Some nonlinear differential inequalities and an application to Hölder continuous almost complex structures

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Abstract

We consider some second order quasilinear partial differential inequalities for real-valued functions on the unit ball and find conditions under which there is a lower bound for the supremum of nonnegative solutions that do not vanish at the origin. As a consequence, for complex-valued functions $f(z)$ satisfying $\partial f/\partial \bar{z} = |f|^\alpha$, $0 < \alpha < 1$, and $f(0) \neq 0$, there is also a lower bound for $\sup |f|$ on the unit disk. For each $\alpha$, we construct a manifold with an $\alpha$-Hölder continuous almost complex structure where the Kobayashi–Royden pseudonorm is not upper semicontinuous.

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1. Introduction

We begin with an analysis of a second order quasilinear partial differential inequality for real-valued functions of $n$ real variables,

$$\Delta u - B|u|^{\varepsilon} \geq 0,$$

(1)

where $B > 0$ and $\varepsilon \in [0, 1)$ are constants. In Section 2, we use a Comparison Principle argument to show that (1) has “no small solutions,” in the sense that there is a uniform lower bound $M > 0$ for the supremum of solutions $u$ which are nonnegative on the unit ball and nonzero at the origin.

We also consider a generalization of (1):

$$u \Delta u - B|u|^{1+\varepsilon} - C|\nabla u|^2 \geq 0,$$

(2)

and find conditions under which there is a similar property of no small solutions, in Theorem 2.4.

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As an application of the results on the inequality (1), we show failure of upper semicontinuity of the Kobayashi–Royden pseudonorm for a family of 4-dimensional manifolds with almost complex structures of regularity $C^{0,\alpha}$, $0 < \alpha < 1$. This generalizes the $\alpha = \frac{1}{2}$ example of [5]; it is known [6] that the Kobayashi–Royden pseudonorm is upper semicontinuous for almost complex structures with regularity $C^{1,\alpha}$.

Our construction of the almost complex manifolds in Section 4 is very similar to that of [5]; we give the details for the convenience of the reader, and to show how the argument breaks down as $\alpha \to 1^-$, due to a shrinking radius of the domain. We also take the opportunity in Section 3 to state some lemmas which allow for a more quantitative description than that of [5].

One of the steps in [5] is a Maximum Principle argument applied to a complex-valued function $h(z)$ satisfying the equation $\partial h / \partial \bar{z} = |h|^{1/2}$, to get the property of no small solutions. The main difference between our paper and [5] is the use of a Comparison Principle in Section 2 instead of the Maximum Principle, and we arrive at this result:

**Theorem 1.1.** For any $\alpha \in (0, 1)$, suppose $h(z)$ is a continuous complex-valued function on the closed unit disk, and on the set $\{ z : |z| < 1,\ h(z) \neq 0 \}$, $h$ has continuous partial derivatives and satisfies

$$\frac{\partial h}{\partial \bar{z}} = |h|^\alpha.$$  \hspace{1cm} (3)

If $h(0) \neq 0$ then $\sup |h| > S_\alpha$, where the constant $S_\alpha > 0$ is defined by:

$$S_\alpha = \left( \frac{2(1 - \alpha)}{2 - \alpha} \right)^{1/(1-\alpha)}. \hspace{1cm} (4)$$

2. Some differential inequalities

Let $D_R$ denote the open ball in $\mathbb{R}^n$ centered at $0$ with radius $R > 0$, and let $\overline{D}_R$ denote the closed ball.

**Lemma 2.1.** Given constants $B > 0$ and $0 \leq \epsilon < 1$, let

$$M = \left( \frac{B(1-\epsilon)^2}{2(2\epsilon + n(1-\epsilon))} \right)^{1/\epsilon} > 0.$$  \hspace{1cm}

Suppose the function $u : \overline{D}_1 \to \mathbb{R}$ satisfies:

- $u$ is continuous on $\overline{D}_1$,
- $u(\bar{x}) \geq 0$ for $\bar{x} \in D_1$,
- on the open set $\omega = \{ \bar{x} \in D_1 : u(\bar{x}) \neq 0 \}$, $u \in C^2(\omega)$,
- for $\bar{x} \in \omega$:

$$\Delta u(\bar{x}) - B(u(\bar{x}))^\epsilon \geq 0.$$ \hspace{1cm} (5)

If $u(0) \neq 0$, then $\sup_{\bar{x} \in D_1} u(\bar{x}) > M$.

