Affine cellularity of affine $q$-Schur algebras

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Abstract

We show that the affine $q$-Schur algebra $U_{r,n,n}$, which has been defined by Lusztig, is affine cellular in the sense of Koenig and Xi. By specializing at $v = 1$, we obtain the affine cellularity of the affine Schur algebra. We also show that $U_{r,n,n}$ is of finite global dimension, its derived module category admits a stratification and is an affine quasi-hereditary algebra when the quantum parameter $v \in \mathbb{C}^*$ is not a root of unity.

Keywords: Affine cellular algebras; Extended affine Hecke algebras; Affine $q$-Schur algebras; Finite global dimension

1 Introduction

In order to approach the fundamental problem of classifying the irreducible representations of a given finite-dimensional algebra, the concept of cellularity, defined by Graham and Lehrer in [GL], has proven extremely useful. Examples of cellular algebras include many finite-dimensional Hecke algebras.

Recently, Koenig and Xi in [KX] has generalized this concept to algebras over a noetherian domain $k$ of not necessarily finite dimension, by introducing the notion of an affine cellular algebra. The most important class of examples of affine cellular algebras, which has been discussed in [KX], is given by the extended affine Hecke algebras of type $A$. Recently, Guilhot and Miemietz have proved that affine Hecke algebras of rank two with generic parameters are affine cellular in [GM]; Kleshchev and Loubert have proved that KLR algebras of finite type are affine cellular in [KL].

The cells attached to the Kazhdan-Lusztig basis of Hecke algebras associated to finite and affine Weyl groups are very useful in understanding the representation of these Coxeter groups and Hecke algebras. The theory of cells attached to the canonical basis for the modified quantum groups $\tilde{U}$, which has been developed by Lusztig, is also very interesting. In the case of quantum groups of finite type, Lusztig has completely described the cells in [L3]. In the same paper, Lusztig has also given a series of conjectures on the cell structure in the case of the level-zero affine quantum groups $\tilde{U}$ corresponding to the degenerate affine root datum, which has been proved by Beck and Nakajima in [BN] (see also [Mc] for type $A_n^{(1)}$).
In [L4], Lusztig defined the affine $q$-Schur algebra $U_{r,n,n}$ and its canonical basis $B_{r}$. In [VV], they have shown that $U_{r,n,n}$ is isomorphic to the affine $q$-Schur algebra $\hat{S}_{q}(n, r)$ in [G], which is defined as the commutator algebra of the right $\mathcal{H}_{r}$-module $\mathcal{T}_{r}$, where $\mathcal{H}_{r}$ is the extended affine Hecke algebra of type $A$ and $\mathcal{T}_{r}$ is an infinite-dimensional vector space.

In [Mc], to obtain the cell structure of $\hat{U}$, McGerty first investigated the structure of cells in $U_{r,n,n}$.

In [Cu1], we have shown that the affine $q$-Schur algebra $U_{r,n,n}$ is an affine cellular algebra when $n > r$. In this note, we will show that the affine $q$-Schur algebra $U_{r,n,n}$ is an affine cellular algebra in the sense of Koenig and Xi. Our proof relies heavily on McGerty’s explicit descriptions of two-sided cells of $U_{r,n,n}$ and the asymptotic algebra of $U_{r,n,n}$. When the quantum parameter $v \in \mathbb{C}^{*}$ is not a root of unity, we also show that all the affine cell ideals are idempotent and have nonzero idempotent elements. Applying [KX, Theorem 4.4], we then obtain, in this case, that $U_{r,n,n}$ is of finite global dimension and its derived module category admits a stratification whose sections are the derived categories of representation rings of products of general linear groups.

The organization of this article is as follows. In Section 2, we introduce affine cellular algebras. In Section 3, we will recall affine $q$-Schur algebras and McGerty’s explicit descriptions of two-sided cells and the asymptotic algebra. In Section 4, we prove our main result Theorem 4.1 and 4.3.

2 Affine cellular algebras

Let $k$ be a noetherian domain. For a $k$-algebra $A$, a $k$-linear anti-automorphism $i$ of $A$ satisfying $i^{2} = id_{A}$ will be called a $k$-involution on $A$. For two $k$-modules $V$ and $W$, we denote by $\tau$ the map $V \otimes W \to W \otimes V$ given by $\tau(v \otimes w) = w \otimes v$. If $B = k[x_{1}, \ldots, x_{t}]/I$ for some ideal $I$ in a polynomial ring in finitely many variables $x_{1}, \ldots, x_{t}$ over $k$, then $B$ is called an affine $k$-algebra.

Definition 2.1. (see [KX, Definition 2.1]) Let $A$ be a unitary $k$-algebra with a $k$-involution $i$. A two-sided ideal $J$ in $A$ is called an affine cell ideal if and only if the following data are given and the following conditions are satisfied:

1. We have $i(J) = J$.
2. There exist a free $k$-module of finite rank and an affine $k$-algebra $B$ with a $k$-involution $\sigma$ such that $\Delta := V \otimes_{k} B$ is an $A$-$B$-bimodule, where the right $B$-module structure is induced by the right regular $B$-module $B$.
3. There is an $A$-$A$-bimodule isomorphism $\alpha : J \to \Delta \otimes_{B} \Delta'$, where
\[ \Delta' = B \otimes_k V \text{ is a } B\text{-}A\text{-bimodule with the left } B\text{-module induced by the left regular } B\text{-module }_BB \text{ and with the right } A\text{-module structure defined by } (b \otimes v)a := \tau(i(a)(v \otimes b)) \text{ for } a \in A, b \in B \text{ and } v \in V, \text{ such that the following diagram is commutative:} \]

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\Delta_B} & \Delta' \\
\downarrow & & \downarrow \\
\Delta_B & \xrightarrow{\Delta_B} & \Delta_B'
\end{array}
\]

The algebra \( A \) together with its \( k \)-involution \( i \) is called affine cellular if and only if there is a \( k \)-module decomposition \( A = J'_1 \oplus J'_2 \oplus \cdots J'_n \) (for some \( n \)) with \( i(J'_l) = J'_l \) for \( 1 \leq l \leq n \), such that, setting \( J_m := \bigoplus_{l=1}^m J'_l \), we have a chain of two-sided ideals of \( A \): \( 0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = A \), where each \( J'_m \) is an affine cell ideal if and only if \( \sigma \) restricted to \( J_m/J_{m-1} \) is an affine cell ideal of \( A/J_{m-1} \) (with respect to the involution induced by \( i \) on the quotient).

