Rationality of an $S_6$-Invariant Quartic 3-Fold

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Abstract
We complete the study of rationality problem for hypersurfaces $X_t \subset \mathbb{P}^4$ of degree 4 invariant under the action of the symmetric group $S_6$.

Keywords  Quartic 3-fold · Ordinary double point · Rationality

Mathematics Subject Classification  14E08 · 14E30 · 14M10

1 Introduction

1.1

Any quartic 3-fold $X_t \subset \mathbb{P}^4$ with a non-trivial action of the group $S_6$ can be given by the equations

$$
\sum x_i = t \sum x_i^4 - \left( \sum x_i^2 \right)^2 = 0
$$

in $\mathbb{P}^5$. Here the parameter $t \in \mathbb{P}^1$ is allowed to vary.

When $t = 2$ one gets the Burkhardt quartic whose rationality is well-known (see e.g. Hunt 1996, 5.2.7). Similarly, $t = 4$ corresponds to the Igusa quartic, which is again rational (see [Prokhorov 2010, Section 3]). On the other hand, it was shown in Beauville (2013) that for all other $t \neq 0, 6, 10/7$ the quartic $X_t$ is non-rational.

Example 1.3 Following (Cheltsov and Shramov 2014, Section 4) let us blow up an $A_6$-orbit of 12 lines in $\mathbb{P}^5$ to get a 3-fold that contracts, $A_6$-equivariantly, onto a quartic threefold with 36 nodes. It follows from Remark in Beauville (2013) that this (Todd) quartic must be $X_{10/7}$. Hence $X_{10/7}$ is rational.
Thus, excluding the trivial case of $t = 0$ it remains to consider only $X_6$, in order to completely determine the birational type of all $S_6$-invariant quartics. Here is the result we obtain in this paper:

**Theorem 1.4** The quartic $X := X_6$ is rational.

Theorem 1.4 is proved in Sect. 4 by, basically, running the equivariant-MMP-type of arguments as in Prokhorov (2012). (Although the proof also uses some computations carried in Sect. 2). Unfortunately, we were not able to apply the results from Kaloghiros (2012), since non-rational $X_t$ all have defect equal 5 (see Beauville 2013, Lemma 2), which seems to contradict either (Kaloghiros 2012, 5.2, Lemma 8) or (Kaloghiros 2012, 5.2, Proposition 3) [compare also with (Kaloghiros 2012, Corollary 1) and the list of cases in (Kaloghiros 2012, Main Theorem)].

The proof of Theorem 1.4 goes as follows. First we show that there exists a $G$-invariant non-Cartier divisor on $X$ for some subgroup $G \subset S_6$ of order 20 (see Proposition 2.6). Next we show that there exists a $G$-equivariant small birational contraction $\phi : Y \rightarrow X$ with terminal $\mathbb{Q}$-factorial 3-fold $Y$ (see Sect. 3). Note that there is a $K_Y$-negative $G$-extremal contraction $\psi : Y \rightarrow Z$ and the proof of Theorem 1.4 is obtained by a detailed analysis of possibilities for $\psi$. Namely, we show that if $\psi$ is birational, with exceptional locus $E$, then it must be a composition of blow-ups at smooth rational curves (see Proposition 4.7). This argument actually allows one to assume $\psi : Y \rightarrow Z$ to be a sequence of $G$-MMP steps and thus reduce the proof of Theorem 1.4 to the case of a $G$-Mori 3-fold $Z$. The cases when $Z$ is a $G\mathbb{Q}$-Fano 3-fold. Rationality in this case is proved in 4.14 and 4.24 by studying the $G$-action on $\text{Sing} Z$ and an appropriate surface $S \in |-K_Z|$ (cf. Lemma 4.21), as well as applying the classification results from Iskovskikh and Prokhorov (1999) to $Z$, based on the estimates for $-K_Z^3$ [see (4.15) and (4.16)].

**Conventions** The ground field is $\mathbb{C}$ and $X$ signifies the quartic $X_6$ in what follows. We will be using freely standard notions and facts from Iskovskikh and Prokhorov (1999) and Kollár and Mori (1998) (yet we recall some of them for convenience).

### 2 Preliminaries

#### 2.1

Recall that the singular locus $\text{Sing} X \subset X$ consists of two $S_6$-orbits, of length 30 and 10, respectively [cf. Remark in Beauville (2013)], where the first orbit contains the point $o := [1 : 1 : w : w : w^2 : w^2]$, $w := \sqrt[3]{1}$, whereas the second one contains the point $o' := [-1 : -1 : -1 : 1 : 1 : 1]$.

Let us consider the local class group $\text{Cl}_{\mathcal{O},X}$ of the 3-fold $X$ at the point $o$. Namely, one identifies the analytic germ of $X \ni o$ with the nodal quadric $X_0 := (xy = zt) \subset \mathbb{C}^4$ and considers various morphisms $\mu : X_0 \rightarrow X'$, where $X'$ is any (not necessarily normal) variety. Then for the Weil divisor $\Delta := (x = z = 0) \subset X_0$, the group...
Cl_{o,X} is generated by the isomorphism classes of sheaves $\mathcal{O}_{X_0}(\Delta)$ and $\mu^*\mathcal{O}_{X'}(H)$ for all Cartier divisors $H$ on $X'$ (note that $\mu^*\mathcal{O}_{X'}(H)$ may no longer be a divisorial sheaf for non-flat $\mu$), with $\mu^*\mathcal{O}_{X'}(H) = 0$ being the only relations in $\text{Cl}_{o,X}$. The group operation “+” in $\text{Cl}_{o,X}$ is induced by the tensor product of $\mathcal{O}_X$-modules. It is immediate from the definition that $\text{Cl}_{o,X} \cong \mathbb{Z}$ (no multiple of $\Delta$ can be the zero locus of a rational function on $X_0$).

2.2

Next we find an effective Weil divisor on $X$ whose restriction to $X_0$ gives a generator of $\text{Cl}_{o,X}$. Consider the subspace $\Pi_1 \subset \mathbb{P}^5$ given by equations

$$x_0 + x_2 + x_5 = x_1 + x_3 + x_4 = 0.$$  

We have $X \cap \Pi = Q_1 + Q_2$, where the quadric $Q_1 \subset \mathbb{P}^3$ is given by

$$x_0^2 + x_0x_2 + x_2^2 + w(x_1^2 + x_1x_3 + x_3^2) = 0,$$

while the equation of $Q_2 \subset \mathbb{P}^3$ is

$$x_0^2 + x_0x_2 + x_2^2 - (w + 1)(x_1^2 + x_1x_3 + x_3^2) = 0.$$  

Note that both $Q_i \subset X$ are non-Cartier divisors because $Q_1 + Q_2 \sim \mathcal{O}_X(1)$.\footnote{“=” denotes the linear equivalence of divisors. We will also sometimes identify a divisor with the corresponding divisorial sheaf.} Furthermore, $Q_i$ are smooth, since they are projectively equivalent to

$$x_0^2 + x_2^2 + x_1^2 + x_3^2 = 0.$$  

In particular, restricting to the tangent cone $X_0$ we get $Q_i \sim \Delta$, so that $Q_i = \pm 1$ in $\text{Cl}_{o,X} = \mathbb{Z}$.