**Proof.** Define a comparison function

$$v(\bar{x}) = M |\bar{x}|^\frac{2\epsilon}{\epsilon + n},$$

so $v \in C^2(\mathbb{R}^n)$ since $0 \leq \epsilon < 1$. By construction of $M$, it can be checked that $v$ is a solution of this nonlinear Poisson equation on the domain $\mathbb{R}^n$:

$$\Delta v(\bar{x}) - B|v(\bar{x})|^{\epsilon} \equiv 0.$$  \hspace{1cm}

Suppose, toward a contradiction, that $u(\bar{x}) \leq M$ for all $\bar{x} \in D_1$. For a point $\bar{x}_0$ on the boundary of $\omega \subseteq \mathbb{R}^n$, either $|\bar{x}_0| = 1$, in which case by continuity, $u(\bar{x}_0) \leq M = v(\bar{x}_0)$, or $0 < |\bar{x}_0| < 1$ and $u(\bar{x}_0) = 0$, so $u(\bar{x}_0) \leq v(\bar{x}_0)$. Since $u \leq v$ on the boundary of $\omega$, the Comparison Principle [4, Theorem 10.1] applies to the subsolution $u$ and the solution $v$ on the domain $\omega$. The relevant hypothesis for the Comparison Principle in this case is that the second term expression
of (5), $-BX^\varepsilon$, is weakly decreasing, which uses $B > 0$ and $\varepsilon \geq 0$. (To satisfy this technical condition for all $X \in \mathbb{R}$, we define a function $c: \mathbb{R} \to \mathbb{R}$ by $c(X) = -BX^\varepsilon$ for $X \geq 0$, and $c(X) = 0$ for $X \leq 0$. Then $c$ is weakly decreasing in $X$, $v$ satisfies $\Delta v(x) + c(v(x)) \equiv 0$ and $u$ satisfies $\Delta u(x) + c(u(x)) \geq 0$.)

The conclusion of the Comparison Principle is that $u \leq v$ on $\omega$, however $\tilde{0} \in \omega$ and $u(\tilde{0}) > v(\tilde{0})$, a contradiction. □

Of course, the constant function $u \equiv 0$ satisfies the inequality (5), and so does the radial comparison function $v$, so the initial condition $u(\tilde{0}) \neq 0$ is necessary.

Example 2.2. In the $n = 1$ case, $M = \left(\frac{B(1-\varepsilon)^2}{2(1+\varepsilon)}\right)^{\frac{1}{\varepsilon+1}}$. For points $c_1, c_2 \in \mathbb{R}, c_1 < c_2$, define a function

$$u(x) = \begin{cases} M(x - c_2)^{\frac{2}{\varepsilon+1}} & \text{if } x \geq c_2, \\ 0 & \text{if } c_1 \leq x \leq c_2, \\ M(c_1 - x)^{\frac{2}{\varepsilon+1}} & \text{if } x \leq c_1. \end{cases}$$

Then $u \in C^2(\mathbb{R})$, and it is nonnegative and satisfies $u'' = B|u|^\varepsilon$ (the case of equality in the $n = 1$ version of (5)). For $c_1 < 0 < c_2$, this gives an infinite collection of solutions of the ODE $u'' = B|u|^\varepsilon$ which are identically zero in a neighborhood of 0, so the ODE does not have a unique continuation property. For $c_1 > 0$ or $c_2 < 0$, the function $u$ satisfies $u(0) \neq 0$ and the other hypotheses of Lemma 2.1, and its supremum on $(0, 1)$ exceeds $M$ even though it can be identically zero on an interval not containing 0.

Example 2.3. In the case $n = 2$, $B = 1$, $\varepsilon = 0$, (5) becomes the linear inequality $\Delta u \geq 1$ and the number $M = \frac{1}{4}$ agrees with Lemma 2 of [5], which was proved there using a Maximum Principle argument.

By applying Lemma 2.1 to the Laplacian of a power of $u$, we get the following generalization.