For an affine \( k \)-algebra \( B \) with a \( k \)-involution \( \sigma \), a free \( k \)-module \( V \) of finite rank and a \( k \)-bilinear form \( \varphi : V \otimes_k V \rightarrow B \), denote by \( \mathbb{A}(V, B, \varphi) \) the (possibly non-unital) algebra given as a \( k \)-module by \( V \otimes_k B \otimes_k V \), on which we impose the multiplication \((v_1 \otimes b_1 \otimes w_1)(v_2 \otimes b_2 \otimes w_2) := v_1 \otimes b_1 \varphi(w_1, v_2)b_2 \otimes w_2\).

**Proposition 2.1.** (see [KX, Proposition 2.2]) Let \( k \) be a noetherian domain, \( A \) a unitary \( k \)-algebra with a \( k \)-involution \( i \). A two-sided ideal \( J \) in \( A \) an affine cell ideal if and only if \( i(J) = J \), there exists an affine \( k \)-algebra \( B \) with a \( k \)-involution \( \sigma \), a free \( k \)-module \( V \) of finite rank and a \( k \)-bilinear form \( \varphi : V \otimes_k V \rightarrow B \), and an \( A \)-\( A \)-bimodule structure on \( V \otimes_k B \otimes_k V \), such that \( J \cong \mathbb{A}(V, B, \varphi) \) as an algebra and an \( A \)-\( A \)-bimodule, and such that under this isomorphism the \( k \)-involution \( i \) restricted to \( J \) corresponds to the \( k \)-involution given by \( v \otimes b \otimes w \mapsto w \otimes \sigma(b) \otimes v \).

Suppose that \( B \) is a \( k \)-algebra and fix a free \( k \)-module \( V \) of finite rank and choose a \( k \)-bilinear form \( \varphi : V \otimes_k V \rightarrow B \). Then the generalized matrix algebra \( (M_n(B), \varphi) \) over \( B \) with respect to \( \varphi \) as a \( k \)-space equals the ordinary matrix algebra \( M_n(B) \) of \( n \times n \) matrices over \( B \), but the multiplication is deformed in the following way \( ab = a \Psi b \) for all \( a, b \in M_n(B) \), where \( \Psi \) is the matrix describing the bilinear form \( \varphi \) with respect to some basis of \( V \).

From this proposition, we can easily get the following proposition about the description of affine cell ideals, which we are going to use.

**Proposition 2.2.** Let \( k \) be a noetherian domain, \( A \) a unitary \( k \)-algebra with a \( k \)-involution \( i \). A two-sided ideal \( J \) in \( A \) an affine cell ideal if and only if \( i(J) = J \), \( J \) is isomorphic to some generalized matrix algebra \( (M_n(B), \varphi) \) for some affine \( k \)-algebra \( B \) with a \( k \)-involution \( \sigma \), a free \( k \)-module \( V \) of
finite rank and a k-bilinear form \( \varphi : V \otimes_k V \to B \). Under the isomorphism, if a basis element \( b \) of \( J \) over \( B \) corresponds to \( E_{jl}(b') \) for some \( b' \in B \), where \( E_{jl}(b') \) denotes a square matrix whose \((j,l)\)-entry is \( b' \) and all the other entries are zero, then \( i(b) \) corresponds to \( E_{jl}(\sigma(b')) \).

**Theorem 2.1.** (see [KX, Theorem 4.4]) Let \( A \) be an affine cellular algebra with a cell chain \( J_0 = 0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A \) such that \( J_j/J_{j-1} = V_j \otimes_k B_j \otimes_k V_j \) as in Definition 2.1. Suppose that each \( B_j \) satisfies \( \text{rad}(B_j) = 0 \). Suppose moreover that each \( J_j/J_{j-1} \) is idempotent and contains a non-zero idempotent element in \( A/J_{j-1} \). Then:

(a) The unbounded derived category \( D(A\text{-Mod}) \) of \( A \) admits a stratification, that is an iterated recollement whose strata are the derived categories of the various affine \( k \)-algebras \( B_j \).

(b) The global dimension \( \text{gldim}(A) \) is finite if and only if \( \text{gldim}(B_j) \) is finite for all \( j \).

Let now \( A \) be an affine cellular algebra with a cell chain \( 0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A \), such that each subquotient \( J_i/J_{i-1} \) is an affine cell ideal of \( A/J_{i-1} \). Then \( J_i/J_{i-1} \) is isomorphic to \( \mathfrak{A}(V_i, B_i, \varphi_i) \) for some free \( k \)-module \( V_i \) of finite rank, an affine \( k \)-algebra \( B_i \) and a \( k \)-bilinear form \( \varphi_i : V_i \otimes_k V_i \to B_i \). Let \( (\varphi_{st}^i) \) be the matrix representing the bilinear form \( \varphi_i \) with respect to some choices of basis of \( V_i \). Then Koenig and Xi obtain a parameterisation of simple modules over an affine cellular algebra by establishing a bijection between isomorphism classes of simple \( A \)-modules and the set

\[
\{(j,m)|1 \leq j \leq n, m \in \text{MaxSpec}(B_i) \text{ such that some } \varphi_{st}^i \notin m\},
\]

where \( \text{MaxSpec}(B_i) \) denotes the maximal ideal spectrum of \( B_j \).

### 3 Affine \( q \)-Schur algebras and its cell structures

#### 3.1 Affine \( q \)-Schur algebras

We first give a description of the affine \( q \)-Schur algebra \( \Phi_{r,n,n} \) following [L4] (see also [GV]). Thus, let \( V_r \) be a free rank \( r \) module over \( k[\epsilon, \epsilon^{-1}] \), where \( k \) is a finite field of \( q \) elements, and \( \epsilon \) is an indeterminate.