2.3

Now we need some equivariant properties of $Q_i$ above. Identify the set $\{x_2, x_0, x_4, x_3, x_1\}$ with $\{1, \ldots, 5\}$ and consider the corresponding action of the group $S_5$. Put $\tau := (13524) \in S_4 \subset S_5$.\footnote{For the set $\{1, \ldots, n\}$, any $n \geq 1$, the symbol $(i_1 \ldots i_n)$, $1 \leq i_j \leq n$, denotes its permutation $\{i_1, \ldots, i_n\}$ (i.e. $1 \mapsto i_1$ and so on). Also, if $i_j = j$ for some $j$, then we will identify (in the obvious way) $\{i_1 \ldots i_n\}$ with the permutation of respective $(n - 1)$-element set.} Then the following (evident) assertion holds:

**Lemma 2.4** $\tau^c(Q_i) \ni \text{of } f \text{ iff } c = 0 \text{ or } 2$.

Consider $h := (23451) \in S_5$. Direct computation again gives the following:

**Lemma 2.5** $h^a\tau^b(Q_i) \ni \text{of } f$ \begin{itemize}
  \item $(a, b) \in \{(0, 0), (3, 0), (4, 0), (0, 2), (3, 3), (1, 2), (4, 2), (1, 1)\}$. More precisely, we have
\end{itemize}
Indeed, otherwise 

\( \tau^2(Q_i) = h^4(Q_i) \equiv o \) and \( \tau^2(Q_i) \neq Q_i; \)

\( h^4 \tau^2(Q_i) = h^3(Q_i) \equiv o \) and \( h^4 \tau^2(Q_i) \neq Q_i, \tau^2(Q_i); \)

\( h^2(Q_i) = Q_i; \)

\( h^3 \tau^3(Q_i) = h \tau(Q_i) \equiv o \) and \( h^3 \tau^3(Q_i) \neq Q_i, \tau^2(Q_i), h^4 \tau^2(Q_i). \)

Let \( G := \langle \tau, h \rangle \) be the group generated by \( \tau \) and \( h. \) Note that the order of \( G \) is divisible by 4 and 5. Then from the classification of subgroups in \( S_5 \) we deduce that \( G \) is the general affine group \( GA(1, 5). \) Note also that \( G = \mathbb{F}_5 \times \mathbb{F}_5^* \) for the field \( \mathbb{F}_5 \) (here \( \mathbb{F}_5, \mathbb{F}_5^* \) are the additive and multiplicative groups, respectively).

Consider the divisor \( D := \sum_{\gamma \in G} \gamma(Q_1). \) By construction (the class of) \( D \) belongs to the subgroup \( Cl^G X \subseteq Cl X \) of classes of \( G \)-invariant divisors on \( X. \) Note that \( O_X(1) \in Cl^G X. \)

The next proposition will be crucial in the forthcoming constructions:

**Proposition 2.6** We have: (i) the class of \( D \) in \( Cl_{o,X} = \mathbb{Z} \) is non-zero and is divisible by 4; (ii) \( \text{rk} Cl^G X > 1. \)

**Proof** By definition of \( \tau, h \) and by Lemmas 2.4, 2.5 we have

\[
D = \sum_{(a,b) \in \{(0,0), \ldots, (1,1)\}} \tau^a h^b(Q_1) = 2h^4(Q_1) + 2h^3(Q_1) + 2Q_1 + 2h \tau(Q_1)
\]

(2.7)

in \( Cl_{o,X}, \) where we have identified \( Q_1 \) with \( O_X(Q_1) \) (same for \( D). \) Also, since \( h^3(Q_1), h^4(Q_1) \equiv o, \) both \( h^3, h^4 \) act on \( Cl_{o,X}. \) Indeed, \( h^3(Q_1) \) and \( h^4(Q_1) \) differ from (a power of) \( Q_1 \) by a suitable \( \mu^k H \) (cf. 2.1).

Since \( h^3 = (h^4)^2, \) we get \( h^3(Q_1) = 1 = (h^3)^3(Q_1) = h^4(Q_1), \) and hence \( D = 4 \) or 8. This means in particular that the product of \( O_X \)-modules

\[
\mathcal{I} := \prod_{\gamma \in G} O_X(-\gamma(Q_1)),
\]

identified with \( D \) as an element in \( Cl_{o,X}, \) is not invertible (otherwise \( D \) will be zero).

Take a \( G \)-equivariant resolution of singularities \( r : W \to X. \) Then the sheaf \( r^* \mathcal{I} \) becomes invertible. Let \( D_{\mathcal{I}} \) be divisor on \( W \) such that \( r^* \mathcal{I} = O_W(D_{\mathcal{I}}). \) Note that \( D_{\mathcal{I}} \) is effective by construction of \( \mathcal{I}. \) Write \( D_{\mathcal{I}} = D' + \Xi, \) where \( D', \Xi \) are effective without common irreducible components, and \( \Xi \) consists of \( r \)-exceptional divisors.

**Lemma 2.8** \( D' \) is not \( r \)-relatively trivial.

**Proof** Indeed, otherwise \( \mathcal{I} \) will be equal to \( \mu_4 O^X(H) \) as in 2.1 by the projection formula, which is impossible as \( D \neq 0 \) in \( Cl_{o,X}. \)

Apply relative \( G \)-equivariant MMP to \( W \) (cf. Shokurov 1992, 9.1). This is a sequence of \( G \)-equivariant birational contractions \( W_i \to W_{i+1} \) over \( X, \) starting
with \( W_0 := W \), which contracts the above divisor \( \Sigma \) and results in a small (no divisorial exceptional part) \( G \)-equivariant contraction \( \phi : Y \rightarrow X \). Furthermore, it follows from Lemma 2.8 that the proper (birational) transform of \( D_{\Sigma} \) on \( Y \) is a \( \phi \)-relatively non-trivial \( G \)-invariant Cartier divisor \( D_Y \), which coincides with the proper transform of \( D' \).

By construction \( \phi(D_Y) \) and \( \mathcal{O}_X(1) \) are non-proportional Weil divisors on \( X \). This implies that \( \text{rk} \mathbb{C}l^G X > 1 \) and completes the proof of Proposition 2.6.

**Remark 2.9** The present definition of \( \mathbb{C}l^G_{o, X} \) differs from the usual (algebraic) one that is via the direct limit of groups \( \mathbb{C}l U / \text{Pic} U \) over all Zariski open subsets \( U \ni o \) in \( X \) (cf. Kollár 2013, Definition 7). A priori there is no natural isomorphism of the latter with \( \mathbb{C}l^G_{o, X} \). At the same time, we have used the fact that \( 0 \neq D \in \mathbb{C}l^G_{o, X} \) in order to construct \( Y \) above, thus proving the existence of some \( G \)-invariant non-Cartier divisor on \( X \).

