MULTIPLICITY ESTIMATES, ANALYTIC CYCLES AND NEWTON POLYTOPES

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ABSTRACT. We consider the problem of estimating the multiplicity of a polynomial when restricted to the smooth analytic trajectory of a (possibly singular) polynomial vector field at a given point or points, under an assumption known as the D-property. Nesterenko has developed an elimination theoretic approach to this problem which has been widely used in transcendental number theory.

We propose an alternative approach to this problem based on more local analytic considerations. In particular we obtain simpler proofs to many of the best known estimates, and give more general formulations in terms of Newton polytopes, analogous to the Bernstein-Kushnirenko theorem. We also improve the estimate’s dependence on the ambient dimension from doubly-exponential to an essentially optimal single-exponential.

1. INTRODUCTION

Let $\xi$ be a polynomial vector field in $M = \mathbb{C}^n$ with the coordinates $(x_1, \ldots, x_n)$,

$$\xi = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \quad \max_i \deg \xi_i = \delta. \quad (1)$$

Let $p \in M$ and $\gamma_p$ a smooth holomorphic $\xi$-invariant curve through $p$ which is not a subset of the singular locus of $\xi$. If $\xi$ is non-singular at $p$ then $\gamma_p$ is the unique trajectory of $\xi$ through $p$; otherwise $\gamma_p$ is a smooth analytic separatrix of $\xi$ through $p$.

Alternatively, using the coordinates $(z, x_1, \ldots, x_n)$ on $M = \mathbb{C}^{n+1}$ we may consider a system of (non-linear) polynomial differential equations

$$\frac{\partial x_i}{\partial z} = \frac{P_i(z, x_1, \ldots, x_n)}{Q_i(z, x_1, \ldots, x_n)}, \quad \deg P_i, Q_i \leq \delta, \quad i = 1, \ldots, n \quad (2)$$

and their solution $f = (f_1, \ldots, f_n)$, viewed as a vector of functions in the variable $z$ defined and holomorphic in some neighborhood of $z_0 \in \mathbb{C}$. We allow $p = (z_0, f(z_0))$ to be a singular point of (2) as long as the graph of $f$ in a neighborhood of $z_0$ lies outside of the singular locus. With this definition, the graph of $f$ forms a smooth holomorphic trajectory $\gamma_p$ of the vector field corresponding to (2).

Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with $\deg P = d$, and suppose that $P|_{\gamma_p} \neq 0$. We consider the following question: can one give an upper bound for $\text{mult}_P P|_{\gamma_p}$ in terms of the parameter $d$? More specifically, one may look for an explicit bound in terms of the parameters $n, \delta, d$, or for a bound involving existential constants depending on $\gamma_p$. More generally, one may attempt to bound the sum of

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the multiplicities of $P$ over a given set of points $p_1, \ldots, p_\nu$, describe the geometry of the locus of points $p$ where the multiplicity is greater than a given value, etc. The answers to various question of this form are referred to as multiplicity estimates.

**Remark 1.** Assuming that $\gamma_p$ is not contained in any proper algebraic set, it follows from basic linear algebra that for an appropriate choice of $P$ we have $\text{mult}_{\gamma_p} P \geq \dim L(d) - 1$, where $L(d)$ denotes the space of polynomials of degree bounded by $d$. Thus with respect to $d$, up to a multiplicative constant, the best possible multiplicity estimate is of order $d^n$.

More generally, one may replace the assumption $\deg P = d$ by a more refined restriction, for instance assigning different degrees to each variable. Two main cases have been considered in the literature: estimates are given either in terms of a single degree $d$ as above, or in terms of the two degrees $d_x = \deg_x P$ and $d_z = \deg_z P$. For simplicity we will refer to the former as pure degree and the latter as mixed degree, although to our knowledge only mixed degrees of the specific type above have been considered in the literature.

In our approach the estimates for different types of degrees are subsumed by a general estimate given in terms of the Newton polytope of $P$, as explained in §1.3.2. Arbitrary mixed degrees are obtained as a special case.

1.1. **Historical review.** Multiplicity estimates have been considered by authors in various areas of mathematics. We list some key contributions below, making no attempt at a comprehensive review.

Multiplicity estimates have been extensively used in transcendental number theory, starting with the work of Siegel [38] and Shidlovskii [37] on the class of E-functions. Nesterenko, motivated by the study of E-functions, introduced elimination theoretic ideas for the study of multiplicity estimates for general linear systems of differential equations in [23, 24]. Further results in the linear context have been studied by Nesterenko [26, 29] and Nguyen [39] using similar methods, and by Bertrand and Beukers [3] using a different approach.

The case of non-linear non-singular systems was considered by Brownawell and Masser in [6, 7] and by Brownawell in [8, 5]. These results were consequently improved to an essentially optimal result by Nesterenko in [28]. The corresponding result for singular systems was established by Nesterenko in [30] (see also [31]), and somewhat generalized by Dolgalev in [9]. We mention also that numerous important results have been obtained in the more refined context of invariant vector fields on commutative group varieties, for instance by Masser and Wüstholz [20, 21] and Philippon [34] (see [19] for a survey).

In control theory, Risler [36] showed how multiplicity estimates could be used in the study of nonholonomic systems, and carried out the program in the planar case, giving a bound for the degree of non-holonomy of a polynomial system. Gabrielov and Risler [13] established a similar result in dimension 3. Multiplicity estimates in arbitrary dimension were given by Gabrielov in [11] and significantly improved in [12], giving for the first time an estimate exhibiting simple exponential growth with respect to the dimension.

In the theory of dynamical systems multiplicity estimates have been studied with the motivation of obtaining bounds on the bifurcation of limit cycles in perturbations of Hamiltonian systems, for instance in the work of Novikov and Yakovenko [33] and Moura [22]. Results about bifurcation of zeros in analytic families have
been established by Yomdin in [40] with the help of Gabrielov's multiplicity estimate.

1.2. The D-property and multiplicity estimates at singular points. We now focus our attention on the case where $p$ is a singular point of $\xi$. In this case, it is in general not possible to give a multiplicity estimate depending only on $n, \delta, d$. For instance, the linear field $x\partial_x + ay\partial_y$ with $a \in \mathbb{N}$ admits a smooth trajectory $\gamma = \{y = x^a\}$ through the origin, and for $P = y$ we have $\text{mult}_0 P|_{\gamma} = a$. However, one may still hope that for a fixed smooth analytic trajectory $\gamma_p$ it is possible to give a good multiplicity estimate with respect to the degree $d$. Toward this end, Nesterenko has introduced the following fundamental definition [25, 30].

Definition 2. Let $\gamma_p$ be a smooth analytic trajectory of $\xi$ through the point $p$. Then $\gamma_p$ is said to satisfy the D-property (with constant $\chi$) if for any $\xi$-invariant variety $V \subset \mathbb{C}^n$, there exists a polynomial $P$ which vanishes identically on $V$ and satisfies $\text{mult}_p P|_{\gamma} \leqslant \chi$.

If $\xi$ is non-singular at $p$ and $\gamma_p$ is not contained in a proper algebraic subset, then the D-property is automatically satisfied with the constant $\chi = 0$: there are no proper $\xi$-invariant algebraic subsets containing $p$ (see §5.3.3 for a discussion of the situation where $\gamma_p$ is contained in a proper algebraic subset).

When $p$ is a singular point of $\xi$ the D-property is non-trivial. Nesterenko has established that the property holds in two main cases (each with different applications in transcendental number theory):

(1) In his study of E-functions, Nesterenko [25] has used differential Galois theory to show that if the system (2) is linear and $f$ is a completely transcendental solution (i.e., $f$ does not satisfy any nontrivial polynomial relation over $\mathbb{C}(z)$) then $f$ automatically satisfies the D-property whenever it is holomorphic, with some suitable constant $\chi$.

More specifically, it is shown in [25] that if (2) admits at least one completely transcendental solution then there exist finitely many $\xi$-invariant proper varieties that are maximal with respect to inclusion. The result easily follows since the graph of $f$ is not contained in any of these varieties, and therefore has finite order contact with them.

(2) In his celebrated work on the algebraic independence of $\pi, e^\pi$ Nesterenko [30] considered the Ramanujan functions

$$P(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)z^n$$

$$Q(Z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)z^n$$

$$R(Z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)z^n$$

and the corresponding system of differential equations (due to Ramanujan [35]),

$$z \frac{\partial P}{\partial z} = \frac{1}{12} (P^2 - Q), \quad z \frac{\partial Q}{\partial z} = \frac{1}{3} (PQ - R), \quad z \frac{\partial R}{\partial z} = \frac{1}{2} (PR - Q^2).$$
In [30] it is proved that the holomorphic solution \( P, Q, R \) satisfies the D-property at \( z = 0 \) with the constant \( \chi = 2 \). Theorem 1 below, applied to the Ramanujan functions, was the main novel ingredient in [30], giving general results on the transcendence properties of modular functions and in particular the algebraic independence of \( \pi, e^\pi, \Gamma(1/4) \).

Nesterenko has developed a powerful technique for proving multiplicity estimates based on elimination theoretic methods (see the appendix for a review). The \( \xi \)-invariant prime ideals play a natural role as a basis for an induction over dimension, and the D-property establishes the inductive hypothesis for this basis. These ideas have been developed in [24, 26, 29] for linear systems, in [28] for non-linear systems at nonsingular points, and culminated in [30] with the following formulation, valid for general non-linear systems with singular points.

**Theorem 1** ([30, Theorem 3]). Let \( z_0 \in \mathbb{C} \) and suppose that \( f(z) \) has the D-property at \( z_0 \). Let \( P \) be a polynomial with \( P(z, f(z)) \not\equiv 0 \), and denote

\[
d_x := \max(\deg_x P, 1) \quad d_z := \max(\deg_z P, 1)
\]

Then

\[
\text{mult}_{z=p} P(z, f_1(z), \ldots, f_n(z)) \leq \alpha_f d_z d_x^n
\]

where \( \alpha_f \) is a constant depending only on \( f \).

Once again, assuming that \( f(z) \) satisfies no algebraic relations over \( \mathbb{C}(z) \), this result is optimal with respect to \( d_z, d_x \) up to the precise multiplicative constant.

The problem of estimating the sum of multiplicities over several points was also considered by various authors [29, 3, 39, 28, 30, 9]. The following result is essentially optimal for the most general case of nonlinear systems with singularities.

