Contextuality Provides Quantum Advantage in Postselected Metrology

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We show that postselection offers a nonclassical advantage in quantum metrology. In every parameter-estimation experiment, the final measurement or the postprocessing incurs some cost. Postselection can improve the rate of Fisher information (the average information learned about an unknown parameter from an experiment) to cost. This improvement, we show, stems from the negativity of a quasi-probability, a quantum extension of a probability. The quasi-probability's nonclassically negative values enable postselected experiments to outperform even postselection-free experiments whose input states and final measurements are optimized. Postselected quantum experiments can yield anomalously large information-cost rates. We prove that this advantage is genuinely nonclassical: no classically noncontextual theory can describe any quantum experiment that delivers an anomalously large Fisher information.

Introduction.—Our ability to deliver new quantum mechanical improvements to technologies relies on a better understanding of the foundation of quantum theory: When is a phenomenon truly nonclassical? We consider two notions of nonclassicality: negativity, which relates to the impossibility of describing quantum states with joint probability distributions [11, 12]; and measurement contextuality, which describes the impossibility of a quantum outcome probability's depending on a unique set of underlying physical states [2, 3].

One field advanced by quantum mechanics is metrology, which concerns the statistical estimation of unknown physical parameters. Quantum metrology relies on quantum phenomena to improve estimations beyond classical bounds [4]. A famous example exploits entanglement [7]. Consider using N separable probe states to evaluate identical systems. The best estimator's error will scale as N−1/2. If the probes are entangled, the error scaling improves to N−1. As Bell's theorem rules out classical (local realist) explanations of entanglement, the improvement is genuinely quantum.

A central quantity in parameter estimation is the Fisher information, F(θ). The Fisher information quantifies the average information learned about an unknown parameter θ from an experiment [8–10]. F(θ) sets a lower bound on the variance of an unbiased estimator θ̂ via the Cramér-Rao inequality: Var(θ̂) ≥ 1/F(θ) [11, 12]. A common metrological task concerns optimally estimating an unknown physical process. The experimental input and the final measurement are optimized to maximize the Fisher information and to minimize the estimator's error.

Classical parameter estimation can benefit from postselecting the output data before postprocessing. Postselection can raise the Fisher information per final measurement or postprocessing event. (See Fig. 1) Postselection can also raise the information-cost rate in a quantum setting. But classical postselection is intuitive, whereas an intense discussion surrounds postselected quantum experiments [13–23]. The ontological nature of postselected quantum states, and the extent to which they exhibit nonclassical behavior, is subject to an ongoing debate. Particular interest has been aimed towards pre- and postselected observable averages. These weak values can lie outside the eigenspectrum of the observable when measured via a weak coupling to a pointer particle [13, 24]. Such values offer metrological advantages in estimations of weak-coupling strengths [15, 20, 25, 26].

In this Letter, we go beyond this restrictive setting and ask: can postselection provide a nonclassical advantage in general quantum parameter-estimation experiments? We conclude that it can. We study metrology experiments for estimating an unknown transformation parameter whose final measurement or postprocessing incurs an experimental cost. Postselection allows the experiment to incur that cost only when the result of the postselected measurement reveals that the Fisher information of the final measurement will be high. We express the Fisher information in terms of a quasi-probability distribution. This expression demonstrates that quantum negativity enables postselection to increase the Fisher information above the values available from standard

FIG. 1. Classical experiment with postselection. A nonoptimal input device initializes a particle in one of two states, with probabilities p and 1−p, respectively. The particle undergoes a stochastic transformation Γθ set by an unknown parameter θ. Only the part of the transformation that acts on particles in the lower path depends on θ. If the final measurement is expensive, the particles in the upper path should be discarded: they possess no information about θ.
input- and measurement-optimized experiments. Such an anomalous Fisher information can improve the rate of information gain to experimental cost, offering a genuine quantum advantage. To round out the understanding of the nonclassicality, we construct a noncontextual model for classical experiments. Within this model, postselection can improve information-cost rates no more than a strategy that uses optimal inputs and final measurements can. We thus conclude that experiments that generate anomalous Fisher-information values prove that quantum mechanics is irreconcilable with noncontextual models.