**Remark 2.10** Let us also comment a bit on the preceding technicalities. The estimate \( \text{rk} \mathbb{C}l^G X > 1 \) will be used, essentially, in the arguments of Sect. 4 below. But we stress that finding a particular \((G\)-invariant\) non-Cartier divisor on \( X \) is not that easy. For instance, in order to find one, we may proceed as follows. Consider the projection \( \pi : X \rightarrow \mathbb{P}^3 \) from either \( o \) or \( o' \), which is a rational map of degree 2, and let \( R \subset X \) be the ramification divisor of \( \pi \). Then one could try to search a non-Cartier irreducible component in \( R \)—for instance the closure \( R_0 \) of the locus on which \( \pi \) is not finite. However, a direct computation shows that \( R \) is reduced, irreducible and \( \dim R_0 \leq 1 \), with \( R \sim \mathcal{O}_X(6) \) being Cartier by construction.

### 3 Auxiliary Results

#### 3.1

Fix \( \phi : Y \rightarrow X \) as in the proof of Proposition 2.6. This is a particular instance of terminal \( G_{\mathbb{Q}} \)-factorial modification (of \( X \)). Namely, in addition to \( \phi \) being \( G \)-equivariant and small, the 3-fold \( Y \) is also terminal and \( G_{\mathbb{Q}} \)-factorial, i.e. every \( G \)-invariant divisor on \( Y \) is \( \mathbb{Q} \)-Cartier.

**Lemma 3.2** \( Y \) is Gorenstein.

**Proof** This follows from the relation \( \phi^* \omega_X = \omega_Y \), the fact that \( \phi \) is small, and the freeness of \( | - K_X | \) for \( -K_X \sim \mathcal{O}_X(1) \).

**Lemma 3.3** One can choose \( Y \) to be such that \( \text{Sing} Y = \emptyset \) or \( G \cdot o' \).

**Proof** Indeed, the divisor \( D \) from 2.3 contains the point \( o \) and the morphism \( \phi \) makes \( X \) \( G_{\mathbb{Q}} \)-factorial near \( o \), which means that one may take \( \phi \) to resolve the singularities in \( G \cdot o \subset D \) (simply run the \( G_{\mathbb{Q}} \)-factorialization procedure as in the proof of Proposition 2.6).

The complement \( \Sigma := \mathcal{O} \backslash G \cdot o \), where \( \mathcal{O} \) is the longest \( S_6 \)-orbit in \( \text{Sing} X \), is also a \( G \)-orbit (of length 10). Furthermore, we have \( s(o) \neq o \in \Sigma \) for \( s := (21) \in S_5 \), and
so the arguments in the proof of Proposition 2.6, with \( s(Q_1) = Q_1 \), apply to show that \( X \) not \( G\mathbb{Q} \)-factorial near \( \Sigma \) as well. Hence we may assume that \( \phi \) also resolves the singularities at \( \Sigma \).

Finally, \( \phi \) either resolves or not the singularities in \( G \cdot o' \), depending on whether there is a \( G \)-invariant non-Cartier divisor passing through \( o' \) or there is no such. \( \square \)

3.4

From now on we will assume that \( Y \) is as in Lemma 3.3.

**Lemma 3.5** \( Y \) is \( \mathbb{Q} \)-factorial with \( \text{rk Pic} \mathcal{O}_Y = 11 \).

**Proof** Note that \( F_5 = \langle h \rangle \) is the unique normal subgroup in \( G = F_5 \rtimes \mathbb{F}_5^* \). Then we have \( Q_i \not\sim h(Q_i) \). Indeed, otherwise \( D \sim 5 \sum_{\gamma \in \langle \tau \rangle} \gamma(Q_i) \), with \( D \) as above. But in this case \( D = 5(Q_1 + \tau^2(Q_1)) \) in \( \text{Cl}_{\mathcal{O}_X} \) (see Lemma 2.5), which is either 0 or 10, thus contradicting Proposition 2.6, (i).

Further, since \( D \) is a \( G \)-orbit of \( Q_1 \), all of its components are linearly independent in \( \text{Cl}_X \otimes \mathbb{R} \). Indeed, otherwise we get \( \sum \gamma(Q_1) = 0 \), which is an absurd. This, together with a computation of the defect in Beauville (2013), yields \( \text{rk Cl}_X = 11 \) for the class group \( \text{Cl}_X \) being generated by \( K_X \) and by the components of \( D \) (the number of these components is 10 because \( Q_1 \not\sim h(Q_1) \)).

Similarly, we find that \( \text{Cl} Y \) is generated by \( K_Y \) and by the components of \( \phi^{-1} D \), all being Cartier according to Lemma 3.3 and the fact that \( D \not\supset o' \). Thus we get \( \text{Cl} Y = \text{Pic} Y \) and the claim follows. \( \square \)

Recall that \( Y \) is terminal and Gorenstein (see Lemma 3.2). This allows for a \( K_Y \)-negative \( G \)-extremal contraction \( \psi : Y \to Z \) (cf. Prokhorov and Shokurov (2009)). For the rest of this section, we will assume that \( \psi \) is birational, with exceptional locus \( E \). Let \( E_i \) be the irreducible components of \( E \). Here are two auxiliary results about these:

**Lemma 3.6** Suppose \( \dim \psi(E) = 0 \). Then \( E \) is a disjoint union of \( E_i \).

**Proof** Since the divisor \( -K_Y \) is nef and big, it follows from Lemmas 3.2, 3.5 and Prokhorov and Shokurov (2009) that the Mori cone \( \overline{NE}(Y) \) is polyhedral, is spanned by extremal rays, and every extremal ray on \( Y \) is contractible. This implies that some family of curves in each \( E_i \) generates an extremal ray because there are no small \( K_Y \)-negative extremal contractions on \( Y \) [see (Kawamata 1988, Lemma 5.1) and Cutkosky (1988)]. In particular, \( E_i \) do not intersect, as \( \dim \psi(E) = 0 \) by assumption. \( \square \)

**Lemma 3.7** Suppose \( \dim \psi(E) = 0 \). Then every surface \( E_i \) is not preserved by the subgroup \( \langle h \rangle \subset G \) (cf. 2.3).

**Proof** Note that \( \text{Cl} X \simeq \text{Cl} Y \) as \( G \)-modules. This induces a natural \( G \)-action on the cone \( \overline{NE}(Y) \). Consider the \( G \)-extremal face of \( \overline{NE}(Y) \) given by \( \psi \). By Lemma 3.6

\( \footnote{\text{“} \phi^{-1} \text{” denotes the proper (birational) transform on} Y \text{of a divisor on} X \text{with respect to} \phi.} \)
this is spanned by a $G$-orbit of some $K_Y$-negative contractible extremal rays $R_i$ corresponding to $E_i$.

Let some $E_j$ be $(h)$-invariant. We have $\phi(E_j) = \mathbb{P}^1 \times \mathbb{P}^1$, a quadratic cone, or $\mathbb{P}^2$ by [Kawamata (1988), Lemma 5.1] and Cutkosky (1988). In particular, there is a projective subspace $\mathbb{P}^3 \subset \mathbb{P}^4 \supset X$ (with $\phi(E_j) \subset X \cap \mathbb{P}^3$), invariant under $\mathbb{F}_5 = \langle h \rangle$. Recall that $h = (23,451)$ permutes $x_0, x_2, x_1, x_3, x_4$. Hence the equation of $\mathbb{P}^3$ is $\sum_{i=0}^{4} x_i = 0$. This implies that $X \cap \mathbb{P}^3 \cap \text{Sing} X = \emptyset$ and so $\phi(E_j)$ is Cartier. But the latter is impossible for otherwise $\phi(E_j)$ would intersect all the curves on $X$ negatively.