Theorem 2.4. Given constants $B > 0$, $C \in \mathbb{R}$, and $\varepsilon < 1$, let

$$M = \begin{cases} \left(\frac{B(1-\varepsilon)^2}{2(1+\varepsilon)+n(1-\varepsilon)}\right)^{\frac{1}{\varepsilon+1}} & \text{if } C \leq \varepsilon, \\ \left(\frac{B(1-\varepsilon)^2}{2n}\right)^{\frac{1}{\varepsilon+1}} & \text{if } C > \varepsilon. \end{cases}$$

Suppose the function $u: \overline{D}_1 \to \mathbb{R}$ satisfies:

\begin{itemize}
  \item $u$ is continuous on $\overline{D}_1$.
  \item $u(x) \geq 0$ for $x \in D_1$.
  \item on the open set $\omega = \{x \in D_1: u(x) \neq 0\}, u \in C^2(\omega)$.
  \item for $x \in \omega$:
    \begin{itemize}
      \item if $u(\tilde{x}) \neq 0$, then $\sup_{x \in D_1} u(x) > M$.
    \end{itemize}
\end{itemize}

Proof. Let $\mu = \min(\varepsilon, C)$, so $\mu \leq \varepsilon < 1$, and on the set $\omega$,

$$u(x) \Delta u(x) \geq B|u(x)|^{1+\varepsilon} + \mu|\nabla u(x)|^2.$$

Consider the function $u^{1-\mu}$ on $\overline{D}_1$, so $u^{1-\mu} \in C^0(\overline{D}_1) \cap C^2(\omega)$, and on the set $\omega$,

$$\Delta(u^{1-\mu}) = (1-\mu)u^{-\mu-1}(u \Delta u - \mu|\nabla u|^2) \geq (1-\mu)u^{-\mu-1}Bu^{1+\varepsilon} = (1-\mu)B(u^{1-\mu})^{(\varepsilon-1)/(1-\mu)}.$$
Since \((1 - \mu)B > 0\), and \(\mu \leq \varepsilon < 1\) \(\Rightarrow 0 \leq \frac{\varepsilon - \mu}{1 - \mu} < 1\), Lemma 2.1 applies to \(u^{1-\mu}\). If \((u(\bar{\Omega}))^{1-\mu} \neq 0\), then
\[
\sup u^{1-\mu} > \left(\frac{(1 - \mu)B(1 - \frac{\varepsilon - \mu}{1 - \mu})^2}{2(2\frac{\varepsilon - \mu}{1 - \mu} + n(1 - \frac{\varepsilon - \mu}{1 - \mu}))}\right)^{\frac{1}{1-\mu}} \Rightarrow \sup u > \left(\frac{B(1 - \varepsilon)^2}{2(2(\varepsilon - \mu) + n(1 - \varepsilon))}\right)^{\frac{1}{1-\mu}}. \quad \Box
\]

Functions satisfying a differential inequality of the form (1) or (2) also satisfy a Strong Maximum Principle; the only condition is \(B > 0\).

**Theorem 2.5.** Given any open set \(\Omega \subseteq \mathbb{R}^n\), and any constants \(B > 0\), \(C, \varepsilon \in \mathbb{R}\), suppose the function \(u : \Omega \rightarrow \mathbb{R}\) satisfies:

- \(u\) is continuous on \(\Omega\),
- on the set \(\omega = \{\bar{x} \in \Omega : u(\bar{x}) > 0\}\), \(u \in C^2(\omega)\),
- on the set \(\omega\), \(u\) satisfies
  \[
u \Delta u - B|u|^{1+\varepsilon} - C|\vec{\nabla} u|^2 \geq 0.\]

If \(u(\bar{x}_0) > 0\) for some \(\bar{x}_0 \in \Omega\), then \(u\) does not attain a maximum value on \(\Omega\).

**Proof.** Note that the constant function \(u \equiv 0\) is the only locally constant solution of the inequality for \(B > 0\). If \(B = 0\) then obviously any constant function would be a solution.

Given a function \(u\) satisfying the hypotheses, \(\omega\) is a nonempty open subset of \(\Omega\). Suppose, toward a contradiction, that there is some \(\bar{x}_1 \in \Omega\) with \(u(\bar{x}) \leq u(\bar{x}_1)\) for all \(x \in \Omega\). In particular, \(u(\bar{x}_1) \geq u(\bar{x}_0) > 0\), so \(\bar{x}_1 \in \omega\). Let \(\omega_1\) be the connected component of \(\omega\) containing \(\bar{x}_1\).

