The Monoidal Eilenberg-Moore Construction and Bialgebroids

Kornél Szlachányi

Abstract. Monoidal functors $U : \mathcal{C} \to \mathcal{M}$ with left adjoints determine, in a universal way, monoids $T$ in the category of oplax monoidal endofunctors on $\mathcal{M}$. Such monads will be called bimonads. Treating bimonads as abstract "quantum groupoids" we derive Tannaka duality between left adjointable monoidal functors and bimonads. Bialgebroids, i.e., Takeuchi’s $\times_R$-bialgebras, appear as the special case when $T$ has also a right adjoint. Street’s 2-category of monads then leads to a natural definition of the 2-category of bialgebroids.

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1. Introduction

In the classical theory of monads [14, 15] or triples [1] one deals with the following construction. If a functor $U: C \rightarrow M$ has a left adjoint $F$ with unit $\eta: M \rightarrow UF$ and counit $\varepsilon: FU \rightarrow C$ then there is a monad $T = \langle UF, U\varepsilon F, \eta \rangle$ on $M$ such that $U$ factorizes as $U = UTK$ through the forgetful functor $UT$ of the category $\mathcal{M}T$ of $T$-algebras with a unique functor $K: C \rightarrow \mathcal{M}T$, called the Eilenberg-Moore comparison functor. For example, if $A$ is a ring, i.e., a monoid in $\mathcal{M} = \text{Ab}$ then the forgetful functor $U: A\mathcal{M} \rightarrow \mathcal{M}$ has a left adjoint, the induction functor $F = A \otimes \mathbb{Z} -$ and the Eilenberg-Moore category $\mathcal{M}T$ is equivalent to the category of left $A$-modules via $K$. As it is the case in this example so in general, if $K$ is an equivalence $U$ is called monadic.

Interpreting the category $\mathcal{M}T$ as the category of $T$-modules the above construction appears as a primitive version of Tannaka reconstruction in which adjointable functors are brought into correspondence with monads on their target categories. Monadicity in turn plays the role of the representation theorem. Pursuing this idea one can investigate the extra structure the monad acquires if the forgetful functor is monoidal. What one obtains is that the monad is oplax monoidal (called lax comonoidal in this paper) in the sense of the underlying functor being lax comonoidal and the natural transformations preserve these comonoidal structures. Such monads will be called bimonads. Bimonads are the abstract versions of bialgebras in the same spirit as monads are related to algebras.

Bimonads first appeared in the work of Moerdijk [16] under the name Hopf monads. He was motivated by generalizing the notion of Hopf operad. In our context the name bimonad seems to be the more natural as it rhymes with bialgebra and bialgebroid. Hopf algebroid could then be the name for a bimonad possessing some sort of antipode.

The motivating example of bimonads is associated to a Takeuchi $\times_R$ algebra [28], also called bialgebroid [12, 10, 22, 26]. For a bialgebroid $A$ over $R$ the algebra $A$ is an extension of $R^e := R \otimes R^{op}$, so $A$ is an $R^e$-$R^e$-bimodule. The underlying functor $T$ of the bimonad is $T = A \otimes R^e -: R\mathcal{M}_R \rightarrow R\mathcal{M}_R$ where we identified $R\mathcal{M}_R$ with $R^e \mathcal{M}$. Such bimonads are obtained by the above mentioned Tannaka reconstruction from the strict monoidal forgetful functors $U: A\mathcal{M} \rightarrow R\mathcal{M}_R$. That bialgebroid structures on $A$ over $R$ are in one-to-one correspondence with the strict monoidal forgetful functors $U_A$ was first pointed out by Schauenburg [23], see also [26]. Of course, the notion of bimonad is much more general than bialgebroids. Hopf algebroid could then be the name for a bimonad possessing some sort of antipode.

Tannaka duality for bialgebroids has recently been proven by Phùng Hô Hai [19] following the tradition of Saavedra [20], Deligne [8] and generalizing the results of Ulbrich [30], Schauenburg [21] for Hopf algebras and of Hayashi [6] for weak Hopf algebras. For more about this theory we refer to [15] and [1] and the references therein. The approach of the present paper does not fit into this series mathematically but perhaps ‘physically’. The categories we think about are module categories and therefore are not small. This forbids to compute the (quantum) group(oid) object as a coend. Also, we do not use any finiteness condition on the images of the functors. Instead we assume that the functors have left adjoints, and at the end also right ones. The question of when the bialgebroid we construct has an antipode, in either Lu’s [12] or Schauenburg’s [22] sense, is not addressed in this paper.
We use the following terminology. A **lax monoidal functor** is a triple \( \langle F, \tau, \iota \rangle \) where \( F \) is a functor between monoidal categories from \( \langle C, \Box, e \rangle \) to \( \langle M, \otimes, i \rangle \), \( \tau_{c,d} : Fc \otimes Fd \to F(c \Box d) \) is a natural transformation and \( \iota : i \to Fe \) is an arrow such that the usual hexagonal and the two square diagrams commute [14]. A **lax comonoidal functor** \( \langle F, \sigma, \upsilon \rangle \) from \( C \) to \( M \) is a lax monoidal functor from \( C^{\text{op}} \) to \( M^{\text{op}} \) (no change in the monoidal product). That is to say, \( \sigma_{c,d} : F(c \Box d) \to Fc \otimes Fd \) and \( \upsilon : Fe \to i \). The commutative diagrams they satisfy are obtained by reversing all the arrows is the usual hexagon and square diagrams of a lax monoidal functor.

A **monoidal functor** is a lax monoidal functor in which the \( \tau_{c,d} \)'s and the \( \iota \) are isomorphisms. **Comonoidal functors** are defined analogously. Of course, monoidal and comonoidal functors are essentially the same: \( \langle F, \tau, \iota \rangle \) is monoidal iff \( \langle F, \tau^{-1}, \iota^{-1} \rangle \) is comonoidal. In later sections where there will be a shortage of the Greek alphabet we shall use the notation \( \langle F, F_2, F_0 \rangle \) for a lax monoidal and \( \langle F, F^2, F^0 \rangle \) for a lax comonoidal functor. There is also a dual analogue of monoidal natural transformations. A monoidal natural transformation between lax comonoidal functors will be called **comonoidal**. So the monoidal categories, the lax comonoidal functors and the comonoidal natural transformations form a 2-category **ComonCat** just like the lax monoidal functors are the 1-cells of the 2-category **MonCat**.

A bimonad on the monoidal category \( \langle M, \otimes, i \rangle \) is a quintuple \( \langle T, \gamma, \pi, \mu, \eta \rangle \) where \( \langle T, \mu : T^2 \to T, \eta : M \to T \rangle \) is a monad on \( M \), \( \langle T, \gamma : T \otimes \to \otimes(T \times T), \pi : Ti \to i \rangle \) is a comonoidal functor and \( \mu \) and \( \eta \) satisfy compatibility conditions with \( \gamma \) and \( \pi \). The 10 commutative diagrams these natural transformations should satisfy are the simplest possible ones in spite of the fact that they are equivalent, at least if \( T \) has a right adjoint, to the somewhat unpleasant bialgebroid axioms that either mix algebra and coalgebra structures in a painful way [28] or use a non-monoidal product in the definition of coalgebras.

At last but not least the bimonad description offers natural ways to define the category of bialgebroids which, even for the special case of weak bialgebras and weak Hopf algebras [6, 7, 20], has not been investigated in detail yet. Unfortunately, it offers two natural ways. At first, since bimonads are monads, one can take Street’s definition [25] of the 2-category **Mnd(ComonCat)** of monads in **ComonCat**. In this framework all functors are lax comonoidal so the monad morphisms \( \langle G, \varphi \rangle \) involve lax comonoidal functors \( G \). This approach was chosen by McCrudden in the preprint [4] many results of which overlap ours. But the present paper uses another method to define the category of bimonads and therefore of bialgebroids. We insist on having lax monoidal functors in monad morphisms since it is more motivated by previous experience with quantum groupoids. So our 2-category **Bmd** of bimonads has objects the bimonads but has lax monoidal functors in the definition of monad morphisms. This choice forces the \( \varphi \) in the monad morphism to be **ambimonoidal** natural transformation (see Definition [14, 17]). With this tentative expression we refer to the unique way of compatibility with the (co)monoidal structures which, however, lies between the usual monoidality and comonoidality of natural transformations.

Having a 2-category of bialgebroids one can imagine the 2-category of Hopf algebroids as a subcategory. The equivalence classes of objects in this 2-category may turn out to be the appropriate objects which characterize a class of monoidal categories uniquely, similarly to the Doplicher-Roberts characterization [17] of certain symmetric monoidal C*-categories as representation categories of uniquely determined compact groups.
2. Monoidal adjunctions

2.1. Lax monoidal functors with left adjoint. The whole content of this paper rests on the following result which can be found in various forms in the categorical literature, in most general form perhaps in [1]. Nevertheless we provide an explicit proof in order for the paper to be self-contained even for non-specialists.

**Theorem 2.1.** Let \((U, \tau, \iota)\) be a lax monoidal functor from \(\langle C, \Box, \varepsilon \rangle\) to \(\langle M, \otimes, \iota \rangle\). If \(U\) has a left adjoint \(F : M \to C\) with counit \(\varepsilon : FU \to C\) and unit \(\eta : M \to UF\) then the natural transformation

\[
\sigma_{x,y} := \varepsilon_{Fx} \circ F\tau_{Fx,Fy} \circ F(\eta_x \otimes \eta_y)
\]

and the arrow

\[
u := \varepsilon_\varepsilon \circ Ft
\]

give rise to a lax comonoidal functor \(\langle F, \sigma, \nu \rangle\).

*Proof.* The expression for \(\sigma\) contains three products: the horizontal and vertical composition of natural transformations and the Cartesian product. So the statement requires 3-categorical computations, in the monoidal 2-category \(\textbf{Cat}\). Denoting the vertical composition by \(\bullet\), the horizontal by juxtaposition and the Cartesian product by \(\times\) and using the, perhaps strange, precedence of horizontal composition coming first and vertical coming last, we can write

\[
\sigma = \varepsilon \Box (F \times F) \bullet F\tau(F \times F) \bullet F \otimes (\eta \times \eta).
\]

Denoting by \(a, l, r\) the associativity and unit coherence isomorphisms in any one of the monoidal categories, the proof of the hexagon

\[
a(F \times F \times F) \bullet \Box (F \times \sigma) \bullet \sigma(M \times \otimes) = \Box (\sigma \times F) \bullet \sigma(\otimes \times M) \bullet Fa
\]

goes as follows.

\[
\Box (\sigma \times F) \bullet \sigma(\otimes \times M) =
\]

\[
= \Box (\varepsilon \Box (F \times F) \times F) \bullet \Box (F\tau(F \times F) \times F) \bullet \Box (F \otimes (\eta \times \eta) \times F)
\]

\[
\bullet \varepsilon \Box (F \times F)(\otimes \times M) \bullet F\tau(F \times F)(\otimes \times M) \bullet F \otimes (\eta \times \eta)(\otimes \times M)
\]

\[
= \varepsilon \Box (\Box \times C)(F \times F) \bullet FU \Box (\varepsilon \Box (F \times F) \times F) \bullet FU \Box (F\tau(F \times F) \times F)
\]

\[
\bullet FU \Box (F \otimes (\eta \times \eta) \times F) \bullet F\tau(F \otimes F) \bullet F \otimes (\eta \otimes \eta)
\]

\[
= \varepsilon \Box (\Box \times C)(F \times F) \bullet FU \Box (\varepsilon \Box (F \times F) \times F) \bullet FU \Box (F\tau(F \times F) \times F)
\]

\[
\bullet F\tau(F \otimes (UF \times UF) \times F) \bullet F \otimes (UF \otimes (\eta \otimes UF) \times UF) \bullet F \otimes (\Box \otimes (\eta \times F) \times UF F)
\]

\[
= \varepsilon \Box (\Box \times C)(F \times F) \bullet FU \Box (\varepsilon \Box (F \times F) \times F) \bullet F\tau(F \times F) \times F)
\]

\[
\bullet F\tau(F \otimes (UF \times UF) \times F) \bullet F \otimes (\eta \otimes (UF \times FF) \times UF F) \bullet F \otimes (\Box \otimes (\eta \times \eta) \times F).
\]
arguing with \( \tau \)

\[ \varepsilon \Box (\Box \times C)(F \times F \times F) \cdot F \tau(\Box(\Box \times F) \times F) \cdot F \otimes (U \varepsilon \Box (\Box \times F) \times UF) \]

\[ \cdot F \otimes (UF \tau(\Box \times F) \times UF) \cdot F \otimes (\eta \otimes (UF \times UF) \times UF) \cdot F \otimes (\otimes (\eta \times \eta) \times \eta) \]

\[ \varepsilon \Box (\Box \times C)(F \times F \times F) \cdot F \tau(\Box(\Box \times F) \times F) \cdot F \otimes (U \varepsilon \Box (\Box \times F) \times UF) \]

\[ \cdot F \otimes (\eta U \Box (\Box \times F) \times UF) \cdot F \otimes (\tau(\Box \times F) \times UF) \cdot F \otimes (\otimes (\eta \times \eta) \times \eta) \]

\[ \varepsilon \Box (\Box \times C)(F \times F \times F) \cdot F \tau(\Box(\Box \times C)(F \times F \times F) \]

\[ \cdot F \otimes (\tau \times U)(F \times F \times F) \cdot F \otimes (\otimes (\eta \times \eta) \times \eta) \]

where in the subsequent equations we used the definition of \( \sigma \), naturality of the \( \varepsilon \) of the 4th term, naturality of the \( \tau \) of the 5th term, naturality of the first \( \eta \) of the 6th term, naturality of the \( \tau \) of the 4th term, naturality of the \( \eta \) of the 5th term, and at last the adjointness \( F \dashv U \) and some cosmetics. A similarly long calculation, or arguing with \( \times^{op} \), yields the formula

\[ \Box(F \times \sigma) \cdot \sigma(M \times \otimes) = \]

\[ \varepsilon \Box (\Box \times C)(F \times F \times F) \cdot F \tau(\Box(\Box \times C)(F \times F \times F) \]

\[ \cdot F \otimes (U \times \tau)(F \times F \times F) \cdot F \otimes (\eta \times \otimes (\eta \times \eta)) \]

Composing the first with \( Fa \) and the second with \( a(F \times F \times F) \) and using the hexagon for \( \tau \)

\[ \tau(\Box \times C) \cdot \otimes (\tau \times U) \cdot a(U \times U \times U) = Ua \cdot \tau(\Box \times C) \cdot \otimes (U \times \tau) \]

one immediately obtains the hexagon for \( \sigma \).

It remained to show the squares of \( \sigma \) and \( \upsilon \)

\[ I_{Fx} \circ (\upsilon \Box Fx) \circ \sigma_{i,x} = \]

\[ r_{Fx} \circ (Fx \Box \upsilon) \circ \sigma_{x,i} = \]

but we will suffice with proving the first.

\[ I_{Fx} \circ (\upsilon \Box Fx) \circ \sigma_{i,x} = \]

\[ = I_{Fx} \circ (\varepsilon_{x} \Box Fx) \circ (F(t \Box Fx) \circ \varepsilon_{Fi} \Box Fx \circ F \tau_{Fi,Fx} \circ F(\eta_{i} \otimes \eta_{x})) \]

\[ = I_{Fx} \circ (\varepsilon_{x} \Box Fx) \circ \varepsilon_{FU \Box Fx} \circ F(t \Box Fx) \circ F \tau_{Fi,Fx} \circ F(\eta_{i} \otimes \eta_{x}) \]

\[ \]

\[ \]

where in the last but one equality the square of \( \tau \) and \( \iota \) was used.

We remark that the converse of the above Theorem holds, too. If \((F, \sigma, \upsilon)\) is lax comonoidal then

\[ \tau_{a,b} := U(\varepsilon_{a} \Box \varepsilon_{b}) \circ U \sigma_{Ua,Ub} \circ \eta_{Ua \otimes Ub} \]

\[ \iota := U \upsilon \circ \eta_{i} \]

defines a lax monoidal structure for \( U \). However, monoidality of \( U \), i.e., invertibility of \( \tau \) and \( \iota \) does not imply comonoidality of \( F \), i.e., invertibility of \( \sigma \) and \( \upsilon \).
2.2. Comonoidality of the unit and counit. We specialize the above Theorem to monoidal $U$. Then we get the following.

**Proposition 2.2.** If $\langle U, \tau, \iota \rangle$ is monoidal and $F$ is a left adjoint of $U$ then the $\langle F, \sigma, \upsilon \rangle$ given in Theorem 2.1 is the unique lax comonoidal structure on $F$ for which the given adjunction data $\varepsilon$ and $\eta$ are comonoidal natural transformations, i.e., for which

\[ FU(a \boxdot b) \xrightarrow{\sigma_{a,b} \circ F_{\tau^{-1}}} FUa \boxdot FUb \xrightarrow{\varepsilon_a \boxdot \varepsilon_b} FUc \xrightarrow{\varepsilon_c} Fc \]

(13)

\[ x \otimes y \xrightarrow{\eta_x \otimes \eta_y} \xrightarrow{\eta_{x \otimes y}} \]

(14)

\[ UFX \otimes UFy \xrightarrow{\sigma_{x,y} \circ F \eta_{x \otimes y}} UFi \xrightarrow{U\upsilon} Ue \]

are commutative.

