theorems, \( \bar{c}_i > 0 \) denote positive constants independence \( t, z, w \).

Here and in what follows, we denote \( \|v_1\|_{L^2(0,T;L^2(\Gamma_1))^N} \) by \( \|v_1\|_{L^2(0,T;\Gamma_1)} \) and \( \|v_2\|_{L^2(0,T;L^2(\Gamma_2))^N} \) by \( \|v_2\|_{L^2(0,T;\Gamma_2)} \) simply.

**Lemma 4.1.** The admissible pair set \( M \) is nonempty and weakly closed subset in the space \( [L^2(\Sigma_1)]^N \times L^2(\Sigma_2) \times Y \).

Proof. From theorem 3.1, we obtain that the admissible pair set \( M \) is nonempty subset in the space \( [L^2(\Sigma_1)]^N \times L^2(\Sigma_2) \times Y \).

Now, we shall prove that \( M \) is a weakly closed set. We assume that \( \{(z_m, w_m), \{v_{1m}, v_{2m}\}\} \subset M \) are satisfied as following;

\[
\begin{align*}
v_{1m} \to v_1, \text{ weakly in } L^2(0,T;L^2(\Gamma_1))^N; & \quad v_{2m} \to v_2, \text{ weakly in } L^2(0,T;L^2(\Gamma_2)) \\
z_m \to z, \text{ weakly in } L^2(0,T;V); & \quad w_m \to w, \text{ weakly in } L^2(0,T;W)
\end{align*}
\]

(4.5)

(4.6)
From (4.4) we obtain that \( \left\| \frac{dz_m}{dt} \right\|_{L^2((0,T;V^*)} \leq \text{const} \). Now, by using compactness theorem of embedding of in [13, 14], we will show that we can chose the subsequence \( \{z_m\} \), which we denote by the same symbols, such that

\[
z_m \to z, \text{ strongly in } L^2(0,T:H)
\]  

(4.7)

Same as above from (4.5) we obtain that \( \left\| \frac{dw_m}{dt} \right\|_{L^2((0,T;W^*)} \leq \text{const} \). Now, by using compactness theorem of embedding of [13, 14], we will show that we can chose the subsequence \( \{w_m\} \), which we denote by the same symbols, such that

\[
w_m \to w, \text{ strongly in } L^2(0,T:H)
\]  

(4.8)

From the (4.5)-(4.8) we can take the limit as \( m \to \infty \) in the equations;

\[
\frac{dz_m}{dt} + wA_i z_m + B(z_m) + \beta \xi w_m = H_1 v_{1m},
\]

\[
\frac{dw_m}{dt} + kA_i w_m + C(z_m, w_m) = H v_{2m}
\]

and we obtain that \( \{z, w\} \) are satisfied the equations (4.1).

We can rewrite the inequalities (4.2), (4.3) for \( \{v_{1m}, v_{2m}\} \), and \( \nu_i \in [\alpha_i(x,t), \beta_i(x,t)] \).

Therefore, \( \{\{z, w\}, \{v_{1i}, v_{2i}\}\} \) is an admissible pair. Thus, proof of lemma is fulfilled. \( \□ \)

**Theorem 2** There exists optimal pair of the optimization problem (1.2).

Proof. We assume that \( \{\{z_m, w_m\}, \{v_{1m}, v_{2m}\}\} \in M \) is the minimization sequences, that is, when \( m \to \infty \)

\[
I[v_m] = I[v_{1m}, v_{2m}] \to \inf I
\]

Then, we have obtain such as estimations

\[
\left\| v_{1m} \right\|_{L^2(\Sigma_1)} \leq \text{const}, \forall m ; \left\| v_{2m} \right\|_{L^2(\Sigma_2)} \leq \text{const}, \forall m
\]

By estimations (4.4), (4.5), we have

\[
\left\| z_m \right\|_{L^2((0,T;V)} \leq \text{const}, \forall m ; \left\| w_m \right\|_{L^2((0,T;W)} \leq \text{const}, \forall m
\]

We can chose the subsequence \( \{w_m\} \), which we denote by the same symbols, by taking the limit as \( m \to \infty \), obtain such that

\[
v_{1m} \to v_1, \text{ weakly in } L^2(\Sigma_1) ; \quad v_{2m} \to v_2, \text{ weakly in } L^2(\Sigma_2)
\]

\[
z_m \to z, \text{ weakly in } L^2(0,T;V) ; \quad w_m \to w, \text{ weakly in } L^2(0,T;W)
\]

Seeing that the admissible pair set M is the nonempty, weakly compactness and cost functional I is continuous, \( \{\{z, w\}, \{v_1, v_2\}\} \) is an optimal pair by generalized Weierstrass theorem. Thus, proof of theorem is fulfilled. \( \square \)

5. Optimal condition

In this section, we derive the first order necessary condition of optimal control problem (1.2). When deriving, the optimal condition in the control problem of nonlinear system, usually additional regularity condition for admissible pair is demanded. Here we assume that regularity condition for optimal state \( \{z, w\} \) in the case three-dimensional domain hold true such that;

\[
z \in L^6(0,T;V) , \quad w \in L^5(0,T;W)
\]  

(5.1)

We assume that this regularity condition is satisfied here and in what follows.

In order to derive the optimal condition of optimal control problem, for arbitrary \( \epsilon > 0 \) we introduce \( \epsilon \)-approximation control problem;
Problem $P_\varepsilon$. 

\[ \inf_{v \in U_\varepsilon} J[v] \quad (5.2) \]

State constraints are given such as:

\[ g_\varepsilon' + vA_\varepsilon g_\varepsilon + B(z_\varepsilon, g_\varepsilon) + B(g_\varepsilon, z_\varepsilon) + \beta g_\varepsilon = H_1(v_1 - v_\varepsilon) \quad (5.3) \]
\[ \eta_\varepsilon' + kA_\varepsilon \eta_\varepsilon + C(z_\varepsilon', \eta_\varepsilon) + C(g_\varepsilon', w_\varepsilon) + \varepsilon \eta_\varepsilon = H_2(v_2 - v_\varepsilon) \quad (5.4) \]
\[ g_\varepsilon(0) = 0, \quad g_\varepsilon \in L^2(0, T; \mathbb{V}), \quad g_\varepsilon' \in L^2(0, T; \mathbb{V}') \quad (5.5) \]
\[ \eta_\varepsilon(0) = 0, \quad \eta_\varepsilon \in L^2(0, T; \mathbb{W}), \quad \eta_\varepsilon' \in L^2(0, T; \mathbb{W}') \quad (5.6) \]

Here, $\{v_\varepsilon, v_\varepsilon\}$ is an optimal control and $\{z_\varepsilon, w_\varepsilon\}$ is optimal state.

