CONDITIONS FOR EXISTENCE AND SMOOTHNESS OF THE DISTRIBUTION DENSITY FOR AN ORNSTEIN-UHLENBECK PROCESS WITH LÉVY NOISE

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Abstract. Conditions are given, sufficient for the distribution of an Ornstein-Uhlenbeck process with Lévy noise to be absolutely continuous or to possess a smooth density. For the processes with non-degenerate drift coefficient, these conditions are a necessary ones. A multidimensional analogue for the non-degeneracy condition on the drift coefficient is introduced.

1. Introduction

The theory of stochastic differential equations (SDE) with jump noise is developed intensively during the last decades. It is stimulated by a wide range of the disciplines that use such types of SDE’s as a models, from climatology (e.g. [1]) to financial mathematics (e.g. [2], [3]). One of the most important point in this theory consists in studying a local properties of the laws of the solutions to such an equations. For instance, an information about the distribution density for the solution allows one to investigate effectively the ergodic properties of this solution (see [4] and discussion therein). This, in turn, allows one to conduct consistent statistical analysis for such a processes, to solve a filtration and optimal control problems for such a processes, etc.

The large variety of publications is devoted to investigation of the properties of the laws of solutions to SDE’s with jump noise (e.g. [5] – [14]). These properties depend essentially both on the structure of the equation and its coefficients, and on the characteristics of the jump noise (i.e., its Lévy measure). Although a wide spectrum of a sufficient conditions is available, these conditions are not completely satisfactory and can not be considered as a definitive ones. On the one hand, it is difficult to compare the available sufficient conditions. On the other hand, it is unclear how close these conditions are to the necessary ones. Therefore, an important (and non simple) question is about the proper form of sufficient conditions for existence and smoothness of the distribution density, close to the necessary ones. In this article, we give an answer to this question in the structurally most simple class of linear SDE’s with additive jump noise. Solutions to such an equations often are called Ornstein-Uhlenbeck processes with Lévy noise.

2. Formulation of the problem

Consider linear SDE in $\mathbb{R}^m$,

\[ X(t) = X(0) + \int_0^t AX(s) \, ds + Z(t), \]

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where \( X(0) \in \mathbb{R}^m, A \) is an \( m \times m \)-matrix, \( Z \) is an \( \mathbb{R}^m \)-valued Lévy process (i.e., a continuous in probability time homogeneous process with independent increments). It is well known (e.g. \cite{15}) that every such a process possesses representation

\[
Z(t) = Z(0) + at + BW(t) + \int_0^t \int_{\|u\|_\mathbb{R}^m > 1} u\nu(ds, du) + \int_0^t \int_{\|u\|_\mathbb{R}^m \leq 1} \tilde{u}\nu(ds, du),
\]

where \( a \in \mathbb{R}^m, B \in \mathbb{R}^{m \times m} \) are deterministic vector and matrix, respectively, \( W \) is the Wiener process in \( \mathbb{R}^m, \nu \) is the Poisson random point measure on \( \mathbb{R}^+ \times \mathbb{R}^m \) with its intensity measure equal \( dt \times \Pi(du) \) (\( \Pi \) is the Lévy measure of the measure \( \nu \)), and \( \tilde{u}\nu(ds, du) = \nu(ds, du) - ds\Pi(du) \) is the corresponding compensated measure, it being known that \( W \) and \( \nu \) are independent.

Equation \((1)\) can be naturally interpreted as a family of a Volterra type integral equations, indexed by the probability variable \( \omega \). Thus, for every measurable process \( Z \) with its trajectories being a.e. locally bounded, this equation possesses unique solution with its trajectories also being a.e. locally bounded. Remark that every Lévy process possesses a modification that satisfies conditions on the process \( Z \) formulated before, and therefore the solution to \((1)\) is well defined. Moreover, this solution has the explicit representation

\[
X(t) = e^{tA}X(0) + \int_0^t e^{(t-s)A}a \, ds + \int_0^t e^{(t-s)A}B \, dW(s) + \int_0^t \int_{\|u\|_\mathbb{R}^m > 1} e^{(t-s)A}u\nu(ds, du) + \int_0^t \int_{\|u\|_\mathbb{R}^m \leq 1} e^{(t-s)A}u\tilde{\nu}(ds, du), \quad t \geq 0,
\]

where \( e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^k}{k!} \), \( t \geq 1 \) is the solution to the matrix-valued differential equation \( dE(t) = AE(t) \, dt, E(0) = I_{\mathbb{R}^m} \) is the unit matrix \( m \times m \). Formula \((2)\) is verified straightforwardly via the Ito formula.

All the summands in \((2)\) are independent and the fourth one takes value 0 on the set \( \{\nu([0, t] \times \{\|u\|_\mathbb{R}^m > 1\}) = 0\} \), the latter having positive probability. Thus \( X(t) \) possesses (smooth) distribution density iff so does the sum in which this summand is absent. Moreover, the first two summands in \((2)\) are deterministic and obviously do not have effect on existence or smoothness of the density. Thus we suppose in a sequel that \( X(0) = 0, a = 0, \Pi(\|u\| > 1) = 0 \).

3. One-dimensional equation

In the one-dimensional case, \( A, B \) are a real numbers and the third summand in \((2)\) is a normal random variable with its second moment equal \( B^2 \int_0^t e^{2(t-s)A} \, ds \). Since the summands in \((2)\) are independent, for \( B \neq 0 \) the law of \( X(t) \) is the convolution of some distribution with a non-degenerate Gaussian one, and thus possesses a smooth density. Further in this section, we consider the case \( B = 0 \).

In a separate case \( A = 0 \) we have \( X(t) = Z(t) - Z(0) \), and the question on the properties of the law of the solution to \((1)\) is exactly the same question for the distribution of the Lévy process that does not contain a diffusion component. The complete answer to this question is not available now. Let us formulate two sufficient conditions.

**Proposition 1.** \cite{16} Denote \( \mu(du) = [u^2 \wedge 1]\Pi(du) \). If \( \Pi(\mathbb{R}) = +\infty \) and, for some \( n \in \mathbb{N} \), the \( n \)-th convolution power of the measure \( \mu \) is absolutely continuous, then the distribution of \( Z(t) \) is absolutely continuous for every \( t > 0 \).
2. If \( \varepsilon^2 \ln \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} u^2 \Pi(du) \to +\infty, \varepsilon \to 0^+ \), then the distribution of \( Z(t) \) possesses the \( C_b^\infty \) density for every \( t > 0 \).

