Global classical solution to 3D compressible magnetohydrodynamic equations with large initial data and vacuum

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Abstract

In this paper, we study the Cauchy problem of the isentropic compressible magnetohydrodynamic equations in $\mathbb{R}^3$. When $(\gamma - 1)^{\frac{1}{6}}E_0^{\frac{2}{3}}$, together with the $\|H_0\|_{L^2}$, is suitably small, a result on the existence of global classical solutions is obtained. It should be pointed out that the initial energy $E_0$ except the $L^2$-norm of $H_0$ can be large as $\gamma$ goes to 1, and throughout the proof of the theorem in the present paper, we make no restriction upon the initial data $(\rho_0, u_0)$. Our result improves the one established by Li-Xu-Zhang in [29], where, with small initial engergy, the existence of classical solution was proved.

Keyword: isentropic compressible magnetohydrodynamic equations, global classical solution, vacuum.

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1 Introduction

In this paper, we consider the following isentropic compressible magnetohydrodynamic equations in $\mathbb{R}^3$ (refer, e.g., [1, 26]):

$$
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = (\nabla \times H) \times H + \mu \Delta u + (\mu + \lambda) \nabla \text{div} u, \\
H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \\
\text{div} H = 0, \quad x \in \mathbb{R}^3, \quad t > 0,
\end{cases}
$$

(1.1)

with the initial data

$$(\rho, u, H)(x, 0) = (\rho_0, u_0, H_0)(x), \quad x \in \mathbb{R}^3,$n

(1.2)

and the far-field behavior

$$(\rho, u, H) \to (0, 0, 0) \quad \text{as} \quad |x| \to \infty, \quad \text{for} \quad t > 0.
$$

(1.3)

Here $\rho = \rho(x, t)$, $u = (u^1, u^2, u^3)(x, t)$, $P$ and $H = (H^1, H^2, H^3)(x, t)$ represent the density, velocity, pressure and magnetic field of the fluid respectively. More precisely, $P$ is given by

$$P(\rho) = A\rho^\gamma,$n

(1.4)

where $\gamma$ is the adiabatic exponent, and $A > 0$ is a constant. Without loss of generality, we assumed that $A = 1$. The viscosity coefficients $\mu$ and $\lambda$ satisfy

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.
$$

(1.5)

The constant $\nu > 0$ is the resistivity coefficient which is inversely proportional to the electrical conductivity and acts as the magnetic diffusivity of magnetic fields.

The magnetohydrodynamic (MHD) model is used to study the dynamics of conducting fluid under the effect of the magnetic field and finds its way in a huge range of physical objects, from liquid metals to cosmic plasmas, refer for example [1, 19, 28, 26, 31]. And for so, there have been a lot of literatures on the MHD system (1.1)-(1.5), see for instance, [2, 3, 8, 9, 10, 11, 14, 15, 16, 17, 21, 22, 23, 32, 33, 34, 39, 40] and references therein. It should be noted that if $H = 0$, i.e., there is no electromagnetic effect, then (1.1) becomes the compressible Navier-Stokes equations, which has been widely studied, refer for example [4, 5, 6, 7, 12, 25, 30, 35, 36, 37] and the references therein. The main difficulty in investigating the issues of well-posedness and dynamical behaviors of MHD system is caused by the strong coupling and interplay interaction between the fluid motion and the magnetic field. Now let’s recall briefly some results on the multi-dimensional compressible MHD system, especially the ones that are closely relative to our topic in the present paper. With large initial data, the local strong solutions to the compressible MHD equations were proved in [33] and [11] for the case $\rho_0 > 0$ and the case $\rho_0 \geq 0$, respectively. When the initial data are small perturbations of a given constant state in $H^3$ -norm, Kawashima in [21] firstly established a result on the global existence of smooth solutions to the general electro-magneto-fluid equations in $\mathbb{R}^2$. The global existence and time decay rate of smooth solutions to the linearized two-dimensional compressible MHD equations was studied by Umeda, Kawashima and Shizuta in [32]. Zhang and Zhao [40] proved the optimal decay estimates of classical solutions to the compressible MHD equations when the initial data are close to a nonvacuum equilibrium. For the case that the initial density is allowed to vanish and even has compact support, Li-Xu-Zhang [29]...
established a result on the existence and large-time behavior of classical solution with regular initial data, which are of small energy but possibly large oscillations, and constant state as far field density which may contain vacuum.

Before stating our main results, we firstly explain the notations and conventions used through this paper.

Notations.

(i) \( \int_{\mathbb{R}^3} f = \int_{\mathbb{R}^3} f\,dx, \int_0^T f = \int_0^T f\,dt. \)

(ii) For \( 1 \leq r \leq \infty \), denote the \( L^r \) spaces and the standard Sobolev spaces as follows:

\[
\begin{align*}
L^r &= L^r(\mathbb{R}^3),
D^{k,r} = \{ u \in L^1_{\text{loc}}(\mathbb{R}^3) | \nabla^k u \in L^r(\mathbb{R}^3) \},
\| u \|_{D^{k,r}} &= \| u \|_{L^r},
\end{align*}
\]

\[
\begin{align*}
D^1 &= D^{1,2},
W^{k,r} &= W^{k,r}(\mathbb{R}^3),
H^k &= W^{k,2},
\end{align*}
\]

\[
\hat{H}^\beta = \left\{ u : \mathbb{R}^3 \to \mathbb{R} \left| \| u \|_{\hat{H}^\beta}^2 = \int_{\mathbb{R}^3} |\xi|^{2\beta} |\hat{u}(\xi)|^2 \,d\xi < \infty \right. \right\},
\]

(iii) \( G \triangleq (2\mu + \lambda)\text{div}u - P - \frac{1}{2}|H|^2 \) is the so-called effective viscous flux, while \( \omega \triangleq \nabla \times u \) is the vorticity.

(iv) \( \dot{h} = h_t + u \cdot \nabla h \) denotes the material derivatives.

(v) \( E_0 = \int_{\mathbb{R}^3} \left( \frac{1}{2}\rho_0|u_0|^2 + \frac{1}{\gamma - 1}\rho_0^\gamma + \frac{1}{2}|H_0|^2 \right) \) is the initial energy.

Now it is the place to state our main theorem.

**Theorem 1.1.** Assume that the initial data \((\rho_0, u_0, H_0)\) satisfy

\[
\begin{align*}
\frac{1}{2}\rho_0|u_0|^2 + \frac{1}{\gamma - 1}\rho_0^\gamma + \frac{1}{2}|H_0|^2 \in L^1, & \quad 0 \leq \rho_0 \leq \bar{\rho}, \\
(\rho_0, P(\rho_0)) \in H^2 \cap W^{2,q}, & \quad u_0 \in D^1 \cap D^2, \\
H_0 \in \cap D^1 \cap D^2, & \quad \|\nabla u_0\|^2_{L^2} \leq M_1, \quad \|H_0\|^2_{D^1} \leq M_2,
\end{align*}
\]

for given constants \( M_i > 0 \) \((i = 1, 2)\), \( \bar{\rho} \geq 1 \) and \( q \in (3, 6) \), and that the compatibility condition holds

\[
-\mu \Delta u_0 - (\lambda + \mu)\text{div}u_0 + \nabla P(\rho_0) + \frac{1}{2}\nabla |H_0|^2 - H_0 \cdot \nabla H_0 = \rho_0^\gamma g,
\]

with \( g \in L^2 \). In addition, we suppose that

\[
(\gamma - 1) \frac{1}{2} E_0 \leq 1, \quad 1 < \gamma \leq \frac{3}{2}.
\]

Then, there exists a unique global classical solution \((\rho, u, H)\) in \( \mathbb{R}^3 \times [0, \infty) \) satisfying

\[
0 \leq \rho(x,t) \leq 2\bar{\rho}, \quad x \in \mathbb{R}^3, \quad t \geq 0,
\]

\[
E_0 \leq \int_{\mathbb{R}^3} \left( \frac{1}{2}\rho|u|^2 + \frac{1}{\gamma - 1}\rho^\gamma + \frac{1}{2}|H|^2 \right) \,dx \leq E_0.
\]
and
\[
\begin{align*}
(r, P) & \in C([0, T]; H^2 \cap W^{2, q}), \\
u & \in C([0, T]; D^1 \cap D^2) \cap \mathcal{L}\infty(\tau, T; D^4 \cap D^3), \\
u_t & \in \mathcal{L}\infty(\tau, T; D^1 \cap D^2) \cap H^1(\tau, T; D^1), \\
H & \in C([0, T]; H^2) \cap \mathcal{L}\infty(\tau, T; H^3), \\
H_t & \in C([0, T]; L^2) \cap H^1(\tau, T; L^2),
\end{align*}
\]
for any \(0 < \tau < T < \infty\), provided that
\[
(\gamma - 1)\frac{3}{2} E_0^{\frac{1}{2}} \leq \varepsilon \triangleq \min \left\{ \left( \frac{\bar{\rho}}{2K_9} \right)^{16}, \varepsilon_5, \frac{\bar{\rho}}{4} \right\}.
\]
(1.12)

Here
\[
\varepsilon_5 = \min \left\{ \varepsilon_4, (C_4 K_9)^{-1}, 1 \right\},
\varepsilon_4 = \min \left\{ \varepsilon_3, \frac{1}{K^{54}}, \left( C(\bar{\rho})(\gamma - 1)\frac{3}{2} K_9^2 \right)^{-\frac{9}{2}}, 1 \right\},
\varepsilon_3 = \min \left\{ \varepsilon_1, \varepsilon_2, 4C_3(\bar{\rho})^{-27}, 1 \right\},
\varepsilon_2 = \min \left\{ \varepsilon_1, (4C_1)^{-\frac{32}{7}} + (4C_2)^{-27} \right\},
\varepsilon_1 = \min \left\{ 1, \left( 4C(M_2^3 + K_1) \right)^{-\frac{9}{2}} \right\}.
\]
(1.13)

**Remark 1.1.** We make no restriction on the initial data \((\rho_0, u_0)\). In fact, it follows from \(1.7\) that \(E_0 \leq C_0(\gamma - 1)^{-\frac{1}{2}}\), then the upper bound of \(E_0\) may go to \(\infty\) as \(\gamma\) goes to \(1\), in spite that \(\|H_0\|_{L^2}\) is small.

**Remark 1.2.** The solution obtained in Theorem 1.1 becomes a classical one away from the initial time. More precisely, we establish a result on the existence of a classical solution to (1.1)-(1.5) under the assumption that \(\parallel\nabla H_{\beta}\parallel_{H^\beta} \leq 1\), and \(\|H\|_{H^\beta} \leq 1\), provided that \(\varepsilon\) in Theorem 1.1 is still applicable (necessarily after some modification for the proof).

**Remark 1.3.** If we remove \(\|\nabla u\|_{L^2} \leq M_1\) and \(\|\nabla H\|_{L^2} \leq M_2\) in (1.7) in Theorem 1.1 and assume instead that \(u_0, H_0 \in H^\beta(\beta \in (\frac{1}{2}, 1))\) with \(\|u\|_{H^\beta} \leq M_1\) and \(\|H\|_{H^\beta} \leq M_2\) for some \(M_i > 0\) \((i = 1, 2)\), Theorem 1.1 will still hold, and the \(\varepsilon\) in Theorem 1.1 will also depend on \(M_i\) instead of \(M_i\) correspondingly. This can be achieved by a similar way in [29].

**Remark 1.4.** It should be noted that when the viscous coefficient \(\mu\) is taken to be suitable large, the initial energy except the \(L^2\)-norm of \(H_0\) could also be large, which together with the conclusion in Theorem 1.1 implies the fact that when \((\gamma - 1)^{\frac{3}{2}} E_0^{\frac{1}{2}} \mu^{-\alpha_1}\) and \(\|H_0\|_{L^2}\) are suitably small for some \(\alpha_1 > 0\), the existence of classical solutions to (1.1)-(1.5) could also be obtained. And this can be done by using a similar method as in [13], which considered the compressible Navier-Stokes equations, we omit it for simplicity in the present paper. Moreover,
when \( H = 0 \), i.e., there is no electromagnetic effect, (1.1) reduces to the compressible Navier-Stokes equations. Roughly speaking, we generalize the result of [13] to the compressible MHD equations.

We now briefly make some comments on the analysis of the present paper. Note that the local existence and uniqueness of classical solutions to problem (1.1)-(1.5) can be proved by combining the arguments in [11] with the higher order estimates in section 4 of [29]. Hence, to extend the classical solution globally in time, we just need some global a priori estimates on the smooth solution \((\rho, u, H)\) in suitable regularity norms. Formally, the key to the proof is to get the time-independent upper bound of the density as well as the time-dependent higher norm estimates of \((\rho, u, H)\). In this paper, the latter one follows in the same way as in [29] (see Lemmas 4.1-4.6), once the former one is achieved. To derive the upper bound of the density, on the one hand, we try to adapt some basic ideas in [12, 18, 29]. However, new difficulties arise in our analysis, since the smallness of \((\gamma - 1)\frac{3}{2}E_0^\frac{3}{2}\) does not result in the small initial energy. One the other hand, compared with compressible Navier-Stokes equations, the strong coupling and interplay interaction between the fluid motion and the magnetic field, such as \(\nabla \times (u \times H)\) and \((\nabla \times H) \times H\), will bring out some new difficulties.

Precisely, in [12, 18, 29] the smallness of the initial energy was used to ensure the smallness of \(\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2\) and \(\|H_0\|_{L^2}\), which play crucial role in the proof of the upper bound of density. Similar to [13, 18, 29], here we need to close the a priori estimates \(A_1\) and \(A_2\). Compared with [13, 18, 29], we not only need to handle the terms \(|\nabla u|^2\), \(|\nabla u|^3\), \(|\nabla u|^4\), \(P|\nabla u|^2\), \(|P\nabla u|^2\), but the terms caused by \(\nabla \times (u \times H)\) and \((\nabla \times H) \times H\), like \(H \cdot \nabla H \cdot u\) and \(\nabla |H|^2 \cdot u\). Adapting the idea developed in [13], we need to derive the smallness of \(\int_0^{\tau(T)} |\nabla u|^2\), but this is not trivial because of the lack of the smallness of \(\|H_0\|_{L^2}\). The key observation to overcome this difficulty is as follows: Looking back to the basic energy \(E_0 = \int_{\mathbb{R}^3} \left( \frac{\rho_0}{2} |u_0|^2 + \frac{1}{\gamma - 1} \rho_0^\gamma + \frac{1}{2} |H_0|^2 \right)\), the smallness of \((\gamma - 1)\frac{3}{2}E_0^\frac{3}{2}\) could remove the smallness restriction upon \(\rho_0\) and the term involving \(u_0\) and \(\rho_0\). But it has nothing to do with \(H_0\). Moreover, For all terms in (1.1), we can never see any term in which \(\rho\) is coupled with \(H\). Hence we assume that \(\|H_0\|_{L^2} \leq (\gamma - 1)\frac{3}{2}E_0\) is small, and then we succeed to derive some estimates on the smallness and boundedness of \(H\) and its derivatives with a key estimate \(\int_0^{\tau(T)} \|\nabla u\|_{L^2}^2\). Similar to [13, 18, 29], we try to estimate \(A_1\) and \(A_2\) and achieve an inequality involving \(|\nabla u|^2\), \(|\nabla u|^3\), \(|\nabla u|^4\), \(P|\nabla u|^2\) and \(|P\nabla u|^2\). And then we handle all these terms one the right hand side of the inequality with two crucial boundedness estimate (see Lemma 5.7). Thus the upper bound of \(\rho\) is obtained by a standard method as in [13, 18, 29], together with some new estimate (see [3.119] and [3.124]). It should be noted that during the process, the estimates obtained for \(H\) always play a key role, especially when controlling the coupled term \(\nabla \times (u \times H)\).