**Proof.** Using invertibility of $\tau_{a,b}$ and $\iota$ the above diagrams can be read as the equations

\[ (\varepsilon_a \boxdot \varepsilon_b) \circ \sigma_{a,b} = \varepsilon_{a \boxdot b} \circ F_{\tau_{a,b}} \]

(15)

\[ \upsilon = \varepsilon_c \circ F \iota \]

(16)

\[ \tau_{F_{x},F_{y}} \circ (\eta_{x} \otimes \eta_{y}) = U\sigma_{x,y} \circ \eta_{x \otimes y} \]

(17)

\[ \iota = U\upsilon \circ \eta_{i} \]

(18)

where $a, b$ run over the objects of $C$ and $x, y$ over those of $M$. Now we are left to show that these equations have a unique solution for $\sigma$ and $\upsilon$. For $\upsilon$ this is obvious from the second equation. In order to obtain $\sigma$ apply $F$ to the third equation and multiply the result with $\varepsilon_{F_{x},F_{y}}$.

\[ \varepsilon_{F_{x},F_{y}} \circ F_{\tau_{F_{x},F_{y}}} \circ F(\eta_{x} \otimes \eta_{y}) = \varepsilon_{F_{x},F_{y}} \circ FU\sigma_{x,y} \circ F\eta_{x \otimes y} \]

\[ = \sigma_{x,y} \circ \varepsilon_{F(x \otimes y)} \circ F\eta_{x \otimes y} = \sigma_{x,y} \]

which is indeed the comonoidal structure of Theorem 2.1. \hfill \Box

In the course of the above proof we have seen that equations (13-18) have unique solutions for $\sigma$ and $\upsilon$ if the $\varepsilon$, $\eta$, $\tau$, $\iota$ are given. This is true even if $U$ is lax monoidal although the comonoidality diagrams (13) and (14) loose their meaning. (The $\varepsilon$ and $\eta$ are ambimonoidal, however, in the sense of Definition 4.7.) Similarly, equations (15-18) can be solved for $\tau$ and $\iota$ if the others are given. This motivates the

**Definition 2.3.** A lax monoidal functor $U$ and a lax comonoidal functor $F$ is called a monoidal adjoint pair, or simply a monoidal adjunction $F \dashv U$, if $F$ is the left adjoint of $U$ as ordinary functors and the counit $\varepsilon: FU \to C$ and the unit $\eta: M \to UF$ can be chosen to satisfy equations (15), (16), (17) and (18).

**Lemma 2.4.** If $U$ is monoidal and $F \dashv U$, $F' \dashv U$ are two monoidal adjunctions then there exists a comonoidal natural isomorphism $F \cong F'$. 

Proof. As in the case of the proof of uniqueness of left adjoints of functors up to natural isomorphisms one takes the natural isomorphism \( \varepsilon F' \bullet F \eta' : F \to F' \) which is made of horizontal and vertical composites of lax comonoidal functors and of comonoidal natural transformations, by Proposition 2.2 so it is itself comonoidal. \( \square \)

3. The monoidal Eilenberg-Moore construction

3.1. Monoidal functors with left adjoints and bimonads. The continuing assumption is that \( \langle U, \tau, i, \rangle \) is a monoidal functor from \( \langle C, \otimes, e \rangle \) to \( \langle M, \otimes, i \rangle \) and that the functor \( U \) has a left adjoint \( F: M \to C \) with counit \( \varepsilon: FU \to C \) and unit \( \eta: M \to UF \). In this situation Proposition 2.2 has the following

Corollary 3.1. If \( \langle U, \tau, i, \rangle \) is a monoidal functor with the underlying functor \( U \) having a left adjoint then the monad \( \langle T, \mu, \eta \rangle \) associated to the adjunction data \( \varepsilon: FU \to C, \eta: M \to UF \) is such that \( T = UF \) is a lax comonoidal functor and \( \mu: T^2 \to T \) and \( \eta: M \to T \) are comonoidal natural transformations.

Proof. By Theorem 2.1 the left adjoint \( F: M \to C \) has a comonoidal structure. \( U \) being monoidal it is also comonoidal and the composition \( T = UF \) of lax comonoidal functors is again lax comonoidal. In Proposition 2.2 we have seen that \( \varepsilon \) and \( \eta \) are comonoidal natural transformations w.r.t. this comonoidal structure on \( U \) and \( F \). Therefore \( \mu = U \varepsilon F \) is also comonoidal since the monoidal categories, the lax comonoidal functors and the comonoidal natural transformations form a 2-category. \( \square \)

The monad we have obtained in the above Corollary suggests the following

Definition 3.2. Let \( M \) be a monoidal category. Then a monoid \( \langle T, \mu, \eta \rangle \) in the monoidal category of lax comonoidal endofunctors \( M \to M \) is called a bimonad in \( M \).

That is to say, a bimonad consists of 6 items,

1. a monoidal category \( \langle M, \otimes, i \rangle \)
2. a functor \( T: M \to M \)
3. a natural transformation \( \gamma_{x,y}: T(x \otimes y) \to Tx \otimes Ty \)
4. an arrow \( \pi: Ti \to i \)
5. a natural transformation \( \mu_x: T^2x \to Tx \)
6. a natural transformation \( \eta_x: x \to Tx \)

subjected to satisfy 6 axioms in the form of 10 commutative diagrams:

BMD 1: \( \gamma \) is coassociative,

\[
\begin{align*}
T(x \otimes (y \otimes z)) & \xrightarrow{\gamma_{x,y} \otimes z} T(x \otimes y) \otimes z \xrightarrow{\gamma_{xy,z}} (Tx \otimes Ty) \otimes Tz \\
\end{align*}
\]

BMD 2: \( \pi \) is a counit for \( \gamma \),

\[
\begin{align*}
T(i \otimes x) & \xrightarrow{\gamma_{i,x}} Ti \otimes Tx \\
\end{align*}
\]

\[
\begin{align*}
T(x \otimes i) & \xrightarrow{\gamma_{x,i}} Tx \otimes Ti \\
\end{align*}
\]

\[
\begin{align*}
T_i & \xrightarrow{i \otimes Tx} T \xrightarrow{\mu \otimes i} T \otimes i \\
\end{align*}
\]

\[
\begin{align*}
T_x & \xrightarrow{\pi \otimes x} T \xrightarrow{\eta \otimes i} T \otimes i \\
\end{align*}
\]

\[
\begin{align*}
T_x & \xrightarrow{\tau_{x,x}} T \xrightarrow{\tau_{x,x}} T \xrightarrow{\tau_{x,x}} T \\
\end{align*}
\]

\[
\begin{align*}
T_x & \xrightarrow{\tau_{x,x}} T \xrightarrow{\tau_{x,x}} T \xrightarrow{\tau_{x,x}} T \\
\end{align*}\]
BMD 3: $\mu$ is comonoidal,

\[
\begin{align*}
T^2(x \otimes y) & \xrightarrow{\gamma_{x,y}} T^2 x \otimes T^2 y & T^2_i \xrightarrow{\pi \circ \pi} i \\
\mu_{x \otimes y} & \xrightarrow{} T(x \otimes y) & \mu_i \xrightarrow{} i
\end{align*}
\]

(21)

\[
\begin{align*}
& \xrightarrow{} \xrightarrow{} & \xrightarrow{}
\end{align*}
\]

BMD 4: $\eta$ is comonoidal,

\[
\begin{align*}
\eta_{x \otimes y} & \xrightarrow{} \eta_{x} \otimes \eta_{y} & \eta_{i} \xrightarrow{} i
\end{align*}
\]

(22)

\[
\begin{align*}
& \xrightarrow{} \xrightarrow{} & \xrightarrow{}
\end{align*}
\]

BMD 5: $\mu$ is associative,

\[
\begin{align*}
T^3_x & \xrightarrow{T_{\mu_x}} T^2 x \\
\mu_{T_x} & \xrightarrow{} \mu_x & \mu_x \xrightarrow{} T_x
\end{align*}
\]

(23)

\[
\begin{align*}
& \xrightarrow{} \xrightarrow{} & \xrightarrow{}
\end{align*}
\]

BMD 6: $\eta$ is a unit for $\mu$,

\[
\begin{align*}
T_x & \xrightarrow{\eta_{T_x}} T^2 x & T_x & \xrightarrow{T_{\eta_x}} T^2 x \\
\mu_x & \xrightarrow{} & \mu_x & \xrightarrow{}
\end{align*}
\]

(24)

\[
\begin{align*}
& \xrightarrow{} \xrightarrow{} & \xrightarrow{}
\end{align*}
\]

Thus Corollary tells us that every monoidal adjunction $F : U$, with $U$ monoidal, determines a bimonad with underlying monad the classical construction $T = \langle UF, U\varepsilon F, \eta \rangle$. Explicitly, if $(U, \tau, \iota)$ is a monoidal functor and $F : U$ is an ordinary adjunction with unit $\eta$ and counit $\varepsilon$ then the associated bimonad is this.

(25) $T := UF$

(26) $\gamma_{x,y} := \tau_{F x, F y}^{-1} \circ U\varepsilon_{F x} \circ F y \circ UF \tau_{F x, F y} \circ UF(\eta_x \otimes \eta_y)$

(27) $\pi := \iota^{-1} \circ U\varepsilon_{\varepsilon} \circ UF \iota$

(28) $\mu_x := U\varepsilon_{F x}$

(29) $\eta_x := \eta_{x}$. 

3.2. The monoidal Eilenberg-Moore category. In this subsection $\mathcal{M}$ is a monoidal category and $(T, \mu, \eta)$ is a bimonad on $\mathcal{M}$.

The Eilenberg-Moore category $\mathcal{M}^T$ has as objects the $T$-algebras, i.e., pairs $\langle x, \alpha \rangle$ where $x$ is an object in $\mathcal{M}$ and $\alpha : T x \rightarrow x$ satisfies

\[
\begin{align*}
T^2x & \xrightarrow{T_{\alpha}} T x & x & \xrightarrow{\eta_x} T x \\
\mu_x & \xrightarrow{} & \alpha & \xrightarrow{} & \alpha
\end{align*}
\]

(30)

\[
\begin{align*}
& \xrightarrow{} \xrightarrow{} & \xrightarrow{}
\end{align*}
\]
The arrows from \( \langle x, \alpha \rangle \) to \( \langle y, \beta \rangle \) are the arrows \( t : x \to y \) in \( \mathcal{M} \) such that

\[
\begin{array}{ccc}
Tx & \xrightarrow{Tt} & Ty \\
\downarrow{\alpha} & & \downarrow{\beta} \\
x & \xrightarrow{t} & y
\end{array}
\]  
(31)

is commutative. The functor

\[
U^T : \mathcal{M}^T \to \mathcal{M} , \quad (x, \alpha) \mapsto x
\]

is called the Eilenberg-Moore forgetful functor.

**Proposition 3.3.** Let \( \langle T, \gamma, \pi, \mu, \eta \rangle \) be a bimonad. Then its Eilenberg-Moore category \( \mathcal{M}^T \) has the following monoidal structure. For \( T \)-algebras \( \langle x, \alpha \rangle \) and \( \langle y, \beta \rangle \) let their tensor product be

\[
\langle x, \alpha \rangle \odot \langle y, \beta \rangle := (x \otimes y, (\alpha \otimes \beta) \circ \gamma_{x,y})
\]

(33)

The tensor product of \( T \)-algebra arrows coincides with their tensor product as arrows in \( \mathcal{M} \). Then \( \odot \) gives rise to a monoidal structure on \( \mathcal{M}^T \) such that the forgetful functor \( U^T : \mathcal{M}^T \to \mathcal{M} \) becomes strictly monoidal.

**Proof.** In order to show that (33) is really a \( T \)-algebra we need to verify the two defining diagrams of [33]). The first of these follow from

\[
(\alpha \otimes \beta) \circ \gamma_{x,y} \circ T(\alpha \otimes \beta) \circ T\gamma_{x,y} = \\
= (\alpha \otimes \beta) \circ (T\alpha \otimes T\beta) \circ \gamma_{Tx,Ty} \circ T\gamma_{x,y} = \\
= (\alpha \otimes \beta) \circ (\mu_x \otimes \mu_y) \circ \gamma_{Tx,Ty} \circ T\gamma_{x,y} = \\
= (\alpha \otimes \beta) \circ \gamma_{x,y} \circ \mu_x \otimes \mu_y
\]

where in the last equation we used comonoidality of \( \mu \). The second diagram follows from

\[
(\alpha \otimes \beta) \circ \gamma_{x,y} \circ \eta_{x \otimes y} = \\
= (\alpha \otimes \beta) \circ (\eta_x \otimes \eta_y) = x \otimes y
\]

where comonoidality of \( \eta \) had to be used.

For \( T \)-algebra arrows \( t : \langle x, \alpha \rangle \to \langle x', \alpha' \rangle \) and \( s : \langle y, \beta \rangle \to \langle y', \beta' \rangle \) we defined

\[
t \odot s := t \otimes s
\]

(34)

which is indeed a \( T \)-algebra arrow because

\[
(\alpha' \otimes \beta') \circ \gamma_{x',y'} \circ T(t \otimes s) = \\
= (\alpha' \otimes \beta') \circ (Tt \otimes Ts) \circ \gamma_{x,y} = \\
= (t \otimes s) \circ (\alpha \otimes \beta) \circ \gamma_{x,y}
\]

This finishes the definition of the functor \( \odot \). As for the monoidal unit we set

\[
i^T := (i, \pi)
\]

(35)

Now we are going to show that the coherence isomorphism \( a, l, r \) of \( \langle \mathcal{M}, \otimes, i \rangle \), when considered as arrows in \( \mathcal{M}^T \), serve as coherence isomorphisms of \( \langle \mathcal{M}^T, \odot, i^T \rangle \). For three \( T \)-algebras \( x^T = \langle x, \alpha_x \rangle \), \( y^T = \langle y, \alpha_y \rangle \) and \( z^T = \langle z, \alpha_z \rangle \) we have

\[
x^T \odot (y^T \odot z^T) = (x \otimes (y \otimes z), (\alpha_x \otimes (\alpha_y \otimes \alpha_z)) \circ (Tx \otimes \gamma_{y,z}) \circ \gamma_{x,y \otimes z} \\
(x^T \odot y^T) \odot z^T = ((x \otimes y) \otimes z, ((\alpha_x \otimes \alpha_y) \otimes \alpha_z) \circ (\gamma_{x,y} \otimes Tz) \circ \gamma_{x \otimes y, z})
\]
and the calculation
\[ a_{x,y,z} \circ (\alpha_x \otimes (\alpha_y \otimes \alpha_z)) \circ (Tx \otimes \gamma_{y,z}) \circ \gamma_{x,y,z} = \]
\[ = ((\alpha_x \otimes \alpha_y) \otimes \alpha_z) \circ a_{T^x,T^y,T^z} \circ (Tx \otimes \gamma_{z,y}) \circ \gamma_{x,y} = \]
proves that \( a_{x,y,z} \) is an isomorphism
\[ x^T \diamond (y^T \diamond z^T) \xrightarrow{a_{x,y,z}} (x^T \diamond y^T) \diamond z^T \]
of \( \mathcal{M}^T \), indeed. In order to show that \( i^T \) is a left unit notice that
\[ i^T \circ x^T = \langle i \otimes x, (\pi \otimes \alpha_x) \circ \gamma_i \rangle \]
and therefore
\[ 1_x \circ (\pi \otimes \alpha_x) \circ \gamma_i = \]
\[ = 1_x \circ (i \otimes \alpha_x) \circ (\pi \otimes Tx) \circ \gamma_i = \]
\[ = T_{1_x} \circ (\pi \otimes Tx) \circ \gamma_i = \]
proves that \( 1_x \) is an isomorphism
\[ i^T \diamond x^T \xrightarrow{1_x} x^T \]
of \( \mathcal{M}^T \). Similarly, \( r_x : x^T \diamond i^T \xrightarrow{\gamma_i} x^T \) for all objects \( x^T \) of \( \mathcal{M}^T \). This finishes the construction of a monoidal structure \( \langle \mathcal{M}^T, \circ, i^T \rangle \) on the Eilenberg-Moore category. It is clear from the construction that the forgetful functor \( U^T \) is strictly monoidal.

**Proposition 3.4.** Let \( T \) be a bimonad. Then the strict monoidal \( U^T \) has a lax comonoidal left adjoint, the free \( T \)-algebra functor
\[ F^T : \mathcal{M} \to \mathcal{M}^T, \quad x \mapsto \langle Tx, T^2x \xrightarrow{\mu_{Tx}} Tx \rangle . \]
such that \( U^TF^T = T \), as lax comonoidal functors.