Quantum Fisher information.—Consider an experiment with outcomes \{i\} and associated probabilities \( p_i(\theta) \), which depend on some unknown parameter \( \theta \). The Fisher information about \( \theta \) is

\[
F(\theta) = \sum_i p_i(\theta) [\partial_\theta \ln(p_i(\theta))]^2 = \sum_i \frac{1}{p_i(\theta)} [\partial_\theta p_i(\theta)]^2.
\]

Repeating the experiment \( N \gg 1 \) times provides an amount \( NF(\theta) \) of information about \( \theta \). The estimator’s variance is bounded by \( \text{Var}(\theta|\theta) \geq 1/[NF(\theta)] \).

Below, we define and compare two types of metrological procedures. In both scenarios, we wish to estimate an unknown parameter \( \theta \) that governs a physical transformation.

Optimized prepare-measure experiment: An input system undergoes the partially unknown transformation, after which the system is measured. Both the input system and the measurement are chosen to provide the largest possible Fisher information.

Postselected prepare-measure experiment: An input system undergoes, first, the partially unknown transformation and, second, a postselection measurement. Conditioned on the postselection’s yielding the desired outcome, the system undergoes an information-optimized final measurement.

Consider a quantum experiment that outputs a state \( \rho_0 = \hat{U}(\theta) \rho_0 \hat{U}(\theta) \), where \( \rho_0 \) is the input state and \( \hat{U}(\theta) \) represents a unitary evolution set by \( \theta \). The quantum Fisher information is defined as the Fisher information maximized over all possible generalized measurements [9, 29, 31]:

\[
F_Q(\theta|\rho_0) = \text{Tr}[\rho_0 \Lambda^2_{\rho_0}]. 
\]

\( \Lambda_{\rho_0} \) is the symmetric logarithmic derivative, implicitly defined by \( \partial_\theta \rho_0 = \frac{1}{2} (\Lambda_{\rho_0} \rho_0 + \rho_0 \Lambda_{\rho_0}) \).

If \( \rho_0 \) is pure, such that \( \rho_0 = |\Psi_\theta\rangle \langle \Psi_\theta| \), the quantum Fisher information can be written as [32]

\[
F_Q(\theta|\rho_0) = 4 \langle \Psi_\theta | \dot{\Psi}_\theta \rangle - 4 | \langle \Psi_\theta | \dot{\Psi}_\theta \rangle |^2, 
\]

where \( \dot{\Psi}_\theta \equiv \partial_\theta \Psi_\theta \equiv \frac{\partial \Psi_\theta}{\partial \theta} \).

In accordance with Stone’s theorem [33], we assume that the evolution can be represented by \( \hat{U}(\theta) \equiv e^{-i \hat{A} \theta} \), where \( \hat{A} \) is a Hermitian operator. For a pure state, the quantum Fisher information equals \( F_Q(\theta|\rho_0) = 4 \text{Var}(\hat{A}|\rho_0) [6] \). Maximizing Eq. 1 over all measurements gives \( F_Q(\theta|\rho_0) \). Similarly, \( F_Q(\theta|\rho_0) \) can be maximized over all input states. For a given unitary \( \hat{U}(\theta) = e^{-i \hat{A} \theta} \), the maximum quantum Fisher information is

\[
\max_{\rho_0} \{ F_Q(\theta|\rho_0) \} = 4 \max_{\rho_0} \{ \text{Var}(\hat{A}|\rho_0) \} = (\Delta a)^2, \quad (4)
\]

where \( \Delta a \) is the difference between the maximum and minimum eigenvalues of \( A \). To summarize, in an optimized quantum prepare-measure experiment, the quantum Fisher information is \( (\Delta a)^2 \).

We now find an expression for the quantum Fisher information in a postselected prepare-measure experiment. A projective postselection occurs after \( \hat{U}(\theta) \) but before the final measurement. Figure 2 shows such a quantum circuit. The renormalized quantum state that passes the postselection is \( |\Psi_{\psi}^{ps}\rangle \equiv |\psi_{\psi}^{ps}\rangle / \sqrt{p_{ps}} \), where we have defined an unnormalized state \( |\psi_{\psi}^{ps}\rangle \equiv \hat{F}|\Psi_\theta\rangle \) and \( p_{ps} \equiv \text{Tr}(\hat{F} \rho_0) \).

\[
\hat{F} = \sum_{j \in F_{ps}} |f_j\rangle \langle f_j| 
\]

is a set of orthonormal basis states allowed by the postselection. Finally, the postselected state undergoes an information-optimal measurement.