**Theorem 2** ([30, Theorem 6]). Let \( z_1, \ldots, z_\nu \in \mathbb{C} \) and suppose that \( f(z) \) has the D-property at \( z_1, \ldots, z_\nu \). Let \( P \) be a polynomial with \( P(z, f(z)) \not\equiv 0 \), and denote

\[
d_x := \max(\deg_x P, 1) \quad d_z := \max(\deg_z P, 1)
\]

Then

\[
\sum_{i=1}^\nu \text{mult}_{z=p_i} P(z, f_1(z), \ldots, f_n(z)) \leq \alpha_f (d_z + \nu) d_x^n
\]

where \( \alpha_f \) is a constant depending only on \( f \) and the points \( p_1, \ldots, p_\nu \).

1.3. **Synopsis of this paper.** In this paper we present an alternative approach to the study of multiplicity estimates. In particular we give new proofs for the multiplicity estimates presented in [12] as well as their generalizations given in [9]. We formulate and prove generalizations of all of these results in terms of Newton polytopes. We also improve the dependence of the multiplicative constants on the dimension \( n \) from a double-exponential dependence to a single-exponential dependence.

In §1.3.1 we give an outline of our approach to the proof of multiplicity estimates in the pure degree case. In §1.3.2 we discuss the generalization of these multiplicity estimates to the context of the theory of Newton polytopes. In §1.3.3 we describe the organization of the paper.

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1. see §5.3.3 for more refined formulations taking algebraic relations among the functions \( f_1, \ldots, f_n \) into account
1.3.1. The pure degree case. We begin by describing our approach in the case of pure degrees. A general paradigm for proving multiplicity estimates, which has been used by different authors in various ways, is as follows:

- Associate to each ideal a multiplicity, such that the multiplicity of the principal ideal generated by $P$ agrees with $\text{mult}_p P|_{\gamma_p}$.
- Show how to construct from a given ideal $I$ a new ideal $J$, of smaller dimension (using the derivative operator $\xi$), such that the multiplicity of $I$ is bounded in terms of the multiplicity of $J$.
- Show that one eventually obtains an ideal for which the multiplicity is known, for example the whole ring.

For instance, in [11, 33] this paradigm was used, where the multiplicity associated to $I$ is $\min_{F \in I} [\text{mult}_p F|_{\gamma_p}]$. In Nesterenko’s approach a more refined notion of multiplicity for unmixed ideals was used (see the appendix for details).

Our approach follows the same paradigm, working with algebraic cycles in place of ideals. We introduce a local analytic notion of the multiplicity of an analytic cycle along $\gamma_p$ (see Definition [4]). Namely, for any germ of an irreducible analytic variety $V \subset M$ at $p$, we define the multiplicity of $V$ along $\gamma_p$, denoted $\text{mult}_{\gamma_p} V$ to be the Samuel multiplicity of the ideal of functions vanishing on $\gamma_p$, restricted to the local ring $O_{V,p}$ of germs of regular functions on $V$ at the point $p$. We extend this notion, by linearity, to arbitrary analytic cycles.

Let $V$ be a germ of an analytic set at $p$ and $f$ the germ of an analytic function with $f|_V \equiv 0$ and $g = \xi f$ with $g|_V \not\equiv 0$. Let $\Gamma$ denote the intersection cycle $V \cdot V(g)$. We prove the estimate $\text{mult}_{\gamma_p} V \leq \text{mult}_p V + \text{mult}_{\gamma_p} \Gamma$ where $\text{mult}_p V$ denotes the multiplicity of the analytic germ $V$ at $p$ (see Lemma [7]). This may be viewed as a local form of Rolle’s lemma, relating the number of zeros of a function (or in the local case, the multiplicity) to the zero locus of its derivative.

We recursively define a forest (union of trees) where each node is an irreducible variety with an associated multiplicity, as follows:

1. The roots are given by the components of $V(P)$ with their multiplicities.
2. If $n[V]$ is a node and $V$ is a point, or is contained in a proper $\xi$-invariant variety, then this node is a leaf.
3. Otherwise, we choose a polynomial $F$ vanishing on $V$ with $G = \xi F$ and $G|_V \not\equiv 0$ (as explained below), and let the children of $n[V]$ be the components of the intersection cycle $n[V] \cdot V(G)$.

Assume now that $n[V]$ is a node of the forest above, which is not contained in any proper $\xi$-invariant variety. In this case we show that one can always choose a polynomial $G$ as in item (3) above, with $\deg G \leq \deg P + \tilde{a}_{n,\delta}$ where $\tilde{a}_{n,\delta}$ is some universal constant depending only on $n, \delta$ and growing singly-exponentially with $n$ (see Lemma [12] and Theorem [5]).

This step is similar to a lemma appearing in the work of Nesterenko (see the appendix for details). However, our approach to the proof is different, relying on local analytic considerations concerning the multiplicities of analytic germs. This allows us to give a shorter and more transparent proof, and improve the growth of the constants from doubly-exponential in Nesterenko’s theorem to single-exponential. Later this also allows us to relatively easily extend our arguments from the pure degree case to general Newton polytopes.

We are now ready to complete the argument. Suppose that $\gamma_p$ satisfies the D-property with constant $\chi$. We show that the multiplicity $\text{mult}_p P|_{\gamma_p}$ is bounded by...
the sum of the (local analytic) multiplicities of all the nodes in the forest above at the point \( p \), where the multiplicities of the leafs are taken with coefficient \( \chi \) (except for isolated points, which may be taken with the coefficient 1). Our proof proceeds by a simple tree induction over the forest, using the following properties for the roots, the recursive step, and the leafs respectively.

- \( \text{mult}_p P|_{\gamma_p} = \text{mult}_p^\gamma V(P) \) (see Proposition 5).
- By the Rolle-type theorem above, if \( \Gamma \) is a node and \( \Gamma' \) denotes the sum of its children, then \( \text{mult}_p^\gamma \Gamma \leq \text{mult}_p \Gamma + \text{mult}_p^\gamma \Gamma' \).
- If \( V \) is a point then \( \text{mult}_p^\gamma V = 1 \), and if \( V \) is contained in a proper \( \xi \)-invariant variety then \( \text{mult}_p^\gamma V \leq \chi \text{mult}_p V \) (see Proposition 8).

The proof of the multiplicity estimate is concluded in a straightforward manner by computing an upper bound for the degrees of all nodes appearing in the forest using the Bezout theorem, thereby in particular bounding their multiplicities at \( p \).

1.3.2. The toric case: estimates in terms of Newton polytopes. For some applications it is not sufficient to give the multiplicity estimate in terms of a pure degree \( d \). For instance, in Nesterenko’s Theorem 1 and its intended application it is important to describe the dependence of the multiplicity function on the \( z \)-degree \( d_z \) and the \( x \)-degree \( d_x \) separately. Estimates in terms of the pure degree \( d \) are naturally tied to the study of varieties in the projective space \( \mathbb{C}P^n \), referred to in the literature as the absolute case. Estimates in the mixed degree case, i.e. for the two separate degrees \( d_z, d_x \) are naturally tied to the study of varieties in the product \( \mathbb{C}P^1 \times \mathbb{C}P^n \), referred to in the literature as the relative case.

It is natural to expect that similar types of estimates should hold for different notions of degree. Rather than consider the different possible compactifications of the affine space corresponding to each type of degree, we present a uniform expression of a multiplicity estimate in the framework of Newton polytope theory. Namely, by the Bernstein-Kusnirenko theorem (see §4.2) it is natural to expect that for a polynomial \( P \) with Newton polytope \( \Delta \), the multiplicity \( \text{mult}_p^\gamma P \) would be essentially bounded by a constant times the volume \( \text{Vol}(\Delta) \). A result of this type would immediately generalize the pure and mixed degree cases: the former corresponds to \( \Delta = d\Delta_z \) and the latter to \( \Delta = d_z\Delta_x + d_x\Delta_z \), where \( \Delta_z, \Delta_x \) denote the standard polytopes in the \( z \) and \( x \) variables respectively.

In §5 we repeat our proof in the context of Newton polytope theory, and obtain the following theorem (this is an easy consequence of the slightly tighter estimate given in Corollary 24).

**Theorem 3.** Let \( p \in (\mathbb{C}^*)^n \) and \( \gamma_p \) a smooth analytic trajectory of \( \xi \) through \( p \). Suppose that \( \gamma_p \) satisfies the D-property with constant \( \chi \). Then for any Laurent polynomial \( P \) with \( \Delta(P) = \Delta \),

\[
\text{mult}_p^\gamma P \leq n!(3 + \chi) \text{Vol}(\Delta + \Delta_{n,\xi})
\]

where \( \Delta_{n,\xi} \) is some explicit polytope depending only on \( n, \xi \). The diameter of \( \Delta_{n,\xi} \) grows as a single-exponential with \( n \).

We remark that the formulation in \((\mathbb{C}^*)^n\) rather than \( \mathbb{C}^n \) is a matter of elegance and technical convenience. If the polytope \( \Delta \) is a convex co-ideal in \( \mathbb{Z}^n_{\geq 0} \) then by a simple translation argument our estimate holds for any point \( p \in \mathbb{C}^n \), as explained in Remark 19.
The forest constructed in our proof describes the behavior of the multiplicity function not only at a single point \( p \), but in fact over any collection of points satisfying the D-property. In \[3\] we show how to derive the multiplicity estimates of \[30, 9\] for the case of multiple points from the geometry of this forest.

1.3.3. Organization of this paper. In \[2\] we review the basic notions related to cycles and their intersections; define the notion of the multiplicity of an analytic cycle along a smooth analytic curve; and prove the basic results concerning this notion. In \[3\] we prove our multiplicity estimates in the most familiar pure degree case. All the key ideas are already present in this context. In \[4\] we give some background on the theory of Newton polytopes including the Bernstein-Kushnirenko theorem, and prove some elementary results in convex geometry that are needed in the sequel. In \[5\] we prove our multiplicity estimates for general Newton polytopes, and show how these results imply Theorem 1 and its various generalizations. In \[6\] we list some general concluding remarks. In the appendix we describe Nesterenko’s approach to multiplicity estimates and point the reader to the appropriate statements in the present paper for comparison.

2. Cycles and multiplicities

For simplicity we present the results of this section in the context of the ambient space \( M = \text{Spec} \, R \), where \( R \) is either \( \mathbb{C}[x_1, \ldots, x_n] \), \( \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \), or the ring \( \mathcal{O}_p \) of germs of holomorphic functions in \( (\mathbb{C}^n, p) \). However, we note that many of the results could be extended without change to more general ambient spaces.