Here and below, we denote by \( C_b^\infty \) the class of infinitely differentiable functions, bounded with all their derivatives. In a sequel, we refer to the conditions formulated in the parts 1 and 2 of Proposition 1 as for the Sato condition and the Kallenberg condition, respectively. We emphasize once more, that both these conditions are sufficient ones, but none of them is necessary. It appears that, for the equation with its drift coefficient being non-degenerate, the necessary and sufficient conditions are available both for existence of the distribution density and for smoothness of this density.

**Proposition 2.** Let \( B = 0, A \neq 0 \). Then the law of \( X(t) \) is absolutely continuous for every \( t > 0 \) iff \( \Pi(\mathbb{R}) = +\infty \).

It is obvious, that the condition \( \Pi(\mathbb{R}) = +\infty \) is necessary: if \( \Pi(\mathbb{R}) = Q < +\infty \), then the law of \( X(t) \) has an atom with its mass equal \( e^{-tQ} \). Sufficiency follows from more general Theorem 4.3 [12] or Theorem A [14].

**Theorem 1.** The following three statements are equivalent:
(i) for every \( t > 0 \), the variable \( X(t) \) possesses a distribution density from the class \( C_b^\infty \);
(ii) for every \( t > 0 \), the variable \( X(t) \) possesses a bounded distribution density;
(iii) \( \varepsilon^2 \ln \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} (u^2 + \varepsilon^2) \Pi(du) \to +\infty, \varepsilon \to 0^+ \).

**Proof.** The implication \( (i) \Rightarrow (ii) \) is obvious. Let us prove that \( (ii) \Rightarrow (iii) \). Denote \( \rho(\varepsilon) = \varepsilon^2 \ln \frac{1}{\varepsilon} \int_{|u| \leq \varepsilon} (u^2 + \varepsilon^2) \Pi(du) \). Take \( \varepsilon \in (0, 1) \) and write

\[
X(t) = \int_0^t \int_{|u| \leq \varepsilon} e^{(t-s)A} u \tilde{\nu}(ds, du) + \int_0^t \int_{|u| \geq \varepsilon} e^{(t-s)A} u \tilde{\nu}(ds, du).
\]

The second moment of the first summand is estimated by \( t e^{2|A|t} \int_{-\varepsilon}^{\varepsilon} u^2 \Pi(du) \). Thus the Cheby-

shev inequality yields that the probability for the modulus of this summand not to exceed \( \sqrt{\varepsilon} \) is not less than

\[
1 - \varepsilon^{-1} t e^{2|A|t} \int_{-\varepsilon}^{\varepsilon} u^2 \Pi(du) \geq 1 - t e^{2|A|t} \left[ \varepsilon \ln \frac{1}{\varepsilon} \right] \rho(\varepsilon).
\]

The second summand is equal \( M(t, \varepsilon) = \int_0^t \int_{|u| \geq \varepsilon} e^{(t-s)A} u \Pi(du) ds \) with probability not less than \( P(\nu((0, t) \times \{|u| \in (\varepsilon, 1]\}) = 0) = \exp[-t \Pi(|u| \in (\varepsilon, 1])] \). The latter term is not less than \( \exp[-t \ln \frac{1}{\varepsilon} \rho(\varepsilon)] = e^{t \rho(\varepsilon)} \). Therefore,

\[
P(X(t) \in [M(t, \varepsilon) - \sqrt{\varepsilon}, M(t, \varepsilon) + \sqrt{\varepsilon}]) \geq e^{t \rho(\varepsilon)} - t e^{2|A|t} \left[ \varepsilon \ln \frac{1}{\varepsilon} \right] \rho(\varepsilon).
\]

Let \( (iii) \) fail, then there exists a sequence \( \varepsilon_n \to 0^+ \) such that \( \rho(\varepsilon_n) \leq C < +\infty \). Then, for \( t < \frac{1}{2C} \), \( x_n = M(t, \varepsilon_n) - \sqrt{\varepsilon_n}, y_n = M(t, \varepsilon_n) + \sqrt{\varepsilon_n} \), it follows from (4) that

\[
P(X(t) \in [x_n, y_n]) \to +\infty, \quad n \to +\infty.
\]

This, in turn, implies that \( (ii) \) fails, also.
Let us prove that (iii) \(\Rightarrow (i)\). We remark that, by general properties of the Fourier transform, the following condition on the characteristic function \(\phi\) of a random vector in \(\mathbb{R}^m\) is sufficient for this vector to possess a \(C^\infty_b\) distribution density:

\[
\forall n \geq 0 \quad \|z\|_{\mathbb{R}^m} \phi(z) \to 0, \quad \|z\|_{\mathbb{R}^m} \to \infty.
\]

This condition is well known; sometimes, it is called a \((C)\)-condition (e.g. [17]).

The value \(X(t)\) is an integral of a deterministic function over the compensated Poisson point measure. Therefore, its characteristic function can be expressed explicitly:

\[
\phi_{X(t)}(z) = \exp \left\{ \int_0^t \int_{\mathbb{R}} \left[ \exp \{iz e^{(t-s)A}u\} - 1 - i z e^{(t-s)A}u \right] \Pi(du)ds \right\}.
\]

Without loss of generality, we can suppose that \(A > 0\). In what follows, we suppose \(t > 0\) to be fixed. Take \(\beta > 0\) in such a way that \(\beta e^{(t-s)A} \leq 1\), \(s \in [0, t]\) (i.e., \(\beta = e^{-At}\)). Denote

\[
I_1(s, z) = \int_{|uz| \leq \beta} \left[ \cos \left( e^{(t-s)A}uz \right) - 1 \right] \Pi(du),
\]

\[
I_2(s, z) = \int_{|uz| > \beta} \left[ \cos \left( e^{(t-s)A}uz \right) - 1 \right] \Pi(du).
\]

Then, by (7),

\[
|\phi_{X(t)}(z)| = \exp \left\{ \int_0^t I_1(s, z) \, ds + \int_0^t I_2(s, z) \right\}, \quad z \in \mathbb{R}.
\]

Let \(C = 1 - \cos 1\), one can verify that \(\cos x - 1 \leq -Cx^2, \ |x| \leq 1\). Then

\[
I_1(s, z) \leq -C \int_{|uz| \leq \beta} (e^{(t-s)A}uz)^2 \Pi(du) = -Cz^2 e^{2(t-s)A} \int_{|uz| \leq \beta} u^2 \Pi(du),
\]

and thus \(\int_0^t I_1(s, z) \, ds \leq -C_1 z^2 \int_{|uz| \leq \beta} u^2 \Pi(du)\) with \(C_1 = C z^2 e^{2A} > 0\). Next,

\[
\int_0^t I_2(s, z) \, ds = \int_{|uz| > \beta} \int_0^t \left[ \cos \left( e^{(t-s)A}uz \right) - 1 \right] ds \Pi(du).
\]