Let \( \mathcal{F}^n \) be the space of \( n \)-step periodic lattices, that is, sequences \( L = (L_i)_{i \in \mathbb{Z}} \) of lattices in our free module \( V_r \) such that \( L_i \subset L_{i+1} \) and \( L_{i-n} = \epsilon L_i \). The group \( G = \text{Aut}(V_r) \) acts on \( \mathcal{F}^n \) in the natural way. Let \( \mathcal{G}_{r,n} \) be the set of nonnegative integer sequences \( (a_i)_{i \in \mathbb{Z}} \) such that \( a_i = a_{i+n} \) and \( \sum_{i=1}^{n} a_i = r \), and let \( \mathcal{G}_{r,n,n} \) be the set of \( \mathbb{Z} \times \mathbb{Z} \) matrices \( A = (a_{i,j})_{i,j \in \mathbb{Z}} \) with nonnegative entries such that \( a_{i,j} = a_{i+n,j+n} \) and \( \sum_{i \in [1,n], j \in \mathbb{Z}} a_{i,j} = r \). The orbits of \( G \) on \( \mathcal{F}^n \) are indexed by \( \mathcal{G}_{r,n} \), where \( L \) is in the orbit \( \mathcal{F}_n \) corresponding to
a if \( a_i = \dim_k (L_i/L_{i-1}) \). The orbits of \( G \) on \( \mathcal{F}^n \times \mathcal{F}^n \) are indexed by the matrices \( \mathfrak{S}_{r,n,n} \), where a pair \((\mathbf{L}, \mathbf{L}')\) is in the orbit \( O_A \) corresponding to \( A \) if

\[
a_{i,j} = \dim \left( \frac{L_i \cap L_j'}{(L_{i-1} \cap L_j') + (L_i \cap L_{j-1})} \right).
\]

For \( A \in \mathfrak{S}_{r,n,n} \), let \( r(A), c(A) \in \mathfrak{S}_{r,n} \) be given by \( r(A)_i = \sum_{j \in \mathbb{Z}} a_{i,j} \) and \( c(A)_j = \sum_{i \in \mathbb{Z}} a_{i,j} \).

Similarly, let \( \mathcal{B}^r \) be the space of complete periodic lattices, that is, sequences of \( \mathbf{L} = (L_i)_{i \in \mathbb{Z}} \) such that \( L_i \subseteq L_{i+1} \) and \( L_{i-r} = cL_i \), and also \( \dim_k (L_i/L_{i-1}) = 1 \) for all \( i \in \mathbb{Z} \). Let \( \mathbf{b}_0 = (\ldots, 1, 1, \ldots) \). The orbits of \( G \) on \( \mathcal{B}^r \times \mathcal{B}^r \) are indexed by matrices \( \mathfrak{S}_{r,r,r} \), where the matrix \( A \) must have \( r(A) = c(A) = \mathbf{b}_0 \).

Let \( \mathfrak{U}_{r,q}, \mathfrak{H}_{r,q} \), and \( \mathfrak{T}_{r,q} \) be the span of the characteristic functions of the \( G \)-orbits on \( \mathcal{F}^n \times \mathcal{F}^n \), \( \mathcal{B}^r \times \mathcal{B}^r \), and \( \mathcal{F}^n \times \mathcal{B}^r \), and \( \mathcal{F}^n \times \mathcal{B}^r \) respectively. Convolution makes \( \mathfrak{U}_{r,q} \) and \( \mathfrak{H}_{r,q} \) algebras and \( \mathfrak{T}_{r,q} \) a \( \mathfrak{U}_{r,q} \)-\( \mathfrak{T}_{r,q} \)-bimodule. For \( A \in \mathfrak{S}_{r,n,n} \), set

\[
d_A = \sum_{i \geq k, j \leq 1; 1 \leq i \leq n} a_{i,j} a_{k,j}.
\]

Let \( \{ [A] \mid A \in \mathfrak{S}_{r,n,n} \} \) be the basis of \( \mathfrak{U}_{r,q} \) given by \( q^{-d_A/2} \) times the characteristic function of the orbit corresponding to \( A \).

All of these spaces of functions are the specialization at \( v = \sqrt{q} \) of modules over \( A = \mathbb{Z}[v, v^{-1}] \), which we denote by \( \mathfrak{U}_r = \mathfrak{U}_{r,n,n}, \mathfrak{H}_r, \) and \( \mathfrak{T}_r \) respectively (here \( v \) is an indeterminate). The \( A \)-algebra \( \mathfrak{U}_r \) is just the affine \( q \)-Schur algebra. Recall that \([L4, \text{Section 4}]\) that \( \mathfrak{U}_r \) possesses a canonical basis \( \mathfrak{B}_r \) consisting of elements \( \{ A \}; A \in \mathfrak{S}_{r,n,n} \). We have

\[
\{ A \} = \sum_{A_1; A_1 \leq A} \Pi_{A_1,A}[A_1],
\]

where \( \leq \) is a natural partial order on \( \mathfrak{S}_{r,n,n} \) and the \( \Pi_{A_1,A} \) are certain polynomials in \( \mathbb{Z}[v^{-1}] \). The algebra \( \mathfrak{U}_r \) has a natural \( A \)-linear anti-automorphism \( \Psi \) which sends \( [A] \) to \( [A'] \) (see \([L4, \text{Lemma 1.11}]\)), then \( \Psi \) sends \( \{ A \} \) to \( \{ A' \} \), where \( A' \) denotes the transpose of \( A \).