2.1. Cycles and their intersections. We give a brief review of the basic notions related to cycles and their intersection product. For a canonical reference see \[10\].

Recall that a \( k \)-cycle is defined to be a formal sum \( \sum n_i[V_i] \) where \( n_i \in \mathbb{Z} \) and \( V_i \subset M \) are irreducible varieties of dimension \( k \). A general cycle is a sum of cycles of various dimensions. In this paper we shall deal only with cycles with positive coefficients.

Two irreducible varieties \( V, W \subset M \) are said to intersect properly at a component \( Z \subset V \cap W \) if \( \text{codim} \, Z = \text{codim} \, V + \text{codim} \, W \). In this case we have a well defined number \( i(Z; V \cdot W; M) \), the multiplicity of \( Z \) in the intersection of \( V \) and \( W \). If every component of the intersection \( V \cap W \) is proper then one has a well defined intersection product

\[
V \cdot W = \sum_{Z \subset V \cap W} i(Z; V \cdot W; M)[Z] \tag{8}
\]

This can be extended by linearity to the product of arbitrary cycles, assuming every pair of components in the product intersect properly. This product is commutative and associative.

To each function \( f \in R \) one can associate its divisor \( V(f) \), which is an \( n - 1 \)-cycle. In particular, if \( \gamma \subset M \) is a curve passing through \( p \) then

\[
i(p; V(f) \cdot \gamma; M) = \text{mult}_p(f|_{\gamma}) \tag{9}
\]

where \( \text{mult} \) above denotes the usual multiplicity of a holomorphic function on a holomorphic curve.

If \( V \subset M \) is an irreducible variety, we denote by \( \mathcal{O}_V \) the ring of regular functions on \( V \) (i.e. \( \mathcal{O}_V := R/I_V \) where \( I_V \) denotes the ideal of functions vanishing on \( V \)). If \( p \in M \), we denote by \( \mathcal{O}_{V,p} \) the corresponding ring of germs of holomorphic functions
on $V$ at the point $p$. Note that we consider holomorphic localization even if $V$ is defined in the algebraic category.

Recall that the multiplicity of $V$ at a point $p \in M$, denoted $\text{mult}_p V$, is defined to be $e(m_p, \mathcal{O}_{V,p})$ where $m_p$ denotes the maximal ideal of $\mathcal{O}_{V,p}$. Geometrically, we have

$$\text{mult}_p V = i(p; V \cdot L; M)$$

(10)

where $L$ denotes a generic affine linear plane of codimension $\dim V$ passing through $p$. We extend this definition by linearity to arbitrary cycles.

We record the following standard fact.

**Proposition 3.** Let $\Gamma$ denote a cycle and $f \in R$, and suppose that $\Gamma$ intersects $V(f)$ properly. Then for any $p \in V(f)$ we have

$$\text{mult}_p \Gamma \leq \text{mult}_p \Gamma \cdot V(f)$$

(11)

**Proof.** By linearity it suffices to prove the inequality for a single irreducible variety $V$ of dimension $k$. If $p \notin V$ there is nothing to prove. Otherwise, the left hand side is given by $e(m_p, \mathcal{O}_{V,p})$ and the right hand side is given by $e(f, \ell_1, \ldots, \ell_{k-1}, \mathcal{O}_{V,p})$ where $\ell_1, \ldots, \ell_{k-1}$ denote $k - 1$ generic linear functionals. The claim follows since the Samuel multiplicity is monotonic with respect to inclusion of ideals.

Alternatively, the reader may prove this claim by arguing in the same manner as in the proof of Lemma 7. □

If $\Gamma = \sum n_i[p_i]$ is a zero dimensional cycle, we define its degree to be $\text{deg} \Gamma := \sum n_i$. If $M$ is $\mathbb{C}^n$ or $(\mathbb{C}^*)^n$ we extend this to cycles of arbitrary dimension by defining $\text{deg} \Gamma := \text{deg}(\Gamma \cdot L)$ where $L$ is a generic affine linear space of dimension complementary to $\Gamma$.

### 2.2. Multiplicity of a cycle along a curve.

We now define the multiplicity of an irreducible variety through a smooth analytic curve.

**Definition 4.** Let $V \subset M$ be an irreducible variety and $\gamma \subset M$ a smooth analytic curve passing through a point $p \in V$, and assume that $\gamma \nsubseteq V$. Let $I_\gamma \subset \mathcal{O}_{V,p}$ denote the restriction to $\mathcal{O}_{V,p}$ of the ideal of functions vanishing on $\gamma$. Then $I_\gamma$ is $m_p$-primary, and we define the multiplicity of $V$ through $\gamma$ at $p$, denoted $\text{mult}_p^\gamma V$, to be $e(I_\gamma, \mathcal{O}_{V,p})$. If $\gamma \subset V$ we define $\text{mult}_p^\gamma V = \infty$.

We extend this definition by linearity to arbitrary cycles.

The preceding definition admits a simple geometric interpretation similar to (10). Indeed, since $\gamma$ is smooth, we may choose analytic coordinates under which $\gamma$ is linear. In this case, we have

$$\text{mult}_p^\gamma V = i(p; V \cdot L; M)$$

(12)

where $L$ denotes a generic affine linear plane of codimension $\dim V$ containing $\gamma$.

The following proposition shows that the multiplicity of a function along a curve $\gamma$ can be interpreted in terms of the corresponding cycle.

**Proposition 5.** Let $f \in R$ and let $\gamma$ be a smooth analytic curve, $f|\gamma \neq 0$. Then

$$\text{mult}_p^\gamma V(f) = \text{mult}_p(f|\gamma)$$

(13)
Proof. Choose analytic coordinates making \( \gamma \) linear. Then since \( \text{codim} \gamma = n - 1 = \dim V \), we have by \( \ref{12} \)

\[
\text{mult}_p^\gamma V(f) = i(p; V(f) \cdot \gamma; M)
\]

(14)
and the proposition follows by \( \ref{19} \).

2.3. Multiplicity of a cycle along a vector field. We now define the multiplicity of a cycle with respect to a vector field. We say that a vector field \( \xi \) is an \( R \)-vector field if its coefficients in the standard coordinates are functions from \( R \).

Definition 6. Let \( \Gamma \subset M \) be a cycle and \( \xi \) be an \( R \)-vector field. Let \( p \in M \) be a nonsingular point of \( \xi \), and let \( \gamma_p \) denote the trajectory of \( \xi \) through \( p \). We define the multiplicity of \( \Gamma \) through \( p \) to be \( \text{mult}_p \xi \Gamma := \text{mult}_p^\gamma \Gamma \).

Note that when a vector field is singular at a point \( p \) and admits a smooth analytic trajectory \( \gamma_p \) through \( p \), we may still talk about the multiplicity of a cycle through the curve \( \gamma_p \). We prove a Rolle-type lemma for cycles (cf. \( \ref{30} \) Lemma 5.1). It applies for singular as well as nonsingular points of a vector field.

Lemma 7. Let \( V \subset M \) be an irreducible variety of dimension \( k \). Let \( \xi \) denote an \( R \)-vector field defined (possibly singular) near \( p \), and let \( \gamma \subset M \) be a smooth analytic trajectory of \( \xi \) passing through \( p \).

Suppose that \( f \in I_V \) and \( g = \xi f \notin I_V \). Then

\[
\text{mult}_p^\gamma V \leq \text{mult}_p V + \text{mult}_p^\gamma (V \cdot V(g)).
\]

Moreover, if \( \xi \) is singular at \( p \) we may omit the \( \text{mult}_p V \) term,

\[
\text{mult}_p^\gamma V \leq \text{mult}_p^\gamma (V \cdot V(g)).
\]

Proof. Choose analytic coordinates making \( \gamma \) linear. If \( L^{k-1} \) is a sufficiently generic affine plane of codimension \( k - 1 \) and \( L \) is a sufficiently generic affine hyperplane, both containing \( \gamma \), then we have by \( \ref{12} \)

\[
\text{mult}_p^\gamma V = i(p; V \cdot (L^{k-1} \cdot L); M) = i(p; C \cdot L; M)
\]

(17)

\[
\text{mult}_p^\gamma (V \cdot V(g)) = i(p; (V \cdot V(g)) \cdot L^{k-1}; M) = i(p; C \cdot (V(g)); M)
\]

(18)
where \( C \) denotes the curve \( V \cdot L^{k-1} \). In deriving this we have used the associativity and commutativity of the intersection product, and the fact that all intersections above are proper for sufficiently generic \( L^{k-1}, L \). Note that since \( L^{k-1} \) is chosen generically from a linear system with base locus \( \gamma \), we may in fact assume (by Bertini’s theorem) that it is a reduced curve without assigned multiplicities (although this does not play a role in our arguments).

It is clear that \( \mu := \text{mult}_p V = \text{mult}_p C \). Let \( \ell \) denote the affine-linear function with \( L = V(\ell) \). Our statement is thus reduced to

\[
i(p; C \cdot V(\ell); M) \leq \mu + i(p; C \cdot V(g); M)
\]

(19)
and similarly, without the \( \mu \) term, for the case when \( \xi \) is singular at \( p \).

Making a linear change of coordinates we may assume that the coordinates are given by \((x_1, \ldots, x_n)\) where \( p \) corresponds to the origin, \( x_1 \) is transversal to \( C \) and \( \gamma \) at the origin, and \( \gamma \) is given by the vanishing of \( x_2, \ldots, x_n \). Then \( C \) admits \( \mu \) real pro-branches

\[
C_i = \{(t, \phi_i^1(t)) := (t, \phi_2^i(t), \ldots, \phi_n^i(t)) : t \in \mathbb{Q}_{\geq 0}\} \quad i = 1, \ldots, \mu
\]

(20)
where the $\phi_j(t)$ admit Puiseux expansions and $\text{ord}_0 \phi_j(t) \geq 1$. By \([1]\) and a well-known formula for the multiplicity of a function on a curve, we have for every analytic function $h$ that

$$i(p; C \cdot V(h); M) = \text{mult}_p(h|_C) = \sum_{i=1}^\mu \text{ord}_0 h(t, \phi_i(t)).$$  

Thus the claim will be proved once we show that for $i = 1, \ldots, \mu$,

$$\text{ord}_0 \ell(t, \phi_i(t)) \leq 1 + \text{ord}_0 g(t, \phi_i(t))$$

and similarly, without the 1 term, for the case when $\xi$ is singular at $p$.