The rest of the paper is organized as follows. In section 2, we first collect some elementary inequality and facts which will be need in the later analysis. In section 3, we devote to derive the necessary lower-order a priori estimates on the classical solution which is independent of time. The time-dependent estimates on the higher-norms of the solutions will be proved in Section 4, and then Theorem [13] is proved.

2 Preliminaries

In this section, we will recall some elementary inequality and results which will be used used frequently later. We begin with the following well-known Gagliardo-Nirenberg inequality (see [27]).
Lemma 2.1. For $2 \leq p \leq 6$, $1 < q < \infty$, and $3 < r < \infty$, there exists a generic constant $C > 0$, depending only on $q$ and $r$, such that for $f \in H^1$ and $g \in L^q \cap D^{1,r}$, we have

$$\|f\|_{L^p} \leq C \|f\|_{L^{6-p}}^{\frac{6-p}{2^p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2^p}},$$  \hspace{1cm} (2.1)$$

$$\|g\|_{L^\infty} \leq C \|g\|_{L^{\frac{4(1-r)}{3r-4}}} \|g\|_{L^2}^{\frac{3r}{3r-4}}.$$  \hspace{1cm} (2.2)

Similar to the compressible Navier-Stokes equations (see, for example [13, 18]), one can easily derive the following elliptic equations from (1.1):

$$\Delta G = \text{div}(\rho \dot{u}) - \text{div}d(H \otimes H), \quad \mu \Delta \omega = \nabla \times (\rho \dot{u} - \text{div}(H \otimes H)).$$  \hspace{1cm} (2.3)

We now state some elementary $L^p$-estimates for the elliptic equations in (2.3) by the virtue of (2.1).

Lemma 2.2. Let $(\rho, u, H)$ be a smooth solution to (1.1)-(1.5) on $\mathbb{R}^3 \times (0, T]$. Then there exists a generic $C > 0$, which may depend on $\mu$ and $\lambda$, such that for any $p \in [2, 6]$,

$$\|\nabla G\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C (\|\rho \dot{u}\|_{L^p} + \|H \cdot \nabla H\|_{L^p}),$$  \hspace{1cm} (2.4)$$

$$\|G\|_{L^p} + \|\omega\|_{L^p} \leq C (\|\rho \dot{u}\|_{L^p} + \|H \cdot \nabla H\|_{L^p}),$$  \hspace{1cm} (2.5)$$

$$\|\nabla u\|_{L^p} \leq C (\|\rho \dot{u}\|_{L^p} + \|P\|_{L^p} + \|\rho \dot{u}\|_{L^p} + \|H \cdot \nabla H\|_{L^p}),$$  \hspace{1cm} (2.6)$$

$$\|\nabla u\|_{L^4} \leq C (\|\nabla u\|_{L^2}^{\frac{3}{2}} \|\rho \dot{u}\|_{L^2}^{\frac{3}{2}} + \|P\|_{L^2}^{\frac{3}{2}} \|\rho \dot{u}\|_{L^2}^{\frac{3}{2}} + \|H\|_{L^4}^{\frac{3}{2}} \|\rho \dot{u}\|_{L^2}^{\frac{3}{2}}$$

$$\quad + C \|\nabla u\|_{L^2}^{\frac{3}{2}} \|H \cdot \nabla H\|_{L^2}^{\frac{3}{2}} + C \|P\|_{L^4}^{\frac{3}{2}} \|H \cdot \nabla H\|_{L^2}^{\frac{3}{2}}$$

$$\quad + C \|H\|_{L^4} \|H \cdot \nabla H\|_{L^2} + C \|P\|_{L^4}).$$  \hspace{1cm} (2.7)$$

Proof. The proof of inequalities (2.1)-(2.3) can be found in Lemma 2.2 in [20]. Here we will prove (2.7). It follows from direct computation that

$$-\Delta u = -\nabla \text{div}u + \nabla \times \omega,$$  \hspace{1cm} (2.8)$$

then the standard $L^p$-estimate for elliptic equation, together with (2.1) and (2.4), leads to

$$\|\nabla u\|_{L^p} \leq C (\|\text{div}u\|_{L^p} + \|u\|_{L^p})$$

$$\leq C \left( \|G\|_{L^p} + \|P\|_{L^p} + \|H\|_{L^p} + \|\nabla u\|_{L^2}^{\frac{6-p}{2^p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2^p}} + \|H \cdot \nabla H\|_{L^2}^{\frac{3p-6}{2^p}} \right)$$

$$\leq C \|\nabla u\|_{L^2}^{\frac{6-p}{2^p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2^p}} + C \|P\|_{L^2}^{\frac{6-p}{2^p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2^p}} + C \|H\|_{L^4}^{\frac{6-p}{2^p}} \|\rho \dot{u}\|_{L^2}^{\frac{3p-6}{2^p}}$$

$$\quad + C \|\nabla u\|_{L^2}^{\frac{6-p}{2^p}} \|H \cdot \nabla H\|_{L^2}^{\frac{3p-6}{2^p}} + C \|P\|_{L^2}^{\frac{6-p}{2^p}} \|H \cdot \nabla H\|_{L^2}^{\frac{3p-6}{2^p}}$$

$$\quad + C \|H\|_{L^4}^{\frac{6-p}{2^p}} \|H \cdot \nabla H\|_{L^2}^{\frac{3p-6}{2^p}} + C \|P\|_{L^p}. \hspace{1cm} (2.9)$$

Let $p = 4$ in (2.9), one gets (2.7). \hfill \Box

To obtain the uniform (in time) upper bound of the density, we need the following Zlotnik inequality.
Lemma 2.3 (see [38]). Assume that the function \( y \) satisfies
\[
y'(t) = g(y) + b'(t) \quad \text{on} \quad [0, T], \quad y(0) = y^0, \tag{2.10}
\]
with \( g \in C(\mathbb{R}) \) and \( y, b \in W^{1,1}(0, T) \). If \( g(\infty) = -\infty \) and
\[
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1), \tag{2.11}
\]
for all \( 0 \leq t_1 \leq t_2 \leq T \) with some \( N_0 \geq 0 \) and \( N_1 \geq 0 \), then
\[
g(\xi) \leq -N_1, \quad \text{for} \quad \xi \geq \bar{\xi}. \tag{2.12}
\]

3 Time-independent estimates

In this section, we will derive the uniform time-independent estimates of the solution to \((1.1)-(1.5)\) and the time-independent upper bound of the density. Assume that \( (\rho, u, H) \) is a smooth solution to \((1.1)-(1.5)\) on \( \mathbb{R}^3 \times (0, T) \) for some positive time \( T > 0 \). Set \( \sigma = \sigma(t) \triangleq \min\{1, t\} \) and define the following functionals:

\[
A_1(T) \triangleq \sup_{0 \leq t \leq T} \sigma \int_{\mathbb{R}^3} \left( |\nabla u|^2 + |\nabla H|^2 \right) \\
+ \int_0^T \sigma \left( |\rho \phi \dot{u}|^2_{L^2} + |\nabla^2 H|^2_{L^2} + |H_t|^2_{L^2} \right),
\]

\[
A_2(T) \triangleq \sup_{0 \leq t \leq T} \sigma^2 \int_{\mathbb{R}^3} \left( \rho |\dot{u}|^2 + |\nabla^2 H|^2 + |H_t|^2 \right) \\
+ \int_0^T \sigma^2 \left( |\nabla \dot{u}|^2_{L^2} + |\nabla H_t|^2_{L^2} \right),
\]

\[
A_3(T) \triangleq \sup_{0 \leq t \leq T} ||H||^2_{L^3} + \int_0^T \int_{\mathbb{R}^3} |H||\nabla H|^2,
\]

\[
A_4(T) \triangleq \sup_{0 \leq t \leq T} \sigma^{\frac{2}{3}} |\nabla u|^2_{L^2},
\]

\[
A_5(T) \triangleq \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho |u|^3.
\]

Throughout this section, for simplicity we denote by \( C \) or \( C_i \) \( (i = 1, 2, \cdots) \) the generic positive constants which may depend on \( \mu, \lambda, \nu, A, \gamma, \bar{\rho}, \tilde{\rho}, C_0, M_i \) \( (i = 1, 2) \) and \( \|\rho_0\|_{L^1} \) but independent of time \( T > 0 \) and \( \gamma - 1 \). Sometimes \( C(\alpha) \) is also used to emphasize the dependent of \( \alpha \).

We state the key proposition in the present paper as follows.

Proposition 3.1. Assume that the initial data \( (\rho_0, u_0, H_0) \) satisfy \((1.4)-(1.9)\). Let \( (\rho, u, H) \) be a smooth solution to problem \((1.1)-(1.5)\) on \( \mathbb{R}^3 \times (0, T) \) satisfying

\[
\left\{ \begin{array}{ll}
0 \leq \rho(x, t) \leq 2\bar{\rho}, & (x, t) \in \mathbb{R}^3 \times [0, T], \\
A_1(T) + A_2(T) \leq 2 \left( (\gamma - 1)\frac{\pi}{2} \right) \|E_0\|^\frac{1}{2}, & A_3(T) \leq 2 \left( (\gamma - 1)\frac{\pi}{2} E_0^\frac{1}{2} \right)^\frac{1}{\gamma}, \\
A_4(\sigma(T)) + A_5(\sigma(T)) \leq 2 \left( (\gamma - 1)\frac{\pi}{2} E_0^\frac{1}{2} \right)^\frac{1}{\gamma},
\end{array} \right. \tag{3.2}
\]
then
\[
\begin{aligned}
0 \leq \rho(x,t) &\leq \frac{7}{4} \bar{\rho}, \ (x,t) \in \mathbb{R}^3 \times [0,T], \\
A_1(T) + A_2(T) &\leq \left( (\gamma - 1) \frac{\bar{\rho}}{2} E_0^{\frac{1}{2}} \right)^{\frac{1}{\gamma - 1}}, \quad A_3(T) \leq \left( (\gamma - 1) \frac{\bar{\rho}}{2} E_0^{\frac{1}{2}} \right)^{\frac{1}{\gamma - 1}}, \\
A_4(\sigma(T)) + A_5(\sigma(T)) &\leq \left( (\gamma - 1) \frac{\bar{\rho}}{2} E_0^{\frac{1}{2}} \right)^{\frac{1}{\gamma - 1}},
\end{aligned}
\]  
(3.3)

provided that
\[
(\gamma - 1) \frac{\bar{\rho}}{2} E_0^{\frac{1}{2}} \leq \varepsilon \equiv \min \left\{ \left( \frac{\bar{\rho}}{2K_9} \right)^{16}, \bar{\rho}, \varepsilon_5 \right\}. 
\]  
(3.4)

Here
\[
\varepsilon_5 = \min \{ \varepsilon_4, (C_4 K_9)^{-1}, 1 \}, \\
\varepsilon_4 = \min \left\{ \varepsilon_3, \frac{1}{K_6}, \left( C(\bar{\rho}) (\gamma - 1) \frac{\bar{\rho}}{2} K_9^{2} \right)^{-\frac{\gamma}{\gamma - 1}}, 1 \right\}, \\
\varepsilon_3 = \min \{ \varepsilon_1, \varepsilon_2, 2C_3(\bar{\rho}) \}^{-27}, 1 \}, \\
\varepsilon_2 = \min \left\{ \varepsilon_1, (4C_1)^{-\frac{27}{2}} + (4C_2)^{-27} \right\}, \\
\varepsilon_1 = \min \left\{ 1, (4C(M_2^2 + K_1))^{-9} \right\}. 
\]  
(3.5)

Proof. Proposition 3.1 can be derived from Lemmas 3.1-3.10 below.

Lemma 3.1. Under the same assumption as in Proposition 3.1, we have
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} P \leq (\gamma - 1) E_0, 
\]  
(3.6)

\[
\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{E_0}{\mu}. 
\]  
(3.7)

Proof. Multiplying (1.1) 1, (1.1) 2 and (1.1) 3 by \( \frac{\tilde{\rho}_t}{\gamma - 1} \), \( u \) and \( H \), respectively, and integrating the resulting equation over \( \mathbb{R}^3 \times (0, T) \), we have
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \left( \frac{P}{\gamma - 1} + \frac{1}{2} \rho|u|^2 + \frac{1}{2} |H|^2 \right) \\
+ \int_0^T \int_{\mathbb{R}^3} \left( \mu|\nabla u|^2 + (\mu + \lambda)|\text{div} u|^2 + \nu|\nabla H|^2 \right) \leq E_0,
\]  
(3.8)

which gives (3.6) and (3.7).

Lemma 3.2. Under the same assumption as in Proposition 3.1, it holds that
\[
\int_0^T \|\nabla u\|^4_{L^2} \leq CK_1 \left( (\gamma - 1) \frac{\bar{\rho}}{2} E_0^{\frac{1}{2}} \right)^{\frac{1}{\gamma - 1}}, 
\]  
(3.9)

where \( K_1 = \left( 1 + (\gamma - 1) \frac{\bar{\rho}}{2} \right) \).
Proof.

\[
\int_{0}^{T} \|
abla u\|_{L^2}^2 \leq \int_{0}^{\alpha(T)} \|
abla u\|_{L^2}^4 + \int_{\sigma(T)}^{T} \sigma \|
abla u\|_{L^2}^4
\]

\[
\leq \sup_{0 \leq t \leq \alpha(T)} \left( \frac{\sigma}{\sigma(T)} \|
abla u\|_{L^2}^2 \right)^2 \int_{0}^{\alpha(T)} \sigma - \frac{2}{3}
\]

\[+ C \left( \sup_{\sigma(T) \leq t \leq T} \|
abla u\|_{L^2}^2 \right) \int_{\sigma(T)}^{T} \|
abla u\|_{L^2}^2
\]

\[\leq C \left( \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{2}{3}} + \left( \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{1}{3}} C E_0^{\frac{1}{2}}
\]

\[\leq C \left( \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{2}{3}} + \left( \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{1}{3}} \left( \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{1}{3}} E_0
\]

\[\leq C \left( \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{1}{3}}, \quad (3.10)
\]

here \((\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} \leq 1\) and \((3.9)\) have been used. Lemma 3.2 is proved. \(\square\)

**Lemma 3.3.** Under the same assumption as in Proposition 3.1, it holds that

\[\sup_{0 \leq t \leq T} \left( \|H\|_{L^2}^2 + \sigma \|
abla H\|_{L^2}^2 \right)
\]

\[+ \int_{0}^{T} \left( \|
abla H\|_{L^2}^2 + \sigma \|H_t\|_{L^2}^2 + \sigma \|
abla^2 H\|_{L^2}^2 \right) \leq CK_2^2 (\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} \quad (3.11)
\]

and

\[\sup_{0 \leq t \leq T} \|
abla H\|_{L^2}^2 + \int_{0}^{T} \left( \|H_t\|_{L^2}^2 + \|
abla^2 H\|_{L^2}^2 \right) \leq CK_2 M_2. \quad (3.12)
\]

where \(K_2 = e^{2K_1}\).