Let \( \mathcal{H}_r \) be the extended affine Hecke algebra of type \( A \), then \( \mathcal{H}_r \) is an algebra over \( \mathbb{Z}[v, v^{-1}] \) generated by symbols \( T_i, X_j^{\pm 1} \), where \( 1 \leq i \leq r - 1 \) and \( 1 \leq j \leq r \), subject to the following relations:

1. \((T_i - v)(T_i + v^{-1}) = 0, T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \text{for } i = 1, 2, \ldots, r - 1;\)
2. \( T_iT_j = T_jT_i, \text{if } |i - j| \geq 2;\)
3. \( X_iX_i^{-1} = X_i^{-1}X_i = 1, \text{for all } i, j;\)
4. \( T_i^{-1}X_iT_i = X_{i+1}, \text{for } i = 1, 2, \ldots, r - 1; T_iX_j = X_jT_i, \text{for } j \neq i, i + 1.\)

This is the Bernstein’s presentation. Let \( W \) be the extended affine Weyl group of type \( A \), that is, \( W \) is the semidirect product of the symmetric group
$S_r$ with $Z^r$. The Iwahori presentation of $H_r$ yields a basis $\{T_w|w \in W\}$, and let $\{C_w|w \in W\}$ be the Kazhdan-Lusztig basis of $H_r$.

For any $c \in \mathfrak{S}_{r,n}$, let $S_c$ denote the parabolic subgroup of $S_r$, that is, the subgroup preserving the subsets $\{1, 2, \ldots, c_1\}, \{c_1 + 1, \ldots, c_1 + c_2\}, \ldots, \{r - c_n + 1, \ldots, r\}$ of $\{1, 2, \ldots, r\}$. As mentioned in [Mc] (see also [Cu2, 3.2]), we see that we can describe an element $A$ of $\mathfrak{S}_{r,n,n}$ uniquely by a tripe consisting of an element $w_A \in W$ together with a pair $a, b \in \mathfrak{S}_{r,n}$. Indeed, $a, b$ are just $r(A)$ and $c(A)$ respectively, and $w_A$ is the element of maximal length in the (finite) double coset of $S_a \setminus W / S_b$ determined by the matrix $A$. Hereafter, we will sometimes identify $A$ with the tripe $(r(A), w_A, c(A))$ and then $A'$ will be identified with $(c(A), w_A^{-1}, r(A))$. This also allows us to describe the structure constants for $\mathfrak{U}_r$ with respect to the basis $\{[A]|A \in \mathfrak{S}_{r,n,n}\}$ of those for $H_r$, with respect to the basis $\{T_w|w \in W\}$ (resp. $\{C_w|w \in W\}$).

More precisely, suppose that we denote the various structure constants for $H_r$ and $\mathfrak{U}_r$ as follows. Let $A, B, C \in \mathfrak{S}_{r,n,n}$, and let $v, w \in W$, write

1. $[A][B] = \sum_C \eta_{A,B}^C[C]$;
2. $\{A\}{B} = \sum_C \nu_{A,B}^C{C}$;
3. $T_vT_w = \sum_x f_{v,w}^x T_x$;
4. $C_vC_w = \sum_x h_{v,w}^x C_x$,

where all the structure constants are in $A = \mathbb{Z}[v, v^{-1}]$.

Then, we have the following relationships between them, which will be crucial in describing the cell structure of the affine $q$-Schur algebra.

**Lemma 3.1.** (see [Mc, Lemma 2.2])

Let $A, B, C \in \mathfrak{S}_{r,n,n}$ and let $w_A, w_B, w_C$ be the corresponding element in $W$. Suppose that $c(A) = r(A) = c$. Let $w_c$ be the longest element of $S_c$ and let

$$p_c = v^{-l(w_c)} \sum_{x \in S_c} v^{2l(x)}$$

be the shifted Poincaré polynomial of $S_c$. Then we have

$$p_c \eta_{A,B}^C = f_{w_A,w_B}^{w_C} \quad \quad p_c \nu_{A,B}^C = h_{w_A,w_B}^{w_C}.$$

### 3.2 Cell structures in $\mathfrak{U}_r$

We will recall (see [Mc]) the definition of cells in $\mathfrak{U}_r$ with respect to the canonical basis $\mathfrak{B}_r = \{\{A\}|A \in \mathfrak{S}_{r,n,n}\}$. Let

$$\{A\}{A'} = \sum_{A''} \nu_{A,A'}^{A''}(v)\{A''\}$$

define the structure constants $\nu_{A,A'}^{A''}(v) \in A$ of $\mathfrak{U}_r$ with respect to the canonical basis $\mathfrak{B}_r$ as before. For $A, A' \in \mathfrak{S}_{r,n,n}$, we say that $A \preceq_L A'$ (resp.
A \preceq_R A' if there is a sequence $A_1 = A', A_2, \ldots, A_N = A$ in $\mathfrak{S}_{r,n,n}$ and a sequence $C_1, C_2, \ldots, C_{N-1} \in \mathfrak{S}_{r,n,n}$ such that $\nu_{A_{s+1},A_s}^{A_{s+1}} \neq 0$ (resp. $\nu_{A_{s+1},C_s}^{A_{s+1}} \neq 0$) for $s = 1, 2, \ldots, N - 1$. We write $A \preceq_{LR} A'$ if either of the above structure constants is nonzero for all $A_s, C_s, A_{s+1}, s = 1, 2, \ldots, N - 1$. We say that $A \sim_L A'$ (resp. $A \sim_R A'$) if and only if $A \preceq_{LR} A'$ and $A' \preceq_{LR} A$ (resp. $A \preceq_R A'$ and $A' \preceq_R A$). The equivalence classes of $\sim_L$ or $\sim_R$ are then called left cells or right cells. We say that $A \sim_{LR} A'$ if and only if $A \preceq_{LR} A'$ and $A' \preceq_{LR} A$. The equivalence classes of $\sim_{LR}$ are then called two-sided cells.

In [Mc, Definition 3.1], McGerty has defined an integer-valued function $a'_r$ on $\mathfrak{S}_{r,n,n}$, which is an analogue of Lusztig’s $a$-function for affine Hecke algebras. Using this function, he has also defined the set $\mathcal{D}_r$ of distinguished elements of $\mathfrak{S}_{r,n,n}$ (see [Mc, Definition 3.5]).