And, we denote $g_\varepsilon = g(x, \tau), \eta_\varepsilon = \eta(x, \tau), \quad g_\varepsilon = \frac{\partial g(x, \tau)}{\partial \tau}, \quad \eta_\varepsilon = \frac{\partial \eta(x, \tau)}{\partial \tau}, \quad \tau \in [0, T]$.

**Lemma 5.1.** We assume that the condition

\[ \frac{\beta \|\xi\|_w (\beta \|\xi\|_w + 1)}{w_1} \leq \frac{k c_1'}{4} \]

is hold. Then, there exists a solution of the equation (5.3)-(5.6).

**(Proof)** We assume that $\{g_m(t)\}, \{\eta_m(t)\}$ is solution for the Galerkin system of equation (5.3), (5.4) that is

\[ g_m + vA_\varepsilon g_m + Q_m^1(B(z_\varepsilon, g_m) + B(g_m, z_\varepsilon)) + \varepsilon B(g_m, g_m) + \beta g_\varepsilon \eta_m = Q_m^1 H_1(v_1 - v_\varepsilon), \quad g_m(0) = 0 \quad (5.7) \]
\[ \eta_m + kA_\varepsilon \eta_m + Q_m^2(C(z_\varepsilon, \eta_m) + C(g_m, w_\varepsilon)) + \varepsilon C(\eta_m, \eta_m) = Q_m^2 H_2(v_2 - v_\varepsilon), \quad \eta_m(0) = 0 \quad (5.8) \]

Here $Q_m^1$ is projection operator from $\mathbb{H}$ to $H_m^1$ and $H_m^1$ is the subspace generated by first $m$ numbers “basis” element of space $\mathbb{H}$. And $Q_m^2$ is projection operator from $\overline{\mathbb{H}}$ to $H_m^2$ and $H_m^2$ is the subspace generated by first $m$ numbers “basis” element of space $\overline{\mathbb{H}}$.

By applying scalar product (5.7) by $g_m$, we obtain

\[ \frac{1}{2} \frac{d}{dt} \|g_m\|^2 + vA_\varepsilon \|g_m\|^2 + \langle B(g_m, z_\varepsilon), g_m \rangle + \langle \beta g_\varepsilon \eta_m, g_m \rangle = \langle H_1(v_1 - v_\varepsilon), g_m \rangle \]

From here, for any given $\varepsilon > 0$, we obtain we can get that

\[ \frac{1}{2} \frac{d}{dt} \|g_m\|^2 + vA_\varepsilon \|g_m\|^2 \leq -\langle B(g_m, z_\varepsilon), g_m \rangle - \langle \beta \eta_\varepsilon, g_m \rangle + \langle H_1(v_1 - v_\varepsilon), g_m \rangle \]
\[ \leq -\langle B(g_m, z_\varepsilon), g_m \rangle + \frac{1}{2} \beta \|\xi\|_w (\frac{1}{2} \|\eta_m\|^2 + \varepsilon^2 \|g_m\|^2) + \frac{1}{2} \frac{1}{\varepsilon^2} \|v_1 - v_\varepsilon\|_{L^2(\mathbb{Z}_1)}^2 + \varepsilon^2 \|g_m\|^2 \]
\[ \leq -\langle B(g_m, z_\varepsilon), g_m \rangle + \frac{1}{2} \beta \|\xi\|_w (\frac{1}{2} \|\eta_m\|^2 + \varepsilon^2 \|g_m\|^2) + \frac{1}{2} \frac{1}{\varepsilon^2} \|v_1 - v_\varepsilon\|_{L^2(\mathbb{Z}_1)}^2 + \frac{1}{2} \frac{1}{\varepsilon^2} \|v_1 - v_\varepsilon\|_{L^2(\mathbb{Z}_1)}^2 \]

From here, we obtain that

\[ \frac{1}{2} \frac{d}{dt} \|g_m\|^2 + |vA_\varepsilon| \frac{1}{2} \beta \|\xi\|_w (\frac{1}{2} \|\eta_m\|^2 + \varepsilon^2 \|g_m\|^2) \leq -\langle B(g_m, z_\varepsilon), g_m \rangle + \frac{1}{2} \beta \|\xi\|_w \frac{1}{\varepsilon^2} \|v_1 - v_\varepsilon\|_{L^2(\mathbb{Z}_1)}^2 + \frac{1}{2} \frac{1}{\varepsilon^2} \|v_1 - v_\varepsilon\|_{L^2(\mathbb{Z}_1)}^2 \]

Now, we put $\varepsilon^2 = \frac{vA_\varepsilon}{(\beta \|\xi\|_w + 1)}$, then obtain that

\[ \frac{1}{2} \frac{d}{dt} \|g_m\|^2 + \frac{1}{2} vA_\varepsilon \|g_m\|^2 \leq c_4 \|v_1 - v_\varepsilon\|_{L^2(\mathbb{Z}_1)}^2 + \frac{1}{2} \beta \|\xi\|_w \frac{(\beta \|\xi\|_w + 1)}{vA_\varepsilon} \|\eta_m\|^2 \]

Here, $c_4 = \frac{1}{2} \frac{(\beta \|\xi\|_w + 1)}{vA_\varepsilon}$.