**Proof.** Left adjointness is proven as in the textbooks. As for the lax comonoidal structure notice that \( \gamma_{x,y} \) provides a \( T \)-algebra arrow from \( F^T(x \otimes y) \) to
\[ F^T x \diamond F^T y = \langle Tx \otimes Ty, (\mu_x \otimes \mu_y) \circ \gamma_{Tx,Ty} \rangle \]
because \( \gamma_{x,y} \circ \mu_{x \otimes y} = (\mu_x \otimes \mu_y) \circ \gamma_{Tx,Ty} \circ T \gamma_{x,y} \) which is precisely the first diagram in \([21]\). Similarly, the second diagram of \([21]\) is the condition for
\[ \hat{\pi} : F^T = \langle Ti, \mu_i \rangle \xrightarrow{\pi} \langle i, \pi \rangle = i^T \]
to be a \( T \)-algebra arrow. Now \([19,20]\) imply that the triple \( \langle F^T, \hat{\gamma}, \hat{\pi} \rangle \) is a lax comonoidal functor. The relations \( U^TF^T = T, U^T \hat{\gamma} = \gamma \) and \( U^T \hat{\pi} = \pi \) are obvious. \( \square \)
3.3. The Tannakian reconstruction. Recall that the Eilenberg-Moore comparison functor $K: C \to \mathcal{M}^T$ maps the objects $c$ of $C$ into the $T$-algebras

\[(40) \quad Kc := \langle Uc, U\varepsilon_c : TUc \to Uc \rangle\]

and the arrows $\psi: c \to d$ to the $T$-algebra morphisms

\[(41) \quad K\psi := \langle Uc, U\varepsilon_c \rangle \xrightarrow{U\psi} \langle Ud, U\varepsilon_d \rangle.\]

This functor allows to factorize the given $U$ through the category of $T$-algebras as $U^TK = U$.

**Proposition 3.5.** Let $U: C \to \mathcal{M}$ be a monoidal functor with left adjoint and $T$ be the associated bimonad. Then the Eilenberg-Moore comparison functor $K: C \to \mathcal{M}^T$ has a unique (lax) monoidal structure such that the factorization $U = U^TK$ is a factorization of monoidal functors.

**Proof.** Since $U^T$ is strict monoidal, if $\langle K, \hat{\tau}, \hat{\iota} \rangle$ is a lax monoidal functor such that $U = U^TK$ then $\hat{\tau} = \tau$ and $\hat{\iota} = \iota$. That is to say, the unique lax monoidal structure, if exists, is monoidal and it is obtained by lifting the arrows $\tau_{c,d}$ and $\iota$ to $T$-algebra arrows. Taking into account formula (26) the action in the tensor product

\[Kc \bowtie Kd = \langle Uc \otimes Ud, (U\varepsilon_c \otimes U\varepsilon_d) \circ \gamma_{Uc,Ud} \rangle\]

can be written as

\[(U\varepsilon_c \otimes U\varepsilon_d) \circ \gamma_{Uc,Ud} = \tau_{c,d}^{-1} \circ U(\varepsilon_c \square \varepsilon_d) \circ U\sigma_{Uc,Ud} \]

\[= \tau_{c,d}^{-1} \circ U\varepsilon_c \circ T\tau_{c,d}\]

where in the last equation the monoidal adjunction (15) has been used. This result, up to multiplying with $\tau_{c,d}$, is precisely the lifting condition for $\tau_{c,d}$ to be $T$-algebra morphism $Kc \bowtie Kd \to K(c \bowtie d)$. As for the unit map $\iota: i \to Ue$, it has a lift to a $T$-algebra morphism

\[\langle i, \pi \rangle \to \langle Ue, U\varepsilon_e \rangle = Ke\]

if and only if $\iota \circ \pi = U\varepsilon_e \circ T\iota$. The right hand side is equal to $UV$ by (16) and the left hand side is equal to $UV$ by (27). \[\square\]

Our main theorem of the Tannakian type relates adjointable monoidal functors to bimonads in the following way.

**Theorem 3.6.** Let $U: C \to \mathcal{M}$ be a monoidal functor possessing a left adjoint. Then there exists a bimonad $T$ on $\mathcal{M}$ and a monoidal functor $K: C \to \mathcal{M}^T$ such that

1. $U$ has the monoidal factorization $U = U^TK$ where $U^T: \mathcal{M}^T \to \mathcal{M}$ is the monoidal Eilenberg-Moore forgetful functor,
2. the pair $(T, K)$ is universal with respect to property (1). That is to say, if $S$ is a bimonad on $\mathcal{M}$ and $L: \mathcal{A} \to \mathcal{M}^S$ is a lax monoidal functor such that $U = U^S L$, as lax monoidal functors, then there exists a unique natural transformation $\varphi: S \to T$ such that
   a. $U^\varphi K = L$ where $U^\varphi$ is the functor mapping a $T$-algebra $\langle x, \alpha \rangle$ to the $S$-algebra $\langle x, \alpha \circ \varphi_x \rangle$,
   b. $\varphi$ is comonoidal,
   c. $\varphi$ is a monad morphism.
Proof. As for the existence of $T$ and $K$ with property (1) one takes for $T$ the bimonad associated to $U$ and to one of its left adjoints $F$ by Corollary 3.1 and for $K$ the Eilenberg-Moore comparison functor. In order to show the universal property (2) we need to prove existence and uniqueness of $\varphi$. Notice that the functors $L$ with property $U^S L = U$ can be written as $Lc = \langle Uc, \beta_c \rangle$ where functoriality implies that $\beta_c : SUc \rightarrow Uc$ is natural in $c \in C$. Furthermore, lax monoidality of $L$, strict monoidality of $U^S$ and monoidality of the factorization $U = U^S L$ implies that $L$ is monoidal and $L_2 : \circ (L \times L) \rightarrow L_{\Box}$ is the lift of $U_2 : \circ (U \times U) \rightarrow U_{\Box}$. Similarly $L_0 : \langle i, S^0 \rangle \rightarrow L e = \langle U e, \beta_e \rangle$ is the lift of $U_0 : i \sim U e$. Since

$$Le \circ \beta_L = \langle Uc \otimes Ud, (\beta_c \otimes \beta_d) \circ SU_{c,d} \rangle$$

the lifting conditions for $U_{c,d}$ and $U_0$, respectively take the form

$$(42) \quad U_{c,d} \circ (\beta_c \otimes \beta_d) \circ SU_{c,d} = \beta_{c,d} \circ SU_{c,d}$$

$$(43) \quad U_0 \circ S^0 = \beta_e \circ SU_0$$

and these are precisely the conditions for $\beta$ to be comonoidal. Let $L(U)$ be the category with objects the pairs $\langle S, \beta \rangle$ where $S : M \rightarrow M$ is a lax comonoidal functor $\langle S, S^2, S^0 \rangle$ and $\beta : SU \rightarrow U$ is a comonoidal natural transformation. The arrows from $\langle R, \alpha \rangle$ to $\langle S, \beta \rangle$ are the comonoidal natural transformations $\varphi : R \rightarrow S$ satisfying $\beta \bullet \varphi U = \alpha$. Now it is standard universal algebra to show that if $U$ has a left adjoint then $L(U)$ has terminal objects. Moreover, in a terminal object $(T, \omega)$ the $T$ is a monad and $\omega$ is an action of $T$. If $\langle S, \beta \rangle$ is an object in which $S$ is a monad and $\beta$ is an action of $S$ on $U$ then the unique arrow $\varphi : \langle S, \beta \rangle \rightarrow \langle T, \omega \rangle$ is a monad morphism, i.e., satisfies

$$\mu^T \bullet T \varphi \bullet \varphi S = \mu^S$$

$$\varphi \bullet \eta^S = \eta^T.$$  

Now it is easy to see that condition (2) is just the expression of the fact that the pair $\langle T, \omega \rangle$, in which $\omega$ corresponds to the comparison functor $K$, is a terminal object in $L(U)$. We omit the details because we shall prove in Theorem 4.19 a more general universality property involving bimonads $S$ on any other monoidal category $N$. (Cf. also the proof of Lemma 4.4 or the literature [18].)

The next theorem serves as a characterization of the forgetful functors of bimonads.

Theorem 3.7. Let $C$ and $M$ be monoidal categories. For a functor $U : C \rightarrow M$ the following conditions are equivalent:

1. There exists a bimonad $T$ on $M$ and a monoidal equivalence $K : C \rightarrow M^T$ such that – via this equivalence – $U$ is isomorphic to the forgetful functor $U^T$.

2. $U$ is monadic and monoidal.

Proof. $(2) \Rightarrow (1)$ Monadicity of $U$ is by definition the requirement that $U$ has a left adjoint and the comparison functor $K$ is a category equivalence. A category equivalence is always part of an adjoint equivalence [14, Theorem IV. 4. 1] so there exists a right adjoint of $K$ with invertible unit and counit. Now $K$ is monoidal, hence comonoidal, therefore the converse of Theorem 2.3 provides a lax monoidal
structure on the right adjoint which, by invertibility of the unit and counit, is actually monoidal. This proves that $K$ is a monoidal equivalence and the rest, $U = U^T K$, is obvious.

(1) ⇒ (2) The Eilenberg-Moore forgetful functor $U^T$ is always monadic because $F^T$ is its left adjoint, $U^T F^T = T$, and the corresponding comparison functor $M^T \to M^T$ sends the object $\langle x, \alpha \rangle$ to

$$\langle U^T(x, \alpha), U^T \varepsilon^T_{(x, \alpha)} \rangle = \langle x, \alpha \rangle.$$  

Therefore the comparison functor is the identity functor. Now we have an equivalence $K : C \to M^T$ and it is easy to see that monadicity of $U^T$ is inherited to $U$ via this $K$. Since $U = U^T K$, this defines a monoidal structure for $U$. □

For structural assumptions on $U$ and $C$ that imply monadicity we refer to the literature [15, 1]. Here we give only a crude consequence of the above Theorem which is still sufficiently general to include as special cases the forgetful functors of bialgebras, to be discussed in Section 2. Therefore it covers also the cases of forgetful functors $U : A M \to k M$ where $A$ is either a weak bialgebra or bialgebra or weak Hopf algebra or Hopf algebra over $k$.

**Corollary 3.8.** Let $C$ be a monoidal category having coequalizers and let $U : C \to M$ be a monoidal functor such that its underlying functor reflects isomorphisms and has a left adjoint and a right adjoint. Then $C$ is monoidally equivalent to the Eilenberg-Moore category $M^T$ of a bimonad.

Necessary and sufficient conditions for a bimonad to be the bimonad of the forgetful functor of a bialgebroid will be given in Section 2.

4. **2-functoriality of the construction of bimonads**

The functors $U$ for which a bimonad can be constructed are the objects of a 2-category $L\text{-}MFunc$. We extend the bimonad construction of the previous Section to a 2-functor $Q : L\text{-}MFunc \to Bmd$ from which a sensible definition for the 2-category $Bmd$ of bimonads emerges. We show that $Q$ is the left adjoint of a 2-functor $EM$ which incorporates the Eilenberg-Moore construction. This adjunction explains and extends the universality result of Theorem 3.6. Finally, the fact that bimonads form a 2-category will enable us to speak about isomorphisms and equivalences of bialgebroids which, in turn, in Section 3, will be shown to be objects of $Bmd$.

4.1. **The 2-category of arrows.** Let $(K, \cdot, o)$ be a 2-category. As before in case of $K = \text{Cat}$ we omit the symbol $o$ for horizontal composition. We define the 2-category of arrows in $K$ as the 2-category $\text{Arr}(K)$ having

- objects $(A, U, M)$ where $U : A \to M$ is a 1-cell of $K$,
- 1-cells $\langle F, \kappa, G \rangle : (A, U, M) \to (B, V, N)$ where $F : A \to B$ and $G : M \to N$ are 1-cells of $K$ and $\kappa : GU \to VF$ is a 2-cell of $K$,
- 2-cells $\langle \vartheta, \nu \rangle : \langle F, \kappa, G \rangle \to \langle F', \kappa', G' \rangle : (A, U, M) \to (B, V, N)$ where $\vartheta : F \to F'$ and $\nu : G \to G'$ are 2-cells of $K$ such that

$$(44) \quad V\vartheta \kappa = \kappa' \nu U$$

The horizontal composition of 1-cells is

$$(45) \quad \langle F, \kappa, G \rangle \circ \langle H, \lambda, I \rangle = \langle FH, \kappa H \bullet G\lambda, GI \rangle$$
the horizontal composition of 2-cells is
\[(\vartheta, \nu) \circ [\vartheta', \nu'] = [\vartheta \vartheta', \vartheta \nu \nu']\]
and the vertical composition of 2-cells is
\[(\vartheta, \nu) \sqcap [\vartheta', \nu'] = [\vartheta \vartheta' \vartheta, \vartheta \nu \nu \nu]\]
whenever they are defined.

Thus we have two 2-functors
\[\mathcal{K} \xleftarrow{\text{dom}} \text{Arr}(\mathcal{K}) \xrightarrow{\text{cod}} \mathcal{K}\]
given respectively by
\[
\begin{align*}
A & \leftarrow (A, U, M) \rightarrow M \\
F & \leftarrow (F, \kappa, G) \rightarrow G \\
\vartheta & \leftarrow [\vartheta, \nu] \rightarrow \nu
\end{align*}
\]
We want to single out a sub-2-category in \(\text{Arr}(\mathcal{K})\) the objects \((A, U, M)\) of which carry a universal action of a monoid \(\langle T, \mu, \eta \rangle\) at \(M\). Such actions \(\alpha : TU \rightarrow U\) will be called left actions as they act on the codomain side of \((A, U, M)\). As it is well known in universal algebra \([18]\) such monoids are readily obtained by universality from a much simpler structure, a 1-cell \(T : M \rightarrow M\) and a 2-cell \(\kappa : TU \rightarrow U\) no condition whatsoever. Existence of universal monoids is guaranteed for example if \(U\) has a left adjoint in \(K\). In the next Definition the usual universality is replaced by a slightly stronger "2-universality" property which we need later but which is also a property of left adjointable \(U\)-s. In the sequel we denote by \(\text{Arr}^{-}(\mathcal{K})\) the sub-2-category in which the \(\kappa\)-s are invertible.

**Definition 4.1.** A left action on a 1-cell \((A, U, M)\) in \(\mathcal{K}\) is a 2-cell in \(\text{Arr}(\mathcal{K})\) of the form \((\alpha, \kappa) : (A, U, M) \rightarrow (A, U, M)\). That is to say a left action on \(U\) consists of a 1-cell \(R : M \rightarrow M\) and a 2-cell \(\alpha : RU \rightarrow U\) in \(\mathcal{K}\). The left action \(\alpha : (A, U, M) \rightarrow (A, U, M)\) is called universal if for any left action \(\beta : (B, \kappa, G) \rightarrow (B, \kappa, G)\) on \((B, V, N)\) and any 1-cells \(\kappa = (F, \kappa, G)\) and \(\kappa' = (F', \kappa', G')\) from \((A, U, M)\) to \((B, V, N)\) in \(\text{Arr}^{-}(\mathcal{K})\) the domain functor gives rise to a bijection of 2-cells
\[(\text{dom}) \rightarrow \mathcal{K}(F, F')\]

That is to say, a left action \(\alpha : RU \rightarrow U\) is universal if for every left action \(\beta : SV \rightarrow V\), 1-cells \(F, F' : A \rightarrow B, G, G' : M \rightarrow N\), 2-cell \(\vartheta : F \rightarrow F'\) and invertible 2-cells \(\kappa : GU \rightarrow VF\) and \(\kappa' : G\psi U \rightarrow VF\) there exists a unique 2-cell \(\psi : SG \rightarrow G'\psi\) such that
\[(49) \quad V\vartheta \bullet \beta F \bullet S\kappa = \kappa' \bullet G'\alpha \bullet \psi U\]

**Remark 4.2.** Taking into account the explicit form of the horizontal composites
\[
\begin{align*}
\langle B, \beta, S \rangle \circ \langle F, \kappa, G \rangle &= \langle F, \beta F \bullet S\kappa, SG \rangle \\
\langle F', \kappa', G' \rangle \circ \langle A, \alpha, R \rangle &= \langle F', \kappa' \bullet G' \alpha, G'R \rangle
\end{align*}
\]
the above equation \((49)\) for \(\psi\) is precisely the condition for the pair \(\vartheta, \psi\) to be a 2-cell
\[(52) \quad [\vartheta, \psi] : \langle B, \beta, S \rangle \circ \langle F, \kappa, G \rangle \rightarrow \langle F', \kappa', G' \rangle \circ \langle A, \alpha, R \rangle.
\]
Therefore, as sets
\[(53) \quad \{\psi\} = \text{cod}_{\beta \bullet \kappa', \alpha} \circ \text{dom}_{\beta \bullet \kappa}^{-1} \circ \text{dom}_{\beta \bullet \kappa, \alpha}^{-1}(\{\vartheta\})\].
The next Lemma secures a familiar class of 1-cells on which universal actions exist.

**Lemma 4.3.** If $U$ is a 1-cell in $K$ which has a left adjoint $\bar{U}$ then there is a universal left action on $U$, namely $\alpha = U\varepsilon : TU \to U$ where $T = U\bar{U}$ and $\varepsilon : \bar{U}U \to A$ is the counit of the adjunction.

