INVARIANT FUNCTIONS ON GRASSMANNIANS

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This article is dedicated to Professor Sigurdur Helgason on the occasion of his 80th birthday

Abstract. It is known, that every function on the unit sphere in $\mathbb{R}^n$, which is invariant under rotations about some coordinate axis, is completely determined by a function of one variable. Similar results, when invariance of a function reduces dimension of its actual argument, hold for every compact symmetric space and can be obtained in the framework of Lie-theoretic consideration. In the present article, this phenomenon is given precise meaning for functions on the Grassmann manifold $G_{n,i}$ of $i$-dimensional subspaces of $\mathbb{R}^n$, which are invariant under orthogonal transformations preserving complementary coordinate subspaces of arbitrary fixed dimension. The corresponding integral formulas are obtained. Our method relies on bi-Stiefel decomposition and does not invoke Lie theory.

Introduction

Integral formulas for semisimple Lie groups and related symmetric spaces constitute a core of geometric analysis. Many such formulas are presented in remarkable books by S. Helgason [H94]-[H01]. They are intimately connected with decompositions of the corresponding Lie algebras and Haar measures, and amount to pioneering works by H. Weyl, É. Cartan, Harish-Chandra; see bibliographical notes in [H00, p. 231]. One of such important formulas is related to the Cartan decomposition $G = KAK$. Its generalization $G = KAH$ is due to Flensted-Jensen [FJ2] in the noncompact case and Hoogenboom [Ho1, Ho2] for $G$ compact.

To be more specific, let $U$ be a connected compact real semisimple Lie group. Let $\theta$ and $\sigma$ be two commuting involutions of $U$, $U^\theta = \{ u \in U \mid \theta(u) = u \}$ (similarly for $U^\sigma$).

Let $K$ and $H$ be closed subgroups of $U$ such that

\[(U^\theta)_0 \subseteq K \subseteq U^\theta \quad \text{and} \quad (U^\sigma)_0 \subseteq H \subseteq U^\sigma,\]

where the subscript $_0$ denotes the corresponding connected component of the unity $e$. Subgroups, which obey (0.1), are called symmetric and the quotient spaces $U/K$ and $U/H$ are compact symmetric spaces. Our interest will be in the double coset space $K \backslash U / H$.

For the Lie algebra $\mathfrak{u}$ of $U$, we consider two Cartan decompositions $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{u} = \mathfrak{h} + \mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are $-1$ eigenspaces of differentials $d\theta$ and $d\sigma$ in $\mathfrak{u}$.

2000 Mathematics Subject Classification. Primary 44A12; Secondary 52A38.

Key words and phrases. Grassmann manifolds, Stiefel manifolds, invariant functions, bi-Stiefel decomposition.

The first author was supported by the NSF grant DMS 0402068. The second author was supported in part by the NSF grant DMS-0556157 and the Louisiana EPSCoR program, sponsored by NSF and the Board of Regents Support Fund.
respectively. Let $a$ be a maximal abelian subalgebra in $p \cap q$, $A = \exp(a)$. Let $M = Z_{K \cap H}(A)$ denote the centralizer of $A$ in $K \cap H$. According to [Ho1, formula (4.12)], there is a nonnegative function $\delta$ on $A$, that can be expressed in terms of sin and cos functions, the restricted roots of $a_C$ in $u_C$, and the multiplicities, so that

$$\int_{U/H} f(uH) \, du = c \int_A \int_{K/M} f(kaH) \, \delta(a) \, dk \, da, \quad f \in C(U/H).$$

The constant $c$ can be explicitly evaluated. Here, as elsewhere in this article, except where clearly stated, the invariant measure on a compact group is normalized to be one.

Formula (0.2) is a consequence of the corresponding decomposition of the Haar measure $du$ on $U$; see [Ho1, Theorem 4.7]. Fundamental results in this direction for noncompact Lie groups were first obtained by Berger [Be]; for a modern account, see Flensted-Jensen [FJ1, FJ2], [Ma]. The method by Hoogenboom [Ho1] gives integral formulas both for the noncompact and compact cases. If $f$ is left $K$-invariant then (0.2) yields

$$\int_{U/H} f(uH) \, du = c \int_A f_0(a) \, \delta(a) \, da,$$

for some function $f_0$ on $A$. The map $f \to f_0$ preserves the smoothness (or integrability) of $f$ up to the weight function $\delta$. Formula (0.3) can be applied to the study of left $K$-invariant functions $f$ on the symmetric space $U/H$.