We now prove another important result to be used in the proof of Theorem 1.4:

**Proposition 3.8** Let $\psi$ be as above. Then $\psi(E)$ is a curve.

**Proof** Assume the contrary. Then it follows from Lemmas 3.6, 3.7 that all $E_i$ are linearly independent in $\text{Pic } Y \otimes \mathbb{R}$, and together with $K_Y$ they generate $\text{Pic } Y$ (cf. the proof of Lemma 3.5).

Let $R_i \subset NE(Y)$ be the extremal ray corresponding to $E_i$. We have $E_i \cdot C \geq 0$ for all $i$ and any $K_Y$-trivial curve $C \subset Y$ because otherwise the class of $C$ belongs to $R_i$ (which is $K_Y$-negative by construction). In particular, there is such $C$ that any other $K_Y$-trivial curve on $Y$ is numerically equivalent to $C + \sum a_i R_i$ for some $a_i \geq 0$, which forces all $a_i = 0$. This implies that all $K_Y$-trivial curves on $Y$ are numerically proportional and so $E_i \cdot C > 0$.

Hence every surface $\phi(E_i) \subset X \cap \mathbb{P}^3$ (of degree $(K_Y)^2 \cdot E_i \leq 2$) contains a $G$-orbit of length at least 30 (see Lemma 3.3). Thus we obtain that $\phi(E_i)$, together with $E_i$, are all $(h)$- invariant because there are no $G$-invariant curves in $\mathbb{P}^3 \cap S_1 \cap S_2$ for two different surfaces $S_i$ of degree $\leq 2$ containing common $G$-orbit of length 30 (cf. Lemma 3.9 below). However, this $(h)$-invariance of $\phi(E_i)$ contradicts Lemma 3.7, and Proposition 3.8 is proved.

We conclude by the following simple (although useful in what follows) observation:

**Lemma 3.9** $G \not\subset \text{GL}(3, \mathbb{C})$.

**Proof** The group $G$ has only one 4-dimensional and four 1-dimensional irreducible representations. The claim follows by decomposing $\mathbb{C}^3$ into the direct sum of irreducible $G$-modules.

4 **Proof of Theorem 1.4**

4.1

We retain the notation of Sect. 3. Consider some $K_Y$-negative $G$-extremal contraction $\psi : Y \longrightarrow Z$. Let us assume for a moment that $\psi$ is birational with exceptional locus $E$. Recall that $E$ is a union of (generically) ruled surfaces $E_i$ contracted by $\psi$ onto some curves (see Proposition 3.8).

**Lemma 4.2** We have $E \cap \text{Sing } Y = \emptyset$. 
Proof Over generic point of $\psi(E_i)$ morphism $\psi$ coincides with the blow-up of a curve [see Cutkosky (1988)]. Then for any ruling $C \subset E_i$ contracted by $\psi$ we have $K_Y \cdot C = -1$. Hence the surfaces $\phi(E_i) \subset X$ are swept out by the lines $\phi(C)$.

Note that $C$ corresponds to a contractible extremal face of $\overline{NE}(Y)$ (cf. the proof of Lemma 3.6). In particular, one may assume that $C$ generates a $K_Y$-negative extremal ray, which shows that $C$ is Cartier on $E_i$ because all scheme fibers of $\psi|_{E_i}$ are smooth (lines) and $C$ varies in a flat family.

Further, it follows from Lemmas 3.2, 3.5 and (Kawamata 1988, Lemma 5.1) that all divisors $E_i$ are Cartier. Now, if $E_i \cap \operatorname{Sing} Y \neq \emptyset$, then $\phi(C)$ is a singular curve for some $C$ as above, which is impossible. Hence $E_i \cap \operatorname{Sing} Y = \emptyset$ and so $E \cap \operatorname{Sing} Y = \emptyset$.

\[\square\]

Remark 4.3 We have $h^{1,2} = 0$ for a resolution of $Y$ according to Remark in Beauville (2013). Then it follows from Cutkosky (1988) and Lemma 4.2 that $\psi(E_i) = \mathbb{P}^1$ for all $i$.

Lemma 4.4 We have $K_Y = \psi^* K_Z + E$ (hence $Z$ is Gorenstein), $K_Y \cdot C = -1$ for any ruling $C \subset E_i$ contracted by $\psi$, and $Z$ is smooth near $\psi(E)$.

Proof One obtains the first two identities by exactly the same argument as in the proof of Lemma 4.2. Further, since the linear system $|-K_Y| = |\phi^*(-K_X)|$ is basepoint-free, generic surface $S \in |-K_Y|$ passing through a given point from $Y \setminus \operatorname{Sing} Y$ is smooth. Then, for $S \cdot C = 1$ we find that the surface $\psi(S) \in |-K_Z|$ is smooth as well, hence $Z$ is smooth near $\psi(E)$.

\[\square\]

Lemma 4.5 $E$ can not consist of only one (connected) surface.

Proof Assume the contrary. Note that $Y$ contains the $G$-orbit of 20 curves $C_j$ contracted by $\phi$ (see Lemma 3.3). In particular, $G$ induces a non-trivial action on the set of these $C_j$, which implies that $(E = E_i) \cap C_j \neq \emptyset$ for all $j$. This yields a faithful $G$-action on the base of the ruled surface $E$. Hence we get $G \subset \operatorname{PGL}(2, \mathbb{C})$ (see Remark 4.3). On the other hand, we have $G \not\subset A_5$ (see Lemma 3.9), a contradiction.

\[\square\]

Here is a refinement of Lemma 4.5:

Lemma 4.6 $E$ is a disjoint union of $G$-orbits, length $\geq 2$, corresponding to extremal faces of $\overline{NE}(Y)$.

Proof Let $E, \tilde{E}$ be two $\psi$-exceptional orbits in question. Choose some connected components $E_j \subset E, \tilde{E}_j \subset \tilde{E}$ and suppose they intersect. Both $E_j, \tilde{E}_j$ are ruled surfaces contracted by the blow-downs, one for each surface (cf. the proof of Lemma 4.2).

Let $\psi_j : Y \longrightarrow Y_j$ be the contraction of $E_j$. Then, given that $E_j \cap \tilde{E}_j \neq \emptyset$, there is a $\psi$-exceptional curve $C \subset \tilde{E}_j$ such that $E_j \cdot C \geq 0$. On the other hand, we have $K_Y = \psi_j^* K_{Y_j} + E_j$ and $K_{Y_j} \cdot \psi_j(C) = -1$ (see Lemma 4.4), which gives either $K_Y \cdot C = -1$ or $K_Y \cdot C = 0$ (recall that $K_Y$ is nef). The latter case is an absurd by construction of $\psi$. In the former case, we get $E_j \cdot C = 0$ and so $\psi_*(E_j \cap \tilde{E}_j) = \psi_* C = 0$, which is impossible for the ruled surfaces $E_j \neq \tilde{E}_j$, since then $0 = E_i \cdot C = (C^2) < 0$ on $E_i$, a contradiction.

\[\square\]
We collect the preceding results into the following:

**Proposition 4.7** Let $\psi$ be the result of running a $G$-MMP on $Y$. Then $\psi$ is a birational contraction that maps its exceptional loci onto 1-dimensional centers and all the intermediate 3-folds are smooth near these centers. In particular, all these 3-folds are $\mathbb{Q}$-factorial, Gorenstein and terminal, with nef and big $-K$, and $\psi$ is composed of blow-ups at smooth rational curves.