**Proof.** Multiply equation (49) from the left by $\kappa^{-1}$, then compose it horizontally from the right with $\bar{U}$, and finally multiply it from the right by $SG\eta$, where $\eta$ is the unit of the adjunction. Thus we obtain

$$\kappa^{-1}U \cdot V\vartheta U \cdot B\bar{F}U \cdot SRU \cdot SG\eta \allowbreak \allowbreak \allowbreak = G'U\varepsilon U \cdot \psi UU \cdot SG\eta \allowbreak \allowbreak \allowbreak = G'U\varepsilon U \cdot G'U\bar{U}\eta \cdot \psi \allowbreak \allowbreak \allowbreak = \psi .$$

□

Let $\text{Arr}_u(K)$ denote the full sub-2-category of $\text{Arr}(K)$ having as objects those objects of $\text{Arr}(K)$ on which a universal left action exists.

4.2. **The construction of the 2-functor $Q$.** We define a 2-functor $Q : \text{Arr}_u(K) \to \text{Mnd}(K)$ as follows. Its object map is provided by the following Lemma.

**Lemma 4.4.** For an object $⟨A, U, M⟩$ of $\text{Arr}_u(K)$ let $⟨A, \omega, T⟩$ be a universal left action. Then there exist unique 2-cells $\mu : TT \to T$ and $\eta : M \to T$ such that

$$\omega \cdot \mu U = \omega \cdot T\omega$$

$$\omega \cdot \eta U = U .$$

The triple $⟨T, \mu, \eta⟩$ is a monad in $K$ on $M$.

**Proof.** Let $L(U)$ be the category of left actions on $U$ which is the subcategory in $\text{Arr}(U, U)$ containing as objects the special 1-cells $⟨A, \alpha, R⟩$ and as arrows the special 2-cells $[A, \nu]$. Then universality of $⟨A, \omega, T⟩$ implies that it is a terminal object in $L(U)$. As a matter of fact if we specialize the universal property to the choice $V = U$, $(F, \kappa, G) = (F', \kappa', G')$ being the identity cell $⟨A, U, M⟩$ and $\theta = A$ then we obtain that for all $R : M \to M$ and all $\alpha : RU \to U$ there exists a unique $\nu : R \to T$ such that $\alpha = \omega \cdot \nu U$.

Now it is clear that the solutions for $\mu$ and $\eta$ of the equations (54-55) provide arrows

$$[A, \mu] : ⟨A, \omega \cdot T\omega, TT⟩ \to ⟨A, \omega, T⟩$$

$$[A, \eta] : ⟨A, U, M⟩ \to ⟨A, \omega, T⟩$$

hence they exist and are unique. The rest is standard universal algebra: One checks that both $\mu \cdot T\mu$ and $\mu \cdot \mu T$ provide arrows from $⟨A, \omega \cdot T\omega \cdot TT\omega, TTT⟩$ to the terminal object, hence $\mu$ is associative. Similarly one proves that $\eta$ is a unit for $\mu$, hence $⟨T, \mu, \eta⟩$ is a monad. □

Given a choice of universal left action $⟨A, \omega, T⟩$ for each object $⟨A, U, M⟩$ of $\text{Arr}_u(K)$ we define

$$Q(A, U, M) := ⟨T, \mu, \eta⟩$$
where the monad on the RHS is obtained from Lemma 4.4.

The definition of Q on 1-cells is provided by the following specialization of the universal property of Definition 4.1. Setting \( \langle F, \kappa, G \rangle = \langle F', \kappa', G' \rangle \) and \( \vartheta = F \) we obtain that if \( \langle A, \alpha, R \rangle \) is a universal left action on \( (A, U, M) \) then for all object \( (B, V, N) \), all left action \( \langle B, \beta, S \rangle \) on \( (B, V, N) \) and all \( \langle F, \kappa, G \rangle : (A, U, M) \to (B, V, N) \) in \( \text{Arr}^\sim(K) \) there exists a unique \( \varphi : SG \to GR \) such that

\[
(59) \quad \beta F \cdot S \kappa = \kappa \cdot G \alpha \cdot \varphi U.
\]

**Lemma 4.5.** Let \( (A, U, M) \) and \( (B, V, N) \) be objects of \( \text{Arr}_u(K) \) and let \( Q(A, U, M) = \langle R, \mu_R, \eta_R \rangle \) and \( Q(B, V, N) = \langle S, \mu_S, \eta_S \rangle \). Then for each 1-cell \( \langle F, \kappa, G \rangle : (A, U, M) \to (B, V, N) \) in \( \text{Arr}^\sim(K) \) the unique solution for \( \varphi \) of equation (59) provides a monad morphism

\[
\langle G, \varphi \rangle : \langle R, \mu_R, \eta_R \rangle \to \langle S, \mu_S, \eta_S \rangle,
\]

i.e., a 1-cell in the 2-category \( \text{Mnd}(K) \) of monads in \( K \). That is to say \( G : M \to N \) is a 1-cell and \( \varphi : SG \to GR \) is a 2-cell satisfying the commutative diagrams

\[
\begin{array}{ccc}
SSG & \xrightarrow{S \varphi} & SGR \\
\downarrow \mu_S G & & \downarrow G \mu_R \\
SG & \xrightarrow{\varphi} & GR
\end{array}
\]

\[
\begin{array}{ccc}
SG & \xrightarrow{\varphi} & GR \\
\downarrow \mu_S G & & \downarrow G \mu_R \\
SG & \xrightarrow{\varphi} & GR
\end{array}
\]

\[
(61)
\]

**Proof.** To prove the first diagram it suffices, by the above special universality property, to show that both \( \varphi' := G \mu_R \cdot \varphi R \cdot S \varphi \) and \( \varphi^\dagger := \varphi \cdot \mu_S G \) are solutions of

\[
(62) \quad \beta^2 F \cdot SS \kappa = \kappa \cdot G \alpha \cdot \Phi U
\]

for \( \Phi : SSG \to GR \) where \( \beta^2 \) stands for \( \beta \cdot S \beta = \beta \cdot \mu_S S : SSB \to B \). As a matter of fact

\[
\kappa \cdot G \alpha \cdot \varphi' U = \kappa \cdot G \alpha \cdot G \mu_R U \cdot \varphi R U \cdot S \varphi U
\]

\[
= \kappa \cdot G \alpha \cdot G \alpha \cdot \varphi U \cdot S \varphi U
\]

\[
= \kappa \cdot G \alpha \cdot \varphi U \cdot SG \alpha \cdot S \varphi U
\]

\[
= \beta F \cdot S \kappa \cdot SG \alpha \cdot S \varphi U
\]

\[
= \beta F \cdot S \beta F \cdot SS \kappa = \beta^2 F \cdot SS \kappa
\]

and

\[
\kappa \cdot G \alpha \cdot \varphi^\dagger U = \kappa \cdot G \alpha \cdot \varphi U \cdot \mu_S GU
\]

\[
= \beta F \cdot S \kappa \cdot \mu_S GA
\]

\[
= \beta F \cdot \mu_S V F \cdot SS \kappa = \beta^2 F \cdot SS \kappa.
\]

Similarly, the proof of the second diagram amounts to showing that both \( G \eta_R \) and \( \varphi \cdot \eta_S G \) solve the equation

\[
(63) \quad \kappa = \kappa \cdot G \alpha \cdot \Upsilon U
\]

for \( \Upsilon : G \to GR \). Indeed,

\[
\kappa \cdot G \alpha \cdot G \eta_R U = \kappa \cdot GU = \kappa
\]
and

\[ \kappa \bullet G \alpha \bullet \varphi U \bullet \eta_S GU = \beta F \bullet S \kappa \bullet \eta_S FU = \beta F \bullet \eta_S V F \bullet \kappa = \kappa. \]

We can therefore define \( Q \) on 1-cells by

\[ Q(\langle F, \kappa, G \rangle) := \langle G, \varphi \rangle \]

where \( \varphi \) is determined by Lemma 4.5.

Before defining \( Q \) on 2-cells we investigate functoriality of \( Q \) on the category of 0-cells and 1-cells. In the following Lemma \( \alpha: RU \to U, \beta: SV \to V \) and \( \gamma: TW \to W \) denote universal actions corresponding to the definition of \( Q \) on the object \( U, V \) and \( W \), respectively.

**Lemma 4.6.** For composable 1-cells in \( \text{Arr}_u(K) \) as in the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\kappa \downarrow & & \lambda \downarrow \\
M & \xrightarrow{G} & N \\
\end{array}
\]

we have \( Q(\langle H, \lambda, I \rangle \circ \langle F, \kappa, G \rangle) = Q(\langle H, \lambda, I \rangle) \circ Q(\langle F, \kappa, G \rangle) \).

**Proof.** Taking into account the formula

\[ \langle I, \chi \rangle \bullet \langle G, \varphi \rangle = \langle IG, I \varphi \bullet \chi G \rangle \]

for composition of monad morphisms, we have to show that if \( \varphi: SG \to GR \) and \( \chi: TI \to IS \) are solutions of the equations

\[ \beta F \bullet S \kappa = \kappa \bullet G \alpha \bullet \varphi U \]
\[ \gamma H \bullet T \lambda = \lambda \bullet I \beta \bullet \chi V \]

then \( \nu = I \varphi \bullet \chi G \) solves the equation

\[ \gamma HF \bullet T \lambda F \bullet TI \kappa = \lambda F \bullet I \kappa \bullet IG \alpha \bullet \varphi U. \]

As a matter of fact,

\[ \lambda F \bullet I \kappa \bullet IG \alpha \bullet I \varphi U \bullet \chi GU = \lambda F \bullet I \beta F \bullet IS \kappa \bullet \chi GU = \lambda F \bullet I \beta F \bullet \chi VF \bullet TI \kappa = \gamma HF \bullet T \lambda F \bullet TI \kappa. \]

The definition of \( Q \) on 2-cells uses the full strength of Definition 4.1. First of all for a 2-cell

\[ [\partial, \nu]: \langle F, \kappa, G \rangle \rightarrow \langle F', \kappa', G' \rangle: (A, U, M) \rightarrow (B, V, N) \]

we set

\[ Q[\partial, \nu] := \nu. \]

The statement that \( \nu: \langle G, \varphi \rangle \rightarrow \langle G', \varphi' \rangle \) is a transformation of monad morphisms, i.e., a 2-cell in \( \text{Mnd}(K) \) is by definition the property

\[ SG \xrightarrow{\varphi} GR \]
\[ S \nu \downarrow \quad \nu R \downarrow \]
\[ SG' \xrightarrow{\varphi'} G'R \]
where \( \langle G, \varphi \rangle = Q(F, \kappa, G) \) and \( \langle G', \varphi' \rangle = Q(F', \kappa', G') \). Commutativity of this diagram follows from universality after noticing that both \( \nu R \cdot \varphi \) and \( \varphi' \cdot S\nu \) are solutions for \( \psi: SG \to G' R \) of the equation (49). Indeed,

\[
\kappa' \cdot G' \alpha \cdot \nu RU \cdot \varphi U = \nu \varphi U \cdot G\alpha \cdot \varphi U = V \varphi U \cdot S\kappa \cdot \beta F \cdot S\kappa
\]

and

\[
\kappa' \cdot G' \alpha \cdot \varphi' U \cdot S\nu U = \beta F' \cdot S\kappa' \cdot S\nu U = V \varphi' U \cdot S\kappa.
\]

Since for 2-cells both the horizontal and vertical composition in \( \text{Mnd}(\mathcal{K}) \) coincides with those of \( \mathcal{K} \), in view of (46) and (47) the \( \mathcal{Q} \) preserves both compositions. This finishes the construction of the 2-functor \( \mathcal{Q}: \text{Arr}^u(\mathcal{K}) \to \text{Mnd}(\mathcal{K}) \).

### 4.3. The monoidal version of \( \mathcal{Q} \)

In this subsection we are interested in the monoidal properties of the 2-functor \( \mathcal{Q} = \mathcal{Q}(\mathcal{K}) \) if the underlying 2-category \( \mathcal{K} \) is monoidal. In order not to drift too far from the main theme of bialgebroids we restrict ourselves to the case \( \mathcal{K} = \text{Cat} \) endowed with the Cartesian product \( \times \) of categories, functors and natural transformations.

The content of this subsection crucially depends on whether \( \alpha: RU \to U \) being a universal action implies \( \alpha \times \alpha: (R \times R)(U \times U) \to U \times U \) is universal, too. Since this property does not seem to be automatic, we shall restrict ourselves to functors \( U \) with left adjoints. In this case \( U \times U \) also has a left adjoint, therefore \( \alpha \times \alpha \) is universal indeed (cf. Lemma 4.3). We denote by \( \text{L-Func} \) the full sub-2-category of \( \text{Arr}^u(\text{Cat}) \) with objects the left adjointable functors.

If \( \mathcal{A} \) and \( \mathcal{M} \) are monoidal categories and the left adjointable \( U: \mathcal{A} \to \mathcal{M} \) is given a monoidal structure \( U_2: \otimes \mathcal{M} (U \times U) \xrightarrow{\sim} U \otimes \mathcal{A}, U_0: i_M \xrightarrow{\sim} U i_A \) then Corollary 1.1 tells us that the monad \( \mathcal{Q}(\mathcal{A}, U, \mathcal{M}) = \langle T, \mu, \eta \rangle \) has a lax comonoidal structure \( T^2: U \otimes \mathcal{A} \to \otimes \mathcal{M}(U \times U), T^0: U i_A \to i_M \) so that \( \langle T, T^2, T^0, \mu, \eta \rangle \) is a bimonad. This result is the object map part of a commutative diagram of 2-functors

\[
\text{L-MFunc} \xrightarrow{\mathcal{Q}} \text{Bmd}
\]

(67)

where the vertical 2-functors forget about (co)monoidal structures, otherwise all items in the first row are yet undefined, including the 2-category \( \text{Bmd} \) of bimonads. Our aim is to define them in such a way that the above diagram commutes.

One solution is obtained by taking \( \mathcal{K} \) to be \( \text{ComonCat} \) in the first row and \( \mathcal{K} = \text{Cat} \) in the second and then apply the procedure of the previous subsection to construct the \( \mathcal{Q} \). This seems to be the most natural choice since bimonads involve lax comonoidal functors. This choice leads to the diagram

\[
\text{L-CFunc} \xrightarrow{\mathcal{Q} \text{(ComonCat)}} \text{Bmd}'
\]

(68)

where \( \text{L-CFunc} \) is the 2-category with
➤ objects the comonoidal functors \( U \) (equivalently: monoidal ones) with left
adjoint,
➤ 1-cells \( (F, \kappa, G) : U \to V \) where \( F, G \) are lax comonoidal functors and
\( \kappa : GU \to VF \) is a comonoidal natural isomorphism,
➤ 2-cells \( [\vartheta, \nu] : (F, \kappa, G) \to (F', \kappa', G') \) where both \( \vartheta : F \to F' \) and \( \nu : G \to G' \)
are comonoidal and satisfy the constraint \( V\vartheta \circ \kappa = \kappa' \circ \nu U \).

Accordingly, the 2-category \( \text{Bmd}' \) involves only (lax) comonoidal functors in place
of 1-cells. Especially, monad morphisms \( (G, \varphi) \) involve lax comonoidal functors
\( G : \mathcal{M} \to \mathcal{N} \). These functors map comonoids into comonoids but does not map
monoids to monoids.

If we want arrows that preserve module algebras over bialgebroids instead of
module coalgebras, we must insist of having lax monoidal functors in the definition
of 1-cells. At first sight this spoils any sensible (co)monoidality of the 2-cell
\( \varphi : SG \to GR \) since \( R \) and \( S \) are lax comonoidal functors but \( G \) is lax monoidal.
Fortunately, the situation is not so bad.

**Definition 4.7.** In the situation of the diagram
\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{G} & \mathcal{N} \\
R & \downarrow & S \\
\mathcal{M}' & \xrightarrow{H} & \mathcal{N}'
\end{array}
\]
with four monoidal categories, lax monoidal functors \( G \) and \( H \) and lax comonoidal
functors \( R \) and \( S \) a natural transformation \( \varphi : SG \to HR \) is called ambimonoidal
if the diagrams
\[
\begin{align}
S \otimes_{\mathcal{N}} (G \times G) & \xrightarrow{S^2(G \times G)} \otimes_{\mathcal{N}} (SG \times SG) \xrightarrow{\otimes_{\mathcal{N}}(\varphi \times \varphi)} \otimes_{\mathcal{N}'}(HR \times HR) \\
SG \otimes_{\mathcal{M}} & \xrightarrow{\varphi \otimes_{\mathcal{M}}} HR \otimes_{\mathcal{M}} \xrightarrow{H^2(R \times R)} H \otimes_{\mathcal{M}'} (R \times R)
\end{align}
\]
are commutative.

Beyond that it is meaningful the motivation for this definition comes from the
following

**Proposition 4.8.** Let \( U \) and \( V \) be monoidal functors with left adjoints, \( F \) and \( G \)
be lax monoidal functors and let \( \kappa : GU \to VF \) be a monoidal isomorphism as in
the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
U & \downarrow \kappa & \downarrow V \\
\mathcal{M} & \xrightarrow{G} & \mathcal{N}
\end{array}
\]
Forgetting the monoidal structures let \( Q(U) = R, Q(V) = S \) and \( Q(F, \kappa, G) = \langle G, \varphi \rangle \). Then \( \varphi : SG \to GR \) is ambimonoidal if \( R \) and \( S \) are considered with lax comonoidal structures by Theorem 2.1.