**Proof.** Multiplying \((1.1)_3\) by \(H_t\), then integrating over \(\mathbb{R}^3\), using \((1.1)_4\), Hölder inequality and Cauchy inequality, we get

\[
\int_{\mathbb{R}^3} H \cdot H_t = \int_{\mathbb{R}^3} \nabla \times (u \times H) \cdot H - \int_{\mathbb{R}^3} \nabla \times (\nu \nabla \times H) \cdot H
\]

\[= \int_{\mathbb{R}^3} [(H \cdot \nabla)u - (u \cdot \nabla)H - (\text{div} u)H] \cdot H + \int_{\mathbb{R}^3} \Delta H \cdot H
\]

\[\leq \int_{\mathbb{R}^3} |u||\nabla H||H| - \nu \int_{\mathbb{R}^3} |\nabla u|^2
\]

\[\leq C \|u\|_{L^6} \|\nabla H\|_{L^2} \|H\|_{L^3} - \nu \int_{\mathbb{R}^3} |\nabla u|^2
\]

\[\leq \epsilon \|\nabla H\|_{L^2}^2 + C \|\nabla u\|_{L^2}^4 \|H\|_{L^2}^2 - \nu \int_{\mathbb{R}^3} |\nabla u|^2, \quad (3.13)
\]

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which implies that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |H|^2 + \int_{\mathbb{R}^3} \nu|\nabla H|^2 \leq C\|\nabla u\|_{L^2}^4 \|H\|_{L^2}^2.
\] (3.14)

An application of Gronwall’s inequality leads to
\[
\|H\|_{L^2}^2 + \int_0^T \nu\|\nabla H\|_{L^2}^2 \leq C K_2 \|H_0\|_{L^2}^2 \leq C e^{K_1(\gamma - 1)\frac{1}{2} E_0^\frac{1}{2}}.
\] (3.15)

Thanks to (1.13) and Lemma 2.1, using integration by parts, we derive that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla H|^2 + \int_{\mathbb{R}^3} |H_t|^2 + \int_{\mathbb{R}^3} |\nabla^2 H|^2 = \int_{\mathbb{R}^3} |H_t - \triangle H|^2 = \int_{\mathbb{R}^3} |H \cdot \nabla u - u \cdot \nabla H - H \text{div} u|^2 \leq C\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 H\|_{L^2}^2,
\] (3.16)

we consequently have
\[
\frac{d}{dt} \|\nabla H\|_{L^2}^2 + (\|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) \leq C\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2.
\] (3.17)

As before, Gronwall’s inequality leads to
\[
\sup_{0 \leq t \leq T} \|\nabla H\|_{L^2}^2 + \int_0^T (\|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) \leq C e^{K_1 M_2}.
\] (3.18)

Multiplying (3.17) by \(\sigma\), one has
\[
\frac{d}{dt} (\sigma \|\nabla H\|_{L^2}^2) + \sigma (\|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) \leq C\sigma\|\nabla u\|_{L^2}^4 \|\nabla H\|_{L^2}^2 + \sigma' \|\nabla H\|_{L^2}^2.
\] (3.19)

Again, using Gronwall’s inequality, we get
\[
\sup_{0 \leq t \leq T} (\sigma \|\nabla H\|_{L^2}^2) + \int_0^T \sigma (\|H_t\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2) \leq C e^{K_1} \int_0^T \|\nabla H\|_{L^2}^2 \leq C e^{2 K_1} \|H_0\|_{L^2}^2.
\] (3.20)

Combining (3.15), (3.18) and (3.20), we finish the proof of Lemma 3.3.

Lemma 3.4. Under the same assumption as in Proposition 3.1, it holds that
\[
\sup_{0 \leq t \leq T} \|H\|_{L^3}^3 + \int_0^T \left( \|H\|_{L^2}^\frac{3}{2} \|\nabla H\|_{L^2}^2 + \|H\|_{L^6}^3 \right) \leq \left( (\gamma - 1)^\frac{1}{2} E_0^\frac{1}{2} \right)^\frac{1}{3},
\] (3.21)

provided \((\gamma - 1)^\frac{1}{2} E_0^\frac{1}{2} \leq \varepsilon_1\), where
\[
\varepsilon_1 = \min \left\{ \left( 4C(M_2^\frac{3}{2} + K_1) \right)^{-\frac{9}{8}}, 1 \right\}.
\] (3.22)
Proof. Multiplying (1.1) by $3|H|H$, using integration by parts as in [29], we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} |H|^3 + 3\nu \int_{\mathbb{R}^3} |H||\nabla H|^2 + 3\nu \int_{\mathbb{R}^3} |H||\nabla|H|^2$$

$$\leq \nu \int_{\mathbb{R}^3} |H||\nabla H|^2 + \nu \int_{\mathbb{R}^3} |H||\nabla|H|^2 + C\|\nabla u\|_{L^2}^\frac{1}{2} \|H\|_{L^3}^\frac{3}{2}. \tag{3.23}$$

Noticing that

$$\|H\|_{L^9}^\frac{3}{2} \leq C\|H\|_{L^6}^\frac{3}{2} \leq C||\nabla H||_{L^2}, \tag{3.24}$$

$$\|H\|_{L^9}^\frac{3}{2} \leq C\|H\|_{L^3}^\frac{1}{2} \|H\|_{L^3}^\frac{1}{2} \leq C\|\nabla H\|^\frac{1}{2} \|H\|^\frac{1}{2}, \tag{3.25}$$

substituting (3.24) and (3.25) into (3.23), using Cauchy inequality, we thus deduce that

$$\frac{d}{dt} \int_{\mathbb{R}^3} |H|^3 + \int_{\mathbb{R}^3} |H||\nabla H|^2 \leq C \left( \frac{\gamma - 1}{2} E_0^\frac{1}{2} \right) \|\nabla u\|_{L^2}^4. \tag{3.26}$$

Integrating (3.26) over $[0, T]$, by the virtue of Sobolev embedding inequality, one can derived that

$$\sup_{0 \leq t \leq T} \|H\|_{L^3}^3 + \int_0^T \int_{\mathbb{R}^3} |H||\nabla H|^2$$

$$\leq C \|H_0\|_{L^2}^\frac{3}{2} \|H_0\|_{D_1}^\frac{3}{2} + CK_1 \left( \frac{\gamma - 1}{2} E_0^\frac{1}{2} \right)^\frac{2}{9}$$

$$\leq CM_2^\frac{3}{2} \left( \frac{\gamma - 1}{2} E_0^\frac{1}{2} \right)^\frac{2}{9} + CK_1 \left( \frac{\gamma - 1}{2} E_0^\frac{1}{2} \right)^\frac{2}{9}$$

$$\leq C(M_2^\frac{3}{2} + K_1) \left( \frac{\gamma - 1}{2} E_0^\frac{1}{2} \right)^\frac{2}{9}$$

$$\leq \frac{\gamma - 1}{2} E_0^\frac{1}{2}, \tag{3.27}$$

provided

$$(\gamma - 1) E_0^\frac{1}{2} E_0^\frac{1}{2} < \min \left\{ 4C(M_2^\frac{3}{2} + K_1)^{-9}, 1 \right\} \triangleq \epsilon_1. \tag{3.28}$$

Then estimate (3.27), together with (3.24), yields (3.21). This ends up the proof of Lemma 3.4. 

**Lemma 3.5.** Under the same assumption as in Lemma 3.4, we have

$$\sup_{0 \leq t \leq \sigma(T)} \|\rho^{\frac{\gamma}{2}} u\|_{L^2}^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq CK_3(\gamma - 1) E_0^\frac{1}{2}, \tag{3.29}$$

where $K_3 = C(\bar{\rho}) \left( 1 + (\gamma - 1)^{\frac{1}{2}} + (\gamma - 1)^{\frac{3}{2}} \right)$. 


Proof. Multiplying (1.2) by \( u \) and then integrating the resulting equality over \( \mathbb{R}^3 \), and using integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^2 + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2)
= \int_{\mathbb{R}^3} P \text{div} u + \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) u.
\] (3.30)

Integrating (3.30) over \((0, \sigma(T))\), one has

\[
\frac{1}{2} \sup_{0 \leq t \leq \sigma(T)} \|\rho^\frac{3}{2} u\|_{L^2}^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) dt
\leq \frac{1}{2} \int_{\mathbb{R}^3} \rho_0 |u_0|^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \text{Pdiv} u + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) u dt
\leq C \|\rho_0\|_{L^\frac{4}{3}}^\frac{3}{4} \|u_0\|_{L^6}^6 + \frac{\mu}{6} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} P^2 + \int_0^{\sigma(T)} \|u\|_{L^6} \|H\|_{L^3} \|\nabla H\|_{L^2}^2 dt
\leq C ((\gamma - 1) E_0)^{\frac{3}{2}} + \frac{\mu}{4} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + C(\bar{\rho})(\gamma - 1) E_0 + \int_0^{\sigma(T)} \|H\|_{L^3} \|\nabla H\|_{L^2}^2 dt
\leq C ((\gamma - 1) E_0)^{\frac{3}{2}} + \frac{\mu}{4} \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + C(\bar{\rho})(\gamma - 1) E_0 + C \left( (\gamma - 1)^{\frac{3}{2}} E_0^{\frac{1}{2}} \right)^{\frac{2}{\gamma + 1}}. \tag{3.31}
\]

It thus holds that

\[
\frac{1}{2} \sup_{0 \leq t \leq \sigma(T)} \|\rho^\frac{3}{2} u\|_{L^2}^2 + \int_0^{\sigma(T)} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 \right) dt
\leq C ((\gamma - 1) E_0)^{\frac{3}{2}} + C(\bar{\rho})(\gamma - 1) E_0 + C \left( (\gamma - 1)^{\frac{3}{2}} E_0^{\frac{1}{2}} \right)^{\frac{2}{\gamma + 1}}
\leq C K_3 (\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}}. \tag{3.32}
\]

Here we have used the condition \((\gamma - 1)^{\frac{3}{2}} E_0^{\frac{1}{2}} \leq 1\). We finish the proof of Lemma 3.5 \( \square \)

**Lemma 3.6.** Under the same assumption as in Proposition 3.1, it holds that

\[
A_1(T) \leq C K_4 (\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2
+ C \int_0^T \sigma \int_{\mathbb{R}^3} |\nabla u|^3 + C \gamma \int_0^T \sigma \int_{\mathbb{R}^3} P |\nabla u|^2 \tag{3.33}
\]

and

\[
A_2(T) \leq C K_5 \left( (\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{11}{18}} + C \gamma^2 \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P \nabla u|^2
+ C \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 + C A_1(T), \tag{3.34}
\]

provided that

\[
(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} \leq \varepsilon_2 \overset{\Delta}{=} \min \left\{ \varepsilon_1, (4C_1)^{-\frac{27}{42}} + (4C_2)^{-27} \right\}, \tag{3.35}
\]

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where \( K_4 \) and \( K_5 \) are given by

\[
\begin{align*}
K_4 &= K_2^3 M_2 + K_2^2 + K_1^2 K_1, \\
K_5 &= K_1 + K_2^2 (\gamma - 1)^{\frac{\alpha}{2}} + K_3^2 M_2.
\end{align*}
\]

(3.36) \hspace{1cm} (3.37)

**Proof.** The basic idea of the proof of this lemma is due to Hoff [12], Huang-Li-Xin [18], Li-Xu-Zhang [29] and Hou-Peng-Zhu [13]. Multiplying (3.1) by \( \sigma \dot{u} \), and integrating over \( \mathbb{R}^3 \), one has

\[
\int_{\mathbb{R}^3} \sigma \rho \dot{u}^2 = - \int_{\mathbb{R}^3} \sigma \dot{u} \cdot \nabla P + \mu \int_{\mathbb{R}^3} \sigma \Delta u \cdot \dot{u} + (\mu + \lambda) \int_{\mathbb{R}^3} \nabla \div \sigma \dot{u} + \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot \sigma \dot{u}
\]

\[
= \sum_{i=1}^{4} \mathcal{E}_i.
\]

(3.38)

Moreover, using integration by parts, we have

\[
\mathcal{E}_1 = \int_{\mathbb{R}^3} \sigma \dot{u} \cdot \nabla P
\]

\[
= \int_{\mathbb{R}^3} \sigma \div u_t P + \int_{\mathbb{R}^3} \sigma \div (u \cdot \nabla) P
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \sigma \div u P - \int_{\mathbb{R}^3} \sigma' \div u P - \int_{\mathbb{R}^3} \sigma \div u_t P + \int_{\mathbb{R}^3} \sigma \div (u \cdot \nabla) P
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \sigma \div u P - \sigma' \int_{\mathbb{R}^3} P \div u + (\gamma - 1) \sigma \int_{\mathbb{R}^3} P |\div u|^2 + \int_{\mathbb{R}^3} \sigma P |\nabla u|^2,
\]

(3.39)

\[
\mathcal{E}_2 = \int_{\mathbb{R}^3} \mu \sigma \Delta u \cdot \dot{u}
\]

\[
= - \mu \int_{\mathbb{R}^3} \sigma \nabla u \cdot \nabla u_t + \mu \int_{\mathbb{R}^3} \sigma \Delta u \cdot (u \cdot \nabla u)
\]

\[
\leq - \left( \frac{\mu}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \right) - \frac{\mu \sigma'}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + 2 \mu \sigma \int_{\mathbb{R}^3} |\nabla u|^3
\]

(3.40)

and

\[
\mathcal{E}_3 = \int_{\mathbb{R}^3} (\mu + \lambda) \sigma \nabla \div u \cdot \dot{u}
\]

\[
\leq - \frac{\mu + \lambda}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \sigma |\div u|^2 + \frac{\mu + \lambda}{2} \sigma' \int_{\mathbb{R}^3} |\div u|^2 + 2 (\mu + \lambda) \sigma \int_{\mathbb{R}^3} |\nabla u|^3.
\]

(3.41)

It remains to estimate \( \mathcal{E}_4 \). In fact,

\[
\mathcal{E}_4 = \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot (\sigma u_t + \sigma u \cdot \nabla u)
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot \sigma u - \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right)_t \cdot \sigma u.
\]
\[- \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot \sigma' u + \int_{\mathbb{R}^3} \sigma \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot (u \cdot \nabla u) \]

\[
\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot \sigma u + C \sigma \|H\|_{L^6} \|H_t\|_{L^2} \|\nabla u\|_{L^3} \\
+ \sigma' \|\nabla u\|_{L^2} \|H\|_{L^4}^2 + C \sigma \|H\|_{L^6} \|\nabla H\|_{L^6} \|u\|_{L^6} \|\nabla u\|_{L^2} 
\]

\[
\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot \sigma u + C \sigma \|\nabla H\|_{L^2}^4 + \sigma \|H_t\|_{L^2}^2 + \sigma' \|\nabla u\|_{L^3}^3 \\
+ C \sigma' \|\nabla u\|_{L^2}^2 + C \sigma' \|\nabla H\|_{L^2}^2 \|H\|_{L^3}^2 + C \sigma \|\nabla^2 H\|_{L^2}^2 + \sigma' \|\nabla u\|_{L^2} \|\nabla H\|_{L^2}^2. \quad (3.42)
\]