From here, we obtain that
\[ |g_m(t)|^2 + w_1 \int_0^t \|g_m(\tau)|^2 \, d\tau \leq 2c_4 |v_1 - v_{11}|_{L^2(E)}^2 + c_{10} \int_0^t \|g_m\|^{N-2} \left| \left| g_m \right|_{L^2(E)} \right|^{N-2} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau + \frac{\beta \|z\|_{L^2}^2}{w_1} + \int_0^t \|\eta_m\|^2 \, d\tau \]

Multiplying (5.8) by \( \eta_m \)
\[
\frac{1}{2} \frac{d}{dt} |\eta_m|^2 + ka_2(\eta_m, \eta_m) + \langle C(g_m, w), \eta_m \rangle = \langle H_2 (v_2 - v_{21}), \eta_m \rangle
\]

Same as above, we obtain that with \( c_5 = \frac{1}{2kc_1} \)
\[
|\eta_m(t)|^2 + c_5 k \int_0^t |\eta_m(\tau)|^2 \, d\tau \leq 2c_5 |v_2 - v_{21}|_{L^2(E)}^2 + c_1 \int_0^t \|g_m\|^{N-2} \left| \left| g_m \right|_{L^2(E)} \right|^{N-2} \bigl( |w|, |\eta_m| \bigr)^{-\frac{N}{2}} \|\eta_m\|^2 \, d\tau
\]

Here we have applied inequality as following;
\[
|\phi|_{L^4} \leq k \|\phi\|_{\mathcal{V}}^{\frac{2}{3}}, \quad \forall \phi \in \mathcal{V} \text{ or } \phi \in \mathcal{W}
\]

\( k > 0 \) is constant.

By adding above two inequalities, we obtain
\[
|g_m(t)|^2 + |\eta_m(t)|^2 + w_1 \int_0^t \|g_m(\tau)|^2 \, d\tau + c_5 k \int_0^t |\eta_m(\tau)|^2 \, d\tau \leq 2c_5 |v_1 - v_{11}|_{L^2(E)}^2 + 2c_5 |v_2 - v_{21}|_{L^2(E)}^2 + 2c_4 |v_1 - v_{11}|_{L^2(E)}^2 + 2c_5 |v_2 - v_{21}|_{L^2(E)}^2 + c_1 \int_0^t \|g_m\|^{N-2} \left| \left| g_m \right|_{L^2(E)} \right|^{N-2} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \bigl( |w|, |\eta_m| \bigr)^{-\frac{N}{2}} \|\eta_m\|^2 \, d\tau + \frac{\beta \|z\|_{L^2}^2}{w_1} + \int_0^t \|\eta_m\|^2 \, d\tau
\]

Here and in what follows, we denote \( \|\cdot\|_{L^\infty(E_1)} \) by \( \|\cdot\|_1 \) and \( \|\cdot\|_{L^\infty(E_1)} \) by \( \|\cdot\|_2 \) simply.

Now, let estimate the terms of right-hand side of (5.9).

We apply Young’s inequality and \( |z| \leq c, |w| \leq c \) in the third term of right-hand side of (5.9). That is, in Young’s inequality \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \), \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) if we put \( p = \frac{8}{4 - N}, \) then \( q = \frac{8}{4 + N} \).

Therefore, we obtain
\[
\int_0^t \|g_m\|^{N-2} \left| \left| g_m \right|_{L^2(E)} \right|^{N-2} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau \leq \int_0^t \|g_m\|^{N-2} \left| \left| g_m \right|_{L^2(E)} \right|^{N-2} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
= \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]

\[
\leq \frac{4 - N}{8} \int_0^t \|g_m\|^{\frac{N}{2} - 1} \|z\|^{\frac{N}{2}} \bigl( |z|, |g_m| \bigr)^{-\frac{N}{2}} \, d\tau
\]
\[
\leq \frac{4 - N}{8} \int_{0}^{t} (|z_{m}| + \frac{2N}{1 - N} |g_{m}|^{2}) d\tau + \frac{4 - N}{8} \int_{0}^{t} |z_{m}|^{2} d\tau + \frac{W_{c}}{4c_{10}} \int_{0}^{t} \|g_{m}\|^{2} d\tau + \frac{c_{10}}{W_{c}} T
\]  
(5.10)

Similarly, by estimating the forth term of right-hand side of (5.9) we obtain

\[
\int_{0}^{t} \|g_{m}\| \|w_{*}\| \|\eta_{m}\|^{1 - \frac{N}{4}} \|\eta_{m}\|^{\frac{N}{4}} d\tau = \int_{0}^{t} (\|w_{*}\| \|\eta_{m}\|^{1 - \frac{N}{4}} \cdot \|w_{*}\|^{\frac{N}{4}} \|g_{m}\| \|\eta_{m}\|^{\frac{N}{4}}) d\tau \leq
\]

\[
\leq \frac{4 - N}{8} \int_{0}^{t} (\|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}} \|\eta_{m}\|^{\frac{N}{4}}) d\tau + \frac{4 + N}{8} \int_{0}^{t} \|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}} \|\eta_{m}\|^{\frac{N}{4}} d\tau
\]

\[
= \frac{4 - N}{8} \int_{0}^{t} (\|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}}) d\tau + \frac{4 + N}{8} \int_{0}^{t} \|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}} \|\eta_{m}\|^{\frac{N}{4}} d\tau
\]

\[
\leq \frac{4 - N}{8} \int_{0}^{t} (\|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}}) d\tau + \frac{W_{c}}{2c_{11}} \frac{\|g_{m}\|^{2}}{8} \int_{0}^{t} \|w_{*}\|^{2} d\tau + \frac{c_{11}}{2W_{c}} \frac{4 + N}{8} \int_{0}^{t} \|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}} \|\eta_{m}\|^{\frac{N}{4}} d\tau
\]

\[
= \frac{4 - N}{8} \int_{0}^{t} (\|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}}) d\tau + \frac{W_{c}}{2c_{11}} \frac{\|g_{m}\|^{2}}{8} \int_{0}^{t} \|w_{*}\|^{2} d\tau + \frac{c_{11}}{2W_{c}} \frac{4 + N}{8} \int_{0}^{t} \|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}} \|\eta_{m}\|^{\frac{N}{4}} d\tau
\]

\[
\leq \frac{4 - N}{8} \int_{0}^{t} (\|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}}) d\tau + \frac{W_{c}}{2c_{11}} \frac{\|g_{m}\|^{2}}{8} \int_{0}^{t} \|w_{*}\|^{2} d\tau + \frac{c_{11}}{2W_{c}} \frac{4 + N}{8} \int_{0}^{t} \|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}} \|\eta_{m}\|^{\frac{N}{4}} d\tau
\]