Let $v(\phi^i) := \min_{j=2, \ldots, n} \text{ord}_0 \phi^i_j(t)$, essentially measuring the asymptotic distance between $\phi^i_j$ and 0 in powers of $t$. Since $\ell$ is a generic linear combination of the $x_2, \ldots, x_n$ coordinates, the left hand side of \(22\) is equal to $v(\phi^i)$ (one only needs to make the choice generic enough to avoid cancellation between the leading terms of $\phi^i_j, j = 2, \ldots, n$). On the other hand, $f(t, \phi^i(t)) \equiv 0$ by assumption, and hence $\text{ord}_0 f(t, 0) \geq v(\phi^i)$. But the $x_1$ axis is a trajectory of $\xi$, and since derivation cannot decrease the order of an analytic function by more than 1 we have

$$\text{ord}_0 g(t, 0) = \text{ord}_0(\xi f)(t, 0) \geq v(\phi^i) - 1.$$  

Finally, translating back to $C_i$, we have $\text{ord}_0 g(t, \phi^i(t)) \geq v(\phi^i) - 1$. In the singular case derivation by $\xi$ does not decrease the order of $f$, and we obtain a similar result without the 1 term, as claimed. $\square$

We also have the following upper bound.

**Proposition 8.** Let $V \subset M$ be an irreducible variety and let $\gamma \subset M$ be a smooth analytic curve passing through $p$.

Suppose that $f \in I_V$ and $\text{mult}_p f|_{\gamma} = \delta$. Then

$$\text{mult}_p^2 V \leq \delta \cdot \text{mult}_p V$$

**Proof.** We keep the notations from the proof of Lemma \(1\) and argue in a similar manner. By the same arguments, it will suffice to prove that $v(\phi^i) \leq \delta$ for $i = 1, \ldots, \mu$. Assuming the contrary, we see that $f(t, \phi^i(t)) \equiv 0$ implies $\text{ord}_0 f(t, 0) > \delta$ contrary to the conditions of the proposition. $\square$

### 2.4. Multiplicities at generic points.

Let $W \subset M$ be an irreducible variety. We introduce the following notation to simplify our exposition. Let $\xi$ be an $R$-vector field, $\Gamma$ a cycle and $f \in R$. For each of the multiplicity functions $\text{mult}_p \Gamma$, $\text{mult}_p^\xi f$ and $\text{mult}_p^\Gamma \Gamma$ we define the multiplicity functions $\text{mult}_W \Gamma$, $\text{mult}_W^\xi f$ and $\text{mult}_W^\Gamma \Gamma$ to denote the value of the corresponding multiplicity function at a generic point $p \in W$.

It is easy to verify that the definition above is well-defined, i.e. that for $p$ outside a set $\Sigma$ of positive codimension in $W$ the multiplicity functions above are constant (and take larger values on $\Sigma$). It is also easy to verify that the two multiplicity functions involving $\Gamma$ above are linear with respect to $\Gamma$.

The following is a simple transversality statement.

**Proposition 9.** Let $V \subset M$ be an irreducible variety and $\xi$ an $R$-vector field. Suppose that $V$ is not invariant under $\xi$. Then $\text{mult}_V^\xi V = 1$. 

Proof. The claim follows since, at a generic point of $V$, $\xi$ must be transversal to $V$. Formally we may argue that by assumption $I_V$ is not a $\xi$-invariant ideal, and applying Lemma 7 with any polynomial $P \in I_V$ such that $\xi P / \in I_V$ proves the claim, since $\text{mult}_V V = 1$ (as $V$ is smooth at a generic point).

\section{3. The multiplicity estimate in the pure degree case}

We now restrict attention to the case $M = \mathbb{C}^n$. We let $\xi$ denote the vector field

$$ \xi = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \quad \max_i \deg \xi_i = \delta. \quad (25) $$

Let $N(n, \delta, d)$ denote the maximal possible (finite) value of $\text{mult}_p P$ for any $p \in M$, $\xi$ as above and $P \in \mathbb{R}$, $\deg P \leq d$. In [4] the following is proved.

\textbf{Theorem 4.} With the notation above,

$$ N(n, \delta, d) \leq 2^{n+1}(d + (n-1)\delta)^n \quad (26) $$

We recall the standard notion of Hilbert functions. For our purposes it is more convenient to work in a given affine chart $x_1, \ldots, x_n$. We define for an affine variety $V \subset \mathbb{C}^n$,

$$ H(V, t) = \dim_{\mathbb{C}} L(V, t), \quad L(V, t) := \{ P|_V : P \in R, \deg P \leq t \} \quad (27) $$

Estimates of the following type were given, in projective form, in [27] (see also [2]). The affine version follows readily from the corresponding projective estimate in homogeneous variables $x_0, \ldots, x_n$ by restricting to the chart $(1 : x_1, \ldots, x_n)$. The reader may also consult the proof of Proposition 18.

\textbf{Proposition 10.} Let $V \subset \mathbb{C}^n$ be an affine variety of dimension $k$. Then

$$ \mathcal{H}(V, t) \leq \deg(V) \cdot t^k + k. \quad (28) $$

The following lemma, a simple corollary of Proposition 10, shows that the ideal of a variety of sufficiently small degree must contain polynomials of bounded degree.

\textbf{Lemma 11.} Let $d > 2n$ and let $V$ be an irreducible variety of dimension $k$ with

$$ \deg V \leq A_n^{-1}d^{n-k} \quad A_n = 2n! \quad (29) $$

Then $I_V$ contains a non-zero polynomial $P$ with $\deg P < d$.

\textbf{Proof.} By Proposition 10

$$ \mathcal{H}(V, d-1) \leq \deg(V)(d-1)^k + k \leq A_n^{-1}d^n + k \leq 2A_n^{-1}d^n \quad (30) $$

On the other hand,

$$ \mathcal{H}(\mathbb{C}^n, d-1) = \binom{d+n-1}{n} > d^n/n! \quad (31) $$

It follows that some non-zero polynomial of degree bounded by $d - 1$ vanishes on $V$, as claimed.

The following lemma plays a key role in our arguments (cf. [30], Lemma 5.4), see the appendix for discussion).
Lemma 12. Let $V \subset M$ be an irreducible variety of dimension $k$ and suppose that $V$ is not contained in a (non-trivial) $\xi$-invariant variety. Let $P$ be a non-zero polynomial of minimal degree in $I_V$. Then
\[
\text{mult}_P I \leq a_{n,\delta}, \quad a_{n,\delta} = N(n, \delta, A_n 2^n n\delta)
\] (32)

Proof. We first note that $\text{mult}_P I$ is finite. Indeed, otherwise $V$ would be contained in the invariant variety defined by $(P, \xi P, \ldots)$ contrary to the conditions of the lemma.

Let $d := \deg P$. Henceforth we assume that $d > A_n 2^n n\delta$. Otherwise, the statement of the lemma follows from Theorem 4.

Claim. Let $W \subset M$ be an irreducible variety of dimension $l \geq k$ with $V \subset W$. Then $\deg W > A_n^{-1} d^{n-1}$.

Proof. Indeed, otherwise by Lemma 11 we have a polynomial of degree smaller than $d$ in $I_W$, and since $I_W \subset I_V$ this contradicts the minimality of $d = \deg P$. \hfill $\square$

We now proceed with the proof of the lemma. We will construct a sequence of cycles $V(P) = \Gamma^1 \supset \cdots \supset \Gamma^{n-k}$ with the following properties:

1. $\text{codim} \Gamma^j = j$.
2. Every component of $\Gamma^j$ contains $V$.
3. For every irreducible variety $W$ with $V \subset W \subset \text{supp} \Gamma^j$ we have
\[
\text{mult}_W \Gamma^j \leq (j-1) \text{mult}_W \Gamma^j + \text{mult}_W \Gamma^j
\] (33)
4. The degree of $\Gamma^j$ is bounded, $\deg \Gamma^j \leq (2d)^j$.

The first cycle $\Gamma^1 := V(P)$ clearly satisfies the conditions (condition (3) follows from Proposition 3 for any point $p$, and certainly for generic points in $W$).

We proceed with the construction by induction. Suppose that $\Gamma^j$ has been constructed, $\Gamma^j = \sum m_i^j [W_i^j]$. By assumption we have $V \subset W_i^j$, and by the preceding claim $\deg W_i^j > A_n^{-1} d^j$. Combining this with condition (4) we see that $m_i^j < A_n 2^j$. By condition (3) and Proposition 9 we have
\[
\text{mult}_{W_i^j} \Gamma^j \leq (j-1) \text{mult}_{W_i^j} \Gamma^j + \text{mult}_{W_i^j} \Gamma^j
\] (34)
\[= (j-1)m_i^j + m_i^j < A_n 2^j
\]

If $\text{mult}_{W_i^j} P = \text{mult}_P \Gamma^j$ then (34) proves the claim of the lemma. Otherwise, let $Q_i^j$ denote the first $\xi$-derivative of $P$ which does not vanish on $W_i^j$ (but still vanishes on $V$). By (34) we have
\[
\deg Q_i^j \leq d + (A_n 2^j)\delta \leq 2d
\] (35)

Let $\tilde{\Gamma}_i^{j+1} = [W_i^j] \cdot V(Q_i^j)$ and $\Gamma_i^{j+1}$ consist of the components of $\tilde{\Gamma}_i^{j+1}$ which contain $V$. Finally, define $\Gamma^{j+1} = \sum_i m_i^{j+1} \Gamma_i^{j+1}$. Conditions (1) and (2) hold by definition, and condition (4) follows inductively by linearity and the Bezout theorem.

We move now to the proof of condition (3). We have for any $V \subset W \subset \text{supp} \Gamma_i^{j+1}$ the inequality
\[
\text{mult}_W W_i^j \leq \text{mult}_W \tilde{\Gamma}_i^{j+1} = \text{mult}_W \Gamma_i^{j+1}
\] (36)
where we used Proposition 3 for generic points in $W$, and the fact that the components of $\tilde{\Gamma}_i^{j+1} - \Gamma_i^{j+1}$ do not contain $V$ (and certainly do not meet generic points
of $W$). Similarly, using Lemma 7 for generic points in $W$ we have
\[
\text{mult}_{W}^{\xi} W_{i}^{j} \leq \text{mult}_{W}^{\xi} W_{i}^{j} + \text{mult}_{W}^{\xi} \tilde{\Gamma}_{i}^{j+1} \\
\leq \text{mult}_{W}^{\xi} \Gamma_{i}^{j+1} + \text{mult}_{W}^{\xi} \tilde{\Gamma}_{i}^{j+1}
\] (37)
Condition (3) now follows inductively from (36) and (37) and linearity of the multiplicity function. This concludes our construction.