For \(\bar{x} \in \omega_1\), \(u\) satisfies the linear, uniformly elliptic inequality
\[
\Delta u(\bar{x}) + (-B(u(\bar{x}))^{\varepsilon-1})u(\bar{x}) + \left(-C \frac{\vec{\nabla} u(\bar{x})}{u(\bar{x})}\right) \cdot \vec{\nabla} u(\bar{x}) \geq 0,
\]
where the coefficients (defined in terms of the given \(u\)) are locally bounded functions of \(\bar{x}\), and \((-B(u(\bar{x}))^{\varepsilon-1})\) is negative for all \(\bar{x} \in \omega\). It follows from the Strong Maximum Principle [4, Theorem 3.5] that since \(u\) attains a maximum value at \(\bar{x}_1\), then \(u\) is constant on \(\omega_1\). Since the only constant solution is 0, it follows that \(u(\bar{x}_1) = 0\), a contradiction. \(\Box\)

The next lemma shows how an inequality like (5) with \(n = 2\) can arise from a first order PDE for a complex-valued function.

**Lemma 2.6.** Consider constants \(\alpha, \gamma \in \mathbb{R}\) with \(0 < \alpha < 1\). Let \(\omega \subseteq \mathbb{C}\) be an open set, and suppose \(h : \omega \rightarrow \mathbb{C}\) satisfies:

- \(h \in C^1(\omega)\),
- \(h(z) \neq 0\) for all \(z \in \omega\),
- \(\frac{\partial h}{\partial \bar{z}} = |h|^\alpha\) on \(\omega\).

Then, the following inequality is satisfied on \(\omega\):
\[
\Delta(|h|^{(1-\alpha)\gamma}) \geq (4(1 - \alpha)\gamma - (2 - \alpha)^2)|h|(1-\alpha)(\gamma-2).
\]

**Remark.** The special case \(\alpha = \frac{1}{2}, \gamma = \frac{3}{2}\) is Lemma 1 of [5]; its proof there is a long calculation in polar coordinates, which can be generalized to some other values of \(\alpha\) by an analogous argument. However, using \(z, \bar{z}\) coordinates allows for a shorter calculation.

**Proof of Lemma 2.6.** We first want to show that \(h\) is smooth on \(\omega\), applying the regularity and bootstrapping technique of PDE to the equation \(\partial h/\partial \bar{z} = |h|^\alpha\). We recall the following fact (for a more general statement, see
Theorem 15.6.2 of [1]): for a nonnegative integer $\ell$, and $0 < \beta < 1$, if $\varphi \in C^{\ell,\beta}_0(\omega)$ and $g : \omega \to \mathbb{C}$ has first derivatives in $L^2_0(\omega)$ and is a solution of $\partial g/\partial z = \varphi$, then $g \in C^{\ell+1,\beta}_0(\omega)$. In our case, $\varphi = |h|^\alpha \in C^1(\omega) \subseteq C^{0,\beta}_0(\omega)$ (since $h \in C^1(\omega)$ and is nonvanishing), and $g = h$ has continuous first derivatives, so we can conclude that $g = h \in C^{1,\beta}_0(\omega)$. Repeating gives that $h \in C^{2,\beta}_0(\omega)$, etc.

Since the conclusion is a local statement, it is enough to express $\omega$ as a union of open subsets $\omega_k$ and establish the conclusion on each subset. For each $z_k \in \omega$, there is a sufficiently small disk $\omega_k$ containing $z_k$, where real exponentiation of $h(z)$ is well defined on $\omega_k$, by choosing a single-valued branch of log to define $h' = \exp(r \log(h))$.