**Proof** It follows from Lemmas 3.2, 3.5, 4.2, 4.4 and (Prokhorov 2005, Proposition-definition 4.5, Corollary 4.9) that each step of $\psi$ produces a $\mathbb{Q}$-factorial, Gorenstein terminal 3-fold, with a $G$-action and nef and big $-K$, unless all exceptional $E_i$ have anticanonical degree $\leq 2$ on this step. In the latter case, arguing as in the proof of Lemma 4.6 one computes that the proper transforms of $E_i$ on $Y$, hence on $X$ as well, will also have degree $\leq 2$. This yields an $(h)$-invariant quadric on $X$ (cf. the proof of Proposition 3.8) and a contradiction with Lemma 3.9.

Similarly, whenever $E_i$ is a quadric or $\mathbb{P}^2$, contracted to a point in both cases (cf. Prokhorov 2005, Proposition-definition 4.5, Corollary 4.9), we get contradiction with Lemma 3.9. Thus on each step $\psi$ can contract $E_i$ only to curves. The final assertion of Proposition 4.7 follows from Cutkosky (1988) and Remark 4.3. $\square$

**4.8**

Let us treat the intermediate case when $E = \emptyset$ (i.e. $\phi$ is non-birational). Recall that $Y$ contains the $G$-orbit of 20 curves $C_j$ contracted by $\phi$ (see Lemma 3.3). Then we get $\text{rk Pic}^G Y = 2$ and $\overline{NE}(Y)$ is generated by (the $G$-orbits of) the classes of $C_j$ and an extremal ray corresponding to some $G$-Mori fibration $\varphi : Y \longrightarrow S$. Note that $\dim S > 0$ by construction.

**Lemma 4.9** Let $\dim S = 1$. Then $Y$ is minimal over $S$ unless it is rational.

**Proof** Suppose there is a surface $\Xi$ which is exceptional for some $K_Y$-negative extremal contraction on $Y/S$. Then $\Xi$ necessarily contains one of $C_j$. Indeed, otherwise $\Xi$ intersects all curves on $Y$ non-negatively by the structure of $\overline{NE}(Y)$, which is impossible. In particular, we find that $\Xi$ must be a minimal ruled surface (same argument as in the proof of Lemma 4.2), with the negative section equal some $C_j$.

We may assume $K_{Y_\eta}^2 \leq 4$ for generic fiber $Y_\eta$ of $\varphi$—otherwise $Y$ is rational (see Graber et al. (2003), Manin (1974)). Moreover, we have $K_{Y_\eta}^2 \neq 1$, since otherwise the group $G \subseteq \text{Aut}(Y_\eta)$ must act faithfully on elliptic curves from $| - K_\eta|$, which is impossible by Lemma 3.9. One also has $K_{Y_\eta}^2 \neq 2$ because the order of the group of automorphisms of del Pezzo surfaces of degree 2 is not divisible by 5 (see e.g. Dolgachev 2012, Table 8.9). Further, if $K_{Y_\eta}^2 = 4$, then contracting $\Xi$ we arrive at a del Pezzo fibration of degree 5, so that $Y$ is rational.

Now, if $K_{Y_\eta}^2 = 3$, then all smooth fibers of $\varphi$ are isomorphic and have $\text{Aut} Y_\eta = S_5$ (see Dolgachev 2012, Table 9.6). Away from the singular fibers $\varphi$ defines a locally trivial (in analytic topology) fibration of smooth cubic surfaces $Y_\eta$. Two charts, $Y_\eta \times S'$
and $Y_{\eta} \times S''$, say (for some analytic subsets $S', S'' \subseteq S$), are glued together via an automorphism $t \in \text{Aut} Y_{\eta}$, which preserves the elements in the $G$-orbit of $\Xi$ and satisfies $tGt^{-1} = G$. Since $G$ is not a normal subgroup in $S\delta$, one gets $t \in G$, and the latter is impossible once $t \neq 1$—by the way $G$ acts on $\Xi$ and $C_j$. Thus we have $t = 1$ and $\varphi$ induces a locally trivial fibration in the Zariski topology, so that $Y$ is rational, and the proof is complete.

Let $\dim S = 1$ and observe that the subgroup $\langle h \rangle \subset G$ must act faithfully on Pic $Y$ in this case. Indeed, otherwise $Q_i \sim h^a(Q_i)$ for all $a, i$, which implies that $Q_i$ contains the orbit $\langle h \rangle \cdot o$, a contradiction (cf. 2.3). Further, from Lemma 4.9 we deduce that either Pic $Y = \mathbb{Z}$ (which contradicts Lemma 3.5), or $\varphi$ contains a fiber with $\geq 5$ irreducible components (interchanged by $\langle h \rangle$). In the latter case, we get $K_{Y_{\eta}}^2 \geq 5$ for generic fiber $Y_{\eta}$, and irrationality of $Y$ follows from Graber et al. (2003), Manin (1974).

Finally, one treats the case when $\varphi$ is a $G$-conic bundle exactly as in the proof of Lemma 4.12 below, which yields rationality of $Y$ (and $X$) whenever $E = \emptyset$.

### 4.10

According to 4.8 we may assume from now on that $E \neq \emptyset$ and $\psi : Y \rightarrow Z$ is as in Proposition 4.7. Then the 3-fold $Z$ is $\mathbb{Q}$-factorial, Gorenstein and terminal, with nef and big $-K_Z$. Furthermore, $Z$ is either a $G$-equivariant del Pezzo fibration, a $G$-conic bundle or a $G\mathbb{Q}$-Fano 3-fold.

**Lemma 4.11** We have $\phi_{\ast}^{-1}Q_j \not\subset E$ for some $j$.

**Proof** Note that $\psi_{\ast}K_Y = K_Z$ because $Z$ has rational singularities. This gives the claim as $-K_Y = \phi_{\ast}^{-1}Q_1 + \phi_{\ast}^{-1}Q_2$. □

**Lemma 4.12** $Z$ is not a $G$-conic bundle.

**Proof** Suppose we are given a $G$-conic bundle structure on $Z$ with generic fiber $C = \mathbb{P}^1$. Then, if $\phi_{\ast}^{-1}Q_1 \not\subset E$, say (see Lemma 4.11), it follows from the definition of $Q_i$ and $G$ in 2.3 that the $G$-orbit of $Q_1$ (hence also of $\phi_{\ast}^{-1}Q_1$) has length $\geq 10$ (cf. the proof of Lemma 3.5). This yields a faithful $G$-action on $C$ which in turn contradicts Lemma 3.9. □