When \( |\Psi_{\psi}^{ps}\rangle \equiv |\psi_{\psi}^{ps}\rangle / \sqrt{p_{ps}} \) is substituted into Eq. 3, the derivatives of \( p_{ps} \) cancel, such that

\[
F_Q(\theta|\Psi_{\psi}^{ps}) = 4 \langle \hat{A}^{ps}_{\psi}|\hat{A}^{ps}_{\psi}\rangle - 4 |\langle \hat{A}^{ps}_{\psi}|\hat{A}^{ps}_{\psi}\rangle |^2 1 | F_{ps}^{2}. \quad (5)
\]

Unsurprisingly, \( F_Q(\theta|\Psi_{\psi}^{ps}) \) can exceed \( F_Q(\theta|\rho_0) \) since, \( p_{ps} \leq 1 \). Also classical systems can achieve such postselected information amplification (see Fig. 1). Unlike the classical case, however, \( F_Q(\theta|\Psi_{\psi}^{ps}) \) can also exceed the Fisher information of an optimized prepare-measure experiment, \( (\Delta a)^2 \). We show how below.

Negative quasiprobabilities.—In classical mechanics, our knowledge of a point particle can be described by a probability distribution for the particle’s position, \( \vec{x} \), and momentum, \( \vec{k} \): \( p(\vec{x}, \vec{k}) \). In quantum mechanics, position and momentum do not commute, and a state cannot generally be represented by a joint probability distribution over observables’ eigenvalues. A quantum state can, however, be represented by a quasiprobability distribution. Many classes of quasiprobability distributions

\footnote{The information-optimal input state is a pure state in an equal superposition of one eigenvector associated with the smallest eigenvalue and one associated with the largest.}
exist. The most famous is the Wigner function \[44, 50, 53\]. Such a distribution satisfies some, but not all, of Kolmogorov’s axioms for probability distributions \[37\]; the entries sum to unity, and marginalizing over the eigenvalues of every observable except one yields a probability distribution over the remaining observable’s eigenvalues. A quasiprobability distribution can however, have negative or nonreal values. Such values signal nonclassical physics, in, for example, quantum computing and quantum chaos \[2, 38–47\].

A cousin of the Wigner function is the Kirkwood-Dirac quasiprobability distribution \[15, 18, 39\]. This distribution resembles the Wigner function for continuous systems but is well-defined for discrete systems, even qubits. We cast the quantum Fisher information for a postselected prepare-measure experiment in terms of a doubly extended Kirkwood-Dirac quasiprobability distribution \[15\]. To begin, we expand \[50, 51\] an arbitrary quantum state \(\rho\) as

\[
\rho = \sum_{a, a', f} |a\rangle \langle f| A_{a, a', f}, \tag{6}
\]

where \(A_{a, a', f} \equiv \langle f|a\rangle \langle a|\rho| a'\rangle \langle a'|f\rangle\) is a doubly extended Kirkwood-Dirac distribution and \(|\{f\}\rangle\), \(|\{a\}\rangle\) and \(|\{a'\}\rangle\) are bases for the Hilbert space of \(\rho\)\[4].

Let \(|\{a\}\rangle\equiv(|\{a'\}\rangle\} denote an eigenbasis of \(A\), and let \(|\{f\}\rangle\) denote an eigenbasis of \(F\). We can express the postselected quantum Fisher information (Eq. 5) in terms of the quasiprobability values \(A_{a, a', f}\) (App. A)

\[
F_Q(\theta|\Psi^\text{ps}_\theta) = 4 \sum_{a, a', f \in F^\text{ps}} A_{a, a', f} / p_{ps} a a' - 4 \sum_{a, a', f \in F^\text{ps}} A_{a, a', f} / p_{ps} a |^{2}, \tag{7}
\]

where \(a\) and \(a'\) denote the eigenvalues associated with \(|a\rangle\) and \(|a'\rangle\), respectively\[3\]. This expression contains a conditional quasiprobability distribution, \(A_{a, a', f} / p_{ps}\). If \(A_{a, a', f} / p_{ps} \in \mathbb{R}^\ast\), the quantum Fisher information is bounded as \(F_Q(\theta|\Psi^\text{ps}_\theta) \leq (\Delta a)^2\). However, if any conditional quasiprobability value is negative, the quantum Fisher information can violate the bound: \(F_Q(\theta|\Psi^\text{ps}_\theta) > (\Delta a)^2\). Moreover, if \(A\) commutes with \(F\), as do classically, then \(A_{a, a', f} \in \mathbb{R}^\ast\). Hence an anomalous postselected Fisher information implies that \(A\) fails to commute with \(F\)\[2\].