**Proof.** The defining equation (59) of \( \varphi \) is equivalent by Remark 4.2 to the condition that

\[
[F, \varphi]: \langle B, \beta, S \rangle \circ \langle F, \kappa, G \rangle \to \langle F, \kappa, G \rangle \circ \langle A, \alpha, R \rangle
\]

is a 2-cell in \( L\text{-Func} \). Monoidality of \( \kappa \),

\[
\kappa \otimes_A G U_2 \otimes (U \times U) = VF_2 \cdot V^2(F \times F) \cdot \otimes_{N'}(\kappa \times \kappa)
\]

is equivalent to

\[
[F_2, G_2]: \langle \otimes_B, V_2, \otimes_{N'} \rangle \circ (\langle F, \kappa, G \rangle \times \langle F, \kappa, G \rangle) \to \langle F, \kappa, G \rangle \circ \langle \otimes_A, U_2, \otimes_{M} \rangle
\]

being a 2-cell in \( L\text{-Func} \). Comonoidality of \( \beta = V_\varepsilon V : SV \to V \),

\[
V_2^{-1} \cdot \beta \otimes_B = \otimes_{N'}(\beta \times \beta) \cdot S^2(V \times V) \cdot SV_2^{-1}
\]

is equivalent to

\[
[\otimes_B, S^2] : \langle B, \beta, S \rangle \circ \langle \otimes_B, V_2, \otimes_{N'} \rangle \to \langle \otimes_B, V_2, \otimes_{N'} \rangle \circ (\langle B, \beta, S \rangle \times \langle B, \beta, S \rangle)
\]

being a 2-cell in \( L\text{-Func} \). Similarly, the \([\otimes_A, R^2] \) is a 2-cell precisely because \( \alpha \) is comonoidal. Therefore one can take the following two parallel vertical composites

\[
\begin{array}{c}
\langle B, \beta, S \rangle \circ \langle \otimes_B, V_2, \otimes_{N'} \rangle \circ (\langle F, \kappa, G \rangle \times \langle F, \kappa, G \rangle) \\
\downarrow \langle \otimes_B, S^2 \rangle \circ \langle F_2, G_2 \rangle \\
\langle B, \beta, S \rangle \circ \langle F, \kappa, G \rangle \circ \langle \otimes_A, U_2, \otimes_{M} \rangle \\
\downarrow \langle F, \varphi \rangle \circ \langle \otimes_A, U_2, \otimes_{M} \rangle \\
\langle F, \kappa, G \rangle \circ \langle \otimes_A, U_2, \otimes_{M} \rangle \circ (\langle A, \alpha, R \rangle \times \langle A, \alpha, R \rangle) \\
\downarrow \langle F, \kappa, G \rangle \circ \langle \otimes_A, U_2, \otimes_{M} \rangle \\
\end{array}
\]

and

\[
\begin{array}{c}
\langle B, \beta, S \rangle \circ \langle \otimes_B, V_2, \otimes_{N'} \rangle \circ (\langle F, \kappa, G \rangle \times \langle F, \kappa, G \rangle) \\
\downarrow \langle \otimes_B, S^2 \rangle \circ (\langle F, \kappa, G \rangle \times \langle F, \kappa, G \rangle) \\
\langle \otimes_B, V_2, \otimes_{N'} \rangle \circ (\langle B, \beta, S \rangle \times \langle B, \beta, S \rangle) \circ (\langle F, \kappa, G \rangle \times \langle F, \kappa, G \rangle) \\
\downarrow \langle \otimes_B, V_2, \otimes_{N'} \rangle \circ (\langle F, \varphi \rangle \times \langle F, \varphi \rangle) \\
\langle F, \kappa, G \rangle \circ \langle \otimes_A, U_2, \otimes_{M} \rangle \circ (\langle A, \alpha, R \rangle \times \langle A, \alpha, R \rangle) \\
\downarrow \langle F_2, G_2 \rangle \circ (\langle A, \alpha, R \rangle \times \langle A, \alpha, R \rangle) \\
\end{array}
\]

Computing the dom of (70) and (71) we obtain the same result, \( F_2 \). By universality of the left action \( \alpha \times \alpha \) their codomains should also be the same. Computing their
we obtain
\[ GR^2 \bullet \varphi \otimes_M S \]
\[ G_2(R \times R) \otimes_N (\varphi \times \varphi) \bullet S^2(G \times G) \]
respectively. They are precisely the LHS and RHS of the ambimonoidality condition (69).

In order to prove the other ambimonoidality condition (80)
\[ GR^0 \bullet \varphi_i M \bullet S \]
we note the following facts. The natural transformations
\[ (81) \]
where 1 is the one element category, are 1-cells in \( L-\text{Func} \) because \( U_0 \) and \( V_0 \) are invertible. Counitality of the comonoidal natural transformation \( \alpha \),
\[ (82) \]
is equivalent to the statement that
\[ (83) \]
is a 2-cell in \( L-\text{Func} \). Similarly, counitality of \( \beta \) is the condition for \( [i_B, S^0] \) to be a 2-cell. Unitality of the monoidal natural transformation \( \kappa \),
\[ (84) \]
in turn is the condition for
\[ (85) \]
to be a 2-cell in \( L-\text{Func} \). Thus one may form the vertical composites
\[ (86) \]
and
\[ (87) \]
which evaluate to be

\[(88) \ [F_0, GR^0 \bullet \varphi_i \mathcal{M} \bullet SG_0] \]
\[(89) \ [F_0, G_0 \bullet S^0] \]

respectively. Since their \texttt{dom} are equal, we can conclude by universality that their \texttt{cod} are equal as well, which in turn are the LHS and RHS of (80). The universal action we use here is the trivial action \((1, 1, 1)\) on the identity functor of the category 1. The cell \(U_0\), using its invertibility, can be absorbed into \(\kappa \circ U_0\) to form the \(\kappa'\) of the universality condition of Definition 4.1. Universality of \((1, 1, 1)\) in turn follows directly from (49) noticing that after inserting a 1-cell for \(\alpha\) and a 0-cell for \(U\) equation (49) immediately gives a unique solution for \(\psi\).

\[\blacksquare\]

Motivated by the above Proposition we can now fill in the missing items in diagram (67).

Definition 4.9. Let \(L\)-\texttt{MFunc} be the 2-category with

- \texttt{objects} the monoidal functors \(U: \mathcal{A} \to \mathcal{M}\) with the underlying functor having a left adjoint,
- \texttt{1-cells} \((\mathcal{A}, U, \mathcal{M})\) \(\to\) \((\mathcal{B}, V, \mathcal{N})\) the triples \((F, \kappa, G)\) where \(F: \mathcal{A} \to \mathcal{B}\), \(G: \mathcal{M} \to \mathcal{N}\) are lax monoidal functors and \(\kappa: GU \to VF\) is a monoidal natural isomorphism and
- \texttt{2-cells} \((F, \kappa, G)\) \(\to\) \((F', \kappa', G')\) the pairs \([\vartheta, \nu]\) where \(\vartheta: F \to F'\) and \(\nu: G \to G'\) are monoidal natural transformations satisfying the constraint \(V\vartheta \bullet \kappa = \kappa' \bullet \nu U\).

All compositions are defined as in \(L\)-\texttt{Func} via forgetting \(L\)-\texttt{MFunc} \(\to\) \(L\)-\texttt{Func}.

This 2-category describes the precise framework in which we are able to associate a "quantum groupoid" to a forgetful functor. The "quantum groupoids" in this generality are the bimonads.

Definition 4.10. Let \(Bmd\) be the 2-category with

- \texttt{objects} the bimonads \(\langle M, T \rangle \equiv \langle M, T, T^2, T^0, \mu, \eta \rangle\) of Definition 3.2,
- \texttt{1-cells} \((\mathcal{A}, R)\) \(\to\) \((\mathcal{N}, S)\) the monad morphisms \((G, \varphi)\) in which the functor \(G: \mathcal{M} \to \mathcal{N}\) is lax monoidal and \(\varphi: SG \to GR\) is ambimonoidal,
- \texttt{2-cells} \((G, \varphi)\) \(\to\) \((G', \varphi')\) the monad transformations \(\nu: G \to G'\) which are monoidal.

All compositions are defined by the forgetting 2-functor \(Bmd \to Mnd\).

In order for \(Bmd\) to be well defined as a 2-category we are still indebted to show that ambimonoidality is preserved by horizontal composition.

Lemma 4.11. Consider two horizontally composable monad morphisms

\[(\mathcal{A}, R) \xrightarrow{(G, \varphi)} (\mathcal{B}, S) \xrightarrow{(H, \chi)} (\mathcal{C}, T)\]

in which the categories are monoidal, the functors are lax monoidal and the natural transformations \(\varphi\) and \(\chi\) are ambimonoidal. Then in the composite

\[(H, \chi) \circ (G, \varphi) = (HG, H\varphi \bullet \chi G)\]

the functor is lax monoidal and the natural transformation is ambimonoidal.
Proof. Lax monoidality of $HG$ is obvious. We have to show that $\xi := H\varphi \cdot \chi G$ satisfies the two diagrams (69) and (70). The proof is this. The first ambimonoidality diagram follows from commutativity of

\[
\begin{array}{ccc}
T \otimes (HG \times HG) & \xrightarrow{TH_2(G \times G)} & TH \otimes (G \times G) & \xrightarrow{THG_2} & THG \otimes \\
T^2(HG \times HG) & \downarrow & \chi \otimes (G \times G) & \downarrow & \chi G \otimes \\
\otimes(THG \times THG) & HS \otimes (G \times G) & \xrightarrow{HSG_2} & HSG \otimes & \\
\otimes(HSG \times HSG) & H_{2}(SG \times SG) & \xrightarrow{HGR_2} & H \otimes (SG \times SG) & HGR \otimes \\
\otimes(HGR \times HGR) & H_{2}(GR \times GR) & \xrightarrow{H_{2}(R \times R)} & HG \otimes (R \times R)
\end{array}
\]

where $\otimes$ in the 1st, 2nd and 3rd column denotes the monoidal product of $C, B$ and $A$, respectively. The second ambimonoidality diagram follows from

\[
\begin{array}{ccc}
Ti_C & \xrightarrow{TH_0} & TH_i_B & \xrightarrow{THG_0} & TH Gi_A \\
\chi i_B & \xrightarrow{i_B} & HSi_B & \xrightarrow{HSG_0} & HSGi_A \\
\chi Gi_A & \xrightarrow{HGi_A} & HGRi_A & \xrightarrow{HGT^0} & \\
\end{array}
\]

Let us summarize what we have obtained so far:

**Theorem 4.12.** Given a 2-functor $Q: L\text{-Func} \to \text{Mnd}$ as in Subsection 4.2 there is a unique 2-functor $Q: L\text{-MFunc} \to \text{Bmd}$ such that (67) is commutative.

Proof. As for the object map of $Q$ we must take the monoid $Q(U)$ and endow it with the comonoidal structure that Corollary 3.1 provides. The arrow map of $Q$ is again uniquely determined by that of $Q$ and it yields bimonad morphisms by Proposition 4.8. For the unique 2-cell map of $Q$ there is nothing to prove.

**4.4. The monoidal Eilenberg-Moore construction as a 2-functor.** Functoriality of the Eilenberg-Moore construction can be formalized as having a 2-functor the object map of which associates forgetful functors to bimonds.

Let $EM: \text{Bmd} \to L\text{-MFunc}$ be the 2-functor defined as follows.
The object map: For a bimonad \( \langle M, T \rangle \) let \( \text{EM}(M, T) := (M^T, U^T, M) \), the strict monoidal forgetful functor of the category \( M^T \) of \( T \)-algebras, see Proposition 4.3.

The arrow map: For a bimonad morphism \( \langle G, \varphi \rangle \) we define
\[
\text{EM} \left( \langle M, T \rangle \xrightarrow{(G, \varphi)} \langle N, S \rangle \right) = \langle G^\varphi, =, G \rangle
\]
where
\[
M^T \xrightarrow{G^\varphi} \mathcal{N}^S \quad (x, \alpha) \mapsto \langle Gx, G\alpha \circ \varphi_x \rangle
\]
which is indeed a functor from \( T \)-algebras to \( S \)-algebras since \( G\beta \circ \varphi_y \circ SG\tau = G\beta \circ GT\tau \circ \varphi_x = GT \circ G\alpha \circ \varphi_x \). The monoidal structure for \( G^\varphi \) is the one given in Lemma 4.13 below.

The 2-cell map: For a transformation \( \nu: \langle G, \varphi \rangle \to \langle G', \varphi' \rangle \) of monad morphisms \( \langle M, T \rangle \to \langle N, S \rangle \) we define
\[
\text{EM}(\nu) := [\hat{\nu}, \nu]: \langle G^\varphi, =, G \rangle \to \langle G'^\varphi, =, G' \rangle
\]
where \( \hat{\nu} \) on the \( T \)-algebra \( (x, \alpha) \) is the lift of \( \nu_x \),
\[
\hat{\nu}_{(x, \alpha)} = \left( \langle Gx, G\alpha \circ \varphi_x \rangle \xrightarrow{\nu_x} \langle G'x, G'\alpha \circ \varphi'_x \rangle \right)
\]
which is indeed an \( S \)-algebra morphism because \( G'\alpha \circ \varphi'_x \circ S\nu_x = G'\alpha \circ \nu_{T_x} \circ \varphi_x = \nu_x \circ G\alpha \circ \varphi_x \).

Lemma 4.13.
\[
G^\varphi_x = \left\{ G^\varphi(x, \alpha) \circ \mathcal{N}^S(y, \beta) \xrightarrow{G_{x,y}} G^\varphi((x, \alpha) \circ \mathcal{N}(y, \beta)) \right\}
\]
\[
G^\varphi_0 = \left\{ (i_{\mathcal{N}, S^0}) \xrightarrow{G_0} G^\varphi(i_{\mathcal{M}, T^0}) \right\}
\]
is the unique monoidal structure on \( G^\varphi \) such that \( U^SG^\varphi =GU^T \), as monoidal functors.

Proof. Since \( U^T \) and \( U^S \) are strict monoidal, the only monoidal structure on \( G^\varphi \) is the one with components that are lifted from the components of \( G_2 \), \( G_0 \), which is precisely the above formula. The hexagon and square identities therefore hold automatically if we can show that the components \( G^\varphi_{x,y} \) and \( G^\varphi_0 \) can indeed be lifted to \( S \)-algebra maps.

\( G^\varphi_{x,y} \) lifts to an arrow in \( \mathcal{N}^S \) if
\[
G(\alpha \otimes_M \beta) \circ GT_{x,y} \circ \varphi_{x \otimes_M y} \circ SG_{x,y} = G_{x,y} \circ (G\alpha \otimes_N G\beta) \circ (\varphi_x \otimes_N \varphi_y) \circ SG_{x,y}
\]
which, after using naturality of \( G_2 \), becomes a consequence of the first ambimonoidality axiom for \( \varphi \).

\( G^\varphi_0 \) is an arrow in \( \mathcal{N}^S \) if \( GT^0 \circ \varphi_{i_M} \circ SG_0 = G_0 \circ T^0 \) which is precisely the second ambimonoidality axiom for \( \varphi \). \( \square \)

4.5. The adjunction \( Q \dashv \text{EM} \) and universality. In this subsection we construct pseudo natural transformations \( \xi \) and \( \zeta \) providing the unit and counit of the adjunction \( Q \dashv \text{EM} \), respectively. Then we show how to restrict these 2-functors to obtain an adjunction in the strict sense, which is needed to establish the universal property of the bimonad \( Q(U) \) of a left adjointable monoidal functor.
4.5.1. The counit $\zeta$. The action of $Q$ on $\text{EM}(M, T)$ does not necessarily return the original monad $T$. Instead it gives $(M, U^T \bar{U}^T)$ where $U^T$ is some left adjoint of $U^T$. If $\bar{U}^T$ were equal to the free $T$-algebra functor $F^T$ then we would get the original monad $T$. Of course, all left adjoints are isomorphic and it is easy to see that the isomorphism $\sigma : F^T \sim \bar{U}^T$ leads to a monad morphism $\zeta = U^T \sigma$ sending $T = U^T F^T$ to $U^T \bar{U}^T$. A closer look gives that we actually have a bimonad isomorphism. This is the content of the next Lemma in which $*$ denotes composition of 2-functors.

**Lemma 4.14.** $(M, \zeta) : Q \ast \text{EM}(M, T) \sim (M, T)$ is a bimonad isomorphism.

**Proof.** Because of uniqueness of left adjoints of monoidal functors up to comonoidal natural isomorphisms (Lemma 2.4), the $\sigma$ can be chosen to be comonoidal. But $U^T$ is also (co)monoidal, so the $\zeta$ is, either. Now the identity functor $M$ being comonoidal, the ambimonoidality condition for $\zeta$ is equivalent to its comonoidality. 