Conditions (1) and (2) guarantee that $\Gamma_{n-k}$ is supported on $V$, and in the notation above $W_{i}^{n-k} = V$. Thus (34) implies the claim of the lemma.

\[\square\]

Remark 13. We have not attempted to optimize the dependence on all parameters in the proof above, which could clearly be improved at a number of points in the argument. Our main goal (as far as explicit estimates are concerned) was to provide a bound admitting single-exponential growth with $n$. Note that one can repeat the proof above without relying on Theorem 4, but the estimates in this case become significantly worse.

\subsection{3.1. The multiplicity forest}

We start with a definition.

\begin{definition}
A cycle forest $T$ is a directed forest (union of directed trees) whose nodes are irreducible varieties with assigned multiplicities, such that:
\begin{enumerate}
\item All nodes of level $k$ have codimension $k$.
\item All children of a node $n[V]$ are subvarieties of $V$.
\end{enumerate}
We will denote by $T_{k}$ the cycle formed by the sum of all nodes of level $k$ (with their assigned multiplicities), and by $L(T)$ the set of leafs of $T$.
\end{definition}

Our principal result is the following theorem.

\begin{theorem}
Let $P \in \mathbb{R}$ with $\deg P = d$. There exists a cycle forest $T_{P}$ with the following properties.
\begin{enumerate}
\item The roots are given by the components of $V(P)$ (with their assigned multiplicities).
\item Every leaf of $T_{P}$ is either an isolated point, or a variety contained in some $\xi$-invariant variety.
\item The degree of $T_{k}$ is bounded,
\[
\deg T_{k} \leq (d + \tilde{a}_{n,\delta})^{k} \quad \tilde{a}_{n,\delta} = \delta a_{n,\delta}
\] (38)
\item Denote by $L^{+}(T)$ (resp. $L^{0}(T)$) the set of leafs of $T$ that have positive dimension (resp. dimension zero). If $\gamma_{p}$ is a smooth analytic trajectory of $\xi$ through $p$ which satisfies the D-property at $p$ with constant $\chi$, then
\[
\text{mult}^{\gamma_{p}}_{p} P \leq \sum_{\Gamma \in T \setminus L^{+}(T_{P})} \text{mult}_{p} \Gamma + \chi \sum_{\Gamma \in L^{+}(T_{P})} \text{mult}_{p} \Gamma
\] (39)
If $\xi$ is singular at $p$ this bound may be tightened to
\[
\text{mult}^{\gamma_{p}}_{p} P \leq \sum_{\Gamma \in L^{0}(T_{P})} \text{mult}_{p} \Gamma + \chi \sum_{\Gamma \in L^{+}(T_{P})} \text{mult}_{p} \Gamma
\] (40)
\end{enumerate}
\end{theorem}

\begin{proof}
We construct $T$ recursively, starting with the roots specified in condition (1). Suppose $n[V]$ is a node. If $V$ is a point, or is contained in an invariant variety, then this node is a leaf. Otherwise, let $\tilde{P}$ be a polynomial of minimal degree in $I_{V}$. Since $V \subset V(P)$ by definition of a cycle forest, we have $P \in I_{V}$, and hence $\deg \tilde{P} \leq d$. By Lemma 12, $\text{mult}_{V}^{\xi} \tilde{P} \leq a_{n,\delta}$. Let $Q$ denote the first derivative of $\tilde{P}$ which does
not vanish identically on $V$. Then $\deg Q \leq d + \tilde{a}_{n,\delta}$. We define the children of $n[V]$ to be the components of $n[V] \cdot V(Q)$ (with their assigned multiplicities).

Conditions (1) and (2) hold by definition. Condition (3) follows inductively by application of the Bezout theorem. Finally, condition (4) follows by a straightforward recursive argument, using Proposition 5 at the root, Lemma 7 at the recursive step, and Proposition 8 at the leaves. It is necessary only to note that for a zero-dimensional leaf $[p]$ we have $\mult^p p = 1$ by definition.

**Corollary 15.** Let $p \in M$ and $\gamma_p$ a smooth analytic trajectory of $\xi$ through $p$. Suppose that $\gamma_p$ satisfies the D-property with constant $\chi$. Then for any $P \in R$ with $\deg P = d$,

$$\mult^p P \leq (d + \tilde{a}_{n,\delta})^n + (2 + \chi)(d + \tilde{a}_{n,\delta})^{n-1} \quad (41)$$

**Proof.** The estimate follows in a straightforward manner from Theorem 5 by noting that the multiplicity of a cycle at a point is bounded by the cycle’s degree.

4. Preliminaries on $(\mathbb{C}^*)^n$ and convex geometry

In this section we consider the case $M = (\mathbb{C}^*)^n$.

Given a Laurent polynomial $P \in R$, we define its support $\text{supp} P \subset \mathbb{Z}^n$ to be the set of exponents appearing with non-zero coefficients in $P$. For any set $A \subset \mathbb{R}^n$ we denote by $\Delta(A)$ the convex hull of $A$. The convex hull of a finite subset $A \subset \mathbb{Z}^n$ is called an integral polytope. Finally, define the Newton polytope of $P$ to be $\Delta(P) := \Delta(\text{supp} P)$.

For each set $A \subset \mathbb{R}^n$ we denote by $L_A$ the linear space of polynomials whose support is contained in $A$.

4.1. Toric classes.

**Definition 16.** A toric $k$-class $T$ is a symmetric map assigning a non-negative number $T(L_{A_1}, \ldots, L_{A_k}) \in \mathbb{Z}_{\geq 0}$ for each collection of finite sets $A_1, \ldots, A_k \subset \mathbb{Z}^n$. We identify the toric 0-classes with $\mathbb{Z}_{\geq 0}$.

If $\Gamma$ is a $k$-cycle, we define the corresponding toric class $\mathcal{T}(\Gamma)$ by associating to each tuple $L_{A_1}, \ldots, L_{A_k}$ the number

$$\mathcal{T}(\Gamma)(L_{A_1}, \ldots, L_{A_k}) := \deg(\Gamma \cdot V(P_1) \cdots V(P_k)) \quad (42)$$

where $P_i$ is a generic element of $L_{A_i}$. This definition (for fixed $\Gamma$) was used in [16] to develop a type of birationally equivalent intersection theory. In particular it is shown in [16] that the number above is well defined.

It is easy to see that the map $\Gamma \to \mathcal{T}(\Gamma)$ is linear. We define the (partial) order relation $\leq$ on the space of toric $k$-classes to be the pointwise ordering. If $A \subset \mathbb{Z}^n$ is finite set and $T$ is a toric $k$-class, we define the product $T \cdot L_A$ to be the toric $k - 1$-class defined by

$$(T \cdot L_A)(L_{A_1}, \ldots, L_{A_{k-1}}) := T(L_A, L_{A_1}, \ldots, L_{A_{k-1}}) \quad (43)$$

This product is linear and respects the order relation.

**Remark 17.** We sometimes identify $L_A$ with the toric $(n-1)$-class $[M] \cdot L_A$.

By definition, if $\Gamma$ is a cycle and $f \in L_A$ is such that $V(f)$ meets $\Gamma$ properly then

$$\mathcal{T}(\Gamma \cdot V(f)) \leq \mathcal{T}(\Gamma) \cdot L_A \quad (44)$$
Proposition 18. Let \( \Delta \subseteq \mathbb{Z} \subset A \). If \( V \subseteq M \) is an irreducible variety of dimension \( k \) then

\[
\mathcal{H}(V, A) \leq \mathcal{H}(V)(L_A)^k + k
\]  

Proof. Denote \( \mathcal{H} := \mathcal{H}(V, A) \). Consider the map \( \phi : V \to \mathbb{C}P^{2k-1} \) whose projective coordinates are given by \( \mathcal{H} \) linearly independent functions from \( (L_A)|_V \). By assumption, \( \phi \) is injective (since \( A \) contains the constant 1 as well as the coordinate functions). Denote by \( W \) the Zariski closure of \( \phi(V) \). Then \( W \) is irreducible and \( \dim W = \dim V = k \).

A generic projective space \( L \) of codimension \( k \) in \( \mathbb{C}P^{2k-1} \) meets \( W \) at points of \( \phi(V) \). Since the pullbacks of the linear forms on \( \mathbb{C}P^{2k-1} \) to \( V \) correspond bijectively to elements of \( L_A \), we have \( \deg W = \mathcal{H}(V)(L_A)^k \).

By definition, \( W \) is not contained in any proper projective subspace of \( \mathbb{C}P^{2k-1} \). The claim now follows from the following classical fact: for any irreducible variety \( W \subseteq \mathbb{C}P^{2k-1} \) which is not contained in a proper projective subspace, \( \mathcal{H} \leq \deg W + \dim W \).

The fact can be proved as follows. First, if \( \dim W > 1 \) then passing to a generic hyperplane section does not affect the inequality (both sides are decreased by 1) and preserves the irreducibility of \( W \). Thus it suffices to prove the claim when \( \dim W = 1 \). In this case, one can certainly choose a hyperplane meeting (any) \( \mathcal{H} - 1 \) points in \( W \). Since this hyperplane does not contain \( W \) by assumption, it follows that indeed \( \deg W \geq \mathcal{H} - \dim W \). \( \square \)

4.2. Mixed volume and the Bernstein-Kushnirenko theorem. Recall that for \( n \) convex bodies \( \Delta_1, \ldots, \Delta_n \) in \( \mathbb{R}^n \), their mixed volume is defined to be

\[
V(\Delta_1, \ldots, \Delta_n) = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} \text{Vol}(\lambda_1 \Delta_1 + \cdots + \lambda_n \Delta_n)|_{\lambda_1 = \cdots = \lambda_n = 0}. \tag{46}
\]

The mixed volume is symmetric and multilinear, and generates the volume function in the sense that \( V(\Delta, \ldots, \Delta) = \text{Vol}(\Delta) \). In fact, these properties completely determine the mixed volume function.

The following result, known as the Bernstein-Kushnirenko theorem, related the number of solutions of a generic system of polynomial equations with prescribed supports to the mixed volume of their Newton polytopes.

