SOME BOUNDS FOR THE PSEUDOCHARACTER OF THE SPACE \( C_\alpha(X,Y) \)

ÇETIN VURAL

Abstract. Let \( C_\alpha(X,Y) \) be the set of all continuous functions from \( X \) into \( Y \) endowed with the set-open topology where \( \alpha \) is a hereditarily closed, compact network on \( X \). We obtain that:

i) \( \psi(f, C_\alpha(X,Y)) \leq w_{\alpha c}(X) \cdot \sup_{A \in \alpha} (\psi(f(A), Y) \cdot \sup_{A \in \alpha} (w(f(A))) \)

ii) \( \psi(f, C_\alpha(X,Y)) \leq w_{\alpha c}(X) \cdot psw_{\psi}(f(X), Y) \).

1. Introduction and Terminology

Let \( X \) and \( Y \) be topological spaces, and let \( C(X,Y) \) denote the set of all continuous mappings from \( X \) into \( Y \). Let \( \alpha \) be a collection of subsets of \( X \). The topology having subbase \( \{[A, V] : A \in \alpha \text{ and } V \text{ is an open subset of } Y \} \) on the set \( C(X,Y) \) is denoted by \( C_\alpha(X,Y) \) where \( [A, V] = \{f \in C(X,Y) : f(A) \subseteq V \} \). If \( \alpha \) consists of all finite subsets of \( X \), then the set \( C(X,Y) \) endowed with that topology is called pointwise convergence topology and denoted by \( C_p(X,Y) \).

The cardinality and the closure of a set is denoted by \(|A|\) and \(\text{cl}(A)\), respectively. The restriction of a mapping \( f : X \to Y \) to a subset \( A \) of \( X \) is denoted by \( f|_A \). \( T(X) \) denotes the set of all non-empty open subsets of a topological space \( X \). \( \text{ord}(x,A) \) is the cardinality of the collection \( \{A \in A : x \in A \} \). Throughout this paper \( X \) and \( Y \) are regular topological spaces, and \( \alpha \) is a hereditarily closed, compact network on the domain space \( X \). (i.e., \( \alpha \) is a network on \( X \) such that each member of it is compact and each closed subset of a member of it is a member of \( \alpha \).)

Without loss of generality, we may assume that \( \alpha \) is closed under finite unions. Recall that the weak \( \alpha \)-covering number of \( X \) is defined to be \( w_{\alpha c}(X) = \min \{|\beta| : \beta \subseteq \alpha \text{ and } \bigcup \beta \text{ is dense in } X \} \). The weight, density and character of a space \( X \) are denoted by \( w(X) \), \( d(X) \) and \( \chi(X) \), respectively. The \( i \)-weight of a topological space \( X \), is the least of cardinals \( w(Y) \) of the Tychonoff spaces \( Y \) which are continuous one-to-one images of \( X \). The pseudocharacter of a space \( X \) at a subset \( A \), denoted by \( \psi(A,X) \), is defined as the smallest cardinal number of the form \( |\mathcal{U}| \), where \( \mathcal{U} \) is a family of open subsets of \( X \) such that \( \bigcap \mathcal{U} = A \). If \( A = \{x\} \) is a singleton, then we

2010 Mathematics Subject Classification. 54C35, 54A25, 54C05.
Key words and phrases. Pseudocharacter, function space.

The author acknowledge Mr. Hasan G"ul for the first draft of manuscript.
write $\psi(x, X)$ instead of $\psi(\{x\}, X)$. The pseudocharacter of a space $X$ is defined to be $\psi(X) = \sup\{\psi(x, X) : x \in X\}$. The diagonal number $\Delta(X)$ of a space $X$ is the pseudocharacter of its square $X \times X$ at its diagonal $\Delta_X = \{(x, x) : x \in X\}$.

The pseudocharacter of the space $C(X, Y)$ has been studied, and some remarkable equalities or inequalities was obtained between the pseudocharacter of the space $C(X, Y)$ for certain topologies and some cardinal functions on the spaces $X$ and $Y$. For instance, in [3], the inequalities $\psi(Y) \leq \psi(C_p(X, Y)) \leq \psi(Y) \cdot d(X)$ and, in [1] and [2], the equalities $\psi(C_p(X, \mathbb{R})) = d(X) = iw(C_p(X, \mathbb{R}))$, and in [6], $\psi(C_\alpha(X, \mathbb{R})) = \Delta(C_\alpha(X, \mathbb{R})) = \omega c(X)$ were obtained. In this paper, when the range space $Y$ is an arbitrary topological space instead of the space $\mathbb{R}$, we obtained some inequalities between the pseudocharacter of the space $C_\alpha(X, Y)$ at a point $f$ and the weak $\alpha$-covering number of the domain space $X$ and some cardinal functions on the range space $Y$.

We assume that all cardinal invariants are at least the first infinite cardinal $\aleph_0$.

Notations and terminology not explained above can be found in [4] and [5].

2. Main Results

First, we give an inequality between the pseudocharacter of a point $f$ in the space $C_\alpha(X, Y)$ and some cardinal functions on spaces $X$ and $Y$.