Next we investigate the naturality properties of $\zeta$. Let $(G, \varphi)$ be a bimonad morphism $(M, T) \to (N, S)$. Then $Q \ast \text{EM}(G, \varphi) = (G, \varphi')$ where $\varphi'$ is determined from the 1-cell

$$
\begin{array}{ccc}
\mathcal{M}^T & \xrightarrow{G^\varphi} & N^S \\
\mathcal{M} & \xrightarrow{G} & N \\
\end{array}
$$

i.e., $\varphi : S' G \to G T'$ is the unique solution of

$$
\beta' G^\varphi = G \alpha' \cdot \varphi' U^T
$$

where $T' = U^T \bar{U}^T$, $S' = U^S \bar{U}^S$ and $\alpha' = U^T \varepsilon_U$, $\beta' = U^S \varepsilon_U$ are the universal actions associated to $U^T$ and $U^S$, respectively, in the definition of $Q$. Denoting by $\alpha = U^T \varepsilon_T$ and $\beta = U^S \varepsilon_S$, respectively, the universal actions associated to them by the Eilenberg-Moore construction, we have

$$
\alpha' = \alpha \cdot \zeta_T^{-1} U^T, \quad \beta' = \beta \cdot \zeta_S^{-1} U^S
$$

so we have to solve

$$
\beta G^\varphi \cdot \zeta_S^{-1} U^S G^\varphi = G \alpha \cdot G \zeta_T^{-1} U^T \cdot \varphi' U^T.
$$

Taking into account the formulae below in which $\langle x, \alpha \rangle$ stands for any $T$-algebra

- $\varepsilon_{G^\varphi}(x, \alpha) = \varepsilon_{G^\varphi(G x, \alpha \circ \varphi_x)} = (SGx, \mu_{Gx}) \xrightarrow{G \alpha \circ \varphi_x} (G x, G \alpha \circ \varphi_x)$
- $\beta_{G^\varphi}(x, \alpha) = SGx \xrightarrow{G \alpha \circ \varphi_x} Gx$
- $G \alpha(x, \alpha) = GT x \xrightarrow{G \alpha} Gx$

the solution is

$$
\varphi' = G \zeta_T \cdot \varphi \cdot \zeta_S^{-1} G
$$

which is equivalent to that

$$
\begin{array}{ccc}
(M, T') & \xrightarrow{(M, \zeta_T)} & (M, T) \\
\langle G, \varphi' \rangle & = & \langle G, \varphi \rangle \\
(N, S') & \xrightarrow{(N, \zeta_S)} & (N, S) \\
\end{array}
$$

(92)
Lemma 4.15. \( \zeta : Q \ast EM \rightarrow Bmd \) is a 2-natural isomorphism, i.e., for any 2-cell \( \nu : (G, \varphi) \rightarrow (G', \varphi') : (M, T) \rightarrow (N, S) \)

\[ \nu \circ \zeta_T = \zeta_S \circ Q \ast EM(\nu). \]

Proof. In the above preparations we have already shown this relation for 1-cells \( \nu \).

If \( \nu \) is a 2-cell then it suffices to check the equation merely as an equality of natural transformations, i.e., as 2-cells in \( \text{Cat} \). Since the functor component of \( \zeta \) is always the identity functor, this equality is the trivial \( \nu = \nu \).

\[ \square \]

4.5.2. The unit \( \xi \). The Eilenberg-Moore comparison functors \( K_U : A \rightarrow M^T \) provide 1-cells

\[ \xi_U := (K_U, =, M) \] is a 2-cell

\[ V \rightarrow EM \ast Q(U) \]

for all objects \( (A, U, M) \) in \( \text{L-MFunc} \). On 1-cells the 2-functor \( EM \ast Q \) acts as

\[ (A, U, M) \rightarrow (M^T, U^T, M) \]

\[ (F, \kappa, G) \rightarrow (G^\varphi, =, G) \]

\[ (B, V, N) \rightarrow (N^S, U^S, N) \]

where \( \varphi : SG \rightarrow GT \) is the unique solution of

\[ \beta F \cdot S \kappa = \kappa \cdot G \alpha \cdot \varphi U \]

where \( T = UU \), \( S = VV \), \( \alpha = U\varepsilon_U \) and \( \beta = V\varepsilon_V \).

Lemma 4.16. The monoidal natural isomorphism \( \kappa : GU \rightarrow VF \) lifts to a monoidal natural isomorphism \( \hat{\kappa} : G^\varphi K_U \rightarrow K_V F \). The pair \( \Xi^\kappa := [\hat{\kappa}, \text{id}] \) is an invertible 2-cell

\[ U \rightarrow EM \ast Q(U) \]

\[ V \rightarrow EM \ast Q(V) \]

in \( \text{L-MFunc} \).

Proof. Computing the effect of the functors on an object \( a \in A \)

\[ K_V F : a \mapsto (VF a, SVF a \beta F a V Fa) \]

\[ G^\varphi K_U : a \mapsto (GU a, SGU a \varphi U a GRU a G a \rightarrow GU a) \]

we see that the lifting property of \( \kappa_a : GU a \rightarrow VF a \) is just the defining equation of \( \varphi \) above. So \( \hat{\kappa} \) has the proper components and it is natural by virtue of the very simple form of the functors \( K_U, K_V \) on arrows. Moreover, \( \hat{\kappa} \) is monoidal since the monoidal structures of \( G^\varphi, K_U, K_V \) are just the lifts of the corresponding structures in \( G, U, V \), respectively. The constraint for \( \Xi^\kappa = [\hat{\kappa}, \text{id}] \) to be a 2-cell is just the lifting property \( U^S \hat{\kappa} = \kappa \).

\[ \square \]
Lemma 4.17. \( \xi : \text{L-MFunc} \to \text{EM} \star \text{Q} \) is a pseudo natural transformation with
\[
\Xi_\kappa : \text{EM} \star \text{Q}(\kappa) \circ \xi_U \overset{\sim}{\to} \xi_V \circ \kappa, \quad \kappa : U \to V.
\]
That is to say, for any 2-cell \( \theta \) of Lemma 4.17.
\[
(\Xi_\kappa)(\xi_U) = (\xi_V \circ \theta) \cdot \Xi_\kappa.
\]
Proof. For \( \theta = [\vartheta, \nu] \) one has \( \text{EM} \star \text{Q}(\theta) = [\vartheta, \nu] \) so one has to check \( \text{dom} \) and \( \text{cod} \) of
\[
[\kappa, =] \circ ([\vartheta, \nu] \circ (K_U, =, \mathcal{M})) = ((K_V, =, \mathcal{N}) \circ [\vartheta, \nu]) \circ [\kappa, =]
\]
which are \( \kappa' \circ \nu K_U = K_V \vartheta \circ \kappa \) and the identity \( \nu = \nu \), respectively. The former is the lift of \( \kappa' \circ \nu U = V \vartheta \circ \kappa \) which is but the the defining equation for the pair \( [\vartheta, \nu] \) to be a 2-cell \( \kappa \to \kappa' \).

4.5.3. The pseudo adjunction \text{Q} \dashv \text{EM}. We want to prove that the 2-functor \text{Q} is the pseudo left adjoint of \text{EM} in the following sense.

Theorem 4.18. There exist pseudo natural transformations
\[
\zeta : \text{Q} \star \text{EM} \to \text{Bmd}, \quad \xi : \text{L-MFunc} \to \text{EM} \star \text{Q}
\]
such that
\[
(\zeta \star \text{Q}) \circ (\text{Q} \star \xi) = \text{Q} \quad (\text{Q} \star \text{EM} \star \zeta) \circ (\zeta \star \text{EM} \star \xi) = \text{EM}
\]
where \( \star \) denotes composition of 2-functors and higher cells, i.e., 2-composition, and \( \circ \) denotes componentwise horizontal composition of natural transformations, i.e., 1-composition in the 3-category \text{2-Cat}.

Proof. We use the pseudo natural transformations \( \zeta \) and \( \xi \) constructed in the Lemmas 4.14, 4.15, 4.16 and 4.17. The effect of the 2-functor \text{Q} on the 1-cell \( \xi_U \) is
\[
\text{Q}(\xi_U) = \text{Q} \langle K_U, =, \mathcal{M} \rangle = \left( (\mathcal{M}, T) \right) \xrightarrow{\langle \mathcal{M}, \zeta \rangle} \left( \mathcal{M}, T' \right)
\]
where \( T = U \bar{U}, T' = U^\ast U^T \) and \( \zeta' : T' \mathcal{M} \to \mathcal{M} T \) is the unique solution of
\[
\alpha' K_U = \alpha \circ \zeta' U
\]
where \( \alpha = U \varepsilon_U \), and \( \alpha' = U^T \varepsilon_{U^T} = U^T \varepsilon^T \circ \zeta^{-1}_{T} U^T \). Since
\[
\alpha' K_U a = \alpha' (U a, \varepsilon_{U a}) = U \varepsilon_U a \circ \zeta^{-1}_{T} U a
\]
\[
\alpha' K_U = \alpha \circ \zeta^{-1}_{T} U,
\]
the solution is \( \zeta' = \zeta^{-1}_{T} \). Since \( T \) here means \( \text{Q}(U) \), we obtain
\[
\left( \left( \langle \text{Q}(U), \xi_U \rangle \xrightarrow{\text{Q}(\xi_U)} \langle \text{EM} \star \text{Q}(U) \rangle \xrightarrow{\zeta_{\text{Q}(U)}} \text{Q}(U) \right) = 1_{\text{Q}(U)} \right)
\]
which is precisely equation (4.15). In order to prove the other adjunction relation we look at
\[
\xi_{\text{EM}(T)} = \langle K_U, =, \mathcal{M} \rangle : (\mathcal{M} \ast, U^T, \mathcal{M}) \to (\mathcal{M} \ast, U^T, \mathcal{M})
\]
where \( T' = U^T U^T \) as before and
\[
K_{U^T} : \langle x, \alpha \rangle \mapsto \langle U^T \langle x, \alpha \rangle, U^T \varepsilon_{U^T} \langle x, \alpha \rangle \rangle = \langle x, \alpha \circ \zeta^{-1}_{x} \rangle.
\]
Let us compare this with
\[
\begin{align*}
\mathsf{EM} \left( (M, T) \xrightarrow{\zeta} (M, T) \right) &= \mathsf{M}T' \xrightarrow{\mathsf{EM}(\zeta)} \mathsf{M}T \\
\mathsf{M} &= \mathsf{M} \\
\end{align*}
\]
where \( \mathsf{M}^\zeta : (x, \alpha) \mapsto (x, \alpha \circ \zeta_x) \), i.e., \( \mathsf{M}^\zeta = (K_{U^T})^{-1} \). Therefore the 1-cell \( \mathsf{EM}(\zeta) \) is the strict inverse of \( \xi_{\mathsf{EM}(T)} \) for any bimonad \( T \). This proves
\[
\left( \mathsf{EM}(T) \xrightarrow{\mathsf{EM}(\zeta)} \mathsf{EM} \ast \mathsf{Q} \ast \mathsf{EM}(T) \xrightarrow{\mathsf{EM}(\zeta)} \mathsf{EM}(T) \right) = 1_{\mathsf{EM}(T)}
\]
for all bimonad \( T \) which is equation (98).

4.5.4. Universality. The fact that \( \mathsf{Q} \) is a left pseudo adjoint of \( \mathsf{EM} \) has the following local description. For each object \( U \) in \( \mathsf{L-MFunc} \) there exist a bimonad \( T = \mathsf{Q}(U) \) and a 1-cell \( \xi : U \to \mathsf{EM}(T) \) satisfying the following property:
\( \mathbf{P} \): If \( S \) is a bimonad and \( \kappa : U \to \mathsf{EM}(S) \) is a 1-cell then there exists \( a \), up to isomorphism unique, bimonad morphism \( \varphi : T \to S \) such that
\[
\mathsf{EM}(\varphi) \circ \kappa \cong \kappa.
\]
As a matter of fact, let \( \varphi := \zeta_S \circ \mathsf{Q}(\kappa) \). Then
\[
\mathsf{EM}(\varphi) \circ \kappa = \mathsf{EM}(\zeta_S) \circ \mathsf{EM} \ast \mathsf{Q}(\kappa) \circ \kappa \xrightarrow{\sim} \mathsf{EM}(\zeta_S) \circ \mathsf{EM}(\mathsf{Q}(\kappa)) \circ \kappa \cong \kappa.
\]
If \( \varphi' : T \to S \) is another monad morphism for which there exists a \( \varphi' : \mathsf{EM}(\varphi') \circ \kappa \xrightarrow{\sim} \kappa \) then
\[
\varphi' = \varphi' \circ \mathsf{Q}(U) \circ \mathsf{Q}(\kappa) \xrightarrow{\sim} \zeta_S \circ \mathsf{Q} \circ \mathsf{EM}(\varphi') \circ \mathsf{Q}(\kappa) \xrightarrow{\sim} \zeta_S \circ \mathsf{Q}(\kappa) = \varphi
\]
Now assume that \( \xi' : \mathsf{EM}(T') \) also satisfies property \( \mathbf{P} \). Then we have 1-cells and invertible 2-cells
\[
\begin{align*}
\varphi : T &\to T' , \quad \phi : \mathsf{EM}(\varphi) \circ \kappa \xrightarrow{\sim} \xi' \\
\varphi' : T' &\to T , \quad \phi' : \mathsf{EM}(\varphi') \circ \kappa \xrightarrow{\sim} \xi
\end{align*}
\]
and it is easy to see that there are invertible 2-cells \( \varphi \circ \varphi' \xrightarrow{\sim} T' \) and \( \varphi' \circ \varphi \xrightarrow{\sim} T \), i.e. \( T \) and \( T' \) are equivalent. This result, however weak, is in complete agreement with the fact that \( \mathsf{Q} \), as a left pseudo adjoint of \( \mathsf{EM} \), is determined only up to pseudo natural isomorphisms.

On the other hand, the way we defined \( \mathsf{Q} \) allowed only the freedom to choose different adjunction data for the functors \( U \), which amounts to \( \mathsf{Q} \) being unique up to 2-natural isomorphisms. Also the universality formulated in Theorem 3.4 suggests that we should find a 2-categorical 2-adjunction generalizing it.

Notice that the image of \( \mathsf{EM} \) lies in a special sub-2-category of \( \mathsf{L-MFunc} \) in which the 1-cells contain identity natural isomorphisms \( \kappa \). Let us call a 1-cell \( \langle F, \kappa, G \rangle \) strict if \( \kappa = 1_{GU} = 1_{VF} \). The 2-category of all objects of \( \mathsf{L-MFunc} \) with only strict 1-cells between them and with all 2-cells between strict 1-cells will be denoted by \( \mathsf{st-L-MFunc} \).
Remember that the counit \( \zeta: Q \ast \mathcal{EM} \to \mathcal{Bmd} \) is a 2-natural transformation. The unit \( \xi: L \text{-}\mathcal{MFuc} \to \mathcal{EM} \ast Q \) is only pseudo natural but the 2-cell \( \Xi_n \) is such that it is the identity for strict 1-cells. Therefore the restriction of \( \xi \) to \( \text{st}-L \text{-}\mathcal{MFuc} \) is also 2-natural. Denoting by \( \mathcal{EM}_M \) and \( Q^{st} \) the corresponding restricted 2-functors we obtain an ordinary 2-adjunction

\[
Q^{st} \dashv \mathcal{EM}_M,
\]

i.e., one in which the unit and counit are 2-natural transformations. Such left adjoints \( Q^{st} \) are already unique up to 2-natural isomorphisms. This is reflected by the following property of the monad \( Q(U) \) of a left adjointable monoidal functor.

**Theorem 4.19.** For each object \( U \) of \( L \text{-}\mathcal{MFuc} \) there exists a bimonad \( T \) and a strict 1-cell \( \xi: U \to \mathcal{EM}(T) \) with the following property:

\( U; \) If \( S \) is a bimonad and \( \iota: U \to \mathcal{EM}(S) \) is a strict 1-cell then there exists a unique monad morphism \( \varphi: T \to S \) such that \( \mathcal{EM}(\varphi) \circ \xi = \iota \).

If another bimonad \( T' \) and another strict 1-cell \( \xi': U \to \mathcal{EM}(T') \) has property \( U \) then there is a bimonad isomorphism \( \psi: T \xrightarrow{\sim} T' \) such that \( \mathcal{EM}(\psi) \circ \xi = \xi' \).

5. Bialgebroids

### 5.1. From bialgebroids to bimonads

Let \( k \) be a commutative ring, \( R \) a (possibly non-commutative) \( k \)-algebra. A Takeuchi \( \times_R \) bialgebra or a left bialgebroid over \( R \) in the sense of \( \Box \) consists of

- a \( k \)-algebra \( A \) with a \( k \)-algebra map \( s \otimes_k t: R \otimes_k R^{op} \to A \) making \( A \) into an \( R \text{-}R \) bimodule via \( r \cdot a \cdot r' := s(r)(r')a \) and
- a comonoid structure \( \langle A, \Delta, \varepsilon \rangle \) on \( A \) in \( R\mathcal{M}_R \)

such that

- **BGD 1.a:** the image of the comultiplication \( \Delta(A) \subset A \otimes_R A \) belongs to the subbimodule

\[
A \times_R A = \{ X \in A \otimes_R A \mid X(1 \otimes s(r)) = X(t(r) \otimes 1), \forall r \in R \}
\]

which has the obvious algebra structure therefore it is meaningful to require that

- **BGD 1.b:** \( \Delta: A \to A \times_R A \) be a \( k \)-algebra map, moreover

- **BGD 2.a:** the counit \( \varepsilon \) preserves the unit, \( \varepsilon(1_A) = 1_R \)

- **BGD 2.b:** and satisfies

\[
\varepsilon(at(\varepsilon(b))) = \varepsilon(ab) = \varepsilon(as(\varepsilon(b)))
\]

for all \( a, b \in A \).