The following proposition shows that all the notions of cells for $\mathfrak{U}_r$ and the $a$-function $a'_r$ on $\mathfrak{S}_{r,n,n}$ can be deduced from those for affine Hecke algebras.

**Proposition 3.1.** (see [Mc, Proposition 3.8]) Let $A, B \in \mathfrak{S}_{r,n,n}$, then the following claims hold:

1. $A \sim_L B$ if and only if $w_A \sim_L w_B$ and $c(A) = c(B)$;
2. $A \sim_R B$ if and only if $w_A \sim_R w_B$ and $r(A) = r(B)$;
3. $A \sim_{LR} B$ if and only if $w_A \sim_{LR} w_B$;
4. $a'_r(A) = a(w_A) - l(w_{c(A)})$, where $w_{c(A)}$ is the longest element of the parabolic subgroup $S_{c(A)}$ and $l$ is the length function.
5. Each left (right) cell contains precisely one distinguished element;
6. If $A \preceq_{LR} B$, then we have $w_A \preceq_{LR} w_B$.

**Proof.** (6) follows from the notion of cells in $\mathfrak{U}_r$ is defined essentially by using a subset of the Kazhdan-Lusztig basis consisting of those elements which are of maximal length in certain double cosets.

Let $\mathcal{P}_r$ be the set of partitions of $r$ and let $\mathcal{P}^n_r$ be the set of partitions of $r$ with at most $n$ parts. It has been shown by Lusztig, based on the work of Shi in [S1], that there is a bijection between the set of two-sided cells of $W$ and $\mathcal{P}_r$ (see [L1]). In fact, this bijection is described by a map $\sigma$ from $W$ to $\mathcal{P}_r$ and the fibers of $\sigma$ are precisely the two-sided cells of $W$. Similarly, in [Mc], McGerty has defined a map $\rho$ from $\mathfrak{S}_{r,n,n}$ to $\mathcal{P}^n_r$ and the fibers of $\rho$ are precisely the two-sided cells of $\mathfrak{S}_{r,n,n}$ (see [Mc, Proposition 4.4]). Thus we have a bijection between the set of two-sided cells of $\mathfrak{S}_{r,n,n}$ and $\mathcal{P}^n_r$. Given a partition $\lambda \in \mathcal{P}^n_r$, we will denote the two-sided cell $\rho^{-1}(\lambda)$ by $c_\lambda$ in what follows.

In [Mc, Definition 4.11], McGerty has defined another integer-valued function $a_r$ on $\mathfrak{S}_{r,n,n}$ and shown that $a'_r$ and $a_r$ coincide in [Mc, Lemma 4.12]. Using this function $a_r$, he has defined the asymptotic ring $J_{c_\lambda}$ for
the two-sided cell \(c_\lambda\). It is equipped with a \(\mathbb{Z}\)-basis \(\{t_A | A \in c_\lambda\}\) and the multiplication is defined by

\[
t_At_B = \sum_C \gamma_{A,B}^C t_C, \quad \text{where } \gamma_{A,B}^C \in \mathbb{Z}.
\]

Let \(D_{c_\lambda} = D_c \cap c_\lambda\). It follows from the affine Hecke algebra case that the \(\mathbb{Z}\)-ring \(J_{c_\lambda}\) has an identity, namely \(1 = \sum_{A \in D_{c_\lambda}} t_A\).

**Lemma 3.2.** Let \(c_\lambda\) is a two-sided cell.

1. Let \(A_1, A_2, A_3 \in c_\lambda\) and let \(A_4, A_5, A_6 \in \mathfrak{D}_{c_\lambda}\), then we have

\[
\sum_{A \in c_\lambda} \nu_{A_1,A_2}(v) \gamma_{A_3,A_5}^A = \sum_{A \in c_\lambda} \nu_{A_1,A_4}(v) \gamma_{A_6,A_2}^A,
\]

2. Let \(A, A', A''\) be in a two-sided cell, if \(\gamma_{A,A'}^A \neq 0\), then \(A \sim_L A'', A'' \sim_R A \sim_L A'\).
3. Suppose that \(A_1, A_2 \in c_\lambda\). Then \(\sum_{A \in D_{c_\lambda}} \gamma_{A_1,A_2}^A = 1\), and \(\gamma_{A_1,A_2}^A = 0\) for any \(A \in D_{c_\lambda}\) if \(A_1 \neq A_2\).

**Proof.** (1) follows from Lemma 3.1 and the affine Hecke algebra case.

(2) follows from \(\gamma_{A,B}^C = \gamma_{A,B}^{C_\lambda} \) and the affine Hecke algebra case.

(3) follows from the fact that \(\sum_{E \in D_{c_\lambda}} t_E\) is the identity of \(J_{c_\lambda}\).

Since \(D_{c_\lambda}\) is a finite set (see [Mc, Prop. 3.8 (5) and Prop. 4.10]), we will use \(\{1, 2, \cdots, n_\lambda\}\) to label these elements in it, where \(n_\lambda = |D_{c_\lambda}|\). From now on, we will always use this fixed label. From [Mc, Prop. 4.13 (ii)], the left cell corresponding to \(E \in D_{c_\lambda}\) will be denoted by \(L_j\), if \(j\) labels \(E\). Then \(L_1, L_2, \cdots, L_{n_\lambda}\) are a list of all left cells in \(c_\lambda\), and \(R_j = L_j^+\), \(1 \leq j \leq n_\lambda\) are a list of all right cells in \(c_\lambda\). Let \(A_{ij} = R_j \cap L_i\) for \(1 \leq i, l \leq n_\lambda\); then \(A_{ij}^r = A_{lj}\) and \(L_j\) is a disjoint union of all \(A_j\). For \(\lambda \in \mathbb{P}^n\) and \(i \in \{1, 2, \cdots, n\}\), let \(\lambda(j) = \lambda_i - \lambda_{i+1}\) (where \(\lambda_{n+1} = 0\)) and let \(G_\lambda = \prod_{i=1}^n GL_{\lambda(i)}(\mathbb{C})\). Let \(\text{Irr } G_\lambda\) be the set of irreducible representations of \(G_\lambda\) and let \(B_\lambda = R(G_\lambda)\) be the representation ring of the algebraic group \(G_\lambda\), which is a product of general linear groups \(GL_{\lambda(i)}\), so \(B_\lambda\) is an affine \(\mathbb{Z}\)-algebra (see [KX, Theorem 5.3]). If \(s \in \text{Irr } G_\lambda\), let us denote by \(s^\#\) the dual representation of \(s\). Since \(B_\lambda\) has a basis consisting of all irreducible representations of \(G_\lambda\), so we can extend \# to an involution anti-automorphism on \(B_\lambda\).

