Propagation of Chaos in a Coagulation Model

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M. Escobedo\textsuperscript{1,2} and F. Pezzotti \textsuperscript{3}

\textbf{Abstract.} A deterministic coalescing dynamics with constant rate for a particle system in a finite volume with a fixed initial number of particles is considered. It is shown that, in the thermodynamic limit, with the constraint of fixed density, the corresponding coagulation equation is recovered and global in time propagation of chaos holds.

\textbf{Key words.} Coagulation equation, BBGKY hierarchy, propagation of chaos.

1 Introduction

Coagulation equations are widespread models describing large sets of coalescing particles. It is argued in such descriptions that the coagulation rate depends on the masses of the particles. In the spatially homogeneous models it is further assumed that spatial fluctuations in the mass density are negligible.

In the first example of this sort, proposed by Smoluchowski \cite{21}, particles of radius $r$ moving by Brownian motion with variance proportional to $1/r$ meet at rate proportional to $(r_1 + r_2)(r_1^{-1} + r_2^{-1})$.

The concentration $f(t, m)$ of particles of mass $m = 1, 2, 3, \cdots$ at time $t$ satisfies the equation:

$$\frac{\partial f}{\partial t}(t, m) = \frac{1}{2} \sum_{k=1}^{m-1} K(m - k, k)f(t, m - k)f(t, k) - f(t, m)\sum_{k=1}^{\infty} K(m, k)f(t, k)$$

$$K(m, k) = \left(m^{1/3} + k^{1/3}\right)\left(m^{-1/3} + k^{-1/3}\right)$$

If masses are considered to take continuous positive values we are generally led to the continuous version:

$$\frac{\partial f}{\partial t}(t, m) = \frac{1}{2}\int_{0}^{m} K(m - m', m')f(t, m - m')f(t, m')dm' - f(t, m)\int_{0}^{\infty} K(m, m')f(t, m')dm' \quad (1.1)$$

Different coagulation kernels $K(m, m')$ may be considered, in particular constant, additive and multiplicative kernels have been widely studied. The concentration $f(t, m)$ is defined as the average

\textsuperscript{1}Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, E–48080 Bilbao, Spain. E–mail: miguel.escobedo@ehu.es

\textsuperscript{2}Basque Center for Applied Mathematics (BCAM), Bizkaia Technology Park, Building 500, E–48160 Derio, Spain.

\textsuperscript{3}Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, E–48080 Bilbao, Spain. E–mail: federica.pezzotti@ehu.es
number of clusters of mass $m$ per unit volume at time $t$ in the discrete case, and as the average number of clusters of mass in $[m, m + dm]$ per unit volume at time $t$ in the continuous case. Our purpose is to deduce the continuous coagulation equation as the limit of equations describing a coalescing system of finite (large) number of particles in a finite spatial volume by means of PDE arguments.

To this end we consider, in a given volume $V$, a system of particles whose number changes in time due to the coagulation process. At any time $t \geq 0$, given any $N \in \mathbb{N}$ and $(m_1, m_2, \ldots, m_N) \in (\mathbb{R}^+)^N$ we consider the mass distribution functions $P_N(t, m_1, ..., m_N)$. Each of these functions describe the state of a system constituted by $N$ particles in a volume $V$ where, at time $t$, one particle has mass between $m_1$ and $m_1 + dm_1$, one has mass between $m_2$ and $m_2 + dm_2$ and so on. We may then define the probability to have $N$ particles in the volume $V$ at time $t = 0$ as the following:

$$P(0, N) = \frac{1}{N!} \int_0^\infty \cdots \int_0^\infty P_N^0(m_1, ..., m_N)dm_1 \cdots dm_N$$

and

$$\sum_{N=1}^\infty P(0, N) = 1$$

Given any $N_0 \in \mathbb{N}^*$ we consider initial data satisfying the following conditions:

$$\left\{ \begin{array}{l}
(i) \quad P_N^0(m_1, ..., m_N) = 0, \text{ for all } N \neq N_0, \\
(ii) \quad P_{N_0}^0(m_1, ..., m_{N_0}) = (N_0)! f_0(m_1) \otimes \cdots \otimes f_0(m_{N_0}), \quad m_i \in (0, +\infty), \quad i = 1, 2, \ldots N_0
\end{array} \right. \quad (1.2)$$

where:  $f_0 \geq 0, \int_0^\infty f_0(m) dm = 1$.

Condition (i) expresses that at $t = 0$ the system has exactly $N_0$ particles. It is easily seen that conditions (i) and (ii) imply $P(0, N) = \delta(N - N_0)$. The average number of particles is then:

$$\sum_{N=1}^\infty N P(N, 0) = N_0.$$ 

as expected.

The starting point of our analysis are the evolution equations satisfied by the mass distribution functions $P_N(t, m_1, ..., m_N)$ throughout the coagulation process. These equations had been obtained in [15] and later in [8], [22] and [14]. It is then possible to deduce a system of equations for the correlation functions:

$$f_j(m_1, \ldots, m_j, t) = \sum_{N=j}^\infty \frac{1}{(N-j)!} \int_0^\infty dm_{j+1} \cdots \int_0^\infty dm_N P_N(m_1, \ldots, m_N, t).$$
where \( j \in \mathbb{N}^* \). Notice that for \( j = 1 \) we have

\[
\overline{N}(t) = \sum_{N=1}^{\infty} N P(t, N) = \sum_{N=1}^{\infty} \frac{1}{(N-1)!} \int_0^\infty \cdots \int_0^\infty \ P_N(t, m_1, m_2, \ldots, m_N) \ dm_1 \cdots dm_N \\
= \int_0^\infty f_1(t, m_1) \ dm_1
\]

and therefore \( f_1(t) \) is the density function associated to the mean number of particles at time \( t \).

Consider now the following rescaled functions:

\[
f^V_j(t, m_1, \ldots, m_j) = f_j(t, m_1, \ldots, m_j) \ V_j. \tag{1.3}
\]

We have, for \( j = 1 \):

\[
\int_0^\infty f^V_1(t, m_1) \ dm_1 = \frac{N(t)}{V} = \text{mean number of particles per unit volume at time } t \geq 0.
\]

We then study the limit of these rescaled functions as

\[
N_0 \to +\infty, \ V \to +\infty, \ \frac{N_0}{V} \to \rho_0. \tag{1.4}
\]

By (1.2), we have

\[
f^V_j(0, m_1, \ldots, m_j) = \frac{N_0(N_0 - 1) \cdots (N_0 - j + 1)}{V_j} f_0^{\otimes j}(m_1, \ldots, m_j) \tag{1.5}
\]

and it easily follows that:

\[
\forall j \in \mathbb{N}^* : \lim_{V, N_0 \to +\infty} \ N_0 \to \rho_0 \ ||f^V_j(t) - (\rho_0)^j f_0^{\otimes j}||_{L^1(\mathbb{R}_+^j)} = 0. \tag{1.6}
\]

The main result of the paper is the following.

**Theorem 1.1** Suppose that the coagulation kernel \( K \) is constant. Let \( \{f^V_j(t)\}_{j \in \mathbb{N}^*} \) be the sequence of functions defined in (1.3) with initial data (1.5). Then,

\[
\forall j \geq 1 : \lim_{V, N_0 \to +\infty} \ N_0 \to \rho_0 \ ||f^V_j(t) - f(t)^{\otimes j}||_{L^1(\mathbb{R}^+_j)} = 0 \tag{1.7}
\]

where \( f \) is the unique solution in \( C([0, +\infty); L^1(\mathbb{R}^+_1)) \) of the coagulation equation (1.1) with initial datum \( \rho_0 f_0 \).

The convergence result (1.7) under the hypothesis (1.5) for the initial data is usually known as propagation of chaos. By the initial condition (1.5) the particles are identically and independently distributed.
at time $t = 0$. The conclusion (1.7) means that at any time $t > 0$, the system of particles has still that property but only asymptotically in the limit (1.4) (cf. [18] and references therein).

The suitable way to describe the finite system of coalescing particles is the stochastic Markov process called Marcus-Lushnikov process. This has been introduced in [15] and later in [8], [22] and [14]. In these four references the authors obtain and pay special attention to the evolution equation satisfied by the probability of the stochastic process, called joint frequency or mass distribution function. They deduce from that equation the system of equations satisfied by the moments of the mass distribution function. The conditions under which this system may be approximated by the coagulation equation are discussed. It is seen in particular that this depends on the no correlation between the numbers of particles of different mass (cf. also [20] for the continuous case).

In [10], [11], [17], [7] the authors consider the Marcus-Lushnikov process itself. The weak convergence of suitably rescaled versions of these processes, called stochastic coalescents, towards the solutions of the coagulation equation is proved under different conditions on the coagulation kernel $K$, both in the discrete and the continuous case.

We believe nevertheless that the questions raised in [15], [8], [22] and [14] about the system of equations satisfied by the moments are of interest. In particular, to study the case of continuous masses via the equations for the mass distribution functions by means of PDE arguments is worthwhile, for itself and for further work [9]. That leads very naturally to a set of equations for the correlation functions $f_j$’s that, as we have already said, has been discussed to some extent in [15], [8], [22] and [14]. The system of rescaled correlation functions $f_j^V$ is similar to the BBGKY hierarchy appearing in the study of many particles hamiltonian systems. Notice that in such cases the underlying microscopic dynamics is deterministic (since it is given by the Newton equations associated with the Hamiltonian under consideration), while the dynamics of the coalescing particles is probabilistic, given by the Marcus-Lushnikov process. Nevertheless, our system of countably coupled equations may be treated along similar lines and our result may be stated in the same language in terms of propagation of chaos. These arguments inspired from many particles hamiltonian systems have already been used in the context of particle’s coalescence in [13] where propagation of chaos is proved for a coagulation equation with spatial dependence and constant kernel.

Let us recall here that for mean field models of classical particles with suitable two body potential, the propagation of chaos holds true [4], [5], [16] and may be seen as a law of large numbers. The method of the proof is based on the use of the so called empirical distribution and, as explained in [9], it relies upon the following facts: the empirical distribution is a weak solution of the Vlasov equation; weak solutions to the Vlasov equation are continuous with respect to the initial datum in the topology of the weak convergence of the measures. In that case no proof of this result based on PDE arguments is known (cf. for example the discussion in Section 1.4 of [18]).