By making the change of the variables \(s \mapsto y = e^{(t-s)A}uz\) and taking into account that the function \(x \mapsto \cos x - 1\) is an even one, we get

\[
\int_0^t I_2(s, z) \, ds = \frac{1}{A} \int_{|uz| \geq \beta} \int_{e^{tA}uz} e^{tA|uz|} \cos y - 1 \, dy \Pi(du) \leq
\]

\[
\leq \frac{1}{A} \int_{|uz| \geq \beta} \frac{1}{|uz|} \int_{e^{tA}|uz|} \left( \cos y - 1 \right) \, dy \Pi(du) =
\]

\[
= \frac{1}{A} \int_{|uz| \geq \beta} \left( \frac{\sin(e^{tA}|uz|) - \sin(|uz|)}{|uz|} - (e^{tA} - 1) \right) \Pi(du).
\]

Denote \(\gamma = \sup_{|y| > (e^{tA-1}A)^2} \left| \frac{\sin y}{y} \right| < 1\). Then, for \(|x| > \beta\),

\[
\frac{\sin(e^{tA}x) - \sin x}{x} = (e^{tA} - 1) \frac{\sin \left( \frac{(e^{tA}-1)x}{2} \right)}{e^{tA-1}x} \cos \left( \frac{(e^{tA}+1)x}{2} \right) \leq \gamma (e^{tA} - 1).
\]
Therefore
\[ \int_0^t I_2(s, z) \, ds \leq -C_2 \Pi (\{|u| > \beta \}), \]
where \( C_2 = \frac{1 - \gamma}{A} (e^{tA} - 1) > 0. \)

Denote \( C_3 = \min (C_1 \beta^2, C_2), \) then the estimates for \( \int_0^t I_{1,2}(s, z) \, ds \) given above yield
\[ |\phi_{X(t)}(z)| \leq \left( \frac{\beta}{|z|} \right)^{-C_3 \rho \left( \frac{\beta}{|z|} \right)}, \quad z \in \mathbb{R}. \]

The latter estimate, under condition (iii), yields (6) and therefore (i). The theorem is proved.

Remark 1. The implication (ii) \( \Rightarrow \) (iii) can be amplified with the following statement: if \( \lim \inf_{\varepsilon \to 0+} \left[ \varepsilon^2 \ln \frac{1}{\varepsilon} \right]^{-1} \int_{\mathbb{R}} (u^2 \wedge \varepsilon^2) \Pi (du) = 0, \) then for every \( t > 0, p > 1 \) the variable \( X(t) \) does not possess a distribution density from the class \( L_p(\mathbb{R}). \)

In order to prove this fact, one should take \( \alpha = \frac{1}{2} + \frac{1}{2p} \in (0, 1) \) and a sequence \( \varepsilon_n \) such that \( \rho (\varepsilon_n) \to 0. \) Then the estimates analogous to those given before provide that
\[ \frac{P(X(t) \in (x_n, y_n))}{(y_n - x_n)^{2-2\alpha}} \to +\infty \]
for \( x_n = M(t, \varepsilon_n) - \varepsilon_n^\alpha, y_n = M(t, \varepsilon_n) + \varepsilon_n^\alpha. \)

The latter convergence, together with Hölder inequality, demonstrates that \( X(t) \) can not possess a distribution density from the \( L_{\frac{1}{2\alpha}}(\mathbb{R}) = L_p(\mathbb{R}). \)

Condition (iii) looks similar to the Kallenberg condition, but the following example shows that these two conditions are remarkably different.

Example 1. Let \( \Pi = \sum_{n \geq 1} n \delta_{\frac{n}{\pi}}. \) Then
\[ \lim \inf_{\varepsilon \to 0+} \rho (\varepsilon) \geq \lim \inf_{\varepsilon \to 0+} \left\{ \left[ \ln \frac{1}{\varepsilon} \right]^{-1} \Pi(|u| > \varepsilon) \right\} \geq \lim \inf_{N \to +\infty} \frac{1}{\ln N} \sum_{n \leq N - 1} n \geq \lim \inf_{N \to +\infty} \frac{N(N - 1)}{2N \ln N} = +\infty, \]
and condition (iii) holds true. One can check that the Kallenberg condition fails, moreover, we will show that the law of \( Z(t) \) is singular for every \( t. \)

In order to do this, it is sufficient to prove that \( E e^{iz Z(t)} \not\to 0, \) \( z \to \infty. \)

But
\[ \lim_{N \to +\infty} \left| E e^{i2\pi N Z(t)} \right| = \lim_{N \to +\infty} \prod_{n > N} \left| \exp \left\{ t n \left( e^{i2\pi n} - 1 - \frac{i2\pi n!}{n!} \right) \right\} \right| = 1, \]
that proves the needed statement. Thus, we have the following interesting effect: the laws of \( Z(t) \) are singular, but the laws of the values of the solution to (11) with non-degenerated drift \( (A \neq 0) \) possesses distribution densities of the class \( C_b^\infty. \)

One can say that the process \( Z \) possess some ”hidden smoothness”, that does not effect to the law of the process itself, but becomes visible when this process is used as a noise in an equation with a non-degenerate drift. Such an effect is possible due to the difference between the Kallenberg condition and (iii).
From Proposition 2 and Theorem 1, one can make the general conclusion that the conditions for existence of the distribution density for $X(t)$ on the one hand, and for smoothness of this density on the other, are essentially different. This difference is well demonstrated by the following example.

**Example 2.** Let $\Pi = \sum_{n \geq 1} \delta_{\frac{1}{n}}$. Then $\Pi(\mathbb{R}) = +\infty$, but
\[
\rho(\varepsilon) \to 0, \quad \varepsilon \to 0.
\]

For $A \neq 0$, the solution to the equation (1) possess the distribution density, but this density is extremely irregular in a sense that it does not belong to any $L^p(\mathbb{R}), p > 1$. These two facts follow from Proposition 2 and Remark 1, respectively. Remark that, in this example, the law of $Z$ is singular: one can show this like it was done in Example 2. Thus, the current example demonstrates one more version of a “regularization” effect for the Lévy process under the SDE with a non-degenerate drift coefficient.