Theorem 6 ([18] [1]). Let \( A_1, \ldots, A_n \subseteq \mathbb{Z}^n \) be finite sets. Then for generic \( P_i \in \text{L}_{A_i} \), the system of equations \( P_i = \cdots = P_n = 0 \) admits exactly \( \mu \) solutions in \((\mathbb{C}^*)^n\), where

\[
\mu = n!V(\Delta(A_1), \ldots, \Delta(A_n)). \tag{47}
\]

In other words,

\[
\text{L}_{A_1} \cdots \text{L}_{A_n} = n!V(\Delta(A_1), \ldots, \Delta(A_n)). \tag{48}
\]

Remark 19. If the integral polytopes \( \Delta_1, \ldots, \Delta_n \subseteq \mathbb{Z}^n_{\geq 0} \) are co-ideals, then the Bernstein-Kushnirenko in fact gives an estimate for the number of solutions \( \mu \) of the corresponding generic system of equation in \( \mathbb{C}^n \). Indeed, the corresponding spaces
are closed under the translation $\vec{x} \rightarrow \vec{x} + \vec{a}$, and one can therefore assume that the solutions of the generic system of equations fall outside of the $x_i$-axes, i.e., the number of solutions in $(\mathbb{C}^*)^n$ is the same as in $\mathbb{C}^n$.

All of our results concerning the torus $(\mathbb{C}^*)^n$ could therefore be extended to the case $\mathbb{C}^n$ under the further assumption that the integral polytopes under consideration are co-ideals.

Let $\Delta^x$ denote the standard simplex in the $x$-variables in $\mathbb{R}^n$. For any convex body $\Delta$ and $j = 0, \ldots, n$ we define the $j$-th (simplicial) quermassintegral as

$$ W_j(\Delta) = V(\Delta, \ldots, \Delta, \Delta^x, \ldots, \Delta^x)_{n-j \text{ times } j \text{ times}}. \quad (49) $$

We note that it is customary to use the Euclidean ball in place of the standard simplex $\Delta^x$, but for our purposes the simplicial normalization is more convenient.

4.3. Two elementary lemmas on volumes and integral volumes. Let $\Pi_n := [-1, 1]^n$ denote the unit cube in $\mathbb{R}^n$. In general we say that $\Pi$ is an integral box if it is of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$ where $a_i, b_i \in \mathbb{Z}$.

For any body $\Delta \subset \mathbb{R}^n$ denote by $\text{Vol}(\Delta)$ the volume of $\Delta$ and by $\#(\Delta)$ the number of integral points in $\Delta$.

**Lemma 20.** For any convex body $\Delta$, we have $\#(\Delta + \Pi_n) \geq \text{Vol}(\Delta)$. Moreover, if $\Pi$ is any integral box then $\#(\Pi \cap (\Delta + \Pi_n)) \geq \text{Vol}(\Pi \cap \Delta)$. 

**Proof.** Let $A$ denote the union of all cubes of the form $z + [0, 1]^n, z \in \mathbb{Z}^n$ that meet $\Delta$. Then one easily checks that

1. $\text{Vol}(\Pi \cap A) \geq \text{Vol}(\Pi \cap \Delta)$.
2. $\#(\Pi \cap A) \geq \text{Vol}(\Pi \cap A)$.
3. $\Pi \cap A \subset \Pi \cap (\Delta + \Pi_n)$.

The claim follows immediately. \qed

**Lemma 21.** Let $\Delta \subset \mathbb{R}^n$ denote a convex polytope such that $n\Pi_n \subset \Delta$. Then $\#(\Delta) \geq \frac{1}{2} \text{Vol}(\Delta)$. Moreover, if $\Pi$ is any integral box then $\#(\Pi \cap \Delta) \geq \frac{1}{4} \text{Vol}(\Pi \cap \Delta)$.

**Proof.** Let $\Delta' := (1 - 1/n)\Delta$. Then $\Delta' + \Pi_n \subset \Delta$ and

$$ \text{Vol}(\Pi \cap \Delta') \geq \text{Vol}((1 - 1/n)(\Pi \cap \Delta)) \geq \frac{1}{4} \text{Vol}(\Pi \cap \Delta) \quad (50) $$

The statement now follows from Lemma 20 applied to $\Delta'$. \qed

5. Multiplicity estimates in the toric case

In this section we consider the ambient space $M = (\mathbb{C}^*)^n$. We let $\xi$ denote the vector field

$$ \xi = \sum_{i=1}^n \xi_i(x) \frac{\partial}{\partial x_i}, \quad \xi_i \in R \quad (51) $$

We define new Newton polytope of $\xi$, denoted $\Delta_\xi$, in the same way we do for polynomials, where to each term $x^\alpha \frac{\partial}{\partial x_i}$ we associate the same point in $\mathbb{Z}^n$ as we do for $x^\alpha x^i$. We suppose for simplicity that $\Delta_\xi$ contains the origin (there is no loss of generality, since all problems considered in this section are invariant under multiplication of $\xi$ by a monomial).

For any polytope $\Delta \subset \mathbb{R}^n$ we define $\text{deg}_\Pi \Delta$ to be the minimal $d \in \mathbb{N}$ such that $\Delta \subset d\Pi_n$. We let $\text{deg}_\Pi P := \text{deg}_\Pi \Delta(P)$. 

5.1. The key lemma in the toric case. Let $\Delta$ be an integral polytope with $n\Pi_n \subset \Delta$. We denote $\Delta_d := \Delta \cap d\Pi_n$.

We begin with a simple lemma analogous to [11]

**Lemma 22.** Let $d > 2n$ and let $V$ be an irreducible variety of dimension $k$ with

$$\mathcal{T}(V)(L_{\Delta_d})^k \leq B_n^{-1}(L_{\Delta_d})^n \quad B_n = 2^4n!$$

Then $I_V$ contains a polynomial $P \in L_{\Delta}$ with $\deg P < d$.

**Proof.** By Proposition [18]

$$\mathcal{T}(V, \Delta_{d-1}) \leq \mathcal{T}(V)(L_{\Delta_{d-1}})^k + k \leq B_n^{-1}(L_{\Delta_d})^n + k =$$

$$B_n^{-1}n! \text{Vol}(\Delta_d) + k \leq (B_n^{-1}n! + k(2n)^{-n}) \text{Vol}(\Delta_d) \leq 2^{-3}\text{Vol}(\Delta_d)$$

and by Lemma [21]

$$\mathcal{H}(M, \Delta_{d-1}) = \#(\Delta_{d-1}) \geq \#((1 - \frac{1}{4})\Delta_d) \geq \frac{1}{4}\text{Vol}((1 - \frac{1}{4})\Delta_d)$$

$$\geq 2^{-3}\text{Vol}(\Delta_d).$$

It follows that some non-trivial element of $L_{\Delta_{d-1}}$ vanishes on $V$, as claimed. $\square$

We are now ready to state the toric analog of the key Lemma [12]

**Lemma 23.** Let $V \subset M$ be an irreducible variety of dimension $k$ and suppose that $V$ is not contained in a (non-trivial) $\xi$-invariant variety. Suppose that

$$B_n2^n\Pi_n \subset \Delta$$

and that $I_V$ intersects $L_{\Delta}$ nontrivially, and let $P$ be a non-zero polynomial with minimal $d := \deg I_V \in \text{the intersection. Then}$

$$\text{mult}_V P \leq b_n, \quad b_n = N(n, \delta, (2n)B_n2^n\Pi_n)$$

**Proof.** The proof is analogous to the proof of Lemma [12] Henceforth we assume that $d > B_n2^n\Pi_n$. Otherwise, the statement of the lemma follows from Theorem [3] applied to the polynomial $(x_1 \cdots x_n)^{\delta}P$. Alternatively one can apply the toric estimate from [3] directly to $P$ to obtain a slightly better estimate.

The claim used in the proof is replaced by

**Claim.** Let $W \subset M$ be an irreducible variety of dimension $l \geq k$ with $V \subset W$. Then $\mathcal{T}W \cdot (L_{\Delta_d})^j > A_n^{-1}(L_{\Delta_d})^n$.

This claim is proved in the same way, based on Lemma [22]. In the definition of the sequence $\Gamma^k$ one replaces condition (4) by

4. The toric class of $\Gamma^j$ is bounded, $\mathcal{T}(\Gamma^j) \leq (L_{2\Delta_d})^j$.

Suppose that $\Gamma^j$ has been constructed, $\Gamma^j = \sum m_i^j [W_i^j]$. By assumption we have $V \subset W_i^j$, and by the preceding claim $\mathcal{T}W_i^j \cdot (L_{\Delta_d})^n \leq B_n^{-1}(L_{\Delta_d})^n$. Using condition (4),

$$m_i^j B_n^{-1}(L_{\Delta_d})^n < \mathcal{T}(\Gamma^j) \cdot (L_{\Delta_d})^{n-j} < (L_{2\Delta_d})^j(L_{\Delta_d})^{n-j} = 2^j(L_{\Delta_d})^n$$

where we used the Bernstein-Kushnirenko theorem and that fact that $\Delta_d, 2\Delta_d$ are integral polytopes for the last step. Therefore $m_i^j < B_n2^j$. By condition (3) and Proposition [9] we have

$$\text{mult}_{W_i^j} P \leq (j-1)\text{mult}_{W_i^j} \Gamma^j + \text{mult}_{W_i^j} \Gamma^j$$

$$= (j-1)m_i^j + m_i^j < B_n2^j$$
If $\mult_{W^i_j} P = \mult_{W^i_j} P$ then (58) proves the claim of the lemma. Otherwise, let $Q^j_i$ denote the first $\xi$-derivative of $P$ which does not vanish on $W^i_j$ (but still vanishes on $V$). By (58) we have

$$\Delta(Q^j_i) \subset \Delta_d + (B_n 2^j j) \Delta_\xi \subset 2\Delta_d$$

(59)

The rest of the proof is entirely analogous to the proof of Lemma 12. We leave the verification of the details to the reader. □

5.2. The multiplicity forest. The following result is the toric analog of Theorem 5.

**Theorem 7.** Let $P \in R$. There exists a cycle forest $T_P$ with the same properties as in Theorem 5, with condition (3) replaced by

3. The toric class of $T^k_P$ is bounded,

$$\mathcal{T}(T^k_P) \leq (L \Delta + \Delta_n, \xi)^k \Delta_n, \xi = (B_n 2^n n + b_{n, \delta}) \Delta_\xi$$