When the coalescing kernel is constant, as it is considered in this work, the proof of the main result is rather simple and clear, due to the fact that the suitable functional space, the set of non negative integrable functions over $\mathbb{R}^+$, is easily seen to be globally preserved along the time evolution of the system. The stochastic coalescent for $K = 1$ is known as Kingman’s coalescent. The construction goes back to Kingman [12] and it has been extensively studied. Some details and references may be found in [2] and [3]. For more general kernels the proofs may become more technical and involved and some other delicate questions may appear (cf. [6]).

The plan of the paper is the following. In Section 2 we briefly recall the description of the finite
particle system and present an existence and uniqueness result for the system of equations satisfied by the mass distribution functions. In Section 3 we introduce the correlation functions and deduce the system of equations that they satisfy. Two simple time asymptotic properties of that system are derived in Section 4. The rescaled system is introduced in Section 5 where the main theorem is proved.

2 The particle system

We want to describe a finite system of particles contained in a finite volume \( V \) whose number \( N \) is not fixed through time. To this end we consider the probability space defined as follows.

The sample space is:

\[
\Omega = \bigcup_{N=1}^{\infty} (N, \mathcal{P}(\mathbb{R}^+)) \subset \mathbb{N}^* \times \mathcal{P}(\mathbb{R}^+) \tag{2.1}
\]

\( \mathcal{P}(\mathbb{R}^+) \) : subsets of \( \mathbb{R}^+ \) with cardinal \( N \).

It is the set of all the pairs \( (N, \omega_N) \) where \( N \in \mathbb{N}^* \) and \( \omega_N \) is any finite subset of \( N \) positive real numbers \( m_1, m_2, \ldots, m_N \). For every time \( t \geq 0 \) the probability distribution of each state \( (N, (m_1, \ldots, m_N)) \in \Omega \) is \( P_N(t, m_1, \ldots, m_N)/N! \) where \( \{P_N(t)\}_{N \in \mathbb{N}^*} \) is a sequence of non negative functions \( P_N(t) = P_N(t, \cdot) \), each of them defined on \( \mathcal{P}(\mathbb{R}^+) \) and normalized according to:

\[
\sum_{N=1}^{\infty} \frac{1}{N!} \int_{0}^{\infty} \cdots \int_{0}^{\infty} dm_N P_N(t, m_1, \ldots, m_N) = 1 \tag{2.3}
\]

The functions \( P_N(t, m_1, \ldots, m_N) \) are assumed to be symmetric with respect to any permutation of the indices \( 1, \ldots, N \) and no restrictions are imposed on the range of mass values other than \( m_i > 0 \) for \( i = 1, \ldots, N \). Each \( P_N = P_N(t, m_1, \ldots, m_N) \) is the mass distribution function of the \( N \)-particle configuration \( (m_1, \ldots, m_N) \). For every \( N \in \mathbb{N}^* \) the function:

\[
P(t, N) = \frac{1}{N!} \int_{0}^{\infty} \cdots \int_{0}^{\infty} dm_N P_N(t, m_1, \ldots, m_N) \tag{2.4}
\]

is the probability that at time \( t \) the system is constituted by \( N \) particles and the normalization (2.3) is natural.

**Remark 2.1** The factor \( 1/N! \) in definition (2.4) is needed to compensate for counting all the \( N! \) physically equivalent ways of arranging the masses \( m_1, \ldots, m_N \) in order of size.

Let \( A_V(m, \mu) \geq 0 \) be the rate (per unit time) at which a single particle of mass \( m \) collides and coalesces with a particle of mass \( \mu \) (given only that each is in the considered spatial region of volume \( V \)). We assume that:

\[
A_V(m, \mu) = A_V(\mu, m), \tag{2.5}
\]

as it is reasonable from a physical point of view.
In a coalescing system the number of particles will be varying along the time evolution (actually it will be decreasing) so what is meaningful is to consider the time evolution of the sequence of mass distribution functions \( \{ P_N(t) \}_{N \in \mathbb{N}} \). As explained in [15], [8], [22] and [14] the evolution equations for the mass distribution functions are:

\[
\frac{\partial}{\partial t} P_N = \frac{1}{2} \sum_{\ell=1}^{N} \int_0^{m_\ell} d\mu \, A_V(m_\ell - \mu, \mu) P_{N+1}(m_1, \ldots, m_{\ell-1}, m_\ell - \mu, m_{\ell+1}, \ldots, m_N, \mu, t) +
\]

\[
- \frac{1}{2} \sum_{\ell \neq q} P_N(m_1, \ldots, m_N, t) A_V(m_\ell, m_q). \quad N = 1, 2, \ldots \tag{2.6}
\]

The first term on the right hand side of (2.6) is the so-called gain term and it describes the positive contribution due to the coagulation of a particle of mass \( \mu \), with \( \mu \leq m_\ell \), with a particle of mass \( m_\ell - \mu \) (in an \( N + 1 \)-particle configuration), giving rise to a particle of mass \( m_\ell \) (in an \( N \)-particle configuration). The second term on the right hand side of (2.6) is the so-called loss term and it describes the negative contribution due to the coagulation of a particle of mass \( m_\ell \) with a particle of mass \( m_q \) (in an \( N \)-particle configuration), giving rise to a particle of mass \( m_\ell + m_q \) (in an \( N - 1 \)-particle configuration). That set of equations completely neglects the contributions that would be due either to many-body collisions (e.g., three-body collisions passing from an \( N + 2 \) configuration to an \( N \) configuration or from an \( N \) configuration to an \( N - 2 \) configuration), or to the occurrence of multiple binary collision (e.g., double binary collision again passing from an \( N + 2 \) (or \( N \)) configuration to an \( N \) (or \( N - 2 \)) configuration).

An important feature of the system (2.6) is that the total mass is preserved. More precisely, denoting by \( M_N \) the total mass of the configuration \( m_1, \ldots, m_N \), i.e. \( m_1 + \cdots + m_N = M_N \), then the only processes that have a non zero contribution in the time variation of the distribution \( P_N(m_1, \ldots, m_N, t) \) are those associated with configurations with the same total mass \( M_N \). In fact, the gain term in (2.6) takes into account \( N + 1 \)-particle configurations with total mass \( m_1 + \cdots + m_\ell - \mu + \cdots + m_N + \mu = M_N \) and the loss term in (2.6) leads to \( N - 1 \) particle configurations in which the total mass is \( m_1 + \cdots + (m_\ell + m_q) + \cdots + m_N = M_N \).

The system of equations (2.6) has been studied in [15], [8], [22] and [14]. For the sake of completeness we present in this Section an existence and uniqueness result that suits our purposes.

### 2.1 The coalescence rate

In general the coagulation kernel \( A_V(m_1, m_2) \) at which two particles of masses \( m_1 \) and \( m_2 \) coalesce in a volume \( V \), has the following form:

\[
A_V(m_1, m_2) = C_V A(m_1, m_2). \tag{2.7}
\]

The function \( A(m_1, m_2) \) encodes only the dependence of the coagulation rate on the masses \( m_1 \) and \( m_2 \). The term \( C_V \) contains the dependence of that rate with respect to the volume \( V \) as well as that on other physical properties of the coalescence process under consideration. An example (considered for instance in [15] and [8]) is the following:

\[
C_V = \frac{1}{V} E(|v_1 - v_2|) \pi \tag{2.8}
\]
where \( E(|v_1 - v_2|) \) is the average relative velocity of the two particles. The volume dependence of \( C_V \) takes into account the fact that the coagulation rate increases as the proportion of the volume occupied by the particles with respect to the total volume \( V \) increases. That gives a dependence inversely proportional to \( V \), i.e. like \( 1/V \). We consider in this paper the simplest possible case compatible with our purpose, that is to study the limit of the finite particle system as \( V \to +\infty \) at fixed density, namely:

\[
A_V(m_1, m_2) = \frac{1}{V} \tag{2.9}
\]

Other kernels are considered in [6].

### 2.2 The initial data

We consider system (2.6) with initial datum:

\[
\begin{align*}
P_0^N(m_N) &= \begin{cases} 
(N_0)! f_0(m_1) \ldots f_0(m_{N_0}), & \text{if } N = N_0, \\
0, & \text{if } N \neq N_0,
\end{cases} 
\end{align*} \tag{2.10}
\]

where the function \( f_0 \) is such that:

\[
f_0(m) \geq 0 \quad \text{a.e.} \quad \int_0^\infty dm f_0(m) = 1. \tag{2.11}
\]

By choosing the initial datum as in (2.10)-(2.11), and defining:

\[
P^0(N) = \frac{1}{N!} \int_0^\infty dm_1 \ldots \int_0^\infty dm_N \ P_0^0(m_1, \ldots, m_N), \tag{2.12}
\]

for every \( N \in \mathbb{N}^* \), we have

\[
\sum_{N=1}^{\infty} P^0(N) = 1 \tag{2.13}
\]

and condition (2.3) is satisfied at time \( t = 0 \). With that choice of initial datum:

\[
P^0(N) = \delta(N - N_0),
\]

i.e. at \( t = 0 \) our system has exactly \( N_0 \) particles.

### 2.3 Well-posedness of the equation for \( P_N(t) \) for the constant kernel

For every \( N \in \mathbb{N}^* \) we define the operator \( G_N \), mapping \( N + 1 \)-particle functions into \( N \)-particle functions, as follows:
\[ G_N(Q_{N+1})(m_N) = \frac{1}{2V} \sum_{\ell=1}^{N} \int_{0}^{m_\ell} dm_{N+1} \; Q_{N+1}(m_1, \ldots, m_{\ell-1}, m_\ell - m_{N+1}, m_{\ell+1}, \ldots, m_N, m_{N+1}), \]  

(2.14)

where \( Q_{N+1} \in L^1((\mathbb{R}^+)^{N+1}) \) and we denote:

\[ m_N := (m_1, \ldots, m_N). \]  

(2.15)

We shall also use the following notation:

\[ \int dt_N := \int_{0}^{t} dt_1 \ldots \int_{0}^{t_{N-1}} dt_N. \]  

(2.16)

and denote \( H \) the set:

\[ H = \{ \{ f_N \}_{N \in \mathbb{N}^*}; \; f_k \in L^1((\mathbb{R}^+)^k), \; k \in \mathbb{N}^* \}. \]  

(2.17)

**Definition 2.2** We say that a sequence \( \{ P_N(t) \}_{N \in \mathbb{N}^*} \in H \) solves the system (2.6) if, for every \( t > 0 \) and every \( N \in \mathbb{N}^* \), each term in (2.6) belongs to \( L^1((\mathbb{R}^+)^N) \) and the equality holds in \( L^1((\mathbb{R}^+)^N) \).