By applying (5.10), (5.11) in (5.9) we obtain

\[
\|g_{m}(t)\|^{2} + \|\eta_{m}(t)\|^{2} + \frac{W_{c}}{4} \int_{0}^{t} \|g_{m}(\tau)\|^{2} d\tau + \frac{k c_{1}'}{2} \int_{0}^{t} \|\eta_{m}(\tau)\|^{2} d\tau \leq 2c_{2} |v_{1} - v_{2}|_{\Sigma} +
\]

\[
2c_{3} |v_{2} - v_{4}|_{\Sigma} + \frac{2N}{8} \int_{0}^{t} \|z_{m}\| \|g_{m}\|^{\frac{2N}{4}} \|\eta_{m}\|^{\frac{N}{4}} d\tau + c_{11} \frac{4 - N}{8} \int_{0}^{t} (\|w_{*}\| \|\eta_{m}\|^{\frac{2N}{4}}) d\tau +
\]

\[
c_{10} \frac{4 - N}{8} \int_{0}^{t} \|z_{m}\|^{2} d\tau + \frac{4 - N}{8} \int_{0}^{t} \|w_{*}\|^{2} d\tau + \frac{c_{10}}{W_{c}} \frac{N}{8} + \frac{c_{11}}{k c_{1}'} T
\]

Now, applying the condition of the theorem, we obtain such as
\[ |g_m(t)|^2 + |\eta_m(t)|^2 + \frac{W_1}{4} \int_0^t \|g_m(\tau)\|^2 d\tau + \frac{k_c^1}{4} \int_0^t \|\eta_m(\tau)\|^2 d\tau \leq 2c_4|v_i - v_{i+1}| + 2c_5|v_2 - v_{2*}|_{L_2} + c_10 \int_0^t \|z_\tau\|^{2N} |g_m|^2 d\tau + c_{11} \int_0^t \|\eta_m(\tau)\|^2 d\tau \leq 2c_4|v_i - v_{i+1}| + 2c_5|v_2 - v_{2*}|_{L_2} + c_10 \int_0^t \|z_\tau\|^{2N} |g_m|^2 d\tau + c_{11} \int_0^t \|\eta_m(\tau)\|^2 d\tau \]

Therefore,

\[ |g_m(t)|^2 + |\eta_m(t)|^2 + \nu \int_0^t \|g_m(\tau)\|^2 d\tau + k_c^1 \int_0^t \|\eta_m(\tau)\|^2 d\tau \leq c_{12} + c_{10} \int_0^t M_1(\tau) |g_m|^2 d\tau + c_{12} \int_0^t M_2(\tau) |\eta_m|^2 d\tau \]

Here,

\[ c_{12} = 2c_4|v_i - v_{i+1}| + 2c_5|v_2 - v_{2*}|_{L_2} + c_10 \int_0^t \|z_\tau\|^{2N} |g_m|^2 d\tau + c_{11} \int_0^t \|\eta_m(\tau)\|^2 d\tau \leq c_{12} + c_{10} \int_0^t M_1(\tau) |g_m|^2 d\tau + c_{12} \int_0^t M_2(\tau) |\eta_m|^2 d\tau \]

and here and in what follows, we denote

\[ M_1(\tau) = c_{10} \frac{4 - N}{8} \|z_\tau(\tau)\|^{2N}, \quad M_2(\tau) = c_{11} \frac{4 - N}{8} \|\eta_m(\tau)\|^{2N} \]

From (5.13), we can obtain

\[ |g_m(t)|^2 + |\eta_m(t)|^2 + \nu \int_0^t \|g_m(\tau)\|^2 d\tau + k_c^1 \int_0^t \|\eta_m(\tau)\|^2 d\tau \leq c_{12} + c_{10} \int_0^t M_1(\tau) |g_m|^2 d\tau + c_{12} \int_0^t M_2(\tau) |\eta_m|^2 d\tau \]

Here, \( c_{13} = \max\{c_{10} \frac{4 - N}{8}, c_{11} \frac{4 - N}{8}\} \).

Then, we can find a nonnegative function \( M_3(\tau) \) that is bounded and integrable in a.e. \( \tau \in [0,T] \) such that

\[ \max\{M_1(\tau), M_2(\tau)\} \leq M_3(\tau), \text{ in a.e. } \tau \in [0,T]. \]

Therefore, from above inequality, we obtain

\[ |g_m(t)|^2 + |\eta_m(t)|^2 + \nu \int_0^t \|g_m(\tau)\|^2 d\tau + k_c^1 \int_0^t \|\eta_m(\tau)\|^2 d\tau \leq c_{14} \int_0^t M_2(\tau) |g_m|^2 d\tau + c_{14} \int_0^t M_3(\tau) |\eta_m|^2 d\tau \]

By (5.15), we obtain

\[ |g_m(t)|^2 + |\eta_m(t)|^2 \leq c_{14} \int_0^t M_2(\tau) |g_m|^2 d\tau + c_{14} \int_0^t M_3(\tau) |\eta_m|^2 d\tau \]

By Gronwall inequality we can obtain

\[ |g_m(t)|^2 + |\eta_m(t)|^2 \leq c_{14} \exp\{c_{14} \int_0^t M_2(\tau) d\tau\} \]

From here, we obtain

\[ \|g_m\|_{L^2(0,T;H)} \leq \text{const}, \quad \|\eta_m\|_{L^2(0,T;L^2(H))} \leq \text{const} \]

Again, replacing \( t \to T \) in (5.15), we obtain

\[ \|g_m\|_{L^2(0,T;V)} \leq \text{const} \quad \|\eta_m\|_{L^2(0,T;W)} \leq \text{const} \]

Accordingly, from (5.7), (5.8) we obtain
\[ \left\| g_m^* \right\|_{L^1(0,T;V^*)} \leq \text{const}, \quad \left\| \eta_m^* \right\|_{L^1(0,T;W^*)} \leq \text{const} \quad (5.18) \]

Therefore, from (5.16)- (5.18) we can conclude that there exists a subsequence of \( \{g_m\}, \{\eta_m\} \) which we denote by the same symbols, such that

- \( g_m \to g^* \) weakly in \( L^\infty(0,T;H) \), \( g_m \to g^* \) weakly in \( L^2(0,T;V) \)
- \( g_m \to g^* \) strongly in \( L^2(0,T;H) \)
- \( \eta_m \to \eta^* \) weakly in \( L^\infty(0,T;\tilde{H}) \), \( \eta_m \to \eta^* \) weakly in \( L^2(0,T;W) \)
- \( \eta_m \to \eta^* \) strongly in \( L^2(0,T;\tilde{H}) \)

And

\[ g^* \in L^\infty(0,T;H) \cap L^2(0,T;V), g^*_m \in L^1(0,T;V^*) \]
\[ \eta^* \in L^\infty(0,T;\tilde{H}) \cap L^2(0,T;W), \eta^*_m \in L^1(0,T;W^*) \].