4. Multidimensional equation

Let us introduce an auxiliary construction. Let a $\sigma$-finite measure $\Pi$ to be defined on $\mathcal{B}(\mathbb{R}^d)$ with some $d \in \mathbb{N}$. Consider the family $\mathcal{L}_\Pi = \{ L \text{ is a linear subspace of } \mathbb{R}^d, \Pi(\mathbb{R}^d \setminus L) < +\infty \}$. It is clear that if $L_1, L_2 \in \mathcal{L}_\Pi$ then $L_1 \cap L_2 \in \mathcal{L}_\Pi$. This yields that there exists a subspace $L_\Pi \in \mathcal{L}_\Pi$ such that $L_\Pi \subset L$ for every $L \in \mathcal{L}_\Pi$.

**Definition 1.** The subspace $L_\Pi$ is called an essential linear support of the measure $\Pi$. The measure $\Pi$ is said to be essentially linearly non-degenerated if $L_\Pi = \mathbb{R}^d$.

Remark that the condition on the measure $\Pi$ to be essentially linearly non-degenerated was imposed first in the paper [18], thus often it is called the Yamasato condition.

In this section, we study the local properties of the law of the solution to the equation of the type
\[
(8) \quad X(t) = \int_0^t AX(s) \, ds + BW(t) + DZ(t), \quad t \geq 0
\]
with $A, B, D$ being an $m \times m, m \times k$- and $m \times d$-matrices respectively, $W$ being a Wiener process in $\mathbb{R}^k$ and the process $Z$ having the form
\[
Z(t) = \int_0^t \int_{\|u\|_{l_2}^d > 1} u\nu(ds, du) + \int_0^t \int_{\|u\|_{l_2}^d \leq 1} u\tilde{\nu}(ds, du), \quad t \geq 0
\]
with the Lévy measure $\Pi$ being essentially linearly non-degenerated. We have already seen that the solution $X$ to equation (1) linearly depends on $Z$: if $Z = Z_1 + Z_2$ then $X = X_1 + X_2$, where $X_{1,2}$ denote the solutions to SDE (1) with $Z$ replaced by $Z_{1,2}$. When, moreover, $Z_1$ and $Z_2$ are independent and the process $Z_2$ is the Lévy process without a diffusion component with its Lévy measure $\Pi_2$ being finite, then the law of $X_2(t)$ has an atom and thus existence or smoothness of the distribution density for $X(t)$ are equivalent to existence or smoothness of the distribution density for $X_1(t)$. This allows one to remove from the process $Z$ the part that is inessential in a sense of Definition 1. Namely, one can put $\nu_1$ equal to the restriction of the point measure $\nu$ to $\mathbb{R}^+ \times (\mathbb{R}^m \setminus L_\Pi)$ and define $Z_1$ by (2) with $a = 0, B = 0$ and $\nu$ replaced by $\nu_1$. One can easily see that equation (1) with $Z$ replaced by $Z_1$ has the form (8) with $k = m, d = \dim L_\Pi$. Thus, equation (1) can be reduced to the form (8), with $Z$ satisfying additionally the Yamasato condition.
If, in the equation (8), $D = 0$ then the following well known Kalman controllability condition is necessary and sufficient for the law of $X(t), t > 0$ to possess a smooth density (e.g., [20]):

$$\text{Rank}[B, AB, \ldots, A^{m-1}B] = m.$$ 

Here $[B, AB, \ldots, A^{m-1}B]$ denotes an $m \times mk$-matrix of a block type, composed from the matrices $B, \ldots, A^{m-1}B$. For the equation (8), we write the analogous condition

$$(\text{H1}) \quad \text{Rank}[B, AB, \ldots, A^{m-1}B, D, AD, \ldots, A^{m-1}D] = m,$$

where $[B, AB, \ldots, A^{m-1}B, D, AD, \ldots, A^{m-1}D]$ is an $m \times m(k + d)$-matrix composed from the matrices $B, \ldots, A^{m-1}B, D, AD, \ldots, A^{m-1}D$.

Below, we denote $S^d = \{ l \in \mathbb{R}^d, \| l \|_{\mathbb{R}^d} = 1 \}$ (the unit sphere in $\mathbb{R}^d$). We introduce the multidimensional analogue of the Kallenberg condition:

$$(9) \quad \left[ \varepsilon^2 \ln \frac{1}{\varepsilon} \right]^{-1} \inf_{l \in S^d} \int_{\| (u, l) \|_{\mathbb{R}^d} \leq \varepsilon} (u, l)^2_{\mathbb{R}^d} \Pi(du) \to +\infty, \quad \varepsilon \to 0 + .$$

We remark that this condition is a new one.

**Theorem 2.** Let the Lévy process $Z$ satisfy (9). Then condition (H1) is sufficient for the law $X(t), t > 0$ to possess a density from the class $C^\infty_0$.

**Proof.** Like in the proof of Theorem 1, we will verify that the characteristic function of $X(t)$ satisfies condition (6). We suppose that $\Pi(\| u \|_{\mathbb{R}^d} > 1) = 0$, this obviously does not restrict generality. The value of $X(t)$ is given as a sum of the (independent) integrals over the Wiener process and the compensated Poisson point measure. Thus the characteristic function of $X(t)$ has the following explicit representation:

$$\phi_{X(t)}(z) = \exp \left\{ \int_0^t \left( -\frac{1}{2} \| B^* e^{(t-s)A^*} z \|_{\mathbb{R}^k}^2 + \int_{\mathbb{R}^d} \left[ \exp \{ i(e^{(t-s)A} Du, z)_{\mathbb{R}^m} \} - 1 - i(e^{(t-s)A} Du, z)_{\mathbb{R}^m} \} \Pi(du) \right] ds \right) \right\}, \quad z \in \mathbb{R}^m,$$

here $Q^*$ denotes adjoint matrix to $Q$ ($Q = A, B, \ldots$). Then

$$(10) \quad |\phi_{X(t)}(z)| = \exp \left\{ \int_0^t \left( -\frac{1}{2} \| B^* e^{(t-s)A^*} z \|_{\mathbb{R}^k}^2 + \int_{\mathbb{R}^d} \left[ \cos \{ e^{(t-s)A} Du, z \}_{\mathbb{R}^m} - 1 \} \Pi(du) \right] ds \right) \right\}.$$