In the present article, we obtain explicit characterization of such functions, when $U$ stands for the orthogonal group $O(n)$, $U/H$ is the Grassmann manifold

$$G_{n,i} = O(n)/(O(n-i) \times O(i)) = SO(n)/S(O(n) \times O(n-i))$$

of $i$-dimensional subspaces of $\mathbb{R}^n$, and the subgroup $K$ has the form

$$K \equiv K_\ell = \left\{ \gamma \in O(n) \mid \gamma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \alpha \in O(n-\ell), \quad \beta \in O(\ell) \right\} \sim O(n-\ell) \times O(\ell).$$

In this setting, $i$ and $\ell$ are arbitrary integers, $1 \leq i, \ell \leq n-1$. Note that $U = O(n)$, $H = O(n-i) \times O(i)$, and $K = O(n-\ell) \times O(\ell)$ are not connected, but one can show that (0.2) and (0.3) are still valid. The subgroup $K_\ell$ is symmetric in the sense that $K_\ell = O(n)^{\theta_\ell}$, where the involution $\theta_\ell$ is defined by

$$\theta_\ell(u) = I_{n-\ell,\ell} u I_{n-\ell,\ell}; \quad u \in O(n), \quad I_{n-\ell,\ell} = \begin{bmatrix} I_{n-\ell} & 0 \\ 0 & -I_\ell \end{bmatrix}.$$

In fact, one can readily see that

$$\theta_\ell \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}.$$
and intertwining operators, then $G_{n,i}$ can also be written as $\text{SL}(n, \mathbb{R})/P_i$, where $P_i$ is the parabolic subgroup

$$P_i = \left\{ \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \text{SL}(n, \mathbb{R}) \mid A \in \text{GL}(i, \mathbb{R}), B \in \text{GL}(n-i, \mathbb{R}), \ X \in \mathfrak{M}_{i,n-i} \right\}$$

and $\mathfrak{M}_{r,s} \simeq \mathbb{R}^{rs}$ stands for the space of $r \times s$ real matrices.

Let us explain the idea of the paper by the simple example $S^{n-1} = \text{SO}(n)/\text{SO}(n-1) = O(n)/O(n-1)$, where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$ with the area $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$. We fix an integer $\ell$, $1 \leq \ell \leq n-1$, and write

\begin{equation}
(0.5) \quad \mathbb{R}^n = \mathbb{R}^{n-\ell} \oplus \mathbb{R}^\ell, \quad \mathbb{R}^{n-\ell} = \bigoplus_{j=1}^{n-\ell} \mathbb{R}e_j, \quad \mathbb{R}^\ell = \bigoplus_{j=n-\ell+1}^{n} \mathbb{R}e_j,
\end{equation}

where $e_1, e_2, \ldots, e_n$ are the coordinate unit vectors. According to (0.5), every point $\theta \in S^{n-1}$ can be represented in bi-spherical coordinates as

\begin{equation}
(0.6) \quad \theta = \begin{bmatrix} u \sin \omega \\ v \cos \omega \end{bmatrix}, \quad u \in S^{n-\ell-1}, \ v \in S^{\ell-1}, \ 0 \leq \omega \leq \frac{\pi}{2},
\end{equation}

so that $d\theta = \sin^{n-\ell-1} \omega \cos^{\ell-1} \omega \, du \, dv \, d\omega$, where $d\theta, du$, and $dv$ denote the relevant (non-normalized) volume elements; see, e.g., [VK]. Clearly, $\cos^2 \omega = \theta^t \sigma \theta = \theta^t \text{Pr}_\mathbb{R} \theta$, where $\text{Pr}_\mathbb{R}$ denotes the orthogonal projection onto $\mathbb{R}^\ell$ and

\begin{equation}
(0.7) \quad \sigma = [e_{n-\ell+1}, \ldots, e_n] = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix}.
\end{equation}

The following statement is an immediate consequence of (0.6).

**Theorem 1.** For $s \in [0,1]$, let

$$dv(s) = s^{\ell/2-1}(1-s)^{(n-\ell)/2-1} \, ds.$$  

An integrable function $f$ on $S^{n-1}$ is $K_\ell$-invariant if and only if there is a function $f_0 \in L^1([0,1]; dv)$ such that $f(\theta) = f_0(s)$, where $s^{1/2} = (\theta^t \text{Pr}_\mathbb{R} \theta)^{1/2}$ is the cosine of the angle between the unit vector $\theta$ and the coordinate subspace $\mathbb{R}^\ell$. Moreover,

\begin{equation}
(0.8) \quad \int_{S^{n-1}} f(\theta) \, d\theta = c \int_0^{\pi/2} f_0(\cos^2 \omega) \sin^{n-\ell-1} \omega \cos^{\ell-1} \omega \, d\omega,
\end{equation}

where

\begin{equation}
(0.8) \quad c = \sigma_{\ell-1} \sigma_{n-\ell-1}.
\end{equation}