The condition $\partial h/\partial z = |h|^\alpha$ can be re-written

$$h\bar{z} = (\bar{h})z = |h|^\alpha = \overline{h^{\alpha/2}h^{\alpha/2}}.$$ 

This leads to

$$h\bar{z} = (h\bar{z})z = \left(h^{\alpha/2}\bar{h}^{\alpha/2}\right)_z$$

$$= \left(h^{(\alpha/2) - 1}\bar{h}^{\alpha/2}h + h^{\alpha}\bar{h}^{\alpha - 1}\right)$$

$$= \left(h^{\alpha/2}\bar{h}^{\alpha/2}\right)_z,$$

which is used in a line of the next step. For an arbitrary exponent $m \in \mathbb{R}$,

$$\left(|h|^m\right)_{\bar{z}} = \left(h^{m/2}\bar{h}^{m/2}\right)_{\bar{z}}$$

$$= \frac{\partial}{\partial z}\left(\frac{m}{2} h^{m/2 - 1}h\bar{z} + h^{m+1/2}\bar{h}^{m - 1} \bar{h}\right)$$

$$= \frac{m}{2} h^{m/2 - 1}h\bar{z} + h^{m+1/2}\bar{h}^{m - 1} \bar{h}\right)$$

$$= \left[\left(\frac{m}{2} + \alpha - 1\right)h^{m/2 - 1}h\bar{z} + \alpha \left(h^{m+1/2}\bar{h}^{m - 1}\bar{h}\right) + \left(h^{m+1/2}\bar{h}^{m - 1}\bar{h}\right)\right].$$

With the aim of applying Lemma 2.1 to the function $|h|^m$, we consider the expression (8), with real constants $B, \epsilon$, and $m \neq 0$. In line (9), we assign

$$\epsilon = \frac{1}{m} (m + 2\alpha - 2)$$

(7)

to be able to combine like terms, and in line (10), we choose $B = 4m - (2 - \alpha)^2$ to complete the square.

$$\Delta(|h|^m) - B(|h|^m)^\epsilon$$

$$= \left[|h|^m\right]_{\bar{z}} - B|h|^m\epsilon$$

$$= 2m \left[\left(\frac{m}{2} + \alpha - 2\right)|h|^{m+\alpha - 4}\bar{h}^2h\bar{z} + \left(\frac{m}{2} + \alpha\right)|h|^{m+2\alpha - 2} + \frac{m}{2}|h|^{m-2}|h\bar{z}|^2\right] - B|h|^m\epsilon$$

$$= \left(m(m + 2\alpha - B)\right)|h|^{m+2\alpha - 2}$$

$$+ \left(2m(m + \alpha) - B\right)|h|^{m+\alpha - 4}\bar{h}^2h\bar{z} + m^2|h|^{m-2}|h\bar{z}|^2$$

$$\geq \left|h|^{m-2}\left((m^2 + 2am - B)|h|^{2\alpha} - 2|m|m + \alpha - 2||h|^m|h\bar{z}| + m^2|h\bar{z}|^2\right)\right]$$

$$= \left|h|^{m-2}(m + \alpha - 2)||h|^\alpha - |m||h\bar{z}|^2\geq 0.$$ (10)

Considering the form of (7), it is convenient to choose $m = (1 - \alpha)\gamma$ for some constant $\gamma \neq 0$. The claim of the lemma follows; the $\gamma = 0$ case can be checked separately. □
The parameter $\gamma$ can be chosen arbitrarily large; to apply Lemma 2.1 to get the “no small solutions” result of Theorem 1.1, we need the RHS coefficient in (6) to be positive, so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$, and also the RHS exponent $(1-\alpha)/(\gamma - 2)$ to be nonnegative, so $\gamma \geq 2$. In contrast, the $\alpha = 1/2$, $\gamma = 3/2$ case appearing in Lemma 1 of [5] has RHS exponent $-\frac{1}{4}$. The approach of Theorem 2 of [5] is to use the negative exponent together with the result of Example 2.3 to show that assuming $h$ has a small solution leads to a contradiction. As claimed, their method can be generalized to apply to other nonpositive exponents, but $\frac{(2-\alpha)^2}{4(1-\alpha)} < \gamma \leq 2$ holds only for $\alpha < 2(\sqrt{2} - 1) \approx 0.8284$.