\[2\]The modifier “doubly extended” comes from the experiment in which one would measure the distribution: One would prepare rho, measure two observables weakly, and measure one observable strongly. The number of weak measurements equals the degree of the extension \[15\].

\[3\]If any \(|f|a\rangle = 0\), we perturb one of the basis states infinitesimally.

\[4\]We have suppressed degeneracy parameters \(\gamma\) in our notation for the states, e.g., \(|a, \gamma\rangle \equiv |a\rangle\).

\[5\]For pure states, the doubly extended quasiprobability distribution can be time-symmetrically expressed in terms of the Kirkwood-Dirac distribution \[14, 49\]: \(A_{a, a', f} = \frac{1}{c^2} \sum_{g} A_{a, f} A_{g, a'}\), where \(A_{a, f} \equiv \langle f|a\rangle \langle a|\rho|f\rangle\) and \(p_f \equiv \langle f|\Psi_\theta\rangle^2\). Hence, a negative value of \(A_{a, a', f}\) implies negative or nonreal values of \(A_{a, f}\), suggesting nonclassicality. See \[16, 52\] for discussions about time-symmetric interpretations of quantum mechanics.
measurements of weak couplings (see \[54\] [62]). Our results show that postselection can improve quantum parameter estimation in a wide range of experiments where \(C_M > C_P + C_{ps}\). Earlier works have identified that nonrenormalized postselected experiments cannot increase the Fisher information \([63\] [64]. Not only the Fisher information, but also measurements’ experimental costs, underlie our result.

**Classical ontic model.**—In classical physics, states correspond to real physical things. In an ontic model of quantum physics, an ontic state (a physically real state) is represented by \(\lambda\). \(\lambda\) may be hidden, or physically inaccessible to the experimentalist. A quantum state \(\rho\) is described by a probability density function \(\mu(\lambda)\) over the set \(\Lambda\) of ontic states \([5\] [65\] [66].

Quantum mechanically, a projective measurement can be generalized to a set \(\{\hat{E}_i = \hat{M}_i^\dagger \hat{M}_i > 0\}\) of positive operators that satisfy the normalization condition \(\sum_i \hat{E}_i = 1\) \([57\] [60]. Under a generalized measurement, the quantum state evolves as \(\rho \rightarrow \hat{M}_i \rho \hat{M}_i^\dagger / \text{Tr} \hat{E}_i \rho\). In a (Spekkens) noncontextual model \([5\], if a state undergoes the generalized measurement \(\{\hat{E}_i\}\), the conditional probability of outcome \(E_i\) is represented by an indicator function \(\xi(\hat{E}_i|\lambda) \in [0, 1]\). Noncontextuality ensures that the indicator function depends on \(\lambda\) and \(E_i\) only and not on contexts, such as the specific Kraus operators \(\hat{M}_i\) or the other measurement elements \(\hat{E}_j \neq \hat{E}_i\) \([5\] [20\]. Moreover, the ontic model’s prediction for the probability of outcome \(E_i\) must agree with the Born rule in quantum theory:

\[
p(\xi\hat{E}_i|\rho) = \text{Tr} \hat{E}_i \rho = \int \xi(\hat{E}_i|\lambda) \mu(\lambda) d\lambda. \quad (8)
\]

Upon measurement, the density function undergoes a state update \([70\]. Suppose the measurement outcome corresponds to \(\hat{E}_i\). The probability density undergoes Bayesian conditioning, \(\mu(\lambda) \rightarrow \mu(\lambda|E_i)\), and then multiplication by a stochastic update matrix, \(\eta_{E_i}(\lambda|\lambda')\). Since only certain ontic states can generate outcome \(E_i\), \(\eta_{E_i}(\lambda|\lambda)\) is well-defined when \(\lambda \in \Theta_{E_i} \equiv \{\lambda \in \Lambda : \xi(\hat{E}_i|\lambda) > 0\}\). The postmeasurement density function is thus represented by

\[
\mu^{E_i}(\lambda'|E_i) = \int_{\Theta_{E_i}} \eta_{E_i}(\lambda'|\lambda) \mu(\lambda|E_i) d\lambda. \quad (9)
\]