**1. Main results**

Let $G_{n,i}$ be the Grassmann manifold of $i$-dimensional linear subspaces $\xi$ of $\mathbb{R}^n$, $1 \leq i \leq n-1$. It is assumed, that $G_{n,i}$ is endowed with the $O(n)$-invariant measure $d\xi$ of total mass 1. For $1 \leq \ell \leq n-1$ let $m = \min\{i, \ell\}$. We will need the simplex

\begin{equation}
(1.1) \quad \Lambda_m = \{ \lambda = (\lambda_1, \ldots, \lambda_m) \mid 1 \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0 \},
\end{equation}

and the Siegel gamma function

\begin{equation}
(1.2) \quad \Gamma_m(\alpha) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2).
\end{equation}
To every subspace $\xi \in G_{n,i}$, we assign a point $\lambda = (\lambda_1, \ldots, \lambda_m)$ in $\Lambda_m$, so that $\lambda_1, \ldots, \lambda_m$ are eigenvalues of the positive semi-definite matrix

$$r = \begin{cases} \Theta^t \text{Pr}_\xi \Theta & \text{if } i \leq \ell, \\ \Psi^t \text{Pr}_\xi \Psi & \text{if } i > \ell. \end{cases}$$

Here $\Theta = (\theta_{i,j})_{n \times i}$ and $\Psi = (\psi_{i,j})_{n \times \ell}$ are arbitrary fixed matrices whose columns form an orthonormal basis in $\xi$ and $\mathbb{R}^\ell$, respectively; $\Theta^t$ and $\Psi^t$ are the corresponding transposed matrices; $\text{Pr}_\xi$ and $\text{Pr}_\xi^\ell$ denote the relevant orthogonal projections.

Clearly, $\lambda$ is independent of the choice of orthonormal bases in $\xi$ and $\mathbb{R}^\ell$.

**Theorem 2.** Assume that $1 \leq i, \ell \leq n - 1$ are such that $i + \ell \leq n$. Let $m = \min\{i, \ell\}$. For $\lambda \in \Lambda_m$, we set

$$d\nu(\lambda) = \prod_{1 \leq j < k \leq m} (\lambda_j - \lambda_k) \prod_{j=1}^m \lambda_j^\alpha (1 - \lambda_j)^\beta d\lambda_j,$$

$$\alpha = (n - \ell - i - 1)/2, \quad \beta = (\ell - i - 1)/2.$$

An integrable function $f$ on $G_{n,i}$ is $K_\ell$-invariant if and only if there is a function $f_0 \in L^1(\Lambda_m; d\nu)$ such that $f(\xi) = f_0(\lambda)$, where $\lambda$ is formed by eigenvalues of matrix (1.3). Moreover,

$$\int_{G_{n,i}} f(\xi) d\xi = c \int_{\Lambda_m} f_0(\lambda) d\nu(\lambda),$$

where

$$c = c_m \left\{ \begin{array}{ll} \Gamma_i(n/2)\Gamma_i(\ell/2)\Gamma_i((n-\ell)/2) & \text{if } i \leq \ell, \\ \Gamma_\ell(n/2)\Gamma_\ell(i/2)\Gamma_\ell((n-i)/2) & \text{if } i \geq \ell, \end{array} \right.$$  

$$c_m = \pi^{(m^2+m)/4} \left( \prod_{j=1}^m \Gamma(j/2) \right)^{-1}.$$

The geometrical meaning of $\lambda_1, \ldots, \lambda_m$ in the equality

$$f(\xi) = f_0(\lambda) \equiv f_0(\lambda_1, \ldots, \lambda_m)$$

is that $\lambda_1 = \cos^2 \omega_1, \ldots, \lambda_m = \cos^2 \omega_m$, where $\omega_1, \ldots, \omega_m$ are canonical angles, which determine the relative position of a subspace $\xi \in G_{n,i}$ with respect to the coordinate subspace $\mathbb{R}^\ell$; see, e.g., [3, 4].

The proof of Theorem 2 relies on the bi-Stiefel decomposition of the Haar measure on the Stiefel manifold [GR, Herz]; see Lemma 5. A simple proof of it, presented in Section 2, is an adaptation of the argument of Zhang [Zh] to the real case.