**Lemma 2.3** For every \( Q_{N+1} \in L^1((\mathbb{R}^+)^{N+1}) \):

\[ ||G_N(Q_{N+1})||_{L^1((\mathbb{R}^+)^N)} \leq \frac{N}{2V} ||Q_{N+1}||_{L^1((\mathbb{R}^+)^{N+1})}. \]  

(2.18)

If \( Q_{N+1} \in L^1((\mathbb{R}^+)^{N+1}) \) and \( Q_{N+1} \geq 0 \) then

\[ ||G_N(Q_{N+1})||_{L^1((\mathbb{R}^+)^N)} = \frac{N}{2V} ||Q_{N+1}||_{L^1((\mathbb{R}^+)^{N+1})}. \]  

(2.19)

**Proof of Lemma 2.3** We start proving (2.18). To this end we write:

\[ ||G_N[Q_{N+1}]||_{L^1((\mathbb{R}^+)^N)} = \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} dm_1 \cdots dm_N \; |G_N[Q_{N+1}](m_1, \ldots, m_N)| \]

\[ \leq \frac{1}{2V} \sum_{\ell=1}^{N} \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} dm_1 \cdots dm_N \; |G_N[Q_{N+1}](m_1, m_2, \ldots, m_\ell - m_{N+1}, \ldots, m_N, m_{N+1})| \]  

(2.20)

\[ = \frac{1}{2V} \sum_{\ell=1}^{N} \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \int_{0}^{\infty} dm_1 \cdots dm_N \; |G_N[Q_{N+1}](m_1, m_2, \ldots, m_\ell - m_{N+1}, \ldots, m_N, m_{N+1})| \]  

where in the last step we have used that the integrals with respect to the variables \( m_i \) for \( i = 1, \ldots, N \) are over independent domains and therefore can be performed in any order. The final and trivial step is to use Fubini’s theorem in the integrals with respect to \( m_\ell \) and \( m_{N+1} \). That may be done since by
hypothesis $Q_{N+1} \in L^1((\mathbb{R}^+)^{N+1})$ and therefore, for almost every $(m_1, m_2, \ldots, m_{\ell-1}, m_{\ell+1}, \ldots, m_N)$ $Q_{N+1}(m_1, m_2, \ldots, m_{\ell}, \ldots, m_N, m_{N+1}) \in L^1((\mathbb{R}^+)^2)$.

It follows from the hypothesis that $Q_{N+1} \geq 0$ the inequality in (2.20) would be an equality and we would have (2.19).

We can state now the existence and uniqueness result of solutions for system (2.6).

**Theorem 2.4** Suppose that the $A_N = V^{-1}$. Then, the Cauchy problem for the system (2.6) with initial data $P_N^0 \in \mathbb{N}^*$ defined in (2.10). (2.11) has a solution $P_N(t) \in \mathcal{H}$ for any $t$ and such that $P_N \in C^\infty([0, +\infty), L^1((\mathbb{R}^+)^N))$ for every $N \in \mathbb{N}^*$. Such a solution is given by

$$P_N(t, m_N) = \begin{cases} \int dt_{N_0-N} \left\{ e^{-\frac{(N-1)}{2V}}(t-t_1)G_N \right\} \cdots \left\{ e^{-\frac{(N_0-1)(N_0-2)}{2V}}(t_{N_0-N-1}t_{N_0-N})G_{N_0-1} \right\} \\ e^{-\frac{(N_0)(N_0-1)}{2V}}t_{N_0-N}P_{N_0}^0(m_{N_0}), \text{ for } 1 \leq N < N_0 \\ e^{-\frac{(N_0)(N_0-1)}{2V}}P_{N_0}^0(m_{N_0}), \text{ for } N = N_0 \\ 0, \text{ for } N > N_0. \end{cases}$$

(2.21)

Moreover, for every $t > 0$, $P_N(t) \geq 0$ and:

$$\sum_{N=1}^{\infty} P(t, N) = 1$$

(2.22)

with $P(t, N) := \frac{1}{N!} \int_0^\infty dm_1 \cdots \int_0^\infty dm_N P_N(t, m_1, \ldots, m_N)$

(2.23)

That solution is unique in the set of sequences $\{P_N(t)\}_{N \in \mathbb{N}^*} \in \mathcal{H}$ such that $P_N \in C^\infty([0, +\infty); L^1((\mathbb{R}^+)^N))$, $P_N \geq 0$ for all $N \in \mathbb{N}^*$ and the function $h_P(t) := \sum_{N=1}^{\infty} P_N(t, N)$ satisfies $h_P \in C^\infty([0, +\infty))$.

**Proof of Theorem 2.4** By our hypothesis $P_0^0 \in L^1((\mathbb{R}^+)^{N_0})$. By Lemma (2.3) we deduce that for every $1 \leq N \leq N_0$:

$$G_N G_{N+1} \cdots G_{N_0-1}[P_{N_0}^0] \in L^1((\mathbb{R}^+)^N).$$

Then, for any $a$ and $\tau$, the function

$$\eta(t, m_N) = e^{a(t-\tau)}G_N G_{N+1} \cdots G_{N_0-1}[P_{N_0}^0](m_N)$$

satisfies:

$$\eta \in C^\infty([0, +\infty), L^1((\mathbb{R}^+)^N)).$$

It then follows that $P_N(t)$ given by (2.21) is such that $P_N \in C^\infty([0, +\infty), L^1((\mathbb{R}^+)^N))$.

We check now that the above sequence $\{P_N(t)\}_{N \in \mathbb{N}^*}$ satisfies the system (2.6). The equations for $N \geq N_0 + 1$ are trivially satisfied. Suppose now that $N \leq N_0 - 1$. Taking the time derivative of
we obtain:

\[
\frac{\partial P_N}{\partial t}(t, m_N) = -\frac{N(N-1)}{2V} P_N(t, m_N) + \\
\int_0^t dt_2 \int_0^{t_2} dt_3 \ldots \int_0^{t_{N_0-1}} dt_{N_0-N} G_N \left\{ e^{-\frac{(N+1)N}{2V} (t-t_2) G_{N_0-1}} \right\} \ldots \\
\ldots \left\{ e^{-\frac{(N_0-1)(N_0-2)}{2V} t_{N_0-N-1} G_{N_0-1}} \right\} e^{-\frac{(N_0)(N_0-1)}{2V} t_{N_0-N} P^0_{N_0}(m_{N_0})}
\]

And, since

\[
\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{N_0-N-2}} dt_{N_0-N-1} \left\{ e^{-\frac{(N-1)}{2V} (t-t_1) G_{N_0-1}} \right\} \ldots \\
\ldots \left\{ e^{-\frac{(N_0-1)(N_0-2)}{2V} t_{N_0-N-2} G_{N_0-1}} \right\} e^{-\frac{(N_0)(N_0-1)}{2V} t_{N_0-N} P^0_{N_0}(m_{N_0})} \equiv P_{N+1}(t, m_{N+1})
\]

we obtain

\[
\frac{\partial P_N}{\partial t}(t, m_N) = -\frac{N(N-1)}{2V} P_N(t, m_N) + G_N [P_{N+1}](t, m_N)
\]

that is nothing but the \(N^{th}\) equation of system (2.6). If \(N = N_0\), we have:

\[
\frac{\partial P_{N_0}}{\partial t}(t, m_{N_0}) = -\frac{N_0(N_0-1)}{2V} P_{N_0}(t, m_{N_0}) \\
\equiv -\frac{N_0(N_0-1)}{2V} P_{N_0}(t, m_{N_0}) + G_{N_0} [P_{N_0+1}](t, m_{N_0})
\]

since, by (2.21), \(P_{N_0+1}(t) \equiv 0\) for any \(t\).

We check now (2.22). To this end we first notice that, by (2.21):

\[
\sum_{N=1}^{\infty} P(t, N) = \sum_{N=1}^{N_0} \frac{1}{N!} \int_0^\infty dm_1 \ldots \int_0^\infty dm_N P_N(t, m_1, \ldots, m_N).
\]
Therefore, using (2.6) and (2.19) (since \( P_N \geq 0 \)):

\[
\frac{d}{dt} \sum_{N=1}^{\infty} P(t, N) = - \sum_{N=1}^{N_0} \frac{N(N-1)}{N!2V}||P_N(t)||_{L^1((R^+)^N)} + \sum_{N=1}^{N_0} \frac{1}{N!}||G_N[P_{N+1}]||_{L^1((R^+)^N)}
\]

\[
= - \sum_{N=1}^{N_0} \frac{N(N-1)}{N!2V}||P_N(t)||_{L^1((R^+)^N)} + \sum_{N=1}^{N_0-1} \frac{N}{N!2V}||P_{N+1}(t)||_{L^1((R^+)^{N+1})}
\]

\[
= - \sum_{N=1}^{N_0} \frac{N(N-1)}{N!2V}||P_N(t)||_{L^1((R^+)^N)} + \sum_{M=2}^{N_0} \frac{M-1}{(M-1)!2V}||P_M(t)||_{L^1((R^+)^M)}
\]

\[
= - \sum_{N=1}^{N_0} \frac{N(N-1)}{N!2V}||P_N(t)||_{L^1((R^+)^N)} + \sum_{M=1}^{N_0} \frac{M(M-1)}{M!2V}||P_M(t)||_{L^1((R^+)^M)} = 0.
\]

Identity (2.22) follows from the fact that, by continuity and (2.13) one has:

\[
\sum_{N=1}^{N_0} P(t, N) = \sum_{N=1}^{N_0} P^0(N) = 1.
\]