Right bialgebroids are defined analogously but using right multiplications with \( s(r), t(r) \) in the definition of the \( R \text{-}R \) bimodule structure of \( A \), so the meaning of \( A \times_R A \) also changes. What is important that in order for the category \( A\mathcal{M} \) of left \( A \) modules to have a monoidal structure one needs a left bialgebroid \( A \) while a right bialgebroid makes \( \mathcal{M}_A \) to be monoidal.

Every left \( A \) module \( \_V \) inherits an \( R \text{-}R \) bimodule structure via the algebra map \( s \otimes_k t \), i.e., if we denote the action of \( a \in A \) on an element \( v \in V \) by \( a \triangleright v \) then \( r \cdot v \cdot r' := s(r)t(r') \triangleright v, r, r' \in R \). This defines the forgetful functor \( U: A\mathcal{M} \to R\mathcal{M}_R.\)
The comultiplication \( \Delta : a \mapsto a_{(1)} \otimes a_{(2)} \) allows to define a monoidal product on \( _A\mathcal{M} \) such that \( U \) becomes strictly monoidal. The monoidal product \( X \otimes Y \) of the \( A \)-modules \( X \) and \( Y \) is the \( R \)-\( R \)-bimodule \( X \otimes_R Y \) equipped with \( A \)-action \( a \triangleright (x \otimes y) = (a_{(1)} \triangleright x) \otimes (a_{(2)} \triangleright y) \) which is well defined due to axiom (BGD 1.a) above.

In the sequel we shall identify \( R \)-\( R \)-bimodules \( X \) with left \( R^e \)-modules via \( (r \otimes r') \cdot a := r \cdot a \cdot r' \), where \( R^e := R \otimes R^{op} \). The left regular \( A \)-module \( A = _A A \) is not only a left \( R^e \)-module but a right \( R^e \)-module, as well. This allows to define a functor \( T := A \otimes_R - : R\mathcal{M}_R \rightarrow R\mathcal{M}_R \).

**Theorem 5.1.** Let \( A \) be a left bialgebroid over \( R \). Then the endofunctor \( T = A \otimes_{R^e} - \) defines a bimonad on \( R\mathcal{M}_R \) with structure maps

\[
\begin{align*}
\mu_X &: A \otimes_{R^e} (A \otimes_{R^e} X) \rightarrow A \otimes_{R^e} X, \quad a \otimes (b \otimes x) \mapsto ab \otimes x \\
\eta_X &: X \rightarrow A \otimes_{R^e} X, \quad x \mapsto 1_A \otimes x \\
\gamma_{X,Y} &: A \otimes_{R^e} (X \otimes_R Y) \rightarrow (A \otimes_{R^e} X) \otimes_R (A \otimes_{R^e} Y), \quad a \otimes (x \otimes y) \mapsto (a_{(1)} \otimes x) \otimes (a_{(2)} \otimes y) \\
\pi &: A \otimes_{R^e} R \rightarrow R, \quad a \otimes r \mapsto \varepsilon(as(r)).
\end{align*}
\]

**Proof.** Since the bimodule structure of \( A \) comes from \( R^e \) being a subalgebra in \( A \), the monoid structure \( A \otimes A \rightarrow A \) in \( \mathcal{M}_k \) determines, via the coequalizer \( A \otimes_k A \rightarrow A \otimes_{R^e} A \), a monoid structure \( A \otimes_{R^e} A \rightarrow A \) in \( R^e \)-\( R^e \). This latter monoid structure makes \( T \) into a monad on \( R^e \mathcal{M} \) with structure maps given in elementwise notation in (100), (101). Thus \( \mu \) and \( \eta \) satisfies the bimonad axioms (BMD 5) and (BMD 6), i.e., \( \langle T, \mu, \eta \rangle \) is a monad.

The comultiplication \( \Delta : A \rightarrow A \times_R A \) defines the comonoidal natural transformation \( \gamma \) by formula (102). It is well defined due to axiom (BGD 1.a) and it satisfies the hexagon of (BMD 1) due to coassociativity of \( \Delta \). The other component \( \pi \) of the comonoidal structure of \( T \) given in (103) is well defined due to that axiom (BGD 2.b) implies \( \varepsilon(\alpha s(r)) = \varepsilon(\alpha t(r)) \) for \( a \in A, r \in R \). It is a counit for \( \gamma \) in the sense of the bimonad axiom (BMD 2) because \( \varepsilon \) is the counit for \( \Delta \). Thus \( \langle T, \gamma, \pi \rangle \) is a comonoidal functor.

The compatibility condition of \( \mu \) with \( \gamma \) follows from the bialgebroid axiom (BGD 1.b) while the compatibility of \( \mu \) with \( \pi \) follows from the counit axiom (BGD 2.b). This proves (BMD 3). Compatibility of \( \eta \) with \( \gamma \) is unitality of \( \Delta \) hence follows from (BGD 1.b) while compatibility of \( \eta \) with \( \pi \) is the section property \( \varepsilon \circ s = id_R \). This proves (BMD 4). \( \square \)

**Remark 5.2.** In this Section we speak about bialgebroids in the category \( \mathcal{M}_k \) of \( k \)-modules including as special cases the category \( \mathcal{A}b \) of Abelian groups or the category \( \text{Vec} K \) of vector spaces over a field \( K \). However, the category \( \mathcal{M}_k \) can be replaced with any symmetric monoidal closed category \( \langle \mathcal{M}, \otimes, i \rangle \) which has coequalizers. The symmetric monoidal structure is required to be able to speak about the monoids \( R, R^{op} \) and \( R^e \) while the coequalizers are needed to define tensor products over such monoids. The tensor product \( \otimes_R \) becomes a monoidal product on \( R\mathcal{M}_R \) if \( \otimes \) preserves coequalizers. This latter property is guarantied if \( \mathcal{M} \) is closed. The definition of bialgebroids as well as the above theorem holds also in this more general setting and should cover non-additive examples.
Example 5.3. The trivial left bialgebroid over $R$ is the bialgebroid $E = R \otimes_k R^{op}$ with comultiplication and counit given respectively by

\[
\Delta_E : E \to E \otimes_R E, \quad r \otimes_k r' \mapsto (r \otimes_k 1_R) \otimes_R (1_R \otimes_k r')
\]

and with source and target maps $s_E(r) = r \otimes_k 1_R$, $t_E(r) = 1_R \otimes_k r$.

If $T$ is the bimonad on $R \mathcal{M}_R$ associated to a bialgebroid $A$ over $R$ then the category of $T$-algebras $R \mathcal{M}^T_R$ is monoidally isomorphic to the category of left $A$-modules $A \mathcal{M}$. 

5.2. Characterizing bimonads of bialgebroids. The bimonad constructed in Theorem 5.1 is special among the bimonads in that $T$ is $k$-linear and has a right adjoint. As a matter of fact, the functor $\text{Hom}(R, A, -)$ maps an $R$-$R$-bimodule $X$ into the $k$-module $\text{Hom}(A, X)$ of $k$-linear maps $f : A \to X$ satisfying $f(s(r)t(r')a) = r \cdot f(a) \cdot r'$ and equipped with $R$-$R$-bimodule structure $r \cdot f \cdot r' = f(-s(r)t(r'))$. This functor is the right adjoint of $T = A \otimes_R -$ with counit and unit

\[
A \otimes_R \text{Hom}(A, X) \to X, \quad a \otimes f \mapsto f(a)
\]

\[
X \to \text{Hom}(A, A \otimes_R X), \quad x \mapsto \{a \mapsto a \otimes x\}.
\]

From now on we never mention $k$-linearity although every functor on $k$-linear categories will be assumed $k$-linear. This means for example that bimonads on $k$-linear categories will be assumed to have $k$-linear underlying functors. Let $\mathbf{Bmd}_k$ be the 2-category of such bimonads.

In this Subsection we will show that the single property of having a right adjoint already characterizes the bialgebroids within the bimonads of $\mathbf{Bmd}_k$. The summary is this.

Theorem 5.4. Let $R$ be a monoid in $k \mathcal{M}$ and $\langle T, \gamma, \pi, \mu, \eta \rangle$ be a bimonad on $R \mathcal{M}_R$. Then $T$ is isomorphic to the bimonad of a bialgebroid over $R$ if and only if it has a right adjoint.

Only sufficiency requires a proof. Nevertheless we will give a detailed proof divided into a series of Lemmas that contain both necessary and sufficient conditions.

Lemma 5.5. Let $E$ be a $k$-algebra and $T : E \mathcal{M} \to E \mathcal{M}$ be an endofunctor on the category of left $E$-modules. Then there exists an $E$-$E$-bimodule $M$ such that $T \cong M \otimes_E -$ if and only if $T$ has a right adjoint.

Proof. Necessity: $M \otimes_E -$ has a right adjoint $\text{hom}(M, -) := \text{Hom}_E( M, - )$ inheriting its left $E$ module structure from the right $E$-action on $M$. The adjunction relation

\[
\text{Hom}_E(X, \text{hom}(M, Y)) \cong \text{Hom}_E(M \otimes_E X, Y)
\]

\[
f \mapsto \{m \otimes x \mapsto f(x)(m)\}
\]

for left $E$-modules $X$ and $Y$ is a standard hom-tensor relation.
Sufficiency: Let $H$ be a right adjoint to $T$. Then, $M := T(EE)$ being an $E$-$E$-bimodule via $E^{op} \cong \text{End}(EE)$ as well as $E$, we have

\[
\begin{align*}
\text{Hom}_E(TX, Y) &\cong \text{Hom}_E(X, HY) \\
&\cong \text{Hom}_E(X, \text{hom}(E, HY)) \\
&\cong \text{Hom}_E(X, \text{hom}(TE, Y)) \\
&\cong \text{Hom}_E(TE \otimes E X, Y)
\end{align*}
\]

implying $TX \cong TE \otimes E X$. \hfill \Box

**Lemma 5.6.** Let $E$ be as before and $(T, \mu, \eta)$ be a monad on $EM$. Then there exists a monoid $A$ in $EM$ and a monad isomorphism $A \otimes E \sim\sim T$ if and only if $T$ has a right adjoint.

**Proof.** Necessity: This is the same as the necessity part of the previous Lemma.

Sufficiency: By the previous Lemma there is a bimodule $A$ and an isomorphism $\nu: T \sim\sim T' := A \otimes E$ of functors. The natural transformations $\mu$ and $\eta$ can be passed to $T'$ via $\nu$ to get a monoid $(T', \mu', \nu')$. Since the powers of $T'$ have hom-functors as right adjoints and the natural transformations between them – by the Yoneda Lemma – arise from bimodule maps between the tensor powers of $A$, it is easy to see that the monad structure on $T'$ is that of arising from a monoid structure on $A$. \hfill \Box

The next Lemma provides an important class of examples of lax monoidal functors.

**Lemma 5.7.** Let $(\mathcal{M}, \otimes, i)$ be a monoidal category with coequalizers and assume that $x \otimes -$ and $- \otimes x$ preserve coequalizers for all objects $x$ of $\mathcal{M}$. Then for any monoid $(R, \nu, \iota)$ in $\mathcal{M}$ the $R$-$R$-bimodules $X = \langle x, \lambda_X: R \otimes x \to x, \rho_X: x \otimes R \to x \rangle$ in $\mathcal{M}$ form a monoidal category $R\mathcal{M}_R$ with monoidal product $\otimes_R$ arising from a choice of coequalizers $x \otimes (R \otimes y) \rightrightarrows x \otimes y \rightrightarrows x \otimes_R y$ for each pair of objects. The forgetful functor $\Phi: R\mathcal{M}_R \to \mathcal{M}$ mapping $\langle x, \lambda_X, \rho_X \rangle$ to $x$ is lax monoidal with

\[
\begin{align*}
\Phi_{X,Y}: \Phi X \otimes \Phi Y &\to \Phi(X \otimes_R Y) \\
\Phi_0: i &\to \Phi R
\end{align*}
\]

being the chosen coequalizer and (110)

Combining the results of the last two Lemma with the fact that a lax monoidal functor $\Phi: \mathcal{B} \to \mathcal{M}$ maps monoids $(A, \mu, \eta)$ in $\mathcal{B}$ into monoids

\[
\langle \Phi A, \Phi(\mu) \circ \Phi A, \Phi(\eta) \circ \Phi_0 \rangle
\]

in $\mathcal{M}$, we immediately obtain

**Corollary 5.8.** Let $E$ be a $k$-algebra and $(T, \mu, \eta)$ be a monad on $EM$. Then there exists a $k$-algebra extension $E \to A$ and a monad isomorphism $A \otimes E \sim\sim T$ if and only if $T$ has a right adjoint.

Now we investigate the coalgebra properties of the $k$-algebra $A$. 

Lemma 5.9. Let \( R \) be a \( k \)-algebra, \( E = R \otimes_k R^{op} \) be its enveloping algebra and let \( \langle T, T^2, T^0 \rangle \) be a lax comonoidal endofunctor on the monoidal category \( _E \mathcal{M} \). Then there exists

- an \( E \)-\( E \)-bimodule \( A \),
- a comonoid \( \langle A, \Delta, \varepsilon \rangle \) in \( _E \mathcal{M} \) in which the \( R \)-\( R \)-bimodule structure of \( A \) comes from \( E A \),
- and a comonoidal natural isomorphism \( A \otimes_E \rightarrow \sim T \)

if and only if \( T \) has a right adjoint.

Proof. Necessity: This holds by Lemma 5.3 even without comonoid structure.

Sufficiency: If \( T \) has a right adjoint then Lemma 5.5 ensures the existence of \( A \otimes_R \) and a natural isomorphism \( A \otimes_E \rightarrow \sim T \). Using this isomorphism we can put a comonoidal structure on \( A \otimes_E \) making the isomorphism into a comonoidal natural isomorphism. Let

\[
\gamma_{X,Y} : A \otimes_E (X \otimes_R Y) \rightarrow (A \otimes_E X) \otimes_R (A \otimes_E Y)
\]

\[
\pi : A \otimes_E R \rightarrow R
\]

be the lax comonoidal structure we obtained that way. Since \( E \) is a generator for \( _E \mathcal{M} \) and \( \gamma \) is natural, the components \( \gamma_{X,Y} \) are completely determined by \( \gamma_{E,E} \). As a matter of fact, for \( x \in X \) let \( f_x : E \rightarrow X \) be defined by \( f(r \otimes r') := r \cdot x \cdot r' \). Similarly, let \( g_y \) be the analogue for \( Y \). Then for all \( x \in X, y \in Y \) and \( a \in A \)

\[
\gamma_{X,Y}(a \otimes_E (x \otimes_R y)) = [(A \otimes_E f_x) \otimes_R (A \otimes_E g_y)] \circ \gamma_{E,E}(a \otimes_E (1_E \otimes_R 1_E))
\]

or, introducing

\[
\Delta(a) := (\rho_{a(1)}^{-1} \otimes_R a_{(2)}) \circ \gamma_{E,E}(a \otimes_E (1_E \otimes_R 1_E))
\]

\[
(1) \quad \pi(a(1) \otimes_R 1_E) \cdot a(2) = a = a(1) \cdot \pi(a(2) \otimes_E 1_R), \quad a \in A.
\]

Therefore \( \Delta \) is counital with counit

\[
\varepsilon(a) := \pi(a \otimes_E 1_R)
\]

and it is left for an exercise to show that \( \varepsilon \) implies that \( \Delta \) is coassociative. \( \square \)

Remark 5.10. The comultiplication and counit of \( A \) can be expressed in terms of the\ coring structure of \( E \) of Example 5.3 and in terms of the comonoidal structure \( \langle T, T^2, T^0 \rangle \) of \( T = A \otimes_E \) as follows. The comultiplication \( \Delta \) is the composite

\[
A \xrightarrow{\rho_A^{-1}} A \otimes_E E \xrightarrow{T \Delta_E} A \otimes_E (E \otimes_R E) \xrightarrow{\gamma_{E,E}} (A \otimes_E E) \otimes_R (A \otimes_E E)
\]

\[
A \otimes_R A
\]

while the counit is the composite

\[
A \xrightarrow{\rho_A^{-1}} A \otimes_E E \xrightarrow{T \varepsilon_E} A \otimes_E R \xrightarrow{\pi} R
\]
in the category of $R$-$R$-bimodules.

It is interesting that the Takesaki $\times_R$ product appears naturally already in the bimodule context, i.e., without the algebra structures, as the next Lemma shows.