We conjecture, that our method extends to the hyperbolic case $\mathbb{H}^n$, when $i$-dimensional planes are substituted by $i$-dimensional totally geodesic submanifolds of $\mathbb{H}^n$. We plan to study this case in the context of related problems of integral geometry in forthcoming publications.
2. The Stiefel manifold and more notation

As before, \( M_{n,m} \simeq \mathbb{R}^{nm} \) denotes the space of real matrices \( x = (x_{i,j}) \) having \( n \) rows and \( m \) columns; \( dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j} \) is the volume element on \( M_{n,m} \). Given a square matrix \( a \), let \( |a| := |\det(a)| \). Let \( S_m \simeq \mathbb{R}^{m(m+1)/2} \) be the space of \( m \times m \) real symmetric matrices \( s = (s_{i,j}) \) with the volume element \( ds = \prod_{i \leq j} ds_{i,j} \). We denote by \( P_m \subset S_m \) the cone of positive definite matrices in \( S_m \). Given \( a \) and \( b \) in \( S_m \), the symbol \( \int_a^b f(s) \, ds \) denotes the integral over the compact set \( (a + P_m) \cap (b - P_m) \) and \( \int_a^\infty f(s) \, ds \) means the integral over \( a + P_m \). The group \( G = GL(m, \mathbb{R}) \) acts transitively on \( P_m \) by the rule \( g \cdot r := grg^t \), \( g \in G \). The corresponding \( G \)-invariant measure on \( P_m \) is

\[
d^* r = |r|^{-d} dr, \quad d = (m + 1)/2,
\]

[11, p. 18]. For \( n \geq m \), let \( V_{n,m} = \{ v \in M_{n,m} \mid v^tv = I_m \} \) be the Stiefel manifold of orthonormal \( m \)-frames in \( \mathbb{R}^n \). The group \( O(n) \) acts transitively on \( V_{n,m} \) by the rule \( \gamma : v \mapsto \gamma v, \gamma \in O(n) \), in the sense of matrix multiplication. Let

\[
\sigma_m = \begin{bmatrix} 0 \\ I_m \end{bmatrix}.
\]

Most of the time we simply write \( \sigma \) for \( \sigma_m \). The stabilizer of \( \sigma \) is \( O(n-m) \simeq \{ \begin{bmatrix} A & 0 \\ 0 & I_m \end{bmatrix} \mid A \in O(n-m) \} \).

Hence \( V_{n,m} = O(n)/O(n-m) \). We fix the corresponding \( O(n) \)-invariant measure \( dv \) on \( V_{n,m} \) so that

\[
\int_{V_{n,m}} dv = \sigma_{n,m} = \frac{\sqrt{\pi}^m \pi^{nm/2}}{\Gamma_m(n/2)},
\]

\( \Gamma_m(\cdot) \) being the Siegel gamma function [12]; see [Mu, p. 70], [11, p. 57], [FK, p. 351]. The measure \( dv \) is also right \( O(m) \)-invariant.

Let

\[
M^*_{n,m} = \{ x \in M_{n,m} \mid \text{rank}(x) = m \}.
\]

Then \( M^*_{n,m} \) is open, dense and of full measure in \( M_{n,m} \). Define

\[
\varphi : V_{n,m} \times P_m \to M^*_{n,m}, \quad (v, r) \mapsto x = vr^{1/2}.
\]

Then \( \varphi \) is surjective and \( r = x^t x \in P_m \) depends smoothly on \( x \). The following lemma describes the measure \( dx \) on \( M_{n,m} \) in terms of \( V_{n,m} \) and \( P_m \).

**Lemma 3.** Assume that \( n \geq m \). Let the notation be as above. Then

\[
dx = 2^{-m} |r|^{n/2} d_r dv.
\]

The polar decomposition [2,1] can be found in many sources, e.g., in [Herz, p. 482], [GK1, p. 93], [Mu, pp. 66, 591], [FT, p. 130].

The next statement, which is actually due to Zhang [Zh], contains a higher-rank generalization of the polar decomposition of the Lebesgue measure in the quarter-plane.
**Lemma 4.** Let $F$ be a function on $\mathcal{P}_m \times \mathcal{P}_m$, $d = (m + 1)/2$. Then

\begin{equation}
(2.5) \quad \int_{\mathcal{P}_m \times \mathcal{P}_m} F(p_1, p_2) \, d\mu_1 \, d\mu_2 = \int_0^I |I_m - r|^{-d} \, ds \int_{\mathcal{P}_m} F(s^{1/2} r s^{1/2}, s^{1/2}(I_m - r)s^{1/2}) \, d\mu_1 \, d\mu_2
\end{equation}

provided that either of these integrals exists in the Lebesgue sense.