From (5.19), (5.20), by taking the limit as \( m \to \infty \) in (5.8), (5.9), we obtain that \( \{g^*_m\}, \{\eta^*_m\} \) is the solution of (5.3)-(5.6). □

**Lemma 5.2.** There is at least one solution \( (g^*_e, \eta^*_e; v^*_e, v^{2e}) \) of problem \( P_e \).

The proof of this lemma is just same as lemma 1.

Now, let derive the optimal condition of approximation control problem \( P_e \).

Let examine \( \{(v^*_e + \varepsilon(v_1 - v^*_e), v^*_e, + \varepsilon(v_2 - v^{2e})), (z^*_e + \varepsilon g^*_e, w^*_e + \varepsilon \eta^*_e)\} \) is admissible pair of problem (1.2).

First of all, \( \{(v^*_e, v^{2e}),(z^*_e, w^*_e)\} \) being admissible pair, we obtain

\[ z^*_e + \nu A^e z^*_e + B(z^*_e, z^*_e) + \beta \xi w^*_e = H_1^e v^*_e, \]
\[ w^*_e + k A^e w^*_e + C(z^*_e, z^*_e) = H_2^e v^{2e} \]

Now, multiplying \( (5.3) \) by \( \varepsilon \) and adding \( (5.21) \), we obtain

\[ (z^*_e + \varepsilon g^*_e)^2 + \nu A^e (z^*_e + \varepsilon g^*_e) + B(z^*_e, z^*_e) \]
\[ + \beta \xi(w^*_e + \varepsilon \eta^*_e) = H_1^e (v^*_e + \varepsilon(v_1 - v^*_e)) \]

Multiplying \( (5.4) \) by \( \varepsilon \) and adding \( (5.22) \), arranging, we obtain

\[ (w^*_e + \varepsilon \eta^*_e)^2 + k A^e (w^*_e + \varepsilon \eta^*_e) + C(z^*_e + \varepsilon g^*_e, w^*_e + \varepsilon \eta^*_e) = H_2^e (v^{2e} + \varepsilon(v_2 - v^{2e})) \]

From (5.23), (5.24), we conclude that \( \{(v^*_e + \varepsilon(v_1 - v^*_e), v^*_e, + \varepsilon(v_2 - v^{2e})), (z^*_e + \varepsilon g^*_e, w^*_e + \varepsilon \eta^*_e)\} \) is admissible pair.

Therefore, we can conclude such that

\[ I(z^*_e + \varepsilon g^*_e, w^*_e + \varepsilon \eta^*_e) - I(z^*_e, w^*_e) \geq 0 \]

Accordingly,

\[ N_1 \int_{0}^{T} r_1(x,t)(z^*_e + \varepsilon g^*_e)_n \frac{\partial}{\partial n}(w^*_e + \varepsilon \eta^*_e)dsdt + N_2 \int_{0}^{T} r_2(x,t) \frac{\partial}{\partial n}(w^*_e + \varepsilon \eta^*_e)dsdt - \]
\[ - N_1 \int_{0}^{T} r_1(x,t)(z^*_e)_n \frac{\partial w^*_e}{\partial n}dsdt - N_2 \int_{0}^{T} r_2(x,t) \frac{\partial w^*_e}{\partial n}dsdt = \]
\[ = \varepsilon N_1 \int_{0}^{T} r_1(x,t)(g^*_e)_n \frac{\partial w^*_e}{\partial n}dsdt + \varepsilon \int_{0}^{T} r_1(x,t)(g^*_e)_n \frac{\partial \eta^*_e}{\partial n}dsdt + \varepsilon N_2 \int_{0}^{T} r_2(x,t) \frac{\partial \eta^*_e}{\partial n}dsdt \geq 0 \]

From here, we obtain the optimal condition of approximation control problem \( P_e \) such as:

\[ N_1 \int_{0}^{T} r_1(x,t)(g^*_e)_n \frac{\partial w^*_e}{\partial n}dsdt + N_2 \int_{0}^{T} r_2(x,t) \frac{\partial \eta^*_e}{\partial n}dsdt \geq 0 \quad (5.25) \]

**Lemma 5.3.** We assume that \( \{(v^*_e, v^{2e}),(z^*_e, w^*_e)\} \) is optimal pair. Then there exists the solution \( \{g, \eta\} \) of equations;

\[ g^* + \nu A^e g + B(z^*_e, g) + \beta \xi \eta = H_1^e (v_1 - v^*_e), \quad g(0) = 0 \quad (5.26) \]
\[ \begin{align*}
\eta g + kA_2 \eta + C(z, \eta) + B(\eta, w_*) &= H_2(v_2 - v_{2*}) , \quad \eta(0) = 0 \\
g &\in L^2(0, T; V) , \quad g' \in L^2(0, T; V^*) \\
\eta &\in L^2(0, T; W) , \quad \eta' \in L^2(0, T; W^*)
\end{align*} \]

and inequalities
\[ N_1 \int_0^T \langle H_1 r_1, g \rangle dt + N_2 \int_0^T \langle H_2 r_2, \eta \rangle dt \geq 0 \]

are satisfied. Here, we denote \( g = g(x, t), \eta = \eta(x, t), g' = \frac{dg(x, t)}{dt}, \eta' = \frac{d\eta(x, t)}{dt} \), \( \tau \in [0, T] \).