In order to prove the uniqueness, we first show that any solution \( \{\overline{P}_N(t)\}_{N \in \mathbb{N}^+} \), such that \( \overline{P}_N(t) \geq 0 \) and \( h\overline{P}(t) = \sum_{N=1}^{\infty} \overline{P}(t, N) \) verifies \( h\overline{P} \in C^1((0, +\infty)) \), satisfies \( \overline{P}_N(t) \equiv 0 \) for all \( t \) and \( N \geq N_0 + 1 \). To this end we write:

\[
\frac{d}{dt} \sum_{N=N_0+1}^{\infty} \overline{P}(t, N) = - \sum_{N=N_0+1}^{\infty} \frac{N(N-1)}{N!2V}||\overline{P}_N(t)||_{L^1((R^+)^N)} + \sum_{N=N_0+1}^{\infty} \frac{1}{N!}||G_N[\overline{P}_{N+1}]||_{L^1((R^+)^N)}
\]

\[
= - \sum_{N=N_0+1}^{\infty} \frac{N(N-1)}{N!2V}||\overline{P}_N(t)||_{L^1((R^+)^N)} + \sum_{N=N_0+1}^{\infty} \frac{N}{N!2V}||\overline{P}_{N+1}(t)||_{L^1((R^+)^{N+1})}
\]

\[
= - \sum_{N=N_0+1}^{\infty} \frac{N(N-1)}{N!2V}||\overline{P}_N(t)||_{L^1((R^+)^N)} + \sum_{N=N_0+2}^{\infty} \frac{N(N-1)}{N!2V}||\overline{P}_N||_{L^1((R^+)^N)}
\]

\[
= - \frac{N_0(N_0+1)}{(N_0+1)!2V}||\overline{P}_{N_0+1}(t)||_{L^1((R^+)^{N_0+1})} < 0
\]

We deduce that:

\[
\sum_{N=N_0+1}^{\infty} \overline{P}(t, N) \leq \sum_{N=N_0+1}^{\infty} \overline{P}(0, N) = \sum_{N=N_0+1}^{\infty} P^0(N). \tag{2.24}
\]

Since, by hypothesis, the right hand side of (2.24) is zero and \( \overline{P}_N(t) \geq 0 \) we deduce that \( \overline{P}_N(t) \equiv 0 \) for every \( N \geq N_0 + 1 \) and \( t \geq 0 \). That leaves only a finite number of equations in the system (2.6), for \( N = 1, \cdots, N_0 \). Moreover the equation for \( N = N_0 \) yields:

\[
P_{N_0}(t, m_{N_0}) = P^0_{N_0}(m_{N_0}) e^{-\frac{S_0(N_0-1)}{2V}}.
\]

Then, the finite system decouples and may be explicitly solved for \( N = 1, \cdots, N_0 - 1 \) to obtain expression (2.21). \( \square \)
Remark 2.5 One would expect that for every initial data \( \{ P_0^N \}_{N \in \mathbb{N}^*} \in \mathcal{H} \) such that \( P_0^N \geq 0 \) for every \( N \) and \( \sum P_0(N) = 1 \) there exists a solution \( \{ P_N(t) \}_{N \in \mathbb{N}^*} \) such that, for every \( N \): \( P_N(t) \geq 0, P_N \in C([0, +\infty); L^1((\mathbb{R}^+)^N)) \cap C^1((0, +\infty); L^1((\mathbb{R}^+)^N)), h_P \in C^1(0, +\infty) \) and \( \sum_{N=1}^{\infty} P(t, N) = 1 \). Since such a result is not necessary for our main purpose, the proof of Theorem 2.4, we only prove the simpler result in Theorem 1.1.

3 Correlation Functions

For any fixed \( j \in \mathbb{N}^* \), we define the \( j \)-particle correlation function \( f_j(m_1, \ldots, m_j, t) \) at time \( t \) as

\[
f_j(m_1, \ldots, m_j, t) = \sum_{N=j}^{\infty} \frac{1}{(N-j)!} \int_0^\infty dm_{j+1} \cdots \int_0^\infty dm_N P_N(m_1, \ldots, m_N, t). \tag{3.1}
\]

Some well known general properties of such functions \( f_j \) are the following. At any time \( t \) the expected (or mean) number of particles \( N(t) \) defined as

\[ N(t) = \sum_{N=1}^{\infty} NP(t, N) \]

satisfies:

\[ N(t) = \int_0^\infty f_1(m_1, t)dm_1. \tag{3.2} \]

The function \( f_1 \) is then the density function associated to the average number of particles. More generally one may define:

\[ \frac{(N(t))(N(t)-1)\ldots(N(t)-j+1)}{(N(t)-1)\ldots(N(t)-j+1)} = \sum_{N=1}^{\infty} N(N-1)\ldots(N-j+1)P(t, N) \]

and one has

\[ \frac{(N(t))(N(t)-1)\ldots(N(t)-j+1)}{(N(t)-1)\ldots(N(t)-j+1)} = ||f_j||_{L^1((\mathbb{R}^+)^j)} \tag{3.3} \]

The functions \( f_j \)'s will be called correlation functions since their definition is very similar to that of the classical correlation functions in statistical mechanics (cf. [19]) and satisfy properties (3.2) and (3.3). Notice nevertheless that they slightly differ from the correlation functions in statistical mechanics since in particular, there is no activity parameter (cf. [19]).

As an immediate Corollary of Theorem 2.4 we have the following.

Corollary 3.1 Consider the solution \( \{ P_N(t) \}_{N \in \mathbb{N}^*} \) of the Cauchy problem (2.6)-(2.10)-(2.11), whose existence and uniqueness have been proved in Theorem 2.4. Then, for every \( j \in \mathbb{N}^* \) the functions \( f_j(t) \) defined as

\[
f_j(m_1 \cdots, m_j, t) = \sum_{N=j}^{\infty} \frac{1}{(N-j)!} \int_0^\infty dm_{j+1} \cdots \int_0^\infty dm_N P_N(t, m_1, \ldots, m_N). \tag{3.4}
\]
are such that:

\begin{enumerate}
\item \( f_j(0) = \frac{(N_0)!}{(N_0-j)!} f_0^\otimes j, \quad \forall j \in \{1, \ldots, N_0\}; \quad f_j(0) \equiv 0, \forall j \geq N_0 + 1. \) \hspace{1cm} (3.5)
\item \( f_j(t) \geq 0 \quad \forall j \geq 1 \) and \( \forall t \geq 0; \quad f_j(t) \equiv 0 \quad \forall t \geq 0 \) if \( j \geq N_0 + 1. \) \hspace{1cm} (3.6)
\item \( f_j \in C^\infty([0, +\infty); L^1((\mathbb{R}^+)^j)). \) \hspace{1cm} (3.7)
\item \( \|f_j(t)\|_{L^1((\mathbb{R}^+)^j)} \leq 2^{j-1} N_0! \left( t \over V \right)^{N_0-j} \sum_{N=0}^{N_0-j} \frac{2^N}{N!} t^{-N} V^N \text{ for } j = 1, \ldots, N_0. \) \hspace{1cm} (3.8)
\end{enumerate}

**Proof of Corollary 3.1** Only (3.8) needs an explanation. It follows from the fact that, by definition:

\[
\|f_j(t)\|_{L^1((\mathbb{R}^+)^j)} = \sum_{N=0}^{N_0-j+1} \frac{1}{(N-j)!} \|P_N(t)\|_{L^1((\mathbb{R}^+)^N)},
\]

where we used that \( P_N(t) \equiv 0 \) for \( N \geq N_0 + 1. \) On the other hand, by Theorem 2.4 for every \( N \leq N_0 \)

\[
\|P_N(t)\|_{L^1((\mathbb{R}^+)^N)} = \int dt_{N_0-N} \int dN e^{-\frac{N(N-1)}{2V}(t-t_1)} G_N \cdots G_N e^{-\frac{(N_0)(N_0-1)}{2V} t_{N_0-N} P_{N_0}(m_{N_0}, 0)}.
\]

Using (2.19) in Lemma 2.9 we deduce:

\[
\|P_N(t)\|_{L^1((\mathbb{R}^+)^N)} \leq \frac{t^{N_0-N}}{(N_0-N)!} 2^{-(N_0-N)} V^{-(N_0-N)} N(N+1) \cdots (N_0-1) \|P_{N_0}(0)\|_{L^1((\mathbb{R}^+)^{N_0})} = \frac{t^{N_0-N}}{(N_0-N)!} 2^{-(N_0-N)} V^{-(N_0-N)} \frac{(N_0-1)!}{(N-1)!} \|P_{N_0}(0)\|_{L^1((\mathbb{R}^+)^{N_0})}.
\]

Using the inequality

\[
\forall a \geq b \geq 1: \quad \frac{(a-1)!}{(a-b)!(b-1)!} \leq 2^{a-1},
\]

we obtain from (3.10):

\[
\|P_N(t)\|_{L^1((\mathbb{R}^+)^N)} \leq \frac{t^{N_0-N}}{(N_0-N)!} 2^{-(N_0-N)} V^{-(N_0-N)} 2^{N_0-1} (N_0-N)! \|P_{N_0}(0)\|_{L^1((\mathbb{R}^+)^{N_0})} = 2^{N-1} \frac{t^{N_0-N}}{(N_0-N)!} V^{-(N_0-N)} N_0!,
\]

where we used assumptions (2.10)-(2.11) on the initial datum \( P_{N_0}(0). \) Estimate (3.3) follows from (3.9) and (3.12). \( \square \)

### 3.1 The system of equations for the functions \( f_j \)

An immediate consequence of the above calculation and Corollary 3.1 is the following:

\[
\]
Theorem 3.2 Suppose that \( \{P_N(t)\}_{N \in \mathbb{N}^*} \) is the unique solution of (2.6) with initial data \( \{P_N^0\}_{N \in \mathbb{N}^*} \) defined in (2.10)-(2.11) and coagulation kernel \( A_V = V^{-1} \). Then the set of functions \( \{f_j(t)\}_{j \in \mathbb{N}^*} \) defined in (3.4) solves the following problem:

\[
\partial_t f_j = \frac{1}{2V} \sum_{\ell=1}^j \int_0^{m_\ell} d\mu f_{j+1}(m_1, \ldots, m_\ell - \mu, \ldots, m_j, \mu, t) + \frac{1}{2V} \frac{j(j-1)}{2} f_j(m_1, \ldots, m_j, t) - \frac{j}{V} \int_0^\infty d\mu f_{j+1}(m_1, \ldots, m_j, \mu, t), \quad j \in \mathbb{N}^*,
\]

Moreover the sequence \( \{f_j(t)\}_{j \in \mathbb{N}^*} \) is the unique solution in \( C([0, +\infty); L^1((\mathbb{R}^+)^j) \cap L^1((\mathbb{R}^+)^j)) \) of (2.13) with initial data given (3.5).