Let \(J_\lambda\) be a free abelian group on the set of triples \((j, l, s)\), where \(j, l \in D_{c_\lambda}\) and \(s \in \text{Irr } G_\lambda\), and the ring structure is given as follows:

\[
(j, l, s_1)(j', l', s_2) = \delta_{j, j'} \sum_{s \in \text{Irr } G_\lambda} c_{s_1, s_2}^s(j, l', s)
\]

where \(c_{s_1, s_2}^s\) is the multiplicity of \(s\) in the tensor product \(s_1 \otimes s_2\).
By using Xi’s results on affine Hecke algebras (see [Xi]), in [Mc, Theorem 4.13 (i)], McGerty has established a ring isomorphism $J_{c_{\lambda}} \to J_\lambda$, which gives a bijection between $c_{\lambda}$ and the set $\{(j, l, s) | j, l \in D_{c_{\lambda}}, s \in \text{Irr } G_{\lambda}\}$ such that $C \in A_{jl}$ if $C$ corresponds to $(j, l, s)$ under this bijection. We will identify $A$ with $(j, l, s)$ under this bijection.

**Theorem 3.1.** (see [Mc, Theorem 4.13 (i)])

The asymptotic ring $J_{c_{\lambda}}$ is isomorphic to an $n_{\lambda} \times n_{\lambda}$ matrix algebra over $B_{\lambda}$. The isomorphism is given by $t_C \mapsto E_{jl}(s)$ for $C = (j, l, s) \in A_{jl}$. We also have $t_C \mapsto E_{jl}(s^\#)$.

**Proof.** The fact $t_C \mapsto E_{jl}(s^\#)$ or $C'$ is identified with $(l, j, s^\#)$ under the above-mentioned bijection follows from [Xi, Theorem 8.2.1].

Thus each element in $J_{c_{\lambda}}$ is a matrix over $B_{\lambda}$. So we identify $t_C$ for $C = (j, l, s) \in A_{jl}$ with $E_{jl}(s)$. We can also label the canonical basis element $\{C\}$ with $C_{\lambda} \in c_{\lambda}$, that is, we write $E_{jl}(s)$ for $\{C\}$ with $C = (j, l, s) \in A_{jl}$. Let $I_{c_{\lambda}}$ be the identity matrix corresponding to the element $\sum_{E \in D_{c_{\lambda}}} E$, and $\tilde{I}_{c_{\lambda}}$ be the matrix corresponding to the element $\sum_{E \in D_{c_{\lambda}}} \{E\}$.

For a two-sided cell $c_{\lambda}$, we define the following $A$-linear map $\Phi : \mathcal{U}_r \rightarrow A \otimes_Z J_{c_{\lambda}}$, due to Lusztig, by

$$\Phi(\{C\}) = \sum_{E \in D_{c_{\lambda}}, B \in c_{\lambda}} \nu^B_{C,E}(v)t_B \quad (C \in \mathfrak{S}_{r,n,n}),$$

which is well-defined since $D_{c_{\lambda}}$ is a finite set and for fixed $C, E$ there are only finitely many $\nu^B_{C,E}(v) \neq 0$.

### 4 Affine cellularity of affine $q$-Schur algebras

Let $c_{\lambda}$ be a two-sided cell. We define by $\mathcal{U}_r^{c_{\lambda}}$ (resp. $\mathcal{U}_r^{\leq c_{\lambda}}$) the free $A$-submodule of $\mathcal{U}_r$ generated by all $\{C\}$ with $C \unlhd_{LR} C'$ (resp. $C \prec_{LR} C'$) for some $C' \in c_{\lambda}$. Then both $\mathcal{U}_r^{c_{\lambda}}$ and $\mathcal{U}_r^{\leq c_{\lambda}}$ are two-sided ideals in $\mathcal{U}_r$. We denote by $\mathcal{U}_r^{-c_{\lambda}}$ the quotient $\mathcal{U}_r^{\leq c_{\lambda}}/\mathcal{U}_r^{\prec c_{\lambda}}$. Thus $\mathcal{U}_r^{c_{\lambda}}$ has an $A$-basis $\{\{C\} | C \in c_{\lambda}\}$ and the multiplication in $\mathcal{U}_r^{c_{\lambda}}$ is given by

$$[[C]][[C']] = \sum_{C'' \in c_{\lambda}} \nu_{C,C''}^{C''}[[C'']], \quad \text{for all } C, C' \in c_{\lambda}. $$

**Lemma 4.1.** $J_{c_{\lambda}}$ is a $J_{c_{\lambda}}\cdot \mathcal{U}_r^{c_{\lambda}}$-bimodule via

$$t_C \cdot [[C']] = \sum_{C'' \in c_{\lambda}} \nu_{C,C''}^{C''}t_{C''}.$$

**Proof.** It follows from Lemma 3.2 (1).
Lemma 4.2. In $J_{c_{\lambda}}$, for all $C, C' \in c_{\lambda}$ we have

\[ t_C \cdot [\{C\}] = t_C \left( \sum_{E \in D_{c_{\lambda}}} t_E \cdot \tilde{I}_{c_{\lambda}} \right) t_{C'}. \]

Proof. Since $\sum_{d \in D_{c_{\lambda}}} t_d$ is the identity of $J_{c_{\lambda}}$, there are, by definition, the following equalities:

\[ t_C \left( \sum_{E \in D_{c_{\lambda}}} t_E \cdot \tilde{I}_{c_{\lambda}} \right) t_{C'} = t_C \cdot \left( \sum_{E \in D_{c_{\lambda}}} [\{E\}] \right) t_{C'} = \sum_{E \in D_{c_{\lambda}}} \nu^A_{E,C'} t_{C'}. \]