Substituting (3.39)-(3.42) into (3.38), we have

\[
\frac{d}{dt} \left\{ \int_{\mathbb{R}^3} \left( \frac{\mu}{2} \sigma \|\nabla u\|_{L^2}^2 + \frac{\mu + \lambda}{2} \sigma \|\text{div} u\|_{L^2}^2 \right) \\
+ \sigma \int_{\mathbb{R}^3} \text{div} u P - \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot \sigma u \right\} + \int_{\mathbb{R}^3} \sigma \rho |\dot{u}|^2 
\]

\[
\leq \frac{\mu \sigma}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + 2 \mu \sigma \int_{\mathbb{R}^3} |\nabla u|^3 + \frac{\mu + \lambda}{2} \sigma' \int_{\mathbb{R}^3} |\text{div} u|^2 + 2(\mu + \lambda) \sigma \int_{\mathbb{R}^3} |\nabla u|^3 \\
+ C \sigma \|\nabla H\|_{L^2}^4 + \sigma \|H_t\|_{L^2}^2 + \sigma' \|\nabla u\|_{L^3}^3 - \sigma' \int_{\mathbb{R}^3} P \text{div} u + \gamma \sigma \int_{\mathbb{R}^3} P |\nabla u|^2 \\
+ C \sigma' \|\nabla u\|_{L^2}^2 + C \sigma' \|\nabla H\|_{L^2}^2 \|H\|_{L^3}^2 + C \sigma \|\nabla^2 H\|_{L^2}^2 + \sigma' \|\nabla u\|_{L^2} \|\nabla H\|_{L^2}^2. \quad (3.43)
\]

Integrate (3.43) over \((0, T)\), we have

\[
\sup_{0 \leq t \leq T} \left( \frac{\mu}{2} \sigma \|\nabla u\|_{L^2}^2 + \frac{\mu + \lambda}{2} \sigma \|\text{div} u\|_{L^2}^2 \right) + \int_0^T \sigma \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 
\]

\[
\leq \sigma \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot u + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + C(4 \mu + \lambda + 1) \int_0^T \int_{\mathbb{R}^3} |\nabla u|^3 \\
+ \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |P \text{div} u| + \gamma \int_0^T \sigma \int_{\mathbb{R}^3} P |\nabla u|^2 + \int_0^T \sigma \|\nabla H\|_{L^2}^4 + \int_0^T \sigma \|H_t\|_{L^2}^2 \\
+ \int_0^{\sigma(T)} \|\nabla H\|_{L^2}^2 \|H\|_{L^3}^2 + C(\gamma - 1) E_0 + \int_0^T \sigma \|\nabla^2 H\|_{L^2}^2 + \int_0^T \sigma \|\nabla H\|_{L^2}^2 \|\nabla u\|_{L^2}^4 \\
\leq \epsilon \mu \sigma \|\nabla u\|_{L^2}^2 + \frac{C(\epsilon)}{\mu} \sigma \|H\|_{L^4}^4 + C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + C \int_0^T \int_{\mathbb{R}^3} |\nabla u|^3 \\
+ C \gamma \int_0^T \sigma \int_{\mathbb{R}^3} P |\nabla u|^2 + C(\tilde{\rho}, \epsilon)(\gamma - 1) E_0 + C K_0\sigma M_2 \left( (\gamma - 1) \frac{1}{2} \gamma_0 \right)^{2} \\
+ C K_2^2 (\gamma - 1) \frac{1}{2} \gamma_0 \gamma_0 + K_2^2 \left( (\gamma - 1) \frac{1}{2} \gamma_0 \gamma_0 \right)^{1+\frac{\gamma}{2}} + K_2^2 K_1 \left( (\gamma - 1) \frac{1}{2} \gamma_0 \gamma_0 \right)^{1+\frac{\gamma}{2}}, \quad (3.44)
\]

which leads to

\[
\sup_{0 \leq t \leq T} \left( \frac{\mu}{4} \sigma \|\nabla u\|_{L^2}^2 + \frac{\mu + \lambda}{4} \sigma \|\text{div} u\|_{L^2}^2 \right) + \int_0^T \sigma \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 dt
\]

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\[
\leq CK_4(\gamma - 1)\frac{1}{2}E_0^2 + C\int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 + C\int_0^T \sigma \int_{\mathbb{R}^3} |\nabla u|^3 \\
+ C\gamma \int_0^T \sigma \int_{\mathbb{R}^3} P|\nabla u|^2.
\] (3.45)

Then, by the virtue of (3.11), we get (3.33).

Next, operating \( \partial_t + \text{div}(u) \) to the both sides of the \( j \)th equation of (1.1)2, yields that

\[
(\rho \dot{u}^j)_t + \text{div}(\rho u \dot{u}^j) - \mu \Delta \dot{u}^j - (\mu + \lambda) \partial_j \text{div} \dot{u} \\
= \mu \partial_t (-\partial_t u \cdot \nabla u^j + \text{div} u \partial_t u^j) - \mu \text{div}(\partial_t u \partial_t u^j) - (\mu + \lambda) \partial_j [\partial_t u \cdot \nabla u^j - |\text{div} u|^2] \\
- \text{div}(\partial_j u (\mu + \lambda) \text{div} u) + (\gamma - 1) \partial_j (P \text{div} u) + \text{div}(P \partial_j u) \\
+ \left[ (H \cdot \nabla H^j)_t + \text{div}(uH \cdot \nabla H^j) \right] - \frac{1}{2} \left[ (\partial_j |H|^2)_t + \text{div}(u\partial_j |H|^2) \right].
\] (3.46)

Multiplying (3.46) by \( \sigma^m \dot{u}^j \) for \( m \geq 0 \), and integrating by parts over \( \mathbb{R}^3 \), we obtain after summing them with respect to \( j \) that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2 + \mu \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + (\mu + \lambda) \int_{\mathbb{R}^3} \sigma^m |\text{div} \dot{u}|^2 \\
= \frac{m}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \mu \int_{\mathbb{R}^3} \sigma^m \partial_t \dot{u}^j (\partial_t u \cdot \nabla u^j - \text{div} u \partial_t u^j) \\
+ \mu \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_k u^j \partial_t u^j + \int_{\mathbb{R}^3} \sigma^m \partial_j \dot{u}^j \left[(\mu + \lambda) \partial_t u \cdot \nabla u^j - \mu |\text{div} u|^2\right] \\
+ (\mu + \lambda) \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_k u^j \partial_t u^j - \int_{\mathbb{R}^3} \sigma^m \partial_k \dot{u}^j \partial_j u^k P \\
- (\gamma - 1) \int_{\mathbb{R}^3} \sigma^m \partial_j \dot{u}^j \partial_k u^k P - \frac{1}{2} \int_{\mathbb{R}^3} \sigma^m \dot{u}^j \left[(\partial_j |H|^2)_t + \text{div}(u\partial_j |H|^2)\right] \\
+ \int_{\mathbb{R}^3} \sigma^m \dot{u}^j \left[(H \cdot \nabla H^j)_t + \text{div}(uH \cdot \nabla H^j)\right] \\
\leq \frac{m}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{\mu}{2} \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + \frac{\mu + \lambda}{2} \int_{\mathbb{R}^3} \sigma^m |\text{div} \dot{u}|^2 \\
+ C\mu \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 + C(\mu + \lambda) \left(1 + \frac{\lambda}{\mu}\right) \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 + C\gamma^2 \int_{\mathbb{R}^3} \sigma^m |P \nabla u|^2 \\
+ C_1 \sigma^m |H|_{L^2}^2 |\nabla H|_{L^2}^2 + C \sigma^m \left(\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4\right) |\nabla^2 H|_{L^2}^2 \\
\leq \frac{m}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{\mu}{2} \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 + \frac{\mu + \lambda}{2} \int_{\mathbb{R}^3} \sigma^m |\text{div} \dot{u}|^2 \\
+ C\mu \int_{\mathbb{R}^3} \sigma^m |\nabla u|^4 + C\gamma^2 \int_{\mathbb{R}^3} \sigma^m |P \nabla u|^2 + C_1 \sigma^m \left(\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4\right) |\nabla^2 H|_{L^2}^2 \\
+ C \sigma^m \left(\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4\right) |\nabla^2 H|_{L^2}^2,
\] (3.47)

which implies that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \sigma^m \rho |\dot{u}|^2 + \int_{\mathbb{R}^3} \sigma^m |\nabla \dot{u}|^2 - C_1 \sigma^m \left(\|\nabla \dot{u}\|_{L^2}^4 + \|\nabla H\|_{L^2}^4\right) |\nabla^2 H|_{L^2}^2
\]
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \sigma^m |H_t|^2 + \nu \int_{\mathbb{R}^3} \sigma^m |\nabla H_t|^2 - \frac{m}{2} \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} |H_t|^2 \\
= \int_{\mathbb{R}^3} \sigma^m (H_t \cdot \nabla u - u \cdot \nabla H_t - H_t \text{div} u) \cdot H_t \\
+ \int_{\mathbb{R}^3} \sigma^m (H \cdot \nabla \dot{u} - \dot{u} \cdot \nabla H - H \text{div} \dot{u}) \cdot H_t \\
- \int_{\mathbb{R}^3} \sigma^m [H \cdot \nabla (u \cdot \nabla u) - (u \cdot \nabla u) \cdot \nabla H - H \text{div}(u \cdot \nabla u)] \cdot H_t \\
\triangleq \sum_{i=1}^{3} N_i. \tag{3.50}
\]

It follows from Cauchy inequality, Sobolev inequality and (3.2) that
\[
N_1 \leq \epsilon \nu \sigma^m \|\nabla H_t\|_{L^2}^2 + C \sigma^m \|H_t\|_{L^2}^2 \|\nabla u\|_{L^2}^4,
\]
\[
N_2 \leq C \|H\|_{L^3} \sigma^m (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)
\]
\[
\leq C_2 \left( (\gamma - 1) \frac{1}{\beta} \overline{E}_0^\frac{1}{\beta} \right) \sigma^m (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2),
\]
\[
N_3 \leq C \sigma^m \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \|\nabla H\|_{L^2} \|\nabla H_t\|_{L^2}. \tag{3.51}
\]

Thanks to (2.46), (3.6) and Sobolev inequality, we deduce that
\[
\|\nabla u\|_{L^6} \leq C (\|\rho \dot{u}\|_{L^2} + \|P\|_{L^2} + \|H \cdot \nabla H\|_{L^2})
\]
\[
\leq C (\overline{\rho}) \left( \|\rho \frac{1}{\beta} \dot{u}\|_{L^2} + \|H\|_{L^3} \|\nabla^2 H\|_{L^2} \right) + C (\overline{\rho}) (\gamma - 1) \frac{1}{\beta} \overline{E}_0^\frac{1}{\beta}. \tag{3.52}
\]

Combining (3.51) and (3.52), we get
\[
N_3 \leq \frac{\nu}{8} \sigma^m \|\nabla H_t\|_{L^2}^2 + C \sigma^m \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \|H\|_{L^3} \|\nabla^2 H\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^2 \left( \|\rho \frac{1}{\beta} \dot{u}\|_{L^2} + \|H\|_{L^3} \|\nabla H\|_{L^2} \right) (\gamma - 1) \frac{1}{\beta} \overline{E}_0^\frac{1}{\beta}. \tag{3.53}
\]

Consequently, substituting (3.51)-(3.53) into (3.50) yields that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \sigma^m |H_t|^2 + \int_{\mathbb{R}^3} \sigma^m |\nabla H_t|^2 - C_2 \nu \sigma^m \left( (\gamma - 1) \frac{1}{\beta} \overline{E}_0^\frac{1}{\beta} \right) \left( \|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \right) \\
\leq \frac{m}{2} C \sigma^{m-1} \sigma' \int_{\mathbb{R}^3} |H_t|^2 + C \sigma^m \|\nabla u\|_{L^2}^2 \|\rho \frac{1}{\beta} \dot{u}\|_{L^2}^2 \|\nabla H\|_{L^2}^2 \\
+ C \sigma^m \left( \|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4 \right) \left( \|H\|_{L^3}^2 \|\nabla^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) + \frac{1}{2} \int_{\mathbb{R}^3} \sigma^m |H_t|^2.
\]
$$+ C_\sigma^m \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 (\gamma - 1)^{\frac{1}{2}} E_0^3.$$  
(3.54)

Thus, taking \( m = 2 \) in (3.48) and (3.54), and integrating the resulting equation over \((0, T)\), we deduce that

$$\sup_{0 \leq t \leq T} \sigma^2 (\|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2) + \int_0^T \sigma^2 (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2)$$

$$\leq C K_5 \left((\gamma - 1)^{\frac{1}{2}} E_0^3\right)^{\frac{11}{16}} + C \gamma^2 \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^2$$

$$+ C \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 + C A_1(T),$$

(3.55)

provided that

$$(\gamma - 1)^{\frac{1}{2}} E_0^3 \leq \varepsilon_2 \triangleq \min \left\{ \varepsilon_1, (4C_1)^{-\frac{44}{27}} + (4C_2)^{-27} \right\},$$

(3.56)

where we have used the following inequalities:

$$\int_0^T \sigma^2 \|\nabla u\|_{L^2}^2 \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 \|\nabla H\|_{L^2}^2$$

$$\leq \sup_{0 \leq t \leq T} \left( \sigma^2 \|\rho^{\frac{1}{2}} \dot{u}\|_{L^2}^2 \right) \int_0^T (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4)$$

$$\leq C K_1 \left((\gamma - 1)^{\frac{1}{2}} E_0^3\right)^{\frac{11}{16}} + C K^3 M_2 \left((\gamma - 1)^{\frac{1}{2}} E_0^3\right)^{\frac{5}{4}},$$

(3.57)

$$\int_0^T \sigma^2 \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 (\gamma - 1)^{\frac{1}{2}} E_0^3 \int_0^T \|\nabla H\|_{L^2}^2$$

$$\leq C K_2 \left((\gamma - 1)^{\frac{1}{2}} E_0^3\right)^{\frac{23}{27}} (\gamma - 1)^{\frac{1}{2}} E_0^3,$$

(3.58)

$$\int_0^T \sigma^2 (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4) (\|H\|_{L^3}^2 \|\nabla^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2)$$

$$\leq C \sup_{0 \leq t \leq T} \left( \sigma^2 \|\nabla^2 H\|_{L^2}^2 \left( \sup_{0 \leq t \leq T} \|H\|_{L^3}^3 \right)^{\frac{2}{3}} \right) \int_0^T \|\nabla u\|_{L^2}^4$$

$$+ \sup_{0 \leq t \leq T} \left( \sigma^2 \|H_t\|_{L^2}^2 \right) \int_0^T \|\nabla H\|_{L^2}^4$$

$$\leq C K_1 \left((\gamma - 1)^{\frac{1}{2}} E_0^3\right)^{\frac{23}{27}} + C K^3 M_2 \left((\gamma - 1)^{\frac{1}{2}} E_0^3\right)^{\frac{5}{4}}$$

$$\leq C (K_1 + K^3 M_2) \left((\gamma - 1)^{\frac{1}{2}} E_0^3\right)^{\frac{11}{16}}.$$

(3.59)

Here, to obtained (3.57 - 3.59), (3.11) and (3.12) have been used.

Furthermore, it holds from (3.13) that

$$\Delta H = H_t - (H \cdot \nabla u - u \cdot \nabla H - \text{div} u H),$$

(3.60)
Due to (3.11), (3.55) and (3.62), we finally get (3.34). And Lemma 3.6 is proved.