**Theorem 2.1.** For each $f \in C_\alpha(X, Y)$, we have

$$\psi(f, C_\alpha(X, Y)) \leq \omega c(X) \cdot \sup_{A \in \alpha} (\psi(f(A), Y)) \cdot \sup_{A \in \alpha} (w(f(A)))$$

**Proof.** Let $\omega c(X) \cdot \sup\{\psi(f(A), Y) : A \in \alpha\} \cdot \sup\{w(f(A)) : A \in \alpha\} = \kappa$. The inequality $\omega c(X) \leq \kappa$ gives us a subfamily $\beta = \{A_i : i \in I\}$ of $\alpha$ such that $|I| \leq \kappa$ and $X = cl(\bigcup \beta) = cl(\bigcup \{A_i : i \in I\})$. Since $\psi(f(A_i), Y) \leq \kappa$ for each $i \in I$, there exists a family $\mathcal{V}_i$ consisting of open subsets of the space $Y$ such that $|\mathcal{V}_i| \leq \kappa$ and $f(A_i) = \bigcap \{V : V \in \mathcal{V}_i\}$ for each $i \in I$. Since $w(f(A_i)) \leq \kappa$ for each $i \in I$, the subspace has a base $\mathcal{B}_i$ with $|\mathcal{B}_i| \leq \kappa$. For each $i \in I$, let $\mathcal{H}_i = \{[A_i, f^{-1}(cl(G)), Y \setminus cl(U)] : G, U \in \mathcal{B}_i \text{ and } cl(G) \cap cl(U) = \emptyset\}$, $\mathcal{R}_i = \{[A_i, V] : V \in \mathcal{V}_i\}$ and $\mathcal{W} = (\bigcup_{i \in I} \mathcal{R}_i) \cup (\bigcup_{i \in I} \mathcal{H}_i)$. It is clear that $|\mathcal{W}| \leq \kappa$ and $f \in \mathcal{W}$ for each $W \in \mathcal{W}$. Now, we shall prove that $\bigcap \mathcal{W} = \{f\}$. Take a $g \in \bigcap \mathcal{W}$. We claim that $g|_{A_i} = f|_{A_i}$ for each $i \in I$. Assume the contrary. Suppose $g|_{A_j} \neq f|_{A_j}$ for some $j \in I$ that is, we have an $x \in A_j$ such that $f(x) \neq g(x)$. Since $g \in \bigcap \mathcal{W}$ and $f(A_j) = \bigcap \{V : V \in \mathcal{V}_j\}$, we have $g(A_j) \subseteq f(A_j)$. Therefore $g(x) \in f(A_j)$ and $f(x) \in f(A_j)$. Since $f(x) \neq g(x)$ and the space $Y$ is regular, there exist $G$ and $U$ in $\mathcal{B}_j$ such that $f(x) \in cl(G)$, $g(x) \in cl(U)$ and $cl(G) \cap cl(U) = \emptyset$. On the other hand, since $[A_j \setminus f^{-1}(cl(G)), Y \setminus cl(U)] \in \mathcal{H}_j$ and
Lemma 2.5. Let \( Z \) be a subspace of the space \( X \). Then we have \( \psi ( Z, X ) \leq \kappa \).

Proof. Let \( A \) be a family of open subsets of \( X \) satisfying \( \psi ( A, Z ) \leq \kappa \) and \( \bigcap \{ V \in A : v \in Z \} = \{ z \} \), for each \( z \in Z \). Let \( K \) be any compact subspace of \( Z \) and let \( \mu = \{ W : W \subseteq V \text{ and } W \text{ is a minimal finite open cover for } K \} \). By Miščenko’s lemma, we have \( |\mu| \leq \kappa \).

Define the family \( \mathcal{O} = \{ \bigcup_{W \in \mu} W : W \in \mu \} \). It is clear that \( |\mathcal{O}| \leq \kappa \) and it can be easily seen that \( \bigcap \mathcal{O} = K \). Hence \( \psi ( K, X ) \leq \kappa \). \( \square \)

Lemma 2.6. Let \( Z \) be subspace of the space \( X \) such that \( \psi ( Z, X ) \leq \kappa \). Then we have \( w ( K ) \leq \kappa \) for each compact subset \( K \) of \( Z \).

Proof. Let \( K \) be a compact subset of \( Z \). Clearly, \( ps w ( K ) \leq ps w ( Z ) \leq ps w ( Z, X ) \). The compactness of \( K \) leads us to the fact that \( w ( K ) = ps w ( K ) \). [in [5], Ch. 1, Theorem 7.4]. Hence the claim. \( \square \)

Now, we are ready to give another bound for the pseudocharacter of the space \( C_\alpha ( X, Y ) \) at a point \( f \).

Theorem 2.7. For each \( f \in C_\alpha ( X, Y ) \), we have

\[
\psi ( f, C_\alpha ( X, Y ) ) \leq \omega c ( X ) \cdot ps w ( f ( X ), Y ) .
\]
Proof. Let $\omega c(X) \cdot psw_{\epsilon}(f(X), Y) = \kappa$, and let $\beta = \{A_i : i \in I\}$ be a subfamily of $\alpha$ such that $cl(\bigcup \beta) = X$ and $|I| \leq \kappa$. The compactness of $A_i$ for each $i \in I$ and the inequality $psw_{\epsilon}(f(X), Y) \leq \kappa$ lead us to the facts that $\psi(f(A_i), Y) \leq \kappa$ and $w(A_i) \leq \kappa$ for each $i \in I$, by lemmas 2.4 and 2.5. Therefore, by Theorem 2.1, we have $\psi(f, C_\alpha(X, Y)) \leq \kappa$. □

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Gazi Universitesi, Fen Fakultesi, Matematik Bolumu, 06500 Teknikokullar, Ankara, Turkey
E-mail address: cvural@gazi.edu.tr