In the experiment we analyze, the initial state undergoes a unitary transformation prior to measurement. In a noncontextual ontic model, a transformation \(T\) is represented by a stochastic transition matrix \(\Gamma_T(\lambda'|\lambda)\). The post-transformation density function is represented by

\[
\mu_T(\lambda') = \int_{\Lambda} \Gamma_T(\lambda'|\lambda) \mu(\lambda) d\lambda. \quad (10)
\]

Finally, a classical model should naturally obey outcome determinism for projective (sharp) measurements: for every projective measurement \(\{\Pi, \mathbb{1} - \Pi\}\), the underlying state \(\lambda\) determines the outcome: \(\xi(\Pi|\lambda) \in \{0, 1\}\). According to noncontextual ontic models, any inability to predict the outcome stems from a lack of knowledge about \(\lambda\).

**Proof of Theorem 1.**—We start by studying the optimized prepare-measure experiment in a noncontextual ontic model. A transformation \(T(\theta)\) encodes \(\theta\) in the density function, which evolves to \(\mu_\theta(\lambda') = \int \Gamma_T(\theta)(\lambda'|\lambda) \mu_\theta(\lambda) d\lambda\). The density function then undergoes an information-optimal measurement \(\{\hat{E}_k\}\), with corresponding probabilities \(p(\hat{E}_k|\theta) = \int \xi(\hat{E}_k|\lambda') \mu_\theta(\lambda') d\lambda'\). Substituting this expression into Eq. 1 and maximizing over all measurements, we obtain an expression for the measurement-optimized Fisher information about \(\theta\) gained in the experiment (App. B):

\[
F(\theta|\mu_\theta(\lambda')) = \frac{\sum_k \left[ \int \xi(\hat{E}_k|\lambda') \mu_\theta(\lambda') d\lambda' \right]^2}{\int \xi(\hat{E}_k|\lambda') \mu_\theta(\lambda') d\lambda'} \leq \int \sum_k \xi(\hat{E}_k|\lambda') \mu(\lambda') \left[ \partial_\theta \ln(\mu(\lambda')) \right]^2 d\lambda' = \int \frac{\partial_\theta \mu(\lambda')^2}{\mu(\lambda')} d\lambda' \quad (11)
\]

The bound comes from applying the Cauchy-Schwarz inequality. Maximizing Eq. 11 over all input density functions \(\mu_\theta(\lambda)\) gives the Fisher information of the optimized prepare-measure experiment for a given \(\Gamma_T(\theta)(\lambda'|\lambda)\).

We now show that postselection for noncontextual models cannot increase the postselected Fisher information beyond the Fisher information obtained from an information-optimized nonpostselected experiment. Before the postselection, the density function is represented by \(\mu_0(\lambda') = \int \Gamma_T(\theta)(\lambda'|\lambda) \mu_0(\lambda) d\lambda\), where \(\Gamma_T(\theta)(\lambda'|\lambda)\) describes the same transformation as above and \(\mu_0(\lambda)\) represents some (not necessarily optimal) input.

The measurement \(\{\hat{F}, \hat{1} - \hat{F}\}\) is conditioned on outcome \(F\). The density function then undergoes a Bayesian update: \(\mu_\theta(\lambda') \rightarrow \mu_\theta(\lambda'|F) = \xi(F|\lambda') \mu_\theta(\lambda') / p_{ps}\), where \(p_{ps}\) denotes the postselection probability.

The postselected measurement stochastically updates the density function to \(\mu_{ps}(\lambda'|F) \equiv \int \eta_F(\lambda'|\lambda') \mu_\theta(\lambda'|F) d\lambda'\). Similarly to Eq. 11

\[
F(\theta|\mu_{ps}(\lambda'|F)) \leq \int \frac{\partial_\theta \mu_{ps}(\lambda'|F)^2}{\mu_{ps}(\lambda'|F)} d\lambda'. \quad (12)
\]

By substituting in the expression for \(\mu_{ps}(\lambda'|F)\), we can find a new bound on the Fisher information (App. C):

\[
F(\theta|\mu_{ps}(\lambda'|F)) \leq \int \frac{\partial_\theta \eta_F(\lambda'|\lambda') \mu_{ps}(\lambda'|F) d\lambda'}{\eta_F(\lambda'|\lambda') \mu_{ps}(\lambda'|F) d\lambda'} d\lambda' \leq \int \frac{\partial_\theta \mu_{ps}(\lambda'|F)^2}{\mu_{ps}(\lambda'|F)} d\lambda' \quad (13)
\]