Under the same assumption as in Proposition 3.1, it holds that

\[ \sup_{0 \leq t \leq T} \sigma^2 \| \nabla^2 H \|^2_{L^2} \leq C \sup_{0 \leq t \leq T} \sigma^2 \left( \| \nabla H \|^2_{L^2} \| \nabla u \|^4_{L^2} + \| H_t \|^2_{L^2} \right) \]

\[ \leq C \sup_{0 \leq t \leq \sigma(T)} \left( \sigma \| \nabla H \|^2_{L^2} \right) \left( \sigma^2 \| \nabla u \|^4_{L^2} \right) + C \sup_{\sigma(T) \leq t \leq T} \sigma^2 \left( \| \nabla H \|^2_{L^2} \| \nabla u \|^4_{L^2} \right) \]

\[ + C \sigma(T) \left( \sigma \| \nabla H \|^2_{L^2} \right) \left( \sigma \| \nabla u \|^4_{L^2} \right) \]

\[ \leq C \sum_{k=5}^{\bar{k}} \left( (\gamma - 1) \frac{1}{5} \left( \int_{R^3} |P\nabla u|^2 \right) \right)^{\frac{11}{18}} + C \gamma^2 \int_0^T \int_{R^3} \sigma^2 |P\nabla u|^2 \]

\[ + C \int_0^T \int_{R^3} \sigma^2 |\nabla u|^4 + C A_1(T) \]

\[ \leq C \sum_{k=5}^{\bar{k}} \left( (\gamma - 1) \frac{1}{5} \left( \int_{R^3} |P\nabla u|^2 \right) \right)^{\frac{11}{18}} + C \gamma^2 \int_0^T \int_{R^3} \sigma^2 |P\nabla u|^2 \]

\[ + C \int_0^T \int_{R^3} \sigma^2 |\nabla u|^4 + C A_1(T). \] (3.62)

Due to (3.11), (3.55) and (3.62), we finally get (3.34). And Lemma 3.6 is proved.

The following lemma will play a crucial role in the proof of the upper bound of the density.

**Lemma 3.7.** Under the same assumption as in Proposition 3.1, it holds that

\[ \sup_{0 \leq t \leq \sigma(T)} \left( \| \nabla u \|^2_{L^2} + \| \text{div} u \|^2_{L^2} \right) + \int_0^{\sigma(T)} \int_{R^3} \rho |\dot{u}|^2 \leq K_6, \] (3.63)

\[ \sup_{0 \leq t \leq \sigma(T)} \int_{R^3} \left( \| \rho \frac{1}{2} |\dot{u}|^2 \|^2_{L^2} + \| H_t \|^2_{L^2} \right) + \int_0^{\sigma(T)} \int_{R^3} \rho |\dot{u}|^2 \leq K_7, \] (3.64)

and

\[ \sup_{0 \leq t \leq T} \| \nabla u \|^2_{L^2} + \int_0^T \rho |\dot{u}|^2 \leq C(K_6 + 1), \] (3.65)

\[ \sup_{0 \leq t \leq T} \sigma \left( \| \rho \frac{1}{2} |\dot{u}|^2 \|^2_{L^2} + \| H_t \|^2_{L^2} \right) + \int_0^T \sigma |\nabla u|^2_{L^2} + \| \nabla H_t \|^2_{L^2} \leq C(K_7 + 1), \] (3.66)

provided that

\[ (\gamma - 1) \frac{1}{5} E_0^\gamma \leq \varepsilon_3 \leq \min \{ \varepsilon_1, \varepsilon_2, 4C_3(\bar{\rho})^{-27}, 1 \}, \] (3.67)
where

\[ K_6 = C(\bar{p})(K_2 M_2 + K_1 + K_2^2 + M_2^2 M_1^2 + 1) + C(\bar{p})(\gamma - 1)^\frac{3}{5} + CK_3(\gamma - 1)^\frac{3}{5} \]
\[ + C(K_2 M_2(\gamma - 1)^\frac{3}{5} + CK_3(\gamma - 1)^\frac{1}{5} + (\gamma - 1)^\frac{1}{5} + C(K_3 + 1)(\gamma - 1), \tag{3.68} \]
\[ K_7 = 2 \max\{K_1^2, K_7^2\}, \tag{3.69} \]
\[ K_1^2 = CK_6 + CK_7^2 + C(K_6^2 K_2 M_2 + CK_6^2(\gamma - 1)^\frac{1}{5} + C(K_6^2 + K_2^2 M_2^2) \]
\[ + C(\gamma - 1)^\frac{1}{5}(K_6^2 + K_2^2 K_2 M_2 + K_6^2(\gamma - 1)^\frac{1}{5}) \frac{1}{5}, \]
\[ K_7^2 = CK_2 M_2 + CK_1^2 + CK_2^2 M_2 + C(K_6^2 + CK_2 M_2^2) K_2^2 + CK_6 K_2^2(\gamma - 1)^\frac{3}{5}. \tag{3.70} \]

**Proof.** Multiplying (3.71) by \( u_t \), we have

\[ \rho |\dot{u}|^2 + \frac{1}{2}(\mu |\nabla u|^2 + (\mu + \lambda)|\text{div} u|^2) - \left( P\text{div} u + \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \cdot u \right)_t \]
\[ = -\left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right)_t \cdot u - P_t \text{div} u + \rho u \cdot \nabla u \cdot \dot{u}. \tag{3.71} \]

Integrating (3.71) over \( \mathbb{R}^3 \), one has

\[ \frac{1}{2} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 dt + \int_{\mathbb{R}^3} \left( \mu |\nabla u|^2 + (\mu + \lambda)|\text{div} u|^2 - P\text{div} u - \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right) \right) \cdot u \]
\[ + \int_{\mathbb{R}^3} \rho |\dot{u}|^2 = \int_{\mathbb{R}^3} \rho \cdot (u \cdot \nabla u) - \int_{\mathbb{R}^3} \left( H \cdot \nabla H - \frac{1}{2} \nabla |H|^2 \right)_t \cdot u - \int_{\mathbb{R}^3} P_t \text{div} u \]
\[ \leq C(\bar{p})\|\rho^\frac{1}{2} \dot{u}\|_{L^2} \|\rho^\frac{1}{2} u\|_{L^2} \|\nabla u\|_{L^6} + \int_{\mathbb{R}^3} \text{div}\text{div}(P u) + (\gamma - 1) \int_{\mathbb{R}^3} P |\text{div} u|^2 \]
\[ + \|H\|_{L^3} \|H_t\|_{L^2} \|\nabla u\|_{L^6} \]
\[ \leq C(\bar{p})\|\rho^\frac{1}{2} \dot{u}\|_{L^2} \|\rho^\frac{1}{2} u\|_{L^2} \left( \|\rho^\frac{1}{2} \dot{u}\|_{L^2} + \|P\|_{L^6} + \|H \cdot \nabla H\|_{L^2} \right) - \int_{\mathbb{R}^3} P u \nabla \text{div} u \]
\[ + C(\bar{p})(\gamma - 1)\|\nabla u\|_{L^2}^2 + \|H\|_{L^3} \|H_t\|_{L^2} \left( \|\rho^\frac{1}{2} \dot{u}\|_{L^2} + \|P\|_{L^6} + \|H \cdot \nabla H\|_{L^2} \right) \]
\[ \leq C(\bar{p})\|\rho^\frac{1}{2} \dot{u}\|_{L^2} \|\rho^\frac{1}{2} u\|_{L^2} A_5^\frac{1}{2} (\sigma(T)) \]
\[
\leq \frac{1}{4} \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 + C_3(\bar{\rho}) \| \rho \frac{1}{2} \dot{u} \|_{L^2}^2 A_0^\frac{1}{4}(\sigma(T)) + C(\bar{\rho}) A_0^\frac{1}{4}(\sigma(T)) (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \\
+ C(\bar{\rho}) A_0^\frac{1}{4}(\sigma(T)) \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \| \nabla^2 H \|_{L^2}^2 + C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \| \nabla u \|_{L^2}^2 \\
+ C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \| \nabla u \|_{L^2}^2 + C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \| \nabla^2 H \|_{L^2}^2 \\
+ C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \| \nabla u \|_{L^2}^2 + C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \| \nabla u \|_{L^2}^2 \\
+ C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \| \nabla u \|_{L^2}^2 + C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \| \nabla^2 H \|_{L^2}^2, \\
\text{(3.72)}
\]

Integrating (3.72) over \((0, \sigma(T))\), we get

\[
\frac{\mu}{2} \sup_{0 \leq t \leq \sigma(T)} \| \nabla u \|_{L^2}^2 + \frac{1}{2} \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \rho \dot{u} \, d^3x \\
\leq C(\bar{\rho}) A_0^\frac{1}{4}(\sigma(T)) (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} + C(\bar{\rho}) A_0^\frac{1}{4}(\sigma(T)) \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \int_{0}^{\sigma(T)} \| \nabla^2 H \|_{L^2}^2 \\
+ C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \int_{0}^{\sigma(T)} \| \nabla u \|_{L^2}^2 + C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \int_{0}^{\sigma(T)} \| \nabla u \|_{L^2}^2 \\
+ C(\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \int_{0}^{\sigma(T)} \| \nabla^2 H \|_{L^2}^2 + (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \int_{0}^{\sigma(T)} \| \nabla u \|_{L^2}^2 \\
+ (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} + C(\bar{\rho}) \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \int_{0}^{\sigma(T)} \| \nabla u \|_{L^2}^2 \\
+ C(\bar{\rho}) \int_{0}^{\sigma(T)} \| \nabla H \|_{L^2}^2 + C(\bar{\rho})(\gamma - 1) \int_{0}^{\sigma(T)} \| \nabla u \|_{L^2}^2 + (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \\
+ C \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \int_{0}^{\sigma(T)} \| H_t \|_{L^2}^2 + C(\bar{\rho}) \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \int_{0}^{\sigma(T)} \| \nabla^2 H \|_{L^2}^2 \\
+ \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 - P \text{div} u - (H \cdot \nabla H - \frac{1}{2} \nabla |H|^2)) \cdot u \bigg|_{t=0} \\
\leq C(\bar{\rho})(K_2 M_2 + K_3 + K_2^\frac{1}{2} + M_2^\frac{3}{2} M_1^\frac{1}{2} + 1) + C(\bar{\rho})(\gamma - 1) \frac{1}{\frac{1}{2}} + CK_3(\gamma - 1) \frac{1}{\frac{1}{2}} \\
+ CK_2 M_2(\gamma - 1) \frac{1}{\frac{1}{2}} + CK_3(\gamma - 1) \frac{1}{\frac{1}{2}} + (\gamma - 1) \frac{1}{\frac{1}{2}} + C(K_3 + 1)(\gamma - 1), \\
\text{(3.73)}
\]

provided \((\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \leq \min\{4C_3(\bar{\rho})^{-27}, 1\}\), then we get (3.63).

Taking \(m = 1\) in (3.43), one has

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho \sigma |\dot{u}|^2 + \int_{\mathbb{R}^3} \sigma |\nabla \dot{u}|^2 - C_1 \sigma \left( (\gamma - 1) \frac{1}{\frac{1}{2}} E_0^\frac{1}{2} \right) \frac{\gamma}{\frac{1}{2}} \| \nabla H_t \|_{L^2}^2 
\]
\[
\leq C\sigma' \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + C\mu \int_{\mathbb{R}^3} \sigma |\nabla u|^4 + C\gamma^2 \int_{\mathbb{R}^3} \sigma |P\nabla u|^2 \\
+ C\sigma (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4) \|\nabla^2 H\|_{L^2}^2.
\]

Integrating (3.74) over \((0, \sigma(T))\), we get

\[
\int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^2 - C_1 \int_{0}^{\sigma(T)} \sigma \left( (\gamma - 1)^{1/4} E_0^{1/4} \right)^{2/3} \|\nabla H_t\|_{L^2}^2 \\
\leq C \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + C\mu \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 + C\gamma^2 \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |P\nabla u|^2 \\
+ C \int_{0}^{\sigma(T)} \sigma (\|\nabla u\|_{L^2}^4 + \|\nabla H\|_{L^2}^4) \|\nabla^2 H\|_{L^2}^2 \\
\leq CK_6 + CK_6^{1/2} K_2 M_2 + CK_6^{1/2} (\gamma - 1)^{1/4} + C \left( \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} P^4 \right)^{1/2} \left( \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{1/2} \\
+ C \left[ \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 \right)^2 + \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla H\|_{L^2}^2 \right)^2 \right] \int_{0}^{\sigma(T)} \sigma \|\nabla^2 H\|_{L^2}^2 \\
\leq CK_6 + CK_6^{1/2} + CK_6^{1/2} K_2 M_2 + CK_6^{1/2} (\gamma - 1)^{1/4} + C(K_6^2 + K_2^2 M_2^2) \\
+ C(\gamma - 1)^{1/4}(K_6^{3/4} + K_6^{3/4} K_2 M_2 + K_6^{3/4} (\gamma - 1)^{3/4})^{1/2} \\
\triangleq K_7^1.
\]

where we have used the condition \((\gamma - 1)^{1/4} E_0^{1/4} \leq 1\) and the following estimate

\[
\int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla u|^4 \\
\leq \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^4}^4 \int_{0}^{\sigma(T)} \sigma \|\nabla u\|_{L^6}^3 \\
\leq CK_6^{1/2} \int_{0}^{\sigma(T)} \sigma \left( \|\rho^{1/2} \dot{u}\|^4_{L^2} + \|P\|^4_{L^6} + \|H \cdot \nabla H\|^3_{L^2} \right) \\
\leq CK_6^{1/2} \left( \sup_{0 \leq t \leq \sigma(T)} \sigma^2 \|\rho^{1/2} \dot{u}\|^2_{L^2} \right)^{1/2} \int_{0}^{\sigma(T)} \|\rho^{1/2} \dot{u}\|_{L^2}^2 + CK_6^{1/2} (\gamma - 1)^{1/4} E_0^{1/4} \\
+ CK_6^{1/2} \int_{0}^{\sigma(T)} \sigma \|H\|_{L^3}^3 \|\nabla^2 H\|_{L^2}^3 \\
\leq CK_6^{1/2} (A_2(T))^{1/4} K_6 + CK_6^{1/2} (\gamma - 1)^{1/4} E_0^{1/4} \\
+ CK_6^{1/2} \left( \sup_{0 \leq t \leq \sigma(T)} \sigma^2 \|\nabla^2 H\|_{L^2}^2 \right)^{1/2} \int_{0}^{\sigma(T)} \|\nabla^2 H\|_{L^2}^2 \\
\leq CK_6^{1/2} (A_2(T))^{1/4} K_6 + CK_6^{1/2} (\gamma - 1)^{1/4} E_0^{1/4} + CK_6^{1/2} (A_2(T))^{1/4} K_2 M_2 \\
\leq CK_6^{3/4} + CK_6^{3/4} K_2 M_2 + CK_6^{3/4} (\gamma - 1)^{3/4}.
\]
Similarly, Taking \( m = 1 \) in (3.54), we get