Denote $B(s, z) = B^* e^{sA^*} z$, $D(s, z) = D e^{sA^*} z$. We restrict the domain of integration w.r.t. $u$ by the set $\{(D(s, z), u)_{\mathbb{R}^d} \leq 1 \}$ and use inequality $1 - \cos x \geq C x^2, |x| \leq 1, C = 1 - \cos 1 > \frac{1}{2}$. We get

$$(12) \quad |\phi_{X(t)}(z)| \leq \exp \left\{ -\frac{1}{2} \int_0^t \left( \| B(s, z) \|_{\mathbb{R}^k}^2 + \int_{\| (D(s, z), u)_{\mathbb{R}^d} \| \leq 1} (D(s, z), u)_{\mathbb{R}^d}^2 \Pi(du) \right) ds \right\}.$$

Denote

$$\Phi(r) = r^2 \inf_{l \in S^m} \int_{\| (u, l) \|_{\mathbb{R}^d} \leq \frac{1}{r}} (u, l)_{\mathbb{R}^d}^2 \Pi(du), \quad r > 0,$$
remark that condition (9) is equivalent to the convergence \( \Phi(r) \frac{1}{\ln r} \to +\infty, \ r \to +\infty \). This notation allows us to rewrite (12) to the form

\[
|\phi_{X(t)}(z)| \leq \exp \left\{ -\frac{1}{2} \int_0^t \left( \|B(s, z)\|_{R_k}^2 + \Phi(\|D(s, z)\|_{R_d}) \right) ds \right\}.
\]

**Lemma 1.** Under condition (H1), for every given \( t > 0 \) there exist \( \alpha, \beta, \gamma > 0 \) such that

\[
\forall l \in S^m \quad \lambda \{ 0 \leq s \leq t : \|B(s, l)\|_{R_k} > \alpha \|D(s, l)\|_{R_d} > \beta \} \geq \gamma,
\]

where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \).

**Proof.** Suppose that the statement of the lemma does not hold true. Then there exists a sequence \( l_n \in S^m, n \geq 1 \) such that

\[
\lambda \left\{ 0 \leq s \leq t : \|B(s, l_n)\|_{R_k} > \frac{1}{n} \text{ or } \|D(s, l_n)\|_{R_d} > \frac{1}{n} \right\} < \frac{1}{n}, \ n \geq 1,
\]

that means that both the sequences \( \{\|B(\cdot, l_n)\|_{R_k}\}, \{\|D(\cdot, l_n)\|_{R_d}\} \) converge in Lebesgue measure to the identical zero. Since \( S^m \) is a compact set, without a restriction of generality one can suppose that \( l_n \to l \in S^m \). But, for every \( s \in [0, t] \), the functions \( B(s, \cdot), D(s, \cdot) \) are a linear and continuous ones, thus the functions \( \|B(\cdot, l)\|, \|D(\cdot, l)\| \) equal zero \( \lambda \)-almost surely. Clearly, these functions are continuous, and thus

\[
B^*e^{sA^*}l = 0, \quad D^*e^{sA*l} = 0, \quad s \in [0, t].
\]

By taking the derivatives of (14) by \( s \) up to the order \( m-1 \) and considering the values of the functions \( B(s, l), D(s, l) \) together with their derivatives for \( s = 0 \), we get

\[
B^*l = B^*A^*l = \cdots = B^*(A^*)^{m-1}l = 0, \quad D^*l = D^*A^*l = \cdots = D^*(A^*)^{m-1}l = 0.
\]

The latter equality means that the rows of the matrix

\[
[B, AB, \ldots, A^{m-1}B, D, AD, \ldots, A^{m-1}D]
\]

are linearly dependent, with the coefficients of the dependence being equal to the coordinates of the vector \( l \). This contradicts to condition (H1). The lemma is proved.

Now, we can complete the proof of Theorem 2. For a given \( z \in \mathbb{R}^m \), we put \( l(z) = \frac{z}{\|z\|_R} \). Then

\[
\lambda \{ 0 \leq s \leq t : \|B(s, z)\|_{R_k} > \alpha \|z\|_{R^m} \|D(s, z)\|_{R_d} > \beta \|z\|_{R^m} \} = \lambda \{ 0 \leq s \leq t : \|B(s, l(z))\|_{R_k} > \alpha \|D(s, l(z))\|_{R_d} > \beta \} \geq \gamma.
\]

The latter inequality and (13) yield the estimate

\[
|\phi_{X(t)}(z)| \leq \exp \left\{ -\gamma \frac{\alpha}{2} \min \left( \alpha \|z\|_{R^m}^2, \Phi(\beta \|z\|_{R^m}^2) \right) \right\},
\]

that, together with (9), guarantees (6). The theorem is proved.

**Remark 2.** One can extend the result of Theorem 2 and describe in a more details the asymptotic behavior of the derivatives of the density \( p_{X(t)} \) for \( \|x\|_{R^m} \to \infty \). In order to make our exposition transparent, we postpone the discussion of this topic to Section 5 below.

**Remark 3.** In [21], the statement is given (Theorem 3.1), being analogous to Theorem 2. However, conditions imposed on the Lévy measure there (Hypothesis 3.1) are somewhat superfluous and less precise than the multidimensional analogue (9) of the Kallenberg condition, used in the current paper.
In [21], Theorem 1.1, it is proved that condition (H1) is sufficient for the law of the solution to (8) to be absolutely continuous, as soon as the jump noise satisfies a multidimensional analogue of Sato condition (in [21], the case $B = 0$ is considered, only). This statement and Theorem 2 of the current paper show that (H1) can be naturally interpreted as the condition on the coefficients of the equation that provides "preservation" of smoothness contained in the additive noise $(W,Z)$. On the other hand, this condition is satisfied for $A = 0, B = 0, D = I_{\mathbb{R}^m}, d = m$. In this case $X(t) = Z(t) - Z(0)$. Therefore, it is clear that condition (H1) does not provide a "regularization" effect, analogous to the one of the one-dimensional equations with non-degenerated drift obtained in the previous section.

Such kind of an effect, at least at the part of existence of the density, is guaranteed by the following condition:

\[(H2) \quad \text{Rank} [AD, \ldots, A^m D] = m.\]

Although this condition contains the matrix $D$ as well as the matrix $A$, we interpret it as an analogue of the condition on the drift coefficient to be non-degenerate. We remark that this condition is a new one, also.

**Theorem 3.** The following statements are equivalent:

(i) condition (H2) holds true;

(ii) for an arbitrary solution to equation (8) with the process $Z$ satisfying Yamazato condition, the law of the random vector $X(t)$ is absolutely continuous for every $t > 0$.