(60)

**Proof.** Let $\Delta' := \Delta + B_n 2^n n \Delta_\xi$. Then Lemma 23 applies to $\Delta'$ and the rest of the proof is analogous to the proof of Theorem 5. We leave the details to the reader. □

**Corollary 24.** Let $p \in M$ and $\gamma_p$ a smooth analytic trajectory of $\xi$ through $p$. Suppose that $\gamma_p$ satisfies the D-property with constant $\chi$. Then for any $P \in R$ with $\Delta(P) = \Delta$, $\mult_{\gamma_p} P \leq n! \Vol(\Delta + \Delta_n, \xi) + n!(2 + \chi)W_1(\Delta + \Delta_n, \xi)$

(61)

**Proof.** The estimate follows in a straightforward manner from Theorem 7. Namely, for any node in the multiplicity forest with toric $k$-class $T$, we estimate its degree by $\mathcal{T} \cdot (L \Delta_x)^k$. The statement follows from (60) using the identity

$$(L \Delta + \Delta_n, \xi)^n = n! \Vol(\Delta + \Delta_n, \xi)$$

(62)

and the inequality

$$(L \Delta + \Delta_n, \xi)^{n-k} \cdot (L \Delta_x)^k \leq n! W_1(\Delta + \Delta_n, \xi)$$

for $k \geq 1$

(63)

both of which follow from the Bernstein-Kushnirenko theorem. □

5.3. Recovering the classical multiplicity estimates. In subsection we show how the toric estimates presented in this section imply the various known multiplicity estimates for mixed degrees as a special case. We thus consider the ambient space $\mathbb{C} \times \mathbb{C}^n$, where we denote the first coordinate by $z$, thought of as the time variable, and the remaining coordinates by $x$, thought of as dependent variables. As usual we denote by $R$ the corresponding polynomial ring.

Consider a vector field $\xi$ of the form

$$\xi = t(z) \frac{\partial}{\partial z} + \sum_{i=1}^n \xi_i(z, x) \frac{\partial}{\partial x_i}.$$  

(64)

We denote by $\Delta_z, \Delta_x$ the standard simplices in the $z$ and $x$ variables, respectively. We fix a trajectory $\gamma = (z, f_1(z), \ldots, f_n(z))$ of $\xi$, with $f_1, \ldots, f_n$ holomorphic in some domain $U \subset \mathbb{C}$. 
5.3.1. Estimate for a single point. We begin with Theorem 1, Nesterenko’s classical estimate for the multiplicity at a single point.

**Theorem.** Let \( p \in U \) and suppose that \( \gamma \) has the D-property at \( p \). Let \( P \in R \) be a polynomial with \( P|_\gamma \not\equiv 0 \), and denote

\[
d_x := \max(\deg_x P, 1) \quad d_z := \max(\deg_z P, 1)
\]

Then

\[
\text{mult}_{z=p} P(z, f_1(z), \ldots, f_n(z)) \leq \alpha_\gamma d_z d_x^n
\]

where \( \alpha_\gamma \) is a constant depending only on \( \gamma \).

**Proof.** Let \( \chi \) denote the D-property constant of \( \gamma \) at \( p \). We have \( \Delta(P) \subset d_x \Delta_x + d_z \Delta_z \), and by Corollary 24 we have

\[
\text{mult}_{z=p} P(z, f_1(z), \ldots, f_n(z)) \leq (n+1)! \text{Vol}(\Delta(P) + \Delta_{n,\xi}) + (n+1)! (2 + \chi) W_1(\Delta(P) + \Delta_{n,\xi})
\]

\[
\leq (n+1)! (3 + \chi) (\deg_{\|\|} \Delta_{n,\xi}) \text{Vol}(d_x \Delta_x + d_z \Delta_z)
\]

\[
\leq \alpha_\gamma d_z d_x^n
\]

for an appropriate constant \( \alpha_\gamma \).

\[\square\]

5.3.2. Estimate for multiple points. We now consider Theorem 2, the analogous result for the case involving multiple points.

**Theorem.** Let \( p_1, \ldots, p_q \in U \) and suppose that \( \gamma \) has the D-property at \( p_i \) for every \( i \). Let \( P \in R \) be a polynomial with \( P|_\gamma \not\equiv 0 \), and denote

\[
d_x := \max(\deg_x P, 1) \quad d_z := \max(\deg_z P, 1)
\]

Then

\[
\sum_{i=1}^q \text{mult}_{z=p_i} P(z, f_1(z), \ldots, f_n(z)) \leq \beta_\gamma (d_z + q)d_x^n
\]

where \( \beta_\gamma \) is a constant depending only on \( \gamma \).

**Proof.** Note that the D-property holds automatically with constant 1 for regular points of \( \xi \), i.e. for any \( p_i \) except for the (finitely many) roots of \( t(z) \). We may thus assume that the D-property holds with a uniform constant \( \chi \) independent of the choice of the points \( p_i \).

Applying Theorem 7 we have a multiplicity forest for \( P \) with the estimate

\[
T(T_P^k) \leq (L_\beta_1(d_x \Delta_x + d_z \Delta_z))^k
\]

for some constant \( \beta_1 \) (depending only on \( \xi \)). We write \( T_P^k := \tilde{T}_P^k + \hat{T}_P^k \) where \( \tilde{T}_P^k \) consists of those component of \( T_P^k \) which are strictly contained in a hyperplane \( z = \text{const} \), and \( \hat{T}_P^k \) the rest.

Each component of \( \tilde{T}_P^k \) can contain at most one point \( p_i \). Thus a simple computation using (69) gives

\[
\sum_{i=1}^q \text{mult}_{p_i} \tilde{T}_P^k \leq \deg \tilde{T}_P^k \leq \beta_2 d_z d_x^n
\]

On the other hand, since the components of \( \hat{T}_P^k \) are not contained in \( z = \text{const} \), we may estimate their multiplicity from above at any point \( p_i \) by intersecting with the
hyperplane \( z = z(p_i) \) and \( n - k \) additional generic hyperplanes. Thus
\[
\text{mult}_{p_i} T^k \leq V(\beta_1 (d_x \Delta_z + d_z \Delta_x), \Delta_z, \Delta_x + \Delta_z) \leq \beta_3 d_x^n
\] (71)
where we expanded the middle expression by multilinearity and used the fact that if the \( \Delta_z \) term appears twice then then mixed volume is zero (since \( \Delta_z \) is one-dimensional).

Finally, using Theorem 7 and (70), (71) the conclusion of the theorem easily follows.

5.3.3. The case of a trajectory satisfying algebraic relations. Let \( Z \) denote the Zariski closure of \( \gamma \). We suppose now that \( Z \neq M \). In this case \( \gamma \) certainly does not satisfy the D-property, because the ideal \( I_Z \) is invariant and any function \( P \in I_Z \) vanishes identically on \( \gamma \). One can avoid such “trivial” counterexamples by considering \( \xi \) as a vector field in the ambient space \( Z \). Indeed, since \( Z \) is the Zariski closure of the irreducible invariant set \( \gamma \), it follows that \( Z \) is itself irreducible and invariant, and \( \xi \) induces a derivation of the ring \( \mathcal{O}_Z \). The D-property in this context states that any non-zero \( \xi \)-invariant prime ideal \( J \subset \mathcal{O}_Z \) (i.e., any \( \xi \)-invariant prime ideal \( J \subset \mathcal{O}_M \) strictly containing \( I_Z \)) contains a function \( F \) with \( \text{mult}_F^m F \leq \chi \).

A result in this context was stated in [30] with a small technical mistake (the proof was only sketched). Dolgalev [9] gave a corrected formulation and a full proof. In this subsection we establish a strengthening of Dolgalev’s results.

Our proof essentially extends to the present context with little changes. One can repeat all considerations within the ambient space \( Z \). Specifically, in Remark 17 we now identify \( L_A \) with the toric class \( \mathcal{H}(Z) \cdot L_A \). Lemma 23 and Theorem 7 as well as their proofs extend literally. The only exception is Lemma 22 where a lower bound for the toric Hilbert function of the ambient space, in our case \((\mathbb{C}^*)^n\), was explicitly used. This bound must be replaced by an appropriate lower bound for \( \mathcal{H}(Z, \Delta_d) \).

We show how this can be carried out in the context of this subsection, namely for \( \Delta \) of the form \( d_x \Delta_x + d_z \Delta_z \). Denote the dimension of \( Z \) by \( m \).

Let \( \pi_x : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n \) denote the projection to the \( x \) variables. We distinguish between two cases for \( Z \):

A. We have \( \dim \pi_x(Z) = \dim Z - 1 \). In this case \( Z = \mathbb{C} \times Z_x \) where \( Z_x \subset \mathbb{C}^n \) is an irreducible variety.

B. We have \( \dim \pi_x(Z) = \dim Z \).

Lemma 25. Denote \( d'_z = \min(d_z, d) \), \( d'_x = \min(d_x, d) \) (these are simply the \( z \) and \( x \) sizes of polytope \( \Delta_d = \Delta \cap \Pi_d \)).

\[
\mathcal{H}(Z, \Delta_d) \geq (m!)^{-1} \cdot \left\{ \begin{array}{ll}
d'_x (d'_z)^{m-1} & \text{in case A} \\
\max(d'_x (d'_z)^{m-1}, (d'_z)^m) & \text{in case B}
\end{array} \right.
\] (72)

Proof. In case A, after a generic linear change in the \( x \) variables we may assume that the projection
\[
\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^{m-1} \quad \pi(z, x_1, \ldots, x_n) = (z, x_1, \ldots, x_{m-1})
\]
is rational dominant. Thus \( I_Z \cap \mathbb{C}[z, x_1, \ldots, x_{m-1}] = \{0\} \), and hence \( \mathcal{H}(Z, \Delta_d) \) is at least the dimension of the space of polynomials in \( z, x_1, \ldots, x_{m-1} \) with \( \deg_z \leq d'_z \) and \( \deg_x \leq d'_x \). The result follows by simple arithmetic.
In case B we argue similarly. In this case, after a generic linear change in the \(x\) variables we may assume that the two projections

\[
\begin{align*}
\pi_1 : \mathbb{C} \times \mathbb{C}^n &\to \mathbb{C} \times \mathbb{C}^{m-1} \\
\pi_2 : \mathbb{C} \times \mathbb{C}^n &\to \mathbb{C}^m
\end{align*}
\]

are both rational dominant. We can thus use either set of variables to produce a lower bound for \(H(Z, \Delta_d)\), and the result follows as above.