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \sigma |H_t|^2 + \int_{\mathbb{R}^3} \sigma |\nabla H_t|^2 - C_2 \sigma \left( (\gamma - 1) \frac{1}{\alpha} E_0^\frac{1}{\alpha} \right) \left( \|\nabla \hat{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \right)
\leq \frac{C}{2} \sigma' \int_{\mathbb{R}^3} |H_t|^2 + C \sigma \|\nabla u\|_{L^2}^2 \|\rho \frac{1}{\alpha} \hat{u}\|_{L^2}^2 \|\nabla H\|_{L^2}^2
+ C \sigma \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \left( \|H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right)
+ C \sigma \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 (\gamma - 1) \frac{1}{\alpha} E_0^\frac{1}{\alpha}
\]

and

\[
\int_{\mathbb{R}^3} \sigma |H_t|^2 + \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma |\nabla H_t|^2 - C_2 \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} \sigma \left( (\gamma - 1) \frac{1}{\alpha} E_0^\frac{1}{\alpha} \right) \left( \|\nabla \hat{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \right)
\leq C \int_{0}^{\sigma(T)} \int_{\mathbb{R}^3} |H_t|^2 + C \int_{0}^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^2 \|\rho \frac{1}{\alpha} \hat{u}\|_{L^2}^2 \|\nabla H\|_{L^2}^2
+ C \int_{0}^{\sigma(T)} \sigma \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \left( \|H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right)
+ C \int_{0}^{\sigma(T)} \sigma \|\nabla u\|_{L^2}^2 \|\nabla H\|_{L^2}^2 (\gamma - 1) \frac{1}{\alpha} E_0^\frac{1}{\alpha}
\]

\[
\leq CK_2 M_2 + C \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 \right) \left( \sup_{0 \leq t \leq \sigma(T)} \sigma \|\nabla H\|_{L^2}^2 \right) \int_{0}^{\sigma(T)} \|\rho \frac{1}{\alpha} \hat{u}\|_{L^2}^2
+ C \left( (\gamma - 1) \frac{1}{\alpha} E_0^\frac{1}{\alpha} \right) \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 \right)^2 \int_{0}^{\sigma(T)} \sigma \|\nabla^2 H\|_{L^2}^2
+ C \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla H\|_{L^2}^2 \right) \left( \sup_{0 \leq t \leq \sigma(T)} \sigma \|\nabla H\|_{L^2}^2 \right) \int_{0}^{\sigma(T)} \|\nabla^2 H\|_{L^2}^2
+ C \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 + \sup_{0 \leq t \leq \sigma(T)} \|\nabla H\|_{L^2}^2 \right)^2 \int_{0}^{\sigma(T)} \sigma \|H_t\|_{L^2}^2
+ C \left( \sup_{0 \leq t \leq \sigma(T)} \sigma \|\nabla H\|_{L^2}^2 \right) \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2}^2 \right) (\gamma - 1) \frac{1}{\alpha} E_0^\frac{1}{\alpha}
\]

\[
\leq CK_2 M_2 + CK_6^2 K_2^2 + CK_2^2 M_2^2 + C(K_6^2 + CK_2^2 M_2^2) K_2^2 + CK_6 K_2^2 (\gamma - 1) \frac{1}{\alpha}
\triangleq K_7.
\]

Combining (3.75) and (3.78), we deduce

\[
\sup_{0 \leq t \leq \sigma(T)} t \left( \|\rho \frac{1}{\alpha} \hat{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) + \int_{0}^{\sigma(T)} t \left( \|\nabla \hat{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \right) \leq K_7,
\]

provided that

\[
(\gamma - 1) \frac{1}{\alpha} E_0^\frac{1}{\alpha} \leq \min \left\{ \varepsilon_2, (4C_1)^{-\frac{27}{2}} + (4C_2)^{-27}, 1 \right\}.
\]
where $K_7 = 2 \max\{K_1^1, K_2^2\}$. And this leads to (3.64). By (3.2), (3.63) and (3.64), we can get (3.65) and (3.66). Then we finish the proof of Lemma 3.7.

Lemma 3.8. Under the same assumption as in Proposition 3.1, we get that

$$A_4(\sigma(T)) + A_5(\sigma(T)) \leq \left( (\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}}, \quad (3.81)$$

provided that

$$(\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \leq \varepsilon_4 \triangleq \min \left\{ \varepsilon_3, \frac{1}{K_6^{54}}, \left( C(\bar{\rho})(\gamma - 1)^{\frac{1}{\gamma}} K_6^{\frac{3}{2}} \right)^{-\frac{9}{8}}, 1 \right\}. \quad (3.82)$$

Proof. It follows from (3.2) and (3.64) that

$$A_4(\sigma(T)) \triangleq \sup_{0 \leq t \leq \sigma(T)} \sigma^{\frac{1}{\gamma}} \|\nabla u\|_{L^2} \leq \left( \sup_{0 \leq t \leq \sigma(T)} \sigma \|\nabla u\|_{L^2} \right)^{\frac{1}{\gamma}} \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla u\|_{L^2} \right) \frac{1}{\gamma} \leq K_6^{\frac{3}{2}} \left( (\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}},$$

$$\leq \left( (\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}}, \quad (3.83)$$

provided $(\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \leq \min \left\{ \varepsilon_3, \frac{1}{K_6^{54}}, 1 \right\}$.

Now, to end up the proof of Lemma 3.8, it remains to estimate $A_5(\sigma(T))$. Due to (3.64), we deduce that

$$A_5(\sigma(T)) = \sup_{0 \leq t \leq \sigma(T)} \int_{\mathbb{R}^3} |u|^3 \rho \, dt \leq \sup_{0 \leq t \leq \sigma(T)} \left( \|\rho\|_{L^2} \|u\|_{L^6}^3 \right) \leq C(\bar{\rho})(\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} K_6^{\frac{3}{2}} \leq C(\bar{\rho})K_6^{\frac{3}{2}} (\gamma - 1)^{\frac{1}{\gamma}} (\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \leq \left( (\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}}, \quad (3.84)$$

provided that

$$(\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \leq \min \left\{ \left( C(\bar{\rho})(\gamma - 1)^{\frac{1}{\gamma}} K_6^{\frac{3}{2}} \right)^{-\frac{9}{8}}, 1 \right\}. \quad (3.85)$$

Lemma 3.9. Under the same assumption as in Proposition 3.1, we get that

$$A_1(T) + A_2(T) \leq \left( (\gamma - 1)^{\frac{1}{\gamma}} E_0^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}}, \quad (3.86)$$
provided

\[(\gamma - 1)^{\frac{3}{4}} E_0^{\frac{1}{4}} \leq \varepsilon_5 \triangleq \min \{\varepsilon_4, (C_4 K_0)^{-1}, 1\}, \quad (3.87)\]

where

\[K_9 = (K_4 + K_5) + K_3 + K_8 \gamma - 1)^{\frac{3}{17}} + \sqrt{K_8^2 K_9 (\gamma - 1)^{\frac{1}{17}}},\]

\[K_1 = C(K_6 + 1)(K_7 + 1)\gamma - 1)^{\frac{4}{9}} + C(\gamma - 1)^{\frac{4}{9}} + C K_2^2 M_2 (K_6 + 1)\]

\[+ C K_2^2 M_2 (K_7 + 1)\gamma - 1)^{\frac{4}{9}} + C K_2^2 M_2^2, \quad (3.88)\]

**Proof.** From (3.33) and (3.34), we have

\[A_1(T) + A_2(T) \leq C(K_4 + K_5) \left( (\gamma - 1)^{\frac{1}{4}} E_0^4 \right)^{\frac{1}{10}} + C \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + C \int_0^T \sigma \int_{\mathbb{R}^3} |\nabla u|^3

+ C \int_0^T \sigma \int_{\mathbb{R}^3} P |\nabla u|^2 + C \gamma^2 \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P u|^2 + C \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4

\leq C(K_4 + K_5) \left( (\gamma - 1)^{\frac{1}{4}} E_0^4 \right)^{\frac{1}{10}} + \sum_{i=1}^5 \mathcal{F}_i. \quad (3.89)\]

An application of (2.47) gives

\[\mathcal{F}_5 = C \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4

\leq C \int_0^T \sigma^2 |\nabla u| L^2 \rho |u|^3 + C \int_0^T \rho^2 \|P L^2 \rho u|^2 + C \int_0^T \rho^2 |H L^2 \rho u|^2

+ C \int_0^T \sigma^2 |\nabla u| L^2 H L^2 |\nabla^2 H|^3 L^2 + C \int_0^T \sigma^2 |P L^2 H L^3 |\nabla^2 H|^2 L^2

+ C \int_0^T \sigma^2 |H L^2 |\nabla H L^2 |H L^3 |\nabla^2 H|^3 L^2 + C \int_0^T \sigma^2 |P|^4 L^4

= \sum_{i=1}^7 J_i. \quad (3.90)\]

Thanks to (1.21), (3.21), (3.65), (3.65) and (3.66), we have

\[J_1 \leq C(\bar{\rho}) \left( \sup_{0 \leq t \leq T} \|\nabla u\|^2 L^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \sigma^2 \|\rho u\|^2 L^2 \right)^{\frac{1}{2}} \int_0^T \sigma \|\rho L^2 |\nabla u|^2 L^2

\leq C(\bar{\rho})(K_6^4 + 1) \left( (\gamma - 1)^{\frac{1}{4}} E_0^4 \right)^{\frac{1}{2}} (\gamma - 1)^{\frac{1}{4}} E_0^2 (K_7 + 1)

\leq C(\bar{\rho})(K_6^4 + 1)(K_7 + 1)(\gamma - 1)^{\frac{1}{4}} E_0^2 \frac{10}{17}.

\]
\[ \leq C(\bar{\rho})(K_0^{\frac{1}{2}} + 1)(K_7 + 1)(\gamma - 1)^{\frac{5}{3}} \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{3}{2}}. \] (3.91)

Similarly, we deduce
\[
J_2 \leq C \int_0^T \sigma^2 \| P \|_{L^2} \| \rho \|_{L^2}^3
\]
\[
\leq C \left( \sup_{0 \leq t \leq T} \sigma^2 \| \rho \|_{L^2} \right) \left( \sup_{0 \leq t \leq T} \| P \|_{L^2} \right) \int_0^T \sigma \| \rho \|_{L^2}^3 \| \nabla u \|_{L^2}^2
\]
\[
\leq C(\bar{\rho})(K_7 + 1)(\gamma - 1)^{\frac{22}{21}} E_0^{\frac{1}{21}}
\]
\[
\leq C(\bar{\rho})(K_7 + 1)(\gamma - 1)^{\frac{7}{5}} \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{3}{2}}. \] (3.92)

It follows from \[1.19\], \[3.2\], \[3.11\], \[3.12\] and \[3.69\]-\[3.66\] that
\[
J_3 \leq C \int_0^T \sigma^2 \| H \|_{L^2} \| \nabla H \|_{L^2} \| \rho \|_{L^2}^3
\]
\[
\leq CK_2^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \sigma^2 \| \rho \|_{L^2} \right) \left( \sup_{0 \leq t \leq T} \| H \|_{L^2} \right)^{\frac{1}{2}} \int_0^T \sigma \| \rho \|_{L^2}^3 \| \nabla u \|_{L^2}^2
\]
\[
\leq CK_2^{\frac{1}{2}} \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}}
\]
\[
\leq CK_2^{\frac{1}{2}} \left( (K_7 + 1)(\gamma - 1)^{\frac{22}{21}} E_0^{\frac{1}{21}}
\]
\[
\leq CK_2^{\frac{1}{2}} \left( (K_7 + 1)(\gamma - 1)^{\frac{7}{5}} \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{3}{2}}, \] (3.93)

Moreover, \( J_4 \) to \( J_6 \) can be estimated, in a similar way, as follows:
\[
J_4 \leq C \int_0^T \sigma^2 \| \nabla u \|_{L^2} \| H \|_{L^2}^3 \| \nabla^2 H \|_{L^2}^3
\]
\[
\leq C \left( \sup_{0 \leq t \leq T} \sigma^2 \| \nabla^2 H \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \| H \|_{L^2}^3 \| \nabla H \|_{L^2}^3 \right) \left( \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2}^2 \right)^{\frac{1}{2}} \int_0^T \sigma \| \nabla^2 H \|_{L^2}^2
\]
\[
\leq CK_2^{\frac{1}{2}} M_2^{\frac{1}{2}} (K_0^{\frac{1}{2}} + 1) \left( \gamma - 1 \right)^{\frac{1}{2}} E_0^{\frac{1}{2}} \right)^{\frac{3}{2}}, \] (3.94)

\[
J_5 \leq C \int_0^T \sigma^2 \| P \|_{L^2} \| H \|_{L^2}^3 \| \nabla^2 H \|_{L^2}^3
\]
\[
\leq C \left( \sup_{0 \leq t \leq T} \sigma^2 \| \nabla^2 H \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \| H \|_{L^2}^3 \| \nabla H \|_{L^2}^3 \right) \left( \sup_{0 \leq t \leq T} \| P \|_{L^2} \right) \int_0^T \sigma \| \nabla^2 H \|_{L^2}^2
\]
\[
\leq CK_2^{\frac{1}{2}} M_2^{\frac{1}{2}} (\gamma - 1)^{\frac{3}{2}} E_0^{\frac{1}{2}} \right)^{\frac{3}{2}} \]
\[
\leq CK_2^3 M_2^3 (\gamma - 1)^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{6}} E_0^\frac{1}{6} \right)^{\frac{3}{4}}, \tag{3.95}
\]

and
\[
J_6 \leq C \int_0^T \sigma^2 \|H\|_L^2 \|\nabla H\|_L^2 \|\nabla^2 H\|_L^2 \|H\|_L^3 \|\nabla^2 H\|_L^3,
\]
\[
\leq C \left( \sup_{0 \leq t \leq T} \sigma^2 \|\nabla^2 H\|_L^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \|H\|_L^3 \right)^{\frac{1}{2}} \left( \int_0^T \|H\|_L^2 \|\nabla H\|_L^2 \right)^{\frac{1}{2}} \int_0^T \sigma \|\nabla^2 H\|_L^2,
\]
\[
\leq CK_2^3 M_2^3 \left( (\gamma - 1)^{\frac{1}{6}} E_0^\frac{1}{6} \right)^{\frac{3}{4}}. \tag{3.96}
\]

In order to obtain desired estimate on \( \mathcal{F}_5 \), it suffices to estimate \( J_7 \). One can deduce from (3.96) that
\[
P_t + u \cdot \nabla P + \gamma P \div u = 0. \tag{3.97}
\]

In terms of the effective viscous flux \( G \), we can rewrite (3.97) as
\[
P_t + u \cdot \nabla P + \frac{\gamma}{\mu + \lambda} PG + \frac{\gamma}{2(2\mu + \lambda)} P|H|^2 + \frac{\gamma}{2\mu + \lambda} P^2 = 0. \tag{3.98}
\]

Multiplying (3.98) by \( 3\sigma^2 P^2 \) and integrating the resulting equality over \( \mathbb{R}^3 \times [0, T] \), we obtain
\[
C \int_0^T \sigma^2 \|P\|_L^4 \leq C \sup_{0 \leq t \leq T} (\sigma^2 \|P\|_L^3) + C \int_0^T \sigma \sigma' \|P\|_L^3 + C \int_0^T \sigma^2 \|P^3 G \|
\]
\[
\leq C(\bar{\rho})(\gamma - 1) E_0 + \epsilon \int_0^T \sigma^2 \|P\|_L^4 + C \int_0^T \sigma^2 \|G\|_L^4
\]
\[
+ \int_0^T \sigma^2 \|H\|_L^8. \tag{3.99}
\]