**Proof.**

\begin{align*}
\text{l.h.s.} & = \int_{\mathcal{P}_m} d\mu_1 \int_{\mathcal{P}_m} F(p_1, p_1 + p_2 - p_1)|p_2|^{-d} \, dp_2 \quad \text{(set } p_1 + p_2 = s) \\
& = \int_{\mathcal{P}_m} d\mu_1 \int_{-p_1}^\infty F(p_1, s - p_1)|s - p_1|^{-d} \, ds \\
& = \int_{\mathcal{P}_m} ds \int_0^s F(p_1, s - p_1)|s - p_1|^{-d} \, dp_1 \quad \text{(set } p_1 = s^{1/2} r s^{1/2}) \\
& = \int_{\mathcal{P}_m} ds \int_0^I F(s^{1/2} r s^{1/2}, s^{1/2}(I_m - r)s^{1/2}) |I_m - r|^{-d} \, ds \\
\end{align*}

and (2.5) follows. \hfill \Box

**Lemma 5.** (bi-Stiefel decomposition) Let $k$, $m$, and $n$ be positive integers satisfying

\[ 1 \leq k \leq n - 1, \quad 1 \leq m \leq n - 1, \quad m \leq \min(k, n - k). \]

Almost all matrices $v \in V_{n,m}$ can be represented in the form

\begin{equation}
(2.6) \quad v = \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_m - r)^{1/2} \end{bmatrix}, \quad u_1 \in V_{n-k,m}, \quad u_2 \in V_{k,m},
\end{equation}

so that

\begin{equation}
(2.7) \quad \int_{V_{n,m}} f(v) \, dv = \int_0^I d\mu(r) \int_{V_{n-k,m}} du_1 \int_{V_{k,m}} f \left( \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_m - r)^{1/2} \end{bmatrix} \right) \, du_2,
\end{equation}

\[ d\mu(r) = 2^{-m} |I_m - r|^{(k-m-1)/2} |r|^{(n-m-k-1)/2} \, dr. \]

**Proof.** For $m=1$, this is a well-known bi-spherical decomposition [VK pp. 12, 22]. For $k = m$, see [Herz p. 495], where the result was obtained using the Fourier transform technique and Bessel functions of matrix argument. In [GR], Herz’s proof was extended to the form presented above and it was conjectured that there is an alternative simple proof that does not need the Fourier transform. Such a proof was given by Zhang [Zh]. For convenience of the reader, we present it here in a slightly different notation.

The result will follow from the bi-polar decomposition of the Lebesgue measure on $\mathcal{M}_{n,m}$. We split $x \in \mathcal{M}_{n,m}$ in two blocks, so that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_1 \in \mathcal{M}_{n-k,m}$, $x_2 \in \mathcal{M}_{k,m}$, and write each block in polar coordinates according to Lemma 3. This
gives
\[ \int_{\mathfrak{m}_{n,m}} f(x) dx = \int_{V_{n-k,m}} du_1 \int_{V_{k,m}} du_2 \]
\[ \times \int_{P_m \times P_m} f \left( \begin{bmatrix} u_1 r^{1/2} \\ u_2 (s^{1/2}/(I_m - r)s^{1/2})^{1/2} \end{bmatrix} \right) h(p_1, p_2) d_1 p_1 d_2 p_2, \]
\[ (2.8) \]
\[ h(p_1, p_2) = 2^{-2m} |p_1|^{(n-k)/2} |p_2|^{k/2}. \]

By (2.5), the integral over \( P_m \times P_m \) transforms as
\[ \int_0^1 dr \int_{P_m} f \left( \begin{bmatrix} u_1 r^{1/2} \\ u_2 (s^{1/2}/(I_m - r)s^{1/2})^{1/2} \end{bmatrix} \right) \tilde{h}(r, s) d_1 s, \]
where \( \tilde{h}(r, s) = 2^{-2m} |I_m - r|^{(k-m-1)/2} |s|^{(n-m-k-1)/2} |s|^{n/2}. \)

Furthermore, one can write
\[ (s^{1/2}r s^{1/2})^{1/2} = \gamma_1 r^{1/2} s^{1/2}, \quad (s^{1/2}/(I_m - r)s^{1/2})^{1/2} = \gamma_2 (I_m - r)^{1/2} s^{1/2}, \]
for some \( \gamma_1, \gamma_2 \in O(m) \) (just note that \( |\gamma_1| = |\gamma_2| = 1 \)). Hence, changing the order \( O(m) \)-invariance of \( du_1 \) and \( du_2 \), we easily get
\[ \int_{\mathfrak{m}_{n,m}} f(x) dx = 2^{-m} \int_{P_m} |s|^{n/2} d_1 s \int_0^1 d_2 r \]
\[ \times \int_{V_{n-k,m}} du_1 \int_{V_{k,m}} f \left( \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_m - r)^{1/2} \end{bmatrix} s^{1/2} \right) d_2 u_2, \]
where \( d_2 r = 2^{-m} |I_m - r|^{(k-m-1)/2} |s|^{(n-m-k-1)/2} dr \). On the other hand, by Lemma \( \ref{lem:transform} \)
\[ \int_{\mathfrak{m}_{n,m}} f(x) dx = 2^{-m} \int_{P_m} |s|^{n/2} d_1 s \int_{V_{n-m}} f(v s^{1/2}) dv. \]

Comparing (2.11) and (2.12), we get the result.

\[ \square \]

3. Proof of Theorem 2

We divide the proof into two parts.