Proof. By applying scalar product (5.3) by \( g_\epsilon \), we have
\[ \frac{1}{2} \frac{d}{dt} \| g_\epsilon \|^2_{L^2(0, T; H)} + \nu a_1 (g_\epsilon, g_\epsilon) + \langle B(g_\epsilon, z_\epsilon), g_\epsilon \rangle + (\beta \xi \eta_\epsilon, g_\epsilon) = \langle H_1 (v_1 - v_{1\epsilon}), g_\epsilon \rangle \]

By applying scalar product (5.4) by \( \eta_\epsilon \), we have
\[ \frac{1}{2} \frac{d}{dt} \| \eta_\epsilon \|^2_{L^2(0, T; H)} + k a_2 (\eta_\epsilon, \eta_\epsilon) + \langle C(g_\epsilon, w_\epsilon), \eta_\epsilon \rangle = \langle H_2 (v_2 - v_{2\epsilon}), \eta_\epsilon \rangle \]

Repeating the process of proof of lemma 5.1, we obtain the estimations same as (5.16)- (5.18) such that;
\[ \| g_\epsilon \|_{L^2(0, T; H)} \leq \text{const}, \quad \| g_\epsilon \|_{L^2(0, T; V)} \leq \text{const} \]
\[ \| \eta_\epsilon \|_{L^2(0, T; H)} \leq \text{const}, \quad \| \eta_\epsilon \|_{L^2(0, T; W)} \leq \text{const} \]
\[ \| g'_\epsilon \|_{L^1(0, T; V^*)} \leq \text{const}, \quad \| \eta'_\epsilon \|_{L^1(0, T; W^*)} \leq \text{const} \]

Therefore, from (5.31)-(5.33) we can conclude that there exists a subsequence of \( \{g_\epsilon\}, \{\eta_\epsilon\} \) which we denote by the same symbols, such that
\[ g_m \rightarrow g_\epsilon, \quad *-\text{weakly in } L^\infty(0, T; H), g_m \rightarrow g_\epsilon \quad \text{weakly in } L^2(0, T; V) \]
\[ g_m \rightarrow g_\epsilon, \quad \text{strongly in } L^2(0, T; H) \]
\[ \eta_m \rightarrow \eta_\epsilon, \quad *-\text{weakly in } L^\infty(0, T; H), \eta_m \rightarrow \eta_\epsilon \quad \text{weakly in } L^2(0, T; W) \]
\[ \eta_m \rightarrow \eta_\epsilon, \quad \text{Strongly in } L^2(0, T; H) \]

And
\[ g_\epsilon \in L^\infty(0, T; H) \cap L^2(0, T; V), g'_\epsilon \in L^1(0, T; V^*) \]
\[ \eta_\epsilon \in L^\infty(0, T; H) \cap L^2(0, T; W), \eta_\epsilon' \in L^1(0, T; W^*) \]

By taking the limit as \( m \rightarrow \infty \) in (5.3), (5.4), from (5.34), (5.35) we obtain that \( \{g, \eta\} \) is the solution of (5.26), (5.27). By considering (5.25), we obtain (5.30) □

Now, let introduce conjugate systems such as;
\[ -p' + \nu A_1 p + B(z, p) + B(p, z) + C(q, w) = H_1 r_1, \quad p(T) = 0 \]
\[ -q' + k A_2 q + C(z, q) + \beta \xi p = H_2 r_2, \quad q(T) = 0 \]

And
\[ g_\epsilon \in L^\infty(0, T; H) \cap L^2(0, T; V), g'_\epsilon \in L^1(0, T; V^*) \]

\[ \eta_\epsilon \in L^\infty(0, T; H) \cap L^2(0, T; W), \eta_\epsilon' \in L^1(0, T; W^*) \]

By taking the limit as \( m \rightarrow \infty \) in (5.36), (5.37) we get that \( \{g, \eta\} \) is the solution of (5.26), (5.27). By considering (5.25), we obtain (5.30) □

\[ \begin{align*}
\text{Lemma 3.4.} \quad \text{There exists solution of conjugate systems (5.36)-(5.38).}
\end{align*} \]

Proof. The existence of solution for linear problem (5.36)-(5.38) is proved by taking the limit as \( m \rightarrow \infty \) in Galerkin system of equation
\[ -p_m' + \nu A_1 p_m + Q_m^1 (B(z, p_m) + B(p_m, z)) + Q_m^2 C(q_m, w_m) = Q_m^1 H r_1, \quad p_m(T) = 0 \]
\[ -q_m' + k A_2 q_m + Q_m^2 C(z, q_m) + \beta \xi p_m = Q_m^2 H r_2, \quad q_m(T) = 0 \]

Here \( Q_m^1 \) is projection operator from \( H \) to \( H_m^1 \) and \( H_m^1 \) is the subspace generated by first \( m \) numbers “basis” element of space \( H \). And \( Q_m^2 \) is projection operator from \( \tilde{H} \) to \( H_m^2 \) and \( H_m^2 \) is the subspace generated by first \( m \) numbers “basis” element of space \( \tilde{H} \). □
Theorem 3. We assume that \( \{v_{y_1}, v_{y_2}\}, \{z_1, w_1\} \) is arbitrary optimal pair. Then there exists \( \{p, q\} \) which satisfy the equations and inequalities such as:

\[
\begin{align*}
 z_1 + \nu A_z z_1 + B(z_1, z_1) + \beta \xi w_1 &= H_1 v_{y_1}, \quad z(0) = z_0 \quad (5.39) \\
 w_1 + kA_2 w_1 + C(w_1, z_1) &= H_2 v_{y_2}, \quad w(0) = w_0 \quad (5.40) \\
 z_1 &\in L^2(0,T;V), w_1 \in L^2(0,T;W) \quad (5.41) \\
 -p + \nu A_z p + B(z_1, p) + B(p, z_1) + C(q, w_1) &= H_1, \quad p(T) = 0 \quad (5.42) \\
 -q + kA_2 q + C(z_1, q) + \beta \xi p &= H_2, \quad q(T) = 0 \quad (5.43) \\
p &\in L^2(0,T;V), q \in L^2(0,T;W) \quad (5.44)
\end{align*}
\]

\[
N_1 \int_0^T p_n (v_1 - v_{y_1}) \text{d}s \text{d}t + N_2 \int_0^T q(v_2 - v_{y_2}) \text{d}s \text{d}t \leq 0, \quad \forall v_1(x, t) \in [\alpha_1(x, t), \beta_1(x, t)], \forall v_2(x, t) \in [\alpha_2(x, t), \beta_2(x, t)]
\quad (5.45)
\]