**Lemma 4.13** $Z$ is not a $G$-del Pezzo fibration unless $Z$ is rational.

**Proof** Argue exactly as in the proof of Lemma 4.9. □

### 4.14

According to Lemmas 4.12 and 4.13 we may assume from now on that $Z$ is a $G\mathbb{Q}$-Fano 3-fold. Note that any two components of the exceptional locus $E$ of $\psi$ can intersect only along the fibers (cf. the proof of Lemma 4.6). Also recall that $\text{rk} \text{Pic} Y = 11$ by Lemma 3.5 and the subgroup $\langle h \rangle \subset G$ acts faithfully on Pic $Y$. Then it follows from Remark 4.3 and Lemmas 4.4, 4.6 that either

$$-K_Z^3 = 4 + 2(-K_Z \cdot \mathbb{P}^1 + 1)$$ (4.15)
for some even $k \leq 10$ when $E_i \subset E$ do not intersect, or

$$-K_3^Z = 4 + 20(-K_Z \cdot \mathbb{P}^1 + 1 - k')$$  \hspace{1cm} \text{(4.16)}$$

for some $k' \leq -K_Z \cdot \mathbb{P}^1$ when some $E_i \subset E$ intersect.

**Lemma 4.17** The linear system $|-K_Z|$ is basepoint-free.

**Proof** Assume the contrary. Then it follows from Jahnke and Radloff (2006) that $Z$ is a $G$-equivariant double cover of the cone over a ruled surface (note that according to (4.15) and (4.16) $-K_3^Z \geq 12$ is divisible by 4). This easily gives $G \subset \text{PGL}(2, \mathbb{C})$ and contradiction with Lemma 3.9. $\square$

**Lemma 4.18** The morphism defined by $|-K_Z|$ is an embedding.

**Proof** Assume the contrary. Then it follows from (Cheltsov et al. 2005, Theorem 1.5) that $Z$ is a $G$-equivariant double cover of either a rational scroll or the cone over a ruled surface. In both cases, arguing similarly as in the proof of Lemma 4.17, one gets contradiction. $\square$

Lemmas 4.17 and 4.18 allow one to identify $Z$ with its anticanonical model $Z_{2g-2} \subset \mathbb{P}^{g+1}$ (here $g := -K_3^Z/2 + 1$ is the genus of $Z$).

**Lemma 4.19** There are no $G$-fixed points and $G$-invariant smooth rational curves on $Z$.

**Proof** Firstly, since $G \not\subset \text{GL}(3, \mathbb{C})$, the group $G$ acts on $Z$ without smooth fixed points. Also, since $Z$ is $G$-isomorphic to $X$ near $\text{Sing} Z$ by construction, we obtain that $G$ does not have fixed points on $Z$ at all.

Further, if $\mathbb{P}^1 \subset Z$ is $G$-invariant, then the action $G \circlearrowright \mathbb{P}^1$ is cyclic, which gives a $G$-fixed point on $\mathbb{P}^1$, a contradiction. $\square$

**Lemma 4.20** $Z$ is singular unless it is rational.

**Proof** Suppose that $Z$ is smooth. Then rationality of $Z$ follows from the fact that $h^{1,2}(Z) = 0$ (see Remark 4.3) and [Iskovskikh and Prokhorov 1999, §§12.2–12.6]. $\square$

According to Lemmas 4.20, 3.3 and Proposition 4.7 we may reduce to the case when $|\text{Sing} Z| = |\text{Sing} Y| = 10$, with the locus $\text{Sing} Z$ being some $G$-orbit.

**Lemma 4.21** Let $S \in |-K_Z|$ be a $G$-invariant hyperplane section such that $S \cap \text{Sing} Z \neq \emptyset$. Then the pair $(Z, S)$ is plt.

**Proof** Lemma 4.19 and the proof of (Prokhorov 2012, Lemma 4.6) show that the pair $(Z, S)$ is log canonical. Moreover, if $(Z, S)$ is not plt, then the same argument as in loc.cit reduces the claim to the case when $S$ is a ruled surface over an elliptic curve, say $B$. On the other hand, since $|S \cap \text{Sing} Z| = 10$, we get either $G \subset \text{PGL}(2, \mathbb{C})$ or a faithful $G$-action on $B$, a contradiction. $\square$

We are ready to prove the following:
Proposition 4.22  $g \leq 9$.

Proof  Let $g > 9$. Note that the linear span of any $G$-orbit in $\text{Sing } Z$ has dimension $\leq 9$. Hence we can consider a $G$-invariant hyperplane section $S \in | -K_Z|$ such that $S \cap \text{Sing } Z \neq \emptyset$.

It follows from Lemma 4.21 and (Shokurov 1992, Corollary 3.8) that $S$ is either normal or reducible. But in the latter case, $-K_Z \sim [\text{some disconnected surface}]$ because $(Z, S)$ is plt, which is impossible.

Thus the surface $S$ is normal with at most canonical singularities. Let us identify $S$ with its $G$-equivariant minimal resolution. In particular, we may assume that $S$ contains a $G$-invariant collection of disjoint $(-2)$-curves $C_i$, $1 \leq i \leq 10$.

It follows from Lemma 4.21 and (Shokurov 1992, Corollary 3.8) that $S$ is either normal or reducible. But in the latter case, $-K_Z \sim [\text{some disconnected surface}]$ because $(Z, S)$ is plt, which is impossible.

Thus the surface $S$ is normal with at most canonical singularities. Let us identify $S$ with its $G$-equivariant minimal resolution. In particular, we may assume that $S$ contains a $G$-invariant collection of disjoint $(-2)$-curves $C_i$, $1 \leq i \leq 10$.

From $G \subseteq \text{Aut } S$ one obtains a $G$-action on the space $H^{2,0}(S) = \mathbb{C}[\omega_S]$. In particular, the subgroup $\langle \tau^2 \rangle \subset G$ preserves the 2-form $\omega_S$, which implies that the quotient $S/\langle \tau^2 \rangle$ has at most canonical singularities. Note also that $\tau^2(C_i) = C_i$ and $h^1(C_i) \neq C_i$ for all $i$.

Let $\tilde{C}_i$ be the image of $C_i$ on $S_{\tau}$. We have $|\tilde{C}_i \cap \text{Sing } S_{\tau}| = 2$ for all $i$ because $(\tilde{C}_i^2) = -1$ by the projection formula. Then for the minimal resolution $S'_{\tau}$ of $S_{\tau}$ we obtain that $S'_{\tau}$ contains $\geq 20$ disjoint $(-2)$-curves. This contradicts $h^{1,1}(S'_{\tau}) = 20$ and finishes the proof of Proposition 4.22. \qed

According to Proposition 4.22 and (4.15), (4.16) we may assume that $-K^3_Z \in \{12, 16\}$. (Note that the case $k = 10$ yields $\text{rk } \text{Pic } Z = 1$ and can be excluded exactly as in the proof of Proposition 4.27 below.)

Remark 4.23  Suppose $Z = Z_{16} \subset \mathbb{P}^{10}$. Then, since the projective $G$-action is induced from the linear one on $\mathbb{C}^{11} = H^0(Z, -K_Z)$, one gets a pencil on $Z$ consisting of $G$-invariant hyperplane sections. In particular, there is such $S$ intersecting $\text{Sing } Z$, so that the arguments in the proof of Proposition 4.22 apply to exclude the case $-K^3_Z = 16$.