Proof of Theorem 3.2 A straightforward computation from (2.6) using the definition (3.4) shows that for any \( j \leq N_0 \) the function \( f_j \) satisfies the equation:

\[
\partial_t f_j(m_j, t) = \frac{1}{2V} \sum_{N=j}^{N_0} \frac{1}{(N-j)!} \sum_{\ell=1}^N \int \, d\mu m_{N,j} \int_0^{m_\ell} dm_{N+1} P_{N+1}(m_1, \ldots, m_\ell - m_{N+1}, \ldots, m_N, t) + \frac{1}{2V} \frac{N(N-1)}{2V} \int \, d\mu m_{N,j} P_N(m_1, \ldots, m_N, t),
\]

where from now on we use the notation:

\[
m_{N,j} := (m_{j+1}, \ldots, m_N) \quad \text{and} \quad \int \, d\mu m_{N,j} := \int_0^\infty dm_{j+1} \ldots \int_0^\infty dm_N.
\]

The first term in the right hand side of (3.14), or gain term, gives the following contributions:

\[
\frac{1}{2V} \sum_{N=j}^{N_0} \frac{1}{(N-j)!} \sum_{\ell=1}^N \int \, d\mu m_{N,j} \int_0^{m_\ell} dm_{N+1} P_{N+1}(m_1, \ldots, m_\ell - m_{N+1}, \ldots, m_N, t)
\]

\[
= \frac{1}{2V} \sum_{\ell=1}^j \int_0^{m_\ell} dm_{N+1} f_{j+1}(m_1, \ldots, m_\ell - m_{N+1}, \ldots, m_j, m_{N+1}, t) + \frac{1}{2V} \frac{N-j}{(N-j)!} \int \, d\mu m_{N,j+1} \int_0^{m_j+1} dm_{N+1} P_{N+1}(m_j, m_{j+1} - m_{N+1}, \ldots, m_N, t),
\]

where we have split the sum with respect to \( \ell \) in two parts, \( 1 \leq \ell \leq j \) and \( j+1 \leq \ell \leq N \), and used the symmetry of \( P_N \) with respect to any permutation of the indeces (that is preserved by the dynamics), to write:

\[
\sum_{\ell=j+1}^N \int_0^{m_\ell} dm_{N+1} P_{N+1}(m_1, \ldots, m_\ell - m_{N+1}, \ldots, m_N, t),
\]

\[
= (N-j) \int_0^{m_{j+1}} dm_{N+1} P_{N+1}(m_1, \ldots, m_{j+1} - m_{N+1}, \ldots, m_N, t).
\]
Using Fubini’s theorem and definition (3.4) we obtain

\[
\frac{1}{2V} \sum_{N=j}^{N_0} \frac{N - j}{(N - j)!} \int_0^\infty dm_{N+1} \int_0^{m_{j+1}} \int_0^{m_{j+1}} dm_{j+1} P_{N+1}(m_j, m_{j+1} - m_{N+1}, \ldots, m_{N+1}, t) = \frac{1}{2V} \int_0^\infty dm_{N+1} \int_0^\infty dm_{j+1} f_{j+2}(m_1, \ldots, m_{j+1}, m_{N+1}, t) \tag{3.17}
\]

On the other hand, by simple algebraic manipulations, the second term in the right hand side of (3.14), or loss term, gives:

\[
\sum_{N=j}^{N_0} \frac{1}{(N - j)!} \frac{N(N - 1)}{2V} \int_0^\infty dm_{j+1} \ldots \int_0^\infty dm_N P_N(m_1, \ldots, m_N, t) = \frac{j(j - 1)}{2V} f_j(m_1, \ldots, m_j, t) + \frac{j}{V} \int_0^\infty dm_{j+1} f_{j+1}(m_1, \ldots, m_{j+1}, t) + \frac{1}{2V} \int_0^\infty dm_{j+1} \int_0^\infty dm_{j+2} f_{j+2}(m_1, \ldots, m_{j+2}, t) \tag{3.18}
\]

By (3.14), (3.16), (3.17) and (3.18), (3.13) follows.

Only uniqueness remains to be proved. To this end we notice that since \(P_N(t) \equiv 0\) for every \(N > N_0\), the same is true for the sequence \(\{f_j(t)\}_{j \in \mathbb{N}}\), namely, \(f_j(t) \equiv 0\) for \(j > N_0\). Therefore the system (3.13) only contains a finite number of non trivial equations and, since the equation for \(j = N_0\) only involves \(f_{N_0}(t)\), the system decouples. We conclude using the same argument used in Theorem 2.4 to prove uniqueness for the mass distribution functions \(\{P_N(t)\}_{N \in \mathbb{N}}\). \(\Box\)

\textbf{Remark 3.3} If the initial data \(f_0\) satisfies not only (2.11) but also:

\[
m_0 = \int_0^\infty m f_0(m) dm < +\infty
\]

then, since, as we have seen above, \(f_1(t)\) is the density function associated to the average number of particles at time \(t\), the quantity:

\[\overline{M}(t) = \int_0^\infty m f_1(m, t) dm \tag{3.19}\]

represents the average mass in the system at time \(t\). If we integrate the equation for \(j = 1\) in system (3.13), a simple application of Fubini’s theorem yields

\[
\frac{d}{dt} \int_0^\infty m f_1(m, t) dm = 0 \tag{3.20}
\]

and then, using (2.11) and (3.3)

\[\overline{M}(t) = \overline{M}(0) = m_0 N_0 \quad \forall t > 0. \tag{3.21}\]

This mass conservation property is a well known feature of the coagulation equation with constant kernel.
The solution of the finite system for \( \{f_j(t)\}_{j=1}^{N_0} \)

In this Section we study the behavior of the sequence \( \{f_j(t)\}_{j=1}^{N_0} \) as \( t \to +\infty \).

**Proposition 4.1** Let \( \{f_j(t)\}_{j=1}^{N_0} \) be the solution of (3.13) with initial datum (3.5) given by Corollary 3.1. Then

\[
\lim_{t \to +\infty} N(t) = 1 \quad \text{and} \quad \lim_{t \to +\infty} \text{var}(N)(t) = 0,
\]

where \( \text{var}(N)(t) = \overline{(N(t))^2} - \overline{N(t)}^2 \) is the variance of the distribution on particle numbers \( P(t, N) \).

The proof of Proposition 4.1 follows from the three following auxiliary results.

**Lemma 4.2** For all \( t > 0 \) and \( j \):

\[
\frac{d}{dt} \| f_j(t) \|_{L^1((R^+)^j)} = -\frac{j}{2V} \| f_{j+1}(t) \|_{L^1((R^+)^{j+1})} - \frac{j(j-1)}{2V} \| f_j(t) \|_{L^1((R^+)^{j})} \leq 0.
\]

**Proof of Lemma 4.2** Since, by definition, \( f_j(t) \geq 0 \) for any \( j \) and \( t \), (the) Lemma 4.2 follows by a simple integration of (3.13) with respect to \( m_1 \ldots m_j \). \( \square \)

From (3.2) and Lemma 4.2 we deduce

**Corollary 4.3** (i) For all \( t > 0 \):

\[
\frac{d}{dt} N(t) \leq 0
\]

(ii) There exists \( N_\infty \in [1, N_0] \) such that \( \overline{N(t)} \to N_\infty \) as \( t \to +\infty \).

**Proof of Corollary 4.3** Only the fact \( N_\infty \geq 1 \) needs perhaps to be explained. From (i) we deduce the existence of \( N_\infty \in [0, N_0] \), the unique limit point of \( \overline{N(t)} \) as \( t \to +\infty \). On the other hand,

\[
\overline{N(t)} = \sum_{N=1}^{N_0} NP(t, N) \geq \sum_{N=1}^{N_0} P(t, N) = 1
\]

from where \( N_\infty \geq 1 \). \( \square \)

We also deduce:

**Corollary 4.4** For all \( t > 0 \) and every \( j \geq 1 \):

\[
(\overline{N(t)})^{(\overline{N(t)}-1)} \ldots (\overline{N(t)}-j+1) \leq e^{-\frac{(j-1)}{2V} t} N_0^j, \quad (4.1)
\]
Proof of Corollary 4.4 By Lemma 4.2:

\[
\frac{d}{dt} ||f_j(t)||_{L^1((\mathbb{R}^+)^j)} = -\frac{j}{2V} ||f_{j+1}(t)||_{L^1((\mathbb{R}^+)^{j+1})} - \frac{j(j-1)}{2V} ||f_j(t)||_{L^1((\mathbb{R}^+)^j)} \leq \frac{j(j-1)}{2V} ||f_j(t)||_{L^1((\mathbb{R}^+)^j)},
\]

(4.2)

that implies, by (3.5)

\[
||f_j(t)||_{L^1((\mathbb{R}^+)^j)} \leq e^{-\frac{j(j-1)}{2V}t} ||f_j(0)||_{L^1((\mathbb{R}^+)^j)} \leq e^{-\frac{j(j-1)}{2V}t} N_0^j,
\]

(4.3)

for any time \( t \). Since, by definition, \((N(t))(N(t) - 1)\ldots(N(t) - j + 1) = ||f_j(t)||_{L^1((\mathbb{R}^+)^j)}\), this concludes the proof. \( \square \)

Proof of Proposition 4.1 By Corollary 4.4 for \( j = 2 \):

\[
||f_2(t)||_{L^1((\mathbb{R}^+)^2)} = (N(t))(N(t) - 1) \leq e^{-\frac{1}{2V}t} N_0^2 \to 0, \quad \text{as} \quad t \to +\infty
\]

(4.4)

and then

\[
\lim_{t \to +\infty} \frac{(N(t))^2}{N(t)} = \lim_{t \to +\infty} \frac{N(t)}{N(t)} = N_\infty.
\]

(4.5)

Therefore, the variance of the distribution on particle numbers for large times is given by:

\[
\lim_{t \to +\infty} \text{var}(N)(t) = \lim_{t \to +\infty} \left(\frac{(N(t))^2}{N(t)} - \frac{N(t)^2}{N(t)}\right) = N_\infty - N_\infty^2.
\]

(4.6)

Since, by definition, the variance is non-negative

\[
N_\infty - N_\infty^2 \geq 0 \Rightarrow N_\infty(1 - N_\infty) \geq 0 \Rightarrow 0 \leq N_\infty \leq 1.
\]

(4.7)

On the other hand by Corollary 4.3 we know that \( N_\infty \geq 1 \), thus it has to be \( N_\infty = 1 \) and therefore

\[
\lim_{t \to +\infty} \text{var}(N)(t) = 0.
\]

(4.8)

Remark 4.5 In other words, the distribution of particle numbers is going to be a delta distribution, as it was at time \( t = 0 \), but now centered at \( N_\infty = 1 \). As expected, due to the coalescence dynamics, the system is going to be constituted by only one large particle of mass \( N_0 m_0 \).