Now, we use Lemma 3.2 (1) to replace $\sum_A \nu^A_{C,E'}$ by $\sum_A \nu^A_{C',E} \in A, C'$, and we get

\[ t_C \left( \sum_{E \in D_{c_{\lambda}}} t_E \cdot \tilde{I}_{c_{\lambda}} \right) t_{C'} = \sum_{E \in D_{c_{\lambda}}} \nu^C_{C',E} t_{C'} = \sum_{E \in D_{c_{\lambda}}} \nu^C_{C',E} \cdot \left( \sum_{E \in D_{c_{\lambda}}} \gamma^A_{E,C'} \right) t_{C'}. \]

It follows from Lemma 3.2 (3) that

\[ t_C \left( \sum_{E \in D_{c_{\lambda}}} t_E \cdot \tilde{I}_{c_{\lambda}} \right) t_{C'} = \sum_{C' \in D_{c_{\lambda}}} \nu^C_{C',C'} = t_C \cdot [\{C\}]. \]

\[ \square \]

When Lemma 4.2 gets translated into matrix language, the left-hand side of the equation expresses the multiplication $\tilde{E}_{jl}(s) \cdot \tilde{E}_{pq}(s')$ in $\mathfrak{U}^\lambda$ for $C = (j, l, s) \in A_{jl}$ and $C' = (p, q, s') \in A_{pq}$, and the right-hand side is just the product $E_{jl}(s) \Psi_{c_{\lambda}} E_{pq}(s')$ in the usual matrix algebra $J_{c_{\lambda}}$, where $\Psi_{c_{\lambda}}$ is the matrix representing $\sum_{E \in D_{c_{\lambda}}} t_E \cdot (\sum_{C \in D_{c_{\lambda}}} [\{C\}])$.

The map given by $[\{C\}] \mapsto [\{C'\}]$ is an $\mathcal{A}$-involution of $\mathfrak{U}^\lambda$. This is induced from the $\mathcal{A}$-linear anti-automorphism $\Psi$ on $\mathfrak{U}_r$ which sends $\{C\}$ to $\{C'\}$ for any $C \in S_{r,n,n}$.

Thus we get the following result.

Proposition 4.1. Let $c_{\lambda}$ be a two-sided cell in $S_{r,n,n}$. Then there is a matrix $\Psi_{c_{\lambda}}$ in $J_{c_{\lambda}}$ such that $\mathfrak{U}^\lambda$ can be identified with the generalized matrix algebra $(M_{n_{\lambda}}(B_{\lambda}), \Psi_{c_{\lambda}})$ such that $\{C\}$ is identified with $\tilde{E}_{jl}(s)$ for $C = (j, l, s) \in c_{\lambda}$. The multiplication in $(M_{n_{\lambda}}(B_{\lambda}), \Psi_{c_{\lambda}})$ is given by

\[ \tilde{E}_{jl}(s) \cdot \tilde{E}_{pq}(s') = E_{jl}(s) \Psi_{c_{\lambda}} E_{pq}(s'). \]
Moreover, the homomorphism \( \Phi_{c_\lambda} \) defined by Lusztig from \( H_{c_\lambda} \) to \( J_{c_\lambda} \) can be identified with the map from \( (M_{n_1}(B_\lambda), \Psi_{c_\lambda}) \) to \( M_{n_1}(B_\lambda) \) by multiplying \( \Psi_{c_\lambda} \) from the right.

It has been proved in [S2] that the dominance order, denoted by \( \preceq \), on \( \mathcal{P}_r \) is compatible with the order \( \preceq_{LR} \) on \( W \); for \( w, u \in W \), we have \( w \preceq_{LR} u \) if and only if \( \sigma(u) \leq \sigma(w) \). From Proposition 3.1 (6), we can get a linear order on the cells \( c_1, c_2, \ldots, c_k \) \((k = |\mathcal{P}_r|)\) such that \( c_j \preceq_{LR} c_l \) implies that \( j \leq l \).

For each \( j \), we define \( \mathcal{C}_j \) to be the \( \mathcal{A} \)-submodule generated by all \( \{C\} \) with \( C \in c_j \), and \( \mathcal{C}_\lambda = \bigoplus_{j=1}^{k} \mathcal{C}_j \). Then \( \mathcal{C}_j \) is invariant under the involution \( \Psi \), \( \mathcal{C}_j \) is a two-sided ideal in \( \mathcal{U}_r \) with \( \mathcal{U}_j / \mathcal{C}_{j-1} = \mathcal{U}^{C_j} \), and the finite chain

\[
\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_k = \mathcal{U}_r
\]

is a cell chain for \( \mathcal{U}_r \) by Prop. 2.2, Theorem 3.1 and Prop. 4.1. Thus we have proved the following theorem.

**Theorem 4.1.** Let \( \mathcal{A} = \mathbb{Z}[v, v^{-1}] \) \((v\) an indeterminate)\). Then the affine \( q \)-Schur algebra \( \mathcal{U}_r \) over \( \mathcal{A} \) is an affine cellular \( \mathbb{Z} \)-algebra with respect to the \( \mathcal{A} \)-involution \( \Psi : \{C\} \mapsto \{C^t\} \) for any \( C \in \mathfrak{S}_{r,n,n} \).

By specializing at \( v = 1 \), we get the affine Schur algebra \( \mathcal{U}_r \), and also obtain the following theorem from [KX, Lemma 2.4].

**Theorem 4.2.** The affine Schur algebra \( \mathcal{U}_r \) over \( \mathbb{Z} \) is an affine cellular \( \mathbb{Z} \)-algebra.

In the following we will consider the affine \( q \)-Schur algebra \( \mathcal{U}_{r,c} = \mathbb{C} \otimes \mathcal{A} \mathcal{U}_r \) over \( \mathbb{C} \), where \( \mathbb{C} \) is regarded as an \( \mathcal{A} \)-module by specializing \( v \) to \( z \) and \( z \in \mathbb{C}^* \) is not a root of unity.