The ideal \(I(Z)\) is generated by polynomials bounded by some degree \(D\), and \(Z\) is a component of a generic combination of these generators. In case A, one can further assume that the generators are independent of \(z\). Thus

\[
\mathcal{T}(Z) \leq (L_D n)^{n-m} \quad \text{in case A}
\]

\[
\mathcal{T}(Z) \leq (L_{D+\Delta_m})^{n-m} \quad \text{in case B}
\]

Lemma 26. Let \(d > 2n\) and let \(V \subset Z\) be an irreducible variety of dimension \(k\) with

\[
\mathcal{H}(V)(L_{\Delta d})^k \leq C_n^{-1}\mathcal{T}(Z)(L_{\Delta d})^m \quad C_{n,Z} = \text{const}(n, Z)
\]

Then \(I_V \setminus I_Z\) contains a polynomial \(P \in L_{\Delta}\) with \(\deg P < d\).

Proof. By Proposition 18,

\[
\mathcal{H}(V)(L_{\Delta d-1})^k \leq \mathcal{T}(V)(L_{\Delta d-1})^k + k \leq C_n^{-1}\mathcal{T}(Z)(L_{\Delta d})^m + k
\]

and using (73) and the Bernstein-Kushnirenko theorem,

\[
\mathcal{H}(V, \Delta_{d-1}) \leq \begin{cases} 
D^{n-m}V(\Delta_d, \Delta_x) & \text{in case A} \\
D^{n-m}V(\Delta_d, \Delta_x + \Delta_z) & \text{in case B}
\end{cases}
\]

One easily checks that in both cases, this agrees (up to a constant depending only on \(n, D\)) with the lower bound for \(\mathcal{H}(Z, \Delta_d)\) given in Lemma 25. The proof can be concluded exactly as in the proof of Lemma 22.

We leave the verification of the details of Lemma 23, Theorem 7 and their corollaries in this context for the reader. In a manner analogous to the result of §5.3.2, one obtains the following theorem, which extends the two main theorems of [9].

Theorem 8. Let \(p_1, \ldots, p_q \in U\) and suppose that \(\gamma\) has the \(D\)-property at \(p_i\) for every \(i\). Recall that \(m\) denotes the dimension of the Zariski closure of \(\gamma\). Let \(P \in R\) be a polynomial with \(P|_{\gamma} \neq 0\), and denote

\[
d_x := \max(\deg_x P, 1) \quad d_z := \max(\deg_z P, 1)
\]

Then

\[
\sum_{i=1}^q \text{mult}_{z=p} P(z, f_1(z), \ldots, f_n(z)) \leq \begin{cases} 
\beta_\gamma(d_z + q)d_x^{m-1} & \text{in case A} \\
\beta_\gamma(d_z + d_x + q)d_x^{m-1} & \text{in case B}
\end{cases}
\]

where \(\beta_\gamma\) is a constant depending only on \(\gamma\).
6. CONCLUDING REMARKS

In this paper we have defined the notion of the multiplicity of a cycle, specifically along a smooth analytic curve. We have restricted our attention to this case in order to simplify our presentation, and because this is the case which is needed for the classical multiplicity estimates that we have sought to strengthen. However, the simple algebraic nature of Definition 4 easily lends itself to generalization.

One interesting direction for such generalization is the study of foliations defined by several commuting vector fields (in place of the single vector field considered in this paper). Let $\xi_1, \ldots, \xi_m$ denote the germs of $m$ commuting polynomial vector fields in $\mathbb{C}^n$, and let $\mathcal{L}_p$ denote the germ of a smooth analytic leaf at the point $p$. One can consider multiplicity estimates of the following types:

- For a polynomial $P$, one may ask about $\text{ord}_p P|_{\mathcal{L}_p}$.
- More generally, for polynomials $P_1, \ldots, P_m$ one may ask about the multiplicity of their common root, $\text{mult}_p(P_1|_{\mathcal{L}_p}, \ldots, P_m|_{\mathcal{L}_p})$.

Multiplicity estimates in the multi-dimensional setting have been used in transcendental number theory, for instance in [20, 21, 34]. They have also been studied in a more geometric context, for instance in [14].

Definition 4 extends to the multi-dimensional setting without change — one should simply replace the ideal of definition of the curve $\gamma_p$ by the ideal of definition of $\mathcal{L}_p$. Moreover, a generalization of the Rolle-type Lemma 7 holds in this context as well. It appears plausible that much of the theory developed in this paper could be carried out for the multi-dimensional setting.

Finally, we would like to mention that multiplicity estimates have also been considered in the related context of functions satisfying Mahler-type functional equations, similarly leading to applications in transcendental number theory. Moreover, Nesterenko’s methods have been successfully applied in this context (see [32] for example). It would be interesting to see whether the ideas developed in this paper could similarly be applied in this context.

APPENDIX: AN OVERVIEW OF NESTERENKO’S APPROACH

Nesterenko’s approach follows the same three-step paradigm outlined in the beginning of §1.3.1. We briefly sketch how each of the steps is realized in his work, and point the reader to the analogous results in the present paper for comparison. When possible we have given references uniformly to [30] in order to allow the reader to follow all references with fixed notations, although many of the statements appear originally in Nesterenko’s earlier works. We refer the reader to [30] for the original references.

We remark generally that in Nesterenko’s approach one considers projective ideals and proves the main results in this context. The results in the original affine context later follow by a projectivization argument. For simplicity we speak of unmixed ideals below without qualification, and it is to be understood that these ideals are taken to be projective ideals, and some technical details related to projectivization are omitted.

Step 1. The basis of Nesterenko’s approach is a notion of a multiplicity associated to an ideal, based on elimination-theoretic ideas. Given a smooth analytic trajectory $\gamma_p$ at the point $p \in M$, Nesterenko associates to each unmixed ideal $I$ the notion of the order of $I$ along $\gamma_p$, denoted $\text{ord} I(\gamma_p)$. The definition of this order
is somewhat involved (see [30, Section 3]; cf. Definition 4). A rough idea (reinterpreted from Nesterenko’s formulation) for the construction is as follows. One first considers the Chow form associated canonically to \( I \). One then associates to this Chow form a *canonical system* of equations for \( I \), following a construction due to Chow and van der Waerden (see [15, 3.2.C]). The order of \( I \) along \( \gamma_p \) is defined to be the minimal order of any of these canonical equations along \( \gamma_p \).

For this notion it is important that the ambient space is \( \mathbb{C}P^n \), so that unmixed ideals can be parametrized by Chow forms. In order to study the case of mixed degrees, where the natural ambient space is rather \( \mathbb{C}P^1 \times \mathbb{C}P^n \), Nesterenko considers it as a projective \( n \)-space over the field \( \mathbb{C}(z) \). One can then consider Chow forms over the field \( \mathbb{C}(z) \) and carry out the preceding construction in a similar manner (although Nesterenko makes some technical modifications).

In the case of \( \mathbb{C}P^n \), each unmixed ideal has a naturally associated *degree*: its degree as a cycle in \( \mathbb{C}P^n \). In the mixed case, each unmixed ideal has two associated numbers: *degree* and *height* (see [30, Section 3]). They essentially correspond to its two cohomological components in the two-dimensional cohomology of \( \mathbb{C}P^1 \times \mathbb{C}P^n \) (with respect to the Kunneth generators). In either case, we shall refer to these numbers as the *degrees* of \( I \).

Nesterenko shows that the order of a principal ideal \( \langle P \rangle \) along \( \gamma_p \) is bounded in terms of the order of \( P \) along \( \gamma_p \) (see [29, Proposition 1]; cf. Proposition 5), thus completing the first step.

**Step 2.** To accomplish this step, Nesterenko uses two main results. First, he proves (see [30, Lemma 5.1]; cf. Lemma 7) that if \( I \) is an unmixed ideal, \( P \in I \) and \( Q = \xi P \notin I \) then there is an unmixed ideal \( J \) whose zeros coincide with the zeros of \( I + \langle Q \rangle \), such that:

- The degrees of \( J \) are appropriately bounded in terms of the degrees of \( I \) and \( Q \).
- \( \text{ord} I(\gamma_p) \leq \text{ord} J(\gamma_p) + C(I, Q) \) where \( C(I, Q) \) denotes a certain expression depending on the degrees of \( I \) and \( Q \).

Next, Nesterenko proves that for any unmixed ideal \( I \) not containing a proper \( \xi \)-invariant ideal, one can always choose a polynomial \( P \) as above with the degrees of \( P \) appropriately bounded in terms of the degrees of \( I \) (see [30, Lemma 5.4]; cf. Lemma 12 and Lemma 23). This lemma is the deepest and most technical part of the proof. It has appeared in various forms of increasing complexity in the work of Nesterenko: in the linear case [29], in the non-linear pure-degree case [28] and finally in the singular case with mixed degrees in [30].

**Remark 27.** The lemma above, in addition to being the deepest part of the proof, also plays the dominant role in determining the size of the multiplicative constant \( \alpha_f \) appearing in Theorem 1. Namely, this constant grows doubly-exponentially with the dimension \( n \). It is known from the work of Gabrielov [12] that the correct growth with respect to \( n \) is singly-exponential, at least in the non-singular case.

Our proof of this lemma follows a different approach, relying on the local nature of Lemma 7 and, at one crucial moment, on Gabrielov’s result (or the more refined form given in [4]). This allows us to give constants growing singly-exponentially with \( n \), and also to extend the result to the case of general Newton polytopes.

The combinations of these two lemmas allows one to construct from an unmixed ideal \( I \) a new unmixed ideal \( J \) of smaller dimension, such that the order of \( \text{ord} I(\gamma_p) \)
is bounded in terms of \(\text{ord} J(\gamma_p)\) and such the degrees of \(J\) are bounded in terms of the degrees of \(I\) — as long as \(I\) is positive dimensional and doesn’t contain a proper \(\xi\)-invariant ideal. This concludes step 2.

Step 3. Assume now that the D-property is satisfied with constant \(\chi\). The final step is accomplished by showing that if \(J\) is a zero dimensional ideal or an unmixed ideal contained in a proper \(\xi\)-invariant variety, then \(\text{ord} J(\gamma_p)\) is bounded by a constant depending on \(\chi\) (this is roughly [30, Lemma 5.3] formulated in the contrapositive; cf. Proposition [8]).

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