5 BBGKY hierarchy of the rescaled correlation functions

Our purpose in this Section is to consider the limit where the volume \( V \) and the initial number of particles \( N_0 \) go to infinity in such a way that

\[
\lim_{V, N_0 \to +\infty} \frac{N_0}{V} = \rho_0 \in (0, +\infty).
\]

(5.1)
To this end let us define the rescaled correlation functions \( \{f^V_j(t)\}_{j=1}^{N_0} \) as
\[
f^V_j(m_1, \ldots, m_j, t) := \frac{f_j(m_1, \ldots, m_j, t)}{V_j}, \quad j = 1, \ldots, N_0.
\] (5.2)

Since, as we have seen in Section 3, the function \( f_1 \) is the number density function, the rescaled function \( f^V_1 \) is the density function associated to the concentration of \( p \) articles (number of particles per unit volume).

By (3.13) the functions \( f^V_j \) satisfy the following set of \( N_0 \) equations:
\[
\partial_t f^V_j = \frac{1}{2} \sum_{\ell=1}^{j} \int_{0}^{m_\ell} d\mu \ f^V_{j+1}(m_1, \ldots, m_\ell - \mu, \ldots, m_j, \mu, t) + \frac{j(j-1)}{2V} f^V_j(m_1, \ldots, m_j, t) - j \int_{0}^{\infty} d\mu f^V_{j+1}(m_1, \ldots, m_j, \mu, t),
\] (5.3)
for \( j = 1, 2, \ldots, N_0 \). We will refer to the family of equations (5.3) as BBGKY hierarchy by analogy with the system arising in the framework of many particles hamiltonian systems.

By (3.5) and (5.2) the rescaled densities at time \( t = 0 \) are
\[
f^V_j(m_1, \ldots, m_j, 0) = \frac{(N_0)!}{(N_0 - j)!} \frac{1}{V_j} f^{\otimes j}_0(m_1, \ldots, m_j), \quad j = 1, 2, \ldots, N_0
\] (5.4)
where the function \( f_0 \) has been defined in (2.11). By (5.4), for every \( j \geq 1 \):
\[
\lim_{V, N_0 \to +\infty, N_0 \to \rho_0} ||f^V_j(0) - \rho_0 f^{\otimes j}_0||_{L^1(R^+)} = 0.
\] (5.5)

In order to state our main result we first recall that, for all non negative initial data in \( L^1(R^+) \), the Cauchy problem for the coagulation equation (1.1) with kernel equal to one has a unique non negative solution in \( C([0, +\infty); L^1(R^+)) \) (cf. [17], Theorem 2.1).

Our main result is then the following:

**Theorem 5.1** Let \( \{f_j(t)\}_{j=1}^{N_0} \) be the solution of system (3.13) with initial data defined in (3.5). Then, if \( \{f^V_j(t)\}_{j=1}^{N_0} \) is the sequence of rescaled densities defined by (5.2):
\[
\forall \ j \geq 1 : \lim_{V, N_0 \to +\infty, N_0 \to \rho_0} ||f^V_j(t) - f(t)^{\otimes j}||_{L^1(R^+)} = 0
\] (5.6)
where \( f \) is the unique solution in \( C([0, +\infty); L^1(R^+)) \) of the coagulation equation (1.1) with kernel \( K = 1 \) and initial datum \( \rho_0 f_0 \), \( f_0 \) given by (2.11).

The proof of Theorem 5.1 is done in two steps. One is to prove that the sequence \( \{f^V_j(t)\}_{j=1}^{N_0} \) converges to a sequence of functions \( \{f^\infty_j(t)\}_{j \in \mathbb{N}} \) that satisfy an infinite set of equations. The second is to prove that for all \( j \geq 1 \) and \( t \geq 0 \), \( f^\infty_j(t) = f(t)^{\otimes j} \). The first uses the explicit expression of
the functions $f_j^V$ as a finite sum and a kind of dominated convergence. The second is done proving the uniqueness of solutions of the new infinite system of equations. We actually start by proving this uniqueness result.

For the sake of notation let us introduce operators $W_j$ defined as follows. Given a function $\varphi \in L^1((\mathbb{R}^+)^{j+1})$ we call $W_j$ the operator such that $W_j[\varphi]$ is:

$$W_j[\varphi](m_j) := V G_j[\varphi](m_j) - j \int_0^\infty \varphi(m, \mu) d\mu, \quad (5.7)$$

where $G_j$ is defined by in (2.14). Therefore, using Lemma (2.3) it is easily seen that $W_j$ is a linear and continuous operator from $L^1((\mathbb{R}^+)^j)$ to $L^1((\mathbb{R}^+)^j)$ whose norm satisfies:

$$\|W_j\| \leq \frac{3j}{2}. \quad (5.8)$$

**Lemma 5.2** Let $f \in C([0, +\infty); L^1(\mathbb{R}^+))$ be the unique non negative solution of the coagulation equation

$$\partial_t f(m, t) = \frac{1}{2} \int_0^m d\mu f(m - \mu, t)f(\mu, t) - f(m, t) \int_0^\infty d\mu f(\mu, t) \quad (5.9)$$

with initial datum $f(0, m) = \rho_0 f_0(m)$, $f_0$ given by (2.11). Then, the sequence of functions defined as

$$f_j^\infty(t, m_j) := f(t)^{\otimes j}(m_j), \quad j \in \mathbb{N}^* \quad (5.10)$$

is the unique non negative solution of the system:

$$\partial_t f_j^\infty(m_j, t) = \frac{1}{2} \sum_{\ell=1}^{j} \int_0^{m_\ell} d\mu f_{j+1}^\infty(m_1, \ldots, m_\ell - \mu, \ldots, m_j, \mu, t) +$$

$$- j \int_0^\infty d\mu f_{j+1}^\infty(m_1, \ldots, m_j, \mu, t), \quad \text{for } j = 1, 2, \ldots \quad (5.11)$$

with initial data $f_j^\infty(0) = \rho_0^j f_0^{\otimes j}$ such that $f_j^\infty \in C([0, +\infty); L^1(\mathbb{R}^+)^j)$. For all $t > 0$, the sequence $\{f_j^\infty(t)\}_{j \in \mathbb{N}^*}$ satisfies:

$$\forall j \geq 1: \quad \|f_j^\infty(t)\|_{L^1(\mathbb{R}^+)^j} \leq \rho_0^j. \quad (5.12)$$

**Proof of Lemma 5.2** A straightforward calculation shows that the sequence $\{f_j^\infty(t)\}_{j \in \mathbb{N}^*}$ defined in (5.10) is indeed a solution of system (5.11) with initial data $\{\rho_0^j f_0^{\otimes j}(m_j)\}_{j \in \mathbb{N}^*}$.

On the other hand, since $f \in C([0, +\infty); L^1(\mathbb{R}^+))$ satisfies (5.9) we have $f \in C^1((0, +\infty); L^1(\mathbb{R}^+))$ and then $f_j^\infty \in C([0, +\infty); L^1(\mathbb{R}^+)^j) \cap C^1((0, +\infty); L^1(\mathbb{R}^+)^j)$. Due to the fact that $\{f_j^\infty(t)\}_{j \in \mathbb{N}^*}$ satisfies (5.11), we obtain, after integration of the $j$–th equation over $(\mathbb{R}^+)^j$:

$$\frac{\partial}{\partial t} ||f_j^\infty(t)||_{L^1((\mathbb{R}^+)^j)} = -j \frac{1}{2} ||f_{j+1}^\infty(t)||_{L^1((\mathbb{R}^+)^{j+1})} \leq 0.$$
and then, for all $t > 0$:

$$
||f_j^\infty(t)||_{L^1((\mathbb{R}^+)^j)} \leq ||f_j^\infty(0)||_{L^1((\mathbb{R}^+)^j)} = \rho_0^j.
$$

(5.13)

That proves (5.12).

In order to prove uniqueness notice that the system (5.11) may be written using operator $W_j$ as

$$\frac{\partial f_j^\infty}{\partial t} = W_j[f_j^\infty]$$

from where

$$f_j^\infty(t) = f_j^\infty(0) + \int_0^t dt_1 W_j f_j^\infty(t_1)$$

(5.14)

Using this same formula for $f_j^\infty$, we obtain after $M$ iterations:

$$f_j^\infty(t) = \sum_{n\geq0} \frac{t^n}{n!} W_j W_{j+1} \ldots W_{j+n-1} f_j^{n}(0) +
\int_0^t dt_1 \ldots \int_0^{t_M} dt_{M+1} W_j \ldots W_{j+M} f_j^{\infty}(m_{j+M+1}; t_{M+1}).$$

(5.15)

Let us assume now that there exist two different non-negative solutions, $\{h_{j,k}(t)\}_{j\in\mathbb{N}^+}$, $k = 1, 2$, of (5.11) such that $h_{j,k} \in C([0, +\infty); L^1(\mathbb{R}^+)^j)$ with the same factorized initial datum $\{\rho_0^j (f_0)^{\otimes j}\}_{j\in\mathbb{N}^+}$. Then by (5.15) and (5.8) we deduce that for every $M > 0$:

$$||h_{j,1}(t) - h_{j,2}(t)||_{L^1((\mathbb{R}^+)^j)} \leq \frac{t^{M+1}}{(M + 1)!} \times
\times||W_j \ldots W_{j+M} (h_{j,M+1,1}(t_{M+1}) - h_{j,M+1,2}(t_{M+1}))||_{L^1((\mathbb{R}^+)^{j+M+1})}
\leq \frac{t^{M+1}}{(M + 1)!} \left(\frac{3}{2}\right)^{M+1} \frac{(j + M)!}{(j - 1)!} \left(||h_{j,M+1,1}(t_{M+1})||_{L^1((\mathbb{R}^+)^{j+M+1})} +
+||h_{j,M+1,2}(t_{M+1})||_{L^1((\mathbb{R}^+)^{j+M+1})}\right).$$