(Proof) Multiplying (5.36) by \( g(t) \) and integrating from 0 to \( T \) for \( t \), we have

\[
\int_0^T \langle -p' + \nu A_z p + B(z_1, p) + B(p, z_1) + C(q, w_1), g(t) \rangle \text{d}t = \int_0^T \langle H_1 r_1, g(t) \rangle \text{d}t
\]

By applying the integration by parts in right hand of this expression, we obtain

\[
\int_0^T \langle g' + \nu A_z g + B(z_1, g) + B(g, z_1), p \rangle \text{d}t + \int_0^T \langle C(q, w_1), g(t) \rangle \text{d}t = \int_0^T \langle H_1 r_1, g(t) \rangle \text{d}t
\quad (5.46)
\]

Multiplying (5.37) by \( \eta(t) \) and integrating from 0 to \( T \) for \( t \), we have

\[
\int_0^T \langle -q' + kA_2 q + C(z_1, q) + \beta \xi p, \eta(t) \rangle \text{d}t = \int_0^T \langle H_2 r_2, \eta(t) \rangle \text{d}t
\]

By applying the integration by parts in right hand of this expression, we obtain

\[
\int_0^T \langle \eta' + kA_2 \eta + C(z_1, \eta) + B(g, z_1), q \rangle \text{d}t + \int_0^T \langle \beta \xi \eta , p \rangle \text{d}t = \int_0^T \langle H_2 r_2, \eta(t) \rangle \text{d}t
\quad (5.47)
\]

By adding (5.46) and (5.47), we obtain

\[
\int_0^T \langle g' + \nu A_z g + B(z_1, g) + B(g, z_1) + \beta \xi \eta , p \rangle \text{d}t + \int_0^T \langle \eta' + kA_2 \eta + C(z_1, \eta) + C(g, w_1), \eta \rangle \text{d}t =
\]

\[
= \int_0^T \langle H_1 r_1, g(t) \rangle \text{d}t + \int_0^T \langle H_2 r_2, \eta(t) \rangle \text{d}t
\]

By applying (5.25)- (5.30) in the right hand of above expression, we have

\[
\int_0^T \langle -H_1 (v_1 - v_{y_1}), p(t) \rangle \text{d}t + \int_0^T \langle -H_2 (v_2 - v_{y_2}), q(t) \rangle \text{d}t \geq 0
\]

From here, by applying the definition of \( H_1 \) and \( H_2 \), we obtain (5.45) \( \Box \)

Corollary 5.1. (Pontryagin's maximum principle in the special case)

We assume that control \( v_2 \equiv 0 \) in optimal control problem (1.2) and \( \{v_{y_1}, \{z_1, w_1\} \) is arbitrary optimal pair. Then optimal control \( v_{y_1} \) satisfy maximal condition in almost everywhere of \( \Sigma_1 \) such as

\[
v_{y_1}(x, t)(p \cdot n)(x, t) = \sup_{\alpha_{i} (x, t) \leq \mu \leq \beta_{i}(x, t)} (\mu \cdot (p \cdot n)(x, t)), \text{a.e.} (x, t) \in \Sigma
\quad (5.48)
\]

Like the preceding, we assume that control \( v_2 \equiv 0 \) in optimal control problem (1.2) and \( \{v_{y_2}, \{z_1, w_1\} \) is arbitrary.
optimal pair. Then optimal control \( v_{1*} \) satisfy maximal condition in almost everywhere of \( \Sigma_2 \) such as
\[
v_{2*}(x,t)q(x,t) = \sup_{\alpha_2(x,t) \leq p_2(x,t) \leq \beta_2(x,t)} (\mu_2 \cdot q(x,t)) \text{, a.e.} \langle x, t \rangle \in \Sigma
\] (5-49)

Here \( (p,q) \) is the solution of conjugate system.

(Proof) First of all, let prove (5-48).

We assume that \( E \) is whole points of \( \Sigma_1 \) which is Lebesgue points for \( p_n \), \( v_{1*} \in L^2(\Sigma_1) \) and \( (x_0,t_0) \in E \).

In the case \( v_{2*} \equiv 0 \), Optimal condition (5.45) is simplified such as
\[
\int_{\Sigma} p_n(v_1 - v_{1*})dxdy \leq 0, \forall v_1(x,t) \in [\beta_1(x,t), \alpha_1(x,t)]. \tag{5.50}
\]

Then, we put in (5.50) \( v_1(x,t) \) such as;
\[
v_1(x,t) = v_{1*} + \chi_j(\mu_1 - v_{1*}(x,t)), \forall \mu_1 \in [\alpha_1(x,t), \beta_1(x,t)]
\]

Here, \( \chi_j \) is characteristic function in certain neighborhood of point \( (x_0,t_0) \) in \( \Sigma \) which converge to point \( (x_0,t_0) \) when \( j \to \infty \).

Now, by dividing (5.50) by \( \chi_j dxdy \), we obtain
\[
\int_{\Sigma} \frac{1}{\chi_j} \cdot \chi_j p_n(\mu_1 - v_{1*}(x,t))dxdy \leq 0
\]

By taking the limit as \( j \to \infty \) in above inequality, we obtain
\[
p_n(\mu_1 - v_{1*}(x,t)) \leq 0, \forall \mu_1 \in [\alpha_1(x,t), \beta_1(x,t)] \text{, a.e.} (x,t) \in \Sigma_1
\]

From here we obtain maximal principle (5.48). By same as above method, we obtain (5.49) □

Remark. In fact, the function \( p_n(x,t) = (p \cdot n) \in L^2(\Sigma_1) \) is switching function of fluid for optimal control.

Seeing that, by (5.48)

if \( p_n(x,t) > 0 \), then \( v_{1*}(x,t) = \beta_1(x,t) \),

if \( p_n(x,t) < 0 \), then \( v_{1*}(x,t) = \alpha_1(x,t) \).