**Lemma 5.11.** Let $R$ and $E$ be as in Lemma 5.4 and let $\langle A, \Delta, \varepsilon \rangle$ be a comonoid in $R\mathcal{M}_R \equiv \mathcal{E}M$ for some $E$-$E$ bimodule $A$ such that the endofunctor $A \otimes E$ on $\mathcal{E}M$ is lax comonoidal. Then $\Delta$ and $\varepsilon$ satisfy

\begin{align*}
\Delta(a_1) \cdot t_E(r) \otimes_R a_2 &= a_1 \otimes_R a_2 \cdot s_E(r) \\
\varepsilon(a \cdot t_E(r)) &= \varepsilon(a \cdot s_E(r))
\end{align*}

for all $a \in A$, $r \in R$.

**Proof.** The proof uses essentially that $\Delta$ and $\varepsilon$ can be expressed in terms of $\gamma_{E,E}$ and $\pi$, see Remark 5.10. First of all, the right action $- \cdot := \rho_A(- \otimes_R \varepsilon)$ of an element of $E$ on $A$ commutes with the left $E$ action therefore it is an $R$-$R$ bimodule map. For a fixed $a \in A$ choose a finite set of $a_{ij}, b_{ik} \in A$ and $e_j, f_k \in E$ such that

$$\gamma_{E,E}(a \otimes E (1_E \otimes_R 1_E)) = (a_{ij} \otimes E e_j) \otimes_R (b_{ik} \otimes E f_k)$$

with summations understood. Then applying naturality of $\gamma_{E,E}$ twice for any $r \in R$ we can write

\begin{align*}
(a_{ij} \otimes E e_j t_E(r)) \otimes_R (b_{ik} \otimes E f_k) &= \gamma_{E,E}(a \otimes E (t_E(r) \otimes_R 1_E)) \\
\gamma_{E,E}(a \otimes E (1_E \otimes_R s_E(r))) &= (a_{ij} \otimes E e_j) \otimes_R (b_{ik} \otimes E f_k s_E(r))
\end{align*}

implying that $\Delta(a) = a_{ij} \cdot e_j \otimes_R b_{ik} \cdot f_k$ satisfies (116). In order to get (117) use

$$\pi(a \cdot t_E(r) \otimes E 1_R) = \pi(a \otimes E t_E(r) \cdot 1_R) = \pi(a \otimes E r) = \pi(a \otimes E s_E(r) \cdot 1_R) = \pi(a \cdot s_E(r) \otimes E 1_R).$$

Now we can finish the proof of Theorem 5.4 as follows.

**Proof.** That the bimonad of a bialgebroid has a right adjoint was shown at the beginning of this subsection. Assume $T$ is a bimonad on $R\mathcal{M}_R$ with a right adjoint. Then by Corollary 5.8 there is an algebra extension $A$ of $R^e$ and a monad isomorphism $\nu: T \iso A \otimes_E -$. Use this $\nu$ to pass the bimonad structure of $T$ to the functor $A \otimes_E -$. Then $\nu$ becomes a bimonad isomorphism. Now $A \otimes_E -$ has a right adjoint therefore by Lemma 5.9 there is a comonoidal natural isomorphism $\chi: A \otimes_E - \iso B \otimes_E -$ for some $R$-coring and $E$-$E$ bimodule $B$. This latter isomorphism determines an $E$-$E$ bimodule isomorphism $A \iso B$ which can be used to make $A$ into a comonoid in $\mathcal{E}M$. Now the bimonad $A \otimes_E -$ has structure maps as in (100, 103) in which $\Delta$ and $\varepsilon$ give rise to an $R$-coring structure on $A$ and satisfy

\begin{align*}
\Delta(a)(t(r) \otimes 1_A) &= \Delta(a)(1_A \otimes_R s(r)) \\
\varepsilon(as(r)) &= \varepsilon(at(r))
\end{align*}

by Lemma 5.11. It remains to use the bimonad axioms (BMD 3) and (BMD 4). Inserting (102) into the first diagrams of (21) and (22), after a little calculation one obtains that $\Delta$ has to be multiplicative and unit preserving, respectively. Substituting (103) into the remaining diagrams of (21) and (22) one immediately arrives to the two bialgebroid axiom (BDG 2.a) and (BDG 2.b). This proves that $A$ is a
left bialgebroid over $R$ and that its bimonad is isomorphic to $T$ via a comonoidal natural isomorphism.

5.3. Tannaka reconstruction for bialgebroids. By characterizing bialgebroids as the bimonads with right adjoints (on bimodule categories) a natural definition arises for what to be the bialgebroid morphisms and transformations.

Definition 5.12. Let $\mathcal{Bgd}_k$ denote the 1-full and 2-full sub-2-category of $\mathcal{Bmd}_k$ the objects of which are the $k$-linear bimonads $T$: $\mathcal{T} \to \mathcal{T}$ with right adjoint where $\mathcal{T}$ is isomorphic to $R\mathcal{M}_R$ for some $k$-algebra $R$. The objects of the form $A \otimes_{R^e} -$ for some bialgebroid $A$ over $R$ are called proper bialgebroids.

Note that we could define $\mathcal{Bgd}_k$ to be 2-replete and not only 1-replete by allowing for objects all bimonads that are equivalent to proper bialgebroid bimonads. Still the above definition works well with the Eilenberg-Moore construction.

Proposition 5.13. Let $T$: $\mathcal{T} \to \mathcal{T}$ be an object in $\mathcal{Bgd}_k$. Then $\mathcal{EM}(T) = UT$: $\mathcal{T}^T \to \mathcal{T}$ is a monoidal functor with both left and right adjoints.

Proof. Strict monoidality of $UT$ and existence of left adjoint follows from Propositions 3.3 and 3.4. Existence of right adjoint follows from [15, Corollary V.8.3].

The converse of the above proposition, namely that a monoidal functor $U: \mathcal{C} \to \mathcal{T}$ with both left and right adjoints determines a bimonad $T = Q(U)$ with right adjoint is obvious since the underlying functor is now a product of two functors $T = UF$ with both $U$ and $F$ having a right adjoint. Therefore appropriate restrictions of the 2-functors $\mathcal{EM}$ and $Q$, denoted by the same letters, provide an adjunction and a Tannakian theory for bialgebroids.

In the following definition $\mathcal{L-MFunc}_k$ denotes the $k$-linear version of $\mathcal{L-MFunc}$ of Section 4 with only strict 1-cells.

Definition 5.14. Let $\mathcal{A-MFunc}_k$ be the 1-full and 2-full sub-2-category of $\mathcal{L-MFunc}_k$ the objects of which are the monoidal functors $U: \mathcal{C} \to \mathcal{T}$ with both left and right adjoints and with target category $\mathcal{T}$ that is isomorphic to some bimodule category over $k\mathcal{M}$.

It follows that the restrictions of $Q$ and $\mathcal{EM}$ to 2-functors $\mathcal{A-MFunc}_k \to \mathcal{Bgd}_k$ and $\mathcal{Bgd}_k \to \mathcal{A-MFunc}_k$, respectively, constitute a 2-adjunction $\mathcal{Q \dashv EM}$. The following corollary is a direct consequence of Theorem 4.19 and of the above definitions.

Corollary 5.15. Let $R$ be a $k$-algebra and $U: \mathcal{C} \to R\mathcal{M}_R$ be a $k$-linear monoidal functor with left and right adjoints. Then there is a bialgebroid $A$ over $R$ and a monoidal functor $K: \mathcal{C} \to A\mathcal{M}$ such that

1. $U$ factorizes as $U = U_A K$ through the strict monoidal forgetful functor $U_A$ of the bialgebroid,

2. if $B$ is another bialgebroid over some $k$-algebra $S$ such that there exist monoidal functors $F: \mathcal{C} \to B\mathcal{M}$ and $G: R\mathcal{M}_R \to S\mathcal{M}_S$ satisfying $GU = U_B F$ then there exists a unique bialgebroid morphism $(G, \varphi): A \to B$, i.e., a unique ambimonoidal natural transformation

$$\varphi: B \otimes_{S^e} G(\cdot) \to G(A \otimes_{R^e} \cdot)$$

such that $G^\varphi K = F$. 

The bialgebroid \( A \) with the above properties is unique up to isomorphisms.

The following representation theorem for bialgebroids, in turn, is a consequence of Theorem 3.7.

**Corollary 5.16.** Let \( C \) be a \( k \)-linear monoidal category and \( R \) be a \( k \) algebra. Then for a \( k \)-linear functor \( \mathcal{C} \rightarrow \mathcal{R} \mathcal{M}_R \) the following conditions are equivalent:

1. There exists a bialgebroid \( A \) over \( R \) and a \( k \)-linear monoidal category equivalence \( K: \mathcal{C} \rightarrow \mathcal{A} \mathcal{M} \) such that \( U_A K = U \).
2. \( U \) is monadic, monoidal and has a right adjoint.

### 5.4. Bialgebroid maps.

According to Definition 6.12 the morphisms \((G, \varphi)\) from a bialgebroid \( A \) over \( R \) to another \( B \) over \( S \) consists of a lax monoidal functor \( G: \mathcal{R} \mathcal{M}_R \rightarrow \mathcal{S} \mathcal{M}_S \) and of an ambimonoidal natural transformation \( \varphi: B \otimes_{S^e} G(\cdot) \rightarrow G(A \otimes_{R^e} \cdot) \) satisfying the two diagrams (61). These conditions are rather complicated for a general functor \( G \) so we can only give some special examples. The simplest are the bialgebroid maps.

Assume that \( G \) arises from a \( k \) algebra homomorphism \( \omega: S \rightarrow R \), i.e., \( G \) is a lax monoidal forgetful functor

\[
G = \Phi^\omega: \mathcal{R} \mathcal{M}_R \rightarrow \mathcal{S} \mathcal{M}_S, \quad rX_R \mapsto \omega(S)X_{\omega(S)}
\]

similar to the \( \Phi \) of Lemma 5.7. In this case the natural transformation

\[
\varphi_X: B \otimes_{S^e} \Phi^\omega(X) \rightarrow \Phi^\omega(A \otimes_{R^e} X)
\]

is completely determined by

\[
\varphi_{R^e}: B \otimes_{S^e} X \rightarrow A \otimes_{R^e} X
\]

since \( R^e \) is a generator. Naturality of \( \varphi_{R^e} \) alone in turn gives

\[
\varphi_{R^e}(b \otimes_{S^e} (r \otimes r')) = \varphi(b) \otimes_{R^e} (r \otimes r')
\]

where the \( S \)-\( S \)-bimodule map \( \varphi: B \rightarrow \Phi^\omega(A) \) is the composite

\[
B \xrightarrow{\sim} B \otimes_{S^e} S^e \xrightarrow{B \otimes_{S^e}(\omega \otimes \omega^\prime)} B \otimes_{S^e} R^e \xrightarrow{\varphi_{R^e}} \Phi^\omega(A) \otimes_{R^e} R^e \xrightarrow{\sim} \Phi^\omega(A)
\]

where note that in \( \Phi^\omega(A) \) only the left \( R^e \) action is forgotten, the right one is intact. Inserting the expression \( \varphi_X(b \otimes x) = \varphi(b) \otimes x \) into the monad morphism axioms (61) we obtain that \( \varphi: B \rightarrow A \) is an algebra map. Since it is also an \( S \)-\( S \) bimodule map by its definition (121), we obtain the identities

\[
\varphi \circ s_B = s_A \circ \omega \\
\varphi \circ t_B = t_A \circ \omega
\]

Now inserting to the ambimonoidality axioms of Definition 5.7 we obtain that \( \varphi: B \rightarrow A \) preserves the coalgebra structure in the sense of the equations

\[
\Phi^\omega(\Delta_A) \circ \varphi = \Phi^\omega_{A,A} \circ (\varphi \otimes_{S^e} \varphi) \circ \Delta_B
\]
\[
\Phi^\omega(\varepsilon_A) \circ \varphi = \Phi^\omega_{0,0} \circ \varepsilon_B
\]

or in Hopf algebraist notation

\[
(126) \quad \varphi(b)_{(1)} \otimes_R \varphi(b)_{(2)} = \varphi(b_{(1)}) \otimes_R \varphi(b_{(2)})
\]
\[
(127) \quad \varepsilon_A(\varphi(b)) = \omega(\varepsilon_B(b))
\]
for all $b \in B$. The equations (122-123-124-125) define what is called a bialgebroid map in [27]. A bialgebroid map is completely determined by $\varphi: B \to A$ since

\[ \omega = \varepsilon_A \circ \varphi \circ s_B. \]

5.5. Bimodule induced bialgebroid morphisms. Another class of bialgebroid morphisms are obtained if we take the functor $G: R \mathcal{M}_R \to S \mathcal{M}_S$ to be $G X = G \otimes_{R^e} X$ for some $S^e\cdot R^e$-bimodule $G$. The natural transformation $\varphi: B \otimes_{S^e} G(-) \to G(A \otimes_{R^e} - )$ then becomes expressed in terms of a bimodule map

\[ \varphi: B \otimes_{S^e} G \to G \otimes_{R^e} A \in S^e \mathcal{M}_{R^e} \]

as $\varphi_X(b \otimes g \otimes x) = \varphi(b \otimes g) \otimes x$. If we insert this expression into the two monad morphism diagram (151) and into the two ambimonoidality diagram (69-70) we obtain four relations between $G$ and $\varphi$ that are reminiscent of the entwining structure of Brzeziński and Majid [4], although not the same.

At first notice that lax monoidality of the functor $(G, G_2, G_0)$ imposes a monoid structure on the bimodule $G$ in the category of $S\cdot S$-bimodules but also satisfies dual analogues of the bialgebroid comultiplication property (BGD 1.b) from the right hand side due to naturality of $G^e$.

Returning to the case of general $R$ and $S$ the bialgebroid morphism $(G \otimes_{R^e} - , \varphi)$ described above is in fact the most general possible if we require it to be an equivalence of the objects $A$ and $B$ in $k$-Bgd. This follows from Morita theory since $R \mathcal{M}_R \xrightarrow{G} S \mathcal{M}_S$ should be an equivalence and using the isomorphisms $R^e \mathcal{M} \cong R \mathcal{M}_R$ and $S^e \mathcal{M} \cong S \mathcal{M}_S$ the $G$ has to be naturally isomorphic to a functor $G \otimes_{R^e} -$ with a Morita equivalence bimodule $S^e G_{R^e}$. That is to say, the rings $R$ and $S$ are $\sqrt{\text{Morita-equivalent}}$ [24]. Ordinary Morita equivalence $R \sim S$ arises under the further assumption that $S^e G_{R^e}$ is the $k$-tensor product of equivalence bimodules $R H_S$ and $S H^e_R$. This latter situation is the Morita base change proposed by Schauenburg [24] while the former was named as $\sqrt{\text{Morita}}$ base change.
5.6. **An exotic example: Hom.** For the tired Reader’s sake let stand here an example of a bimonad that is not a bialgebroid. It shows that every set is a bimonad in a canonical way.

Let $\mathbf{Set}$ be the category of small sets equipped with the Cartesian closed monoidal structure $(\mathbf{Set}, \times, 1)$ with some one element set $1 = \{\ast\}$. Every object $C$ in $\mathbf{Set}$ is a comonoid in a unique way, namely by the diagonal mapping $\Delta_C : x \mapsto \langle x, x \rangle$ and by the constant mapping $\varepsilon_C : C \to 1$. This comonoid structure makes the endofunctor $T := \text{Hom}(C, -)$ into a monad with multiplication and unit
\[
\mu_A : T^2 A \to TA, \quad \mu_A(f)(c) = f(c)(c)
\]
\[
\eta_A : A \to TA, \quad \eta_A(a)(c) = a
\]
respectively. Moreover, $T$ is comonoidal with
\[
\gamma_{A,B} : T(A \times B) \longrightarrow T(A) \times T(B), \quad f \mapsto \langle p_1 \circ f, p_2 \circ f \rangle
\]
\[
\pi : T(1) \longrightarrow 1, \quad f \mapsto \ast
\]
where $p_1$ are the projections of the product $A \times B$. Now it is an easy exercise to check that both $\mu$ and $\eta$ are comonoidal natural transformations, so $(T, \gamma, \pi, \mu, \eta)$ is a bimonad.

A $T$-algebra for this bimonad is a set $A$ and a function $\alpha : \text{Hom}(C, A) \to A$ such that the two diagrams of (30) commute. For a finite set $C$ with $n$ elements such a function $\alpha$ can be identified with an $n$ variable function on $A$ with values in $A$. Then the $T$-algebra conditions become the following equations for $\alpha$.
\[
\alpha(a(a_{11}, \ldots, a_{1n}), \ldots, a(a_{n1}, \ldots, a_{nn})) = \alpha(a_{11}, \ldots, a_{nn})
\]
\[
\alpha(a, \ldots, a) = a
\]
for all $a_{ij} \in A$ and $a \in A$. There are solutions that are evaluations at an element of $C$, let’s say, $\alpha(a_1, \ldots, a_n) = a_i$. But there are solutions that are not evaluations, the free $T$-algebras for example. A free $T$-algebra $(\text{Hom}(C, A), \mu_A)$ is a product set $\Pi^n A$ with action
\[
\alpha(a_{11}, \ldots, a_{1n}), \ldots, a_{nn}) = \langle a_{11}, \ldots, a_{nn}\rangle
\]
Of course, the solutions form a monoidal category $\mathbf{Set}^T$ by Proposition 3.3, otherwise the general solution for $T$-algebras is not known to the author.

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