(5.16)

By our assumptions on $\{h_{j,k}(t)\}_{j\in\mathbb{N}^+}$, $k = 1, 2$, they both satisfy (5.12). Thus, plugging (5.12) into (5.10) we deduce:

$$||h_{j,1}(t) - h_{j,2}(t)||_{L^1((\mathbb{R}^+)^j)} \leq 2\rho_0^j \frac{t^{M+1}}{(M + 1)!} \left(\frac{3}{2}\right)^{M+1} \frac{(j + M)!}{(j - 1)!}$$

(5.17)

Using (3.11) we obtain

$$||h_{j,1}(t) - h_{j,2}(t)||_{L^1((\mathbb{R}^+)^j)} \leq 2\rho_0^j (3\rho_0 t)^{M+1}$$

(5.18)

and then, for every $t < 1/(3\rho_0)$:

$$||h_{j,1}(t) - h_{j,2}(t)||_{L^1((\mathbb{R}^+)^j)} \leq \lim_{M\to+\infty} 2\rho_0^j (3\rho_0 t)^{M+1} = 0$$

(5.19)
and the uniqueness follows but only for $t \in [0, 1/3 \rho_0)$. This argument may be now iterated as follows.

Suppose that (5.19) holds for $t \in [0, T)$ for some $T > 0$. Consider now (5.10) for $t \in [T, T + 1/3 \rho_0)$ (since it holds for all $t > 0$). Since (5.12) holds also true for all $t > 0$, it follows as before that (5.19) also holds for $t \in [T, T + 1/3 \rho_0)$. Therefore, global uniqueness is proven. This ends the proof of Lemma 5.2.

\[\square\]

**Lemma 5.3** Under the same hypothesis than in Theorem 5.1 for all $T > 0$ and all $j \geq 1$:

\[
\lim_{V, N_0 \to +\infty} \sup_{0 \leq t < T} \|f_j^V(t) - g_j(t)\|_{L^1((\mathbb{R}^+)^{\nu})} = 0. \tag{5.20}
\]

where

\[
g_j(t, m_j) = \sum_{n=0}^{\infty} \rho_0 \frac{t^n}{n!} W_j \circ W_{j+1} \cdots \circ W_{j+n-1}[f_0^{\otimes j+n}](m_j) \tag{5.21}
\]

is such that $g_j \in C([0, +\infty); L^1((\mathbb{R}^+)^{\nu}) \cap C^1((0, \infty); L^1((\mathbb{R}^+)^{\nu}))$ and $g_j \geq 0$. Moreover the sequence \{\{g_j\}_{j \in \mathbb{N}}\} satisfies system (5.11) for $t \in (0, \infty)$.

**Proof of Lemma 5.3** Let us prove first that the series in (5.21) defines a function $g_j \in C([0, \tau_1); L^1((\mathbb{R}^+)^{\nu}) \cap C^1((0, \tau_1); L^1((\mathbb{R}^+)^{\nu}))$ for some $\tau_1 > 0$. To this end we first deduce the following estimate from (5.3) and the the explicit expression (5.4) of the initial datum:

\[
\left\|W_j \circ \cdots \circ W_{j+n-1}[f_0^{\otimes j+n}]\right\|_{L^1((\mathbb{R}^+)^{\nu})} \leq \left(\frac{3}{2}\right)^n j(j+1) \cdots (j+n-1) \|f_0\|_{L^1((\mathbb{R}^+)^{\nu})}^j \tag{5.22}
\]

from where, using again (5.11) we deduce, for all $j \geq 1$, $n \geq 0$ and $t > 0$:

\[
\rho_0 \frac{t^n}{n!} \left\|W_j \circ \cdots \circ W_{j+n-1}[f_0^{\otimes j+n}]\right\|_{L^1((\mathbb{R}^+)^{\nu})} \leq 2^{j-1} \rho_0 (3 \rho_0 t)^n. \tag{5.23}
\]

It then follows that the series in (5.21) defines a continuous function $g_j \in C([0, 1/(3 \rho_0)]; L^1((\mathbb{R}^+)^{\nu}))$. A similar argument shows that $g_j \in C^1([0, 1/(3 \rho_0)]; L^1((\mathbb{R}^+)^{\nu}))$ and that

\[
\frac{\partial g_j(t)}{\partial t} = \sum_{n=1}^{\infty} \rho_0 \frac{t^{n-1}}{(n-1)!} W_j \circ W_{j+1} \cdots \circ W_{j+n-1}[f_0^{\otimes j+n}](m_j).
\]

By construction the sequence \{\{f_j^V(t)\}_{j \in \mathbb{N}}\} satisfies, for every $t > 0$:

\[
f_j^V(t, m_j) = e^{-\frac{j(t-1)}{2\nu}} f_j^V(m_j; 0) - \int_0^t dt_1 e^{-\frac{j(t-1)}{2\nu}(t-t_1)} j \int_0^\infty d\mu f_{j+1}^V(m_j, \mu; t_1) + \int_0^t dt_1 e^{-\frac{j(t-1)}{2\nu}(t-t_1)} \sum_{\ell=1}^j \int_0^{m_\ell} d\mu f_{j+1}^V(m_1, \ldots, m_\ell - \mu, \ldots, m_j, \mu; t_1). \tag{5.24}
\]

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that, using the operator $W_j$ defined by (5.7), can be rewritten as:

$$f_j^V(t) = e^{-\frac{j(j-1)}{2V}t} f_j^V(0) + \int_0^t dt_1 e^{-\frac{j(j-1)}{2V} (t-t_1)} W_j f_j^{V+1}(t_1)$$

$$= \sum_{n \geq 0} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n e^{-\frac{j(j-1)}{2V} (t-t_1)} W_j e^{-\frac{j(j+1)}{2V} (t_1-t_2)} W_{j+1} \cdots$$

$$\cdots e^{-\frac{j(n-2)(j+n-1)}{2V} (t_{n-1}-t_n)} W_{j+n-1} e^{-\frac{j(j+n-1)(j+n)}{2V} t_n} f_j^{V+1}(0),$$

(5.25)

where we set $t_{-1} \equiv 0$, $t_0 \equiv t$.

Property (5.20) follows from the two following facts. The first is that the sum:

$$\lim_{V, N_0 \to +\infty} \left( \int dt_n e^{-\frac{j(j-1)}{2V} (t-t_1)} W_j \cdots e^{-\frac{j(n-2)(j+n-1)}{2V} (t_{n-1}-t_n)} W_{j+n-1} e^{-\frac{j(j+n-1)(j+n)}{2V} t_n} f_j^{V+1}(0) \right)$$

(5.26)

in $L^1((\mathbb{R}^+)^i)$, uniformly for $t \in [0, T]$ for any $T > 0$.

We start proving the first. By (5.4) and (5.8) we obtain:

$$\|W_j \circ \cdots \circ W_{j+n-1} f_j^{V+1}(0)\|_{L^1((\mathbb{R}^+)^i)} \leq \left( \frac{3}{2} \right)^n j(j+1) \cdots (j+n-1) \|f_j^{V+1}(0)\|_{L^1((\mathbb{R}^+)^i)}$$

$$\leq \left( \frac{3}{2} \right)^n j(j+1) \cdots (j+n-1) \left( \frac{N_0}{V} \right)^{j+n}.$$  

(5.27)

Since in the expression (5.25) $t_n < t_{n-1} < \cdots < t_1 < t_0 \equiv t$, it follows that $t_{k-1} - t_k > 0$ for any $k = 1, \ldots, n$ and by (5.27) we deduce:

$$\int dt_n \left\| e^{-\frac{j(j-1)}{2V} (t-t_1)} W_j \cdots e^{-\frac{j(n-2)(j+n-1)}{2V} (t_{n-1}-t_n)} W_{j+n-1} e^{-\frac{j(j+n-1)(j+n)}{2V} t_n} f_j^{V+1}(0) \right\|_{L^1((\mathbb{R}^+)^i)}$$

$$= \int dt_n e^{-\frac{j(j-1)}{2V} (t-t_1)} \cdots e^{-\frac{j(n-2)(j+n-1)}{2V} (t_{n-1}-t_n)} e^{-\frac{j(j+n-1)(j+n)}{2V} t_n} \times$$

$$\times \left\| W_j \circ \cdots \circ W_{j+n-1} f_j^{V+1}(0) \right\|_{L^1((\mathbb{R}^+)^i)}$$

$$\leq \frac{t^n}{n!} \left( \frac{3}{2} \right)^n j(j+1) \cdots (j+n-1) \left( \frac{N_0}{V} \right)^{j+n}.$$  

(5.28)

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Using again (5.11) and the hypothesis (5.1) we deduce, for all \( j \geq 1, n \geq 0 \) and \( t > 0 \):
\[
\int dt_n \left\| e^{-\frac{j(j-1)}{2V}(t-t_n)} W_j \cdots e^{-\frac{(j+n-2)(j+n-1)}{2V}(t_{n-1}-t_n)} W_{j+n-1} e^{-\frac{(j+n-1)(j+n)}{2V}t_n} f_{j+n}(0) \right\|_{L^1((\mathbb{R}^+)^j)} \\
\leq 4^{j-1} (6\rho_0 t)^n. \tag{5.29}
\]

Estimate (5.29) shows that the sum giving \( f_j^V(t) \) in (5.25) is dominated, uniformly for \( t \in [0, \tau_1) \), \( \tau_1 = \frac{1}{6\rho_0} \), by a convergent series.