A segment \( s \) with center \( a \in \mathbb{C}^* \) is by definition an ordered sequence

\[
s = (az^{-k+1}, az^{-k+3}, \ldots, az^{-1}) \in (\mathbb{C}^*)^k,
\]

here \( k \) is called the length of the segment, denoted by \(|s|\). If \( s = \{s_1, \ldots, s_p\} \) is an unordered collection of segments, define \( \varphi(s) \) to be the partition associated with the sequence \((|s_1|, \ldots, |s_p|)\). That is, \( \varphi(s) = (|s_1|, \ldots, |s_p|) \) with \(|s_i| \geq \cdots \geq |s_p|\), where \(|s_1|, \ldots, |s_p|\) is a permutation of \(|s_1|, \ldots, |s_p|\). We also call \(|s| = |s_1| + \cdots + |s_p|\) the length of \( s \). Let \( \mathcal{S}_r \) be the set of unordered collection of segments \( s \) with \(|s| = r \) and let \( \mathcal{S}_r^{(n)} = \bigcup_{\lambda \in \mathcal{P}_r^n} \mathcal{S}_r^{(\lambda)} \), where \( \mathcal{S}_r^{(\lambda)} = \{s \in \mathcal{S}_r | \varphi(s) = \lambda\} \).

Let \( c_\lambda \) be a two-sided cell corresponding to a partition \( \lambda \in \mathcal{P}_r^n \) and let \( \mathcal{U}_{r,c_\lambda} \) be the free \( \mathbb{C} \)-submodule of \( \mathcal{U}_{r,c} \), which is spanned by \( \{A\}, A \in c_\lambda \).

**Lemma 4.3.** When \( z \in \mathbb{C}^* \) is not a root of unity, \( \mathcal{U}_{r,c_\lambda} \) is idempotent and has a nonzero idempotent element.
Proof. Let $J_C = \bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{C} \otimes_\mathbb{Z} J_{c_{\lambda}}$ be the asymptotic ring for $\mathfrak{U}_{r,C}$. It follows from Theorem 3.1 that all the simple modules of the $\mathbb{C}$-algebra $\mathbb{C} \otimes_\mathbb{Z} J_{c_{\lambda}}$ all have dimension $n_{\lambda}$, and that the set of isomorphism classes of such modules is in bijection with the semisimple conjugacy classes of $G_{\lambda}$. It follows from [L2, Corollary 3.6] and [Ro], [Z] that the set of isomorphism classes of $J_C$-modules is in bijection with the set $\mathcal{S}_r(n)$ when $z \in \mathbb{C}^*$ is not a root of unity. According to [DDF, Theorem 4.3.4 and 4.5.3], when $z \in \mathbb{C}^*$ is not a root of unity, the set of non-isomorphic simple $\mathfrak{U}_{r,C}$-modules is also parameterized by the set $\mathcal{S}_r(n)$. So we have obtained the following bijection:

\[ \{ \text{simple } \mathfrak{U}_{r,C} \text{-modules up to isomorphism} \} \leftrightarrow \{ \text{simple } J_C \text{-modules up to isomorphism} \} \]

According to [KX, Theorem 4.1], we get the idempotence of $\mathfrak{U}_{r,C}$. Recall that McGerty has defined a map $\rho$ from $\mathfrak{S}_{r,n,n}$ to $\mathcal{P}_n$ in [Mc, Definition 4.3]. Let $A = (\lambda, w_{0_{\lambda}}, \lambda)$, where $w_{0_{\lambda}}$ is the longest element in the standard Young subgroup $W_{\lambda} := \mathfrak{S}_{(\lambda_1, \ldots, \lambda_n)} \subseteq \mathfrak{S}_r$. Then it follows from the definition of $\rho$ that we have $A \in c_{\lambda}$ and it is also easy to see that $\{A\} \{A\} = \{A\}$. We have proved the claims.

Applying Theorem 2.1, from Theorem 4.1 and Lemma 4.3 we can get the following theorem.

**Theorem 4.3.** Assume that $z \in \mathbb{C}^*$ is not a root of unity. Then all cells in the cell chain of $\mathfrak{U}_{r,C}$ correspond to idempotent ideals, which all have idempotent generators. Moreover, $\mathfrak{U}_{r,C}$ is of finite global dimension and its derived module category admits a stratification whose sections are the derived module categories of the various algebras $B_{\lambda}$.

**Remark 4.1.** Based on an algebraic study of extension algebras, S. Kato in [Ka] has obtained that the affine $q$-Schur algebra $\mathfrak{U}_{r,C}$ has finite global dimension. Derived module categories and stratifications are not considered in [Ka].

**Remark 4.2.** When $n > r$, we have proved in [Cu1, Theorem 4.3] that the affine $q$-Schur algebra $\mathfrak{U}_{r,k}$ over a noetherian domain $k$ always has finite global dimension provided that $k$ has that. In [Cu2, §6], we have discussed homological properties of some relatively simple affine $q$-Schur algebras.

**Remark 4.3.** Very recently, Kleshchev has introduced the affine highest weight categories and affine quasihereditary algebras in [Kle] for graded algebras. In fact, in [Koe, Page. 531] Koenig has introduced the concept of the affine quasi-hereditary algebras for infinite-dimensional algebras (see [Koe, Theorem 8.4]). From Theorem 4.1 and Theorem 4.3, we immediately get the affine $q$-Schur algebra $\mathfrak{U}_{r,C}$ is an affine quasi-hereditary algebra in the sense of Koenig (see also [KX]) when the parameter $z \in \mathbb{C}^*$ is not a root of unity. In fact, From [KX, Theorem 4.3] and its proof, it is easy to see...
that $\mathfrak{U}_{r,\mathbb{C}}$, when $z \in \mathbb{C}^*$ is not a root of unity, is an affine quasihereditary algebra in the sense of Kleshchev if we don’t consider the gradings.

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