4.24

It follows from Lemma 4.7 and Namikawa (1997) that there is a 1-parameter family $s : Z \rightarrow \Delta$ over a small disk $\Delta \subset \mathbb{C}$ of smooth Fano 3-folds $Z_t$, $t \neq 0$, deforming to $Z_0 = Z$. Since $H^i(Z_t, nK_{Z_t}) = 0$ for all $n \leq 0$, $i > 1$ and $t$, we deduce that the sheaf $s_*(\varepsilon) = (-K_Z)$ is locally free.

Further, the cone $\overline{\text{NE}}(Z)$ is polyhedral, with contractible extremal rays (cf. the proof of Lemma 3.6). Let $H$ be a nef divisor on $Z$ that determines one of these contractions. Then Jahnke and Radloff (2011) and (Lazarsfeld 2004, Proposition 1.4.13) imply that $H$ varies in the family $H_t$ of nef divisors on $Z_t$.

Proposition 4.25  $\text{rk } \text{Pic } Z \neq 2$.

Proof  Assume the contrary. It follows from the condition $\text{rk } \text{Pic }^G Z = 1$ that both of the extremal contractions on each $Z_t$ above must be either birational or Mori fibrations. Now [Iskovskikh and Prokhorov 1999, §12.3] and Remark 4.23 show that $Z$ can only be a divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, 2)$. Let us show that $Z$ is smooth (this will contradict $|\text{Sing } Z| = 10$ and prove Proposition 4.25).
Let \( x_i \) (resp. \( y_j \)) be coordinates on the first (resp. second) factor of \( \mathbb{P}^2 \times \mathbb{P}^2 \). Let also \( f(x, y) = 0 \) be the equation of \( Z \) (so that it defines a conic in \( \mathbb{P}^2 \) whenever \( x := [x_0 : x_1 : x_2] \) or \( y \) is fixed).

Note that projections to the \( \mathbb{P}^2 \)-factors induce conic bundle structures on \( Z \). These are interchanged by \( G \) (because of \( \text{rk} \text{Pic}^G Z = 1 \)) and are \( \langle h, \tau^2 \rangle \)-invariant. One may assume that \( \text{Sing} Z \) belongs to the affine chart \( x_0 = y_0 = 1 \) on \( \mathbb{P}^2 \times \mathbb{P}^2 \). Then, after a coordinate change, we obtain that \( f(x, y) = x_1 x_2 y_1 y_2 + x_1 x_2 + y_1 y_2 + 1 \) in this chart, with \( h \) acting diagonally on \( x_i \) and \( y_j \).

Now, differentiating \( f(x, y) \) by \( x_1, x_2 \) we get \( x_i = -y_1 y_2 \), and similarly \( y_i = -x_1 x_2 \). This gives \( x_1 = x_2, y_1 = y_2 \in \{-1, -w\} \), which contradicts \( f(x, y) = 0 \). \( \square \)

Note that there is a \( G \)-invariant surface \( S \in |-K_Z| \), since \( \mathbb{P}^8 = \mathbb{P}(\mathbb{C}^9) \supset Z \), similarly as in Remark 4.23.

**Lemma 4.26** The pair \((Z, S)\) is plt.

**Proof** As in the proof of Lemma 4.21, it suffices to exclude the case when (the normalization of) the surface \( S \) is ruled, over some base curve \( B \) of genus \( \leq 1 \).

Note that any line \( L \) passing through two points from \( \text{Sing} Z \) is contained in \( Z \) (as \( Z \) is an intersection of quadrics). In particular, we have \( S \cdot L > 0 \) for \( > 10 \) of such \( L \), which yields either \( G \subset \text{PGL}(2, \mathbb{C}) \) or a faithful \( G \)-action on \( B \), a contradiction. \( \square \)

**Proposition 4.27** \( \text{rk Pic} Z \neq 1 \).

**Proof** Assume the contrary. Then \( Z_t \subset \mathbb{P}^8 \) from the beginning of 4.24 are Fano 3-folds of the principal series.

It follows from Lemma 4.26 that \( S \) is normal and connected. Further, we have \( k \leq 2 \) and \( -K_Z : \mathbb{P}^1 \leq 2 \) in (4.15), which means (cf. Lemma 4.19) that the exceptional locus of \( \psi : Y \to Z \) consists of two disjoint surfaces, say \( E_1, E_2 \), so that \( L_i := \psi(E_i) \) are two lines on \( Z \). In particular, there is a \( G \)-invariant subspace \( \mathbb{P}^3 \subset \mathbb{P}^8 \), with \( Z \cap \mathbb{P}^3 = L_1 \cup L_2 \), such that \( X \) is obtained from \( Z \) via the linear projection from \( \mathbb{P}^3 \) (recall that both \( X \) and \( Z \) are anticanonically embedded).

We may assume that \( Z \cap \mathbb{P}^3 \subset S \) (otherwise there is a pencil as in Remark 4.23). Hence \( S \) contains the \((-2)\)-curve \( L_1 \) (we have identified \( S \) with its minimal resolution). Note that \( L_1 \) is preserved by the group \( \langle h \rangle \).

Consider the quotient \( S_h := S/\langle h \rangle \). Then the image of \( L_1 \) on \( S_h \) has self-intersection \( = -2/5 \) by the projection formula. On the other hand, this self-intersection \( \in \mathbb{Z}[0.5] \) (for \( S_h \) has at most canonical singularities due to \( h^* (\omega_S) = \omega_{S_h} \)), a contradiction.

Proposition 4.27 is completely proved. \( \square \)

It follows from Propositions 4.25 and 4.27 that \( \text{rk Pic} Z > 2 \). Now, since \( -K_Z^3 = 12 \), from Namikawa (1997), Jahnke and Radloff (2011) and [Iskovskikh and Prokhorov 1999, §§12.4–12.6] we obtain that \( Z \) is a deformation of \( Z_r = \mathbb{P}^1 \times [\text{del Pezzo surface of degree} 2] \) or a double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \), ramified along a divisor of tridegree \((2, 2, 2)\). In both cases, \( Z \) is hyperelliptic (cf. the beginning of the proof of Proposition 4.25), which contradicts Lemma 4.18.

The proof of Theorem 1.4 is finished.
5 Concluding Discussion

5.1

Equations (1.2) and the results of Cynk (2001) show that any $S_6$-invariant quartic $X_t$ is not $\mathbb{Q}$-factorial. In turn, as we saw in Sect. 3, it is indispensable to compute the group $\text{Cl} X_t = H_4(X_t, \mathbb{Z})$ (e.g. for the arguments of Sect. 4 to carry on).

This amazing interrelation between topology and (birational) geometry of $X_t$ provides one with a hint for studying the birational type of $X_t$ by “topological” means. In this regard, let us give a sketch of an argument, showing that $X_t$ is unirational for generic $t \in \mathbb{R}$, hence for (again generic) $t \in \mathbb{C}$ (cf. de Fernex and Fusi 2013, Proposition 2.3).