**Proof of Theorem 1.1.** Given a continuous $h : \overline{D}_1 \to \mathbb{C}$ satisfying the hypotheses of Theorem 1.1, on the set $\omega = \{z \in D_1 : h(z) \neq 0\}$, $h \in C^1(\omega)$, and the conclusion of Lemma 2.6 can be re-written:

$$\Delta((h(1-\alpha)\gamma) \geq (4(1-\alpha)\gamma - (2-\alpha)^2)((h(1-\alpha)\gamma)^{1/\gamma} - \frac{1}{\gamma}).$$

(11)

The hypotheses of Lemma 2.1 are satisfied with $n = 2$, $u(x, y) = |h(x + iy)|^{1/(1-\alpha)\gamma}$, and $u(0) \neq 0$, when the RHS of (11) has a positive coefficient (so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$) and the quantity $\varepsilon = 1 - \frac{2}{\gamma}$ is in $[0, 1)$ (for $\gamma \geq 2$). The conclusion of Lemma 2.1 is:

$$\sup_{z \in D_1} |h(z)|^{1/(1-\alpha)\gamma} > M = \left(\frac{1}{4} \cdot (4(1-\alpha)\gamma - (2-\alpha)^2) \cdot \left(\frac{2}{\gamma}\right)^2\right)^{\gamma/2}$$

$$\Rightarrow \sup_{z \in D_1} |h(z)| > \left(\frac{4(1-\alpha)\gamma - (2-\alpha)^2}{\gamma^2}\right)^{1/(1-\alpha)\gamma}.$$

We can optimize this lower bound, using elementary calculus to show that the maximum value of $\frac{4(1-\alpha)\gamma - (2-\alpha)^2}{\gamma^2}$ is achieved at the critical point $\gamma = \frac{(2-\alpha)^2}{2(1-\alpha)} > \max\{2, \frac{(2-\alpha)^2}{4(1-\alpha)}\}$, and the lower bound for the sup is $S_{\alpha}$ as appearing in (4). □

Note that $S_{\alpha}$ is decreasing for $0 < \alpha < 1$, with $S_{1/2} = \frac{4}{7}$, $S_{2/3} = \frac{1}{8}$, and $S_{\alpha} \to 0$ as $\alpha \to 1^-$. This theorem is used in the proof of Theorem 4.3.

**Example 2.7.** As noted by [5], a 1-dimensional analogue of Eq. (3) in Theorem 1.1 is the well-known (for example, [2, §I.9]) ODE $u'(x) = B|u(x)|^\alpha$ for $0 < \alpha < 1$ and $B > 0$, which can be solved explicitly. By an elementary separation of variables calculation, the solution on an interval where $u \neq 0$ is $|u(x)| = (\pm(1-\alpha)(Bx + C))^{1/\alpha}$. The general solution on the domain $\mathbb{R}$ is, for $c_1 < c_2$,

$$u(x) = \begin{cases} (1-\alpha)^{1/\alpha} (B(x - c_2))^{1/\alpha} & \text{if } x \geq c_2, \\ 0 & \text{if } c_1 \leq x \leq c_2, \\ -(1-\alpha)^{1/\alpha} (B(c_1 - x))^{1/\alpha} & \text{if } x \leq c_1. \end{cases}$$

So $u \in C^1(\mathbb{R})$, and if $u(0) \neq 0$, then $\sup_{-1 < x < 1} |u(x)| > ((1-\alpha)B)^{1/\alpha}$.

### 3. Lemmas for holomorphic maps

We continue with the $D_R$ notation for the open disk in the complex plane centered at the origin. The following quantitative lemmas on inverses of holomorphic functions $D_R \to \mathbb{C}$ are used in a step of the proof of Theorem 4.3 where we put a map $D_r \to \mathbb{C}^2$ into a normal form, (14).

**Lemma 3.1.** (See [3, Exercise I.1].) Suppose $f : D_1 \to D_1$ is holomorphic, with $f(0) = 0$, $|f'(0)| = \delta > 0$. For any $\eta \in (0, \delta)$, let $s = (\frac{\delta - \eta}{1 - \eta})^\eta$; then the restricted function $f : D_\eta \to D_1$ takes on each value $w \in D_s$ exactly once. □

The hypotheses imply $\delta \leq 1$ by the Schwarz Lemma.
Lemma 3.2. For a holomorphic map $Z_1 : D_r \to D_2$ with $Z_1(0) = 0$, $Z_1'(0) = 1$, if $r > \frac{4\sqrt{r}}{3}$ then there exists a continuous function $\phi : \overline{D}_1 \to D_r$ which is holomorphic on $D_1$ and which satisfies $(Z_1 \circ \phi)(z) = z$ for all $z \in \overline{D}_1$.