Part I. Let \( i \leq \ell \), let \( \sigma_i \) be as in (2.2), and let
\[ \kappa_i : V_{n,i} = O(n)/O(n-i) \to G_{n,i} = O(n)/O(n-i) \times O(i), \quad g \cdot \sigma_i \to g\sigma_i \]
be the canonical map. It is obviously \( O(n) \)-equivariant. We use this to identify a \( K_\ell \)-invariant function \( f \) on \( G_{n,i} \) with the left \( K_\ell \)-invariant and right \( O(i) \)-invariant function \( \varphi = f \circ \kappa_i \) on the Stiefel manifold \( V_{n,i} \). Then
\[ (3.1) \]
\[ \int_{G_{n,i}} f(\xi) d\xi = \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi(\Theta) d\Theta. \]

By Lemma (with \( m = i, k = \ell \)), almost all \( \Theta \in V_{n,i} \) can be represented as
\[ \Theta = \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_i - r)^{1/2} \end{bmatrix}, \quad u_1 \in V_{n-\ell,i}, \quad u_2 \in V_{\ell,i}, \quad r \in (0, I_i). \]

\[ \Theta = \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_i - r)^{1/2} \end{bmatrix}, \quad u_1 \in V_{n-\ell,i}, \quad u_2 \in V_{\ell,i}, \quad r \in (0, I_i). \]
Since \( \varphi \) is left \( K_\ell \)-invariant, it follows that \( \varphi \) is independent of the bi-Stiefel coordinates \( u \) and \( v \) in (3.2). Thus we can write \( \varphi(\Theta) \equiv \varphi_0(r) \). By (2.7) and (2.3) we get

\[
(3.3) \quad \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi(\Theta) \, d\Theta = c \int_0^{I_i} \varphi_0(r) \, dv_1(r),
\]

where

\[
(3.4) \quad dv_1(r) = |I_i - r|^{(\ell - i - 1)/2} |r|^{(n - \ell)/2} \, ds r,
\]

\[
(3.5) \quad c = \frac{\sigma_{n-i,1} \sigma_{\ell,i}}{2^i \sigma_{n,i}} = \frac{\Gamma_i(n/2)}{\Gamma_i(\ell/2) \Gamma_i((n - \ell)/2)}
\]

Since \( r \) can be expressed through \( \Theta \) as \( r = \Theta^t \sigma \Theta \) and \( \varphi \) is right \( O(i) \)-invariant we obtain

\[
\varphi_0(r) = \varphi_0(\Theta^t \sigma \Theta) = \varphi(\Theta) = \varphi(\Theta g) = \varphi_0(\gamma^t \Theta^t \sigma \Theta \gamma) = \varphi_0(\gamma^t r \gamma)
\]

for any \( \gamma \in O(i) \). Hence, if we write \( r \) in polar coordinates

\[
(3.6) \quad r = \gamma^t \lambda \gamma, \quad \gamma \in O(i), \quad \lambda = \text{diag}(\lambda_1, \ldots, \lambda_i),
\]

where \( \lambda_1, \ldots, \lambda_i \) are eigenvalues of \( r \), we get \( \varphi_0(r) = \varphi_0(\lambda) \). Moreover, using the known formula for the invariant measure

\[
(3.7) \quad ds r = c_i \prod_{1 \leq j < k \leq i} (\lambda_j - \lambda_k) \left( \prod_{j=1}^i \lambda_j^{-(i+1)/2} \, d\lambda_j \right) \, d\gamma
\]

where

\[
c_i = \pi^{(i^2 + i)/4} \left( \prod_{j=1}^i j \Gamma(j/2) \right)^{-1},
\]

(see [1] p. 23, 43), we obtain

\[
\int_0^{I_i} \varphi_0(r) \, dv_1(r) = c_i \int_{\Lambda_i} \varphi_0(\lambda) \prod_{1 \leq j < k \leq i} (\lambda_j - \lambda_k)
\]

\[
\times \prod_{j=1}^i \lambda_j^{(n - \ell - i - 1)/2} (1 - \lambda_j)^{(\ell - i - 1)/2} \, d\lambda_j.
\]

Combining this formula with (3.1) and (3.3), we obtain (3.4).