*Proof.* Let us prove the implication (i) $\Rightarrow$ (ii) under supposition that $B = 0, \Pi(\|u\|_{\mathbb{R}^d} > 1) = 0$. It was already shown that such a supposition does not restrict generality since the solution depends on the noise linearly. We use the sufficient condition for absolute continuity of the law of a solution to SDE with jump noise, given in Theorem 1.1 [13]. This condition is based on the construction proposed in [12]. We remark that, in [13], a general class of (non-linear) SDE’s with jump noise is investigated under a specific moment condition (1.1). This condition is used in [12],[13] in the proof of the differentiability of the variable $X(t)$ w.r.t. certain group of transformations of the Poisson point measure. For the equations with an additive noise, such a differentiability holds true without a specific moment condition (see [14],[19]). Thus, we can apply the results obtained in [13] to the solution to (8), not requiring the moment condition (1.1) [13] to hold true.

Statement A of Theorem 1.1 [13] is formulated in the terms of a certain subspace generated by a sequence of vector fields, associated with the initial equation. In the partial case of a linear equation (8), these fields are defined as

$$\Delta(u) = ADu, \quad \mathcal{L}(u) = \text{Span}\{\Lambda^k \Delta(u), k \in \mathbb{Z}_+\}, \quad u \in \mathbb{R}^d, \quad \Lambda v = -Av.$$ 

By statement A of Theorem 1.1 [13], if for every $l \in S^m$

$$(16) \quad \Pi\left( u : l \text{ is not orthogonal to } \mathcal{L}(u) \right) = +\infty,$$

then the law of the solution to (8) is absolutely continuous.

Under condition (H2), for every $l \in S^m$ there exists proper subspace $L_l \subset \mathbb{R}^d$ such that

$$u \notin L_l \Rightarrow \exists k \in \{1, \ldots, m\} : A^k Du \not\perp l.$$ 

Then

$$\Pi\left( u : l \text{ is not orthogonal to } \mathcal{L}(u) \right) \geq \Pi\left( \mathbb{R}^d \setminus L_l \right).$$
This, together with the Yamazato condition, provides (16). The implication (i) $\Rightarrow$ (ii) is proved.

Now, let us prove the inverse implication (ii) $\Rightarrow$ (i). We put $B = 0$. Let us prove that there exists a non-zero vector $l \in \mathbb{R}^m$ such that

$$\tag{17} (X(t), l)_{\mathbb{R}^m} = (Z(t) - Z(0), D^*l)_{\mathbb{R}^d}, \quad t \geq 0.$$ 

If $D = 0$, then (17) trivially holds for every $l \in \mathbb{R}^m$. Thus, we suppose further that $D \neq 0$. Under this supposition, $\text{Ker} D^*$ is a proper subspace of $\mathbb{R}^m$. If (H2) does not hold, then there exists a non-zero vector $l \in \mathbb{R}^m$ such that

$$\tag{18} D^*A^*l = \cdots = D^*(A^*)^ml = 0,$$

that means that the vectors $A^*l, \ldots, (A^*)^ml$ belong to the subspace $\text{Ker} D^*$. Since the dimension of this subspace does not exceed $m - 1$, there exist $k \leq m, c_1, \ldots, c_{k-1} \in \mathbb{R}$:

$$\tag{19} (A^*)^kl = \sum_{j=1}^{k-1} c_j(A^*)^jl.$$ 

By multiplying both sides of (19) on $(A^*)^{m+1-k}$ from the left, and taking into account that $(A^*)^{m+1-k}/\text{Ker} D^*, j \leq k - 1$, we get $(A^*)^ml \in \text{Ker} D^*$. Repeating these considerations, we obtain that $(A^*)^ml \in \text{Ker} D^*, n \in \mathbb{N}$ and thus $e^{(t-s)A^*}l - l \in \text{Ker} D^*, 0 \leq s \leq t$. Then

$$\begin{align*}
(X(t), l)_{\mathbb{R}^m} &= \int_0^t \int_{\|u\|_{\mathbb{R}^d} \leq 1} (u, D^*e^{(t-s)A^*}l)_{\mathbb{R}^d} \tilde{\nu}(ds, du) = \\
&= \int_0^t \int_{\|u\|_{\mathbb{R}^d} \leq 1} (u, D^*l)_{\mathbb{R}^d} \tilde{\nu}(ds, du),
\end{align*}$$

that proves (17).

If $D^*l = 0$, then (17) immediately guarantees singularity of the law $X(t)$ for every $Z$. Let us consider the case $D^*l \neq 0$. Take the orthonormal basis $e_1, \ldots, e_d$ in $\mathbb{R}^d$ in such a way that $e_1$ has the same direction with $D^*l$. Denote by $\gamma(r), r \geq 0$ the point in $\mathbb{R}^d$ with its coordinates (in this basis) equal $r, r^2, \ldots, r^d$. A standard argument using Vandermonde determinant provide that the curve $\{\gamma(r), r \in \mathbb{R}^+\}$ has at most $d$ intersection points with every hyperplane in $\mathbb{R}^d$. Thus the measure $\Pi = \sum_{k \in \mathbb{R}} \delta_{\gamma(k)}$ satisfies Yamazato condition. On the other hand, the law of the variable $(X(t), e_1)_{\mathbb{R}^m}$ coincides with the law of the variable $Z(t)$ constructed in Example 2, and therefore is singular. Thus, the law of $X(t)$ is singular, also. The theorem is proved.

Remark 4. Condition (H1) involves non-trivially all the matrices $A, B, D$, and thus the smoothness of the distribution density of the solution to (8) is provided by the diffusion and jump noise conjointly. This differs from the statement (ii) of Theorem 3, where both the process $Z$ and the matrix $B$ are arbitrary. The analogue of the condition (H2) for the equation with the fixed matrix $B$ has the form

$$(\text{H2}’) \quad \text{Rank} [B, \ldots, A^{m-1}B, AD, \ldots, A^mD] = m.$$ 

The proof of necessity for this condition is totally analogous to the proof of necessity for (H2), given in Theorem 3. We can not use the results from the papers [12],[13] in order to prove sufficiency of this condition, since the equations with a diffusion component are not considered in these papers. For the linear SDE, the approach developed in [12],[13], without an essential changes, can be extended to SDE’s with a diffusion component, and using this approach one
can prove sufficiency of (H2'). However, we do not give a detailed exposition of this proof here, since, for such an exposition, we would need to repeat the noticeable part of [12], [13].

Remark 5. It is well known that Kalman controllability condition is, in fact, a version of the Hörmander hypoellipticity condition, formulated for a separate class of linear diffusions. Our condition (H2) also has such an interpretation: in the proof of sufficiency part of Theorem 3, we refer to Theorem 1.1 [13]. The conditions given in the latter theorem can be naturally considered as an analogue of the Hörmander hypoellipticity condition for SDE’s with a jump noise.