Remark. It follows from the Schwarz Lemma that $r \leq 2$, and it follows from the fact that $\phi$ is an inverse of $Z_1$ that $\phi(0) = 0$ and $\phi'(0) = 1$.

Proof of Lemma 3.2. Define a new holomorphic function $f : D_1 \to D_1$ by

$$f(z) = \frac{1}{2} \cdot Z_1(r \cdot z),$$

so $f(0) = 0$, $f'(0) = \frac{1}{2}$, and Lemma 3.1 applies with $\delta = \frac{1}{4}$. If we choose $\eta = \frac{3r}{8}$, then $s = \frac{3r^2}{64 - 12r}$, and the assumption $r > \frac{4\sqrt{r}}{3}$ implies $s > \frac{1}{2}$. It follows from Lemma 3.1 that there exists a function $\psi : D_3 \to D_2$ such that $(f \circ \psi)(z) = z$ for all $z \in \overline{D}_{1/2} \subseteq D_r$; this inverse function $\psi$ is holomorphic on $D_{1/2}$. The claimed function $\phi : \overline{D}_1 \to D_r \subseteq D_r$ is defined by $\phi(z) = r \cdot \psi\left(\frac{1}{2} \cdot z\right)$, so for $z \in \overline{D}_1$,

$$Z_1(\phi(z)) = Z_1\left(r \cdot \psi\left(\frac{1}{2} \cdot z\right)\right) = 2 \cdot f\left(\psi\left(\frac{1}{2} \cdot z\right)\right) = 2 \cdot \frac{1}{2} \cdot z = z. \quad \Box$$

4. J-holomorphic disks

For $S > 0$, consider the bidisk $\Omega_S = D_2 \times D_S \subseteq \mathbb{C}^2$, as an open subset of $\mathbb{R}^4$, with coordinates $\vec{x} = (x_1, y_1, x_2, y_2) = (z_1, z_2)$ and the trivial tangent bundle $T \Omega_S \subseteq T \mathbb{R}^4$. Consider an almost complex structure $J$ on $\Omega_S$ given by a complex structure operator on $T_{\vec{x}} \Omega_S$ of the following form:

$$J(\vec{x}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ \lambda & 0 & 1 & 0 \end{pmatrix}, \quad (12)$$

where $\lambda : \Omega_S \to \mathbb{R}$ is any function.

A differentiable map $Z : D_r \to \Omega_S$ is a $J$-holomorphic disk if $dZ \circ J_{std} = J \circ dZ$, where $J_{std}$ is the standard complex structure on $D_r \subseteq \mathbb{C}$. Let $z = x + iy$ be the coordinate on $D_r$. For $J$ of the form (12), if $Z(z)$ is defined by complex-valued component functions,

$$Z : D_r \to \Omega_S : Z(z) = (Z_1(z), Z_2(z)), \quad (13)$$

then the $J$-holomorphic property implies that $Z_1 : D_r \to D_2$ is holomorphic in the standard way.

Example 4.1. If the function $\lambda(z_1, z_2)$ satisfies $\lambda(z_1, 0) = 0$ for all $z_1 \in D_2$, then the map $Z : D_2 \to \Omega_S : Z(z) = (z, 0)$ is a $J$-holomorphic disk.

Definition 4.2. The Kobayashi–Royden pseudonorm on $\Omega_S$ is a function $T \Omega_S \to \mathbb{R} : (\vec{x}, \vec{v}) \mapsto \| (\vec{x}, \vec{v}) \|_K$, defined on tangent vectors $\vec{v} \in T_{\vec{x}} \Omega_S$ to be the number

$$\mathrm{glb} \left\{ \frac{1}{r} : \exists \text{ a } J\text{-holomorphic } Z : D_r \to \Omega_S, \ Z(0) = \vec{x}, \ dZ(0) \left( \frac{\partial}{\partial x} \right) = \vec{v} \right\}.$$

Under the assumption that $\lambda \in C^{0,\alpha}(\Omega_S)$, $0 < \alpha < 1$, it is shown by [6] and [7] that there is a nonempty set of $J$-holomorphic disks through $\vec{x}$ with tangent vector $\vec{v}$ as in the definition, so the pseudonorm is a well-defined function. Further, each such disk satisfies $Z \in C^1(D_r)$.