Conversely, to every function \( f_0 \) on \( \Lambda_i \), we can assign a function \( \varphi_0 \) on \((0, I_i)\) by the rule \( \varphi_0(r) = f_0(\lambda_1, \ldots, \lambda_i) \), where \( \lambda_1, \ldots, \lambda_i \) are eigenvalues of \( r \) arranged as \( 1 \geq \lambda_1 \geq \cdots \geq \lambda_i \geq 0 \). The function \( \varphi_0 \) is \( O(i) \)-invariant, i.e.,

\[
(3.8) \quad \varphi_0(r) = \varphi_0(g^t r g) \quad \text{for any} \quad g \in O(i),
\]

because \( r \) and \( g^t r g \) have the same eigenvalues. Next we define a function \( \varphi \) on \( V_{n,i} \) by \( \varphi(\Theta) = \varphi_0(\Theta^t \sigma \Theta) \), where \( \sigma = \sigma_i \in V_{n,\ell} \). The function \( \varphi \) is left \( K_\ell \)-invariant, because for any \( \gamma \in K_\ell \) we have \( \gamma^t \sigma \gamma = \sigma \gamma^t \) and, therefore,

\[
\varphi(\gamma \Theta) = \varphi_0(\Theta^t (\gamma \sigma \gamma) \Theta) = \varphi_0(\Theta^t \sigma \Theta) = \varphi(\Theta).
\]

It remains to note that, owing to (3.8), \( \varphi \) is right \( O(i) \)-invariant and can therefore be identified with a \( K_\ell \)-invariant function on \( G_{n,i} \).
Part II. Let \( i \geq \ell \). Suppose that \( f \) is a \( K_\ell \)-invariant function on \( G_{n,i} \), which is identified with a function \( \varphi(\Theta) \) on \( V_{n,i} \) as in Part I. Let \( \Theta = \begin{bmatrix} a \\ b \end{bmatrix} \), \( a \in \mathfrak{m}_{n-\ell,i} \), \( b = \sigma^t \Theta \in \mathfrak{m}_{n,i} \). Since \( n-\ell \geq i \), by Lemma 3, we can write \( a \) in polar coordinates as \( a = vr^{1/2}, v \in V_{n-\ell,i}, r = a^t a = I - b'b \). Fix any \( v_0 \) in \( V_{n-\ell,i} \) and set \( v = av_0 \), \( \alpha \in \text{O}(n-\ell) \). Then

\[
\Theta = \begin{bmatrix} \alpha v_0 r^{1/2} \\ b \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & I_\ell \end{bmatrix} \begin{bmatrix} v_0(I - b'b)^{1/2} \\ b \end{bmatrix}.
\]

Since \( \varphi \) is left \( K_\ell \)-invariant, then

\[
\varphi(\Theta) = \varphi \left( \begin{bmatrix} v_0(I - b'b)^{1/2} \\ b \end{bmatrix} \right) \equiv \varphi_1(b), \quad b = \sigma^t \Theta.
\]

We write the transpose of \( b \) in polar coordinates

\[
b^t = us^{1/2}, \quad u \in V_{i,\ell}, \quad s = bb^t \in \mathcal{P}_\ell.
\]

Then we fix any \( u_0 \in V_{i,\ell} \) and replace \( u \) by \( \beta u_0 \) for some \( \beta \in \text{O}(i) \). This gives \( b = s^{1/2}u_0^t \beta^t \) and therefore, since \( \varphi \) is right \( \text{O}(i) \)-invariant,

\[
\varphi(\Theta) = \varphi(\Theta \beta) = \varphi_1(\sigma^t \Theta \beta) = \varphi_1(b \beta) = \varphi_1(s^{1/2}u_0^t \beta^t \beta) = \varphi_1(s^{1/2}u_0^t).
\]

It means that \( \varphi(\Theta) \) is actually a function of \( s \). Denote it by \( \varphi_0(s) \). Since \( b = \sigma^t \Theta \) the positive definite matrix \( s = bb^t = \sigma^t \Theta \Theta^t \sigma = \sigma^t \text{Pr}_{\xi} \sigma \) lies in the “interval” \((0, I_\ell)\). Here \( \text{Pr}_{\xi} \sigma \) denotes the orthogonal projection of \( \sigma \in V_{n,i} \) onto the subspace \( \xi \in \text{G}_{n,i} \). Thus

\[
f(\xi) \equiv \varphi(\Theta) = \varphi_0(\sigma^t \Theta \Theta^t \sigma) = \varphi_0(\sigma^t \text{Pr}_{\xi} \sigma).
\]

Since \( \varphi \) is left-invariant under left translation by \( \tilde{\beta} = \begin{bmatrix} I_{n-\ell} & 0 \\ 0 & \beta \end{bmatrix}, \beta \in \text{O}(\ell), \) i.e.,

\[
f(\tilde{\beta} \xi) = f(\xi), \text{ it follows that } \varphi_0(\sigma^t \text{Pr}_{\tilde{\beta} \xi} \sigma) = \varphi_0(\sigma^t \text{Pr}_{\xi} \sigma), \text{ and therefore}
\]

\[
\varphi_0(s) = \varphi_0(\sigma^t \text{Pr}_{\xi} \sigma) = \varphi_0(\sigma^t \text{Pr}_{\tilde{\beta} \xi} \sigma) = \varphi_0(\sigma^t \tilde{\beta} \Theta \Theta^t \tilde{\beta}^t \sigma).
\]