To handle the terms on the right hand side of (3.99), we have by (2.2), (3.11)-(3.12) and (3.21) that
\[
\int_0^T \sigma^2 \|H\|_L^8 \leq C \int_0^T \sigma^2 \|H\|_L^6 \|H\|_L^4 \|H\|_L^3 \|H\|_L^3
\]
\[
\leq C \int_0^T \sigma^2 \|\nabla H\|_L^2 \|\nabla^2 H\|_L^2 \|\nabla^2 H\|_L^2 \|H\|_L^3 \|H\|_L^3
\]
\[
\leq C \left( \sup_{0 \leq t \leq T} \sigma^2 \|\nabla^2 H\|_L^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \sigma \|\nabla H\|_L^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \|\nabla^2 H\|_L^2 \right)^{\frac{1}{2}} \int_0^T \|H\|_L^3
\]
\[
\leq CK_2^3 M_2^3 \left( (\gamma - 1)^{\frac{1}{6}} E_0^\frac{1}{6} \right)^{\frac{3}{4}}. \tag{3.100}
\]
Furthermore, we have also

\[ C \int_0^T \sigma^2 \| G \|_{L^4}^4 \]

\[ \leq C \int_0^T \sigma^2 \| G \|_{L^2} \| G \|_{L^6}^3 \]

\[ \leq C \int_0^T \sigma^2 ((2\mu + \lambda)\| \nabla u \|_{L^2} + \| P \|_{L^2}) \left( \| \rho \|_{L^3}^3 + \| H \cdot \nabla H \|_{L^2}^3 \right) \]

\[ + C \int_0^T \sigma^2 \| H \|_{L^2}^\frac{3}{2} \| \nabla H \|_{L^2}^\frac{5}{2} \left( \| \rho \|_{L^3}^3 + \| H \cdot \nabla H \|_{L^2}^3 \right) \]

\[ \leq C \int_0^T \sigma^2 \| \nabla u \|_{L^2} \| \rho \|_{L^3}^3 \]

\[ + C \int_0^T \sigma^2 \| \nabla u \|_{L^2} \| H \|_{L^3}^\frac{3}{2} \| \nabla^2 H \|_{L^2}^\frac{3}{2} \]

\[ + C \int_0^T \sigma^2 \| H \|_{L^2}^\frac{3}{2} \| \nabla H \|_{L^2}^\frac{3}{2} \| \rho \|_{L^3}^3 \]

\[ \leq C \left( \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2} \right)^\frac{1}{2} \left( \sup_{0 \leq t \leq T} \sigma^2 \| \rho \|_{L^3}^3 \right)^\frac{1}{2} \int_0^T \sigma \| \rho \|_{L^2}^2 \]

\[ + C \left( \sup_{0 \leq t \leq T} \| P \|_{L^2} \right) \left( \sup_{0 \leq t \leq T} \sigma^2 \| \rho \|_{L^3}^3 \right)^\frac{1}{2} \int_0^T \sigma \| \rho \|_{L^2}^2 \]

\[ + C \left( \sup_{0 \leq t \leq T} \| \nabla u \|_{L^2} \right)^\frac{1}{2} \left( \sup_{0 \leq t \leq T} \| H \|_{L^3}^\frac{3}{2} \| \nabla H \|_{L^2}^\frac{3}{2} \right) \left( \sup_{0 \leq t \leq T} \sigma^2 \| \nabla^2 H \|_{L^2}^\frac{3}{2} \right)^\frac{1}{2} \int_0^T \sigma \| \nabla^2 H \|_{L^2}^2 \]

\[ + C \left( \sup_{0 \leq t \leq T} \| P \|_{L^2} \right) \left( \sup_{0 \leq t \leq T} \| H \|_{L^3}^\frac{3}{2} \| \nabla H \|_{L^2}^\frac{3}{2} \right) \left( \sup_{0 \leq t \leq T} \sigma^2 \| \nabla^2 H \|_{L^2}^\frac{3}{2} \right)^\frac{1}{2} \int_0^T \sigma \| \nabla^2 H \|_{L^2}^2 \]

\[ + C \left( \sup_{0 \leq t \leq T} \| H \|_{L^2} \right)^\frac{1}{4} \left( \sup_{0 \leq t \leq T} \| \nabla H \|_{L^2}^\frac{3}{2} \right)^\frac{3}{4} \left( \sup_{0 \leq t \leq T} \sigma^2 \| \rho \|_{L^3}^3 \right)^\frac{1}{2} \int_0^T \sigma \| \rho \|_{L^2}^4 \]

\[ + C \left( \sup_{0 \leq t \leq T} \| H \|_{L^2} \right)^\frac{1}{4} \left( \sup_{0 \leq t \leq T} \| \nabla H \|_{L^2}^\frac{3}{2} \right)^\frac{3}{4} \left( \sup_{0 \leq t \leq T} \| H \|_{L^2} \| \nabla H \|_{L^2} \right) \]

\[ \cdot \left( \sup_{0 \leq t \leq T} \sigma^2 \| \nabla^2 H \|_{L^2}^\frac{3}{2} \right)^\frac{1}{2} \int_0^T \sigma \| \nabla^2 H \|_{L^2}^2 \]

\[ \leq C (K_6^\frac{3}{2} + 1)(K_7 + 1) \left( (\gamma - 1) \frac{1}{\sigma} E_0^\frac{3}{2} \right) \left( (\gamma - 1) \frac{1}{\sigma} E_0^\frac{3}{2} \right) \]

\[ + C(K_7 + 1) \left( (\gamma - 1) \frac{1}{\sigma} E_0^\frac{3}{2} \right)^\frac{1}{2} \left( (\gamma - 1) \frac{7}{8} E_0^\frac{7}{8} \right) \]

\[ + C K_2^\frac{3}{2} M_2^\frac{3}{2} (K_6^\frac{3}{2} + 1) \left( (\gamma - 1) \frac{1}{\sigma} E_0^\frac{3}{2} \right)^\frac{1}{2} + C K_2^\frac{3}{2} M_2^\frac{3}{2} (\gamma - 1) \frac{1}{\sigma} E_0^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\sigma} E_0^\frac{1}{2} \right)^\frac{3}{2} \]
\[ + CK_2^\frac{7}{2} M_2^\frac{3}{2} (K_7 + 1) \left( (\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma} \right) \frac{3}{\gamma} \]

\[ + CK_2^\frac{7}{2} M_2^\frac{3}{2} \left( (\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma} \right) \frac{3}{\gamma} \]

\[ \leq K_8^\frac{1}{2} \left( (\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma} \right) \frac{3}{\gamma}, \quad (3.101) \]

provided \((\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma} \leq 1\), where

\[ K_8^1 = C(K_6^\frac{1}{2} + 1)(K_7 + 1)(\gamma - 1) \frac{1}{\gamma} + C(K_7 + 1)(\gamma - 1) \frac{1}{\gamma} + CK_2^\frac{7}{2} M_2^\frac{3}{2} (K_6^\frac{1}{2} + 1) \]

\[ + CK_2^\frac{7}{2} M_2^\frac{3}{2} (\gamma - 1) \frac{1}{\gamma} + CK_2^\frac{7}{2} M_2^\frac{3}{2} (K_7 + 1)(\gamma - 1) \frac{1}{\gamma} + CK_2^\frac{7}{2} M_2^\frac{3}{2}. \quad (3.102) \]

Substituting (3.100) and (3.101) into (3.99), assuming \((\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma} \leq 1\), we have

\[ J_7 = C \int_0^T \sigma^2 \|P\|_{L^4} \]

\[ \leq C(\bar{\rho})(\gamma - 1) E_0^\frac{1}{\gamma} + C \int_0^T \sigma^2 \|G\|_{L^4} + \int_0^T \sigma^2 \|H\|_{L^8} \]

\[ \leq K_8^2 \left( (\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma} \right) \frac{3}{\gamma}, \quad (3.103) \]

where

\[ K_8^2 = C(\bar{\rho})(\gamma - 1) \frac{1}{\gamma} + CK_2^\frac{12}{3} M_2^\frac{3}{2} + K_8^1. \quad (3.104) \]

Combining (3.90)-(3.96) and (3.103), we consequently get that

\[ \mathcal{T}_5 = C \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \leq K_8 \left( (\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma} \right) \frac{3}{\gamma}, \quad (3.105) \]

where

\[ K_8 = C(\bar{\rho})(K_6^\frac{1}{2} + 1)(K_7 + 1)(\gamma - 1) \frac{1}{\gamma} + C(\bar{\rho})(K_7 + 1)(\gamma - 1) \frac{1}{\gamma} \]

\[ + CK_2^\frac{1}{2} (K_7 + 1)(\gamma - 1) \frac{1}{\gamma} + CK_2^\frac{9}{2} M_2^\frac{3}{2} (K_6^\frac{1}{2} + 1) \]

\[ + CK_2^\frac{9}{2} M_2^\frac{3}{2} (\gamma - 1) \frac{1}{\gamma} + CK_2^\frac{9}{2} M_2^\frac{3}{2} + K_8^2. \quad (3.106) \]

It holds from (3.29) that

\[ \mathcal{T}_1 = C \int_0^{\sigma(T)} \int_{\mathbb{R}^3} |\nabla u|^2 \leq CK_3 (\gamma - 1) \frac{1}{\gamma} E_0^\frac{1}{\gamma}. \quad (3.107) \]

To estimate \( \mathcal{T}_2 \), due to Hölder inequality, (3.7), (1.9) and (3.105), we deduce

\[ \mathcal{T}_2 = C \int_0^T \sigma \int_{\mathbb{R}^3} |\nabla u|^3 \]

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\[\begin{align*}
\leq C \left( \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} \sigma^2 |\nabla u|^4 \right)^{\frac{1}{2}} \\
\leq CE_0^\frac{1}{2} K_8^\frac{1}{2} \left( \gamma - 1 \right)^{\frac{1}{6}} E_0^\frac{1}{6}
\leq CK_8^\frac{1}{2} \left( \gamma - 1 \right)^{\frac{1}{24} + \frac{1}{12}} \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{1}{12}}.
\end{align*}\]  

(3.108)

As for \( \mathcal{T}_4 \), by Hölder inequality, (3.103) and (3.105), we get

\[\begin{align*}
\mathcal{T}_4 &= C \gamma^2 \int_0^T \int_{\mathbb{R}^3} \sigma^2 |P \nabla u|^2 \\
&\leq C \left( \int_0^T \sigma^2 ||P||_{L^4}^4 \right)^{\frac{1}{4}} \left( \int_0^T \sigma^2 ||\nabla u||_{L^4}^4 \right)^{\frac{1}{4}} \\
&\leq C \sqrt{K_8^2 K_8^2} \left( \gamma - 1 \right)^{\frac{1}{6}} E_0^\frac{1}{6}
\right)^{\frac{3}{2}}.
\end{align*}\]  

(3.109)

Now it remains to estimate \( \mathcal{T}_3 \). It follows from (3.7), (3.103) and (3.105) that

\[\begin{align*}
\mathcal{T}_3 &= C \gamma \int_0^T \int_{\mathbb{R}^3} \sigma \int_{\mathbb{R}^3} P |\nabla u|^2 \\
&\leq C \left( \int_0^T \sigma^2 ||P||_{L^4}^4 \right)^{\frac{1}{4}} \left( \int_0^T \sigma^2 ||\nabla u||_{L^4}^4 \right)^{\frac{1}{4}} \\
&\leq C E_0^\frac{1}{4} \sqrt{K_8^2 K_8} \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}
\right)^{\frac{3}{2}} \\
&\leq C \sqrt{K_8^2 K_8} \left( \gamma - 1 \right)^{\frac{1}{12} + \frac{1}{24}} \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{7}{12}}.
\end{align*}\]  

(3.110)

Finally, we deduce from (3.105), (3.107) - (3.110) that

\[\begin{align*}
A_1(T) + A_2(T) &\leq C (K_4 + K_5) \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{11}{18}} + CK_3(\gamma - 1)^{\frac{1}{6} E_0^\frac{1}{6}}
\leq CK_8^\frac{1}{2} \left( \gamma - 1 \right)^{\frac{1}{12} + \frac{1}{24}} \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{7}{12}} \\
&+ C \sqrt{K_8^2 K_8} (\gamma - 1)^{\frac{1}{12} + \frac{1}{24}} \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{3}{2}} + K_8 \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{9}{12}}
\leq C K_9 \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{11}{18}} \\
&\leq \left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}}\right)^{\frac{1}{2}}, \quad \text{(3.111)}
\end{align*}\]

provided that

\[\left( \gamma - 1 \right)^{\frac{1}{6} E_0^\frac{1}{6}} \leq \min \{ (C K_9)^{-9}, 1 \}. \quad \text{(3.112)}\]
where $K_9$ is given by

$$
K_9 = (K_4 + K_5) + K_3 + K_2^{\frac{1}{2}} (\gamma - 1)^{\frac{1}{2}} + K_2^{\frac{1}{2}} K_8 (\gamma - 1)^{\frac{1}{2}} + \sqrt{K_2^{\frac{1}{2}} K_8 + K_8}.
$$

(3.113)

And to get (3.111), we have used the facts that $(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} \leq 1$. Then we finish the proof of Lemma 3.9.

Now we are ready to prove the upper bound of the density.