Namely, differentiating (1.2) one interprets this system of equations as the graph of a Morse function $F : \mathbb{R}P^4 \rightarrow \mathbb{R}$, so that $X_t^{\mathbb{R}} = F^{-1}(t)$ are smooth level sets for $t \notin \{\infty, 0, 10/7, 2, 4, 6\}$, while the rest of $t \notin \{0, 4\}$ correspond to critical level sets of (maximal) index 3 (here $X_t^{\mathbb{R}}$ denotes the real locus of $X_t$).

We may replace $\mathbb{R}P^4$ with its universal cover $\mathbb{S}^4$. Then $F$ lifts to a Morse function on $\mathbb{S}^4$ and thus all smooth $X_t^{\mathbb{R}}$ are homotopy $\mathbb{R}P^3$. Actually generic $X_t^{\mathbb{R}}$ is diffeomorphic to $\mathbb{R}P^3$ (note that this $X_t^{\mathbb{R}}$ is smooth and connected).

Further, $X_t^{\mathbb{R}}$ is contained in an affine space $\mathbb{R}^N$, some $N$, because $\sum x_i^4 \neq 0$ over $\mathbb{R}$. Then the function $F_p := \text{dist}(\cdot, p)$ defines a Morse function on $X_t^{\mathbb{R}}$ for very general points $p \in \mathbb{R}^N$. (Here $\text{dist}(x, y) := \|x - y\|^2$ is the standard Euclidean distance.)

The layers of $F_p$ yield a vector field on $X_t^{\mathbb{R}}$, which is non-degenerate and normal to these layers outside two points, where this field vanishes. We thus obtain a (Hopf) fibration on $X_t^{\mathbb{R}}$ with a section $F_p^{-1}(o) \setminus \{2 \text{ points } o_1, o_2\} = \mathbb{R}P^2$ such that $F_p^{-1}(o) \subset X_t^{\mathbb{R}}$ as an algebraic subset. It remains to apply a diffeomorphism over $F_p^{-1}(o) \setminus \{o_1, o_2\}$ which makes $X_t^{\mathbb{R}} \setminus \{F_p^{-1}(o_1), F_p^{-1}(o_2)\} = \mathbb{R}P^1 \times F_p^{-1}(o) \setminus \{o_1, o_2\}$ as algebraic varieties.

The upshot of the above discussion is that $X_t^{\mathbb{R}}$ (hence $X_t$) admits many cancellations in the sense of Bogomolov et al. (2013). This implies that $X_t$ is unirational.

5.2

We conclude with the following questions:

- What is the Fano 3-fold which the quartic $X_6$ is $G$-birationally isomorphic to (cf. Sect. 4)?
- Are there non-trivial $G$-birational modifications of $X_6$ for other subgroups $G \subset S_6$?
- Is $X_t$ unirational over a number field field?\(^4\)
- Does the set of $\mathbb{Q}$-points on $X_t$ satisfy the potential density property?
- Does $X_t$ carry a pencil of ( birationally) Abelian surfaces?\(^5\)

\(^4\) Note that all rational quartics are $\mathbb{Q}$-rational.
\(^5\) Again this holds for rational $X_t$. 
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References

Beauville, A.: Non-rationality of the $S_6$-symmetric quartic threefolds. Rend. Sem. Mat. Univ. Politec. Torino 71(3–4), 385–388 (2013)

Bogomolov, F., Karzhevanov, I., Kuyumzhiyan, K.: Unirationality and existence of infinitely transitive models. In: Bogomolov, F., Brendan, H., Tschinkel, Y. (eds.) Birational Geometry, Rational Curves, and Arithmetic. Simons Symposia. Springer, New York, pp. 77–92 (2013)

Cheltsov, I., Przhiyalkovski, V., Shramov, C.: Hyperelliptic and trigonal Fano threefolds. Izv. Ross. Akad. Nauk Ser. Mat. 69(2), 145–204 (2005), translation in Izv. Math. 69 (2005), no. 2, 365 – 421

Cheltsov, I., Shramov, C.: Five embeddings of one simple group. Trans. Am. Math. Soc. 366(3), 1289–1331 (2014)

Cutkosky, S.: Elementary contractions of Gorenstein threefolds. Math. Ann. 280(3), 521–525 (1988)

Cynk, S.: Defect of a nodal hypersurface. Manuscr. Math. 104(3), 325–331 (2001)

Dolgachev, I.V.: Classical Algebraic Geometry. Cambridge University Press, Cambridge (2012)

de Fernex, T., Fusi, D.: Rationality in families of threefolds. Rend. Circ. Mat. Palermo (2) 62(1), 127–135 (2013)

Graber, T., Harris, J., Starr, J.: Families of rationally connected varieties. J. Am. Math. Soc. 16(1), 57–67 (2003). (electronic)

Hunt, B.: The Geometry of Some Special Arithmetic Quotients. Lecture Notes in Mathematics, vol. 1637. Springer, Berlin (1996)

Iskovskikh, V.A., Prokhorov, Y.G.: Fano varieties. In: Algebraic geometry, V, Encyclopaedia Math. Sci., 47. Springer, Berlin, pp. 1–247 (1999)

Jahnke, P., Radloff, I.: Gorenstein Fano threefolds with base points in the anticanonical system. Compos. Math. 142(2), 422–432 (2006)

Jahnke, P., Radloff, I.: Terminal Fano threefolds and their smoothings. Math. Z. 269(3–4), 1129–1136 (2011)

Kaloghiros, A.-S.: A classification of terminal quartic 3-folds and applications to rationality questions. Math. Ann. 354(1), 263–296 (2012)

Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. Math. (2) 127(1), 93–163 (1988)

Kollár, J.: Grothendieck-Lefschetz type theorems for the local Picard group. J. Ramanujan Math. Soc. 28A, 267–285 (2013)

Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties, Translated from the 1998 Japanese Original, Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998)

Lazarsfeld, R.: Positivity in Algebraic Geometry. I, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, Springer, Berlin (2004)

Manin, Y.I.: Cubic Forms: Algebra, Geometry, Arithmetic, Translated from the Russian by M. Hazewinkel, North-Holland Amsterdam (1974)

Namikawa, Y.: Smoothing Fano 3-folds. J. Algebraic Geom. 6(2), 307–324 (1997)

Prokhorov, Y. G.: Fields of invariants of finite linear groups. In: Cohomological and Geometric Approaches to Rationality Problems, Progr. Math., 282, Birkhäuser Boston, Boston, pp. 245–273 (2010)

Prokhorov, YuG: Simple finite subgroups of the Cremona group of rank 3. J. Algebraic Geom. 21(3), 563–600 (2012)

Prokhorov, YuG: The degree of Fano threefolds with canonical Gorenstein singularities. Mat. Sb. 196(1), 81–122 (2005), translation in Sb. Math. 196 (2005), no. 1-2, 77 – 114

Prokhorov, YuG, Shokurov, V.V.: Towards the second main theorem on complements. J. Algebraic Geom. 18(1), 151–199 (2009)
Shokurov, V.V.: Three-dimensional log perestroikas. Izv. Ross. Akad. Nauk Ser. Mat. 56(1), 105–203 (1992). translation in Russian Acad. Sci. Izv. Math. 40 (1993), no. 1, 95 – 202

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