We have applied the Cauchy-Schwarz inequality, used Tonelli’s theorem, and maximized over all measurements and \(\eta_F(\lambda'|\lambda')\).
In App. [D] we substitute in the expression for $\mu_0'(\lambda'|F)$ and prove an upper bound on Eq. [13]

$$\int d\lambda' \frac{[\partial_0 \mu_0'(\lambda'|F)]^2}{\mu_0(\lambda'|F)} \leq \frac{1}{p_{pe}} \int d\lambda' \frac{[\xi(F|\lambda') \partial_0 \mu_0'(\lambda')]^2}{\xi(F|\lambda') \mu_0'(\lambda')}.$$  

(14)

Because $\{\hat{F}, 1 - \hat{F}\}$ corresponds to a projective measurement, we can invoke outcome determinism to write $\mu_0'(\lambda') = p_{pe} \mu_0^F(\lambda') + (1 - p_{pe}) \mu_0^{1-F}(\lambda')$, where $\mu_0^F(\lambda') \equiv \xi(F|\lambda') \mu_0'(\lambda')$ and $\mu_0^{1-F}(\lambda') \equiv \xi(1 - F|\lambda') \mu_0'(\lambda')$ are ontologically distinct. We can rewrite the right-hand side of Eq. [14] as

$$\int d\lambda' \frac{[\partial_0 \mu_0^F(\lambda')]^2}{\mu_0^F(\lambda')}.$$  

(15)

Recall that $\mu_0^F(\lambda') \equiv \int \Gamma_{T(\theta)}(\lambda'|\lambda) \mu_0^F(\lambda) d\lambda$. The stochastic transformation $\Gamma_{T(\theta)}(\lambda'|\lambda)$ cannot increase the variational distance between probability distributions [66]. Thus, $\mu_0^F(\lambda')$ and $\mu_0^{1-F}(\lambda')$ must have been mapped to by the transformation from densities $\mu_0^F$ and $\mu_0^{1-F}$ that were ontologically distinct with respect to some projective measurement $\{F_0, 1 - F_0\}$. Hence, inputting $\mu_0^F(\lambda')$ into Eq. [11] (from the postselection-free scenario) yields the same upper bound for the Fisher information as in the postselected scenario (Eq. [14]).

Consequently, the Fisher information from a postselected prepare-measure experiment can never exceed the Fisher information from the corresponding optimized prepare-measure experiment in any noncontextual ontic model. In contrast, as shown in Eq. [7], quantum measurements’ contextuality—as highlighted by quasiprobability negativity—can increase the postselected Fisher information more than the optimization of input states and final measurements can.

**Discussion.**—From a practical perspective, our results highlight an important quantum asset for parameter-estimation experiments with expensive final measurements. In some scenarios, the postselection’s costs exceed the final measurement’s costs, as an unsuccessful postselection might require fast feed-forward to block the final measurement. But in single-particle experiments, the postselection can be virtually free and, indeed, unavoidable: an unsuccessful postselection can destroy the particle, precluding the final measurement [71]. Thus, current single-particle metrology experiments could benefit from postselected improvements of the Fisher information.

From a fundamental perspective, our results highlight the strangeness of quantum mechanics as a contextual theory. Classically, an increase of the Fisher information via postselection can be understood as the a posteriori selection of a better input distribution. But it is nonintuitive that quantum mechanical postselection can enable a quantum state to carry more Fisher information than the best possible input state could. Postselection can distort an anomalously large Fisher information into a few selected states. A quantum experiment that shows such anomalous information values proves that quantum mechanics is contextual.

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Appendix A: Derivation of Eq. 7

The nonrenormalized postselected quantum state is $|\psi^{ps}_0\rangle = \hat{F}\hat{U}(\theta)|\Psi_0\rangle$, where $|\Psi_0\rangle \equiv \rho_0$. The first term of the quantum Fisher information (Eq. 5) is

$$\frac{4}{p_{ps}} \langle \psi^{ps}_0 | \psi^{ps}_0 \rangle = \frac{4}{p_{ps}} \text{Tr}((\hat{F}\hat{U}(\theta)\rho_0\hat{U}^\dagger(\theta)\hat{F}^\dagger)) = \frac{4}{p_{ps}} \text{Tr}\left(\hat{F}\hat{A}\rho_0\hat{A}^\dagger\right)$$  \hspace{1cm} (A1)