**Lemma 3.10.** Under the same assumption as in Proposition 3.1, it holds that

$$
\sup_{0 \leq t \leq T} \|\rho\|_{L^\infty} \leq \frac{7}{4} \bar{\rho},
$$

(3.114)

provided that

$$
(\gamma - 1)^{\frac{1}{2}} E_0^{\frac{1}{2}} \leq \min \left\{ \varepsilon_5, \left( \frac{\bar{\rho}}{2K_9} \right)^{\frac{16}{3}}, \frac{\bar{\rho}}{4C(\bar{\rho})(1 + K_2^{\frac{1}{2}})} \right\},
$$

(3.115)

where

$$
K_{10} = K_5^{\frac{1}{2}} + K_2^{\frac{1}{2}} M_2^{\frac{1}{2}} + K_3^{\frac{1}{2}} M_2^{\frac{1}{2}} + K_3^{\frac{1}{2}} M_2^{\frac{1}{2}} K_2^{\frac{1}{2}}.
$$

(3.116)

**Proof.** Let $D_t \triangleq \partial_t + u \cdot \nabla$ denote the material derivative operator. Then, in terms of the effective viscous flux $G$, we can rewrite (1.1) as

$$
D_t \rho = g(\rho) + b'(\rho),
$$

where

$$
 g(\rho) \triangleq \frac{\rho P}{2\mu + \lambda}, \quad b(t) = -\frac{1}{2\mu + \lambda} \int_0^t \left( \rho G + \frac{1}{2} |\rho| H^2 \right).
$$

(3.117)

Moreover, it follows from Lemmas 2.1-2.2, (2.4) and (2.5) that

$$
\|G\|_{L^\infty} \leq C \|G\|^\frac{1}{3}_{L^6} \|\nabla G\|^\frac{1}{3}_{L^6}
$$

\[ \leq C \left( \|\rho \dot{u}\|^\frac{1}{2}_{L^2} + \|\nabla H\|^\frac{1}{2}_{L^2} \|\nabla^2 H\|^\frac{1}{2}_{L^2} \right) \left( \|\nabla \dot{u}\|^\frac{1}{2}_{L^2} + \|\nabla H\|^\frac{1}{2}_{L^2} \|\nabla^2 H\|^\frac{1}{2}_{L^2} \right)
$$

\[ \leq C \|\rho \dot{u}\|^\frac{1}{2}_{L^2} \|\nabla \dot{u}\|^\frac{1}{2}_{L^2} + C \|\rho \dot{u}\|^\frac{1}{2}_{L^2} \|\nabla H\|^\frac{1}{2}_{L^2} \|\nabla^2 H\|^\frac{3}{2}_{L^2}
$$

\[ + C \|\nabla \dot{u}\|^\frac{1}{2}_{L^2} \|\nabla H\|^\frac{3}{2}_{L^2} \|\nabla^2 H\|^\frac{3}{2}_{L^2} + C \|\nabla H\|_{L^2} \|\nabla^2 H\|_{L^2} \] (3.118)

For $t \in [0, \sigma(T)]$, one can deduce that for all $0 \leq t_1 \leq t_2 \leq \sigma(T)$,

$$
|b(t_2) - b(t_1)| \leq C \int_0^{\sigma(T)} (\|G\|_{L^\infty} + \|H\|^2_{L^\infty})
$$

\[ \leq C \int_0^{\sigma(T)} \|\rho \dot{u}\|^\frac{1}{2}_{L^2} \|\nabla \dot{u}\|^\frac{1}{2}_{L^2} + C \int_0^{\sigma(T)} \|\nabla H\|_{L^2} \|\nabla^2 H\|_{L^2}
$$
\[
\begin{align*}
&+ \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 \|\nabla H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 + \int_0^{\sigma(T)} \|\nabla \dot{u}\|_{L^2}^2 \|\nabla H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 \\
&\leq C \left( \int_0^{\sigma(T)} \|\rho \dot{u}\|_{L^2}^2 t^{-\frac{1}{3}} \right)^{\frac{3}{4}} \left( \int_0^{\sigma(T)} t^{\frac{1}{2}} \|\nabla \dot{u}\|_{L^2}^2 \right)^{\frac{1}{4}} \\
&+ C \left( \int_0^{\sigma(T)} \|\nabla H\|_{L^2}^2 \right)^{\frac{3}{8}} \left( \int_0^{\sigma(T)} \|\nabla^2 H\|_{L^2}^2 \right)^{\frac{1}{8}} \\
&+ \left( \sup_{0 \leq t \leq \sigma(T)} t^2 \|\rho \dot{u}\|_{L^2}^2 \right)^{\frac{3}{8}} \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla H\|_{L^2}^2 \right) \int_0^{\sigma(T)} t^{-\frac{2}{3}} \|\nabla^2 H\|_{L^2}^2 \\
&+ C \left( \sup_{0 \leq t \leq \sigma(T)} \|\nabla H\|_{L^2}^2 \right)^{\frac{3}{8}} \left( \sup_{0 \leq t \leq \sigma(T)} t^2 \|\nabla^2 H\|_{L^2}^2 \right) \int_0^{\sigma(T)} \left( t^\frac{3}{4} \|\nabla \dot{u}\|_{L^2}^2 \right) t^{-\frac{3}{4}} \\
&\leq CK_T^{\frac{1}{8}} \left( \int_0^{\sigma(T)} \left( \|\rho \dot{u}\|_{L^2}^2 t \right)^{\frac{3}{4}} t^{-\frac{3}{4}} \right) + CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
&+ CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{8}} \left( \int_0^{\sigma(T)} \|\nabla^2 H\|_{L^2}^2 \right)^{\frac{1}{8}} \left( \int_0^{\sigma(T)} t^{-\frac{3}{4}} \right)^{\frac{1}{8}} \\
&+ CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{8}} \int_0^{\sigma(T)} \left( t^\frac{3}{4} \|\nabla \dot{u}\|_{L^2}^2 \right) \int_0^{\sigma(T)} \left( t^\frac{3}{4} t^{-\frac{3}{4}} \right)^{\frac{3}{4}} \\
&\leq CK_T^{\frac{1}{8}} \left( \sup_{0 \leq t \leq \sigma(T)} \left( \|\rho \dot{u}\|_{L^2}^2 t \right)^{\frac{1}{16}} \right) \left( \int_0^{\sigma(T)} \left( \|\rho \dot{u}\|_{L^2}^2 t \right)^{\frac{3}{4}} t^{-\frac{3}{4}} \right)^{\frac{3}{4}} \\
&+ CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{2}} + CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
&+ CK_2 M_2^{\frac{1}{4}} K_7^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{16}} \\
&\leq CK_T^{\frac{5}{10}} \left( \int_0^{\sigma(T)} t \|\rho \dot{u}\|_{L^2}^2 \right)^{\frac{3}{10}} \left( \int_0^{\sigma(T)} t^{-\frac{3}{4}} \right)^{\frac{1}{10}} + CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{4}} \\
&+ CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{4}} + CK_2 M_2^{\frac{3}{4}} K_7^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{10}} \\
&\leq CK_T^{\frac{5}{10}} A_1(\sigma(T))^{\frac{1}{10}} + CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{2}} + CK_2 M_2^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
&+ CK_2 M_2^{\frac{3}{4}} K_7^{\frac{1}{4}} \left( (\gamma - 1)^{\frac{1}{4}} E_0^{\frac{1}{2}} \right)^{\frac{1}{16}}
\end{align*}
\]
\[\leq C \left( K_1^\frac{3}{N} + K_2^\frac{2}{N} M_1^\frac{1}{2} + K_2^\frac{1}{2} M_2^\frac{3}{2} + K_2^\frac{3}{2} M_2^\frac{2}{2} K_7^\frac{1}{2} \right) \left( (\gamma - 1)^\frac{1}{\beta} E_0^\frac{1}{2} \right) \leq K_{10} \left( (\gamma - 1)^\frac{1}{\beta} E_0^\frac{1}{2} \right) \frac{1}{\mu}, \] 

(3.119)

where (3.12), (3.61), (3.86) and (3.118) have been used. Therefore, for \( t \in [0, \sigma(T)] \), we can choose \( N_0 \) and \( N_1 \) in Lemma 2.3 as follows

\[ N_1 = 0, \quad N_0 = K_{10} \left( (\gamma - 1)^\frac{1}{\beta} E_0^\frac{1}{2} \right) \frac{1}{\mu}, \] 

(3.120)

and \( \tilde{\zeta} = 0 \). Then

\[ g(\zeta) = -\frac{\zeta \rho}{2 \mu + \lambda} \leq -N_1 = 0 \text{ for all } \zeta \geq \tilde{\zeta} = 0. \] 

(3.121)

We thus have

\[ \sup_{0 \leq t \leq \sigma(T)} \| \rho \|_{L^\infty} \leq \max \{ \bar{\rho}, 0 \} + N_0 \leq \bar{\rho} + K_{10} \left( (\gamma - 1)^\frac{1}{\beta} E_0^\frac{1}{2} \right) \leq \frac{3\bar{\rho}}{2}, \] 

(3.122)

provided

\[ (\gamma - 1)^\frac{1}{\beta} E_0^\frac{1}{2} \leq \min \left\{ \left( \frac{\bar{\rho}}{2K_g} \right)^{16}, \varepsilon_5 \right\}. \] 

(3.123)

Furthermore, due to Lemmas 2.21, 2.22 for \( t \in [\sigma(T), T] \), we can derive

\[ |b(t_2) - b(t_1)| \leq C \int_{t_1}^{t_2} (\| G \|_{L^\infty} + \| H \|_{L^2}^2) \]

\[ \leq \frac{C(\bar{\rho})}{2 \mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) \int_{\sigma(T)}^{T} \| G \|_{L^\infty}^4 + C \int_{\sigma(T)}^{T} \| \nabla H \|_{L^2} \| \nabla^2 H \|_{L^2} \]

\[ \leq \frac{C(\bar{\rho})}{2 \mu + \lambda} (t_2 - t_1) + C \left( \int_{\sigma(T)}^{T} \| \nabla H \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \int_{\sigma(T)}^{T} \| \nabla^2 H \|_{L^2}^2 \right)^{\frac{1}{2}} \]

\[ + C(\bar{\rho}) \int_{\sigma(T)}^{T} \| \rho \tilde{\rho} \|_{L^2} \| \nabla \tilde{u} \|_{L^2}^2 + C \int_{\sigma(T)}^{T} \| \rho \tilde{u} \|_{L^2} \| \nabla^2 H \|_{L^2}^2 \]

\[ + C \int_{\sigma(T)}^{T} \| \nabla \tilde{u} \|_{L^2} \| \nabla H \|_{L^2}^3 \| \nabla^2 H \|_{L^2} \]

\[ + C \int_{\sigma(T)}^{T} \| \nabla \tilde{u} \|_{L^2} \| \nabla^2 H \|_{L^2}^3 \| \nabla^2 H \|_{L^2} \]

\[ \leq \frac{C(\bar{\rho})}{2 \mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) K_2^2 (\gamma - 1)^\frac{1}{\beta} E_0^\frac{1}{2} + C(\bar{\rho}) A_2(T) \int_{\sigma(T)}^{T} \| \nabla \tilde{u} \|_{L^2}^2 \]

\[ + C A_2(T)^\frac{3}{2} A_1(T)^\frac{3}{2} + A_1(T)^\frac{3}{2} A_2(T)^\frac{3}{2} \int_{\sigma(T)}^{T} \| \nabla \tilde{u} \|_{L^2}^2 + C A_1(T)^2 \]

\[ \leq \frac{C(\bar{\rho})}{2 \mu + \lambda} (t_2 - t_1) + C(\bar{\rho}) (\gamma - 1)^\frac{1}{\beta} E_0^\frac{1}{2} + C(\bar{\rho}) K_2^2 A_2(T)^2 \]
Lemma 4.1. The following estimates hold:
\[ + C A_2(T)^{\frac{3}{2}} A_1(T)^{\frac{1}{2}} + CA_1(T)^2 \]
\[ \leq \frac{C(\bar{\rho})}{2\mu + \lambda} (t_2 - t_1) + C(\bar{\rho})(1 + K_2^2)(\gamma - 1)\frac{1}{6} E_0^{\frac{1}{6}}. \]  
(3.124)

Consequently, for \( t \in [\sigma(T), T] \), we can choose \( N_0 \) and \( N_1 \) in Lemma 2.3 as follows
\[ N_1 = \frac{1}{2\mu + \lambda}, \quad N_0 = C(\bar{\rho})(1 + K_2^2)(\gamma - 1)\frac{1}{6} E_0^{\frac{1}{6}}. \]  
(3.125)

Noticing that
\[ g(\zeta) = -\frac{\zeta P(\zeta)}{2\mu + \lambda} \leq -N_1 = -\frac{1}{2\mu + \lambda} \text{ for all } \zeta \geq 1, \]  
(3.126)

one can set \( \tilde{\zeta} = 1 \). Thus
\[ \sup_{\sigma(T) \leq t \leq T} \|\rho\|_{L^\infty} \leq \max\{\frac{3}{2}\tilde{\rho}, 1\} + N_0 \leq \frac{3}{2}\tilde{\rho} + C(\bar{\rho})(1 + K_2^2)(\gamma - 1)\frac{1}{6} E_0^{\frac{1}{6}} \leq \frac{7\tilde{\rho}}{4}, \]  
(3.127)

provided
\[ (\gamma - 1)\frac{1}{6} E_0^{\frac{1}{6}} \leq \min\left\{ \epsilon_5, \frac{\tilde{\rho}}{4C(\bar{\rho})(1 + K_2^2)} \right\}. \]  
(3.128)

4 Proof of Theorem 1.1

In this section, we devote to prove the main result of this paper. First, from now on we always assume that the conditions in Theorem 1.1 hold. Moreover, we denote the generic constant by \( C \) which may depends on \( T, \mu, \lambda, \nu, \gamma, \tilde{\rho}, \bar{\rho}, M_1, M_2, g \) and some other initial data. Here \( g \in L^2 \) is the function in the compatibility condition (4.1). The following higher-order a priori estimates of the smooth solutions which are needed to guarantee the classical solutions \((\rho, u, H)\) to be global ones have been proved in [29], so we omit their proof here.

Lemma 4.1. The following estimates hold:
\[ \sup_{0 \leq t \leq T} (\|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \]
\[ + \int_0^T \left( \|\rho^{\frac{1}{2}} u\|_{L^2}^2 + \|H\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 \right) \leq C(T), \]  
(4.1)

\[ \sup_{0 \leq t \leq T} (\|\rho^{\frac{1}{2}} u\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|H\|_{L^2}^2) \]
\[ + \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \leq C(T), \]  
(4.2)

\[ \sup_{0 \leq t \leq T} (\|\nabla \rho\|_{L^2} + \|\nabla u\|_{H^1}^2) + \int_0^T \|\nabla u\|_{L^\infty} \leq C(T), \]  
(4.3)

\[ \sup_{0 \leq t \leq T} (\|\rho^{\frac{1}{2}} u_t\|_{L^2}^2) + \int_0^T \|\nabla u_t\|_{L^2}^2 \leq C(T), \]  
(4.4)

\[ \sup_{0 \leq t \leq T} (\|\nabla \rho\|_{H^1} + \|\nabla P\|_{H^1}^2) + \int_0^T \|\nabla^2 u\|_{H^1}^2 \leq C(T), \]  
(4.5)
\[
\sup_{0 \leq t \leq T} \left( \| \rho_t \|_{H^1} + \| P_t \|_{H^1} \right) + \int_0^T \left( \| \rho_{tt} \|_{L^2} + \| P_{tt} \|_{L^2} \right) \leq C(T), \quad (4.6)
\]
\[
\sup_{0 \leq t \leq T} \sigma \left( \| \nabla u \|_{H^2}^2 + \| \nabla u_t \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 + \| \nabla H_t \|_{L^2}^2 \right) + \int_0^T \sigma \left( \| \rho_{tt} \|_{L^2}^2 + \| \nabla u_t \|_{H^1}^2 + \| H_{tt} \|_{L^2}^2 \right) \leq C(T), \quad (4.7)
\]
\[
\sup_{0 \leq t \leq T} \left( \| \nabla \rho \|_{W^{1,q}} + \| \nabla P \|_{W^{1,q}} \right) + \int_0^T \left( \| \nabla u_t \|_{L^q}^{p_0} + \| \nabla^2 u \|_{W^{1,q}}^{p_0} \right) \leq C(T), \quad (4.8)
\]

for fixed \( q \in (3, 6) \), where \( 1 \leq p_0 < \frac{4q}{3q-6} \in (1, 2) \).

\[
\sup_{0 \leq t \leq T} \sigma \left( \| \rho_{tt} \|_{L^2} + \| \nabla^2 u_t \|_{L^2} + \| \nabla^2 u \|_{W^{1,q}} + \| \nabla^2 H \|_{H^2} + \| H_{tt} \|_{L^2} + \| H_{tt} \|_{L^2} \right) + \int_0^T \sigma^2 \left( \| \nabla u_t \|_{L^2}^2 + \| \nabla H_{tt} \|_{L^2}^2 \right) \leq C(T). \quad (4.9)
\]

Thanks to all the a priori estimates established above, we now are ready to prove Theorem 1.1. In fact, this can be done in a method the same as that in [29], we omit it here for simplicity.

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