At this point we pick $\alpha \in (0, 1)$ and set $\lambda(z_1, z_2) = -2|z_2|^\alpha$. Let $S = S_\alpha > 0$ be the constant defined by formula (4) from Theorem 1.1. Then, $(\Omega_S, J)$ is an almost complex manifold with the following property:
Theorem 4.3. If $0 \neq b \in D_S$ then $\|(0, b), (1, 0)\|_K \geq \frac{3}{4\sqrt{2}}$.

Remark. Since $\frac{3}{4\sqrt{2}} \approx 0.53$, and $\|(0, 0), (1, 0)\|_K \leq \frac{1}{2}$ by Example 4.1, the theorem shows that the Kobayashi–Royden pseudonorm is not upper semicontinuous on $T\Omega_S$.

Proof. Consider a $J$-holomorphic map $Z : D_r \to \Omega_S$ of the form (13), and suppose $Z(0) = (0, b) \in \Omega_S$ and $dZ(0)(\frac{\partial}{\partial z}) = (1, 0)$. Then the holomorphic function $Z_1 : D_r \to D_2$ satisfies $Z_1(0) = 0$, $Z'_1(0) = 1$, and $Z_2 \in C^1(D_r)$ satisfies $Z_2(0) = b$.

Suppose, toward a contradiction, that there exists such a map $Z$ with $b \neq 0$ and $r > \frac{4\sqrt{2}}{1}$ Then Lemma 3.2 applies to $Z_1$; there is a re-parametrization $\phi$ which puts $Z$ into the following normal form:

$$(Z \circ \phi) : D_1 \to \Omega_S,$$

$$z \mapsto (Z_1(\phi(z)), Z_2(\phi(z))) = (z, f(z)), \quad (14)$$

where $f = Z_2 \circ \phi : D_1 \to D_S$ satisfies $f \in C^1(D_1) \cap C^1(D_1)$. From the fact that $Z \circ \phi$ is $J$-holomorphic on $D_1$, it follows from the form (12) of $J$ that if $f(z) = u(x, y) + iv(x, y)$, then $f$ satisfies this system of nonlinear Cauchy–Riemann equations on $D_1$:

$$\frac{du}{dy} = -\frac{dv}{dx} \quad \text{and} \quad \frac{du}{dx} + \lambda(z, f(z)) = \frac{dv}{dy} \quad (15)$$

with the initial conditions $f(0) = b$, $u_x(0) = u_y(0) = v_x(0) = 0$ and $v_y(0) = \lambda(0, b) = -2|b|^\alpha$. The system of equations implies

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} (u + iv) + i \frac{\partial}{\partial y} (u + iv) \right)$$

$$= \frac{1}{2} \left( u_x - v_y + iv_x + u_y \right)$$

$$= -\frac{1}{2} \lambda(z, f(z)) = |f|^\alpha. \quad (16)$$

So, Theorem 1.1 applies, with $f = h$. The conclusion is that

$$\sup_{z \in D_1} |f(z)| > S_\alpha,$$

but this contradicts $|f(z)| < S = S_\alpha$. $\square$

The previously mentioned existence theory for $J$-holomorphic disks shows there are interesting solutions of Eq. (16), and therefore also the inequality (11).

Example 4.4. For $0 < \alpha < 1$, $(\Omega_S, J)$, $\lambda(z_1, z_2) = -2|z_2|^\alpha$ as above, a map $Z : D_r \to \Omega_S$ of the form $Z(z) = (z, f(z))$ is $J$-holomorphic if $f(x, y) = u(x, y) + iv(x, y)$ is a solution of (15). Again generalizing the $\alpha = \frac{1}{2}$ case of [5], examples of such solutions can be constructed (for small $r$) by assuming $v \equiv 0$ and $u$ depends only on $x$, so (15) becomes the ODE $u'(x) - 2|u(x)|^\alpha = 0$. This is the equation from Example 2.7; we can conclude that $J$-holomorphic disks in $\Omega_S$ do not have a unique continuation property.

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