$$= \frac{4}{p_{ps}} \text{Tr}\left(\sum_a |a\rangle \langle a| \hat{A}\rho_0\hat{A} \sum_{a'} |a'\rangle \langle a'| \sum_f |f\rangle \langle f| \hat{F} \right)$$  \hspace{1cm} (A2)

$$= \frac{4}{p_{ps}} \text{Tr}\left(\sum_a |a\rangle \langle a| \rho_0 \sum_{a'} |a'\rangle \langle a'| \sum_{f \in F_{ps}} |f\rangle \langle f| \hat{F} \right).$$  \hspace{1cm} (A3)

where we have inserted three resolutions of unity. This expression can be rewritten in terms of the doubly extended Kirkwood-Dirac quasiprobability distribution (Eq. 6):

$$\frac{4}{p_{ps}} \sum_{a,a',f \in F_{ps}} \text{Tr}\left(aa' A_{a,a',f} \frac{|a\rangle \langle a|}{|f\rangle \langle f|} \right) = \frac{4}{p_{ps}} \sum_{a,a',f \in F_{ps}} A_{a,a',f} aa'.$$  \hspace{1cm} (A4)

Similarly, the second term of Eq. 5 is

$$\left| \frac{4}{p_{ps}} \langle \psi^{ps}_0 | \psi^{ps}_0 \rangle \right|^2 = \frac{4}{p_{ps}} \left| \text{Tr}(\hat{A}\rho_0\hat{F}) \right|^2 = \frac{4}{p_{ps}} \left| \sum_{a,a',f \in F_{ps}} A_{a,a',f} a \right|^2. \hspace{1cm} (A5)$$

Combining the expressions above gives Eq. 7

$$F_Q(\theta | \Psi^{ps}_0) = 4 \sum_{a,a',f \in F_{ps}} \frac{A_{a,a',f} aa'}{p_{ps}} - 4 \sum_{a,a',f \in F_{ps}} \frac{A_{a,a',f}}{p_{ps}} a^2.$$  \hspace{1cm} (A6)

Appendix B: Derivation of Eq. 11

Here, we provide a more elaborate derivation of Eq. 11

$$F(\theta | \mu_0(\lambda')) = \sum_k \frac{\left[ \partial_\theta \int \xi(E_k|\lambda')\mu_0(\lambda')d\lambda' \right]^2}{\int \xi(E_k|\lambda')\mu_0(\lambda')d\lambda'}$$  \hspace{1cm} (B1)

$$= \sum_k \left[ \int \xi(E_k|\lambda')\mu_0(\lambda') \partial_\lambda \ln(\mu_0(\lambda'))d\lambda' \right]^2$$  \hspace{1cm} (B2)

$$= \sum_k \left[ \int \xi(E_k|\lambda')\mu_0(\lambda')^{\frac{1}{2}} \xi(E_k|\lambda')\mu_0(\lambda')^{\frac{1}{2}} \partial_\lambda \ln(\mu_0(\lambda'))d\lambda' \right]^2$$  \hspace{1cm} (B3)

$$= \sum_k \left[ \int \xi(E_k|\lambda')\mu_0(\lambda') \partial_\lambda \ln(\mu_0(\lambda'))d\lambda' \right] \times \left[ \int \xi(E_k|\lambda')\mu_0(\lambda') \partial_\lambda \ln(\mu_0(\lambda'))d\lambda' \right]$$  \hspace{1cm} (B4)

$$= \sum_k \int \xi(E_k|\lambda')\mu_0(\lambda') \partial_\lambda \ln(\mu_0(\lambda'))d\lambda'$$  \hspace{1cm} (B5)

$$= \sum_k \int \xi(E_k|\lambda')\mu_0(\lambda') \partial_\lambda \ln(\mu_0(\lambda'))d\lambda'$$  \hspace{1cm} (B6)

$$\leq \sum_k \int \xi(E_k|\lambda')\mu_0(\lambda')d\lambda' \times \int \xi(E_k|\lambda')\mu_0(\lambda')d\lambda'$$  \hspace{1cm} (B7)

$$= \int \mu_0(\lambda')d\lambda'$$  \hspace{1cm} (B8)

where the inequality comes from applying the Cauchy-Schwarz inequality.
Appendix C: Derivation of Eq. 13