Example 3. ([13], Example 1.1) Consider the system of equations
\[
\begin{align*}
    dX_1(t) &= X_1(t) \, dt + dZ(t) \\
    dX_2(t) &= X_1(t) \, dt.
\end{align*}
\]

When \( Z \) is replaced by \( W \) in this system, one get the well known Kolmogorov’s example of a two-dimensional diffusion possessing smooth distribution density and being generated by a one-dimensional Brownian motion. Initial system has the form [5] with \( m = 2, d = 1, B = 0, \)

\[
A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [D, AD] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad [AD, A^2D] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Condition (H1) holds, but condition (H2) does not hold true. Thus, in the Kolmogorov’s example, the ”preservation of smoothness” takes place, but the ”regularization effect” does not come into play.

Let us modify the Kolmogorov’s example and consider the system of equations
\[
\begin{align*}
    dX_1(t) &= X_2(t) \, dt + dZ(t) \\
    dX_2(t) &= X_1(t) \, dt.
\end{align*}
\]

This system has the form [5] with \( m = 2, d = 1, B = 0, \)

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [AD, A^2D] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

For this system, condition (H2) holds true. Thus, for every process \( Z \) with an infinite Lévy measure (i.e., for a process that has an infinite number of the jumps on every time interval), the law of \( X(t) = (X_1(t), X_2(t)) \) in \( \mathbb{R}^2 \) is absolutely continuous.

5. ASYMPTOTIC PROPERTIES OF THE DERIVATIVES OF THE DISTRIBUTION DENSITIES

Together with the question on existence and smoothness of the distribution density \( p_{X(t)}(x) \), \( x \in \mathbb{R}^m \), it is natural to study the limit behavior of the derivatives of this density for \( \|x\|_{\mathbb{R}^m} \to +\infty \). In [21], Remark 3.1, the problem of integrability of the derivatives of the density \( p_{X(t)} \) is formulated in the connection with the investigation of the smoothing properties of the semigroup generated by the process \( X \). In this section, we give a more strong version of Theorem 2, that solve this problem completely.

Below, we denote by \( \mathcal{S}(\mathbb{R}^m) \) the Schwarz space of infinitely differentiable functions \( f : \mathbb{R}^m \to \mathbb{R} \) such that every derivative of the function \( f \), as \( \|x\|_{\mathbb{R}^m} \to \infty \), tends to 0 faster than \( \|x\|_{\mathbb{R}^m}^-n \) for every \( n \).
Theorem 4. Consider equation (8). If conditions (9) and (H1) hold, then, for every \( j_1, \ldots, j_r \in \{1, \ldots, m\}, r \in \mathbb{N}, t > 0, \)

\[
\frac{\partial}{\partial x_{j_1} \ldots \partial x_{j_m}} p_{X(t)} \in L_1(\mathbb{R}^m).
\]

If, additionally, the Lévy measure of the process \( Z \) satisfies the condition

\[
\int_{\|u\|_{\mathcal{L}^d}>1} \|u\|^n_{\mathbb{R}^d} \Pi(du) < +\infty, \quad n \in \mathbb{N},
\]

then \( p_{X(t)} \in \mathcal{S}(\mathbb{R}^m), t > 0. \)

Proof. Let us consider first the case where the Lévy measure satisfies (21). The Fourier transform is a bijective mapping \( \mathcal{S}(\mathbb{R}^m) \rightarrow \mathcal{S}(\mathbb{R}^m) \) (e.g., [22], §6.1). Thus, for the proof of Theorem, it is sufficient to prove that every derivative of the characteristic function \( \phi_{X(t)}(z), z \in \mathbb{R}^m \) tends to 0, as \( \|z\|_{\mathbb{R}^m} \rightarrow \infty, \) faster than \( \|z\|_{\mathbb{R}^m}^{-n} \) for every \( n. \) This function has the representation, analogous to (4): \( \phi_{X(t)} = \exp[\psi_{X(t)}], \)

\[
\psi_{X(t)}(z) = \int_0^t \left( -\frac{1}{2} \|B^*e^{(t-s)A}z\|^2_{\mathbb{R}^k} + \int_{\|u\|_{\mathcal{L}^d}>1} \left[ \exp\{i(e^{(t-s)A}Du, z)_{\mathbb{R}^m}\} - 1 \right] \Pi(du) - 
\right)
\]

\[
- \int_{\|u\|_{\mathcal{L}^d} \leq 1} \left[ \exp\{i(e^{(t-s)A}Du, z)_{\mathbb{R}^m}\} - 1 - i(e^{(t-s)A}Du, z)_{\mathbb{R}^m} \right] \Pi(du) \right) ds, \quad z \in \mathbb{R}^m.
\]

Thus, every derivative of the function \( \phi_{X(t)} \) has the form \( R \cdot \phi_{X(t)} \), where \( R \) is some polynomial of the derivatives of the function \( \psi_{X(t)}. \) We have already proved in Theorem 2 that, under conditions (9) and (H1),

\[
\phi_{X(t)}(z) = o(\|z\|_{\mathbb{R}^m}^{-n}), \quad \|z\|_{\mathbb{R}^m} \rightarrow \infty, \quad n \in \mathbb{N}.
\]

Thus, it is enough to verify that every derivative of the function \( \psi_{X(t)} \) has at most polynomial growth as \( \|z\|_{\mathbb{R}^m} \rightarrow \infty. \) We have

\[
\frac{\partial}{\partial z_j} \psi_{X(t)}(z) = - \int_0^t (B^*e^{(t-s)A}z, B^*e^{(t-s)A}e_j)_{\mathbb{R}^k} ds + 
\]

\[
+ \int_0^t \int_{\|u\|_{\mathcal{L}^d} \leq 1} i(e^{(t-s)A}Du, e_j)_{\mathbb{R}^m} \left[ \exp\{i(e^{(t-s)A}Du, z)_{\mathbb{R}^m}\} - 1 \right] \Pi(du) ds 
\]

\[
+ \int_0^t \int_{\|u\|_{\mathcal{L}^d}>1} i(e^{(t-s)A}Du, e_j)_{\mathbb{R}^m} \exp\{i(e^{(t-s)A}Du, z)_{\mathbb{R}^m}\} \Pi(du) ds, \quad j = 1, \ldots, m,
\]

where \( e_j \) is the \( j \)-th basis vector in \( \mathbb{R}^m. \) Taking into account the inequality \( |e^z - 1| \leq |z|, \) we get

\[
(22) \quad \left| \frac{\partial}{\partial z_j} \psi_{X(t)}(z) \right| \leq C_1 \left( \|z\|_{\mathbb{R}^m} + \|z\|_{\mathbb{R}^m} \int_{\|u\|_{\mathcal{L}^d} \leq 1} \|u\|^2_{\mathbb{R}^d} \Pi(du) + \int_{\|u\|_{\mathcal{L}^d}>1} \|u\|^2_{\mathbb{R}^d} \Pi(du) \right).
\]