Since

\[
\tilde{\beta}^t \sigma = \begin{bmatrix} I_{n-\ell} & 0 \\ 0 & \beta^t \end{bmatrix} \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \begin{bmatrix} 0 \\ \beta^t \end{bmatrix} = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \beta^t = \sigma \beta^t,
\]

equation 10 implies that for all \( \beta \in \text{O}(\ell) \) we have

\[
\varphi_0(s) = \varphi_0(\beta \sigma^t \Theta \Theta^t \sigma \beta^t) = \varphi_0(\beta s \beta^t).
\]

Thus, \( \varphi_0 \) depends only on the eigenvalues \( \lambda_1, \ldots, \lambda_\ell \) of \( s = \sigma^t \text{Pr}_{\xi} \sigma \), \( \varphi_0(s) = \varphi_0(\text{diag}(\lambda_1, \ldots, \lambda_\ell)) \). Finally, we write

\[
f(\xi) = \varphi_0(\text{diag}(\lambda_1, \ldots, \lambda_\ell)) \equiv f_0(\lambda_1, \ldots, \lambda_\ell).
\]
For the corresponding integral formula we have
\[
I = \int_{G_{n,i}} f(\xi) \, d\xi = \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi(\Theta) \, d\Theta
\]
\[
= \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi_0(\sigma^t \Theta \Theta^t \sigma) \, d\Theta
\]
(replace \( \Theta \) by \( g \sigma_i, \ g \in O(n), \ \sigma_i = \begin{bmatrix} 0 & I_i \end{bmatrix} \in V_{n,i} \))
\[
= \int_{O(n)} \varphi_0(\sigma^t g \sigma_i \sigma^t g \sigma) \, dg = \frac{1}{\sigma_{n,\ell}} \int_{V_{n,\ell}} \varphi_0(\Psi^t \sigma_i \sigma^t \Psi) \, d\Psi.
\]
The last integral can be written in bi-Stiefel coordinates by setting
\[
\Psi = \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_{\ell} - r)^{1/2} \end{bmatrix}, \quad u_1 \in V_{n-i,\ell}, \quad u_2 \in V_{i,\ell}, \quad r = \Psi^t \sigma_i \sigma^t \Psi.
\]
Then Lemma 5 gives
\[
(3.12) \quad I = c \int_0^{I_{\ell}} \varphi_0(r) \, dv_2(r),
\]
where
\[
(3.13) \quad dv_2(r) = |I_{\ell} - r|^{(i-\ell-1)/2} |r|^{(n-\ell-i-1)/2} \, dr,
\]
\[
(3.14) \quad c = \frac{\sigma_{n-i,\ell} \sigma_{i,\ell}}{2^\ell \sigma_{n,\ell}} = \frac{\Gamma((n/2)) \Gamma((n-i)/2)}{\Gamma((i/2)) \Gamma((n-\ell)/2)}.
\]
Since \( \varphi_0 \) is \( O(\ell) \)-invariant (see (3.11)), we can write (3.12) in polar coordinates in the required form (1.4).

Conversely, as in Part I, every function \( f_0 \) on \( \Lambda_{\ell} \) can be associated with an \( O(\ell) \)-invariant function \( \varphi_0(0, I_{\ell}) \), and the latter generates a function \( \varphi(\Theta) = \varphi_0(\sigma^t \Theta \Theta^t \sigma) \). This function is left \( K_{\ell} \)-invariant on \( V_{n,i} \). Indeed, let \( \gamma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in K_{\ell}, \ alpha \in O(n-\ell), \ \beta \in O(\ell) \). Then \( \varphi(\gamma \Theta) = \varphi_0(\sigma^t \gamma \Theta \gamma^t \sigma) \). Since
\[
\sigma^t \gamma = \begin{bmatrix} 0 & I_{\ell} \\ \alpha & 0 \\ 0 & \beta \end{bmatrix} = [0, \beta] = \beta \sigma^t,
\]
then \( \varphi(\gamma \Theta) = \varphi_0(\beta \sigma^t \Theta \sigma^t \beta^t) = \varphi_0(\sigma^t \Theta \sigma) = \varphi(\Theta) \). Furthermore, \( \varphi(\Theta) \) is obviously right \( O(i) \)-invariant, and therefore, can be identified with a \( K_{\ell} \)-invariant function on \( G_{n,i} \).

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