Similarly, we obtain

if \( q(x,t) > 0 \), then \( v_{2*}(x,t) = \beta_2(x,t) \),

if \( p_n(x,t) < 0 \), then \( v_{2*}(x,t) = \alpha_2(x,t) \).

Therefore, the function \( q(x,t) \in L^2(\Sigma_2) \) is switching function of heat flow for optimal control.

In the case \( p_n|_{\Sigma_1} \neq 0 \), a.e. \( (x,t) \in \Sigma_1 \), \( q|_{\Sigma_2} \neq 0 \), a.e. \( (x,t) \in \Sigma_2 \), equalities (5.48),(5.49) denote “Bang-Bang Principle” which well known in the optimal control theory.

References

[1] S. A. Lorca, J.L. Boldrini, The initial value problem for a generalized Boussinesq model, Nonlinear Analysis, 36(4), 457-480, 1999
[2] G.Galiano, Spatial and time localization of solutions of the Boussinesq system with nonlinear thermal diffusion, Nonlinear Analysis, 42, 423-438, 2000
[3] J. I. Diaz, G. Galiano, On the Boussinesq system with nonlinear thermal diffusion, Nonlinear Analysis, 30, 3255-3263, 1997
[4] S. A. Lorca, J.L. Boldrini, The initial value problem for a generalized Boussinesq model:
regularity and global existence of strong solutions, Matemática Contemporânea, 11, 175-201, 1996
[5] S.A. Lorca, J.L. Boldrini, Stationary solutions for generalized Boussinesq model, J. Differential Equations, 124(2), 201-225, 1996
[6] K. Óeda, On the initial value problem for the heat convection equation of Boussinesq approximation in a time-dependent domain, Proc. Japan Acad.64, Sec.A143-146, 1988
[7] H. Mormoto, Non-stationary Boussinesq equations, J. Fac. Sci. Univ. Tokyo, Sec.1A Math., 39 61-75, 1992
[8] Tang Xianjiang, On the existence and uniqueness of the solution to the Navier-Stokes Equations, Acta Mathematica Scientia, 15(3), 342-351, 1995
[9] R. Temam, Navier-Stokes Equations, Studies in Mathematics and its Applications, 2 North-Holland, Amsterdam, 1984.
[10] M.Shinbrot,W.P. Kotorynski, The initial value problem for a viscous heat-conducting fluid, J. Math. Anal.Appl.45, 1-22, 1974
[11] M.Shinbrot, W.P. Kotorynski, The initial value problem for a viscous heat-conducting fluid, J. Math. Anal. Apple. 45, 1-22, 1974
[12] J. L. Lions, Controle des systemes distibues singuliers, Gauthier-villars, Bordas, Paris, 1983.
[13] R. Temam, Navier-Stokes Equations, Studies in Mathematics and its Applications, North-Holland, Amsterdam, 1984
[14] J. L. Lions, Quelques methodes de resolution des problems aux non lineares, Dunod, paris, 1968
[15] H.M.Park, W.S.Jung, The Karhunen-Loeve Galerkin method for the inverse natural convection problems, International Journal of Heat and Mass transfer, 44, 155-167, 2001
[16] H.M.Park et al, An inverse natural convection problem of estimating the strength of a heat source, International Journal of Heat and Mass transfer, 42, 4259 - 4273, 1999.
[17] A.A.Illarionov, Asymptotics of solution to the optimal control problem for time-independent Navier-Stokes Equations, Computational Mathematics and Mathematical Physics, 41(7), 1045-1056, 2001
[18] L. Steven Hou, Thomas P. Svobodny T.P., Optimization problem for the Navier-Stokes equations with regular boundary conditions, Journal of Mathematical Analysis and Applications, 177 (2), 342-367, 1993.
[19] Thomas Bewley, Roger Temam, Mohammed Ziane, Existence and uniqueness of optimal control to the Navier-Stokes equations, C. R. Acad. Sci. Paris, t.330, Serie 1, 1007-1011, 2000.
[20] Wang, Gengsheng, Stabilization of the Boussinesq equation via internal feedback controls, Nonlinear Analysis, 52(2), 2002, 485-508,
[21] Wang, Lijuan; Wang, Gengsheng, Local internal controllability of the Boussinesq system, Nonlinear Analysis, 53(5), 637-655, 2003
[22] Shugang Li, Gengsheng Wang, The time optimal control of the Boussinesq equations Numerical Functional Analysis and Optimization, 24(1-2), 163-180, 2003
[23] Jose Luiz Boldrini, Enrique Fernández-Cara, Marko Antonio Rojas-Medar, An optimal control problem for a generalized Boussinesq model: The time dependent case, Rev. Mat. Complut. 20(2), 339-366, 2007
[24] Gennady Alekseev, Dmitry Tereshko, Stability of optimal controls for the stationary, International Journal of Differential Equations, Volume 2011, 1-28, 2011
[25] Gennady Alekseev1, Dmitry Tereshko1, Vladislav Pukhnachev, Boundary control problems for Oberbeck–Boussinesq model of heat and mass transfer, Advanced Topics in Mass Transfer, Edited by Mohamed El-Amin, 485-512, 2011
[26] G. V. Alekseev , R. V. Brizitskii, Control problems for stationary magnetohydrodynamic equations of a viscous heat-conducting fluid under mixed boundary conditions, Computational Mathematics and Mathematical Physics, 45(12), 2049–2065, 2005
[27] R. Temam, Navier-Stokes Equations: theory and numerical analysis, AMS Chelsea Publishing, American Mathematical Society-Providence, Rhodel Island, 2001
[28] G.V. Alekseev and R.V. Brizitskii, Control problems for stationary magnetohydrodynamic
equations of a viscous heat-conducting fluid under mixed boundary conditions, Comput. Math.
Math. Phys., 45, 2049-2065, 2005
[29] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, AMS, Providence,
Rhode Island, 2001.
[30] G.V. Alekseev, Solvability of control problems for stationary equations of magneto
hydrodynamics of a viscous fluid, Siberian Mathematical Journal, 45(2), 197–213, 2004
[31] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, AMS, Providence,
Rhode Island, 2001.