In order to deduce (5.26) for every \( j \geq 1 \) and every \( n \geq 1 \) uniformly for \( t \in [0, \tau_1) \) we notice first of all that, as \( V \to +\infty \):
\[
\int dt_n e^{-\frac{j(j-1)}{2V}(t-t_n)} \cdots e^{-\frac{(j+n-2)(j+n-1)}{2V}(t_{n-1}-t_n)} e^{-\frac{(j+n-1)(j+n)}{2V}t_n} \to \int dt_n = \frac{t^n}{n!} \tag{5.30}
\]
uniformly for \( t \) in any compact of \([0, +\infty)\). On the other hand by (5.5) and the continuity of the operators \( W_k \) we have
\[
\lim_{N_0/V \to +\infty} \left\| W_j \circ \cdots \circ W_{j+n-1} \left[ f_{j+n}^{V}(0) - \rho_0^j \frac{t^n}{n!} f_{j+n}^{0} \right] \right\|_{L^1((\mathbb{R}^+)^j)} = 0 \tag{5.31}
\]

From (5.30) and (5.31), (5.26) follows and therefore:
\[
f_j^V(t) \to \sum_{n \geq 0} \frac{t^n}{n!} \rho_0^j \frac{t^n}{n!} W_j \circ \cdots \circ W_{j+n-1} [ f_{j+n}^{0} ] = g_j(t)
\]
in \( C([0, \tau_1); L^1(\mathbb{R}^+)) \cap C^1((0, \tau_1); L^1(\mathbb{R}^+)) \). Since \( f_j^V(t) \geq 0 \) for all \( j \in \mathbb{N}^* \) and \( t \geq 0 \) it follows that \( g_j(t) \geq 0 \) for \( t \in [0, \tau_1) \) and all \( j \).

A straightforward computation shows that \( \{ g_j \}_{j \in \mathbb{N}^*} \) satisfies system (5.11) for \( t \in [0, \tau_1) \) and this proves the Lemma (5.3) for \( T \in [0, \tau_1) \).

In order to extend this result for \( T > \tau_1 \) we first notice that for all \( t > 0 \):
\[
\begin{align*}
 f_j^V \left( t + \frac{\tau_1}{2} \right) & = e^{-\frac{j(j-1)}{2V} \tau_1/2} f_j^V(\tau_1/2) + \int_0^t dt_1 e^{-\frac{j(j-1)}{2V}(t-t_1)} W_j f_{j+n}^V \left( \tau_1/2 \right) \\
 & = \sum_{n \geq 0} \int_0^t dt_1 \cdots \int_0^{t_n-1} dt_n \cdots \int_0^{(j+n-2)(j+n-1)} dt_{n-1} \cdots \int_0^{(j+n-1)(j+n)} dt_{n-2} \cdots \int_0^{(j+n-2)(j+n-1)} dt_{n-3} \cdots \int_0^{(j+n-1)(j+n)} dt_{n-4} \cdots \int_0^{(j+1)(j+1)} \frac{(j+1)}{2V} W_{j+n-1} e^{-\frac{j(j-1)}{2V}(t_{n-1}-t_n)} \\
 & \cdots e^{-\frac{(j+n-2)(j+n-1)}{2V}(t_{n-1}-t_n)} W_{j+n-1} e^{-\frac{(j+n-1)(j+n)}{2V}t_n} f_{j+n}^V(\tau_1/2) \tag{5.32}
\end{align*}
\]

In order to pass to the limit as \( V \to +\infty \) and \( N_0/V \to \rho_0 \) in (5.32) we use the same two arguments as for (5.25). More precisely, using Lemma (4.2) we obtain:
\[
\| W_j \circ \cdots \circ W_{j+n-1} [ f_{j+n}^V(\tau_1/2) ] \|_{L^1((\mathbb{R}^+)^j)} \leq \left( \frac{3}{2} \right)^n j(j+1) \cdots (j+n-1) \| f_{j+n}^V(0) \|_{L^1((\mathbb{R}^+)^j)} \\
\leq \left( \frac{3}{2} \right)^n j(j+1) \cdots (j+n-1) \left( \frac{N_0}{V} \right)^{j+n} \tag{5.33}
\]
and then, arguing as before:

\[
\int dt_n \left\| e^{-\frac{j(t-1)}{2V}(t-t_1)} W_j \cdots e^{-\frac{j(n-2)}{2V}(t_n-1) L} W_{j+n-1} e^{-\frac{j(n-1)}{2V}L} f_{j+n}(\tau_1/2) \right\|_{L^1([0,\tau_1])} \leq 4j^{j-1}(6\rho_0 t)^n. \tag{5.34}
\]

On the other hand, by the continuity of the operators \( W_k \) and the convergence [5.20] that we have just proved for \( T \in [0, \tau_1) \):

\[
\lim_{N_0 \to +\infty} \| W_j \circ \cdots \circ W_{j+n-1} \left[ f_{j+n}(\tau_1/2) - g_j(\tau_1/2) \right] \|_{L^1([0,\tau_1])} = 0 \tag{5.35}
\]

From (5.34) and (5.35) we deduce that, as \( V \to \infty \), \( N_0/V \to \rho_0 \)

\[
f_j^V \left( t + \frac{\tau_1}{2} \right) \to h_j(t) \quad \text{in} \quad C([0, \tau_1); L^1(\mathbb{R}^+)) \cap C^1((0, \tau_1); L^1(\mathbb{R}^+))
\]

\[
h_j(t) = \sum_{n \geq 0} \frac{t^n}{n!} \rho_0^j W_j \circ \cdots \circ W_{j+n-1}[g_j(\tau_1/2)] \geq 0 \quad \forall t \in [0, \tau_1).
\]

We define now the function \( g_j \) for \( t \in [\tau_1, 3\tau_1/2) \) as:

\[
g_j(t) = h_j(t - \tau_1/2), \quad \forall t \in [\tau_1, 3\tau_1/2).
\]

Since \( \{g_j(t)\}_{j \in \mathbb{N}^\ast} \) satisfies system [5.11] for \( t \in [0, \tau_1) \) it is such that:

\[
g \left( t + \frac{\tau_1}{2} \right) = \sum_{n \geq 0} \frac{t^n}{n!} \rho_0^j W_j \circ \cdots \circ W_{j+n-1}[g_j(\tau_1/2)] \quad \forall t \in [0, \tau_1/2)
\]

It then follows \( h_j(t) = g_j(t + \tau_1/2) \) for all \( t \in [0, \tau_1/2) \). We deduce that \( g_j \in C([0, 3\tau_1/2); L^1(\mathbb{R}^+)) \cap C^1((0, 3\tau_1/2); L^1(\mathbb{R}^+)) \) and

\[
f_j^V(t) \to g_j(t) \quad \text{in} \quad C([0, \tau_1/2); L^1(\mathbb{R}^+)) \cap C^1((0, 3\tau_1/2); L^1(\mathbb{R}^+))
\]

as \( N_0 \to \infty \), \( V \to \infty \), \( N_0/V \to \rho_0 \).

Since, as a plain calculation shows again, the sequence \( \{h_j(t)\}_{j \in \mathbb{N}^\ast} \) satisfies system [5.11] for \( t \in [0, \tau_1) \) and \( h_j(0) = g_j(\tau_1/2) \) the sequence \( \{g_j\}_{j \in \mathbb{N}^\ast} \) satisfies system [5.11] for \( t \in [0, 3\tau_1/2) \).

We have extended in that way the previous result on \([0, \tau_1)\) to the interval \([0, 3\tau_1/2)\). This procedure may be repeated to obtain the result in all finite interval \([0, T)\) from where Lemma 5.3 follows.

\[ \square \]

**Proof of Theorem 5.3** From Lemma 5.3, for any \( T > 0 \), \( f_j^V \) converges to \( g_j \) in \( C([0, T); L^1(\mathbb{R}^+)^j) \) and \( \{g_j\}_{j \in \mathbb{N}^\ast} \) is a non negative solution of system [5.11] with initial data \( \{\rho_0^j f_0^j\}_{j \in \mathbb{N}^\ast} \). By Lemma 5.2 we have \( g_j = f_j^\infty \) for all \( j \geq 1 \) and Theorem 5.3 follows.

\[ \square \]
Remark 5.4 We have considered a factorized initial data for the finite system of particles (cf. (2.10)). We could also have considered a more general initial data of the form:

\[ P_{N_0}(m_{N_0},0) = (N_0)! \Psi_{N_0}(m_{N_0}), \quad P_N(m_N,0) = 0 \quad \text{for} \quad N \neq N_0 \]

with

\[ \Psi_{N_0}(m_{N_0}) \geq 0, \quad \int dm_{N_0} \Psi_{N_0}(m_{N_0}) = 1, \quad (5.36) \]

Then, for any \( j = 1, \ldots, N_0 \), the correlation functions at time \( t = 0 \) would have been

\[ f_j(m_j,0) = \sum_{N=j}^{\infty} \frac{1}{(N-j)!} \int dm_{j+1} \ldots \int dm_N P_N(m_N,0) = \frac{(N_0)!}{(N_0-j)!} F_j^{(N_0)}(m_j), \quad (5.37) \]

where \( F_j^{(N_0)}(m_j) \) is the \( j \)-particle marginal distribution (in the sense of probability measures) of \( P_{N_0}(m_{N_0}) \), namely:

\[ F_j^{(N_0)}(m_j) := \int dm_{j+1} \ldots \int dm_{N_0} \Psi_{N_0}(m_{N_0}) \]

and, by definition (see (5.36)), we have:

\[ \int dm_j F_j^{(N_0)}(m_j) = \int dm_j \int dm_{j+1} \ldots \int dm_{N_0} \Psi_{N_0}(m_{N_0}) = 1. \]

Then, the rescaled correlation functions at time \( t = 0 \) would have satisfied

\[ \lim_{V,N_0 \to +\infty} \frac{||f_j^V(0)||_{L^1((\mathbb{R}^+)^j)}}{\rho_0^j} = \rho_0^j, \quad (5.38) \]

Now, assuming

\[ \lim_{N_0 \to +\infty} ||F_j^{(N_0)}(0) - f_0^{\otimes j}||_{L^1((\mathbb{R}^+)^j)} = 0, \quad (5.39) \]

for some \( f_0(m) \) such that

\[ f_0(m) \geq 0, \quad \int dm f_0(m) = 1, \]

we could have proved exactly the same results that we got for the a priori factorized case. Choosing an initial datum as in (5.39) would have meant to assume propagation of chaos to hold at time \( t = 0 \). On the other hand, what we did has been to assume hypotheses of molecular chaos to hold at time \( t = 0 \).

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