Here and below, \( C_r, r = 1, 2, \ldots \) are some constants that depend on coefficients \( A, B, D \) and time moment \( t. \) Next,

\[
\frac{\partial^2}{\partial z_{j_1} \partial z_{j_2}} \psi_{X(t)}(z) = - \int_0^t (B^*e^{(t-s)A}e_{j_1}, B^*e^{(t-s)A}e_{j_2})_{\mathbb{R}^k} ds + 
\]
\[ + \int_0^t \int_{\mathbb{R}^d} i^2 (e^{(t-s)A} Du, e_j)_{\mathbb{R}^m} (e^{(t-s)A} Du, e_j)_{\mathbb{R}^m} \exp\{i(e^{(t-s)A} Du, z)_{\mathbb{R}^m}\} \Pi(du) ds, \]

\[ j_1, j_2 \in \{1, \ldots, m\}. \] Thus

\[ \frac{\partial^2}{\partial z_{j_1} \partial z_{j_2}} \psi(z) X(t)(z) \leq C_2 \left( 1 + \int_{\mathbb{R}^d} \|u\|_{\mathbb{R}^d}^2 \Pi(du) \right), \quad j_1, j_2 \in \{1, \ldots, m\}. \] (23)

At last, the partial derivatives of the order \( r \geq 3 \) have the form

\[ \frac{\partial^r}{\partial z_{j_1} \cdots \partial z_{j_r}} \psi X(t)(z) = \int_0^t \int_{\mathbb{R}^d} i^r \prod_{l=1}^r (e^{(t-s)A} Du, e_j)_{\mathbb{R}^m} \exp\{i(e^{(t-s)A} Du, z)_{\mathbb{R}^m}\} \Pi(du) ds, \]

\[ j_1, \ldots, j_r \in \{1, \ldots, m\}, \] that implies the estimate

\[ \left| \frac{\partial^r}{\partial z_{j_1} \cdots \partial z_{j_r}} \psi X(t)(z) \right| \leq C_r \left( \int_{\|u\|_{\mathbb{R}^d} \leq 1} \|u\|_{\mathbb{R}^d}^2 \Pi(du) + \int_{\|u\|_{\mathbb{R}^d} > 1} \|u\|^r_{\mathbb{R}^d} \Pi(du) \right). \] (24)

It follows from (22) – (24) that the first derivatives of \( \psi X(t) \) have at most linear grows, while all the higher derivatives are even bounded. Thus \( \phi X(t) \in S(\mathbb{R}^m) \) and therefore \( p_{X(t)} \in S(\mathbb{R}^m) \).

Now, let us consider the general case. Write \( W = W_1 + W_2, Z = Z_1 + Z_2, W_2 = 0, Z_2(t) = \int_0^t \int_{\|u\|_{\mathbb{R}^d} > 1} \nu(ds, du) \), and denote \( X_1, Z \) the solutions to SDE of the type (8) with \( W, Z \) replaced by \( W_1, Z_1 \), respectively. Then the solution to (8) has the form \( X = X_1 + X_2 \), and \( X_1, X_2 \) are independent. The density \( p_{X(t)} \) is equal

\[ p_{X(t)}(x) = \int_{\mathbb{R}^m} p_{X_1(t)}(x - y) \mu_{X_2(t)}(dy), \quad x \in \mathbb{R}^m, \]

where \( \mu_{X_2(t)} \) denotes the law of \( X_2(t) \). We have already proved that \( p_{X_1(t)} \in S(\mathbb{R}^m) \). Thus,

\[ \frac{\partial^r}{\partial x_{j_1} \cdots \partial x_{j_m}} p_{X(t)} = \int_{\mathbb{R}^m} \frac{\partial^r}{\partial x_{j_1} \cdots \partial x_{j_m}} p_{X_1(t)}(\cdot - y) \mu_{X_2(t)}(dy), \]

\[ j_1, \ldots, j_r \in \{1, \ldots, m\}, r \in \mathbb{N}, t > 0, \] and

\[ \left\| \frac{\partial^r}{\partial x_{j_1} \cdots \partial x_{j_m}} p_{X(t)} \right\|_{L_1(\mathbb{R}^m)} \leq \int_{\mathbb{R}^m} \left\| \frac{\partial^r}{\partial x_{j_1} \cdots \partial x_{j_m}} p_{X_1(t)}(\cdot - y) \right\|_{L_1(\mathbb{R}^m)} \mu_{X_2(t)}(dy) = \left\| \frac{\partial^r}{\partial x_{j_1} \cdots \partial x_{j_m}} p_{X_1(t)} \right\|_{L_1(\mathbb{R}^m)}. \]

Theorem is proved.

Remark 6. If (21) does not hold, then \( E\|Z(t)\|_{\mathbb{R}^d}^n = +\infty \) for some \( n \in \mathbb{N} \); the typical example here is provided by the stable process with the index \( \alpha \in (0, 2) \). Taking \( d = m, A = 0, B = 0, D = I_{\mathbb{R}^m} \), we get \( E\|X(t)\|_{\mathbb{R}^m}^n = +\infty \). Therefore, the condition (21) is, in fact, necessary for the distribution density of the solution \( X(t) \) to belong to the Schwarz space \( S(\mathbb{R}^m) \).
Conclusions

In the paper conditions are established, allowing one to separate several questions that arise naturally when the local properties of the laws of the solutions to SDE’s with jump noise are studied. The questions on existence and smoothness of the distribution density appear to be essentially different. Smoothness of the density is closely related to the conditions on the behavior of the Lévy measure of the noise in the vicinity of the point 0 (Kallenberg condition and its analogue (9), condition (iii) of Theorem 1). Conditions, necessary or sufficient for the density to exist, in general, are much weaker. Moreover, the case of the equation that contains a non-degenerate drift coefficient, appears to differ essentially from the general one. For the equations with a non-degenerate drift, on the contrary to the general ones, the criteria for existence and smoothness of the distribution densities are available. In addition, non-degeneracy of the drift coefficient makes possible the ”regularization” of the distribution of the Lévy noise.

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