Here, we provide a more elaborate derivation of Eq. 13

\[
F(\theta|\mu_0^p(\lambda'|F)) \leq \int \frac{[\partial_0\mu_0^p(\lambda'|F)]^2}{\mu_0^p(\lambda'|F)} d\lambda' \tag{C1}
\]

\[
= \int \frac{[\partial_0 \int \eta_F(\lambda'|\lambda)\mu_0^p(\lambda|F)d\lambda']^2}{\eta_F(\lambda'|\lambda)|\mu_0^p(\lambda|F)|d\lambda'} d\lambda' \tag{C2}
\]

\[
= \int \frac{[\int \eta_F(\lambda''|\lambda')\mu_0^p(\lambda'|F)\partial_0 \ln(\mu_0^p(\lambda'|F))]^2}{\eta_F(\lambda''|\lambda')|\mu_0^p(\lambda'|F)|d\lambda'} d\lambda' \tag{C3}
\]

\[
\leq \int \frac{\eta_F(\lambda''|\lambda')\mu_0^p(\lambda'|F)d\lambda'}{\eta_F(\lambda''|\lambda')|\mu_0^p(\lambda'|F)|d\lambda'} \times \int \eta_F(\lambda''|\lambda')\mu_0^p(\lambda'|F)|\partial_0 \ln(\mu_0^p(\lambda'|F))|^2 d\lambda' d\lambda'' \tag{C4}
\]

\[
= \int \int \eta_F(\lambda''|\lambda')\mu_0^p(\lambda'|F)|\partial_0 \ln(\mu_0^p(\lambda'|F))|^2 d\lambda' d\lambda'' \tag{C5}
\]

\[
= \int \int \eta_F(\lambda''|\lambda')\mu_0^p(\lambda'|F)\partial_0 (\mu_0^p(\lambda'|F))|^2 d\lambda' d\lambda'' \tag{C6}
\]

\[
= \mu_0^p(\lambda'|F)|\partial_0 \ln(\mu_0^p(\lambda'|F))|^2 d\lambda' \tag{C7}
\]

\[
= \int \frac{[\partial_0\mu_0^p(\lambda'|F)]^2}{\mu_0^p(\lambda'|F)} d\lambda'. \tag{C8}
\]

We have applied the Cauchy-Schwarz inequality, used Tonelli’s theorem, and maximized over all measurements and \(\eta_F(\lambda''|\lambda')\).

Appendix D: Proof of Eq. 14

Here, we prove the inequality in Eq. 14

\[
\int \frac{[\partial_0\mu_0^p(\lambda'|F)]^2}{\mu_0^p(\lambda'|F)} d\lambda' = \int \left[ \frac{\partial_0 \xi(\lambda'|\mu_0^p(\lambda'))}{\mu_0^p(\lambda'|F)} \right]^2 \frac{p_{ps}}{\xi(\lambda'|\mu_0^p(\lambda'))} d\lambda' \tag{D1}
\]

\[
= \int \left[ \frac{\xi(\lambda'|\lambda')\mu_0^p(\lambda')}{p_{ps}^2} - \frac{\xi(\lambda'|\lambda')\mu_0^p(\lambda')p_{ps}}{p_{ps}^2} \right]^2 \frac{p_{ps}}{\xi(\lambda'|\mu_0^p(\lambda'))} d\lambda' \tag{D2}
\]

\[
= \int \left[ \frac{[\xi(\lambda'|\lambda')\mu_0^p(\lambda')]^2}{p_{ps}^2} - 2 \frac{\xi(\lambda'|\lambda')\mu_0^p(\lambda')p_{ps}}{p_{ps}^2} + \frac{\xi(\lambda'|\lambda')\mu_0^p(\lambda')p_{ps}}{p_{ps}^3} \right]^2 \frac{p_{ps}}{\xi(\lambda'|\mu_0^p(\lambda'))} d\lambda' \tag{D3}
\]

\[
= \frac{1}{p_{ps}} \int \frac{[\xi(\lambda'|\lambda')\mu_0^p(\lambda')]^2}{\xi(\lambda'|\lambda')\mu_0^p(\lambda')} d\lambda' - \frac{p_{ps}^2}{p_{ps}^2} \tag{D4}
\]

\[
\leq \frac{1}{p_{ps}} \int \frac{[\xi(\lambda'|\lambda')\mu_0^p(\lambda')]^2}{\xi(\lambda'|\lambda')\mu_0^p(\lambda')} d\lambda', \tag{D5}
\]

where we used \(\int \xi(\lambda'|\lambda')\mu_0^p(\lambda')d\lambda' = p_{ps}\). □